Hausdorff operators on compact abelian groups

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Abstract
Necessary and sufficient conditions are given for the boundedness of Hausdorff operators on the generalized Hardy spaces $H^p_{E}(G)$, real Hardy space $H^1_{R}(G)$, BMO($G$), and BMOA($G$) for compact Abelian group $G$. Surprisingly, these conditions turned out to be the same for all groups and spaces under consideration. Applications to Dirichlet series are given. The case of the space of continuous functions on $G$ and examples are also considered.

KEYWORDS
BMO, compact Abelian group, Dirichlet series, Hausdorff operator, Hardy space

MSC (2020)
47B90, 47B38, 30B50

1 | INTRODUCTION

Hausdorff operators are closely connected with classical harmonic analysis (see, e.g., [3, 5, 12, 18], or [33]). The modern stage in the development of this theory begins with the work by Liflyand and Mricz [19]. (See also [15].) The concept of a Hausdorff operator in the general framework of topological groups was introduced by the author in [25] as a generalization of the classical definition in Euclidean spaces [4, 16] and the definition in $p$-adic spaces [34] (see Definition 3.1).

In [25], sufficient conditions were given for the boundedness of a Hausdorff operator on the atomic (real) Hardy space over a locally compact metrizable group satisfying the so-called doubling property. Generalizations to homogeneous spaces of Lie groups appeared in [28]. The case of locally compact groups with local doubling property and their homogeneous spaces was considered in [27].

But there are compact connected Abelian groups that are not metrizable (e.g., the Bohr compactification of reals $b\mathbb{R}$ (the Bohr compactum)), or metrizable but without local doubling property (e.g., the infinite-dimensional torus $\mathbb{T}^\infty$). The aim of this work is to give necessary and sufficient conditions for the boundedness of Hausdorff operators on Hardy spaces and BMO for this case, as well. Surprisingly, these conditions turned out to be the same for all groups and spaces under consideration. The case of the space of continuous functions and examples that may be of interest in their own way are also considered. It should be noted that Hausdorff operators on Hardy spaces $H^1$ and BMO over Euclidean spaces $\mathbb{R}^d$ were studied in [16].

Due to the well-known identification of Bohr, a lot of theory of ordinary Dirichlet series may be seen a sub-theory of Fourier analysis on the infinite-dimensional torus. This observation and especially the seminal results of Hedenmalm, Lindqvist, and Seip [13] (see also [8, Corollary 5.3] or [30, Proposition 6.5.3, p. 159]) and Bayart [1] gave us an opportunity for an application of our results to ordinary Dirichlet series. In the case of general Dirichlet series, we use results obtained
by Defant and Schoolmann in [7, 32], and [6]. Based on these results, classes of bounded Hausdorff operators that act in some natural spaces of Dirichlet series (ordinary and general) are introduced.

2 | PRELIMINARIES

This section collects all preliminary information we need in the next parts of the paper.

In the following, unless otherwise stated, $G$ stands for a compact and connected Abelian group with normalized Haar measure $\nu$, and a total order (which agrees with the group structure) is fixed on its dual group $\hat{X}$. In turn, $X$ is the dual group for $G$ by the Pontryagin–van Kampen theorem. Let $X_+ := \{ \chi \in X : \chi \geq 1 \}$ be the positive cone in $X$ (1 denotes the unit character). In other words, $X_+$ is a subsemigroup of $X$ such that $X_+^{-1} \cup X_+ = X$, and $X_+^{-1} \cap X_+ = \{ 1 \}$ (see, e.g., [31, Chapter 8]). We put also $X_- := X \setminus X_+$. Then, $X_- = X_+^{-1} \setminus \{ 1 \}$.

As is well known, a (discrete) Abelian group $X$ can be totally ordered if and only if it is torsion-free, which in turn is equivalent to the condition that its character group $\hat{X}$ is connected. In general, the group $X$ may possess many different total orderings.

In applications, frequently $X$ is a dense subgroup of $\mathbb{R}^d$ endowed with the discrete topology and $G = bX$ is its Bohr compactification, or $X = \mathbb{Z}^d$ so that $\hat{X} = \mathbb{T}^d$ is the $d$-torus ($\mathbb{T}$ is the circle group and $\mathbb{Z}$ is the group of integers). Other interesting examples are the infinite-dimensional torus $\mathbb{T}^\infty$ (see Section 6 and Examples 1 and 6), an $a$-adic solenoids $\Sigma_a$ (see Example 7), and their finite and countable products (see, e.g., [7]).

We denote by $\text{Aut}(H)$ the group of topological automorphisms of a topological group $H$ endowed with its natural topology (see, e.g., [14]). If $H$ is Abelian and $A \in \text{Aut}(H)$, the dual automorphism $A^* \in \text{Aut}(\hat{H})$ is defined by the rule

$$ A^*(\xi) := \xi \circ A, \quad \xi \in \hat{H} $$

(here $\hat{H}$ denotes the dual group; see, e.g., [14, (24.37), (24.41)]).

In the following, $\text{Aut}_+(X)$ stands for the subset of $\text{Aut}(X)$ consisting of all ordered automorphisms of the group $X$ with respect to the given order. By definition, these automorphisms preserve the order (equivalently, these automorphisms map the positive cone $X_+$ into itself).

We also denote by $\text{Aut}(G)^+$ the set of such $A \in \text{Aut}(G)$ that $A^+ \in \text{Aut}_+(X)$.

The following simple lemma will be useful.

Lemma 2.1.

1) Let $G$ be a locally compact Abelian group. If $A \in \text{Aut}(G)$ then $(A^+)^{-1} = (A^{-1})^*$.

2) Let $G$ be a compact and connected Abelian group. Then, $\text{Aut}_+(X)$ is a subgroup of $\text{Aut}(X)$.

3) Let $G$ be a compact and connected Abelian group. Then, $\text{Aut}(G)^+$ is a subgroup of $\text{Aut}(G)$.

Proof. The map $A \mapsto A^+$ is a topological anti-isomorphism of $\text{Aut}(G)$ onto $\text{Aut}(X)$ [14, Theorem (26.9)]. It follows that $(A^{-1})^+ = (A^+)^{-1}$ for $A \in \text{Aut}(G)$.

2) We will show that if $\tau \in \text{Aut}_+(X)$ then $\tau^{-1} \in \text{Aut}_+(X)$, as well. Since $X_- = X_+^{-1} \setminus \{ 1 \}$, the map $\chi \mapsto \chi^{-1}$ is a bijection of $X_+ \setminus \{ 1 \}$ onto $X_-$. Let us assume that $\tau \in \text{Aut}_+(X)$ and $\chi \in X_+ \setminus \{ 1 \}$, but $\xi := \tau^{-1}(\chi) \in X_-$. Then, $\xi^{-1} \in X_+ \setminus \{ 1 \}$, and $\chi = \tau(\xi) = (\tau(\xi^{-1}))^{-1} \in (X_+ \setminus \{ 1 \})^{-1} \subset X_+$, a contradiction.

3) This is an immediate consequence of (1) and (2). 

We denote by $\hat{\varphi}$ the Fourier transform of $\varphi \in L^1(G)$, and by $\| \cdot \|_\infty$ the norm in $L^\infty(G)$. We put also

$$ \| f \|_p = \left( \int_G |f|^p \, d\nu \right)^{1/p} $$

for $f \in L^p(G) (0 < p < \infty)$. 

In the following, the compliment $X \setminus E$ of the subset $E \subset X$ will be denoted by $E^c$.

The next class of spaces is important, in particular, for general Hilbert transform \cite{22} and for the theory of Dirichlet series (see \cite{1, 7, 13} and Section 8).

**Definition 2.2** \cite{7, 22}. Let $G$ be a compact Abelian group, $1 \leq p \leq \infty$, and $E \subset X$ a non-empty set. The generalized Hardy space $H^p_E(G)$ is the closed subspace of $L^p(G)$ defined as follows:

$$H^p_E(G) = \{f \in L^p(G) : \hat{f}(\chi) = 0 \forall \chi \in E^c\}.$$ 

The case where $G$ is connected and $E = X_+$ is due to Helson and Lowdenslager (see, e.g., \cite{31}). We shall write $H^p(G)$ instead of $H^p_{X_+}(G)$ in this case. In particular, $H^2(G)$ is the subspace of $L^2(G)$ with Hilbert basis $X_+$. We denote by $P_+$ the orthogonal projection $L^2(G) \rightarrow H^2(G)$, and $P_- = I - P_+$.

Of course, the space $H^p(G)$ (as well as the spaces $H^p_\mathbb{R}(G)$, BMO(G), and BMOA(G) considered below) depends on the chosen order in $X$.

For every $u \in L^2(G, \mathbb{R})$, there is a unique $\tilde{u} \in L^2(G, \mathbb{R})$ such that $\hat{\tilde{u}}(1) = 0$ and $u + i \tilde{u} \in H^2(G)$. The linear continuation of the mapping $u \mapsto \tilde{u}$ to the complex space $L^2(G)$ is called a Hilbert transform on $G$. This operator extends to a bounded operator $\varphi \mapsto \tilde{\varphi}$ on $L^p(G)$ for $1 < p < \infty$ (generalized Marcel Riesz’s inequality), in particular, $\|\tilde{\varphi}\|_2 \leq \|\varphi\|_2$ for every $\varphi \in L^2(G)$ \cite{31, 8.7}, \cite{22, Theorem 8, Corollary 20}. Let $Pf = \hat{f}$ be the Fourier transform on $G$. Then, the formula

$$\hat{\tilde{f}} = -i \text{sgn}_{X_+} \hat{f}$$

holds, where $\text{sgn}_{X_+}(\chi) = 1$ for $\chi \in X_+ \setminus \{1\}$, $\text{sgn}_{X_+}(1) = 0$, and $\text{sgn}_{X_+}(\chi) = -1$ for $\chi \in X \setminus X_+$ (see \cite{22} for details).

