Quantum Fractal Fluctuations

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We numerically analyse quantum survival probability fluctuations in an open, classically chaotic system. In a quasi-classical regime, and in the presence of classical mixed phase space, such fluctuations are believed to exhibit a fractal pattern, on the grounds of semiclassical arguments. In contrast, we work in a classical regime of complete chaos, and in a deep quantum regime of strong localization. We provide evidence that fluctuations are still fractal, due to the slow, purely quantum algebraic decay in time produced by dynamical localization. Such findings considerably enlarge the scope of the existing theory.

PACS numbers: 05.45.Df, 05.40.-a, 05.45.Mt

The study of mesoscopic conductance fluctuations has gained important insights from the analysis of the underlying classical dynamics \[^1\]. In particular, it has been found that some statistical properties of quantum fluctuations are semiclassically determined by the classical law which rules the decay in time of the survival probability inside the open chaotic system \[^2\]. Building on this general relation, it was eventually predicted that, if the classical decay law is algebraic, due to the presence of residual stable islands in the classical phase space, then the quantum fluctuation pattern should display a fractal structure with a fractal dimension directly related to the classical decay exponent \[^3\]. Such fractal structure of quantum fluctuations has been interpreted as a fundamental aspect of classical fractality associated with hierarchical structures of surviving stable islands. This theoretical prediction has received experimental support \[^4\], and has been numerically confirmed on a dissipative model system, where coupling to continuum is simulated by complete absorption of that part of the wave packets which propagates outside the interaction region \[^5\].

In such a theoretical frame, quasi-classicality of the quantum dynamics appears essential in order that slow classical decay be quantally reproduced over sufficiently long time scales. In contrast, in a deep quantum regime, localization effects would become dominant, rapidly facing any memory of the underlying classical dynamics.

In this Letter we provide evidence that fractality of quantum fluctuations nevertheless survives even in such strongly non-classical situations. In the model systems presently to be described, the classical phase space has no significant stable islands, the classical motion is diffusive, and the classical decay is exponential; therefore, in the quantum weakly localized regime, one observes non-fractal, Lorentz-correlated conductance fluctuations \[^6\]. In contrast, we work in a strongly localized regime, and observe fractal fluctuations, apparently unrelated to any classical phase-space structure. Decisive in this respect is the fact that the quantum decay, albeit highly non-classical, is algebraic due to purely quantum localization effects, as recently predicted \[^7\]. As a matter of fact, we find that the fractal dimension of quantum fluctuations is connected to the algebraic decay exponent in exactly the same way predicted by the semiclassical theory \[^8\]. It therefore appears that the quantum fluctuation pattern bears the same relation to the quantum probability decay, which was originally predicted by semiclassical arguments, even though such decay is not by any means quasi-classical.

We consider the kicked rotator model \[^10\] with absorbing boundary conditions. Classical dynamics is described by the standard map:

$$\hat{I} = I + k \sin \theta, \quad \hat{\theta} = \theta + T \hat{I},$$

(1)

with absorption for $\hat{I} < 0$ and for $\hat{I} > N$. Classical dynamics only depends on the scaling parameter $K = kT$. The corresponding quantum map is obtained by the substitution $I \rightarrow \hbar \hat{n} = -i\hbar \partial / \partial \theta$. We denote $\psi_t$ the state vector immediately before the $t$-th kick, $T$ the kicking period, and we set $\hbar = 1$. Throughout this Letter, we will treat the kick counter $t$ as a discrete time variable in units of $T$. The discrete-time evolution from time $t$ to time $t+1$ is described in the classical case by \[^1\] and in the quantum case by :

$$\tilde{\psi}_{t+1} = \hat{P} \psi_{t+1} = \hat{P} e^{-i T(n+\phi)^2/2} e^{-i k \cos \theta} \psi_t,$$

(2)

The state $\tilde{\psi}_{t+1}$ is obtained from $\psi_t$ by means of the unitary one-period propagator of the kicked rotor. The operator $\hat{P}$ then projects $\tilde{\psi}_{t+1}$ over states in a fixed interval of $n$ values ($0 \leq n \leq N$), so it describes complete deletion (absorption) of the part of the wave packet which propagates outside the given interval. The parameter $\phi$
can be interpreted as an Aharonov-Bohm flux through the ring parametrized by the coordinate $\theta$.

