The Symmetric Stable Lévy Flights and the Feynman Path Integral

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Abstract

We determine the solution of the fractional spatial diffusion equation in \( n \)-dimensional Euclidean space for a “free” particle by computing the corresponding propagator. We employ both the Hamiltonian and Lagrangian approaches which produce exact results for the case of jumps governed by symmetric stable Lévy flights.
1 Introduction

In nature there is a broad variety of systems in which the correlations in space or time give rise to anomalous transport whose probability density function (pdf) is non-Gaussian and the squared displacement is non-linear in time or diverges. A typical example is an asymmetric particle plume which spreads at a rate inconsistent with the classical model. We will concentrate on the power-law pattern $E X_t^2 \sim K^{\alpha} t^\alpha$ in the super-diffusion case $1 < \alpha < 2$, which is manifested in a diverse number of systems [1].

In a continuous time random walk framework [2] if the jump length and the waiting time are considered to be independent random variables the joint pdf $\psi(x,t)$ can be written as a product of the marginal pdf’s for each random variable. It can be shown that if the waiting time satisfies a Poisson distribution while the jump length follows a symmetric stable Lévy distribution (see Appendix A for the definition) then a Markov process (the first moment of the waiting time is finite) is generated and satisfies the fractional spatial diffusion equation [3]

$$\frac{\partial P(x,t)}{\partial t} = K^{\alpha} D_{+}^{\alpha} P(x,t).$$

In [1] the real parameter $\alpha$ ranges into the interval $(1, 2]$, the generalised diffusion constant has dimensions $[K^{\alpha}] = [L]^{\alpha}$ with $[L],[T]$ being the dimensions of length and time, and $D_{+}^{\alpha}$ is the Weyl operator as defined in the Appendix A. The solution of (1) with the sharp initial condition $\lim_{t \to 0^+} P(x,t) = \delta(x)$ can be obtained analytically in terms of the H-function [4] (the general definition of Fox functions is given in Appendix B)

$$P(x,t;\alpha) = \frac{1}{\alpha |x|} H_{2,2}^{1,1} \left[ \frac{|x|}{(K^{\alpha}t)^{\frac{\alpha}{2}}} \right| \left( \begin{array}{c} 1, \frac{1}{\alpha} \\ 1, 1 \end{array} \right) \left( 1, \frac{1}{\alpha} \right) \right].$$

(2)

Its structure depends exclusively on the underlying geometry of the space. Taking in [2] the $\lim_{\alpha \to 2^{-}} P(x,t;\alpha)$ the classical Gaussian solution is recovered by standard theorems of the Fox functions.

The present work aims at the construction of the quantum mechanical path integral over Lévy flights. The outline of the paper is as follows.

In Section 2 the matrix elements of the time evolution operator in real time are calculated by inserting complete sets of eigenstates of momentum and position operators. Using the property that plane waves remain eigenstates of the Hermitian operator $(-\nabla^2)^{\frac{\alpha}{2}}$ and performing the integrations over the intermediate positions we end up with an analytic expression for the propagator of the “free” particle in Euclidean space in one- and three-dimensions. The propagators are then written in closed form in terms of the Fox functions. We also derive the asymptotic behaviour of the transition probability density in one dimension for large and short distances. In two- and n-dimensions ($n > 3$) the propagators are written in integral form using the Bessel functions of the first kind whose order is half an odd integer.

In Section 3 we write the propagator in a path integral representation in which all paths contribute in the quantum mechanical evolution but they are weighted with the complex weight $e^{i S_M}$, with $S_M$ representing the action of the particle. Descritizing and decomposing the paths into classical and quantum fluctuating trajectories and using the binomial expansion we derive the asymptotic behaviour of the propagator in the long time limit at fixed position. Our result is then confirmed by employing the steepest descent method to the result extracted from the Hamiltonian approach.
In Appendix we give all the mathematical background needed for understanding some notions of the probability theory as well as integral relations which help to manipulate the calculations of the text. We also prove a proposition that justifies the use of plane waves as eigenfunctions for the operator \((-\nabla^2)^{\frac{\alpha}{2}}\).

2 The Hamiltonian approach

We consider the time independent, one-dimensional Hamiltonian operator

\[
\hat{H}(\hat{p}; \alpha) = \left(\frac{\hat{p}^2}{2m}\right)^{\frac{\alpha}{2}}, \quad \alpha \in (1, 2)
\]

and investigate the matrix elements of the time evolution operator \(e^{-\frac{\hat{p}}{\bar{\hbar}}\hat{H}}\), the so-called propagator

\[
P(x_{in}, x_f; \alpha) = <x_f|e^{-\frac{\hat{p}}{\bar{\hbar}}\hat{H}}|x_{in}>.
\]

In (4) \(|x_{in}\rangle\) and \(|x_{f}\rangle\) stand for the initial and final states of the particle which are eigenstates of the position operator \(\hat{x}\)

\[
\hat{x}|x_{in}\rangle = x_{in}|x_{in}\rangle, \quad <x_{f}\hat{x} = x_{f} < x_{f} >.
\]

The position and momentum eigenstates form a complete and orthonormal set of states

\[
\int dx|x><x| = I = \int dp|p><p|<x|y> = \delta(x - y); \quad <p|p'> = \delta(p - p')
\]

with inner product

\[
<x|p> = \frac{1}{2\pi\hbar} e^{i\hat{p}\cdot\hat{x}}
\]

where the angle between \(\hat{p}\) and \(\hat{x}\) is zero or \(\pi\). We insert in (4) \(N\) complete sets of momentum eigenstates and \(N-1\) complete sets of position eigenstates. The time lapse between the initial and final states has been discretized according to \(\beta = N\epsilon > 0\). We then obtain

\[
P(x_0, x_N; \beta; \alpha) = <x_N|e^{-\frac{N\epsilon}{\hbar}\hat{H}}|x_0> = \prod_{k=1}^{N} \int dp_k dx_{k-1} <x_k|e^{-\frac{N\epsilon}{\hbar}\hat{H}(\hat{p})}|p_k><p_k|x_{k-1}>
\]

\[
= \frac{1}{(2\pi\hbar)^N} \prod_{k=1}^{N} \int dp_k dx_{k-1} e^{i\hat{p}_k(x_{k-1}-x_k)} e^{-\frac{\hbar}{2}\hat{H}(\hat{p}_k)}
\]

with the identifications \(|x_{in}\rangle = |x_0\rangle\) and \(|x_{f}\rangle = |x_N\rangle\). In (8) we have used the property that plane waves are eigenfunctions of the Hamiltonian with eigenvalues \(|p|^{\alpha}\) (see Appendix A) therefore

\[
e^{-\frac{i}{\hbar}\hat{H}(\hat{p})}|p_k> = e^{-\frac{i}{\hbar}\hat{H}(\hat{p}_k)}|p_k>.
\]
Performing the integrations over $x_k$’s we obtain

$$P(x_0, x_N, \beta; \alpha) = \frac{\hbar^{N-1}}{2\pi \hbar^N} \int_{-\infty}^{\infty} \prod_{k=1}^{N} dp_k e^{ip_k(x_k-x_{k-1})-\frac{\beta}{\hbar}H(p_k)} \prod_{l=2}^{N} \delta(p_l - p_{l-1}).$$  \hspace{1cm} (10)$$

