LIFTING MORPHISMS BETWEEN GRADED GROTHENDIECK GROUPS OF LEAVITT PATH ALGEBRAS

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Abstract. We show that any pointed, preordered module map $\mathcal{B}_{gr}(E) \to \mathcal{B}_{gr}(F)$ between Bowen-Franks modules of finite graphs can be lifted to a unital, graded, diagonal preserving ∗-homomorphism $L_{\ell}(E) \to L_{\ell}(F)$ between the corresponding Leavitt path algebras over any commutative unital ring with involution $\ell$. Specializing to the case when $\ell$ is a field, we establish the fullness part of Hazrat’s conjecture about the functor from Leavitt path $\ell$-algebras of finite graphs to preordered modules with order unit that maps $L_{\ell}(E)$ to its graded Grothendieck group. Our construction of lifts is of combinatorial nature; we characterize the maps arising from this construction as the scalar extensions along $\ell$ of unital, graded ∗-homomorphisms $L_{\mathbb{Z}}(E) \to L_{\mathbb{Z}}(F)$ that preserve a sub-∗-semiring introduced here.

Keywords: Leavitt path algebras, graded $K$-theory, Hazrat’s conjectures.

1. Introduction

Let $E$ be a directed graph, and let $L_{\ell}(E)$ be its associated Leavitt path algebra over a commutative ring $\ell$ with involution ([16, Definition 2.5]). This is a ∗-algebra which is graded over $\mathbb{Z}$. All graphs considered will have finitely many vertices and edges. We write $K_{\mathbb{Z}}^{gr}(L_{\ell}(E))$ for the graded Grothendieck group of $L_{\ell}(E)$, that is, the group completion of the monoid of finitely generated projective $\mathbb{Z}$-graded $L_{\ell}(E)$-modules. This is a preordered module with order unit $[L_{\ell}(E)]$; see Subsections 2.3 and 2.4 for definitions of these terms and further details. In [11] Hazrat conjectures that, when $\ell$ is a field, the graded Grothendieck group classifies Leavitt path algebras as graded algebras.

Conjecture 1.1 ([11, Conjecture 1]). Suppose that $\ell$ is a field. Given graphs $E$ and $F$, the algebras $L_{\ell}(E)$ and $L_{\ell}(F)$ are graded isomorphic if and only if there is a preordered module isomorphism $K_{\mathbb{Z}}^{gr}(L_{\ell}(E)) \xrightarrow{\sim} K_{\mathbb{Z}}^{gr}(L_{\ell}(F))$ mapping $[L_{\ell}(E)]$ to $[L_{\ell}(F)]$.

Conjecture 1.2 ([11, Conjecture 3]). Let $\ell$ be a field. The graded Grothendieck group is a fully faithful functor from the category of unital Leavitt path $\ell$-algebras with graded homomorphisms modulo inner-automorphisms to the category of preordered abelian groups with order unit.

In the same article, Hazrat proves Conjecture 1.1 for a class of graphs called polycephaly graphs ([11, Definition 3.6]). A weaker version of Conjecture 1.1 was proven by Ara and Pardo in [3] for finite graphs with no sinks or sources. They also show that the faithfulness part of Conjecture 1.2 fails to hold in full generality ([3, Example 6.7]).

The objective of this article is to address the fullness part of Conjecture 1.2. In the particular case of a field, our main result reads as follows.

Theorem 1.3. Let $\ell$ be a field. Given finite graphs $E$ and $F$ and $\phi: K_{\mathbb{Z}}^{gr}(L_{\ell}(E)) \to K_{\mathbb{Z}}^{gr}(L_{\ell}(F))$ a morphism of preordered modules such that $\phi([L_{\ell}(E)]) = [L_{\ell}(F)]$, there exists a unital, diagonal preserving $\mathbb{Z}$-graded ∗-homomorphism $\varphi: L_{\ell}(E) \to L_{\ell}(F)$ such that $K_{\mathbb{Z}}^{gr}(\varphi) = \phi$.

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Here diagonal preserving means that the map \( \varphi \) above sends a distinguished commutative subalgebra of \( L_\ell(E) \), the diagonal subalgebra, to that of \( L_\ell(F) \).

For an arbitrary commutative ring with involution \( \ell \), we turn our attention to the Bowen-Franks module \( \mathcal{B}\mathcal{F}_G(\ell) \) of a graph \( E \), a notion closely related to that of the graded Grothendieck group. There are a distinguished element \( 1_E \in \mathcal{B}\mathcal{F}_G(\ell) \) and a comparison map

\[
\text{can}: \mathcal{B}\mathcal{F}_G(\ell) \rightarrow K^\ell_0(L_\ell(E)), \quad 1_E \mapsto [L_\ell(E)],
\]

which is an isomorphism whenever \( K^\ell_0(\ell) = \mathbb{Z} \) ([5, Corollary 5.4]). Theorem 1.3 is then implied by the more general statement below.

**Theorem 1.4** (Theorem 6.1). Let \( E \) and \( F \) be finite graphs. If \( \phi: \mathcal{B}\mathcal{F}_G(E) \rightarrow \mathcal{B}\mathcal{F}_G(F) \) is a morphism of preordered modules mapping \( 1_E \mapsto 1_F \), then there exists a unital, \( \mathbb{Z} \)-graded, diagonal preserving \(*\)-homomorphism \( \varphi: L_\ell(E) \rightarrow L_\ell(F) \) such that the following diagram commutes.

\[
\begin{array}{ccc}
K^\ell_0(L_\ell(E)) & \xrightarrow{K^\ell_0(\phi)} & K^\ell_0(L_\ell(F)) \\
\text{can} & & \text{can} \\
\mathcal{B}\mathcal{F}_G(E) & \xrightarrow{\phi} & \mathcal{B}\mathcal{F}_G(F)
\end{array}
\]

A natural question that arises from the theorem above is whether the lift of an isomorphism between Bowen-Franks modules is necessarily an algebra isomorphism. Using recent results of Carlsen, Dor-On and Eilers [7], we relate this to a question on the associated graph \( C^* \)-algebras (Question 6.15), which can be also be interpreted as a question in terms of shift equivalences and strong shift equivalences of matrices. In Theorem 6.17 we show that an affirmative answer to this question would imply that, for Leavitt path \( \mathbb{C} \)-algebras, the functor \( K^\ell_0 \) does not reflect isomorphisms. We also note that a counterexample to the Williams conjecture by Kim and Roush [14] yields a pair of graphs satisfying almost all of the conditions in Question 6.15; see Question 6.16 and Remark 6.18 for further details.

The construction of lifts given in Theorem 1.4 is of combinatorial nature; we translate the information encoded by a Bowen-Franks module map to the existence of certain partitions of paths in a graph and bijections between them. We call the homomorphisms that arise in such a way *tidy*; see Definition 7.1 for a precise statement. We prove a characterization of tidy maps in terms of a sub-*\*-semiring of a Leavitt path \( \mathbb{Z} \)-algebra introduced here, which we call its **positive cone** (see Definition 7.2). A morphism \( L_\ell(E) \rightarrow L_\ell(F) \) will be said to be order preserving if it maps the positive cone of \( L_\ell(E) \) to that of \( L_\ell(F) \). The theorem below lets us conclude, in particular, that the composite of two tidy homomorphisms is again tidy (Corollary 7.7).

**Theorem 1.5** (Theorem 7.4). Let \( E \) and \( F \) be finite graphs and \( \varphi: L_\ell(E) \rightarrow L_\ell(F) \) a unital \( \ell \)-algebra homomorphism. The following statements are equivalent.

i) The morphism \( \varphi \) is tidy.

ii) The morphism \( \varphi \) is the scalar extension along \( \ell \) of a unital, order preserving, \( \mathbb{Z} \)-graded \(*\)-homomorphism \( L_\ell(E) \rightarrow L_\ell(F) \).

The rest of the article is organized as follows. In Section 2 we recall the basic notions of Leavitt path algebras and Bowen-Franks modules. Section 3 is dedicated to auxiliary results regarding the relation between the graded Grothendieck group and homogeneous idempotents. In Section 4 we establish some properties of Bowen-Franks modules using a presentation introduced in [5]; this is then used in Section 5 to give a characterization of maps between Bowen-Franks modules in terms of vertex indexed matrices with nonnegative integer coefficients (Proposition 5.1). With this in place, in Section 6 we prove Theorem 1.4 as Theorem 6.1. Theorem 1.5 is proven in Section 7 as Theorem 7.4.
Note. We point out that, simultaneously and independently, Lia Vaš has proved a lifting result [17, Theorem 3.2] in the case when \( \ell \) is a field, similar to Theorem 1.3. The lifts obtained in loc. cit. are not shown to be involution nor diagonal preserving. We also remark that [17, Theorem 3.2] holds for graphs with finitely many vertices and countably many edges. In [17, Section 5.2], some non-constructive steps of the proof of [17, Theorem 3.2] are discussed, and the question of whether one can explicitly produce a lift is posed. We note that the proofs of Proposition 5.1 and Theorem 6.1 give an algorithm to construct lifts; however, we exhibit no bound for the finitely many steps needed to obtain Equations (5.9) and (5.8). We would also like to remark that the implications (1) \( \Rightarrow (5^+) \Rightarrow (5) \) of [17, Theorem 4.1] can be recovered by setting \( L = 1 \) in Corollary 3.5.

Notation 1.6. In this article the natural numbers do not include zero; we will write \( \mathbb{N} := \mathbb{Z}_{\geq 1} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). The letter \( \sigma \) will denote the generator of the infinite cyclic group \( \mathbb{C}_\infty \simeq \mathbb{Z} \), written multiplicatively. Its group ring will be denoted \( \mathbb{Z}[\sigma] := \mathbb{Z}[C_\infty] \).

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2. Preliminaries

2.1. Graphs. A graph \( E \) consists of two sets \( E^0 \) of vertices and \( E^1 \) of edges, together with source and range functions \( s, r \colon E^1 \to E^0 \). All graphs in this paper will be assumed to be finite, meaning that \( E^0 \) and \( E^1 \) are finite sets.

A vertex \( v \in E^0 \) is regular if \( s^{-1}(v) \neq \emptyset \) and a sink otherwise. The set of regular vertices will be denoted \( \text{reg}(E) \subset E^0 \); its complement is \( \text{sink}(E) = E^0 \setminus \text{reg}(E) \). We say that \( E \) is regular if \( E^0 = \text{reg}(E) \).

Given \( v, w \in E^0 \), we will write \( E_{v,w} = s^{-1}(v) \cap r^{-1}(w) \) for the set of edges with source \( v \) and range \( w \). The adjacency matrix of a graph \( E \) is the matrix \( A_E \in \mathbb{Z}^{E^0 \times E^0} \), whose \( (v, w) \)-entry is the amount of edges in \( E \) with source \( v \) and range \( w \). \( (A_E)_{v,w} = \# E_{v,w} \). The reduced adjacency matrix \( A_E \in \mathbb{Z}^{\text{reg}(E) \times E^0} \), \( (A_E)_{v,w} = (A_E)_{v,w} \), is the matrix obtained from removing the rows of \( A_E \) corresponding to sinks.

