Bäcklund transformations and the Atiyah–Ward ansatz for non-commutative anti-self-dual
Yang–Mills equations

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We present Bäcklund transformations for the non-commutative anti-self-dual Yang–Mills equation where the gauge group is \( G = GL(2) \), and use it to generate a series of exact solutions from a simple seed solution. The solutions generated by this approach are represented in terms of quasi-determinants. We also explain the origins of all the ingredients of the Bäcklund transformations within the framework of non-commutative twistor theory. In particular, we show that the generated solutions belong to a non-commutative version of the Atiyah–Ward ansatz.

Keywords: non-commutative integrable systems; anti-self-dual Yang–Mills equation; quasi-determinant solutions; Atiyah–Ward ansatz; Penrose–Ward transformation

1. Introduction

In both mathematics and physics, a non-commutative extension is a natural generalization of a commutative theory that sometimes leads to a new and deeper understanding of that theory. While matrix (or more general, non-Abelian) generalizations have been studied for a long time, the generalization to non-commutative flat spaces, triggered by developments in string theory (e.g. Douglas & Nekrasov 2002; Szabo 2003), has become a hot topic recently. These generalizations are realized by replacing all products in the commutative theory with associative but non-commutative Moyal products. In gauge theories, such a non-commutative extension is equivalent to the presence of a background magnetic field. Many successful applications of the analysis of non-commutative solitons to D-brane dynamics have been made. (Here, we use the word ‘soliton’ as a stable configuration that possesses localized energy densities and hence includes static configurations such as instantons.)

Integrable systems and soliton theory can also be extended to a non-commutative setting and yield interesting results and applications. (For reviews, see Kupershmidt 2000; Hamanaka 2003, 2005a; Hamanaka & Toda 2003; Dimakis & Müller-Hoissen 2004; Tamassia 2005; Mazzanti 2007; Lechtenfeld 2008.) Among them, the non-commutative anti-self-dual Yang–Mills (ASDYM)
equation in four-dimensions is important because in the Euclidean signature $(++++)$, the ADHM construction can be used to find all exact instanton solutions and gives rise to new physical objects such as $U(1)$ instantons (Nekrasov & Schwarz 1998). In the split signature $(++--)$, many non-commutative integrable equations can be derived from the non-commutative ASDYM equation by a reduction process (see Hamanaka 2005b, 2006 and references therein). Integrable aspects of the non-commutative ASDYM equation can be understood in the geometrical framework of non-commutative twistor theory (Hannabuss 2001; Kapustin et al. 2001; Takasaki 2001; Horváth et al. 2002; Lechtenfeld & Popov 2002; Ihl & Uhlmann 2003; Brain 2005; Brain & Majid 2008). Therefore, it is worth studying the integrable aspects of the non-commutative ASDYM equation in detail for the applications both to lower dimensional integrable equations and to the corresponding physical situations in the framework of a non-commutative analogue of $N = 2$ string theory (Lechtenfeld et al. 2001a,b). Here solitons are not, in general, static, and suggest the existence of some kinds of new configurations. For these purposes, Bäcklund transformations play an important role in constructing exact solutions and revealing a (infinite-dimensional) symmetry of the solution space in terms of the transformation group. Also, a twistor description is useful for a discussion of the origin of the transformations and for checking whether or not the group action is transitive.

In the present paper, we give Bäcklund transformations for the non-commutative ASDYM equation where the gauge group is $G = GL(2)$ and use them to generate a sequence of exact solutions from a simple seed solution. This approach gives both finite action solutions (instantons) and infinite action solutions (such as nonlinear plane waves). The solutions obtained are written in terms of quasi-determinants (Gelfand & Retakh 1991, 1992), which appear also in the construction of exact soliton solutions in lower dimensional non-commutative integrable equations such as the Toda equation (Etingof et al. 1997, 1998; Li & Nimmo 2008, 2009), the KP and KdV equations (Etingof et al. 1997; Dimakis & Müller-Hoissen 2007; Gilson & Nimmo 2007; Hamanaka 2007), the Hirota–Miwa equation (Nimmo 2006; Gilson et al. 2007; Li et al. 2009), the mKP equation (Gilson et al. 2008a,b), the Schrödinger equation (Goncharenko & Veselov 1998; Samsonov & Pecheritsin 2004), the Davey–Stewartson equation (Gilson & Macfarlane 2009), the dispersionless equation (Hassan 2009), and the chiral model (Haider & Hassan 2008), in which they play the role that determinants do in the corresponding commutative integrable systems. We also clarify the origin of the results from the viewpoint of non-commutative twistor theory by using non-commutative Penrose–Ward correspondence or by solving a non-commutative Riemann–Hilbert problem. It is shown that the solutions generated belong to a non-commutative version of the Atiyah–Ward ansatz (Atiyah & Ward 1977).

The discussion and strategy used in this paper are simple non-commutative generalizations of those used in the commutative case (Corrigan et al. 1978a,b; Mason et al. 1988a,b; Mason & Woodhouse 1996). In the commutative limit, our results coincide in part with the known results, but, in the non-commutative case, there are several non-trivial points. First, in §3(b) we show that quasi-determinants are ideally suited to the non-commutative extension of the known results and greatly simplify the proofs of the Bäcklund transformations even in the commutative limit. The simple quasi-determinant representations of
Yang’s $J$-matrix are new and imply the important result that the Bäcklund transformation is not just a gauge transformation. It is possible for the non-commutative twistor description to work as it does in the commutative setting because one of the three local coordinates can be taken to be a commutative variable.

In our treatment, all dependent variables belong to a ring, which has an associative but not necessarily commutative product. Hence, the results we obtain are available in any non-commutative setting such as the Moyal-deformed, matrix or quaternion-valued ASDYM equations.

2. The non-commutative ASDYM equation

Let us consider non-commutative Yang–Mills theories in four dimensions in which the gauge group is $GL(N)$ and the real coordinates are $x^\mu$, $\mu = 0, 1, 2, 3$. In the rest of the paper, we follow the conventions of notation given in Mason & Woodhouse (1996).

(a) The non-commutative ASDYM equation

The non-commutative ASDYM equation is derived from the compatibility condition of the linear system

\[
\begin{align*}
L\psi &:= (D_w - \zeta D\tilde{z})\psi = (\partial_w + A_w - \zeta (\partial_{\tilde{z}} + A_{\tilde{z}}))\psi = 0, \\
M\psi &:= (D_{\tilde{z}} - \zeta D\tilde{w})\psi = (\partial_{\tilde{z}} + A_{\tilde{z}} - \zeta (\partial_{\tilde{w}} + A_{\tilde{w}}))\psi = 0,
\end{align*}
\]

where $A_z, A_w, A_{\tilde{z}}, A_{\tilde{w}}$ and $D_z, D_w, D_{\tilde{z}}, D_{\tilde{w}}$ denote gauge fields and covariant derivatives in Yang–Mills theory, respectively. The (commutative) variable $\zeta$ is a local coordinate of a one-dimensional complex projective space $CP_1$, and is called the spectral parameter. We note that $\psi$ is not regular at $\zeta = \infty$ because, if it were regular, by Liouville’s theorem it would be a constant function and the gauge fields would be flat (e.g. Mason & Woodhouse 1996).

