Research Article

High-Order Total Bounded Variation Model and Its Fast Algorithm for Poissonian Image Restoration

Jun Zhang,1,2 Mingxi Ma,2 Zhaoming Wu,1 and Chengzhi Deng1

1 Jiangxi Province Key Laboratory of Water Information Cooperative Sensing and Intelligent Processing, Nanchang Institute of Technology, Nanchang 330099, Jiangxi, China
2 College of Science, Nanchang Institute of Technology, Nanchang 330099, Jiangxi, China

Correspondence should be addressed to Chengzhi Deng; dengcz@nit.edu.cn

Received 12 November 2018; Revised 19 January 2019; Accepted 30 January 2019; Published 11 February 2019

1. Introduction

Being signal-dependent, Poisson noise is inevitable in various applications such as positron emission tomography [1], electronic microscopy [2], and astronomical imaging [3]. In particular, the degree of Poisson noise depends on the peak value of pixel intensity values, in the sense that the smaller peak value yields the higher intensity of Poisson noise. Moreover, the Poisson noise magnitude in an image increases with the pixel intensity of the region of this image. Since Poisson noise is different from the Gaussian noise, the restoration models and related algorithms proposed for Gaussian white noise are not effective in removing Poisson noise.

A traditional method for Poisson denoising was proposed by Richardson and Lucy (RL) [4, 5]. However, the RL algorithm converges slowly with noise remained in restored images. A standard approach to remove the noise involves incorporating a regularization term, such as Tikhonov regularization [6], total variation (TV) regularization [7], and high-order regularization [8]. In particular, the TV-based Poisson denoising model [9] can be described as

$$\min_u \|\nabla u\|_1 + \beta \int_{\Omega} (u - f \log u) \, dx,$$  \hspace{1cm} (1)

where $\Omega$ is a bounded open domain in $\mathbb{R}^2$ and $\beta > 0$ is a regularization parameter. For this model (1), a number of fast and efficient numerical methods have been proposed, for instance, the dual algorithm [10], the augmented Lagrangian method [11, 12], the split Bregman algorithm [13], the multilevel algorithm [14], and a semi-implicit gradient descent method [15]. These algorithms have also been successfully applied to remove noisy images corrupted by Gaussian, impulse, and multiplicative noises [16–21]. In [11, 13], the following TV-based Poisson restoration model has also been studied:

$$\min_u \|\nabla u\|_1 + \beta \int_{\Omega} (Ku - f \log Ku) \, dx.$$  \hspace{1cm} (2)

For solving this model, in [11], the authors proposed an augmented Lagrangian method called PIDAL. However, it needs to solve a TV denoising subproblem and thus needs an inner iteration loop. Furthermore, the nonnegative property of the sequence $u^k$ cannot be guaranteed. To deal with these problems, Setzer et al. [13] proposed a split Bregman algorithm called PIDSplit+ r. Subsequently, Wen et al. [22] proposed primal–dual algorithms for model (2). Their algorithms require less memory and no need to solve any linear systems,
and thus are fast. However, they did not state related theories for this Poisson restoration model in their papers.

Replacing the TV regularization term in (2) by $\|\nabla u\|_1 + (\alpha/2)\|u\|_2^2$ [23], Liu and Huang [24] proposed the total bounded variation-based Poisson restoration model as follows:

$$\min_{u \in BV^2(\Omega), \log K \in L^1(\Omega)} \|\nabla u\|_1 + \frac{\alpha}{2} \|u\|_2^2 + \beta \int_\Omega (Ku - f \log Ku) \, dx,$$

where $\alpha \geq 0, \beta > 0$ are parameters, and $K$ is a bounded linear operator representing the convolution kernel. Owing to the strongly convex property of $(\alpha/2)\|u\|_2^2$, uniqueness of the solution of this model can be guaranteed. Further, the convergence of the proposed algorithm was proved in [24].

The TV-based restoration model performs very well for preserving edges while removing noise. However, it often causes staircase effects in flat regions. To overcome the staircase effects, some high-order models [25–28] have been introduced in image restoration. For restoring blurred images corrupted by Poisson noise, the hybrid regularizations combining TV with high-order variation were also considered [29, 30]. In this paper, we propose the following model by replacing the TV term in (3) with second order term $\|\nabla^2 u\|_1$:

$$\min_{u \in BV^2(\Omega), \log K \in L^1(\Omega)} E(u) = \|\nabla^2 u\|_1 + \frac{\alpha}{2} \|u\|_2^2 + \beta \int_\Omega (Ku - f \log Ku) \, dx,$$

where $BV^2(\Omega)$ is a Banach space. Its definition and related properties were introduced in [17]. For this model, we prove existence and uniqueness of its solution. We use a split Bregman algorithm to compute the resulting minimization problem by introducing new auxiliary variables. Besides, we prove the convergence of our proposed algorithm. The experimental results show that our model (4) avoids undesirable staircase effects. Compared with the split Bregman algorithm for solving the total bounded variation-based Poisson restoration model (3), our method is faster.