Using the identity

$$\int_{-\infty}^{\infty} \delta(p_k - p_{k-1})e^{-\frac{\beta}{\hbar}H(p_{k-1})}\delta(p_{k-1} - p_{k-2})dp_{k-1} = \delta(p_k - p_{k-2})e^{-\frac{\beta}{\hbar}H(p_k)}$$  \hspace{1cm} (11)$$

and successively performing the integrations over $p_{N-1}, \ldots, p_1$ we arrive at the transition probability density

$$P(x_N - x_0, \beta; \alpha) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} e^{ipN(x_N-x_0)-\frac{\beta}{\hbar}H(pN)} dpN = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipN(x_N-x_0)-c\beta(|pN|^2)\frac{\beta}{\hbar}} dpN. \hspace{1cm} (12)$$

In (12) the momentum has been rescaled and the parameter $c = \frac{\hbar^{\alpha-1}}{2m} \bar{\pi}$ will play the role of the diffusion constant. We recognize expression (12) to be the Fourier transform of the characteristic function of the stable law $S_{\alpha}(0, \frac{\hbar^{\alpha-1}}{2m} \bar{\pi}, 0)$.

One can prove that (12) is the solution of the fractional spatial diffusion equation

$$\frac{\partial P(x, t)}{\partial t} = -c \left(-\nabla^2\right)^{\frac{\alpha}{2}} P(x, t).$$  \hspace{1cm} (13)$$

To prove this we set for simplicity $x_0 = 0, x_N = x, \beta = t$ and partial integrate (12) with respect to time in which case it reveals that

$$\frac{\partial P(x, t)}{\partial t} = -\frac{1}{2\pi} c \int_{-\infty}^{\infty} e^{ipx} |p|^\alpha e^{-c|p|^2\frac{\beta}{\hbar}} dp = -c \mathcal{F}^{-1} \left[ |p|^\alpha e^{-c|p|^2\frac{\beta}{\hbar}} \right] = -c \left(-\nabla^2\right)^{\frac{\alpha}{2}} P(x, t). \hspace{1cm} (14)$$

In deriving the second equality in (14) we used the result (A.17) of Appendix A.

One can proceed by writing the $\beta$-dependent integrand of (12) in terms of the H-Fox function as

$$e^{-\frac{\beta}{\hbar}H(pN)\frac{\beta}{\hbar}} = \frac{1}{\alpha} H_{0,1}^{1,0} \left[ \left( \frac{\beta}{\hbar} \right)^{\frac{1}{\alpha}} \sqrt{\frac{2m}{\hbar^2}} |pN| \right] \left( 0, \frac{1}{\alpha} \right).$$  \hspace{1cm} (15)$$

thus having

$$I(|x_N - x_0|; \beta, \alpha) = 2 \int_{0}^{\infty} \cos \left( \frac{1}{\hbar} pN(x_N - x_0) \right) e^{-\frac{\beta}{\hbar}H(pN)} dpN$$

$$= \frac{2\pi \hbar}{\alpha |x_N - x_0|} H_{2,2}^{1,1} \left[ \sqrt{\frac{2m}{\hbar^2}} \left( \frac{\hbar}{\beta} \right)^{\frac{1}{\alpha}} h |x_N - x_0| \right] \left( 0, \frac{1}{\alpha} \right) \left( 1, 1 \right) \left( 1, \frac{1}{2} \right). \hspace{1cm} (16)$$

The propagator is then given by

$$P(x_N - x_0, \beta; \alpha) = \frac{1}{2\pi \hbar} I(|x_N - x_0|; \beta, \alpha)$$

$$= \frac{1}{\alpha |x_N - x_0|} H_{2,2}^{1,1} \left[ \sqrt{\frac{2m}{\hbar^2}} \left( \frac{\hbar}{\beta} \right)^{\frac{1}{\alpha}} h |x_N - x_0| \right] \left( 0, \frac{1}{\alpha} \right) \left( 1, 1 \right) \left( 1, \frac{1}{2} \right). \hspace{1cm} (17)$$
a result that was already discussed in [2] by solving (1) in Fourier-Laplace space.

The asymptotic behaviour of the propagator for \(|x_N - x_0| \to \infty\) or \(|x_N - x_0| \to 0\) at fixed time, can be determined by partial integrating (12) and rescaling the momentum leading to the expression

\[
P(x_N - x_0, \beta; \alpha) = \frac{\beta \alpha}{\pi |x_N - x_0|^{\alpha+1}} \int_0^\infty \xi^{\alpha-1} \sin \xi e^{-\beta c \left( \frac{\xi}{|x_N - x_0|} \right)^\alpha} d\xi.
\]  

(18)

- **Large x expansion**
  The propagator with the help of (18) is
  \[
P(x_N - x_0, \beta; \alpha) \approx \frac{\beta \alpha}{\pi |x_N - x_0|^{\alpha+1}} \Gamma\left(\frac{\alpha}{2}\right) \sin\left(\frac{\pi \alpha}{2}\right), \quad \alpha < 2
\]  

(19)

which exhibits the known power-law tail of the Lévy distribution. Due to this property, the mean squared displacement diverges \(EX_t^2 \to \infty\).

