Relative differential $K$-characters

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Abstract. We define a group of relative differential $K$-characters associated with a smooth map between two smooth compact manifolds. We show that this group fits into a short exact sequence as in the non-relative case. Some secondary geometric invariants are expressed in this theory.

Key words: geometric $K$-homology; differential $K$-characters

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1 Introduction

Cheeger and Simons [10] introduced the notion of differential characters to express some secondary geometric invariants of a principal $G$-bundle in the base space. This theory has been appearing more and more frequently in quantum field and string theories (see [7, 15, 13]). On the other hand, it was shown recently (see [4, 16, 17]) that $K$-homology of Baum–Douglas [5] is an appropriate arena in which various aspects of $D$-branes in superstring theory can be described.

In [8] we have defined with M.T. Benameur the notion of differential characters in $K$-theory on a smooth compact manifold. Our original motivation was to explain some secondary geometric invariants coming from the Chern–Weil and Cheeger–Simons theory in the language of $K$-theory. To do this, we have used the Baum–Douglas construction of $K$-homology. As a result, we obtained the eta invariant of Atiyah–Patodi–Singer as a $\mathbb{R}/\mathbb{Z}$-differential $K$-character, while it is a $\mathbb{R}/\mathbb{Q}$-invariant in the works of Cheeger and Simons. Recall that a geometric $K$-cycle of Baum–Douglas over a smooth compact manifold $X$ is a triple $(M, E, \phi)$ such that: $M$ is a closed smooth Spin$^c$-compact manifold with a fixed Riemannian structure; $E$ is a Hermitian vector bundle over $M$ with a fixed Hermitian connection $\nabla_E$ and $\phi : M \to X$ is a smooth map. Let $\mathcal{C}_*(X)$ be the semi-group for the disjoint union of equivalence classes of $K$-cycles over $X$ generated by direct sum and vector bundle modification [5]. A differential $K$-character on $X$ is a homomorphism of semi-group $\varphi : \mathcal{C}_*(X) \to \mathbb{R}/\mathbb{Z}$ such that it is restriction to the boundary is given by the formula

$$\varphi(\partial(M, E, \phi)) = \int_M \phi^*(\omega) \text{Ch}(E) \text{Td}(M) \mod \mathbb{Z},$$

where $\omega$ is a closed form on $X$ with integer $K$-periods [8], $\text{Ch}(E)$ is the Chern form of the connection $\nabla_E$ and $\text{Td}(M)$ is the Todd form of the tangent bundle of $M$. This can be assembled into a group which is denoted by $\hat{K}^*(X)$ and called the group of differential $K$-characters. We showed then that many secondary invariants can be expressed as a differential $K$-characters, and the group $K^*(X, \mathbb{R}/\mathbb{Z})$ of $K$-theory of $X$ with coefficients in $\mathbb{R}/\mathbb{Z}$ [2] is injected in $\hat{K}^*(X)$.

The aim of this work is to define the group $\hat{K}^*(\rho)$ of relative differential $K$-characters associated with a smooth map $\rho : Y \to X$ between two smooth compact manifolds $Y$ and $X$ following [9] [12] and [13]. We show that this group fits into a short exact sequence as in the non-relative case. The paper is organized as follows:
In Section 2 we define a group of relative geometric $K$-homology $K_\ast (\rho )$ adapted to this situation and study some of its properties. This generalizes the works of Baum–Douglas [6] for $Y$ a submanifold of $X$. Section 3 is concerned with the definition and the study of the group $\hat{K}^\ast (\rho )$ of relative differential $K$-characters. An odd relative group $K^{-1}(\rho ; \mathbb{R}/\mathbb{Z})$ of $K$-theory with coefficients in $\mathbb{R}/\mathbb{Z}$ is also defined here. We prove the following short exact sequence

$$0 \to K^{-1}(\rho ; \mathbb{R}/\mathbb{Z}) \hookrightarrow \hat{K}^{-1}(\rho ) \xrightarrow{\delta_1} \Omega_0^{\text{even}}(\rho ) \to 0,$$

where $\Omega_0^{\text{even}}(\rho )$ is the group of relative differential forms (Definition 5) with integer $K$-periods. We show then that some secondary geometric invariants can be expressed in this theory.

2 Relative geometric $K$-homology

Let $Y$ and $X$ be smooth compact manifolds and $\rho : Y \to X$ a smooth map. In this section, we define the relative geometric $K$-homology $K_\ast (\rho )$ for the triple $(\rho , Y, X)$. This construction generalizes the relative geometric $K$-homology group $K_\ast (X, Y)$ of Baum–Douglas for $Y$ being a closed submanifold of $X$. We recall the definition of the geometric $K$-homology of a smooth manifold following the works of Baum and Douglas. This definition is purely geometric. For a complete presentation see [3, 6] and [17].

