HEAT KERNELS AND ANALYTICITY OF NON-SYMMETRIC JUMP DIFFUSION SEMIGROUPS

ZHENG-QING CHEN AND XICHENG ZHANG

Abstract. Let $d \geq 1$ and $\alpha \in (0, 2)$. Consider the following non-local and non-symmetric Lévy-type operator on $\mathbb{R}^d$:

$$L_\alpha^* f(x) := \text{p.v.} \int_{\mathbb{R}^d} (f(x+z) - f(x)) \frac{\kappa(x,z)}{|z|^{d+\alpha}} \, dz,$$

where $0 < \kappa_0 \leq \kappa(x,z) \leq \kappa_1$, $\kappa(x,z) = \kappa(x,-z)$, and $|\kappa(x,z) - \kappa(y,z)| \leq \kappa_2 |x-y|^\beta$ for some $\beta \in (0, 1)$. Using Levi’s method, we construct the fundamental solution (also called heat kernel) $p_\alpha^*(t,x,y)$ of $L_\alpha^*$, and establish its sharp two-sided estimates as well as its fractional derivative and gradient estimates of the heat kernel. We also show that $p_\alpha^*(t,x,y)$ is jointly Hölder continuous in $(t,x)$. The lower bound heat kernel estimate is obtained by using a probabilistic argument. The fundamental solution of $L_\alpha^*$ gives rise a Feller process $\{X, \mathbb{P}_x, x \in \mathbb{R}^d\}$ on $\mathbb{R}^d$. We determine the Lévy system of $X$ and show that $\mathbb{P}$ solves the martingale problem for $(L_\alpha^*, C^\infty_0(\mathbb{R}^d))$. Furthermore, we obtain the analyticity of the non-symmetric semigroup associated with $L_\alpha^*$ in $L^p$-spaces for every $p \in [1, \infty)$. A maximum principle for solutions of the parabolic equation $\partial_t u = L_\alpha^* u$ is also established.

Keywords and Phrases: Heat kernel estimate, fractional derivative estimate, non-symmetric stable-like operator, Levi’s method, martingale problem, Lévy system

1. Introduction

Let $\mathcal{L}$ be a second order elliptic differential operator in $\mathbb{R}^d$ given by

$$\mathcal{L} f(x) = \sum_{i,j=1}^{d} \partial_i \left( a_{ij}(x) \partial_j f(x) \right) + \sum_{i=1}^{d} b_i(x) \partial_i f(x), \quad (1.1)$$

where $(a^{ij}(x))_{1 \leq i,j \leq d}$ is a bounded measurable (not necessarily symmetric) $d \times d$-matrix-valued function on $\mathbb{R}^d$ that is uniformly elliptic, and $b_i(x)$, $1 \leq i \leq d$, are bounded measurable functions on $\mathbb{R}^d$. Here $\partial_i f(x)$ stands for the partial derivative $\frac{\partial f(x)}{\partial x_i}$. It is well known that there is a diffusion process $X$ having $\mathcal{L}$ as its infinitesimal generator; see [19]. The celebrated DeGiorgi-Nash-Moser-Aronson theory asserts that every bounded parabolic function of $\mathcal{L}$ (or equivalently, of $X$) is locally Hölder continuous and the parabolic Harnack inequality holds for non-negative parabolic functions of $\mathcal{L}$. Moreover, $\mathcal{L}$ has a jointly continuous heat kernel (or equivalently, transition density function of $X$) $p(t,x,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ that enjoys the Aronson’s Gaussian type estimates.

Quite a lot progress has been made during the last decade in developing DeGiorgi-Nash-Moser-Aronson type theory for symmetric non-local operators; see, e.g., [3, 8, 14, 15, 9] and the references therein. In particular, it is shown in Chen and Kumagai [14] that, for every $0 < \alpha < 2$ and for any symmetric measurable function $c(x,y)$ on $\mathbb{R}^d \times \mathbb{R}^d$ that is bounded

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between two positive constants \(\kappa_0\) and \(\kappa_1\), the symmetric non-local operator
\[
\mathcal{L} f(x) = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d;|y|>|x|} (f(y) - f(x)) \frac{c(x,y)}{|x - y|^{d+\alpha}} \, dy \tag{1.2}
\]
defined in the distributional sense admits a jointly Hölder continuous heat kernel \(p(t, x, y)\) with respect to the Lebesgue measure on \(\mathbb{R}^d\), which satisfies
\[
C^{-1} \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}} \leq p(t, x, y) \leq C \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}} \tag{1.3}
\]
for every \(t > 0\) and \(x, y \in \mathbb{R}^d\), where \(C \geq 1\) is a constant that depends only on \(d, \alpha, \kappa_0\) and \(\kappa_1\).

The operator \(\mathcal{L}\) in (1.2) is symmetric in the sense that
\[
\int_{\mathbb{R}^d} g(x) \mathcal{L} f(x) \, dx = \int_{\mathbb{R}^d} f(x) \mathcal{L} g(x) \, dx \quad \text{for } f, g \in C_c^\infty(\mathbb{R}^d),
\]
where \(C_c^\infty(\mathbb{R}^d)\) denotes the space of smooth functions on \(\mathbb{R}^d\) with compact support. When \(c(x,y)\) is a positive constant, \(\mathcal{L}\) above is a constant multiple of the fractional Laplacian \(\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}\) on \(\mathbb{R}^d\), which is the infinitesimal generator of a (rotationally) symmetric \(\alpha\)-stable process on \(\mathbb{R}^d\). The symmetric non-local stable-like operator \(\mathcal{L}\) defined by (1.2) is the analog to \(\Delta^{\alpha/2}\) of a symmetric uniformly elliptic divergence form operator to Laplacian \(\Delta\). Estimate (1.3) can be viewed as an Aronson type estimate for symmetric stable-like operator \(\mathcal{L}\) of (1.2).

The purpose of this paper is to study heat kernels and their sharp two-sided estimates for non-symmetric and non-local stable-like operators of the following form:
\[
\mathcal{L}^\alpha_c f(x) := \text{p.v.} \int_{\mathbb{R}^d} (f(x + z) - f(x)) \frac{\kappa(x,z)}{|z|^{d+\alpha}} \, dz, \tag{1.4}
\]
where p.v. stands for the Cauchy principle value; that is
\[
\mathcal{L}^\alpha_c f(x) = \lim_{\varepsilon \to 0} \int_{\{|x|,|z| > \varepsilon\}} (f(x + z) - f(x)) \frac{\kappa(x,z)}{|z|^{d+\alpha}} \, dz.
\]
Here \(d \geq 1, 0 < \alpha < 2\), and \(\kappa(x,z)\) is a measurable function on \(\mathbb{R}^d \times \mathbb{R}^d\) satisfying
\[
0 < \kappa_0 \leq \kappa(x,z) \leq \kappa_1, \quad \kappa(x,z) = \kappa(x,-z), \tag{1.5}
\]
and for some \(\beta \in (0, 1)\)
\[
|\kappa(x,z) - \kappa(y,z)| \leq \kappa_2 |x - y|^{\beta}. \tag{1.6}
\]
That \(\kappa(x,z)\) is symmetric in \(z\) is a commonly assumed condition in the literature of non-local operators; see [3] for example. Due to this symmetry condition, we may write
\[
\mathcal{L}^\alpha_c f(x) = \frac{1}{2} \int_{\mathbb{R}^d} (f(x + z) + f(x - z) - 2f(x)) \frac{\kappa(x,z)}{|z|^{d+\alpha}} \, dz.
\]

We point out here that, unlike the operator \(\mathcal{L}\) of (1.2), the operator \(\mathcal{L}^\alpha_c\) defined by (1.4) is typically non-symmetric. The relation between \(\mathcal{L}^\alpha_c\) of (1.4) to \(\mathcal{L}\) of (1.2) is analogous to that of elliptic operators of non-divergence form to elliptic operators of divergence form.

The following is the main result of this paper.

**Theorem 1.1.** Under (1.3) and (1.6), there exists a unique nonnegative continuous function \(p^\alpha(t, x, y)\) on \((0, 1] \times \mathbb{R}^d \times \mathbb{R}^d\) solving
\[
\partial_t p^\alpha(t, x, y) = \mathcal{L}^\alpha_c p^\alpha(t, x, y)(x), \tag{1.7}
\]
and satisfying the following four properties:

(i) (Upper bound) There is a constant \(c_1 > 0\) so that for all \(t \in (0, 1]\) and \(x, y \in \mathbb{R}^d\),
\[
p^\alpha(t, x, y) \leq c_1 t (t^{1/\alpha} + |x - y|)^{-d-\alpha}. \tag{1.8}
\]
We can restate Estimate (1.10) as
\[
|\partial_s p^\kappa_\alpha(t,x,y)| \leq c_3(t^{1/\alpha} + |x - y|)^{-d-\alpha}.
\] (1.17)

This together with (1.9) and (1) of Theorem 1.1 yields that for \(0 < s < t\) and \(x, x', y \in \mathbb{R}^d\),
\[
|p^\kappa_\alpha(s, x, y) - p^\kappa_\alpha(t, x', y)|
\leq \tilde{c}_2 \left( |t - s| + |x - x'|^{1 - \frac{\alpha}{\gamma}} \right) \left( s^{1/\alpha} + |x - y| \wedge |x' - y| \right)^{-d-\alpha},
\]
where $c_2 = c_2 + c_3$.

To the best of the authors’ knowledge, Theorem 1 is the first result on heat kernels and their estimates for a general class of non-symmetric and non-local stable-like operators under Hölder continuous condition in $x \mapsto \kappa(x, z)$. We mention that in the framework of pseudodifferential operator theory, Kochubei [28] (see also [21]) has already studied the existence of fundamental solutions for $\mathcal{L}_\alpha^\kappa$ by using Levi’s method. But strong smoothness of $\kappa(x, y)$ in $y$ and $\alpha \in [1, 2)$ are required. In Chen and Wang [16], fractional Laplacian $\Delta^{\alpha/2}$ perturbed by lower order non-local operator is studied, which corresponds to the case when $\kappa(x, z) = a + b(x, z)|z|^{\alpha-\delta}$ for some constant $a > 0$ and $\alpha > 0$. The uniqueness and non-negativeness of the heat kernels is recently studied in Bass and Ren [4] (see Theorem 5.3 there) for symmetric stable-like operators. Strong stability of the heat kernels in terms of the maximal distance between jumping kernels is assumed in $x \mapsto b(x, z)$, $\mathcal{L}_\alpha^\kappa$ has a unique jointly continuous heat kernel $p_\alpha^y(t, x, y)$ and it enjoys the two-sided estimates (1.8) and (1.14).

Although quite a lot is known for symmetric non-local operators, there are very limited results in literature on heat kernel estimates for non-symmetric and non-local operators. In [6], Bogdan and Jakubowski studied the estimates of heat kernel of $\Delta^{\alpha/2}$ perturbed by a gradient operator with $\alpha \in (1, 2)$ (see also [33] for some extension). Jakubowski and Szczypkowski [27] considered the time-dependent gradient perturbation of $\Delta_\alpha$, while Jakubowski [25] established the global time estimate of heat kernel of $\Delta^{\alpha/2}$ under small singular drifts. In [11, 12, 13], Chen, Kim and Song obtained sharp two-sided estimates for the Dirichlet heat kernel of $\Delta^{\alpha/2}$ as well as of its gradient and Feynman-Kac perturbations. Global as well as Dirichlet heat kernel estimates for non-local operators $\Delta + \Delta^{\alpha/2} + b \cdot \nabla$ and for $m - (m^{2/\alpha} - \Delta)^{\alpha/2} + b \cdot \nabla$ have been investigated in [10] and [17], respectively. In the critical case of $\alpha = 1$, the sharp two-sided heat kernel estimates of $\Delta^{1/2} + b \cdot \nabla$ with Hölder continuous drift $b$ was obtained recently in [34] by using a Levi’s method. In [31], Maekawa and Miura obtained the upper bounds estimates for the fundamental solutions of general non-local diffusions with divergence free drift.

We next briefly describe the approach of this paper. For the construction and upper bound estimates of the heat kernel, we use a method based on Levi’s freezing coefficients argument (cf. [30, 25]). However, in contrast to the previous work [34], a new way to freeze the coefficient $\kappa(x, z)$ is needed (see Section 3). This causes quite many new challenges. In particular, we need to estimate the fractional derivative of the freezing heat kernel and to prove the continuous dependence of heat kernels with respect to the kernel function $\kappa$ (see Subsections 2.3 and 2.4). Strong stability of the heat kernels in terms of the maximal distance between jumping kernels has recently been studied in Bass and Ren [4] (see Theorem 5.3 there) for symmetric stable-like operators (1.2). But here we need a more refined stability results on the heat kernels and their derivatives; see Theorem 2.5 below. To show the uniqueness and non-negativeness of the heat kernel, we establish a maximum principle for solutions of the parabolic equation $\partial_t u(t, x) = \mathcal{L}_\alpha^\kappa u(t, x)$; see Theorem 4.1. For the lower bound estimate (1.14) on the heat kernel, we use a probabilistic approach. The heat kernel $p_\alpha^y(t, x, y)$ determines a strong Feller process $X = \{X_t, t \geq 0; \mathbb{P}_x, x \in \mathbb{R}^d\}$ on $\mathbb{R}^d$. We show that for each $x \in \mathbb{R}^d$, $\mathbb{P}_x$ solves the martingale problem for $(\mathcal{L}_\alpha^\kappa, C_c^\alpha(\mathbb{R}^d))$ with initial value $x$; see (4.24) below. We then deduce from it the Lévy system of $X$, which tells us that $k(x, z)|z|^{-(d+\alpha)}$ is the jump intensity of $X$ making a jump from $x$ with size $z$. The lower bound estimate for $p_\alpha^x$ can then be obtained by a probabilistic argument involving the use of the Lévy system of $X$.

**Remark 1.3.** It will be shown in a subsequent paper [18] that solution to the martingale problem for $(\mathcal{L}_\alpha^\kappa, C_c^\alpha(\mathbb{R}^d))$ is unique. (In fact it will be established for a more general class of non-local operators.) Thus the heat kernel $p_\alpha^y(t, x, y)$ in Theorem 4.1 can also be regarded as
the (unique) transition density function of the unique solution to the martingale problem for \((L_0, C_c^\kappa(\mathbb{R}^d))\).

Notion of analyticity of a \(C_0\)-semigroup plays a central role in the semigroup theory of evolution equations (cf. \([22, 24, 32]\)). For differential operators \(L\) of (1.1), it is well-known that its associated \(C_0\)-semigroup is analytic in \(L^p\)-spaces for every \(p \in (1, \infty)\) at least when \(a_{ij}\) are smooth (cf. \([32\text{, Chapter 7}])\). The proof of this fact is based upon the following deep a priori estimate:

\[
\|\partial_t \partial_j f\|_p \leq C(\|L_{1/2} f\|_p + \|f\|_p), \quad f \in \mathbb{H}^{1/2,p}(\mathbb{R}^d),
\]

which is a consequence of singular integral operator theory. For nonlocal operator \(L_0^\kappa\) of (1.2), under some additional assumptions on \(\kappa(x, z)\), it was shown in \([35, 36]\) that for any \(p \in (1, \infty)\) and \(\alpha \in (0, 2)\),

\[
c_0 \|f\|_{\mathbb{H}^{\alpha, p}} \leq \|L_0^\kappa f\|_p + \|f\|_p \leq c_0^{-1} \|f\|_{\mathbb{H}^{\alpha, p}}, \quad f \in \mathbb{H}^{\alpha, p},
\]

where \(\mathbb{H}^{\alpha, p} = (I - \Delta)^{-\alpha/2}(L^p)\) is the usual Bessel potential space. In this case, it is possible to show the analyticity of its associated semigroup \((P_t^\kappa)_{t \geq 0}\) by using Agmon’s method \([22]\). However in this paper we are able to establish the analyticity of the semigroup \((P_t^\kappa)_{t \geq 0}\) without these additional assumptions. We achieve this by establishing the inequality \(\|L_0^\kappa P_t f\|_p \leq ct^{-1}\|f\|_p\) for every \(t > 0\) and \(p \in [1, \infty)\).

