ON THE SPECTRAL STABILITY OF STANDING WAVES OF THE ONE-DIMENSIONAL $M^5$-MODEL

SALVADOR CRUZ-GARCÍA AND CATHERINE GARCÍA-REIMBERT

Departamento de Matemáticas y Mecánica
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas
Universidad Nacional Autónoma de México
Apdo. Postal 20-726, C.P. 01000 México D.F., México

(Communicated by Thomas Hillen)

Abstract. We consider the spectral stability problem for a family of standing pulse and wave front solutions to the one-dimensional version of the $M^5$-model formulated by Hillen [T. Hillen, $M^5$ mesoscopic and macroscopic models for mesenchymal motion, J. Math. Biol., 53 (2006), 585–616], to describe the mesenchymal cell motion inside tissue. The stability analysis requires the definition of spectrum, which is divided into two disjoint sets: the point spectrum and the essential spectrum. Under this partition the eigenvalue zero belongs to the essential spectrum and not to the point spectrum. By excluding the eigenvalue zero we can bring the spectral problem into an equivalent scalar quadratic eigenvalue problem. This leads, naturally, to deduce the existence of a negative eigenvalue which also turns out to belong to the essential spectrum. Beyond this result, the scalar formulation enables us to use the integrated equation technique to establish, via energy methods, that the point spectrum is empty. Our main result is that the family of standing waves is spectrally stable. To prove it, we go back to the original scalar problem and show that the rest of the essential spectrum is a subset of the open left-half complex plane.

1. Introduction. The ability of cancer cells to invade locally the host stroma and its eventual metastatic dissemination to distant organs has attracted considerable attention to the study of the process of cell migration. Friedl and Wolf [3] reported that mesenchymal-type movement is found in cells from connective tissue tumors. Mesenchymal motion is a class of individual cell migration that takes place inside the network of collagen fibres which form a supporting framework for tissues (the extracellular matrix, abbreviated as ECM). The interaction between cells and ECM plays an important role in the process of mesenchymal migration; during this process, directional information is provided by the orientation of the matrix fibres (a process known as contact guidance) and at the same time cells degrade these fibres through the secretion of matrix proteolytic enzymes to create a path to facilitate migration [21].

2010 Mathematics Subject Classification. Primary: 34L05, 35B35, 35A18; Secondary: 92C17, 35L40, 35B40.

Key words and phrases. Mesenchymal motion, standing pulse and wave front, spectral stability, integrated eigenvalue equation, Gap lemma.

The work of SCG was supported by CONACYT through the Programa de Becas de Posgrado, grant no. 220870.

* Corresponding author: cgr@mym.iimas.unam.mx.

1079
In [6], a transport model to describe the mesenchymal cell movement was put forward by Hillen. The $M^5$-model introduced by the author is written in the form of two coupled integro-differential equations. The system consists of a transport equation for the changes in space and time of the moving cell population and a dynamic equation for the directional distribution of tissue fibres, in both equations cell-ECM interaction terms are included. In this model a distinction is made between undirected and directed tissue types. In undirected tissues the fibres are symmetric, so that the cells are unable to distinguish between two opposite directions of migration. In directed tissues the fibres are asymmetric and the two ends have a specific polarity.

Painter proposed in [11] an extended version of the $M^5$-model for undirected tissues. Through numerical simulations in 2D, the author investigated the effect of contact guidance and the joint action of focussed proteolysis and new matrix assembly. The results obtained suggest that these processes are capable of producing stable steady network patterns. These patterns are composed by a network of interconnected dense cell-chains sustained by highly aligned fibres parallel to the direction of cell movement, which is surrounded by zones of nearly unaligned ECM where the cell density is lower. The formation of these numerically stable networks offered Hillen et al. [7] a source of motivation to investigate analytically the steady states of the $M^5$-model for the undirected case. Having constructed a suitable solution framework, Hillen and coworkers found that, in $\mathbb{R}^n$, cells and fibres uniformly distributed in space and directions (homogeneous tissue) constitute a steady state, in which they observed that in any steady state the cell orientation is completely given by the direction of the fibres. More interesting tissue configurations with regions of completely aligned fibres required the use of Dirac delta functions to represent fibres orientation. With the aim of designing network type steady states, Hillen et al. introduced the concepts of weak steady state and pointwise steady state. The first concept allowed them indentify the arrangements of strictly aligned matrix fibres (strictly aligned tissues) as steady states of the model. The second concept permits building patchy steady states and steady states of network type in $\mathbb{R}^2$. The former consists of homogeneous tissue disposed over disjoint open sets, which are divided by curves of finite length possibly closed but without intersections between them, the orientation of cells and fibres along the curves is determined by the direction of the tangent vector at each point. The latter is similar, differing in that intersections are admissible. At the end of [7] the question about the stability of the steady states is put on the table, this question also concerns if solutions can evolve to a traveling wave as $t \to \infty$. As far as we know the question still remains without an answer.

Before the publication of [7], Wang et al. worked on the one-dimensional version of the $M^5$-model [20], which corresponds to the case where the mesenchymal cell population moves within a tissue made up of totally aligned fibres. During their investigation they found that standing and traveling pulse and front solutions are admitted when the motion takes place in directed tissues, and that in undirected tissues these have no chance to exist. In this paper the spectral stability of the standing pulse and front solutions is investigated. Here we do not mean to answer the question put by Hillen et al., but we hope to make a path for future research into the asymptotic stability of the one-dimensional traveling pulse and front solutions or even of the steady states identified in [7].
The one-dimensional $M^5$-model reads [6]:

$$
\begin{align*}
 p_t^+ + sp_x^+ & = -\mu p^+ + \mu q^+ (p^+ + p^-), \\
 p_t^- - sp_x^- & = -\mu p^- + \mu q^- (p^+ + p^-), \\
 q_t^+ & = \kappa (p^+ - p^-) (q^- - q^+ + 1) q^+, \\
 q_t^- & = \kappa (p^+ - p^-) (q^- - q^+ - 1) q^-,
\end{align*}
\tag{1}
$$

with $(x, t) \in \mathbb{R} \times [0, \infty)$. The constants $\mu > 0$ and $\kappa > 0$ denote the turning rate and the rate of matrix degradation, respectively. The function $p^\pm(x, t)$ describes the density of cells moving to the right/left with constant speed $\pm s$. The probability density for a cell to choose to move to the right/left is denoted by $q^\pm(x, t)$.

In order to see what kind of movement patterns are used by cells in the process of tissue invasion, Wang et al. [20] sought traveling wave solutions for the system (1). They showed that $q^+ + q^- = 1$ is an invariant manifold of the system (1). Using this fact they could rewrite the system (1) as a system of equations for the total cell population $p = p^+ + p^-$, the population flux $j = s(p^+ - p^-)$, and the probability of moving to the right $q^+$. The equivalent model is composed of the following system

$$
\begin{align*}
 p_t + j_x & = 0, \\
 j_t + s^2 p_x & = -\mu j + \mu s (2q^+ - 1) p, \\
 q_t^+ & = \frac{2\kappa}{s} jq^+ (1 - q^+).
\end{align*}
\tag{2}
$$

By applying the traveling wave ansatz

$$
p(x, t) = \bar{p}(z), \quad j(x, t) = \bar{j}(z), \quad q^+(x, t) = \bar{q}^+(z), \quad z = x - ct,
$$

where $c \geq 0$ is the wave speed, Wang et al. [20] obtained the traveling wave system:

$$
\begin{align*}
 -c\bar{p}_z + \bar{j} & = 0, \\
 -c\bar{j}_z + s^2 \bar{p}_z & = -\mu \bar{j} + \mu s (2\bar{q}^+ - 1) \bar{p}, \\
 -c\bar{q}^+_z & = \frac{2\kappa}{s} \bar{j} \bar{q}^+ (1 - \bar{q}^+) .
\end{align*}
\tag{3}
$$

The solutions of interest are those that satisfy the boundary conditions

$$
\bar{p}(\pm \infty) = \bar{j}(\pm \infty) = 0, \quad \bar{q}^+(-\infty) = \bar{q}^+_i \quad \text{and} \quad \bar{q}^+ (+\infty) = \bar{q}^+_f,
\tag{4}
$$

where $\bar{q}^+_i$ and $\bar{q}^+_f$ are constants that satisfy $0 < \bar{q}^+_i, \bar{q}^+_f \leq 1$ and $\bar{q}^+_i > \bar{q}^+_f$. These solutions correspond to pulses for the quantities $\bar{p}$ and $\bar{j}$, and a wave front for $\bar{q}^+$. Integration of the first equation in (3) together with the boundary conditions lead to the invariant of motion

$$
\bar{j} = c\bar{p}.
\tag{5}
$$

Substitution of (5) into the last two equations of system (1) reduces to

$$
\begin{align*}
 (c^2 - s^2) \bar{p}_z & = \mu \bar{p} \left[ c - s(2\bar{q}^+ - 1) \right], \\
 \bar{q}^+_z & = -\frac{2\kappa}{s} \bar{p} (1 - \bar{q}^+) \bar{q}^+. \tag{6}
\end{align*}
$$

Wang et al. [20] assume that $c \neq s$, since when $c = s$ the problem (6) becomes singular with homogeneous steady solution $\bar{q}^+ = 1$, which does not satisfy the above
boundary conditions. The system (6) finally takes the form
\[
\begin{align*}
\bar{p}_z &= \frac{\mu}{c^2 - s^2} \bar{p} [c - s (2q^+ - 1)], \\
\bar{q}^+_z &= -\frac{2\kappa}{s} \bar{p} (1 - \bar{q}^+) \bar{q}^+.
\end{align*}
\]
(7)

The analysis of (7) performed by Wang and collaborators revealed that there exists a family of standing and traveling pulse solutions for the total cell population \(\bar{p}\), and a family of standing and traveling front solutions for the probability density \(\bar{q}^+\). They noted that the system (7) has a continuum of steady states \((\bar{p}, \bar{q}^+) = (0, \theta)\) with \(0 \leq \theta \leq 1\). They first proved that each fixed \(c\) on the segment \(0 \leq c < s\), determines a constant \(\theta^* = \frac{c + s}{2s}\) with the property that the steady state \((0, \theta)\) is stable for all \(0 \leq \theta < \theta^*\), and unstable for all \(\theta^* < \theta \leq 1\). Thereafter, the authors showed that for each left state \(q_l\) with \(\theta^* < q^+_l < 1\), the trajectory leaving the point \((0, q^+_l)\) finishes, as \(z \to +\infty\), at some right steady state \((0, q^+_r)\) with \(0 < q^+_r < \theta^*\), giving rise to a nonnegative heteroclinic connection. The authors stress out the fact that neither a heteroclinic trajectory starting from \((0, 1)\), nor a heteroclinic trajectory ending at \((0, 0)\), can exist. For a detailed discussion we refer the reader to [20].

