Ultra-cold dipolar gases

Chiara Menotti\textsuperscript{1,2} and Maciej Lewenstein\textsuperscript{1,3}

\textsuperscript{1}ICFO – Institut de Ciències Fotòniques, E-08860 Castelldefels, Barcelona, Spain
\textsuperscript{2}CNR-INFM-BEC and Dipartimento di Fisica, Università di Trento, I-38050 Povo, Italy
\textsuperscript{3}ICREA – Institució Catalana de Recerca i Estudis Avançats, E-08010 Barcelona, Spain

We present a concise review of the physics of ultra-cold dipolar gases, based mainly on the theoretical developments in our own group. First, we discuss shortly weakly interacting ultra-cold trapped dipolar gases. Dipolar Bose-Einstein condensates exhibit non-standard instabilities and the physics of both Bose and Fermi dipolar gases depends on the trap geometry. We focus then the second part of the paper on strongly correlated dipolar gases and discuss ultra-cold dipolar gases in optical lattices. Such gases exhibit a spectacular richness of quantum phases and metastable states, which may perhaps be used as quantum memories. We comment shortly on the possibility of superchemistry aiming at the creation of dipolar heteronuclear molecules in lattices. Finally, we turn to ultra-cold dipolar gases in artificial magnetic fields, and consider rotating dipolar gases, that provide in our opinion the best option towards the realization of the fractional quantum Hall effect and quantum Wigner crystals.

PACS numbers: 03.75.Hh, 05.30.Jp, 32.80.Qk, 42.50.Vk

INTRODUCTION

This paper has been presented as an invited lecture at the International Conference on Recent Progress on Many Body Theories, RPMBT 2007, held in Barcelona in July 2007, in the session devoted to ultra-cold atoms. This conference puts traditionally a lot of emphasis on the development of new methodologies, analytic and numerical methods for many body problems. The lecture given by M. Lewenstein was a little different in character: instead of talking about some new specific method, M. Lewenstein gave a broad review of the topic, trying to convince the audience that ultra-cold dipolar gases provide a fantastic playground to apply modern many body theory.

This topic belongs to one of the hottest areas of modern AMO physics, and giving a full review in a one hour lecture is impossible. M. Lewenstein based his presentation mainly on the activities of his own group, which fortunately touch most of the aspects of the physics of ultra-cold dipolar gases (UDG), mentioning only some selected important contributions from other groups. In this sense the present paper is not a review in the strict sense; it is more like a review of important subtopics within the main topic.

Outline. The outline of this paper is thus the following. It consists of three, relatively independent parts. In the first part we introduce UDGs, and argue why they are so interesting and how to realize them in the laboratory. We sketch very briefly the physics of weakly interacting trapped dipolar Bose and Fermi gases, and talk about the influence of the trap geometry on the physical properties of the UDGs.

The second subject of the paper concerns ultra-cold dipolar gases in optical lattices, that are examples of strongly correlated systems. Such gases exhibit a spectacular richness of quantum phases (Mott insulators and insulating checkerboard phase, superfluid and supersolid phases), as well as an extravagantly large variety of metastable states, which may perhaps be used as quantum memories. We comment here shortly on the possibility of superchemistry aiming at the creation of dipolar heteronuclear molecules in lattices.

In the third and last part, we turn to ultra-cold dipolar gases in artificial magnetic fields and consider rotating dipolar gases, that provide in our opinion the best option towards the realization of the fractional quantum Hall effect and quantum Wigner crystals.

WEAKLY INTERACTING TRAPPED DIPOLAR GASES

Why dipolar gases? Some of the most fascinating experimental and theoretical challenges of modern atomic and molecular physics arguably concern ultra-cold dipolar quantum gases\textsuperscript{1}. The recent experimental realization of a quantum degenerate dipolar Bose gas of Chromium\textsuperscript{2}, and the progress in trapping and cooling of dipolar molecules\textsuperscript{3} have opened the path towards ultra-cold quantum gases with dominant dipole interactions. In particular just before the Barcelona RPMBT conference, the group of Tilman Pfau in Stuttgart realized a UDG of Chromium with dominant magnetic dipole interactions, employing a Feshbach resonance to turn off the short range Van der Waals forces\textsuperscript{4}. Several groups have reported in 2007 enormous progresses in trapping and manipulating mixtures of different atomic...
Dipole-dipole interaction

\[ V(r) = \frac{d^2}{r^3} \left( 1 - 3 \cos^2 \theta \right) \]

Figure 1: Schematic representation of the anisotropic character of dipole-dipole interaction between dipoles oriented vertically (red arrows): for relative distances along the orientation of the dipoles the interaction is attractive (green arrows), while for relative distances perpendicular to the orientation of the dipoles the interaction is repulsive (blue arrows).

species in an optical lattice (cf. [3]). Such systems realize the first step towards the superchemistry of UDGs, that we discuss in the second part of this paper.

