Hamiltonian Structure of
Gauge-Invariant Variational Problems

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Abstract
Let $C \to M$ be the bundle of connections of a principal bundle on $M$. The solutions to Hamilton-Cartan equations for a gauge-invariant Lagrangian density $\Lambda$ on $C$ satisfying a weak condition of regularity, are shown to admit an affine fibre-bundle structure over the set of solutions to Euler-Lagrange equations for $\Lambda$. This structure is also studied for the Jacobi fields and for the moduli space of extremals.

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1 Introduction
Let $p: E \to M$ be a fibred manifold and let $\Lambda$ be a Lagrangian density on $J^1E$. The solutions to the Hamilton-Cartan (H-C, in short) equations (e.g., see [7], [8], or [14]) for $\Lambda$, are the sections $\bar{s}: M \to J^1E$ of the canonical projection
2 Notations and preliminaries

2.1 Jet bundles

The bundle of $r$-jets of a fibred manifold $p: E \rightarrow M$ is denoted by $p_r: J^rE \rightarrow M$ with projections $p_{rk}: J^rE \rightarrow J^kE$, $r \geq k$, the $r$-jet extension of a section $s: M \rightarrow E$ of $p$ being denoted by $j^r s$. Every $p$-fibred coordinate system $(x^i, y^a)$, $1 \leq i \leq n = \dim M$, $1 \leq \alpha \leq m = \dim E - n$, defined on an open subset $U \subseteq E$, induces a coordinate system $(x^i, y^a_{\alpha}(x))$, $0 \leq |I| \leq r$, on $(p_r)^{-1}(U) = J^rU$; namely, $y^a_{\alpha}(j^r s)(x) = (\partial^I(y^a \circ s)/\partial x^I)(x)$, with $y^a_0 = y^a$. Every fibred map $\Phi: E \rightarrow E'$ whose induced map $\phi: M \rightarrow M'$ on the base manifold is a diffeomorphism, induces a map $\Phi^{(r)}: J^rE \rightarrow J^rE'$ by setting, $\Phi^{(r)}(j^r s) = j^r(\Phi \circ s \circ \phi^{-1})$, $\forall j^r s \in J^rE$.

An automorphism of $p$ is a pair of diffeomorphisms $\Phi: E \rightarrow E$, $\phi: M \rightarrow M$ such that $p \circ \Phi = \phi \circ p$. The set of all automorphisms of $p$ is a group denoted by $\text{Aut } E$.

For every—not necessarily $p$-projectable—vector field $Y \in \mathfrak{X}(E)$ a unique vector field $Y^{(1)} \in \mathfrak{X}(J^1E)$ exists (called the 1-jet prolongation of $Y$) such that, 1st) $Y^{(1)}$ is $p_{10}$-projectable onto $Y$, and 2nd) $L_{Y^{(1)}} C \subseteq C$, where $C$ is the
contact system; i.e., the differential system generated by the contact 1-forms \( \theta^\alpha = dy^\alpha - y^\alpha_i dx^i \) in \( \Omega^1(J^1E) \).

If \( Y \) is a \( p \)-vertical vector field (the only case that we consider below), then the formulas of 1-jet prolongation are as follows (cf. [14], Section 2.4):

\[
Y^{(1)} = v^\alpha \frac{\partial}{\partial y^\alpha} + v^\alpha_i \frac{\partial}{\partial y^\alpha_i}, \quad v^\alpha_i = \frac{\partial v^\alpha}{\partial x^i} + \frac{\partial v^\alpha}{\partial y^\beta} y^i_\beta,
\]

\[
Y = v^\alpha \frac{\partial}{\partial y^\alpha}, \quad v^\alpha \in C^\infty(E).
\]

**Lemma 2.1** ([14]). For every \( p \)-vertical vector field \( Y \in \mathfrak{X}^v(E) \) the value of the vector field \( Y^{(1)} \) at a point \( j^1_x(Y \circ s) \).

## 2.2 Bundles of connections

An automorphism of a principal \( G \)-bundle \( \pi: P \to M \) is an equivariant diffeomorphism \( \Phi: P \to P \); i.e., \( \Phi(u \cdot g) = \Phi(u) \cdot g, \forall u \in P, \forall g \in G \). We denote by \( \text{Aut} P \) the group of all automorphisms of \( P \) under composition. Every \( \Phi \in \text{Aut} P \) determines a unique diffeomorphism \( \phi: M \to M \), such that \( \pi \circ \Phi = \phi \circ \pi \). If \( \phi \) is the identity map, then \( \Phi \) is said to be a gauge transformation (cf. [14], Section 2.4]). We denote by \( \text{Gau} P \subset \text{Aut} P \) the subgroup of all gauge transformations.

A vector field \( X \in \mathfrak{X}(P) \) is said to be \( G \)-invariant if \( R_g \cdot X = X, \forall g \in G \). If \( \Phi_t \) is the flow of a vector field \( X \in \mathfrak{X}(P) \), then \( X \) is \( G \)-invariant if and only if \( \Phi_t \in \text{Aut} P, \forall t \in \mathbb{R} \). Because of this we denote the Lie subalgebra of \( G \)-invariant vector fields on \( P \) by \( \text{aut} P \subset \mathfrak{X}(P) \). Each \( G \)-invariant vector field on \( P \) is \( \pi \)-projectable. Similarly, a \( \pi \)-vertical vector field \( X \in \mathfrak{X}(P) \) is \( G \)-invariant if and only if \( \Phi_t \in \text{Gau} P, \forall t \in \mathbb{R} \). We denote by \( \text{gau} P \subset \text{aut} P \) the ideal of all \( \pi \)-vertical \( G \)-invariant vector fields on \( P \), which is called the gauge algebra of \( P \).

The group \( G \) acts on \( TP \) by setting \( X \cdot g = (R_g)_*(X), \forall X \in TP, \forall g \in G \). The global sections of the quotient vector bundle \( T(P)/G \) can naturally be identified to \( \text{aut} P \); i.e., \( \text{aut} P \cong \Gamma(M, T(P)/G) \). Similarly, \( \text{gau} P \cong \Gamma(M, \text{ad} P) \), where \( \text{ad} P \) denotes the adjoint bundle: The bundle associated to \( P \) by the adjoint representation of \( G \) on its Lie algebra \( \mathfrak{g} \), denoted by \( \pi_\mathfrak{g}; \text{ad} P \to M \); that is, \( \text{ad} P = (P \times \mathfrak{g})/G \), where the action of \( G \) on \( P \times \mathfrak{g} \) is defined by

\[
(u, B) \cdot g = (u \cdot g, \text{Ad}_g^{-1}(B)), \quad \forall u \in P, \forall B \in \mathfrak{g}, \forall g \in G.
\]

The \( G \)-orbit in \( \text{ad} P \) of a pair \((u, B) \in P \times \mathfrak{g}\) is denoted by \((u, B)_\text{ad}\).

An exact sequence of vector bundles over \( M \) (the so-called Atiyah sequence) holds, \( 0 \to \text{ad} P \to T(P)/G \xrightarrow{\pi} TM \to 0 \). The fibres \( \text{ad} P_x \) are endowed with a Lie-algebra structure determined by \([u, B]_\text{ad} = (u, -[B, B'])_\text{ad}\), for all \( u \in \pi^{-1}(x), B, B' \in \mathfrak{g} \), where \([\cdot, \cdot]\) denotes the bracket in \( \mathfrak{g} \). The sign of the bracket above is needed in order to ensure that the natural identification \( \text{gau} P \cong \Gamma(M, \text{ad} P) \) is a Lie-algebra isomorphism, when \( \text{gau} P \) is considered as a Lie subalgebra of \( \mathfrak{X}(P) \).
Let \( X^{hr} \in \mathfrak{X}(P) \) be the horizontal lift of a vector field \( X \in \mathfrak{X}(M) \) with respect to a connection \( \Gamma \) on \( \pi: P \to M \). The vector field \( X^{hr} \) is \( G \)-invariant and projects onto \( X \) (cf. [10] II. Proposition 1.2]). Hence we have a splitting of the Atiyah sequence, \( s_\Gamma: TM \to T(P)/G \), \( s_\Gamma(X) = X^{hr} \). Conversely, any splitting \( \sigma: TM \to T(P)/G \) of the Atiyah sequence (i.e., \( \sigma \) is a vector bundle homomorphism such that \( \pi_* \circ \sigma = \text{id}_{TM} \)) comes from a unique connection on \( P \) so that there is a natural bijection between connections on \( P \) and splittings of the Atiyah sequence. We thus define the bundle of connections \( p: C = C(P) \to M \) as the sub-bundle of \( \text{Hom}(TM, T(P)/G) \) determined by all \( \mathbb{R} \)-linear mappings \( \lambda: T_xM \to (T(P)/G)_x \) such that \( \pi_* \circ \lambda = \text{id}_{TM} \). Connections on \( P \) can be identified to the global sections of \( p: C \to M \). We also denote by \( s_\Gamma: M \to C \) the section of the bundle of connections induced by \( \Gamma \).

