LINEAR PERTURBATION OF THE YAMABE PROBLEM ON MANIFOLDS WITH BOUNDARY

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Abstract. We build blowing-up solutions for linear perturbation of the Yamabe problem on manifolds with boundary, provided the dimension of the manifold is \( n \geq 7 \) and the trace-free part of the second fundamental form is non-zero everywhere on the boundary.

1. Introduction

Given \((M, g)\) a smooth compact Riemannian manifold without boundary, the Yamabe problem is to find, in the conformal class of \( g \), a metric of constant scalar curvature. The geometric problem has a PDE formulation, i.e. the metric \( \tilde{g} = u^{\frac{4}{n-2}} g \) has the required properties if the function \( u \) is a smooth positive solution to the critical equation

\[
L_g u = \kappa u^{\frac{n+2}{n-2}} \quad \text{in} \quad M,
\]

for some constant \( \kappa \). Here \( L_g := \Delta_g - \frac{n-2}{4(n-1)} R_g \) is the conformal Laplacian, \( \Delta_g \) is the Laplace Beltrami operator and \( R_g \) is the scalar curvature of \((M, g)\). Solutions to (1) are critical points of the functional

\[
E(u) := \frac{\int_M \left( |\nabla u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) dv_g}{\left( \int_{\partial M} |u|^{\frac{2n}{n-2}} d\sigma \right)^{\frac{n-2}{n}}}, \quad u \in H^1_{\tilde{g}}(M),
\]

were \( dv_g \) denotes the volume form on \( M \) and \( \partial M \). The exponent \( \frac{2n}{n-2} \) is critical for the Sobolev embedding \( H^1_{\tilde{g}}(M) \hookrightarrow L^{\frac{2n}{n-2}}(\partial M) \). The existence of a minimizing solution to the Yamabe problem is well-known and follows from the combined works of Yamabe [23], Trudinger [22], Aubin [1] and Schoen [20].

One of the generalizations of this problem on manifolds \((M, g)\) with boundary was proposed by Escobar in [10] and it consists of finding in the conformal class of \( g \), a scalar-flat metric of constant boundary mean curvature. Also in this case the geometric problem has a PDE formulation, i.e. the metric \( \tilde{g} = u^{\frac{4}{n-2}} g \) has the required properties if the function \( u \) is a smooth positive solution to the critical boundary value problem

\[
\begin{cases}
L_g u = 0 \quad \text{in} \quad M \\
\partial_\nu u + \frac{n-2}{2} H_g u = \kappa u^{\frac{2(n-1)}{4(n-2)-1}} \quad \text{on} \quad \partial M.
\end{cases}
\]

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for some constant $\kappa$. Here $\nu$ is the outward unit normal vector to $\partial M$ and $H_g$ is the mean curvature on $\partial M$ with respect to $g$.

Solutions to (2) are critical points of the functional

$$Q(u) := \int_{\partial M} \left( |\nabla u|^2 + \frac{n-2}{n(n-1)} R_g u^2 \right) d\sigma_g + \int_{\partial \Omega} |u|^2 d\sigma_g H_g,$$

were $d\sigma_g$ and $d\sigma_g$ denote the volume forms on $M$ and $\partial M$, respectively, and the space

$$H := \{ u \in H^1_0(M) : u \neq 0 \text{ on } \partial M \}.$$

Escobar in [10] introduced the Sobolev quotient

$$Q(M, \partial M) := \inf_H Q(u),$$

which is conformally invariant and always satisfies

$$Q(M, \partial M) \leq Q(B^n, \partial B^n),$$

where $B^n$ is the unit ball in $\mathbb{R}^n$ endowed with the euclidean metric $g_0$.

Following Aubin’s approach (see [4]), Escobar proved that if $Q(M, \partial M)$ is finite and the strict inequality in (4) holds, i.e.

$$Q(M, \partial M) < Q(B^n, \partial B^n),$$

then the infimum (3) is achieved and a solution to problem (2) does exist.

In the negative case, i.e. $Q(M, \partial M) \leq 0$, it is clear that (5) holds. The positive case, i.e. $Q(M, \partial M) > 0$, is the most difficult one and the proof of the validity of (5) has required a lot of works. Assume $(M, g)$ is not conformally equivalent to $(B^n, g_0)$. (5) has been proved by Escobar in [10] if

- $n = 3$,
- $n = 4, 5$ and $\partial M$ is umbilic,
- $n \geq 6$, $\partial M$ is umbilic and $M$ is locally conformally flat
- $n \geq 6$ and $M$ has a non-umbilic point

by Marques in [11, 12] if

- $n = 4, 5$ and $\partial M$ is not umbilic,
- $n \geq 8$, Weyl$_g(\xi) \neq 0$ for some $\xi \in \partial M$
- $n \geq 9$, Weyl$_g(\xi) \neq 0$ for some $\xi \in \partial M$

by Almaraz in [3] if

- $n = 6, 7, 8$, $\partial M$ is umbilic and Weyl$_g(\xi) \neq 0$ for some $\xi \in \partial M$.

We remind that a point $\xi \in \partial M$ is said to be umbilic if the tensor $T_{ij} = h_{ij} - H_g g_{ij}$ vanishes at $\xi$, where $h_{ij}$ are the coefficients of the second fundamental form and $H = \frac{1}{n} g^{ij} h_{ij}$ is the mean curvature. The boundary $\partial M$ is said to be umbilic if all its points are umbilical. Moreover, Weyl$_g(\xi)$ denotes the Weyl tensor of the restriction of the metric to the boundary.

The strategy to prove that the strict inequality (5) holds consists in finding good test functions, which involve the minimizer of the Sobolev quotient in $\mathbb{R}^n_+ := \{(x, t) : x \in \mathbb{R}^{n-1}, t > 0 \}$, namely the so-called bubble

$$U_{\delta, y}(x, t) := \delta^{-\frac{n-2}{2}} U \left( \frac{x-y}{\delta}, \frac{t}{\delta} \right), \quad \delta > 0, \quad x, y \in \mathbb{R}^{n-1}, \ t > 0,$$

where

$$U(x, t) := \frac{1}{((1+t)^2 + |x|^2)^{n/2}}.$$

Indeed Beckner in [5] and Escobar [11] proved that

\[ Q(\mathbb{B}^n, \partial \mathbb{B}^n) = \inf \left\{ \frac{\int_{\mathbb{R}^n_+} |\nabla u|^2 \, dx}{\int_{\partial \mathbb{R}^n_+} |u|^\frac{2(n-1)}{n-2} \, dx} : u \in H^1(\mathbb{R}^n_+), u \neq 0 \text{ on } \partial \mathbb{R}^n_+ \right\}. \]

The infimum is achieved by the functions \( U_{\delta, y} \) which are the only positive solutions to the limit problem

\[ \begin{cases} 
\Delta u = 0 & \text{in } \mathbb{R}^n_+ \\
\partial_\nu u + \frac{n-2}{2} H_g u + \varepsilon \gamma u &= \frac{2(n-1)}{n-2} u^{n-2} \partial_{\partial M}. 
\end{cases} \]

Once the existence of solutions of problems (1) or (2) is settled, a natural question concerns the structure of the full set of positive solutions of (1) or (2). Concerning the Yamabe problem on manifold without boundary, Schoen (see [21]) raised the question of compactness of the set of solutions of problem (1). The question has been recently resolved by S. Brendle, M. A. Khuri, F. C. Marques and R. Schoen in a series of works [6, 7, 15] (see also the survey by Marques 16). By their results, the set of solutions for the Yamabe problem (1) is compact on any compact manifold of dimension \( n \leq 24 \), while it is not compact on some compact manifold of dimension \( n \geq 25 \).

Therefore, it is natural to address the question of compactness of the set of positive solutions of (2). If \( Q(M, \partial M) < 0 \) the solution is unique and if \( Q(M, \partial M) = 0 \) the solution is unique up to a constant factor. If \( Q(M, \partial M) > 0 \) the situation turns out to be more delicate. Indeed in the case of the euclidean ball \( (\mathbb{B}^n, g_0) \) the set of solutions is not compact! Felli and Ould-Ahmedou [13] proved that compactness holds when \( n \geq 3 \), \( (M, g) \) is locally conformally flat and \( \partial M \) is umbilic. Almaraz in [2] proved that compactness also holds if \( n \geq 7 \) and the trace-free second fundamental form of \( \partial M \) is non zero everywhere. This last assumption is generic as a transversality argument shows. Up to our knowledge, the only non-compactness result is due to Almaraz. In [11] he constructs a sequence of blowing-up conformal metrics with zero scalar curvature and constant boundary mean curvature on a ball of dimension \( n \geq 25 \). It is unknown if the dimension 25 is sharp for the compactness, namely if \( n \leq 24 \) the problem (2) is compact or not.

In this paper we are interested in the existence of blowing-up solutions to problems which are linear perturbation of the geometric problem (2). More precisely, the question we address is the following. Does the problem

\[ \left\{ \begin{array}{l}
L_g u = 0 \text{ in } M \\
\partial_\nu u + \frac{n-2}{2} H_g u + \varepsilon \gamma u = \frac{2(n-1)}{n-2} u^{n-2} \text{ on } \partial M.
\end{array} \right. \]

where \( \gamma \in C^2(M) \), have positive blowing-up solutions as the positive parameter \( \varepsilon \) approaches zero?

We give a positive answer under suitable geometric assumptions on \( M \) and on the sign of the linear perturbation term \( \gamma \). Our main result reads as follows.

**Theorem 1.** Assume \( n \geq 7 \), \( Q(M, \partial M) > 0 \) and the trace-free second fundamental form of \( \partial M \) is non zero everywhere. If the function \( \gamma \in C^1(M) \) is strictly positive, then for \( \varepsilon > 0 \) small there exists a positive solution \( u_\varepsilon \) of (2) such that \( \|u_\varepsilon\|_{H^1} \) is bounded and \( u_\varepsilon \) blows-up at a suitable point \( q_0 \in \partial M \) as \( \varepsilon \to 0 \).
Remark 2. The proof of our result relies on a Ljapunov-Schmidt procedure. We build solutions to (9) which at the main order looks like the bubble (6) centered at a point $q_0$ on the boundary. As usual the blowing-up point $q_0$ turns out to be a critical point of the reduced energy whose leading term is a function (see (47)) defined on the boundary, which cannot be explicitly written in terms of the geometry quantities of the boundary. The difficulty comes from the fact that we cannot find an explicit expression of the correction term we need to add to the bubble to have a good approximation. The correction term solves the linear problem (18) and it gives a significant contribution to the reduced energy (see (35)). Actually, we conjecture that the term (35) (up to a constant factor) is nothing but the trace-free second fundamental form at $q_0$ and so the blowing-up point $q_0$ is a critical point of the function

$$q \rightarrow \frac{\|\text{the trace-free second fundamental form at } q\|}{\gamma^2(q)}, \quad q \in \partial M.$$ 

Remark 3. Theorem 1 states that problem (9) is not compact if the linear perturbation term is strictly positive in $\partial M$. We strongly believe that the compactness is recovered if the linear perturbation is negative somewhere in $\partial \Omega$. This is what happens in the case of linear perturbation of the Yamabe problem (1). Indeed, if we consider the perturbed problem

$$(10) \quad L_g u + \varepsilon fu = \kappa u^{\frac{2n}{n-2}} \text{ in } M,$$

where $\varepsilon$ is a positive parameter and $f \in C^2(M)$. Druet in [8] shows that if $f \leq 0$ in $M$, blow-up does not occur if $3 \leq n \leq 5$. When $f$ is positive somewhere in $M$, blow-up is possible as showed by Druet and Hebey in [9] in the case of the sphere and by Esposito, Pistoia, and Vétois in [12] on general compact manifolds.

