Nonlinear Dynamics of Accelerator via Wavelet Approach

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Abstract. In this paper we present the applications of methods from wavelet analysis to polynomial approximations for a number of accelerator physics problems. In the general case we have the solution as a multiresolution expansion in the base of compactly supported wavelet basis. The solution is parametrized by the solutions of two reduced algebraical problems, one is nonlinear and the second is some linear problem, which is obtained from one of the next wavelet constructions: Fast Wavelet Transform, Stationary Subdivision Schemes, the method of Connection Coefficients. According to the orbit method and by using construction from the geometric quantization theory we construct the symplectic and Poisson structures associated with generalized wavelets by using metaplectic structure. We consider wavelet approach to the calculations of Melnikov functions in the theory of homoclinic chaos in perturbed Hamiltonian systems and for parametrization of Arnold–Weinstein curves in Floer variational approach.

INTRODUCTION.

In this paper we consider the following problems: the calculation of orbital motion in storage rings, some aspects of symplectic invariant approach to wavelet computations, Melnikov functions approach in the theory of homoclinic chaos, the calculation of Arnold-Weinstein curves (periodic loops) in Hamiltonian systems. The key point in the solution of these problems is the use of the methods of wavelet analysis, relatively novel set of mathematical methods, which gives us a possibility to work with well-localized bases in functional spaces and with the general type of operators (including pseudodifferential) in such bases. Our problem as many related problems in the framework of our type of approximations of complicated physical nonlinearities is reduced to the problem of the solving of the systems of differential equations with polynomial nonlinearities with or without some constraints. In this paper we consider as the main example the particle motion in storage rings in standard approach, which is based on consideration in [1], [2]. Starting from Hamiltonian, which described classical dynamics in storage rings
\[ H(\vec{r}, \vec{P}, t) = c\{\pi^2 + m_0^2 c^2\}^{1/2} + e\phi \]  

(1)

and using Serret–Frenet parametrization, we have the following Hamiltonian for orbital motion in machine coordinates:

\[
H(x, p_x, z, p_z, \sigma, p_\sigma; s) = p_\sigma - \left[1 + f(p_\sigma) \right] \cdot [1 + K_x \cdot x + K_z \cdot z] \times \left\{ 1 - \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]^2} \right\}^{1/2} + \frac{1}{2} \cdot [1 + K_x \cdot x + K_z \cdot z]^2 - \frac{1}{2} \cdot g \cdot (z^2 - x^2) - N \cdot x z + \frac{\lambda}{6} \cdot (x^3 - 3 x z^2) + \frac{\mu}{24} \cdot (z^4 - 6 x^2 z^2 + x^4) + \frac{1}{\beta^2_0} \cdot \frac{L}{2 \pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos \left[ h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi \right]
\]

(2)

Then, after standard manipulations with truncation of power series expansion of square root we arrive to the following approximated Hamiltonian for particle motion:

\[
H = \frac{1}{2} \cdot \frac{[p_x + H \cdot z]^2 + [p_z - H \cdot x]^2}{[1 + f(p_\sigma)]} + p_\sigma - [1 + K_x \cdot x + K_z \cdot z] \cdot f(p_\sigma) + \frac{1}{2} \cdot [K_x^2 + g] \cdot x^2 + \frac{1}{2} \cdot [K_z^2 - g] \cdot z^2 - N \cdot x z + \frac{\lambda}{6} \cdot (x^3 - 3 x z^2) + \frac{\mu}{24} \cdot (z^4 - 6 x^2 z^2 + x^4) + \frac{1}{\beta^2_0} \cdot \frac{L}{2 \pi \cdot h} \cdot \frac{eV(s)}{E_0} \cdot \cos \left[ h \cdot \frac{2\pi}{L} \cdot \sigma + \varphi \right]
\]

(3)

and the corresponding equations of motion:

\[
\frac{d}{ds} x = \frac{\partial H}{\partial p_x} = \frac{p_x + H \cdot z}{[1 + f(p_\sigma)]};
\]

\[
\frac{d}{ds} p_x = -\frac{\partial H}{\partial x} = \frac{[p_z - H \cdot x]}{[1 + f(p_\sigma)]} \cdot H - [K_x^2 + g] \cdot x + N \cdot z + K_x \cdot f(p_\sigma) - \frac{\lambda}{2} \cdot (x^2 - z^2) - \frac{\mu}{6} (x^3 - 3 x z^2); \]

\[
\frac{d}{ds} z = \frac{\partial H}{\partial p_z} = \frac{p_z - H \cdot x}{[1 + f(p_\sigma)]};
\]

\[
\frac{d}{ds} p_z = -\frac{\partial H}{\partial z} = -\frac{[p_x + H \cdot z]}{[1 + f(p_\sigma)]} \cdot H - [K_z^2 - g] \cdot z + N \cdot x + K_z \cdot f(p_\sigma) - \lambda \cdot x z - \frac{\mu}{6} (z^3 - 3 x^2 z); \]