The organization of this paper is as follows. In Section 2, we show existence and uniqueness of the solution of our proposed model (4). Section 3 presents a split Bregman algorithm for solving this model. We also study the convergence of the proposed algorithm in this section. In Section 4, experiments are carried out to show the efficiency of our proposed algorithm. We end the paper with some conclusions in Section 5.

2. Existence and Uniqueness

In this section, we prove the existence and uniqueness of the solution for proposed model (4).

**Theorem 1.** Model (4) has a minimizer in $BV^2(\Omega)$. Furthermore, if $K$ is injective, the minimizer is unique.

**Proof.** Let $G(q) = \int_\Omega h(q) \, dx$ with $h(q) = q - f \log q$ and $q = Ku$. For the given $f > 0$, $\nabla^2 h(q) = f/2 > 0$. Thus $h(q)$ is strictly convex. Then it holds $G(q) = G(Ku) \geq G(Kf) = \int_\Omega Kf - f \log Kf \, dx$. So we can choose a minimizing sequence $\{u_n\}_{n=1}^{\infty}$ for $E(u)$. By using Hölder’s inequality and Jensen’s inequality, we have

$$G(q_n) \geq \|q_n\|_1 - \|f\|_\infty \log \|q_n\|_1,$$

and thus $\|q_n\|_1$ is bounded. Meanwhile, considering that $K \in \mathcal{L}(BV^2(\Omega))$ and the boundedness property of $\|\nabla^2 u\|_1$, we know that $\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in $BV^2(\Omega)$. By the compactness theory, there is $u^* \in BV^2(\Omega)$ such that a subsequence $\{u_{n_k}\}_{k=1}^{\infty}$ converges to $u^*$ a.e. By the lower semicontinuity of the $BV^2$ norm, $\|\nabla^2 u^*\|_1 \leq \liminf_{k \to \infty} \|\nabla^2 u_{n_k}\|_1$ and $\|u_{n_k}\|_2 \geq \|u^*\|_2$. Further, by using Fatou’s lemma, we get

$$\inf \left\{ \|\nabla^2 u\|_1 + \frac{\alpha}{2} \|u\|_2^2 + \beta \int_\Omega (Ku - f \log Ku) \, dx \right\} \geq \liminf_{k \to \infty} \left\{ \|\nabla^2 u_{n_k}\|_1 + \frac{\alpha}{2} \|u_{n_k}\|_2^2 + \beta \int_\Omega (Ku_{n_k} - f \log Ku_{n_k}) \, dx \right\} \geq \|\nabla^2 u^*\|_1 + \frac{\alpha}{2} \|u^*\|_2^2 + \beta \int_\Omega (Ku^* - f \log Ku^*) \, dx.$$

Thus $u^*$ is the minimizer of model (4).

Obviously, if $K$ is injective, the high-order total variation term and the fidelity term in (4) are convex. Since $(\alpha/2)\|u\|_2^2$ is a strongly convex function for a given constant $\alpha$, $E(u)$ is strongly convex. Therefore, its minimizer is unique.

3. The Split Bregman Algorithm

We describe the split Bregman algorithm (SBA) for solving the proposed model (4), together with convergence analysis. The SBA is an efficient method, which was initially introduced by Goldstein and Osher to solve the general $L_1$-regularized problems [31]. It has been successfully applied to minimization problems that involve nondifferentiable and high-order functionals [19, 24, 32].

3.1. Algorithm Description. It is straightforward that the proposed model (4) is equivalent to the following constrained optimization problem:

$$\min_{u \in \mathcal{P}} \|p\|_1 + \frac{\alpha}{2} \|u\|_2^2 + \beta \int_\Omega (q - f \log q) \, dx,$$

s.t. $p = \nabla^2 u,$

$q = Ku.$

To solve this problem, we convert it into an unconstrained problem by applying the penalty method here:
\[
\begin{align*}
\min_{\mathbf{p}, q} & \quad \frac{\alpha}{2} \lVert \mathbf{u} \rVert_2^2 + \frac{\beta}{2} \int_{\Omega} (q - f \log q) \, dx \\
& \quad + \frac{\gamma_1}{2} \lVert \mathbf{p} - \nabla^2 \mathbf{u} - \mathbf{b}_k \rVert_2^2 \\
& \quad + \frac{\gamma_2}{2} \lVert q - K \mathbf{u} \rVert_2^2,
\end{align*}
\]

where \(\gamma_1, \gamma_2 > 0\) are two penalty parameters. Consequently, the SBA for solving (4) can be described as follows:

\[
\begin{align*}
u^{k+1} &= \arg\min_{\mathbf{u}} \frac{\alpha}{2} \lVert \mathbf{u} \rVert_2^2 + \frac{\gamma_1}{2} \lVert \mathbf{p} - \nabla^2 \mathbf{u} - \mathbf{b}_k \rVert_2^2 \\
& \quad + \frac{\gamma_2}{2} \lVert q - K \mathbf{u} \rVert_2^2,
\end{align*}
\]

\[
p^{k+1} = \arg\min_{\mathbf{p}} \lVert \mathbf{p} \rVert_1 + \frac{\gamma_1}{2} \lVert \mathbf{p} - \nabla^2 \mathbf{u}^{k+1} - \mathbf{b}_k \rVert_2^2,
\]

\[
q^{k+1} = \arg\min_{q} \int_{\Omega} (q - f \log q) \, dx
\]

where the analysis of the unconstrained split Bregman iterations was presented in detail.

**3.2. Convergence Analysis.** Since our model (4) is strictly convex, there exists a unique solution as guaranteed by Theorem 1. Below we give a rigorous proof that the proposed algorithm (Algorithm 2) converges to the unique solution.

Note that the following analysis is stemmed from [24, 33], where the analysis of the unconstrained split Bregman iteration was presented in detail.

**Theorem 3.** Assume that \(K\) is injective and \(\alpha, \beta, \gamma_1, \gamma_2 > 0\). Then the sequence \(\{\mathbf{u}^k\}\) generated by Algorithm 2 converges to the unique solution \(\mathbf{u}^*\) of model (4).

**Proof.** Let \(\{u^k, \mathbf{p}^k, q^k, \mathbf{b}_1^k, \mathbf{b}_2^k\}\) be the sequence generated by Algorithm 2. It follows from the first-order optimality conditions at \(u^{k+1}, \mathbf{p}^{k+1}, q^{k+1}\) that

\[
0 = \alpha u^{k+1} - \gamma_1 (\nabla^2)^T (\mathbf{p}^k - \nabla^2 \mathbf{u}^{k+1} - \mathbf{b}_k^k) - \gamma_2 K^T (q^k - K \mathbf{u}^{k+1} - \mathbf{b}_2^k)
\]

and the solution of \(q\)-subproblem is obtained by using the quadratic formula, i.e.,

\[
q^{k+1} = \frac{K \mathbf{u}^{k+1} + b_2^k - \beta / \gamma_2}{2} \quad \sqrt{\left(\frac{K \mathbf{u}^{k+1} + b_2^k - \beta / \gamma_2}{2}\right)^2 + \frac{\beta f}{\gamma_2}}.
\]

We summarize the overall algorithm in Algorithm 2.

**Algorithm 2 (SBA for solving (4)).**

**Step 1.** Input \(\alpha, \beta, \gamma_1, \gamma_2\). Initialization: \(u^0 = f, \mathbf{p}^0, q^0, \mathbf{b}_1^0, \mathbf{b}_2^0\). Let \(k = 0\).

**Step 2.** Compute \(u^{k+1}\) by (11).

**Step 3.** Compute \(\mathbf{p}^{k+1}\) by (12).

**Step 4.** Compute \(q^{k+1}\) by (13).

**Step 5.** Update \((\mathbf{b}_1^{k+1}, \mathbf{b}_2^{k+1})\) by the last two equations of (9).

**Step 6.** Stop if the stopping criterion is satisfied. Otherwise, let \(k := k + 1\) and go to Step 2.
where \( g^* \in \partial \| p^* \|_1 \) with \( p^* = V^2 u^* \) and \( h^* = 1 - f/q^* \) with \( q^* = Ku^* \). We further denote \( b^*_i = (1/\gamma_i)g^*_i, b^*_e = (\beta/\gamma_e)h^* \) so as to get the following set of conditions:

\[
0 = \alpha u^* - \gamma_1 \left( V^2 \right)^T (p^* - V^2 u^* - b^*_1) \\
- \gamma_2 K^T (q^* - Ku^* - b^*_e),
\]

\[
0 = g^* + \gamma_1 \left( p^* - V^2 u^* - b^*_i \right),
\]

\[
0 = \beta \left( 1 - \frac{f}{q^*} \right) + \gamma_2 (q^* - Ku^* - b^*_e),
\]

\[
b^*_1 = b^*_1 + V^2 u^* - p^*,
\]

\[
b^*_e = b^*_e + Ku^* - q^*.
\]