An element \( \lambda: T_xM \to (T(P)/G)_x \) of the bundle \( C \) over a point \( x \in M \) is nothing but a ‘connection at a point \( x \); i.e., \( \Lambda \) induces a complementary subspace \( H_u \) of the vertical subspace \( V_u(P) \subset T_u(P) \) for every \( u \in \pi^{-1}(x) \). Any other connection at \( x \) can be written as \( \lambda' = h + \lambda \), where \( h: T_xM \to (\text{ad}P)_x \) is a linear map. Hence \( C \) is an affine bundle modelled over the vector bundle \( \text{Hom}(TM, \text{ad}P) \cong T^*M \otimes \text{ad}P \).

Let \( (U; x^i) \) be a coordinate open domain in \( M \) such that \( \pi^{-1}(U) \cong U \times G \). For every \( \hat{B} \in \mathfrak{g} \) we define a flow of gauge transformations over \( U \) by setting \( \varphi_t^\hat{B}(x, g) = (x, \exp(t\hat{B}) \cdot g), x \in U \). Let \( \hat{B} \) be the corresponding infinitesimal generator. If \( (B_1, \ldots, B_m) \) is a basis of \( \mathfrak{g} \), then \( \hat{B}_1, \ldots, \hat{B}_m \) is a basis of \( \Gamma(U, \text{ad}P) \).

Let \( p: C \to M \) be the bundle of connections of \( P \). The horizontal lift with respect to \( \Gamma \) of the basic vector field \( \partial/\partial x^i \) is given as follows:

\[
s_\Gamma\left( \frac{\partial}{\partial x^i} \right) = \left( \frac{\partial}{\partial x^i} \right)^{hr} = \frac{\partial}{\partial x^i} - (A^\alpha_i \circ s_\Gamma) \hat{B}_\alpha.
\]

The functions \( (x^i, A^\alpha_j), i, j = 1, \ldots, n = \dim M, 1 \leq \alpha \leq m = \dim G \), induce a coordinate system on \( p^{-1}(U) = C(\pi^{-1}U) \) (cf. [2]).

Each automorphism \( \Phi \in \text{Aut} P \) acts on connections of \( P \) by pulling back connection forms; that is, \( \Gamma' = \Phi(\Gamma) \) where \( \omega_{\Gamma'} = (\Phi^{-1})^* \omega_{\Gamma} \) (cf. [10] II. Proposition 6.2-(b)]). If \( \Psi \in \text{Aut} P \) is another automorphism, then \( (\Psi \circ \Phi)(\Gamma) = \Psi(\Phi(\Gamma)) \).

For each \( \Phi \in \text{Aut} P \) there exists a unique diffeomorphism \( \Phi_C: C \to C \) such that \( p \circ \Phi_C = \Phi \circ p \), where \( \Phi: M \to M \) is the diffeomorphism induced by \( \Phi \) on the base manifold. We thus obtain a group homomorphism \( \text{Aut} P \to \text{Diff} C \). For every connection \( \Gamma \) on \( P \) we have \( \Phi_C \circ s_\Gamma = s_{\Phi(\Gamma)} \).

If \( \Phi_t \) is the flow of a \( G \)-invariant vector field \( X \in \text{aut} P \), then \( \Phi_t \) is a one-parameter group in \( \text{Diff} C \) with infinitesimal generator denoted by \( X_C \), and the map \( \text{aut} P \to \mathfrak{X}(C) \), \( X \mapsto X_C \) is a Lie-algebra homomorphism.

### 2.3 Affine-bundle structures

Let \( V(p) = \{ X \in TE : p_*X = 0 \} \) be the vertical subbundle of a fibred manifold \( p: E \to M \).

(a) Let \( p: E \to M \) be an affine bundle modelled over the vector bundle \( p_W: W \to M \). The directional derivative determines an isomorphism
of vector bundles over $E$, $p^*W \cong V(p)$, $(e, w) \mapsto X_{e, w}$, $p(e) = p_W(w) = x$, where $X_{e, w}$ is the tangent vector at $t = 0$ to the curve $t \mapsto tw + e$, which takes values in the fibre $p^{-1}(x)$. In coordinates, $X_{e, w} = w^\alpha (\partial/\partial e^\alpha)_e$.

(b) Moreover, if $p: E \to M$ is an arbitrary surjective submersion, then the projection $p_{10}: J^1E \to E$ is endowed with an affine-bundle structure modelled over $p^*TM \otimes V(p)$. In fact, every jet $j^1s \in (p_{10})^{-1}(e)$, with $s(x) = e$, can be identified with the section $s_{e, e}: J^1M \to T_eE$ of $p_{e, e}: T_eE \to T_xM$. Hence, if $j^1s' \in (p_{10})^{-1}(e)$ is another jet, then $p_{e, e} \circ (s_{e, e}' - s_{e, e}) = 0$ and accordingly $s_{e, e}' - s_{e, e}$ takes values into $V_e(p)$. Therefore, $s_{e, e}' - s_{e, e}$ determines an element in $\text{Hom}(T_xM \otimes V_e(p)) = T_{x, e}^*M \otimes V_e(p)$. From (a) it follows an isomorphism, $p_{10}^*(p^*TM \otimes V(p)) = T^*M \otimes J^1C V(p) \cong V(p_{10})$.

2.4 The Hessian metric

If $p: E \to M$ be an affine bundle modelled over the vector bundle $p_W: W \to M$, then, according to the item (a) in Section 2.3, every $w \in (p_W)^{-1}(x)$ induces a vector field along the fibre $X_w \in \mathfrak{X}(p^{-1}(x))$, $X_w(e) = X_{e, w}$, $\forall e \in p^{-1}(x)$. For every $f \in C^\infty(E)$ and every $e \in E$, with $x = p(e)$, a bilinear form
\[ \text{Hess}_e(f): V_e(p) \times V_e(p) \to \mathbb{R} \]

can be defined as follows: $\text{Hess}_e(f)(w_1, w_2) = X_{w_2}(e)((d_E/M)f X_{w_1})$, where the canonical isomorphism $W_x \cong V_e(p)$, defined in the item (a) in Section 2.3, has been used and $d_E/M$ denotes the fibred derivative, e.g., see [8]. As this form is proved to be symmetric, $e \mapsto \text{Hess}_e(f)$ defines a section of the vector bundle $S^2V^*(p) \cong p^*S^2W^*$.

3 Hamilton-Cartan equations

Let $p: E \to M$ be a fibre manifold, $\dim M = n$, $\dim E = m + n$, where $M$ is assumed to be connected and oriented by a volume form $v$. Below, Latin indices run from 1 to $n$, and Greek indices run from 1 to $m$. The solutions to the Hamilton-Cartan equations for a density $\Lambda = L v$, $L \in C^\infty(J^1E)$ on $p$, are the sections $\tilde{s}: M \to J^1E$ of the canonical projection $p_1: J^1E \to M$ such that,

\[ (2) \quad \tilde{s}^*(i_X d\Theta_\Lambda) = 0, \quad \forall X \in \mathfrak{X}^v(J^1E), \]

where

(i) $\Theta_\Lambda = (-1)^{s-1}(\partial L/\partial y^\alpha_i) \theta^\alpha \wedge v_i + L v$ is the Poincaré-Cartan form attached to $\Lambda$ (cf. [3, 13]),

(ii) $\mathfrak{X}^v(J^1E)$ denotes the Lie algebra of $p_1$-vertical vector fields,

(iii) $\theta^\alpha = dy^\alpha - y^\alpha_i dx^i$ are the standard contact forms on the 1-jet bundle, and
(iv) \((x^i, y^\alpha, y^\beta_i)\) is the coordinate system on \(J^1E\) induced by a fibred coordinate system \((x^i, y^\alpha)\) for the submersion \(p\), adapted to the given volume form; i.e., \(v = dx^1 \wedge \ldots \wedge dx^n\) and \(v_i = (-1)^{i-1}i_{\partial/\partial x^i}v\).

**Lemma 3.1.** A section \(\bar{s}: M \to J^1E\) of \(p_1: J^1E \to M\) is a solution to the H-C equations \(\text{[2]}\) if and only if the following equations hold:

\[
(s^\beta_j - \bar{s}^\beta_j)\left(\frac{\partial^2 L}{\partial y^\alpha_i \partial y^\beta_j} \circ \bar{s}\right) = 0, \quad 1 \leq i \leq n, 1 \leq \alpha \leq m,
\]

\[
-\frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial y^\alpha_j} \circ \bar{s}\right) + \frac{\partial L}{\partial y^\alpha} \circ \bar{s} + (s^\beta_j - \bar{s}^\beta_j)\left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta_j} \circ \bar{s}\right) = 0,
\]

for \(1 \leq \alpha \leq m\), where \(s^\alpha = y^\alpha \circ \bar{s}\), \(s^\alpha_i = \partial s^\alpha / \partial x^i\), and \(\bar{s}^\alpha_i = y^\alpha_i \circ \bar{s}\).

