ON MULTITYPE BRANCHING PROCESSES WITH LOGISTIC
COMPETITION

MARÍA CLARA FITTIPALDI, SANDRA PALAU

Abstract. Motivated by the stochastic Lotka-Volterra model, we introduce continuous-time
discrete-state interacting multitype branching processes (both through intratype and inter-
type competition or cooperation). We show that these processes can be obtained as the sum
of a multidimensional random walk with a Lamperti-type change proportional to the popu-
lation size; and a multidimensional Poisson process with a time-change proportional to the
pairwise interactions. Then, we define the analogous continuous-state process as the unique
strong solution of a multidimensional stochastic differential equation. Finally, we prove that
a large population scaling limits of the discrete-state process correspond to its continuous
counterpart. In addition, we show that the continuous-state model can be constructed as a
generalized Lamperti-type transformation of multidimensional Lévy processes.

Keywords: multitype branching process; Lévy processes; time-change equation; Stochastic
differential equation, scaling limits; Lamperti representation.
MSC2020 subject classifications: 60F17; 60G51; 60H20; 60J25; 60J80; 92D25.

1. Introduction

In this article, we define a continuous-time and discrete-state interacting multitype branch-
ing process, where the interaction is intratype or intertype and corresponds to competition or
cooperation. We prove that it can be obtained as the sum of two time changed processes; a
multidimensional random walk with a Lamperti-type change proportional to the population
size and a multidimensional Poisson process with a time change proportional to the pairwise
interactions. The Markovian population dynamics associated with this process is the following

Model 1.1. Discrete-space interacting multitype branching process (DIMBP) Consider a population with d different types of individuals. When there are \( u_1, \ldots, u_d \) individuals
of types \( 1, \ldots, d \), each individual of type \( i \in \{1, \ldots, d\} \) dies at rate \( \lambda_i \) and leaves behind
\( v_1, \ldots, v_d \) offspring of types \( 1, \ldots, d \) with probability \( \mu_i(v_1, \ldots, v_d) \), independently of others.
Additionally, each individual of type \( i \) chooses at rate \( |c^{i,j}| \), an individual of type \( j \). If \( c^{i,j} < 0 \),
the chosen individual is killed. Otherwise an independent replica of the chosen individual is
incorporated into the population.

There are some works in similar models. In discrete time, we can interpret the dynamic of
our process as a particular case of a population size-dependent multitype branching process
studied in Gonzalez et. al. [10, 9]. In continuous time, Champagnat and Villemonais [5]
presented a model where an individual generates one child of the same type with birth rate
dependent on the whole population, and there is a non-linear interspecie and intraspecie
competition.

Also, we construct also the continuous-state analogous model. In this case, the continuum
mass of each type reproduces within its own population type and allows seeding of mass into
other population types. In addition, there is both intra and intertype pairwise interactions
proportional to the product of their type-population masses. We call it continuous-state
interacting multitype branching process (CIMBP). We prove that it can be characterized as the unique strong solution of a stochastic differential equation.

Our process generalize the Lotka-Volterra model studied by Cattiaux and Méleard, [3] which consist in two independent continuous-state branching processes with continuous trajectories that have both pairwise interactions. There are other related models with branching mechanisms, interactions or mutations, for example [4, 11].

Our main result is that a CIMBP is the scaling limit of a sequence of renormalized DIMBPs. We prove it by using a sequence of piecewise multitype branching processes approximation on a time-population grid. As in the DIMBP case, with this scaling limit representation we show that a CIMBP can be seen as a generalized Lamperti-type transformation of multidimensional Lévy processes.

The remainder of the paper is structured as follows. In Section 2 we study the discrete-state model and its Lamperti-type transformation. In Section 3 we define the continuous-state processes. Our construction features multiparameter time-changes as the ones introduced in [6]. For simplicity of the exposition, the proofs of the auxiliary results are presented in Section 4.

2. THE DISCRETE SPACE MODEL

Consider a DIMBP \((Z_i, t \geq 0)\) associated with Model 1.1 i.e. a \(Z^d_+\)-valued Markovian process with the following dynamics

- the transition rate from \(\mathbf{u}\) to \(\mathbf{u} - \mathbf{e}_i + \mathbf{v}\) following a reproduction event of the \(i\)-th type is equals to \(\lambda_i u_i \mu_i(\mathbf{v})\);
- the transition rate from \(\mathbf{u}\) to \(\mathbf{u} + \text{sgn}(c^{i,j}) \mathbf{e}_j\) following an interaction event from type \(i\) to type \(j\) is equals to \(|c^{i,j}| u_i u_j\).

To be clear, for each \(i \in D := \{1, \ldots, d\}\), \(\mu_i\) is a distribution on \(\mathbb{N}^d\), \(\lambda_i > 0\) and \(c^{i,j} \in \mathbb{R}\). Without loss of generality, we now assume that \(\mu_i(\mathbf{e}_i) = 0\).

We let \(\mathbf{C}\) be the matrix \((c^{i,j}, i, j \in D)\). Note that if \(\mathbf{C}\) is the zero matrix, we obtain a multitype continuous-time Galton-Watson processes. If \(\mathbf{C}\) has negative entries, our model is a multitype extension of the logistic branching process introduced in [13]. The multitype extension includes both intra and intertype pairwise competition (leading to quadratic intratype competition). We obtain the discrete version of the predator-prey (or Lotka-Volterra) branching processes studied in [3] when \(d = 2\), \(\mathbf{C}\) has zero diagonal, the off-diagonal entries have different signs, and there is no intratype reproduction.

Let us turn to a construction of the above DIMBP in terms of random walks and Poisson processes. Our construction features multparameter time-changes as the ones introduced in [12] and used in the branching process setting in [9].

Consider \(d\) independent random walks \(X^1, \ldots, X^d\) with values in \(Z^d\), where the jump rate of \(X^i\) is \(\lambda_i \tilde{\mu}_i\), with \(\tilde{\mu}_i(\mathbf{v}) = \mu_i(\mathbf{v} + \mathbf{e}_i)\). Let \(X^{i,j}\) be the \(j\)-entry of \(X^i\). Note that \(X^{i,i}\) is downwards skip-free, i.e. its jumps belong to \(-1, 1, 2, \ldots\). In addition, \(X^{i,j}\) is non-decreasing for \(j \neq i\). Let \(N^{i,j}\) be unit rate Poisson processes. Suppose that all the processes are independent. Consider the process \(Z = \{(Z_1, \ldots, Z_d), t \geq 0\}\) defined as

\[
Z_t^j = z^j + \sum_{i=1}^d X_t^{i,j} \mathbf{1}_{Z_t}; dr + \sum_{i=1}^d \text{sgn}(c^{i,j}) N_t^{i,j} \mathbf{1}_{[c^{i,j}] \mathbb{N}^d} \mathbf{1}_{Z_t}; dr, \quad j \in D,
\]

with starting value \(z = (z^i, i \in D) \in Z^d_+\).
Observe that the vector-valued equation has to be solved simultaneously for all \( j \in D \). The existence and uniqueness of a solution follows from the piecewise constant paths of \( X^{i,j} \) and \( N^{i,j} \), \( i, j \in D \), which implies the same for every \( Z^j \).