By comparing (14) and (16), we can conclude that \( \{u^*, p^*, q^*, b^*_1, b^*_e\} \) is a fixed point of (9). We then introduce new variables to represent the errors, i.e., \( u^k = u^k - u^*, p^k = p^k - p^*, q^k = q^k - q^*, b^k_e = b^*_e - b^*_1 \) and \( b^k_e = b^*_e - b^*_1 \). By subtracting each equation in (16) from the corresponding one in (14), we get

\[
0 = \alpha u^k_{e^1} - \gamma_1 \left( V^2 \right)^T \left( p^k - V^2 u^k_{e^1} - b^k_e \right) \\
- \gamma_2 K^T (q^k - Ku^k_{e^1} - b^k_e),
\]

\[
0 = \left( g^k_{e^1} - g^* \right) + \gamma_1 \left( p^k_{e^1} - V^2 u^k_{e^1} - b^k_{e^1} \right),
\]

\[
0 = \beta f \left( \frac{q^k_{e^1}}{q_{e^1}} \right) + \gamma_2 (q^k_{e^1} - Ku^k_{e^1} - b^k_{e^1}),
\]

\[
b^k_{e^1} = b^k_{e^1} + V^2 u^k_{e^1} - p^k_{e^1},
\]

\[
b^k_{e^2} = b^k_{e^2} + Ku^k_{e^1} - q^k_{e^1}.
\]

Then for (17), taking the inner product of the first three equations with respect to \( u^k_{e^1}, p^k_{e^1}, \) and \( q^k_{e^1} \), respectively, and squaring both sides of the last two equations, we obtain

\[
0 = \alpha \| u^k_{e^1} \|_2^2 + \gamma_1 \left( \left( V^2 \right)^T V^2 u^k_{e^1} , u^k_{e^1} \right) \\
- \gamma_1 \left( \left( V^2 \right)^T \left( p^k - b^k_{e^1} \right) , u^k_{e^1} \right) \\
+ \gamma_2 \left( K^T Ku^k_{e^1} , u^k_{e^1} \right) \\
- \gamma_2 \left( K^T \left( q^k_{e^1} - b^k_{e^1} \right) , u^k_{e^1} \right),
\]

\[
0 = \langle \left( g^k_{e^1} - g^* \right) \left( p^k_{e^1} - b^k_{e^1} \right) , u^k_{e^1} \rangle + \gamma_1 \| p^k_{e^1} \|_2^2 \\
- \gamma_1 \langle V^2 u^k_{e^1} + b^k_{e^1} , p^k_{e^1} \rangle,
\]

\[
0 = \beta f \| q^k_{e^1} \|_2^2 + \gamma_2 \| q^k_{e^1} \|_2^2
\]

\[
- \gamma_2 \langle Ku^k_{e^1} + b^k_{e^1} , q^k_{e^1} \rangle,
\]

\[
\| b^k_{e^1} \|_2^2 = \| b^k_{e^1} \|_2^2 + \| V^2 u^k_{e^1} - p^k_{e^1} \|_2^2
\]

\[
+ 2 \langle b^k_{e^1} , V^2 u^k_{e^1} - p^k_{e^1} \rangle,
\]

\[
\| b^k_{e^2} \|_2^2 = \| b^k_{e^2} \|_2^2 + \| Ku^k_{e^1} - q^k_{e^1} \|_2^2
\]

\[
+ 2 \langle b^k_{e^2} , Ku^k_{e^1} - q^k_{e^1} \rangle.
\]

By combining (20) with the first three equations in (19), we have

\[
\frac{\gamma_2}{2} \left( \| b^k_{e^1} \|_2^2 - \| b^k_{e^1} \|_2^2 \right) = -\frac{\gamma_2}{2} \| V^2 u^k_{e^1} - p^k_{e^1} \|_2^2
\]

\[
- \gamma_1 \left( \| b^k_{e^1} , V^2 u^k_{e^1} - p^k_{e^1} \|_2^2 \right),
\]

\[
\frac{\gamma_2}{2} \left( \| b^k_{e^2} \|_2^2 - \| b^k_{e^2} \|_2^2 \right) = -\frac{\gamma_2}{2} \| Ku^k_{e^1} - q^k_{e^1} \|_2^2
\]

\[
- \gamma_2 \left( \| b^k_{e^2} , Ku^k_{e^1} - q^k_{e^1} \|_2^2 \right).
\]

Summing (21) bilaterally from \( k = 0 \) to \( k = N \) yields

\[
\frac{\gamma_2}{2} \left[ \| b^0_{e^1} \|_2^2 - \| b^N_{e^1} \|_2^2 \right] + \frac{\gamma_2}{2} \left[ \| b^0_{e^2} \|_2^2 - \| b^N_{e^2} \|_2^2 \right]
\]

\[
= \alpha \sum_{k=0}^N \| u^k_{e^1} \|_2^2 + \sum_{k=0}^N \left\langle g^k - g^* , p^k \right\rangle
\]

\[
+ \sum_{k=0}^N \beta f \left\| q^k \right\|_2^2 + \frac{\gamma_1}{2} \sum_{k=0}^N \| V^2 u^k_{e^1} - p^k_{e^1} \|_2^2
\]

\[
+ \frac{\gamma_2}{2} \sum_{k=0}^N \| Ku^k_{e^1} - q^k_{e^1} \|_2^2.
\]

