Detecting Majorana fermions in quasi-1D topological phases using non-local order parameters

Yasaman Bahri¹ and Ashvin Vishwanath¹

¹Department of Physics, University of California at Berkeley, Berkeley, CA 94720, USA
(Dated: February 6, 2014)

Topological phases which host Majorana fermions cannot be identified via local order parameters. We give simple non-local order parameters to distinguish quasi-1D topological superconductors of spinless fermions, for any interacting model in the absence of time reversal symmetry. These string or brane-type order parameters are amenable to measurements via quantum gas microscopy in cold atom systems, which could serve as an alternative route towards Majorana fermion detection that involves bulk rather than edge degrees of freedom. We also discuss our attempts at accessing 2D topological superconductors via the quasi-1D limit of coupling N identical chains with \( \mathbb{Z}_N \) translation symmetry. We classify the resulting symmetric topological phases and discuss general rules for constructing non-local order parameters that distinguish them. The phases include quasi-1D analogs of (i) the \( p + ip \) topological superconductor, which can be distinguished up to the 2D Chern number \( \mod 2 \), and (ii) the 2D weak topological superconductor. The non-local order parameters for some of these phases simply involve a product of the string order parameters for the individual chains. Finally, we sketch a physical picture of the topological phases as a condensate of Ising charged domain walls, which motivates the form of the non-local order parameter.

Quantum phases with emergent Majorana fermion excitations have received much attention in the past several years, with recent experimental evidence suggesting they have been realized in solid-state systems. Majorana fermions are known to appear at the boundaries and topological defects of exotic 1D ²² and 2D topological superconductors.²³ Cold atom realizations of such phases would serve as a new platform for studying and manipulating Majorana fermions.

At the same time, a general framework for classifying quantum phases continues to be developed. Important achievements include the classification of free fermion systems,²⁴ an understanding of how interactions can modify these results in certain symmetry classes,²⁵ and general methods for many of the symmetry-protected bosonic or fermionic systems with interactions.²⁶⁻²⁸ In 1D, where matrix-product states provide a framework for describing ground states, the gapped symmetry-protected topological phases have been completely classified.²⁹ ³¹ A natural resulting question is how such phases may be distinguished via accessible quantities, in any dimension. Fully symmetric phases are distinct but have no broken symmetries and hence are immune to a local order parameter description.

The insight obtained from classification enables design of non-local order parameters that extract the defining quantities, associated with the cohomology group of the symmetry group, which characterize a symmetry-protected phase. This problem has been largely addressed in 1D for bosons,³³ ³⁴ but remains open in higher dimensions. These 1D bosonic non-local order parameters (OPs) are robust in that they are strictly symmetric rather than wavefunction dependent; their extension to fermions to our knowledge has not been discussed. In contrast, conventional AKLT string order ³⁵⁻³⁷ relevant for the \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) spin rotation symmetry protected Haldane phase of 1D bosonic systems, measures extra-
in cold atom systems have been discussed\cite{28-33}—here, however, we will be concerned with generic interacting systems, in particular topological superconductors. As an example of a quantity measurable with current experimental techniques, we describe a system of two identical chains for which a topological phase can be detected via fermion parity only.

We then add $\mathbb{Z}_N$ translation symmetry to $N$ identical chains with interactions and describe the protected symmetric phases, which can scale to interesting 2D phases. The classification appears to distinguish certain topological indices in the case of free fermions. We justify simple symmetry transformation rules which the edge operators of a non-local OP should satisfy in order to uniquely distinguish the symmetric phases. As further illustration, we discuss a condensate of Ising defects.

Our results can be supported by either working in bosonic or fermionic variables; in the main text we mainly take the bosonic point of view and discuss the fermionic description, also sketching a derivation of "selection rules" for non-local OPs, in the appendices. Throughout the paper, we use the term non-local OP to include string and brane-type order.

I. INTERACTING SPINLESS FERMION TOPOLOGICAL SUPERCONDUCTORS

Example: Single Majorana Chain. To illustrate the general form of the order parameters, we first consider Kitaev’s spinless p-wave topological superconductor on an open chain with Hamiltonian:

$$H_{KH} = \sum_i (-t a_i^\dagger a_{i+1} + |\Delta| a_i a_{i+1} + h.c.) - \mu (a_i^\dagger a_i - \frac{1}{2})$$

(1a)

$$= \frac{i}{2} \sum_i \left\{ (-t + |\Delta|) \chi_i \chi_{i+1} + (t + |\Delta|) \chi_i^* \chi_{i+1} - \mu \chi_i \chi_i^* \right\}$$

(1b)

with site fermions $a_i = \frac{1}{2}(\chi_i + i \chi_i^*$) and Majorana operators $\chi_i$, $\chi_i^*$. The phase of the superconducting order parameter $\Delta = |\Delta| e^{i\theta}$ has been gauged away. For $|\Delta| < 1$, there are gapped topological phases if $|\Delta| \neq 0$ and a gapless normal phase if $|\Delta| = 0$. If $|\Delta| > 1$ for any $|\Delta|$, the phase is gapped and trivial. Let the Jordan-Wigner mapping be $\sigma^x_j = e^{i\pi n_j}, \sigma^y_j = \sum_{i<j} e^{i\pi n_j} \chi_j, \sigma^z_j = -\sum_{i<j} e^{i\pi n_j} \chi_j$. The Majorana chain maps onto an XY-type spin model in a transverse magnetic field. Fermion parity corresponds to a $\mathbb{Z}_2$ spin symmetry $\prod_i \sigma^z_i$. This symmetry is broken or unbroken, respectively, in the spin system when the corresponding fermionic system is gapped and in a topological or trivial phase. This is in fact a more general correspondence between 1D fermionic and bosonic systems.

Consider an Ising limit (e.g. set $t = |\Delta|$) of the spin Hamiltonian obtained from the Kitaev model: $H_{spin} = \sum_i -J (\sigma^z_i \sigma_{i+1}^z + g \sigma^y_i)$ with $J = |\Delta|$ and $g = -\frac{|\Delta|}{2}$. A two-point spin correlation function is finite in the spin ordered phase and vanishes in the disordered phase. It maps to a string OP which distinguishes the topological phase from the trivial in this limit:

$$\langle \sigma^z_i \sigma^z_k \rangle = \left( \langle (-i \chi_i) \prod_{j=i+1}^{k-1} e^{i\pi n_j} \chi_j \rangle \right)$$

(2)

As fermion parity cannot be broken, the fermion ground states in the topological phase correspond to linear combinations of the $\mathbb{Z}_2$ breaking spin ground states. Two-point correlations are, however, insensitive to the choice of linear combination.

The Kitaev model Eq. (2) has additional symmetries, for instance time reversal $\chi \to \chi, \bar{\chi} \to -\bar{\chi}$; this reduces the possible two-point spin correlations which can be chosen. $\sigma^x$ or $\sigma^y$ correlations are finite in the broken symmetry regimes of $t > 0$ or $t < 0$, respectively, so that we have either $(\chi_i, \chi_k)$ string termination operators, as in Eq. (2) or $(\chi_i, \bar{\chi}_k)$. For fermion models with strictly no other symmetries, these constraints will not occur.

On the other hand, the self-duality of the quantum transverse isometric model under mapping $\tau^z_{i+\frac{1}{2}} = \sigma^z_i \sigma^z_{i+1}, \tau^z_{i+\frac{3}{2}} = \prod_{j>i} \sigma^z_j$ to domain wall variables on bonds yields $H_{dual} = \sum_i (-|\Delta| \tau^z_{i+\frac{1}{2}} + \frac{1}{2} \mu \tau^z_{i+\frac{3}{2}} \tau^z_{i+\frac{1}{2}})$ in the thermodynamic limit. A two-point correlation in the $\tau$ variables distinguishes the two phases. In the $t = |\Delta|$ limit, this yields a fermion string OP which is finite in the trivial phase and vanishes in the topological:

$$\langle \tau^z_{i+\frac{1}{2}} \tau^z_{j+\frac{1}{2}} \rangle \sim \langle \prod_k \sigma^z_k \rangle = \langle \prod_k e^{i\pi n_k} \rangle$$

(3)

General Form. We used the Kitaev model, and in particular, an Ising limit, to illustrate a more general correspondence which holds for quasi-1D topological superconductors of spinless fermions with interactions and no symmetries. These fermionic phases were classified in Ref.\cite{44} by considering the bosonic phases protected by a global bosonic $\mathbb{Z}_2$ symmetry corresponding to fermion parity. There are only two gapped phases, which have the $\mathbb{Z}_2$ symmetry unbroken or broken; via the Jordan-Wigner mapping, they correspond to fermion symmetric phases that are, respectively, trivial or topological, with boundary Majorana zero modes. The models we cite have translation symmetry along their infinite direction, which multiplies the number of possible phases by a factor but we neglect this, focusing only on topological distinctions. In this case, there are two distinct phases.