**Proof.** As a simple computation shows, we have

\[
d\Theta_\Lambda = \theta^\beta \wedge \left((-1)^j d\left(\frac{\partial L}{\partial y^\beta_j}\right) \wedge v_j + \frac{\partial L}{\partial y^\beta} v\right).
\]

Hence

\[
\bar{s}^* \left(i_{\partial/\partial y^\alpha}d\Theta_\Lambda\right) = (-1)^{j-1}\left(\frac{\partial^2 L}{\partial y^\beta_j \partial y^\alpha} \circ \bar{s}\right)\bar{s}^* \theta^\beta \wedge v_j,
\]

\[
\bar{s}^* \left(i_{\partial/\partial y^\alpha}d\Theta_\Lambda\right) = \left(-\frac{\partial}{\partial x} \left(\frac{\partial L}{\partial y^\beta_j} \circ \bar{s}\right) + \frac{\partial L}{\partial y^\alpha} \circ \bar{s}\right) v
\]

\[
\quad + (-1)^j \left(\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta_j} \circ \bar{s}\right) \bar{s}^* \theta^\beta \wedge v_j,
\]

and the formulas in the statement follow. \(\square\)

If \(\bar{s} = j^1s\) is a holonomic section, then \(\bar{s}\) is a solution to the H-C equations if and only if \(s\) is a solution to the Euler-Lagrange equations. If \(L\) is regular, then the converse holds true: Every solution to the H-C equations, is of the form \(\bar{s} = j^1s\), \(s\) being a solution to the E-L equations. Hence, for regular variational problems, H-C equations are equivalent to E-L equations; but this is no longer true for non-regular densities, as is the case for the Yang-Mills Lagrangian.

4 **Jacobi fields**

4.1 **Jacobi fields introduced**

Let \(p: E \to M\) be a fibred manifold and let \(\Omega^1(E/M) = \Gamma(M, V^*(p))\). Let

\[
\mathcal{E}^\Lambda: \Gamma(p) \to \Omega^1(E/M) \otimes_{C^\infty(M)} \Omega^n(M)
\]
be the Euler-Lagrange operator of \( \Lambda = L\nu \), \( L \in C^\infty(J^1E) \), which is the second-order differential operator locally given on a fibred coordinate system \( (U; x^i, y^\alpha) \) for the submersion \( p \) and for every section \( s \) of \( p|U \) by,

\[
\mathcal{E}_s^\Lambda(t) = (\mathcal{E}_s^\Lambda \circ j^2 s) \, d_{E/M} y^\alpha \otimes \nu,
\]

where the functions \( \mathcal{E}_s^\Lambda \in C^\infty(J^2E) \) are defined as follows:

\[
\mathcal{E}_s^\Lambda (j^2 s) = \frac{\partial L}{\partial y^\alpha} (j^1 s) - \frac{\partial}{\partial x^j} \left( \frac{\partial L}{\partial y^\alpha} \circ j^1 s \right) (x).
\]

The linearisation of \( \mathcal{E}_s^\Lambda \) at \( s \in \Gamma(p) \) is the operator

\[
\mathcal{L}_s \mathcal{E}_s^\Lambda : T_s \Gamma(p) = \Gamma(M, s^* V(p)) \to \Omega^1(E/M) \otimes_{C^\infty(M)} \Omega^n(M)
\]

defined as follows.

If \( S : (\varepsilon, \varepsilon) \times U \to E \) is a one-parameter family of sections, i.e., \( p \circ S_t = \text{id}_U \), \( |t| < \varepsilon \), then a vector field \( X \in \Gamma(U, s^* V(p)) \) along \( s = S_0 \)—called the ‘initial velocity’ of \( S \)—is defined to be the tangent vector \( X(x) \in V_{s(x)}(p) \) at \( t = 0 \) to the curve \( t \mapsto S_t(x) \), which takes values in the fibre \( p^{-1}(x) \) for every point \( x \in U \). Expanding \( y^\alpha \circ S_t \) up to second order, we obtain \( y^\alpha \circ S_t = y^\alpha \circ s + tv^\alpha + t^2 f^\alpha \), for certain functions \( v^\alpha \in C^\infty(U) \), \( f^\alpha \in C^\infty((-\varepsilon, \varepsilon) \times U) \). Hence,

\[
X(x) = v^\alpha(x) (\partial / \partial y^\alpha)_{s(x)}, \quad \forall x \in U.
\]

Therefore, every vector field \( X \in \Gamma(U, s^* V(p)) \) is the initial velocity of a one-parameter family of sections \( S \), and we define,

\[
\mathcal{L}_s \mathcal{E}_s^\Lambda(X) = \frac{\partial}{\partial t} \bigg|_{t=0} \mathcal{E}_s^\Lambda(S_t).
\]

The definition makes sense as it does not depend on the particular one-parameter family of sections chosen. In fact,

\[
\frac{\partial}{\partial t} \bigg|_{t=0} (\mathcal{E}_s^\Lambda \circ j^2 S_t) = \left( \frac{\partial \mathcal{E}_s^\Lambda}{\partial y^\alpha} \circ j^2 s \right) v^\beta + \left( \frac{\partial \mathcal{E}_s^\Lambda}{\partial y^\beta} \circ j^2 s \right) \frac{\partial v^\beta}{\partial x^j} + \left( \frac{\partial \mathcal{E}_s^\Lambda}{\partial y^\alpha} \circ j^2 s \right) \frac{\partial^2 v^\beta}{\partial x^j \partial x^k}.
\]

This expression also shows that \( \mathcal{L}_s \mathcal{E}_s^\Lambda(X) \) depends linearly on \( X \).

**Definition 4.1.** A vector field \( X \in \Gamma(M, s^* V(p)) \) defined along an extremal \( s \) of a Lagrangian density \( \Lambda \) on \( p \) is called a Jacobi field if \( X \in \ker \mathcal{L}_s \mathcal{E}_s^\Lambda \).

**Proposition 4.2.** A vector field \( X \in \Gamma(M, s^* V(p)) \) defined along an extremal \( s \) of a Lagrangian density \( \Lambda \) on \( p \) is a Jacobi field if and only if the following equation holds:

\[
(j^1 s)^* (i_Y L_{\tilde{X}} \omega^\alpha) = 0, \quad \forall Y \in \mathfrak{X}^\nu(J^1E),
\]

where \( \tilde{X} \in \mathfrak{X}^\nu(E) \) is an arbitrary \( p \)-vertical extension of \( X \).
Proof. The equation (4) does not depend on the vertical extension chosen. In fact,
\[(5) \quad (j^1 s)^* (i_Y L_{\tilde{X}(1)} d(\Theta \Lambda)) = \Upsilon (j^1 s)^* (i_Y d(\Theta \Lambda)) + (j^1 s)^* (i_{[Y,X]} d(\Theta \Lambda)),\]
and the second term on the right-hand side vanishes, as \(s\) is an extremal. Hence
\[(j^1 s)^* (i_Y L_{\tilde{X}(1)} d(\Theta \Lambda)) = \Upsilon (j^1 s)^* (i_Y d(\Theta \Lambda)),\]
and we conclude by simply applying Lemma 2.1.

Moreover, from the formula (3) the following identity is obtained:
\[
d(\Theta \Lambda) = \Upsilon E_{\Lambda} \theta^\alpha \wedge v + (\Upsilon - 1) i_{\partial_j \partial_j L} \theta^\alpha \wedge \theta_j \wedge v,\]
where \(\theta_j = dy^j - y^j (\partial k) dx^k\). If \(\Phi_0\) is the flow of \(\tilde{X}\), then \(j^1 S_t = J^1 (\Phi_t) \circ j^1 s\) and from the previous formula we have
\[
(j^1 s)^* \left\{ J^1 (\Phi_t)^* (i_Y d(\Theta \Lambda)) \right\} = (j^1 S_t)^* (i_Y d(\Theta \Lambda)) = (E_{\Lambda} \circ j^2 S_t) \theta^\alpha(Y) v.
\]
Taking derivatives with respect to \(t\) at \(t = 0\) in this formula, we have
\[
(j^1 s)^* (L_{\tilde{X}(1)} (i_Y d(\Theta \Lambda))) = L_s E_{\Lambda} (X) v,
\]
and we can conclude by simply applying the formula (5) recalling that \(s\) is an extremal. \(\square\)

Let \(S_\Lambda\) (resp. \(\tilde{S}_\Lambda\)) denote the set of solutions to the E-L equations (resp. H-C equations) attached to a Lagrangian density \(\Lambda = L v\), \(L \in C^\infty(J^1 E)\).

