Robust Eigenvectors of Symmetric Tensors

Tommi Muller\textsuperscript{1,2}, Elina Robeva\textsuperscript{2}, and Konstantin Usevich\textsuperscript{3}

\textsuperscript{1}University of Oxford
\textsuperscript{2}University of British Columbia
\textsuperscript{3}CNRS and Université de Lorraine

Abstract

The tensor power method generalizes the matrix power method to higher order arrays, or tensors. Like in the matrix case, the fixed points of the tensor power method are the eigenvectors of the tensor. While every real symmetric matrix has an eigen-decomposition, the vectors generating a symmetric decomposition of a real symmetric tensor are not always eigenvectors of the tensor.

In this paper we show that whenever an eigenvector is a generator of the symmetric decomposition of a symmetric tensor, then (if the order of the tensor is sufficiently high) this eigenvector is robust, i.e., it is an attracting fixed point of the tensor power method. We exhibit new classes of symmetric tensors whose symmetric decomposition consists of eigenvectors. Generalizing orthogonally decomposable tensors, we consider equiangular tight frame decomposable and equiangular set decomposable tensors. Our main result implies that such tensors can be decomposed using the tensor power method.

1 Introduction

With the rising demand for techniques to handle massive, high-dimensional datasets, many scientists have turned to finding adaptations of matrix algorithms to high-order arrays, known as tensors. The main obstacle is that determining quantities such as the rank, singular values, and eigenvalues [13, 22, 26] of a general tensor is an NP-hard problem [12]. Nonetheless some heuristics have been proposed for computing such quantities [1, 2, 19, 20, 24] and efficient algorithms exist for many families of tensors. For symmetric tensors efficient algorithms for decomposition exist in the low-rank case [10, 11, 16]. Furthermore, the decomposition, approximations, eigenvectors, and algebraic characterization have been thoroughly studied in the special case of orthogonally decomposable tensors [1, 3, 18, 21, 23, 28].

The computation of eigenvectors and singular vectors is particularly important because it is tightly linked to the best rank-one approximation problem [6]. A recurring tool that has been used in several of the works cited here is the tensor power method [36], which generalizes the well-known matrix power method. For non-symmetric tensors, the tensor power method is globally convergent [34], with speed of convergence established in [14]. For the symmetric case, fewer results are available on the convergence of the power method; examples are known when the method does not converge at all [6, 17].
In this paper, we first show that if a vector in the symmetric decomposition of a symmetric tensor is an eigenvector, then for sufficiently high orders, it is robust, i.e., it is an attracting fixed point of the tensor power method (see Theorem 3.1). We then exhibit several families of tensors whose symmetric decomposition consists of (robust) eigenvectors. Generalizing the class of orthogonally decomposable tensors, we study tensors generated by linear combinations of tensor powers of vectors which form an equiangular tight frame (ETF), or, more generally, an equiangular set (ES).

The rest of the paper is organized as follows. In Section 2 we provide background on tensor decompositions and the tensor power method. In Section 3 we present our main result, Theorem 3.1. In Section 4 we introduce and provide a detailed study of ETF and ES decomposable tensors; this includes the study of not only eigenvectors and their robustness, but also regions of convergence of the tensor power method. In Section 5 we conclude with a discussion and some open problems.

2 Background

Denote by $[n]$ the set $\{1, ..., n\}$. We write unbolded lowercase letters for scalars in $\mathbb{F} = \mathbb{R}, \mathbb{C}$, such as $\lambda$, bolded lowercase letters for vectors, such as $\mathbf{v}$, bolded uppercase letters for matrices, such as $\mathbf{M}$, and script letters for $d$-tensors where $d \geq 3$, such as $\mathcal{T}$. An order $d$ tensor with dimensions $n_1, ..., n_d$ is an element $\mathcal{T} \in \mathbb{F}^{n_1 \times \cdots \times n_d}$. The $(i_1, ..., i_d)$-th entry of the tensor $\mathcal{T}$ will be denoted by $T_{i_1, ..., i_d}$ where $i_1 \in [n_1], ..., i_d \in [n_d]$. An order $d$ tensor $\mathcal{T} \in \mathbb{F}^{n_1 \times \cdots \times n}$ is said to be symmetric if for all permutations $\sigma \in S_d$ of $[d]$,

$$T_{i_1, ..., i_d} = T_{i_{\sigma(1)}, ..., i_{\sigma(d)}}.$$ 

We denote the set of all symmetric tensors of order $d$ and dimension $n$ by $S^d(\mathbb{F}^n)$.

**Definition 2.1.** A symmetric decomposition of a symmetric tensor $\mathcal{T} \in S^d(\mathbb{F}^n)$ is an expression of $\mathcal{T}$ of the form

$$\mathcal{T} = \sum_{i=1}^{r} \lambda_i \mathbf{v}_i \otimes \cdots \otimes \mathbf{v}_i,$$

where $\lambda_1, ..., \lambda_r \in \mathbb{F}$, $\mathbf{v}_1, ..., \mathbf{v}_r \in \mathbb{F}^d$ are unit-norm vectors, and $\mathbf{v} \otimes \cdots \otimes \mathbf{v}$ ($d$-times) is a symmetric rank-one tensor. We say that $\mathcal{T}$ is generated by the vectors $\mathbf{v}_1, ..., \mathbf{v}_r$ and the coefficients $\lambda_1, ..., \lambda_r$. The smallest $r$ for which such a decomposition exists is called the symmetric rank of $\mathcal{T}$.

The rank of any symmetric matrix is always at most $n$. For tensors $\mathcal{T} \in S^d(\mathbb{C}^n)$, this is not the case since the rank can be much larger. The Alexander-Hirschowitz Theorem [4] states that, with probability 1, the symmetric rank of a random tensor $\mathcal{T} \in S^d(\mathbb{C}^n)$ (drawn from an absolutely continuous probability distribution) is $\lfloor \frac{1}{n} (n + d - 1) \rfloor$ except for a few special values of $d$ and $n$ where the rank is 1 more than this number.

A vector $\mathbf{v} \in \mathbb{F}^n$ is an eigenvector of $\mathcal{T}$ with eigenvalue $\mu \in \mathbb{F}$ if

$$\mathcal{T} \cdot \mathbf{v} = \mu \mathbf{v},$$
where $\mathcal{T} \cdot \mathbf{v}^{d-1}$ is a vector defined by contracting $\mathcal{T}$ by $\mathbf{v}$ along all of its modes except for one, i.e. the $i$-th entry of $\mathcal{T} \cdot \mathbf{v}^{d-1}$ is

$$(\mathcal{T} \cdot \mathbf{v}^{d-1})_i = \sum_{i_1,\ldots,i_{d-1}=1}^{n} \mathcal{T}_{i_1\ldots i_{d-1}i} v_{i_1} \cdots v_{i_{d-1}}.$$ 

Since $\mathcal{T}$ is symmetric, it does not matter which $d-1$ modes of $\mathcal{T}$ we contract. The eigenvectors of $\mathcal{T}$ are the fixed points (up to sign) of an iterated method called the tensor power method given by

$$x_{k+1} \mapsto \frac{\mathcal{T} \cdot x_k^{d-1}}{\|\mathcal{T} \cdot x_k^{d-1}\|}.$$ 

Yet another important characterization of the eigenvectors is that they are the critical points of the symmetric best rank-one approximation problem:

$$\min_{c,\mathbf{v}} \|\mathcal{T} - c \mathbf{v}^{\otimes d}\|_F^2,$$  

(2.2)

see e.g., [6, §6] for a related discussion. Note that while it is known that a non-symmetric best rank-one approximation of a symmetric tensor can be always chosen symmetric [9], this does not give us information about all critical points; in particular the results about the convergence of the non-symmetric power method [34] cannot be applied.

We call the vector $\mathbf{x}_0$ an initializing vector of the tensor power method. Note that we are interested in real tensors $\mathcal{T}$ and their real eigenvectors and investigate the convergence behavior of the tensor power method for real, non-zero initializing vectors. We are also not interested in eigenvectors of $\mathcal{T}$ that have eigenvalue 0, since in that case, the tensor power method is not applicable. A robust eigenvector of $\mathcal{T}$ is an eigenvector $\mathbf{v}$ that is an attracting fixed point of the tensor power method, i.e. there exists an $\epsilon > 0$ such that the tensor power method converges to $\mathbf{v}$ for all initializing vectors $\mathbf{x}_0 \in B_{\epsilon}(\mathbf{v})$ in the ball of radius $\epsilon$ centered at $\mathbf{v}$. This means that an eigenvector is robust if it can be reliably obtained from the tensor power method.

A tensor $\mathcal{T}$ is said to be orthogonally decomposable, or odeco, if it has a symmetric decomposition of the form

$$\mathcal{T} = \sum_{i=1}^{n} \lambda_i \mathbf{v}_i^{\otimes d},$$  

(2.3)

where $\mathbf{v}_1,\ldots,\mathbf{v}_n$ form an orthonormal basis of $\mathbb{R}^n$. Since there are at most $n$ orthogonal vectors in $\mathbb{R}^n$, the symmetric rank of an odeco tensor is at most $n$. Odeco tensors have been thoroughly characterized and display a number of remarkable properties [1, 3, 28, 29]. One, of interest here, is that the robust eigenvectors of an odeco tensor (2.3) are precisely $\mathbf{v}_1,\ldots,\mathbf{v}_n$ [1].

