Approximation of BV by SBV functions in metric spaces *

Panu Lahti
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Abstract

In a complete metric space that is equipped with a doubling measure and supports a Poincaré inequality, we show that functions of bounded variation (BV functions) can be approximated in the strict sense and pointwise uniformly by special functions of bounded variation, without adding significant jumps. As a main tool, we study the variational 1-capacity and its BV analog.

1 Introduction

In the theory of functions of bounded variation, one is often interested in approximating a BV function by more regular functions, see e.g. [1, 11, 12, 26]. Already the definition of the total variation in metric spaces is based on such approximations. The variation measure of a BV function can be decomposed into three parts: the absolutely continuous part, the Cantor part, and the jump part. Of these, the absolutely continuous part is of the same dimension as the space, and the jump part is of dimension one less than the space. The Cantor part can be of any dimension between these, making it often more difficult to analyze than the other two parts. A function is said to be in the SBV class (special functions of bounded variation), first introduced in [3], if its variation measure has no Cantor part. The recent paper [13] (as well as the earlier papers [11, 12]) studied how SBV function in Euclidean spaces can be approximated in the BV norm by piecewise smooth functions. This is based on the fact that outside its jump set, an SBV function is essentially a Sobolev function, and then it is possible to construct convolution approximations that are close to the original function in the Sobolev/BV norm.

Due to the lack of structure of the Cantor part, it is in some way a rather more subtle problem to approximate a general BV function by SBV functions, and little seems to be known in this direction. It is impossible to find such approximations in the BV norm (see Example 5.14) but we show in this paper that such approximations

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can be obtained in the following sense; this will be given (with some more details) in Corollary 5.15.

**Theorem 1.1.** Let $\Omega$ be an open set and let $u \in BV(\Omega)$. Then there exists a sequence $(u_i) \subset SBV(\Omega)$ such that

- $u_i \to u$ in $L^1(\Omega)$ and $\|Du_i\|(\Omega) \to \|Du\|(\Omega)$,
- $u_i \to u$ uniformly in $\Omega$,
- $\mathcal{H}(S_{u_i} \setminus S_u) = 0$ for all $i \in \mathbb{N}$.

The first condition is often expressed by saying that $(u_i)$ converges to $u$ strictly in $BV(\Omega)$. The last condition expresses the fact that the approximation procedure does not add any significant jump set; see Section 2 for definitions. It is also possible to ensure that $u_i \geq u$ and that the $u_i$’s have the same “boundary values” as $u$. Thus our result shows that it is sufficient to infimize various functionals defined for the BV class, possibly involving also boundary values and obstacles, only over the SBV class. We discuss some implications of the result at the end of the paper.

In order to prove the approximation result, we first study some properties of a class of BV functions with zero boundary values, which was previously studied in [33]. This is done in Section 3. Then in Section 4 we establish the key tool needed for the approximation result, namely a result on capacities that should be also of independent interest. The *variational $p$-capacity* $\text{cap}_p$ is an essential concept in nonlinear potential theory, see e.g. the monographs [6, 21, 35]. In the case $p = 1$, it is natural to also consider the BV analog $\text{cap}_{BV}$ of the variational 1-capacity, and such a notion has been studied in the metric setting in [20, 25, 27]. In [20] the authors considered a slightly different definition of this capacity compared to ours, but nonetheless it follows from [20, Theorem 4.3, Corollary 4.7] that

$$\text{cap}_{BV}(A, D) \simeq \text{cap}_{lip, 1}(A, D)$$

when $A$ is a compact subset of an open set $D$, and $\text{cap}_{lip, 1}$ is a Lipschitz version of the variational 1-capacity. By the sign “$\simeq$” we mean that the quantities are comparable, with constants of comparison depending only on the space. In [33, Theorem 4.23] it was then shown that in fact equality holds. In particular, this implies that

$$\text{cap}_{BV}(A, D) = \text{cap}_1(A, D).$$

In this paper we show that this equality holds much more generally, namely whenever $A$ is a *quasiclosed* set and $D$ is a *quasiopen* set. This is given in Theorem 4.5.

Recently, there has been much interest in studying BV functions and other topics of analysis in the abstract setting of metric measure spaces, see e.g. [2, 5, 36]. The standard assumptions in this setting are that $(X, d, \mu)$ is a complete metric space equipped with *doubling* Radon measure $\mu$, and that the space supports a *Poincaré inequality*. While our results seem to be mostly new even in Euclidean spaces, in this paper we also work in such a metric space setting.
2 Preliminaries

In this section we introduce the definitions, assumptions, and some standard background results used in the paper.

Throughout this paper, \((X, d, \mu)\) is a complete metric space that is equipped with a metric \(d\) and a Borel regular outer measure \(\mu\) satisfying a doubling property, meaning that there exists a constant \(C_d \geq 1\) such that

\[
0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty
\]

for every ball \(B(x, r) := \{y \in X : d(y, x) < r\}\). Given a ball \(B = B(x, r)\) and \(\beta > 0\), we sometimes abbreviate \(\beta B := B(x, \beta r)\). When we want to state that a constant \(C\) depends on the parameters \(a, b, \ldots\), we write \(C = C(a, b, \ldots)\). When a property holds outside a set of \(\mu\)-measure zero, we say that it holds almost everywhere, abbreviated a.e.

All functions defined on \(X\) or its subsets will take values in \([-\infty, \infty]\). A complete metric space equipped with a doubling measure is proper, that is, closed and bounded sets are compact. Given a \(\mu\)-measurable set \(A \subset X\), we define \(L^1_{\text{loc}}(A)\) as the class of functions \(u\) on \(A\) such that for every \(x \in A\) there exists \(r > 0\) such that \(u \in L^1(A \cap B(x, r))\). Other local spaces of functions are defined similarly. For an open set \(\Omega \subset X\), a function is in the class \(L^1_{\text{loc}}(\Omega)\) if and only if it is in \(L^1(\Omega^\prime)\) for every open \(\Omega^\prime \subset \Omega\). Here \(\Omega^\prime \subset \Omega\) means that \(\Omega^\prime\) is a compact subset of \(\Omega\).

