The boundary value problem for the mean field equation on a compact Riemann surface

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Abstract Let $(\Sigma, g)$ be a compact Riemann surface with smooth boundary $\partial \Sigma$, $\Delta_g$ be the Laplace-Beltrami operator, and $h$ be a positive smooth function. Using a min-max scheme introduced by Djadli and Malchiodi (2008) and Djadli (2008), we prove that if $\Sigma$ is non-contractible, then for any $\rho \in (8k \pi, 8(k + 1) \pi)$ with $k \in \mathbb{N}^*$, the mean field equation

$$
\begin{cases}
\Delta_g u = \rho \frac{h e^u}{\int_{\Sigma} h e^u d\sigma_g} & \text{in } \Sigma, \\
u = 0 & \text{on } \partial \Sigma
\end{cases}
$$

has a solution. This generalizes earlier existence results of Ding et al. (Ann Inst H Poincaré Anal Non Linéaire, 1999) and Chen and Lin (2003) in the Euclidean domain. Also we consider the corresponding Neumann boundary value problem. If $h$ is a positive smooth function, then for any $\rho \in (4k \pi, 4(k + 1) \pi)$ with $k \in \mathbb{N}^*$, the mean field equation

$$
\begin{cases}
\Delta_g u = \rho \left( \frac{h e^u}{\int_{\Sigma} h e^u d\sigma_g} - \frac{1}{|\Sigma|} \right) & \text{in } \Sigma, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Sigma
\end{cases}
$$

has a solution, where $\nu$ denotes the unit normal outward vector on $\partial \Sigma$. Note that in this case we do not require the surface to be non-contractible.

Keywords mean field equation, topological method, min-max scheme

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1 Introduction

As a basic problem of mathematical physics, the mean field equation has attracted the interest of many mathematicians for at least half a century. In addition to the prescribed Gaussian curvature problem [4,7,8,10,22], it also arises in Onsager’s vortex model for turbulent Euler flows [27, p. 256] and in Chern-Simons-Higgs models [6,14,16,29,33,35].

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Let Ω be a smooth bounded domain in \( \mathbb{R}^2 \). It was proved by Ding et al. [15] that if the complement of Ω contains a bounded region, and \( h : \overline{\Omega} \rightarrow \mathbb{R} \) is a positive function, then the mean field equation

\[
\begin{cases}
-\Delta_{\mathbb{R}^2} u = \rho \frac{h^u}{\int_{\Omega} h^u \, dx} & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]  

(1.1)

has a solution for all \( \rho \in (8\pi, 16\pi) \), where \( \Delta_{\mathbb{R}^2} = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 \) is the standard Laplacian operator in \( \mathbb{R}^2 \). The proof is based on a compactness result of Li and Shafrir [24], the monotonicity technique used by Struwe [28] in dealing with harmonic maps, and a general min-max theorem [34, Theorem 2.8].

It was pointed out by Li [23] that the Leray-Schauder degree for the mean field equation should depend only on the topology of the domain and \( k \in \mathbb{N} \) satisfying \( \rho \in (8k\pi, 8(k+1)\pi) \). To illustrate this point, he calculated the simplest case \( \rho < 8\pi \). Later, by computing the topological degree, Chen and Lin [9] improved Ding-Jost-Li-Wang’s result to the following: if Ω is not simply connected, and \( h \) is positive on \( \overline{\Omega} \), then (1.1) has a solution for all \( \rho \in (8k\pi, 8(k+1)\pi) \). Also they were able to compute the topological degree for the mean field equation on the compact Riemann surface \((\Sigma, g)\) without boundary, namely,

\[
\Delta_g u = \rho \left( \frac{h^u}{\int_\Sigma h^u \, dv_g} - \frac{1}{|\Sigma|} \right) \text{ in } \Sigma, 
\]

(1.2)

where \( \Delta_g \) denotes the Laplace-Beltrami operator, and \( |\Sigma| \) stands for the area of Σ with respect to the metric \( g \). Precisely, the degree-counting formula for (1.2) is given by \( (k-\chi(\Sigma)) \) for \( \rho \in (8k\pi, 8(k+1)\pi) \).

As a consequence, if the Euler characteristic \( \chi(\Sigma) \leq 0 \), then (1.2) has a solution.

Note that solutions of (1.2) are critical points of the functional

\[
J_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 \, dv_g - \rho \log \int_\Sigma h^u \, dv_g + \rho \int_\Sigma u \, dv_g, \quad u \in W^{1,2}(\Sigma).
\]

A direct method of variation leads to that \( J_\rho \) has critical points for \( \rho < 8\pi \). When \( \rho = 8\pi \), Ding et al. [13] found a critical point of \( J_{8\pi} \) under certain conditions on \( \Sigma \) and \( h \). For the cases where \( \rho = 8\pi \), \( h \geq 0 \) or \( h \) changes the sign, we refer the readers to [31,32,39].

In a celebrated paper, Djadli [17] was able to find a solution of (1.2) for all \( \rho \in (8k\pi, 8(k+1)\pi) (k \in \mathbb{N}^*) \) and arbitrary genera of Σ, by adapting a min-max scheme introduced by Djadli and Malchiodi [18]. In particular, Chen-Lin’s existence result for (1.2) was improved by Djadli [17] to arbitrary possible \( \chi(\Sigma) \). Let us summarize the procedure in [17]. Denote the family of formal sums by

\[
\Sigma_k = \left\{ \sum_{i=1}^k t_i \delta_{x_i} : t_i \geq 0, \sum_{i=1}^k t_i = 1, x_i \in \Sigma \right\},
\]

endowed with the weak topology of distributions, say the topology of \((C^1(\Sigma))^*\). This is known in the literature as the formal set of barycenters of Σ. The first and most important step is to construct two continuous maps \( \Psi \) and \( \Phi_\lambda \) between \( \Sigma_k \) and sub-levels of \( J_\rho \), say,

\[
\Sigma_k \xrightarrow{\Phi_\lambda} J_\rho^{-1}([-\infty, -8k\pi]) \xrightarrow{\Psi} \Sigma_k
\]

for \( \lambda \geq \lambda_L = e^{L/(\rho-8k\pi)} \) and large \( L > 0 \); moreover \( \lim_{\lambda \to +\infty} \Psi \circ \Phi_\lambda = \text{Id} \), and in particular, \( \Psi \circ \Phi_\lambda \) is homotopic to the identity on \( \Sigma_k \) provided \( \lambda \geq \lambda_L \). Here, \( J_\rho^\alpha \) stands for a set of all the functions \( u \in W^{1,2}(\Sigma) \) with \( J_\rho(u) \leq \alpha \) for any real number \( \alpha \). The second step is to set the suitable min-max value for \( J_\rho \), namely,

\[
\alpha_{\lambda, \rho} = \inf_{\gamma \in \Gamma_\lambda} \sup_{(\sigma, t) \in \Sigma_k} J_\rho(\gamma(\sigma, t)),
\]

where \( \tilde{\Sigma}_k = \Sigma_k \times [0, 1]/(\Sigma_k \times \{0\}) \) is a topological cone, and \( \Gamma_\lambda \) is a set of paths

\[
\Gamma_\lambda = \{ \gamma \in C^0(\tilde{\Sigma}_k, W^{1,2}(\Sigma)) : \gamma(\sigma, 1) = \Phi_\lambda(\sigma), \forall \sigma \in \Sigma_k \}.
\]
Theorem 1.1. Let \( \Sigma \) be a compact Riemann surface with smooth boundary \( \partial \Sigma \). Our first aim is to generalize the results of Ding et al. \cite{15} and Chen and Lin \cite{9}. Precisely, we have the following theorem. Marchis et al. \cite{12} employed it to find critical points of a Trudinger-Moser functional. therein. Recently, Sun et al. \cite{30} extended Djadli’s result to the case of a generalized mean field equation. For Toda systems, we refer the readers to \cite{1,3,25,26} and the references of \cite{8}. \( \rho \) is a positive integer, the Dirichlet problem

\[
\begin{align*}
\Delta \rho u &= \rho \frac{he^u}{\int_{\Sigma} he^udv} \quad & \text{in } \Sigma, \\
u &= 0 \quad & \text{on } \partial \Sigma
\end{align*}
\]  

\hspace{1cm} (1.3)

has a solution.

The proof of Theorem 1.1 is based on the min-max theorem \cite[Theorem 2.8]{34}, which was also used by Ding et al. \cite{15} and Djadli \cite{17}, compactness analysis, and an improved Trudinger-Moser inequality. All of the three parts are quite different from those of \cite{15,17}. On the choice of the metric space, we use \( \tilde{\Sigma}_{\epsilon,k} \) (see (2.44) below) instead of \( \Sigma_{\epsilon,k} \) or \( \tilde{\Sigma}_{\epsilon,k} \); on compactness analysis, we use a reflection method different from that of Chen and Lin \cite{9} to show that the blow-up phenomenon cannot occur on the boundary \( \partial \Sigma \); moreover, we need to prove an improved Trudinger-Moser inequality for functions with boundary value zero, nor is it the original one in \cite{11}.

We also consider the Neumann boundary value problem on the mean field equation. In this regard, our second result is the following theorem.

Theorem 1.2. Let \( (\Sigma,g) \) be a compact Riemann surface with smooth boundary \( \partial \Sigma \), \( v \) be the unit normal outward vector on \( \partial \Sigma \), \( \Delta \rho \) be the Laplace-Beltrami operator, and \( h: \Sigma \to \mathbb{R} \) be a positive smooth function. If \( \rho \in (4k\pi,4(k+1)\pi) \) with \( k \) a positive integer, then the Neumann boundary value problem

\[
\begin{align*}
\Delta \rho u &= \rho \left( \frac{he^u}{\int_{\Sigma} he^udv} - \frac{1}{|\Sigma|} \right) \quad & \text{in } \Sigma, \\
\partial u/\partial v &= 0 \quad & \text{on } \partial \Sigma
\end{align*}
\]  

\hspace{1cm} (1.4)

has a solution.

We remark that in Theorem 1.2, \( \Sigma \) does not need to be non-contractible. For the proof of Theorem 1.2, we choose a metric space \( \tilde{\Sigma}_{\epsilon,k} \) (see (3.27) below), which is non-contractible, whether \( \Sigma \) is non-contractible or not. Concerning the compactness of solutions to (1.4), if it has a sequence of blow-up solutions, then we show that \( \rho = 4k\pi \) for \( k \in \mathbb{N}^* \). Also we derive an improved Trudinger-Moser inequality for functions with integral mean zero, which is important in our analysis.

Before ending Section 1, we mention a recent result of Zhang et al. \cite{40}. Using the min-max scheme of Djadli \cite{17} and Djadli and Malchiodi \cite{18}, they obtained the existence of solutions to the equation

\[
\begin{align*}
\Delta \rho u &= 0 \quad & \text{in } \Sigma, \\
\partial u/\partial v &= \rho \frac{he^u}{\int_{\partial \Sigma} he^uds} \quad & \text{on } \partial \Sigma
\end{align*}
\]
for any $\rho \in (2k\pi, 2(k+1)\pi)$, $k \in \mathbb{N}^*$, and any positive smooth function $h$. This improved an early result of Guo and Liu [19].

The rest of this paper is organized as follows. Theorems 1.1 and 1.2 are proved by the min-max method in Sections 2 and 3, respectively. Throughout this paper, the sequence and the subsequence are not distinguished. We often denote various constants by the same $C$ from line to line, even in the same line. Sometimes we write $C_k, C_{k,\epsilon}, C(\epsilon), \ldots$ to emphasize the dependence of these constants.

2 The Dirichlet boundary value problem

In this section, Theorem 1.1 is proved. This will be divided into several subsections. In Subsection 2.1, we analyze the compactness of solutions to the Dirichlet problem (1.3). In Subsection 2.2, we derive an improved Trudinger-Moser inequality for functions $u \in W^{1,2}_0(\Sigma)$. In Subsection 2.3, we construct two continuous maps between sub-levels $J_{\rho}^{-L}$ with sufficiently large $L$ and the topological space $\Sigma_k$. In Subsection 2.4, we construct min-max levels of $J_{\rho}$, and ensure these min-max levels are finite. In Subsection 2.5, several uniform estimates on min-max levels of $J_{\rho}$ are obtained. In Subsection 2.6, adapting the argument of [15, Lemma 3.2], we prove that $J_{\rho}$ has a critical point for $\rho$ in a dense subset of $(8k\pi, 8(k+1)\pi)$. In Subsection 2.7, using compactness of solutions to the Dirichlet problem (1.3), we conclude that $J_{\rho}$ has a critical point for any $\rho \in (8k\pi, 8(k+1)\pi)$.

2.1 Compactness analysis

Let $(\rho_n)$ be a sequence of numbers tending to $\rho$, $(h_n)$ be a function sequence converging to $h$ in $C^1(\Sigma)$, and $(u_n)$ be a sequence of solutions to

$$
\begin{align*}
\Delta_g u_n = \rho_n \frac{h_n e^{u_n}}{\int_{\Sigma} h_n e^{u_n} dv_g} & \quad \text{in } \Sigma, \\
 u_n = 0 & \quad \text{on } \partial \Sigma.
\end{align*}
$$

Define $v_n = u_n - \log \int_{\Sigma} h_n e^{u_n} dv_g$. Then $\Delta_g v_n = \rho_n h_n e^{u_n} \text{and } \int_{\Sigma} h_n e^{v_n} dv_g = 1$.

**Lemma 2.1.** Assume that $\rho$ is a positive number and $h$ is a positive function. Up to a subsequence, one of the following alternatives holds:

(i) $(u_n)$ is bounded in $L^\infty(\Sigma)$;
(ii) $(u_n)$ converges to $-\infty$ uniformly in $\Sigma$;
(iii) there exists a finite singular set $S = \{p_1, \ldots, p_m\} \subset \Sigma$ such that for any $1 \leq j \leq m$, there is a sequence of points $\{p_{j,n}\} \subset \Sigma$ satisfying $p_{j,n} \to p_j$, $u_n(p_{j,n}) \to +\infty$, and $v_n$ converges to $-\infty$ uniformly on any compact subset of $\Sigma \setminus S$ as $n \to \infty$. Moreover,

$$
\rho_n \int_{\Sigma} h_n e^{v_n} dv_g \to 8m\pi.
$$

**Proof.** Note that $\Sigma = \Sigma \cup \partial \Sigma$, where $\Sigma$ is an open set including all the inner points of $\Sigma$, and $\partial \Sigma$ is its boundary. The compactness analysis on $(u_n)$ will be divided into two parts.

**Part I.** Analysis in the interior domain $\Sigma$.

According to an observation in [37, Subsection 4.1] (compared with [2, Theorem 4.17]), by the Green representation formula for functions with boundary value zero, we have

$$
\|u_n\|_{W^{1,q}_0(\Sigma)} \leq C_q, \quad \forall 1 < q < 2.
$$

We claim that there exists some constant $c_0 > 0$ such that for all $n \in \mathbb{N}$,

$$
\int_{\Sigma} h_n e^{u_n} dv_g \geq c_0.
$$
Suppose the claim is not true. By Jensen’s inequality,
\[ e \frac{1}{|\Sigma|} \int_{\Sigma} u_n^\epsilon \, dv_g \leq \frac{1}{|\Sigma|} \int_{\Sigma} e^{u_n} \, dv_g \to 0. \]
Thus \( \int_{\Sigma} u_n \, dv_g \to -\infty \), which contradicts (2.2), and concludes our claim (2.3).