Remark 4. Almaraz in [2] studied the compactness of problem (2) when the exponent in the non-linearity of the boundary is below the critical exponent and he proved the following result.

**Theorem 5.** Assume $n \geq 7$, $Q(M, \partial M) > 0$ and the trace-free second fundamental form of $\partial M$ is non zero everywhere. Then the problem

$$(11) \quad \begin{cases} L_g u = 0 \text{ in } M \\ \partial_\nu u + \frac{n-2}{2} H_g u = u^{\frac{2(n-1)}{n-2} - 1} - \varepsilon \text{ on } \partial M. \end{cases}$$

is compact, namely there exist $\varepsilon_0 > 0$ and a positive constant $C$ such that for any $\varepsilon \in (0, \varepsilon_0)$ any positive solution $u_\varepsilon$ of (11) satisfies $\|u_\varepsilon\|_{C^{2,\alpha}(M)} \leq C$ for some $\alpha \in (0,1)$.

In other words, problem (11) does not have any blowing-up solutions as the positive parameter $\varepsilon$ approaches zero. Let us point out that combining our argument with some ideas developed in a previous paper [14] we can also obtain the existence of blowing-up solutions for problem (11) when the parameter $\varepsilon$ is negative and small. Then the compactness result Theorem 5 is sharp, namely the problem (11) is compact if the exponent in the non-linearity of the boundary approaches the critical exponent from below and it is non-compact if the exponent approaches the critical exponent from above.

The paper is organized as follows. In Section 2 we set the problem in a suitable scheme, in Section 3 we perform the finite-dimensional reduction, in Section 4 we study the reduced problem and in Section 5 we prove Theorem 1. The Appendix contains some technical results.
2. Variational framework and preliminaries

It is well known \[10\] that there exists a global conformal transformation which maps the manifold \( M \) in a manifold for which the mean curvature of the boundary is identically zero, so we can choose a metric \((M, g)\) such that \( H_g \equiv 0 \). This can be done, by a global conformal transformation \( g = \varphi^{4/n} \tilde{g} \), where \( \varphi_1 \) is the positive eigenvector of the first eigenvalue \( \lambda_1 \) of the problem

\[
\begin{aligned}
-L_g \varphi + \lambda_1 \varphi &= 0 & \text{on } M; \\
B_g \varphi &= 0 & \text{on } \partial M.
\end{aligned}
\]

It is useful to point out that if \( \pi \) denotes the second fundamental form related to \( g \) and \( q \in \partial M \) then \( \pi(q) \) is non-zero if and only if the trace-free second fundamental form related to \( \tilde{g} \) at the point \( q \) is non-zero.

By the assumption \( Q(M, \partial M) > 0 \) we have \( K > 0 \) in \( [2] \), so we can normalize it to be \( (n-2) \). Moreover, to gain in readability, we set \( a = \frac{n-2}{4(n-1)} R_g \), so Problem \( [3] \) reads as

\[
\begin{aligned}
-\Delta g u + au &= 0 & \text{on } M; \\
\frac{\partial u}{\partial \nu} + \varepsilon \gamma u &= (n-2)(u^+) \frac{\partial}{\partial r} & \text{on } \partial M.
\end{aligned}
\]

Since \( Q(M, \partial M) > 0 \), we can endow \( H^1(M) \) with the following equivalent scalar product

\[
\langle (u, v) \rangle_H = \int_M (\nabla_g u \nabla_g v + auv) d\mu_g
\]

which leads to the equivalent norm \( \| \cdot \|_H \). We have the well know maps

\[
i^* : H^1(M) \rightarrow L^1(\partial M)
\]

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\]

for \( 1 \leq t \leq \frac{2(n-1)}{n-2} \) (and for \( 1 \leq t < \frac{2(n-1)}{n-2} \) the embedding \( i \) is compact).

Given \( f \in L^{\frac{2(n-1)}{n-2}}(\partial M) \) there exists a unique \( u \in H^1(M) \) such that

\[
(u, \varphi)_H = \int_{\partial M} f \varphi d\sigma \quad \text{for all } \varphi.
\]

(13)

The functional defined on \( H^1(M) \) associated to (12) is

\[
J_\varepsilon(u) := \frac{1}{2} \int_M |\nabla_g u|^2 + au^2 d\mu_g + \frac{1}{2} \int_{\partial M} \varepsilon \gamma u^2 d\sigma - \frac{(n-2)^2}{2(n-1)} \int_{\partial M} (u^+) \frac{2(n-1)}{n-2} \frac{\partial}{\partial r} d\sigma.
\]

To solve problem (12) is equivalent to find \( u \in H^1(M) \) such that

\[
u = i^*(f(u) - \varepsilon \gamma u)
\]

(14)

where \( f(u) = (n-2)(u^+) \frac{\partial}{\partial r} \). We remark that, if \( u \in H^1 \), then \( f(u) \in L^{\frac{2(n-1)}{n-2}}(\partial M) \).

Given \( q \in \partial M \) and \( \psi_q^\delta : \mathbb{R}^n_+ \rightarrow M \) the Fermi coordinates in a neighborhood of \( q \); we define

\[
W_{\delta, q}(\xi) = U_\delta \left((\psi_q^\delta)^{-1}(\xi)\right) \chi \left((\psi_q^\delta)^{-1}(\xi)\right) = \frac{1}{\delta} \left(U \left(\frac{y}{\delta}\right)\right) \chi(y) = \frac{1}{\delta} \frac{1}{\delta} U(x) \chi(\delta x)
\]

where \( y = (z, t) \), with \( z \in \mathbb{R}^{n-1} \) and \( t \geq 0 \), \( \delta x = y = (\psi_q^\delta)^{-1}(\xi) \) and \( \chi \) is a radial cut off function, with support in ball of radius \( R \).
Here $U_\delta(y) = \frac{1}{\delta^{n/2}} U \left( \frac{y}{\delta} \right)$ is the one parameter family of solution of the problem

\[
(15) \quad \begin{cases}
-\Delta U_\delta = 0 & \text{on } \mathbb{R}_+^n; \\
\frac{\partial U_\delta}{\partial t} = -(n-2)U_\delta \frac{\partial}{\partial y} & \text{on } \partial \mathbb{R}_+^n.
\end{cases}
\]

and $U(z,t) := \frac{1}{(1+t)^{n/2} + |z|^2}^{n/2}$ is the standard bubble in $\mathbb{R}_+^n$.

Moreover, we consider the functions

\[
j_i = \frac{\partial U}{\partial x_i}, \quad i = 1, \ldots, n-1 \quad j_n = \frac{n}{2}U + \sum_{i=1}^n y_i \frac{\partial U}{\partial y_i}
\]

which are solutions of the linearized problem

\[
(16) \quad \begin{cases}
-\Delta \phi = 0 & \text{on } \mathbb{R}_+^n; \\
\frac{\partial \phi}{\partial t} + n U \frac{\partial \phi}{\partial y} = 0 & \text{on } \partial \mathbb{R}_+^n.
\end{cases}
\]

Given $q \in \partial M$ we define, for $b = 1, \ldots, n$

\[
Z_{b,q}^\delta(\xi) = \frac{1}{\delta} j_b \left( \frac{1}{\delta} (\psi_q^\delta)^{-1}(\xi) \right) \chi \left( (\psi_q^\delta)^{-1}(\xi) \right)
\]

and we decompose $H^1(M)$ in the direct sum of the following two subspaces

\[
K_{\delta,q} = \text{Span} \left( Z_{1,q}^\delta, \ldots, Z_{n,q}^\delta \right)
\]

\[
K_{\delta,q}^\perp = \{ \phi \in H^1(M) : \langle \phi, Z_{b,q}^\delta \rangle_M = 0, \quad b = 1, \ldots, n \}
\]

and we define the projections

\[
\Pi = H^1(M) \to K_{\delta,q} \quad \Pi^\perp = H^1(M) \to K_{\delta,q}^\perp.
\]

Given $q \in \partial M$ we also define in a similar way

\[
V_{\delta,q}(\xi) = \frac{1}{\delta} v_q \left( \frac{1}{\delta} (\psi_q^\delta)^{-1}(\xi) \right) \chi \left( (\psi_q^\delta)^{-1}(\xi) \right),
\]

and

\[
(17) \quad (v_q)_q(y) = \frac{1}{\delta^{n/2}} v_q \left( \frac{y}{\delta} \right);
\]

here $v_q : \mathbb{R}_+^n \to \mathbb{R}$ is the unique solution of the problem

\[
(18) \quad \begin{cases}
-\Delta v = 2h_{ij}(q)t\frac{\partial^2}{\partial y_i \partial y_j} U & \text{on } \mathbb{R}_+^n; \\
\frac{\partial v}{\partial t} + n U \frac{\partial v}{\partial y} = 0 & \text{on } \partial \mathbb{R}_+^n.
\end{cases}
\]

such that $v_q$ is $L^2(\mathbb{R}_+^n)$-ortogonal to $j_b$ for all $b = 1, \ldots, n$ Here $h_{ij}$ is the second fundamental form and we use the Einstein convention of repeated indices. We remark

\[
|\nabla^r v_q(y)| \leq C(1 + |y|)^{3-r-n} \quad \text{for } r = 0, 1, 2,
\]

\[
(20) \quad \int_{\partial \mathbb{R}_+^n} U \frac{\partial v_q}{\partial y} = 0
\]

and

\[
(21) \quad \int_{\partial \mathbb{R}_+^n} \Delta v_q v_q dzdt \leq 0,
\]

(see [2] Proposition 5.1 and estimate (5.9)).

