Hot scalar radiation setting bounds on the curvature coupling parameter

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Abstract
This paper addresses the interplay between vacuum and thermal local averages for massless scalar radiation near a plane wall of a large cavity where the Dirichlet boundary condition is assumed to hold. The main result is that stable thermodynamic equilibrium is possible only if the curvature coupling parameter is restricted to a certain range. In more than three spacetime dimensions such a range contains the conformal coupling, but it does not contain the minimal coupling. Since this same range for possible values of the curvature coupling parameter also applies to massive scalar radiation, it may be relevant in settings where arbitrarily coupled scalar fields are present.

Keywords: scalar radiation, curvature coupling, finite temperature, quantum field theory

1. Introduction

When the physics of blackbody radiation in a cavity is derived in statistical mechanics textbooks, the energy of the vacuum is usually ignored. The argument is that since the vacuum is by definition the state of minimum energy of a system, and one is interested only in the energy that can be extracted from the cavity, then the energy of the vacuum is effectively zero. In this context, electromagnetic radiation in thermodynamic equilibrium with the walls of a large cavity at temperature $T$ is addressed either by considering it as an ideal gas of massless bosons or an ensemble of harmonic oscillators. Any of these tools leads to the energy density $u$ and pressure $P$ (equation of state) given by,
\[ u = \pi^2 \left( \frac{k_B T}{\hbar c} \right)^4, \quad P = \frac{u}{3}, \]  

(1)

where the fundamental constants \( \hbar \) and \( c \) in equation (1) point out the quantum-relativistic nature of the phenomenon.

In another different but presumably equivalent approach, one considers an ensemble of infinite cubic cavities each containing the quantum field—quantum field at finite temperature. When techniques are applied to the electromagnetic field, it results in the following ensemble average for the stress–energy–momentum tensor,

\[ \langle T^{\mu \nu} \rangle = \text{diag}(u, P, P, P), \]  

(2)

with \( u \) and \( P \) given precisely by equation (1). If one of the walls of the cavity is brought isothermally from infinity and parallel to the opposite fixed wall, \( \langle T^{\mu \nu} \rangle \) changes radically, losing its isotropic character although remaining uniform [1]. When the distance between these two walls is small enough, \( \langle T^{\mu \nu} \rangle \) is such that it hardly depends on \( T \); in fact it does not even vanish when \( T = 0 \). At this stage, the two walls are attracted to each other—the Casimir effect—and that is a surprising dynamical manifestation of the vacuum state in quantum field theory (see reviews [2, 3]).

There are also manifestations of the electromagnetic vacuum when the walls are far apart. When a large cavity has curved walls [4, 5], the behaviour of electromagnetic radiation in the bulk is still essentially given by equations (1) and (2). Approaching a wall one sees that \( \langle T^{\mu \nu} \rangle \), in fact, is not uniform. In particular near the wall the expression for \( u \) in equation (1) becomes a subleading contribution in \( \langle T^{00} \rangle \). The corresponding leading contribution is a temperature independent term, which diverges as the curved wall is approached if the perfect conductor boundary conditions are taken on it. This non integrable divergence has long been known in the literature, and it is commonly understood as a consequence of overidealizing a real conductor (see also [6, 7], for an alternative interpretation). When a ‘cut off procedure’ is considered for modelling a real conductor, \( \langle T^{\mu \nu} \rangle \) remains finite as the wall is approached, while keeping its Planckian thermal behaviour.

So far only electromagnetic radiation has been addressed. Considering now massless scalar radiation, deep in the bulk of a large cavity equations (1) and (2) hold, after halving \( u \) (consistent with the absence of polarization) [6]. In analogy with electromagnetic radiation in a cavity with curved walls, one will eventually notice that \( \langle T^{\mu \nu} \rangle \) is not uniform by moving toward a wall, but this time regardless of whether the wall is curved or not. As has been shown in [6], the thermal behaviour of \( \langle T^{00} \rangle \) near a reflecting plane wall will still be Planckian, but the corresponding ‘Stefan constant’ will depend on the kind of boundary condition—Dirichlet or Neumann—that is assumed to hold on the wall. Again, (zero temperature [6]) non integrable divergences may appear in \( \langle T^{\mu \nu} \rangle \) on the wall, and ‘cut off procedures’ to eventually cure them will have to be judiciously chosen to avoid violation of the principle of virtual work [8].