Obviously, the last two equations of (19) can be rewritten as

\[
\frac{\gamma_2}{2} \left( \| b^k_{e^1} \|_2^2 - \| b^k_{e^1} \|_2^2 \right) = -\frac{\gamma_2}{2} \| V^2 u^k_{e^1} - p^k_{e^1} \|_2^2
\]

\[
- \gamma_1 \left( \| b^k_{e^1} , V^2 u^k_{e^1} - p^k_{e^1} \|_2^2 \right),
\]

\[
\frac{\gamma_2}{2} \left( \| b^k_{e^2} \|_2^2 - \| b^k_{e^2} \|_2^2 \right) = -\frac{\gamma_2}{2} \| Ku^k_{e^1} - q^k_{e^1} \|_2^2
\]

\[
- \gamma_2 \left( \| b^k_{e^2} , Ku^k_{e^1} - q^k_{e^1} \|_2^2 \right).
\]
And note that \( \{d^k\}_{k=1}^{\infty} \) is a bounded sequence in \( L^1(\Omega) \). Therefore, all terms in (23) are nonnegative. Letting \( N \to \infty \), inequality (23) suggests the following conclusions:

\[
\sum_{k=0}^{\infty} \|u^k\|_2^2 < +\infty,
\]

\[
\sum_{k=0}^{\infty} \langle g^k - g^*, p^k - p^* \rangle < +\infty,
\]

\[
\sum_{k=0}^{\infty} \|q^k - q^*\|_2^2 < +\infty,
\]

\[
\sum_{k=0}^{\infty} \|\nabla^2 u^k - p^k\|_2^2 < +\infty,
\]

\[
\sum_{k=0}^{\infty} \|K u^k - q^k\|_2^2 < +\infty,
\]

which implies that

\[
\lim_{k \to +\infty} \|u^k - u^*\|_2 = 0,
\]

\[
\lim_{k \to +\infty} \langle g^k - g^*, p^k - p^* \rangle = 0,
\]

\[
\lim_{k \to +\infty} \|g^k - q^*\|_2 = 0,
\]

\[
\lim_{k \to +\infty} \|\nabla^2 u^k - p^k\|_2 = 0,
\]

\[
\lim_{k \to +\infty} \|K u^k - q^k\|_2 = 0.
\]

For any convex function \( J \), the Bregman distance satisfies

\[
B_J^p(u, v) + B_J^q(v, u) = \langle q - p, u - v \rangle,
\]

\[
\forall p \in \partial J(v), \forall q \in \partial J(u).
\]

This together with the second equality of (26) and the nonnegativity of the Bregman distance implies that

\[
\lim_{k \to +\infty} \|p^k\|_2 = \|p^*\|_2 = \langle g^*, p^k - p^* \rangle = 0.
\]

Recall that \( p^* = \nabla^2 u^* \). And by the fourth equality of (26), we obtain

\[
\lim_{k \to +\infty} \|\nabla^2 u^k\|_1 - \|\nabla^2 u^*\|_1 - \langle g^*, \nabla^2 u^k - \nabla^2 u^* \rangle = 0.
\]

In the same way, we also have

\[
\lim_{k \to +\infty} \|u^k\|_2^2 - \|u^*\|_2^2 - \langle 2u^*, u^k - u^* \rangle = 0,
\]

\[
\lim_{k \to +\infty} G(K u^k) - G(K u^*) - \langle h^*, K u^k - K u^* \rangle = 0.
\]

Combining this with (29) yields

\[
\lim_{k \to +\infty} \left( \|\nabla^2 u^k\|_1 + \frac{\alpha}{2} \|u^k\|_2^2 + \beta G(K u^k) \right) - \left( \|\nabla^2 u^*\|_1 + \frac{\alpha}{2} \|u^*\|_2^2 + \beta G(K u^*) \right) = 0.
\]

Finally, this and equality (15), we get

\[
\lim_{k \to +\infty} \|\nabla^2 u^k\|_1 + \frac{\alpha}{2} \|u^k\|_2^2 + \beta G(K u^k) = \|\nabla^2 u^*\|_1 + \frac{\alpha}{2} \|u^*\|_2^2 + \beta G(K u^*).
\]

\[\square\]