**Definition 1.** A $K$-chain over $X$ is a triple $(M, E, \phi )$ such that:

- $M$ is a smooth Spin$^c$-compact manifold which may have non-empty boundary $\partial M$, and with a fixed Riemannian structure;
- $E$ is a Hermitian vector bundle over $M$ with a fixed Hermitian connection $\nabla^E$;
- $\phi : M \to X$ is a smooth map.

Denote that $M$ is not supposed connected and the fibres of $E$ may have different dimensions on the different connected components of $M$. Two $K$-chains $(M, E, \phi )$ and $(M', E', \phi ')$ are said to be isomorphic if there exists a diffeomorphism $\psi : M \to M'$ such that:

- $\phi ' \circ \psi = \phi$;
- $\psi^* E' \cong E$ as Hermitian bundles over $M$.

A $K$-cycle is a $K$-chain $(M, E, \varphi )$ without boundary; that is $\partial M = \emptyset$. The boundary $\partial (M, E, \varphi )$ of the $K$-chain $(M, E, \varphi )$ is the $K$-cycle $(\partial M, E|_{\partial M}, \varphi |_{\partial M})$. The set of $K$-chains is stable under disjoint union.

2.1 Vector bundle modification

Let $(M, E, \phi )$ be $K$-chain over $X$, and let $H$ be a Spin$^c$-vector bundle over $M$ with even dimensional fibers and a fixed Hermitian structure. Let $l = M \times \mathbb{R}$ be the trivial bundle and $\hat{M} = S(H \oplus l)$ the unit sphere bundle. Let $\rho : \hat{M} \to M$ the natural projection. The Spin$^c$-structure on $M$ and $H$ induces a Spin$^c$-structure on $\hat{M}$.

Let $S = S_- \oplus S_+$ be the $\mathbb{Z}/2\mathbb{Z}$-grading Clifford module associated with the Spin$^c$-structure of $H$. We denote by $H_0$ and $H_1$ the pullback of $S_-$ and $S_+$ to $H$. Then $H$ acts on $H_0$ and $H_1$ by Clifford multiplication: $H_0 \sigma H_1$.

The manifold $\hat{M}$ can be thought of as two copies, $B_0(H)$ and $B_1(H)$, of the unit ball glued together by the identity map of $S(H)$

$$\hat{M} = B_0(H) \cup_{S(H)} B_1(H).$$
The vector bundle \( \hat{H} \) on \( \hat{M} \) is obtained by putting \( H_0 \) on \( B_0(H) \) and \( H_1 \) on \( B_1(H) \) and then clutching these two vector bundles along \( S(H) \) by the isomorphism \( \sigma \).

The \( K \)-chain \( (M, \hat{H} \otimes \rho^*E, \hat{\phi} = \rho \circ \phi) \) is called the Bott \( K \)-chain associated with the \( K \)-chain \( (M, E, \phi) \) and the Spin\(^c\)-vector bundle \( H \).

The boundary of the Bott \( K \)-chain \( (M, \hat{H} \otimes \rho^*E, \hat{\phi}) \) associated with the \( K \)-chain \( (M, E, \phi) \) and the Spin\(^c\)-vector bundle \( H \) is the Bott \( K \)-cycle of the boundary \( \partial(M, E, \phi) \) with the restriction of \( H \) to \( \partial M \).

**Definition 2.** We denote by \( C_*(X) \) the set of equivalence classes of isomorphic \( K \)-cycles over \( X \) up to the following identifications:

- we identify the disjoint union \( (M, E, \phi) \amalg (M, E', \phi) \) with the \( K \)-cycle \( (M, E \oplus E', \phi); \)
- we identify a \( K \)-cycle \( (M, E, \phi) \) with the Bott \( K \)-cycle \( (\hat{M}, \hat{H} \otimes \rho^*E, \hat{\phi}) \) associated with any Hermitian vector bundle \( H \) over \( M \).

We can easily show that disjoint union then respects these identifications and makes \( C_*(X) \) into an Abelian semi-group which splits into \( C_0(X) \oplus C_1(X) \) with respect to the parity of the connected components of the manifolds in (the equivalence classes of) the \( K \)-cycles.

**Definition 3.** Two \( K \)-cycles \( (M, E, \phi) \) and \( (M', E', \phi') \) are bordant if there exists a \( K \)-chain \( (\overline{N}, \mathcal{E}, \psi) \) such that

\[
\partial(\overline{N}, \mathcal{E}, \psi) \quad \text{is isomorphic to} \quad (M, E, \phi) \amalg (-M', E', \phi'),
\]

where \(-M' \equiv M' \) with the Spin\(^c\)-structure reversed [5].