For any \(0 < R < \infty\), the codimension one Hausdorff content of a set \(A \subset X\) is

\[
\mathcal{H}_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq R \right\}.
\]

The codimension one Hausdorff measure is then defined as

\[
\mathcal{H}(A) := \lim_{R \to 0} \mathcal{H}_R(A).
\]

By a curve we mean a nonconstant rectifiable continuous mapping from a compact interval of the real line into \(X\). The length of a curve \(\gamma\) is denoted by \(\ell_\gamma\). We will assume every curve to be parametrized by arc-length, which can always be done (see e.g. [17, Theorem 3.2]). A nonnegative Borel function \(g\) on \(X\) is an upper gradient of a function \(u\) on \(X\) if for all curves \(\gamma\), we have

\[
|u(x) - u(y)| \leq \int_0^{\ell_\gamma} g(\gamma(s)) \, ds,
\]

where \(x\) and \(y\) are the end points of \(\gamma\). We interpret \(|u(x) - u(y)| = \infty\) whenever at least one of \(|u(x)|, |u(y)|\) is infinite. Upper gradients were originally introduced in [22].

We say that a family of curves \(\Gamma\) is of zero 1-modulus if there is a nonnegative Borel function \(\rho \in L^1(X)\) such that for all curves \(\gamma \in \Gamma\), the curve integral \(\int_{\gamma} \rho \, ds\) is infinite. A property is said to hold for 1-almost every curve if it fails only for a curve family with zero 1-modulus. If \(g\) is a nonnegative \(\mu\)-measurable function on \(X\)
and (2.1) holds for 1-almost every curve, we say that $g$ is a 1-weak upper gradient of $u$. By only considering curves $\gamma$ in $A \subset X$, we can talk about a function $g$ being a (1-weak) upper gradient of $u$ in $A$.

Given a $\mu$-measurable set $H \subset X$, we let

$$\|u\|_{N^{1,1}(H)} := \|u\|_{L^1(H)} + \inf \|g\|_{L^1(H)},$$

where the infimum is taken over all 1-weak upper gradients $g$ of $u$ in $H$. The substitute for the Sobolev space $W^{1,1}$ in the metric setting is the Newton-Sobolev space

$$N^{1,1}(H) := \{ u : \|u\|_{N^{1,1}(H)} < \infty \},$$

which was first introduced in [38]. We understand a Newton-Sobolev function to be defined at every $x \in H$ (even though $\| \cdot \|_{N^{1,1}(H)}$ is then only a seminorm). It is known that for any $u \in N^{1,1}_{\text{loc}}(X)$ there exists a minimal 1-weak upper gradient of $u$ in $H$, always denoted by $g_u$, satisfying $g_u \leq g$ a.e. in $H$ for any 1-weak upper gradient $g \in L^1_{\text{loc}}(H)$ of $u$ in $H$, see [6, Theorem 2.25].

We will assume throughout the paper that $X$ supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for every ball $B(x, r)$, every $u \in L^1_{\text{loc}}(X)$, and every upper gradient $g$ of $u$, we have

$$\int_{B(x, r)} |u - u_{B(x, r)}| \, d\mu \leq C_P r \int_{B(x, \lambda r)} g \, d\mu,$$

where

$$u_{B(x, r)} := \frac{1}{\mu(B(x, r))} \int_{B(x, r)} u \, d\mu.$$

The 1-capacity of a set $A \subset X$ is defined by

$$\text{Cap}_1(A) := \inf \|u\|_{N^{1,1}(X)},$$

where the infimum is taken over all functions $u \in N^{1,1}(X)$ such that $u \geq 1$ on $A$. We know that $\text{Cap}_1$ is an outer capacity, meaning that

$$\text{Cap}_1(A) = 0 \text{ if and only if } \text{Cap}_1(W) = 0,$$

for any $A \subset X$, see e.g. [6, Theorem 5.31].

If a property holds outside a set $A \subset X$ with $\text{Cap}_1(A) = 0$, we say that it holds 1-quasieverywhere, or 1-q.e. If $H \subset X$ is $\mu$-measurable, then

$$u = v \text{ 1-q.e. implies } \|u - v\|_{N^{1,1}(H)} = 0,$$

see [6, Proposition 1.61].

By [19, Theorem 4.3, Theorem 5.1], we know that for $A \subset X$, $\text{Cap}_1(A) = 0$ if and only if $\mathcal{H}(A) = 0$.

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, following [36]. See also the monographs [4, 14, 15, 16, 39] for
the classical theory in the Euclidean setting. We will always denote by \( \Omega \subset X \) an open set. Given a function \( u \in L^1_{\text{loc}}(\Omega) \), we define the total variation of \( u \) in \( \Omega \) by

\[
\|Du\|(\Omega) := \inf \left\{ \liminf_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu : u_i \in N^{1,1}_{\text{loc}}(\Omega), u_i \to u \text{ in } L^1_{\text{loc}}(\Omega) \right\},
\]

where each \( g_{u_i} \) is the minimal 1-weak upper gradient of \( u_i \) in \( \Omega \). (In \cite{36}, local Lipschitz constants were used in place of upper gradients, but the theory can be developed similarly with either definition.) We say that a function \( u \in L^1(\Omega) \) is of bounded variation, and denote \( u \in BV(\Omega) \), if \( \|Du\|(\Omega) < \infty \). For an arbitrary set \( A \subset X \), we define

\[
\|Du\|(A) := \inf \{ \|Du\|(W) : A \subset W, W \subset X \text{ is open} \}.
\]

In general, we understand the expression \( \|Du\|(A) < \infty \) to mean that there exists some open set \( \Omega \supset A \) such that \( u \) is defined on \( \Omega \) with \( u \in L^1_{\text{loc}}(\Omega) \) and \( \|Du\|(\Omega) < \infty \). If \( u \in L^1_{\text{loc}}(\Omega) \) and \( \|Du\|(\Omega) < \infty \), \( \|Du\|(-) \) is a Radon measure on \( \Omega \) by \cite[Theorem 3.4]{36}. A \( \mu \)-measurable set \( E \subset X \) is said to be of finite perimeter if \( \|D\chi_E\|(X) < \infty \), where \( \chi_E \) is the characteristic function of \( E \). The perimeter of \( E \) in \( \Omega \) is also denoted by

\[
P(E, \Omega) := \|D\chi_E\|(\Omega).
\]