We now give an application of Theorem 1.1 to stochastic differential equations driven by (rotationally) symmetric stable processes. Suppose that \(A(x) = (a_{ij}(x))_{1 \leq i, j \leq d}\) is a bounded continuous \(d \times d\)-matrix-valued function on \(\mathbb{R}^d\) that is nondegenerate at every \(x \in \mathbb{R}^d\), and \(Y\) is a (rotationally) symmetric \(\alpha\)-stable process on \(\mathbb{R}^d\) for some \(0 < \alpha < 2\). It is shown in Bass and Chen \([1\text{, Theorem 7.1}]\) that for every \(x \in \mathbb{R}^d\), SDE

\[
dX_t = A(X_{t-})dY_t, \quad X_0 = x, \quad (1.18)
\]

has a unique weak solution. (Although in \([1]\) it is assumed \(d \geq 2\), the results there are valid for \(d = 1\) as well.) The family of these weak solutions forms a strong Markov process \(\{X, \mathbb{P}_x, x \in \mathbb{R}^d\}\). Using Itô’s formula, one deduces (see the display above (7.2) in \([1]\)) that \(X\) has generator

\[
\mathcal{L} f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(x + A(x)y) - f(x)) \frac{c_{d, \alpha}}{|y|^{d+\alpha}} dy, \quad (1.19)
\]

where \(c_{d, \alpha}\) is a positive constant that depends on \(d\) and \(\alpha\). A change of variable formula \(z = A(x)y\) yields

\[
\mathcal{L} f(x) = \text{p.v.} \int_{\mathbb{R}^d} (f(x + z) - f(x)) \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz, \quad (1.20)
\]

where

\[
\kappa(x, z) = \frac{c_{d, \alpha}}{|\det A(x)|} \left(\frac{|z|}{|A(x)^{-1}z|}\right)^{d+\alpha}. \quad (1.21)
\]

Here \(\det(A(x))\) is the determinant of the matrix \(A(x)\) and \(A(x)^{-1}\) is the inverse of \(A(x)\). As an application of the main result of this paper, we have

**Corollary 1.4.** Suppose that \(A(x) = (a_{ij}(x))\) is uniformly bounded and elliptic (that is, there are positive constants \(\lambda_0\) and \(\lambda_1\) so that \(\lambda_0 I_{d \times d} \leq A(x) \leq \lambda_1 I_{d \times d}\) for every \(x \in \mathbb{R}^d\)) and there are \(\beta \in (0, 1)\) and \(\lambda_2 > 0\) so that

\[
|a_{ij}(x) - a_{ij}(y)| \leq \lambda_2 |x - y|^\beta \quad \text{for } 1 \leq i, j \leq d.
\]
Then the strong Markov process $X$ formed by the unique weak solution to SDE (1.18) has a jointly continuous transition density function $p(t, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ and there is a constant $C > 0$ that depends only on $(d, \alpha, \beta, \lambda_0, \lambda_1)$ so that

$$C^{-1} \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}} \leq p(t, x, y) \leq C \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}}$$

for every $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$.

The remainder of the paper is organized as follows. In Section 2, we prepare some necessary results about the estimates of the heat kernel of spatial-independent symmetric Lévy operators. In Section 3, we construct the heat kernel of spatial-dependent Lévy operators by using Levi’s method. Lastly, in Section 4 we present the proof of the main result of this paper, Theorem 1.1.

We conclude this section by introducing the following conventions. The letter $C$ with or without subscripts will denote a positive constant, whose value is not important and may change in different places. We write $f(x) \lesssim g(x)$ to mean that there exists a constant $C_0 > 0$ such that $f(x) \leq C_0 g(x)$; and $f(x) \asymp g(x)$ to mean that there exist $C_1, C_2 > 0$ such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$. We will also use the abbreviation $f(x \pm z)$ for $f(x + z) + f(x - z)$. For $p \geq 1$, $L^p$-norm of $L^p(\mathbb{R}^d) = L^p(\mathbb{R}^d; dx)$ will be denoted as $\|f\|_p$. We use “:=” to denote a definition. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

2. Preliminaries

Throughout this paper, we shall fix $\alpha \in (0, 2)$ and assume $(t, x) \in (0, 1] \times \mathbb{R}^d$.

For $\gamma, \beta \in \mathbb{R}$, we introduce the following function on $(0, 1] \times \mathbb{R}^d$ for later use:

$$G_\gamma^\beta(t, x) := t^{\frac{\gamma}{\alpha}}(|x|^{\beta} \wedge 1)(t^{1/\alpha} + |x|)^{-d-\alpha}.$$  \hfill (2.1)

2.1. Convolution inequalities. The following lemma will play an important role in the sequel, which is similar to [29] Lemma 1.4 and [34] Lemma 2.3.

**Lemma 2.1.** (i) For all $\beta \in [0, \frac{\alpha}{2}]$ and $\gamma \in \mathbb{R}$, we have

$$\int_{\mathbb{R}^d} G_\gamma^\beta(t, x) dx \leq t^{\frac{\gamma+\beta}{\alpha}} (t^{1/\alpha} + |x|)^{-d-\alpha}, \quad (t, x) \in (0, 1) \times \mathbb{R}^d. \hfill (2.2)$$

(ii) For all $\beta_1, \beta_2 \in [0, \frac{\alpha}{2}]$ and $\gamma_1, \gamma_2 \in \mathbb{R}$, we have

$$\int_{\mathbb{R}^d} G_{\gamma_1}^{\beta_1}(t - s, x - z) G_{\gamma_2}^{\beta_2}(s, z) dz \leq \left( (t - s) \frac{\gamma_1 + \beta_1}{\alpha} \frac{s^{\gamma_1}}{s^{\gamma_1 + \beta_1}} + (t - s) \frac{\gamma_2 + \beta_2}{\alpha} \frac{s^{\gamma_2}}{s^{\gamma_2 + \beta_2}} \right) G_0^0(t, x)$$

$$+ (t - s) \frac{\gamma_1 + \beta_1}{\alpha} \frac{s^{\gamma_1}}{s^{\gamma_1 + \beta_1}} G_0^{\beta_1}(t, x) + (t - s) \frac{\gamma_2 + \beta_2}{\alpha} \frac{s^{\gamma_2}}{s^{\gamma_2 + \beta_2}} G_0^{\beta_2}(t, x). \hfill (2.3)$$

(iii) If $\gamma_1 + \beta_1 > 0$ and $\gamma_2 + \beta_2 > 0$, then

$$\int_0^t \int_{\mathbb{R}^d} G_{\gamma_1}^{\beta_1}(t - s, x - z) G_{\gamma_2}^{\beta_2}(s, z) dz ds$$

$$\leq B\left(\frac{\gamma_1 + \beta_1}{\alpha}, \frac{\gamma_2 + \beta_2}{\alpha}\right) \left( G_0^{\beta_1}(t, x) + G_0^{\beta_2}(t, x) + G_0^{\beta_1 + \beta_2}(t, x) \right), \hfill (2.4)$$

where $B(\gamma, \beta)$ is the usual Beta function defined by

$$B(\gamma, \beta) := \int_0^1 (1 - s)^{\gamma-1} s^{\beta-1} ds, \quad \gamma, \beta > 0.$$
Proof. (i) Notice that
\[ \int_{\mathbb{R}^d} |x|^\beta (t^{1/\alpha} + |x|)^{d+\alpha} dx \leq \int_0^\infty \frac{r^{\beta+d-1}}{(t^{1/\alpha} + r)^{d+\alpha}} dr = \left( \int_0^{q_{1/\alpha}} + \int_{q_{1/\alpha}}^\infty \right) \frac{r^{\beta+d-1}}{(t^{1/\alpha} + r)^{d+\alpha}} dr \]
\[ \leq \int_0^{q_{1/\alpha}} \frac{d\beta-1}{t^{d+\alpha}} dr + \int_{q_{1/\alpha}}^\infty \frac{\beta-1}{\alpha} dr = \frac{d\beta + \beta}{\alpha - \beta}, \]
which implies (2.2) by definition.

(ii) In view of
\[ (t^{1/\alpha} + |x|)^{d+\alpha} \leq C_{d,\alpha} \left( (t - s)^{1/\alpha} + |x - z|^{d+\alpha} + (s^{1/\alpha} + |z|)^{d+\alpha} \right), \]
we have
\[ \mathcal{G}_0^0(t - s, x - z) \mathcal{G}_0^0(s, z) \leq C_{d,\alpha} \left( \mathcal{G}_0^0(t - s, x - z) + \mathcal{G}_0^0(s, z) \right) \mathcal{G}_0^0(t, x). \tag{2.5} \]

Noticing that by \((a + b)^\beta \leq a^\beta + b^\beta\) for \(\beta \in (0, 1),\)
\[ (|x - z|^\beta_1 \wedge 1)(|z|^\beta_2 \wedge 1) \leq (|x - z|^\beta_1 \wedge 1)((|x - z|^\beta_2 \wedge 1) \wedge 1) \leq |x - z|^\beta_1 \wedge 1 + (|x - z|^\beta_1 \wedge 1)(|x|^\beta_2 \wedge 1), \]
\[ (|x - z|^\beta_1 \wedge 1)(|z|^\beta_2 \wedge 1) \leq ((|x|^\beta_1 \wedge 1)|z|^\beta_2 \wedge 1) \leq |z|^\beta_1 \wedge 1 + (|x|^\beta_1 \wedge 1)(|z|^\beta_2 \wedge 1), \]
we have
\[ \mathcal{G}_{\gamma_1}^{\beta_1}(t - s, x - z) \mathcal{G}_{\gamma_2}^{\beta_2}(s, z) = (t - s)^{\beta_1} s^{\beta_2} \left( |x - z|^\beta_1 \wedge 1 \right) \mathcal{G}_0^0(t - s, x - z) \mathcal{G}_0^0(s, z) \]
\[ \leq (t - s)^{\beta_1} s^{\beta_2} \left( |x - z|^\beta_1 \wedge 1 \right) \left( |x|^\beta_2 \wedge 1 \right) \mathcal{G}_0^0(t - s, x - z) \mathcal{G}_0^0(t, x) \]
\[ + (t - s)^{\beta_1} s^{\beta_2} \left( |z|^\beta_1 \wedge 1 \right) \left( |z|^\beta_2 \wedge 1 \right) \mathcal{G}_0^0(s, z) \mathcal{G}_0^0(t, x) \]
\[ \leq s^{\beta_1} \left( \mathcal{G}_{\gamma_1}^{\beta_1}(t - s, x - z) \mathcal{G}_0^0(t, x) + \mathcal{G}_{\gamma_1}^{\beta_1}(t - s, x - z) \mathcal{G}_{\gamma_2}^{\beta_2}(s, z) \right) \]
\[ + (t - s)^{\beta_1} \left( \mathcal{G}_{\gamma_1}^{\beta_1}(s, z) \mathcal{G}_0^0(t, x) + \mathcal{G}_{\gamma_2}^{\beta_2}(s, z) \mathcal{G}_0^0(t, x) \right). \]

Integrating both sides with respect to \(z\) and using (i), we obtain (ii).

(iii) Observe that for \(\gamma, \beta > 0,\)
\[ \int_0^t (t - s)^{\gamma - 1} s^{\beta - 1} ds = t^{\gamma + \beta - 1} \mathcal{B}(\gamma, \beta). \tag{2.6} \]

Integrating both sides of (2.3) with respect to \(s\) from 0 to \(t,\) we obtain
\[ \int_0^t \int_{\mathbb{R}^d} \mathcal{G}_{\gamma_1}^{\beta_1}(t - s, x - z) \mathcal{G}_{\gamma_2}^{\beta_2}(s, z) dz \]
\[ \leq t^{\gamma_1 + \gamma_2 + \beta_1 + \beta_2} \mathcal{B}(\gamma_1^{\beta_1 + \beta_2}, \gamma_2^{\beta_1 + \beta_2}) + \mathcal{B}(\gamma_2^{\beta_1 + \beta_2}, \gamma_1^{\beta_1 + \beta_2}) \]
\[ + t^{\gamma_1 + \gamma_2 + \beta_1 + \beta_2} \mathcal{B}(\gamma_1^{\beta_1 + \beta_2}, \gamma_2^{\beta_1 + \beta_2}) \mathcal{G}_0^0(t, x) + t^{\gamma_1 + \gamma_2 + \beta_1 + \beta_2} \mathcal{B}(\gamma_2^{\beta_1 + \beta_2}, \gamma_1^{\beta_1 + \beta_2}) \mathcal{G}_0^0(t, x), \]
which implies (2.4) by \(\beta_1, \beta_2 < \alpha\) and that \(\mathcal{B}(\gamma, \beta)\) is symmetric and non-increasing with respect to variables \(\gamma\) and \(\beta.\) \(\square\)
2.2. Some estimates of heat kernel of $\Delta_{\alpha}^x$. Let $(Z_{t}^{(\alpha)})_{t \geq 0}$ be a rotationally invariant $d$-dimensional $\alpha$-stable process, and $p_{\alpha}(t,x)$ its probability transition density function with respect to the Lebesgue measure on $\mathbb{R}^{d}$. By the scaling property of $Z_{t}^{(\alpha)} = t^{1/\alpha}Z_{1}^{(\alpha)}$, it is easy to see that

$$p_{\alpha}(t,x) = t^{-d/\alpha}p_{\alpha}(1,t^{-1/\alpha}x). \tag{2.7}$$

Let $(W_{t})_{t \geq 0}$ be a $d$-dimensional standard Brownian motion, and $S_{t}^{(\alpha)}$ an $\alpha/2$-stable subordinator. It is well-known that $Z_{t}^{(\alpha)}$ can be realized as

$$Z_{t}^{(\alpha)} = W_{S_{t}^{(\alpha)}}.$$

Let $\eta_{i}(s)$ be the density of $S_{t}^{(\alpha)}$. By subordination, we have

$$p_{\alpha}(t,x) = \int_{0}^{\infty} (2\pi s)^{-\frac{d}{2}} e^{-\frac{|x|^{2}}{2s}} \eta_{i}(s)ds.$$

By [5] Theorem 2.1], one knows that

$$p_{\alpha}(t,x) = 2\alpha^{d/\alpha}p_{\alpha}(0,t|x|^{d/\alpha}). \tag{2.8}$$

The following obvious inequality will be used frequently:

$$(t^{1/\alpha} + |x + z|)^{-\gamma} \leq 2^{\gamma}(t^{1/\alpha} + |x|)^{-\gamma}, \quad |z| \leq t^{1/\alpha} \vee (|x|/2). \tag{2.9}$$

Below, for a function $f$ defined on $\mathbb{R}^{+} \times \mathbb{R}^{d}$, we shall simply write

$$\delta_{f}(t,x; z) := f(t,x+z) + f(t,x-z) - 2f(t,x). \tag{2.10}$$

We need the following lemma.

