Canonical Submersions in Nearly Kähler Geometry

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We explore submersions introduced by reducible holonomy representations of connections with parallel skew torsion. A submersion theorem extending previous less general results is given. As our main application we show that parallel 3-(α, δ)-Sasaki manifolds admit 1-dimensional submersions onto nearly Kähler orbifolds. As a secondary application we reprove that a certain class of nearly Kähler manifolds submerges onto quaternionic Kähler manifolds. This new proof gives an direct expression for the quaternionic structure on the base.

1 Introduction

The investigation of holonomy groups has played a key role in Riemannian geometry over the last century. Thanks to DeRham splitting one can reduce the investigation to irreducible holonomy representations. Only therefore it is possible to classify them in the shape of Berger’s list of special holonomies. However, these cases are narrow and many, in particular odd dimensional, geometries fail to be represented. An early generalization due to Gray were weak holonomy groups, extending to the classes of nearly Kähler, nearly parallel $G_2$ and nearly parallel Spin(9) manifolds [Gra71; Fri01]. We are instead considering geometries that admit a metric connection with skew symmetric torsion, or geometries with skew torsion for short. These include the aforementioned classes of weak holonomies and are also of widespread interest in type II string theory, compare [GMW04; FI02; Fri03].

Unlike with the Levi-Civita connection, for connections with skew torsion there exist no general DeRham splitting so reducible representations are very much to be considered. In these cases the reducibility tells us a great deal about the geometry. In fact, under the assumption of parallel torsion R. Cleyton, A. Moroianu and U. Semmelmann in [CMS21] show that they always admit a locally defined Riemannian submersion. We refine their theorem giving us control over the submersion motivated by the case of 3-(α, δ)-Sasaki manifolds in [ADS21]. We will call them canonical submersions.
3-(\(\alpha, \delta\))-Sasaki manifolds were defined by the condition \(d\eta_i = 2\alpha\Phi_i + 2(\alpha - \delta)\xi_j \wedge \xi_k\) in [AD20] as a generalization of 3-Sasaki manifolds when \(\alpha = \delta = 1\). Pivotaly they admit a canonical connection \(\nabla\) with skew-torsion. In [ADS21] it was shown that \(\nabla\) gives rise to a canonical submersion in above sense, whose base admits a quaternionic Kähler, resp. Hyperkähler, structure. This divides them into positive, negative and degenerate 3-(\(\alpha, \delta\))-Sasaki structures. In the positive realm there are various interesting cases of parameters. Apart from aforementioned 3-\(\alpha\)-Sasakian if \(\alpha = \delta = 1\) there is the secondary Einstein metric or so-called squashed 3-Sasaki Einstein metric if \(\delta = (2n + 3)\alpha\), [AD20]. We are particularly interested in the parallel case \(\delta = 2\alpha\). In this case \(\nabla\) parallelizes the Reeb vector fields. We will show that this yields apart from the, in most cases maximal, submersion discussed in [CMS21] and [ADS21] another minimal canonical submersion onto nearly Kähler spaces. This is the first instance investigated of canonical submersions whose base is a geometry with non-vanishing torsion showing that this possibility is not an inconvenience, but a feature of the theory.

Nearly Kähler manifolds are Hermitian, non Kähler manifolds, known to exist on the twistor spaces of quaternionic Kähler manifolds, [ES85]. It therefore shouldn’t come as a surprise they can be obtained as a quotient of the Konishi space. However, our proof shows that the nearly Kähler structure and it’s canonical Hermitian connection are tightly locked with the parallel 3-(\(\alpha, \delta\))-Sasaki structure and it’s canonical connection. A second application closes the circle showing how a class of nearly Kähler manifolds, essentially those obtained by above construction, admit a further canonical submersion onto quaternionic Kähler spaces. This result was shown previously in the decomposition of nearly Kähler spaces of dim \(\geq 6\) by P.-A. Nagy in [Nag02] and locally in [Ale06]. However, we include it since the proof fits neatly into our established framework of canonical submersions.

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**2 Canonical Submersions**

On a Riemannian manifold \((M, g)\) a metric connection \(\nabla\) on the tangent bundle is uniquely characterized by its torsion tensor

\[ T^\nabla(X, Y, Z) := g(\nabla_X Y - \nabla_Y X - [X, Y], Z). \]

By definition the torsion is skew-symmetric in the first two entries. If, however it is totally skew-symmetric also in the last entry we say that \(\nabla\) has *skew torsion*. In this case the connection is given by

\[ g(\nabla_X Y, Z) = g(\nabla^\nabla_X Y, Z) + \frac{1}{2}T^\nabla(X, Y, Z) \]
where we denote by $\nabla^g$ the Levi-Civita connection of $(M, g)$. We require additionally that our connection $\nabla$ has parallel torsion, i.e. $\nabla T = 0$. This yields many simplifications, among them such connections satisfy the following Bianchi-Identity:

$$X,Y,Z \sigma R(X,Y,Z,V) = \sigma T(X,Y,Z,V) := X,Y,Z g(T(X,Y),T(Z,V)), \quad (1)$$

where the shorthand notation $\sigma$ denotes the sum over all cyclic permutations of $X,Y,Z$.

With those preliminaries we pose our first main theorem. Given a manifold admitting a connection with parallel skew-torsion and reducible holonomy. Then the Canonical Submersion Theorem answers the question how such a manifold can be simplified. Earlier more restrictive version of this theorem were presented in [CMS21], [ADS21] and [Ste21]. However, since the assumptions are simplified we include the proof.