- **Short x expansion**
  Taylor expanding the cosine and changing the integration variable to \(w = \beta c |p|^\alpha\) in (12) we have
  \[
P(x_N - x_0, \beta; \alpha) \approx \frac{1}{\pi \alpha (\beta c)^{\frac{1}{\alpha}}} \Gamma\left(\frac{1}{\alpha}\right).
\]  

(20)

The generalization of the propagator in n-dimensions can be achieved by considering the Hamiltonian operator

\[
\hat{H}(\hat{p}; \alpha) = \left( \frac{\hat{p}^2}{2m} \right)^{\frac{\alpha}{2}}.
\]  

(21)

In this case the transition probability density is

\[
P(|\vec{x}_N - \vec{x}_0|, \beta; \alpha) = \frac{1}{(2\pi \hbar)^n} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dV \ e^{\frac{i}{\hbar} \vec{p}_N \cdot (\vec{x}_N - \vec{x}_0)} e^{-\frac{i}{\hbar} H(p_N)}
\]  

(22)

where \(dV = \prod_{i=1}^{n} dp_i\) in Cartesian coordinates, \(H(p_N) = \left( \frac{\delta_{ij} p_i p_j}{2m} \right)^{\frac{\alpha}{2}}\) and repeated indices are summed over. In deriving (22) we used the proposition of the Appendix A.

It is obvious that expression (22) depends only on the magnitude but not on the direction of \(\vec{x}_N - \vec{x}_0\). We distinguish the following cases

1. **Two-dimensions**

   We adopt plane polar coordinates, choose the first axis of the \(\vec{p}_N\)-space in the direction of \(\vec{x}_N - \vec{x}_0\), and find
   \[
P(|\vec{x}_N - \vec{x}_0|, \beta; \alpha) = \frac{1}{(2\pi \hbar)^2} \int_0^{\infty} dp_N \ p_N e^{-\frac{i}{\hbar} H(p_N)} \int_0^{2\pi} d\theta \ e^{\frac{i}{\hbar} p_N |\vec{x}_N - \vec{x}_0| \cos \theta}.
\]  

(23)

But \(\int_0^{2\pi} d\theta \ e^{ic \cos \theta} = 2\pi J_0(c)\), where \(J_0\) is the standard Bessel function defined in Appendix C. Thus

\[
P(|\vec{x}_N - \vec{x}_0|, \beta; \alpha) = \frac{1}{2\pi \hbar^2} \int_0^{\infty} dp_N \ p_N e^{-\frac{i}{\hbar} H(p_N)} J_0\left(\frac{p_N}{\hbar} |\vec{x}_N - \vec{x}_0|\right).
\]  

(24)
2. Three-dimensions

Adopting spherical coordinates we find

\[
P(|\vec{x}_N - \vec{x}_0|, \beta; \alpha) = \frac{1}{(2\pi \hbar)^3} \int_0^\infty \int_{S^{n-1}} dp_N \, p_N^2 \, e^{-\frac{\tilde{p}}{\hbar} H(p_N)} \int_1^1 (d \cos \theta) \, e^{\tilde{p} p_N (|\vec{x}_N - \vec{x}_0| \cos \theta)} \int_0^{\pi} d\phi
\]

\[
= -\frac{1}{2\pi^2 \hbar} \frac{1}{|\vec{x}_N - \vec{x}_0| \, d|\vec{x}_N - \vec{x}_0|} \int_0^\infty dp_N e^{-\frac{\tilde{p}}{\hbar} H(p_N)} \cos \left( \frac{p_N}{\hbar} |\vec{x}_N - \vec{x}_0| \right)
\]

\[
= -\frac{1}{2\pi^2 \hbar} \frac{1}{|\vec{x}_N - \vec{x}_0| \, d|\vec{x}_N - \vec{x}_0|}
\]

\[
\left[ \left. \frac{1}{\alpha |\vec{x}_N - \vec{x}_0|} H_{1/2}^2 \left( \sqrt{2m} \left( \frac{h}{\beta} \right)^{1/2} |\vec{x}_N - \vec{x}_0| \right) \right| \begin{pmatrix} 1 \, 1 \cr 1, 1 \end{pmatrix} \right] \right].
\]

(25)

3. n-dimensions, \( n \geq 3 \)

Let \( dr \) denotes the uniform probability measure on the unit sphere \( S^{n-1} = \{ x \in \mathbb{R}^n : ||x||^2 = 1 \} \). Then \( dV = \frac{2^n}{\Gamma(n/2)} \, dp_N \, d\sigma \) where the constant factor represents the area of the sphere in \( n \)-dimensions. The squares \( \{ \sigma_j \} \) have a Dirichlet \( Di(\frac{1}{2}, \cdots, \frac{1}{2}) \) joint distribution \(^1\), so each \( \sigma_j \) is distributed as the square root of a \( Be(\frac{1}{2}, \frac{n-1}{2}) \)^2 random variable. Thus the integral becomes

\[
P(|\vec{x}_N - \vec{x}_0|, \beta; \alpha) = \frac{2 \pi^{\frac{n}{2}}}{(2\pi \hbar)^n \Gamma \left( \frac{n}{2} \right)} \int_0^\infty \int_{S^{n-1}} dp_N \, d\sigma \, p_N^{n-1} \cos \left( \frac{\tilde{p} \cdot (\vec{x}_N - \vec{x}_0)}{\hbar} \right) e^{-\frac{\tilde{p}}{\hbar} H(p_N)}
\]

\[
= \frac{2 \pi^{\frac{n}{2}}}{(2\pi \hbar)^n \Gamma \left( \frac{n-1}{2} \right)} \int_0^\infty dp_N \, p_N^{n-1} \, e^{-\frac{\tilde{p}}{\hbar} H(p_N)} F(p_N, |\vec{x}_N - \vec{x}_0|, n)
\]

(26)

where

\[
F(p_N, |\vec{x}_N - \vec{x}_0|, n) = \int_0^1 du \cos \left( \frac{p_N |x_N - x_0|}{\hbar} \sqrt{u} \right) u^{-\frac{1}{2} \left( 1 - \frac{n}{2} \right)}
\]

\[
= \sqrt{\pi} \Gamma \left( \frac{n-1}{2} \right) \left( \frac{2h}{p_N |x_N - x_0|} \right)^{\frac{n-2}{2}} J_{\frac{n-1}{2}} \left( \frac{1}{\hbar} p_N |x_N - x_0| \right).
\]

(27)

Combining (26) and (27) we finally have

\[
P(|\vec{x}_N - \vec{x}_0|, \beta; \alpha) = \frac{1}{(2\pi \hbar)^{\frac{n}{2}} |x_N - x_0|^{\frac{n-1}{2}}} G(|x_N - x_0|, n)
\]

(28)

where \( G(|x_N - x_0|, n) = \int_0^\infty dp_N \, p_N \hbar e^{-\frac{\tilde{p}}{\hbar} H(p_N)} J_{\frac{n-1}{2}} \left( \frac{1}{\hbar} p_N |x_N - x_0| \right) \).