Remark 4.3. If \(S_t \in S_\Lambda\) is a one-parameter family of extremals, then its initial velocity \(X\) is readily seen to be a Jacobi field along the extremal \(s = S_0\); in this case, \(X\) is said to be ‘integrable’ (e.g., see [11, Definition 1.2], [12, Section 2.6]). Although important examples of non-integrable Jacobi fields exist, usually Jacobi fields along \(s\) are considered as the tangent space \(T_s S_\Lambda\) at an extremal \(s \in S_\Lambda\) to the ‘manifold’ of solutions to the E-L equations for \(\Lambda\). By the same token, we give the following

Definition 4.4. The tangent space \(T_s S_\Lambda\) at a solution \(s \in S_\Lambda\) to the ‘manifold’ of solutions to the H-C equations for \(\Lambda\) is defined to be the space of vector fields \(\tilde{X} \in \Gamma(M, s^* V(p_1))\) that satisfy the equation
\[
\tilde{s}^* (i_Y L_{\tilde{X}} d(\Theta \Lambda)) = 0, \quad \forall Y \in \mathcal{X}^v(J^1 E),
\]
where \(\tilde{X} \in \mathcal{X}^v(J^1 E)\) is a \(p_1\)-vertical extension of \(\tilde{X}\).
4.2 The embedding $T_s S_\Lambda \hookrightarrow T_{j^1 s} \hat{S}_\Lambda$

**Proposition 4.5.** For every $s \in S_\Lambda$, there is an embedding

$$T_s S_\Lambda \hookrightarrow T_{j^1 s} \hat{S}_\Lambda,$$

$$X \mapsto X^{(1)} \circ j^1 s,$$

where $\hat{X}$ is any $p$-vertical extension of $X \in \Gamma(M, s^* V(p))$ to $E$. If $\Lambda$ is regular, then $T_s S_\Lambda \cong T_{j^1 s} \hat{S}_\Lambda$, $\forall s \in S_\Lambda$.

**Proof.** As a straightforward—but rather long—computation shows, a vector field $\hat{X} \in \Gamma(M, s^* V(p_1))$ with local expression

$$\hat{X} = v^\alpha \frac{\partial}{\partial y^\alpha} \big|_{\bar{s}} + v_i^\alpha \frac{\partial}{\partial y_i^\alpha} \big|_{\bar{s}}, \quad v^\alpha, v_i^\alpha \in C^\infty(M),$$

belongs to $T_j \hat{S}_\Lambda$ if and only if the following two equations hold:

$$0 = \left( \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \circ \bar{s} - \frac{\partial^3 L}{\partial x^i \partial y^\alpha \partial y_i^\rho} \circ \bar{s} + \left( \frac{\partial s^\beta}{\partial x^i} - \bar{s}^\beta_i \right) \left( \frac{\partial^3 L}{\partial y^\alpha \partial y^\beta \partial y_i^\rho} \circ \bar{s} \right) v^\alpha \right.$$

$$+ \left( \frac{\partial s^\beta}{\partial x^j} - \bar{s}^\beta_i \right) \left( \frac{\partial^3 L}{\partial y^\alpha \partial y_i^\rho \partial y_j^\rho} \circ \bar{s} \right) v_i^\alpha - \left( \frac{\partial^2 L}{\partial y^\alpha \partial y_i^\rho} \circ \bar{s} \right) \left( \frac{\partial s^\beta}{\partial y_{i\rho}^\alpha} - \bar{s}^\beta_i \right) \left( \frac{\partial^3 L}{\partial y^\alpha \partial y_i^\rho \partial y_j^\rho} \circ \bar{s} \right) v_i^\alpha$$

$$+ \left( \frac{\partial s^\beta}{\partial x^j} - \bar{s}^\beta_i \right) \left( \frac{\partial^3 L}{\partial y^\alpha \partial y_i^\rho \partial y_j^\rho} \circ \bar{s} \right) v_i^\alpha - \left( \frac{\partial^2 L}{\partial y^\alpha \partial y_i^\rho} \circ \bar{s} \right) \left( \frac{\partial s^\beta}{\partial y_{i\rho}^\alpha} - \bar{s}^\beta_i \right) \left( \frac{\partial^3 L}{\partial y^\alpha \partial y_i^\rho \partial y_j^\rho} \circ \bar{s} \right) v_i^\alpha,$$

$$+ \left( \frac{\partial s^\beta}{\partial x^j} - \bar{s}^\beta_i \right) \left( \frac{\partial^3 L}{\partial y^\alpha \partial y_i^\rho \partial y_j^\rho} \circ \bar{s} \right) v_i^\alpha - \left( \frac{\partial^2 L}{\partial y^\alpha \partial y_i^\rho} \circ \bar{s} \right) \left( \frac{\partial s^\beta}{\partial y_{i\rho}^\alpha} - \bar{s}^\beta_i \right) \left( \frac{\partial^3 L}{\partial y^\alpha \partial y_i^\rho \partial y_j^\rho} \circ \bar{s} \right) v_i^\alpha \right) \big|_{\bar{s}}.$$

(6)

$$0 = \left( \frac{\partial^2 L}{\partial y^\alpha \partial y_i^\rho} \circ \bar{s} \right) \left( \frac{\partial v^\alpha}{\partial x^i} - v_i^\alpha \right) + \left( \frac{\partial s^\beta}{\partial x^i} - \bar{s}^\beta_i \right) \left( \frac{\partial^3 L}{\partial y^\alpha \partial y_i^\rho \partial y_j^\rho} \circ \bar{s} \right) v_i^\alpha,$$

(7)

where $s = p_{10} \circ \bar{s}$, $s^\alpha = y^\alpha \circ \bar{s} = y^\alpha \circ s$, and $\bar{s}^\alpha = y_i^\alpha \circ \bar{s}$. Along a holonomic section $\bar{s} = j^1 s$, the equations (6) and (7) become respectively,
From the formula (8) we deduce that a vector field \( X \) is regular, then the equation (9) is equivalent to saying that \( v_i^\alpha = \partial v^\alpha / \partial x^i \), and from the formula (11) and Lemma 2.1 we conclude that \( X \) is the 1-jet prolongation of a Jacobi field, i.e., \( \tilde{X} = X^{(1)} \).

\( \tilde{X} = X^{(1)} \)

Remark 4.6. From the formula (8) we deduce that a vector field

\[
X \in \Gamma(M, s^*V(p)),
\]

\[
X = v^\alpha \left. \frac{\partial}{\partial y^\alpha} \right|_s, \quad v^\alpha \in C^\infty(M),
\]

belongs to \( T_s\mathcal{S}_A \) if and only if the following equations hold:

\[
0 = \left( \frac{\partial^2 L}{\partial y^\alpha \partial y^\gamma} \circ j^1s - \frac{\partial^3 L}{\partial x^i \partial y^\alpha \partial y^\gamma} \circ j^1s - \frac{\partial s^\gamma}{\partial x^i} \left( \frac{\partial^2 L}{\partial y^\alpha \partial y^\gamma} \circ j^1s \right) \right) v^\alpha
\]

\[
- \frac{\partial^2 s^\gamma}{\partial x^b \partial x^i} \left( \frac{\partial^3 L}{\partial y^\alpha \partial y^\gamma \partial y^\delta} \circ j^1s \right) v^\alpha
\]

\[
+ \left( \frac{\partial^2 L}{\partial y^\alpha \partial y^\gamma} \circ j^1s - \frac{\partial^2 L}{\partial y^\alpha \partial y^\gamma} \circ j^1s \right) \frac{\partial v^\alpha}{\partial x^i} - \left( \frac{\partial^3 L}{\partial y^\alpha \partial y^\gamma \partial y^\delta} \circ j^1s \right) \frac{\partial v^\alpha}{\partial x^i}
\]

\[
+ \left( \frac{\partial^2 L}{\partial y^\alpha \partial y^\gamma} \circ j^1s \right) \frac{\partial^2 v^\alpha}{\partial x^i \partial x^j}
\]
5 H-C equations and gauge-invariance

5.1 Gauge-invariant Lagrangians

Definition 5.1. A smooth function $L: J^1 C \rightarrow \mathbb{R}$ is said to be gauge invariant if $L \circ \Phi_C^{(1)} = L$ for every $\Phi \in \text{Gau}_P$.

This equation obviously implies $X^{(1)}_C L = 0$ for every $X \in \text{gau}_P$. The converse also holds if the group $G$ is connected. As every $\Phi \in \text{Gau}_P$ induces the identity map on $M$, the function $L$ is gauge invariant if and only if the gauge group is a group of symmetries for the Lagrangian density $\Lambda = L\nu$, where $\nu$ is an arbitrary volume form on the base manifold.