**Proposition 6.** The map $q \mapsto v_q$ is in $C^2(\partial M)$. 

Proof. Let \( q_0 \in \partial M \). If \( q \in \partial M \) is sufficiently close to \( q_0 \), in Fermi coordinates we have \( q = q(y) = \exp_{q_0} y \), with \( y \in \mathbb{R}^{n-1} \). So \( v_q = v_{\exp_{q_0} y} \) and we define

\[
\Gamma_i = \frac{\partial}{\partial y_i} v_{\exp_{q_0} y} \bigg|_{y=0}.
\]

We prove the result for \( \Gamma_1 \), being the other cases completely analogous. By (18) we have that \( \Gamma_1 \) solves

\[
\left\{ \begin{array}{l}
-\Delta \Gamma_1 = 2 \left( \frac{\partial}{\partial y_1} (h_{ij}(q(y))) \right) \bigg|_{y=0} t \partial^2_y U & \text{on } \mathbb{R}^n_+;
\quad \frac{\partial R}{\partial x_1} + nU \pi^2 \Gamma_1 = 0 & \text{on } \partial \mathbb{R}^n_+.
\end{array} \right.
\]

and, by the result of [2], we know that \( \Gamma_1 \) exists. We can proceed in analogous way for the second derivative. \( \square \)

We define the useful integral quantity

\[
I_m^\alpha = \int_0^\infty \frac{\rho^\alpha}{(1 + \rho^2)^m} d\rho
\]

and in the appendix (Remark [7]) we recall some useful estimates of these integrals.

Finally, we have to recall the Taylor expansion for the metric \( g \) and for the volume form on \( M \), expressed by the Fermi coordinates.

Since, without loss of generality, we have chosen a manifold for which \( H_0 \equiv 0 \), we have the following expansions in a neighborhood of \( y = 0 \), with the usual notation \( y = (z, t) \), where \( z \in \mathbb{R}^n \) and \( t \geq 0 \). Here and in the following, we use the Einstein convention on the sum of repeated indices. Moreover, we use the convention that \( a, b, c, d = 1, \ldots, n \) and \( i, j, k, l = 1, \ldots, n - 1 \).

(22) \( |g(y)|^{1/2} = 1 - \frac{1}{2} \left( ||\pi||^2 + \text{Ric}_\eta(0) \right) t^2 - \frac{1}{6} R_{ij}(0) z_i z_j + O(|y|^3) \)

\[
g^{ij}(y) = \delta_{ij} + 2 h_{ij}(0) t + \frac{1}{3} R_{ijkl}(0) z_k z_l + \frac{2}{3} \frac{\partial h_{ij}(0) t z_k}{\partial z_k}
\]

(23) \( [R_{imjn}(0) + 3 h_{ik}(0) h_{kj}(0)] t^2 + O(|y|^3) \)

(24) \( g^{an}(y) = \delta_{an} \)

where \( \pi \) is the second fundamental form and \( h_{ij}(0) \) are its coefficients, \( R_{ijkl}(0) \) and \( R_{abcd}(0) \) are the curvature tensor of \( \partial M \) and \( M \), respectively, \( R_{ij}(0) = R_{ikj}(0) \) are the coefficients of the Ricci tensor, and \( \text{Ric}_\eta(0) = R_{nimn}(0) = R_{nn}(0) \) (see [10]).

3. Finite dimensional reduction

We look for a good approximation for the solution of problem [14], then we look for solution with the form

\( u = W_{\delta, q} + \delta V_{\delta, q} + \Phi \), with \( \Phi \in K_{\delta, q}^+ \).

and we project [14] on \( K_{\delta, q}^+ \) and \( K_{\delta, q} \) obtaining

(25) \( \Pi^+ \{ W_{\delta, q} + \delta V_{\delta, q} + \Phi - i^* (f(W_{\delta, q} + \delta V_{\delta, q} + \Phi) - \varepsilon \gamma(W_{\delta, q} + \delta V_{\delta, q} + \Phi)) \} = 0 \)

(26) \( \Pi \{ W_{\delta, q} + \delta V_{\delta, q} + \Phi - i^* (f(W_{\delta, q} + \delta V_{\delta, q} + \Phi) - \varepsilon \gamma(W_{\delta, q} + \delta V_{\delta, q} + \Phi)) \} = 0 \).

To solve (25) we define the linear operator \( L = L_{\delta, q} : K_{\delta, q}^+ \to K_{\delta, q}^+ \) as

(27) \( L(\Phi) = \Pi^+ \{ \Phi - i^* (f(W_{\delta, q} + \delta V_{\delta, q})(\Phi)) \} \)
and a nonlinear term \( N(\Phi) \) and a remainder term \( R \) as

\[
N(\Phi) = \Pi^{-1}(i^* (f(W_{\delta,q} + \delta V_{\delta,q} + \Phi) - f(W_{\delta,q} + \delta V_{\delta,q}) - f'(W_{\delta,q} + \delta V_{\delta,q})[\Phi]))
\]

so eq (25) rewrites as

\[
L(\Phi) = N(\Phi) + R - \Pi^{-1}(i^* (\varepsilon \gamma (W_{\delta,q} + \delta V_{\delta,q} + \Phi))).
\]

Lemma 7. Let \( \delta = \varepsilon \lambda \) for \( a, b \in \mathbb{R}, 0 < a < b \) there exists a positive constant \( C = C(a, b) \) such that, for \( \varepsilon \) small, for any \( q \in \partial M \), for any \( \lambda \in [a, b] \) and for any \( \phi \in K_{\delta,q} \) there holds

\[
\|L_{\delta,q}(\phi)\|_H \geq C\|\phi\|_H.
\]

The proof of this lemma is postponed in the appendix.

Lemma 8. Assume \( n \geq 7 \) and \( \delta = \lambda \varepsilon \), then it holds

\[
\|R\|_H = O(\varepsilon^2)
\]

\( C^0 \)-uniformly for \( q \in \partial M \) and \( \lambda \) in a compact set of \((0, +\infty)\).

Proof. We recall that there is a unique \( \Gamma \) such that

\[
\Gamma = i^* (f(W_{\delta,q} + \delta V_{\delta,q})),
\]

that is, according to (13) equivalent to say that there exists a unique \( \Gamma \) solving

\[
\begin{cases}
-\Delta_p \Gamma + a\Gamma = 0 & \text{on } M; \\
\frac{\partial}{\partial \nu} \Gamma = (n - 2)(W_{\delta,q} + \delta V_{\delta,q})^{1/2} - \frac{\partial}{\partial \nu}(W_{\delta,q} + \delta V_{\delta,q}) & \text{on } \partial M.
\end{cases}
\]

By definition of \( i^* \) we have that

\[
\|R\|_H^2 = \|\Gamma - W_{\delta,q} - \delta V_{\delta,q}\|_H^2
\]

\[
= \int_M [-\Delta_p(\Gamma - W_{\delta,q} - \delta V_{\delta,q}) + a(\Gamma - W_{\delta,q} - \delta V_{\delta,q})(\Gamma - W_{\delta,q} - \delta V_{\delta,q})]d\mu_q
\]

\[
+ \int_{\partial M} \left[ \frac{\partial}{\partial \nu}(\Gamma - W_{\delta,q} - \delta V_{\delta,q}) \right] (\Gamma - W_{\delta,q} - \delta V_{\delta,q})d\sigma
\]

\[
= \int_M [\Delta_p(\Gamma - W_{\delta,q} - \delta V_{\delta,q}) - a(\Gamma - W_{\delta,q} - \delta V_{\delta,q})] Rd\mu_q
\]

\[
\int_{\partial M} \left[ (n - 2)(W_{\delta,q} + \delta V_{\delta,q})^{1/2} - \frac{\partial}{\partial \nu}(W_{\delta,q} + \delta V_{\delta,q}) \right] Rd\sigma
\]

We have

\[
\int_M aW_{\delta,q} Rd\mu_q \leq c\|W_{\delta,q}\|_{L^{\frac{2n}{n+2}}(M)}\|R\|_{L^{\frac{2n}{n+2}}(M)} \leq c\delta^2\|U\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)}\|R\|_H
\]

and \( \|U\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)} \) is bounded since \( n > 6 \). Moreover

\[
\delta \int_M aV_{\delta,q} Rd\mu_q \leq c\delta\|V_{\delta,q}\|_{L^2(M)}\|R\|_{L^2(M)} \leq c\delta^2\|v_q\|_{L^2(\mathbb{R}^n)}\|R\|_H
\]

We have

\[
\int_{\partial M} \left[ (n - 2)W_{\delta,q}^{1/2} - \frac{\partial}{\partial \nu}W_{\delta,q} \right] Rd\sigma \leq \|\int_{\partial M} \left[ (n - 2)W_{\delta,q}^{1/2} - \frac{\partial}{\partial \nu}W_{\delta,q} \right] Rd\sigma \|_{L^{\frac{2(n-2)}{n-2}}(\partial M)}\|R\|_H
\]

\[
\leq c\delta^2\|R\|_H
\]
since $U$ is a solution of (15). In fact
\[
\begin{align*}
\left\| (n-2)W_{\delta,q}^{-\frac{n}{n-2}} - \frac{\partial}{\partial \nu} W_{\delta,q} \right\|_{L^{2(n-1)}(\partial M)} &= \\
\left( \int_{\partial M} |g(\delta z, 0)|^{\frac{n}{n-2}} \left[ (n-2)U_{\delta,q}^{-\frac{n}{n-2}}(\delta z, 0) - \chi(\delta z, 0) \frac{\partial U}{\partial \nu}(z, 0) \right] \frac{2(n-1)}{n} dz \right)^{\frac{n}{2(n-1)}} \leq C \left( \int_{\mathbb{R}^{n-1}} \left[ (n-2)U_{\delta,q}^{-\frac{n}{n-2}}(\delta z, 0) - \chi(\delta z, 0) \right] \frac{2(n-1)}{n} dz \right)^{\frac{n}{2(n-1)}} = O(\delta^2),
\end{align*}
\]

Now we estimate
\[
\begin{align*}
\left\| (n-2) \left[ \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n}{n-2}} - W_{\delta,q}^{\frac{n}{n-2}} \right] - \delta \frac{\partial V_{\delta,q}}{\partial \nu} \right\|_{L^{2(n-1)}(\partial M)} \leq \left( n-2 \right) \left[ \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n}{n-2}} - W_{\delta,q}^{\frac{n}{n-2}} \right] + o(\delta^2)
\end{align*}
\]
and, by Taylor expansion and by definition of the function $v_q$ (see (18))
\[
\left\| (n-2) \left[ \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n}{n-2}} - W_{\delta,q}^{\frac{n}{n-2}} \right] - \delta \frac{\partial V_{\delta,q}}{\partial \nu} \right\|_{L^{2(n-1)}(\partial M)} \leq \left\| \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n}{n-2}} v_q + \frac{\partial v_q}{\partial t} \right\|_{L^{2(n-1)}(\partial M)} + o(\delta^2)
\]
\[
= \delta \left\| \left( W_{\delta,q} + \delta V_{\delta,q} \right)^{\frac{n}{n-2}} v_q - U_{\delta,q}^{\frac{n}{n-2}} v_q \right\|_{L^{2(n-1)}(\partial M)} + o(\delta^2).
\]