A path in \( E \) is a finite sequence of edges \( \alpha = e_1 \ldots e_n \) such that \( r(e_i) = s(e_{i+1}) \) for each \( i \in \{1, \ldots, n-1\} \). The source of \( \alpha \) is \( s(\alpha) = s(e_1) \) and its range is \( r(\alpha) = r(e_n) \). The length of \( \alpha \) is \( |\alpha| = n \). We will consider a vertex \( v \in E^0 \) as a path of length zero with source and range \( v \). The set of paths of \( E \) will be denoted \( E^\infty \). For each \( k \geq 0 \) and \( v, w \in E^0 \), we shall write \( E^k_{v,w} = \{ \alpha \in E^\infty : s(\alpha) = v, r(\alpha) = w, |\alpha| = k \} \) and \( E^k_{v,v} = \bigcup_{w \in E^0} E^k_{v,w} = \{ \alpha \in E^\infty : r(\alpha) = w, |\alpha| = k \} \).

For convenience, we recall that \( \# E^k_{v,w} = (A^k)_{v,w} \) and thus

\[
\# E^k_{v,v} = \sum_{v \in E^0} (A^k)_{v,v}. \tag{2.1.1}
\]

We shall also need to distinguish between paths of a certain length ending at sinks or regular vertices; given \( n \geq 0 \) we put \( R_n(E) = \bigcup_{w \in \text{reg}(E)} E^n_{w,v} \) and \( S_n(E) = \bigcup_{w \in \text{sink}(E)} \bigcup_{i=0}^n E^i_{v,w} \).

2.2. Leavitt path algebras. Throughout the paper, we fix a commutative unital ring \( \ell \) equipped with an involution \( * \colon \ell \to \ell \).

Let \( E \) be a graph and \( L(E) := L_\ell(E) \) its Leavitt path algebra over \( \ell \) [16, Definition 2.5] equipped with its canonical \( \mathbb{Z} \)-grading [16, Proposition 4.7]. Recall that by [16, Theorem 8.1] there is an isomorphism \( L(E) = \ell \otimes_{\mathbb{Z}} L_\mathbb{Z}(E) \) and, in particular, when \( \ell \) is a field this definition
agrees with that of [1, Definition 1.2.3]. We consider the canonical involution on \(L_Z(E)\) as defined in [16, Remark 4.1] and equip \(L(E)\) with the tensor product involution.

Unless specified otherwise, whenever we refer to [1], we shall implicitly mean that the cited argument holds true for a Leavitt path algebra over any commutative unital ring. We record the following lemmas for future use.

**Lemma 2.2.1.** Let \(\alpha\) and \(\beta\) be two paths in a finite graph \(E\). The following identities hold in \(L(E)\):

1. If \(|\alpha| = |\beta|\), then \(\alpha^*\beta = \delta_{\alpha,\beta}r(\alpha)\).
2. If \(r(\alpha) \in \text{reg}(E), r(\beta) \in \text{sink}(E)\) and \(|\beta| \leq |\alpha|\), then \(\alpha^*\beta = \beta^*\alpha = 0\).
3. If \(r(\alpha), r(\beta) \in \text{sink}(E)\), then \(\alpha^*\beta = \delta_{\alpha,\beta}r(\alpha)\).
4. For each \(N \geq 0\), the set \(\{xx^* : x \in \mathcal{R}_N(E) \cup \mathcal{S}_N(E)\}\) consists of homogeneous orthogonal projections.

**Proof.** Straightforward from [1, Lemma 1.2.12]. \(\square\)

**Lemma 2.2.2.** Let \(E\) be a finite graph. For each \(N \in \mathbb{N}_0\),

\[
1 = \sum_{\alpha \in \mathcal{R}_N(E)} \alpha\alpha^* + \sum_{\beta \in \mathcal{S}_N(E)} \beta\beta^*
\]

**Proof.** The lemma is true when \(N = 0\) by [1, Lemma 1.2.12 (iv)]. The general case follows inductively from [1, Relation (CK2) in Definition 1.2.3]. \(\square\)

Recall that given a graph \(E\), the diagonal of \(L(E)\) is the sub*-algebra \(D(E) = \text{span}_\mathbb{C}\{\alpha^* : \alpha \in E^\infty\} \subseteq L(E)_0\). An algebra homomorphism \(\varphi : L(E) \to L(F)\) is diagonal preserving if \(\varphi(D(E)) \subseteq D(F)\). Writing

\[
L(E)_{0,n} = \text{span}_\mathbb{C}\{\alpha^* : \alpha \in \mathcal{R}_n(E)\} \oplus \text{span}_\mathbb{C}\{\beta^* : \alpha, \beta \in \mathcal{S}_n(E)\}
\]

and

\[
D(E)_n = \text{span}_\mathbb{C}\{\alpha^* : \alpha \in \mathcal{R}_n(E) \cup \mathcal{S}_n(E)\},
\]

each algebra \(L(E)_{0,n}\) is matricial and \(D(E)_n\) is its diagonal subalgebra. We have increasing unions \(L(E)_0 = \bigcup_{n \geq 0} L(E)_{0,n}, D(E) = \bigcup_{n \geq 0} D(E)_n\).

### 2.3. Preordered \(\mathbb{Z}[\sigma]\)-modules

A preordered \(\mathbb{Z}[\sigma]\)-module is a \(\mathbb{Z}[\sigma]\)-module \(M\) together with an additive submonoid \(M_+\) such that \(\sigma M_+ \subseteq M_+\) and \(\sigma^{-1}M_+ \subseteq M_+\). For \(m, n \in M_+\), we will write \(m \geq n\) to mean that \(m - n \in M_+\). An order unit in a preordered module \(M\) is an element \(u \in M\) such that for all \(m \in M\), there exist \(x \in \mathbb{N}_0[\sigma]\) such that \(xu \geq m\). A morphism of preordered \(\mathbb{Z}[\sigma]\)-modules with order unit is a \(\mathbb{Z}[\sigma]\)-linear map \(f : (M, u) \to (N, v)\) such that \(f(M_+) \subseteq N_+\) and \(f(u) = v\).

### 2.4. Bowen-Franks modules

Let \(E\) be a graph and \(I \in \mathbb{Z}^{E_0 \times \text{reg}(E)}\) the matrix defined by \(I_{i,v} = \delta_{i,v}\). The Bowen-Franks \(\mathbb{Z}[\sigma]\)-module of \(E\) is

\[
\mathfrak{B}_\Delta(E) := \text{coker}(I - \sigma A_E^t) = \text{coker}(v - \sigma \sum_{e \in v} r(e) : v \in \text{reg}(E))
\]

There is a canonical preordered module structure on \(\mathfrak{B}_\Delta(E)\) given by the submonoid \(\mathfrak{B}_\Delta(E)_+\) generated by the elements \(\sigma^i v\) for each \(v \in E^0\) and \(i \in \mathbb{Z}\) (this is known as the talented monoid of \(E\); see [13]). The element \(1_E := \sum_{v \in E^0} v\) is an order unit with respect to this preordered module structure.

Write \(K_0^+ (L(E))\) for the group completion of the monoid of projective, finitely generated, graded \(L(E)\)-modules. This group is equipped with a canonical preordered \(\mathbb{Z}[\sigma]\)-module structure.
3. Graded idempotents and Murray-von Neumann equivalence

Let \( R \) be a ring. Recall that the idempotent elements \( \text{idem}(R) \) of \( R \) are partially ordered by defining \( e \leq f \) whenever \( ef = fe = e \) and that \( e, f \in \text{idem}(R) \) are said to be Murray-von Neumann equivalent, denoted \( e \sim f \), if there exist \( x, y \in R \) such that \( xy = e, yx = f \). This is an equivalence relation, put \( \mathcal{V}(R) = \text{idem}(R)/\sim \) and \( \mathcal{V}_\infty(R) := \text{idem}(\mathcal{M}_\infty(R))/\sim \). We also recall that the block sum of matrices makes \( \mathcal{V}_\infty(R) \) into a commutative monoid.

Remark 3.1. Recall that a ring \( R \) has local units if for each finite subset \( F \subset R \) there exists \( e \in \text{idem}(R) \) satisfying \( F \subset eRe \). If \( R \) is a ring with local units, then by [4, Section 4A] the monoid \( \mathcal{V}_\infty(R) \) is isomorphic to the monoid of isomorphism classes of finitely generated projective unital \( R \)-modules. Let \( S \) be a unital \( \mathbb{Z} \)-graded ring and let \( \mathbb{Z} \ltimes S \) be its crossed product as defined in [5, Subsection 2.5]. By [4, Section 2C] (see also [5, Section 3.1]), we have that \( \mathcal{V}_\infty(\mathbb{Z} \ltimes S) \) is naturally isomorphic to the monoid \( \mathcal{V}^\mathbb{Z}(S) \) of isomorphism classes of finitely generated \( \mathbb{Z} \)-graded projective \( S \)-modules. In particular \( K_0^\mathbb{Z}(S) \) is the group completion of \( \mathcal{V}_\infty(\mathbb{Z} \ltimes S) \).

Lemma 3.2. Let \( R \) be a ring and \( e, f, g \in \text{idem}(R) \) such that \( e, f \leq g \). The following statements are equivalent:

i) There exist \( x, y \in gRg \) such that \( xy = e, yx = g \).

ii) The idempotents \( e \) and \( f \) are Murray-von Neumann equivalent.

Proof. Since \( gRg \subset R \), we have that i) \( \Rightarrow \) ii). To prove ii) \( \Rightarrow \) i), consider \( x, y \in R \) such that \( xy = e, yx = f \) and put \( x' = exf, y' = fye \). Then \( x'y' = e, y'x' = f \) and \( x' \in eRf = geRfg \subset gRg, y \in gRg \).

Lemma 3.3. If \( R \) is a ring and \( g \in \text{idem}(R) \), then the inclusion induced map \( \mathcal{V}(gRg) \to \mathcal{V}_\infty(R) \) is injective.

Proof. Write \( \iota_k : gRg \to M_k R \) for the upper-left corner inclusion. Let \( e, f \in gRg \) be two idempotents that are Murray-von Neumann equivalent as elements of \( M_k R \). There exists \( n \geq 1 \) such that \( \iota_n(e) \sim \iota_n(f) \) in \( M_n R \). Since \( e, f \leq g \), it follows that \( \iota_n(e), \iota_n(f) \leq \iota_n(g) \); by Lemma 3.2 there must exist \( a, b \in \iota_n(g)(M_n R) \iota_n(g) = \iota_n(gRg) \) such that \( ab = \iota_n(e), ba = \iota_n(f) \). Let \( x, y \in gRg \) be such that \( a = \iota_n(x), b = \iota_n(y) \). Since \( \iota_n \) is injective, it follows that \( xy = e, yx = f \), thus proving that \( e \sim f \).

Corollary 3.4. If \( R \) is a \( \mathbb{Z} \)-graded unital ring, the canonical map \( \mathcal{V}(R_0) \to \mathcal{V}_\infty(\mathbb{Z} \ltimes R) \) is injective.

Proof. Apply Lemma 3.3 to \( \mathbb{Z} \ltimes R \) and the idempotent \( g = \chi_0 \ltimes 1 \), and notice that \( R_0 \simeq g(\mathbb{Z} \ltimes R)g \) via \( r \mapsto \chi_0 \ltimes r \).
Corollary 3.5. Suppose that $\ell$ is a field. Let $E$ and $F$ be finite graphs and $f, g: L(E) \to L(F)$ two unital $\mathbb{Z}$-graded maps. If $K_0^\mathbb{Z}(f) = K_0^\mathbb{Z}(g)$, then for each $L \geq 0$ there exists a degree zero unit $u_L \in L(E)$ such that $\text{ad}(u_L) \circ f$ and $g$ coincide on $D(E)L = \text{span}_F \{xx^*: x \in R_L \cup S_L \}$.