\(^1\)The pdf of the Dirichlet distribution with parameters \( \{ \alpha_1, \cdots, \alpha_{n+1} \} \) is

\[
f(x_1, \cdots, x_n; \alpha_1, \cdots, \alpha_{n+1}) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_{n+1})}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_{n+1})} x_1^{\alpha_1-1} \cdots x_n^{\alpha_n-1} \left( 1 - \sum_{i=1}^n x_i \right)^{\alpha_{n+1}-1} I_{A_n}(y)
\]

where \( A_n = \{ x : x_i \geq 0, \sum_{i=1}^n x_i \leq 1 \} \).

\(^2\)The pdf of the beta distribution with parameters \( \{ \alpha, \beta \} \) is \( f(x; \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha) \Gamma(\beta)} x^{\alpha-1} (1 - x)^{\beta-1} I_{(0,\infty)}(x) \).
An alternative approach following similar steps as for two and three-dimensions is developed in Appendix C. Note that (28) for $\alpha \to 2^-$ reduces to the multivariable Brownian motion with propagator

$$\lim_{\alpha \to 2^-} P(|\vec{x}_N - \vec{x}_0|, \beta; \alpha) = \left(\frac{m}{2\pi \hbar \beta}\right)^{\frac{\beta}{2}} e^{-\frac{m|x_N-x_0|^2}{2\hbar \beta}}.$$ (29)

The propagator (28) is the solution of the spatial fractional diffusion equation

$$\frac{\partial P(\vec{x},t)}{\partial t} = -\frac{\hbar}{(2m)^{\frac{\alpha}{2}}} (\nabla)^{\alpha} P(\vec{x},t)$$ (30)

with the condition $P(\vec{x},t; \vec{x}',t) = \delta(\vec{x} - \vec{x}')$ at equal times and $P \to 0$ as $|\vec{x} - \vec{x}'| \to \infty$.

3 The Langrangian Approach

Consider the action of the one-dimensional “free” particle in Minkowski space

$$S_M = f(\alpha) \int_{t_i}^{t_f} \left[(\dot{x})^2\right]^{\frac{\alpha}{2(\alpha-1)}} dt$$ (31)

where the function $f(\alpha)$ has dimensions $[E](\frac{L}{T})^{\frac{\alpha}{\alpha-1}}$ (with $[E]$, $[L]$, $[T]$ representing the dimensions of energy, length and time respectively) and is given by

$$f(\alpha) = (\alpha - 1) \left(\frac{(2m)^{\frac{\alpha}{2}}}{\alpha \alpha R_\alpha}\right)^{1 - \frac{1}{\alpha-1}}.$$ (32)

In (32) $R_\alpha$ is a constant with dimensions $[R_\alpha] = [E]^{1-\frac{\alpha}{2}}$. The Lagrangian is invariant under time translations. Applying the transformation $t \to t - t_f$ together with a rescaling by $\beta = t_f - t_i$ the action becomes

$$S_M = f(\alpha) \left(\frac{1}{\beta}\right)^{\frac{1}{\alpha-1}} \int_{-1}^{0} \left[(\dot{x})^2\right]^{\frac{\alpha}{2(\alpha-1)}} dt.$$ (33)

We decompose now the orbits $x(t)$ of the point particle into a classical path $x_{cl}(t)$ and the deviations $q(t)$ which are referred onwards as the quantum fluctuations of the particle orbit. The split and the boundary conditions $x(t)$ satisfy are

$$x(t) = x_{cl}(t) + q(t); \quad x(0) = x(-1) = 0.$$ (34)

The classical path correpsonds to the solution of the Euler-Lagrange equation

$$\ddot{x}_{cl}(\dot{x}_{cl})^{\frac{2-\alpha}{\alpha-1}} = 0$$ (35)

with boundary conditions $x_{cl}(0) = x_{in}$ and $x_{cl}(-1) = x_f$. The solution of (35) with the prescribed conditions is

$$x_{cl}(t) = x_{in} + t(x_i - x_f).$$ (36)
The vanishing of the paths $x(t), x_{cl}(t)$ at the endpoints of the time interval $[-1, 0]$ forces the quantum fluctuations to obey the conditions $q(0) = q(-1) = 0$. Inserting the decomposition (34) into the action we have

$$S_M = f(\alpha) \left( \frac{1}{\beta} \right)^{\frac{1}{\alpha-1}} \int_{-1}^{0} (\dot{x}_{cl} + \dot{q})^{\frac{\alpha}{\alpha-1}} \, dt. \quad (37)$$

Using the binomial expansion $(a + b)^r = \sum_{k=0}^{\infty} \binom{r}{k} a^{r-k} b^k$, with $r$, $(k)$ be a positive real (nonnegative integer) number, $\binom{r}{k} = \frac{r(r-1)\cdots(r-k+1)}{k!}$, and keeping only terms quadratic in $q$’s we obtain

$$S_M = f(\alpha) \left( \frac{1}{\beta} \right)^{\frac{1}{\alpha-1}} \int_{-1}^{0} \left[ (\dot{x}_{cl})^{\frac{\alpha}{\alpha-1}} + \left( \frac{\alpha}{\alpha - 1} \right) (\dot{x}_{cl})^{\frac{1}{\alpha-1}} \dot{q} + \frac{\alpha}{2(\alpha - 1)^2} (\dot{x}_{cl})^{\frac{2\alpha}{\alpha-1}} \dot{q}^2 + O(q^3) \right] \, dt. \quad (38)$$

The first term is dominated by the contribution along the classical path and the midterm vanishes identically after partial integration with respect to $q(t)$ and using the equation of motion (35). The action then is written as

$$S_M = f(\alpha) \left( \frac{1}{\beta} \right)^{\frac{1}{\alpha-1}} \left[ (x_i - x_f)^{\frac{\alpha}{\alpha-1}} + \frac{\alpha}{2(\alpha - 1)^2} (x_i - x_f)^{\frac{2\alpha}{\alpha-1}} \int_{-1}^{0} (\dot{q})^2 \right] \, dt. \quad (39)$$

The “free” particle amplitude in configuration space reads

$$< x_f, t_f | x_i, t_i > = \int_{(x_i,t_i) \rightarrow (x_f,t_f)} \mathcal{D}x \, e^{\frac{i}{\hbar} S_M} = e^{\frac{i}{\hbar} f(\alpha) \left( \frac{1}{\beta} \right)^{\frac{1}{\alpha-1}} (x_i - x_f)^{\frac{\alpha}{\alpha-1}} \int\mathcal{D}q \, e^{\frac{i}{\hbar} f(\alpha) \left( \frac{1}{\beta} \right)^{\frac{1}{\alpha-1}} \left( x_i - x_f \right)^{\frac{\alpha}{\alpha-1}} \int_{-1}^{0} (\dot{q})^2 \, dt}. \quad (40)$$

Expression (40) implies a natural factorization of the amplitude into one factor stemming from the contribution of the classical path and another arising from the quantum fluctuations.

