Lightlike shell solitons of extremal space-time film

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Abstract

New exact solution class of Born—Infeld type nonlinear scalar field model is obtained. The variational principle of this model has a specific form which is characteristic for extremal four-dimensional hypersurface or hyper-film in five-dimensional space-time. Obtained solutions are singular solitons propagating with speed of light and having energy, momentum, and angular momentum which can be calculated for explicit conditions. Such solitons will be called the lightlike ones. The soliton singularity has a form of moving two-dimensional surface or shell. The lightlike soliton can have a set of tubelike singular shells with the appropriate cavities. A twisted lightlike soliton is considered. It is notable that its energy is proportional to its angular momentum in high-frequency approximation. A case with one tubelike cavity is considered. In this case the soliton shell is diffeomorphic to a cylindrical surface with threads by multifilar helix. The shell transverse size of the appropriate finite energy soliton can be converging to zero at infinity. The ideal gas of such lightlike solitons with minimal twist parameter is considered in a finite volume. Explicit conditions provide that the angular momentum of each soliton in the volume equals Planck constant. The equilibrium energy spectral density for the solitons is obtained. It has the form of Planck distribution in some approximation. A beam of the twisted lightlike solitons is considered. The representation of arbitrary polarization for the beam with the twisted lightlike solitons is discussed. It is shown that the effect of mechanical angular momentum transfer to absorbent by the circularly polarized beam can be provided. This effect is well known for photon beam. Thus the soliton solution which have determinate likeness with photon is obtained in particular.

1. Introduction

A nonlinear space-time scalar field model considered here is known for a long time sufficiently. This model is related to well known Born—Infeld nonlinear electrodynamics [1, 2]. It is sometimes called also Born—Infeld type scalar field model [3].

This model is attractive because it has comparatively simple and geometrically clear form. It can be considered as a relativistic generalization of the minimal surface or minimal thin film model in three-dimensional space.

In this generalization we have an extremal four-dimensional film in five-dimensional space-time. But the model equation appears as differential one for a scalar field in four-dimensional space-time.

On the other hand, this model can describe the observable physical effects, which is a necessary requirement for the realistic filed model.

In particular, the model under consideration has a static spherically symmetric solution, which is identical in form to a zero component of the electromagnetic four-vector potential for dyon solution of Born—Infeld electrodynamics [4, 5]. This static solution of the scalar model gives the appropriate moving soliton solution with the aid of Lorentz transform.

As it was shown in the work [4], in the case of nonlinear electrodynamics there are the conformity between the long-range interaction of solitons and two known long-range interactions of physical particles, those are...
electromagnetic and gravitational ones. But the methods which was used for the investigation of soliton long-range interaction are independent of the field model. The appropriate instruments are the integral conservation laws and the characteristic equation.

These methods applying to the scalar model under consideration give the results, which are similar to ones for the nonlinear electrodynamics \[6, 7\]. Here we briefly discuss the interaction of such scalar solitons in the next section.

An essential difference of the scalar field model from the nonlinear electrodynamics is obviously caused by the different tensor character of the fields. In particular, a weak single-component scalar wave can not describe the transverse polarization of an electromagnetic wave.

But at the present time we can consider the light as a photon flux but not a weak electromagnetic wave with a constant amplitude. The photon beam could be represented by an appropriate scalar soliton beam. In this case an essential space-time nonhomogeneous of soliton solution may provide the necessary symmetry properties for the beam.

Thus at the present work we consider the model of extremal space-time film. We obtain its singular exact soliton solutions propagating with the speed of light and having energy, momentum, and angular momentum which can be calculated for explicit conditions. Such solitons will be called the lightlike ones.

Then we investigate in detail a lightlike soliton solution having a rotation about the direction of propagation that is twisted lightlike soliton.

We consider the ideal gas of such twisted lightlike solitons. Using explicit assumptions we obtain Planck distribution formula in some approximation.

At last we consider a beam with the twisted lightlike solitons. We show that this beam can represent the photon one. In this case the beam can have the polarization property and can provide the effect of the mechanical angular momentum transfer to absorbent for the circularly polarization.

2. Extremal space-time film

Let us consider the following action integral which has the world volume form:

\[ \mathcal{A} = \int \sqrt{|\Omega|} \, (dx)^4, \]  
\[(2.1a)\]

where \( \Omega \triangleq \det(\Omega_{\mu\nu}) \), \((dx)^4 \triangleq dx^0 dx^1 dx^2 dx^3\), \( \nabla \) is space-time volume,

\[ \Omega_{\mu\nu} = \gamma_{\mu\nu} + \chi^2 \frac{\partial \Phi}{\partial x^\mu} \frac{\partial \Phi}{\partial x^\nu}, \]  
\[(2.1b)\]

\( m_{\mu\nu} \) are the components of the metric tensor for the flat four-dimensional space-time, \( \Phi \) is a scalar real field function, \( \chi \) is a dimensional constant. The Greek indices take the values \( \{0, 1, 2, 3\} \).

As we have mentioned in introduction, the action \((2.1)\) is a generalization of the appropriate expression for the mathematical model of two-dimensional minimal thin film in the tree-dimensional space of our everyday experience. Such generalized model is considered, for example, in the monograph \[8\].

It is worth emphasizing that the choice of such model is consistent with the trend of geometrization of physics, which has already proven its fruitfulness in description of processes and phenomena of the material world. Here we have in mind the conception of general relativity. From the standpoint of discussion for the questions related to geometrization of physical world picture, the monographs \[9–11\] should be noted.

A natural feature of the world volume action \((2.1)\) is that his density does not vanish for zero derivatives of the field function \( \partial \Phi / \partial x^\mu = 0 \) and, consequently, for zero hypersurface curvature. In just the same way an area element of a two-dimensional thin film in the three-dimensional space does not vanish where the curvature of the minimal surface becomes zero.

Thus the variational principle \( \delta \mathcal{A} = 0 \) with action \((2.1)\) corresponds to extremal four-dimensional film \( \Phi(\{x^\mu\}) \) in a five-dimensional space-time \( \{\Phi, x^0, x^1, x^2, x^3\} \).

Determinant \( \Omega \) in \((2.1)\) can be represented in the form

\[ \Omega = m \left( 1 + \chi^2 m_{\mu\nu} \frac{\partial \Phi}{\partial x^\mu} \frac{\partial \Phi}{\partial x^\nu} \right), \]  
\[(2.2)\]

where \( m \triangleq \det(m_{\mu\nu}) \).

Taking into account \((2.2)\) we can write the model action \((2.1)\) in the form

\[ \mathcal{A} = \int \mathcal{L} \, d\nabla, \]  
\[(2.3a)\]
where \( dV = \sqrt{|m|} \) (d\( x \))^4 is a four-dimensional volume element,

\[
\mathcal{L} = \sqrt{1 + \chi^2 m^{\mu\nu} \frac{\partial \Phi}{\partial x^\mu} \frac{\partial \Phi}{\partial x^\nu}}. \tag{2.3b}
\]

The variational principle with action (2.3) gives the following model equation:

\[
\frac{1}{\sqrt{|m|}} \frac{\partial}{\partial x^\mu} \sqrt{|m|} \nabla^\mu = 0, \tag{2.4a}
\]

where

\[
\nabla^\mu \equiv \frac{\Phi^\mu}{\mathcal{L}}, \quad \Phi^\mu = m^{\mu\nu} \Phi_\nu,
\]

\[
\Phi_\nu \equiv \frac{\partial \Phi}{\partial x^\nu}. \tag{2.4c}
\]

Note that the action (2.1), (2.3) and the equation (2.4) are valid for the field outside of singularities. The presence of field singularities requires an addition of the appropriate singular densities to the action (2.1), (2.3) and in the right side of the equation (2.4). This aspect was considered for the case of nonlinear electrodynamics in the work \[4\] and it is expanded in the monograph \[5\]. But here we do not introduce the singular densities as inessential for the subsequent analysis.

We have the following evident relations from (2.4c):

\[
\frac{\partial \Phi_\nu}{\partial x^\nu} - \frac{\partial \Phi_\mu}{\partial x^\mu} = 0. \tag{2.5}
\]

Inversion for relations (2.4b) gives

\[
\Phi^\mu = \frac{\nabla^\mu}{\mathcal{L}}, \tag{2.6a}
\]

where we put

\[
\mathcal{L} = \sqrt{|1 - \chi^2 m^{\mu\nu} \nabla^\mu \nabla^\nu|}. \tag{2.6b}
\]

It must be noted here that the presence of modulus in the both expressions (2.3b) and (2.6b) stipulates the following feature of the model under consideration. The signature of space-time metric tensor can be changed on singular sets of solutions. We shall consider this feature in more detail below.

Let us write also the following useful relation, which is obtained from (2.3b) and (2.6):

\[
\mathcal{L} \mathcal{L} = 1. \tag{2.7}
\]

For the case when the field invariant \( \Phi^\mu \Phi_\mu \) is relatively small \( (\chi^2 | \Phi^\mu \Phi_\mu | \ll 1) \) we can represent the action density \( \mathcal{L} \) by two first terms of the formal power series in \( \chi^2 \):

\[
\mathcal{L} = 1 + \chi^2 \frac{1}{2} m^{\mu\nu} \frac{\partial \Phi}{\partial x^\mu} \frac{\partial \Phi}{\partial x^\nu} + \mathcal{O}(\chi^4)_{\chi \to 0}. \tag{2.8}
\]

The appropriate linearized equation has the form

\[
\frac{1}{\sqrt{|m|}} \frac{\partial}{\partial x^\mu} \left( \sqrt{|m|} m^{\mu\nu} \frac{\partial \Phi}{\partial x^\nu} \right) + \mathcal{O}(\chi^2)_{\chi \to 0} = 0. \tag{2.9}
\]

Also let us write the linearized relation for (2.4b) as:

\[
\nabla^\mu = \Phi^\mu + \mathcal{O}(\chi^2)_{\chi \to 0}. \tag{2.10}
\]

The nonlinear differential equation of second order (2.4) for the function \( \Phi \) can be represented in the form of the first order differential equation system for four-vectors \( \Phi_\mu \) or \( \nabla^\mu \). In this case we have differential field equations (2.4a) and (2.5) accordingly. In addition to this we must consider algebraical relations (2.4b) or (2.6).

As it can be seen, the model action (2.3) is susceptible to the choice of a metric signature. Here we will consider both the signatures \{+, −, −, −\} and \{−, +, +, +\}. Thus we use the following designations for Minkowski metric:

\[
\bar{m}^{00} = 1, \quad \bar{m}^{0i} = 0, \quad \bar{m}^{ij} = -\delta^{ij} \tag{2.11a}
\]

\[
\bar{m}^{00} = -1, \quad \bar{m}^{0i} = 0, \quad \bar{m}^{ij} = \delta^{ij} \tag{2.11b}
\]

where \( \delta^{ij} \) is Kronecker symbol. The Latin indices take values \{1, 2, 3\}. 


The choice between the two metric signatures (2.11) can be made if a comparison of theoretical results and experiments provides this opportunity. We can be also guided by some formal mathematical reasonings.

As mentioned above, we assume that the signature of metric can be changed on singular sets of solutions for the model under consideration. As will be shown (see section 4 below), this assumption allows to obtain more physically acceptable solutions in the case of the presence of singularities.

The signature of the metric (2.11a) allows the same spherically symmetric solution of the model that was obtained for the nonlinear electrodynamics by M Born and L Infeld in their classical work [1]:

\[
\bar{\mathcal{V}}_r = \frac{\bar{q}}{r^2}, \quad \frac{\partial \Phi}{\partial r} = \frac{\bar{q}}{\sqrt{r^4 + \bar{r}^4}},
\]

where \( \bar{q} \) is a constant, \( r \) is the radial spherical coordinate, \( \bar{r} = \sqrt[1/2]{|\bar{q}|} \).

It is evident that the solution (2.12) give birth to the class of soliton solutions through Lorentz transformations. Such solutions in this model also can be considered as point charged particles because their long-range interaction has features of electromagnetic one in particular.

