The backward $\lambda$-Lemma and Morse filtrations

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Abstract
Consider the infinite dimensional hyperbolic dynamical system provided by the (forward) heat semi-flow on the loop space of a closed Riemannian manifold $M$. We use the recently discovered backward $\lambda$-Lemma and elements of Conley theory to construct a Morse filtration of the loop space whose cellular filtration complex represents the Morse complex associated to the downward $L^2$-gradient of the classical action functional. This paper is a survey. Details and proofs will be given in [6].

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1 Introduction
Consider a closed smooth manifold $M$ of dimension $n \geq 1$ equipped with a Riemannian metric and the Levi-Civita connection $\nabla$. Pick a smooth function $V : S^1 \times M$ and set $V_t(q) := V(t, q)$. Here and throughout we identify $S^1 = \mathbb{R}/\mathbb{Z}$.

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For smooth maps $\mathbb{R} \times S^1 \to M : (s,t) \mapsto u(s,t)$ consider the heat equation
\[
\partial_s u - \nabla u \partial_t u - \nabla V_t(u) = 0.
\]
It corresponds to the downward $L^2$-gradient equation of the action given by
\[
\mathcal{S}_V(x) = \int_0^1 \left( \frac{1}{2} |\dot{x}(t)|^2 - V(t,x(t)) \right) dt
\]
for any element $x : S^1 \to M$ of the free loop space $\Lambda^a M := \{ \mathcal{S}_V < a \}$. The critical points of $\mathcal{S}_V$ are the solutions $x \in \Lambda^a M$ of the ODE $-\nabla \dot{x} - \nabla V_t(x) = 0$, that is the (perturbed) closed geodesics. Throughout this paper we fix a regular value $a$ of $\mathcal{S}_V$ and assume that the Morse-Smale condition holds true below level $a$. Consider the sublevel set $\Lambda^a M := \{ \mathcal{S}_V < a \}$. In this case the action is a Morse function on $\Lambda^a M$ and the set of solutions to (1) that converge to critical points $x^+ \in \Lambda^a M$, as $s \to \pm \infty$, carries the structure of a smooth manifold whose dimension is given by the Morse index difference $\text{ind} V(x) - \text{ind} V(y)$. Moreover, the number $n_a$ of elements of the set $\text{Crit}$ of critical points of $\mathcal{S}_V$ in $\Lambda^a M$ is finite. By $\text{Crit}_k$ we denote the set of critical points in $\Lambda^a M$ of Morse index $k$. For each $x \in \text{Crit}$ pick an orientation of the largest subspace $E_x$ of the Hilbert space
\[
X := T_x \Lambda^a M = W^{1,2}(S^1, x^*TM)
\]
on which the Hessian of $\mathcal{S}_V$ at $x$ is negative definite. (The dimension of $E_x$ is finite and called the Morse index of $x$.)

**Heat flow homology [4]**

By definition the Morse chain groups $\text{CM}_k = \text{CM}_k(\Lambda^a M, \mathcal{S}_V; \mathbb{Z})$ are the free abelian groups generated by the (perturbed) closed geodesics $x$ of Morse index $k$ and below level $a$, that is $\mathbb{Z}^{\text{Crit}_k}$. Set $\text{CM}_k = \{0\}$ in case of the empty set. The chosen orientations provide the characteristic sign $n_u \in \{\pm 1\}$ for each heat flow solution $u$ of (1) between critical points of index difference one. Up to shift in the time variable $s$, there are only finitely many such $u$. Counting them with signs $n_u$ provides the Morse boundary operator $\partial_k : \text{CM}_k \to \text{CM}_{k-1}$. By $\text{HM}_k$ we denote the corresponding homology groups.