**Lemma 2.2.** There is a constant $C = C(d, \alpha) > 0$ so that for every $t > 0$, $x, x', z \in \mathbb{R}^{d}$,

$$|\nabla^{k}p_{\alpha}(t,x)| \leq C t^{(1/\alpha) - d/\alpha - k}, \quad k \in \mathbb{N}, \tag{2.11}$$

$$|p_{\alpha}(t,x) - p_{\alpha}(t,x')| \leq C ((t^{1/\alpha}|x - x'|) \wedge 1)(p_{\alpha}(t,x) + p_{\alpha}(t,x')) \tag{2.12}$$

$$|\delta_{p_{\alpha}}(t,x; z)| \leq C ((t^{1/\alpha}|x - x'|) \wedge 1)(p_{\alpha}(t,x \pm z) + p_{\alpha}(t,x)), \tag{2.13}$$

$$|\delta_{p_{\alpha}}(t,x; z) - \delta_{p_{\alpha}}(t,x'; z)| \leq C ((t^{1/\alpha}|x - x'|) \wedge 1)((t^{1/\alpha}|z|^{2}) \wedge 1)$$

$$\times (p_{\alpha}(t,x \pm z) + p_{\alpha}(t,x) + p_{\alpha}(t,x' \pm z) + p_{\alpha}(t,x')). \tag{2.14}$$

**Proof.** By the scaling property (2.7), it suffices to prove these estimates for $t = 1$.

(i) Noticing that (cf. [20] Theorem 37.1])

$$\eta_{i}(s) \leq s^{1-\frac{d}{2}}e^{-s^{\alpha/2}} \leq s^{1-\frac{d}{2}},$$

we have for $|x| > 1$,

$$|\nabla p_{\alpha}(1,x)| \leq |x| \int_{0}^{\infty} s^{\frac{d}{2} - \frac{d}{2}} e^{-\frac{|x|^{2}}{s}} ds = |x|^{-d-1} \int_{0}^{\infty} u^{|\frac{d}{2}} e^{-\frac{|x|^{2}}{u}} du.$$

Hence,

$$|\nabla p_{\alpha}(1,x)| \leq (1 + |x|)^{-d-1}, \quad x \in \mathbb{R}^{d},$$

which gives (2.11) for $k = 1$. The estimates of higher order derivatives are similar.

(ii) Observe that

$$p_{\alpha}(1,x) = p_{\alpha}(1,x') = \int_{0}^{1} \nabla_{x-x'} p_{\alpha}(1,x + \theta(x' - x))d\theta. \tag{2.15}$$
If \(|x - x'| \leq 1\), then by (2.11), we have
\[
|p_a(1, x) - p_a(1, x')| \leq |x - x'| \int_0^1 (1 + |x + \theta(x' - x)|)^{-d-a-1} \, d\theta
\]
(2.9) \leq |x - x'|(1 + |x|)^{-d-a-1} \leq |x - x'|p_a(1, x).
So,
\[
|p_a(1, x) - p_a(1, x')| \leq (|x - x'| \wedge 1)\{p_a(1, x) + p_a(1, x')\}.
\]

Estimate (2.12) follows.

(iii) By using (2.15) twice, we have
\[
\delta_{p_a}(1, x; z) = p_a(1, x + z) + p_a(1, x - z) - 2p_a(1, x)
\]
\[
= \int_0^1 (\nabla p_a(1, x + \theta z) - \nabla p_a(1, x - \theta z)) \, d\theta
\]
\[
= \int_0^1 \int_0^1 \theta \nabla \nabla p_a(1, x + (1 - 2\theta')\theta z) \, d\theta' \, d\theta.
\]
(2.16)

If \(|z| > 1\), then
\[
|\delta_{p_a}(1, x; z)| \leq p_a(1, x + z) + p_a(1, x - z) + 2p_a(1, x).
\]
If \(|z| \leq 1\), then by (2.11), we have
\[
|\delta_{p_a}(1, x; z)| \leq |z|^2 \int_0^1 \int_0^1 |\nabla^2 p_a(1, x + (1 - 2\theta')\theta z)| \, d\theta' \, d\theta
\]
\[
\leq |z|^2 \int_0^1 \int_0^1 (1 + |x + (1 - 2\theta')\theta z|)^{-d-a-2} \, d\theta' \, d\theta
\]
\[
\leq |z|^2 p_a(1, x).
\]
Hence,
\[
|\delta_{p_a}(1, x; z)| \leq (|z|^2 \wedge 1)\{p_a(1, x \pm z) + p_a(1, x)\},
\]
(2.17)

which yields (2.13).

(iv) If \(|z| \leq 1\) and \(|x - x'| \leq 1\), then by (2.16) and (2.11), we have
\[
|\delta_{p_a}(1, x; z) - \delta_{p_a}(1, x'; z)|
\]
\[
\leq |x - x'| \cdot |z|^2 \int_0^1 \int_0^1 \int_0^1 |\nabla^3 p_a(1, x + (1 - 2\theta')\theta z + \theta'(x' - x))| \, d\theta'' \, d\theta' \, d\theta
\]
\[
\leq |x - x'| \cdot |z|^2 \int_0^1 \int_0^1 \int_0^1 (1 + |x + (1 - 2\theta')\theta z + \theta'(x' - x)|)^{-d-a-3} \, d\theta'' \, d\theta' \, d\theta
\]
\[
\leq |x - x'| \cdot |z|^2 (1 + |x|)^{-d-a-3} \leq |x - x'| \cdot |z|^2 p_a(1, x).
\]
(2.18)

If \(|z| > 1\) and \(|x - x'| \leq 1\), then we have
\[
|\delta_{p_a}(1, x; z) - \delta_{p_a}(1, x'; z)| \leq |x - x'| \int_0^1 |\nabla p_a(1, x \pm z + \theta(x' - x))| \, d\theta
\]
\[
+ |x - x'| \int_0^1 |\nabla p_a(1, x + \theta(x' - x))| \, d\theta
\]
\[
\leq |x - x'| (1 + |x \pm z|)^{-d-a-1} + (1 + |x|)^{-d-a-1}
\]
Combining (2.17), (2.18) and (2.19), we obtain

\[ |\delta p_a(1, x; z) - \delta p_a(1, x'; z)| \leq (|x - x'|) (|z|^\alpha \land 1) (p_a(1, x \pm z) + p_a(1, x) + p_a(1, x' \pm z) + p_a(1, x')), \]

which implies (2.14). The proof is complete. \(\square\)

2.3. Fractional derivative estimate of heat kernel of \(L_a^\kappa\). Let \(\kappa(z)\) be a measurable function on \(\mathbb{R}^d\) with

\[ \kappa(z) = \kappa(-z), \quad 0 < \kappa_0 \leq \kappa(z) \leq \kappa_1. \]  

Consider the following nonlocal symmetric operator

\[ L_a^\kappa f(x) := \text{p.v.} \int_{\mathbb{R}^d} (f(x + z) - f(x)) \kappa(z)|z|^{-d-\alpha} \, dz = \frac{1}{2} \int_{\mathbb{R}^d} \delta_f(x; z) \kappa(z)|z|^{-d-\alpha} \, dz, \]

where \(\delta_f(x; z)\) is defined in a similar way as in (2.10) but with function \(f\) not containing \(t\) variable. It is the infinitesimal generator of a symmetric Lévy process that is stable-like. Let \(p_a^\kappa(t, x)\) be the heat kernel of operator \(L_a^\kappa\), i.e.,

\[ \partial_t p_a^\kappa(t, x) = L_a^\kappa p_a^\kappa(t, x), \quad \lim_{t \to 0} p_a^\kappa(t, x) = \delta_0(x). \]

Under (2.20), it is well-known from the inverse Fourier transform that

\[ p_a^\kappa \in C(\mathbb{R}_+; C^\infty_b(\mathbb{R}^d)). \]  

Moreover, it follows from [14, Theorem 1.1] that

\[ p_a^\kappa(t, x) \asymp \mathcal{E}_a^0(t, x) = t^{1/\alpha} |x|^{-d-\alpha}. \]

If we set

\[ \tilde{\kappa}(z) := \kappa(z) - \frac{\kappa_0}{2}, \]

then by the construction of the Lévy process, one can write

\[ p_a^\kappa(t, x) = \int_{\mathbb{R}^d} p_a^{\kappa_0/2}(t, x - y) p_a^\kappa(t, y) \, dy = \int_{\mathbb{R}^d} p_a^{\kappa_0/2}(t, x) - \frac{\kappa_0}{2}, x - y) p_a^\kappa(t, y) \, dy. \]

The following lemma is an easy consequence of (2.22), (2.23) and Lemma 2.2.

**Lemma 2.3.** Under (2.20), there exists a constant \(C = C(d, \alpha, \kappa_0, \kappa_1, \kappa_2) > 0\) such that

\[ |p_a^\kappa(t, x) - p_a^\kappa(t, x')| \leq C((r^{-1/\alpha} |x - x'|) \land 1) \left( \mathcal{E}_a^0(t, x) + \mathcal{E}_a^0(t, x') \right), \]

\[ |\nabla p_a^\kappa(t, x)| \leq Cr^{-1/\alpha} \mathcal{E}_a^0(t, x), \]

\[ |\delta p_a^\kappa(t, x; z)| \leq C \left( f^{-\hat{\kappa}} |z|^\alpha \land 1 \right) \left( \mathcal{E}_a^0(t, x \pm z) + \mathcal{E}_a^0(t, x) \right), \]

\[ |\delta p_a^\kappa(t, x; z) - \delta p_a^\kappa(t, x'; z)| \leq C \left( (r^{-1/\alpha} |x - x'|) \land 1 \right) \left( f^{-\hat{\kappa}} |z|^\alpha \land 1 \right) \times \left( \mathcal{E}_a^0(t, x \pm z) + \mathcal{E}_a^0(t, x) + \mathcal{E}_a^0(t, x' \pm z) + \mathcal{E}_a^0(t, x') \right). \]
Proof. By (2.23) and (2.12), we have
\[
|p^\alpha(t, x) - p^\alpha(t, x')| \leq ((t^{-1/\alpha}|x - x'|) \wedge 1) \int_{\mathbb{R}^d} \left\{ p_{\alpha}(\frac{\kappa_1}{2}, x - y) + p_{\alpha}(\frac{\kappa_2}{2}, x' - y) \right\} p^\alpha(t, y)dy
\]
\[
= ((t^{-1/\alpha}|x - x'|) \wedge 1) \left\{ p^\alpha(t, x) + p^\alpha(t, x') \right\}
\]
(2.22)
\[
\leq ((t^{-1/\alpha}|x - x'|) \wedge 1) \left\{ \mathcal{G}_\alpha^0(t, x) + \mathcal{G}_\alpha^0(t, x') \right\}. 
\]
Similarly, we have (2.25), (2.26) and (2.27) by (2.23), (2.11), (2.13), (2.14) and (2.22). \(\square\)

Now, we can prove the following fractional derivative estimate of \(p^\alpha(t, x)\).

Theorem 2.4. Under (2.20), there exists a constant \(C = C(d, \alpha, \kappa_0, \kappa_1, \kappa_2) > 0\) such that
\[
\int_{\mathbb{R}^d} |\delta_{p^\alpha}(t, x; z)| \cdot |z|^{-d-\alpha} dz 
\leq C \mathcal{G}_0^0(\alpha, t, x), \tag{2.28}
\]
\[
\int_{\mathbb{R}^d} |\delta_{p^\alpha}(t, x; z) - \delta_{p^\alpha}(t, x'; z)| \cdot |z|^{-d-\alpha} dz 
\leq C((t^{-1/\alpha}|x - x'|) \wedge 1) \left\{ \mathcal{G}_0^0(t, x) + \mathcal{G}_0^0(t, x') \right\}. \tag{2.29}
\]

Proof. By (2.26), we have
\[
\int_{\mathbb{R}^d} |\delta_{p^\alpha}(t, x; z)| \cdot |z|^{-d-\alpha} dz \leq \int_{\mathbb{R}^d} ((t^{-\frac{2}{d}}|z|^2) \wedge 1) \mathcal{G}_\alpha^0(t, x \pm z)|z|^{-d-\alpha} dz 
+ \mathcal{G}_\alpha^0(t, x) \int_{\mathbb{R}^d} ((t^{-\frac{2}{d}}|z|^2) \wedge 1)|z|^{-d-\alpha} dz =: I_1 + I_2.
\]
For \(I_1\), we have
\[
I_1 \leq t^{-\frac{2}{d}} \int_{|z| \leq t^{1/\alpha}} \mathcal{G}_\alpha^0(t, x \pm z)|z|^{2-d-\alpha} dz + \int_{|z| > t^{1/\alpha}} \mathcal{G}_\alpha^0(t, x \pm z)|z|^{-d-\alpha} dz =: I_{11} + I_{12}.
\]
For \(I_{11}\), by (2.9), we have
\[
I_{11} \leq t^{1-\frac{2}{d}} \int_{|z| \leq t^{1/\alpha}} (t^{1/\alpha} + |x \pm z|)^{-d-\alpha} |z|^{2-d-\alpha} dz
\leq t^{1-\frac{2}{d}} (t^{1/\alpha} + |x|)^{-d-\alpha} \int_{|z| \leq t^{1/\alpha}} |z|^{2-d-\alpha} dz \leq \mathcal{G}_0^0(t, x).
\]
For \(I_{12}\), if \(|x| \leq 2t^{1/\alpha}\), then
\[
I_{12} \leq t \int_{|z| > t^{1/\alpha}} (t^{1/\alpha} + |x \pm z|)^{-d-\alpha} |z|^{-d-\alpha} dz
\leq t^{-d/\alpha} \int_{|z| > t^{1/\alpha}} |z|^{-d-\alpha} dz \leq t^{-d/\alpha} \leq \mathcal{G}_0^0(t, x);
\]
if \(|x| > 2t^{1/\alpha}\), then
\[
I_{12} \leq \left( \int_{|z| > 2t^{1/\alpha}} + \int_{|z| > t^{1/2}} \right) \mathcal{G}_\alpha^0(t, x \pm z) \cdot |z|^{-d-\alpha} dz
\leq t \int_{|z| > 2t^{1/\alpha}} (t^{1/\alpha} + |x \pm z|)^{-d-\alpha} |z|^{-d-\alpha} dz + |x|^{-d-\alpha} \int_{|z| > t^{1/2}} \mathcal{G}_\alpha^0(t, x \pm z) dz
\leq t(t^{1/\alpha} + |x|)^{-d-\alpha} \int_{|z| > t^{1/\alpha}} |z|^{-d-\alpha} dz + |x|^{-d-\alpha} \int_{\mathbb{R}^d} \mathcal{G}_\alpha^0(t, x \pm z) dz
\leq (t^{1/\alpha} + |x|)^{-d-\alpha} + |x|^{-d-\alpha} \leq \mathcal{G}_0^0(t, x).
\]
For $I_2$, we have
\[
I_2 = \int_{\mathbb{R}^d} (|z|^2 + 1)|z|^{-d-\alpha} dz \leq C_0(t, x).
\]

Combining the above calculations, we obtain (2.28).

By (2.27), as above, we have
\[
\int_{\mathbb{R}^d} |\delta_{p_{\alpha}^0}(t, x; z) - \delta_{p_{\alpha}^0}(t, x'; z)| \cdot |z|^{-d-\alpha} dz \leq ((t^{-1/\alpha}|x-x'|) \wedge 1)
\times \left\{ \int_{\mathbb{R}^d} ((t^{-1/\alpha}|x-x'|) \wedge 1)|\mathcal{G}_{\alpha}^0(t, x + z) + \mathcal{G}_{\alpha}^0(t, x' + z)||z|^{-d-\alpha} dz\right\}
+ |\mathcal{G}_{\alpha}^0(t, x) + \mathcal{G}_{\alpha}^0(t, x')| \int_{\mathbb{R}^d} ((t^{-1/\alpha}|x-x'|) \wedge 1)|z|^{-d-\alpha} dz
\leq ((t^{-1/\alpha}|x-x'|) \wedge 1)\left\{ \mathcal{G}_{\alpha}^0(t, x) + \mathcal{G}_{\alpha}^0(t, x') \right\}.
\]

The proof is complete. □

2.4. **Continuous dependence of heat kernels with respect to $\kappa$.** In this subsection, we prove the following continuous dependence of the heat kernel with respect to the kernel function $\kappa$, which seems to be new.