Indeed to investigate the soliton interactions we can use the method based on the integral conservation law of momentum (for Born—Infeld nonlinear electrodynamics see [4, 5]). Let us consider the long-range interaction for the rest soliton-particles of the type (2.12). In this case the method gives the pure electrostatic interaction between the particles. Then we can transform the obtained law of motion for a particle to a moving reference frame. In this case Lorentz transform of the obtained electrical force gives its magnetic component. In this connection it should be noted that the Lorentz transform of the force was presented by A Einstein in last section of his classical work on special relativity [12].

It should be mentioned that using the another metric signature (2.11b) for action (2.3) leads to the following spherically symmetric solution instead of (2.12a):

\[
\bar{\mathcal{V}}_r = \frac{\tilde{q}}{r^2}, \quad \frac{\partial \Phi}{\partial r} = \frac{\tilde{q}}{\sqrt{|r^4 - \tilde{r}^4|}}.
\]

In this case we have infinity values for \( \Phi \) and \( \mathcal{F} \) on the sphere \( r = \tilde{r} \). But, as it can be shown, the field function \( \Phi \) of the solution is finite on this sphere.

More detail investigation of the long-range interaction for the solitons of type (2.12) is presented in the work [7].

Instead relations (2.4b) and (2.6) between four-vectors \( \{ \Phi_i \} \) and \( \{ \bar{\mathcal{V}}^\alpha \} \) we can consider relations between quadruples of components \( \{ \Phi_0, \, \bar{\mathcal{V}}_0, \, \bar{\mathcal{V}}_i, \, \bar{T}_i \} \) and \( \{ \Phi_0, \, \Phi_i, \, \bar{T}_0, \, \bar{T}_i \} \) in Minkowski metric (2.11). This representation can be preferable for some problems.

The appropriate solution of equations (2.4b) gives the following relations:

\[
\bar{\mathcal{V}}_0 = \left( \frac{1 \pm \chi^2 \bar{T}_0 \bar{T}_i}{1 \pm \chi^2 \Phi_0^2} \right) \Phi_0, \quad \Phi_i = \left( \frac{1 \pm \chi^2 \Phi_i^2}{1 \pm \chi^2 \bar{\mathcal{V}}_0} \right) \bar{\mathcal{V}}_0.
\]

\[
\Phi_0 = \left( \frac{1 \pm \chi^2 \Phi_0 ^2}{1 \pm \chi^2 \bar{T}_0} \right) \Phi_0, \quad \bar{\mathcal{V}}_0 = \left( \frac{1 \pm \chi^2 \bar{T}_0}{1 \pm \chi^2 \Phi_0} \right) \Phi_i.
\]

Here and below top and bottom signs are appropriate to metrics (2.11a) and (2.11b) accordingly, unless otherwise specified.

The comparison of the relations (2.13) and (2.14) gives the following expressions for the action density \( \mathcal{F} \) in Cartesian coordinates:

\[
\mathcal{F} = \sqrt{\frac{1 \pm \chi^2 \Phi_0^2}{1 \pm \chi^2 \bar{T}_0}}, \quad \bar{\mathcal{V}}_0 = \sqrt{\frac{1 \pm \chi^2 \Phi_0^2}{1 \pm \chi^2 \bar{T}_0}}.
\]

Now let us write the field equation (2.4) in Cartesian coordinates with the metric (2.11). After the differentiation \( \bar{\mathcal{V}}^{\mu} (2.4a) \) in (2.4a) and the multiplication the equation by \( L^2 \) we obtain

\[
(m^{\mu\nu}(1 + \chi^2 m^{\alpha\beta} \Phi_0, \Phi_0)) \frac{\partial^2 \Phi}{\partial x^\mu \partial x^\nu} = 0.
\]

As we see, the obtained equation does not include radicals. It is significant that the general form of this homogeneous equation does not depend on a sign of the expression \( 1 + \chi^2 m^{\alpha\beta} \Phi_0, \Phi_0 \) in the action density (2.36).

It is evident that the model under consideration keeps the invariance for the space-time rotation and the scale transformation. Thus any solution gives birth to the appropriate class of solutions with the following transform:
\[ \Phi(\{x^\mu\}) \rightarrow a \Phi(\{L_\mu^\nu x^\nu/a\}), \tag{2.16} \]

where \( L_\mu^\nu \) are the components of the space-time rotation matrix, \( a \) is a scale parameter.

### 3. Energy-momentum and angular momentum

The customary method gives the following expression for canonical energy-momentum density tensor of the model outside of singularities in Cartesian coordinates

\[ T^{\mu\nu} = \frac{1}{4\pi \chi^2} \mathcal{E} (\chi^2 \Phi^\nu \Phi^\mu - m^{\mu\nu} (1 + \chi^2 m^{\alpha\beta} \Phi_\alpha \Phi_\beta)). \tag{3.1} \]

The possible field singularities can require a modification of the energy-momentum tensor (3.1) or the appropriate differential conservation law by including singular densities. But there are solutions with singularities for which this modification is inessential because the appropriate integral conservation law is unchanged. Here we consider such solutions only.

Notice also that the canonical energy-momentum density tensor is defined here uniformly for both cases of the representation for the expression under modulus \( \pm (1 + \chi^2 m^{\alpha\beta} \Phi_\alpha \Phi_\beta) \) in the action density (2.3b). This can provide the positive definiteness of the energy density in space-time regions where the expression under modulus has negative values.

Here we use an allowed mathematical arbitrariness in the definition of the energy-momentum tensor for a field model, which does not violate its differential conservation law. But, of course, the cases for negative values of the expression under modulus in the action density (2.3b) must be considered individually from a viewpoint of their physical relevance.

As we see, the canonical tensor (3.1) is symmetrical.

To obtain the finite integral characteristics of solutions in infinite space-time we introduce the regularized energy-momentum density tensor in the following way:

\[ \tilde{T}^{\mu\nu} = T^{\mu\nu} - \tilde{T}^{\mu\nu}, \tag{3.2} \]

where \( \tilde{T}^{\mu\nu} \) is a regularizing symmetrical energy-momentum density tensor which can be defined depending on the class of solutions under consideration. Here we will use the constant regularizing tensor

\[ \tilde{T}^{\mu\nu} = - \frac{1}{4\pi \chi^2} m^{\mu\nu}. \tag{3.3} \]

We have the conservation law for the regularized energy-momentum density tensor in Cartesian coordinates

\[ \frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0. \tag{3.4} \]

Let us define the angular momentum density tensor by customary way. We have the following appropriate conservation law:

\[ \frac{\partial \text{M}^{\mu\nu}}{\partial x^\nu} = 0, \tag{3.5} \]

where

\[ \text{M}^{\mu\nu} = x^\mu \tilde{T}^{\nu\rho} - x^\nu \tilde{T}^{\mu\rho}. \tag{3.6} \]

We introduce the following special designations for energy, momentum vector, and angular momentum vector densities: \( \mathcal{E}, \mathcal{P}, \mathcal{F} \). Let us write the appropriate expressions taking into account relations (2.3b), (2.4b), and (2.14):

\[ \mathcal{E} \equiv \mathcal{E}^{00} = \frac{1}{4\pi \chi^2} (\chi^2 (\Phi_1 \Phi_2) + m^{00} (\mathcal{E} - 1)), \tag{3.7a} \]

\[ \mathcal{P}^i \equiv \mathcal{P}^{0i} = \mathcal{E}^{0i} = \frac{1}{4\pi} \frac{\Phi^0 \Phi^i}{\mathcal{E}} = \frac{1}{4\pi} \Phi^0 \gamma^i = \frac{1}{4\pi} \Phi^i \gamma^0, \tag{3.7b} \]

\[ \mathcal{F}_i \equiv \epsilon_{ijk} x^j \mathcal{P}^k, \tag{3.7c} \]

where \( \epsilon_{ijk} \) is Levi-Civita symbol (\( \epsilon_{123} = 1 \)).
Let us define energy, momentum, and angular momentum of the field in a three-dimensional volume $V$:

$$
E_V = \int_V E \, dV, \quad P_V = \int_V P \, dV, \quad J_V = \int_V J \, dV.
$$

(3.8)

4. General lightlike soliton

Let us consider the solution in a form of wave propagating along $x^3$ axis of Cartesian coordinate system with the speed of light. Let this solution be have some transverse and longitudinal field distributions. Thus we can write

$$
\Phi = \Phi(\theta, x^1, x^2), \quad \theta = \omega x^0 - k_3 x^3, \quad k_3^2 = \omega^2, \quad \omega > 0.
$$

(4.1a)

(4.1b)

Substitution (4.1) to field equation (2.15) gives the following equation:

$$
\left(1 \mp \chi^2 \left( \frac{\partial^2 \Phi}{\partial x^2} \right)^2 \right) \frac{\partial^2 \Phi}{\partial x^2} \pm 2 \chi^2 \frac{\partial^2 \Phi}{\partial x^1 \partial x^2} \frac{\partial^2 \Phi}{\partial x^1 \partial x^2} + \left(1 \mp \chi^2 \left( \frac{\partial^2 \Phi}{\partial x^2} \right)^2 \right) \frac{\partial^2 \Phi}{\partial x^2} = 0,
$$

(4.2)

where top and bottom signs are appropriate to Minkowski metrics (2.11a) and (2.11b) accordingly.

As we see the obtained equation (4.2) does not include derivatives on phase of wave $\Phi(4.1b)$.

The equation (4.2) is elliptical with the following condition:

$$
1 \mp \chi^2(\Phi_{x^1}^2 + \Phi_{x^2}^2) > 0.
$$

(4.3)

The similar in form (4.2) equations were considered. About this topic see the paper by R. Ferraro [13] and references therein.

In particular, the similar in form but different in type equation was considered by B M Barbashov and N A Chernikov [14]. A Lax representation (see, for example, [15]) for this equation was presented in an article by J C Brunelli and A. Das [16].

The monograph by G.B. Whitham [15] contains relatively simple way for obtaining the Barbashov—Chernikov solution with the help of hodograph transformation (see, for example, [17]).

Here we use in outline the Whitham method but for the elliptic (for condition (4.3)) equation (4.2). The qualitative difference between hyperbolic and elliptic equations causes the appropriate difference in the solution way.

Let us introduce new independent variables

$$
\xi = x^3 + r x^2, \quad *\xi = x^3 - r x^2,
$$

(4.4)

where $r^2 = -1$.

Also we will use cylindrical coordinates $\{\rho, \varphi, x^3\}$. We have the following evident relations:

$$
\xi = \rho e^{i \varphi}, \quad *\xi = \rho e^{-i \varphi},
$$

(4.5a)

$$
\rho = \sqrt{\xi *\xi}, \quad \varphi = -i \ln(\xi / \sqrt{\xi *\xi}).
$$

(4.5b)

Using new variables (4.4) we obtain from (4.2) the following equation:

$$
\left(1 \mp 2 \chi^2 \frac{\partial^2 \Phi}{\partial \xi \partial *\xi} \frac{\partial^2 \Phi}{\partial \xi \partial *\xi} \right) \frac{\partial^2 \Phi}{\partial \xi \partial *\xi} \pm \chi^2 \left( \frac{\partial^2 \Phi}{\partial \xi \partial *\xi} \right)^2 \pm \chi^2 \left( \frac{\partial^2 \Phi}{\partial \xi \partial *\xi} \right)^2 = 0.
$$

(4.6)

Equation (4.6) is hyperbolic with the following condition:

$$
1 \mp 4 \chi^2(\Phi_{\xi}^2 + \Phi_{*\xi}^2) > 0.
$$

(4.7)

As it was noted in section 2 the field model under consideration is invariant by space-time rotations and scale transformations. But the equation (4.2) does not contain derivatives with respect to coordinates $\{x^0, x^3\}$. Because this here we have space-time rotation and scale invariance in the planes $\{x^1, x^2\}$ and $\{x^0, x^3\}$ with mutually independent parameters. Thus equation (4.2) is invariant with respect to rotation about $x^3$ axis and scale transformation in $\{x^1, x^2\}$ plane.
Taking into account relations (4.5) and (2.16), these two types of invariance applied to equation (4.6) are provided by the following general substitution:

\[ \Phi(\zeta, \eta) \rightarrow \sqrt{\sigma} \Phi(\zeta/\sigma, \eta/\sigma), \]

where \( \sigma \) is arbitrary complex constant with respect to coordinates \( \{ \zeta, \eta \} \), \( \bar{\sigma} \) is complex conjugate to \( \sigma \) quantity. The constant \( \sigma \) will be called the scale-rotation parameter of solution in the plane \( \{ x^1, x^2 \} \).

