NATURAL TRANSFORMATIONS
FROM CONSTRUCTIBLE FUNCTIONS TO
HOMOLOGY

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Abstract. For complex projective varieties, all natural transformations from constructible functions to homology (modulo torsion) are linear combinations of the MacPherson-Schwartz-Chern classes. In particular, the total Chern class is the only such natural transformation $c$ such that for all projective spaces $P$, the top component of $c(1_P)$ is the fundamental class of $P$.

1. Introduction

The MacPherson-Schwartz-Chern class natural transformation $c$ from the constructible functions functor to homology satisfies a remarkably stringent normalization requirement: for each nonsingular variety $X$, the element $c(1_X)$ of homology assigned to the characteristic function of $X$ is the total Chern class $c(TX) \sim [X]$. Now suppose that we entirely abandon this requirement. Then each individual component $c_i$ of the Chern class natural transformation (assigning to the characteristic function $1_X$ of a nonsingular variety the homology class $c_i(TX) \sim [X]$) is likewise a natural transformation, as is any linear combination of these components.

We will show that, modulo torsion, these linear combinations are the only natural transformations between the two functors. In particular, the MacPherson-Schwartz-Chern class is the only such natural transformation satisfying this weak normalization requirement: for each projective space $P$, the top-dimensional component of $c(1_P)$ is the fundamental class $[P]$. We conjecture that the same statements are valid even for integral homology. We want to note two features of our proofs. First, we never appeal to resolution of singularities. Second,
several of our arguments are similar to those in the proof (attributed to A. Landman) of the last Proposition of [2].

2. THE FUNCTORS

Consider the category of complex projective algebraic varieties. If $X$ is such a variety, the characteristic function of a subvariety $Z$ is the function $1_Z$ on $X$ whose value on $Z$ is 1 and whose value elsewhere is 0. If $f : X \to Y$ is a morphism, then $f_*1_Z$ is the constructible function on $Y$ whose value at $y$ is the Euler characteristic of $f^{-1}(y) \cap Z$ (a subvariety of $X$). A finite linear combination (over $\mathbb{Z}$) of characteristic functions of subvarieties of $X$ is called a constructible function on $X$. We define the pushforward of an arbitrary constructible function by extending the previous definition by linearity, and thus define the constructible functions functor $C$ to the category of abelian groups.

We also want to work with an appropriate homology theory. This will be either of the following:

- **Ordinary singular or simplicial homology.** The fundamental class of an $n$-dimensional variety $X$ is an element of $H_{2n}(X)$.
- **Algebraic cycles modulo rational equivalence.** The standard reference is [3, Ch. 1]. The fundamental class of an $n$-dimensional variety $X$ is an element of the cycle class group $A_n(X)$.

3. PROJECTIVE SPACES

Suppose that $\tau$ is a natural transformation from $C$ to $H_{2i}$ or to $A_i$. Suppose that $n > i$; then $\mathbb{P}^i$ is naturally a linear subspace of $\mathbb{P}^n$.

**Theorem 1.** $\tau(1_{\mathbb{P}^n}) = \binom{n+1}{i+1} \tau(1_{\mathbb{P}^i})$.

**Proof.** Consider the morphism $f : \mathbb{P}^n \to \mathbb{P}^n$,

$$f : [x_0, x_1, \ldots, x_n] \mapsto [x_0^2, x_1^2, \ldots, x_n^2].$$

Let

$$L_k = \{ [x_0, x_1, \ldots, x_n] \mid \text{at least } n-k \text{ of the coordinates equal zero} \}$$

and

$$U_k = \{ [x_0, x_1, \ldots, x_n] \mid \text{exactly } n-k \text{ of the coordinates equal zero} \}.$$ 