The four complex coordinates $z, \tilde{z}, w, \tilde{w}$ are double null coordinates (Mason & Woodhouse 1996). By imposing the corresponding reality conditions, we can realize real spaces with different signatures, that is

(i) the Euclidean space, obtained by putting $\tilde{w} = -\tilde{w}; \tilde{z} = \tilde{z}$, for example

\[
\begin{bmatrix}
\tilde{z} \\
\tilde{w}
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
x^0 + ix^1 \\
x^2 + ix^3
\end{bmatrix}, \quad \begin{bmatrix}
x^0 - ix^3 \\
x^2 - ix^1
\end{bmatrix}.
\]

(ii) the ultrahyperbolic space, obtained by putting $\tilde{w} = \tilde{w}; \tilde{z} = \tilde{z}$, for example

\[
\begin{bmatrix}
\tilde{z} \\
\tilde{w}
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
x^0 + ix^1 \\
x^2 + ix^3
\end{bmatrix}, \quad \text{or} \quad z, w, \tilde{z}, \tilde{w} \in \mathbb{R}.
\]

The compatibility condition $[L, M] = 0$ gives rise to a quadratic polynomial in $\zeta$ and each coefficient yields the non-commutative ASDYM equations, with explicit representations
\[ F_{wz} = \partial_w A_z - \partial_z A_w + [A_w, A_z] = 0, \quad F_{\bar{w}\bar{z}} = \partial_{\bar{w}} A_{\bar{z}} - \partial_{\bar{z}} A_{\bar{w}} + [A_{\bar{w}}, A_{\bar{z}}] = 0, \]
\[ F_{\bar{z} w} - F_{w \bar{z}} = \partial_{\bar{z}} A_{\bar{w}} - \partial_{\bar{w}} A_{\bar{z}} + [A_{\bar{w}}, A_{\bar{z}}] - [A_w, A_{\bar{w}}] = 0, \]
\[ F_{\bar{z} z} - F_{z \bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}] - [A_{\bar{w}}, A_z] = 0, \]

which is equivalent to the ASD condition for gauge fields \( F_{\mu \nu} = - \ast F_{\mu \nu} \) in the real representation where the symbol \( \ast \) is the Hodge dual operator.

When the compatibility conditions are satisfied, the linear system (equation (2.1)) has \( N \) independent solutions. Hence, the solution \( \psi(x; \zeta) \) can be interpreted as an \( N \times N \) matrix whose columns are the \( N \) independent solutions.

Gauge transformations act on the linear system (equation (2.1)) as
\[ L \mapsto g^{-1} L g, \quad M \mapsto g^{-1} M g, \quad \psi \mapsto g^{-1} \psi, \quad g \in G. \]

(b) The non-commutative Yang equation and J, K-matrices

Here, we discuss the different potential forms of the non-commutative ASDYM equations such as the non-commutative \( J, K \)-matrix formalisms and the non-commutative Yang equation, which were already presented by, for example, Takasaki (2001).

Let us first discuss the \( J \)-matrix formalism of the non-commutative ASDYM equation (2.4). The first and second equations of (2.4) are the compatibility conditions of
\[ D_z h = 0, \quad D_w h = 0, \quad \text{and} \quad D_{\bar{z}} \tilde{h} = 0, D_{\bar{w}} \tilde{h} = 0, \]
respectively. Here \( h \) and \( \tilde{h} \) are \( N \times N \) matrices, whose \( N \) columns of \( h \) and \( \tilde{h} \) are independent solutions of the linear systems. Their existence is formally proved in the case of the Moyal deformation by Takasaki (2001), and presumed here. These equations can be satisfied by choosing
\[ A_z = - (\partial_z h) h^{-1}, \quad A_w = - (\partial_w h) h^{-1}, \quad A_{\bar{z}} = - (\partial_{\bar{z}} \tilde{h}) \tilde{h}^{-1}, \quad A_{\bar{w}} = - (\partial_{\bar{w}} \tilde{h}) \tilde{h}^{-1}. \]

By defining \( J = \tilde{h}^{-1} h \), the third equation of (2.4) becomes
\[ \partial_z (J^{-1} \partial_z J) - \partial_{\bar{w}} (J^{-1} \partial_{\bar{w}} J) = 0. \]
This equation is called the non-commutative Yang equation and the matrix \( J \) is called Yang’s \( J \)-matrix.

Gauge transformations act on \( h \) and \( \tilde{h} \) as
\[ h \mapsto g^{-1} h, \quad \tilde{h} \mapsto g^{-1} \tilde{h}, \quad g \in G. \]

Hence Yang’s \( J \)-matrix is gauge invariant. Gauge fields are obtained from a solution \( J \) of the non-commutative Yang’s equation via a decomposition \( J = \tilde{h}^{-1} h \), and (equation (2.7)). The different decompositions correspond to different choices of gauge.

There is another potential form of the non-commutative ASDYM equation, known as the \( K \)-matrix formalism. In the gauge in which \( A_w = A_{\bar{z}} = 0 \), the third equation of (2.4) becomes \( \partial_z A_z - \partial_{\bar{w}} A_{\bar{w}} = 0 \). This implies the existence of a potential \( K \) such that \( A_z = \partial_w K, A_{\bar{w}} = \partial_{\bar{z}} K \). Then the second equation of (2.4) becomes
\[ \partial_z \partial_{\bar{z}} K - \partial_w \partial_{\bar{w}} K + [\partial_w K, \partial_z K] = 0. \quad (2.10) \]

This gauge is suitable for the discussion of (binary) Darboux transformations for the (non-commutative) ASDYM equation (Gilson et al. 1998; Nimmo et al. 2000; Saleem et al. 2007).

3. Bäcklund transformation for the non-commutative ASDYM equation

In this section, we present two kinds of Bäcklund transformations that leave the non-commutative Yang equation for \( G = GL(2) \) invariant. This is a non-commutative version of the Corrigan–Fairlie–Yates–Goddard transformation (Corrigan et al. 1978a,b). This transformation generates a class of exact solutions that belong to a non-commutative version of the Atiyah–Ward ansatz (Atiyah & Ward 1977) labelled by a non-negative integer \( l \in \mathbb{Z}_{\geq 0} \). The origin of these results will be clarified in the next section.

In order to discuss Bäcklund transformations for the non-commutative Yang equation, we parameterize the \( 2 \times 2 \) matrix \( J \) as

\[
J = \begin{bmatrix} f - g_{\bar{z}} b^{-1} e & -g_{\bar{z}} b^{-1} \\ b^{-1} e & b^{-1} \end{bmatrix}. \quad (3.1)
\]

This parameterization is always possible when \( f \) and \( b \) are invertible. In contrast to the commutative case, where only \( f \) appears, in the non-commutative setting, we need to introduce another variable, \( b \). In the commutative limit, we may choose \( b = f \).