Why are dipole interactions interesting? Because of their very nature. For most of the systems studied so far, one assumes the dipole moment to be polarised, i.e. oriented in the same direction via applying either magnetic or electric fields. In such case the dipolar potential between two particles is

\[ V(r) = \frac{d^2}{r^3} (1 - 3 \cos^2 \theta) \]

where \( r \) is the inter-particle distance, \( \theta \) is the angle between the direction of the dipole moment and the vector connecting the particles. The interaction is anisotropic and partially attractive. In particular, if the two dipoles are on top of one another, they attract themselves, if they are aside, they repel each other (see Fig. 1). Locating the dipoles in a vertical cigar-shaped trap leads to a mainly attractive gas that should therefore exhibit collapse. Locating the dipoles in a horizontal pancake trap leads to a mainly repulsive gas and should allow to at least partially stabilise the system.

Several groups have attacked the theory of the UDGs starting from 1999 (for reviews see [1, 6]). Particularly important were the pioneering papers by Góral, Pfau and Rzȩzewski [7], the L. You group (cf. [8]), or the G. Kurizki group [9].

Trapped dipolar Bose gases The dependence of the physics of trapped dipolar gases on geometry is quite strong and leads to dramatic effects [10], as described in the following and summarised in Fig 2. Let us for simplicity consider a dipolar gas of \( N \) Bose particles trapped in a harmonic trap, with all of the dipoles \( d \) oriented in the same direction, and assume that the particles interact dominantly via dipole-dipole interaction. This means, for instance, that we are dealing with heteronuclear molecules with sufficiently large electric dipoles, or a Chromium gas with small magnetic moments interactions, but even smaller \( s \)-wave scattering length. At sufficiently low temperatures such a gas will undergo condensation, and its behaviour is well described by the Gross-Pitaevskii equation (this fact is by no means obvious).

Let us first consider cigar-shaped (\( \omega_\rho \geq \omega_z \)) and “soft” pancake-shaped traps with \( (\omega_\rho \leq \omega_z) \). There appears a kind of quantum phase transition as a function of the trap aspect ratio \( \lambda^* = (\omega_\rho/\omega_z)^{1/2} \simeq 0.4 \), above which the sign of the energy of the mean dipole interaction \( V \) changes from positive to increasingly negative, as we increase \( N d^2 \) for \( \lambda^* < \lambda \leq 1 \), and remains always increasingly negative for \( \lambda > 1 \). The condensate becomes more and more
cigar-shaped, until it undergoes a collapse, somewhat similar to what is occurring for a gas with negative scattering length, i.e. effectively attractive Van der Waals interactions \[11\].

For hard pancake traps with \(\lambda < \lambda^*\), \(V\) grows as we increase \(Nd^2\) and the gas is dominantly repulsive. There is no standard collapse and BEC with much larger values of \(N\) are stable. The condensate aspect ratio decreases with \(Nd^2\). For \(\omega_p \ll \omega_z\), one can distinguish two regimes:

i) for \(\omega_p \ll V \ll \omega_z\), we deal with a quasi-2D Bose gas with repulsive interactions, which attains radially a parabolic Thomas-Fermi profile;

ii) for \(V \geq \hbar \omega_z\), we deal with the 3D gas in the Thomas-Fermi regime. Here the gas does feel the attractive part of the dipolar interactions and undergoes a short wavelength instability, which leads to a roton-maxon minimum and then instability in the excitations spectrum \[12\] (see Fig. 3).

**Trapped dipolar Fermi gases** The dependence of the physics of trapped dipolar gases on the geometry leads also to a quantum phase transition in the case of Fermi gases \[13\]. Let us consider again for simplicity a dipolar gas of \(N\) Fermi particles trapped in a harmonic trap, with all of the dipoles \(d\) oriented in the same direction, and

---

**Figure 2:** Mean dipole-dipole interaction energy \(V\) versus \(\sigma \propto Nd^2\) for various regimes: (a) \(\lambda = 10\), (b) \(\lambda = 1\) show collapse for a cigar-shaped and isotropic trap, and (c) \(\lambda = 0.1\) shows a stable BEC for hard pancake trap. In the inset the figure (c) is depicted in a larger scale.