4. Numerical Experiments

We present experimental results to show the performance of our proposed method (4), referred to as BLLTSB, in comparison with the total bounded variation-based Poisson restoration model (3), referred to as BTVSB. All the experiments were conducted in MATLAB environment on a PC with a 3.4GHz CPU processor and 16G RAM. To assess the restoration performance qualitatively, we use the peak signal to noise ratio (PSNR) defined by

\[
\text{PSNR} = 10 \log_{10} \frac{255^2}{(1/MN) \sum_{i=1}^{M} \sum_{j=1}^{N} (I_{ij} - I_{ij}^*)^2},
\]

where \( I_{ij} \) and \( I_{ij}^* \) are the pixel values of the original and restored images, respectively. We also use the structural similarity (SSIM) [34] for quantitative comparison. The stopping criterion in all the experiments is

\[
\frac{\|u^{k+1} - u^k\|_2}{\|u^k\|_2} < 10^{-4},
\]

or the iteration reaches 500.
We choose Lena (256 × 256), Train (512 × 357), and Barbara (512 × 512) as the test images, as shown in Figure 1, and two types of blurring kernels, Gaussian blur and motion blur (we use the MATLAB commands `fspecial ('gaussian', [5,5], √3)` and `fspecial ('motion', 10,45) to generate Gaussian blur and motion blur, respectively). The blurry and noisy images are simulated as follows. The original image $I$ is first convolved with the blur kernel $K$, followed by additive Poisson noise (the Poisson noise is generated by using the MATLAB function `poissrnd`). Since Poisson noise is data-dependent, the noise level of the observed images depends on the pixel intensity. To test different noise levels, we use $\text{poissrnd}(B/\sigma) \ast \sigma$, where $B$ is the blurred image and $\sigma$ is set to be 1 or 5.

The choice of parameters significantly affects the convergence rate and other characteristics of the methods. In our experiments, we study the influence of parameters $\alpha, \beta, \gamma_1, \gamma_2$ on the BTVSB. Concretely speaking, after taking a reasonable guess for the parameters $\alpha, \gamma_1, \gamma_2$, we consider different values of $\beta$. We observe which value of $\beta$ could contribute to the biggest value of PSNR, then we set it to be the value of $\beta$. And in turn, we use the same strategy for the choice of other parameters. Similar strategy is adopted in our BLLTSB for the parameters. After the stopping criterion is satisfied, the algorithms are terminated for the parameter tuning.

**Experiment 1.** In this example, we use the Lena image as the test image. We first consider to restore the image corrupted by the Gaussian blur and Poisson noise with $\sigma = 1$. For the BTVSB, we set $\alpha = 10^{-3}, \beta = 61, \gamma_1 = 4.5, \gamma_2 = 0.012$. For our BLLTSB, we use $\alpha = 10^{-3}, \beta = 98, \gamma_1 = 0.05, \gamma_2 = 0.8$. Then the Poisson noise with $\sigma = 5$ is added to the blurred image. For the BTVSB and the BLLTSB, we choose $\alpha = 10^{-3}, \beta = 17, \gamma_1 = 2.8, \gamma_2 = 0.005$ and $\alpha = 10^{-3}, \beta = 24, \gamma_1 = 0.06, \gamma_2 = 0.2$, respectively. The numerical results are arranged in Table 1, while the responding visual results are shown in Figure 2.

**Experiment 2.** In our second example, the Train image is adopted as the test image. Similar to the above example, the original image is degraded by the same Gaussian blur and Poisson noise with $\sigma = 1$. The parameters are set to be $\alpha = 5 \cdot 10^{-4}, \beta = 72, \gamma_1 = 1, \gamma_2 = 0.06$ for the BTVSB and $\alpha = 5 \cdot 10^{-4}, \beta = 135, \gamma_1 = 0.03, \gamma_2 = 0.4$ for the BLLTSB. Then we consider more Poisson noise with $\sigma = 5$. The parameters for the BTVSB are $\alpha = 10^{-4}, \beta = 18, \gamma_1 = 1.4, \gamma_2 = 0.007$. And for our proposed BLLTSB, $\alpha = 10^{-4}, \beta = 25, \gamma_1 = 0.09, \gamma_2 = 0.1$ are chosen. Numerical results are also listed in Table 1. The visual results are presented in Figure 3.

Since the proposed model (4) has higher order than the previous model (3), the BLLTSB spends more time per iteration than the BTVSB. But our BLLTSB needs less iterations for the same stopping criterion. Thus, it is faster as a whole, as shown in Table 1. From the plots of PSNR and relative error versus CPU time in Figures 2 and 3, we can also conclude that our BLLTSB is faster than the BTVSB. In terms of PSNR and SSIM, the restoration results by these two methods are similar. But from Figures 2 and 3, we can see that the restored images by the BTVSB suffer from staircase effects. Moreover, the higher the noise level, the more obvious the staircase effects. It is easier to observe from the face of Lena and the head of the train.

To further show the advantages of our BLLTSB, we consider to restore an image degraded by the motion blur in the following experiment.

