POINTWISE $C^{2,\alpha}$ ESTIMATES AT THE BOUNDARY FOR THE MONGE-AMPERE EQUATION

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ABSTRACT. We obtain pointwise $C^{2,\alpha}$ estimates at boundary points for solutions to the Monge-Ampere equation under appropriate local conditions on the right hand side and boundary data.

1. Introduction

Boundary estimates for solutions of the Dirichlet problem for the Monge-Ampere equation

$$
\begin{cases}
\det D^2 u = f & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega,
\end{cases}
$$

were obtained in the classical paper by Caffarelli, Nirenberg and Spruck [CNS] in the case when $\partial \Omega$, $\varphi$ and $f$ are sufficiently smooth. When $f$ is less regular, i.e $f \in C^\alpha$, the global $C^{2,\alpha}$ estimates were obtained by Trudinger and Wang [TW] for $\varphi$, $\partial \Omega \in C^3$. In this paper we discuss pointwise $C^{2,\alpha}$ estimates at boundary points under appropriate local conditions on the right hand side and boundary data. Our main result can be viewed as a natural extension up to the boundary of the pointwise interior $C^{2,\alpha}$ estimate of Caffarelli in [C2].

We start with the following definition (see [CC]).

Definition: Let $0 < \alpha \leq 1$. We say that a function $u$ is pointwise $C^{2,\alpha}$ at $x_0$ and write

$$
u \in C^{2,\alpha}(x_0)
$$

if there exists a quadratic polynomial $P_{x_0}$ such that

$$u(x) = P_{x_0}(x) + O(|x - x_0|^{2+\alpha}).$$

We say that $u \in C^2(x_0)$ if

$$u(x) = P_{x_0}(x) + o(|x - x_0|^2).$$

Similarly one can define the notion for a function to be $C^k$ and $C^{k,\alpha}$ at a point for any integer $k \geq 0$.

It is easy to check that if $u$ is pointwise $C^{2,\alpha}$ at all points of a Lipschitz domain $\bar{\Omega}$ and the equality in the definition above is uniform in $x_0$ then $u \in C^{2,\alpha}(\bar{\Omega})$ in the classical sense. Precisely, if there exists $M$ and $\delta$ such that for all points $x_0 \in \bar{\Omega}$

$$|u(x) - P_{x_0}(x)| \leq M|x - x_0|^{2+\alpha} \quad \text{if } |x - x_0| \leq \delta, \ x \in \bar{\Omega}$$

then

$$[D^2 u]_{C^{\alpha}(\bar{\Omega})} \leq C(\delta, \Omega)M.$$
Caffarelli showed in [C2] that if $u$ is a strictly convex solution of
$$\det D^2 u = f$$
and $f \in C^\alpha(x_0)$, $f(x_0) > 0$ at some interior point $x_0 \in \Omega$, then $u \in C^{2,\alpha}(x_0)$. Our main theorem deals with the case when $x_0 \in \partial \Omega$.

**Theorem 1.1.** Let $\Omega$ be a convex domain and let $u : \bar{\Omega} \to \mathbb{R}$ convex, continuous, solve the Dirichlet problem for the Monge-Ampere equation
\[\begin{cases}
\det D^2 u = f & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega,
\end{cases}\]
with positive, bounded right hand side i.e
$$0 < \lambda \leq f \leq \Lambda,$$
for some constants $\lambda$, $\Lambda$.

Assume that for some point $x_0 \in \partial \Omega$ we have
$$f \in C^\alpha(x_0), \quad \varphi, \partial \Omega \in C^{2,\alpha}(x_0),$$
for some $\alpha \in (0, 1)$. If $\varphi$ separates quadratically on $\partial \Omega$ from the tangent plane of $u$ at $x_0$, then
$$u \in C^{2,\alpha}(x_0).$$

The way $\varphi$ separates locally from the tangent plane at $x_0$ is given by the tangential second derivatives of $u$ at $x_0$. Thus the assumption that this separation is quadratic is in fact necessary for the $C^{2,\alpha}$ estimate to hold. Heuristically, Theorem 1.1 states that if the tangential pure second derivatives of $u$ are bounded below then the boundary Schauder estimates hold for the Monge-Ampere equation.

A more precise, quantitative version of Theorem 1.1 is given in section 7 (see Theorem 7.1).

Given the boundary data, it is not always easy to check the quadratic separation since it involves some information about the slope of the tangent plane at $x_0$. However, this can be done in several cases (see Proposition 3.2). One example is when $\partial \Omega$ is uniformly convex and $\varphi$, $\partial \Omega \in C^3(x_0)$. The $C^3$ condition of the data is optimal as it was shown by Wang in [W]. Other examples are when $\partial \Omega$ is uniformly convex and $\varphi$ is linear, or when $\partial \Omega$ is tangent of second order to a plane at $x_0$ and $\varphi$ has quadratic growth near $x_0$.

As a consequence of Theorem 1.1, we obtain a pointwise $C^{2,\alpha}$ estimate in the case when the boundary data and the domain are pointwise $C^3$. As mentioned above, the global version was obtained by Trudinger and Wang in [TW].

**Theorem 1.2.** Let $\Omega$ be uniformly convex and let $u$ solve (1.1). Assume that
$$f \in C^\alpha(x_0), \quad \varphi, \partial \Omega \in C^3(x_0),$$
for some point $x_0 \in \partial \Omega$, and some $\alpha \in (0, 1)$. Then $u \in C^{2,\alpha}(x_0)$.

We also obtain the $C^{2,\alpha}$ estimate in the simple situation when $\partial \Omega \in C^{2,\alpha}$ and $\varphi$ is constant.

**Theorem 1.3.** Let $\Omega$ be a uniformly convex domain and assume $u$ solves (1.1) with $\varphi \equiv 0$. If $f \in C^\alpha(\bar{\Omega})$, $\partial \Omega \in C^{2,\alpha}$, for some $\alpha \in (0, 1)$ then $u \in C^{2,\alpha}(\bar{\Omega})$. 
The key step in the proof of Theorem 1.1 is a localization theorem for boundary points which was recently proved in [S]. It states that under natural local assumptions on the domain and boundary data, the sections

\[ S_h(x_0) = \{ x \in \overline{\Omega} \mid u(x) < u(x_0) + \nabla u(x_0) \cdot (x - x_0) + h \}, \]

with \( x_0 \in \partial \Omega \) are “equivalent” to ellipsoids centered at \( x_0 \).

**Theorem 1.4.** Let \( \Omega \) be convex and \( u \) satisfy \((1.1)\), and assume \( \partial \Omega, \varphi \in C^{1,1}(x_0) \).

If \( \varphi \) separates quadratically from the tangent plane of \( u \) at \( x_0 \), then for each small \( h > 0 \) there exists an ellipsoid \( E_h \) of volume \( h^{n/2} \) such that

\[ cE_h \cap \overline{\Omega} \subset S_h(x_0) - x_0 \subset CE_h \cap \overline{\Omega}, \]

with \( c, C \) constants independent of \( h \).

Theorem 1.4 is an extension up to the boundary of the localization theorem at interior points due to Caffarelli in [C1]. For completeness we provide also its proof in the current paper.

The paper is organized as follows. In section 2 we discuss briefly the compactness of solutions to the Monge-Ampere equation which we use later in the paper (see Theorem 2.7). For this we need to consider also solutions with possible discontinuities at the boundary. In section 3 we give a quantitative version of the Localization Theorem (see Theorem 3.1). In sections 4 and 5 we provide the proof of Theorem 3.1. In section 6 we obtain a version of the classical Pogorelov estimate in half-domain (Theorem 6.4). Finally, in section 7 we use the previous results together with a standard approximation method and prove our main theorem.

## 2. Solutions with discontinuities on the boundary

Let \( u : \Omega \to \mathbb{R} \) be a convex function with \( \Omega \subset \mathbb{R}^n \) bounded and convex. Denote by

\[ U := \{(x, x_{n+1}) \in \Omega \times \mathbb{R} \mid x_{n+1} \geq u(x)\} \]

the upper graph of \( u \).

**Definition 2.1.** We define the values of \( u \) on \( \partial \Omega \) to be equal to \( \varphi \) i.e

\[ u|_{\partial \Omega} = \varphi, \]

if the upper graph of \( \varphi : \partial \Omega \to \mathbb{R} \cup \{\infty\} \)

\[ \Phi := \{(x, x_{n+1}) \in \partial \Omega \times \mathbb{R} \mid x_{n+1} \geq \varphi(x)\} \]

is given by the closure of \( U \) restricted to \( \partial \Omega \times \mathbb{R} \),

\[ \Phi := \overline{U} \cap (\partial \Omega \times \mathbb{R}). \]

From the definition we see that \( \varphi \) is lower semicontinuous.

If \( u : \Omega \to \mathbb{R} \) is a viscosity solution to

\[ \det D^2 u = f(x), \]

with \( f \geq 0 \) continuous and bounded on \( \Omega \), then there exists an increasing sequence of subsolutions, continuous up to the boundary,

\[ u_n : \Omega \to \mathbb{R}, \quad \det D^2 u_n \geq f(x) \]
with

\[ \lim u_n = u \quad \text{in } \Omega, \]

where the values of \( u \) on \( \partial \Omega \) are defined as above.

Indeed, let us assume for simplicity that \( 0 \in \Omega, \ u(0) = 0, \ u \geq 0 \). Then, on each ray from the origin \( u \) is increasing, hence \( v_\varepsilon : \Omega \to \mathbb{R}, \)

\[ v_\varepsilon(x) = u((1-\varepsilon)x) \]

is an increasing family of continuous functions as \( \varepsilon \to 0 \), with

\[ \lim v_\varepsilon = u \quad \text{in } \Omega. \]

In order to obtain a sequence of subsolutions we modify \( v_\varepsilon \) as

\[ u_\varepsilon(x) := v_\varepsilon(x) + w_\varepsilon(x), \]

with \( w_\varepsilon \leq -\varepsilon \), convex, so that

\[ \det D^2 w_\varepsilon \geq |f(x) - (1-\varepsilon)^2n f((1-\varepsilon)x)|, \]

thus

\[ \det D^2 u_\varepsilon(x) = \det(D^2v_\varepsilon + D^2w_\varepsilon) \geq \det D^2v_\varepsilon + \det D^2w_\varepsilon \geq f(x). \]

The claim is proved since as \( \varepsilon \to 0 \) we can choose \( w_\varepsilon \) to converge uniformly to 0.

**Proposition 2.2** (Comparison principle). Let \( u, v \) be defined on \( \Omega \) with

\[ \det D^2u \geq f(x) \geq \det D^2v \]

in the viscosity sense and

\[ u|_{\partial \Omega} \leq v|_{\partial \Omega}. \]

Then

\[ u \leq v \quad \text{in } \Omega. \]

**Proof.** Since \( u \) can be approximated by a sequence of continuous functions on \( \bar{\Omega} \) it suffices to prove the result in the case when \( u \) is continuous on \( \bar{\Omega} \) and \( u < v \) on \( \partial \Omega \). Then, \( u < v \) in a small neighborhood of \( \partial \Omega \) and the inequality follows from the standard comparison principle.

\[ \square \]

A consequence of the comparison principle is that a solution \( \det D^2u = f \) is determined uniquely by its boundary values \( u|_{\partial \Omega} \).

Next we define the notion of convergence for functions which are defined on different domains.

**Definition 2.3.** a) Let \( u_k : \Omega_k \to \mathbb{R} \) be a sequence of convex functions with \( \Omega_k \) convex. We say that \( u_k \) converges to \( u : \Omega \to \mathbb{R} \) i.e

\[ u_k \to u \]

if the upper graphs converge

\[ \bar{U}_k \to \bar{U} \quad \text{in the Hausdorff distance.} \]

In particular it follows that \( \bar{\Omega}_k \to \bar{\Omega} \) in the Hausdorff distance.

b) Let \( \varphi_k : \partial \Omega_k \to \mathbb{R} \cup \{\infty\} \) be a sequence of lower semicontinuous functions. We say that \( \varphi_k \) converges to \( \varphi : \partial \Omega \to \mathbb{R} \cup \{\infty\} \) i.e

\[ \varphi_k \to \varphi \]
if the upper graphs converge
\[ \Phi_k \to \Phi \] in the Hausdorff distance.

c) We say that \( f_k : \Omega_k \to \mathbb{R} \) converge to \( f : \Omega \to \mathbb{R} \) if \( f_k \) are uniformly bounded and
\[ f_k \to f \]
uniformly on compact sets of \( \Omega \).

**Remark:** When we restrict the Hausdorff distance to the nonempty closed sets of a compact set we obtain a compact metric space. Thus, if \( \Omega_k, u_k \) are uniformly bounded then we can always extract a convergent subsequence \( u_{k_m} \to u \). Similarly, if \( \Omega_k, \varphi_k \) are uniformly bounded we can extract a convergent subsequence \( \varphi_{k_m} \to \varphi \).

**Proposition 2.4.** Let \( u_k : \Omega_k \to \mathbb{R} \) be continuous and
\[ \det D^2 u_k = f_k, \quad u_k = \varphi_k \quad \text{on } \partial \Omega_k. \]
If
\[ u_k \to u, \quad \varphi_k \to \varphi, \quad f_k \to f, \]
then
\[ (2.1) \quad \det D^2 u = f, \quad u = \varphi^* \quad \text{on } \partial \Omega, \]
where \( \varphi^* \) is the convex envelope of \( \varphi \) on \( \partial \Omega \) i.e \( \Phi^* \) is the restriction to \( \partial \Omega \times \mathbb{R} \) of the convex hull generated by \( \Phi \).

