Lorentz-Invariant Non-Commutative Space-Time
Based On DFR Algebra

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Abstract

It is argued that the familiar algebra of the non-commutative space-time with c-number $\theta_{\mu\nu}$ is inconsistent from a theoretical point of view. Consistent algebras are obtained by promoting $\theta_{\mu\nu}$ to an anti-symmetric tensor operator $\hat{\theta}_{\mu\nu}$. The simplest among them is Doplicher-Fredenhagen-Roberts (DFR) algebra in which the triple commutator among the coordinate operators is assumed to vanish. This allows us to define the Lorentz-covariant operator fields on the DFR algebra as operators diagonal in the 6-dimensional $\theta$-space of the hermitian operators, $\hat{\theta}_{\mu\nu}$. It is shown that we then recover Carlson-Carone-Zobin (CCZ) formulation of the Lorentz-invariant non-commutative gauge theory with no need of compactification of the extra 6 dimensions. It is also pointed out that a general argument concerning the normalizability of the weight function in the Lorentz metric leads to a division of the $\theta$-space into two disjoint spaces not connected by any Lorentz transformation so that the CCZ covariant moment formula holds true in each space, separately. A non-commutative generalization of Connes’ two-sheeted Minkowski space-time is also proposed. Two simple models of quantum field theory are reformulated on $M_4 \times Z_2$ obtained in the commutative limit.
§1. Introduction

The Lorentz invariance is one of the fundamental symmetries in (relativistic) quantum field theory (QFT). Even if the Minkowski space-time would become no longer a continuum but could be described by a non-commutative geometry at, say, Planck scales, the Lorentz invariance should be maintained simply because only Lorentz-covariant fields are qualified to represent interactions between matters and fields. There is a long history\textsuperscript{1} about the existence of a minimum length in relation with quantum mechanics, special relativity and general relativity. Such a minimum length, if exists, should be introduced into the theory without conflict to relativistic invariance.

From this point of view the popular non-commutative space-time characterized by the algebra,

\[ [\hat{x}^{\mu}, \hat{x}^{\nu}] = i\theta^{\mu\nu}, \]  

where \( \hat{x}^{\mu} \) is the coordinate operators and \( \theta^{\mu\nu} \) is a real, anti-symmetric constant, is unacceptable as a possible space-time structure at short distances because it violates the Lorentz symmetry. The apparent lack of the Lorentz covariance of the algebra is cured by considering a set of similar algebras as if \( \theta^{\mu\nu} \) is a 2-tensor. Namely, if one assumes the algebra (1.1) in a particular Lorentz frame, the algebra in a general Lorentz frame connected by a Lorentz transformation (\( \Lambda_{\nu}^{\mu} \)) with the original one could be given by

\[ [\hat{x'}^{\mu}, \hat{x'}^{\nu}] = i\theta'^{\mu\nu}, \]  

where \( \hat{x'} = \Lambda^{-1}\hat{x} \) and \( \theta' = (\Lambda^{-1})^2\theta \). If interpreted this way, we call (1.1) the c-number \( \theta \)-algebra. As we shall see in the next section, this interpretation of the algebra (1.1) is still not consistent theoretically. It is clear, however, that, in any case, (1.1) automatically implies the vanishing triple commutator,

\[ [\hat{x}^{\mu}, [\hat{x}^{\nu}, \hat{x}^{\rho}]] = 0, \]  

which becomes a nontrivial constraint in a non-commutative space-time in which the commutator \([\hat{x}^{\mu}, \hat{x}^{\nu}]\) is no longer assumed to be a c-number.

In these respects it should be recalled that Snyder\textsuperscript{2} was the first to realize that the Lorentz invariance does not necessarily require a continuum space-time. Instead he invented a non-commutative space-time with a fundamental length, where the coordinates are no longer commutative but obey the following commutation relations,

\[ [\hat{x}^{\mu}, \hat{x}^{\nu}] = ia^2 \hat{M}^{\mu\nu}, \]
\[
\left[ \hat{M}^{\mu \nu}, \hat{x}^\rho \right] = i\left( g^{\nu \rho} \hat{x}^\mu - g^{\rho \mu} \hat{x}^\nu \right),
\]
\[
\left[ \hat{M}^{\mu \nu}, \hat{M}^{\rho \sigma} \right] = i\left( g^{\nu \rho} \hat{M}^{\mu \sigma} - g^{\rho \sigma} \hat{M}^{\mu \nu} - g^{\mu \rho} \hat{M}^{\nu \sigma} + g^{\mu \sigma} \hat{M}^{\nu \rho} \right).
\]

(1.4)

Here, \(a\) is a fundamental length in the theory and \(\hat{M}^{\mu \nu}\) are the infinitesimal generators of the Lorentz group, the metric being given by \((g^{\mu \nu}) = (+1, -1, -1, -1)\). It is important to recognize that the commutator \([\hat{x}^\mu, \hat{x}^\nu]\) is a \(q\)-number unlike (1.1). A short look at (1.4) shows that Snyder’s quantized space-time is Lorentz-invariant. The relativistic QFT on it was developed in Ref. 3 in which a simple geometric interpretation of Snyder’s quantized space-time in terms of curved momentum space of constant curvature was also described. We here note that, in contrast to (1.3), the triple commutator in Snyder’s quantized space-time is no longer vanishing,
\[
\left[ \hat{x}^\mu, \left[ \hat{x}^\nu, \hat{x}^\rho \right] \right] = -a^2\left( g^{\nu \rho} \hat{x}^\mu - g^{\rho \mu} \hat{x}^\nu \right).
\]

(1.5)

Such a \(q\)-number triple commutator is not a general feature of a Lorentz-invariant non-commutative space-time.

In fact, Doplicher, Fredenhagen and Roberts (DFR) were led to propose a new algebra of a non-commutative space-time through considerations on the space-time uncertainty relations derived from quantum mechanics and general relativity. In the DFR algebra \(\theta^{\mu \nu}\) in (1.1) is replaced with an anti-symmetric tensor operator \(\hat{\theta}^{\mu \nu}\) which simply defines the commutator \(-i[\hat{x}^\mu, \hat{x}^\nu]\). DFR further assumed the vanishing of the triple commutator as in (1.3). It is then easy to prove the commutativity between the operators \(\hat{\theta}^{\mu \nu}\), whence they can be simultaneously diagonalized. The DFR algebra defines a Lorentz-invariant non-commutative space-time different from Snyder’s quantized space-time.

Quite recently, Carlson, Carone and Zobin (CCZ) rederived the DFR algebra by a certain contraction of Snyder’s algebra and formulated the non-commutative gauge theory (NCGT) in a Lorentz-invariant way. In their formulation of NCGT, the old \(\theta^{\mu \nu}\) becomes an extra 6-dimensional coordinate of the fields, leading to an integration over it in the non-commutative action which, however, contains no derivatives of the field quantities with respect to the extra variables \(\theta^{\mu \nu}\), so that the fields do not propagate into the \(\theta\)-space of the hermitian operators, \(\hat{\theta}^{\mu \nu}\), with no need of compactification of the extra dimensions. It turns out that the constant \(\theta\)-algebra violates the

\*\*\* Snyder’s motivation to introduce a quantized space-time is to modify point interactions between fields and matters so that UV divergence trouble in relativistic QFT may be avoided. However, a storm of applause in the success of the renormalization theory puts aside Snyder’s important paper which was largely ignored in physics community.

\*\*\* CCZ’s contraction process is questionable in view of the fact that Snyder’s coordinates have a simple geometric interpretation, making it insufficient to compare only an algebraic similarity between Snyder’s and the DFR algebras. In our opinion there is no connection between the two algebras.
Lorentz invariance because it singles out only one point in the $\theta$-space, the whole of which is needed to maintain the Lorentz invariance. The $\theta$-integration is assumed to be controlled by a weight function $W(\theta)$, which is normalized. One (K. M.) of the authors applied CCZ formulation to construct a realistic non-commutative QED and to derive an invariant damping factor.

We continue in this paper our study on the Lorentz-invariant non-commutative space-time based on the DFR algebra. The purpose of the present paper is four-fold. The first is to present a simple way of defining a non-commutative space-time based on a representation of the coordinate operators in the momentum space. Our derivation of the DFR algebra incidentally clarifies the reason that the $c$-number $\theta$-algebra is inconsistent theoretically. The second is to put CCZ formulation into a firmer basis by defining the operator fields on the DFR algebra as operators diagonal in the $\theta$-space. Our method resorts to a completeness relation of the state vectors obtained by diagonalizing $\hat{\theta}^{\mu\nu}$. The third is to point out that a general argument concerning the normalizability of the weight function in the Lorentz metric requires that the $\theta$-space be divided into two disjoint (orthogonal) spaces not connected by any Lorentz transformation. CCZ covariant moment formula should then be applied in each space, separately. The fourth is to propose a non-commutative generalization of Connes’ $M_4 \times \mathbb{Z}_2$ in the standard model. Two simple QFT models are reformulated on $M_4 \times \mathbb{Z}_2$ obtained in the commutative limit.