**Main result: The natural isomorphism to singular homology [6]**

The idea to use cellular filtrations to calculate Morse homology goes back at least to Milnor [3]. One needs to construct a cellular filtration $\mathcal{F}$ of $\Lambda^a M$ whose cellular filtration complex $(C_*, \mathcal{F}, \partial_*)$ precisely represents the Morse complex, up to natural identification. In this case we are done, since
\[
\text{HM}_k \equiv H_*((C_*, \mathcal{F}, \partial_*)) \simeq H_*(\Lambda^a M)
\] (2)
where the isomorphism is provided by algebraic topology given any cellular filtration of $\Lambda^a M$ (related to the Morse complex or not); see e.g. [2].
2 Morse filtrations and Conley pairs

Definition 2.1 (Cellular filtration and homology). Assume \( \mathcal{F} = \{F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_\mu \} \) is a nested sequence of open subsets of \( \Lambda^aM \) such that relative singular homology \( H_k(F_k, F_{k-1}) \) is trivial whenever \( \ell \neq k \) and where \( F_{-1} := \emptyset \). In this case \( \mathcal{F} \) is a cellular filtration of \( \Lambda^aM \). For the algebraic topology used in this section we refer to [2]. The cellular chain complex consists of the cellular chain groups \( C_k\mathcal{F} := H_k(F_k, F_{k-1}) \) together with the triple boundary operators \( \partial_k : H_k(F_k, F_{k-1}) \to H_{k-1}(F_{k-1}, F_{k-2}) \). A cellular filtration \( \mathcal{F} \) is called a Morse filtration, if \( C_k\mathcal{F} = CM_k \) for every \( k \in \mathbb{N} \), that is each relative homology group \( H_k(F_k, F_{k-1}) \) is generated precisely by the critical points of Morse index \( k \).

Remark 2.2. To establish (2) we need to a) construct a Morse filtration \( \mathcal{F} \) of \( \Lambda^aM \) and b) show that the associated triple boundary operator counts heat flow lines according to their characteristic signs between critical points of index difference one. How to solve these two problems is known for flows; cf. [3] or [1, thm. 2.11]. The solution to b) carries over to our semi-flow situation, since restricted to the (finite dimensional) unstable manifolds the semi-flow turns into a flow. It remains to construct a Morse filtration \( \mathcal{F} \) of \( \Lambda^aM \).

The Abbondandolo-Majer construction for flows [1]

In their construction of a Morse filtration \( \mathcal{F}' \) of \( \Lambda^aM \) openness of the sets \( F_k' \) follows from openness of the time-T-map and the Morse property is a consequence of forward flow invariance of the open sets \( F_k' \). Start by setting \( N_0 \) equal to the union of open local sublevel sets, one for each local minimum \( x_0 \). Set \( F_0' := N_0 \). Next choose a small open ball about each index one critical point and denote their (disjoint) union by \( N_1' \). Then take the union of \( F_0' \) and the whole forward flow of \( N_1' \) and call it \( F_1' := F_0' \cup \varphi_0(\varphi_{0,\infty})N_1' \). Similarly define \( F_2' \) and \( F_3', \ldots, F_n' \).

A construction for semi-flows using Conley pairs [6]

The Cauchy problem associated to the heat equation (1) for maps \( [0, \infty) \to \Lambda^aM : s \mapsto u_s = u(s, \cdot) \) is well posed and leads to the continuous semi-flow

\[
\varphi : [0, \infty) \times \Lambda^aM \to \Lambda^aM
\]

called the heat flow. In fact \( \varphi \) is of class \( C^1 \) on \( (0, \infty) \). A characteristic feature of the heat flow is its extremely regularizing nature, namely \( \varphi, \gamma \in C^\infty(S^1, M) \) whenever \( \gamma \in \Lambda M \) and \( s > 0 \). Observe that the set of nonsmooth elements is dense\(^1\) in \( \Lambda M \). Hence \( \varphi_s \) is not an open map for \( s > 0 \) and the Abbondandolo-Majer method does not work. Instead we propose the following construction.

