Boundary values in $R^t(K, \mu)$-spaces and invariant subspaces

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Abstract

For $1 \leq t < \infty$, a compact subset $K$ of the complex plane $\mathbb{C}$, and a finite positive measure $\mu$ supported on $K$, $R^t(K, \mu)$ denotes the closure in $L^t(\mu)$ of rational functions with poles off $K$. The paper examines the boundary values of functions in $R^t(K, \mu)$ for certain compact subset $K$ and extends the work of Aleman, Richter, and Sundberg on nontangential limits for the closure in $L^t(\mu)$ of analytic polynomials (Theorem A and Theorem C in [Aleman et al. (2009)]). We show that the Cauchy transform of an annihilating measure has some continuity properties in the sense of capacitary density. This allows us to extend Aleman, Richter, and Sundberg's results for $R^t(K, \mu)$ and provide alternative short proofs of their theorems as special cases.

1 Introduction

Let $\mathcal{P}$ denote the set of polynomials in the complex variable $z$. For a compact subset $K$ of the complex plane $\mathbb{C}$, let $\operatorname{Rat}(K)$ be the set of rational functions with poles off $K$. For $1 \leq t < \infty$ with conjugate exponent $t' = \frac{t}{t-1}$ and a finite positive measure $\mu$ supported on $K$, let $R^t(K, \mu)$ denote the closure in $L^t(\mu)$ of $\operatorname{Rat}(K)$. In the case that $K$ is polynomially convex, $R^t(K, \mu) = P^t(\mu)$, the closure of $\mathcal{P}$ in $L^t(\mu)$. Multiplication by $z$ defines a bounded linear operator on $R^t(K, \mu)$ which we will denote by $S_z$. A rationally invariant subspace of $R^t(K, \mu)$ is a closed linear subspace $M \subset R^t(K, \mu)$ such that $r(S_z)M \subset M$ for $r \in \operatorname{Rat}(K)$. For a subset $A \subset \mathbb{C}$, we set $\bar{A}$ for its closure, $A'$ for its complement, and $\chi_A$ for its characteristic function. For $\lambda \in \mathbb{C}$ and $\delta > 0$, we set $B(\lambda, \delta) = \{z : |z - \lambda| < \delta\}$ and $D = B(0,1)$. Let $m$ be the normalized Lebesgue measure $\frac{du}{|z|}$ on $\partial D$. For a compactly supported finite measure $\nu$ on $\mathbb{C}$, we denote the support of $\nu$ by $\operatorname{spt}(\nu)$. For a compact subset $K$, we denote the boundary of $K$ by $\partial K$. The inner boundary of $K$, denoted by $\partial_0 K$, is the set of boundary points which do not belong to the boundary of any connected component of $\mathbb{C} \setminus K$.

For $\lambda \in K$, we denote evaluation on $\operatorname{Rat}(K)$ at $\lambda$ by $\epsilon_\lambda$, i.e. $\epsilon_\lambda(r) = r(\lambda)$ for $r \in \operatorname{Rat}(K)$. $\lambda$ is a bounded point evaluation (bpe) for $R^t(K, \mu)$ if $\epsilon_\lambda$ extends to a bounded linear functional on $R^t(K, \mu)$, which we will also denote by $\epsilon_\lambda$. We denote the set of bounded point evaluations for $R^t(K, \mu)$ by $\operatorname{bpe}(R^t(K, \mu))$ and set $M_\lambda = \|\epsilon_\lambda\|_{R^t(K, \mu)}$. For $\lambda_0 \in K$, if there is a neighborhood of $\lambda_0$, $B(\lambda_0, \delta)$, consisting entirely of bpe’s for $R^t(K, \mu)$ with $\lambda \to \epsilon_\lambda(f)$ analytic in $B(\lambda_0, \delta)$ for all $f \in R^t(K, \mu)$, then we say that $\lambda_0$ is an analytic bounded point evaluation (abpe) for $R^t(K, \mu)$. We denote the set of abpe’s for $R^t(K, \mu)$ by $\operatorname{abpe}(R^t(K, \mu))$. Clearly analytic bounded point evaluations are contained in the interior of $K$.

Thomson [1991] proves a remarkable structural theorem for $P^t(\mu)$: There is a Borel partition $\{\Delta_i\}_{i=0}^\infty$ of $\operatorname{spt} \mu$ such that the space $P^t(\mu|_{\Delta_i})$ contains no nontrivial characteristic functions and

$$P^t(\mu) = L^t(\mu|_{\Delta_0}) \oplus \{\oplus_{i=1}^\infty P^t(\mu|_{\Delta_i})\}.$$ 

Furthermore, if $U_i$ is the open set of analytic bounded point evaluations for $P^t(\mu|_{\Delta_i})$ for $i \geq 1$, then $U_i$ is a simply connected region and the closure of $U_i$ contains $\Delta_i$. 

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Because of Thomon’s decomposition, the study of general $P^d(\mu)$ can be reduced to the case where $P^d(\mu)$ is irreducible (contains no nontrivial characteristic functions) and $abpe(P^d(\mu))$ is a nonempty simply connected open set whose closure contains $\text{spt} \mu$. Olin and Yang [1993] shows that one can use the Riemann Mapping Theorem to further reduce to the case where $abpe(P^d(\mu)) = \mathbb{D}$. In this case, Aleman et al. [2009] obtained the following remarkable structural theorem.

**Aleman-Richter-Sundberg’s Theorem.** Suppose that $\mu$ is supported in $\mathbb{D}$ and is such that $abpe(P^d(\mu)) = \mathbb{D}$ and $P^d(\mu)$ is irreducible, and that $\mu(\partial \mathbb{D}) > 0$. Then:

a) If $f \in P^d(\mu)$ then the nontangential limit $f^*(z)$ of $f$ exists for $\mu(\partial \mathbb{D})$ almost all $z$, and $f^* = f|_{\partial \mathbb{D}}$ as elements of $L^1(\mu|_{\partial \mathbb{D}})$.

b) Every nonzero invariant subspace of $P^d(\mu)$ has index 1.

Conway and Elias [1993] extends some results of Thomson’s Theorem to the space $R^d(K, \mu)$. Brennan [2008] expresses $R^d(K, \mu)$ as a direct sum as the following: With the assumption that the diameters of the components of $\mathbb{C} \setminus K$ are bounded away from zero, there exists a Borel partition $\{ \Delta_i \}_{i=0}^{\infty}$ of $\text{spt} \mu$ and matching compact subsets $\{ K_i \}_{i=0}^{\infty}$ of $K$ such that $\Delta_i \subset K_i$ and

$$R^d(K, \mu) = L^1(\mu|_{\Delta_0}) \oplus \{ \bigoplus_{i=1}^{\infty} R^d(K_i, \mu|_{\Delta_i}) \},$$  \hspace{1cm} (1-1)

where for each $i \geq 1$ the corresponding summand $R^d(K_i, \mu|_{\Delta_i})$ is irreducible in the sense that it contains no non-trivial characteristic function. Furthermore, if $U_i = abpe(R^d(K_i, \mu|_{\Delta_i}))$ for $i \geq 1$, then $U_i$ is a connected region and the closure of $U_i$ contains $\Delta_i$. The results includes both Thomson’s theorem and results of Conway and Elias [1993].

It is evident that some restriction on the nature of $\mathbb{C} \setminus K$ is necessary in order ensure (1-1) to be valid in general. Because of Brennan’s decomposition under some additional conditions for $\mathbb{C} \setminus K$, it is reasonable to assume, in the study of general $R^d(K, \mu)$, that $R^d(K, \mu)$ is irreducible and $abpe(R^d(K, \mu))$ is a nonempty connected open set whose closure contains $\text{spt} \mu$. It is the purpose of this paper to explore the boundary values of functions and indices of rationally invariant subspaces for $R^d(K, \mu)$ and to extend Aleman-Richter-Sundberg’s Theorem.

Notice that it is possible for two compact sets, $K_1$ and $K_2$, to contain the support of $\mu$ and satisfy $R^d(K_1, \mu) = R^d(K_2, \mu)$. Thus giving conditions on a compact set $K$ is inappropriate unless attention is focused on the smallest compact set which yields the same set of functions. Since $K \supset \sigma(S_\mu)$, the spectrum of $S_\mu$, $\sigma(S_\mu)$ is the smallest set. We will always assume that $K = \sigma(S_\mu)$.

For readability purpose, in section 2, we consider an important special case that the boundary of unbounded component of $\mathbb{C} \setminus K$ is the unit circle. Proposition 4 which locally estimates the boundary values of Cauchy transform of an annihilating measure in the sense of capacity density, plays a key role in proving Theorem 1 that extends Aleman-Richter-Sundberg’s Theorem. As a consequence, our approach provides an alternative short proof of Aleman-Richter-Sundberg’s Theorem. The main difficulty in their original proof, in Aleman et al. [2009], is the proof of the following inequality:

$$\lim_{\lambda \to z} (1 - |\lambda|^2)^\frac{n}{2} M_{\lambda} \leq \frac{C}{h(z)^\frac{1}{q}}$$  \hspace{1cm} (1-2)

nontangentially for $m$-almost all $z \in \partial \mathbb{D}$, where $C$ is some constant. Our proof does not depend on the inequality (1-2). However, we will also develop a more general version of (1-2) in section 3 (see Theorem 5). Proposition 2 which estimates the upper bound of Cauchy transform of an annihilating measure, is used to prove Theorem 2 that extends Theorem C in Aleman et al. [2009].

To facilitate the discussion of further results for more general $K$, we provide the following example.

**Example.** Let $0 < \epsilon < \frac{1}{2}$, $M = \{ z : -\frac{1}{2} < \text{Re}(z) < -\frac{1}{2}, \text{Im}(z) = 0 \}$, $U_\nu = \{ z : -\frac{1}{2} < \text{Re}(z) < \frac{1}{2}, \frac{1}{2\nu}(\frac{1}{2} - \epsilon) < \text{Im}(z) < \frac{1}{2\nu}(\frac{1}{2} + \epsilon) \}$, and $L_\nu = \{ z : -\frac{1}{2} < \text{Re}(z) < \frac{1}{2}, \frac{1}{2\nu}(\frac{1}{2} - \epsilon) < \text{Im}(z) < \frac{1}{2\nu}(\frac{1}{2} + \epsilon) \}$. Let $K_1 = \overline{\mathbb{D}} \setminus (\bigcup_{n=1}^{\infty} U_n)$
and

\[ K_2 = \overline{\mathbb{D}} \setminus ((\bigcup_{n=1}^{\infty} U_n) \cup (\bigcup_{n=1}^{\infty} L_n)) \]

Let \( \mu \) and \( \nu \) be positive finite measures with \( \text{spt}(\mu) \subset K_1 \) and \( \text{spt}(\nu) \subset K_2 \) so that \( R^1(K_1, \mu) \) and \( R^1(K_2, \nu) \) are irreducible. Suppose that \( \text{albe}(R^1(K_1, \mu)) = \text{Int}(K_1) \) (for example, \( \mu = \text{Area}_{\text{Int}(K_1)} + m|_{\lambda} \), where \( m|_{\lambda} \) is Lebesgue measure on \( M \)) and \( \text{albe}(R^1(K_2, \nu)) = \text{Int}(K_2) \).

By the Radon-Nikodym theorem, we can write \( \mu = \mu_a + \mu_s \) and \( \nu = \nu_a + \nu_s \), where \( \mu_a << m|_{\lambda}, \mu_s \perp m|_{\lambda}, \nu_a << m|_{\lambda}, \) and \( \nu_s \perp m|_{\lambda} \).

In this example, \( M \) is the inner boundary of \( K_1 \). It is natural to explore nontangential limits of functions of \( R^1(K_1, \mu) \) on the inner boundary \( M \) (from below) with respect to \( \mu_a \). What can we say about \( R^1(K_2, \nu) \)?

The purpose of section 3 is to investigate the boundary behaviors of the functions in \( R^1(K, \mu) \) for the boundaries other than the unit circle in section 2. Theorem 3 proves if \( R^1(K, \mu) \) is irreducible and there are 'big parts' of \( C \setminus K \) near both sides of \( E \subset \partial K \), then \( \mu(E) = 0 \). In the above example, the inner boundary \( M \) of \( K_2 \) satisfies the property, so our result implies \( \nu_a(E) = 0 \). Therefore, it is not needed to investigate the values of functions in \( R^1(K_2, \nu) \) for the boundary \( M \). Theorem 3 can also be applied to those \( K \) for which the diameters of all components of \( C \setminus K \) are bounded away from zero. For example, if \( K \) in Theorem 1 or Theorem 2 satisfies the property, then the carrier of \( \mu_{|\partial D} \) is away from \( D \setminus K \). In the case, the nontangential limits of functions in \( R^1(K, \mu) \) can be defined with respect to \( \mu_{|\partial D} \). Theorem 1 generalizes Theorem 1.

Finally, Theorem 5 generalizes the inequality (1-2) ((1.4) in Aleman et al. (2009)).

Before closing this section, we mention here a few related papers. For a compactly supported complex measure \( \nu \) of \( C \), by estimating analytic capacity of the set \( \{ \lambda : \text{Cap}(\lambda) \geq c \} \), where \( \text{Cap} \) is Cauchy transform of \( \nu \) (see section 2 for definition), Brennan (2006, English), Aleman et al. (2009), and Aleman et al. (2010) provide interesting alternative proofs of Thomson’s theorem. Both their proofs rely on X. Tolsa’s deep results on analytic capacity. The author refines the estimations for Cauchy transform, in Lemma 4 of Yang (2015), to study the bounded point evaluations for rationally multicyclic subnormal operators. Also see the work of Akeroyd (2001), Akeroyd (2003), Aleman and Richter (1997), Miller and Smith (1990), Miller et al. (1993), Olin and Thomson (1988), Thomson and Yang (1993), Trent (1979a), Trent (1979b), Wu and Yang (1998), Yang (1995a), and Yang (1995b).

2 Outer boundary of \( K \) is the unit circle

In this section, we will concern the special cases where the outer boundary of \( K \) is the unit circle \( \partial \mathbb{D} \). Consequently, we provide alternative proofs of Theorem A and Theorem C in Aleman et al. (2009).