**Theorem 2.5.** Let $\kappa$ and $\bar{\kappa}$ be two functions on $\mathbb{R}^d$ satisfying (2.20). For any $\gamma \in (0, \alpha \wedge 1)$, there exists a constant $C = C(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \gamma) > 0$ such that
\[
|p_{\alpha}^0(t, x) - p_{\alpha}^0(t, x)| \leq C||\kappa - \bar{\kappa}||_{\infty}(\mathcal{G}_{\alpha}^0 + \mathcal{G}_{\alpha-\gamma}^0)(t, x),
\]
(2.30)
\[
|\nabla p_{\alpha}^0(t, x) - \nabla p_{\alpha}^0(t, x)| \leq C||\kappa - \bar{\kappa}||_{\infty}t^{-1/\alpha}(\mathcal{G}_{\alpha}^0 + \mathcal{G}_{\alpha-\gamma}^0)(t, x),
\]
(2.31)
and
\[
\int_{\mathbb{R}^d} |\delta_{p_{\alpha}^0}(t, x; z) - \delta_{p_{\alpha}^0}(t, x; z)| \cdot |z|^{-d-\alpha} dz \leq C||\kappa - \bar{\kappa}||_{\infty}(\mathcal{G}_{\alpha}^0 + \mathcal{G}_{\alpha-\gamma}^0)(t, x).
\]
(2.32)

**Proof.** (i) Note that the heat kernel $p_{\alpha}^0(t, x)$ is an even function in $x$. We have
\[
p_{\alpha}^0(t, x) - p_{\alpha}^0(t, x) = \int_0^t \frac{d}{ds} \left( \int_{\mathbb{R}^d} p_{\alpha}^0(s, y)p_{\alpha}^0(t-s, x-y)dy \right) ds
= \int_0^t \left( \int_{\mathbb{R}^d} \left( \mathcal{L}_{\alpha}^{\kappa}p_{\alpha}^0(s, y)p_{\alpha}^0(t-s, x-y) - p_{\alpha}^0(s, y)\mathcal{L}_{\alpha}^{\kappa}p_{\alpha}^0(t-s, x-y) \right) dy \right) ds
= \int_0^t \left( \int_{\mathbb{R}^d} p_{\alpha}^0(s, y)(\mathcal{L}_{\alpha}^{\kappa} - \mathcal{L}_{\alpha}^{\bar{\kappa}})p_{\alpha}^0(t-s, x-y) dy \right) ds
= \int_0^t \left( \int_{\mathbb{R}^d} (\mathcal{L}_{\alpha}^{\kappa} - \mathcal{L}_{\alpha}^{\bar{\kappa}})p_{\alpha}^0(t-s, x-y) (p_{\alpha}^0(s, y) - p_{\alpha}^0(s, x)) dy \right) ds,
\]
where the third equality is due to the symmetry of the operator $\mathcal{L}_{\alpha}^{\kappa}$, (2.22), (2.28) and (2.3), and the fourth equality is due to
\[
\int_{\mathbb{R}^d} p_{\alpha}^0(t-s, x-y) dy = 1.
\]
Thus, by (2.24) and (2.28), we have
\[
|p_{\alpha}^0(t, x) - p_{\alpha}^0(t, x)| \leq ||\kappa - \bar{\kappa}||_{\infty} \int_0^t \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\delta_{p_{\alpha}^0}(t-s, x-y, z)| \cdot |z|^{-d-\alpha} dz \right) \times |p_{\alpha}^0(s, y) - p_{\alpha}^0(s, x)| dy ds
\]
which gives (2.30).

(ii) By (2.23), (2.25) and (2.30), we have

\[
|\nabla p^\alpha_x(t, x) - \nabla p^\gamma_x(t, x)| = \left| \int_{\mathbb{R}^d} \nabla p_\alpha(\frac{y-t}{2}, x-y)(p_\alpha(t, y) - p_\gamma(t, y))dy \right|
\]

\[
\leq \|\kappa - \tilde{\kappa}\|_{\infty} \int_{\mathbb{R}^d} \mathcal{G}_\alpha(t, x-y)(\mathcal{G}_\alpha^\gamma + \mathcal{G}_\alpha^\gamma)(t, y)dy
\]

\[
\leq \|\kappa - \tilde{\kappa}\|_{\infty} t^{-1/\alpha} \int_{\mathbb{R}^d} \mathcal{G}_\alpha(t, x-y)(\mathcal{G}_\alpha^\gamma + \mathcal{G}_\alpha^\gamma)(t, y)dy
\]

\[
\leq \|\kappa - \tilde{\kappa}\|_{\infty} t^{-1/\alpha} (\mathcal{G}_\alpha^\gamma + \mathcal{G}_\alpha^\gamma)(2t, x),
\]

which gives (2.31).

(iii) By (2.23), (2.26) and (2.30), we have

\[
|\delta p^\alpha_x(t, x; z) - \delta p^\gamma_x(t, x; z)| = \left| \int_{\mathbb{R}^d} \delta p_\alpha(\frac{y-t}{2}, x-y; z)(p_\alpha(t, y) - p_\gamma(t, y))dy \right|
\]

\[
\leq \|\kappa - \tilde{\kappa}\|_{\infty} (t^{-\frac{2}{\alpha}}|z|^2) \leq 1)
\]

\[
\times \int_{\mathbb{R}^d} \left\{ \mathcal{G}_\alpha(t, x-y \pm z) + \mathcal{G}_\alpha^\gamma(t, x-y) \right\}(\mathcal{G}_\alpha^\gamma + \mathcal{G}_\alpha^\gamma)(t, y)dy
\]

\[
\leq \|\kappa - \tilde{\kappa}\|_{\infty} (t^{-\frac{2}{\alpha}}|z|^2) \leq 1)
\]

\[
\times \left\{ (\mathcal{G}_\alpha^\gamma + \mathcal{G}_\alpha^\gamma)(2t, x \pm z) + (\mathcal{G}_\alpha^\gamma + \mathcal{G}_\alpha^\gamma)(2t, x) \right\}.
\]

Using the same argument as in estimating (2.29), we obtain (2.32).

\[\square\]

3. Levi’s construction of heat kernels

In this section we consider the spatial dependent operator \(\mathcal{L}_\alpha^x\) defined by (1.4), with the kernel function \(\kappa(x, z)\) satisfying conditions (1.5)–(1.6). In order to reflect the dependence on \(x\), we also write

\[
\mathcal{L}_\alpha^{x}(x)f(x) = \mathcal{L}_\alpha^x f(x) = \frac{1}{2} \int_{\mathbb{R}^d} \delta_f(x; z)\kappa(x,z)|z|^{-d-\alpha}dz.
\]

For fixed \(y \in \mathbb{R}^d\), let \(\mathcal{L}_\alpha^{y}\) be the freezing operator

\[
\mathcal{L}_\alpha^{y}(x)f(x) = \frac{1}{2} \int_{\mathbb{R}^d} \delta_f(x; z)\kappa(y,z)|z|^{-d-\alpha}dz.
\]

Let \(p_\alpha(t, x) := p_\alpha^{y}(t, x)\) be the heat kernel of operator \(\mathcal{L}_\alpha^{y}\), i.e.,

\[
\partial_t p_\alpha(t, x) = \mathcal{L}_\alpha^{y} p_\alpha(t, x), \lim_{t \to 0} p_\alpha(t, x) = \delta_0(x),
\]

where, with a little abuse of notation, \(\delta_0(x)\) denotes the usual Dirac function.
Now, we want to seek the heat kernel $p^\kappa_0(t, x, y)$ of $L^\kappa_0$ with the following form:

$$p^\kappa_0(t, x, y) = p_\gamma(t, x - y) + \int_0^t \int_{\mathbb{R}^d} p_\gamma(t - s, x - z)q(s, z, y)dzds. \quad (3.2)$$

The classical Levi’s method suggests that $q(t, x, y)$ solves the following integral equation:

$$q(t, x, y) = q_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z)q(s, z, y)dzds, \quad (3.3)$$

where

$$q_0(t, x, y) := (L^\kappa_0 - L^\kappa_0) p_\gamma(t, x - y) = \int_{\mathbb{R}^d} \delta_{\gamma}(t, x - y; z)(\kappa(x, z) - \kappa(y, z))|z|^{-d-\alpha} dz.$$

In fact, we formally have

$$\frac{\partial p^\kappa_0(t, x, y)}{\partial t} = L^\kappa_0 p^\kappa_0(t, x, y) + q(t, x, y) + \int_0^t \int_{\mathbb{R}^d} \frac{\partial p_\gamma(t - s, x - z)}{\partial t} q(s, z, y)dzds$$

$$= L^\kappa_0 p^\kappa_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} L^\kappa_0 p_\gamma(t - s, x - z)q(s, z, y)dzds$$

$$= L^\kappa_0 p^\kappa_0(t, x, y). \quad (3.4)$$

Thus, the main aims of this section are to solve equation (3.3), and to make the calculations in (3.4) rigorous.

3.1. **Solving equation (3.3)**. In this subsection, we use Picard’s iteration to solve (3.3).

**Theorem 3.1.** For $n \in \mathbb{N}$, define $q_n(t, x, y)$ recursively by

$$q_n(t, x, y) := \int_0^t \int_{\mathbb{R}^d} q_0(t - s, x, z)q_{n-1}(s, z, y)dzds. \quad (3.5)$$

Under (1.5) and (1.6), the series $q(t, x, y) := \sum_{n=0}^\infty q_n(t, x, y)$ is absolutely convergent and solves the integral equation (3.3). Moreover, $q(t, x, y)$ has the following estimates: there is a constant $C_1 = C_1(d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2) > 0$ so that

$$|q(t, x, y)| \leq C_1(|\delta_\beta + \eta_\beta^0)(t, x - y), \quad (3.6)$$

and for any $\gamma \in (0, \beta)$, there is a constant $C_2 = C_2(d, \alpha, \beta, \gamma, \kappa_0, \kappa_1, \kappa_2) > 0$ so that

$$|q(t, x, y) - q(t, x', y)| \leq C_2 \left(|x - x'|^{\beta - \gamma} 1 \right) \left(|\delta_\gamma^0 + \eta_\gamma^0 + \eta_\gamma^0| + |\delta_{\gamma - \beta}^0 + \eta_{\gamma - \beta}^0| \right)(t, x' - y). \quad (3.7)$$

**Proof.** Without loss of generality, we assume $\beta \in (0, \frac{1}{2}]$. We divide the proof into three steps.

(Step 1). First of all, by (1.5), (1.6) and (2.28), we have

$$|q_0(t, x, y)| \leq (|x - y|^{\beta} 1 \right) \int_{\mathbb{R}^d} \delta_{\gamma}(t, x - y; z) \cdot |z|^{-d-\alpha} dz \leq (|x - y|^{\beta} 1 \right) \delta_\beta^0(t, x - y). \quad (3.8)$$

For $n = 1$, by definition (3.5) and (2.4), there exits a constant $C_{d, \gamma} > 0$ such that

$$|q_1(t, x, y)| \leq C_{d, \alpha} \mathcal{B}(\beta, \beta) \left| \delta_{\beta_{2\gamma}}^0 + \eta_{\beta_{2\gamma}}^0 \right|(t, x - y). \quad (3.9)$$

Suppose now that

$$|q_n(t, x, y)| \leq \gamma_n \left| \delta_{\gamma_{(n+1)\beta}}^0 + \eta_{\gamma_{(n+1)\beta}}^0 \right|(t, x - y),$$

where $\gamma_n > 0$ will be determined below. By (2.4), we have

$$|q_{n+1}(t, x, y)| \leq C_{d, \gamma n} \mathcal{B}(\beta, (n + 1)\beta) \left| \delta_{\gamma_{(n+2)\beta}}^0 + \eta_{\gamma_{(n+2)\beta}}^0 \right|(t, x - y)$$

$$\leq \gamma_{n+1} \left| \delta_{\gamma_{(n+2)\beta}}^0 + \eta_{\gamma_{(n+1)\beta}}^0 \right|(t, x - y). \quad (14)$$
where

\[ \gamma_{n+1} = C_{d,\alpha} \gamma_n \mathcal{B}(\beta, (n+1)\beta). \]

Hence, by \( \mathcal{B}(\gamma, \beta) = \frac{\Gamma(\gamma)\Gamma(\beta)}{\Gamma(\gamma+\beta)} \), where \( \Gamma \) is the usual Gamma function, we obtain

\[ \gamma_n = C_{d,\alpha}^{n+1} \mathcal{B}(\beta, \beta) \mathcal{B}(\beta, 2\beta) \cdots \mathcal{B}(\beta, n\beta) = \frac{(C_{d,\alpha} \Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)}. \]

Thus,

\[ |q_n(t, x, y)| \leq \frac{(C_{d,\alpha} \Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)} \left( \mathcal{E}^0_{\gamma,\beta} + \mathcal{G}^\beta_{n\beta} \right)(t, x - y), \quad (3.10) \]

which in turn implies that

\[ \sum_{n=0}^{\infty} |q_n(t, x, y)| \leq \sum_{n=0}^{\infty} \frac{(C_{d,\alpha} \Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)} \left( \mathcal{E}^0_{\gamma,\beta} + \mathcal{G}^\beta_{n\beta} \right)(t, x - y) \]

\[ \leq \sum_{n=0}^{\infty} \frac{(C_{d,\alpha} \Gamma(\beta))^{n+1}}{\Gamma((n+1)\beta)} \left( \mathcal{E}^0_{\beta} + \mathcal{G}^\beta_0 \right)(t, x - y) \leq \left( \mathcal{E}^0_{\beta} + \mathcal{G}^\beta_0 \right)(t, x - y). \]

Thus, (3.6) is proven. Moreover, by (3.5), we have

\[ \sum_{n=0}^{m+1} q_n(t, x, y) = q_0(t, x, y) + \int_0^t \int_{\mathbb{R}^d} q_0(t-s, x, z) \sum_{n=0}^m q_n(s, z, y) dz ds, \]

which yields (3.3) by taking limits \( m \to \infty \) for both sides.

(Step 2). In this step, we prove the following estimate:

\[ |q_0(t, x, y) - q_0(t, x', y)| \leq (|x - x'|^\beta - \gamma \wedge 1) \left( \mathcal{E}^\beta_{\gamma,\beta} + \mathcal{G}^\beta_{\gamma,\beta} \right)(t, x - y) + \left( \mathcal{E}^\beta_0 + \mathcal{G}^\beta_0 \right)(t, x' - y). \quad (3.11) \]

In the case of \( |x - x'| > 1 \), we have

\[ |q_0(t, x, y)| \leq \mathcal{G}^\beta_0(t, x - y) \leq \mathcal{G}^\beta_{\gamma,\beta}(t, x - y) \]

and

\[ |q_0(t, x', y)| \leq \mathcal{G}^\beta_0(t, x' - y) \leq \mathcal{G}^\beta_{\gamma,\beta}(t, x' - y). \]

In the case of \( 1 \geq |x - x'| > t^{1/\alpha} \), by (3.8), we have

\[ |q_0(t, x, y)| \leq \mathcal{G}^\beta_0(t, x - y) = t^\frac{\beta - \alpha}{\alpha} \mathcal{G}^\beta_{\gamma,\beta}(t, x - y) \leq |x - x'|^{\beta - \gamma} \mathcal{G}^\beta_{\gamma,\beta}(t, x - y), \]

and also

\[ |q_0(t, x', y)| \leq |x - x'|^{\beta - \gamma} \mathcal{G}^\beta_{\gamma,\beta}(t, x' - y). \]

Suppose now that

\[ |x - x'| \leq t^{1/\alpha}. \quad (3.12) \]

By definition and Theorem 2.4, we have

\[ |q_0(t, x, y) - q_0(t, x', y)| = \left| \int_{\mathbb{R}^d} \delta_{p_\gamma}(t, x - y, z)(\kappa(x, z) - \kappa(y, z))|z|^{-\alpha} dz \right| \]

\[ - \int_{\mathbb{R}^d} \delta_{p_\gamma}(t, x' - y, z)(\kappa(x', z) - \kappa(y, z))|z|^{-\alpha} dz \right| \]

\[ \leq (|x - y|^\beta \wedge 1) \int_{\mathbb{R}^d} |\delta_{p_\gamma}(t, x - y, z) - \delta_{p_\gamma}(t, x' - y, z)| \cdot |z|^{-\alpha} dz \]
which yields (3.7) by summing up in Lemma 3.2.