The above bordism relation induces a well defined equivalence relation on \( C_*(X) \) that we denote by \( \sim_\partial \). The quotient \( C_*(X)/\sim_\partial \) turns out to be an Abelian group for the disjoint union. The inverse of \((M, E, \phi) \sim_\partial (\overline{N}, \mathcal{E}, \psi) \) is \((-M, E, \phi) \).

**Definition 4** (Baum–Douglas). The quotient group of \( C_*(X) \) by the equivalence relation \( \sim_\partial \) is denoted by \( K_*(X) \) and is called the geometric \( K \)-homology group of \( X \). It can be decomposed into

\[
K_*(X) = K_0(X) \oplus K_1(X).
\]

A smooth map \( \varphi : Y \to X \) induces a group morphism

\[
\varphi_* : K_*(Y) \to K_*(X),
\]

given by \( \varphi_*(f)(M, E, \phi) = f(M, E, \varphi \circ \phi) \). The \( K_\ast \) is a covariant functor from the category of smooth compact manifolds and smooth maps to that of Abelian groups and group homomorphisms.

In the same way we can form a semi-group \( \mathcal{L}_*(X) \) out of \( K \)-chains \( (\overline{N}, \mathcal{E}, \psi) \), say with the same definition as \( C_*(X) \) and the boundary \( \partial(\overline{N}, \mathcal{E}, \psi) = (\partial \overline{N}, \mathcal{E}|_{\partial \overline{N}}, \psi \circ i) \), where \( i : \partial \overline{N} \to \overline{N} \). This gives a well defined map \( \partial : \mathcal{L}_*(X) \to C_*(X) \subset \mathcal{L}_*(X) \).

The Hermitian structure of the complex vector bundle \( \mathcal{E}|_{\partial \overline{N}} \) is the restricted one.

The group of \( K \)-cochains with coefficients in \( \mathbb{Z} \) denoted by \( \mathcal{L}^*_*(X) \) is the group of semi-group homomorphisms \( f \) from \( \mathcal{L}_*(X) \) to \( \mathbb{Z} \). On the group \( \mathcal{L}^*_*(X) \) there is a coboundary map defined by transposition \( \delta(f)(\overline{N}, \mathcal{E}, \psi) = f(\partial(\overline{N}, \mathcal{E}, \psi)) \).

The set of \( K \)-cocycles is the subset \( \mathcal{C}^*_*(X) \) of \( \mathcal{L}^*_*(X) \) of those \( K \)-cochains that vanish on boundaries, i.e. the kernel of \( \delta \). The set of \( K \)-coboundaries is the image of \( \delta \) in \( \mathcal{L}^*_*(X) \).
2.2 The relative geometric group $K_\ast (\rho)$

Let $Y$ and $X$ be smooth compact manifolds and $\rho : Y \to X$ a smooth map.

The set $\mathcal{L}_\ast (\rho)$ of relative $K$-chains associated with the triple $(\rho, Y, X)$ is by definition

$$\mathcal{L}_{\ast +1}(\rho) = \mathcal{L}_{\ast +1}(X) \times \mathcal{L}_\ast (Y).$$

The boundary $\partial : \mathcal{L}_{\ast +1}(\rho) \to \mathcal{L}_\ast (\rho)$ is given by

$$\partial (\sigma, \tau) = (\partial \sigma + \rho \tau, -\partial \tau).$$

We will denote by $\mathcal{C}_\ast (\rho)$ the set of relative $K$-cycles in $\mathcal{L}_\ast (\rho)$, i.e., the kernel of $\partial$. A $K$-cycle in $\mathcal{L}_\ast (\rho)$ is then a pair $(\sigma, \tau)$ where $\tau$ is a $K$-cycle over $Y$ and $\sigma$ is $K$-chain over $X$ with boundary in the image of $\rho : \mathcal{C}_\ast (Y) \to \mathcal{C}_\ast (X)$. The set $\mathcal{C}_\ast (\rho)$ is a semi-group for the sum

$$(\sigma, \tau) + (\sigma', \tau') = (\sigma \cup \sigma', \tau \cup \tau'),$$

where $\cup$ is the disjoint union. We say that two relatives $K$-cycles $(\sigma, \tau)$ and $(\sigma', \tau')$ are bordant and we write $(\sigma, \tau) \sim_\partial (\sigma', \tau')$ if there exists a relative $K$-chain $(\overline{\sigma}, \overline{\tau})$ such that

$$\partial (\overline{\sigma}, \overline{\tau}) = (\sigma, \tau) + (-\sigma', -\tau'),$$

where $-x$ denotes the relative $K$-cycle $x$ with the reversed Spin$^c$-structure of the underlying manifold.

**Definition 5.** The relative geometric $K$-homology group denoted by $K_\ast (\rho)$ is the quotient group $\mathcal{C}_\ast (\rho) \sim_\partial$.