**Remark 2.1.** When \( Z^j_t = 0 \), the values \( X^{j,j}_{\{0\}^d \times [t]} \) and \( N^{i,j}_{[e^j] \{0\}^d \times [t]} \) become constant and since \( X^{i,j}, i \neq j \) are non-decreasing, then \( Z^j_{t+} \) is non-negative. Moreover, the vector zero is an absorbing point for \( Z \).

**Proposition 2.2.** The stochastic process \( Z \) given in (1) is a DIMBP.

**Proof.** The reader can note that any solution of equation (1) is non-negative and has the correct jump rates. In particular, the transition from \( u \) to \( u - e_1 + \mathbf{v} \) following a reproduction event of the \( i \)-th type is governed by the jumps of vector \( X^i \); while the transition from \( u \) to \( u + \text{sgn}(e^j) e_j \) following an interaction event from type \( i \) to type \( j \) is governed by \( N^{i,j} \). □

### 3. Continuous-state model

For every \( i \in D \), let \( W^{(i)}_t \) be a standard Brownian motion and let \( \mathcal{N}^{(i)}(ds, dr, du) \) be a Poisson random measure on \( \mathbb{R}^{d+2}_+ \) with intensity measure \( dsm^{(i)}(dr)du \), where \( m^{(i)} \) is a Borel measure on \( \mathbb{R}^d_+ \) satisfying

\[
\int_{\mathbb{R}^d_+ \setminus \{0\}} \left[ ||r|| \wedge ||r||^2 \right] + \sum_{j=1,j \neq i}^d r_j \] \( m^{(i)}(dr) < \infty \).

Denote by \( \hat{\mathcal{N}}^{(i)} \) the compensated measure of \( \mathcal{N}^{(i)} \). We assume that all the process are independent. Let \( \sigma = (\sigma^i, i \in D) \in \mathbb{R}^d_+ \), \( C = (c^{ij}, i, j \in D) \in \mathbb{R}^{d \times d} \), \( B = (b^{ij}, i, j \in D) \in \mathbb{R}^{d \times d} \), the set of essentially non-negative \( d \times d \) matrices.

Let us consider the \( d \)-dimensional stochastic differential equation (SDE for short)

\[
Y^j_t = y^j + \sum_{i=1}^d \int_0^t c^{ij} Y^i_s Y^j_s ds + \sum_{i=1}^d \int_0^t b^{ij} Y^i_s ds + \int_0^t \sqrt{2\sigma^{ij}} Y^j_s dW^j_s
\]

(2)

\[ + \sum_{i=1}^d \int_0^t \int_{\mathbb{R}^d_+} \int_0^\infty r_j 1_{\{s \leq Y^j_s\}} \hat{\mathcal{N}}^{(i)}(ds, dr, du) \]

with starting value \( y = (y^i, i \in D) \in \mathbb{R}^d_+ \).

**Remark 3.1.** If \( C = 0 \), there exist a unique strong solution which correspond to a multitype continuous-state branching process (MCBP for short) studied by [1]. If \( C \neq 0 \), the associated non-linear terms can be interpreted as pairwise interactions.

We define a continuous-state interacting multitype branching process (CIMBP) as the vector-valued Markov process \( Y = \{Y^i_t, \ldots, Y^d_t\}, t \geq 0 \) that is the unique strong solution to (2). The following theorem guarantees the existence of such solution. It is an extension of the papers by Barczy, Li and Pap [1] and Ma [14]. Its proof can be found in Section [5].
Theorem 3.2. There exists a unique non-negative strong solution to (2). Moreover, the associated process \( Y = \{ Y_t^1, \ldots, Y_t^d \}, t \in \mathbb{R}_+ \) has infinitesimal generator

\[
\mathcal{A}F(x) = \left\langle \text{diag}(Cxx^T) + Bx, F'(x) \right\rangle + \sum_{i=1}^d \sigma_i^j x_i F''_{ij}(x)
\]

\[
+ \sum_{i=1}^d x_i \int_{\mathbb{R}_+} \left[ F(x + r) - F(x) - \left\langle r, F'(r) \right\rangle \right] m(i)(dr),
\]

for \( F \in C^2_c(\mathbb{R}^d_+, \mathbb{R}) \) and \( x \in \mathbb{R}^d_+ \), where \( F'_i \) and \( F''_{ij} \), \( i \in D \), denote the first and second order partial derivatives of \( F \) with respect to its \( i \)-th variable, respectively, and \( F'(x) = (F'_1(x), \ldots, F'_d(x))^T \).

We recall that, by equation (1), a discrete-space interacting multitype branching process can be defined in terms of random walks and Poisson processes. The scaling limits of random walks are known to be Lévy processes and, by the strong law of large numbers, the scaling limits of Poisson processes are deterministic Lévy processes. Hence, it is natural to think that the scaling limits of DIMBPs can be constructed in terms of Lévy processes.

Let \( X^1, \ldots, X^d \) be independent \( \mathbb{R}^d \)-valued Lévy processes. We suppose that \( X^{i,j} \) has no negative jumps and that \( X^{i,j} \) is a subordinator for \( i \neq j \). Let \( Z = \{ (Z_t^1, \ldots, Z_t^d), t \geq 0 \} \) be a solution to the equation

\[
Z_t^j = z_t^j + \sum_i \left[ X_{b_t^i,0}^{i,j} z_t^i + \sigma_{i,j} \int_0^t Z_{s}^i Z_{s}^j ds \right], \quad j \in D,
\]

with starting value \( z = (z_t^i, i \in D) \in \mathbb{R}^d_+ \).

When \( C \equiv 0 \), the authors in [2] proved, by using analytical techniques, the existence and uniqueness of a solution to the multi-parameter time-change equation (3). In the next subsection, we show the existence of a process that is a solution to (3). Such process is obtained by a scaling limit representation, and is a CIMBP.

3.1. Scaling limits of DIMBP. The main result of this work is the convergence of re-normalized discrete models to the continuous one.

Let \( (X^{(n)}, 1 \leq i \leq d)_{n \in \mathbb{N}} \) be a sequence of random walks such that \( X^{(n),i,i} \) is downwards skip-free and \( X^{(n),i,j} \) is non-decreasing for \( j \neq i \). Assume that for each \( i \in D \), \( (X^{(n),i,j}, j \in D)_{n \in \mathbb{N}} \) can be scaled to converge to a Lévy process \( X^i = (X^{i,j}, j \in D) \). That is, assume the existence of constants \( (a_n)_{n \in \mathbb{N}} \) and \( (b_n^i, i \in D)_{n \in \mathbb{N}} \) such that

\[
(\text{CL}) \quad \left( \frac{a_n}{b_n^i} X^{(n),i,j}_{b_n^i t}, \ t \geq 0, j \in D \right) \xrightarrow[n \to \infty]{} \left( X_t^{i,j}, \ t \geq 0, j \in D \right), \quad i \in D,
\]

where \( a_n \to \infty \) and \( b_n^i/a_n \to \infty \). The convergence is almost surely in the Skohohod space.