Proof. Since $V_\mathbb{Z}^\mathbb{Z}(L(F))$ is a cancellative monoid when $\ell$ is a field by [4, Corollary 5.8], using Remark 3.1 and Corollary 3.4 we get that the map $V(L(F)_0) \to K_0^\mathbb{Z}(L(F))$ is injective. Since $[f(xx^*)] = [g(xx^*)]$ in $K_0^\mathbb{Z}(L(F))$ for each $x \in X := R_L \cup S_L$, there exist $p_x \in f(xx^*)L(F)_{0g(xx^*)}$, $q_x \in g(xx^*)L(F)_0f(xx^*)$ such that $p_xq_x = f(xx^*)$, $q_xp_x = g(xx^*)$. Now put $u_L := \sum_{x \in X} q_x$ and proceed as in [8, Lemma 9.7]. 

Lemma 3.6. Let $E$ be a finite graph and can as in (2.4.1). If $\alpha$ is a path of length $N \in \mathbb{N}_0$ in $E$, then

$$\text{can}(\sigma^N r(\alpha)) = [\alpha\alpha^*].$$

Proof. Via the identification $K_0^\mathbb{Z}(L(E)) = V_\infty(\mathbb{Z} \ast L(E))^+$, the element $[L(E)\alpha\alpha^*] \in K_0^\mathbb{Z}(L(E))$ corresponds to the class of the idempotent $\chi_0 \cong \alpha\alpha^* \in \mathbb{Z} \ast L(E)$. By [5, Theorem 5.2], the element $\text{can}(\sigma^N r(\alpha))$ corresponds to the class of the idempotent element $\chi_n \cong v \in \mathbb{Z} \ast L(E)$. It suffices then to note that the elements $\chi_0 \cong \alpha$, $\chi_n \cong \alpha\alpha^* \in \mathbb{Z} \ast L(E)$ implement a Murray-von Neumann equivalence between $\chi_n \cong v$ and $\chi_0 \cong \alpha\alpha^*$.

4. A characterization of Bowen-Franks modules

Let $E$ be a finite graph. We first recall from [5] a characterization of $\mathfrak{B}_gf(E)$ and establish further properties of this presentation. For each $k \in \mathbb{Z}$ and $v \in E^0$, write $v_k = (v, k)$. Given $x = \sum_{v \in E^0} x_v v \in \mathbb{Z} E^0$, we will write

$$x \otimes \sigma^k = \sum_{v \in E^0} x_v v_k \in \mathbb{Z} E^0 \times \{k\}.$$

Put $V_n := \{u_i : u \in \text{sink}(E), |i| \leq n \} \cup \{w_n : w \in \text{reg}(E)\}$ for each $n \geq 0$ and define $\mathbb{Z}$-linear maps $\iota_n : \mathbb{Z} V_n \to \mathbb{Z} V_{n+1}$ via

$$\iota_n(u_i) := u_i, \quad \iota_n(w_n) := \sum_{v \in E^0} (A_E)_{w,v} v_{n+1}.$$

The maps above form an $\mathbb{N}_0$-indexed filtered system of abelian groups. Given two non-negative integers $i < j$, we will write $\iota_{i,j} = \iota_{j-1} \circ \cdots \circ \iota_i$ for the transition map. We also put $\iota_{i,i} = \text{id}_{\mathbb{Z} V_i}$. The following proposition follows from [5, proof of Theorem 5.2].

Proposition 4.1. Let $E$ be a finite graph. There is a $\mathbb{Z}[\sigma]$-module isomorphism

$$(4.2) \quad \mathfrak{B}_gf(E) \cong \text{colim}(\mathbb{Z} V_0 \xrightarrow{\iota_0} \mathbb{Z} V_1 \to \cdots), \quad \sigma^nv \mapsto [v_n],$$

that maps $\mathfrak{B}_gf(E)_+$ to $\text{colim}(\mathbb{N} V_0 \xrightarrow{\iota_0} \mathbb{N} V_1 \to \cdots)$ and $1_E$ to $\sum_{v \in E^0} [v_0]$. \hfill \Box

Remark 4.3. Let $E$ be a finite graph. Since $\mathfrak{B}_gf(E)$ is a filtering colimit, it follows that if $x \in \mathbb{Z} V_n$ and $y \in \mathbb{Z} V_m$ are such that $[x] = [y] \in \mathfrak{B}_gf(E)$ then there exists $k_0 \geq n, m$ such that $\iota_{n,k}(x) = \iota_{m,k}(y)$ for all $k \geq k_0$. Moreover, if $x_1 \in \mathbb{Z} V_{n_1}, \ldots, x_j \in \mathbb{Z} V_{n_j}$, $y_1 \in \mathbb{Z} V_{m_1}, \ldots, y_j \in \mathbb{Z} V_{m_j}$ are such that $[x_i] = [y_i]$ for all $i \in \{1, \ldots, j\}$, there exists $k_0 \geq n_1, \ldots, n_j, m_1, \ldots, m_j$ such that $\iota_{n_i,k}(x_i) = \iota_{m_j,k}(y_k)$ for all $1 \leq i \leq j$ and $k \geq k_0$.

Definition 4.4. Let $E$ be a finite graph. We will write $B_E \in \mathbb{Z} [\text{reg}(E) \times \text{reg}(E)]$ and $C_E \in \mathbb{Z} [\text{sink}(E) \times \text{reg}(E)]$ for the matrices obtained from projecting $A'_E$ onto $\mathbb{Z} [\text{reg}(E)]$ and $\mathbb{Z} [\text{sink}(E)]$, respectively,

$$(B_E)_{v,w} = (A_E)_{w,v}, \quad (C_E)_{v,u} = (A_E)_{u,v} \quad (u \in \text{sink}(E), \; v, w \in \text{reg}(E)).$$
In particular we have $A_E^r x = B_E x + C_E x$ for each $x \in \mathbb{Z}^\text{reg}(E)$ and if $s(-N), \ldots, s(N) \in \mathbb{Z}^\text{sink}(E)$, $r \in \mathbb{Z}^\text{reg}(E)$ then
\begin{equation}
\label{eq:4.5}
I_N \left( \sum_{i=-N}^N s^{(i)} \otimes \sigma^i + r \otimes \sigma^N \right) = \sum_{i=-N}^N s^{(i)} \otimes \sigma^i + A_E^r \otimes \sigma^{N+1} = \sum_{i=-N}^N s^{(i)} \otimes \sigma^i + C_E r \otimes \sigma^{N+1} + B_E r \otimes \sigma^{N+1}.
\end{equation}

The following identities are straightforward from (4.5).

**Lemma 4.6.** Let $E$ be a finite graph and $N \in \mathbb{N}$. Let $s(-N), \ldots, s(N) \in \mathbb{Z}^\text{sink}(E)$, $r \in \mathbb{Z}^\text{reg}(E)$. If $x = \sum_{i=-N}^N s^{(i)} \otimes \sigma^i + r \otimes \sigma^N \in \mathbb{Z}^V$, then:

(i) $\sigma^{-1} [x] = \sum_{i=-N}^N s^{(i)} \otimes \sigma^{i-1} + [A_E^r \otimes \sigma^N] \in \mathfrak{B}_{\mathfrak{gr}}(E)$.

(ii) If $M \geq 0$, then
\begin{equation}
\label{eq:4.6}
I_{N,N+M} (x) = \sum_{i=-N}^N s^{(i)} \otimes \sigma^i + \sum_{j=0}^{M-1} C_E B_E^j r \otimes \sigma^{N+j+1} + B_E^M r \otimes \sigma^{N+M}.
\end{equation}

(iii) If $M \geq 0$, then
\begin{equation}
\label{eq:4.7}
I_{N,N+M} (A_E^r \otimes \sigma^N) = \sum_{j=0}^M C_E B_E^j r \otimes \sigma^{N+j} + B_E^{M+1} r \otimes \sigma^{N+M}
\end{equation}

\hfill \square

We now turn to characterizing the representatives of the order unit of $\mathfrak{B}_{\mathfrak{gr}}(E)$ for a finite graph $E$. For each $n \geq 0$, let
\begin{equation}
\label{eq:4.8}
u_n := u_{0,n} (1_E) \in \mathbb{Z}^V.
\end{equation}

**Lemma 4.8.** Let $E$ be a finite graph. For each $N \in \mathbb{N}_0$,
\begin{equation}
\label{eq:4.9}
u_N = \sum_{u \in \text{sink}(E)} \sum_{i=0}^N \# E^i_u u_i + \sum_{v \in \text{reg}(E)} \# E^N v v_N.
\end{equation}

\textbf{Proof.} Since the lemma holds for $N = 0$ by definition of $u_0$ and $1_E$, we may suppose that $N \geq 1$.

By Lemma 4.6 (ii), we have to show that $\# E^i_u = (C_E B_E^{i-1}) u_{1 \text{reg}(E)}$ for each $u \in \text{sink}(E)$, $i \in \{1, \ldots, N\}$ and $\# E^N v = B_E^{N+1} v_\text{reg}(E)$, for each $v \in \text{reg}(E)$.

Observe that viewed as an endomorphism of $\mathbb{Z}^\text{sink}(E) \oplus \mathbb{Z}^\text{reg}(E)$, the matrix $A_E^r$ has the form
\[
A_E^r = \begin{pmatrix}
0 & C_E \\
0 & B_E
\end{pmatrix}
\] and inductively,
\[
(A_E^r)^i = \begin{pmatrix}
0 & C_E B_E^{i-1} \\
0 & B_E^i
\end{pmatrix}
\] $(1 \leq i \leq N)$.

To conclude we note that Equation (2.1.1) can be restated as $\# E^i_u = ((A_E^r)^i 1_E)_w$ for each vertex $w \in E^0$. \hfill \square

**Lemma 4.10.** Let $E$ be a finite graph and $N \in \mathbb{N}$. Let $s(-N), \ldots, s(N) \in \mathbb{Z}^\text{sink}(E)$, $r \in \mathbb{Z}^\text{reg}(E)$ and $x = \sum_{i=-N}^N s^{(i)} \otimes \sigma^i + \sum_{v \in \text{reg}(E)} r_v \otimes \sigma^N$. If $[1_E] = [x]$ in $\mathfrak{B}_{\mathfrak{gr}}(E)$, then $s^{(i)} = 0$ for all $i \in \{-N, \ldots, -1\}$.

\textbf{Proof.} By Remark 4.3, there exists $M \in \mathbb{N}$ such that $I_{N,M} (x) = u_{0,N+M} (1_E) = u_{N+M} \in \mathbb{Z}^V$.

