Abstract

A general form for ladder operators is used to construct a method to solve bound-state Schrödinger equations. The characteristics of supersymmetry and shape invariance of the system are the start point of the approach. To show the elegance and the utility of the method we use it to obtain energy spectra and eigenfunctions for the one-dimensional harmonic oscillator and Morse potentials and for the radial harmonic oscillator and Coulomb potentials.

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I. INTRODUCTION

Algebraic methods applied to quantum mechanical problems have a long history [1]. The development of the supersymmetric quantum mechanics [2,3] has contributed to enlarge these approaches. Examples of this contribution is the concept of shape invariance introduced by Gendenshtein [4], and the study of coherent states for one-dimensional potentials using a superalgebra formalism [5–7].

Recently, Balantekin has shown that shape invariance has an underlying algebraic structure [8]. That approach was applied to studied barrier penetration [9] and exactly solvable coupled-channels problems [10].

In this article we show that the algebraic structure presented by Balantekin can be applied as a generalized method to solve one- and three-dimensional Schrödinger equations exactly. When one introduces shape invariance added to supersymmetric formalism it is possible to obtain a general approach to construct generalized ladder operators. The one-dimensional harmonic oscillator and Morse potential, the radial equation for the three-dimensional harmonic oscillator and Coulomb potential are analyzed using this algebraic approach. Different forms of raising and lowering operators, not defined in the shape invariance context, have been proposed [11–13] for these potentials.

The usual approach to solve Schrödinger equations using superalgebra formalism is by constructing a hierarchy of Hamiltonians [14]. For the method showed in this paper that construction is not necessary. The $n$-th eigenfunction for a given shape invariant potential can be deduced from the ground state wave function and the appropriated ladder operators.

In the next section we introduce the formalism to construct the generalized ladder operators. In the section III the one-dimensional potentials are solved using this approach, as examples, the harmonic oscillator and the Morse potential are explicitly solved. In the section IV the radial Schrödinger equation is studied and the spectrum and eigenfunctions for the three-dimensional harmonic oscillator and Coulomb potential are obtained.

II. THE GENERALIZED LADDER OPERATORS

From a bound-state Hamiltonian as ($\hbar$ and $2m = 1$)

$$H = -\frac{d^2}{dx^2} + V(x)$$

with

$$H\Psi_n(x) = E_n\Psi_n(x) ,$$  \hspace{1cm} (2.2)

the supersymmetric quantum mechanics can be constructed if $H$ can be factorized in the bosonic operators:

$$A^\pm(x;a) = \pm \frac{d}{dx} + W(x;a) ,$$ \hspace{1cm} (2.3)

where $W(x;a)$ is well known as the superpotential, which is a function of the position variable and a set of parameters, $a$, that represent space-independent properties of the original potential $V(x)$. If so, we obtain the first supersymmetric partner Hamiltonian
\[ H_+ = H - E_0 = A^+ A^- \]  

where \( E_0 \) is the ground state energy eigenvalue. The second partner Hamiltonian is

\[ H_- \equiv A^- A^+ \]  

The Hamiltonians \( H_+ \) and \( H_- \) have the same energy spectra except the ground state of \( H_+ \), for which there is no corresponding state in the spectra of \( H_- \). We can see that, from Eq. (2.4) the superpotential satisfies the Ricatti equation:

\[ W(x; a) - W'(x; a) = V(x) - E_0 = V_+(x) . \]  

It was introduced in 1983 by Gendenshtein [4] the shape invariance condition

\[ R(a_1) = V_-(x; a_0) - V_+(x; a_1) , \]  

where \( R(a_1) \) is independent of any dynamical variable and \( a_1 \) is a function of \( a_0 \). Potentials which satisfy this condition are exactly solvable. However, shape invariance is not the most general integrability condition, once there are exactly solvable potentials that seen do not show the shape invariance property [15]. In this work we consider shape invariance involving translations of the parameters \( a \):

\[ a_1 = a_0 + \eta , \]  

where \( \eta \) is the translation step. The other kind of relation between the parameters so far studied, scaling [16], is not broached here because only for the case of translations the potentials are explicitly known and can be written in a closed form in terms of simple functions, whereas for the case of scaling the potentials can only be written as a Taylor series [3]. In this way, it is possible to define translation operators \( T(a_0) \) as

\[
T(a_0) = \exp \left( \eta \frac{\partial}{\partial a_0} \right) \]  

\[
T^{-1}(a_0) = T^\dagger(a_0) = \exp \left( -\eta \frac{\partial}{\partial a_0} \right) . \]  

These operators only act on objects defined in the parameters space.