Theorem 3.3. Assume hypothesis (CL). Let \((z_n^i, i \in D)_{n \in \mathbb{N}}\) and \((C_n)_{n \in \mathbb{N}} = (c_{i,j}^n, i, j \in D)_{n \in \mathbb{N}}\) be such that \( z_n^i a_n/b_n^i \to z_i^j \) and \( c_{i,j}^n b_n^i \to c_{i,j}^i \). Let \( Z^{(n)} = \{ (Z_t^{(n),1}, \ldots, Z_t^{(n),d}), t \geq 0 \} \) be a DIMBP that satisfies equation (1) for \( (X^{(n)}, i \in D) \) and \( N = (N^{i,j}, i, j \in D) \), with starting value \( z_n = (z_n^i, i \in D) \) and competition parameters \( C_n \). Then,

\[
\left( \frac{a_n}{b_n^i} Z_{a_n t}^{(n),i}, \ t \geq 0, i \in D \right) \xrightarrow[n \to \infty]{} \left( Z_t^i, \ t \geq 0, i \in D \right)
\]
almost surely in the Skorohod space, where \( Z = \{ (Z^1_t, \ldots, Z^d_t) \}, \ t \geq 0 \) is the solution to (2) and it is also a solution of (3) with starting value \( z = (z^i, \ i \in D) \).

To prove this result, first we approximate our interacting processes by a sequence of piecewise multitype continuous-state branching processes (MCBP) on a time-population grid. Then, we use the results of [2] for every MCBP to prove the scaling limits for the approximations. Our construction allows us to see that a CIMBP is also a generalized Lamperti-type transformation of multidimensional Lévy processes. With these tools, we conclude the desired convergence.

4. Proof of Theorem 3.3 by using a time-population grid approximation

We start by constructing a sequence of piecewise multitype continuous-state branching processes on a time-population grid which approximate a CIMBP. We are going to use some auxiliary results, whose proofs can be found in Section 5.

Let \( \epsilon, \delta > 0 \) and \( Z = (z^i, \ i \in D) \). For each \( j \in D \), we define \( k^{0,j} := [\frac{z^j}{\delta}] \). By [2, Theorem 1] the time-change equation

\[
Z^{1,j}_t = z^j + \sum_{i=1}^d c^{ij} k^{0,j} \delta \int_0^t Z^{1,i}_s ds, \quad t \geq 0, \quad j \in D,
\]

has a unique solution, which is a MCBP. Therefore, by using [1, Theorem 4.6], \( Z^1 \) is also the unique strong solution to

\[
Y^{1,j}_t = z^j + \sum_{i=1}^d c^{ij} k^{0,j} \delta \int_0^t Y^{1,i}_s ds + \sum_{i=1}^d \int_0^t b^{ij} Y^{1,i}_s ds + \int_0^t \sqrt{2\sigma^{ij} Y^{1,j}_s} dW^j_s,
\]

\[
+ \sum_{i=1}^d \int_0^t \int_0^\infty r_j 1_{\{u \in \mathbb{N}^{(i)}\}}(ds, dr, du)
\]

where \( (W^i, \ i \in D) \) are standard Brownian motions and \( (\mathbb{N}^{(i)}, \ i \in D) \) are Poisson random measures on \( \mathbb{R}^{d+2}_+ \) defined as in equation (2).

Given \( (Z^1_t, t \geq 0) \), for each \( i \in D \), the vector-valued process defined by

\[
X^{(1),i,j}_t := X^{(1),i,j}_0 Z^{1,i}_s ds + X^{(1),i,j}_0 Z^{1,i}_s ds, \quad j \in D,
\]

is a Lévy process in \( \mathbb{R}^d \). If we set \( k^{1,j} := [\frac{Z^{1,j}}{\delta}] \), again the time-change equation

\[
Z^{2,j}_t = Z^{1,j}_t + \sum_{i=1}^d X^{(1),i,j}_0 Z^{2,j}_s ds + \sum_{i=1}^d c^{ij} (k^{1,j} \delta) \int_0^t Z^{2,i}_s ds, \quad t \geq 0, \quad j \in D.
\]

has an unique solution, which is a MCBP starting from \( Z^1 \).

Inductively, for each \( m \geq 2 \), given \( (Z^{m-1}_t, t \geq 0) \), we define \( k^{m-1,j} := [\frac{Z^{m-1,j}}{\delta}] \). The time-change equation

\[
Z^{m,j}_t = Z^{m-1,j}_t + \sum_{i=1}^d X^{(m-1),i,j}_0 Z^{m,j}_s ds + \sum_{i=1}^d c^{ij} (k^{m-1,j} \delta) \int_0^t Z^{m,i}_s ds, \quad t \geq 0, \quad j \in D,
\]

has an unique solution, where

\[
X^{(m-1),i,j}_t = X^{(m-2),i,j}_0 Z^{m-1,j}_s ds + X^{(m-1),i,j}_0 Z^{m-1,j}_s ds, \quad r \geq 0.
\]
Since the solution is a MCBP starting from $Z_\varepsilon^{m-1}$, also satisfies

$$Y_t^{m,j} = Z_\varepsilon^{m-1,j} + \sum_{i=1}^d c^{ij}_{m-1,j} \int_0^t Y_s^{m,j}ds + \sum_{i=1}^d \int_0^t b^{ij}_{m,j} ds$$

$$+ \int_0^t \sqrt{2\sigma} Y_s^{m,j} dW^{m-1,j}_s + \sum_{i=1}^d \int_0^t \int_{\mathbb{R}^d} \int_{0}^{\infty} r_j \mathbb{1}_{\{u \leq Y_s^{m,j}\}} \hat{N}^{(m-1,i)}(ds,dr,du),$$

where $W^{m-1,j}_t = W^{(j)}_{t+(m-1)\varepsilon} - W^{(j)}_{(m-1)\varepsilon}$ is a Brownian motion and the measure $\hat{N}^{(m-1,i)}$ is given by $\hat{N}^{(m-1,i)}(ds,dr,du) = \hat{N}^{(i)}(ds + (m-1)\varepsilon,dr,du).$

Using the processes constructed above, we define the process $Z^{\varepsilon,\delta}$ as

$$Z_t^{\varepsilon,\delta,j} := Z_{t/\varepsilon - [(t/\varepsilon)]}^{\varepsilon,j}, \quad t \geq 0, j \in D,$$

which satisfies the equation

$$Z_t^{\varepsilon,\delta,j} = z^j + \sum_{i=1}^d x_{\varepsilon,\delta,i}^{(j)} \int_{0}^{\varepsilon \wedge \delta} \int_{0}^{\varepsilon \wedge \delta} \frac{Y_s^{\varepsilon,\delta,i}}{\delta} ds + \sum_{i=1}^d \int_{0}^{\varepsilon \wedge \delta} b^{ij}_{\varepsilon,\delta,i} ds$$

$$+ \int_{0}^{\varepsilon \wedge \delta} \sqrt{2\sigma} Y_s^{\varepsilon,\delta,j} dW^{(j)}_s + \sum_{i=1}^d \int_{0}^{\varepsilon \wedge \delta} \int_{\mathbb{R}^d} \int_{0}^{\infty} r_j \mathbb{1}_{\{u \leq Y_s^{\varepsilon,\delta,j}\}} \hat{N}^{(i)}(ds,dr,du).$$

The following proposition states that the above piecewise MCBPs converge to a CIMBP. It also establishes that a CIMBP can be seen as a generalized Lamperti-type transformation of a given multidimensional Lévy processes.