Note also that the Hilbert transform is a continuous map from $L^1(G)$ to $L^p(G)$ for $0 < p < 1$ (see, e.g., \cite{31, Theorem 8.7.6}).

**Definition 2.3** \cite{10}. We define the space $BMO(G)$ of functions of bounded mean oscillation on $G$ and its subspace $BMOA(G)$ as follows:

$$BMO(G) := \{f + \tilde{g} : f, g \in L^\infty(G)\},$$

$$BMOA(G) := BMO(G) \cap H^1(G),$$

$$\|\varphi\|_{BMO} := \inf\{\|f\|_\infty + \|g\|_\infty : \varphi = f + \tilde{g}, f, g \in L^\infty(G)\}$$

for $\varphi \in BMO(G)$.

**Lemma 2.4** (\cite{24, Lemma 1}). The following equalities hold:

1) $BMO(G) = P_- L^\infty(G) + P_+ L^\infty(G)$, with an equivalent norm

$$\|\varphi\|_* := \inf\{\max(\|f_1\|_\infty, \|g_1\|_\infty) : \varphi = P_- f_1 + P_+ g_1, f_1, g_1 \in L^\infty(G)\};$$

2) $BMOA(G) = P_+ L^\infty(G)$. Moreover, for the norm

$$\|\varphi\|_* = \inf\{\|h\|_\infty : \varphi = P_+ h, h \in L^\infty(G)\}$$

in this space, the following inequalities take place:

$$\frac{2}{3} \|\varphi\|_{BMO} \leq \|\varphi\|_* \leq 2 \|\varphi\|_{BMO}.$$
**Definition 2.5** [24]. We define the space $H^1_{\mathbb{R}}(G)$ (the real $H^1$ space on $G$) as the completion of the space $\text{Pol}(G, \mathbb{R})$ of real-valued trigonometric polynomials on $G$ with respect to the norm

$$\|q\|_{1^*} := \|P^- q\|_1 + \|P^+ q\|_1.$$  

We denote the norm in $H^1_{\mathbb{R}}(G)$ by $\| \cdot \|_{1^*}$, too.

The notation $H^1_{\mathbb{R}}(G)$ should not lead to the confusion with $H^p_{E}(G)$ in Definition 2.2.

**Lemma 2.6** [24], Proposition 1.

(i) Projectors $P_\pm$, and the Hilbert transform are bounded operators on $H^1_{\mathbb{R}}(G)$;

(ii) restrictions $P_\pm|\text{Pol}(G, \mathbb{R})$ extend to bounded operators $P^1_\pm$ from $H^1_{\mathbb{R}}(G)$ to $L^1(G)$ and

$$\|f\|_{1^*} = \|P^- f\|_{1^*} + \|P^+ f\|_{1^*} = \|P^1_- f\|_1 + \|P^1_+ f\|_1 \quad (f \in H^1_{\mathbb{R}}(G));$$

(iii) $H^1_{\mathbb{R}}(G) = \text{Im} P_- + \text{Im} P_+$ (the direct sum of closed subspaces);

(iv) $\bigcup_{p>1} L^p(G, \mathbb{R}) \subset H^1_{\mathbb{R}}(G) \subset L^1(G, \mathbb{R})$;

(v) $\|f\|^- := \|f\|_1 + \|\tilde{f}\|_1$ is an equivalent norm in $H^1_{\mathbb{R}}(G)$;

(vi) $H^1_{\mathbb{R}}(G) = \text{Re} H^1(G)$.

### 3 | HAUSDORFF OPERATORS ON GENERAL GROUPS

In [25], the following definition was proposed.

**Definition 3.1** [25]. Let $(\Omega, \mu)$ be a measure space, $G$ a topological group, $A : \Omega \to \text{Aut}(G)$ a measurable map, and $\Phi$ a locally $\mu$-integrable function on $\Omega$. We define the Hausdorff operator with a kernel $\Phi$ over the group $G$ by the formula

$$(H_{\Phi,A} f)(x) = \int_{\Omega} \Phi(u) f(A(u)(x)) d\mu(u).$$

In particular, we get a class of discrete Hausdorff operators of the form

$$f \mapsto \sum_{u \in \Omega} \Phi(u)(f \circ A(u)),$$

where $\Omega$ is a countable set endowed with the counting measure.

Throughout, we denote by $\mathcal{L}(Y)$ the space of linear bounded operators on a normed space $Y$.

By [25, Lemma 1], the operator $H_{\Phi,A}$ is bounded on $L^p(G)$ ($1 \leq p \leq \infty$), for a locally compact group $G$, provided $\Phi(u)(\mod A(u))^{-1/p} \in L^1(\Omega, \mu)$, and

$$\|H_{\Phi,A}\|_{\mathcal{L}(L^p(G))} \leq \int_{\Omega} |\Phi(u)|(\mod A(u))^{-1/p} d\mu(u). \quad (3.1)$$

**Example 1.** Let $\mathbb{T}^\infty$ be the infinite-dimensional torus (the product of a countably many copies of the circle group) and let $C := \{-1, 1\}^\infty$ be a Cantor group endowed with a regular Borel measure $\mu$ (e.g., $\mu$ be the normalized Haar measure of the compact group $C$). The group $C$ acts on $\mathbb{T}^\infty$ by coordinate-wise automorphisms $A(u)(x) = x^u := (x_j^u)_{j \in \mathbb{N}}$, where $u = (u_j)_{j \in \mathbb{N}}$, $u_j \in \{-1, 1\}$, and $x = (x_j)_{j \in \mathbb{N}}$, $x_j \in \mathbb{T}$. Thus, we get a Hausdorff operator

$$H_\Phi f(x) = \int_C \Phi(u) f(x^u) d\mu(u).$$
Since $\mathbb{T}^\infty$ is unimodular, $\text{mod} A(u) = 1$ and hence this operator is bounded on $L^p(\mathbb{T}^\infty)$ ($1 \leq p \leq \infty$), for $\Phi \in L^1(\mu)$ and $\| H_\Phi \|_{L(L^p(\mathbb{T}^\infty))} \leq \| \Phi \|_{L^1(\mu)}$.

The next proposition shows that Hausdorff operators in a sense of Definition 3.1 possess a regularity property and thus give us a family (for various $\Phi$, $A(u)$, and $\Omega$) of generalized limits at infinity for functions on $G$.

In the case $G = \mathbb{R}$ and $\Omega = [0,1]$, this can be found in [12, Theorem 217] (cf. [26]).

**Proposition 3.2.** Suppose that the conditions of Definition 3.1 are fulfilled and the group $G$ is $\sigma$-compact. In order that the transformation $H_{\Phi,A}$ should be regular (i.e., that for a locally bounded measurable function $f$ on $G$, the condition $f(x) \to l$ as $x \to \infty$ should imply $H_{\Phi,A}f(x) \to l$), it is necessary and sufficient that

$$\int_{\Omega} \Phi(u) d\mu(u) = 1.$$

**Proof.** If $f(x) \equiv 1$, then $H_{\Phi,A}f(x) = \int_{\Omega} \Phi(u) d\mu(u)$. Thus, $\int_{\Omega} \Phi(u) d\mu(u) = 1$ is a necessary condition.

To prove the sufficiency, first note that since $A(u)$ has continuous inverse, $f(x) \to l$ as $x \to \infty$ implies $f(A(u)(x)) \to l$ as $x \to \infty$. Indeed, if $f(x) \to l$ as $x \to \infty$ then for each $\varepsilon > 0$ there is such compact $C_\varepsilon \subset G$ that $|f(x) - l| < \varepsilon$ for $x \in G \setminus C_\varepsilon$. Now if $x \in G \setminus A(u)^{-1}(C_\varepsilon)$, we get $A(u)(x) \in G \setminus C_\varepsilon$ and therefore $|f(A(u)(x)) - l| < \varepsilon$. If, in addition, $f$ is locally bounded it is bounded on $G$ and therefore $H_{\Phi,A}f(x) \to l$ by the Lebesgue theorem.

\[\square\]

## 4 | COMMUTING RELATIONS FOR HAUSDORFF OPERATORS

In this section, we will show that Hausdorff operator commutes in a sense both with the Fourier transform and the Hilbert transform.

