Role of Mathematics in Physical Sciences

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The role of mathematics in physical sciences is discussed, particularly how higher mathematics found applications in empirical problems. Several examples are given to illustrate this role.

Before I discuss the role of mathematics in physical sciences, let me first define: What is Mathematics? And What is Science?

I. WHAT IS MATHEMATICS?

Mathematics is a fusion of skillful operations with concepts and rules invented just for this purpose [1]. The principal emphasis is on invention of concepts, which go beyond those contained in the axioms. “These are defined with a view of permitting ingenious logical operations which appeal to our aesthetic sense both as operations and also in their results of great generality and simplicity” [1]. Without concepts a mathematician would not go far. These may or may not be suggested by the actual world. “Mathematics is independent of material objects. In mathematics the word ‘exist’ can have only one meaning; it signifies exemption from contradiction [2].” In fact most of the advanced concepts in mathematics are those on which a mathematician can demonstrate his ingenuity and sense of beauty. Take for example “complex numbers”; the introduction of which cannot be suggested by physical observations. A mathematician’s interest in complex numbers lies in that many beautiful theorems in analytical functions owe their origin to the introduction of complex numbers. It so happened that much later complex numbers became essential in the formulation of quantum mechanics where they are not a calculational trick of applied mathematics. Indeed “Mathematics can not be defined without acknowledging its most obvious feature: namely, that it is interesting” [3].

It is appropriate to mention Cambridge Mathematician Godfrey Hardy and his book: “A Mathematician’s Apology” [4], the message of which is that pure mathematics is the only kind of mathematics worthy of respect. He wrote “A mathematician, like a painter or a poet is a maker of patterns. If his patterns are more permanent than theirs, it is because they are made in the ideas. The mathematician’s patterns like the painter’s or the poet’s must be beautiful; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test, there is no place in the world for ugly mathematics.” As we shall see the role which mathematics plays in physical sciences, where one is concerned to understand basic mysteries of nature, has also beauty of its own and a source of joy and excitement to its practitioners. Indeed “a scientist worthy of name, above all a mathematician, experiences in his work the same impression as an artist. His pleasure is as great and of the same nature” [2].

II. WHAT IS SCIENCE (PHYSICS)?

Science is a fusion of philosophical thinking, which supplies concepts and skilled crafts, which supply tools. The two are intimately connected. Concepts are needed to explain old things (in the form of empirical data) in new ways. Tools are needed to discover new things that have to be explained (in terms of concepts) or to discover things predicted by a concept-driven theory so as to verify or discard that theory. The concept driven revolutions have been rare. Taking quantum mechanics as a prime example of a concept-driven revolution, Thomas Kuhn in his book “The Structure of Scientific revolution”, has listed, in addition to quantum mechanics, only six major concept driven revolutions in the last 500 years, associated with the names of Copernicus, Newton, Darwin, Maxwell, Freud and Einstein. According to F.J. Dyson [5], during the same period, there have been about twenty tool-driven revolutions, some in physics itself, but mostly in biology and astronomy, using tools created by physics. Physics has had great success in creating new tools that have started revolution in biology, computer science, engineering, astronomy and medicines. Two prime examples are [5] the Galileean revolution resulting from the use of the telescope in astronomy and the Crick-Watson revolution.

*Based on the talk given at the first national seminar on mathematics organized by Pakistan Mathematical Society
(1950) resulting from the use of X-ray crystallography to determine the structure of DNA in biology. Another tool driven revolution having a great impact on society was based in the invention of transistor resulting in the advent of computers and memory banks in the 1960's. Electronic data processing and simulation revolutionized every branch of science, increasing the power of scientific theories to interpret and predict new phenomena. Computers, becoming cheaper and smaller, have become personal and are used for variety of purposes, from toys to highly sophisticated scientific work. They have revolutionized the communication, the mode of information and finance.

III. ROLE OF MATHEMATICS IN PHYSICAL THEORIES:

First there is a mundane role which is to facilitate for the physicists the numerical calculation of certain constants or the integration of certain differential equations. Mathematics, does however, play a more sovereign role in which we will be concerned and bring out how higher mathematics found applications in subtle empirical problems.

A. The laws of nature are written in the language of mathematics.

(This is attributed to Galilleo, more than 300 years ago). “All laws are deduced from experiment, but to enunciate them, a special language is needful, ordinary language is too poor, it is besides too vague, to express relations so delicate, so rich and so precise” [2]. Let us discuss some examples:

A): The basic axioms of Newtonian mathematical physics is stated in the preface of the first edition of the Principia: rational mechanics ought to address “motion” with the same precision as geometry handles the size and shape of idealized objects. The association of “motion” (particularly the change in motion) with “mathematics” was a stroke of genius. The mathematical language in which it was formulated contained the concept of second derivative - not a very immediate concept. The act of writing down a fundamental law is a rather singular and rare event. It is a miracle that in spite of the baffling complexity in the world, certain regularities in the events could be discovered. A monumental example of such a law is Newton’s law of gravitation - a single law which explained everything from planetary motion to the terrestrial motion of pendulums and which appears simple to the mathematicians and which proved accurate beyond all reasonable expectations but still it is a law of limited scope.