**Figure 3:** Dispersion law \(\epsilon_0(q)\) for various values of the ratio between on-site and dipole interaction strength \(\beta\) and \(\mu/\hbar \omega\): \(\beta = 0.53\), \(\mu/\hbar \omega = 46\) (upper curve) and \(\beta = 0.47\), \(\mu/\hbar \omega = 54\) (lower curve). The solid curves show the numerical results, and the dotted curves the result of an analytic perturbative approach (see \[12\]).
interactions being dominantly of dipole-dipole kind. The question is whether at sufficiently low temperatures such a gas will undergo a transition to a superfluid state (Bardeen-Cooper-Schriefer (BCS) state), and whether its behaviour is well described by the BCS equations (again, this latter fact is by no means obvious). Pioneering papers on this subject were written by the groups of L. You and H. Stoof, who have looked at the possibility of $p$-wave pairing \cite{14}, and the group of K. Rzążewski, who studied the Thomas-Fermi theory \cite{15}. The BCS theory in homogeneous dipolar gas was investigated in detail by Baranov and Shlyapnikov \cite{16}.

In \cite{13}, we have looked at the BCS transition in a trap, and have shown indeed the existence of a critical aspect ratio, similarly as in the case of a Bose gas. The phase diagram is presented in Fig. 4 as a function of $\lambda^{-1}$ and dipole interactions in units of Fermi energy $\Gamma$. It can be viewed in two ways: for a given $\lambda$ the systems undergoes a transition from the normal to the superfluid state as the dipole interactions grow. Conversely, for a fixed dipole interactions the system undergoes the normal-superfluid transition as $\lambda$ decreases. For very small dipole interactions this transitions occurs in a region of parameters that goes beyond the applicability of our theory.

**DIPOLAR BOSE GASES IN OPTICAL LATTICES**

Ultra-cold gases in optical lattices  

Ultra-cold atomic gases in optical lattices (OL) are nowadays the subject of very intensive studies, since they provide an unprecedented and unique possibility to study numerous challenges of quantum many body physics (for reviews see \cite{17, 18}). In particular, such systems allow to realize various versions of Hubbard models \cite{19}, a paradigmatic example of which is the Bose-Hubbard model \cite{20}. This model exhibits a superfluid (SF) - Mott insulator (MI) quantum phase transition \cite{21}, and its atomic realization has been proposed in the seminal paper of Ref. \cite{22}, followed by the seminal experiments of Ref. \cite{23}. Several aspects and modifications of the SF-MI quantum phase transition, or better to say crossover \cite{24}, have been intensively studied recently (cf. \cite{18, 25}).

Ultra-cold dipolar gas in a lattice  

We have proposed to look at the UDG in a 2D lattice in 2002 \cite{26}. The Hamiltonian of the system differs from the standard Hubbard-Bose model described by the Hamiltonian

$$
H = -t \sum_i [b_i^\dagger b_{i-1} + h.c.] + \frac{U}{2} \sum_i n_i (n_i - 1) - \mu N,
$$

where $N = \sum_i n_i = \sum_i b_i^\dagger b_i$ is the atom number operator, $t$ is the hopping term, and $U$ denotes the strength of on-site interactions. A UDG in a lattice is described by an extended Bose-Hubbard Hamiltonian

Figure 4: Phase diagram for a trapped dipolar Fermi gas as a function of $\lambda^{-1}$ showing the critical dipole interactions in units of Fermi energy $\Gamma$ above which BCS takes place. The upper (lower) curve corresponds to $N = 10^6$ ($N = 2 \times 10^6$).
$H = -t \sum_i [b_i \dagger b_{i-1} + \text{h.c.}] + \frac{U}{2} \sum_i n_i (n_i - 1) + \frac{U_1}{2} \sum_{\langle i,j \rangle} n_i n_j + \frac{U_2}{2} \sum_{\langle\langle i,j \rangle \rangle} n_i n_j + ... - \mu N,$

where the sum over \((i,j)\) pertains to nearest neighbours, the one over \(\langle\langle i,j \rangle\rangle\) to next-nearest neighbours, etc., and \(U_i\) are determined by dipole-dipole interactions. To a good approximation, assuming that all dipoles are perpendicular to the plane of the lattice, \(U_n = d^2/r_{ij}^3\), where \(r_{ij}\) are the distances between the involved sites; generally it is given by the expression in Eq. (1). The resulting model exhibits a rich variety of quantum phases: apart from the standard superfluid and Mott insulator states, it can form a checkerboard phase at half filling, or close to it in the parameter space a supersolid state, i.e. superfluid with a periodic density modulation in the density and in the order parameter. It can also form a collapsing state if the interactions are too attractive \([26]\). The possibility of realizing a supersolid state with ultra-cold atoms is at present particularly attractive because, to our knowledge, its existence, claimed in $^4\text{He}$ experiments \([27]\), is still controversial \([28]\). Experiments with ultra-cold dipolar atoms might provide a much cleaner environment for the creation and observation of such phases.