**Theorem 2.1.** Suppose $\nabla$ is a metric connection with parallel skew torsion $T$ on $(M, g)$ and $TM = \mathcal{H} \oplus \mathcal{V}$ splits orthogonally as representation of the holonomy group $\text{Hol}_0(\nabla)$. Assume further that $T \in \Lambda^3 \mathcal{H} \oplus \Lambda^2 \mathcal{H} \wedge \mathcal{V} \oplus \Lambda^3 \mathcal{V} \subseteq \Lambda^3 TM$, \quad (2)

i.e. the $\Lambda^2 \mathcal{V} \wedge \mathcal{H}$-part of $T$ vanishes.

Then there exists a locally defined Riemannian submersion $\pi: (M, g) \to (N, g_N)$ with totally geodesic fibers tangent to $\mathcal{V}$, the purely horizontal part of the torsion $T^\mathcal{H}$ is projectable, $\pi^* T = T^\mathcal{H}$, and $\nabla^\mathcal{H} = \nabla^g_N + \frac{1}{2} T$ is a connection with parallel skew torsion on $N$ satisfying

$$\nabla^\mathcal{H}_X Y = \pi_*(\nabla_X Y), \quad (3)$$

where $\overline{X}, \overline{Y}$ denote the horizontal lifts of $X,Y \in TN$.

**Proof.** We note that by (2) and the invariance of $\mathcal{V}$ under $\nabla$ for any vertical vector fields $V,W \in \mathcal{V}$ we have

$$\nabla^\mathcal{V}_V W = \nabla^\mathcal{V}_W V - \frac{1}{2} T(V,W) \in \mathcal{V}.$$ 

Therefore, the distribution $\mathcal{V}$ is integrable and a curve is geodesic on the integral submanifold tangent to $\mathcal{V}$ if and only if it is a geodesic in $M$. The integral submanifolds give rise to a foliation and hence to a submersion $\pi$ from a small neighborhood $U \subset M$ to a local transverse section $S$. We show that the metric restricted to $\mathcal{H} \times \mathcal{H}$ is constant along vertical vector fields and therefore projectable. For $X,Y \in \mathcal{H}$ we have

$$(L_V g)(X,Y) = V(g(X,Y)) - g([V,X],Y) - g(X,[V,Y]) = g(\nabla^\mathcal{V}_X V,Y) + g(X,\nabla^\mathcal{V}_X Y)$$

$$= g(\nabla^\mathcal{V}_X Y) + g(X,\nabla^\mathcal{V}_Y V) = 0$$

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since $\mathcal{V}$ is preserved by $\nabla^T$. This proves that $\pi$ is a Riemannian submersion.

To prove the second assertion we denote $T^H = \text{pr}_{\Lambda^3H}T$. If we show that $T^H$ is constant along the fibers it projects to a well-defined 3-form $\sigma$. Let $V \in \mathcal{V}$. Then

$$X,Y,Z \otimes T^H(X,Y,T(V,Z)) = 0 \quad (4)$$

whenever either $X,Y,Z \in \mathcal{V}$. Indeed, by (2) we have that $T(V,Z) \in \mathcal{V}$ if $Z \in \mathcal{V}$ and, hence, $T(V,Z) \otimes T^H = 0$. Now consider $X,Y,Z \in \mathcal{H}$. Since $\nabla^T$ preserves $\mathcal{V}$ and $\mathcal{H}$ the curvature $R^{\nabla^T}(X,Y,Z,V) = 0$ and the Bianchi identity for connections with parallel skew torsion, (1), implies

$$0 = (\nabla^T_{X}T^H(X,Y,Z,V) = -\sigma_T(X,Y,Z,V) = -X,Y,Z \otimes g(T(X,Y),T(Z,V))$$

where the last step used again that $T(V,Z) \in \mathcal{H}$. Thus, (4) holds for any $X,Y,Z \in TM$.

As the subspaces $\mathcal{V}$, $\mathcal{H}$ are preserved by $\nabla^T$ so are the components of tensors on $TM$. In particular, $\nabla^T T = 0$ implies $\nabla^T T^H = 0$. Use (4) and $\nabla^T V \in \mathcal{V}$ to obtain

$$L_V T^H(X,Y,Z) = V(T^H(X,Y,Z)) - X,Y,Z \otimes T^H(X,Y,L_V Z)$$

$$= (\nabla^T V T^H)(X,Y,Z) + X,Y,Z \otimes T^H(X,Y,\nabla^T V Z)$$

$$- X,Y,Z \otimes T^H(X,Y,\nabla^T V Z - \nabla^T Z - T(V,Z))$$

$$= X,Y,Z \otimes T^H(X,Y,T(V,Z)) = 0.$$

Equation (3) follows directly from $\nabla^T_X Y = \pi_*(\nabla^T_X Y)$ for Riemannian submersions. Finally we conclude that $\mathcal{T}$ is $\nabla^T$-parallel:

$$(\nabla^T_X \mathcal{T})(X,Y,Z) = A(\mathcal{T}(X,Y,Z)) - X,Y,Z \otimes \mathcal{T}(X,Y,\nabla^T_X Z)$$

$$= A(\pi \circ T^H(\mathcal{X},\mathcal{Y},\mathcal{Z})) - X,Y,Z \otimes \mathcal{T}(X,Y,\pi_*(\nabla^T_X Z))$$

$$= \mathcal{A}(T^H(\mathcal{X},\mathcal{Y},\mathcal{Z})) - X,Y,Z \otimes T^H(\mathcal{X},\mathcal{Y},\nabla^T_X Z)$$

$$= (\nabla^T_X T^H)(X,Y,Z) = 0 \quad \square$$

Observe that compared to earlier versions this yields the following well known generalization of DeRham splitting as an immediate consequence.

