Plectic points and Hida-Rankin $p$-adic $L$-functions

Víctor Hernández Barrios and Santiago Molina Blanco

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Abstract

Plectic points were introduced by Fornea and Gehrman in [9] as certain tensor products of local points on elliptic curves over arbitrary number fields $F$. In rank $r < [F : \mathbb{Q}]$-situations, they conjecturally come from $p$-adic regulators of basis of the Mordell-Weil group defined over dihedral extensions of $F$.

In this article we define two variable anticyclotomic $p$-adic $L$-functions attached to a family of overconvergent modular symbols defined over $\mathbb{Q}$ and a quadratic extension $K/F$. Their restriction to the weight space provide Hida-Rankin $p$-adic $L$-functions.

If such a family passes through an overconvergent modular symbol attached to a modular elliptic curve $E$, we obtain a $p$-adic Gross-Zagier formula that computes higher derivatives of such Hida-Rankin $p$-adic $L$-functions in terms of plectic points. This result generalizes that of Bertolini and Darmon in [3], which has been key to demonstrating the rationality of Darmon points.

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1 Introduction

Let $E$ be an elliptic curve over a number field $F$ of degree $d = [F : \mathbb{Q}]$. The most important breakthrough towards the Birch and Swinnerton-Dyer conjecture is the well-known Gross-Zagier formula, which relates the first derivative of a Rankin-Selberg $L$-function attached to $E$ with the height of a classical Heegner point. It applies when $E$ is modular over $F$ totally real, and the corresponding Heegner points are defined over abelian extensions of a CM quadratic extension $K/F$. Theory of Heegner points together with Gross-Zagier formula are the main ingredients to prove BSD conjecture for rank 0 and 1, which is the best progress towards the conjecture we have so far.

Due to the success of Heegner points, such a construction has been generalized to other situations, for instance, the cases where $F$ is not totally real or the extension $K/F$ is just quadratic not necessarily CM. The first of these constructions is due to Darmon [7], and it covers the case where $F = \mathbb{Q}$ and $K$ is a real quadratic field. Although these Darmon points appear in rank one situations, they have their own importance because they give rise to an explicit Class Field Theory for real quadratic fields. The original construction has been generalized to arbitrary $F$ and arbitrary quadratic extensions $K/F$ in [8], [12], [10], [14] and [15].

Very recently, a new construction that works for rank $r > 1$ situations has emerged due to Fornea and Gehrmann (see [9]). Such a construction applies when there are $r$ prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_r$ of $F$ above a rational prime $p$ where the curve $E$ has multiplicative reduction and the quadratic extension $K/F$ is non-split. The construction provides, for any given character $\xi$ of the absolute Galois group of $K$, an element

$$P_\xi^S \in \bigotimes_{j=1}^r \hat{E}(K_{\mathfrak{p}_j})_\xi, \quad S = \{\mathfrak{p}_1, \cdots, \mathfrak{p}_r\}, \quad \hat{E}(K_{\mathfrak{p}_j})_\xi = E(K_{\mathfrak{p}_j})_\xi \otimes_\mathbb{Z} \mathbb{Z}_p,$$

where the tensor product is taken as $\mathbb{Z}_p$-modules, the field $K_{\mathfrak{p}_j}$ is the completion of $K$ at the prime ideal above $\mathfrak{p}_j$, and the subindex $\xi$ denotes the $\xi$-isotypical component. When $r = 1$, we recover the above mentioned Darmon points.

As can be seen, such constructions are $p$-adic. The results of the construction are (tensor product of) points defined over algebraic extensions of $\mathbb{Q}_p$. Following the philosophy of the BSD conjecture, it is conjectured that such points are defined over certain explicit number fields. More precisely, it is conjectured that, if the rank of $E(K)_\xi$ is $r$, there exist $r$ points $P_1, \cdots, P_r \in E(K)_\xi$ such that

$$P_\xi^S = \det(P_1 \wedge \cdots \wedge P_r),$$

where

$$\det : \bigwedge^r E(K)_\xi \longrightarrow \bigotimes_{j=1}^r \hat{E}(K_{\mathfrak{p}_j})_\xi; \quad \det(P_1 \wedge \cdots \wedge P_r) := \det \begin{pmatrix} \iota_{\mathfrak{p}_1}(P_1) & \cdots & \iota_{\mathfrak{p}_r}(P_1) \\ \vdots & \ddots & \vdots \\ \iota_{\mathfrak{p}_1}(P_r) & \cdots & \iota_{\mathfrak{p}_r}(P_r) \end{pmatrix},$$

and $\iota_\mathfrak{p} : E(K)_\xi \to \hat{E}(K_{\mathfrak{p}})_\xi$ is the natural morphism provided by the inclusion $K \subset K_{\mathfrak{p}}$.

If we go back to the case $r = 1$, some progress towards rationality of Darmon points has been made. In [3] and [18] a strategy for $F = \mathbb{Q}$ and $K$ real quadratic is carried out that consist of realising Darmon points as derivatives of Hida–Rankin $p$-adic L-functions. This point of view leads in loc. cit. to show that certain linear combination of Darmon points with coefficients given by values of genus characters of $K$ comes from a global point defined over the Hilbert class field of $K$. Moreover, it also provides the crucial bridge between Darmon points and generalised Kato classes arising from $p$-adic families of diagonal cycles. The results of [3] and [18] can likewise be exploited to make a similar comparison with Heegner points, and the explicit comparison between Heegner or Darmon points and generalised Kato classes is the main tool of [4] to finally prove rationality of Darmon points.

If we want to follow the footsteps of [4] in order to prove rationality of plectic points, we need to extend the $p$-adic Gross-Zagier formula that realizes plectic points as derivatives of Hida–Rankin $p$-adic L-functions. In this note, we will construct such Hida–Rankin $p$-adic L-functions as restrictions to the weight space of two variable $p$-adic L-functions interpolating anticyclotomic $p$-adic L-functions.

Let $\phi_E^p$ be the automorphic modular symbol associated to the elliptic curve $E$ (see §4 for a precise description of $\phi_E^p$). In [9] and [16] an anticyclotomic $p$-adic L-function associated to $E$ and a quadratic extension $K/F$ is...
constructed by means of \( \phi_p^\Lambda \). Such \( p \)-adic L-function is though as a \( p \)-adic measure \( \mu_{\phi_p^\Lambda} \) of the Galois group \( \mathcal{G}_T \) of the anticyclotomic abelian extension of \( K \). Moreover, the main result in loc. cit. provides a \( p \)-adic Gross-Zagier formula that computes higher derivatives of \( \mu_{\phi_p^\Lambda} \) in terms of plectic points and \( L \)-invariants. Following the philosophy of Mazur-Tate in \cite{[19]}, we understand the evaluation of a measure at a character \( \xi \) as its image through the ring homomorphism

\[
\varphi_{\xi} : \text{Meas}(\mathcal{G}_T) \longrightarrow C_p,
\varphi_{\xi}(\mu) = \int_{\mathcal{G}_T} \xi(\gamma) d\mu(\gamma).
\]

Therefore, if we write \( I_\xi = \ker(\varphi_{\xi}) \) for the augmentation ideal, the measure \( \mu_{\phi_p^\Lambda} \) lies in \( I_\xi \) when it has a zero of order \( r \) at \( \xi \). If this is the case, the \( r \)-th derivative of \( \mu_{\phi_p^\Lambda} \) at \( \xi \) is understood to be its image in \( I_\xi^r/I_\xi^{r+1} \). There is a natural way to send elements of \( \bigotimes_{j=1}^{\infty} \hat{E}(K_p) / I_\xi^j \) to \( I_\xi^r \) and the \( p \)-adic Gross-Zagier formula relates \( \mu_{\phi_p^\Lambda} \) with the image of \( P_\xi^S \) modulo \( I_\xi^{r+1} \). In this paper we will obtain a similar result but exchanging \( \mu_{\phi_p^\Lambda} \) with a new \( p \)-adic L-function \( L_p(\phi_p^\Lambda, \xi, k) \) depending on the weight \( \xi + 2 \in (2\mathbb{N})^d \) of the automorphic modular symbol. For this purposes, we need to extend the definition of \( \mu_{\phi_p^\Lambda} \) to automorphic modular symbols \( \phi_p^\Lambda \) of arbitrary even weight and interpolate the measures \( \mu_{\phi_p^\Lambda} \) in families.

Let

\[
\Lambda_F = \mathbb{Z}_p[[O_{F_p}^\times]], \quad O_{F_p} := O_F \otimes \mathbb{Z}_p,
\]

be the Iwasawa algebra associated with \( O_{F_p}^\times \). It classifies continuous characters \( \chi : O_{F_p}^\times \rightarrow \mathbb{R}^\times \), for any complete \( \mathbb{Z}_p \)-algebra \( R \). In particular, any even weight \( k = (k_p) \in (2\mathbb{N})^d \) is seen as a point in the spectrum of \( \Lambda_F \) because it defines the character \( x \mapsto x^{k} := \prod_{p} \sigma(x)^{k_p} \). These types of weights will usually be referred to as classical weights. The idea behind the construction of the Hida-Rankin \( p \)-adic L-function \( L_p(\phi_p^\Lambda, \xi, k) \) is to compute the image through \( \varphi_{\xi} \) of a family that interpolates the measures \( \mu_{\phi_p^\Lambda} \). This will produce an element in \( \Lambda_F \) whose classical specializations are values at \( \xi \) of anticyclotomic measures associated to automorphic forms for different weights.

In this paper we give a general construction of locally analytic anticyclotomic measures associated with a non-critical automorphic form of even weight \( \ell+2 \), where \( \ell \in (2\mathbb{N})^d \). Write \( \pi \) for the automorphic representation it generates. The Eichler-Shimura isomorphism provides modular symbols \( \phi_p^\Lambda \) attached to our automorphic forms. We will generalize our construction of \( \mu_{\phi_p^\Lambda} \) described in \cite{[16]} for ordinary weight 2 modular symbols and extend it to the non-critical arbitrary even weight setting. We will study in \S 5.5 the relation between \( \mu_{\phi_p^\Lambda} \) and the Rankin-Selberg L-function \( L(s, \pi, \xi) \) attached to the representation \( \pi \) twisted by a locally polynomial character \( \xi \).

Once we have extended the construction of \( \mu_{\phi_p^\Lambda} \) to arbitrary even weight, one needs to interpolate the measures as the weight varies. The first step is to introduce the concept of overconvergent modular symbols and relate it with our measures \( \mu_{\phi_p^\Lambda} \). In \S 6 we will prove that any non-critical modular symbol \( \phi_p^\Lambda \) extends to a unique overconvergent modular symbol \( \hat{\phi}_p^\Lambda \). Moreover, in Theorem 6.7 we will show that \( \mu_{\phi_p^\Lambda} \) can be also obtained as the cap-product of the overconvergent modular symbol \( \hat{\phi}_p^\Lambda \) with a fundamental class associated with the quadratic extension \( K/F \). This result is similar to the classical result of Pollack-Stevens in \cite{[23]}.

Finally, in \S 7 we will introduce our concept of family of modular symbols \( \Phi_p^\Lambda \). We will restrict ourselves to the ordinary case, hence these will be cohomology classes of \( \Lambda_F \)-valued distributions analogous to classical Hida families. We believe that everything can be extended to the non-critical setting of Coleman families using standard techniques. Although our concept of family \( \Phi_p^\Lambda \) differs from the classical one treated for instance in \cite{[1]} using local systems, we will prove in \S 7.1 that both concepts coincide.

Let us go back to the case of parallel weight 2, thus \( \phi_p^\Lambda \) will be an ordinary modular symbol attached to an elliptic curve \( E/F \) and \( \hat{\phi}_p^\Lambda \) will be the uniquely determined overconvergent modular symbol associated with it. We would like construct a Hida family \( \Phi_p^\Lambda \) passing through \( \hat{\phi}_p^\Lambda \). Since the existence of such a family is ultimately based on the geometry of the corresponding eigenvariety, and these concepts are beyond the scope of these paper, we will directly assume that \( \Phi_p^\Lambda \) exists. However, the analysis of \S 7.1 together with the results of \cite{[1]} ensure the existence of \( \Phi_p^\Lambda \) if \( F \) is totally real. Given the family of overconvergent modular symbols, one can naturally construct a \( \Lambda_F \)-valued measure \( \mu_{\phi_p^\Lambda} \) using formalism of Theorem 6.7. We may think of the measure \( \mu_{\phi_p^\Lambda} \) as a two variable \( p \)-adic L-function where, in addition to the level variable we add the weight variable.
Finally, we define \( L_p(\phi_L^n, \xi, k) \) as the restriction of \( \mu_{qF}^n \) to the weight space, more precisely,

\[
L_p(\phi_L^n, \xi, k) = \varphi_\xi(\mu_{qF}^n) = \int_{G_F} \xi(\gamma) d\mu_{qF}^n(\gamma) \in \Lambda_F \otimes_\mathbb{Z} \mathbb{Q}.
\]

The specialization at parallel weight 2 provides a morphism

\[ \rho_1 : \Lambda_F \to \mathbb{Z}_p. \]

Thus, the evaluation of \( L_p(\phi_L^n, \xi, k) \) at weight 2 is given by its image through \( \rho_1 \). It has zero of order \( r \) at weight 2 if \( L_p(\phi_L^n, \xi, k) \in I^r \), where \( I = \ker \rho_1 \) is the augmentation ideal, and its \( r \)-th derivative is given by its image in \( I^r/I^{r+1} \). The main result of the paper computes such a higher derivative in terms of plectic points:

**Theorem 1.1 (Theorem 8.2).** Let \( S \) be the set of places \( v \) above \( p \) where \( E \) has multiplicative reduction and \( v \) does not split at \( K \). Assume that \( r = \#S \neq 0 \). Then we have that

\[
L_p(\phi_L^n, \xi, k) \in I^r \otimes_\mathbb{Z} \mathbb{Q}.
\]

Moreover,

\[
L_p(\phi_L^n, \xi, k) \equiv C_\ell \cdot \prod_{v \in S} \varepsilon_p(\pi_v, \xi_p) \cdot \ell_v \circ \text{Tr}(P_v^\ast) \pmod{I^{r+1} \otimes_\mathbb{Z} \mathbb{Q}).
\]

where \( C_\ell \) is an explicit constant depending on \( T \), \( \text{Tr} : \bigotimes_{v \in S} E(K_v) \to \bigotimes_{v \in S} E(F_v) \) is the usual trace map, and \( \varepsilon_p(\pi_v, \xi_p) \) are explicit Euler factors.

The natural morphism \( \ell_v : \bigotimes_{v \in S} E(F_v)_\xi \to I^r \) will be explained in §7.4 and it can be thought as the tensor product of the classical \( p \)-adic logarithms of the groups \( E(F_v)_\xi \).

## 2 Setup and notation

For any field \( L \) we will write \( O_L \) its integer ring. Sometimes we denote \( O_{\mathbb{Q}} \) by \( \mathbb{Z} \) and \( O_{\mathbb{Q}_p} \) by \( \mathbb{Z}_p \). If \( L \) is a number field and \( p \) is a rational prime, we will write \( O_{L,p} = O_L \otimes_\mathbb{Z} \mathbb{Z}_p \).

Let \( F \) be a number field and let \( r_F, s_F \) be the number of real and complex places, respectively, so that \( [F : \mathbb{Q}] = r_F + 2s_F \). Let \( K/F \) be a quadratic extension and let \( \Sigma_{un}(K/F) \) be the set of infinite places of \( F \) that split at \( K \). Then

\[
\Sigma_{un}(K/F) = \Sigma_{\mathbb{R}}(K/F) \cup \Sigma_{\mathbb{C}}(K/F)
\]

where \( \Sigma_{\mathbb{R}}(K/F) \) is the set of real places of \( F \) which remain real in \( K \) and \( \Sigma_{\mathbb{C}}(K/F) \) is the set of complex places of \( F \). Let \( u, r_{K/F, \mathbb{R}} \) and \( s_F \) be the cardinality of these three sets, hence \( u = r_{K/F, \mathbb{R}} + s_F \). Analogously, we write \( \Sigma_{\mathbb{C}}(K/F) \) for the set of places \( \infty \setminus \Sigma_{un}(K/F) \). If a place \( \sigma \in \infty \) is given by the class of an embedding \( \bar{\sigma} : F \hookrightarrow \mathbb{C} \), we write \( \bar{\sigma} | \sigma \).

For any finite set of places \( S \) of \( F \) we write \( F_S = \prod_{v \in S} F_v \). We denote by \( A_S \) the ring of adèles of \( F \), and by \( A_S^\circ \) the ring of adèles outside \( S \), namely, \( A_S^\circ = A_S \cap \prod_{v \notin S} F_v \). We will often write \( \infty \) for the set of infinite places of \( F \) and \( p \) for the set of places of \( F \) above a rational prime \( p \). For a finite place \( v \), we will write \( \omega_v \) for a fixed uniformizer, \( v : F_v^\ast \to \mathbb{Z} \) for the \( p \)-adic valuation such that \( v_p(\omega_v) = 1 \), and \( q_v \) for the cardinal of the residue field \( O_F/v \). Similarly, we write \( v_p : \mathbb{C}_p^\ast \to \mathbb{Q} \) for the \( p \)-adic valuation such that \( v_p(p) = 1 \).

Let \( B/F \) be a quaternion algebra for which there exists an embedding \( K \hookrightarrow B \), that we fix now. Let \( \Sigma_B \) be the set of archimedean classes of \( F \) splitting the quaternion algebra \( B \). We can define \( G \) an algebraic group associated to \( B^\ast/F^\ast \) as follows: \( G \) represents the functor that sends any \( O_{\mathbb{Q}} \)-algebra \( R \) to

\[
G(R) = (O_{B \otimes_{O_{\mathbb{Q}}} R})^\ast/R^\ast,
\]

where \( O_B \) is a maximal order in \( B \) that we fix once and for all. Similarly, we define the algebraic group \( T \) associated to \( K^\ast/F^\ast \) by

\[
T(R) = (O_{\mathbb{C} \otimes_{O_{\mathbb{Q}}} R})^\ast/R^\ast,
\]

where \( O_{\mathbb{C}} := O_B \cap K \) is an order of conductor \( c \) in \( O_K \). Note that \( T \subset G \). We denote by \( G(F_\infty)_+ \) and \( T(F_\infty)_+ \) the connected component of the identity of \( G(F_\infty) \) and \( T(F_\infty) \), respectively. We also define \( T(F)_+ := T(F) \cap T(F_\infty)_+ \) and \( G(F)_+ := G(F) \cap G(F_\infty)_+ \).
Let $M$ be a $G(A^\mathbb{Q}_\mathfrak{p})$-representation over a field $L$, and let $\rho$ be an irreducible $G(A^\mathbb{Q}_\mathfrak{p})$-representation over $L$ with $S' \subseteq S$. We will write

$$M_p := \text{Hom}_{G(A^\mathbb{Q}_\mathfrak{p})}(\rho \mid_{G(A^\mathbb{Q}_\mathfrak{p})}, M).$$

as representations over $L$.

For any ring $R$ and an even number $k$, let $\mathcal{P}(k)_R$ be the $R$-module of homogeneous polynomials in two variables with coefficients in $R$, together with the following $\text{GL}_2(R)$-action; given $\gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$

$$(\gamma \cdot P)(x, y) := (\det \gamma)^{-k/2} \cdot P \left( \begin{array}{c} a \cdot x + b \\ c \cdot y + d \end{array} \right)$$

Note that the factor $(\det \gamma)^{-k/2}$ makes the central action trivial.

Denote by $V(k)_R$ the dual space $\text{Hom}_R(\mathcal{P}(k)_R, R)$. In the case $R = \mathbb{C}$ we set $V(k) := V(k)_{\mathbb{C}}$. Given a vector $\underline{k} = (k_i) \in (2\mathbb{N})^n$ we define

$$V(\underline{k})_R := \bigotimes_{i=1}^n V(k_i)_R.$$

Notice that, if $n = d = [F : \mathbb{Q}]$ and $\underline{k} = (k_i) \in (2\mathbb{N})^d$ is indexed by the embeddings $\tilde{\sigma} : K \hookrightarrow \mathbb{C}$ of $F$, then $V(k) := V(\underline{k})_{\mathbb{C}}$ has a natural action of $G(F_{\infty})$. The subspace $V(\underline{k})_{\mathbb{C}}$ is fixed by the action of the subgroup $G(F) \subseteq G(F_{\infty})$.

Note that, if we fix $\bar{Q} \hookrightarrow \mathbb{C}_p$ and denote by $\Sigma_p$ the set of embeddings $\bar{\sigma} : F \hookrightarrow \mathbb{C}_p$. We have fixed a bijection between embeddings $\tilde{\bar{\sigma}} : F \hookrightarrow \mathbb{C}$ and $\bar{\sigma} : F_p \hookrightarrow \mathbb{C}_p$ for all $p | p$. Thus, for any $\underline{k} \in (2\mathbb{N})^d$ we have also a natural action of $\text{PGL}_2(F_p)$ on $V(\underline{k})_{\mathbb{Q}_p}$.

Let $p$ be any place above $p$, and assume that $G(F_p) = \text{PGL}_2(F_p)$. The fixed embedding $\iota : K^\times_p \hookrightarrow \text{GL}_2(F_p)$ provides two eigenvectors $v^1_p, v^2_p \in (\bar{F}_p)^2$ satisfying

$$v^1_p(t\bar{F}_p) = \lambda_{t_p} v^1_p, \quad v^2_p(t\bar{F}_p) = \tilde{\lambda}_{t_p} v^2_p, \quad \lambda_{t_p}, \tilde{\lambda}_{t_p} \in \bar{F}_p.$$ 

Notice that the quotient $\lambda_{t_p}/\tilde{\lambda}_{t_p}$ depends on the class $t_p \in K^\times_p/F^\times_p = T(F_p)$ of $t_p$. By abuse of notation we will denote $\lambda_{t_p}/\tilde{\lambda}_{t_p} \in \bar{F}_p$ also by $t_p$. Write $K^\times_p$ for the minimum extension of $F_p$ where all $\lambda_{t_p}$ lie, hence $K^\times_p = F_p$ if $p$ splits and $K^\times_p = K_p$ otherwise. For any $\sigma \in \Sigma_p$, we will fix once for all $\tilde{\bar{\sigma}} : K^\times_p \hookrightarrow \mathbb{C}_p$ such that $\tilde{\bar{\sigma}} |_{F_p} = \sigma$. Also by abuse of notation, we will write

$$\sigma(t_p) := \tilde{\bar{\sigma}}(t_p) \in \mathbb{C}_p,$$

$$t_p \in T(F_p).$$

We will write the open compact subsets

$$U_\sigma(a, n) := a \left( \left( O_{F_p} + \omega_p^{-n \cdot \varphi(a)} O_{K^\times_p} \right)^\times \cap O^\times_{F_p} \right) \subseteq K^\times_p/F^\times_p = T(F_p),$$

for any $a \in T(F_p)$. Notice that for all $p | p$, the $U_\sigma(a, n)$ generate a basis for the topology of $T(F_p)$. Moreover, it can be described as

$$U_\sigma(a, n) = \{ t \in T(F_p); \sqrt{a} \mid (t - a) \}.$$

We will assume that $n \in \mathbb{N}$ if $T(F_p)$ does not ramify. If $T(F_p)$ ramifies we will choose by convenience that $n \in \frac{1}{2} + \mathbb{N}$, understanding that $\sqrt{a}^\frac{1}{2}$ is the uniformizer of $\mathfrak{v}^\mathfrak{v}$, the unique prime ideal of $K_p$ above $\mathfrak{v}^\mathfrak{v}$.

### 3 Fundamental classes of tori

In this section we will define certain fundamental classes associated with the torus $T$.

#### 3.1 The fundamental class $\eta$ of the torus

Let $U = \prod_{\mathfrak{v}} T(O_{F_{\mathfrak{v}}})$ and denote by $O := T(F) \cap U$ the group of relative units. Similarly, we define $O^+ := O \cap T(F_{\infty})^+$ to be the group of totally positive relative units. By an straightforward argument using Dirichlet Units Theorem

$$\text{rank}_2 O = \text{rank}_2 O^+ = (2r_{K/F, \mathbb{R}} + r_{K/F, \mathbb{C}} + 2s_F - 1) - (r_F + s_F - 1) = r_{K/F, \mathbb{R}} + s_F = u.$$
We also define the class group

\[ \text{Cl}(T)_+ := T(A\ell_F)/(T(F) \cdot U \cdot T(F\infty)_+) = T(A\ell_F^\text{reg})/(T(F)_+ \cdot U). \]

Recall that we have the usual homomorphism \( T(F\infty)_+ \to \mathbb{R}^n \) given by \( z \mapsto (\log |z|)_{\ell \in \Sigma_{\infty}(K/F)} \). Moreover, under this isomorphism the image of \( O_+ \) is a \( \mathbb{Z} \)-lattice \( \Lambda \subset \mathbb{R}^n \), as in the proof of Dirichlet’s Unit Theorem. We can identify \( \Lambda \) with its preimage in \( O_+ \). Thus, we can define a fundamental class \( \xi \in H_{\text{u}}(\Lambda, \mathbb{Z}) \).