To proceed, we assume that \( \rho_n h_n e^{u_n} \, dv_g \) converges to some nonnegative measure \( \mu \). If \( \mu(x^*) < 4\pi \) for some \( x^* \in \Sigma \), then there exist two positive constants \( c_0 \) and \( r_0 \) such that
\[ \int_{B_{r_0}(r_0)} \rho_n h_n e^{u_n} \, dv_g \leq 4\pi - c_0. \]
In view of (2.1), by Brezis-Merle’s theorem [5, Theorem 1] and elliptic estimates, we have that (2.5) does not hold. In view of (2.3), there is a constant \( c_1 \) such that up to a subsequence,
\[ 0 < c_0 \leq \int_{\Sigma} h_n e^{u_n} \, dv_g \leq c_1. \]
Choose \( x_0 \in \mathcal{S} \) and \( 0 < r_0 < \text{dist}(x_0, \partial\Sigma) \) satisfying \( B_{x_0}(r_0) \cap \mathcal{S} = \{ x_0 \} \). Note that \( (u_n) \) is locally uniformly bounded in \( \Sigma \setminus \mathcal{S} \). There exists a positive constant \( c_2 \) depending on \( x_0 \) and \( r_0 \) such that \( |v_n(x)| \leq c_2 \) for all \( x \in \partial B_{x_0}(r_0) \). Let \( w_n \) be a solution to
\[ \begin{cases} 
\Delta_g w_n = \rho_n h_n e^{v_n} & \text{in } B_{x_0}(r_0), \\
\delta w_n = c_2 & \text{on } \partial B_{x_0}(r_0).
\end{cases} \]
Then the maximum principle implies that \( w_n \leq v_n \) in \( B_{x_0}(r_0) \). By the Green formula, \( w_n \) converges to \( w \) weakly in \( W^{1,q}(B_{x_0}(r_0)) \) and a.e. in \( B_{x_0}(r_0) \). Moreover, \( w \) is a solution of
\[ \begin{cases} 
\Delta_g w = \mu & \text{in } B_{x_0}(r_0), \\
\delta w = c_2 & \text{on } \partial B_{x_0}(r_0).
\end{cases} \]
Let \( G_{x_0} \) be a distributional solution of
\[ \begin{cases} 
\Delta_g G_{x_0} = 4\pi \delta_{x_0} & \text{in } B_{x_0}(r_0), \\
G_{x_0} = c_2 & \text{on } \partial B_{x_0}(r_0).
\end{cases} \]
Clearly, \( G_{x_0} \) is represented by
\[ G_{x_0}(x) = -2 \log \text{dist}(x, x_0) + A_{x_0} + o(1), \]
where \( A_{x_0} \) is a constant, and \( o(1) \to 0 \) as \( x \to x_0 \). Since
\[ \begin{cases} 
\Delta_g (w - G_{x_0}) \geq 0 & \text{in } B_{x_0}(r_0), \\
w - G_{x_0} = 0 & \text{on } \partial B_{x_0}(r_0),
\end{cases} \]
it follows from the maximum principle that
\[ w(x) \geq G_{x_0}(x) \text{ for all } x \in B_{x_0}(r_0) \setminus \{ x_0 \}. \]
Combining (2.6), (2.7) and the fact that $w_n \to w$ a.e. in $B_{r_0}(r_0)$, by Fatou’s lemma, we calculate
\[ +\infty = \int_{B_{r_0}(r_0)} e^{G_n + x} \, dv_y \leq \liminf_{n \to \infty} \int_{B_{r_0}(r_0)} e^{\mu_n + x} \, dv_y \leq \liminf_{n \to \infty} \int_{B_{r_0}(r_0)} e^{\mu_n + x} \, dv_y \leq C. \]
This is impossible and excludes the possibility of (2.5). Hence we conclude (2.4).

We may assume $\mathcal{S} = \{x_1, \ldots, x_m\}$. Then it holds that $\mu(x_i) = 8\pi$ for all $1 \leq i \leq m$. Without loss of generality, it suffices to prove $\mu(x_1) = 8\pi$. Choose an isothermal coordinate system $\phi : U \to B(0)$ near $x_1$. In such coordinates, the metric $g$ and the Laplace-Beltrami operator $\Delta_g$ are represented by $g = e^{\psi(y)}(dy_1^2 + dy_2^2)$ and $\Delta_g = -\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ denotes the standard Laplacian on $\mathbb{R}^2$. Set $\tilde{u} = u \circ \phi^{-1}$ for any function $u : U \to \mathbb{R}$. Since $(u_n)$ is a sequence of solutions to (2.1), $\tilde{u}_n = u_n \circ \phi^{-1}$ satisfies
\[ -\Delta_g \tilde{u}_n(y) = e^{\psi(y)} \rho_n \tilde{h}_n(y) e^{\tilde{\psi}_n(y)}, \quad y \in B_1. \] (2.8)
Multiplying both sides of (2.8) by $y \cdot \nabla \tilde{u}_n(y)$, by integration by parts, we have
\[ \frac{r}{2} \int_{\partial B_r} |\nabla \tilde{u}_n|^2 \, d\sigma - r \int_{\partial B_r} \langle \nabla \tilde{u}_n, \nu \rangle^2 \, d\sigma = \int_{B_r} e^{\psi_n} \rho_n \tilde{h}_n e^{\tilde{\psi}_n} \, d\sigma - \int_{B_r} e^{\psi_n} \rho_n (\nabla \tilde{G}(\tilde{h}_n), y) \, dy \]
\[ -2 \int_{B_r} e^{\psi_n} \rho_n \tilde{h}_n e^{\tilde{\psi}_n} \, d\sigma, \] (2.9)
where $B_r = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 < r\}$, $\partial B_r = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 = r\}$, and $\nu$ denotes the unit outward vector on $\partial B_r$. In view of (2.4), $(u_n)$ converges to a Green function $G(x_, \cdot)$ weakly in $W^{1, q}_0(\Sigma)$ and in $C_0^\infty(\Sigma \setminus \mathcal{S})$. Locally, $G(x_1, \cdot)$ satisfies
\[ \Delta_g \tilde{G}(x_1, z) = \mu(x_1) \delta_{x_1}(z), \quad \forall z \in \phi^{-1}(B_1). \]
Clearly,
\[ \tilde{G}(y) = G(x_1, \phi^{-1}(y)) = -\frac{\mu(x_1)}{2\pi} \log |y| + \eta(y) \]
for some $\eta \in C^2(B_1)$. Passing to the limit $n \to \infty$ first, and then $r \to 0$ in (2.9), we obtain
\[ \mu(x_1) = \lim_{r \to 0} \frac{r}{2} \int_{\partial B_r} (\nabla \tilde{G}, \nu)^2 \, d\sigma - \frac{r}{4} \int_{\partial B_r} |\nabla \tilde{G}|^2 \, d\sigma = \frac{(\mu(x_1))^2}{8\pi}. \]
This immediately leads to $\mu(x_1) = 8\pi$.

**Part II.** Analysis on the boundary $\partial \Sigma$.

Let $x^* \in \Sigma$ be fixed. Note that $\rho_n \tilde{h}_n e^{\psi_n} \, dv_y$ converges to the nonnegative Radon measure $\mu$ on $\Sigma$. If $\mu(x^*) < 2\pi$, there exist a neighborhood $V$ of $x^*$ and a number $\gamma_0 > 0$ such that
\[ \int_V \rho_n \tilde{h}_n e^{\psi_n} \, dv_y \leq 2\pi - \gamma_0. \] (2.10)
Without loss of generality, we take an isothermal coordinate system $(V, \phi, \{y_1, y_2\})$ such that $\phi(x^*) = (0, 0)$, and $\phi : V \to B_1^+ \cup \Gamma = \{(y_1, y_2) : y_2 \geq 0\}$, where $\Gamma = \{(y_1, y_2) : |y_1| < 1, y_2 = 0\}$. Moreover, in this coordinate system, the metric $g = e^{\psi(y)}(dy_1^2 + dy_2^2)$ and the Laplace-Beltrami operator $\Delta_g = -e^{-\psi(y)} \Delta_{\mathbb{R}^2}$, where $\psi : B_1^+ \cup \Gamma \to \mathbb{R}$ is a smooth function with $\psi(0, 0) = 0$. For more details about isothermal coordinates on the boundary, we refer the readers to [38]. Now the local version of (2.1) is
\[ \begin{cases} -\Delta_{\mathbb{R}^2}(u_n \circ \phi^{-1})(y) = e^{\psi(y)} \rho_n (h_n \circ \phi^{-1})(y) e^{(\psi_n \circ \phi^{-1})(y)} & \text{in } B_1^+, \\ u_n \circ \phi^{-1}(y) = 0 & \text{on } \Gamma. \end{cases} \] (2.11)
For any function $u : V \to \mathbb{R}$, we define a function $\tilde{u} : B_1^+ \cup \Gamma \to \mathbb{R}$ by
\[ \tilde{u}(y_1, y_2) = \begin{cases} u \circ \phi^{-1}(y_1, y_2), & \text{if } y_2 \geq 0, \\ -u \circ \phi^{-1}(y_1, -y_2), & \text{if } y_2 < 0. \end{cases} \] (2.12)
One can easily check that \( \tilde{u}_n \) is a distributional solution of
\[
-\Delta_{\mathbb{R}^2} \tilde{u}_n(y) = \tilde{f}_n(y), \quad y \in \mathbb{B}_1, \tag{2.13}
\]
where \( \tilde{f}_n \) is defined as in (2.12) and for \( y \in \mathbb{B}_1^+ \cup \Gamma \),
\[
f_n \circ \phi^{-1}(y) = e^{\phi(y)} \rho_n (h_n \circ \phi^{-1})(y) e^{(\nu_n \circ \phi^{-1})(y)}.
\]
In view of (2.10) and the fact that \( \psi(0,0) = 0 \), there would exist a number \( r_0 \in (0,1) \) such that
\[
\int_{\mathbb{B}_{r_0}} |\tilde{f}_n(y)| \, dy \leq 4\pi - \gamma_0.
\]
Let \( w_n \) be a solution of
\[
\begin{cases}
-\Delta_{\mathbb{R}^2} w_n = \tilde{f}_n & \text{in } \mathbb{B}_{r_0}, \\
w_n = 0 & \text{on } \partial \mathbb{B}_{r_0}.
\end{cases}
\]
By Brezis-Merle’s theorem [5, Theorem 1], there exists some constant \( C \) depending only on \( \gamma_0 \) and \( r_0 \) such that
\[
\int_{\mathbb{B}_{r_0}} \exp \left( \frac{(4\pi - \gamma_0/2)\|w_n\|}{\|f_n\|_{L^1(\mathbb{B}_{r_0})}} \right) \, dy \leq C.
\]
Hence, there exists some \( q_0 > 1 \) such that
\[
\|e^{w_n}\|_{L^{q_0}(\mathbb{B}_{r_0})} \leq C. \tag{2.14}
\]
Let \( \eta_n = \tilde{u}_n - w_n \). Then \( \eta_n \) satisfies
\[
\begin{cases}
-\Delta_{\mathbb{R}^2} \eta_n = 0 & \text{in } \mathbb{B}_{r_0}, \\
\eta_n = \tilde{u}_n & \text{on } \partial \mathbb{B}_{r_0}.
\end{cases} \tag{2.15}
\]
Noticing (2.2) and (2.14), we have by applying elliptic estimates to (2.15) that
\[
\|\eta_n\|_{L^\infty(\mathbb{B}_{r_0}/2)} \leq C. \tag{2.16}
\]
Combining (2.3), (2.14) and (2.16), we conclude \( \|\tilde{f}_n\|_{L^w(\mathbb{B}_{r_0}/2)} \leq C \). Applying elliptic estimates to (2.13), we obtain that \( \|\tilde{u}_n\|_{L^\infty(\mathbb{B}_{r_0}/4)} \leq C \), which implies \( \|u_n\|_{L^\infty(\phi^{-1}(\mathbb{B}_{r_0}/4))} \leq C \). In conclusion, we have that if \( \mu(x^*) < 2\pi \), then \( (u_n) \) is uniformly bounded near \( x^* \). This also leads to \( \mu(x^*) = 0 \).

If \( \mu(x^*) \geq 2\pi \), in the same coordinate system \((V, \phi, \{y_1, y_2\})\) as above, \( f_n(y) \, dy \) converges to a Radon measure \( \tilde{\mu} \) with \( \tilde{\mu}(0,0) > 4\pi \). Obviously, there exists some \( r_1 > 0 \) such that for any \( x \in \mathbb{B}_{r_1} \setminus \{(0,0)\} \), \( \tilde{\mu}(x) = 0 \). Using the same argument as the proof of (2.4), we conclude that for any compact set \( A \subset \mathbb{B}_{r_1} \setminus \{(0,0)\} \), \( \tilde{v}_n \) converges to \( -\infty \) uniformly in \( A \). This leads to \( \tilde{f}_n(y) \, dy \) converging to the Dirac measure \( \tilde{\mu}(0,0)\delta(0,0)(y) \). Recalling (2.2), we have that \( \tilde{u}_n \) converges to \( \tilde{G}_0 \) weakly in \( W^{1,q}(\mathbb{B}_{r_1}) \) and a.e. in \( \mathbb{B}_{r_1} \), where \( \tilde{G}_0 \) satisfies
\[
-\Delta_{\mathbb{R}^2} \tilde{G}_0(y) = \tilde{\mu}(0,0)\delta(0,0)(y), \quad y \in \mathbb{B}_{r_1}.
\]
Clearly, \( \tilde{G}_0 \) is represented by
\[
\tilde{G}_0(y) = \frac{-\tilde{\mu}(0,0)}{2\pi} \log |y| + A_0 + O(|y|) \tag{2.17}
\]
as \( y \to 0 \), where \( A_0 \) is a constant. Noting that \( \tilde{v}_n \) converges to \( -\infty \) locally uniformly in \( \mathbb{B}_{r_1} \setminus \{(0,0)\} \), we have by applying elliptic estimates to (2.13) that
\[
\tilde{u}_n \to \tilde{G}_0 \quad \text{in } C^1_{\text{loc}}(\mathbb{B}_{r_1} \setminus \{(0,0)\}). \tag{2.18}
\]
By (2.11), \( \tilde{u}_n(y_1,0) = 0 \) for all \( |y_1| < 1 \), which together with (2.18) leads to \( \tilde{G}_0(y_1,0) = 0 \) for all \( 0 < |y_1| < r_1 \). This contradicts (2.17). Therefore,
\[
\{ x \in \partial \Sigma : \mu(x) \geq 2\pi \} = \emptyset.
\]
Combining Parts I and II, we conclude the lemma. \( \square \)
2.2 An improved Trudinger-Moser inequality

In this subsection, we shall derive an improved Trudinger-Moser inequality, which is an analog of that of Chen and Li [11]. It is known (see, for example, [20]) that

\[
\log \int_{\Sigma} e^u dv_g \leq \frac{1}{16\pi} \int_{\Sigma} |\nabla_g u|^2 dv_g + C, \quad \forall u \in W^{1,2}_0(\Sigma). \tag{2.19}
\]

**Lemma 2.2.** Let \( b_0 > 0 \) and \( \gamma_0 > 0 \) be two constants, and \( \Omega_1, \ldots, \Omega_k \) be \( k \) domains of \( \Sigma \) satisfying \( \text{dist}(\Omega_i, \Omega_j) \geq b_0 \) for all \( 1 \leq i < j \leq k \). Then for any \( \epsilon > 0 \), there exists some constant \( C \) depending only on \( b_0, \gamma_0, k \) and \( \epsilon \) such that

\[
\log \int_{\Sigma} e^u dv_g \leq \frac{1}{16k\pi - \epsilon} \int_{\Sigma} |\nabla_g u|^2 dv_g + C \tag{2.20}
\]

for all \( u \in W^{1,2}_0(\Sigma) \) with

\[
\int_{\Omega_i} e^u dv_g \geq \gamma_0 \int_{\Sigma} e^u dv_g, \quad i = 1, \ldots, k. \tag{2.21}
\]