**Theorem 4.1** cf. [18], Theorem 4.4, [20].

(i) Let $G$ be a compact (not necessary connected) Abelian group, $f \in L^1(G)$, and $\Phi \in L^1(\mu)$. Then

$$(H_{\Phi,A}f)^\wedge = H_{\Phi,(A^\gamma)^{-1}}\hat{f}.$$  

(ii) Let $G$ be a compact and connected Abelian group, $f \in L^2(G)$, $\Phi \in L^1(\mu)$, and $A(u) \in \text{Aut}(G)^+$ for $\mu$-a. e. $u \in \Omega$. Then

$$H_{\Phi,A}f = (H_{\Phi,A}f)^\wedge.$$

**Proof.**

(i) By the Fubini theorem,

$$\int_G (\int_{\Omega} \Phi(u) f(A(u)(x)) d\mu(u)) \overline{\chi(x)} d\nu(x)$$

$$= \int_{\Omega} \Phi(u) \left( \int_G f(A(u)(x)) \overline{\chi(x)} d\nu(x) \right) d\mu(u).$$

Moreover, since $G$ is unimodular, we have $\text{mod} A(u) = 1$, and putting $y = A(u)(x)$, we get

$$\int_G f(A(u)(x)) \overline{\chi(x)} d\nu(x) = \int_G f(y) \overline{\chi(A(u)^{-1}(y))} d\nu(y) = \hat{f}((A(u)^{-1})^{-1}(\chi)).$$

It follows that

$$(H_{\Phi,A}f)^\wedge = H_{\Phi,(A^\gamma)^{-1}}(f^\wedge).$$
(ii) Note that $\tilde{f} \in L^2(G)$. Then, in view of (i), one has, for all $\chi \in X$, that

$$F(H_{\Phi,A} \tilde{f})(\chi) = H_{\Phi,(A^*)^{-1}} \hat{f}(\chi) = (H_{\Phi,(A^*)^{-1}}(-i \text{sgn}_{X_+} \hat{f}))(\chi)$$

$$= -i \int_{\Omega} \Phi(u) \text{sgn}_{X_+}((A^*)^{-1}(\chi)) \hat{f}(A(u)^{-1}(\chi)) d\mu(u).$$

Since $(A(u)^{-1})$ is an order automorphism for $\mu$-a.e. $u \in \Omega$, one has $\text{sgn}_{X_+}((A^*)^{-1}(\chi)) = \text{sgn}_{X_+}(\chi)$ a.e. This yields (again by (i))

$$F(H_{\Phi,A} \tilde{f})(\chi) = -i \text{sgn}_{X_+}(\chi) \int_{\Omega} \Phi(u) \hat{f}(A(u)^{-1}(\chi)) d\mu(u)$$

$$= -i \text{sgn}_{X_+}(\chi) F(H_{\Phi,A}f)(\chi) = F(H_{\Phi,A}f)^-(\chi),$$

which completes the proof.

\[\square\]

**Corollary 4.2.** Let $\Phi \in L^1(\mu)$ and $A(\cdot) \in \text{Aut}(G)^+$ for $\mu$-a.e. $u \in \Omega$. Then, the range of $H_{\Phi,A}$ in the space $L^2(G)$ is invariant with respect to the Hilbert transform.

## 5. HAUSDORFF OPERATORS ON SPACES $H^p_E(G)$ AND $\text{BMOA}(G)$

The following theorem deals with general compact Abelian groups.

**Theorem 5.1.** Let $G$ be a compact (not necessary connected) Abelian group, $E \subset X$, and $(A(u)^*)^{-1} : E^c \to E^c$ for $\mu$-a.e. $u \in \Omega$. The Hausdorff operator $H_{\Phi,A}$ is bounded on $H^p_E(G)$ ($1 \leq p \leq \infty$) if $\Phi \in L^1(\mu)$. In this case,

$$\|H_{\Phi,A}\|_{L(H^p_E)} \leq \|\Phi\|_{L^1(\mu)}.$$

**Proof.** Let $\Phi \in L^1(\mu)$. Since $G$ is unimodular, $\text{mod}A(u) = 1$. Thus, as mentioned in the introduction, the operator $H_{\Phi,A}$ is bounded in $L^p(G)$ and formula (3.1) holds with $\text{mod}A(u) = 1$. Therefore, it suffices to show that $H_{\Phi,A}$ acts in $H^p_E(G)$. In other words, it suffices to show that for each $f \in H^p_E(G)$, the Fourier transform of $H_{\Phi,A}f$ is concentrated on $E$. But by Theorem 4.1 (i),

$$(H_{\Phi,A}f)^+(\chi) = \int_{\Omega} \Phi(u) \hat{f}(A(u)^{-1}(\chi)) d\mu(u).$$

Let $\chi \in E^c$. Since $\hat{f}$ is concentrated on $E$, we have $\hat{f}(A(u)^{-1}(\chi)) = 0$ for $\mu$-a.e. $u \in \Omega$. It follows that $(H_{\Phi,A}f)^+(\chi) = 0$, too. This completes the proof.

\[\square\]

**Corollary 5.2.** Let $G$ be a compact and connected Abelian group, and $A(\cdot) \in \text{Aut}(G)^+$ for $\mu$-a.e. $u \in \Omega$. The Hausdorff operator $H_{\Phi,A}$ is bounded on $H^p(G)$ ($1 \leq p \leq \infty$) if and only if $\Phi \in L^1(\mu)$. In this case,

$$\|H_{\Phi,A}\|_{L(H^p)} \leq \|\Phi\|_{L^1(\mu)}.$$

**Proof.** Since $1 \in H^p(G)$, the “only if” part is obvious. Now, let $\Phi \in L^1(\mu)$. In our case $E = X_+$. Therefore, it suffices to show that $(A(u)^*)^{-1} : X_- \to X_-$ for $\mu$-a.e. $u \in \Omega$. Indeed, let $\chi \in X_- = X \setminus X_+$. Then, $\chi^{-1} \in X_+ \setminus \{1\}$ and we have $(A(u)^*)^{-1}(\chi) = A(u)^*(\chi^{-1}) \in X_+ \setminus \{1\}$. Thus, $A(u)^*(\chi) \in X \setminus X_+$. This completes the proof.

\[\square\]

From now on we denote by $Y^*$ the dual of a normed space $Y$ and by $B^*$ the adjoint of an operator $B \in \mathcal{L}(Y)$.

For the proof of our next theorem, we need the following.
Theorem 5.3 ([24], Theorem 1). For every \( \varphi \in BMOA(G) \), the formula
\[
F_\varphi(f) = \int_G f \overline{\varphi} \, d\nu
\]
defines a linear functional on \( H^\infty(G) \), and this functional extends uniquely to a continuous linear functional \( F_\varphi \) on \( H^1(G) \). Moreover, the correspondence \( \varphi \mapsto F_\varphi \) is an isometrical isomorphism of \((BMOA(G), \| \cdot \|_*)\) and \( H^1(G)^* \), and a topological isomorphism of \((BMOA(G), \| \cdot \|_{BMO})\) and \( H^1(G)^* \).

Theorem 5.4. Let \( A(u) \in \text{Aut}(G)^+ \) for \( \mu \)-a.e. \( u \in \Omega \). The Hausdorff operator \( H_{\varphi, A} \) is bounded on the space \( BMOA(G) \) if and only if \( \Phi \in L^1(\mu) \). Moreover,
\[
\| H_{\varphi, A} \|_{L(E,BMOA)} \leq \| \Phi \|_{L^1(\mu)}.
\]

Proof. Since \( 1 \in BMOA(G) \), the “only if” part is obvious. Now, let \( \Phi \in L^1(\mu) \). In view of Theorem 5.1, for the proof, it suffices to show that \( H_{\varphi, A} = H_{\Phi, A}^{-1} \) where \( H_{\Phi, A}^{-1} \) is considered in \( H^1(G) \). To this end, we employ Theorem 5.3. Let \( f \in H^\infty(G) \). Then, it is clear that \( H_{\Phi, A}^{-1} f \in H^\infty(G) \), too, and for every \( \varphi \in BMOA(G) \), we have
\[
H_{\Phi, A}^{-1} (F_\varphi)(f) = F_\varphi (H_{\Phi, A}^{-1} f) = \int_G \left( \int_\Omega \Phi(u) f(A(u)^{-1}(x)) d\mu(u) \right) \overline{\varphi(x)} \, d\nu(x)
\]
\[
= \int_\Omega \Phi(u) \left( \int_G f(A(u)^{-1}(x)) \overline{\varphi(x)} \, d\nu(x) \right) d\mu(u),
\]
by the Fubini theorem.

Further, as in the proof of Theorem 4.1, putting \( y = A(u)(x) \), we get
\[
\int_G f(A(u)^{-1}(x)) \overline{\varphi(x)} \, d\nu(x) = \int_G f(y) \overline{\varphi(A(u)(y))} \, d\nu(y).
\]
Thus (again by the Fubini theorem),
\[
H_{\Phi, A}^{-1} (F_\varphi)(f) = \int_G f(y) \left( \int_\Omega \Phi(u) \overline{\varphi(A(u)(y))} d\mu(u) \right) d\nu(y)
\]
\[
= \int_G f(y) H_{\Phi, A} \varphi(y) d\nu(y) = F_\varphi(f),
\]
where \( \psi = H_{\Phi, A} \varphi \). Since by Theorem 5.3 every continuous linear functional on \( H^1(G) \) is uniquely defined by its values on \( H^\infty \), it follows that \( H_{\Phi, A}^{-1} (F_\varphi) = F_\psi \). If we identify (again by Theorem 5.3) \( F_\varphi \) with \( \varphi \) and \( F_\psi \) with \( \psi \), we have
\[
H_{\Phi, A} \varphi = H_{\Phi, A}^{-1} \varphi,
\]
which completes the proof. \( \square \)

For the next corollary, we need the following.

Definition 5.5 [31]. We call a subset \( E \subset X^+ \) lacunary (in the sense of Rudin) if there is a constant \( K_E \) such that the number of terms of the set \( \{ \xi \in E : \chi \leq \xi \leq \chi^2 \} \) does not exceed \( K_E \) for every \( \chi \in X^+ \).

