A ratio of finitely many gamma functions and its properties with applications

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Abstract
In the paper, the authors establish an inequality involving exponential functions and sums, introduce a ratio of finitely many gamma functions, discuss properties, including monotonicity, logarithmic convexity, (logarithmically) complete monotonicity, and the Bernstein function property, of the newly introduced ratio, and construct two inequalities of multinomial coefficients and multivariate beta functions.

Keywords  Ratio · Gamma function · Bernstein function · Completely monotonic function · Logarithmically completely monotonic function · Inequality · Multinomial coefficient · Multivariate beta function · Logarithmic derivative · Logarithmic convexity · Integral representation · Open problem

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In memory of the first author’s mother, Ji-Rong Zhang, who passed away in December 1995.

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1 Preliminaries

A real-valued function \( F(x) \) defined on a finite or infinite interval \( I \subseteq \mathbb{R} \) is said to be completely monotonic on \( I \) if and only if \((-1)^k F^{(k)}(x) \geq 0 \) for all \( k \in \{0\} \cup \mathbb{N} \) and \( x \in I \). A positive function \( F(x) \) defined on a finite or infinite interval \( I \subseteq \mathbb{R} \) is said to be logarithmically completely monotonic on \( I \) if and only if \((-1)^k [\ln F(x)]^{(k)} \geq 0 \) for all \( k \in \mathbb{N} \) and \( x \in I \). A nonnegative function \( F(x) \) defined on a finite or infinite interval \( I \) is called a Bernstein function if its derivative \( F'(x) \) is completely monotonic on \( I \). In the paper [2] and the monograph [41, pp. 66–68, Comments 5.29], it is pointed out that the terminology “logarithmically completely monotonic function” was explicitly defined in [23, 24] for the first time. The logarithmically complete monotonicity is weaker than the Stieltjes function, but stronger than the complete monotonicity [2, 10, 29]. For more information on this topic, please refer to [22, 41, 43] and closely related references therein.

Recall from [42, p. 51, (3.9)] that the classical Euler gamma function \( \Gamma(z) \) can be defined by

\[
\Gamma(z) = \lim_{n \to \infty} \frac{n! z^n}{(z)_n},
\]

where \( z \neq 0, -1, -2, \ldots \) and

\[
(z)_n = \prod_{\ell=0}^{n-1} (z + \ell) = \begin{cases} 
(z + 1) \cdots (z + n - 1), & n \geq 1 \\
1, & n = 0
\end{cases}
\]

for \( z \in \mathbb{C} \) and \( n \in \{0\} \cup \mathbb{N} \) is called the rising factorial. The logarithmic derivative \( \psi(x) = [\ln \Gamma(x)]' = \frac{\Gamma'(x)}{\Gamma(x)} \) of the gamma function \( \Gamma(z) \) and \( \psi^{(k)} \) for \( k \in \mathbb{N} \) are usually called in sequence the digamma function, the trigamma function, the tetragamma function, and the like.

With the help of the gamma function \( \Gamma(z) \), the binomial coefficient \( \binom{m}{n} = \frac{m!}{n!(m-n)!} \) can be generalized as the multinomial coefficient

\[
\binom{\alpha_1 + \alpha_2 + \cdots + \alpha_m}{\alpha_1, \alpha_2, \ldots, \alpha_m} = \frac{\Gamma(1 + \sum_{i=1}^{m} \alpha_i)}{\prod_{i=1}^{m} \Gamma(1 + \alpha_i)} \frac{1}{\prod_{i=1}^{m} \Gamma(\alpha_i)},
\]

and the classical beta function \( B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \) can be generalized as the multivariate beta function

\[
B(\alpha_1, \alpha_2, \ldots, \alpha_m) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_m)}{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_m)},
\]

where \( \mathfrak{M}(\alpha_1), \mathfrak{M}(\alpha_2), \ldots, \mathfrak{M}(\alpha_m) \geq 0 \). See [1, Section 24.1.2] and [9, Section II.2].