**Proof.** We modify an argument of Chen and Li [11]. Take smooth functions \( \phi_1, \ldots, \phi_k \) defined on \( \Sigma \) satisfying

\[
\text{supp} \phi_i \cap \text{supp} \phi_j = \emptyset, \quad \forall 1 \leq i < j \leq k,
\]

\[
\phi_i \equiv 1 \quad \text{on} \ \Omega_i, \quad 0 \leq \phi_i \leq 1 \quad \text{on} \ \Sigma, \quad \forall 1 \leq i \leq k,
\]

and for some positive constant \( b_1 \) depending only on \( b_0 \) and \( g \),

\[
|\nabla_g \phi_i| \leq b_1, \quad \forall 1 \leq i \leq k. \tag{2.24}
\]

For any \( u \in W^{1,2}_0(\Sigma) \) satisfying (2.21), we have \( \phi_i u \in W^{1,2}_0(\Omega) \) for all \( 1 \leq i \leq k \), and thus (2.19) implies

\[
\int_{\Sigma} e^u dv_g \leq \frac{1}{\gamma_0} \int_{\Omega_i} e^{\phi_i u} dv_g \leq \frac{1}{\gamma_0} \int_{\Sigma} e^{\phi_i u} dv_g \leq \frac{1}{\gamma_0} \exp \left( \frac{1}{16\pi} \| \nabla_g (\phi_i u) \|^2_{L^2(\Sigma)} + C \right).
\]

Recall an elementary inequality: if \( a \leq a_i \) for nonnegative numbers \( a \) and \( a_i, i = 1, \ldots, k \), then \( a \leq (a_1 \cdots a_k)^{1/k} \). In view of (2.22)–(2.24), we have

\[
\int_{\Sigma} e^u dv_g \leq \frac{1}{\gamma_0} \left( \prod_{i=1}^k \exp \left( \frac{1}{16\pi} \| \nabla_g (\phi_i u) \|^2_{L^2(\Sigma)} + C \right) \right)^{1/k}
\]

\[
= \frac{e^C}{\gamma_0} \exp \left( \frac{1}{16\pi} \sum_{i=1}^k \| \nabla_g (\phi_i u) \|^2_{L^2(\Sigma)} \right)
\]

\[
= \frac{e^C}{\gamma_0} \exp \left( \frac{1}{16\pi} \left| \nabla_g \left( \sum_{i=1}^k \phi_i \right) \right|^2_{L^2(\Sigma)} \right)
\]

\[
\leq C \exp \left( \frac{1}{16\pi} \| \nabla_g u \|^2_{L^2(\Sigma)} + C(\epsilon_1) \| u \|^2_{L^2(\Sigma)} \right). \tag{2.25}
\]

Let \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots \) be all the eigenvalues of the Laplace-Beltrami operator with respect to the Dirichlet boundary condition with \( \lambda_i \to +\infty \) as \( i \to \infty \), and \( \{ \epsilon_i \}_{i=1}^\infty \) be the corresponding unit normal eigenfunctions, i.e., \( \Delta_g \epsilon_i = \lambda_i \epsilon_i \) and \( \int_{\Omega_i} \epsilon_i \epsilon_j dv_g = \delta_{ij} \) for \( i, j = 1, 2, \ldots \). It is known that
$W^{1,2}_0(\Sigma) = E_\ell \oplus E_\ell^*$, where $E_\ell = \text{span}\{e_1, \ldots, e_\ell\}$ and $E_\ell^* = \{e_{\ell+1}, e_{\ell+2}, \ldots\}$. Let $u \in W^{1,2}_0(\Sigma)$ be as above. Write $u = v + w$ with $v \in E_\ell$ and $w \in E_\ell^*$. Thus the Poincaré inequality implies

$$\|v\|_{C^0(\Sigma)} \leq \sum_{i=1}^\ell \|e_i\|_{C^0(\Sigma)} \int_\Sigma |u| |e_i| dv_g \leq C_\ell \|\nabla v\|_{L^2(\Sigma)},$$

while by the definition of the $(\ell + 1)$-th eigenvalue,

$$\int_\Sigma w^2 dv_g \leq \frac{1}{\lambda_{\ell+1}} \int_\Sigma |\nabla w|^2 dv_g.$$

Having the above two estimates and applying (2.25) to $w$, we have

$$\int_\Sigma e^w dv_g \leq e^{C_\ell \|\nabla u\|_{L^2(\Sigma)}} \int_\Sigma e^w dv_g \leq Ce^{C_\ell \|\nabla u\|_{L^2(\Sigma)}} \exp \left( \frac{1}{16k\pi} (1 + \epsilon_1) \|\nabla w\|_{L^2(\Sigma)}^2 + \frac{C(\epsilon_1)}{\lambda_{\ell+1}} \|\nabla w\|_{L^2(\Sigma)}^2 \right) \leq Ce^{C_\ell \|\nabla u\|_{L^2(\Sigma)}} \exp \left( \frac{1}{16k\pi} (1 + \epsilon_1) \frac{C(\epsilon_1)}{\lambda_{\ell+1}} \|\nabla w\|_{L^2(\Sigma)}^2 \right).$$

This together with Young’s inequality gives

$$\log \int_\Sigma e^u dv_g \leq \frac{1}{16k\pi} \left( 1 + \epsilon_1 \frac{C(\epsilon_1)}{\lambda_{\ell+1}} + \epsilon_1 \right) \int_\Sigma |\nabla u|^2 dv_g + C_{\ell,k,\epsilon_1}. \quad (2.26)$$

Let $\epsilon > 0$ be any given number. Choosing $\epsilon_1 = \epsilon/(32k\pi - 2\epsilon)$, and then taking a sufficiently large $\ell$ such that $C(\epsilon_1)/\lambda_{\ell+1} \leq 1$ in (2.26), we immediately have

$$\log \int_\Sigma e^u dv_g \leq \frac{1}{16k\pi} - \epsilon \int_\Sigma |\nabla u|^2 dv_g + C,$$

where $C$ is a constant depending only on $b_0$, $\gamma_0$, $k$ and $\epsilon$. This is exactly (2.20). \hfill \Box

For any $\rho > 0$, we define a functional $J_\rho : W^{1,2}_0(\Sigma) \to \mathbb{R}$ by

$$J_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla u|^2 dv_g - \rho \log \int_\Sigma e^u dv_g. \quad (2.27)$$

Clearly $J_\rho \in C^2(W^{1,2}_0(\Sigma), \mathbb{R})$. To find solutions of the mean field equation (1.3), it suffices to find critical points of $J_\rho$. For any $a \in \mathbb{R}$, the sub-level of $J_\rho$ is written as $J_\rho^a = \{ u \in W^{1,2}_0(\Sigma) : J_\rho(u) \leq a \}$.

Let $\Sigma_k$ be the formal set of barycenters of $\Sigma$ (of order $k$), which is

$$\Sigma_k = \left\{ \sum_{i=1}^k t_i \delta_{x_i} : t_i \geq 0, x_i \in \Sigma, \sum_{i=1}^k t_i = 1 \right\}. \quad (2.28)$$

It is endowed with the weak topology of distributions. In computation, we use on $\Sigma_k$ the metric given by $(C^1(\Sigma))^*$ inducing the same topology. Similarly, we may define

$$\Sigma_k = \left\{ \sum_{i=1}^k t_i \delta_{x_i} : t_i \geq 0, x_i \in \Sigma, \sum_{i=1}^k t_i = 1 \right\}. \quad (2.29)$$

### 2.3 Continuous maps between sub-levels of $J_\rho$ and $\Sigma_k$

Let $J_\rho$, $\Sigma_k$ and $\Sigma_k$ be defined as in (2.27)–(2.29), respectively. In this subsection, we shall construct continuous maps between sub-levels of $J_\rho$ and $\Sigma_k$ (or $\Sigma_k$).

**Lemma 2.3.** For any $k \geq 1$, both $\Sigma_k$ and $\Sigma_k$ are non-contractible.
Proof. Since $\Sigma_k$ is homotopic to $\Sigma_k$, we only need to prove that $\Sigma_k$ is non-contractible. Let $\chi(\Sigma)$ be the Euler characteristic of $\Sigma$. By [21, Corollary 1.4(a)],

$$
\chi(\Sigma_k) = 1 - \frac{1}{k!} (1 - \chi(\Sigma)) \cdots (k - \chi(\Sigma)).
$$

(2.30)

Denote the genus of $\Sigma$ by $g$ and the number of the connected components of $\partial \Sigma$ by $m$. Notice that $\Sigma$ is simply connected if and only if the Euler characteristic of $\Sigma$ equals 1. By the assumption, it then follows that

$$
\chi(\Sigma) = 2 - 2g - m \leq 0.
$$

(2.31)

Inserting (2.31) into (2.30), we have

$$
\chi(\Sigma_k) \leq 0.
$$

(2.32)

On the other hand, it holds that

$$
\chi(\Sigma_k) = \sum_{j=0}^{2k-1} (-1)^j \dim H_j(\Sigma_k, \mathbb{Z}).
$$

(2.33)

Suppose that $\Sigma_k$ is contractible. Then $\dim H_j(\Sigma_k, \mathbb{Z}) = 0$ for all $j \geq 1$, while $\dim H_0(\Sigma_k, \mathbb{Z}) = 1$, since $\Sigma_k$ is connected. Hence (2.33) gives $\chi(\Sigma_k) = 1$, contradicting (2.32). Hence $\Sigma_k$ is non-contractible. \hfill $\square$

Lemma 2.4. Let $\rho \in (8k\pi, 8(k+1)\pi)$. Then for any sufficiently large $L > 0$, there exists a continuous retraction

$$
\Psi : J^{-L}_{\rho} = \{ u \in W^{1,2}_{0}(\Sigma) : J_{\rho}(u) \leq -L \} \to \Sigma_k.
$$

Moreover, if $(u_n) \subset W^{1,2}_{0}(\Sigma)$ satisfies $\frac{e^{\rho_n}}{\int_{\Sigma} e^{\rho_n} dv_g} dv_g \to \sigma \in \Sigma_k$, then $\Psi(u_n) \to \sigma \in \Sigma_k$.

Proof. By [17, Proposition 4.1], for any $\epsilon > 0$, there exists an $L_0 > 0$ such that for all $u \in J^{-L_0}_{\rho}$,

$$
\frac{e^{\rho_n}}{\int_{\Sigma} e^{\rho_n} dv_g} dv_g \in \{ \sigma \in D(\Sigma) : d(\sigma, \Sigma_k) < \epsilon \},
$$

where $D(\Sigma)$ denotes the set of all the distributions on $\Sigma$. If $\epsilon_0 > 0$ is sufficiently small, then there exists a continuous retraction

$$
\psi_k : \{ \sigma \in D(\Sigma) : d(\sigma, \Sigma_k) < \epsilon_0 \} \to \Sigma_k.
$$

(2.34)

For sufficiently large $L > 0$, we set

$$
\Psi(u) = \psi_k \left( \frac{e^{\rho_n}}{\int_{\Sigma} e^{\rho_n} dv_g} dv_g \right), \quad \forall u \in J^{-L}_{\rho}.
$$

As a consequence, we have a continuous map $\Psi : J^{-L}_{\rho} \to \Sigma_k$.

Moreover, if $(u_n) \subset W^{1,2}_{0}(\Sigma)$ satisfies $\frac{e^{\rho_n}}{\int_{\Sigma} e^{\rho_n} dv_g} dv_g \to \sigma \in \Sigma_k$, then as $n \to \infty$,

$$
\Psi(u_n) = \psi_k \left( \frac{e^{\rho_n}}{\int_{\Sigma} e^{\rho_n} dv_g} dv_g \right) \to \psi_k(\sigma) = \sigma,
$$

as desired. \hfill $\square$

Let $\sigma = \sum_{i=1}^{k} t_i \delta_{x_i} \in \Sigma_k$ be fixed. Take a smooth increasing function $\eta : \mathbb{R} \to \mathbb{R}$ satisfying $\eta(t) = t$ for $t \leq 1$, and $\eta(t) = 2$ for $t \geq 2$. Set $\eta_r(t) = r \eta(t/r)$ for $r > 0$. For $\lambda > 0$ and $x \in \Sigma$, we define

$$
\tilde{\phi}_{\lambda, \sigma}(x) = \log \left( \sum_{i=1}^{k} t_i \left( \frac{8\lambda^2}{(1 + \lambda^2 \eta_r^2(\text{dist}(x, x_i)))^2} \right) \right)
$$

(2.35)

and

$$
\phi_{\lambda, \sigma}(x) = \tilde{\phi}_{\lambda, \sigma}(x) - \log \left( \frac{8\lambda^2}{(1 + 4\lambda^2 \eta_r^2)^2} \right).
$$

(2.36)
Lemma 2.5. Let $\rho \in (8k\pi, 8(k+1)\pi)$ and $\epsilon > 0$. If $\lambda > 0$ is chosen sufficiently large, and $r > 0$ is chosen sufficiently small, then for any $\sigma \in \Sigma_k$ with $\text{dist}(\text{supp}\sigma, \partial \Sigma) \geq \epsilon$, it holds that
\[
J_\rho(\phi_{\lambda, \sigma}) \leq (8k\pi - \rho) \log \lambda
\] (2.37)
and
\[
\int_\Sigma e^{\phi_{\lambda, \sigma}} v_g \, dv_g \to \sigma \quad \text{as} \quad \lambda \to +\infty.
\] (2.38)

Proof. Given $\sigma \in \Sigma_k$, without loss of generality, we assume $\text{supp}\sigma = \{x_1, \ldots, x_k\} \subset \Sigma$. Let $\tilde{\phi}_{\lambda, \sigma}$ and $\phi_{\lambda, \sigma}$ be defined as in (2.35) and (2.36), respectively, where $\lambda > 0$ and $0 < r < \epsilon/4$. Write $r_i = r_i(x) = \text{dist}(x, x_i)$ for $x \in \Sigma$. A simple observation gives
\[
\tilde{\phi}_{\lambda, \sigma}(x) = \begin{cases} 
\log \left( \frac{8\lambda^2}{(1 + 4\lambda^2r^2)^2} \right) & \text{for } x \in \Sigma \setminus \bigcup_{i=1}^k B_{2r}(x_i), \\
\log \left( \frac{8\lambda^2 t_i}{(1 + 4\lambda^2 r^2)^2} + \frac{8\lambda^2 (1 - t_i)}{(1 + 4\lambda^2 r^2)^2} \right) & \text{for } x \in B_{2r}(x_i).
\end{cases}
\] (2.39)

As a consequence, $\phi_{\lambda, \sigma} \in W_0^{1,2}(\Sigma)$. For $x \in B_{2r}(x_i)$, a straightforward calculation shows
\[
\nabla_g \tilde{\phi}_{\lambda, \sigma}(x) = \frac{8\lambda^2 t_i}{(1 + 4\lambda^2 r^2)^2} + \frac{8\lambda^2 (1 - t_i)}{(1 + 4\lambda^2 r^2)^2},
\]
and thus
\[
|\nabla_g \tilde{\phi}_{\lambda, \sigma}(x)| \leq \frac{4\lambda^2 - \eta^i(r_i) \eta^i(r_i)}{1 + \lambda^2 \eta_i^2(r_i)}.
\]

