A modular equality for $m$-ovoids of elliptic quadrics

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Abstract
An $m$-ovoid of a finite polar space $P$ is a set $\mathcal{O}$ of points such that every maximal subspace of $P$ contains exactly $m$ points of $\mathcal{O}$. In the case when $P$ is an elliptic quadric $Q^- (2r + 1, q)$ of rank $r$ in $\mathbb{F}_q^{2r+2}$, we prove that an $m$-ovoid exists only if $m$ satisfies a certain modular equality, which depends on $q$ and $r$. This condition rules out many of the possible values of $m$. Previously, only a lower bound on $m$ was known, which we slightly improve as a byproduct of our method. We also obtain a characterization of the $m$-ovoids of $Q^- (7, q)$ for $q = 2$ and $(m, q) = (4, 3)$.

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1 | INTRODUCTION

Let $\Gamma$ be a finite connected regular graph on $v$ vertices and of valency $k$, $Y$ a proper subset of the vertex set of $\Gamma$. If $\theta^+$ and $\theta^-$ denote the second largest and least eigenvalues of $\Gamma$, respectively, then the number $N$ of ordered pairs of adjacent vertices of $Y$ satisfies (see [11, Proposition 3.8; 13, Theorem 2.1])

$$\theta^- |Y| + \frac{k-\theta^-}{v} |Y|^2 \leq N \leq \theta^+ |Y| + \frac{k-\theta^+}{v} |Y|^2. \tag{1.1}$$

The case of equality in Equation (1.1) often gives rise to interesting combinatorial objects; in particular, when $\Gamma$ is related to incidence structures in finite geometry.

Let $PG(n, q)$ denote the projective space of dimension $n$ with underlying vector space $V := \mathbb{F}_q^{n+1}$ over the finite field $\mathbb{F}_q$ with $q$ elements. For a nondegenerate quadratic (or reflexive...
sesquilinear) form $f$ on $V$, the classical polar space $P$ associated with $f$ is the incidence structure formed by the totally singular (or totally isotropic, respectively) subspaces with respect to $f$; their incidence is defined by symmetrized containment $[8,19]$. We consider the elements of $P$ as subspaces of $\text{PG}(n, q)$, so they are projective points, lines, and so on. A maximal subspace of $P$ has dimension $r - 1$, where $r$ is the Witt index of $f$, also called the rank of $P$; such a subspace is called a generator. (For further details, the reader is referred to Section 2.)

A set $\mathcal{O}$ of points of $P$ is an $m$-ovoid if every generator of $P$ meets $\mathcal{O}$ in exactly $m$ points. Equivalently, if $\Gamma$ is the collinearity graph of a finite polar space $P$, then a set $Y$ that attains equality in the left-hand inequality of Equation (1.1) is an $m$-ovoid of $P$, for some natural $m$ (see $[3,11]$). The notion of $m$-ovoids, which goes back to a classical work of B. Segre [24], was introduced by Thas for generalized quadrangles in [26], and extended to finite polar spaces of higher rank in [25]. A set of points of $P$ is called tight [12] if it attains equality in the right-hand side of Equation (1.1). In [3], a uniform algebraic framework for ovoids and tight sets was developed, and their connections with various geometric objects were explored.

The central problem concerning $m$-ovoids in a polar space $P$ is to determine the values of $m$, for which $P$ possesses an $m$-ovoid. In this paper, we focus on the elliptic polar spaces (quadrics) $Q^-(2r + 1, q)$ of rank $r$, which arise from a nondegenerate orthogonal form of Witt index $r$ in the $(2r + 2)$-dimensional vector space over $\mathbb{F}_q$. An $m$-ovoid of $Q^-(2r + 1, q)$ has $m(q^{r+1} + 1)$ points, and it is said to be trivial if it is empty ($m = 0$) or consists of all points of the space ($m = q^r - 1$). As the complement of an $m$-ovoid is an $(\frac{q^r - 1}{q-1} - m)$-ovoid, one may assume $m \leq \frac{q^r - 1}{2(q-1)}$.

In 1965 Segre [24, p. 162] proved that if an elliptic quadric $Q^-(5, q)$ of $\text{PG}(5, q)$, $q$ odd, has an $m$-ovoid, then $m = \frac{q + 1}{2}$, and he called such a $(\frac{q+1}{2})$-ovoid a hemisystem. He also constructed a hemisystem of $Q^-(5, 3)$, admitting a group isomorphic to $\text{PSL}(3,4)$. In [7], by extending Segre’s result, it was shown that $Q^-(5, q)$, $q$ even, possesses no nontrivial $m$-ovoids, see also [18, section 19.3]. Several constructions of hemisystems of $Q^-(5, q)$ have been presented in the literature in the last 15 years [1, 2, 4, 9, 10, 23]. However, not much is known about $m$-ovoids of $Q^-(2r + 1, q)$ for $r > 2$. It was shown in [3, Theorem 13] that if an $m$-ovoid of $Q^-(2r + 1, q)$ exists, then $m \geq (\sqrt{4q^{r+1} + 9} - 3)/(2q - 2)$ (see also Remark 1).

Here, we obtain the following (nonexistence) result for $m$-ovoids of elliptic quadrics.

**Theorem 1.1.** If $Q^-(2r + 1, q)$ possesses an $m$-ovoid, then

$$F(m) \equiv 0 \pmod{q + 1},$$

where

$$F(m) = \begin{cases} m^2 - m & \text{if } r \text{ is odd}, \\ m^2 & \text{if } r \text{ is even and } q \text{ is even}, \\ m^2 + \frac{q+1}{2}m & \text{if } r \text{ is even and } q \text{ is odd}. \end{cases}$$

An analysis of Equation (1.2) in Section 3 shows that the admissible values of $m$ for an $m$-ovoid of $Q^-(2r + 1, q)$ are asymptotically rare.

Note that certain properties of $m$-ovoids of elliptic quadrics mirror those of tight sets of hyperbolic quadrics $Q^+(2r + 1, q)$; in fact, a result of a similar spirit as Theorem 1.1 was shown for tight sets of hyperbolic quadrics [15–17]. Tight sets, the counterpart of $m$-ovoids, have been studied intensely in recent years (perhaps, with the main focus on those in the hyperbolic quadric.
Q+(5,q), also known as the Cameron–Liebler line classes in PG(3,q), see [14] and references therein).