Corollary 5.6. Let a subset \( E \subset X^+ \) be lacunary, \( A(u) \in \text{Aut}(G)^+ \) for \( \mu \)-a.e. \( u \in \Omega \), and \( \Phi \in L^1(\mu) \). Then, \( H_{\Phi, A} \) is a bounded operator from \( H^2_E(G) \) into \( BMOA(G), \| \cdot \|_{BMO} \) and
\[
\| H_{\Phi, A} \|_{H^2_E \rightarrow BMOA} \leq 3 \sqrt{K_E} \| \Phi \|_{L^1}.
\]
Proof. Let $\text{Pol}_E(G) : = \text{span}_\mathbb{C}(E)$ be the space of $E$-polynomials. It is known [23, Lemma 1] that $\text{Pol}_E(G)$ is a dense subspace of $H^p_E(G)$ for all $p \in [1, \infty)$. Since $E \subset X_+$, we have $H^p_E(G) \subset H^p(G)$. Let $\varphi \in \text{Pol}_E(G)$. Then, $\varphi \in H^1_E(G) \cap H^2(G)$, and by [24, Theorem 3], one has $\varphi \in \text{BMOA}(G)$ and $\|\varphi\|_{\text{BMO}} \leq 3 \sqrt{K_E} \|\varphi\|_{H^2}$. Now, Theorem 5.4 yields

$$\|H_{\varphi, A}\|_{\text{BMO}} \leq \|\varphi\|_{L^1} \|\varphi\|_{\text{BMO}} \leq 3 \sqrt{K_E} \|\varphi\|_{L^1} \|\varphi\|_{H^2},$$

and the result follows.

Concluding this section, we show the necessity of the condition in Corollary 5.2 of Theorems 5.1 and 5.4.

**Proposition 5.7.** Let $G$ be metrizable. Assume in addition to the assumptions of Definition 3.1 that $\int E \varphi \, d\mu \neq 0$ for every measurable $E \subset \Omega$, $\mu(E) > 0$. If the Hausdorff operator $H_{\varphi, A}$ acts in $H^1(G)$ or $\text{BMOA}(G)$, then $A(u) \in \text{Aut}(G)^+$ for $\mu$-a.e. $u \in \Omega$.

**Proof.** Since $X_+ \subset \text{BMOA}(G) \subset H^1(G)$, we have, for every $\chi \in X_+$ and every $\xi \in X_- = X \setminus X_+$, by Theorem 4.1, that

$$(H_{\varphi, A} \chi)^\wedge(\xi) = (H_{\varphi, (A^*)^{-1}} \hat{\chi})(\xi) = 0.$$ 

On the other hand, the orthogonality of characters of $G$ implies that $\hat{\chi} = 1_{\{\chi\}}$, where $1_A$ stands for the indicator function of a subset $A$ of $X$. Thus,

$$0 = (H_{\varphi, (A(u)^*)^{-1}} 1_{\{\chi\}})(\xi) = \int_{E(\chi, \xi)} \Phi(u) \, d\mu(u),$$

where

$$E(\chi, \xi) = \{u \in \Omega : (A(u)^*)^{-1}(\xi) = \chi\} = \{u \in \Omega : A(u)^*(\chi) = \xi\}.$$ 

Therefore, $\mu(E(\chi, \xi)) = 0$ for an arbitrary $\chi \in X_+$ and $\xi \in X_-$. Moreover,

$$\{u \in \Omega : A(u)^*: X_+ \to X_+\} = \cup E(\chi, \xi) : \chi \in X_+, \xi \in X_-.$$ 

Since $G$ is metrizable, $X$ is countable (see, e.g., [29, Corollary of Theorem 29]). It follows that $A(u)^*: X_+ \to X_+$ for $\mu$-a.e. $u$, which completes the proof.

### 6 | HAUSDORFF OPERATORS ON SPACES BMO(G) AND H^{1}_{\R}(G)

**Theorem 6.1.** Let $A(u) \in \text{Aut}(G)^+$ for $\mu$-a.e. $u \in \Omega$. The Hausdorff operator $H_{\varphi, A}$ is bounded on $\text{BMO}(G)$ if and only if $\Phi \in L^1(\mu)$. In this case,

$$\|H_{\varphi, A}\|_{\ell(\text{BMO})} \leq \|\Phi\|_{L^1(\mu)}.$$ 

**Proof.** The necessity is obvious. Let $\Phi \in L^1(\mu)$. Every function $\varphi \in \text{BMO}(G)$ has the form $\varphi = f + \tilde{g}$ where $f, g \in L^\infty(G)$. Then, by Theorem 4.1,

$$H_{\varphi, A} \varphi = H_{\varphi, A} f + (H_{\varphi, A} g)^\wedge.$$ 

Note that $H_{\varphi, A} f, H_{\varphi, A} g \in L^\infty(G)$. Thus,

$$\|H_{\varphi, A} \varphi\|_{\text{BMO}} \leq \|H_{\varphi, A} f\|_\infty + \|H_{\varphi, A} g\|_\infty \leq \|\Phi\|_{L^1(\mu)} (\|f\|_\infty + \|g\|_\infty),$$

and the result follows.

Below we shall use the following.
Theorem 6.2 [24], Theorem 2. For every \( \phi \in BMO(G, \mathbb{R}) \), the linear functional

\[
F_\phi(q) = \int_G q \phi \, d\nu \quad (6.1)
\]
on \( Pol(G, \mathbb{R}) \) extends uniquely to a continuous linear functional \( F_\phi \) on \( H^1_{\mathbb{R}}(G) \). Moreover, the correspondence \( \phi \mapsto F_\phi \) is an isometrical isomorphism of \((BMO(G, \mathbb{R}), \| \cdot \|_{BMO}) \) and \( H^1_{\mathbb{R}}(G)^* \), and a topological isomorphism of \((BMO(G, \mathbb{R}), \| \cdot \|_{BMO}) \) and \( H^1_{\mathbb{R}}(G)^* \).

Corollary 6.3. The last theorem is valid with \( q \in L^2(G, \mathbb{R}) \) in place of \( q \in Pol(G, \mathbb{R}) \).

Proof. Since \( Pol(G, \mathbb{R}) \subset L^2(G, \mathbb{R}) \) and \( Pol(G, \mathbb{R}) \) is dense in \( H^1_{\mathbb{R}}(G) \), it suffices to show that the right-hand side in Equation (6.1) is continuous on the set \( L^2(G, \mathbb{R}) \) with respect to the \( H^1_{\mathbb{R}}(G) \) norm. Let \( \phi = \tilde{P} - g + \tilde{P} + h \), where \( g, h \in L^\infty(G) \). Then, for every \( q \in L^2(G, \mathbb{R}) \), one has that (\( q \) is real valued)

\[
\int_G q \phi \, d\nu = \int_G P^- q g \, d\nu + \int_G P^+ q h \, d\nu = \int_G g P^- q \, d\nu + \int_G h P^+ q \, d\nu.
\]

This yields

\[
\left| \int_G q \phi \, d\nu \right| \leq \max(\|g\|_{\infty}, \|h\|_{\infty})(\|P^- q\|_1 + \|P^+ q\|_1).
\]

So, \( \left| \int_G q \phi \, d\nu \right| \leq \|\phi\|_* \|q\|_{1*} \) (we used Lemma 2.6 and the fact that \( P^\pm q = P \pm q \) for \( q \in L^2(G, \mathbb{R}) \)), and the proof is complete.

Theorem 6.4. Let \( A(u) \in \text{Aut}(G)^+ \) for \( \mu \)-a.e. \( u \in \Omega \). The Hausdorff operator \( H_{\phi, A} \), with real-valued \( \Phi \), is bounded on the real Hardy space \( H^1_{\mathbb{R}}(G) \) if and only if \( \Phi \in L^1(\mu) \). Moreover,

\[
\| H_{\phi, A} \|_{L(H^1_{\mathbb{R}})} \leq \| \Phi \|_{L^1(\mu)}.
\]

Proof. As above, the “only if” part is obvious. Let \( \Phi \in L^1(\mu) \). We employ Theorem 5.1 and the fact that \( H^1_{\mathbb{R}}(G) = \text{Re} H^1(G) \) (Lemma 2.6). Let \( g \in H^1_{\mathbb{R}}(G) \). Then, \( g = f + \bar{f} \), where \( f \in H^1(G) \). But since \( \Phi \) is real, we have

\[
\overline{H_{\phi, A} f(x)} = \int_\Omega \Phi(u) \overline{f(A(u)(x))} \, d\mu(u).
\]

Since \( H_{\phi, A} \) is linear in \( L^1(G) \), it follows that

\[
H_{\phi, A} g = H_{\phi, A} f + H_{\phi, A} \bar{f} = H_{\phi, A} f + \overline{H_{\phi, A} f} \in \text{Re} H^1(G) = H^1_{\mathbb{R}}(G).
\]

Thus, \( H_{\phi, A} \) acts in \( H^1_{\mathbb{R}}(G) \). Now, we will apply the closed graph theorem. Let \( f_n \to f \) and \( H_{\phi, A} f_n \to g \) in \( H^1(G) \). Since there is a continuous embedding \( H^1_{\mathbb{R}}(G) \subset L^1(G, \mathbb{R}) \) (Lemma 2.6), it follows that \( f_n \to f \) and \( H_{\phi, A} f_n \to H_{\phi, A} f \) in \( L^1(G) \). Thus, \( g = H_{\phi, A} f \), and the proof of the continuity of \( H_{\phi, A} \) is complete.