In view of (2.39), it holds that $\nabla_g \tilde{\phi}_{\lambda, \sigma}(x) = 0$ for $x \in \Sigma \setminus \bigcup_{i=1}^k B_{2r}(x_i)$. Hence,
\[
\int_\Sigma |\nabla_g \tilde{\phi}_{\lambda, \sigma}|^2 v_g = \int_{\bigcup_{i=1}^k B_{2r}(x_i)} |\nabla_g \tilde{\phi}_{\lambda, \sigma}|^2 v_g \\
\leq \int_{\bigcup_{i=1}^k B_{2r}(x_i)} \left( \frac{4\lambda^2 \eta^i(r_i) \eta_i^i(r_i)}{1 + \lambda^2 \eta_i^2(r_i)} \right)^2 dv_g \\
= \sum_{i=1}^k 16\pi (1 + O(r^2)) \left( \log(1 + \lambda^2 r^2) + \frac{1}{1 + \lambda^2 r^2} - 1 \right) + O(1) \\
\leq 16k \pi (1 + O(r^2)) \log \lambda^2 + C
\] (2.40)
for some constant $C$ independent of $r$ and $\lambda$. Moreover, for any $s$ with
\[
0 < s < \min \left\{ r, \min_{1 \leq i < j \leq k} \text{dist}(x_i, x_j) \right\},
\]
it holds that
\[
\int_{\bigcup_{i=1}^k B_{2r}(x_i)} e^{\phi_{\lambda, \sigma}} v_g = \int_{\bigcup_{i=1}^k B_{s}(x_i)} e^{\phi_{\lambda, \sigma}} v_g + O\left( \frac{1}{\lambda^2 s^2} \right) \\
= \sum_{i=1}^k \int_{B_{s}(x_i)} \frac{8\lambda^2 t_i}{(1 + \lambda^2 r^2)^2} dv_g + O\left( \frac{1}{\lambda^2 s^2} \right) \\
= 8\pi (1 + O(s^2)) + O\left( \frac{1}{\lambda^2 s^2} \right)
\]
and
\[
\int_{\Sigma \setminus \bigcup_{i=1}^k B_{2r}(x_i)} e^{\phi_{\lambda, \sigma}} v_g = O\left( \frac{1}{\lambda^2 r^2} \right).
\]
It follows that
\[
\int_\Sigma e^{\delta_{\lambda,\sigma}} dv_g = 8\pi (1 + O(s^2)) + O\left(\frac{1}{\lambda^2 r^2}\right) + O\left(\frac{1}{\lambda^2 r^4}\right).
\] (2.41)

Passing to the limit \(\lambda \to +\infty\) first, and then \(s \to 0^+\), we have
\[
\lim_{\lambda \to +\infty} \int_\Sigma e^{\delta_{\lambda,\sigma}} dv_g = 8\pi.
\] (2.42)

Hence,
\[
\int_\Sigma e^{\phi_{\lambda,\sigma}} dv_g = \int_\Sigma e^{\delta_{\lambda,\sigma}} dv_g \left(1 + \frac{4\lambda^2 r^2}{8\lambda^2}\right) = (8\pi + o_\lambda(1))\lambda^2 r^4.
\] (2.43)

Combining (2.40) and (2.43), we obtain
\[
J_\rho(\phi_{\lambda,\sigma}) = \frac{1}{2} \int_\Sigma |\nabla_g \phi_{\lambda,\sigma}|^2 dv_g - \rho \log \int_\Sigma h e^{\phi_{\lambda,\sigma}} dv_g \leq (16k\pi - 2\rho + O(r^2)) \log \lambda + C_r.
\]

Since \(\rho > 8k\pi\), choosing \(r > 0\) sufficiently small and \(\lambda > 0\) sufficiently large, we conclude (2.37).

Finally, we prove (2.38). Let \(\sigma = \sum_{i=1}^k t_i \delta_{x_i} \in \Sigma_k\) be as above. For any \(\varphi \in C^1(\Sigma)\), similar to (2.41), we calculate
\[
\int_{\Sigma_k} \varphi e^{\delta_{\lambda,\sigma}} dv_g = 8\pi \sum_{i=1}^k t_i \varphi(x_i) + O(s^2) + O\left(\frac{1}{\lambda^2 r^2}\right) + O\left(\frac{1}{\lambda^2 r^4}\right).
\]

Letting \(\lambda \to +\infty\) first, and then \(s \to 0^+\), we obtain
\[
\lim_{\lambda \to +\infty} \int_{\Sigma_k} \varphi e^{\delta_{\lambda,\sigma}} dv_g = 8\pi \sum_{i=1}^k t_i \varphi(x_i).
\]

This together with (2.42) implies (2.38).

\[\Box\]

**Lemma 2.6.** Let \(\Psi\) and \(L > 0\) be as in Lemma 2.4. If \(\lambda > 0\) is chosen sufficiently large, there exists a continuous map \(\Phi_\lambda : \Sigma_k \to J_{-L}^\rho\) such that \(\Psi \circ \Phi_\lambda\) is homotopic to the identity map \(\text{Id} : \Sigma_k \to \Sigma_k\).

**Proof.** Let \(\phi_{\lambda,\sigma}\) be constructed as in Lemma 2.5. For any \(\sigma \in \Sigma_k\), we define \(\Phi_\lambda(\sigma) = \phi_{\lambda,\sigma}\). Clearly, the map \(\Phi_\lambda : \Sigma_k \to W_0^{1,2}(\Sigma)\) is continuous. By (2.37), if \(\lambda \geq e^{L/(\rho - 8k\pi)}\), then \(J_\rho(\phi_{\lambda,\sigma}) \leq -L\). Thus \(\Phi_\lambda(\sigma) \in J_{-L}^\rho\). By Lemma 2.4 and (2.38), it holds that
\[
\Psi \circ \Phi_\lambda(\sigma) = \Psi(\phi_{\lambda,\sigma}) = \psi_k \left(\frac{e^{\phi_{\lambda,\sigma}}}{\int_\Sigma e^{\phi_{\lambda,\sigma}} dv_g}\right) \to \sigma
\]
as \(\lambda \to +\infty\). Hence, \(\Psi \circ \Phi_\lambda\) is homotopic to \(\text{Id} : \Sigma_k \to \Sigma_k\).

\[\Box\]

### 2.4 Min-max values

In this subsection, we shall construct the suitable min-max value of \(J_\rho\) for \(\rho \in (8k\pi, 8(k+1)\pi)\), \(k \in \mathbb{N}^*\). Recalling that \(\Sigma\) is non-contractible, we can take a sufficiently small \(\epsilon > 0\) such that
\[
\Sigma_\epsilon = \{x \in \Sigma : \text{dist}(x, \partial \Sigma) \geq \epsilon\}
\]
is non-contractible. Let
\[
\Sigma_{\epsilon,k} = \{\sigma \in \Sigma_k : \text{dist}(\text{supp} \sigma, \partial \Sigma) \geq \epsilon\}.
\]

According to Lemma 2.3, we see that \(\Sigma_{\epsilon,k}\) is also non-contractible. Let
\[
\hat{\Sigma}_{\epsilon,k} = \Sigma_{\epsilon,k} \times [0,1]/(\Sigma_{\epsilon,k} \times \{0\})
\] (2.44)
be the topological cone over $\Sigma_{r,k}$. A path set associated with the metric space $\hat{\Sigma}_{r,k}$ is defined by
\[ \Gamma_\lambda = \{ \gamma \in C^0(\hat{\Sigma}_{r,k}, W^{1,2}_0(\Sigma)) : \gamma|_{\Sigma_{r,k} \times \{1\}} \in \Gamma_{\lambda,0} \}, \]
where $\Gamma_{\lambda,0}$ is given by
\[ \Gamma_{\lambda,0} = \{ \gamma \in C^0(\Sigma_{r,k} \times \{1\}, W^{1,2}_0(\Sigma)) : \gamma(\sigma,1) = \Phi_\lambda(\sigma), \forall \sigma \in \Sigma_{r,k} \}. \]
If we write a path $\gamma : \hat{\Sigma}_{r,k} \to W^{1,2}_0(\Sigma)$ by $\gamma(\sigma,t) = t\phi_{\lambda,\sigma}$, then $\gamma \in \Gamma_{\lambda}$, and thus $\Gamma_{\lambda} \neq \emptyset$.

For any real numbers $\lambda$ and $\rho$, as above. If $\lambda$ is chosen sufficiently small, then
\[ \Phi_\lambda(\sigma) \]
where
\[ \gamma(\sigma,t) = t\phi_{\lambda,\sigma} \]
and $\lambda > 0$ and sufficiently small $r > 0$, it holds that $\Phi_\lambda(\sigma) \in J_{\rho}^{-2L}$ for all $\sigma \in \Sigma_{r,k}$. This together with (2.47) implies
\[ \beta_{\lambda,\rho} \leq -4L. \] (2.48)

Now we claim $\alpha_{\lambda,\rho} > -2L$. If not, $\alpha_{\lambda,\rho} \leq -2L$. By the definition of $\alpha_{\lambda,\rho}$, namely (2.46), there exists some $\gamma_1 \in \Gamma_{\lambda}$ such that $\sup_{(\sigma,t) \in \Sigma_{r,k}} J_{\rho}(\gamma_1(\sigma,t)) \leq -3L/2$. As a consequence,
\[ J_{\rho}(\gamma_1(\sigma,t)) \leq -\frac{3}{2}L \quad \text{for all } (\sigma,t) \in \hat{\Sigma}_{r,k}. \]

Since $\Psi : J_{\rho}^{-2L} \to \Sigma_k$ is continuous, the map $\Psi \circ \gamma_1 : \hat{\Sigma}_{r,k} \to \Sigma_k$ is also continuous. Note that $\gamma_1(\sigma,1) = \Phi_\lambda(\sigma)$ and $\gamma_1(\sigma,0) \equiv u_0 \in J_{\rho}^{-2L}$ for all $\sigma \in \Sigma_{r,k}$. If we let $\pi : \Sigma_k \to \Sigma_{r,k}$ be a continuous projection, then $\pi \circ \Psi \circ \Phi_\lambda : \Sigma_{r,k} \to \Sigma_k$ is homotopic to a constant map $\pi \circ \Phi_\lambda \gamma_1(\cdot,0) : \Sigma_{r,k} \to \Sigma_k$. Moreover, by Lemma 2.6, $\pi \circ \Psi \circ \Phi_\lambda$ is homotopic to $\text{Id} : \hat{\Sigma}_{r,k} \to \Sigma_{r,k}$. Hence the identity map $\text{Id} : \Sigma_{r,k} \to \Sigma_{r,k}$ is homotopic to the constant map $\pi \circ \Psi \circ \gamma_1(\cdot,0) : \Sigma_{r,k} \to \Sigma_{r,k}$, which contradicts the fact that $\Sigma_{r,k}$ is non-contractible. Therefore,
\[ \alpha_{\lambda,\rho} > -2L. \] (2.49)

Since $J_{\rho} \in C^2(W^{1,2}_0(\Sigma), \mathbb{R})$ and $\hat{\Sigma}_{r,k}$ is a compact metric space, we immediately have that $\beta_{\lambda,\rho} > -\infty$ and $\alpha_{\lambda,\rho} < +\infty$. This together with (2.48) and (2.49) concludes the lemma. \hfill \Box

To proceed, we need several uniform estimates for functionals $J_{\rho}$.

### 2.5 Uniform estimates with respect to $\rho$

Let $[a,b] \subset (8k\pi, 8(k+1)\pi)$ be any closed interval. Let $L > 0$ be sufficiently large such that
\[ \Psi : J_{\rho}^{-L} \to \Sigma_k \]
(2.50)
is a continuous map defined as in Lemma 2.4. Let $\Sigma_{r,k}$ be given as in the previous subsection. Choose a sufficiently large $\lambda > 0$ such that for all $\sigma \in \Sigma_{r,k}$, $\Phi_\lambda(\sigma) = \phi_{\lambda,\sigma}$ satisfies
\[ J_{a}(\phi_{\lambda,\sigma}) \leq -(a - 8k\pi)\log \lambda \leq -4L, \] (2.51)where $\phi_{\lambda,\sigma}$ is defined as in (2.36). It should be remarked that the choice of $\lambda$ depends not only on $L$, $k$ and $a$, but also on $\epsilon$. Let $\Gamma_\lambda$ and $\alpha_{\lambda,\rho}$ be given as in (2.45) and (2.46), respectively.
Lemma 2.8. Let $\rho \in [a, b]$. Then $\Psi : J_{\rho}^{-L} \to \Sigma_k$ is well defined uniformly with respect to $\rho$. Moreover, for all $\rho \in [a, b]$, it holds that

$$J_{\rho}(\phi, \sigma) \leq -4L, \quad \forall \sigma \in \Sigma_{\epsilon, k}.$$ 

Proof. Let $\rho \in [a, b]$. If $u \in J_{\rho}^{-L}$, then $J_{\rho}(u) \leq -L$. This implies

$$\log \int_{\Sigma} he^u dv_g > 0.$$ 

It follows that $J_{\rho}(u) \leq J_{\rho}(u) \leq -L$ and $u \in J_{\rho}^{-L}$. As a consequence $J_{\rho}^{-L} \subset J_{\rho}^{-L}$, and thus by (2.50), $\Psi : J_{\rho}^{-L} \to \Sigma_k$ is well defined.

Let $\sigma \in \Sigma_{\epsilon, k}$ and $\phi_{\lambda, \sigma}$ satisfy (2.51). As above, we have that by (2.51),

$$J_{\rho}(\phi, \sigma) = \frac{1}{2} \int_{\Sigma} |\nabla g \phi_{\lambda, \sigma}|^2 dv_g - \rho \int_{\Sigma} he^{\phi_{\lambda, \sigma}} dv_g$$

$$\leq \frac{1}{2} \int_{\Sigma} |\nabla g \phi_{\lambda, \sigma}|^2 dv_g - a \int_{\Sigma} he^{\phi_{\lambda, \sigma}} dv_g$$

$$= J_{\rho}(\phi, \sigma) \leq -4L.$$ 

This ends the proof of the lemma. □

For simplicity, we denote $\alpha_{\lambda, \rho}$ by $\alpha_{\rho}$. By Lemmas 2.7 and 2.8, $\alpha_{\rho}$ is a real number for any $\rho \in [a, b]$.

Then we have an analog of [15, Lemma 2.4].

Lemma 2.9. $\alpha_{\rho}/\rho$ is decreasing in $\rho \in [a, b]$.

Proof. Let $a \leq \rho_1 < \rho_2 \leq b$. Then for any $(\sigma, t) \in \hat{\Sigma}_{\epsilon, k}$ and any $\gamma \in \Gamma_{\lambda}$, it holds that

$$\frac{J_{\rho_1}(\gamma, \sigma, t)}{\rho_1} - \frac{J_{\rho_2}(\gamma, \sigma, t)}{\rho_2} = \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \int_{\Sigma} |\nabla g \gamma(\sigma, t)| dv_g \geq 0.$$ 

It then follows that $\alpha_{\rho_1}/\rho_1 \geq \alpha_{\rho_2}/\rho_2$. □

By Lemma 2.9, $\alpha_{\rho}/\rho$ is differentiable almost everywhere in $[a, b] \subset (8k\pi, 8(k + 1)\pi)$. Define

$$\Lambda_{a, b} = \left\{ \rho \in (a, b) : \frac{\alpha_{\rho}}{\rho} \text{ is differentiable at } \rho \right\}.$$ (2.52)

Then $\Lambda_{a, b}$ is a dense subset of $[a, b]$.

2.6 Existence for a dense set

In this subsection, we shall prove that $J_{\rho}$ has a critical point for any $\rho \in \Lambda_{a, b}$. The argument we shall use is adapted from Ding et al. [15]. For readers’ convenience, we provide the details here.

Lemma 2.10. If $\rho \in \Lambda_{a, b}$, then $\alpha_{\rho}$ is differentiable at $\rho$. In particular, if $\rho \in \Lambda_{a, b}$, then we have

$$\alpha_{\rho} = \alpha_{\rho} + O(\hat{\rho} - \rho) \quad \text{as} \ \hat{\rho} \to \rho.$$ 

Proof. In view of (2.52), it suffices to notice that $\alpha_{\rho} = \rho(\alpha_{\rho}/\rho)$. □

As an analog of [15, Lemma 3.2], we have the following lemma.