It is a very simple—but far reaching—observation that by continuity of \( \varphi_s \), preimages of open sets are open. Define \( N_0 \) as above. Observe that the preimage \( (\varphi_T)^{-1}N_0 \)

\(^1\)Pick \( \gamma \in \Lambda M \) and a nonsmooth \( \xi \in W^{1,2}(S^1, \pi^* TM) \). For large integers \( j \) set \( \exp_j(\frac{1}{j} \xi) \).
is open and semi-flow invariant. Pick any index one critical point \( x_1 \). The (one-dimensional) unstable manifold of \( x_1 \) necessarily enters \( N_0 \). Consequently our preimage gets very close to \( x_1 \) for \( T \) very large, however, it never contains \( x_1 \). To get over the barrier \( x_1 \) assume we had an open neighborhood \( N_{x_1} \) of \( x_1 \) containing no other critical points and an open subset \( L_{x_1} \subset N_{x_1} \) whose closure does not contain \( x_1 \). Assume further that \( L_{x_1} \) is semi-flow invariant in \( N_{x_1} \) and every element leaving \( N_{x_1} \) under the semi-flow necessarily runs through \( L_{x_1} \) first. Such a pair \((N_{x},L_{x})\) is called a Conley pair for \( x \in \text{Crit} \) and \( L_{x} \) is called an exit set for the Conley set \( N_{x} \).

Pick \( x \in \text{Crit} \) and set \( c := \mathcal{A}_{\mathcal{V}}(x) \). For \( \varepsilon > 0 \) small and \( \tau > 0 \) large the sets

\[
N_i = N_i^{c,\tau} := \{ y \in \Lambda^c \mathcal{V} | \mathcal{A}_{\mathcal{V}}(\phi_{\tau} y) > c - \varepsilon \},
\]

\[
L_s = L_s^{c,\tau} := \{ y \in N_s | \mathcal{A}_{\mathcal{V}}(\phi_{2\tau} y) < c - \varepsilon \}
\]

form a Conley pair for \( x \). Here \( \{\ldots\} \) indicates the path connected component that contains \( x \). By Theorem 3.2 (d) the sets \( N_s \) corresponding to different critical points \( x \) are pairwise disjoint. For \( k \in \{0,\ldots,n_a\} \) set

\[
N_k := \cup_{x \in \text{Crit}_k} N_k, \quad L_k := \cup_{x \in \text{Crit}_k} L_k.
\]

Consider the sets

\[
F_k := (\phi_{T_k})^{-1}(F_{k-1} \cup N_k) \supset L_{k+1}, \quad k = 0,\ldots,n_a - 1,
\]

where the constant \( T_k \) is chosen sufficiently large\(^2\) such that the inclusion holds true: see Figure 1. Because there are no critical points in the complemen of \( F_{n_a-1} \cup N_{n_a} \) in \( \Lambda^c \mathcal{V} \), there is a constant \( T_{n_a} \) such that \( \Lambda^c \mathcal{V} \) is equal to \( F_{n_a} := (\phi_{T_{n_a}})^{-1}(N_{n_a} \cup F_{n_a-1}) \).

Observe that each set \( F_k \) is open, because \( N_k \) and \( F_{k-1} \) are. Furthermore, although \( N_k \) is not semi-flow invariant the union \( N_k \cup F_{k-1} \) is, because the exit set \( L_k \) of \( N_k \) is contained in \( F_{k-1} \). Openness and semi-flow invariance heavily enter the calculation (5) in the proof of the Morse filtration property.

\(^2\)By Palais-Smale and \( \mathcal{F} \) being Morse \( y_k := \lim_{\tau \to +\infty} \phi_{\tau} y \) always exists and lies in \( \text{Crit} \). If \( y \in W^s(x_1) \) and \( y \neq x_1 \), then \( y_k \in C_{00} \) by Morse-Smale.

\(^3\)Here Palais-Smale, Morse-Smale on neighborhoods, and \( \mathcal{F} \) being bounded below enter.
\[ \gamma \in S_u \]
\[ D \gamma (x) \]
\[ \infty \]
\[ \tau_2 \]
\[ \{ S \tau = c + \epsilon \} \]
\[ W^u(x) \]
\[ D(x) \]
\[ \gamma \in S^c_u(x) \]
\[ L_x \]
\[ N_x \]
\[ \phi_T \]
\[ \{ S \tau = c - \epsilon \} \]

\[ \text{Figure 2: Conley pair } (N_x, L_x) \text{ foliated by equal time disks } (\phi_T)^{-1} \mathcal{S}_T(x) \]