The organization of this paper is as follows. In the next section we argue that the $c$-number $\theta$-algebra is inconsistent from a theoretical point of view. It is followed in §3 by defining the DFR algebra and the fields on it. A comparison with Snyder’s algebra is also made. The section 4 gives a general argument concerning the normalizability of the weight function in the Lorentz metric such that CCZ covariant moment formula should be applied in two disjoint spaces not connected by any Lorentz transformation, separately. We propose a two-sheeted structure of the Lorentz-invariant non-commutative space-time based on the DFR algebra in §5. We formulate in §6 two simple QFT models on a discrete space-time $M_4 \times \mathbb{Z}_2$ obtained in the commutative limit. One of them relating the Higgs mechanism to the discrete manifold à la Connes was earlier discussed in a different geometrical terminology. The last section is devoted to discussions. We show in Appendix A that the Lorentz generators have different expressions for Snyder’s and the DFR algebras.

§2. Inconsistency of $c$-number $\theta$-algebra

The canonical commutation relation among the momentum and coordinate operators,

$$[\hat{p}^\mu, \hat{x}^\nu] = ig^{\mu\nu},$$

\[(2.1)\]
with *commuting* momentum variables,

\[ [\hat{p}^\mu, \hat{p}^\nu] = 0, \tag{2.2} \]

does not necessarily imply that the coordinate operators form an Abelian group of the translations in the momentum space. Instead we have, in general,

\[ \hat{x}^\mu = -i \frac{\partial}{\partial p_\mu} + f^\mu(p), \tag{2.3} \]

in the \( p \)-space, where \( f^\mu(p) \) is a 4-vector function of the momentum. Then the commutator between the coordinate operators become,

\[ [\hat{x}^\mu, \hat{x}^\nu] = -i \left( \frac{\partial f^\nu(p)}{\partial p_\mu} - \frac{\partial f^\mu(p)}{\partial p_\nu} \right). \tag{2.4} \]

If the function \( f^\mu(p) \) is given by a gradient,

\[ f^\mu(p) = \frac{\partial f(p)}{\partial p_\mu}, \tag{2.5} \]

the right-hand side of (2.4) vanishes so that the coordinate operators commute with each other. Conversely, if the coordinate operators commute with each other, the function \( f^\mu(p) \) must be a gradient, and the function \( f(p) \) can be eliminated through the redefinition of the wave function, \( \langle p|\psi\rangle \to e^{-if(p)/2}\langle p|\psi\rangle \) up to a constant phase factor.

We now suppose that \( f^\mu(p) \) is no longer a gradient. The simplest assumption is that it contains no derivative with respect to the momentum,

\[ f^\mu(p) = f^{\mu\nu}(p)p_\nu, \tag{2.6} \]

where \( f^{\mu\nu}(p) \) is a 2-tensor. There are two symmetric tensors available, \( p^\mu p^\nu \) and the metric tensor. However, both \( p^\mu p_\mu^2 \) and \( p^\mu \) are pure gradients and should be excluded from consideration by the assumption that \( f^\mu(p) \) is not a gradient. Consequently, we must assume that \( f^{\mu\nu}(p) \) is anti-symmetric. Let \( \theta^{\mu\nu} \) be an anti-symmetric tensor which can not depend on \( p \). Then we have, putting \( f^{\mu\nu} = \theta^{\mu\nu}/2, \)

\[ \hat{x}^\mu = -i \frac{\partial}{\partial p_\mu} + \frac{1}{2} \theta^{\mu\nu} p_\nu, \tag{2.7} \]

which *formally* reproduces the algebra (1.1) with the 2-tensor \( \theta^{\mu\nu} \). Unfortunately, however, the Lorentz covariance can not be maintained for the solution (2.7) unless \( \theta^{\mu\nu} = 0 \) despite of its appearance. To see this we take the commutator of \( \hat{M}^{\mu\nu} \) with (1.1), which gives an identity,

\[ -\theta^{\mu\sigma} g^{\nu\rho} + \theta^{\mu\rho} g^{\nu\sigma} + \theta^{\nu\sigma} g^{\mu\rho} - \theta^{\nu\rho} g^{\mu\sigma} = 0. \tag{2.8} \]
Putting $\mu = \sigma \neq \nu, \rho$ and using the anti-symmetry of $\theta^{\mu\nu}$ we obtain,

$$\theta^{\nu\rho} = 0. \quad (2.9)$$

This result is obtained as long as the commutator between the coordinate operators is assumed to be a $c$-number $\theta^{\mu\nu}$, either regarded as a constant or a 2-tensor. Consequently, the algebra (1.1) for the nonvanishing $\theta^{\mu\nu}$ violates the Lorentz covariance. Usually one ignores this difficulty, looking for observable effects of the Lorentz violation based on the constant $\theta$-algebra. However, this point of view is not allowed theoretically. Any non-commutative space-time should be characterized by a consistent algebra which is closed under the commutation relations for the operators $\{\hat{x}^\mu, \hat{M}^{\mu\nu}\}$. There is an example (1.4) that obeys this requirement. The commutation relation (1.1) and the last two ones of (1.4) are closed. But the commutator of $\hat{M}^{\mu\nu}$ with (1.1) is inconsistent unless $\theta^{\mu\nu} = 0$.

§3. DFR algebra and fields on it

Let us now try to restore the Lorentz covariance. The next simplest case would be to suppose that $\theta^{\mu\nu}$ in (2.7) is regarded as an anti-symmetric 2-tensor operator $\hat{\theta}^{\mu\nu}$,

$$f^\mu(p) = \frac{1}{2} \hat{\theta}^{\mu\nu} p_\nu, \quad (3.1)$$

where

$$[\hat{M}^{\mu\nu}, \hat{\theta}^{\rho\sigma}] = i(g^{\nu\rho} \hat{\theta}^{\mu\sigma} - g^{\nu\sigma} \hat{\theta}^{\mu\rho} - g^{\mu\rho} \hat{\theta}^{\nu\sigma} + g^{\mu\sigma} \hat{\theta}^{\nu\rho}). \quad (3.2)$$

Because the operator $\hat{\theta}^{\mu\nu}$ is independent of the momentum, it commutes with the momentum operator,

$$[\hat{\theta}^{\mu\nu}, \hat{p}^\rho] = 0. \quad (3.3)$$

If we assume that the operator $\hat{\theta}^{\mu\nu}$ commute with each other,

$$[\hat{\theta}^{\mu\nu}, \hat{\theta}^{\rho\sigma}] = 0, \quad (3.4)$$

*) This Lorentz violation is to be contrasted to a possible Lorentz violation considered in recent literature in which small violations of the Lorentz invariance are described in terms of the Lorentz covariant fields. On the contrary, the constant $\theta$ algebra does not allow us to define the Lorentz covariant fields unless the Seiberg-Witten map is exploited. See the next section.
we immediately find the commutator

\[ [\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \] (3.5)

Conversely, if we assume (3.5) together with the commutator,

\[ [\hat{\theta}^{\mu\nu}, \hat{x}^\rho] = 0, \] (3.6)

the commutator (3.4) is obtained by the Jacobi identity, \([\hat{\theta}^{\mu\nu}, [\hat{x}^\rho, \hat{x}^\sigma]] + \text{cyclic}=0\). Similarly, the commutator (3.3) is derived from the Jacobi identity, \([\hat{p}^\mu, [\hat{x}^\nu, \hat{x}^\rho]] + \text{cyclic}=0\). Any quantity that is independent of the canonical variables is not dynamical, so is not \(\hat{\theta}^{\mu\nu}\).

The commutation relations (3.5), (3.6) and (3.4) define the DFR algebra. For this algebra an inconsistency discussed in the previous section regarding the \(c\)-number \(\theta\)-algebra (1.1) is absolutely absent: the commutator of the Lorentz generators \(\hat{M}^{\mu\nu}\) with (3.5) only reproduces (3.2). By construction the DFR algebra is Lorentz-covariant and closed under the commutation relations for the operators \(\{\hat{x}^\mu, \hat{p}^\mu, \hat{\theta}^{\mu\nu}, \hat{M}^{\mu\nu}\}\).

