High-Q optical microresonators have emerged over the last two decades as a revolutionary new platform for nonlinear optics\(^1\). Their ultra-high finesse and small modal volume enable nonlinear interactions to be driven with unprecedented efficiency, permitting, for example, harmonic generation at microwatt input levels\(^2\). Resonators dominated by third-order Kerr nonlinearities have attracted particular attention\(^3\), and notably have allowed the development of chip-scale generators of coherent optical frequency combs\(^4\). The application of such Kerr frequency combs now constitutes a major subject of research, with impressive demonstrations reported in contexts such as spectroscopy\(^5\)–\(^8\), optical ranging\(^9\)–\(^12\) and telecommunications\(^13\),\(^14\).

The nonlinear optical phenomenon that underlies the formation of Kerr combs is known as four-wave mixing (FWM). In addition to comb formation, FWM can also be harnessed for the generation of isolated pairs of new optical frequencies that exhibit large frequency shifts from the pump and that can be widely tuned by small changes in the pump wavelength\(^15\). While such Kerr parametric sources have been extensively studied using optical fibres as the nonlinear medium\(^16\)–\(^27\), the low nonlinearity and finesse of silica fibre-based systems impose severe limitations on power efficiency and spectral coverage. High-Q Kerr microresonators are ideally positioned to overcome these limitations, offering an intriguing possibility for the development of a new type of widely tunable optical source.

Several pioneering studies have demonstrated the generation of large-frequency-shift parametric sidebands in Kerr nonlinear optical microresonators\(^28\)–\(^32\). The ability to tune the wavelengths of the generated sidebands has, however, remained elusive, with the majority of studies reporting little\(^33\) or no\(^34\)–\(^36\) tunability. The largest tuning range reported so far was observed in a fused-silica microsphere, where sidebands tunable from 1,200 nm to 1,900 nm were generated\(^31\). However, the high attenuation of fused silica at wavelengths beyond 1,900 nm fundamentally prevents further wavelength-scaling of this type of oscillator, obstructing access to the spectroscopically rich mid-infrared (mid-IR). Moreover, the broad Raman gain of fused silica invariably results in the concomitant generation of numerous parasitic spectral components, which, together with the unwieldy microsphere geometry, critically diminishes the attractiveness of this system. To fully unleash the potential of microresonator Kerr parametric oscillators, a platform capable of overcoming these deficiencies is needed.

Here, we present experimental demonstrations of widely tunable parametric oscillation in crystalline Kerr microdisk resonators made of MgF\(_2\). Because MgF\(_2\) remains transparent to wavelengths up to 6,000 nm, the material loss restriction encountered in ref. \(^30\) is lifted. In addition, thanks to the narrowband Raman gain spectrum characteristic of crystalline materials, pure Kerr FWM signals can be generated. We experimentally consider several different resonators with dispersion profiles carefully engineered to give access to different wavelength regions, observing clean and low-noise sidebands that can be tuned by hundreds of nanometres with a standard C-band telecommunications laser as the pump. Combining three distinct resonators, we realize over an octave of narrowband, continuous-wave (c.w.) tunable output ranging from 1,083 nm to 2,670 nm, at an input pump power of only 100 mW. Moreover, we observe signatures of spectral components at wavelengths as high as 3,800 nm, thus demonstrating the scheme’s potential for mid-IR applications. Our work paves the way for future low-cost, low-power, widely tunable sources of electromagnetic radiation based on Kerr microresonators.

**Results**

**Overall scheme and phase-matching.** Figure 1a presents a schematic illustration of our experiment. A c.w. laser with frequency \(\omega_p\) is coupled into a high-Q MgF\(_2\) microdisk resonator, resulting in the generation of large frequency shift signal and idler sidebands through degenerate FWM. Energy conservation dictates that the sidebands are symmetrically detuned around the pump,
\( \omega_p = \omega_p \pm \Omega. \) To enable wide tunability, we operate in a regime where small adjustments of the pump frequency give rise to very large changes in the sideband frequency shift \( \Omega. \) This is achieved by phase-matching the nonlinear wave mixing process through higher-order dispersion\(^{24,30}. \) Assuming all interacting waves share the same mode family, the phase-matching condition can be approximately written as (see Methods)

\[
\frac{\beta_2 \Omega^2 L}{2} + \frac{\beta_4 \Omega^4 L}{24} + 2 \gamma P L \delta_0 = 0 \tag{1}
\]

where \( \beta_2 \) and \( \beta_4 \) are, respectively, the second- and fourth-order group-velocity dispersion (GVD) coefficients of the resonator mode family under study (evaluated at the pump frequency), \( L \) is the resonator circumference, \( P \) is the intracavity power at the pump frequency, \( \gamma \) is the nonlinear interaction coefficient of the mode family and \( \delta_0 \) is the phase detuning between the pump field and the closest linear cavity resonance.

