Photonic Wheels and Polarization Möbius Strips in Highly-Confined Trigonometric Beams

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Abstract—A photonic wheel describes a special structure in which the electric vector spins (with time) in a plane containing the propagation direction, while around a C-point (a point with circular polarization) the polarization Möbius strip can be formed topologically in three-dimensional (3D) space. These two interesting but very different structures are both unique to 3D structured fields, and in this article they are studied together and also their connections are analyzed. By highly confining the trigonometric beams, the three-dimensional (3D) fields with a wide area of photonic wheels are constructed, and the characteristics of the photonic wheels in different regions are analyzed. The expression for the C-lines of the photonic wheels is derived, which provides a simple way to get the exact position of the C-lines and also is a basis for observing polarization Möbius strips. Along a C-line, the behaviors of the polarization Möbius strips, including two typical (generic) Möbius strips with opposite indices, and two special topological events seemingly violating the topological law are examined. Our results show that the trick and the paradox in the two special events actually are the manifestations of the trace-dependent property and the observer-dependent property of the polarization Möbius strips respectively. The finding also suggests that even in a short segment of a C-line rich topological events related to polarization Möbius strips can be observed. Our research provides a way to observe the photonic wheels, the polarization Möbius strips and their connection, also supplies a theoretical foundation for special structures in 3D fields.

Index Terms—Photonic wheel, spin density vector, polarization Möbius strip, spin angular momentum, orbital angular momentum.

I. INTRODUCTION

THREE-dimensional (3D) structured fields have been studied extensively within the last decades [1], with the research areas ranging from the classical optics, such as high-resolution imaging [2], [3], to quantum information technologies [4], [5]. In these areas, spin angular momentum (SAM), orbital angular momentum (OAM) and spin-orbit interactions (SOIs) play crucial roles since they enrich the structures of the 3D fields and extend the freedom of the manipulations [6], [7], [8], [9], [10]. The SAM density vector (spin density vector for short) is a feature physical quantity in 3D structured fields measuring both the density of SAM and the polarization states in 3D space [11], [12]. Studies on spin density vectors are growing rapidly recently because their important optical effects in nanoscale [13], [14], such as in controlling the light-matter interactions [15]. Also, the structures of the spin density vectors in 3D space are of special interest to help understand the physical nature of 3D optical fields, for instance the spiral twisting structures of spin density vectors [16], [17], [18], [19], [20], [21], which has been found to be a result of SOIs [16], [17], [20], and more specially, the purely transverse structures of spin density vectors which lead to an interesting optical phenomenon—‘photonic wheels’ [22], [23], [24], [25], [26], [27], [28], [29], [30], implying that light can spin in the plane with the propagation direction like a wheel [23], [26], [31]. It has been proved that the photonic wheels not only are fundamentally meaningful, such as their strong connection with the geometrical spin Hall effect of light [32], [33], [34], [35] but also have various applications, for instance in observing the SOIs [36], in on-chip and interchip optical circuitry and quantum computing [22], [23], [26]. Constructing optical field with photonic wheels is a key step in all of these related researches, and a lot of methods have been proposed and investigated [22], [25], [27], [28], [29]. Among these methods, the most convenient way may be to tailor field parameters of the incident beam in a highly-confined field, such as to tailor the polarization [22], [25], [27] and the amplitude and/or phase of the incident beams [28], [29]. While, in these methods, the tailored beams usually have one certain structure, which is not flexible in practice. In this article, we will propose a group of tailored beams with different orders as the incident beams in a high numerical aperture (NA) system, and demonstrate that any order of them can have rich structures of photonic wheels. Additionally, we will also explore the connection of the photonic wheels to a very peculiar topological structure—the polarization Möbius strips in our constructed fields.

