ON THE STABILITY OF A HYPERBOLIC FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

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Abstract. In this paper, by means of the Gronwall inequality, the $\psi$-Riemann-Liouville fractional partial integral and the $\psi$-Hilfer fractional partial derivative are introduced and some of its particular cases are recovered. Using these results, we investigate the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the solutions of a fractional partial differential equation of hyperbolic type in a Banach space $(E, |\cdot|)$, real or complex.

Keywords: Hyperbolic fractional partial differential equation; $\psi$-Riemann-Liouville fractional partial integral; $\psi$-Hilfer fractional partial derivative; Ulam-Hyers stability; Ulam-Hyers-Rassias stability.

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1. Introduction

Fractional calculus is currently one of the subjects most studied by mathematicians and also by researchers in physics, chemistry, engineering, among others, due to its innumerable applications in modeling real phenomena. Indeed, by considering non-integer order derivatives, it has been possible, in several cases, to better match theoretical models to experimental data, allowing better predictions of the future dynamics of processes [1, 2, 3, 4, 5]. One problem in this field is the great number of possible different definitions of fractional operators, making the choice of the best operator for each particular system a crucial issue. One way to overcome this problem is to consider more general definitions, of which the usual ones can be considered particular cases [4, 5, 6]. In this sense, Sousa and Oliveira [7] recently introduced a fractional derivative of one variable, the so-called $\psi$-Hilfer fractional derivative, from which it is possible to obtain a wide class of fractional derivatives already well established. Therefore, one of the objectives of this paper is to introduce the $\psi$-Riemann-Liouville fractional partial integral of a function with respect to another function and the $\psi$-Hilfer fractional partial derivative of $N$ variables.

On the other hand, the use of fractional derivatives in systems of differential equations has found several applications in mathematical models for population dynamics, erythrocyte sedimentation rate and others [1, 8, 9, 10, 11]. In addition, there has been a significant development in the study of ordinary differential equations involving derivatives of fractional order (see [12] and references therein). In particular, many studies have focused on the study
of the Ulam-Hyers stability of solutions of integrodifferential equations of fractional order \[13, 14, 15, 16\].

In the last years, Lungu et al., Brzdek et al. and Rus et al. published several papers on the study of the Ulam stability of the solution of partial differential equations of hyperbolic and pseudoparabolic types \[17, 18, 19, 20\]. In some of these cases, the authors made use of the Gronwall inequality and studied the Ulam-Hyers stability in Banach spaces \[17, 18\]. On the other hand, Abbas and Benchohra \[21, 22, 23\], Elmad and Rezapour \[24\], Abbas et al. \[25\], among others \[12, 26\], devoted themselves to studying the existence, uniqueness and the Ulam-Hyers stability of solutions of differential equations of fractional order. We can also mention the study of functional partial differential equations and the Darboux problem for partial differential equations of fractional order \[27, 28\].

In this paper, we consider the following fractional partial differential equation of hyperbolic type:

\[
\frac{\partial^2 \alpha \beta \psi u}{\partial \beta \psi x^\alpha \partial \beta \psi y^\alpha} (x, y) = f \left( x, y, u (x, y), \frac{\partial^\alpha \beta \psi u}{\partial \beta \psi x^\alpha} (x, y), \frac{\partial^\alpha \beta \psi u}{\partial \beta \psi y^\alpha} (x, y) \right),
\]

where \( \frac{\partial^2 \alpha \beta \psi u}{\partial \beta \psi x^\alpha \partial \beta \psi y^\alpha} (\cdot, \cdot) \) is the \( \psi \)-Hilfer fractional partial derivative with \( \frac{1}{2} < \alpha \leq 1, 0 \leq \beta \leq 1 \) and \( 0 \leq x < a, 0 \leq y < b \) and \( f \in C ([0, a) \times [0, b) \times B^3, \mathbb{B}) \), where \((\mathbb{B}, |\cdot|)\) is a real or complex Banach space.

The main objective of this paper is to introduce the \( \psi \)-Riemann-Liouville fractional partial integral and the \( \psi \)-Hilfer fractional partial derivative of \( N \) variables, in order to study the Ulam-Hyers and Ulam-Hyers-Rassias stabilities of the solutions of Eq. (1.1) by means of the \( \psi \)-Hilfer fractional partial derivative and the Gronwall inequality.

The paper is organized as follows: In section 2, we present definitions and results considered important for the development of the article. In section 3, we introduce the \( \psi \)-Riemann-Liouville fractional integral of a function with respect to another function of \( N \) variables and the \( \psi \)-Hilfer fractional partial derivative and study some of its particular cases, highlighting relevant points. Moreover, using the \( \psi \)-Hilfer fractional partial derivative, we present new versions for the definitions of Ulam-Hyers and Ulam-Hyers-Rassias stabilities. In section 4, by means of Theorem 4 we study the Ulam-Hyers stability, while in section 5, through Theorem 5 we discuss the generalized Ulam-Hyers-Rassias stability. Concluding remarks close the paper.

2. Preliminaries

In this section we present the definitions of the \( \psi \)-Riemann-Liouville fractional integral and the \( \psi \)-Hilfer fractional derivative; we introduce the Gronwall inequality by means of a lemma and present a particular case of Eq. (2.1).
Definition 1. [7] Let \((a, b) (-\infty \leq a < b \leq \infty)\) be a finite (or infinite) interval of the real line \(\mathbb{R}\) and let \(\alpha > 0\). Also let \(\psi (t)\) be an increasing and positive monotone function on \((a, b)\) having a continuous derivative \(\psi'(t)\) on \((a, b)\). We denote its first derivative as \(\frac{d}{dt}\psi(t) = \psi'(t)\). The left-sided fractional integral of a function \(f\) with respect to a function \(\psi\) on \([a, b]\) is defined by

\[
I_{a+}^{\alpha; \psi} f(t) = \frac{1}{\Gamma (\alpha)} \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha - 1} f(s) \, ds.
\]

The right-sided fractional integral is defined in an analogous form.

Definition 2. [7] Let \(n - 1 < \alpha < n\) with \(n \in \mathbb{N}\); let \(I = [a, b]\) be an interval such that \(-\infty \leq a < b \leq \infty\) and let \(f, \psi \in C^n [a, b]\) be two functions such that \(\psi\) is increasing and \(\psi'(t) \neq 0\) for all \(t \in I\). The left-sided \(\psi\)-Hilfer fractional derivative, denoted by \(H\mathbb{D}_{a+}^{\alpha, \beta; \psi} \cdot \), of a function \(f\), of order \(\alpha\) and type \(0 \leq \beta \leq 1\), is defined by

\[
H\mathbb{D}_{a+}^{\alpha, \beta; \psi} f(t) = I_{a+}^{(n-\alpha)\psi} \left( \frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a+}^{(1-\beta)(n-\alpha)\psi} f(t).
\]

The right-sided \(\psi\)-Hilfer fractional derivative is defined in an analogous form.

The following Gronwall lemma is an important tool in proving the main results of this paper.

Lemma 1. [29] We assume that:

(1) \(u, v, h \in C \left( \mathbb{R}_+^n, \mathbb{R}_+ \right)\);
(2) \(\psi(t)\) is increasing and \(\psi'(t)\) nondecreasing, for all \(t \in \mathbb{R}_+^n\) and for any \(t \geq a\),

\[
u(t) \leq v(t) + h(t) \int_a^t \psi'(s) (\psi(t) - \psi(s))^{\alpha - 1} u(s) \, ds;
\]

(3) \(h(t)\) is nonnegative and nondecreasing.

Then, we have

\[
u(t) \leq v(t) \mathbb{E}_a [h(t) \Gamma (\alpha) (\psi(t) - \psi(a))^\alpha],
\]

for any \(t \geq a\), where \(\mathbb{E}_a(\cdot)\) is the one-parameter Mittag-Leffler function.

Lemma 2. [7] Let \(\alpha > 0\) and \(\delta > 0\). If \(f(x) = (\psi(x) - \psi(a))^\delta\), then

\[
I_{a+}^{\alpha; \psi} f(x) = \frac{\Gamma (\delta)}{\Gamma (\alpha + \delta)} (\psi(x) - \psi(a))^{\alpha + \delta - 1}.
\]

3. \(\psi\)-Hilfer fractional partial derivative

In this section we introduce the \(\psi\)-Riemann-Liouville fractional partial integral and the \(\psi\)-Hilfer fractional partial derivative of a function of \(N\) variables with respect to another function, as well as some particular cases. Using these definitions, we introduce new versions
of the Ulam-Hyers and Ulam-Hyers-Rassias stabilities. In addition, some important results are discussed.