**Remark:** If \( \Omega \) is strictly convex then \( \varphi^* = \varphi \).

**Proof.** Since
\[ \bar{U}_k \to \bar{U}, \quad \Phi_k \to \Phi, \quad \Phi_k \subset \bar{U}_k, \]
we see that \( \Phi \subset \bar{U} \). Thus, if \( K \) denotes the convex hull generated by \( \Phi \), then \( \Phi^* \subset K \subset \bar{U} \). It remains to show that \( \bar{U} \cap (\partial \Omega \times \mathbb{R}) \subset K \).

Indeed consider a hyperplane
\[ x_{n+1} = l(x) \]
which lies below \( K \). Then
\[ u_k - l \geq 0 \quad \text{on } \partial \Omega_k \]
and by Alexandrov estimate we have that
\[ u_k - l \geq -Cd_k^{1/n} \]
where \( d_k \) represents the distance to \( \partial \Omega_k \). By taking \( k \to \infty \) we see that
\[ u - l \geq -Cd^{1/n} \]
thus no point on \( \partial \Omega \times \mathbb{R} \) below the hyperplane belongs to \( \bar{U} \).

\[ \square \]

**Proposition 2.4** says that given any \( \varphi \) bounded and lower semicontinuous, and \( f \geq 0 \) bounded and continuous we can always solve uniquely the Dirichlet problem
\[
\begin{cases}
\det D^2 u = f & \text{in } \Omega, \\
u = \varphi & \text{on } \partial \Omega
\end{cases}
\]
by approximation. Indeed, we can find sequences \( \varphi_k, f_k \) of continuous, uniformly bounded functions defined on strictly convex domains \( \Omega_k \) such that \( \varphi_k \to \varphi \) and

**Remark:**
Then the corresponding solutions $u_k$ are uniformly bounded and continuous up to the boundary. Using compactness and the proposition above we see that $u_k$ must converge to the unique solution $u$ in (2.1).

We extend the Definition 2.1 in order to allow a boundary data that is not necessarily convex.

**Definition 2.5.** Let $\varphi : \partial \Omega \to \mathbb{R}$ be a lower semicontinuous function. When we write that a convex function $u$ satisfies $u = \varphi$ on $\partial \Omega$

we understand

$$u|_{\partial \Omega} = \varphi^*$$

where $\varphi^*$ is the convex envelope of $\varphi$ on $\partial \Omega$.

Whenever $\varphi^*$ and $\varphi$ do not coincide we can think of the graph of $u$ as having a vertical part on $\partial \Omega$ between $\varphi^*$ and $\varphi$.

It follows easily from the definition above that the boundary values of $u$ when we restrict to the domain

$$\Omega_h := \{ u < h \}$$

are given by

$$\varphi_h = \varphi \text{ on } \partial \Omega \cap \{ \varphi \leq h \} \subset \partial \Omega_h$$

and $\varphi_h = h$ on the remaining part of $\partial \Omega_h$.

By Proposition 2.2, the comparison principle still holds. Precisely, if $u = \varphi, \ v = \psi, \ \varphi \leq \psi$ on $\partial \Omega$,

$$\det D^2 u \geq f \geq \det D^2 v \text{ in } \Omega,$$

then

$$u \leq v \text{ in } \Omega.$$

The advantage of introducing the notation of Definition 2.5 is that the boundary data is preserved under limits.

**Proposition 2.6.** Assume

$$\det D^2 u_k = f_k, \ u_k = \varphi_k \text{ on } \partial \Omega_k,$$

with $\Omega_k, \varphi_k$ uniformly bounded and

$$\varphi_k \to \varphi, \ f_k \to f.$$

Then

$$u_k \to u$$

and $u$ satisfies

$$\det D^2 u = f, \ u = \varphi \text{ on } \partial \Omega.$$

**Proof.** Using the compactness of solutions we may assume that $u_k$ converges to a limit $u$ and it remains to prove that $u = \varphi$ on $\partial \Omega$.

Denote by $\tilde{u}_k$ the restriction of $u_k$ to the set

$$\tilde{\Omega}_k = \{ x \in \Omega_k | \ \text{dist}(x, \partial \Omega_k) \geq \varepsilon_k \}.$$
Notice that for fixed $k$, as $\varepsilon_k \to 0$ then $\tilde{u}_k \to u_k$ and $\tilde{\varphi}_k \to \varphi^*_k$. On the other hand, from the hypotheses we obtain that $\varphi^*_k \to \varphi^*$. Thus, we can choose a sequence of $\varepsilon_k \to 0$ such that

\[ \tilde{u}_k \to u, \quad \tilde{\varphi}_k \to \varphi^*, \quad \tilde{f}_k \to f. \]

Now, since $\tilde{u}_k$ are continuous up to the boundary, the conclusion follows from Proposition 2.4.

Finally, we state a version of the last proposition for solutions with bounded right-hand side i.e

\[ \lambda \leq \det D^2 u \leq \Lambda, \]

where the two inequalities are understood in the viscosity sense.

**Theorem 2.7.** Assume

\[ \lambda \leq \det D^2 u_k \leq \Lambda, \quad u_k = \varphi_k \text{ on } \partial \Omega_k, \]

and $\Omega_k$, $\varphi_k$ uniformly bounded.

Then there exists a subsequence $k_m$ such that

\[ u_{k_m} \to u, \quad \varphi_{k_m} \to \varphi \]

with

\[ \lambda \leq \det D^2 u \leq \Lambda, \quad u = \varphi \text{ on } \partial \Omega. \]

3. The Localization Theorem

In this section we state the quantitative version of the localization theorem at boundary points (Theorem 3.1).

Let $\Omega$ be a bounded convex set in $\mathbb{R}^n$. We assume that

\[ (3.1) \quad B_{\rho}(\rho e_n) \subset \Omega \subset \{ x_n \geq 0 \} \cap B_{\rho}, \]

for some small $\rho > 0$, that is $\Omega \subset (\mathbb{R}^n)^+$ and $\Omega$ contains an interior ball tangent to $\partial \Omega$ at $0$.

Let $u : \overline{\Omega} \to \mathbb{R}$ be continuous, convex, satisfying

\[ (3.2) \quad \det D^2 u = f, \quad 0 < \lambda \leq f \leq \Lambda \text{ in } \Omega. \]

We extend $u$ to be $\infty$ outside $\overline{\Omega}$.

After subtracting a linear function we assume that

\[ (3.3) \quad x_{n+1} = 0 \text{ is the tangent plane to } u \text{ at } 0, \]

in the sense that

\[ u \geq 0, \quad u(0) = 0, \]

and any hyperplane $x_{n+1} = \varepsilon x_n, \varepsilon > 0$, is not a supporting plane for $u$.

We investigate the geometry of the sections of $u$ at 0 that we denote for simplicity of notation

\[ S_h := \{ x \in \overline{\Omega} : u(x) < h \}. \]

We show that if the boundary data has quadratic growth near $\{ x_n = 0 \}$ then, as $h \to 0$, $S_h$ is equivalent to a half-ellipsoid centered at 0.

Precisely, our theorem reads as follows.
Theorem 3.1 (Localization Theorem). Assume that $\Omega, u$ satisfy (3.1)-(3.3) above and for some $\mu > 0$,

\[ \mu |x|^2 \leq u(x) \leq \mu^{-1} |x|^2 \quad \text{on } \partial \Omega \cap \{ x_n \leq \rho \}. \]

Then, for each $h < c(\rho)$ there exists an ellipsoid $E_h$ of volume $h^{n/2}$ such that

\[ kE_h \cap \Omega \subset \Omega \subset k^{-1}E_h \cap \Omega. \]

Moreover, the ellipsoid $E_h$ is obtained from the ball of radius $h^{1/2}$ by a linear transformation $A_h^{-1}$ (sliding along the $x_n = 0$ plane)

\[ A_h E_h = h^{1/2} B_1 \]

\[ A_h(x) = x - \nu x_n, \quad \nu = (\nu_1, \nu_2, \ldots, \nu_{n-1}, 0), \]

with

\[ |\nu| \leq k^{-1} |\log h|. \]

The constant $k$ above depends on $\mu, \lambda, \Lambda, n$ and $c(\rho)$ depends also on $\rho$.

The ellipsoid $E_h$, or equivalently the linear map $A_h$, provides information about the behavior of the second derivatives near the origin. Heuristically, the theorem states that in $S_h$ the tangential second derivatives are bounded from above and below and the mixed second derivatives are bounded by $|\log h|$.

The hypothesis that $u$ is continuous up to the boundary is not necessary, we just need to require that (3.1) holds in the sense of Definition 2.5.

Given only the boundary data $\varphi$ of $u$ on $\partial \Omega$, it is not always easy to check the main assumption (3.4) i.e. that $\varphi$ separates quadratically on $\partial \Omega$ (in a neighborhood of $\{x_n = 0\}$) from the tangent plane at 0. Proposition 3.2 provides some examples when this is satisfied depending on the local behavior of $\partial \Omega$ and $\varphi$ (see also the remarks below).

Proposition 3.2. Assume (3.1), (3.2) hold. Then (3.4) is satisfied if any of the following holds:

1) $\varphi$ is linear in a neighborhood of 0 and $\Omega$ is uniformly convex at the origin.

2) $\partial \Omega$ is tangent of order 2 to $\{x_n = 0\}$ and $\varphi$ has quadratic growth in a neighborhood of $\{x_n = 0\}$.

3) $\varphi, \partial \Omega \in C^3(0)$, and $\Omega$ is uniformly convex at the origin.

Proposition 3.2 is standard (see [CNS], [W]). We sketch its proof below.

Proof. 1) Assume $\varphi = 0$ in a neighborhood of 0. By the use of standard barriers, the assumptions on $\Omega$ imply that the tangent plane at the origin is given by

\[ x_{n+1} = -\mu x_n \]

for some bounded $\mu > 0$. Then (3.4) clearly holds.

2) After subtracting a linear function we may assume that

\[ \mu |x'|^2 \leq \varphi \leq \mu^{-1} |x'|^2 \]

on $\partial \Omega$ in a neighborhood of $\{x_n = 0\}$. Using a barrier we obtain that $b_0$, the tangent plane at the origin, has bounded slope. But $\partial \Omega$ is tangent of order 2 to $\{x_n = 0\}$,
thus $l_0$ grows less than quadratic on $\partial \Omega$ in a neighborhood of $\{x_n = 0\}$ and (3.4) is again satisfied.

3) Since $\Omega$ is uniformly convex at the origin, we can use barriers and obtain that $l_0$ has bounded slope. After subtracting this linear function we may assume $l_0 = 0$. Since $\varphi, \partial \Omega \in C^3(0)$ we find that
\[
\varphi = Q_0(x') + o(|x'|^3)
\]
with $Q_0$ a cubic polynomial. Now $\varphi \geq 0$, hence $Q_0$ has no linear part and its quadratic part is given by, say
\[
\sum_{i<n} \frac{\mu_i^2}{2} x_i^2, \quad \text{with} \quad \mu_i \geq 0.
\]
We need to show that $\mu_i > 0$.

If $\mu_1 = 0$, then the coefficient of $x_1^3$ is 0 in $Q_0$. Thus, if we restrict to $\partial \Omega$ in a small neighborhood near the origin, then for all small $h$ the set $\{\varphi < h\}$ contains
\[
\{ |x_1| \leq r(h) h^{1/3} \} \cap \{|x'| \leq ch^{1/2}\}
\]
for some $c > 0$ and with
\[
r(h) \to \infty \quad \text{as} \quad h \to 0.
\]
Now $S_h$ contains the convex set generated by $\{\varphi < h\}$ thus, since $\Omega$ is uniformly convex,
\[
|S_h| \geq c'(r(h) h^{1/3})^3 h^{(n-2)/2} \geq c'' r(h)^3 h^{n/2}.
\]
On the other hand, since $u$ satisfies (3.2) and
\[
0 \leq u \leq h \quad \text{in} \quad S_h
\]
we obtain (see (4.4))
\[
|S_h| \leq C h^{n/2},
\]
for some $C$ depending on $\lambda$ and $n$, and we contradict the inequality above as $h \to 0$.

\begin{remark}
The proof easily implies that if $\partial \Omega, \varphi \in C^3(\Omega)$ and $\Omega$ is uniformly convex, then we can find a constant $\mu$ which satisfies (3.4) for all $x \in \partial \Omega$.
\end{remark}

\begin{remark}
From above we see that we can often verify (3.4) in the case when $\varphi, \partial \Omega \in C^{1,1}(0)$ and $\Omega$ is uniformly convex at 0. Indeed, if $l_\varphi$ represents the tangent plane at 0 to $\varphi : \partial \Omega \to \mathbb{R}$ (in the sense of (3.3)), then (3.4) holds if either $\varphi$ separates from $l_\varphi$ quadratically near 0, or if $\varphi$ is tangent to $l_\varphi$ of order 3 in some tangential direction.
\end{remark}

\begin{remark}
Given $\varphi, \partial \Omega \in C^{1,1}(0)$ and $\Omega$ uniformly convex at 0, then (3.4) holds if $\lambda$ is sufficiently large.
\end{remark}

4. Proof of Theorem 3.1 (I)

We prove Theorem 3.1 in the next two sections. In this section we obtain some preliminary estimates and reduce the theorem to a statement about the rescalings of $u$. This statement is proved in section 5 using compactness.