The 3D structured fields, from the topological view, also can exhibit marvelous topological structures, such as the phase/polarization knots [37], [38], the topological 3D skyrmionic hopfin [39], twisted ribbons [40], [41], [42], [43], while the most interesting one may be the polarization Möbius strip [27], [40], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55]. Since it was first proposed, a lot of effort has gone into observing the polarization Möbius strip [27], [50], [53],...
[54], finding the same structures in other types of waves [56], and exploring its topological features [40], [45], [46], [47], [48], [49], [51], [55], for instance the topological trace-dependent property of the polarization Möbius strip [47], [53], [55], and the topological observer-dependent property [48], [51]. It is also interesting to see that the polarization Möbius strip can become a ‘Möbius-like topology’ due to a point with photonic wheel (i.e. purely transverse spin density vector) [27]. That raises a question: if the observed field is full of photonic wheels, how does the Möbius strip behave along the C-lines with photonic wheels? Further, in this case, can the typical polarization Möbius strips and their related special topological events be observed along one C-line? In this article, we will try to answer these questions and unveil new properties of the polarization Möbius strips.

II. THEORY

A. Trigonometric Beam and Spin Density Vector

A trigonometric beam, is a kind of optical beam with the complex amplitude described by a trigonometric function [57], [58]. For a Gaussian type trigonometric beam, its profile can be simply written as

$$U_m(r, \phi) = r^me^{-\rho^2/w_0^2}\cos m\phi,$$  \hspace{1cm} (1)

where $m (m \in \mathbb{N})$ denotes the order of the beam, $\rho = \sqrt{x^2 + y^2}$ and $\phi = \arg[y/x]$ are the radial direction and azimuthal direction respectively. $w_0$ represents the beam width. Examples of the trigonometric beams described by (1) are shown in Fig. 1, where the first row shows the intensity distribution and the second row displays the phase distributions. Here $m$ is chosen as 1 in the left column, 2 in the middle column and 3 in the right column. One can see that the intensity pattern has a flower-shape with $2m$ folds, and along each division-line of the folds there is an edge-type dislocation of the phase, i.e. a line of phase singularities, where the phase undergoes a $\pi$ shift (see the phase plots). In plots (d) (e) and (f) of Fig. 1, the white lines denote these phase dislocations.

Since the interesting intensity patterns, the trigonometric beams sometimes are also called as the amplitude-tailored beams [29]. From the view of OAM, the trigonometric beams can be treated as a combination of two vortex beams with opposite topological charge, for instance, the $U_2$ can be obtained by the superposition of the Gaussian vortex beams $\rho^2e^{-\rho^2/w_0^2}e^{2i\phi}$ and $\rho^2e^{-\rho^2/w_0^2}e^{-2i\phi}$.

For the trigonometric beams above, we only discuss the amplitude and phase of the electric field in 2D scalar fields. While, the electric field of a general optical wave is 3D, $E = (e_x, e_y, e_z)$, which implies that besides its amplitude and phase, it also has the polarization state. Usually, a 3D fully polarized electric field can be expressed by the generalized Stokes parameters, as [16], [59], [60]

$$\begin{align*}
\Lambda_0 &= |e_x|^2 + |e_y|^2 + |e_z|^2, \\
\Lambda_1 &= 3|e_x||e_y|\cos \phi_{yx}, \\
\Lambda_2 &= 3|e_x||e_y|\sin \phi_{yx}, \\
\Lambda_3 &= \frac{3}{2}(|e_x|^2 - |e_y|^2), \\
\Lambda_4 &= 3|e_x||e_z|\cos \phi_{xz}, \\
\Lambda_5 &= 3|e_x||e_z|\sin \phi_{xz}, \\
\Lambda_6 &= \frac{3\sqrt{2}}{2}(|e_x|^2 + |e_y|^2 - 2|e_z|^2), \\
\Lambda_7 &= \frac{e_0}{4\omega} \text{Im}(E^* \times E), \\
\Lambda_8 &= \frac{e_0}{6\omega} |e_x||e_y|\sin \phi_{zy},
\end{align*}$$  \hspace{1cm} (2)

where $\phi_{ij} = \arg[e_i] - \arg[e_j]$ is the phase difference between two field components. (Note: here we adopt $\phi_{zx}$ instead of $\phi_{xz}$ in [16], [59].) These 9 generalized Stokes parameters describe the amplitude, phase and polarization of the electric field in 3D space. If only the 3D polarization states are discussed, another physical quantity can be used for the description. That is the spin density vector of electric field, $s_E$, with its absolute value reflecting the shape of the polarization ellipse and its direction showing the orientation and the handedness of the polarization ellipse. The spin density vector is defined as [12], [16], [23]

