A novel analysis approach of uniform persistence for a COVID-19 model with quarantine and standard incidence rate

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Abstract

A coronavirus disease 2019 (COVID-19) model with quarantine and standard incidence rate is first developed, then a novel analysis approach for finding the ultimate lower bound of COVID-19 infectious individuals is proposed, which means that the COVID-19 pandemic is uniformly persistent if the control reproduction number \( R_c > 1 \). This approach can be applied to other related biomathematical models, and some existing works can be improved by using that. In addition, the COVID-19-free equilibrium \( V_0 \) is locally asymptotically stable (LAS) if \( R_c < 1 \) and linearly stable if \( R_c = 1 \), respectively; while \( V_0 \) is unstable if \( R_c > 1 \).

Keywords: Uniform persistence, COVID-19 model, control reproduction number, quarantine measure

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1. Introduction

At present, the COVID-19 caused by the severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2), which emerged in December 2019 has spread around the globe. As of September 2, 2022, there have been cumulatively 601,189,435 confirmed cases in the world, of which 6,475,346 deaths \cite{35}. The COVID-19 not only inflicts a global public health crisis, but also has a major impact on the normal life of humans \cite{28}. In the early stages of the COVID-19 pandemic, some large-scale activities exacerbated the spread of the epidemic \cite{36}. Following World Health Organization (WHO) report, COVID-19 can be spread by contact and droplets, airborne and contaminant transmission, among other means. Available evidence suggests that SARS-CoV-2 is passed from human-to-human mainly through respiratory droplets and contact routes \cite{34}. If domestic animals or wild animals become the host of SARS-CoV-2, then COVID-19 will pose a greater threat to humans \cite{27}.

In the process of epidemic prevention and control, mathematical modeling methods can help us understand the interaction between different epidemiological factors, thereby helping to control the transmission of this epidemic \cite{29}. Infected individuals are divided into symptomatic infections, and asymptomatic infections who have a positive nucleic acid test but do not show any symptoms \cite{11}. Since asymptptomatically infected individuals do not know that they have been infected by the virus, the transmission caused by these people accounts for the vast majority \cite{12}. Thus, the mathematical model of COVID-19 with asymptomatic transmission will be more reasonable. Analyzing the dynamic behavior of the infectious disease model helps us comprehend...
the long-term behavior of the mathematical model so as to more effectively control the spread of the disease. Khouach et al. [23] established a SQEAIHR (S: susceptible individuals, Q: quarantined individuals, E: exposed individuals, A: asymptomatically infected individuals, I: symptomatically infected individuals, H: hospitalized individuals, R: recovered individuals) mathematical model for COVID-19 and demonstrated that this model is uniformly persistent if $R_0 > 1$, which means that COVID-19 will persist in the population. Zhang et al. [38] developed a stochastic model of COVID-19 and found some sufficient conditions for the persistence or the extinction of the disease. Cui et al. [7] gave a thorough analysis for the global stability of equilibria of a SIQS (I: infected individuals) model with quarantine measure under some conditions. Jiang et al. [27] used SEIAQR models to assess qualitatively the effects of joint measures led by Fangfang shelter hospitals, H: designated hospitals) models to assess qualitatively the effects of joint measures led by Fangfang shelter hospitals in response to COVID-19 pandemic in Wuhan, China. Mohsen et al. [26] believed that one of the reasons for the spread of COVID-19 is immigration, thus they proposed a system that takes into account the impact of immigration and quarantine. Their findings suggest that the disappearance of the disease is due to the implementation of quarantine measures.

Recently, Bai et al. [1] established the SEIAQR model for the spread of mumps:

$$
\begin{align*}
\dot{S}(t) &= \lambda - \beta S(t) (aE(t) + I(t) + bA(t)) - dS(t), \\
\dot{E}(t) &= \beta S(t) (aE(t) + I(t) + bA(t)) - (c + d)E(t), \\
\dot{I}(t) &= pcE(t) - (q + r + d)I(t), \\
\dot{A}(t) &= (1 - p)cE(t) - (r + d)A(t), \\
\dot{Q}(t) &= qI(t) - (r + d)Q(t), \\
\dot{R}(t) &= rI(t) + rA(t) + rQ(t) - dR(t),
\end{align*}
$$

(1)

and gave a complete analysis for the global stability of the disease-free equilibrium and the unique endemic equilibrium of model [1]. In this model [1], $q$ is the quarantined rate of symptomatic infections and $r$ stands for the recovery rate, and the descriptions of all other parameters are listed in Tab. [1]. From the transmission characteristics of COVID-19, the disease can be transmitted by exposed individuals, symptomatically and asymptomatically infected individuals [3]. In fact, model [1] is also in compliance with the propagation mechanism of COVID-19. Following the discussion of McCallum et al. [25], the standard incidence rate can better reflect the transmission of a pathogen. Thus, we will develop a COVID-19 model [2] (also see Fig. 1) with standard incidence rate on the basis of [1].