**Morse filtration property**

Constructing suitable homotopy equivalences and applying the excision axiom of relative homology one shows that

\[
H_{\ell}(F_k, F_{k-1}) \simeq H_{\ell}(N_k, L_k) \simeq \bigoplus_{x \in \text{Crit}_k} H_{\ell}(N_x, L_x). \tag{5}
\]

Here the final step uses that \( N_k \) is a union of pairwise disjoint sets \( N_x \). So in order to prove that the nested sequence \( \mathcal{F} \) consisting of the open semi-flow invariant sets \( F_k \) defined by (4) is a Morse filtration of \( \Lambda^M \)—thereby concluding the proof of (2) via Remark 2.2—it remains to show that

\[
H_{\ell}(N_x, L_x) \simeq H_{\ell}(D^k, \partial D^k) \simeq \begin{cases} 
\mathbb{Z} , & \ell = k, \\
0 , & \text{otherwise}, 
\end{cases} \tag{6}
\]

for every \( x \in \text{Crit}_k \). To prove the first isomorphism was precisely the problem which inspired us to come up with the backward \( \lambda \)-Lemma in [5]: Since the part of \( N_x \) in the unstable manifold \( W^u(x) \) is a \( k \)-disk and the corresponding part of \( L_x \) is homotopy equivalent to the disk boundary, it remains to deformation retract \( (N_x, L_x) \) to its part in \( W^u(x) \). A very simple, but crucial, observation is that the semi-flow \( \phi \) deforms the *ascending disk* \( W^s_\epsilon(x) := W^s(x) \cap \mathcal{N}^{+\epsilon} \mathcal{M} = W^s(x) \cap N_x \) to \( x \), as \( s \to \infty \). Clearly this fails on other parts of \( N_x \). Note that \( W^s_\epsilon(x) \) is a \( C^1 \) graph over its tangent space denoted by, say \( X^+ \). The idea is to *foliate all of \( N_x \) by copies of \( W^s_\epsilon(x) \), more precisely \( C^1 \) graphs over \( X^+ \), then extend \( \phi \) artificially to all of \( N_x \) using the graph maps*; see (8) and Figure 4.

To see the foliation assign to each point of \( N_x \) the time \( T \) at which it hits the level surface \( \{ S_T = c - \epsilon \} \); see Figure 2. This suggests that \( N_x \) is foliated by (pieces of) the equal time hypersurfaces \( (\phi_T)^{-1} \{ S_T = c - \epsilon \} \) for \( T \in (\tau, \infty) \). For \( T = \infty \) one obtains the codimension \( k \) ascending disk \( W^s_\epsilon(x) \). Of course, the leaves of a foliation need to be of the same codimension: Consider the tubular neighborhood \( \mathcal{D}(x) \to S^c_u(x) \) associated to the (sufficiently small) radius \( a \) normal bundle of the descending sphere \( S^c_u(x) := W^u(x) \cap \{ S_T = c - \epsilon \} \) in the Hilbert manifold \( \{ S_T = c - \epsilon \} \). Each fiber \( \mathcal{D}(x) \) is a codimension \( k \) disk.
3 Backward $\lambda$-Lemma and stable foliations

Fix $x \in \text{Crit}_k$ and set $c := \mathcal{V}(x)$. Since $N_k = N_k^{\epsilon, \tau}$ fits into any neighborhood of $x$ for $\epsilon > 0$ small and $\tau > 0$ large we use local coordinates about $x \in \Lambda M$.