The inverse of the $K$-cycle $x$ is $-x$. The equivalence relation on the relative $K$-cycle $(\sigma, \tau)$ preserves the dimension modulo 2 of the $K$-cycles $\sigma$ and $\tau$. Hence, there is a direct sum decomposition

$$K_\ast (\rho) = K_0(\rho) \oplus K_1(\rho).$$

The construction of the group $K_\ast (\rho)$ is functorial in the sense that for a commutative diagram

$$\begin{array}{ccc}
Y & \xrightarrow{\rho} & X \\
\downarrow f & & \downarrow g \\
Y' & \xrightarrow{\rho'} & X'
\end{array}$$

the map $F_\ast = (f_\ast, g_\ast) : \mathcal{L}_\ast (\rho) \to \mathcal{L}_\ast (\rho')$ is compatible with the equivalence relation on the relative $K$-cycles and induces a homomorphism from $K_\ast (\rho)$ to $K_\ast (\rho')$. As in the homology theory, we have the long exact sequence for the triple $(\rho, Y, X)$

$$\begin{array}{ccc}
K_0(Y) & \xrightarrow{\rho_\ast} & K_0(X) \\
\downarrow \partial & & \downarrow \partial \\
K_1(\rho) & \xleftarrow{\varsigma_\ast} & K_1(X) \xrightarrow{\rho_\ast} K_1(Y)
\end{array}$$

The boundary map $\partial$ associates to a relative $K$-cycle $(\sigma, \tau)$ the cycle $\tau$ whose image $\rho_\ast \tau$ is a boundary in $X$ and $\varsigma_\ast (\sigma) = (\sigma, 0)$. The exactness of the diagram is an easy check.

There is a differential $\delta$ on the group $\mathcal{L}^\ast (\rho) = \text{Hom}(\mathcal{L}_\ast (\rho), \mathbb{Z})$ given by

$$\delta (h, e) = (\delta h, \rho^\ast h - \delta e).$$

The relative Baum–Douglas $K$-group is

$$K^\ast (\rho) = \frac{\ker (\delta : \mathcal{L}^\ast (\rho) \to \mathcal{L}^{\ast +1}(\rho))}{\text{Im}(\delta : \mathcal{L}^{\ast -1}(\rho) \to \mathcal{L}^\ast (\rho))}.$$
Remark 1. The relative topological $K$-homology group $K^*_\rho(X,Y)$ can be constructed in the same way for normal topological spaces $X$ and $Y$, and $\rho: Y \to X$ is a continuous map. Let $K^*_\rho(X)$ be the relative topological $K$-homology group defined by Baum–Douglas in [6] for $Y \subset X$ is a closed subset of a $X$. We can easily show that $K^*_\rho(X,Y) = K^*_\rho(Y)$, where $\rho$ is the inclusion of $Y$ in $X$.

3 Relative differential $K$-characters

This section is concerned with the definition and the study the notion of relative differential $K$-characters [8]. This is a $K$-theoretical version of the works of [9, 12] and [13].

Let $X$ be a smooth compact manifold. The graded differential complex of real differential forms on the manifold $X$ will be denoted by

$$
\Omega^s(X) = \oplus_{k \geq 0} \Omega^k(X), \quad \Omega^k(X) \xrightarrow{d} \Omega^{k+1}(X) \quad \text{with} \quad d^2 = 0,
$$

where $d$ denotes the de Rham differential on $X$.

Furthermore, we denote by $\Omega^s_0(X)$ the subgroup of closed forms on the manifold $X$ with integer $K$-periods [8].

In the remainder of this section we fix $\rho: Y \to X$ a smooth map and we consider the complex

$$
\Omega^s(\rho) = \Omega^s(X) \times \Omega^{s-1}(Y)
$$

with differential $\delta(\omega, \theta) = (d\omega, \rho^* \omega - d\theta)$.

We can view $\Omega^s(\rho)$ as a subgroup of the the group $\text{Hom}(L^s(\rho), \mathbb{R})$ via integration

$$(\omega, \theta)(\sigma, \tau) = \omega(\sigma) + \theta(\tau),$$

where for $\sigma = (M, E, f)$ and $\tau = (N, F, g)$

$$
\omega(\sigma) = \int_M f^*(\omega) \text{Ch}(E) \text{Td}(M) \quad \text{and} \quad \theta(\tau) = \int_N g^*(\theta) \text{Ch}(F) \text{Td}(N).
$$

Let

$$j: \Omega^s(\rho) \to \text{Hom}(L^s(\rho), \mathbb{R})$$

such that

$$j(\omega, \theta)(\sigma, \tau) = \omega(\sigma) + \theta(\tau).$$

Definition 6. Let $(\omega, \theta) \in \Omega^s(\rho)$ be a pair of real differential forms.