We consider the case $K = kT = 7$, $k = 5$, with absorption for $n < 0$ and $n > N = 250$. With such parameter values, the classical phase space has no significant island of stability [5]. We consider a statistical ensemble of orbits starting at $t = 0$ with $I = 0$ and randomly distributed phases $\theta$. The survival probability $P(t)$ is the fraction of the ensemble which hasn’t been absorbed at time $t$; equivalently, $P(t)$ is the integrated distribution of exit times of orbits in the ensemble.

\[ P(t) = \frac{1}{N} \sum_{n=0}^{N-1} \exp(-\gamma I_n), \]

with $\gamma \approx 1/t_D \approx 2.2 \times 10^{-3}$. In contrast, quantum dynamics is strongly localized, as the localization length $\ell \approx k^2$ is much less than the “sample size” $N$. Such strong localization causes a slow decay of the quantum survival probability $P_\phi(t)$. The latter is defined as the total probability on states $0 < n < 250$ at time $t$, given that at $t = 0$ the state was $n_0 = 0$. After the “Heisenberg time” $t_H$, it decays proportional to $1/t$ up to a time $t_{max} \sim \exp(2N/t)$ [3], as shown in Fig.1 (in this case, $t_{max} \sim 5 \times 10^8$).

Though the exponent 1 of algebraic quantum decay is independent of the value of $\phi$, the quantum survival probability $P_\phi(t)$ at a fixed time $t$ is sensitively dependent on $\phi$. The fluctuation pattern exhibited by the graph of $P_\phi(t)$ versus $\phi$ at fixed $t$ (two examples are given in Fig.2) is precisely the object of our analysis, at values of $t$ up to $10^4$; and for $10^4$ values of $\phi$ in the interval $[0, 0.1]$. First we have computed autocorrelation functions:

\[ C(\delta \phi) = \langle P_\phi(t) P_{\phi+\delta \phi}(t) \rangle_{\phi}. \]  \hspace{1cm} (3)

Such correlations play a key role in the existing semiclassical theory of fractal fluctuations, which consists of three steps:

(i) correlation functions are related via Fourier transform to the classical decay of the canonical variable conjugate to $\phi$ [3], notably, to the function $P_{cl}(\Theta)$ yielding the probability distribution of the angle $\Theta$ accumulated by the rotor at the time of exit from the finite sample (“accumulated” means multiples of $2\pi$ included). In particular, if $P_{cl}(\Theta)$ decays as $\Theta^{-\alpha}$ at large $\Theta$, then

\[ C(\delta \phi) \sim C(0) - \text{const}|\delta \phi|^\alpha \]  \hspace{1cm} (4)

at small $\delta \phi$.

(ii) It is physically intuitive, and numerically confirmed [5], that the decay exponent $\alpha$ coincides with the decay exponent of the classical survival probability.

(iii) A signal-theoretic argument [12] allows to conclude from (i) and (ii) that the fluctuation graph has a fractal dimension given by

\[ D = 2 - \frac{\alpha}{2} \]  \hspace{1cm} (5)

where $\alpha$ is the time decay exponent of the classical survival probability. Step (iii) actually invokes certain statistical properties of fluctuations, which were assumed to be ensured by the chaotic nature of classical dynamics [13].