Then the non-commutative Yang equation (2.8) is decomposed as

\[
\begin{align*}
\partial_z (f^{-1} g_z b^{-1}) - \partial_w (f^{-1} g_{\bar{w}} b^{-1}) &= 0, \\
\partial_{\bar{z}} (b^{-1} e_z f^{-1}) - \partial_{\bar{w}} (b^{-1} e_w f^{-1}) &= 0, \\
\partial_z (b^{-1} f_z) - \partial_w (b^{-1} f_{\bar{w}}) &= f^{-1} g_z b^{-1} e_z + f^{-1} g_{\bar{w}} b^{-1} e_w = 0,
\end{align*}
\]

where subscripts denote partial derivatives.

(a) The non-commutative Corrigan–Fairlie–Yates–Goddard transformation

The non-commutative Corrigan–Fairlie–Yates–Goddard transformation is a composition of the following two Bäcklund transformations for the non-commutative Yang equations (3.2).

(i) \( \beta \)-transformation (Mason & Woodhouse 1996)

\[
\begin{align*}
e_{\bar{w}}^{\text{new}} &= -f^{-1} g_{\bar{z}} b^{-1}, & e_z^{\text{new}} &= -f^{-1} g_{\bar{w}} b^{-1}, & g_{\bar{z}}^{\text{new}} &= -b^{-1} e_w f^{-1}, \\
g_{\bar{w}}^{\text{new}} &= -b^{-1} e_z f^{-1}, & f^{\text{new}} &= b^{-1}, & b^{\text{new}} &= f^{-1}.
\end{align*}
\]

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The first four equations can be interpreted as integrability conditions for the first two equations in (3.2). We can easily check that the last two equations in (3.2) are invariant under this transformation.

(ii) $\gamma_0$-transformation (Gilson et al. 2009)

\[
\begin{bmatrix}
  f_{\text{new}} \\
  e_{\text{new}} \\
  g_{\text{new}} \\
  b_{\text{new}}
\end{bmatrix} =
\begin{bmatrix}
  b & e \\
  g & f \\
  e & b \\
  f & g
\end{bmatrix}^{-1} =
\begin{bmatrix}
  (b - ef^{-1}g)^{-1} & (g - fe^{-1}b)^{-1} \\
  (e - bg^{-1}f)^{-1} & (f - gb^{-1}e)^{-1}
\end{bmatrix}.
\] (3.4)

This follows from the fact that the transformation $\gamma_0 : J \mapsto J_{\text{new}}$ is equivalent to the simple conjugation $J_{\text{new}} = C_0^{-1}JC_0$, $C_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which clearly leaves the non-commutative Yang equation (2.8) invariant. The relation (3.4) is derived by comparing elements in this transformation.

It is easy to see that $\beta \circ \beta = \gamma_0 \circ \gamma_0 = \text{id}$, the identity transformation.

**(b) Exact non-commutative Atiyah–Ward ansatz solutions**

Now we construct exact solutions by using a chain of Bäcklund transformations from a seed solution. Let us consider $b = e = f = g = \Delta_0^{-1}$. We can easily find that the decomposed non-commutative Yang equation is reduced to a non-commutative linear equation $(\partial_z \partial_{\tilde{w}} - \partial_w \partial_{\tilde{w}})\Delta_0 = 0$. (We note that, for the Euclidean space, this is the non-commutative Laplace equation because of the reality condition $\tilde{w} = -\tilde{\bar{w}}$.) Hence, we can generate two series of exact solutions $R_l$ and $R'_l$ by iterating the $\beta$- and $\gamma_0$-transformations one after the other as follows:

\[
\begin{align*}
R_0 \xrightarrow{\alpha} & R_1 \xrightarrow{\alpha} R_2 \xrightarrow{\alpha} R_3 \xrightarrow{\alpha} R_4 \rightarrow \cdots \\
R_1' \xrightarrow{\alpha'} & R_2' \xrightarrow{\alpha'} R_3' \xrightarrow{\alpha'} R_4' \rightarrow \cdots,
\end{align*}
\]

where $\alpha = \gamma_0 \circ \beta : R_l \rightarrow R_{l+1}$ and $\alpha' = \beta \circ \gamma_0 : R'_l \rightarrow R'_{l+1}$. These two kinds of series of solutions in fact arise from some class of non-commutative Atiyah–Ward ansatz. The explicit form of the solutions $R_l$ or $R'_l$ can be represented in terms of quasi-determinants whose elements $\Delta_i$ ($i = -l + 1, -l + 2, \ldots, l - 1$) satisfy

\[
\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial \tilde{w}}, \quad \frac{\partial \Delta_i}{\partial \tilde{w}} = \frac{\partial \Delta_{i+1}}{\partial z}, \quad -l + 1 \leq i \leq l - 2 \quad (l \geq 2),
\] (3.5)

which imply that every element $\Delta_i$ is a solution of the non-commutative linear equation $(\partial_z \partial_{\tilde{w}} - \partial_w \partial_{\tilde{w}})\Delta_i = 0$. A brief introduction of quasi-determinants is given in appendix A.

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The results are as follows.

(i) Non-commutative Atiyah–Ward ansatz solutions $R_l$

The elements in $J_l$ are given explicitly in terms of quasi-determinants of the same $(l+1) \times (l+1)$ matrix,

$$b_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \end{vmatrix}^{-1}, \quad f_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \end{vmatrix}^{-1},$$

$$e_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \end{vmatrix}^{-1}, \quad g_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \end{vmatrix}^{-1}.$$

In the commutative limit, we can easily check by using equation (A2) that $b_l = f_l$. The ansatz $R_0$ leads again to the so-called Corrigan–Fairlie–’t Hooft–Wilczek ansatz (Corrigan & Fairlie 1977; Wilczek 1977; G. ’t Hooft 1976, unpublished data).

(ii) Non-commutative Atiyah–Ward ansatz solutions $R'_l$

The elements in $J'_l$ are given explicitly in terms of quasi-determinants of the $l \times l$ matrices,

$$b'_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 \end{vmatrix}, \quad f'_l = \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 \end{vmatrix},$$

$$e'_l = \begin{vmatrix} \Delta_{-1} & \Delta_{-2} & \cdots & \Delta_{-l} \\ \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{l-2} & \Delta_{l-3} & \cdots & \Delta_{-1} \end{vmatrix}, \quad g'_l = \begin{vmatrix} \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\ \Delta_2 & \Delta_1 & \cdots & \Delta_{3-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 \end{vmatrix}.$$

In the commutative case, $b'_l = f'_l$ also holds. For $l = 1$, we get $b'_1 = f'_1 = \Delta_0, \ e'_1 = \Delta_{-1}, \ g'_1 = \Delta_1$ and then the relation (3.5) implies that $e'_{1,z} = f'_{1,w}, e'_{1,w} = f'_{1,z}, b'_1 = g'_1, b'_1 = g'_{1,z}$, and leads to the Corrigan–Fairlie–’t Hooft–Wilczek ansatz as first pointed out by Yang (1977).

