We construct pairs of surfaces in symplectic 4-manifolds that are topologically isotopic yet are not equivalent under any ambient diffeomorphism, including the first such examples of holomorphic disks in the unit ball in $\mathbb{C}^2$, symplectic 2-spheres, and symplectic surfaces of nonzero self-intersection, as well as Lagrangian 2-spheres that are homotopic but not topologically equivalent. The underlying construction provides infinitely many pairs of exotic ribbon disks in the 4-ball, and we provide an entirely topological exposition of such disks.

1 Introduction

A pair of smooth surfaces in a smooth 4-manifold $X$ are said to be exotic (or exotically knotted) if the surfaces are topologically isotopic but not smoothly isotopic. Much of the progress in studying knotted surfaces has been sparked by questions from algebraic geometry and symplectic topology. In particular, if $X$ is a closed, simply connected complex surface, then any two smoothly embedded complex curves that are homologous are in fact smoothly isotopic [21, §1]. But if the surfaces are only required to be smooth, then there are many examples of infinite families of surfaces that are pairwise topologically but not smoothly isotopic. The majority of such examples are obtained using variants of the “rim surgery” technique pioneered by Fintushel-Stern [20] and further developed by several authors, though a handful of other constructions have arisen [2, 4, 5, 45].

The symplectic setting offers a compromise between this rigidity and flexibility. In $\mathbb{C}P^2$, for example, it is conjectured that any two smoothly embedded symplectic surfaces in the same homology class are symplectically isotopic; to date, this has been proven for homology classes of degree at most 17 [47]. And in any closed symplectic 4-manifold, a symplectic 2-sphere with self-intersection greater than $-2$ is unique up to isotopy in its homology class (cf [36, Proposition 3.2]).

However, the principle fails in general. Works of several authors, including Fintushel-Stern [21] and Park-Poddar-Vidussi [41], provide an array of symplectic 4-manifolds containing symplectic surfaces of positive genus that are homologous but not smoothly isotopic, with similar results for Lagrangian tori [52, 22]; see [17] for a concise survey. While many of these surfaces are distinguished by the fundamental groups of their complements, others require more sensitive tools, such as Seiberg-Witten invariants. Moreover, a subset of such tori can be constructed so that the surfaces have simply connected complements, implying that they are topologically isotopic [23, 8, 48] and thus smoothly exotic.
To date, all methods of constructing smoothly exotic surfaces $\Sigma, \Sigma' \subset X$ suffer from at least two of the following three limitations: (1) the surfaces have positive genus, (2) the topological isotopy is only ensured when $X \setminus \Sigma$ and $X \setminus \Sigma'$ have finite cyclic fundamental group, and (3) the surfaces $\Sigma$ and $\Sigma'$ are not both symplectic. In this paper, we fill these gaps with a new construction of smoothly exotic symplectic surfaces, including the first examples of smoothly exotic symplectic 2-spheres and smoothly exotic symplectic surfaces with nonzero self-intersection (cf [17, §8]).

**Theorem 1.1** For any integer $g \geq 0$, there exist infinitely many symplectic 4-manifolds each of which contains a pair of smooth symplectic surfaces of genus $g$ that are topologically isotopic yet are not equivalent under any ambient diffeomorphism.

By construction, our exotic surfaces $S, S' \subset X$ agree outside of an embedded 4-ball in $X$, which they intersect along a pair of disks $D, D' \subset B^4$ that are topologically isotopic rel boundary. The ambient 4-manifolds $X$ are simple Stein domains homotopy equivalent to $S$ and $S'$; we expect that similar results can be obtained in closed symplectic 4-manifolds. We distinguish the surface complements $X \setminus S$ and $X \setminus S'$ using the adjunction inequality for surfaces in Stein domains [37] and an analysis of the branched covers of $X$ along $S$ and $S'$. For additional context, we compare this approach with previous constructions of exotic smooth and symplectic surfaces at the end of this introduction.

The underlying construction also yields pairs of exotic disks in the 4-ball:

**Theorem 1.2** There are infinitely many knots in $S^3$ such that each bounds a pair of properly embedded disks in $B^4$ that are pairwise topologically isotopic rel boundary, but there is no diffeomorphism of $B^4$ taking one to the other.

Our proof of Theorem 1.2 is geared towards the topologically-minded reader, without the trappings of symplectic topology, and employs an obstruction from knot Floer homology. In order to prove Theorem 1.1, we further arrange for the exotic disks $D, D' \subset B^4$ to be symplectic. In fact, by work of Rudolph [43] and Boileau-Orevkov [6], the resulting symplectic disks are isotopic to compact pieces of algebraic curves in $B^3 \subset \mathbb{C}^2$.

**Theorem 1.3** There exist infinitely many pairs of holomorphic disks in $B^4 \subset \mathbb{C}^2$ that are topologically isotopic but are not equivalent under any ambient diffeomorphism, and the double branched covers of $B^4$ over these disks yield pairs of contractible Stein domains with the same contact boundary that are homeomorphic but not diffeomorphic.

A related construction yields Lagrangian 2-spheres that are homotopic yet are not topologically equivalent, filling a gap in the literature noted in [36, §1].

**Theorem 1.4** There exist infinitely many symplectic 4-manifolds $X$ containing a pair of smooth Lagrangian 2-spheres $S$ and $S'$ that are homotopic yet are not equivalent under any ambient homeomorphism. Moreover, $X$ may be taken so that each embedding $S, S' \hookrightarrow X$ induces a homotopy equivalence.
For additional motivation, we recall that any smooth Lagrangian 2-sphere $\Sigma$ in a symplectic 4-manifold has a Weinstein neighborhood $N(\Sigma)$ symplectomorphic to $T^*S^2$. By work of Lalonde-Sikorav [34] and Eliashberg-Polterovich [16], all homologically nontrivial Lagrangian surfaces in $N(\Sigma) \cong T^*S^2$ are isotopic to the zero-section $\Sigma$ and hence to each other. In Theorem 1.4, the ambient 4-manifold can be viewed as a Weinstein neighborhood $N(\Sigma)$ of a singular Lagrangian 2-sphere $\Sigma$ whose unique singularity is modeled on the cone of a Legendrian knot in $(S^3, \xi_{\text{st}})$; see §4. Therefore, in contrast with the results of [34] and [16], Theorem 1.4 shows that Weinstein neighborhoods $N(\Sigma)$ of singular Lagrangian 2-spheres $\Sigma$ can contain smooth Lagrangian 2-spheres that are homotopic to $\Sigma$ yet are not isotopic to each other. This also provides an alternative answer to [16, Question 1.3A], complementing earlier work of Seidel [46] and Vidussi [52].