**Proposition 4.1.** Let $(Z_t^{\varepsilon,\delta}, t \geq 0)$ be the unique solution to (8), which coincides with the unique solution to (6). Then, as $\varepsilon, \delta$ goes to zero, $Z^{\varepsilon,\delta}$ converges almost surely in the Skorohod space to $Z$, the unique solution of (2). Furthermore, $Z$ satisfies equation (2).

Now, for any fixed pair $\varepsilon, \delta > 0$, we prove the scaling limits for the previous approximations. The main idea, as one can see in Section 3, is to use the results of 2 for every piecewise MCBP.

**Lemma 4.2. Scaling limits on the $\varepsilon, \delta$-grid.** Given $\varepsilon, \delta > 0$, let $(Z_t^{\varepsilon,\delta,n,j}, t \geq 0)$ be the process defined as the unique solution to

$$Z_t^{\varepsilon,\delta,n,j} = z_n^j + \sum_{i=1}^d x_{\varepsilon,\delta,i}^{(n),j} \int_{0}^{\varepsilon \wedge \delta} \int_{0}^{\varepsilon \wedge \delta} \frac{Y_s^{(n),i,j}}{\delta} ds + \sum_{i=1}^d \sum_{m=1}^{n-\varepsilon \wedge \delta} d_{i,j}^{(n),i,j} \int_{0}^{\varepsilon \wedge \delta} \int_{0}^{\varepsilon \wedge \delta} \frac{Z_{t+u}^{(n),i,j}}{\delta} ds + \sum_{i=1}^d \int_{0}^{\varepsilon \wedge \delta} b^{ij}_{\varepsilon,\delta,i} ds$$

$$+ \int_{0}^{\varepsilon \wedge \delta} \sqrt{2\sigma} Y_s^{\varepsilon,\delta,j} dW^{(j)}_s + \sum_{i=1}^d \int_{0}^{\varepsilon \wedge \delta} \int_{\mathbb{R}^d} \int_{0}^{\infty} r_j \mathbb{1}_{\{u \leq Y_s^{\varepsilon,\delta,j}\}} \hat{N}^{(i)}(ds,dr,du).$$

for $j \in D$, driven by $(X^{(n),i}, i \in D)$ and $N = (N^{i,j}, i, j \in D)$, with starting value $z_n = (z_n^j, j \in D)$ and competition parameters $C_n$ as in Theorem 3.3. Assume hypothesis (CL). Then,

$$\left(\begin{array}{c}
\alpha_n Z_{n,t}^{\varepsilon,\delta,n,i} \\
b_{n,t}^{\varepsilon,\delta,n,i}
\end{array}\right), \quad t \geq 0, i \in D$$

almost surely in the Skorohod space, where $Z^{\varepsilon,\delta}$ is the unique solution to (6).
Finally, for each pair \( \varepsilon, \delta > 0 \) we can write
\[
\frac{a_n}{b_n} Z_{a_n t}^{(n), i} - Z_t^i - \frac{a_n}{b_n} \left( Z_{a_n t}^{(n), j} - Z_t^{\varepsilon, \delta, (n), j} \right) + \left( \frac{a_n}{b_n} Z_{a_n t}^{\varepsilon, \delta, (n), j} - Z_t^{\varepsilon, \delta, j} \right) + \left( Z_t^{\varepsilon, \delta, j} - Z_t^j \right),
\]
for \( Z_t^{\varepsilon, \delta, (n)} \) as in (10) and \( Z_t^{\varepsilon, \delta} \) the unique solution to (8). Thanks to Proposition 4.1 and Lemma 4.2 it only remains to prove that the second term in the right-hand side of the equation converges to zero. This is asserted in the following lemma.

**Lemma 4.3.** Let \( \{Z_t^{n,i}, t \geq 0\} \) be the process referred in Theorem 3.3 and let \( \{Z_t^{\varepsilon, \delta, (n), i}, t \geq 0\} \) be the process referred in Lemma 4.2. Then,
\[
\lim_{\varepsilon, \delta \to 0} \lim_{n \to \infty} \frac{a_n}{b_n} \left( Z_{a_n t}^{(n), i} - Z_t^{\varepsilon, \delta, (n), i} \right) = 0
\]
almost surely in the Skorohod space.

Therefore, from Proposition 4.1, Lemma 4.2 and Lemma 4.3 we conclude that the renormalized DIMBPs converge to a CIMBP, i.e.
\[
\left( \frac{a_n}{b_n} Z_{a_n t}^{n,i}, t \geq 0, i \in D \right) \xrightarrow{n \to \infty} \left( Z_t^i, t \geq 0, i \in D \right).
\]

\( \Box \)

### 5. Proofs

In this section, we prove Theorem 3.2, Proposition 4.1, Lemma 4.2 and Lemma 4.3.

**Proof of Theorem 3.2.** Following the same ideas used in the proof of [8, Proposition 2.4], the existence of an unique strong solution to
\[
Y_t^j = y^j + \sum_{i=1}^{d} \int_0^t e^{ij} (Y_s^i \wedge n) (Y_s^j \wedge n) ds + \sum_{i=1}^{d} \int_0^t b^{ij} (Y_s^i \wedge n) ds
\]
\[
\int_0^t \sqrt{2 \sigma} (Y_s^j \wedge n) dW_s^{(j)}, \quad j \in D,
\]
\[
+ \sum_{i=1}^{d} \int_0^t \int_{\mathbb{R}_+^d} \int_0^\infty (r_j \wedge n) 1_{\{a \leq (Y_s^i \wedge n)\}} \tilde{N}(ds, dr, dn)
\]
for each \( n \geq 1 \), implies the existence of an unique strong solution to (11).

By [15, Example 134], we know that there exists a unique weak solution for equation (11). Therefore, by [15, Theorem 137], it is enough to show pathwise uniqueness to assure the existence of a unique strong solution. To show the pathwise uniqueness for equation (11), we will adapt the proof of [7, Theorem 2.1].

Let \( a_0 = 1 \) and choose a decreasing sequence \( a_k \to 0 \) such that \( \int_{a_k}^{a_{k+1}} z^{-1} dz = k \) for \( k \geq 1 \). Let \( x \mapsto \psi_k(x) \) be a non-negative continuous function on \( \mathbb{R} \) with support in \( (a_k, a_{k-1}) \) that satisfies \( \int_{a_k}^{a_{k+1}} \psi_k(x) dx = 1 \) and \( 0 \leq \psi_k(x) \leq 2k^{-1} x^{-1} \) for \( a_k < x < a_{k-1} \). For each \( k \geq 1 \), we define the non-negative and twice continuously differentiable function
\[
\varphi_k(z) = \int_0^{|z|} dy \int_0^y \psi_k(x) dx.
\]
We observe that the sequence of functions \( \{\varphi_k, k \geq 1\} \) satisfies the following properties:
(i) \( \varphi_k(z) \to |z| \) non-decreasingly as \( k \to \infty \);
(ii) \( 0 \leq \varphi_k(z) \leq 1 \) for \( z \geq 0 \) and \( -1 \leq \varphi_k(z) \leq 0 \) for \( z \leq 0 \);
(iii) \( \varphi''_k(z) \geq 0 \) for every \( z \in \mathbb{R} \). In addition, as \( k \to \infty \),

\[
\varphi''_k(x - y)(\sqrt{x} - \sqrt{y})^2 \to 0.
\]