Proof.

and

Some estimates about $q_n(t, x, y)$

Combining the above calculations, we obtain (3.11).

(Step 3). By definition (3.5) and (3.10), (3.11), we have for $n \in \mathbb{N}$,

$$\left| q_n(t, x, y) - q_n(t, x', y) \right| \leq \int_{0}^{t} \int_{\mathbb{R}^d} \left| q_0(t - s, x, z) - q_0(t - s, x', z) \right| q_{n-1}(s, z, y) dz ds$$

$$\leq \frac{(C_d \Gamma(\beta))^{\alpha}}{\Gamma(n\beta)} \left| x - x' \right|^{\beta - \gamma} \left| 1 \right| \int_{0}^{t} \int_{\mathbb{R}^d} \left( g_0^0 + g_{n-1}^\beta \right)(s, z, y) \times \left( g_0^0 + g_{\gamma - \beta}^\beta(t, x - z) + (g_0^0 + g_{\gamma - \beta}^\beta)(t, x' - z) \right) dz ds$$

$$\leq \frac{(C_d \Gamma(\beta))^{\alpha}}{\Gamma(n\beta)} \left| x - x' \right|^{\beta - \gamma} \left| 1 \right| \left( g_0^0 + g_{\gamma - \beta}^\beta(t, x - y) + (g_0^0 + g_{\gamma - \beta}^\beta)(t, x' - y) \right),$$

which yields (3.7) by summing up in $n$. 

3.2. Some estimates about $p_y(t, x - y)$. In this subsection, we prepare some important estimates for later use.

Lemma 3.2. Under (1.5) and (1.6), there exists a constant $C = C(d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2) > 0$ such that for all $\varepsilon > 0$, $x, y \in \mathbb{R}^d$ and $t > 0$,

$$\left| \int_{\mathbb{R}^d} \left( \int_{|w| > \varepsilon} \delta_{p_x}(t, x - y; w) \kappa(x, w) |w|^{-\alpha} dw \right) dy \right| \leq C t^{\frac{\varepsilon}{d - 1}},$$

(3.13)

and

$$\left| \int_{\mathbb{R}^d} \nabla p_x(t, \cdot)(x - y) dy \right| \leq C t^{\frac{d-1}{d^2}}.$$

(3.14)

Proof. Since

$$\int_{\mathbb{R}^d} p_x(t, \xi - y) dy = 1, \quad \forall \xi \in \mathbb{R}^d,$$

(3.15)

by definition of $\delta_{p_x}(t, x - y; w)$, we have

$$\int_{\mathbb{R}^d} \delta_{p_x}(t, x - y; w) dy = 0, \quad \forall w \in \mathbb{R}^d.$$

Thus, by Fubini’s theorem and (2.32), we have for any $\gamma \in (0, \alpha \land 1),$

$$\left| \int_{\mathbb{R}^d} \left( \int_{|w| > \varepsilon} \delta_{p_x}(t, x - y; w) \kappa(x, w) |w|^{-\alpha} dw \right) dy \right|$$

$$= \left| \int_{\mathbb{R}^d} \left( \int_{|w| > \varepsilon} \left( \delta_{p_x}(t, x - y; w) - \delta_{p_x}(t, x - y; w) \right) \kappa(x, w) |w|^{-\alpha} dw \right) dy \right|$$

$$\leq \kappa_1 \int_{\mathbb{R}^d} \left( \int_{|w| > \varepsilon} \left| \delta_{p_x}(t, x - y; w) - \delta_{p_x}(t, x - y; w) \right| \cdot |w|^{-\alpha} dw \right) dy$$
Proof. Under (1.5) and (1.6), there is a constant $C$ for (3.18), by (3.15), we have for any

which gives (3.13).

As for (3.14), it is similar by (3.15) and (2.31) that

The proof is complete.

Lemma 3.3. Under (1.5) and (1.6), there is a constant $C = C(d, \alpha, \beta, \kappa_0, \kappa_1, \kappa_2) > 0$ so that

\[
\left| \int_{\mathbb{R}^d} \mathcal{L}_x^{-\alpha}(p_\gamma(t, \cdot))(x-y)dy \right| \leq Ct^\beta - 1, \quad (3.16)
\]

\[
\left| \int_{\mathbb{R}^d} \partial_\gamma p_\gamma(t, x - y)dy \right| \leq Ct^\beta - 1, \quad (3.17)
\]

\[
\limsup_{t/\alpha \to 0} \left| \int_{\mathbb{R}^d} p_\gamma(t, x - y)dy - 1 \right| = 0. \quad (3.18)
\]

Proof. Estimate (3.16) follows by (3.13). For (3.17), by (3.1) we have

\[
\left| \int_{\mathbb{R}^d} \partial_\gamma p_\gamma(t, x - y)dy \right| = \left| \int_{\mathbb{R}^d} \mathcal{L}_x^{-\alpha}(p_\gamma(t, \cdot))(x-y)dy \right|
\]

\[
\leq \left| \int_{\mathbb{R}^d} (\mathcal{L}_x^{-\alpha} - \mathcal{L}_x^{-\alpha})(p_\gamma(t, \cdot))(x-y)dy \right| + \left| \int_{\mathbb{R}^d} \mathcal{L}_x^{-\alpha}(p_\gamma(t, \cdot))(x-y)dy \right|
\]

\[
\leq \int_{\mathbb{R}^d} \mathcal{L}_x^{-\alpha}(\beta_0(t, x - y)dy + t^\beta - 1 \leq t^\beta - 1. \quad (3.8)
\]

For (3.18), by (3.15), we have for any $\gamma \in (0, \alpha \wedge 1)$,

\[
\sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_\gamma(t, x - y)dy - 1 \right| \leq \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} |p_\gamma(t, x - y) - p_\gamma(t, x - y)|dy
\]

\[
\leq \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} \|k(y, \cdot) - k(x, \cdot)\|_\infty (\alpha_\gamma(t, x - y) + \alpha_\gamma^{-\gamma}(t, x - y))dy
\]

\[
\leq \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d} (\alpha_\gamma(t, x - y) + \alpha_\gamma^{-\gamma}(t, x - y))dy \leq t^\beta \to 0,
\]

as $t \to 0$. The proof is complete.
3.3. **Smoothness of** $p_\alpha^t(x, y)$. In this subsection, we give a rigorous proof about (3.4). Below, for the simplicity of notation, we write

$$\phi_y(t, x, s) := \int_{\mathbb{R}^d} p_z(t-s, x-z)q(s, z, y)dz,$$

(3.19)

and

$$\varphi_y(t, x) := \int_0^t \phi_y(t, x, s)ds = \int_0^t \int_{\mathbb{R}^d} p_z(t-s, x-z)q(s, z, y)dzds.$$  

(3.20)

First of all, we have

**Lemma 3.4.** For all $y \in (0, \alpha \wedge 1)$, there is a constant $C = C(d, \alpha, \beta, \gamma, \kappa_0, \kappa_1, \kappa_2) > 0$ so that

$$|p_\alpha^t(x, y) - p_\alpha^t(x', y)| \leq C|x - x'|^\gamma \{|G_0^{\gamma}(t, x-y) + G_0^{\gamma}(t, x'-y)|.$$

**Proof.** First of all, by (2.24), we have

$$|p_\alpha^t(x, y) - p_\alpha^t(x', y)| \leq ((t^{-1/\alpha}|x - x'|) \wedge 1) \{G_0^{\gamma}(t, x-y) + G_0^{\gamma}(t, x'-y)|$$

$$\leq |x - x'|^\gamma \{G_0^{\gamma}(t, x-y) + G_0^{\gamma}(t, x'-y)|.$$

On the other hand, by (3.6) we also have

$$|\varphi_y(t, x) - \varphi_y(t, x')| \leq \int_0^t \int_{\mathbb{R}^d} |p_z(t-s, x-z) - p_z(t-s, x'-z)| \cdot |q(s, z, y)|dzds$$

$$\leq \int_0^t \int_{\mathbb{R}^d} ((t-s)^{-1/\alpha}|x - x'| \wedge 1) \{G_0^{\gamma}(t-s, x-z) + G_0^{\gamma}(t-s, x'-z)$$

$$\times \{G_0^{\gamma}(s, z-y) + G_0^{\gamma}(s, z-y)|dzds$$

$$\leq |x - x'|^\gamma \int_0^t \int_{\mathbb{R}^d} \{G_0^{\gamma}(t-s, x-z) + G_0^{\gamma}(t-s, x'-z)$$

$$\times \{G_0^{\gamma}(s, z-y) + G_0^{\gamma}(s, z-y)|dzds$$

$$\leq |x - x'|^\gamma \{(G_0^{\gamma} + G_0^{\gamma})(t, x-y) + (G_0^{\gamma} + G_0^{\gamma})(t, x'-y)|.$$

Combining the above two estimations, we obtain the desired estimate. \[\square\]

**Lemma 3.5.** For all $x \neq y \in \mathbb{R}^d$, the mapping $t \mapsto \varphi_y(t, x)$ is absolutely continuous, and

$$\partial_t \varphi_y(t, x) = q(t, x) + \int_0^t \int_{\mathbb{R}^d} \mathcal{L}_\alpha^{\gamma}(z)p_z(t-s, \cdot)(x-z)q(s, z, y)dzdx.$$

(3.21)

**Proof.** We divide the proof into four steps.

(Step 1). In this step we prove that for any $s \in (0, t)$,

$$\partial_t \phi_y(t, x, s) = \int_{\mathbb{R}^d} \partial_t p_z(t-s, x-z)q(s, z, y)dz.$$  

(3.22)

Notice that

$$\frac{\phi_y(t + \varepsilon, x, s) - \phi_y(t, x, s)}{\varepsilon} = \frac{1}{\varepsilon} \int_{\mathbb{R}^d} (p_z(t + \varepsilon - s, x-z) - p_z(t-s, x-z))q(s, z, y)dz$$

$$= \int_{\mathbb{R}^d} \int_0^1 \partial_t p_z(t + \theta \varepsilon - s, x-z) d\theta q(s, z, y)dz.$$

By (3.1) and (2.28), we have for $|s| < \frac{t-s}{2}$,

$$|\partial_t p_z(t + \theta \varepsilon - s, x-z)| = |\mathcal{L}_\alpha^{\gamma}(z)(t + \theta \varepsilon - s)(x-z)|$$

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which together with (3.6) yields

\[
\frac{1}{a} \leq (|x - z| + t + \theta s - s)^{-d-a}
\]

\[
\leq (|x - z| + t - s)^{-d-a}
\]

\[
= \frac{1}{a}^0(t - s, x - z),
\]

By (3.22), we have

\[
\int \phi_{\gamma}(r, x, s) - \phi_{\gamma}(t, x, s) \frac{\varepsilon}{\varepsilon} = \int \partial_{t}p_{z}(t - s, x - z)q(s, z, y)dz,
\]

and (3.22) is proven.

(Step 2). In this step we prove that for all $x \neq y$ and $t > 0$,

\[
\int_{0}^{r} \int_{0}^{\pi} |\partial_{t}\phi_{\gamma}(r, x, s)|dsdr < +\infty.
\]

By (3.22), we have

\[
|\partial_{t}\phi_{\gamma}(r, x, s)| \leq \int_{\mathbb{R}^d} |\partial_{t}p_{z}(r - s, x - z)| \cdot |q(s, z, y) - q(s, x, y)|dz + |q(s, x, y)| \int_{\mathbb{R}^d} \partial_{t}p_{z}(r - s, x - z)dz =: Q^{(1)}_{\gamma}(r, x, s) + Q^{(2)}_{\gamma}(r, x, s).
\]

For $Q^{(1)}_{\gamma}(r, x, s)$, by (3.7) and (2.28), we have

\[
\int_{0}^{r} \int_{0}^{\pi} Q^{(1)}_{\gamma}(r, x, s)dsdr \leq \int_{0}^{r} \int_{0}^{\pi} \int_{\mathbb{R}^d} |\mathcal{L}_{\alpha}^{\phi_{\gamma}}(r - s, x - z)| \cdot (|x - z|^\beta - \gamma \wedge 1)
\times (\phi_{\gamma}^0 + \phi_{\gamma}^\beta(s, x - y) + (\phi_{\gamma}^0 + \phi_{\gamma}^\beta)(s, z - y))dzdsdr
\]

\[
\leq \int_{0}^{r} \int_{0}^{\pi} \int_{\mathbb{R}^d} \phi_{\gamma}^{0 - \gamma}(r - s, x - z)\phi_{\gamma}^0(s, x - y)dzdsdr + \int_{0}^{r} \int_{0}^{\pi} \int_{\mathbb{R}^d} \phi_{\gamma}^{0 - \gamma}(r - s, x - z)\phi_{\gamma}^\beta(s, z - y)dzdsdr
\]

\[
\leq \int_{0}^{r} \int_{0}^{\pi} (r - s)^{\frac{\beta - \gamma}{\alpha - \gamma} - 1}(\phi_{\gamma}^0 + \phi_{\gamma}^\beta)(s, x - y)dsdr + \int_{0}^{r} (\phi_{\gamma}^0 + \phi_{\gamma}^\beta)(r, x - y)dr
\]

\[
\leq \frac{1}{|x - y|^{d + \alpha}} \int_{0}^{r} \int_{0}^{\pi} (r - s)^{\frac{\beta - \gamma}{\alpha - \gamma} - 1}(s^{\frac{x}{\alpha}} + s^{\frac{y}{\alpha}})dsdr + \frac{1}{|x - y|^{d + \alpha}} \int_{0}^{r} (r^{\frac{x}{\alpha}} + 1 + r^{\frac{y}{\alpha}})dr < +\infty.
\]

(3.25)
Thus, on the other hand, we have
\[
\int_0^t \int_0^r Q_y^{(2)}(r, x, s) dr dr \leq \int_0^t \int_0^r (\varphi_0^0 + \varphi_0^1)(s, x - y)(r - s)^{\frac{d}{d - \alpha} - 1} ds dr < +\infty. \tag{3.26}
\]
Combining (3.24)–(3.26), we obtain (3.23).