In order to introduce the generalized creation and annihilation operators we compose the translation operators \( T \) and the bosonic operators \( A \) as

\[
B_+(a_0) = A^+(a_0)T(a_0) \]  

\[
B_-(a_0) = T^\dagger(a_0)A^-(a_0) . \]  

These operators \( B \) present the necessary algebraic structure [8] to identify them as ladder operators. As such they are generalizations of the harmonic oscillator ladder operators.

The method to solve Schrödinger equation with shape-invariant potentials is similar to a factorization method applied to the case of a harmonic oscillator potential. Thus, the ground state, \( \Psi_0(x; a_0) \), must obey
\[ B_-(a_0)\Psi_0(x; a_0) = A^-(a_0)\Psi_0(x; a_0) = 0, \quad (2.11) \]

or

\[ \Psi_0(x; a_0) \propto \exp \left( -\int^x W(\bar{x}; a_0) d\bar{x} \right). \quad (2.12) \]

Eq. (2.12) is a relation between the ground state and the superpotential commonly found in the supersymmetric approach \[ [3,14] \]. The excited states are obtained by the repeated action of the creation operator on the ground state

\[ \Psi_n(x; a_0) = [B_+(a_0)]^n\Psi_0(x; a_0). \quad (2.13) \]

At this point we want to stress that, since the operators \( T \) do not commute with any \( a \)-dependent object and the operators \( A \) do not commute with any \( x \)-dependent object then, the operators \( B \) act on objects defined in the dynamical variable space only by the bosonic operators, and on objects defined in the parameters space by the translation operators. The normalization constant that transform Eq. (2.12) into a relation of equality, in general, depends on the parameters \( a \). Thus, this constant is also affected by the action of the operators \( B \) on the eigenfunctions. The normalization constant is not explicitly computed in this work, it can be included during the process of the use of the operators \( B \) or determined from the normalization condition of the eigenfunctions.

In order to determine the energy eigenvalues, it is important to understand the rule played by the remainder \( R(a_n) \), Eq. (2.7), in this context. From the translation operators defined in Eqs. (2.9), we can write

\[ R(a_n) = T(a_0)R(a_{n-1})T^\dagger(a_0), \quad (2.14) \]

where

\[ a_n = a_0 + n\eta, \quad (2.15) \]

is a generalization of the relation given by Eq. (2.8). The superalgebra permits to obtain a simple relation between the energy spectrum and the remainders \( R(a_n) \). To see how it is possible we use Eq. (2.14) to write down

\[ R(a_n)B_+(a_0) = B_+(a_0)R(a_{n-1}). \quad (2.16) \]

From Eqs. (2.14) and (2.16) we have the following commutation relation

\[ [H_+, (B_+)^n] = \left( \sum_{k=1}^{n} R(a_k) \right) (B_+)^n. \quad (2.17) \]

Now we apply this equation on \( \Psi_0(x; a_0) \) to have

\[ H_+ \{ [B_+(a_0)]^n\Psi_0(x; a_0) \} = \left( \sum_{k=1}^{n} R(a_k) \right) \{ [B_+(a_0)]^n\Psi_0(x; a_0) \}, \quad (2.18) \]

since \( \Psi_0(x; a_0) \) is the ground state of \( H_+ \) with a null eigenvalue. Thus, \( [B_+(a_0)]^n\Psi_0(x; a_0) \) is an eigenfunction of \( H_+ \) with eigenvalue \( \epsilon_n = \sum_{k=1}^{n} R(a_k) \). Therefore we have for the energy spectrum
\[ E_n = E_0 + \epsilon_n = E_0 + \sum_{k=1}^{n} R(a_k) , \quad (2.19) \]

where \( E_0 \) can be obtained directly from the factorization of the original Hamiltonian, Eq. (2.4) or Eq. (2.6).