Another new ingredient that thermal scalar radiation brings about, and that has been overlooked in the literature, is the thermal dependence of \( \langle T^{\mu \nu} \rangle \) on the curvature coupling parameter \( \xi \). As is well known, in flat spacetime \( \xi \) does not appear in the wave equation; but it does appear in the corresponding stress–energy–momentum tensor. The reason is that the variation of the action with respect to the metric that leads to \( T^{\mu \nu} \) is made before solving the Einstein equations, which yield the flat geometry.

Thermal behaviour of \( \langle T^{\mu \nu} \rangle \) for massless scalar fields near plane walls in four spacetime dimensions has been investigated in [6] and [9]. The present work extends the investigation by examining the thermal dependence of \( \langle T^{\mu \nu} \rangle \) on \( \xi \) near a reflecting plane wall of an infinite
cavity in flat spacetime with an arbitrary number of dimensions. The organization of the paper will be as follows. In section 2 the Feynman propagator at finite temperature is obtained, and used according to the ‘point splitting procedure’ [10] to derive, in section 3, the asymptotic behaviours (i.e., at low and high temperatures) of $\langle T^{\mu\nu} \rangle$. An analysis of certain local properties of the hot scalar radiation is presented in section 4. The main result in the paper is in section 5, namely, by requiring stable thermodynamic equilibrium it is shown that only values of $\xi$ restricted to a certain range are acceptable. A few final points are addressed in section 6 that also contains a summary.

Before proceeding to the next section, a word of caution is in order. This paper concerns the thermal behaviour of local quantities such as the energy density and stresses. It differs from papers (e.g., [11, 12]) that investigate global quantities, among them the Helmholtz free energy and the internal energy. In principle, global quantities can be obtained by integrating the corresponding local quantities; but as mentioned in [6] it would be erroneous to assume that one can infer the thermal behaviour of the latter looking at the thermal behaviour of the former.

In the rest of the text, unless stated otherwise, $\kappa = \hbar = c = 1$.

2. The scalar propagator

A plane wall is taken at $x = 0$, where the massless scalar field $\phi$ vanishes—Dirichlet’s boundary condition. All the other walls of the $(N - 1)$-dimensional cavity are at infinity. Radiation in the cavity is assumed to be in thermodynamic equilibrium with the wall at temperature $T$, so one uses the formalism of analytically continuing time $t$ to imaginary values, taking it periodic with period $1/T$. Observing these prescriptions the equation [10]

$$\Box_{\mu} D_{\mu}(x, x') = -\delta^{\mathbf{N}}(x - x')$$

is solved to obtain the scalar propagator. The solution is well known for $N = 4$ (see [10], and references therein). When $N > 3$,

$$D_{\mu}(x, x') = -\frac{i}{4\pi^{N/2}} \Gamma\left(\frac{N - 2}{2}\right) \sum_{n = -\infty}^{\infty} \left( (-\sigma_-)^{2-N}/2 - (-\sigma_+)^{2-N}/2 \right).$$

where $\sigma_+ := (t - t' - in/T)^2 - (x + x')^2 - (y - y')^2 - (z - z')^2 - \ldots$ plus an infinitesimal negative imaginary term. Noting that the term corresponding to $n = 0$ and involving $\sigma_-$ in equation (4) is simply the zero temperature propagator in ordinary Minkowski spacetime, direct application of $\Box_{\mu}$ to equation (4) promptly yields equation (3). In the following, the Minkowski propagator will be dropped in order to implement renormalization. It is worth mentioning that for $N \leq 3$ familiar divergences arise [13]. As will be seen shortly, such divergences do not bother $\langle T^{\mu\nu} \rangle$ though when $N \geq 2$.