**Metastable states.** Most recently, we have pointed out for the first time that a lattice system with long-range interactions presents many insulating metastable states in the low tunneling part of the phase diagram \([29]\). The metastable states arise as local minima of the energy. We access them using a mean-field approach and a time dependent Gutzwiller Ansatz, which allows to study the dynamics of the system both in real and imaginary time.

The imaginary time evolution, which mimics dissipation in the system, converges unambiguously to the ground state of the system for the Bose-Hubbard model in presence of on-site interaction only. In the presence of long-range interaction it shows a strikingly different behaviour and converges often to different configurations, depending on the exact initial conditions. In this way, we clearly get a feeling of the existence of metastable states in the system. In the real time evolution, their stability is confirmed by typical small oscillations around a local minimum of the energy. For all the insulating metastable configurations, we calculate the insulating lobes in the \(J - \mu\) phase space, using a mean-field perturbative approach. This results in a much more complex phase diagram, as shown in Fig. 5.

The metastable states have a finite lifetime due to the tunneling to different metastable states. We have used a path integral approach in imaginary time, combined with a dynamical variational method to estimate this lifetime, which results to be very long for small tunneling parameter \(J\) and large systems.

However, for large systems the number of the metastable states and the variety of their patterns is so large, and their energy separation so small, that it turns out to be very difficult to control the presence of defects. We have checked that by using superlattices one can prepare the atoms in configurations of preferential symmetry with very small uncertainty. If a given configuration corresponds to a metastable state, it survives also once the superlattice is removed, due to dipole-dipole interaction.

For the detection scheme we have presently in mind, it is also essential (not in line of principle, but practically for present experimental possibilities) to create a given configuration in a reproducible way. In fact, the spatially modulated structures characterising the metastable states can be detected via the measurement of the noise correlations of the expansion pictures \([30, 31, 32]\), which equal the modulus square of the Fourier transform of the density distribution in the lattice and is in principle able to recognise the periodic modulations or the defects in the density distribution. However, since the signal to noise ratio required for single defect recognition is beyond the present experimental possibilities, one should average over a finite number of different experimental runs producing the same spatial distribution of atoms in the lattice, and hence accurate reproducibility is required. In Fig. 6 we show the noise correlations for the metastable configurations at filling factor 1/2 shown in Fig. 5 (I) to (III).

Presently we are studying the possibility of transferring in a controlled way those systems from one configuration to another. This, together with the capability of initialising and reading out the state of the lattice, may make those systems useful for applications as quantum memories.

"Superchemistry“ of dipolar heteronuclear molecules in an optical lattice. The idea is to create heteronuclear molecules starting from a Mott insulating phase with exactly one atom per species per site \([33]\). Polar dimers would be then formed by photo-association or by using a Feshbach resonance. With an appropriate choice of scattering lengths, such that the two species do not remain immiscible, one can demonstrate that the two-species Mott state is the ground state of the system and can be reached starting by a two-component superfluid and slowly ramping up the optical lattice potential (see Fig. 7).

Before the creation of molecules, the system is described by a two-species Bose-Hubbard Hamiltonian with local interactions...
Figure 5: Phase diagram for weak and strong dipole-dipole interaction and interaction range up to the 4th nearest neighbour: $U/U_1 = 20$ (a) and $U/U_1 = 2$ (b). The thick lines are the ground state lobes, found (for increasing chemicals potential) for filling factors equal to all multiples of 1/8. The thin lines are the metastable states, found at all filling factors equal to multiples of 1/16. Some of the metastable configurations at filling factor 1/2 (I to III) and corresponding ground state (IV). Empty sites are light and sites occupied with 1 atom are dark.

Figure 6: Spatial noise correlation patterns for configurations (I) to (III) in Fig. 5 assuming a localised gaussian density distribution at each lattice site.

$$H = \sum_{(i,j)} \left[ J_{a_i}a_i^\dagger a_j + J_{b_i}b_i^\dagger b_j \right] + U_{ab} \sum_i n_{a_i}n_{b_i} + \frac{1}{2} \sum_i \left[ U_{0a}n_{a_i}(n_{a_i} - 1) + U_{0b}n_{b_i}(n_{b_i} - 1) \right],$$  \hspace{1cm} (4)

with $a$ and $b$ denoting the two species, while, after the creation of the molecules, assuming molecules with non negligible dipole moment, the Hamiltonian is the one of a single molecular component gas with long-range interactions, as written in Eq. (4). Finally, this Mott insulating state of dipolar molecules can be melted to a superfluid heteronuclear molecular condensate, as shown in Fig. 5.