Local coordinates about $x \in \Lambda M$

The nonlinear part of the heat equation (1) determines a closed radius $\rho_0$ ball $B_{\rho_0}$ about $0 \in X$ such that the following is true. Paths $s \mapsto u(s)$ in $\Lambda M$ near $x$ and $s \mapsto \xi(s)$ in $B_{\rho_0}$ uniquely correspond to each other via the identity $u(s) = \exp_x \xi(s)$ pointwise for every $t \in S^1$. In the new coordinates $\xi$ the Cauchy problem associated to (1) turns into the equivalent Cauchy problem

$$\xi'(s) + A\xi(s) = f(\xi(s)), \quad \xi(0) = z \in B_{\rho_0},$$

(7)

for maps $\xi : [0, T] \to B_{\rho_0} \subset X$. Here $A = A_z$ is the Jacobi operator associated to the (perturbed) closed geodesic $x$. The semi-flow $\varphi$ turns into a local semi-flow $\phi$ on $B_{\rho_0} \subset X$. The nondegenerate critical point $x$ corresponds to the hyperbolic fixed point 0 of $\phi$. Furthermore, there is the orthogonal splitting

$$X := T_x \Lambda M \simeq T_x W^u(x) \oplus T_x W^s(x) =: X^- \oplus X^+.$$

Here $X^-$ is of finite dimension $k = \text{ind}_V(x)$ and consists of smooth loops along $x$. By $\pi_\pm : X \to X^\pm$ we denote the associated orthogonal projections. For coordinate representatives of global objects we shall use the global notation omitting $x$, for example $W^u(x)$ becomes $W^u$. By $\mathcal{F}$ we denote the representative of $\mathcal{F}_V$. Via a (standard) change of coordinates one achieves that locally near zero $W^u$ is contained in $X^-$. By $B_R^+$ we denote the radius $R$ ball about $0 \in X^+$. The spectral gap $d > 0$ is the distance between 0 and the spectrum of $A_z$.

**Theorem 3.1** (Backward $\lambda$-Lemma, [5]). Pick $\mu \in (0, d)$ and a hypersurface $\mathcal{D} \subset B_{\rho_0}$ of the form $S_{\mu}^\alpha \times B_{\rho_0}^+$. Then the following is true (see Figure 3). There is a ball $B^+$ about $0 \in X^+$, a constant $T_0 > 0$, and a Lipschitz continuous map

$$\mathcal{F} : (T_0, \infty) \times S_{\mu}^\alpha \times B^+ \to W^u \times B^+ \subset B_{\rho_0}$$

$$(T, \gamma, z_+) \mapsto (G^T_\gamma(z_+), z_+) =: \mathcal{F}^T_\gamma(z_+)$$

of class $C^1$. Each map $\mathcal{F}^T_\gamma : B^+ \to X$ is bi-Lipschitz, a diffeomorphism onto its image, and $\mathcal{F}^T_\gamma(0) = \phi_{-T} \gamma =: \gamma_T$. The graph of $G^T_\gamma$ consists of those $z \in B_{\rho_0}$ which satisfy $\pi_+ z \in B^+$ and reach the fiber $\mathcal{D}_T = \{ \gamma \} \times B^+_{\gamma T}$ at time $T$, that is

$$\mathcal{F}^T_\gamma(B^+) = (\phi_T)^{-1} \mathcal{D}_T \cap (X^- \times B^+).$$

Furthermore, the graph map $\mathcal{F}^T_\gamma$ converges uniformly, as $T \to \infty$, to the stable manifold graph map $\mathcal{F}_\infty$. More precisely, the estimates

$$\|\mathcal{F}^T_\gamma(z_+) - \mathcal{F}_\infty(z_+)\|_{W^{1,1}} \leq e^{-T} \|\mathcal{F}^T_\gamma(z_+)\|_{W^{1,1}}, \quad \|d\mathcal{F}^T_\gamma(z_+)v\|_2 \leq 2 \|v\|_2,$$

$$\|d\mathcal{F}^T_\gamma(z_+)v - d\mathcal{F}_\infty(z_+)v\|_2 \leq e^{-T} \|\mathcal{F}^T_\gamma(z_+)\|_{L^2} \|v\|_2$$

hold true for all $T > T_0$, $\gamma \in S_{\mu}^\alpha$, $z_+ \in B^+$, and $v$ in the $L^2$ closure of $X^+$.  

Theorem 3.1 is based on the observation that the Cauchy problem for a heat flow line $\xi : [0, T] \to X$ with $\xi(0) = z$ is equivalent to a mixed Cauchy problem with data $(T, \gamma, z_+)$. Namely, there is a unique heat flow line $\xi : [0, T] \to X$ with $\pi_+ \xi(0) = z_+$ and $\pi_- \xi(T) = \gamma$.