2 Motivation

Motivated by the papers [14, 34] and related texts in the survey article [22], by establishing the inequality

\[
\sum_{i=1}^{m} \frac{1}{x^{1/v_i} - 1} + \sum_{j=1}^{n} \frac{1}{x^{1/\tau_j} - 1} > \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{x^{1/\kappa_{ij}} - 1},
\]
where \( x > 1, 0 < \lambda_{ij} \leq 1, \)

\[
v_i = \sum_{j=1}^{n} \lambda_{ij}, \quad \tau_j = \sum_{i=1}^{m} \lambda_{ij}, \quad \sum_{i=1}^{m} v_i = \sum_{j=1}^{n} \tau_j = 1,
\]

Ouimet obtained in [15] that the ratio

\[
g(t) = \frac{\prod_{i=1}^{m} \Gamma(v_i t + 1) \prod_{j=1}^{n} \Gamma(\tau_j t + 1)}{\prod_{i=1}^{m} \prod_{j=1}^{n} \Gamma(\lambda_{ij} t + 1)} \tag{2.2}
\]

is logarithmically completely monotonic function on \((0, \infty)\), where \(m, n \in \mathbb{N}\) and \(0 < \nu_i, \tau_j, \lambda_{ij} \leq 1\) such that

\[
\sum_{j=1}^{n} \lambda_{ij} = v_i, \quad \sum_{i=1}^{m} \lambda_{ij} = \tau_j, \quad \sum_{i=1}^{m} v_i = \sum_{j=1}^{n} \tau_j = \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} \lambda_{ij} \leq 1.
\]

We observe that

(1) the proof of the inequality (2.1) is lengthy and complicated;
(2) the inequality (2.1) can be refined and extended;
(3) in the third paragraph on [25, p. 12], Theorem 2.1 in [15] has been demonstrated to be wrong; because the variable \(t\) was missed inside three logarithms in the second line of the equation (8) on page 3 in the arXiv preprint [15], the proof of [15, Theorem 2.1] is wrong.

In this paper, we will

(1) refine and extend the inequality (2.1) and supply a concise proof of the refinement and extension;
(2) motivated by \(g(t)\) in (2.2), formulate a new ratio and prove its properties;
(3) construct inequalities of multinomial coefficients and multivariate beta functions.

3 A new inequality involving exponential functions and sums

Now we present a new inequality which refines and extends the inequality (2.1).

**Theorem 3.1** Let \( x > 0 \) and \( \lambda_{ij} > 0 \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Then

\[
\sum_{i=1}^{m} \frac{1}{e^{x/v_i} - 1} + \sum_{j=1}^{n} \frac{1}{e^{x/\tau_j} - 1} \geq 2 \sum_{i=1}^{m} \sum_{j=1}^{n} \frac{1}{e^{x/\lambda_{ij}} - 1}, \tag{3.1}
\]

where \( v_i = \sum_{j=1}^{n} \lambda_{ij} \) and \( \tau_j = \sum_{i=1}^{m} \lambda_{ij} \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \).

**Proof** Recall from [13, p. 650] that

(1) a function \( \varphi : [0, \infty) \to \mathbb{R} \) is said to be star-shaped if \( \varphi(\nu x) \leq \nu \varphi(x) \) for all \( \nu \in [0, 1] \) and \( x \geq 0 \);
(2) a real function \( \varphi \) defined on a set \( S \subset \mathbb{R}^n \) is said to be super-additive if \( x, y \in S \) implies \( x + y \in S \) and \( \varphi(x + y) \geq \varphi(x) + \varphi(y) \);
(3) if \( \varphi : [0, \infty) \to \mathbb{R} \) is star-shaped, then \( \varphi \) is super-additive;
(4) if \( \varphi \) is a real function defined on \([0, \infty), \varphi(0) \leq 0\), and \( \varphi \) is convex, then \( \varphi \) is star-shaped.
Let \( h(x) = \frac{1}{e^x - 1} \) for \( x > 0 \). Then the inequality (3.1) can be rearranged as
\[
\sum_{i=1}^{m} h\left(\frac{x}{\nu_i}\right) + \sum_{j=1}^{n} h\left(\frac{x}{\tau_j}\right) > 2 \sum_{i=1}^{m} \sum_{j=1}^{n} h\left(\frac{x}{\lambda_{ij}}\right). 
\] (3.2)