On the other hand, notice that the group \( \text{Cl}(T)_+ \) fits in the following exact sequence

\[ 0 \to T(F)_+/O_+ \to T(A\ell_F^\text{reg})/U \to \text{Cl}(T)_+ \to 0. \]

We fix preimages \( \bar{t}_i \in T(A\ell_F^\text{reg}) \) for every element \( t_i \in \text{Cl}(T)_+ \) and we consider the compact set

\[ \mathcal{F} = \bigcup_{i} \bar{t}_i U \subset T(A\ell_F^\text{reg}). \]

Write \( \eta \) for the cap product

\[ \eta = \mathbb{1}_F \cap \xi \in H_u(O_+, C(\mathcal{F}, \mathbb{Z})), \]

where \( \mathbb{1}_F \in H^0(O_+, C(\mathcal{F}, \mathbb{Z})) \) is the indicator function and \( \xi \in H_u(O_+, \mathbb{Z}) \) is the image of \( \xi \) through the corestriction morphism. By [16, Lemma 3.3], we have an isomorphism of \( T(F) \)-modules \( \text{Ind}_{O_+}^{T(F)}(C(\mathcal{F}, \mathbb{Z})) \cong C_0^0(T(A\ell_F), \mathbb{Z}) \), where \( C_0^0 \) stands for the set of compactly supported locally constant functions. Thus, by Shapiro’s lemma one may regard

\[ \eta \in H_u(T(F), C_0^0(T(A\ell_F), \mathbb{Z})). \]

### 3.1.1 The \( S \)-fundamental classes \( \eta^S \)

As shown in [22], by means of the fundamental class \( \eta \) we can compute certain periods related with certain critical values of classical \( L \)-functions. In this section we are going to define different fundamental classes \( \eta^S \) that will be useful to define plectic points. At the end of the section we will relate both fundamental classes.

Let \( S \) be a set of places \( p \) of \( F \) above \( p \) where \( T \) does not split. Similarly as above, we consider:

\[ \mathcal{F}^S = \bigcup_{i} \bar{t}_i U^S, \quad U^S = \prod_{v \notin S} T(O_{F,v}), \]

where \( \bar{t}_i \in T(A\ell_F^{S,\text{reg}}) \) are representatives of the elements of \( \text{Cl}(T)^S_+ \) and

\[ \text{Cl}(T)^S_+ = T(A\ell_F^{S,\text{reg}})/(U^S T(F)_+). \]

Let us consider also the set of totally positive relative \( S \)-units \( O^S_+ := U^S \cap T(F)_+ \). Note that we have an exact sequence

\[ 0 \to T(F_S)/O_S^ST(O_{F,S}) \to \text{Cl}(T)_+ \to \text{Cl}(T)^S_+ \to 0 \]

where \( O_{F,S} := \prod_{v \in S} O_{F,v} \).

It is clear that the quotient \( T(F_S)/T(O_{F,S}) \) is finite. Moreover, we have the natural exact sequence

\[ 0 \to O_+ \to O^S_+ \to T(F_S)/T(O_{F,S}). \]

This implies that the \( \mathbb{Z} \)-rank of \( O^S_+ \) is \( u \). Hence, similarly as above, we consider the image \( \xi^S \in H_u(O^S_+, \mathbb{Z}) \) of \( \xi \) through the corestriction morphism. We have as well

\[ C_0^0(T(A\ell_F^{S,\text{reg}}), \mathbb{Z}) = \text{Ind}_{O^S_+}^{T(F)_+}(C(\mathcal{F}^S, \mathbb{Z})). \]

Thus, the cap product \( \mathbb{1}_{FS} \cap \xi^S \) provides an element

\[ \eta^S := \mathbb{1}_{FS} \cap \xi^S \in H_u(O^S_+, C(\mathcal{F}^S, \mathbb{Z})) = H_u(T(F)_+, C_0^0(T(A\ell_F^{S,\text{reg}}), \mathbb{Z})). \]

The following result relates the previously defined fundamental classes

**Lemma 3.1.** [16, Lemma 3.4] We have that

\[ [O^S_+: O_+] \cdot \eta = \eta^S \cap \mathbb{1}_{T(F_S)} \]

where \( \mathbb{1}_{T(F_S)} \in H^0(T(F), C_0^0(T(F_S), \mathbb{Z})) \) is the constant function.
4 Cohomology classes attached to automorphic forms

In this section we will describe cohomology classes associated with automorphic forms by means of the Eichler-Shimura map. Let \( H \subset G(F) \) be a subgroup, \( R \) a topological ring, and \( S \) a finite set of places of \( F \) above \( p \). The subgroup \( H \) will usually be \( G(F), G(F)_+, T(F) \) or \( T(F)_+ \). For any \( R[H] \)-modules \( M, N \) let

\[
\mathcal{A}^S_{\text{loc}}(M, N) := \left\{ \phi : G(\mathcal{A}^S_{\text{loc}}) \to \text{Hom}_R(M, N), \begin{array}{l}
\text{there exists an open compact} \\
\text{subgroup } U \subset G(\mathcal{A}^S_{\text{loc}}) \\
\text{with } \phi(U) = \phi(0) \end{array} \right\},
\]

and let also \( \mathcal{A}^S_{\text{loc}}(N, N) := \mathcal{A}^S_{\text{loc}}(R, N) \). Then \( \mathcal{A}^S_{\text{loc}}(M, N) \) has a natural action of \( H \) and \( G(\mathcal{A}^S_{\text{loc}}) \), namely

\[
(h\phi)(x) := h \cdot \phi(h^{-1}x), \quad (y\phi)(x) := \phi(xy),
\]

where \( h \in H \) and \( x, y \in G(\mathcal{A}^S_{\text{loc}}) \).

4.1 Automorphic cohomology classes

Let \( E/F \) be a modular elliptic curve. Hence, attached to \( E \), we have an automorphic form for \( \text{PGL}_2/F \) of parallel weight \( 2 \). Let us assume that such form admits a Jacquet-Langlands lift to \( G \), and denote by \( \pi \) the corresponding automorphic representation. Let \( s := \# \Sigma_p \). As shown in [20], once fixed a character \( \lambda : (\mathbb{Q}/\mathbb{Z})/\mathbb{Z} \to \pm 1 \), the image through the Eichler-Shimura isomorphism of such Jacquet-Langlands lift provides a cohomology class in \( H^s(G(F)_+, \mathcal{A}^\infty(\mathbb{C}))^\lambda \), where the super-index \( \lambda \) stands for the subspace where the natural action of \( G(F)/G(F)_+ \) on the cohomology groups is given by the character \( \lambda \). Moreover, since the coefficient ring of the automorphic forms is \( \mathbb{Z} \), such a class is the extension of scalars of a class

\[
\phi_{\lambda} \in H^s(G(F)_+, \mathcal{A}^\infty(\mathbb{Z}))^\lambda.
\]

Indeed, \( H^s(G(F)_+, \mathcal{A}^\infty(\mathbb{Z})) \otimes_{\mathbb{Z}} \mathbb{C} = H^s(G(F)_+, \mathcal{A}^\infty(\mathbb{C})) \) (see for example [26, Proposition 4.6]). For general automorphic forms of arbitrary even weight \( (k + 2) \in (2\mathbb{N})^d \), the Eichler-Shimura morphism provides a class

\[
\phi_{\lambda} \in H^s(G(F)_+, \mathcal{A}^\infty(V(\mathbb{Q}))^\lambda.
\]

For any set \( S \) of places above \( p \), write \( V_S := \rho |_{G(F)_+} \), with \( V_S = \bigotimes_{v \in S} V_v \), and for any ring \( R \) we denote by \( V^R_S = \bigotimes_{v \in S} V^R_v \) the \( R \)-module generated by \( \phi_{\lambda} \). By [15, Remark 2.1] we have that

\[
\Phi_{\lambda} \in H^s(G(F)_+, \mathcal{A}^\infty(V(k)))_{\mathbb{Q}_{\mathbb{Q}}}^\lambda = H^s(G(F)_+, \mathcal{A}^S_{\text{loc}}(V_S, V(k)))_{\mathbb{Q}_{\mathbb{Q}}}^\lambda.
\]

For any \( x^S \in \rho^S := \rho |_{G(\mathcal{A}^S_{\text{loc}})} \), the image \( \Phi_{\lambda}(x^S) \) defines an element

\[
\phi^S_{\lambda} \in H^s(G(F)_+, \mathcal{A}^S_{\text{loc}}(V_S, V(k)))_{\mathbb{Q}_{\mathbb{Q}}}^\lambda.
\]

We will usually treat \( \phi^S_{\lambda} \) as an element of the cohomology group \( H^s(G(F)_+, \mathcal{A}^S_{\text{loc}}(V_S, V(k)))^\lambda \) since

\[
H^s(G(F)_+, \mathcal{A}^S_{\text{loc}}(V_S^\mathbb{Q}, V(k))) \otimes_{\mathbb{Q}_{\mathbb{Q}}} \mathbb{C} = H^s(G(F)_+, \mathcal{A}^S_{\text{loc}}(V_S, V(k)))/\mathbb{C},
\]

again by [26, Proposition 4.6]. The classes \( \phi^S_{\lambda} \) are essential in our construction of anti-cyclotomic \( p \)-adic L-functions, and plectic points.

If \( \phi^1 \) is associated with an elliptic curve as above and \( x^S \in \rho^S \) is now an element of \( \mathbb{Z} \)-module generated by the translations of \( \phi_{\lambda} \), we can think \( \phi^S_{\lambda} \) as an element

\[
\phi^S_{\lambda} \in H^s(G(F)_+, \mathcal{A}^S_{\text{loc}}(V^\mathbb{Q}_S, \mathbb{Z}))^\lambda,
\]

since the coefficient ring of \( \pi \) is \( \mathbb{Z} \).
4.2 Pairings

In order to relate the $p$-arithmetic cohomology groups defined in §4.1 and the homology groups defined in §3, we will define certain pairings that will allow us to perform cap products. For this purpose, we assume the following hypothesis:

**Hypothesis 4.1.** Assume that $\Sigma_B = \Sigma_{\text{un}}(K/F)$. Hence, in particular, $u = s$ and $G(F)/G(F)_\epsilon = T(F)/T(F)_\epsilon$.

As above, let $S$ be a set of primes $\mathfrak{p}$ above $p$. For any $T(F)$-modules $M$ and $N$, let us consider the $T(F)_\epsilon$-equivariant pairing

$$
\langle \cdot , \cdot \rangle : C^0_\epsilon(T(\mathcal{A}_F^{\text{cyc}}), M) \times \mathcal{A}^{\text{cyc}}(M, N) \to N, \\
(f, \phi) \mapsto \langle f, \phi \rangle := \int_{T_\epsilon(\mathcal{A}_F^{\text{cyc}})} \phi(t)(f(t))d\gamma t,
$$

where $d\gamma t$ is the corresponding Haar measure. This implies that, once fixed a character $\lambda : G(F)/G(F)_\epsilon = T(F)/T(F)_\epsilon \to \pm 1$, it induces a well-defined $T(F)$-equivariant pairing

$$
\langle \cdot , \cdot \rangle : \text{Ind}^{T(F)}_{T(\mathcal{A}_F^{\text{cyc}})} C^0_\epsilon(T(\mathcal{A}_F^{\text{cyc}}), M) \times \mathcal{A}^{\text{cyc}}(M, N)(\lambda) \to N, \\
\left( \sum \tau_{\epsilon(T(F)/T(\mathcal{A}_F^{\text{cyc}}))} f_\tau \otimes t, \phi \right) := [T(F) : T(F)_\epsilon]^{-1} \sum \tau_{\epsilon(T(F)/T(\mathcal{A}_F^{\text{cyc}}))} t \cdot (f_\tau, t^{-1} \phi),
$$

where $\mathcal{A}^{\text{cyc}}(M, N)(\lambda)$ is the twist of the $T(F)$-representation $\mathcal{A}^{\text{cyc}}(M, N)$ by the character $\lambda$.

Assume that $M = C^0_\epsilon(T(F_S), R) \otimes_R V$, for a finite rank $R$-module $V$,

$$
\text{Ind}^{T(F)}_{T(\mathcal{A}_F^{\text{cyc}})} C^0_\epsilon(T(\mathcal{A}_F^{\text{cyc}}), M) = C^0_\epsilon(T(\mathcal{A}_F), R) \otimes_R V.
$$

This implies that (10) provides a final $T(F)$-equivariant pairing

$$
\langle \cdot , \cdot \rangle : C^0_\epsilon(T(\mathcal{A}_F), R) \otimes_R V \times \mathcal{A}^{\text{cyc}}(C^0_\epsilon(T(F_S), R) \otimes_R V, N)(\lambda) \to N, \\
\langle f_\lambda \otimes f^\phi \otimes v, \phi \rangle = [T(F) : T(F)_\epsilon]^{-1} \sum \tau_{\epsilon(T(F)/T(\mathcal{A}_F^{\text{cyc}}))} f_\lambda^\phi(x) \cdot \phi(t)(f_\lambda \otimes v)d\gamma t.
$$

All the pairings above induce cap products in $H^\epsilon(\text{co})$-homology by their $H$-equivariance. Now denote by $f_\lambda$ the projection of $f$ to the subspace

$$
C^0_\epsilon(T(\mathcal{A}_F), R)_\lambda := \left\{ f \in C^0_\epsilon(T(\mathcal{A}_F), R) | f|_{T(\mathcal{F}_\epsilon)} = \lambda \right\}.
$$

One easily computes that

$$
\langle f \otimes v, \phi \rangle = \langle f_\lambda \otimes v, \phi \rangle = \langle f_\lambda |_{T_\epsilon(\mathcal{A}_F^{\text{cyc}})} \otimes v, \phi \rangle, \quad v \in V, \quad f \in C^0_\epsilon(T(\mathcal{A}_F), R), \quad \phi \in \mathcal{A}^{\text{cyc}}(C^0_\epsilon(T(F_S), R), N)(\lambda).
$$

Since we can identify $H^\epsilon(G(F)_\epsilon, \bullet) \cong H^\epsilon(G(F), \bullet)(\lambda)$, we deduce that for all $f \otimes v \in H^\epsilon_\epsilon(T(F), C^0_\epsilon(T(\mathcal{A}_F), R) \otimes_R V)$ and $\phi \in H^\epsilon_\epsilon(T(F), \mathcal{A}^{\text{cyc}}(C^0_\epsilon(T(F_S), R) \otimes_R V, N))(\lambda)$,

$$
(f \otimes v) \cap \phi = (f_\lambda \otimes v) \cap \phi = (f_\lambda |_{T(\mathcal{A}_F^{\text{cyc}})} \otimes v) \cap \text{res}^{T(F)}_{T(F)_\epsilon} \phi \in N,
$$

where $\text{res}^{T(F)}_{T(F)_\epsilon}$ is the restriction morphism and the cap products are the induced by (9), (11), respectively.

5 Anticyclotomic $p$-adic L-functions

In this section we will define the anticyclotomic $p$-adic L-functions associated with $p$, $T$ and the automorphic cohomology class $\phi^\lambda_A$ (here we choose $S = p$). From this point on we will assume that hypothesis 4.1 is fulfilled.

5.1 Defining the distribution

Let $C_\epsilon(T(F_p), \overline{\mathbb{Q}}_p)$ be the space of $\overline{\mathbb{Q}}_p$-valued locally polynomial functions of $T(F_p)$ of degree less that $k$. These correspond to the set of functions $f : T(F_p) \to \overline{\mathbb{Q}}_p$ such that in an small neighbourhood $U$ of $t_p = (t_p)_{\mathfrak{p} \mid p} \in T(F_p)$

$$
f(s_p) = \sum_{|m_i| \leq M} a_m(U) \prod_{\mathfrak{p} \mid p} \prod_{i \in \Sigma_p} \sigma(s_p)^{m_i}.
$$
where \( \mathbf{m} = (m_\sigma) \in \mathbb{Z}^d, a_{\mathbf{m}}(U) \in \overline{\mathbb{Q}}_p \) and \( s_p \in U \). If we denote by \( C^0 \) the set of locally constant functions, there is an isomorphism

\[
\iota : \mathcal{C}^0(T(F_p), \overline{\mathbb{Q}}) \otimes \mathcal{P}(k) \otimes \overline{\mathbb{Q}}_p \longrightarrow \mathcal{C}_{\mathcal{L}}(T(F_p), \overline{\mathbb{Q}}_p)
\]  

(13)

\[\text{defined by} \]

\[
h \otimes \bigotimes_{\sigma} P_{\sigma} \mapsto \left( t_p = (t_p)_{p|\sigma} \mapsto h(t_p) \prod_{p|\sigma} P_{\sigma}(\tilde{\alpha}(v_1^p + t_pv_2^p)) \sigma(t_p)^{\frac{1}{2}} \right).\]

Notice that it is \( T(F_p) \)-equivariant, indeed for any \( x_p = (x_p) \in T(F_p)\),

\[
x_p \cdot \left( h \otimes \bigotimes_{\sigma} P_{\sigma} \right)(t_p) = \iota \left( h \otimes \bigotimes_{\sigma} P_{\sigma} \right)(x_p^{-1} t_p) = h(x_p^{-1} t_p) \prod_{p|\sigma} P_{\sigma} \left( \tilde{\alpha} \left( v_1^p + \frac{\tilde{\lambda}_\sigma}{\lambda_\sigma} t_pv_2^p \right) \right) \tilde{\alpha} \left( \frac{1}{2} \right).\]

Let \( \mathcal{G}_T \) be the Galois group of the abelian extension of \( K \) associated with \( T \). By class field theory, there is a continuous morphism

\[
\rho : \left( T(F_{\infty})/T(F_{\infty}), \times T(\mathbb{A}_F^{\infty}) \right) / T(F) \rightarrow \mathcal{G}_T.
\]  

(14)

Let us consider the subset of \( \mathcal{C}(T(\mathbb{A}_F), \overline{\mathbb{Q}}_p) \):

\[
\mathcal{C}_{\mathcal{L}}(T(\mathbb{A}_F), \overline{\mathbb{Q}}_p) := \{ f : T(\mathbb{A}_F) \rightarrow \mathcal{C}_{\mathcal{L}}(T(F_p), \overline{\mathbb{Q}}_p), \text{ locally constant} \},
\]

and write also \( \mathcal{C}_{\mathcal{L}}(\mathcal{G}_T, \overline{\mathbb{Q}}_p) \) for the subspace of continuous functions \( f : \mathcal{G}_T \rightarrow \overline{\mathbb{Q}}_p \) such that \( \rho'(f) \in \mathcal{C}_{\mathcal{L}}(T(\mathbb{A}_F), \overline{\mathbb{Q}}_p) \). The pullback of \( \rho \) together with the cap product by \( \eta \) give the following morphism

\[
\delta : \mathcal{C}_{\mathcal{L}}(\mathcal{G}_T, \overline{\mathbb{Q}}_p) \xrightarrow{\rho'} H^0(T(F), \mathcal{C}_{\mathcal{L}}(T(\mathbb{A}_F), \overline{\mathbb{Q}}_p)) \xrightarrow{\cap \eta} H_0(T(F), \mathcal{C}_{\mathcal{L}}(T(\mathbb{A}_F), \overline{\mathbb{Q}}_p)).
\]  

(15)

where \( \mathcal{C}_{\mathcal{L}}(T(\mathbb{A}_F), \overline{\mathbb{Q}}_p) \) is the subspace of functions in \( \mathcal{C}_c(T(\mathbb{A}_F), \overline{\mathbb{Q}}_p) \) which are compactly supported when restricted to \( T(\mathbb{A}_F^{\infty}) \). Notice that \( i \) of (13) provides an isomorphism

\[
\mathcal{C}_{\mathcal{L},c}(T(\mathbb{A}_F^{\infty}), \overline{\mathbb{Q}}_p)) = \mathcal{C}_0(T(\mathbb{A}_F^{\infty}), \overline{\mathbb{Q}}) \otimes \mathcal{P}(k) \otimes \overline{\mathbb{Q}}_p.
\]  

(16)

In order to define our distribution we need to construct a \( T(F_p) \)-equivariant morphism:

\[
\delta_p = (\delta_p)_{p|\sigma} : \mathcal{C}_0(T(F_p), \overline{\mathbb{Q}}) \longrightarrow V_p,
\]

\[
\delta_p : \mathcal{C}_0(T(F_p), \overline{\mathbb{Q}}) \longrightarrow V_p.
\]  

(17)

Given such a \( \delta_p \), we can directly define our distribution associated with \( \phi^{\mathcal{L}}_A \in H^u(G(F)_+ \mathcal{A}^{\mathcal{L}, 0}(V_p, V(k)_{\overline{\mathbb{Q}}})) \) as follows:

\[
\int_{\mathcal{G}_T} \check{g} d\mu_{\mathcal{L}, \phi^{\mathcal{L}}_A} := \delta(g) \cap \delta_p \phi^{\mathcal{L}}_A, \quad \text{for all } g \in \mathcal{C}_{\mathcal{L}}(\mathcal{G}_T, \overline{\mathbb{Q}}_p)
\]  

(18)

where the cap product is induced by the pairing (11) and

\[
\delta_p : \mathcal{A}^{\mathcal{L}, 0}(V_p, V(k)_{\overline{\mathbb{Q}}}) \longrightarrow \mathcal{A}^{\mathcal{L}, 0} \left( \mathcal{C}_0(T(F_p), \overline{\mathbb{Q}}) \otimes \mathcal{P}(k)_{\overline{\mathbb{Q}}} \right)
\]

is the corresponding \( T(F) \)-equivariant pullback.
5.2 Admissibility

A continuous function $f \in C(T(F_p), \mathbb{C}_p)$ is locally analytic if, for any $x_p = (x_p)_q \in T(F_p)$ there exists a neighbourhood $U$ of $t_p$ and $a_n(U) \in \overline{\mathbb{C}_p}$ such that

$$f \mid_U (s_p) = \sum_{n \in \mathbb{N}} a_n(U) \prod_{q \in \mathbb{C}_p} (\sigma(s_p) - \sigma(x_p))^m,$$

for all $s_p = (s_p)_q \in U \subset T(F_p)$.

Write $C_{an}(T(F_p), \mathbb{C}_p)$ for the subspace of locally analytic functions. Notice that $C_{an}(T(F_p), \mathbb{C}_p) \subset C_{an}(T(F_p), \mathbb{C}_p)$. Indeed, it is clear that $\sigma(t_p)^{m_n} \in C_{an}(T(F_p), \mathbb{C}_p)$ when $m_n$ is positive, but when $m_n < 0$,

$$\sigma(t_p)^{m_n} = \sum_{i \geq 0} \binom{m_n}{i} \sigma(t_p)^{m_n-i} (\sigma(t_p) - \sigma(x_p))^i \in C_{an}(T(F_p), \mathbb{C}_p).$$

Similarly as above, we consider

$$C_{an}(T(A_F), \mathbb{C}_p) := \{ f : T(A_F) \to C_{an}(T(F_p), \mathbb{C}_p), \text{ locally constant} \} \subset C(T(\Delta), \mathbb{C}_p),$$

and $C_{an}(\mathcal{G}_T, \mathbb{C}_p) := (\sigma^*)^{-1} C_{an}(T(A_F), \mathbb{C}_p)$.

In the previous section we have constructed a locally polynomial distribution

$$\mu_{\phi_\alpha} \in \text{Dist}(\mathcal{G}_T, \overline{\mathbb{C}_p}) := \text{Hom}(C_{\Delta}(\mathcal{G}_T, \overline{\mathbb{C}_p}), \overline{\mathbb{C}_p}).$$

In this section we aim to extend it to a locally analytic distribution

$$\mu_{\phi_\alpha} \in \text{Dist}_{an}(\mathcal{G}_T, \mathbb{C}_p) := \text{Hom}_{an}(C_{an}(\mathcal{G}_T, \mathbb{C}_p), \mathbb{C}_p).$$

Write also $\text{Dist}(\mathcal{G}_T, \overline{\mathbb{C}_p}) := \text{Hom}(C_{\Delta}(\mathcal{G}_T, \overline{\mathbb{C}_p}), \overline{\mathbb{C}_p})$ and $\text{Dist}_{an}(\mathcal{G}_T, \mathbb{C}_p) := \text{Hom}_{an}(C_{an,c}(T(F_p), \mathbb{C}_p), \mathbb{C}_p)$.

**Definition 5.1.** For any $\nu | \nu$, a locally polynomial distribution $\mu \in \text{Dist}(\mathcal{G}_T, \overline{\mathbb{C}_p})$ is $h_{\nu}$-admissible at $\nu$ if for every $a \in T(F_p)$ there exists a fixed constant $A_{\nu} \in \mathbb{C}_p$ only depending on $\nu, T$ and a neighbourhood of such that

$$\int_{U_\nu(a,n)} g \, d\mu \in A_{\nu} p^{-nh_{\nu}} \mathcal{O}_{\mathbb{C}_p},$$

for any $n \in \mathbb{N}$ and $g \in C_{\nu}(T(F_p), \mathcal{O}_{\mathbb{C}_p})$.

**Proposition 5.2.** Let $e_p$ be the ramification index of $\nu$. If $e_p h_{\nu} < \min\{k_{\sigma} + 1, \sigma \in \Sigma_\nu\}$, a distribution $\mu \in \text{Dist}(\mathcal{G}_T, \overline{\mathbb{C}_p})$ that is $h_{\nu}$-admissible at every $\nu | p$ can be extended to a unique locally analytic measure in $\text{Dist}_{an}(T(F_p), \mathbb{C}_p)$ such that

$$\int_{U_\nu(a,n)} g \, d\mu \in A_{\nu} p^{-nh_{\nu}} \mathcal{O}_{\mathbb{C}_p},$$

for any $g \in C_{an}(T(F_p), \mathcal{O}_{\mathbb{C}_p})$ which is analytic in $U_\nu(a,n)$.