Our derivation of the DFR algebra makes it easy to compare with Snyder’s algebra. Let us now suppose that the function \(f^\mu(p)\) in (2.3) is linear in the derivative, \(\partial/\partial p\),

\[ f^\mu(p) = f^{\mu\nu}(p)\frac{\partial}{\partial p^\nu}, \] (3.7)

where \(f^{\mu\nu}(p)\) is a 2-tensor function of \(p\). In this case the commutator \([\hat{p}^\mu, \hat{x}^\nu]\) gets no longer canonical,

\[ [\hat{p}^\mu, \hat{x}^\nu] = ig^{\mu\nu} - f^{\nu\mu}. \] (3.8)

This deviation from the canonical commutation relations is only allowed if the boundary condition that the extra term \(-f^{\nu\mu}\) in (3.8) vanishes in the commutative limit is satisfied. Then the generalized Heisenberg uncertainty relations can be proved for (3.8) (see, for instance, Ref.13)). Because \(f^{\mu\nu}(p)\) can not be constant,\(^{**)\) the simplest possibility is to take,

\[ f^{\mu\nu}(p) = ia^2 p^\mu p^\nu, \] (3.9)

where \(a\) has a dimension of length. The commutative limit corresponds to \(a \to 0\). The coordinate operators are then given by,

\[ \hat{x}^\mu = -i\frac{\partial}{\partial p_\mu} + ia^2 p^\rho p^\nu \frac{\partial}{\partial p^\nu}. \] (3.10)

\(^*)\) The following commutator should be added, \([\hat{M}^{\mu\nu}, \hat{p}^\rho] = i(g^{\nu\rho}\hat{p}^\mu - g^{\mu\rho}\hat{p}^\nu).\)

\(^**)\) There exists no constant anti-symmetric tensor. If \(f^{\mu\nu}(p)\) is a constant symmetric tensor \(g^{\mu\nu}\), the coordinate operators commute with each other. We consider a contrary case in what follows.
This is nothing but a geometrical expression for Snyder’s coordinates given by Kadyshevskii. In fact, we can recover Snyder’s algebra (1.4) if we identify $\hat{M}^{\mu \nu}$ with the Lorentz generators in the $p$-space,

$$\hat{M}^{\mu \nu} = i(p^\mu \frac{\partial}{\partial p_\nu} - p^\nu \frac{\partial}{\partial p_\mu}).$$  \hspace{1cm} (3.11)

The fundamental length $a$ in (1.4) equals $a$ in (3.9). We also note that the commutator $[\hat{p}^\mu, \hat{x}^\nu]$ given by (3.8) with (3.9) is identical to that obtained by Snyder. We thus find two notable differences between Snyder’s and the DFR algebras,

\[
\begin{align*}
[\hat{x}^\mu, [\hat{x}^\nu, \hat{x}^\rho]] & = 0 \quad \text{(DFR algebra)}, \\
\neq 0 & \quad \text{(Snyder’s algebra)}, \\
[\hat{p}^\mu, \hat{x}^\nu] & = ig^{\mu \nu} \quad \text{(DFR algebra)}, \\
\neq ig^{\mu \nu} & \quad \text{(Snyder’s algebra)}.
\end{align*}
\hspace{1cm} (3.12)
\]

In what follows we exclusively employ the DFR algebra for its simpler nature, although it contains a non-dynamical quantity $\hat{\theta}^{\mu \nu}$ for the Lorentz invariance.

Because of (3.4) it is possible to diagonalize $\hat{\theta}^{\mu \nu}$ simultaneously,

$$\hat{\theta}^{\mu \nu}|\theta\rangle = \theta^{\mu \nu}|\theta\rangle, \hspace{1cm} (3.13)$$

where the eigenvalue $\theta^{\mu \nu}$ transforms as an antisymmetric 2-tensor. The states $|\theta\rangle$ are orthogonal for different eigenvalues and normalized as follows,

$$\langle \theta | \theta \prime \rangle = \delta^6(\theta - \theta \prime)/W(\theta), \hspace{1cm} (3.14)$$

where $\delta^6(\theta - \theta \prime) \equiv \delta(\theta^{01} - \theta^{01})\delta(\theta^{02} - \theta^{02})\delta(\theta^{03} - \theta^{03})\delta(\theta^{12} - \theta^{12})\delta(\theta^{23} - \theta^{23})\delta(\theta^{31} - \theta^{31})$ and $W(\theta)$ is introduced for later purpose. We also assume the completeness relation,

$$\int d^6\theta W(\theta) |\theta\rangle \langle \theta | = 1. \hspace{1cm} (3.15)$$

Now the commutator (1.1) is valid as a ’weak’ relation

$$\langle \theta | [\hat{x}^\mu, \hat{x}^\nu]|\theta\rangle = i\theta^{\mu \nu} \langle \theta | \theta \rangle, \hspace{1cm} (3.16)$$

with due care of the unnormalizability of the state $|\theta\rangle$. The quantity $\theta^{\mu \nu}$ here is a 2-tensor specifying a point in the $\theta$-space, the whole of which constitutes the Lorentz-invariant manifold. On the contrary,
the constant $\theta^{\mu\nu}$ in (1.1) singles out one point in the $\theta$-space, and, hence, (1.1) violates the Lorentz invariance of the theory. Even if the algebra (1.1) is considered to be Lorentz-covariant as discussed below it, it still violates the Lorentz invariance: the Lorentz covariance of the relation (3.16) is derived from the Lorentz transformation property of the operator $\hat{\theta}^{\mu\nu}$ and, hence, the state vector $|\theta\rangle$ which is missing in the $c$-number algebra. That is, the algebra (1.2) should be regarded as a ‘weak’ relation of the commutator (3.5) sandwiched between the Lorentz-transformed states, $\langle \theta' |$ and $| \theta' \rangle$, with $\theta' = (A^{-1})^2 \theta$.

Now the Weyl representation of the DFR algebra is given by the operators

$$\hat{T}(p, \sigma) = e^{ip\hat{x}^\mu + i\sigma_{\mu\nu}\hat{\theta}^{\mu\nu}} \equiv e^{ip\hat{x} + i\sigma\hat{\theta}},$$

with the multiplication law

$$\hat{T}(p_1, \sigma_1)\hat{T}(p_2, \sigma_2) = e^{-\frac{i}{2}(p_1 \times p_2)\hat{\theta}} \hat{T}(p_1 + p_2, \sigma_1 + \sigma_2)$$

$$= \hat{T}(p_1 + p_2, \sigma_1 + \sigma_2 - \frac{1}{2} p_1 \times p_2).$$

Here we introduced the shorthand notation, $(p_1 \times p_2)\hat{\theta} = (p_1 \times p_2)_{\mu\nu}\hat{\theta}^{\mu\nu}$ with $(p_1 \times p_2)_{\mu\nu} \equiv (1/2)(p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu})$. If we take the expectation value of (3.18) between the states $|\theta\rangle$ using the relations (3.4), (3.6) and the completeness relation (3.15), and put

$$\langle \theta | \hat{T}(p, \sigma) | \theta \rangle = \hat{T}_\theta(p)e^{i\sigma\hat{x}} \langle \theta | \theta \rangle, \quad \hat{T}_\theta(p) = e^{ip\hat{x}},$$

we find the following multiplication law of the operators $\hat{T}_\theta(p)$,

$$\hat{T}_\theta(p_1)\hat{T}_\theta(p_2) = e^{-\frac{i}{2}(p_1 + p_2)\hat{\theta}} \hat{T}_\theta(p_1 + p_2).$$

The subscript of the operators $\hat{T}_\theta(p)$ reminds of the ‘weak’ relation (3.16). In this way we reproduce the multiplication law of the operators $\hat{T}_\theta(p)$ for the $c$-number $\theta$-algebra (1.1). Consequently, if we define the operator field on the DFR algebra in the Weyl representation,

$$\hat{\varphi}(\hat{x}, \hat{\theta}) = \frac{1}{(2\pi)^4} \int d^4p d^4\sigma \hat{\varphi}(p, \sigma) \hat{T}(p, \sigma),$$

and take the the expectation value between the states $|\theta\rangle$ using the relations (3.4) and (3.6), we obtain

$$\langle \theta | \hat{\varphi}(\hat{x}, \hat{\theta}) | \theta \rangle = \hat{\varphi}(\hat{x}, \theta) \langle \theta | \theta \rangle,$$
where the operator field diagonal in the $\theta$-space is put into the form,

$$
\hat{\varphi}(\hat{x}, \theta) = \frac{1}{(2\pi)^4} \int d^4pd^6\sigma \tilde{\varphi}(p, \sigma)e^{ip\hat{x} + i\sigma \theta}. \tag{3.23}
$$

As usual, the Weyl symbol with respect to the operator coordinates is defined by replacing them with the commuting coordinates,

$$
\varphi(x, \theta) = \frac{1}{(2\pi)^4} \int d^4pd^6\sigma \tilde{\varphi}(p, \sigma)e^{ipx + i\sigma \theta}. \tag{3.24}
$$

Then, the product of operators corresponds to the Moyal $*$-product,

$$
\hat{\varphi}_1(\hat{x}, \hat{\theta}) \hat{\varphi}_2(\hat{x}, \hat{\theta}) = \frac{1}{(2\pi)^4} \int d^4pd^6\sigma \tilde{\varphi}_{12}(p, \sigma)e^{ip\hat{x} + i\sigma \hat{\theta}},
$$

$$
\varphi_{12}(x, \theta) = \frac{1}{(2\pi)^4} \int d^4pd^6\sigma \tilde{\varphi}_{12}(p, \sigma)e^{ipx + i\sigma \theta}
$$

$$
= e^{i\hat{\theta}^\mu \frac{\partial}{\partial x^\mu}} e^{i\theta^\mu \frac{\partial}{\partial \theta^\mu}} \varphi_1(x, \theta)\varphi_2(y, \theta)|_{x=y} \equiv \varphi_1(x, \theta) \ast \varphi_2(x, \theta). \tag{3.25}
$$

This correspondence needs no explicit proof in the present formalism. As in deducing (3.20) from (3.18) through (3.19) it is easy to prove that the product of operators is also diagonal in the $\theta$-space:

$$
\langle \theta | \hat{\varphi}_1(\hat{x}, \hat{\theta}) \hat{\varphi}_2(\hat{x}, \hat{\theta})|\theta'\rangle = \hat{\varphi}_1(\hat{x}, \theta) \hat{\varphi}_2(\hat{x}, \theta)|\theta\rangle|_{\theta'}. \tag{3.26}
$$

It then follows from (3.23) that the product of the operators, $\hat{\varphi}_1(\hat{x}, \theta) \hat{\varphi}_2(\hat{x}, \theta)$, corresponds to the Moyal product with the deformation parameter, $\theta^\mu \nu$. Only difference lies in the additional dependence of the operators on the same $\theta^\mu \nu$. This is observed by Carlson, Carone and Zobin without considering the $\theta$-space spanned by the basis vectors $|\theta\rangle$. It gives a logical basis for the $\theta$-expansion.