3. The stress–energy–momentum tensor

The expectation value of the stress–energy–momentum tensor can be obtained by applying the differential operator [10]

$$D^{\mu\nu} := (1 - 2\xi)\partial^{\mu}\partial^{\nu} - 2\xi\partial^{\mu}\partial^{\nu} + (2\xi - 1/2)\eta^{\mu\nu}\partial^{\lambda}\partial_{\lambda};$$
to the renormalized propagator,
\[ \langle T^{\mu \nu} \rangle = i \lim_{x' \to x} D^{\mu \nu} D_{\rho} (x, x'), \]
with the result that \( \langle T^{\mu \nu} \rangle \) is conserved, traceless when \( \xi = \xi_0 \) (conformal coupling),
\[ \xi_N = \frac{N - 2}{4(N - 1)}, \]
(5)
and with diagonal form,
\[ \langle T^{\mu \nu} \rangle = \text{diag} \{ \rho, P_\perp, P_\parallel, \ldots, P_\parallel \}. \]
(6)
The only component that is uniform in equation (6) is the pressure perpendicular to the wall,
\[ P_\perp = \frac{1}{\pi^{N/2}} \text{\Gamma} \left( \frac{N}{2} \right) \zeta(N) T^N. \]
(7)
The interest here is on the asymptotic behaviours of the quantities in equation (6). For convenience formulas will be specialized to non negative \( x \). In the bulk (\( T_x \gg 1 \)) one finds for the energy density,
\[ \rho = (N - 1)P_\perp + \rho_{\text{class}}, \]
(8)
where
\[ \rho_{\text{class}} = \frac{2^{2-N}}{\pi^{(N-1)/2}} (N - 2) F \left( \frac{N - 1}{2} \right) (\xi - \frac{1}{4}) \frac{T}{x^{N-1}}, \]
(9)
and for the pressure parallel to the wall (\( N \geq 3 \)),
\[ P_\parallel = P_\perp + P_{\text{class}}, \]
(10)
where \( P_{\text{class}} \) is given by the negative of equation (9) with \( \xi = 1/4 \) replaced by \( \xi - \xi_{N-1} \).
Equations (8) and (10) are exact up to exponentially small corrections, and at high temperatures they apply also near the wall.
Moving now near the wall (\( T_x \ll 1 \)), \( P_\perp \) is still given by equation (7) since it is uniform. Neglecting terms with higher powers of \( T_x \), it results that
\[ \rho = \frac{2}{(4\pi)^{N/2}} (N - 1) F \left( \frac{N}{2} \right) (\xi - \xi_{N-1}) x^{-N} + \frac{1}{\pi^{N/2}} F \left( \frac{N}{2} \right) \zeta(N)(1 - 4\xi) T^N, \]
(11)
and that
\[ P_\parallel = -\rho. \]
(12)
Clearly, equations (11) and (12) also hold everywhere at low temperatures.
Part of the material in [6] is concerned with thermal scalar radiation near a reflecting plane wall. By setting in equation (8) \( N = 4 \) and \( \xi = 0 \) (minimal coupling), or \( \xi = 1/6 \) (conformal coupling when \( N = 4 \), see equation (5)), the results in [6] are successfully reproduced. Equation (11) also reproduces the corresponding expression in [6], when \( N = 4 \) and \( \xi = 1/6 \). [9] has investigated thermal scalar radiation between two parallel reflecting walls separated by unity, providing a formula for \( \langle T^{\mu \nu} \rangle \) with arbitrary \( \xi \) and \( N = 4 \). It has been checked that, by reintroducing arbitrary distance between the walls and taking it to infinity, the resulting asymptotic behaviours of \( \langle T^{\mu \nu} \rangle \) in [9] agree with those above, after setting \( N = 4 \).
4. Some local features of the hot radiation

The expression for $P_\perp$ in equation (7), which holds everywhere at arbitrary $T$, is precisely the usual blackbody radiation pressure in $N$ dimensions. Deep in the bulk ($x \to \infty$) $P_{\text{class}}$ and $P_{\text{class}}$ in equations (8) and (10) can be neglected, resulting in the usual proper relations of the uniform and isotropic blackbody radiation.