Let \( \nu \) be a compactly supported finite measure on \( C \). The Cauchy transform of \( \nu \) is defined by

\[ \mathcal{C} \nu(z) = \int \frac{1}{w - z} d\nu(w) \]

for all \( z \in C \) for which \( \int \frac{d\nu(z)}{|w - z|} < \infty \). A standard application of Fubini’s Theorem shows that \( \mathcal{C} \nu \in L^1_{loc}(C) \) for \( 0 < s < 2 \), in particular, it is defined for area-almost all \( z \), and clearly \( \mathcal{C} \nu \) is analytic in \( C_{\infty} \setminus \text{spt} \nu \), where \( C_{\infty} = C \cup \{ \infty \} \).

For a compact \( K \subset C \) we define the analytic capacity of \( K \) by

\[ \gamma(K) = \sup |f'(\infty)| \]

where the sup is taken over those functions \( f \) analytic in \( C_{\infty} \setminus K \) for which \( |f(z)| \leq 1 \) for all \( z \in C_{\infty} \setminus K \), and \( f'(\infty) = \lim_{z \to \infty} z[f(z) - f'(\infty)] \). The analytic capacity of a general \( E \subset C \) is defined to be

\[ \gamma(E) = \sup \{ \gamma(K) : K \subset E, \text{ K compact} \} \]

Good sources for basic information about analytic capacity are Garnett (1972), Chapter VIII of Gamelin (1969), Chapter V of Conway (1991), and Tolsa (2014).
A related capacity, \( \gamma_+ \), is defined for \( E \subset \mathbb{C} \) by
\[
\gamma_+(E) = \sup \|\mu\|
\]
where the sup is taken over positive measures \( \mu \) with compact support contained in \( E \) for which \( \|C\mu\|_{L^\infty} \leq 1 \). Since \( C\mu \) is analytic in \( \mathbb{C} \setminus \text{spt} \mu \) and \( (C\mu)'(\infty) = \|\mu\| \), we have
\[
\gamma_+(E) \leq \gamma(E)
\]
for all \( E \subset \mathbb{C} \). Tolsa (2003) proves the astounding result (Tolsa’s Theorem) that \( \gamma_+ \) and \( \gamma \) are actually equivalent. That is, there is an absolute constant \( A_T \) such that
\[
\gamma(E) \leq A_T \gamma_+(E) \tag{2-1}
\]
for all \( E \subset \mathbb{C} \). The following semiadditivity of analytic capacity is a conclusion of Tolsa’s Theorem.
\[
\gamma \left( \bigcup_{i=1}^m E_i \right) \leq A_T \sum_{i=1}^m \gamma(E_i) \tag{2-2}
\]
where \( E_1, E_2, ..., E_m \subset \mathbb{C} \).

Let \( \nu \) be a compactly supported finite measure on \( \mathbb{C} \). For \( \epsilon > 0 \), \( C\epsilon \nu \) is defined by
\[
C\epsilon \nu(z) = \int_{|w-z|>\epsilon} \frac{1}{w-z} d\nu(w),
\]
and the maximal Cauchy transform is defined by
\[
C^* \nu(z) = \sup_{\epsilon>0} |C\epsilon \nu(z)|.
\]
The 1-dimensional radial maximal operator of \( \nu \) (see also (2.7) in Tolsa (2014)) is defined by
\[
M_R \nu(z) = \sup_{r>0} \frac{|\nu|(B(z,r))}{r}.
\]

**Lemma 1.** There is an absolute positive constant \( C_T \), for \( a > 0 \), we have
\[
(1) \quad \gamma(\{C \nu \geq a\}) \leq \frac{C_T}{a} \|\nu\|, \tag{2-3}
\]
\[
(2) \quad m(\{M_R \nu \geq a\}) \leq \frac{C_T}{a} \|\nu\|.
\]
In this case, if we define
\[
MV(\nu) = \{e^{i\theta} : M_R \nu(e^{i\theta}) = +\infty\}, \tag{2-4}
\]
then \( m(MV(\nu)) = 0 \).

**Proof:** (1) follows from Proposition 2.1 of Tolsa (2002) and Tolsa’s Theorem (2-1) (also see Tolsa (2014) Proposition 4.16). Theorem 2.6 in Tolsa (2014) implies (2).

For \( 0 < \sigma < 1 \) and \( z \in \partial D \), we define the nontangential approach region \( \Gamma_{\sigma}(z) \) to be the interior of the convex hull of \( \{z\} \cup B(0, \sigma) \). It is well known that the existence of nontangential limits on a set \( E \subset \partial D \) is independent of \( \sigma \) up to sets of \( m \)-measure zero, so we will write \( \Gamma(z) = \Gamma_{\frac{1}{2}}(z) \) a nontangential approach region. The following lemma is due to Lemma 1 in Kriete and Trent (1977).

**Lemma 2.** Suppose \( \nu \) is a finite positive measure supported on \( \mathbb{D} \), define
\[
IV(\nu) = \{e^{i\theta} : \lim_{\Gamma(e^{i\theta}) \ni \lambda \to e^{i\theta}} \int_{\mathbb{D}} \frac{1 - |\lambda|^2}{|1 - \lambda z|^2} d\nu(z) > 0\}, \tag{2-5}
\]
then \( m(IV(\nu)) = 0 \).
For a finite compactly supported measure $\nu$, denote

$$U(\nu) = \{ \lambda \in \mathbb{C} : \int \frac{1}{|z - \lambda|} d|\nu|(z) < \infty \}. $$

Then $\text{Area}(U(\nu)^c) = 0$.

**Lemma 3.** Let $\nu$ be a finite measure supported in $\overline{D}$ and $|\nu|(|\partial D|) = 0$. Let $1 < p \leq \infty$, $q = \frac{p}{p-1}$, $f \in C(\overline{D})$, and $g \in L^q(\nu)$. Define

$$EV(|g|^q|\nu|) = MV(|g|^q|\nu|) \cup IV(|g|^q|\nu|)$$

(2-6)

where $MV(|g|^q|\nu|)$ and $IV(|g|^q|\nu|)$ are defined as in (2-4) and (2-5), respectively. Suppose that $a > 0$ and $e^{i\theta} \in \partial \overline{D} \setminus EV(|g|^q|\nu|)$, then there exist $\frac{\lambda}{2} < r_\theta < 1$, $E_3 \subset B(e^{i\theta}, \delta)$, and $\epsilon(\delta) > 0$, where $0 < \delta < 1 - r_\theta$, such that

$$\lim_{\delta \to 0} \epsilon(\delta) = 0,$$

$$\gamma(E_3^{\delta}) < \epsilon(\delta),$$

and for $|\lambda_0 - e^{i\theta}| = \frac{\delta}{2}$ and $\lambda_0 \in \Gamma(e^{i\theta})$,

$$\left| C \left( \frac{(1 - \lambda_0 z)^{\frac{p}{2}}}{(1 - |\lambda_0|^2)^{\frac{p}{2}}} f g \nu \right)(\lambda) - C \left( \frac{(1 - \lambda_0 z)^{\frac{p}{2}}}{(1 - |\lambda_0|^2)^{\frac{p}{2}}} f g \nu \right) \left( \frac{1}{\lambda_0} \right) \right| \leq a \|f\|_{L^p(\nu)}$$

for all $\lambda \in (B(e^{i\theta}, \delta) \setminus E_3^{\delta}) \cap U(\nu)$. Notice that $E_3^{\delta}$ depends on $f$ and all other parameters are independent of $f$.

**Proof:** For $e^{i\theta} \in \partial \overline{D} \setminus EV(|g|^q|\nu|)$, by Lemma 1 and 2 we conclude that $m(EV(|g|^q|\nu|)) = 0$, $M_1 = M_B(|g|^q|\nu|)(e^{i\theta}) < \infty$, and there exists $\frac{\lambda}{2} < r_\theta < 1$ such that

$$\left( \int_{\overline{D}} \frac{1 - |\lambda_0|^2}{|1 - \lambda_0 z|^2} d|\nu| \right)^{\frac{1}{2}} \leq \frac{a}{256}$$

(2-7)

for $\delta < 1 - r_\theta$. Let $\nu_3 = \frac{x_{B(e^{i\theta}, \lambda), \lambda}}{(1 - \lambda_0 z)^{\frac{1}{2}} r_\theta^2} f g \nu$. For $\epsilon < \delta$, $N > 2$, and $\lambda \in B(e^{i\theta}, \delta)$, we get:

$$2(1 - |\lambda_0|) \leq \delta \leq 4(1 - |\lambda_0|),$$

$$B(\lambda, \epsilon) \subset B(e^{i\theta}, 2\delta) \subset B(e^{i\theta}, N\delta),$$
and

\[
\left| C_\epsilon \left( (1 - \lambda_0 z)^{\frac{2}{\sigma} - \frac{1}{\beta}} f \right) (\lambda) - C \left( (1 - \lambda_0 z)^{\frac{2}{\sigma} - \frac{1}{\beta}} f \right) \left( \frac{1}{\lambda_0} \right) \right|
\]

\[
\leq \left\| \frac{\sin \left( -\sqrt{\frac{\lambda_0}{a} \left( 1 - \lambda_0 z \right)} \right)}{\sqrt{\frac{\lambda_0}{a} \left( 1 - \lambda_0 z \right)}} \right\| \int_{D_{\epsilon, \lambda, \delta}} \left| f \right| \left| d\nu \right|
\]

\[
\leq 2\delta \sum_{k=0}^{\infty} \int_{B(e^{\theta}, 2\lambda \delta)} \left| f \right| \left| d\nu \right|
\]

\[
\leq 4(N + 2)^{1 + \frac{1}{2}} \sum_{k=0}^{\infty} \frac{2^{2M_1}}{(N - 1)(N - 2)} \left\| f \right\|_{L^p(\nu)} + 2\delta C_\epsilon \nu_\delta(\lambda)
\]

Let

\[
N = 6 + \left( \frac{256}{a} \sum_{k=0}^{\infty} 2^{\frac{M_1}{8}} \right) M_1
\]

then together with (2-7), we get

\[
\left| C_\epsilon \left( (1 - \lambda_0 z)^{\frac{2}{\sigma} - \frac{1}{\beta}} f \right) (\lambda) - C \left( (1 - \lambda_0 z)^{\frac{2}{\sigma} - \frac{1}{\beta}} f \right) \left( \frac{1}{\lambda_0} \right) \right|
\]

\[
\leq \frac{a}{8} \left\| f \right\|_{L^p(\nu)} + 2\delta C_\epsilon \nu_\delta(\lambda)
\]

for \( \lambda \in B(e^{\theta}, \delta) \). Let

\[
E'_\delta = \{ \lambda : C_\epsilon \nu_\delta(\lambda) \geq \frac{a}{16\delta} \}
\]

From (2-3) and Holder’s inequality, we get

\[
\gamma(E'_\delta) \leq \frac{16C_\epsilon \gamma \delta}{a \left\| f \right\|_{L^p(\nu)}} \int_{B(e^{\theta}, 2\lambda \delta)} \left| f \right| \left| d\nu \right|
\]

\[
\leq \frac{16C_\epsilon \gamma \delta}{a \left\| f \right\|_{L^p(\nu)}} \int_{B(e^{\theta}, 2\lambda \delta)} \left| \frac{\sin \left( -\sqrt{\frac{\lambda_0}{a} \left( 1 - \lambda_0 z \right)} \right)}{\sqrt{\frac{\lambda_0}{a} \left( 1 - \lambda_0 z \right)}} \right| \left| d\nu \right|
\]

\[
\leq \frac{16C_\epsilon \gamma (N + 2) \delta}{a \left\| f \right\|_{L^p(\nu)}} \int_{B(e^{\theta}, 2\lambda \delta)} \left| \frac{\sin \left( -\sqrt{\frac{\lambda_0}{a} \left( 1 - \lambda_0 z \right)} \right)}{\sqrt{\frac{\lambda_0}{a} \left( 1 - \lambda_0 z \right)}} \right| \left| d\nu \right|
\]

\[
\leq \frac{64C_\epsilon \gamma (N + 2) \delta}{a} \left( \int_{D_{\epsilon, \lambda, \delta}} \left| \frac{\sin \left( -\sqrt{\frac{\lambda_0}{a} \left( 1 - \lambda_0 z \right)} \right)}{\sqrt{\frac{\lambda_0}{a} \left( 1 - \lambda_0 z \right)}} \right| \left| d\nu \right| \right) \]

\[
6
\]
Set
\[ \epsilon(\delta) = \frac{65C_T(N + 2)}{a} \left( \int_{\Omega} \frac{1 - |\lambda_0|^2}{|1 - \lambda_0 z|^2} |g|^2 d\nu \right)^{\frac{1}{2}}, \]
then \( \lim_{\delta \to 0} \epsilon(\delta) = 0 \) and
\[ \gamma(E_0^\prime) < \epsilon(\delta). \]
From (2-8) and the definition of \( E_0^\prime \), for \( \lambda \in B(e^{i\theta}, \delta) \setminus E_0^\prime \) and \( \epsilon < \delta \), we conclude that
\[
\left| C_{\nu} \left( (1 - \lambda_0 z)^{\frac{\delta}{2}} f g \nu \right)(\lambda) \right| - \mathcal{C}_{\nu} \left( (1 - \lambda_0 z)^{\frac{\delta}{2}} f g \nu \right)(\lambda) \leq 4 \left| \mathcal{C}_{\nu} \left( (1 - \lambda_0 z)^{\frac{\delta}{2}} f g \nu \right)(\lambda) \right|
\leq 4a \| \mathcal{L}_{\nu} \|_{L^p(\nu)}
\]
\[ \leq \epsilon(\delta). \]

The lemma follows since the limit of \( C_{\nu} \), when \( \epsilon \to 0 \), exists for \( \lambda \in (B(e^{i\theta}, \delta) \setminus E_0^\prime) \cap U(\nu) \).