**ULTRA-COLD GASES IN ARTIFICIAL GAUGE FIELDS**

**Rotating ultra-cold gases** It is well known that rapidly rotating harmonically trapped gases of neutral atoms exhibit effects completely analogous to charged particles in uniform magnetic fields (for a recent overview, see for instance [34]). In particular, one should be able to realize analogues of the fractional quantum Hall effect (FQHE) in such systems [35, 36]. Particularly interesting in this context are rotating dipolar gases (RDG). Bose-Einstein
Figure 7: Creating a two species Mott insulator in a real time starting from a superfluid phase of $^{41}$K (solid line) and $^{87}$Rb (dashed line); the upper plot shows the value of the order parameters $|\langle a_i \rangle|$, $|\langle b_i \rangle|$ (constant for all lattice sites) for both species, while the lower one depicts the variance $\text{Var}(n) = \sqrt{\langle n^2 \rangle - \langle n \rangle^2}$ of the on-site occupation.

Figure 8: Quantum melting of $^{41}$K$-^{87}$Rb dimers initially in the MI phase towards the ultra-cold dipolar molecular BEC; the plot shows the time evolution of the molecular superfluid order parameter $|\langle b_i \rangle|$ (solid line), which is the same for all lattice sites. The dashed line refers to static calculations of the ground state of the dipolar molecules placed in the lattice.

condensates of RDGs exhibit novel forms of vortex lattices, e. g., square, stripe- and bubble-“crystal” lattices \[37\]. The stability of these phases in the lowest Landau level was recently investigated \[38\]. We have demonstrated that the quasi-hole gap survives the large $N$ limit for fermionic RDGs \[39\]. This property makes them perfect candidates to approach the strongly correlated regime, and to realize Laughlin liquids (cf. \[40\]) at filling $\nu = 1/3$, and quantum Wigner crystals at $\nu \leq 1/7$ \[41\] for a mesoscopic number of atoms $N \simeq 50 - 100$. Lately, Rezayi et al. \[42\] have shown that the presence of a small amount of dipole-dipole interactions stabilises the so-called bosonic Rezayi-Read state at $\nu = 3/2$ whose excitations are both fractional and non-Abelian.

**Ordered structures in rotating Bose gases** In the recent two years our group has concentrated on studies of small samples of rotating atoms using exact diagonalization methods. These early studies dealt with the description of ordered structures in rotating ultra-cold Bose gases (by looking at the single particle density matrix and pair correlation function \[34\]), and by studying symmetry breaking in small rotating clouds of trapped ultra-cold Bose atoms \[43\]. The characterisation of small samples of cold bosonic atoms in rotating micro-traps has recently attracted increasing interest due to the possibility to deal with a few number of particles per site in optical lattices. In the Ref. \[34\] we considered two-dimensional systems of few cold Bose atoms confined in a harmonic trap in the $XY$ plane, and...
submitted to strong rotation around the $Z$ axis. By means of exact diagonalization, we analysed the evolution of the ground state structures as the rotational frequency $\Omega$ increases. Various kinds of ordered structures were observed. In some cases, hidden interference patterns exhibit themselves only in the pair correlation function; in some other cases explicit broken-symmetry structures appear that modulate the density. The standard scenario, valid for large systems (i.e., nucleation of vortices into an Abrikosov lattice, melting of the lattice, and subsequent appearance of fractional quantum Hall type states up to the Laughlin state), is absent for small systems of $N < 10$ atoms, and only gradually recovered as $N$ increases. On the one hand, the Laughlin state in the strong rotational regime contains ordered structures much more similar to a Wigner crystal or a molecule than to a fermionic quantum liquid. This result has some similarities to electronic systems, extensively analysed previously. On the other hand, in the weak rotational regime, the possibility to obtain equilibrium states whose density reveals an array of vortices is restricted to some critical values of the rotation frequency $\Omega$.

**Rotational symmetry breaking** In Ref. [43] we have studied the signatures of rotational and phase symmetry breaking in small rotating clouds of trapped ultra-cold Bose atoms by looking at the rigorously defined condensate wave function. Rotational symmetry breaking occurs in narrow frequency windows, where energy degeneracy between the lowest energy states of different total angular momentum takes place, and leads to a complex condensate wave function that exhibits vortices clearly seen as holes in the density, as well as characteristic vorticities. Phase symmetry (or gauge symmetry) breaking, on the other hand, is clearly manifested in the interference of two independent rotating clouds.