Our model differs from model [1] in three ways. Firstly, the contributions of the interaction among susceptible individuals $S$, exposed individuals $E$, symptomatically infected individuals $I$ and asymptomatically infected individuals $A$ to the growth rate of exposed individuals are no longer accounted for by the mass action term $S(t)(aE(t) + I(t) + bA(t))$, which has been replaced with the standard incidence term $S(t)(aE(t) + I(t) + bA(t))/N(t)$, where

$$
N(t) = S(t) + E(t) + I(t) + A(t) + Q(t) + R(t).
$$

Secondly, the quarantined rate of asymptomatic infections is added and different from that of symptomatic infections. Thirdly, the recovery rates of symptomatic infections, asymptomatic infections and quarantine are different. At present, numerical results of COVID-19 models with standard incidence rates are abundant, while dynamics analysis is rare. Our purpose is to present a more refined approach of uniform persistence of model [2] by using a thorough analysis, which can give some refined estimates to the ultimate lower bounds of solutions of the model.
The rest of this paper is structured as follows. In Section 2, the model formulation is given. In Section 3, the control reproduction number $R_c$ is calculated and the existence condition of the COVID-19 equilibrium is obtained. In Section 4, the stability of the COVID-19-free equilibrium is analyzed, and a complete analysis approach is proposed for the uniform persistence of model (2). Meanwhile, some explicit estimations on the ultimate lower bound of COVID-19 individuals are acquired, and some examples are given to illustrate our main result. Finally, a brief conclusions section completes this paper.

2. Model formulation

We divide the total population $N$ into six subclasses: susceptible individuals $S$, exposed individuals $E$, symptomatically infected individuals $I$, asymptomatically infected individuals $A$, quarantined individuals $Q$ and recovered individuals $R$. To this end, a flow chart of COVID-19 transmission model is shown in Fig. 1, where all parameters of this model are positive and their definitions are listed in Tab. 1, and $p \in (0, 1)$.

From Fig. 1 the COVID-19 transmation model is as follows,

$$
\dot{S}(t) = \lambda - \beta \frac{S(t)}{N(t)} (aE(t) + I(t) + bA(t)) - dS(t),
$$

$$
\dot{E}(t) = \beta \frac{S(t)}{N(t)} (aE(t) + I(t) + bA(t)) - (c + d)E(t),
$$

$$
\dot{I}(t) = pcE(t) - (q_1 + r_1 + d)I(t),
$$

$$
\dot{A}(t) = (1 - p)cE(t) - (q_2 + r_2 + d)A(t),
$$

$$
\dot{Q}(t) = q_1 I(t) + q_2 A(t) - (r_3 + d)Q(t),
$$

$$
\dot{R}(t) = r_1 I(t) + r_2 A(t) + r_3 Q(t) - dR(t).
$$

In virtue of the general theory of ordinary differential equations (see, e.g., [5, 20]), we know that model (2) is well-posed and dissipative in the set

$$
D = \left\{ \phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6)^T \in \mathbb{R}_+^6 : \sum_{i=1}^6 \phi_i > 0 \right\}
$$

positively invariant for the model system, where $\mathbb{R}_+ = [0, \infty)$. Thus, we will analyze the global dynamics of system (2) in $D$. 

Fig. 1. Flow chart of the COVID-19 transmission model.
Tab. 1. Definition of parameters in model (2).

| Parameter | Definition |
|-----------|------------|
| $\lambda$ | The birth rate of susceptible individuals |
| $d$ | The natural death rate |
| $\beta$ | The transmission rate of COVID-19 |
| $a$ | The regulatory factor for infection probability of exposed individuals |
| $b$ | The regulatory factor for infection probability of asymptomatically infected individuals |
| $c$ | The transfer rate of exposed individuals to other infected individuals |
| $p$ | The transition probability of symptomatically infected individuals |
| $q_1$ | The quarantined rate of symptomatically infected individuals |
| $q_2$ | The quarantined rate of asymptomatically infected individuals |
| $r_1$ | The recovery rate of symptomatically infected individuals |
| $r_2$ | The recovery rate of asymptomatically infected individuals |
| $r_3$ | The recovery rate of quarantined individuals |

3. Existence of pandemic equilibrium

It is clear to see that system (2) always has a COVID-19-free equilibrium $V^0 = (S^0, 0, 0, 0, 0, 0)^T$, where $S^0 = \lambda / d$. To get the existence of the COVID-19 equilibrium $V^* = (S^*, E^*, I^*, A^*, Q^*, R^*)^T$ of system (2), we first calculate the control reproduction number

$$R_c = \frac{a\beta}{c + d} + \frac{pc\beta}{(c + d)B_1} + \frac{bc\beta(1 - p)}{(c + d)B_2},$$

by using the method in [8], where $B_i := q_i + r_i + d$, $i = 1, 2$. Here, the first term can be expressed as that an exposed individual can averagely infect $a\beta$ susceptible individuals in a unit time, and the average duration of the exposure period is $1/(c + d)$. And the second term can be expressed as that the exposed individuals with $pc/(c + d)$ can be transformed into the symptomatically infected individuals, a symptomatically infected individual can averagely infect $\beta$ susceptible individuals in a unit time, and the average duration of symptomatic infection is $1/B_1$. While the third term can be expressed as that the exposed individuals with $(1 - p)c/(c + d)$ can be transformed into the asymptotically infected individuals, an asymptotically infected individual can averagely infect $b\beta$ susceptible individuals in a unit time, and the average duration of asymptomatic infection is $1/B_2$.