This brings a few important points. In general, a symmetric tensor with a symmetric decomposition (2.1), firstly, may not have $\mathbf{v}_j$ as an eigenvector for some $j$, secondly, may have eigenvectors that are not robust, and lastly, may have robust eigenvectors that are not one of the $\mathbf{v}_j$’s. In comparison, the only robust eigenvectors of a generic symmetric matrix is the
one whose eigenvalue is largest in absolute value. Additionally, unlike symmetric matrices, symmetric tensors can have several robust eigenvectors.

In the following Section 3, we present our main result which essentially says that if a term \( v_i \otimes d_i \) is part of the symmetric decomposition (2.1) of a symmetric tensor \( T_d \), then, for \( d \) sufficiently large, if \( v_i \) is an eigenvector, then it is robust. In Section 4, we introduce a family of tensors, called equiangular tensors, which generalize odico tensors. These tensors share the property that the vectors \( v_i \) in the symmetric decomposition (2.1) are eigenvectors. We apply our main result to study the robustness of these eigenvectors. We leave the study of robust eigenvectors that do not generate the decomposition of the symmetric tensor as an open problem in the conclusion.

3 Main Theorem

We now proceed to our main result which gives a condition on when an eigenvector is robust.

**Theorem 3.1.** For \( d \in \mathbb{N} \), let \( T_d \in S^d(\mathbb{R}^n) \) be a tensor with symmetric decomposition

\[
T_d = \sum_{i=1}^{r} \lambda_i v_i \otimes d_i ,
\]

with \( \|v_i\| = 1 \) for all \( i \). Then there exists a \( D \in \mathbb{N} \) such that for all \( d \geq D \), if \( v_j \) is an eigenvector of \( T_d \) with non-zero eigenvalue, then \( v_j \) is a robust eigenvector of \( T_d \).

As we will see in the sections to follow, this result allows us to use the tensor power method in order to decompose certain classes of tensors.

The following lemma is used in the proof of Theorem 3.1.

**Lemma 3.2.** [27, Theorem 3.5] Let \( x_* \in \mathbb{R}^n \) be a fixed point of a \( C^1(U, \mathbb{R}^n) \) function \( \phi : U \to \mathbb{R}^n \) where \( U \subseteq \mathbb{R}^n \) is an open set, and let \( J : U \to \mathbb{R}^{n \times n} \) be the Jacobian matrix of \( \phi \). Then \( x_* \) is an attracting fixed point of the iterative method \( x_{k+1} = \phi(x_k) \) if \( \rho(J(x_*)) < 1 \), where \( \rho(J(x_*)) \) is the spectral radius of the matrix \( J(x_*) \). Furthermore, if \( \rho(J(x_*)) > 0 \), then for \( x_0 \) sufficiently close to \( x_* \), the rate of convergence of this iterative method is linear.

We will also need the following lemma about the structure of the Jacobian matrix of the tensor power method iteration.

**Lemma 3.3.** Let \( T_d \in S^d(\mathbb{R}^n) \) and let \( \phi : U \to \mathbb{R}^n \) be the tensor power method iteration map

\[
\phi(x) = \frac{T_d \cdot x^{d-1}}{\|T_d \cdot x^{d-1}\|} \quad (3.2)
\]

where \( U \subseteq \mathbb{R}^n \) is an open set. Assume that the vector \( v \in \mathbb{R}^n \) is a unit-norm eigenvector of \( T_d \) with non-zero eigenvalue \( \mu \in \mathbb{R} \). Then the Jacobian matrix of \( \phi \) at \( v \), \( J(v) \), is symmetric and has the following form:

\[
J(v) = \frac{(d-1)}{\mu} \left( T \cdot v^{d-2} - \mu vv^T \right) .
\]
Proof. Denote \( \phi_1(x) = \frac{x}{(x^\top x)^2} \) and \( \phi_2(x) = T_d \cdot x^{d-1} \) so that \( \phi(x) = \phi_1(\phi_2(x)) \). Then

\[
\phi'_1(x) = \frac{x^\top x I_{n \times n} - xx^\top}{(x^\top x)^{\frac{3}{2}}}
\]

and

\[
\phi'_2(x) = (d - 1) (T \cdot x^{d-2}).
\]

Next, we can express

\[
\phi'_1(\phi_2(x)) = \frac{\|T \cdot x^{d-1}\|^2 I_{n \times n} - (T \cdot x^{d-1}) (T \cdot x^{d-1})^\top}{(\|T \cdot x^{d-1}\|)^3}
\]

and therefore, by the chain rule,

\[
J(x) = \phi'(x) = \phi'_1(\phi_2(x))\phi'_2(x) = (d - 1) \frac{\|T \cdot x^{d-1}\|^2 (T \cdot x^{d-2}) - (T \cdot x^{d-1}) (T \cdot x^{d-1})^\top (T \cdot x^{d-2})}{(\|T \cdot x^{d-1}\|)^3},
\]

Now let us evaluate the expression at the unit-norm eigenvector \( v \) corresponding to an eigenvalue \( \mu \neq 0 \) (i.e., satisfying \( T \cdot v^{d-1} = \mu v \)). Then we have that

\[
J(v) = (d - 1) \frac{\mu^2 (T \cdot v^{d-2}) - \mu^2 vv^\top (T \cdot v^{d-2})}{\mu^3} = (d - 1) \frac{(T \cdot v^{d-2} - \mu vv^\top)}{\mu},
\]

where we used the fact that \( v^\top (T \cdot v^{d-2}) = (T \cdot v^{d-1})^\top = \mu v^\top \).

We now proceed with the proof of our theorem.

**Proof of Theorem 3.1.** We may assume that no \( \lambda_i \) is 0 and no two vectors \( v_k \) and \( v_\ell \) are colinear, or else we may rewrite \( T_d \) as a sum of a smaller number \( r \) of symmetric rank-1 tensors.

Contracting \( T_d \) with \( d - 1 \) copies of \( v_j \), since \( v_j \) is an eigenvector of \( T_d \) with eigenvalue \( \mu_{j,d} \neq 0 \), we have

\[
T_d \cdot v_j^{d-1} = \sum_{i=1}^{r} \lambda_i \langle v_i, v_j \rangle^{d-1} v_i = \lambda_j v_j + \sum_{i \in [r] \setminus \{j\}} \lambda_i \alpha_{i,j}^{d-1} v_i = V \Lambda (V^\top v_j)^{\odot(d-1)} = \mu_{j,d} v_j,
\]

where \( \alpha_{i,j} := \langle v_i, v_j \rangle \), and contracting \( T_d \) with \( d - 2 \) copies of \( v_j \), we have

\[
T_d \cdot v_j^{d-2} = \sum_{i=1}^{r} \lambda_i \langle v_i, v_j \rangle^{d-2} v_i v_j^\top = V \Lambda D(x) V^\top,
\]

where \( \Lambda = \text{diag} (\lambda_1, ..., \lambda_r) \), \( D(x) = \text{diag} ((V^\top x)^{\odot(d-2)}) \), and \( x^\odot m = (x_1^m, ..., x_n^m)^\top \) is the \( m \)-th Hadamard power of the vector \( x \). Hence, by Lemma 3.3, we have that the Jacobian matrix of \( \phi \) at \( v_j \) is

\[
J(v_j) = \frac{d - 1}{\mu_{j,d}} (V \Lambda D(v_j)V^\top - \mu_{j,d} v_j v_j^\top).
\]

(3.4)
Now multiplying both sides of (3.3) by $v_j^\top$ on the right, we have
\[
\lambda_j v_j v_j^\top + \sum_{i \in [r] \setminus \{j\}} \lambda_i \alpha_i^{-d-1} v_i v_j^\top = \mu_{j,d} v_j v_j^\top,
\]
and hence
\[
\lambda_j v_j v_j^\top - \mu_{j,d} v_j v_j^\top = - \sum_{i \in [r] \setminus \{j\}} \lambda_i \alpha_i^{-d-1} v_i v_j^\top.
\] (3.5)

Next, we are going to bound the spectral radius of $J(v_j)$, which is equal to $\|J(v_j)\|_2$ because $J(v_j)$ is symmetric. Due to (3.5), we can express
\[
\text{VAD}(v_j)V^\top - \mu_{j,d} v_j v_j^\top = \sum_{i \in [r]} \lambda_i \alpha_i^{-d-2} v_i v_i^\top - \mu_{j,d} v_j v_j^\top
\]
\[
= \sum_{i \in [r] \setminus \{j\}} \lambda_i \alpha_i^{-d-2} v_i v_i^\top - \sum_{i \in [r] \setminus \{j\}} \lambda_i \alpha_i^{-d-1} v_i v_j^\top
\]
\[
= \sum_{i \in [r] \setminus \{j\}} \lambda_i \alpha_i^{-d-2} v_i (v_i - \alpha_i v_j) v_j^\top = \left( \sum_{i \in [r] \setminus \{j\}} \lambda_i \alpha_i^{-d-2} v_i v_i^\top \right) (I - v_j v_j^\top).
\]
Therefore, since $I - v_j v_j^\top$ is an orthogonal projector with $\|I - v_j v_j^\top\|_2 = 1$, and by submultiplicativity of the spectral norm, we get
\[
\rho(J(v_j)) = \left\| \text{VAD}(v_j)V^\top - \mu_{j,d} v_j v_j^\top \right\|_2 \leq \left\| \sum_{i \in [r] \setminus \{j\}} \lambda_i \alpha_i^{-d-2} v_i v_i^\top \right\|_2
\] (3.6)
\[
\leq \left| \frac{d-1}{\mu_{j,d}} \right| \sum_{i \in [r] \setminus \{j\}} \left\| \lambda_i \alpha_i^{-d-2} v_i v_i^\top \right\|_2 = \left| \frac{d-1}{\mu_{j,d}} \right| \sum_{i \in [r] \setminus \{j\}} |\lambda_i| \alpha_i^{-d-2}
\]
\[
\leq \left| \frac{d-1}{\mu_{j,d}} \right| (r-1) \left( \max_{i \in [r] \setminus \{j\}} |\lambda_i| \right) \left( \max_{i \in [r] \setminus \{j\}} |\alpha_i| \right)^{d-2},
\]
where we used the triangle inequality and the Cauchy-Schwarz inequalities.