Next proposition was proved by Trudinger and Wang in [TW]. It states that the volume of $S_h$ is proportional to $h^{n/2}$ and after an affine transformation (of controlled norm) we may assume that the center of mass of $S_h$ lies on the $x_n$ axis. Since our setting is slightly different we provide its proof.
Proposition 4.1. Under the assumptions of Theorem 3.1 for all \( h \leq c(\rho) \), there exists a linear transformation (sliding along \( x_n = 0 \))

\[
A_h(x) = x - \nu x_n,
\]

with

\[
\nu_n = 0, \quad |\nu| \leq C(\rho)h^{\frac{\mu}{\mu+1}},
\]

such that the rescaled function

\[
\tilde{u}(A_h x) = u(x),
\]

satisfies in

\[
\tilde{S}_h := A_h S_h = \{ \tilde{u} < h \}
\]

the following:

(i) the center of mass of \( \tilde{S}_h \) lies on the \( x_n \)-axis;

(ii) \( k_0 h^{n/2} \leq |\tilde{S}_h| = |S_h| \leq k_0^{-1} h^{n/2}; \)

(iii) the part of \( \partial \tilde{S}_h \) where \( \{ \tilde{u} < h \} \) is a graph, denoted by

\[
\tilde{G}_h = \partial \tilde{S}_h \cap \{ \tilde{u} < h \} = \{ (x', g_h(x')) \}
\]

that satisfies

\[
g_h \leq C(\rho)|x'|^2
\]

and

\[
\frac{\mu}{2}|x'|^2 \leq \tilde{u} \leq 2\mu^{-1}|x'|^2 \quad \text{on } \tilde{G}_h.
\]

The constant \( k_0 \) above depends on \( \mu, \lambda, \Lambda, n \) and the constants \( C(\rho), c(\rho) \) depend also on \( \rho \).

In this section we denote by \( c, C \) positive constants that depend on \( n, \mu, \lambda, \Lambda \). For simplicity of notation, their values may change from line to line whenever there is no possibility of confusion. Constants that depend also on \( \rho \) are denote by \( c(\rho), C(\rho) \).

Proof. The function

\[
v := \mu|x'|^2 + \frac{\lambda}{\mu n - 1} x_n^2 - C(\rho) x_n
\]

is a lower barrier for \( u \) in \( \Omega \cap \{ x_n \leq \rho \} \) if \( C(\rho) \) is chosen large.

Indeed, then

\[
v \leq u \quad \text{on } \partial\Omega \cap \{ x_n \leq \rho \},
\]

\[
v \leq 0 \leq u \quad \text{on } \Omega \cap \{ x_n = \rho \},
\]

and

\[
\det D^2 v > \Lambda.
\]

In conclusion,

\[
v \leq u \quad \text{in } \Omega \cap \{ x_n \leq \rho \},
\]

hence

\[
S_h \cap \{ x_n \leq \rho \} \subset \{ v < h \} \subset \{ x_n > c(\rho)(\mu|x'|^2 - h) \}.
\]
Let \( x^*_h \) be the center of mass of \( S_h \). We claim that
\[
(4.2) \quad x^*_h \cdot e_n \geq c_0(\rho) h^\alpha, \quad \alpha = \frac{n}{n+1},
\]
for some small \( c_0(\rho) > 0 \).

Otherwise, from (4.1) and John’s lemma we obtain
\[
S_h \subset \{ x_n \leq C(n) c_0 h^{\alpha} \leq h^{\alpha} \} \cap \{ |x'| \leq C_1 h^{\alpha}/2 \},
\]
for some large \( C_1 = C_1(\rho) \). Then the function
\[
w = \varepsilon x_n + \frac{h}{2} \left( \frac{|x'|}{C_1 h^{\alpha/2}} \right)^2 + \Lambda \frac{C_2(n-1)}{h^{\alpha}} \left( \frac{x_n}{h^{\alpha}} \right)^2
\]
is a lower barrier for \( u \) in \( S_h \) if \( c_0 \) is sufficiently small.

Indeed,
\[
w \leq \frac{h}{4} + \frac{h}{2} + \Lambda \frac{C_2(n-1)}{h^{\alpha}} (C(n) c_0)^2 h < h \quad \text{in} \quad S_h,
\]
and for all small \( h \),
\[
w \leq \varepsilon x_n + \frac{h^{1-\alpha}}{C_1} |x'|^2 + C(\rho) h c_0 \frac{x_n}{h^{\alpha}} \leq \mu |x'|^2 \leq u \quad \text{on} \quad \partial \Omega,
\]
and
\[
\det D^2 w = 2\Lambda.
\]

Hence
\[
w \leq u \quad \text{in} \quad S_h,
\]
and we contradict that 0 is the tangent plane at 0. Thus claim (4.2) is proved.

Now, define
\[
A_h x = x - \nu x_n, \quad \nu = \frac{x^*_h \cdot e_n}{x^*_h \cdot e_n},
\]
and
\[
\tilde{u}(A_h x) = u(x).
\]

The center of mass of \( \tilde{S}_h = A_h S_h \) is
\[
\tilde{x}_h^* = A_h x_h^*
\]
and lies on the \( x_n \)-axis from the definition of \( A_h \). Moreover, since \( x_h^* \in S_h \), we see from (4.1), (4.2) that
\[
|\nu| \leq C(\rho) \left( \frac{x^*_h \cdot e_n}{x^*_h \cdot e_n} \right)^{1/2} \leq C(\rho) h^{-\alpha/2},
\]
and this proves (i).

If we restrict the map \( A_h \) on the set on \( \partial \Omega \) where \( \{ u < h \} \), i.e. on
\[
\partial S_h \cap \partial \Omega \subset \{ x_n \leq \frac{|x'|}{\rho} \} \cap \{ |x'| < C h^{1/2} \}
\]
we have
\[
|A_h x - x| = |\nu| x_n \leq C(\rho) h^{-\alpha/2} |x'|^2 \leq C(\rho) h^{\frac{\alpha-1}{2}} |x'|,
\]
and part (iii) easily follows.

Next we prove (ii). From John’s lemma, we know that after relabeling the \( x' \) coordinates if necessary,
\[
(4.3) \quad D_h B_1 \subset \tilde{S}_h - \tilde{x}_h^* \subset C(n) D_h B_1
\]
where
\[ D_h = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix}. \]

Since
\[ \tilde{u} \leq 2\mu^{-1}|x'|^2 \quad \text{on } \tilde{G}_h = \{(x', g_h(x'))\}, \]
we see that the domain of definition of \( g_h \) contains a ball of radius \((\mu h/2)^{1/2}\). This implies that
\[ d_i \geq c_1 h^{1/2}, \quad i = 1, \cdots, n-1, \]
for some \( c_1 \) depending only on \( n \) and \( \mu \). Also from (4.2) we see that
\[ \tilde{x}_h^* \cdot e_n = x_h^* \cdot e_n \geq c_0(\rho)h^\alpha \]
which gives
\[ d_n \geq c(n)\tilde{x}_h^* \cdot e_n \geq c(\rho)h^\alpha. \]

We claim that for all small \( h \),
\[ \prod_{i=1}^{n} d_i \geq k_0 h^{n/2}, \]
with \( k_0 \) small depending only on \( \mu, n, \Lambda \), which gives the left inequality in (ii).

To this aim we consider the barrier,
\[ w = \varepsilon x_n + \sum_{i=1}^{n} ch \left( \frac{x_i}{d_i} \right)^2. \]

We choose \( c \) sufficiently small depending on \( \mu, n, \Lambda \) so that for all \( h < c(\rho) \),
\[ w \leq h \quad \text{on } \partial \tilde{S}_h, \]
and on the part of the boundary \( \tilde{G}_h \), we have \( w \leq \tilde{u} \) since
\[ w \leq \varepsilon x_n + \frac{c}{c_1} |x'|^2 + ch \frac{x_n}{d_n} \]
\[ \leq \frac{\mu}{4} |x'|^2 + chC(n) \frac{x_n}{d_n} \]
\[ \leq \frac{\mu}{4} |x'|^2 + ch^{1-\alpha}C(\rho)|x'|^2 \]
\[ \leq \frac{\mu}{2} |x'|^2. \]

Moreover, if our claim does not hold, then
\[ \det D^2 w = (2ch)^n (\prod d_i)^{-2n} > \Lambda, \]
thus \( w \leq \tilde{u} \) in \( \tilde{S}_h \). By definition, \( \tilde{u} \) is obtained from \( u \) by a sliding along \( x_n = 0 \), hence \( 0 \) is still the tangent plane of \( \tilde{u} \) at \( 0 \). We reach again a contradiction since \( \tilde{u} \geq w \geq \varepsilon x_n \) and the claim is proved.

Finally we show that
\[ |\tilde{S}_h| \leq C h^{n/2} \]
for some \( C \) depending only on \( \lambda, n \). Indeed, if
\[ v = h \quad \text{on } \partial \tilde{S}_h, \]
and

\[ \det D^2 v = \lambda \]

then

\[ v \geq u \geq 0 \quad \text{in } \tilde{S}_h. \]

Since

\[ h \geq h - \min_{\tilde{S}_h} v \geq c(n, \lambda)|\tilde{S}_h|^{2/n} \]

we obtain the desired conclusion.

\[ \square \]

In the proof above we showed that for all \( h \leq c(\rho) \), the entries of the diagonal matrix \( D_h \) from \((4.3)\) satisfy

\[ d_i \geq ch^{1/2}, \quad i = 1, \ldots, n - 1 \]

\[ d_n \geq c(\rho) h^\alpha, \quad \alpha = \frac{n}{n + 1} \]

\[ ch^{n/2} \leq \prod d_i \leq Ch^{n/2}. \]

The main step in the proof of Theorem 1.1 is the following lemma that will be completed in Section 5.

**Lemma 4.2.** There exist constants \( c, c(\rho) \) such that

\[ (4.5) \quad d_n \geq ch^{1/2}, \]

for all \( h \leq c(\rho) \).

Using Lemma 4.2 we can easily finish the proof of our theorem.

**Proof of Theorem 1.1.** Since all \( d_i \) are bounded below by \( ch^{1/2} \) and their product is bounded above by \( Ch^{n/2} \) we see that

\[ Ch^{1/2} \geq d_i \geq ch^{1/2} \quad i = 1, \ldots, n \]

for all \( h \leq c(\rho) \). Using \((4.3)\) we obtain

\[ \tilde{S}_h \subset Ch^{1/2} B_1. \]

Moreover, since

\[ \tilde{x}_h^* \cdot e_n \geq d_n \geq ch^{1/2}, \quad (\tilde{x}_h^*)' = 0, \]

and the part \( \tilde{G}_h \) of the boundary \( \partial \tilde{S}_h \) contains the graph of \( \tilde{g}_h \) above \( |x'| \leq ch^{1/2} \), we find that

\[ ch^{1/2} B_1 \cap \tilde{\Omega} \subset \tilde{S}_h, \]

with \( \tilde{\Omega} = A_h \Omega, \tilde{S}_h = A_h S_h. \) In conclusion

\[ ch^{1/2} B_1 \cap \tilde{\Omega} \subset A_h S_h \subset Ch^{1/2} B_1. \]

We define the ellipsoid \( E_h \) as

\[ E_h := A_h^{-1} (h^{1/2} B_1), \]

hence

\[ cE_h \cap \tilde{\Omega} \subset S_h \subset CE_h. \]
Comparing the sections at levels $h$ and $h/2$ we find
\[ cE_{h/2} \cap \Omega \subset CE_h \]
and we easily obtain the inclusion
\[ A_h A_{h/2}^{-1} B_1 \subset CB_1. \]
If we denote
\[ A_h x = x - \nu_h x_n \]
then the inclusion above implies
\[ |\nu_h - \nu_{h/2}| \leq C, \]
which gives the desired bound
\[ |\nu_h| \leq C |\log h| \]
for all small $h$.

In order to prove Lemma 4.2 we introduce a new quantity $b(h)$ which is proportional to $d_n h^{-1/2}$ and is appropriate when dealing with affine transformations.

**Notation.** Given a convex function $u$ we define
\[ b_u(h) = h^{-1/2} \sup_{S_n} x_n. \]
Whenever there is no possibility of confusion we drop the subindex $u$ and use the notation $b(h)$.

Below we list some basic properties of $b(h)$.

1) If $h_1 \leq h_2$ then
\[ \left( \frac{h_1}{h_2} \right)^{\frac{1}{2}} \leq \frac{b(h_1)}{b(h_2)} \leq \left( \frac{h_2}{h_1} \right)^{\frac{1}{2}}. \]

2) A rescaling
\[ \tilde{u}(Ax) = u(x) \]
given by a linear transformation $A$ which leaves the $x_n$ coordinate invariant does not change the value of $b$, i.e
\[ b_{\tilde{u}}(h) = b_u(h). \]

3) If $A$ is a linear transformation which leaves the plane $\{x_n = 0\}$ invariant the values of $b$ get multiplied by a constant. However the quotients $b(h_1)/b(h_2)$ do not change values i.e
\[ \frac{b_{\tilde{u}}(h_1)}{b_{\tilde{u}}(h_2)} = \frac{b_u(h_1)}{b_u(h_2)}. \]

4) If we multiply $u$ by a constant, i.e.
\[ \tilde{u}(x) = \beta u(x) \]
then
\[ b_{\tilde{u}}(\beta h) = \beta^{-1/2} b_u(h), \]
and

\[ \frac{b_0(\beta h_1)}{b_0(\beta h_2)} = \frac{b_u(h_1)}{b_u(h_2)}. \]

From (4.3) and property 2 above,

\[ c(n)d_n \leq b(h)h^{1/2} \leq C(n)d_n, \]

hence Lemma 4.2 will follow if we show that \( b(h) \) is bounded below. We achieve this by proving the following lemma.

**Lemma 4.3.** There exist \( c_0, c(\rho) \) such that if \( h \leq c(\rho) \) and \( b(h) \leq c_0 \) then

\[ b(h) > 2, \]

for some \( t \in [c_0, 1] \).

This lemma states that if the value of \( b(h) \) on a certain section is less than a critical value \( c_0 \), then we can find a lower section at height still comparable to \( h \) where the value of \( b \) doubled. Clearly Lemma 4.3 and property 1 above imply that \( b(h) \) remains bounded for all \( h \) small enough.