We can distinguish the phases in the bosonic variables and map the result to fermions. Any two-point correlation function $\langle O^l O^l \rangle$ with $O/O'$ local operators which are odd under the $\mathbb{Z}_2$ spin operation is generically finite as $|i-j| \to \infty$ in the the spin ordered phase and vanishes in the disordered phase. Hence, the two-point function
maps to a fermionic string OP (for one chain) or brane OP (for two or more) whose bulk measures fermion parity and which is terminated by fermionic operators.

The spin disordered (i.e. symmetric) phase is not susceptible to local order parameters. Rather, utilizing bosonic selection rules proposed in \cite{5}, we conclude that a non-local OP which is finite in this phase should apply the local \( \mathbb{Z}_2 \) symmetry over a domain in the bulk, and the domain should be terminated by operators which are \( \mathbb{Z}_2 \) invariant. In Appendix I, we argue that this order parameter vanishes in the ordered phase of spins. Mapped to fermions, an order parameter which is finite in the trivial phase and zero in the topological phase would consist of a bulk which measures fermion parity and is terminated by local bosonic operators.

The appearance of fermionic or bosonic terminations for a non-local OP is a fermionic selection rule, analogous to those described in \cite{5} for bosonic systems, which distinguishes the two phases. As an alternative to the Jordan-Wigner mapping, in Appendix III we try to justify fermionic selection rules for non-local OPs from fermions directly based on ideas from fermion classification.

To summarize, order parameters for the two topologically distinct phases of interacting spinless fermion topological superconductors can be constructed with the form:

\[
S_{\text{top}} = \langle O_{FL} \prod_{j \in \Omega} e^{i \pi n_j} O_{FR} \rangle \tag{4a}
\]

\[
S_{\text{triv}} = \langle O_{BL} \prod_{j \in \Omega} e^{i \pi n_j} O_{BR} \rangle \tag{4b}
\]

where \( O_{FL/R}, O_{BL/R} \) are local fermionic or bosonic operators near the left, right edges of region \( \Omega \). \( S_{\text{top}} \) is non-zero in the topological phase and vanishes elsewhere; the behavior of \( S_{\text{triv}} \) is reversed. These are the generic values, as we now discuss.

Remark. While the order parameters proposed throughout this paper can be used for general interacting models, their values depend in part on the state, as it requires evaluating matrix elements of certain local operators. This is no different than tailoring an order parameter for a symmetry breaking theory: certain operators may be more “optimal” for detecting the broken symmetry because they yield larger values, while specific models may have larger symmetry groups, which we can identify from the outset. For instance, the symmetry group of the quantum Ising model with \( \sigma^z \) nearest-neighbor couplings includes time reversal of spins followed by \( \pi \) rotation about \( y \), so that \( \langle \sigma^y_j \rangle = 0 \). Our order parameters distinguish the phases simply, theoretically and experimentally, and give an interpretation to the distinct order in the phases. Their general form, which is robust to interactions, will be given by Eqs. (4).

Microscopic Picture. We explain why fermionic or bosonic operator ends distinguish the topologically distinct fermionic phases. To illustrate, we specialize to string order in the Ising limit of the single Kitaev chain. Introduce bond fermions \( \hat{a}_j = \frac{1}{2} (\chi_{j+1} + i \tilde{\chi}_j) \) of re-paired Majoranas, neglecting the non-local fermion \( \hat{a}_{NL} = \frac{1}{2} (\chi_1 + i \tilde{\chi}_N) \) by working on an infinite chain. This basis exactly solves the \( t = |\Delta|, \mu = 0 \) limit. The topological and trivial phase string OPs (2),(3) can be rewritten (\( k \geq i + 1 \)):

\[
S_{\text{top}} = \left( -i \tilde{\chi}_i \right) \prod_{j=i+1}^{k-1} e^{i \pi n_j} \chi_k \propto \prod_{j=i}^{k-1} e^{i \pi n_j} \tag{5a}
\]

\[
S_{\text{triv}} = \prod_{j=i}^{k} e^{i \pi n_j} \propto (\hat{a}_{i-1} + \hat{a}_{i-1}^\dagger) \prod_{j=i}^{k-1} e^{i \pi n_j} (\hat{a}_k - \hat{a}_k^\dagger) \tag{5b}
\]

Evidently, the fermionic or bosonic nature of the terminations depends on the basis used. The topological ground states at \( t = |\Delta|, \mu = 0 \) have uniform filling in the bulk in the bond fermion basis \( \hat{a}_i \) so that \( S_{\text{top}} \), which measures their parity, is finite. Weak perturbations \( \mu \neq 0 \) drive the system away from this “bond-centered” ordering; they favor on-site Majorana pairings and hence create localized pairs of bond fermion defects with respect to the unperturbed state. But because the defects are pairs and are localized, they weakly modify the bond fermion parity \( S_{\text{top}} \) measured in a region. They are more likely to fall into the bulk of \( S_{\text{top}} \), in which case they do not modify the bond fermion parity measured, rather than cross its ends. Hence, the value of \( S_{\text{top}} \) remains finite in the topological phase. A dual picture holds for site fermions, which are a good basis to use (i.e. the wavefunction is simple). This explains the finite value of \( S_{\text{triv}} \) in the trivial phase.

On the other hand, such string OPs vanish in the complementary phases. To understand how this occurs, consider perturbatively evaluating \( S_{\text{top}} \) in the trivial phase

\[\text{FIG. 1. Single chain at top shows two Majorana fermions } \chi, \tilde{\chi} \text{ per site (circles with same color) with non-trivial pairing (boxes). The non-local order parameter } S_{\text{top}} \text{ for the topological phase measures this pairing (green line). Bottom chain shows a phase with trivial Majorana pairings, which is measured by } S_{\text{triv}}.\]
of the $t = |\Delta|$ Kitaev model. The ground state at the point $H_0 = \frac{4}{\beta} \sum_j (a_j + a_j^\dagger)$ with $\mu < 0$ is the site fermion vacuum $|0\rangle$ which is then corrected by the perturbation $V = |\Delta| \sum_j i \chi_j \chi_{j+1} = |\Delta| \sum_j (a_j - a_j^\dagger)(a_j + a_j^\dagger)$. $V$ corrects $|0\rangle$ by creating localized pairs of “defects” relative to $|0\rangle$; these become delocalized, or new ones are created, with higher orders of perturbation theory. But, locality is a strong constraint on physically allowed operators; $V$ preserves fermion parity not just globally but also “locally,” in a certain sense. On the other hand, $S_{\text{top}}$ must connect states which differ in site fermion parity at two widely separated points $i, k$. Such states cannot arise through the effects of a local and fermion parity preserving perturbation applied to an initial state with uniform parity throughout. Hence, $S_{\text{top}}$ should asymptotically stay zero away from the point $H_0$. We anticipate that this argument holds for the entire phase because of the strong constraint of locality on physically allowed fermion operators. $S_{\text{triv}}$ can also be argued to remain zero in the topological phase by a similar, dual argument.

II. NON-LOCAL ORDER AND QUANTUM GAS MICROSCOPY

Enabled by advances in single-site resolved imaging of optical lattices,\textsuperscript{26,27} non-local measurements in cold atom systems are now possible and were recently demonstrated\textsuperscript{28} for string order in bosonic Mott insulators\textsuperscript{24,34}. Similarly,

$$S_{\text{triv}} = \prod_{j \in \Omega} e^{i \pi n_j} \quad (6)$$

which yields a finite value in the trivial phase, is accessible with current experimental techniques.

We consider how one might measure OPs with more complex terminations. For instance,

$$S_{\text{top}} = (-i \chi_1) \prod_{j=2}^{N-1} e^{i \pi n_j} \chi_N \quad (7)$$

directly detects the topological phase by generically yielding a finite value. The difficulty with measuring non-local OPs such as $S_{\text{top}}$ is that they are off-diagonal in the site fermion basis imaged in experiments. We suggest a scheme for measuring a string OP such as $S_{\text{top}}$ on the interval $[1, N]$ in the bulk of a long Majorana chain. The idea is that by evolving the ground state in a controlled manner, such as with fermion tunneling, we may extract the additional information needed to reconstruct the string OP (Fig. 2).