We first fix the integer \( n \geq 1 \). Let \( Y = \{(Y^1_t, \ldots, Y^d_t), \ t \geq 0\} \) and \( \tilde{Y} = \{(Y^1_t, \ldots, Y^d_t), \ t \geq 0\} \) be two weak solution of (11). For each \( j \in D \), we define \( \zeta^j_t = Y^j_t - \tilde{Y}^j_t \) for \( t \geq 0 \). Then, by Itô’s formula, for each \( j \in D \)

\[
\varphi_k(\zeta^j_t) = \sum_{i=1}^{d} \int_0^t \varphi'_k(\zeta^j_s) \left( b^i_j(\zeta^j_s) + c^i_j(\tilde{Y}^j_s \wedge n) \right) \left( Y^i_s \wedge n - \tilde{Y}^i_s \wedge n \right) ds
\]

\[
+ \sum_{i=1}^{d} \int_0^t \varphi'_k(\zeta^j_s) c^i_j(Y^i_s \wedge n) \left( Y^i_s \wedge n - \tilde{Y}^i_s \wedge n \right) ds
\]

\[
+ \int_0^t \varphi_k(\zeta^j_s) \sqrt{2\sigma^j} \left[ \sqrt{Y^j_s \wedge n - \tilde{Y}^j_s \wedge n} \right] dW^{(j)}_s
\]

\[
+ \int_0^t \varphi_k(\zeta^j_s) \sigma^j \left[ \sqrt{Y^j_s \wedge n - \tilde{Y}^j_s \wedge n} \right]^2 ds
\]

\[
+ \sum_{i=1}^{d} \int_0^t \int_{\mathbb{R}_+^d} \int_{0}^{\infty} \left[ 1_{\{u \in Y^i_s \wedge n\}} - 1_{\{u \in \tilde{Y}^i_s \wedge n\}} \right] \Delta_{r_j \wedge n} \varphi_k(\zeta^j_s) \tilde{N}^{(i)}(ds, dr, da)
\]

\[
+ \sum_{i=1}^{d} \int_0^t \int_{\mathbb{R}_+^d} \left( Y^i_s \wedge n - \tilde{Y}^i_s \wedge n \right) D_{r_j \wedge n} \varphi_k(\zeta^j_s) ds m^{(i)}(dr)
\]

where \( \Delta_r \varphi(z) = \varphi(z + r) - \varphi(z) \) and \( D_r \varphi(z) = \Delta_r \varphi(z) - \varphi'(z) r \). Note that

\[
\left( Y^i_s \wedge n - \tilde{Y}^i_s \wedge n \right) D_{r_j \wedge n} \varphi_k(\zeta^j_s)
\]

\[
= \left( Y^i_s \wedge n - \tilde{Y}^i_s \wedge n \right) (r_j \wedge n)^2 \int_0^1 \psi_k(|\zeta^j_s + t(r_j \wedge n)|)(1 - t) dt
\]

\[
\leq \left( Y^i_s \wedge n - \tilde{Y}^i_s \wedge n \right) (r_j \wedge n)^2 \int_0^1 \frac{2(1 - t)}{k|\zeta^j_s + t(r_j \wedge n)|} dt
\]

\[
\leq \begin{cases} 
\frac{(r_j \wedge n)^2}{k} & \text{if } \zeta^j_s > 0 \\
0 & \text{if } \zeta^j_s \leq 0
\end{cases}.
\]

and also

\[
D_{r_j \wedge n} \varphi_k(\zeta^j_s) = \Delta_{r_j \wedge n} \varphi_k(\zeta^j_s) + \varphi'_k(\zeta^j_s)(r_j \wedge n) \leq 2(r_j \wedge n).
\]
By (12) we have that
\[
\varphi_k(\zeta^j_t) \leq \sum_{i=1}^{d} \int_0^t \varphi_k(\zeta^j_{s-}) \left( b^{ij} + c^{ij}(\tilde{Y}^j_t \wedge n) \right) \left( Y^{i}_s \wedge n - \tilde{Y}^{i}_s \wedge n \right) ds
\]
\[
+ \sum_{i=1, i \neq j}^{d} \int_0^t \varphi_k(\zeta^j_{s-})c^{ij}(Y^i_t \wedge n) \left( Y^j_s \wedge n - \tilde{Y}^j_s \wedge n \right) ds
\]
\[
+ \int_0^t \varphi_k''(\zeta^j_{s-}) \sigma^2 \left[ \sqrt{Y^j_t \wedge n - \sqrt{\tilde{Y}^j_s \wedge n}} \right]^2 ds
\]
\[
+ \sum_{i=1, i \neq j}^{d} \int_0^t \left( Y^i_s \wedge n - \tilde{Y}^i_{s-} \wedge n \right) \int_{\mathbb{R}_+^d} 2(r_j \wedge n) m^{(i)}(dr)ds
\]
\[
+ \int_0^t \left( Y^j_s \wedge n - \tilde{Y}^j_{s-} \wedge n \right) \int_{\mathbb{R}_+^d} 2(r_j \wedge n) 1_{\{r_j \geq 1\}} m^{(j)}(dr)ds
\]
\[
+ \int_0^t \int_{\mathbb{R}_+^d} \frac{(r_j \wedge n)^2}{k} 1_{\{r_j \leq 1\}} m^{(j)}(dr)ds + \text{ local mart.}
\]

If we take expectation and let $k \to \infty$, we obtain
\[
\mathbb{E} \left( |\zeta^j_t| \right) \leq \sum_{i=1}^{d} \int_0^t \left[ |b^{ij} + c^{ij} n| \mathbb{E} \left( |Y^i_s \wedge n - \tilde{Y}^i_s \wedge n| \right) + |c^{ij} n| \mathbb{E} \left( |Y^j_s \wedge n - \tilde{Y}^j_s \wedge n| \right) \right] ds
\]
\[
+ 2n \sum_{i=1, i \neq j}^{d} \left[ \int_{\mathbb{R}_+^d} m^{(i)}(dr) \right] \int_0^t \mathbb{E} \left( |Y^i_s \wedge n - \tilde{Y}^i_s \wedge n| \right) ds
\]
\[
+ 2n \left[ \int_{\mathbb{R}_+^d} 1_{\{r_j \geq 1\}} m^{(j)}(dr) \right] \int_0^t \mathbb{E} \left( |Y^j_s \wedge n - \tilde{Y}^j_s \wedge n| \right) ds
\]
\[
\leq C_j \sum_{i=1}^{d} \int_0^t \mathbb{E} \left( |Y^i_s \wedge n - \tilde{Y}^i_s \wedge n| \right) ds \leq C_j \sum_{i=1}^{d} \int_0^t \mathbb{E} \left( |\zeta^i_s| \wedge n \right) ds
\]
\[
\leq C_j \sum_{i=1}^{d} \int_0^t \mathbb{E} \left( |\zeta^i_s| \right) ds
\]
for a big enough constant $C_j$. Thus, by summing over $j$, we have
\[
\mathbb{E} \left( \sum_{j=1}^{d} |\zeta^j_t| \right) \leq \bar{C} \int_0^t \mathbb{E} \left( \sum_{j=1}^{d} |\zeta^j_s| \right) ds,
\]
where $\bar{C} = \sum_{j=1}^{d} C_j$. Therefore, by Gronwall’s inequality,
\[
\mathbb{E} \left( \sum_{j=1}^{d} |\zeta^j_t| \right) = 0 \quad \text{for all } t \geq 0.
\]