The complex constant \( \sigma \) can be written in the form

\[ \sigma = \bar{\sigma} e^{i \varphi}, \]

where \( \bar{\sigma} \) and \( \varphi \) are real constants with respect to coordinates \( \{ x^1, x^2 \} \). But in general case these constants can be depend on phase of soliton \( \theta \) (4.1b):

\[ \bar{\sigma} = \bar{\sigma}(\theta), \quad \varphi = \varphi(\theta). \]

Because this we will call \( \sigma(\theta) \) the scale-rotation function of the soliton. It is evident that the function \( \sigma(\theta) \) defines the phase dependence of transversal scale and the function \( \varphi(\theta) \) defines the phase dependence of rotation about \( x^3 \) axis.

Thus if we have a solution \( \Phi(\zeta, \eta) \) to equation (4.6) then by means of invariant substitution (4.8) we obtain wave propagating along \( x^3 \) axis and preserving its transversal form. Longitudinal form of the wave defined by scale-rotation phase function \( \sigma(\theta) \) is also preserved.

As a result, we have in (4.8) a wave packet propagating with speed of light and preserving its shape. It will be called the lightlike soliton.

Hereafter up to formulas (4.23) we consider the equation (4.6) for the case of top signs that is for the metric (2.11a). But the obtained solution will be represented in general form which is appropriate to both metrics (2.11).

Thus the equation (4.6) with top signs is equivalent to the following first order system:

\[ \frac{\partial \Phi}{\partial \xi} - \frac{\partial \Phi_{\xi \zeta}}{\partial \zeta} = 0, \]

\[ (1 - 2 \chi^2 \Phi_{\xi} \Phi_{\zeta}) \frac{\partial \Phi}{\partial \xi} + \chi^2 \Phi_{\xi} \frac{\partial \Phi_{\xi \zeta}}{\partial \zeta} + \chi^2 \Phi_{\zeta} \frac{\partial \Phi_{\xi \zeta}}{\partial \zeta} = 0, \]

where

\[ \Phi_{\xi} \equiv \frac{\partial \Phi}{\partial \xi}, \quad \Phi_{\xi \zeta} \equiv \frac{\partial \Phi_{\xi \zeta}}{\partial \zeta}. \]

By interchanging the roles of the dependent and independent variables in (4.9) we obtain the linear system

\[ \frac{\partial \zeta}{\partial \Phi_{\xi \zeta}} - \frac{\partial \Phi_{\xi \zeta}}{\partial \Phi_{\xi \zeta}} = 0, \]

\[ (1 - 2 \chi^2 \Phi_{\xi} \Phi_{\zeta}) \frac{\partial \Phi}{\partial \xi} - \chi^2 \Phi_{\xi} \frac{\partial \Phi_{\xi \zeta}}{\partial \zeta} - \chi^2 \Phi_{\zeta} \frac{\partial \Phi_{\xi \zeta}}{\partial \zeta} = 0, \]

which is equivalent to the single equation

\[ (1 - 2 \chi^2 \Phi_{\xi} \Phi_{\zeta}) \frac{\partial^2 \zeta}{\partial \Phi_{\xi \zeta}^2} - \chi^2 \Phi_{\xi} \frac{\partial^2 \Phi_{\xi \zeta}}{(\partial \Phi_{\xi \zeta})^2} - \chi^2 \Phi_{\zeta} \frac{\partial^2 \Phi_{\xi \zeta}}{(\partial \Phi_{\xi \zeta})^2} - 2 \chi^2 \Phi_{\xi} \frac{\partial \zeta}{\partial \Phi_{\xi \zeta}} - 2 \chi^2 \Phi_{\zeta} \frac{\partial \zeta}{\partial \Phi_{\xi \zeta}} = 0. \]

Let us introduce new independent variables in (4.12)

\[ \eta_1 = \sqrt{1 - 4 \chi^2 \Phi_{\xi} \Phi_{\zeta}} - 1, \]

\[ \eta_2 = -i \frac{\sqrt{1 - 4 \chi^2 \Phi_{\xi} \Phi_{\zeta}} - 1}{2 \chi \Phi_{\xi}}. \]

The inversion for relations (4.13) gives

\[ \Phi_{\xi} = \frac{i \eta_1}{\chi (1 + \eta_1 \eta_2)}, \]
\[ \Phi_{\xi} = - \frac{1}{\chi} \eta_1 \eta_2. \] (4.14b)

Substituting (4.13) and (4.14) into (4.11), we obtain
\[ \eta_1^2 \frac{\partial \xi}{\partial \eta_1} + \frac{\partial \xi}{\partial \eta_2} = 0, \] (4.15a)
\[ \frac{\partial \xi}{\partial \eta_2} + \eta_2^2 \frac{\partial \xi}{\partial \eta_1} = 0. \] (4.15b)

The sequential elimination each of the dependent variables \( \xi(\eta_1, \eta_2) \) and \( \xi(\eta_1, \eta_2) \) from system (4.15) gives two simple equations
\[ \frac{\partial^2 \xi}{\partial \eta_1 \partial \eta_2} = 0, \quad \frac{\partial^2 \xi}{\partial \eta_1 \partial \eta_2} = 0, \] (4.16)
solutions of which have the form
\[ \xi = \xi(\eta_1) + \xi(\eta_2), \quad \xi = \xi(\eta_1) + \xi(\eta_2), \] (4.17)
where \( \xi(\eta_1) \), \( \xi(\eta_2) \), \( \xi(\eta_1) \), \( \xi(\eta_2) \) are arbitrary functions.

Substitution (4.17) to (4.15) gives
\[ \eta_1^2 \xi_1' + \xi_4' = 0, \quad \xi_3' + \eta_2^2 \xi_2' = 0. \] (4.18)

Taking into consideration (4.17) and (4.18), we can write the following relations for general solution of the system (4.15):
\[ d\xi = \xi_1 - \eta_1^2 \xi_2' d\eta_2 = d\xi_1 - \eta_2^2 d\xi_2', \] (4.19a)
\[ d\xi = \xi_2 - \eta_2^2 \xi_1' d\eta_1 = d\xi_2 - \eta_2^2 d\xi_1'. \] (4.19b)

Thus the general solution of the system (4.15) contains only two arbitrary functions \( \xi(\eta_1) \) and \( \xi(\eta_2) \).

Using (4.14) and (4.19), we obtain
\[ \frac{\partial \Phi}{\partial \eta_1} = \Phi_{\xi} \frac{\partial \xi}{\partial \eta_1} + \Phi_{\xi} \frac{\partial \xi}{\partial \eta_1} = \frac{1}{\chi} \eta_1 \xi_1', \] (4.20a)
\[ \frac{\partial \Phi}{\partial \eta_2} = \Phi_{\xi} \frac{\partial \xi}{\partial \eta_2} + \Phi_{\xi} \frac{\partial \xi}{\partial \eta_2} = \frac{1}{\chi} \eta_2 \xi_2'. \] (4.20b)

From (4.20) we have
\[ d\Phi = \frac{1}{\chi} (\eta_1 \xi_1' d\eta_1 - \eta_2 \xi_2' d\eta_2) = \frac{1}{\chi} (\eta_1 d\xi_1 - \eta_2 d\xi_2). \] (4.21)

Here the variables \( \eta_1 \) and \( \eta_2 \) in last expression must be considered as inverse functions for \( \xi(\eta_1) \) and \( \xi(\eta_2) \) that are \( \eta_1 = \eta_1(\xi) \) and \( \eta_2 = \eta_2(\xi) \).

Let us introduce the designations
\[ \frac{1}{\chi} \eta_1(\xi) \equiv \frac{d\Xi_1}{d\xi}, \quad \frac{1}{\chi} \eta_2(\xi) \equiv \frac{d\Xi_2}{d\xi}, \] (4.22)
where functions \( \Xi_1(\xi) \) and \( \Xi_2(\xi) \) are arbitrary because of arbitrariness of the functions \( \xi(\eta_1) \) and \( \xi(\eta_2) \).

Then, using (4.21) and (4.22), we have the general solution of equation (4.6) in the form
\[ \Phi = \Xi_1(\xi_1) + \Xi_2(\xi_2). \] (4.23a)

Here the arbitrariness of the functions \( \Xi_1(\xi) \) and \( \Xi_2(\xi) \) is restricted by reality of the field function \( \Phi \). The connection between variables \( \{\xi, \xi'\} \) and \( \{\xi, \xi^*\} \) is defined (for both metrics (2.11)) by relations
\[ \left( \begin{array}{c} d\xi \\ d\xi^* \end{array} \right) = \left( \begin{array}{cc} 1 & \pm \chi^2 (\Xi_1')^2 \\ \pm \chi^2 (\Xi_2')^2 & 1 \end{array} \right) \left( \begin{array}{c} d\xi_1 \\ d\xi_2 \end{array} \right) \] (4.23b)
which (for top signs) are obtained from (4.19) with (4.22).

Expressions (4.23) with scale-rotation transformation (4.8) represent the general lightlike soliton of the model under consideration.

Relations (4.23b) can be inverted on the assumption of nonsingularity of the transformation matrix:
\[ 1 - \chi^4 (\Xi_1')^2 (\Xi_2')^2 = 0, \] (4.24a)
\[
\left( \frac{d\xi}{d\zeta} \right)_a = \frac{1}{1 - \chi^4 \left( \Xi_a^2 \right)^2} \left( \frac{1}{1 + \chi^2 \left( \Xi_a^2 \right)^2} \right)^{\frac{1}{2}} \left( \frac{1}{1 + \chi^2 \left( \Xi_a^2 \right)^2} \right)^{\frac{1}{2}} \frac{d\xi}{d\zeta}.
\] (4.24b)

The obtained solution (4.23) can be checked directly. Substitution (4.23a) to equation (4.6) and using (4.24) reduce to identity.

One could say that relations (4.23b) define the transformation of independent variables \( \{\zeta, \xi\} \) to \( \{\tilde{\zeta}, \tilde{\xi}\} \) for the equation (4.6). But the definition of the transformation with differential relations is not complete. The direct connection between the variables can be obtained by the path integration of relations (4.23b) in a nonsingular area, that is for the condition (4.24a). At the same time we must define an initial correspondence between the variables \( \{\zeta, \xi\} \) and \( \{\tilde{\zeta}, \tilde{\xi}\} \).

It is useful to introduce polar coordinates \( \{\rho, \varphi\} \) and \( \{\tilde{\rho}, \tilde{\varphi}\} \) for the variables \( \{\tilde{\zeta}, \tilde{\xi}\} \) by analogy with the polar coordinates \( \{\rho, \varphi\} \) (4.5) for the variables \( \{\zeta, \xi\} \):

\[
\tilde{\zeta}_1 = \tilde{\rho}_1 e^{i \tilde{\varphi}_1}, \quad \tilde{\zeta}_2 = \tilde{\rho}_2 e^{i \tilde{\varphi}_2}.
\] (4.25)

We can consider the simplest case by taking \( \chi = 0 \) in (4.23b). In this case we can put

\[
\begin{pmatrix}
\tilde{\xi}_1 \\
\tilde{\xi}_2
\end{pmatrix} = \begin{pmatrix}
\xi_1 \\
\xi_2
\end{pmatrix},
\] (4.26)

and expression (4.23a) becomes evident solution of the appropriate to (4.6) linear equation when \( \chi = 0 \).

In general case let us consider relation (4.26) as asymptotic for \( \rho \to \infty \). Then for an area including infinity we can use the following designations:

\[
\tilde{\zeta}_1 = \zeta, \quad \tilde{\zeta}_2 = \tilde{\xi}, \quad \tilde{\rho}_1 = \tilde{\rho}_2 = \tilde{\rho}, \quad \tilde{\varphi}_1 = -\tilde{\varphi}_2 = \tilde{\varphi}.
\] (4.27)

The general solution in the form of lightlike soliton depending on the phase \( \theta \) (4.1b) can be obtained from (4.23) by the invariant substitution (4.8).