Hence $u_{N+M} = I_{N,M} (x) = \sum_{i=-N}^N s^{(i)} \otimes \sigma^i + t_{N,M} (r \otimes \sigma^N)$. Writing $P = V_{N+M} \setminus (\text{sink}(E) \times$
\{- (N + M), \ldots, -1\}, notice that \(u_{N+M}, t_{N,M}(r \otimes \sigma^N)\) and \(s^{(i)} \otimes \sigma^i\) for non-negative \(i\) belong to \(\mathbb{Z}^P\). Thus,
\[
\sum_{i=-N}^{-1} s^{(i)} \otimes \sigma^i = u_{N+M} - t_{N,M}(r \otimes \sigma^N) - \sum_{i=0}^{N} s^{(i)} \otimes \sigma^i \in \mathbb{Z}^P \cap \mathbb{Z}^{V_{N+M} \setminus P} = 0.
\]
This completes the proof.

5. Maps between Bowen-Franks modules

The main result of this section is the following characterization of morphisms between Bowen-Franks modules.

**Proposition 5.1.** Let \(E\) and \(F\) be a finite graphs and \(\phi : \mathcal{B}\mathcal{D}_{\mathcal{G}}(E) \to \mathcal{B}\mathcal{D}_{\mathcal{G}}(F)\) a morphism of preorder \(\mathbb{Z}[\sigma]\)-modules such that \(\phi(1_E) = 1_F\).

For each \(L_0 \in \mathbb{N}_0\) there exist \(L \geq L_0\) and matrices \(S^{(0)}, S^{(1)}, \ldots, S^{(L)} \in \mathbb{N}_0^{E_0^{\cap} \times \text{sink}(F)}, R \in \mathbb{N}_0^{E_0 \times \text{reg}(F)}\) such that, for each \(v \in E_0\),
\[
\phi([v]) = \sum_{u \in \text{sink}(F)} \sum_{i=0}^{L} S^{(i)}_{u, v} [u] + \sum_{w \in \text{reg}(F)} R_{v, w} [w].
\]
Moreover, for each \(v \in \text{reg}(F), w \in \text{reg}(F), u \in \text{sink}(F), i \in \{0, \ldots, L\}\), we have the following equations:
\[
\# F^L_v = \sum_{z \in E_0} R_{z, w}, \quad \# F^i_v = \sum_{z \in E_0} S^{(i)}_{z, w},
\]
\[
S^{(0)}_{v, w} = 0, \quad S^{(i)}_{v, w} = (A_E S^{(i-1)})_{v, u} (1 \leq i \leq L), \quad (A_E S^{(L)})_{v, u} = (RA_F)_{v, u},
\]
\[
(A_E R)_{v, w} = (RA_F)_{v, w}.
\]

**Proof.** Since \(\phi\) is order preserving and \(E_0\) is finite, by (4.2) there exist \(N \geq L_0\) and a family \((x_v)_{v \in E_0}\) with \(x_v \in \mathbb{N}_0^V\) such that \(\phi([v]) = [x_v]\) for all \(v \in E_0\). Thus, for each \(v \in E_0\) and \(i \in \{-N, \ldots, N\}\), there exist vectors \(s_v^{(-N)}, \ldots, s_v^{(N)} \in \mathbb{N}_0^{\text{sink}(F)}, \tilde{r}_v \in \mathbb{N}_0^{\text{reg}(F)}\) such that
\[
\phi([v]) = \sum_{i=-N}^{N} [s_v^{(i)} \otimes \sigma^i] + [\tilde{r}_v \otimes \sigma^N].
\]
From here we see that
\[
[1_F] = \sum_{v \in E_0} \phi([v]) = \sum_{i=-N}^{N} | \sum_{v \in E_0} s_v^{(i)} \otimes \sigma^i | + | \sum_{v \in E_0} \tilde{r}_v \otimes \sigma^N |.
\]
By Lemma 4.10, this implies that \(\sum_{v \in E_0} s_v^{(i)} = 0\) for each \(v \in E_0\) and negative \(i\). Since the coefficients of each vector \(s_v^{(i)}\) are non-negative, we obtain that \(s_v^{(i)} = 0\) for all \(v \in E_0, i \in \{-N, \ldots, -1\}\). As a consequence, the equations above simplify to \(\phi([v]) = \sum_{i=0}^{N} [s_v^{(i)} \otimes \sigma^i] + [\tilde{r}_v \otimes \sigma^N]\) and
\[
[u_N] = [1_F] = \sum_{v \in E_0} \phi([v]) = \sum_{i=0}^{N} | \sum_{v \in E_0} s_v^{(i)} \otimes \sigma^i | + | \sum_{v \in E_0} \tilde{r}_v \otimes \sigma^N |.
\]
Now, for each \(v \in \text{reg}(E)\) the equation \(\sigma^{-1} \phi([v]) = \phi(\sigma^{-1}[v])\) and Lemma 4.6 (i) yield
\[
\sum_{i=0}^{N} [s_v^{(i)} \otimes \sigma^{-1}] + [A_E \tilde{r}_v \otimes \sigma^N] = \sum_{i=0}^{N} \sum_{x \in E_0} (A_E)_{v, x} s_x^{(i)} \otimes \sigma^i + \sum_{x \in E_0} (A_E)_{v, x} \tilde{r}_x \otimes \sigma^N.
\]
By Remark 4.3 there exists \( M \geq 0 \) such that the representatives involved in Equations (5.6) and (5.7) become equal upon applying \( \iota_{\mathbb{N},\mathbb{N}+M} \). Let \( L := N + M \) and put \( r_v := B_F^M \tilde{r}_v \in \mathbb{N}_0^{\text{reg}(F)} \), \( s_v^{N+1+j} := C_F B_F^j \tilde{r}_v \in \mathbb{N}_0^{\text{sink}(F)} \) for each \( v \in E^0 \) and \( 0 \leq j \leq M - 1 \). Set \( S^{(i)}_{v, u} := (s_v^{(i)})_u \) and \( R_{v, w} = (r_v)_w \) for each \( v \in E^0 \), \( w \in \text{reg}(F) \), \( u \in \text{sink}(F) \) and \( i \in \{0, \ldots, L\} \). With this notation in place, by Lemma 4.6 we have the following equality in \( \mathbb{Z}^L \):

\[
u_L = \sum_{i=0}^L \sum_{v \in E^0} s_v^{(i)} \otimes \sigma^i + \sum_{v \in E^0} r_v \otimes \sigma^L.
\]

The identities of (5.3) follow by comparing (5.8) with (4.9) applied to \( F \). By the same argument, for each \( v \in \text{reg}(E) \) we obtain

\[
\sum_{i=0}^L s_v^{(i)} \otimes \sigma^{i-1} + C_F r_v \otimes \sigma^L + B_F r_v \otimes \sigma^L = \sum_{i=0}^L \sum_{x \in E^0} (A_E)_{v, x} s_x^{(i)} \otimes \sigma^i + \sum_{x \in E^0} (A_E)_{v, x} r_x \otimes \sigma^L.
\]

The identities of (5.4) and (5.5) follow by comparing the coefficients of \( \sigma^i \) for \( i = 0, \ldots, L \) in both sides of (5.9) and taking coordinates.

**Corollary 5.10.** Let \( E \) and \( F \) be finite graphs and \( \phi : \mathfrak{B}_{\mathfrak{gr}}(E) \to \mathfrak{B}_{\mathfrak{gr}}(F) \) morphism of preordered \( \mathbb{Z}[\sigma] \)-modules such that \( \phi(1_E) = 1_F \). If \( E \) is regular, then so is \( F \).

**Proof.** By Proposition 5.1, there exists a matrix \( S^{(0)} \in \mathbb{N}_0^{E^0 \times \text{sink}(F)} \) such that for each \( v \in \text{reg}(E) \) and \( u \in \text{sink}(F) \) we have \( s_v^{(0)}_u = 0 \) and \( 1 = \#E_u = \sum_{z \in E^0} S_z^{(0)}_u \). Since \( E = \text{reg}(E) \), the existence of an element \( u \in \text{sink}(F) \) would imply that \( 1 = \sum_{z \in E^0} S_z^{(0)}_u = \sum_{z \in \text{reg}(E)} S_z^{(0)}_u = 0 \), a contradiction.

We conclude the section by showing that isomorphisms between Bowen-Franks modules restrict to a bijection between sinks.

A vertex \( v \) of a finite graph \( E \) is a line-point if either \( v \) is a sink or there exist edges \( e_1, \ldots, e_n \) such that \( s(e_1) = v \), \( \#s^{-1}(s(e_i)) = 1 \) for all \( i \in \{1, \ldots, n\} \) and \( r(e_n) \in \text{sink}(E) \). We remark that this definition is equivalent to the one given in [13].

**Remark 5.11.** If \( E \) is a finite graph and \( v \in E^0 \) is a line point, there exists \( i \in \mathbb{N}_0 \) and \( u \in \text{sink}(E) \) such that \( v = \sigma^i u \) in \( \mathfrak{B}_{\mathfrak{gr}}(E) \).

In [13], line-points are characterized in terms of the positive cone of the Bowen-Franks module of a graph. Recall that an element \( x \in \mathfrak{B}_{\mathfrak{gr}}(E)_+ \) is aperiodic if the set \( \{\sigma^i x\}_{i \in \mathbb{Z}} \) is infinite and minimal if \( y \leq x \) implies \( y = x \) for all \( y \neq 0 \). We shall also recall from [13, Section 2.2] that in \( \mathfrak{B}_{\mathfrak{gr}}(E)_+ \) an element \( x \) is minimal if and only if it is an atom, meaning that if \( x = z + y \) for some \( z, y \geq 0 \) then either \( y = 0 \) or \( z = 0 \).

**Proposition 5.12 ([13, Lemma 5.6 ii])**. Let \( E \) be a finite graph. A vertex \( v \in E^0 \) is a line-point if and only if it is a minimal and aperiodic element in \( \mathfrak{B}_{\mathfrak{gr}}(E)_+ \).

Before proving the proposition below, we remark that if \( E \) is a finite graph, then Lemma 4.6 implies that the map \( \text{sink}(E) \to \mathfrak{B}_{\mathfrak{gr}}(E), u \mapsto [u \otimes 1] \) is an injection.

**Proposition 5.13.** Let \( E \) and \( F \) be finite graphs. If \( \phi : \mathfrak{B}_{\mathfrak{gr}}(E) \to \mathfrak{B}_{\mathfrak{gr}}(F) \) is an isomorphism of preordered \( \mathbb{Z}[\sigma] \)-modules such that \( \phi(1_E) = 1_F \), then \( \phi \) restricts to a bijection \( \phi : \text{sink}(E) \to \text{sink}(F) \).

**Proof.** Given that \( \phi \) is an isomorphism, it follows from (2.4.3) that \( \#\text{sink}(E) = \text{rk} \mathfrak{B}_{\mathfrak{gr}}(E) = \text{rk} \mathfrak{B}_{\mathfrak{gr}}(F) = \#\text{sink}(F) \). Since \( \phi \) is injective, it will suffice to see that it restricts to a map \( \text{sink}(E) \to \text{sink}(F) \).
Let $x \in \text{sink}(E)$ and, using Proposition 5.1, write
\[ \phi(x) = \sum_{i=0}^{N} \sum_{w \in \text{sink}(F)} S_{x,u}^{(i)} [u \otimes \sigma^i] + \sum_{w \in \text{reg}(F)} R_{x,w} [w \otimes \sigma^N]. \]

Since $x$ is an aperiodic atom by Proposition 5.12, the element $\phi(x)$ is an aperiodic atom in $\mathfrak{B}(F)$. In particular, in the equality above the right hand side must consist of exactly one summand and so $\phi(x) = [z \otimes \sigma^i]$ for some $z \in E^0$ and $i \geq 0$. Using once again that $\phi(x) = [z \otimes \sigma^i]$ is an aperiodic atom we obtain the same conclusion for $[z \otimes 1]$; Proposition 5.12 then tells us that $z$ must be a line point. Hence $[z \otimes 1] = [u \otimes \sigma^i]$ for some $u \in \text{sink}(F)$ and $j \geq 0$, as per Remark 5.11.