**Proposition 1.** Let \( \nu \) be a finite complex measure with support in \( K \subset \mathbb{D} \). Suppose that \( \nu \perp \text{Rat}(K) \) and \( \nu_{|\partial D} = h_{|\partial D} \). Then for \( b > 0 \) and \( m \)-almost all \( e^{i\theta} \in \partial D \), there exist \( \frac{1}{2} < r_\theta < 1 \), \( E_0 \subset B(e^{i\theta}, \delta) \), and \( \epsilon(\theta) > 0 \), where \( 0 < \delta < 1 - r_\theta \), such that \( \lim_{\delta \to 0} \epsilon(\delta) = 0 \), \( \gamma(E_0) < \epsilon(\delta) \), and
\[ \left| \mathcal{C}_{\nu}(\lambda) - e^{-i\theta} h(e^{i\theta}) \right| \leq 2b \]
for all \( \lambda \in (B(e^{i\theta}, \delta) \setminus E_0) \cap U(\nu) \).

**Proof:** Let \( \nu_1 = \nu_{|\partial D} \) and \( \nu_2 = \nu_{|\partial D} = h_{|\partial D} \). Using Plemelj’s formula (see page 56 of Cima et al. [2006] or Theorem 8.8 in Tolsa [2014]), we can find \( E_1 \subset \partial D \) with \( m(E_1) = 0 \) such that
\[ \lim_{\Gamma(e^{i\theta}) \ni z \to e^{i\theta}} C_{\nu_2}(z) - \lim_{\Gamma(e^{i\theta}) \ni z \to e^{i\theta}} C_{\nu_2}(\frac{1}{z}) = e^{-i\theta} h(e^{i\theta}) \]
for \( e^{i\theta} \in \partial D \setminus E_1 \). Set \( E_0 = E_1 \cup EV(\nu_1) \), where \( EV(\nu_1) \) is defined as in (2-6) and \( m(EV) = 0 \).

We now apply Lemma 3 for \( p = \infty \), \( q = 1 \), \( f = 1 \), \( g = 1 \), and \( a = \frac{b}{2} \). For \( e^{i\theta} \in \partial D \setminus E_0 \), there exist \( \frac{1}{2} < r_\theta < 1 \), \( E_0 \subset B(e^{i\theta}, \delta) \), and \( \epsilon(\theta) > 0 \), where \( 0 < \delta < 1 - r_\theta \), such that \( \lim_{\delta \to 0} \epsilon(\delta) = 0 \), \( \gamma(E_0) < (\delta) \delta \), and for \( \lambda_0 \in (\partial B(e^{i\theta}, \frac{1}{2})) \cap \Gamma(e^{i\theta}) \),
\[ \left| \mathcal{C}_{\nu_1}(\lambda) - \mathcal{C}_{\nu_1}(\frac{1}{\lambda_0}) \right| \leq \frac{b}{2} \]
for all \( \lambda \in (B(e^{i\theta}, \delta) \setminus E_0) \cap U(\nu) \). Moreover, from (2-11), \( r_\theta \) can be chosen so that
\[ \left| \mathcal{C}_{\nu_2}(\lambda) - \mathcal{C}_{\nu_2}(\frac{1}{\lambda_0}) - e^{-i\theta} h(e^{i\theta}) \right| \leq \frac{b}{2} \]
for \( \lambda \in B(e^{i\theta}, \delta) \cap \Gamma(e^{i\theta}) \). Since \( \mathcal{C}_{\nu}(\frac{1}{\lambda_0}) = 0 \), we get
\[ \left| \mathcal{C}_{\nu}(\lambda) - e^{-i\theta} h(e^{i\theta}) \right| \leq \left| \mathcal{C}_{\nu_1}(\lambda) - \mathcal{C}_{\nu_1}(\frac{1}{\lambda_0}) \right| + \left| \mathcal{C}_{\nu_2}(\lambda) - \mathcal{C}_{\nu_2}(\frac{1}{\lambda_0}) - e^{-i\theta} h(e^{i\theta}) \right| \leq b \]
all \( \lambda \in (B(e^{i\theta}, \delta) \cap \Gamma(e^{i\theta}) \setminus E_0) \cap U(\nu) \). The proposition is proved.

Let \( R = \{ z : -1/2 < \Re(z), \Im(z) < 1/2 \} \) and \( Q = \mathbb{D} \setminus R \). For a bounded Borel set \( E \subset \mathbb{C} \) and \( 1 \leq p \leq \infty \), \( L^p(E) \) denotes the \( L^p \) space with respect to the area measure \( dA \) restricted to \( E \). The following Lemma is a simple application of Thomson’s coloring scheme.

**Lemma 4.** There is an absolute constant \( \epsilon_1 > 0 \) with the following property. If \( \gamma(\mathbb{D} \setminus K) < \epsilon_1 \), then
\[ |f(\lambda)| \leq \| f \|_{L^\infty(Q \setminus K)} \]
for \( \lambda \in R \) and \( f \in A(\mathbb{D}) \), the uniform closure of \( \mathcal{T} \) in \( C(\mathbb{D}) \).
Proof: Let $S$ be a closed square whose edges are parallel to x-axis and y-axis. $S$ is defined to be light if $Area(S \cap K) = 0$. $S$ is heavy if it is not light.

We now sketch our version of Thomson’s coloring scheme for $Q$ with a given a positive integer $m$. We refer the reader to Thomson (1991) and Thomson (1993) section 2 for details.

For each integer $k > 3$ let $\{S_k\}$ be an enumeration of the closed squares contained in $C$ with edges of length $2^{-k}$ parallel to the coordinate axes, and corners at the points whose coordinates are both integral multiples of $2^{-k}$ (except the starting square $S_m$, see (3) below). In fact, Thomson’s coloring scheme is just needed to be modified slightly as the following:

1. Use our definition of a light $\epsilon$ square.
2. A path to $\infty$ means a path to any point that is outside of $Q$ (replacing the polynomially convex hull of $\Phi$ by $Q$).
3. The starting yellow square $S_m$ in the $m$-th generation is $R$. Notice that the length of $S_m$ in $m$-th generation is 1 (not $2^{-m}$).

We will borrow notations that are used in Thomson’s coloring scheme such as $\{\gamma_n\}_{n \geq m}$ and $\{\Gamma_n\}_{n \geq m}$, etc. We denote

$$ \text{YellowBuffer}_m = \sum_{k=m+1}^{\infty} k^2 2^{-k}. $$

Suppose the scheme terminates, in our setup, this means Thomson’s coloring scheme reaches a square $S$ in $n$-th generation that is not contained in $Q$. One can construct a polygonal path $P$, which connects the centers of adjacent squares, from the center of a square (contained in $Q$) adjacent to $S$ to the center of a square adjacent to $R$ so that the orange (non green in Thomson’s coloring scheme) part of length is no more than $\text{YellowBuffer}_m$. Let $GP = \cup S_j$, where $\{S_j\}$ are all light squares with $P \cap S_j \neq \emptyset$. By Tolsa’s Theorem (2-2), we see

$$ \gamma(P) \leq Ar(\gamma(\text{Int}(GP)) + \text{YellowBuffer}_m). $$

Since $P$ is a connected set, $\gamma(P) \geq \frac{1}{40A_T}$ (Theorem 2.1 on page 199 of Gamelin (1969)). We can choose $m$ to be large enough so that

$$ \gamma(GP) \geq \frac{1}{40A_T} - \text{YellowBuffer}_m = \epsilon_m > 0. $$

Now by Lemma 3 in Brennan (2006, English) (or the proof of Case I of Lemma B in Aleman et al. (2009) on page 462-464), there exists a constant $\epsilon_0 > 0$ such that

$$ \gamma(GP \setminus K) \geq \epsilon_0 \gamma(GP) \geq \epsilon_0 \epsilon_m. \tag{2-13} $$

So we have prove if the scheme terminates, then (2-13) holds.

Set $\epsilon_1 = \epsilon_0 \epsilon_m$. By assumption $\gamma(D \setminus K) < \epsilon_1$, we must have $\gamma(GP \setminus K) \leq \gamma(D \setminus K) < \epsilon_1$. Therefore, the scheme will not terminate since (2-13) does not hold. In this case, one can construct a sequence of heavy barriers inside $Q$, that is, $\{\gamma_n\}_{n \geq m}$ and $\{\Gamma_n\}_{n \geq m}$ are infinite.

Let $f \in A(D)$, by the maximal modulus principle, we can find $z_n \in \gamma_n$ such that $|f(\lambda)| \leq |f(z_n)|$ for $\lambda \in R$. By the definition of $\gamma_n$, we can find a heavy square $S_n$ with $z_n \in S_n \cap \gamma_n$. Since $Area(S_n \cap K) > 0$, we can choose $w_n \in S_n$ with $|f(w_n)| = \|f\|_{L^\infty(S_n \cap K)}$. $\frac{f(w_n) - f(z_n)}{w_n - z_n}$ is analytic in $D$, therefore, by the maximal modulus principle again, we get

$$ \left| \frac{f(w_n) - f(z_n)}{w_n - z_n} \right| \leq \sup_{w \in \gamma_n+1} \left| \frac{f(w) - f(z_n)}{w - z_n} \right| \leq \frac{2 \|f\|_{L^\infty(D)}}{\text{dist}(z_n, \gamma_n+1)}. $$

Therefore,

$$ |f(\lambda)| \leq |f(z_n)| \leq |f(w_n)| + \frac{2 |z_n - w_n| \|f\|_{L^\infty(D)}}{\text{dist}(z_n, \gamma_n+1)} \leq \|f\|_{L^\infty(Q \cap K)} + \frac{2 \sqrt{2} 2^n \|f\|_{L^\infty(D)}}{n^2 2^{-n}}, $$

for $\lambda \in R$. The lemma follows by taking $n \to \infty$. 

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Corollary 1. There is an absolute constant $\epsilon_1 > 0$ with the following property. If $\lambda_0 \in \mathbb{C}$, $\delta > 0$, and $\gamma(B(\lambda_0, \delta) \setminus K) < \epsilon_1 \delta$, then

$$|f(\lambda)| \leq \|f\|_{L^\infty(B(\lambda_0, \delta) \cap K)}$$

for $\lambda \in B(\lambda_0, \frac{\delta}{2})$ and $f \in A(B(\lambda_0, \delta))$, the uniform closure of $\mathcal{P}$ in $C(B(\lambda_0, \delta))$.

Now we assume that $R^t(K, \mu)$ is irreducible and $\Omega$ is a connected region satisfying:

$$abpe(R^t(K, \mu)) = \Omega, \quad K = \overline{\Omega}, \quad \Omega \subset \mathbb{D}, \quad \partial \mathbb{D} \subset \partial \Omega. \quad (2-14)$$

It is well known that, in this case, $\mu|_{\partial \mathbb{D}} < \frac{1}{m}$. So we assume $\mu|_{\partial \mathbb{D}} = \rho m$.

For $0 < \delta < 1$ and $e^{i\theta} \in \partial \mathbb{D}$, define $\Gamma_\sigma^0(e^{i\theta}) = \Gamma_\sigma(e^{i\theta}) \cap B(e^{i\theta}, \delta)$. In order to define a nontangential limit of a function in $R^t(K, \mu)$ at $e^{i\theta} \in \partial \Omega$, one needs $\Gamma_\sigma^0(e^{i\theta}) \subset \Omega$ for some $\delta$. Therefore, we define the strong outer boundary of $\Omega$ as the following:

$$\partial_{so, \Omega} = \{ e^{i\theta} \in \partial \Omega : \exists \delta > 0, \Gamma_\sigma^0(e^{i\theta}) \subset \Omega \}, \quad \partial_{so} \Omega = \partial_{so, \Omega}. \quad (2-15)$$

It is known that $\partial_{so, \Omega}$ is a Borel set (i.e., see Lemma 4 in Olin and Thomson [1980]) and $m(\partial_{so, \Omega} \setminus \partial_{so, \sigma_2} \Omega) = 0$ for $\sigma_1 \neq \sigma_2$. From Theorem 3 in section 3, if $R^t(K, \mu)$ is irreducible and the diameters of all components of $\mathbb{C} \setminus K$ are bounded away from zero, then $\mu(\partial \mathbb{D} \setminus \partial_{so} \Omega) = 0$. This means that the carrier of $\mu|_{\partial \mathbb{D}}$ is a subset of $\partial_{so} \Omega$ and the nontangential limit of a function at $e^{i\theta} \in \partial \mathbb{D} \setminus \partial_{so} \Omega$ is not defined.

From Lemma VII.1.7 in Conway [1991], we find a function $G \in R^t(K, \mu) \subset L^t(\mu)$ such that $G(z) \neq 0$ for $\mu$-almost every $z$. Every $f \in R^t(K, \mu)$ is analytic on $\Omega$ and

$$f(\lambda) \mathcal{C}(G\mu)(\lambda) = \int \frac{f(z)}{z - \lambda} G(z) d\mu(z) = \mathcal{C}(f \mu)(\lambda) \quad (2-16)$$

for $\lambda \in \Omega \cap U(G\mu)$.

**Theorem 1.** Suppose that $\mu$ is a finite positive measure supported in $K$ and is such that $abpe(R^t(K, \mu)) = \Omega$ and $R^t(K, \mu)$ is irreducible, where $\Omega$ is a connected region satisfying (2-14), $\mu|_{\partial \mathbb{D}} = \rho m$, and $\mu(\partial_{so} \Omega) > 0$. Then:

(a) If $f \in R^t(K, \mu)$ then the nontangential limit $f^*(z)$ of $f$ exists for $\mu|_{\partial_{so} \Omega}$-almost all $z$, and $f^* = f|_{\partial_{so} \Omega}$ as elements of $L^t(\mu|_{\partial_{so} \Omega})$.

(b) Every nonzero rationally invariant subspace $M$ of $R^t(K, \mu)$ has index 1, that is, dim$(M/(S_\mu - \lambda_0)M) = 1$, for $\lambda_0 \in \Omega$.

If the diameters of all components of $\mathbb{C} \setminus K$ are bounded away from zero, then by Theorem 4 in section 3, the above $\partial_{so} \Omega$ can be replaced by $\partial \mathbb{D}$.