Note that since no two vectors $v_k$ and $v_j$ are colinear, $|\alpha_i| < 1$ for all $i \in [r] \setminus \{j\}$. Therefore, rearranging (3.3) and applying the triangle inequality,
\[
|\mu_{j,d} - \lambda_j| \leq \sum_{i \in [r] \setminus \{j\}} |\lambda_i| |\alpha_i|^{-d-1},
\]
we see that $\mu_{j,d}$ converges to $\lambda_j$ as $d$ becomes large. This shows that for sufficiently large $d$, $\rho(J(v_j)) < 1$ and hence the result follows by Lemma 3.2.

\[
\square
\]

4 Equiangular Tensors

The main result Theorem 3.1 compels us to find sufficient conditions for when a generating vector of a symmetric tensor $T$ in (2.1) is an eigenvector. We will see that $T$ generated by a certain class of vectors will have this property.
4.1 Equiangular sets and equiangular tight frames

Definition 4.1. An equiangular set (ES) is a collection of vectors $v_1, \ldots, v_r \in \mathbb{R}^n$ with $r \geq n$ if there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = |\langle v_i, v_j \rangle|, \forall i \neq j \quad \text{and} \quad \|v_i\| = 1, \forall i.$$ (4.1)

An ES is an equiangular tight frame (ETF) if, in addition,

$$\begin{pmatrix} v_1 & \cdots & v_r \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_r \end{pmatrix} = \frac{r}{n} I_n$$ (4.2)

where $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix.

Note that if $v_1, \ldots, v_r$ form an ES, and if $\sigma_{i,j} = \text{sgn}(\langle v_i, v_j \rangle) \in \{1, -1\}$ for $i \neq j$, then

$$\begin{pmatrix} -v_1 & -v_r \end{pmatrix} \begin{pmatrix} v_1 & \cdots & v_r \end{pmatrix} = \begin{pmatrix} 1 & \sigma_{1,2} & \ldots & \sigma_{1,r-1} & \sigma_{1,r} \\ \sigma_{2,1} & 1 & \ldots & \sigma_{2,r-1} & \sigma_{2,r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{r-1,1} & \sigma_{r-1,2} & \ldots & 1 & \sigma_{r-1,r} \\ \sigma_{r,1} & \sigma_{r,2} & \ldots & \sigma_{r,r-1} & 1 \end{pmatrix}.$$ (4.3)

Suppose $v_1, \ldots, v_r$ form an ETF. Then a number of additional results can be deduced. If $u_1, \ldots, u_r \in \mathbb{R}^n$ with $r \geq n$ is a collection of vectors, then the following always holds

$$\max_{i,j \in [r]} |\langle u_i, u_j \rangle| \geq \sqrt{\frac{r - n}{n(r - 1)}}$$ (4.4)

with equality if and only if $u_1, \ldots, u_r$ is an ETF [7]. Thus, $\alpha = \sqrt{\frac{r - n}{n(r - 1)}}$ in (4.1). Furthermore, the matrix $V^T V \in \mathbb{R}^{r \times r}$ in (4.3), known as the Gram matrix, has rank $n$ (Proposition 3, [33]). The Gram matrix gives a canonical representation of an equiangular tight frame. This results in a one-to-one correspondence between ETFs up to orthogonal transformation and their corresponding Gram matrix [35], which means that $V$ also has rank $n$.

ESs correspond to sets of lines in $\mathbb{R}^n$ passing through the origin such that the angle between every pair of lines is the same. Determining the maximum number of equiangular lines in $\mathbb{R}^n$ for each $n$ is an old problem that has recently seen significant progress by [15], who determined an asymptotically tight upper bound.

ETFs with $r$ vectors in $\mathbb{R}^n$ do not exist for many values of $r$ and $n$, making ETFs quite rare [32]. Nonetheless, they have attracted a wide interest for a number of reasons. ETFs are a natural generalization of orthonormal sets of vectors where the number of vectors in the set is allowed to exceed the dimension of the space they lie in. ETFs minimize the maximum coherence between the vectors, attaining equality in what is known as the Welch bound (4.4). ETFs can also be formulated for $\mathbb{C}^n$ and have found numerous applications in signal processing [8], coding theory [31], and quantum information processing [30].
4.2 Eigenvectors of equiangular tensors

We call a tensor (2.1) generated by vectors from an ES equiangular set decomposable, or equiangular for short. We begin with some general results on equiangular tensors.

Theorem 4.2. Let \( \mathcal{T} \) be a tensor generated by an ES \( \mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{R}^n \) and coefficients \( \lambda_1, \ldots, \lambda_r \in \mathbb{R} \). If for some \( j \in [r] \), there exists \( \mu_j \in \mathbb{R} \) such that

\[
(\lambda_1 \sigma_{1,j}^{d-1}, \ldots, \lambda_{d-1} \sigma_{d-1,j}^{d-1}, \mu_j, \lambda_{d+1} \sigma_{d+1,j}^{d-1}, \ldots, \lambda_r \sigma_{r,j}^{d-1}) \in \text{Ker}(V)
\]

(4.5)

where \( V \in \mathbb{R}^{n \times r} \) is the matrix whose columns are \( \mathbf{v}_1, \ldots, \mathbf{v}_r \), then \( \mathbf{v}_j \) is an eigenvector.

In particular, if \( d \) is odd and \( (\lambda_1, \ldots, \lambda_r) \in \text{Ker}(V) \), then, all of \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) are eigenvectors of \( \mathcal{T} \). Furthermore, in this case, all of these vectors are robust eigenvectors if

\[
\|VV^\top\|_2 \alpha^{d-2}(d-1) \left( \min_{i \in [r]} |\lambda_i| \right) (1 - \alpha^{d-1}) < 1,
\]

(4.6)

which always holds when \( d \) is large enough.

Proof. We observe that for \( j \in [r] \),

\[
\mathcal{T} \cdot \mathbf{v}_j^{d-1} = \sum_{i=1}^{r} \lambda_i \langle \mathbf{v}_i, \mathbf{v}_j \rangle^{d-1} \mathbf{v}_i = \lambda_j \mathbf{v}_j + \alpha^{d-1} \sum_{i \in [r]\setminus\{j\}} \lambda_i \sigma_{i,j}^{d-1} \mathbf{v}_i,
\]

and hence if (4.5) holds, then

\[
\lambda_j \mathbf{v}_j + \alpha^{d-1} \sum_{i \in [r]\setminus\{j\}} \lambda_i \sigma_{i,j}^{d-1} \mathbf{v}_i = \lambda_j \mathbf{v}_j - \alpha^{d-1} \mu_j \mathbf{v}_j = (\lambda_j - \alpha^{d-1} \mu_j) \mathbf{v}_j.
\]

Therefore, \( \mathbf{v}_j \) is an eigenvector of \( \mathcal{T} \).