$$s_E = \begin{pmatrix}
\frac{e_0}{4\omega} \text{Im}(E^* \times E) \\
\frac{e_0}{6\omega} |e_x||e_y|\sin \phi_{zy}
\end{pmatrix},$$  \hspace{1cm} (3)

where $e_0$ denotes the permittivity of free space and $\omega$ is the angular frequency. Im and $^*$ represent the imaginary part and the complex conjugate, respectively. The spin density vector has a close relation with the generalized Stokes parameters, which can be expressed as

$$s_E = \begin{pmatrix}
\frac{e_0}{2\omega} (|e_y||e_z|\sin \phi_{zy} - |e_x||e_z|\sin \phi_{xz}) \\
|e_x||e_y|\sin \phi_{zy}
\end{pmatrix} = \begin{pmatrix}
\Lambda_7 \\
\Lambda_8
\end{pmatrix}.$$  \hspace{1cm} (4)

B. Constructed 3D Vector Fields

In this part, we will use the trigonometric beams to construct 3D vector fields in a high numerical aperture (NA) system.

We first consider an aplanatic, high NA focusing system with the semi-aperture angle $\alpha$ and the focal length $f$, which can be seen in Fig. 2. The aplanatic system means the Abbe sine condition $\rho = f \sin \theta$ and the apodization function $\sqrt{\cos \theta}$. Assume that the trigonometric beam expressed by (1) is $x$-polarized and incident upon this focusing system with the beam waist coincident with the entrance plane. According to the Richards-Wolf vectorial diffraction theory [61], the 3D vector field can be
constructed in the focal region, and also by applying the Abbe sine condition and the apodization function, the electric field of this 3D vector field at an observation point \(P(\rho_s, \phi_s, z_s)\) can be written as

\[
E^{(f)}(\rho_s, \phi_s, z_s) = \begin{bmatrix} e_x^{(f)} \\ e_y^{(f)} \\ e_z^{(f)} \end{bmatrix} = \frac{ik}{2\pi} \int_0^\alpha \int_0^{2\pi} U_m(\rho, \phi) \sqrt{\cos \theta} \sin \theta e^{ikz_s \cos \theta} \times \left[ \begin{array}{c} \cos \theta + \sin^2 \phi (1 - \cos \theta) \\ (\cos \theta - 1) \cos \phi \sin \phi \\ -\sin \theta \cos \phi \end{array} \right] e^{ik\rho_s \sin \theta \cos(\phi - \phi_s)} d\phi d\theta, \tag{5} \]

where \(k\) is the wave number with \(k = 2\pi/\lambda\) (\(\lambda\) is the wavelength) and \((\rho_s, \phi_s, z_s)\) are the cylindrical coordinates in image space \((\rho_s = \sqrt{x_s^2 + y_s^2})\). After integrating with respect to \(\phi_s\), we can get

\[
e_x^{(f)}(\rho_s, \phi_s, z_s) = -ik \int_0^\alpha \sqrt{\cos \theta} (f \sin \theta)^{m+1} e^{-f^2 \sin^2 \theta / w_0^2} \times \left( I_x^{(m)} + I_x^{(m+2)} + I_x^{(m-2)} \right) e^{ikz_s \cos \theta} d\theta, \tag{6} \]

\[
e_y^{(f)}(\rho_s, \phi_s, z_s) = -ik \int_0^\alpha \sqrt{\cos \theta} (f \sin \theta)^{m+1} e^{-f^2 \sin^2 \theta / w_0^2} \times \left( I_y^{(m+2)} + I_y^{(m-2)} \right) e^{ikz_s \cos \theta} d\theta, \tag{7} \]

\[
e_z^{(f)}(\rho_s, \phi_s, z_s) = -ik \int_0^\alpha \sqrt{\cos \theta} (f \sin \theta)^{m+1} e^{-f^2 \sin^2 \theta / w_0^2} \times \left( I_z^{(m+1)} + I_z^{(m-1)} \right) e^{ikz_s \cos \theta} d\theta, \tag{8} \]

with

\[
I_x^{(m)} = \frac{im}{2} (1 + \cos \theta) \cos m \phi_s J_m(k \rho_s \sin \theta), \tag{9} \]