Here one may ask whether the model can maintain the pulses and wave fronts during cell spreading. This is a matter of orbital asymptotic stability. This issue refers to whether a solution initially close to a wave profile approach, as \(t \to \infty\), some translation thereof. In a notable study on stability of traveling waves in nonstrictly hyperbolic systems, Rottmann-Matthes [16, 17] proves orbital stability due to the presence of spectral stability. In [17], the orbital stability problem is tackled by reformulating it as a partial differential algebraic equation. In general, spectral stability does not necessarily imply orbital stability, but in view of the results of Rottmann-Matthes, we believe that establishing spectral stability is a good advance towards our more ambitious goal of achieving orbital stability (see the Discussion section). As a first step in this direction, we focus on the family of standing pulse and wave front solutions, carrying out a careful analysis of its spectral stability. In this work we prove the following.

**Theorem 1.1.** The family of standing waves is spectrally stable.

In order to prove this result, we have organized the paper as follows. In Section 2 we state the main existence results of [20]. Then we prove that higher amplitude pulses travel with a slower speed than smaller amplitude pulses. We end by showing that the wave profiles converge exponentially to their end states. This last property will be of fundamental importance to the subsequent analysis. In Section 3 we formulate the spectral problem and define the resolvent set and the spectrum, as well as spectral stability of the wave profiles; the spectrum is defined to be the disjoint union of the point spectrum and the essential spectrum. We then prove that the eigenvalue zero belongs to the essential spectrum, thus remaining outside the point spectrum. The spectral problem is recast as an equivalent scalar quadratic eigenvalue problem in Section 4, where, we show that the minus value of the turning rate parameter, namely \(-\mu\), is an eigenvalue which is also an element of the essential spectrum. Section 5 is devoted to analyse in detail the spectrum of the scalar operator. The scalar spectral problem is reformulated as a integrated eigenvalue problem, and this approach helps us to show that the point spectrum of the original spectral problem is empty. The main result of this paper is stated in Section 6, this
establishes that the family of standing wave profiles is spectrally stable. The proof consists in showing that the essential spectrum is a subset of the open left-half complex plane.

**Notation.** Throughout the paper, let \( L^2 \) denote the space of all square integrable functions on \( \mathbb{R} \) with norm \( \| f \|_{L^2} = (\int_\mathbb{R} |f(x)|^2\, dx)^{1/2} \). We denote by \( H^n \) \((n \geq 1)\) the usual Sobolev space on \( \mathbb{R} \) with norm \( \| f \|_{H^n} = (\sum_{i=0}^n \| \partial_x^i f \|_{L^2}^2)^{1/2} \).

2. Existence and structure of standing and traveling wave profiles. We state the results obtained by Wang et al. \cite{20}, concerning the existence of standing and traveling wave solutions for the system (7).

**Theorem 2.1.** \cite{20} Let us consider the system (7) given traveling speed \( c \) with \( 0 \leq c < s \) and \( \theta^* = \frac{c + s}{2s} \). Then for any equilibrium \((0, c_1)\) with \( \theta^* < c_1 < 1 \) there exists another equilibrium \((0, c_2)\) with \( 0 < c_2 < \theta^* \) such that there is a bounded, nonnegative, heteroclinic orbit connecting \((0, c_1)\) to \((0, c_2)\). That is, there exists a traveling solution \((\bar{p}, \bar{q}^+\)\) of the system (7) connecting two equilibria. Particularly, the system (7) admits a standing wave for \( c = 0 \).

**Lemma 2.2.** \cite{20} Given a speed \( c \) satisfying \( 0 \leq c < s \), the left and right equilibria \((0, q_i^+)\) and \((0, q_r^+)\) are related as

\[
\left( \frac{1 - q_i^+}{1 - q_r^+} \right)^{s-c} = \left( \frac{q_r^+}{q_i^+} \right)^{s+c}, \quad 0 \leq c < s. \tag{8}
\]

**Remark 1.** It is worth noting that for each fixed \( c \), the left state \( q_i^+ \) is a free parameter whose variation generates a family of traveling wave solutions, and that, in turn, the wave speed \( c \) parametrizes a family of traveling waves for a given \( q_i^+ \).

In \cite{20}, Wang et al. obtained an explicit expression for the \((\bar{p}, \bar{q}^+)\) heteroclinic trajectory, namely

\[
\bar{p}(\bar{q}^+) = \frac{\mu s}{2\kappa} \ln \left[ \left( \frac{1 - \bar{q}^+}{1 - q_i^+} \right)^{\frac{1}{s+c}} \left( \frac{\bar{q}^+}{q_i^+} \right)^{\frac{1}{s+c}} \right]. \tag{9}
\]

Using this result they computed the maximal value of \( \bar{p} \). They found that the maximum \( \bar{p} \), \( p_{\text{max}} \), occurs at \( \bar{q}^+ = \theta^* \) and is given by

\[
p_{\text{max}} = \frac{\mu s}{2\kappa} \ln \left[ \left( \frac{1 - \theta^*}{1 - q_i^+} \right)^{\frac{s}{2}} \left( \theta^* \right)^{\frac{1}{2s}} \right], \quad \theta^* = \frac{c + s}{2s}. \tag{10}
\]

From (10), Wang et al. observed that \( p_{\text{max}} \) is an increasing function of the left state \( q_i^+ \). This observation can be extended to the dependence of \( p_{\text{max}} \) on the wave speed \( c \). We have noticed that pulses of higher amplitude are slower than those of a lower amplitude. To make this idea precise, we establish the following result.

**Proposition 1.** Let \( q_i^+ \) be a given left state. Then the maximum value \( p_{\text{max}} \) is a decreasing function of the wave speed.

**Proof.** Taking a derivative in the parameter \( c \) in (10), we get

\[
\frac{\partial p_{\text{max}}}{\partial c} = \frac{\mu s}{2\kappa} \left( \frac{1}{(c + s)^2} \left[ \ln(1 - q_i^+) - \ln(1 - \theta^*) \right] + \frac{1}{(s - c)^2} \left[ \ln(\theta^*) - \ln(q_i^+) \right] \right).
\]

Since \( q_i^+ > \theta^* \), we have that \( \ln(1 - q_i^+) < \ln(1 - \theta^*) \) and \( \ln(\theta^*) < \ln(q_i^+) \) for all \( c \in [0, s] \), therefore \( \partial p_{\text{max}} / \partial c < 0 \). \( \square \)
A fundamental structural feature of the wave profiles is that the pulse $\bar{p}$ and the wave front $\bar{q}^+$ converge with an exponential rate to their end states.

**Lemma 2.3.** Traveling wave solutions $\bar{p}$ and $\bar{q}^+$ satisfy

\[
|d^i/dz^i \left( \bar{q}^+ (z) - q^+_i \right)| \sim C \exp \left( -\frac{(c + s - 2sq^+_i)\mu}{s^2 - c^2} z \right), \quad \text{as } z \to +\infty,
\]

\[
|d^i/dz^i \left( \bar{q}^+ (z) - q^+_i \right)| \sim C \exp \left( -\frac{(c + s - 2q^+_i)\mu}{s^2 - c^2} z \right), \quad \text{as } z \to -\infty,
\]

\[
|d^i/dz^i \left( \bar{p} (z) \right)| \sim C \exp \left( -\frac{(c + s - 2q^+_i)\mu}{s^2 - c^2} z \right), \quad \text{as } z \to +\infty,
\]

\[
|d^i/dz^i \left( \bar{p} (z) \right)| \sim C \exp \left( -\frac{(c + s - 2q^+_i)\mu}{s^2 - c^2} z \right), \quad \text{as } z \to -\infty,
\]

for $i = 0, 1$, and some uniform $C > 0$.

**Proof.** We begin by obtaining an uncoupled differential equation for $\bar{q}^+$. Upon substituting formula (9) into the equation for $\bar{q}^+$ in (7) we arrive at

\[
\bar{q}^+_z = -\mu \ln \left[ \frac{1 - \bar{q}^+_z}{1 - q^+_i} \right] \left( \frac{\bar{q}^+}{q^+_i} \right)^{1/\eta} (1 - \bar{q}^+) \bar{q}^+.
\]

Computing the Taylor series of the right-hand side of this equation we find that, as $z \to +\infty$,

\[
\bar{q}^+_z \approx -\frac{(c + s - 2sq^+_i)\mu}{s^2 - c^2} (\bar{q}^+ - q^+_i), \quad (12)
\]

and, that

\[
\bar{q}^+_z \approx -\frac{(c + s - 2q^+_i)\mu}{s^2 - c^2} (\bar{q}^+ - q^+_i), \quad (13)
\]

as $z \to -\infty$. Hence, from (12) we obtain

\[
|\bar{q}^+ (z) - q^+_1 | \sim C_1 \exp \left( -\frac{(c + s - 2sq^+_i)\mu}{s^2 - c^2} z \right), \quad (14)
\]

as $z \to +\infty$, for some constant $C_1 > 0$. In addition, taking the absolute value of (12) and substituting (14), we can conclude

\[
|d/dz \left( \bar{q}^+ (z) - q^+_1 \right)| \sim \frac{(c + s - 2sq^+_i)\mu}{s^2 - c^2} C_1 \exp \left( -\frac{(c + s - 2q^+_i)\mu}{s^2 - c^2} z \right),
\]

as $z \to +\infty$.