B): The concepts of modern physics are abstract. “Mathematics is the tool specially suited for dealing with abstract concepts of any kind and there is no limit in its power in this field” [6]. In this context let us consider two of the great theories of the last century: Relativity and Quantum Mechanics, both of which involve mathematics of transformations. This is because the important quantities in nature appear as the invariant or having simple transformation properties under these transformations. Let us consider them one by one:

i): General Theory of Relativity:

Einstein gave a new concept of gravity. Gravity cannot be switched off at will. Einstein argued that because of its permanency, gravity must be related to some intrinsic feature of space-time. He identified this feature as the geometry of space-time – only that that this geometry is unusual. Existence of matter causes the fabric of space-time to warp somewhat like the effect of a bowling ball placed on foam. Such distortion to the fabric of space-time transmits the force of gravity from one place to another. Gravity resides in the curvature of space-time. The geometry which describes curved spaces is known as Riemann geometry.

ii): Quantum Mechanics:

There are two basic concepts in quantum mechanics: States and Observables. The states, which have no classical analogue, are vectors in Hilbert space. The observables are dynamical variables, which although appear in classical mechanics, are treated in quantum mechanics as hermitian operators on state vectors. Let us also remind ourselves that Hilbert space of quantum mechanics is complex with a hermitian scalar product and as such the use of complex numbers is necessary in the formulation of laws of quantum mechanics.

In many cases mathematical concepts were independently developed by the physicist and recognized then as having been conceived before by the mathematician. Quantum mechanics is a good example of this where Dirac invented his own mathematics in his formulation of quantum mechanics. Einstein, on the other hand, recognised Riemann Geometry as tailor-made for implementing his view of gravitational force.
B. Mathematical Symmetry and Analogies

Let us consider this role of mathematics by discussing some examples:

a): Maxwell’s equations:

The laws of electrodynamics are described by following equations which expressed all known facts at the time Maxwell began his work:

\[ \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J} \]
\[ \nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \]
\[ \nabla \cdot \mathbf{D} = 4\pi \rho \]
\[ \nabla \cdot \mathbf{B} = 0 \]

In the absence of sources (\( \rho = 0, \mathbf{J} = 0 \)) Maxwell noticed that the first two equations lack symmetry as \( \nabla \times \mathbf{H} = 0 \) while \( \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \). Maxwell removed this lack of symmetry by modifying the first equation to

\[ \nabla \times \mathbf{H} = \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \]

It was not a new experiment, which came to invalidate the equations. But in looking at them under a new perspective, Maxwell saw that the equations become more symmetrical when \( \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t} \) is added. In that Maxwell was twenty years ahead of experiment since his “a priori” views awaited twenty years for an experimental verification. He formulated these views because he “was profoundly steeped in the sense of mathematical symmetry” [2]. Maxwell unified electricity and magnetism and as a result the electromagnetic radiation in the form of light, radiowaves and X-rays provide many of the conveniences of modern life — lights, television, telephones etc. Furthermore the requirement of mutual compatibility of Newtonian Mechanics and Maxwellian Electrodynamics leads to the foundation of special theory of relativity.

b): Dirac Equation

Dirac combined special theory of relativity with quantum mechanics, which resulted in his famous equation for electron (or any fundamental fermion having spin 1/2). The Schrödinger equation in non-relativistic quantum mechanics does not satisfy the basic requirement of relativity, namely that space and time must be treated on equal footing. This is because it involves first order time derivative and second other space derivatives. Thus Dirac looked for an equation which is linear in time and space derivatives. In order to satisfy this requirement, an analogy with Maxwell’s equations, written earlier, which are Lorentz invariant, may be useful. Maxwell’s equations are first order in time and space derivatives. Moreover, the vector potential \( A_\mu \) to which electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) can be related satisfies in free space the second order wave equation:

\[ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} A_\mu - \nabla^2 A_\mu = 0 \]