\[
I_x^{(m+2)} = \frac{im}{4} (1 - \cos \theta) \cos (m + 2) \phi_s J_{m+2}(k \rho_s \sin \theta), \tag{10} \]

\[
I_x^{(m-2)} = \frac{im}{4} (1 - \cos \theta) \cos (m - 2) \phi_s J_{m-2}(k \rho_s \sin \theta), \tag{11} \]

\[
I_y^{(m+2)} = \frac{im}{4} (1 - \cos \theta) \sin (m + 2) \phi_s J_{m+2}(k \rho_s \sin \theta), \tag{12} \]

\[
I_y^{(m-2)} = \frac{im}{4} (1 - \cos \theta) \sin (m - 2) \phi_s J_{m-2}(k \rho_s \sin \theta), \tag{13} \]

\[
I_z^{(m+1)} = \frac{im+1}{2} \sin \theta \cos (m + 1) \phi_s J_{m+1}(k \rho_s \sin \theta), \tag{14} \]

\[
I_z^{(m-1)} = \frac{im-1}{2} \sin \theta \cos (m - 1) \phi_s J_{m-1}(k \rho_s \sin \theta), \tag{15} \]

where \(J_n(x)\) is the Bessel function of first kind with order \(n\). In the following, based on the expressions of this 3D electric field, we will discuss the photonic wheels and the polarization Möbius strips.

III. RESULTS AND DISCUSSIONS

A. Photonic Wheels and Purely Transverse Spin Density Vectors

Photonic wheels refer to a special 3D polarization state, in which the polarization ellipse lies in the plane with propagation direction (rather than the plane perpendicular to the propagation direction), and the electric vector spins around a transverse axis with time in analogy to a rolling mechanical wheel. Thus, applying the expressions of generalized Stokes parameters, (2), the photonic wheels can be generated if the conditions, \(A_2 = 0\) and \(|A_7| + |A_8| \neq 0\) are satisfied. From (4), these conditions are equivalent to null longitudinal component of the spin density vector, i.e. \(s_{E}^{(z)} = 0\). So, the photonic wheels also can be treated as the purely transverse spin density vectors in a 3D optical field. In the 3D vector fields constructed by the trigonometric beams, as it will be shown, the photonic wheels are formed in a very wide area. This area can be generally divided into three regions as follows.

Region 1: For any value of \(m\), the field with \(\phi_s = 0, \pi\) (i.e. on the \(x, Oz_s\) plane) carries photonic wheels. From (6–15), one can find that if \(\phi_s = 0, \pi\) (for any value of \(m\)), \(e_y^{(f)} = 0\), which means that the polarization ellipses of the field on the \(x, Oz_s\) plane are parallel to the propagation direction, i.e. the photonic wheels are formed on this plane. By substituting \(e_y^{(f)} = 0\) into the expression of the spin density vector, (4), we get \(s_{E}^{(z)} = 0\) (with \(s_{E}^{(x)} = 0\) and \(s_{E}^{(y)} \neq 0\)), which shows that the spin density vectors for \(\phi_s = 0, \pi\) are purely transverse. The photonic wheels (i.e. polarization ellipses) on the \(x, Oz_s\) plane are shown in Fig. 3, where (a) (b) (c) are plots for the beams with order \(m = 1, m = 2\) and \(m = 3\) respectively. (d) depicts the relation of the spin density vector and the polarization ellipse on the \(x, Oz_s\) plane. In this figure the spin density vectors always point to the \(y_s\)-axis. Note that in order to show more shapes of photonic wheels in Fig. 3, the positions of the ellipses in plots (a), (b) and (c) are drawn from \((-2.1\lambda, -2.1\lambda)\) to \((1.9\lambda, 1.9\lambda)\) with an interval \(0.4\lambda\), i.e., the distribution of these ellipses is not symmetric with respect to the \(z_s\) axis or with respect to the \(x_s\) axis.
which are very different from the photonic wheels in region 1 (on the \(x_zO_z\) plane).