The BV norm is defined by

\[
\|u\|_{BV(\Omega)} := \|u\|_{L^1(\Omega)} + \|Du\|(\Omega).
\]

The measure-theoretic interior of a set \( E \subset X \) is defined by

\[
I_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} = 0 \right\},
\]

and the measure-theoretic exterior by

\[
O_E := \left\{ x \in X : \lim_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0 \right\}.
\]

The measure-theoretic boundary \( \partial^* E \) is defined as the set of points \( x \in X \) at which both \( E \) and its complement have strictly positive upper density, i.e.

\[
\limsup_{r \to 0} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0. \tag{2.4}
\]

Given an open set \( \Omega \subset X \) and a \( \mu \)-measurable set \( E \subset X \) with \( P(E, \Omega) < \infty \), we know that for any Borel set \( A \subset \Omega \),

\[
P(E, A) = \int_{\partial^* E \cap A} \theta_E \, d\mathcal{H}, \tag{2.5}
\]

where \( \theta_E : X \to [\alpha, C_d] \) with \( \alpha = \alpha(C_d, C_P, \lambda) > 0 \), see \cite[Theorem 5.3, Theorem 5.4]{2} and \cite[Theorem 4.6]{5}. The following coarea formula is given in \cite[Proposition 4.2]{36}: if \( \Omega \subset X \) is an open set and \( u \in BV(\Omega) \), then for any Borel set \( A \subset \Omega \),

\[
\|Du\|(A) = \int_{-\infty}^{\infty} P(\{u > t\}, A) \, dt. \tag{2.6}
\]
If \( \|Du\|_1(\Omega) < \infty \), from (2.5) and (2.6) we get the absolute continuity
\[
\|Du\| \ll \mathcal{H} \text{ on } \Omega.
\] (2.7)

If \( u, v \in L^1_{\text{loc}}(\Omega) \), then
\[
\|D\min\{u, v\}\|_1(\Omega) + \|D\max\{u, v\}\|_1(\Omega) \leq \|Du\|_1(\Omega) + \|Dv\|_1(\Omega);
\] (2.8)

for a proof see e.g. [33, Lemma 3.1]. Moreover, for any \( u, v \in L^1_{\text{loc}}(\Omega) \), it is straightforward to show that
\[
\|D(u + v)\|_1(\Omega) \leq \|Du\|_1(\Omega) + \|Dv\|_1(\Omega).
\] (2.9)

Since \( \text{Lip}_{\text{loc}}(\Omega) \) is dense in \( N^{1,1}_{\text{loc}}(\Omega) \), see [6, Theorem 5.47], it follows that
\[
\|Du\|_1(\Omega) \leq \int_{\Omega} g_u \, d\mu \text{ for every } u \in N^{1,1}_{\text{loc}}(\Omega).
\] (2.10)

The lower and upper approximate limits of a function \( u \) on \( \Omega \) are defined respectively by
\[
\begin{align*}
\underline{u}(x) := & \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x, r) \cap \{u < t\})}{\mu(B(x, r))} = 0 \right\} \\
\overline{u}(x) := & \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0} \frac{\mu(B(x, r) \cap \{u > t\})}{\mu(B(x, r))} = 0 \right\},
\end{align*}
\]
for \( x \in \Omega \). We then define the jump set as
\[
S_u := \{u^\wedge < u^\vee\}.
\]

Note that since we understand \( u^\wedge \) and \( u^\vee \) to be defined only on \( \Omega \), also \( S_u \) is a subset of \( \Omega \).

Unlike Newton-Sobolev functions, we understand BV functions to be \( \mu \)-equivalence classes. To consider fine properties, we need to consider the pointwise representatives \( u^\wedge \) and \( u^\vee \). The following fact clarifies the relationship between the different pointwise representatives; it essentially follows from the Lebesgue point result for Newton-Sobolev functions given in [23].

**Proposition 2.11** ([33, Proposition 3.10]). Let \( \Omega \subset X \) be an open set and let \( u \in N^{1,1}(\Omega) \). Then \( u = u^\wedge = u^\vee \mathcal{H}\text{-a.e. in } \Omega \).

By [5, Theorem 5.3], the variation measure of a BV function can be decomposed into the absolutely continuous and singular part, and the latter into the Cantor and jump part (which are all Radon measures), as follows. Given an open set \( \Omega \subset X \) and \( u \in \text{BV}(\Omega) \), we have for any Borel set \( A \subset \Omega \)
\[
\|Du\|_1(A) = \|Du\|^a_1(A) + \|Du\|^s_1(A) \\
= \|Du\|^a_1(A) + \|Du\|^c_1(A) + \|Du\|^j_1(A) \\
= \int_A \theta d\mu + \|Du\|^c_1(A) + \int_{A \cap S_u} \int_{u^\wedge(x)} u^\vee(x) \, dt \, d\mathcal{H}(x),
\] (2.12)
where $a \in L^1(\Omega)$ is the density of the absolutely continuous part and the functions $\theta_{\{u>t\}} \in [\alpha, C_a]$ are as in (2.5). It follows that $S_u$ is $\sigma$-finite with respect to $\mathcal{H}$. Moreover, $\|Du\|^{c}(S) = 0$ for any $S \subset \Omega$ that is $\sigma$-finite with respect to $\mathcal{H}$. If $\|Du\|^{c}(\Omega) = 0$, we say that $u \in \text{SBV}(\Omega)$.

**Definition 2.13.** We say that a set $A \subset H$ is 1-quasiopen with respect to a set $H \subset X$ if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\text{Cap}_1(G) < \varepsilon$ and $A \cup G$ is relatively open in the subspace topology of $H$.