Direct computation gives
\[
\frac{d}{dx} h\left(\frac{1}{x}\right) = \frac{e^{1/x}}{(e^{1/x} - 1)^2 x^2},
\]
\[
\frac{d^2}{dx^2} h\left(\frac{1}{x}\right) = \frac{e^{1/x} \left[ e^{1/x} (1 - 2x) + 2x + 1 \right]}{(e^{1/x} - 1)^3 x^4},
\]
\[
\left[ e^{1/x} (1 - 2x) + 2x + 1 \right]' = 2 - e^{1/x} \left( \frac{1}{x^2} - \frac{2}{x} + 2 \right) \to 0, \quad x \to \infty,
\]
\[
\left[ e^{1/x} (1 - 2x) + 2x + 1 \right]' = \frac{e^{1/x} x^4}{x^4} > 0,
\]
\[
\lim_{x \to \infty} \left[ e^{1/x} (1 - 2x) + 2x + 1 \right]' = 0
\]
for \( x > 0 \). Consequently, combining these with \( \lim_{x \to 0^+} h\left(\frac{1}{x}\right) = 0 \) reveals that the function \( h\left(\frac{1}{x}\right) \) is convex, then star-shaped, and then super-additive on \((0, \infty)\). As a result, it follows that
\[
\frac{x}{\nu_i} = \frac{x}{\sum_{j=1}^{n} \lambda_{ij}} \geq \sum_{j=1}^{n} h\left(\frac{x}{\lambda_{ij}}\right)
\]
and
\[
\frac{x}{\tau_j} = \frac{x}{\sum_{i=1}^{m} \lambda_{ij}} \geq \sum_{i=1}^{m} h\left(\frac{x}{\lambda_{ij}}\right).
\]

Substituting these two inequalities into the left hand side of (3.2) results in
\[
\sum_{i=1}^{m} h\left(\frac{x}{\nu_i}\right) + \sum_{j=1}^{n} h\left(\frac{x}{\tau_j}\right) \geq \sum_{i=1}^{m} \sum_{j=1}^{n} h\left(\frac{x}{\lambda_{ij}}\right) + \sum_{j=1}^{n} \sum_{i=1}^{m} h\left(\frac{x}{\lambda_{ij}}\right)
\]
\[
\geq 2 \sum_{i=1}^{m} \sum_{j=1}^{n} h\left(\frac{x}{\lambda_{ij}}\right).
\]

The proof of the inequality (3.1), and then the proof of Theorem 3.1, is thus complete. \( \square \)

4 A new ratio of finitely many gamma functions and its properties

In this section, we formulate a new ratio of finitely many gamma functions and find its properties.

**Theorem 4.1** Let \( \rho \in \mathbb{R} \) and \( \lambda_{ij} > 0 \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), let \( v_i = \sum_{j=1}^{n} \lambda_{ij} \) and \( \tau_j = \sum_{i=1}^{m} \lambda_{ij} \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), and let
\[
f(t) = \frac{\prod_{i=1}^{m} \Gamma(1 + v_i t) \prod_{j=1}^{n} \Gamma(1 + \tau_j t)}{[\prod_{i=1}^{m} \prod_{j=1}^{n} \Gamma(1 + \lambda_{ij} t)]^\rho}.
\] (4.1)
Then the following conclusions are valid:

(1) when \( \rho \leq 2 \), the second derivative \( [\ln f(t)]'' \) is a completely monotonic function of \( t \in (0, \infty) \) and maps from \( (0, \infty) \) onto the open interval

\[
\left(0, \frac{\pi^2}{6} \left( \sum_{i=1}^{m} v_i^2 + \sum_{j=1}^{n} \tau_j^2 - \rho \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}^2 \right) \right);
\]

(2) when \( \rho = 2 \), the logarithmic derivative \( [\ln f(t)]' = \frac{f'(t)}{f(t)} \) is a Bernstein function of \( t \in (0, \infty) \) and maps from \( (0, \infty) \) onto the open interval

\[
\left(0, \ln \frac{\prod_{i=1}^{m} v_i \prod_{j=1}^{n} \tau_j}{\left( \prod_{i=1}^{m} \prod_{j=1}^{n} \lambda_{ij}^2 \right)^2} \right);
\]

(3) when \( \rho < 2 \), the logarithmic derivative \( [\ln f(t)]' \) is increasing, concave, and from \( (0, \infty) \) onto the open interval

\[
\left(-\gamma (2 - \rho) \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}, \infty \right),
\]

where \( \gamma = 0.57721566 \ldots \) is the Euler–Mascheroni constant;

(4) when \( \rho = 2 \), the function \( f(t) \) is increasing, logarithmically convex, and from \( (0, \infty) \) onto the open interval \( (1, \infty) \);

(5) when \( \rho < 2 \), the function \( f(t) \) has a unique minimum, is logarithmically convex, and satisfies \( \lim_{t \to 0^+} f(t) = 1 \) and \( \lim_{t \to \infty} f(t) = \infty \).