Now if \( d \) is odd and \( (\lambda_1, \ldots, \lambda_r) \in \text{Ker}(V) \), we have

\[
\mathcal{T} \cdot \mathbf{v}_j^{d-1} = \sum_{i=1}^{r} \lambda_i \langle \mathbf{v}_i, \mathbf{v}_j \rangle^{d-1} \mathbf{v}_i = \lambda_j \mathbf{v}_j + \sum_{i \in [r]\setminus\{j\}} \lambda_i (\sigma_{i,j} \alpha)^{d-1} \mathbf{v}_i = \lambda_j \mathbf{v}_j + \alpha^{d-1} \sum_{i \in [r]\setminus\{j\}} \lambda_i \mathbf{v}_i
\]

\[
= \lambda_j \mathbf{v}_j + \alpha^{d-1}(-\lambda_j \mathbf{v}_j) = \lambda_j (1 - \alpha^{d-1}) \mathbf{v}_j,
\]

i.e., all of \( \mathbf{v}_1, \ldots, \mathbf{v}_r \) are eigenvectors. In addition, using inequality (3.6),

\[
\rho(J(\mathbf{v}_j)) \leq \frac{d - 1}{|\lambda_j|(1 - \alpha^{d-1}) \alpha^{d-2}} \left\| \sum_{i \in [r]\setminus\{j\}} \mathbf{v}_i \mathbf{v}_i^\top \right\|_2
\]

\[
\leq \frac{d - 1}{|\lambda_j|(1 - \alpha^{d-1}) \alpha^{d-2}} \|VV^\top\|_2 \alpha^{d-2}(d-1) \left( \min_{i \in [r]} |\lambda_i| \right) (1 - \alpha^{d-1})^{-1}.
\]

When the above quantity is less than 1, \( \mathbf{v}_j \) is a robust eigenvector, for all \( j \in [r] \). \( \Box \)

When a tensor is generated by the vectors in an ETF, it is called an ETF decomposable tensor. Such tensors are a special case of fradeco tensors, which were studied in [25].
Theorem 4.3. If \( v_1, \ldots, v_r \in \mathbb{R}^n \) form an ETF, then 
\[
\sum_{i \in [r] \setminus \{j\}} \sigma_{ij} v_i = C v_j \text{ for some } C \in \mathbb{R},
\]
for all \( j \in [r] \). In particular, all of \( v_1, \ldots, v_r \) are eigenvectors of the tensor

\[
T = \sum_{i=1}^r v_i \otimes d_i
\]

(4.7)

when \( d \) is even. Furthermore, in this case, all of these vectors are robust eigenvectors if

\[
\frac{\sqrt{n} \alpha^{d-2}(d-1)}{(1 + \alpha^{d-2}(\frac{n}{n} - 1))} < 1,
\]

(4.8)

which always holds when \( d \) is large enough.

Proof. Starting with the Gram matrix of the ETF,

\[
\begin{pmatrix}
1 & \sigma_{1,2} & \ldots & \sigma_{1,r-1} & \sigma_{1,r} \\
\sigma_{2,1} & 1 & \ldots & \sigma_{2,r-1} & \sigma_{2,r} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sigma_{r-1,1} & \sigma_{r-1,2} & \ldots & 1 & \sigma_{r-1,r} \\
\sigma_{r,1} & \sigma_{r,2} & \ldots & \sigma_{r,r-1} & 1
\end{pmatrix}
= V^T V
\]

we subtract the identity matrix on both sides of the equation to obtain

\[
\begin{pmatrix}
0 & \sigma_{1,2} & \ldots & \sigma_{1,r-1} & \sigma_{1,r} \\
\sigma_{2,1} & 0 & \ldots & \sigma_{2,r-1} & \sigma_{2,r} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sigma_{r-1,1} & \sigma_{r-1,2} & \ldots & 0 & \sigma_{r-1,r} \\
\sigma_{r,1} & \sigma_{r,2} & \ldots & \sigma_{r,r-1} & 0
\end{pmatrix}
= V^T V - I_{r \times r}.
\]

Multiplying on the left of both sides of the equation by \( V \),

\[
V \begin{pmatrix}
0 & \sigma_{1,2} & \ldots & \sigma_{1,r-1} & \sigma_{1,r} \\
\sigma_{2,1} & 0 & \ldots & \sigma_{2,r-1} & \sigma_{2,r} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\sigma_{r-1,1} & \sigma_{r-1,2} & \ldots & 0 & \sigma_{r-1,r} \\
\sigma_{r,1} & \sigma_{r,2} & \ldots & \sigma_{r,r-1} & 0
\end{pmatrix}
= V V^T V - VI_{r \times r} = (V V^T) V - V
\]

and using (4.2), we obtain

\[
\frac{r}{n} I_{r \times r} V - V = \frac{r}{n} V - V = \left( \frac{r}{n} - 1 \right) V.
\]
Dividing by $\alpha$, we have

$$V = \frac{1}{\alpha} \left( \begin{array}{cccc} 0 & \sigma_{1,2} \alpha & \ldots & \sigma_{1,r-1} \alpha & \sigma_{1,r} \alpha \\ \sigma_{2,1} \alpha & 0 & \ldots & \sigma_{2,r-1} \alpha & \sigma_{2,r} \alpha \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{r-1,1} \alpha & \sigma_{r-1,2} \alpha & \ldots & 0 & \sigma_{r-1,r} \alpha \\ \sigma_{r,1} \alpha & \sigma_{r,2} \alpha & \ldots & \sigma_{r,r-1} \alpha & 0 \end{array} \right) = \frac{1}{\alpha} \left( \begin{array}{c} \sigma_{1,1} \alpha \\ \sigma_{1,2} \alpha \\ \vdots \\ \sigma_{r-1,1} \alpha \\ \sigma_{r,1} \alpha \end{array} \right),$$

so $C = \frac{1}{\alpha} \left( \frac{r}{n} - 1 \right)$.

When $d$ is even, we now show that all of $v_1, \ldots, v_r$ are eigenvectors of $T$. We have

$$T \cdot v_j^{d-1} = \sum_{i=1}^{r} \langle v_i, v_j \rangle^{d-1} v_i = v_j + \sum_{i \in [r] \setminus \{j\}} (\sigma_{i,j} \alpha)^{d-1} v_i = v_j + \alpha^{d-1} \sum_{i \in [r] \setminus \{j\}} \sigma_{i,j} v_i$$

$$= v_j + \alpha^{d-1} \left( \frac{1}{\alpha \left( \frac{r}{n} - 1 \right)} \right) v_j = \left( 1 + \alpha^{d-2} \left( \frac{r}{n} - 1 \right) \right) v_j.$$

Therefore, by (3.6), we obtain

$$\rho(J(v_j)) \leq \frac{d - 1}{\left( 1 + \alpha^{d-2} \left( \frac{r}{n} - 1 \right) \right)} \alpha^{d-2} \|VV^\top\|_2 = \frac{\frac{r}{n} \alpha^{d-2}(d-1)}{\left( 1 + \alpha^{d-2} \left( \frac{r}{n} - 1 \right) \right)},$$

where the last equality follows from (4.2). Thus, $v_j$ is a robust eigenvector for any $j \in [r]$ and all even $d$ for which the quantity above is strictly less than 1 (which holds when $d$ is large enough).

**Example 4.4. (Orthogonally Decomposable Tensors)** An ETF $v_1, \ldots, v_n \in \mathbb{R}^n$ with $r = n$ is clearly an orthonormal set of vectors, with constant $\alpha = 0$, and thus we may choose $\sigma_{i,j} = 1$ for all $i, j \in [r]$. Tensors $T$ generated by this ETF are called orthogonally decomposable tensors and their properties have been studied in [28, 29]. It is not hard to see that an orthogonally decomposable tensor has all of $v_1, \ldots, v_n$ as eigenvectors, and the spectral radius bound (3.6) is trivially equal to 0 and therefore less than 1, meaning all of $v_1, \ldots, v_n$ are also robust eigenvectors of $T$, for all $d \geq 2$.

In the following sections, we first study in more detail the robust eigenvectors of tensors generated by particular ETFs, and then by an ES which is not an ETF.
Figure 1: The Mercedes-Benz frame (4.9) on the xy-plane.

4.3 Regular Simplex Tensors

An ETF $v_1, \ldots, v_{n+1} \in \mathbb{R}^n$ with $r = n + 1$ always consists of the vertices of a regular simplex in $\mathbb{R}^n$ (page 623, [32]), called a regular $n$-simplex frame, or regular simplex frame for short, with constant $\alpha = \frac{1}{n}$ and signs $\sigma_{i,j} = -1$ for all $i, j \in [r], i \neq j$. A particular example of a regular simplex is given by the Mercedes-Benz frame in $\mathbb{R}^2$, as shown in Figure 1, consisting of the vectors

$$v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}, v_3 = \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}. \quad (4.9)$$

We will call a tensor generated by a regular simplex frame a regular $n$-simplex tensor, or regular simplex tensor for short. Let $V \in \mathbb{R}^{n \times (n+1)}$ be the matrix whose columns are $v_1, \ldots, v_{n+1}$. We observe for the regular simplex frame that

$$VV^T \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{n} \\ 1 \\ \vdots \\ -\frac{1}{n} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ -\frac{1}{n} & 1 & \cdots & -\frac{1}{n} & -\frac{1}{n} \\ 1 & -\frac{1}{n} & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & -\frac{1}{n} & \cdots & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = 0,$$

and since $V$ has rank $n$, the kernel of $V$ is the span of $(1, \ldots, 1)^T \in \mathbb{R}^{n+1}$. Thus, consider the regular simplex tensor

$$T = \sum_{i=1}^{n+1} v_i^\otimes d \quad (4.10)$$

for $d \geq 2$. Then all of $v_1, \ldots, v_{n+1}$ are eigenvectors of $T$ by Theorems 4.2 and 4.3.

There is a systematic method of generating regular simplex frames in $\mathbb{R}^n$ for all $n \geq 2$ as follows: if $e_1, \ldots, e_n \in \mathbb{R}^n$ are the standard basis vectors and $1_n = (1, \ldots, 1) \in \mathbb{R}^n$, then set

$$v_i = \sqrt{1 + \frac{1}{n}} e_i - \frac{1}{n^{\frac{3}{2}}} (\sqrt{n+1} - 1) 1_n, \quad i \in [n]$$

$$v_{n+1} = -\frac{1}{\sqrt{n}} 1_n. \quad (4.11)$$

These vectors are constructed by projecting the standard basis vectors in $\mathbb{R}^{n+1}$ onto the subspace orthogonal to the vector $1_{n+1}$ with an appropriate rotation and rescaling.
We now present a theorem which shows that all of $v_1, ..., v_{n+1}$ are also robust eigenvectors of $T$ for many values of $n$ and $d$.