By reintroducing dimensionful $\hbar$ in equations (8) and (10), one sees that the ‘classical’ corrections $\rho_{\text{class}}$ and $P_{\text{class}}$ carry $\hbar^0$, whereas the blackbody quantities carry a negative power of $\hbar$, namely, $\hbar^{-N}$. Although the terminology ‘classical’ is at some extent justified, by setting $\hbar \to 0$ the ‘classical’ corrections effectively vanish since the blackbody quantities diverge. In the context of $N = 4$, [6] has pointed out that the only relevant correction to the blackbody contribution in equation (8) is linear in $T$. Now it can be seen from equation (9) that this is so regardless of the number of dimensions $N$. This fact resembles the equipartition principle of energy, and it could not be reached simply by using dimensional arguments. At this point it should be noted that ‘classical’ contributions also appear in the context of Casimir’s effect at finite temperature [1, 4] (see also [2, 3], and references therein).

Whereas only the subleading contribution in the expression for $\rho$ in the bulk (see equation (8)) carries dependence on $\xi$, both terms in equation (11) depend on it. As the wall is approached, the vacuum energy density in equation (11) diverges for $\xi \neq \xi_0$ [14], a well known fact. Such a limitation can be seen as a consequence of replacing a real reflecting plane wall (which has some thickness) by a boundary condition on a plane [5]. Roughly speaking, certain ‘cut off procedures’ result in inserting a small parameter $\epsilon$ in the formula for the vacuum energy density in equation (11), such that when $x = 0$ (keeping $\epsilon \neq 0$), the final formula is finite. It also follows that when $x \gg \epsilon$ (with $Tx \ll 1$ still holding), equation (11) is taken as a good approximation. Another way of tackling the non integrable divergence on the reflecting wall is the ‘renormalization procedure’ proposed in [6]. According to this procedure the zero temperature contribution in equation (11) holds literally out of the wall, but a $\delta$-function contribution is added to it, giving rise to a surface energy that cancels the troublesome divergent contribution in the total vacuum energy after integration is performed over space. (This ‘renormalization procedure’ has successfully been extended to the electromagnetic field in [7].)

Unlike the vacuum energy density, the ‘Planckian contribution’ in equation (11) is divergence free, and this fact has been stressed in [6]. Although such a temperature dependent contribution does not depend on $x$, strictly speaking it should not be taken as purely thermal. Its mixed nature (vacuum-thermal) is reflected in the fact that it depends both on $\xi$ and $T$. As has been pointed out in [6] for $N = 4$, and mentioned earlier in the text, the behaviour in equation (11) resembles that for electromagnetic radiation near a curved wall where the perfect conductor boundary condition is assumed to hold [4]. An important difference though is that the subleading contribution in the case of the electromagnetic radiation is the very blackbody energy density, and thus is purely thermal. Another feature near the wall (or everywhere at low temperatures) that should also be remarked on is that whereas $P_\parallel$ in equation (7) is always positive, each term of $P_\parallel$ in equation (12) can be negative depending on $\xi$.

The expression for the ‘specific heat’ per unit of volume in the bulk (see equation (8)) is given by (quotation marks stand for the fact that $c_V$ below is simply the rate of change with temperature of the energy density, rather than the rate of change with temperature of the total energy in the cavity divided by its volume)
\[
\pi = \frac{N(N - 1)}{\pi N^2} \Gamma \left( \frac{N}{2} \right) \zeta(N) T^{N-1} + \frac{\rho_{\text{class}}}{T},
\]
(13)

where the factor multiplying \( NT^{N-1} \) is the familiar ‘Stefan constant’ in \( N \) spacetime dimensions. Near the wall (see equation (11)),
\[
\pi = \frac{N}{\pi N^2} \Gamma \left( \frac{N}{2} \right) \zeta(N)(1 - 4\xi) T^{N-1},
\]
(14)

which also has Planckian form, but now the ‘Stefan constant’ depends on \( \xi \). Examining these equations one sees that \( \pi \) in the bulk (or everywhere at high temperatures) is positive, but near the wall (or everywhere at low temperatures) it can be negative.