We say that a set $A \subset H$ is 1-quasiclosed with respect to $H$ if $H \setminus A$ is 1-quasiopen with respect to $H$, or equivalently, if for every $\varepsilon > 0$ there is an open set $G \subset X$ such that $\text{Cap}_1(G) < \varepsilon$ and $A \setminus G$ is relatively closed in the subspace topology of $H$.

When $H = X$, we omit mention of it.

Given $H \subset X$, we say that $u$ is 1-quasi (lower/upper semi-)continuous on $H$ if for every $\varepsilon > 0$ there exists an open set $G \subset X$ such that $\text{Cap}_1(G) < \varepsilon$ and $u|_{H \setminus G}$ is real-valued (lower/upper semi-)continuous.

It is a well-known fact that Newton-Sobolev functions are quasicontinuous, see [10, Theorem 1.1] or [6, Theorem 5.29]. This is also true in quasiopen sets; the following is a special case of [9, Theorem 1.3]. Note that 1-quasiopen sets are $\mu$-measurable by [7, Lemma 9.3].

**Theorem 2.14.** Let $U \subset X$ be 1-quasiopen and let $u \in N^{1,1}_{\text{loc}}(U)$. Then $u$ is 1-quasicontinuous on $U$.

BV functions have the following partially analogous quasi-semicontinuity property.

**Proposition 2.15.** Let $\Omega \subset X$ be open, let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|(\Omega) < \infty$, and let $\varepsilon > 0$. Then $u^\wedge$ is 1-quasi lower semicontinuous and $u^\vee$ is 1-quasi upper semicontinuous on $\Omega$.

**Proof.** This follows from [32, Corollary 4.2], which is based on [34, Theorem 1.1].

We also have the following.

**Theorem 2.16** ([30, Theorem 4.3]). Let $U \subset X$ be 1-quasiopen. If $\|Du\|(U) < \infty$, then

$$\|Du\|(U) = \inf \left\{ \liminf_{i \to \infty} \int_G g_{u_i} \, d\mu, \ u_i \in N^{1,1}_{\text{loc}}(U), \ u_i \to u \text{ in } L^1_{\text{loc}}(U) \right\},$$

where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $U$.

For any $D \subset H \subset X$, with $H \mu$-measurable, the space of Newton-Sobolev functions with zero boundary values is defined as

$$N^{1,1}_{\text{loc}}(D, H) := \{u|_D : u \in N^{1,1}(H) \text{ and } u = 0 \text{ on } H \setminus D \}.$$

The space is a subspace of $N^{1,1}(D)$ when $D$ is $\mu$-measurable, and it can always be understood to be a subspace of $N^{1,1}(H)$. If $H = X$, we omit it from the notation.
Similarly, for $D \subset \Omega \subset X$, with $\Omega$ open, we define the class of BV functions with zero boundary values as

$$\text{BV}_0(D, \Omega) := \{ u|_D : u \in \text{BV}(\Omega), \ u^+(x) = u^-(x) = 0 \text{ for } \mathcal{H}\text{-a.e. } x \in \Omega \setminus D \}.$$  

This class was previously considered in [33]. Functions in $\text{BV}_0(D, \Omega)$ can also be understood to be defined on the whole of $\Omega$, and we will do so without further notice. Moreover, if $\Omega = X$, we omit it from the notation. By (2.10), Proposition 2.11, and (2.3) we see that

$$N^{1,1}_0(D, \Omega) \subset \text{BV}_0(D, \Omega). \quad (2.17)$$

Next we define the fine topology in the case $p = 1$.

**Definition 2.18.** We say that $A \subset X$ is 1-thin at the point $x \in X$ if

$$\lim_{r \to 0} r \frac{\text{cap}_1(A \cap B(x,r), B(x,2r))}{\mu(B(x,r))} = 0.$$  

We also say that a set $U \subset X$ is 1-finely open if $X \setminus U$ is 1-thin at every $x \in U$. Then we define the 1-fine topology as the collection of 1-finely open sets on $X$.

We denote the 1-fine interior of a set $H \subset X$, i.e. the largest 1-finely open set contained in $H$, by fine-int $H$. We denote the 1-fine closure of $H$, i.e. the smallest 1-finely closed set containing $H$, by $\overline{H}^1$. The 1-fine boundary of $H$ is $\partial^1 H := \overline{H}^1 \setminus \text{fine-int} H$.

See [29, Section 4] for discussion on this definition, and for a proof of the fact that the 1-fine topology is indeed a topology. By [27, Lemma 3.1], 1-thinness implies zero measure density, i.e.

$$\text{If } A \text{ is 1-thin at } x, \text{ then } x \in O_A. \quad (2.19)$$

**Theorem 2.20 ([31, Corollary 6.12]).** A set $U \subset X$ is 1-quasiopen if and only if it is the union of a 1-finely open set and a $\mathcal{H}$-negligible set.

**Lemma 2.21 ([30, Lemma 4.9]).** Let $U \subset X$ be 1-quasiopen and let $A \subset X$ be $\mathcal{H}$-negligible. Then $U \setminus A$ and $U \cup A$ are 1-quasiopen sets.

### 3 BV functions with zero boundary values

In this section we consider some questions related to the class $\text{BV}_0(D, \Omega)$, which will be needed in later sections. We will always denote by $\Omega$ a nonempty open set.

The support of a function $u$ defined on a subset of $\Omega$ (usually the entire $\Omega$, except in Lemma 3.14 below) is the relatively closed (in the subspace topology of $\Omega$) set

$$\text{spt}_\Omega u := \{ x \in \Omega : \mu(B(x,r) \cap \{ u \neq 0 \}) > 0 \text{ for all } r > 0 \}.$$ 

**Theorem 3.1 ([33, Theorem 3.16]).** Let $D \subset \Omega \subset X$, and let $u \in \text{BV}(\Omega)$. Then the following are equivalent:
(1) \( u \in BV_0(D, \Omega) \).