(4)
\[ \frac{d \sigma}{ds} = \frac{\partial H}{\partial p} = 1 - [1 + K_x \cdot x + K_z \cdot z] \cdot f'(p) - \frac{1}{2} \cdot \left[ p_x + H \cdot z \right]^2 + \left[ p_z - H \cdot x \right]^2 \cdot f'(p) \]
\[ \frac{d p}{ds} \sigma = -\frac{\partial H}{\partial \sigma} = \frac{1}{\beta_0} \cdot \frac{eV(s)}{E_0} \cdot \sin \left[ \frac{h \cdot 2\pi}{L} \cdot \sigma + \varphi \right] \]

Then we use series expansion of function \( f(p) \) from [2]:
\[ f(p) = f(0) + f'(0)p + f''(0) \frac{1}{2}p^2 + \ldots = p - \frac{1}{\gamma_0} \cdot \frac{1}{2}p^2 + \ldots \]

and the corresponding expansion of RHS of equations (4). In the following we take into account only an arbitrary polynomial (in terms of dynamical variables) expressions and neglecting all nonpolynomial types of expressions, i.e. we consider such approximations of RHS, which are not more than polynomial functions in dynamical variables and arbitrary functions of independent variables ("time" in our case, if we consider our system of equations as dynamical problem).

I POLYNOMIAL DYNAMICS

Introduction.

The first main part of our consideration is some variational approach to this problem, which reduces initial problem to the problem of solution of functional equations at the first stage and some algebraical problems at the second stage. We consider also two private cases of our general construction. In the first case (particular) we have for Riccati equations (particular quadratic approximations) the solution as a series on shifted Legendre polynomials, which is parameterized by the solution of reduced algebraical (also Riccati) system of equations. This is only an example of general construction. In the second case (general polynomial system) we have the solution in a compactly supported wavelet basis. Multiresolution expansion is the second main part of our construction. The solution is parameterized by solutions of two reduced algebraical problems, one as in the first case and the second is some linear problem, which is obtained from one of the next wavelet construction: Fast Wavelet Transform (FWT), Stationary Subdivision Schemes (SSS), the method of Connection Coefficients (CC).

Variational method.

Our problems may be formulated as the systems of ordinary differential equations
\[ dx_i/dt = f_i(x_j, t), \quad (i, j = 1, \ldots, n) \]
with fixed initial conditions \( x_i(0) \), where \( f_i \) are
not more than polynomial functions of dynamical variables $x_j$ and have arbitrary dependence of time. Because of time dilation we can consider only next time interval: $0 \leq t \leq 1$. Let us consider a set of functions $\Phi_i(t) = x_i dy_i/dt + f_i y_i$ and a set of functionals $F_i(x) = \int_0^1 \Phi_i(t) dt - x_i y_i \big|_0^1$, where $y_i(t)(y_i(0) = 0)$ are dual variables. It is obvious that the initial system and the system $F_i(x) = 0$ are equivalent. In the last part we consider the symplectization of this approach. Now we consider formal expansions for $x_i, y_i$:

$$x_i(t) = x_i(0) + \sum_k \lambda_i^k \varphi_k(t) \quad y_j(t) = \sum_r \eta_j^r \varphi_r(t),$$

(5)

where because of initial conditions we need only $\varphi_k(0) = 0$. Then we have the following reduced algebraical system of equations on the set of unknown coefficients $\lambda_i^k$ of expansions (5):

$$\sum_k \mu_{kr} \lambda_i^k - \gamma_i^r(\lambda_j) = 0$$

(6)

Its coefficients are $\mu_{kr} = \int_0^1 \varphi_k'(t) \varphi_r(t) dt$, $\gamma_i^r = \int_0^1 f_i(x_j, t) \varphi_r(t) dt$. Now, when we solve system (6) and determine unknown coefficients from formal expansion (5) we therefore obtain the solution of our initial problem. It should be noted if we consider only truncated expansion (5) with $N$ terms then we have from (6) the system of $N \times n$ algebraical equations and the degree of this algebraical system coincides with degree of initial differential system. So, we have the solution of the initial nonlinear (polynomial) problem in the form

$$x_i(t) = x_i(0) + \sum_{k=1}^N \lambda_i^k X_k(t),$$

(7)

where coefficients $\lambda_i^k$ are roots of the corresponding reduced algebraical problem (6). Consequently, we have a parametrization of solution of initial problem by solution of reduced algebraical problem (6). But in general case, when the problem of computations of coefficients of reduced algebraical system (6) is not solved explicitly as in the quadratic case, which we shall consider below, we have also parametrization of solution (4) by solution of corresponding problems, which appear when we need to calculate coefficients of (6). As we shall see, these problems may be explicitly solved in wavelet approach.