Then $L_k$ is a union of $\binom{n+1}{k+1}$ linear subspaces of dimension $k$, and $U_k = L_k \setminus L_{k-1}$. The restriction of $f$ to $U_k$ is a self-covering map of degree $2^k$. Hence the restriction of $f$ to each component of $L_k$ is a branched self-cover of the same degree, and the induced map $f_* : H_{2k}(\mathbb{P}^n) \to H_{2k}(\mathbb{P}^n)$, or the induced map $f_* : A_k(\mathbb{P}^n) \to A_k(\mathbb{P}^n)$, is multiplication.
by $2^k$. Consider the constructible function $1_{U_k} = 1_{L_k} - 1_{L_{k-1}}$. By naturality of $\tau$ we have

$$2^k \tau(1_{U_k}) = \tau(2^k 1_{U_k}) = \tau f_*(1_{U_k}) = f_* \tau(1_{U_k}) = 2^k \tau(1_{U_k}).$$

If $i \neq k$, then $\tau(1_{U_k}) = 0$. Therefore

$$\tau(1_{P^n}) = \tau \left( 1_{L_i} + \sum_{k=i+1}^n 1_{U_k} \right) = \tau(1_{L_i}).$$

And

$$\tau(1_{L_i}) = \tau \left( \sum_{K} 1_K \right),$$

where the sum is taken over the components of $L_i$, since the two functions in question differ only on a variety of dimension $i - 1$. Since $L_i$ is a union of $\binom{n+1}{i+1}$ linear subspaces of dimension $i$, the statement of the theorem follows.

4. Galois coverings

A morphism $X \to Y$ of (irreducible) varieties is called Galois if it is finite and surjective, and the corresponding field extension $K(Y) \subset K(X)$ is Galois.

**Lemma 1.** Suppose that $X$ is a projective variety of dimension $n$. Then there exists a normal projective variety $Z$, a finite morphism $Z \to X$, and a finite morphism $X \to P^n$, such that the composition $Z \to P^n$ is Galois.

**Proof.** To construct a finite morphism $X \to P^n$, first embed $X$ in a projective space $P^N$ and then project it from a general $P^N - n - 1$. The field extension $K(P^n) \subset K(X)$ is finite; hence it can be obtained by adjoining a single element satisfying a minimal polynomial $p$. Let $K$ be the splitting field of $p$. Then $K$ is a normal extension of $K(P^n)$ [7, Thm. 8.4, p. 82]. Let $Z$ be the normalization of $X$ in $K$ [7, III.8, Thm. 3]. Then $Z$ is projective [7, III.8, Thm. 4]. Since $K(Z) = K$, the morphism $Z \to P^n$ is Galois.

**Lemma 2.** If $Z$ and $Y$ are normal varieties and $\gamma : Z \to Y$ is Galois, with Galois group $G$, then $Y$ is isomorphic to $Z/G$.

**Proof.** The finite group $G$ acts by birational maps. Let $g : Z \to Z$ be any one of these maps; let $W \subset Z \times Z$ be the closure of its graph. The projection $W \to Z$ onto the first factor is a birational morphism with finite fibers. By Zariski’s Main Theorem [7, Ch. 3, sec. 9] this projection is an isomorphism. Hence $g$ is in fact a morphism.
The morphisms $\gamma$ and $\gamma \circ g$ agree on an open dense subset of $Z$; hence they are equal. The quotient $Z/G$ is a projective variety \[5, \text{pp. 126–127}\]. By the universal property of $Z/G$, there is a morphism to $Y$. Again by Zariski, this morphism is an isomorphism. \[\square\]

**Lemma 3.** Under the same hypotheses, $\gamma_*$ maps the invariant subgroup $(H_\ast Z \otimes \mathbb{Q})^G$ isomorphically to $H_\ast(Z/G) \otimes \mathbb{Q}$. Likewise $\gamma_*$ maps $(A_\ast Z \otimes \mathbb{Q})^G$ isomorphically to $A_\ast(Z/G) \otimes \mathbb{Q}$.