This implies the pathwise uniqueness for equation (11) for all $n \geq 1$, and from here we deduce that there exists an unique strong solution for (2).
Finally, by Itô's formula for every $F \in C^2(\mathbb{R}_+^d)$ we have

$$
F(Y_t) = F(Y_0) + \int_0^t \sum_{i=1}^d \left\{ F_i''(Y_{s-})\sigma_i Y_s^i + F_i'(Y_{s-}) \sum_{j=1}^d \left[ b_{ij}^j Y_s^j + c_{ij}^j Y_s^j Y_s^j \right] \right\} ds \\
+ \sum_{i=1}^d \int_0^t \int_{\mathbb{R}_+^d} Y_s^i \left[ F(Y_{s-} + r) - F(Y_{s-}) - \sum_{j=1}^d F_j'(Y_{s-})r_j \right] m(i)(dr)ds + M_t,
$$

where $M_t$ is a local martingale. Then, $Y$ has the desired infinitesimal generator. 

\[ \square \]

**Proof of Proposition 4.1** As some of the techniques are the same as those used in the previous proof, we only give the short version.

Let $Y = \{ (Y^1_t, \ldots, Y^d_t), \ t \geq 0 \}$ be the unique strong solution of (2) and for every $\varepsilon, \delta > 0$ let $Y_{\varepsilon,\delta} = \{ (Y^1_{\varepsilon,\delta,t}, \ldots, Y^d_{\varepsilon,\delta,t}), \ t \geq 0 \}$ be the strong solution of (3). For each $j \in \{1, \ldots, d\}$, we define $\Delta^j_t = \Delta^{\varepsilon,\delta}_t Y^j_t = Y^j_{\varepsilon,\delta,t} - Y^j_t$.

Let $\tau_M := \inf\{ t \geq 0 : Y^i_t > M \text{ or } Y^i_{\varepsilon,\delta,t} > M \text{ for some } i \in D \}$. Take a sequence of functions $\{\varphi_k\}_{k \in \mathbb{N}}$ as in the proof of Theorem 3.2. By Itô’s formula, we have

$$
\varphi_k(\Delta^j_{t\wedge \tau_M}) = \sum_{i=1}^d \int_0^{t\wedge \tau_M} \varphi_k'(\Delta^j_s) \left( b_{ij}^j + c_{ij}^j Y_s^j \right) \Delta^j_s ds + \sum_{i=1}^d \int_0^{t\wedge \tau_M} \varphi_k''(\Delta^j_s) c_{ij}^j Y_{\varepsilon,\delta,t}^j \Delta^j_s ds \\
+ \sum_{i=1}^d \int_0^{t\wedge \tau_M} \varphi_k''(\Delta^j_s) \sigma^j \left[ \sqrt{Y_s^j} - \sqrt{Y^j_{\varepsilon,\delta,t}} \right]^2 ds \\
+ \sum_{i=1}^d \int_0^{t\wedge \tau_M} \Delta^i_s D_{\varphi_k}(\Delta^j_{s-}) ds m(i)(dr) + \text{local mart}.
$$

By using the properties of $\{\varphi_k\}_{k \in \mathbb{N}}$, taking expectation and letting $k \to \infty$, we obtain

$$
\mathbb{E}\left( |\Delta^j_{t\wedge \tau_M}| \right) \leq \sum_{i=1}^d \int_0^{t\wedge \tau_M} \left[ |b_{ij}^j + c_{ij}^j M| \mathbb{E}\left( |\Delta^j_s| \right) + |c_{ij}^j| M \mathbb{E}\left( |\Delta^j_s| \right) \right] ds \\
+ dM \sum_{m=1}^{\lfloor t\wedge \tau_M \rfloor} \int_{(m-1)\varepsilon}^{m\varepsilon} \mathbb{E}\left( |Y^j_{s-} - \frac{Y^j_{(m-1)\varepsilon} - Y^j_{(m-1)\varepsilon}}{\delta}| \right) ds \\
+ 2M \sum_{i \neq j} \int_{(m-1)\varepsilon}^{m\varepsilon} m(i)(dr) \int_0^{t\wedge \tau_M} \mathbb{E}\left( |\Delta^j_s| \right) ds \\
+ 2M \int_{(m-1)\varepsilon}^{m\varepsilon} 1_{\{r, s \leq 1\}} m(j)(dr) \int_0^{t\wedge \tau_M} \mathbb{E}\left( |\Delta^j_s| \right) ds \\
\leq C_j \sum_{i=1}^d \int_0^{t\wedge \tau_M} \mathbb{E}\left( |\Delta^i_{s\wedge \tau_M}| \right) ds + dM \sum_{m=1}^{\lfloor t\wedge \tau_M \rfloor} \int_{(m-1)\varepsilon}^{m\varepsilon} \mathbb{E}\left( |Y^j_{s-} - \frac{Y^j_{(m-1)\varepsilon} - Y^j_{(m-1)\varepsilon}}{\delta}| \right) ds
$$
for a big enough constant $C_j$. Thus we have

$$
E \left( \sum_{j=1}^{d} |\Delta_{t,\tau_M}^j| \right) \leq \bar{C} \int_{0}^{t} E \left( \sum_{j=1}^{d} \left| \Delta_{s,\tau_M}^j \right| \right) ds + dM \sum_{j=1}^{d} \sum_{m=1}^{\left| \tau_{m-1} \right|} \int_{(m-1)\varepsilon}^{m\varepsilon} E \left( \left| Y_{s,\delta,j}^{\varepsilon} - Y_{(m-1)\varepsilon,\delta,j}^{\varepsilon} \right| \right) ds \\
+ dM \sum_{j=1}^{d} \sum_{m=1}^{\left| \tau_{m-1} \right|} \varepsilon \mathbb{E} \left( \left| Y_{(m-1)\varepsilon,\delta,j}^{\varepsilon} - \left[ \frac{Y_{(m-1)\varepsilon,\delta,j}^{\varepsilon}}{\delta} \right] \right| \right),
$$

where $\bar{C} = \sum_{j=1}^{d} C_j$. By Gronwall’s inequality, we have that

$$
E \left( \sum_{j=1}^{d} |\Delta_{t,\tau_M}^j| \right) \leq e^{\bar{C}t} dM \sum_{j=1}^{d} \sum_{m=1}^{\left| \tau_{m-1} \right|} \int_{(m-1)\varepsilon}^{m\varepsilon} E \left( \left| Y_{s,\delta,j}^{\varepsilon} - Y_{(m-1)\varepsilon,\delta,j}^{\varepsilon} \right| \right) ds + \varepsilon \mathbb{E} \left( \left| Y_{(m-1)\varepsilon,\delta,j}^{\varepsilon} - \left[ \frac{Y_{(m-1)\varepsilon,\delta,j}^{\varepsilon}}{\delta} \right] \right| \right) \cdot \delta Z_{\varepsilon,\delta,i} ds.
$$

Hence, by using the càdlàg properties of the process, we deduce

$$
\lim_{\varepsilon, \delta \to 0} E \left( \sum_{j=1}^{d} |\Delta_{t,\tau_M}^j| \right) = 0, \quad \text{for all } t \geq 0.
$$