(2) There exists a sequence \((u_k) \subset BV(\Omega)\) such that each \(spt_\Omega u_k\) is a bounded subset of \(D\), and \(u_k \to u\) in \(BV(\Omega)\).

The following lemma, though slightly technical, simply shows that we can apply the definition of the total variation to find approximating locally Lipschitz functions that converge suitably in the \(L^1\)-norm.

**Lemma 3.2.** Let \(\Omega_1 \subset \Omega_2 \subset \ldots \subset \bigcup_{j=1}^\infty \Omega_j = \Omega\) be open sets, let \(\Omega_0 := \emptyset\), and let \(\eta_j \in \text{Lip}_c(\Omega_{j+1})\) such that \(0 \leq \eta_j \leq 1\) on \(X\) and \(\eta_j = 1\) on \(\Omega_j\) for each \(j \in \mathbb{N}\), and \(\eta_0 \equiv 0\). Moreover, let \(u \in L^1_{\text{loc}}(\Omega)\) with \(||Du||(\Omega) < \infty\), and let \((u_i) \subset \text{Lip}_{\text{loc}}(\Omega)\) such that \(u_i \to u\) in \(L^1_{\text{loc}}(\Omega)\) and

\[
\lim_{i \to \infty} \int_{\Omega} g_{u_i} \, d\mu = ||Du||(\Omega),
\]

where each \(g_{u_i}\) is the minimal \(1\)-weak upper gradient of \(u_i\) in \(\Omega\). Finally, let \(\delta_j > 0\) for each \(j \in \mathbb{N}\), and let \(\varepsilon > 0\). Then, passing to a suitable subsequence of \((u_i)\) (not relabeled), and with the understanding that terms can be repeated) and defining

\[
v := \sum_{i=1}^\infty (\eta_i - \eta_{i-1})u_i, \tag{3.3}
\]

we have \(||v - u||_{L^1(\Omega_j \setminus \Omega_{j-1})} < \delta_j\) for all \(j \in \mathbb{N}\) and \(\int_{\Omega} g_u \, d\mu < ||Du||(\Omega) + \varepsilon\).

**Proof.** By the definition of the total variation, we have \(||Du||(W) \leq \liminf_{i \to \infty} \int_{W} g_{u_i} \, d\mu\) for any open \(W \subset \Omega\), and thus \(||Du||(F) \geq \limsup_{i \to \infty} \int_{F} g_{u_i} \, d\mu\) for any closed \(F \subset \Omega\), and so

\[
\limsup_{i \to \infty} \int_{\Omega_{j+1} \setminus \Omega_{j-1}} g_{u_i} \, d\mu \leq ||Du||(\Omega_{j+1} \setminus \Omega_{j-1})
\]

for each \(j \in \mathbb{N}\). Denote by \(L_j > 0\) (some) Lipschitz constants of the functions \(\eta_j\); we can take this to be an increasing sequence. By passing to a subsequence of \((u_i)\) (not relabeled), we can assume that

\[
||u_{i-1} - u||_{L^1(\Omega_i)} < \min\{\delta_{i-1}, \delta_i, 2^{-i+1}\varepsilon/L_{i-1}\}/2 \tag{3.4}
\]

and

\[
\int_{\Omega} g_{u_1} \, d\mu \leq ||Du||(\Omega) + \varepsilon, \quad \int_{\Omega_{i+1} \setminus \Omega_{i-1}} g_{u_i} \, d\mu \leq ||Du||(\Omega_{i+1} \setminus \Omega_{i-1}) + 2^{-i} \varepsilon \tag{3.5}
\]

for all \(i = 2, 3, \ldots\). We can also assume that for \(k \in \mathbb{N}\) to be chosen later, \(u_1 = \ldots = u_k\). We have

\[
||v - u||_{L^1(\Omega_j \setminus \Omega_{j-1})} = \sum_{i=1}^\infty (\eta_i - \eta_{i-1})u_i - u||_{L^1(\Omega_j \setminus \Omega_{j-1})}
\]

\[
= ||\eta_{j-1}u_{j-1} + (1 - \eta_{j-1})u_j - u||_{L^1(\Omega_j \setminus \Omega_{j-1})}
\]

\[
\leq ||u_{j-1} - u||_{L^1(\Omega_j \setminus \Omega_{j-1})} + ||u_j - u||_{L^1(\Omega_j \setminus \Omega_{j-1})}
\]

\[
< \delta_j
\]

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as desired. Let \( v_1 := u_1 \) and recursively \( v_{i+1} := \eta_i v_i + (1 - \eta_i) u_{i+1} \). We see that \( v = \lim_{i \to \infty} v_i \). By a Leibniz rule [6, Lemma 2.18], the minimal 1-weak upper gradient of \( v_2 \) in \( \Omega \) satisfies

\[
g_{v_2} \leq g_{\eta_1} (u_1 - u_2) + \eta_1 g_{u_1} + (1 - \eta_1) g_{u_2}.
\]

Inductively, we get

\[
g_{v_i} \leq \sum_{j=1}^{i-1} g_{\eta_j} (u_j - u_{j+1}) + \sum_{j=1}^{i-1} (\eta_j - \eta_{j-1}) g_{u_j} + (1 - \eta_i) g_{u_i};
\]

to prove this, assume that it holds for the index \( i \). Then we have by applying a Leibniz rule as above, and noting that \( g_{\eta_i} \) can be nonzero only in \( \Omega_{i+1} \setminus \Omega_i \) (see [6, Corollary 2.21]), where \( v_i = u_i \),

\[
g_{v_{i+1}} \leq g_{\eta_i} (v_i - u_{i+1}) + \eta_i g_{v_i} + (1 - \eta_i) g_{u_{i+1}}
\leq g_{\eta_i} (u_i - u_{i+1}) + \sum_{j=1}^{i-1} g_{\eta_j} (u_j - u_{j+1})
\quad + \sum_{j=1}^{i-1} (\eta_j - \eta_{j-1}) g_{u_j} + (\eta_i - \eta_{i-1}) g_{u_i} + (1 - \eta_i) g_{u_{i+1}}.
\]