Inspection of equation (1) reveals that large-frequency-shift, widely tunable sidebands can be expected when the pump experiences normal GVD (\( \beta_2 > 0 \)) and a negative fourth-order dispersion coefficient (\( \beta_4 < 0 \)). In this regime, the negative fourth-order dispersion can compensate for the positive second-order dispersion at large frequency shifts \( \Omega \approx -\frac{2 \beta_4}{\beta_2}. \) For typical parameters, shifts of the order of 10–100 THz can be expected\(^{25-28,30}\). Moreover, the precise frequency shift can be tuned over wide regions by adjusting the values of the dispersion coefficients \( \beta_2 \) and \( \beta_4 \), which can be readily achieved through small changes of the pump frequency (see Methods).

**Resonator dispersion.** The resonators used in our experiments are made of MgF\(_2\), which exhibits a very low loss over the target spectral region (1,000–4,000 nm) and is amenable to fabrication using established single-point turning techniques\(^{33-35}. \) We wish to drive the resonators using a pump source in the telecommunications C-band (1,530–1,565 nm) so as to take advantage of the low-cost, high-quality lasers and components available at these wavelengths. Unfortunately, the zero-dispersion (\( \beta_2 = 0 \)) wavelength (ZDW) of bulk MgF\(_2\) is located at ~1,300 nm, with normal dispersion at shorter wavelengths\(^{-} \). To achieve normal dispersion at 1,550 nm, as required for large-frequency-shift sideband generation, additional waveguide dispersion is necessary to shift the ZDW to longer wavelengths.
We first performed detailed modelling using finite-element software (COMSOL Multiphysics) to identify resonator dimensions that satisfy the phase-matching conditions required for widely tunable parametric oscillation (see Methods). We consider a microdisk resonator geometry characterized by major and minor radii \( R \) and \( r \), respectively, as shown in Fig. 1b. Our modelling reveals that, for the range of parameters accessible with our fabrication (\( R \approx 100–500\,\mu\text{m} \) and \( r \approx 50–250\,\mu\text{m} \)), the ZDW can be straightforwardly increased by reducing the resonator’s major radius, whereas changing the minor radius has a much smaller effect. Illustrative results are shown in Fig. 1c, where we plot the ZDW of the fundamental transverse electric (TE) mode as a function of the major radius \( R \) for a minor radius \( r = 130\,\mu\text{m} \) (all the modelling results shown use this value). The results show that the ZDW can be shifted to 1,550 nm by reducing the major radius \( R \leq 500\,\mu\text{m} \), thus alluding to the feasibility of widely tunable parametric oscillation with a C-band pump.

Figure 1d shows the modelled wavelength-dependence of the \( \beta_p \) coefficient for two resonators with major radii of 300\,\mu m and 200\,\mu m. Two important points are immediately discernable. First, both resonators exhibit normal dispersion over our intended range of pump wavelengths around 1,550 nm. Second, the curvatures of \( \beta_p \) are negative, implying \( \beta_p < 0 \). These resonators thus fulfill both requirements for achieving widely tunable parametric oscillation. Figure 1e shows the phase-matched sideband wavelengths predicted by equation (1) as a function of the pump wavelength for these two resonators (other parameters are listed in the caption). As can be seen, pumping in the normal dispersion regime (below the ZDW) is expected to result in parametric sidebands with very large frequency shifts that change dramatically in response to small changes in the pump wavelength. For the 300\,\mu m resonator (solid orange curve), the tuning range predicted as the pump is tuned by 80 nm from 1,500 nm to 1,580 nm is well in excess of an octave (1,000–2,500 nm). Given that our experimental implementation is, however, limited to pump wavelengths in the C-band, octave-tunability is not accessible with a single resonator. We lift this limitation by fabricating resonators with different major radii, which provides a simple route to control the position of the ZDW and hence the entire phase-matching curve. Indeed, as shown by the dashed blue curve in Fig. 1e, a resonator with a major radius of 200\,\mu m is predicted to give access to significantly larger frequency shifts for the same range of pump wavelengths compared to the 300\,\mu m resonator.