The proof of Theorem 1.1 occupies Sections 2 and 3, and it is based on the following approach. Fix a maximal flag $P_0 \subset \ell_0 \subset \pi_0 \subset \ldots \subset \Pi_0$ in $Q^{-}(2r+1,q)$ and define a sequence of quotient polar spaces: $Q_{r-1}$ in $P_0^\perp / P_0$, $Q_{r-2}$ in $\ell_0^\perp / \ell_0$, and so on, induced by $Q_r := Q^{-}(2r+1,q)$. Suppose that $\mathcal{O}$ is an $m$-ovoid of $Q_r$, and $\mu_0 : Q_r \rightarrow \mathbb{Z}$ is the characteristic function of $\mathcal{O}$. In Section 2, we show that $\mu_0$ induces a function $\mu_1 : Q_{r-1} \rightarrow \mathbb{Z}$, called a weighted ovoid of $Q_{r-1}$, which in some sense generalizes the notion of an $m$-ovoid. Furthermore, such a function $\mu_i$ induces a weighted ovoid $\mu_{i+1} : Q_{r-i-1} \rightarrow \mathbb{Z}$, for every $i = 1, \ldots, r-2$. Put $\| \mu_i \|^2 := \sum_{P \in Q_{r-i}} (\mu_i(P))^2$, and note that $\| \mu_0 \|^2$ is simply equal to $|\mathcal{O}| = m(q^{r+1} + 1)$. We prove that, for $i = 0, \ldots, r-2$, $\| \mu_i \|^2$ can be expressed via $\| \mu_{i+1} \|^2$. Arguing by induction on $i$ for $i = 1, \ldots, r-2$ implies that $\| \mu_i \|^2 \equiv G(q,r,m) \pmod{q+1}$, where $G$ is a certain function. On the other hand, as $\| \mu_0 \|^2 = m(q^{r+1} + 1)$, applying the induction step with $i = 0$ shows that $\| \mu_1 \|^2$ is another function in $q,r,m$, say $\| \mu_1 \|^2 = E(q,r,m) \pmod{q+1}$ should have an integer solution in $m$, which, as shown in Section 3, gives the conclusion of Theorem 1.1.

Finally, in Section 4, by using the technique developed in Section 2, we slightly improve the above-mentioned lower bound for $m$ from [3, Theorem 13]. We also provide a complete classification of the $m$-ovoids of $Q^{-}(7,2)$ and a characterization of the 4-ovoids of $Q^{-}(7,3)$.

**Remark 1.** In [22], it is shown that the so-called field reduction allows one to construct:
- an $(m^{2e-1}q^{-1}-1,q)$-ovoid of $Q^{-}(2e(r+1) - 1,q)$ from an $m$-ovoid of $Q^{-}(2r + 1,q^e)$,
- an $(m^{2e-1}q^{-1})$-ovoid of $Q^{-}(2e(2r+1) - 1,q)$ from an $m$-ovoid of a Hermitian variety $H(2r,q^{2e})$.

However, apart from a $(q^{(4r+2)/3}-1)/q^2-1)$-ovoid of $H(2r,q^2)$, $r \equiv 1 \pmod{3}$, see [19, Corollary 7.39], which in turn is obtained by the field reduction from a 1-ovoid of $H(2,q^{(4r+2)/3})$, no nontrivial $m$-ovoids of $H(2r,q^2)$ are known to exist. Thus, the only examples of nontrivial $m$-ovoids of elliptic quadrics of rank at least 3 arise by applying the field reduction to all points of $Q^{-}(2r+1,q^e)$, $r \geq 1, e \geq 2$, or to a hemisystem of $Q^{-}(5,q^e), e \geq 2$, or to all points of $H(2r,q^{2e}), r \geq 1, e \geq 2$.

**2 | PRELIMINARY RESULTS**

In this section, we prepare technical results needed for the proof of Theorem 1.1. First, we recall some basic properties of elliptic polar spaces; further details can be found in [8, 19].

Let us consider an elliptic polar space (quadric) $Q_r := Q^{-}(2r+1,q)$ of rank $r \geq 1$, formed by the set of projective points of PG$(2r+1,q)$ satisfying $f(x) = 0$, where

$$f(x) := f(x_0, \ldots, x_{2r+1}) = x_0x_1 + \cdots + x_{2r-2}x_{2r-1} + g(x_{2r}, x_{2r+1}), \quad x \in \mathbb{F}_q^{2r+2},$$

and $g$ is a homogeneous irreducible polynomial of degree 2 over $\mathbb{F}_q$. The number of points in $Q_r$ is

$$k_r := \frac{(q^r-1)(q^{r+1} + 1)}{q-1}.$$

The associated bilinear form $B(x,y) := f(x+y) - f(x) - f(y)$ defines the polarity $\perp$ of PG$(2r+1,q)$. Two points $X, Y$ of PG$(2r+1,q)$ represented by vectors $x, y$ are said to be
orthogonal if $B(x, y) = 0$. Moreover, two orthogonal points $X, Y \in Q_r$ either coincide, $X = Y$, or are collinear, which means that the projective line joining $X, Y$ is entirely contained in $Q_r$.

For a point $P$, denote by $P^\perp$ the set of points of $PG(2r + 1, q)$ orthogonal with $P$ (clearly, $R \in P^\perp$ if and only if $P \in R^\perp$); such a set is a hyperplane of $PG(2r + 1, q)$ that is either tangent or not according as $P$ belongs to $Q_r$ or not. Note that $P \in P^\perp$ if and only if $P \in Q_r$, and, for a point set (or a subspace) $\Pi$, let $\Pi^\perp$ denote $\cap_{P \in \Pi} P^\perp$. We use the term $(j$-)space to denote a $(j$-dimensional) projective subspace of the ambient projective space. Recall that the quotient space $PG(2r + 1, q) / \Pi \cong PG(2r - j, q)$ is a projective space whose points, lines, and so on, are the subspaces of $PG(2r + 1, q)$ of dimension $j + 1, j + 2$, and so on, containing a $j$-space $\Pi$.