Let $t_0 = 0 > t_1 > \cdots > t_N = -1$ be a partition of the time interval $[-1, 0]$ by points of subdivision $t_0, t_1, \cdots, t_N$. The quantum fluctuations are discretized according to $q_i = q(t_i)$, $i = 0, \cdots, N$ with $q_0 = q_N = 0$. The normalization factor of fluctuations occurring in (40) will be evaluated by writing the functional integral as an infinite product of integrals over the discretized fields $q_k$ as follows

$$\int\mathcal{D}q \, e^{\frac{i}{\hbar} g(x_i - x_f; \alpha) \int_{-1}^{0} (\dot{q})^2 \, dt} = \lim_{N \rightarrow \infty} \left( 2\pi \hbar \right)^{-N} \left( \frac{2\pi \hbar g(x_i - x_f; \alpha)}{i\epsilon} \right)^{\frac{N}{2}} \prod_{k=1}^{N-1} \int_{-\infty}^{\infty} dq_k \, e^{\frac{i}{\hbar} S_N} \quad (41)$$

where $g(x_i - x_f; \alpha) = f(\alpha) \left( \frac{1}{\beta} \right)^{\frac{1}{\alpha-1}} \left( x_i - x_f \right)^{\frac{2\alpha}{\alpha-1}}$ and $S_N = \frac{1}{2} g(x_i - x_f; \alpha) \sum_{k=1}^{N} (q_k - q_{k-1})^2$.

To diagonalize the discretized action $S_N$ we expand the quantum fluctuations in Fourier modes

$$q_k = \sqrt{\frac{2}{N}} \sum_{m=1}^{N-1} \sin \left( \frac{k m \pi}{N} \right) r_m = \sum_{m=1}^{N-1} O_k^m r_m; \quad k = 1, \cdots, N - 1. \quad (42)$$

This expansion also appears in the continuum time limit by solving the Sturm-Liouville problem: $\dot{q} + \lambda q = 0; \quad q(0) = q(-1) = 0$, with the unique solution $q(t) = \sum_{n=1}^{\infty} r_n \sqrt{2} \sin(n\pi t)$. 

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Substituting (42) into the discretized action and employing the orthonormality relation \(\sum_{m=1}^{N-1} O_j^m O_k^m = \delta_{j,k}\) we get a product of independent Gaussian integrals which can be calculated giving the result
\[
\prod_{k=1}^{N-1} \int_{-\infty}^{\infty} dq_k \ e^{\frac{2i\pi h}{\beta}(x_{1-f,\alpha})} \sum_{k=1}^{N} (q_k-a_{k-1})^2 = \left(\frac{2i\pi h}{g(x_i-x_f,\alpha)}\right)^{\frac{N-1}{2}} \frac{1}{\sqrt{N}}. \tag{43}
\]
Combining (40), (41) and (43) the amplitude is given by
\[
< x_f, t_f | x_i, t_i > = \frac{1}{\sqrt{2\pi h}} f(\alpha)^{\frac{1}{2}} \frac{1}{\beta^{\frac{1}{2(\alpha-1)}}} \left(\frac{x_i-x_f}{2(\alpha-1)}\right)^{\frac{1}{2(\alpha-1)}} e^{\frac{i}{h}(\frac{1}{\beta})^\alpha} \bar{f} \left(\frac{x_i-x_f}{\beta}\right)^{\frac{1}{2(\alpha-1)}}. \tag{44}
\]
If we analytically continue \(\beta\) from Minkowski to Euclidean space using \(\beta = e^{-i\varphi(\alpha-1)}\beta_E\) the Euclidean amplitude becomes
\[
< x_f, t_f | x_i, t_i > = \frac{1}{\sqrt{2\pi h}} f(\alpha)^{\frac{1}{2}} \frac{1}{\beta^{\frac{1}{2(\alpha-1)}}} \left(\frac{x_i-x_f}{2(\alpha-1)}\right)^{\frac{1}{2(\alpha-1)}} e^{\frac{1}{h}(\frac{1}{\beta})^\alpha} \bar{f} \left(\frac{x_i-x_f}{\beta}\right)^{\frac{1}{2(\alpha-1)}}. \tag{45}
\]
We will show now that this amplitude coincides with the long time limit of the propagator \(\sqrt{E}\) at fixed position. We write it as
\[
P(|x_N-x_0|, \beta; \alpha) = \frac{1}{2\pi|x_N-x_0|} \int_{-\infty}^{\infty} e^{iz} e^{-\rho |z|^\alpha} dz \tag{46}
\]
where \(\rho = \frac{\beta h^{\alpha-1}}{(2m)^{\frac{1}{2}}|x_N-x_0|^\alpha}\). To get the best asymptotic approximation of (46) with \(\beta \rightarrow \infty\) we apply the method of steepest descent. For this we write
\[
I(\rho; \alpha) = \int_{i\gamma} e^{iz-\rho |z|^\alpha} dz = \int_{i\gamma} e^{h(z)} dz \tag{47}
\]
where \(\gamma\) is the real axis in the complex \(z\)-plane. We deform the original contour of integration onto \(\gamma'\) so that it takes a steepest descent path through the saddle point of \(h(z)\) avoiding singularities and branch cuts. Then on this contour, most of the contribution to the integral comes from the region around the saddle point, and possibly the endpoints.