This completes the induction. Thus in each \( \Omega_i \), where \( v = v_{i+1} \), the minimal 1-weak upper gradient of \( v \) in \( \Omega_i \) satisfies

\[
g_v = g_{v_{i+1}} \leq \sum_{j=1}^{\infty} g_{\eta_j} (u_j - u_{j+1}) + \sum_{j=1}^{\infty} (\eta_j - \eta_{j-1}) g_{u_j}
\quad = \sum_{j=1}^{\infty} g_{\eta_j} (u_j - u_{j+1}) + \eta_k g_{u_1} + \sum_{j=k+1}^{\infty} (\eta_j - \eta_{j-1}) g_{u_j},
\]

since \( u_1 = \ldots = u_k \). Thus

\[
\int_{\Omega_i} g_v \, d\mu
\leq \sum_{j=1}^{\infty} \int_{\Omega} g_{\eta_j} (u_j - u_{j+1}) \, d\mu + \int_{\Omega} \eta_k g_{u_1} \, d\mu + \sum_{j=k+1}^{\infty} \int_{\Omega} (\eta_j - \eta_{j-1}) g_{u_j} \, d\mu
\]

\[
\leq \sum_{j=1}^{\infty} L_j \| u_j - u_{j+1} \|_{L^1(\Omega_{j+1} \setminus \Omega_j)} + \| Du \|(\Omega) + \| Du \|_{L^1(\Omega_j \setminus \Omega_{j+1})} + \sum_{j=k+1}^{\infty} \int_{\Omega_{j+1} \setminus \Omega_j} g_{u_j} \, d\mu.
\]

\[
(3.5)
\leq 3\varepsilon + \| Du \|(\Omega) + \sum_{j=k+1}^{\infty} \| Du \|_{L^1(\Omega_j \setminus \Omega_{j+1})}
\leq 3\varepsilon + \| Du \|(\Omega) + 3\| Du \|(\Omega \setminus \Omega_k)
\leq \| Du \|(\Omega) + 4\varepsilon,
\]
if we choose $k$ large enough. Note that $g_v$ does not depend on $i$, see [6, Lemma 2.23], and so it is well defined on $\Omega$. Since $g_v$ is the minimal 1-weak upper gradient of $v$ in each $\Omega_i$, it is clearly also (the minimal) 1-weak upper gradient of $v$ in $\Omega$. Then by Lebesgue’s monotone convergence theorem,

$$\int_{\Omega} g_v \, d\mu \leq \|Du\|_0(\Omega) + 4\varepsilon.$$  

\[\square\]

In a rather similar way, we prove the following lemma which we will need later.

**Lemma 3.6.** Let $u \in L^1_{\text{loc}}(\Omega)$ with $\|Du\|_0(\Omega) < \infty$ and let $(u_i) \subset L^1_{\text{loc}}(\Omega)$ such that $\|u_i - u\|_{L^\infty(\Omega)} \to 0$ and

$$\lim_{i \to \infty} \|Du_i\|_0(\Omega) = \|Du\|_0(\Omega).$$

Let $\varepsilon > 0$. Then we find a function $v \geq u$ such that $\|v - u\|_{L^1(\Omega)} < \varepsilon$, $\|v - u\|_{L^\infty(\Omega)} < \varepsilon$, $\|Dv\|_0(\Omega) < \|Du\|_0(\Omega) + \varepsilon$,

$$\lim_{\Omega \ni y \to x} |v - u|(y) = 0 \quad \text{for all } x \in \partial \Omega,$$

and $S_v \subset \bigcup_{i=1}^\infty S_{u_i}$. Moreover, $\|Dv\|_0(\Omega) = 0$ if $\|Du_i\|_0(\Omega) = 0$ for all $i \in \mathbb{N}$.

**Proof.** Take nonempty open sets $\Omega_1 \Subset \Omega_2 \Subset \ldots \Subset \bigcup_{j=1}^\infty \Omega_j = \Omega$, and $\Omega_0 := \emptyset$. Also take functions $\eta_j \in \text{Lip}_c(\Omega_{j+1})$ such that $0 \leq \eta_j \leq 1$ on $X$ and $\eta_j = 1$ on $\Omega_j$ for each $j \in \mathbb{N}$, and $\eta_0 \equiv 0$.

By replacing the functions $u_i$ with $u_i + \|u_i - u\|_{L^\infty(\Omega)}$, we can assume that $u_i \geq u$ on $\Omega$ for each $i \in \mathbb{N}$. By passing to a subsequence (not relabeled), we can assume that for each $i \in \mathbb{N}$,

$$\|u_i - u\|_{L^\infty(\Omega)} < 2^{-i} \varepsilon \min \left\{1, \mu(\{\eta_i > 0\})^{-1}, \int_{\Omega} g_{\eta_i-1} \, d\mu, \int_{\Omega} g_{\eta_i} \, d\mu \right\}. \quad (3.7)$$

From the fact that $\lim_{i \to \infty} \|Du_i\|_0(\Omega) = \|Du\|_0(\Omega)$ and from the lower semicontinuity of the total variation in open sets, it follows that for each $j \in \mathbb{N}$ (see [4, Proposition 1.80])

$$\lim_{i \to \infty} \int_{\Omega} (1 - \eta_{j-1}) \, d\|Du_i\| = \int_{\Omega} (1 - \eta_{j-1}) \, d\|Du\|$$

and

$$\lim_{i \to \infty} \int_{\Omega} (\eta_j - \eta_{j-1}) \, d\|Du_i\| = \int_{\Omega} (\eta_j - \eta_{j-1}) \, d\|Du\|.$$ 

Thus we can also assume that for each $i \in \mathbb{N}$,

$$\int_{\Omega} (1 - \eta_i) \, d\|Du_i\| < \int_{\Omega} (1 - \eta_i) \, d\|Du\| + 2^{-i} \varepsilon \quad \text{ (3.8)}$$

and

$$\int_{\Omega} (\eta_i - \eta_{i-1}) \, d\|Du_i\| < \int_{\Omega} (\eta_i - \eta_{i-1}) \, d\|Du\| + 2^{-i} \varepsilon. \quad (3.9)$$
Let 
\[ v := \sum_{j=1}^{\infty} (\eta_j - \eta_{j-1}) u_j. \]  
(3.10)

Then \( v \geq u \) and
\[ \|v - u\|_{L^1(\Omega)} = \| \sum_{j=1}^{\infty} (\eta_j - \eta_{j-1}) (u_j - u) \|_{L^1(\Omega)} \leq \sum_{j=1}^{\infty} \mu(\{\eta_j > 0\}) \|u_j - u\|_{L^\infty(\Omega)} < \varepsilon \]
by (3.7). Clearly also \( \|v - u\|_{L^\infty(\Omega)} < \varepsilon \).

