Non-deterministic Chaos

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Non-deterministic chaos is a form of low-dimensional dynamics which is characterized by the existence of a countable set of sensitive decision points (SDP’s). Away from these points, the dynamics is well-behaved. Near these points, however, perturbations (e.g., thermal noise) may cause the outgoing trajectory to be chosen randomly. An example of a non-deterministic chaotic system is given, and a statistical method of analyzing the resultant dynamics is developed.

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Introduction

Solutions of non-linear ordinary differential equations (ODE’s) sometimes exhibit the phenomenon of deterministic chaos, displaying sensitive dependence on initial conditions and long-term unpredictability [1]. The term “deterministic” is used because the ODE’s are described by functions of the dynamical variables and time which are well-defined in the dynamical domain. An oft-cited examples is the kicked harmonic oscillator, which may be described by the equation

\[ \ddot{x} + \omega^2 x = Af(x, t) \sum_{n=1}^{\infty} \delta(t - nT), \]  

where \( \omega \) is the frequency of the oscillator, \( A \) is some constant, and \( f(x, t) \) is some arbitrary well-defined function. The solution of Eq. (1) is simply that of a simple harmonic oscillator (an ellipse in the phase plane) until the kick at \( t = nT \) occurs. At this point the solution jumps from one ellipse to another as determined by the magnitude \( Af(x, nT) \) of the kick at this point. For an appropriate choice of \( A \) and \( f(x, t) \), the solution \( x(t) \) of Eq. (1) is chaotic [1].

Let us now suppose that \( f(x, t) \) is not well-defined, but rather is a random function of time. By “random”, we mean that any point on this function has zero correlation with all other points on the function. The effect of this is to give the oscillator an unpredictable kick whenever \( t = nT \). The solutions will thus be similar to those described above, with the important difference that the kick-induced jump from one phase plane ellipse to another is non-deterministic, and cannot be predicted.

We will not examine the particular case of the randomly kicked harmonic oscillator (RKHO) here. We note, however, that the random function \( f(x, t) \) is expected to be describable by some distribution about a mean value (\( \langle f(x, t) \rangle \)). If this distribution is everywhere > 0, we are then led to the following observations:

1. The dynamical trajectory after the kick at \( t = nT \) is completely uncorrelated with the trajectory before.
2. We may make statistical predictions of what will occur after each kick based on the distribution of \( f(x, t) \). From this, we should be able to find some probability density that the dynamical trajectory will visit a particular point in phase space.

Realization

The randomly kicked harmonic oscillator, while providing a useful illustrative example, requires a random driving force of large amplitude, which seems rather contrived. We seek a more general paradigm where seemingly insignificant random fluctuations can induce the type of non-determinism seen above. To this end we design a “toy” dynamical system, whose orbits are all circles parameterized by their radius, \( r \). The key feature of our toy system is that all of these circles share a common tangent point at the origin (see Fig. [1]). A set of ODE’s whose solutions satisfy these criteria is easily found to be \( \ddot{x} = y \)

\[ \dot{y} = \frac{y^2}{2x} - \frac{1}{2} x. \]  

We will not examine Eqs. (2) in detail except to note that while the origin is a common point of all dynamical trajectories, it is not a fixed point. A particular circle of radius \( r \) (for positive \( x \)) is given by
\[(x - r)^2 + y^2 = r^2, \quad (3)\]

or,
\[y^2 = 2xr - x^2. \quad (4)\]

The equation for \(\dot{y}\) may then be re-expressed as
\[\dot{y} = \frac{2xr - x^2}{2x} - \frac{1}{2}x = r - x. \quad (5)\]

In the limit that \(x, y \to 0\), we find simply that \(\dot{y} = r\). Excepting the case of the trivial circle \((r = 0)\), we see that the origin is not a fixed point and any dynamical trajectory will reach the origin in finite time.