**Lemma 3.1.** System (2) possesses a unique COVID-19 equilibrium $V^*$ if and only if $R_c > 1$.

**Proof.** Let the right-hand sides of system (2) equal zero, it follows $N = S^0$. Thus, we have

$$S = \frac{\lambda S^0 B_1 B_2}{B_1 B_2 (dS^0 + \beta a E) + E B_2 p + B_1 b (1 - p) \beta c} = \frac{\lambda - (c + d) E}{d},$$

$$I = \frac{pcE}{B_1}, \quad A = \frac{(1 - p)cE}{B_2}, \quad Q = \frac{q_1 pcE}{B_1 (r_3 + d)} + \frac{q_2 (1 - p)cE}{B_2 (r_3 + d)},$$

$$R = \frac{r_1 pcE}{dB_1} + \frac{r_2 (1 - p)cE}{dB_2} \frac{1}{dB_2} + \frac{r_3 q_1 pcE}{dB_1 (r_3 + d)} + \frac{r_3 q_2 (1 - p)cE}{dB_2 (r_3 + d)}$$

According to [4], there holds
\[(c + d) B_1 B_2 E (a_1 E - a_2) = 0,\]

where

\[a_1 = (c + d) R_c > 0, a_2 = \lambda (R_c - 1).\]

Therefore, system (2) possesses a unique pandemic equilibrium \(V^* \gg 0\) if and only if \(0 < E^* = \frac{a_2}{a_1} < \frac{\lambda}{c+\lambda_1}\).

**Remark 3.1.** It is not difficult to find that \(S^0 = S^* + E^* + I^* + A^* + Q^* + R^*\) and \(R_c = S^0 / S^*\) for \(R_c > 1\).

**4. Stability and uniform persistence**

In this section, we study the asymptotic stability of COVID-19-free equilibrium \(V^0\) for \(R_c < 1\) and the uniform persistence of system (2) for \(R_c > 1\).

**Theorem 4.1.** The COVID-19-free equilibrium \(V^0\) is LAS if \(R_c < 1\) and unstable if \(R_c > 1\).

**Proof.** The characteristic equation of the corresponding linearized system of system (2) at \(V^0\) can be taken by

\[F(\Lambda) = (\Lambda + d)^2 (\Lambda + r_3 + d) \left( \Lambda^3 + b_1 \Lambda^2 + b_2 \Lambda + b_3 \right),\]

where

\[b_1 = B_1 + B_2 + (c + d) \left[ 1 - R_c + \frac{pc\beta}{(c + d)B_1} + \frac{bc\beta(1 - p)}{(c + d)B_2} \right],\]

\[b_2 = (B_1 + B_2) (c + d) (1 - R_c) + \frac{B_2pc\beta}{B_1} + \frac{B_1bc\beta(1 - p)}{B_2} + B_1B_2,\]

\[b_3 = B_1B_2 (c + d) (1 - R_c),\]

and \(R_c\) is calculated as in (3). Obviously, \(F(\Lambda) = 0\) has a root \(\Lambda = -(r_3 + d)\) and a double root \(\Lambda = -d\). For \(R_c < 1\), it is not difficult to find that \(b_1 > 0, b_3 > 0\) and \(b_1b_2 > b_3\). Therefore, from the Routh-Hurwitz criterion it follows that any root of the equation \(\Lambda^3 + b_1 \Lambda^2 + b_2 \Lambda + b_3 = 0\) has negative real part. That is to say, each root of \(F(\Lambda) = 0\) has negative real part, and then \(V^0\) is LAS.

Obviously, it holds that \(b_3 < 0\) for \(R_c > 1\). Hence, there can be found a positive \(\Lambda^*\) such that \(F(\Lambda^*) = 0\). In consequence, \(V^0\) is unstable. This completes the proof.

From the above discussion, the linear stability of \(V^0\) follows immediately.

**Corollary 4.1.** If \(R_c = 1\), then the COVID-19-free equilibrium \(V^0\) is linearly stable.

The uniform persistence of system (2) has important implications for controlling the COVID-19 pandemic, which hints that the COVID-19 pandemic will be persistent with long-term basis. Let \(\Omega = \{ \phi \in \mathbb{R}_+^6 : \phi_2 > 0 \}\) and \(u(t) \equiv (S(t), E(t), I(t), A(t), Q(t), R(t))^T\) be the solution of system (2) with any \(\phi \in \Omega\). We can obtain easily that \(\Omega \subseteq D\) is positively invariant for system (2), and \(u(t) \geq 0\) for \(t > 0\).