**Corollary 2.2.** Let $\nabla^T$ be a connection with parallel skew-torsion $T$ and $TM = V_1 \oplus V_2$ a holonomy-invariant decomposition. Then locally

$$(M,g,\nabla^T) = (M_1,g_1,\nabla^{T_1}) \times (M_2,g_2,\nabla^{T_2})$$

is a product if and only if the torsion is decomposable, i.e. $T = T_1 + T_2$ with $T_i \in \Lambda^3V_i$. 
Proof. Here (2) is satisfied for $V = V_1$, $H = V_2$ and vice versa. Hence, both distributions are integrable and we obtain locally defined Riemannian projection maps to their respective integral submanifolds. The converse is clear.

We now show that a further splitting of the holonomy representation can manifest a splitting on the base.

**Proposition 2.3.** Suppose the holonomy representation of $\nabla^T$ splits into orthogonal subspaces $TM = V \oplus H_1 \oplus H_2$ such that $V$ and $H = H_1 \oplus H_2$ satisfy condition (2) of Theorem 2.1. Let $\pi : M \to N$ denote the canonical submersion and suppose further that $H_1$ and $H_2$ are projectable. Then the representation $TN$ of the holonomy $\text{Hol}_0(\nabla^T)$ is reducible into modules $\pi_*H_1 \oplus \pi_*H_2$.

Proof. By the Ambrose-Singer theorem the holonomy algebra at $x \in N$ is generated by elements of the form

$$(P_{\gamma}^{\nabla^T})^{-1} \circ R_{\nabla^T}^{\nabla^T}(X \wedge Y) \circ P_{\gamma}^{\nabla^T}$$

where $P_{\gamma}^{\nabla^T}$ denotes parallel transport along a piecewise smooth curve $\gamma$ from $x$ to $p$ and $R_{\nabla^T}^{\nabla^T}(X \wedge Y) \in \Lambda^2 T_pN \cong \text{End}(T_pN)$ is the evaluation of the curvature operator at $X, Y \in T_pN$. For any such curve let $\overline{\gamma}$ be the horizontal lift starting at a point $x_0 \in \pi^{-1}(x)$. Then let $X(t)$ be the unique parallel vector field along $\gamma$. By (3)

$\pi_*(\nabla_{\overline{\gamma}X(t)}^T) = \nabla_{\overline{\gamma}X}^T X(t) = 0$

and hence $\nabla_{\overline{\gamma}X(t)}^T X(t) \in V$. However, $\nabla_{\overline{\gamma}X(t)}^T X(t) \in H$ since $\overline{X}(t) \in H$ and $H$ is invariant under the holonomy of $\nabla$. Therefore $\nabla_{\overline{\gamma}X}^T X(t) = 0$ and

$$P_{\overline{\gamma}}^{\nabla^T} X = \overline{P_{\gamma}^{\nabla^T} X}.$$  

In particular, parallel transport with respect to $\nabla^T$ preserves $\pi_*H_1$ and $\pi_*H_2$. Now let $X, Y \in T_pN$, then

$$R_{\nabla^T}^{\nabla^T}(X \wedge Y)Z = \nabla_{\overline{\gamma}X}^T \overline{\gamma}Y^T Z - \nabla_{\overline{\gamma}Y}^T \overline{\gamma}X^T Z - \nabla_{\overline{\gamma}[X,Y]}^T Z$$

$$= \pi_*(\nabla_{\overline{\gamma}X}^T \overline{\gamma}Y^T Z - \nabla_{\overline{\gamma}Y}^T \overline{\gamma}X^T Z - \nabla_{\overline{\gamma}[X,Y]}^T Z)$$

preserves the modules $\pi_*H_1$ and $\pi_*H_2$ as well.  

The following computational lemma has already been used in [ADS21]. However, it holds for any canonical submersion.
Lemma 2.4. Let $\nabla$ be a connection as in Theorem 2.1. Then
\[
g(\nabla_X Y, Z) = T(X, Y, Z)
\]
for any vertical vector $X \in \mathcal{V}$, horizontal vector $Z \in \mathcal{H}$, and basic vector field $Y \in \mathcal{H}$. In particular, the expression is tensorial.

Proof. Since $Y$ is basic we have $[X, Y] \in \mathcal{V}$ and $\nabla_X Y \in \mathcal{V}$ as the decomposition is $\text{hol}(\nabla)$-invariant. Therefore
\[
g(\nabla_X Y, Z) = g(\nabla_Y X, Z) + g([X, Y], Z) + T(X, Y, Z) = T(X, Y, Z). \quad \Box
\]

3 Parallel $3-(\alpha, \delta)$-Sasaki and Nearly Kähler Manifolds

We like to introduce the structures involved in the application of canonical submersions we consider in this paper, in particular $3-(\alpha, \delta)$-Sasakian and nearly Kähler manifolds. An almost contact metric manifold $(M^{2n+1}, g, \xi, \eta, \phi)$ is given by an odd dimensional Riemannian manifold, a unit length vector field $\xi \in T M$ called Reeb vector field, its metric dual 1-form $\eta$ and an almost Hermitian structure $\phi \in \text{End}(T M)$ on $\ker \eta$ satisfying
\[
\phi \xi = 0, \quad \eta \circ \phi = 0, \quad \phi^2 = -\text{id} + \xi \wedge \eta,
\]
\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).
\]
These are considered the odd dimensional analog of almost Hermitian manifolds. As for the latter we define the fundamental 2-form $\omega(X, Y) = g(X, \phi Y)$ for a given almost contact metric manifold.