Let

$$\Omega: J^1 C \rightarrow \bigwedge^2 T^* M \otimes \text{ad} P$$
$$\Omega(j^1 x) = (\Omega_x)$$

be the curvature map. Here, the curvature form $\Omega_x$ of the connection $\Gamma$ corresponding to a section $s_\Gamma$ of $p$, is seen to be a two form on $M$ with values in the adjoint bundle $\text{ad} P$. On the vector bundle $\bigwedge^2 T^* M \otimes \text{ad} P$ we consider the coordinate systems $(x^i; P_{jk}^\alpha)$, $j < k$, induced by a coordinate system $(U; x^i)$ on $M$, and a basis $(B_\alpha)$ of $g$, as follows:

$$\eta_2 = \sum_{j<k} \left( R^\alpha_{jk} (\eta_2) dx^j \wedge dx^k \otimes \tilde{B}_\alpha \right), \quad \forall \eta_2 \in \bigwedge^2 T^* M \otimes (\text{ad} P)_x.$$  

The geometric formulation of Utiyama’s Theorem (e.g., see [1]) states that a Lagrangian $L: J^1 C \rightarrow \mathbb{R}$ is gauge invariant if and only $L$ factors through $\Omega$ as $L = \tilde{L} \circ \Omega$, where

$$\tilde{L}: \bigwedge^2 T^* M \otimes \text{ad} P \rightarrow \mathbb{R}$$

is a differentiable function which is invariant under the adjoint representation of $G$ on the curvature bundle. As the curvature map (10) is surjective, the function $\tilde{L}$ is unique.

5.2 Projecting $\bar{S}_\Lambda$ onto $S_\Lambda$

From Lemma 3.1 in Section 3, we readily obtain

Proposition 5.2. The H-C equations of a Lagrangian $L$ on the bundle of connections $p: C \rightarrow M$ of a principal bundle $\pi: P \rightarrow M$, read as follows:

$$\left(s^{\beta}_{h,j} - \bar{s}^{\beta}_{h,j}\right) \left(\frac{\partial^2 L}{\partial A^{\alpha}_{i,k} \partial A^{\beta}_{j,h}} \circ \bar{s}\right) = 0, \quad \forall \alpha, i, k,$$

$$-\frac{\partial}{\partial x^j} \left(\frac{\partial L}{\partial A^{\alpha}_{i,j}} \circ \bar{s}\right) + \frac{\partial L}{\partial A^{\alpha}_{i}} \circ \bar{s} + \left(s^{\beta}_{h,j} - \bar{s}^{\beta}_{h,j}\right) \left(\frac{\partial^2 L}{\partial A^{\alpha}_{i} \partial A^{\beta}_{h,j}} \circ \bar{s}\right) = 0, \quad \forall \alpha, i,$$
where $\bar{s}: M \to J^1C$ is a section of $p_1: J^1C \to M$, and we have set
\[
s^\alpha_i = A^\alpha_i \circ p_{10} \circ \bar{s}, \quad s^\alpha_{i,j} = \frac{\partial s^\alpha_i}{\partial x^j}, \quad \bar{s}^\alpha_{i,j} = A^\alpha_{i,j} \circ \bar{s}.
\]

**Lemma 5.3.** The Hessian metric of a gauge-invariant Lagrangian $L$ on the bundle of connections $p: C \to M$ of a principal bundle $\pi: P \to M$, is singular; i.e.,
\[
\det \left( \frac{\partial^2 L}{\partial A^\alpha_{i,i} \partial A^\beta_{k,l}} \right)_{i,j} = 0.
\]

**Proof.** As $L = \bar{L} \circ \Omega$, we have $\partial L / \partial A^\alpha_{i,i} = 0$, taking the curvature equations into account, i.e.,
\[
R^\alpha_{ij} \circ \Omega = A^\alpha_{i,j} - A^\alpha_{j,i} - c^\alpha_{\beta\gamma} A^\beta_i A^\gamma_j.
\]

Hence $\partial^2 L / \partial A^\alpha_{i,i} \partial A^\beta_{k,l} = 0$, for all indices $\beta, k, l$. \qed

**Definition 5.4.** A Lagrangian $L \in C^\infty(J^1C)$ is said to be regular if the Hessian $\text{Hess}(L)$ is non-singular.

A gauge-invariant Lagrangian $L \in C^\infty(J^1C)$ is said to be weakly regular if the Hessian metric $\text{Hess}(\bar{L})$ of the function in (11) associated to $L$ according to Utiyama’s theorem, is non-singular.

In terms of the coordinate system $(x^i; R^\alpha_{jk})$, $j < k$, on $\Lambda^2 T^*M \otimes \text{ad}P$ introduced in Section 5.1, this means
\[
\det \left( \frac{\partial^2 \bar{L}}{\partial R^\alpha_{i,j} \partial R^\beta_{k,l}} \right)_{i,j} \neq 0.
\]

**Remark 5.5.** The inequation (12) imposes a generic condition on $\bar{L}$. In fact, most of gauge-invariant Lagrangians in the field theory satisfy the weak regularity condition (12); for example, the general Yang-Mills Lagrangian on the bundle of connections of a principal bundle $P \to M$ with semisimple Lie group $G$ over a pseudo-Riemannian manifold $(M, g)$ (even when constructed by using a non-degenerate adjoint-invariant pairing on the Lie algebra other than the Cartan-Killing pairing, see [3, 5]) is weakly regular. More generally, any quadratic function as in (11), which is simultaneously invariant under the adjoint representation and under the action of the gauge group of the principal bundle of $g$-orthonormal linear frames, gives rise to a weakly regular Lagrangian, see [4]. Similarly, Born-Infeld Lagrangians (e.g., see [15, 16]) are also weakly regular. We remark on the fact that some special Lagrangians are not weakly regular; for example, if we let the matter field vanish in the Seiberg-Witten Lagrangian (e.g., see [9]) then we obtain a Lagrangian on the bundle of connections, which is not weakly regular (basically, because it involves only the self-dual part of the curvature). Finally, we should also remark that non-gauge invariant Lagrangians (in the sense of Section 5.1) may produce gauge invariant actions, as the Chern-Simons Lagrangian. All of them are not considered below.
Given an arbitrary fibred manifold \( p: E \to M \), we recall that the projection \( p_{10}: J^1 E \to E \) is endowed with an affine-bundle structure modelled over \( T^* M \otimes E V(p) = p^* T^* M \otimes V(p) \); see the item (b) in Section 2.3. In the particular case of the bundle of connections, which is itself an affine bundle modelled over \( T^* M \otimes \text{ad} P \), we conclude that \( p_{10}: J^1 C \to C \) is an affine bundle modelled over

\[
T^* M \otimes_C V(p) = p^*(\otimes^2 T^* M \otimes \text{ad} P).
\]

Hence, sections of \( T^* M \otimes_C V(p) \) can be considered as \( \text{ad} P \)-valued covariant tensors of degree 2 on \( M \) with coefficients in \( C \).

**Theorem 5.6.** Let \( \overline{S}_\Lambda \) (resp. \( S_\Lambda \)) denote the set of solutions to H-C (resp. E-L) equations of a weakly regular gauge-invariant Lagrangian \( L \) on the bundle of connections \( \pi: P \to M \). If \( \bar{s}: M \to J^1 C \) belongs to \( \overline{S}_\Lambda \), then the section \( s = p_{10} \circ \bar{s} \) belongs to \( S_\Lambda \). Hence a natural projection exists \( \bar{\pi}: \overline{S}_\Lambda \to S_\Lambda \), \( \bar{\pi}(\bar{s}) = p_{10} \circ \bar{s} \), which is an affine bundle modelled as follows:

\[
\bar{\pi}^{-1}(s) = \{ j^1 s + t : t \in \Gamma(S^2 T^* M \otimes \text{ad} P) \}, \quad \forall s \in S_\Lambda.
\]

**Proof.** We begin with the first H-C equation in Proposition 5.2. As \( L = \tilde{L} \circ \Omega \), we obtain

\[
\frac{\partial L}{\partial A_{i,j}^\alpha} = 0, \quad \frac{\partial L}{\partial A_{i,k}^\alpha} = \frac{\partial \tilde{L}}{\partial A_{i,k}^\alpha} \circ \Omega,
\]

where we have set \( R_{i,k}^\alpha = -R_{k,i}^\alpha \) for \( i > k \). Hence

\[
(s_{h,j}^\beta - \bar{s}_{h,j}^\beta) \left( \frac{\partial^2 \tilde{L}}{\partial R_{i,k}^\alpha \partial R_{h,j}^\beta} \circ (\Omega \circ \bar{s}) \right) = 0.
\]

If we assume the weak regularity condition (12) holds, then the previous equation yields

\[
(s_{h,j}^\beta - \bar{s}_{h,j}^\beta) - (s_{j,h}^\beta - \bar{s}_{j,h}^\beta) = 0, \quad \forall \beta, h, j.
\]

If we write \( \bar{s} = j^1 s + t \), for a 2-tensor \( t \), the condition above means that \( t \) is symmetric; that is, \( t \) is a section of \( S^2 T^* M \otimes \text{ad} P \to M \).