**Proof.** Notice that any locally analytic function is topologically generated by functions of the form

$$P_{\nu}^{a,n}(x) := \prod_{\omega \in \Sigma} \left( \frac{x_{p} - a}{\omega_{p}} \right)^{m} x_{p}^{-\frac{1}{2}}, \quad a \in T(F_p),$$

where $m \in \mathbb{N}^{\Sigma}$ and

$$\left( \frac{x_{p} - a}{\omega_{p}} \right)^{m} x_{p}^{-\frac{1}{2}} := \prod_{\sigma \in \Sigma} \left( \frac{x_{p} - a}{\omega_{p}} \right)^{m_{\sigma}} \sigma(x_{p})^{-\frac{1}{2}}.$$

By definition, we have the values $\mu(P_{\nu}^{a,n})$ for every $m_{\sigma} \leq k_{\sigma}$. If there exist $\tau \in \Sigma_{\nu}$ such that $m_{\tau} > e_{\nu} h_{\nu}$, we define

$$\mu(P_{\nu}^{m,n}) = \lim_{n \to \infty} a_n,$$
and

\[
\left( \frac{m}{f} \right) = \prod_{\alpha \in \Sigma_p} \left( \frac{m_\alpha}{f_\alpha} \right),
\]

\[
\alpha_{\nu, P}^{j(n-N)} := \prod_{\alpha \in \Sigma_p} \sigma(\omega\nu)_{\alpha}^{j(n-N)}
\]

The definition agrees with \( \mu \) when there exist \( \tau \in \Sigma_p \) such that \( e_\nu h_\nu < m_\tau \leq k_\tau \) because \( \alpha_{\nu, P}^{j(n-N)} \rightarrow 0 \) when \( f_\tau > e_\nu h_\nu \).

It can be checked similarly as in [21, Proposition 2] that the sequence \( a_n \) is Cauchy, hence the limit exists. It is clear by the definition that \( \mu(P^{n-N}) \in A_\nu \cdot p^{-N}h_\nu \), for all \( \nu \) and \( N \). Hence, it extends to a locally analytic measure as described.

Let \( M_T = \left( \begin{smallmatrix} \nu_1 & \nu_2 \\ \bar{v}_1 & \bar{v}_2 \end{smallmatrix} \right) \in GL_2(\overline{\mathbb{Q}}_p) \) be the matrix given by the eigenvectors \( v_\nu^1 \). Recall that \( M_T \in GL_2(K_\nu^\mathbb{A}) \) acts also on \( \mathcal{P}(k)_\nu^\mathbb{A} \) and \( \mathcal{V}(k)_\nu^\mathbb{A} \) since we have fixed embeddings \( \mathfrak{P} : K_\nu^\mathbb{A} \hookrightarrow \overline{\mathbb{Q}}_p \) extending each \( \sigma \). In the non-split situation, we will fix \( v_\nu^1 = (1, -\bar{v}_\nu) \) and \( v_\nu^2 = (-1, \tau_\nu) \), where \( \tau_\nu, \bar{v}_\nu \in K_\nu^\mathbb{A} \) are the points fixed by the action of \( \iota(K_\nu^\mathbb{A}) \) on \( \mathcal{P}_\nu^1(K_\nu) \) given by linear fractional transformations. Notice that

\[
M_T^{-1} \cdot t_\nu = \begin{pmatrix} \tau_\nu & \bar{v}_\nu \\ -\bar{v}_\nu & 1 \end{pmatrix} = \begin{pmatrix} \tau_\nu t_\nu + \bar{v}_\nu \\ t_\nu + 1 \end{pmatrix}.
\]

Since \( \bar{v}_\nu = t_\nu^{-1} \) where \( (\cdot) \) stands for the non-trivial automorphism of \( \text{Gal}(K_\nu/F_\nu) \), we obtain

\[
M_T^{-1} \cdot t_\nu = \begin{pmatrix} \bar{v}_\nu t_\nu^{-1} + \tau_\nu \\ t_\nu^{-1} + 1 \end{pmatrix} = M_T^{-1} \cdot t_\nu,
\]

which is in \( \mathcal{P}_\nu^1(F_\nu) \) as claimed.

**Proposition 5.3.** Assume that \( G(F_\nu) = \text{PGL}_2(F_\nu) \) for all \( \nu \mid p \) and there exists \( a_\nu \in \mathbb{C}_p^\times \) such that the \( T(F_\nu) \)-equvariant morphism \( \delta_\nu = (\delta_\nu)_\nu \) of (17) satisfies for \( n \) big enough

\[
\left\{ \begin{array}{l}
\frac{1}{a_\nu^\beta} M_T \delta_\nu(1\mathbb{I}_{\mu, (a, n)}) = \frac{1}{a_\nu^\beta} c(a, n)V, \\
\delta_\nu(1\mathbb{I}_{\mu, (a, n)}) = \frac{1}{a_\nu^\beta} c(a, n)V,
\end{array} \right.
\]

where \( m \) is the \( p \)-valuation of \( \tau_\nu - \bar{v}_\nu \beta = [v_\nu(1-a)] \), if \( s \in \mathcal{O}_{F_\nu} \), and \( \beta = [v_\nu(at_\nu - \bar{v}_\nu)] \), if \( s^{-1} \in \mathfrak{p} \). Moreover, \( V \in V_\nu \) do not depend either \( a \) nor \( n \), and \( c(a, n) \) has \( p \)-adic valuation only depending on a neighbourhood of \( a \). If

\[
e_\nu v_\nu(a_\nu^\beta) < \min\{k_\nu + 1, \sigma \in \Sigma_p\}, \quad \text{where} \quad \alpha^* = a_\nu a_\nu^\beta,
\]

then the distribution \( \mu_{\nu, a_\nu} \) extends to a unique locally analytic \( p \)-adic distribution.

**Remark 5.4.** Notice that

\[
\tau_\nu - \bar{v}_\nu = (s + \tau_\nu)(1-a) = (1 + \tau_\nu s^{-1})(a \tau_\nu - \bar{v}_\nu).
\]

This implies that \( |\beta| \) is bounded. Indeed, it is easy to show that \( \beta \leq \max\{m, m - v_\nu(\tau_\nu - \bar{v}_\nu)\} \).

**Proof.** Let \( C_p = (C_p) \subset G(\mathcal{O}_{F_\nu}) \) be finite index subgroup such that \( V \in V_{F_\nu}^{C_p} \) for all \( \nu \mid p \). Notice that we have a well defined \( G(F_\nu) \)-equivariant morphism

\[
\theta_V: \text{Hom} \left( V, V(\mathbb{Q}^\mathbb{A}_F) \right) \rightarrow \text{colInd}^{G(F_\nu)}_{C_p}(\mathcal{V}(k)_\nu^\mathbb{A}) = \left\{ f: G(F_\nu) \rightarrow V(\mathbb{Q}^\mathbb{A}_F), f(gc) = c^{-1} f(g), c \in C_p \right\},
\]

\[
\theta_V(\phi)(g)(P) := \phi(gV)(gP).
\]

Thus, we have a \( G(F_\nu) \)-equivariant morphism

\[
\theta_V: \mathcal{A}^{\text{colInd}}_{C_p}(V, V(k)_\nu^\mathbb{A}) \rightarrow \mathcal{A}^{\text{colInd}}(\text{colInd}^{G(F_\nu)}_{C_p}(\mathcal{V}(k)_\nu^\mathbb{A})) \cong \mathcal{A}^{\text{sur}}(V(k)_\nu^\mathbb{A})^{C_p}.
\]
By [26, Proposition 4.6] we have that $H^n\left(G(F)_+, \mathcal{A}_\infty(V(k)_{S_p})\right)^{\mathcal{L}, C_p} \simeq H^n\left(G(F)_+, \mathcal{A}_\infty(V(k)_{S_p})\right)^{\mathcal{L}, C_p}$, $\frac{\omega_{S_p}}{\omega_{S_p}}$, hence (up-to-constant) we can assume that

$$\theta_V \phi^p_A \in H^n\left(G(F)_+, \mathcal{A}_\infty(V(k)_{S_p})\right)^{\mathcal{L}, C_p} = H^n\left(G(F)_+, \mathcal{A}_\infty^{(p, \infty)}(\text{coind}_{C_p}^{G(F)}(V(k)_{S_p}))\right)^{\mathcal{L}, C_p}.$$

(Notice that $V(k)_{S_p}$ admits an action of $C_p \subseteq G(O_{F_p})$).

On the other hand, let $\varphi \in \text{Hom}(V_p, V(k)_{S_p})$ such that $\theta_V(\varphi) \in \text{coind}_{C_p}^{G(F)}(V(k)_{S_p})$. We compute for all $m \in \mathbb{N}^{C_p}$ with $m_\sigma \leq k_\sigma$,

$$\int_{U_d(a, n)} \left(\frac{u_p - a}{\omega_p^n}\right)^m u_p^{-k} d\delta_p^* \varphi(u_p) = \varphi(\delta_p \mathbb{I}_{U_d(a, n)} \left(M^{-1}_{\sigma \in \Sigma_p} \left(\frac{y_\sigma - \sigma(a)x_\sigma}{\sigma(\omega_p)^n}\right)^m \chi_{\kappa_{\sigma} - m_\sigma}\right) \cdot \det(M_T)^{-\frac{1}{2}}) = \varphi(\delta_p \mathbb{I}_{U_d(a, n)} \left(M^{-1}_{\sigma \in \Sigma_p} \left(\frac{a}{\omega_p^n}\right)^{-1} \left(y_\sigma - \sigma(a)x_\sigma\right)^m \chi_{\kappa_{\sigma} - m_\sigma}\right) \cdot \det(M_T)^{-\frac{1}{2}} \cdot \omega_p^{-\frac{1}{2}})$$

If $T(F_p)$ splits, we have $\gamma_{a, n} = \left(\frac{a}{\omega_p^n}\right)^m M_T \in \text{GL}_2(F_p)$, hence

$$\int_{U_d(a, n)} \left(\frac{u_p - a}{\omega_p^n}\right)^m u_p^{-k} d\delta_p^* \varphi(u_p) = \frac{c(a, n)}{\left(\frac{a}{\omega_p^n}\right)^{\gamma_{a, n}^{-1}V}} \cdot \varphi(\gamma_{a, n}^{-1} \left(\bigotimes_{\sigma \in \Sigma_p} y_\sigma^{m_\sigma} \chi_{\kappa_{\sigma} - m_\sigma}\right)) = \frac{c(a, n)}{\left(\frac{a}{\omega_p^n}\right)^{\gamma_{a, n}^{-1}V}} \cdot \theta_V(\varphi)(\gamma_{a, n}^{-1} \left(\bigotimes_{\sigma \in \Sigma_p} y_\sigma^{m_\sigma} \chi_{\kappa_{\sigma} - m_\sigma}\right)) \in \frac{A_p}{p^{|\gamma_{a, n}^{-1}V|}} \mathcal{O}_{C_p},$$

where $A_p = \frac{c(a, n)}{\det(M_T)^{-\frac{1}{2}}}$. Since all such $\frac{c(a, n)}{\det(M_T)^{-\frac{1}{2}}} \mathbb{I}_{U_d(a, n)}(u_p)$ generate the space $C_k(T(F_p), \mathcal{O}_{C_p})$, he obtain that $\delta_p^* \varphi$ is $\nu_p(\mathcal{A}_p^*)$-admissible for all $p | p$.

If $T(F_p)$ does not split, we write $s = -M_T^{-1} \cdot (-a) = \frac{\gamma_{s, n}^{-1}V - \gamma_{a, n}^{-1}V}{\gamma_{s, n}^{-1} - \gamma_{a, n}^{-1}}$. Assume that $n > 1$, then we have decompositions

$$\left(\frac{a}{\omega_p^n}\right)^m M_T = (A(n, a))_{s, s},$$

where

$$\gamma_{s, n} := \left(\frac{\omega_p^\beta}{\omega_p^n}, \frac{s \omega_p^\beta}{\omega_p^n + m - \beta}\right) \in \text{GL}_2(F_p), \quad A(n, a) := \frac{1-a}{\omega_p^n - \omega_p^n - \omega_p^n + m - \beta}$$

and

$$\gamma_{s, n} := \left(\frac{s^{-1} \omega_p^\beta}{\omega_p^n - m - \beta}, \frac{\omega_p^\beta}{\omega_p^n + m - \beta}\right) \in \text{GL}_2(F_p), \quad A(n, a) := \frac{1-a}{\omega_p^n - \omega_p^n - \omega_p^n + m - \beta}.$$
In summary, since $\theta_v \phi^*_P \in H^u \left( G(F)_+, \mathcal{A}^{\text{loc}} \left( \text{coInd}_{C_p}^G(V(K/F_p)) \right) \right)$, we deduce that

$$\delta^*_p \phi^*_P \in H^u \left( T(F)_+, \mathcal{A}^{\text{loc}} \left( \text{Dist}_T(T(F_p), \overline{\mathbb{Q}}_p)_{h_p} \right) \right), \quad h_p = (v_p(\alpha^*_p)),$$

where superindex $h_p$ means $v_p(\alpha^*_p)$-admissible for all $p \mid p$. The result follows by Proposition 5.2.

\subsection*{5.3 The morphism $\delta_p$}

Assume that $G(F_p) = \text{PGL}_2(F_p)$. As seen in the previous §, we want to construct a morphism

$$\delta_p = (\delta_v)_{v \mid p} : C_v^0(T(F_p), \overline{\mathbb{Q}}) \longrightarrow V_p, \quad \delta_p : C_v^0(T(F_p), \overline{\mathbb{Q}}) \longrightarrow V_p.$$

satisfying relation (19).

Let us fix a place $p \mid p$, and let $\pi_p$ be the local representation. From now on we will do the following assumptions:

**Hypothesis 5.5.** Let $P$ be the subgroup of upper triangular matrices, then we assume that $P \cap (K^\infty_p) = F^\infty_p$. Moreover we will assume that $\pi_p$ is either principal series $\pi(\tilde{\lambda}_p, \tilde{\lambda}_p^{-1})$ or Steinberg $\sigma(\tilde{\lambda}_p, \tilde{\lambda}_p^{-1})$.

By the previous assumption, $V_p$ is a quotient of

$$\text{Ind}_G^P(\tilde{\lambda}_p)^0 = \left\{ f \in \text{GL}_2(F_p) \rightarrow \overline{\mathbb{Q}}, \text{locally constant } f \left( \begin{pmatrix} x_1 & y \\ x_2 & 0 \end{pmatrix} \right) = \tilde{\lambda}_p \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \cdot f(g) \right\},$$

for a locally constant character $\tilde{\lambda}_p$. We construct

$$\delta_p : C_v^0(T(F_p), \overline{\mathbb{Q}}) \longrightarrow \text{Ind}_G^P(\tilde{\lambda}_p)^0$$

where

$$\delta_p(f) (g) := \left\{ \begin{array}{ll} \tilde{\lambda}_p \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \cdot f(t^{-1}) , & g = \begin{pmatrix} x_1 & y \\ 0 & 0 \end{pmatrix} \in P \, (K^\infty_p) \\ 0, & g \notin P \, (K^\infty_p) \end{array} \right.$$ (20)

It is clearly $T(F_p)$-equivariant. Moreover, it induces the wanted $T(F_p)$-equivariant morphism

$$\delta_p : C_v^0(T(F_p), \overline{\mathbb{Q}}) \longrightarrow V_p.$$

Let us consider $X_p$ to be

$$X_p := \left\{ \begin{array}{l} \mathcal{D}_p = K_p \setminus F_p, \text{ if } T \text{ does not split at } p, \\ \mathbb{P}(1, F_p), \text{ if } \text{splits at } p. \end{array} \right.$$ 

In both cases $X_p$ comes equipped with a natural action of $G(F_p)$ given by fractional linear transformations. We write $\tau_p$ and $\tilde{\tau}_p$ for the two fixed points by $i(T(F_p))$ in $X_p$. Since $i(T(F_p)) \cap P = F^\infty_p$, in the split case $\tau_p$, $\tilde{\tau}_p \neq \infty$. In fact, $\tau_p$ and $\tilde{\tau}_p$ correspond to the two simultaneous eigenvectors of all matrices $i(t)$ because of the following identity.

$$\begin{bmatrix} \tau_p \\ 1 \end{bmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} \tau_p \\ 1 \end{bmatrix} = (ct \tau_p + d) \begin{pmatrix} \tau_p \\ c \end{pmatrix} = \lambda_p \begin{pmatrix} \tau_p \\ c \end{pmatrix}, \quad \begin{bmatrix} \tilde{\tau}_p \\ 1 \end{bmatrix} = \tilde{\lambda}_p \begin{pmatrix} \tilde{\tau}_p \\ 1 \end{bmatrix}.$$ (21)

where again $t \mapsto \tilde{t}$ is the non-trivial automorphism in $\text{Gal}(K_p/F_p)$. The following result is a generalization of [16, Lemma 5.2] for more general induced representations:

**Lemma 5.6.** The morphism $\delta_p$ is given by

$$\delta_p : C_v(T(F_p), \overline{\mathbb{Q}}) \longrightarrow \text{Ind}_G^P(\tilde{\lambda}_p)^0; \quad \delta_p(f) \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \tilde{\lambda}_p \left( \begin{pmatrix} ad - bc \\ (ct \tau_p + d)(ct \tilde{\tau}_p + d) \end{pmatrix} \right) \cdot f \left( \begin{pmatrix} c \tau_p + d \\ c \tilde{\tau}_p + d \end{pmatrix} \right).$$

In particular, if $T(F_p)$ does not split then $\delta_p$ is bijective.
Proof. From the relations
\[
\tau(t) \left( \frac{\tau_p}{1} \right) = \lambda_t \left( \frac{\tau_p}{1} \right), \quad \tau(t) \left( \frac{\bar{\tau}_p}{1} \right) = \bar{\lambda}_t \left( \frac{\tau_p}{1} \right);
\]
we deduce
\[
\tau(t) = \frac{1}{\tau_p - \tau} \left( \frac{\lambda_t \tau_p - \bar{\lambda}_t \bar{\tau}_p}{\lambda_t - \bar{\lambda}_t} \frac{\tau_p \bar{\tau}_p(\lambda_t - \bar{\lambda}_t)}{\lambda_t \bar{\tau}_p - \lambda_t \tau_p} \right).
\]
Hence, if we have
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x_1 & y \\ x_2 & x \end{pmatrix}, \quad \tau(t) = \frac{\lambda_t}{\tau_p - \tau} \begin{pmatrix} x_1 & y \\ x_2 & x \end{pmatrix} \left( \frac{t \tau_p - \tau}{t - 1} \frac{\tau_p \bar{\tau}_p(1 - t)}{t \tau_p - \tau} \right)
\]
where \( t = \lambda_t / \bar{\lambda}_t \in T(F_p) \), we obtain the identities
\[
- \frac{d}{c} = \frac{t^{-1} \tau_p - \tau}{t^{-1} - 1}, \quad ad - bc = x_1 x_2 \lambda_t \bar{\lambda}_t, \quad t^{-1} = \frac{\tau_p + d}{\tau_p + d}, \quad c = \frac{x_2 (t - 1) \lambda_t}{\tau_p - \tau_p}.
\]
Thus, we obtain
\[
\begin{align*}
\delta_\varphi(f) \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \lambda_p \frac{(ad - bc)}{c^2} \frac{(t - 1)^2}{(\tau_p - \tau)^2} \cdot \frac{\tau_p + d}{c \tau_p + d} \\
&= \lambda_p \left( \frac{(ad - bc)}{c^2} \frac{(t - 1)^2}{(\tau_p - \tau)^2} \cdot \frac{\tau_p + d}{c \tau_p + d} \right),
\end{align*}
\]
and the result follows. \( \square \)

Let us check that it satisfies the relations (19): We aim to compute for \( n \) big enough:
\[
\gamma_{a,n} \delta_\varphi (\mathbb{I}_{L(a,n)}) \cdot (t(t) \mathbb{M}_{T})^{-1}
\]
\[
\gamma_{a,n} = \begin{cases} 1 & a = \frac{1}{a_p^n} \mathbb{M}_T, \quad \text{if } T(F_p) \text{ splits}, \\
\omega_p \mathbb{M}_T, & a \neq \frac{1}{a_p^n} \mathbb{M}_T, \quad \text{if } T(F_p) \text{ is non-split and } s = -M_T(-a) \in \mathcal{O}_{F_p}, \\
\omega_p \mathbb{M}_T, & a \neq \frac{1}{a_p^n} \mathbb{M}_T, \quad \text{if } T(F_p) \text{ is non-split and } s = -M_T(-a) \notin \mathcal{O}_{F_p}.
\end{cases}
\]

5.3.1 Split case

Assuming that \( T(F_p) \) splits, we can choose \( M_T = \left( \frac{\lambda_t}{1 - \tau_p} \right) \). Thus, we have
\[
\gamma_{a,n} \delta_\varphi (\mathbb{I}_{L(a,n)}) \cdot (t(t) \mathbb{M}_{T})^{-1} = \delta_\varphi(\mathbb{I}_{L(a,n)}) \left( \frac{1}{(\tau_p - \tau)^2} \frac{\lambda_t}{\lambda_t - \tau_p} \frac{\tau_p \tau_p(\lambda_t - \bar{\lambda}_t)}{\lambda_t \bar{\tau}_p - \lambda_t \tau_p} \right) \left( \frac{1}{\tau_p} \frac{\omega_p}{\tau_p} \right) \left( \frac{1}{\tau_p} \frac{1}{\omega_p} \right)
\]
\[
= \delta_\varphi(\mathbb{I}_{L(a,n)}) \left( \frac{\lambda_t}{(\tau_p - \tau)^2} \frac{(t \tau_p - \tau) \tau_p \bar{\tau}_p(1 - t)}{(t - 1) \tau_p \bar{\tau}_p} \right) \left( \frac{1}{\tau_p} \frac{\omega_p}{\tau_p} \right) \left( \frac{1}{\tau_p} \frac{1}{\omega_p} \right)
\]
\[
= \lambda_p \left( \frac{\omega_p}{t \tau_p + \omega_p + 1} \right) \cdot \mathbb{I}_{L(a,n)} \cdot (a + \omega_p(t^{-1} - 1) = \frac{\lambda_p}{\lambda_p(a)} \cdot \mathbb{I}_{L(a,n)} \cdot (a + \omega_p(t^{-1} - 1),
\]
if \( n \) is bigger then the conductor of \( \chi_p \). We obtain that \( \delta_\varphi \) satisfies (19) with \( \alpha_p = \frac{\lambda_p}{\lambda_p(a)} \) and \( V = M_T V_0 \) where
\[
V_0(t(t)) = \frac{\lambda_p(t^{-1}) \cdot \mathbb{I}_{O_{F_p}(t^{-1})}}{\mathcal{O}_{F_p}(t^{-1})}.
\]
5.3.2 Non-split case with \( s = -M_T(-a) \in O_{F_p} \)

Write the bijective map

\[
\varphi : T(F_p) \rightarrow \mathbb{P}^1(F_p); \quad t \mapsto t(t) \cdot \infty = \frac{tt_T - t_T}{t - 1}.
\]

We compute for \( n - \beta \) bigger than the conductor of \( \chi \)

\[
\begin{align*}
\left( \frac{\alpha^\epsilon_P}{\alpha^s_P} \cdot \frac{s^\epsilon_P}{s^s_P} \right) \delta_P \left( \mathbb{I}_{U_d(a,n)} \right) (t(t)) &= \delta_P \left( \mathbb{I}_{U_d(a,n)} \right) \left( \frac{1}{tt_T - t_T} \right) \\
&= \delta_P \left( \mathbb{I}_{U_d(a,n)} \right) \left( \frac{t(t) - t_T}{t - 1} \right) \left( \frac{\alpha^\epsilon_P}{\alpha^s_P} \cdot \frac{s^\epsilon_P}{s^s_P} \right) \\
&= \delta_P \left( \mathbb{I}_{U_d(a,n)} \right) \left( \frac{t(t) - t_T}{t - 1} \right) \left( \frac{\alpha^\epsilon_P}{\alpha^s_P} \cdot \frac{s^\epsilon_P}{s^s_P} \right) \\
&= \hat{\lambda}_p \left( \frac{\alpha^P}{\alpha^s_P} \right) \left( \frac{(a - 1)^2}{\alpha^P} \right) \cdot \mathbb{I}_{U_d(a,n)} \left( \frac{t(t) - t_T}{t - 1} \right) \left( \frac{\alpha^\epsilon_P}{\alpha^s_P} \cdot \frac{s^\epsilon_P}{s^s_P} \right)
\end{align*}
\]

where \( \epsilon = 0 \) if \( T(F_p) \) is inert and \( \epsilon = 1 \) if it ramifies. Since \( v_P \left( \frac{(a - 1)^2}{\alpha^s_P} \right) = \epsilon \), we obtain that \( \delta_P \) satisfies (19) with \( \alpha_P = \hat{\lambda}_p (\alpha_P)^{-1} \) and

\[
V(t(t)) = \hat{\lambda}_p \left( \frac{t(t) - t_T}{t - 1} \right) \cdot \mathbb{I}_{U_d(a,n)} \left( \frac{t(t) - t_T}{t - 1} \right).
\]

5.3.3 Non-split case with \( s = -M_T(-a) \notin O_{F_p} \)

We compute for \( \max\{ n - \beta, n - \beta + v_P(t) \} \) bigger than the conductor of \( \chi_p \)

\[
\begin{align*}
\left( \frac{s^{-1}a^\epsilon_P}{\alpha^{-s}_P} \cdot \frac{a^s_P}{a^s_P} \right) \delta_P \left( \mathbb{I}_{U_d(a,n)} \right) (t(t)) &= \delta_P \left( \mathbb{I}_{U_d(a,n)} \right) \left( \frac{a - 1}{a - 1} \right) \left( \frac{t(t) - t_T}{t - 1} \right) \left( \frac{s^{-1}a^\epsilon_P}{\alpha^{-s}_P} \cdot \frac{a^s_P}{a^s_P} \right) \\
&= \hat{\lambda}_p \left( \frac{a - 1}{a - 1} \right) \left( \frac{t(t) - t_T}{t - 1} \right) \left( \frac{s^{-1}a^\epsilon_P}{\alpha^{-s}_P} \cdot \frac{a^s_P}{a^s_P} \right) \\
&= \hat{\lambda}_p \left( \frac{a - 1}{a - 1} \right) \left( \frac{t(t) - t_T}{t - 1} \right) \left( \frac{s^{-1}a^\epsilon_P}{\alpha^{-s}_P} \cdot \frac{a^s_P}{a^s_P} \right) \\
&= \hat{\lambda}_p (\alpha_P)^{P + m} \cdot \hat{\lambda}_p \left( \frac{(a - 1)^2}{a^P} \right) \cdot \mathbb{I}_{U_d(a,n)} \left( \frac{t(t) - t_T}{t - 1} \right) \left( \frac{s^{-1}a^\epsilon_P}{\alpha^{-s}_P} \cdot \frac{a^s_P}{a^s_P} \right)
\end{align*}
\]

Again since \( v_P \left( \frac{t(t) - t_T}{t - 1} \right) = \epsilon \), we obtain that \( \delta_P \) satisfies (19) with \( \alpha_P \) and \( V \) as above.