Furthermore, when a point $P$ belongs to $Q_r$, $P^\perp$ is a tangent hyperplane $\cong PG(2r, q)$ and $P^\perp / P$ is the quotient projective space $\cong PG(2r - 1, q)$. Then the subspaces of $Q_r$ of dimension $1, 2$, and so on, containing $P$, induce the projective subspace $P^\perp / P$, which is projectively equivalent to an elliptic polar space $Q_{r-1}$ of rank $r - 1$. For the sake of simplicity, we will simply denote this polar space by $Q_{r-1}$. In particular, the set $L(P)$ of lines of $Q_r$ through $P$ can be identified with the point set of $Q_{r-1}$ induced in $P^\perp / P$ by $Q_r$.

Tight sets and $m$-ovoids share the property that they exhibit precisely two intersection numbers with respect to tangent hyperplanes (see [3]). Namely, if $\mathcal{O}$ is an $m$-ovoid of $Q_r$, then for every point $P \in Q_r$ we have

$$ |P^\perp \cap \mathcal{O}| = \begin{cases} (m - 1)(q^r + 1) + 1 & \text{if } P \in \mathcal{O}, \\ m(q^r + 1) & \text{if } P \in Q_r \setminus \mathcal{O}. \end{cases} \tag{2.1} $$

Let $\mu : PG(2r + 1, q) \to \mathbb{Z}$ be a function defined on the points of $PG(2r + 1, q)$ such that $\mu(P) = 0$ if $P \in PG(2r + 1, q) \setminus Q_r$. For every subset $X$ of the point set of $PG(2r + 1, q)$, we define $\mu(X) = \sum_{P \in X} \mu(P)$; in particular, $\mu(X) = \mu(X \cap Q_r)$. Such a map $\mu$ is said to be a weighted $m$-ovoid of $Q_r$, for some natural $m$, if the following property is satisfied:

$$ \mu(P^\perp) + q^r \mu(P) = m(q^r + 1) \text{ for every point } P \in Q_r, \tag{2.2} $$

and it immediately follows from Equation (2.1) that the $(0,1)$-characteristic function of an $m$-ovoid of $Q_r$ is a weighted $m$-ovoid. Part (c) of the next lemma shows that Equation (2.2) generalizes to arbitrary subspaces of the ambient projective space, and this fact will often be used in the proof of our main result.

**Lemma 2.1.** Let $\mu$ be a weighted $m$-ovoid of $Q_r$.

(a) $\mu(Q_r) = m(q^{r+1} + 1)$.

(b) $\mu(H) = m(q^r + 1)$ for every nontangent hyperplane $H$.

(c) $\mu(\Pi^\perp) + q^{-j} \mu(\Pi) = m(q^{r-j} + 1)$ for every $j$-space $\Pi$ of $PG(2r + 1, q)$.

**Proof.**

(a) Computing in two ways the sum of $\mu(P_2)$ over all pairs $(P_1, P_2)$, where $P_1, P_2$ are (not necessarily distinct) points of $Q_r$ and $P_2 \in P_1^\perp$, we obtain

$$ \sum_{P_1 \in Q_r} \mu(P_1^\perp) = \sum_{P_2 \in Q_r} \mu(P_2)(1 + qk_{r-1}) $$

$$ = \mu(Q_r)(1 + qk_{r-1}). $$
By Equation (2.2), the left-hand side equals \( m(q^r + 1)k_r - q^r \sum_{P_1 \in Q_r} \mu(P_1) = m(q^r + 1)k_r - q^r \mu(Q_r) \). Simplifying and using that \((q^r + qk_{r-1} + 1)(q^{r+1} + 1) = k_r(q^r + 1)\), we get the result.

(b) Let \( H \) be a nontangent hyperplane of \( PG(2r + 1, q) \). Note that \( H \cap Q_r \) induces a parabolic quadric of rank \( r \) in the projective space \( H = PG(2r, q) \) and \( \mu(H \cap Q_r) = \mu(H) \). Computing in two ways the sum of \( \mu(P_2) \) over all pairs \((P_1, P_2)\), where \( P_1 \in H \cap Q_r, P_2 \in Q_r \) and \( P_2 \in P_\perp \), we obtain
\[
\sum_{P_1 \in H \cap Q_r} \mu(P_1 \perp) = \sum_{P_2 \in Q_r} \mu(P_2) |H \cap Q_r \cap P_\perp|.
\]

By Equation (2.2), the left-hand side equals \(|H \cap Q_r| \cdot m(q^r + 1) - \mu(H)q^r\). As \(|H \cap Q_r| = \frac{q^{2r-1}-1}{q-1}\) and \(|H \cap Q_r \cap P_\perp|\) equals \(k_{r-1}\) if \(P_2 \in Q_r \setminus H\) or \(\frac{q^{2r-1}-1}{q-1}\) if \(P_2 \in H \cap Q_r\), we have that
\[
\left( q^r + \frac{q^{2r-1}-1}{q-1} - k_{r-1} \right) \mu(H) = m \left( (q^r + 1) \frac{q^{2r-1}-1}{q-1} -(q^{r+1} + 1)k_{r-1} \right),
\]
whence the result follows.

(c) Let \( \Pi \) be a \( j \)-space of \( PG(2r + 1, q) \). Consider the \( \frac{q^{2r-j+1}-1}{q-1} \) hyperplanes \( R_\perp \) with \( R \in \Pi_\perp \). Every point of \( \Pi \) lies in all these hyperplanes and every other point lies in \( \frac{q^{2r-j}-1}{q-1} \) of these hyperplanes. Computing in two ways the sum of \( \mu(P) \) over all pairs \((R_\perp, P)\), where \( R \in \Pi_\perp, P \in R_\perp \cap Q_r \), shows that
\[
\sum_{R \in \Pi_\perp} \mu(R_\perp) = \frac{q^{2r-j+1}-1}{q-1} \mu(\Pi) + \frac{q^{2r-j}-1}{q-1} \mu(Q_r \setminus \Pi). \tag{2.3}
\]

To calculate the left-hand side of Equation (2.3), consider a point \( R \) of \( \Pi_\perp \). If \( R \) is not a point of the quadric \( Q_r \), then \( \mu(R_\perp) = m(q^r + 1) \) by (b). If \( R \) is a point of \( Q_r \), we apply Equation (2.2). This shows that the left-hand side of Equation (2.3) is equal to
\[
m(q^r + 1)|\Pi_\perp| - q^r \sum_{R \in \Pi_\perp \cap Q_r} \mu(R) = m(q^r + 1) \frac{q^{2r-j+1}-1}{q-1} - q^r \mu(\Pi_\perp).
\]

On the other hand, the right-hand side of Equation (2.3) is equal to
\[
q^{2r-j} \mu(\Pi) + \frac{q^{2r-j}-1}{q-1} \mu(Q_r).
\]
By (a), the assertion follows.