This implies the convergence in distribution in Skorohod space. By the strong uniqueness of the processes (as solutions of SDEs), we have that $(Z_{t}, t \geq 0)$ converges almost surely to $(Z_{t}, t \geq 0)$. This implies that

$$
\int_{0}^{t} Z_{s,\delta,i} ds \xrightarrow{a.s.} \int_{0}^{t} Z_{s,\delta,i} ds \quad \text{and} \quad \sum_{m=1}^{\left| \tau_{m-1} \right|} \int_{(m-1)\varepsilon}^{m\varepsilon} \left[ \frac{Z_{s,\varepsilon,i}^{\delta}}{\delta} \right] ds Z_{\varepsilon,\delta,i} ds \xrightarrow{a.s.} \int_{0}^{t} Z_{s,\delta,i} ds.
$$

Therefore, $(Z_{t}, t \geq 0)$ is a solution to (3) for the Lévy process $X$. □

**Proof of Lemma 4.2.** First, we observe that by using a recursive procedure, $Z_{t}^{\varepsilon,\delta,(n)}$ can be constructed in an analogous way to $Z_{t}^{\varepsilon,\delta}$. Let $k^{0,(n),j} := [z_{j}^{(n)}/\delta]$. We define $Z_{t}^{1,(n)}$ as the unique solution of

$$
Z_{t}^{1,(n),j} = z_{n}^{j} + \sum_{i=1}^{d} X_{t}^{1,(n),i,j} ds + \sum_{i=1}^{d} \text{sgn}(c^{i,j}) N_{[c^{i,j}k^{0,(n),j}\delta_{0}^{r}]} Z_{t}^{1,(n),i} dr,
$$

where $(\hat{X}_{t}^{0,(n),i}, i \in D)$ are independent $d$-dimensional random walks, defined by

$$
\hat{X}_{t}^{0,(n),i,j} = X_{t}^{0,(n),i,j} + \text{sgn}(c^{i,j}) N_{[c^{i,j}k^{0,(n),j}\delta t]}.
$$

Observe that, as $\frac{a_n}{b_n} k^{0,(n),j} \to k^{0,j} = [z_{j}/\delta],

$$
\left( \frac{a_n}{b_n} \hat{X}_{t}^{0,(n),i,j}, t \geq 0, i, j \in D \right) \xrightarrow{n \to \infty} \left( X_{t}^{i,j} + c^{i,j} k^{0,j} \delta t, t \geq 0, i, j \in D \right)
$$

in the Skorohod space. Therefore, by Corollary 1 in [2], we can see that

$$
(13) \quad \left( \frac{a_n}{b_n} Z_{n,\varepsilon,\delta,}^{1,(n),i}, t \geq 0, i \in D \right) \xrightarrow{n \to \infty} \left( Z_{t}^{1,i}, t \geq 0, i \in D \right)
$$
almost surely in the Skorohod space, where $Z^1$ is the unique solution of (1).

Now, given $Z^{1,(n)}$, the vector-valued process defined by

$$X^{(1),(n),i,j}_r := X^{(n),i,j}_r - X^{(n),i,j}_{t_0^+,j}, \quad j \in D$$

is a random walk in $\mathbb{R}^d$ for each $i \in D$, and

$$N^{(1),i,j}_r := N^{i,j}_{[c^2,j]1,(n),j\delta t_0^+,j} - N^{i,j}_{[c^2,j]k(0,(n),j\delta t_0^+,j} Z^{1,(n),i,j}_r, \quad i, j \in D$$

is a Poisson process. We set $k^{1,(n),j} := [\frac{Z^{1,(n),j}_{t_0^+,j}}{\delta}]$, and we define $Z^{2,(n)}$ as the unique solution of

$$Z^{2,(n),i,j}_t = Z^{1,(n),i}_t + \sum_{i=1}^d X^{1,(n),i,j}_t \mathbb{E}_t Z^{2,(n),i,j}_t, \quad t \geq 0, \quad i, j \in D,$$

where

$$X^{1,(n),i,j}_t = X^{(1),(n),i,j}_t + \text{sgn}(c^i,j) N^{(1),i,j}_{[c^2,j]1,(n),j\delta t}.$$

By taking $t = \varepsilon$ in (13), we define

$$\left( \frac{a_n}{b_n} Z^{1,(n),j}_t, \quad j \in D \right) \rightarrow (Z^{1,j}_\varepsilon, \quad j \in D).$$

Additionally,

$$\left( \frac{a_n}{b_n} \tilde{X}^{1,(n),i,j}_{b_n t}, \quad t \geq 0, \quad i, j \in D \right) \rightarrow \left( X^{1,j}_t + c^i,j \tilde{k}^{1,j}\delta t, \quad t \geq 0, \quad i, j \in D \right)$$

in the Skorohod space, where $(X^{1,j}_t, t \geq 0)$ is the process defined in (13) and $\tilde{k}^{1,j} = [\frac{Z^{1,j}_0}{\delta}]$. Using again Corollary 1 in [2], we have that

$$\left( \frac{a_n}{b_n} \tilde{Z}^{2,(n),i,j}_{b_n t}, \quad t \geq 0, \quad i \in D \right) \rightarrow (Z^{2,i}_t, \quad t \geq 0, \quad i \in D),$$

where $Z^{2,i}$ is the unique solution of (6).

Inductively, for each $n > 2$, given $(Z^{m-1,(n),i,j}_t, t \geq 0)$ with $k^{m-1,(n),j} = [\frac{Z^{m-1,(n),j}_{t_0^+,j}}{\delta}]$, we define $Z^{m,(n)}$ as the unique solution of

$$Z^{m,(n),i,j}_t = Z^{m-1,(n),i}_t + \sum_{i=1}^d X^{m-1,(n),i,j}_t \mathbb{E}_t Z^{m,(n),i,j}_t, \quad t \geq 0, \quad j \in D,$$

where

$$X^{m-1,(n),i,j}_t = X^{m-2,(n),i,j}_t - X^{m-1,(n),i,j}_{t_0^+,j} Z^{m,(n),i,j}_t,$$

and

$$N^{m-1,(i,j)}_r := N^{m-2,(i,j)}_{\mathbb{E}_t Z^{m-1,(n),i,j}_t, t \geq 0, \quad i, j \in D \rightarrow (Z^{m,i}_t, \quad t \geq 0, \quad i \in D)}$$

in the Skorohod space, where $Z^{m,i}$ is the unique solution of (7).
Using the processes constructed above, we define the process $Z^{ε,δ}_{s,t}$ as

$$Z^{ε,δ}_{s,t} := Z^{[s/a_n ε]_i,(n),i}_{t/a_n ε - [t/a_n ε]}, \quad t \geq 0, \; i \in D,$$

which is the unique solution to (10). By the previous Skorohod convergences, we can conclude that

$$\left( a_n Z^{ε,δ}_{a_n t,i}, \; t \geq 0, \; i \in D \right) \xrightarrow{n \to \infty} \left( Z^{ε,δ}_{t,i}, \; t \geq 0, \; i \in D \right)$$

in the Skorohod space, where $Z^{ε,δ}$ satisfies (8), and hence is the unique solution of (9). □