Experimental set-up and results. Following the analysis above, we fabricated five MgF\(_2\) microdisk resonators with progressively decreasing dimensions (Fig. 2a) so as to access a wide range of sideband wavelengths. The resonators were first shaped on a lathe using diamond point turning, and then mechanically polished with diamond abrasives to achieve high finesse. After polishing, the major radii of the five resonators were measured to be 515, 400, 265, 190 and 165\,\mu m, with the minor radii about one-third...
of the corresponding major radius. The measured finesse of each resonator is ~50,000, corresponding to ultra-high Q-factors ranging from 0.5 × 10^9 to 1.5 × 10^9.

The resonators were driven with a standard C-band external cavity diode laser (ECDL), amplified with an erbium-doped optical amplifier (EDFA) (Fig. 2b and Methods). The ensuing pump light had an average power of ~100 mW and a wavelength that was continuously tunable between 1,530 nm and 1,570 nm. The pump light was coupled to the microresonators using a silica optical fibre taper with a waist diameter of ~1 μm.

Focusing first on the results obtained for the 515 μm resonator, Fig. 2c presents experimentally measured spectra at the taper output for five different pump wavelengths (corresponding to different driven cavity modes from the same mode family). At each wavelength, the initially blue-detuned pump laser was tuned into resonance until parametric oscillation was observed (thermal locking allows for stable operation). As the pump wavelength decreases, the parametric frequency shifts are seen to increase rapidly, as expected for phase-matching via higher-order dispersion. Each of the measured spectra comprise of only a single pair of FWM sidebands, with no evidence of the strong stimulated Raman components observed in earlier experiments with silica microspheres. We attribute this to the narrow Raman gain spectrum of crystalline MgF_2, which is typically not resonant with the mode family under investigation (unless specifically engineered to be so).

We repeated the spectral measurements for a range of pump wavelengths in the C-band, advancing the pump wavelength by about three free-spectral ranges for each data point. Figure 2d shows the wavelengths of the parametric sidebands extracted from each measurement. As can be seen, just 30 nm of pump tuning in the C-band yields more than 650 nm of signal tunability, with sidebands ranging from 1,270 nm to 1,920 nm observed. Also shown as the solid lines in Fig. 2d are theoretical fits to the phase-matching condition given by equation (1), with the dispersion coefficients β_2, β_4, and β_6 treated as fitting parameters (see Methods). The agreement is outstanding, confirming that the FWM process is phase-matched via higher-order dispersion. From this fit, we are able to extract the dispersion coefficients and the ZDW of the mode under study (the extracted ZDW of 1,558 nm is highlighted with a dotted line in Fig. 2c,d). We also performed delayed self-heterodyne measurements (see Methods) to confirm the narrow linewidths of selected sidebands (Fig. 2e). We observe the sideband linewidths to be ~100 kHz, which is comparable to the original pump linewidth of ~70 kHz. These measurements show that the parametric sidebands share the noise characteristics of the pump, which is an important feature for applications requiring signals with narrow optical linewidth.

The results reported in Fig. 2 show that a MgF_2 microdisk resonator with a 515 μm major radius can deliver parametric sidebands with over 650 nm of tunability when driven with a standard C-band pump.
laser. Remarkably, we find that significantly larger frequency shifts can be achieved by using resonators with smaller major radii (hence, larger ZDWs). Following a similar procedure as for the 515 µm resonator, we measured the wavelengths of the parametric sidebands generated in each of the five resonators fabricated. The extreme frequency shifts achieved in our resonators necessitated the use of two different optical spectrum analysers (the first measuring from 600 nm to 1,750 nm and the second from 1,200 nm to 2,400 nm) as well as a Fourier-transform infrared spectrometer (FTIR, used to measure signals above 2,400 nm). Figure 3a shows the experimental results obtained for three of our largest devices (with major radii of 515, 400 and 265 µm), together with fits to the theoretical phase-matching curves defined by equation (1). Note that, for the 265 µm resonator, we identified two distinct mode families producing large frequency-shift sidebands. Figure 3a shows the tuning curves pertaining to both families as obtained from independent experiments where the pump laser was separately tuned to the spectrally distinct resonances of the respective mode families. We also note that the detection of sidebands with wavelengths larger than ~2,300 nm required us to reduce the length of the silica fibre after the coupling region to only 5 cm in an effort to minimize the very large attenuation of silica glass at these wavelengths.