For instance, let a Kitaev chain ground state be $| \psi \rangle = \sum_{i,j,k} \beta_{i,j,k} | n_i^1 \rangle | n_j^0 \rangle | n_k \rangle$. Here, $| n_i^1 \rangle$ is a site fermion configuration indexed by $i$ for the inner region sites 2 to $N - 1$, $| n_j^0 \rangle$ indexes states for the region outside $[1, N]$, while $| n_k \rangle$ is a configuration for the string end sites 1, $N$, with

\begin{align*}
\langle \prod_j e^{i \pi n(j)} \rangle 
\end{align*}

FIG. 2. Top figure shows single Majorana chain (sites are circles) and the order parameter Eq. 6 finite in the trivial phase, which can be measured with current cold atom techniques since it only involves fermion parity (highlighted green circles). Bottom shows a potential experimental scheme for measuring order parameters such as Eq. 7 with complex terminations. In the bulk (green), fermion parity is still measured, but additional measurements for the end sites labeled 1, N (blue circles) must be made (see text) to extract the string order parameter value.

$$\langle \{ | n_k \rangle \}_{k=1}^4 = \{ | 0 \rangle, a_1^\dagger | 0 \rangle, a_N^\dagger | 0 \rangle, a_N^\dagger a_1^\dagger | 0 \rangle \}$. The measured value is

$$\langle S_{\text{top}} \rangle = \sum_{ij} 2 P_i (\text{Re} \beta_{i,j+1} \bar{\beta}_{i,j} + \text{Re} \beta_{i,j+2} \bar{\beta}_{i,j+1}) \quad (8)$$

where $P_i$ is the parity of configuration $i$ for sites $[2, N-1]$. The additional information, beyond amplitudes $| \beta_{i,j,k} |$, needed in order to reconstruct the expectation value is certain relative phases, such as those in $\beta_{i,j+1} \bar{\beta}_{i,j}$ and $\beta_{i,j+2} \bar{\beta}_{i,j+1}$.

We imagine consistently starting the system in a fixed Majorana chain ground state $| \psi \rangle$. A tunneling Hamiltonian $H_T$ which for instance couples only sites 1, N is turned on rapidly, preserving the state. We may consider changing the experimental geometry to have the single chain folded into two in order to couple 1, N. After dynamic evolution with $H_T$, the site fermion occupations are measured at specified times. Accurate knowledge of the coupling Hamiltonian parameters and amplitudes $| \beta_{i,j,k} |$, determined from previous repeated measurements, may enable extraction of the necessary relative phases and reconstruction of the string OP value.

The practicality of the suggested scheme for current systems would need to be determined. A general challenge appears to be the number of measurements needed, as a ground state for N sites in the deepest regions of the topological phase consists of an exponential in N number of states in the site fermion basis, all with equal magnitude weights. Design of a protocol to enable extraction of the off-diagonal interference terms, if possible, would be interesting.
**Example: Two Identical Chains.** We consider the case of two identical Kitaev chains A, B each with parameters \((t_e, |\Delta|, \mu)\) and coupled with an interchain hopping \(t_L\). This high symmetry model is a special case of the general N chain system with \(\mathbb{Z}_N\) translational symmetry considered in the subsequent sections, but we emphasize it here because of its potential experimental relevance.

The phases of this system can be easily seen by switching to momentum \(k_y = 0, \pi\) in the transverse direction. The resulting Hamiltonian consists of two decoupled Kitaev models for the \(k_y = 0, \pi\) degrees of freedom (DOF) \(\{a_0(i)\} \cup \{a_\pi(i)\}\), with modified chemical potentials \(\mu_\pm = \mu \pm 2t_L\):

\[
H = H_{Kit,A} + H_{Kit,B} - 2t_L \sum_i (a_i^\dagger a_i + h.c.) \quad (9a)
\]

\[
= H_{Kit,0}(\mu_+) + H_{Kit,\pi}(\mu_-) \quad (9b)
\]

The phases of the system for \(|\Delta| \neq 0\) have two, one, or zero Majorana zero modes per edge as the interchain coupling \(t_L\) is increased (the boundaries are the same as those in Fig. 4). The phase with two Majorana zero modes per edge is protected by the exchange symmetry. To distinguish the phases we need only independently test whether the \(k_y = 0, \pi\) DOF are in the topological or trivial phases using Kitaev model string OPs. In regimes where only one of \(k_y = 0, \pi\) DOF are in the topological phase, we use string termination operators such as \(\chi_0/\chi_\pi \sim \chi_A \pm \chi_B\) or those built out of \(\bar{\chi}\) operators. When both \(k_y = 0, \pi\) DOF are in the topological phase, the terminating operators are for instance \(\chi_0/\chi_\pi \sim \chi_A \chi_B\).

In other words, two copies of the topological phase order parameter of the Kitaev model, one for each of the \(k_y = 0, \pi\) momentum DOF, detects the weakly-coupled regime of this two chain system, in which each end has two Majorana zero modes. This is equivalent to a product of topological string OPs for each chain:

\[
\langle \bar{\chi}_0(i)\chi_\pi(i) \prod_{j=1}^{k-1} e^{i\pi(n_0(j)+n_\pi(j))} \chi_0(k)\chi_\pi(k) \rangle \to \quad (10)
\]

\[
\langle \bar{\chi}_A(i)\chi_B(i) \prod_{j=1}^{k-1} e^{i\pi(n_A(j)+n_B(j))} \chi_A(k)\chi_B(k) \rangle
\]

Taking products of string order parameter works here because of the additional protecting symmetry.

Alternatively, we ask whether it is possible to devise an order parameter for the two chain system which involves only fermion parity but which detects a non-trivial phase, with Majorana zero modes. That this might be possible is suggested by the form of Eq. 10 which is terminated by bosonic operators such as \(\bar{\chi}_A\chi_B\). In fact, we will later give selection rules which the terminating operators of a non-local OP should obey in order to uniquely detect a symmetric phase among other symmetric phase (see Table 1). For the phase with two Majorana zero modes per edge, the terminating operator should be bosonic (even under parity) but odd under exchange. Operators with other transformation rules under the symmetries (fermion parity and translation) detect the other symmetric phases:

\[
\langle [n_A(i) - n_B(i)] \prod_{j=i+1}^{k-1} e^{i\pi(n_A(j)+n_B(j))} [n_A(k) - n_B(k)] \rangle \quad (11)
\]

has the proper transformation rules to detect the phase with two Majorana zero modes per edge. This non-local OP works, for instance, for a model of two identical chains with intrachain pairing and interchain diagonal hopping. While it vanishes for the special model Eqs. 9 because of its larger symmetry group, we expect that for models with no additional symmetries this order parameter will detect the special topological phase.

**III. PHASES WITH ADDED \(\mathbb{Z}_N\) TRANSLATION SYMMETRY**

**N Chain Systems.** Consider a system of N identical, interacting topological superconducting chains of spinless fermions with \(\mathbb{Z}_N\) translation symmetry transverse to the infinite chain length. The geometry is that of a cylinder with finite circumference N. We first seek to...
understand the symmetric fermionic phases, which cannot be distinguished from each other through local measurements. One way to identify the gapped quantum phases is to classify the corresponding bosonic phases using results from the group cohomology approach to classification. Alternatively, the fermionic phases can be identified directly by analyzing fermionic symmetry operators, as in \[^{10}\] We give a correspondence between the two descriptions specific to \(\mathbb{Z}_2 \times \mathbb{Z}_N\). We then illustrate with models for the phases.

**Bosonic Classification.** 1D gapped bosonic phases with local interactions are in correspondence with the unbroken subgroups \(G'\) of symmetry group \(G\) and their second cohomology group \(H^2(G', \mathbb{C})\). \[^{11,12}\] \[^{13}\] \[^{14}\] \[^{15}\] \[^{16}\] \[^{17}\] \[^{18}\] \[^{19}\] \[^{20}\] \[^{21}\] \[^{22}\] \[^{23}\] Therefore, of the solutions \(\Gamma\) for \(N\) odd. These gauge-invariant phases are quantized and hence, of the solutions \(U, U'\) with factor systems \(\omega, \omega'\) are equivalent if they differ by redefinitions of the representations, that is, if \(U'(g) = \beta(g)U(g)\) with \(\beta \in \mathbb{C}\). Certain complex phases associated with a projective representation \(\hat{\rho}\) are invariant under these gauge changes and are therefore characteristic of an equivalence class and of a quantum phase. These gauge-invariant quantities distinguish the symmetry-protected topological phases. The bosonic phases we obtain may be fully symmetric or involve symmetry breaking; as we will see, both classes can map to fully symmetric fermionic phases.