The location of the saddle points of \(h(z)\) is determined by setting the first derivative of \(h(z)\) to zero. Thus we find
\[
z_0(\alpha) = \frac{1}{(\rho\alpha)^{\frac{1}{\alpha-1}}} e^{\frac{\pi i}{2(\alpha-1)}}. \tag{48}
\]
Expanding \(h(z)\) in a Taylor series around \(z_0\) we have
\[
h(z) = h(z_0) + \frac{1}{2!} (z-z_0)^2 h''(z_0) + \cdots \tag{49}
\]
\[
= h(z_0) + \frac{1}{2!} (z-z_0)^2 e^{i\theta} |h''(z_0)| + \cdots \tag{50}
\]
where \(e^{i\theta}\) is the phase of \(h''(z_0)\). The linear part of the new path \(\gamma'\) is
\[
z = z_0 + re^{i\phi} \tag{51}
\]
so that

\[ dz = e^{i\phi} dr. \]  

(52)

Here \( e^{i\phi} \) controls the direction of the path. We choose the direction of the path to decrease as rapidly as possible such that the function \( h(z) \) along this path has the form

\[ h(z) = h(z_0) - \frac{1}{2!} r^2 |h''(z_0)| + \cdots \]  

(53)

Expression (53) coincides to (50) provided that

\[ \phi = \frac{(\pi - \theta)}{2} \]  

(54)

Thus the integral \( I(\rho; \alpha) \) becomes

\[ I(\rho; \alpha) = e^{h(z_0)} \int_0^\infty e^{-\frac{1}{2} |h''(z_0)| r^2} \left( \frac{dz}{dr} \right)_{z_0} \]  

(55)

Substituting in (55) \( h(z_0) = \left( \frac{1}{\rho\alpha} \right)^{\frac{1}{\alpha-1}} \left( 1 - \frac{1}{\alpha} \right) e^{\frac{i\pi}{2} \left( \frac{1}{\alpha} \right)} \), \( |h''(z_0)| = (\alpha - 1) (\rho\alpha)^{-\frac{1}{\alpha-1}} \), \( \phi_0 = \frac{\pi}{4} \left( \frac{2-\alpha}{\alpha-1} \right) \) and \( \alpha = \frac{2(1+2k)}{1+4k} \), with \( k \in \mathbb{Z} \), we get

\[ P(|x_N - x_0|, \beta; \alpha) = \frac{1}{\sqrt{2\pi h \beta}} \frac{f(\alpha)}{\beta^2^{(\alpha-1)}} \left( \frac{\alpha-1}{\alpha} \right)^{\frac{1}{\alpha-1}} |x_N - x_0|^{2-\alpha} e^{-\frac{1}{2} f(\alpha) \left( \frac{1}{\beta E} \right)^{\frac{1}{\alpha-1}} |x_N - x_0|^{\frac{\alpha}{\alpha-1}}}. \]  

(56)

As a check one can prove that

\[ \lim_{\alpha \to 2^{-}} P(|x_N - x_0|, \beta; \alpha) = \left( \frac{m}{2\pi h \beta} \right)^{\frac{1}{2}} e^{-\frac{m|x_N - x_0|^2}{2\beta \alpha}} \]  

(57)

since the binomial expansion becomes a perfect square and hence all higher order contributions vanish in this limit.

4 Conclusion

In the super-diffusion case \( 1 < \alpha < 2 \), we have generalized the transition probability density of a particle governed by a symmetric and stable Lévy flight in Euclidean \( n \)-dimensional space. Following the Hamiltonian approach the propagator is written in terms of the Fox function and its derivative in one- and three-dimensions correspondingly. In \( n = 2 \) and \( n > 3 \) the propagator is given in integral form using the Bessel functions. The Lagrangian approach on the other hand provides the asymptotic behaviour of the propagator in the long time limit at fixed position. The result produced by the Lagrangian method has also been crossed checked using the steepest descent applied to the exact result derived from the Hamiltonian approach.
Appendix A

Definition 1
Let \( X \) be a random vector on \( \mathbb{R}^n \) with probability distribution \( \mu \) and characteristic function \( \omega \). If \( \mu^m = \mu * \cdots * \mu \) denotes the \( m \)-fold convolution of \( \mu \) with itself then we say that \( X \) is infinitely divisible if for each \( m \in \mathbb{N} \) there exist \( X_{1,m}, X_{2,m}, \ldots, X_{m,m} \) mutually independent and identically distributed random variables such that the sum \( X_{1,m} + X_{2,m} + \cdots + X_{m,m} \) has the same distribution as \( X \).

Hence if \( X_{i,m} \) has distribution \( \mu_{i,m} \) and characteristic function \( \omega_{i,m} \) then \( \mu = \mu_{1,m} \) and \( \omega = \omega_{1,m} \). The Lévy-Khintchine representation states that a probability measure \( \mu \) on \( \mathbb{R}^n \) is infinitely divisible iff we can write the characteristic function \( \omega(p) = E[e^{i<p,X>}] \) in the form \( \exp(\psi(p)) \) where

\[
\psi(p) = i < p, c > - \frac{1}{2} < p, Ap > + \int_{\mathbb{R}_0^n} \left( e^{i<p,x>} - 1 - \frac{i < p, x >}{1 + ||x||^2} \right) M(dx)
\]

where \( \mathbb{R}_0^n := \mathbb{R}^n / \{0\}, \ c \in \mathbb{R}^n \), \( A \) is a symmetric nonnegative-definite \( n \times n \) matrix (called the Gaussian covariance matrix) and \( M \) is a \( \sigma \)-finite Borel measure on \( \mathbb{R}_0^n \) (called a Lévy measure) such that

\[
\int_{\mathbb{R}_0^n} \min\{1, ||x||^2\} M(dx) < \infty.
\]

If \( A = 0 \) then \( \mu \) is said to be purely non-Gaussian. The triplet \([c, A, M]\) is unique and will be called the generating triplet of the infinitely divisible distribution \( \mu \).

A special case of infinitely divisible laws are the stable laws in \( n = 1 \). Suppose that \( X, X_1, \ldots, X_m \) denote mutually independent random variables with a common distribution \( F \) and \( S_m = \sum_{i=1}^m X_i \).

Definition 2
The distribution \( F \) is stable if for each \( m \in \mathbb{N} \) there exist constants \( c_m > 0 \) and \( \gamma_m \in \mathbb{R} \) such that

\[
S_m \overset{d}{=} c_m X + \gamma_m
\]

and \( F \) is not concentrated at one point. \( F \) is stable in the strict sense if \( \gamma_m = 0 \).

The symbol \( \overset{d}{=} \) means that the distributions of \( S_m \) and \( X \) differ by location and scale parameters. The norming constants are of the form \( c_m = m^\frac{\alpha}{2} \) with \( 1 < \alpha \leq 2 \) and the constant \( \alpha \) is called characteristic exponent of \( F \) or index of the stable law.