The frequency shifts reported in Fig. 3a are remarkable. Hundreds of nanometres of sideband tunability is achieved in each resonator, with the combined output of the three devices representing more than an octave of quasi-continuous tunability from 1,083 nm to 2,670 nm. Yet, our experiments signal that even larger frequency shifts can be realized by further reducing the resonator dimensions. Indeed, as shown in Fig. 3b, the shortest wavelengths measured in our smallest resonators (with major radii of 190 and 165 µm) are well below 1,000 nm (down to 957 nm), implying the generation of mid-IR wavelengths well above 3,000 nm (up to 3,860 nm). Note that an additional L-band laser source was used here to interrogate the tuning curves over an extended range of pump wavelengths (see Methods). Because our silica coupling fibre is incapable of efficiently coupling out mid-IR light, such long wavelengths cannot be directly observed in our experiments; rather, the positions of the mid-IR sidebands shown in Fig. 3b were inferred from the corresponding short-wavelength signals. We expect this issue can be overcome by selectively coupling out the long-wavelength sidebands using a prism that is transparent in the mid-IR.

Figure 3c–e presents details of typical spectra measured from three of our largest devices (Fig. 3a). In stark contrast with observations made in silica microsphere systems90, each spectrum displays only a single pair of FWM sidebands, with no signatures of parasitic components associated with stimulated Raman scattering or other nonlinear processes. Combined with our other experiments, these results unequivocally show that MgF2 microdisk resonators—driven with standard low-power telecommunications lasers—can deliver clean and low-noise parametric sidebands that can be tuned over an entire optical octave.

We must note that the wideband tunability demonstrated in Figs. 2 and 3 is inherently discrete due to the requirement that the pump frequency must always be (almost) resonant with a cavity mode. To achieve continuous tunability, the cavity mode spectrum must be tuned in tandem with the pump frequency; this can be achieved, for example, by controlling the resonator temperature84,89 or by applying a mechanical strain on the resonator80,81. To realize a proof-of-concept demonstration of continuous sideband tunability, we exploited the microresonator’s intrinsic thermo-optic nonlinearity85,86. The frequency of an initially blue-detuned pump laser was scanned into resonance, causing the cavity modes to shift as the build-up of intracavity power changed the resonator temperature. By following the thermally shifted cavity mode, the pump could remain (almost) resonant over an extended range of ~12 GHz, corresponding to ~5,000 cold cavity linewidths. Figure 4 shows the change in frequency of the parametric sidebands as the pump frequency changes, as measured with a high-precision wave meter (see Methods). As can be seen, the change in each sideband's frequency exactly follows the change in frequency of the pump, thus demonstrating ~12 GHz of continuous tunability. We envisage that the range of continuous tunability can be significantly extended by externally heating the resonator89.

**Discussion**

Our work shows that parametric oscillation in crystalline Kerr microresonators represents a viable route for the generation of widely tunable laser light. However, the typical conversion efficiencies observed in our work are still rather small, ranging from $10^{-3}$ to $10^{-1}$ for sidebands within the range of our OSAs. We believe that such a low level of conversion is not a fundamental limit, but arises due to unoptimized coupling and attenuation in our detection scheme. Extensive numerical and analytic investigations indicate that, with further optimization, conversion efficiencies in excess of 10% to each sideband should be feasible (see Supplementary Information). These investigations reveal, in particular, that the outcoupling efficiency plays a very important role in determining the strength of the parametric sidebands outside the cavity (and hence the conversion efficiency). As noted above, the silica fibre taper used in our experiments is designed to efficiently couple light around 1,550 nm into the microresonator, but it does not allow for efficient outcoupling of the large-frequency shift sidebands. We suspect this is the main explanation for the comparatively low conversion efficiencies observed. We expect that selective outcoupling will enable significant improvements to the observed conversion efficiency.

To conclude, we have shown that crystalline Kerr microresonators provide an ideal platform for widely tunable parametric oscillation. Thanks to their ultra-high finesse, narrow Raman gain
spectrum and attractive dispersion characteristics, clean and low-noise sidebands separated by more than an octave can be generated—and tuned over hundreds of nanometres—using a low-power telecommunications pump laser. The ability to fabricate resonators from materials (such as MgF\(_2\)) that remain transparent deep into the mid-IR presents alluring opportunities for low-cost sources of light in that spectral region. Moreover, the use of materials such as silicon or silicon nitride could allow for the scheme to be realized in an integrated, on-chip format\(^{42-44}\). We note in this context that tunable mid-IR generation has recently been achieved by launching intense femtosecond pulses into single-pass silicon nitride waveguides\(^{45}\), alluding to the possibility of fabricating on-chip resonators with suitable dispersion characteristics.