We take a winding path in the Jordan-Wigner mapping to transform the fermionic cylinder into a 1D infinite bosonic chain; each unit cell accounts for one cylindrical ring and contains \(N\) spin-1/2 degrees of freedom (DOF). The \(\mathbb{Z}_N\) fermion translation symmetry for the circumference, \(a_{i,j} \rightarrow a_{i,(j+1) \mod N}\), maps to a \(\mathbb{Z}_N\) symmetry internal to the unit cell, which we note is not translation of the spins.

The possible symmetric bosonic phases depends on the parity of \(N\), since \(H^2(\mathbb{Z}_2 \times \mathbb{Z}_N, \mathbb{C}) = \mathbb{Z}_2\) or \(\mathbb{Z}_1\) for \(N\) even or odd. To see this, consider a translation-invariant matrix product state \((\Gamma_j, \Lambda)\) with physical DOF per site indexed by \(j\). If the state is symmetric under a symmetry \(\Sigma(g)\) with \(g \in G\), the matrices transform as \(\Sigma_j \Sigma(g) \Sigma_j' = e^{i\theta} \Sigma_j U_1^{\dagger} \Gamma_j U_2^{\dagger}\). For our \(\mathbb{Z}_2 \times \mathbb{Z}_N\) symmetry generated by the bosonic versions of fermion parity \(P\) and translation \(T\), the overall phase of the generator projections \(U_P, U_T\) cannot be gauged such that \(U_P^N = U_T^N = 1\), but the phase in \(U_P U_T = e^{i\theta} U_T U_P\) cannot be eliminated and must satisfy \(e^{i2\theta} = e^{iN\theta} = 1\). Hence, of the solutions \(\phi = 0, \pi\), the latter is forbidden for \(N\) odd. These gauge-invariant phases are quantized and so are preserved under smooth, gap-preserving deformations to the wavefunctions. A product state can be represented by \(\Gamma_j\) and hence \(U_P, U_T\) all scalars, so \(\phi = 0\) describes the trivial phase. \(\phi = \pi\) characterizes a topologically non-trivial phase.

The relevant symmetry breaking bosonic phases are those identified with the proper cyclic subgroups generated by translation \(T\) and also (for \(N\) even) the product of fermion parity and translation \(PT\) when mapped to spins; we denote these by \(G' = (T)\) and \(G' = (PT)\) respectively. While parity symmetry \(P\) is broken in these bosonic phases, it must be restored for fermions; these two classes retain enough symmetry so that, as a consequence, all other broken bosonic symmetries in \(\mathbb{Z}_2 \times \mathbb{Z}_N\) are also restored for fermions. That is, the resulting fermionic phases are symmetric.

For a complementary description of the phases, we also apply the approach developed in \[^{22}\] for fermions directly. On a certain subspace, fermionic symmetry operators acquire an effective form \((P, T)\) consisting of two “fractional” pieces; again, it is the commutations of these pieces which identify the quantum phases. We elaborate on this in Appendix II for our case and mention the result here. Two angles \(\mu, \mu'\), defined such that if symmetry operators acquire effective forms \(\hat{P} \sim P, P_R, \hat{T} \sim T, T_R\) then \(P \hat{P}_R = e^{i\mu} P \hat{P}_R\) and \(T \hat{T}_R = e^{i\mu} T \hat{T}_R\), are sufficient to characterize the quantum phases.

The correspondence between the bosonic and fermionic classifications is given in Table I. In general, fractionalization of fermionic P into fermionic pieces means it is broken in bosonic variables\[^{22}\] This is not true for other fermionic symmetries (such as \(\mathbb{Z}_2\) translation) whose behavior in bosonic variables depends in part on those of parity (see Appendix II).

![Table I](image)

**Representative Models for Symmetric Fermonic Phases.** We give models for which the symmetry operators take simple effective forms, thereby allowing matching to the correct phase. The fermionic operators \(P\) and \(T\) are:

\[
P = \prod_{i,k_y} e^{i\pi n_{k_y}(i)} \quad T = \prod_{i,k_y} e^{i\Phi_{n_{k_y}}(i)}
\]

where \(i, k_y\) respectively index lattice sites along the cylinder length \(x\) and momentum around the circumference \(y\). \(n_{k_y}(i)\) measures the occupation of the mode \(n_{k_y}(i) = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{i k_y \phi_{a_{i,j}}} a_{i,j}\). A decomposition into Majorana operators \(\chi_{k_y}(i), \tilde{\chi}_{k_y}(i)\) is \(\tilde{a}_{k_y}(i) = \frac{1}{2} \left[ \chi_{k_y}(i) + i \tilde{\chi}_{k_y}(i) \right]\).
We can view the set of operators \( \{a_{k_y}(i)\}_{i} \) for fixed \( k_y \) as DOF for a single Majorana wire. To construct a model for Class 3 (symmetry breaking bosonic phase with \( G' = \langle T' \rangle \)), we consider fixing the ground state occupations of all the \( k_y \neq 0 \) DOF (i.e. put all such \( k_y \neq 0 \) chains into the trivial phase) so that \( T \) will act as a scalar in the ground state subspace. By treating the remaining \( k_y = 0 \) DOF as a topological Majorana wire, \( P \) splits into \( P_L, P_R \) components which are fermionic. Therefore

\[
H_3 = \sum_i i\bar{\chi}_0(i)\chi_0(i+1) + \sum_{i,k_y \neq 0} i\bar{\chi}_{k_y}(i)\chi_{k_y}(i) 
\]  

serves as a representative for this class. The symmetry effective forms on a subsystem spanning sites \( i \in [1, L] \) of a closed system are \( \bar{P} = i\bar{\chi}_{0}(1)\chi_{0}(L), \bar{T} = 1 \). On the cylinder, systems in Class 3 generically have a ground state degeneracy which can be determined from symmetry breaking for bosons, namely \( |G/G'| = 2 \) with \( G' = \langle T' \rangle \).

To construct a model for Class 4 (for even \( N \)), we simply switch the treatments of the \( k_y = 0, \pi \) DOF.

\[
H_4 = \sum_i i\bar{\chi}_\pi(i)\chi_\pi(i+1) + \sum_{i,k_y \neq \pi} i\bar{\chi}_{k_y}(i)\chi_{k_y}(i) 
\]

serves as a representative. Class 4 systems on the cylinder also have two-fold ground state degeneracy generically since \( |G/\langle PT \rangle| = 2 \).

We can construct a model for Class 2 (for even \( N \)) by placing both the \( k_y = 0 \) and \( k_y = \pi \) DOF into the topological phase and fixing the site occupations of the remaining \( k_y \neq 0, \pi \) DOF. A representative model is therefore

\[
H_2 = \sum_i (i\bar{\chi}_\pi(i)\chi_\pi(i+1) + i\bar{\chi}_{0}(i)\chi_{0}(i+1)) + \sum_{i,k_y \neq 0, \pi} i\bar{\chi}_{k_y}(i)\chi_{k_y}(i) 
\]  

For a subsystem spanning sites \( i \in [1, L] \) of a closed system, the effective forms on the subsystems are \( \bar{P} = i\bar{\chi}_{0}(1)\chi_{0}(L), \bar{T} = i\chi_\pi(1)\chi_\pi(L) \), so that \( \mu_\pi = (0, \pi) \). This particular model has four-fold ground state degeneracy, but it is physically plausible that Majorana zero modes can be gapped out in pairs on the edges without the system undergoing a topological transition. We expect that four is the minimal ground state degeneracy of this phase.