**The solutions**

Next we consider the construction of explicit time solution for our problem. The obtained solutions are given in the form (7), where in our first case we have $X_k(t) = Q_k(t)$, where $Q_k(t)$ are shifted Legendre polynomials and $\lambda_i^k$ are roots of reduced quadratic system of equations. In wavelet case $X_k(t)$ correspond to
multiresolution expansions in the base of compactly supported wavelets and $\lambda_i$ are the roots of corresponding general polynomial system (6) with coefficients, which are given by FWT, SSS or CC constructions. According to the variational method to give the reduction from differential to algebraical system of equations we need compute the objects $\gamma^j_\ell$ and $\mu_{ji}$, which are constructed from objects:

$$
\sigma_i \equiv \int_0^1 X_i(\tau) d\tau = (-1)^{i+1}, \quad \nu_{ij} \equiv \int_0^1 X_i(\tau) X_j(\tau) d\tau = \sigma_i \sigma_j + \frac{\delta_{ij}}{(2j+1)},
$$

$$
\mu_{ji} \equiv \int X_i^*(\tau) X_j(\tau) d\tau = \sigma_j F_1(i,0) + F_1(i,j),
$$

$$
F_1(r,s) = [1 - (-1)^{r+s}] \hat{s}(r-s-1), \quad \hat{s}(p) = \begin{cases} 1, & p \geq 0 \\ 0, & p < 0 \end{cases}
$$

$$
\beta_{klj} \equiv \int_0^1 X_k(\tau) X_l(\tau) X_j(\tau) d\tau = \sigma_k \sigma_l \sigma_j + \frac{\sigma_k \delta_{jl}}{2j+1} + \frac{\sigma_l \delta_{kj}}{2k+1} + \frac{\sigma_j \delta_{kl}}{2l+1},
$$

$$
\alpha_{klj} \equiv \int_0^1 X_k^* X_l^* X_j^* d\tau = \frac{1}{(j+k+l+1)R(1/2(i+j+k))} \times 
\frac{R(1/2(j+k-l))R(1/2(j-k+l))R(1/2(-j+k+l))}{R(1/2(i+j+k))},
$$

if $j+k+l = 2m, m \in Z$, and $\alpha_{klj} = 0$ if $j+k+l = 2m+1$; where $R(i) = (2i)!/(2^i i!)^2$, $Q_i = \sigma_i + P_i^*$, where the second equality in the formulae for $\sigma, \nu, \mu, \beta, \alpha$ hold for the first case.

Wavelet computations.

Now we give construction for computations of objects (8) in the wavelet case. We use some constructions from multiresolution analysis: a sequence of successive approximation closed subspaces $V_j$: \ldots $V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \subset \ldots$ satisfying the following properties: \( \bigcap_{j \in Z} V_j = 0, \bigcup_{j \in Z} V_j = L^2(\mathbb{R}), f(x) \in V_j \iff f(2x) \in V_{j+1} \).

There is a function $\varphi \in V_0$ such that \( \{ \varphi_{0,k}(x) = \varphi(x-k) \}_{k \in \mathbb{Z}} \) forms a Riesz basis for $V_0$. We use compactly supported wavelet basis: orthonormal basis for functions in $L^2(\mathbb{R})$. As usually $\varphi(x)$ is a scaling function, $\psi(x)$ is a wavelet function, where $\varphi_i(x) = \varphi(x-i)$. Scaling relation that defines $\varphi, \psi$ are

$$
\varphi(x) = \sum_{k=0}^{N-1} a_k \varphi(2x-k) = \sum_{k=0}^{N-1} a_k \varphi_k(2x), \quad \psi(x) = \sum_{k=-1}^{N-2} (-1)^k a_{k+1} \varphi(2x+k)
$$

Let be $f : \mathbb{R} \rightarrow \mathbb{C}$ and the wavelet expansion is

$$
f(x) = \sum_{i \in \mathbb{Z}} c_i \varphi_i(x) + \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}} c_{jk} \psi_{jk}(x)
$$

(9)
The indices \( k, \ell \) and \( j \) represent translation and scaling, respectively
\[
\varphi_{jl}(x) = 2^{j/2} \varphi(2^j x - \ell), \psi_{jk}(x) = 2^{j/2} \psi(2^j x - k)
\]
The set \( \{ \varphi_{j,k} \}_{k \in \mathbb{Z}} \) forms a Riesz basis for \( V_j \). Let \( W_j \) be the orthonormal complement of \( V_j \) with respect to \( V_{j+1} \). Just as \( V_j \) is spanned by dilation and translations of the scaling function, so are \( W_j \) spanned by translations and dilation of the mother wavelet \( \psi_{jk}(x) \). If in formulae (9) \( c_{jk} = 0 \) for \( j \geq J \), then \( f(x) \) has an alternative expansion in terms of dilated scaling functions only \( f(x) = \sum_{\ell \in \mathbb{Z}} c_{J\ell} \varphi_{J\ell}(x) \). This is a finite wavelet expansion, it can be written solely in terms of translated scaling functions. We use wavelet \( \psi(x) \), which has \( k \) vanishing moments \( \int x^k \psi(x) \text{d}x = 0 \), or equivalently \( x^k = \sum c_{\ell} \psi_{\ell}(x) \) for each \( k, 0 \leq k \leq K \). Also we have the shortest possible support: scaling function \( DN \) (where \( N \) is even integer) will have support \([0, N - 1]\) and \( N/2 \) vanishing moments.