**Ultra-cold rotating dipolar Fermi gases** Armed by the experience on rotating Bose gases with short range interactions, we considered in the recent Letter [44] a system of $N$ dipolar fermions rotating in an axially symmetric harmonic trapping potential strongly confined in the direction of the axis of rotation. Along this $z$-axis, the dipole moments, as well as spins are assumed to be aligned. Various ways of experimental realization of ultra-cold dipolar gases are discussed in [1]. In case of low temperature $T$ and weak chemical potential $\mu$ with respect to the axial confinement $\omega_z$, the gas is effectively 2D, and the Hamiltonian of the system in the rotating reference frame reads

$$H = \sum_{j=1}^{N} \frac{1}{2M} (\vec{p}_j - M\vec{e}_z \times \vec{r}_j)^2 + \frac{M}{2} (\omega_0^2 - \Omega^2) r_j^2 + V_d.$$  

Here, $\omega_0 \ll \omega_z$ is the radial trap frequency, $\Omega$ is the frequency of rotation, $M$ is the mass of the particles, $V_d = \sum_{j<k} \frac{d^2}{|\vec{r}_j - \vec{r}_k|^2}$ is the dipolar interaction potential (rotationally invariant with respect to the $z$-axis), $d$ is the dipole moment, and $\vec{r}_j = x_j\vec{e}_x + y_j\vec{e}_y$ is the position vector of the $j$-th particle. The first term of (5) is formally equivalent to the Landau Hamiltonian of particles with mass $M$ and charge $e$ moving in a constant magnetic field of strength $B = 2M\omega_c/e$ perpendicular to their plane of motion. The eigenvectors of $H_{\text{Landau}}$ span Landau levels (LL) with energies $\varepsilon_n = n\omega_c (n + 1/2)$ where $n \equiv N_{\text{LL}}/2\pi l^2$ the number of states per unit area in each LL, where $l = \sqrt{\hbar/M\omega_c}$ is the magnetic length. Given a fermionic density $n_f$, the filling factor $\nu = 2\pi l^2 n_f$ refers to the fraction of occupied LLs. Even though the above definition applies to infinite homogeneous systems, it may be used for finite systems as a suitable truncation of the Hilbert space at specific angular momenta. The second term in (5) accounts for a rotationally induced effective reduction of the trap strength. For $\Omega \rightarrow \omega_0$, the confining potential vanishes.

Our results may be summarised as follows: we have studied in detail ground and excited states of quasi-2D ultra-cold rotating dipolar Fermi gases. By exact diagonalization methods, we studied systems up to 12 particles. We have identified novel kinds of strongly correlated states in the intermediate regime, i. e., "boosted" pseudo-hole ground states, which appear alternatively as $\Omega$ grows (see Fig. 9). The calculation of the substantial gap in the excitation spectrum of the dipolar Laughlin state at $\nu = 1/3$ (see Fig. 11) proves the accessibility of fractional quantum Hall states in these microsystems. At lower fillings, interactions favour crystalline order. Rotating dipolar gases are thus very suitable candidates to realize Laughlin-like and more exotic quantum liquids, as well as their crossover behaviour to Wigner crystals.

**Wigner crystals** Finally, in Ref. [41] we have discussed the existence of a Wigner crystal phase in a rapidly rotating gas of polarised dipolar fermions. It is shown that for sufficiently low filling factors $\nu < 1/7$ the Wigner crystal has a lower energy than the Laughlin liquid (see Fig. 11). We have also examined the stability of the Wigner crystal state by incorporating phonon-phonon interactions and identified the quantum melting point with the appearance of imaginary frequencies. While magnetic fields the critical filling factor is a constant and we formulate the Lindemann criterion. Note that in the considered case of a rotating dipolar gas, the crystal phase exists at lower densities, contrary to a non-rotating gas, in which the ground state has a crystal order at high densities, see Ref. [40]. In fact detailed studies of this latter possibility using quantum Monte Carlo methods have been performed by the
Figure 9: Ground state density-density correlation functions $\hat{\rho}(\vec{r}, \vec{r}_0)$ for $N = 10$ dipolar fermions at $L_z = (\text{top})$ 45, 80, (center) 90, 93, (bottom) 103, 117 with $\vec{r}_0$ set to the maximum of the density, which occurs at the edge.

Figure 10: Density-density correlation function $\hat{\rho}(\vec{r}, \vec{r}_0)$ of the Laughlin state for $N = 12$ dipolar fermions with $\vec{r}_0$ chosen at the maximum of the density, which occurs at the edge.

The group of J. Boronat in Barcelona, together with G. Astrakharchik. These authors have in particular studied weakly interacting two-dimensional systems of dipoles and limitations of mean-field theory [47], Wigner crystallization and quantum phase transition in a two-dimensional system of dipoles [48], and also 1D dipolar gases [49].