**Physical Models.** The previous models become increasingly non-local for large \( N \); we connect them to phases of a local non-interacting model. A nearest neighbor interchain hopping \( t_\perp \) and, for \( N > 2 \), a nearest neighbor interchain pairing \( \Delta_\perp \) are allowed by translational invariance. We consider the simple lattice \( p + ip \) topological superconductor (TSC) studied \( 35 \). The Hamiltonian is \( N > 2 \), considering \( N \) even:

\[
H = \sum_{j=1}^{N} (-t_{a_j}^\dagger a_{j+1,j} + |\Delta|a_{j+1,j}a_{j+1,j} + h.c.) 
\]  

\[ -\mu(n_{i,j} - \frac{1}{2}) + (\Delta_\perp a_{i,j}a_{i,j+1} - t_{a_j}^\dagger a_{j+1,j} + h.c.) \]

with \( i,j \) indexing sites along the cylinder length and circumference, respectively. We take fixed parameters \( |Im(\Delta_\perp)| > 0, Re(\Delta_\perp) = 0 \) and \( |\Delta| > 0 \). In this case, there are transitions between phases including quasi-1D versions of the 2D weak and strong TSCs as \( \frac{\mu}{\mu'}, \frac{t}{\mu} \) are varied.\( ^{35} \)

The phases of the system can be seen by rewriting the Hamiltonian \( 10 \) as

\[
H = H_{K,0}(\mu_+) + H_{K,\pi}(\mu_-) + H'_{K,\pi} \]

where \( H_{K,0}, H_{K,\pi} \) are the Kitaev Hamiltonians \( 11 \) with \( \chi_0, \chi_\pi \) Majorana DOF, respectively, and, as before, \( \mu_\pi = \mu + 2t_\perp \). To analyze the remaining piece \( H'_{K,\pi} \) containing all \( k_y \neq 0, \pi \) DOF, we transform the Majorana basis by recombining the four Majoranas \( \chi \), \( \bar{\chi} \) into the Majorana zero modes of each chain: \( \chi_0, \chi_\pi \). There is an “interchain” coupling in this basis which is proportional to \( Im(\Delta_\perp) \) and which gaps out the Majorana zero modes of each chain:

\[
H'_{K,\pi} = \sum_{k_0 \in (0, \pi)} H_{K,\pi}(\eta_{k_0}, \bar{\eta}_{k_0})(\mu_{k_0}) + \sum_{(\delta_{k_0}, \bar{\delta}_{k_0})}(\mu_{k_0}) 
\]  

\[ - \sum_{k_0 \in (0, \pi), i} iIm(\Delta_\perp) \sin(k_0)(\eta_{k_0}\delta_{k_0} + \bar{\eta}_{k_0}\bar{\delta}_{k_0})(i) \]

Consequently, the phases of the system are determined by whether the independent \( k_y = 0, \pi \) DOF are topologically non-trivial. The \( k_y = 0 \) DOF are in the topological phase when the associated chemical potential is sufficiently weak \( |t_\perp + \frac{\mu}{2}| < |t| \). Here the 1D \( \mathbb{Z}_2 \) invariant \( \nu_{k_y} = 0 \).\text{35} Likewise, the \( k_y = \pi \) DOF are topological when \( \nu_{k_y} = \pi \) for \( |t_\perp - \frac{\mu}{2}| < |t| \). Hence, for weak \( |\frac{\mu}{\mu'}| \) and \( |\frac{t}{\mu'}| > \frac{1}{2} \) the system has two Majorana zero modes per edge, at \( k_y = 0, \pi \). This phase scales to a 2D weak TSC as \( N \to \infty \) in Eq. \( 10 \), as in this regime the 2D \( \mathbb{Z} \) invariant (Chern number) is \( \nu = 0 \). For intermediate values \( \frac{|t|}{|\mu|} \sim \frac{|t|}{|\mu|} \) and weak chemical potential the system has a single Majorana zero mode per edge (either at \( k_y = 0 \) or \( \pi \)) and will scale to a 2D strong TSC as \( N \to \infty \) since the 2D \( \mathbb{Z} \) invariant \( |\nu| = 1 \). When interchain hopping \( t_\perp \) dominates over intrachain hopping \( |\frac{t_\perp}{\mu'}| > |\frac{t}{\mu'}| + \frac{1}{2} \), the system is a weak TSC in the \( x \)-direction. Fig. \( 11 \) gives a phase diagram.
Since the three sets of DOF \( k_y = 0, k_y = \pi \), and \( k_y \neq 0, \pi \) decouple, we can tune each into topological or trivial phases independently while maintaining translational invariance. For instance, if the \( k_y = 0 \) DOF form a non-trivial state, treating this as a single chain with no other symmetries, we can find a path connecting to the model \( \sum_i i \hat{\chi}_0(i) \chi_0(i+1) \) which preserves translation symmetry since it only involves \( k_y = 0 \) operators. Hence, the quasi-1D phases of Eq. \( \ref{eq:1} \) namely the 2D weak TSC associated with \( y \)-direction layering and the two strong TSCs with \( (\nu_{k_y=0, \nu_{k_y=\pi}}) = (1, 0) \) or \( (0, 1) \), would fall into Classes 2, 3, and 4, respectively, of our classification. It appears that for free fermions our classification identifies the 1D \( \mathbb{Z}_2 \) invariants \( \nu_{k_y=0} \) and \( \nu_{k_y=\pi} \) and consequently \( \nu \mod 2 \) since \( \nu_{k_y=0} + \nu_{k_y=\pi} = \nu \mod 2 \).

The weak TSC with \( \nu_{k_y} \neq 0 \) for \( k_x = 0 \) or \( \pi \), which is associated with layering in the \( x \)-direction, appears as a trivial phase. This phase results with strong \( t_\perp \), but in the momentum \( k_y \) basis this coupling is an effective chemical potential, \( \mu_\parallel = \mu \pm 2t_\perp \), which favors on-site pairing of \( y \)-momentum Majoranas, driving the \( k_y = 0, \pi \) Majorana chains away from non-trivial pairing. It is natural that our classification is unable to detect the topological index associated with \( x \)-translation symmetry along the cylinder length, as only \( y \)-translation has been included.

### IV. NON-LOCAL ORDER PARAMETERS FOR THE SYMMETRIC PHASES

We construct non-local order parameters to distinguish the \( \mathbb{Z}_2 \times \mathbb{Z}_N \) protected symmetric phases of the previous section from each other; we consider even \( N \), which encompass the results for odd \( N \). The construction is based on general distinctions made apparent by the classifications; working in the 1D thermodynamic limit, we use the bosonic point of view for illustration.

The bosonic description has revealed hidden structure (breaking of certain symmetry operators) which can be used, along with recently derived selection rules for bosonic non-local OPs, to identify bosonic operators which distinguish the four phases and then map back to fermions.

In the infinite bosonic chain, there is a natural unit cell which makes the \( \mathbb{Z}_N \) translation symmetry on-site. To distinguish symmetric bosonic phases, we chose a symmetry to apply over many unit cells of the chain (a string), and we terminate the domain with operators obeying proper symmetry transformation rules. Mapped back to fermions, the non-local OP consists of a cylindrical brane-type region in the bulk over which a symmetry is applied, and terminating operators reside on the domain edges. We model this general form by writing the OP as the long-distance limit of \( \langle O_L \prod_{j=1}^L \Sigma_j O_R \rangle \). \( O_L/O_R \) are possibly different operators acting near the left, right bosonic string (fermionic brane) edges and \( \Sigma_j \) is a symmetry operation on a bosonic unit cell (fermionic cylin-

| Phase (Bosonic variables) | P Trans. | T Trans. | Example |
|---------------------------|----------|----------|---------|
| 1. Trivial symmetric \( G' = G \) | Even | Even | \( \langle S_1 \rangle \neq 0 \) |
| 2. Non-trivial symmetric \( G' = G \) | Even | Odd | \( \langle S_2 \rangle \neq 0 \) |
| 3. Symmetry breaking \( G' = \langle T \rangle \) | Odd | Even | \( \langle S_3 \rangle \neq 0 \) |
| 4. Symmetry breaking \( G' = \langle PT \rangle \) | Odd | Odd | \( \langle S_4 \rangle \neq 0 \) |

\[ S_1 \equiv \prod_{k_y,i=1}^{i=L} e^{i\pi \nu_{k_y}(i)} \]

The expectation \( \langle S_1 \rangle \), taken with any choice of ground state, vanishes in Classes 3, 4 which have \( P \) broken in the bosonic variables (Appendix I). For Classes 1 and 2, \( P \) remains a symmetry for the bosons. Typically, applying a symmetry over a large domain of a symmetric state would yield a finite answer.