The quotient in (4.6) is the same for \( \tilde{u} \) which is defined in Proposition 4.1. We normalize the domain \( \tilde{S}_h \) and \( \tilde{u} \) by considering the rescaling

\[ v(x) = \frac{1}{h} \tilde{u}(h^{1/2}Ax) \]

where \( A \) is a multiple of \( D_h \) (see (4.3)), \( A = \gamma D_h \) such that

\[ \det A = 1. \]

Then

\[ ch^{-1/2} \leq \gamma \leq Ch^{-1/2}, \]

and the diagonal entries of \( A \) satisfy

\[ a_i \geq c, \quad i = 1, 2, \ldots, n-1, \]

\[ cb_u(h) \leq a_n \leq Cb_u(h). \]

The function \( v \) satisfies

\[ \lambda \leq \det D^2v \leq \Lambda, \]

\[ v \geq 0, \quad v(0) = 0, \]

is continuous and it is defined in \( \bar{\Omega}_v \) with

\[ \Omega_v := \{ v < 1 \} = h^{-1/2}A^{-1}\bar{S}_h. \]

Then

\[ x^* + cB_1 \subset \Omega_v \subset CB_1^+, \]

for some \( x^* \), and

\[ ct^{n/2} \leq |S_t(v)| \leq Ct^{n/2}, \quad \forall t \leq 1, \]

where \( S_t(v) \) denotes the section of \( v \). Since

\[ \tilde{u} = h \quad \text{in} \quad \partial\bar{S}_h \cap \{ x_n \geq C(\rho)h \}, \]
then
\[ v = 1 \quad \text{on } \partial \Omega_v \cap \{ x_n \geq \sigma \}, \quad \sigma := C(\rho)h^{1-\alpha}. \]

Also, from Proposition 4.1 on the part \( G \) of the boundary of \( \partial \Omega_v \) where \( \{ v < 1 \} \)
we have
\[
(4.7) \quad \frac{1}{2} \mu \sum_{i=1}^{n-1} a_i^2 x_i^2 \leq v \leq 2 \mu^{-1} \sum_{i=1}^{n-1} a_i^2 x_i^2.
\]

In order to prove Lemma 4.3 we need to show that if \( \sigma, a_n \) are sufficiently small depending on \( n, \mu, \lambda, \Lambda \) then the function \( v \) above satisfies
\[
(4.8) \quad b_v(t) \geq 2b_v(1)
\]
for some \( 1 > t \geq c_0 \).

Since \( \alpha < 1 \), the smallness condition on \( \sigma \) is satisfied by taking \( h < c(\rho) \) sufficiently small. Also \( a_n \) being small is equivalent to one of the \( a_i, 1 \leq i \leq n-1 \) being large since their product is 1 and \( a_i \) are bounded below.

In the next section we prove property (4.8) above by compactness, by letting \( \sigma \to 0, a_i \to \infty \) for some \( i \) (see Proposition 5.1).

5. Proof of Theorem 3.1 II

In this section we consider the class of solutions \( v \) that satisfy the properties above. After relabeling the constants \( \mu \) and \( a_i \), and by abuse of notation writing \( u \) instead of \( v \), we may assume we are in the following situation.

Fix \( \mu \) small and \( \lambda, \Lambda \). For an increasing sequence
\[ a_1 \leq a_2 \leq \ldots \leq a_{n-1} \]
with
\[ a_1 \geq \mu, \]
we consider the family of solutions
\[ u \in \mathcal{D}_{\mu}^\nu(a_1, a_2, \ldots, a_{n-1}) \]
of convex functions \( u : \Omega \to \mathbb{R} \) that satisfy
\[
(5.1) \quad \lambda \leq \det D^2u \leq \Lambda \quad \text{in } \Omega, \quad 0 \leq u \leq 1 \quad \text{in } \Omega;
\]
\[
(5.2) \quad 0 \in \partial \Omega, \quad B_\mu(x_0) \subset \Omega \subset B_{1/\mu}^+ \quad \text{for some } x_0;
\]
\[
(5.3) \quad \mu h^{n/2} \leq |S_h| \leq \mu^{-1} h^{n/2}.
\]

Moreover we assume that the boundary \( \partial \Omega \) has a closed subset \( G \)
\[
(5.4) \quad G \subset \{ x_n \leq \sigma \} \cap \partial \Omega
\]
which is a graph in the \( e_n \) direction with projection \( \pi_n(G) \subset \mathbb{R}^{n-1} \) along \( e_n \)
\[
(5.5) \quad \{ \mu^{-1} \sum_{i=1}^{n-1} a_i^2 x_i^2 \leq 1 \} \subset \pi_n(G) \subset \{ \mu \sum_{i=1}^{n-1} a_i^2 x_i^2 \leq 1 \},
\]
and (see Definition 2.5), the boundary values of \( u = \varphi \) on \( \partial \Omega \) satisfy
\[
(5.6) \quad \varphi = 1 \quad \text{on } \partial \Omega \setminus G;
\]
and
\begin{equation}
\mu \sum_{i=1}^{n-1} a_i^2 x_i^2 \leq \varphi \leq \min \{ 1, \mu^{-1} \sum_{i=1}^{n-1} a_i^2 x_i^2 \} \quad \text{on } G.
\end{equation}

In this section we prove

**Proposition 5.1.** For any $M > 0$ there exists $C_*$ depending on $M, \mu, \lambda, \Lambda, n$ such that if $u \in D_{\sigma}^{\mu}(a_1, a_2, \ldots, a_{n-1})$ with
\[ a_{n-1} \geq C_*, \quad \sigma \leq C_*^{-1} \]
then
\[ b(h) = (\sup_{S_h} x_n) h^{-1/2} \geq M \]
for some $h$ with $C_*^{-1} \leq h \leq 1$.

Property (4.8) (hence Theorem 3.1), easily follows from this proposition. Indeed, by choosing $M = 2 \mu^{-1} \geq 2 b(1)$ in Proposition 5.1 we prove the existence of a section $S_h$ with $h \geq c_0$ such that
\[ b(h) \geq 2 b(1). \]

Clearly the function $v$ of the previous section satisfies the hypotheses above (after renaming the constant $\mu$) provided that $\sigma, a_n$ are sufficiently small.

We prove Proposition 5.1 by compactness. We introduce the limiting solutions from the class $D_{\sigma}^{\mu}(a_1, \ldots, a_{n-1})$ when $a_{k+1} \to \infty$ and $\sigma \to 0$.

If $\mu \leq a_1 \leq \ldots \leq a_k$, we denote by
\[ D_{\sigma}^{\mu}(a_1, \ldots, a_k, \infty, \infty, \ldots, \infty), \quad 0 \leq k \leq n-2 \]
the class of functions $u$ that satisfy properties (5.1)-(5.2)-(5.3) with,
\begin{equation}
G \subset \{ x_i = 0, \ i > k \} \cap \partial \Omega
\end{equation}
and if we restrict to the space generated by the first $k$ coordinates then
\begin{equation}
\{ \mu^{-1} \sum_{i=1}^{k} a_i^2 x_i^2 \leq 1 \} \subset G \subset \{ \mu \sum_{i=1}^{k} a_i^2 x_i^2 \leq 1 \}.
\end{equation}

Also, $u = \varphi$ on $\partial \Omega$ with
\begin{equation}
\varphi = 1 \quad \text{on } \partial \Omega \setminus G;
\end{equation}
\begin{equation}
\mu \sum_{i=1}^{k} a_i^2 x_i^2 \leq \varphi \leq \min \{ 1, \mu^{-1} \sum_{i=1}^{k} a_i^2 x_i^2 \} \quad \text{on } G.
\end{equation}

The compactness theorem (Theorem 2.7) implies that if $u_m \in D_{\sigma_m}^{\mu}(a_1^m, \ldots, a_{n-1}^m)$ is a sequence with
\[ \sigma_m \to 0 \quad \text{and} \quad a_{k+1}^m \to \infty \]
for some fixed $0 \leq k \leq n-2$, then we can extract a convergent subsequence to a function $u$ (see Definition 2.3) with
\[ u \in D_{\sigma}^{\mu}(a_1, \ldots, a_l, \infty, \ldots, \infty), \]
for some $l \leq k$ and $a_1 \leq \ldots \leq a_l$. 
Proposition 5.1 follows easily from the next proposition.

**Proposition 5.2.** For any \( M > 0 \) and \( 0 \leq k \leq n - 2 \) there exists \( c_k \) depending on \( M, \mu, \lambda, \Lambda, n, k \) such that if

\[
(5.12) \quad u \in \mathcal{D}_0^\mu(a_1, \ldots, a_k, \infty, \ldots, \infty)
\]

then

\[
b(h) = (\text{sup}_{S_h} x_n) h^{-1/2} \geq M
\]

for some \( h \) with \( c_k \leq h \leq 1 \).

Indeed, if Proposition 5.1 fails for a sequence of constants \( C_* \to \infty \) then we obtain a limiting solution \( u \) as in (5.12) for which \( b(h) \leq M \) for all \( h > 0 \). This contradicts Proposition 5.2 (with \( M \) replaced by \( 2M \)).

We prove Proposition 5.2 by induction on \( k \). We start by introducing some notation.

Denote

\[
x = (y, z, x_n), \quad y = (x_1, \ldots, x_k) \in \mathbb{R}^k, \quad z = (x_{k+1}, \ldots, x_{n-1}) \in \mathbb{R}^{n-1-k}.
\]

**Definition 5.3.** We say that a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^n \) is a sliding along the \( y \) direction if

\[
Tx := x + \nu_1 z_1 + \nu_2 z_2 + \ldots + \nu_{n-k-1} z_{n-k-1} + \nu_{n-k} x_n
\]

with

\[
\nu_1, \nu_2, \ldots, \nu_{n-k} \in \text{span}\{e_1, \ldots, e_k\}
\]

We see that \( T \) leaves the \((z, x_n)\) components invariant together with the subspace \((y, 0, 0)\). Clearly, if \( T \) is a sliding along the \( y \) direction then so is \( T^{-1} \) and

\[
\det T = 1.
\]

The key step in the proof of Proposition 5.2 is the following lemma.

**Lemma 5.4.** Assume that

\[
u \geq p(|z| - qx_n),
\]

for some \( p, q > 0 \) and assume that for each section \( S_h \) of \( u, h \in (0, 1) \), there exists \( T_h \) a sliding along the \( y \) direction such that

\[
T_h S_h \subset C_0 h^{1/2} B^+_1,
\]

for some constant \( C_0 \). Then

\[
u \notin \mathcal{D}_0^\mu(1, \ldots, 1, \infty, \ldots, \infty).
\]

**Proof.** Assume by contradiction that \( u \in \mathcal{D}_0^\mu \) and it satisfies the hypotheses with \( q \leq q_0 \) for some \( q_0 \). We show that

\[
(5.13) \quad u \geq p'(|z| - q'x_n), \quad q' = q - \eta,
\]

for some \( 0 < p' \ll p \), where the constant \( \eta > 0 \) depends only on \( q_0 \) and \( \mu, C_0, \Lambda, n \).

Then, since \( q' \leq q_0 \), we can apply this result a finite number of times and obtain

\[
u \geq \varepsilon(|z| + x_n),
\]

for some small \( \varepsilon > 0 \). This gives \( S_h \subset \{x_n \leq \varepsilon^{-1} h\} \) hence

\[
T_h S_h \subset \{x_n \leq \varepsilon^{-1} h\}
\]
and by the hypothesis above
\[ |S_h| = |T_h S_h| = O(h^{(n+1)/2}) \quad \text{as } h \to 0, \]
and we contradict (5.3).

Now we prove (5.13). Since \( u \in D^\mu_0 \) as above, there exists a closed set
\[ G_h \subset \partial S_h \cap \{z = 0, x_n = 0\} \]
such that on the subspace \( (y, 0, 0) \)
\[ \{\mu^{-1}|y|^2 \leq h\} \subset G_h \subset \{\mu|y|^2 \leq h\}, \]
and the boundary values \( \varphi_h \) of \( u \) on \( \partial S_h \) satisfy (see Section 2)
\[ \varphi_h = h \quad \text{on } \partial S_h \setminus G_h; \]
\[ \mu|y|^2 \leq \varphi_h \leq \min\{h, \mu^{-1}|y|^2\} \quad \text{on } G_h. \]

Let \( w \) be a rescaling of \( u \),
\[ w(x) := \frac{1}{h} u(h^{1/2}T_h^{-1}x) \]
for some small \( h \ll a \). Then
\[ S_1(w) := \Omega_w = h^{-1/2}T_h S_h \subset B^+_{C_0} \]
and our hypothesis becomes
\[ (5.14) \quad w \geq \frac{p}{h^{1/2}}(\|z\| - q x_n). \]
Moreover the boundary values \( \varphi_w \) of \( w \) on \( \partial \Omega_w \) satisfy
\[ \varphi_w = 1 \quad \text{on } \partial \Omega_w \setminus G_w \]
\[ \mu|y|^2 \leq \varphi_w \leq \min\{1, \mu^{-1}|y|^2\} \quad \text{on } G_w := h^{-1/2}G_h. \]
Next we show that \( \varphi_w \geq v \) on \( \partial \Omega_w \) where \( v \) is defined as
\[ v := \delta|t|^2 + \frac{\Lambda}{\delta^{n-1}}(z_1 - q x_n)^2 + N(z_1 - q x_n) + \delta x_n, \]
and \( \delta \) is small depending on \( \mu \) and \( C_0 \), and \( N \) is chosen large such that
\[ \frac{\Lambda}{\delta^{n-1}}t^2 + Nt \]
is increasing in the interval \( |t| \leq (1 + q_0)C_0 \).
From the definition of \( v \) we see that
\[ \det D^2 v > \Lambda. \]
On the part of the boundary \( \partial \Omega_w \) where \( z_1 \leq q x_n \) we use that \( \Omega_w \subset B_{C_0} \) and obtain
\[ v \leq \delta(|t|^2 + x_n) \leq \varphi_w. \]
On the part of the boundary \( \partial \Omega_w \) where \( z_1 > q x_n \) we use (5.14) and obtain
\[ 1 = \varphi_w \geq C(|z| - q x_n) \geq C(z_1 - q x_n) \]
with \( C \) arbitrarily large provided that \( h \) is small enough. We choose \( C \) such that the inequality above implies
\[ \frac{\Lambda}{\mu^{n-1}}(z_1 - q x_n)^2 + N(z_1 - q x_n) < \frac{1}{2}. \]
Then

\[ \varphi_w = 1 > \frac{1}{2} + \delta(|x|^2 + x_n) \geq v. \]