Now, from (13) we have that

\[
|\bar{q}^+ (z) - q^+_1 | \sim C_2 \exp \left( -\frac{(c + s - 2q^+_i)\mu}{s^2 - c^2} z \right), \quad (15)
\]

as $z \to -\infty$, for some constant $C_2 > 0$. In view of this, we use (13) and (15) to get

\[
|d/dz \left( \bar{q}^+ (z) - q^+_1 \right)| \sim -\frac{(c + s - 2q^+_i)\mu}{s^2 - c^2} C_2 \exp \left( -\frac{(c + s - 2q^+_i)\mu}{s^2 - c^2} z \right),
\]

as $z \to -\infty$.

Proceeding analogously as before, it results from equation (9) that

\[
\bar{p}(\bar{q}^+) \approx \frac{\mu s (c + s - 2sq^+_i)}{2sq^+_i (1 - q^+_i) (s^2 - c^2)} (\bar{q}^+ - q^+_i), \quad (16)
\]
as \( z \to +\infty \), and also that
\[
\bar{p}(\bar{q}^+ \approx \frac{\mu s(c + s - 2sq_1^+)}{2\kappa q_1^+(1 - q_1^+)(s^2 - c^2)}(\bar{q}^+ - q_1^+),
\]
as \( z \to -\infty \).

We substitute (14) into the absolute value of (16) to obtain
\[
|\bar{p}(z)| \sim \frac{\mu s(c + s - 2sq_1^+)}{2\kappa q_1^+(1 - q_1^+)(s^2 - c^2)} \exp \left( \frac{(c + s - 2sq_1^+}\mu}{s^2 - c^2} \right),
\]
as \( z \to +\infty \). Similarly, from (15) and (17) we can deduce that, as \( z \to -\infty \),
\[
|\bar{p}(z)| \sim -\frac{\mu s(c + s - 2sq_1^+)}{2\kappa q_1^+(1 - q_1^+)(s^2 - c^2)} \exp \left( \frac{(c + s - 2sq_1^+}\mu}{s^2 - c^2} \right),
\]
Given that \( \bar{p} \) and \( \bar{q}^+ \) converge exponentially fast to their stationary states, we can see that the dominant term in the right-hand side of (7) is the linear one. Thus, from (18) we infer that
\[
\left| \frac{dp}{dz} \right| \sim \frac{\mu^2 s(c + s - 2sq_1^+)}{2\kappa q_1^+(1 - q_1^+)(s^2 - c^2)(s - c)} \exp \left( \frac{(c + s - 2sq_1^+}\mu}{s^2 - c^2} \right),
\]
as \( z \to +\infty \). Likewise, (19) leads to
\[
\left| \frac{dp}{dz} \right| \sim -\frac{\mu^2 s(c + s - 2sq_1^+)}{2\kappa q_1^+(1 - q_1^+)(s^2 - c^2)(s - c)} \exp \left( \frac{(c + s - 2sq_1^+}\mu}{s^2 - c^2} \right),
\]
as \( z \to -\infty \).

Finally, we let \( C \) be the upper bound of all constant terms that multiply the exponential functions.

\textbf{3. Spectral problem.} This section is concerned with the formulation of the spectral stability problem for the family of standing wave solutions of (2), whose existence and structure has been established by the results of Theorem 2.1, Lemma 2.2, and by formula (5). All members of this family consist of a function identically zero, \( j \equiv 0 \), and a profile \((\bar{p}, \bar{q}^+)\) that satisfies system (7); for \( c = 0 \) the latter reads
\[
\begin{align*}
\bar{p}_x &= \frac{\mu}{s} \bar{p}(2\bar{q}^+ - 1), \\
\bar{q}^+_x &= -\frac{2\kappa}{s} \bar{p} (1 - \bar{q}^+) \bar{q}^+.
\end{align*}
\]
For a standing front with amplitude \( \varepsilon = q_l^+ - q_r^+ \), \( 0 < \varepsilon < 1 \), one can infer from (8) that the connected left and right endpoints are
\[
q_l^+ = \frac{1}{2}(1 + \varepsilon) \quad \text{and} \quad q_r^+ = \frac{1}{2}(1 - \varepsilon).
\]

\textbf{Remark 2.} If we attempt to directly solve the stationary version of system (2), it results from the boundary conditions (4) that \( j \equiv 0 \). As a consequence, any function \( \bar{q}^+ \) solves the third stationary equation in (2). Thus the problem reduces to solving the scalar equation
\[
p_x = \frac{\mu}{s} (2\bar{q}^+ - 1) p,
\]
for a given function \( \bar{q}^+ \) satisfying (4). Therefore, it is possible that \( \bar{q}^+ \) is not a wave front or that it is a front that does not satisfy the second equation in (20). In this work we do not deal with these kind of solutions.
In order to investigate the spectral stability of the family of standing waves, we consider solutions to system (2) of the form

\[ p(x, t) = \tilde{p}(x) + \tilde{p}(x, t) \], \quad \tilde{j}(x, t) = 0 + \tilde{j}(x, t) \quad \text{and} \quad q^+(x, t) = \tilde{q}^+(x) + \tilde{q}^+(x, t). \] (22)

Substituting (22) into (2) and neglecting nonlinear terms in the perturbations, we obtain the linearized system about the waves:

\[
\begin{align*}
\tilde{p}_t &= -\tilde{j}_x, \\
\tilde{j}_t &= -s^2 \tilde{p}_x - \mu \tilde{j} + \mu s \left[ (2q^+ - 1) \tilde{p} + 2p\tilde{q}^+ \right], \\
\tilde{q}_t &= \frac{2\kappa}{s} q^+ (1 - \tilde{q}^+) \tilde{j}.
\end{align*}
\]

Assuming that perturbations are of the form \( e^{\lambda t}p(x), e^{\lambda t}j(x) \) and \( e^{\lambda t}q^+(x) \) with \( \lambda \in \mathbb{C} \), substitution yields the spectral problem

\[
\begin{align*}
\lambda p &= -j_x, \\
\lambda j &= -s^2 p_x - \mu j + \mu s \left[ (2q^+ - 1) p + 2p\tilde{q}^+ \right], \\
\lambda q^+ &= \frac{2\kappa}{s} q^+ (1 - q^+) j.
\end{align*}
\] (23)

We are interested in solutions of problem (23) in the space \( H^1(\mathbb{R}; \mathbb{C}^3) \). We start our study by giving some definitions used in the stability theory [10].

**Definition 3.1.** Let \( \mathcal{X} \) be a Banach space and let \( \mathcal{L} : D(\mathcal{L}) \to \mathcal{X} \) be a linear operator with dense domain \( D(\mathcal{L}) \subset \mathcal{X} \). The **resolvent set** of \( \mathcal{L} \) is the set of all numbers \( \lambda \in \mathbb{C} \) such that the operator \( \mathcal{L} - \lambda \mathcal{I} \) has a bounded inverse. The complement of the resolvent is called the **spectrum** \( \sigma(\mathcal{L}) \). We say that \( \lambda \in \sigma(\mathcal{L}) \) is an **eigenvalue** of \( \mathcal{L} \) if \( \mathcal{L} - \lambda \mathcal{I} \) has a nontrivial kernel.

**Definition 3.2.** Let \( \mathcal{L} : D(\mathcal{L}) \to \mathcal{X} \) be a linear, closed, densely defined operator. Its spectrum is divided into the **point spectrum** \( \sigma_{pt}(\mathcal{L}) \), which is composed of those eigenvalues \( \lambda \) such that \( \mathcal{L} - \lambda \mathcal{I} \) is Fredholm with index zero, and the **essential spectrum** \( \sigma_{ess}(\mathcal{L}) \) which is the remaining part; \( \sigma_{ess}(\mathcal{L}) = \sigma(\mathcal{L}) \setminus \sigma_{pt}(\mathcal{L}) \).

Recall that a linear operator \( \mathcal{L} : \mathcal{X} \to \mathcal{Y} \) is said to be a **Fredholm** operator whenever its range \( R(\mathcal{L}) \) is closed, and \( \dim[\ker(\mathcal{L})] \) and \( \dim[R(\mathcal{L})] \) are both finite.

**Remark 3.** In general, the point spectrum does not represent the entire set of eigenvalues since some eigenvalues may belong to the essential spectrum. Indeed, we show in Lemma 3.4 immediately below that \( \lambda = 0 \) is an eigenvalue, but nevertheless it does not lie in the point spectrum; it belongs to the essential spectrum instead, because the operator fails to be Fredholm. The same situation occurs in combustion fronts [4], KdV solitons [14] and fronts in isothermal autocatalytic chemical reactions [22], just to mention a few.

The spectral problem (23) can be recast as

\[
\mathcal{L} \begin{pmatrix} p \\ j \\ q^+ \end{pmatrix} = \lambda \begin{pmatrix} p \\ j \\ q^+ \end{pmatrix}, \quad \begin{pmatrix} p \\ j \\ q^+ \end{pmatrix} \in H^1(\mathbb{R}; \mathbb{C}^3), \quad (24)
\]

where the operator \( \mathcal{L} : H^1(\mathbb{R}; \mathbb{C}^3) \to L^2(\mathbb{R}; \mathbb{C}^3) \) is defined by

\[
\mathcal{L} = \begin{pmatrix}
0 & -\partial_x & 0 \\
-\lambda^2 \partial_x + \mu s(2q^+ - 1) & -\mu & 2\mu s\tilde{p} \\
0 & 2\mu s\tilde{q}^+(1 - q^+) & 0
\end{pmatrix}.
\]
**Definition 3.3.** We say that the standing waves \( \bar{p}, \bar{q}^+ \) and the identically zero function \( j \) are spectrally stable if
\[
\sigma(\mathcal{L}) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq 0 \} = \{ 0 \}.
\]

**Lemma 3.4.** \( \lambda = 0 \) is an eigenvalue of \( \mathcal{L} \) embedded in the essential spectrum with an infinite dimensional eigenspace.