According to CC method [12] we use the next construction. When \( N \) in scaling equation is a finite even positive integer the function \( \varphi(x) \) has compact support contained in \([0, N - 1]\). For a fixed triple \((d_1, d_2, d_3)\) only some \( \Lambda_{\ell m}^{d_1 d_2 d_3} \) are nonzero:
\[
2 - N \leq \ell \leq N - 2, \quad 2 - N \leq m \leq N - 2, \quad |\ell - m| \leq N - 2.
\]
There are \( M = 3N^2 - 9N + 7 \) such pairs \((\ell, m)\). Let \( \Lambda_{\ell m}^{d_1 d_2 d_3} \) be an \( M \)-vector, whose components are numbers \( \Lambda_{\ell m}^{d_1 d_2 d_3} \). Then we have the first key result: \( \Lambda \) satisfy the system of equations \((d = d_1 + d_2 + d_3)\)
\[
A \Lambda_{d_1 d_2 d_3} = 2^{1-d} \Lambda_{d_1 d_2 d_3}, \quad A_{\ell, m; q, r} = \sum_{p} \alpha_p \alpha_{q-2\ell+p} \alpha_{r-2m+p}
\]
By moment equations we have created a system of \( M + d + 1 \) equations in \( M \) unknowns. It has rank \( M \) and we can obtain unique solution by combination of LU decomposition and QR algorithm. The second key result gives us the 2-term connection coefficients:

\[
\Lambda^{d_1 d_2} = 2^{1-d} \Lambda^{d_1 d_2}, \quad d = d_1 + d_2, \quad A_{\ell, q} = \sum_p a_p a_{q - 2\ell + p}
\]

For nonquadratic case we have analogously additional linear problems for objects (10). Also, we use FWT and SSS for computing coefficients of reduced algebraic systems. We use for modelling D6,D8,D10 functions and programs RADAU and DOPRI for testing.

As a result we obtained the explicit time solution (7) of our problem. In comparison with wavelet expansion on the real line which we use now and in calculation of Galerkin approximation, Melnikov function approach, etc also we need to use periodized wavelet expansion, i.e. wavelet expansion on finite interval. Also in the solution of perturbed system we have some problem with variable coefficients. For solving last problem we need to consider one more refinement equation for scaling function \( \phi_2(x) \): \( \phi_2(x) = \sum_{k=0}^{N-1} a_k^2 \phi_2(2x - k) \) and corresponding wavelet expansion for variable coefficients \( b(t) \): \( \sum_k B_k^j(b) \phi_2(2^j x - k) \), where \( B_k^j(b) \) are functionals supported in a small neighborhood of \( 2^{-j} k \).

The solution of the first problem consists in periodizing. In this case we use expansion into periodized wavelets defined by \( \phi_{-j,k}^{\text{per}}(x) = 2^{j/2} \sum_{\ell} \phi(2^j x + 2^j \ell - k) \). All these modifications lead only to transformations of coefficients of reduced algebraic system, but general scheme remains the same.

II METAPLECTIC WAVELETS

In this part we continue the application of powerful methods of wavelet analysis to polynomial approximations of nonlinear accelerator physics problems. In part 1 we considered our main example and general approach for constructing wavelet representation for orbital motion in storage rings. But now we need take into account the Hamiltonian or symplectic structure related with system (4). Therefore, we need to consider generalized wavelets, which allow us to consider the corresponding symplectic structures, instead of compactly supported wavelet representation. By using the orbit method and constructions from the geometric quantization theory we consider the symplectic and Poisson structures associated with Weyl–Heisenberg wavelets by using metaplectic structure and the corresponding polarization. In the next part we consider applications to construction of Melnikov functions in the theory of homoclinic chaos in perturbed Hamiltonian systems.
In wavelet analysis the following three concepts are used now: 1). a square integrable representation \( U \) of a group \( G \), 2). coherent states over \( G \), 3). the wavelet transform associated to \( U \).

We have three important particular cases:

a) the affine \((ax + b)\) group, which yields the usual wavelet analysis

\[
[\pi(b, a)f](x) = \frac{1}{\sqrt{a}} f\left(\frac{x - b}{a}\right)
\]

b). the Weyl-Heisenberg group which leads to the Gabor functions, i.e. coherent states associated with windowed Fourier transform.

\[
[\pi(q, p, \varphi)f](x) = \exp(i\mu(\varphi - p(x - q)))f(x - q)
\]

In both cases time-frequency plane corresponds to the phase space of group representation.

c). also, we have the case of bigger group, containing both affine and Weyl-Heisenberg group, which interpolate between affine wavelet analysis and windowed Fourier analysis: affine Weyl–Heisenberg group \([13]\). But usual representation of it is not square–integrable and must be modified: restriction of the representation to a suitable quotient space of the group (the associated phase space in that case) restores square – integrability: \( G_{aW H} \to \) homogeneous space. Also, we have more general approach which allows to consider wavelets corresponding to more general groups and representations \([14], [15]\). Our goal is applications of these results to problems of Hamiltonian dynamics and as consequence we need to take into account symplectic nature of our dynamical problem. Also, the symplectic and wavelet structures must be consistent (this must be resemble the symplectic or Lie-Poisson integrator theory). We use the point of view of geometric quantization theory (orbit method) instead of harmonic analysis. Because of this we can consider (a) – (c) analogously.