In fact, bosonic selection rules can modify this conclusion. The operators \( O_L/O_R \) which terminate the bosonic string (fermionic brane) can be chosen to transform under symmetries in such a way as to select a quantum phase. The distinction between the two phases Classes 1 and 2 in the bosonic description are certain physical angles \( \phi = 0, \pi \). \( O \) must be even under parity (because it is the symmetry used in the bulk of the order parameter) but transform as \( e^{i\phi} \) under translation in order to be finite in the quantum phase labeled by
The OP will vanish in the other symmetric bosonic phase. Mapped back to the fermionic system, the brane termination operators should be bosonic but should be even or odd under translation so that the OP is finite in Classes 1 or 2, respectively.

$S_1$ is, for instance, an order parameter which is finite in Class 1 but vanishes in Class 2 since translation invariant operators terminate its bulk. To construct a candidate with reversed behavior, we can choose operators such as $O(i) = \chi_0(i)\chi_x(i)$ or $i\chi_0(i)\bar{\chi}_x(i)$, which are parity invariant but translation odd, to terminate the fermionic brane. Hence, candidate order parameters which uniquely give a finite value for Class 2 include

$$S_2 \equiv \chi_0(1)\bar{\chi}_x(1)\prod_{k_y,i=2}^{L-1} e^{i\pi n_{k_y}(i)}\chi_0(L)\bar{\chi}_x(L)$$

(20)

and a similarly constructed $S'_2$ with fermionic ends $\chi_0(1)\bar{\chi}_x(1)$ and $\chi_0(L)\bar{\chi}_x(L)$.

Finally, we utilize the fact that certain symmetries are broken in the bosonic variables to construct OPs that are finite in a symmetry breaking bosonic phase but vanish elsewhere. We use two-point functions in the bosonic variables $(U_i, V_j)$ with $|i-j| \rightarrow \infty$: if $U$, $V$ are odd under $P$ but even under $T$, the result is finite in Class 3 but vanishes in the other symmetric fermionic phases. Likewise, operators odd under $P$ but even under $PT$ yield OPs which can detect Class 4. Mapped back to fermions, a few such candidates are:

$$S_3 \equiv -i\chi_0(1)\prod_{k_y,i=2}^{L-1} e^{i\pi n_{k_y}(i)}\chi_0(L)$$

(21)

$$S_4 \equiv -i\bar{\chi}_x(1)\prod_{k_y,i=2}^{L-1} e^{i\pi n_{k_y}(i)}\bar{\chi}_x(L)$$

(22)

or similar constructions defined as $S'_3$, $S'_4$, with fermionic ends $-i\chi_0(1), \chi_0(L)$ and $-i\bar{\chi}_x(1), \bar{\chi}_x(L)$, respectively.

Application. We apply these order parameters to the cylinder $p+ip$ model. For instance, let us work in the $t = |\Delta|$ limit of the strong TSC with only the $k_y = 0$ DOF in the topological phase. Evaluations decouple into $k_y = 0, \pi$, and $k_0 \in (0, \pi)$ contributions: $\langle S_3 \rangle = \langle S_{\text{stop}} \rangle_{k_y=0} = \langle S_{\text{triv}} \rangle_{k_y=\pi} \langle S_{\text{triv}} \rangle_{k_0 \in (0, \pi)} \neq 0$. This OP would vanish in the other phases because the behavior of the $k_y = 0, \pi$ DOF (i.e. viewed as topological or trivial Majorana chains) would be different. Regions where the order parameters take finite values are shown in Fig. 4.

The fermion parity operator used in the bulk of the OPs given in the previous section can be reduced to parity for just the $k_y = 0, \pi$ DOF, since the remaining momenta DOF always remain trivial in the $p+ip$ model. This reduction may not be applicable in general, as when interactions are added.

We emphasize that this model, as in the case of the single Kitaev Majorana chain, has additional symmetries beyond $\mathbb{Z}_2 \times \mathbb{Z}_N$ which puts constraints on the construction of the OP. For instance, a choice of $O_{L/R} \propto (\chi, \bar{\chi})$ is different from $(\bar{\chi}, \chi)$, as we saw in the single chain. However, as discussed previously, for models which only have $\mathbb{Z}_2 \times \mathbb{Z}_N$ symmetry, only these symmetries need to be accounted for; therefore, order parameters constructed using the general principles described work generically. For instance, $S_3, S'_3$ and similar order parameters are all suitable choices to distinguish Class 3 from the other symmetric fermionic phases.

Ising Condensate for Non-trivial Symmetric Bosonic Phase. We give a simple bosonic description of the non-trivial symmetric bosonic phase (Class 2) for the case $N=2$, i.e. $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry.

Let $\sigma, \tau$ be two Ising variables with the $\mathbb{Z}_2$ symmetries $\prod_j \sigma_x(i,j), \prod_j \tau_x(i,j)$. Consider beginning in the partially ordered phase with $\langle \sigma_x(i) \rangle \neq 0$ but $\langle \tau_x(i) \rangle = 0$. Here, the domain walls of $\sigma$, created at site $j$ by $\prod_{i<j} \sigma_x(i)$, and the $\mathbb{Z}_2$ quanta of $\tau$, created at site $j$ by the spin flip $\tau_x(j)$, are both gapped. Condensing the former realizes the completely disordered phase of $\sigma$ and $\tau$, while condensing the latter results in the completely ordered phase. Instead, consider condensing a bound state of the two defects, the object $\rho(j) = \prod_{i<j} \sigma_x(i)\tau_x(j)$. Since $\sigma$ domain walls have been condensed, the Ising symmetry of $\sigma$ is restored. Moreover, though we condense $\tau$ quanta, this does not order $\tau$ since the order is non-local. Hence, both $\mathbb{Z}_2$ symmetries are preserved. The string order parameter for this phase is the correlation of the condensed objects:

$$\langle \rho(i)\rho(j) \rangle = \langle \tau_x(i) \prod_{i \leq k < j} \sigma_x(k)\tau_x(j) \rangle$$

(23)

The order parameter is of the general form discussed previously. It consists of applying one symmetry over
the bulk ($\prod_k \sigma_z(k)$) and terminating with operators ($\tau_z$) which are even under this symmetry and odd under the other ($\prod_k \tau_z(k)$).

This phase is topological with protected edge states, which we can motivate with the following space-time picture. In the path integral representation for this system with a spatial boundary, the world lines of the condensate sometimes intersect the boundary; in these cases, since the condensate carries a spin flip $\tau_z$, the spins at the edges fluctuate in time.

Based on this description, we write a Hamiltonian which realizes this phase. Consider starting at the critical point of a pair of a decoupled 1D Ising models:

$$H_0 = -\sum_i (\sigma_z(i) + \tau_z(i) + \sigma_z(i)\sigma_z(i+1) + \tau_z(i)\tau_z(i+1))$$

and adding correlations for the charge and domain wall bound state in order to induce condensation of the bound state:

$$H_1 = -\sum_i (\sigma_z(i)\sigma_z(i+1)\tau_z(i+1) + \tau_z(i)\tau_z(i+1)\sigma_z(i))$$

The Hamiltonian $H(\lambda) = H_0 + \lambda H_1$ realizes the non-trivial topological phase for $\lambda > 1$ with non-local order parameter $\rho$; this can be seen by making a dual transformation on one of the $Z_2$ variables and mapping onto the quantum Ashkin-Teller model. Moreover, $H_1$ itself is exactly solvable and its ground state is a so-called cluster state. There is a four-fold degeneracy on a chain with sites 1 to L. On each edge, we can construct a spin-1/2 algebra with local operators: for instance, $\sigma_z(1)\tau_x(1)$, $\sigma_x(1)\tau_z(2)$, and $\tau_y(1)\tau_x(1)\tau_z(2)$ operate on the left edge while $\sigma_z(L)\tau_z(L)$, $\sigma_z(L-1)\tau_\tau_z(L)$, and $\tau_y(L)\tau_z(L)\tau_y(L)$ operate on the right edge. Since we can map within the ground state manifold via edge and not bulk operators, the distinction between the degenerate states is topological rather than associated with symmetry breaking.

V. CONCLUSION

We have given string or brane-type non-local order parameters to distinguish the phases of quasi-1D topological superconductors of spinless fermions with interactions and no symmetries. These order parameters measure fermion parity in their bulk and are terminated by fermionic or bosonic operators at their edges; we illustrated how they probe the different natures of the Majorana pairings in the topological and trivial phases. They would be an alternative way to detect Majorana fermions via quantum gas microscope measurements in cold atom systems. We also gave an example of an order parameter for two chains which only involves fermion parity and hence would be measurable using current experimental techniques but which detects a topological phase.