**Remark 5.7.** By Proposition 5.3, using the above \( \delta_P \) our \( p \)-adic distribution extends to a locally analytic measure if \( e_P v_P(\alpha_P^*) < \min_{i \in \Sigma_o} (k_o + 1) \) where \( \alpha_P^* = \hat{\lambda}_p (\alpha_P)^{-1} \alpha_P^s \). If this is the case we say that \( \psi_P \) has non-critical slope.
5.4 Local integrals

Given the morphism $\delta_p$ defined in §5.3, we aim to calculate in this section the following integrals

$$\int_{T(F_p)} \xi_p(t)(t \delta_p(1_H), \delta_p(1_H)) d^\times t,$$

(22)

where $\xi_p$ is a locally constant character, $H \subset T(O_{F_p})$ is an open and compact subgroup small enough so that $\xi_p$ is $H$-invariant, $d^\times$ is a Haar measure of $T(F_p)$, and $\langle \cdot, \cdot \rangle$ is the natural $G(F_p)$-equivariant pairing on $V_p$.

Since $\pi_p$ is unitary, by [6, Proposition 4.6.11] the character $\hat{\pi}_p | \cdot |^{3/2}$ is either unitary or real. In our setting, the second case corresponds to the Steinberg representations, namely $\hat{\pi}_p = \pm 1$.

If $\hat{\pi}_p | \cdot |^{3/2}$ is unitary, by [6, Proposition 4.5.5] and [5, Corollary 8.2] we have a $G(F_p)$-invariant pairing

$$\langle \cdot, \cdot \rangle : \text{Ind}^G_p(\hat{\pi}_p)^0 \times \text{Ind}^G_p(\hat{\pi}_p)^0 \to \mathbb{C}; \quad \langle f_1, f_2 \rangle_{\hat{\pi}_p} = \int_{T(F_p)} f_1(t(\tau)) \cdot \overline{f_2(t(\tau))} d^\times \tau.$$

We deduce

$$\langle t \delta_p(1_H), \delta_p(1_H) \rangle = \int_{T(F_p)} \delta_p(1_H)(t(\tau)) \cdot \overline{\delta_p(1_H)(t(\tau))} d^\times \tau = \int_{T(F_p)} 1_{H}(\tau^{-1}) \cdot 1_{H}(\tau^{-1}) d^\times \tau = \text{vol}(H) \cdot 1_{H}(t).$$

Thus we obtain

$$\int_{T(F_p)} \xi_p(t)(t \delta_p(1_H), \delta_p(1_H)) d^\times t = \text{vol}(H) \int_{H} \xi_p(t) d^\times t = \text{vol}(H)^2.$$

If $\hat{\pi}_p = \pm 1$ the computation is much more complicated but it can be found in [5, §3.5]. Up to a constant depending on $T$, we have

$$\int_{T(F_p)} \xi_p(t)(t \delta_p(1_H), \delta_p(1_H)) d^\times t = \begin{cases} \text{vol}(H)^2 \cdot L(\frac{1}{2}, \pi_p, \xi_p) \cdot L(-\frac{1}{2}, \pi_p, \xi_p)^{-1}, & \text{cond}(\xi_p) = 0, \\
\text{vol}(H)^2 \cdot q_n^\pi, & \text{cond}(\xi_p) = n_{\xi_p},
\end{cases}$$

where, if we write $\alpha_{\xi_p} = \hat{\alpha}_{\xi_p}(\omega_p) = \pm 1$

$$L(s, \pi_p, \xi_p) = \begin{cases} (1 - \alpha_{\xi_p} \xi_p(\omega_p)q^{-s-\frac{1}{2}})^{-1}(1 - \alpha_{\xi_p} \xi_p(\omega_p)^{-1}q^{s-\frac{1}{2}})^{-1}, & T(F_p) \text{ splits}, \\
(1 - q^{1 - 2s})^{-1}, & T(F_p) \text{ inert}, \\
(1 - \alpha_{\xi_p} \xi_p(\omega_p)q^{-s-\frac{1}{2}})^{-1}, & T(F_p) \text{ ramifies}.
\end{cases}$$

(23)

Recall that in the ramified case $\hat{\alpha}_{\xi_p}^{\pm}$ denotes the uniformizer of $K_p$.

Let $J \in G(F)$ such that $J \cdot t(\tau) = t(\tau) \cdot J$ for all $t \in T(F)$. We write $J = (t_{1j} \cdot t_{j})$ for some $t_{1j} \in T(F_p)$ and $J_{j} \in F_p^\times$. We compute

$$J \delta_p(f)(t(\tau)) = \delta_p(f)(t(\tau) \cdot J) = \delta_p(f)(\tau^{-1} t_{1j}^{-1} \cdot t_{j}^{-1}) = \hat{\chi}_p \left( \begin{array}{c} t_{1j} \\ -j 
\end{array} \right) \cdot \delta_p(f)(\tau^{-1} t_{1j}^{-1} \cdot t_{j}^{-1}) = \hat{\chi}_p \left( \begin{array}{c} t_{1j} \\ j 
\end{array} \right) \cdot \delta_p(f^*)(t(\tau)),$$

where $f^*(t) = f(t^{-1} t_{j})$. Since $J^2 \in F^\times$, we deduce that $\hat{\chi}_p \left( \begin{array}{c} t_{1j} \\ j 
\end{array} \right) = \pm 1$. We aim to compute as well the integral

$$\int_{T(F_p)} \xi_p(t)(t \delta_p(1_H), J \delta_p(1_H)) d^\times t.$$

Using the above computation

$$\int_{T(F_p)} \xi_p(t)(t \delta_p(1_H), J \delta_p(1_H)) d^\times t = \pm \int_{T(F_p)} \xi_p(t)(t \delta_p(1_H), \delta_p(1_{jH})) d^\times t = \pm \int_{T(F_p)} \xi_p(t)(t \delta_p(1_{jH}), J_{j} \delta_p(1_H)) d^\times t$$

$$= \pm \xi_p(\tau) \cdot \int_{T(F_p)} \xi_p(\tau)(t \delta_p(1_H), \delta_p(1_{jH})) d^\times t.$$

Thus, we obtain

$$\int_{T(F_p)} \xi_p(t)(t \delta_p(1_H), J \delta_p(1_H)) d^\times t = \begin{cases} \pm \xi_p(\tau) \cdot \text{vol}(H)^2 \cdot L(\frac{1}{2}, \pi_p, \xi_p) \cdot L(-\frac{1}{2}, \pi_p, \xi_p)^{-1}, & \text{cond}(\xi_p) = 0, \\
\pm \xi_p(\tau) \cdot \text{vol}(H)^2 \cdot q_n^\pi, & \text{cond}(\xi_p) = n_{\xi_p}.
\end{cases}$$
5.5 Interpolation properties

As we have previously emphasized, we have to think of \( \mu_{\phi^p} \) as a generalization of Bertolini-Darmon anticyclotomic \( p \)-adic \( L \)-function. Hence it needs to have a link with the classical \( L \)-function, namely, an interpolation property.

**Definition 5.8.** Let \( \xi \in C_{\mathbb{A}}(G_T, \mathbb{C}_p) \) be a locally polynomial character. Thus, in a neighbourhood \( U \) of 1 in \( T(F_p) \)

\[
\rho^* \xi |_{U} (t_p) = \prod_{\mathfrak{p} \mid \mathfrak{p} \in U} \sigma(t_{\mathfrak{p}})^m, \quad m = (m_\mathfrak{p}); \quad \frac{k}{2} \leq m \leq \frac{k}{2}.
\]

We define the archimedean avatar of \( \xi \):

\[
\tilde{\xi} : T(\mathbb{A}_F)/T(F) \rightarrow \mathbb{C}^\times; \quad \tilde{\xi}(t) = \rho^* \xi(t) \cdot \prod_{\mathfrak{p} \mid \mathfrak{p} \in U} \sigma(t_{\mathfrak{p}})^{-m}, \quad \prod_{v \notin U} \prod_{t \in \mathbb{T}} \sigma(t_v)^{m_v},
\]

once identified the set of embeddings \( \sigma : F \hookrightarrow \mathbb{C} \) with \( \bigcup_{\mathfrak{p}} \Sigma_{\mathfrak{p}} \).

We will write \( \rho^* \xi = \prod_{v} \xi_v \) and \( \xi^p = \prod_{v \mid p} \xi_v \). The following results provides the interpolation property of the distribution \( \mu_{\phi^p} \):

**Theorem 5.9.** Given a locally polynomial character \( \xi \in C_{\mathbb{A}}(G_T, \mathbb{C}_p) \), we have that

\[
\int_{G_T} \xi d\mu_{\phi^p} = \begin{cases} K(x^p, \xi^p) \cdot \left( \frac{k}{k-m} \cdot C(k) \cdot e_p(\pi_p, \xi_p) \cdot L(1/2, \pi, \tilde{\xi}) \right), & \rho^* \xi |_{T(F_p)} = \lambda, \\
0, & \rho^* \xi |_{T(F_p)} \neq \lambda,
\end{cases}
\]

where

\[
\left( \frac{k}{k-m} \right) = \prod_{v} \left( \frac{k_v}{k_v - m_v} \right); \quad C(k) = \prod_{v \notin U} \left( \frac{\sum_{v \mid p}(k_v + 1)!}{\prod_{v \mid p} k_v!} \right).
\]

\( K(x^p, \xi^p) \) is an explicit constant depending on \( \xi^p \) and the image of \( \phi^p \) in \( \pi^{T(\mathfrak{p}, \infty)} \).

\[
eq_p(\pi_p, \xi_p) = \prod_{\mathfrak{p} \mid \mathfrak{p} \in U} \epsilon_p(\pi_p, \xi_p); \quad \epsilon_p(\pi_p, \xi_p)^2 = \begin{cases} L(1/2, \pi_p, \xi_p)^{-1}, & \pi_p \neq \text{St}_\mathbb{C}(F_p)(\pm), \\
L(-1/2, \pi_p, \xi_p)^{-1}, & \pi_p = \text{St}_\mathbb{C}(F_p)(\pm), \text{cond} \xi_p = 0, \\
q^p L(1/2, \pi_p, \xi_p)^{-1}, & \pi_p = \text{St}_\mathbb{C}(F_p)(\pm), \text{cond} \xi_p = n_{\xi_p}
\end{cases}
\]

and \( \text{St}_\mathbb{C}(F_p)(\pm) \) denotes the Steinberg representation twisted by the character \( g \mapsto (\pm)^{\text{det}(g)} \).

**Proof.** Let us consider the \((F,T)\)-equivariant morphism

\[
\varphi : (C^0(T(\mathbb{A}_F), \mathbb{C}) \otimes \mathcal{P}(k)) \otimes \mathcal{A}_\infty(V(k))(\lambda) \rightarrow C^0(T(\mathbb{A}_F), \mathbb{C});
\]

\[
\varphi(f \otimes P) \otimes \varphi(z, t) := (z, t) \cdot \lambda(z)^{-1} \cdot \varphi(t(P),
\]

for all \( z \in T(F_\infty) \) and \( t \in T(\mathbb{A}_F) \), and the natural pairing \( \langle \cdot, \cdot \rangle_T : C^0(T(\mathbb{A}_F), \mathbb{C}) \times C^0(T(\mathbb{A}_F), \mathbb{C}) \rightarrow \mathbb{C} \) given by the Haar measure

\[
\langle f_1, f_2 \rangle_T := \frac{1}{n} \sum_{x \in \mathbb{C}} \int_{T(A_{F,F})} f_1(x, t) \cdot f_2(x, t) d^4x, \quad n := [T(F) : T(F_\infty)].
\]

For any \( f \in C^0(T(\mathbb{A}_F), \mathbb{C}) \) and \( f_2 \in C^0(T(\mathbb{A}_F), \mathbb{C}) \), write \( f \cdot f_2 = f \otimes f_2 \) and let \( H \subseteq T(F_p) \) be a small enough open compact subgroup so that \( f_\mathfrak{p} \) is \( H \)-invariant, namely, \( f_\mathfrak{p} = \sum_{t \in \mathbb{T}(F_\mathfrak{p})/H} f_\mathfrak{p}(t_p) 1_t, H \). If we consider the \((G,F)\)-equivariant morphism \( \theta : \mathcal{A}(G^p, V(k))(\lambda) \rightarrow \mathcal{A}_\infty(V(k))(\lambda) \) defined by

\[
\theta(\phi)(g_F, g^p) := \phi(g^p)(g_F \delta_p(1_H)); \quad g_F \in G(F_p), g^p \in G(A_{F,F}^p),
\]

we compute using the concrete description of \( \langle \cdot, \cdot \rangle \) given in (11):
for all \( \phi \in \mathcal{A}^\infty_{\text{loc}}(V_p, \mathbb{C}), \mathbb{C}(\lambda) \). Hence, we obtain by definition

\[
\int_{\mathcal{G}_S} \xi d\mu_{\mathcal{G}_S} = (\phi^* \xi \cap \eta) \cap \delta_p^* \phi_{\mathcal{A}}^* = \frac{1}{\text{vol}(H)} (\tilde{\xi} \cup \theta \phi_{\mathcal{A}}^* \cap \eta),
\]

where the cap and cup products in the third identity correspond to the pairings \( \langle \cdot, \cdot \rangle_T \) and \( \varphi \). In [22, Theorem 4.25] and expression for \( (\tilde{\xi} \cup \theta \phi_{\mathcal{A}}^*) \cap \eta \) is obtained in terms of the classical L-function:

\[
(\tilde{\xi} \cup \theta \phi_{\mathcal{A}}^*) \cap \eta = \left\{ K \cdot \left( \frac{1}{2} \right)^{-1} \cdot C(k) \cdot L(1/2, \pi, \tilde{\xi}) \frac{\lambda}{\prod_{v \nmid \infty} \alpha(v)} \cdot \rho^* \xi \mid_{T(\varphi)} = \lambda, \rho^* \xi \mid_{T(\varphi)} \neq \lambda \right\},
\]

where \( K \in \mathbb{C} \) is a non-zero explicit constant only depending on \( T \) and \( \pi, x_v \in \pi_v \) is the image of \( \theta \phi_{\mathcal{A}}^* \) in the corresponding local representation, and

\[
\alpha(x) = \frac{L(1, \pi \varphi)}{\zeta(1)} \cdot L(1/2, \pi, \tilde{\xi}) \int_{T(\varphi)} \xi(t) \langle \pi(t) x_v, \pi_v(f_v) x_v \rangle d^x t,
\]

are usual local factors appearing in classical Waldspurger formulas (see [28] and [17] for more details).

The product \( K(x^\varphi, \xi^\varphi) \) is a constant only depending on \( T \) and \( \xi^\varphi \). We can apply the formulas obtained in §5.4 to compute that, up to a constant factor depending on \( T(F_p) \),

\[
\frac{\alpha(\tilde{\pi}(1_H))}{\text{vol}(H)^2} = \left\{ \begin{array}{ll}
L(1/2, \pi, \tilde{\xi})^{-1}, & \pi \neq \text{St}(\varphi)(\pm), \\
L(-1/2, \pi, \tilde{\xi})^{-1}, & \pi = \text{St}(\varphi)(\pm), \text{cond} \chi_p = 0,
\end{array} \right.
\]

and the result follows. \( \square \)

**Remark 5.10.** If \( \pi_p = \text{St}(\varphi)(\pm) \) and \( \xi_p \) is unramified, we observe in (23) that if \( \xi_p(\alpha_p) = \pm 1 \) we obtain that \( L(-1/2, \pi, \tilde{\xi})^{-1} \neq 0 \). This phenomena is known as exceptional zero and the aim of the following chapters is to relate this exceptional zeroes with points in the extended Mordell-Weil group.

6 Overconvergent modular symbols

In this section we extend non-critical modular symbols to overconvergent modular symbols. We will describe our admissible distributions in terms of the corresponding overconvergent modular symbols.

6.1 Distributions

Let us consider

\[ L_p := \{ (x, y) \in O_{F_p} \times O_{F_p}; (x, y) \notin p \times p \}, \quad L_p := \prod_p L_p. \]

For any complete \( \mathbb{Z}_p \)-algebra \( R \), and any continuous character \( \chi_p : O_{F_p}^\times \rightarrow R^\times \), let us consider the space of homogeneous functions

\[ C_{\chi_p}(L_p, R) = \left\{ f : L_p \rightarrow R, \text{continuous s.t. } f(ax, ay) = \chi_p(a) \cdot f(x, y), \text{ for } a \in O_{F_p}^\times \right\}. \]

We write \( \mathcal{D}_{\chi_p}(R) \) for the \( R \)-dual space of \( C_{\chi_p}(L_p, R) \), namely,

\[ \mathcal{D}_{\chi_p}(R) := \text{Hom}(C_{\chi_p}(L_p, R), R). \]

For any continuous extension \( \hat{\chi}_p : F_p^\times \rightarrow R^\times \) of \( \chi_p \), we can consider the induced representation

\[ \text{Ind}_{\chi_p}(\hat{\chi}_p) = \left\{ f : G(F_p) \rightarrow R, \text{continuous, } f\left( \begin{pmatrix} x_1 & y \\ x_2 & y \\
\end{pmatrix} \right) = \hat{\chi}_p \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \cdot f(g) \right\}, \]

Notice that a choice of the extension \( \hat{\chi}_p \) provides a natural \( G(F_p) \)-action on \( C_{\chi_p}^{-1}(L_p, R) \) since we have an isomorphism

\[ q_p : C_{\chi_p}(L_p, R) \rightarrow \text{Ind}_{\chi_p}(\hat{\chi}_p), \]

\[ q_p(f)\left( \begin{pmatrix} x_1 & y \\ x_2 & y \end{pmatrix} k \right) = \hat{\chi}_p \left( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right) \cdot f(c, d) \cdot \chi_p(\det(k)), \quad k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(O_{F_p}). \]

This provides a well defined \( G(F_p) \)-action on \( \mathcal{D}_{\chi_p}^{-1}(R) \) depending on the extension \( \hat{\chi}_p \).
**Definition 6.1.** To provide an extension \( \hat{\chi}_p : F_p^\times \to \mathbb{R}^\times \) amounts to choosing a tuple \( \alpha^* = (\alpha_p)^\oplus \), where \( \alpha_p \in \mathbb{R}^\times \). Indeed, the extension depends on a choice \( \alpha^*_p = \hat{\chi}_p(\alpha_p)^{-1} \in \mathbb{R}^\times \) for the fixed uniformizers \( \alpha_p \). We will denote by \( D_{\chi_p^*}(R)_M \) the space \( D_{\chi_p^*}(R) \) with the action of \( G(F) \) provided by the corresponding extension.

**Remark 6.2.** Given the extension \( \hat{\chi}_p : F_p^\times \to \mathbb{R}^\times \), we can directly describe the action of \( g \in G(F_p) \) on \( f \in C_{\chi_p^*}(\mathcal{L}_p, R) \) compatible with \( \varphi_p^* \). Indeed, for \( (c, d) \in \mathcal{L}_p \),

\[
(gf)(c, d) = \hat{\chi}_p(x)^{-2} \cdot \hat{\chi}_p(\det g) \cdot f(x^{-1}(c, d)g),
\]

where \( x \in F_p^\times \) is such that \( x^{-1}(c, d)g \in \mathcal{L}_p \).

Assume that \( R \subseteq \mathbb{C} \) and \( \chi_p \) is locally analytic. We can define the subspace \( C_{\chi_p^*}(\mathcal{L}_p, R) \) of locally analytic functions. If we also assume that \( \chi_p(a) = a^{\frac{\bar{k}}{2}} \chi_0^0(a) \), for some \( k \in 2\mathbb{N}^d \) and some locally constant character \( \chi_0^0 \), then we can consider the subspace \( C_{\chi_p^*}^0(\mathcal{L}_p, R) \) of locally polynomial functions of homogeneous degree \( k \). If \( \chi_p^* = \chi_0^0 \) then the subspace is \( C_{\chi_p^*}^0(\mathcal{L}_p, R) \) the set of locally constant functions. For any extension \( \hat{\chi}_p \) as above, we define

\[
\text{Ind}_{\chi}^G(\hat{\chi}_p)^* = \varphi_p(\mathcal{C}_{\chi_p^*}(\mathcal{L}_p, R)), \quad \text{where } * = 0, \text{an}, \frac{k}{2}.
\]

We also write \( D_{\chi_p^*}(R) \) for the dual space of \( C_{\chi_p^*}(\mathcal{L}_p, R) \), where \( * = k \), 0, and write \( D_{\chi_p^*}^\text{an}(R) \) for the continuous dual of \( C_{\chi_p^*}^0(\mathcal{L}_p, R) \). Notice that, if we write \( \hat{\chi}_p(a) = a^{\frac{\bar{k}}{2}} \chi_0^0(a) \) for some locally constant character \( \hat{\chi}_0^0 \), we have \( G(F_p) \)-equivariant isomorphisms

\[
\kappa : \text{Ind}_{\chi_p^*}^G(\hat{\chi}_p)^0 \otimes_R \mathcal{P}(\ell)_R \xrightarrow{\cong} \text{Ind}_{\chi_p^*}^G\hat{\chi}_p, \quad \kappa (f \otimes P) \begin{pmatrix} a & b \\ c & d \end{pmatrix} = f \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P(c, d) \cdot (ad - bc)^{-\frac{k}{2}},
\]

\[
\kappa^* : D_{\chi_p^*}^\text{an}(R)_M \xrightarrow{\cong} \text{Hom}\left( \text{Ind}_{\chi_p^*}^G(\hat{\chi}_p)^0, \mathcal{V}(\ell)_R \right),
\]

where \( \omega^* = (\alpha_p)^* \) with \( \alpha^*_p = \chi_0^0(\alpha_p)^{-1} \chi_0^0 \).

### 6.2 Admissibility

As above, we fix a locally polynomial character \( \chi_p = (\chi_p) : \mathcal{O}_{F_p}^\times \to \mathbb{C} \) where \( \chi_p(a) = a^{\frac{\bar{k}}{2}} \chi_0^0(a) \), for some \( k \in 2\mathbb{N}^d \) and some locally constant character \( \chi_0^0 = \chi_0^0(a) \). For any \( b \in \mathcal{O}_{F_p}, m \in \mathbb{N}^d \), and \( n \in \mathbb{N} \), let us consider the homogeneous locally analytic functions \( f_{b,n}^m, g_{b,n}^m \in C_{\chi_p^*}^n(\mathcal{L}_p, \mathcal{O}_{F_p}) \):

\[
f_{b,n}^m(x, y) = \left( \frac{y - bx}{\alpha_p^n} \right)^m \chi_{b}^{\frac{k}{2}} \cdot \chi_0^0(x)^{-2} \cdot \mathbb{I}_{U_p(b, n)}(x, y), \quad g_{b,n}^m(x, y) = \left( \frac{x - by}{\alpha_p^n} \right)^m y^{\omega^m} \cdot \chi_0^0(y)^{-2} \cdot \mathbb{I}_{V_p(b, n)}(x, y),
\]

where

\[
U_p(b, n) = \{(x, y) \in \mathcal{L}_p; x \in \mathcal{O}_{F_p}, yx^{-1} \equiv b \mod \alpha_p^n \}, \quad V_p(b, n) = \{(x, y) \in \mathcal{L}_p; y \in \mathcal{O}_{F_p}, xy^{-1} \equiv b \mod \alpha_p^n \}.
\]

It is clear that if \( m \leq k \) the functions \( f_{b,n}^m \) and \( g_{b,n}^m \) form a basis of \( C_{\chi_p^*}(\mathcal{L}_p, \mathbb{C}) \). Moreover, any locally analytic function in \( C_{\chi_p^*}^n(\mathcal{L}_p, \mathbb{C}) \) with support in \( U_p(b, n) \) (resp. \( V_p(b, n) \)) can be written as a series \( \sum_m a_m f_{b,n}^m \) (resp. \( \sum_m a_m g_{b,n}^m \)), where the coefficients \( a_m \) tend to 0.