**Proof** Taking logarithm and differentiating give

\[
\ln f(t) = \sum_{i=1}^{m} \ln \Gamma(1 + v_i t) + \sum_{j=1}^{n} \ln \Gamma(1 + \tau_j t) - \rho \sum_{i=1}^{m} \sum_{j=1}^{n} \ln \Gamma(1 + \lambda_{ij} t),
\]

\[
[\ln f(t)]' = \sum_{i=1}^{m} v_i \psi(1 + v_i t) + \sum_{j=1}^{n} \tau_j \psi(1 + \tau_j t) - \rho \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} \psi(1 + \lambda_{ij} t),
\]

\[
[\ln f(t)]'' = \sum_{i=1}^{m} v_i^2 \psi'(1 + v_i t) + \sum_{j=1}^{n} \tau_j^2 \psi'(1 + \tau_j t) - \rho \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}^2 \psi'(1 + \lambda_{ij} t).
\]

It is easy to see that

\[
\lim_{t \to 0^+} [\ln f(t)]'' = \frac{\pi^2}{6} \left( \sum_{i=1}^{m} v_i^2 + \sum_{j=1}^{n} \tau_j^2 - \rho \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}^2 \right).
\]

Making use of the integral representation

\[
\psi^{(n)}(z) = (-1)^{n+1} \int_{0}^{\infty} \frac{r^n}{1 - e^{-rt}} e^{-zr} dr, \quad \Re(z) > 0
\]

in [1, p. 260, 6.4.1] leads to

\[
[\ln f(t)]'' = \sum_{i=1}^{m} v_i^2 \int_{0}^{\infty} \frac{s}{1 - e^{-s}} e^{-(1 + v_i t)s} ds + \sum_{j=1}^{n} \tau_j^2 \int_{0}^{\infty} \frac{s}{1 - e^{-s}} e^{-(1 + \tau_j t)s} ds.
\]
and completely monotonic with respect to $t$.

By virtue of Lemma 3.1, when $\rho \leq 2$, we conclude that the second derivative $[\ln f(t)]''$ is completely monotonic with respect to $t \in (0, \infty)$.

Since the second derivative $[\ln f(t)]''$ is completely monotonic with respect to $t \in (0, \infty)$, the logarithmic derivative $[\ln f(t)]'$ is increasing and concave on $(0, \infty)$, with the limits

$$\lim_{t \to 0^+} [\ln f(t)]' = -\gamma(2 - \rho) \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} = \left\{ \begin{array}{ll}
0, & \rho = 2 \\
-\gamma(2 - \rho) \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}, & \rho < 2
\end{array} \right.$$  

and

$$\lim_{t \to \infty} [\ln f(t)]' = \lim_{t \to \infty} \left[ \sum_{i=1}^{m} v_i \psi(1 + v_i t) + \sum_{j=1}^{n} \tau_j \psi(1 + \tau_j t) - \rho \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} \psi(1 + \lambda_{ij} t) \right]$$

$$= \sum_{i=1}^{m} v_i \lim_{t \to \infty} \left[ \psi(1 + v_i t) - \ln(1 + v_i t) \right]$$

$$+ \sum_{j=1}^{n} \tau_j \lim_{t \to \infty} \left[ \psi(1 + \tau_j t) - \ln(1 + \tau_j t) \right]$$

$$- \rho \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij} \lim_{t \to \infty} \left[ \psi(1 + \lambda_{ij} t) - \ln(1 + \lambda_{ij} t) \right]$$

$$+ \ln \lim_{t \to \infty} \frac{\prod_{i=1}^{m} (1 + v_i t)^{\psi} \prod_{j=1}^{n} (1 + \tau_j t)^{\psi}}{\prod_{i=1}^{m} \prod_{j=1}^{n} (1 + \lambda_{ij})^{\psi(\lambda_{ij})}}$$