Lemma 2.11. If $\rho \in \Lambda_{a, b}$, then $\alpha_{\rho}$ is a critical value of $J_{\rho}$.

Proof. Let $\langle \rho_n \rangle \subset [a, b]$ be an increasing sequence converging to $\rho \in \Lambda_{a, b}$. By the definition of $\alpha_{\rho_n}$, there must be a path $\gamma_n \in \Gamma_{\lambda}$ such that

$$\sup_{u \in \gamma_n(\Sigma_{\epsilon, k})} J_{\rho_n}(u) \leq \alpha_{\rho_n} + \rho - \rho_n.$$ (2.53)

Also we use the definition of $\alpha_{\rho}$ to find some $u_n \in \gamma_n(\hat{\Sigma}_{\epsilon, k}) \subset W^{1, 2}_0(\Sigma)$ with

$$J_{\rho}(u_n) \geq \alpha_{\rho} - (\rho - \rho_n).$$ (2.54)
For the above $u_n$, by Lemma 2.10, we have
\[
\frac{1}{2} \int_\Sigma |\nabla g u_n|^2 dv_g = \frac{J_{\rho_n}(u_n)}{\rho_n} - \frac{J_{\rho}(u_n)}{\rho}
\leq \rho \frac{\alpha_{\rho_n} - \alpha_{\rho}}{\rho} + \left(1 - \frac{1}{\rho_n}\right) \alpha_{\rho} + \rho_n + \rho
\leq c_0
\]  
(2.55)

for some constant $c_0$ depending only on $\rho$, $\alpha_{\rho}$ and $(\alpha_{\rho}/\rho)'$. Moreover, by Lemmas 2.9 and 2.10, and the estimate (2.53), one finds
\[
J_{\rho}(u_n) \leq \frac{\rho}{\rho_n} J_{\rho_n}(u_n) \leq \frac{\rho}{\rho_n} (\alpha_{\rho_n} + \rho - \rho_n) \leq \alpha_{\rho} + C (\rho - \rho_n)
\]  
(2.56)

for some constant $C$ independent of $n$.

Suppose that $\alpha_{\rho}$ is not a critical value of $J_{\rho}$. Since any bounded Palais-Smale sequence must converge to a critical point of $J_{\rho}$ (see [15, Lemma 3.1]), there would exist a $\delta > 0$ such that
\[
\|dJ_{\rho}(u)\|_{(W^{1,2}_0(\Sigma))^*} \geq 2\delta
\]  
(2.57)

for all $u \in \mathcal{A}_\delta$, where
\[
\mathcal{A}_\delta = \left\{ u \in W^{1,2}_0(\Sigma) : \int_\Sigma |\nabla g u|^2 dv_g \leq 2c_0, |J_{\rho}(u) - \alpha_{\rho}| < \delta \right\}.
\]  
(2.58)

It follows from (2.54)–(2.56) that $\mathcal{A}_\delta \neq \emptyset$. Let $X_{\rho} : \mathcal{A}_\delta \to W^{1,2}_0(\Sigma)$ be a pseudo-gradient vector field for $J_{\rho}$ in $\mathcal{A}_\delta$, namely, a locally Lipschitz vector field satisfying $\|X_{\rho}\|_{W^{1,2}_0(\Sigma)} \leq 1$ and
\[
dJ_{\rho}(u)(X_{\rho}(u)) \leq -\delta.
\]  
(2.59)

Here, we have used (2.57). One can check that as $n \to \infty$, $dJ_{\rho_n}(u)$ converges to $dJ_{\rho}(u)$ in $(W^{1,2}_0(\Sigma))^*$ uniformly in $u$ with $\int_\Sigma |\nabla g u|^2 dv_g \leq C$. Thus, $X_{\rho}$ is also a pseudo-gradient vector field for $J_{\rho_n}$ in $\mathcal{A}_{\delta}$. Moreover, it holds that for all $u \in \mathcal{A}_{\delta}$ and sufficiently large $n$,
\[
dJ_{\rho_n}(u)(X_{\rho}(u)) \leq -\delta/2.
\]  
(2.60)

Take a Lipschitz continuous cut-off function $\eta$ such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $\mathcal{A}_{\delta}/2$, and $\eta \equiv 0$ outside $\mathcal{A}_{\delta}$. Let $\psi : W^{1,2}_0(\Sigma) \times [0, +\infty)$ be the flow generated by $\eta X_{\rho}$, which satisfies
\[
\begin{cases}
\frac{\partial}{\partial s} \psi(u, s) = \eta(\psi(u, s)) X_{\rho}(\psi(u, s)), \\
\psi(u, 0) = u.
\end{cases}
\]

This flow has long time existence because it always remains stationary outside $\mathcal{A}_{\delta}$. It follows from (2.59) that for all $u \in \mathcal{A}_{\delta}/2$,
\[
\left. \frac{d}{ds} \right|_{s=0} J_{\rho}(\psi(u, s)) = dJ_{\rho}(u)(X_{\rho}(u)) \leq -\delta.
\]  
(2.61)

In view of (2.49) and (2.58), one easily sees $\phi_{\lambda, \sigma} \notin \mathcal{A}_{\delta}$ for all $\sigma \in \Sigma_{\epsilon, k}$. Since $J_{\rho}(\psi(\phi_{\lambda, \sigma}, s))$ is decreasing in $s$, it holds that $J_{\rho}(\psi(\phi_{\lambda, \sigma}, s)) \leq -4L$ for all $s \in [0, +\infty)$. Hence $\psi(\gamma_{n}(\sigma, 1), s) \notin \mathcal{A}_{\delta}$, and thus $\psi(\gamma_{n}(\sigma, 1), s) \equiv \psi(\gamma_{n}(\sigma, 1), 0) = \phi_{\lambda, \sigma}$ for all $\sigma \in \Sigma_{\epsilon, k}$ and all $s \in [0, +\infty)$. As a consequence, if we write $\psi_{s}(\cdot) = \psi(\cdot, s)$, then we have $\psi_{s} \circ \gamma_{n} \in \Gamma_{\lambda}$. By (2.56) and the monotonicity of $J_{\rho}(\psi_{s}(u))$ in $s$, we obtain
\[
\alpha_{\rho} \leq \sup_{u \in \psi_{s} \circ \gamma_{n}(\Sigma_{\epsilon, k})} J_{\rho}(u) \leq \sup_{u \in \gamma_{n}(\Sigma_{\epsilon, k})} J_{\rho}(u) \leq \alpha_{\rho} + C(\rho - \rho_n).
\]  
(2.62)
We now claim that
\[ \sup_{u \in \psi \circ \gamma_n(\Sigma, k)} J_\rho(u) \text{ is achieved in } \mathcal{M}_{\delta/2}. \]  
(2.63)

In fact, since \( \hat{\Sigma}_{\epsilon, k} \) is a compact metric space, the continuous function \( J_\rho(\psi \circ \gamma_n(\cdot, \cdot)) \) attains its supremum at some \((\sigma_0, t_0) \in \hat{\Sigma}_{\epsilon, k}\). As a result, the function \( u_{s,n} = \psi_s \circ \gamma_n(\sigma_0, t_0) \) achieves \( \sup_{u \in \psi \circ \gamma_n(\Sigma, k)} J_\rho(u) \). If \( n \) is chosen sufficiently large, (2.62) implies that \( \alpha_\rho \leq J_\rho(u_{s,n}) \leq \alpha_\rho + \delta/2 \). By (2.60), \( J_{\rho_n}(\psi_s(u)) \) is decreasing in \( s \), which together with (2.53) gives \( J_{\rho_n}(u_{s,n}) \leq \alpha_{\rho_n} + \rho - \rho_n \). It then follows that
\[
\begin{align*}
\frac{1}{2} \int_{\Sigma} |\nabla g u_{s,n}|^2 dv_g &= \left( \frac{1}{\rho_n} - \frac{1}{\rho} \right)^{-1} \left( \frac{J_{\rho_n}(u_{s,n})}{\rho_n} - J_\rho(u_{s,n}) \right) \\
&\leq \left( \frac{1}{\rho_n} - \frac{1}{\rho} \right)^{-1} \left( \frac{\alpha_{\rho_n} + \rho - \rho_n}{\rho_n} - \frac{\alpha_\rho}{\rho} \right) \\
&\leq c_0,
\end{align*}
\]
where \( c_0 \) is the same constant as in (2.55). Therefore, \( u_{s,n} \in \mathcal{M}_{\delta/2} \), and our claim is confirmed.

Let \( \delta > 0 \) and \( \Sigma_n = \psi_s \circ \gamma_n \). Then by (2.61) and (2.63), we have
\[
\frac{d}{ds} \sup_{s = \delta/2} J_\rho(\psi_s \circ \gamma_n(\sigma, t)) = \frac{d}{ds} \sup_{s = \delta/2} J_\rho(\psi_s \circ \gamma_n(\sigma, t)) \\
= \frac{d}{ds} \sup_{s = \delta/2} J_\rho(\psi_s \circ \gamma_n(\sigma, t)) \\
\leq \sup_{u \in \mathcal{M}_{\delta/2}} \frac{d}{ds} J_\rho(u) \\
\leq -\delta.
\]
(2.64)

Using the Newton-Leibniz formula, we conclude from (2.62) and (2.64) that
\[
\sup_{u \in \psi \circ \gamma_n(\Sigma, k)} J_\rho(u) < \alpha_\rho,
\]
if \( s > 0 \) is sufficiently large. This contradicts the definition of \( \alpha_\rho \), and ends the proof of the lemma. \( \Box \)

### 2.7 Existence for all \( \rho \in (8k\pi, 8(k + 1)\pi) \)

In this subsection, we use the previous analysis to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** For any \( \rho \in (8k\pi, 8(k + 1)\pi) \) and \( k \in \mathbb{N}^* \), there are two constants \( a \) and \( b \) with \( 8k\pi < a < b < 8(k + 1)\pi \). By Lemma 2.11, we may take an increasing sequence of numbers \( (\rho_n) \subset \Lambda_{a,b} \) such that \( \rho_n \to \rho \) and \( \alpha_{\rho_n} \) is achieved by \( u_n \in W^{1,2}_0(\Sigma) \). Moreover, \( u_n \) satisfies
\[
\Delta_g u_n = \rho_n \int_{\Sigma} h u_n dv_g \quad \text{in } \Sigma.
\]
(2.65)

By Lemma 2.9,
\[
\alpha_{\rho_n} \leq \frac{b}{a} \alpha_a.
\]
(2.66)

Denoting \( v_n = u_n - \log \int_{\Sigma} h u_n dv_g \), by (2.65), we have
\[
\begin{cases}
\Delta_g v_n = \rho_n h u_n, \\
\int_{\Sigma} h u_n dv_g = 1.
\end{cases}
\]
By Lemma 2.1, \((u_n)\) is bounded in \(L^\infty(\Sigma)\). Let \(\Omega_1, \ldots, \Omega_{k+1}\) be disjoint sub-domains of \(\Sigma\). By Lemma 2.2,
\[
\log \int_{\Sigma} e^{u_n} dv_{\Sigma} \leq \frac{1}{16(k+1)\pi} \int_{\Sigma} |\nabla u_n|^2 dv_{\Sigma} + C_\epsilon
\]
for any \(\epsilon > 0\) and some constant \(C_\epsilon > 0\). This together with (2.66) implies that for \(0 < \epsilon < 16(k+1)\pi - 2b\),
\[
\frac{1}{2} \int_{\Sigma} |\nabla u_n|^2 dv_{\Sigma} = J_{\rho_n}(u_n) + \rho_n \log \int_{\Sigma} h e^{u_n} dv_{\Sigma} \\
\leq \frac{b}{16(k+1)\pi - \epsilon} \int_{\Sigma} |\nabla u_n|^2 dv_{\Sigma} + C.
\]
It then follows that \((u_n)\) is bounded in \(W^{1,2}_0(\Sigma)\). Without loss of generality, we assume that \(u_n\) converges to \(u_\rho\) weakly in \(W^{1,2}_0(\Sigma)\), strongly in \(L^p(\Sigma)\) for any \(p > 1\), and almost everywhere in \(\Sigma\). Moreover, \(e^{u_n}\) converges to \(e^{u_\rho}\) strongly in \(L^p(\Sigma)\) for any \(p > 1\). By (2.65), \(u_\rho\) is a distributional solution of (1.3). Hence, \(u_\rho\) is a critical point of \(J_{\rho}\).

\[\square\]

3 The Neumann boundary value problem

In this section, we shall prove Theorem 1.2 by the min-max method. Since part of the proof is analogous to that of Theorem 1.1, we only give its outline but stress the difference. In Subsection 3.1, we prove a compactness result for solutions of (1.4). In Subsection 3.2, we derive an improved Trudinger-Moser inequality for functions \(u \in W^{1,2}(\Sigma)\) with \(\int_{\Sigma} udv_{\Sigma} = 0\). In Subsection 3.3, we construct two continuous maps between sub-levels of \(J_\rho\) and the topological space \(\mathcal{A}_k\), where \(J_\rho\) and \(\mathcal{A}_k\) are defined as in (3.26) and (3.27), respectively. In Subsection 3.4, we construct min-max levels of \(J_\rho\). The remaining part of the proof of Theorem 1.2 is outlined in Subsection 3.5.

3.1 Compactness analysis

Let \((\rho_n)\) be a number sequence tending to \(\rho \in \mathbb{R}\), \((h_n)\) be a function sequence converging to \(h \in C^{1}(\Sigma)\), and \((u_n)\) be a sequence of solutions to
\[
\begin{align*}
\Delta_g u_n &= \rho_n \left( \frac{h_n e^{u_n}}{\int_{\Sigma} h_n e^{u_n} dv_{\Sigma}} - \frac{1}{|\Sigma|} \right) \quad \text{in } \Sigma, \\
\partial u_n / \partial v &= 0 \quad \text{on } \partial \Sigma, \\
\int_{\Sigma} u_n dv_{\Sigma} &= 0.
\end{align*}
\]
(3.1)

Define
\[
v_n = u_n - \log \int_{\Sigma} h_n e^{u_n} dv_{\Sigma}.
\]
Then \(\Delta_g v_n = \rho_n (h_n e^{v_n} - 1/|\Sigma|)\) and \(\int_{\Sigma} h_n e^{v_n} dv_{\Sigma} = 1\). Concerning the compactness of \((u_n)\), we have an analog of Lemma 2.1.

**Lemma 3.1.** Assume that \(\rho\) is a positive number and \(h\) is a positive function. Up to a subsequence, one of the following alternatives holds:
(i) \((u_n)\) is bounded in \(L^\infty(\Sigma)\);
(ii) \((v_n)\) converges to \(-\infty\) uniformly in \(\Sigma\);
(iii) there exists a finite singular set \(S = \{p_1, \ldots, p_m\} \subset \Sigma\) such that for any \(1 \leq j \leq m\), there is a sequence of points \(\{p_{j,n}\} \subset \Sigma\) satisfying \(p_{j,n} \to p_j, u_n(p_{j,n}) \to +\infty\), and \(v_n\) converges to \(-\infty\) uniformly on any compact subset of \(\Sigma \setminus S\) as \(n \to \infty\). Moreover, if \(S\) has \(\ell\) points in \(\Sigma\) and \((m - \ell)\) points on \(\partial \Sigma\), then
\[
\rho_n \int_{\Sigma} h_n e^{v_n} dv_{\Sigma} \to 4(m + \ell)\pi.
\]
Proof. We modify arguments in the proof of Lemma 2.1, and divide the proof into two parts.