**Proof:** (a) Let $1 > \epsilon > 0$ and $\epsilon_0 = \frac{\epsilon}{2A_T}$, where $\epsilon_1$ is as in Lemma 3 and $A_T$ is from (2-2). For $f \in R^t(K, \mu)$, from Proposition 1 we see that for $\mu$-almost all $e^{i\theta} \in \partial_{so} \Omega$ with $\Gamma_\sigma(e^{i\theta}) \subset \Omega$ and $G(e^{i\theta}) h(e^{i\theta}) \neq 0$, $b = \frac{\epsilon}{2(1 + T(f \mu \cap \mathcal{C}(G\mu)))} > 0$, there exist $r_0, r_1 < r_0 < 1$, $E_2^1 \subset B(e^{i\theta}, \delta)$, $E_2^2 \subset B(e^{i\theta}, \delta)$, and $\epsilon(\delta) > 0$, where $0 < \delta < 1 - r_0$, such that lim$_{\delta \to 0} \epsilon(\delta) = 0$, $\gamma(E_2^1) < \epsilon(\delta) \delta$, $\gamma(E_2^2) < \epsilon(\delta) \delta$,

$$\left| \mathcal{C}(G\mu)(\lambda) - G(e^{i\theta}) e^{-i\theta} h(e^{i\theta}) \right| \leq b$$

for all $\lambda \in (B(e^{i\theta}, \delta) \cap \Gamma(e^{i\theta}) \setminus E_2^1) \cap U(G\mu)$, and

$$\left| \mathcal{C}(f \mu)(\lambda) - f(e^{i\theta}) G(e^{i\theta}) e^{-i\theta} h(e^{i\theta}) \right| \leq b$$

for $\lambda \in (B(e^{i\theta}, \delta) \cap \Gamma(e^{i\theta}) \setminus E_2^2) \cap U(G\mu)$. Now choose $\delta$ small enough so that $\epsilon(\delta) < \epsilon_0$. Set $E_\delta = E_2^1 \cup E_2^2$, then from the semi-additivity (2-2), we get

$$\gamma(E_\delta) \leq A_T(\gamma(E_2^1) + \gamma(E_2^2)) < \epsilon_1 \frac{\delta}{16}.$$
Therefore, by (2-16), for \( \lambda \in (B(e^{i\theta}, \delta) \cap \Gamma(e^{i\theta}) \setminus E_3) \cap U(G\mu) \),

\[
|f(\lambda) - f(e^{i\theta})| \\
\leq \frac{|C(fG\mu)(\lambda) - f(e^{i\theta})C(G\mu)(\lambda)|}{C(G\mu)(\lambda)} \\
\leq \frac{2|C(fG\mu)(\lambda) - f(e^{i\theta})G(e^{i\theta})e^{-i\theta}h(e^{i\theta})|}{|G(e^{i\theta})h(e^{i\theta})|} + \frac{2|C(G\mu)(\lambda) - G(e^{i\theta})e^{-i\theta}h(e^{i\theta})||f(e^{i\theta})|}{|G(e^{i\theta})h(e^{i\theta})|} \\
\leq \epsilon.
\]

For \( \lambda_0 \in (\partial B(e^{i\theta}, \frac{\delta}{2}) \cap \Gamma\frac{1}{4}(e^{i\theta}) \), we see that \( B(\lambda_0, \frac{\delta}{2\pi}) \subset B(e^{i\theta}, \delta) \cap \Gamma(e^{i\theta}) \). Using Lemma [4] for \( f - f(e^{i\theta}) \), we get

\[
|f(\lambda) - f(e^{i\theta})| \leq \| f - f(e^{i\theta}) \|_{L^\infty(B(\lambda_0, \frac{\delta}{2\pi}) \setminus E_3)} \leq \epsilon
\]

for every \( \lambda \in B(\lambda_0, \frac{\delta}{2\pi}) \). Hence,

\[
\lim_{\delta \to 0} \Gamma\frac{1}{4}(e^{i\theta}) \cap \lambda \to e^{i\theta} f(\lambda) = f(e^{i\theta}).
\]

We turn to prove (b). Let \( M \) be a nonzero rationally invariant subspace of \( R^2(K, \mu) \). Without loss of generality, we assume \( \lambda_0 = 0 \) and \( 0 \in \Omega \). We must show that \( \dim(M/S_M) = 1 \). Let \( n \) be the smallest integer such that \( f(z) = z^n f_0(z) \) for every \( f \in M \) and there exists \( g \in M \) with

\[
g(z) = z^n g_0(z) \quad \text{and} \quad g_0(0) \neq 0.
\]

We only need to show that for \( \phi \in M^+ \subset L^\infty(M\mu) \), the function

\[
\Phi(\lambda) = \int \frac{g(\lambda)f(z) - f(\lambda)g(z)}{z - \lambda} \phi(z) d\mu(z),
\]

which is analytic in \( \Omega \), is identically zero. In fact, the proof is similar to that of (a). Let \( E \subset \partial_{\infty} \Omega \) so that for \( e^{i\theta} \in E \), \( f \) and \( g \) have nontangential limits at \( e^{i\theta} \), and \( h(e^{i\theta}) > 0 \). By Theorem [1] (a), \( m(E) > 0 \). For \( 1 > \epsilon > 0 \) and \( \epsilon_0 = \frac{\epsilon}{2\pi \Lambda_1} \), applying Proposition [1] for \( f \phi \mu, \overline{g} \phi \mu \) with \( \lambda \in \Omega \), \( \Gamma\frac{1}{4}(e^{i\theta}) \subset \Omega \) and \( b = \frac{1}{1 + |e^{i\theta}| + |g(e^{i\theta})|1 + |\phi(e^{i\theta})|h(e^{i\theta})^\epsilon} \), there exist \( \max(r_0, \frac{\delta}{2}) \subset E_3 \subset B(e^{i\theta}, \delta) \), \( E_3^2 \subset B(e^{i\theta}, \delta) \), and \( \delta(\epsilon) > 0 \), where \( 0 < \delta < 1 - r_0 \), such that \( \lim_{\delta \to 0} \epsilon(\delta) = 0, \gamma(E_3) < \epsilon(\delta) \), \( \gamma(E_3^2) < \epsilon(\delta) \),

\[
\left| C(f\phi \mu)(\lambda) - f(e^{i\theta}) \phi(e^{i\theta}) e^{-i\theta} h(e^{i\theta}) \right| \leq b
\]

for all \( \lambda \in (B(e^{i\theta}, \delta) \cap \Gamma(e^{i\theta}) \setminus E_3^2) \cap U(G\mu) \),

\[
\left| C(g\phi \mu)(\lambda) - g(e^{i\theta}) \phi(e^{i\theta}) e^{-i\theta} h(e^{i\theta}) \right| \leq b.
\]

for all \( \lambda \in (B(e^{i\theta}, \delta) \cap \Gamma(e^{i\theta}) \setminus E_3^2) \cap U(G\mu) \), \( |f(\lambda) - f(e^{i\theta})| < b \) and \( |g(\lambda) - g(e^{i\theta})| < b \) on \( B(e^{i\theta}, \delta) \cap \Gamma(e^{i\theta}) \). Choose \( \delta \) small enough so that \( \epsilon(\delta) < \epsilon_0 \). Set \( E_3 = E_3^1 \cup E_3^2 \), then by the semi-additivity (2-2) again, we have \( \gamma(E_3) < \epsilon(\delta) \). Therefore, for all \( \lambda \in (B(e^{i\theta}, \delta) \cap \Gamma(e^{i\theta}) \setminus E_3^2) \cap U(G\mu) \),

\[
\left| \Phi(\lambda) \right| \\
\leq \left| g(\lambda) \right| \left| C(f\phi \mu)(\lambda) - f(e^{i\theta}) \phi(e^{i\theta}) e^{-i\theta} h(e^{i\theta}) \right| + \left| f(\lambda) \right| \left| C(g\phi \mu)(\lambda) \right| - g(e^{i\theta}) \phi(e^{i\theta}) e^{-i\theta} h(e^{i\theta}) + \left| f(\lambda) \right| \left| g(e^{i\theta}) - g(\lambda) \right| \left| f(e^{i\theta}) \right| \left| \phi(e^{i\theta}) \right| h(e^{i\theta}) \\
\leq (b + |f(e^{i\theta})| + |g(e^{i\theta})| (1 + |\phi(e^{i\theta})| h(e^{i\theta})) b \\
\leq \epsilon.
\]

For \( \lambda_0 \in (\partial B(e^{i\theta}, \frac{\delta}{2}) \cap \Gamma\frac{1}{4}(e^{i\theta}) \), \( B(\lambda_0, \frac{\delta}{2\pi}) \subset B(e^{i\theta}, \delta) \cap \Gamma(e^{i\theta}) \). Using Lemma [4] for \( \Phi \), we get

\[
\left| \Phi(\lambda) \right| \leq \left| \Phi \right|_{L^\infty(B(\lambda_0, \frac{\delta}{2\pi}) \setminus E_3)} \leq \epsilon
\]
for every $\lambda \in B(\lambda_0, \frac{4}{2\pi})$. Hence,

$$\lim_{\epsilon \to 0, \epsilon \in \mathbb{R}} \Phi(\lambda) = 0.$$ 

Let $V = \cup_{e^{i\theta} \in E} \Gamma^\epsilon_{\frac{1}{2}}(e^{i\theta})$. Since $m(E) > 0$, there exists a connected component $V_0$ of $V$ with $m(\partial V_0 \cap \partial D) > 0$. $\partial V_0$ is a rectifiable Jordan curve and $\Phi(\lambda)$ is analytic in $V_0$. Therefore $\Phi(\lambda) = 0$ since $\Omega$ is a connected region. This completes the proof.

**Proposition 2.** Let $\mu$ be a finite positive measure with support in $K \subset \overline{D}$ and $\mu|_{\partial D} = 0$. Then $1 < p < \infty$, $q = \frac{1}{p-1}$, $f \in C(\overline{D})$, $g \in L^p(\mu)$, and $fg\mu \perp \text{Rat}(K)$. Then for $0 < \beta < \frac{1}{16}$, $b > 0$, and $m$-almost all $e^{i\theta} \in \partial D$, there exist $\frac{1}{2} < r_0 < 1$, $E_0^\beta \subset B(e^{i\theta}, \delta)$, and $\epsilon(\delta) > 0$, where $0 < \delta < 1 - r_0$, such that $\lim_{\lambda \to 0} \epsilon(\delta) = 0$, $\gamma(E_0^\beta) < \epsilon(\delta)\delta$, and for $\lambda_0 \in (\partial B(e^{i\theta}, \frac{1}{2})) \cap \Gamma^\epsilon_{\frac{1}{2}}(e^{i\theta})$,

$$\left| \mathcal{C}\left( \frac{(1 - \lambda_0 z)^{\frac{2}{p}}}{(1 - |\lambda_0|^2)^{\frac{2}{p}}} f g \mu \right) \right| \leq \left( b + \frac{1 + 4\beta}{1 - 4\beta} \left( \int_{\partial D} \frac{1 - |\lambda_0|^2}{1 - \lambda_0 z^2} \|g\|_d\mu \right) \right)^\frac{1}{p} \|f\|_{L^p(\mu)}$$

(2.17)

for all $\lambda \in (B(\lambda_0, \beta \delta) \setminus E_0^\beta) \cap U(g\mu)$. 

**Proof:** Let $\nu = \mu|_{\partial D}$. We now apply Lemma 3 for $p$, $q$, $f$, and $g$, and $a = b$. For $e^{i\theta} \in \partial D \setminus EV([g]^q\nu)$ (as in (2.6) and $m(EV([g]^q\nu)) = 0$), there exist $\frac{1}{2} < r_0 < 1$, $E_0^\beta \subset B(e^{i\theta}, \delta)$, and $\epsilon(\delta) > 0$, where $0 < \delta < 1 - r_0$, such that $\lim_{\lambda \to 0} \epsilon(\delta) = 0$, $\gamma(E_0^\beta) < \epsilon(\delta)\delta$, and for $\lambda_0 \in (\partial B(e^{i\theta}, \frac{1}{2})) \cap \Gamma^\epsilon_{\frac{1}{2}}(e^{i\theta})$,

$$\left| \mathcal{C}\left( \frac{(1 - \lambda_0 z)^{\frac{2}{p}}}{(1 - |\lambda_0|^2)^{\frac{2}{p}}} f g \mu \right) \right| \leq \left( b + \frac{1 + 4\beta}{1 - 4\beta} \left( \int_{\partial D} \frac{1 - |\lambda_0|^2}{1 - \lambda_0 z^2} \|g\|_d\mu \right) \right)^\frac{1}{p} \|f\|_{L^p(\mu)}$$

for all $\lambda \in (B(e^{i\theta}, \beta \delta) \setminus E_0^\beta) \cap U(g\mu)$.

$$\mathcal{C}\left( \frac{(1 - \lambda_0 z)^{\frac{2}{p}}}{(1 - |\lambda_0|^2)^{\frac{2}{p}}} f g \mu \right) \left( \frac{1}{\lambda_0} \right) = 0$$

since $fg\mu \perp \text{Rat}(K)$. From Lemma 3 for all $\lambda \in (B(\lambda_0, \beta \delta) \setminus E_0^\beta) \cap U(g\mu)$, we get

$$\left| \mathcal{C}\left( \frac{(1 - \lambda_0 z)^{\frac{2}{p}}}{(1 - |\lambda_0|^2)^{\frac{2}{p}}} f g \mu \right) \right| \leq \int_{\partial D} \left| f \right| \left| \frac{1}{z - \lambda} - \frac{1}{\lambda_0} \right| \left( \frac{1}{1 - |\lambda_0|^2} \right)^\frac{2}{p} d\mu$$

$$\leq b \|f\|_{L^p(\mu)}$$

where the last step follows from

$$\left| \frac{1 - \lambda_0 z}{|z - \lambda|} \right| \leq \frac{1 - |\lambda_0|^2 + |\lambda - \lambda_0|}{|z - \lambda_0| - |\lambda - \lambda_0|} \leq \frac{(1 + 4\beta)(1 - |\lambda_0|^2)}{|z - \lambda_0| - 4\beta(1 - |\lambda_0|)} \leq \frac{(1 + 4\beta)(1 - |\lambda_0|^2)}{(1 - 4\beta)|1 - \lambda_0 z|}$$

for $z \in \partial D$. The corollary now follows from Holder’s inequality.
Theorem 2. Suppose that $\mu$ is a finite positive measure supported in $K$ and is such that $\text{abpe}(R^2(K, \mu)) = \Omega$ and $R^2(K, \mu)$ is irreducible, where $\Omega$ is a connected region satisfying (2-14), $\mu|_{\partial \Omega} = h_m$, and $\mu(\partial \omega) > 0$. Then for $t > 1$,

$$\lim_{t \to \infty} \frac{1}{\frac{1}{t} \frac{1}{2} + \lambda - e^{it\theta}} M_{\lambda} = \frac{1}{h(e^{it\theta})}$$

for $\mu$-almost all $e^{it\theta} \in \partial \omega \Omega$. If the diameters of all components of $\mathbb{C} \setminus K$ are bounded away from zero, then by Theorem 3 (in section 3), the above $\partial \omega \Omega$ can be replaced by $\partial \Omega$.