We remark that a notable paper on exotic surfaces by Juhász, Miller, and Zemke [33] recently appeared while the later sections of this paper were in preparation. They use a variant of rim surgery to produce infinite collections of smoothly embedded surfaces of positive genus in the 4-ball that are topologically isotopic rel boundary yet are not ambiently diffeomorphic. To distinguish these surfaces, they compare the maps the surfaces induce on knot Floer homology. This naturally raises the following question:

**Question 1.5** Can the exotic disks in $B^4$ described above be distinguished using the maps they induce on knot Floer homology?

We conclude this introduction by comparing our proof of Theorem 1.1 with previous approaches to the construction of exotic smooth and symplectic surfaces. Our underlying topological construction of pairs of potentially exotic disks $D, D' \subset B^4$ is based on a variant of cork twisting, strengthening and systemizing an example of a related phenomenon found by Akbulut [1]. In contrast with rim surgery, the explicit nature of this construction makes it simple to describe the resulting surfaces using a sequence of link diagrams. Under suitable conditions, we can apply results from [27] (see also [18]) to realize the resulting disks $D$ and $D'$ symplectically in $(B^4, \omega_{\text{st}})$. We then obtain closed symplectic surfaces $S, S'$ by capping off these disks $D, D' \subset B^4$ with symplectic surfaces inside a larger Weinstein domain $X$. As mentioned above, the complements $X \setminus S$ and $X \setminus S'$ are distinguished using the adjunction inequality for embedded surfaces in Stein domains [37] and an analysis of the branched covers of $X$ along $S$ and $S'$.

In contrast, the earlier strategy for constructing and detecting smoothly exotic 2-spheres in [2] and [4] is not well-suited to the symplectic setting, nor to the study of surfaces of positive genus. The underlying strategy, also applied earlier in [42], is based on the existence of homeomorphic but nondiffeomorphic 4-manifolds $X$ and $X'$ whose blowups $X\#\mathbb{C}P^2$ and $X'\#\mathbb{C}P^2$ are diffeomorphic. If $S$ and $S'$ denote the 2-spheres given by $\mathbb{C}P^1$ inside the $\mathbb{C}P^2$-summand of $X\#\mathbb{C}P^2$ and $X'\#\mathbb{C}P^2$, respectively, then there is a homeomorphism from $X\#\mathbb{C}P^2$ to $X'\#\mathbb{C}P^2$ carrying $S$ to $S'$. However, any diffeomorphism carrying $S$ to $S'$ would descend to a diffeomorphism of $X$ and $X'$ after blowing down the 2-spheres $S$ and $S'$ (which each have self-intersection +1), a contradiction.
In the symplectic setting, however, there can be no smoothly exotic symplectic 2-spheres of self-intersection \( \pm 1 \). Indeed, as mentioned above, any symplectic 2-sphere of self-intersection greater than \(-2\) in a closed symplectic 4-manifold is unique up to isotopy in its homology class [36, Proposition 3.2]. Moreover, it was pointed out to the author by T.-J. Li [35] that this result can be extended to symplectic 4-manifolds with convex boundary using the tools surveyed in [54, §9]. In theory, it remains possible that a pair of symplectic 2-spheres in a 4-manifold \( Z \) could be distinguished by a rational blowdown, which consists of replacing a tubular neighborhood of the 2-sphere with a rational homology ball that has the same boundary [19]. Such a rational homology ball exists precisely when the 2-sphere has self-intersection \( \pm 1 \) or \( \pm 4 \). While the case of self-intersection \(-4\) is not ruled out above, the manner in which the gauge-theoretic invariants of \( Z \) determine those of its rational blowdowns makes it difficult to distinguish the resulting 4-manifolds.

Organization. We begin in §2 by explaining the topological construction underlying our main results and proving Theorem 1.2. In §3, we describe the tools needed to construct the desired symplectic surfaces and then prove Theorems 1.1 and 1.3. We close by turning to Lagrangian surfaces in §4, where we prove Theorem 1.4.

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2 Exotic ribbon disks in the 4-ball

In this section, we outline the topological construction underlying our pairs of smoothly exotic disks and prove Theorem 1.2. To begin, let \( L \) be a three-component link of unknots \( A, B, \) and \( C \) such that \( A \cup B \) is a Hopf link and each of the links \( A \cup C \) and \( B \cup C \) is a two-component unlink. For a running example that will be used to prove Theorem 1.2, see Figure 1(a); in this and all other figures, a boxed integer indicates positive full twists. We may view \( C \) as a knot \( K \subset S^3 \) using the non-standard surgery description of \( S^3 \) given by zero-surgery on both \( A \) and \( B \). Moreover, by viewing \( A \) as a dotted circle (i.e. carving out a neighborhood of a standard slice disk for \( A \) from \( B^4 \) to yield \( S^1 \times B^3 \) [26]) and attaching a zero-framed 2-handle along \( B \), we obtain a non-standard handle diagram for \( B^4 \); see Figure 1(b).

Observe that, since \( K \) is unknotted in this diagram and does not run over the 1-handle, it naturally bounds a slice disk in \( B^4 \). More precisely, the link component \( C \) underlying \( K \) is split from \( A \) (when ignoring \( B \)), so \( C \) bounds an embedded disk in \( S^3 \setminus A \) that is unique up to isotopy. The interior of this disk can be pushed into the interior of \( S^1 \times B^3 \),
Exotic ribbon disks and symplectic surfaces

Figure 1

i.e. the exterior of the slice disk for \( A \). The resulting disk is disjoint from the 2-handle attached along \( B \), giving rise to the desired disk \( D \subset B^4 \) bounded by \( K \subset S^3 \). By hypothesis, we may reverse the roles of \( A \) and \( B \) in this construction, yielding another disk \( D' \) in \( B^4 \) bounded by \( K \); see Figure 1(c).

Our goal is to produce examples in which the resulting disks are topologically but not smoothly equivalent. By construction, any disks arising as above will be ribbon; that is, they can be arranged so that the restriction of the radial distance function from \( B^4 \) to the disk is a Morse function with no local maxima. When the resulting ribbon disks’ complements have infinite cyclic fundamental group, the topological equivalence is obtained using the following result.

**Theorem 2.1** (cf [10, Theorem 1.1]) Any pair of ribbon disks \( D, D' \subset B^4 \) with the same boundary and \( \pi_1(B^4 \setminus D) \cong \pi_1(B^4 \setminus D) \cong \mathbb{Z} \) are topologically isotopic rel boundary.