5. Bounds on \( \xi \)

Consider now the local conservation law of energy and momentum,
\[
\mathcal{T}^{\mu\nu} = 0.
\]
(15)

In order to make the discussion that follows regarding the allowed values of \( \xi \) consistent with stable thermodynamic equilibrium as clear as possible, the four dimensional case will be treated first. Thus, in equation (15), \( \mu \) and \( \nu \) run from 0 to 3 corresponding to \( t, x, y \) and \( z \), respectively. The energy flux density \( S = (T^{01}, T^{02}, T^{03}) \) and the momentum \( P \) in a volume \( V \) are related by,
\[
P = \int_V S d^3x.
\]
(16)

The global conservation of energy follows from equation (15),
\[
\frac{d}{dt} \int_V T^{00} d^3x = -\oint_S T^{00} n^i da = -\oint_S S \cdot nda,
\]
(17)

where \( n \) is the outward normal to the boundary \( S \) of \( V \). Denoting,
\[
F = \frac{dP}{dt},
\]
(18)

the global conservation of momentum is also obtained from equation (15),
\[
F^i = -\oint_S T^{i j} n^j da,
\]
(19)

where, as in equation (17), \( i \) and \( j \) run from 1 to 3, and repeated indices indicates summation.

To show that \( \xi \) has bounds, one begins by considering a tiny cubic region of the scalar radiation in the cavity, and say, facing the walls of the cavity. Each side of the cube has area \( A \) and is imagined to be contained in a rectangular parallelepiped with volume \( A\delta \), where \( \delta \) is as small as one likes. There are, therefore, six of these very thin rectangular parallelepipeds. Proceeding, consider the three sides of the cube for which the outward normal \( n \) coincides with the usual unity coordinate vectors \( i, j \) and \( k \). Below, these sides and the corresponding rectangular parallelepipeds will be labelled by subscripts (1), (2) and (3), respectively.

It is further assumed that the temperature \( T_{in} \) inside the cube may be slightly different from the temperature \( T_{out} \) outside, and that these temperatures are low enough for equations (11), (12) and (14) to hold. Setting \( N = 4 \), it follows then from equations (6) and (19) that the time rates of variation of the momentum in each rectangular parallelepiped are,
\[ F_{(1)} = \frac{\pi^2}{90} A \left( T_{\text{in}}^4 - T_{\text{out}}^4 \right) \mathbf{i}, \quad F_{(2)} = \frac{\pi^2}{90} A (1 - 4\xi) \left( T_{\text{in}}^4 - T_{\text{out}}^4 \right) \mathbf{j}, \]
\[ F_{(3)} = -\frac{\pi^2}{90} A (1 - 4\xi) \left( T_{\text{in}}^4 - T_{\text{out}}^4 \right) \mathbf{k}. \] (20)

If at some instant \( S \) vanishes inside the tiny cubic region, after a short time interval \( dt \),
\[ S_{(i)} = \frac{1}{A \delta} F_{(i)} dt, \] (21)
where equations (16) and (18) have been used. Corresponding to equations (20) and (21), the energy flux per unit of time out of the cube,
\[ \oint S \cdot dS, \]
is given by
\[ \Phi = [1 - 2(1 - 4\xi)] \left( T_{\text{in}}^4 - T_{\text{out}}^4 \right), \] (22)
up to an overall positive factor.