(Step 3). For fixed \(s, x, y\), we have
\[
\lim_{\delta \downarrow 0} \phi_y(t, x, s) = q(s, x, y). \tag{3.27}
\]
By (3.18), it suffices to prove that
\[
\lim_{\delta \downarrow 0} \left| \int_{\mathbb{R}^d} p_z(t - s, x - z)(q(s, z, y) - q(s, x, y)) dz \right| = 0.
\]
Notice that for any \(\delta > 0\),
\[
\left| \int_{\mathbb{R}^d} p_z(t - s, x - z)(q(s, z, y) - q(s, x, y)) dz \right| \leq \int_{|x - z| \leq \delta} p_z(t - s, x - z)|q(s, z, y) - q(s, x, y)| dz \]
\[+ \int_{|x - z| > \delta} p_z(t - s, x - z)|q(s, z, y) - q(s, x, y)| dz =: J_1(\delta, t, s) + J_2(\delta, t, s).
\]
For any \(\varepsilon > 0\), by (3.7), there exists a \(\delta = \delta(s, x, y) > 0\) such that for all \(|x - z| \leq \delta\),
\[|q(s, z, y) - q(s, x, y)| \leq \varepsilon.
\]
Thus,
\[
J_1(\delta, t, s) \leq \varepsilon \int_{|x - z| \leq \delta} p_z(t - s, x - z) dz \]
\[\leq \varepsilon \int_{\mathbb{R}^d} p_z(t - s, x - z) dz \]
\[\leq \varepsilon \int_{\mathbb{R}^d} \varphi_0^0(t - s, x - z) dz \leq \varepsilon. \tag{2.2}
\]
On the other hand, we have
\[
J_2(\delta, t, s) \leq (t - s) \int_{|x - z| > \delta} \frac{|q(s, z, y)| + |q(s, x, y)|}{|x - z|^{d + \alpha}} dz \]
\[\leq (t - s) \left( \delta^{-\alpha} \int_{\mathbb{R}^d} |q(s, z, y)| dz + |q(s, x, y)| \int_{|z| > \delta} |z|^{-d - \alpha} dz \right),
\]
which, by (3.6) and (2.2), converges to zero as \(t \downarrow r\). Thus, (3.27) is proved.

(Step 4). Now, by the integration by parts formula and (3.27), we have
\[
\int_s^t \partial_x \phi_y(r, x, s) dr = \phi_y(t, x, s) - q(s, x, y).
\]
Integrating both sides with respect to \(s\) from 0 to \(t\), and then by (3.23) and Fubini’s theorem, we obtain
\[
\varphi_y(t, x) - \int_0^t q(s, x, y) ds = \int_0^t \int_s^t \partial_x \phi_y(r, x, s) dr ds = \int_0^t \int_0^r \partial_x \phi_y(r, x, s) ds dr.
\]
By definition of Lemma 3.6.

\[ 3.22 \quad \int_0^t \int_{\mathbb{R}_d} \mathcal{L}_\alpha^{(x)} p_z(t - s, \cdot) (x - z) q(s, z, y) dz ds, \]

which in turn implies (3.21) by the Lebesgue differential theorem. \(\square\)

**Lemma 3.6.** For all \( t > 0 \) and \( x \neq y \), we have

\[ 3.28 \quad \mathcal{L}_\alpha^{(x)} \varphi_y(t, x) = \int_0^t \int_{\mathbb{R}_d} \mathcal{L}_\alpha^{(x)} p_z(t - s, \cdot) (x - z) q(s, z, y) dz ds, \]

and if \( \beta > (1 - \alpha) \vee 0 \), then

\[ 3.29 \quad \nabla \varphi_y(t, x) = \int_0^t \int_{\mathbb{R}_d} \nabla p_z(t - s, \cdot) (x - z) q(s, z, y) dz ds, \]

where the integrals are understood in the sense of iterated integrals. Moreover, for any \( x \neq y \),

\[ 3.30 \quad t \mapsto \mathcal{L}_\alpha^{(x)} \varphi_y(t, x) \text{ is continuous on } (0, 1). \]

**Proof.** We only prove (3.28), and (3.29) is analogue by using (3.14). First of all, for fixed \( s \in (0, t) \), since

\[ x \mapsto p_y(t - s, x - y) \in C^\infty_b (\mathbb{R}^d \times \mathbb{R}^d), \]

and

\[ z \mapsto q(s, z, y) \in C^\infty_b (\mathbb{R}^d), \]

by (2.28) and Fubini’s theorem, it is easy to see that

\[ 3.31 \quad \mathcal{L}_\alpha^{(x)} \varphi_y(t, x, s) = \int_{\mathbb{R}_d} \mathcal{L}_\alpha^{(x)} p_z(t - s, \cdot) (x - z) q(s, z, y) dz. \]

By definition of \( \phi_y \) and Fubini’s theorem, we have for \( \varepsilon \in (0, 1) \)

\[ I_{\varepsilon}(t, x, s, y) := \int_{|w| > \varepsilon} \delta_{\varepsilon, y}(t, x, s, w) \kappa(x, w) |w|^{d-\alpha} dw \]

\[ = \int_{|w| > \varepsilon} \left( \int_{\mathbb{R}_d} \delta_{\varepsilon, y}(t - s, x - z, w) q(s, z, y) dz \right) \kappa(x, w) |w|^{d-\alpha} dw \]

\[ = \int_{\mathbb{R}_d} \left( \int_{|w| > \varepsilon} \delta_{\varepsilon, y}(t - s, x - z, w) \kappa(x, w) |w|^{d-\alpha} dw \right) q(s, z, y) dz \]

\[ \leq \int_{\mathbb{R}_d} \left( \int_{|w| > \varepsilon} |\delta_{\varepsilon, y}(t - s, x - z, w)| |w|^{d-\alpha} dw \right) |q(s, z, y) - q(s, x, y)| dz \]

\[ + \int_{\mathbb{R}_d} \left( \int_{|w| > \varepsilon} \delta_{\varepsilon, y}(t - s, x - z, w) \kappa(x, w) |w|^{d-\alpha} dw \right) dz \cdot |q(s, x, y)|. \]

Using (2.28), (3.13), (3.6) and (3.7), we further have

\[ I_{\varepsilon}(t, x, s, y) \leq \int_{\mathbb{R}_d} \varepsilon - \gamma(t - s, x - z) (\varepsilon_0^0 + \varepsilon_0^\beta, \gamma - \beta)(s, z - y) dz \]

\[ + \left( \int_{\mathbb{R}_d} \varepsilon - \gamma(t - s, x - z) dz \right) \left( \varepsilon_0^0 + \varepsilon_0^\beta, \gamma - \beta \right)(s, x - y) \]

\[ + \left( t - s \right)^{\varepsilon - 1} (\varepsilon_0^0(s, x - y) + \varepsilon_0^\beta(s, x - y)) \]

\[ \leq \int_{\mathbb{R}_d} \varepsilon - \gamma(t - s, x - z) (\varepsilon_0^0 + \varepsilon_0^\beta, \gamma - \beta)(s, z - y) dz \]

\[ + \left( t - s \right)^{\varepsilon - 1} (\varepsilon_0^0(s, x - y) + \varepsilon_0^\beta(s, x - y)) \]

\[ + \left( t - s \right)^{\varepsilon - 1} (\varepsilon_0^0(s, x - y) + \varepsilon_0^\beta(s, x - y)). \]
which implies that for some \( p > 1 \),
\[
\sup_{\varepsilon \in (0, 1)} \int_0^1 |u(t, x, s, y)|^p \, ds < +\infty.
\] (3.32)

Now, by Fubini’s theorem again, we obtain
\[
\mathcal{L}^{\alpha(x)}(x, s) = \lim_{\varepsilon \downarrow 0} \int_0^s \int_{|x-y| < \varepsilon} \delta_{\phi}(t, x, s; w)\kappa(x, w)|w|^{-d-\alpha} \, dw \, ds
\]
\[
= \lim_{\varepsilon \downarrow 0} \int_0^s \int_{|x-y| < \varepsilon} \delta_{\phi}(t, x, s; w)\kappa(x, w)|w|^{-d-\alpha} \, dw \, ds
\]
\[
= \int_0^s \lim_{\varepsilon \downarrow 0} \int_{|x-y| < \varepsilon} \delta_{\phi}(t, x, s; w)\kappa(x, w)|w|^{-d-\alpha} \, dw \, ds
\]
\[
= \int_0^s \mathcal{L}^{\alpha(x)}(x, s) \, ds,
\]
which together with (3.31) yields (3.28).

As for (3.30), it follows by (3.28) and a direct calculation. \( \square \)

4. Proofs of Theorem 1.1 and Corollary 1.4

4.1. A nonlocal maximal principle. In this subsection, we prove a nonlocal maximal principle (cf. [35]). Notice that the current assumptions are weaker than [35].

**Theorem 4.1.** Let \( u(t, x) \in C_b([0, 1] \times \mathbb{R}^d) \) with
\[
\lim_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} |u(t, x) - u(0, x)| = 0.
\] (4.1)

Suppose that for each \( x \in \mathbb{R}^d \),
\[
t \mapsto \mathcal{L}^{\alpha}_t u(t, x)
\]
\[\text{is continuous on } (0, 1],\] (4.2)
and for any \( \varepsilon \in (0, 1) \) and some \( \gamma \in ((\alpha - 1) \lor 0, 1) \),
\[
\sup_{t \in (0, 1)} |u(t, x) - u(t, x')| \leq C_{\gamma}|x - x'|^\gamma.
\] (4.3)

If \( u(t, x) \) satisfies the following equation: for all \( (t, x) \in (0, 1) \times \mathbb{R}^d \),
\[
\partial_t u(t, x) = \mathcal{L}^{\alpha}_t u(t, x),
\]
then for all \( t \in (0, 1) \),
\[
\sup_{x \in \mathbb{R}^d} u(t, x) \leq \sup_{x \in \mathbb{R}^d} u(0, x).
\]

**Proof.** First of all, by (4.1), it suffices to prove that for any \( \varepsilon \in (0, 1) \),
\[
\sup_{x \in \mathbb{R}^d} u(t, x) \leq \sup_{x \in \mathbb{R}^d} u(\varepsilon, x), \quad \forall t \in (\varepsilon, 1).
\] (4.4)

Below, we shall fix \( \varepsilon \in (0, 1) \). Let \( \chi(x) : \mathbb{R}^d \rightarrow [0, 1] \) be a smooth function with \( \chi(x) = 1 \) for \(|x| \leq 1\) and \( \chi(x) = 0 \) for \(|x| > 2\). For \( R > 0 \), define the following cutoff function
\[
\chi_R(x) := \chi(x/R).
\]

For \( R, \delta > 0 \), consider
\[
u_R^\delta(t, x) := u(t, x)\chi_R(x) - (t - \varepsilon)\delta.
\]
Then
\[
\partial_t \nu_R^\delta(t, x) = \mathcal{L}^{\alpha}_t \nu_R^\delta(t, x) + g_R^\delta(t, x),
\] (4.5)
where
\[ g_\delta^\beta(t, x) := \mathcal{L}_a^\alpha u(t, x)\chi(x) - \mathcal{L}_a^\alpha (u\chi_R)(t, x) - \delta. \]

Our aim is to prove that for each \( \delta > 0 \), there exists an \( R_0 \geq 1 \) such that for all \( t \in (\varepsilon, 1) \) and \( R > R_0 \),
\[
\sup_{x \in \mathbb{R}^d} u_R^\delta(t, x) \leq \sup_{x \in \mathbb{R}^d} u_R^\delta(\varepsilon, x) \leq \sup_{x \in \mathbb{R}^d} u(\varepsilon, x).
\] (4.6)

If this is proven, then taking \( R \to \infty \) and \( \delta \to 0 \), we obtain (4.4).

We first prove the following claim:

**Claim:** For \( \beta \in (0, \alpha \wedge 1) \), there exists a constant \( C_\varepsilon > 0 \) such that for all \( R \geq 1 \),
\[
\sup_{(t, x) \in [\varepsilon, 1] \times \mathbb{R}^d} |\mathcal{L}_a^\alpha u(t, x)\chi_R(x) - \mathcal{L}_a^\alpha (u\chi_R)(t, x)| \leq \frac{C_\varepsilon}{R^\beta}.
\] (4.7)

Moreover, for each \( x \in \mathbb{R}^d \),
\[
\int t \mapsto \mathcal{L}_a^\alpha u_R^\delta(t, x) \text{ and } g_\delta^\beta(t, x) \text{ are continuous on } (\varepsilon, 1).
\] (4.8)

**Proof of Claim:** Notice that by definitions,
\[
\mathcal{L}_a^\alpha (u\chi_R)(t, x) - \mathcal{L}_a^\alpha u(t, x)\chi_R(x) - u(t, x)\mathcal{L}_a^\alpha \chi_R(x)
= \int_{\mathbb{R}^d} (u(t, x + z) - u(t, x))(\chi_R(x + z) - \chi_R(x))\kappa(x, z)|z|^{-d-\alpha} \, dz.
\] (4.9)

Thus,
\[
|\mathcal{L}_a^\alpha (u\chi_R)(t, x) - \mathcal{L}_a^\alpha u(t, x)\chi_R(x) - u(t, x)\mathcal{L}_a^\alpha \chi_R(x)|
\leq \|\kappa\|_\infty \int_{|z| \geq 1} |u(t, x + z) - u(t, x)| \cdot |\chi_R(x + z) - \chi_R(x)| \cdot |z|^{-d-\alpha} \, dz,
\]
\[
+ \|\kappa\|_\infty \int_{|z| < 1} |u(t, x + z) - u(t, x)| \cdot |\chi_R(x + z) - \chi_R(x)| \cdot |z|^{-d-\alpha} \, dz = I_1 + I_2.
\]

For \( I_1 \), we have
\[
I_1 \leq 2\|\kappa\|_\infty\|u\|_\infty \int_{|z| > 1} (2\|\chi_R\|_\infty)^{1-\beta} \|\chi_R'\|_\infty \|z\|^{\beta-\alpha} \, dz \leq \|\kappa\|_\infty\|u\|_\infty (2\|\chi\|_\infty)^{1-\beta}\|\chi'\|_\infty / R^\beta.
\] (4.10)

For \( I_2 \), by (4.5), we have
\[
I_2 \leq \|\kappa\|_\infty C_\varepsilon \int_{|z| < 1} \|\chi_R'\|_\infty \|z\|^{1+\gamma-\alpha} \, dz \leq \|\kappa\|_\infty C_\varepsilon \|\chi\|_\infty / R.
\] (4.11)

Moreover, it is also easy to see that
\[
\|\mathcal{L}_a^\alpha \chi_R\|_\infty \leq \frac{C}{R^\beta}.
\] (4.12)

Combining (4.9)-(4.12), we obtain (4.7). As for (4.8), it follows by (4.2), (4.9) and the dominated convergence theorem.

We now use the contradiction argument to prove (4.6). Fix
\[
R > (2C_\varepsilon / \delta)^{1/\beta}.
\] (4.13)

Suppose that (4.6) does not hold, then there exists a \((t_0, x_0) \in (\varepsilon, 1) \times \mathbb{R}^d\) such that
\[
\sup_{(t, x) \in (\varepsilon, 1) \times \mathbb{R}^d} u_R^\delta(t, x) = u_R^\delta(t_0, x_0).
\] (4.14)
Thus, by (4.5), we have for any \( h \in (0, t_0 - \varepsilon) \),
\[
0 \leq \frac{u_R^\delta(t_0, x_0) - u_R^\delta(t_0 - h, x_0)}{h} = \frac{1}{h} \int_{t_0-h}^{t_0} L_a^\kappa u_R^\delta(s, x_0) ds + \frac{1}{h} \int_{t_0-h}^{t_0} g_R^\delta(s, x_0) ds,
\]
which implies by (4.8) and letting \( h \to 0 \) that
\[
0 \leq L_a^\kappa u_R^\delta(t_0, x_0) + g_R^\delta(t_0, x_0). \tag{4.15}
\]
On the other hand, by definition of \( L_a^\kappa \) and (4.14), we have
\[
L_a^\kappa u_R^\delta(t_0, x_0) = \int_{\mathbb{R}^d} \delta_u^\kappa(t_0, x; z) \kappa(x, z)|z|^{-\alpha} dz \leq 0, \tag{4.16}
\]
and by the claim and (4.13),
\[
g_R^\delta(t_0, x_0) \leq \frac{C_\kappa}{R^\delta} - \delta \leq -\frac{\delta}{2}. \tag{4.17}
\]
Combining (4.15)-(4.17), we obtain a contradiction, and the proof is complete. \( \Box \)

4.2. Fractional derivative and gradient estimates of \( p_\alpha^\kappa \). We prove two lemmas about the fractional derivative and gradient estimates of \( p_\alpha^\kappa \).