It is notable that the action density for the obtained solution does not contain radical. The substitution solution (4.8) with (4.23a) into (2.3b) with (4.4) and (4.24b) gives the expression

\[
\mathcal{E} \equiv \begin{vmatrix}
1 + \chi^2 \Xi_1^2 \\
1 + \chi^2 \Xi_2^2
\end{vmatrix}.
\] (4.28)

As we see, explicit dependence on phase \( \theta \), which we have in (4.8), here is absent.

As a consequence of probes of various arbitrary functions \( \Xi_1(\tilde{\zeta}) \) and \( \Xi_2(\tilde{\zeta}) \) we have obtained that the solutions under consideration have tubelike shells, where the condition (4.24a) is broken. Note that the lightlike soliton can have a set of such shells. Thus the obtained solution can be called the shell lightlike soliton.

In order to identify some features in the use of the formulas (4.23) and (4.24), let us consider a simplest solution which does not depend on polar angle \( \varphi \).

This cylindrically symmetric solution is similar to the spherically symmetric one (2.12) and it can be obtained directly without the use of formulas (4.23) and (4.24).

Relations (2.3b) and (2.4) written in cylindrical coordinates give in this case the following expressions for the action density and the radial derivative of the field function:

\[
\begin{align*}
\mathcal{E} &= \sqrt{\left[1 + \chi^2 \Phi^2_{\rho} \right]} \quad \text{(4.29)} \\
\gamma_{\rho} &= -\frac{2 \tilde{q}}{\rho} \quad \Rightarrow \quad \frac{\partial \Phi}{\partial \rho} = \frac{2 \tilde{q}}{\sqrt{\rho^2 + \tilde{p}^2}}, \quad \tilde{p} \equiv |2 \tilde{q} \chi|.
\end{align*}
\] (4.30)

where \( \tilde{q} \) is a real constant representing a linear charge density.

An integration of the expression of derivative \( \Phi_{\rho} \) (4.30) for the cases of top and bottom signs gives accordingly

\[
\Phi = \begin{cases} 
-2 \tilde{q} \ln \frac{\rho + \sqrt{\rho^2 + \tilde{p}^2}}{\tilde{p}}, & \text{if } \rho \geq \tilde{p}, \\
-2 i \tilde{q} \ln \frac{\rho + i \sqrt{\rho^2 - \tilde{p}^2}}{\tilde{p}} = 2 \tilde{q} \arccos(\rho/\tilde{p}), & \text{if } \rho \leq \tilde{p}.
\end{cases}
\] (4.31a)

\[
\Phi = \begin{cases} 
-2 \tilde{q} \ln \frac{\rho + \sqrt{\rho^2 - \tilde{p}^2}}{\tilde{p}}, & \text{if } \rho \geq \tilde{p}, \\
-2 i \tilde{q} \ln \frac{\rho + i \sqrt{\rho^2 - \tilde{p}^2}}{\tilde{p}} = 2 \tilde{q} \arccos(\rho/\tilde{p}), & \text{if } \rho \leq \tilde{p}.
\end{cases}
\] (4.31b)
In the case of the first solution in (4.31) the field function \( F \) (4.31a) and its derivative \( F' \) (4.30, top sign) are finite and regular at any finite point (for \( \rho < \infty \)). The appropriate plots are shown on figure 1. It can be shown that the energy of this solution is logarithmically divergent for \( \rho \to \infty \).

In the case of the second solution in (4.31) the field function \( F \) (4.31b) is finite for \( \rho < \infty \) and it becomes zero at \( \rho = \tilde{\rho} \) while remaining continuous everywhere. But its derivative \( F' \) (4.30, bottom sign) becomes infinite at \( \rho = \tilde{\rho} \).

Thus there is the singular surface \( \rho = \tilde{\rho} \) in this case. The appropriate plots are shown on figure 2. The energy of this solution is logarithmically divergent for \( \rho \to \infty \) only. It can be shown that the energy in the region \( 0 \leq \rho < \tilde{\rho} \) is finite and positive.

It should be noted that the choice of the top sign in (4.30) and the appropriate solution (4.31a) correspond to the signature of metric (2.11a) in the whole space.

The choice of the bottom sign in (4.30) and the appropriate solution (4.31b) correspond to the signature of metric (2.11b) in the region \( \rho > \tilde{\rho} \). In the region \( 0 < \rho < \tilde{\rho} \), the solution (4.31b) corresponds to the signature of metric (2.11a). In this case, in the region \( 0 < \rho < \tilde{\rho} \), the alternative of representation of modulus in (4.30) \( \rho^2 - \tilde{\rho}^2 \) corresponds to the action density (4.29) in the form \( \mathcal{E} = \sqrt{1 - \chi^2 \Phi_{\rho}^2} = \sqrt{\chi^2 \Phi_{\rho}^2 - 1} \).

In consequence, as we see on figure 2, the field function of the solution (4.31b) is continuous for \( \rho < \infty \) and the appropriate configuration of the space-time film represents a smooth surface for \( 0 < \rho < \infty \).

Thus in this case, the change of the signature of metric on the singular set allows to obtain the solution in the form of a smooth surface for \( \rho > 0 \).

Now, to obtain the solution (4.31) with the help of the relations (4.23), we take the arbitrary functions of the general solution (4.23) in the following form:

\[
\xi_1 = -\frac{\xi_1}{\chi} \ln \left( \frac{\xi_1}{\rho} \right), \quad \xi_2 = -\frac{\xi_2}{\chi} \ln \left( \frac{\xi_2}{\rho} \right),
\]

\[
\xi'_1 = -\frac{\xi_1}{\chi \xi_1}, \quad \xi'_2 = -\frac{\xi_2}{\chi \xi_1},
\]

where \( \{ \xi_1, \xi_2 \} \) are complex constants, \( \rho \) is a real positive constant having the physical dimension of length and providing the dimensionless for argument of logarithm.

The substitution of these functions (4.32) into relations (4.23b) give the following dependence between the differentials of old and new independent variables:
\[ d\xi = d\xi_1 \pm \frac{\xi_2^2}{\xi_1^2} \, d\xi_2, \quad d^*\xi = \pm \frac{\xi_1^2}{\xi_2^2} \, d\xi_1 + d\xi_2. \]  

(4.33)

The condition for nonsingularity of the transformation matrix to the variables \( \{\xi, \, *\xi\} \) (4.24a) has the following form in the case under consideration (4.33):

\[ \frac{\xi_1^2}{\xi_2^2} \frac{\xi_2^2}{\xi_1^2} \equiv 1. \]  

(4.34)

To obtain the functions \( \tilde{\xi} (\xi) \) and \( \tilde{\xi} (\xi^*) \), first we must integrate relations (4.33). For this purpose, it is convenient to use the polar coordinates \( \{\rho, \, \varphi\} \), \( \{\rho, \, \varphi\} \) (4.25), and \( \{\rho, \, \varphi\} \) (4.5).

Let us take the paths of integration in the planes \( \{\tilde{\rho}, \, \tilde{\varphi}\} \), \( \{\tilde{\rho}, \, \tilde{\varphi}\} \), and \( \{\rho, \, \varphi\} \) with sufficiently far beginning points from coordinate origin. Let the starting points in these planes be \( \{\rho_{1\infty}, \, 0\} \), \( \{\rho_{2\infty}, \, 0\} \), and \( \{\rho_{\infty}, \, 0\} \). In the plane \( \{\tilde{\rho}, \, \tilde{\varphi}\} \), for example, we can integrate by the following path: from \( \{\rho_{1\infty}, \, 0\} \) to \( \{\rho_1, \, 0\} \) for \( \tilde{\varphi}_1 = \text{const} \) and from \( \{\rho_1, \, 0\} \) to \( \{\tilde{\rho}, \, \tilde{\varphi}_1\} \) for \( \rho_1 = \text{const} \). Similarly, we integrate in the plane \( \{\tilde{\rho}, \, \tilde{\varphi}\} \) as well as the left parts of equalities (4.33) in the plane \( \{\rho, \, \varphi\} \).

Then we take that \( \rho_{1\infty} \to \infty, \rho_{2\infty} \to \infty, \) and \( \rho_{\infty} \to \infty \). Using again the variables \( \{\tilde{\xi}, \, \tilde{\xi}\} \), \( \{\tilde{\xi}, \, \tilde{\xi}\} \), and \( \{\xi, \, *\xi\} \), as a result we have

\[ \xi = \tilde{\xi}_1 = \frac{\xi_2^2}{\xi_1^2}, \quad *\xi = \tilde{\xi}_2 = \frac{\xi_1^2}{\xi_2^2}. \]  

(4.35)

Two solutions for the system of equations (4.35) with respect to variables \( \{\tilde{\xi}, \, \tilde{\xi}\} \) can be written in an explicit form. But here we write these solution for the special case

\[ \xi_1^2 = \xi_2^2. \]  

(4.36a)

Using the representation of the variables \( \{\xi, \, *\xi\} \) in polar coordinates \( \{\rho, \, \varphi\} \) (4.5) we have

\[ \tilde{\xi}_1 = \frac{e^{i\varphi}}{2} (\rho + \sqrt{\rho^2 \pm 4 \xi_1^2}), \quad \tilde{\xi}_2 = \frac{e^{-i\varphi}}{2} (\rho + \sqrt{\rho^2 \pm 4 \xi_1^2}), \]  

(4.36b)

\[ \tilde{\xi}_1 = \frac{e^{i\varphi}}{2} (\rho - \sqrt{\rho^2 \pm 4 \xi_1^2}), \quad \tilde{\xi}_2 = \frac{e^{-i\varphi}}{2} (\rho - \sqrt{\rho^2 \pm 4 \xi_1^2}). \]  

(4.36c)

As can be easy seen, the solution (4.36b) satisfies the asymptotic relations \( \tilde{\xi} \to \xi \) and \( \tilde{\xi} \to *\xi \) for \( \rho \to \infty \).

But the solution (4.36c) gives the asymptotic relations \( \tilde{\xi} \to 0 \) and \( \tilde{\xi} \to 0 \) for \( \rho \to \infty \).

Let us consider the solution (4.36) with equal and real constants \( \{\xi, \, \xi\} \) such that

\[ \xi_1 = \xi_2 = \overline{\rho} \chi, \quad |\overline{\rho} \chi| = \frac{\rho}{2} \equiv \rho. \]  

(4.37)

Here \( \rho \) is the real positive constant contained also in the formulas for the functions \( \Xi_1 \) and \( \Xi_2 \) (4.32a). The constants \( \overline{\rho} \) and \( \rho \) are contained in the solutions (4.31).

In this case (4.37), the consecutive substitution (4.36b) to (4.32a) and (4.32a) to (4.23a) gives the solution (4.31a) for the signature (2.1.1a) and the solution (4.31b) in the region \( \rho \geq \rho \) for the signature (2.1.1b). The second solution (4.36c) gives the same field function \( \Phi \) but with the opposite sign.