Proof: By Proposition 4 and 2 for $\mu$-almost all $e^{it\theta} \in \partial \omega \Omega$ with $G(e^{it\theta})h(e^{it\theta}) \neq 0$ and $\Gamma_{e^{it\theta}}(e^{it\theta}) \subset \Omega$, $0 < \beta < \frac{1}{16}$, $b > 0$, and $f \in \text{Rat}(K)$, there exist $\max(\frac{1}{8}, r_0) < r_\theta < 1$, $E_\delta \subset B(e^{it\theta}, \delta)$, $E_\delta' \subset B(e^{it\theta}, \delta)$, and $\epsilon(\delta) > 0$, where $0 < \epsilon(\delta) < 1 - r_\theta$, such that $\lim_{\epsilon \to 0} \epsilon(\delta) = 0$, $\gamma(E_\delta) < \epsilon(\delta)\delta$, $\gamma(E_\delta) < \epsilon(\delta)\delta$.

$$\left| \mathcal{C}(G\mu)(\lambda) - e^{-it\theta}G(e^{it\theta})h(e^{it\theta}) \right| \leq b \tag{2-18}$$

for all $\lambda \in (B(e^{it\theta}, \delta) \cap \Gamma(e^{it\theta}) \setminus E_\delta) \cap U(G\mu)$, and

$$\left| \mathcal{C} \left( \frac{(1 - \lambda_0)^{\frac{1}{2}}}{1 - |\lambda_0|^2} fG\mu \right)(\lambda) \right| \leq b + \frac{\beta}{1 - 4\beta} \left( \int_{\partial \Omega} \frac{1}{1 - |\lambda_0|^2} |G|^2 \, d\mu \right)^{\frac{1}{2}} \|f\|_{L^2(\mu)} \tag{2-19}$$

for $\lambda_0 \in \partial B(e^{it\theta}, \frac{1}{2}) \cap \Gamma_\frac{1}{4}(e^{it\theta})$ and all $\lambda \in (B(\lambda_0, \beta\delta) \setminus E_\delta') \cap U(G\mu)$. From semi-additivity of (2-2), we get

$$\gamma(E_\delta \cup E_\delta') \leq A_T(\gamma(E_\delta) + \gamma(E_\delta')) \leq 2A_T(\epsilon(\delta)\delta).$$

Let $\delta$ be small enough so that $\epsilon(\delta) < \frac{1}{24T}\epsilon_1$, where $\epsilon_1$ is as in Corollary 4. From (2-16), (2-18), and (2-19), for $\lambda_0 \in \partial B(e^{it\theta}, \delta) \cap \Gamma_\frac{1}{4}(e^{it\theta})$ and all $\lambda \in (B(\lambda_0, \beta\delta) \setminus (E_\delta \cup E_\delta')) \cap U(G\mu)$, we have the following calculation:

$$|1 - \lambda_0\lambda| \geq 1 - |\lambda_0|^2 - |\lambda - \lambda_0||\lambda_0| \geq 1 - |\lambda_0|^2 - \beta\delta|\lambda_0|$$

and

$$(1 - |\lambda_0|^2)^{\frac{1}{2}}|f(\lambda)| \leq \frac{|(1 - \lambda_0\lambda)(1 - |\lambda_0|^2)|^{-\frac{1}{2}} |f(\lambda)|}{(1 - \beta \frac{\delta|\lambda_0|}{1 - |\lambda_0|^2})^{\frac{1}{2}}} \left( \frac{\mathcal{C} \left( \frac{(1 - \lambda_0^{\frac{1}{2}})}{1 - (\lambda_0)^{\frac{1}{2}}} fG\mu \right)(\lambda)}{\mathcal{C}(G\mu)(\lambda)} \right) \leq \frac{b + \frac{\beta}{1 - 4\beta} \left( \int_{\partial \Omega} \frac{1}{1 - |\lambda_0|^2} |G|^2 \, d\mu \right)^{\frac{1}{2}}}{(1 - 4\beta)^{\frac{1}{2}} (|G(e^{it\theta})| h(e^{it\theta}) - b)} \|f\|_{L^2(\mu)}.$$

Since $\gamma(E_\delta \cup E_\delta') < \epsilon_1\delta$, from Corollary 4 we conclude

$$M_{\lambda_0} \leq \sup_{f \in \text{Rat}(K) \atop \|f\|_{L^2(\mu)} = 1} |f(\lambda_0)| \leq \sup_{f \in \text{Rat}(K) \atop \|f\|_{L^2(\mu)} = 1} \|f\|_{L^\infty(B(\lambda_0, \beta\delta) \setminus (E_\delta \cup E_\delta'))}$$

for $\lambda_0 \in \partial B(e^{it\theta}, \frac{1}{2}) \cap \Gamma_\frac{1}{4}(e^{it\theta})$. Hence,

$$\lim_{t \to \infty} \frac{1}{\frac{1}{t} \frac{1}{2} + \lambda - e^{it\theta}} M_{\lambda_0} \leq \frac{b + \frac{1 + 4\beta}{1 - 4\beta} \left( |G(e^{it\theta})| h(e^{it\theta}) \right) \frac{1}{2}}{(1 - 4\beta)^{\frac{1}{2}} (|G(e^{it\theta})| h(e^{it\theta}) - b)}$$
since \( \frac{1 - |\lambda|^2}{1 - \lambda e^{-i\theta}} \) is the Poisson kernel. Taking \( b \to 0 \) and \( \beta \to 0 \), we get
\[
\lim_{\Gamma} \frac{1}{(e^{i\theta})_{|\lambda| \to \infty}} (1 - |\lambda|^2)^{\frac{1}{2}} M_\lambda \leq \frac{1}{b(e^{i\theta})^{\frac{1}{2}}}
\]
The reverse inequality is from [Kriete and Trent (1977)](applying Lemma 2 to testing function \((1 - \lambda_0 z)^{-\frac{1}{2}}\)). This completes the proof.

### 3 Boundary values of \( R^t(K, \mu) \) for certain \( K \)

In this section, we are concerning the boundary behaviors of functions in \( R^t(K, \mu) \) near the boundary of \( K \) (not necessarily outer boundary as in last section), in particular, the inner boundary of \( K \). Our approach in estimating Cauchy transform, in section 2, is concentrating on the local behavior of the transform. This makes it possible to extend our methodology to more general \( K \). In order to apply our approach, the following requirements are needed.

(A) Plemelj’s formula must hold for the boundary points under consideration;

(B) Lemma 3 (2) and Lemma 2 shall be extended.

For (A), it is known that Plemelj’s formula holds for a Lipschitz graph (see Theorem 8.8 in [Tolsa (2014)])). So we will restrict our attention to the boundary of \( K \) which is a part of a Lipschitz graph although Plemelj’s formula may hold for more general rectifiable curves.

We define the open cone (with vertical axis)
\[
\Gamma(\lambda, \alpha) = \{ z \in \mathbb{C} : |\text{Re}(z) - \text{Re}(\lambda)| < \alpha |\text{Im}(z) - \text{Im}(\lambda)| \},
\]
and the half open cones
\[
\Gamma^+(\lambda, \alpha) = \{ z \in \Gamma(\lambda, \alpha) : \text{Im}(z) > \text{Im}(\lambda) \}
\]
and
\[
\Gamma^-(\lambda, \alpha) = \{ z \in \Gamma(\lambda, \alpha) : \text{Im}(z) < \text{Im}(\lambda) \}.
\]
Set \( \Gamma^+(\lambda, \alpha) = B(\lambda, \delta) \cap \Gamma^+(\lambda, \alpha) \) and \( \Gamma^-(\lambda, \alpha) = B(\lambda, \delta) \cap \Gamma^-(\lambda, \alpha) \). \( \Gamma^+(\lambda, \alpha) \) (or \( \Gamma_+^-(\lambda, \alpha) \)) is called upper cone. \( \Gamma^-(\lambda, \alpha) \) (or \( \Gamma^-_+(\lambda, \alpha) \)) is called lower cone.

Let \( A : \mathbb{R} \to \mathbb{R} \) be a Lipschitz function and let \( LG \) be its graph. Observe that if \( \alpha < \frac{1}{A''(z)} \).

then, for every \( \lambda \in LG \), \( \Gamma^+(\lambda, \alpha) \subset \{ z \in \mathbb{C} : \text{Im}(z) > A(\text{Re}(z)) \} \) and \( \Gamma^-(\lambda, \alpha) \subset \{ z \in \mathbb{C} : \text{Im}(z) < A(\text{Re}(z)) \} \). On the graph of \( A \), we consider the usual complex measure
\[
dz_{LG} = \frac{1 + iA'(\text{Re}(z))}{1 + A'(\text{Re}(z))^2} d\mathcal{H}^1|_{LG} = (L(z))^{-1} d\mathcal{H}^1|_{LG} \quad (3-1)
\]
where \( \mathcal{H}^1 \) is one dimensional Hausdorff measure. Notice that \( |L(z)| = 1 \). For \( 1 \leq p < \infty \) and \( f \in L^p(\mathcal{H}^1|_{LG}) \), the nontangential limits
\[
C_+(f dz_{LG})(\lambda) = \lim_{\Gamma^+(\lambda, \alpha) \ni z \to \lambda} C(f dz_{LG}(z))
\]
and
\[
C_-(f dz_{LG})(\lambda) = \lim_{\Gamma^-(\lambda, \alpha) \ni z \to \lambda} C(f dz_{LG}(z))
\]
exist \( \mathcal{H}^1|_{LG} \)-almost everywhere. Moreover,
\[
\frac{1}{2\pi i} C_+(f dz_{LG}(\lambda)) - \frac{1}{2\pi i} C_-(f dz_{LG}(\lambda)) = f(\lambda) \quad (3-2)
\]
(see Theorem 8.8 in [Tolsa (2014)].

Suppose that \( R^t(K, \mu) \) is irreducible and \( \Omega \) is a connected region satisfying:
\[
\text{abpe}(R^t(K, \mu)) = \Omega, \ K = \tilde{\Omega}. \quad (3-3)
\]
Let \( G \in R^1(K, \mu) \subset L^1(\mu) \) such that \( G(z) \neq 0 \) for \( \mu \)-almost every \( z \).

In order to apply our approach, we need to impose some constraints on \( K \) and define type I and II boundaries for \( K \). Upper cone \( \Gamma^+(\lambda, \alpha) \) (or lower cone \( \Gamma^-(-\lambda, \alpha) \)) is outer for \( \lambda \in LG \cap \partial K \) if there exist \( \delta \), \( \epsilon \), \( \lambda > 0 \) such that for every \( \delta < \lambda \),

\[
B(\lambda_\alpha \epsilon, \epsilon \delta) \subset K^c \cap \Gamma^+_\epsilon(\lambda, \alpha) \quad \text{or} \quad B(\lambda_\alpha \epsilon, \epsilon \delta) \subset K^c \cap \Gamma^-_\epsilon(\lambda, \alpha).
\]

(3-4)

\( \lambda \in LG \cap \partial K \) is a type I boundary point of \( LG \cap \partial K \) if either upper cone \( \Gamma^+(\lambda, \alpha) \) or lower cone \( \Gamma^-(-\lambda, \alpha) \) is outer. The type I boundary \( \partial_{I, \lambda, \alpha}K \) is the set of all type I boundary points of \( LG \cap \partial K \). For example, if \( V \) is a component of \( K \) and \( \partial V \) is a Lipschitz graph, then \( \partial V \) is a type I boundary.

Upper cone \( \Gamma^+(\lambda, \alpha) \) (or lower cone \( \Gamma^-(-\lambda, \alpha) \)) is inner for \( \lambda \in LG \cap \partial K \) if there exists \( \delta > 0 \) such that

\[
\Gamma^+_\epsilon(\lambda, \alpha) \subset \Omega \quad \text{or} \quad \Gamma^-_\epsilon(\lambda, \alpha) \subset \Omega.
\]

(3-5)

\( \lambda \in LG \cap \partial K \) is a type II boundary point of \( LG \cap \partial K \) if \( \lambda \) is type I and either upper cone \( \Gamma^+(\lambda, \alpha) \) or lower cone \( \Gamma^-(-\lambda, \alpha) \) is inner. The type II boundary \( \partial_{II, \lambda, \alpha}K \) is the set of all type II boundary points of \( LG \cap \partial K \). The strong outer boundary of \( \Omega \) defined in the section 2 is type II boundary of \( K \).

Without loss of generality, for type I boundary point \( \lambda \), we usually assume upper cone \( \Gamma^+(\lambda, \alpha) \) is outer, and for type II boundary point \( \lambda \), we usually assume lower cone \( \Gamma^-(-\lambda, \alpha) \) is inner.

**Lemma 5.** Both \( \partial_{I, \lambda, \alpha}K \) and \( \partial_{II, \lambda, \alpha}K \) are Borel sets.

**Proof:** Let \( \epsilon_0 = \frac{1}{m} \) and define \( A_{nm} \) to be the set of \( \lambda \in LG \cap \partial K \) such that for every \( 0 < \delta < \frac{1}{m} \), there exists \( \lambda_0 \) with

\[
B(\lambda_0, \epsilon_0 \delta) \subset K^c \cap \Gamma^+_\epsilon(\lambda, \alpha).
\]

One sees that \( A_{nm} \) is a closed set and \( \partial_{I, \lambda, \alpha}K = \bigcup A_{nm} \). If we define \( B_{nmk} \) to be the set of \( \lambda \in A_{nm} \) such that \( \Gamma^+_\epsilon(\lambda, \alpha) \subset Int(K) \), then it is straightforward to verify that \( B_{nmk} \) is a closed set and \( \partial_{II, \lambda, \alpha}K = \bigcup B_{nmk} \).