Note that if \( \mathcal{O} \) is an \( m \)-ovoid of \( Q_r \) and \( \Pi_j \) is a \( j \)-space of \( PG(2r + 1, q) \), then it immediately follows from Lemma 2.1(c) that
\[
|\Pi_\perp \cap \mathcal{O}| + q^{r-j} |\Pi_j \cap \mathcal{O}| = m(q^{r-j} + 1),
\]
and this result is a counterpart of [5, Lemma 2.1].
For a point \( P_0 \in Q_r \), consider the quadric \( Q_{r-1} \) induced in the projective space \( P_0^\perp / P_0 \cong \PG(2r - 1, q) \) by \( Q_r \). Recall that the set \( L(P_0) \) of lines of \( Q_r \) through \( P_0 \) can be identified with the point set of \( Q_{r-1} \). With this in mind, given a weighted ovoid \( \mu \) of \( Q_r \), define a function \( \mu_P^\perp : \PG(2r - 1, q) \to \mathbb{Z} \) by

\[
\mu_P^\perp(R) = \begin{cases} 
0 & \text{if } R \neq \ell / P_0 (\forall \ell \in L(P_0)), \\
\sum_{P \in \ell \setminus \{P_0\}} \mu(P) = \mu(\ell) - \mu(P_0) & \text{if } R = \ell / P_0, \ \ell \in L(P_0).
\end{cases}
\] (2.4)

Moreover, in what follows by \( \mu_P^\perp(\ell) \) we mean \( \mu_P^\perp(R) \) if \( R = \ell / P_0 \) for some \( \ell \in L(P_0) \), and this convention naturally extends to an arbitrary subspace \( U \) on \( P_0 \) by \( \mu_P^\perp(U) := \sum_{\ell \in L(P_0), \ell \subset U} \mu_P^\perp(\ell) \).

**Lemma 2.2.** Let \( P_0 \) be a point of \( Q_r \) and let \( \mu \) be a weighted \( m \)-ovoid of \( Q_r \). Then \( \mu_P^\perp \) is a weighted \((m - \mu(P_0))\)-ovoid of \( Q_{r-1} \).

**Proof.** For every \( \ell \in L(P_0) \), we have that

\[
\mu_P^\perp(\ell^\perp) := \sum_{\ell_1 \in L(P_0), \ell_1 \subset \ell^\perp} \mu_P^\perp(\ell_1) = \mu(\ell^\perp) - \mu(P_0) = m(q^{r-1} + 1) - q^{r-1} \mu(\ell) - \mu(P_0) \quad \text{[by Lemma 2.1(c)]}
\]

\[
= (m - \mu(P_0))(q^{r-1} + 1) - q^{r-1} \mu_P^\perp(\ell) \quad \text{[by Equation (2.4)]},
\]

which shows that \( \mu_P^\perp \) satisfies Equation (2.2); thus, the result follows. \( \square \)

For a weighted ovoid \( \mu \) of \( Q_r \), let \( \|\mu\|^2 \) denote the squared norm of \( \mu \), that is,

\[
\|\mu\|^2 := \sum_{P \in \PG(2r + 1, q)} \mu(P)^2 = \sum_{P \in Q_r} \mu(P)^2,
\]

where we omit the notation for \( r \), as it should be clear from the context. The next lemma relates \( \|\mu\|^2 \) and \( \|\mu_P^\perp\|^2 \).

**Lemma 2.3.** Let \( \mu \) be a weighted \( m \)-ovoid of \( Q_r \). Then, for any point \( P_0 \in Q_r \), the following equality holds:

\[
\|\mu\|^2 = \mu(P_0)^2 + (\mu(P_0) + m(q - 1))^2 + (q + 1) \cdot \sum_{P_1 \in P_0^\perp \setminus \{P_0\}} \mu(P_1)^2 - \|\mu_P^\perp\|^2.
\]

**Proof.** Let \( E \) denote the set of pairs \( (P, R) \) such that \( P \in P_0^\perp \cap R^\perp \) and \( R \not\in P_0^\perp \). We will count in two ways the following quantity

\[
S = \sum_{(P, R) \in E} \mu(P)\mu(R).
\]
Given a point \( R \in Q_r \setminus P_0^\perp \), the line \( \langle P_0, R \rangle \) is a 2-secant to \( Q_r \), hence \( \mu(\langle P_0, R \rangle) = \mu(R) + \mu(P_0) \). Thus, applying Lemma 2.1(c) to this line gives

\[
\sum_{(P,R) \in \mathcal{E}} \mu(P) = m(q^r-1) + 1 - q^r-1(\mu(R) + \mu(P_0)).
\]

As \( \sum_{R \notin P_0^\perp} \mu(R) = q^r(m(q-1) + \mu(P_0)) \) holds by Lemma 2.1, we obtain

\[
S = \sum_{R \notin P_0^\perp} \left( m(q^r-1) + 1 - q^r-1\mu(R) - q^r-1\mu(P_0) \right) \mu(R)
= q^r-1 \left( m(q^r-1) + 1 - q^r-1\mu(P_0) \right) (mq(q-1) + q\mu(P_0)) - \sum_{R \notin P_0^\perp} \mu(R)^2. \tag{2.5}
\]