**Experiment 3.** In our third example, we test the Barbara image. Different from the above two examples, the original image is corrupted by the motion blur and then Poisson
Figure 2: (a1) Blurry and noisy Lena image with $\sigma = 1$; (a2), (a3) restored Lena images by BTVSB and BLLTSB, respectively; (a4), (a5) the plots of PSNR and relative error, respectively; (b1)-(b5) the corresponding case with $\sigma = 5$. 
Figure 3: (a1) Blurry and noisy Train image with $\sigma = 1$; (a2), (a3) restored Train images by BTVSB and BLLTSB, respectively; (a4), (a5) the plots of PSNR and relative error, respectively; (b1)-(b5) the corresponding case with $\sigma = 5$. 
noise with $\sigma = 1$. Here, parameters for the BTVSB and our BLLTSB are set to be $\alpha = 10^{-3}, \beta = 60, \gamma_1 = 7, \gamma_2 = 0.008$ and $\alpha = 10^{-3}, \beta = 75, \gamma_1 = 0.08, \gamma_2 = 0.1$, respectively. Results of this experiment are given in Table 1 and Figure 4.

As shown in Figures 4(a4) and 4(a5), our BLLTSB is faster than BTVSB. It can also be seen from Table 1. Besides, we can easily see from Figures 4(a2) and 4(a3) that our model successfully overcomes the staircase effects and produces relative natural result.

5. Conclusion

In this paper, we have proposed a new model based on high-order total bounded variation for restoring blurred images under Poisson noise. We proved existence and uniqueness of its solution. Furthermore, we proposed a split Bregman algorithm to solve this model and proved its convergence. Compared with the previous method for total bounded variation-based Poissonian image restoration model, experimental results show that our model can overcome the staircase effects and the proposed algorithm is fast.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

We would like to thank Dr. Zhi-Feng Pang of Henan University for his useful suggestion. This work was supported by the Science and Technology Project of Jiangxi Provincial Department of Education (GJJ171015), the China Scholarship Council (201708360066), the NNSF of China Grants...
References

[1] Y. Vardi, L. A. Shepp, and L. A. Kaufman, “Statistical model for positron emission tomography,” *Journal of the American Statistical Association*, vol. 80, no. 389, pp. 8–20, 1985.

[2] N. Dey, L. Blanc-Féraud, C. Zimmer et al., “3D microscopy deconvolution using Richardson-Lucy algorithm with total variation regularization,” *INRIA*, p. 71, 2004.

[3] H. Lantéri and C. Theys, “Restoration of astrophysical images: the case of Poisson data with additive Gaussian noise,” *EURASIP Journal on Applied Signal Processing*, vol. 2005, pp. 2500–2513, 2005.

[4] L. B. Lucy, “An iterative technique for the rectification of observed distributions,” *The Astronomical Journal*, vol. 79, p. 745, 1974.

[5] W. H. Richardson, “Bayesian-based iterative method of image restoration,” *Journal of the Optical Society of America*, vol. 62, no. 1, pp. 55–59, 1972.

[6] A. N. Tikhonov, A. Goncharsky, V. V. Stepanov, and A. G. Yagola, *Numerical Methods for the Solution of Ill-Posed Problems*, Kluwer Academic, Dordrecht, The Netherlands, 2013.

[7] L. I. Rudin, S. Osher, and E. Fatemi, “Nonlinear total variation based noise removal algorithms,” *Physica D: Nonlinear Phenomena*, vol. 60, no. 1–4, pp. 259–268, 1992.

[8] T. Chan, A. Marquina, and P. Mulet, “High-order total variation-based image restoration,” *SIAM Journal on Scientific Computing*, vol. 22, no. 2, pp. 503–516, 2000.

[9] T. Le, R. Chartrand, and T. J. Asaki, “A variational approach to reconstructing images corrupted by Poisson noise,” *Journal of Mathematical Imaging and Vision*, vol. 27, no. 3, pp. 257–263, 2007.

[10] A. Sawatzky, C. Brune, J. Müller, and M. Burger, “Total variation processing of images with poisson statistics,” in *Computer Analysis of Images and Patterns*, vol. 5702 of Lecture Notes in Computer Science, pp. 533–540, Springer, Berlin, Heidelberg, 2009.

[11] M. A. T. Figueiredo and J. M. Bioucas-Dias, “Deconvolution of Poissonian images using variable splitting and augmented Lagrangian optimization,” in *Proceedings of the IEEE/SP 15th Workshop on Statistical Signal Processing*, pp. 733–736, 2009.

[12] M. A. Figueiredo and J. M. Bioucas-Dias, “Restoration of Poissonian images using alternating direction optimization,” *IEEE Transactions on Image Processing*, vol. 19, no. 12, pp. 3133–3145, 2010.

[13] S. Setzer, G. Steidl, and T. Teuber, “Deblurring Poissonian images by split Bregman techniques,” *Journal of Visual Communication and Image Representation*, vol. 21, no. 3, pp. 193–199, 2010.