**Theorem 4.5.** Let

$$T = \sum_{i=1}^{n+1} v_i^{\otimes d}$$

be a tensor generated by a regular simplex frame $v_1, ..., v_{n+1} \in \mathbb{R}^n$. Then all of $v_1, ..., v_{n+1}$ are robust eigenvectors for $T$ for $n \geq 2$ and $d \geq 3$ such that $n + d \geq 7$.

**Proof.** For a regular $n$-simplex frame, we have $r = n + 1$, $\alpha = \frac{1}{n}$, and $\sigma_{i,j} = -1$ for all $i, j \in [n+1], i \neq j$. Since $(1, ..., 1)^\top \in \text{Ker}(V)$, both bounds (4.6) and (4.8) apply. Thus,

$$\frac{\tau_n \alpha^{d-2}(d-1)}{1 + \alpha^{d-2} (\frac{\tau_n}{n} - 1)} = \frac{(n + 1)(d - 1)}{n^{d-1} + 1} < \frac{(n + 1)(d - 1)}{n^{d-1} - 1} = \frac{\tau_n \alpha^{d-2}(d-1)}{(\min_{i \in [r]} |\lambda_i|) (1 - \alpha^{d-1})}.$$

Hence, regardless of the parity of $d$, it suffices to find values of $n$ and $d$ for which $\frac{(n+1)(d-1)}{n^{d-1} - 1} < 1$. This happens if and only if the following quantity is positive ($\gamma(n, d) > 0$):

$$\gamma(n, d) = n^{d-1} + n - d - dn.$$

We can easily check that $\gamma(n, d)$ is positive for the following values:

$$\gamma(2, 5) = 3, \quad \gamma(3, 4) = 14, \quad \gamma(4, 3) = 5$$

Moreover the partial derivatives

$$\frac{\partial}{\partial n} \gamma(n, d) = (d - 1)(n^{d-2} - 1), \quad \frac{\partial}{\partial d} \gamma(n, d) = \ln(n)n^{d-1} - (n + 1)$$

are positive for $n \geq 2, d \geq 3$. This guarantees that $\gamma(n, d) > 0$ whenever

$$(n, d) \geq (2, 5) \text{ or } (n, d) \geq (3, 4) \text{ or } (n, d) \geq (4, 3),$$

and thus all of $v_1, ..., v_{n+1}$ are robust eigenvectors for $n \geq 2$ and $d \geq 3$ with $n + d \geq 7$. \qed

### 4.4 Regular 2-Simplex Tensors

In fact, for regular 2-simplex tensors, we can prove even stronger results compared to Theorem 4.5. The next theorem concerns not only robustness of the vectors in the regular simplex frame, but regions of convergence of the tensor power method, for tensors of order $d$ when $d \geq 4$ is even.

**Theorem 4.6.** Let $v_1, v_2, v_3 \in \mathbb{R}^2$ be vectors of a regular 2-simplex frame. If $x_0 \in \mathbb{R}^2$ and there is a unique $v \in \{v_1, -v_1, v_2, -v_2, v_3, -v_3\}$ which maximizes $\langle v, x_0 \rangle$, then the tensor power method with initializing vector $x_0$ applied to the tensor

$$T = v_1^{\otimes d} + v_2^{\otimes d} + v_3^{\otimes d}.$$

will converge to $v$, for all even $d \geq 6$. If $d = 4$, then any initializing vector $x_0$ is a fixed point of the tensor power method.
We leave the proof of this theorem in the Appendix.

The results of Theorems 4.5 and 4.6 can be visualized in Table 1. We denote by a green tick mark (✓) convergence which is guaranteed by Theorem 4.5, and by a red cross mark (✗) failure of convergence for the case \( n = 2 \) and \( d = 4 \) where every initializing vector is a fixed point of the tensor power method, due to Theorem 4.6.

In addition, we performed the following two numerical experiments.

The first numerical experiment concerns robustness. Let \( T \) be the tensor (4.10) generated by the regular simplex frame (4.11). We can observe the values of \( n \) and \( d \) for which the tensor power method applied to \( T \) converges to vectors in the frame. Using MATLAB, we choose an initial vector \( x_0 \) drawn from a uniform distribution on the unit sphere in \( \mathbb{R}^n \) and apply the tensor power method

\[
x_{k+1} = \frac{T(x_k, ..., x_k)}{\|T(x_k, ..., x_k)\|}
\]

for 100 iterations. We denote by a black cross mark (✗) a lack of convergence, which occurs if \( \|x_{100} - v_j\| > 10^{-10} \) for all eigenvectors \( v_j \) of \( T \). We then performed this experiment for \( 2 \leq n, d \leq 10 \). In the cases with the green cross mark, we did not observe the tensor power method converging to any vectors other than \( v_1, ..., v_{n+1} \). This suggests that the frame vectors may be the only robust eigenvectors, which we leave as a conjecture:

**Conjecture 4.7.** *The robust eigenvectors of a regular simplex tensor (4.10) are precisely the vectors in the frame.*

The second numerical experiment concerns the regions of convergence of the tensor power method in Theorem 4.6. Figure 2 shows regions of convergence to the eigenvectors \( v_1, v_2, \) and \( v_3 \), starting from an initial vector \( x_0 \) of the tensor power method applied to the regular simplex tensor \( T \) in (4.10) generated by the vectors in a Mercedes-Benz frame (4.9). For \( x_0 \) in the blue, red, and green regions, the method will converge to \( v_1, v_2, \) and \( v_3 \), respectively. As Theorem 4.6 predicts, when \( d \geq 6 \) is even, the regions of convergence form a partition of the unit disk into sectors. One can also observe a fractal subdivision of the regions of convergence for odd values of \( d \), which we lack an explanation for. As a consequence of the Theorem 3.1, however, larger values of \( d \) result in greater robustness of the eigenvectors, and hence the observed thinning of these fractal subdivisions.
Figure 2: Regions of convergence on the unit disk in \( \mathbb{R}^2 \) of the tensor power method for different values of \( d \) of the Mercedez-Benz tensor.

For the tensor \( \mathcal{T} \) generated by a Mercedes-Benz frame \( (n = 2) \) for \( 3 \leq d \leq 10 \) given by the vectors in (4.9), we can find its eigenvectors and their corresponding eigenvalues and multiplicities, which we show in Tables 2 and 3.

The eigenvectors can be found by solving two polynomial equations in two variables \( u \) and \( v \) over \( \mathbb{C} \):

\[
\begin{align*}
\det \begin{pmatrix}
\mathcal{T}
& \begin{pmatrix}\vdots \end{pmatrix}
& \begin{pmatrix} u \\ \vdots \\ u \end{pmatrix}
& \begin{pmatrix} u \\ \vdots \\ u \end{pmatrix}
\end{pmatrix}
&= \det \begin{pmatrix}
\begin{pmatrix} u \\ \vdots \\ u \end{pmatrix}
& \begin{pmatrix} \sqrt{3} \left( \frac{\sqrt{3}}{2} u - \frac{1}{2} v \right)^{d-1} \\
& - \sqrt{3} \left( -\frac{\sqrt{3}}{2} u - \frac{1}{2} v \right)^{d-1}
\end{pmatrix}
& \begin{pmatrix} \frac{\sqrt{3}}{2} u - \frac{1}{2} v \\
& -\frac{\sqrt{3}}{2} u - \frac{1}{2} v
\end{pmatrix}
\begin{pmatrix} u \\ \vdots \\ u \end{pmatrix}
\end{pmatrix} = 0,
\end{align*}
\]

where the first equation encodes the information that \( (u, v) \top \in \mathbb{C}^2 \) is an eigenvector of \( \mathcal{T} \), and the second equation imposes that \( (u, v) \top \), the unique representative of an eigenvector, has norm 1.