Now, for instance, suppose that \( T_{\text{in}} > T_{\text{out}} \). If \( \xi > 1/4 \), equation (22) yields \( \Phi \) > 0, and then the energy inside the tiny cube decreases (see equation (17)). Noting equation (14), when \( \xi > 1/4, \psi \) > 0 resulting that \( T_{\text{in}} \) becomes even higher and \( T_{\text{out}} \) lower, i.e., the system is taken away from thermodynamic equilibrium. It follows that \( \xi \) must not be greater than 1/4. As \( T_{\text{in}} > T_{\text{out}} \) and \( \xi < 1/4 \) by assumption, the energy inside the cube must decrease (see equation (14)), otherwise the system will run away from thermodynamic equilibrium. Accordingly, \( \Phi \) in equation (22) must be positive, i.e., \( 1 - 2(1 - 4\xi) > 0 \). Putting all this together, \( \xi \) needs to be such that
\[ \frac{1}{8} < \xi < \frac{1}{4}. \] (23)

The whole argument can be carried to arbitrary \( N \) in a straightforward way. In so doing, equation (22) is replaced by
\[ \Phi = [1 - (N - 2)(1 - 4\xi)] (T_{\text{in}}^N - T_{\text{out}}^N). \]
Thus, by requiring stable thermodynamic equilibrium, one finds that \( \xi \) needs to be such that
\[ \xi < \frac{1}{4}, \] (24)
for \( N = 2 \), and
\[ \xi_{N-1} < \xi < \frac{1}{4}, \] (25)
for \( N > 2 \). It is rather curious that the lower bound in equation (25) is the conformal coupling in one less dimension (see equation (5)), and that the upper bound is the limit of \( \xi_{N} \) as \( N \to \infty \). Thus the conformal coupling \( \xi_{N} \) fits in equation (25), but the minimal coupling \( \xi = 0 \) does not. As can be readily seen, to eventually include equalities in equations (24) and (25), further corrections in equations (11) and (12) are required. It should be additionally noted that when \( N \to \infty \), the interval in equation (25) narrows to a point, encapsulating the conformal coupling.

A connection between equation (25) and the ‘classical’ corrections in equations (9) and (10) is worth pointing out. The ‘classical’ corrections are related by
\[ (\xi - \frac{1}{4}) \rho_{\text{class}} = (\xi_{N-1} - \xi) \rho_{\text{class}}, \] (26)
and therefore equation (25) implies that \( \rho_{\text{class}} \) and \( \rho_{\text{class}} \) have the same sign. Now, whereas equation (26) is a feature of the behaviour of the scalar radiation in the bulk (or everywhere at
high temperatures), the permissible range in equation (25) arises by considering its behaviour near the wall (or everywhere at low temperatures).

Mathematically speaking, a wall is a boundary condition assumed to hold on a surface to prevent flux of energy across the surface. As equations (24) and (25) were obtained assuming the Dirichlet boundary condition, it is pertinent to ask if another commonly used boundary condition, namely the Neumann boundary condition, leads to the same result. In other words one may wish to know how dependent on the type of boundary condition equations (24) and (25) are. A preliminary answer can be obtained from [9]. After manipulating formulas in [9] concerning the Neumann boundary condition, one finds a permissible range $-5/4 < \xi < 7/8$, which is to be compared with equation (23). It follows that, in this case, the permissible range corresponding to the Neumann boundary condition not only includes the minimal coupling $\xi = 0$, but also contains the range corresponding to the Dirichlet boundary condition. The Dirichlet boundary condition is therefore more restrictive.

Another pertinent question that could be raised is how dependent on the mass $m$ of the scalar field equations (24) and (25) are. It is expected that for high temperatures ($T \gg m$) massive scalar radiation behaves as if it were massless, and therefore equations (24) and (25) should hold. In fact, it has been checked that these equations do hold for arbitrary $m$. When $m \neq 0$, modified Bessel functions of the second kind come into play, but they do so without affecting the range of permissible values of $\xi$.