**Metaplectic Group and Representations.**

Let \( Sp(n) \) be symplectic group, \( Mp(n) \) be its unique two- fold covering – metaplectic group. Let \( V \) be a symplectic vector space with symplectic form \((, )\), then \( R \oplus V \) is nilpotent Lie algebra - Heisenberg algebra:

\[
[R, V] = 0, \quad [v, w] = (v, w) \in R, \quad [V, V] = R.
\]

\( Sp(V) \) is a group of automorphisms of Heisenberg algebra.

Let \( N \) be a group with Lie algebra \( R \oplus V \), i.e. Heisenberg group. By Stone– von Neumann theorem Heisenberg group has unique irreducible unitary representation in which \( 1 \mapsto i \). This representation is projective: \( U_{g_1}U_{g_2} = c(g_1, g_2) \cdot U_{g_1g_2} \), where \( c \) is a map: \( Sp(V) \times Sp(V) \to S^1 \), i.e. \( c \) is \( S^1 \)-cocycle.
But this representation is unitary representation of universal covering, i.e. metaplectic group $Mp(V)$. We give this representation without Stone-von Neumann theorem. Consider a new group $F = N' \rtimes Mp(V), \rtimes$ is semidirect product (we consider instead of $N = R \oplus V$ the $N' = S^1 \times V, S^1 = (R/2\pi Z)$). Let $V^*$ be dual to $V$, $G(V^*)$ be automorphism group of $V^*$. Then $F$ is subgroup of $G(V^*)$, which consists of elements, which acts on $V^*$ by affine transformations. This is the key point!

Let $q_1, \ldots, q_n; p_1, \ldots, p_n$ be symplectic basis in $V$, $\alpha = pdq = \sum p_i dq_i$ and $d\alpha$ be symplectic form on $V^*$. Let $M$ be fixed affine polarization, then for $a \in F$ the map $a \mapsto \Theta_a$ gives unitary representation of $G$: $\Theta_n : H(M) \to H(M)$

Explicitly we have for representation of $N$ on $H(M)$:

$$(\Theta_f)^*(x) = e^{-iqx} f(x), \quad \Theta_p f(x) = f(x - p)$$

The representation of $N$ on $H(M)$ is irreducible. Let $A_q, A_p$ be infinitesimal operators of this representation

$A_q = \lim_{t \to 0} \frac{1}{t} [\Theta - tq - I], \quad A_p = \lim_{t \to 0} \frac{1}{t} [\Theta - tp - I],$

then $A_q f(x) = i(qx) f(x), \quad A_p f(x) = \sum p_j \frac{\partial f}{\partial x_j}(x)$

Now we give the representation of infinitesimal basic elements. Lie algebra of the group $F$ is the algebra of all (nonhomogeneous) quadratic polynomials of $(p,q)$ relatively Poisson bracket (PB). The basis of this algebra consists of elements $1, q_1, \ldots, q_n, p_1, \ldots, p_n, q_i q_j, q_i p_j, p_i p_j, \quad i, j = 1, \ldots, n, \quad i \leq j$,

$$PB \ {f, g} = \sum \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_j} \quad \text{and} \quad \{1, g\} = 0 \quad \text{for all } g,$$

$$\{p_i, q_j\} = \delta_{ij}, \quad \{p_i q_j, q_k\} = \delta_{ik} q_j, \quad \{p_i q_j, p_k\} = -\delta_{jk} p_i, \quad \{p_i p_j, p_k\} = 0,$$

$$\{p_i p_j, q_k\} = \delta_{ik} p_j + \delta_{jk} p_i, \quad \{q_i q_j, q_k\} = 0, \quad \{q_i q_j, p_k\} = -\delta_{ik} q_j - \delta_{jk} q_i$$

so, we have the representation of basic elements $f \mapsto A_f : 1 \mapsto i, q_k \mapsto ix_k$,

$$p_l \mapsto \frac{\delta}{\delta x^l}, p_i q_j \mapsto x^i \frac{\partial}{\partial x^j} + \frac{1}{2} \delta_{ij}, \quad p_k p_l \mapsto \frac{1}{i} \frac{\partial^k}{\partial x^k \partial x^l}, q_k q_l \mapsto ix^k x^l$$

This gives the structure of the Poisson manifolds to representation of any (nilpotent) algebra or in other words to continuous wavelet transform.

The Segal-Bargman Representation.