We observe that, chosen a large positive $R$, we have $U + \delta v_q > 0$ in $B(0, R)$ for some $\delta$. Moreover, on the complementary of this ball, we have $\frac{\partial U}{\partial \nu} \leq U(y) \leq \frac{C}{|y|^{n-2}}$ and $|v_q| \leq C |y|^{-\frac{n-2}{2}}$ for some positive constants $c, C, C_1$. So it is possible to prove that, for $\delta$ small enough, $U + \delta v_q > 0$ if $|y| \leq 1/\delta$. At this point
\[
\begin{align*}
\int_{\partial M} \left[ \left( U + \delta v_q \right)^{\frac{n}{n-2}} \right] \frac{2(n-1)}{n} dz &= \int_{U + \delta v_q > 0} \left[ \left( U + \delta v_q \right)^{\frac{n}{n-2}} - U^{\frac{n}{n-2}} \right] \frac{2(n-1)}{n} dz + \int_{U + \delta v_q \leq 0} \left[ \left( U + \delta v_q \right)^{\frac{n}{n-2}} - U^{\frac{n}{n-2}} \right] \frac{2(n-1)}{n} dz \\
&= \delta \frac{2(n-1)}{n} \int_{U + \delta v_q > 0} \left( U + \delta v_q \right)^{\frac{n-2}{n-2}(n-1)} \frac{2(n-1)}{n} dz + \int_{U + \delta v_q \leq 0} \left( U + \delta v_q \right)^{\frac{n-2}{n-2}(n-1)} \frac{2(n-1)}{n} dz \\
&\leq \delta \frac{2(n-1)}{n} \int_{U + \delta v_q > 0} \left( U + \delta v_q \right)^{\frac{n-2}{n-2}(n-1)} \frac{2(n-1)}{n} dz + \int_{|z| > \delta} \left( U + \delta v_q \right)^{\frac{n-2}{n-2}(n-1)} \frac{2(n-1)}{n} dz
\end{align*}
\]
We recall that in local charts the Laplace Beltrami operator is

\[
\Delta_g W = \Delta_{\text{euc}} (U \delta(u) \chi(y)) + [g^{ij}(y) - \delta_{ij}] \partial_{ij}^2 (U \delta(u) \chi(y)) - g^{ij}(y) \Gamma_{ij}^k(y) \partial_k (U \delta(u) \chi(y))
\]

where \( i, k = 1, \ldots, n - 1 \), \( \Delta_{\text{euc}} \) is the euclidean Laplacian, and \( \Gamma_{ij}^k \) are the Christoffel symbols. Notice that, by (15) and (23) we have that \( \Gamma_{ij}^k(y) = O(|y|) \). Now, by (15) and (23) we have, in variables \( y = \delta x \),

\[
\Delta_g W_{\delta,q} = U \delta(u) \Delta_{\text{euc}} (\chi(y)) + 2 \nabla U \delta(u) \nabla \chi(y) + [g^{ij}(y) - \delta_{ij}] \partial_{ij}^2 (U \delta(u) \chi(y)) - g^{ij}(y) \Gamma_{ij}^k(y) \partial_k (U \delta(u) \chi(y))
\]

\[
= \frac{1}{\delta^n} \left( 2h_{ij}(0) \delta x_n \frac{1}{\delta} \partial_{ij} U(x) + g^{ij}(x) \Gamma_{ij}^k(x) \frac{1}{\delta} \partial_k U(x) + o(\delta) c(x) \right)
\]

(32)

where, with abuse of notation, we call \( c(x) \) a suitable function such that \( \left| \int_{\mathbb{R}^n} c(x) dx \right| \leq C \) for some \( C \in \mathbb{R}^+ \).

In a similar way, by (18) and by (24) we have

\[
\delta \Delta_g V_{\delta,q} = \frac{\delta}{\delta^n} \left( \frac{1}{\delta^2} \Delta_{\text{euc}} v_q(x) + \frac{1}{\delta^2} [g^{ij} - \delta_{ij}] \partial_{ij}^2 v_q(x) + \delta g(x) \Gamma_{ij}^k(x) \frac{1}{\delta} \partial_k v_q(x) + o(\delta^2) c(y) \right)
\]

\[
= \frac{1}{\delta^n} \left( -2h_{ij}(0) x_n \partial_{ij}^2 U(y) + O(\delta) c(x) \right)
\]

(33)

Thus, in local chart by (32) and (33) we get

\[
||\Delta_g (W_{\delta,q} + \delta V_{\delta,q})||_{L^{2n/(n+1)}(M)} = \delta^n \frac{n+1}{n} \frac{1}{\delta^2} O(\delta) = O(\delta^2)
\]

and we obtain the proof, once we set \( \delta = \lambda \varepsilon \).

Remark 9. We have that the nonlinear operator \( N \) (see (23)) is a contraction. By the properties of \( \varepsilon^* \) and using the expansion of \( f(\varepsilon (W_{\delta,q} + \Phi_1 + \delta V_{\delta,q}) \) centered in \( W_{\delta,q} + \Phi_2 + \delta V_{\delta,q} \) we have

\[
||N(\Phi_1) - N(\Phi_2)||_H \\
\leq ||f'(W_{\delta,q} + \Phi_1 + (1 - \theta) \Phi_2 + \delta V_{\delta,q} - f'(W_{\delta,q} + \delta V_{\delta,q})) [\Phi_1 - \Phi_2]||_{L^{2n/(n+1)}(\partial M)}
\]

and

\[
||f'(W_{\delta,q} + \Phi_1) - f'(W_{\delta,q} + \delta V_{\delta,q})||_{L^{2n/(n+1)}(\partial M)} \\
\leq \frac{1}{\delta^n} \frac{n+1}{n} O(\delta^2)
\]

where

\[
f'(W_{\delta,q} + \Phi_1 + (1 - \theta) \Phi_2 + \delta V_{\delta,q} - f'(W_{\delta,q} + \delta V_{\delta,q})) [\Phi_1 - \Phi_2] \\
= \frac{1}{\delta^n} \frac{n+1}{n} (1 - \theta) \delta V_{\delta,q} [\Phi_1 - \Phi_2]
\]

and

\[
||\delta V_{\delta,q}||_{L^{2n/(n+1)}(\partial M)} \\
\leq \frac{1}{\delta^n} \frac{n+1}{n} \delta V_{\delta,q}
\]
and, since $|\phi_1 - \phi_2| \leq C \varepsilon$ and, since $L^{\frac{n-1}{2}}(\partial M)$ and $f_\varepsilon(\cdot) \in L^{\frac{n-1}{2}}(\partial M)$, we have

$$\|N(\phi_1) - N(\phi_2)\|_H \leq \|f'(W_{\delta,q} + \theta \phi_1 + (1 - \theta)\phi_2 + \delta V_{\delta,q}) - f'(W_{\delta,q} + \delta V_{\delta,q})\| L^{\frac{n-1}{2}}(\partial M) \|\phi_1 - \phi_2\|_H$$

where

$$\beta = \|f'(W_{\delta,q} + \theta \phi_1 + (1 - \theta)\phi_2 + \delta V_{\delta,q}) - f'(W_{\delta,q} + \delta V_{\delta,q})\| L^{\frac{n-1}{2}}(\partial M) < 1,$$

provided $\|\phi_1\|_H$ and $\|\phi_2\|_H$ sufficiently small.

In the same way we can prove that $\|N(\phi)\|_H \leq \beta \|\phi\|_H$ with $\beta < 1$ if $\|\phi\|_H$ is sufficiently small.

**Proposition 10.** Let $\delta = \varepsilon \lambda$ for $a, b \in \mathbb{R}$, $0 < a < b$ there exists a positive constant $C = C(a, b)$ such that, for $\varepsilon$ small, for any $q \in \partial M$, for any $\lambda \in [a, b]$ there exists a unique $\Phi = \Phi_{\varepsilon, \delta, q} \in K_{\delta,q}$ which solves (25) such that

$$\|\Phi\|_H \leq C \varepsilon^2$$

**Proof.** By Remark 8 we have that $N$ is a contraction. Moreover, by Lemma 7 and Lemma 8 there exists $C > 0$ such that

$$\|L^{-1}(N(\phi) + R - \Pi^\perp \{i^* (\varepsilon \gamma (W_{\delta,q} + \delta V_{\delta,q} + \phi))\})\|_H \leq C ((\beta + \varepsilon) \|\phi\|_H + \varepsilon^2).$$

In fact, we have

$$\|i^* (\varepsilon \gamma (W_{\varepsilon \lambda,q} + \varepsilon \lambda V_{\varepsilon \lambda,q} + \phi))\|_H \leq \varepsilon \left(\|W_{\varepsilon \lambda,q} + \varepsilon \lambda V_{\varepsilon \lambda,q}\| L^{\frac{n-1}{2}} + \|\phi\|_H\right)$$

$$\leq C (\varepsilon^2 + \varepsilon \|\phi\|_H)$$

Notice that, given $C > 0$, in Remark 9 it is possible (up to choose $\|\phi\|_H$ sufficiently small) to choose $0 < C(\beta + \varepsilon) < 1/2$.

Now, if $\|\phi\|_H \leq 2C \varepsilon^2$ then the map

$$T(\phi) := L^{-1}(N(\phi) + R - \Pi^\perp \{i^* (\varepsilon \gamma (W_{\delta,q} + \delta V_{\delta,q} + \phi))\})$$

is a contraction from the ball $\|\phi\|_H \leq 2C \varepsilon^2$ in itself, so, by the fixed point Theorem, there exists a unique $\Phi$ with $\|\Phi\|_H \leq 2C \varepsilon^2$ solving (25). The regularity of the map $q \mapsto \Phi$ can be proven via the implicit function Theorem. 