Part 1. Analysis in the interior domain $\Sigma$.

Let $(u_n)$ be a sequence of solutions to (3.1). By the Green representation formula (see [38]), we have
\[
\|u_n\|_{W^{1,q}(\Sigma)} \leq C_q, \quad \forall 1 < q < 2. \tag{3.2}
\]
Since $(h_n)$ converges to $h > 0$ in $C^1(\Sigma)$, there exists some constant $C > 0$ such that for all $n \in \mathbb{N},$
\[
\int_{\Sigma} e^{u_n} dv_\Sigma \leq C. \tag{3.3}
\]
Moreover, Jensen's inequality implies
\[
\liminf_{n \to \infty} \int_{\Sigma} e^{u_n} dv_\Sigma \geq |\Sigma|.
\]
Without loss of generality, we assume that $\rho_n h_n e^{v_n} dv_\Sigma$ converges to some nonnegative measure $\mu$ on $\Sigma$. If $\mu(x^*) < 4\pi$ for some $x^* \in \Sigma$, then there exist two positive constants $\epsilon_0$ and $r_0$ verifying
\[
\int_{B_{r^*}(r_0)} \rho_n h_n e^{v_n} dv_\Sigma \leq 4\pi - \epsilon_0.
\]
In view of (3.1), by a result of Brezis and Merle [5, Theorem 1] and elliptic estimates, we have that $\mu$ is bounded in $L^\infty(B_{r^*}(r_0/2))$. This leads to $\mu(x^*) = 0$. Define a set $\mathcal{S} = \{x \in \Sigma : \mu(x) \geq 4\pi\}$. If $\mathcal{S} \neq \emptyset$, then by almost the same argument as the proof of (2.4), we conclude that for any compact set $A \subset \Sigma \setminus \mathcal{S}$, it holds that
\[
v_n \to -\infty \text{ uniformly in } x \in A. \tag{3.4}
\]
Assume $\mathcal{S} = \{x_1, \ldots, x_j\}$ for some positive integer $j$. We shall show that $\mu(x_i) = 8\pi$ for all $1 \leq i \leq j$. Without loss of generality, it suffices to prove $\mu(x_1) = 8\pi$. For this purpose, we choose an isothermal coordinate system $\phi : U \to \mathbb{B}_1 = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 < 1\}$ near $x_1$. In such coordinates, the metric $g$ and the Laplace-Beltrami operator $\Delta_\phi$ are represented by $g = e^{\psi(y)}(dy_1^2 + dy_2^2)$ and $\Delta_\phi = -e^{-\psi(y)}\Delta_{\mathbb{R}^2}$, respectively, where $\psi$ is a smooth function with $\psi(0, 0) = 0$, and $\Delta_{\mathbb{R}^2} = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2$ denotes the standard Laplacian on $\mathbb{R}^2$. Set $\tilde{u} = u \circ \phi^{-1}$ for any function $u : U \to \mathbb{R}$. Since $(u_n)$ satisfies (3.1), $\tilde{u}_n = u_n \circ \phi^{-1}$ satisfies
\[
-\Delta_{\mathbb{R}^2} \tilde{u}_n(y) = e^{\psi(y)}\rho_n(\tilde{h}_n(y))e^{\tilde{u}_n(y)} - |\Sigma|^{-1}, \quad y \in \mathbb{B}_1. \tag{3.5}
\]
Multiplying both sides of (3.5) by $y \cdot \nabla \mathbb{R}^2 \tilde{u}_n(y)$, by integration by parts, we have
\[
\frac{r}{2} \int_{\partial \mathbb{B}_r} |\nabla_{\mathbb{R}^2} \tilde{u}_n|^2 d\sigma - r \int_{\partial \mathbb{B}_r} \langle \nabla_{\mathbb{R}^2} \tilde{u}_n, \nu \rangle e^{\tilde{u}_n} d\sigma = r \int_{\mathbb{B}_r} e^\psi \rho_n \tilde{h}_n e^{v_n} d\sigma - \int_{\mathbb{B}_r} e^\gamma \rho_n \langle \nabla_{\mathbb{R}^2} e^\psi \tilde{h}_n, y \rangle dy - 2 \int_{\mathbb{B}_r} e^\psi \rho_n \tilde{h}_n e^{v_n} dy + \frac{\rho_n}{|\Sigma|} \int_{\mathbb{B}_r} e^\psi y \cdot \nabla_{\mathbb{R}^2} \tilde{u}_n dy, \tag{3.6}
\]
where $\mathbb{B}_r = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 < r\}, \partial \mathbb{B}_r = \{y \in \mathbb{R}^2 : y_1^2 + y_2^2 = r\}$, and $\nu$ denotes the unit outward vector on $\partial \mathbb{B}_r$. In view of (3.4), $(u_n)$ converges to a Green function $G(x_1, \cdot)$ weakly in $W^{1,q}(\Sigma)$ and in $C^2_{\text{loc}}(\Sigma \setminus \mathcal{S})$. Locally, $G(x_1, \cdot)$ satisfies
\[
\Delta_{\mathbb{R}^2} G(x_1, z) = \mu(x_1) \delta_{x_1}(z) - \rho |\Sigma|^{-1}, \quad \forall z \in \phi^{-1}(\mathbb{B}_1).
\]
Clearly,
\[
\tilde{G}(y) = G(x_1, \phi^{-1}(y)) = -\frac{\mu(x_1)}{2\pi} \log |y| + \eta(y)
\]
for some $\eta \in C^2(\mathbb{B}_1)$. Passing to the limit $n \to \infty$ first, and then $r \to 0$ in (3.6), we obtain
\[
\mu(x_1) = \lim_{r \to 0} \left( \frac{r}{2} \int_{\partial \mathbb{B}_r} \langle \nabla_{\mathbb{R}^2} \tilde{G}, \nu \rangle^2 d\sigma - \frac{r}{4} \int_{\partial \mathbb{B}_r} |\nabla_{\mathbb{R}^2} \tilde{G}|^2 d\sigma \right) = \frac{(\mu(x_1))^2}{8\pi}. \tag{3.7}
\]
This immediately leads to $\mu(x_1) = 8\pi$. In conclusion, we have

$$\mu(x_i) = 8\pi \quad \text{for all } 1 \leq i \leq j.$$  \hspace{1cm} (3.8)

**Part II.** Analysis on the boundary $\partial \Sigma$.

Let $x^* \in \Sigma$ be fixed. Note that $\rho_n h_n e^{\psi} dv_\Sigma$ converges to the nonnegative Radon measure $\mu$ on $\Sigma$ as $n \to \infty$. If $\mu(x^*) < 2\pi$, there exist a neighborhood $V$ of $x^*$ and a number $\gamma_0 > 0$ such that

$$\int_V \rho_n h_n e^{\psi} dv_\Sigma \leq 2\pi - \gamma_0.$$  \hspace{1cm} (3.9)

Without loss of generality, we take an isothermal coordinate system $(V, \phi, \{y_1, y_2\})$ such that $\phi(x^*) = (0, 0)$, and $\phi : V \to \mathbb{B}_1^+ \cap \Gamma = \{(y_1, y_2) : y_1^2 + y_2^2 < 1, y_2 \geq 0\}$, where $\Gamma = \{(y_1, y_2) : |y_1| < 1, y_2 = 0\}$. Moreover, in this coordinate system, the metric $g = e^{\psi(y)}(dy_1^2 + dy_2^2)$ and the Laplace-Beltrami operator $\Delta_g = -e^{-\psi(y)} \Delta$, where $\psi : \mathbb{B}_1^+ \cap \Gamma \to \mathbb{R}$ is a smooth function with $\psi(0, 0) = 0$; moreover, $\partial/\partial \nu = e^{-\psi(y)/2}\partial/\partial y_2$. For more details about isothermal coordinates on the boundary, we refer the readers to [38, Section 2]. Now the local version of (3.1) is

$$\begin{cases}
-\Delta_{\mathbb{B}_1^+}(u_n \circ \phi^{-1})(y) = e^{\psi(y)} \rho_n ((h_n \circ \phi^{-1})(y)e^{(u_n \circ \phi^{-1})(y)} - |\Sigma|^{-1}) & \text{in } \mathbb{B}_1^+, \\
\frac{\partial}{\partial y_2}(u_n \circ \phi^{-1})(y) = 0 & \text{on } \Gamma.
\end{cases}$$  \hspace{1cm} (3.10)

For any function $u : V \to \mathbb{R}$, we define a function $\tilde{u} : \mathbb{B}_1 \to \mathbb{R}$ by

$$\tilde{u}(y_1, y_2) = \begin{cases} u \circ \phi^{-1}(y_1, y_2), & \text{if } y_2 \geq 0, \\ u \circ \phi^{-1}(y_1, -y_2), & \text{if } y_2 < 0. \end{cases}$$  \hspace{1cm} (3.11)

One can easily derive from (3.10) that $\tilde{u}_n$ is a distributional solution of

$$-\Delta_{\mathbb{B}_1^+} \tilde{u}_n(y) = \tilde{f}_n(y), \quad y \in \mathbb{B}_1,$$  \hspace{1cm} (3.12)

where $\tilde{f}_n$ is defined as in (3.11) and for $y \in \mathbb{B}_1^+ \cup \Gamma$,

$$f_n \circ \phi^{-1}(y) = e^{\psi(y)} \rho_n ((h_n \circ \phi^{-1})(y)e^{(u_n \circ \phi^{-1})(y)} - |\Sigma|^{-1}).$$

In view of (3.9) and the fact $\psi(0, 0) = 0$, there exists a number $r_0 \in (0, 1)$ such that

$$\int_{\mathbb{B}_{r_0}} |\tilde{f}_n(y)| dy \leq 4\pi - \gamma_0.$$

Let $w_n$ be a solution of

$$\begin{cases}
-\Delta_{\mathbb{B}_{r_0}} w_n = \tilde{f}_n & \text{in } \mathbb{B}_{r_0}, \\
w_n = 0 & \text{on } \partial\mathbb{B}_{r_0}.
\end{cases}$$

By [5, Theorem 1], there exists some constant $C$ depending only on $\epsilon_0$ and $r_0$ such that

$$\int_{\mathbb{B}_{r_0}} \exp \left( \frac{(4\pi - \gamma_0/2)|w_n|}{\|\tilde{f}_n\|_{L^1(\mathbb{B}_{r_0})}} \right) dy \leq C.$$  \hspace{1cm} (3.13)

Hence, there exists some $q_0 > 1$ such that

$$\|e^{w_n}\|_{L^{q_0}(\mathbb{B}_{r_0})} \leq C.$$  \hspace{1cm} (3.14)

Let $\eta_n = \tilde{u}_n - w_n$. Then $\eta_n$ satisfies

$$\begin{cases}
-\Delta_{\mathbb{B}_{r_0}} \eta_n = 0 & \text{in } \mathbb{B}_{r_0}, \\
\eta_n = \tilde{u}_n & \text{on } \partial\mathbb{B}_{r_0}.
\end{cases}$$
Noticing (3.2) and (3.13), we have by applying elliptic estimates to (3.14) that
\[
\|\eta_n\|_{L^\infty(B_{r_0}/2)} \leq C. \tag{3.15}
\]
Combining (3.3), (3.13) and (3.15), we conclude \(\|\tilde{\eta}_n\|_{L^\infty(B_{r_0}/4)} \leq C\). Applying elliptic estimates to (3.12), we obtain \(\|u_n\|_{L^\infty(B_{r_0}/4)} \leq C\), which implies \(\|u_n\|_{L^\infty(\partial^{-1}(\mathcal{A}^+_{r_0/4}))} \leq C\). In conclusion, we have that if \(\mu(x^*) < 2\pi\), then \((u_n)\) is uniformly bounded near \(x^*\). This also leads to \(\mu(x^*) = 0\).

If \(\mu(x^*) \geq 2\pi\), in the same coordinate system \((V, \phi, (y_1, y_2))\) as above, \(\tilde{f}_n(y) dy\) converges to a Radon measure \(\tilde{\mu}\) with \(\tilde{\mu}(0,0) = 2\mu(x^*) \geq 4\pi\). Obviously, there exists some \(r_1 > 0\) such that for any \(x \in \mathbb{B}_{r_1} \setminus \{(0,0)\}\), \(\tilde{\mu}(x) = 0\). Using the same argument as the proof of (3.4), we conclude that for any compact set \(A \subset \mathbb{B}_{r_1} \setminus \{(0,0)\}\), \(\tilde{\mu}_n\) converges to \(-\infty\) uniformly in \(A\). This leads to that \(\tilde{f}_n(y) dy\) converges to the Dirac measure \(\tilde{\mu}(0,0)\delta_{(0,0)}(y)\). Recalling (3.2), we have that \(\tilde{u}_n\) converges to \(\tilde{G}_0\) weakly in \(W^{1,q}(B_{r_1})\) and a.e. in \(\mathbb{B}_{r_1}\), where \(\tilde{G}_0\) satisfies
\[
-\Delta_{\mathbb{R}^2}\tilde{G}_0(y) = \tilde{\mu}(0,0)\delta_{(0,0)}(y) - \rho|\Sigma|^{-1}, \quad y \in \mathbb{B}_{r_1}.
\]
Clearly, \(\tilde{G}_0\) is represented by
\[
\tilde{G}_0(y) = -\frac{\tilde{\mu}(0,0)}{2\pi} \log |y| + A_0 + O(|y|)
\]
as \(y \to 0\), where \(A_0\) is a constant. Noting that \(\tilde{u}_n\) converges to \(-\infty\) locally uniformly in \(\mathbb{B}_{r_1} \setminus \{(0,0)\}\), we have by applying elliptic estimates to (3.12) that
\[
\tilde{u}_n \to \tilde{G}_0 \quad \text{in} \quad C^2_{\text{loc}}(\mathbb{B}_{r_1} \setminus \{(0,0)\}).
\]
Multipling both sides of (3.12) by \(y \cdot \nabla_{\mathbb{R}^2} \tilde{u}_n(y)\), completely analogous to (3.6) and (3.7), we obtain \(\tilde{\mu}(0,0) = 8\pi\), and thus
\[
\mu(x^*) = 4\pi. \tag{3.16}
\]

Note that if \(\mu(x^*_i) > 0\) for some \(x_i \in \Sigma\), then there must exist \(x_{i,n} \subset \Sigma\) satisfying \(u_n(x_{i,n}) \to +\infty\). If not, \((u_n)\) would be uniformly bounded near \(x\), which leads to \(\mu(x) = 0\). The lemma then follows from (3.8) and (3.16) immediately. \(\Box\)

### 3.2 An improved Trudinger-Moser inequality

For a compact surface with smooth boundary, it was proved by Yang [36] that
\[
\sup_{u \in W^{1,2}(\Sigma), \|\nabla u\|^2_{L^\infty(\Sigma)} \leq 1} \int_{\Sigma} e^{2\pi u^2} dv_g < \infty. \tag{3.17}
\]
Define \(\varpi = \frac{1}{|\Sigma|} \int_{\Sigma} u dv_g\). By (3.17) and Young’s inequality, we obtain
\[
\log \int_{\Sigma} e^{u-\varpi} dv_g \leq \log \int_{\Sigma} e^{2\pi \frac{u-\varpi}{\|\nabla u\|^2}} \frac{1}{\|\nabla u\|^2} dv_g
\]
\[
= \frac{1}{8\pi} \int_{\Sigma} \|\nabla u\|^2 dv_g + \log \int_{\Sigma} e^{2\pi \frac{u-\varpi}{\|\nabla u\|^2}} dv_g
\]
\[
\leq \frac{1}{8\pi} \int_{\Sigma} \|\nabla u\|^2 dv_g + C.
\]
Hence,
\[
\log \int_{\Sigma} e^{u} dv_g \leq \frac{1}{8\pi} \int_{\Sigma} \|\nabla u\|^2 dv_g + \frac{1}{|\Sigma|} \int_{\Sigma} u dv_g, \quad \forall u \in W^{1,2}(\Sigma). \tag{3.18}
\]
Lemma 3.2. Let $b_0$ and $\gamma_0$ be two positive constants, and $\Omega_1, \ldots, \Omega_k$ be $k$ domains of $\Sigma$ with $\text{dist}(\Omega_i, \Omega_j) \geq b_0$ for all $1 \leq i < j \leq k$. Then for any $\epsilon > 0$, there exists some constant $C$ depending only on $b_0$, $\gamma_0$, $k$ and $\epsilon$ such that

$$\log \int_{\Sigma} e^u dv_g \leq \frac{1}{8k\pi - \epsilon} \int_{\Sigma} |\nabla_g u|^2 dv_g + \frac{1}{|\Sigma|} \int_{\Sigma} udv_g + C$$