Online content
Any methods, additional references, Nature Research reporting summaries, source data, statements of code and data availability and associated accession codes are available at https://doi.org/10.1038/s41566-019-0485-4.

Received: 31 October 2018; Accepted: 29 May 2019; Published online: 8 July 2019.

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Phase-matching considerations. The nonlinear dynamics of a driven Kerr (micro) resonator can be modelled using the generalized Lugiato–Lefever equation (LLE)\(^{35,36}\). A standard linear stability analysis can be used to determine the per-roundtrip amplitude gain \(g(\Omega)\) experienced by a small sideband perturbation with frequency shift \(\Omega\) superimposed on top of a cw steady-state solution of the system:\(^{35,36}\)

\[
g(\Omega) = -\alpha + \sqrt{\beta L^2 p^2 - (\delta_0 - D_2(\omega_p)\frac{\Omega}{2})^2} \quad (2)
\]

Here, \(\alpha\) is half of the power lost per round-trip and \(D_2(\omega_p)\) describes the dispersion-induced phase mismatch of the parametric interaction. This latter parameter depends only on the even orders of dispersion:

\[
D_2(\omega_p) = \sum_{k \geq 2} \frac{\beta_{2k}}{(2k)!} \quad (3)
\]

where \(\beta_{2k} = d^k\beta/d\omega^k\) is the \(k\)th derivative of the mode propagation constant \(\beta(\omega)\) evaluated at the pump frequency \(\omega_p\).

Parametric gain can be achieved at frequencies for which \(g(\Omega) > 0\). It is easy to show that the gain is maximized at the phase-matched frequency that satisfies

\[
D_2(\omega_p) L + 2\beta L - \delta_0 = 0 \quad (4)
\]

Truncating the dispersion operator \(D_2(\omega_p)\) to its first two terms (fourth order in dispersion) gives the simplified phase-matching equation (1) introduced in the main text. Although this truncation may not be entirely accurate for some of the larger frequency shifts observed in our experiments, we expect the resulting deviations to be qualitatively valid. Indeed, we note that the phase-matching curves observed experimentally are well fitted by the simplified phase-matching equation (1). These curves are also in reasonable qualitative agreement with resonator dispersion characteristics extracted from finite-element simulations. In this context, we note that quantitative agreement between our experiments and finite-element simulations is not to be expected due to (1) uncertainties in the bulk refractive index of MgF\(_2\), over the extreme wavelength ranges accessed in our experiments and (2) uncertainties in the precise morphology of our resonators. Regardless, our finite-element simulations readily predict the general trends that underpin our work, namely the shifting of the resonator ZDW to longer wavelengths with increasing major radius \(R\).

Steady-state parametric oscillations can be expected close to the phase-matched frequency that satisfies equation (4). An additional restriction stems from the requirement that the parametric sidebands must be resonant with the cavity. Thanks to the non-zero gain bandwidth described by equation (2), it is generally possible to satisfy both conditions simultaneously. However, as noted in ref. \(^35\), the parametric gain bandwidth tends to decrease with increasing sideband frequency shift. As a consequence, for sufficiently large frequency shifts, the parametric gain may fall in between cavity modes, prohibiting oscillation or reducing its efficiency.

The sideband wavelengths can be tuned by controlling the pump frequency. This changes the dispersion operator \(D_2(\omega_p)\) and hence the frequency shift \(\Omega\) that satisfies the phase-matching condition. For example, when truncating the dispersion at fourth-order, a small pump frequency shift \(\delta_0\) transforms the GVD coefficient as

\[
\beta_1 \rightarrow \beta_1 + \beta_2 \delta_0 + \frac{\beta_3}{2} \delta_0^2 \quad (5)
\]

Resonator modelling. To model the dispersion characteristics of our resonators, we use commercial finite-element software (COMSOL Multiphysics) to calculate the resonant eigen-frequencies \(\omega_n\) of the fundamental TE mode of the resonator. From the resonant frequencies, we obtain the mode propagation constant \(\beta(\omega)\) by using the resonance condition \(\beta(\omega_n) L = 2\pi m\). A Taylor series expansion at a pump frequency \(\omega_p\) of interest finally yields the dispersion coefficients \(\beta_n\).