As emphasized in Ref. 6) the Lorentz-covariant operator fields can be defined only if the underlying algebra among the operator coordinates is Lorentz-covariant. For instance, the condition for the scalar field

$$
\varphi'(x', \theta') = \varphi(x, \theta), \tag{3.27}
$$

where $x' = \Lambda x$ and $\theta' = \Lambda^2 \theta$, is translated into that of the scalar operator field

$$
\hat{\varphi}'(\hat{x}, \hat{\theta}) = \hat{\varphi}(\hat{x}', \hat{\theta}'). \tag{3.28}
$$
where \( \hat{x}' = \Lambda^{-1} \hat{x} \) and \( \hat{\theta}' = (\Lambda^{-1})^2 \hat{\theta} \), and vice versa. Conversely, if the operator coordinates do not obey the Lorentz-covariant algebra, one is unable to define the Lorentz-covariant (operator) fields unless the Seiberg-Witten map is exploited, which amounts to expressing the non-commutative fields in terms of the Lorentz-covariant commutative fields through the nonlinear field redefinition.

Finally, we employ the trace property

\[
\text{tr} \hat{T}(p) = (2\pi)^4 \delta^4(p),
\]

and the completeness relation (3.15) to obtain the CCZ trace formula,

\[
\text{tr} \hat{\varphi}(\hat{x}, \hat{\theta}) = \int d^4x d^6 \theta W(\theta) \varphi(x, \theta).
\]

A Lorentz-invariant non-commutative action is then defined as in Ref. 6):

\[
\hat{S} = \int d^4x d^6 \theta W(\theta) L(\varphi(x, \theta), \partial_\mu \varphi(x, \theta)),
\]

where the subscript of the Lagrangian means that the \(*\)-product should be taken for all products of the field variables. It is to be recalled that, although the fields \( \varphi(x, \theta) \) depend on \( x^\mu \) as well as \( \theta^{\mu\nu} \), the action thus constructed contains the derivative with respect to \( x \) only, so that the fields do not propagate into the extra \( \theta \)-space. Hence we do not need any compactification of the extra dimensions.

## §4. Modified completeness relation in the \( \theta \)-space

We now investigate the completeness relation (3.15) in more detail. We slightly change the previous notation by noting the fact that the commutator \([\hat{x}^\mu, \hat{x}^\nu]\) should vanish in the commutative limit, \( a \to 0 \), where \( a \) is a fundamental length in the theory as in the case of Snyder's quantized space-time. By dimensional argument we put

\[
\hat{\theta}^{\mu\nu} = a^2 \tilde{\theta}^{\mu\nu},
\]

where \( \tilde{\theta}^{\mu\nu} \) is a dimensionless operator. We write the state vector as

\[
\hat{\theta}^{\mu\nu} | \hat{\theta} \rangle = \tilde{\theta}^{\mu\nu} | \tilde{\theta} \rangle.
\]

\( ^* \) The trace in the \( \theta \)-space is defined up to an irrelevant factor as in (3.29). An explicit proof of the trace formula (3.29) with an irrelevant factor absorbed can be seen, for instance, in Appendix A of Ref. 15).\( ^{**} \) At the present time it is still optional to choose the non-commutative scale so that we are free to determine it in this paper provided no conflict with experiment arises.
so that

\[ \hat{\theta}^{\mu\nu}|\bar{\theta}\rangle = \theta^{\mu\nu}|\bar{\theta}\rangle, \]  

(4.3)

with

\[ \theta^{\mu\nu} = a^2 \bar{\theta}^{\mu\nu}. \]  

(4.4)

Since \( a \) is Lorentz scalar, \( \bar{\theta}^{\mu\nu} \) is a dimensionless anti-symmetric 2-tensor.

The normalization condition (3.14) reads

\[ \langle \bar{\theta} | \bar{\theta}' \rangle = \delta^6(\bar{\theta} - \bar{\theta}')/w(\bar{\theta}), \]  

(4.5)

where we put

\[ W(\theta) = a^{-12}w(\bar{\theta}). \]  

(4.6)

The completeness condition (3.15) then becomes,

\[ \int d^6\bar{\theta} w(\bar{\theta}) |\bar{\theta}\rangle\langle \bar{\theta}| = 1. \]  

(4.7)

We also note that the non-commutative action (3.31) can be expanded in terms of the parameter \( a \) which is considered a fundamental length in the theory:

\[ \hat{S} = S^{(0)} + S^{(2)} + S^{(4)} + \cdots, \]  

(4.8)

where \( S^{(2n)} \) is of order \( a^{2n} \), \( n = 0, 1, 2, \ldots \). We do not consider any question of the convergence property of this expansion. Through (4.4) and the \( \theta \)-integration in (3.31) this expansion is obtained from the \( \theta \)-expansion \( [3,4] \) of the Lagrangian \[ [4] \]

\[ \mathcal{L}(\varphi(x,\theta), \partial_\mu \varphi(x,\theta))_* = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \cdots, \]  

(4.9)

where \( \mathcal{L}^{(n)} \) is of order \( n \) in \( \theta^{\mu\nu} \), \( n = 0, 1, 2, \ldots \). The reason that the odd terms \( \mathcal{L}^{(2n+1)} \) disappear in the expansion \( [1,8] \) stems from the fact that \( W(\theta) \) is an even function:

\[ W(-\theta) = W(\theta). \]  

(4.10)

*) In the old version based on (1.1) the \( \theta \)-expansion of the Lagrangian is the same as that of the non-commutative action, but in the CCZ Lorentz-invariant formulation the \( \theta \)-expansion defined for the Lagrangian turns into the \( a \)-expansion of the non-commutative action via the \( \theta \)-integration.
This property is obvious from the Lorentz invariance of \(W(\theta)\). We recover QFT action in the commutative limit, \(a \to 0\),

\[
S \equiv S^{(0)} = \int d^4x \mathcal{L}^{(0)}(\phi(x), \partial_\mu \phi(x)) \equiv \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)),
\]

provided

\[
\int d^6\theta W(\theta) = 1. \tag{4.12}
\]

This is CCZ normalization condition which is equivalent to,

\[
\int d^6\bar{\theta}w(\bar{\theta}) = 1. \tag{4.13}
\]

Dynamical content of the weight function is revealed through the substitution of the expansion (4.9) into the definition (3.31). Neglecting the odd-order terms in (4.9) as explained above, we encounter the integrals,

\[
\int d^6\bar{\theta}w(\bar{\theta}) \prod_{i=1}^{2n} \bar{\theta}^{\mu_i \nu_i}, \quad n = 0, 1, 2, \cdots \tag{4.14}
\]

These integrals can be expressed in terms of the invariant moments,

\[
\langle \bar{\theta}^{2n} \rangle = \int d^6\bar{\theta}w(\bar{\theta})(\bar{\theta}^{\mu \nu} \bar{\theta}_{\mu \nu})^n, \quad n = 0, 1, 2, \cdots \tag{4.15}
\]

The normalization condition (4.12) implies \(\langle 1 \rangle = 1\), corresponding to the case, \(n = 0\). For \(n = 1\), we have

\[
\int d^6\bar{\theta}w(\bar{\theta}) \bar{\theta}^{\mu_1 \nu_1} \bar{\theta}^{\mu_2 \nu_2} = \frac{\langle \bar{\theta}^2 \rangle}{12} (g^{\mu_1 \mu_2} g^{\nu_1 \nu_2} - g^{\mu_1 \nu_2} g^{\mu_2 \nu_1}), \tag{4.16}
\]

for \(n = 2\), we get

\[
\int d^6\bar{\theta}w(\bar{\theta}) \bar{\theta}^{\mu_1 \nu_1} \bar{\theta}^{\mu_2 \nu_2} \bar{\theta}^{\mu_3 \nu_3} \bar{\theta}^{\mu_4 \nu_4}
\]

\[
= \frac{\langle \bar{\theta}^4 \rangle}{192} [(g^{\mu_1 \mu_2} g^{\nu_1 \nu_2} - g^{\mu_1 \nu_2} g^{\mu_2 \nu_1})(g^{\mu_3 \mu_4} g^{\nu_3 \nu_4} - g^{\mu_3 \nu_4} g^{\mu_4 \nu_3})
\]

\[
+ (g^{\mu_1 \nu_3} g^{\mu_2 \nu_4} - g^{\mu_1 \nu_4} g^{\mu_2 \nu_3})(g^{\mu_3 \mu_4} g^{\nu_3 \nu_4} - g^{\mu_3 \nu_4} g^{\mu_4 \nu_3})
\]

\[
+ (g^{\mu_1 \mu_4} g^{\nu_1 \nu_4} - g^{\mu_1 \nu_4} g^{\mu_1 \nu_4})(g^{\mu_3 \mu_4} g^{\nu_3 \nu_4} - g^{\mu_3 \nu_4} g^{\mu_3 \nu_4})], \tag{4.17}
\]
and so on.