We have thus seen that there exists $k \geq 0$ such that $\phi(x) = [u \otimes \sigma^k]$. By applying the same argument to $\phi^{-1}$ and $u$, there exists $x' \in \text{sink}(E)$ and $k' \geq 0$ such that $\phi^{-1}(u) = [x' \otimes \sigma^{k'}]$. From here it follows that
\[ [x \otimes 1] = \phi^{-1}(\phi(x)) = [x' \otimes \sigma^{k+k'}], \]
which implies $x' = x$ and $k + k' = 0$. Hence $k = k'$ and $\phi(x) = [u \otimes 1]$, as desired. \qed

Remark 5.14. The proof of Proposition 5.13 says in particular that if $\phi: \mathfrak{B}(E) \rightarrow \mathfrak{B}(F)$ is a preordered $\mathbb{Z}[]$-module isomorphism $L(E) \rightarrow L(F)$, then in the description of Proposition 5.1 we have $\sigma(u, w) = 0$, $S^0_{u, w'} = 0$ and $S_{u, w'}^0 = \delta_{u, \phi(w)}$ for all $u \in \text{sink}(E)$, $w \in \text{reg}(F)$, $w' \in \text{sink}(F)$ and $i \in \{1, \ldots, L\}$.

6. Lifting maps between Bowen-Franks modules to algebra maps

This section will be devoted to the proof of a lifting result concerning Bowen-Franks modules and their corresponding Leavitt path algebras.

**Theorem 6.1.** Let $E$ and $F$ be finite graphs. If $\phi: \mathfrak{B}(E) \rightarrow \mathfrak{B}(F)$ is a morphism of preordered $\mathbb{Z}[]$-modules such that $\phi(1_E) = 1_F$, then there exists a unital, $\mathbb{Z}$-graded, diagonal preserving $*$-homomorphism $\psi: L(E) \rightarrow L(F)$ such that the following diagram commutes.

\[
\begin{array}{ccc}
K^0_{E}(L(E)) & \xrightarrow{K^0_{E}(\phi)} & K^0_{F}(L(F)) \\
\text{can} \uparrow & & \uparrow \text{can} \\
\mathfrak{B}(E) & \xrightarrow{\phi} & \mathfrak{B}(F)
\end{array}
\]

**Proof.** Since $L_1(E) = \ell \otimes \mathbb{Z} L(E)$, $L_1(F) = \ell \otimes \mathbb{Z} L(F)$, and all $*$-homomorphisms between Leavitt path algebras over $\mathbb{Z}$ are diagonal preserving ([6, Corollary 5]), it suffices to show that there exists a unital, $\mathbb{Z}$-graded, diagonal preserving $*$-homomorphism $\psi: L_0(E) \rightarrow L_0(F)$ satisfying $K^0_{E}(\phi) \circ \text{can} = \text{can} \circ \phi$.

By Proposition 5.1, there exist $L \in \mathbb{N}_0$ and matrices $S^{(0)}, \ldots, S^{(L)} \in \mathbb{N}_0^{E^0 \times \text{sink}(F)}$, $R \in \mathbb{N}_0^{E^0 \times \text{reg}(F)}$ satisfying Equations (5.2), (5.3), (5.4), and (5.5). This implies, for each $w \in \text{reg}(F)$ and $u \in \text{sink}(F)$, $i \in \{0, \ldots, L\}$, the existence of partitions
\[ F^L_w = \bigsqcup_{x \in E^n} \Gamma_{x, w}, \quad F^i_u = \bigsqcup_{x \in E^n} \Sigma^{i}_{x, u} \]

such that $\#\Gamma_{x, w} = R_{x, w}$ and $\#\Sigma^{i}_{x, u} = S^{(i)}_{x, u}$. Moreover, if $w \in \text{reg}(F)$ then $\Sigma^0_{x, u} = \emptyset$ and there exist bijections
\[
\begin{align*}
&\zeta^{i}_{x, u}: \{(\beta, \alpha) : \beta \in S^{-1}(v), \beta \in \Sigma^{i}_{x, u}\} \xrightarrow{\sim} \Sigma^{i+1}_{v, u}, & (0 \leq i \leq L - 1), \\
&\zeta^{L}_{x, u}: \{(\beta, \alpha) : \beta \in S^{-1}(v), \beta \in \Sigma^{L}_{x, u}\} \xrightarrow{\sim} \{(\alpha f : f \in F^1, r(f) = u, \alpha \in \Gamma_{v, s(f)}\}, \\
&\xi_{v, w}: \{(\epsilon, \alpha) : \epsilon \in S^{-1}(v), \alpha \in \Gamma_{r(v), w}\} \xrightarrow{\sim} \{(\alpha f : f \in F^1, r(f) = w, \alpha \in \Gamma_{v, s(f)}\}.
\end{align*}
\]
We shall identify the images of the morphisms above with paths in $L(F)$. We will omit the indices in the functions above since they can be deduced from the element at which the function is being evaluated, namely $\xi(e,\alpha) = \xi_{s(e),r(e)}(e,\alpha)$ and $\zeta^i(e,\beta) = \zeta^i_{s(e),r(e)}(e,\beta)$.

For each $v \in E^0$ and $e \in E^1$, we define

\begin{equation}
\varphi(v) = \sum_{u \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{v,u}} \alpha \alpha^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma_{v,u}} \beta \beta^*, \tag{6.6}
\end{equation}

\begin{equation}
\varphi(e) = \sum_{w \in \text{reg}(F)} \sum_{\xi \in \Gamma_{v,w}} \xi(e,\alpha) \alpha^* + \sum_{u \in \text{sink}(F)} \sum_{\zeta \in \Sigma_{v,u}} \zeta^i(e,\beta) \beta^*. \tag{6.7}
\end{equation}

We will show that the prescriptions above define a graded $*$-homomorphism. Note that for each $v \in E^0$ and $e \in E^1$ we have $\varphi(e) \in L(F)_1, \varphi(v) \in L(F)_0$ and, by Lemma 2.2.2,

$$\sum_{v \in E^0} \varphi(v) = \sum_{u \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{v,u}} \alpha \alpha^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma_{v,u}} \beta \beta^*$$

$$= \sum_{w \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{w,v}} \alpha \alpha^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma_{w,v}} \beta \beta^* = 1.$$

Thus, to show that the assignments (6.6) and (6.7) define a unital, graded $*$-homomorphism, it suffices to verify the following relations:

(P) $\varphi(v) = \varphi(v)^*$, \hspace{1cm} (v \in E^0)

(V) $\varphi(v) \varphi(v') = \delta_{v,v'} \varphi(v)$, \hspace{1cm} (v, v' \in E^0)

(E) $\varphi(s(e)) \varphi(e) = \varphi(e) \varphi(r(e)) = \varphi(e)$, \hspace{1cm} (e \in E^1)

(CK1) $\varphi(g)^* \varphi(e) = \delta_{e,e} \varphi(r(e))$, \hspace{1cm} (g, e \in E^1)

(CK2) $\varphi(v) = \sum_{e \in s^{-1}(v)} \varphi(e) \varphi(e)^*$. \hspace{1cm} (v \in \text{reg}(E))

Relations (P) and (V) follow directly from Lemma 2.2.1. We turn our attention to (E). First, we will compute $\varphi(e) \varphi(r(e))$. By Lemma 2.2.1,

$$\varphi(e) \varphi(r(e)) = \sum_{w \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{r(e),w}, \lambda \in \Gamma_{e,w}} \xi(e,\alpha) \lambda \lambda^* + \sum_{u \in \text{sink}(F)} \sum_{\zeta \in \Sigma_{r(e),u}} \zeta^i(e,\beta) \beta^* \zeta^i(e,\beta)^*$$

$$= \sum_{w \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{r(e),w}} \xi(e,\alpha) \alpha^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma_{r(e),u}} \zeta^i(e,\beta) \beta^* = \varphi(e).$$

Using Lemma 2.2.1 once again, we see that $\varphi(s(e)) \varphi(e)$ coincides with the following:

\begin{equation}
\sum_{w,w' \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{r(e),w}, \lambda \in \Gamma_{e,w'}} \lambda \lambda^* \xi(e,\alpha) \alpha^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma_{r(e),u}} \gamma \gamma^* \zeta^i(e,\beta) \beta^* + \sum_{u \in \text{sink}(F), w' \in \text{reg}(F) \beta \in \Sigma_{r(e),u}} \lambda \lambda^* \zeta^L(e,\beta) \beta^*. \tag{6.8}
\end{equation}
For a fixed $w \in \text{reg}(E)$ and $\alpha \in \Gamma_{\epsilon(e), w}$, we know that $\xi(e, \alpha) = \varepsilon f$ for some $f \in F^1$ and $\varepsilon \in \Gamma_{s(e), s(f)}$. Therefore if $\lambda \in \Gamma_{s(e), w'}$ we must have

$$\lambda^* \xi(e, \alpha) \alpha^* = \delta_{\lambda, \epsilon} \lambda f \alpha^* = \delta_{\lambda, \epsilon} \varepsilon f \alpha^* = \delta_{\lambda, \epsilon} \xi(e, \alpha) \alpha^*.$$ 

In the same fashion, fix $u \in \text{sink}(E)$, and $\beta \in \Sigma^i_{r(e), u}$. If $i < L$ we have that $\gamma^* \xi^i(e, \beta) \beta^* = \delta_{\gamma, \xi^i(e, \beta)} \xi^i(e, \beta) \beta^*$ for each $\gamma \in \Sigma^i_{s(e), u}$. If $i = L$, then there is a unique $\lambda \in \Gamma_{s(e), w'}$ such that $\lambda \lambda^* \xi^L(e, \beta) \beta^*$ is nonzero, in which case it equals $\xi^L(e, \beta) \beta^*$. Hence Equation (6.8) agrees with

$$\sum_{w \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{\epsilon(e), w}} \xi(e, \alpha) \alpha^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma^i_{r(e), u}} \xi^i(e, \beta) \beta^* = \varphi(e)$$

as desired.