The first interesting and important result of this paper is the \( \psi \)-Riemann-Liouville fractional partial integral of a function of \( N \) variables with respect to another function. The motivation for this extension comes from Eq.(2.1).

**Definition 3.** Let \( \theta = (\theta_1, \theta_2, ..., \theta_N) \) and \( \alpha = (\alpha_1, \alpha_2, ..., \alpha_N) \), where \( 0 < \alpha_1, \alpha_2, ..., \alpha_N < 1 \), \( N \in \mathbb{N} \). Also put \( \bar{I} = I_1 \times I_2 \times \cdots \times I_N = [\theta_1, a_1] \times [\theta_2, a_2] \times \cdots \times [\theta_N, a_N] \), where \( a_1, a_2, ..., a_N \) and \( \theta_1, \theta_2, ..., \theta_N \) are positive constants. Also let \( \psi (\cdot) \) be an increasing and positive monotone function on \( (\theta_1, a_1) \times (\theta_2, a_2) \times \cdots \times (\theta_N, a_N) \), having a continuous derivative \( \psi' (\cdot) \) on \( (\theta_1, a_1), (\theta_2, a_2), ..., (\theta_N, a_N) \). The \( \psi \)-Riemann-Liouville partial integral of a function of \( N \) variables \( u = (u_1, u_2, ..., u_N) \in L^1 (\bar{I}) \) is defined by

\[
I_{\theta,x}^{\alpha;\psi} u (x) = \frac{1}{\Gamma (\alpha)} \int \cdots \int_{\bar{I}} \psi' (s_j) (\psi (x_j) - \psi (s_j))^{\alpha_j - 1} u (s_j) \, ds_j,
\]

with \( \psi' (s_j) (\psi (x_j) - \psi (s_j))^{\alpha_j - 1} = \psi' (s_1) (\psi (x_1) - \psi (s_1))^{\alpha_1 - 1} \psi' (s_2) (\psi (x_2) - \psi (s_2))^{\alpha_2 - 1} \cdots \psi' (s_N) (\psi (x_N) - \psi (s_N))^{\alpha_N - 1} \) and where we use the notations \( \Gamma (\alpha) = \Gamma (\alpha_1) \Gamma (\alpha_2) \cdots \Gamma (\alpha_N) \), \( u (s_j) = u (s_1) u (s_2) \cdots u (s_N) \) and \( ds_j = ds_1 ds_2 \cdots ds_N \), \( j \in \{1, 2, ..., N\} \) with \( N \in \mathbb{N} \).

Particular cases of this fractional partial integral, Eq.\( (3.1) \), are presented below.

**Remark 1.**

(1) If we consider \( \psi (x_i) = x_i \) in Eq.\( (3.1) \), we obtain the Riemann-Liouville fractional partial integral of a function of \( N \) variables, given by

\[
I_{\theta,x}^{\alpha} u (x) = \frac{1}{\Gamma (\alpha)} \int \cdots \int_{\bar{I}} (x_j - s_j)^{\alpha_j - 1} u (s_j) \, ds_j,
\]

with \( (x_j - s_j)^{\alpha_j - 1} = (x_1 - s_1)^{\alpha_1 - 1} (x_2 - s_2)^{\alpha_2 - 1} \cdots (x_N - s_N)^{\alpha_N - 1} \), \( \Gamma (\alpha) = \Gamma (\alpha_1) \Gamma (\alpha_2) \cdots \Gamma (\alpha_N) \), \( u (s_j) = u (s_1) u (s_2) \cdots u (s_N) \) and \( ds_j = ds_1 ds_2 \cdots ds_N \), \( j \in \{1, 2, ..., N\} \) with \( N \in \mathbb{N} \).

(2) If we consider \( \psi (x_i) = \ln x_i \) and \( \theta_i = 0 \) for \( i = 1, 2, ..., N \) in Eq.\( (3.1) \), we obtain the Hadamard fractional partial integral of a function of \( N \) variables, given by

\[
I_{\theta,x}^{\alpha} u (x) = \frac{1}{\Gamma (\alpha)} \int \cdots \int_{\bar{I}} \left( \ln \frac{x_j}{s_j} \right)^{\alpha_j - 1} u (s_j) \, \frac{ds_j}{s_j},
\]

with \( \left( \ln \frac{x_j}{s_j} \right)^{\alpha_j - 1} = \left( \ln \frac{x_1}{s_1} \right)^{\alpha_1 - 1} \left( \ln \frac{x_2}{s_2} \right)^{\alpha_2 - 1} \cdots \left( \ln \frac{x_N}{s_N} \right)^{\alpha_N - 1} \), \( \Gamma (\alpha) = \Gamma (\alpha_1) \Gamma (\alpha_2) \cdots \Gamma (\alpha_N) \), \( u (s_j) = u (s_1) u (s_2) \cdots u (s_N) \) and \( ds_j = ds_1 ds_2 \cdots ds_N \), \( j \in \{1, 2, ..., N\} \) with \( N \in \mathbb{N} \).
(3) If we consider \( \psi(x_i) = x_i \) and \( \theta_i = -\infty \) for \( i = 1, 2, \ldots, N \) in Eq. (3.1), we obtain the Liouville fractional partial integral of a function of \( N \) variables, given by

\[
I_{\theta, x}^\alpha u(x) = \frac{1}{\Gamma(\alpha)} \int \cdots \int_{I_{\infty}} (x_j - s_j)^{\alpha_j - 1} u(s_j) ds_j,
\]

with \((x_j - s_j)^{\alpha_j - 1} = (x_1 - s_1)^{\alpha_1 - 1} (x_2 - s_2)^{\alpha_2 - 1} \cdots (x_N - s_N)^{\alpha_N - 1}\), \( \Gamma(\alpha_j) = \Gamma(\alpha_1) \Gamma(\alpha_2) \cdots \Gamma(\alpha_N) \), \( u(s_j) = u(s_1) u(s_2) \cdots u(s_N) \) and \( ds_j = ds_1 ds_2 \cdots ds_N, j \in \{1, 2, \ldots, N\} \) with \( N \in \mathbb{N} \).

It is possible to obtain other fractional partial integrals, that is, Erdélyi-Kober fractional partial integral, Katugampola fractional partial integral, Weyl fractional partial integral, among others, as well as Sousa and Oliveira [7], in a recent work, introduced the \( \psi \)-Hilfer fractional integral, using different choices for \( \psi(\cdot) \) and parameters \( d_i \). Moreover, each fractional partial integral obtained here, is an extension of its respective fractional integral [3 4 5 7].

As an application, taking \( N = 2 \) and \( \theta_1 = \theta_2 = 0 \) in Eq. (3.1) we have the fractional partial integral that will be used in what follows:

\[
I_{\theta, x}^{\alpha \psi} u(x_1, x_2) = \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^{x_1} \int_0^{x_2} \psi'(s_1) \psi'(s_2) (\psi(x_1) - \psi(s_1))^{\alpha_1 - 1} (\psi(x_2) - \psi(s_2))^{\alpha_2 - 1} u(s_1, s_2) ds_1 ds_2,
\]

with \( 0 < \alpha_1, \alpha_2 \leq 1 \).

Also, we have

\[
I_{0+, x_1}^{\alpha_1 \psi} u(x_1, x_2) = \frac{1}{\Gamma(\alpha_1)} \int_0^{x_1} \psi'(s_1) (\psi(x_1) - \psi(s_1))^{\alpha_1 - 1} u(s_1, s_2) ds_1
\]

and

\[
I_{0+, x_2}^{\alpha_2 \psi} u(x_1, x_2) = \frac{1}{\Gamma(\alpha_2)} \int_0^{x_2} \psi'(s_2) (\psi(x_2) - \psi(s_2))^{\alpha_2 - 1} u(s_1, s_2) ds_2,
\]

with \( 0 < \alpha_1, \alpha_2 \leq 1 \).