Next, we study the second equation in Proposition 5.2. Taking the equations (14) into account, we have

\[
- \frac{\partial}{\partial x^2} \left( \frac{\partial L}{\partial A_{i,j}^\alpha} \circ \bar{s} \right) + \frac{\partial L}{\partial A_{i,k}^\alpha} \circ \bar{s} + (s_{h,j}^\beta - \bar{s}_{h,j}^\beta) \left( \frac{\partial \tilde{L}}{\partial A_{i}^\alpha} \left( \frac{\partial \tilde{L}}{\partial R_{h,j}^\beta} \circ \Omega \right) \circ \bar{s} \right) = 0.
\]

The last term vanishes identically as \( s_{h,j}^\beta - \bar{s}_{h,j}^\beta \) is symmetric by virtue of (13) and we have \( \frac{\partial L}{\partial R_{h,j}^\beta} = -\frac{\partial L}{\partial R_{j,h}^\beta} \). Then, the second equation reduces to

\[
- \frac{\partial}{\partial x^2} \left( \frac{\partial L}{\partial A_{i,j}^\alpha} \circ \bar{s} \right) + \frac{\partial L}{\partial A_{i,k}^\alpha} \circ \bar{s} = 0.
\]
which is precisely the E-L equation, but evaluated at \( s \) instead of \( j^1s \). Nevertheless, the following formula is readily checked:

\[
\frac{\partial L}{\partial A^\alpha_i} = 2\left( c^\beta_{\alpha \gamma} A^\gamma_j \frac{\partial \tilde{L}}{\partial R^\beta_{ij}} \right) \circ \Omega.
\]

Moreover, from (15) we deduce

\[
\Omega \circ \tilde{s} = (s^\alpha_{i,j} - s^\alpha_{j,i}, - c^\alpha_{\beta \gamma} s^\beta_i s^\gamma_j) dx^i \wedge dx^j \otimes \tilde{B}_\alpha,
\]

\[
= (s^\alpha_{i,j} - s^\alpha_{j,i} - c^\alpha_{\beta \gamma} s^\beta_i s^\gamma_j) dx^i \wedge dx^j \otimes \tilde{B}_\alpha
\]

\[
= \Omega \circ j^1s.
\]

Therefore, from the formulas (14) and (17) we conclude that the equation (16) coincides with the E-L equation for \( s \).

**Remark 5.7.** As mentioned in the introduction, the formula (18) shows that the curvature remains constant along the fibre of \( \tilde{g} \) over any \( j^1s \in S_\Lambda \).

**Corollary 5.8.** The H-C equations of a weakly-regular gauge-invariant Lagrangian \( L \) on the bundle of connections \( p: C \to M \) of a principal bundle \( \pi: P \to M \), are equivalent to the following system:

\[
(s^\alpha_{i,j} - s^\alpha_{j,i}) - (s^\beta_{i,j} - s^\beta_{j,i}) = 0, \quad \forall \alpha, h, j.
\]

\[
- \frac{\partial}{\partial x^1} \left( \frac{\partial L}{\partial \omega^\alpha_{i,j}} \circ j^1s \right) + \frac{\partial L}{\partial A^\alpha_i} \circ j^1s = 0, \quad \forall \alpha, i,
\]

where \( \tilde{s}: M \to J^1C \) is a section of \( p_1: J^1C \to M \), and \( s^\alpha_i, s^\alpha_{i,j}, \) and \( s^\alpha_{i,j} \) are as in Proposition 5.2.

**Remark 5.9.** For every section \( \tilde{s}: M \to J^1C \) of \( p_1: J^1C \to M \), let \( s: M \to C \) be the section of \( p: C \to M \) defined by \( s = p_{10} \circ \tilde{s} \). As the points \( \tilde{s}(x), j^1s \in J^1C \) lie over the same fibre of \( p_{10}: J^1C \to C \) and this map admits an affine-bundle structure modelled over the vector bundle (13), a map

\[
\delta_C: J^1(p_1) \to \otimes^2 T^* M \otimes \text{ad} P
\]

exists such that, \( \delta_C(j^1s) = \tilde{s}(x) - j^1s \). If \( \text{alt}: \otimes^2 T^* M \otimes \text{ad} P \to \wedge^2 T^* M \otimes \text{ad} P \) denotes the anti-symmetrization operator, then the first group of H-C equations in Corollary 5.8 means that \( j^1s \) takes values into the subbundle \( \text{ker}(\text{alt} \circ \delta_C) \).

**Proposition 5.10.** If \( \Lambda \) is a gauge-invariant Lagrangian density on the bundle of connections of a principal bundle \( \pi: P \to M \) and \( \tilde{s} \in S_\Lambda \) (resp. \( s \in S_\Lambda \)), then \( \Phi_C^{(1)} \circ \tilde{s} \in S_\Lambda \) (resp. \( \Phi_C \circ s \in S_\Lambda \)) for every \( \Phi \in \text{Gau}P \). Accordingly, the gauge group of \( P \) acts (on the left) on \( S_\Lambda \) (resp. \( S_\Lambda \)) by setting \( \Phi \cdot \tilde{s} = \Phi_C^{(1)} \circ \tilde{s} \) (resp. \( \Phi \cdot s = \Phi_C \circ s \)), \( \forall \tilde{s} \in S_\Lambda \) (resp. \( \forall s \in S_\Lambda \)), \( \forall \Phi \in \text{Gau}P \).
Proof. We prove that the section $\Phi_C^{(1)} \circ \bar{s}$ of $p_1$ is a solution to H-C equation 2. For every $Z \in \mathfrak{X}^v(J^1E)$, we set

$$Y = \left(\Phi_C^{(1)}\right)^{-1} \cdot Z \in \mathfrak{X}^v(J^1E).$$

As $\Lambda$ is gauge invariant, we have $\left(\Phi_C^{(1)}\right)^* \Lambda = \Lambda$, from the functorial character of the Poincaré-Cartan form (see [7]) we obtain

$$\left(\Phi_C^{(1)} \circ \bar{s}\right)^* (i_Z d\Theta) = \bar{s}^* i_Y d\left(\left(\Phi_C^{(1)}\right)^* \Lambda\right) = \bar{s}^* i_Y d\Lambda = 0.$$

The proof for the solutions to E-L equations, is similar and therefore it is omitted.

Theorem 5.6 shows that the set of solutions of H-C trivially fibers over the set of solutions of E-L. On the other hand, for gauge-invariant problems, the moduli space of solutions under the action of the (restricted) gauge group plays a relevant role. We now study the relationship between the moduli of H-C and E-L showing that the first fibers over the second, but not necessarily in a trivial way. First, note that the gauge group $\text{Gau}_P$ acts on the adjoint bundle $\text{ad}P$ by setting $\Phi_{\text{ad}}((u, B)_{\text{ad}}) = (\Phi(u), B)_{\text{ad}}$, $\forall \Phi \in \text{Gau}_P$, $\forall (u, B) \in P \times \mathfrak{g}$, and this action obviously induces another action on $\mathcal{S}^2 T^* M \otimes \text{ad}P$ as follows:

$$\Phi_{\text{ad}P} (w_1 \odot w_2 \otimes v) = w_1 \odot w_2 \otimes \Phi_{\text{ad}P}(v),$$

$\forall w_1, w_2 \in T_x^* M$, $\forall v \in (\text{ad}P)_x$, where the symbol $\odot$ denotes symmetric product.

Proposition 5.11. Given a point $x_0 \in M$, let $\text{Gau}_{x_0}P$ be the subgroup of gauge transformations $\Phi \in \text{Gau}_P$ such that, $\Phi(u) = u$, $\forall u \in \pi^{-1}(x_0)$. Then

(i) For every gauge-invariant Lagrangian density $\Lambda$ on the bundle of connections $\pi: P \to M$, the quotient map $\kappa_P: \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda/\text{Gau}_{x_0}P$ is a set-theoretical principal $\text{Gau}_{x_0}P$-bundle.

(ii) In addition, if $\Lambda$ is weakly regular, the projection $\varrho: \mathcal{S}_\Lambda \to \mathcal{S}_\Lambda$ defined in Theorem 5.6 induces a mapping

$$\varrho_{\text{Gau}_{x_0}P}: \mathcal{S}_\Lambda/\text{Gau}_{x_0}P \to \mathcal{S}_\Lambda/\text{Gau}_{x_0}P,$$

$$\varrho_{\text{Gau}_{x_0}P} (\bar{s} \mod \text{Gau}_{x_0}P) = \varrho(\bar{s}) \mod \text{Gau}_{x_0}P,$$

which is the vector bundle associated to the principal bundle $\kappa_P$ by the action on $\mathcal{S}^2 \Omega^1(M) \otimes \text{gau}P$ induced on the sections of $\mathcal{S}^2 T^* M \otimes \text{ad}P$ by the action of $\text{Gau}_P$ defined in the formula (19) above.
Proof. As is known, Gau$_{x_0}P$ acts freely on the space of connections, i.e., on the sections of $p: C \to M$ and, in particular, on $S_{\Lambda}$ and on $S_{\Lambda}$ (e.g., see [6] Theorem 2.2.4, [13] III.C]). Nevertheless, the quotients $S_{\Lambda}/$Gau$_{x_0}P$ and $S_{\Lambda}/$Gau$_{x_0}P$ may be singular, e.g., see [6] p. 134). Because of this, we consider such structure from the set-theoretical point of view only.