Now, we are in a position to discuss the persistence of system (2) in \(\Omega\). Following the definition in [2] [10], system (2) is said to be uniformly persistent if there exists a \(\rho > 0\) independent of the initial data such that \(\rho \leq \liminf_{t \to \infty} \psi(t)\), where \(\psi = S, E, I, A, Q, R\). Based on some analysis methods in [3] [13] [14] [19] [30], we will give an explicit eventual lower bound of COVID-19. Now let \(R_c > 1, \eta \in (0, 1)\) and

\[\mathcal{S}(\varepsilon) \equiv \frac{\lambda}{\eta \beta (aE^* + I^* + bA^*) / (S^0 - \varepsilon + d)} \cdot e \in (0, S^0 (1 - \eta)).\]
Then there is an \( \varepsilon_0 \in (0, S^0(1 - \eta)) \) such that
\[
\frac{S^*}{S^0} < \frac{\delta (\varepsilon)}{S^0 + 2\varepsilon}
\]
for any \( \varepsilon \in (0, \varepsilon_0) \). Note that
\[
N(t) = \lambda - dN(t),
\]
we thus have \( \lim_{t \to \infty} N(t) = S^0 \). Let \( \lambda S^* = \lim \inf_{t \to \infty} S(t) \). Then for any \( \varepsilon \in (0, \varepsilon_0) \), there exists a \( T_0 = T_0(\varepsilon, \phi) > 0 \) such that for all \( t \geq T_0 \), we have
\[
I(t) < S^0 + \varepsilon, A(t) < S^0 + \varepsilon, N(t) > S^0 - \varepsilon, N(t) < S^0 + \varepsilon, S(t) > S^\infty - \varepsilon.
\]
Let \( B = \min\{B_1, B_2\} \) and \( m = \max\{a, b, 1\} \). Then \((B + c)/(c + d)\) is strictly decreasing with respect to \( c \), and it yields that
\[
1 < \frac{S^0}{S^*} = \mathcal{R}_c \leq \beta m \frac{B + c}{(c + d)B} < \frac{\beta m}{d}.
\]
Hence, for all \( t \geq T_0 \), it follows from the first equation of system (2) that
\[
\dot{S}(t) > \lambda - \left[ \beta m \left( 1 - \frac{S(t)}{N(t)} \right) + d \right] S(t) > \lambda - \left[ \beta m \left( 1 - \frac{S^\infty - \varepsilon}{S^0 + \varepsilon} \right) + d \right] S(t),
\]
which leads to
\[
S^\infty \geq \frac{\lambda}{\beta m (1 - S^\infty / S^0) + d}.
\]
Solving the resulting inequality for \( S^\infty \), we can obtain \( S^\infty \geq \lambda / \beta m \) by means of (6).

To start the uniform persistence of system (2), the following lemmas are needed.

**Lemma 4.1.** Assume that \( \mathcal{R}_c > 1, \theta \in (0, 1) \), and there is a \( t_0 \geq T_0 \) such that \( E(t) \leq \theta E^* \) for \( t \geq t_0 \). Then
\[
\frac{S(t)}{N(t)} > \frac{\bar{k}(\varepsilon) \bar{S}(\varepsilon)}{S^0 + \varepsilon} > \frac{S^*}{S^0}
\]
for \( t \geq t_0 + \bar{T}_1(\varepsilon) + \bar{T}_2(\varepsilon) \), where \( \eta \in (\theta, 1) \),
\[
\bar{T}_1(\varepsilon) = \max \left\{ \frac{-1}{B_1} \ln \frac{(\eta - \theta)I^*}{S^0 + \varepsilon - \theta I^*}, \frac{-1}{B_2} \ln \frac{(\eta - \theta)A^*}{S^0 + \varepsilon - \theta A^*} \right\},
\]
\[
\bar{T}_2(\varepsilon) = -\frac{\bar{S}(\varepsilon)}{\lambda} \ln \frac{(1 - \bar{k}(\varepsilon)) \bar{S}(\varepsilon)}{\bar{S}(\varepsilon) + \frac{\bar{S}(\varepsilon)}{\varepsilon - \lambda / \beta m}} = \frac{\bar{S}(\varepsilon) (S^0 + 2\varepsilon)}{\bar{S}(\varepsilon)}.
\]