Next we prove (CK1). In view of (P), (V) and (E), we can assume without loss of generality that $s(g) = s(e)$. Using that for each $w \in \text{reg}(F), u \in \text{sink}(F)$ the functions $\xi_{s(e), w}$ and $\xi^i_{s(e), u}$ are bijections, we obtain the following equalities:

$$\xi(g, \lambda)^* \xi(e, \alpha) = \delta_{\xi(g, \lambda), \xi(e, \alpha)} r(\xi(e, \alpha)) = \delta_{\xi(g, \lambda), \lambda} w,$$

$$\xi^i(g, \gamma)^* \xi^i(e, \beta) = \delta_{\xi^i(g, \gamma), \xi^i(e, \beta)} r(\xi^i(e, \beta)) = \delta_{\xi^i(g, \gamma), \gamma} u.$$

Consequently,

$$\varphi(g) \varphi(e) = \sum_{w \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{\epsilon(e), w}} \delta_{\xi(g, \lambda)} \alpha \alpha^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma^i_{r(e), u}} \delta_{\xi^i(g, \gamma)} \beta \beta^* = \delta_{\xi(g, \lambda)} \varphi(r(e)).$$

At last, we prove (CK2). Fix $v \in \text{reg}(E)$. By hypothesis, we know that $\Sigma^0_{v, u} = \emptyset$ for all $u \in \text{sink}(F)$ and thus

$$\varphi(v) = \sum_{w \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{\epsilon(e), w}} \alpha \alpha^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma^i_{r(e), u}} \beta \beta^*$$

$$= \sum_{w \in \text{reg}(F)} \sum_{f \in \xi^{-1}(w)} \sum_{\alpha \in \Gamma_{\epsilon(e), w}} \alpha f(\alpha)^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma^i_{r(e), u}} \beta \beta^*$$

$$= \sum_{w \in \text{reg}(F)} \sum_{f \in \xi^{-1}(w)} \sum_{\alpha \in \Gamma_{\epsilon(e), w}} \alpha f(\alpha)^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma^i_{r(e), u}} \beta \beta^*$$

$$= \sum_{f \in F^1} \sum_{\alpha \in \Gamma_{\epsilon(e), w}} \alpha f(\alpha)^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma^i_{r(e), u}} \beta \beta^*.$$

(6.9)

Given $e \in E^1$, a similar reasoning as the one used to prove (E) goes to show that

$$\varphi(e) \varphi(e)^* = \sum_{w \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{\epsilon(e), w}} \xi(e, \alpha) \xi(e, \alpha)^* + \sum_{u \in \text{sink}(F)} \sum_{\beta \in \Sigma^i_{r(e), u}} \xi^i(e, \beta) \xi^i(e, \beta) \beta \beta^*.$$ 

Summing the expression above for each $e \in s^{-1}(v)$ we obtain the following:

$$\sum_{w \in \text{reg}(F)} \sum_{e \in s^{-1}(v)} \sum_{\alpha \in \Gamma_{\epsilon(e), w}} \xi(e, \alpha) \xi(e, \alpha)^* + \sum_{u \in \text{sink}(F)} \sum_{e \in s^{-1}(v)} \sum_{\beta \in \Sigma^i_{r(e), u}} \xi^i(e, \beta) \xi^i(e, \beta) \beta \beta^*.$$
Notice that given \( w \in \text{reg}(F) \) and \( u \in \text{sink}(F) \), we are summing over the domains of definition of the bijections \( \zeta^1_{v,u} \) and \( \xi^1_{v,u} \) respectively. From this and (6.10) we see that

\[
\sum_{e \in s^{-1}(v)} \varphi(e)\varphi(e)^* = \sum_{z \in E^0} \sum_{f \in r^{-1}(z)} \sum_{\alpha \in \Gamma_{v,s}(f)} \alpha f(\alpha f)^* + \sum_{u \in \text{sink}(F)} \sum_{L=1}^{n} \sum_{\beta \in \Sigma^1_{v,u}} \beta \beta^*.
\]

The expression above is precisely (6.9), which equals \( \varphi(v) \); this completes the proof of (CK2).

Finally, let us see that \( K^v_{\alpha^+}(\varphi) = \text{can} \phi \). Fix \( v \in E^0 \). Applying Lemma 2.2.1 we obtain

\[
K^v_{\alpha^+}(\varphi) \text{ can}(v) = [\varphi(v)] = \sum_{w \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{v,w}} \text{can}(w_L) \sum_{\beta \in \Sigma^1_{v,u}} \text{can}(\beta \beta^*).
\]

If \( \alpha \in \Gamma_{v,w} \) and \( \beta \in \Sigma^1_{v,u} \), by Lemma 3.6 we have that \( [\alpha^*] = \text{can}([r(\alpha)|_\alpha]) = \text{can}([w_L]) \) and likewise \( [\beta \beta^*] = \text{can}([u_i]) \). Therefore

\[
K^v_{\alpha^+}(\varphi) \text{ can}(v) = \sum_{w \in \text{reg}(F)} \sum_{\alpha \in \Gamma_{v,w}} \text{can}(w_L) \sum_{\beta \in \Sigma^1_{v,u}} \text{can}(\beta \beta^*)
\]

The last term in the chain of equalities above agrees with \( (\text{can} \circ \phi)(v) \) by Equation (5.2). \( \square \)

**Example 6.11.** We go through the construction of the proof of Theorem 6.1 in a concrete example. Consider the following graphs:

\[
E = \begin{array}{c}
\bullet \\
\bullet
\end{array}, \quad A_E = (2), \quad F = \begin{array}{c}
\bullet \\
\bullet
\end{array}, \quad A_F = \begin{pmatrix} 1 & 1 \\
1 & 1 \end{pmatrix},
\]

There is a preordered \( \mathbb{Z}[\sigma] \)-module isomorphism \( \phi : \mathfrak{B}_{\text{gr}}(E) \cong \mathfrak{B}_{\text{gr}}(F) \) determined by \( 1_E = [z_0] \mapsto [u_0] \) and \( 1_F = [u_0] \). Setting \( L = 0 \), the matrix \( R = (1 \quad 1) \) satisfies the equations of Proposition 5.1. Thus, there exist partitions of paths of length \( L = 0 \) ending at each regular vertex, namely \( \Gamma_{z,u} = \{ u \} \), \( \Gamma_{z,v} = \{ v \} \), and bijections

\[
\xi_{z,u} : \{ x_1, x_2 \} \times \{ u \} \rightarrow \{ u \} \times \{ e_1 \} \cup \{ v \} \times \{ f_2 \}, \quad \xi_{z,v} : \{ x_1, x_2 \} \times \{ v \} \rightarrow \{ u \} \times \{ e_2 \} \cup \{ v \} \times \{ f_1 \}.
\]

For example, we may set

\[
\xi_{z,u}(x_1, u) = e_1, \quad \xi_{z,u}(x_2, u) = f_2, \quad \xi_{z,v}(x_1, v) = e_2, \quad \xi_{z,v}(x_2, v) = f_1.
\]

From this choice of bijections we obtain a lift \( \varphi : L_\ell(E) \rightarrow L_\ell(F) \) of \( \phi \) determined by the following assignments:

\[
\varphi(1) = 1, \quad \varphi(x_1) = e_1 + e_2, \quad \varphi(x_2) = f_1 + f_2.
\]
Remark 6.12. If \( \phi: \mathcal{B}_{\text{gr}}(E) \to \mathcal{B}_{\text{gr}}(F) \) is an ordered \( \mathbb{Z}[\sigma] \)-module isomorphism mapping \( 1_E \mapsto 1_F \) and \( \varphi: L(E) \to L(F) \) is constructed as in the proof of Theorem 6.1, then by Remark 5.14 the map \( \varphi \) restricts to a bijection \( \text{sink}(E) \rightarrow \text{sink}(F) \).

Remark 6.13. Corollary 5.10 may also be derived from Theorem 6.1. Indeed, let \( \phi: \mathcal{B}_{\text{gr}}(E) \to \mathcal{B}_{\text{gr}}(F) \) be a preorderd \( \mathbb{Z}[\sigma] \)-module map such that \( \phi(1_E) = 1_F \). If \( E \) is regular then \( L(E) \) is strongly graded, by [10, Theorem 3.15]. Now, Theorem 6.1 implies the existence of a \( \mathbb{Z} \)-graded \( * \)-homomorphism \( \varphi: L(E) \to L(F) \) and so \( L(F) \) must be strongly graded as well; see [12, Proposition 1.1.15 (4)]. Using [10, Theorem 3.15] once again, we conclude that \( F \) is regular.

We shall presently explore the question of whether there exist graded unital maps \( \varphi: L_C(E) \to L_C(F) \) between Leavitt path algebras of finite graphs such that \( K_0^{gr}(\varphi) \) is an isomorphism but \( \varphi \) is not.

To this end, we recall some facts regarding graph \( C^* \)-algebras; we refer the reader to [1, Section 5.2] and [2, Sections 1-4] for further details. Given a graph \( E \), its \( C^* \)-algebra \( C^*(E) \) is given by a completion of \( L_C(E) \) in a suitable norm ([2, Proposition 3.1]). A \( * \)-homomorphism between Leavitt path \( C \)-algebras can be extended upon completion to one between the corresponding graph \( C^* \)-algebras ([2, Proposition 4.4]).

The algebra \( C^*(E) \) is equipped with a so-called gauge action of the circle \( S^1 \) ([2, Definition 2.13]). We can view the automorphism associated to multiplication by \( z \in S^1 \) as the completion of the \( * \)-automorphism given by
\[
\mu_z: L_C(E) \to L_C(E), \quad v \mapsto v, \quad e \mapsto ze \quad (v \in E^0, e \in E^1).
\]
Notice that for each homogeneous element \( x \in L_C(E) \) of degree \( k \) we have \( \mu_z(x) = z^k x \). Hence, a \( \mathbb{Z} \)-graded \( * \)-homomorphism \( L_C(E) \to L_C(F) \) is equivariant with respect to the actions defined above. In particular, its completion yields an \( S^1 \)-equivariant map \( C^*(E) \to C^*(F) \).

Recall that a graph \( E \) is irreducible if for each \( v, w \in E^0 \) there exists a path from \( v \) to \( w \), and non-trivial if it does not consist of a single cycle.

Remark 6.14. If \( E \) is an irreducible, non-trivial graph with at least one edge, then by [1, Lemma 6.3.14] and [9, Proposition 4.5 and Theorem 4.13] it follows that \( D(E) \) is a maximal commutative subalgebra of \( L_C(E) \). In particular, if \( E \) and \( F \) are irreducible, non-trivial graphs with at least one edge and \( \varphi: L_C(E) \to L_C(F) \) a diagonal preserving isomorphism, then \( \varphi(D(E)) = D(F) \). In particular, for \( D(E) \) the closure of \( D(E) \) in \( C^*(E) \), the completion of \( \varphi \) is an \( S^1 \)-equivariant isomorphism which maps \( D(E) \) bijectively onto \( D(F) \).

Write \( \mathcal{O} \) for the category of pointed preorderd \( \mathbb{Z}[\sigma] \)-modules and \( \text{Leavitt}_f \) for the full subcategory of \( \mathbb{Z} \)-graded \( f \)-algebras generated by Leavitt path algebras. We may view the graded Grothendieck group as a functor \( K_0^{gr}: \text{Leavitt}_f \to \mathcal{O} \) sending \( L(E) \) to \( (K_0^{gr}(L(E)), K_0^{gr}(L(E))_+, [L(E)]) \).

Question 6.15. Do there exist finite, irreducible, non-trivial graphs \( E \) and \( F \) with at least one edge satisfying the following two conditions?

(i) there exists an order preserving \( \mathbb{Z}[\sigma] \)-module isomorphism \( \phi: \mathcal{B}_{\text{gr}}(E) \xrightarrow{\sim} \mathcal{B}_{\text{gr}}(F) \) mapping \( 1_E \mapsto 1_F \);

(ii) there are no \( S^1 \)-equivariant isomorphisms \( \varphi: C^*(E) \to C^*(F) \) such that \( \varphi([D(E)]) = [D(F)] \).