Furthermore, the mapping $\varrho_{\text{Gau}_{x_0}P}$ is well defined as

$$\varrho \left( \Phi^{(1)}_C \circ \bar{s} \right) = \left( p_{10} \circ \Phi^{(1)}_C \right) \circ \bar{s} = \left( \Phi_C \circ p_{10} \right) \circ \bar{s} = \Phi_C \circ \varrho(\bar{s}),$$

and from Theorem 5.6 it follows that every $\bar{s}$ in $S_{\Lambda}$ can be uniquely written as $\bar{s} = j^1s + t$, where $s = \varrho(\bar{s})$ and $t \in S^2\Omega^1(M) \otimes \text{gau}P$. Hence $\bar{s}$ can be identified to the pair $(s, t)$, i.e., $S_{\Lambda} \cong S_{\Lambda} \times S^2\Omega^1(M) \otimes \text{gau}P$. Recalling that $\Phi_C: C \to C$ is an affine-bundle morphism whose associated vector-bundle is $\text{id}_T^* \otimes \Phi_{\text{ad}}: T^*M \otimes \text{ad}P \to T^*M \otimes \text{ad}P$, we have

$$\Phi^{(1)}_C \circ \bar{s} = \Phi^{(1)}_C \circ (j^1s + t) = J^1(\Phi_C \circ s) + \Phi_{\text{ad}} \cdot t,$$

thus concluding the proof. \hfill \Box

**Theorem 5.12.** With the same notations as in Section 4 and the same assumptions as in Theorem 5.6 if $\bar{X} \in T_sS_{\Lambda}$, then $(p_{10})_* \circ \bar{X} \in T_sS_{\Lambda}$, where $s = p_{10} \circ \bar{s}$. Hence, the natural map $\varrho: S_{\Lambda} \to S_{\Lambda}$ induces a linear map $\varrho_*: T_sS_{\Lambda} \to T_sS_{\Lambda}$, $\varrho_* (\bar{X}) = (p_{10})_* \circ \bar{X}$. Moreover, $\ker \varrho_* \cong \Gamma(S^2T^*M \otimes \text{ad}P)$.

Proof. We first begin with the second Jacobi equation (7) for a gauge-invariant Lagrangian $L: J^1C \to \mathbb{R}$ and a Jacobi vector field

$$\bar{X} = v^\alpha_i \frac{\partial}{\partial A^\alpha_i} + v^\alpha_{i,j} \frac{\partial}{\partial A^\alpha_{i,j}}, \quad v^\alpha_i, v^\alpha_{i,j} \in C^\infty(M),$$

along a solution $\bar{s} \in S_{\Lambda}$. We have

$$0 = \left( \frac{\partial^2 L}{\partial A^\alpha_{r,i} \partial A^\alpha_{r,j}} \circ \bar{s} \right) \left( \frac{\partial v^\alpha_i}{\partial x^l} - v^\alpha_{r,i} \right)$$

$$+ \left( \frac{\partial v^\alpha_i}{\partial x^l} - s^\alpha_{r,i} \right) \left( \frac{\partial^3 L}{\partial A^\alpha_i \partial A^\alpha_{r,i} \partial A^\alpha_{h,j}} \circ \bar{s} \right) v^\beta_j$$

$$+ \left( \frac{\partial v^\alpha_i}{\partial x^l} - s^\alpha_{r,i} \right) \left( \frac{\partial^3 L}{\partial A^\alpha_{r,i} \partial A^\alpha_{l,j} \partial A^\alpha_{h,j}} \circ \bar{s} \right) v^\beta_{l,k},$$

for any $\sigma, h, j$. From Theorem 5.6 $j^1s - \bar{s}$ is a symmetric tensor. Moreover, taking the formula (13) into account, the last two summands of (20) vanish. We thus obtain

$$0 = \left( \frac{\partial^2 L}{\partial A^\alpha_{r,i} \partial A^\alpha_{s,j}} \circ \bar{s} \right) \left( \frac{\partial v^\alpha_i}{\partial x^l} - v^\alpha_{r,i} \right).$$
which, assuming the weak regularity of $L$, implies

$$v^\alpha_{r,i} = \frac{\partial v^\alpha_r}{\partial x^i} + t^\alpha_{r,i},$$

where $t^\alpha_{r,i}$ are the components of a symmetric tensor $t \in \Gamma(S^2T^*M \otimes \text{ad}P)$. From the formula (1) and Lemma 2.1 we conclude $\bar{L} = X^{(1)} + t$, where $X^{(1)}$ is the 1-jet prolongation of the vector field along $s$ given by $X = v^\alpha_i \partial / \partial A^\alpha_i$.

Next, we consider the first Jacobi equation [3], from which we obtain

$$0 = \left( \frac{\partial^2 L}{\partial A^\alpha_i \partial A^\alpha_q} \circ \bar{s} - \frac{\partial^3 L}{\partial x^i \partial A^\alpha_i \partial A^\alpha_q} \circ \bar{s} \right) \frac{\partial v^\alpha_r}{\partial x^i} + \left( s^\beta_{t,i} - \bar{s}^\beta_{t,i} \right) \left( \frac{\partial^3 L}{\partial A^\alpha_i \partial A^\alpha_q \partial A^\alpha_{q,i}} \circ \bar{s} \right) \frac{\partial v^\alpha_r}{\partial x^i},$$

$$0 = \left( \frac{\partial^2 L}{\partial A^\alpha_i \partial A^\alpha_q} \circ \bar{s} - \frac{\partial^3 L}{\partial x^i \partial A^\alpha_i \partial A^\alpha_q} \circ \bar{s} \right) \frac{\partial v^\alpha_r}{\partial x^i} + \left( s^\beta_{t,i} - \bar{s}^\beta_{t,i} \right) \left( \frac{\partial^3 L}{\partial A^\alpha_i \partial A^\alpha_q \partial A^\alpha_{q,i}} \circ \bar{s} \right) \frac{\partial v^\alpha_r}{\partial x^i}.$$  (22)

Again taking the symmetry of the differences $s^\beta_{t,i} - \bar{s}^\beta_{t,i}$ (and their derivatives) into account, the previous equation reduces to the following:

$$0 = \left( \frac{\partial^2 L}{\partial A^\alpha_i \partial A^\alpha_q} \circ \bar{s} - \frac{\partial^3 L}{\partial x^i \partial A^\alpha_i \partial A^\alpha_q} \circ \bar{s} \right) \frac{\partial v^\alpha_r}{\partial x^i} - \frac{\partial^2 s^\gamma_{t,i}}{\partial x^i \partial x^j} \left( \frac{\partial^3 L}{\partial A^\alpha_i \partial A^\alpha_{q,i} \partial A^\alpha_{q,j}} \circ \bar{s} \right) \frac{\partial v^\alpha_r}{\partial x^i} + \left( \frac{\partial^3 L}{\partial A^\alpha_i \partial A^\alpha_{q,i} \partial A^\alpha_{q,j}} \circ \bar{s} \right) \frac{\partial v^\alpha_r}{\partial x^i}.$$  (23)

By taking derivatives in (21), we obtain

$$\frac{\partial v^\alpha_r}{\partial x^i} = \frac{\partial v^\alpha_r}{\partial x^j} \frac{\partial x^j}{\partial x^i} + \frac{\partial t^\alpha_{r,j}}{\partial x^i},$$

and substituting these expressions into the equation (23), again by virtue of the
skew-symmetry of $\partial L/\partial A_{r,i}^\alpha$, we have
\[
0 = \left( \frac{\partial^2 L}{\partial A_{r}^\alpha \partial A_{q}^\beta} \circ \bar{s} - \frac{\partial^3 L}{\partial x^i \partial A_{r}^\alpha \partial A_{q,j}^\beta} \circ \bar{s} - s_{t,j} \left( \frac{\partial^3 L}{\partial A_{r}^\alpha \partial A_{q,j}^\beta} \circ \bar{s} \right) - \frac{\partial^3 L}{\partial x^i \partial x^j} \circ \bar{s} \right) v^\alpha_r + \left( \frac{\partial^3 L}{\partial A_{r}^\alpha \partial A_{q,j}^\beta} \circ \bar{s} + s_{t,j} \left( \frac{\partial^3 L}{\partial A_{r}^\alpha \partial A_{q,j}^\beta} \circ \bar{s} \right) \right) \frac{\partial v^\alpha_r}{\partial x^j} \circ \bar{s}.
\]

Finally, taking the equations $\Omega \circ \bar{s} = \Omega \circ j^1 s$, $L = \tilde{L} \circ \Omega$, into account, we conclude that $\bar{s}$ can be replaced by $j^1 s$ into the previous equation and we can end the proof by simply applying Remark 4.6.

**Corollary 5.13.** A Jacobi field $\bar{X} \in T_{\bar{s}}\bar{S}_\Lambda$ is integrable if and only if the Jacobi field $\varrho_* (\bar{X}) = X \in T_{\bar{s}}S_\Lambda$, $s = p_{10} \circ \bar{s}$, is integrable.