**Definition 6.3.** A distribution \( \mu \in D_{\chi_p^*}(\mathbb{C}) \) is \( h_p \)-admissible at \( p \) if for every \( b \in \mathcal{O}_p \), there exists a fixed constant \( A_p \in \mathbb{C} \) only depending on \( p \) such that

\[
\int_{U_p(b, n)} f \, d\mu \in A_p p^{-n h_p} \mathcal{O}_{F_p}, \quad \int_{V_p(b, n)} f \, d\mu \in A_p p^{-n h_p} \mathcal{O}_{F_p},
\]

for any \( n \in \mathbb{N} \) and \( f \in C_{\chi_p^*}^n(\mathcal{L}_p, \mathcal{O}_{F_p}) \).
We choose \( \omega^* = (\alpha^*_p)_{\nu} \), with \( \alpha^*_p \in \mathbb{C}^\nu_p \), and we consider the \( G(F_p) \)-representation \( \mathbb{D}_{\lambda^p}^k(C_p)^{h_p} \).

**Lemma 6.4.** Let \( h_p = (h_p)_p \) where \( h_p = \nu(p)(\alpha^*_p) \). We write \( \mathbb{D}_{\lambda^p}^k(C_p)^{h_p} \subseteq \mathbb{D}^k_{\lambda^p}(C_p)^{\xi} \) for the subspace of \( h_p \)-admissible distributions at every \( \nu \mid p \). Then \( \mathbb{D}_{\lambda^p}^k(C_p)^{h_p} \) is \( G(F_p) \)-invariant.

**Proof.** By (28) the action of \( G(O_p) \) on \( \mathbb{C}_{\lambda^p}(L_p, C_p) \) fixes \( \mathbb{C}_{\lambda^p}(L_p, O_p) \) (notice that \( \chi_p(O_p) \in O^\infty_p \)). Moreover,

\[
\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mathbb{U}(\beta, \nu) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \nu(\beta, \nu) & b \\ c & d \end{pmatrix} \mathbb{U}(\beta, \nu) = \begin{pmatrix} a \nu(\beta, \nu) & b \\ c & d \end{pmatrix} \mathbb{U}(\beta, \nu),
\]

for any \( \nu \in C_p \).

Together, this implies that \( G(O_p) \cdot \mathbb{D}_{\lambda^p}^k(C_p)^{h_p} \subseteq \mathbb{D}_{\lambda^p}^k(C_p)^{h_p} \).

Again by equation (28),

\[
\begin{pmatrix} \alpha_p \\ 1 \end{pmatrix}^m \mathbb{L}(F_p, y) = \sum_{\nu \in C_p} \langle \mathbb{L}(\nu, y) \rangle^m F(x, y) = \sum_{\nu \in C_p} \langle \mathbb{L}(\nu, y) \rangle^m F(x, \omega_p^{-m} y);
\]

we obtain for every \( f \in C_{\lambda^p}(L_p, C_p) \),

\[
\begin{pmatrix} \alpha_p \\ 1 \end{pmatrix}^m \mathbb{L}(F_p, y) = \sum_{\nu \in C_p} \langle \mathbb{L}(\nu, y) \rangle^m F(x, \omega_p^{-m} y).
\]

Thus, we obtain for every \( f \in C_{\lambda^p}(L_p, C_p) \),

\[
\begin{pmatrix} \alpha_p \\ 1 \end{pmatrix}^m \mathbb{L}(F_p, y) = \sum_{\nu \in C_p} \langle \mathbb{L}(\nu, y) \rangle^m F(x, \omega_p^{-m} y).
\]

We conclude that \( \mathbb{D}_{\lambda^p}^k(C_p)^{h_p} \subseteq \mathbb{D}_{\lambda^p}^k(C_p)^{h_p} \).

By Cartan decomposition

\[
G(F_p) = \bigcup_{\nu \in C_p} G(O_p) \begin{pmatrix} \alpha_p \\ 1 \end{pmatrix}^m G(O_p).
\]

Hence, the result follows. \( \square \)

**Lemma 6.5.** If we assume that \( e_p h_p < \min\{k_\sigma + 1, \sigma \in \Sigma_p\} \) for all \( \nu \mid p \), then any \( \mu \in \mathbb{D}_{\lambda^p}^k(C_p)^{h_p} \) lifts to a unique locally analytic distribution \( \mu \in \mathbb{D}_{\lambda^p}^k(C_p)^{h_p} \).

**Proof.** The proof is completely analogous to that of Proposition 5.2. By definition, the values \( \mu(f_n^{m_0}) \) and \( \mu(S_{a_{m_0}}^{m_0}) \) are given for every \( m_0 \leq k \). We proceed to define \( \mu(f_n^{m_0}) \) when there exist \( \tau \in \Sigma_p \) such that \( m_\tau > e_p h_p \), and the rest of values \( \mu(S_{a_{m_0}}^{m_0}) \) can be defined analogously: We write \( \mu(f_n^{m_0}) = \lim_{m \to \infty} a_n \), where

\[
a_n = \sum_{b \text{ mod } \omega_p^{m_0}} \sum_{j \leq e_p h_p} \sum_{a \text{ mod } \omega_p^{m_0}} \left( \frac{m - j}{j} \right) \frac{a - b}{\omega_p} \mu^{(j-n)}(f_n^{j_0}),
\]

The definition agrees with \( \mu \) when there exist \( \tau \in \Sigma_p \) such that \( e_p h_p < m_\tau \leq k_\sigma \), because \( \omega_p^{m_0} \mu^{(j-N)}(f_n^{j_0}) \to 0 \) when \( j > e_p h_p \). The usual computations show that the sequence \( a_n \) is Cauchy, hence the limit exists. Since the functions \( f_n^{m_0} \) and \( S_{a_{m_0}}^{m_0} \) topologically generate \( C_{\lambda^p}(L_p, C_p) \), this defines a locally analytic measure extending our locally polynomial distribution \( \mu \). \( \square \)
6.3 Lifting the modular symbol

Assume that $\pi_\psi$ is principal series or special for all $\psi \mid p$. Thus, we have a projection

$$r : \text{Ind}^G_{\psi_k} (\hat{\chi}_p^0)^0 \to V_{\psi_k}^0 = \bigotimes_{\psi \mid p} V_{\psi_k}^0,$$

as $G(F_p)$-representations for some locally constant character $\hat{\chi}_p^0 : F_p^\times \to \mathbb{C}^\times$. Write $\chi_p^0$ for the restriction of $\hat{\chi}_p^0$ to $\mathcal{O}_p^\times$. Moreover, we consider the locally polynomial character $\hat{\chi}_p = z^{-1/2} \hat{\chi}_p^0 : F_p^\times \to \mathbb{C}_p^\times$ and $\chi_p$ its restriction to $\mathcal{O}_p^\times$.

Under the above assumptions, our modular symbol

$$\phi_{\lambda}^p \in H^u \left( G(F)_+, \mathcal{A}^{\text{pl,ios}}(V_\psi, V(\mathfrak{k})_{\mathcal{O}_p}) \right)^{\lambda}$$

satisfies

$$r^* \phi_{\lambda}^p \in H^u \left( G(F)_+, \mathcal{A}^{\text{pl,ios}} \left( \text{Ind}^G_{\psi_k} (\hat{\chi}_p^0)^0, V(\mathfrak{k})_{\mathcal{O}_p} \right) \right)^{\lambda} \approx H^u \left( G(F)_+, \mathcal{A}^{\text{pl,ios}} \left( \nabla_{\chi}^\lambda(\mathcal{O}_p)_{\mathcal{O}_p} \right) \right)^{\lambda},$$

where $\varphi^* = (\alpha_\psi^*)_p$ with $\alpha_\psi^* = \hat{\chi}_p^0(\varphi_\psi)^{-1} \alpha_\psi^* = \hat{\chi}_p(\varphi_\psi)^{-1}$.

**Proposition 6.6.** Assume that for all $\psi \mid p$ we have $e_\psi \cdot v_\psi(\alpha_\psi^*) < \min \{ k_\sigma + 1, \sigma \in \Sigma_\psi \}$, then any cohomology class $\phi_{\lambda}^p \in H^u \left( G(F)_+, \mathcal{A}^{\text{pl,ios}}(V_\psi, V(\mathfrak{k})_{\mathcal{O}_p}) \right)^{\lambda}$ extends to a unique

$$\hat{\phi}_{\lambda}^p \in H^u \left( G(F)_+, \mathcal{A}^{\text{pl,ios}}(\nabla_{\chi}^\lambda(\mathcal{O}_p)_{\mathcal{O}_p}) \right)^{\lambda}.$$

Namely,

$$\kappa^\ast \hat{\phi}_{\lambda}^p = r^* \phi_{\lambda}^p \in H^u \left( G(F)_+, \mathcal{A}^{\text{pl,ios}} \left( \text{Ind}^G_{\psi_k} (\hat{\chi}_p^0)^0, V(\mathfrak{k})_{\mathcal{O}_p} \right) \right)^{\lambda}.$$

**Proof.** We write $\chi_p^0 = \prod_{\psi} \chi_\psi^0$. Notice that, if $V = \chi_p^0(x)^{-2} \cdot \mathbb{1}_{\mathcal{O}_p^\times} \cdot (x, y) \in C_0^0(\mathbb{A}^2, (\mathbb{L}_p, \mathcal{O}_p))$, by equation (28)

$$\hat{\chi}_p(\omega_\psi)^{-n} \cdot f_{\mathfrak{m}, \mu} = \left( \frac{b}{\omega_\psi^p} \right)^{-1} (V \cdot y^{\omega_\psi^p} \chi_\psi^0) \cdot (V \cdot y^{\omega_\psi^p} \chi_\psi^0),$$

As in the proof of Proposition 5.3, let us consider the $G(F_p)$-equivariant morphism

$$\theta_V : D_{\chi, \mu}^\lambda(C_{\mathcal{O}_p})_{\mathbb{A}} \cong \text{Hom} \left( \text{Ind}^G_{\psi_k} (\hat{\chi}_p^0)^0, V(\mathfrak{k})_{\mathcal{O}_p} \right) \to \text{coInd}_{C_{\mathcal{O}_p}}^{G(F_p)}(V(\mathfrak{k})_{\mathcal{O}_p}),$$

where $C_p \subseteq G(O_{\mathcal{O}_p})$ is a small enough subgroup of classes of matrices upper triangular modulo $p$. We compute, for any $\mu \in D_{\chi, \mu}^\lambda(C_{\mathcal{O}_p})_{\mathbb{A}}$, such that $\theta_V(\mu) \in \text{coInd}_{C_p}^{G(F_p)}(V(\mathfrak{k})_{\mathcal{O}_p})$,

$$\int_{U(1,b,n)} f_{\mathfrak{m}, \mu}^{\mathfrak{m}}(x) \, dx = \left( \frac{1}{\alpha_\psi} \right)^n \int_{\mathfrak{S}} \left( \frac{1}{\alpha_\psi} \right)^n (V \cdot y^{\omega_\psi^p} \chi_\psi^0) \, dx = \left( \frac{1}{\alpha_\psi} \right)^n \theta_V(\mu) \left( \frac{b}{\omega_\psi^p} \right)^{-1} (y^{\omega_\psi^p} \chi_\psi^0),$$

$$\int_{V(1,b,n)} g_{\mathfrak{m}, \mu}^{\mathfrak{m}}(x) \, dx = \left( \frac{1}{\alpha_\psi} \right)^n \int_{\mathfrak{S}} \left( \frac{1}{\alpha_\psi} \right)^n (V \cdot y^{\omega_\psi^p} \chi_\psi^0) \, dx = \left( \frac{1}{\alpha_\psi} \right)^n \theta_V(\mu) \left( \frac{b}{\omega_\psi^p} \right)^{-1} (y^{\omega_\psi^p} \chi_\psi^0),$$

where $h_\psi = v_\psi(\alpha_\psi^*)$. Since $f_{\mathfrak{m}, \mu}$ and $g_{\mathfrak{m}, \mu}$ generate the functions in $C_{\chi, \mu}^\lambda(\mathbb{L}_p, \mathcal{O}_p)$ with support in $U(1,b,n)$ and $V(1,b,n)$ respectively, we deduce $\mu \in D_{\chi, \mu}^\lambda(C_{\mathcal{O}_p})_{\mathbb{A}}$, with $h_\psi = (h_\psi)_p$.

By [26, Proposition 4.6] we have that $H^u \left( G(F)_+, \mathcal{A}^{\psi}(V(\mathfrak{k})_{\mathcal{O}_p}) \right)^{\lambda, C_p} \cong H^u \left( G(F)_+, \mathcal{A}^{\psi}(V(\mathfrak{k})_{\mathcal{O}_p}) \right)^{\lambda, C_p}$, hence (up-to-constant) we can assume that

$$\theta_V r^* \phi_{\lambda}^p \in H^u \left( G(F)_+, \mathcal{A}^{\psi}(V(\mathfrak{k})_{\mathcal{O}_p}) \right)^{\lambda, C_p} = H^u \left( G(F)_+, \mathcal{A}^{\psi}(\text{coInd}_{C_{\mathcal{O}_p}}^{G(F_p)}(V(\mathfrak{k})_{\mathcal{O}_p})) \right)^{\lambda}. $$
By the above computation, this implies that
\[ r^* \phi_{\Lambda_A}^P \in H^u \left( G(F)_+, \mathcal{R}^{p_{\text{Ioo}}} \left( \mathbb{D}_{\mathfrak{y}}^{k+1} \mathcal{R}_{\mathfrak{y}}^{k+2} (R) \right) \right) \Lambda. \]
Hence, the result follows from Lemma 6.5. \qed

6.4 Relation with \( p \)-adic L-functions

Attached to our modular symbol \( \phi_{\Lambda_A}^P \), we have a unique overconvergent cohomology class
\[ \hat{\phi}_{\Lambda_A}^P \in H^u \left( G(F)_+, \mathcal{R}^{p_{\text{Ioo}}} (\mathbb{D}_{\mathfrak{y}}^{k+2} (C_p)) \right) \Lambda. \]
On the other side, we have a fundamental class
\[ \eta \in H_a(T(F), C_c(T(\mathcal{A}_F), \mathbb{Z})). \]
Notice that the formula to define \( \delta_p \) in (20) extends to
\[ \delta_p : \mathcal{C}_{an,c}(T(F_p), C_p) \longrightarrow \text{Ind}_{\mathfrak{y}}^\mathbb{Z}_{\mathfrak{y}}(\hat{\chi}_p)^{an} \hat{\chi}_p^\mathfrak{y} \cong \mathbb{C}_{\mathfrak{y}}(\mathcal{L}_p, C_p). \]
Hence, for any \( f \in \mathcal{C}_{an}(\mathcal{G}_T, C_p) \), we can define the cap-product with respect to the pairings of §4.2
\[ ((p^* f) \cap \eta) \cap \delta_p(\hat{\phi}_{\Lambda_A}^P), \]
where
\[ p^* f \in H^0(T(F), \mathcal{C}_{an}(T(\mathcal{A}_F), C_p)), \]
\[ \delta_p(\hat{\phi}_{\Lambda_A}^P) \in H^u \left( G(F)_+, \mathcal{R}^{p_{\text{Ioo}}} (\text{Dist}_{an}(T(F_p), C_p)) \right) \Lambda, \]
\[ p^* f \cap \eta \in H_a(T(F), \mathcal{C}_{an,c}(T(\mathcal{A}_F), C_p) = H_a \left( T(F), \text{Ind}_{T(F)}^{T(F_p)} C_c \left( T(\mathcal{A}_F), C_{an,c}(T(F_p), C_p) \right) \right). \]
The following result follows directly from the definitions:

**Theorem 6.7.** Assume that for all \( p \mid p \) we have \( e_\sigma \cdot v_p(\alpha_\mathfrak{p}^*) = e_\sigma \cdot v_p(\hat{\chi}_p(\omega_\mathfrak{p})^{-1}) < \min \{ k_\sigma + 1, \sigma \in \Sigma_p \} \), and let \( \phi_{\Lambda_A}^P \in H^u \left( G(F)_+, \mathcal{R}^{p_{\text{Ioo}}} (\mathbb{D}_{\mathfrak{y}}^{k+2} (C_p)) \right) \Lambda \) be the extension of \( \phi_{\Lambda_A}^P \) provided by Proposition 6.6. Then we have that
\[ \int_{\mathcal{G}_T} f \text{d} \mu_{\eta, p} = ((p^* f) \cap \eta) \cap \delta_p(\hat{\phi}_{\Lambda_A}^P), \]
for all locally analytic function \( f \in \mathcal{C}_{an}(\mathcal{G}_T, C_p) \).

7 Hida families

From now on we will assume that our modular symbol \( \phi_{\Lambda_A}^P \in H^u(G(F)_+, \mathcal{R}^{p_{\text{Ioo}}}(V_p, V(\mathfrak{y}))) \Lambda \) is ordinary, namely, the attached distribution is 0-admissible. By Proposition 5.3 this amounts to saying that the valuations of \( \alpha_\mathfrak{p}^* = \hat{\chi}_p(\omega_\mathfrak{p})^{-1} = \omega_\mathfrak{p}^{k/2} \hat{\chi}_p(\omega_\mathfrak{p})^{-1} \) are zero for all \( p \mid p \). We are convinced that our work can be generalized to the finite slope situation by working with locally analytic distributions. From now on we will work with continuous functions and measures instead of locally analytic functions and admissible distributions.

Let \( \Lambda_F \) be the Iwasawa algebra associated with \( \mathcal{O}_F^{\infty} \), and let
\[ \kappa_{\mathfrak{p}} : \mathcal{O}_F^{\infty} \longrightarrow \Lambda_F^*, \]
be the universal character. Recall that \( \kappa_{\mathfrak{p}} \) is characterized by the following property: For any complete \( \mathbb{Z}_p \)-algebra \( R \), and any continuous character \( \chi_{\mathfrak{p}} : \mathcal{O}_F^{\infty} \longrightarrow R^* \), there exists a morphism \( \rho_{\chi_{\mathfrak{p}}} : \Lambda_F \rightarrow R \) such that \( \chi_{\mathfrak{p}} = \rho_{\chi_{\mathfrak{p}}} \circ \kappa_{\mathfrak{p}} \).

Let \( \mathfrak{a} = (a_{\mathfrak{p}})_{\mathfrak{p}} \), where \( a_{\mathfrak{p}} \in \Lambda_F^* \). Notice that \( \mathbb{D}_p^{-2}(\Lambda_F \mathfrak{a}) \) satisfies that, for any such a pair \( (R, \chi_{\mathfrak{p}}) \), we have a natural \( G(F_{p}) \)-equivariant morphism
\[ \mathbb{D}_p^{-2}(\Lambda_F \mathfrak{a}) \otimes_{\rho_{\chi_{\mathfrak{p}}}} R \longrightarrow \mathbb{D}_p^{-2}(R) \mathfrak{a}, \]
where \( \mathfrak{a}^* = \rho_{\chi_{\mathfrak{p}}}(\mathfrak{a}) \).
7.1 Lifting the form to a family

In this ordinary setting, Proposition 6.6 amounts to saying that our modular symbol $\phi_A^p$ lifts to an overconvergent modular symbol $\hat{\phi}_A^p \in H^u \left( G(F)_+, \mathcal{A}^{p,loc}(\mathbb{D}_{k^p}(\Lambda_F)_A) \right)^A$. Assume that there exists $a = (a_\nu)_\nu$, with $a_\nu \in \Lambda_F^\circ$ such that $p_\nu \Lambda_F^\circ(a_\nu) = a_\nu$. This defines an extension $\hat{k}_p$ of the universal character $k_p$. Hence, the specialization map $\rho_{\hat{k}_p}$ provides a morphism

$$
H^u \left( G(F)_+, \mathcal{A}^{p,loc}(\mathbb{D}_{k^p}(\Lambda_F)_A) \right)^A \xrightarrow{\rho_{\hat{k}_p}} H^u \left( G(F)_+, \mathcal{A}^{p,loc}(\mathbb{D}_{k^p}(\Lambda_F)_A) \right)^A \otimes_{\rho_{\hat{k}_p}} O_{C_{\hat{k}_p}} \quad (30)
$$

In this section we will discuss the existence of both $\hat{a}_\nu$ and a class $\Phi_A^p \in H^u \left( G(F)_+, \mathcal{A}^{p,loc}(\mathbb{D}_{k^p}(\Lambda_F)_A) \right)^A$ lifting the overconvergent modular symbol $\phi_A^p$.

7.1.1 Local systems and group cohomology

Given a compact subgroup $C \subseteq G(A_F^\circ)$, we can construct the locally symmetric space

$$
Y_C := G(F)_+ \backslash (G(F_\infty)_+ \times G(A_F^\circ))/C_\infty C,
$$

where $C_\infty$ is the maximal compact subgroup of $G(F_\infty)_+$. Notice that when $F$ is totally real and $C$ is small enough $Y_C$ is in correspondence with the set of complex points of a Shimura variety.

Let $C_p := C \cap G(F_p)$. Given a $C_p$-module $V$, we can define the local system

$$
\mathcal{V} := G(F)_+ \backslash (G(F_\infty)_+ \times G(A_F^\circ) \times V)/C_\infty C \rightarrow Y_C,
$$

where the left $G(F)_+$-action and right $C_\infty C$-action on $G(A_F) \times V$ is given by

$$
g(\gamma g^\infty, v)(c_\infty, c) = (\gamma g^\infty c, \gamma g^\infty c, c_p^{-1} v),
$$

being $c_p \in C_p$ the $p$-component of $c \in C$.

A locally constant section of the local system $\mathcal{V}$ amounts to a function $s : G(A_F^\circ) \rightarrow V$ such that $s(g^\infty) = \epsilon_p s(\gamma g^\infty c)$, for all $\gamma \in G(F)_+$, and $c \in C$. Indeed, such a function provides the well defined section

$$
Y_C \rightarrow \mathcal{V}; \quad g \mapsto (g_\infty, g^\infty, s(g^\infty)).
$$

Let us consider the coinduced representation

$$
\text{coInd}_{C_p}^{G(F_p)} V = \{ f : G(F_p) \rightarrow V; \ f(g_p c_p) = c_p^{-1} f(g_p), \ g_p \in G(F_p), \ c_p \in C_p \},
$$

with $G(F_p)$-action $(h_p f)(g_p) = f(h_p^{-1} g_p)$, $h_p \in G(F_p)$. Thus, to provide such an $s$ is equivalent to provide an element

$$
\hat{s} \in H^u \left( G(F)_+, \mathcal{A}^{p,loc}(\text{coInd}_{C_p}^{G(F_p)} V) \right)^{C_p}, \quad \hat{s}(g^p)(g_p) := s(g_p, g^p),
$$

where $C_p := C \cap G(A_F^{p,loc})$. Hence, we can identify the cohomology of the sheaf of local sections of $\mathcal{V}$ with the $C^p$-invariant subgroup of the group cohomology of $\mathcal{A}^{p,loc}(\text{coInd}_{C_p}^{G(F_p)} V)$, namely,

$$
H^k(Y_C, \mathcal{V}) = H^k \left( G(F)_+, \mathcal{A}^{p,loc}(\text{coInd}_{C_p}^{G(F_p)} V) \right)^{C_p}. \quad (31)
$$
7.1.2 Families and the eigencurve

Given a \(\mathbb{Z}_p\)-algebra \(R\), write \(C(O_{F_p}, R)\) and \(D(O_{F_p}, R)\) for the set of \(R\)-valued continuous functions and its continuous dual, respectively. Write \(I_p \subset G(F_p)\) the usual Iwahori subgroup

\[
I_p = \prod_{p|I} I_p, \quad I_p := \text{PGL}_2(O_{F_p}) \cap \left( \frac{O_{F_p}^\times}{\omega_p O_{F_p}} \big/ O_{F_p}^\times \right).
\]

Given a continuous character \(\chi_p : O_{F_p}^\times \to R^\times\), one can define an \(I_p\)-action on \(C(O_{F_p}, R)\) by means of the formula

\[
(i_p f)(x) = f \left( \frac{b + dx}{a + cx} \right) \cdot \chi_p(\det i_p) \cdot \chi_p^{-2}(a + cx), \quad i_p = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in I_p, \quad f \in C(O_{F_p}, R).
\]

This provides the usual action on \(D(O_{F_p}, R)\) given by \((i_p \mu)(f) = \mu(i_p^{-1} f)\). We write \(D_{\chi_p} (O_{F_p}, R)\) for the space endowed with the above action of \(I_p\). Notice that such an action can be extended to the semigroup \(\Sigma_p^{-1}\) of inverses of

\[
\Sigma_p := \prod_{p|I} \Sigma_p, \quad \Sigma_p := \text{PGL}_2(O_{F_p}) \cap \left( \frac{O_{F_p}^\times}{\omega_p O_{F_p}} \big/ O_{F_p}^\times \right).
\]

Indeed, if we write \(\langle \alpha \rangle = \frac{\alpha}{\prod_{p|I} \omega_p^{\varepsilon_p}} \in O_{F_p}^\times\) for any \(\alpha \in F_p^\times\), the action is defined by

\[
(g_p f)(x) = f \left( \frac{b + dx}{a + cx} \right) \cdot \chi_p \left( \langle \det g_p \rangle \right) \cdot \chi_p^{-2}(a + cx), \quad g_p = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Sigma_p.
\]

Hence it makes sense to consider the action of the matrices \(\left( \omega_p \frac{a_i}{c_i} \right)\), \(i \in O_{F_p}/p\), defining Hecke operators \(U_p\).