A tuple $(M, g, \xi_i, \phi_i, \eta_i)_{i=1,2,3}$ of three almost contact metric structures on the same underlying Riemannian manifold is called almost 3-contact metric manifold if they additionally satisfy the compatibility conditions
\[
\phi_i \xi_j = \xi_k, \quad \eta_i \circ \phi_j = \eta_k, \quad \phi_i \phi_j = \phi_k + \xi_i \otimes \eta_j.
\]
These properties guarantee that the endomorphisms act as imaginary quaternions on the horizontal distribution $\mathcal{H} := \bigcap \ker \eta_i$. Accordingly the complementary distribution $\mathcal{V} := \mathcal{H}^\perp = \text{span}\{\xi_1, \xi_2, \xi_3\}$ is called vertical.

Definition 3.1 ([AD20]). An almost 3-contact metric manifold is called $3-(\alpha, \delta)$-Sasaki manifold for real constants $\delta$ and $\alpha \neq 0$ if
\[
d\eta_i = 2\alpha \Phi_i + 2(\alpha - \delta)\eta_j \wedge \eta_k
\]
for every even permutation $(ijk)$ of $(123)$. 6
Most prominently, if $\alpha = \delta = 1$ the manifold is 3-Sasakian. A second class often considered is the second Einstein metric with parameters $\delta = (2n + 3)\alpha$ where $\dim M = 4n + 3$. Our work highlights the subclass of parallel 3-(\(\alpha, \delta\))-Sasaki manifolds, that is if $\delta = 2\alpha$. Their most prominent feature is highlighted with regards to the canonical connection as defined via the following theorem.

**Theorem 3.2** ([AD20]). A 3-(\(\alpha, \delta\))-Sasaki manifold admits a unique metric connection $\nabla$ with skew torsion such that

$$\nabla_X \varphi_i = \beta(\eta_k(X)\varphi_j - \eta_j(X)\varphi_k),$$  \hspace{1cm} (6)

for every even permutation $(ijk)$ of $(123)$ and some function $\beta \in C^\infty(M)$.

The function is a constant given by $\beta = 2(\delta - 2\alpha)$. The connection $\nabla$ preserves the splitting $TM = V \oplus H$ and its torsion is given by

$$T = 2\alpha \sum_{i=1}^{3} \eta_i \wedge \Phi_i - 2(\alpha - \delta)\eta_{123}.$$  \hspace{1cm} (7)

In particular, the torsion is parallel $\nabla T = 0$.

Hence for parallel 3-(\(\alpha, \delta\))-Sasaki manifolds the canonical connection parallelizes all structure tensors. On homogeneous 3-(\(\alpha, \delta\))-Sasaki manifolds it was shown that the parallel metric is naturally reductive with respect to the standard representation as a quotient of the automorphism group, see [ADS21]. In that case the canonical and Ambrose-Singer connections coincide.

The canonical connection also gave rise for an initial instance of canonical submersions over quaternionic Kähler spaces in [ADS21]. Recall that quaternionic Kähler manifolds $(\tilde{\mathbb{N}}^4, g_{\tilde{\mathbb{N}}}, Q)$, $n \geq 2$ are equipped with a 3-dimensional subbundle $Q \subset \text{End}(T\tilde{\mathbb{N}})$ locally generated by a triple of almost Hermitian structures that is invariant under $\text{Hol}(\nabla g_{\tilde{\mathbb{N}}})$. Equivalently, they are the class of Riemannian manifolds with exceptional holonomy $\text{Sp}(1)\text{Sp}(n)$. For $\dim \tilde{\mathbb{N}} = 4$ we require $\tilde{\mathbb{N}}$ to be Einstein and anti-self-dual.

**Theorem 3.3** ([ADS21]). Let $(M, g, \xi_i, \varphi_i, \eta_i)_{i=1,2,3}$ a 3-(\(\alpha, \delta\))-Sasaki manifold. Then there exists a locally defined Riemannian submersion $\pi: M \to \tilde{\mathbb{N}}$ with totally geodesic fibers tangent to $V$ such that $\nabla^g_X Y = \pi_*(\nabla_X Y)$. The base space $\tilde{\mathbb{N}}$ admits a quaternionic Kähler structure locally generated by the 3 almost Hermitian structures

$$I_i = \pi_* \circ \varphi_i \circ s_*,$$

for any locally defined section $s: \tilde{\mathbb{N}} \to M$.

We should remark that in the following we will fix $\mathcal{H}$ and $\mathcal{V}$ to be the horizontal and vertical distributions for a 3-(\(\alpha, \delta\))-Sasaki manifold while the horizontal/vertical subspaces in further applications of Theorem 2.1 will be denoted appropriate to the situation.

For later use we recall the following Lie derivatives along vertical vectors.
Proposition 3.4 ([AD20]). For any 3-(α, δ)-Sasaki manifold and any even permutation (ijk) of (123) we have the following identities

\begin{align}
\mathcal{L}_{\xi_i} \varphi_i &= 0, & \mathcal{L}_{\xi_i} \varphi_j &= -\mathcal{L}_{\xi_j} \varphi_i = 2\delta \varphi_k \\
\mathcal{L}_{\xi_i} \xi_i &= 0, & \mathcal{L}_{\xi_i} \xi_j &= -\mathcal{L}_{\xi_j} \xi_i = 2\delta \xi_k
\end{align}

With that let us shift attention to nearly Kähler manifolds.