**Proof.** It is not difficult to see that \( \bar{k}(\varepsilon) < 1 \) from (5) and \( S^* > \lambda / \beta m \) from (6). By the third equation of system (2), we have
\[
\dot{I}(t) \leq pc\theta E^* - B_1 I(t) \quad \text{for} \quad t \geq t_0,
\]
which implies that
\[
I(t) \leq \theta I^* + \left( I(t_0) - \theta I^* \right) e^{B_1(t_0 - t)} \leq \theta I^* + \left( S^0 + \varepsilon - \theta I^* \right) e^{B_1(t_0 - t)},
\]
where \( I^* = pcE^* / B_1 \). For \( t \geq t_0 + \bar{T}_1(\varepsilon) \), it holds \( I(t) \leq \eta I^* \). Similarly, we have \( A(t) \leq \eta A^* \) for \( t \geq t_0 + \bar{T}_1(\varepsilon) \). As a result, it follows that
\[
\frac{aE(t) + I(t) + bA(t)}{N(t)} \leq \frac{\eta (aE^* + I^* + bA^*)}{S^0 - \varepsilon}
\]
for \( t \geq t_0 + \bar{T}_1(\epsilon) \), and thus there holds
\[
\dot{S}(t) = \lambda - \left[ \frac{\beta(aE(t) + I(t) + bA(t))}{N(t)} + d \right] S(t) \\
\geq \lambda - \left[ \frac{\eta \beta \left( aE^* + I^* + bA^* \right)}{S^0 - \epsilon} + d \right] S(t) \\
= \lambda - \frac{\lambda}{\bar{S}(\epsilon)} S(t).
\]

Consequently, for \( t \geq t_0 + \bar{T}_1(\epsilon) + \bar{T}_2(\epsilon) \), we have
\[
S(t) \geq \bar{S}(\epsilon) + \left( S(t_0 + \bar{T}_1(\epsilon)) - \bar{S}(\epsilon) \right) e^{-\frac{\lambda}{\bar{S}(\epsilon)} (t - t_0 - \bar{T}_1(\epsilon))} \\
> \bar{S}(\epsilon) + \left( \frac{\lambda}{\beta m} - \epsilon - \bar{S}(\epsilon) \right) e^{-\frac{\lambda}{\bar{S}(\epsilon)} (t - t_0 - \bar{T}_1(\epsilon))} \\
\geq \tilde{k}(\epsilon) \bar{S}(\epsilon).
\]

Hence, for \( t \geq t_0 + \bar{T}_1(\epsilon) + \bar{T}_2(\epsilon) \), it comes to the conclusion that
\[
\frac{S(t)}{N(t)} > \frac{\tilde{k}(\epsilon) \bar{S}(\epsilon)}{S^0} > \frac{S^*}{S^0}.
\]

**Lemma 4.2.** Under the assumptions of Lemma 4.1, it holds that \( E(t) \geq \bar{v} = \bar{v}(\epsilon, t_0) \equiv E(t_0)e^{C(t)T(t)} \) for \( t \geq t_0 \), where
\[
\bar{T}(\epsilon) \equiv \max\{\bar{T}_1(\epsilon) + \bar{T}_2(\epsilon), \tilde{a}(\epsilon)\}, \quad \tilde{a}(\epsilon) = -\ln \left( 1 - \tilde{k}(\epsilon) \right), \quad \bar{C}(\epsilon) = \frac{\beta a(\lambda / \beta m - \epsilon)}{S_0 + \epsilon} - c - d.
\]

**Proof.** First, by the second equation of system (2), we have
\[
\dot{E}(t) > \bar{C}(\epsilon) E(t).
\]
(7)

For \( t > t_0 \), it follows
\[
E(t) > E(t_0)e^{C(t)(t-t_0)}.
\]
(8)

Let
\[
\tilde{v} = \bar{v}(\epsilon, t_0) \equiv E(t_0)e^{C(t)T(t)}.
\]

Then it follows from (8) that \( E(t) > \tilde{v} \) for \( t \in [t_0, t_0 + \bar{T}(\epsilon)] \). For \( t > t_0 + \bar{T}(\epsilon) \), we can obtain \( E(t) \geq \bar{v} \). In fact, if not, then there is a \( T_2 > 0 \) such that \( E(t) \geq \bar{v} \) for \( t \in [t_0, \bar{v}] \), where \( \bar{v} = t_0 + \bar{T}(\epsilon) + T_2, E(\bar{v}) = \tilde{v} \) and \( \dot{E}(\bar{v}) \leq 0 \). Subsequently, we can claim that \( I(\bar{v}) > \tilde{k}(\epsilon)pc\bar{v}/B_1 \) and \( A(\bar{v}) > \tilde{k}(\epsilon)(1-p)c\bar{v}/B_2 \). Indeed, for \( t \in [t_0, \bar{v}] \), it holds that
\[
I(t) = pcE(t) - B_1I(t) \geq pc\bar{v} - B_1I(t), \\
A(t) = (1-p)cE(t) - B_2A(t) \geq (1-p)c\bar{v} - B_2A(t).
\]

And hence,
\[
I(t) \geq \frac{pc\bar{v}}{B_1} + \left( I(t_0) - \frac{pc\bar{v}}{B_1} \right) e^{-B_1(t-t_0)} > \frac{pc\bar{v}}{B_1} \left( 1 - e^{-B_1(t-t_0)} \right), \\
A(t) \geq \frac{(1-p)c\bar{v}}{B_2} + \left( A(t_0) - \frac{(1-p)c\bar{v}}{B_2} \right) e^{-B_2(t-t_0)} > \frac{(1-p)c\bar{v}}{B_2} \left( 1 - e^{-B_2(t-t_0)} \right).
\]

Thus, for \( t \in [t_0 + \tilde{a}(\epsilon), \bar{v}] \), we have
\[
I(t) > \frac{\tilde{k}(\epsilon)pc\bar{v}}{B_1}, \quad A(t) > \frac{\tilde{k}(\epsilon)(1-p)c\bar{v}}{B_2}.
\]
The claim is proved.