The multiplicity of an eigenvector is the multiplicity of its solution in this system of equations. In [5], it was shown that if a \( n \times \ldots \times n \) \( (d \) times) tensor with complex entries has finitely many eigenvectors in \( \mathbb{C}^n \) up to scaling, then this number is \( \frac{(d-1)^n-1}{d-2} \) counted with multiplicity. In this setting, we have \( \frac{(d-1)^n-1}{d-2} = d \), and as can be seen in the tables, the sum of the multiplicities of each eigenvector for each \( d \) is equal to \( d \), meaning that these tables form a complete list of the eigenvectors of \( \mathcal{T} \). The table also suggests another conjecture:
| $d$ | eigenvector | eigenvalue | multiplicity |
|-----|-------------|------------|--------------|
| 3   | $(0,1)\top$ | $\frac{3}{4}$ | 1            |
|     | $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{3}{4}$ | 1            |
|     | $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{3}{4}$ | 1            |
|     | $(0,1)\top$ | $\frac{5}{8}$ | 1            |
| 4   | $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{9}{16}$ | 1            |
|     | $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{9}{16}$ | 1            |
|     | $(1,0)\top$ | $\frac{5}{8}$ | 1            |
| 5   | $(0,1)\top$ | $\frac{33}{32}$ | 1            |
|     | $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{33}{32}$ | 1            |
|     | $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{33}{32}$ | 1            |
|     | $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)\top$ | $\frac{27}{32}$ | 1            |
|     | $\left(-\frac{1}{2}, \frac{\sqrt{2}}{2}\right)\top$ | $\frac{27}{32}$ | 1            |
|     | $(1,0)\top$ | $\frac{33}{32}$ | 1            |
| 6   | $(0,1)\top$ | $\frac{63}{64}$ | 1            |
|     | $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{63}{64}$ | 1            |
|     | $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{63}{64}$ | 1            |
|     | $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\top$ | $\frac{0}{2}$ | 2            |
|     | $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\top$ | $\frac{0}{2}$ | 2            |
| 7   | $(0,1)\top$ | $\frac{129}{128}$ | 1            |
|     | $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{129}{128}$ | 1            |
|     | $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{129}{128}$ | 1            |
|     | $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\top$ | $\frac{3}{16}$ | 1            |
|     | $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\top$ | $\frac{3}{16}$ | 1            |
|     | $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\top$ | $\frac{81}{128}$ | 1            |
|     | $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\top$ | $\frac{81}{128}$ | 1            |
|     | $(1,0)\top$ | $\frac{129}{128}$ | 1            |
| 8   | $(0,1)\top$ | $\frac{253}{256}$ | 1            |
|     | $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{253}{256}$ | 1            |
|     | $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)\top$ | $\frac{253}{256}$ | 1            |
|     | $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\top$ | $\frac{1}{2}$ | 1            |
|     | $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\top$ | $\frac{1}{2}$ | 1            |
|     | $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\top$ | $\frac{1}{2}$ | 1            |
|     | $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\top$ | $\frac{1}{2}$ | 1            |
|     | $(1,0)\top$ | $\frac{253}{256}$ | 1            |
| 9   | $\mathbf{x} \approx (1/Z)(0.393942 - 0.624439i, 1)^\top$ | $\lambda_1 \approx -0.234194 - 0.107117i$ | 1 |
|     | $\mathbf{y} \approx (1/Z)(1.965672i, 1)^\top$ | $\lambda_2 \approx 0.257529$ | 1 |
|     | $\mathbf{x}$ | $\lambda_1$ | 1 |
|     | $(-\mathbf{x}_1, \mathbf{x}_2)^\top$ | $\lambda_1$ | 1 |
|     | $(-\mathbf{x}_1, \mathbf{x}_2)^\top$ | $\lambda_1$ | 1 |
|     | $(-\mathbf{y}_1, \mathbf{y}_2)^\top$ | $\lambda_2$ | 1 |

Table 2: The unit-norm eigenvectors and their eigenvalues with multiplicities of the tensor $\mathbf{T} = \sum_{i=1}^{3} (\mathbf{(0,1)})^\otimes d + (\mathbf{(\sqrt{3}/2, -1/2)})^\otimes d + (\mathbf{(-\sqrt{3}/2, -1/2)})^\otimes d$, where $Z \in \mathbb{R}$ is a normalizing constant.
Conjecture 4.8. All eigenvectors of a regular simplex tensor are complex, except for the vectors in the frame, and when \( d \geq 6 \) and even, the vectors on the boundary of the regions of convergence.

### 4.5 Other Equiangular Tensors

If \( V \in \mathbb{R}^{n \times r} \) is the matrix of an ETF whose columns are \( v_1, \ldots, v_r \in \mathbb{R}^n \), then more examples of ETFs include the diagonals of a cube in \( \mathbb{R}^3 \),

\[
V = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix},
\]

the diagonals of a regular icosahedron in \( \mathbb{R}^3 \) (Figure 3),

\[
V = \frac{1}{\sqrt{1 + \varphi^2}} \begin{pmatrix} 0 & 0 & 1 & -1 & \varphi & -\varphi \\ 1 & -1 & \varphi & \varphi & 0 & 0 \\ \varphi & \varphi & 0 & 0 & 1 & 1 \end{pmatrix}, \tag{4.12}
\]

where \( \varphi = \frac{1+\sqrt{5}}{2} \), and the following 16 equiangular lines in \( \mathbb{R}^6 \),

\[
V = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}.
\]

**Example 4.9. (Icosahedral Frame)** Consider the tensor \( \mathcal{T} \) generated by the icosahedral frame (4.12),

\[
\mathcal{T} = \sum_{i=1}^{16} v_i \otimes v_i.
\]
for $d \geq 2$. Here, $n = 3$, $r = 6$, $\alpha = \frac{1}{\sqrt{3}}$, and the Gram matrix is

$$
V^\top V = \begin{pmatrix}
1 & \alpha & \alpha & \alpha & \alpha & \alpha \\
\alpha & 1 & -\alpha & -\alpha & \alpha & \alpha \\
\alpha & -\alpha & 1 & \alpha & -\alpha & \alpha \\
\alpha & -\alpha & \alpha & 1 & -\alpha & \alpha \\
\alpha & \alpha & -\alpha & \alpha & 1 & -\alpha \\
\alpha & \alpha & \alpha & -\alpha & \alpha & 1
\end{pmatrix}. 
$$

(4.13)

By Theorem 4.3, all of $v_1, ..., v_6$ are robust eigenvectors of $T$ for all even $d \geq 6$ where the bound

$$
\frac{\frac{r}{n} \alpha d^2 (d-1)}{1 + \alpha^{d-2} (\frac{r}{n} - 1)} = \frac{2(d-1) \left(\frac{1}{\sqrt{5}}\right)^{d-2}}{1 + \left(\frac{1}{\sqrt{5}}\right)^{d-2}}
$$

(4.14)

from (4.7) is strictly less than 1. However, when $d$ is odd, one can verify that indeed, $v_2, ..., v_6$ are not eigenvectors of $T$. But $v_1$ is still an eigenvector of $T$ because $\sigma_{1,1} = ... = \sigma_{6,1}$ (they are all equal to 1) as seen from the Gram matrix (4.13). Thus, the bound (4.7) still applies for $v_1$ and (4.14) is strictly less than 1 for all $d \geq 5$, regardless of the parity of $d$, and hence for all these values of $d$, $v_1$ is a robust eigenvector.

**Example 4.10. (An ES that is not an ETF)** Below are 6 equiangular lines in $\mathbb{R}^4$, forming an ES that is not an ETF:

$$
V = \begin{pmatrix}
1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} \\
0 & 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\
0 & 0 & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 \\
0 & 0 & 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3}
\end{pmatrix}.
$$

Then, Theorem 4.3 fails, and indeed, if $T$ is the following tensor generated by this ES,

$$
T = \sum_{i=1}^{6} v_i^\otimes d,
$$

then none of $v_1, ..., v_6$ are eigenvectors of $T$ for all $d \geq 2$. But if $T$ is the following tensor,
\[ \mathcal{T} = \sum_{i=1}^{6} \lambda_i v_i^{\otimes d}, \]

where \((\lambda_1, ..., \lambda_6) \in \text{Ker}(\mathbf{V})\), then Theorem 4.2 still holds with \(\mu_j = \lambda_j\) and \(d\) odd, so that all of \(v_1, ..., v_6\) are eigenvectors. When \(d\) is even, \(v_1\) is an eigenvector with \(\mu_j = \lambda_j\) since \(\sigma_{1,1} = ... = \sigma_{6,1}\) (they are all equal to 1), while \(v_2, ..., v_6\) are not.

5 Conclusion

We have given sufficient conditions for when an eigenvector in a symmetric decomposition of a tensor is robust. We have then explored robustness for a family of tensors that are generated by sets of vectors that are mutually equiangular. We leave behind an open problem whose solution would entirely solve the task of characterising eigenvectors that can be retrieved by the tensor power method:

**Problem 5.1.** Under what conditions are the elements \(v_1, ..., v_r\) of an ES precisely the real, robust eigenvectors of a tensor \(\mathcal{T}\) they generate? Is there an efficient method to distinguish the vectors of the ES and the other (if any) robust eigenvectors of \(\mathcal{T}\)?

In addition, it would be interesting to extend our results to tensors generated by non-equiquiangular sets.

In [23] it was shown that the eigenvectors of orthogonally decomposable tensors are stable under perturbation of the tensor, i.e., if a tensor with small Frobenious norm is added to an orthogonally decomposable tensor, then the eigenvectors of this new tensor can be retrieved by the tensor power method and they will be close to the original eigenvectors. From other numerical experiments we performed, we strongly believe that this is also true for equiangular tight frame decomposable tensors:

**Conjecture 5.2.** The eigenvectors of an equiangular tight frame decomposable tensor are stable under perturbation and can be approximately recovered using the tensor power method.

Acknowledgements

We wish to thank Jeffery Zhang and Kevin Shen for creating Figure 2. Elina Robeva was supported by an NSERC Discovery grant (DGECR-2020-00338). Konstantin Usevich was supported by the ANR grant LeaFleT (ANR-19-CE23-0021).