We can see from (4.36), the tilde variables \( \{\tilde{\xi}, \, \tilde{\xi}\} \) are mutually complex conjugated in this case such that the designations without indexes \( \{\xi, \, \xi\} \) (4.27) can be used. Then according to (4.37), the condition for nonsingularity of the transformation matrix (4.34) takes the form

\[ \bar{\rho} = \rho \iff \bar{\rho} = \frac{\rho}{2}. \]  

(4.38)

Thus the determinant of matrix for the transformation of variables \( \{\tilde{\xi}, \, *\xi\} \to \{\xi, \, *\xi\} \) (4.23b) for the functions (4.32a) is zero at the ring with radius \( \bar{\rho} = \rho = \frac{\rho}{2} \) in the variables \( \{\tilde{\xi}, \, *\xi\} \). According to equations (4.35), in the case of the signature of metric (2.1.1a) (top sign in (4.35)) this ring is reflected to the origin of the initial coordinate system \( \rho = 0 \). But in the case of signature of metric (2.1.1b) (bottom sign in (4.35)) the ring \( \rho = \rho = \frac{\rho}{2} \) in the variables \( \{\xi, \, \xi\} \). The formulas (4.36) give the appropriate inverse mappings. Note that for the signature of metric (2.1.1b) we must use the substitution \( \sqrt{\bar{\rho}} \to t \sqrt{\rho} \) in (4.36). Thus we have the following mutually mappings of the regions for the two signatures of metric (2.1.1a) and (2.1.1b):

\[ \rho = 0 \iff \bar{\rho} = \rho = \frac{\rho}{2}. \]  

(4.39a)
\[ \rho = 2 \varphi = \tilde{\rho} \quad \leftrightarrow \quad \tilde{\rho} = \rho = \frac{\tilde{\rho}}{2}. \]  \hfill (4.39b)

To obtain the solution (4.31b) in the region \( \rho \leq \tilde{\rho} \), we consider the case of equal and purely imaginary constants in (4.36):

\[ \xi_1 = \xi_2 = i \tilde{\varphi} \chi, \quad |\tilde{\varphi} \chi| = \frac{\tilde{\rho}}{2} = \rho. \]  \hfill (4.40)

Now the tilde variables \( \tilde{\xi}_1 \) and \( \tilde{\xi}_2 \) are not mutually complex conjugated in general.

Substituting (4.40) into (4.36b) and using relation \( \sqrt{-1} = i \), we obtain the solution for the system of equations (4.35) in the following form:

\[ \tilde{\xi}_1 = \frac{e^{i\varphi}}{2}(\rho + i\sqrt{\tilde{\rho}^2 - \rho^2}), \quad \tilde{\xi}_2 = \frac{e^{-i\varphi}}{2}(\rho + i\sqrt{\tilde{\rho}^2 - \rho^2}). \]  \hfill (4.41)

The substitution (4.40) and (4.41) into expressions (4.32a) for the functions \( \Xi_1 \) and \( \Xi_2 \) gives the solution (4.23a) which is real for the signature of metric (2.11a) (top sign in (4.41)) in the region \( \rho \leq \tilde{\rho} \). This solution coincides with (4.31b) for \( \rho \leq \tilde{\rho} \).

It should be noted also that top and bottom signs in the expressions (4.33), (4.35), and (4.36) are reversed when the type of the constants \( \tilde{\xi}_1 \) and \( \tilde{\xi}_2 \) for solution (4.32) is changed from (4.37) to (4.40).

Now let us obtain the energy, momentum, and angular momentum densities for a lightlike soliton. For this purpose we substitute solution (4.23a) with scale-rotation transformation (4.8) to formulas (3.7).

Using relations (4.4) and (4.24b), we obtain the expressions for energy, momentum, and angular momentum densities containing some common functions, which we denote by \( f^\xi_0 \). Then we have

\[ E = f^\xi_0 + \omega^2((\varphi')^2 f^\xi_0 + (\varphi')^2 (f^\xi_0 / \varphi + f^\xi_0 / \varphi + f^\xi_0 / \varphi + f^\xi_0 / \varphi)), \]  \hfill (4.42a)

\[ P^\varphi = k \omega ((\varphi')^2 f^\xi_0 + (\varphi')^2 (f^\xi_0 / \varphi + f^\xi_0 / \varphi + f^\xi_0 / \varphi + f^\xi_0 / \varphi)), \]  \hfill (4.42b)

\[ J^\varphi = \omega (\varphi' f^\xi_0 + \varphi' (f^\xi_0 / \varphi + f^\xi_0 / \varphi)), \]  \hfill (4.42c)

where \( k_1^2 = \omega^2 \) according to (4.1b).

Here we write explicitly only two functions:

\[ \frac{f^\xi}{2\pi} = \frac{1}{2\pi} \frac{\Xi_1 \Xi_2}{1 + \chi^2 \Xi_1 \Xi_2}, \quad \frac{f^\xi}{2} = \frac{1}{4\pi} \frac{(\xi e^{-i\varphi} \Xi_1 - \xi e^{i\varphi} \Xi_2)\varphi}{1 - \chi^2 (\Xi_1 \Xi_2)^2} \]  \hfill (4.43)

These functions play the main role in the area, where the scale function \( \rho(\varphi) \) is almost constant: \( \varphi \to 0 \). We have from (4.42) the following notable relation for the case \( \varphi' \to 0 \):

\[ E - f^\xi_0 = |P_\varphi| = \omega |\varphi' J^\varphi|. \]  \hfill (4.44)

The arbitrary functions \( \varphi(\varphi) \) and \( \varphi(\varphi) \) (4.8c) define scale and rotation in the plane \( \{x^1, x^2\} \) respectively. Using (5.2), (4.8), and (4.1b), we can show that the case \( \varphi' > 0 \) corresponds to positive rotation by angle \( \varphi \) in time \( x^0 \) and in \( x^3 \) axis for \( k_1 > 0 \).

Thus for right-handed coordinate system \( \{x^1, x^2, x^3\} \), the cases \( \varphi' > 0 \) and \( \varphi' < 0 \) correspond to right and left local twist of the soliton accordingly.

It is interesting to consider the solitons with constant twist:

\[ \varphi' = \text{const} \neq 0. \]  \hfill (4.45)

Such solitons can be called the uniformly twisted ones. For conciseness we will call them the twisted solitons.

As we see in (4.44), for the case (4.45) the soliton energy density \( E \) is proportional to its angular momentum density \( J \) in high-frequency approximation, that is for \( \omega |\varphi' J^\varphi| \gg |f^\xi_0| \). The appropriate proportionality relation between soliton energy and its angular momentum is notable property of the twisted lightlike soliton.

To obtain integral characteristics of the soliton it is necessary to integrate the functions \( \{f^\xi_0, \ldots, f^\xi_6\} \) in the plane \( \{x^1, x^2\} \).

Considering (4.23b), we can see that the appropriate integrands (4.43) have notable simple form in the tilde variables \( \{\tilde{\xi}_1, \tilde{\xi}_2\} \). Here we obtain these expressions for the case of their mutually complex conjugation (4.27) such that \( \{\tilde{\xi}_1, \tilde{\xi}_2\} = \{\tilde{\xi}, \tilde{\xi}\} \).

We must take into consideration also that the functions \( \{\tilde{\xi}, \tilde{\xi}\} \) and, accordingly, \( \{\Xi_1, \Xi_2\} \) depends on arguments \( \{\xi / \sigma, \xi / \sigma\} \) after the scale-rotation transformation (4.8).

Thus making additional substitution \( \{\xi / \sigma, \xi / \sigma\} \to \{\xi, \xi\} \) and using the polar coordinates \( \{\rho, \varphi\} \) and \( \{\tilde{\rho}, \varphi\} \), we obtain the following integrands:
where top and bottom signs are appropriate to metrics (2.11a) and (2.11b) accordingly.

5. Twisted lightlike soliton

For further calculations, we define the arbitrary functions \( \Xi_i \) and \( \Xi_2 \). Let us take power function with integer negative exponent for the tilde variables \( \tilde{\xi}_1, \tilde{\xi}_2 \). Introducing also the appropriate multiplicative complex constants \( \xi_1, \xi_2 \) and considering the necessity for concordance of physical dimension, we can write

\[
\Xi_1 = \xi_1^{m+1} \frac{\tilde{\xi}_1^{-m}}{\chi^m}, \quad \Xi_2 = \xi_2^{m+1} \frac{\tilde{\xi}_2^{-m}}{\chi^m},
\]

(5.1a)

\[
\Xi'_1 = -\frac{1}{\chi} \left( \frac{\xi_1}{\tilde{\xi}_1} \right)^{m+1}, \quad \Xi'_2 = -\frac{1}{\chi} \left( \frac{\xi_2}{\tilde{\xi}_2} \right)^{m+1},
\]

(5.1b)

where \( m \) is natural number.

Then the formula (4.23a) representing the solution of equation (4.6) gives the following expression:

\[
\Phi = \frac{1}{\chi^m} (\xi_1^{m+1} \tilde{\xi}_1^{-m} + \xi_2^{m+1} \tilde{\xi}_2^{-m}).
\]

(5.2)

By analogy with derivation of formulas (4.33), (4.34), and (4.35), we obtain the following relations:

\[
d\xi_1 = d\tilde{\xi}_1 \pm \left( \frac{\xi_2}{\xi_1} \right)^{2(m+1)} d\tilde{\xi}_2, \quad d\xi_2 = \pm \left( \frac{\xi_2}{\xi_1} \right)^{2(m+1)} d\tilde{\xi}_1 + d\tilde{\xi}_2,
\]

(5.3)

\[
\left( \frac{\xi_1}{\xi_2} \right)^{2(m+1)} = 1,
\]

(5.4)

\[
\xi_1 = \tilde{\xi}_1 = \frac{\xi_2^{2(m+1)}}{2m+1}, \quad \xi_2 = \tilde{\xi}_2 = \frac{\xi_1^{2(m+1)}}{2m+1}.
\]

(5.5)

It should be noted that the formulas (5.1b) and (5.3)–(5.5) for \( m = 0 \) give the formulas (4.32b) and (4.33)–(4.35) accordingly.

Without restricting generality we can assume that the complex constants \( \xi_1, \xi_2 \) have identical modulus \( \rho \) with physical dimension of length. Let us represent it in the following form:

\[
\xi_1 = \rho e^{i \varphi_1}, \quad \xi_2 = \rho e^{i \varphi_2}.
\]

(5.6a)

According to the asymptotic condition (4.26), the tilde variables are mutually complex-conjugated \( \{ \tilde{\xi}_1, \tilde{\xi}_2 \} = \{ \tilde{\xi}_1, \tilde{\xi}_2 \} \) (4.27) in the region including infinity. For this case the phase constants \( \{ \varphi_1, \varphi_2 \} \) in (5.6a) obey the relation

\[
\varphi_1 = -\varphi_2 = \varphi.
\]

(5.6b)

Taking into account the scale-rotation transformation (4.8) we must make the following changes in the formulas (5.1)–(5.5):
\[ \Phi \rightarrow \varrho \Phi, \quad \xi \rightarrow \frac{e^{-4 \pi}}{\varrho} \xi, \quad \varphi \rightarrow \frac{e^{+4 \pi}}{\varrho} \varphi. \quad (5.7) \]

Because \( \xi \sim \tilde{\xi} \) and \( \varphi \sim \tilde{\varphi} \) at infinity \( \varrho \rightarrow \infty \), we have from (5.2), (5.6), and (5.7) the following asymptotic solution:

\[ \Phi \sim \frac{2(\varrho^2 \varrho^m + 1)}{\chi m \varrho^m \cos (m(\varphi - \varphi)) - (m + 1) \varphi} \quad \text{at} \ \varrho \rightarrow \infty. \quad (5.8) \]

In view of dependence on phase \( \varphi(\theta) \) and \( \varphi(\theta) \), the formula (5.8) describes the propagating wave along the \( x^3 \) axis. The dependence \( \varphi(\theta) \) in (5.8) describes also the twist of this wave about the propagation direction.

Let us consider the twisted lightlike soliton with constant twist (4.45). We can put for this case

\[ \varphi = \frac{\pm \theta}{m}, \quad (5.9) \]

where the signs ‘+’ and ‘−’ correspond to right and left twisted soliton accordingly.

In addition, let us assume that the scale function \( \varphi(\theta) \) is almost constant: \( \varphi' \sim 0 \). As we can see in (5.8) with (5.9), in this case \( \omega \) in (4.1b) is a radian frequency of the soliton wave with the wave length \( 2\pi / |k| \).

Now let us consider the projection of the singular surface of the lightlike soliton to the transverse plane, that is a line on which the condition (5.4) is violated.

In the plane of the tilde variables \( \{ \tilde{\xi}, \tilde{\varphi} \} \), the singular line is a circle of radius \( \varrho \), just as for the cylindrically symmetric solution considered in the previous section (4.39):

\[ \tilde{\varrho} = \varrho. \quad (5.10) \]

It can be shown that, the action density (4.28) vanishes or becomes infinite on this line for the signatures of metric (2.11a) and (2.11b) respectively. Also, accordingly, the vector components \( \Phi^t \) (4.24b) or \( \Phi^0 \) (2.6) become infinite on it.