From Lemma 4.1, Remark 3.1 and the second equation of system (2), it follows

\[ \dot{E}(t) = \beta \frac{S(t)}{N(t)} (aE(t) + I(t) + bA(t)) - (c + d)E(t) \]

\[ > (c + d) \left( \frac{\tilde{k}(e)S(e)}{S_0 + \varepsilon} \right) \nabla \]

\[ > (c + d) \left( \frac{S^*}{S_0 R_c - 1} \right) \nabla = 0, \]

which contradicts \( \dot{E}(t) \leq 0 \). In consequence, \( E(t) \geq \nabla \) for \( t \geq t_0 \).

**Lemma 4.3.** Let \( R_c > 1 \) and \( \theta \in (0, 1) \). Then \( \limsup_{t \to \infty} E(t) \geq \theta E^* \).

**Proof.** We prove the statement by contradiction. Assume that this is not true. Then, there exists a \( t_0 \geq T_0 \) such that \( E(t) \leq \theta E^* \) for any \( t \geq t_0 \). Now, we define a function as follows,

\[ L(\phi) = \phi_2 + \frac{\beta \tilde{k}(e)S(e)}{B_1 (S_0 + \varepsilon)} \phi_3 + \frac{b \beta \tilde{k}(e)S(e)}{B_2 (S_0 + \varepsilon)} \phi_4, \phi \in \Omega. \]

Then by Lemma 4.1, the derivative of \( L \) along the solution \( u(t) \) for \( t \geq t_0 + \tilde{T}(\varepsilon) \) can be taken as

\[ L(u(t)) = \beta \frac{S(t)}{N(t)} (aE(t) + b \beta \tilde{k}(e)S(e)) \frac{p \varepsilon}{1 - \theta} E(t) - (c + d)E(t) \]

\[ + \beta \left( \frac{S(t)}{N(t)} - \frac{\tilde{k}(e)S(e)}{S_0 + \varepsilon} \right) \tilde{I}(t) + \beta b \left( \frac{S(t)}{N(t)} - \frac{\tilde{k}(e)S(e)}{S_0 + \varepsilon} \right) \tilde{A}(t) \]

\[ \geq (c + d) \left( \frac{\tilde{k}(e)S(e)}{S_0 + \varepsilon} R_c - 1 \right) E(t). \]

Consequently, for \( t \geq t_0 + \tilde{T}(\varepsilon) \), it follows from Lemma 4.2 that

\[ \dot{L}(u(t)) \geq (c + d) \left( \frac{\tilde{k}(e)S(e)}{S_0 + \varepsilon} R_c - 1 \right) \nabla > 0, \]

which hints \( L(u(t)) \to \infty \) as \( t \to \infty \). Accordingly, this contradicts the boundedness of \( L(u(t)) \).

**Theorem 4.2.** Suppose \( R_c > 1 \), \( \theta \in (0, 1) \) and \( \eta \in (\theta, 1) \). Then the solution \( u(t) \) of system (2) with any \( \phi \in \Omega \) satisfies that

\[ \liminf_{t \to \infty} E(t) \geq \theta E^* e^{(\frac{\eta}{\theta} - c - d)T} = \frac{\theta \lambda (R_c - 1)}{(c + d) R_c} e^{(\frac{\eta}{\theta} - c - d)T} \equiv \nabla, \]

where

\[ T = \max\{T_1 + T_2, \alpha\}, \]

\[ T_1 = \max \left\{ -\frac{1}{B_1} \ln \frac{\eta / \theta - 1}{(1/c + 1/d) B_1 / p (1 - 1/R_c) \theta - 1} -\frac{1}{B_2} \ln \frac{\eta / \theta - 1}{(1/c + 1/d) B_2 / (1 - p) (1 - 1/R_c) \theta - 1} \right\}, \]

\[ T_2 = -\frac{1}{B} \ln \left[ \frac{\eta (R_c - 1) + 1}{1 - d [\eta (R_c - 1) + 1]} / \beta m' \right], \]

\[ \alpha = -\frac{1}{B} \ln [(1 - \eta) (1 - 1/R_c)]. \]