Let \( v_1 := u_1 \) and then recursively \( v_{i+1} := \eta_i v_i + (1 - \eta_i) u_{i+1} \). Then \( v = \lim_{i \to \infty} v_i \).

By induction, we show that in \( \Omega \),
\[ d\|Dv_i\| \leq \sum_{j=1}^{i-1} g_{\eta_j} |u_j - u_{j+1}| d\mu + \sum_{j=1}^{i-1} (\eta_j - \eta_{j-1}) d\|Du_j\| + (1 - \eta_{i-1}) d\|Du_i\| ; \]  
(3.11)

this clearly holds for \( i = 1 \). Note that \( g_{\eta_i} \) can be nonzero only in \( \Omega_{i+1} \setminus \Omega_i \) (see [6, Corollary 2.21]), where \( v_i = u_i \). Assuming that (3.11) holds for the index \( i \), by a Leibniz rule (see [18, Lemma 3.2]) we get
\[ d\|Dv_{i+1}\| \leq g_{\eta_i} |v_i - u_{i+1}| d\mu + \eta_i d\|Dv_i\| + (1 - \eta_i) d\|Du_{i+1}\| \]
\[ \leq g_{\eta_i} |u_i - u_{i+1}| d\mu + \sum_{j=1}^{i-1} g_{\eta_j} |u_j - u_{j+1}| d\mu + \sum_{j=1}^{i-1} (\eta_j - \eta_{j-1}) d\|Du_j\| \]
\[ + (\eta_i - \eta_{i-1}) d\|Du_i\| + (1 - \eta_i) d\|Du_{i+1}\|. \]

This completes the induction. Thus since \( v_i \to v \) in \( L^1_{\text{loc}}(\Omega) \), we get
\[ \|Dv\| (\Omega) \leq \liminf_{i \to \infty} \|Dv_i\| (\Omega) \]
\[ \leq \sum_{j=1}^{\infty} \int_{\Omega} g_{\eta_j} |u_j - u_{j+1}| d\mu + \sum_{j=1}^{\infty} \int_{\Omega} (\eta_j - \eta_{j-1}) d\|Du_j\| + \liminf_{i \to \infty} (1 - \eta_{i-1}) d\|Du_i\| \]
\[ < 2 \sum_{j=1}^{\infty} 2^{-j} \varepsilon + \sum_{j=1}^{\infty} \left( \int_{\Omega} (\eta_j - \eta_{j-1}) d\|Du\| + 2^{-j} \varepsilon \right) \quad \text{by (3.7), (3.9), (3.8)} \]
\[ = 3 \varepsilon + \|Du\| (\Omega), \]
as desired. Next, note that (3.10) is a locally finite sum. If \( x \notin S_{u_j} \), clearly \( x \notin S_{(\eta_j - \eta_{j-1})u_j} \). Thus \( S_v \subset \bigcup_{j=1}^{\infty} S_{u_j} \).

If \( \|Du_i\|^c (\Omega) = 0 \) for all \( i \in \mathbb{N} \), we show that \( \|Dv\|^c (\Omega) = 0 \) as follows. Let \( F \subset \Omega \) be a \( \mu \)-negligible set such that \( \|Dv\|^c (\Omega \setminus F) = 0 \). Note that \( \|Dv\| = \|Dv_{i+1}\| \) in \( \Omega_i \), and so by (3.11), in \( \Omega_i \) we have
\[ d\|Dv\| = d\|Dv_{i+1}\| \leq \sum_{j=1}^{\infty} g_{\eta_j} |u_j - u_{j+1}| d\mu + \sum_{j=1}^{\infty} (\eta_j - \eta_{j-1}) d\|Du_j\|. \]  
(3.12)
Since this inequality holds in every $\Omega_i$, it holds in $\Omega$. By the discussion after (2.12),
\[
\|Dv\|^{c}(F) = \|Dv\|^{c}\left(F \setminus \bigcup_{i=1}^{\infty} S_{u_i}\right) \\
\leq \sum_{j=1}^{\infty} \|Du_j\|\left(F \setminus \bigcup_{i=1}^{\infty} S_{u_i}\right) \quad \text{by (3.12)} \\
= 0
\]
since $\|Du_j\|^{a}(F) = 0$ and $\|Du_j\|^{j}(\Omega \setminus S_{u_j}) = 0$. Thus $\|Dv\|^{c}(\Omega) = 0$. 

The next simple lemma shows the existence of suitable cutoff functions.

**Lemma 3.13.** Let $W \subset \Omega \subset X$ be open sets and let $H \subset W$ be relatively closed (in the subspace topology of $\Omega$). Then there is a function $\eta \in \text{Lip}_{\text{loc}}(\Omega)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $H$, and $\text{spt}_{\Omega} \eta \subset W$.

Moreover, if $H$ is bounded, also $\text{spt}_{\Omega} \eta$ is bounded.

**Proof.** Take open sets $\Omega_1 \subset \Omega_2 \subset \ldots \subset \bigcup_{j=1}^{\infty} \Omega_j = \Omega$ and $\Omega_0 := \emptyset$. Note that for each $j = 0, 1, \ldots, H \cap \overline{\Omega_{j+1}} \setminus \Omega_j$ is a compact subset of the open set $W \cap \Omega_{j+2} \setminus \overline{\Omega_{j-1}}$. Take $\eta_j \in \text{Lip}_{\text{loc}}(W \cap \Omega_{j+2} \setminus \overline{\Omega_{j-1}})$ such that $0 \leq \eta_j \leq 1$ and $\eta_j = 1$ on $H \cap \overline{\Omega_{j+1}} \setminus \Omega_j$.