On the other hand, for a fixed point \( P \in Q_r \cap (P_0^\perp \setminus \{P_0\}) \), the quantity \( \sum_{(P,R) \in \mathcal{E}} \mu(R) \) equals \( \mu(P^\perp) - \mu(\ell^\perp_P) \), where \( \ell^\perp_P \) denotes the line of \( Q_r \) joining \( P_0 \) and \( P \). Set

\[
S_1 = \sum_{P \in P_0^\perp \setminus \{P_0\}} \mu(P) \mu(P^\perp) \quad \text{and} \quad S_2 = \sum_{P \in P_0^\perp \setminus \{P_0\}} \mu(P) \mu(\ell^\perp_P).
\]

As \( \mu(P_0^\perp \setminus \{P_0\}) = (m - \mu(P_0))(q^r + 1) \) holds by Equation (2.2), we evaluate \( S_1 \) as follows:

\[
S_1 = \sum_{P \in P_0^\perp \setminus \{P_0\}} \left( m(q^r-1) + 1 - q^r\mu(P) \right) \mu(P)
= m(q^r-1)(m - \mu(P_0)) - q^r \cdot \sum_{P \in P_0^\perp \setminus \{P_0\}} \mu(P)^2.
\]

To evaluate \( S_2 \), observe that \( \mu(\ell^\perp_P) = m(q^r-1) + 1 - q^r-1\mu(\ell^\perp_P) \) by Equation (2.2). Therefore,

\[
S_2 = \sum_{P \in P_0^\perp \setminus \{P_0\}} \mu(P) \left( m(q^r-1) + 1 - q^r-1\mu(\ell^\perp_P) \right)
= m(q^r-1)(m - \mu(P_0))(q^r + 1) - q^r-1 \cdot \sum_{P \in P_0^\perp \setminus \{P_0\}} \mu(P) \mu(\ell^\perp_P).
\]

Further,

\[
\sum_{P \in P_0^\perp \setminus \{P_0\}} \mu(P) \mu(\ell^\perp_P) = \sum_{\ell \in L(P_0)} \left( \mu(\ell) - \mu(P_0) \right) \mu(\ell)
= \sum_{\ell \in L(P_0)} \left( \mu(P_0)(\mu(\ell) - \mu(P_0)) + \mu(P_0^\perp)(\ell)^2 \right)
= \mu(P_0)(\mu(P_0^\perp) - \mu(P_0)) + \sum_{\ell \in L(P_0)} \mu(P_0^\perp)(\ell)^2
= \mu(P_0)(m - \mu(P_0))(q^r - 1) + \| \mu(P_0^\perp) \|^2.
\]
Thus, we finally obtain
\[
S_2 = m(q^{r-1} + 1)(m - \mu(P_0))(q^{r} + 1) - q^{r-1} \left( \mu(P_0)(m - \mu(P_0))(q^{r} + 1) + \|\mu_{P_0}^1\|^2 \right),
\]
so
\[
S = S_1 - S_2
= m(m - \mu(P_0))q^{r-1}(q^{r} + 1)(q - 1) + \mu(P_0)(m - \mu(P_0))q^{r-1}(q^{r} + 1)
- q^r \cdot \sum_{P \in P_0^\perp} \mu(P)^2 + q^{r-1} \cdot \|\mu_{P_0}^1\|^2.
\]
(2.6)

Equating Equations (2.5) and (2.6) and simplifying the result completes the proof of the lemma.

\[\square\]

**Corollary 2.4.** For \(r \geq 1\) and every weighted \(m\)-ovoid \(\mu\) of \(Q_r\), one has
\[
\|\mu\|^2 \equiv \begin{cases} 
-2qm^2 + (q + 1)(q^r + 1)m & \text{if } r \text{ is even} \\
(q^2 + 1)m^2 & \text{if } r \text{ is odd} 
\end{cases} \pmod{2(q + 1)}.
\]

**Proof.** In this proof, \(\equiv\) stands for equivalence modulo \(2(q + 1)\). We prove the assertion by induction on \(r\). For \(r = 1\), we have \(\mu(P) = m\) for each of the \(q^2 + 1\) points of \(Q_1\) and the claim follows.

Now suppose that \(r \geq 2\), let \(P_0\) be any point of \(Q_r\) and put \(x := \mu(P_0)\). For each integer \(t\) we have \((q + 1)t^2 \equiv (q + 1)t\) and hence, by Equation (2.2),
\[
\sum_{P \in P_0^\perp \setminus \{P_0\}} (q + 1)\mu(P)^2 \equiv \sum_{P \in P_0^\perp \setminus \{P_0\}} (q + 1)\mu(P) \equiv (q + 1)(q^{r} + 1)(m - x).
\]

Lemma 2.3 shows thus
\[
\sum_{P \in Q_r} \mu(P)^2 \equiv x^2 + (x + m(q - 1))^2 + (q + 1)(q^r + 1)(m - x) - \sum_{\ell \in L(P_0)} \mu_{P_0}^1(\ell)^2.
\]

Now we apply the induction hypothesis to the quadric \(P_0^\perp / P_0\) (with point-set \(L(P_0)\)) induced by \(Q_r\) and the weighted \((m - x)\)-ovoid \(\mu_{P_0}^1\) of \(Q_{r-1}\). When \(r\) is even, this gives
\[
\sum_{P \in Q_r} \mu(P)^2 \equiv x^2 + (x + m(q - 1))^2 + (q + 1)(q^r + 1)(m - x) - (q^2 + 1)(m - x)^2
\equiv (1 - q^2)x^2 + 2xm(q + 1) - x(q + 1)(q^r + 1) - 2qm^2 + (q + 1)(q^r + 1)m
\equiv -2qm^2 + (q + 1)(q^r + 1)m,
\]
where we use \((1 - q^2)x^2 \equiv (1 - q^2)x\) in the last step. When \(r\) is odd, we find instead
\[
\sum_{P \in Q_r} \mu(P)^2 \equiv x^2 + (x + m(q - 1))^2 + (q + 1)(q^r + 1)(m - x)
+ 2q(m - x)^2 - (q + 1)(q^{r-1} + 1)(m - x)
\]
\[ \equiv x^2 + (x + m(q - 1))^2 + 2q(m - x)^2 \]
\[ \equiv 2(q + 1)x^2 - 2xm(q + 1) + m^2(q^2 + 1) \equiv m^2(q^2 + 1) \]
as desired. \qed

3 A MODULAR EQUALITY FOR \( m \)

In this section, we prove Theorem 1.1. Let \( O \) be an \( m \)-ovoid of \( Q_r, r \geq 2 \), and fix a point \( P_0 \in Q_r \). Recall that the (0,1)-characteristic function \( \chi \) of \( O \) is a weighted \( m \)-ovoid of \( Q_r \), and \( \chi_{P_0} \) is a weighted \((m - \chi(P_0))\)-ovoid of \( Q_{r-1} \) by Lemma 2.2.