6. Final remarks

An important issue to be addressed here is to which extent the atmosphere near the wall can be considered as an ideal gas of massless scalar bosons at temperature $T$. Taking for convenience $N = 4$, one may ask to which extent a standard formula such as (reintroducing dimensionful constants)

$$\rho = \int \frac{d^3p}{h^3} \frac{pc}{e^{pc/h\kappa T} - 1}$$

still holds in the present situation (see equation (6)). It should be noted that the right hand side of equation (27) is in fact equal to $U/V$, and strictly speaking the equality might hold only if the internal energy $U$ were uniformly distributed in the cavity, which is clearly not the case for the scalar radiation. Nevertheless, deep in the bulk ($x \to \infty$) equation (27) and its corresponding equation of state $p = \rho/3$ hold as follows from the discussion opening section 4. Similarly one might guess that the temperature dependent term in equation (11) (see also equation (14)), which holds near the wall (or everywhere at low temperatures), could be associated to an effective Planck’s distribution as in equation (27). However, recalling how the pressure of a gas on the walls of a cavity is obtained from the distribution function, one sees clearly that such an effective Planck distribution would fail in reproducing the pressure in equations (7) and (12). This conclusion has to do with the already mentioned mixed nature (vacuum-thermal) of the scalar radiation at finite temperature near the reflecting wall.

Perhaps this duality can be better appreciated by saying, as in [4], that the atmosphere near the wall is a mixture of virtual and real bosons. Indeed, if initially at any point in the cavity the ‘specific heat’ is given by the nearly blackbody expression in equation (13), and then the temperature is lowered such that eventually $c_v$ will be given by equation (14), as $T \to 0$, $c_v \to 0$ and $P_0 \to 0$, resulting that at $T = 0$ only virtual bosons are left in the cavity from where no further energy can be extracted—only vacuum energy is left behind.
In contrast to the scalar radiation, the thermal behaviour of the electromagnetic energy density $\rho$ in the bulk and near a curved wall of a large cavity is the same [4]. Thus an expression like (precisely twice that in) equation (27) yields the thermal contribution in the bulk and also near the wall, corresponding to the usual radiation pressure (see equation (1)). It follows then that, in this case, one can think of an ideal gas of massless bosons (photon gas) in the bulk, as well as near the wall.

It should be noticed that by dropping the vacuum energy density, one can check that the integral over the space of $\rho$ in equation (6) does not depend on $\xi$, for $N > 2$. This has also been pointed out in [9] when $N = 4$, for a configuration of two parallel walls, and it is consistent with the fact that standard blackbody expressions do not carry dependence on $\xi$. (Clearly, if the vacuum energy density is included in the integrand, a ‘cut off procedure’ must be used to deal with the non integrable divergence [5, 6, 8].)

It is worth pointing out that to derive equations (24) and (25) the expressions in section 3 were assumed to hold in a situation slightly out of thermodynamic equilibrium, since the temperatures $T_{\text{in}}$ and $T_{\text{out}}$ in section 5 were allowed to be slightly different from each other. This assumption is entirely plausible and, indeed, it is the kind of assumption one makes to show that the familiar electromagnetic radiation is in stable thermodynamic equilibrium in a cavity.

Recapitulating, this paper investigated locally the vacuum-thermal nature of the scalar radiation. The main conclusion is that the statistical mechanics of a scalar field restrict the values of the curvature coupling parameter $\xi$. Corresponding to the Dirichlet boundary condition on a plane wall, such a restriction is characterized by the inequalities in equations (24) and (25). A further still unclear role seems to be played by the bounds in equation (25). For example, the symmetrical appearance of these bounds in the ‘classical’ corrections to the blackbody expressions in the bulk needs more investigation.

A particularly interesting issue to address in the future is the possible effects of a more realistic wall on inequalities such as those in equations (24) and (25), and the use of the Robin boundary condition [6, 14] (which interpolates between Dirichlet and Neumann) may be a good start. Before ending, it should be mentioned that the present paper fits in a class of works that constrain couplings involving curvature by invoking consistency conditions in various contexts (e.g., [15–17]).

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