Sousa and Oliveira [7] used the Riemann-Liouville fractional integral with respect to a function (one variable) to introduce the \( \psi \)-Hilfer fractional derivative (one variable); we follow here a similar procedure, starting with a fractional integral of \( N \) variables, to introduce the \( \psi \)-Hilfer fractional partial derivative of \( N \) variables.

**Definition 4.** Let \( \theta = (\theta_1, \theta_2, \ldots, \theta_N) \) and \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \), where \( 0 < \alpha_1, \alpha_2, \ldots, \alpha_N < 1 \), \( N \in \mathbb{N} \). Put \( \mathbb{I} = I_{\alpha_1} \times I_{\alpha_2} \times \cdots \times I_{\alpha_N} = [\theta_1, a_1] \times [\theta_2, a_2] \times \cdots \times [\theta_N, a_N] \), where \( a_1, a_2, \ldots, a_N \) and \( \theta_1, \theta_2, \ldots, \theta_N \) are positive constants. Also, let functions \( u, \psi \in C^n(\mathbb{I}, \mathbb{R}) \) and such that \( \psi \) is increasing and \( \psi'(x_i) \neq 0 \), \( i \in \{1, 2, \ldots, N\}, x_i \in \mathbb{I}, N \in \mathbb{N} \). The \( \psi \)-Hilfer fractional
partial derivative of a function of $N$ variables, of order $\alpha$ and type $0 \leq \beta_1, \beta_2, ..., \beta_N \leq 1$, denoted by $D_{\theta,x}^{\alpha;\beta;\psi}(\cdot)$, is defined by

\[
D_{\theta,x}^{\alpha;\beta;\psi} u(x) = I_{\theta,x_j}^{(1-\alpha)j;\psi} \left( \frac{1}{\psi'(x_j)} \frac{\partial^N}{\partial x_j} \right) I_{\theta,x_j}^{(1-\beta)(1-\alpha)j;\psi} u(x_j),
\]

with $\partial x_j = \partial x_1 \partial x_2 \cdots \partial x_N$ and $\psi'(x_j) = \psi'(x_1) \psi'(x_2) \cdots \psi'(x_N)$, $j \in \{1,2,...,N\}$, $N \in \mathbb{N}$.

In what follows a few particular cases of fractional partial derivatives are presented. It is important to note that the $\psi$-Hilfer fractional partial derivative contains a wide class of fractional partial derivatives, each one is an extension of the fractional derivative of one variable.

Remark 2.

1. Taking the limit $\beta \to 0$ on both sides of Eq. (3.5), we have the $\psi$-Riemann-Liouville fractional partial derivative of $N$ variables, given by

\[
\mathcal{R}_\theta \mathcal{D}_{\theta,x}^{\alpha;\psi} u(x) = \left( \frac{1}{\psi'(x_j)} \frac{\partial^N}{\partial x_j} \right) I_{\theta,x_j}^{1-\alpha;\psi} u(x_j),
\]

with $\partial x_j = \partial x_1 \partial x_2 \cdots \partial x_N$ and $\psi'(x_j) = \psi'(x_1) \psi'(x_2) \cdots \psi'(x_N)$, $j \in \{1,2,...,N\}$, $N \in \mathbb{N}$.

2. Taking the limit $\beta \to 1$ on both sides of Eq. (3.5), we have the $\psi$-Caputo fractional partial derivative of $N$ variables, given by

\[
\mathcal{C}_\theta \mathcal{D}_{\theta,x}^{\alpha;\psi} u(x) = \left( \frac{1}{\psi'(x_j)} \frac{\partial^N}{\partial x_j} \right) I_{\theta,x_j}^{1-\alpha;\psi} u(x_j),
\]

with $\partial x_j = \partial x_1 \partial x_2 \cdots \partial x_N$ and $\psi'(x_j) = \psi'(x_1) \psi'(x_2) \cdots \psi'(x_N)$, $j \in \{1,2,...,N\}$, $N \in \mathbb{N}$.

3. Taking the limit $\beta \to 0$ on both sides of Eq. (3.5) and choosing $\psi(x_j) = x_j$, $j \in \{1,2,...,N\}$, we have the Riemann-Liouville fractional partial derivative of $N$ variables, given by

\[
\mathcal{R}_\theta \mathcal{D}_{\theta,x}^{\alpha;\psi} u(x) = \left( \frac{\partial^N}{\partial x_j} \right) I_{\theta,x_j}^{1-\alpha;\psi} u(x_j),
\]

with $\partial x_j = \partial x_1 \partial x_2 \cdots \partial x_N$, for $N \in \mathbb{N}$.

4. Taking the limit $\beta \to 1$ on both sides of Eq. (3.5) and choosing $\psi(x_j) = x_j$, $j \in \{1,2,...,N\}$, we have the Caputo fractional partial derivative of $N$ variables, given by

\[
\mathcal{C}_\theta \mathcal{D}_{\theta,x}^{\alpha;\psi} u(x) = \left( \frac{\partial^N}{\partial x_j} \right) I_{\theta,x_j}^{1-\alpha;\psi} u(x_j),
\]

with $\partial x_j = \partial x_1 \partial x_2 \cdots \partial x_N$, for $N \in \mathbb{N}$.
We have presented only a few particular cases of fractional partial derivatives. It is possible to obtain many other fractional partial derivatives, starting from different choices for function \( \psi (\cdot) \) and taking the limits \( \beta \to 0 \) and \( \beta \to 1 \), e.g. the Hadamard fractional partial derivative, Caputo-Hadamard fractional partial derivative, Caputo-Katugampola fractional partial derivative, among others. In all these cases, each fractional partial derivative obtained is an extension of its corresponding fractional derivative \[3, 4, 5, 6, 7\].

Now, choosing \( N = 2 \) in Eq.(3.5), we find the fractional partial derivative that will be used in this paper:

\[
(3.6)\quad H \mathbb{D}^{\alpha,\beta,\psi}_{\theta} u (x_1, x_2) = I^{(1-\alpha)\psi}_\theta \left( \frac{1}{\psi'(x_1) \psi'(x_2)} \frac{\partial^2}{\partial x_1 \partial x_2} \right) I^{(1-\beta)(1-\alpha)\psi}_\theta u (x_1, x_2).
\]

We shall also use the following notation:

\[
(3.7)\quad H \mathbb{D}^{\alpha,\beta,\psi}_{\theta} u (x_1, x_2) = \frac{\partial^{2\alpha}}{\partial \beta \psi x^\alpha \partial \beta \psi y^\alpha} (x_1, x_2).
\]

Let \( a, b \in (0, \infty) \), \( \varepsilon > 0 \), \( \varphi \in C ([0,a) \times [0,b), \mathbb{R}_+) \) and let \( \mathbb{B}, |\cdot| \) be a real or complex Banach space.

Consider also the following inequalities, of paramount importance, which will be used to introduce the Ulam-Hyers and Ulam-Hyers-Rassias stabilities:

\[
(3.8)\quad \left| \frac{\partial^{2\alpha}}{\partial \beta \psi x^\alpha \partial \beta \psi y^\alpha} (x, y) - f \left( x, y, v (x, y), \frac{\partial^{\alpha}}{\partial \beta \psi x^\alpha} (x, y), \frac{\partial^{\alpha}}{\partial \beta \psi y^\alpha} (x, y) \right) \right| \leq \varepsilon,
\]

\( x \in [0,a) \), \( y \in [0,b) \).

\[
(3.9)\quad \left| \frac{\partial^{2\alpha}}{\partial \beta \psi x^\alpha \partial \beta \psi y^\alpha} (x, y) - f \left( x, y, v (x, y), \frac{\partial^{\alpha}}{\partial \beta \psi x^\alpha} (x, y), \frac{\partial^{\alpha}}{\partial \beta \psi y^\alpha} (x, y) \right) \right| \leq \varphi (x, y),
\]

\( x \in [0,a) \), \( y \in [0,b) \).

\[
(3.10)\quad \left| \frac{\partial^{2\alpha}}{\partial \beta \psi x^\alpha \partial \beta \psi y^\alpha} (x, y) - f \left( x, y, v (x, y), \frac{\partial^{\alpha}}{\partial \beta \psi x^\alpha} (x, y), \frac{\partial^{\alpha}}{\partial \beta \psi y^\alpha} (x, y) \right) \right| \leq \varepsilon \varphi (x, y),
\]

\( x \in [0,a) \), \( y \in [0,b) \), with \( \frac{1}{2} < \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \).