The $\gamma_0$-transformation is proved simply, using the non-commutative Jacobi identity (equation (A 5)) applied to the four corner elements. For example,
In this, it follows from equation (A6) that the first two and last two factors are
\[
(f'_l - g'_l b'_l^{-1} e'_l).
\]

The proof of the \(\beta\)-transformation uses both the non-commutative Jacobi identity (equation (A 5)) and also the homological relations (equation (A 6)). We will consider the first equation in the \(\beta\)-transformation,

\[
e'_{l,w} = f'^{-1}_{l-1} g_{l-1,\bar{z}} b'^{-1}_{l-1}.
\]

The RHS is equal to

\[
-b'_l g_{l-1} (g'^{-1}_{l-1}) \bar{z} g_{l-1} f'
\]

In this, it follows from equation (A 6) that the first two and last two factors are

\[
b'_l g_{l-1} = \begin{bmatrix}
0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\
0 & \Delta_0 & \cdots & \Delta_{2-l} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \Delta_{l-3} & \cdots & \Delta_{-1} \\
1 & \Delta_{l-2} & \cdots & \Delta_0
\end{bmatrix}, \quad g_{l-1} f'_l = \begin{bmatrix}
\Delta_0 & \Delta_{-1} & \cdots & \Delta_{2-l} & \Delta_{1-l} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Delta_{l-2} & \Delta_{l-3} & \cdots & \Delta_0 & \Delta_{-1} \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix}.
\]

Next, from equation (A 7), we have

\[
(g'^{-1}_{l-1}) \bar{z} = \begin{bmatrix}
\Delta_0, \bar{z} & \Delta_{-1} & \cdots & \Delta_{1-l} \\
\Delta_{1,\bar{z}} & \Delta_0 & \cdots & \Delta_{2-l} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta_{l-2,\bar{z}} & \Delta_{l-3} & \cdots & \Delta_{-1} \\
\Delta_{l-1,\bar{z}} & \Delta_{l-2} & \cdots & \Delta_0
\end{bmatrix} + \sum_{k=1}^{l-1} \Delta_{l-2-k,\bar{z}} \Delta_{l-3} \cdots \Delta_{-1} \\
\times \begin{bmatrix}
\Delta_{-k,\bar{z}} & \Delta_{-1} & \cdots & \Delta_{-k} & \Delta_{1-l} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Delta_{l-2} & \Delta_{l-3} & \cdots & \Delta_{l-2-k} & \Delta_{-1} \\
0 & 0 & \cdots & 1 & \cdots & 0
\end{bmatrix}.
\]
The effect of the left and right factors on this expression is to move expansion points as specified in equation (A6), obtaining

\[
f_i^{-1} g_{l-1,z} b_{l-1}^{-1} = - \begin{vmatrix}
\Delta_{-1} & \cdots & \Delta_{1-l} \\
\Delta_0 & \cdots & \Delta_{2-l} \\
\vdots & \ddots & \vdots \\
\Delta_{l-3} & \cdots & \Delta_{-1} \\
\Delta_{l-2} & \cdots & \Delta_0 \\
\end{vmatrix}
\begin{vmatrix}
\Delta_{-1} & \cdots & \Delta_{1-l} \\
\Delta_0 & \cdots & \Delta_{2-l} \\
\vdots & \ddots & \vdots \\
\Delta_{l-2} & \cdots & \Delta_0 \\
\end{vmatrix} - \sum_{k=0}^{l-2} \begin{vmatrix}
\Delta_{-1} & \cdots & \Delta_{1-l} \\
\Delta_0 & \cdots & \Delta_{2-l} \\
\vdots & \ddots & \vdots \\
\Delta_{l-2} & \cdots & \Delta_0 \\
\end{vmatrix}.
\]

On the other hand,

\[
e_i'_{l,w} = \begin{vmatrix}
\Delta_{-1} & \cdots & \Delta_{1-l} \\
\Delta_0 & \cdots & \Delta_{2-l} \\
\vdots & \ddots & \vdots \\
\Delta_{l-3} & \cdots & \Delta_{-1} \\
\Delta_{l-2} & \cdots & \Delta_0 \\
\end{vmatrix} + \sum_{k=0}^{l-2} \begin{vmatrix}
\Delta_{-1} & \cdots & \Delta_{1-l} \\
\Delta_0 & \cdots & \Delta_{2-l} \\
\vdots & \ddots & \vdots \\
\Delta_{l-2} & \cdots & \Delta_0 \\
\end{vmatrix} + \begin{vmatrix}
\Delta_{-1} & \cdots & \Delta_{1-l} \\
\Delta_0 & \cdots & \Delta_{2-l} \\
\vdots & \ddots & \vdots \\
\Delta_{l-2} & \cdots & \Delta_0 \\
\end{vmatrix}.
\]

and then the result follows immediately from \( \Delta_{i,w} = \Delta_{i+1,z} \) in equation (3.5).

We can find that the proof of these results relies on using quasi-determinant identities alone. Thus, we can conclude that non-commutative Bäcklund transformations are identities of quasi-determinants.

We can also present a compact form of the whole of Yang’s J-matrix in terms of a single quasi-determinant expanded by a \( 2 \times 2 \) submatrix,

\[
\begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{bmatrix} := \begin{bmatrix}
a & b \\
d & e \\
g & h \\
\end{bmatrix} \begin{bmatrix}
a \\
d \\
g \\
\end{bmatrix}.
\]
The solutions for the $J$-matrix can be presented as follows.

(i) Non-commutative Atiyah–Ward ansatz solutions $R_l$

\[
J_l = \begin{bmatrix}
0 & -1 & 0 & \cdots & 0 & 0 \\
1 & \Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} \\
0 & \Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} \\
0 & \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 \\
\end{bmatrix}, \quad J_l^{-1} = \begin{bmatrix}
\Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} \\
\Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} \\
\Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 \\
0 & 0 & \cdots & 0 & -1 \\
\end{bmatrix}.
\]

(ii) Non-commutative Atiyah–Ward ansatz solutions $R'_l$

\[
J'_l = \begin{bmatrix}
\Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} & \Delta_{-l} \\
\Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 & \Delta_{-1} \\
\Delta_l & \Delta_{l-1} & \cdots & \Delta_1 & \Delta_0 \\
1 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}, \quad J'_l^{-1} = \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & \Delta_0 & \cdots & \Delta_{2-l} & \Delta_{1-l} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0 \\
-1 & \Delta_l & \Delta_{l-1} & \cdots & \Delta_1 \\
\end{bmatrix}.
\]

Because $J$ is gauge invariant, this shows that the present Bäcklund transformation is not just a gauge transformation but a non-trivial transformation.