(i) The set of $K$-periods of $(\omega, \theta)$ is the subset of $\mathbb{R}$ image of the map $j(\omega, \theta)$ restricted to $C_s(\rho)$.

(ii) We denoted by $\Omega^s_0(\rho)$ the set of differential forms $(\omega, \theta)$ of integer $K$-periods.

$\Omega^s_0(\rho)$ is an Abelian group for the sum of differential forms.

Lemma 1. Let $(\omega, \theta) \in \Omega^s_0(\rho)$. Then

1) $\delta(\omega, \theta) = 0$ in the complex $\Omega^s(\rho)$;

2) $\omega \in \Omega^0_0(X)$. 

Example 2. An interesting situation is obtained by differential forms. If \((\omega, \theta) \in \Omega^*\) such that \(\Delta \omega = 0\), then we define \(f_{(\omega, \theta)}\) by letting for \(\sigma = (M, E, f)\) and \(\tau = (N, F, g)\)

\[
f_{(\omega, \theta)}(\sigma, \tau) = \left[ \int_M f^* (\omega) \text{Ch}(E) \text{Td}(M) \right] + \left[ \int_N g^* (\theta) \text{Ch}(F) \text{Td}(N) \right].
\]

We have

\[
\delta_1 (f_{(\omega, \theta)}) = (d\omega, \rho^* \omega - d\theta).
\]
Example 3. Suppose \( Y \) be submanifold of \( X \) and \( \rho : Y \hookrightarrow X \) is a smooth inclusion. Let \( \omega \in \Omega^\ast(X) \) with trivial restriction to \( Y \) and \( f_\omega \in \hat{K}(X) \) – the associated differential \( K \)-character \([8]\). Let \( \psi \in \hat{K}(Y) \) be any differential \( K \)-character on \( Y \). We have \( f_\omega(\mathcal{C}_sY) = 0 \). The map \( \phi_{\omega,\psi} \) defined on \( \mathcal{C}_s(\rho) \) by

\[
\phi_{\omega,\psi}(\sigma, \tau) = f_\omega(\sigma) + \psi(\tau)
\]

is a relative differential \( K \)-character with \( \delta_1(\phi_{\varphi,\psi}) = (d\omega, -\delta_1(\psi)) \).

3.1 Relative \( \mathbb{R}/\mathbb{Z} \)-\( K \)-theory

Let \( X \) be a smooth manifold, \( E \) a Hermitian vector bundle on \( X \) and \( \nabla^E \) a Hermitian connection on \( E \). The geometric Chern form \( Ch(\nabla^E) \) of \( \nabla^E \) is the closed real even differential form given by

\[
Ch(\nabla^E) = \text{tr} e^{-\frac{(\nabla^E)^2}{2\pi}}.
\]

The cohomology class of \( Ch(\nabla^E) \) does not depend on the choice of the connection \( \nabla^E \) \([14]\). Let \( \nabla_1^E \) and \( \nabla_2^E \) be two Hermitian connections on \( E \). There is a well defined Chern–Simons form \([14]\) \( CS(\nabla_1^E, \nabla_2^E) \in \Omega^{\text{odd}}(X) \otimes \mathbb{C} \) such that

\[
dCS(\nabla_1^E, \nabla_2^E) = Ch(\nabla_1^E) - Ch(\nabla_2^E),
\]

and

\[
CS(\nabla_1^E, \nabla_3^E) = CS(\nabla_1^E, \nabla_2^E) + CS(\nabla_2^E, \nabla_3^E).
\]

Given a short exact sequence of complex Hermitian vector bundles on \( X \)

\[
0 \to E_1 \xrightarrow{i} E_2 \xrightarrow{j} E_3 \to 0,
\]

and choose a splitting map \( s : E_3 \to E_2 \). Then \( i \circ s : E_1 \oplus E_3 \to E_2 \) is an isomorphism. For \( \nabla^{E_1}, \nabla^{E_2} \) and \( \nabla^{E_3} \) are Hermitian connection on \( E_1, E_2 \) and \( E_3 \) respectively, we set

\[
CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) = CS((i \circ s)^*\nabla^{E_2}, \nabla^{E_1} \oplus \nabla^{E_3}).
\]

The form \( CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) \) is independent of the choice of the splitting map \( s \) and

\[
dCS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) = Ch(\nabla_2^E) - Ch(\nabla_1^E) - Ch(\nabla_3^E).
\]

Definition 8. Let \( X \) be a smooth manifold. A \( \mathbb{R}/\mathbb{Z} \)-\( K \)-generator of \( X \) is a quadruple

\[
\mathcal{E} = (E, h^E, \nabla^E, \omega),
\]

where \( E \) is a complex vector bundle on \( X \), \( h^E \) is a positive definite Hermitian metric on \( E \), \( \nabla^E \) is a Hermitian connection on \( E \), \( \omega \in \Omega^{\text{odd}}(X) \text{ im}(d) \) which satisfies \( d\omega = Ch(\nabla^E) - \text{rank}(E) \), where \( \text{rank}(E) \) is the rank of \( E \).