In conclusion \( \varphi_w \geq v \) on \( \partial\Omega_w \) hence the function \( v \) is a lower barrier for \( w \) in \( \Omega_w \). Then

\[ w \geq N(z_1 - qx_n) + \delta x_n \]

and, since this inequality holds for all directions in the \( z \)-plane, we obtain

\[ w \geq N(|z| - (q - \eta)x_n), \quad \eta := \frac{\delta}{N}. \]

Scaling back we get

\[ u \geq p'(|z| - (q - \eta)x_n) \quad \text{in} \quad S_h. \]

Since \( u \) is convex and \( u(0) = 0 \), this inequality holds globally, and (5.13) is proved. \( \square \)

**Lemma 5.5.** Proposition 5.2 holds for \( k = 0 \).

**Proof.** By compactness we need to show that there does not exist \( u \in D_0^\mu(\infty, \ldots, \infty) \) with \( b(h) \leq M \) for all \( h \). If such \( u \) exists then \( G = \{0\} \). Let

\[ v := \delta(|x'| + \frac{1}{2}|x'|^2) + \Lambda x_n - Nx_n \]

with \( \delta \) small depending on \( \mu \), and \( N \) large so that

\[ \frac{\Lambda}{\delta^{n-1}}x_n^2 - Nx_n \leq 0 \]

in \( B^+_{1/\mu} \). Then

\[ v \leq \varphi \quad \text{on} \quad \partial\Omega, \quad \det D^2v > \Lambda, \]

hence

\[ v \leq u \quad \text{in} \quad \Omega. \]

This gives

\[ u \geq \delta|x'| - Nx_n, \]

and we obtain

\[ S_h \subset \{|x'| \leq C(x_n + h)\}. \]

Since \( b(h) \leq M \) we conclude

\[ S_h \subset Ch^{1/2}B^+_1, \]

and we contradict Lemma 5.4 for \( k = 0 \). \( \square \)

Now we prove Proposition 5.2 by induction on \( k \).

**Proof of Proposition 5.2**. In this proof we denote by \( c, C \) positive constants that depend on \( M, \mu, \Lambda, n \) and \( k \).

We assume that the proposition holds for all nonnegative integers up to \( k - 1 \), \( 1 \leq k < n - 2 \), and we prove it for \( k \). Let

\[ u \in D_0^\mu(a_1, \ldots, a_k, \infty, \ldots, \infty). \]

By the induction hypotheses and compactness we see that there exists a constant

\[ C_k(\mu, M, \Lambda, n) \]
such that if $a_k \geq C_k$ then $b(h) \geq M$ for some $h \geq C_k^{-1}$. Thus, it suffices to consider only the case when $a_k < C_k$.

If no $c_{k+1}$ exists then we can find a limiting solution that, by abuse of notation, we still denote by $u$ such that

\[(5.15) \quad u \in \mathcal{D}^{\tilde{\mu}}_0(1,1,\ldots,1,\infty,\ldots,\infty)\]

with

\[(5.16) \quad b(h) \leq Mh^{1/2}, \quad \forall h > 0\]

where $\tilde{\mu}$ depends on $\mu$ and $C_k$.

We show that such a function $u$ does not exist.

Denote as before $x = (y,z,x_n)$, $y = (x_1,\ldots,x_k) \in \mathbb{R}^k$, $z = (x_{k+1},\ldots,x_{n-1}) \in \mathbb{R}^{n-1-k}$.

On $\partial \Omega$ we have

\[\varphi(x) \geq \delta|z'|^2 + \delta|z| + \frac{\Lambda}{\delta^{n-1}}x_n^2 - N x_n\]

where $\delta$ is small depending on $\tilde{\mu}$, and $N$ is large so that

\[\frac{\Lambda}{\delta^{n-1}}x_n^2 - N x_n \leq 0\]

in $B^+_1$. As before we obtain that the inequality above holds in $\Omega$, hence

\[(5.17) \quad u(x) \geq \delta|z| - N x_n.\]

From (5.16) we see that the section $S_h$ of $u$ satisfies

\[(5.18) \quad S_h \subset \{|z| < \delta^{-1}(N x_n + \tilde{h})\} \cap \{x_n \leq Mh^{1/2}\} \subset S_h - x_n^* \subset C(n)E_h, \quad |E_h| = |S_h|,\]

with $x_n^*$ the center of mass of $S_h$.

From John’s lemma we know that $S_h$ is equivalent to an ellipsoid $E_h$ of the same volume i.e

\[(5.19) \quad c(n)E_h \subset S_h - x_n^* \subset C(n)E_h, \quad |E_h| = |S_h|,\]

with $x_n^*$ the center of mass of $S_h$.

For any ellipsoid $E_h$ in $\mathbb{R}^n$ of positive volume we can find $T_h$, a sliding along the $y$ direction (see Definition 5.3), such that

\[(5.20) \quad T_h E_h = |E_h|^{1/n} AB_1,\]

with a matrix $A$ that leaves the $(y,0,0)$ and $(0,z,x_n)$ subspaces invariant, and $\det A = 1$. By choosing an appropriate system of coordinates in the $y$ and $z$ variables we may assume in fact that

\[A(y,z,x_n) = (A_1 y, A_2(z,x_n))\]

with

\[A_1 = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_k \end{pmatrix}\]

with $0 < \beta_1 \leq \cdots \leq \beta_k$, and
\begin{equation}
A_2 = \begin{pmatrix}
\gamma_{k+1} & 0 & \cdots & 0 & \theta_{k+1} \\
0 & \gamma_{k+2} & \cdots & 0 & \theta_{k+2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \gamma_{n-1} & \theta_{n-1} \\
0 & 0 & \cdots & 0 & \theta_n
\end{pmatrix}
\end{equation}

with \( \gamma_j, \theta_n > 0 \).

The \( h \) section \( \tilde{S}_h = T_h S_h \) of the rescaling
\( \tilde{u}(x) = u(T_h^{-1} x) \)
satisfies (5.18) and since \( u \in D_0^\mu \), there exists \( \tilde{G}_h = G_h \),
\( \tilde{G}_h \subset \{z = 0, x_n = 0\} \cap \partial \tilde{S}_h \)
such that on the subspace \((y, 0, 0)\)
\( \{\mu^{-1}|y|^2 \leq h\} \subset \tilde{G}_h \subset \{\mu|y|^2 \leq h\} \),
and the boundary values \( \tilde{\varphi}_h \) of \( \tilde{u} \) on \( \partial \tilde{S}_h \) satisfy
\( \tilde{\varphi}_h = h \) on \( \partial \tilde{S}_h \setminus \tilde{G}_h ; \)
\( \mu|y|^2 \leq \tilde{\varphi}_h \leq \min \{h, \mu^{-1}|y|^2\} \) on \( \tilde{G}_h \).

Moreover, using that
\( |S_h| \sim h^{n/2} \)
in (5.19), (5.20) and that \( 0 \in \partial S_h \), we obtain
(5.21) \[ \tilde{x}_h^* + ch^{1/2}A_1 \subset \tilde{S}_h \subset Ch^{1/2}AB_1, \quad \text{det} \ A = 1, \]
for the matrix \( A \) as above and with \( \tilde{x}_h^* \) the center of mass of \( \tilde{S}_h \).

Next we use the induction hypothesis and show that \( \tilde{S}_h \) is equivalent to a ball.

**Lemma 5.6.** There exists \( C_0 \) such that
\( T_h S_h = \tilde{S}_h \subset C_0 h^{n/2} B_1^+ \).

**Proof.** We need to show that
\( |A| \leq C. \)

Since \( \tilde{S}_h \) satisfies (5.18) we see that
\( \tilde{S}_h \subset \{(z, x_n)| \leq Ch^{1/2}\}, \)
which together with the inclusion (5.21) gives \( |A_2| \leq C \) hence
\( \gamma_j, \theta_n \leq C, \quad |\theta_j| \leq C. \)

Also, since
\( \tilde{G}_h \subset \tilde{S}_h, \)
we find from (5.21)
\( \beta_i \geq c > 0, \quad i = 1, \cdots, k. \)

We define the rescaling
\( w(x) = \frac{1}{h} \tilde{u}(h^{1/2} Ax) \)
defined in a domain \( \Omega_w = S_1(w) \). Then (5.24) gives
\( B_c(x_0) \subset \Omega_w \subset B_c^+, \)
and \( w = \varphi_w \) on \( \partial \Omega_w \) with
\[
\varphi_w = 1 \quad \text{on } \partial \Omega_w \setminus G_w, \\
\tilde{\mu} \sum_{i=1}^{k} \beta_i^2 x_i^2 \leq \varphi_w \leq \min \{ 1, \tilde{\mu}^{-1} \sum_{i=1}^{k} \beta_i^2 x_i^2 \} \quad \text{on } G_w := h^{-1/2} A^{-1} \tilde{C} h.
\]
This implies that
\[
w \in D^\bar{\mu}_0(\beta_1, \beta_2, \ldots, \beta_k, \infty, \ldots, \infty)
\]
for some small \( \tilde{\mu} \) depending on \( \mu, M, \lambda, \Lambda, n, k \).

We claim that \( b_u(h) \geq c_* \).

First we notice that
\[
b_u(h) = b_{\tilde{u}}(h) \sim \theta_n.
\]
Since
\[
\theta_n \prod \beta_i \prod \gamma_j = \det A = 1
\]
and
\[
\gamma_j \leq C,
\]
we see that if \( b_u(h) \) (and therefore \( \theta_n \)) becomes smaller than a critical value \( c_* \) then
\[
\beta_k \geq C_k(\tilde{\mu}, \tilde{M}, \lambda, \Lambda, n),
\]
with \( \tilde{M} := 2\tilde{\mu}^{-1} \), and by the induction hypothesis
\[
b_w(\tilde{h}) \geq \tilde{M} \geq 2 b_w(1)
\]
for some \( \tilde{h} > C_k^{-1} \). This gives
\[
\frac{b_u(h\tilde{h})}{b_u(h)} = \frac{b_{\tilde{u}}(\tilde{h})}{b_{\tilde{u}}(1)} \geq 2,
\]
which implies \( b_u(h\tilde{h}) \geq 2b_u(h) \) and our claim follows.

Next we claim that \( \gamma_j \) are bounded below by the same argument. Indeed, from the claim above \( \theta_n \) is bounded below and if some \( \gamma_j \) is smaller than a small value \( \tilde{c}_* \) then
\[
\beta_k \geq C_k(\tilde{\mu}, \tilde{M}_1, \lambda, \Lambda, n)
\]
with
\[
\tilde{M}_1 := \frac{2M}{\tilde{\mu}c_*}.
\]
By the induction hypothesis
\[
b_w(\tilde{h}) \geq \tilde{M}_1 \geq 2M c_*/b_w(1),
\]
hence
\[
\frac{b_u(h\tilde{h})}{b_u(h)} \geq \frac{2M}{c_*}
\]
which gives \( b_u(h\tilde{h}) \geq 2M \), contradiction. In conclusion \( \theta_n, \gamma_j \) are bounded below which implies that \( \beta_i \) are bounded above. This shows that \( |A| \) is bounded and the lemma is proved.
End of the proof of Proposition 5.2.
The proof is finished since Lemma 5.6, (5.15), (5.17) contradict Lemma 5.4.

6. Pogorelov estimate in half-domain

In this section we obtain a version of Pogorelov estimate at the boundary (Theorem 6.4 below). A similar estimate was proved also in [TW]. We start with the following a priori estimate.

**Proposition 6.1.** Let \( u : \bar{\Omega} \to \mathbb{R}, u \in C^4(\bar{\Omega}) \) satisfy the Monge-Ampere equation
\[
\det D^2 u = 1 \quad \text{in } \Omega.
\]
Assume that for some constant \( k > 0 \),
\[
B_k^+ \subset \Omega \subset B_{k-1}^+,
\]
and
\[
\begin{aligned}
&u = \frac{1}{2}|x'|^2 \quad \text{on } \partial \Omega \cap \{x_n = 0\} \\
&u = 1 \quad \text{on } \partial \Omega \cap \{x_n > 0\}.
\end{aligned}
\]
Then
\[
\|u\|_{C^3(\{u < \frac{k^2}{2}\})} \leq C(k, n).
\]

**Proof.** We divide the proof into four steps.

**Step 1:** We show that
\[
|\nabla u| \leq C(k, n) \quad \text{in the set } \ D := \{u < k^2/2\}.
\]

For each
\[
x_0 \in \{|x'| \leq k, \ x_n = 0\},
\]
we consider the barrier
\[
w_{x_0}(x) := \frac{1}{2}|x_0|^2 + x_0 \cdot (x - x_0) + \delta|x' - x_0|^2 + \delta^{1-n}(x_n^2 - k^{-1}x_n),
\]
where \( \delta \) is small so that
\[
w_{x_0} \leq 1 \quad \text{in } \ B_{k-1}^+.
\]
Then
\[
w_{x_0}(x_0) = u(x_0), \quad w_{x_0} \leq u \quad \text{on } \partial \Omega \cap \{x_n = 0\},
\]
\[
w_{x_0} \leq 1 = u \quad \text{on } \partial \Omega \cap \{x_n > 0\},
\]
and
\[
\det D^2 w_{x_0} > 1,
\]
thus in \( \Omega \)
\[
u \geq w_{x_0} \geq u(x_0) + x_0 \cdot (x - x_0) - \delta^{1-n}k^{-1}x_n.
\]

This gives a lower bound for \( u_n(x_0) \). Moreover, writing the inequality for all \( x_0 \) with \( |x_0| = k \) we obtain
\[
D \subset \{x_n \geq c(|x'| - k)\}.
\]
From the values of \( u \) on \( \{x_n = 0\} \) and the inclusion above we obtain a lower bound on \( u_n \) on \( \partial D \) in a neighborhood of \( \{x_n = 0\} \). Since \( \Omega \) contains the cone generated by \( kc_n \) and \( \{|x'| \leq 1, x_n = 0\} \) and \( u \leq 1 \) in \( \Omega \), we can use the convexity of \( u \) and obtain also an upper bound for \( u_n \) and all \( |u_i|, 1 \leq i \leq n - 1, \) on \( \partial D \) in a neighborhood of \( \{x_n = 0\} \). We find
\[
|\nabla u| \leq C \quad \text{on } \partial D \cap \{x_n \leq c_0\},
\]
where $c_0 > 0$ is a small constant depending on $k$ and $n$. We obtain a similar bound on $\partial D \cap \{x_n \geq c_0\}$ by bounding below

$$\text{dist}(\partial D \cap \{x_n \geq c_0\}, \partial \Omega)$$

by a small positive constant. Indeed, if

$$y \in \partial \Omega \cap \{x_n \geq c_0/2\},$$

then there exists a linear function $l_y$ with bounded gradient so that

$$u(y) = l_y(y), \quad u \geq l_y \quad \text{on} \quad \partial \Omega.$$ 

Then, using Alexandrov estimate for $(u - l_y)$ we obtain

$$u(x) \geq l_y(x) - Cd(x)^{1/n}, \quad d(x) := \text{dist}(x, \partial \Omega)$$

hence $D$ stays outside a fixed neighborhood of $y$.