Definition 3.5. A nearly Kähler manifold \((N^{2m}, g_N, J)\) is an almost Hermitian manifold such that \((\nabla^N g)_{XJ}X = 0\) for all \(X \in T_N\).

As initially observed in [Gra76] a nearly Kähler manifold admits a particularly nice connection \(\nabla^c\). We will denote \(\nabla^c\) the characteristic connection of a nearly Kähler manifold as it is the unique connection with skew torsion preserving the \(U(m)\)-structure, compare [Agr06]. In the literature it is often also called Bismut or canonical connection. In fact their usual definitions agree on nearly Kähler manifolds as it is the unique Hermitian connection and has torsion

\[ T^c(X, Y, Z) = g_N((\nabla^N g_X Y, Z). \]

Furthermore, \(T^c\) is parallel with respect to \(\nabla^c\).

4 Nearly Kähler orbifolds from parallel 3-(α, δ)-Sasaki manifolds

Let \((M, g, \xi_i, \varphi_i, \eta_i)_{i=1,2,3}\) be a 3-(α, δ)-Sasaki manifold and \(\nabla\) its canonical connection with parallel skew-torsion \(T\). If \(\beta = 0 \iff \delta = 2\alpha\) we have \(\nabla \varphi = 0\) for all \(\varphi\) in the associated sphere. In this case also \(\nabla \xi = 0\) and \(\nabla \eta = 0\) so the decomposition \(TM = \langle \xi \rangle \oplus \ker \eta = \langle \xi \rangle \oplus \langle \xi \rangle^\perp\) is holonomy invariant. Note that (2) is trivial for any one dimensional vertical distribution. In the following we apply Theorem 2.1 to this setup. We fix the notation \(V\) and \(H\) for the vertical and horizontal spaces of the initial parallel 3-(α, δ)-manifold and denote the vertical and horizontal spaces with respect to this newly obtained submersion by \(\langle \xi \rangle\) and \(\langle \xi \rangle^\perp\) respectively.

Theorem 4.1. Let \((M, g, \xi_i, \varphi_i, \eta_i)_{i=1,2,3}\) be a parallel 3-(α, δ)-Sasaki manifold and fix an almost contact metric structure \((\xi, \varphi, \eta)\) inside the associated sphere.

a) Then there exists a locally defined Riemannian submersion \(\pi: (M, g) \rightarrow (N, g_N)\) along the orbits of \(\xi\).

b) Set \(\tilde{\varphi} := \varphi|_H - \varphi|_V\) and \(J = \pi_* \circ \tilde{\varphi} \circ s_*\) with an arbitrary section \(s: N \rightarrow M\) of \(\pi\). Then \((N, g_N, J)\) is nearly Kähler.

c) The characteristic connection on \((N, g_N, J)\) agrees with the connection obtained from the canonical connection on \(M\). In particular, \(\tilde{T}(X, Y, Z) = g_N((\nabla^N g_X Y, Z)\).
Proof. By the argument ahead the assumptions in Theorem 2.1 are satisfied. We may in the following assume $\xi = \xi_1$. Theorem 2.1 then implies that there is a locally defined Riemannian submersion $\pi: (M, g) \rightarrow (N, g_N)$ along the orbits of $\xi_1$ and horizontal space $\langle \xi_1 \rangle^\perp = \langle \xi_2, \xi_3 \rangle \oplus \mathcal{H}$. Further, there is a connection $\nabla^T = \nabla^{g_N} + \frac{1}{2} \tilde{T}$ on $TN$ with parallel skew torsion given by

$$\pi_* \tilde{T} = T^{\langle \xi \rangle^\perp} = 2\alpha(\eta_2 \land \Phi_2^H + \eta_3 \land \Phi_3^H)$$

(10)

where we have used (7). This connection satisfies

$$\nabla_X^Y = \pi_*(\nabla_X Y).$$

Now consider $J = \pi_* \circ \tilde{\phi} \circ s_*$. Due to (8) we have $\mathcal{L}_\xi \tilde{\phi} = 0$ and since $\mathcal{L}_\xi$ preserves $\mathcal{H}$ and $\mathcal{V}$, also $\mathcal{L}_\xi \tilde{\phi} = 0$. In particular, we have that $J$ is independent of the choice of $s$. The compatibility with $g_N$ follows immediately as $\pi_*: \langle \xi \rangle^\perp \rightarrow TN$ and $pr_{\langle \xi \rangle^\perp} \circ s_* : TN \rightarrow \langle \xi \rangle^\perp$ are isometric. We check that $J^2 = -id$. Since $s$ is a section of $\pi$ we have $s_* \circ \pi_* = id$ on the image of $s$ and thus

$$J^2 = \pi_* \circ \tilde{\phi} \circ s_* \circ \pi_* \circ \tilde{\phi} \circ s_* = \pi_* \circ \tilde{\phi}^2 \circ s_* = \pi_* \circ (\text{id} + \eta \otimes \xi) \circ s_* = -id.$$

where we have used that all involved endomorphisms preserve the orthogonal splitting $\mathcal{H} \oplus \langle \xi_2, \xi_3 \rangle$.