**Proof.** Take \( \lambda = 0 \) in the spectral system (23); it immediately follows that \( j \equiv 0 \). Therefore, it all comes down to solve the differential equation
\[
- s^2 p_x + \mu s [ (2q^+ - 1) p + 2\bar{p}q^+] = 0.
\]
(25)
The problem consists in finding nontrivial solutions \((p, q^+)\) in the space \( H^1(\mathbb{R}; \mathbb{C}^2) \).

Differentiating the second equation in (20), we find
\[
\text{Applying H"older's inequality to (26) and using (27), we obtain}
\]
\[
\text{Thereafter we solve (25) and verify that the solutions}
\]
\[
\text{solution of (25). In order to obtain such solutions, we first show that}
\]
\[
\text{in order to obtain such solutions, we first show that}
\]
\[
\text{We say that the standing waves } \bar{p}, \bar{q}^+ \text{ and the identically zero function } j \text{ are spectrally stable if}
\]
\[
\sigma(\mathcal{L}) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq 0 \} = \{ 0 \}.
\]

Now we solve equation (25) for the variable \( p \). From the equation for \( \bar{p} \) in (20) we have
\[
\frac{\bar{p}_x}{\bar{p}} = \frac{\mu}{s} (2\bar{q}^+ - 1),
\]
(28)
let $q^+ = \bar{q}^+_x$, substitution of (28) into (25) yields

$$p_x - \bar{p}_x \frac{p}{\bar{p}} = \frac{2\mu}{s} \bar{p} q^+_x. \tag{29}$$

Solving for $p$ we get

$$p(x) = \bar{p}(x) \left( C + \frac{2\mu}{s} (\bar{q}^+(x) - q^+_i) \right), \quad C \in \mathbb{C} \tag{30}$$

We readily note that there is a value of $C$ that allows us to recover the solution $p = \bar{p}$.

Next, we show that $p$ is an element of $H^1(\mathbb{R}; \mathbb{C})$. To this end, we make estimates for $p$ and $p_x$. Thus

$$\|p\|_{L^2}^2 = \int_{\mathbb{R}} \left( C + \frac{2\mu}{s} (\bar{q}^+(x) - q^+_i) \right)^2 dx,$$

hence the right-hand side of the above equation is bounded above by

$$\left( |C| + \frac{2\mu}{s} \varepsilon \right)^2 \|\bar{p}\|_{L^2}^2.$$

That $\bar{p} \in L^2(\mathbb{R}; \mathbb{C})$ is a consequence of Lemma 2.3. Therefore, the above estimate implies that $p \in L^2(\mathbb{R}; \mathbb{C})$.

Now, from (29) and the triangle inequality we have that

$$\|p_x\|_{L^2}^2 \leq \int_{\mathbb{R}} \left( \left( \frac{\bar{p}_x}{\bar{p}} \right)^2 |p|^2 + \frac{4\mu}{s} \|p\|_{L^2} \|p_0\|_{L^2} + \frac{4\mu^2}{s^2} \|p^2 \|_{L^2} \right) dx. \tag{31}$$

Hölder’s inequality applied to (31) and the absolute value of (28) combined with (27) give

$$\|p_x\|_{L^2}^2 \leq \left( \frac{\mu \varepsilon}{s} \right)^2 \|p\|_{L^2}^2 + \frac{4\mu^2 \varepsilon}{s^2} \bar{p}_{\text{max}} \|p\|_{L^2} \|\bar{q}^+_x\|_{L^2} + \frac{4\mu^2}{s^2} \|\bar{q}^+_x\|_{L^2}^2.$$

This shows that $p \in H^1(\mathbb{R}; \mathbb{C})$.

The infinite dimension of the eigenspace is a direct consequence of the linear independence of the solutions given by formula (30): let $p_1$ and $p_2$ be two solutions corresponding to two different values of $C$, say $C_1$ and $C_2$; it is not difficult to check that the Wronskian of these solutions is $W(p_1, p_2)(x) = \frac{2\mu}{s} \bar{p}^2 \bar{q}^+_x (C_1 - C_2)$. Since $\bar{p} > 0$ and $0 < \bar{q}^+_x < 1$ on the whole real line, we see from the equation for $\bar{q}^+_x$ in (20) that $\bar{q}^+_x < 0$ for all $x \in \mathbb{R}$. Then it holds that $W(p_1, p_2)(x) \neq 0$ for all $x \in \mathbb{R}$. Therefore, we conclude that equation (25) has infinitely many linearly independent solutions.

To finish the proof we argue as follows. Previously, we have found that the problem

$$\mathcal{L}_0 p := -s^2 p_x + \mu s \left[ (2\bar{q}^+_x - 1) p + 2\bar{p} q^+_x \right] = 0$$

has an infinite number of solutions, then according to definition of Fredholm operator, this means that the operator $\mathcal{L}_0$ is not Fredholm. Thus, it follows that $\lambda = 0$ is an eigenvalue that belongs to the essential spectrum; see Definition 3.2. This completes the proof.
4. The quadratic eigenvalue problem. With the purpose of characterizing the whole spectrum, we assume \( \lambda \neq 0 \) in (23). We then multiply the second equation in (23) to obtain
\[
\lambda^2 j = -s^2 \lambda p_x - \mu \lambda j + \mu s[(2q^+ - 1)\lambda p + 2\bar{p}\lambda q^+],
\]
from the first and third equations in (23), we have
\[
\lambda^2 j = s^2 j_{xx} - \mu \lambda j - \mu s \left[ (2\bar{q}^+ - 1)j_x - 2\bar{p} \left( \frac{2\kappa}{s} q^+(1 - q^+) \right) j \right],
\]
by substituting the equation for \( \bar{q}^+ \) in (20) into (32), we get the quadratic eigenvalue problem
\[
s^2 j_{xx} - \mu s[(2\bar{q}^+ - 1)j_x + 2\bar{q}^+_x j] - (\lambda^2 + \mu \lambda) j = 0, \quad \lambda \neq 0.
\]  
\[ (33) \]

**Lemma 4.1.** \(-\mu\) is an eigenvalue of \( L \) embedded in the essential spectrum associated with the one-dimensional eigenspace spanned by \( \bar{q} \).

**Proof.** We begin by noting that (33) can be written in the form
\[
s^2 j_{xx} - \mu s \frac{d}{dx} \left[(2\bar{q}^+ - 1) j_x \right] = (\lambda^2 + \mu \lambda) j,
\]
using (28) to substitute \( 2\bar{q}^+ - 1 = s\bar{p}_x/\mu \bar{p} \) into (34) gives us
\[
s^2 \frac{d}{dx} \left( j_x - \frac{\bar{p}_x}{\bar{p}} j \right) = (\lambda^2 + \mu \lambda) j.
\]
Letting \( \lambda = -\mu \), leads us to the equation
\[
\frac{d}{dx} \left( j_x - \frac{\bar{p}_x}{\bar{p}} j \right) = 0.
\]
By integrating this equation, we have
\[
j_x - \frac{\bar{p}_x}{\bar{p}} j = C,
\]
for an arbitrary constant \( C \).
Recalling (21), we deduce from (28) that \( \bar{p}_x/\bar{p} \rightarrow \mp \varepsilon \mu / s \) as \( x \rightarrow \pm \infty \). From this, and the requirement \( j, j_x \rightarrow 0 \) as \( x \rightarrow \pm \infty \), we infer that the left-hand side of (35) tends to 0 as \( x \rightarrow \pm \infty \), which implies that \( C = 0 \).
We multiply (35) by \( 1/\bar{p} \) to arrive at
\[
\frac{d}{dx} \left( \frac{j}{\bar{p}} \right) = 0.
\]
Thus, the solution is \( j = C_0 \bar{p} \), for some constant \( C_0 \).

Now, substituting \( \lambda = -\mu \) and \( j = \bar{p} \) into the first and third equation in (23) we obtain that \( p = \bar{p}_x/\mu \) and \( q^+ = \bar{q}_x^+ / \mu \). To obtain \( q^+ \) it is necessary to use the second equation of (20). We conclude therefore that \(-\mu\) is an eigenvalue of (23) associated with the eigenspace spanned by \( (\bar{p}_x/\mu, \bar{p}, \bar{q}_x/\mu)^t \).

The fact that \(-\mu\) is an element of the essential spectrum is because the differential operator defined by (33) is not Fredholm when \( \lambda = -\mu \). For convenience, we provide this result at the end of Section 6 below. \[ \square \]
5. **The spectrum.** To treat the problem (33), it is useful to introduce the parameter \( \lambda := \lambda^2 + \mu \lambda \). Thus, the eigenvalue problem now reads

\[
\lambda^2 + \mu \lambda - \tilde{\lambda} = 0.
\]

so that the eigenvalues of (33) and consequently, those of (23), are given by solutions of the equation

\[
\lambda^2 + \mu \lambda - \tilde{\lambda} = 0.
\]

**Remark 4.** From Lemma 4.1, \( \tilde{\lambda} = 0 \) is an eigenvalue of \( L \) associated with the eigenfunction \( \tilde{p} \).

5.1. **The essential spectrum.** In order to analyze the spectrum of \( L \), we proceed as Alexander et al. [1] by rewriting (36) as the first-order system

\[
Y_x = A(x, \tilde{\lambda})Y
\]

where \( Y = (j, j_x)^\dagger \) and

\[
A(x, \tilde{\lambda}) = \begin{pmatrix}
\tilde{\lambda} + 2\mu q^+ & 1 \\
\frac{\mu}{s} (2q^+ - 1) & 1
\end{pmatrix}.
\]

The aim of the reformulation is to use the exponential dichotomies enjoyed by (37). Hence, we introduce the definition of exponential dichotomy and Morse index given by Sandstede and Scheel in [19].