If we consider \( A = 0 \) and the Lévy measure to be of the form \( M(r, \infty) = pCr^{-\alpha} \) and \( M(-\infty, -r) = qCr^{-\alpha} \) where \( r, C > 0, \ p, q \geq 0, \ p + q = 1 \) for some \( 0 < \alpha \leq 2 \) then \( \mu \) has characteristic function given by

\[
\psi(p) = i \gamma p - \begin{cases} \left| c \right|^\alpha \left[ 1 - ib \text{sgn}(p) \tan\left( \frac{\pi \alpha}{2} \right) \right]; & \alpha \neq 1 \\ C \frac{\alpha}{2} \left[ 1 + \frac{2}{\pi} \text{sgn}(p) \ln(|p|) \right]; & \alpha = 1 \end{cases}
\]

where \( b = p - q, \ c = C^{\frac{\Gamma(2-\alpha)}{1-\alpha}} \cos\left( \frac{\pi \alpha}{2} \right) \) with \(-1 \leq b \leq 1, \ c \geq 0 \) and \( \gamma \in \mathbb{R} \). The parameters \( \alpha, b, c, \gamma \) characterize the asymptotic behaviour, the skewness, the scale and the location of the peak of the stable distribution respectively. Moreover the collection \((\alpha, b, c, \gamma)\) is called the stable law parameters. A stable law generated by \((\alpha, b, c, \gamma)\) is often denoted by \( S_\alpha(b, c, \gamma) \). In the present work we set the degree of asymmetry to zero \((b = 0)\).
The \( \alpha \)th order Weyl derivatives on the infinite axis are defined as \(^6\)

\[
(-\infty \mathcal{D}_{\pm}^\alpha f)(x) = (\mathcal{D}_{\pm}^\alpha f)(x) = \left\{ \begin{array}{ll} \frac{d^m}{dx^m} \left( t_+^{1-\alpha} f(x) \right) & \text{for } x > -\infty \\ \frac{d^m}{dx^m} \left( t_-^{1-\alpha} f(x) \right) & \text{for } x < \infty \end{array} \right.
\]

where \( \alpha = \lceil \alpha \rceil + \{ \alpha \} \) with \( \lceil \alpha \rceil \), \( \{ \alpha \} \) standing for the integral and fractional part \((0 < \{ \alpha \} < 1)\) of the real number \( \alpha \). Also \( m = \lceil \alpha \rceil + 1 \) and \( \mathcal{D}_+^\alpha f(I_+^{1-\alpha}) \) are the left- and right-handed fractional derivatives (integrals) and \( \Gamma \) the Euler’s gamma function. It can be shown that for \( f \in C_0^\infty(\Omega) \), \( \Omega \subset \mathbb{R} \) the Fourier transforms of \((A.5)\) satisfy for \( 0 < \alpha \leq 2 \)

\[
\mathcal{F} \left( \mathcal{D}_{\pm}^\alpha f(x) \right) = (\mp ip)^\alpha \mathcal{F} (f(x)) = (\mp ip)^\alpha \hat{f}(p)
\]

where \( \mathcal{F} (f(x)) = \hat{f}(p) = \int_{\mathbb{R}} e^{i p x} f(x) dx \) and \((\mp ip)^\alpha = |p|^\alpha e^{\mp i \alpha \text{sgn}(p)}\). One can prove that

\[
D_+^{\tilde{\alpha}} D_-^{\tilde{\alpha}} f(x) = \mathcal{F}^{-1} \left[ (-i p)^\alpha (ip)^\alpha \hat{f}(p) \right] = \mathcal{F}^{-1} \left[ |p|^\alpha \hat{f}(p) \right] = (-\nabla^2)^{\tilde{\alpha}} f(x)
\]

where \( \nabla^2 \) is the one dimensional Laplacian.

**Proposition 1**

Let \( f(x) = e^{i \tilde{p} \tilde{x}} \) where \( \tilde{p}, \tilde{x} \in \mathbb{R}^n \). Then

\[
(-\nabla^2)^{\tilde{\alpha}} f(x) = (p^2)^{\tilde{\alpha}} f(x)
\]

where \( p^2 = \sum_{i=1}^n p_i^2 \) and \( p_i \in \mathbb{R} \).

**Proof**

Define the translation operator \( T_{\tilde{h}} \) of a function \( f(\tilde{x}) \) by

\[
(T_{\tilde{h}} f)(\tilde{x}) = f(\tilde{x} - \tilde{h})
\]

and the finite difference of order \( l \) of \( f(\tilde{x}) \) with step \( \tilde{h} \) and center at the point \( \tilde{x}, \Delta_{\tilde{h}}^l \), by

\[
(\Delta_{\tilde{h}}^l f)(\tilde{x}) = (I - T_{\tilde{h}})^l f(\tilde{x}) = \sum_{k=0}^l \binom{l}{k} (-1)^k f(\tilde{x} - k\tilde{h})
\]

where \( I \) is the identity operator. It can be shown that the Fourier transform of the hypersingular integral

\[
D^\alpha f = \int_{\mathbb{R}^n} \frac{\Delta_{\tilde{y}}^l f(\tilde{x})}{|\tilde{y}|^{n+\alpha}} d^n\tilde{y}
\]

is given by

\[
\mathcal{F} (D^\alpha f(\tilde{x})) = |p|^\alpha d_{n,l}(\alpha) \mathcal{F} (f(\tilde{x}))
\]

where the constant \( d_{n,l}(\alpha) \) is

\[
d_{n,l}(\alpha) = \int_{\mathbb{R}^n} d^n u \frac{(1 - e^{i \tilde{p} \cdot \tilde{u}})^l}{|u|^{n+\alpha}}.
\]
In (A.13) \( \hat{p} \) is the unit vector in the direction of \( \vec{p} \). Using as \( f(x) = e^{i\vec{p} \cdot \vec{x}} \) we have

\[
(-\nabla^2)^{\frac{\alpha}{2}} e^{i\vec{p} \cdot \vec{x}} = \frac{1}{d_{n,l}(\alpha)} D^\alpha e^{i\vec{p} \cdot \vec{x}}
\]

\[
= \frac{1}{d_{n,l}(\alpha)} e^{i\vec{p} \cdot \vec{x}} \int_{\mathbb{R}^n} d^n y \frac{1}{|y|^{n+\alpha}} \sum_{k=0}^l \left( \frac{l}{k} \right) (-1)^k e^{-ik\vec{p} \cdot \vec{y}}
\]

\[
= \frac{1}{d_{n,l}(\alpha)} e^{i\vec{p} \cdot \vec{x}} \int_{\mathbb{R}^n} d^n y \left( 1 - e^{i\vec{p} \cdot \vec{y}} \right)^l
\]

\[
= (p^2)^{\frac{\alpha}{2}} e^{i\vec{p} \cdot \vec{x}}.
\]