**Proof of Lemma 4.3** Define the process $\gamma_{a_n t} = |Z^{ε,δ}_{a_n t} - Z^{ε,δ}_{a_n t}|$ and

$$\tau_M = \inf \{ s \geq 0 : \frac{a_n}{b_n} Z^{ε,δ}_{a_n s,i} \geq M \text{ or } \frac{a_n}{b_n} Z^{ε,δ}_{a_n s,i} \geq M \text{ for some } i \in \{1, \ldots, d\} \}.$$

We have that

$$E \left[ \frac{a_n}{b_n} \xi^{ε,δ}_{a_n t} \tau_M \right] \leq \frac{a_n}{b_n} \sum_{i=1}^d E \left[ \frac{\xi^{ε,δ}_{a_n t,i}}{b_n} \right] \leq \frac{a_n}{b_n} \sum_{i=1}^d \left( \left[ \frac{t}{ε} + 1 \right] \sum_{m=1}^{(m+1) \epsilon t \leq a_n t \leq τ} Z^{ε,δ}_{a_n t,i} \right) + \frac{a_n}{b_n} \sum_{i=1}^d \left( \left[ \frac{t}{ε} + 1 \right] \sum_{m=1}^{(m+1) \epsilon t \leq a_n t \leq τ} Z^{ε,δ}_{a_n t,i} \right) \delta ds.$$

Hence,

$$E \left[ \frac{a_n}{b_n} \xi^{ε,δ}_{a_n t} \tau_M \right] \leq \frac{a_n}{b_n} \sum_{i=1}^d \left( \left[ \frac{t}{ε} + 1 \right] \sum_{m=1}^{(m+1) \epsilon t \leq a_n t \leq τ} Z^{ε,δ}_{a_n t,i} \right) + \frac{a_n}{b_n} \sum_{i=1}^d \left( \left[ \frac{t}{ε} + 1 \right] \sum_{m=1}^{(m+1) \epsilon t \leq a_n t \leq τ} Z^{ε,δ}_{a_n t,i} \right) \delta ds.$$

If we take $C_n = \max_{i,j \in \{1, \ldots, d\}} \frac{|c_{i,j}| b_n^2}$ and $\Delta(n) = \max_{i,j \in \{1, \ldots, d\}} E \left[ \frac{a_n}{b_n} \Delta X^{ε,δ}_{b_n t} \right]$, we have

$$\sum_{j=1}^d E \left[ \frac{a_n}{b_n} \xi^{ε,δ}_{a_n t} \tau_M \right] \leq d \left[ \Delta(n) + 2MC_n \right] \left[ \left[ \frac{t}{ε} + 1 \right] \sum_{m=1}^{(m+1) \epsilon t \leq a_n t \leq τ} Z^{ε,δ}_{a_n t,i} \right] \delta ds$$

$$+ MC_n \sum_{j=1}^d \sum_{m=1}^{(m+1) \epsilon t \leq a_n t \leq τ} E \left[ \frac{a_n}{b_n} \Delta X^{ε,δ}_{a_n t,i} \right] \delta ds.$$
By taking the upper limit and using Lemma 4.2, we have that
\[
\limsup_n \sum_{j=1}^d \mathbb{E}\left[ \frac{a_{n,\epsilon} \Lambda \delta}{b_{n,\epsilon}} \right] = \mathbb{E}\left[ \sum_{j=1}^d \int_{e^{(m+1)\epsilon+t}} \ldots \right] ds
\]
where \( C = \max_{i,j \in [1, \ldots, d]} \left| c_{i,j} \right| \) and \( \Delta = \max_{i,j \in [1, \ldots, d]} \mathbb{E}\left[ \left| \Delta X^{i,j} \right| \right] \).

Finally, by the càdlàg properties of the process \( Z^{\epsilon, \delta} \), we have that
\[
\lim_{\epsilon, \delta \to 0} \lim_{n \to \infty} \sum_{j=1}^d \mathbb{E}\left[ \frac{a_{n,\epsilon} \Lambda \delta}{b_{n,\epsilon}} \right] = 0 \quad \forall t \geq 0.
\]
Using the uniqueness of the processes, we deduce the almost surely convergence in Skorohod space.

\[\square\]

Acknowledgements

Both authors would like to thank Gerónimo Uribe Bravo for collaborating on an early draft of this paper. SP’s research is supported by PAPIIT IA103220.

References

[1] Mátys Barczy, Zenghu Li, and Gyula Pap. Stochastic differential equation with jumps for multi-type continuous state and continuous time branching processes with immigration. ALEA Lat. Am. J. Probab. Math. Stat., 12(1):129–169, 2015.
[2] M. Emilia Caballero, José Luis Pérez Garmendia, and Gerónimo Uribe Bravo. Affine processes on \( \mathbb{R}^m \times \mathbb{R}^n \) and multiparameter time changes. Ann. Inst. Henri Poincaré Probab. Stat., 53(3):1280–1304, 2017.
[3] Patrick Cattiaux and Sylvie Méléard. Competitive or weak cooperative stochastic Lotka-Volterra systems conditioned on non-extinction. J. Math. Biol., 60(6):797–829, 2010.
[4] Nicolas Champagnat, Régis Ferrière, and Sylvie Méléard. From individual stochastic processes to macroscopic models in adaptive evolution. Stoch. Models, 24(suppl. 1):2–44, 2008.
[5] Nicolas Champagnat and Denis Villemonais. Quasi-stationary distribution for multi-dimensional birth and death processes conditioned to survival of all coordinates. arXiv preprint [arXiv:1508.03161], 2015.
[6] Loïc Chaumont. Breadth first search coding of multitype forests with application to Lamperti representation. In In memoriam Marc Yor—Séminaire de Probabilités XLVII, volume 2137 of Lecture Notes in Math., pages 561–584. Springer, Cham, 2015.
[7] Donald A. Dawson and Zenghu Li. Stochastic equations, flows and measure-valued processes. Ann. Probab., 40(2):813–857, 2012.
[8] Zongfei Fu and Zhenli Li. Stochastic equations of non-negative processes with jumps. Stochastic Process. Appl., 120(3):306–330, 2010.
[9] M. González, R. Martínez, and M. Mota. Population size dependent generalized multitype branching processes. Stoch. Anal. Appl., 23(6):1179–1197, 2005.
[10] Miguel González, Rodrigo Martínez, and Manuel Mota. Multitype population size-dependent branching processes with dependent offspring. Statist. Probab. Lett., 71(2):145–154, 2004.
[11] Alexandre Hening and Dang H. Nguyen. Coexistence and extinction for stochastic Kolmogorov systems. Ann. Appl. Probab., 28(3):1893–1942, 2018.
[12] Thomas G. Kurtz. Representations of Markov processes as multiparameter time changes. Ann. Probab., 8(4):682–715, 1980.
[13] Amaury Lambert. The branching process with logistic growth. Ann. Appl. Probab., 15(2):1506–1535, 2005.
[14] Rugang Ma. Stochastic equations for two-type continuous-state branching processes with immigration and competition. Statist. Probab. Lett., 91:83–89, 2014.
[15] Rong Situ. Theory of stochastic differential equations with jumps and applications. Mathematical and Analytical Techniques with Applications to Engineering. Springer, New York, 2005. Mathematical and analytical techniques with applications to engineering.
(MCF) Facultad de Ciencias, Universidad Nacional Autónoma de México, México. email: mcfittipaldi@ciencias.unam.mx

(SP) Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas, Universidad Nacional Autónoma de México, México. email: sandra@sigma.iimas.unam.mx