(3.19)

for all $u \in W^{1,2}(\Sigma)$ with

$$\int_{\Omega_i} e^u dv_g \leq \frac{1}{\gamma_0} \int_{\Sigma} e^u dv_g, \quad i = 1, \ldots, k.$$  

(3.20)

Proof. We follow the lines of Chen and Li [11]. Take smooth functions $\phi_1, \ldots, \phi_k$ defined on $\Sigma$ satisfying

$$\text{supp} \phi_i \cap \text{supp} \phi_j = \emptyset, \quad \forall \ 1 \leq i < j \leq k,$$

(3.21)

$$\phi_i \equiv 1 \text{ on } \Omega_i, \quad 0 \leq \phi_i \leq 1 \text{ on } \Sigma, \quad \forall \ 1 \leq i \leq k,$$

(3.22)

and for some positive constant $b_1$ depending only on $b_0$ and the metric $g$,

$$|\nabla_g \phi_i| \leq b_1, \quad \forall \ 1 \leq i \leq k.$$  

(3.23)

For any $u \in W^{1,2}(\Sigma)$ satisfying (3.20), we have $\phi_i u \in W^{1,2}(\Omega)$ for all $1 \leq i \leq k$, and thus (3.18) implies

$$\int_{\Sigma} e^u dv_g \leq \frac{1}{\gamma_0} \int_{\Omega_i} e^\phi_i u dv_g \leq \frac{1}{\gamma_0} \int_{\Sigma} e^\phi_i u dv_g \leq \frac{1}{\gamma_0} \exp \left( \frac{1}{8\pi} \|\nabla_g (\phi_i u)\|_{L^2(\Sigma)}^2 + \frac{1}{|\Sigma|} \int_{\Sigma} \phi_i u dv_g + C \right).$$

Note that (3.21) gives

$$\sum_{i=1}^k \|\nabla_g (\phi_i u)\|_{L^2(\Sigma)}^2 = \|\nabla_g \left( u \sum_{i=1}^k \phi_i \right)\|_{L^2(\Sigma)}^2,$$

and (3.22) implies

$$\sum_{i=1}^k \int_{\Sigma} \phi_i u dv_g \leq \int_{\Sigma} |u| dv_g \leq \frac{1}{|\Sigma|^{1/2}} \|u\|_{L^2(\Sigma)}.$$

Combining the above three estimates, (3.23), Young’s inequality and an elementary inequality

$$a \leq (a_1 \cdots a_k)^{1/k}, \quad \text{if } 0 \leq a \leq a_i, \quad i = 1, \ldots, k,$$

we obtain

$$\int_{\Sigma} e^u dv_g \leq \frac{1}{\gamma_0} \left( \prod_{i=1}^k \exp \left( \frac{1}{8\pi} \|\nabla_g (\phi_i u)\|_{L^2(\Sigma)}^2 + \frac{1}{|\Sigma|} \int_{\Sigma} \phi_i u dv_g + C \right) \right)^{1/k}$$

$$= \frac{e^C}{\gamma_0} \exp \left( \frac{1}{8k\pi} \sum_{i=1}^k \|\nabla_g (\phi_i u)\|_{L^2(\Sigma)}^2 + \frac{1}{|\Sigma|} \sum_{i=1}^k \int_{\Sigma} \phi_i u dv_g + C \right)$$

$$= \frac{e^C}{\gamma_0} \exp \left( \frac{1}{8k\pi} \left( u \sum_{i=1}^k \phi_i \right) \|\nabla_g \|_{L^2(\Sigma)}^2 + \frac{1}{|\Sigma|} \sum_{i=1}^k \int_{\Sigma} \phi_i u dv_g + C(\epsilon_1) \|u\|_{L^2(\Sigma)}^2 \right).$$

(3.24)

Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \lambda_{k+1} \leq \cdots$ be all the eigenvalues of the Laplace-Beltrami operator with respect to the Neumann boundary condition. Clearly $\lambda_i \to +\infty$ as $i \to \infty$. Let $\{\epsilon_i\}_{i=1}^\infty$ be the
corresponding unit normal eigenfunctions, i.e., \( \Delta_g e_i = \lambda_i e_i \), \( \int_\Sigma e_i dv_g = 0 \) and \( \int_\Sigma e_i e_j dv_g = \delta_{ij} \) for \( i, j = 1, 2, \ldots \). It is known that \( \mathcal{H} := \{ u \in W^{1,2}(\Sigma) : \pi = 0 \} = E_\ell + E^\perp_\ell \), where \( E_\ell = \text{span}\{ e_1, \ldots, e_\ell \} \) and \( E^\perp_\ell = \text{span}\{ e_{\ell+1}, e_{\ell+2}, \ldots \} \). Let \( u \in W^{1,2}(\Sigma) \) be given as above. We decompose \( u - \pi = v + w \) with \( v \in E_\ell \) and \( w \in E^\perp_\ell \). Thus, the Poincaré inequality implies

\[
\|v\|_{C^0(\Sigma)} \leq \sum_{i=1}^\ell \|e_i\|_{C^0(\Sigma)} \int_\Sigma |u - \pi| e_i dv_g \leq C\ell\|\nabla g u\|_{L^2(\Sigma)},
\]

while by the definition of the \((\ell + 1)\)-th eigenvalue \( \lambda_{\ell+1} \),

\[
\int_\Sigma w^2 dv_g \leq \frac{1}{\lambda_{\ell+1}} \int_\Sigma |\nabla g w|^2 dv_g.
\]

Having the above two estimates and applying (3.24) to \( w \), we have

\[
\int_\Sigma e^{u-\pi} dv_g \leq C\ell\|\nabla g u\|_{L^2(\Sigma)} \int_\Sigma e^u dv_g \leq C\ell\|\nabla g u\|_{L^2(\Sigma)} \exp \left( \frac{1}{8k\pi} (1 + \epsilon_1) \|\nabla g w\|_{L^2(\Sigma)}^2 + \frac{C(\epsilon_1)}{\lambda_{\ell+1}} \|\nabla g w\|_{L^2(\Sigma)}^2 \right)
\]

\[
\leq C\epsilon \|\nabla g u\|_{L^2(\Sigma)} \exp \left( \frac{1}{8k\pi} \left( 1 + \epsilon_1 + \frac{C(\epsilon_1)}{\lambda_{\ell+1}} \right) \|\nabla g u\|_{L^2(\Sigma)}^2 \right).
\]

This together with Young’s inequality gives

\[
\log \int_\Sigma e^{u-\pi} dv_g \leq \frac{1}{8k\pi} \left( 1 + \epsilon_1 + \frac{C(\epsilon_1)}{\lambda_{\ell+1}} + \epsilon_1 \right) \int_\Sigma |\nabla g u|^2 dv_g + C_{\ell,k,\epsilon_1}.
\]  

(3.25)

Let \( 0 < \epsilon < 8k\pi \) be any given number. Choosing \( \epsilon_1 = \epsilon/(16k\pi - 2\epsilon) \), and then taking a sufficiently large \( \ell \) such that \( C(\epsilon_1)/\lambda_{\ell+1} \leq 1 \), by (3.25), we have

\[
\log \int_\Sigma e^{u-\pi} dv_g \leq \frac{1}{8k\pi} \epsilon \int_\Sigma |\nabla g u|^2 dv_g + C,
\]

where \( C \) is a constant depending only on \( b_0, \gamma_0, k \) and \( \epsilon \). This is exactly (3.19). \( \square \)

Define a functional \( J_\rho : W^{1,2}(\Sigma) \rightarrow \mathbb{R} \) by

\[
J_\rho(u) = \frac{1}{2} \int_\Sigma |\nabla g u|^2 dv_g - \rho \log \int_\Sigma e^u dv_g - \rho \int_\Sigma u dv_g.
\]  

(3.26)

From now on to the end of this section, \( J_\rho \) is always given as in (3.26).

### 3.3 Continuous maps between sub-levels of \( J_\rho \) and \( \Sigma_k \)

Let \( \Sigma_k \) be defined as in (2.29), and \( (\partial \Sigma)_k \) be defined by

\[
(\partial \Sigma)_k = \left\{ \sum_{i=1}^k t_i e_{x_i} : t_i \geq 0, \sum_{i=1}^k t_i = 1, x_i \in \partial \Sigma \right\}.
\]

**Lemma 3.3.** Let \( \rho \in (4k\pi, 4(k+1)\pi) \). Then for any sufficiently large \( L > 0 \), there exists a continuous retraction

\[
\Psi : J^{-L}_\rho = \{ u \in W^{1,2}(\Sigma) : J_\rho(u) \leq -L \} \rightarrow \Sigma_k.
\]

Moreover, if \( (u_n) \subset W^{1,2}(\Sigma) \) satisfies \( \int_\Sigma e^{\rho u_n} dv_g \rightarrow \sigma \in \Sigma_k \), then \( \Psi(u_n) \rightarrow \sigma \in \Sigma_k \).

**Proof.** Since the proof is almost the same as that of Lemma 2.4, we omit the details here. \( \square \)

For any finite set \( E \), we denote the number of all the distinct points of \( E \) by \( \sharp E \). We define

\[
\mathcal{F}_k = \{ \sigma \in \Sigma_k : \sharp(\text{supp } \sigma \cap \Sigma) + \sharp(\text{supp } \sigma) \leq k \}.
\]  

(3.27)
Let us explain what $\mathcal{K}_k$ means. Clearly, $\mathcal{K}_1 = (\partial \Sigma)_1$. For $k = 2$,

$$\mathcal{K}_2 = \{ \delta_2 : x \in \Sigma \} \cup \{ t \delta_2, + (1 - t) \delta_2 : 0 \leq t \leq 1, x_1, x_2 \in \partial \Sigma \} = (\partial \Sigma)_2 \cup \Sigma.$$ 

For $k = 3$, we write

$$\mathcal{K}_3 = \{ t \delta_3, + (1 - t) \delta_2 : 0 \leq t \leq 1, x_1, x_2, x_3 \in \partial \Sigma \} \cup \{ t_1 \delta_3 + t_2 \delta_2 + t_3 \delta_2 : 0 \leq t_1, t_2, t_3 \leq 1, x_1, x_2, x_3 \in \partial \Sigma, 1 \leq i \leq 3, t_1 + t_2 + t_3 = 1 \} = (\partial \Sigma)_3 \cup \mathcal{A}_3,$$

where $\mathcal{A}_3 = \{ t \delta_3, + (1 - t) \delta_2 : 0 \leq t \leq 1, x_1, x_2 \in \partial \Sigma \}$. We observe that $\dim \mathcal{A}_3 < \dim (\partial \Sigma)_3 = 5$, since $(\partial \Sigma)_3 \backslash (\partial \Sigma)_2$ is a smooth 5-dimensional manifold, and $\dim \mathcal{A}_3 \leq 4$.

**Lemma 3.4.** For any $k \geq 1$, $\mathcal{K}_k$ is non-contractible.

**Proof.** Obviously, $\mathcal{K}_1 = \partial \Sigma$ is non-contractible. For $k \geq 2$, based on the above observation, an induction argument shows $\mathcal{K}_k = (\partial \Sigma)_k \cup \mathcal{A}_k$, where $\dim \mathcal{A}_k < \dim (\partial \Sigma)_k = 2k - 1$. Though $(\partial \Sigma)_k$ is a combination of several branches of different dimensions, we still denote the maximum dimension of those branches by $\dim (\partial \Sigma)_k$. Arguing as in [17, Lemma 4.7], we have that $(\partial \Sigma)_k$ is non-contractible. (This was also noticed by Zhang et al. [40].) In fact, we have

$$H_{2k - 1}((\partial \Sigma)_k, Z_2) \neq \{0\}. \quad (3.28)$$

Give any $(2k - 1)$-dimensional closed chain $C_{2k - 1}$ and any $(2k - 1)$-dimensional boundary chain $\mathcal{E}_{2k - 1}$ of $(\partial \Sigma)_k$. Since $(\partial \Sigma)_k$ is a closed sub-topological space in $\mathcal{K}_k$, and $\dim ((\partial \Sigma)_k) = \dim (\mathcal{K}_k) = 2k - 1$, one easily sees that $C_{2k - 1}$ is also a closed chain of $\mathcal{K}_k$ and $\mathcal{E}_{2k - 1} = 0$ is also the boundary $(2k - 1)$-chain of $\mathcal{K}_k$. Hence,

$$H_{2k - 1}((\partial \Sigma)_k, Z_2) \subset H_{2k - 1}(\mathcal{K}_k, Z_2),$$

which together with (3.28) implies that $\mathcal{K}_k$ is non-contractible. \hfill $\square$

Take a smooth increasing function $\eta : \mathbb{R} \to \mathbb{R}$ satisfying $\eta(t) = t$ for $t \leq 1$ and $\eta(t) = 2$ for $t \geq 2$. Set $\eta_r(t) = r \eta(t/r)$ for $r > 0$. For $\lambda > 0, x \in \Sigma$, and $1 \leq \ell < m$, we define

$$\tilde{\phi}_{\lambda, \sigma}(x) = \log \left( \sum_{i=1}^{\ell} \frac{8\lambda^2}{2(1 + \lambda^2 \eta_{\epsilon}^2(\text{dist}(x, x_i)))^2} + \sum_{i=\ell+1}^{m} \frac{8\lambda^2}{(1 + \lambda^2 \eta_{\epsilon}^2(\text{dist}(x, x_i)))^2} \right) \quad (3.29)$$

and

$$\phi_{\lambda, \sigma}(x) = \tilde{\phi}_{\lambda, \sigma}(x) - \frac{1}{|\Sigma|} \int_{\Sigma} \tilde{\phi}_{\lambda, \sigma} dv_g. \quad (3.30)$$

**Lemma 3.5.** Let $\rho \in (4k\pi, 4(k + 1)\pi)$. If $\lambda > 0$ is chosen sufficiently large, and $r > 0$ is chosen sufficiently small, then for any $\sigma \in \mathcal{K}_k$, it holds that

$$J_{\rho}(\phi_{\lambda, \sigma}) \leq (4k\pi - \rho) \log \lambda \quad (3.31)$$

and

$$\int_{\Sigma} e^{\phi_{\lambda, \sigma}} dv_g \rightarrow \sigma \quad \text{as} \; \lambda \rightarrow +\infty. \quad (3.32)$$