We would like to close this section by emphasizing that this algebraic approach is self-consistent and it permits to determine the energy eigenvalues and eigenfunctions of a bound-state Schrödinger equation from supersymmetric and shape invariance properties of the system. In this way this approach is a method for exact resolution of that equation.

III. ONE-DIMENSIONAL APPLICATIONS

To illustrate the method discussed in the preceding section we first analyse one-dimensional shape-invariant potentials.

A. Harmonic Oscillator

This is the simplest example of a shape-invariant potential. In this sense, it is interesting in order to become the approach clearer. The original Hamiltonian is

\[ H = -\frac{d^2}{dx^2} + x^2 \quad (3.1) \]

and the factorization is

\[ H_+ = A^+ A^- = \left( -\frac{d}{dx} + ax \right) \left( \frac{d}{dx} + ax \right) = -\frac{d^2}{dx^2} + a^2 x^2 - a , \quad (3.2) \]

thus the superpotential is simply given by

\[ W(x; a) = ax . \quad (3.3) \]

The supersymmetric partner Hamiltonian is

\[ H_- = A^- A^+ = -\frac{d^2}{dx^2} + a^2 x^2 + a , \quad (3.4) \]

and the shape invariance condition, Eq. (2.7), establishes that

\[ R(a_1) = a_0^2 x^2 + a_0 - a_1^2 x^2 + a_1 . \quad (3.5) \]

Since \( R(a_1) \) must not depend on the \( x \)-coordinate then \( a_0 = a_1 \) and a comparison of \( H_+ \), Eq. (3.2), with the original Hamiltonian \( H \), Eq. (3.1), results that, in this case, all parameters are equal to 1. So we get \( a_n = a_0 = 1, \eta = 0, R(a_k) = 2 \) and for the ground state energy, that is given by the constant term in Eq. (3.2), \( E_0 = 1 \).

The harmonic oscillator is the simplest case of a shape-invariant potential because the parameter step is null (\( \eta = 0 \)) and from Eqs. (2.3) and (2.10) the generalized ladder operators are simply the same as the bosonic ones (\( B^\pm = A^\pm \)). Thus, we recover the standard
approach of the ladder operators method applied to the harmonic oscillator. In this way, from Eqs. (2.11) and (2.19) we have

\[ B_- (a_0) \Psi_0(x; a_0) = A^- (a_0) \Psi_0(x; a_0) = 0 \quad \Rightarrow \quad \Psi_0(x; a_0) \propto e^{-x^2/2}, \quad (3.6) \]

\[ E_n = E_0 + \sum_{k=1}^{n} R(a_k) = 1 + \sum_{k=1}^{n} 2 = 2n + 1, \quad (3.7) \]
as it should be.

**B. Morse Potential**

Algebraic approaches, but not developed within the context of shape invariance, for one-dimensional Morse potential can be found in Refs. [11,13,17]. In Ref. [8] it is explored the shape invariance property to determine the SU(1,1) structure of the algebra associated to that potential.

Here we start with the original bound-state Hamiltonian for the Morse potential,

\[ H = -\frac{d^2}{dx^2} + \lambda^2 \left(1 - e^{-x}\right)^2. \quad (3.8) \]

Factorization of this Hamiltonian gives

\[ H_+ = H - E_0 = \left(\frac{d}{dx} + \lambda \left(1 - e^{-x}\right) - a\right) \left(\frac{d}{dx} + \lambda \left(1 - e^{-x}\right) - a\right) \]
\[ = -\frac{d^2}{dx^2} + \lambda^2 \left(1 - e^{-x}\right)^2 - 2a\lambda + a^2 + \lambda (2a - 1) e^{-x}, \quad (3.9) \]

where the superpotential is

\[ W(x; a) = \lambda \left(1 - e^{-x}\right) - a. \quad (3.10) \]