**Remark 7.1.** Notice that we have a morphism

\[
C(O_{F_p}, R) \longrightarrow C_{\chi_p^2}(L_p, R); \quad f \longmapsto \hat{f}(x, y) = \chi_p(x)^2 \cdot f \left( \frac{y}{x} \right) \cdot \mathbb{1}_{O_{F_p}^\times \times O_{F_p}^\times}(x, y),
\]

satisfying for all \(k = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in G(O_{F_p})\)

\[
(k \hat{f})(x, y) = \chi_p(\det k) \cdot \hat{f}((x, y) k) = f \left( \frac{bx + dy}{ax + cy} \right) \cdot \chi_p(\det k) \cdot \chi_p^{-2}(ax + cy) \cdot \mathbb{1}_{O_{F_p}^\times \times O_{F_p}^\times}(x, y).
\]

In particular it is \(I_p\)-equivariant with respect to the above action. More generally, given \(g = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \in \Sigma_p\)

\[
g^{-1} \hat{f}(x, y) = \hat{\hat{f}}(x, y) = \left( \begin{smallmatrix} a & \hat{b} \\ c & \hat{d} \end{smallmatrix} \right) \cdot \hat{f}(\alpha^{-1}(x, y) g^{-1}) = \left( \begin{smallmatrix} a & \hat{b} \\ c & \hat{d} \end{smallmatrix} \right) \cdot \hat{f}(\alpha^{-1}(x, y) g^{-1}) = \hat{\hat{f}} \left( \frac{1}{\alpha \det(g)} (dx - cy, ay - bx) \right),
\]

by (28), where \(\alpha \in F_p^\times\) is such that \(\alpha^{-1}(x, y) g^{-1} \in L_p\). Notice that, since \(\alpha \in O_{F_p}^\times\) and \(c \in \prod_p \omega_p\), a necessary condition for \((dx - cy, ay - bx)\) being in \(F_p^\times(O_{F_p}^\times \times O_{F_p}^\times)\) is \(x \in O_{F_p}^\times\). Since the support of \(\hat{f}\) is precisely \(O_{F_p}^\times \times O_{F_p}^\times\), we conclude that \(x \in O_{F_p}^\times\) and

\[
v_p \left( \frac{y}{x} - \frac{b}{a} \right) = v_p(ay - bx) \geq v_p(dx - cy) = v_p \left( d - c \frac{y}{x} \right) = v_p \left( \frac{\det(g)}{d} + c \left( \frac{b}{a} - \frac{y}{x} \right) \right).
\]

Thus, \(\frac{y}{x} \notin \frac{b}{a} + \det(g)O_{F_p}\) and \(\alpha = 1\). Hence, if we write \(U(g) := \frac{b}{a} + \det(g)O_{F_p}\), we compute,

\[
g^{-1} \hat{f}(x, y) = \hat{\hat{f}} = \left( \begin{smallmatrix} a \cdot \det(g) & b \cdot \det(g) \\ c \cdot \det(g) & d \cdot \det(g) \end{smallmatrix} \right) \cdot \hat{f}(\frac{y}{x}) = \hat{\hat{f}} \left( \frac{y}{x} \right) = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \cdot \hat{f}(\frac{y}{x}).
\]
The classical strategy to construct the eigenvariety is to consider finite slope subspaces for the action of \( \mathcal{U}_\mathfrak{a} \) of the locally analytic analogues of cohomology spaces \( H^i(Y_C, \mathcal{D}_\mathfrak{b}(O_{F_F}, \Lambda_F)) \), where \( \mathcal{D}_\mathfrak{b}(O_{F_F}, \Lambda_F) \) is the local system associated with \( D_{\mathfrak{b}}(O_{F_F}, \Lambda_F) \) and \( C \subset G(\mathbb{A}_F^{inf}) \) is a compact open subgroup such that \( C_F = I_F \). Since our setting is ordinary, it is enough for us to consider \( D_{\mathfrak{b}}(O_{F_F}, \Lambda_F) \). A connected component of the eigenvariety passing through \( \pi \) provides a system of eigenvalues for the Hecke operators in \( \Lambda_F \).

Once we have a system of eigenvalues, in particular eigenvalues \( a_\mathfrak{a} \) for the \( \mathcal{U}_\mathfrak{a} \)-operators, a different and challenging problem is to provide eigenvectors living in our space of cohomology \( H^i(Y_C, \mathcal{D}_\mathfrak{b}(O_{F_F}, \Lambda_F)) \). If \( F \) is totally real, then one can make use of the étaleness of the eigenvariety to prove the existence of such families in middle degree \( k = u \) (see [1, Theorem 2.14] for the case \( G = \text{PGL}_2 \), and notice that techniques of [1, §2.5] can be extended to general \( G \)). For arbitrary number fields \( F \) the situation is more complicated (see [11] or [2]). Since these questions are beyond the scope of this paper, we will directly assume the existence of a family \( \phi_\Lambda^F \in H^u(Y_C, \mathcal{D}_\mathfrak{b}(O_{F_F}, \Lambda_F))^{{\lambda}, U = a_\mathfrak{a}} \) and we will address the reader to the previous references for details in each concrete situation. By Equation (31)

\[
H^u(Y_C, \mathcal{D}_\mathfrak{b}(O_{F_F}, \Lambda_F))^{{\lambda}, U = a_\mathfrak{a}} = H^u\left( \mathcal{G}(F)+, \mathcal{A}_{F,F}^{\text{fil}}(\text{coInd}_{I_F}^{G(F)} D_{\mathfrak{b}}(O_{F_F}, \Lambda_F)^{{\lambda}, U = a_\mathfrak{a}}) \right)^{\Lambda_F},
\]

where the action of \( \mathcal{U}_\mathfrak{a} \) on \( \phi \in \text{coInd}_{I_F}^{G(F)} D_{\mathfrak{b}}(O_{F_F}, \Lambda_F) \) is given by

\[
\mathcal{U}_\mathfrak{a} \phi(g_{\mathfrak{a}}) = \sum_{i \in \mathcal{O}_F / \mathfrak{a}} \left( \frac{1}{a_\mathfrak{a}} \right)^i \phi \left( g_{\mathfrak{a}} \left( \frac{1}{a_\mathfrak{a}} \right)^{-i} \right).
\]

**Remark 7.2.** For any \( a \in O_{F_F} / \mathfrak{a}^u \) write \( g_{\mathfrak{a}} = (\frac{1}{\alpha_\mathfrak{a}}) \). For any \( \phi \in \text{coInd}_{I_F}^{G(F)} D_{\mathfrak{b}}(O_{F_F}, \Lambda_F)^{{\lambda}, U = a_\mathfrak{a}} \), by the definition of \( \mathcal{U}_\mathfrak{a} \). Applying this fact inductively, we deduce

\[
\int_{a + \mathcal{O}_F} d\phi(g) = \frac{1}{\alpha_\mathfrak{a}} \int_{\mathcal{O}_F} d\phi(g g_{\mathfrak{a}}^{-1}),
\]

for all \( a \in O_{F_F} / \mathfrak{a}^u \). This implies that the integrals \( \int_{\mathcal{O}_F} d\phi(g) \) characterize the element \( \phi \).

**Proposition 7.3.** We have that a \( G(F_F) \)-equivariant isomorphism.

\[
\mathcal{D}_{\mathfrak{b}}^a(\Lambda_F)_{\mu} \cong \text{coInd}_{I_F}^{G(F_F)} D_{\mathfrak{b}}(O_{F_F}, \Lambda_F)^{{\lambda}, U = a_\mathfrak{a}},
\]

where \( a = (a_\mathfrak{a})_\mathfrak{a} \).

**Proof.** Notice that, by Remark 7.1, we have a \( I_F \)-equivariant morphism

\[
\text{res} : \mathcal{D}_{\mathfrak{b}}(\Lambda_F)_{\mu} \longrightarrow D_{\mathfrak{b}}(O_{F_F}, \Lambda_F); \quad \int_{\mathcal{O}_F^\mu} f(z) d(\text{res}_{\mu})(z) := \int_{\mathcal{O}_F^\mu \times \mathcal{O}_F} \hat{f}(x, y) d\mu(x, y).
\]

Hence it provides a well defined \( G(F_F) \)-equivariant morphism

\[
\varphi : \mathcal{D}_{\mathfrak{b}}(\Lambda_F)_{\mu} \longrightarrow \text{coInd}_{I_F}^{G(F_F)} D_{\mathfrak{b}}(O_{F_F}, \Lambda_F), \quad \varphi(\mu)(g_{\mathfrak{a}}) := \text{res}(g_{\mathfrak{a}}^{-1} \mu).
\]

Let us check that the image lies in the subspace where \( \mathcal{U}_\mathfrak{a} \) acts like \( a_\mathfrak{a} \). If we write \( g_{\mathfrak{a}} = (\frac{1}{\alpha_\mathfrak{a}}) \), then by Remark 7.1,

\[
\int_{\mathcal{O}_F} f(z) d\mathcal{U}_\mathfrak{a}(\varphi(\mu))(g_{\mathfrak{a}})(z) = \sum_{i \in \mathcal{O}_F / \mathfrak{a}} \int_{\mathcal{O}_F} f(z) d\left( \frac{1}{\alpha_\mathfrak{a}} \varphi(\mu)(g_{\mathfrak{a}}^{-1} \mu) \right) = \sum_{i \in \mathcal{O}_F / \mathfrak{a}} \int_{\mathcal{O}_F} g_{\mathfrak{a}}^{-1} f(z) d(g_{\mathfrak{a}}^{-1} \mu) = \alpha_\mathfrak{a} \cdot \sum_{i \in \mathcal{O}_F / \mathfrak{a}} \int_{\mathcal{O}_F} f(z) d(\varphi(\mu))(g_{\mathfrak{a}})(z) = a_\mathfrak{a} \cdot \int_{\mathcal{O}_F} f(z) d(\varphi(\mu))(g_{\mathfrak{a}})(z).
\]
Thus \( U_\mu(\varphi(\mu)) = a_\mu \cdot \varphi(\mu) \). This implies that we have a well defined \( G(F) \)-equivariant morphism

\[
\varphi : \mathcal{H}_{k_\mu}(\Lambda_F) \rightarrow \text{coInd}_{\mathcal{K}_p}^{(G(F))} D_{k_\mu}(O_{F_p}, \Lambda_F)^{U_\mu = a_\mu}.
\]

It is clearly injective since the vanishing of \( \varphi(\mu) \) in particular implies that the distribution \( \mu \) vanishes when restricted to \( (O_{F_p}^\times \times O_{F_p}) G(O_{F_p}) = L_p \).

Let us consider the function \( f_0 \in C_k \) defined by \( f_0(x, y) = k_\mu^{-1}(x) \cdot \mathbb{1}_{O_{F_p}}(y) \). Notice that \( \mathbb{1}_{O_{F_p}} = f_0 \).

Then it is clear that the translates \( g_p f_0 \), where \( g_p \in G(F_p) \), topologically generate \( C_k \). Thus, we can define

\[
\psi : \text{coInd}_{\mathcal{K}_p}^{(G(F))} D_{k_\mu}(O_{F_p}, \Lambda_F)^{U_\mu = a_\mu} \rightarrow \mathcal{H}_{k_\mu}(\Lambda_F) \quad \int_{\mathcal{L}_p} (g_p f_0) d\psi(\phi) = \int_{O_{F_p}} d\phi(g_p).
\]

It is easy to check that the morphism \( \psi \) is \( G(F_p) \)-equivariant, indeed, for \( h_p \in G(F_p) \)

\[
\int_{\mathcal{L}_p} (g_p f_0) d\psi(h_p \phi) = \int_{O_{F_p}} d\phi(h_p^{-1} g_p) = \int_{\mathcal{L}_p} (h_p^{-1} g_p f_0) d\psi(\phi) = \int_{\mathcal{L}_p} (g_p f_0) d (h_p \psi(\phi)) .
\]

But we have to check that it is well defined, namely, given any linear relation \( \sum_i c_i h_i f_0 = \sum_i h_i f_0 \) we have that

\[
\sum_i c_i \int_{\mathcal{L}_p} h_i f_0 d\psi(\phi) = \sum_i c_i \int_{\mathcal{L}_p} h_i f_0 d\psi(\phi).
\]

But such relations are \( G(F_p) \)-generated by

\[ f_0 = \frac{1}{\alpha_p} \sum_{i \in O_{F_p}/p} \left( \alpha_p \frac{i}{1} \right) f_0 = \frac{1}{\alpha_p} \sum_{i \in O_{F_p}/p} g_i^{-1} f_0 , \]

Hence, by Remark 7.2, the morphism \( \psi \) is well defined since

\[
\int_{\mathcal{L}_p} f_0 d\psi(\phi) = \int_{O_{F_p}} d\phi(1) = \sum_{i \in O_{F_p}/p} \int_{\mathcal{L}_p} d\phi(1) = \frac{1}{\alpha_p} \sum_{i \in O_{F_p}/p} \int_{O_{F_p}} d\phi(g_i^{-1}) = \frac{1}{\alpha_p} \sum_{i \in O_{F_p}/p} \int_{\mathcal{L}_p} (g_i^{-1} f_0) d\psi(\phi).
\]

We compute

\[
\int_{\mathcal{L}_p} (g_p f_0) d (\psi \circ \varphi(\mu)) = \int_{O_{F_p}} d\varphi(\mu)(g_p) = \int_{O_{F_p} \times O_{F_p}} d\mathbb{1}_{O_{F_p}} (g_p^{-1} \mu) = \int_{\mathcal{L}_p} (g_p f_0) d\mu,
\]

hence \( \psi \circ \varphi(\mu) = \mu \). Moreover, for all \( g_p \in G(F_p) \),

\[
\int_{a + ps O_{F_p}} d\varphi \circ \psi(\phi)(g_p) = \frac{1}{\alpha_p} \int_{O_{F_p}} d\varphi \circ \psi(\phi)(g_p g_a^{-1}) = \frac{1}{\alpha_p} \int_{\mathcal{L}_p} d\mathbb{1}_{O_{F_p}} (g_p g_a^{-1} \psi(\phi)) = \frac{1}{\alpha_p} \int_{\mathcal{L}_p} (g_p g_a^{-1} f_0) d\psi(\phi) = \frac{1}{\alpha_p} \int_{O_{F_p}} d\phi(g_p g_a^{-1}) = \int_{a + ps O_{F_p}} d\phi(g_p),
\]

by Remark 7.2. Hence \( \varphi \circ \psi(\phi) = \phi \) and the result follows.

The above result together with equation (31) implies that we have an isomorphism

\[ H^u \left( G(F)_+, \mathcal{A}^p_{\mathcal{K}_p} \right) \rightarrow H^u(\mathcal{Y}_C, D_{k_\mu}(O_{F_p}, \Lambda_F))^{U_\mu = a_\mu} \]

Thus, our assumption on the existence of a classical family \( k_\mu P^\mu \in H^u(\mathcal{Y}_C, D_{k_\mu}(O_{F_p}, \Lambda_F))^{U_\mu = a_\mu} \) ensures the existence of our lift \( \Phi^\mu_\lambda \in H^u \left( G(F)_+, \mathcal{A}^p_{\mathcal{K}_p} \right)^{U_\mu = a_\mu} \).
7.2 Extensions of the Steinberg representation

Let us assume that the local representation \( \pi_{p} \) is Steinberg. Thus \( \pi_{p} \) is the quotient of the induced representation \( \text{Ind}_{\mathbb{C}^{1}}^{\mathbb{C}}1 \) modulo the constant functions. Since \( G(F_{p})/P \simeq \mathbb{P}^{1}(F_{p}) \), we can give a simple description of \( V_{p}^{Z} = \text{St}_{\mathbb{Z}}(F_{p}) \):

\[
\text{St}_{\mathbb{Z}}(F_{p}) = \mathcal{C}^{0}(\mathbb{P}^{1}(F_{p}), \mathbb{Z})/\mathbb{Z},
\]

where \( \mathcal{C}^{0}(\mathbb{P}^{1}(F_{p}), \mathbb{Z}) \) is the set of locally constant \( \mathbb{Z} \)-valued functions of the projective line, with the natural action of \( G(F_{p}) = \text{PGL}_{2}(F_{p}) \) induced by the action on the projective given by fractional linear transformations

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax + b}{cx + d}.
\]

Denote by \( \text{Cov}(X) \) the poset of open coverings of a topological space \( X \) ordered by refinement. Let us consider \( \mathcal{S}_{p} = K_{p} \setminus F_{p} \) the \( p \)-adic upper half plane. Let \( \Delta_{p} := \text{Div}(\mathcal{S}_{p}) \) and \( \Delta_{p}^{0} := \text{Div}^{0}(\mathcal{S}_{p}) \subset \Delta_{p} \) the set of divisors and degree zero divisors. Notice that the multiplicative integral provides a morphism

\[
\psi \mapsto \left( z_{2} - z_{1} \mapsto \int_{\mathbb{P}^{1}(F_{p})} \frac{x-z_{2}}{x-z_{1}} d\mu_{\psi}(x) = \lim_{U \to \text{Cov}(\mathbb{P}^{1}(F_{p}))} \prod_{U \subseteq U} \left( \frac{x-u-z_{2}}{x-u-z_{1}} \right)^{\psi(1)}(1) \right),
\]

where each \( u \in U \).

In [15] this multiplicative integral is described alternatively as follows: For any topological group \( M \), we write \( \text{St}_{M} := \mathcal{C}(\mathbb{P}^{1}(F_{p}), M)/M \). Then we have a natural morphism

\[
\psi_{\text{unv}} : \text{Hom}(\text{St}_{\mathbb{Z}}(F_{p}), \mathbb{Z}) \to \text{Hom}(\text{St}_{\mathbb{Z}}^{\mathbb{C}}(F_{p}^{\mathbb{C}}), \mathbb{Z}), \quad \psi_{\text{unv}}(\psi)(f) := \lim_{U \to \text{Cov}(\mathbb{P}^{1}(F_{p}))} \prod_{U \subseteq U} f(x_{U})^{\psi(1)}(1).
\]

Moreover, we can consider the universal extension \( \mathcal{E}_{K_{p}^{\mathbb{C}}} = \mathcal{E}(\text{Id}) \) of \( \text{St}_{\mathbb{Z}}^{\mathbb{C}} \), where for any topological group \( M \) (written multiplicatively) and any continuous character \( \ell : F_{p}^{\mathbb{C}} \to M \)

\[
\mathcal{E}(\ell) := \left\{ (\phi, y) \in \mathcal{C}(\text{GL}_{2}(F_{p}), M) \times \mathbb{Z} : \phi \left( \begin{pmatrix} s & x \\ t & y \end{pmatrix} g \right) = \ell(t)^{y} \cdot \phi(g) \right\}/(M, 0).
\]

Notice that we have a natural morphism \( \psi : \Delta_{p} \to \mathcal{E}_{K_{p}^{\mathbb{C}}} \) making the following diagram commutative

\[
\begin{array}{ccc}
0 & \to & \Delta_{p}^{0} \\
\downarrow \psi & & \downarrow \psi \\
\text{St}_{K_{p}^{\mathbb{C}}} & \to & \mathcal{E}_{K_{p}^{\mathbb{C}}}
\end{array}
\]

Indeed, we can define

\[
\psi(z) := (\phi_{z}, 1), \quad \phi_{z} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) := cz + d \in K_{p}^{\mathbb{C}}.
\]

We can describe the morphism \( \psi_{\text{unv}} \) as the composition \( \psi_{\text{unv}} \circ \phi_{\text{unv}} \), where \( \phi_{\text{unv}} \) is the associated pull-back.

**Remark 7.4.** Notice that \( \mathcal{E}_{K_{p}^{\mathbb{C}}} \) is universal in the sense that, for any character \( \ell \) as above, the morphism \( (y, \ell \phi) \mapsto (y, \ell \phi) \) provides a morphism \( \ell : \mathcal{E}_{K_{p}^{\mathbb{C}}} \to \mathcal{E}(\ell) \).

Let \( \Lambda_{p} := \mathbb{Z}[\{\mathcal{O}_{p}^{\mathbb{C}}\}] \) the \( p \)-Iwasawa algebra with universal character \( k_{p} \). For any continuous character \( \chi_{p} : \mathcal{O}_{p}^{\mathbb{C}} \to \mathbb{R}^{\mathbb{C}}, \) we consider the space of homogeneous measures

\[
\mathbb{D}_{\chi_{p}}(R) = \mathbb{H}_{\text{univ}} \cdot \mathcal{C}_{\chi_{p}}(\mathcal{L}_{p}, R), \quad \mathbb{D}_{\chi_{p}}(\Lambda_{p})^{0} = \{ \mu \in \mathbb{D}(\Lambda_{p}) : (\rho_{1}, \mu)(1) = 0 \},
\]

where the specializations \( \rho_{1} : \mathbb{D}_{\chi_{p}}(\Lambda_{p}) \to \mathbb{D}_{\chi_{p}}(R) \) are defined analogously as in \( (29) \). Recall that any extension \( \tilde{k}_{p} : F_{p}^{\mathbb{C}} \to \Lambda_{p}^{\mathbb{C}} \) of the universal character provides an isomorphism \( \mathbb{C}_{\chi_{p}}(\mathcal{L}_{p}, \Lambda_{p})^{\psi} \simeq \text{Ind}_{\mathbb{C}}^{\psi}(\tilde{k}_{p}) \) as in \( (27) \), and this provides an action of \( G(F_{p}) \) on \( \mathbb{D}_{\chi_{p}}(\Lambda_{p}) \). If we write \( a_{p} = \tilde{k}_{p}(a_{p})^{-1} \) as usual, then we denote the space with such an action by \( \mathbb{D}_{\chi_{p}}(\Lambda_{p})_{a_{p}} \).

**Lemma 7.5.** If \( \rho_{1}(a_{p}) = 1 \), then the subspace \( \mathbb{D}_{\chi_{p}}(\Lambda_{p})^{0} \) is \( G(F_{p}) \)-invariant. We will denote the subspace with the corresponding action by \( \mathbb{D}_{\chi_{p}}(\Lambda_{p})_{a_{p}} \).
Proof. Given $\mu \in \mathbb{D}_{k_0}(\Lambda_\varphi)^0$, we have to check that $g\mu \in \mathbb{D}_{k_0}(\Lambda_\varphi)^0$ for all $g \in G(F_\varphi)$. By $G(F_\varphi)$-equivariance

$$(\rho_1 g\mu)(1) = (g\rho_1 \mu)(1) = (\rho_1 \mu)(g^{-1}1).$$

But if $\rho_1(a_\varphi) = 1$ we have $\rho_1 C_\Lambda(L_\varphi, \Lambda_\varphi) \cong \text{Ind}_{\mu}^\varphi(1)$, and $\varphi_\varphi(1)$ is $G(F_\varphi)$-invariant. Hence $g^{-1}1 = 1$ and the result follows. \hfill $\Box$

Throughout the rest of this section we will assume that $\rho_1(a_\varphi) = 1$, hence we are in the setting of the above Lemma. Write $I_\varphi$ for the augmentation ideal

$$I_\varphi := \ker \left( \Lambda_\varphi \to Z_\varphi \right).$$

Notice that there is a natural isomorphism

$$O_{F_\varphi}^\times \otimes_\mathbb{Z} Z_\varphi := \hat{O}_{F_\varphi}^\times \to I_\varphi/I_\varphi^2; \quad \alpha \mapsto (k_\varphi(\alpha) - 1) + I_\varphi^2.$$

Since $\rho_1(a_\varphi) = 1$, the above isomorphism can be extended to a character

$$\ell_\varphi : K_\varphi^\times \to I_\varphi/I_\varphi^2; \quad \alpha \mapsto (k_\varphi(\alpha) - 1) + I_\varphi^2.$$

Remark 7.6. Notice that for any $y \in \mathbb{Z}$, $f \in C_{k_\varphi}^\times(L_\varphi, O_{F_\varphi}^\times) \subset C_{k_\varphi}^\times(L_\varphi, \Lambda_\varphi)$ and any $\mu \in \mathbb{D}_{k_\varphi}(\Lambda_\varphi)^0$,

$$\int_{L_\varphi} f d\rho_{k_\varphi} \mu \in I_\varphi \subset \Lambda_\varphi.$$

Indeed, since $f$ has values in $O_{F_\varphi}^\times$, we have that $\rho_1 f = 1$. Hence

$$\rho_1 \left( \int_{L_\varphi} f d\rho_{k_\varphi} \mu \right) = \int_{L_\varphi} 1 d\rho_1 \mu = 0,$$

by the definition of $\mathbb{D}_{k_\varphi}(\Lambda_\varphi)^0$.

Lemma 7.7. Given $f_1 \in C_{k_\varphi}^\times(L_\varphi, O_{F_\varphi}^\times)$ and $f_2 \in C_{k_\varphi}^\times(L_\varphi, O_{F_\varphi}^\times)$, we have

$$\int_{L_\varphi} (f_1 \cdot f_2) d\rho_{k_\varphi} \mu = \int_{L_\varphi} f_1 d\rho_{k_\varphi} \mu + \int_{L_\varphi} f_2 d\rho_{k_\varphi} \mu \mod I_\varphi^2$$

for any $\mu \in \mathbb{D}_{k_\varphi}(\Lambda_\varphi)^0$.