4. The reduced functional

**Lemma 11.** Assume $n \geq 7$ and $\delta = \varepsilon \lambda$. It holds

$$J_\varepsilon(W_{\delta,q} + \delta V_{\delta,q} + \Phi) - J_\varepsilon(W_{\delta,q} + \delta V_{\delta,q}) = o(\varepsilon^2)$$

$C^0$-uniformly for $q \in \partial M$ and $\lambda$ in a compact set of $(0, +\infty)$. 
Proof. We know that $\|\Phi\|_H = O(\varepsilon^2)$, so we estimate, for some $\theta \in (0, 1)$$J_{\varepsilon}(W_{\delta,q} + \delta V_{\delta,q} + \Phi) - J_{\varepsilon}(W_{\delta,q} + \delta V_{\delta,q}) = J_{\varepsilon}(W_{\delta,q} + \delta V_{\delta,q})(\Phi) + \frac{1}{2} J''_{\varepsilon}(W_{\delta,q} + \delta V_{\delta,q} + \theta\Phi)(\Phi, \Phi)$

$$= \int_M (\nabla_g W_{\delta,q} + \delta \nabla_g V_{\delta,q}) \nabla \Phi + a (W_{\delta,q} + \delta V_{\delta,q}) \Phi d\mu_g$$

$$+ \int_{\partial M} \varepsilon \gamma (W_{\delta,q} + \delta V_{\delta,q}) \Phi d\sigma - (n - 2) \int_{\partial M} \left((W_{\delta,q} + \delta V_{\delta,q})^+\right)^\frac{\varepsilon}{2} \Phi d\sigma$$

$$+ \frac{1}{2} \int_M |\nabla \Phi|^2 + a \Phi^2 d\mu_g + \frac{1}{2} \int_{\partial M} \varepsilon \gamma \Phi^2 d\sigma$$

$$- \frac{n}{2} \int_{\partial M} ((W_{\delta,q} + \delta V_{\delta,q} + \theta\Phi)^+)^\frac{\varepsilon}{2} \Phi^2 d\sigma.$$}

Immediately we have, by Holder inequality, and setting $\delta = \varepsilon \lambda$,

$$\frac{1}{2} \int_M |\nabla \Phi|^2 + a \Phi^2 d\mu_g + \int_{\partial M} \varepsilon \gamma \Phi^2 d\sigma \leq C \|\Phi\|_H^2 = o(\varepsilon^2);$$

$$\int_M aW_{\delta,q} \Phi d\mu_g \leq C \|W_{\delta,q}\|_{L^\frac{2}{n-1}(M)} \|\Phi\|_{L^\frac{2}{n-1}(M)} \leq C \delta^2 \|\Phi\|_H = o(\varepsilon^2);$$

$$\frac{1}{2} \int_M aV_{\delta,q} \Phi d\mu_g \leq C \|V_{\delta,q}\|_{L^2(M)} \|\Phi\|_{L^2(M)} \leq C \delta^2 \|\Phi\|_H = o(\varepsilon^2);$$

$$\int_{\partial M} \varepsilon \gamma (W_{\delta,q} + \delta V_{\delta,q}) \Phi d\sigma \leq C \varepsilon \|W_{\delta,q} + \delta V_{\delta,q}\|_{L^\frac{2(n-1)}{n}(\partial M)} \|\Phi\|_{L^\frac{2(n-1)}{n}(\partial M)} \leq \varepsilon C \delta \|\Phi\|_H = o(\varepsilon^2)$$

$$\int_{\partial M} ((W_{\delta,q} + \delta V_{\delta,q} + \theta\Phi)^+)^\frac{\varepsilon}{2} \Phi^2 d\sigma \leq C \|\Phi\|_H^2 \left(\|W_{\delta,q} + \delta V_{\delta,q} + \theta\Phi\|_{L^\frac{2(n-1)}{n}(\partial M)}\right)$$

$$\leq C \|\Phi\|_H^2 = o(\varepsilon^2);$$

By integration by parts we have

$$\int_M (\nabla_g W_{\delta,q} + \delta \nabla_g V_{\delta,q}) \nabla \Phi d\mu_g = - \int_M \Delta_g (W_{\delta,q} + \delta V_{\delta,q}) \Phi d\mu_g$$

$$+ \int_{\partial M} \left(\frac{\partial}{\partial \nu} W_{\delta,q} + \delta \frac{\partial}{\partial \nu} V_{\delta,q}\right) \Phi d\mu_g.$$
In fact, by (17), (18) and by taylor expansion we have
\[
\int_{\partial M} \left[ (n-2) \left( (W_{\delta,q} + \delta V_{\delta,q})^+ \right)^{\frac{2}{n-2}} - \frac{\partial}{\partial t} W_{\delta,q} \right]^{\frac{2(n-1)}{n}} d\sigma \\
\leq \int_{\partial R^+} \left[ (n-2) \left( (U_{\delta} + \delta (v_q))^{+} \right)^{\frac{2}{n-2}} + \frac{\partial}{\partial t} U_{\delta} \right]^{\frac{2(n-1)}{n}} dz + o(1) \\
\leq \int_{\partial R^+} \left[ n \left( (U_{\delta} + \theta \delta (v_q))^{+} \right)^{\frac{2}{n-2}} \delta (v_q) \right]^{\frac{2(n-1)}{n}} d\sigma + o(1) = o(1),
\]
which concludes the proof.

**Proposition 12.** Assume \( n \geq 7 \) and \( \delta = \lambda \varepsilon \). It holds
\[
J_\varepsilon(W_{x,q} + \lambda \varepsilon V_{x,q}) = A + \varepsilon^2 \left[ \lambda B \gamma(q) + \lambda^2 \varphi(q) \right] + o(\varepsilon^2),
\]
\( C^0 \)-uniformly for \( q \in \partial M \) and \( \lambda \) in a compact set of \((0, +\infty)\), where (see (21))
\[
\varphi(q) = \frac{1}{2} \int_{R^+} \Delta v_q v_q dz dt - \frac{(n-6)(n-1)\omega_{n-1}I_{n-1}^n}{4(n-1)^2} ||\pi(q)||^2 \leq 0.
\]
and
\[
A = \frac{1}{2} \int_{R^+} |\nabla U(z,t)|^2 dz dt - \frac{(n-2)^2}{2(n-1)} \int_{\partial R^+} U(z,0)^{\frac{2(n-1)}{n-2}} dz \\
= \frac{n^2 - 2n}{2(n-1)^2} \omega_{n-1} I_{n-1}^n > 0.
\]

**Remark 13.** Notice that \( A \) is the energy level \( J_\infty(U) = \inf_{u \in H^1(R^+_n)} J_\infty(u) \), where \( J_\infty \) is the functional associated to the limit equation (15).

**Proof.** We expand in \( \delta \) the functional
\[
J_\varepsilon(W_{\delta,q} + \delta V_{\delta,q}) - \frac{1}{2} \int_M |\nabla g W_{\delta,q} + \delta \nabla g V_{\delta,q}|^2 d\mu_g + \frac{1}{2} \int_M a (W_{\delta,q} + \delta V_{\delta,q})^2 d\mu_g \\
+ \frac{1}{2} \int_{\partial M} \varepsilon \gamma (W_{\delta,q} + \delta V_{\delta,q})^2 d\sigma \\
- \frac{(n-2)^2}{2(n-1)} \int_{\partial M} \left[ (W_{\delta,q} + \delta V_{\delta,q})^{+} \right]^{\frac{2(n-1)}{n-2}} - (W_{\delta,q})^{\frac{2(n-1)}{n-1}} d\sigma \]
\[
- \frac{(n-2)^2}{2(n-1)} \int_{\partial M} (W_{\delta,q})^{\frac{2(n-1)}{n-1}} d\sigma = I_1 + I_2 + I_3 + I_4 + I_5.
\]

For the term \( I_2 \), by Remark 17 in the appendix, we have, by change of variables,
\[
I_2 = \frac{1}{2} \int_{R^+_n} \delta^2 \int_{R^+_n} \hat{a}(\delta y) (U(y)\gamma(\delta y) + \delta v_q(\delta y)\gamma(\delta y)) y^{1/2} dy \\
= \frac{1}{2} \int_{R^+_n} \delta^2 a(q) U(y)^2 dy + o(\delta^2) \\
= \frac{1}{2} \int_{R^+_n} \delta^2 a(q) \frac{n-2}{n-1} \omega_{n-1} I_{n-1}^n + o(\delta^2)
\]
(36)
in fact by Remark 17 we have
\[
\int_{R^+_n} U(y)^2 dy = \frac{1}{n-4} \omega_{n-1} I_{n-1}^{n-2} = \frac{2(n-2)}{(n-4)(n-1)} \omega_{n-1} I_{n-1}^n
\]
For the term $I_3$, recalling that $y = (z, t)$ with $z \in \mathbb{R}^{n-1}$, $t \geq 0$, we have, by Remark [17]

$$I_3 = \frac{\varepsilon \delta}{2} \int_{\mathbb{R}^{n-1}} \gamma(0, \delta z) \left( U(0, z) \chi(0, \delta z) + \delta \nu_q(0, z) \chi(0, \delta z)^2 \right) |g(0, \delta z)|^{1/2} dz$$

$$= \frac{\varepsilon \delta}{2} \gamma(q) \int_{\mathbb{R}^{n-1}} U(0, z)^2 dz + o(\varepsilon \delta) = \frac{\varepsilon \delta}{2} \gamma(q) \int_{0}^{\infty} \frac{1}{1 + |z|^2} dz$$

$$= \varepsilon \delta \gamma(q) \frac{n-2}{n-1} \omega_{n-1} I_{n-1}^{n-2}$$

(37)

For the term $I_5$, by [22] we have

$$I_5 = -\frac{(n-2)^2}{2(n-1)} \int_{\mathbb{R}^{n-1}} \delta y_2 U(0, z) (\chi(0, \delta z)) |g(0, \delta z)|^{1/2} dz$$

$$= -\frac{(n-2)^2}{2(n-1)} \int_{\mathbb{R}^{n-1}} U(0, z) \left( 1 - \frac{\delta^2}{6} \tilde{R}_{ij}(q) z_i z_j \right) dz + o(\delta^2);$$

by Remark [17] it holds

$$\int_{\mathbb{R}^{n-1}} U(0, z)^{\frac{2(n-1)}{n-3}} = \omega_{n-1} I_{n-1}^{n-2}$$

and, by symmetry reasons,

$$\tilde{R}_{ij}(q) \int_{\mathbb{R}^{n-1}} U(0, z)^{\frac{2(n-1)}{n-3}} z_i z_j dz = \frac{1}{n-1} \sum_{i=1}^{n-1} \tilde{R}_{ii}(q) \int_{\mathbb{R}^{n-1}} U(0, z)^{\frac{2(n-1)}{n-3}} z_i^2 dz$$

$$= \frac{\tilde{R}_{ii}(q)}{n-1} \int_{\mathbb{R}^{n-1}} |z|^2 dz$$

Thus, since $I_{n-1}^{n-2} = \frac{n-3}{n-1} I_{n-1}^{n-1}$ by Remark [17]

$$I_5 = -\frac{(n-2)^2}{2(n-1)} \left( I_{n-1}^{n-2} - \frac{\delta^2}{6(n-1)} \tilde{R}_{ii}(q) \omega_{n-1} I_{n-1}^{n-1} \right)$$

(38)