In the plane \( \{ \xi, \varphi \} \), in conformity with (5.5)–(5.7) and (5.10), the singular line is described by the formula

\[ \tilde{\xi} = \varrho e^{i(\varphi + \varphi)} \left( 1 \pm e^{i \frac{2(\varphi - (m + 1)\varphi)}{2m + 1}} \right). \quad (5.11) \]

Here the function \( \tilde{\xi} = \tilde{\xi}(\varphi) \) represents a parametric expression for the singular line in the complex plane of the variable \( \xi \). The phases \( \varphi \) and \( \varphi \) define a rotating of this closed curve as a whole.

The curve \( \tilde{\xi}(\varphi) \) (5.11) is an epicycloid with 2 \( m \) cusps. For \( m = 1 \) this line is shown in figure 3. These figures was obtained also by R Ferraro [18, 19] for mathematically similar but another problem.

The change of the signature of metric in (5.11) from (2.11a) (top sign, \( \mathfrak{m}^{00} = 1 \)) to (2.11b) (bottom sign, \( \mathfrak{m}^{00} = -1 \)) implies a turn of the whole curve to the angle \( \pi / (2 \ m) \).

In the present investigation, the system (5.5) with \( \varrho = 1 \) for given values of the parameter \( m \) and the variables \( \{ \xi, \varphi \} \) is solved numerically with respect to the variables \( \{ \tilde{\xi}, \tilde{\varphi} \} \) in all characteristic areas of the plane \( \{ x^1, x^2 \} \).

In the region of the plane \( \{ x^1, x^2 \} \) outside of the singular line (5.11), we have one-to-one mapping \( \{ \tilde{\xi}, \tilde{\varphi} \} \leftrightarrow \{ \xi, \varphi \} \) with the condition (4.26) at infinity \( \rho \rightarrow \infty \).

But the solutions of equations (5.5) give a multivalued mapping \( \{ \xi, \varphi \} \rightarrow \{ \tilde{\xi}, \tilde{\varphi} \} \) in the interior of the singular line (5.11) on the plane \( \{ x^1, x^2 \} \).

The class of solutions under consideration (5.1) is characterized by zero value of the field function \( \Phi \) at the origin of the coordinates \( \{ x^1, x^2 \} \). In the scope of relations (5.1)–(5.5), it is possible to make a composite continuous everywhere solution consisting of sectorial pieces and satisfying this condition \( \Phi(0, 0) = 0 \). But its absolute value increases at infinity \( (\rho \rightarrow \infty) \).

The values of the field function \( \Phi \) on the line (5.11) for this inner solution are distinct from ones for the external solution with asymptotics (5.8). Thus the composite solution consisting of these external and inner ones would have a finite discontinuity on the singular line (5.11).

But in this case we could take also the trivial solution \( \Phi = 0 \) for the inner area of the singular line (5.11).

To obtain an everywhere continuous solution, the various kinds of composite solutions can be used. One of them will be considered below (see figures 7 and 8).

Apart from this one case we shall consider the solutions of the exterior domain with respect to the singular line (5.11). The region of space in the interior of the singular line will be excluded from the space of the problem.
In this case we have a soliton with an appropriate cavity. Such soliton has all characteristic properties of the twisted soliton. It is possible to generate of such solitons, in particular, by sources having cavities also.

Thus, according to (5.10) and (5.11), we have the following condition for the space of the solution:

$$\rho \geq |\xi(\varphi(\theta))| \varphi(\theta), \quad \hat{\rho} \geq \rho,$$

(5.12)

where the dependence $\xi = \xi(\varphi(\theta))$ corresponds to the rotation of the singular contour in the plane $\{x^1, x^2\}$ by the angle $\varphi(\theta)$ (4.8c).

Thus, according to (5.12), we have the soliton with an inner shell.

The results of numerical calculations for the field function $\Phi$ (5.2) for $m = 1$ on the singular line are shown in figure 4. The appropriate results for the field function $\Phi$ on the plane $\{x^1, x^2\}$ are shown in figures 5 and 6. The points $\{A_{i1}, B_{i1}, C_{i1}, D_{i1}\}$ are corresponding in figures 4–6.

It should be noted that the values of the parameters $\varphi$ and $\varphi$, which are contained in the formulas (5.6) and (5.7), define a turning of the whole field configuration only and do not change the solution qualitatively.

To construct an everywhere continuous solution, let us consider two solutions with shifted singular line. In these cases the constants are added to the variables $\{\xi, \xi, \varphi, \varphi\}$. Its internal radius is defined in (5.12). Let its external radius and length be designated as $\rho_\infty$ and $l_f$ accordingly. Thus in addition to condition (5.12) we have

$$\rho \leq \rho_\infty, \quad -\frac{l_f}{2} \leq x^3 \leq \frac{l_f}{2},$$

(5.13)

First we calculate the integrals on right-hand parts of relations (4.46) by variables $\{\hat{\rho}, \tilde{\varphi}\}$ in area $\{(\rho, \rho_\infty; [-\pi, \pi])\}$. This integration corresponds to one of the left-hand parts for the relations (4.46) by the variables $\{\rho, \varphi\}$ in the outside area of the singular line $\xi$ (5.11) and bounded by the line $\xi(\rho_\infty \exp^{-ix}, \rho_\infty \exp^ix)$, where the function $\xi(\xi, \xi)$ is defined in (5.5).

Making the integration in the plane $\{x^1, x^2\}$ we can put the rotation parameter (5.9) be zero: $\varphi = 0$. Also we put $\varphi = 0$. Let us substitute (5.1) and (5.5) with (4.23) and (4.27) to the right-hand parts of (4.46). We change the integration by variable $x^3$ to one by phase $\theta$ (4.1b).

As a result, we have the following expressions for energy and absolute values of momentum and angular momentum:

$$E = P + \mathbb{E},$$

(5.14a)
Figure 4. The field function $\Phi$ on the singular line of the plane $\{x^1, x^2\}$ for $m = 1$.

Figure 5. The field function $\Phi$ on the plane $\{x^1, x^2\}$ for $m = 1$ and $m^{00} = 1$.

Figure 6. The field function $\Phi$ on the plane $\{x^1, x^2\}$ for $m = 1$ and $m^{00} = -1$.

\[
\mathbb{E} = \frac{\rho^2}{\omega^2 \chi^2} C_0 \mathcal{I}_0 = \mathbb{P} \frac{1}{\rho^2 \omega^2} \frac{C_0 \mathcal{I}_0}{C_0 \mathcal{I}_1 + C_1 \mathcal{I}_2}, \quad (5.14b)
\]

\[
\mathbb{P} = \frac{\omega^2}{\chi^2} C_0 (C_1 \mathcal{I}_1 + C_3 \mathcal{I}_2), \quad (5.14c)
\]
where \( \mathcal{E} \) is the part of soliton energy obtained from the part \( \mathcal{F} \) of energy density \( \mathcal{E} \) (4.42a),

\[
J = \frac{\rho^4}{x^2} C_1 C_2 |\mathcal{I}_3|,
\]

(5.14d)

\[
\mathcal{I}_0 \equiv \int_{-\omega t/2}^{\omega t/2} \bar{p} \varphi d\theta, \quad \mathcal{I}_1 \equiv \int_{-\omega t/2}^{\omega t/2} \bar{p} (\varphi')^2 d\theta, \\
\mathcal{I}_2 \equiv \int_{-\omega t/2}^{\omega t/2} \bar{p}^2 (\varphi')^2 d\theta, \quad \mathcal{I}_3 \equiv \int_{-\omega t/2}^{\omega t/2} \bar{p}^4 \varphi' d\theta,
\]

(5.15)

\[
C_0 = \frac{1}{2} \left( \frac{1}{m} + \frac{1}{2m + 1} \right) - \frac{(\rho/\bar{\rho}_{\text{sc}})^2 m}{2m} \pm \frac{(\rho/\bar{\rho}_{\text{sc}})^2 (m-1)}{2(2m + 1)},
\]

(5.16a)

\[
C_1 = \ln \left( \frac{\bar{\rho}_{\text{sc}}}{\bar{\rho}} \right) \pm \frac{1}{6} \left( 1 - \frac{\rho^4}{\bar{\rho}_{\text{sc}}^4} \right) + \frac{1}{72} \left( 1 - \frac{\rho^4}{\bar{\rho}_{\text{sc}}^4} \right) \text{ for } m = 1,
\]

(5.16b)

\[
C_2 = \frac{1}{2m^2} \left( \frac{2}{2m + 1} \left( \frac{2m(3m^2 + 4m + 2)}{(2m + 1)(m - 1)(3m + 1)} \pm \frac{1}{2m} \right) - \frac{(\rho/\bar{\rho}_{\text{sc}})^{2(m-1)}}{m - 1} \right) \pm \frac{(\rho/\bar{\rho}_{\text{sc}})^4 m}{m(2m + 1)} - \frac{(\rho/\bar{\rho}_{\text{sc}})^{2(m+1)}}{(2m + 1)^2(3m + 1)},
\]

for \( m \geq 2 \).

(5.16c)

\[
C_2 \div m^2, \quad C_3 \div (m + 1)^2.
\]

(5.16d)
Here in (5.16) we have the different expressions for the two metric signatures (2.11a) and (2.11b) (top and bottom signs respectively).

The value of the $x_3$ momentum projection is defined by the sign of wave vector projection $k_3$ (4.1b): $p_3 = \pm p$.

In general case the $x_3$ angular momentum projection is defined by the integral $I_3$ (5.15), which can be called the integral twist of the soliton with weight $r_4$:

$$I_3 = \pm I_3$$

Let us write the appropriate to (5.14) expressions for the twisted soliton with constant twist (5.9). Using the condition (5.9) and formulas (5.15), we have

$$|I_3| = \frac{1}{m} \hat{r}_3, \quad I_3 = \frac{1}{m^2} \hat{r}_3, \quad \hat{r}_3 \equiv \int_{-\omega_1/2}^{\omega_1/2} p^3 d\theta.$$  \hspace{1cm} (5.17)

Using (5.17) and (5.16), we obtain from (5.14) the following expressions for the twisted soliton:

$$E = P = \omega J,$$  \hspace{1cm} (5.18a)

$$P = \omega \sqrt{\frac{1}{m} \left( 1 + \frac{C_0 I_0}{I_1} \right)}.$$  \hspace{1cm} (5.18b)

$$J = \frac{p^3}{\chi^2} m C_4 \hat{r}_4.$$  \hspace{1cm} (5.18c)

For the twisted soliton let us consider the case for slowly varying scale function $p(\theta)$, such that $I_2 \to 0$ (5.15). Also we suppose that the frequency $\omega$ is sufficiently high, such that $\omega \to \infty$. According to expressions (5.18), in this case we have the following relations:

$$E = P = \omega J,$$  \hspace{1cm} (5.19a)

where

$$\omega = \frac{\omega}{m}$$  \hspace{1cm} (5.19b)

is the angular velocity of the twisted soliton.
Let us consider the twisted soliton with scale function in the form of Gaussian curve:

\[
\rho = \exp\left(-\frac{\theta^2}{2\bar{\theta}^2}\right),
\]

(5.20)

where \(\bar{\theta}\) is the characteristic length of the soliton measured in radians and numerically equals to a total angle of twist on the characteristic length of the soliton along \(x^3\) axis. A twist angle \(2\pi/m\) corresponds to soliton wavelength along \(x^3\) axis.