**Lemma 4.2.** We have
\[
|L_a^\kappa p^\kappa_\alpha(t, y)(x)| \leq \ell_0^0(t, x - y), \tag{4.18}
\]
and if \( \alpha \in [1, 2) \), then
\[
|\nabla p^\kappa_\alpha(t, y)| \leq t^{\frac{\alpha-1}{2}} \ell_0^0(t, x - y). \tag{4.19}
\]

**Proof.** (i) First of all, by (2.28), it is easy to see that
\[
|L_a^\kappa p_\gamma(t, \cdot)(x - y)| \leq \ell_0^0(t, x - y).
\]
Recalling (3.20), by (3.28), we can write
\[
L_a^\kappa \varphi(t, x) = \int_0^t \int_{\mathbb{R}^d} L_a^\kappa p_\gamma(t - s, \cdot)(x - z)(q(s, z, y) - q(s, x, y)) dz ds + \int_0^t \left( \int_{\mathbb{R}^d} L_a^\kappa p_\gamma(t - s, \cdot) dz \right) q(s, x, y) ds + \int_0^t \int_{\mathbb{R}^d} L_a^\kappa p_\gamma(t - s, \cdot) (x - z) q(s, z, y) dz ds.
\]
For \( Q_1(t, x, y) \), by (2.28) and (3.7), we have for any \( \gamma \in (0, \beta) \),
\[
Q_1(t, x, y) \leq \int_0^t \int_{\mathbb{R}^d} \ell_0^0(t - s, x - z)(q_0^\gamma + g_0^\gamma)(s, x - y) dz ds + \int_0^t \int_{\mathbb{R}^d} \ell_0^0(t - s, x - z)(q_0^\gamma + g_0^\gamma)(s, z - y) dz ds \leq \int_0^t (t - s)^{\frac{\alpha-1}{2}} (q_0^\gamma + g_0^\gamma)(s, x - y) ds + (q_0^\gamma + g_0^\gamma + \ell_0^0)(t, x - y) \leq \ell_0^0(t, x - y).
\]
For $Q_2(t, x, y)$, by (3.16), we have
\[ Q_2(t, x, y) \leq \int_t^\prime \int_{\mathbb{R}^d} (t - s)^{\frac{d-1}{2}} (q^0_\beta + q^0_\alpha)(s, x - y)ds \leq q_0^0(t, x - y). \]

For $Q_3(t, x, y)$, by (2.28), (3.6) and (2.3), we have
\[ Q_3(t, x, y) \leq \int_0^t \int_{\mathbb{R}^d} q_0^0(t - s, x - z)(q^0_\beta + q^0_\alpha)(s, z - y)dzds \leq q_0^0(t, x - y). \]

Combining the above calculations and by (3.2), we obtain (4.18).

(ii) By (2.25), we have
\[ |\nabla p_\gamma(t, \cdot)(x - y)| \leq t^{\frac{d-1}{2}} q_0^0(t, x - y). \]

By (3.29), we can write
\[
\nabla \varphi_\gamma(t, x) = \int_t^\prime \int_{\mathbb{R}^d} \nabla p_\gamma(t - s, \cdot)(x - z)(q(s, z, y) - q(s, x, y))dzds \\
+ \int_t^\prime \left( \int_{\mathbb{R}^d} \nabla p_\gamma(t - s, \cdot)(x - z)dz \right) q(s, x, y)ds \\
+ \int_0^t \int_{\mathbb{R}^d} \nabla p_\gamma(t - s, \cdot)(x - z)q(s, z, y)dzds \\
=: R_1(t, x, y) + R_2(t, x, y) + R_3(t, x, y).
\]

For $R_1(t, x, y)$, by (2.25), (3.1) and Lemma 2.1 in view of $\alpha \in [1, 2)$, we have for any $\gamma \in (0, \beta)$,
\[
R_1(t, x, y) \leq \int_t^\prime \int_{\mathbb{R}^d} q_0^{\beta - \gamma}(t - s, x - z)(\xi_\gamma^0 + \xi_{\gamma - \beta}^0)(s, x - y)dzds \\
+ \int_t^\prime \int_{\mathbb{R}^d} q_0^{\beta - \gamma}(t - s, x - z)(\xi_\gamma^0 + \xi_{\gamma - \beta}^0)(s, z - y)dzds \\
\leq \int_t^\prime (t - s)^{\frac{d-1}{2}} (q_\gamma^0 + q_{\beta - \gamma}^0)(s, x - y)ds \\
+ (q_0^{\beta + \alpha - 1} + q_0^{\beta - \gamma} + q_0^{\beta})(t, x - y) \leq q_0^0(t, x - y).
\]

For $R_2(t, x, y)$, by (3.14), we have
\[
R_2(t, x, y) \leq \int_t^\prime (t - s)^{\frac{d-1}{2}} (q_\beta^0 + q_0^\alpha)(s, x - y)ds \leq q_0^0(t, x - y).
\]

For $R_3(t, x, y)$, by (2.28), (3.6) and (2.3), we have
\[
R_3(t, x, y) \leq \int_0^t \int_{\mathbb{R}^d} q_0^{\alpha - 1}(t - s, x - z)(\xi_\beta^0 + \xi_0^0)(s, z - y)dzds \leq q_0^0(t, x - y).
\]

Combining the above calculations and by (3.2), we obtain (4.19). □

Below, we write
\[
P_\alpha^\beta f(x) := \int_{\mathbb{R}^d} p_\alpha^\beta(t, x, y)f(y)dy.
\]

**Lemma 4.3.** For any bounded and Hölder continuous function $f$, we have
\[
\mathcal{L}_\alpha^\beta \left( \int_0^t P_\alpha^\beta f(\cdot)ds \right)(x) = \int_0^t \mathcal{L}_\alpha^\beta P_\alpha^\beta f(x)ds, \quad x \in \mathbb{R}^d.
\]

(4.20)
Proof. By definition of $\mathcal{L}_a^p$ and Fubini’s theorem, we have

$$\mathcal{L}_a^p\left(\int_0^\infty P_a^tf ds\right)(x) = \lim_{\varepsilon \downarrow 0} \int_{|w| > \varepsilon} \left( \int_0^\infty \delta_{P_a^f}(x, w) ds \right) \kappa(x, w)|w|^{-d-\alpha} dw = \lim_{\varepsilon \downarrow 0} \int_0^\infty I_\varepsilon(s, x) ds,$$

where

$$I_\varepsilon(s, x) := \int_{|w| > \varepsilon} \delta_{P_a^f}(x, w) \kappa(x, w)|w|^{-d-\alpha} dw.$$ 

Using the same argument as in proving (3.32), one can prove that for some $p > 1$,

$$\sup_{\varepsilon \in (0, 1)} \int_0^\infty |I_\varepsilon(s, x)|^p ds < +\infty.$$ 

Hence, we can interchange the limit and integral, and obtain (4.20). \qed

**Lemma 4.4.** For any $p \in [1, \infty)$ and $f \in L^p(\mathbb{R}^d)$, $(0, 1) \ni t \mapsto \mathcal{L}_a^p P_t^f f \in L^p(\mathbb{R}^d)$ is continuous. In the case of $p = \infty$, i.e., if $f$ is a bounded measurable function on $\mathbb{R}^d$, then for each $x \in \mathbb{R}^d$, $t \mapsto L_a^\infty P_t^f(x)$ is a continuous function on $(0, 1)$. Moreover, for any $p \in [1, \infty)$, there exists a constant $C = C(p, d, \alpha, \beta, k_0, k_1, k_2) > 0$ such that for all $f \in L^p(\mathbb{R}^d)$ and $t > 0$,

$$\|L_a^p P_t^f\|_p \leq Ct^{-1}\|f\|_p.$$ 

(4.21)

**Proof.** For any $p \in [1, \infty)$, by Lemma 4.2 and Young’s inequality, we have

$$\|L_a^p P_t^f\|_p \leq \left( \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} g_0^\alpha(t, x-y)f(y)dy \right|^p dx \right)^{1/p} \leq \|g_0^\alpha(t)\|_p \|f\|_p \leq t^{-1}\|f\|_p.$$ 

Thus, we obtain (4.21).

On the other hand, for any $\varepsilon \in (0, 1)$, by Lemma 4.2, we have

$$\sup_{\varepsilon \in (0, 1)} |L_a^p P_t^f(t, x, y)| \leq \sup_{\varepsilon \in (0, 1)} g_0^\alpha(t, x-y) \leq g_0^\alpha(\varepsilon, x-y).$$

Since for fixed $x \neq y \in \mathbb{R}^d$, the mapping $t \mapsto L_a^\infty P_t^f(t, x, y)$ is continuous by (3.30), the desired continuity of $t \mapsto L_a^p P_t^f(x)$ follows by the dominated convergence theorem. \qed

4.3. **Proof of Theorem 1.1.** After the above preparation, we are now in a position to give the proof of Theorem 1.1. First of all, using Lemmas 3.5 and 3.6, one sees that the calculations in (3.4) make sense, and thus, we obtain (1.7).

(i) Notice that by (2.22), (3.6) and (2.4),

$$\int_0^\infty \int_{\mathbb{R}^d} p_\varepsilon(t-s, x-z)q(s, z, y)dzds \leq \int_0^\infty \int_{\mathbb{R}^d} \phi_\alpha(t-s, x-z)(\phi_\beta^\alpha + \phi_\beta^\beta)(s, z-y)dzds \leq (\phi_\alpha + \phi_\beta)(t, x-y),$$

(4.22)

which in turn gives estimate (1.8) by equation (3.2) and (2.22), where the constant $c_1$ can be chosen to depend only on $(d, \alpha, \beta, k_0, k_1, k_2)$.

(ii) Estimate (1.9) follows by Lemma 3.4.

(iii) Estimate (1.10) follows by (4.18). The continuity of $t \mapsto L_a^p P_t^f(t, \cdot, y)(x)$ follows by (3.30).

(iv) Let $f$ be a bounded and uniformly continuous function. For any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $|x - y| \leq \delta$,

$$|f(x) - f(y)| \leq \varepsilon.$$ 

By (3.18) and (2.22), we have

$$\limsup_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_\varepsilon(t, x-y)f(y)dy - f(x) \right|$$

By (3.18) and (2.22), we have

$$\limsup_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} p_\varepsilon(t, x-y)f(y)dy - f(x) \right|$$

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Secondly, by (1.9) we have for any
which implies that
\[
\limsup_{t \to 0} \int_{\mathbb{R}^d} p_y(t, x - y) \cdot |f(y) - f(x)| dy \\
\leq \varepsilon + 2\|f\|_{\infty} \limsup_{t \to 0} \int_{|x-y|<\delta} \varphi^0_y(t, x - y) dy \leq \varepsilon,
\]
which implies that
\[
\limsup_{t \to 0} \left| \int_{\mathbb{R}^d} p_y(t, x - y) f(y) dy - f(x) \right| = 0.
\]
Moreover, by (4.22), we also have
\[
\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} p_z(t - s, x - z) q(s, z, y) f(y) dz ds dy \right| \\
\leq \int_{\mathbb{R}^d} (\varphi^0_{\alpha+\beta} + \rho^\beta)(t, x - y) dy \overset{2.2}{\leq} t^\beta \to 0, \ t \downarrow 0.
\]
Thus, (1.11) is proven by equation (5.2).

We now show that kernels that satisfy (1.7)–(1.11) is unique. For this, let \( \overline{p}_\alpha(t, x, y) \) be any kernel that satisfies (1.7)–(1.11) and, for \( f \in C_c^\infty(\mathbb{R}^d) \), define \( \overline{u}_f(t, x) := \int_{\mathbb{R}^d} \overline{p}_\alpha(t, x, y) f(y) dy \).

First of all, by (iv), one sees that
\[
\overline{u}_f \in C_b([0, 1] \times \mathbb{R}^d), \quad \limsup_{t \to 0} |\overline{u}_f(t, x) - f(x)| = 0.
\]
Secondly, by (1.9) we have for any \( \gamma \in (0, \alpha \wedge 1) \),
\[
|\overline{u}_f(t, x) - \overline{u}_f(t, x')| \leq \|f\|_{\infty} \int_{\mathbb{R}^d} |\overline{p}_\alpha(t, x, y) - \overline{p}_\alpha(t, x', y)| dy \\
\leq \|f\|_{\infty} |x - x'|^{\gamma} \int_{\mathbb{R}^d} \left( \varphi^0_{\alpha-\gamma}(t, x - y) + \varphi^0_{\alpha-\gamma}(t, x' - y) \right) dy \\
\overset{2.2}{\leq} \|f\|_{\infty} |x - x'|^{\gamma} t^{-\frac{\beta}{2}}.
\]
The same holds for \( u_f(t, x) := \int_{\mathbb{R}^d} p_\alpha(t, x, y) f(y) dy \). Thus in view of (1.7) and (iii), \( w(t, x) := u_f(t, x) - \overline{u}_f(t, x) \) satisfies all the conditions of Theorem 4.1 with \( w(0, x) = 0 \) for every \( x \in \mathbb{R}^d \).

Applying Theorem 4.1 to both \( w \) and \( -w \) yields \( w(t, x) = 0 \) for every \( t > 0 \) and \( x \in \mathbb{R}^d \). Consequently, we have \( \overline{p}_\alpha(t, x, y) = p_\alpha(t, x, y) \).

(1) has already been proved in the above.

(2) Applying the maximum principle Theorem 4.1 to \( u_f \) with \( f \in C_c^\infty(\mathbb{R}^d) \) and \( f \leq 0 \) implies that \( p_\alpha(t, x, y) \geq 0 \). Moreover, since constant function \( u(t, x) = 1 \) solves the equation \( \partial_t u(t, x) = L^\alpha u(t, x) \) with initial value 1, we have (1.12).

(3) This follows from the uniqueness of the solution to \( \partial_t u(t, x) = L^\alpha u(t, x) \), implied by Theorem 4.1.

(4) will be proven in the next subsection.

(5) If \( \alpha \in [1, 2) \), then estimate (1.15) follows by (4.19).

(6) For \( f \in C_b^2(\mathbb{R}^d) \), define
\[
u(t, x) := f(x) + \int_0^t P_s L^\alpha f(x) ds.
\]
By (4.20) we have
\[ \mathcal{L}_\alpha^k u(t, x) = \mathcal{L}_\alpha^k f(x) + \int_0^t \mathcal{L}_\alpha^k P_s^x \mathcal{L}_\alpha^k f(x) \, ds \]
\[ = \mathcal{L}_\alpha^k f(x) + \int_0^t \partial_s (P_s^x \mathcal{L}_\alpha^k f)(x) \, ds \]
\[ = P_t^x \mathcal{L}_\alpha^k f(x) = \partial_t u(t, x). \]
Moreover, it is easy to see that (4.1), (4.2) and (4.3) are satisfied for \( u \). Thus, by Theorem 4.1 we obtain
\[ P_t^x f(x) = u(t, x) = f(x) + \int_0^t P_s^x \mathcal{L}_\alpha^k f(x) \, ds, \] (4.23)
which in turn implies that
\[ \lim_{t \downarrow 0} \frac{1}{t} (P_t^x f(x) - f(x)) = \lim_{t \downarrow 0} \frac{1}{t} \int_0^t P_s^x \mathcal{L}_\alpha^k f(x) \, ds \overset{(1.1)}{=} \mathcal{L}_\alpha^k f(x) \]
and the convergence is uniform.