The proof of these representations is given by using the non-commutative Jacobi identity, homological relations and the inversion formula for $J$,

\[
J^{-1} = \begin{bmatrix}
f^{-1} & f^{-1}g \\
f^{-1} & b - ef^{-1}g \\
\end{bmatrix},
\]

or simply using the formula (A.8). (For a detailed proof, see appendix A in Gilson et al. (2009).)

(c) Some explicit examples

By solving the non-commutative linear equation $(\partial_z \partial_{\bar{z}} - \partial_w \partial_{\bar{w}})\Delta_0 = 0$ for the seed solution of the Bäcklund transformations, we can obtain exact solutions explicitly.

For example, in Euclidean space, the non-commutative linear equation is just the four-dimensional non-commutative Laplace equation whose solutions include a non-commutative version of the fundamental solution: $\Delta_0 = 1 + \sum_{p=1}^{k}(a_p/(z\bar{z} - w\bar{w}))$ ($a_p$ are constants), which leads to non-commutative instanton solutions whose instanton number is $k$ (Nekrasov & Schwarz 1998; Correa et al. 2001; Lechtenfeld & Popov 2002). The Bäcklund transformations do not increase the instanton number.
There is also a simple new solution,
\[ \Delta_0 = c \exp(az + b\tilde{z} + aw + b\tilde{w}), \] (3.10)
where \( a, b \) and \( c \) are constants. This leads to a non-commutative version of nonlinear plane wave solutions (de Vega 1988). These solutions behave as standard solitons in lower dimension and do not decay at infinity, which implies that this gives an infinite value for the Yang–Mills action. By following the analysis used in Hamanaka (2007) for the non-commutative KP equation, the asymptotic behaviour of these solutions can be shown to be the same as the corresponding commutative ones.

Other solutions are also easily obtained and a more detailed discussion on this topic will be reported elsewhere.

4. Twistor descriptions of the non-commutative ASDYM equations

In this section, we explain the origin of the Bäcklund transformations for the non-commutative ASDYM equations and non-commutative Atiyah–Ward ansatz solutions from the geometrical viewpoint of non-commutative twistor theory. Here, we just need a one-to-one correspondence between a solution of the non-commutative ASDYM equations and a patching matrix \( P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) \) of a non-commutative holomorphic vector bundle on a non-commutative three-dimensional projective space, which is called the non-commutative Penrose–Ward correspondence. This correspondence is established in the Moyal-deformed case by Kapustin et al. (2001), Takasaki (2001), Lechtenfeld & Popov (2002), Brain (2005) and Brain & Majid (2008); and here we apply their formal procedure to general non-commutative situations. Such twistor treatments are useful not only for constructing exact solutions but also for checking whether the Bäcklund transformations act on the solution spaces transitively.

In order to review this correspondence briefly, we introduce another linear system defined on another local patch whose (commutative) coordinate is \( \tilde{\zeta} = 1/\zeta \),
\[ (\tilde{\zeta} D_w - D_{\tilde{z}})\tilde{\psi} = 0, \quad (\tilde{\zeta} D_z - D_{\tilde{w}})\tilde{\psi} = 0. \] (4.1)
A non-trivial solution \( \tilde{\psi} \) (\( N \times N \) matrix) of the linear system (4.1) is supposed to exist and is not regular at \( \tilde{\zeta} = \infty \) (or equivalently \( \zeta = 0 \)) as discussed earlier for \( \psi \).

Any solution of the non-commutative ASDYM equation determines solution \( \psi \) and \( \tilde{\psi} \) that are unique up to gauge transformation, and then the corresponding patching matrix is given by
\[ P(x; \zeta) = \tilde{\psi}^{-1}(x; \zeta)\psi(x; \zeta). \] (4.2)
Conversely, if a patching matrix \( P = P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) \) is factorized as
\[ P(\zeta w + \tilde{z}, \zeta z + \tilde{w}, \zeta) = \tilde{\psi}^{-1}(x; \zeta)\psi(x; \zeta), \] (4.3)
where \( \psi \) and \( \tilde{\psi} \) are regular near \( \zeta = 0 \) and \( \zeta = \infty \), respectively, then \( \psi \) and \( \tilde{\psi} \) are solutions of the linear system (2.1) for the non-commutative ASDYM equation. Then we can recover the ASDYM gauge fields in terms of \( h \) and \( \tilde{h} \) by using equation (2.7) and the fact that \( h(x) = \psi(x, \zeta = 0), \tilde{h}(x) = \tilde{\psi}(x, \zeta = \infty) \). (We can easily understand this by comparing the linear systems (2.1) and (4.1) with (2.6).)
In the commutative case, if $P$ is holomorphic w.r.t. $\xi$, then the factorization is guaranteed by the Birkhoff factorization theorem. In the case of the Moyal deformation, this is formally proved by Takasaki (2001). Here, we will see that, under the Atiyah–Ward ansatz for the patching matrix, the factorization problem (the Riemann–Hilbert problem) is solved.

In this section, we fix the gauge to be, what we call in this paper, the Mason–Woodhouse gauge

$$
J = \begin{bmatrix}
    f - gb^{-1}e & -gb^{-1} \\
    b^{-1}e & -gb^{-1}
\end{bmatrix} = \begin{bmatrix}
    1 & g \\
    0 & b
\end{bmatrix}^{-1} \begin{bmatrix}
    f & 0 \\
    e & 1
\end{bmatrix} = h_{MW}^{-1} h_{MW}.
$$

(4.4)

We note that the gauge transformation $g = \text{diag}(f^{1/2}, b^{1/2})$ connects the Mason–Woodhouse gauge with a non-commutative version of Yang’s $R$-gauge (Yang 1977),

$$
J = \begin{bmatrix}
    f - gb^{-1}e & -gb^{-1} \\
    b^{-1}e & -gb^{-1}
\end{bmatrix} = \begin{bmatrix}
    f^{-1/2} & f^{-1/2} \\
    0 & b^{1/2}
\end{bmatrix}^{-1} \begin{bmatrix}
    f^{1/2} & 0 \\
    b^{-1/2}e & b^{-1/2}
\end{bmatrix} = h_{R}^{-1} h_{R},
$$

(4.5)

where the square root is considered to be any quantity that satisfies $f^{1/2} f^{1/2} = f$, $f^{-1/2} := (f^{1/2})^{-1}$, and, whenever this notation is used, it is assumed to exist.