It is easy to verify that \( \mathcal{H}^1(LG(\partial_{I, \lambda, \alpha}K \setminus \partial_{II, \lambda, \alpha}K) = 0 \) and \( \mathcal{H}^1(LG(\partial_{II, \lambda, \alpha}K \setminus \partial_{II, \lambda, \alpha}K) = 0 \) for \( \alpha_1 \neq \alpha_2 \). Therefore, we will fix \( 0 < \alpha < \frac{1}{\|a\|_{\infty}} \) and use \( \partial_{I, \lambda, \alpha}K \), \( \partial_{II, \lambda, \alpha}K \) for \( \partial_{I, \lambda, \alpha}K \), \( \partial_{II, \lambda, \alpha}K \), respectively.

For (B), Lemma 5 and Corollary 2 below extend Lemma 1(2) and Lemma 2. From now on, we use \( LG \) for a fixed Lipschitz graph as above.

**Lemma 6.** Let \( \nu \) be a finite complex measure with compact support. Suppose \( \nu \) is singular to \( \mathcal{H}^1|_{LG} \). Then

\[
\mathcal{H}^1|_{LG}(\{ \lambda : M_R\nu(\lambda) \geq a \}) \leq \frac{C}{\alpha} \|\nu\|
\]

where \( C \) is an absolute constant. In this case,

\[
\mathcal{H}^1|_{LG}(\{ \lambda : M_R\nu(\lambda) = \infty \}) = 0.
\]

(3-5)

(2)

(3-6)

**Proof:** As the same as Lemma 1(2), (1) follows from Theorem 2.6 in Tolks, 2014.

(2) Let \( E_0 \) be a Borel set such that \( \mathcal{H}^1|_{LG}(E_0) = 0 \) and \( \|\nu\| (E_0) = 0 \) (since \( \nu \) \perp \mathcal{H}^1|_{LG} \).

Let \( \epsilon, \eta > 0 \) and let \( E \subset \{ \lambda : \lim_{\delta \to 0} \frac{\nu(B(\lambda, \delta))}{\delta} > \frac{1}{\eta} \} \cap E_0 \) be a compact subset. Let \( O \) be an open set containing \( E \) with \( \nu(O) < \eta \). Let \( x \in E \), then there exists \( 0 < \delta_x < \frac{1}{\eta} \) such that \( \nu(B(x, \delta_x)) > \frac{1}{\eta} \delta_x \) and \( B(x, \delta_x) \subset O \). Since \( E \subset \bigcup_{x \in E} B(x, \delta_x) \), we can choose a finite...
subset \( \{x_i\}_{i=1}^m \) so that \( E \subset \bigcup_{i=1}^m B(x_i, \delta_{x_i}) \). From 3\(\varepsilon\)-covering theorem (see Theorem 2.1 in Tolsa (2014)), we can further select a subset \( \{x_{ij}\}_{j=1}^n \) such that \( \{B(x_{ij}, \delta_{x_{ij}})\} \) are disjoint and

\[
E \subset \bigcup_{i=1}^m B(x_i, \delta_{x_i}) \subset \bigcup_{j=1}^n B(x_{ij}, 3\delta_{x_{ij}}).
\]

Therefore,

\[
\mathcal{H}^1_\varepsilon(E) \leq 3 \sum_{j=1}^n \delta_{x_{ij}} \leq \frac{3}{N} \sum_{j=1}^n |\nu|(B(x_{ij}, \delta_{x_{ij}})) \leq \frac{3}{N} |\nu|(\bigcup_{j=1}^n B(x_{ij}, \delta_{x_{ij}})) \leq \frac{3}{N} |\nu|(O) < \frac{3}{N} \eta.
\]

This implies \( \mathcal{H}^1 |_{\mathcal{L}G}(E) = 0 \). The lemma is proved.

**Corollary 2.** Let \( \nu \) be a positive finite compactly supported measure on \( \mathcal{C} \) and \( \nu \) is singular to \( \mathcal{H}^1 |_{\mathcal{L}G} (\nu \perp \mathcal{H}^1 |_{\mathcal{L}G}) \). For \( \mathcal{H}^1 |_{\mathcal{L}G} \)-almost all \( w \in \mathcal{L}G \), if there exists \( \delta_w, \epsilon_w > 0 \) such that

\[
B(\lambda \delta, \epsilon_w \delta) \subset (spt(\nu))^c \cap B(w, \delta)
\]

for \( 0 < \delta < \delta_w \), then

\[
\lim_{\delta \to 0} \int \frac{\delta}{|z - \lambda\delta|^2} d\nu(z) = 0.
\]

**Proof:** From (3-5) and (3-6), we assume that

\[
M_R(w) < \infty, \quad \lim_{\delta \to 0} \frac{\nu(B(w, \delta))}{\delta} = 0.
\]

Hence, for \( N > 2 \),

\[
\int \frac{\delta}{|z - \lambda\delta|^2} d\nu(z) \leq \int_{B(w, N\delta)} \frac{\delta}{|z - \lambda\delta|^2} d\nu(z) + \int_{B(w, N\delta)^c} \frac{\delta}{|z - \lambda\delta|^2} d\nu(z)
\]

\[
\leq \frac{N}{\epsilon_w^2} \frac{\nu(B(w, N\delta))}{N\delta} + \sum_{k=0}^{\infty} \int_{2^k N\delta \leq |z - w| < 2^{k+1} N\delta} \frac{\delta}{|z - \lambda\delta|^2} d\nu(z)
\]

\[
\leq \frac{N}{\epsilon_w^2} \frac{\nu(B(w, N\delta))}{N\delta} + \sum_{k=0}^{\infty} \frac{2^{k+1} N\delta^2 \nu(B(w, 2^{k+1} N\delta))}{2^{k+1} N\delta}
\]

\[
\leq \frac{N}{\epsilon_w^2} \frac{\nu(B(w, N\delta))}{N\delta} + \frac{4N}{(N - 1)^2} M_R(w)
\]

The second term is small for \( N \) large and for a given \( N \), the first term is small if \( \delta \) is small enough. Therefore, (3-7) holds.

Now we state our generalized version of Lemma 3 below. Notice that there is no corresponding function \( (1 - \lambda_0 z)^2 \) for a boundary point \( w \) of an arbitrary \( K \), in particular, for an inner boundary point \( w \).

**Lemma 7.** Let \( \nu \) be a finite measure supported in \( K \) and \( |\nu| \perp \mathcal{H}^1 |_{\partial^G K} \). Let \( 1 < p \leq \infty \), \( q = \frac{p}{p-1} \), \( f \in C(K) \), and \( g \in L^q(|\nu|) \). Define

\[
EVG(|g|^q|\nu|) = \{ \lambda \in \partial^G K : M_R(|g|^q|\nu|)(\lambda) = \infty \text{ or } \lim_{\delta \to 0} \int \frac{\leq |\nu|}{|z - \lambda_\delta|^2} d\nu(z) > 0 \}
\]

where \( \lambda_\delta \) is defined as in (3.4). Then \( \mathcal{H}^1 |_{\partial^G K} (EVG(|g|^q|\nu|)) = 0 \) (Lemma 6 and Corollary 3). Suppose that \( a > 0 \), \( w \in \partial^G K \setminus EV(|g|^q|\nu|) \), and upper cone \( \Gamma^+_\delta(w, \alpha) \) is outer, then there exist \( \delta_w > 0 \), \( E^\prime_w \subset \hat{B}(w, \delta) \), and \( \epsilon(\delta) > 0 \), where \( 0 < \delta < \delta_w \), such that \( \lim_{\delta \to 0} \epsilon(\delta) = 0 \), \( \Gamma^+_\delta(w, \alpha) \leq \epsilon(\delta) \), and

\[
|C(fg\nu)(\lambda) - C(fg\nu)(\lambda_\delta)| \leq M\delta^\frac{p}{p-1} \|f\|_{L^p(|\nu|)}
\]

for all \( \lambda \in (B(w, \delta) \setminus E^\prime_w) \cap U(\nu) \). Notice that \( E^\prime_w \) depends on \( f \) and all other parameters are independent of \( f \).
Proof: We just need to make the following slight modifications to the proof of Lemma 3. 

1. Replace \( \frac{1}{N} \) by \( \lambda_3 \).
2. Use Lemma 1 (1) instead of Lemma 1 (2) and use Corollary 2 instead of Lemma 2.
3. Replace \( \nu_3 \) by \( \nu_3 = \frac{1}{2} \lambda_{B_1(w, \nu_3)}^{\perp} f \nu_3 \).
4. (2-8) becomes
   \[
   \delta^+ |C_{\nu}(\lambda) - C_{\nu}(\lambda_3)| \leq \frac{a}{2} \|f\|_{L^p(\nu)} + 2C_{\nu}(\lambda_3).
   \]
5. Define
   \[
   E_3' = \{ \lambda : C_{\nu}(\lambda) > \frac{a\|f\|_{L^p(\nu)}}{4\delta} \} \cap B(w, \delta).
   \]
   (2-9) becomes
   \[
   \gamma(E_3) \leq \frac{4C_{\nu}\delta}{\|f\|_{L^p(\nu)}} \int_{B(w, N\delta)} \frac{\delta^{|g|d\nu|}}{|z - \lambda_3|} < \epsilon(\delta).\]

where \( \epsilon_w \) is as in (3-4) and
   \[
   \epsilon(\delta) = \frac{5(N + 1)C_{\nu}}{a\epsilon_w} \left( \int \delta^{|g|d\nu|} \right)^{\frac{1}{2}}.
   \]

The proof is completed.

Proposition 3. Let \( \nu \) be a finite complex measure with support in \( K \). Suppose that \( \nu \perp \text{Rat}(K) \) and \( \nu = \nu_0 + \nu_3 \), is the Radon-Nikodym decomposition with respect to \( H^1|_{\partial^G K} \), where \( \nu_0 = \frac{1}{2\pi} hH^1|_{\partial^G K} \) and \( \nu_3 \perp H^1|_{\partial^G K} \). Suppose upper cone \( \Gamma_3^{-1}(w, \alpha) \) is outer for \( w \in \partial^G K \). Then for \( b > 0 \) and \( H^1|_{\partial^G K} \)-almost all \( w \in \partial^G K \), there exist \( \delta_0 \) and \( \delta_3 \), such that \( \lim_{\delta \to 0} \epsilon(\delta) = 0 \), \( \gamma(E_3) < \epsilon(\delta) \), and
   \[
   |C_{\nu}(\lambda) - L(w)h(w)| \leq b
   \]

for all \( \lambda \in (\Gamma_3^{-1}(w, \alpha) \setminus E_3) \cap U(\nu) \).

Proof: We just need to replace Plemelj’s formula (2-11) in the proof of Proposition 1 by (3-2).

The following Lemma is from Lemma B in [Aleman et al., 2004] (also see Lemma 3 in [Yang, 2018]).

Lemma 8. There are absolute constants \( \epsilon_1 > 0 \) and \( C_1 < \infty \) with the following property. For \( R > 0 \), let \( E \subset B(\lambda_0, R) \) with \( \gamma(E) < R\epsilon_1 \). Then
   \[
   |p(\lambda)| \leq \frac{C_1}{R^2} \int_{B(\lambda_0, R) \setminus E} |p| \frac{dA}{\pi}
   \]

for all \( \lambda \in B(\lambda_0, \frac{R}{2}) \) and \( p \in A(B(\lambda_0, R)) \), the uniform closure of \( P \) in \( C(B(\lambda_0, R)) \).

Set
   \[
   \alpha(\alpha) = \frac{1}{8} \sin\left( \frac{\tan^{-1}(\alpha)}{2} \right).
   \]

Clearly, for \( \lambda_0 \in \Gamma_3^{-1}(w, \frac{\pi}{2}) \cap (\partial B(w, \frac{\pi}{2})) \),
   \[
   B(\lambda_0, 2\alpha(\alpha)\delta) \subset \Gamma_3^{-1}(w, \alpha).
   \]

The following theorem indicates that the carrier of \( \mu_3 \), for irreducible \( R'(K, \mu) \), does not intersect the boundary points for which both upper and lower cones contain a big portion of \( C \setminus K \).

Theorem 3. Suppose that \( \mu \) is a finite positive measure supported in \( K \) and is such that \( abp(B(\lambda_3, \mu)) = \Omega \) and \( \mu_{\perp} R'(K, \mu) \) is irreducible, where \( \Omega \) satisfies (3-3). Suppose that upper cone \( \Gamma_3^{-1}(w, \alpha) \) is outer for all \( w \in \partial^G K \) and \( \mu = \mu_0 + \mu_3 \) is the Radon-Nikodym decomposition
with respect to $H^1_{\partial \Gamma \cup \partial K}$, where $\mu_a = \frac{1}{\pi \delta} h H^1_{\partial \Gamma \cup \partial K}$ and $\mu_a \perp H^1_{\partial \Gamma \cup \partial K}$.

(a) Define

$$E = \{ w \in \partial \Gamma \cup \partial K \colon \lim_{\delta \to 0} \frac{\gamma(\Gamma^-(w,\alpha) \cap K)}{\delta} > 0 \},$$

then $\mu_a(E) = 0$.

(b) If the diameters of all components of $C \setminus K$ are bounded away from zero, then

$$\mu_a(\partial \Gamma \cup \partial K \cap \partial \Gamma \cup \partial K) = 0.$$

Proof: (a) Let $G \in R^2(K,\mu)^+$ and $G(z) \neq 0 \mu$ a.e. as above. Suppose $\mu_a(E) > 0$, then there exists $w \in E$ such that

(1) $G(w)h(w) \neq 0$.

(2) Proposition 3 holds for $w$, that is, for $b = \frac{|G(w)h(w)|}{\delta}$, there exist $\delta_w > 0$, $E_\delta \subset B(w,\delta)$, and $\epsilon(\delta) > 0$, where $0 < \delta < \delta_w$, such that $\lim_{\delta \to 0} \epsilon(\delta) = 0$, $\gamma(E_\delta) < \epsilon(\delta)\delta$, and

$$|C(G\mu)(\lambda) - L(w)G(w)h(w)| \leq b \quad (3-9)$$

for all $\lambda \in (\Gamma^-(w,\alpha) \cap E_\delta) \cap U(G\mu)$.