$$= \ln \lim_{t \to \infty} \frac{\prod_{i=1}^{m} (1/t + v_i)^{\psi} \prod_{j=1}^{n} (1/t + \tau_j)^{\psi}}{\prod_{i=1}^{m} \prod_{j=1}^{n} (1/t + \lambda_{ij})^{\psi(\lambda_{ij})}}$$
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\[ + \ln \lim_{t \to \infty} t^{\sum_{i=1}^{m} v_i + \sum_{j=1}^{n} \tau_j - \rho \sum_{j=1}^{n} \lambda_{ij}} \]

\[ = \ln \left( \frac{\prod_{i=1}^{m} v_i \prod_{j=1}^{n} \tau_j}{\left( \prod_{i=1}^{m} \prod_{j=1}^{n} \lambda_{ij} \right)^{\rho}} \right) + \ln \lim_{t \to \infty} t^{(2-\rho) \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}} \]

\[ = \ln \left( \frac{\prod_{i=1}^{m} v_i \prod_{j=1}^{n} \tau_j}{\left( \prod_{i=1}^{m} \prod_{j=1}^{n} \lambda_{ij} \right)^{\rho}} \right) \]

\[ = \{ 0, \quad \rho = 2; \]

\[ \infty, \quad \rho < 2, \]

where we used the limit \( \lim_{x \to \infty} [\psi(x) - \ln x] = 0 \) in [11, Theorem 1] and [12, Section 1.4]. Accordingly,

(1) when \( \rho = 2 \), the logarithmic derivative \( [\ln f(t)]' \) is positive and increasing and maps from \((0, \infty)\) onto

\[ \left( 0, \ln \left( \frac{\prod_{i=1}^{m} v_i \prod_{j=1}^{n} \tau_j}{\left( \prod_{i=1}^{m} \prod_{j=1}^{n} \lambda_{ij} \right)^{\rho}} \right) \right). \]

(2) when \( \rho < 2 \), the logarithmic derivative \( [\ln f(t)]' \) is increasing, does not keep the same sign, and maps from \((0, \infty)\) onto

\[ \left( -\gamma(2 - \rho) \sum_{i=1}^{m} \sum_{j=1}^{n} \lambda_{ij}, \infty \right). \]

In conclusion, the logarithmic derivative \( [\ln f(t)]' \) is a Bernstein function and the function \( f(t) \) for \( \rho < 2 \) has a minimum on \((0, \infty)\).

It is easy to see that \( \lim_{t \to 0^+} f(t) = 1. \)

In [42, p. 62, (3.20)], it was given that

\[ \ln \Gamma(z + 1) = \left( z + \frac{1}{2} \right) \ln z - z + \frac{1}{2} \ln(2\pi) + \int_{0}^{\infty} \vartheta(s)e^{-zs} ds, \]

where

\[ \vartheta(s) = \frac{1}{s} \left( \frac{1}{e^s - 1} - \frac{1}{s} + \frac{1}{2} \right). \]

Then direct computation acquires

\[ \lim_{t \to \infty} \ln f(t) = \lim_{t \to \infty} \left( \sum_{i=1}^{m} \left[ (v_i t + \frac{1}{2}) \ln(v_i t) - v_i t + \frac{1}{2} \ln(2\pi) \right] \right. \]

\[ + \int_{0}^{\infty} \vartheta(s)e^{-v_i ts} ds \right) + \sum_{j=1}^{n} \left[ \left( \tau_j t + \frac{1}{2} \right) \ln(\tau_j t) \right. \]

\[ - \tau_j t + \frac{1}{2} \ln(2\pi) + \int_{0}^{\infty} \vartheta(s)e^{-\tau_j ts} ds \left. \right] \]

\[ - \rho \sum_{i=1}^{m} \sum_{j=1}^{n} \left[ (\lambda_{ij} t + \frac{1}{2}) \ln(\lambda_{ij} t) - \lambda_{ij} t \right. \]

\[ + \frac{1}{2} \ln(2\pi) + \int_{0}^{\infty} \vartheta(s)e^{-(\lambda_{ij} t)s} ds \left. \right) \]
The proof of Theorem 4.1 is complete.