Let $z = 1/\sqrt{2} \cdot (p - iq), \quad \bar{z} = 1/\sqrt{2} \cdot (p + iq), \quad p = (p_1, \ldots, p_n)$. $F_n$ is the space of holomorphic functions of $n$ complex variables with $(f, f) < \infty$, where

$$(f, g) = (2\pi)^{-n} \int f(z) \overline{g(z)} e^{-|z|^2} dp dq$$
Consider a map $U : H \rightarrow F_n$, where $H$ is with real polarization, $F_n$ is with complex polarization, then we have

$$(U\Psi)(a) = \int A(a, q)\Psi(q) dq,$$

where $A(a, q) = \pi^{-n/4} e^{-1/2(a^2 + q^2)} + \sqrt{2}aq$

i.e. the Bargmann formula produce wavelets. We also have the representation of Heisenberg algebra on $F_n$:

$$U \frac{\partial}{\partial q_j} U^{-1} = \frac{1}{\sqrt{2}} \left( z_j - \frac{\partial}{\partial z_j} \right), \quad U q_j U^{-1} = -\frac{i}{\sqrt{2}} \left( z_j + \frac{\partial}{\partial z_j} \right)$$

and also: $\omega = d\beta = dp \wedge dq$, where $\beta = \bar{i} \bar{z} dz$.

**Orbital Theory for Wavelets.**

Let coadjoint action be $< g \cdot f, Y >= < f, Ad(g)^{-1} Y >$, where $<,>$ is pairing $g \in G, \ f \in g^\ast, \ Y \in G$. The orbit is $O_f = G \cdot f \equiv G/G(f)$. Also, let $A = A(M)$ be algebra of functions, $V(M)$ is $A$-module of vector fields, $A^p$ is $A$-module of $p$-forms. Vector fields on orbit is

$$\sigma(O, X)_f(\phi) = \frac{d}{dt} (\phi(\exp tXf))|_{t=0}$$

where $\phi \in A(O), \ f \in O$. Then $O_f$ are homogeneous symplectic manifolds with 2-form $\Omega(\sigma(O, X)_f, \sigma(O, Y)_f) = < f, [X, Y] >$, and $d\Omega = 0$. PB on $O$ have the next form \{ $\Psi_1, \Psi_2$ \} = $p(\Psi_1)\Psi_2$ where $p$ is $A^1(O) \rightarrow V(O)$ with definition $\Omega(p(\alpha), X) = i\langle X, \alpha \rangle$. Here $\Psi_1, \Psi_2 \in A(O)$ and $A(O)$ is Lie algebra with bracket \{ \}. Now let $N$ be a Heisenberg group. Consider adjoint and coadjoint representations in some particular case. $N = (z, t) \in C \times R, z = p + iq; \ compositions$ in $N$ are $(z, t) \cdot (z', t') = (z + z', t + t' + B(z, z'))$, where $B(z, z') = pq' - qp'$. Inverse element is $(-t, -z)$. Lie algebra $n$ of $N$ is $(\zeta, \tau) \in C \times R$ with bracket $[[(\zeta, \tau), (\zeta', \tau')] = (0, B(\zeta, \zeta'))$. Centre is $\bar{z} \in n$ and generated by $(0,1); \ Z$ is a subgroup $\exp \bar{z}$. Adjoint representation $N$ on $n$ is given by formula $Ad(z, t)(\zeta, \tau) = (\zeta, \tau + B(z, \zeta))$ Coadjoint: for $f \in n^\ast, \ g = (z, t), (g \cdot f)(\zeta, \tau) = f(\zeta, \tau - B(z, \zeta))f(0, 1)$ then orbits for which $f|_z \not= 0$ are plane in $n^\ast$ given by equation $f(0, 1) = \mu$. If $X = (\zeta, 0), \ Y = (\zeta', 0), \ X, Y \in n$ then symplectic structure is

$$\Omega(\sigma(O, X)_f, \sigma(O, Y)_f) = < f, [X, Y] > = f(0, B(\zeta, \zeta'))\mu B(\zeta, \zeta')$$

Also we have for orbit $O_\mu = N/Z$ and $O_\mu$ is Hamiltonian $G$-space.
Kirillov Character Formula or Analogy of Gabor Wavelets.

Let $U$ denote irreducible unitary representation of $N$ with condition $U(0,t) = \exp(it\ell) \cdot 1$, where $\ell \neq 0$, then $U$ is equivalent to representation $T_\ell$ which acts in $L^2(\mathbb{R})$ according to

$$T_\ell(z,t)\phi(x) = \exp(i\ell(t + px)) \phi(x - q)$$

If instead of $N$ we consider $E(2)/\mathbb{R}$ we have $S^1$ case and we have Gabor functions on $S^1$.

Oscillator Group.