We check that $J$ is parallel with respect to $\nabla^T$. Remark that the horizontal lift of any vector field on $TN$ is basic and $\pi_*(\tilde{\phi}_1(s_*Y)) = (\tilde{\phi}_1(s_*Y))_\mathcal{H}$ wherever the right side is defined, that is on the image $s(N)$. Set $X := \tilde{X} - s_*X$ the vertical part of $-s_*X$. Then

$$(\nabla^T_X J)Y = \nabla^T_X (JY) - J(\nabla^T_X Y) = \pi_*(\nabla_X^{\tilde{Y}}) - J(\pi_*(\nabla_X^{\tilde{Y}}))$$

$$= \pi_*(\nabla_X^{\tilde{Y}}(\pi_*(\tilde{\phi}_1(s_*Y))) - \tilde{\phi}_1(s_* \pi_*(\nabla_X^{\tilde{Y}})))$$

$$= \pi_*(\nabla_X^{\tilde{Y}}(\pi_*(\tilde{\phi}_1(s_*Y))) + \nabla_{s_*X}(\pi_*(\tilde{\phi}_1(s_*Y)))) - \tilde{\phi}_1(\nabla_X Y - \tilde{\phi}_1(\nabla_{s_*X} Y)).$$

Remark that the horizontal lift of any vector field on $TN$ is basic so we may employ Lemma 2.4. In our case

$$g(\nabla_{\xi_1}, H, Z) = T(\xi_1, H, Z) = 2\alpha \left( \sum_{i=1}^3 \eta_i \land \Phi_i^H - 2\eta_{123} \right) (\xi_1, H, Z)$$

$$= 2\alpha(\Phi_1^H(H, Z) - 2(\eta_{23}(H, Z))),$$

where $H$ is either $\pi_*(\tilde{\phi}_1(s_*Y))$ or $\tilde{Y}$. Note that we apply $\pi_*$ in the end so it suffices to
assume $Z \in \langle \xi \rangle$ in the following:

\[
g(\nabla_{\hat{X}}(\pi_*(\phi_1(s, sY))), Z) = 2\alpha \eta_1(\hat{X})(\Phi_1^h(\pi_*(\phi_1(sY))), Z) - 2\eta_{23}(\pi_*(\phi_1(s, sY))), Z))
\]

\[
= 2\alpha \eta_1(\hat{X})(\Phi_1^h(\phi_1(s, sY)), Z) - 2\eta_{23}(\phi_1(s, sY)), Z)
\]

\[
= 2\alpha \eta_1(\hat{X})(\Phi_1^h(\phi_1(s, sY)), Z) + 2\eta_{23}(\phi_1(s, sY)), Z)
\]

\[
= 2\alpha \eta_1(\hat{X})(g((s, sY)_H, Z) - 2(\eta_3(s, sY)\eta_2(Z) + \eta_2(s, sY)\eta_2(Z)))
\]

\[
= 2\alpha \eta_1(\hat{X})(g((s, sY)_H, Z) - 2g((s, sY)_{(\xi_2, \xi_3)})), Z))
\]

\[
g(\hat{X}_\pi(\nabla_{\hat{X}}Y), Z) = -2\alpha \eta_1(\hat{X})(\Phi_1^h(Y, \phi_1 Z) - 2\eta_{23}(Y, \phi_1 Z))
\]

\[
= -2\alpha \eta_1(\hat{X})(\Phi_1^h(Y, \phi_1 Z) + 2\eta_{23}(Y, \phi_1 Z))
\]

\[
= 2\alpha \eta_1(\hat{X})(g(Y, Z) - 2(\eta_2(Y)\eta_2(Z) + \eta_2(Y)\eta_3(Z)))
\]

\[
= 2\alpha \eta_1(\hat{X})(g(Y, Z) - 2g((Y)_{(\xi_2, \xi_3)}), Z))
\]

Since $(s, sY)_H = Y$ both terms cancel. Further, both $\phi_1$ and $\nabla$ preserve the splitting

\[
TM = \mathbb{R}_{\xi_1} \oplus (\xi_2, \xi_3) \oplus H, \quad \text{thus } \nabla_{s, X}sY - \nabla_{s, X}Y \in V
\]

\[
(\nabla_{\hat{X}}J)Y = \pi_*(\nabla_{s, X}(\phi_1(s, sY))) - \phi_1(\nabla_{s, X}sY) = \pi_*((\nabla_{s, X}sY)_H) = 0. \quad (11)
\]

In order to control the covariant derivative of $J$ with respect to the Levi-Civita connection we need to compute

\[
g((\hat{X} \cdot J)Y, Z) = g(\hat{X} \cdot (JY), Z) - g(J(\hat{X} \cdot Y), Z) = \hat{T}(X, JY, Z) + \hat{T}(X, Y, JZ).
\]

This is linear so we may compute it for any combination of $X, Y, Z$ in either $\pi_*H$ or $\pi_*V$ individually. Note that $J$ preserves this splitting and $\pi^*T \in V \wedge \Lambda^2H$ by (10). Thus we only need to check for these combination of vectors. Further, note that $g((\hat{X} \cdot J)Y, Z)$ is skew-symmetric in $Y, Z$. Two cases remain. For $X \in \pi_*H$ and $Y, Z \in \pi_*H$ we have

\[
g((\hat{X} \cdot J)Y, Z) = T(X, JY, Z) + \hat{T}(X, Y, JZ)
\]

\[
= 2\alpha \sum_{i=1,2,3} \eta_i(X)(\Phi_i(\phi_1 s, sY, Z) + \Phi_i(Y, \phi_1 s, Z))
\]

\[
= 2\alpha \sum_{i=1,2,3} \eta_i(X)(g(Y, \phi_1 s, Z) - g(Y, \phi_1 s, Z))
\]

\[
= -4\eta_2(X)\Phi_3(Y, Z) - \eta_3(X)\Phi_2(Y, Z).
\]