To distinguish the disks in Theorem 1.2 up to ambient diffeomorphism, we show that there is a knot in \( S^3 \setminus K \) that bounds a smoothly embedded, once-punctured torus in \( B^4 \setminus D' \) but not in \( B^4 \setminus D \) (even up to ambient diffeomorphism). In \( B^4 \setminus D \), such a surface is obstructed using the invariant \( \tau \) derived from knot Floer homology [40]. To describe the obstruction, we recall that the \( n \)-trace of a knot \( K \) in \( S^3 \) is the 4-manifold \( X_n(K) \) obtained by attaching an \( n \)-framed 2-handle to \( B^4 \) along \( K \). The \( n \)-shake genus \( g^0_{sh}(K) \) of \( K \) is defined to be the minimal genus of a smoothly embedded surface generating the second homology of \( X_n(K) \). It can be shown that \( \tau \) is a lower bound on the zero-shake genus; see Remark 4.10 or Theorem 1.6 of [29].

**Theorem 2.2** (cf [29]) If \( K \) is a knot in \( S^3 \), then \( |\tau(K)| \leq g^0_{sh}(K) \).

These tools in hand, we can prove Theorem 1.2.

**Proof of Theorem 1.2** Let \( D, D' \subset B^4 \) denote the ribbon disks depicted in Figure 1 corresponding to a choice of \( m \in \mathbb{Z} \). We claim that \( D \) and \( D' \) are topologically isotopic rel boundary for all \( m \in \mathbb{Z} \) but are smoothly inequivalent for \( m = 0 \) and \( m \ll 0 \).

By construction, these ribbon disks have the same boundary \( K \subset S^3 \). Therefore, to establish the topological isotopy using Theorem 2.1, it suffices to show \( \pi_1(B^4 \setminus D) \cong \pi_1(B^4 \setminus D') \cong \mathbb{Z} \).
Figure 2: Handle diagrams for the slice disk exteriors, decorated to simplify the calculation of their fundamental groups.

\[ \pi_1(B^4 \setminus \tilde{N}(D)) \cong \mathbb{Z} \] for \( m = 0 \) and \( m \ll 0 \); see [28] for additional documentation regarding this calculation. It follows that every self-diffeomorphism of \( S^3 \setminus K \) is isotopic to the identity, hence we may further isotope \( f \) so that it restricts to the identity on \( S^3 \setminus \tilde{N}(K) \) for \( m = 0 \) or \( m \ll 0 \).

Now let \( g \) denote the diffeomorphism of the disk exteriors induced by \( f \). If \( m = 0 \) or \( m \ll 0 \), then \( g \) sends the knot \( J \) in the boundary of \( B^4 \setminus \tilde{N}(D) \) in Figure 3 to the knot in the boundary of \( B^4 \setminus \tilde{N}(D') \) shown on the right. It is clear that the latter bounds an embedded once-punctured torus in \( B^4 \setminus \tilde{N}(D') \). We claim that \( J \) does not bound such a surface in \( B^4 \setminus \tilde{N}(D) \). To see this, we construct a new 4-manifold \( X \) by attaching a pair of 2-handles to \( B^4 \setminus \tilde{N}(D) \) as shown in Figure 4, one of which is attached along \( J \). After performing the handle calculus in Figure 4, we see that \( X \) is the zero-trace of a knot that we will denote by \( L_m \).
Exotic ribbon disks and symplectic surfaces

Figure 3: Any diffeomorphism exchanging the disks induces a diffeomorphism of the disk exteriors’ boundaries fixing the knot $J$.

Figure 4: The 4-manifold $X$ from the proof of Theorem 1.2 is depicted in part (a). Passing from (a) to (b) corresponds to a handleslide; (b) to (c) is handle cancellation and minor isotopy; (c) to (d) is isotopy; (d) to (e) consists of three handleslides and a handle cancellation, yielding the zero-trace of the knot $L_m$. Part (f) depicts the knot $L_0$, which is obtained from $L_m$ by removing the $m$ positive twists.

If $J$ bounds a once-punctured torus in $B^4 \setminus \tilde{N}(D)$, then this can be capped off with the core of the 2-handle attached along $J$ to yield a closed torus embedded in $X = X_0(L_m)$. Looking at Figure 4(a-d), it is easy to see that such a surface represents a generator of $H_2(X)$. However, we claim that no such torus can exist. To see this, we note that $L_m$ is obtained from $L_0$ by adding $m$ positive full-twists along a pair of oppositely-oriented strands, hence $L_m$ can be turned into $L_0$ by performing $m$ positive-to-negative crossing...
changes. By [40, Corollary 1.5] (or [38, Corollary 3]), it follows that \( \tau(L_m) \geq \tau(L_0) \) for all \( m \leq 0 \). Next, by a direct calculation using [50], we calculate \( \tau(L_0) = 2 \); see [28] for additional documentation. This implies that \( \tau(L_m) \geq 2 \) for all \( m \leq 0 \), hence Theorem 2.2 implies that the second homology of \( X_0(L_m) \) cannot be generated by an embedded torus; this provides the desired contradiction.

\[ \square \]

3 Exotic symplectic surfaces

In this section, we adapt the topological construction above to produce smoothly exotic pairs of symplectic surfaces in symplectic 4-manifolds. In particular, we build pairs of ribbon disks associated to a family of links that generalize the link \( L14n40949 \), realize these as symplectic disks in \( (B^4, \omega_{st}) \), and then cap them off to yield closed surfaces in other symplectic 4-manifolds. The key constructive tools are explained in §3.1, followed by the proofs of Theorems 1.1 and 1.3 in §3.2. For background on contact, symplectic, and Stein manifolds, we refer the reader to [39, 25, 9].

3.1 Building symplectic surfaces

Our primary tool for realizing a prescribed smooth surface in \( B^4 \) as a symplectic surface with respect to \( \omega_{st} \) is the following lemma. It has likely been known to experts for some time, forming a symplectic analog of a construction of Lagrangian cobordisms due to Ekholm-Honda-Kálmán [14] and Rizell [12]; see also Theorem 4.2 and Figure 5 of [7]. For a proof, see Example 4.7 and Lemma 5.1 of [27] (cf [18, Lemma 2.7]).

**Lemma 3.1** If a smooth surface \( \Sigma \) in \( B^4 \) can be presented by a sequence of transverse link diagrams in \( (S^3, \xi_{sa}) \) related by transverse isotopy and the diagram moves in Figure 5, then \( \Sigma \) is smoothly isotopic rel boundary to a symplectic surface in \( (B^4, \omega_{st}) \).

For a key example illustrating Lemma 3.1, see Figure 6, which presents a family of pairs of symplectic disks in \( B^4 \) that will be used in the proof of Theorems 1.1 and 1.2.

**Remark 3.2** Lemma 3.1 also holds for surfaces in a compact piece \( Y \times [a,b] \) of the symplectization \( Y \times \mathbb{R} \) of any contact 3-manifold \( Y \), where the diagram moves occur in local Darboux charts.