**Proof.** Both the cases where $\text{supp} \sigma \cap \Sigma = \emptyset$ and $\text{supp} \sigma \cap \Sigma \neq \emptyset$ can be dealt with in the same way. Give $\sigma \in \mathcal{K}_k$. Without loss of generality, we assume $\text{supp} \sigma = \{x_1, \ldots, x_m\} \subset \Sigma$, $\text{supp} \sigma \cap \Sigma = \{x_1, \ldots, x_{\ell}\}$ and $m + \ell \leq k$. Let $\phi_{\lambda, \sigma}$ and $\phi_{\lambda, \sigma}$ be defined as in (3.29) and (3.30), respectively, where $\lambda > 0$ and $r > 0$. Write $r_i = r_i(x) = \text{dist}(x, x_i)$ for $x \in \Sigma$. A simple observation gives

$$\tilde{\phi}_{\lambda, \sigma}(x) = \begin{cases} \log(\frac{8\lambda^2}{(1 + 4\lambda^2 r_i^2)^2}) & \text{for} \; x \in \Sigma \setminus \bigcup_{i=1}^{k} B_{2r_i}(x_i), \\ \log\left(\frac{8\lambda^2}{(1 + \lambda^2 \eta_{\epsilon}^2(r_i))^2} + \frac{8\lambda^2}{(1 + 4\lambda^2 r_i^2)^2}\right) & \text{for} \; x \in B_{2r_i}(x_i). \end{cases} \quad (3.33)$$
where \( L_i = t_i / 2 \) for \( 1 \leq i \leq \ell, \) \( L_i = t_i \) for \( \ell + 1 \leq i \leq m, \) and \( B_{2r}(x_i) = \{ x \in \Sigma : \text{dist}(x, x_i) < 2r \} \) denotes a geodesic ball centered at \( x_i \) with radius \( 2r. \) One easily sees \( \phi_{\lambda, \sigma} \in W^{1,2}(\Sigma) \) and \( \int_\Sigma \phi_{\lambda, \sigma} dv_g = 0. \) For \( x \in B_{2r}(x_i) \) and \( i = 1, \ldots, m, \) a straightforward calculation shows

\[
\nabla_g \tilde{\phi}_{\lambda, \sigma}(x) = \frac{4 \lambda^2 r_i}{(1 + \lambda^2 \eta^2_i(r_i))} \left( \frac{4 \lambda^2 \eta_i(r_i) \eta'_i(r_i)}{1 + \lambda^2 \eta^2_i(r_i)} \right) \nabla_g r_i,
\]

and thus,

\[
|\nabla_g \tilde{\phi}_{\lambda, \sigma}(x)| \leq \frac{4 \lambda^2 \eta_i(r_i) \eta'_i(r_i)}{1 + \lambda^2 \eta^2_i(r_i)}.
\]

In view of (3.33), it holds that \( \nabla_g \tilde{\phi}_{\lambda, \sigma}(x) = 0 \) for \( x \in \Sigma \setminus \bigcup_{i=1}^m B_{2r}(x_i). \) We calculate that for \( 1 \leq i \leq \ell, \)

\[
\int_{B_{2r}(x_i)} \frac{4 \lambda^2 \eta_i(r_i) \eta'_i(r_i)}{1 + \lambda^2 \eta^2_i(r_i)} dv_g = 16 \pi (1 + O(r^2)) \left( \log(1 + \lambda^2 r^2) + \frac{1}{1 + \lambda^2 r^2} - 1 \right),
\]

and for \( \ell + 1 \leq i \leq m, \)

\[
\int_{B_{2r}(x_i)} \left( \frac{4 \lambda^2 \eta_i(r_i) \eta'_i(r_i)}{1 + \lambda^2 \eta^2_i(r_i)} \right) dv_g = 8 \pi (1 + O(r^2)) \left( \log(1 + \lambda^2 r^2) + \frac{1}{1 + \lambda^2 r^2} - 1 \right).
\]

For a fixed \( r > 0, \) since \( x_1, \ldots, x_m \) are arbitrary, one sees that \( \{B_{2r}(x_1), \ldots, B_{2r}(x_m)\} \) may have nonempty intersections, and thus,

\[
\int_\Sigma |\nabla_g \tilde{\phi}_{\lambda, \sigma}|^2 dv_g = \sum_{i=1}^m \int_{B_{2r}(x_i)} |\nabla_g \tilde{\phi}_{\lambda, \sigma}|^2 dv_g \\
\leq \sum_{i=1}^m \int_{B_{2r}(x_i)} \left( \frac{4 \lambda^2 \eta_i(r_i) \eta'_i(r_i)}{1 + \lambda^2 \eta^2_i(r_i)} \right)^2 dv_g \\
\leq (16 \pi^2 \ell + 8 \pi (m - \ell)) (1 + O(r^2)) \left( \log(1 + \lambda^2 r^2) + \frac{1}{1 + \lambda^2 r^2} - 1 \right) + O(1) \\
\leq 8 k \pi (1 + O(r^2)) \log \lambda^2 + C
\]

(3.34)

for some constant \( C \) depending only on \( r. \) Moreover, for any \( s \) and

\[
0 < s < \min \left\{ r^2 \frac{1}{2} \min_{1 \leq i < j \leq m} \text{dist}(x_i, x_j) \right\},
\]

it holds that

\[
\int_{\bigcup_{i=1}^m B_{2r}(x_i)} e^{\tilde{\phi}_{\lambda, \sigma}} dv_g = \int_{\bigcup_{i=1}^m B_{2r}(x_i)} e^{\tilde{\phi}_{\lambda, \sigma}} dv_g + O \left( \frac{1}{\lambda^2 s^2} \right) \\
= \sum_{i=1}^m \int_{B_{2r}(x_i)} \frac{8 \lambda^2 r_i}{(1 + \lambda^2 \eta^2_i(r_i))} dv_g + O \left( \frac{1}{\lambda^2 s^2} \right) \\
= 4 \pi (1 + O(s^2)) + O \left( \frac{1}{\lambda^2 s^2} \right)
\]

and

\[
\int_\Sigma e^{\tilde{\phi}_{\lambda, \sigma}} dv_g = O \left( \frac{1}{\lambda^2 r^4} \right).
\]

It follows that

\[
\int_\Sigma e^{\tilde{\phi}_{\lambda, \sigma}} dv_g = 4 \pi (1 + O(s^2)) + O \left( \frac{1}{\lambda^2 s^2} \right) + O \left( \frac{1}{\lambda^2 r^4} \right).
\]
Passing to the limit $\lambda \to +\infty$ first, and then $s \to 0+$, we have
\[
\lim_{\lambda \to +\infty} \int_{\Sigma} e^{\tilde{\phi}_{\lambda,\sigma}} dv_g = 4\pi. \tag{3.36}
\]
Note that there exists some constant $C$ depending only on $r$ such that
\[
\frac{1}{|\Sigma|} \int_{\Sigma} \tilde{\phi}_{\lambda,\sigma} dv_g \leq -\log \lambda^2 + C. \tag{3.37}
\]
Hence by (3.36) and (3.37),
\[
\int_{\Sigma} e^{\tilde{\phi}_{\lambda,\sigma}} dv_g \geq C(1 + o(1)) \lambda^2.
\]
This together with (3.34) gives
\[
J_\rho(\phi_{\lambda,\sigma}) = \frac{1}{2} \int_{\Sigma} |\nabla_g \phi_{\lambda,\sigma}|^2 dv_g - \rho \log \int_{\Sigma} h e^{\tilde{\phi}_{\lambda,\sigma}} dv_g \leq (4k\pi - \rho + O(r^2)) \log \lambda^2 + C_r.
\]
Since $\rho > 4k\pi$, choosing $r > 0$ sufficiently small and $\lambda > 0$ sufficiently large, we conclude (3.31).

Finally we prove (3.32). Let $\sigma = \sum_{i=1}^m t_i \delta_{x_i} \in \mathcal{K}$ be as above. For any $\varphi \in C^1(\Sigma)$, similar to (3.35), we calculate
\[
\int_{\Sigma} \varphi e^{\tilde{\phi}_{\lambda,\sigma}} dv_g = 4\pi \sum_{i=1}^k t_i \varphi(x_i) + O(s^2) + O\left(\frac{1}{\lambda^2 s^2}\right) + O\left(\frac{1}{\lambda^2 r^4}\right).
\]
Letting $\lambda \to +\infty$ first, and then $s \to 0+$, we obtain
\[
\lim_{\lambda \to +\infty} \int_{\Sigma} \varphi e^{\tilde{\phi}_{\lambda,\sigma}} dv_g = 4\pi \sum_{i=1}^k t_i \varphi(x_i).
\]
This together with (3.36) implies (3.32).

Similar to (2.34), for a sufficiently small $\epsilon_0 > 0$, we have a continuous retraction
\[
p : \{\sigma \in D(\Sigma) : d(\sigma, \mathcal{K}) < \epsilon_0\} \to \mathcal{K}.
\]

**Lemma 3.6.** Let $\Psi$ and $L$ be as in Lemma 3.3. If $\lambda > 0$ is chosen sufficiently large, then there exists a continuous map $\Phi_\lambda : \mathcal{K} \to J_\rho^{-L}$ such that $p \circ \Psi \circ \Phi_\lambda : \mathcal{K} \to \mathcal{K}$ is homotopic to the identity map $Id : \mathcal{K} \to \mathcal{K}$.

**Proof.** Let $\phi_{\lambda,\sigma}$ be constructed as in Lemma 3.5. For any $\sigma \in \mathcal{K}$, we define $\Phi_\lambda(\sigma) = \phi_{\lambda,\sigma}$ for large $\lambda > 0$. Clearly, the map $\Phi_\lambda : \mathcal{K} \to W^{1,2}(\Sigma)$ is continuous. By (3.31), if $\lambda \geq e^{L/(\rho - 4k\pi)}$, then $J_\rho(\phi_{\lambda,\sigma}) \leq -L$. Thus $\Phi_\lambda(\sigma) \in J_\rho^{-L}$. By Lemma 3.3 and (3.32), it holds that
\[
p \circ \Psi \circ \Phi_\lambda(\sigma) = p \circ \Psi(\phi_{\lambda,\sigma}) = p \circ \psi_k \left(\frac{e^{\phi_{\lambda,\sigma}}}{\int_{\Sigma} e^{\phi_{\lambda,\sigma}} dv_g}\right) \to \sigma
\]
as $\lambda \to +\infty$. Hence, $p \circ \Psi \circ \Phi_\lambda$ is homotopic to $Id : \mathcal{K} \to \mathcal{K}$. 

**3.4 Min-max values**

Let
\[
\mathcal{H} = \left\{ u \in W^{1,2}(\Sigma) : \int_{\Sigma} u dv_g = 0 \right\}
\]
and
\[ \mathcal{F}_k = \mathcal{F} 	imes [0,1]/(\mathcal{F} \times \{0\}) \]
be the topological cone over \( \mathcal{F} \). A path set associated with the metric space \( \mathcal{F}_k \) is defined by
\[ \Gamma_\lambda = \{ \gamma \in C^0(\mathcal{F}_k, \mathcal{H}) : \gamma |_{\mathcal{F} \times \{1\} } \in \Gamma_{\lambda,0} \}, \]
where \( \Gamma_{\lambda,0} \) is given by
\[ \Gamma_{\lambda,0} = \{ \gamma \in C^0(\mathcal{F} \times \{1\}, \mathcal{H}) : \gamma(\sigma,1) = \Phi_\lambda(\sigma), \forall \sigma \in \mathcal{F} \}. \]
If we write a path \( \gamma : \mathcal{F}_k \to \mathcal{H} \) by \( \gamma(\sigma,t) = t\phi_{\lambda,\sigma} \), then \( \gamma \in \Gamma_\lambda \), and thus \( \Gamma_\lambda \neq \emptyset \).

### Lemma 3.7
Let \( \rho \in (4k\pi, 4(k+1)\pi) \). If \( \lambda \) is chosen sufficiently large, and \( r \) is chosen sufficiently small, then \( -\infty < \beta_{\lambda,\rho} < \alpha_{\lambda,\rho} < +\infty \).

**Proof.** The proof is very similar to that of Lemma 2.7. It suffices to use Lemma 3.6 instead of the fact that \( \pi \circ \Psi \circ \Phi_\lambda : \Sigma_{e,k} \to \Sigma_{e,k} \) is homotopic to \( \text{Id} : \Sigma_{e,k} \to \Sigma_{e,k} \), and use Lemma 3.4 instead of the non-contractibility of \( \Sigma_{e,k} \).

#### 3.5 Completion of the proof of Theorem 1.2
Define \( \alpha_\rho = \alpha_{\lambda,\rho} \) for sufficiently large \( \lambda > 0 \). Similar to Lemma 2.9, \( \alpha_\rho/\rho \) is decreasing in \( \rho \in (4k\pi, 4(k+1)\pi) \). Let
\[ \Lambda_k = \left\{ \rho \in (4k\pi, 4(k+1)\pi) : \frac{\alpha_\rho}{\rho} \text{ is differentiable at } \rho \right\}. \]
In view of an analog of Lemma 2.11, \( \alpha_\rho \) is a critical value of \( J_\rho \) for any \( \rho \in \Lambda_k \).

Now we let \( \rho \in (4k\pi, 4(k+1)\pi) \). Take an increasing sequence of numbers \( (\rho_n) \subset \Lambda_k \) such that \( \rho_n \to \rho \), \( (\rho_n) \subset [a,b] \subset (4k\pi, 4(k+1)\pi) \), and \( \alpha_{\rho_n} \) is achieved by \( u_n \in \mathcal{H} \). Moreover, \( u_n \) satisfies the Euler-Lagrange equation
\[ \Delta_g u_n = \rho_n \left( \frac{h e^{u_n}}{\int_{\Sigma} h e^{u_n} d\nu_g} - \frac{1}{|\Sigma|} \right). \]
(3.38)

Since \( \alpha_\rho/\rho \) is decreasing in \( \rho \in [a,b] \),
\[ \alpha_{\rho_n} \leq \frac{b}{a} \alpha_a. \]
(3.39)

Define \( v_n = u_n - \log \int_{\Sigma} h e^{u_n} d\nu_g \), and we have
\[ \left\{ \begin{aligned}
\Delta_g v_n &= \rho_n (h e^{v_n} - |\Sigma|^{-1}), \\
\int_{\Sigma} h e^{v_n} d\nu_g &= 1.
\end{aligned} \right. \]

By Lemma 3.1, \( (u_n) \) is bounded in \( L^\infty(\Sigma) \). Let \( \Omega_1, \ldots, \Omega_{k+1} \) be disjoint closed sub-domains of \( \Sigma \). It follows from Lemma 3.2 that
\[ \log \int_{\Sigma} e^{u_n} d\nu_g \leq \frac{1}{8(k+1)\pi - \epsilon} \int_{\Sigma} |\nabla_g u_n|^2 d\nu_g + C_\epsilon \]
for any \( \epsilon > 0 \) and some constant \( C_\epsilon > 0 \). This together with (3.39) implies that for \( 0 < \epsilon < 8(k+1)\pi - 2b \),
\[
\frac{1}{2} \int_\Sigma |\nabla g u_n|^2 dv_g = J_{\rho_n}(u_n) + \rho_n \log \int \Sigma h e^{u_n} dv_g \\
\leq \frac{b}{8(k+1)\pi - \epsilon} \int_\Sigma |\nabla g u_n|^2 dv_g + C.
\]

Then it follows that \((u_n)\) is bounded in \( \mathcal{H} \). Without loss of generality, we assume that \( u_n \) converges to \( u_0 \) weakly in \( \mathcal{H} \), strongly in \( L^p(\Sigma) \) for any \( p > 1 \), and almost everywhere in \( \Sigma \). Moreover, \( e^{u_n} \) converges to \( e^{u_0} \) strongly in \( L^p(\Sigma) \) for any \( p > 1 \). By (3.38), \( u_0 \) satisfies
\[
\Delta_g u_0 = \rho \left( \frac{h e^{u_0}}{\int \Sigma h e^{u_0} dv_g} - \frac{1}{|\Sigma|} \right)
\]
in the distributional sense. In particular, \( u_0 \) is a critical point of \( J_{\rho} \).

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