Lemma 3.1. The following holds:
\[ \|\chi_{P_0}\|^2 = \chi(P_0) + (\chi(P_0) + m(q - 1))^2 - \chi(P_0)(q + 1)(q^- + 1) + m(q^- + q). \] (3.1)

Proof. The result follows from Lemma 2.3 applied to \( \chi \) in the role of \( \mu \) (observe that \((\chi(P))^2 = \chi(P)\) for any point \( P \)). \qed

The following lemma immediately follows from Corollary 2.4 applied to \( \chi_{P_0} \).

Lemma 3.2. Let \( \equiv \) denote equivalence modulo \( 2(q + 1) \). The following holds:
\[ \|\chi_{P_0}\|^2 \equiv \begin{cases} -2q(m - \chi(P_0))^2 + (q + 1)(q^- + 1)(m - \chi(P_0)) & \text{if } r \text{ is odd}, \\ (q^2 + 1)(m - \chi(P_0))^2 & \text{if } r \text{ is even}. \end{cases} \] (3.2)

We are now in a position to prove our main result, Theorem 1.1.

Proof. Fix a point \( P_0 \in Q_r \setminus O \). By Lemmas 3.1 and 3.2, we have two equalities for \( \|\chi_{P_0}\|^2 \).

Suppose that \( r \) is odd. Equating (modulo \( 2(q + 1) \)) Equations (3.1) and (3.2) gives
\[ (q^2 + 1)m^2 - (q^- + 1)m \equiv 0 \mod 2(q + 1), \]
which is equivalent to either \((2(m^2 - m) \equiv 0 \mod 2(q + 1)) \) or \((q + 3)(m^2 - m) \equiv 0 \mod 2(q + 1)) \), according as \( q \) is odd or even, respectively. In the latter case, note that \((q + 3)(m^2 - m) \equiv 2(m^2 - m) \mod 2(q + 1)\); hence \(2(m^2 - m) \equiv 0 \mod 2(q + 1)\) holds in the even characteristic case as well.

Similarly, if \( r \) is even, we obtain
\[ 2m^2 + (q^- + q)m \equiv 0 \mod 2(q + 1), \]
and the result follows. \qed

We now determine the number of solutions of Equation (1.2).
Lemma 3.3. Let \( q + 1 = p_1^{k_1} \cdot \ldots \cdot p_t^{k_t}, p_1 < \ldots < p_t \), be the prime factorization of \( q + 1 \). Then the following hold.

(a) There are \( 2^t \) integers \( m \), with \( 0 \leq m \leq q \), such that \( m^2 - m \equiv 0 \pmod{q+1} \).

(b) If \( q \) is even, there are \( p_1^{\left\lfloor \frac{k_1}{2} \right\rfloor} \cdot \ldots \cdot p_t^{\left\lfloor \frac{k_t}{2} \right\rfloor} \) integers \( m \), with \( 0 \leq m \leq q \), such that \( m^2 \equiv 0 \pmod{q+1} \).

(c) If \( q \equiv -1 \pmod{4} \), then there are \( p_1^{\left\lfloor \frac{k_1}{2} \right\rfloor} \cdot \ldots \cdot p_t^{\left\lfloor \frac{k_t}{2} \right\rfloor} \) integers \( m \), with \( 0 \leq m \leq q \), such that \( m^2 + \frac{q+1}{2} m \equiv 0 \pmod{q+1} \).

(d) If \( q \equiv 1 \pmod{4} \), then there are \( 2 \cdot p_2^{\left\lfloor \frac{k_2}{2} \right\rfloor} \cdot \ldots \cdot p_t^{\left\lfloor \frac{k_t}{2} \right\rfloor} \) integers \( m \), with \( 0 \leq m \leq q \), such that \( m^2 + \frac{q+1}{2} m \equiv 0 \pmod{q+1} \).

Proof. Let \( f(m) = m^2 + am \), for some integer \( a \). From the Chinese Remainder Theorem, we have that

\[
f(m) \equiv 0 \pmod{q+1}
\]

has a solution if and only if each of the equations,

\[
f(m) \equiv 0 \pmod{p_i^{k_i}}, \quad 1 \leq i \leq t,
\]

has a solution. Moreover, if Equation (3.4) has \( n_i \) solutions, then Equation (3.3) has \( n_1 \cdot \ldots \cdot n_t \) solutions. As \( m^2 - m \equiv 0 \pmod{p_i^{k_i}} \) has two solutions and \( m^2 \equiv 0 \pmod{p_i^{k_i}} \) admits \( p_i^{\left\lfloor \frac{k_i}{2} \right\rfloor} \) solutions, \( 1 \leq i \leq t \), statements (a) and (b) follow.

If \( q \) is odd, then, assuming that \( p_1 = 2 \), one has \( \frac{q+1}{2} \equiv 0 \pmod{p_i^{k_i}} \), \( 2 \leq i \leq t \), and \( \frac{q+1}{2} \equiv 2^{k_i-1} \pmod{2^{k_i}} \) or \( \frac{q+1}{2} \equiv 1 \pmod{2} \), according as \( q \equiv -1 \pmod{4} \) or \( q \equiv 1 \pmod{4} \). The fact that \( m^2 + m \equiv 0 \pmod{2} \) has two solutions and that \( m^2 + 2^{k_i-1} m \equiv 0 \pmod{2^{k_i}} \) admits \( 2^{\left\lfloor \frac{k_i}{2} \right\rfloor} \) solutions, shows (c) and (d).