In our case, semiclassical arguments are invalid, because of strong quantum localization. In addition, there is no classical decay exponent $\alpha$, because classical decay is exponential. So the above theory doesn’t apply; still, numerical results to be described below demonstrate that (i), (ii) and (iii) are still legitimate in our highly non semiclassical case, too, provided one replaces the classical distribution functions $P(t), P_{cl}(\Theta)$ by the corresponding
quantum distributions \( P_\phi(t), P_\phi(\Theta) \), and \( \alpha \) by the exponent of the quantum algebraic decay. In order to compute \( P_\phi(\Theta) \) we perform a lifting of our model \( \hat{\Theta} \), that is, we translate the model \( \hat{\Theta} \) of a kicked particle on a circle into the model of a kicked particle on the line, described by the coordinate \( \Theta \). To simulate the corresponding dynamics we restrict \( \Theta \) inside a large, yet finite box, with periodic boundary conditions. At every evolution step \( t \) we compute the function \( |\psi_t(\Theta) - \psi_t(\Theta)|^2 \), which yields the probability loss occurring at time \( t \) and position \( \Theta \) due to absorption. Summing this quantity over \( t \) gives the total probability lost at position \( \Theta \), that is, the probability distribution of the angle accumulated at the exit time. The corresponding integrated distribution is shown in Fig.3 and displays a decay \( \propto \Theta^{-1} \). The faster decay in the rightmost part of Fig.3 is due to the finite total time \( t \). The corresponding integrated distribution up to that time can be assumed to decay proportional to \( \Theta^{-1} \) in a range \( \Theta_{\min} < \Theta < \Theta_{\max,t} \), with \( \Theta_{\max,t} \) increasing with \( t \) as long as \( t < t_{\text{max}} \). One can estimate \( \Theta_{\min} \) as the angle accumulated until the time \( t_{\text{H}} \approx \ell \). This gives \( \Theta_{\min} \approx \int_0^{t_{\text{H}}} \sqrt{(D/t)\tau} d\tau \sim (2/3)t_{\text{H}}^2T \). Assuming a maximal momentum \( \sim \ell \) for the rotor started at \( n_0 = 0 \), a rough estimate for \( \Theta_{\max,t} \) is provided by \( \ell tT \) (\( \Theta_{\min} \approx 2 \times 10^2 \) and \( \Theta_{\max,t} \approx 2.1 \times 10^4 \) for the case of Fig.3).

![Log-log plot of survival probability as a function of angle](image)

**FIG. 3.** Quantum integrated distribution of the accumulated angle \( \Theta \) at the exit time, computed over a total time \( t = 600 \), parameter values as in Fig.1. The straight line has slope one.

Numerically computed correlations \( \Theta_0 \) shown in the inset of Fig.4, at two different times, indicate that relation \( \Theta_{\min} \) is still valid in our strongly localized case \( \Theta_{\max} \). We have computed the fractal dimensions of the corresponding graphs of \( P_\phi(t) \) versus \( \phi \) shown in Fig.2. The results are presented in Fig.4 and demonstrate that the graph has a fractal dimension \( \approx 1.5 \) over a significant range of \( \delta \phi \) scales, as predicted by equation (3). This range roughly lies in between scales \( \delta \phi_{\min,t}, \delta \phi_{\max} \). The latter scale is inverse to \( \Theta_{\min} \). The former scale can be estimated by \( \Theta_{\max,t}^{-1} \) (hence it decreases as \( t \) increases), as long as it remains above a minimal \( \delta \phi_0 \sim 2\pi/(N^2T) \) scale. This scale is imposed by the finiteness of the sample, and can be explained as follows. The evolution operator \( \hat{\Theta} \) has complex eigenvalues inside the unit circle. Varying \( \phi \) causes these eigenvalues to move, and their maximal shift under a change \( \phi_0 \) is estimated by \( TN\delta \phi_0 \). Any time one moving eigenvalue comes close to the unit circle, a local maximum in the graph of \( P_\phi(t) \) vs \( \phi \) appears. The number of peaks produced by a single moving eigenvalue as \( \phi \) changes from 0 to 1 is at most \( \approx TN/(2\pi) \).