![Diagram](a)

![Diagram](b)

Figure 5: Elementary birth and saddle moves between transverse link diagrams.
Figure 6: For $m \leq 0$, a transverse knot $K$ is shown in part (a). Passing from (a) to (b) or (c-1) corresponds to a transverse saddle move from Lemma 3.1 and Figure 5(b). It is easy to see that (b) depicts a standard two-component transverse unlink, which bounds a pair of disks via Figure 5(a). The same is true of the link in (c-1); for this link, we include additional diagrams in parts (c-2)-(c-4) indicating the transverse isotopy that makes the standard two-component transverse unlink more apparent.
To produce closed symplectic surfaces, we take symplectic surfaces with transverse boundary constructed via Lemma 3.1 and cap them off with certain standard pieces. Recall that if \( X \) is a symplectic 4-manifold with convex boundary and \( L \subset \partial X \) is a Legendrian knot, then we may extend the symplectic structure to the 4-manifold \( X' \) obtained from \( X \) by attaching a Weinstein 2-handle to \( X \) along \( L \) (whose underlying framing is \(-1\) relative to the contact framing of \( L \)) [53]. Moreover, when \( X \) is a Stein domain, its Stein structure extends to \( X' \) [15]. The following lemma allows us to cap off symplectic surfaces in \( X \) using the core disk of a Weinstein 2-handle.

**Lemma 3.3** If \( L \) is a Legendrian knot in the convex boundary of a symplectic 4-manifold \( X \) and \( X' \) is obtained by attaching a Weinstein 2-handle to \( X \) along \( L \), then any symplectic surface in \( X \) bounded by a transverse pushoff \( K \) of the Legendrian \( L \) can be capped off with a perturbation of the core of the 2-handle to yield a closed symplectic surface in \( X' \).

**Proof** The core \( D^2 \times 0 \) in the Weinstein 2-handle \( D^2 \times D^2 \) is a Lagrangian disk meeting \( \partial X \) along \( L \). This can be perturbed (by a \( C^0 \)-small isotopy) to yield a symplectic disk meeting \( \partial X \) along another transverse pushoff \( K' \) of \( L \); see, for example, the proof of [18, Theorem 1.1]. Then \( K \) and \( K' \) are transversely isotopic, so we can construct a symplectic cylinder in a compact piece \( \partial X \times [0, c] \) of the symplectization of \( \partial X \) meeting \( \partial X \times 0 \) along \( K \) and \( \partial X \times c \) along \( K' \). We may thicken \( X \) by attaching the collar \( \partial X \times [0, c] \) and extend any symplectic surface \( \Sigma \) bounded by \( K \) and \( \Sigma' \) with the symplectic perturbation of the core disk \( D^2 \times 0 \). □

The above tools are sufficient for constructing closed symplectic surfaces: construct a symplectic surface (of arbitrary genus) in \((B^4, \omega_{\text{std}})\), attach a Weinstein 2-handle along a Legendrian approximation of its boundary, then cap off the surface with a symplectic disk in the 2-handle. However, there is another option for producing closed symplectic surfaces of positive genus using “higher genus” Weinstein handles. To explain this, we borrow the following definition and discussion from [30, §3].

**Definition 3.4** For any integer \( g \geq 0 \), a genus \( g \) handle is a copy of \( F \times D^2 \), where \( F \) is a compact genus \( g \) surface with one boundary component, attached to the boundary of an oriented 4-manifold \( X \) by an embedding \( \varphi : \partial F \times D^2 \to \partial X \).

As with traditional handle attachments, a genus \( g \) handle attachment is determined by an attaching sphere \( K \subset \partial X \) along which \( \partial F \times 0 \subset \partial F \times D^2 \) is attached and a framing \( n \in \mathbb{Z} \) used to identify a tubular neighborhood of \( K \) with \( \partial F \times D^2 \). The following lemma determines a sufficient condition for a Stein structure on \( X \) to extend over a genus \( g \) handle attached along a Legendrian knot in \( \partial X \).
Lemma 3.5 ([30, Lemma 3.6]) If $X'$ is obtained by attaching a genus $g$ handle to a Stein domain $X$ along a Legendrian knot $L \subset \partial X$ with framing $2g - 1$ relative to the contact framing of $L$, then $X'$ admits a Stein structure.

For later use, we recall that a Stein handle diagram for $X'$ can be obtained from one for $X$ by attaching $g$ pairs of Stein 1-handles in a neighborhood of $L$ and, near each such pair of 1-handles, modifying $L$ as shown in Figure 7 by replacing a subarc of $L$ with the indicated arc passing through the pair of 1-handles.

Lemma 3.6 Let $L$ be a Legendrian knot in the boundary of a Stein domain $X$, and suppose $X'$ is obtained by attaching a genus $g$ handle $F \times D^2$ to $X$ along $L$ with framing $2g - 1$ relative to the contact framing of $L$. Then any symplectic surface in $X$ bounded by a transverse pushoff $K$ of $L$ can be capped off with a symplectic surface in $F \times D^2$ obtained by perturbing the core surface $F \times 0$.

Proof If $g = 0$, then this is simply Lemma 3.3. If $g \geq 1$, then we begin by attaching $g$ pairs of Weinstein 1-handles to $X$ in a neighborhood of the Legendrian knot $L$ as discussed following Lemma 3.5; the genus $g$ Weinstein handle is then obtained by attaching a Weinstein 2-handle to $X_{2g}^2S^1 \times B^3$ along a modified Legendrian knot $L'$ that passes over each pair of 1-handles as shown in Figure 7. Note that the Legendrian knot $L$ and its transverse pushoff $K$ naturally embed in the boundary of $X_{2g}^2S^1 \times B^3$. As illustrated in Figure 8, there is a genus $g$ cobordism from $K$ to a transverse pushoff $K'$ of the modified Legendrian $L'$ constructed using transverse isotopy and the moves from Lemma 3.1. By that lemma (and Remark 3.2), we can extend $\Sigma$ to a symplectic surface $\Sigma' \subset X_{2g}^2S^1 \times B^3$ of genus $g(\Sigma) + g$ bounded by $K'$. Since $X'$ is obtained by attaching a standard Weinstein 2-handle along the Legendrian $L'$, this reduces to the case of $g = 0$, completing the proof.

3.2 Distinguishing the surfaces

Recall from §2 that the $n$-trace of a knot $K$ in $S^3$ is the 4-manifold $X_n(K)$ obtained by attaching an $n$-framed 2-handle to $B^4$ along $K$. Given a pair of slice disks $D, D' \subset B^4$
bounded by the same knot $K$, we obtain a pair of 2-spheres $S, S' \subset X_n(K)$ by capping off each disk $D, D' \subset B^4$ with the core of the 2-handle in $X_n(K)$. Our goal will be to distinguish such 2-spheres by studying surfaces in their complements. In our case, this will be simplified by passing to the double branched covers of $X_n(K)$ along $S$ and $S'$.

The following lemma describes the handle structure of such branched covers.