For $X, Z \in \pi_*H$ and $Y \in \pi_*V$

\[
g((\hat{X} \cdot J)Y, Z) = T(X, JY, Z) + \hat{T}(X, Y, JZ)
\]

\[
= -2\alpha \sum_{i=2,3} \eta_i(X)(\phi_1 s, sY, Z) + \eta_i(Y)\Phi_i(X, \phi_1 s, Z)
\]

\[
= -2\eta_2(Y)\Phi_3(X, Z) - \eta_2(Y)\Phi_3(X, Z)
\]

\[
= -2\eta_2(Y)\Phi_3(X, Z) + \eta_3(Y)\Phi_2(X, Z))
\]

\[
= 4\eta_2(Y)\Phi_3(X, Z) - \eta_3(Y)\Phi_2(X, Z).
\]
This implies that \((\nabla_X^{g_N} J)X = (\nabla_X^{\hat{T}} J)X - \frac{1}{2}(\hat{T}_X \cdot J)X = 0\). Indeed, (13) proves that the 
\(\pi_*H \times \pi_*H\)-part is skew, and the sign difference between (12) and (13) shows that we are skew for mixed terms as well. Therefore \((N, g_N, J)\) is nearly Kähler.

Corollary 4.2. The locally defined Riemannian submersion \(\pi\) gives rise to a globally defined submersion \(\pi: M \to N\) where \((N, g_N, J)\) is a nearly Kähler orbifold.

Proof. We need to prove that the \(\mathbb{R}\)-action generated by \(\xi\) acts locally free. Now \(\xi\) generates a 1-dimensional subgroup of the group \(SU(2)\) generated by \(V\). Therefore the orbits of \(\xi\) are compact \(S^1\), in particular the action is locally free.

Coming from parallel \(3-(\alpha, \delta)\)-Sasaki manifolds these nearly Kähler spaces are rather special, inheriting additional properties.

Proposition 4.3. The nearly Kähler spaces obtained through Theorem 4.1 have reducible characteristic holonomy.

Proof. As in the proof of Theorem 4.1 we may assume that \(\xi = \xi_1\). We show that the holonomy representation \(TM = \langle \xi_1 \rangle \oplus \langle \xi_2, \xi_3 \rangle \oplus H\) satisfies the conditions of Proposition 2.3. Since \(\nabla\) preserves each Reeb vector field individually the aforementioned decomposition is \(\text{Hol}_0(\nabla)\)-invariant. By (9) the distribution \(\langle \xi_2, \xi_3 \rangle\) is invariant under \(\xi_1\) and, thus, projectable. As \(\xi_1\) is Killing the same is true for \(H\).

Remark 4.4. This shows that projectability is essential in Proposition 2.3 as both \(\xi_2, \xi_3\) are parallel with respect to \(\nabla\) but their projections individually are not.

Complete strictly nearly Kähler 6-folds with reducible characteristic holonomy were investigated by [BM01]. They show that the only such manifolds are the twistor spaces \(CP^3\) and \(F(1,2)\) with their standard nearly Kähler structures. More generally we see that \((N, g_N, J)\) is of special algebraic torsion in the notation of [Nag02]. Indeed, from (10) we have that the projections of \(\hat{T}\) to \(\Lambda^3\pi_*\langle \xi_2, \xi_3 \rangle\), \(\Lambda^2\pi_*\langle \xi_2, \xi_3 \rangle \wedge \pi_*H\) and \(\Lambda^3\pi_*H\) all vanish. Additionally, we can show that the tensor \(F: H \to H\) defined in [Nag02] is given by

\[
F := \sum_{i=2,3} (\nabla_{\xi_i}^{g_N} J)^2 = \sum_{i=2,3} (\xi_i \cdot \hat{T})^2 = 4\alpha^2(\varphi_2^2 + \varphi_3^2) = -8\alpha^2
\]

where we used \((\nabla J)J = -J(\nabla J)\) and (10).

In his classification of nearly Kähler manifolds, [Nag02], Nagy shows that such a manifold, if complete, is either homogeneous of type 3 in his notation or the twistor space of a quaternionic Kähler manifold. We proof with canonical submersions a local version of that theorem similar to the local version in [Ale06].
Theorem 4.5. Let \((N, g^N, J)\) be a nearly Kähler manifold with Hermitian connection \(\nabla^{\mathbb{T}^N}\). Assume the tangent space \(TN = V \oplus H\) splits into \(\text{Hol}(\nabla^{\mathbb{T}^N})\)- and \(J\)-invariant subsets such that the characteristic torsion satisfies \(T^N \in \Lambda^2 H \wedge V\). If \(F = -\sum(\nabla^{g^N}_\gamma J)^2|_\mathcal{H} = \text{kid}_H\), \(k > 0\), and \(\dim V = 2\) then there is a locally defined Riemannian submersion \(\pi: N \to \check{N}\) along \(V\). Furthermore, \(\check{N}\) admits a quaternionic Kähler structure locally defined by

\[
I_1 = \pi_* \circ J \circ s_*, \quad I_2 = \sqrt{\frac{2}{k}} \pi_* \circ (JV \mathcal{J} T^N) \circ s_*, \quad I_3 = \sqrt{\frac{2}{k}} \pi_* \circ (V \mathcal{J} T^N) \circ s_*
\]

for any section \(s: \check{N} \to N\) of \(\pi\) and \(V \in \mathcal{V}\) of norm 1.