The wave functions $\psi$ and $\tilde{\psi}$ can be expanded by $\xi$ and $\tilde{\xi} = 1/\xi$, respectively,

$$
\psi = h + O(\xi) = \begin{bmatrix}
    h_{11} + \sum_{i=1}^{\infty} a_i \xi^i \\
    h_{21} + \sum_{i=1}^{\infty} c_i \xi^i
\end{bmatrix} \begin{bmatrix}
    h_{12} + \sum_{i=1}^{\infty} b_i \xi^i \\
    h_{22} + \sum_{i=1}^{\infty} d_i \xi^i
\end{bmatrix},
$$

$$
\tilde{\psi} = \tilde{h} + O(\tilde{\xi}) = \begin{bmatrix}
    \tilde{h}_{11} + \sum_{i=1}^{\infty} \tilde{a}_i \tilde{\xi}^i \\
    \tilde{h}_{21} + \sum_{i=1}^{\infty} \tilde{c}_i \tilde{\xi}^i
\end{bmatrix} \begin{bmatrix}
    \tilde{h}_{12} + \sum_{i=1}^{\infty} \tilde{b}_i \tilde{\xi}^i \\
    \tilde{h}_{22} + \sum_{i=1}^{\infty} \tilde{d}_i \tilde{\xi}^i
\end{bmatrix},
$$

(4.6)

(a) Riemann–Hilbert problem for non-commutative Atiyah–Ward ansatz

From now on, we restrict ourselves to $G = GL(2)$. In this case, we can take a simple ansatz for the patching matrix $P$, which is called the Atiyah–Ward ansatz in the commutative case (Atiyah & Ward 1978). The non-commutative generalization of this ansatz is straightforward, and actually leads to a solution of the factorization problem. The $l$-th order non-commutative Atiyah–Ward ansatz ($l = 0, 1, 2, \ldots$) is specified by choosing the patching matrix to be

$$
P_l(x; \xi) = \begin{bmatrix}
    0 & \xi^{-l} \\
    \xi^l & \Delta(x; \xi)
\end{bmatrix}.
$$

(4.7)

(The standard representation of the ansatz is not $P_l$ but $C_0 P_l$. Both representations are essentially the same.) We note that the coordinate dependence of $P_l = P_l(\xi w + \bar{z}, \xi z + \bar{w}, \xi)$ implies that $(\partial_w - \xi (\partial_z)) \Delta = 0$, $(\partial_z - \xi (\partial_w)) \Delta = 0$. Hence, the Laurent expansion of $\Delta$ w.r.t. $\xi$,

$$
\Delta(x; \xi) = \sum_{i=-\infty}^{\infty} \Delta_i(x) \xi^{-i},
$$

(4.8)
gives rise to the following relationships among the coefficients:

\[
\frac{\partial \Delta_i}{\partial z} = \frac{\partial \Delta_{i+1}}{\partial w}, \quad \frac{\partial \Delta_i}{\partial w} = \frac{\partial \Delta_{i+1}}{\partial \bar{z}},
\]

(4.9)

which coincide with the recurrence relation (3.5). We will soon see that the coefficients \(\Delta_i(x)\) are the scalar functions in the solutions generated by the Bäcklund transformations in the previous section.

We will now solve the factorization problem \(\tilde{\psi} F_i = \psi\) for the non-commutative Atiyah–Ward ansatz. In explicit form this is

\[
\begin{bmatrix}
\tilde{\psi}_{11} & \tilde{\psi}_{12} \\
\tilde{\psi}_{21} & \tilde{\psi}_{22}
\end{bmatrix}
\begin{bmatrix}
0 & \zeta^{-l} \\
\zeta^l & \Delta(x; \zeta)
\end{bmatrix}
= \begin{bmatrix}
\psi_{11} & \psi_{12} \\
\psi_{21} & \psi_{22}
\end{bmatrix},
\]

(4.10)

where \(\psi_{ij}\) is the \((i,j)\)-th element of \(\psi\), and so,

\[
\begin{align*}
\tilde{\psi}_{12} \zeta^l &= \psi_{11}, \quad \tilde{\psi}_{22} \zeta^l = \psi_{21}, \\
\tilde{\psi}_{11} \zeta^{-l} + \tilde{\psi}_{12} \Delta &= \psi_{12}, \quad \tilde{\psi}_{21} \zeta^{-l} + \tilde{\psi}_{22} \Delta = \psi_{22}.
\end{align*}
\]

(4.11, 4.12)

From equations (4.6) and (4.11), we find that some entries in \(\psi\) and \(\tilde{\psi}\) are polynomials w.r.t. \(\zeta\) and \(\tilde{\zeta} = \zeta^{-1},

\[
\begin{align*}
\psi_{11} &= h_{11} + a_1 \zeta + a_2 \zeta^2 + \ldots + a_{l-1} \zeta^{l-1} + \tilde{h}_{12} \zeta^l, \\
\psi_{21} &= h_{21} + c_1 \zeta + c_2 \zeta^2 + \ldots + c_{l-1} \zeta^{l-1} + \tilde{h}_{22} \zeta^l, \\
\tilde{\psi}_{12} &= \tilde{h}_{12} + a_{l-1} \zeta^{-1} + a_{l-2} \zeta^{-2} + \ldots + a_1 \zeta^{1-l} + h_{11} \zeta^{-l}, \\
\tilde{\psi}_{22} &= \tilde{h}_{22} + c_{l-1} \zeta^{-1} + c_{l-2} \zeta^{-2} + \ldots + c_1 \zeta^{1-l} + h_{21} \zeta^{-l}.
\end{align*}
\]

(4.13)

By substituting these formulas into equation (4.12), we get the following sets of equations for \(h\) and \(\tilde{h}\) in the coefficients of \(\zeta^0, \zeta^{-1}, \ldots, \zeta^{-l}:

\[
\begin{align*}
(h_{11}, a_1, \ldots, a_{l-1}, \tilde{h}_{12}) D_{l+1} &= (-\tilde{h}_{11}, 0, \ldots, 0, h_{12}), \\
(h_{21}, c_1, \ldots, c_{l-1}, \tilde{h}_{22}) D_{l+1} &= (-\tilde{h}_{21}, 0, \ldots, 0, h_{22}),
\end{align*}
\]

(4.14)

where

\[
D_l := \begin{bmatrix}
\Delta_0 & \Delta_{-1} & \cdots & \Delta_{1-l} \\
\Delta_1 & \Delta_0 & \cdots & \Delta_{2-l} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta_{l-1} & \Delta_{l-2} & \cdots & \Delta_0
\end{bmatrix}.
\]

(4.15)

These non-commutative linear equations can be solved in terms of quasi-determinants (cf. equation (A1)) as

\[
\begin{align*}
h_{11} &= h_{12} |D_{l+1}|_{1,1}^{-1}_{1,1} - \tilde{h}_{11} |D_{l+1}|_{1,1}^{-1}, \\
h_{21} &= h_{22} |D_{l+1}|_{1,1}^{-1}_{1,1} - \tilde{h}_{21} |D_{l+1}|_{1,1}^{-1}, \\
\tilde{h}_{12} &= h_{12} |D_{l+1}|_{l+1,1}^{-1}_{1,1} - \tilde{h}_{11} |D_{l+1}|_{l+1,1}^{-1}, \\
\tilde{h}_{22} &= h_{22} |D_{l+1}|_{l+1,1}^{-1}_{1,1} - \tilde{h}_{21} |D_{l+1}|_{l+1,1}^{-1}.
\end{align*}
\]