Let us now add some very small random fluctuations to our system. Equations (2) then become
\[\dot{x} = y + \epsilon f(t)\]
\[\dot{y} = \frac{y^2}{2x} - \frac{1}{2}x + \epsilon g(t), \quad (6)\]

where \(\epsilon \ll 1\) and \(f(t)\) and \(g(t)\) are identically distributed random functions (we have named them differently to denote that they take on different functional values for the same \(t\)). Usually, if \(\epsilon\) is small enough, we ignore such fluctuations, as they have little effect on the dynamics. Indeed, as long as the dynamical trajectory is away from the origin, the average solution of Eqs. (6) will correspond to a solution of the unperturbed Eqs. (2). Near the origin, however, such fluctuations may have a dramatic effect, since any small change in \(x\) and \(y\) could land on a circle with any radius ranging from 0 to \(\infty\).

The dynamics of Eqs. (1) are then expected to be similar to those of the RKHO. The solution (on average) moves along a circle of particular \(r\) until it reaches some neighborhood about the origin where the fluctuations become prevalent. The trajectory will leave this neighborhood on a circle of some different \(r\) which was randomly selected by the action of the fluctuations. Like the RKHO, we see that circles before and after the origin are uncorrelated, and that the dynamics is correspondingly unpredictable. Unlike the RKHO, the size of the fluctuations may be arbitrarily small, yet the resultant behavior will be equally non-deterministic (in essence, the random kick has been replaced by the infinite divergence at the origin). We term this dynamical behavior non-deterministic chaos. The key feature of the underlying dynamical system is a point such as the origin for Eqs. (2), which is reachable in finite time, but is a common point among many dynamical trajectories. We denote such a point as a sensitive decision point (SDP).

It should be noted here that the type of behavior described above is not unique to the particular conditions we have put forth. The case of noise induced instability, as described by Chen [3], also produces large-scale stochastic behavior in the presence of small random fluctuations. The underlying causes, however, are somewhat different. Noise induced instability occurs when the random perturbations drive the dynamical trajectory over the potential barrier created by a driving force [3]. Non-deterministic chaos is essentially the result of a singularity in the equations of motion (i.e., infinite divergence at a point), which allows dynamical orbits comprising a finite region of the phase plane to intersect at a single point.

**Statistical Properties**  In the case of the RKHO, the statistics of the orbit were obviously dependent on the statistics of the driving function. This is not so obvious in the case at hand, but we should be able to make some statistical predictions about the dynamics. More precisely, we would like to know the probability that a circle of given \(r\) is chosen when the orbit leaves some neighborhood about the origin. Let us define this neighborhood as a disk of radius \(\delta\), and note that an orbit leaving this neighborhood does so with angle \(\theta\), which we take as measured from the \(y\)-axis. Given that we expect the fluctuations to be isotropic, the probability density of picking a particular \(\theta\) is constant, i.e.,

\[p(\theta)d\theta \propto d\theta. \quad (7)\]

Next we note that everywhere except at the origin the Existence and Uniqueness Theorem applies to solutions of Eqs. (2), thus each circle of radius \(r\) is associated with a unique \(\theta\), and we may write \(\theta\) as a function of \(r\). Substituting into \(p(\theta)\), we find
\[ p(r)dr \propto \frac{\partial \theta(r)}{\partial r} dr. \quad (8) \]

The probability of getting a circle between \( r \) and \( r + \Delta r \) is simply

\[ P(r, \Delta r) \propto \int_{r}^{r+\Delta r} \frac{\partial \theta(r)}{\partial r} dr = \theta(r + \Delta r) - \theta(r). \quad (9) \]

For the case at hand, we find

\[ P(r, \Delta r) \propto \arccos \frac{\delta}{2(r + \Delta r)} - \arccos \frac{\delta}{2r}. \quad (10) \]

**Discussion**

To date, there is one candidate for a physical non-deterministic chaotic system. This is a model of neutron star dynamics \([4]\) which exhibits a sensitive decision point and is expected to behave as described above \([2,5]\).

Whether other physical examples of non-deterministic chaos exist is not clear at this time. However, it is known that apparently complex systems may often successfully be modeled by a low-dimensional deterministic chaotic system \([1]\). We cautiously advance the possibility of an analogous situation for non-deterministic chaos. This is attractive, since the analysis and modeling of a non-deterministically chaotic signal is reduced to finding the SDP’s, doing some curve fitting of the signal between SDP’s, and performing some straightforward statistical analysis as described above. We are continuing to study such possibilities.

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**FIG. 1.** Some examples of circular orbits of different radii, all sharing a common point at the origin.