Finally, due to Corollary 6.3, as in the proof of Theorem 5.4 we have \( H_{\phi, A}^* = H_{\phi, A}^{-1} \), where \( H_{\phi, A}^{-1} \) is considered in \( BMO(G) \). Then, by Theorem 6.1,

\[
\| H_{\phi, A} \|_{L(H^1_{\mathbb{R}})} = \| H_{\phi, A}^{-1} \|_{L(BMO)} \leq \| \Phi \|_{L^1(\mu)}.
\]

Remark 1. It is clear that \( 1 \in H^1_{\mathbb{R}}(G) \) and \( \| 1 \|_{1*} = 1 \). If \( \Phi \geq 0 \), we have \( H_{\phi, A} 1 = \| \Phi \|_{L^1(\mu)} 1 \). Thus, \( \| H_{\phi, A} \|_{L(H^1_{\mathbb{R}})} = \| \Phi \|_{L^1(\mu)} \). Then, formula (6.2) shows that \( \| H_{\phi, A} \|_{L(BMO)} = \| \Phi \|_{L^1(\mu)} \), as well. For \( \Phi \geq 0 \), similar equalities hold for the spaces \( H^p(G) \) (\( 1 \leq p < \infty \)), \( H^1_{\mathbb{R}}(G) \), and \( BMOA(G) \).
ON THE ACTION OF $H_{\Phi,A}$ IN $C(G)$

The next simple proposition gives sufficient conditions for the boundedness of a Hausdorff operator in $C(G)$.

Proposition 7.1. Let $G$ be a compact (not necessary connected) Abelian group, and one of the following two conditions holds:

1) $G$ is first-countable;
2) $\Omega$ is a completely regular topological space with a bounded Radon measure $\mu$, $\Phi$ is a bounded and continuous function on $\Omega$, and the map $\Omega \times G \to G$, $(u,x) \mapsto A(u)(x)$ is continuous.

Then, $H_{\Phi,A}$ acts in the space $C(G)$ and is bounded if and only if $\Phi \in L^1(\mu)$, and in this case $\|H_{\Phi,A}\| \leq \|\Phi\|_{L^1(\mu)}$.

Proof. The necessity is obvious. In the case (1), the sufficiency follows from the Lebesgue theorem, and in the case (2) this follows, e.g., from [2, Chapter IX, Section 5, Corollary of Proposition 13].

The following examples show that the conditions of the previous proposition are essential, because in general $H_{\Phi,A}$ does not act in $C(G)$.

Example 2. Let $G = \mathfrak{b}R$ be the Bohr compactification of the reals (see, e.g., [31, Section 1.8]). This means that $G$ is the dual group of the additive group $X := \mathbb{R}_d$, where the group $\mathbb{R}$ of reals is endowed with the discrete topology and the usual order. Then, $X$ is the dual group of $\mathfrak{b}R$ by the Pontryagin–van Kampen theorem. The map $\tau_u(\gamma) := uy$ belongs to $\text{Aut}(X)$ for every $u \in \mathbb{R}$, $u \neq 0$.

For each $t \in \mathbb{R}$, let $\hat{\gamma} = e^{-it\gamma}$ be the corresponding continuous character of $\mathbb{R}$ ($\gamma \in \mathbb{R}$). Then, the map $\hat{\beta} : \mathbb{R} \to \mathfrak{b}R$, $t \mapsto \hat{t}$ is a continuous isomorphism of $\mathbb{R}$ onto a dense subgroup of $\mathfrak{b}R$ (see, e.g., [31, 1.8.2]). Therefore, we identify $\hat{t}$ with $t \in \mathbb{R}$ and consider $\mathbb{R}$ as a dense subgroup of $\mathfrak{b}R$.

The space $\text{AP}(\mathbb{R})$ of uniformly almost periodic functions on $\mathbb{R}$ (endowed with the sup norm) is isometrically isomorphic to $C(\mathfrak{b}R)$ via the restriction map $C(\mathfrak{b}R) \to \text{AP}(\mathbb{R})$, $g \mapsto g|\mathbb{R}$ (see, e.g., [31, 1.8.4], [21, Chapter VIII, Section 41]).

Let $\Omega = \mathbb{R}$, $d\mu(u) = du$, $\Phi \in L^1(\mathbb{R})$. If we assume that the Hausdorff operator

$$H_{\Phi,A}g(x) = \int_{\mathbb{R}} \Phi(u)g(\tau_u^*(x))du$$

acts in $C(\mathfrak{b}R)$, then the classical one-dimensional Hausdorff operator

$$H_{\Phi,f}(t) := \int_{\mathbb{R}} \Phi(u)f(ut)du$$

acts in $\text{AP}(\mathbb{R})$.

For the proof, it suffices to show that $H_{\Phi,f} = (H_{\Phi,\tau^*_u g})|\mathbb{R}$, where $g \in C(\mathfrak{b}R)$, $f := g|\mathbb{R}$. But for $t \in \mathbb{R}$, one has

$$\tau^*_u(\hat{t})(\gamma) = \hat{t}(uy) = e^{-iuy} = \hat{u}t(\gamma) \ (\gamma \in \mathbb{R}).$$

Thus, $\tau^*_u(\hat{t}) = \hat{u}t$. It follows that for $t \in \mathbb{R}$,

$$H_{\Phi,\tau^*_u g}(t) = H_{\Phi,\tau^*_u g}(\hat{t}) = \int_{\mathbb{R}} \Phi(u)g(\tau^*_u(\hat{t}))du$$

$$= \int_{\mathbb{R}} \Phi(u)f(\hat{u}t)du = \int_{\mathbb{R}} \Phi(u)f(ut)du$$

(recall that we identify $\hat{t}$ with $t \in \mathbb{R}$), which completes the proof.

In particular, taking $f(t) = e^{-it}$, we get that for $\Phi \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, the Fourier transform $\hat{\Phi}$ belongs to $\text{AP}(\mathbb{R})$. But it is known (see, e.g., [11, Theorem 3]) that in this case the measure $\Phi(u)du$ ought to be discrete, and we arrive at a contradiction.
Remark 2. As follows from the above considerations, the classical one-dimensional Hausdorff operator $H_b$ acts in $AP(\mathbb{R})$ if and only if the measure $\Phi(u)du$ is discrete and finite. Indeed, the necessity of this condition was shown above. Conversely, if $(\phi_k) \in \ell^1(\mathbb{Z})$, $u_k \in \mathbb{R}$, the series $\sum_k \phi_k f(u_k x)$ converges uniformly for each $f \in AP(\mathbb{R})$.

8 | APPLICATIONS TO DIRICHLET SERIES

8.1 | Ordinary Dirichlet series

In this subsection, we consider the action of a Hausdorff operator on ordinary Dirichlet series

$$ D = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. $$

Let $\mathbb{Z}^\infty$ be the additive group of infinite sequences of integers with finite support, $\mathbb{Z}^\infty_+ := \{ \alpha \in \mathbb{Z}^\infty : \forall k \alpha_k \geq 0 \}$. Since every natural $n$ has the prime number decomposition $n = p^\alpha := p_1^{\alpha_1} \cdots p_N^{\alpha_N}$, where $\alpha \in \mathbb{Z}^\infty_+$ and $p = (2,3,5,\ldots)$ is the sequence of all primes, one can identify the series $D$ with the corresponding coefficient function $\mathbb{Z}^\infty_+ \ni \alpha \mapsto a(\alpha)$. In this case, the action of a Hausdorff operator on $D$ means the action on the function $a(\alpha)$, and Definition 3.1 takes the form

$$ (H_{\Phi,\tau} a(\alpha)) = \int_{\Omega} \Phi(u) a(\tau_\alpha u) \, d\mu(u). \quad (8.1) $$

(Since a function $a(\alpha)$ is supported in $\mathbb{Z}^\infty_+$, one can consider only such automorphisms $\tau_\alpha$ of $\mathbb{Z}^\infty_+$ that $\tau_\alpha : \mathbb{Z}^\infty_+ \to \mathbb{Z}^\infty_+$ a.e. $u \in \Omega$.)

We show that a certain class of such operators acts in some Banach spaces of ordinary Dirichlet series.

Bayart [1] introduced the Banach spaces $\mathcal{S}^p (1 \leq p \leq \infty)$ of ordinary Dirichlet series as the isometric image of $H^p_{\mathbb{Z}^\infty_+}(\mathbb{T}^\infty)$ under the map $f_a \mapsto D$, where $f_a$ is a function from $H^p_{\mathbb{Z}^\infty_+}(\mathbb{T}^\infty)$ with the Fourier transform $\hat{f_a}(\alpha) = a(\alpha)$ ($\alpha \in \mathbb{Z}^\infty_+$).

In particular, $\mathcal{S}^2$ is the Hilbert space of such Dirichlet series $D$ that $(a_n) \in \ell^2(\mathbb{N})$ introduced and studied in [13].

In addition, the Banach space $\mathcal{S}^\infty$ coincides with the space $\mathcal{S}^\infty_\infty$ in [7] of all ordinary Dirichlet series $D$ which converge and define a bounded and holomorphic function $D(\cdot)$ on the half-plane $\{ \text{Re } s > 0 \}$ ($\mathcal{S}^\infty_\infty$ is endowed with the supremum norm $\| \cdot \|_\infty$ on $\{ \text{Re } s > 0 \}$).