For the term $I_1$ we write

$$I_1 = \frac{1}{2} \int_M |W_\delta W_\delta q|^2 + \frac{1}{2} \int_M 2 \delta \nabla W_\delta q \nabla V_\delta q + \delta^2 |\nabla_q V_\delta q|^2 d\mu_q = I'_1 + I''_1 + I'''_1$$

and we proceed by estimating each term separately. By [22], [23], [28], we have (here $a, b = 1, \ldots, n$ and $i, j, m, l = 1, \ldots, n-1$)

$$I'_1 = \frac{1}{2} \int_{\mathbb{R}^n} g^a(\delta y) \frac{\partial}{\partial y_a} (U(y) \chi(\delta y)) \frac{\partial}{\partial y_b} (U(y) \chi(\delta y)) |g(\delta y)|^{1/2} dy$$

$$= \int_{\mathbb{R}^n} \left[ \frac{|U|^2}{2} + \left( \delta h_{ij} \frac{\partial}{\partial z_i} \delta h_{ik} \frac{\partial}{\partial z_k} \right) t_{zj} + \delta^2 \frac{\partial h_{ij}}{\partial z_k} t_{zk} + \frac{\delta^2}{2} \frac{1}{[1 + t^2 + |z|^2]^{\frac{n-1}{2}}} \right] \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j}$$

$$\times \left( 1 - \frac{\delta^2}{6} \frac{1}{[1 + t^2 + |z|^2]^{\frac{n-1}{2}}} \right) dz + o(\delta^2).$$

Since $\frac{\partial U}{\partial z_i} = (2-n) \frac{t z_i}{(1 + t^2 + |z|^2)^{\frac{n-1}{2}}}$, by symmetry reasons and since $h_{ii} \equiv 0$ we have that

$$h_{ij}(q) \int_{\mathbb{R}^n} \frac{t z_i}{(1 + t^2 + |z|^2)^{\frac{n-1}{2}}} dz = 0$$

$$\frac{\partial h_{ij}}{\partial z_k}(q) \int_{\mathbb{R}^n} t z_k \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} dz = (2-n) \frac{\partial h_{ij}}{\partial z_k}(q) \int_{\mathbb{R}^n} \frac{t z_k}{(1 + t^2 + |z|^2)^{\frac{n-1}{2}}} dz = 0;$$
in a similar way, using the symmetries of the curvature tensor one can check that
\[
\tilde{R}_{ijkl}(q) \int_{\mathbb{R}^n_+} z_i z_j U' U'' dz dt = \tilde{R}_{ijkl}(q) \int_{\mathbb{R}^n_+} z_i z_j z_k z_l dz dt = \frac{\alpha}{3} (R_{kkk}(q) + R_{kkk}(q) + R_{iiij}(q)) = 0
\]
where \(\alpha = \int_{\mathbb{R}^n_+} z_i z_j dz dt.\) Thus, using again symmetry

\[
I_1' = \int_{\mathbb{R}^n_+} \left[ \frac{\nabla U'^2}{2} + \frac{\delta^2}{2} [R_{lnjn} + 3h_{lk}b_{kj}] l^2 \right] \frac{\partial U}{\partial z} \frac{\partial U}{\partial z} dz dt + \frac{n-1}{2} \int_{\mathbb{R}^n_+} \frac{|z|^2 t^2 dz dt}{(1+t)^2 + |z|^2} + \frac{\delta^2}{4} \int_{\mathbb{R}^n_+} \frac{|z|^2 |\pi|^2 +\frac{\delta^2}{6} \tilde{R}_{lnjn}(0)z_l z_n dz dt + o(\delta^2)}{(n-2)^2 \int_{\mathbb{R}^n_+} (1+t)^2 + |z|^2}}
\]

Thus, by Remark 17

\[
I_1' = \frac{(n-2)\omega_{n-1} l^2}{2} + \frac{\delta^2}{2} \frac{(n-2)\omega_{n-1} l^2}{2} \frac{(n-2)(n-3)(n-4)}{\omega_{n-1} l^2} [\text{Ric}_q(q) + 3||\pi(q)||^2] \]

For the term \(I_1''\), by (22), (23), (24) and by definition of \(V_{\delta,q}\) and \(v_q\) we have

\[
I_1'' = \int_M \nabla W_{\delta,q} \nabla V_{\delta,q} \mu_g = \int_{\mathbb{R}^n_+} g^{a\beta}(\delta y) \frac{\partial}{\partial y_a} (U(y) \chi(\delta y)) \frac{\partial}{\partial y_\beta} (v_q(y) \chi(\delta y)) ||g(\delta y)||^{1/2} dy
\]

\[
= \int_{\mathbb{R}^n_+} (U''(q)) \frac{\partial}{\partial y_\beta} (v_q(q) \chi(\delta y)) ||g(\delta y)||^{1/2} dy + o(\delta^2)
\]

(40)
in fact

\[ \int_{\mathbb{R}_+^n} \nabla U \nabla v_q \, d\mu_g = - \int_{\mathbb{R}_+^n} U \Delta v_q \, d\mu_g + \int_{\partial \mathbb{R}_+^n} U(0, z) \frac{\partial v_q}{\partial t} \, dz \]

\[ = 2h_{ij} \int_{\mathbb{R}_+^n} U t \frac{\partial^2 U}{\partial z_i \partial z_j} - n \int_{\partial \mathbb{R}_+^n} U(0, z) \left( U(0, z) \frac{\partial v_q}{\partial t} \right) \, dz = 0 \]

since the first term is zero by symmetry and using that \( h_{ii} = 0 \), and the second term is zero by (18) and (26).

For the term \( I''_1 \), immediately we have

\[ I''_1 = \frac{\delta^2}{2} \int_M |\nabla V_{\delta,q}|^2 \, d\mu_g = \frac{\delta^2}{2} \int_{\mathbb{R}_+^n} |\nabla v_q|^2 \, dz \]

so

\[ I'_1 + I''_1 = \delta^2 2h_{ij} (q) \int_{\mathbb{R}_+^n} U t \frac{\partial U}{\partial z_i} \frac{\partial v_q}{\partial z_j} \, dz + \frac{\delta^2}{2} \int_{\mathbb{R}_+^n} |\nabla v_q|^2 \, dz + o(\delta^2) \]

For the term \( I_4 \), by (20) and (22), and recalling that \( y = (z, t) \) we have

\[ I_4 = - \frac{(n-2)^2}{2(n-1)} \int_{\partial \mathbb{R}_+^n} \left[ (U + \delta v_q)^+ \right] \frac{\partial (n-1)}{\partial \nu} - U \frac{\partial (n-1)}{\partial \nu} |g(0, \delta z)|^2 \, dz + o(\delta^2) \]

\[ = - \delta(n-2) \int_{\partial \mathbb{R}_+^n} U \frac{\partial v_q}{\partial z} \, dz - \delta^2 \frac{n}{2} \int_{\partial \mathbb{R}_+^n} ((U + \delta v_q)^+) \frac{\partial v_q}{\partial z} \, dz + o(\delta^2) \]

\[ = - \delta^2 \frac{n}{2} \int_{\partial \mathbb{R}_+^n} U \frac{\partial v_q}{\partial z} \, dz + o(\delta^2). \]

At this point we observe that

\[ 2h_{ij} (q) \int_{\mathbb{R}_+^n} \frac{\partial U}{\partial z_i} \frac{\partial v_q}{\partial z_j} \, dz - n \int_{\mathbb{R}_+^n} U \frac{\partial v_q}{\partial z} \, dz = - \int_{\mathbb{R}_+^n} |\nabla v_q|^2 \, dz 
\]

in fact, by (18) we get

\[ 2h_{ij} (q) \int_{\mathbb{R}_+^n} \frac{\partial U}{\partial z_i} \frac{\partial v_q}{\partial z_j} \, dz = - 2h_{ij} (q) \int_{\mathbb{R}_+^n} \frac{\partial^2 U}{\partial z_i \partial z_j} v_q \, dz = - \int_{\mathbb{R}_+^n} |\nabla v_q|^2 \, dz + \int_{\partial \mathbb{R}_+^n} v_q \frac{\partial v_q}{\partial \nu} \, dz \]

\[ = - \int_{\mathbb{R}_+^n} |\nabla v_q|^2 \, dz + n \int_{\partial \mathbb{R}_+^n} U \frac{\partial v_q}{\partial z} \, dz. \]

Hence by (12), (15), (16) and (18) it holds

\[ I'_1 + I''_1 + I_4 = \delta^2 \left( - \frac{1}{2} \int_{\mathbb{R}_+^n} |\nabla v_q|^2 \, dz + \frac{n}{2} \int_{\partial \mathbb{R}_+^n} U \frac{\partial v_q}{\partial z} \, dz \right) + o(\delta^2) \]

\[ = \frac{1}{2} \delta^2 \int_{\mathbb{R}_+^n} \Delta v_q v_q \, dz + o(\delta^2). \]
In light of [13], [37], [38], [39], [1], finally we get
\[
J_\varepsilon(W_{\delta,q} + \delta V_{\delta,q}) = \frac{(n-2)(n-3)}{2(n-1)^2} \omega_{n-1} R^n_{n-1} + \varepsilon \delta \gamma(q) \frac{n-2}{n-1} \omega_{n-1} I^n_{n-1} + \frac{1}{2} \delta^2 \int_{B^n_+} \Delta \nu_g v_q dzdt + \delta^2 a(q) \frac{n-2}{(n-1)(n-4)} \omega_{n-1} I^n_{n-1} - \delta^2 \frac{(n-2)^2}{4(n-1)^2} \omega_{n-1} I^n_{n-1} \left[2 \text{Ric}_q(q) + 2 \frac{n-4}{n-2} \|\pi(q)\|^2 + \bar{R}_4(q)\right] + o(\delta^2)
\]

Now, we choose \( \delta = \lambda \varepsilon \), where \( \lambda \in [\alpha, \beta] \), with \( \alpha, \beta \) Recalling that \( a = \frac{n-2}{4(n-1)} R_g \) and that \( R_g(q) = 2 \text{Ric}_q(q) + \bar{R}_4(q) + \|\pi(q)\|^2 \) (see [10]) we have the proof.

\[ \square \]

5. PROOF OF THEOREM 1

Lemma 14. If \((\tilde{\lambda}, \tilde{q}) \in (0, +\infty) \times \partial M\) is a critical point for the reduced functional
\[
I_\varepsilon(\lambda, q) := J_\varepsilon(W_{\epsilon \lambda,q} + \varepsilon \lambda V_{\epsilon \lambda,q} + \Phi_{\epsilon \lambda,q})
\]
then the function \( W_{\epsilon \lambda,q} + \varepsilon \lambda V_{\epsilon \lambda,q} + \Phi \) is a solution of (13). Here \( \Phi_{\epsilon \lambda,q} = \Phi_{\epsilon, \lambda, q} \) is defined in Proposition 1.