The addition of translation to the system as a protecting symmetry distinguished among certain interesting 2D phases in the quasi-1D limit. We elaborated on how two 1D $Z_2$ invariants are distinguished by the classification in the case of free fermions; in particular, this allows us to distinguish the 2D Chern number mod 2, for instance the $p+i\rho$ strong topological superconductor and the weak topological superconductor.

We gave simple general rules which the terminating operators of a non-local order parameter should satisfy in order to uniquely distinguish among the fermionic symmetric phases (four for N even and two for N odd), even in the case of interactions. We sketched a direct fermionic derivation of these rules, which may be extendable to other symmetry classes. We illustrated the construction by giving a non-local order parameter for the $p+i\rho$ topological superconductor which distinguishes it from the weak topological superconductor or the trivial phase in the quasi-1D limit and which is robust to interactions. Attempts at extending string to brane order for coupled chains have been discussed in other contexts; here, we note that taking products of single chain string order parameters can work because of the additional protecting $Z_N$ symmetry.

In summary, we have given uniquely identifying non-local order parameters for the symmetric phases of finitely many coupled topological superconducting spinless fermion chains under interactions. We considered both the case of (i) no protecting symmetries and (ii) transverse translation symmetry. The extension of our ideas to incorporate time reversal symmetry, as well as extraction of useful 2D information, would be interesting.

Acknowledgements. We thank E. Altman, X. Chen, E. Demler, L. Fidkowski, and A. Turner for helpful conversations and L. Fidkowski for feedback on the manuscript. Financial support from NSF GRFP under Grant No. DGE 1106400 (Y.B.) and the Army Research Office with funding from the DARPA Optical Lattice Emulator program and NSF-DMR 0645691 (A.V.).

APPENDIX

I. Applying a Broken Symmetry on a Domain.

Consider a quasi-1D spin system with an infinite dimension indexed by $j$ and which has a discrete broken symmetry operator $u = \prod_j \psi^\dagger_{-\infty} \psi_j$. We argue that any state $|\psi_0\rangle$ in the ground state manifold obeys

$$\lim_{N\to\infty} \langle\psi_0| \prod_{j=-N}^N u_j |\psi_0\rangle \to 0,$$

with a special ordering of the limits. It applies even when mapped to fermions because it considers arbitrary ground state choices.

Let $\{ |\eta_k\rangle \}_{i=1}^M$ be the broken symmetry states which are mapped to each other under $u$. We explain that

$$\lim_{N\to\infty} \langle\eta_k| \prod_{j=-N}^N u_j |\eta_k\rangle \to 0$$

for any $i, k$. If $i = k$, $u$ creates a finite-sized domain which is orthogonal to the original state as the domain size increases $N \to \infty$. For instance, for the quantum Ising model with a $Z_2$
broken symmetry, $\lim_{N \to \infty} \langle \uparrow, \downarrow | \prod_{j=-N}^{N} \sigma_j^z | \uparrow, \downarrow \rangle \to 0$, where $| \uparrow \rangle, | \downarrow \rangle$ denote the broken symmetry states in the thermodynamic limit. Off-diagonal matrix elements $i \neq k$ also vanish due to the order of our limiting procedures; since the thermodynamic limit precedes $N \to \infty$ there is always an infinite region outside the domain $[-N,N]$ where the broken symmetry states are orthogonal. Practically, this means that the system size must be much larger than the domain over which the broken symmetry is applied in order to yield an asymptotically vanishing value. We expect that our description can be formalized with matrix-product states $\{ A_i^a \}_{a=1}^d$ associated with $| \eta_i \rangle$ (d is the local physical dimension) by considering the eigenproblem of the transfer matrices $E_{ij} = \sum_{a=1}^{d} A_i^a \otimes (A_j^a)^*$, $1 \leq i, j \leq M$, which govern the behavior of state overlaps.

**II. Fermionic Classification.** We follow the approach developed in [10] for 1D fermionic and bosonic systems. We consider the system $\Omega = \Omega_S \cup \Omega_E$ with periodic boundary conditions and a unique gapped ground state. Let an observable be $O$. Consider the effective action $\mathcal{O}$ of this operator in the space spanned by the low entanglement (EE) Schmidt states of a subsystem $\Omega_S$ obtained by a partition of the ground state into $\Omega_S$ and the environment $\Omega_E$. Ref. [10] observed that the action reduces to that of two operators $O_L, O_R$ acting locally near the left and right edges, respectively, of $\Omega_S$, i.e., $O \sim O_L O_R$. That is, in this subspace spanned by low EE states, states are distinguished by physics near their edges (as observables have “fractionalized” into two spatially separated pieces) but behave similarly in their bulk. Symmetry-protected phases are distinguished by the commutation relations obeyed by the edge operators.

Our two $Z_2 \times Z_N$ commuting symmetry generators are parity and translation $P, T$. They fractionalize as $P \sim P_L P_R$ and $T \sim T_L T_R$. We fix $P^2 = P_L^2 = P_R^2 = 1$ and $T^N = T_L^N = T_R^N = 1$. (The proportionality constants depend on the following relations). Define angles $\mu, \mu'$ with $P_L P_R = e^{i\mu} T_L P_L T_R = e^{i\mu'} T_R T_L T_R$ which are 0, mod 2$\pi$ since fractional pieces can be fermionic or bosonic.

We claim that $\mu, \mu'$, along with an additional assumption that $P_T L = e^{i\mu'} T_L P_L$ are sufficient to distinguish the quantum phases, since the other commutations follow from these. The complete operator $P$ determines whether operators such as $T_L$ are bosonic or fermionic (value of $\mu'$ for $T_L$) and it is natural to assume that its effective form does also. Parity is in this way a more fundamental operator for fermionic systems relative to other symmetries. Other commutation relations follow, such as $P_L T = T L e^{i\mu}$.

An equation such as $P_L T = T L P_L e^{i\mu'}$ imposes a constraint since $P_L^2 = T^N = 1$; namely, $\mu' = \pi$ is not allowed if $N$ is odd. Hence, we recover the same symmetric fermionic phases as in the bosonic group cohomology classification: $(Z_2)^2 \times N$ for even and $Z_2 \times N$ odd. We additionally have a direct fermionic description of the phases based on effective forms. Finally, to establish a correspondence between the bosonic and fermionic descriptions, we should understand when the Jordan-Wigner mapped versions of the fermionic symmetry operators are broken or unbroken in the bosonic variables. This leads us to find that parity is broken when $\mu = \pi$, while translation is broken when $\mu' = \pi$ and $\mu = \pi$; this is summarized in Table [II].

**III. Sketch of Fermionic Selection Rules.** We sketch a proof that the terminating operators should satisfy certain transformation (selection) rules in order for the non-local order parameter to potentially remain finite in one symmetric fermionic phase and vanish in the others; the result is Eqs. [30]. We will evaluate the long-distance limit of the string or brane OP $\langle O_L \prod_{j \in \Omega_S} \Sigma_j \rangle O_R$, with $O_L, O_R$ local terminating operators and $\Sigma_j$ an on-site symmetry. The asymptotic form of a non-local order parameter in a symmetric phase is really a two-point function of certain operators because symmetries reduce to acting on the edges of the domain over which they are applied. We use the effective forms for fermionic symmetries from the fermionic classification; though they are state dependent, we only rely on properties of the phases.