(A.14)

**Appendix B**

The class of Fox or H-functions \( [7] \) comprises a large class of special functions known in mathematical physics. It is defined in terms of the Mellin-Barnes path integral

\[
H_{m,n}^{p,q}(z) = H_{p,q}^{m,n} \left[ z \right| (a_1, A_1), (a_2, A_2), \ldots, (a_p, A_p) \right| (b_1, B_1), (b_2, B_2), \ldots, (b_q, B_q) \right] = \frac{1}{2\pi i} \int_L \chi(s) z^s ds \quad (B.1)
\]

with the integral density

\[
\chi(s) = \frac{\prod_{i=1}^m \Gamma(b_i - B_i) \prod_{j=1}^p \Gamma(1 - a_i + A_i)}{\prod_{i=m+1}^q \Gamma(1 - b_i + B_i) \prod_{j=n+1}^p \Gamma(a_j - A_j)}.
\]

(B.2)

The H-function possesses a number of interesting properties \( [8] \) which we list the ones we employed in our presentation.

- If \( k > 0 \) then,

\[
H_{p,q}^{m,n}(z) = k H_{p,q}^{m,n} \left[ z \right| (a_1, A_1), (a_2, A_2), \ldots, (a_p, A_p) \right| (b_1, B_1), (b_2, B_2), \ldots, (b_q, B_q) \right].
\]

(B.3)

- Under Fourier cosine transformation, the H-function transforms as

\[
\int_0^\infty H_{p,q}^{m,n} \left[ x \right| (a_p, A_p) \right| (b_q, B_q) \right] \cos(kx) dx = \frac{\pi}{k} H_{q+1,p+2}^{n+1,m} \left[ k \right| (1 - b_q, B_q), (1, \frac{1}{2}) \right| (1, 1), (1 - a_p, A_p), (1, \frac{1}{2}) \right].
\]

(B.4)

- The function \( z^b e^{-z} \) can be represented in terms of the H-function as

\[
z^b e^{-z} = H_{0,1}^{1,0} \left[ z \right| (0, 0) \right| (b, 1) \right].
\]

(B.5)
Appendix C

Using spherical coordinates in (22) one has

\[ P(|\vec{x}_N - \vec{x}_0|, \beta; \alpha) = \frac{1}{(2\pi \hbar)^n} \int_0^\infty dp_N \ p_N^{n-1} e^{-\frac{\beta}{\hbar} H(p_N)} F(p_N, |\vec{x}_N - \vec{x}_0|, n) \]  

(C.1)

where

\[ F(p_N, |\vec{x}_N - \vec{x}_0|, n) = \int_0^{2\pi} d\theta_1 \left( \prod_{k=2}^{n-2} \int_0^\pi d\theta_k \sin^{k-1} \theta_k \right) \int_0^\pi d\theta_{n-1} \sin^{n-2} \theta_{n-1} e^{\frac{i}{\hbar} p_N |\vec{x}_N - \vec{x}_0| \cos \theta_{n-1}}. \]  

(C.2)

Performing the integrations over the angles the integral representation of the propagator is

\[ P(|\vec{x}_N - \vec{x}_0|, \beta; \alpha) = \left( \frac{1}{2\pi \hbar} \right)^{\frac{n}{2}} \frac{1}{|\vec{x}_N - \vec{x}_0|^{\frac{n}{2} - 1}} \int_0^\infty dp_N \ p_N^{n} e^{-\frac{\beta}{\hbar} H(p_N)} J_{\frac{n}{2} - 1}(\frac{p_N}{\hbar} |\vec{x}_N - \vec{x}_0|). \]  

(C.3)

The calculation of the integrals over the angles is based on the formulas [9]

\[ \int_0^\pi d\theta \sin^m \theta = \frac{\Gamma(\frac{1}{2}(m+1))}{\Gamma(\frac{1}{2}(m+2))}, \quad m \geq 0 \]

\[ \int_0^\pi d\theta \sin^m \theta e^{ic \cos \theta} = \int_1^1 dx (1 - x^2)^{\frac{m-1}{2}} e^{icx} = 2 \frac{\Gamma(m)}{\Gamma(m/2)} e^{ic} J_{\frac{m}{2}}(c) \]

\[ \int_0^1 (1 - x^2)^{\nu - \frac{1}{2}} \cos(ax) dx = \sqrt{\frac{\pi}{2}} \left( \frac{2}{a} \right)^{\nu} \Gamma(\nu + \frac{1}{2}) J_\nu(a) \]

\[ \int_0^\infty x^\nu e^{-ax^2} J_\nu(\beta x) dx = \frac{\Gamma(\nu + \mu + 12}{\beta a^{\frac{\nu}{2}} \Gamma(\nu + 1) e^{-\frac{a^2}{12}} M_{\frac{\nu}{2}, \frac{1}{2}}(\frac{\beta^2}{4a})} \]

\[ 2^{2\nu - 1} \Gamma(\nu) \Gamma(\nu + 1 + 2x) = \sqrt{\pi} \Gamma(2x). \]  

(C.4)

In (C.4) \( J_\nu \) is the Bessel function of the first kind of order \( \nu \) defined by the equations [10]

\[ J_\nu(z) = \frac{1}{2\pi i} \left( \frac{z}{2} \right)^{\nu} \int_{-\infty}^{+\infty} t^{-\nu-1} e^{zt - \frac{z^2}{4t}} dt \]

\[ = \left( \frac{z}{2} \right)^{\nu} \sum_{k=0}^{\infty} (-1)^k \frac{\frac{z^{2k}}{2^{2k} k!}}{\Gamma(\nu + k + 1)} \]  

(C.5)

or in a more suitable form by

\[ J_{\frac{n}{2} + \frac{1}{2}}(z) = (-1)^n z^{\frac{n}{2} + \frac{1}{2}} \sqrt{\frac{2}{\pi (zdz)^n}} \sin z \]  

(C.6)

where \( n \) is a natural number. For completeness we give the explicit expressions for \( J_0 \) and \( J_{\frac{1}{2}} \) used for the propagators in two and three dimensions. They are

\[ J_0(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\frac{z^{2k}}{2^{2k} k!}}{\Gamma(\nu + k + 1)} \]

\[ J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \sin z. \]  

(C.7)
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