Proof. Set \( q = q(y) = \phi_{\frac{\partial}{\partial y}}(y) \). Since \((\tilde{\lambda}, \tilde{q})\) is a critical point for the \( I_\varepsilon(\lambda, q) \) we have, for \( h = 1, \ldots, n-1, \)
\[
0 = \left. \frac{\partial}{\partial y_h} I_\varepsilon(\lambda, q(y)) \right|_{y=0} = \langle W_{\epsilon \lambda,q}(y) + \varepsilon \lambda V_{\epsilon \lambda,q}(y) + \Phi_{\epsilon \lambda,q}(y), -i^* f(W_{\epsilon \lambda,q}(y)} + \varepsilon \lambda V_{\epsilon \lambda,q}(y) + \Phi_{\epsilon \lambda,q}(y)) \rangle + \varepsilon \gamma(W_{\epsilon \lambda,q}(y) + \varepsilon \lambda V_{\epsilon \lambda,q}(y) + \Phi_{\epsilon \lambda,q}(y), \frac{\partial}{\partial y_h} (W_{\epsilon \lambda,q}(y) + \varepsilon \lambda V_{\epsilon \lambda,q}(y) + \Phi_{\epsilon \lambda,q}(y))) \rangle \bigg|_{y=0} = n \sum_{i=1}^{n} c_i \langle Z^i_{\epsilon \lambda,q}(y), \frac{\partial}{\partial y_h} (W_{\epsilon \lambda,q}(y) + \varepsilon \lambda V_{\epsilon \lambda,q}(y) + \Phi_{\epsilon \lambda,q}(y)) \rangle \bigg|_{y=0} + \varepsilon \lambda \sum_{i=1}^{n} c_i \langle Z^i_{\epsilon \lambda,q}(y), \frac{\partial}{\partial y_h} V_{\epsilon \lambda,q}(y) \rangle \bigg|_{y=0} + \sum_{i=1}^{n} c_i \langle \frac{\partial}{\partial y_h} Z^i_{\epsilon \lambda,q}(y), \Phi_{\epsilon \lambda,q}(y) \rangle \bigg|_{y=0}
\]
using that \( \Phi_{\epsilon \lambda,q}(y) \) is a solution of (25) and that
\[
\langle Z^i_{\epsilon \lambda,q}(y), \frac{\partial}{\partial y_h} \Phi_{\epsilon \lambda,q}(y) \rangle \rangle = \langle \frac{\partial}{\partial y_h} Z^i_{\epsilon \lambda,q}(y), \Phi_{\epsilon \lambda,q}(y) \rangle \bigg|_{y=0}
\]
since \( \Phi_{\epsilon \lambda,q}(y) \in K^\perp_{\epsilon \lambda,q}(y) \) for any \( y \).

Arguing as in Lemma 6.1 and Lemma 6.2 of [19] we have
\[
\left\| \frac{\partial}{\partial y_h} Z^i_{\epsilon \lambda,q}(y) \right\|_H = O \left( \frac{1}{\varepsilon} \right) \quad \left\| \frac{\partial}{\partial y_h} W_{\epsilon \lambda,q}(y) \right\|_H = O \left( \frac{1}{\varepsilon} \right) \quad \left\| \frac{\partial}{\partial y_h} V_{\epsilon \lambda,q}(y) \right\|_H = O \left( \frac{1}{\varepsilon} \right)
\]
so we get
\[ \langle Z_{\varepsilon\lambda,q}(y), \partial_{\partial y} W_{\varepsilon\lambda,q}(y) \rangle H = \frac{1}{\lambda} \langle Z_{\varepsilon\lambda,q}(y), Z_{\varepsilon\lambda,q}(y) \rangle H + o(1) = \frac{\delta_{ih}}{\lambda} + o(1) \]
\[ \langle Z_{\varepsilon\lambda,q}(y), \partial_{\partial y} V_{\varepsilon\lambda,q}(y) \rangle H \leq \| Z_{\varepsilon\lambda,q}(y) \|_{H} \| \partial_{\partial y} V_{\varepsilon\lambda,q}(y) \|_{H} = O\left(\frac{1}{\varepsilon}\right) \]
\[ \| Z_{\varepsilon\lambda,q}(y) \| \leq \| \Phi_{\varepsilon\lambda,q}(y) \|_{H} = o(1). \]
We conclude that
\[ 0 = \frac{1}{\lambda} \sum_{i=1}^{n} c_{i}^{2} (\delta_{ih} + O(1)) \]
and so \( c_{i} = 0 \) for \( i = 1, \ldots, n \).

Analogously we proceed for \( \frac{\partial}{\partial x} I_{\varepsilon}(\lambda, \bar{q}) \mid_{\lambda=\bar{\lambda}}. \)

For the sake of completeness, we recall the definition of \( C^{0}\)-stable critical point before proving Theorem 1.

**Definition 15.** Let \( f : \mathbb{R}^{n} \to \mathbb{R} \) be a \( C^{1} \) function and let \( K = \{ \xi \in \mathbb{R}^{n} : \nabla f(\xi) = 0 \} \).
We say that \( \xi_{0} \in \mathbb{R}^{n} \) is a \( C^{0}\)-stable critical point if \( \xi_{0} \in K \) and there exist \( \Omega \) neighborhood of \( \xi_{0} \) with \( \partial \Omega \cap K = \emptyset \) and a \( \eta > 0 \) such that for any \( g : \mathbb{R}^{n} \to \mathbb{R} \) of class \( C^{1} \) with \( \| g - f \|_{C^{0}(\Omega)} \leq \eta \) we have a critical point of \( g \) near \( \Omega \).

**Proof of Theorem 1.** Let us call
\[ G(\lambda, q) = \lambda B\gamma(q) + \lambda^{2}\varphi(q). \]
If we find a \( C^{0}\)-stable critical point for \( G(\lambda, q) \) then we find a critical point for \( I_{\varepsilon}(\lambda, \bar{q}) := J_{\varepsilon}(W_{\lambda\varepsilon,q} + \lambda \varepsilon V_{\lambda\varepsilon,q} + \Phi) \) for \( \varepsilon \) small enough (see Lemma 11 and Proposition 12), hence a solution for Problem 12, by Lemma 14.

Since we assumed the trace-free second fundamental form to be nonzero everywhere, we have \( \| \pi \|^{2} > 0 \), so \( \varphi(q) < 0 \).

Also, we assumed \( \gamma(q) \) to be strictly positive on \( \partial M \), so there exists \( (\lambda_{0}, q_{0}) \) maximum point of \( G(\lambda, q) \) with \( \lambda_{0} > 0 \). Moreover, \( (\lambda_{0}, q_{0}) \) is a \( C^{0}\)-stable critical point of \( G(\lambda, q) \). Then, for any sufficiently small \( \varepsilon > 0 \) there exists \( (\lambda_{\varepsilon}, q_{\varepsilon}) \) critical point for \( I_{\varepsilon}(\lambda, q) \) and we completed the proof of our main result, in fact we found a sequence \( \lambda_{\varepsilon} \) bounded away from zero, a sequence of points \( q_{\varepsilon} \in \partial M \) and a sequence of positive functions
\[ u_{\varepsilon} = W_{\lambda_{\varepsilon},q_{\varepsilon}} + \lambda_{\varepsilon} \varepsilon V_{\lambda_{\varepsilon},q_{\varepsilon}} + \Phi \]
which are solution for 12 with \( q_{\varepsilon} \to q_{0}. \)

**Remark 16.** We give another example of function \( \gamma(q) \) such that problem 12 admits a positive solution. Let \( q_{0} \in \partial M \) be a maximum point for \( \varphi \). This point exists since \( \partial M \) is compact. Now choose \( \gamma \in C^{2}(\partial M) \) such that \( \gamma \) has a positive local maximum in \( q_{0} \). Then the pair \( (\lambda_{0}, q_{0}) = \left( -\frac{B\gamma(q_{0})}{2\varphi(q_{0})}, q_{0} \right) \) is a \( C^{0}\)-stable critical point for \( G(\lambda, q) \).

In fact, we have
\[ \nabla_{\lambda,q} G = (B\gamma + 2\lambda \varphi, \lambda B \nabla_{q} \gamma + \lambda^{2} \nabla_{q} \varphi) \]
which vanishes for \( (\lambda_{0}, q_{0}) = \left( -\frac{B\gamma(q_{0})}{2\varphi(q_{0})}, q_{0} \right) \). Moreover the Hessian matrix is
\[ G_{\lambda,q}^{\cdot\cdot} \left( -\frac{B\gamma(q_{0})}{2\varphi(q_{0})}, q_{0} \right) = \left( \begin{array}{cc}
2\varphi(q_{0}) & 0 \\
0 & -\frac{B\gamma(q_{0})}{2\varphi(q_{0})} \gamma_{q}^{\cdot\cdot}(q_{0}) + \frac{B^{2}\gamma(q_{0})}{2\varphi^{2}(q_{0})} \varphi^{\cdot\cdot}(q_{0})
\end{array} \right) \]
which is negative definite. Thus \((\lambda_0, q_0) = \left(-\frac{B\gamma(q_0)}{2\gamma(q_0)} , q_0\right)\) is a maximum \(C^0\)-stable point for \(G(\lambda, q)\).

### 6. Appendix

**Proof of Lemma 3** We argue by contradiction. We suppose that there exist two sequence of real numbers \(\varepsilon_m \to 0, \lambda_m \in [a, b]\) a sequence of points \(q_m \in \partial M\) and a sequence of functions \(\phi_{\varepsilon_m \lambda_m, q_m} \in K_{\varepsilon_m \lambda_m, q_m}\) such that
\[
\|\phi_{\varepsilon_m \lambda_m, q_m}\|_H = 1 \quad \text{and} \quad \|L_{\varepsilon_m \lambda_m, q_m}(\phi_{\varepsilon_m \lambda_m, q_m})\|_H \to 0 \text{ as } m \to +\infty.
\]

For the sake of simplicity, we set \(\delta_m = \varepsilon_m \lambda_m\) and we define
\[
\tilde{\phi}_m := \delta_m^{-\frac{n-2}{2}} \phi_{\delta_m, q_m}(\psi_{q_m}(\delta_m y)) for \(y = (z, t) \in \mathbb{R}_+^n\), with \(z \in \mathbb{R}_+^{n-1}\) and \(t \geq 0\)
\]
Since \(\|\phi_{\varepsilon_m \lambda_m, q_m}\|_H = 1\), by change of variables we easily get that \(\left\{\tilde{\phi}_m\right\}_m\) is bounded in \(D^{1,2}(\mathbb{R}_+^n)\) (but not in \(H^1(\mathbb{R}_+^n)\)). Thus there exists \(\tilde{\phi} \in D^{1,2}(\mathbb{R}_+^n)\) such that \(\tilde{\phi}_m \rightharpoonup \tilde{\phi}\) weakly in \(D^{1,2}(\mathbb{R}_+^n)\), in \(L_{\text{loc}}^{2(n-1)}(\partial \mathbb{R}_+^n)\) and in \(L_{\text{loc}}^{\frac{2n}{n-2}}(\partial \mathbb{R}_+^n)\), strongly in \(L^2(\partial \mathbb{R}_+^n)\) for \(s \leq \frac{2(n-1)}{n}\) and almost everywhere.