**Lemma 3.7** Let $D$ be a properly embedded disk in $B^4$ bounded by $K$ in $S^3$ and, for any $n \in \mathbb{Z}$, let $S$ be the 2-sphere in the knot trace $X = X_n(K)$ obtained by capping off $D$ with the core of the 2-handle. Then

(a) $\pi_1(X \setminus S)$ is normally generated by the meridian to $S$, which has order $n$,

(b) $H_1(X \setminus S)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, and

(c) for any positive integer $k$ dividing $n$, the $k$-fold cyclic branched cover of $X$ along $S$, denoted $\Sigma_k(X, S)$, is obtained from the $k$-fold cyclic branched cover of $B^4$ along $D$, denoted $\Sigma_k(B^4, D)$, by attaching a 2-handle to the lift $\tilde{K}$ in $\Sigma_k(S^3, K) = \partial \Sigma_k(B^4, D)$ with framing $n/k$.

**Proof** We will decompose $X \setminus S$ as a union $A \cup B$ as depicted in Figure 9. Begin by decomposing the knot trace $X$ as a union $E \cup B$, where $E$ is a tubular neighborhood of $S$ and $B$ is the disk exterior $B^4 \setminus \tilde{N}(D)$. More specifically, let $E$ be constructed as a union of the 2-handle $D^2 \times D^2$ and a tubular neighborhood $N(D)$ of the disk $D$ in $B^4 \subset X$. Observe that $E$ is diffeomorphic to the disk bundle over $S^2$ with Euler number $n$. Letting $A$ denote the complement of the 2-sphere $S$ in $E$ (which is identified with the zero-section of the disk bundle), we have $X \setminus S = A \cup B$. Note that $E$ is compact and has boundary diffeomorphic to the lens space $L(n, 1)$. The subset $A = E \setminus S$ is diffeomorphic to $\partial E \times (0, 1] \cong L(n, 1) \times (0, 1]$, and $\partial A$ coincides with $\partial E \cong L(n, 1)$.

Turning to $B$, note that its boundary $\partial B$ is diffeomorphic to $S^3_0(K)$, the 3-manifold obtained by zero-framed Dehn surgery on $K$. The subsets $A$ and $B$ meet along a solid torus $V = \partial A \cap \partial B$, which can be viewed as the unit normal bundle to $D$ in $B^4$. Note that the meridian of $D$ (which is a meridian of $S$) is isotopic to the core of the solid torus $V$. We will denote this curve by $\mu$, and we claim that it generates $\pi_1(X \setminus S)$.

Viewing $V$ in $\partial A$, we see that $V$ forms half of a Heegaard splitting of $\partial A \cong L(n, 1)$, and we note that the curve $\mu$ generates

$$\pi_1(A) \cong \pi_1(\partial A) \cong \langle \mu \mid \mu^n = 1 \rangle \cong \mathbb{Z}/n\mathbb{Z}.$$
Lemma 3.8  Let $D \subset B^4$ and $K \subset S^3$ be as above, and let $S$ be the genus $g$ surface in $X^n_g(K)$ obtained by capping off $D$ with the core surface of the genus $g$ handle. Then
for any positive integer $k$ dividing $n$, the $k$-fold cyclic branched cover of $X_n^g(K)$ along $S$ is obtained from the $k$-fold cyclic branched cover of $B^4$ along $D$ by attaching a genus $g$ handle to the lift $\tilde{K} \subset \partial \Sigma(B^4,D)$ of $K \subset S^3$ with framing $n/k$.

We proceed to the proof of our main result:

**Proof of Theorem 1.1** Fix integers $m,n \leq 0$ and let $D,D' \subset B^4$ denote the disks with common boundary $K \subset S^3$ shown in Figure 6. We begin with the case $g = 0$, i.e. pairs of smoothly exotic symplectic 2-spheres. Our ambient 4-manifold will be the knot trace $X = X_n(K)$ depicted in Figure 10, and the 2-spheres $S,S' \subset X$ are obtained by capping off $D,D' \subset B^4$ with the core of the 2-handle in $X$. We first claim that

(a) $D$ and $D'$ are topologically isotopic (rel boundary), and

(b) $D$ and $D'$ can be made symplectic by a smooth isotopy (rel boundary).

To prove (a), we note that the disks $D,D' \subset B^4$ are ribbon, hence it suffices to show that their complements have infinite cyclic fundamental group, at which point the claim follows from Theorem 2.1. We find handle diagrams for the disk exteriors using the recipe from [26] in Figures 11 and 12, then compute their fundamental groups:

$$\pi_1(B^4 \setminus D) \cong \langle x,y \mid x^{-1}y^{-1}xy^{-1}y = 1 \rangle \cong \langle x,y \mid x^{-1}y^{-1}x = 1 \rangle \cong \mathbb{Z}$$

$$\pi_1(B^4 \setminus D') \cong \langle x,y \mid y^{-1}xyx^{-1}yy^{-1} = 1 \rangle \cong \langle x,y \mid xy^{-1}x^{-1} = 1 \rangle \cong \mathbb{Z}.$$
Figure 11: Simplifying a handle diagram for the exterior of $D$ in $B^4$. Passing from (b) to (c) corresponds to sliding one 1-handle over the other; other steps are isotopy.

(c) $S$ and $S'$ are topologically isotopic,

(d) for any integer $n \leq -3$, the 4-manifold $X = X_n(K)$ admits a Weinstein structure with respect to which $S$ and $S'$ are isotopic to symplectic 2-spheres, and

(e) for any even integer $n \ll 0$ and either $m = 0$ or $m \ll 0$, the exteriors of $S$ and $S'$ are not diffeomorphic, so there is no diffeomorphism of $X$ carrying $S$ to $S'$.

The claim in (c) follows immediately from (a) because $S$ and $S'$ are obtained by capping off $D$ and $D'$ with the core of the 2-handle attached along their common boundary $K$.

To prove (d), we note that $K$ can be realized as the transverse pushoff of a Legendrian knot $L$ with Thurston-Bennequin number $tb(L) = -2$; see Figure 13. After negative Legendrian stabilization of $L$ (inducing transverse isotopy of $K$), we can ensure $tb(L) - 1 = n$ for any $n \leq 3$. By Lemma 3.3, $X$ admits a Weinstein structure and the symplectic disks $D, D' \subset B^4$ can be capped off to yield closed symplectic 2-spheres $S, S' \subset X$.

Before proving (e), we make a simplifying observation about the 4-manifold $X$: For
Figure 12: Finding and simplifying a handle diagram for the exterior of $D'$ in $B^4$. 
Figure 13: The transverse knot $K$ (right) is the pushoff of a Legendrian knot $L$ (left) with Thurston-Bennequin number $tb(L) = -2$ and rotation number $r(L) = -1$.