We now evaluate the integral (4.15) by assuming that the weight function $w(\bar{\theta})$ is a function of only the invariant

$$\bar{\alpha} \equiv \frac{1}{2} \tilde{g}^{\mu \nu} \tilde{g}_{\mu \nu}. \tag{4.18}$$

Using the parametrization

$$\bar{\theta}^{01} = \rho \sin \theta \cos \phi, \quad \bar{\theta}^{02} = \rho \sin \theta \sin \phi, \quad \bar{\theta}^{03} = \rho \cos \theta,$$

$$\bar{\theta}^{12} = \sigma \sin \vartheta \cos \phi, \quad \bar{\theta}^{23} = \sigma \sin \vartheta \sin \phi, \quad \bar{\theta}^{31} = \sigma \cos \vartheta, \tag{4.19}$$

we find

$$\langle \bar{\theta}^{2n} \rangle = 16 \pi^2 \int_0^\infty \rho^2 \, d\rho \int_0^\infty \sigma^2 \, d\sigma w(-\rho^2 + \sigma^2)(-\rho^2 + \sigma^2)^n. \tag{4.20}$$

For this integral to converge for all $n$ the function $w(-\rho^2 + \sigma^2)$ must act as an exponential damping factor. Because of the negative sign in front of $\rho^2$, which is unavoidable in the Lorentz metric, any exponential damping factor becomes powerless on the line $\rho = \sigma$, which corresponds to the light-like region $\bar{\alpha} = 0$. To see this more explicitly we put,

$$\rho = r \cos \psi, \quad \sigma = r \sin \psi, \tag{4.21}$$

so that the region $\bar{\alpha} > 0$ corresponds to

$$0 < r < \infty, \quad \psi = \frac{\pi}{4} + \epsilon \rightarrow \psi = \frac{\pi}{2}, \tag{4.22}$$

while the region $\bar{\alpha} < 0$ corresponds to

$$0 < r < \infty, \quad \psi = 0 \rightarrow \psi = \frac{\pi}{4} - \epsilon. \tag{4.23}$$

The equation $\psi = \frac{\pi}{4}$ defines the line $\rho = \sigma$ to be excluded from the integration region. Consequently, the parameter $\epsilon$ is to be chosen positive. For $\bar{\alpha} > 0$ we get

$$\langle \bar{\theta}^{2n} \rangle_+ \equiv 4 \pi^2 \int_0^\infty r^{(5+2n)} \, dr \int_{\pi/4+\epsilon}^{\pi/2} d\psi (-\cos 2\psi)^n \left( \sin^2 2\psi \right) w(-r^2 \cos 2\psi)$$

$$= \pi^2 B_{1-\epsilon}(\frac{3}{2},-1) \int_0^\infty dRR^{5+2n} w(R^2), \tag{4.24}$$
where $R^2 = -r^2 \cos 2\psi$ is a Lorentz-invariant variable and,

$$
B_x(p, q) = \int_0^t dt t^{p-1} (1 - t)^{q-1},
$$

is the incomplete beta function. Similarly, we obtain

$$
\langle \bar{\theta}^{2n} \rangle_+ \equiv 4\pi^2 \int_0^\infty r^{(5+2n)} dr \int_0^{\pi/4-\epsilon} d\psi (-\cos 2\psi)^n (\sin^2 2\psi) w(-r^2 \cos 2\psi)
$$

$$
= \pi^2 B_{1-\epsilon}(\frac{3}{2}, -1) \int_0^\infty dRR^{5+2n}(-1)^n w(-R^2).
$$

Here $R^2 = r^2 \cos 2\psi$. In both expressions (4.24) and (4.26) $R^2$ is positive. The divergent factor $B_{1-\epsilon}(\frac{3}{2}, -1)$, which, at first sight, seems to violate the Lorentz invariance, can be eliminated through the normalization condition,

$$
\langle 1 \rangle = \langle 1 \rangle_+ + \langle 1 \rangle_-
$$

$$
= \pi^2 B_{1-\epsilon}(\frac{3}{2}, -1) \int_0^\infty dRR^{5} \left( w(R^2) + w(-R^2) \right) = 1.
$$

Then one can express all the moments as,

$$
\langle \bar{\theta}^{2n} \rangle = \langle \bar{\theta}^{2n} \rangle_+ + \langle \bar{\theta}^{2n} \rangle_-
$$

$$
= \frac{\int_0^\infty dRR^{5+2n} \left( w(R^2) + (-1)^n w(-R^2) \right)}{\int_0^\infty dRR^{5} \left( w(R^2) + w(-R^2) \right)}.
$$

This integral converges only if both $w(R^2)$ and $w(-R^2)$ diminish exponentially at $R^2 \to \infty$. In general, however, this is not the case. For instance, if we put

$$
w(\bar{\theta}) = Ne^{-b\bar{\alpha}^2+c\bar{\alpha}} = Ne^{-bR^2+cR^2} \equiv w(R^2),
$$

there are two possible choices, $b > 0$ or $b = 0$. For $b > 0$, both $w(R^2)$ and $w(-R^2)$ go to 0 at $R^2 \to \infty$, while, for $b = 0$, $w(R^2)$ and $w(-R^2)$ vanish exponentially at $R^2 \to \infty$ if $c < 0$ and $c > 0$, respectively. Namely, we have, in general, to choose the weight function separately according to the sign of the invariant $\bar{\alpha}$.

This is in accord with the following situation. Since the two regions with different sign of the
invariant $\bar{\alpha}$ can not be connected by any (including improper) Lorentz transformation, one can define
the field variable $\varphi(x, \theta)$ independently in the two regions, denoted $\varphi_{\pm}(x, \theta)$, where $\epsilon = \pm$ corresponds
to positive and negative $\bar{\alpha}$, respectively. There are no Lorentz-invariant interactions between the fields
$\varphi_{\epsilon}(x, \theta)$ for different $\epsilon = \pm$. This suggests a division of the $\theta$-space into two disjoint spaces according
to the sign of $\bar{\alpha}$. This Lorentz-invariant decomposition precisely corresponds to the necessity of the
separate choice of the weight function according to the sign of $\bar{\alpha}$. Consequently, we have to define
the invariant moments separately in each disjoint space,

$$\langle \bar{\theta}^{2n} \rangle_{\pm} \equiv \int_{\pm} d^6 \bar{\theta} w_{\pm}(\bar{\theta}), \quad (4.30)$$

where the integral $f_{\pm}$ denotes that the integration region is restricted to positive $\bar{\alpha}$ and negative $\bar{\alpha}$
as in (4.24) and (4.26), respectively, while the weight function $w_{\pm}(\pm R^2)$ vanishes exponentially at
large $R^2$. It turns out that

$$\langle \bar{\theta}^{2n} \rangle_{+} = \frac{\int_{0}^{\infty} dRR^{5+2n}w_{+}(R^2)}{\int_{0}^{\infty} dRR^5w_{+}(R^2)},$$

$$\langle \bar{\theta}^{2n} \rangle_{-} = (-1)^n \frac{\int_{0}^{\infty} dRR^{5+2n}w_{-}(-R^2)}{\int_{0}^{\infty} dRR^5w_{-}(-R^2)}, \quad (4.31)$$

where we assumed $\langle 1 \rangle_{+} = \langle 1 \rangle_{-} = 1$. Also, the covariant moment formula (4.16) is modified as

$$\int_{\pm} d^6 \bar{\theta} w_{\pm}(\bar{\theta}) \bar{\theta} \mu_1 \nu_1 \bar{\theta} \mu_2 \nu_2 = \frac{\langle \bar{\theta}^{2} \rangle_{\pm}}{12} (g_{\mu_1 \nu_2} g_{\nu_1 \mu_2} - g_{\mu_1 \nu_2} g_{\nu_1 \mu_2}), \quad (4.32)$$

and similarly for the higher-order covariant moment formulae.