**Proof.** From Lemma 4.3, we will consider (3) in two cases: \( E(t) \geq \theta E^* \) or \( E(t) \) oscillates around \( \theta E^* \) for sufficiently large \( t \). We thus only need to discuss \( E(t) \) oscillates around \( \theta E^* \). In consequence, we assume that \( t_1, t_2 \geq T_0 \) such that

\[ E(t) < \theta E^* \] for \( t \in (t_1, t_2) \) and \( E(t_1) = E(t_2) = \theta E^*. \]
When $t_2 \leq t_1 + \bar{T}(\varepsilon)$, it follows from (7) that

$$E(t) > E(t_1)e^{C(t-t_1)} \geq \theta E^*e^{C(t)} = \bar{v}(\varepsilon, t_1) = \bar{v} > 0$$

for $t \in (t_1, t_2]$. When $t_2 > t_1 + \bar{T}(\varepsilon)$, it holds $E(t) \geq \bar{v}$ for $t \in [t_1, t_1 + \bar{T}(\varepsilon)]$. For $t \in [t_1 + \bar{T}(\varepsilon), t_2]$, then proceeding exactly as in the proof of Lemma 4.2 we have $E(t) \geq \bar{v}$. Consequently, $E(t) \geq \bar{v}$ for $t \in [t_1, t_2]$. Consider that this kind of interval $[t_1, t_2]$ is chosen arbitrarily. Thus, $E(t) \geq \bar{v}$ for sufficiently large $t$, which implies $\lim_{t_1 \to \infty} E(t) \geq \bar{v}$. Note that $\varepsilon$ is given arbitrarily, we thus have $\lim_{t_1 \to \infty} E(t) \geq v$. In fact, by Lemma 3.1 and Remark 3.1 we have

$$E^* = \frac{\lambda (R_c - 1)}{(c + d) R_c}, I^* = \frac{pc \lambda (R_c - 1)}{B_1 (c + d) R_c} \quad A^* = \frac{(1 - p)c \lambda (R_c - 1)}{B_2 (c + d) R_c}, \quad R_c = \frac{S^0}{\varphi}.$$

Therefore, it follows

$$\lim_{\varepsilon \to 0^+} \bar{T}_1(\varepsilon) = \max \left\{ \frac{-1}{B_1} \ln \frac{\eta - \theta}{S^0 - \theta I^*}, \frac{-1}{B_2} \ln \frac{(\eta - \theta) A^*}{S^0 - \theta I^*} \right\} = T_1,$$

$$\lim_{\varepsilon \to 0^+} \bar{T}_2(\varepsilon) = -\frac{\tilde{S}(0)}{\lambda} \ln \frac{1 - \tilde{k}(0)}{\tilde{S}(0)} = T_2,$$

$$\lim_{\varepsilon \to 0^+} \tilde{\alpha}(\varepsilon) = -\frac{\ln (1 - \tilde{k}(0))}{B} = \alpha,$$

where

$$\tilde{S}(0) := \lim_{\varepsilon \to 0^+} \frac{S^0}{\eta (R_c - 1) + 1}, \quad \tilde{k}(0) := \lim_{\varepsilon \to 0^+} \frac{\eta (R_c - 1) + 1}{R_c}.$$

By Theorem 4.2 we have the following result immediately.

**Theorem 4.3.** If $R_c > 1$, then the solution $u(t)$ of system (2) with any $\phi \in \Omega$ is uniformly persistent, and satisfies

$$\liminf_{t \to \infty} S(t) \geq \frac{\lambda}{\beta m}, \quad \liminf_{t \to \infty} E(t) \geq v, \quad \liminf_{t \to \infty} f(t) \geq \frac{pc \nu}{B_1} = v_1, \quad \liminf_{t \to \infty} A(t) \geq \frac{(1 - p)c \nu}{B_2} = v_2,$$

$$\liminf_{t \to \infty} Q(t) \geq \frac{q_1 v_1 + q_2 v_2}{\bar{r}_3 d} = v_3, \quad \liminf_{t \to \infty} R(t) \geq \frac{r_1 v_1 + r_2 v_2 + r_3 v_3}{d}.$$

In the following, we will exhibit a case to illustrate the distinction of analysis method of Theorem 4.2. We reconsider the uniform persistence for microorganism concentration $m(t)$ of the following microorganism flocculation model proposed in [19],

$$\begin{align*}
\dot{n}(t) &= 1 - n(t) - \frac{\gamma n(t)m(t)}{1 + \rho m(t)}, \\
\dot{m}(t) &= \frac{\gamma n(t)m(t)}{1 + \rho m(t)} - m(t) - \frac{\mu t(t)}{1 + \sigma m(t)}, \\
\dot{f}(t) &= 1 - f(t) - \frac{\delta t(t) f(t)}{1 + \sigma m(t)},
\end{align*}$$

(10)

where $n(t), m(t)$ and $f(t)$ represent the concentrations of nutrient, microorganisms, and flocculants at time $t$, respectively, the parameters $\rho, \sigma \geq 0$ and $\tau \geq 0$ is time delay, and all other parameters are positive. Let $u(t) = (n(t), m(t), f(t))^T$ be the solution of model (10) with any $\phi \in X = \{ \phi \in C([-\tau, 0], \mathbb{R}^+) : \phi_2(0) > 0 \}$ and the threshold $R_0 = \mu / (\gamma + 1) > 1$ of model (10). Then there exists an $\varepsilon_1 > 1$ such that for any $\varepsilon \in (1, \varepsilon_1)$, it follows that