Region 3: When \(m\) is odd, the photonic wheels can be formed on the plane with \(\phi_s = \pm \pi/2\) (\(y_zO_z\) plane). For \(\phi_s = \pm \pi/2\), from (6)–(15), it can be got that \(e_{z} = 0\) with odd values of \(m\). So the electric vectors spin on the \(y_{z}O_{z}\) plane, and from (4) it also means that \(s_{E}^{(x)} = 0\) (with \(s_{E}^{(y)} \neq 0\) and \(s_{E}^{(z)} \neq 0\)), i.e. the spin density vectors are purely transverse on the \(y_{z}O_{z}\) plane. When \(m\) is even, similarly, from (6)–(15) and (4), we can calculate that the photonic wheels can be formed on the plane with \(\phi_s = \pm N_1 \pi/(m+2) = \pm N_2 \pi/(m-2)\) \((N_1, N_2 \in \mathbb{N})\), where \(N_1\) and \(N_2\) satisfy the condition \(0 < N_1/(m-2) < 1\) and \(0 < N_1/(m+2) < 1\). This condition leads to a result that the even values of \(m\) only can be 2 and 4. So, for \(m = 2, 4\) the photonic wheels for \(\pm \pi/2\) become lines since \(e_{z} = 0\) too, which also means the spin density is null there. The photonic wheels in region 3 are similar to those in region 1 which can be produced by using the same method, so that here those photonic wheels will no more be shown.

B. Polarization Möbius Strips of Photonic Wheels

In this part, we will examine the polarization Möbius strips of the photonic wheels. As it is discussed above, there are three main regions existing rich photonic wheels, while in region 1 and region 3 the photonic wheels lie in the planes carrying them (i.e. the photonic wheels are parallel to these planes), which will lead to a result that the topological structure of a C-point of the photonic wheels will be 2D, i.e. the structure there will be Lemon-, Monstar- or Star-pattern rather than a Möbius strip. While, the photonic wheels in region 2 (on the focal plane) exhibit different features, i.e., they are perpendicular to the focal plane, which can lead to 3D topological structures, such as the Möbius strips. So, here we will focus on the photonic wheels on the focal plane and analyze the polarization Möbius strips there.

In order to observe the polarization Möbius strips, we need first to find the C-points. The C-points in 3D space, actually exhibit in lines, i.e. the C-lines. To calculate the exact position of a line usually is complicated, since the phases and the intensities of three field components need to satisfy the C-line condition at the same time (i.e. six equations are needed). Fortunately, because the special phase relations between the components of the field with the photonic wheels, here we can derive a simple condition for the C-lines.

Assume there is a 3D vector field \(E^v\), and we can use its real part \(P^v\) and imaginary part \(Q^v\) to express it, as [11]
\[
E^v = P^v + iQ^v.
\]
A complex scalar field \(\varphi^v\) based on \(E^v\) can be defined as [11]
\[
\varphi^v \equiv E^v \cdot \bar{E}^v = (P^v)^2 - (Q^v)^2 + i2P^v \cdot Q^v.
\]
Thus, according to their definition, the C-lines can be found if the following condition is satisfied:
\[
\varphi^v = 0,
\]

Fig. 3. Photonic wheels on the \(x_zO_z\) plane.

Fig. 4. Purely transverse spin density vectors on the focal plane.
which implies that both \((P^v)^2 - (Q^v)^2 = 0\) and \(P^v \cdot Q^v = 0\). While this condition is still not easy to solve. However, since the existence of the photonic wheels, we can get that the real part and the imaginary part of \(E^v\) actually correspond to the longitudinal component and the transverse component of the field on the focal plane, respectively, i.e.

\[
P^v = \tilde{x}e_x^{(f)} ,
\]

\[
Q^v = \tilde{x}e_x^{(f)} + \tilde{y}e_y^{(f)} ,
\]

where \(\tilde{x}, \tilde{y}\) and \(\tilde{z}\) represent the unit vectors along \(x_s, y_s\) and \(z_s\) axis respectively. Since the perpendicular relation between the longitudinal component and the transverse component, this can be got immediately:

\[
P^v \cdot Q^v = 0 .
\]

Thus, on the focal plane the condition for C-lines becomes:

\[
(P^v)^2 - (Q^v)^2 = 0 ,
\]

and by substituting (19) and (20) into above equation, we can obtain

\[
|e_x^{(f)}|^2 + |e_y^{(f)}|^2 - |e_z^{(f)}|^2 = 0 ,
\]

which is the condition for C-lines on the focal plane with photonic wheels. (23) indicates that we only need to calculate the intensity to find the position of C-lines instead of the complicated calculations of the phases and the amplitudes for three field components. Fig. 5 shows the C-lines on the focal plane drawn by using (23), and it is also can be seen that the distribution of C-lines resembles the symmetric distribution of the spin density vectors (Fig. 4) of the same order \(m\).