An \( \mathbb{R}/\mathbb{Z} \)-\( K \)-relation is given by three \( \mathbb{R}/\mathbb{Z} \)-\( K \)-generators \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \), along with a short exact sequence of Hermitian vector bundles

\[
0 \to E_1 \xrightarrow{i} E_2 \xrightarrow{j} E_3 \to 0,
\]

such that \( \omega_2 = \omega_1 + \omega_3 + CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) \).
The notation $\bar{\tau}_E$ sequence is split. Let

**Proposition 1.**

Since the image of $\rho, h, E, \omega$ and let the exact sequence

\[
\begin{array}{cccc}
K^0(Y, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\rho^*} & K^0(X, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\phi^*} \text{Hom}(K_0(\rho), \mathbb{R}/\mathbb{Z}) \\
\downarrow \partial_0^* & & \downarrow \partial_1^* & \\
\text{Hom}(K_1(\rho), \mathbb{R}/\mathbb{Z}) & \xrightarrow{\phi^*} & K^{-1}(X, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\rho^*} K^{-1}(Y, \mathbb{R}/\mathbb{Z})
\end{array}
\]

obtained from the one in p. 4 and after identification of the groups $K^*(Y, \mathbb{R}/\mathbb{Z})$ and $\text{Hom}(K_*(Y), \mathbb{R}/\mathbb{Z})$ following Proposition 4 of [3].

We denote by $\tilde{L}^{-1}(\rho, \mathbb{R}/\mathbb{Z})$ the subgroup of $\text{Hom}(K_1(\rho), \mathbb{R}/\mathbb{Z})$ image of the morphism $\partial_0^*$ and $\tilde{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z})$ the subgroup of $K^{-1}(X, \mathbb{R}/\mathbb{Z})$ the kernel of the morphism $\rho_1^*$.

**Definition 10.** Let $Y$ and $X$ be two smooth compact manifolds and $\rho : Y \to X$ a smooth map. The group $K^{-1}(\rho, \mathbb{R}/\mathbb{Z})$ is by definition the product of the groups $\tilde{L}^{-1}(\rho, \mathbb{R}/\mathbb{Z})$ and $\tilde{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z})$

\[K^{-1}(\rho, \mathbb{R}/\mathbb{Z}) = \tilde{L}^{-1}(\rho, \mathbb{R}/\mathbb{Z}) \times \tilde{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z}).\]

**Proposition 1.** The groups $K^{-1}(\rho, \mathbb{R}/\mathbb{Z})$ and $\text{Hom}(K_{-1}(\rho), \mathbb{R}/\mathbb{Z})$ are isomorphic.

**Proof.** Since the image of $\phi^*$ is the kernel of $\rho_1^*$, it is enough to show that the short exact sequence

\[0 \to \tilde{L}^{-1}(\rho, \mathbb{R}/\mathbb{Z}) \hookrightarrow \text{Hom}(K_1(\rho), \mathbb{R}/\mathbb{Z}) \xrightarrow{\phi^*} \tilde{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z}) \to 0\]

is split. Let $E$ be an element of $K^{-1}(\rho, \mathbb{R}/\mathbb{Z})$ and $(E_{\pm}, h_{E_{\pm}}, \nabla_{E_{\pm}}, \omega)$ be a relative $\mathbb{Z}/2\mathbb{Z}$-graded cocycle associated to $E$. Let $(\sigma, \tau)$ be a relative $K$-cycle in $C_{-1}(\rho)$. For $\sigma = (M, E, \phi)$ and $\tau = (N, F, \psi)$ we set

\[\alpha(\sigma, \tau)) = \tilde{\eta}_{\phi^* E_{\pm} \otimes E} - \tilde{\eta}_{\phi^* E_{\mp} \otimes E} - \tilde{f}_\omega(\sigma),\]

where the notation $\tilde{\eta}_V = \eta(D_V) + \text{dim} \ker D_V \mathbb{Z}$ is the reduced eta invariant [2] of Atiyah–Patodi–Singer of the Dirac operator associated to the Spin$^c$-structure of $M$ with coefficients in the vector bundle $V$ and

\[\tilde{f}_\omega(M, E, \phi) = \left[ \int_M \phi^* (\omega) \text{Ch}(E) \text{Td}(M) \right].\]