We identify a function $D(\cdot) \in \mathcal{S}^p$ with a coefficient function $a(\alpha)$ as mentioned above and put $\| a(\cdot) \| := \| D(\cdot) \|_{\mathcal{S}^p}$.

Theorem 8.1. Let $\Phi \in L^1(\mu)$, and let an automorphic family $(\tau_\alpha)_{\alpha \in \Omega}$ of automorphisms of $\mathbb{Z}^\infty$ enjoy the property $\tau_\alpha : (\mathbb{Z}^\infty_+)^c \to (\mathbb{Z}^\infty_+)^c$ a.e. $u \in \Omega$. Then, the Hausdorff operator (8.1) acts in $\mathcal{S}^p$ ($1 \leq p \leq \infty$) and $\| H_{\Phi,\tau} \|_{L^1(\mathcal{S}^p)} \leq \| \Phi \|_{L^1}$.

Proof. The group $\mathbb{Z}^\infty$ can be identified with the dual of the infinite-dimensional torus $\mathbb{T}^\infty$ via the map $\alpha \mapsto \chi_\alpha$, where the character $\chi_\alpha(t) = t^\alpha := t_1^{\alpha_1} \cdots t_N^{\alpha_N}$ and $\alpha = (\alpha_1,\ldots,\alpha_N,0,0,\ldots) \in \mathbb{Z}^\infty$.

According to [1, 13] (see also [8, Corollary 5.3] or [30, Theorem 6.2.3, p. 145]), the map $\Psi$ taking a function $a(\alpha)$ from $\mathcal{S}^p$ to a function $f_a$ on $\mathbb{T}^\infty$ with the Fourier transform $\hat{f_a}(\alpha) = a(\alpha)$ ($\alpha \in \mathbb{Z}^\infty$) is an isometric isomorphism of Banach spaces $\mathcal{S}^p$ and $H^p_{\mathbb{Z}^\infty_+}(\mathbb{T}^\infty)$.

Now, Theorem 4.1 with $G = \mathbb{T}^\infty$ shows that for $\alpha \in \mathbb{Z}^\infty_+$, one has

$$ (H_{\Phi,\tau})^{-1} f_a(\alpha) = (H_{\Phi,\tau} \hat{f_a})(\alpha) = (H_{\Phi,\tau} a(\alpha))(\alpha). $$

Putting $A(u) = (\tau_u^*)^{-1} E = \mathbb{Z}^\infty_+$ in Theorem 5.1 we get

$$ H_{\Phi,\tau}^{(\alpha)} f_a = f_b, $$

where $f_b \in H^p_{\mathbb{Z}^\infty_+}(\mathbb{T}^\infty)$ and therefore $(H_{\Phi,\tau})^{-1} f_a = \hat{f_b}$. Since $\hat{f_b}(\alpha) = b(\alpha)$ for all $\alpha \in \mathbb{Z}^\infty_+$, it follows that

$$ H_{\Phi,\tau} a(\alpha) = b(\alpha), $$

that is, $H_{\Phi,\tau}$ acts in $\mathcal{S}^p$. 

Finally, for the isometric isomorphism \( \Psi : \mathcal{S}^p \to H^p_{\mathbb{Z}^\omega}(\mathbb{T}^\omega) \), we have \( \Psi^{-1} f_a = a(p^{(i)}) \) for each \( f_a \in H^p_{\mathbb{Z}^\omega}(\mathbb{T}^\omega) \). Consequently,
\[
\Psi H_{\Phi, r} \Psi^{-1} f_a = \Psi H_{\Phi, r} a(p^{(i)}) = \Psi b(p^{(i)}) = f_b.
\]
Thus, \( \Psi H_{\Phi, r} \Psi^{-1} = H_{\Phi,(r^{-1})} \) and therefore
\[
\| H_{\Phi, r} \|_{\mathcal{L}(\mathcal{S}^p)} = \| H_{\Phi,(r^{-1})} \|_{\mathcal{L}(H^p_{\mathbb{Z}^\omega})} \leq \| \Phi \|_{L^1},
\]
which completes the proof. \( \square \)

**Example 3.** Fix \( u \in \Omega \) and let \( \mu = \delta_u \) be the Dirac measure, \( \Phi \equiv 1 \). Let \( r \in \text{Aut}(\mathbb{Z}^\omega) \) be such that \( r : (\mathbb{Z}^\omega)^c \to (\mathbb{Z}^\omega)^c \), \( \mathbb{Z}^\omega \to \mathbb{Z}^\omega \). For every natural \( n = \gamma_{\alpha} \), let \( r_u(n) := \gamma_{r(\alpha)} \) \( (\alpha \in \mathbb{Z}^\omega) \). Then, by Theorem 8.1, if a function \( D = \sum_n a(n)/n^s \) belongs to \( \mathcal{S}^p \), the rearrangement \( H_{\Phi, r} D = \sum_n a(r_u(n))/r_u(n)^s \) belongs to \( \mathcal{S}^p \), too (at the same time the initial series \( D \) should not be absolutely convergent, see, e.g., [8, Theorem 4.1]).

The following corollary is a generalization of Bohr’s theorem (see, e.g., [31, p. 224]).

**Corollary 8.2.** Let \( \Phi \in L^1(\mu) \), and let a family \( (\tau_u)_{u \in \Omega} \) of automorphisms of \( \mathbb{Z}^\omega \) enjoy the property \( \tau_u : (\mathbb{Z}^\omega)^c \to (\mathbb{Z}^\omega)^c \) a.e. \( u \in \Omega \). Let \( E \) be the set of all \( \alpha \in (\mathbb{Z}^\omega)^+ \) with \( \sum_j \alpha_j = 1 \). Then for every \( D(\cdot) \in \mathcal{S}^\omega \) with the coefficient function \( a(p^{(i)}) \), we have
\[
\sum_{\alpha \in E} |(H_{\Phi, r} a(p^{(i)}))(\alpha)| \leq \| \Phi \|_{L^1} \| a(p^{(i)}) \|.
\]

**Proof.** By Theorem 8.1, the function \( \phi \) on \( \{\text{Res} > 0\} \), which is a sum of a Dirichlet series with the coefficient function
\[
c(p^{(i)}) := H_{\Phi, r} a(p^{(i)}),
\]
belongs to \( D_\infty \). Then, by the theorem of Bohr mentioned above and Theorem 8.1,
\[
\sum_{\alpha \in E} |(H_{\Phi, r} a(p^{(i)}))(\alpha)| = \sum_{\alpha \in E} |c(p^{(i)})| = \sum_{p \in \mathbb{P}} |c(p)| \leq \| \phi \|_{\infty} = \| c(p^{(i)}) \|
\]
\[
= \| H_{\Phi, r} a(p^{(i)}) \|_{\infty} \leq \| H_{\Phi, r} \|_{\mathcal{L}(D_\infty)} \| a(p^{(i)}) \| \leq \| \Phi \|_{L^1} \| a(p^{(i)}) \|,
\]
as required. \( \square \)

**Remark 3.** An obvious way to construct the family of automorphisms of \( \mathbb{Z}^\omega \) that meet the condition of Theorem 8.1 is as follows. Let \( \Omega = \text{Sym}(\mathbb{N}) \) be the set of all permutations of \( \mathbb{N} \). (It is known that \( \text{Card}(\Omega) = 2^{\aleph_0} \).) Then for every \( u \in \text{Sym}(\mathbb{N}) \), the map
\[
\sigma_u(\alpha) := (\sigma_u(\alpha)_k)_{k \in \mathbb{N}} \quad (\alpha = (\alpha_k)_{k \in \mathbb{N}} \in \mathbb{Z}^\omega)
\]
belongs to \( \text{Aut}(\mathbb{Z}^\omega) \) and maps both \( (\mathbb{Z}^\omega)^c \) and \( \mathbb{Z}^\omega \) onto itself. Hence, we get the following.

**Corollary 8.3.** Let \( \mu \) be some measure on \( \text{Sym}(\mathbb{N}) \) and \( \Phi \in L^1(\mu) \). Then, a Hausdorff operator \( H_{\Phi, \sigma} \) acts in \( \mathcal{S}^p \) and
\[
\| H_{\Phi, \sigma} \|_{\mathcal{L}(\mathcal{S}^p)} \leq \| \Phi \|_{L^1}.
\]

We now consider another concrete family of automorphisms of \( \mathbb{Z}^\omega \) that meet the condition of Theorem 8.1.

Let \( \Omega = \mathbb{Z}^\omega_+ \) (with the counting measure). For each \( u \in \mathbb{Z}^\omega_+ \), define the map \( \rho_u : \mathbb{Z}^\omega \to \mathbb{Z}^\omega \) as follows:
\[
\rho_u(\alpha) := (\alpha_1, -u_1\alpha_1 + \alpha_2, \ldots, -u_{k-1}\alpha_{k-1} + \alpha_k, \ldots).
\]
Then, \( \rho_u \in \text{Aut}(\mathbb{Z}^\omega) \), and its inverse is given by the rule \( \rho_u^{-1}(\beta) := (\alpha \in \mathbb{Z}^\omega \text{ and } \alpha \in \mathbb{Z}^\omega \text{ satisfies the following recurrent relation: } \alpha_1 := \beta_1, \alpha_k := \beta_k + u_{k-1}\alpha_{k-1} (k \geq 2) \).
**Corollary 8.4.** Let $\Phi \in C^1(\mathbb{Z}^\infty_+)$. Then, the discrete Hausdorff operator

$$
(H_{\Phi, \rho_a}(p^u)(\alpha)) = \sum_{u \in \mathbb{Z}^\infty_+} \Phi(u) a(p^{\rho_u(\alpha)})
$$

acts in $\mathcal{S}^p$ and $\|H_{\Phi, \rho}\|_{L^p(\mathcal{S})} \leq \|\Phi\|_{C^1}$.