Proof. \((N, g, \nabla^{\mathbb{T}^N})\) satisfies the conditions in Theorem 2.1 so we obtain a locally defined Riemannian submersion \(\pi: N \to \check{N}\) with totally geodesic fibers tangent to \(\mathcal{V}\) such that

\[
\nabla_{X}^{g^N} Y = \pi_* \left( \nabla_{X}^{\mathbb{T}^N} Y \right).
\]

We check that \(I_1, I_2, I_3\) satisfy the quaternion relations. As in Theorem 4.1 we see immediately \(I_1^2 = -\text{id}\). Observe that

\[
(\nabla^{g^N}_V J)^2 = (\nabla^{g^N}_V J)(\nabla^{g^N}_V J) = -(\nabla^{g^N}_V J)^2 J^2 = (\nabla^{g^N}_V J)^2 = -\frac{1}{2} F = -\frac{k^2}{2} \text{id}
\]

(14)

since \(V, JV\) form an orthonormal base of \(\mathcal{V}\). It follows that

\[
(V \mathcal{J} T^N)^2 = ((\nabla^{g^N}_V J)^2 J)^2 = -\frac{k^2}{2} \text{id}
\]

and analogously for \((JV \mathcal{J} T^N)^2\). Therefore \(I_2, I_3\) are almost complex structures on \(\check{N}\). The quaternionic relations follow immediately from \(J(\nabla^{g^N}_V J) = -(\nabla^{g^N}_V J) J = -(\nabla^{g^N}_V J)\) and (14).

It remains to show that \(\nabla^{g^N}\) preserves the subbundle of \(\text{End}(TM)\) generated by \(I_1, I_2, I_3\). We proceed as in Theorem 4.1 and set \(\check{X} := X - s_* X\). Then

\[
(\nabla^{g^N}_X I_1) Y = \nabla^{g^N}_X (I_1 Y) - I_1 (\nabla^{g^N}_X Y) = \pi_* \left( \nabla^{\mathbb{T}^N}_X \left( \pi_* (J(s_* Y)) \right) \right) - J(s_* (\pi_* (\nabla^{\mathbb{T}^N}_X \check{Y})))
\]

\[
= \pi_* \left( (\nabla^{\mathbb{T}^N}_{s_* X} (J(s_* Y))) + \nabla^{\mathbb{T}^N}_X (\pi_* (J(s_* Y))) \right) - J(s_* (\pi_* (\nabla^{\mathbb{T}^N}_X \check{Y})))
\]

\[
= \pi_* ((\nabla^{\mathbb{T}^N}_{s_* X} J)(s_* Y) + (\check{X} \mathcal{J} T^N)(J(s_* Y))) - J(\check{X} \mathcal{J} T^N)(s_* Y))
\]

\[
= 2 \pi_* ((J \check{X} \mathcal{J} T^N)(s_* Y))
\]

where we made use of Lemma 2.4. This shows \(\nabla^{g^N} I_1 \in \langle I_2, I_3 \rangle\) since \(\check{X} \in \mathcal{V} = (V, JV)\). We play the game once more for \(I_3\). Then the covariant derivative of \(I_2\) is computed
completely analogous.

\[
\sqrt{\frac{k}{2}}(\nabla^g_X I_3)Y = \sqrt{\frac{k}{2}}(\nabla^g_X (I_3Y) - I_3\nabla^g_X Y)
\]

\[
= \pi_*(\nabla^T_N s_*(V \mathcal{J} T^N)(s_*Y)) - (V \mathcal{J} T^N)(s_*(\pi_*(\nabla^T_N Y)))
\]

\[
= \pi_*(\nabla^T_N s_*(V \mathcal{J} T^N)(s_*Y)) + \nabla^T_X (\pi_*(\nabla^T_N (s_*Y)))
\]

\[
- (V \mathcal{J} T^N)(\nabla^T_N Y) - (V \mathcal{J} T^N)(s_*(\pi_*(\nabla^T_N Y)))
\]

\[
= \pi_*(\nabla^T_N s_*(V \mathcal{J} T^N)(s_*Y) + (\mathcal{J} \, T^N)((V \mathcal{J} T^N)(s_*Y))
\]

\[
- (V \mathcal{J} T^N)(\mathcal{J} \, T^N)(s_*Y))
\]

\[
= \pi_*(\nabla^T_N s_*(V \mathcal{J} T^N)(s_*Y)) + \sqrt{2kg(JV, \mathcal{J} I_1}
\]

Now the result follows as \(\nabla^T_N s_*(V \in V = \langle V, J V \rangle\).

\[\square\]

**Remark 4.6.** The expressions for \(\nabla^g_X I_2\) and \(\nabla^g_X I_3\) do not look particularly nice. However, suppose that \(V\) is a Killing field for \(g_N\), for instance, if \(V = \pi_\xi_2\) from Theorem 4.1. Then \(g(\nabla^T_N s_*(X V, V) = g(\nabla^g_X V, V) + \frac{1}{2} T^N (s_*X, V, V) = 0\) and thereby \(\nabla^g_X I_2 \in \langle I_1, I_3 \rangle\).

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