**Step 2:** We show that

$$\|D^2 u\| \leq C(k, n) \quad \text{on} \quad E := \{x_n = 0\} \cap \{|x'| \leq k/2\}.$$ 

It suffices to prove that $|u_{in}|$ are bounded in $E$ with $i = 1, \ldots, n - 1$. Let

$$L \phi := u^{ij} \phi_{ij}$$

denote the linearized Monge-Ampere operator for $u$. Then

$$Lu_i = 0, \quad u_i = x_i \quad \text{on} \quad \{x_n = 0\},$$

and if we define $P(x) = \delta|x'|^2 + \delta^{1-n}x_n^2$ then

$$LP = Tr((D^2 u)^{-1} D^2 P)$$

$$\geq n (\det(D^2 u)^{-1} \det D^2 P)^{\frac{1}{n}}$$

$$\geq n.$$ 

Fix $x_0 \in E$. We compare $u_i$ and

$$v_{x_0}(x) := x_i + \gamma_1 \left[ \delta|x'|^2 + \delta^{1-n}(x_n^2 - \gamma_2 x_n) - (u - l_{x_0}) \right],$$

where $l_{x_0}$ denotes the supporting linear function for $u$ at $x_0$, $\delta = 1/4$, and $\gamma_1$, $\gamma_2 \geq 0$. Clearly,

$$Lv_{x_0} \geq 0,$$

and, since $u$ is Lipschitz in $D$ we can choose $\gamma_1$, $\gamma_2$ large, depending only on $k$ and $n$ such that

$$v_{x_0} \leq u_i \quad \text{on} \quad \partial D.$$ 

This shows that the inequality above holds also in $D$ and we obtain a lower bound on $u_{in}(x_0)$. Similarly we obtain an upper bound.

**Step 3:** We show that

$$\|D^2 u\| \leq C \quad \text{on} \quad \{u < k^2/8\}.$$ 

We apply the classical Pogorelov estimate in the set

$$F := \{u < k^2/4\}. $$
Precisely if the maximal value of
$$\log \left( \frac{1}{4}k^2 - u \right) + \log u_{ii} + \frac{1}{2}u_i^2$$
occurs in the interior of $F$ then this value is bounded by a constant depending only on $n$ and $\max_F |\nabla u|$ (see [C2]). From step 2, the expression is bounded above on $\partial F$ and the estimate follows.

**Step 4.** The Monge-Ampere equation is uniformly elliptic in $\{u < k^2/8\}$ and by Evans-Krylov theorem and Schauder estimates we obtain the desired $C^{3,1}$ bound.

**Remark 6.2.** Assume the boundary values of $u$ are given by
$$\begin{cases}
  u = p(x') & \text{on } \partial \Omega \cap \{x_n = 0\} \\
  u = 1 & \text{on } \partial \Omega \cap \{x_n > 0\},
\end{cases}$$
with $p(x')$ a quadratic polynomial that satisfies
$$\rho|x'|^2 \leq p(x') \leq \rho^{-1}|x'|^2,$$
for some $\rho > 0$. Then
$$\|u\|_{C^{3,1}(\{u < k^2/16\})} \leq C(\rho, k, n).$$
Indeed, after an affine transformation we can reduce the problem to the case $p(x') = |x'|^2/2$.

**Remark 6.3.** Proposition 6.1 holds as well if we replace the half-space $\{x_n \geq 0\}$ with a large ball of radius $\varepsilon^{-1}$
$$B_{\varepsilon} := \{ |x - \varepsilon^{-1}e_n| \leq \varepsilon^{-1} \}.$$
Precisely, if
$$B_k \cap B_\varepsilon \subset \Omega \subset B_{k-1} \cap B_\varepsilon,$$
and the boundary values of $u$ satisfy
$$\begin{cases}
  u = \frac{1}{2}|x'|^2 & \text{on } B_1 \cap \partial B_\varepsilon \subset \partial \Omega \\
  u = [1, 2] & \text{on } \partial \Omega \setminus (B_1 \cap \partial B_\varepsilon),
\end{cases}$$
then for all small $\varepsilon$,
$$\|u\|_{C^{3,1}(\{u < k^2/16\})} \leq C,$$
with $C$ depending only on $k$ and $n$.

The proof is essentially the same except that in the barrier functions $w_{x_0}$, $v_{x_0}$ we need to replace $x_n$ by $(x - x_0) \cdot \nu_{x_0}$ where $\nu_{x_0}$ denotes the inner normal to $\partial \Omega$ at $x_0$, and in step 2 we work (as in [CNS]) with the tangential derivative
$$T_i := (1 - \varepsilon x_n)\partial_{x_i} + \varepsilon x_i\partial_{x_n},$$
instead of $\partial_{x_i}$.

As a consequence of the Proposition 6.1 and the remarks above we obtain

**Theorem 6.4.** Let $u : \Omega \to \mathbb{R}$ satisfy the Monge-Ampere equation
$$\det D^2 u = 1 \quad \text{in } \Omega.$$
Assume that for some constants $\rho, k > 0$,
$$B_k^+ \subset \Omega \subset B_{k-1}^+,$$
and (see Definition 2.5) the boundary values of \( u \) are given by
\[
\begin{cases}
u = p(x') & \text{on } \{ p(x') \leq 1 \} \cap \{ x_n = 0 \} \subset \partial \Omega \\
u = 1 & \text{on the rest of } \partial \Omega,
\end{cases}
\]
where \( p \) is a quadratic polynomial that satisfies
\[
\rho |x'|^2 \leq p(x') \leq \rho^{-1} |x'|^2.
\]
Then
\[
(6.1) \quad \| u \|_{C^{3,1}(B_\rho^+)} \leq c_0^{-1},
\]
with \( c_0 > 0 \) small, depending only on \( k, \rho \) and \( n \).

**Proof.** We approximate \( u \) on \( \partial \Omega \) by a sequence of smooth functions \( u_m \) on \( \partial \Omega_m \), with \( \Omega_m \) smooth, uniformly convex, so that \( u_m, \Omega_m \) satisfy the conditions of Remark 6.3 above. Notice that \( u_m \) is smooth up to the boundary by the results in [CNS], thus we can use Proposition 6.1 for \( u_m \). We let \( m \to \infty \) and obtain (6.1) since \( B_\rho^+ \subset \{ u < k^2/16 \} \), by convexity.

\[ \Box \]

7. **Pointwise \( C^{2,\alpha} \) estimates at the boundary**

Let \( \Omega \) be a bounded convex set with
\[
B_\rho(\rho e_n) \subset \Omega \subset \{ x_n \geq 0 \} \cap B_\rho,
\]
for some small \( \rho > 0 \), that is \( \Omega \subset (\mathbb{R}^n)^+ \) and \( \Omega \) contains an interior ball tangent to \( \partial \Omega \) at 0.

Let \( u : \Omega \to \mathbb{R} \) be convex, continuous, satisfying
\[
det D^2 u = f, \quad 0 < \lambda \leq f \leq \Lambda \quad \text{in } \Omega,
\]
and
\[
x_{n+1} = 0 \text{ is a tangent plane to } u \text{ at } 0,
\]
in the following sense:
\[
u \geq 0, \quad u(0) = 0,
\]
and any hyperplane \( x_{n+1} = \varepsilon x_n, \varepsilon > 0 \) is not a supporting plane for \( u \).

We also assume that on \( \partial \Omega \), in a neighborhood of \( \{ x_n = 0 \} \), \( u \) separates quadratically from the tangent plane \( \{ x_{n+1} = 0 \} \),
\[
(7.4) \quad \rho |x|^2 \leq u(x) \leq \rho^{-1} |x|^2 \quad \text{on } \partial \Omega \cap \{ x_n \leq \rho \}.
\]

Our main theorem is the following.

**Theorem 7.1.** Let \( \Omega, u \) satisfy (7.1) - (7.4) above with \( f \in C^\alpha \) at the origin, i.e
\[ |f(x) - f(0)| \leq M |x|^\alpha \quad \text{in} \quad \Omega \cap B_\rho, \]
for some \( M > 0 \), and \( \alpha \in (0,1) \). Suppose that \( \partial \Omega \) and \( u|_{\partial \Omega} \) are \( C^{2,\alpha} \) at the origin, i.e we assume that on \( \partial \Omega \cap B_\rho \) we satisfy
\[
|x_n - q(x')| \leq M |x'|^{2+\alpha},
\]
\[
|u - p(x')| \leq M |x'|^{2+\alpha},
\]
where \( p(x'), q(x') \) are quadratic polynomials.
Then \( u \in C^{2, \alpha} \) at the origin, that is there exists a quadratic polynomial \( P_0 \) such that
\[
\det D^2 P_0 = f(0), \quad \|D^2 P_0\| \leq C(M), 
\]
such that
\[
|u - P_0| \leq C(M)|x|^{2+\alpha} \quad \text{in} \quad \Omega \cap B_\rho,
\]
where \( C(M) \) depends on \( M, \rho, \lambda, \Lambda, n, \alpha \).

From (7.1) and (7.4) we see that \( p, q \) are homogenous of degree 2 and
\[
\|D^2 p\|, \|D^2 q\| \leq \rho^{-1}.
\]

A consequence of the proof of Theorem 7.1 is that if \( f \in C^{\alpha} \) near the origin, then \( u \in C^{2, \alpha} \) in any cone \( C_\theta \) of opening \( \theta < \pi/2 \) around the \( x_n \)-axis, i.e.
\[
C_\theta := \{ x \in (\mathbb{R}^n)^+ \mid |x'| \leq x_n \tan \theta \}.
\]

**Corollary 7.2.** Assume \( u \) satisfies the hypotheses of Theorem 7.1 and
\[
\|f\|_{C^{\alpha}(\overline{\Omega})} \leq M.
\]
Given any \( \theta < \pi/2 \) there exists \( \delta(M, \theta) \) small, such that
\[
\|u\|_{C^{2, \alpha}(C_\theta \cap B_\delta)} \leq C(M, \theta).
\]

We also mention the global version of Theorem 7.1.

**Theorem 7.3.** Let \( \Omega \) be a bounded, convex domain and let \( u : \overline{\Omega} \rightarrow \mathbb{R} \) be convex, Lipschitz continuous, satisfying
\[
\det D^2 u = f, \quad 0 < \lambda \leq f \leq \Lambda \quad \text{in} \ \Omega.
\]
Assume that
\[
\partial \Omega, \quad u|_{\partial \Omega} \in C^{2, \alpha}, \quad f \in C^\alpha(\overline{\Omega}),
\]
for some \( \alpha \in (0, 1) \) and there exists a constant \( \rho > 0 \) such that
\[
u(y) - u(x) - \nabla u(x) \cdot (y - x) \geq \rho|y - x|^2 \quad \forall x, y \in \partial \Omega,
\]
where \( \nabla u(x) \) is understood in the sense of (7.3). Then \( u \in C^{2, \alpha}(\overline{\Omega}) \) and
\[
\|u\|_{C^{2, \alpha}(\overline{\Omega})} \leq C,
\]
with \( C \) depending on \( \|\partial \Omega\|_{C^{2, \alpha}}, \|u|_{\partial \Omega}\|_{C^{2, \alpha}}, \|u\|_{C^{0, 1}(\overline{\Omega})}, \|f\|_{C^\alpha(\overline{\Omega})}, \rho, \lambda, \Lambda, n, \alpha \).

In general, the Lipschitz bound is easily obtained from the boundary data \( u|_{\partial \Omega} \). We can always do this if for example \( \Omega \) is uniformly convex.

The proof of Theorem 7.1 is similar to the proof of the interior \( C^{2, \alpha} \) estimate from (7.2), and it has three steps. First we use the localization theorem to show that after a rescaling it suffices to prove the theorem only in the case when \( M \) is arbitrarily small (see Lemma 7.4). Then we use Pogorelov estimate in half-domain (Theorem 6.4) and reduce further the problem to the case when \( u \) is arbitrarily close to a quadratic polynomial (see Lemma 7.3). In the last step we use a standard iteration argument to show that \( u \) is well-approximated by quadratic polynomials at all scales.

We assume for simplicity that
\[
f(0) = 1,
\]
otherwise we divide \( u \) by \( f(0) \).