**Proof.** Triangulate the quotient map \[4\]. The $G$-invariant simplicial homology of $Z$ is isomorphic to the homology of the complex of $G$-invariant simplicial chains of $Z$. (The proof is by averaging over the group $G$.) And the $G$-invariant simplicial chain complex of $Z$ is isomorphic to the simplicial chain complex of $Z/G$.

For the statement about rational equivalence groups, see \[3, \text{Example 1.7.6}\]. \[\square\]

5. The natural transformations

**Theorem 2.** Suppose that $\sigma$ is a natural transformation from the constructible function functor $C$ to $H_\ast \otimes \mathbb{Q}$, ordinary singular homology with rational coefficients, or to $A_\ast \otimes \mathbb{Q}$, rational equivalence theory with rational coefficients. Suppose that for each projective space $P$ the top-dimensional component of $\sigma_P(1_P)$ vanishes. Then $\sigma$ is identically zero.

**Corollary 1.** Suppose that $\tau$ is a natural transformation from $C$ to $H_\ast \otimes \mathbb{Q}$ or to $A_\ast \otimes \mathbb{Q}$. Then $\tau$ is a rational linear combination $\sum_{i=0}^\infty r_i c_i$ of the components of the MacPherson-Schwartz-Chern class.

Note that although the sum appearing in the corollary is nominally infinite, for any particular variety it is a finite sum.

**Proof of corollary.** Define $r_i$ to be the rational number for which

$$\tau(1_{P^i}) = r_i [P^i] + \text{terms of lower dimension}.$$  

Then apply the theorem to

$$\sigma = \tau - \sum_{i=0}^\infty r_i c_i.$$  

**Proof of theorem.** Suppose that $Z$ is a subvariety of $X$. If $\sigma(1_Z)$ is zero in the homology group of $Z$, then by naturality it is likewise zero in the homology group of $X$. Hence it will suffice to show that, for each projective variety,

$$\sigma(1_X) = 0 \quad (1)$$
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in the homology group of \( X \).

If \( X \) is a projective space, then by hypothesis the top-dimensional component of \( \sigma(1_X) \) vanishes. From Theorem 1 we deduce equation \( \text{[1]} \).

In general we use induction on the dimension of \( X \), together with the Galois covering techniques of section 4. Let \( n \) be the dimension of \( X \); suppose that \( \sigma(1_W) = 0 \) for all varieties of smaller dimension. By Lemma 1, we can construct a normal projective variety \( Z \), a finite morphism \( \pi: Z \to X \), and a finite morphism \( \rho: X \to \mathbb{P}^n \), such that the composition \( \gamma: Z \to \mathbb{P}^n \) is Galois. By Lemma 2, \( \mathbb{P}^n \) is the quotient of \( Z \) by the Galois group \( G \). The characteristic function of \( Z \) is fixed by the action of the group; hence by naturality it is an element of the invariant homology group. By naturality and the inductive hypothesis

\[
\gamma_*\sigma(1_Z) = \sigma(|G| \cdot 1_{\mathbb{P}^n} + \text{function supported on varieties of smaller dimension}) = 0.
\]

By Lemma 3, \( \gamma_* \) maps the invariant homology isomorphically to the homology of \( \mathbb{P}^n \). Hence \( \sigma(1_Z) \) must be zero. Let \( d \) be the degree of \( \pi \).

Then

\[
0 = \pi_*\sigma(1_Z) = \sigma(d \cdot 1_X) + \sigma(\text{function supported on varieties of smaller dimension}).
\]

By the inductive hypothesis the second term is zero. Hence \( \sigma(1_X) = 0 \).

Finally we wish to remark that the theorem and corollary are also valid if we interpret all functors as emanating from the larger category of compact complex algebraic varieties. Indeed, by Chow’s lemma [8, p. 282] an arbitrary compact variety is the image of a projective variety via a birational morphism. Thus we may prove an extended version of Theorem 2 by an induction on dimension.

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