**Definition 5.** A function \( u \) is a solution of Eq.(1.1), if \( u \in C^1 ([0,a) \times [0,b), \mathbb{B}) \) and \( \frac{\partial^{2\alpha}}{\partial \beta \psi x^\alpha \partial \beta \psi y^\alpha} (x, y) \in C ([0,a) \times [0,b), \mathbb{B}) \), with \( \frac{1}{2} < \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \).

**Definition 6.** A solution of Eq.(1.1) admits Ulam-Hyers stability if there exist real numbers \( C^1_f, C^2_f \) and \( C^3_f \) such that, for any \( \varepsilon > 0 \) and for any solution \( v \) to the inequality Eq.(3.8), we have
\[ |v(x, y) - u(x, y)| \leq C_1^1 \epsilon, \quad \forall x \in [0, a), \quad \forall y \in [0, b), \]
\[ \left| \frac{\partial_\beta \partial_\psi v}{\partial_\beta \partial_\psi \alpha} (x, y) - \frac{\partial_\beta \partial_\psi u}{\partial_\beta \partial_\psi \alpha} (x, y) \right| \leq C_2^0 \epsilon, \quad \forall x \in [0, a), \quad \forall y \in [0, b), \]
\[ \left| \frac{\partial_\beta \partial_\psi v}{\partial_\beta \partial_\psi \alpha} (x, y) - \frac{\partial_\beta \partial_\psi u}{\partial_\beta \partial_\psi \alpha} (x, y) \right| \leq C_3^3 \epsilon, \quad \forall x \in [0, a), \quad \forall y \in [0, b), \]

with \( \frac{1}{2} < \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \).

**Definition 7.** A solution of Eq. (1.1) admits generalized Ulam-Hyers-Rassias stability if there exist real numbers \( C_1^1 \), \( C_2^0 \) and \( C_3^3 > 0 \) such that, for any \( \epsilon > 0 \) and for any solution \( v \) to the inequality Eq. (3.9), we have

\[ |v(x, y) - u(x, y)| \leq C_1^1 \phi (x, y), \quad \forall x \in [0, a), \quad \forall y \in [0, b), \]
\[ \left| \frac{\partial_\beta \partial_\psi v}{\partial_\beta \partial_\psi \alpha} (x, y) - \frac{\partial_\beta \partial_\psi u}{\partial_\beta \partial_\psi \alpha} (x, y) \right| \leq C_2^0 \phi (x, y), \quad \forall x \in [0, a), \quad \forall y \in [0, b), \]
\[ \left| \frac{\partial_\beta \partial_\psi v}{\partial_\beta \partial_\psi \alpha} (x, y) - \frac{\partial_\beta \partial_\psi u}{\partial_\beta \partial_\psi \alpha} (x, y) \right| \leq C_3^3 \phi (x, y), \quad \forall x \in [0, a), \quad \forall y \in [0, b), \]

with \( \frac{1}{2} < \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \).

**Remark 3.** A function \( v \) is a solution to the inequality Eq. (3.8) if, and only if, there exists a function \( g \in C([0, a) \times [0, b), \mathbb{B}) \), which depends on \( v \), such that

1. For all \( \epsilon > 0 \), \( |g(x, y)| \leq \epsilon \), for all \( x \in [0, a) \), for all \( y \in [0, b) \);
2. For all \( x \in [0, a) \), for all \( y \in [0, b) \),

\[ \frac{\partial_\beta \partial_\psi v}{\partial_\beta \partial_\psi \alpha} (x, y) = f \left( x, y, v(x, y), \frac{\partial_\beta \partial_\psi v}{\partial_\beta \partial_\psi \alpha} (x, y), \frac{\partial_\beta \partial_\psi v}{\partial_\beta \partial_\psi \alpha} (x, y) \right) + g(x, y) \]

with \( \frac{1}{2} < \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \).

**Remark 4.** A function \( v \) is a solution to the inequality Eq. (3.9) if, and only if, there exists a function \( g \in C([0, a) \times [0, b), \mathbb{B}) \), which depends on \( v \), such that

1. \( |g(x, y)| \leq \phi (x, y) \), for all \( x \in [0, a) \), for all \( y \in [0, b) \);

\[ \frac{\partial_\beta \partial_\psi v}{\partial_\beta \partial_\psi \alpha} (x, y) = f \left( x, y, v(x, y), \frac{\partial_\beta \partial_\psi v}{\partial_\beta \partial_\psi \alpha} (x, y), \frac{\partial_\beta \partial_\psi v}{\partial_\beta \partial_\psi \alpha} (x, y) \right) + g(x, y) \]
(2) For all \( x \in [0,a) \), for all \( y \in [0,b) \),

\[
\frac{\partial^{2\alpha} v}{\partial^{\alpha} x^{\alpha} \partial^{\alpha} y^{\alpha}}(x,y) = f(x,y,v(x,y), \frac{\partial^{\alpha} v}{\partial^{\alpha} x^{\alpha}}(x,y), \frac{\partial^{\alpha} v}{\partial^{\alpha} y^{\alpha}}(x,y)) + g(x,y)
\]

with \( \frac{1}{2} < \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \).

Remark 5. A function \( v \) is a solution to the inequality Eq. (3.10) if, and only if, there exists a function \( g \in C([0,a) \times [0,b), \mathbb{R}) \), which depends on \( v \), such that

1. For all \( \varepsilon > 0 \), \( |g(x,y)| \leq \varepsilon \varphi(x,y) \), for all \( x \in [0,a) \), for all \( y \in [0,b) \);  
2. For all \( x \in [0,a) \), for all \( y \in [0,b) \),

\[
\frac{\partial^{2\alpha} v}{\partial^{\alpha} x^{\alpha} \partial^{\alpha} y^{\alpha}}(x,y) = f(x,y,v(x,y), \frac{\partial^{\alpha} v}{\partial^{\alpha} x^{\alpha}}(x,y), \frac{\partial^{\alpha} v}{\partial^{\alpha} y^{\alpha}}(x,y)) + g(x,y)
\]

with \( \frac{1}{2} < \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \).

We recall that, in this paper, we have used the notation

\[
\begin{align*}
\frac{\partial^{\alpha} v}{\partial^{\alpha} x^{\alpha}}(x,y) &= u_1(x,y) = \frac{\partial^{\alpha} v}{\partial^{\alpha} x^{\alpha}}(x,y) \\
\frac{\partial^{\alpha} v}{\partial^{\alpha} y^{\alpha}}(x,y) &= u_2(x,y) = \frac{\partial^{\alpha} v}{\partial^{\alpha} y^{\alpha}}(x,y) \\
\frac{\partial^{\alpha} v}{\partial^{\alpha} x^{\alpha}}(x,y) &= v_1(x,y) = \frac{\partial^{\alpha} v}{\partial^{\alpha} x^{\alpha}}(x,y) \\
\frac{\partial^{\alpha} v}{\partial^{\alpha} y^{\alpha}}(x,y) &= v_2(x,y) = \frac{\partial^{\alpha} v}{\partial^{\alpha} y^{\alpha}}(x,y)
\end{align*}
\]

with \( \frac{1}{2} < \alpha \leq 1 \) and \( 0 \leq \beta \leq 1 \).

Lemma 3. Let \( 0 < \alpha_1, \alpha_2, \ldots, \alpha_N < 1 \) and \( \delta_1, \delta_2, \ldots, \delta_N > 0 \). If \( u(x_j) = (\psi(x_j) - \psi(0))^{\delta_j} - 1 \) with \( u(x) = u(x_1, x_2, \ldots, x_N) \) and \( (\psi(x_j) - \psi(0))^{\delta_j} - 1 = (\psi(x_1) - \psi(0))^{\delta_1} (\psi(x_2) - \psi(0))^{\delta_2} - 1. \ldots (\psi(x_N) - \psi(0))^{\delta_N} - 1 \), then

\[
I^{\alpha_\psi}_0 u(x) = \prod_{j=1}^{N} \frac{\Gamma(\delta_j)}{\Gamma(\alpha_j + \delta_j)} (\psi(x_j) - \psi(0))^{\alpha_j + \delta_j - 1},
\]

with \( N \in \mathbb{N} \).