(3) There are a sequence of $\{\delta_n\}$ with $\delta_n \to 0$ and $\epsilon_0 > 0$ such that

$$\gamma(\Gamma^-_n(w,\alpha) \cap K) \geq \epsilon_0 \delta_n.$$

Choose $N$ large enough so that $\epsilon(\delta_0) < \frac{\epsilon_0}{2}$. For $\lambda \in \Gamma^-_N(w,\alpha) \cap K$, we see that $\lambda \in U(G\mu)$ and (3-9) does not hold since $C(G\mu)(\lambda) = 0$. That implies

$$\Gamma^-_n(w,\alpha) \cap K \subset E_{\delta_n}.$$

Hence,

$$\gamma(\Gamma^-_n(w,\alpha) \cap K) \leq \gamma(\epsilon_0 \delta_n) \leq \frac{\epsilon_0}{2} \delta_n.$$  

This contradicts (3).

We now turn to prove (b). Let $lb > 0$ be less than the diameters of all components of $C \setminus K$. Let $E_1$ be the set of $w \in \partial \Gamma \cup \partial K$ such that there exists a sequence of $\{\delta_n\}$ with $\delta_n \to 0$ and $\Gamma^-_n(w,\frac{\delta_n}{2}) \cap \partial K \neq \emptyset$. For a given $w \in E_1$, there exists a component $V_n$ of $C \setminus K$ so that $\Gamma^-_n(w,\frac{\delta_n}{2}) \cap V_n \neq \emptyset$. Let $\lambda_n \in (\Gamma^-_n(w,\frac{\delta_n}{2}) \cap V_n$, then

$$B(\lambda_n, a(\alpha)\delta_n) \cap V_n \subset (\Gamma^-_n(w,\alpha) \cap K,$$

where $a(\alpha)$ is defined as in (3-8). Hence,

$$\frac{1}{A} \min(a(\alpha)\delta_n, lb) \leq \frac{1}{A} \text{diameter}(B(\lambda_n, a(\alpha)\delta_n) \cap V_n) \leq \gamma(B(\lambda_n, a(\alpha)\delta_n) \cap V_n) \leq \gamma(\Gamma^-_n(w,\alpha) \cap K),$$

where the second inequality is implied by Theorem 2.1 on page 199 of [Gamelin 1969]. This implies

$$\frac{1}{A} \lim_{r \to 0} \frac{\gamma(\Gamma^-_n(w,\alpha) \cap K)}{\delta} \geq \frac{a(\alpha)}{8}.$$

So $E_1 \subset E$, from (a), we conclude $\mu_a(E_1) = 0$. We have shown that $\Gamma^-_n(w,\frac{\delta_n}{2}) \cap \partial K = \emptyset$ for $w \in \partial \Gamma \cup \partial K \setminus E_1$ with $\mu_a(E_1) = 0$ as $\delta$ is close to zero enough, in this case, $\Gamma^-_n(w,\frac{\delta_n}{2}) \subset \text{Int}(K)$.

Let $w \in \partial \Gamma \cup \partial K \setminus E_1$ so that we can apply Proposition 3 for $w$ and $b = \frac{|G(w)h(w)|}{\delta}$. There exist $\delta_w > 0$, $E_\delta \subset B(w,\delta)$, and $\epsilon(\delta) > 0$, where $0 < \delta < \delta_w$, such that $\lim_{\delta \to 0} \epsilon(\delta) = 0$, $\gamma(E_\delta) < \epsilon(\delta)\delta$, and

$$|C(G\mu)(\lambda)| \geq \frac{|G(w)h(w)|}{2} \quad (3-10)$$

for all $\lambda \in (\Gamma^-_n(w,\alpha) \cap E_\delta) \cap U(G\mu)$. Now choose $\delta$ to be small enough so that $\epsilon(\delta) < a(\frac{\delta}{2})\epsilon_1$, where $\epsilon_1$ is as in Lemma 3 and $a(\alpha)$ is defined in (2-8). Let $\lambda_0 \in \Gamma^-_n(w,\frac{\delta}{2})$ with $|\lambda_0 - w| = \frac{\delta}{2},$
then $B(\lambda_0, a(\frac{\delta}{2})\delta) \subset \Gamma^*_\delta (w, \frac{\delta}{2}) \subset \text{Int}(K)$, where $\delta$ is small enough. Since $\gamma(B(\lambda_0, a(\frac{\delta}{2})\delta) \cap E_\delta) < \epsilon_1 a(\frac{\delta}{2})\delta$, from Lemma 3 and (3-10), we conclude $\lambda \in B(\lambda_0, a(\frac{\delta}{2})\delta)$,

$$|r(\lambda)| \leq \frac{C_1}{\pi(a(\frac{\delta}{2})\delta)^2} \int_{\partial B(\lambda_0, a(\frac{\delta}{2})\delta) \setminus E_\delta} |r(z)|dA(z)$$

$$\leq \frac{2C_1}{\pi(G(w)h(w)a(\frac{\delta}{2})\delta)^2} \int_{\partial B(\lambda_0, a(\frac{\delta}{2})\delta) \setminus E_\delta} |C(rG\mu)(z)|dA(z)$$

$$\leq \frac{C_1}{\pi(G(w)h(w)a(\frac{\delta}{2})\delta)^2} \int_{\partial B(\lambda_0, a(\frac{\delta}{2})\delta) \setminus E_\delta} \frac{1}{|z - \lambda|}dA(z)|rG|d\mu(\lambda)$$

$$\leq \frac{C^2}{\delta} ||G||_L^*(\mu)||r||_{L^*(\mu)}$$

where $r \in \text{Rat}(K)$ and $C_2$ is a constant. Thus, $B(\lambda_0, a(\frac{\delta}{2})\delta) \subset \Omega$. This implies $\Gamma^*_\delta (w, \frac{\delta}{2}) \subset \Omega$ for $\delta$ small enough. Let

$$F(\alpha) = \{z \in \partial^I_{\Omega} \setminus (E_0 \cup E_1) : \exists \delta > 0, \text{ such that } \Gamma^*_\delta (z, \alpha) \subset \Omega\},$$

then $w \in F(\frac{\alpha}{4})$ and there exists a $H^1|_{\partial^I_{\Omega} \setminus K}$ zero set $E_0$ such that

$$\partial^I_{\Omega} \setminus (E_0 \cup E_1) \subset F(\frac{\alpha}{4}).$$

It is easy to verify $H^1|_{\partial^I_{\Omega} \setminus K}(F(\alpha_1) \setminus F(\alpha_2)) = 0$ for $\alpha_1 \neq \alpha_2$. Let $E_2 = F(\frac{\alpha}{4}) \setminus F(\alpha)$, then $H^1|_{\partial^I_{\Omega} \setminus K}(E_2) = 0$ and

$$\partial^I_{\Omega} \setminus (E_0 \cup E_1 \cup E_2) \subset \partial^I_{\Omega} \setminus K.$$

The theorem is proved.

The following example is an interesting application of above theorem.

**Example.** A Swiss cheese $K$ can be constructed as

$$K = \overline{\mathbb{D}} \setminus \bigcup_{n=1}^{\infty} B(a_n, r_n),$$

where $B(a_n, r_n) \subset \mathbb{D}$, $B(a_i, r_i) \cap B(a_j, r_j) = \emptyset$ for $i \neq j$, $\sum_{n=1}^{\infty} r_n < \infty$, and $K$ has no interior points. Let $\mu$ be the sum of the arc length measures of $\partial \mathbb{D}$ and all $\partial B(a_n, r_n)$. Let $\nu$ be the sum of $dz$ on $\partial \mathbb{D}$ and all $-dz$ on $\partial B(a_n, r_n)$. For $f \in \text{Rat}(K)$, we have

$$\int fd\nu = 0.$$

Clearly $\frac{dv}{du} > 0$, a.e. $\mu$ and $\frac{dv}{du} \bot R^2(K, \mu)$, so $R^2(K, \mu)$ is irreducible. From Theorem 3, we conclude that

$$\lim_{\delta \to 0} \gamma(\Gamma^\delta(e^{i\theta}) \setminus K) = 0$$

$m$-almost all $e^{i\theta} \in \partial \mathbb{D}$, where $\Gamma^\delta(e^{i\theta})$ is defined in section 2 (right before Theorem 4).

The example indicates although swiss cheese $K$ has no interior, the portion of $D \setminus K$ near $\partial \mathbb{D}$ is very small.

**Theorem 4.** Suppose that $\mu$ is a finite positive measure supported in $K$ and is such that $abpe(R^2(K, \mu)) = \Omega$ and $R^2(K, \mu)$ is irreducible, where $\Omega$ is a connected region satisfying $(3-3)$. Suppose that upper cone $\Gamma^\delta_w (w, \alpha)$ is outer for all $w \in \partial^I_{\Omega} K$ and $\mu = \mu_0 + \mu_+ \mu_\perp$ is the Radon-Nikodym decomposition with respect to $H^1|_{\partial^I_{\Omega} \setminus K}$, where $\mu_0 = \frac{1}{\pi} hR^1|_{\partial^I_{\Omega} \setminus K}$ and $\mu_\perp \bot H^1|_{\partial^I_{\Omega} \setminus K}$, and $\mu_0(\partial^I_{\Omega} \setminus K) > 0$. Then:

(a) If $f \in R^2(K, \mu)$ then the nontangential limit $f^*(z)$ of $f$ exists for $\mu_0|_{\partial^I_{\Omega} \setminus K}$ almost all $z$, and

$$f^* = f|_{\partial^I_{\Omega} \setminus K}$$

as elements of $L^1(\mu_0|_{\partial^I_{\Omega} \setminus K})$.

(b) Every nonzero rationally invariant subspace $M$ of $R^2(K, \mu)$ has index 1, that is, if $\lambda_0 \in \Omega$, then $\dim(M/(S_\mu - \lambda_0)M) = 1$.

If the diameters of all components of $C \setminus K$ are bounded away from zero, then by Theorem 3 the above $\partial^I_{\Omega} \setminus K$ can be replaced by $\partial^I_{\Omega} \setminus K$. 

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Proof: The proof is the same as in Theorem \[1\] if we apply Proposition \[2\] instead of Proposition \[1\].

The following lemma is an easy exercise.

**Lemma 9.** Let \(B(\lambda, \delta) \subset \Gamma_\delta^*(w, \alpha)\) (or \(\Gamma_\delta^+(w, \alpha)\)). Then there are constants \(c(\alpha)\), \(C(\alpha) > 0\) that only depend on \(\alpha\) and \(\|A\|_\infty\) such that

\[
\min(e, c(\alpha))(|\alpha + |Re(z - w)|| \leq |z - \lambda| \leq C(\alpha)(|\alpha + |Re(z - w)||)
\]

for \(z \in LG\).

**Proof:** In fact, \(C(\alpha) = 1 + \sqrt{1 + \|A\|_\infty^2}\) and \(c(\alpha) = \frac{1-\alpha\|A\|_\infty}{\sqrt{1 + \|A\|_\infty^2}}\). We leave the details to the reader.

Because we do not have an analogous testing function (such as \((1 - \lambda_0 z)^{-\frac{q}{2}}\) in Proposition \[2\]) in general, we are not able to get an estimation of the Cauchy transform as in Proposition \[2\]. However, our following proposition is enough for us to estimate an upper bound as in (1-2) ((1.4) in Aleman et al. (2009)). We define a set

\[
B\Gamma_\delta^*(w, \alpha) = \bigcup_{\lambda_0 \in \Gamma_\delta^*(w, \alpha) \cap (\partial B(w, \frac{2}{3}))} B(\lambda_0, a(\alpha)\delta)
\]

where \(a(\alpha)\) is defined as in (3-8).

**Proposition 4.** Let \(\mu\) be a finite complex measure with support in \(K\). Suppose that \(\mu = \mu_a + \mu_s\) is the Radon-Nikodym decomposition with respect to \(H^1|_{\partial^+_K}\), where \(\mu_a = \frac{1}{\pi} \partial H^1|_{\partial^+_K}\) and \(\mu_s \perp H^1|_{\partial K}\). Suppose upper cone \(\Gamma_\delta^+(w, \alpha)\) is outer for \(w \in \partial^+_K\). Let \(1 < p < \infty\), \(q = \frac{p}{p-1}\), \(f \in C(K)\), \(g \in L^q(\mu)\), and \(fg\mu \perp \text{Rad}(K)\). Then for \(b > 0\), and \(H^1|_{\partial^+_K}\)-almost all \(w \in \partial^+_K\), there exist \(\delta_w > 0\), \(E_b^\delta \subset B(w, \delta)\), and \(\gamma(\delta) > 0\), where \(0 < \delta < \delta_w\), such that \(\lim_{\delta \to 0} \gamma(\delta) = 0\), \(\gamma(E_b^\delta) \leq \epsilon(\delta)\delta^2\), and for \(\lambda_\delta\) as in (3-4),

\[
|\mathcal{C}(fg\mu)|(\lambda) \leq b\delta^{-\frac{p}{q}}\|f\|_{L^p(\mu)} + \frac{2C(\alpha)\delta^{-\frac{p}{q}}\|f\|_{L^p(\mu)}}{\epsilon_w^c c_0(\alpha)} \left(\int \frac{\delta}{|Re(z - w) - \lambda_\delta - w|^2}\|g\|d\mu_a\right)^{\frac{1}{q}}
\]

for all \(\lambda \in (B\Gamma_\delta^*(w, \alpha) \setminus E_b^\delta) \cap U(g)\), where \(\epsilon_w^c = \min(\epsilon_w, c(\alpha))\) and \(c_0(\alpha) = \min(a(\alpha), c(\alpha))\), where \(\epsilon_w\) is as in (3-4), \(a(\alpha)\) is from (3-8), and \(c(\alpha)\), \(C(\alpha)\) are from Lemma \[3\].