According to the division of the $\theta$-space into two disjoint spaces not connected by any Lorentz
transformation, we write the state vectors as $|\bar{\theta}_{\pm}\rangle$, where the sign $\pm$ indicates the signature of the
invariant $\bar{\alpha}$. The completeness condition (3.15) then reads

$$\sum_{\epsilon = \pm} \int_{\epsilon} d^6 \bar{\theta} w(\bar{\theta}) |\bar{\theta}_{\epsilon}\rangle \langle \bar{\theta}_{\epsilon}| = 1. \quad (4.33)$$

Hence, the state vectors of different signatures are orthogonal:

$$\epsilon \langle \bar{\theta}|\bar{\theta}'\rangle_{\epsilon'} = \delta^{6}(\bar{\theta} - \bar{\theta}')\delta_{\epsilon \epsilon'}/w(\bar{\theta}). \quad (4.34)$$

Because the states $|\bar{\theta}_{\pm}\rangle$ are not connected by any Lorentz transformation and are orthogonal to each
other, the expectation values of the operator fields are diagonal also in the signatures,

$$\epsilon \langle \bar{\theta}|\hat{\varphi}(\hat{x}, \bar{\theta})|\bar{\theta}'\rangle_{\epsilon'} = \hat{\varphi}_{\epsilon}(\hat{x}, \theta) \delta_{\epsilon \epsilon'} \epsilon \langle \bar{\theta}|\bar{\theta}'\rangle_{\epsilon}, \quad (4.35)$$

*) The delta function $\delta^{6}(\bar{\theta} - \bar{\theta}')$ in the equation below already implies the Kronecker delta $\delta_{\epsilon \epsilon'}$ in the same equation. For later convenience we write both deltas.
where $\epsilon, \epsilon' = \pm$. This implies no interactions between the fields $\varphi_\epsilon(x, \theta)$ for different $\epsilon$. Hence it is a matter of convention to choose the plus sign or the minus sign.

§5. Non-commutative generalization of Connes’ space-time $M_4 \times Z_2$

Let us suppose that we live in the space of positive $\bar{\alpha}$ and drop the plus sign attached to the state vector, $|\bar{\theta} \rangle \equiv |\bar{\theta} \rangle_+$, and the operator field $\hat{\varphi}(\hat{x}, \theta) \equiv \hat{\varphi}_+(\hat{x}, \theta)$.

In connection with the Lorentz-invariant non-commutative space-time based on the DFR algebra we recall that Connes assumed a discrete space-time $M_4 \times Z_2$ to be the underlying space-time behind the standard model. Without entering into details, Connes’ idea is the following. The standard model is a spontaneously broken gauge theory, containing matter, gauge and Higgs fields. Since the existence of the gauge fields have a geometrical interpretation, the Higgs fields might also have a geometric origin. Although the gauge fields are defined on a continuous manifold, one can not introduce the Higgs fields as a kind of gauge fields on the same continuous manifold, because the gauge fields are related to the propagating nature of the matter fields, while the Higgs fields are introduced into the theory in relation to the mass term of fermions, which contains no derivative. Consequently, a radical change of the space-time geometry is needed to understand that all force-mediating fields may be interpreted as originating from the underlying geometry. A concrete space-time structure proposed by Connes is $M_4 \times Z_2$ which is a discrete manifold. In this space the Higgs fields can be interpreted as a gauge field in the discrete direction, while the ordinary gauge fields are defined on the continuous manifolds, two copies of the Minkowski space-time. The number 2 is related to the dichotomic nature of the chirality of the fermions.

It is then natural to ask if Connes’ idea works also in the non-commutative space-time. We affirmatively answer this question in this section. To this purpose we introduce the additional quantum number $y = \pm$ to characterize the state $|\bar{\theta} \rangle$, which is then denoted,

$$|\bar{\theta}, y \rangle, \quad y = \pm.$$  \hfill (5.1)

The two states, $|\bar{\theta}, \pm\rangle$, for the same $\bar{\theta}$ are further assumed to be non-orthogonal. This assumption means that there exist interactions between the diagonal fields defined by the expectation value,

$$\langle \bar{\theta}, y | \varphi(\hat{x}, \theta) | \bar{\theta}', y \rangle = \varphi(\hat{x}, \theta, y, \bar{\theta}) \langle \bar{\theta}, y | \bar{\theta}', y \rangle \equiv \varphi(\hat{x}, \theta, y, \bar{\theta}), \quad y = \pm.$$  \hfill (5.2)

\textsuperscript{\textdagger} Without modifying the underlying space-time, Sogami proposed to unify the gauge and the Higgs fields in terms of the generalized covariant derivative acting on the total fermion field including the chiral leptons and quarks.
Such interactions must be mediated by the off-diagonal fields,

$$
\langle \bar{\theta}, y | \varphi(\bar{x}, \bar{\theta}) | \varphi'(y), -y \rangle = \varphi(\bar{x}, \varphi, y, -y) \langle \bar{\theta}, y | \varphi'(y), -y \rangle, \ y = \pm.
$$

(5.3)

According to our assumption, any relativistic field defined on the non-commutative space-time based on the DFR algebra has the following c-number structure,

- monogenic fields \( \varphi_{\pm}(x, \theta, y), \ y = \pm \),
- dichotomic fields \( \varphi_{\pm}(x, \theta, y, -y), \ y = \pm \).

(5.4)

In the commutative limit, \( a \to 0 \), we are left with the local fields, \( \varphi(x, y) = \varphi(x, \theta = 0, y) \), which are supposed to describe the chiral matter fields and the chiral gauge fields, and the local fields \( \varphi(x, y, -y) \equiv \varphi(x, \theta = 0, y, -y) \), which are assumed to be the Higgs fields responsible for the masses of the monogenic fields. Consequently, the Higgs fields on our non-commutative space-time have a geometric origin as a kind of the non-commutative gauge fields. A detailed model construction will be postponed to later publications. In the next section we present a plausible argument to support this interpretation in the commutative limit based on our earlier work.

The same discussion is also valid for the case, \( \bar{\alpha} < 0 \).

§6. QFT on \( M_4 \times Z_2 \)

As alluded to above, to make our presentation as simple as possible we restrict ourselves to the commutative limit \( a \to 0 \) in this section still working in the space with positive \( \bar{\alpha} \).

The free Dirac action on \( M_4 \times Z_2 \) is given by,

$$
S_{D_0} = \sum_{y=\pm} \int d^4 x \bar{\psi}(x, y)(i\gamma^\mu \partial_\mu + i\gamma^5 \partial_y)\psi(x, y),
$$

(6.1)

where we define the derivative in the discrete direction as,

$$
\partial_y \psi(x, y) = M(y, -y)\psi(x, -y).
$$

(6.2)

It should be noted that the derivative in the discrete direction is required in relation with the mass term of fermion, which has no continuous derivative. If we assume the hermiticity condition,

$$
M^\dagger(y, -y) = M(-y, y),
$$

(6.3)

\(^*)\) Hence, there is no *-product in what follows.

\(^**)\) The matrix \( \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \) is inserted for later convenience.
the action (6.1) is hermitian. We assign the left-handed and the right-handed spinors on the different sheets,
\begin{equation}
\psi_L(x) = \psi(x, +), \quad \psi_R(x) = \psi(x, -).
\end{equation}

Upon transforming $\psi_L(x) \rightarrow e^{i\pi/4}\psi_L(x)$ and $\psi_R(x) \rightarrow e^{-i\pi/4}\psi_R(x)$, we recover the well-known free Dirac action,
\begin{equation}
S_{D_0} = \int d^4x \left( \bar{\psi}(x)i\gamma^\mu \partial_\mu \psi(x) - \bar{\psi}(x)M\psi(x) \right), \quad M \equiv M(+, -).
\end{equation}

The gauge transformation may or may not depend on the variable $y$. Let us first consider the following gauge transformation independent of $y$,
\begin{equation}
\psi(x, y) \rightarrow g\psi(x, y) = g(x)\psi(x, y),
\end{equation}
where $g(x)$ is unitary. The gauge-invariant Dirac action $S_D$ is obtained by the replacement,
\begin{equation}
\partial_\mu \rightarrow D_\mu = \partial_\mu + A_\mu(x),
\end{equation}
in (6.1), with the gauge transformation property
\begin{equation}
A_\mu(x) \rightarrow gA_\mu(x) = g(x)A_\mu(x)g^\dagger(x) + g(x)\partial_\mu g^\dagger(x).
\end{equation}
The derivative $\partial_y$ is unchanged because $\partial_y g(x) = 0$, the fermion mass being gauge-invariant. Writing the gauge field as
\begin{equation}
A(x) = i\gamma^\mu A_\mu(x),
\end{equation}
we define the exterior derivative by
\begin{equation}
dA(x) = \frac{1}{2}i\gamma^\mu \wedge i\gamma^\nu(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)),
\end{equation}
and the wedge product by
\begin{equation}
A(x) \wedge A(x) = \frac{1}{2}i\gamma^\mu \wedge i\gamma^\nu[A_\mu(x), A_\nu(x)],
\end{equation}
to obtain the field strength through
\begin{equation}
F(x) = dA(x) + A(x) \wedge A(x) = \frac{1}{2}i\gamma^\mu \wedge i\gamma^\nu F_{\mu\nu}(x),
\end{equation}
\begin{equation}
F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + [A_\mu(x), A_\nu(x)].
\end{equation}
Here the wedge product of $\gamma$ matrices is defined by

$$\gamma^\mu \wedge \gamma^\nu = \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \equiv -i\sigma^{\mu\nu}. \quad (6.13)$$

The bosonic action identical to the YM action is then given by

$$S_B = -\frac{1}{8} \int d^4x \text{Tr} \frac{1}{g^2} F^2(x), \quad (6.14)$$

where $g^2$ is a matrix of the coupling constants squared, Tr includes both the traces over the Dirac matrices and the internal symmetry matrices. For the gauge group $U(1)$ the sum $S_D + S_B$ defines spinor QED.