References

[1] A. Anandkumar, R. Ge, S. Kakade D. Hsu, and M. Telgarsky. Tensor decompositions for learning latent variable models. *Journal of Machine Learning Research*, 15(80):2773–2832, 2014.
[2] Animashree Anandkumar, Rong Ge, and Majid Janzamin. Learning overcomplete latent variable models through tensor methods. In Peter Grünwald, Elad Hazan, and Satyen Kale, editors, *Proceedings of The 28th Conference on Learning Theory*, volume 40 of *Proceedings of Machine Learning Research*, pages 36–112, Paris, France, 03–06 Jul 2015. PMLR.

[3] A. Boralevi, J. Draisma, E. Horobet, , and E. Robeva. Orthogonal and unitary tensor decomposition from an algebraic perspective. *Israel Journal of Mathematics*, 222(1):223–260, 2017.

[4] Maria Chiara Brambilla and Giorgio Ottaviani. On the Alexander–Hirschowitz theorem. *Journal of Pure and Applied Algebra*, 212(5):1229–1251, 2008.

[5] Dustin Cartwright and Bernd Sturmfels. The number of eigenvalues of a tensor. *Linear Algebra and its Applications*, 438(2):942–952, 2013. Tensors and Multilinear Algebra.

[6] Jie Chen and Yousef Saad. On the tensor svd and the optimal low rank orthogonal approximation of tensors. *SIAM Journal on Matrix Analysis and Applications*, 30(4):1709–1734, 2009.

[7] Ole Christensen, Somantika Datta, and Rae Young Kim. Equiangular frames and generalizations of the welch bound to dual pairs of frames. *Linear and Multilinear Algebra*, 68(12):2495–2505, 2020.

[8] Matthew Fickus, Dustin G. Mixon, and Janet C. Tremain. Steiner equiangular tight frames. *Linear Algebra and its Applications*, 436(5):1014–1027, 2012.

[9] Shmuel Friedland. Best rank one approximation of real symmetric tensors can be chosen symmetric. *Frontiers of Mathematics in China*, 8(1):19–40, 2013.

[10] Karim Halaseh, Tommi Muller, and Elina Robeva. Orthogonal decomposition of tensor trains. *Linear and Multilinear Algebra*, 0(0):1–31, 2021.

[11] Richard A. Harshman. Foundations of the parafac procedure: Models and conditions for an “explanatory” multi-model factor analysis. 1970.

[12] C. Hillar and L.-H. Lim. Most tensor problems are NP-hard. *Journal of the ACM*, 60(6), 2013.

[13] F. L. Hitchcock. The expression of a tensor or a polyadic as a sum of products. *Journal of Mathematics and Physics*, 6:164–189, September 1927.

[14] Shenglong Hu and Guoyin Li. Convergence rate analysis for the higher order power method in best rank one approximations of tensors. *Numerische Mathematik*, 140(4):993–1031, 2018.

[15] Zilin Jiang, Jonathan Tidor, Yuan Yao, Shengtong Zhang, and Yufei Zhao. Equiangular lines with a fixed angle. *arXiv: Combinatorics*, 2019.
[16] Joe Kileel and João M. Pereira. Subspace power method for symmetric tensor decomposition and generalized PCA, 2021.

[17] Eleftherios Kofidis and Phillip A. Regalia. On the best rank-1 approximation of higher-order supersymmetric tensors. *SIAM Journal on Matrix Analysis and Applications*, 23(3):863–884, 2002.

[18] Tamara G. Kolda. Symmetric orthogonal tensor decomposition is trivial, 2015.

[19] Tamara G. Kolda and Brett W. Bader. Tensor decompositions and applications. *SIAM Review*, 51(3):455–500, Sep 2009.

[20] Tamara G. Kolda and Jackson R. Mayo. Shifted power method for computing tensor eigenpairs. *SIAM Journal on Matrix Analysis and Applications*, 32(4):1095–1124, 2011.

[21] Jianze Li, Konstantin Usevich, and Pierre Comon. Globally convergent jacobi-type algorithms for simultaneous orthogonal symmetric tensor diagonalization. *SIAM Journal on Matrix Analysis and Applications*, 39(1):1–22, 2018.

[22] Lek-Heng Lim. Singular values and eigenvalues of tensors: a variational approach. *Proceedings of IEEE Workshop on Computational Advances in Multisensor Adaptive Processing*, 1:129–132, 2005.

[23] C. Mu, D. Hsu, and D. Goldfarb. Successive rank-one approximations for nearly orthogonally decomposable symmetric tensors. *SIAM J. Matrix Anal. Appl.*, 36(4):1638–1659, 2015.

[24] Jiawang Nie and Li Wang. Semidefinite relaxations for best rank-1 tensor approximations. *SIAM Journal on Matrix Analysis and Applications*, 35(3):1155–1179, 2014.

[25] Luke Oeding, Elina Robeva, and Bernd Sturmfels. Decomposing tensors into frames. *Advances in Applied Mathematics*, 73:125–153, 2016.

[26] Liqun Qi. Eigenvalues of a real supersymmetric tensor. *Journal of Symbolic Computation*, 40(6):1302–1324, 2005.

[27] Werner C. Rheinboldt. *Methods for Solving Systems of Nonlinear Equations*. Society for Industrial and Applied Mathematics, 1998.

[28] E. Robeva. Orthogonal decomposition of symmetric tensors. *SIAM Journal on Matrix Analysis and Applications*, 37(1):86–102, 2016.

[29] Elina Robeva and Anna Seigal. Singular vectors of orthogonally decomposable tensors. *Linear and Multilinear Algebra*, 65(12):2457–2471, 2017.

[30] A. J. Scott. Tight informationally complete quantum measurements. *Journal of Physics A*, 39:13507–13530, 2006.
Proof of Theorem 4.6. Suppose \( v_1, v_2, v_3 \) is the Mercedes-Benz frame (4.9). Let \( x_0 \in \mathbb{R}^2 \) be an initializing vector for the tensor power method applied to \( T \). Then

\[
x_{k+1} := T \cdot x_k^{d-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (x_k^2)^{d-1} + \left( \frac{\sqrt{3}}{3} \right) \left( \frac{\sqrt{3}}{2} x_k - \frac{1}{2} x_k^2 \right)^{d-1} + \left( -\frac{\sqrt{3}}{2} \right) \left( -\frac{\sqrt{3}}{2} x_k^2 - \frac{1}{2} x_k \right)^{d-1}
\]

\[
= \begin{pmatrix} 0 \\ 1 \end{pmatrix} c_k^{d-1} + \left( \frac{\sqrt{3}}{2} \right) a_k^{d-1} + \left( -\frac{\sqrt{3}}{2} \right) b_k^{d-1} = \left( \frac{\sqrt{3}}{2} \right) (a_k^{d-1} - b_k^{d-1}) + \left( -\frac{\sqrt{3}}{2} \right) (a_k^{d-1} + b_k^{d-1})
\]

where

\[
a_k := \langle x_k, v_2 \rangle, \quad b_k := \langle x_k, v_3 \rangle, \quad c_k := \langle x_k, v_1 \rangle.
\]

If we can find a neighbourhood of \( v_1 \) (or \(-v_1\)) for \( x_0 \) to lie in so that

\[
\lim_{k \to \infty} \frac{\langle x_{k+1}, v_2 \rangle}{\langle x_{k+1}, v_1 \rangle} = -\frac{1}{2}, \quad \lim_{k \to \infty} \frac{\langle x_{k+1}, v_3 \rangle}{\langle x_{k+1}, v_1 \rangle} = -\frac{1}{2},
\]

then we will have proven that the tensor power method converges to \( v_1 \) (or \(-v_1\)) because this ratio of inner products is invariant under scaling of \( x_{k+1} \). Inner products are also invariant under rotations on the plane, and therefore this will also prove the robustness of \( v_2 \) and \( v_3 \) and for any regular simplex frame in \( \mathbb{R}^2 \), not just the Mercedez-Benz frame, since we can rotate a frame so that \( x_0 \) lies in a neighbourhood of \((0, 1)^T \) (or \((0, -1)^T \)). We see that

\[
\frac{\langle x_{k+1}, v_2 \rangle}{\langle x_{k+1}, v_1 \rangle} = \frac{3}{2} (a_k^{d-1} - b_k^{d-1}) - \frac{1}{2} c_k^{d-1} + \frac{1}{4} (a_k^{d-1} + b_k^{d-1}) = \frac{a_k^{d-1}}{2 a_k^{d-1} - \frac{1}{2} b_k^{d-1}} - \frac{b_k^{d-1}}{2 b_k^{d-1} + c_k^{d-1}}.
\]
= \left(\frac{a_k}{c_k}\right)^{d-1} - \frac{1}{2} \left(\frac{b_k}{c_k}\right)^{d-1} - \frac{1}{2} = \left(\frac{a_k}{c_k}\right)^{d-1} - \frac{1}{2} \left(\frac{b_k}{c_k}\right)^{d-1} + 1
\right)

= \left(\frac{\alpha_k}{c_k}\right)^{d-1} - \frac{1}{2} \left((-1 - \alpha_k)\right)^{d-1} - \frac{1}{2}

where we define
\alpha_k := -\frac{a_k}{c_k}, \quad 1 - \alpha_k = \frac{c_k + a_k}{c_k} = -\frac{b_k}{c_k}.