Let $O$ be an oscillator group, i.e. semidirect product of $\mathbb{R}$ and Heisenberg group $N$. Let $H, P, Q, I$ be standard basis in Lie algebra $o$ of the group $O$ and $H^*, P^*, Q^*, I^*$ be dual basis in $o^*$. Let functional $f = (a, b, c, d)$ be $aI^* + bP^* + cQ^* + dH^*$. Let us consider complex polarizations $h = (H, I, P + iQ)$, $\bar{h} = (I, H, P - iQ)$ Induced from $h$ representation, corresponding to functional $f$ (for $a > 0$), unitary equivalent to the representation

$$W(t, n) f(y) = \exp(i(t(h - 1/2))) \cdot U_a(n) V(t),$$

where

$$V(t) = \exp[-it(P^2 + Q^2)/2a], \quad P = -d/dx, \quad Q = iax,$$

and $U_a(n)$ is irreducible representation of $N$, which have the form $U_a(z) = \exp(iaz)$ on the center of $N$. Here we have: $U(n=(x,y,z))$ is Schrödinger representation, $U_t(n) = U(t(n))$ is the representation obtained from previous by automorphism (time translation) $n \rightarrow t(n)$; $U_t(n) = U(t(n))$ is also unitary irreducible representation of $N$. $V(t) = \exp(i(P^2 + Q^2 + h - 1/2))$ is an operator, which according to Stone–von Neumann theorem has the property $U_t(n) = V(t)U(n)V(t)^{-1}$.

This is our last private case, but according to our approach we can construct by using methods of geometric quantization theory many "symplectic wavelet constructions" with corresponding symplectic or Poisson structure on it. Very useful particular spline–wavelet basis with uniform exponential control on stratified and nilpotent Lie groups was considered in [15].

III MELNIKOV FUNCTIONS APPROACH

In this part we continue the application of the methods of wavelet analysis to polynomial approximations of nonlinear accelerator physics problems. Now we consider one problem of nontrivial dynamics related with complicated differential geometrical and topological structures of system (4). We give some points of applications of wavelet methods from the preceding parts to Melnikov approach in the theory of homoclinic chaos in perturbed Hamiltonian systems.
Routes to Chaos

Now we give some points of our program of understanding routes to chaos in some Hamiltonian systems in the wavelet approach [3]-[11]. All points are:

1. A model.
2. A computer zoo. The understanding of the computer zoo.
3. A naive Melnikov function approach.
4. A naive wavelet description of (hetero) homoclinic orbits (separatrix) and quasiperiodic oscillations.
5. Symplectic Melnikov function approach.
6. Splitting of separatrix... → stochastic web with magic symmetry, Arnold diffusion and all that.

1. As a model we have two frequencies perturbations of particular case of system (4):

\[
\begin{align*}
\dot{x}_1 &= x_2, \quad \dot{x}_3 = x_4, \quad \dot{x}_5 = 1, \quad \dot{x}_6 = 1, \\
\dot{x}_2 &= -ax_1 - b[\cos(rx_5) + \cos(sx_6)]x_1 - dx^3_1 - mdx_1x^2_3 - px_2 - \varphi(x_5) \\
\dot{x}_4 &= ex_3 - f[\cos(rx_5) + \cos(sx_6)]x_3 - gx^3_3 - kx^2_1x_3 - gx_4 - \psi(x_5)
\end{align*}
\]

or in Hamiltonian form

\[
\dot{x} = J \cdot \nabla H(x) + \varepsilon g(x, \Theta), \quad \dot{\Theta} = \omega, \quad (x, \Theta) \in \mathbb{R}^4 \times T^2, \quad T^2 = S^1 \times S^1,
\]

for \(\varepsilon = 0\) we have:

\[
\dot{x} = J \cdot \nabla H(x), \quad \dot{\Theta} = \omega \quad (11)
\]

2. For pictures and details one can see [5], [10]. The key point is the splitting of separatrix (homoclinic orbit) and transition to fractal sets on the Poincare sections.

3. For \(\varepsilon = 0\) we have homoclinic orbit \(\bar{x}_0(t)\) to the hyperbolic fixed point \(x_0\). For \(\varepsilon \neq 0\) we have normally hyperbolic invariant torus \(T_\varepsilon\) and condition on transversally intersection of stable and unstable manifolds \(W^s(T_\varepsilon)\) and \(W^u(T_\varepsilon)\) in terms of Melnikov functions \(M(\Theta)\) for \(\bar{x}_0(t)\).

\[
M(\Theta) = \int_{-\infty}^{\infty} \nabla H(\bar{x}_0(t)) \wedge g(\bar{x}_0(t), \omega t + \Theta) dt
\]

This condition has the next form:

\[
M(\Theta_0) = 0, \quad \sum_{j=1}^{2} \omega_j \frac{\partial}{\partial \Theta_j} M(\Theta_0) \neq 0
\]
According to the approach of Birkhoff-Smale-Wiggins we determined the region in parameter space in which we observe the chaotic behaviour [5], [10].

4. If we cannot solve equations (11) explicitly in time, then we use the wavelet approach from part 1 for the computations of homoclinic (heteroclinic) loops as the wavelet solutions of system (11). For computations of quasiperiodic Melnikov functions

\[
M_{m/n}(t_0) = \int_0^{mT} DH(x_\alpha(t)) \wedge g(x_\alpha(t), t + t_0) dt
\]

we used periodization of wavelet solution from part 1.