Proof. The proof is realized by induction on \( N \). According to Lemma 2, Eq. (3.12) is valid for \( N = 1 \).
For $N = 2$, we have

$$I_{\theta}^{\alpha;\psi} u (x_1, x_2) = \frac{1}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^{x_1} \int_0^{x_2} \psi' (s_1) \psi' (s_2) (\psi (x_1) - \psi (s_1))^{\alpha_1 - 1} (\psi (x_2) - \psi (s_2))^{\alpha_2 - 1} \psi (s_1) - \psi (0))^{\delta_1 - 1} (\psi (s_2) - \psi (0))^{\delta_2 - 1} ds_2 ds_1$$

$$= \frac{\Gamma(\delta_2) (\psi (x_2) - \psi (0))^{\alpha_2 + \delta_2 - 1}}{\Gamma(\alpha_2 + \delta_2)} \int_0^{x_1} \psi' (s_1) (\psi (x_1) - \psi (s_1))^{\delta_1 - 1} ds_1$$

$$= \frac{\Gamma(\delta_1) \Gamma(\delta_2)}{\Gamma(\alpha_1 + \delta_1) \Gamma(\alpha_2 + \delta_2)} (\psi (x_1) - \psi (0))^{\alpha_1 + \delta_1 - 1} (\psi (x_2) - \psi (0))^{\alpha_2 + \delta_2 - 1}.$$ 

Suppose it is true for $N - 1$, i.e.,

$$I_{\theta}^{\alpha;\psi} u (x) = \frac{\Gamma(\delta_1)}{\Gamma(\alpha_1 + \delta_1)} (\psi (x_1) - \psi (0))^{\alpha_1 + \delta_1 - 1} \frac{\Gamma(\delta_2)}{\Gamma(\alpha_2 + \delta_2)} (\psi (x_2) - \psi (0))^{\alpha_2 + \delta_2 - 1} \cdots \frac{\Gamma(\delta_1)}{\Gamma(\alpha_{N-1} + \delta_{N-1})} (\psi (x_{N-1}) - \psi (0))^{\alpha_{N-1} + \delta_{N-1} - 1}.$$

Let’s prove that it holds for $N$. Indeed, for $u (x_j) = u (x_1, x_2, \ldots, x_N)$, $\Gamma (\alpha_j) = \Gamma (\alpha_1) \Gamma (\alpha_2) \cdots \Gamma (\alpha_{N-1})$, $\psi' (s_j) (\psi (x_j) - \psi (s_j))^{\alpha_j - 1} = \psi' (s_1) (\psi (x_1) - \psi (s_1))^{\alpha_1 - 1} \cdots \psi' (s_{N-1}) (\psi (x_{N-1}) - \psi (s_{N-1}))^{\alpha_{N-1} - 1}$ we have

$$I_{\theta, x}^{\alpha;\psi} u (x_j) = \frac{1}{\Gamma(\alpha_j) \Gamma(\alpha_N)} \int_0^{x_j} \int_0^{x_N} \psi' (s_j) (\psi (x_j) - \psi (s_j))^{\alpha_j - 1} (\psi (x_j) - \psi (0))^{\delta_j - 1} \psi' (s_N) (\psi (x_N) - \psi (s_N))^{\alpha_N - 1} (\psi (s_N) - \psi (0))^{\delta_N - 1} ds_j ds_N$$

$$= \frac{\Gamma(\delta_1) \Gamma(\delta_2) \cdots \Gamma(\delta_{N-1})}{\Gamma(\alpha_1 + \delta_1) \Gamma(\alpha_2 + \delta_2) \cdots \Gamma(\alpha_{N-1} + \delta_{N-1}) \Gamma(\alpha_N)} (\psi (x_1) - \psi (0))^{\alpha_1 + \delta_1 - 1} \times$$

$$\int_0^{x_N} \psi' (s_N) (\psi (x_N) - \psi (s_N))^{\alpha_N - 1} (\psi (s_N) - \psi (0))^{\delta_N - 1} ds_N$$

$$= \prod_{j=1}^{N} \frac{\Gamma(\delta_j)}{\Gamma(\alpha_j + \delta_j)} (\psi (x_j) - \psi (0))^{\alpha_j + \delta_j - 1}.$$
Theorem 1. If \( v \) is a solution to the inequality Eq. (3.8), then \( (v, v_1, v_2) \) satisfy the following system of integral inequalities:

\[
\begin{align*}
|v(x, y) - (\psi(y) - \psi(0))^{\gamma^{-1}}\frac{\Gamma(\gamma)}{\Gamma(\gamma)}v(x, 0) - (\psi(x) - \psi(0))^{\gamma^{-1}}\frac{\Gamma(\gamma)}{\Gamma(\gamma)}v(0, y)| + v(0, 0) - I^{\alpha_2, \psi}_0 f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \\
&\leq \varepsilon \frac{(\psi(y) - \psi(0))^{\alpha_2}}{\Gamma(\alpha_2 + 1)} \frac{(\psi(x) - \psi(0))^{\alpha_1}}{\Gamma(\alpha_1 + 1)},
\end{align*}
\]

\[
\begin{align*}
|v_1(x, y) - (\psi(y) - \psi(0))^{\gamma^{-1}}v_1(x, 0) - I^{\alpha_2, \psi}_{0+} f(x, y, v(x, y), v_1(x, y), v_2(x, y))| \\
&\leq \varepsilon \frac{(\psi(y) - \psi(0))^{\alpha_2}}{\Gamma(\alpha_2 + 1)},
\end{align*}
\]

\[
\begin{align*}
|v_2(x, y) - (\psi(x) - \psi(0))^{\gamma^{-1}}v_2(0, y) - I^{\alpha_1, \psi}_{0+} f(x, y, v(x, y), v_1(x, y), v_2(x, y))| \\
&\leq \varepsilon \frac{(\psi(x) - \psi(0))^{\alpha_1}}{\Gamma(\alpha_1 + 1)},
\end{align*}
\]

Proof. From Remark 3 we have that

\[
v(x, y) = \frac{(\psi(y) - \psi(0))^{\gamma^{-1}}}{\Gamma(\gamma)}v(x, 0) + \frac{(\psi(x) - \psi(0))^{\gamma^{-1}}}{\Gamma(\gamma)}v(0, y) - v(0, 0) + I^{\alpha_2, \psi}_0 f(x, y, v(x, y), v_1(x, y), v_2(x, y)) + I^{\alpha_2, \psi}_0 g(x, y);
\]

\[
v_1(x, y) = \frac{(\psi(y) - \psi(0))^{\gamma^{-1}}}{\Gamma(\gamma)}v(x, 0) + I^{\alpha_2, \psi}_{0+} f(x, y, v(x, y), v_1(x, y), v_2(x, y)) + I^{\alpha_2, \psi}_{0+} g(x, y);
\]

\[
v_2(x, y) = \frac{(\psi(x) - \psi(0))^{\gamma^{-1}}}{\Gamma(\gamma)}v(0, y) + I^{\alpha_1, \psi}_{0+} f(x, y, v(x, y), v_1(x, y), v_2(x, y)) + I^{\alpha_1, \psi}_{0+} g(x, y).
\]

The following also hold:

\[
\begin{align*}
|v(x, y) - (\psi(y) - \psi(0))^{\gamma^{-1}}v(x, 0) - (\psi(x) - \psi(0))^{\gamma^{-1}}v(0, y)| + v(0, 0) - I^{\alpha_2, \psi}_0 f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \\
&\leq I^{\alpha_2, \psi}_0 |g(x, y)| \leq \varepsilon I^{\alpha_2, \psi}_0 (1).
\end{align*}
\]

(3.13)
Note that, by Lemma 3, we have
\[
I_\theta^{\alpha;\psi} 1 = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y \psi'(s)(\psi(x) - \psi(s))^{\alpha_1-1}(\psi(y) - \psi(t))^{\alpha_2-1} dt ds
\]
\[(3.14) = \frac{(\psi(y) - \psi(0))^{\alpha_2}(\psi(x) - \psi(0))^{\alpha_1}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)}.\]

Substituting Eq. (3.14) in Eq. (3.13), we get
\[
|v(x, y) - \frac{(\psi(y) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v(x, 0) - \frac{(\psi(x) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v(0, y)|
+ v(0, 0) - I_{\theta}^{\alpha;\psi} f(x, y, v(x, y), v_1(x, y), v_2(x, y))
\leq \varepsilon \frac{(\psi(y) - \psi(0))^{\alpha_2}(\psi(x) - \psi(0))^{\alpha_1}}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)}.
\]

On the other hand, we also have
\[
|v_1(x, y) - \frac{(\psi(y) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v_1(x, 0) - I_{0+\gamma}^{\alpha_2;\psi} f(x, y, v(x, y), v_1(x, y), v_2(x, y))| 
\leq \varepsilon \frac{(\psi(y) - \psi(0))^{\alpha_2}}{\Gamma(\alpha_2 + 1)}
\]
and
\[
|v_2(x, y) - \frac{(\psi(x) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v_2(0, y) - I_{0+\gamma}^{\alpha_1;\psi} f(x, y, v(x, y), v_1(x, y), v_2(x, y))|
\leq \varepsilon \frac{(\psi(x) - \psi(0))^{\alpha_1}}{\Gamma(\alpha_1 + 1)}.
\]

The proofs of the following theorems will be optimized here, since they follow the same reasoning of the previous one.