**Definition 5.1.** Consider the differential equation

\[
u_x = A(x, \lambda)\nu, \quad \nu \in \mathbb{C}^n.
\]

Let \( I = \mathbb{R}^+, \mathbb{R}^- \) or \( \mathbb{R} \), and fix \( \lambda_* \in \mathbb{C} \). We say that (39), with \( \lambda = \lambda_* \) fixed, has an exponential dichotomy on \( I \) if there exist positive constants \( K, k^* \) and \( k^u \) and a family of projections \( P(x) \) defined and continuous for \( x \in I \) such that the following is true:

1. For any fixed \( y \in I \) and \( u_0 \in \mathbb{C}^n \), there exists a solution \( \varphi^s(x, y)u_0 \) of (39) with initial value \( \varphi^s(y, y)u_0 = P(y)u_0 \) for \( x = y \), and

\[
|\varphi^s(x, y)| \leq Ke^{-k^*|x-y|}, \quad \text{for all } x \geq y, \; x, y \in I.
\]

2. For any fixed \( y \in I \) and \( u_0 \in \mathbb{C}^n \), there exists a solution \( \varphi^u(x, y)u_0 \) of (39) with initial value \( \varphi^u(y, y)u_0 = (I - P(y))u_0 \) for \( x = y \), and

\[
|\varphi^u(x, y)| \leq Ke^{-k^u|x-y|}, \quad \text{for all } x \leq y, \; x, y \in I.
\]

3. The solutions \( \varphi^s(x, y)u_0 \) and \( \varphi^u(x, y)u_0 \) satisfy

\[
\varphi^s(x, y)u_0 \in \text{R}(P(x)) \quad \text{for all } x \geq y, \; x, y \in I,
\]

\[
\varphi^u(x, y)u_0 \in \text{N}(P(x)) \quad \text{for all } x \leq y, \; x, y \in I.
\]

The \( x \)-independent dimension \( \text{N}(P(x)) \) is referred to as the Morse index \( i(\lambda_*) \) of the exponential dichotomy on \( I \). If (39) has exponential dichotomies on \( \mathbb{R}^+ \) and on \( \mathbb{R}^- \), the associated Morse indices are denoted by \( i_+(\lambda_*) \) and \( i_- (\lambda_*) \), respectively.

Following the ideas of Flores and Plaza [2], and Sandstede [18], we consider the family of operators

\[
T(\tilde{\lambda}) : H^1(\mathbb{R}; \mathbb{C}^2) \rightarrow L^2(\mathbb{R}; \mathbb{C}^2), \quad \nu \mapsto \nu_x - A(x, \tilde{\lambda})\nu,
\]

for \( \tilde{\lambda} \in \mathbb{C} \).
By Lemma 2.3, $A(x, \lambda) \to A_\pm(\lambda)$ exponentially fast as $x \to \pm \infty$, with
\[
A_\pm(\lambda) = \begin{pmatrix} 0 & 1 \\
\frac{1}{\eta} & \mp \xi \end{pmatrix}.
\] (41)

The operators $L - \tilde{\lambda}$ and $T(\lambda)$ are linked by their Fredholm properties. In this sense, if one operator is Fredholm so is the other, in addition to having the same Fredholm index (see [18] and its references). In turn, $T(\lambda)$ is Fredholm if and only if (37) has an exponential dichotomy on both half-lines $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^- = (-\infty, 0]$, [12, 13]. In such case the Fredholm index is computed by
\[
\text{ind} \, T(\lambda) = i_-(\lambda) - i_+(\lambda).
\]
Here is where the asymptotic matrices $A_\pm(\lambda)$ come into play. An exponential dichotomy on $\mathbb{R}^+$ exists if and only if $A_+(\lambda)$ is hyperbolic, in which case, the Morse index $i_+(\lambda)$ is equal to the dimension of the unstable eigenspace of $A_+(\lambda)$. Likewise, the hyperbolicity of $A_-(\lambda)$ determines the existence of an exponential dichotomy on $\mathbb{R}^-$ and $i_-(\lambda)$ is given by the dimension of the unstable eigenspace of $A_+(\lambda)$ (cf. [18]).

In the light of all this, the essential spectrum of $L$ comprises all $\lambda$ for which (37) has exponential dichotomies on both $\mathbb{R}^+$ and $\mathbb{R}^-$ with distinct Morse indices, that is $\text{ind} \, T(\lambda) \neq 0$, and those $\lambda$ such that (37) has no an exponential dichotomy on at least one half-line.

We begin by identifying the set of $\lambda$ where the asymptotic matrices $A_\pm(\lambda)$ are not hyperbolic. Thus, consider the sets
\[
\{ \lambda \in \mathbb{C} \mid \det(A_\pm(\lambda) - aiI) = 0, \ 	ext{for} \ a \in \mathbb{R} \}.
\] (42)
By straightforward calculations, we obtain that the elements of (42) are the algebraic curves
\[
\{ \lambda_\pm(a) = -s^2 a^2 \mp \mu \varepsilon a i, \ 	ext{for} \ a \in \mathbb{R} \}.
\] (43)
Note that the curves $\lambda_\pm(a)$ describe one single parabola. Along the parabola the limiting matrices $A_\pm(\lambda)$ have at least one purely imaginary eigenvalue, outside, the matrices are hyperbolic. Denote $\Omega$ to be the open set in the complex plane bounded on the left by the parabola (43), and let $\Theta$ denote the complement of the closure of $\Omega$. We further denote by $E^+_\pm(\lambda)$ and $E^-_\pm(\lambda)$ the stable and unstable eigenspaces of $A_\pm(\lambda)$, respectively.

**Proposition 2.** The following statements are true.

(i) For each $\lambda \in \Omega$, $\dim[E^+_\pm(\lambda)] = \dim[E^-_\pm(\lambda)] = 1$.

(ii) For all $\lambda \in \Theta$, $\dim[E^+_\pm(\lambda)] = \dim[E^-_\pm(\lambda)] = 2$.

**Proof.** The eigenvalues of $A_-(\lambda)$ and $A_+(\lambda)$ are given by
\[
\eta_{1,2}^- (\lambda) = \frac{1}{2s} \left( \mu \varepsilon \pm \sqrt{\mu^2 \varepsilon^2 + 4 \lambda} \right) \quad \text{and} \quad \eta_{1,2}^+ (\lambda) = \frac{1}{2s} \left( -\mu \varepsilon \pm \sqrt{\mu^2 \varepsilon^2 + 4 \lambda} \right) .
\] (44)
Clearly $\Re \eta_1^- (\lambda) > 0$ and $\Re \eta_2^+ (\lambda) < 0$ for all $\lambda \in \Omega$. Since $\eta_2^- (\lambda) = -\eta_1^+ (\lambda)$, the first statement of the proposition will be proved as soon as we have shown that $\Re \eta_2^- (\lambda) < 0$ for all $\lambda \in \Omega$. 


Consider \( \lambda \in \Omega \) with negative real part. Taking the real part of \( \eta_2(\lambda) \), we have

\[
\Re \eta_2(\lambda) = \frac{1}{2s} \left( \mu \varepsilon - \Re \sqrt{\mu^2 \varepsilon^2 + 4\lambda} \right),
\]

from this expression we see that \( \Re \eta_2(\lambda) < 0 \) if and only if

\[
\left( \Re \sqrt{\mu^2 \varepsilon^2 + 4\lambda} \right)^2 > \mu^2 \varepsilon^2.
\]

(45)

Computing the left-hand side of the above equation gives

\[
\frac{1}{2} \left( \mu^2 \varepsilon^2 + 4\Re \lambda + \left( \mu^2 \varepsilon^2 + 4\Re \lambda \right)^2 + 16(\Im \lambda)^2 \right) > \mu^2 \varepsilon^2.
\]

After some calculations we get that the inequality (45) is satisfied if and only if

\[
(\Im \lambda)^2 > -\mu^2 \varepsilon^2 \Re \lambda.
\]

(46)

Since \( \Re \lambda < 0 \), we have that \( \Re \lambda = \Re \lambda_\pm(a) = -a^2 s^2 \) for some \( a \in \mathbb{R} \), thus, because \( \lambda \in \Omega \) it happens that \( (\Im \lambda)^2 > (\Im \lambda_\pm(a))^2 = a^2 s^2 \mu^2 \varepsilon^2 \), that is to say, \( (\Im \lambda)^2 > -\mu^2 \varepsilon^2 \Re \lambda \), hence the inequality (45) holds and consequently \( \Re \eta_2^-(\lambda) < 0 \). By connectedness of the set \( \Omega \) and continuity of \( \eta_2^-(\lambda) \) in \( \lambda \), the sign of \( \Re \eta_2^-(\lambda) \) must remain constant on that region. Therefore, we conclude that \( \Re \eta_2^-(\lambda) < 0 \) for all \( \lambda \in \Omega \).

Every \( \tilde{\lambda} \in \Theta \) has real part \( \Re \tilde{\lambda} = \Re \tilde{\lambda}_\pm(a) = -a^2 s^2 \) with \( a \in \mathbb{R} \setminus \{0\} \), then, because \( (\Im \tilde{\lambda})^2 < (\Im \tilde{\lambda}_\pm(a))^2 \) for all \( \lambda \in \Theta \) and all \( a \in \mathbb{R} \setminus \{0\} \), it follows that

\[
(\Im \tilde{\lambda})^2 < (\Im \tilde{\lambda}_\pm(a))^2 = a^2 s^2 \mu^2 \varepsilon^2 = -\mu^2 \varepsilon^2 \Re \lambda, \quad \forall \lambda \in \Theta, \quad \forall a \in \mathbb{R} \setminus \{0\}.
\]

By virtue of this result, inequality (45) is not satisfied, thus \( \Re \eta_2^-(\tilde{\lambda}) > 0 \) and \( \Re \eta_2^+(\tilde{\lambda}) < 0 \) for all \( \lambda \in \Theta \). This proves the second statement of the proposition, since \( \Re \eta_2^+(\tilde{\lambda}) > 0 \) and \( \Re \eta_2^-(\tilde{\lambda}) < 0 \) is also true inside \( \Theta \).

\[ \square \]

**Lemma 5.2.** The essential spectrum of \( L \) is the region to the left of the parabola described by (43), including the parabola.