**Proof.** The automorphism $\rho_u$ of $\mathbb{Z}^\infty$ does map the set $(\mathbb{Z}^\infty)^c = \{\alpha \in \mathbb{Z}^\infty : \exists k \alpha_k < 0\}$ into itself (indeed, if $\alpha_k$ is the first negative entry of $\alpha \in (\mathbb{Z}^\infty)^c$ and $\beta = \rho_u(\alpha)$, then $\beta_k < 0$). It remains to note that in our case the operator $H_{\Phi, \rho}$ has the form (8.2) because the function $a(p^u)$ is supported in $\mathbb{Z}^\infty_+$.

For another result in this direction, see Corollary 8.7.

### 8.2 General Dirichlet series

To formulate and prove similar results on general Dirichlet series, we need some notation, definitions, and results from [6, 7, 32].

Let $\lambda = (\lambda_n)$ be a non-negative strictly increasing sequence of real numbers tending to $\infty$ (“a frequency”). The value $L(\lambda) := \limsup_{n \to \infty} (\log n)/\lambda_n$ (the maximal width of the strip of convergence and non-absolutely convergence of the corresponding Dirichlet series) is associated with a frequency $\lambda$.

A compact Abelian group $G$ is called a $\lambda$-Dirichlet group if there is a continuous homomorphism $\beta : \mathbb{R} \to G$ with dense range such that every continuous character $\hat{\lambda}_n = e^{-i\lambda_n \cdot}$ of $\mathbb{R}$ has an “extension” $h_{\lambda_n} \in X$ (which then is unique) such that $h_{\lambda_n} \circ \beta = \hat{\lambda}_n$.

We consider formal general Dirichlet series

$$D_\lambda = \sum_{n=1}^{\infty} a_n(D)e^{-\lambda_n s}.$$  

In [7], the next two spaces were introduced.

$$D_\infty(\lambda) := \{D_\lambda : D_\lambda \text{ converge to a function from } H^\infty(\{\text{Re} > 0\})\},$$

and $D_\infty^\text{ext}(\lambda)$ of all somewhere convergent $\lambda$-Dirichlet series, which have a holomorphic and bounded extension to the right half-plane $\{\text{Re} > 0\}$. In general, $D_\infty(\lambda) \subseteq D_\infty^\text{ext}(\lambda)$ and Theorem 2.2 from [7] gives sufficient conditions for the equality here. Moreover, if $L(\lambda) < \infty$, then the space $D_\infty^\text{ext}(\lambda)$ is complete with respect to the supremum norm over $\{\text{Re} > 0\}$ [32, Theorem 5.1].

Let $(G, \beta)$ be a $\lambda$-Dirichlet group. Following [7] for $f \in L^1(G)$, we consider formal general Dirichlet series of the form

$$D_{f, \lambda} = \sum_{n=1}^{\infty} \hat{f}(h_{\lambda_n})e^{-\lambda_n s}.$$  

(8.3)

If the space $D_\infty^\text{ext}(\lambda)$ is complete, one has

$$D_\infty^\text{ext}(\lambda) = D_\infty(\lambda) = \{D_{f, \lambda} : f \in H_1^\infty(G) \text{ where } E = \{h_{\lambda_n} : n \in \mathbb{N}\}\}$$

(8.4)

(see [6, Theorem 4.1], and references therein).

We introduce a Hausdorff operator on (formal) general Dirichlet series of the form (8.3) as follows.

**Definition 8.5.** Let $(G, \beta)$ be a $\lambda$-Dirichlet group, $\Phi \in C^1(\mu)$, and $\{\tau_u : u \in \Omega\} \subset \text{Aut}(X)$. For $f \in L^1(G)$, we put

$$H_{\Phi, \tau} D_{f, \lambda} := D_{g, \lambda},$$

where $g = H_{\Phi, (\tau^*)^{-1}} f$.  

This definition is correct, because \( g \in L^1(G) \).

Since \( \hat{g} = H_{\Phi, \tau} \hat{f} \) by Theorem 4.1, Definition 8.5 means that

\[
H_{\Phi, \tau} : \sum_{n=1}^{\infty} \hat{f}(h_{\lambda_n}) e^{-\lambda_n s} \mapsto \sum_{n=1}^{\infty} (H_{\Phi, \tau}(\hat{f}))(h_{\lambda_n}) e^{-\lambda_n s}.
\]

**Theorem 8.6.** Let \( (G, \beta) \) be a \( \lambda \)-Dirichlet group, \( E := \{ h_{\lambda_n} : n \in \mathbb{N} \} \), \( \tau_u : E^c \rightarrow E^c \) for all \( u \in \Omega \), and \( \Phi \in L^1(\mu) \). If \( L(\lambda) < \infty \), then \( H_{\Phi, \tau} \) acts in \( D_\infty(\lambda) \) and \( \| H_{\Phi, \tau} \|_{\mathcal{L}(D_\infty)} \leq \| \Phi \|_{L^1}. \)

**Proof.** Since \( L(\lambda) < \infty \), we have that \( D_\infty^{ext}(\lambda) \) is complete by [32, Theorem 5.1]. Therefore, Equation (8.4) holds. Let \( D \mathcal{F}, \lambda \in D_\infty^{ext}(\lambda) \). Then, \( f \in H^\infty(E(\gamma)) \) and the function \( g = H_{\Phi, \tau}(\mathcal{F}) \) belongs to the space \( H^\infty(E(\gamma)) \), too, by Theorem 5.1. Thus, the operator \( H_{\Phi, \tau} \) acts in \( D_\infty^{ext}(\lambda) \).

Following [7], consider the Bohr map \( B : H^{\infty}_E(G) \rightarrow D_\infty(\lambda), f \mapsto D \mathcal{F}, \lambda \).

As mentioned, in our case \( D_\infty^{ext}(\lambda) = D_\infty(\lambda) \). Therefore, Theorem 4.12 in [7] states that \( B \) is an isometrical isomorphism of Banach spaces. But the equality \( H_{\Phi, \tau} \mathcal{F} = \mathcal{F} \mathcal{G} \mathcal{H}_{\Phi, \tau}(\mathcal{F})^{-1} \mathcal{F} \) for all \( f \in H^{\infty}_E(G) \). In other words,

\[
H_{\Phi, \tau} = B \mathcal{H}_{\Phi, \tau}(\mathcal{F})^{-1} B^{-1}.
\]

It follows that \( \| H_{\Phi, \tau} \|_{\mathcal{L}(D_\infty)} = \| H_{\Phi, \tau}(\mathcal{F})^{-1} \|_{\mathcal{L}(H^{\infty}_E)} \leq \| \Phi \|_{L^1}. \) This completes the proof. \( \square \)

One can apply Theorem 8.6 to the space \( D_\infty = D_\infty((\log n)) \) of ordinary Dirichlet series and obtain the following.

**Corollary 8.7.** Let \( G = \mathfrak{b} \mathbb{R} \) be the Bohr compactum, \( \Omega = \{ 1/q : q \in \mathbb{N} \} \), and \( \tau_u(\gamma) = u \gamma \) for \( u \in \Omega \). If \( \Phi \in \ell^1(\Omega) \), then the discrete Hausdorff operator \( H_{\Phi, \tau} \) acts in the space \( D_\infty \) of ordinary Dirichlet series and \( \| H_{\Phi, \tau} \|_{\mathcal{L}(D_\infty)} \leq \| \Phi \|_{L^1}. \)

**Proof.** First, we note that by [7, Example 3.19] the Bohr compactum \( (\mathfrak{b} \mathbb{R}, \beta) \), where \( \beta(\mathfrak{t}) = \hat{\mathfrak{t}} \), is a \( \lambda \)-Dirichlet group for any frequency \( \lambda \). Further, in our case \( \lambda_n = \log n \). If we identify the group \( \mathbb{R}^d \) with the dual for \( \mathfrak{b} \mathbb{R} \), then \( E = \{ h_{\lambda_n} : n \in \mathbb{N} \} = \{ \log n : n \in \mathbb{N} \} \). Since \( \tau_u : E^c \rightarrow E^c \) for all \( u \in \Omega \) and \( L((\log n)) = 1 \), the result follows from Theorem 8.6. \( \square \)

### 9 EXAMPLES IN THE CASE OF ORDERED DUAL

**Example 4.** Let \( G = \mathfrak{b} \mathbb{R} \) be the Bohr compactification of the reals, \( X = \mathbb{R}_d \) as in Example 2. The map \( \tau_u : X \rightarrow X, \gamma \mapsto u \gamma \) belongs to \( \text{Aut}_+(X) \) for \( u \in (0, \infty) \). Since \( \tau_u(\gamma) = u \gamma \) for \( u \in \Omega \) and \( \tau_u : E^c \rightarrow E^c \) for all \( u \in \Omega \) and \( L((\log n)) = 1 \), the result follows from Theorem 8.6.