$n \ll 0$ and either $m = 0$ or $m \ll 0$, every self-diffeomorphism of $\partial X$ is isotopic to the identity. To see this, we first observe that $\partial X$ is diffeomorphic to $S^3_{n,-1/m}(K_0 \cup \gamma)$, the 3-manifold obtained by performing Dehn surgery on the knots $K_0$ and $\gamma$ in $S^3$ with framing $n$ and $-1/m$, respectively, as shown on the right side of Figure 10. Using SnapPy and Sage [11, 51], we verify that $S^3 \setminus (K_0 \cup \gamma)$ admits a hyperbolic structure with trivial isometry group for $m = 0$ and $m \ll 0$; see [28] for additional documentation regarding this calculation. Hence for $|n| \gg 0$ and either $m = 0$ or $m \ll 0$, the surgered 3-manifold $\partial X$ also admits a hyperbolic structure with trivial isometry group; see [13, Lemma 2.2]. By [24], every self-diffeomorphism of a hyperbolic 3-manifold is isotopic to an isometry, hence every self-diffeomorphism of $\partial X$ is isotopic to the identity.

To prove (e), we assume for the sake of contradiction that there exists a diffeomorphism of $X$ carrying $S'$ to $S$. By the above, we may assume that the diffeomorphism restricts to the identity on $\partial X$. Let $J$ denote the knot in $\partial X$ induced by the dashed curve in Figure 10. We claim that $J$ bounds a disk in $X$ that is disjoint from $S'$. To see this, we consider the knot in $S^3$ induced by the underlying dashed curve from Figure 10. By construction, this knot is disjoint from $K$. Moreover, viewed in the disk exterior $B^4 \setminus N(D')$ as depicted in Figure 12(b), we see that this knot does not pass over any 1-handles, hence it bounds a disk in $B^4$ that is disjoint from $D'$. This disk includes into $X = X_n(K)$ as a disk bounded by $J$ that is disjoint from $S'$, as desired.

Since the supposed diffeomorphism of $X$ fixes $J \subset \partial X$ and carries $S'$ to $S$, it carries the disk $J$ bounds in $X \setminus S'$ to a disk that $J$ bounds in $X \setminus S$. We will prove that no such disk can exist, producing the desired contradiction. To that end, we consider $\Sigma(X,S)$, the branched double cover of $X$ along $S$. Per Lemma 3.7, $\Sigma(X,S)$ is obtained from $\Sigma(B^4,D)$, the branched double cover of $B^4$ along $D$, by attaching a 2-handle to the lift $\bar{K}$ of $K = \partial D$ with framing $n/2$.

Following the recipe from [26], we find a handle diagram for $\Sigma(B^4,D)$. This is carried out in parts (a) through (h) of Figure 15. Throughout parts of this figure, we track two
curves: the lift $\tilde{K}$ of $K$ and a curve that becomes a preferred lift $\tilde{J}$ of $J$ (after attaching the 2-handle along $\tilde{K}$ with framing $n/2$). In Figure 15(i), we show that $\Sigma(B^4, D)$ admits a Stein structure by recasting the diagram from part (h) in the “standard form” of [25].

A schematic handle diagram for $\Sigma(X, S)$ itself is depicted in Figure 14. Though it is not drawn in detail, the knot $\tilde{K}$ can be represented by a Legendrian knot in standard form in the diagram; the attaching curve for the 0-framed 2-handle and the dashed curve representing $\tilde{J}$ are drawn as shown, though $\tilde{K}$ may have crossings with these curves. Thus, for all even integers $n$ such that $n/2$ is less than the Thurston-Bennequin number of the chosen Legendrian representative of $\tilde{K}$, we see that $\Sigma(X, S)$ admits a Stein structure. (For $n/2 < tb - 1$, the Legendrian representative of $\tilde{K}$ will need to be stabilized before the 2-handle is attached.)

From here, our argument follows a well-tread path that dates back at least to [3]. Observe that the lifted knot $\tilde{J}$ has a Legendrian representative in $\partial \Sigma(X, S)$ whose Thurston-Bennequin number is zero. It follows that the 4-manifold obtained by attaching a $(-1)$-framed 2-handle to $\Sigma(X, S)$ along $\tilde{J}$ also admits a Stein structure. Since the curve $J$ in $\partial X$ bounds a disk in $X$ that is disjoint from $S$, the lift $\tilde{J}$ bounds a disk in the branched cover $\Sigma(X, S)$. After attaching the $(-1)$-framed 2-handle to $\tilde{J}$, the disk bounded by $\tilde{J}$ gives rise to a smoothly embedded 2-sphere with self-intersection number $-1$ in the resulting Stein domain. However, this contradicts [37, Proposition 2.2], which states that any homologically essential 2-sphere in a Stein domain has self-intersection at most $-2$.

We conclude that $J$ cannot bound a smoothly embedded disk in $X$ that is disjoint from $S$, hence there can be no diffeomorphism of $X$ carrying $S'$ to $S$.

This completes the proof for surfaces of genus $g = 0$. For $g \geq 1$, the proof is nearly identical, so we focus on the ways in which it differs. To begin, our ambient 4-manifold is now the 4-manifold $X = X^g_b(K)$ obtained from $B^4$ by attaching an $n$-framed, genus $g$ handle along $K \subset \partial B^4$ (as in Definition 3.4). The surfaces $S, S' \subset X$ are obtained by capping off the disks $D, D' \subset B^4$ with the core surface of the genus $g$ handle in $X$. Recall from above that $D$ and $D'$ are smoothly isotopic (rel boundary) to symplectic...
Exotic ribbon disks and symplectic surfaces

Figure 15: Simplifying the diagram of $\Sigma(B^4, D)$ using handle slides and cancellation.
Figure 15 - Continued: Further simplifying the diagram of $\Sigma(B^4, D)$ by isotopy.
surfaces in \((B^4, \omega_{st})\). Therefore, for \(n \leq tb(L) + 2g - 1\), the 4-manifold \(X\) is a Stein domain admitting a symplectic structure with respect to which the surfaces \(S\) and \(S'\) are smoothly isotopic to symplectic surfaces by Lemma 3.6.

The boundary of \(X\) is no longer given by Dehn surgery on \(K\). Instead, it is obtained from \(S^3 \setminus \tilde{N}(K)\) by gluing in \(F \times S^1\), where \(F\) is a compact surface of genus \(g\) with one boundary component. Here the gluing takes \(\partial F \times \{pt\}\) to an \(n\)-framing curve for \(K\) in the boundary of \(S^3 \setminus \tilde{N}(K)\). The torus \(T = \partial N(K)\) is incompressible and separates \(\partial X\) into the two pieces of its JSJ decomposition \([31, 32]\): a hyperbolic piece \(S^3 \setminus \tilde{N}(K)\) and a Seifert-fibered piece \(F \times S^1\). Every self-diffeomorphism of \(\partial X\) can be isotoped to preserve \(T\) setwise and preserves the JSJ pieces on each side of \(T\) (because they are distinct). For \(m = 0\) and \(m \ll 0\), we already showed above that every self-diffeomorphism of \(S^3 \setminus \tilde{N}(K)\) is isotopic to the identity, hence we may assume that any self-diffeomorphism of \(\partial X\) restricts to the identity on \(S^3 \setminus \tilde{N}(K) \subset \partial X\).