Combining Lemma 3.3 together with Theorem 1.1, we get that the number of admissible values of \( m \) for an \( m \)-ovoid of \( Q_r \) equals:

\[
2^t \cdot (q^{r-2} + q^{r-4} + \ldots + q) + 1, \quad \text{if } r \text{ is odd},
\]

\[
p_1^{\left\lfloor \frac{k_1}{2} \right\rfloor} \cdot \ldots \cdot p_t^{\left\lfloor \frac{k_t}{2} \right\rfloor} \cdot (q^{r-2} + q^{r-4} + \ldots + q^2 + 1), \quad \text{if } r \text{ is even and } q \not\equiv 1 \pmod{4},
\]

\[
2 \cdot p_2^{\left\lfloor \frac{k_2}{2} \right\rfloor} \cdot \ldots \cdot p_t^{\left\lfloor \frac{k_t}{2} \right\rfloor} \cdot (q^{r-2} + q^{r-4} + \ldots + q^2 + 1), \quad \text{if } r \text{ is even and } q \equiv 1 \pmod{4}.
\]

4 | A LOWER BOUND FOR \( m \) AND SOME CHARACTERIZATION RESULTS

In this section, we slightly improve on the lower bound \( m \geq \frac{\sqrt{4q^r+1} + 9 - 3}{2q - 2} \) for an \( m \)-ovoid of \( Q_r \), which was shown in \[3, Theorem 13\].

In particular, an \( m \)-ovoid of \( Q_r(7, q) \) may exist only when \( m \geq q + 1 \). All known examples of \((q + 1)\)-ovoids of \( Q_r(7, q) \) arise by applying the field reduction to the points of \( Q_r(3, q^2) \), see \[22\]. In this case, the \((q + 1)\)-ovoid consists of the points of \( q^4 + 1 \) pairwise disjoint lines \( \ell_1, \ldots, \ell_{q^4+1} \).
forming a 1-system. Recall that a 1-system in \( Q^{-}(7, q) \) is a set of \( q^4 + 1 \) pairwise disjoint lines \( \ell_1, \ldots, \ell_{q^4+1} \) such that every plane of \( Q^{-}(7, q) \) containing \( \ell_i \) is disjoint from \( \bigcup_{j=1, j \neq i}^{q^4+1} \ell_j \). The 1-systems in \( Q^{-}(7, q) \) are unique [20, 21]. Conversely, we will show that a \( (q + 1) \)-ovoid of \( Q^{-}(7, q) \), \( q \in \{2, 3\} \), consists of the points covered by the lines of a 1-system of \( Q^{-}(7, q) \).

**Theorem 4.1.** If \( Q^{-}(2r + 1, q) \) possesses an \( m \)-ovoid with \( m > 0 \), then

\[
m \geq \frac{\sqrt{4q^{r+1} + 4q + 1} - 3}{2(q - 1)}.
\]

**Proof.** Let \( \mathcal{O} \) be an \( m \)-ovoid of \( Q_r \), \( \chi \) the characteristic function of \( \mathcal{O} \). Fix a point \( P_0 \in \mathcal{O} \), and for every line \( \ell \in L(P_0) \), define \( t_\ell := \chi_{P_0}^{\perp}(\ell) = |\ell \cap \mathcal{O}| - \chi(P_0) \). Then \( |P_0^{\perp} \cap \mathcal{O}| = m(q^r + 1) - q^r \) holds by Equation (2.1), and hence

\[
\sum_{\ell \in L(P_0)} t_{\ell} = |P_0^{\perp} \cap \mathcal{O}| - 1 = (m - 1)(q^r + 1).
\]

Moreover, by Lemma 3.1, we have

\[
\sum_{\ell \in L(P_0)} t_{\ell}^2 = 1 + (1 + m(q - 1))^2 - (q + 1)(q^r + 1) + m(q^r + q).
\]

Therefore, subtracting Equation (4.1) from Equation (4.2), we obtain

\[
\sum_{\ell \in L(P_0)} t_{\ell}(t_{\ell} - 1) = m^2(q - 1)^2 + 3m(q - 1) - q^{r+1} - q + 2.
\]

The left-hand side of Equation (4.3) is nonnegative, whereas its right-hand side is a quadratic polynomial in \( m \) with positive leading coefficient, whose largest root is

\[
m_1 = \frac{\sqrt{4q^{r+1} + 4q + 1} - 3}{2(q - 1)}.
\]

Hence, \( m \geq m_1 \), which completes the proof. \( \Box \)

For \( r = 3 \), we find that \( m \geq q + 1 \). If \( m = q + 1 \) and \( P_0 \in \mathcal{O} \), then Equation (4.3) reads as

\[
\sum_{\ell \in L(P_0)} t_{\ell}(t_{\ell} - 1) = q(q - 1).
\]

It is readily seen by Equation (4.4) that if a line on \( P_0 \) is contained in \( \mathcal{O} \), then every other line of \( Q^{-}(7, q) \) through \( P_0 \) meets \( \mathcal{O} \) in one or two points. Suppose that this occurs for every point of \( \mathcal{O} \). Then it turns out that there are exactly \( q^4 + 1 \) lines contained in \( \mathcal{O} \) and that the set \( L \) of these lines forms a partition of the \( (q + 1) \)-ovoid. Furthermore, in this case a plane of \( Q^{-}(7, q) \) that contains a line of \( L \) does not meet any other line of \( L \), as it intersects \( \mathcal{O} \) in exactly \( q + 1 \) points. Therefore, \( L \) is a 1-system of \( Q^{-}(7, q) \). In the next two theorems, we will show that this is the case when \( q \in \{2, 3\} \).
Theorem 4.2. The elliptic quadric $Q^-(7, 2)$ has a nontrivial m-ovoid only for $m = 3$ and $m = 4$. Moreover, every 3-ovoid of $Q^-(7, 2)$ is the union of the lines of a 1-system of $Q^-(7, 2)$.