Further confirmation for the validity of the above illustrated scenario was obtained by analyzing the following variant of the basic model \( \hat{\Theta} \):

\[
\psi_{t+1} = \hat{P} e^{-iT(h+\phi)^2/2} e^{-ik|\cos \hat{\delta}|} \psi_t.
\]

which differs from \( \hat{\Theta} \) because the kicking potential has now a discontinuous derivative. Localization is in this case algebraic, as eigenfunctions decay like \( 1/n^2 \). The argument used to predict the \( t^{-1} \) decay of the survival probability in the case of exponential localization \( \hat{\Theta} \) yields an

![Fractal analysis for the kicked rotator model](image)

**FIG. 4.** Fractal analysis for the kicked rotator model, with parameter values as in Fig.1. The graph of \( P_\phi(t) \) vs \( \phi \) was covered by boxes of side \( \delta \phi_0 \) and the largest excursion of \( P_\phi(t) \) in each strip was recorded; summing over all strips and dividing the result by \( \delta \phi_0 \), we obtained \( N(\delta \phi_0) \). Data are shown for \( t = 10^3 \) (circles) and \( t = 10^4 \) (diamonds). The straight lines correspond to the fractal dimension \( D = 1.5 \) expected for \( \alpha = 1 \). The inset shows the correlation function \( C(\delta \phi) \) at the same times. The straight lines correspond to \( 1 - C(\delta \phi)/C(0) \propto |\delta \phi|^\alpha \), with \( \alpha = 1 \).
asymptotic decay \( \propto t^{-3/4} \). The numerical analysis illustrated above for the case of exponential localization was replicated for this model, too, yielding the results shown in Fig. 5. Fractal analysis of the fluctuation graph yields good agreement over a broad range with the theoretical \( D = 2 - \alpha/2 = 13/8 \) predicted using equation (5) and the decay exponent \( \alpha = 3/4 \).

In conclusion, we have provided numerical evidence for fractal fluctuations of the quantum survival probability in a classically chaotic system in the regime of strong localization, and in the absence of significant classical critical structures. Whereas the original theory predicted fractal conductance fluctuations, our present results are about fluctuations of a different quantity, namely survival probability. As both quantities ultimately reflect the fluctuations of the scattering matrix, their fluctuations should be similar in nature. As a matter of fact, the very same semiclassical arguments used to predict conductance fluctuations could be adapted to the survival probability, too. The main difference is that a scan of \( P_\delta(t) \) at finite however large \( t \) cannot achieve the \( \delta \)-resolution exhibited by conductance at fixed (quasi-)energy. On account of the effect of increasing \( t \) reported earlier in this Letter, such coarsening is unlikely to “fractalize” otherwise non-fractal fluctuation patterns. In any case, survival probability is a meaningful quantity in a broader class of problems than electronic transport in semiconductor structures. The scenario we have analyzed in this Letter can also be exported to other realistic problems, where ‘conductance’ is instead a problematic concept. As an example we quote microwave ionization of Rydberg atoms. This may open a way for experimental observation of localization-induced fractal fluctuations in atomic physics.

Our results show that quantum fluctuations may be fractal even in situations where the existing semiclassical theory does not apply. This does not command a reinterpretation of existing experimental data, which were obtained in situations far from the strongly localized regime considered in this Letter. Nevertheless it signals that the current understanding of the deep quantum mechanisms responsible for fractal fluctuations is still far from complete.

Support from the Progetto Avanzato INFM “Quantum transport and classical chaos” is gratefully acknowledged.

\[ \begin{align*}
\text{FIG. 5. Same as in Fig.1 (upper inset) and Fig.4 but for the discontinuous map (1). The straight lines correspond to a power law decay of the quantum survival probability with } \alpha = 3/4 \text{ (upper inset), a correlation function } C(\delta \phi) \propto |\delta \phi|^\alpha \text{ (lower inset) and a fractal dimension } D = 2 - \alpha/2 = 13/8. \text{ The x-axis in the lower inset is the same as in the main figure.}
\end{align*} \]

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