Observe that \(X\) has a handle diagram obtained from Figure 10 by replacing the \(n\)-framed 2-handle with an \(n\)-framed genus \(g\) handle. Let \(J\) be the knot in \(S^3 \setminus \tilde{N}(K) \subset \partial X\) induced by the dashed curve in Figure 10. Just as in the case of \(g = 0\), \(J\) is seen to bound a smooth disk in the exterior of \(S'\). If there exists a diffeomorphism of \(X\) carrying \(S'\) to \(S\), then it can be assumed to fix \(J \subset S^3 \setminus \tilde{N}(K)\), so it carries the disk bounded by \(J\) in \(X\setminus S'\) to a disk bounded by \(J\) in \(X\setminus S\). To obstruct the existence of such a disk, we again consider a lift \(\tilde{J}\) of \(J\) to the double branched cover \(\Sigma(X,S)\). By Lemma 3.8, \(\Sigma(X,S)\) is obtained from \(\Sigma(B^4,D)\) by attaching a genus \(g\) handle along \(\tilde{K}\) with framing \(n/2\). It follows that \(\Sigma(X,S)\) has a handle diagram obtained from Figure 14 by replacing the 2-handle attaching along \(\tilde{K}\) with a genus \(g\) handle. Mirroring the argument from above, we see that \(\Sigma(X,S)\) admits a Stein structure for all \(n/2 \ll 2g\) by Lemma 3.5. As above, if \(J\) bounds a disk in \(X\setminus S\), then \(\tilde{J}\) bounds a disk in \(\Sigma(X,S)\). Attaching a \((-1)\)-framed 2-handle to \(\Sigma(X,S)\) along \(J\) then yields a Stein domain that contains a smoothly embedded 2-sphere with self-intersection number \(-1\), again contradicting [37, Proposition 2.2]. We conclude that there is no diffeomorphism of \(X\) taking \(S'\) to \(S\). □

**Proof of Theorem 1.3** By [6, Theorem 1], any symplectic surface in \((B^4, \omega_{st})\) with transverse boundary \(K\) in \((S^3, \xi_{st})\) is diffeomorphic (up to rounding corners) to a “positively braided surface” in \(D^2 \times D^2\) with boundary in \(\partial D^2 \times D^2\). In fact, the proof of [6, Theorem 1] shows that the diffeomorphism may be constructed so that the boundary of the braided surface is any chosen braid that is transversely isotopic to \(K\) when viewed in \((S^3, \xi_{st})\). By [43] (as interpreted in [44, §4]), every positively braided surface in the bidisk \(D^2 \times D^2 \subset \mathbb{C}^2\) is isotopic to a compact piece of a smooth algebraic curve. It can be seen that the isotopy of the surface restricts to braid isotopy along its boundary. Moreover, after applying the transformation of \(\mathbb{C}^2\) given by \((z,w) \mapsto (z, w/r)\) for \(r \gg 0\), this algebraic curve intersects the contact boundary of the unit \(B^4 \subset \mathbb{C}^2\) in a transverse link that is transversely isotopic to the chosen braid representative of \(K\).
Applying these results to the symplectic disks $D, D' \subset B^4$ constructed in the proof of Theorem 1.1, we obtain a pair of holomorphic curves $C$ and $C'$ in $B^4 \subset \mathbb{C}^2$ that are diffeomorphic to $D$ and $D'$, respectively. Moreover, $\partial C$ and $\partial C'$ are transversely isotopic to $K = \partial D = \partial D'$. Since $C$ and $C'$ are ribbon disks with isotopic boundary and whose exteriors have infinite cyclic fundamental group, they are topologically isotopic.

The disks $C$ and $C'$ are holomorphic, so the branched covers $W = \Sigma(B^4, C)$ and $W' = \Sigma(B^4, C')$ are Stein domains. Since $\partial C$ and $\partial C'$ are transversely isotopic to $K$, the Stein domains $W$ and $W'$ fill isotopic contact structures on $\Sigma(S^3, K)$. And since $C$ and $C'$ are topologically isotopic, $W$ and $W'$ are homeomorphic. Moreover, $\Sigma(B^4, D)$ and $\Sigma(B^4, D')$ are not diffeomorphic. In particular, the knot in $S^3$ corresponding to the dashed curve in Figure 10 lifts to a pair of curves in $\Sigma(S^3, K)$, each of which bounds a smoothly embedded disk in $\Sigma(B^4, D')$ but not in $\Sigma(B^4, D)$. It follows that $W$ and $W'$ are not diffeomorphic. Finally, by inspecting Figure 15(i), we see that $\Sigma(B^4, D)$ is contractible, hence so are $W$ and $W'$.

4 Lagrangian 2-spheres

Applying the construction from §2 to the link $L_{8a16}$ (or its mirror $m(L_{8a16})$, depending on conventions) gives rise to a well-studied pair of slice disks for the knot $m(9_{46})$. These slice disks are known not to be topologically isotopic rel boundary [4, 10]. They are also well-known to be realized as Lagrangian disks in $(B^4, \omega_{st})$. We expand on this construction to produce examples of homotopic but topologically inequivalent Lagrangian 2-spheres in symplectic 4-manifolds.

We first clarify our notion of singular Lagrangian surfaces with cone points. Consider $(B^4, \omega_{st})$ and its radial Liouville vector field $v = \frac{1}{2}r \partial_r$. Given any Legendrian knot $L$ in the boundary $(S^3, \xi_{st})$ of $(B^4, \omega_{st})$, the image of $L$ under the flow of $-v$ is a Lagrangian cylinder in $B^4 \setminus 0$. Taking the union of this cylinder with $0 \in B^4$, we obtain a piecewise-linear Lagrangian disk $\Delta$ in $(B^4, \omega_{st})$. We say that a surface $\Sigma$ in a symplectic 4-manifold has a singularity $p \in \Sigma$ modeled on the cone of a Legendrian knot $L$ in $(S^3, \xi_{st})$ if there exist symplectic Darboux coordinates centered at $p$ in which $\Sigma$ coincides with $\Delta$.

Lemma 4.1 Let $\Sigma$ be a Lagrangian 2-sphere in a symplectic 4-manifold that is smooth away from a singular point modeled on the cone of a Legendrian knot $L$ in $(S^3, \xi_{st})$. Then $\Sigma$ has a Weinstein neighborhood $N$ obtained from a round 4-ball in $(\mathbb{R}^4, \omega_{st})$ by attaching a Weinstein 2-handle along $L$ in $(S^3, \xi_{st})$ with framing $tb(L) - 1$.