Let us check that $\alpha(\sigma, \tau)) = 0$ in $\mathbb{R}/\mathbb{Z}$ for any $K$-chain $\sigma$ over $X$ and any $K$-chain $\tau$ over $Y$. Recall that $\partial(\sigma, \tau) = (\partial \sigma + \rho^* \tau, -\partial \tau)$. Furthermore, the invariant $\tilde{\eta}$ and $\tilde{f}_\omega$ defines $K$-cochains over $X$ [3]. We have then

\[\alpha(\sigma, \tau)) = \alpha(\partial \sigma, -\partial \tau) + \alpha(\rho^* \tau, 0).\]

The index theorem of APS (see [1] [2] [3]) implies that

\[\tilde{\eta}(\phi^* E_{\pm} \otimes E) | \partial M - \tilde{\eta}(\phi^* E_{-} \otimes E) | \partial M - \tilde{f}_\omega(\sigma) = \text{ind}(D_+ \otimes \phi^* E_{\pm} \otimes E) - \text{ind}(D_+ \otimes \phi^* E_{-} \otimes E),\]
is entire, where \( \text{ind}(D_+ \otimes \phi^* E_+ \otimes E) \) is the index of the Dirac type operator associated to the Spin\(^c\)-structure of \( M \) with coefficients in the bundle \( \phi^* E_+ \otimes E \). On the other hand, we have
\[
\alpha(\mathcal{E})(\rho^* \tau, 0)) = \alpha(\rho^* \mathcal{E})(0, \tau) = 0.
\]

This construction defines a homomorphism \( \alpha : \hat{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z}) \to \text{Hom}(K_1(\rho), \mathbb{R}/\mathbb{Z}) \) which is a split of \( \varsigma_1 \). In fact, let us consider the following commutative diagram
\[
\begin{array}{ccc}
\hat{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\alpha} & \text{Hom}(K_1(\rho), \mathbb{R}/\mathbb{Z}) \\
\downarrow i^* & & \downarrow \varsigma_1 \\
K^{-1}(X, \mathbb{R}/\mathbb{Z}) & \xrightarrow{\alpha_X} & \text{Hom}(K_1(X), \mathbb{R}/\mathbb{Z})
\end{array}
\]
where \( i^* \) is the embedding of \( \hat{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z}) \) in \( K^{-1}(X, \mathbb{R}/\mathbb{Z}) \) and \( \alpha_X \) is the restriction of \( \alpha \) to \( C_*(X) \times \{0\} \) which is an isomorphism \([14]\). For any \( \mathcal{E} \in \hat{K}^{-1}(\rho, \mathbb{R}/\mathbb{Z}) \), we have \( \varsigma_1(\alpha(\mathcal{E})) = i^*(\mathcal{E}) = \mathcal{E} \).

**Theorem 1.** The following sequence is exact:
\[
0 \to K^{-1}(\rho, \mathbb{R}/\mathbb{Z}) \hookrightarrow \hat{K}^{-1}(\rho) \xrightarrow{\delta_1} \Omega_{0}^{\text{even}}(\rho) \to 0.
\]

**Proof.** From Proposition \([11]\) \( K^{-1}(\rho, \mathbb{R}/\mathbb{Z}) \) is isomorphic to \( \text{Hom}(K_1(\rho), \mathbb{R}/\mathbb{Z}) \) which obviously injects in \( \hat{K}^{-1}(\rho) \) with trivial \( \delta_1 \). It is clear that a relative differential \( K \)-character \( f \) with \( \delta_1(f) = 0 \), induces a homomorphism from \( K_1(\rho) \) to \( \mathbb{R}/\mathbb{Z} \). Hence the sequence is exact at \( K^{-1}(\rho) \). It remains to show the surjectivity of \( \delta_1 \).

Let \( (\omega, \theta) \in \Omega_{0}^{\text{even}}(\rho) \) and \( f_{\omega, \theta} : \mathcal{C}_*(\rho) \to \mathbb{R}/\mathbb{Z} \) defined by
\[
f_{\omega, \theta}(\sigma, \tau) = f_\omega(\sigma) + f_\theta(\tau).
\]

The map \( f_{\omega, \theta} \) is trivial on \( \mathcal{C}_1(\rho) \). Therefore, we define an element \( \chi \in \text{Hom}(\mathcal{B}_1(\rho), \mathbb{R}/\mathbb{Z}) \) by setting
\[
\chi(\partial(\sigma, \tau)) = f_{\omega, \theta}(\sigma, \tau),
\]
where \( \mathcal{B}_1(\rho) \) is the image of the boundary map \( \partial : \mathcal{L}_0(\rho) \to \mathcal{C}_1(\rho) \).

Since \( \mathbb{R}/\mathbb{Z} \) is divisible, \( \chi \) can be extended to a relative differential \( K \)-character \( \chi : \mathcal{C}_1(\rho) \to \mathbb{R}/\mathbb{Z} \) with \( \delta_1(\chi) = (\omega, \theta) \).