**Proof.** Suppose that the matrices \( A_\pm(\lambda) \) are hyperbolic. By Theorem 3.3 in [18], this leads to the existence of exponential dichotomies for the equation (37) on \( \mathbb{R}^+ \) and \( \mathbb{R}^- \). Additionally, Theorem 3.3 tells us that the Morse indices \( i_\pm(\lambda) \) are equal to \( \dim [E_\pm^s(\lambda)] \). According to Lemma 4.2 of [12], this means that \( T(\lambda) \) is Fredholm with index

\[
\text{ind } T(\lambda) = i_-(\lambda) - i_+(\lambda).
\]

Statement (i) of Proposition 2 yields that \( \text{ind } T(\lambda) = 0 \) for all \( \lambda \in \Omega \), which implies that \( \sigma_{\text{ess}}(L) \subset \mathbb{C} \setminus \Omega \). To show that \( \sigma_{\text{ess}}(L) \) covers \( \mathbb{C} \setminus \Omega \) we argue as follows. From statement (ii) of Proposition 2 we obtain that \( \text{ind } T(\lambda) = 2 \) for all \( \lambda \in \Theta \), therefore \( \Theta \subset \sigma_{\text{ess}}(L) \). In accordance with Theorem 3.3, the lack of hyperbolicity \( A_\pm(\lambda) \) on the set (43) results in the absence of exponential dichotomies of (37). By Palmer’s Theorem in [13], this entails that \( T(\lambda) \) is not Fredholm, thereby the parabola described by (43) is a subset of \( \sigma_{\text{ess}}(L) \). Since

\[
\mathbb{C} \setminus \Omega = \Theta \cup \left\{ \lambda_\pm(a) = -s^2 a^2 \pm \mu \varepsilon a i, \text{ for } a \in \mathbb{R} \right\},
\]

we deduce that \( \mathbb{C} \setminus \Omega \subset \sigma_{\text{ess}}(L) \), and hence that \( \sigma_{\text{ess}}(L) = \mathbb{C} \setminus \Omega \).  \[ \square \]
From the above lemma we observe that the point spectrum of (40) must only contain complex numbers that belong to the region \( \Omega \). Accordingly, \( \lambda = 0 \) is an eigenvalue that does not belong to the point spectrum, \( \sigma_{pt}(L) \).

We show below that any eigenfunction of the operator \( T(\tilde{\lambda}) \) must have exponential decay. To this end, as in [10, 14], we rewrite equation (37) in the following form

\[
\mathbf{Y}_x = [\mathbf{A}_- (\tilde{\lambda}) + \mathbf{R}(x, \tilde{\lambda})]\mathbf{Y} \quad \text{for} \quad x < 0 \quad \text{and} \quad \mathbf{Y}_x = [\mathbf{A}_+ (\tilde{\lambda}) + \mathbf{R}(x, \tilde{\lambda})]\mathbf{Y} \quad \text{for} \quad x \geq 0,
\]

where

\[
\mathbf{R}(x, \tilde{\lambda}) = \begin{cases} 
\mathbf{A}(x, \tilde{\lambda}) - \mathbf{A}_-(\tilde{\lambda}), & x < 0, \\
\mathbf{A}(x, \tilde{\lambda}) - \mathbf{A}_+(\tilde{\lambda}), & x \geq 0.
\end{cases}
\]

We have by Lemma 2.3 that

\[
|\mathbf{R}(x, \tilde{\lambda})| = |\mathbf{A}(x, \tilde{\lambda}) - \mathbf{A}_\pm (\tilde{\lambda})| \leq C e^{-\frac{\alpha}{2}|x|} \quad \text{as} \quad |x| \to \infty,
\]

for all \( \tilde{\lambda} \in \Omega \).

According to the Gap Lemma [24], if \( V_j^{\pm}(\tilde{\lambda}) \) are eigenvectors of \( \mathbf{A}_\pm (\tilde{\lambda}) \) associated with the eigenvalues \( \eta_j^{\pm}(\tilde{\lambda}) \), \( j = 1, 2 \), the decay estimate (47) implies that for all \( \alpha < \frac{\mu}{2\pi} \), the system (37) has a set of solutions \( \mathbf{Y}_j^\pm(x, \tilde{\lambda}) \), \( j = 1, 2 \), that satisfy

\[
\mathbf{Y}_j^-(x, \tilde{\lambda}) = \left( V_j^-(\tilde{\lambda}) + O \left( e^{-\alpha|x|} |V_j^-(\lambda)| \right) \right) e^{\eta_j^-(\tilde{\lambda})x}, \quad (j = 1, 2) \quad x < 0,
\]

\[
\mathbf{Y}_j^+(x, \tilde{\lambda}) = \left( V_j^+(\tilde{\lambda}) + O \left( e^{-\alpha|x|} |V_j^+(\lambda)| \right) \right) e^{\eta_j^+(\tilde{\lambda})x}, \quad (j = 1, 2) \quad x > 0,
\]

for any \( \tilde{\lambda} \in \Omega \). The importance of these relations stems from the fact that they allow us to characterize the asymptotic behaviour of the eigenfunctions of \( T(\tilde{\lambda}) \). Indeed, we have found previously that \( \Re \eta_1^+(\tilde{\lambda}) > 0 \) and \( \Re \eta_2^+(\tilde{\lambda}) < 0 \) provided that \( \tilde{\lambda} \in \Omega \), since we are interested in solutions to (37) in \( H^1(\mathbb{R}; \mathbb{C}^2) \), we observe from (48) that one can construct such solutions only if they decay exponentially to zero as \( |x| \to +\infty \). We summarize this result as follows.

**Proposition 3.** Let \( \tilde{\lambda} \) be an element of the point spectrum of \( T(\tilde{\lambda}) \) and assume that \( \mathbf{Y}(x, \tilde{\lambda}) \) is the associated eigenfunction. Then \( \mathbf{Y}(x, \tilde{\lambda}) \) decays exponentially fast as \( |x| \to +\infty \), satisfying

\[
\mathbf{Y}(x, \tilde{\lambda}) \to V_1^-(\tilde{\lambda}) e^{\eta_1^-(\tilde{\lambda})x} \quad \text{as} \quad x \to -\infty,
\]

\[
\mathbf{Y}(x, \tilde{\lambda}) \to V_2^+(\tilde{\lambda}) e^{\eta_2^+(\tilde{\lambda})x} \quad \text{as} \quad x \to +\infty,
\]

where \( V_1^-(\tilde{\lambda}) \) and \( V_2^+(\tilde{\lambda}) \) are eigenvectors associated to the unstable and stable eigenvalues \( \eta_1^-(\tilde{\lambda}) \) and \( \eta_2^+(\tilde{\lambda}) \), respectively.

### 5.2. Integrated equation.

Suppose that \( \tilde{\lambda} \in \sigma_{pt}(L) \) is an eigenvalue with a corresponding eigenfunction \( j \in H^2(\mathbb{R}; \mathbb{C}) \). Let us rewrite equation (36) in the form

\[
s^2 j_{xx} - \mu s (2q^+ - 1) j = \tilde{\lambda} j,
\]

from which we obtain

\[
s^2 \frac{d}{dx} \left( j_x - \frac{\mu}{s} (2q^+ - 1) j \right) = \tilde{\lambda} j.
\]

Applying the technique conceived by Goodman [5], we introduce the **integrated variable**

\[
w(x) := \int_{-\infty}^{x} j(y) dy.
\]
We integrate (49) from \(-\infty\) to \(x\) and obtain
\[
\tilde{\lambda}w(x) = \tilde{\lambda} \int_{-\infty}^{x} j(y)dy = s^2 \left( j_x(x) - \frac{\mu}{s} (2q^+ - 1) j(x) \right),
\]
and then, we substitute \(j\) for \(w\) in (51) in order to obtain the integrated eigenvalue equation
\[
s^2 \left( w_{xx} - \frac{\mu}{s} (2q^+ - 1) w_x \right) = \mathcal{L} w = \tilde{\lambda} w.
\]

The significance of the above equation arises from the fact that the point spectrum of \(L\) and \(\mathcal{L}\) is the same (see Proposition 4 below). This result will prove very useful for characterizing the point spectrum of \(L\) as the integrated eigenvalue problem (52) will provide the required information. The same approach has been carried out by Zumbrun [23] and Humpherys [9] in the context of viscous conservation laws.

The family of operators associated with (52) is given by
\[
T^I(\tilde{\lambda}): H^1(R; C^2) \to L^2(R; C^2),
\]
where \(W = (w, w_x)^t\) and
\[
A^I(x, \tilde{\lambda}) = \begin{pmatrix} 0 & 1 \\ \frac{\mu}{2s} (2q^+ - 1) \end{pmatrix}.
\]

Remark 5. We point out that \(A^I(x, \tilde{\lambda})\) has the same asymptotic limits as \(A(x, \tilde{\lambda})\). Thus the essential spectrum of \(L\) and \(\mathcal{L}\) coincide, and therefore the point spectrum of \(\mathcal{L}\) is also contained in the set \(\Omega\). We may use arguments similar to those that led to Proposition 3, to conclude that for a given \(\tilde{\lambda} \in \sigma_{pt}(\mathcal{L})\), the corresponding eigenfunction \(W(x, \tilde{\lambda})\) has the asymptotic behavior
\[
W(x, \tilde{\lambda}) \to V_1^- (\tilde{\lambda}) e^{\eta_- (\tilde{\lambda})x} \quad \text{as} \quad x \to -\infty,
\]
\[
W(x, \tilde{\lambda}) \to V_2^+ (\tilde{\lambda}) e^{\eta_+ (\tilde{\lambda})x} \quad \text{as} \quad x \to +\infty.
\]

Proposition 4. The point spectrum of \(L\) and point spectrum of \(\mathcal{L}\) coincide.