Figure 11: Energy per particle for the Wigner crystal (dotted line) and for the Laughlin state (solid line) as a function of the filling factor for $l = 0$. Insert shows the critical filling factor as a function of the extension in the $z$-direction.
Conclusions We conclude with the standard conclusion of M. Lewenstein and the motto to the review \cite{18}, which is a citation from William Shakespeare’s “Hamlet”:

There are more thing in heaven and earth, Horatio, than are dreamt of in your philosophy.

In the present context, it expresses our never ending curiosity, enthusiasm, joy, and excitement of working in atomic physics in general, and physics of ultra-cold dipolar atoms in particular.

We thank all the co-authors of the papers on dipolar gases and related subjects. We acknowledge support from EU IP Programme ”SCALA”, ESF PESC QUEDIS, MEC (Spanish Government) under contracts FIS 2005-04627, Consolider Ingeni 2010 “QOIT”, and Acciones Integradas ICF-O Hannover. C.M. acknowledges financial support by the EU through an EIF Marie-Curie Action.

[1] M. Baranov, L. Dobrek, K. Góral, L. Santos, and M. Lewenstein, *Physica Scripta* **102**, p. 74 (2002).
[2] A. Griesmaier, J. Werner, S. Hensler, J. Stuhler, and T. Pfau, *Phys. Rev. Lett.* **94**, p. 160401 (2005).
[3] Special issue on *Ultracold Polar Molecules: Formation and Collisions*, *Eur. Phys. J. D* **31** (2004).
[4] T. Lahaye, T. Koch, B. Fröhlich, M. Fattori, J. Metz, A. Griesmaier, S. Giovanazzi, and T. Pfau, *Nature* **448**, p. 672-675 (2007).
[5] Particularly impressive are the results of C. Ospelkaus, S. Ospelkaus, L. Humbert, P. Ernst, K. Sengstock, K. Bongs, *Phys. Rev. Lett.* **97**, p. 120402 (2006).
[6] M.A. Baranov, to be published in *Physics Reports*.
[7] K. Góral, K. Rzązewski, and T. Pfau, *Phys. Rev. A* **61**, p. 051601 (2000).
[8] S. Yi and L. You, *Phys. Rev. A* **61**, 041604 (2000) and *Phys. Rev. A* **63**, p. 053607 (2001).
[9] D. O’Dell, S. Giovanazzi, G. Kurizki, and V.M. Akulin, *Phys. Rev. Lett.* **84**, p. 5687 (2000).
[10] L. Santos, G. Shlyapnikov, P. Zoller, and M. Lewenstein, *Phys. Rev. Lett.* **85**, p. 1791 (2000).
[11] C.A. Sackett, J.M. Gerton, M. Welling, and R.G. Hulet, *Phys. Rev. Lett.* **92**, p. 876 (1999).
[12] L. Santos, G. Shlyapnikov, and M. Lewenstein, *Phys. Rev. Lett.* **90**, p. 250403 (2003); see also D.H. O’Dell, S. Giovanazzi, and G. Kurizki, *Phys. Rev. Lett.* **90**, p. 110402 (2003).
[13] M. Baranov, L. Dobrek, and M. Lewenstein, *Phys. Rev. Lett.* **92**, p. 250403 (2004).
[14] M. Houbiers and H.T. Stoot, *Phys. Rev. A* **59**, p. 1556 (1999); L. You and M. Marinescu, *Phys. Rev. A* **60**, p. 2324 (1999).
[15] K. Góral, B.-G. Englert, and K. Rzązewski, *Phys. Rev. A* **63**, p. 033606 (2001).
[16] M.A. Baranov, M.S. Mar’enko, S. Van Rynckov, and G.V. Shlyapnikov, *Phys. Rev. A* **66**, p. 013606 (2002).
[17] I. Bloch and M. Greiner, *Adv. At. Molec. Opt. Phys.* **52**, p. 1 (2005).
[18] M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen De, and U. Sen, *Advances in Physics*, Vol. **56**, Nos. 1-2, January-April 2007, p. 243-379.\[
\textit{cond-mat/0606771}
\]
[19] D. Jaksch and P. Zoller, *Ann. Phys.* (N.Y.) **315**, p. 52 (2005).
[20] M.P.A. Fisher, P.B. Weichman, G. Grinstein, and D.S. Fisher, *Phys. Rev. B* **40**, p. 546 (1989).