The supersymmetric partner of \( H_+ \) is

\[ H_- = A^- A^+ = -\frac{d^2}{dx^2} + \lambda^2 \left(1 - e^{-x}\right)^2 - 2a\lambda + a^2 + \lambda (2a + 1) e^{-x}. \quad (3.11) \]

The shape invariance condition, Eq. (2.7), for this potential can be written as

\[ R(a_1) = (2a_0 - 2a_1 + 2) \lambda e^{-x} - 2 (a_0 - a_1) \lambda - a_1^2 + a_0^2. \quad (3.12) \]

In order to have \( R(a_1) \) independent of the variable \( x \), it is necessary that \( a_1 = a_0 + 1 \) then, from Eq. (2.8), \( \eta = 1 \). A comparison between Eq. (3.9) and Eq. (3.8) shows that \( a_0 = 1/2 \) and \( E_0 = \lambda - 1/4 \). The shape invariance condition gives to \( R(a_k) \) the following form:

\[ R(a_k) = -2 (a_{k-1} - a_k) \lambda - a_k^2 + a_{k-1}^2 = 2\lambda - 2k. \quad (3.13) \]

For this case the energy spectrum, given by Eq. (2.19), is
\[ E_n = E_0 + \sum_{k=1}^{n} (2\lambda - 2k) = 2\lambda \left( n + \frac{1}{2} \right) - \left( n + \frac{1}{2} \right)^2 , \] (3.14)

that is the exact result \[12\] \[13\].

From Eqs. (2.10), the generalized ladder operators can be written as

\[
B_+(a_0) = A^+(a_0)T(a_0) = \left[ -\frac{d}{dx} + \lambda \left( 1 - e^{-x} \right) - a_0 \right] \exp \left( \frac{\partial}{\partial a_0} \right) \]  \hspace{1cm} (3.15a)

\[
B_-(a_0) = T^\dagger(a_0)A^-(a_0) = \exp \left( -\frac{\partial}{\partial a_0} \right) \left[ \frac{d}{dx} + \lambda \left( 1 - e^{-x} \right) - a_0 \right] . \]  \hspace{1cm} (3.15b)

Substitution of Eq. (3.10) into Eq. (2.12), gives for the ground state eigenfunction the following expression

\[
\Psi_0(x; a_0) \propto e^{a_0 x} \exp \left[ -\lambda \left( x + e^{-x} \right) \right] . \]  \hspace{1cm} (3.16)

The excited states are constructed by the repeated action of the creation operator \( B_+ \), Eq. (3.15a), on the ground state Eq. (3.16). For example, for the first excited state we have

\[
\Psi_1(x; a_0) \propto B_+(a_0)\Psi_0(x; a_0) = \exp \left[ (-\lambda + a_0 + 1) x \right] \exp \left[ -\lambda e^{-x} \right] \left\{ 2\lambda - 2\lambda e^{-x} - 2a_0 - 1 \right\} . \]  \hspace{1cm} (3.17)

These same results for the Morse potential eigenfunctions \[12\] can be obtained by other exact methods, just remembering that \( a_0 = 1/2 \).

**IV. THREE-DIMENSIONAL POTENTIALS**

The algebraic approach developed in the section II is not restricted to one-dimensional potentials. As a matter of fact, it can immediately be extended to radial equations for central force potentials in more dimensions. In particular, here we present the results for the radial harmonic oscillator and the Coulomb potentials in three dimensions.