Proof. We begin by proving that \(\sigma_{pt}(L) \subset \sigma_{pt}(\mathcal{L})\). Observe that the existence of the eigenpair \((j, \tilde{\lambda})\) of (36) gives rise to a solution \((w, \tilde{\lambda})\) of equation (52), then the problem consists in checking that \(w\) belongs to \(H^2(R; C)\). Since \(j \in H^2(R; C)\), it is clear that \(w_x, w_{xx} \in L^2(R; C)\). Thus, only need to show that \(\tilde{w} \in L^2(R; C)\). By Plancherel’s theorem, it suffices to show that \(\tilde{w} \in L^2(R; C)\). For this purpose, we differentiate (50) and take the Fourier transform to obtain \(ik \tilde{w}(k) = \tilde{j}(k)\). So we have that
\[
\|\tilde{w}\|_{L^2}^2 = \int_R \frac{|\tilde{j}(k)|^2}{k^2} dk.
\]
The above integral may be split into three parts
\[
\int_R \frac{|\tilde{j}(k)|^2}{k^2} dk = \int_{-\infty}^{-a} \frac{|\tilde{j}(k)|^2}{k^2} dk + \int_{-a}^{a} \frac{|\tilde{j}(k)|^2}{k^2} dk + \int_{a}^{\infty} \frac{|\tilde{j}(k)|^2}{k^2} dk,
\]
with \(a > 1\). The first and the last integral converge because are both bounded above by \(\|\tilde{j}\|_{L^2}^2\). Given that \(\tilde{j}(k)\) is a continuous function, then to establish the
convergence of the second integral we only need to show that \( \hat{j}(k)/k \) tends to a finite limit as \( k \to 0 \). First note that

\[
w(\pm \infty) = \int_{-\infty}^{\infty} j(y) \, dy = \sqrt{2\pi} \hat{j}(0).
\]

On the other hand, since \( \lambda \neq 0 \) and, \( j \) and \( j_x \) decay to zero at \( x = \pm \infty \), it follows from (51) that \( w \) approaches zero as \( x \to \pm \infty \). Which implies that \( \hat{j}(0) = 0 \). Hence, using L'Hospital's rule, we get

\[
\lim_{k \to 0} \frac{\hat{j}(k)}{k} = \lim_{k \to 0} \frac{d}{dk} \hat{j}(k) = \int_{\mathbb{R}} y j(y) \, dy.
\]

The fact that \( j \) decays exponentially fast to zero as \( |y| \to \infty \) ensures the convergence of the integral.

Next we show that \( \sigma_{pt}(\mathcal{L}) \subset \sigma_{pt}(L) \). Let \( w \in H^2(\mathbb{R}; \mathbb{C}) \) be an eigenfunction of (52) for \( \tilde{\lambda} \in \sigma_{pt}(\mathcal{L}) \). Setting \( j = w_x \), it is readily seen that

\[
w(x) = \int_{-\infty}^{x} j(y) \, dy.
\]

Substituting \( w_x = j \) and (53) into (52), and differentiating, we obtain

\[
s^2 w_{xxx} = (2\mu s \tilde{q}_x^+ + \mu^2 (2\tilde{q}_x^+ - 1)^2 + \tilde{\lambda}) w_x + \frac{\mu}{s} \tilde{\lambda} (2\tilde{q}_x^+ - 1) w.
\]

Therefore, we use the exponential decay of \( w \) and \( w_x \) together with the boundedness of \( \tilde{q}_x^+ \) and \( \tilde{q}_x^+ \), to conclude that \( w_{xxx} \in L^2(\mathbb{R}; \mathbb{C}) \).

5.3. Energy estimates. In what follows we use energy methods \([2, 4, 8]\) to prove that the point spectrum of the operator \( \mathcal{L} \) is the empty set.

**Lemma 5.3.** The point spectrum of \( \mathcal{L} \) is empty.

**Proof.** We use (28) to substitute \( 2\tilde{q}_x^+ - 1 = s\tilde{p}_x/\mu \tilde{p} \) into (52). This gives

\[
s^2 \left( w_{xx} - \frac{\tilde{p}_x}{\tilde{p}} w_x \right) = \tilde{\lambda} w.
\]

By multiplying (54) by the integrating factor \( 1/\tilde{p} \), we obtain that \( w \) satisfies

\[
s^2 \frac{d}{dx} \left( \frac{w_x}{\tilde{p}} \right) = \tilde{\lambda} \frac{w}{\tilde{p}}.
\]

We now multiply (55) by the complex conjugate \( w^* \) and integrate over \( \mathbb{R} \) to obtain

\[
s^2 \int_{\mathbb{R}} \frac{d}{dx} \left( \frac{w_x}{\tilde{p}} \right) w^* \, dx = \tilde{\lambda} \int_{\mathbb{R}} \frac{|w|^2}{\tilde{p}} \, dx.
\]

**Claim.** \( w/\sqrt{\tilde{p}}, w_x/\sqrt{\tilde{p}} \to 0 \) exponentially as \( |x| \to \infty \).
Indeed, from Lemma 2.3 and Remark 5 we have that
\[ p^{-\frac{1}{2}} W(x, \bar{\lambda}) \sim CV_1^- (\bar{\lambda}) e^{(\tau_1 (\bar{\lambda}) - \frac{\pi}{2}) x} = CV_1^- (\bar{\lambda}) e^{\pm \sqrt{\mu^2 s^2 + 4\lambda} x} \quad \text{as} \quad x \to -\infty, \]
\[ p^{-\frac{1}{2}} W(x, \bar{\lambda}) \sim CV_2^+ (\bar{\lambda}) e^{(\tau_2 (\bar{\lambda}) + \frac{\pi}{2}) x} = CV_2^+ (\bar{\lambda}) e^{-\frac{1}{2} \sqrt{\mu^2 s^2 + 4\lambda} x} \quad \text{as} \quad x \to +\infty. \]
Recall that \( W = (w, w_x)^t \). Thus, noting that
\[ \Re e \sqrt{\mu^2 s^2 + 4\lambda} = |\mu^2 s^2 + 4\bar{\lambda}|^{\frac{1}{2}} \cos\left(\arg(\mu^2 s^2 + 4\bar{\lambda})/2\right) \]
is positive for all \( \bar{\lambda} \in \sigma_{\text{pt}}(L) \subset \Omega \), we obtain the exponential convergence to zero.

In view of the claim, we can integrate the left-hand side by parts. We infer that
\[ -s^2 \int_{\mathbb{R}} \frac{|w_x|^2}{p} \, dx = \tilde{\lambda} \int_{\mathbb{R}} \frac{|w|^2}{p} \, dx. \]
This shows that \( \tilde{\lambda} < 0 \). But this contradicts \( \bar{\lambda} \in \sigma_{\text{pt}}(L) \subset \Omega \), because of the fact that the negative half-real line is a subset of \( \sigma_{\text{ess}}(L) = \mathbb{C} \setminus \Omega \) (see Lemma 5.2). Therefore, there is no point spectrum for \( L \).

5.4. The point spectrum. The results of Proposition 4 and Lemma 5.3 come together in the following theorem.

Theorem 5.4. The point spectrum of the quadratic eigenvalue problem (33) is the empty set.

6. Spectral stability. We begin the section with our main result, previously stated in the Introduction.

Theorem 1.1. The family of standing waves is spectrally stable.

In view of Theorem 5.4, the point spectrum of \( L \) is empty, meaning that the spectrum is made up completely of essential spectrum. This being so, we must show that the rest of the essential spectrum, namely the non-zero elements, is composed of complex numbers with negative real part. In this fashion, we finish the proof of the main Theorem. For this purpose, we will show that the essential spectrum of the equivalent problem (33) is a subset of the stable half plane. As before, we may associate (33) with the family of operators
\[ \mathcal{T}(\lambda^2 + \mu \lambda) = Y_x - A(x, \lambda^2 + \mu \lambda) Y, \quad \text{for} \quad \lambda \in \mathbb{C} \setminus \{0\}. \]

Proposition 5. Let \( \Omega^S = \{ \lambda \in \mathbb{C} \setminus \{0\} \mid \Re e \lambda \geq 0 \} \). Then, for all \( \lambda \in \Omega^S \), the matrices \( A_\pm (\lambda^2 + \mu \lambda) \) are hyperbolic with one-dimensional eigenspaces \( E^\pm_\pm (\lambda^2 + \mu \lambda) \) and \( E^\pm_\pm (\lambda^2 + \mu \lambda) \).

Proof. We begin by proving that the set
\[ \{ \lambda \in \mathbb{C} \setminus \{0\} \mid \det(A_\pm (\lambda^2 + \mu \lambda) - aI) = 0, \text{ for } a \in \mathbb{R} \} \]
contains only complex numbers with negative real part.

Computing the determinant, one may write (56) as
\[ \{ \lambda \in \mathbb{C} \setminus \{0\} \mid \lambda^2 + B(a) \lambda + C(a) = 0, \text{ for } a \in \mathbb{R} \}, \]
where \( B(a) = \mu \) and \( C(a) = a^2 s^2 \mp a \mu s i \).

Solving the characteristic polynomial, we find that (56) consists of the curves
\[ \lambda_{1,2}(a) = \frac{-B(a) \pm \sqrt{D(a)}}{2}, \quad a \in \mathbb{R}, \]
where \( D(a) = \mu^2 - 4(a^2 s^2 + \alpha s \varepsilon i) \).