where, when \( \rho = 2 \), we used the fact that

\[
\frac{\prod_{i=1}^{m} \nu_i^{\lambda_{ij}} \prod_{j=1}^{n} \tau_j^{\lambda_{ij}}}{\left( \prod_{i=1}^{m} \prod_{j=1}^{n} \lambda_{ij}^{2} \right)^{2}} = \prod_{i=1}^{m} \nu_i^{\lambda_{ij}} \prod_{j=1}^{n} \tau_j^{\lambda_{ij}}
\]

\[
= \prod_{i=1}^{m} \nu_i^{\lambda_{ij}} \prod_{j=1}^{n} \frac{\sum_{\ell=1}^{n} \lambda_{i\ell}^{\lambda_{ij}}}{\lambda_{ij}^{\lambda_{ij}}} \prod_{j=1}^{n} \frac{\sum_{\ell=1}^{m} \lambda_{\ell j}^{\lambda_{ij}}}{\lambda_{ij}^{\lambda_{ij}}}
\]

\[
= \prod_{i=1}^{m} \prod_{j=1}^{n} \left( \frac{\sum_{\ell=1}^{n} \lambda_{i\ell}}{\lambda_{ij}} \right)^{\lambda_{ij}} \prod_{j=1}^{n} \prod_{i=1}^{m} \left( \frac{\sum_{\ell=1}^{m} \lambda_{\ell j}}{\lambda_{ij}} \right)^{\lambda_{ij}}
\]

\[
> 1 \times 1
\]

\[
= 1.
\]

The proof of Theorem 4.1 is complete. \( \square \)

5 Two inequalities for multinomial coefficients

In this section, as did in [34, Sections 3 and 4], by applying the fourth conclusion in Theorem 4.1, we derive two inequalities of multinomial coefficients \((a_1 + a_2 + \cdots + a_m)\) and of multivariate beta functions \(B(a_1, a_2, \ldots, a_m)\).

When \( \rho = 2 \), the function \( f(t) \) defined by (4.1) can be rearranged as

\[
f(t) = \prod_{i=1}^{m} \Gamma \left(1 + \sum_{j=1}^{n} \lambda_{ij} t \right) \prod_{j=1}^{n} \Gamma \left(1 + \sum_{i=1}^{m} \lambda_{ij} t \right)
\]

\[
= \prod_{i=1}^{m} \left( \sum_{j=1}^{m} \lambda_{ij} t \right) \prod_{j=1}^{n} \left( \sum_{i=1}^{n} \lambda_{ij} t \right).
\]
For $a_i > 0$ and $i \in \mathbb{N}$, multinomial coefficients and multivariate beta functions are connected by

\[
\left( \sum_{i=1}^{m} a_i \right) = \frac{\sum_{i=1}^{m} a_i}{\prod_{i=1}^{m} a_i} B(a_1, a_2, \ldots, a_m).
\]

Therefore, we have

\[
f(t) = \frac{1}{t^{2mn-m-n}} \prod_{j=1}^{m} \sum_{i=1}^{n} \lambda_{ij} \prod_{i=1}^{n} \frac{1}{\lambda_{ij}} \prod_{i=1}^{m} B(\lambda_{ij} t, \lambda_{ij} t, \ldots, \lambda_{ij} t).
\]

Let $\ell \in \mathbb{N}$ and $\theta_k \in (0, 1)$ satisfy $\sum_{k=1}^{\ell} \theta_k = 1$. Let $\lambda = (\lambda_{ij})_{1 \leq i \leq m}$ be a matrix such that $\lambda_{ij} > 0$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. By virtue of the fourth conclusion in Theorem 4.1, the function $f(t)$ is logarithmically convex on $(0, \infty)$. Hence,

\[
f\left( \sum_{k=1}^{\ell} \theta_k y_k \right) \leq \prod_{k=1}^{\ell} f^{\theta_k}(y_k).
\]