Notice that \( \mathbf{E} \) and \( \mathbf{B} \), which satisfy first order equation, have more components than \( A_\mu \) which satisfies second order equation. One should expect the increase in the number of components as a price one has to pay for the first order equations. Note also that the above second order equation is also satisfied by each of the components of \( \mathbf{E} \) and \( \mathbf{B} \). The above considerations suggest that the most general equation we can write is

\[ \frac{1}{c} \frac{\partial \psi}{\partial t} + \sum_{n=1}^{N} (\alpha)_{l,n} \cdot \nabla \psi_n + \frac{imc}{\hbar} \sum_{n} \rho_n \psi_n = 0 \]

where \( \alpha = (\alpha_x, \alpha_y, \alpha_z) = \alpha_j, j = 1, 2, 3 \) and \( l, n = 1, \cdots, N \), \( N \) being the number of components we have for the state function \( \psi \). Further the requirement that each component of \( \psi \) satisfies a second order equation (just as \( \mathbf{E} \) and \( \mathbf{B} \) do)

\[ \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi + \frac{m^2c^2}{\hbar^2} \psi = 0 \]
shows that four quantities $\alpha_x$, $\alpha_y$, $\alpha_z$ and $\beta$ anticommute in pairs and their square is unity. They cannot be numbers and can be expressed in terms of matrices. One can show that $N$ must be even and if in the interest of simplicity we require the representation to have as low a rank as possible, we need to go to $4 \times 4$ matrices. In this way Dirac obtained his equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi = (-c\alpha \cdot c\hbar \nabla + \beta mc^2)\psi$$

where $\psi$ has four components and transforms under Lorentz transformation in a way different from a vector and is called Dirac spinor. The Dirac equation has profound consequences: it naturally comes out that particle it represents has spin $1/2$; antimatter must exist, to each particle there is an antiparticle, and gave a new meaning to vacuum in the microscopic world. Both Maxwell’s equations and Dirac equation give much more than what was put in, purely from mathematical symmetry and analogies. Moreover they reveal to a physicist the hidden symmetry of things in making him see them in a new way.

c): Symmetry Principles and Group Theory:

Until the twentieth century group theory played a little role in theoretical physics. This had a background. "Most classical physicists expected infinitesimal analysis to be the natural mathematics for all of physics, with priority accorded to the differential equations of Newtonian mechanics. It was a widely held tacit assumption that this must be the way mathematics enters microphysics [7]". Partly it may also be due to new and unfamiliar mathematics comprising group theory. As Wigner noted, "There is a great reluctance among physicists towards accepting group theoretical arguments" [8]. Wigner’s early experience in X-ray crystallography led him to a programme of applying the theory of group representations to atomic and molecular spectra as well as nuclear physics. He also gave the infinite unitary representations of the Lorentz group and laid foundation both for the application of group theory to quantum mechanics and for the role of symmetry in microphysics.

In the last three decades Lie groups and Lie algebras played a major role in applying symmetry principles in containing allowable dynamical laws mainly for the reason that in many cases in subatomic phenomena, the dynamical laws were not known a priori.

Let us consider a group element of a continuous special unitary group $SU(N) \exp[i\alpha_\alpha T_a]$ where $\alpha_\alpha$ is a set of real parameters ($\alpha = 1, \ldots, N$), $\det \alpha = 1$ and $T_a$ are generators of the symmetry group satisfying the commutation relations

$$[T^a , T^b ] = i f^{abc} T^c$$

and the Jacobi identity

$$[T^a , [T^b , T^c ] ] + [T^b , [T^c , T^a ] ] + [T^c , [T^a , T^b ] ] = 0$$

where $f^{abc}$ are structure constants, antisymmetric in $a, b, c$. The above equations define the Lie algebra associated with the group $SU(N)$. The group $SU(3)$ has been used in the classification of physical “particles” which exist in multi-plets and can be regarded as belonging to irreducible representations of $SU(3)$. The unitary groups have also been used in more fundamental way when we consider local gauge symmetries where the parameters $\alpha_\alpha$ are functions of space-time. Here an object transforms as

$$\psi (x) \rightarrow \exp[i\alpha_\alpha (x) T_a] \psi (x)$$

Under an infinitesimal transformation, the derivative

$$dx^\mu \partial_\mu \psi (x) = [\psi (x + dx) - \psi (x)]$$

does not make sense since $\psi (x + dx)$ transforms differently from $\psi (x)$ under the group. It is necessary to introduce a vector field $A^\mu$, called gauge field in physics. In the language of differential geometry $A^\mu$ forms a connection and defines a parallel displacement of geometrical objects belonging to representation spaces of the group [9]. A parallel displacement of $\psi$ from $x$ to $x + dx$ is defined through

$$\psi_u (x) = U (x + dx, x, A) \psi (x) = [1 + idx^\mu A^\mu_a T^a] \psi (x)$$
The definition of covariant derivative then naturally follows

\[ dx^\mu D_\mu \psi (x) = \psi (x + dx) - \psi (x) = dx^\mu \left( \partial_\mu - iA_\mu^a T^a \right) \psi (x) \]

This concept unifies fundamental particles with fundamental forces through which particles interact; those forces are mediated by the quanta of the gauge field A. As C.N. Yang stated: “Symmetry dictates interaction”. Further, as seen above that the gauge symmetry is based on a sophisticated geometrical concept, gives it a deep and beautiful foundation.