**A. Radial Harmonic Oscillator**

The original Hamiltonian for a radial harmonic oscillator is

\[ H = -\frac{d^2}{dr^2} + r^2 + \frac{l(l+1)}{r^2} . \]  \hspace{1cm} (4.1)

Following the same development used in one-dimensional cases, the Hamiltonian given by Eq. (4.1) can be factorized, for each value of \( l \), as

\[
H_+ = H - E_0^{(l)} = \left( \frac{d}{dr} + r - \frac{l+1}{r} \right) \left( \frac{d}{dr} + r - \frac{l+1}{r} \right) = -\frac{d^2}{dr^2} + r^2 + \frac{l(l+1)}{r^2} - 2l - 3 , \]  \hspace{1cm} (4.2)
where the superpotential is
\[ W(r; l) = r - \frac{l + 1}{r}. \]  
(4.3)

The supersymmetric partner of \( H_+ \), Eq. (4.2), is
\[ H_- = A^- A^+ = -\frac{d^2}{dr^2} + r^2 + \frac{(l + 1)(l + 2)}{r^2} - 2l - 1. \]  
(4.4)

For this case, the shape invariance condition, Eq. (2.7), can be written as
\[ R(l_1) = r^2 + \frac{(l_0 + 1)(l_0 + 2)}{r^2} - 2l_0 - 1 - r^2 - \frac{l_1(l_1 + 1)}{r^2} + 2l_1 + 3. \]  
(4.5)

Note that here, the parameter \( a \) gives place to the angular momentum number \( l \). Imposed that the right-hand side terms of Eq. (4.5) must not depend on the variable \( r \), we obtain
\[ l_1 = l_0 + 1. \]

Then, \( \eta = 1 \) and \( R(l_1) = 4 \), and from Eq. (4.2) \( E_0^{(l)} = 2l + 3 \). As in the one-dimensional harmonic oscillator case, the function \( R(l_k) \) is constant. Thus, the energy spectrum for the 3-D harmonic oscillator is
\[ E_n^{(l)} = E_0^{(l)} + \sum_{k=1}^{n} R(l_k) = 2l + 3 + 4n. \]  
(4.6)

The eigenfunctions can be obtained from the generalized ladder operators:
\[ B_+(l) = A^+(l)T(l) = \left[ -\frac{d}{dr} + r - \frac{l + 1}{r} \right] \exp \left( \frac{\partial}{\partial l} \right), \]  
(4.7a)
\[ B_-(l) = T^\dagger(l)A^-(l) = \exp \left( -\frac{\partial}{\partial l} \right) \left[ \frac{d}{dr} + r - \frac{l + 1}{r} \right]. \]  
(4.7b)

The minimum energy state for a fixed \( l \) \((n = 0)\), is obtained by the substitution of Eq. (4.7b) into Eq. (2.11), giving
\[ \Psi_0^{(l)}(r) \propto r^{l+1} e^{-r^2/2}. \]  
(4.8)

The other eigenfunctions (for \( n \geq 1 \)) can be obtained by repeated action of the creation operator, Eq. (4.7a) on the function given by Eq. (4.8). As an example, we have for \( n = 1 \)
\[ \Psi_1^{(l)}(r) \propto B_+(l)\Psi_0^{(l)}(r) = \left( 2r^2 - 2l - 3 \right) r^{l+1} e^{-r^2/2}. \]  
(4.9)

These results are obtained by other methods, for instance, in Ref. [18].

B. Radial Coulomb Potential

The radial Hamiltonian for the Coulomb potential is
\[ H = -\frac{d^2}{dr^2} - \frac{1}{r} + \frac{l(l + 1)}{r^2}, \]  
(4.10)
which is factorized as

\[ H_+ = H - E_0^{(l)} = \left( \frac{d}{dr} + \frac{1}{2(l+1)} - \frac{l+1}{r} \right) \left( \frac{d}{dr} + \frac{1}{2(l+1)} - \frac{l+1}{r} \right) \]

\[ = -\frac{d^2}{dr^2} - \frac{1}{r} + \frac{l(l+1)}{r^2} + \frac{1}{4(l+1)^2}, \quad (4.11) \]

where

\[ W(r; l) = \frac{1}{2(l+1)} - \frac{l+1}{r}. \quad (4.12) \]

The supersymmetric partner of the Hamiltonian (4.11) is

\[ H_- = A^- A^+ = -\frac{d^2}{dr^2} - \frac{1}{r} + \frac{(l+1)(l+2)}{r^2} + \frac{1}{4(l+1)^2}. \quad (4.13) \]