Accordingly, by simplification, it follows that

\[
\prod_{j=1}^{n} \left( \sum_{k=1}^{\ell} \theta_k y_k \right) \leq \prod_{k=1}^{\ell} \left( \sum_{i=1}^{m} \lambda_{ij} \prod_{j=1}^{n} \left( \sum_{k=1}^{\ell} \theta_k y_k \right)^{\theta_k} \right)^{\frac{\sum_{j=1}^{n} \lambda_{ij} \prod_{i=1}^{m} \sum_{k=1}^{\ell} \theta_k y_k}{\sum_{i=1}^{m} \sum_{k=1}^{\ell} \theta_k y_k}}
\]

and

\[
\prod_{j=1}^{n} B(\lambda_{ij} \sum_{k=1}^{\ell} \theta_k y_k, \lambda_{ij} \sum_{k=1}^{\ell} \theta_k y_k, \ldots, \lambda_{ij} \sum_{k=1}^{\ell} \theta_k y_k) \leq \prod_{i=1}^{m} B(\lambda_{ij} \sum_{k=1}^{\ell} \theta_k y_k, \lambda_{ij} \sum_{k=1}^{\ell} \theta_k y_k, \ldots, \lambda_{ij} \sum_{k=1}^{\ell} \theta_k y_k) \prod_{i=1}^{m} B(\lambda_{ij} \sum_{k=1}^{\ell} \theta_k y_k, \lambda_{ij} \sum_{k=1}^{\ell} \theta_k y_k, \ldots, \lambda_{ij} \sum_{k=1}^{\ell} \theta_k y_k)
\]

We note that inequalities for multinomial coefficients are seldom.

### 6 Three open problems

In this section, we pose three open problems.
6.1 First open problem

The logarithmically complete monotonicity is stronger than the complete monotonicity \[2,10,29\]. This means that a logarithmically completely monotonic function must be completely monotonic. Completely monotonic functions on the infinite interval \((0, \infty)\) have a characterization \[43, p. 161, Theorem 12b\]: a function \(f(t)\) defined on the infinite interval \((0, \infty)\) is completely monotonic if and only if the integral

\[
    f(t) = \int_0^{\infty} e^{-ts} d\sigma(s) \quad (6.1)
\]

converges for \(0 < t < \infty\), where \(\sigma(s)\) is nondecreasing. In other words, a function \(f(t)\) is completely monotonic on \((0, \infty)\) if and only if it is a Laplace transform of a nondecreasing measure \(\sigma(s)\) on \((0, \infty)\).

Under conditions of Theorem 4.1, the second derivative \([\ln f(t)]''\) is completely monotonic with respect to \(t \in (0, \infty)\). Motivated by the integral representation (6.1), we now pose the first open problem: can one find a closed expression of the nondecreasing measure \(\sigma_{m,n;\lambda}(s)\) such that the integral \([\ln f(t)]'' = \int_0^{\infty} e^{-ts} d\sigma_{m,n;\lambda}(s)\) converges for \(0 < t < \infty\)?

6.2 Second open problem

Recall from [41, Theorem 3.2] that a function \(F: (0, \infty) \to [0, \infty)\) is a Bernstein function if and only if it admits the representation

\[
    F(t) = a + bt + \int_0^{\infty} \left(1 - e^{-ts}\right) d\sigma(s), \quad (6.2)
\]

where \(a, b \geq 0\) and \(\sigma(s)\) is a measure on \((0, \infty)\) satisfying \(\int_0^{\infty} \min\{1, s\} d\sigma(s) < \infty\). By Theorem 4.1, the logarithmic derivative \([\ln f(t)]'\) is a Bernstein function on \((0, \infty)\). Motivated by the integral representation (6.2), we now pose the second open problem: can one find the values of \(a, b\) and present a closed expression of the measure \(\sigma_{m,n;\lambda}(s)\) such that

\[
    [\ln f(t)]' = a + bt + \int_0^{\infty} \left(1 - e^{-ts}\right) d\sigma_{m,n;\lambda}(s)
\]

and \(\int_0^{\infty} \min\{1, s\} d\sigma_{m,n;\lambda}(s) < \infty\) hold?

For example, when \(\rho = 2\), since \(\lim_{t \to 0^+} [\ln f(t)]' = 0\) and

\[
    \lim_{t \to \infty} [\ln f(t)]' = \ln \left( \frac{\prod_{i=1}^m v_i \prod_{j=1}^n \tau_j}{\prod_{i=1}^m \prod_{j=1}^n \lambda_{ij}^{\frac{1}{2}}} \right),
\]

we can derive that \(a = b = 0\).

In order to solve the above two open problems, we suggest readers to refer to the papers [2–8,17,20,21,26–28,30,32,35–40] and closely related references therein.