(4.16)
In equation (4.16) there are four equations but eight unknowns and so, in order to solve them, it is necessary to impose four conditions corresponding to a choice of gauge. In particular, the Mason–Woodhouse gauge \((h_{12} = \tilde{h}_{21} = 0, \tilde{h}_{11} = h_{22} = 1)\) leads to simple representations,

\[
\begin{align*}
  h_{11} &= -\begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \\
  \end{vmatrix}^{-1}, \\
  h_{21} &= \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \\
  \end{vmatrix}^{-1}, \\
  \tilde{h}_{12} &= -\begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-2} & \cdots & \Delta_0 \\
  \end{vmatrix}^{-1}, \\
  \tilde{h}_{22} &= \begin{vmatrix} \Delta_0 & \Delta_{-1} & \cdots & \Delta_{-l} \\ \Delta_1 & \Delta_0 & \cdots & \Delta_{1-l} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_l & \Delta_{l-1} & \cdots & \Delta_0 \\
  \end{vmatrix}^{-1}, \\
\end{align*}
\]

which coincides exactly with the solutions \(R_l\) generated by the Bäcklund transformation in the previous section, except for signs in \(f_l\) and \(g_l\). (The mismatch of the signs is not essential because it can be absorbed into the reflection symmetry \(f \mapsto -f, g \mapsto -g\) of the non-commutative Yang equation (3.2).) That is why we call them the non-commutative Atiyah–Ward ansatz solutions. The class of solutions \(R'_{l}\) is also obtained in the same way by starting with the Atiyah–Ward ansatz \(C^{-1}_0 P_l C_0\).

(b) *Origin of the non-commutative Corrigan–Fairlie–Yates–Goddard transformation*

Finally, let us discuss the origin of the non-commutative Corrigan–Fairlie–Yates–Goddard transformation, constructed from the \(\beta\)-transformation and the \(\gamma_0\)-transformation, and give a generalization of it. Such geometrical understanding is useful when discussing whether the group action of the Bäcklund transformations is transitive, and hence to find the symmetry of the non-commutative ASDYM equation. The present results are essentially due to Mason et al. (1988a, b) and Mason & Woodhouse (1996). These transformations can be viewed as adjoint actions of the patching matrix \(P\),

\[
\beta : P \mapsto P^{\text{new}} = B^{-1} PB, \quad \gamma_0 : P \mapsto P^{\text{new}} = C_0^{-1} PC_0, \quad (4.18)
\]

where

\[
B = \begin{bmatrix} 0 & 1 \\ \xi^{-1} & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \quad (4.19)
\]

It is obvious that \(\beta \circ \beta = id\), \(\gamma_0 \circ \gamma_0 = id\).

The composition of these transformations actually maps the \(l\)-th Atiyah–Ward ansatz to the \((l+1)\)-th one,

\[
P_l \mapsto C_0^{-1} B^{-1} \begin{bmatrix} 0 & \xi^{-l} \\ \xi^l & \Delta \end{bmatrix} BC_0 = \begin{bmatrix} 0 & \xi^{-(l+1)} \\ \xi^{l+1} & \Delta \end{bmatrix} = P_{l+1}. \quad (4.20)
\]

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The action of $C_0$ leads to $h \mapsto hC_0, \tilde{h} \mapsto \tilde{h}C_0$ and hence to the $\gamma_0$-transformation, and the action of $B$ is defined at the level of $\psi$ and $\tilde{\psi}$ as follows:

$$\psi^{\text{new}} = g^{-1}\psi B, \quad \tilde{\psi}^{\text{new}} = g^{-1}\tilde{\psi} B,$$

where

$$g^{-1} = \begin{bmatrix} 0 & \xi b^{-1} \\ f^{-1} & 0 \end{bmatrix}.$$  \hspace{1cm} (4.22)

The gauge transformation $g$ is needed to maintain the regularity of $\psi$ and $\tilde{\psi}$, w.r.t $\zeta$ and $\tilde{\zeta}$, respectively, in the factorization of $P$.

The explicit calculation gives

$$\psi^{\text{new}} = \begin{bmatrix} b^{-1}\psi_{22} & \xi b^{-1}\psi_{21} \\ \xi^{-1}f^{-1}\psi_{12} & f^{-1}\psi_{11} \end{bmatrix}, \quad (4.23)$$

In the $\zeta \to 0$ limit, this reduces to

$$h^{\text{new}} = \begin{bmatrix} f^{\text{new}} & 0 \\ e^{\text{new}} & 1 \end{bmatrix} = \begin{bmatrix} b^{-1} & 0 \\ f^{-1}k_{12} & 1 \end{bmatrix}, \quad (4.24)$$

where $\psi = h + k\zeta + O(\zeta^2)$.

Here, we note that the linear system (2.1) can be represented in terms of $b, f, e, g$ as

$$L\psi = (\partial_w - \zeta \partial_{\tilde{z}})\psi + \begin{bmatrix} -f_wf^{-1} & \xi g_{z}b^{-1} \\ -e_wf^{-1} & \xi b_{\tilde{z}}b^{-1} \end{bmatrix} \psi = 0,$$

$$M\psi = (\partial_{\tilde{z}} - \zeta \partial_w)\psi + \begin{bmatrix} -f_{\tilde{z}}f^{-1} & \xi g_{\tilde{w}}b^{-1} \\ -e_{\tilde{z}}f^{-1} & \xi b_{\tilde{w}}b^{-1} \end{bmatrix} \psi = 0.$$  \hspace{1cm} (4.25)

By considering the first-order term of $\zeta$ in the (1,2) component of the first equation, we find that

$$\partial_w(f^{-1}k_{12}) = -f^{-1}g_{\tilde{z}}b^{-1}.$$  \hspace{1cm} (4.26)

Hence from the (1,1) and (2,1) components of equation (4.24), we have

$$f^{\text{new}} = b^{-1}, \quad \partial_w e^{\text{new}} = \partial_w (f^{-1}k_{12}) = -f^{-1}g_{\tilde{z}}b^{-1},$$

which are just parts of the $\beta$-transformation (equation (3.3)). In a similar way, we can get the other ones. Therefore, the $\beta$-transformation (equation (3.3)) can be interpreted as the transformation of the patching matrix $P \mapsto B^{-1}PB$ together with the gauge transformation $g$. The results presented in this section and the previous one lead to a simpler proof of the results in §3.