The motivation for Question 6.15 stems from Theorem 6.17 below. Conditions (i) and (ii) are related to the notions of shift equivalence and strong shift equivalence of matrices (see e.g. [7, Definition 1.1]).

By [7, Theorem 7.3 and Remark 7.5], we know that finite, irreducible, non-trivial graphs whose adjacency matrices are shift equivalent but not strong shift equivalent satisfy all the conditions of
Question 6.15 with the exception of the unitality requirement in condition (i). Such an example is that of the graphs $E_{KR}$ and $F_{KR}$ with adjacency matrices $A$ and $B$ as in the counterexample of Kim and Roush to the irreducible case of the Williams conjecture [14, Section 7]. (See also [7, proof of Theorem 7.4]; in loc. cit. the notation $G_A$ and $G_B$ is employed for $E_{KR}$ and $F_{KR}$ respectively.) This motivates the following question.

**Question 6.16.** Does there exist an ordered $\mathbb{Z}[\sigma]$-module isomorphism $\mathfrak{B}_{gr}(E_{KR}) \xrightarrow{\cong} \mathfrak{B}_{gr}(F_{KR})$ mapping $1_{E_{KR}}$ to $1_{F_{KR}}$?

Note that an affirmative answer to Question 6.16 would, in particular, answer Question 6.15 in the affirmative. Some partial results on Question 6.16 are collected in Remark 6.18 below.

**Theorem 6.17.** If the answer to Question 6.15 is affirmative, then the functor $K^0_{\ell^*}: \text{Leavitt}_\mathbb{C} \to \mathcal{O}$ does not reflect isomorphisms.

**Proof.** We shall use that, over $\mathbb{C}$, the graded Grothendieck group of a Leavitt path algebra can be identified as a pointed, preordered $\mathbb{Z}[\sigma]$-module with its Bowen-Franks module. Suppose that there exist two finite, irreducible, non-trivial graphs $E$ and $F$ satisfying conditions (i) and (ii) of Question 6.15. Then, by Theorem 6.1, there exists a $\mathbb{Z}$-graded, diagonal preserving *-homomorphism $\varphi: L(E) \to L(F)$ such that $K^0_{\ell}(\varphi) = 0$. If $\varphi$ were an isomorphism, then Remark 6.14 would imply that there exists an $S^1$-equivariant isomorphism $C^*(E) \to C^*(F)$ mapping $D(E)$ to $D(F)$, contradicting (ii).

**Remark 6.18.** Consider the matrices $A$ and $B$ in [14, Section 7], and the matrices $R$ and $S$ defined in loc. cit. that implement their shift equivalence over $\mathbb{Z}$. The argument of [15, proof of Theorem 7.3.6] can be used to obtain a shift equivalence between $A$ and $B$ over $\mathbb{N}_0$. Namely, the matrices $\tilde{S} := S \cdot B^6$ and $\tilde{R} := R \cdot A^6$ are non-negative and such that $A^{13} = \tilde{S} \tilde{R}, B^{11} = \tilde{R} \tilde{S}, \tilde{R}A = B\tilde{R}, A\tilde{S} = \tilde{S}B$. Identifying $\{1, \ldots, 7\}$ with $E_{KR}$ and $F_{KR}$, we obtain ordered $\mathbb{Z}[\sigma]$-module homomorphisms

\[
\begin{align*}
\phi: \mathfrak{B}_{gr}(E_{KR}) &\to \mathfrak{B}_{gr}(F_{KR}), & x \mapsto \tilde{S}^t \cdot x, \\
\psi: \mathfrak{B}_{gr}(F_{KR}) &\to \mathfrak{B}_{gr}(E_{KR}), & x \mapsto \tilde{R}^t \cdot x.
\end{align*}
\]

We recall that these maps, induced by the shift equivalence above, are isomorphisms. To see this, note that $\psi \circ \phi$ coincides with multiplication by $(A^1)^{13}$, which can be identified with multiplication by $\sigma^{-13}$ on $\mathfrak{B}_{gr}(E_{KR})$. Likewise $\phi \circ \psi$ coincides with multiplication by $\sigma^{-13}$ on $\mathfrak{B}_{gr}(F_{KR})$.

Neither of these isomorphisms is unital. To check this, we first observe that $A^t$ and $B^t$ are invertible, since their determinant is $-1$. Now, in view of Remark 4.3, non-unitality of $\phi$ and $\psi$ follows from the fact that neither $\tilde{R}$ nor $\tilde{S}$ fix the vector $v := (1, 1, 1, 1, 1, 1, 1)^t$. This is because both $\tilde{R}$ and $\tilde{S}$ have a column with all entries greater than 1. Moreover, a direct computation shows that both $u := \tilde{S}^t \cdot (1, 1, 1, 1, 1, 1, 1)^t$ and $v := \tilde{R}^t \cdot (1, 1, 1, 1, 1, 1, 1)^t$ have all coordinates greater than one.

One could also consider shift equivalences defined by $S \cdot B^{6+j}$ and $R \cdot A^{6+j}$ for $j \in \mathbb{N}$. The induced homomorphisms are given by the matrices $(B^j)^t \tilde{S}^t$ and $(A^j)^t \tilde{R}^t$ respectively; these also fail to be unital by the following argument.

Since $E_{KR}$ and $F_{KR}$ are regular, both $(B^j)^t$ and $(A^j)^t$ are non-negative with no zero columns. Using once again that all coordinates of $u$ and $v$ are greater than one, it follows that $(B^j)^t \tilde{S}^t \cdot (1, 1, 1, 1, 1, 1, 1)^t = (B^j)^t u$ has a coordinate which is greater than one; the same conclusion holds for $(A^j)^t \tilde{R}^t \cdot (1, 1, 1, 1, 1, 1, 1)^t$.

**Remark 6.19.** We remark that, by [5, Theorem 5.6], when $\ell$ is a field the functor $K^0_{\ell^*}: \text{Leavitt}_\ell \to \mathcal{O}$ does reflect monomorphisms.
7. Tidy maps

We now characterize the type of morphisms constructed in the course of the proof of Theorem 6.1.

**Definition 7.1.** We say that an algebra homomorphism \( \varphi : L(E) \to L(F) \) is tidy if there exist \( L \geq 0 \) and, for each \( w \in \text{reg}(F), \ u \in \text{sink}(F) \) and \( i \in \{0, \ldots, L\} \), partitions \( \{\Gamma_{v,w}\}_{v \in \mathcal{E}^0} \) of \( F_{w}^L \) and \( \{\Sigma_{v,u}^i\}_{v \in \mathcal{E}^0} \) of \( F_{u}^L \) together with bijections (6.3), (6.4), and (6.5) such that (6.6) and (6.7) hold for all \( v \in \mathcal{E}^0 \) and \( e \in E^1 \).

We shall relate the notion of tidy homomorphism with that of order preserving maps as defined below.

**Definition 7.2.** Let \( E \) be a finite graph. The positive cone of \( L_{Z}(E) \) is defined to be the sub-\(*\)-semiring of \( L_{Z}(E) \) given by

\[
PC(E) = \left\{ \sum_{i=1}^{n} \lambda_i \alpha_i \beta_i^* : \alpha_i, \beta_i \in E^\infty, r(\alpha_i) = r(\beta_i), \lambda_i \in \mathbb{N}_0, n \in \mathbb{N} \right\}.
\]

An algebra homomorphism \( \varphi : L_{Z}(E) \to L_{Z}(F) \) is order preserving if \( \varphi(PC(E)) \subset PC(F) \).

**Lemma 7.3.** If \( E \) is a finite graph and \( D(E) \) the diagonal subalgebra of \( L_{Z}(E) \), then

\[
D(E) \cap PC(E) = \left\{ \sum_{\alpha \in \mathcal{R}(E) \cup \mathcal{S}(E)} \lambda_\alpha \alpha^* : \lambda_\alpha \in \mathbb{N}_0, n \in \mathbb{N}_0 \right\}.
\]

**Proof.** Let \( x \in D(E) \cap PC(E) \). Since \( x \in PC(E) \), we may write \( x = \lambda_1 \alpha_1 \beta_1^* + \cdots + \lambda_k \alpha_k \beta_k^* \) for \( \alpha_i, \beta_i \in E^\infty \) such that \( r(\alpha_i) = r(\beta_i) \) and coefficients \( \lambda_i \in \mathbb{N}_0 \). Fix \( M \in \mathbb{N} \) for which \( x \in D(E)_M \). Applying [1, Relation (CK2) in Definition 1.2.3] if necessary, we may assume that there exists \( N \geq M \) for which \( \beta_i \in \mathcal{R}(E) \cup \mathcal{S}(E) \) for all \( i \). In particular we have that \( x \in D(E)_N \). Writing \( x \) as an \( \ell \)-linear combination of elements \( \gamma \gamma^* \) with \( \gamma \in \mathcal{R}(E)_N \cup \mathcal{S}(E)_N \), the coefficient \( c_\gamma \in \mathbb{Z} \) accompanying each projection \( \gamma \gamma^* \) satisfies the following equation:

\[
c_\gamma r(\gamma) = \gamma^* x \gamma = \left( \sum_{i=1}^{k} \delta_{\gamma,\alpha_i \beta_i} \right) r(\gamma).
\]

Thus, it follows that \( c_\gamma \geq 0 \); this completes the proof. \( \square \)

**Theorem 7.4.** Let \( E \) and \( F \) be finite graphs and \( \varphi : L(E) \to L(F) \) a unital \( \ell \)-algebra homomorphism. The following statements are equivalent.

i) The morphism \( \varphi \) is tidy.

ii) The morphism \( \varphi \) is the scalar extension along \( \ell \) of a unital, order preserving, \( \mathbb{Z} \)-graded \(*\)-homomorphism \( L_{Z}(E) \to L_{Z}(F) \).

**Proof.** Implication \( i) \Rightarrow ii) \) follows from the proof of Theorem 6.1, we prove the converse. We recall that by [6, Corollary 5] all \(*\)-homomorphisms between Leavitt path \( \mathbb{Z} \)-algebras are diagonal preserving. Since \( \varphi \) is the scalar extension of a graded and order preserving homomorphism \( L_{Z}(E) \to L_{Z}(F) \), for each \( e \in E^1 \) there must exist paths \( \alpha_{e,1}, \ldots, \alpha_{e,n_e}, \beta_{e,1}, \ldots, \beta_{e,n_e} \) in \( F \) and scalars \( \lambda_{e,1}, \ldots, \lambda_{e,n_e} \in \mathbb{N}_0 \) such that \( r(\alpha_{e,i}) = r(\beta_{e,i}) \), \( |\alpha_{e,i}| = 1 + |\beta_{e,i}| \) and \( \varphi(e) = \sum_{i=1}^{n_e} \lambda_{e,i} \alpha_{e,i} \beta_{e,i}^* \). By a repeated application of [1, Relation (CK2) in Definition 1.2.3], there exists \( N \in \mathbb{N}_0 \) such that for all \( e \in E^1 \), we have \( \beta_{e,i} \in \mathcal{R}(F)_N \cup \mathcal{S}(F)_N \).