Let us next consider the gauge transformation depending on the variable $y$,

$$\psi(x, y) \rightarrow g\psi(x, y) = g(x, y)\psi(x, y), \quad (6.15)$$

where $g(x, y)$ is unitary. The gauge-invariant Dirac action $S_D$ is obtained by the replacement,

$$\partial_\mu \rightarrow D_\mu = \partial_\mu + A_\mu(x, y),$$

$$\partial_y \rightarrow D_y = \partial_y + \Phi(x, y, -y), \quad (6.16)$$

in (6.1), where the gauge field $\Phi(x, y, -y)$ is assumed to flip the sign of $y$ of the spinor on the right,

$$D_y\psi(x, y) = \partial_y\psi(x, y) + \Phi(x, y, -y)\psi(x, -y), \quad (6.17)$$

in accordance with the definition (6.2) of the derivative $\partial_y$. The gauge transformation properties of the gauge fields are obtained as follows,

$$A_\mu(x, y) \rightarrow ^gA_\mu(x, y) = g(x, y)A_\mu(x, y)g^\dagger(x, y) + g(x, y)\partial_\mu g^\dagger(x, y),$$

$$\Phi(x, y, -y) \rightarrow ^g\Phi(x, y, -y) = g(x, y)\Phi(x, y, -y)g^\dagger(x, -y) - (\partial_y g(x, y))g^\dagger(x, -y). \quad (6.18)$$

The derivative $\partial_y g(x, y)$ can be determined by the Leibniz rule,

$$\partial_y(g(x, y)\psi(x, y)) = M(y, -y)g(x, -y)\psi(x, -y)$$

$$= g(x, y)\partial_y\psi(x, y) + (\partial_y g(x, y))\psi(x, -y), \quad (6.19)$$

$^*$) The differential geometry on $M_4 \times Z_2$ was developed in Ref. 8) and we can use it for the case under consideration. The following presentation is only a translation of the result in Ref. 8).
where the first equality comes from the definition (6.2) and the second equality is the Leibniz rule. Using (6.2) we obtain

$$\partial_y g(x, y) = M(y, -y)g(x, -y) - g(x, y)M(y, -y).$$

(6.20)

Inserting this equation into the second equation of (6.18) and putting,

$$H(x, y, -y) \equiv \Phi(x, y, -y) + M(y, -y),$$

(6.21)

we find

$$gH(x, y, -y) \equiv g\Phi(x, y, -y) + M(y, -y) = g(x, y)H(x, y, -y)g^\dagger(x, -y).$$

(6.22)

The gauge-invariant Dirac action $S_D$ is then given by

$$S_D = \sum_{y = \pm} \int d^4x \left[ \bar{\psi}(x, y)i\gamma^\mu(\partial_\mu + A_\mu(x, y))\psi(x, y) + \bar{\psi}(x, y)i\gamma^5H(x, y, -y)\psi(x, -y) \right].$$

(6.23)

Setting

$$A_\mu^L(x) = A_\mu(x, +), \quad A_\mu^R(x) = A_\mu(x, -),$$

$$H(x) = H(+, -), \quad H^\dagger(x) = H(-, +),$$

(6.24)

and performing the phase rotation $\psi_L(x) \rightarrow e^{i\pi/4}\psi_L(x)$ and $\psi_R(x) \rightarrow e^{-i\pi/4}\psi_R(x)$, (6.23) can be put into the well-known form with Yukawa coupling,

$$S_D = \int d^4x \left[ \bar{\psi}_L(x)i\gamma^\mu(\partial_\mu + A_\mu^L(x))\psi_L(x) + \bar{\psi}_R(x)i\gamma^\mu(\partial_\mu + A_\mu^R(x))\psi_R(x) - \bar{\psi}_L(x)H(x)\psi_R(x) - \bar{\psi}_R(x)H^\dagger(x)\psi_L(x) \right].$$

(6.25)

The product rule of the field quantities and the gauge functions are now obvious. Only the functions with the same $x$ and the same sign of the variable $y$ can be multiplied. For instance, there appears no product like $H(x, y, -y)\psi(x, y)$. Similarly, only the product like $(\partial_y \Phi(x, y, -y))\psi(y)$ appears. The same rule should be respected in constructing the bosonic action.

To determine the bosonic Lagrangian we define the generalized gauge field,

$$A(x, y, -y) = i\gamma^\mu A_\mu(x, y) + i\gamma^5\Phi(x, y, -y),$$

$$\gamma^0 A^\dagger(x, y, -y)\gamma^0 = A(x, y) + i\gamma^5\Phi(x, -y, y).$$

(6.26)
The corresponding generalized field strength is obtained by

\[ A(x, y, -y) = dA(x, y, -y) + A(x, y, -y) \wedge A(x, y, -y). \]  \quad (6.27)

Here, the generalized exterior derivative is defined as

\[ dA(x, y, -y) = \frac{1}{2} i \gamma^\mu \wedge i \gamma^\nu (\partial_\mu A_\nu(x, y) - \partial_\nu A_\mu(x, y)) \]
\[ + i \gamma^5 \wedge i \gamma^\mu \partial_y A_\mu(x, y) + i \gamma^5 \wedge i \gamma^\mu \partial_y \Phi(x, y, -y) \]
\[ + i \gamma^5 \wedge i \gamma^5 \partial_y \Phi(x, y, -y), \]  \quad (6.28)

with the wedge product of \( \gamma \) matrices including \( \gamma^5 \) being defined in Ref. 18), so that \( \gamma^5 \wedge \gamma^\mu = -\gamma^\mu \wedge \gamma^5 = \gamma^5 \gamma^\mu \) and \( \gamma^5 \wedge \gamma^5 = 1 \). Although the derivative \( \partial_y A_\mu(x, y) \) is similar to the derivative (6.20), the derivative \( \partial_y \Phi(x, y, -y) \) is different. To explain it we assume the Leibniz rule

\[ \partial_y (\Phi(y, x, -y) \psi(x, y)) = M(y, -y) \Phi(x, y, -y) \psi(x, y) \]
\[ = -\Phi(x, y, -y) \partial_y \psi(x, y, -y) + (\partial_y \Phi(x, y, -y)) \psi(x, y). \]  \quad (6.29)

Note the minus sign in front of the first term of the last expression. It is now trivial to obtain

\[ \partial_y \Phi(x, y, -y) = M(y, -y) \Phi(x, y, -y) + \Phi(x, y, -y) M(-y, y). \]  \quad (6.30)

We can thus evaluate the exterior derivative \( dA(x, y) \).

The second term in (6.27) is obtained as

\[ A(x, y, -y) \wedge A(x, y, -y) = \frac{1}{2} i \gamma^\mu \wedge i \gamma^\nu [A_\mu(x, y), A_\nu(x, y)] \]
\[ + i \gamma^5 \wedge i \gamma^\mu A_\mu(x, y) \Phi(x, y, -y) \]
\[ + i \gamma^5 \wedge i \gamma^5 \Phi(x, y, -y) \Phi(x, y, -y). \]  \quad (6.31)

Note a consistent sign change of \( y \) in accordance with the product rule mentioned above. By adding (6.31) to (6.28) we get

\[ F(x, y, -y) = \frac{1}{2} i \gamma^\mu \wedge i \gamma^\nu F_{\mu\nu}(x, y) + i \gamma^5 \wedge i \gamma^\mu F_{\mu}(x, y, -y) + i \gamma^5 \wedge i \gamma^5 F(x, y, -y), \]  \quad (6.32)
where

\[ F_{\mu\nu}(x, y) = \partial_\mu A_\nu(x, y) - \partial_\nu A_\mu(x, y) + [A_\mu(x, y), A_\nu(x, y)], \]

\[ F_\mu(x, y, -y) = \partial_\mu \Phi(x, y, -y) + A_\mu(x, y) \Phi(x, y, -y) + A_\mu(x, y) M(y, -y) \]

\[ -\Phi(x, y, -y) A_\mu(x, y) - M(y, -y) A_\mu(x, y) \equiv D_\mu H(x, y, -y), \]

\[ F(x, y, -y) = \Phi(x, y, -y) + M(y, -y) \Phi(x, -y, y) + \Phi(x, y, -y) M(-y, y) \equiv H(x, y, -y) H(x, -y, y) - M(y, -y) M(-y, y). \] (6.33)