6.3 Third open problem

Is the inequality (3.1) in Theorem 3.1 sharp except for \(m = n = 1\)? Equivalently speaking, can the number 2 in the right hand side of (3.1) be replaced by a larger constant except for \(m = n = 1\)?
7 Remarks

Finally, we list several remarks.

Remark 7.1 An anonymous referee pointed out that, as enriching references as much as possible is very important for readers, the recently published papers [44, 46–48] should be mentioned as follows.

In [44], among other things, Yang and Tian proved that,

(1) for \( a \geq \frac{31}{98} \), the function

\[
fa(x) = \ln \Gamma \left( x + \frac{1}{2} \right) - x \ln x + x - \frac{1}{2} \ln(2\pi) + \frac{x(x^2 + c)}{24(x^4 + ax^2 + b)}
\]

with \( b = \frac{7}{120}(a - \frac{31}{98}) \) and \( c = a - \frac{7}{120} \) is increasing and concave on \((0, \infty)\) if and only if \( a \geq \frac{5281}{6068} \), while \( fa(x) \) is decreasing and convex on \((0, \infty)\) if and only if \( a = \frac{31}{98} \);

(2) the function

\[
F_a(x) = -\left( x^4 + ax^2 + \frac{98a - 31}{1680} \right)fa(x)
\]

is completely monotonic on \((0, \infty)\) if and only if \( a \geq \frac{5281}{6068} \).

In [46], among other things, with the monotonicity rule for the ratio of two Laplace transforms [45, Lemma 4] and other techniques, Yang and Tian determined that the double inequality

\[
\exp\left(-\frac{x}{24x^2 + \beta}\right) < \frac{\Gamma(x + 1/2)}{\sqrt{2\pi}(x/e)^x} < \exp\left(-\frac{x}{24x^2 + \alpha}\right)
\]

holds for \( x > x_0 \geq 0 \) if and only if \( \alpha \geq \frac{7}{5} \) and \( \beta \leq -f(x_0) \), where

\[
f(x) = 24x^2 + \frac{x}{\ln(x + 1/2) - x \ln x + x - \ln \sqrt{2\pi}}
\]

is decreasing from \((0, \infty)\) onto \((-\frac{7}{5}, 0)\).

In [47], three authors obtained an integral, an asymptotic expansion, and a Maclaurin series representation for generalized Gaussian ratio, discovered their various related properties such as complete monotonicity and some inequalities, and derived several simple approximations for the inverse function of generalized Gaussian ratio.

In [48], Zhu established several inequalities for \( A_p(x) = \frac{1}{x} \int_0^x e^{-t^p} \, dt \), \( B_q(x) = 1 - \frac{1 - e^{-qtx}}{q(p+1)} \), and some ratios involving \( A_p(x) \) and \( B_q(x) \).

Remark 7.2 The monotonicity rule for the ratio of two Laplace transforms [45, Lemma 4] reads that, if the ratio \( \frac{A(x)}{B(x)} \) is increasing, then the ratio \( \frac{\int_0^\infty A(x)e^{-xt^q} \, dx}{\int_0^\infty B(x)e^{-xt^q} \, dx} \) is decreasing on \((0, \infty)\). This monotonicity rule for the ratio of two Laplace transforms has been generalized in [19, Lemma 2.7 and Remark 6.3] as follows: when the functions \( U(x) \), \( V(x) > 0 \), and \( W(x, t) > 0 \) are integrable in \( x \in (a, b) \),

(1) if the ratios \( \frac{\bar{W}(x,t)}{W(x,t)} \) and \( \frac{U(x)}{V(x)} \) are both increasing or both decreasing in \( x \in (a, b) \), then the ratio

\[
R(t) = \frac{\int_a^b U(x)W(x,t) \, dx}{\int_a^b V(x)W(x,t) \, dx}
\]

is increasing in \( t \);
(2) if one of the ratios $\frac{\partial W(x,t)}{\partial t}$ and $\frac{U(x)}{V(x)}$ is increasing and the other is decreasing in $x \in (a, b)$, then the ratio $R(t)$ is decreasing in $t$.

**Remark 7.3** This paper is a slightly revised version of the arXiv preprint [31] and a companion of the series of papers [14,16,18,25,33,34] and closely related references therein.

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