That the $(k$-dimensional) unstable manifolds carry backward time information is evident from their definition. In contrast, Theorem 3.1 provides backward time information on open sets.

**Stable foliation of Conley set**

Theorem 3.1 foliates neighborhoods of $x$ by (globally meaningless) codimension $k$ disks. The next result provides global information in various directions. By definition the descending disk $W^u(x)$ is given by $W^u(x) \cap \{S V > c - \varepsilon\}$.

**Theorem 3.2** ([6]). Given $\mu \in (0, d)$ there are constants $\varepsilon_1, \tau_1, a > 0$ such that the following is true. Assume $\tau > \tau_1$ and $\varepsilon \in (0, \varepsilon_1)$ and consider the radius a tubular neighborhood $\mathcal{D}(x) \to S^u(x)$ defined in the paragraph preceding section 3.

a) The Conley set $N_x = N_x^{\mathcal{D}}$ carries the structure of a codimension $k$ foliation whose leaves are parametrized by the disk $\varphi_{-T} W^u(x)$. The leaf over $x$ is the ascending disk $W^s(x)$ and the other leaves are given by the disks

$$N_x(\gamma_T) = \{(\varphi_T)^{-1} \mathcal{D}(x) \cap \{S < c + \varepsilon\} \}_{\gamma_T}, \quad \gamma_T := \varphi_{-T} \gamma,$$

whenever $T > \tau$ and $\gamma \in S^u(x)$.

b) Leaves and semi-flow are compatible in the sense that

$$z \in N_x(\gamma_T) \Rightarrow \varphi_\sigma z \in N_x(\varphi_\sigma \gamma_T), \quad \forall \sigma \in [0, T - \tau).$$

c) The leaves converge uniformly to the ascending disk in the sense that

$$\text{dist}_{W^u}(N_x(\gamma_T), W^u(x)) \leq e^{-T \mu}$$

for all $T > \tau$ and $\gamma \in S^u(x)$. Furthermore, if $U$ is a $\delta$-neighborhood of $W^u(x)$ in $\Lambda M$, then $N_x^{\mathcal{D}, \tau_1} \subset U$ for some constant $\tau_1$.

d) Assume $U$ is an open neighborhood of $x$ in $\Lambda M$. Then there are constants $\varepsilon_*$ and $\tau_*$ such that $N_x^{\mathcal{D}, \tau_*} \subset U$. 

Figure 3: Backward $\lambda$-Lemma
4 Strong deformation retract

Pick \( x \in \text{Crit}_k \). It remains to prove (6). If \( k = 0 \), then \( L_x = \emptyset \) and \( W^u(x) = \{x\} \) is a strong deformation retract of \( W^u_T(x) = N_x \). The retraction is provided by the semi-flow \( \varphi \), and we are done. Assume \( k > 0 \). Consider the local setup of section 3 and denote the representative of \( N_x \) by \( N \); similarly for other quantities. Fix \( \rho_0 > 0 \) so small that the only critical point in \( \mathcal{R}_{\rho_0} \) is 0.

**Definition 4.1.** By Theorem 3.2 each \( z \in N \) lies on a leaf \( N(\gamma_T) \) for some time \( T > 0 \) and some point \( \gamma \) in the descending disk \( S^u_\gamma \) where \( \gamma_T := \phi_{-T} \gamma \). The continuous leaf preserving map \( \theta : (0, \infty) \times N \to N \) defined by

\[
\theta_s z := \mathcal{G}_T^\pi_+ \phi_s \mathcal{F}_{\mathcal{F}} \pi_+ z
\]

is called the *induced semi-flow on N*; see Figure 4. It is of class \( C^1 \) on \((0, \infty) \times N\).