Constants depending on \( \rho, \lambda, \Lambda, n \) and \( \alpha \) are called universal. We denote them by \( C, c \) and they may change from line to line whenever there is no possibility of
confusion. Constants depending on universal constants and other parameters i.e $M, \sigma, \delta$, etc. are denoted as $C(M, \sigma, \delta)$.

We denote linear functions by $l(x)$ and quadratic polynomials which are homogeneous and convex we denote by $p(x'), q(x'), P(x)$.

The localization theorem says that the section $S_h$ is comparable to an ellipsoid $E_h$ which is obtained from $B_{h^{1/2}}$ by a sliding along $\{x_n = 0\}$. Using an affine transformation we can normalize $S_h$ so that it is comparable to $B_1$. In the next lemma we show that, if $h$ is sufficiently small, the corresponding rescaling $u_h$ satisfies the hypotheses of $u$ in which the constant $M$ is replaced by an arbitrary small constant $\sigma$.

**Lemma 7.4.** Given any $\sigma > 0$, there exist small constants $h = h_0(M, \sigma)$, $k > 0$ depending only on $\rho, \lambda, \Lambda, n$, and a rescaling of $u$

$$u_h(x) := \frac{u(h^{1/2}A_h^{-1} x)}{h}$$

where $A_h$ is a linear transformation with

$$\det A_h = 1, \quad ||A_h^{-1}||, ||A_h|| \leq k^{-1} |\log h|.$$ 

so that

a) $B_k \cap \bar{\Omega}_h \subset S_1(u_h) \subset B_{k^{-1}}^{+}, \quad S_1(u_h) := \{u_h < 1\}$,

b) $\det D^2 u_h = f_h, \quad |f_h(x) - 1| \leq \sigma |x|^{\alpha}$ in $\Omega_h \cap B_{k^{-1}}$,

c) On $\partial \Omega_h \cap B_{k^{-1}}$ we have

$$|x_n - q_h(x')| \leq \sigma |x'|^{2+\alpha}, \quad |q_h(x')| \leq \sigma,$$

$$|u_h - p(x')| \leq \sigma |x'|^{2+\alpha},$$

where $q_h$ is a quadratic polynomial.

**Proof.** By the localization theorem Theorem 3.1 for all $h \leq c$,

$$S_h := \{u < h\} \cap \bar{\Omega},$$

satisfies

$$k E_h \cap \bar{\Omega} \subset S_h \subset k^{-1} E_h,$$

with

$$E_h = A_h^{-1} B_{h^{1/2}}, \quad A_h x = x - \nu_h x_n$$

$$\nu_h \cdot e_n = 0, \quad ||A_h^{-1}||, ||A_h|| \leq k^{-1} |\log h|.$$ 

Then we define $u_h$ as above and obtain

$$S_1(u_h) = h^{-1/2} A_h S_h,$$

hence

$$B_k \cap \bar{\Omega}_h \subset S_1(u_h) \subset B_{k^{-1}}^{+},$$

where

$$\Omega_h := h^{-1/2} A_h \Omega.$$ 

Then

$$\det D^2 u_h = f_h(x) = f(h^{1/2} A_h^{-1} x),$$

where $f_h$ is a quadratic polynomial.
and

$$|f_h(x) - 1| \leq M|h^{1/2}A_h^{-1}x|^\alpha \leq M(h^{1/2}k^{-1}|\log h|)^\alpha|x|^{\alpha} \leq \sigma|x|^{\alpha}$$

if $h_0(M, \sigma)$ is sufficiently small.

Next we estimate $|x_n - h^{1/2}q(x')|$ and $|u_n - p(x')|$ on $\partial\Omega_h \cap B_{k^{-1}}$. We have

$$x \in \partial\Omega_h \Leftrightarrow y := h^{1/2}A_h^{-1}x \in \Omega,$$

or

$$h^{1/2}x_n = y_n, \quad h^{1/2}x' = y' - \nu_h y_n.$$

If $|x| \leq k^{-1}$ then

$$|y| \leq k^{-1}h^{1/2}|\log h||x| \leq h^{1/4},$$

if $h_0$ is small hence, since $\Omega$ has an interior tangent ball of radius $\rho$, we have

$$|y_n| \leq \rho^{-1}|y'|^2.$$

Then

$$|\nu_h y_n| \leq k^{-1}|\log h||y'|^2 \leq |y'|/2,$$

thus

$$\frac{1}{2}|y'| \leq |h^{1/2}x'| \leq \frac{3}{2}|y'|.$$

We obtain

$$|x_n - h^{1/2}q(x')| \leq h^{-1/2}|y_n - q(y')| + h^{1/2}|q(h^{-1/2}y') - q(x')|$$

$$\leq Mh^{-1/2}|y'|^{2+\alpha} + Ch^{1/2}(|x'||^\nu_h x_n| + |\nu_h x_n|^2)$$

$$\leq 2Mh^{(\alpha+1)/2}|x'|^{2+\alpha} + Ch^{1/2}\left(h^{1/2}|\log h||x'|^3 + h|\log h|^2|x'|^4\right)$$

$$\leq \sigma|x'|^{2+\alpha},$$

if $h_0$ is chosen small. Hence on $\partial\Omega_h \cap B_{k^{-1}}$,

$$|x_n - q_h(x')| \leq \sigma|x'|^{2+\alpha}, \quad q_h := h^{1/2}q(x'),$$

and also

$$|u_h - p(x')| \leq h^{-1}|u(y) - p(y')| + |p(h^{-1/2}y') - p(x')|$$

$$\leq Mh^{-1}|y'|^{2+\alpha} + C(|x'||^\nu_h x_n| + |\nu_h x_n|^2)$$

$$\leq 2Mh^{\alpha/2}|x'|^{2+\alpha} + C\left(h^{1/2}|\log h||x'|^3 + h|\log h|^2|x'|^4\right)$$

$$\leq \sigma|x'|^{2+\alpha}.$$

□

In the next lemma we show that if $\sigma$ is sufficiently small, then $u_h$ can be well-approximated by a quadratic polynomial near the origin.
Lemma 7.5. For any $\delta_0, \varepsilon_0$ there exist $\sigma_0(\delta_0, \varepsilon_0), \mu_0(\varepsilon_0)$ such that for any function $u_h$ satisfying properties a), b), c) of Lemma 7.4 with $\sigma \leq \sigma_0$ we can find a rescaling

$$\tilde{u}(x) := \frac{(u_h - l_h)(\mu_0 x)}{\mu_0^2},$$

with

$$l_h(x) = \gamma_h x_n, \quad |\gamma_h| \leq C_0, \quad C_0 \text{ universal,}$$

that satisfies

a) in $\tilde{\Omega} \cap B_1$,

$$\det D^2 \tilde{u} = \tilde{f}, \quad |\tilde{f}(x) - 1| \leq \delta_0 \varepsilon_0 |x|^{\alpha} \quad \text{in} \quad \tilde{\Omega} \cap B_1,$$

and

$$|\tilde{u} - P_0| \leq \varepsilon_0 \quad \text{in} \quad \tilde{\Omega} \cap B_1,$$

for some $P_0$, quadratic polynomial,

$$\det D^2 P_0 = 1, \quad \|D^2 P_0\| \leq C_0;$$

b) On $\partial \tilde{\Omega} \cap B_1$ there exist $\tilde{p}_0, \tilde{q}_0$ such that

$$|x_n - \tilde{q}_0(x')| \leq \delta_0 \varepsilon_0 |x'|^{2+\alpha}, \quad |\tilde{q}_0(x')| \leq \delta_0 \varepsilon_0,$$

and

$$|\tilde{u} - \tilde{p}_0(x')| \leq \delta_0 \varepsilon_0 |x'|^{2+\alpha}, \quad \tilde{p}_0(x') = P_0(x'), \quad \frac{\rho}{2} |x'|^2 \leq \tilde{p}_0(x') \leq 2\rho |x'|^2.$$

Proof. We prove the lemma by compactness. Assume by contradiction that the statement is false for a sequence $u_m$ satisfying a), b), c) of Lemma 7.4 with $\sigma_m \to 0$. Then, we may assume after passing to a subsequence if necessary that

$$p_m \to p_\infty, \quad q_m \to 0 \quad \text{uniformly on} \quad B_{k-1},$$

and

$$u_m : S_1(u_m) \to \mathbb{R}$$

converges to (see Definition 2.3)

$$u_\infty : \Omega_\infty \to \mathbb{R}.$$

Then, by Theorem 2.3 $u_\infty$ satisfies

$$B_k^+ \subset \Omega_\infty \subset B_{k-1}^+, \quad \det D^2 u_\infty = 1,$$

$$\left\{ \begin{array}{ll} u_\infty = p_\infty(x') & \text{on} \quad \{p_\infty(x') < 1\} \cap \{x_n = 0\} \subset \partial \Omega_\infty \\ u_\infty = 1 & \text{on the rest of} \quad \partial \Omega_\infty. \end{array} \right.$$}

From Pogorelov estimate in half-domain (Theorem 6.6) there exists $c_0$ universal such that

$$|u_\infty - l_\infty - P_\infty| \leq c_0^{-1} |x|^3 \quad \text{in} \quad B_{c_0}^+, \quad$$

where

$$l_\infty := \gamma_\infty x_n, \quad |\gamma_\infty| \leq c_0^{-1},$$

and $P_\infty$ is a quadratic polynomial such that

$$P_\infty(x') = p_\infty(x'), \quad \det D^2 P_\infty = 1, \quad \|D^2 P_\infty\| \leq c_0^{-1}.$$
Choose $\mu_0$ small such that 
\[ \varepsilon_0^{-1} \mu_0 = \varepsilon_0 / 32, \]
hence 
\[ |u_\infty - l_\infty - P_\infty| \leq \frac{1}{4} \varepsilon_0 \mu_0^2 \quad \text{in} \quad B_{2\mu_0}^+, \]
which together with $p_m \to p_\infty$ implies that for all large $m$ 
\[ |u_m - l_\infty - P_\infty| \leq \frac{1}{2} \varepsilon_0 \mu_0^2 \quad \text{in} \quad S_1(u_m) \cap B_{\mu_0}^+. \]
Then, for all large $m$, 
\[ \tilde{u}_m := \frac{(u_m - l_\infty)(\mu_0 x)}{\mu_0^2} \]
satisfies in $\tilde{\Omega}_m \cap B_1$ 
\[ |\tilde{u}_m - P_\infty| \leq \varepsilon_0 / 2, \]
and 
\[ \det D^2 \tilde{u}_m = \tilde{f}_m(x) = f_m(\mu_0 x), \]
\[ |\tilde{f}_m(x) - 1| \leq \sigma_m(\mu_0|x|)^\alpha \leq \delta_0 \varepsilon_0 |x|^\alpha. \]
We define 
\[ \tilde{q}_m := \mu_0 q_m, \quad \tilde{p}_m := p_m - \gamma_{\infty} q_m, \]
and clearly 
\[ \tilde{p}_m \to p_\infty, \quad \tilde{q}_m \to 0 \quad \text{uniformly in} \quad B_1. \]
On $\partial \tilde{\Omega}_m \cap B_1$ we have 
\[ |x_n - \tilde{q}_m(x')| = \mu_0^{-1} |\mu_0 x_n - q_m(\mu_0 x')| \]
\[ \leq \mu_0^{-1} \sigma_m |\mu_0 x'|^{2+\alpha} \]
\[ \leq \delta_0 \varepsilon_0 |x'|^{2+\alpha}, \]
and 
\[ |\tilde{u}_m - \tilde{p}_m(x')| = \mu_0^{-2} |(u_m - l_\infty)(\mu_0 x) - p_m(\mu_0 x') + \gamma_{\infty} q_m(\mu_0 x')| \]
\[ \leq \mu_0^{-2} (|(u_m - p_m)(\mu_0 x')| + |\gamma_{\infty}||\mu_0 x_n - q_m(\mu_0 x')|) \]
\[ \leq \sigma_m \mu_0^\alpha (1 + |\gamma_{\infty}|)|x'|^{2+\alpha} \]
\[ \leq \delta_0 \varepsilon_0 |x'|^{2+\alpha}. \]
Finally, we let $P_m$ be a perturbation of $P_\infty$ such that 
\[ P_m(x') = \tilde{p}_m(x'), \quad \det D^2 P_m = 1, \quad P_m \to P_\infty \quad \text{uniformly in} \quad B_1. \]
Then $\tilde{u}_m, P_m, \tilde{p}_m, \tilde{q}_m$ satisfy the conclusion of the lemma for all large $m$, and we reached a contradiction. \[ \square \]

From Lemma 7.4 and Lemma 7.5 we see that given any $\delta_0, \varepsilon_0$ there exist a linear transformation 
\[ T := \mu_0 h_0^{1/2} A_{h_0}^{-1} \]
and a linear function 
\[ l(x) := \gamma x_n \]
with 
\[ |\gamma|, \|T^{-1}\|, \|T\| \leq C(M, \delta_0, \varepsilon_0), \]
such that the rescaling
\[ \tilde{u}(x) := \frac{(u - l)(Tx)}{(\det T)^{2/n}}, \]
defined in \( \tilde{\Omega} \subset \mathbb{R}^n \) satisfies
1) in \( \tilde{\Omega} \cap B_1 \)
\[ \det D^2 \tilde{u} = \tilde{f}, \quad |\tilde{f} - 1| \leq \delta_0 \epsilon_0 |x|^{\alpha}, \]
and
\[ |\tilde{u} - P_0| \leq \epsilon_0, \]
for some \( P_0 \) with
\[ \det D^2 P_0 = 1, \quad \|D^2 P_0\| \leq C_0; \]
2) on \( \partial \tilde{\Omega} \cap B_1 \) we have \( \tilde{p}, \tilde{q} \) so that
\[ |x_n - \tilde{q}(x')| \leq \delta_0 \epsilon_0 |x'|^{2+\alpha}, \quad |\tilde{q}(x')| \leq \delta_0 \epsilon_0, \]
\[ |\tilde{u} - \tilde{p}(x')| \leq \delta_0 \epsilon_0 |x'|^{2+\alpha}, \quad \frac{\tilde{P}}{2} |x'|^2 \leq \tilde{p}(x') = P_0(x') \leq 2\tilde{P}|x'|^2. \]
By choosing \( \delta_0, \epsilon_0 \) appropriately small, universal, we show in Lemma 7.6 that there exist \( \tilde{l}, \tilde{P} \) such that
\[ |\tilde{u} - \tilde{l} - \tilde{P}| \leq C|x|^{2+\alpha} \text{ in } \tilde{\Omega} \cap B_1, \quad |\nabla \tilde{l}|, \|D^2 \tilde{P}\| \leq C, \]
with \( C \) a universal constant. Rescaling back, we obtain that \( u \) is well approximated by a quadratic polynomial at the origin i.e
\[ |u - l - P| \leq C(M)|x|^{2+\alpha} \text{ in } \Omega \cap B_r, \quad |\nabla l|, \|D^2 P\| \leq C(M) \]
which, by (7.3), proves Theorem 7.1

Since \( \alpha \in (0, 1) \), in order to prove that \( \tilde{u} \in C^{2,\alpha}(0) \) it suffices to show that \( \tilde{u} \) is approximated of order \( 2 + \alpha \) by quadratic polynomials \( l_m + P_m \) in each ball of radius \( r_0^m \) for some small \( r_0 > 0 \), and then \( \tilde{l} + \tilde{P} \) is obtained in the limit as \( m \to \infty \) (see [C2], [C]). Thus Theorem 7.1 follows from the next lemma.