The topological feature of a C-point in 3D vector fields can be deciphered by the structure of the major axes \(\alpha\) (or the minor axes \(\beta\) of the polarization ellipses around this point in an observation plane, and in this focused field \(\alpha\) and \(\beta\) can be defined as [44], [45]

\[
\alpha(\rho_s, \phi_s, z_s) = \frac{1}{|\sqrt{E^{(f)} \cdot E^{(f)*}}|} \text{Re} \left( E^{(f)} \sqrt{E^{(f)* \cdot E^{(f)*}}} \right) ,
\]

\[
\beta(\rho_s, \phi_s, z_s) = \frac{1}{|\sqrt{E^{(f)} \cdot E^{(f)*}}|} \text{Im} \left( E^{(f)} \sqrt{E^{(f)* \cdot E^{(f)*}}} \right) ,
\]

where \(\text{Re}(u)\) means the real part of \(u\) and \(E^{(f)*}\) denotes the complex conjugate of \(E^{(f)}\). Here we adopt the major axis \(\alpha\) to observe the topological structure of a C-point.

Since the topological structure of a Möbius strip is observer-dependent [48], [51], it is also necessary to determine the observing angle, in other words, the observation plane. Here, we define the observation plane as the plane of the polarization ellipse of the C-point, i.e. the plane with its normal direction coincident with the direction of the spin density vector there, see Fig. 6. In this figure, the blue circle is the polarization ellipse of a C-point and the red arrow denotes its spin density vector \(s_E\), so that the yellow plane is the observation plane where the normalized \(s_E\) also represents the normal direction of this plane. This definition indicates that the observation plane may be different if the position of the C-point is changed, \((\rho', \phi', z')\) are the coordinates in the observation space, which are not fixed, but vary with \(s_E\). The direction of \(z'\) is oriented to the direction of \(s_E\) and the center of the coordinates \((\rho', \phi', z')\) is chosen at the C-point.

Now, let us observe the behaviors of polarization Möbius strips along the C-line. Here we choose some typical points along a C-line for \(m = 2\), see Fig. 7, where the spin density vectors are also shown. (Note that this chosen is not special, and the following phenomena also can be observed along other C-lines for \(m = 1, 3\).) From the directions of the spin density vectors, we can get that the observation planes of the C-points are all perpendicular to the focal plane and the planes of the
points, for instance the points $A$ to $D$, have a very small angle to each other.

The point $A$ is located at $x_s = 0.8008\lambda$, $y_s = 0.8979\lambda$, $z_s = 0$ and the normalized spin density vector at this point is $s_E^{(x)} = -0.1096$, $s_E^{(y)} = 0.9940$, $s_E^{(z)} = 0$. The major axes of the polarization ellipses around this point in 3D space are shown in Fig. 8 for trace radii of $r_t = 0.01\lambda$ and $0.1\lambda$, where the semi-axes are colored in blue and red. The yellow plane here also denotes the observation plane defined in Fig. 6, and the projection of the major axes onto the observation plane is shown in gray. It is quite clear that the topological structure around $A$ is a typical (generic) Möbius strip with topological index $\tau = -1/2$, and its projection has 3 lobes. Another typical (generic) Möbius strip with topological index $\tau = +1/2$ is illustrated in Fig. 9, which is the topological structure around another C-point, $D$ (with $x_s = 0.9304\lambda$, $y_s = 0.8635\lambda$, $z_s = 0$ and $s_E^{(x)} = -0.2606$, $s_E^{(y)} = 0.9655$, $s_E^{(z)} = 0$). For $\tau = +1/2$, the number of the lobes of this Möbius strip projection is 1. In both figures, Figs. 8 and 9, the topological features (i.e. the index $\tau$) do not change with the trace radius $r_t$, which also verifies the topological stability of the C-points. While, that is not always the case and in the following we will see two specialties.