[14] R. H. Chan and K. Chen, “Multilevel algorithm for a Poisson noise removal model with total-variation regularization,” *International Journal of Computer Mathematics*, vol. 84, no. 8, pp. 1183–1198, 2007.

[15] W. Wang and C. He, “A fast and effective algorithm for a poisson denoising model with total variation,” *IEEE Signal Processing Letters*, vol. 24, no. 3, pp. 269–273, 2017.

[16] J. M. Bioucas-Dias and M. A. Figueiredo, “Multiplicative noise removal using variable splitting and constrained optimization,” *IEEE Transactions on Image Processing*, vol. 19, no. 7, pp. 1720–1730, 2010.

[17] H.-Z. Chen, J.-P. Song, and X.-C. Tai, “A dual algorithm for minimization of the LLT model,” *Advances in Computational Mathematics*, vol. 31, no. 1–3, pp. 115–130, 2009.

[18] F. F. Dong, H. L. Zhang, and D. X. Kong, “Nonlocal total variation models for multiplicative noise removal using split Bregman iteration,” *Mathematical and Computer Modelling*, vol. 55, no. 3–4, pp. 939–954, 2012.

[19] C. Wu and X.-C. Tai, “Augmented Lagrangian method, dual methods, and split Bregman iteration for ROF, vectorial TV, and high order models,” *SIAM Journal on Imaging Sciences*, vol. 5, no. 3, pp. 300–339, 2010.

[20] C. Wu, J. Zhang, and X.-C. Tai, “Augmented Lagrangian method for total variation restoration with non-quadratic fidelity,” *Inverse Problems and Imaging*, vol. 5, no. 1, pp. 237–261, 2011.

[21] S. Zhang, J. Li, H.-C. Li, C. Deng, and A. Plaza, “Spectral-spatial weighted sparse regression for hyperspectral image unmixing,” *IEEE Transactions on Geoscience and Remote Sensing*, vol. 56, no. 6, pp. 3265–3276, 2018.

[22] Y. Wen, R. H. Chan, and T. Zeng, “Primal-dual algorithms for total variation based image restoration under Poisson noise,” *Science China Mathematics*, vol. 59, no. 1, pp. 141–160, 2016.

[23] G. Chavent and K. Kunisch, “Regularization of linear least squares problems by total bounded variation,” *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 2, pp. 359–376, 1997.

[24] X. Liu and J. Huang, “Total bounded variation-based Poissonian images recovery by split Bregman iteration,” *Mathematical Methods in the Applied Sciences*, vol. 35, no. 5, pp. 520–529, 2012.

[25] M. Lysaker, A. Lundervold, and X. C. Tai, “Noise removal using fourth-order partial differential equation with applications to medical magnetic resonance images in space and time,” *IEEE Transactions on Image Processing*, vol. 12, no. 12, pp. 1579–1589, 2003.

[26] J. Zhang, R. Chen, C. Deng, and S. Wang, “Fast linearized augmented Lagrangian method for Euler’s elastica model,” *Numerical Mathematics: Theory, Methods and Applications*, vol. 10, no. 1, pp. 98–115, 2017.

[27] J. Zhang and Y.-F. Yang, “Nonlinear multigrid method for solving the LLT model,” *Applied Mathematics and Computation*, vol. 219, no. 10, pp. 4964–4976, 2013.

[28] W. Zhu and T. Chan, “Image denoising using mean curvature of image surface,” *SIAM Journal on Imaging Sciences*, vol. 5, no. 1, pp. 1–32, 2012.

[29] L. Jiang, J. Huang, X-G. Lv, and J. Liu, “Restoring poissonian images by a combined first-order and second-order variation approach,” *Journal of Mathematics*, vol. 2013, Article ID 274573, 11 pages, 2013.

[30] L. Jiang, J. Huang, X.-G. Lv, and J. Liu, “Alternating direction method for the high-order total variation-based Poisson noise removal problem,” *Numerical Algorithms*, vol. 69, no. 3, pp. 495–516, 2015.

[31] T. Goldstein and S. Osher, “The split Bregman method for L1-regularized problems,” *SIAM Journal on Imaging Sciences*, vol. 2, no. 2, pp. 323–343, 2009.

[32] X. C. Tai and C. Wu, “Augmented Lagrangian method, dual methods and split Bregman iteration for ROF model,” *SSVM*, vol. LNCS 5567, pp. 502–513, 2009.

[33] J. Cai, S. Osher, and Z. Shen, “Split bregman methods and frame based image restoration,” *Multiscale Modeling and Simulation: A SIAM Interdisciplinary Journal*, vol. 8, no. 2, pp. 337–369, 2009.
[34] Z. Wang, A. C. Bovik, H. R. Sheikh, and E. P. Simoncelli, “Image quality assessment: from error visibility to structural similarity,” *IEEE Transactions on Image Processing*, vol. 13, no. 4, pp. 600–612, 2004.