Lemma 7.6. Assume \( \tilde{u} \) satisfies the properties 1), 2) above. There exist \( \epsilon_0, \delta_0, r_0 \) small, universal, such that for all \( m \geq 0 \) we can find \( l_m, P_m \) so that
\[ |\tilde{u} - l_m - P_m| \leq \epsilon_0 r^{2+\alpha} \text{ in } \tilde{\Omega} \cap B_r, \quad \text{with } r = r_0^m. \]

Proof. We prove by induction on \( m \) that the inequality above is satisfied with
\[ l_m = \gamma_m x_n, \quad |\gamma_m| \leq 1, \]
\[ P_m(x') = \tilde{p}(x') - \gamma_m \tilde{q}(x'), \quad \det D^2 P_m = 1, \quad \|D^2 P_m\| \leq 2C_0. \]
From properties 1), 2) above we see that this holds for \( m = 0 \) with \( \gamma_0 = 0 \).

Assume the conclusion holds for \( m \) and we prove it for \( m + 1 \). Let
\[ v(x) := \frac{(\tilde{u} - l_m)(rx)}{r^2}, \quad \text{with } r := r_0^m, \]
and define
\[ \epsilon := \epsilon_0 r^{\alpha}. \]
Then
\[ |v - P_m| \leq \epsilon \text{ in } \Omega_v \cap B_1, \quad \Omega_v := r^{-1}\tilde{\Omega}, \]
\[ |\det D^2 v - 1| = |f(rx) - 1| \leq \delta_0 \epsilon. \]
On $\partial \Omega_v \cap B_1$ we have
\[
\left| \frac{x_n}{r} - \tilde{q}(x') \right| = r^{-2}|rx_n - \tilde{q}(rx')| \\
\leq \delta_0 \varepsilon |x'|^{2+\alpha}, \\
\leq \delta_0 \varepsilon.
\]
which also gives
\[
(x_n) \leq 2\delta_0 \varepsilon \quad \text{on} \quad \partial \Omega_v \cap B_1.
\]
From the definition of $v$ and the properties of $P_m$ we see that in $B_1$
\[
|v - P_m| \leq r^{-2}|(\tilde{u} - \tilde{p})(rx)| + |\gamma_m||x_n/r - \tilde{q}| + 2nC_0|x_n|,
\]
and the inequalities above and property 2) imply
\[
|v - P_m| \leq C_1 \delta_0 \varepsilon \quad \text{in} \quad \partial \Omega_v \cap B_1,
\]
with $C_1$ universal constant (depending only on $n$ and $C_0$).

We want to compare $v$ with the solution
\[
w : B_{1/8}^+ \to \mathbb{R}, \quad \det D^2w = 1,
\]
which has the boundary conditions
\[
\begin{align*}
& \{ \begin{array}{ll}
  w = v & \text{on} \quad \partial B_{1/8}^+ \cap \Omega_v, \\
  w = P_m & \text{on} \quad \partial B_{1/8}^+ \setminus \Omega_v.
\end{array} \}
\end{align*}
\]

In order to estimate $|u - w|$ we introduce a barrier $\phi$ defined as
\[
\phi : \tilde{B}_{1/2} \setminus B_{1/4} \to \mathbb{R}, \quad \phi(x) := c(\beta) \left(4^\beta - |x|^{-\beta}\right),
\]
where $c(\beta)$ is chosen such that $\phi = 1$ on $\partial B_{1/2}$ and $\phi = 0$ on $\partial B_{1/4}$.

We choose the exponent $\beta > 0$ depending only on $C_0$ and $n$ such that for any symmetric matrix $A$
\[
(2C_0)^{1-n}I \leq A \leq (2C_0)^{n-1}I,
\]
we have
\[
\text{Tr} \ A(D^2\phi) \leq -\eta_0 < 0,
\]
for some $\eta_0$ small, depending only on $C_0$ and $n$.

For each $y$ with $y_n = -1/4$, $|y| \leq 1/8$ the function
\[
\phi_y(x) := P_m + \varepsilon(C_1 \delta + \phi(x - y))
\]
satisfies
\[
\det D^2\phi_y \leq 1 - \frac{\eta_0}{2} \varepsilon \quad \text{in} \quad B_{1/2}(y) \setminus B_{1/4}(y),
\]
if $\varepsilon \leq \varepsilon_0$ is sufficiently small. From (7.5), (7.7) we see that
\[
v \leq \phi_y \quad \text{on} \quad \partial(\Omega_v \cap B_{1/2}(y)),
\]
and if $\delta_0 \leq \eta_0/2$,
\[
\det D^2v \geq \det D^2\phi_y.
\]
This gives
\[
v \leq \phi_y \quad \text{in} \quad \Omega_v \cap B_{1/2}(y),
\]
and using the definition of $w$ we obtain
\[
w \leq \phi_y \quad \text{on} \quad \partial B_{1/8}^+.
\]
The maximum principle yields
\[ w \leq \phi_y \quad \text{in} \quad B^+_{1/8}, \]
and by varying \( y \) we obtain
\[ w(x) \leq P_m + \varepsilon(C_1\delta_0 + Cx_n) \quad \text{in} \quad B^+_{1/8}. \]
Recalling (7.6), this implies
\[ w - P_m \leq 2C_1\delta_0 \varepsilon \quad \text{on} \quad B^+_{1/8} \setminus \Omega_v. \]
The opposite inequality holds similarly, hence
\[ |w - P_m| \leq 2C_1\delta_0 \varepsilon \quad \text{on} \quad B^+_{1/8} \setminus \Omega_v. \]
From the definition of \( w \) and (7.7) we also obtain
\[ |v - w| \leq 3C_1\delta_0 \varepsilon \quad \text{on} \quad \partial(\Omega_v \cap B^+_{1/8}). \]
Now we claim that
\[ |v - w| \leq C_2\delta_0 \varepsilon \quad \text{in} \quad \Omega_v \cap B^+_{1/8}, \quad C_2 \text{ universal}. \]
For this, we use the following inequality. If \( A \geq 0 \) is a symmetric matrix with
\[ 1/2 \leq \det A \leq 2, \]
and \( a \geq 0 \), then
\[ \det(A + aI) = \det A \det(I + aA^{-1}) \]
\[ \geq \det A(1 + \text{Tr}(aA^{-1})) \]
\[ \geq \det A(1 + a/2) \]
\[ \geq \det A + a/4. \]
This and (7.9) give that in \( \Omega_v \cap B^+_{1/8} \)
\[ w + 2\delta_0 \varepsilon(|x|^2 - 2C_1) \leq v, \]
\[ v + 2\delta_0 \varepsilon(|x|^2 - 2C_1) \leq w, \]
and the claim (7.10) is proved.

Next we approximate \( w \) by a quadratic polynomial near 0. From (7.6), (7.8), (7.10) we can conclude that
\[ |w - P_m| \leq 2\varepsilon \quad \text{in} \quad B^+_{1/8}, \]
if \( \delta_0 \) is sufficiently small. Since \( w = P_m \) on \( \{x_n = 0\} \), and
\[ \frac{P}{4}|x'|^2 \leq P_m(x') \leq 4\rho|x'|^2, \quad \det D^2P_m = 1, \quad \|D^2P_m\| \leq 2C_0, \]
we conclude from Pogorelov estimate (Theorem 6.4) that
\[ \|D^2w\|_{C^{1,1}(B^+_{c_0})} \leq c_0^{-1}, \]
for some small universal constant \( c_0 \). Thus in \( B^+_{c_0} \), \( w - P_m \) solves a uniformly elliptic equation
\[ \text{Tr } A(x)D^2(w - P_m) = 0, \]
with the \( C^{1,1} \) norm of the coefficients \( A(x) \) bounded by a universal constant. Since
\[ w - P_m = 0 \quad \text{on} \quad \{x_n = 0\}, \]
we obtain 
\[ \|w - P_m\|_{C^{1,1}(B_{r_0/2})} \leq C_3\|w - P_m\|_{L^\infty(B_{r_0}^+)} \leq 2C_3\varepsilon, \]
with \(C_3\) a universal constant. Then 
\[ |w - P_m - \hat{\gamma}_m - \hat{\gamma}_m| \leq 2C_3\varepsilon|x|^3 \text{ if } |x| \leq c_0/2, \]
with 
\[ \hat{\gamma}_m(x') = 0, \quad \hat{\gamma}_m = \hat{\gamma}_m x_n, \quad |\hat{\gamma}_m|, \|D^2\hat{\gamma}_m\| \leq 2C_3\varepsilon. \]
Since \(\tilde{l}_m + P_m + \hat{\gamma}_m\) is the quadratic expansion for \(w\) at 0 we also have 
\[ \det D^2(P_m + \hat{\gamma}_m) = 1. \]
We define 
\[ P_{m+1}(x) := P_m(x) + \hat{\gamma}_m(x') + \sigma_m x_n^2 \]
with \(\sigma_m\) so that 
\[ \det D^2P_{m+1} = 1, \]
and let 
\[ l_{m+1}(x) := \gamma_{m+1} x_n, \quad \gamma_{m+1} = \gamma_m + r\hat{\gamma}_m. \]
Notice that 
\[ |\gamma_{m+1} - \gamma_m|, \|D^2P_{m+1} - D^2P_m\| \leq C_4\varepsilon = C_4\varepsilon_0 r_0^{m_0}, \]
and 
\[ \|D^2P_{m+1} - D^2(P_m + \hat{\gamma}_m)\| \leq C_4\delta_0\varepsilon, \]
for some \(C_4\) universal. From the last inequality and \(7.10\) we find 
\[ |v - \tilde{l}_m - P_{m+1}| \leq (2C_3r_0^3 + C_2\delta_0 + C_4\delta_0)\varepsilon \text{ in } \Omega \cap B_{r_0}^+. \]
This gives 
\[ |v - \tilde{l}_m - P_{m+1}| \leq \varepsilon r_0^{2+\alpha} \text{ in } \Omega \cap B_{r_0}^+, \]
if we first choose \(r_0\) small (depending on \(C_3\)) and then \(\delta_0\) depending on \(r_0, C_2, C_4\), hence 
\[ |\tilde{u} - l_{m+1} - P_{m+1}| \leq \varepsilon r_0^{2+\alpha} = \varepsilon_0(r_0)^{\alpha} \text{ in } \Omega \cap B_{rr_0}^+. \]
Finally we choose \(\varepsilon_0\) small such that \(7.12\) and 
\[ \gamma_0 = 0, \quad \|D^2P_0\| \leq C_0, \]
guarantee that 
\[ |\gamma_m| \leq 1, \quad \|D^2P_m\| \leq 2C_0 \]
for all \(m\). This shows that the induction hypotheses hold for \(m + 1\) and the lemma is proved. 

\[ \square \]

**Remark 7.7.** The proof of Lemma \(7.6\) applies also at interior points. More precisely, if \(\tilde{u}\) satisfies the hypotheses in \(B_1(x_0) \subset \Omega\) instead of \(B_1 \cap \Omega\) then the conclusion holds in \(B_1(x_0)\). The proof is in fact simpler since, in this case we take \(w\) so that 
\[ w = v \text{ on } \partial B_1(x_0), \]
and then \(7.9\) is automatically satisfied, so there is no need for the barrier \(\phi\). Also, at the end we apply the classical interior estimate of Pogorelov instead of the estimate in half-domain.
Now we can sketch a proof of Corollary 7.2 and Theorem 7.3. If $u$ satisfies the conclusion of Theorem 7.1 then, after an appropriate dilation, any point in $C_{\theta} \cap B_{\delta}$ becomes an interior point $x_0$ as in Remark 7.7 above for the rescaled function $\tilde{u}$. Moreover, the hypotheses of Lemma 7.6 hold in $B_1(x_0)$ for some appropriate $\varepsilon \leq \varepsilon_0$. Then Corollary 7.2 follows easily from Remark 7.7.

If $u$ satisfies the hypotheses of Theorem 7.3 then we obtain as above that

$$\|u\|_{C^{2,\alpha}(D_\delta)} \leq C, \quad D_\delta := \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \leq \delta \},$$

for some $\delta$ and $C$ depending on the data. We combine this with the interior $C^{2,\alpha}$ estimate of Caffarelli in [C2] and obtain the desired bound.

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