Proof Let $X$ denote the ambient symplectic 4-manifold and let $p \in \Sigma$ denote the unique singular point. We will construct $N$ as the union of a Weinstein neighborhood $W$ of $p$ containing the singular disk $\Delta = \Sigma \cap W$ and a Weinstein neighborhood of the smooth
Lagrangian disk $\Sigma \setminus \Delta$. More precisely, there exists a Darboux neighborhood $W \subset X$ of $p \in \Sigma$ such that $W$ is symplectomorphic to a 4-ball of some radius $\epsilon > 0$ in $(\mathbb{R}^4, \omega_{st})$, where the symplectomorphism carries $\Delta = \Sigma \cap W$ onto the singular disk in $B^4 \subset \mathbb{R}^4$ given by the cone on $L \subset (S^3, \xi_{st})$. Outside this neighborhood, the usual construction of Weinstein neighborhoods provides a neighborhood of the smooth Lagrangian disk $\Sigma \setminus \Delta$ in $X \setminus W$; this neighborhood is diffeomorphic to a 2-handle $D^2 \times D^2$ and is symplectomorphic to $T^* D^2$. After smoothing corners, the union of $W$ and this 2-handle is equivalent to $B^4$ with a standard Weinstein 2-handle attached along the Legendrian knot $L$; see [39, §7.2], which also establishes the claim that the 2-handle framing is $tb(L) - 1$.

We obtain Theorem 1.4 as a corollary of the following:

**Theorem 4.2** There exist infinitely many Legendrian knots $L$ in $(S^3, \xi_{st})$ such that if $\Sigma$ is a Lagrangian 2-sphere in a symplectic 4-manifold that is smooth away from a single cone point modeled on the Legendrian knot $L$, then $\Sigma$ has a Weinstein neighborhood $N$ containing a pair of smooth Lagrangian 2-spheres $S, S'$ that are homotopic in $N$ yet are not equivalent under any homeomorphism of $N$. Moreover, each embedding $S, S' \hookrightarrow N$ induces a homotopy equivalence.

**Proof** Fix an integer $m \leq 0$ and let $L$ be the Legendrian knot shown in Figure 16(a), which we note has Thurston-Bennequin number $tb(L) = -1$. We claim that $L$ satisfies the statement of the theorem for all sufficiently negative $m \ll 0$. By Lemma 4.1, if $\Sigma$ is a Lagrangian 2-sphere in a symplectic 4-manifold with a single cone point modeled on $L$, then $\Sigma$ has a Weinstein neighborhood $N$ given by the knot trace $X_{-2}(L)$.

Parts (b) and (c) of Figure 16 illustrate two pairs of standard Legendrian three-component unlinks, each obtained from $L$ by a pair of saddle moves. By [7, Theorem 4.2] (cf [14, 12]), this gives rise to a pair of Lagrangian disks in $(B^4, \omega_{st})$ bounded by $L$, which we denote by $D$ and $D'$, respectively. Let $S$ and $S'$ be closed Lagrangian 2-spheres in $N$ obtained by capping off $D$ and $D'$ with the Lagrangian core of the Weinstein 2-handle attached
along $L$; it is clear that the embeddings $S, S' \hookrightarrow N$ induce homotopy equivalences. For $m \ll 0$, we claim that all self-diffeomorphisms of $\partial N = S^2_{1,2}(L)$ are isotopic to the identity; this calculation is made using SnapPy [11] and Sage [51] and the results of [13], with full details of the calculation available in [28].

Now consider the knot $J$ in $S^3 \setminus L$ depicted in Figure 16(a). We claim that the knot in $\partial N$ induced by $J$ is nullhomotopic in $N \setminus S'$ but its image under any homeomorphism of $\partial N \to \partial N$ can extend to a homotopy equivalence of $N \setminus S'$ and $N \setminus S$, hence $N \setminus S$ is not homeomorphic to $N \setminus S'$.

As a first step, we draw handle diagrams for $B^4 \setminus \tilde{N}(D)$ and $B^4 \setminus \tilde{N}(D')$; see Figure 17. The knot $J$ in $S^3 \setminus L$ does not pass over any 1-handles in $B^4 \setminus \tilde{N}(D')$, so it is nullhomotopic in $B^4 \setminus D'$. Including $B^4 \setminus D'$ into $N \setminus S'$, it follows that the knot in $\partial N$ induced by $J$ is nullhomotopic in $N \setminus S'$. Turning to $B^4 \setminus \tilde{N}(D)$, we further simplify the handle diagram and decorate it with generators of $\pi_1(B^4 \setminus D)$ in Figure 18. We obtain the following fundamental group presentation:

$$\pi_1(B^4 \setminus D) \cong \langle x, y, z \mid xyx^{-1}y^{-1}xy = 1, \quad z^{-1}zyz^{-1} = 1 \rangle$$

$$\cong \langle x, y, z \mid xyx^{-1} = y^{-1}x^{-1}y, \quad y = z \rangle$$

$$\cong \langle x, y \mid xyx^{-1} = yx^{-1}y^{-1} \rangle.$$

Figure 17: Handle diagrams for the exteriors of the disks $D$ and $D'$ in $B^4$.

Figure 18: Setting up the calculation of the fundamental group of $B^4 \setminus D$. 
In this presentation, the free homotopy class of the curve $J$ corresponds to the conjugacy class of the element $xy$. Similarly, we choose the element $x$ as a representative of the free homotopy class of the meridian $\mu$ of $D$. Therefore, by Lemma 3.7, a presentation for $\pi_1(N \setminus S)$ is obtained from $\pi_1(B^4 \setminus D)$ by introducing the relation $x^2 = 1$, i.e. $x = x^{-1}$. Note that this implies $y = y^{-1}$ because
\[
y^2 = x^2y^2x^2 = x(xy)(xy)x = x(yy^{-1})(yy^{-1})x = 1.
\]
This gives the equivalence $xyx^{-1}y^{-1}xy = xyyyy = (xy)^3$, so we can write
\[
\pi_1(N \setminus S) \cong \langle x, y \mid (xy)^3 = x^2 = y^2 = 1 \rangle.
\]
The group $\pi_1(N \setminus S)$ admits a homomorphism — in fact, an isomorphism — onto the symmetric group $S_3$ mapping $x$ to $(1 2)$, $y$ to $(2 3)$, and $xy$ to $(1 2 3)$ (cf [4, Proposition 4.4]). Since the knot in $\partial N$ induced by $J$ corresponds to the element $xy$ and $(1 2 3)$ is a nonzero element of $S_3$, we conclude that the induced knot represents a nonzero element of $\pi_1(N \setminus S)$.

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