Since \( B(a) < 0 \), then clearly \( \Re \lambda_2(a) < 0 \) for all \( a \in \mathbb{R} \). On the other hand, observe that \( \Re \lambda_1(a) < 0 \) if and only if

\[
(\Re B(a))^2 > \left(\Re \sqrt{D(a)}\right)^2 = \frac{1}{2} \left(\Re D(a) + \sqrt{(\Re D(a))^2 + (\Im D(a))^2}\right),
\]

for all \( a \in \mathbb{R} \setminus \{0\} \). We deduce from this observation that \( \Re \lambda_1(a) \) is negative if and only if

\[
(\Im D(a))^2 + 4(\Re B(a))^2 \Re D(a) - 4(\Re B(a))^4 < 0, \quad a \in \mathbb{R} \setminus \{0\}.
\]

We have excluded \( a = 0 \) because \( \lambda(0) = 0 \).

After some elementary calculations, we have

\[
(\Im D(a))^2 + 4(\Re B(a))^2 \Re D(a) - 4(\Re B(a))^4 = -16\mu^2 a^2 s^2 (1 - \varepsilon^2),
\]

for all \( a \in \mathbb{R} \setminus \{0\} \), therefore we can conclude (59) because \( 0 < \varepsilon < 1 \). Thus, it follows that matrices \( A_\pm(\lambda^2 + \mu \lambda) \) are hyperbolic on \( \Omega^S \).

It remains to verify that the stable and unstable eigenspaces are one-dimensional. Consider \( \lambda = \zeta \in \mathbb{R}^+ \), it is clear that \( \zeta^2 + \mu \zeta \in \Omega \), then by Proposition 2, \( \dim[E_\pm^S(\zeta^2 + \mu \zeta)]=1 \) and \( \dim[E^u_\pm(\zeta^2 + \mu \zeta)] = 1 \). Therefore, from the fact that \( \Omega^S \) is connected and \( \eta^\pm_{1,2}(\cdot) \) are continuous, it follows that \( \dim[E^S_\pm(\lambda^2 + \mu \lambda)] = 1 \) and \( \dim[E^u_\pm(\lambda^2 + \mu \lambda)] = 1 \) for all \( \lambda \in \Omega^S \).

**Lemma 6.1.** The rest of the essential spectrum of \( \mathcal{L} \) is a subset of the open left-half complex plane.

**Proof.** By similar arguments as used in the proof of Lemma 5.2, one can show that

\[
\sigma_{\text{ess}}(\mathcal{L}) \setminus \{0\} \subset \{ \lambda \in \mathbb{C} \mid \Re \lambda < 0 \}.
\]

\[ \square \]

6.1. **End of the proof of Lemma 4.1.** Observe that \( (\lambda, a) = (-\mu, 0) \) is a solution of the characteristic polynomial in (57), this means that 0 is an eigenvalue of the asymptotic matrices \( A_\pm(0) \), or in other words, that \( A_\pm(0) \) are nonhyperbolic. As consequence of the Theorem 3.3 of [18] and Palmer’s Theorem in [13], the operator \( T(0) \) is not Fredholm, therefore \( -\mu \) belongs to the essential spectrum of \( \mathcal{L} \).

7. **Discussion.** Through our investigation we have shown spectral stability for all members of the family of standing wave solutions. It was found that the spectrum of the linearized perturbation problem (24) consists of pure essential spectrum. The latter belongs to the left-half complex plane and intersects the imaginary axis only at zero. We proved that \( \lambda = 0 \) is an eigenvalue within the essential spectrum whose eigenspace has an infinite dimension, which means that the operator \( \mathcal{L} \) is not Fredholm. As we mentioned in the introduction, the work of Rottmann-Matthes shows that spectral stability implies orbital stability for a large class of hyperbolic systems. In our case, we cannot apply Rottmann-Matthes’s theory because one of the main assumptions in [16, 17] is that there must exist a spectral gap, that is a separation between the boundary of the essential spectrum and the imaginary axis. As we have showed, this hypothesis is not satisfied here. A standard technique to circumvent this difficulty is the use of exponential weighted spaces. Nevertheless, in this case, zero remains in the boundary of the essential spectrum, because it keeps an eigenspace with infinite dimension, independently of the weight function that is used (calculations not presented here). To show orbital stability we must rely
on more sophisticated techniques (see e.g. [23, 24] for viscous shocks), for which spectral stability is a fundamental starting point. Another remaining open problem is the stability of traveling pulses and fronts with \( c > 0 \). In a work in preparation, we apply Evans function techniques [15] to analyze the case of small wave speeds \( 0 < c \ll 1 \), using perturbation arguments to show that the point spectrum at \( c = 0 \) is preserved on a neighborhood thereof; the Evans function is an analytic function whose zeroes away from the essential spectrum correspond to the point spectrum of the linearized operator. In the traveling case, the same problem of the absence of a gap between the eigenvalue zero and the boundary of the essential spectrum is present. However, the situation there is very different, as it is possible to build an appropriate weighted space for which the essential spectrum lies in the open left-half complex plane with a gap between its boundary and the imaginary axis.

There is still a long way to go before drawing conclusions about the persistence of a propagating pulse of mesenchymal cells and its respective wave front of aligned fibres. Properties like the wave speed and the amplitude of the wave front could be fundamental for the orbital stability.

Acknowledgments. The authors thank Professors Thomas Hillen and Ramón G. Plaza for helpful discussions and constructive comments. SCG is grateful to Professor Ramón G. Plaza for awakening his interest in the study of stability of traveling waves. We also thank Ana Pérez Arteaga for computational support. Thanks to the anonymous referees for the great help to improve the manuscript.

REFERENCES

[1] J. Alexander, R. Gardner and C. K. R. T. Jones, A topological invariant arising in the stability analysis of travelling waves, *J. Reine Angew. Math.*, 410 (1990), 167–212.

[2] G. Flores and R. G. Plaza, Stability of post-fertilization traveling waves, *J. Differential Equations*, 247 (2009), 1529–1590.

[3] P. Friedl and K. Wolf, Tumor cell invasion and migration: Diversity and escape mechanisms, *Nat. Rev. Cancer*, 3 (2003), 362–374.

[4] A. Ghazaryan and C. K. R. T. Jones, On the stability of high lewis number combustion fronts, *Discrete Contin. Dyn. Sys. A.*, 24 (2009), 809–826.

[5] J. Goodman, Nonlinear asymptotic stability of viscous shock profiles for conservation laws, *Arch. Rational Mech. Anal.*, 95 (1986), 325–344.

[6] T. Hillen, \( M^2 \) mesoscopic and macroscopic models for mesenchymal motion, *J. Math Biol.*, 53 (2006), 585–616.

[7] T. Hillen, P. Hinow and Z. A. Wang, Mathematical analysis of a kinetic model for cell movement in network tissues, *Discrete Contin. Dyn. Sys. B.*, 14 (2010), 1055–1080.

[8] J. Humpherys, On the shock wave spectrum for isentropic gas dynamics with capillarity, *J. Differential Equations*, 246 (2009), 2938–2957.

[9] J. Humpherys, *Spectral Energy Methods and the Stability of Shock Waves*, Ph.D thesis, Indiana University, 2002.

[10] T. Kapitula and K. Promislov, *Spectral and Dynamical Stability of Nonlinear Waves*, Springer, New York, 2013.

[11] K. J. Painter, Modelling cell migration strategies in the extracellular matrix, *J. Math. Biol.*, 58 (2009), 511–543.

[12] K. J. Palmer, Exponential dichotomies and transversal homoclinic points, *J. Differential Equations*, 55 (1984), 225–256.

[13] K. J. Palmer, Exponential dichotomies and Freholm operators, *Proc. Amer. Math. Soc.*, 104 (1988), 149–156.

[14] R. L. Pego and M. I. Weinstein, Asymptotic stability of solitary waves, *Commun. Math. Phys.*, 164 (1994), 305–349.

[15] R. G. Plaza and K. Zumbrun, An Evans function approach to spectral stability of small-amplitude shock profiles, *Discr. and Cont. Dynam. Syst.*, 10 (2004), 885–924.
[16] J. Rottmann-Matthes, Linear stability of traveling waves in first-order hyperbolic PDEs, *J. Dyn. Diff. Equat.*, 23 (2011), 365–393.

[17] J. Rottmann-Matthes, Stability and freezing of nonlinear waves in first-order hyperbolic PDEs, *J. Dyn. Diff. Equat.*, 24 (2012), 341–367.

[18] B. Sandstede, Stability of travelling waves, in *Handbook of Dynamical Systems* (ed. B. Fiedler), North-Holland, Amsterdam, 2 (2002), 983–1055.

[19] B. Sandstede and A. Scheel, Absolute and convective instabilities of waves on unbounded and large bounded domains, *Physica D*, 145 (2000), 233–277.

[20] Z. A. Wang, T. Hillen and M. Li, Mesenchymal motion models in one dimension, *SIAM J. Appl. Math.*, 69 (2008), 375–397.

[21] K. Wolf, I. Mazo, H. Leung, K. Engelke, U. H. von Andrian, E. I. Deryugina, A. Y. Strongin, E.-B. Bröcker and P. Friedl, Compensation mechanism in tumor cell migration: Mesenchymal-amoeboïd transition after blocking of pericellular proteolysis, *J. Cell Biol.*, 160 (2003), 267–277.

[22] Y. Wu and X. Xing, Stability of traveling waves with critical speeds for $p$-degree Fisher-type equations, *Discrete Contin. Dyn. Sys.*, 20 (2008), 1123–1139.

[23] K. Zumbrun, Stability and dynamics of viscous shock waves, in *Nonlinear Conservation Laws and Applications* (eds. A. Bressan et al.), Springer, 153 (2011), 123–167.

[24] K. Zumbrun, Stability of large-amplitude shock waves of compressible Navier-Stokes equations, in *Handbook of Mathematical Fluid Dynamics* (eds. S. Friedlander and D. Serre), North-Holland, Amsterdam, 3 (2004), 311–533.

Received April 2015; revised November 2015.

E-mail address: s.c.garcia.fr@gmail.com
E-mail address: cgr@mym.iimas.unam.mx