For the first special case, let us see the polarization topological structure around a C-point, $A$, which is displayed in Fig. 10. Here the trace radius is chosen as $r_t = 0.02\lambda$ in plot (a) and $0.1\lambda$ in plot (b). It is seen that when $r_t$ is small (i.e. $0.02\lambda$), the topological structure is a typical Möbius strip with $\tau = -1/2$ (having 3 lobes), while as $r_t$ is increased to $0.1\lambda$ it becomes a simple ribbon which has 2 lobes and the topological index $\tau = 0$. This phenomenon appears to violate the topological stability of the C-point, while in fact it does not. The trick is that as the trace radius $r_t$ is increased to $0.1\lambda$, there is another C-point with index $\tau = +1/2$ involved in the observation plane resulting in a different topological structure. The polarization topology in Fig. 10(b) actually is for two C-points with opposite topological indices (i.e. $\tau_1 + \tau_2 = -1/2 + 1/2 = 0$), thus the structure is a simple ribbon with $\tau = 0$. So, this phenomenon not only keeps the topological stability but also obeys the topological conservation law. In a word, the first special case is caused by more than one C-points being involved as the trace radius varies.

The other special case is shown in Fig. 11, which depicts the polarization topological structure around a C-point, $C$. It is very surprising to see that as the trace radius $r_t$ changes from $0.01\lambda$ to $0.1\lambda$, the topological structure of point $C$ is always a simple ribbon with $\tau = 0$ (having two lobes in the projection plane). Also, we can demonstrate that in the observation plane there is no other C-points being involved. It is known that around one C-point, the topological structure should be the Möbius-type.
A ‘paradox’ seems to have arisen: a C-point has a topological structure for the non-C-points. To explain this paradox, let us review the definition of the observation plane and the position of this C-point, C. The point C is located at \( x_s = 0.8501\lambda, y_s = 0.8979\lambda, z_s = 0 \) and the normalized spin density vector has values of \( s_{E}^{(x)} = -0.0818, s_{E}^{(y)} = 0.9967, s_{E}^{(z)} = 0 \). After doing some calculations, we can obtain that at this point \( s_E \) is perpendicular to the C-line. In other words, the observation plane is tangential to the C-line at point C. So that, the observation plane of this point actually does not include this C-point. That is why the topological structure of point C is a ribbon with \( \tau = 0 \). The point C is indeed a ‘turning point’ of this C-line [48], [51].

IV. CONCLUSION

The photonic wheels and the polarization Möbius strips are studied in this article. We show that by highly confining the trigonometric beams, the photonic wheels can be generated in a wide area including three different regions. Though analyzing the characteristics of the photonic wheels in each region, we find that for any mode \( m \) (i.e. the mode of the trigonometric beam), the photonic wheels always can be formed in region 1 (the \( x_sOz_s \) plane) and region 2 (the focal plane), while the exact place of region 3 is dependent on the parity of the mode \( m \). Applying the special phase and amplitude relation in the field with the photonic wheels, a simple expression for the condition of the C-lines in the focal plane (full of photonic wheels) is derived, which provides a convenient and accurate way to calculate the position of the C-lines. Based on the exact position of the C-lines, the polarization Möbius strips of the photonic wheels are observed through defining the observation plane. Two types of typical (generic) polarization Möbius strips are analyzed, and more importantly two special Möbius-related topological events, which violate the topological law, are discussed. Our result shows that the first special event is essentially caused by the new C-point being involved with the trace radius, which reflects the trace-dependent property of the polarization Möbius strips, while the paradox in the second special event can be explained by the ‘turning point’ of the C-lines, which actually is a manifestation of the observer-dependent property of the polarization Möbius strips. Our research not only provides a way to generate the photonic wheels and polarization Möbius strips in a wide area, but also shows the connection between these two interesting optical structures. The finding in this article also suggests that the rich topological events related to the polarization Möbius strips can be found easily in the field with the photonic wheels, even in a short segment of a C-line there.

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