IV. MODERN TRENDS: QUANTUM GEOMETRY:

We have two great theories of the last century: the quantum mechanics and the theory of relativity. The two theories have their roots in mutually exclusive groups of phenomena. Quantum Mechanics provides a theoretical framework for understanding the universe on the smallest of scales: molecules, atoms and all the way to subatomic particles like electrons and quarks. General relativity provides a theoretical framework for understanding the universe on the largest of scales: stars, galaxies, clusters of galaxies, and beyond to the immense expanse of the universe itself. The two theories operate with different mathematical concepts - infinite dimensional Hilbert space and the four dimensional Riemann space, respectively. In most situations their union is not even required. This is because in most situations as mentioned above the domains (like atoms and their constituents) in which quantum mechanics is interested and domains like (status and galasions) in which general theory of relativity is relevant have no overlap. There are, however, situations where both theories become relevant. For instance, in a black hole an enormous mass is crumped to a very small size. At the moment of big bang the whole of the universe erupted from a microscopic nugget, compared to which even the grain of sand looks enormous. These are domains that are tiny and yet incredibly massive and as such require Quantum Theory and General Theory of Relativity simultaneously. Until recently the two theories could not be united i.e. no mathematical formulation exists to which both of these theories are approximations.

It turns out that to achieve this one needs (i) a higher dimensional space-time for getting both general coordinate invariance of the Einstein and the Yang-Mills gauge transformations (mentioned earlier) corresponding to internal degrees of freedom (ii) supersymmetry to avoid tachyons (i.e. the particles which move faster term the speed of light) and for taming infinities (iii) going beyond point field theory i.e. the most fundamental entities are not point-like but extended one dimensional objects. (This too helps to tame the infinities). The above three ingredients are incorporated in superstring theory. It naturally contains a massless spin 2 particle which could be identified by graviton, the mediator of gravitational interaction just as spin 1 photon is a mediator of electromagnetic interaction.

First a words about supersymmetry. Supersymmetry incorporates boson-fermion symmetry. Such theories predict a new kind of matter in the form of supersymmetric partners of all observed elementary particles. The observation of these partners would provide the first experimental evidence for supersymmetry. But there is so far no experimental evidence for such particles. The experimental situation will become clear in about 5 years time when the world’s largest accelerator being developed at CERN, Geneva become operational. The mathematics of supersymmetry involves use of Clifford algebra and Grassman numbers which unlike ordinary numbers, anticommute. It turned-out that dynamics of superstring theory can be formulated in 10-dimensional space-time: four familiar space-time dimensions and six extra dimensions. The extra-spatial dimensions of string theory are to be “crumped” up (to the size of Planck length \(10^{-35}\) m) in a particular class of 6-dimensional geometrical shapes known as Calabi–Yau shapes. The mathematics of Calabi–Yau shapes is studied in a field called Algebraic Geometry - a relatively new field that combines algebra and geometry. Towards the end of last century, it led to some of the crowning achievements of pure mathematics, including the solution of Fermat’s last theorem, Mordell conjecture and the Weil conjectures [10]. It is now being used by string theorists leading to a new branch of Physics and Mathematics, which may be called Quantum Geometry [11].

Let me end this article by quoting from E. P. Wigner’s thought provoking article “Unreasonable effectiveness of Mathematics in the Natural Sciences”:

“The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning”.

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[4] G.H. Hardy, A Mathematician’s Apology, Canto edition, Cambridge University Press 1992, pp. 84-85.

[5] F.J. Dyson, in Imagined Worlds, Harvard University Press, 1998, Chapter 2.

[6] P.A.M. Dirac, in Preface to the first edition of his book: Principles of Quantum Mechanics, Cambridge University Press, Cambridge, 1930.

[7] L. Tisza, Remembering Eugene Wigner and pondering his legacy, Europhysics news, March/April 2003; see also D.J. Gross, in Symmetry in Physics: Wigner’s Legacy, Physics Today, December 1995.

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[9] See for example, C.Itzykson and J. Zuber, Quantum Field Theory, McGraw Hill Co, 1980, p.564.

[10] P.A. Griffiths, in Mathematics for a New Millennium, talk given in TWAS 9th General Conference and TWAS 20th Anniversary Celebrations, held in Beijing, 14-19 October 2003.

[11] For a non-technical account of Superstring Theory and Hidden Dimensions, see for example. B, Green, The Elegant Universe, W.W. Norton and Company, New York London, 1999.