Since \( E^0 \) is finite and contained in \( D(E) \cap PC(E) \), increasing \( N \) if necessary, by Lemma 7.3 we may write

\[
\varphi(v) = \sum_{\alpha \in \mathcal{R}(F)} \lambda_{v,\alpha} \alpha^* + \sum_{\beta \in \mathcal{S}(F)} \lambda_{v,\beta} \beta^*, \quad \left( \lambda_{v,\alpha}, \lambda_{v,\beta} \in \mathbb{N}_0 \right)
\]
for all \( v \in E^0 \). By the fact that \( \varphi(v) \) is an idempotent together with Lemma 2.2.1, we obtain that the coefficients \( \lambda_{v,\alpha} \) and \( \lambda_{v,\beta} \) must lie in Idem(\( Z \)) = \{0, 1\}.

Put \( \Gamma_{v,w} = \{ \alpha \in E_N^0 : \lambda_{v,\alpha} = 1 \} \) for each \( w \in \text{reg}(E) \) and \( \Sigma_i^{v,u} = \{ \beta \in E_u^i : \lambda_{v,\beta} = 1 \} \) for each \( u \in \text{sink}(E) \), \( i \in \{1, \ldots, N\} \). With this notation in place,

\[
(7.5) \quad \varphi(v) = \sum_{w \in \text{reg}(E)} \sum_{\alpha \in \Gamma_{v,w}} \alpha^* + \sum_{u \in \text{sink}(E)} \sum_{i=0}^{N} \sum_{\beta \in \Sigma_i^{u,v}} \beta \beta^*.
\]

Next we show that the sets \( \Gamma_{v,w} \) and \( \Sigma_i^{v,u} \) define the desired partitions. Given two distinct vertices \( v \neq v' \),

\[
0 = \varphi(v) \varphi(v') = \sum_{w \in \text{reg}(E)} \sum_{\alpha \in \Gamma_{v,w} \cap \Gamma_{v',w}} \alpha^* + \sum_{u \in \text{sink}(E)} \sum_{i=0}^{N} \sum_{\beta \in \Sigma_i^{u,v} \cap \Sigma_i^{u,v'}} \beta \beta^*,
\]

and thus \( \Gamma_{v,w} \cap \Gamma_{v',w} = \emptyset \) and \( \Sigma_i^{v,u} \cap \Sigma_i^{v',u} = \emptyset \). Now, by unitarity of \( \varphi \),

\[
1 = \sum_{v \in E^0} \varphi(v) = \sum_{v \in E^0} \sum_{w \in \text{reg}(E)} \sum_{\alpha \in \Gamma_{v,w}} \alpha^* + \sum_{u \in \text{sink}(E)} \sum_{i=0}^{N} \sum_{\beta \in \Sigma_i^{u,v}} \beta \beta^* = \sum_{w \in \text{reg}(E)} \sum_{\alpha \in \Gamma_{v,w}} \alpha^* + \sum_{u \in \text{sink}(E)} \sum_{i=0}^{N} \sum_{\beta \in \Sigma_i^{u,v}} \beta \beta^*.
\]

Thus \( R_N(F) = \bigcup_{v \in E^0} \Gamma_{v,w} \) and \( S_N(F) = \bigcup_{v \in E^0} \Sigma_i^{v,u} \). By intersecting these equalities with \( E_N^0 \) and \( E_u^i \) we obtain that \( E_N^0 = \bigcup_{v \in E^0} \Gamma_{v,w} \) and \( E_u^i = \bigcup_{v \in E^0} \Sigma_i^{v,u} \) respectively.

Fix \( e \in E^1 \). By our previous observations, we may write

\[
\varphi(e) = \sum_{\alpha \in R_N(F)} x_{\alpha} \alpha^* + \sum_{\beta \in S_N(F)} y_{\beta} \beta^*
\]

with \( x_{\alpha} \) a finite sum of paths of length \( N + 1 \) with range \( r(\alpha) \) and \( y_{\beta} \) a finite sum of paths of length \( |\beta| + 1 \) with range \( r(\beta) \). Moreover, since \( \varphi(e) = \varphi(e) \varphi(e) \), by (7.5) we get

\[
\varphi(e) = \sum_{w \in \text{reg}(E)} \sum_{\alpha \in \Gamma_{v,w}} x_{\alpha} \alpha^* + \sum_{u \in \text{sink}(E)} \sum_{i=0}^{N} \sum_{\beta \in \Sigma_i^{u,v}} y_{\beta} \beta^*.
\]

For each \( \alpha, \beta \) as above, write \( x_{\alpha} = \sum_{j=0}^{n_{\alpha}} \gamma_{\alpha,j} \gamma_{\alpha,j} \) with \( n_{\alpha} \geq 0 \), \( \gamma_{\alpha,j} \in \mathbb{N} \), and distinct paths \( \gamma_{\alpha,j} \in E_r^{N+1} \). Likewise write \( y_{\beta} = \sum_{j=0}^{n_{\beta}} \gamma_{\beta,j} \gamma_{\beta,j} \) with \( n_{\beta} \geq 0 \), \( \gamma_{\beta,j} \in E_r^{N+1} \). Then

\[
x_{\alpha'} x_{\alpha''} = \left( \sum_{i \in [n_{\alpha'}] \cup [n_{\alpha''}]} \delta_{\alpha,i,\gamma_{\alpha,j}, \gamma_{\alpha,j} \cdot \gamma_{\alpha,j}} \right) r(\alpha) \quad \text{and} \quad y_{\beta'} y_{\beta''} = \left( \sum_{i \in [n_{\beta'}] \cup [n_{\beta''}]} \delta_{\beta,i,\gamma_{\beta,j}, \gamma_{\beta,j} \cdot \gamma_{\beta,j}} \right) r(\beta).
\]

Since

\[
\varphi(r(e)) = \varphi(e)^* \varphi(e) = \sum_{w \in \text{reg}(E)} \sum_{\alpha, \alpha' \in \Gamma_{v,w}} x_{\alpha'} x_{\alpha''} \alpha^* + \sum_{u \in \text{sink}(E)} \sum_{i=0}^{N} \sum_{\beta, \beta' \in \Sigma_i^{u,v}} y_{\beta'} y_{\beta''} \beta^*.
\]

necessarily \( n_{\alpha} = n_{\beta} = 1 \) and \( c_{\alpha,1} = c_{\beta,1} = 1 \) for all \( \alpha, \beta \). Hence \( x_{\alpha} y_{\beta} \in E^\infty \) and moreover the maps \( \alpha \mapsto x_{\alpha} \) and \( \beta \mapsto y_{\beta} \) have to be injective, since \( x_{\alpha'} x_{\alpha''} \) and \( y_{\beta'} y_{\beta''} \) ought to be zero for \( \alpha \neq \alpha' \) and \( \beta \neq \beta' \).
We can now define $\xi(e, \alpha) := x_\alpha$ for each $\alpha \in \Gamma_{r(e), w}$ and $\zeta^i(e, \beta) := y_\beta$ for each $\beta \in \Sigma^i_{r(e), u}$, so that

$$\varphi(e) = \sum_{w \in \text{reg}(E)} \sum_{\alpha \in \Gamma_{r(e), w}} \xi(e, \alpha) \alpha^* + \sum_{u \in \text{sink}(E)} \sum_{\beta \in \Sigma_{r(e), u}} \zeta^i(e, \beta) \beta^*.$$  

Notice that $\varphi(s(e))\xi(e, \alpha)\alpha^*$ will be zero if $\xi(e, \alpha)$ does not start at a path of length $N$ belonging to $\bigcup_{w \in \text{reg}(E)} \Gamma_{s(e), w}$, and it will coincide with $\xi(e, \alpha)\alpha^*$ otherwise. Since $\varphi(s(e))\varphi(e) = \varphi(e)$, this establishes that $\xi(e, \alpha) \in \bigcup_{w \in \text{reg}(E)} \Gamma_{s(e), w} \times F_{w, r(\alpha)}$ and in a similar fashion that $\zeta^N(e, \beta) \in \bigcup_{w \in \text{reg}(F)} \Gamma_{s(e), w} \times F_{w, r(\beta)}$ and $\zeta^i(e, \beta) \in \Sigma_{r(e), r(\beta)}$ when $|\beta| < N$.

The proof will be concluded once we see that for vertices $v \in \text{reg}(E), w \in \text{reg}(F), u \in \text{sink}(F)$ the maps

$$\zeta_{v, w} := \bigcup_{x \in E^0} E_{v, x} \times \Gamma_{x, u} \to \bigcup_{z \in \text{reg}(F)} \Gamma_{v, z} \times F_{z, w},$$

$$\zeta^N_{v, w} := \bigcup_{x \in E^0} E_{v, x} \times \Sigma^N_{z, u} \to \bigcup_{z \in \text{reg}(F)} \Gamma_{v, z} \times F_{z, u},$$

$$\zeta^i_{v, w} := \bigcup_{x \in E^0} E_{v, x} \times \Sigma^i_{z, u} \to \Sigma^{i+1}_{v, u}, \quad (0 \leq i \leq N - 1).$$

are bijective.

We have already observed that for a fixed $e \in E^1$, the assignments $\alpha \mapsto \xi_{s(e), r(\alpha)}(e, \alpha)$ and $\beta \mapsto \zeta^i_{s(e), r(\beta)}(e, \beta)$ are injective. The fact that if $e, f \in E^1$ are distinct edges starting at $v$ and $\alpha \in \Gamma_{v, w}, \alpha' \in \Gamma_{v, w}$ then $\xi_{v, w}(e, \alpha) \neq \xi_{v, w}(f, \alpha')$ follows from the equation $\varphi(f)^* \varphi(e) = 0$. The same argument applies to prove the injectivity of each function $\zeta^i_{v, w}$.

To conclude we show surjectivity. Note that by [1, Relation (CK2) in Definition 1.2.3] and (7.5), the element $\varphi(v)$ must coincide with the sum of all expressions $\gamma f(\gamma f)^*$ with $\gamma \in \bigcup_{z \in \text{reg}(F)} \Gamma_{v, z} \times F_{z, w}$ and $\delta \delta^*$ with $\delta \in \bigcup_{w \in \text{reg}(E)} \bigcup_{u \in \text{sink}(F)} \Sigma^i_{v, u}$. At the same time $\varphi(v) = \sum_{s(e) = v} \varphi(e)\varphi(e)^*$ must coincide with the sum of expressions $\xi(e, \alpha)\xi(e, \alpha)^*$ and $\zeta^i(e, \beta)\zeta^i(e, \beta)^*$ for each $i, \alpha \in \mathcal{R}(F). N, \beta \in \mathcal{S}(F). N$. This together with restrictions of the respective codomains implies that each map is surjective.

\begin{example}
If $E = \mathcal{R}_1$ is the graph with one vertex and one loop, then $L(E) \simeq \ell[t, t^{-1}]$ where $t$ has degree 1 and $t^* = t^{-1}$. For each $N \geq 0$, there is only one path of length $N$ in $E$, which corresponds under the isomorphism above to the monomial $t^N$. Thus the only tidy map $f : L(E) \to L(E)$ is the identity. In particular, there exist unital $\mathbb{Z}$-graded, diagonal preserving, involution preserving maps that are not tidy; for example, the one determined by $t \mapsto -t$.
\end{example}

\begin{corollary}
The composite of two tidy homomorphisms is again tidy.
\end{corollary}

\begin{proof}
The conditions of ii) in Theorem 7.4 are preserved by composition.
\end{proof}

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