5. We also used symplectic Melnikov function approach

\[
M_i(z) = \lim_{j \to \infty} \int_{-T_j}^{T_j} \{h_i, \hat{h}\}_{\Psi(t, z)} dt
\]

\[
d_i(z, \varepsilon) = h_i(z^u_\varepsilon) - h_i(z^s_\varepsilon) = \varepsilon M_i(z) + O(\varepsilon^2)
\]

where \{, \} is the Poisson bracket, \(d_i(z, \varepsilon)\) is the Melnikov distance. So, we need symplectic invariant wavelet expressions for Poisson brackets. The computations are produced according to part 2.

6. Some hypothesis about strange symmetry of stochastic web in multi-degree-of-freedom Hamiltonian systems [11].

**IV SYMPLECTIC TOPOLOGY AND WAVELETS**

Now we consider another type of wavelet approach which gives us a possibility to parametrize Arnold–Weinstein curves or closed loops in Hamiltonian systems by generalized refinement equations or Quadratic Mirror Filters equations.

**Wavelet Parametrization in Floer Approach.**

Now we consider the generalization of our wavelet variational approach to the symplectic invariant calculation of closed loops in Hamiltonian systems [16]. We also have the parametrization of our solution by some reduced algebraical problem but in contrast to the general case where the solution is parametrized by construction based on scalar refinement equation, in symplectic case we have parametrization of the solution by matrix problems – Quadratic Mirror Filters equations [17].

The action functional for loops in the phase space is [16]

\[
F(\gamma) = \int_\gamma pdq - \int_0^1 H(t, \gamma(t)) dt
\]

The critical points of \(F\) are those loops \(\gamma\), which solve the Hamiltonian equations associated with the Hamiltonian \(H\) and hence are periodic orbits. By the way, all
critical points of $F$ are the saddle points of infinite Morse index, but surprisingly this approach is very effective. This will be demonstrated using several variational techniques starting from minimax due to Rabinowitz and ending with Floer homology. So, $(M, \omega)$ is symplectic manifolds, $H : M \to \mathbb{R}$, $H$ is Hamiltonian, $X_H$ is unique Hamiltonian vector field defined by

$$\omega(X_H(x), v) = -dH(x)(v), \quad v \in T_xM, \quad x \in M,$$

where $\omega$ is the symplectic structure. A $T$-periodic solution $x(t)$ of the Hamiltonian equations

$$\dot{x} = X_H(x) \quad \text{on } M$$

is a solution, satisfying the boundary conditions $x(T) = x(0), T > 0$. Let us consider the loop space $\Omega = C^\infty(S^1, \mathbb{R}^{2n})$, where $S^1 = \mathbb{R}/\mathbb{Z}$, of smooth loops in $\mathbb{R}^{2n}$. Let us define a function $\Phi : \Omega \to \mathbb{R}$ by setting

$$\Phi(x) = \int_0^1 \frac{1}{2} < -J\dot{x}, x > dt - \int_0^1 H(x(t)) dt, \quad x \in \Omega$$

The critical points of $\Phi$ are the periodic solutions of $\dot{x} = X_H(x)$. Computing the derivative at $x \in \Omega$ in the direction of $y \in \Omega$, we find

$$\Phi'(x)(y) = \frac{d}{d\epsilon} \Phi(x + \epsilon y)|_{\epsilon=0} = \int_0^1 < -J\dot{x} - \nabla H(x), y > dt$$

Consequently, $\Phi'(x)(y) = 0$ for all $y \in \Omega$ iff the loop $x$ satisfies the equation

$$-J\dot{x}(t) - \nabla H(x(t)) = 0,$$

i.e. $x(t)$ is a solution of the Hamiltonian equations, which also satisfies $x(0) = x(1)$, i.e. periodic of period 1. Periodic loops may be represented by their Fourier series:

$$x(t) = \sum_{k \in \mathbb{Z}} e^{ik2\pi Jt} x_k, \quad x_k \in \mathbb{R}^{2k},$$

where $J$ is quasicomplex structure. We give relations between quasicomplex structure and wavelets in [11]. But now we use the construction [17] for loop parametrization. It is based on the theorem about explicit bijection between the Quadratic Mirror Filters (QMF) and the whole loop group: $LG : S^1 \to G$. In particular case we have relation between QMF-systems and measurable functions $\chi : S^1 \to U(2)$ satisfying

$$\chi(\omega + \pi) = \chi(\omega) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

in the next explicit form
\[
\begin{bmatrix}
\dot{\Phi}_0(\omega) & \dot{\Phi}_0(\omega + \pi) \\
\dot{\Phi}_1(\omega) & \dot{\Phi}_1(\omega + \pi)
\end{bmatrix} = \chi(\omega) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \chi(\omega + \pi) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\]

where
\[
|\dot{\Phi}_i(\omega)|^2 + |\dot{\Phi}_i(\omega + \pi)|^2 = 2, \quad i = 0, 1.
\]

Also, we have symplectic structure on \( LG \)
\[
\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle d\theta
\]
So, we have the parametrization of periodic orbits (Arnold–Weinstein curves) by reduced QMF equations.

Extended version and related results may be found in [3]-[11].

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