**Theorem 2.** If \(v\) is a solution to the inequality Eq. (3.9), then \((v, v_1, v_2)\) satisfy the following integral inequalities:
\[
|v(x, y) - \frac{(\psi(y) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v(x, 0) - \frac{(\psi(x) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v(0, y)|
+ v(0, 0) - I_{\theta}^{\alpha;\psi} f(x, y, v(x, y), v_1(x, y), v_2(x, y))
\leq I_{\theta}^{\alpha;\psi} \phi(x, y),
\]
ON THE STABILITY OF A HYPERBOLIC FRACTIONAL PARTIAL DIFFERENTIAL EQUATION

\[ v_1(x, y) - \frac{(\psi(y) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v_1(x, 0) - I_{0+}^{\alpha_2;\psi} f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \]
\[ \leq I_{0+}^{\alpha_2;\psi} \phi(x, y), \]

\[ v_2(x, y) - \frac{(\psi(x) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v_2(0, y) - I_{0+}^{\alpha_1;\psi} f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \]
\[ \leq I_{0+}^{\alpha_1;\psi} \phi(x, y), \]

with \( \frac{1}{2} < \alpha \leq 1 \), \( 0 \leq \gamma < 1 \) (\( \gamma = \alpha + \beta(\alpha - 1) \)), for all \( x \in [0, a) \) and \( y \in [0, b) \).

**Theorem 3.** If \( v \) is a solution to the inequality Eq. (3.10), then \( (v, v_1, v_2) \) satisfy the following integral inequalities:

\[ v(x, y) - \frac{(\psi(y) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v(x, 0) - \frac{(\psi(x) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v(0, y) \]
\[ + v(0, 0) - I_{0}^{\alpha_1;\psi} f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \]
\[ \leq \varepsilon I_{0}^{\alpha_1;\psi} \phi(x, y), \]

\[ v_1(x, y) - \frac{(\psi(y) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v_1(x, 0) - I_{0+}^{\alpha_2;\psi} f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \]
\[ \leq \varepsilon I_{0+}^{\alpha_2;\psi} \phi(x, y), \]

\[ v_2(x, y) - \frac{(\psi(x) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v_2(0, y) - I_{0+}^{\alpha_1;\psi} f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \]
\[ \leq \varepsilon I_{0+}^{\alpha_1;\psi} \phi(x, y), \]

with \( \frac{1}{2} < \alpha \leq 1 \), \( 0 \leq \gamma < 1 \) (\( \gamma = \alpha + \beta(\alpha - 1) \)), for all \( x \in [0, a) \) and \( y \in [0, b) \).

4. **Ulam-Hyers stability**

In this section we present a result on the existence and uniqueness of the solution to Eq. (1.1) and we deduce a result on Ulam-Hyers stability for the same equation in the case \( a < \infty \) and \( b < \infty \).
Theorem 4. We assume that
(i) \( a < \infty \) and \( b < \infty \);
(ii) \( f \in C ([0, a] \times [0, b] \times \mathbb{B}^3, \mathbb{B}) \);
(iii) There exists \( L_f > 0 \) such that
\[
|f(x, y, z_1, z_2, z_3) - f(x, y, t_1, t_2, t_3)| \leq L_f \max_{i \in \{1, 2, 3\}} |z_i - t_i|
\]
for all \( x \in [0, a] \) and \( y \in [0, b] \) and \( z_1, z_2, z_3, t_1, t_2, t_3 \in \mathbb{B} \).

Then
(1) For \( \phi \in C^1 ([0, a], \mathbb{B}) \) and \( \xi \in C^1 ([0, b], \mathbb{B}) \), Eq. (1.1) has a unique solution satisfying
\[
\begin{align*}
I_0^{1-\gamma \psi} u(x, 0) &= \phi(x), \quad \forall x \in [0, a] \\
I_0^{1-\gamma \psi} u(0, y) &= \xi(y), \quad \forall y \in [0, b]
\end{align*}
\]
with \( \gamma = \alpha + \beta (\alpha - 1) (0 \leq \gamma < 1) \).

(2) The solution of Eq. (1.1) is Ulam-Hyers stable.

Proof. 1. Let us consider problem Eq. (1.1) with conditions Eq. (1.1) as a fixed point problem. If \( u \) is a solution of the problem Eq. (1.1) and Eq. (4.1), then \( u, \frac{\partial u}{\partial \beta \psi x^\alpha}, \frac{\partial^2 u}{\partial \beta \psi y^\alpha} \) is a solution of the following system:
\[
\begin{align*}
u(x, y) &= \frac{(\psi(y) - \psi(0))^{\gamma - 1}}{\Gamma(\gamma)} \varphi(x) + \frac{(\psi(x) - \psi(0))^{\gamma - 1}}{\Gamma(\gamma)} \xi(y) - \varphi(0) \\
u_1(x, y) &= \frac{(\psi(y) - \psi(0))^{\gamma - 1}}{\Gamma(\gamma)} \varphi(x) + \frac{1}{0^+ \psi} f(x, y, u(x, y), u_1(x, y), u_2(x, y)) \\
u_2(x, y) &= \frac{(\psi(x) - \psi(0))^{\gamma - 1}}{\Gamma(\gamma)} \xi(y) + \frac{1}{0^+ \psi} f(x, y, u(x, y), u_1(x, y), u_2(x, y))
\end{align*}
\]
with \( \frac{1}{2} < \alpha_1, \alpha_2 \leq 1 \) and \( 0 \leq \gamma < 1 \). We can write this in a general form:
\[
\begin{align*}
u(x, y) &= \frac{(\psi(y) - \psi(0))^{\gamma - 1}}{\Gamma(\gamma)} \varphi(x) + \frac{(\psi(x) - \psi(0))^{\gamma - 1}}{\Gamma(\gamma)} \xi(y) - \varphi(0) \\
u_1(x, y) &= \frac{1}{0^+ \psi} f(x, y, u(x, y), u_1(x, y), u_2(x, y)) \\
u_2(x, y) &= \frac{1}{0^+ \psi} f(x, y, u(x, y), u_1(x, y), u_2(x, y))
\end{align*}
\]
with \( \frac{1}{2} < \alpha_1, \alpha_2 \leq 1 \) and \( 0 \leq \gamma < 1 \). We can write this in a general form:
\[
\begin{align*}
u(x, y) &= \frac{(\psi(y) - \psi(0))^{\gamma - 1}}{\Gamma(\gamma)} \varphi(x) + \frac{(\psi(x) - \psi(0))^{\gamma - 1}}{\Gamma(\gamma)} \xi(y) - \varphi(0) \\
u_1(x, y) &= \frac{1}{0^+ \psi} f(x, y, u(x, y), u_1(x, y), u_2(x, y)) \\
u_2(x, y) &= \frac{1}{0^+ \psi} f(x, y, u(x, y), u_1(x, y), u_2(x, y))
\end{align*}
\]
with \( \frac{1}{2} < \alpha_1, \alpha_2 \leq 1 \) and \( 0 \leq \gamma < 1 \). We can write this in a general form:
\[
\begin{align*}
u(x, y) &= \frac{(\psi(y) - \psi(0))^{\gamma - 1}}{\Gamma(\gamma)} \varphi(x) + \frac{(\psi(x) - \psi(0))^{\gamma - 1}}{\Gamma(\gamma)} \xi(y) - \varphi(0) \\
u_1(x, y) &= \frac{1}{0^+ \psi} f(x, y, u(x, y), u_1(x, y), u_2(x, y)) \\
u_2(x, y) &= \frac{1}{0^+ \psi} f(x, y, u(x, y), u_1(x, y), u_2(x, y))
\end{align*}
\]
From Theorem 1, hypothesis (iii) and from Gronwall Lemma 1, it follows that
\[
(4.3)
\]
which satisfies the following conditions:

\[
\begin{align*}
\tau > 0 & \quad \text{such that} \\
\|A_1 (\overline{u}, \overline{u_1}, \overline{u_2}) - A_1 (\overline{u}, \overline{u_1}, \overline{u_2})\|_B & \leq \frac{L_f}{\tau} \|\overline{u}, \overline{u_1}, \overline{u_2}\|_B.
\end{align*}
\]