Our final task is to construct the bosonic Lagrangian through the generalized field strength \( F(x, y, -y) \). A method leading to (6.14) no longer works because the expression

\[ S'_B = -\frac{1}{8} \sum_{y=\pm} \int d^4x \text{Tr} \frac{1}{g_y^2} F^2(x, y, -y), \] (6.34)

contains fewer parameters than those required by the renormalizability of the model. To avoid the shortcomings we introduce the associated field strength with the same gauge transformation property as that of \( F(x, y, -y) \),

\[ \tilde{F}(x, y, -y) = \sum_A h_A^2 \Gamma_A F(x, y, -y) \Gamma^A, \] (6.35)

where the sum over \( A \) runs over \( S, V, A, T, P \) corresponding to \( \Gamma^A = 1, \gamma^\mu, \gamma^5, \gamma^\mu, \sigma^\mu\nu, \gamma^5 \), respectively. We define the positive parameters \( \alpha^2 \) and \( \beta^2 \) in place of \( h_A^2 \) to write down the bosonic Lagrangian by

\[ S_B = -\frac{1}{8} \sum_{y=\pm} \int d^4x \text{Tr} \frac{1}{g_y^2} \tilde{F}(x, y, -y) F(x, y, -y) \]

\[ = -\frac{1}{4} \sum_{y=\pm} \int d^4x \text{tr} \frac{1}{g_y^2} F^\dagger_{\mu\nu}(x, y) F^{\mu\nu}(x, y) \]

\[ + \frac{\alpha^2}{2} \sum_{y=\pm} \int d^4x \text{tr} \frac{1}{g_y^2} F_{\mu}(x, -y, y) F^\mu(x, y, -y) \]

\[ - \frac{\beta^2}{2} \sum_{y=\pm} \int d^4x \text{tr} \frac{1}{g_y^2} F(x, -y, y) F(x, y, -y). \] (6.36)

* As noted in Ref. 8) the component \( F(x, y, -y) \) is not gauge-covariant unless \( M(y, -y) M(-y, y) \) commutes with \( g(x, y) \). In the following we assume that the matrix \( M(y, -y) M(-y, y) \) is proportional to the unit matrix. This assumption is also necessary to prove the gauge-Bianchi identity for the generalized field strength.
It would be instructive to consider a simple chiral gauge group \( U(1) \times U(1) \), namely, the Abelian Higgs model coupled to fermion. After rescaling of the gauge and the Higgs fields with \( g_+ = g \) and \( g_- = g' \), and using the notation of (6.24) we obtain from (6.37),

\[
S_B = \int d^4x \left[ -\frac{1}{4} F^{L\mu\nu}_\mu(x) F^{L,\mu\nu}_\mu(x) -\frac{1}{4} F^{R\mu\nu}_\mu(x) F^{R,\mu\nu}_\mu(x) \right. \\
+ (D^\mu H(x)) \dagger D^\mu H(x) - \frac{\lambda}{4} (H \dagger H(x) - \frac{v^2}{2})^2, \\
D^\mu H(x) = (\partial^\mu - ig A^L_\mu(x) + ig' A^R_\mu(x)) H(x).
\] (6.37)

The covariant derivatives for the chiral spinors are given by

\[
D^L_\mu \psi_L(x) = (\partial^\mu - ig A^L_\mu(x)) \psi_L(x), \\
D^R_\mu \psi_R(x) = (\partial^\mu - ig' A^R_\mu(x)) \psi_R(x),
\] (6.38)

and the fermion mass is proportional to the parameter \( v \). The gauge field,

\[
B^\mu(x) = \frac{1}{\sqrt{g^2 + g'^2}} (g A^L_\mu(x) - g' A^R_\mu(x)),
\] (6.39)

becomes massive with the mass squared, \( M^2 = v^2(g^2 + g'^2) \), while the orthogonal combination,

\[
A^\mu(x) = \frac{1}{\sqrt{g^2 + g'^2}} (g' A^L_\mu(x) + g A^R_\mu(x)),
\] (6.40)

remains massless.

\[\text{§7. Conclusions}\]

As pointed out by Kadyshevskii\(^\text{3}\), there are two kinds of Snyder’s quantized space-time where the triple commutator between the operator coordinates does not vanish. In one of them, which was originally proposed by Snyder\(^\text{2}\), the spatial coordinates have a discrete spectrum of eigenvalues of the form, \( na \), where \( n \) is an integer, while the time coordinate has a continuous spectrum. The other is opposite: the spectrum of the time coordinate is discrete, while that of the spatial coordinates is continuous. The two possible cases may correspond to the existence of the \( \pm \) spaces introduced in §4. As emphasized there, the \( \pm \) spaces are disjoint not connected by any Lorentz transformation. The orthogonal spaces similar in nature is utilized in thermofield dynamics which introduces non-tilde and tilde states to express the ensemble averages in terms of pure states.
In the DFR algebra, the triple commutator between the operator coordinates is assumed to vanish, keeping the canonical commutator $[\hat{p}^\mu, \hat{x}^\nu] = ig^{\mu\nu}$ intact. This assumption allows us to introduce the non-commutative fields which are subject to the Lorentz-invariant non-local interactions. Although the non-commutative fields have (4+6)-dimensional coordinates, the compactification of the extra 6-dimensions is not needed, because the fields do not propagate into the extra dimensions. We modified CCZ covariant moment formula in a nontrivial way in §4. We also showed that it is possible to generalize Connes’ two-sheeted structure, $M_4 \times Z_2$, to the non-commutative space-time. Connes’ interpretation that the Higgs field can be regarded as a kind of the gauge fields in the discrete direction was explicitly displayed in §6 by introducing a set of consistent rules borrowed from the differential geometry on $M_4 \times Z_2$. In spite of the fact that we only rederived the known QFT models, we recall that the discrete space-time, $M_4 \times Z_2$, proposed by Connes in relation with the standard model fits nicely to QFT without introducing extra physical degrees of freedom. Our introduction of the two-sheeted structure into the Lorentz-invariant non-commutative space-time will permit us to construct a non-commutative standard model with the non-commutative Higgs fields being interpreted as a kind of the gauge fields in the discrete direction. In this sense Connes’ theory of the standard model is related to the Lorentz-invariant NCGT.

From our lesson in §6 it is apparent that the set of the monogenic and the dichotomic fields with appropriate internal quantum numbers can be regarded as the set of the fields considered by Connes in his reformulation of the standard model on $M_4 \times Z_2$. In this respect it should be remembered that there exist only two non-Abelian charges, flavor and color, in the standard model. This simple fact is mathematically interpreted that the chiral matter fields can be regarded as a bi-module over the flavor-color algebra. In the non-commutative regime this concept would be generalized to the non-commutative bi-module as explained in Ref. 6). Realistic model construction along this line of thought will be a theme in future works.

Appendix A

Lorentz Generators of Snyder’s and The DFR Algebras

The generators of the Lorentz group for the Snyder’s algebra were given by (3·11). Using the operator coordinates (3·10) they have the usual expression in terms of coordinates and momenta,

$$\hat{M}^{\mu\nu} = \hat{x}^\mu \hat{p}^\nu - \hat{x}^\nu \hat{p}^\mu.$$
This is no longer the case for the DFR algebra. In fact, the orbital angular momentum,

\[ \hat{L}^{\mu\nu} = \hat{x}^\mu \hat{p}^\nu - \hat{x}^\nu \hat{p}^\mu, \]

commutes with the operator \( \hat{\theta}^{\mu\nu} \) because of (3-3) and (3-6). The tensor nature (3-2) of the operator \( \hat{\theta}^{\mu\nu} \), therefore, comes from another source. In this connection we recall that the generators of the Lorentz group consist of two terms, the orbital angular momentum and the spin angular momentum. Thus we may write

\[ \hat{M}^{\mu\nu} = \hat{L}^{\mu\nu} + \hat{S}^{\mu\nu}, \]
\[ \hat{L}^{\mu\nu} = \hat{x}^\mu \hat{p}^\nu - \hat{x}^\nu \hat{p}^\mu, \]

where the spin part \( \hat{S}^{\mu\nu} \) is chosen such that the extra terms in the commutator,

\[ [\hat{L}^{\mu\nu}, \hat{x}^\rho] = i(g^{\rho\nu} \hat{x}^\mu - g^{\rho\mu} \hat{x}^\nu) + [\hat{x}^\mu, \hat{x}^\rho] \hat{p}^\nu - [\hat{x}^\nu, \hat{x}^\rho] \hat{p}^\mu; \]

are precisely cancelled to reproduce the correct commutation relations, the second equation of (1-4),

\[ [\hat{S}^{\mu\nu}, \hat{x}^\rho] = -[\hat{x}^\mu, \hat{x}^\rho] \hat{p}^\nu + [\hat{x}^\nu, \hat{x}^\rho] \hat{p}^\mu. \]

Similarly from the commutator \([\hat{L}^{\mu\nu}, \hat{p}^\rho]\) we should have

\[ [\hat{S}^{\mu\nu}, \hat{p}^\rho] = 0. \]

The ordinary spin angular momentum in \( \hat{S}^{\mu\nu} \) commutes with the operator coordinates. It is then easy to prove (3-2) by computing the commutator \([\hat{M}^{\mu\nu}, \hat{\theta}^{\rho\sigma}] = [\hat{S}^{\mu\nu}, \hat{\theta}^{\rho\sigma}]\).

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