$$\frac{e^2 \gamma + 1}{(\gamma + 1) R_0} < 1, \quad \liminf_{t \to \infty} n(t) \geq \frac{\gamma (\gamma + 1) R_0 + \xi}{(\gamma + \xi)(\gamma + 1) R_0} > \frac{e^2 (\gamma + 1) R_0 + \xi}{e^2 (\gamma + \xi)(\gamma + 1) R_0}.$$
Corollary 4.2. If $R_0 > 1$, then it holds that

$$\liminf_{t \to \infty} m(t) \geq \frac{\theta}{(\xi + \zeta) (\gamma + 1)} e^{-(\gamma+1)(T+\tau)},$$

where

$$T = \frac{1 + \varphi/ (\xi + \zeta) (\gamma + 1)}{1 + \varphi/ (\gamma + 1)} \ln \frac{\gamma/\theta + 1/ [1 + \varphi/ (\gamma + 1)]}{\varphi/\zeta + 1/ [1 + \varphi/ (\gamma + 1)]}.$$ 

Remark 4.1. In fact, Corollary 4.2 is an improvement of [19] Theorem 4.1. By using the method employed in the proof of Theorem 4.2, the main results on persistence in [3, 9, 13–18, 24, 31, 37] can be improved.

Next, we give a numerical example with Matlab to illustrate the availability of our work.

Example 4.1. In system (2), if we take

$$\lambda = 1100, \beta = 0.12, a = 0.0116, b = 0.063, d = 9.6 \times 10^{-5}, c = 1.2 \times 10^{-5},$$

$$p = 0.74, q_1 = 0.03, q_2 = 0.6, r_1 = 0.76, r_2 = 0.17, r_3 = 0.1.$$ 

Thereby we can obtain $E^* \approx 9.396 \times 10^6$ and $R_c \approx 12.902 > 1$ using Matlab. Let the initial data be selected as $S_0 = 2.1 \times 10^7, I_0 = 2.3 \times 10^5, E_0 = 4.56 \times 10^3, A_0 = 5.76 \times 10^3, Q_0 = 1.8 \times 10^5, R_0 = 1.8 \times 10^5$. Then it follows $\liminf_{t \to \infty} E(t) \approx E^*$. Take $\theta = 0.9, \eta = 0.901$, we have $\liminf_{t \to \infty} E(t) \geq \nu \approx 6.725 \times 10^6$ and $\nu/\liminf_{t \to \infty} E(t) \approx 72\%$. This implies that Theorem 4.2 is valid, and numerical simulations suggest that the COVID-19 equilibrium $V^*$ may be globally attractive if $R_c > 1$ in $\Omega$. Among the given parameters, there can be found $\theta$ and $\eta$ such that $0 < \theta < \eta < 1$ and $\nu$ is a better estimate of the lower bound on $\liminf_{t \to \infty} E(t)$. Therefore, $\nu$ is not only a good explicit estimate of $\liminf_{t \to \infty} E(t)$, but also it has many practical meanings.

5. Conclusions

In this paper, a COVID-19 system (2) with nonlinear incidence rate is considered. In system (2), standard incidence rate, instead of bilinear incidence rate, is used to account for the population growth rate of exposed individuals, and different quarantined rates and recovery rates for the symptomatic and asymptomatic infected individuals are introduced. System (2) admits a unique COVID-19 equilibrium $V^*$ if and only if the control reproduction number $R_c > 1$. Then the local asymptotic stability of the COVID-19-free equilibrium $V^0$ of system (2) is proceeded. It shows that if $R_c < 1$ (the COVID-19 equilibrium $V^*$ is not viable), the COVID-19-free equilibrium $V^0$ is LAS, which implies that COVID-19 pandemic will disappear; if $R_c = 1$ (the COVID-19 equilibrium $V^*$ is also not viable), the linearized system of system (2) at $V^0$ is stable; if $R_c > 1$, $V^0$ is unstable.

For persistence dynamics of system (2), it shows that $V^*$ is viable, i.e., $R_c > 1$, the COVID-19 pandemic is uniformly persistent. Furthermore, a more refined analysis method is proposed to better estimate the ultimate lower bound of COVID-19 infected individuals if $R_c > 1$, which also non-trivially improves some analysis techniques for system persistence in [3, 9, 13–18, 24, 31, 37] as well as can be applied to other related mathematical models in biology. It is not difficult to find that $R_c$ is a decreasing function with respect to the quarantined rates $q_1$ and $q_2$. We thus can strengthen the quarantine, which also effectively reduce COVID-19 transmission. In addition, numerical simulations show that the COVID-19-free equilibrium $V^0$ and the COVID-19 equilibrium $V^*$ may be globally attractive for $R_c < 1$ in $D$ and $R_c > 1$ in $\Omega$, respectively. Therefore, the global stability problems of $V^0$ and $V^*$ are very practical and challenging, which means that the COVID-19 pandemic will die out or be persistent under certain conditions, we will settle these problems in future work.
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Conflict of interest and data availability statements

The authors declare that they have no conflict of interest, and data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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