We consider again a closed system $\Omega$ partitioned into a subsystem $\Omega_S$ over which the symmetry $\Sigma$ acts and an environment $\Omega_E$, on whose edges $O_L, O_R$ act. The ground state has Schmidt decomposition $| \psi \rangle = \sum_a e^{-E_a} | \phi_a \rangle | \eta_a \rangle$ where $\phi_a, \eta_a$ are for $\Omega_S, \Omega_E$, respectively. We specialize to the case of interest where fermion parity $P_{135}$ is applied in the bulk. The idea of [10] is that $\langle \phi_a | P_{135} \phi_{a'} \rangle = \langle \phi_a | P_{135,L} P_{135,R} \phi_{a'} \rangle$ (with effective forms $P_{135,L}, P_{135,R}$ on $\Omega_S$) for states with low entanglement energy (EE), so that $a, a' < \chi$ with $\chi$ a cutoff. The forms $P_{135,L}, P_{135,R}$ are localized to a distance $l$ near the edges of $\Omega_S$ which increases with $\chi$. While the replacement by effective forms is approximate, it is good because states with high EE contribute less to evaluations of observables. Hence:

$$\langle \psi | O_L P_{135} O_R | \psi \rangle \approx \sum_{a,a' < \chi} e^{-E_a - E_{a'}} \langle \phi_a | O_L P_{135,L} P_{135,R} | \phi_{a'} \rangle \langle \eta_a | O_L O_R | \eta_{a'} \rangle$$

$$= \sum_{a,a' < \chi} e^{-E_a - E_{a'}} \langle \phi_a | O_L P_{135,L} P_{135,R} O_R | \phi_{a'} \rangle \langle \eta_a | O_L O_R | \eta_{a'} \rangle$$

$$\equiv \langle \tilde{\psi} | O_L P_{135,L} P_{135,R} O_R | \tilde{\psi} \rangle$$

(26)

where $| \tilde{\psi} \rangle = \sum_{a,a' < \chi} e^{-E_a} | \phi_a \rangle | \eta_{a'} \rangle$ is a good approximation to ground state $\psi$. We first take the thermodynamic limit of the closed system and then $\Omega_S$ so that the evaluations at the left and right boundaries of $\Omega_S$ near $\Omega_E$ decouple; the non-local order parameter reduces to an evaluation of local operators:

$$\langle \psi | O_L P_{135} O_R | \psi \rangle \approx \langle \tilde{\psi} | O_L P_{135,L} | \tilde{\psi} \rangle \langle \tilde{\psi} | P_{135,R} O_R | \tilde{\psi} \rangle$$

(27)

We then take the limit $\chi, l \to \infty$, so $P_{135,L}, P_{135,R}$ penetrate further into the bulk of the (infinite) subsystem $S$,
Consider the transformation properties of just one edge evaluation, for instance \( \langle \psi | O_L P_{0s,L} | \psi \rangle \). \( P_{0s,L} \) has known transformation rules under the symmetries, characteristic of the quantum phase; how must \( O_L \) transform in order to force the expression to vanish, or possibly remain finite? \( | \psi \rangle \) is approximately an eigenstate of the effective forms of the total symmetries \( P_{0}, T_{0} \), becoming exact in the above limits. Consider introducing translation, for instance:

\[
\langle \psi | O_L P_{0s,L} | \psi \rangle = \langle \psi | T_{0}^{|} O_L P_{0s,L} T_{0} | \psi \rangle
\]  

(28)

\[
T_{0}^{|} O_L P_{0s,L} T_{0} = (T_{0}^{|} O_L T_{0s}) (T_{0}^{|} P_{0s,L} T_{0s})
\]  

(29)

(Note that \( T_{0s}, P_{0s} \) are bosonic). From Appendix II, we have \( T_{0s}^{|} P_{0s,L} T_{0s} = e^{i\mu} P_{0s,L} \). In order to keep the expression invariant and hence possibly finite, we need \( T_{0s}^{|} O_L T_{0s} = e^{-i\mu} O_L \). Applying the same argument with parity symmetry and using \( P_{0s}^{|} P_{0s,L} P_{0s} = e^{i\mu} P_{0s,L} \) implies \( P_{0s}^{|} O_L P_{0s} = e^{-i\mu} O_L \) is needed also. When the end operator satisfies transformation laws different from the one governed by the quantum phase \((\mu, \mu')\) of the system, the local evaluation \( \langle \psi | O_L P_{0s,L} | \psi \rangle \) will vanish asymptotically. In summary, we need:

\[
P_{0s,L} = e^{i\mu} O_L
\]

(30a)

\[
T_{0s,L} O_L T = e^{-i\mu} O_L
\]

(30b)

so that the non-local order parameter vanishes in the fermionic symmetric phases characterized by a set of angles different from \((\mu, \mu')\). These selection rules support the conclusions reached using bosonic selection rules and local order parameters for bosonic symmetry breaking. For instance, a non-local order parameter for Class 2 \((\mu, \mu') = (0, \pi)\) should have \( O_L, O_R \) chosen to be even under fermion parity and odd under translation, as mentioned also in the main text.

1. J. Alicea, Rep. Prog. Phys. 75 (2012).
2. V. Mourik, K. Zuo, S. Frolov, S. Plissard, E. Bakkers, and L. Kouwenhoven, Science 336 (2012).
3. A. Kitaev, Phys.-Usp. 44 (2001).
4. N. Read and D. Green, Phys. Rev. B 61, 10267 (2000).
5. X.-L. Qi, T. L. Hughes, S. Raghu, and S.-C. Zhang, Phys. Rev. Lett. 102, 187001 (2009).
6. A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, Phys. Rev. B 78, 195125 (2008).
7. A. Kitaev, in Advances in Theoretical Physics (Proceedings of the L.D. Landau Memorial Conference, 2008), arXiv:0901.2686.
8. L. Fidkowski and A. Kitaev, Phys. Rev. B 81, 134509 (2010).
9. L. Fidkowski and A. Kitaev, Phys. Rev. B 83, 075103 (2011).
10. A. M. Turner, F. Pollmann, and E. Berg, Phys. Rev. B 83, 075102 (2011).
11. X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 84, 233528 (2011).
12. X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, arXiv:1106.4772.
13. N. Schuch, D. Perez-Garcia, and I. Cirac, Phys. Rev. B 84, 165319 (2011).
14. Z.-C. Gu and X.-G. Wen, arXiv:1201.2648.
15. Y.-M. Lu and A. Vishwanath, Phys. Rev. B 86, 125119 (2012).
16. A. Vishwanath and T. Senthil, Phys. Rev. X 3, 011016 (2013).
17. J. Haegeman, D. Perez-Garcia, I. Cirac, and N. Schuch, Phys. Rev. Lett. 109, 050402 (2012).
18. F. Pollmann and A. M. Turner, Phys. Rev. B 86, 125441 (2012).
19. M. den Nijs and K. Rommelse, Phys. Rev. B 40, 4709 (1989).
20. T. Kennedy and H. Tasaki, Phys. Rev. B 45, 304 (1992).
21. M. Oshikawa, J. Phys.: Condens. Matter 4 (1992).
22. D. Perez-Garcia, M. M. Wolf, M. Sanz, F. Verstraete, and J. I. Cirac, Phys. Rev. Lett. 100, 167202 (2008).
23. F. Anfuso and A. Rosch, Phys. Rev. B 76, 085124 (2007).
24. E. Berg, E. G. Dalla Torre, T. Giamarchi, and E. Altman, Phys. Rev. B 77, 245119 (2008).
25. F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. B 85, 075125 (2012).
26. W. Bakr, J. I. Gillen, A. Peng, S. Fölling, and M. Greiner, Nature 462, 74 (2009).
27. W. Bakr, A. Peng, M. Tai, R. Ma, J. Simon, J. Gillen, S. Fölling, L. Pollet, and M. Greiner, Science 329, 547 (2010).
28. M. Endres, M. Cheneau, T. Fukuhara, C. Weitenberg, P. Schauss, C. Gross, L. Mazza, M. Banuls, L. Pollet, I. Bloch, et al., Science (2011).
29. E. Alba, X. Fernandez-Gonzalvo, J. Mur-Petit, J. K. Pachos, and J. J. Garcia-Ripoll, Phys. Rev. Lett. 107, 235301 (2011).
30. E. Zhao, N. Bray-Ali, C. J. Williams, I. B. Spielman, and I. I. Satija, Phys. Rev. A 84, 063629 (2011).
31. N. Goldman, E. Anisimovas, F. Gérber, P. Öhberg, I. Spielman, and G. Juzeliūnis, Phys. Rev. A 84, 063629 (2011).
32. H. M. Price and N. R. Cooper, Phys. Rev. A 85, 033620 (2012).
33. M. Atala, M. Aidelsburger, J. T. Barreiro, D. Abanin, T. Kitagawa, E. Demler, and I. Bloch, arXiv:1212.0572.
34. E. G. Dalla Torre, E. Berg, and E. Altman, Phys. Rev. Lett. 97, 260401 (2006).
35. D. Asahi and N. Nagaosa, Phys. Rev. B 86, 100504 (2012).
36. M. Kohmoto, M. den Nijs, and L. P. Kadanoff, Phys. Rev. B 24, 5229 (1981).
37. M. Yamanaka and M. Kohmoto, Phys. Rev. B 52, 1138 (1995).
38. W. Son, L. Amico, and V. Vedral, Quantum Information Processing 11, 1961 (2012).