Proof. Let $f_0 \in C_{k_\varphi}(L_\varphi, O_{F_\varphi}^\times)$. For example, we can choose

$$f_0(c, d) = \begin{cases} k_\varphi(c), & \text{if} \ (c, d) \in O_{F_\varphi}^\times \times O_{F_\varphi} \\ k_\varphi(d), & \text{otherwise.} \end{cases}$$

Notice that, if $\varphi : L_\varphi \to \mathbb{P}^1(F_\varphi)$ is the natural projection,

$$\int_{L_\varphi} f d\rho_{k_\varphi} \mu = \lim_{\mu \in \text{Cov}(\mathbb{P}^1(F_\varphi))} \sum_{\mathfrak{p} \in \mathfrak{p}} \left( \frac{f_1}{f_0} \right) (x_{\mathfrak{p}}) \int_{\varphi^{-1}(\mathfrak{p})} f_0^\varphi d\rho_{k_\varphi} \mu = \lim_{\mu \in \text{Cov}(\mathbb{P}^1(F_\varphi))} \sum_{\mathfrak{p} \in \mathfrak{p}} \left( \frac{f_1}{f_0} \right) (x_{\mathfrak{p}}) \cdot \rho_\varphi \int_{\varphi^{-1}(\mathfrak{p})} f_0 d\mu,$$

since $\rho_{k_\varphi} f_0 = f_0^\varphi$. For any $\beta \in I_\varphi$, write $\overline{\beta}$ for its image in $I_\varphi/I_\varphi^2$, and write $M = \mu(f_0) \in I_\varphi$. Since $\overline{a\beta} = \rho_1(\alpha) \cdot \overline{\beta}$ for any $\alpha \in \Lambda_\varphi$, we obtain

$$\int_{L_\varphi} f d\rho_{k_\varphi} \mu = \lim_{\mu \in \text{Cov}(\mathbb{P}^1(F_\varphi))} \sum_{\mathfrak{p} \in \mathfrak{p}} \left( \frac{f_1}{f_0} \right) (x_{\mathfrak{p}}) - 1 \right) \cdot \rho_1 \mu(1).$$

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Hence
\[\int_{L_p} (f_1 \cdot f_2) \, d\rho_{k_{\mathbb{V}}^{\mathbb{I}+\mathbb{II}}} - \rho_{k_{\mathbb{V}}^{\mathbb{I}+\mathbb{II}}} \mathcal{M} = \lim_{u \to \infty} \left( \sum_{u \in \mathcal{U}} \left( \int_{L_p} \left( f_1 \cdot f_2 \right) \left( x^u \right) - 1 \right) \cdot \rho_{1}(1^u) \right) \] 
\[= \lim_{u \to \infty} \left( \sum_{u \in \mathcal{U}} \left( \int_{L_p} \left( \frac{f_1}{f_{0^u}} \right) \left( x^u \right) - 1 \right) \cdot \rho_{1}(1^u) \right) \] 
\[= \int_{L_p} f_1 \, d\rho_{k_1} - \rho_{k_1} \mathcal{M} + \int_{L_p} f_2 \, d\rho_{k_2} - \rho_{k_2} \mathcal{M}.\]

Finally, result follows from the fact that \(\rho_{k_i} \mathcal{M} = y \mathcal{M}\). Indeed, under the group isomorphism \(\hat{O}_{F_p}^\mathfrak{h} \cong \mathbb{I}_{\mathbb{V}}^2\) the specialization \(\rho_{k_i} \mathcal{M}\) on \(I_{\mathbb{I}}/I_{\mathbb{V}}^2\) corresponds to raising to the \(y\)-th power on \(\hat{O}_{F_p}^\mathfrak{h}\).

\[\square\]

**Remark 7.8.** Notice that, since \(O_{F_p}^\mathfrak{h} = \mu_{q_{\mathbb{V}} - 1} \cdot (1 + p)\), where \(\mu_{q_{\mathbb{V}} - 1}\) are the \((q_{\mathbb{V}} - 1)\)-th roots of unity, we have

\[\hat{O}_{F_p}^\mathfrak{h} = O_{F_p}^\mathfrak{h} \otimes_{\mathcal{O}_F} \mathbb{Z}_{\mathbb{V}} \cong 1 + p \subset O_{F_p}^\mathfrak{h} \hookrightarrow \Lambda_{\mathbb{V}}^\mathfrak{h}.\]

Thus, we can think of \(I_{\mathbb{V}}\) and the functions in \(\mathcal{E}(I_{\mathbb{V}})\) as having values in \(\hat{O}_{F_p}^\mathfrak{h} \subset \Lambda_{\mathbb{V}}^\mathfrak{h}\).

**Lemma 7.9.** We have a well defined \(G(F_p)\)-equivariant morphism

\[r_p : \mathcal{D}_{k_{\mathbb{V}}^{\mathbb{I}+\mathbb{II}}} \rightarrow \text{Hom}_{\mathcal{O}_F}(\mathcal{E}(I_{\mathbb{V}}), I_{\mathbb{V}}/I_{\mathbb{V}}^2); \quad r_p(\mu)\phi, y = \int_{L_p} \phi \left( \frac{a}{c} \frac{b}{d} \right) \, d\rho_{k_{\mathbb{V}}^{\mathfrak{h}}} \mu(c, d) \mod I_{\mathbb{V}}^2,\]

where \(\left( \frac{a}{c} \frac{b}{d} \right) \in \text{GL}_2(O_{F_p})\).

**Proof.** The morphism is well defined since for any other \(\left( \frac{a'}{c'} \frac{b'}{d'} \right) \in \text{GL}_2(O_{F_p})\) there exist \(\left( \frac{\mathbb{I}'}{\mathbb{V}} \right) \in \text{GL}_2(O_{F_p})\) such that \(\left( \frac{a}{c} \frac{b}{d} \right) = \left( \frac{\mathbb{V}}{\mathbb{I}} \right) \left( \frac{\mathbb{I}'}{\mathbb{V}} \right)\). Moreover \(r_p(\mu)\) is a group homomorphism by Lemma 7.7.

For any \((\phi, y) \in \mathcal{E}(I_{\mathbb{V}})\), we write \(\hat{\phi}(c, d) = \phi (\frac{a}{c} \frac{b}{d}) \in \mathcal{O}_{k_{\mathbb{V}}^{\mathbb{I}+\mathbb{II}}}(L_{\mathbb{V}}, O_{F_p})\). If we fix \((c, d) \in L_{\mathbb{V}}\), and \(g \in G(F_p)\), we choose \(k \in \text{GL}_2(O_{F_p})\) such that \(\left( \frac{a}{c} \frac{b}{d} \right) \in \mathcal{E}(I_{\mathbb{V}}), \) \((a, c, g) = (x_1, y_2) + k\) for some \(x_1, y, x_2 \in F_p\). Thus, we obtain

\[(\hat{\phi}g)(c, d) = (\hat{\phi})^{-1}(g \hat{\phi}) (c, d) = g \hat{\phi} (\hat{\phi})^{-1} (a \frac{b}{d} \frac{c}{d}) = \phi (\frac{a}{c} \frac{b}{d}) = \phi (\frac{a}{c} \frac{b}{d}).\]

Since \(\rho_{1} \hat{\phi}(\text{det}(g)) = 1\), we obtain,

\[r_p(g\phi)(c, d) = \int_{L_p} \phi \left( \frac{a}{c} \frac{b}{d} \right) \, d\rho_{k_{\mathbb{V}}^{\mathfrak{h}}} \mu(c, d) = \int_{L_p} (g^{-1}) \phi(c, d) \, d\rho_{k_{\mathbb{V}}^{\mathfrak{h}}} \mu(c, d)\]

Hence the \(G(F_p)\)-equivariance follows.

\[\square\]

### 7.3 Relation between Iwasawa algebras

At the beginning of §7 we have introduced the general Iwasawa algebra \(\Lambda_{\mathbb{V}}\), and in §7.2 we have defined the \(p\)-Iwasawa algebra \(\Lambda_{\mathbb{V}}\). In this section we will explore relations between both.
Since we have a natural continuous character
\[ \mathcal{O}_F^\chi \longrightarrow \mathcal{O}_F^\chi K_p \longrightarrow \Lambda_F^\chi, \]
we obtain using the universal property of \( \Lambda_F \) another algebra morphism \( \varphi_p : \Lambda_F \rightarrow \Lambda_F \). Similarly, the character
\[ \mathcal{O}_F^\chi \longrightarrow \mathcal{O}_F^\chi K_p \longrightarrow \Lambda_F^\chi, \]
provides another algebra morphism \( d_p : \Lambda_F \rightarrow \Lambda_F \). It is clear that \( \varphi_p \) is a section of \( d_p \), namely, \( d_p \circ \varphi_p = \text{id} \). The existence of such sections implies that \( \Lambda_F^\chi \simeq \prod_{\mathfrak{p} \neq \mathfrak{p}} \Lambda_F^\chi \).

Notice that if \( I \subset \Lambda_F \) is the augmentation ideal
\[ I = \ker \left( \Lambda_F \longrightarrow \mathbb{Z}_p \right), \]
then \( \varphi_p(I_p) \subset I \). Indeed, the morphism \( \Lambda_F \rightarrow \Lambda_F \) is natural, since we have a natural continuous character \( \lambda \). This will imply that the induced morphism
\[ \varphi_p : \Lambda_F \rightarrow \Lambda_F \]
is \( G(F_p) \)-equivariant. In fact, under our assumption
\[ \hat{\lambda}_p(x) = \prod_{\mathfrak{p} \neq \mathfrak{p}} \varphi_p \left( \hat{\lambda}_p(x_\mathfrak{p}) \right), \quad \text{for all } x = (x_\mathfrak{p}) \in F_F^\chi. \]

For any set \( S \) of primes dividing \( p \) and any \( n_S = (n_\mathfrak{p})_{\mathfrak{p} \in S} \in \mathbb{Z}^S \), write
\[ k_p^{n_S} : \mathcal{O}_F^\chi \longrightarrow \prod_{\mathfrak{p} \in S} \mathcal{O}_F^\chi \longrightarrow \prod_{\mathfrak{p} \in S} \mathcal{O}_F^\chi K_p \longrightarrow \Lambda_F^\chi. \]
Thus, we can consider the subspaces
\[ \mathbb{D}_{k_p^2}(\Lambda_F)^S = \{ \mu \in \Lambda_F^\chi : \rho_p^{k_p^2} \mu = 0, \text{ for all } \mathfrak{p} \in S \}, \]
where \( 1_p \in \mathbb{Z}^{(\mathfrak{p}, \mathfrak{p})} \) is the element with all components 1. By Lemma 7.5, if we assume that \( \rho_1(a_\mathfrak{p}) = 1 \) for all \( \mathfrak{p} \in S \), this defines a \( G(F_p) \)-invariant subspace denoted by \( \mathbb{D}_{k_p^2}(\Lambda_F)^S_{\mathcal{A}} \).

**Proposition 7.10.** Write \( \rho_1 \hat{\lambda}_p = (\hat{\lambda}_p)_p \) and assume that \( \hat{\lambda}_p = 1 \) for all \( \mathfrak{p} \in S \) (equivalently \( \rho_1(a_\mathfrak{p}) = 1 \)). We have a well defined \( G(F_p) \)-equivariant morphism
\[ r_S : \mathbb{D}_{k_p^2}(\Lambda_F)^S_{\mathcal{A}} \longrightarrow \text{Hom}_{\mathcal{A}} \left( \bigotimes_{\mathfrak{p} \in S} \text{Ind}_{\mathfrak{p}}^F(\hat{\lambda}_p) \otimes \bigotimes_{\mathfrak{p} \in S} \mathcal{E}(\hat{\lambda}_p^2), \mathfrak{I}^{r+1} \right); \]
\[ r_S(\mu) \left( \bigotimes_{\mathfrak{p} \in S} \phi_{\mu}^{(f)}(c_\mathfrak{p}^{(f)}) \otimes \bigotimes_{\mathfrak{p} \in S} \phi_{\mu}^{(y)}(c_\mathfrak{p}^{(y)}) \right) = \int_{\mathcal{L}_F} \prod_{\mathfrak{p} \in S} f_\mathfrak{p}(c_\mathfrak{p}, d_\mathfrak{p}) \prod_{\mathfrak{p} \in S} \phi_\mathfrak{p}^{(a_\mathfrak{p}, b_\mathfrak{p})}(c_\mathfrak{p}) d_\mathfrak{p} \rho_{k_p^2}^\chi \mu(c, d) \mod \mathfrak{I}^{r+1}, \]
where \( (a_\mathfrak{p}, b_\mathfrak{p}) \in \text{GL}_2(\mathcal{O}_F), r = \# S \) and \( y := (y_\mathfrak{p})_{\mathfrak{p} \in S} \in \mathbb{Z}^S \).
Proof. We write \( \mathcal{L}_p := \prod_{q \notin p} \mathcal{L}_p \). Notice that we have
\[
\int_{\mathcal{L}_p} \prod_{v \in S} f_u(c_v, d_v) \prod_{v \in S} \phi_v \left( \frac{a_v}{c_v}, \frac{b_v}{d_v} \right) d\rho_{k_u-z_v}(c, d) = \int_{\mathcal{L}_p} \prod_{v \in S} f_u(c_v, d_v) \prod_{v \in S} \phi_v \left( \frac{a_v}{c_v}, \frac{b_v}{d_v} \right) d\rho_{k_u-z_v}(c, d),
\]
and it is clear by definition that \( d\rho_{k_u-z_v}(c, d) |_{L_v} \in D_{k_v}(\Lambda_v) \circ \otimes \Lambda_F \). Thus
\[
\int_{\mathcal{L}_p} \phi_v \left( \frac{a_v}{c_v}, \frac{b_v}{d_v} \right) d\rho_{k_u-z_v}(c, d) \in \iota_v(I_v)\Lambda_F \otimes \Lambda_v D_{k_v}(\Lambda_v) \circ \otimes \Lambda_F, \quad q \in S \setminus \{v\}.
\]
Applying a straightforward induction we obtain that
\[
\int_{\mathcal{L}_p} \prod_{v \in S} f_u(c_v, d_v) \prod_{v \in S} \phi_v \left( \frac{a_v}{c_v}, \frac{b_v}{d_v} \right) d\rho_{k_u-z_v}(c, d) \in \prod_{v \in S} \iota_v(I_v)\Lambda_F \subset I'.
\]
Moreover, by Lemma 7.9 the expression \( r_\Sigma \) defines a well defined \( G(F_p) \)-equivariant group homomorphism. □

Write \( \Delta_{F_p} \subset \Delta_p \) for the subgroup of \( \text{Gal}(K_p/F_p) \)-invariant divisors, namely, the even degree divisors generated by those of the form \( \tau + \bar{\tau} \). Write \( \Delta_0^{F_p} \) for the degree zero subgroup. Let
\[
\hat{\Omega}_S := \bigotimes_{p \in \mathcal{S}} \hat{\Omega}_p, \quad E(\ell_p^2) := \bigotimes_{p \in \mathcal{S}} E(\ell_p^2), \quad \hat{\mathcal{F}}^\times := \bigotimes_{p \in \mathcal{S}} (F_p^0 \otimes \mathbb{Z}_p), \quad \hat{\mathcal{K}}^\times := \bigotimes_{p \in \mathcal{S}} (K_p^0 \otimes \mathbb{Z}_p),
\]
where tensor products are taken with respect to \( \mathbb{Z}_p \), and
\[
\Delta_S := \bigotimes_{p \in \mathcal{S}} \Delta_p, \quad \Delta_0^0 := \bigotimes_{p \in \mathcal{S}} \Delta_0^0, \quad \Delta_F := \bigotimes_{p \in \mathcal{S}} \Delta_F, \quad \Delta_0^F := \bigotimes_{p \in \mathcal{S}} \Delta_0^F, \quad \mathcal{S}_F := \bigotimes_{p \in \mathcal{S}} \mathcal{S}_F, \quad \mathcal{E}_F := \bigotimes_{p \in \mathcal{S}} \mathcal{E}_F,
\]
where the tensor products are with respect to \( \mathbb{Z} \). Notice that, since \( \hat{\Omega}_p^\times \simeq I_p/I_p^2 \), we have a well defined morphism
\[
\hat{\Omega}_S^\times \to I'/I'^{+1}.
\]
This provides via \( \ell_\alpha : F^\times_p \to \hat{\Omega}_p^\times \) a morphism
\[
\ell_\alpha : \hat{\mathcal{F}}^\times_S \otimes \ell_p \to \hat{\Omega}_S^\times \to I'/I'^{+1}. \tag{37}
\]
Note that \( \varphi_{uv} \) and \( ev \) of (33) and (36) extend to
\[
\varphi_{uv} : \text{Hom}(\text{St}_{Z_p}(F_p), \mathbb{Z}_p) \to \text{Hom}(\text{St}_{Z_p}^\times, \hat{\mathcal{F}}^\times_S), \quad ev_S : \Delta_S \to \mathcal{E}_F^\times,
\]
Similarly as in (32), the composition \( ev_S \circ \varphi_{uv} \) provides a morphism
\[
\text{Hom}(\text{St}_{Z_p}(F_p), \mathbb{Z}_p) \to \text{Hom}(\Delta_0^F, \hat{\mathcal{K}}^\times_S)
\]
which extends to
\[
\psi \longmapsto \left( \bigotimes_{p \in \mathcal{S}} (z_{2,p}-z_{1,p}) \mapsto \lim_{\{U_p\}_p \in \text{Cov}(\mathbb{P}^1(F_p))} \bigotimes_{p \in \mathcal{S}} \prod_{U_p \in \mathcal{U}_p} (z_{1,p}-z_{2,p}) \right) \psi(\ell_{U_p}(z_{2,p}-z_{1,p})), \tag{38}
\]
where \( \text{Cov}(\mathbb{P}^1(F_p)) := \prod_{p \in \mathcal{S}} \text{Cov}(\mathbb{P}^1(F_p)) \) and each \( x_{U_p} \in U_p \). For any \( z_p \in \Delta_p \), we recall the functions \( \varphi_{z_p} : G(F_p) \to K_p \) of (36).

Proposition 7.11. Write \( \ell_\alpha \hat{\mathcal{K}}_p = (\hat{\mathcal{K}}_p)_p, V^S = \bigotimes_{p \in \mathcal{S}} \text{Ind}_{\mathcal{Q}_p}(\hat{\mathcal{K}}_p), \) and assume that \( \hat{\mathcal{K}}_p = 1 \) for all \( p \in \mathcal{S} \). We have \( G(F_p) \)-equivariant morphisms
\[
\varphi_{uv} : \text{Hom}(V^S \otimes E(\ell_p^2), I'/I'^{+1}) \to \text{Hom}(V^S \otimes \Delta_F, I'/I'^{+1}), \quad \varphi \longmapsto \left( V^S \otimes \bigotimes_{p \in \mathcal{S}} (z_p + z_{p'}) \mapsto \varphi \left( V^S \otimes \bigotimes_{p \in \mathcal{S}} (\ell_\alpha(\varphi_{z_p} \cdot \varphi_{z_p})), 1 \right) \right),
\]
making the following diagram commutative.
\[
\begin{align*}
\text{Hom}(V^S \otimes \Delta_{F_p}, \mathcal{I}'/\mathcal{I}'^{+}) & \xrightarrow{\epsilon_{\text{unv}} \circ \epsilon_{V, \text{unv}}} \text{Hom}(V^S \otimes \Delta_{F_p}, \mathcal{I}'/\mathcal{I}'^{+}) \\
\mathbb{D}_{k'(\Lambda_F)^S} & \xrightarrow{\rho_1} \text{Hom}_{\mathcal{I}_p}(V^S \otimes \text{St}_{\mathcal{I}_p}(F_p), \mathbb{Z}_p) & \text{ev}_{\text{unv}} \circ \epsilon_{V, \text{unv}} \rightarrow \text{Hom}(V^S \otimes \Delta_{F_p}'', \tilde{\mathcal{I}}_p') \end{align*}
\]

**Proof.** The \(G(F_p)\)-equivariance of \(\epsilon_{V, \text{unv}}\) is clear by the definitions. Given \(\mu \in \mathbb{D}_{k'(\Lambda_F)^S}\), we compute

\[
\begin{align*}
\ell_{\mathbf{a}} \left( (\epsilon_{V, \text{unv}} \circ \rho_1)(\mu) \left( V^S \otimes \bigotimes_{\mathbf{p} \in S} (z_{\mathbf{p}}, -z_{\mathbf{p}} + \bar{z}_{\mathbf{p}}, 1 - z_{\mathbf{p}}, 2) \right) \right) &= \\
= \lim_{(\mathcal{U}_\mathbf{p}) \in \text{Conv}(\mathbb{P}^1(F_p))} \left( \prod_{\mathbf{p} \in S} \left( \frac{\phi_{\mathbf{p}}(z_{\mathbf{p}}, -z_{\mathbf{p}} + \bar{z}_{\mathbf{p}}, 1 - z_{\mathbf{p}}, 2)}{\phi_{\mathbf{p}}(z_{\mathbf{p}}, -z_{\mathbf{p}} + \bar{z}_{\mathbf{p}}, 1 - z_{\mathbf{p}}, 2)} \left( x_{\mathcal{U}_\mathbf{p}} \right)^{\rho_1(\mu)(V^S \otimes 1_{\mathcal{U}_\mathbf{p}})} \right) \mod I'^{+1} \\
= \int_{\mathbb{P}^1(F_p)} \prod_{\mathbf{p} \in S} \ell_{\mathbf{a}} \left( \frac{\phi_{\mathbf{p}}(z_{\mathbf{p}}, -z_{\mathbf{p}} + \bar{z}_{\mathbf{p}}, 1 - z_{\mathbf{p}}, 2)}{\phi_{\mathbf{p}}(z_{\mathbf{p}}, -z_{\mathbf{p}} + \bar{z}_{\mathbf{p}}, 1 - z_{\mathbf{p}}, 2)} \left( x_{\mathcal{U}_\mathbf{p}} \right)^{\rho_1(\mu)(V^S \otimes 1_{\mathcal{U}_\mathbf{p}})} \right) d\rho_1(\mu)(V^S) \mod I'^{+1} \\
= r_5(\mu) \left( V^S \otimes \bigotimes_{\mathbf{p} \in S} \left( \ell_{\mathbf{a}} \left( \frac{\phi_{\mathbf{p}}(z_{\mathbf{p}}, -z_{\mathbf{p}} + \bar{z}_{\mathbf{p}}, 1 - z_{\mathbf{p}}, 2)}{\phi_{\mathbf{p}}(z_{\mathbf{p}}, -z_{\mathbf{p}} + \bar{z}_{\mathbf{p}}, 1 - z_{\mathbf{p}}, 2)} \right), 0 \right) \right) = \epsilon_{V, \text{unv}} \circ r_5(\mu) \left( V^S \otimes \bigotimes_{\mathbf{p} \in S} (z_{\mathbf{p}}, -z_{\mathbf{p}} + \bar{z}_{\mathbf{p}}, 1 - z_{\mathbf{p}}, 2) \right).
\end{align*}
\]

Hence, we obtain the required commutativity of the diagram. \(\Box\)

### 7.4 \(p\)-adic periods and L-invariants

Fix a prime \(p \mid p\) and let \(\phi^p_\lambda \in H^2(G(F), \mathcal{A}^{\text{pl,jo}}(S_{\mathcal{I}_p}(F_p), \mathbb{Z}))\) be a modular symbol associated with an elliptic curve \(E/F\) with multiplicative reduction at \(p\). The short exact sequence

\[
0 \rightarrow \Delta_p^0 \rightarrow \Delta_p \rightarrow \mathbb{Z} \rightarrow 0,
\]

provides a connection morphism

\[
H^2(G(F)_p, \mathcal{A}^{\text{pl,jo}}(\Delta_p^0, \mathbb{K}_p^\times)) \rightarrow H^{2+1}(G(F)_p, \mathcal{A}^{\text{pl,jo}}(\mathbb{K}_p^\times))\).
\]

Moreover, the commutative diagram (35) shows that

\[
c(\epsilon_{V^S} \circ \rho_{\text{unv}}) \phi^p_\lambda \in H^{2+1}(G(F)_p, \mathcal{A}^{\text{pl,jo}}(q_{F_p}^Z)) \subset H^{2+1}(G(F)_p, \mathcal{A}^{\text{pl,jo}}(F_p))\),
\]

for some \(q_p \in F_p^\times\). The generalized Oda’s conjecture ([15, Conjecture 3.8]) asserts that, in fact, the elliptic curve \(E/F_p\) is isogenous to the Tate curve defined by the quotient \(\tilde{F}_p/q_pZ\).

**Theorem 7.12.** Let \(\Phi^p_\lambda \in H^2(G(F)_p, \mathcal{A}^{\text{pl,jo}}(\mathbb{D}_{k'(\Lambda_F)}))\) be a family lifting \(\phi^p_\lambda\). Then

\[
\ell_{\mathbf{a}}(q_p) = 1 \in \hat{\mathcal{O}}_{F_p}^\times.
\]

**Proof.** Since \(\Phi^p_\lambda\) specializes to the Steinberg representation, we have in fact

\[
\Phi^p_\lambda \in H^2(G(F)_p, \mathcal{A}^{\text{pl,jo}}(\mathbb{D}_{k'(\Lambda_F)}))\).
\]

If we consider the exact sequence

\[
H^2(G(F)_p, \mathcal{A}^{\text{pl,jo}}(\Delta_{F_p}, F_p)) \rightarrow \text{res} \rightarrow H^2(G(F)_p, \mathcal{A}^{\text{pl,jo}}(\Delta_{F_p}^0, F_p)) \rightarrow \text{res} \rightarrow H^{2+1}(G(F)_p, \mathcal{A}^{\text{pl,jo}}(F_p^\times))\),
\]

by Proposition 7.11,

\[
\text{res}(\epsilon_{V^S} \circ \rho_p) \Phi^p_\lambda = \ell_{\mathbf{a}}(\epsilon_{V^S} \circ \rho_{\text{unv}}) \Phi^p_\lambda |_{\Delta_p^0}.
\]

This implies that

\[
\ell_{\mathbf{a}}(\epsilon_{V^S} \circ \rho_{\text{unv}}) \Phi^p_\lambda |_{\Delta_p^0} = 0.
\]

Since \(c(\epsilon_{V^S} \circ \rho_{\text{unv}}) \Phi^p_\lambda |_{\Delta_p^0}\) lies in \(H^{2+1}(G(F)_p, \mathcal{A}^{\text{pl,jo}}(q_{F_p}^Z))\), we deduce \(\ell_{\mathbf{a}}(q_p) = 1\) from the fact that \(\hat{\mathcal{O}}_{F_p}^\times\) is torsion free. \(\Box\)
Remark 7.13. As showed in §7.1, the elements $a_p \in \Lambda_F$ are the eigenvalues of the $U_p$-operators acting on the Hida family. By definition

$$\ell_{a_p}(\omega_p) = a_p^{-1} - 1 \in I_p/I_p^2.$$ 

Thus, the relation $\ell_{a_p}(q_p) = 1$ implies

$$\text{ord}_p(q_p)(a_p^{-1} - 1) + \log_p(q_p) \equiv 0 \mod I_p^2.$$ 

We obtain that the image of $a_p - 1$ in $I_p/I_p^2$ is given by the $L$-invariant $L_p = \log_q(q_p)/\text{ord}_p(q_p)$. With the formalism previously described, we have showed that the derivative of $a_p - 1$ with respect to the weight variable is given by the $L$-invariant $L_p$. In the classical setting, this is a key result due to Greenberg-Stevens towards the exceptional zero conjecture (see [13]).