**Proof.** If $\bar{X} \in T_{\bar{s}}\bar{S}_\Lambda$ is an integrable Jacobi field, then $\bar{X} = d/d\varepsilon |_{\varepsilon=0} \bar{s}_\varepsilon$, where $\bar{s}_\varepsilon = j^1 s + \varepsilon t$, with $\pi_{10} \circ \bar{s} = s_\varepsilon \in S_\Lambda$, $t_\varepsilon$ being a symmetric tensor. Then,
\[
\varrho_* (\bar{X}) = (\pi_{10})_* \bar{X} = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \pi_{10} \circ \bar{s}_\varepsilon = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} s_\varepsilon.
\]
Hence $X = \varrho_* (\bar{X})$ is integrable. Conversely, assume $X = d/d\varepsilon |_{\varepsilon=0} s_\varepsilon$, $s_\varepsilon \in S_\Lambda$, and $\bar{X} \in (\varrho_*)^{-1}X$. We know that $\bar{X} = X^{(1)} + t$, where $t$ is a symmetric tensor. Then, $\bar{X} = d/d\varepsilon |_{\varepsilon=0} \bar{s}_\varepsilon$ with $\bar{s}_\varepsilon = j^1 s + \varepsilon t \in \bar{S}_\Lambda$, and $\bar{X}$ is integrable.

**6** H-C self-dual and anti-self-dual connections

Let $(M, g)$ be a pseudo-Riemannian $n$-dimensional oriented connected manifold of signature $(n^+, n^-)$, $n = n^+ + n^-$, and let $v_g = \sqrt{\left| \det (g_{ij}) \right|} dx^1 \wedge \ldots \wedge dx^n$, $g_{ij} = g(\partial/\partial x^i, \partial/\partial x^j)$, be its pseudo-Riemannian volume form. The canonical duality isomorphism attached to $g$ is denoted by $T_* M \to T_*^* M$, $X \mapsto X^\flat$, with inverse map $T^*_* M \to T_* M$, $w \mapsto w^\sharp$. Let $g^{(r)}$ be the metric on $\Lambda^r T^* M$ given by $g^{(r)}(w^1 \wedge \ldots \wedge w^r, \bar{w}^1 \wedge \ldots \wedge \bar{w}^r) = \det (g((w^j)^j, ((\bar{w})^j)^j))_{j=1}^r$.

Let $V \to M$ be a vector bundle. The Hodge star can be extended to $V$-valued forms as follows: $*(\omega_r \otimes v) = (\ast \omega_r) \otimes v$, $\forall \omega_r \in \Lambda^r_t T^*_* M$, $\forall v \in V_x$.

Let $\pi: P \to M$ be a principal $G$-bundle and let $\pi_{ad} P: \text{ad} P \to M$ be the adjoint bundle; i.e., the bundle associated with $P$ under the adjoint representation of $G$ on its Lie algebra $\mathfrak{g}$. For every $B \in \mathfrak{g}$ and every $u \in P$, let $(u, B)_G$ be the coset of $(u, B) \in P \times \mathfrak{g}$ modulo $G$. A symmetric bilinear form $\langle \cdot, \cdot \rangle \in S^2 \mathfrak{g}^\ast$ is
said to be invariant under the adjoint representation if the following equation holds: \( \langle \text{Ad}_g B, \text{Ad}_g C \rangle = \langle B, C \rangle, \forall g \in G, \forall B, C \in \mathfrak{g}. \) By taking derivatives on this equation we obtain \( \langle [A, B], C \rangle + \langle B, [A, C] \rangle = 0, \forall A, B, C \in \mathfrak{g}. \) If the group \( G \) is connected, then both equations above are equivalent.

Every symmetric bilinear form \( \langle \cdot, \cdot \rangle \) \( \in \mathcal{S}_g^2 \) invariant under the adjoint representation induces a fibred metric \( \langle \cdot, \cdot \rangle \) : \( \text{ad}P \oplus \text{ad}P \to \mathbb{R} \) by setting

\[
\langle (u, B)G, (u, C)G \rangle = \langle B, C \rangle, \quad \forall u \in P, \forall B, C \in \mathfrak{g}.
\]

We further assume that the pairing (24) is non-degenerate.

Every pseudo-Riemannian metric \( g \) on \( M \) and every fibred \( \langle \cdot, \cdot \rangle \) on \( \text{ad}P \) induce a fibred metric on the vector bundle of \( \text{ad}P \)-valued differential \( r \)-forms on \( M \) as follows: \( \langle (\alpha_r \otimes a, \beta_r \otimes b) \rangle = g^r(a, b) \langle \alpha_r, \beta_r \rangle \), \( \forall \alpha_r, \beta_r \in \bigwedge^r T^*M, \) and \( \forall a, b \in (\text{ad}P)_x. \) Moreover, the pairing (24) defines an exterior product (see [1]),

\[
\langle \Lambda(\omega, \wedge) \rangle \to \bigwedge^r T^*M, \quad \langle \alpha_q \otimes a \rangle \Lambda(\beta_r \otimes b) = \langle \alpha_q \wedge \beta_r \rangle \langle a, b \rangle.
\]

Let \( p : C \to M \) be the bundle of connections of \( P. \) According to the previous definitions, a pseudo-Riemannian metric \( g \) on \( M \) and an adjoint-invariant symmetric bilinear form \( \langle \cdot, \cdot \rangle \) allow one to define a quadratic Lagrangian density \( \Lambda = \Lambda^g \) on \( J^1C \) by setting,

\[
\Lambda \left( j^1_s \right) = \langle \Omega^s(x), \Omega^s(x) \rangle \nu_g(x) = \Omega^s(x) \wedge \ast \Omega^s(x),
\]

where \( s \) is a local section of \( p \) defining a principal connection whose curvature form is denoted by \( \Omega^s. \) In [3] it is proved that the E-L equations of the Lagrangians above are seen to be independent of the pairing (24) and they coincide with the classical Yang-Mills equations: \( \nabla \ast \Omega^s = 0. \)

**Theorem 6.1.** In addition to the hypotheses above, assume \( \dim M = 4 \) and \( g \) is a Riemannian metric. Let \( \mathcal{S}_g^+ \) (resp. \( \mathcal{S}_g^- \)) be the set of self-dual (resp. anti-self-dual) connections with respect to the Lagrangian density (24). A section \( s \) of \( p_1 : J^1C \to M \) belongs to the fibre \( g^{-1}(s) \), with \( s \in \mathcal{S}_g^+ \) (resp. \( s \in \mathcal{S}_g^- \)) if and only if the following equations hold:

\[
\text{alt}(s - j^1_s) = 0, \quad (26)
\]

\[
\ast(\Omega \circ s) = (\Omega \circ s) \quad \text{(resp. } \ast (\Omega \circ s) = -(\Omega \circ s)), \quad (27)
\]

where \text{alt} : \( \bigwedge^2 T^*M \otimes \text{ad}P \to \bigwedge^2 T^*M \otimes \text{ad}P \) is the alternating operator.

**Remark 6.2.** The equation (26) is a first-order differential equation, whereas (27) is a purely algebraic equation. The condition \( \text{alt}(s - j^1_s) = 0 \) is not specific of the Yang-Mills Lagrangian but general for any weakly regular gauge-invariant Lagrangian. In fact, it defines the subset of \( \Gamma(p_1) \) given by

\[
\{ j^1_s + t : s \in \Gamma(p), t \in \Gamma(S^2T^*M \otimes \text{ad}P) \}.
\]
It is thus interesting to note that the group of equations really defined by self-dual (resp. anti-self-dual) connections, is not longer a differential equation but an algebraic constrain on the subset \( \mathcal{S}_\Lambda \).

**Proof of Theorem 6.1.** As \( L \) is weakly regular, if \( \bar{s} \in \mathcal{S}_\Lambda \), then \( \text{alt} (\bar{s} - j^1 s) = 0 \) by virtue of Remark 5.9. Moreover, as

\[
-\frac{\partial}{\partial x^j} \left( \frac{\partial L}{\partial A^\alpha_{i,j} \circ \bar{s}} \circ \bar{s} \right) + \frac{\partial L}{\partial A^\alpha_i \circ \bar{s}} = -\frac{\partial}{\partial x^j} \left( \frac{\partial L}{\partial A^\alpha_{i,j} \circ j^1 s} \circ j^1 s \right) + \frac{\partial L}{\partial A^\alpha_i} \circ j^1 s,
\]

as \( \Omega \circ \bar{s} = \Omega \circ j^1 s \), we also obtain \( \nabla^s \ast \Omega^s = 0 \), where \( \Omega^s = \Omega \circ j^1 s \) and \( s = p_{\Omega} \circ \bar{s} \). Hence, the definition of a self-dual connection (i.e., \( \ast(\Omega \circ \bar{s}) = (\Omega \circ j^1 s) \)) can be written as \( \ast(\Omega \circ \bar{s}) = (\Omega \circ \bar{s}) \) for every \( \bar{s} \in \varrho^{-1}(S^*_\Lambda). \) Similarly for \( S^*_\Lambda \). \( \Box \)

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