**3.2 Application**

Let \( G \) be an almost connected Lie group and \( M \) be a smooth compact manifold. Let \( \pi : Y \to M \) be a compact principal \( G \)-bundle with connection \( \nabla \). We denote by \( I^*(G) \) the ring of symmetric multilinear real functions on the Lie algebra of \( G \) which are invariant under the adjoint action of \( G \) \([11]\). Let \( \Omega \) be the curvature form. For any \( P \in I^*(G) \), there is a well defined closed form \( P(\Omega) \) on \( M \). The pullback \( \pi^* P(\Omega) \) is an exact form on \( Y \) \([11]\). For \( P \in I^*(G) \), let \( TP(\nabla) \) be such that \( \pi^* P(\Omega) = dTP(\nabla) \). The form \( \omega = (\pi^* P(\Omega), dTP(\nabla)) \) is a closed form in the complex \( (\Omega^*(\pi), \delta) \). The relative differential \( K \)-character \( f_\omega \) has a trivial \( \delta_1 \) and defines consequently an element of the group \( K^{-1}(\pi, \mathbb{R}/\mathbb{Z}) \). This gives an additive map from \( I^*(G) \) to \( K^{-1}(\pi, \mathbb{R}/\mathbb{Z}) \) which can be looked as a home of secondary geometric invariants of the principal \( G \)-bundle with connection \( (M, Y, \nabla) \) analogous to the Chern–Weil theory.

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References

[1] Atiyah M.F., Patodi V.K., Singer I.M., Spectral asymmetry and Riemannian geometry. I, *Math. Proc. Cambridge Philos. Soc.* **77** (1975), 43–69.

[2] Atiyah M.F., Patodi V.K., Singer I.M., Spectral asymmetry and Riemannian geometry. II, *Math. Proc. Cambridge Philos. Soc.* **78** (1975), 405–432.

[3] Atiyah M.F., Patodi V.K., Singer I.M., Spectral asymmetry and Riemannian geometry. III, *Math. Proc. Cambridge Philos. Soc.* **79** (1976), 71–99.

[4] Asahawa T., Surgimoto S., Terashima S., D-branes, matrix theory and $K$-homology, *J. High Energy Phys.* **2002** (2002), no. 3, 034, 40 pages, hep-th/0108085.

[5] Baum P., Douglas R., $K$-homology and index theory, in Operator Algebras and Applications, *Proc. Sympos. Pure Math.*, Vol. 38, Amer. Math. Soc., Providence, R.I., 1982, 117–173.

[6] Baum P., Douglas R., Relative $K$-homology and $C^*$-algebras, *K-theory* **5** (1991), 1–46.

[7] Bunke U., Turner P., Willerton S., Gerbes and homotopy quantum field theories, *Algebr. Geom. Topol.* **4** (2004), 407–437.

[8] Benameur M.T., Maghfoul M., Differential characters in $K$-theory, *Differential Geom. Appl.* **24** (2006), 417–432.

[9] Brightwell M., Turner P., Relative differential characters, *Comm. Anal. Geom.* **14** (2006), 269–282, math.AT/0408333.

[10] Cheeger J., Simons J., Differential characters and geometric invariants, in Geometry and Topology (1983/84, College Park, Md.), *Lecture Notes in Math.*, Vol. 1167, Springer, Berlin, 1985, 50–80.

[11] Chern S.S., Simons J., Characteristic forms and geometric invariants, *Ann. of Math. (2)* **79** (1974), 48–69.

[12] Harvey R., Lawson B., Lefschetz–Pontrjagin duality for differential characters, *An. Acad. Brasil. Ciênc.* **73** (2001), 145–159.

[13] Hopkins M.J., Singer I.M., Quadratic functions in geometry, topology and M-theory, *J. Differential Geom.* **70** (2005), 329–452, math.AT/0211216.

[14] Lott J., $\mathbb{R}/\mathbb{Z}$-index theory, *Comm. Anal. Geom.* **2** (1994), 279–311.

[15] Lupercio E., Uribe B., Differential characters on orbifolds and string connection. I. Global quotients, in Gromov–Witten Theory of Spin Curves and Orbifolds (May 3–4, 2003, San Francisco, CA, USA), Editors T.J. Jarvis et al., Amer. Math. Soc., Providence, *Contemp. Math.* **403** (2006), 127–142, math.DG/0311008.

[16] Periwal V., $D$-branes charges and $K$-homology, *J. High Energy Phys.* **2000** (2000), no. 7, 041, 6 pages, hep-th/9805170.

[17] Reis R.M., Szabo R.J., Geometric $K$-homology of flat $D$-branes, *Comm. Math. Phys.* **266** (2006), 71–122, hep-th/0507043.