We note that an alternative description of dispersion—often encountered in studies of microresonators—involves expanding \(\omega_n\) (rather than \(\beta(\omega)\)) as a Taylor series around a particular resonance frequency \(\omega_{n0}\):

\[
\omega_n = \omega_{n0} + \sum_{s=2} D_s (m-m_0)^s \quad (6)
\]

where \(D_s\) are the Taylor series coefficients. Although the use of this description may seem more natural, it is worth noting that the parametric sidebands are symmetrically detuned with respect to the pump frequency \(\omega_p\) which does not, in general, coincide with a resonant frequency (for example, \(\omega_{n0}\)). As a consequence, while the phase-matching conditions can be expressed in terms of the Taylor series coefficients \(D_s\), the odd orders of dispersion no longer cancel out in general, resulting in slightly more complex expressions.

Additional experimental details. The driving laser used in most of our experiments is a commercially available C-band laser (New Focus, Velocity TBL-6728) with a narrow linewidth of \(\sim 70\) kHz. The laser is amplified by a C-band EDFA, and then filtered by a tunable 1 nm passband filter to remove the majority of the amplifier’s amplified spontaneous emission noise. Another laser (New Focus, Velocity TBL-6730), tunable in the L-band (1,550 nm–1,630 nm), was used in combination with an L-band EDFA to extend the pump wavelength range up to 1,590 nm for the measurements shown in Fig. 3a.

To measure the linewidth of the resonator modes, we phase modulate the driving laser to provide an absolute frequency reference, and then scan the laser frequency across the mode of interest in the low-power linear regime. The finesse is obtained by dividing the free-spectral range estimated from the resonator geometry with the measured resonance linewidth.

Our resonators support several different mode families, only some of which exhibit dispersion characteristics suitable for large-frequency-shift sidebands. To identify appropriate modes, a part of the light coupled out from the resonator is passed through an optical filter that transmits only wavelengths below 1,520 nm. Monitoring the signal transmitted by this filter as the laser wavelength is scanned allows us to straightforwardly single out modes generating large-frequency-shift sidebands. Once suitable modes have been identified, we continuously tune the laser into resonance from the blue-detuned side so as to leverage thermal locking\(^2\).

The linewidth measurements reported in Fig. 2e were obtained using a delayed self-heterodyne measurement. Here the sideband of interest is first spectrally isolated using a tunable optical filter, then divided into two parts. One part is frequency-shifted by 80 MHz with an AOM, while the other part passes through 30 km of fibre so as to decorrelate the two beams. Superimposing the beams on a photodetector allows the linewidth to be estimated from the recorded beat signal.

Two OSAs and a FTIR spectrometer were used to gather the data shown in Fig. 3. The first OSA (Yokogawa AQ6370D) measures from 600 nm to 1,750 nm and the second (Yokogawa AQ6373B) from 1,200 nm to 2,400 nm, and the FTIR spectrometer (Bristol Instruments 771B) notes that this device can operate as a high-precision wave meter or a FTIR spectrometer) was used to record spectra of sidebands with wavelengths above 2,400 nm. Because the dynamic range of the FTIR spectrometer is limited to 45 dB, a bandpass filter with a centre wavelength of \(-2,500\) nm and a pass-band of \(500\) nm was placed just before the spectrometer so as to filter out the strong pump line. The two different OSAs and the FTIR were calibrated with a signal at 1,550 nm to construct the spectra shown in Fig. 3c–e.

To demonstrate continuous sideband tunability (Fig. 4), we first tune the pump laser into an appropriate resonance so as to generate parametric sidebands. We then isolate the pump and the sidebands using appropriate spectral filters, and measure their wavelengths using a high-precision wave meter (Bristol Instruments 771B) with wavelength accuracy of 0.75 parts per million. We then continuously decrease the frequency of the pump and re-measure all the wavelengths. This procedure is repeated for as long as the thermo-optic nonlinearity allows the pump to remain thermally locked to the resonance. We must emphasize that, because our wave meter only allows one wavelength to be measured at a time, there is a short (\(-15\) s) delay between the measurement of the pump wavelength and the sideband wavelengths. During this delay, the pump wavelength can exhibit small fluctuations, giving rise to an additional uncertainty in the precise pump wavelength for which each sideband wavelength was measured. This uncertainty is included in the horizontal error bars in Fig. 4.

Data availability

The data that support the plots within this paper and other findings of this study are available from the corresponding author upon reasonable request.

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