[21] S. Sachdev, *Quantum Phase Transitions*, (Cambridge University Press, Cambridge, 1999).
[22] D. Jaksch, C. Bruder, J.I. Cirac, C.W. Gardiner, and P. Zoller, *Phys. Rev. Lett.* **81**, p. 3108 (1998).
[23] M. Greiner, O. Mandel, T. Esslinger, T.W. Hänisch and I. Bloch, *Nature* **415**, p. 30 (2002).
[24] G.G. Batrouni, V. Rousseau, R.T. Scalettar, M. Rigol, A. Muramatsu, P.J.H. Denteneer, and M. Troyer, *Phys. Rev. Lett.* **89**, p. 117203 (2002); G.G. Batrouni, F.F. Assaad, R.T. Scalettar, and P.J.H. Denteneer, *Phys. Rev. A* **72**, p. 031601(R) (2005).
[25] I. Bloch, *Physics World* **17**, p. 25 (2004).
[26] K. Góral, L. Santos, and M. Lewenstein, *Phys. Rev. Lett.* **88**, p. 170406 (2002).
[27] E. Kim and M.H.W. Chan, *Nature* **427**, p. 225 (2004); *Science* **305**, p. 1941 (2004); *Phys. Rev. Lett.* **97**, p. 115302 (2006).
[28] S. Sasaki, F. Caupin, and S. Balibar, arXiv:0707.3110.
[29] C. Menotti, C. Trefeger, and M. Lewenstein, *Phys. Rev. Lett.* **98**, p. 235301 (2007).
[30] E. Altman, E. Demler, and M.D. Lukin, *Phys. Rev. A* **70**, p. 013603 (2004).
[31] S. Fölling, F. Gerbier, A. Widera, O. Mandel, T. Gericke, and I. Bloch, *Nature* **434**, p. 481 (2005).
[32] V.W. Scarola, E. Demler, and S. Das Sarma, *Phys. Rev. A* **73**, p. 051601(R) (2006).
[33] B. Damski, L. Santos, E. Tiemann, M. Lewenstein, S. Kotochigova, P. Juliennne, and P. Zoller, *Phys. Rev. Lett.* **90**, p. 110401 (2003).
[34] N. Barberán, M. Lewenstein, K. Osterloh, and D. Dagnino, *Phys. Rev. A* **73**, p. 063623 (2006).
[35] N.K. Wilkin, J.M.F. Gunn, and R.A. Smith, *Phys. Rev. Lett.* **80**, p. 2265 (1998); N.K. Wilkin and J.M.F. Gunn, *Phys. Rev. Lett.* **84**, p. 6 (2000).
[36] B. Paredes, P. Fedichev, J.I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **87**, p. 010402 (2001).
[37] N. R. Cooper, E. H. Rezayi, and S. H. Simon, *Phys. Rev. Lett.* **95**, p. 200402 (2005).
[38] S. Komineas and N.R. Cooper, \textit{cond-mat/0610702}.
[39] M. A. Baranov, K. Osterloh, and M. Lewenstein, *Phys. Rev. Lett.* **94**, p. 070404 (2005).
[40] R. E. Prange and S. M. Girvin (editors), *The Quantum Hall Effect*, New York: Springer Verlag (1987).
[41] M. A. Baranov, H. Fehrmann, and M. Lewenstein, cond-mat/0612592.
[42] E. H. Rezayi, N. Read and N. R. Cooper, Phys. Rev. Lett. 95, p. 160404 (2005).
[43] D. Dagnino, N. Barberán, K. Osterloh, A. Riera, and M. Lewenstein, Phys. Rev. A 76, p. 013625 (2007).
[44] K. Osterloh, N. Barberán, and M. Lewenstein, Phys. Rev. Lett. 99, p. 160403 (2007).
[45] V.M. Bedanov, G.V. Gadiyak, and Yu.E. Lozovik, Physics Letters A 109, p. 289 (1985).
[46] Y.E. Lozovik, D.R. Musin, and V.I. Yudson, Sov. Phys. Solid State 21, p. 1132 (1979); Y.E. Lozovik, V.M. Furztdinov, and A. Abdullaev, J. Phys. C: Solid State Phys. 18, p. L807 (1985).
[47] G.E. Astrakharchik, J. Boronat, J. Casulleras, I.L. Kurbakov, and Yu.E. Lozovik, arXiv:cond-mat/0612691.
[48] G.E. Astrakharchik, J. Boronat, I.L. Kurbakov, and Yu.E. Lozovik, Phys. Rev. Lett. 98, p. 060405 (2007); see also H.P. Büchler, E. Demler, M. Lukin, A. Micheli, N. Prokof’ev, G. Pupillo, and P. Zoller, Phys. Rev. Lett. 98, p. 060404 (2007).
[49] A.S. Arkhipov, G.E. Astrakharchik, A.V. Belikov, Yu.E. Lozovik, arXiv:cond-mat/0505700 see also R. Citro, E. Orignac, S. De Palo, and M.-L. Chiofalo, Phys. Rev. A 75, p. 051602 (2007).