Strong coupling expansion of circular Wilson loops and string theories in $\text{AdS}_5 \times S^5$ and $\text{AdS}_4 \times \mathbb{C}P^3$

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arXiv:2007.08512
\[ \frac{1}{2} \] BPS circular WL's in \( \mathcal{N} = 4 \) SYM and ABJM:

[Ericson, Semenoff, Zarembo; Drukker, Gross; Pestun]  [Drukker, Marino, Putrov]

SYM: \( \mathcal{W} = \text{Tr} \ P e^{\int (iA + \Phi)} \), in planar limit \( \langle \mathcal{W} \rangle = N \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \)

\( \lambda \gg 1 \): \( \langle \mathcal{W} \rangle = \frac{N}{\lambda^{3/4}} \sqrt{\frac{\pi}{2}} e^{\sqrt{\lambda}} + ... \)

compare to string theory: \( \text{Tr} (...) \rightarrow Z_{\text{str}} \) (no \( \frac{1}{N} \) in \( \mathcal{W} \) [Lewkowycz, Maldacena])

disk partition function near AdS\(_2\) minimal surface

\( \langle \mathcal{W} \rangle = Z_{\text{str}} = \frac{1}{g_s} Z_1 + \mathcal{O}(g_s) \), \quad Z_1 = \int [dx] ... e^{-T \int d^2 \sigma L} \)

SYM: \( \langle \mathcal{W} \rangle = \sqrt{\frac{2}{\pi}} \frac{N}{\lambda^{3/4}} e^{\sqrt{\lambda}} + ... \), \quad g_s = \frac{\lambda}{4\pi N} \), \quad T = \frac{\sqrt{\lambda}}{2\pi} \), \quad \lambda = g_{\text{YM}}^2 N

ABJM: \( \langle \mathcal{W} \rangle = \frac{N}{4\pi \lambda} e^{\pi\sqrt{2\lambda}} + ... \), \quad g_s = \frac{\sqrt{\pi} (2\lambda)^{5/4}}{N} \), \quad T = \frac{\sqrt{2\lambda}}{2} \), \quad \lambda = \frac{N}{k}

both \( \langle \mathcal{W} \rangle \) have remarkably universal form at strong coupling:
reason – dual string theories have similar structure
\[ \langle \mathcal{W} \rangle = W_1 \left[ 1 + O(T^{-1}) \right] + O(g_s) \]

\[ W_1 = c_1 \frac{\sqrt{T}}{g_s} e^{2\pi T}, \quad c_1 = \frac{1}{(\sqrt{2\pi})^{n-3}} = \frac{1}{\sqrt{2\pi}} \bar{c}_1, \quad n = 5, 4 \]

dual string theory in AdS\(_n \times M^{10-n}\): AdS\(_5 \times S^5\) and AdS\(_4 \times CP^3\)

- \(e^{2\pi T} = e^{-TV_{AdS_2}}\) – area of AdS\(_2\) minimal surface [Berenstein, Corrado, Fischler, Maldacena]

- \(\bar{c}_1 = \frac{1}{(\sqrt{2\pi})^{n-4}}\) – numerical constant in \(Z_1\) [Drukker, Gross, AT; Kruczenski, Tirziu; Buchbinder, AT; ... ]

- \(\sqrt{T}\) prefactor – from universal dependence of \(Z_1\) on AdS radius

- extra \(\frac{1}{\sqrt{2\pi}}\) remains to be explained

sensitive to defn of string path integral measure; implicitly checked by computing ratio of \(\frac{1}{2}\) and \(\frac{1}{4}\) BPS WL’s [Medina-Rincon, Zarembo, AT]
1-loop AdS$_n \times M^{10-n}$ superstring partition function

near AdS$_2$ minimal surface (with AdS scale R) in static gauge

$$\log Z_1 = -\frac{1}{2} \log \frac{[\det(-\nabla^2 + 2)]^{n-2} [\det(-\nabla^2)]^{10-n}}{[\det(-\nabla^2 + \frac{1}{2})]^{2n-2} [\det(-\nabla^2 - \frac{1}{2})]^{10-2n}}$$

$$\log Z_1 = B_2 \log (R \Lambda) + \log \bar{c}_1, \quad B_2 = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} R^{(2)} = \chi$$

- $B_2 = \zeta_{\text{tot}}(0) = \chi$ is universal and $n$-independent

$$Z_1 \sim (\sqrt{T})^\chi, \quad (Z_1)_{\text{disk}} \sim \sqrt{T}, \quad T = \frac{R^2}{2\pi\alpha'}$$