Since \(\phi_{\delta_m, q_m} \in K_{\delta_m, q_m}\), and taking in account \(19\) we get, for \(i = 1, \ldots, n\),
\[
(48) \quad o(1) = \int_{\mathbb{R}_+^n} \nabla \tilde{\phi} \nabla j_i dz dt = n \int_{\partial \mathbb{R}_+^n} U^\frac{n}{n-2}(z, 0) j_i(z, 0) \tilde{\phi}(z, 0) dz.
\]

Indeed, by change of variables we have
\[
0 = \left\langle \phi_{\delta_m, q_m}, Z_{\delta_m, q_m}\right\rangle_H = \int_M (\nabla_{\partial M} \phi_{\delta_m, q_m} \nabla_g Z_{\delta_m, q_m} + \alpha \phi_{\delta_m, q_m} Z_{\delta_m, q_m}) \, d\mu_g
\]
\[
= \int_{\mathbb{R}_+^n} \left[ \frac{\partial}{\partial \eta_x} j_i(y) \frac{\partial}{\partial \eta_x} \phi_{\delta_m, q_m}(\psi_{q_m}(\delta_m y)) \right] dy + \int_{\mathbb{R}_+^n} \eta a(\psi_{q_m}(\delta_m y)) j_i(y) \phi_{\delta_m, q_m}(\psi_{q_m}(\delta_m y)) dy + o(1)
\]
\[
= \int_{\mathbb{R}_+^n} \nabla j_i(y) \nabla \phi_{\delta_m}(y) + \delta^2 a(q_m) j_i(y) \phi_{\delta_m}(y) dy + o(1)
\]
\[
= \int_{\mathbb{R}_+^n} \nabla j_i(y) \nabla \tilde{\phi}(y) + o(1).
\]

By definition of \(L_{\delta_m, q_m}\) we have
\[
(49) \quad \phi_{\delta_m, q_m} = i^* \left( f^*(W_{\delta_m, q_m} + \delta_m V_{\delta_m, q_m})[\phi_{\delta_m, q_m}] - L_{\delta_m, q_m} \phi_{\delta_m, q_m} \right) = \sum_{i=1}^n c_i Z_{\delta_m, q_m}^i.
\]

We want to prove that, for all \(i = 1, \ldots, n\), \(c_i \to 0\) while \(m \to \infty\). Multiplying equation \((49)\) by \(Z_{\delta_m, q_m}^i\) we obtain, by definition \((13)\) of \(i^*\),
\[
\sum_{i=1}^n c_i \left\langle \left\langle Z_{\delta_m, q_m}^i, Z_{\delta_m, q_m}^j \right\rangle_H \right\rangle_H = \left\langle \left\langle i^* \left( f^*(W_{\delta_m, q_m} + \delta_m V_{\delta_m, q_m})[\phi_{\delta_m, q_m}] \right), Z_{\delta_m, q_m}^j \right\rangle_H \right\rangle_H = \int_{\partial M} f^*(W_{\delta_m, q_m} + \delta_m V_{\delta_m, q_m})[\phi_{\delta_m, q_m}] Z_{\delta_m, q_m}^j d\sigma
\]
Now
\[
\int_{\partial M} f'(W_{\delta_m,q_m} + \delta_m V_{\delta_m,q_m})[\phi_{\delta_m,q_m}]Z_{\delta_m,q_m}^k d\sigma
= n \int_{\partial M} \left((W_{\delta_m,q_m} + \delta_m V_{\delta_m,q_m})^+\right) \frac{\delta}{\sqrt{\delta}} \phi_{\delta_m,q_m} Z_{\delta_m,q_m}^k d\sigma
\]
\[
= n \int_{\partial R^n_+} \left((U + \delta_m v_{q_m})^+\right) \frac{\delta}{\sqrt{\delta}} \phi_m j d\sigma + o(1) = n \int_{\partial R^n_+} \left(U\right) \frac{\delta}{\sqrt{\delta}} \phi_m j d\sigma + o(1) = o(1)
\]
since \(\tilde{\phi}_m \to \tilde{\phi}\) weakly \(L^{2n/(n+1)}(\partial R^n_+)\), \(\|v_{q_m}\|_{L^\infty}\) is bounded independently on \(q_m\) by (49) and by equation (48). At this point, since
\[
\left\langle \left\langle Z_{\delta_m,q_m}^1, Z_{\delta_m,q_m}^j \right\rangle \right\rangle = C\delta_{ij} + o(1),
\]
we conclude that \(c'_{m} \to 0\) while \(m \to \infty\) for each \(i = 1, \ldots, n\). By (49) and recalling \(\|L_{\epsilon_m,\lambda_m,q_m}(\phi_{\epsilon_m,\lambda_m,q_m})\|_H \to 0\) this implies
\[
(50) \quad \|\phi_{\delta_m,q_m} - \ast^i (f'_{\epsilon_m}(W_{\delta_m,q_m} + \delta_m V_{\delta_m,q_m})[\phi_{\delta_m,q_m}])\|_H
= \sum_{i=0}^{n-1} c'_{m}\|Z^i\|_H + o(1) = o(1)
\]
Now, choose a smooth function \(\varphi \in C_0^\infty(R^n_+)\) and define
\[
\varphi_m(x) = \frac{1}{\delta_m^{-n/2}} \varphi \left(\frac{1}{\delta_m} \left(\psi_{q_m}\right)^{-1}(x)\right) \chi \left(\left(\psi_{q_m}\right)^{-1}(x)\right)
\text{ for } x \in M.
\]
We have that \(\|\varphi_m\|_H\) is bounded and, by (50), that
\[
\left\langle \left\langle \phi_{\delta_m,q_m}, \varphi_m \right\rangle \right\rangle = H = \int_{\partial M} f'_{\epsilon_m}(W_{\delta_m,q_m} + \delta_m V_{\delta_m,q_m})[\phi_{\delta_m,q_m}]\varphi_m d\sigma
\]
\[
+ \left\langle \left\langle \phi_{\delta_m,q_m} - \ast^i \left(f'_{\epsilon_m}(W_{\delta_m,q_m} + \delta_m V_{\delta_m,q_m})[\phi_{\delta_m,q_m}]\right), \varphi_m \right\rangle \right\rangle_\partial R^n_+
\]
\[
= \int_{\partial M} f'_{\epsilon_m}(W_{\delta_m,q_m} + \delta_m V_{\delta_m,q_m})[\phi_{\delta_m,q_m}]\varphi_m d\sigma + o(1)
\]
\[
= n \int_{\partial R^n_+} \left((U + \delta_m v_{q_m})^+\right) \frac{\delta}{\sqrt{\delta}} \phi_m j d\sigma + o(1)
\]
by the strong \(L^t_{\text{loc}}(\partial R^n_+)\) convergence of \(\tilde{\phi}_m\) for \(t < \frac{2(2n+1)}{n+2}\). On the other hand
\[
\left\langle \left\langle \phi_{\delta_m,q_m}, \varphi_m \right\rangle \right\rangle_\partial R^n_+ = \int_{R^n_+} \nabla \tilde{\phi} \nabla \varphi d\eta + o(1),
\]
so \(\tilde{\phi}\) is a weak solution of (16) and we conclude that
\[
\tilde{\phi} \in \text{Span } \{j_1, \ldots, j_n\}.
\]
This, combined with (18) gives that \(\tilde{\phi} = 0\). Proceeding as before we have
\[
\left\langle \left\langle \phi_{\delta_m,q_m}, \phi_{\delta_m,q_m} \right\rangle \right\rangle = \int_{\partial M} f'_{\epsilon_m}(W_{\delta_m,q_m} + \delta_m V_{\delta_m,q_m})[\phi_{\delta_m,q_m}][\phi_{\delta_m,q_m}] d\sigma + o(1)
\]
\[
= n \int_{\partial R^n_+} \left((U + \delta_m v_{q_m})^+\right) \frac{\delta}{\sqrt{\delta}} \phi_m j d\sigma + o(1)
\]
\[
= n \int_{\partial R^n_+} U \frac{\delta}{\sqrt{\delta}} \phi_m j d\sigma + o(1) = o(1)
\]
and, by explicit computation, by the previous formula, we obtain:

\[ I_m^\alpha := \int_0^\infty \frac{\rho^\alpha}{(1 + \rho^2)^m} d\rho = \frac{2m}{\alpha + 1} I_{m+1}^{\alpha+2} \text{ for } \alpha + 1 < 2m \]

\[ I_m^\alpha = \frac{2m}{\alpha - 1} I_{m+1}^\alpha \text{ for } \alpha + 1 < 2m \]

\[ I_m^\alpha = \frac{2m}{\alpha - 3} I_{m+2}^{\alpha+2} \text{ for } \alpha + 3 < 2m. \]

In particular we have

\[ I_n^0 = \frac{n-3}{2(n-1)} I_{n-1}^0, \quad I_{n-1}^{n-2} = \frac{n-3}{n-1} I_{n-1}^n, \quad I_{n-2}^{n-2} = \frac{2(n-2)}{n-1} I_{n-1}^n. \]

Moreover, for \( m > k + 1, m, k \in \mathbb{N} \), we have

\[
\int_0^\infty \frac{t^k}{(1 + t)^m} dt = \frac{k!}{(m-1)(m-2)\cdots(m-k-1)}
\]

and, by explicit computation, by the previous formula, we obtain:

\[
\int_{\mathbb{R}^n_+} \frac{dz dt}{[(1 + t)^2 + |z|^2]^{m-1}} = \frac{\omega_{n-1} I_{n-1}^{n-2}}{(n-2)}
\]

\[
\int_{\mathbb{R}^n_+} \frac{|z|^2 t^2 dz dt}{[(1 + t)^2 + |z|^2]^{m-1}} = \frac{2\omega_{n-1} I_{n-1}^n}{(n-2)(n-3)(n-4)}
\]

\[
\int_{\mathbb{R}^n_+} \frac{t^2 dz dt}{[(1 + t)^2 + |z|^2]^{m-1}} = \frac{2\omega_{n-1} I_{n-1}^{n-2}}{(n-2)(n-3)(n-4)}
\]

\[
\int_{\mathbb{R}^n_+} \frac{|z|^2 dz dt}{[(1 + t)^2 + |z|^2]^{m-1}} = \frac{\omega_{n-1} I_{n-1}^n}{(n-4)}
\]

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