**Proof:** Using Lemma \[3\] we have the following calculation:

\[
|\mathcal{C}(fg\mu_a)|(\lambda) \leq |\lambda - \lambda_\delta| |fg|d\mu_a
\]

\[
\leq \frac{2\delta}{\epsilon_w c_0(\alpha)} \int \frac{1}{|Re(z - w) + \delta|^2}|fg|d\mu_a
\]

\[
\leq \frac{2\delta^{-\frac{p}{q}}\|f\|_{L^p(\mu)}}{\epsilon_w^c c_0(\alpha)} \left(\int \frac{\delta}{|Re(z - w) + \delta|^2}\|g\|d\mu_a\right)^{\frac{1}{q}}
\]

where the last step also follows Lemma \[3\]. The rest of proof is the same as in the proof of Proposition \[2\].
Theorem 5. Suppose that $\mu$ is a finite positive measure supported in $K$ and is such that $\text{abpe}(R^t(K,\mu)) = \Omega$ and $R^t(K,\mu)$ is irreducible, where $\Omega$ is a connected region satisfying (3-3). Suppose that the upper cone $\Gamma^t_\lambda(w,\alpha)$ is outer for all $w \in \partial^t K$ and $\mu = \mu_0 + \mu_s$ is the Radon-Nikodym decomposition with respect to $\mathcal{H}^1|_{\partial^t K}$, where $\mu_0 = \frac{1}{\pi} h \mathcal{H}^1|_{\partial^t K}$ and $\mu_s \perp \mathcal{H}^1|_{\partial^t K}$, and $\mu_0(\partial^t K) > 0$. Then:
(a) For $t = 1$, there are constants $C(w) > 0$ (depending on $G$) such that
\[
\lim_{(w, \frac{2}{3}) \rightarrow \lambda \rightarrow w} |\lambda - w|M_\lambda \leq \frac{C(w)}{h(w)}
\]
for $\mu_\alpha$-almost all $w \in \partial^t G K$.
(b) For $t > 1$, there are constants $C_0(\alpha) > 0$ (depending on $\alpha$ and $\|A'\|_\infty$) such that
\[
\lim_{(w, \frac{2}{3}) \rightarrow \lambda \rightarrow w} |\lambda - w|M_\lambda \leq \frac{C_0(\alpha)}{h(w)}
\]
for $\mu_\alpha$-almost all $w \in \partial^t G K$, where $\epsilon_{(w)}$ is as in (3-4) and $\epsilon_{(w)}$ is from Proposition 4.
If the diameters of all components of $C \setminus K$ are bounded away from zero, then by Theorem 3 the above $\partial^t G K$ can be replaced by $\partial^t G K$.

Proof: (a) Let $w \in \partial^t G K$ so that we can apply Proposition 3 for $w$ and $b = \frac{|G(w)h(w)|}{G(w)h(w)}$. There exist $\delta > 0$, $E_\delta \subset B(w, \delta)$, and $\epsilon(\delta) > 0$, where $0 < \delta < \delta_0$, such that $\lim_{\delta \rightarrow 0} \epsilon(\delta) = 0$, $\gamma(E_\delta) < \epsilon(\delta)\delta$, and
\[
|\mathcal{C}(G\mu)(\lambda)| \geq \frac{|G(w)h(w)|}{2}
\]
for all $\lambda \in (\Gamma^t_\delta(w, \alpha) \setminus E_\delta) \cap U(G\mu)$, where $\Gamma^t_\delta(w, \alpha) \subset \Omega$. Now choose $\delta$ to be small enough so that $\epsilon(\delta) < a(\alpha)e_1$, where $e_1$ is as in Lemma 8 and $a(\alpha)$ is from (3-8). Let $\lambda_0 \in \Gamma^t_\delta(w, \frac{2}{3})$ and $|\lambda_0 - w| = \frac{1}{2}$, then $\mathcal{B}(\lambda_0, a(\alpha)\delta) \subset \Gamma^t_\delta(w, \alpha) \subset \Omega$, where $\delta$ is small enough. Since $\mathcal{G}(\lambda_0, a(\alpha)\delta) \cap E_\delta) < \epsilon_1a(\alpha)\delta$, from Lemma 8 (2-16), and (3-13), we conclude that for $\lambda \in B(\lambda_0, \frac{u(\alpha)\delta}{2})$ and $r \in \text{Rad}(K)$, we have
\[
|r(\lambda)| \leq \frac{C_1}{a(\alpha)\delta^2} \int_{B(\lambda_0, a(\alpha)\delta) \setminus E_\delta} |r(z)| \frac{dA(z)}{\pi} \leq \frac{C_1}{a(\alpha)\delta^2} \int_{B(\lambda_0, a(\alpha)\delta) \setminus E_\delta} \frac{|\mathcal{C}(G\mu)(z)|}{|\mathcal{C}(G\mu)(z)|} dA(z)
\]
\[
= \frac{2C_1}{a(\alpha)\delta^2} \frac{|\mathcal{C}(G\mu)(z)|}{|\mathcal{C}(G\mu)(z)|} \int_{B(\lambda_0, a(\alpha)\delta)} \frac{1}{|z - u|} dA(z) \int_{B(\lambda_0, a(\alpha)\delta)} |r(u)||G(u)|d\mu(u)
\]
\[
\leq \frac{C_2}{|\mathcal{C}(G\mu)(\lambda)|} \int_{B(\lambda_0, a(\alpha)\delta)} |r(u)||G(u)|d\mu(u),
\]
where $C_1, C_2, C_3, \ldots$ stand for absolute constants, and hence,
\[
|\lambda - w||r(\lambda)| \leq \frac{C_5|G||L^\infty(\mu)|}{|G(\mu)|} \int_{B(\lambda_0, a(\alpha)\delta)} |r(u)||G(u)|d\mu(u)
\]
for $\lambda \in \Gamma^t_\delta(w, \frac{2}{3})$ and $\lambda - w = \frac{1}{2}$. Let $C(w) = \frac{C_5|G||L^\infty(\mu)|}{|G(\mu)|}$, we get
\[
\lim_{(w, \frac{2}{3}) \rightarrow \lambda \rightarrow w} |\lambda - w|M_\lambda \leq \frac{C(w)}{h(w)}.
\]
(b) By Proposition 3 and Proposition 4 for $b > 0$, and $\mathcal{H}^1|_{\partial^t G K}$-almost all $w \in \partial^t G K$, there exist $\delta > 0$, $E_\delta \subset B(w, \delta)$, $E_\delta \subset B(w, \delta)$, and $\epsilon(\delta) > 0$, where $0 < \delta < \delta_0$, such that $\lim_{\delta \rightarrow 0} \epsilon(\delta) = 0$, $\gamma(E_\delta) < \epsilon(\delta)\delta$, $\gamma(E_\delta) < \epsilon(\delta)\delta$, and $|\mathcal{C}(G\mu)(\lambda) - L(w)G(w)h(w)| \leq b$.

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for all $\lambda \in (\Gamma^+(w, \alpha) \setminus E_\lambda) \cap U(G\mu)$, and for $\lambda_\delta$ as in (3-4),
\[
|C(rg\mu)(\lambda)| \leq b\delta^{-\frac{1}{2}}\|\tau\|_{L^1(\mu)}^\frac{1}{2} \sup_{\lambda \in (\Gamma^+(w, \alpha) \setminus E_\lambda) \cap U(G\mu)} \left( \sum_{\lambda \in (\Gamma^+(w, \alpha) \setminus E_\lambda) \cap U(G\mu)} \frac{\delta}{|\tau(\lambda)||\lambda - w|} |G|^{l'} d\mu_\lambda \right)^{\frac{1}{2}}.
\]
\[
(3-15)
\]
for all $\lambda \in (B_{\Gamma^+_\lambda}(w, \alpha) \cup E_\lambda^+) \cap U(|G|^{l'} \mu)$.

From Plemelj’s formula (3-2), we have the following calculation:
\[
\lim_{\delta \rightarrow 0} \int \frac{d\delta}{|\tau(\lambda)||\lambda - w|^2} |G|^{l'} d\mu_\lambda
\]
\[
= \lim_{\delta \rightarrow 0} \frac{i\delta}{2} \Re \left( \frac{1}{2\pi i} \int \frac{C(|G|^{l'} h \sqrt{1 + (A')^2} dx)(\lambda - w - \Re(w))}{|\lambda - w - \Re(w)|^2} \right)
\]
\[
- \frac{1}{2\pi i} C(|G|^{l'} h \sqrt{1 + (A')^2} dx)(\lambda - w - \Re(w))
\]
\[
\leq \frac{|G(w)|^{l'} h(w) \sqrt{1 + (A')^2}}{2\epsilon_{w}}.
\]
\[
(3-16)
\]
Therefore, for $\eta > 0$, if $\delta$ is small enough, we conclude
\[
\int \frac{d\delta}{|\tau(\lambda)||\lambda - w|^2} |G|^{l'} d\mu_\lambda
\]
\[
< \frac{|G(w)|^{l'} h(w) \sqrt{1 + (A')^2}}{2\epsilon_{w}} + \eta.
\]
\[
(3-17)
\]
Combining (3-14), (3-15), and (3-17), for $\delta$ small enough, $\lambda_\delta \in (\partial B(w, \frac{\delta}{2})) \cap \Gamma^+(w, \alpha)$, $B(\lambda_\delta, a(\alpha) \delta) \subset \Gamma^+(w, \alpha)$, and $\lambda \in (B(\lambda_\delta, a(\alpha) \delta) \setminus (E_\lambda \cup E_\lambda^+)) \cap U(|G|^{l'} \mu)$, we get
\[
\frac{|\tau(\lambda)|}{||\tau||_{L^1(\mu)}} \leq \frac{b\delta^{-\frac{1}{2}} + \frac{2C(\alpha)\delta^{-\frac{1}{2}}}{\epsilon^\alpha_c(\alpha)} \frac{|G(w)|^{l'} h(w) \sqrt{1 + (A')^2}}{2\epsilon_{w}} + \eta}{|G(w)| h(w) - b}.
\]
\[
(3-18)
\]
From semi-additivity of (2-2), we see
\[
\gamma(E_\lambda \cup E_\lambda^+) \leq A_T(\gamma(E_\lambda) + \gamma(E_\lambda^+)) \leq 2A_T \epsilon_1 \delta.
\]
Let $\delta$ be small enough so that $\epsilon(\delta) < \frac{a(\alpha)}{2A_T} \epsilon_1$, where $\epsilon_1$ is as in Corollary [1]. From Corollary [1] we conclude that (3-18) holds for all $\lambda \in (B(\lambda_0, a(\alpha) \delta) \cap \Gamma^+(w, \alpha)$).

Hence, for $\delta$ small enough, $\lambda \in (\partial B(w, \frac{\delta}{2})) \cap \Gamma^+(w, \alpha)$,
\[
|\lambda - w|^\frac{1}{2} M_\lambda \leq \frac{b + \frac{2C(\alpha)\delta^{-\frac{1}{2}}}{\epsilon^\alpha_c(\alpha)} \frac{|G(w)|^{l'} h(w) \sqrt{1 + (A')^2}}{2\epsilon_{w}} + \eta}{|G(w)| h(w) - b}.
\]
\[
(3-19)
\]
Therefore, there exists a constant $C_0(\alpha) > 0$ that only depends on $\alpha$ and $\|A'||\infty$ so that
\[
\lim_{\Gamma^+(w, \alpha) \cap \lambda \rightarrow w} |\lambda - w|^\frac{1}{2} M_\lambda \leq \frac{C_0(\alpha)}{\epsilon_w \epsilon^\alpha_c(\alpha)}.
\]
\[
(3-20)
\]
for $\mathcal{H}^{1}_{L^1} \cap \mathcal{H}^{1}_{L^1} \cap \mathcal{H}^{1}_{L^1}$-almost all $w \in \partial \mathcal{H}^{1}_{L^1} K$.

For the lower bound, we do have testing functions $f_1^1(z) = (z - \lambda_\delta)^{-1} \in R^1(K, \mu)$ and $f_2^1(z) = (z - \lambda_\delta)^{-1} \in R^2(K, \mu)$. The following proposition estimates their norms.

Proposition 5. Let $\mu$ be a finite positive measure with support in $K$. Suppose that $\mu = \mu_a + \mu_s$ is the Radon-Nikodym decomposition with respect to $\mathcal{H}^{1}_{\alpha} \cap \mathcal{H}^{1}_{\beta} K$, where $\mu_a = \frac{1}{\Delta_\mu^\alpha} h \mathcal{H}^{1}_{\alpha} \cap \mathcal{H}^{1}_{\beta} K$ and $\mu_s \perp \mathcal{H}^{1}_{\alpha} \cap \mathcal{H}^{1}_{\beta} K$, and $\mu_a(\partial \mathcal{H}^{1}_{L^1} K) > 0$. Suppose that $\Gamma^+(w, \alpha)$ is outer for $w \in \partial \mathcal{H}^{1}_{L^1} K$, then there exists a constant $C_1(\alpha) > 0$ that only depends on $\alpha$ and $\|A'||\infty$ such that
\[
\lim_{\delta \rightarrow 0} \int \frac{d\delta}{|z - \lambda_\delta|^2} d\mu \leq \frac{C_1(\alpha)}{\epsilon_w \epsilon^\alpha_c(\alpha)} h(w).
\]
\[
(3-21)
\]
for $\mu_a$-almost all $w \in \partial \mathcal{H}^{1}_{L^1} K$, where $\lambda_\delta$ and $\epsilon_w$ are from (3-4).
Proof: The proposition follows from Corollary 2, Lemma 9, and the same proof of (3-16). So we have lower bounds for $R^1(K, \mu)$ and $R^2(K, \mu)$ as the following:

For $t = 1$,

$$\lim_{\Gamma^- (w, \frac{\partial}{\partial \lambda}) \lambda \to w} |\lambda - w|M_\Delta \geq \lim_{\delta \to 0} \frac{|f^1_t(\lambda)|}{\|f^1_t\|_{L^2(\mu)}} \geq \frac{\epsilon_w (\epsilon_w^*)^2}{4C_1(\alpha) h(w)}.$$ 

For $t = 2$,

$$\lim_{\Gamma^- (w, \frac{\partial}{\partial \lambda}) \lambda \to w} |\lambda - w|^2 M_\Delta \geq \lim_{\delta \to 0} \frac{|f^2_t(\lambda)|}{\|f^2_t\|_{L^2(\mu)}} \geq \frac{\sqrt{\epsilon_w \epsilon_w^*}}{2\sqrt{C_1(\alpha) h(w)}}.$$ 

For $t \neq 1$ and $t \neq 2$, if $w$ is a boundary point of $\mathbb{C} \setminus K$, then we can define a similar testing function and have corresponding lower bounds. However, if $w$ is an inner boundary point, we do not have such a testing function to estimate the lower bounds.

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