That \( \theta_\ast \) preserves the central leaf \( N(0) = W^u_0 \) is due to the downward \( L^2 \)-gradient nature of the heat equation. The proof for a general leaf \( N(\gamma_T) \) turns out to be surprisingly complex although the idea is once more simple: Show that the map \( s \mapsto \mathcal{F}(\theta_s z) \) strictly decreases whenever \( z \) lies in the (topological) boundary of a leaf. This implies preservation of leaves as follows. Firstly, note that \( \theta \) is actually defined on a neighborhood of \( N(\gamma_T) \) in \( \mathcal{F}_T \mathcal{F}^+ \). Secondly, the (topological) boundary of a leaf lies on action level \( c + \epsilon \) whereas the leaf itself lies strictly below that level. Thus the induced semi-flow points inside along the boundary of each leaf—which is a disk by Theorem 3.2. So \( \theta_\ast \) preserves leaves, thus \( N \) and \( L \) by Theorem 3.2. Moreover, it continuously deforms both topological spaces to their respective part in the unstable manifold and this concludes the proof of (6). Therefore \( \mathcal{F} \) defined by (4) is indeed a Morse filtration for \( \Lambda^aM \) and by Remark 2.2 this establishes the desired natural isomorphism (2).

It remains to show that \( \frac{d}{ds} \mathcal{F}(\theta_s z) < 0 \) whenever \( z \) lies in the (topological) boundary of a leaf. Note that grad \( \mathcal{F} \) is defined on loops whose regularity is at least \( W^{2,2} \). Consider the neighborhood \( \mathcal{W} := \mathcal{B}_{\rho_0} \cap \{ \mathcal{F} \leq c + \epsilon/2 \} \) of 0 illustrated by Figure 5. By Palais-Smale the constant defined by

\[
\alpha := \inf_{z \in (\mathcal{B}_{\rho_0} \cap W^{2,2}) \setminus \mathcal{W}} \| \text{grad} \mathcal{F}(z) \|_2 > 0
\]

![Figure 4: The induced flow \( \theta_\ast \) on N](image-url)
Figure 5: The neighborhood $\mathcal{W}$ of 0 used to define $\alpha > 0$

is strictly positive. A rather technical argument, see [6], involving a long calculation which uses heavily the estimates provided by Theorem 3.1 shows that for all $\epsilon > 0$ small and $\tau > 0$ large the following is true. If $T > \tau$ and $\gamma \in S_{\nu}(\epsilon)$, then

$$\frac{d}{ds} \mathcal{J}(\theta_s z) = d\mathcal{J}|_{\theta_s z} dG_T^{1}(z_+^{(s)}) \pi_+(\phi_s \mathcal{H}\tau \pi_+ z)$$

$$= -\left< \text{grad}\mathcal{J}|_{\theta_s z} dG_T^{1}(z_+^{(s)}) \pi_+ \text{grad}\mathcal{J}|_{\phi_s \mathcal{H}\tau_+ z} \right>_{L^2}$$

$$\leq -\frac{1}{4} \alpha^2$$

for all $z \in \partial N(\gamma_T)$ and $s > 0$ small. It is precisely this calculation where we need convergence in $W^{1,4}$ and the extension to $L^2$ of the linearized graph map $dG_T^{1}(z_+)$ in Theorem 3.1. (The nonlinear part $f$ of (1) maps $W^{1,4}$ to $L^2$.)

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**References**

[1] A. Abbondandolo and P. Majer, Lectures on the Morse complex for infinite dimensional manifolds, in Morse theoretic methods in nonlinear analysis and in symplectic topology, pp. 1-74, NATO Science Series II: Mathematics, Physics and Chemistry, P. Biran, O. Cornea, and F. Lalonde Eds, Springer, 2006.

[2] A. Dold, Lectures on Algebraic Topology, Grundlehren Math. Wiss. 200, Springer-Verlag, Berlin, second edition 1980.

[3] J. Milnor, Lectures on the $h$-Cobordism Theorem, Princeton Univ. Press, 1965.

[4] J. Weber, Morse homology for the heat flow, Math. Z. (2012). doi: 10.1007/s00209-012-1121-x

[5] J. Weber, A backward $\lambda$-Lemma for the forward heat flow. arXiv:1210.3897

[6] J. Weber, Stable foliations and the homology of the loop space. In preparation