(7) Fix \( p \in [1, \infty) \). By (iv), (2) and (4.21), it is easy to see that \( (P_t^x)_{t \geq 0} \) is a \( C_0 \)-semigroup in \( L^p(\mathbb{R}^d) \). On the other hand, for any \( f \in L^p(\mathbb{R}^d) \), by equation (1.7) and Lemma 4.4 one sees that \( P_t^x f \) is differentiable in \( L^p(\mathbb{R}^d) \) for any \( t > 0 \), i.e.,
\[ \lim_{\varepsilon \downarrow 0} \frac{||P_{t+\varepsilon}^x f - P_t^x f - e \mathcal{L}_\alpha^k P_t^x f||_p}{\varepsilon} \leq \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} ||\mathcal{L}_\alpha^k P_{t+s} f - \mathcal{L}_\alpha^k P_t^x f||_p \, ds = 0. \]
The analyticity of \( C_0 \)-semigroup \( (P_t^x)_{t \geq 0} \) follows by (4.21) and [32, p.61 Theorem 5.2 (d)].

4.4. **Proof of lower bound estimate of \( p_s^x(t, x, y) \).** From the previous subsection, one sees that \( (P_t^x)_{t \geq 0} \) is a Feller semigroup. Hence, it determines a Feller process \( (\Omega, \mathcal{F}, (\mathbb{F}_s)_{s \in \mathbb{R}^d}, (X_t)_{t \geq 0}) \). For any \( f \in C^2_0(\mathbb{R}^d) \), it follows from (4.23) and the Markov property of \( X \) that under \( \mathbb{P}_x \), with respect to the filtration \( \mathcal{F}_t := \sigma\{X_s, s \leq t\}, \)
\[ M_t^f := f(X_t) - f(X_0) - \int_0^t \mathcal{L}_\alpha^k f(X_s) \, ds \] is a martingale. (4.24)
In other words, \( \mathbb{P}_x \) solves the martingale problem for \( (\mathcal{L}_\alpha^k, C^2_0(\mathbb{R}^d)) \). Thus \( \mathbb{P}_x \) in particular solves the martingale problem for \( (\mathcal{L}_\alpha^k, C^\infty_0(\mathbb{R}^d)) \).

We now derive a Lévy system of \( X \) by following an approach from [12]. By (4.24), one can derive that \( X_t = (X_t^1, \ldots, X_t^d) \) is a semi-martingale. By Itô’s formula, we have that, for any \( f \in C^\infty_0(\mathbb{R}^d), \)
\[ f(X_t) - f(X_0) = \sum_{i=1}^d \int_0^t \partial_i f(X_{s-}) \, dX_t^i + \sum_{s \neq j} \eta_s(f) + \frac{1}{2} A_s(f), \] (4.25)
where
\[ \eta_s(f) = f(X_s) - f(X_{s-}) - \sum_{i=1}^d \partial_i f(X_{s-})(X_s^i - X_{s-}^i) \] (4.26)
and
\[ A_s(f) = \sum_{i,j=1}^d \int_0^s \partial_i \partial_j f(X_{s-}) \, d\langle (X^j) \rangle_s. \] (4.27)
Now suppose that $A$ and $B$ are two bounded closed subsets of $\mathbb{R}^d$ having a positive distance from each other. Let $f \in C_c^\infty(\mathbb{R}^d)$ with $f = 0$ on $A$ and $f = 1$ on $B$. Clearly $N_t^f := \int_0^t 1_A(X_s^-)dM_s^f$ is a martingale. Define
\[ J(x, y) = k(x, y - x)/|y - x|^{d+\alpha}, \] (4.28)
so $L_\alpha$ can be re-written as
\[ L_\alpha f(x) = \lim_{\varepsilon \to 0} \int_{|y-x|<\varepsilon} (f(y) - f(x))J(x, y)dy. \] (4.29)

We get by (4.24)–(4.27) and (4.29),
\[ N_t^f = \sum_{s \leq t} 1_A(X_s^-)(f(X_s) - f(X_s^-)) - \int_0^t 1_A(X_s)\mathcal{L}_t^kf(X_s)ds \]
\[ = \sum_{s \leq t} 1_A(X_s^-)f(X_s) - \int_0^t 1_A(X_s) \int_{\mathbb{R}^d} f(y)J(X_s, y)dyds. \]

By taking a sequence of functions $f_n \in C_c^\infty(\mathbb{R}^d)$ with $f_n = 0$ on $A$, $f_n = 1$ on $B$ and $f_n \downarrow 1_B$, we get that, for any $x \in \mathbb{R}^d$,
\[ \sum_{s \leq t} 1_A(X_s^-)1_B(X_s) - \int_0^t 1_A(X_s) \int_B J(X_s, y)dyds \]
is a martingale with respect to $\mathbb{P}_x$. Thus,
\[ \mathbb{E}_x \left[ \sum_{s \leq t} 1_A(X_s^-)1_B(X_s) \right] = \mathbb{E}_x \left[ \int_0^t \int_{\mathbb{R}^d} 1_A(X_s)1_B(y)J(X_s, y)dyds \right]. \]

Using this and a routine measure theoretic argument, we get
\[ \mathbb{E}_x \left[ \sum_{s \leq t} f(X_s, X_s) \right] = \mathbb{E}_x \left[ \int_0^t \int_{\mathbb{R}^d} f(X_s, y)J(X_s, y)dyds \right] \]
for any non-negative measurable function $f$ on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x = y\}$. Finally following the same arguments as in \[14, Lemma 4.7\] and \[15, Appendix A\], we get

**Theorem 4.5.** $X$ has a Lévy system $(L, t)$ as $X$, that is, for any $x \in \mathbb{R}^d$ and any non-negative measurable function $f$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ vanishing on $\{(s, x, y) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d : x = y\}$ and $(\mathcal{F}_s)$-stopping time $T$,
\[ \mathbb{E}_x \left[ \sum_{s \leq T} f(s, X_s, X_s) \right] = \mathbb{E}_x \left[ \int_0^T \left( \int_{\mathbb{R}^d} f(s, X_s, y)J(X_s, y)dy \right)ds \right]. \] (4.30)

For a set $K \subset \mathbb{R}^d$, denote
\[ \sigma_K := \inf\{t \geq 0 : X_t \in K\}, \quad \tau_K := \inf\{t \geq 0 : X_t \notin K\}. \]

Let $B(x, r)$ be the ball with radius $r$ and center $x$. We need the following lemma (see \[2\] and \[14\]).

**Lemma 4.6.** For each $\gamma \in (0, 1)$, there exists $A_0 > 0$ such that for every $A > A_0$ and $r \in (0, 1)$,
\[ \mathbb{P}_x (\tau_{B(x, rA)} \leq r^\delta) \leq \gamma. \] (4.31)

**Proof.** Without loss of generality, we assume that $x = 0$. Given $f \in C_b^2(\mathbb{R}^d)$ with $f(0) = 0$ and $f(x) = 1$ for $|x| \geq 1$, we set
\[ f_r(x) := f(x/r), \quad r > 0. \]
By the definition of $f_r$, we have
\[
\mathbb{P}_0(\tau_{B(0,Ar)} \leq r^\alpha) \leq \mathbb{E}_0 \left[ f_{Ar}(X_{\tau_{B(0,Ar)}},r^\alpha) \right] = 4.24 \leq \mathbb{E}_0 \left( \int_0^{\tau_{B(0,Ar)}} \mathcal{L}_{\alpha}^k f_{Ar}(X_s) ds \right). \tag{4.32}
\]
On the other hand, by the definition of $\mathcal{L}_{\alpha}^k$, we have for $\lambda > 0$,
\[
|\mathcal{L}_{\alpha}^k f_{Ar}(x)| = \frac{1}{2} \left| \int_{\mathbb{R}^d} \left( f_{Ar}(x+z) + f_{Ar}(x-z) - 2f_{Ar}(x) \right) \kappa(x,z)|z|^{-d-\alpha} dz \right|
\leq \frac{\kappa_1 \| \nabla^2 f_{Ar} \|_\infty}{2} \int_{|z| < Ar} |z|^{-d-\alpha} dz + 2\kappa_1 \| f_{Ar} \|_\infty \int_{|z| \geq Ar} |z|^{-d-\alpha} dz
\leq \kappa_1 \| \nabla^2 f \|_\infty \frac{(Ar)^{2-\alpha}}{2(2-\alpha)} s_1 + 2\kappa_1 \| f \|_\infty \frac{(Ar)^{2-\alpha}}{2(2-\alpha)} s_1
= \kappa_1 s_1 \left( \frac{\| \nabla^2 f \|_\infty}{A^2} \frac{\lambda^{2-\alpha}}{2(2-\alpha)} + 2\| f \|_\infty \frac{\lambda^{2-\alpha}}{\alpha} \right) r^{-\alpha},
\]
where $s_1$ is the sphere area of the unit ball. Substituting this into (4.32), we get
\[
\mathbb{P}_0(\tau_{B(0,Ar)} \leq r^\alpha) \leq \kappa_1 s_1 \left( \frac{\| \nabla^2 f \|_\infty}{A^2} \frac{\lambda^{2-\alpha}}{2(2-\alpha)} + 2\| f \|_\infty \frac{\lambda^{2-\alpha}}{\alpha} \right).
\]
Choosing first $\lambda$ large enough and then $A$ large enough yield the desired estimate. \qed

Now we can give

**Proof of lower bound of $p^*_{\alpha}(t,x,y)$.** By Lemma 4.6, there is a constant $\lambda \in (0, \frac{1}{2})$ such that for all $t \in (0, 1)$,
\[
\mathbb{P}_x(\tau_{B(x,t^{1/\alpha}/2)} \leq \lambda t) \leq \frac{1}{2}. \tag{4.33}
\]
By (3.2), (2.22) and (4.22), there is a time $t_0 \in (0, 1)$ such that
\[
p^*_{\alpha}(t,x,y) \geq r^{d/\alpha} \text{ for all } t \in (0, t_0) \text{ and } |x-y| \leq 3t^{1/\alpha}.
\]

By C-K equation (1.13) and iterating $[1/t_0] + 1$ times, we conclude that
\[
p^*_{\alpha}(t,x,y) \geq r^{d/\alpha} \text{ for all } t \in (0, 1) \text{ and } |x-y| \leq 3t^{1/\alpha}. \tag{4.34}
\]
Below, we assume
\[
|x-y| > 3t^{1/\alpha}. \tag{4.35}
\]
For the given $\lambda$ in (4.33), by the strong Markov property, we have
\[
\mathbb{P}_x \left( X_{\lambda t} \in B(y, t^{1/\alpha}) \right) \geq \mathbb{P}_x \left( \sigma := \sigma_{B(y,t^{1/\alpha}/2)} \leq \lambda t; \sup_{s \in [\sigma, \sigma + t]} |X_s - X_{\sigma}| < t^{1/\alpha}/2 \right)
= \mathbb{E}_x \left( \mathbb{P}_z \left( \sup_{s \in [0, \lambda t]} |X_s - z| < t^{1/\alpha}/2 \right) \right)_{\sigma = X_{\sigma}} \mathbb{P}_x \left( \sigma_{B(y,t^{1/\alpha}/2)} \leq \lambda t \right)
\geq \inf_{z \in B(y,t^{1/\alpha}/2)} \mathbb{E}_z \left( \mathbb{P}_z \left( \tau_{B(z,t^{1/\alpha}/2)} > \lambda t \right) \mathbb{P}_x \left( \sigma_{B(y,t^{1/\alpha}/2)} \leq \lambda t \right) \right)
\geq \frac{1}{2} \mathbb{P}_x \left( \sigma_{B(y,t^{1/\alpha}/2)} \leq \lambda t \right)
\geq \frac{1}{2} \mathbb{P}_x \left( X_{\lambda t} \in B(y, t^{1/\alpha}/2) \right) \tag{4.36}
\]
Noticing that
\[
X_s \not\in B(y, t^{1/\alpha}/2) \subset B(x, t^{1/\alpha}), \quad s < \lambda t \wedge \tau_{B(z,t^{1/\alpha})},
\]
we have
\[
1_{\Omega_{Br(y,t^{1/n})/2}} = \sum_{s \in \Omega_{Br(y,t^{1/n})/2}} 1_{x \in B(y,t^{1/n}/2)}.
\]
Thus, by (4.30) we have
\[
\mathbb{P}_x \left[ X_{\Delta_n \tau_{Br(y,t^{1/n})/2}} \in B(y,t^{1/n}/2) \right] = \mathbb{E}_x \left[ \int_0^{\Delta_n \tau_{Br(y,t^{1/n})/2}} \int_{B(y,t^{1/n}/2)} J(X_s,u)du \right] \\
\geq \mathbb{E}_x \left[ \int_0^{\Delta_n \tau_{Br(y,t^{1/n})/2}} \int_{B(y,t^{1/n}/2)} \frac{\kappa_0}{|X_s - u|^{d+\alpha}} du \right] \\
\geq \mathbb{E}_x \left[ \Delta_n \tau_{Br(y,t^{1/n})/2} \right] \int_{B(y,t^{1/n}/2)} \frac{\kappa_0}{|x - y|^{d+\alpha}} \\
\geq \left( \lambda \kappa_0(2/3)^{d+\alpha} 2^{-d-1} s_1 \right) |x - y|^{1/d+\alpha}, \tag{4.37}
\]
where \(s_1\) is the sphere area of the unit ball.

Now, by Chapman-Kolmogorov’s equation again, we have
\[
p_h(t,x,y) \geq \int_{B(y,t^{1/n})} p_h(t, x, z)p_h((1 - \lambda)t, z, y)dz \\
\inf_{z \in B(y,t^{1/n})} p_h((1 - \lambda)t, z, y) \int_{B(y,t^{1/n})} p_h(t, x, z)dz \\
\geq t^{-d/\alpha} \mathbb{P}_x \left[ X_{\Delta_n} \in B(y,t^{1/n}) \right] \tag{4.36} \geq t|x - y|^{-d-\alpha} \tag{4.37}
\]
which, combining with (4.34), gives the lower bound estimate of \(p_h(t, x, y)\).

\[\square\]

4.5. **Proof of Corollary 1.4** Since \(\lambda_0 I_{d \times d} \leq A(x) \leq \lambda_1 I_{d \times d}\) and \(|a_{ij}(x) - a_{ij}(y)| \leq \lambda_2 |x - y|^\beta\) for each \(1 \leq i, j \leq d\), the function \(\kappa(x, z)\) defined by (1.21) satisfies the conditions (1.5)-(1.6) with \(\kappa_i, i = 0, 1, 2\), depend only on \(d, \alpha, \beta, \lambda_0, \lambda_1\) and \(\lambda_2\). Thus by Theorem 1.1 there is a jointly continuous heat kernel \(p(t, x, y)\) for the non-local operator \(\mathcal{L} = \mathcal{L}_\alpha^x\) of (1.20) corresponding to this \(\kappa(x, z)\). Let \((\tilde{X}, \mathbb{P}_x, x \in \mathbb{R}^d)\) be the Feller process having \(p(t, x, y)\) as its transition density function. As we observed in the beginning of subsection 4.4, \(\mathbb{P}_x\) solves the martingale problem for \((\mathcal{L}, C_c^\infty(\mathbb{R}^d))\). On the other hand, it is shown in §7 of [11] (see Theorem 7.1 and its proof as well as Theorems 4.1 and 6.3 there) that the law of the unique weak solution \(X\) to SDE (1.18) is the unique solution to the martingale problem for \((\mathcal{L}, C_c^\infty(\mathbb{R}^d))\). Hence \(\tilde{X}\) and \(X\) have the same distribution. Therefore \(p(t, x, y)\) is the transition density function of \(X\). The conclusion of the corollary now follows from Theorem 1.1. \[\square\]

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ZHEN-QING CHEN: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WA 98195, USA, EMAIL: zqchen@uw.edu

XICHENG ZHANG: SCHOOL OF MATHEMATICS AND STATISTICS, WUHAN UNIVERSITY, WUHAN, HUBEI 430072, P.R.CHINA, EMAIL: XichengZhang@gmail.com