8 Plectic points and weight $p$-adic L-functions

8.1 Plectic points

Let $\phi^S_A \in H^u(G(F), S^u_{\text{Log}}(V^\mathbb{Z}_p, \mathbb{Z})^1$ be the modular symbol associated with an elliptic curve $E/F$, and assume that all primes $p \in S$ are non-split at $T$ the representation $V_p^\mathbb{Z}$ is $\text{St}_{2}(F_p)(\epsilon_p)$, the twist of $\text{St}_{2}(F_p)$ by the character $\epsilon_p : G(F_p) \to \{\pm 1\}$ given by $g \mapsto (\pm 1)^{v_p(\text{det} g)}$. Notice that the restriction $\epsilon_p : T(F_p) \to \{\pm 1\}$ is non trivial only if $\epsilon_p \neq 0$ and $T$ ramifies at $p$.

We observe that, for all $p \in S$, morphism (32) gives rise to an analogous $G(F_p)$-invariant morphism

$$i_p : \text{Hom}(\text{St}_2(F_p)(\epsilon_p), \mathbb{Z}) \to \text{Hom}(\Lambda^0_\mathbb{Z}, K^p_{\mathbb{Z}}(\epsilon_p),$$

The composition $\text{ev}^* \circ q_{\text{univ}}$ of (38) provides a morphism

$$\text{ev}^* \circ q_{\text{univ}} : \text{Hom}(\text{St}_{2}(F_p)(\epsilon_p), \mathbb{Z}_p) \to \text{Hom}(\Lambda^0_\mathbb{Z}, \hat{K}^p_{\mathbb{Z}}(\epsilon_p),$$

where $\epsilon_S = \prod_{p \in S} \epsilon_p : G(F_S) \to \{\pm 1\}$.

If we write $H_{v_p}/K_p$ for the extension cut out by $\epsilon_p$, by [25, Corollary 5.4]

$$E(K_p) = \left\{ u \in H_{v_p} : u^{-v_p(\lambda)}u\tau \in q_p^\mathbb{Z} \right\}, \quad 1 \neq \tau \in \text{Gal}(H_{v_p}/K_p).$$

(39)

Hence, we no longer have a Tate uniformization $K^p_v/q_p^\mathbb{Z} \sim E(K_p)$ but an isomorphism

$$K^p_v/q_p^\mathbb{Z} \sim E(K_p)_{v_p} = \{ P \in E(H_{v_p}) : P^\tau = \epsilon_p(\tau) \cdot P \}.$$ 

By Hypothesis, $V^\mathbb{Z}_p = \text{St}_{2}(F_p)(\epsilon_S)$. Hence, given a class $\phi^S_A \in H^u(G(F), S^u_{\text{Log}}(V^\mathbb{Z}_p, \mathbb{Z}_p))^1$ generating our automorphic representation $\rho \mid_{(G(F)_{v_p})}$ we can consider

$$\text{ev}^* \circ q_{\text{univ}}(\phi^S_A) \in H^u(G(F), S^u_{\text{Log}}(\Lambda^0_\mathbb{Z}, \hat{K}^p_{\mathbb{Z}})(\epsilon_S))^1,$$

where $\epsilon_S$ is now seen as a character of $G(F)$ by means of the composition $G(F) \hookrightarrow G(F_S) \twoheadrightarrow \{\pm 1\}$. Fornea and Gehrmann show in [9] (see also [16]) that this class can be uniquely extended to a class

$$\psi^S_A \in H^u(G(F), S^u_{\text{Log}}(\Lambda_S, \hat{E}(K_S)_{v_p})(\epsilon_S))^1,$$

$$\hat{E}(K_S)_{v_p} := \bigotimes_{p \in S} (E(K_p)_{v_p} \otimes_{\mathbb{Z}} \mathbb{Z}_p) = \bigotimes_{p \in S} \hat{K}_p^v \otimes_{q_p^\mathbb{Z}},$$

generating $\rho \mid_{(G(F)_{v_p})}$.

For all $p \in S$, let $\tau_p \in S_p$ be the point fixed by $T(F)$ as in §5.3. Notice that we have a well-defined morphism of $G(F)_v$-modules

$$\mathbb{Z}[G(F)/T(F)] \to \Lambda_S; \quad n \cdot gT(F) \mapsto n \left( \bigotimes_{p \in S} g\tau_p \right).$$

(40)

It induces a $G(F)_v$-equivariant morphism

$$\mathcal{A}^u_{\text{Log}}(\Lambda_S, \hat{E}(K_S)_{v_p})(\epsilon_S) \to \mathcal{A}^u_{\text{Log}}(\mathbb{Z}[G(F)/T(F)], \hat{E}(K_S)_{v_p})(\epsilon_S), \quad \psi \mapsto \psi \mid_{(G(S)_p).}$$
Since we have (see [15, Lemma 4.1])
\[ \mathcal{A}_{S}^{\text{LooS}}(\mathbb{Z}[G(F)_{+}, T(F)_{+}], \hat{E}(K_{S})_{\varepsilon_{S}})(\varepsilon_{S}) = \text{Ind}_{T(F)_{+}}^{G(F)_{+}}(\mathcal{A}_{S}^{\text{LooS}}(\hat{E}(K_{S})_{\varepsilon_{S}}(\varepsilon_{S}))), \]
we obtain by Shapiro’s lemma
\[ \psi_{\Lambda}^{S} \mid_{T_{\varepsilon_{S}}} \in H^{u}(T(F)_{+}, \mathcal{A}_{S}^{\text{LooS}}(\hat{E}(K_{S})_{\varepsilon_{S}})(\varepsilon_{S}))^{\Lambda}. \]

Let us consider now the subspace of functions
\[ C(\mathcal{G}_{T}, \mathbb{Q})^{\varepsilon_{S}} := \{ f \in C(\mathcal{G}_{T}, \mathbb{Q}), \rho f \mid_{T(F)_{\varepsilon_{S}}} = \varepsilon_{S} \}, \]
where as above \( \varepsilon_{S} : T(F)_{+} \rightarrow \pm 1 \) is the product of the local \( \varepsilon_{p} \). In this situation, the Artin map provides a decomposition:
\[ C(\mathcal{G}_{T}, \mathbb{Q})^{\varepsilon_{S}} = \bigoplus_{\lambda, \varepsilon_{S} = \pm 1} H^{0}(T(F)_{+}, C^{0}(T, \mathcal{A}_{T}^{\text{LooS}}(\mathbb{Q})(\varepsilon_{S}))), \tag{41} \]

Given a locally constant character \( \xi \in C(\mathcal{G}_{T}, \mathbb{Q})^{\varepsilon_{S}} \) we can consider \( \xi_{\lambda} \in H^{0}(T(F)_{+}, C^{0}(T, \mathcal{A}_{T}^{\text{LooS}}(\mathbb{Q})(\varepsilon_{S}))) \) its \( \lambda \)-component. Then we can define a twisted plectic point
\[ p_{\xi}^{S} = (\eta \cap \xi_{\lambda}) \cap \psi_{\Lambda}^{S} \mid_{T_{\varepsilon_{S}}} \in \hat{E}(K_{S})_{\varepsilon_{S}} \otimes_{\mathbb{Z}} \mathbb{Z}, \]
where again the cap product is with respect to the pairing \( (\cdot, \cdot)_{\varepsilon} \) of (9) twisted by \( \varepsilon_{S} \).

### 8.2 2 variable \( p \)-adic L-functions

Let \( \phi_{\Lambda}^{p} \in H^{u}(G(F)_{+}, \mathcal{A}_{S}^{\text{LooS}}(V_{p}, V(\hat{k}_{p})))^{\Lambda} \) be an ordinary cohomology class generating an automorphic representation \( \pi \). By Proposition 6.6, \( \phi_{\Lambda}^{p} \) provides an overconvergent cohomology class
\[ \delta_{\Lambda}^{p} \in H^{u}(G(F)_{+}, \mathcal{A}_{S}^{\text{LooS}}(\mathbb{D}_{\chi_{p}^{-2}}(C_{p}\hat{k}_{p})))^{\Lambda}, \]
for some locally polynomial character \( \chi_{p}^{-1} \). Assume that we have the extension \( \hat{k}_{p} \) of the universal character \( k_{p} \) and an element
\[ \Phi_{\Lambda}^{p} \in H^{u}(G(F)_{+}, \mathcal{A}_{S}^{\text{LooS}}(\mathbb{D}_{k_{p}^{-2}}(\Lambda_{T}_{p})))^{\Lambda}, \]

mapping to \( \phi_{\Lambda}^{p} \) through the morphism \( \rho_{\lambda_{p}} \) of (30). Similarly as in Theorem 6.7, we can define the distribution \( \mu_{\phi_{\Lambda}^{p}} \) as
\[ \int_{\mathcal{G}_{T}} f d\mu_{\phi_{\Lambda}^{p}} = (\langle \rho_{p}^{f} \cap \eta \rangle \cap \delta_{\Lambda}^{p}(\Phi_{\Lambda}^{p})), \quad f \in C(\mathcal{G}_{T}, \Lambda_{F}), \]
where \( \delta_{p} \) is defined as in (20):
\[ \delta_{p} : C_{c}(T(F_{p}), \Lambda_{F}) \rightarrow \text{Ind}_{T_{p}}^{C_{c}}(\mathcal{G}_{p}) \cong C_{k_{p}^{-2}}(\mathcal{L}_{p}, \Lambda_{F}). \]

Indeed, we can think \( \delta_{\Lambda}^{p}(\Phi_{\Lambda}^{p}) \) and \( \rho_{p}^{f} \cap \eta \) as elements
\[ \delta_{\Lambda}^{p}(\Phi_{\Lambda}^{p}) \in H^{u}(G(F)_{+}, \mathcal{A}_{S}^{\text{LooS}}(\text{Meas}(T(F_{p}), \Lambda_{F})))^{\Lambda}, \]
\[ \rho_{p}^{f} \cap \eta \in H_{u}(T(F), C_{c}(T, \Lambda_{T}), \Lambda_{F}) = H^{u}(T(F), \text{Ind}_{T(F)_{+}}^{T(F)_{+}} C^{0}(T, \mathcal{A}_{T}^{\text{LooS}}(\mathbb{Q})(\varepsilon_{S}))), \]

hence the pairing (10) applies and the cap-product is well defined. The measure \( \mu_{\phi_{\Lambda}^{p}} \) is considered as the 2-variable \( p \)-adic L-function since its specialization at different weights \( \theta_{p} : \mathcal{O}_{F}^{\varepsilon} \rightarrow \mathcal{C}_{p} \) provides different measures \( \rho_{\theta_{p}}(\mu_{\phi_{\Lambda}^{p}}) \) of \( \mathcal{G}_{T} \). In particular
\[ \rho_{\lambda_{p}}(\mu_{\phi_{\Lambda}^{p}}) = \mu_{\phi_{\Lambda}^{p}}, \]

by Theorem 6.7. Thus \( \mu_{\phi_{\Lambda}^{p}} \) interpolates \( \mu_{\phi_{\Lambda}^{p}} \) as the weight varies.
8.3 Hida-Rankin $p$-adic L-function

Let $\xi : G_T \to \mathbb{Z}^\times$ be a locally constant character such that $\rho^* \xi \mid_{\Gamma(F_p)} = \lambda$. Using the natural ring homomorphism $\mathbb{Z} \subset \Lambda_F \otimes \mathbb{Z}$, it can be seen as a function in $C(G_T, \Lambda_F \otimes \mathbb{Z})$. We consider

$$L_p(\phi^p_\Lambda, \xi, k) := \int_{G_T} \xi d\mu_{\phi^p_\Lambda} \in \Lambda_F \otimes \mathbb{Z}. $$

For any weight $\theta_p \in \Lambda_F(C_p) = \text{Hom}(\Lambda_F, C_p)$, we write

$$L_p(\phi^p_\Lambda, \xi, \theta_p) := \rho_{\theta_p}(L_p(\phi^p_\Lambda, \xi, k)) \in C_p.$$

We can think of $L_p(\phi^p_\Lambda, \xi, k)$ as a restriction of the two variable $p$-adic L-function $\mu_{\phi^p_\Lambda}$ to the weight variable.

Assume that $\hat{\phi}^p_\Lambda = \rho_1(\Phi^p_\Lambda)$ is associated with an elliptic curve $E/F$. Similarly as in §8.1, let $S$ be a set of primes $p$ above $\mathfrak{p}$ such that:

- $T$ does not split at any $p \in S$.
- The representation $V^S_p$ is $\text{St}_F(F_p)(\varepsilon_p)$ for all $p \in S$.
- We have that $\rho^* \xi \mid_{\Gamma(F_p)} = \varepsilon_S \mid_{\Gamma(F_p)}$.

As discussed in §7.4, we will assume that for any $p \in S$ one can identify $E(K_p)_{\varepsilon_p}$ with $K_p^*/q_p^S$. For any point $P \in E(K_p)_{\varepsilon_p}$, we can think its trace $P + \tilde{P} \in E(F_p)_{\varepsilon_p}$ as an element

$$P + \tilde{P} \in F_p^*/q_p^S \subset E(F_p)_{\varepsilon_p} := \{ P \in E(H_{\varepsilon_p}); P^* = \varepsilon_p(\tau) \cdot P, \tilde{P} = P \},$$

where $H_{\varepsilon_p}/K_p$ is the extension cut out by $\varepsilon_p$. Hence, by Theorem 7.12 it makes sense to consider

$$\ell_a \circ \text{Tr}(P) := \ell_a(P + \tilde{P}) \in I'/I'^{r}.$$

This implies that, given a plectic point $P^S_\xi \in \hat{E}(K_S)_{\varepsilon_S} \otimes \mathbb{Z}$, we can consider

$$\ell_a \circ \text{Tr}(P^S_\xi) := \left( \prod_{p \in S} \ell_a \circ \text{Tr}(P^S_\xi) \right) \in I'/I'^{r} \otimes \mathbb{Z},$$

where $r := \# S$.

Remark 8.1. Recall that $\phi^p_\Lambda \in H^a(G(F), \mathcal{A}^{\text{ proj}}(V_S \otimes \mathcal{V}_p, V^S_p, \mathbb{Z}_p)(\Lambda))$, where $V^S_p$ is a product of Steinberg representations and $V^S = \bigotimes_{q \notin S} V^Z_q$. Since for any $v^S \in V^S$ we have a $G(F)$-equivariant morphism

$$\mathcal{A}^{\text{ proj}}(V_S \otimes \mathcal{V}_p, V^S_p, \mathbb{Z}_p) \rightarrow \mathcal{A}^{\text{ proj}}(V^S_p, \mathbb{Z}_p); \phi \mapsto \phi(v^S),$$

with

$$\phi(v^S)(g^S)(v_S) := \phi(g^p)(g_{p|S} v^S \otimes v_S), \quad v_S \in V_S, \quad g^S = (g_{p|S}, g^p) \in G(S^{\text{proj}}),$$

we can consider $\phi^p_\Lambda(v^S) \in H^a(T(F), \mathcal{A}^{\text{ proj}}(V_S, \mathbb{Z}_p)(\Lambda))$. By means of $\phi^p_\Lambda(v^S)$ and a character $\xi$ as in §8.1, we can construct the corresponding plectic point (depending on $v^S$)

$$P^S_\xi(v^S) \in \hat{E}(K_S)_{\varepsilon_S} \otimes \mathbb{Z}.$$

The following theorem is the main result of the paper. Recall the pairing $\alpha$ of (25) and the Euler factors $\varepsilon_p(\pi_\mathfrak{p}, \varepsilon_p)$ introduced in §5.5.

Theorem 8.2. Assume that $r = \# S \neq 0$. Then we have that

$$L_p(\phi^p_\Lambda, \xi, k) \in I' \otimes \mathbb{Q}.$$ 

Moreover,

$$L_p(\phi^p_\Lambda, \xi, k) \equiv \frac{(-1)^r}{[O^\zeta : \mathbb{Z}]} \cdot \prod_{q \in S} \varepsilon_q(\pi_q, \xi_q) \cdot \ell_{a_q} \circ \text{Tr}(P^S_\xi(v^S_0)) \pmod{I'^{r+1} \otimes \mathbb{Q}},$$

where $v^S_0 \in V^S$ is such that $\alpha(v^S_{0,a}) = 1$ for all $q \in p \setminus S$. 

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Proof. Notice that, since $\Phi_\lambda^p \in H^u(G(F), \mathcal{A}^{p,\text{loc}}(\mathbb{D}_{k_\varphi^2}(\Lambda_T),\lambda)(\lambda))$ is such that

$$\rho_1(\Phi_\lambda^p) \in H^u(G(F), \mathcal{A}^{p,\text{loc}}(V^S \otimes_{Z_p} \text{St}_{Z_p}(F_\lambda(\mathbb{S}), Z_p)(\lambda)); V^S = \bigotimes_{\alpha \in S} V_\alpha,$$

the family lies in fact in

$$\Phi_\lambda^p \in H^u(G(F), \mathcal{A}^{p,\text{loc}}(\mathbb{D}_{k_\varphi^2}(\Lambda_T),\lambda)(\lambda)) = \bigotimes_{\alpha \in S} \hat{\varphi}_\lambda^p,$$

Hence we can apply $\Phi_\lambda^p$ where

$$\hat{\varphi}^p = (\varphi^p_{\lambda_1}) \cdot a_v,$$

and

$$\Delta = \mathbb{Z}_\varnothing \cup \infty$$

is the set of degree zero divisors in $\Delta_T$. Thus, for all $v^S \in V^S$ we obtain the following diagram for $M = \hat{P}_s^\varphi$ or $M = \bigotimes_{\alpha \in S} \hat{P}_s^\varphi/\varpi_{Z_p}^\varphi \subseteq \bigotimes_{\alpha \in S} \hat{E}(F_\lambda)_s = \hat{E}(F_\lambda)_s$

$$\phi^p_\lambda(v^S) \in H^u(G(F), \mathcal{A}^{s,\text{loc}}(\text{St}_{Z_p}(F_\lambda(\mathbb{S}), Z_p)(\lambda))$$

Let $\Delta_T := \mathbb{Z}[G(F)/T(F)]$. Since $\lambda_s$ and $T_\lambda$ are $T(F)$-invariant, one can construct a $G(F)$-module morphism

$$\Delta_T \xleftarrow{\Delta_{F_\lambda}} \Delta_{F_\lambda} = \bigotimes_{\alpha \in S} \Delta_{F_\lambda} \subseteq \bigotimes_{\alpha \in S} \Delta_{F_\lambda}$$

The morphism $\text{ev}^s_\lambda \circ \varphi_{\text{univ}}$ restricts to a $G(F)$-equivariant morphism $\text{Hom}(\text{St}_{Z_p}(F_\lambda(\mathbb{S}), Z_p)) \to \text{Hom}(\text{St}_{Z_p}(F_\lambda(\mathbb{S}), Z_p))$ where $\Delta^0_T$ is the set of degree zero divisors in $\Delta_T$. Thus, for all $v^S \in V^S$ we obtain the following diagram for $M = \hat{P}_s^\varphi$ or $M = \bigotimes_{\alpha \in S} \hat{P}_s^\varphi/\varpi_{Z_p}^\varphi$, we have that $\varphi_{\text{univ}}^p \varphi_\lambda^p(v^S)$ extends to an element $\tilde{\phi}_\lambda^p(v^S) \in H^u(G(F), \mathcal{A}^{s,\text{loc}}(\text{St}_{Z_p}(F_\lambda(\mathbb{S}), Z_p)(\lambda))$. It is clear that

$$\text{Tr} \left( \tilde{P}_s^\varphi(v^S) \right) = \eta^S \cap \xi_\lambda \cap \text{ev} \mid_{\Delta_T} \tilde{\phi}_\lambda^p(v^S) \in \hat{E}(F_\lambda)_s \otimes \hat{\mathbb{Z}}.$$
\[(9): \]

\[
\langle f^S, \text{ev} |_{\Delta_r} r_S \Phi(\delta_p^S(\mathbb{I}_H)) \rangle_+ = \int_{T(\mathbb{A}_F^{p\text{-comp}})} (f_p \otimes f^p)(t) \cdot \text{ev} |_{\Delta_r} r_S \Phi(\delta_p^S(\mathbb{I}_H))(t) d^\times t
\]

\[
= \int_{T(\mathbb{A}_F^{p\text{-comp}})} f^p(z) \int_{T(F_p \mathbb{A})} f_p(x) \cdot (\sigma_S \otimes r_S)(\Phi)(z) \left( x \delta_p^S(\mathbb{I}_H) \otimes \bigotimes_{p \neq S} (\tau_p + \Upsilon_p) \right) d^\times x d^\times z
\]

\[
= \text{vol}(H) \int_{T(\mathbb{A}_F^{p\text{-comp}})} f^p(z) \cdot r_S \Phi(z) \left( \delta_p^S(f_p) \otimes \bigotimes_{p \neq S} (\ell_p(\phi_{\tau_p}, \phi_{\Upsilon_p}), 1) \right) d^\times z
\]

\[
= \text{vol}(H) \int_{T(\mathbb{A}_F^{p\text{-comp}})} f^p(z) \cdot \frac{\Delta_p^S(f_p) \cdot \prod_{p \neq S} \ell_p((c \tau_p + d)(c \bar{\tau}_p + d)) \cdot d\rho_{k_p} \Phi(z) d^\times z}{(-1)^S \sigma_S} + \text{vol}(H) \left( f^S \otimes 1_{T(f_{F_p})} \delta_p^S(\mathbb{I}_H) \right)
\]

where the fifth equality follows from Corollary 7.7, the sixth equality follows from the fact that \( k_{\ell}(x) - k_{\ell}(x) \in I^2 \) by definition of \( k_{\ell} \), and the seventh equality follows from Lemma 5.6. Hence by Lemma 3.1 and relation (12), we obtain

\[
L_p(\phi_p^S, \xi, k) = \frac{(p^S \xi \otimes 1) \otimes \delta_p^S(\Phi_p)}{[O_p^S : O_+]} = \frac{(p^S \xi \otimes 1) \otimes \delta_p^S(\Phi_p)}{[O_p^S : O_+] \cdot \text{vol}(H)} = \frac{(1)^r \cdot \text{vol}(H)}{[O_p^S : O_+] \cdot \text{vol}(H)} \cdot \ell_{\Delta_r} \circ \text{Tr} \left( p^S \xi \right).
\]

Since \( \phi_p^S(t v^S) = t \phi_p^S(v_S) \), for all \( t \in T(F_p \mathbb{A}) \), the morphism

\[
\psi^S : V^S \rightarrow [I'/I'^{r+1} \otimes \overline{Q}], \quad v^S \mapsto \ell_{\Delta_r} \circ \text{Tr} \left( p^S \xi (v^S) \right),
\]

satisfies \( \psi^S(t v^S) = p^S \xi(t)^{-1} \cdot \psi^S(v^S) \) for all \( t \in T(F_p \mathbb{A}) \). By Saito-Tunnel (see [24] and [27]) the space

\[
\text{Hom}_{T(F_p \mathbb{A})}(V^S \otimes p^S \xi, [I'/I'^{r+1} \otimes \overline{Q}])
\]

is at most one dimensional. Moreover, in §5.5 we have introduced the pairing

\[
a(v_1, v_2) = \int_{T(F_p \mathbb{A})} p^S \xi(t)(v_1, f v_2) d^\times t \in \text{Hom}_{T(F_p \mathbb{A})}(V^S \otimes p^S \xi, \overline{Q})^\otimes \approx \overline{Q}.
\]

By Saito-Tunnel, we have that

\[
\psi^S(\delta_p^S(\mathbb{I}_H)) = \psi^S(v^S_0) \cdot \alpha(\delta_p^S(\mathbb{I}_H), v^S_0).
\]

Moreover, also by Saito-Tunnel,

\[
a(v^S, \delta_p^S(\mathbb{I}_H)) = a(v^S_0, \delta_p^S(\mathbb{I}_H)) \cdot \alpha(v^S, v^S_0).
\]

From the symmetry of \( \alpha \), we deduce \( \alpha(\delta_p^S(\mathbb{I}_H), v^S_0) = a(\delta_p^S(\mathbb{I}_H), \delta_p^S(\mathbb{I}_H)) \cdot \alpha^\dagger(v^S, v^S_0) \), and the result follows from the computations of §5.4.

\[ \square \]

**References**

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