We note that the $\gamma_0$-transformation can be generalized to the following transformation (the $\gamma$-transformation):

$$\gamma : P \mapsto P^{\text{new}} = C^{-1}PC,$$

where $C$ is an arbitrary constant matrix. The actions of $\beta$- and $\gamma$-transformations generate the action of the loop group $LGL(2)$ on $P$ by conjugation. Therefore, the symmetry group of the non-commutative ASDYM equation includes the loop group $LGL(2)$ as a subgroup.
5. Conclusion and discussion

In this paper, we have presented Bäcklund transformations for the non-commutative ASDYM equation with \( G = GL(2) \) and constructed from a simple seed solution a series of exact non-commutative Atiyah–Ward ansatz solutions expressed explicitly in terms of quasi-determinants. We have found that the Bäcklund transformations generate a wide class of new solutions. We have also given the origin of the Bäcklund transformation and the generated solutions in the framework of non-commutative twistor theory and generalized them.

The present results could be taken as the starting point to reveal an infinite-dimensional symmetry of the non-commutative ASDYM equation in terms of some infinite-dimensional algebra. We have to prove that the Atiyah–Ward ansatz covers all solutions of the non-commutative ASDYM equation and generalize the Bäcklund transformations \( \beta \) and \( \gamma \), so that they should act on the solution space transitively.

Furthermore, investigation of the non-commutative extension of a bilinear form approach to the ASDYM equation (Gilson et al. 1998; Sasa et al. 1998; Wang & Wadati 2004) would be beneficial because many aspects in these papers are close to ours. The relationship with non-commutative Darboux and non-commutative binary Darboux transformations (Saleem et al. 2007) is also interesting.

M.H. would like to thank L. Mason and J.J.C.N. for hospitality (and many helpful comments from L. Mason) during his stay at the Mathematical Institute, University of Oxford, and at the Department of Mathematics, University of Glasgow, respectively. The work of M.H. was supported by Grant-in-Aid for Young Scientists (no. 18740142), the Nishina Memorial Foundation and the Daiko Foundation.

Appendix A. Brief review of quasi-determinants

In this section, we give a brief introduction to quasi-determinants, introduced by Gelfand & Retakh (1991), in which a few of the key properties that play important roles in §3 are described. More detailed discussion is seen in the survey (Gelfand et al. 2005).

Quasi-determinants are defined in terms of inverse matrices and we suppose the existence of all matrix inverses to which reference is made. Let \( A = (a_{ij}) \) be an \( n \times n \) matrix and \( B = (b_{ij}) \) be the inverse matrix of \( A \), that is, \( AB = BA = 1 \). Here, the matrix entries belong to a non-commutative ring. Quasi-determinants of \( A \) are defined formally as the inverses of the entries in \( B \)

\[
|A|_{ij} := b_{ji}^{-1}. \quad (A1)
\]

In the case that variables commute, this is reduced to

\[
|A|_{ij} = (-1)^{i+j} \frac{\det A}{\det A^{ij}}, \quad (A2)
\]

where \( A^{ij} \) is the matrix obtained from \( A \) by deleting the \( i \)-th row and the \( j \)-th column.
We can also write down a more explicit definition of quasi-determinants. In order to see this, let us recall the following formula for the inverse of a square $2 \times 2$ block square matrix

$$
\begin{bmatrix}
A & B \\
C & d
\end{bmatrix}^{-1} = \begin{bmatrix}
A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\
-S^{-1}CA^{-1} & S^{-1}
\end{bmatrix},
$$

where $A$ is a square matrix, $d$ is a single element and $B$ and $C$ are column and row vectors of appropriate length and $S = d - CA^{-1}B$ is called a Schur complement. In fact, this formula is valid for $A$, $B$, $C$ and $d$ in any ring, and not just for matrices. Thus, the quasi-determinant associated with the bottom right element is simply $S$.

By choosing an appropriate partitioning, any entry in the inverse of a square matrix can be expressed as the inverse of a Schur complement, and, hence, quasi-determinants can also be defined recursively by

$$
|A|_{ij} = a_{ij} - \sum_{i',j' \neq i,j} a_{ii'}(A^{ij^{-1}})_{i'j'}a_{j'j} = a_{ij} - \sum_{i',j' \neq i,j} a_{ii'}(|A|^{ij}_{j'j'})^{-1}a_{j'j}, \quad (A3)
$$

It is sometimes convenient to use the following alternative notation in which a box is drawn about the corresponding entry in the matrix:

$$
|A|_{ij} = \begin{vmatrix}
a_{11} & \cdots & a_{ij} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
a_{11} & \cdots & a_{ij} & \cdots & a_{nn}
\end{vmatrix}.
$$

Quasi-determinants have various interesting properties similar to those of determinants. Among them, the following play important roles in this paper. In the block matrices given in these results, lower case letters denote single entries and upper case letters denote matrices of compatible dimensions so that the overall matrix is square.

(i) Non-commutative Jacobi identity.

A simple and useful special case of the non-commutative Sylvester's theorem (Gelfand & Retakh 1991) is

$$
\begin{vmatrix}
A & B & C \\
D & f & g \\
e & h & i
\end{vmatrix} = \begin{vmatrix}
A & C \\
E & i
\end{vmatrix} - \begin{vmatrix}
A & B \\
E & h
\end{vmatrix} \begin{vmatrix}
A & B^{-1} & A \\
D & f & g
\end{vmatrix} = \begin{vmatrix}
A & C \\
D & g
\end{vmatrix}.
$$

(ii) Homological relations (Gelfand & Retakh 1991) are given by

$$
\begin{vmatrix}
A & B & C \\
D & f & g \\
e & h & i
\end{vmatrix} = \begin{vmatrix}
A & B & C \\
D & f & g \\
e & h & i
\end{vmatrix} = \begin{vmatrix}
A & B & C \\
D & f & g \\
e & h & i
\end{vmatrix} = \begin{vmatrix}
A & B & C \\
D & f & g \\
e & h & i
\end{vmatrix}.
$$

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(iii) A derivative formula for quasi-determinants (Gilson & Nimmo 2007) is
\[
\begin{vmatrix} A & B \\ C & d \end{vmatrix}' = \begin{vmatrix} A & B' \\ C & d' \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} A & (A_k)' \\ C & (C_k)' \end{vmatrix} \begin{vmatrix} A & B \\ C & d \end{vmatrix}
\]
where \(A_k\) is the \(k\)th column of a matrix \(A\) and \(e_k\) is the column \(n\)-vector \((\delta_{ik})\) (i.e. 1 in the \(k\)th row and 0 elsewhere).

(iv) A special formula of inverse of a quasi-determinant (Gilson et al. 2007) is
\[
\begin{vmatrix} a & B & c & \alpha \\ D & E & F & 0 \\ g & H & i & 0 \\ \beta & 0 & 0 & 0 \end{vmatrix}^{-1} = \begin{vmatrix} 0 & 0 & 0 & \gamma \\ 0 & 0 & a & B \\ 0 & \delta & g & H \\ i & 0 & 0 & 0 \end{vmatrix},
\]
with \(\alpha \beta = \gamma \delta = -1, \alpha + \gamma = 0\), where lower case letters denote single entries, upper case letters denote matrices of compatible dimensions and Greek letters are scalars (i.e. commute with everything).

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