Proof. For an m-ovoid of $Q^-(7, 2)$ we have that 3 divides $m(m-1)$ by Theorem 1.1. It follows that a nontrivial m-ovoid can exist only for $m \in \{1, 3, 4, 6\}$. The case $m = 1$ does not occur by Theorem 4.1. Hence, $m \neq 6$, as the complement of a 6-ovoid is a 1-ovoid. Let $\mathcal{O}$ be a 3-ovoid of $Q^-(7, 2)$ and let $P_0 \in \mathcal{O}$. Note that if $\ell \in L(P_0)$, then $t_{\ell} = |\ell \cap \mathcal{O}| - 1 \in \{0, 1, 2, 3\}$ and $\sum_{\ell \in L(P_0)} t_{\ell}(t_{\ell} - 1) = 2$ by Equation (4.4). Therefore, every point of $\mathcal{O}$ lies on exactly one line contained in $\mathcal{O}$ and hence the 3-ovoid arises from a 1-system of $Q^-(7, 2)$. (Note that a 4-ovoid is the complement of a 3-ovoid.)

\[ \]

Theorem 4.3. A nontrivial m-ovoid of $Q^-(7, 3)$ can exist only for $m \in \{4, 5, 8, 9\}$. Moreover, every 4-ovoid of $Q^-(7, 3)$ is the union of the lines of a 1-system of $Q^-(7, 3)$.

Proof. Theorems 1.1 and 4.1 imply that a nontrivial m-ovoid of $Q^-(7, 3)$ can exist only for $m \in \{4, 5, 8, 9\}$. Let $\mathcal{O}$ be a 4-ovoid of $Q^-(7, 3)$ and let $P_0 \in \mathcal{O}$. If $\ell \in L(P_0)$, then $t_{\ell} = |\ell \cap \mathcal{O}| - 1 \in \{0, 1, 2, 3\}$ and $\sum_{\ell \in L(P_0)} t_{\ell}(t_{\ell} - 1) = 6$ by Equation (4.4). It follows that two possibilities arise: either there is exactly one line of $Q^-(7, 3)$ through $P_0$ contained in $\mathcal{O}$ and the remaining lines of $L(P_0)$ have at most two points in common with $\mathcal{O}$ or there are exactly three lines of $Q^-(7, 3)$ through $P_0$ intersecting $\mathcal{O}$ in three points and the remaining lines of $L(P_0)$ have at most two points in common with $\mathcal{O}$. If the latter case does not occur, then every point of $\mathcal{O}$ lies on exactly one line contained in $\mathcal{O}$ and hence the 4-ovoid arises from a 1-system of $Q^-(7, 3)$.

Assume by way of contradiction that the second case occurs for some point $P_0$ of $\mathcal{O}$ and let $\ell_1, \ell_2, \ell_3$ be the three lines of $Q^-(7, 3)$ on $P_0$ such that $|\ell_i \cap \mathcal{O}| = 3$, that is, $t_{\ell_i} = 2$. We will evaluate in two different ways the number of lines $\ell_i$ belonging to $L(P_0)$ and such that $t_{\ell_i} = 0$, that is, meeting $\mathcal{O}$ exactly in $P_0$. As the point $P_0$ lies on 112 lines of $Q^-(7, 3)$ and $\sum_{\ell \in L(P_0)} t_{\ell} = 84$ by Equation (4.1), it follows that there are exactly

\[ 112 - (84 - 6) - 3 = 31 \]

lines of $L(P_0)$ meeting $\mathcal{O}$ exactly in $P_0$.

On the other hand, each of the 10 planes on $\ell_i$ meets $\mathcal{O}$ in four points and hence in each of these planes there are precisely two lines through $P_0$ that meet $\mathcal{O}$ only in $P_0$. This shows that $P_0$ lies on 20 lines of the quadric that lie in $\ell_i^\perp$ and meet $\mathcal{O}$ only in $P_0$. Moreover, the plane $\sigma_{i,j} = \langle \ell_i, \ell_j \rangle$, $1 \leq i < j \leq 3$ meets $Q^-(7, 3)$ only in $\ell_i \cup \ell_j$ and the 4-space $\sigma_{i,j}^\perp$ meets $Q^-(7, 3)$ in a cone having as vertex the point $P_0$ and as base a $Q^-(3, 3)$. As $|\sigma_{i,j} \cap \mathcal{O}| = 5$, by Lemma 2.1(c) we have that $|\sigma_{i,j} \cap \mathcal{O}| = 1$, and hence $\sigma_{i,j}^\perp$ meets $\mathcal{O}$ only in $P_0$. Similarly, the solid $\Pi = \langle \ell_1, \ell_2, \ell_3 \rangle$ as well as the solid $\Pi^\perp$ intersect $Q^-(7, 3)$ in a quadratic cone having as vertex $P_0$; as $|\Pi \cap \mathcal{O}| \geq 7$, Lemma 2.1(c) gives $|\Pi^\perp \cap \mathcal{O}| \leq 1$, and hence $\Pi^\perp$ meets $\mathcal{O}$ only in $P_0$. Let $a_i$ denote the number of lines of $Q^-(7, 3)$ on $P_0$ that meet $\mathcal{O}$ only in $P_0$ and that are contained in $\ell_i^\perp$ for all $i \in I$. Thus,

\[
\begin{align*}
a_{i} &= 20 & \text{for } i = 1, 2, 3, \\
\sum_{i \neq j} a_{i} &= 10 & \text{for } 1 \leq i < j \leq 3, \\
\sum_{i = 1}^{3} a_{i} &= 4. 
\end{align*}
\]
Hence, $P_0$ lies on at least

$$3 \cdot 20 - 3 \cdot 10 + 4 = 34$$

lines of $Q^-(7, 3)$ that meet $\mathcal{O}$ only in $P_0$, a contradiction.

A natural question arising from Theorem 4.3 concerns the existence of a 5-ovoid of $Q^-(7, 3)$. Note that the complement of a 5-ovoid is an 8-ovoid; hence one could ask whether or not it is possible to obtain an 8-ovoid of $Q^-(7, 3)$ by gluing together two disjoint 4-ovoids of $Q^-(7, 3)$. Some computations performed with Magma [6] show that the sets of points covered by two distinct 1-systems of a $Q^-(7, 3)$ always have in common at least four points.

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