Therefore, we need to find a neighbourhood of \(a_0\) around \(\frac{1}{2}\) for which \(\lim_{k \to \infty} \alpha_k = \frac{1}{2}\). If we can do this, then we will also have proven \(\lim_{k \to \infty} \frac{\langle \mathbf{x}_{k+1}, \mathbf{v}_0 \rangle}{\langle \mathbf{x}_{k+1}, \mathbf{v}, \mathbf{v}_1 \rangle} = -\frac{1}{2}\) since \(\frac{\langle \mathbf{x}_{k+1}, \mathbf{v}_0 \rangle}{\langle \mathbf{x}_{k+1}, \mathbf{v}_{1} \rangle} = -(1 - \alpha_{k+1})\).

Note that the hypothesis in the theorem about the uniqueness of a maximizer ensures that \(\alpha_0 \neq 0\). We claim that the neighbourhood where \(0 < \alpha_0 < 1\) will suffice. This is the neighbourhood \(S\) of \(\mathbf{v}_1\) and \(-\mathbf{v}_1\) where for every \(\mathbf{x}_0 \in S\), \(\|\mathbf{v}, \mathbf{x}_0\|\) is maximized among \(\mathbf{v} \in \{\mathbf{v}_1, -\mathbf{v}_1, \mathbf{v}_2, -\mathbf{v}_2, \mathbf{v}_3, -\mathbf{v}_3\}\) by \(\mathbf{v} = \pm \mathbf{v}_1\). Thus, assuming \(0 < \alpha_0 < 1\), \(0 < 1 - \alpha_0 < 0\), and since

\[
\alpha_{k+1} = \frac{(-\alpha_k)^{d-1} - \frac{1}{2}(-(1 - \alpha_k))^{d-1} - \frac{1}{2}}{1 + \frac{1}{2}\alpha_k^{d-1} + \frac{1}{2}(1 - \alpha_k)^{d-1}} = \frac{\alpha_k^{d-1} - \frac{1}{2}(1 - \alpha_k)^{d-1} + \frac{1}{2}}{1 + \frac{1}{2}\alpha_k^{d-1} + \frac{1}{2}(1 - \alpha_k)^{d-1}}
\]

when \(d\) is even, by the induction, the numerator and denominator are both positive, and hence \(0 < \alpha_{k+1}\). Furthermore,

\[
0 < \frac{1}{2} - \frac{1}{2} \alpha_k^{d-1} + (1 - \alpha_k)^{d-1}
\]

by induction, and hence

\[\alpha_k^{d-1} - \frac{1}{2}(1 - \alpha_k)^{d-1} + \frac{1}{2} < 1 + \frac{1}{2}\alpha_k^{d-1} + \frac{1}{2}(1 - \alpha_k)^{d-1},\]

which shows that \(\alpha_{k+1} < 1\) by (5.2), and thus \(0 < \alpha_k < 1\) for all \(k\).

Now we show that \((\alpha_{k+1} - \frac{1}{2}) - (\alpha_k - \frac{1}{2}) = C_{d,k} (\alpha_k - \frac{1}{2})\), where \(C_{d,k}\) is a continuous function of \(\alpha_k\), with \(-1 < C_{d,k} < 0\) when \(d \geq 6\) is even. We will also show that \(C_{d,k} = 0\) if \(\alpha_k = 0\) or \(\alpha_k = 1\). Thus, assuming \(d \geq 6\) is even and \(0 < \alpha_k < 1\), there then exists a constant, namely \(C_d = \sup_{k \geq 0} C_{d,k}\), such that \(-1 < C_{d,k} \leq C_d < 0\) for all \(k\). Thus, if we can prove these statements, then rearranging the equality, we would obtain

\[\left|\alpha_{k+1} - \frac{1}{2}\right| = C_{d,k} \left|\alpha_k - \frac{1}{2}\right|
\]

where \(0 < C_{d,k} \leq C_d + 1 < 1\), which would imply that \(\lim_{k \to \infty} \alpha_k = \frac{1}{2}\) when \(d \geq 6\) and even. When \(d = 4\), we will see that \(C_{d,k} = 0\) and hence every initializing vector \(\mathbf{x}_0\) is a fixed point of the tensor power method when \(d = 4\). We have

\[
\left(\alpha_{k+1} - \frac{1}{2}\right) - \left(\alpha_k - \frac{1}{2}\right) = \left(\alpha_k^{d-1} - \frac{1}{2}(1 - \alpha_k)^{d-1} + \frac{1}{2} \right) - \left(\alpha_k - \frac{1}{2}\right) = \left(\frac{2\alpha_k^{d-1} - (1 - \alpha_k)^{d-1} + 1 - \frac{1}{2}\alpha_k^{d-1} - \frac{1}{2}(1 - \alpha_k)^{d-1}}{2(1 + \frac{1}{2}\alpha_k^{d-1} + \frac{1}{2}(1 - \alpha_k)^{d-1})} \right) - \left(\alpha_k - \frac{1}{2}\right)
\]

22
\[ \begin{align*}
\frac{2}{2 + \alpha_k^{d-1} + (1 - \alpha_k)^{d-1}} &= \frac{2\alpha_k^{d-1} - (1 - \alpha_k)^{d-1} - 2\alpha_k - \alpha_k^d - \alpha_k(1 - \alpha_k)^{d-1} + 1 + \frac{1}{2}\alpha_k^{d-1} + \frac{1}{2}(1 - \alpha_k)^{d-1}}{2 + \alpha_k^{d-1} + (1 - \alpha_k)^{d-1}} \\
&= \frac{2\alpha_k^{d-1} - (1 - \alpha_k)^{d-1} - 2\alpha_k + \alpha_k^d - \alpha_k(1 - \alpha_k)^{d-1}}{2 + \alpha_k^{d-1} + (1 - \alpha_k)^{d-1}} \\
&= \left( -2 + 4 \sum_{j=0}^{d-2} \alpha_k^{d-2-j}(1 - \alpha_k)^j - 2(1 - \alpha_k)^{d-1} - 2\alpha_k \sum_{j=0}^{d-2} \alpha_k^{d-2-j}(1 - \alpha_k)^j \right) \left( \alpha - \frac{1}{2} \right) \\
&= 2(2 - \alpha_k) \sum_{j=0}^{d-2} \alpha_k^{d-2-j}(1 - \alpha_k)^j - \sum_{j=0}^{d-2} (-1)^j(1 - \alpha_k)^j \\
&= \sum_{j=0}^{d-2} \left( \alpha_k^{d-2-2j}(1 - \alpha_k)^{2j} + \alpha_k^{d-2-(2j+1)}(1 - \alpha_k)^{2j+1} \right) + (1 - \alpha_k)^{d-2} \\
&\quad - \sum_{j=0}^{d-2} \left( (1 - \alpha_k)^{2j} - (1 - \alpha_k)^{2j+1} \right) - (1 - \alpha_k)^{d-2} \\
&= \sum_{j=0}^{d-4} \alpha_k^{d-2-(2j+1)}(1 - \alpha_k)^{2j} - \sum_{j=0}^{d-4} \alpha_k(1 - \alpha_k)^{2j} = \sum_{j=0}^{d-4} \left( \alpha_k^{d-2-(2j+1)} - \alpha_k \right) (1 - \alpha_k)^{2j}.
\end{align*} \]

Notice that \( C_{d,k} \) is a continuous function of \( \alpha_k \) and when \( \alpha_k = 0 \) or \( \alpha_k = 1 \), \( C_{d,k} = 0 \). We observe that
\[
\sum_{j=0}^{d-2} \alpha_k^{d-2-j}(1 - \alpha_k)^j - \sum_{j=0}^{d-2} (-1)^j(1 - \alpha_k)^j
\]

When \( d = 4 \), this sum is 0 and hence \( C_{d,k} = 0 \). When \( d \geq 6 \) and even, each term has \( \left( \alpha_k^{d-2-(2j+1)} - \alpha_k \right) (1 - \alpha_k)^{2j} < 0 \), and thus \( C_{d,k} < 0 \).

Starting with the inequality
\[
((2\alpha_k - 1) + 2(2 - \alpha_k)) \sum_{j=0}^{d-2} \alpha_k^{d-2-j}(1 - \alpha_k)^j = 3 \sum_{j=0}^{d-2} \alpha_k^{d-2-j}(1 - \alpha_k)^j > 0,
\]
we find that
\[
\left( \alpha_k^{d-1} - (1 - \alpha_k)^{d-1} \right) + 2(2 - \alpha_k) \sum_{j=0}^{d-2} \alpha_k^{d-2-j}(1 - \alpha_k)^j > 0
\]
\[
2 \left( (2 - \alpha_k) \sum_{j=0}^{d-2} \alpha_k^{d-2-j}(1 - \alpha_k)^j - (1 + (1 - \alpha_k)^{d-1}) \right) > -2 - \alpha_k^{d-1} - (1 - \alpha_k)^{d-1}
\]

23
\[
C_{d,k} = 2 \frac{(2 - \alpha_k) \sum_{j=0}^{d-2} \alpha_k^{d-2-j} (1 - \alpha_k)^j - (1 + (1 - \alpha_k)^{d-1})}{2 + \alpha_k^{d-1} + (1 - \alpha_k)^{d-1}} > -1
\]
when \( d \geq 6 \) and even.