Let us consider the case of infinite space with the conditions

\[
\frac{\hat{\mathcal{E}}_\infty}{\rho} \to \infty, \quad \omega I_1 \to \infty.
\]

(5.21)

Then the calculation of the essential integrals in (5.15) and (5.17) for the function (5.20) gives

\[
\mathcal{I}_0 = \theta \sqrt{\pi}, \quad \mathcal{I}_1 = \theta \frac{\pi}{2}, \quad \mathcal{I}_2 = \frac{1}{4\bar{\theta}} \frac{\pi}{2}.
\]

(5.22)

As we see in (5.14) and (5.16) with (5.21) and (5.22), for the case of infinite space we have the finite values of energy, momentum, and angular momentum if \(m \gg 2\). Using (5.18) with (5.16), (5.17), (5.21), and (5.22), let us write the appropriate expressions for \(m = 2\) and the metric signature (2.11a):

\[
E = \frac{560\sqrt{2}}{129(9 + 4\bar{\theta}^2)\rho^2 \omega^2}.
\]

(5.23a)

\[
\mathcal{P} = \omega J \left(\frac{1}{2} + \frac{9}{4\bar{\theta}^2}\right)\sqrt{\pi} \frac{\rho^2}{2} \frac{\bar{\theta}}{\chi^2}.
\]

(5.23b)

\[
J = \frac{387}{1400} \sqrt{\pi} \frac{\rho^2}{2} \frac{\bar{\theta}}{\chi^2}.
\]

(5.23c)

It is evident that the case \(\bar{\theta} \gg 1\) and \(\rho \omega \gg 1\) for expressions (5.23) gives relations (5.19).

The shell of the twisted soliton with Gaussian scale phase functions is shown on figures 11 and 12. It is significant that the twist parameter \(m\) is a topological invariant for diffeomorphism. The shell of twisted lightlike soliton is diffeomorphic to a cylindrical surface with threads by multihelix, where the number of continuous threads is \(2m\). These threads correspond to the singular lines on the shell, which we can see on figures 11 and 12.

The field function \(\Phi\) of the Gaussian twisted soliton in the plane section \(\{x^1, x^3\}\) for \(x^2 = 0\) is shown on figures 13 and 14.

At last we show zero level surfaces of the field function \(\Phi\) for the Gaussian twisted soliton with \(m = 1\) (figure 15) and \(m = 2\) (figure 16). The twist of the solitons is well seen also on these figures. We have two-sheeted helical surface with excluded cavity for \(m = 1\) and we have four-sheeted one for \(m = 2\).

All images 11–16 are appropriate to the solitons twisted on the right.

Here we have considered the simplest arbitrary functions \(\Xi_1\) and \(\Xi_2\), which give the twisted shell lightlike soliton with one cavity. For more complicated cases we can have the appropriate solitons with a set of cavities. But we will have the notable asymptotic relation between energy, momentum, and angular momentum (5.19) for these cases, because of the appropriate relation for the densities (4.44).
Figure 12. The shell of Gaussian twisted soliton for $m = 2$.

Figure 13. The field function $\Phi$ of Gaussian twisted soliton for $m = 1$ and $m^0 = 1$ on the plane $(x^1, x^3)$ for $x^2 = 0$.

Figure 14. The field function $\Phi$ of Gaussian twisted soliton for $m = 2$ and $m^0 = 1$ on the plane $(x^1, x^3)$ for $x^2 = 0$.

Figure 15. Zero level surfaces of the field function $\Phi$ for the Gaussian twisted soliton with $m = 1$. 
6. Relation to photons

In view of the obtained notable connection (5.19) between energy, momentum, and angular momentum for the twisted lightlike solitons, it is reasonable to consider their relation to photons.

For this purpose first we consider an ideal gas of these solitons in bounded three-dimensional volume $V$.

As it is known, the ideal gas model is characterized by the zero interaction between the particles. But an interaction of the particles with the walls of the bounding volume provides thermodynamic equilibrium of the ideal gas.

Thus we consider a great number of the twisted lightlike solitons in the three-volume $V$ for the case of negligible their interaction with each other because of the adequate mutual remoteness.

Let us suppose that absorptive and emissive capacities of the walls are provided by soliton-particles having the following constant absolute value of angular momentum

$$j = \frac{h}{2},$$

where $h$ is Planck constant.

We suppose also that each lightlike soliton can interact simultaneously with only one soliton-particle of the wall. We assume the angular momentum conservation for the combination of lightlike soliton with the soliton-particle of the wall in an absorption and an emission event.

Then, because of the angular momentum conservation, an absorption or an emission of a twisted lightlike soliton is possible only when the angular momentum of the soliton-particle in the wall is oppositely directed to the angular momentum of the lightlike soliton. The soliton-particle angular momentum is reversed in the absorption and the emission event.

According to known theses of quantum physics, these conditions correspond exactly to absorption and emission processes of photons by electrons of real walls. The fixity of the angular momentum absolute value of the electron (6.1) is the principal feature here. The electrons interacting with photons can be conduction ones or it can be bound in atoms. In any case the electrons interacting also with a lattice take part in a very complicated motion. As a result of such complex interactions we have that any incident photon is absorbed by some electron for a sufficient thickness of the wall. The electrons, in one’s turn, emit the photons. We suppose the same situation for the twisted lightlike solitons interacting with the wall.

Thus the absolute value of angular momentum of any twisted lightlike soliton in the volume $V$ must be equal to $h$ in the case of thermodynamic equilibrium.

The structure of twisted lightlike solitons depends on the structure and states of emissive and absorbent soliton-particles. We must define the value of twist parameter $m$ and the scale phase function $\rho(\varphi)$ for the twisted lightlike solitons in the volume $V$.

Let us consider the case

$$m = 1.$$  \hfill (6.2a)

As we see in (5.16b), in this case the energy of the soliton is logarithmically divergent in infinite space. But here we consider the finite volume, where its energy is finite.

Strictly speaking, the obtained soliton solutions must be modified for the finite volume. But here we consider the integral characteristics of the solitons only. Thus we can consider the soliton solutions of infinite space for the finite volume in some approximation.
Let us suppose also that the scale phase function $\pi(\theta)$ is slow variable:

$$\pi' \to 0. \quad (6.2b)$$

Thus, taking into account (5.14)–(5.17) and (6.2), we have the following relations for any twisted lightlike soliton in the volume $V$:

$$\mathcal{E} = \mathcal{P} + \mathcal{E}, \quad (6.3a)$$

$$\mathcal{P} = \omega \ h, \quad (6.3b)$$

$$\mathcal{J} = h, \quad (6.3c)$$

where

$$\mathcal{E} = \int_V f_0^E \ dV, \quad (6.3d)$$

$f_0^E$ is the static part of energy density $\mathcal{E}$ for the lightlike soliton in expression (4.42a).

As we see in (4.42a), the static part of energy $\mathcal{E}$ is independent explicitly of the soliton frequency $\omega$. But the condition (6.3c) with expressions (5.18c) and (5.17) gives a dependence of the soliton transversal size (characterized by the parameter $\rho$) from the soliton frequency $\omega$. Thus according to (5.14b) and (5.15) the static energy $\mathcal{E}$ is implicitly dependent on the frequency $\omega$.

An estimation for this dependence will be made below. But at first for simplicity we consider that the static energy $\mathcal{E}$ is approximately constant:

$$\mathcal{E} \approx \text{const}. \quad (6.4)$$

The finiteness of the volume under consideration confines the set of possible frequencies of the solitons. As it is known, the field in any finite volume can be represented by the appropriate mode expansion. In the case of cuboid we have the simple space-time Fourier components, which satisfies the periodic boundary conditions.

In the case of arbitrary volume with cavities, the finding of volume modes looks very complicated. Here we assume that the cavities inside the soliton shells are sufficiently small to neglect of their influence. Also we take that each soliton in the volume has one of its allowed frequencies.

Hereafter up to formulas (6.16) we obtain the equilibrium distribution function by soliton frequencies. The appropriate derivation of formulas is similar to ones represented in the classical works by S. Bose [20], A. Einstein [21, 22], and contained in monographs (see, for example, [23]).

As distinct from the cited works, here we use the natural energy cells instead of finite phase space cells. Complete deduction is expounded to show that all assumptions are in the framework of the real soliton dynamics only.

For simplicity let us consider the volume $V$ in cubic form with side $l_v$. Then the allowed frequencies are defined by formula

$$\omega_i = \frac{2\pi}{\lambda_i} = \frac{2\pi}{l_v} \bar{n}_i = \frac{2\pi}{l_v} \sqrt{n_1^2 + n_2^2 + n_3^2}, \quad (6.5)$$

where $\{n_1, n_2, n_3\}$ are integer numbers, excepting the case when all number are zero, $i$ is the index for different frequencies.

According to (6.5) we have the following minimal frequency in the volume:

$$\omega_{\text{min}} = \frac{2\pi}{l_v}. \quad (6.6)$$

If there are $N_i$ solitons with frequency $\omega_i$ in the volume $V$, then the full energy of the solitons is given by the formula

$$U = \sum_{i=1}^{\infty} N_i \ E_i, \quad (6.7a)$$

where $E_i$ is the energy of the soliton with frequency $\omega_i$,

$$E_i = \omega_i h + \mathcal{E}, \quad (6.7b)$$

$$N = \sum_{i=1}^{\infty} N_i \quad (6.7c)$$

$N$ is a total number of solitons in the volume $V$.

Because there is the minimal frequency $\omega_{\text{min}}$ (6.6) for the solitons in the volume $V$, then according to (6.7) we have the following expression for their maximal quantity:
If we suppose that a full angular momentum as well as a full momentum of the soliton gas in the volume \( V \) are zero, then the total number of solitons must be even. Thus their minimal quantity is 2 and we have from (6.7) the maximal value for frequency:

\[
N_{\min} = 2, \quad \omega_{\max} = \frac{U - 2 \bar{E}}{2 \hbar}.
\]

Among all the possible distributions by soliton frequencies \( \{N_i\} \) there is their part providing an identical total energy of the gas \( U \). According to general principles of statistical physics such distributions are considered as equally probable.

Let us introduce the size of energy cell \( E_i \), which are the quantity of solitons having the energy \( \bar{E}_i \) and the corresponding frequency \( \omega_i \). Different states in the sell are defined with the set of numbers \( \{n_1, n_2, n_3\} \) in (6.5) for frequency \( \omega_i \) and two directions of twist (right and left).

Let us count the number of ways to provide the part of total energy \( U \) produced by the solitons with energy \( \bar{E}_i \) that is \( N_i \bar{E}_i \) (6.7a). According to known representation we line up \( N_i \) solitons \( \sigma \) and \( (E_i - 1) \) dividing walls (\( | \) in random order:

\[
o|o|oo| |o|00o|o| \cdots \cdots |o|00\cdots |o|.
\]

Here the dividing walls (\( | \) separate the different soliton states (numbers \( \{n_1, n_2, n_3\} \) and twist direction).

In that case, the permutation number \( (N_i + E_i - 1)! \) is a total number of distributions for solitons with energy \( \bar{E}_i \). Then we take into account that \( N_i! \) permutations of solitons and \( (E_i - 1)! \) permutations of dividing walls correspond to one state. As a result, we have the sought number of ways to provide the part \( N_i \bar{E}_i \) of total energy \( U \):

\[
W_i = \frac{(N_i + E_i - 1)!}{N_i!(E_i - 1)!}.
\]

We obtain the total number of ways providing the energy \( U \) by multiplication of the numbers \( W_i \):

\[
W = \prod_{i=1}^{\infty} W_i = \prod_{i=1}^{\infty} \frac{(N_i + E_i - 1)!}{N_i!(E_i - 1)!}.
\]

According to the usual method, we take into account that the most probable distribution provided by the maximum number of the ways \( W \) corresponds to equilibrium. The total number of solitons \( N \) is not fixed here.

Let us solve the problem for the maximization of number \( W \) with the fixed total energy \( U \). (6.7a). For this purpose the method of Lagrange multipliers is used. For the convenience we maximize the natural logarithm of number \( W \). Thus the problem for finding of the equilibrium distribution \( \{N_i\} \) take the form:

\[
S = \ln W - T^{-1} U, \quad S \rightarrow \text{max},
\]

where \( T^{-1} \) is Lagrange multiplier, the parameter \( T \) has a physical dimension of energy.