Thus, if \( \tau > 0 \) is such that \( \frac{L_f}{\tau} < 1 \), then operator \( A \) is a contraction and \( A \) is a Picard operator. Therefore, Eq. (1.1) and Eq. (4.1) have a unique solution.

2. Let \( v \) be a solution to the inequality Eq. (3.8) and let \( u \) be the unique solution to Eq. (1.1), which satisfies the following conditions:

\[
(4.3) \quad \begin{cases} I_{\alpha}^{\gamma-\psi} u(x,0) = v(x,0), \forall x \in [0,a] \\ I_{\alpha}^{\gamma-\psi} u(0,y) = u(0,y), \forall y \in [0,b] \end{cases}
\]

From Theorem 1, hypothesis (iii) and from Gronwall Lemma 1, it follows that

\[
\begin{align*}
|v(x,y) - u(x,y)| & = \left| v(x,y) - \frac{(\psi(y) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} v(x,0) - \frac{(\psi(x) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)} v(0,y) \right| \\
& \leq \frac{\varepsilon(\psi(x) - \psi(0))^{\alpha_1}(\psi(y) - \psi(0))^{\alpha_2}}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} + L_f I_{\theta}^{\alpha;\psi} \left\{ \max_{i\in\{1,2,3\}} |v_i(x,y) - u_i(x,y)| \right\} \\
& \leq \frac{\varepsilon(\psi(a) - \psi(0))^{\alpha_1}(\psi(b) - \psi(0))^{\alpha_2}}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} \times \\
& \quad \mathbb{E}_\alpha \left[ L_f \Gamma(\alpha_1) \Gamma(\alpha_2) (\psi(a) - \psi(0))^{\alpha_1} (\psi(b) - \psi(0))^{\alpha_2} \right] \\
& = \varepsilon C_f^i,
\end{align*}
\]

where

\[
C_f^i := \frac{(\psi(a) - \psi(0))^{\alpha_1}(\psi(b) - \psi(0))^{\alpha_2}}{\Gamma(\alpha_1 + 1) \Gamma(\alpha_2 + 1)} \mathbb{E}_\alpha \left[ L_f \Gamma(\alpha_1) \Gamma(\alpha_2) (\psi(a) - \psi(0))^{\alpha_1} (\psi(b) - \psi(0))^{\alpha_2} \right],
\]
where $\mathbb{E}_a(\cdot)$ is the one-parameter Mittag-Leffler function.

Similarly, we get,

$$
|v_1(x, y) - u_1(x, y)| \\
\leq |v_1(x, y) - \frac{(\psi(y) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v_1(x, 0) - I_{0+}^{\alpha_2;\psi} f(x, y, v(x, y), v_1(x, y), v_2(x, y))| \\
+ I_{0+}^{\alpha_2;\psi} |f(x, y, v(x, y), v_1(x, y), v_2(x, y)) - f(x, y, u(x, y), u_1(x, y), u_2(x, y))| \\
\leq \varepsilon \frac{(\psi(y) - \psi(0))^{\alpha_2}}{\Gamma(\alpha_2 + 1)} + L_f I_{0+}^{\alpha_2;\psi} \left\{ \max_{i \in \{1, 2, 3\}} |v_i(x, y) - u_i(x, y)| \right\} \\
\leq \varepsilon \frac{(\psi(b) - \psi(0))^{\alpha_2}}{\Gamma(\alpha_2 + 1)} \mathbb{E}_a[L_f \Gamma(\alpha_2) (\psi(b) - \psi(0))^{\alpha_2}] \\
= \varepsilon C_f^2
$$

where

$$
C_f^2 := \frac{(\psi(b) - \psi(0))^{\alpha_2}}{\Gamma(\alpha_2 + 1)} \mathbb{E}_a[L_f \Gamma(\alpha_2) (\psi(b) - \psi(0))^{\alpha_2}]
$$

and

$$
|v_2(x, y) - u_2(x, y)| \\
\leq |v_2(x, y) - \frac{(\psi(x) - \psi(0))^{\gamma-1}}{\Gamma(\gamma)}v_2(0, y) - I_{0+}^{\alpha_1;\psi} f(x, y, v(x, y), v_1(x, y), v_2(x, y))| \\
+ I_{0+}^{\alpha_1;\psi} |f(x, y, v(x, y), v_1(x, y), v_2(x, y)) - f(x, y, u(x, y), u_1(x, y), u_2(x, y))| \\
\leq \varepsilon \frac{(\psi(x) - \psi(0))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} + L_f I_{0+}^{\alpha_1;\psi} \left\{ \max_{i \in \{1, 2, 3\}} |v_i(x, y) - u_i(x, y)| \right\} \\
\leq \varepsilon \frac{(\psi(b) - \psi(0))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \mathbb{E}_a[L_f \Gamma(\alpha_1) (\psi(b) - \psi(0))^{\alpha_1}] \\
= \varepsilon C_f^3
$$

where

$$
C_f^3 := \frac{(\psi(a) - \psi(0))^{\alpha_1}}{\Gamma(\alpha_1 + 1)} \mathbb{E}_a[L_f \Gamma(\alpha_1) (\psi(a) - \psi(0))^{\alpha_1}].
$$

Remark 6.

(1) In general, if $a = \infty$ or $b = \infty$, then the solution of Eq. (1.1) is not Ulam-Hyers stable;

(2) Taking the limit $\beta \to 0$ on both sides of Eq. (1.1), we have a hyperbolic fractional partial differential equation in the $\psi$-Riemann-Liouville sense. Consequently, Theorem 4 is valid for the $\psi$-Riemann-Liouville fractional derivative.
In this section, we prove the generalized Ulam-Hyers-Rassias stability of the hyperbolic fractional partial differential equation Eq.(1.1) in the case a in the fractional partial differential equation Eq.(1.1). We consider Eq.(1.1) and inequality Eq.(3.9)

We assume that:

Proof. Let $v$ be a solution to the inequality Eq.(3.9). Denote by $u$ the unique solution of the fractional Darboux problem:

$$
\left\{
\begin{array}{ll}
\frac{\partial^{2\alpha} u}{\partial \beta;\psi x^\alpha \partial \beta;\psi y^\alpha} (x,y) &= f (x,y,u(x,y),u_1(x,y),u_2(x,y)), \forall x,y \in [0,\infty) \\
I^{1-\gamma;\psi}_\beta u(x,0) &= v(x,0), \forall x \in [0,\infty) \\
I^{1-\gamma;\psi}_\beta u(0,y) &= v(0,y), \forall y \in [0,\infty).
\end{array}
\right.
$$

5. Generalized Ulam-Hyers-Rassias stability

In this section, we prove the generalized Ulam-Hyers-Rassias stability of the hyperbolic fractional partial differential equation Eq.(1.1). We consider Eq.(1.1) and inequality Eq.(3.9) in the case $a = \infty$ and $b = \infty$.