Here again the quantum number \( l \) is used as the parameter \( a \) \( (a_0 = l_0) \). Using the shape invariance condition, we obtain

\[ R(l_1) = -\frac{1}{r} + \frac{(l_0 + 1)(l_0 + 2)}{r^2} + \frac{1}{4(l_0 + 1)^2} + \frac{1}{r} - \frac{l_1(l_1 + 1)}{r^2} - \frac{1}{4(l_1 + 1)^2}. \quad (4.14) \]

In order to have \( R(l_1) \) independent on \( r \) we must have \( l_1 = l_0 + 1 \) and thus, \( \eta = 1 \) and

\[ R(l_1) = \frac{1}{4(l_0 + 1)^2} - \frac{1}{4(l_1 + 1)^2}. \quad (4.15) \]

For the \( k \)-th parameter, the shape invariance condition gives

\[ R(l_k) = \frac{1}{4(l_{k-1} + 1)^2} - \frac{1}{4(l_k + 1)^2} \]

\[ = \frac{1}{4} \left[ \frac{1}{(l+n)^2} - \frac{1}{(l+n+1)^2} \right]. \quad (4.16) \]

The energy eigenvalue \( E_0^{(l)} \) is obtained from the constant term in Eq. (4.11);

\[ E_0^{(l)} = \frac{1}{4(l+1)^2}. \quad (4.17) \]

Thus, the energy spectrum for this case is given by

\[ E_n^{(l)} = E_0^{(l)} + \sum_{k=1}^{n} \frac{1}{4} \left[ \frac{1}{(l+n)^2} - \frac{1}{(l+n+1)^2} \right] = -\frac{1}{4(n+l+1)^2}. \quad (4.18) \]

The generalized ladder operators for the radial Coulomb potential are

\[ B_+(l) = A^+(l) T(l) = \left[ -\frac{d}{dr} + \frac{1}{2(l+1)} - \frac{l+1}{r} \right] \exp \left( \frac{\partial}{\partial l} \right) \quad (4.19a) \]

\[ B_-(l) = T^+(l) A^-(l) = \exp \left( -\frac{\partial}{\partial l} \right) \left[ -\frac{d}{dr} + \frac{1}{2(l+1)} - \frac{l+1}{r} \right]. \quad (4.19b) \]
The lowest energy level eigenfunction for each $l$ is obtained using Eqs. (2.11) and (4.19b) or by the substitution of Eq. (4.12) into Eq. (2.12)

$$
\Psi_0^{(l)}(r) \propto r^{l+1} e^{-r/2(l+1)}.
$$

(4.20)

The other eigenfunctions (for $n \geq 1$) are calculated by the action of the creation operator, Eq. (4.19a) on the above eigenfunction. As an example, for $n = 1$ we have the following eigenfunction:

$$
\Psi_1^{(l)}(r) \propto B_+^{(l)} \Psi_0^{(l)}(r) = \left\{ \frac{1}{2} \left[ \frac{1}{l + 2} + \frac{1}{l + 1} \right] - 2l - 3 \right\} r^{l+1} e^{-r/2(l+2)}.
$$

(4.21)

These results for the radial Coulomb potential are also obtained by other exact methods [12]. Usually for this case, the principal quantum number adopted is $N = n + l + 1$.

V. CONCLUDING REMARKS

A general constructive method to solve bound-state Schrödinger equations for shape invariant potentials is presented. This algebraic method is based on a generalization of harmonic oscillator raising and lowering operators. The introduction of hierarchy of Hamiltonians, usual in this kind of algebraic treatment, is no longer necessary for the present approach. Solutions for one-dimensional harmonic oscillator and Morse potentials and for three-dimensional radial harmonic oscillator and Coulomb potentials are explicitly obtained.

Solutions for other shape invariant potentials can also be obtained by this approach. In particular, radial harmonic oscillator and Coulomb potential in arbitrary dimensions [19, 20] are simple extensions of the results appointed here.

An interesting discussion is concerning coherent states for one-dimensional potentials and supersymmetric quantum mechanics [7]. In this way one has a elegant and compact method to introduce ladder operators and define coherent states from these operators.

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