Let us consider the case when the numbers \( N_i \) and \( E_i \) are sufficiently great. In this case we use the Stirling formula for factorial of number. Thus for \( N_i \gg 1 \) and \( E_i \gg 1 \) we have

\[
\ln W \approx \sum_{i=1}^{\infty} ((N_i + E_i) \ln (N_i + E_i) - N_i \ln N_i - E_i \ln E_i).
\]

Considering the sequence of numbers \( N_i \) as quasicontinuous, we have the following necessary conditions for maximum of the function \( S \):

\[
\frac{\partial S}{\partial N_i} = 0.
\]

From (6.14) with (6.12), (6.13), and (6.7a) we have the following equilibrium distribution:

\[
N_i = \frac{E_i}{e^{E_i/T} - 1}.
\]
the two directions of twist and proceeding to the limit $\Delta \omega \to 0$, we obtain

$$E_{\omega, \Delta \omega} \approx 2 \cdot 4 \pi n^2 \Delta n \approx \frac{l^3}{\pi^2} \omega^2 \Delta \omega \to E_{\omega} \ d\omega = \frac{l^3}{\pi^2} \omega^2 \ d\omega.$$  \hfill (6.16a)

$$N_\omega = \frac{l^3}{\pi^2} \frac{\omega^2}{\hbar^3} e^{\frac{\hbar \omega}{k_B T}} - 1,$$  \hfill (6.16b)

where

$$E_\omega = \hbar \omega + E.$$  \hfill (6.16c)

Then we integrate the expressions $E_\omega$, $N_\omega$, and $N$ with substitution $E_\omega$ from (6.16) over frequency from $\omega = 0$ to infinity. As a result, we obtain the following expressions for total energy and number of solitons in the volume $V$:

$$U = \frac{l^3}{\pi^2} T^4 \text{Li}_4(e^{-\frac{E}{k_B T}}) + N E,$$  \hfill (6.17a)

$$N = \frac{l^3}{\pi^2} T^3 \frac{2 \text{Li}_3(e^{-\frac{E}{k_B T}})}{\hbar^3},$$  \hfill (6.17b)

where $\text{Li}_i(z)$ is polylogarithm function.

For connection between energy parameter $T$ of distribution $\{N_i\}$ (6.15) and absolute temperature $T$ we take

$$T = k_B T,$$  \hfill (6.18)

where $k_B$ is Boltzmann constant.

The relation (6.18) can be validated by means of comparison between statistical determination for entropy $S$ and its thermodynamic one for the case of constant volume ($V = \text{const}$):

$$S = k_B \ln W,$$  \hfill (6.19a)

$$dS = \frac{dU}{T},$$  \hfill (6.19b)

But because the equivalence of these determinations must be postulated, it is reasonable here to postulate the relation (6.18).

Let us write the equilibrium energy spectral density for the twisted lightlike solitons in the volume $V$. According to (6.16) and (6.18) we have

$$u(\omega, T) \approx \frac{E_\omega N_\omega}{V} = \frac{\omega^2}{\pi^2} \frac{\hbar \omega + E}{\exp \frac{\hbar \omega + E}{k_B T} - 1}. $$  \hfill (6.20)

For the case of negligible static soliton energy $E \to 0$, we have from (6.20) the following known Planck formula for photons:

$$u(\omega, T) = \frac{\omega^2}{\pi^2} \frac{\hbar \omega}{\exp \frac{\hbar \omega}{k_B T} - 1}. $$  \hfill (6.21)

Thus we can consider the relation between the twisted lightlike solitons and photons.

Now let us estimate the possible values of soliton parameters in the volume $V$ using certain suppositions. Taking into account (6.2b), we put $\rho' = 0$ and without the loss of generality $\rho = 1$.

Let the longitudinal size of the soliton in (5.15) and the external diameter of the cylindrical integration domain in (5.16) be equal to the side of the considered cubic volume:

$$l = 2 \tilde{r}_\infty = l_v.$$  \hfill (6.22)

Then, taking into account (6.2a), we have the following values contained in (5.15)–(5.17) for the metric signature (2.11a):

$$T_0 = \tilde{T}_1 = \omega l_v, \quad T_2 = 0,$$  \hfill (6.23a)

$$C_3 = \frac{1}{3} - \frac{2}{l_v^2} + \frac{32 \rho^6}{3 l_v^6}, \quad C_2 = 1,$$  \hfill (6.23b)

$$C_1 = \ln \left( \frac{l_v}{2 \rho} \right) + \frac{13}{72} - \frac{8 \rho^4}{3 l_v^4} - \frac{32 \rho^6}{9 l_v^6}.$$  \hfill (6.23b)
The condition (6.3c) with expression (5.18c) gives relation
\[ \rho^4 \omega L = \chi^2 \hbar. \]  
(6.24)

Thus, by virtue of the fixedness of the angular momentum of the soliton (6.3c), the radius of its shell \( \rho \) depends on frequency \( \omega \). But to calculate \( \rho \) we must have the value of the constant \( \chi \).

Nevertheless, to make a very rough estimate, we assume that for a visible light frequency \( \omega \) the shell radius depends on frequency \( \omega \). But to calculate we must have the value of the constant \( c \).

Let
\[ \omega = k \approx 10^7 \text{ m}^{-1}, \]  
(6.25a)
\[ \rho \approx 3 \cdot (10^{-15} \div 10^{-7}) \text{ m}, \quad L \sim 0.1 \text{ m}. \]  
(6.25b)
Expressions (6.23b) with (6.25b) give
\[ C_0 \approx \frac{1}{3}, \quad C_1 \sim (12 \div 31), \quad C_2 = 1. \]  
(6.25c)

The standard value of Planck constant must be multiply by the velocity of light for used unit of frequency (6.25a):
\[ \hbar \approx 2 \cdot 10^{-7} \text{ eV} \cdot \text{m}. \]  
(6.26)

Then the relation (6.24) with (6.25) and (6.26) gives
\[ \chi \sim (1 \cdot 10^{-22} \div 7 \cdot 10^{-7}) \text{ m}^{3/2} \cdot \text{eV}^{-1/2}, \]  
(6.27a)
\[ \chi^{-2} \sim (2 \cdot 10^{12} \div 1 \cdot 10^{49}) \frac{\text{eV}}{\text{m}^2} \sim (3 \cdot 10^{-7} \div 2 \cdot 10^{23}) \frac{\text{J}}{\text{m}^3}, \]  
(6.27b)
where the minimal value of \( \chi \) in (6.27a) and the maximal value of \( \chi^{-2} \) in (6.27b) correspond to the minimal value of \( \rho \) in (6.25b).

According to the formula (5.14b) and taking into account (6.23), (6.25), and (6.27), we have the following values for the static part of soliton energy:
\[ \mathbb{E} \approx \frac{\rho^2 L}{3 \chi^2} \sim (6 \cdot 10^{-3} \div 2 \cdot 10^{13}) \text{eV}, \]  
(6.28)
where the minimal value of \( \mathbb{E} \) corresponds to the minimal value of \( \chi^{-2} \) in (6.27b) and the maximal value of \( \rho \) in (6.25b).

Thus to provide the condition \( \mathbb{E} \ll \omega \hbar \), the diameter of soliton shell 2 \( \rho \) must be closer to the soliton wavelength than to the electron classical diameter.

Expressing \( \rho \) from (6.24) and substituting it to formula for \( \mathbb{E} \) in (6.28), we obtain from (6.3) the following formula for soliton energy:
\[ E \approx \hbar \omega + \frac{1}{3} \chi \sqrt{\frac{\hbar L}{\omega C_1}}, \]  
(6.29)
where \( C_1 \) is considered to be constant. Here we disregard the dependence \( C_1(\rho) \) (6.23b) what is justified for the used approximation.

The dependence (6.29) is shown on figure 17 for the explicit values of parameters. Of course, it must be considered mainly for a qualitative analysis.

As we see on figure 17, the distinction of soliton energy function from the linear one \( \hbar \omega \) (dashed line) can be noticeable in a low-frequency region. We see also a confirmation of the approximate condition (6.4) at the center of the plot.

The question arises as to whether there is a static part of energy for real photons. The appropriate experimental check may be possible with the help of the extrinsic photoeffect. If the photon energy not exactly equals to \( \hbar \omega \), then the frequency dependence of photoelectron energy may have a weak nonlinearity near photoemission threshold. The substances with low photoemission threshold is preferable for such experiments.

Let us next consider all values of the twist parameter \( m \) for the lightlike solitons. For \( m = 1 \) we have obtained the known expression for photon energy in the case \( \hbar \omega \gg \mathbb{E} \).

Thus for \( m \gg 2 \) here we could be considered a fractional photon with the following energy expression, according to (5.19):
\[ E = \frac{\hbar \omega}{m} + \mathbb{E}. \]  
(6.30)

But we must pay attention once again to the fact that the twisted lightlike solitons with \( m \gg 2 \) have the qualitative distinction from ones with \( m = 1 \) in the part of the energy representation.
The energy of longitudinally limited twisted lightlike soliton with \( m = 1 \) logarithmically diverges in infinite space, but for \( m \gg 2 \) its energy is finite. In this point of view the solitons with \( m = 1 \) more closely resemble the plane waves with constant amplitude, the energy of which also diverges in infinite space.

Let us consider the representation of the polarization property of light by twisted lightlike solitons. A beam of these solitons with right or left twist has a necessary symmetry of right or left circularly polarized light wave accordingly. This beam, in particular, can provide the Sadovskii effect \(^{24}\), which is a mechanical angular momentum transfer to absorbent by circularly polarized electromagnetic wave. This effect has the experimental verification \(^{25,26}\), including one for electromagnetic centimeter waves \(^{27}\).

As it is known, the plane circularly polarized electromagnetic wave with constant amplitude does not have angular momentum \(^{24}\). Thus this wave does not provide the Sadovskii effect. But the twisted lightlike solitons as well as photons have angular momentum and can provide this effect respectively.

The elliptical polarization and, as limiting case, linear one of the soliton beam could be provided by a coherent combining of solitons twisted to the right and to the left.

This representation for elliptical polarization conforms to one in the beam of photons, which have two helicity states only.

The peculiarity of the value \( m = 1 \) for the twist parameter becomes apparent here. According to the solution symmetry for this case (see figures 5 and 6), the coherent combining of equal quantities of such right and left twisted solitons can give a beam having a crystal like symmetry with axes of the first order. This case can be interpreted as a linear polarization.

But for the case of the solitons with higher values for the twist parameter, we have for the same conditions the appropriate crystal like symmetry with axes of \( m \gg 2 \) order (see figures 9 and 10). This case can not be interpreted as a linear polarization.

Thus the lightlike solitons with the twist parameter \( m = 1 \) can be considered as usual photons in some approximation. But the solitons of the higher twist \( m \gg 2 \) have qualitative differences from the solitons of the lowest twist \( m = 1 \).

### 7. Conclusions

Thus we have considered the field model of extremal space-time film, which is sometimes called Born—Infeld type scalar field model.

We have obtained the new class of exact solutions for this model that is the class of lightlike solitons. We have considered the significant subclass of these solutions that are twisted lightlike solitons. It is notable that the energy of these solitons is proportional to its angular momentum in high-frequency approximation.

The soliton under consideration has a singularity which is a moving two-dimensional tubelike surface or shell. The lightlike soliton can have a set of such tubelike shells with the appropriate cavities.

A relatively simple twisted lightlike soliton with one cavity was considered in details. This soliton is characterized, in particular, by a twist parameter \( m \) which is a natural number. The energy of longitudinally limited this soliton in infinite space is finite for \( m \gg 2 \), but for \( m = 1 \) its energy is logarithmically divergent. For
the case $m = 1$ we have the asymptotic relation between soliton energy, momentum, and angular momentum, which is characteristic for photon.

Then we have investigated relations of the twisted lightlike solitons with $m = 1$ to photons. The model of ideal gas of the twisted lightlike solitons in a bounded volume has considered for this purpose. Planck formula for the soliton energy spectral density in the volume has obtained with explicit assumptions in some approximation.

An experimental check for a validity of the obtained soliton energy true formula for real photon is proposed.

A beam of twisted lightlike solitons was considered. It was noted that this beam can provide the effect of mechanical angular momentum transfer to an absorbent by the circularly polarized beam. This effect well known for photon beam.

It has been found that a beam of the twisted lightlike solitons with $m = 1$ can provide the polarization property of light as well as photon beam.

Thus we have a correspondence between photon and the lightlike twisted soliton with the minimal value of the twist parameter.

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