**Theorem 5.** We assume that:

1. $f \in C ([0,\infty) \times [0,\infty) \times \mathbb{B}^3, \mathbb{B})$;
2. There exists $L_f \in C^1 ([0,\infty) \times [0,\infty), \mathbb{R}_+)$ such that
   $$
   |f (x,y,z_1,z_2,z_3) - f (x,y,t_1,t_2,t_3)| \leq L_f (x,y) \max_{i \in \{1,2,3\}} |z_i - t_i|
   $$
   for all $x,y \in [0,\infty)$.
3. There exist $\lambda_1^\varphi, \lambda_2^\varphi, \lambda_3^\varphi > 0$ such that
   $$
   \left\{
   \begin{array}{l}
   I^{\alpha;\psi}_{\varphi} \varphi (x,y) \leq \lambda_1^\varphi \varphi (x,y), \forall x,y \in [0,\infty) \\
   I^{\alpha;\psi}_{0+} \varphi (x,y) \leq \lambda_2^\varphi \varphi (x,y), \forall x,y \in [0,\infty) \\
   I^{\alpha;\psi}_{0+} \varphi (x,y) \leq \lambda_3^\varphi \varphi (x,y), \forall x,y \in [0,\infty).
   \end{array}
   \right.
   $$
4. $\varphi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is increasing.

Then the solution of Eq.(1.1) ($a = \infty$ and $b = \infty$) is generalized Ulam-Hyers-Rassias stable.

**Proof.** Let $v$ be a solution to the inequality Eq.(3.9). Denote by $u$ the unique solution of the fractional Darboux problem:

$$
\left\{
\begin{array}{ll}
\frac{\partial^{2\alpha} u}{\partial \beta;\psi x^\alpha \partial \beta;\psi y^\alpha} (x,y) &= f (x,y,u(x,y),u_1(x,y),u_2(x,y)), \forall x,y \in [0,\infty) \\
I^{1-\gamma;\psi}_\beta u(x,0) &= v(x,0), \forall x \in [0,\infty) \\
I^{1-\gamma;\psi}_\beta u(0,y) &= v(0,y), \forall x \in [0,\infty).
\end{array}
\right.
$$
If \( u \) is a solution for the fractional problem Eq.(5.1), then \((u, u_1, u_2)\) is a solution of the following problem:

\[
\begin{align*}
\frac{(\psi(y) - \psi(0))^{-1}}{\Gamma(\gamma)} v(x, 0) + \frac{(\psi(x) - \psi(0))^{-1}}{\Gamma(\gamma)} v(0, y) - v(0, 0) + I^\psi_\theta f(x, y, u(x, y), u_1(x, y), u_2(x, y)) \\
u(x, y) = \frac{(\psi(y) - \psi(0))^{-1}}{\Gamma(\gamma)} v_1(x, 0) + I^\alpha_0 \psi f(x, y, u(x, y), u_1(x, y), u_2(x, y)) \\
u_1(x, y) = \frac{(\psi(x) - \psi(0))^{-1}}{\Gamma(\gamma)} v_2(0, y) + I^{\alpha_1} \psi f(x, y, u(x, y), u_1(x, y), u_2(x, y))
\end{align*}
\]

From Theorem 2 and hypothesis (iii), it follows that

\[
\begin{align*}
\left| v(x, y) - \frac{(\psi(y) - \psi(0))^{-1}}{\Gamma(\gamma)} v(x, 0) - \frac{(\psi(x) - \psi(0))^{-1}}{\Gamma(\gamma)} v(0, y) + v(0, 0) - I^\psi_\theta f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \right| \\
\leq I^\psi_\theta \varphi(x, y, \gamma, \varphi(x, y), x, y \in [0, \infty)) \\
\left| v_1(x, y) - \frac{(\psi(y) - \psi(0))^{-1}}{\Gamma(\gamma)} v_1(x, 0) - I^\psi_0 \psi f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \right| \\
\leq I^\psi_0 \psi \varphi(x, y, \gamma, \varphi(x, y), x, y \in [0, \infty)) \\
\left| v_2(x, y) - \frac{(\psi(x) - \psi(0))^{-1}}{\Gamma(\gamma)} v_2(0, y) - I^{\psi_1}_0 \psi f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \right| \\
\leq I^{\psi_1}_0 \psi \varphi(x, y, \gamma, \varphi(x, y), x, y \in [0, \infty)).
\end{align*}
\]

Using Eq.(5.2), Eq.(5.3), Eq.(5.4), Eq.(5.5), Eq.(5.6) and Eq.(5.7), we get

\[
\begin{align*}
\left| v(x, y) - u(x, y) \right| \\
\leq \left| v(x, y) - \frac{(\psi(y) - \psi(0))^{-1}}{\Gamma(\gamma)} v(x, 0) - \frac{(\psi(x) - \psi(0))^{-1}}{\Gamma(\gamma)} v(0, y) + v(0, 0) - I^\psi_\theta f(x, y, v(x, y), v_1(x, y), v_2(x, y)) \right| \\
+ I^\psi_\theta | f(x, y, v(x, y), v_1(x, y), v_2(x, y)) - f(x, y, u(x, y), u_1(x, y), u_2(x, y)) | \\
\leq \lambda^1_\varphi \varphi(x, y) + I^\psi_\theta \left\{ L f(x, y) \max_{i \in \{1, 2, 3\}} | v_i(x, y) - u_i(x, y) | \right\}.
\end{align*}
\]
From Lemma 1, it follows that
\[ |v(x, y) - u(x, y)| \leq \lambda_1^{\psi} E_\alpha [L_f(x, y) \Gamma(\alpha_1) \Gamma(\alpha_2) (\psi(b) - \psi(t))^{\alpha_2} (\psi(a) - \psi(s))^{\alpha_1}] \]
\[ = C^1_{f,\psi} (x, y) \]
where
\[ C^1_{f,\psi} := \lambda_1^{\psi} E_\alpha [L_f(x, y) \Gamma(\alpha_1) \Gamma(\alpha_2) (\psi(\infty) - \psi(0))^{\alpha_2} (\psi(\infty) - \psi(0))^{\alpha_1}] \]
\[ x, y \in [0, \infty), \frac{1}{2} < \alpha \leq 1 \text{ and } \psi(\infty) < \infty. \]

Similarly, we have
\[ |v_1(x, y) - u_1(x, y)| \leq \lambda_2^{\psi} E_\alpha [L_f(x, y) \Gamma(\alpha_1) (\psi(\infty) - \psi(0))^{\alpha_1}] \]
\[ = C^2_{f,\psi} (x, y) \]
where
\[ C^2_{f,\psi} := \lambda_2^{\psi} E_\alpha [L_f(x, y) \Gamma(\alpha_1) (\psi(\infty) - \psi(0))^{\alpha_1}] \]
\[ x, y \in [0, \infty), \frac{1}{2} < \alpha \leq 1 \text{ and } \psi(\infty) < \infty. \]

Also,
\[ |v_2(x, y) - u_2(x, y)| \leq \lambda_3^{\psi} E_\alpha [L_f(x, y) \Gamma(\alpha_2) (\psi(\infty) - \psi(0))^{\alpha_2}] \]
\[ = C^3_{f,\psi} (x, y) \]
where
\[ C^3_{f,\psi} := \lambda_3^{\psi} E_\alpha [L_f(x, y) \Gamma(\alpha_2) (\psi(\infty) - \psi(0))^{\alpha_2}] \]
\[ x, y \in [0, \infty), \frac{1}{2} < \alpha \leq 1 \text{ and } \psi(\infty) < \infty. \]

Therefore, the solution of Eq. (1.1) is generalized Ulam-Hyers-Rassias stable.

6. Concluding remarks

By means of the \(\psi\)-Riemann-Liouville fractional partial integral and the \(\psi\)-Hilfer fractional partial derivative of \(N\) variables, we investigated a fractional partial differential equation of hyperbolic type and showed that its solutions admits Ulam-Hyers and Ulam-Hyers-Rassias stabilities in a Banach space \(B\). Besides, we have also made some remarks which we consider extremely important, in particular the ones concerning the wide class of fractional partial integrals and derivatives obtained from these new extensions, as in Remarks 1 and 2.
It is important to note that the extension of the $\psi$-Hilfer fractional derivative in one variable to the case of $N$ variables shows that the use of fractional calculus can be naturally extended in the case of models involving partial differential equations, in the study of the existence and uniqueness of solutions of such PDEs and in the analytical continuation of solutions of partial differential equations. In a forthcoming work, we study the Volterra integral equations and various types of stability of partial differential equations and other properties for $\psi$-Hilfer fractional partial derivative.

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