**Example 5.** Let \( G = \mathbb{T}^d \) be the \( d \)-dimensional torus (\( d \geq 2 \)). Let \( \Omega \) be the subgroup of the arithmetic group \( \text{GL}(d, \mathbb{Z}) \) which consists of matrices \( u = (u_{ij})_{i,j=1}^{d} \) with \( \det u = \pm 1 \). Then, every map

\[
A(u)(z) = z^u := (z_1^{u_{11}} z_2^{u_{12}} \cdots z_d^{u_{1d}}, \ldots, z_1^{u_{d1}} z_2^{u_{d2}} \cdots z_d^{u_{dd}})
\]
\((z = (z_j)_{j=1}^d \in \mathbb{T}^d)\) belongs to \(\text{Aut}(\mathbb{T}^d)\) (see, e.g., [14, (26.18)(h)]). Thus, the corresponding Hausdorff operator over \(\mathbb{T}^d\) is of the form

\[
(H_{\Phi,f})(z) = \int_{\mathbb{T}^d} \Phi(u)f(z^u)d\mu(u),
\]

where \(\mu\) stands for a regular Borel measure on \(\Omega\) (e.g., \(\mu\) is the Haar measure of the group \(\Omega\)).

Every character of \(\mathbb{T}^d\) is of the form \(\chi_n(z) = z_1^{n_1} \cdots z_d^{n_d}\), where \(n = (n_1, \ldots, n_d) \in \mathbb{Z}^d\). Thus, the dual of \(\mathbb{T}^d\) can be identified with the group \(\mathbb{Z}^d\) via the map \(\chi_n \mapsto n\). We endow \(\mathbb{Z}^d\) with the lexicographic order. For this order, the positive cone is

\[
X_+ = \{n \in \mathbb{Z}^d : n_1 > 0\} \cup \{n \in \mathbb{Z}^d : n_1 = 0, n_2 > 0\} \cup \cdots \cup \{n \in \mathbb{Z}^d : n_1 = \cdots = n_{d-1} = 0, n_d > 0\} \cup \{0\}.
\]

Consider the arithmetic strict lower triangular group \(T_1(d, \mathbb{Z})\). This group consists of matrices \(u \in \text{SL}(d, \mathbb{Z})\) such that \(u_{ii} = 1\), and \(u_{ij} = 0\) for \(i < j\). Then, the map

\[
\tau_u(n) := un^T = (n_1, u_{21}n_1 + n_2, \ldots, u_{d1}n_1 + \cdots + u_{d,d-1}n_{d-1} + n_d)
\]

(here \(n^T \in \mathbb{Z}^d\) is a column vector) belongs to \(\text{Aut}_+(X)\).

Example 6. Let \(\mathbb{T}^\infty\) be the infinite-dimensional torus and let \(X = \mathbb{Z}^\infty_{\text{lex}}\) be the additive group of infinite sequences of integers with finite support endowed with the lexicographic order. For this order, by definition the positive cone is

\[
X_+ = \{0\} \cup \{x \in \mathbb{Z}^\infty : x_1 > 0\} \cup \{x \in \mathbb{Z}^\infty : x_1 = 0, x_2 > 0\} \cup \cdots
\]

In other words, \(X_+\) consists of sequences whose first non-zero entry is positive and the zero sequence. As above, we identify the group \(\mathbb{Z}^\infty\) with the dual group of \(\mathbb{T}^\infty\) via the map \(\alpha \mapsto \chi_\alpha\), where \(\chi_\alpha(z) = z_1^{\alpha_1} z_2^{\alpha_2} \cdots (z \in \mathbb{T}^\infty)\). Let \(J(\infty, \mathbb{Z})\) consist of infinite lower two-diagonal matrices \(u\) of integers such that \(u_{ii} = 1\), \(u_{ij} = 0\) for \(i < j\), and \(u_{k,1} = \cdots = u_{k,k-2} = 0\) for \(k \geq 3\).

Then, the map

\[
\tau_u(\alpha) := u\alpha^T = (\alpha_1, u_{21}\alpha_1 + \alpha_2, u_{32}\alpha_2 + \alpha_3, \ldots, u_{k,1}\alpha_{k-1} + \alpha_k, \ldots)
\]

\((u \in J(\infty, \mathbb{Z}), \text{we have that } \alpha \in \mathbb{Z}^\infty)\) belongs to \(\text{Aut}_+(\mathbb{Z}^\infty_{\text{lex}})\). Since

\[
\tau_u(z) = z^u := (z_1, z_1^{u_{21}} z_2, \ldots, z_{k-1}^{u_{k,k-1}} z_k, \ldots),
\]

in this case,

\[
(H_{\Phi,f})(z) = \int_{J(\infty, \mathbb{Z})} \Phi(u)f(z^u)d\mu(u)
\]

where \(\mu\) is a regular Borel measure on \(J(\infty, \mathbb{Z})\).
This operator is bounded on $H^p(\mathbb{T}^\infty)$ $(1 \leq p \leq \infty)$, BMOA$(\mathbb{T}^\infty)$, $H^1_\mathbb{R}(\mathbb{T}^\infty)$ (for real-valued $\Phi$), and BMO$(\mathbb{T}^\infty)$ if and only if $\Phi \in L^1(\mu)$ and its norm does not exceed $\|\Phi\|_{L^1(\mu)}$.

**Example 7.** Let $a = (2, 3, 4, \ldots)$. Then, the $a$-adic solenoid $\Sigma_a$ (see, e.g., [14, (10.12)]) is a compact and connected topological group which is topologically isomorphic to the character group $\hat{\mathbb{Q}}_d$ of the discrete additive group $X = \mathbb{Q}_d$ of rationals [14, (25.4)]. On the other hand, by [14, (25.5)] the group $\hat{\mathbb{Q}}_d$ can be identified with some subgroup $G$ of the infinite-dimensional torus $\mathbb{T}^\infty$ in the following way. Let the sequence $\alpha = (\alpha_n)_{n \in \mathbb{N}} \in \mathbb{T}^\infty$ be such that $\alpha_n = \alpha_{n+1}$ for all $n \in \mathbb{N}$. Then, it produces a character of $\mathbb{Q}_d$ via the rule

$$\chi_\alpha\left(\frac{m}{n}\right) = \alpha_n^m \quad (m \in \mathbb{Z}, n \in \mathbb{N}).$$

Moreover, each character of $\mathbb{Q}_d$ can be identified with such a sequence $\alpha$ and we get an isomorphism $\alpha \mapsto \chi_\alpha$ of the subgroup $G := \{\alpha\} \subset \mathbb{T}^\infty$ and $\Sigma_a$. Thus, one can obtain the group $G$ with $\Sigma_a$. Further, for each $q \in \mathbb{Q}$, $q > 0$, the map $l_q(x) = qx$ is an order automorphism of the group $\mathbb{Q}_d$ endowed with the usual order. It follows that the corresponding dual automorphism $l^*_q$ of the dual group $G = \Sigma_a$ belongs to $\text{Aut}(\Sigma_a)^+$. This implies that for every measurable map $k : \Omega \to \mathbb{Q}_d \setminus \{0\}$, the corresponding Hausdorff operator

$$H_{\Phi,l^*_q}f(\alpha) = \int_{\Omega} \Phi(u)f(l^*_q(u)(\alpha))d\mu(u)$$

is bounded on $H^p(\Sigma_a)$ $(1 \leq p \leq \infty)$, BMOA$(\Sigma_a)$, $H^1_\mathbb{R}(\Sigma_a)$ (for real valued $\Phi$), and BMO$(\Sigma_a)$ if and only if $\Phi \in L^1(\mu)$, and its norm does not exceed $\|\Phi\|_{L^1(\mu)}$.

**Example 8.** Let $G$ be a compact and connected Abelian group with totally ordered dual and $\Omega$ a compact subgroup of $\text{Aut}(G)$ with normalized Haar measure $\mu$. The *generalized shift operator of Delsarte* [9, 17, Ch. I, Section 2] (also the terms “generalized translation operator of Delsarte”, or “generalized displacement operator of Delsarte” are used) is defined to be

$$T^hf(x) = \int_{\Omega} f(hu(x))d\mu(u) \quad (x, h \in G).$$

Then, $T^h = H_1S_h$, where

$$H_1f(x) := \int_{\Omega} f(u(x))d\mu(u)$$

is a Hausdorff operator on $G$ with $\Phi \equiv 1$, $A(u) = u$, and $S_hf(x) := f(hx)$. Let $u \in \text{Aut}(G)^+$ for $\mu$-a.e. $u \in \Omega$. Then for every fixed $h$, the generalized shift operator of Delsarte is bounded on $H^p(G)$ $(1 \leq p \leq \infty)$, BMOA$(G)$, BMO$(G)$, and $H^1_\mathbb{R}(G)$. In addition, its norm in this space equals to $\mu(\Omega) = 1$ (Remark 1).

**ACKNOWLEDGMENTS**

The author is partially supported by the State Program of Scientific Research of Republic of Belarus, project No. 20211776 and by the Ministry of Education and Science of Russia, agreement No. 075-02-2023-924.

**CONFLICT OF INTEREST STATEMENT**

The author declares no conflict of interests.

**ENDNOTES**

1. The author is indebted to Professor A. Bendikov for this observation.
2. Here, we correct a typo made in [10, p. 139].

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