- computing determinants on the disk ($V_{\text{AdS}_2} = -2\pi$)

$$\bar{c}_1 = \exp \left[ -\frac{1}{2} \int_0^\infty dv v \left( \tanh(\pi v) \left[ (n-2) \ln(v^2 + \frac{9}{4}) + (10-n) \ln(v^2 + \frac{1}{4}) \right] - \coth(\pi v) \left[ (2n-2) \ln(v^2 + 1) + (10-2n) \ln(v^2) \right] \right) \right] = \frac{1}{(\sqrt{2\pi})^{n-4}}$$
• disk with $h$ handles: $g_s^{-1} \rightarrow g_s^\chi$, $\sqrt{T} \rightarrow (\sqrt{T})^\chi$, $\chi = 1 - 2h = \text{Euler}$

$$\langle \mathcal{W} \rangle = e^{2\pi T} \sum_{h=0}^{\infty} c_{h+1} \left( \frac{g_s}{\sqrt{T}} \right)^{2h-1} \left[ 1 + \mathcal{O}(T^{-1}) \right]$$

• remarkably consistent with structure of $\frac{1}{N}$ corrections in SYM for $N \gg 1$, $\lambda \gg 1$ [Drukker, Gross; Pestun]

$$\langle \mathcal{W} \rangle = e^{\lambda} \sum_{h=0}^{\infty} \frac{\sqrt{2}}{96^{h} \sqrt{\pi} h!} \frac{\lambda^{\frac{6h-3}{4}}}{N^{2h-1}} \left[ 1 + \mathcal{O}(\frac{1}{\sqrt{\lambda}}) \right]$$

$$\rightarrow c_{h+1} = \frac{1}{2\pi h!} \left( \frac{\pi}{12} \right)^h , \quad c_1 = \frac{1}{2\pi}$$

• $\frac{g_s}{\sqrt{T}}$ is natural expansion parameter; for above $c_{p+1}$ powers exponentiate:

$$\langle \mathcal{W} \rangle = W_1 e^H \left[ 1 + \mathcal{O}(T^{-1}) \right] , \quad W_1 = \frac{1}{2\pi} \frac{\sqrt{T}}{g_s} e^{2\pi T} , \quad H \equiv \frac{\pi}{12} \frac{g_s^2}{T}$$

• $H$= "handle insertion operator": $\exp H$ expected in "dilute handle gas" approximation (thin far-separated handles) relevant for large $T$
**another interpretation of exponentiation** [Drukker, Fiol]:

start with circular WL in $k$-symmetric $SU(N)$ representation for large $k$, $N$ and $\lambda$ with $\kappa = \frac{k\sqrt{\lambda}}{4N} =$ fixed

$\langle \mathcal{W} \rangle \sim \exp(-S_{D3})$ – determined by action of classical D3-brane solution

for $1 \ll k \ll N$ should apply also to WL in $k$-fundamental rep described by minimal string surface ending on $k$-wrapped circle

$S_{D3} = Nf(\kappa) = -k\sqrt{\lambda} - \frac{k^3\lambda^{3/2}}{96N^2} + ... = -2\pi kT - k^3 \frac{\pi}{12} \frac{g_s^2}{T} + \mathcal{O}(\frac{g_s^4}{T^3})$

extrapolating to $k = 1$ gives: $\langle \mathcal{W} \rangle \sim \exp(-S_{D3}) \rightarrow \exp(2\pi T + H)$

**similar structure of topological expansion for $\frac{1}{2}$ BPS circular WL in ABJM?**

$\langle \mathcal{W} \rangle$ should be series in $\frac{\sqrt{T}}{g_s} = \frac{N}{\sqrt{8\pi \lambda}} = \frac{k}{\sqrt{8\pi}} \sim$ CS level

leading and first subleading $\frac{1}{N}$ corrections found explicitly in [Drukker, Marino, Putrov]

$\langle \mathcal{W} \rangle = e^{\pi\sqrt{2\lambda} \left( \frac{N}{4\pi\lambda} + \frac{\pi\lambda}{6N} + ... \right)} = W_1 \left( 1 + \frac{\pi}{12} \frac{g_s^2}{T} + ... \right), \quad W_1 = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{T}}{g_s} e^{2\pi T}$

same (!) $H = \frac{\pi}{12} \frac{g_s^2}{T}$ term at $\frac{1}{N^2}$ ("disk with one handle") order as in SYM
• $H$ exponentiates as in SYM case?
D2-brane description of exponentiation? (cf. [Drukker, Plefka, Young; Cookmeyer, Liu, Pando Zayas])
so far does not seem so ... [Beccaria, Giombi, AT]

• hope for progress in detailed understanding $\frac{1}{N}$ corrections on string side