Let
\[
\eta := \sup_{j \in \mathbb{N}} \eta_j.
\]
Now it is straightforward to check that $\eta$ has the required properties. If $H$ is bounded, we can also choose the $\eta_j$’s so that $\eta_j = 0$ outside a 1-neighborhood of $H$, ensuring that $\text{spt}_{\Omega} \eta$ is bounded. 

**Lemma 3.14.** Let $W \subset \Omega \subset X$ be open sets and let $u \in \text{BV}(W)$ such that $\text{spt}_{\Omega} u \subset W$. Then there exists a sequence $(u_i) \subset \text{Lip}_{\text{loc}}(W)$ such that $\text{spt}_{\Omega} u_i$ are subsets of $W$ and bounded if $\text{spt}_{\Omega} u$ is, $u_i \to u$ in $L^1(W)$, and
\[
\|Du\|(W) = \lim_{i \to \infty} \int_{W} g_{u_i} \, d\mu, \tag{3.15}
\]
where each $g_{u_i}$ is the minimal 1-weak upper gradient of $u_i$ in $W$. Moreover, if $u \in \text{BV}_0(W, \Omega)$ with $\|Du\|(W) = \|Du\|(\Omega)$ (by zero extension to $\Omega \setminus W$), and then (3.15) holds also with $W$ replaced by $\Omega$.

**Proof.** Fix $\varepsilon > 0$. By Lemma 3.13 we find a function $\eta \in \text{Lip}_{\text{loc}}(\Omega)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ on $\text{spt}_{\Omega} u$, and $\text{spt}_{\Omega} \eta$ is a subset of $W$ and bounded if $\text{spt}_{\Omega} u$ is. Take open sets $W_1 \subset W_2 \subset \ldots \subset \bigcup_{j=1}^{\infty} W_j = W$ and $W_0 := \emptyset$. Denote by $L_j > 0$ the Lipschitz constant of $\eta$ in $W_j$. By Lemma 3.2 we find a function $v \in \text{Lip}_{\text{loc}}(W)$ such that $\int_{W} g_{v} \, d\mu \leq \|Du\|(W) + \varepsilon$ and $\|v-u\|_{L^1(W \setminus W_{j-1})} < 2^{-j} \varepsilon \min\{1, L_j^{-1}\}$ for every $j \in \mathbb{N}$. Then
\[
\|\eta v - u\|_{L^1(W)} = \|\eta v - \eta u\|_{L^1(W)} \leq \|v - u\|_{L^1(W)} < \varepsilon.
\]
Moreover, \( g_\eta = 0 \) on \( \text{spt}_\Omega u \) by [6, Corollary 2.21], and then by the Leibniz rule [6, Theorem 2.15],

\[
\int_W g_\eta v \, d\mu \leq \int_W g_\eta v \, d\mu + g_\eta |v - u| \, d\mu \\
= \int_W g_\eta v \, d\mu + \sum_{j=1}^{\infty} \int_{W_j \setminus W_{j-1}} g_\eta |v - u| \, d\mu \\
\leq \int_W g_\eta v \, d\mu + \sum_{j=1}^{\infty} L_j \|v - u\|_{L^1(W_j \setminus W_{j-1})} \\
\leq \|Du\|(W) + 2\varepsilon.
\]

We find the desired functions by letting \( u_i := \eta v \) with the choices \( \varepsilon = 1/i \). To prove the second claim, denote by \( u, u_i \) also the zero extensions of these functions to \( \Omega \setminus W \). Obviously \( u_i \to u \) in \( L^1(\Omega) \). Note that the minimal 1-weak upper gradient \( g_{u_i} \) (now as a function defined on \( \Omega \)) is clearly the zero extension of \( g_{u_i} \) (as a function defined only on \( W \)), and so we have

\[
\|Du\|(\Omega) \leq \liminf_{i \to \infty} \int_\Omega g_{u_i} \, d\mu = \liminf_{i \to \infty} \int_W g_{u_i} \, d\mu = \|Du\|(W).
\]

Thus \( u \in BV(\Omega) \) and then clearly \( u \in BV_0(W, \Omega) \).

Now we can show that Lipschitz functions with zero boundary values are dense in the class \( BV_0(W, \Omega) \) in the following weak sense.

**Proposition 3.16.** Let \( W \subset \Omega \subset X \) be open sets and let \( u \in BV_0(W, \Omega) \). Then there exists a sequence \( (u_i) \subset \text{Lip}_{\text{loc}}(\Omega) \) such that each \( \text{spt}_\Omega u_i \subset W \) is bounded, \( u_i \to u \) in \( L^1(\Omega) \), and

\[
\lim_{i \to \infty} \int_\Omega g_{u_i} \, d\mu = \|Du\|(\Omega).
\]

**Proof.** By Theorem 3.1, we find a sequence \( (v_i) \subset BV(\Omega) \) such that \( \text{spt}_\Omega v_i \subset W \) are bounded and \( \|v_i - u\|_{BV(\Omega)} < 1/i \) for each \( i \in \mathbb{N} \). Then by Lemma 3.14, for each \( i \in \mathbb{N} \) we find \( u_i \in \text{Lip}_{\text{loc}}(\Omega) \) such that \( \text{spt}_\Omega u_i \subset W \) is bounded, \( \|u_i - v_i\|_{L^1(\Omega)} < 1/i \), and

\[
\left| \int_\Omega g_{u_i} \, d\mu - \|Dv_i\|(\Omega) \right| < 1/i.
\]

We conclude that \( \|u_i - u\|_{L^1(\Omega)} < 2/i \) and

\[
\left| \int_\Omega g_{u_i} \, d\mu - \|Du\|(\Omega) \right| < 2/i.
\]

\( \square \)
The analog of the following result is well known for Newton-Sobolev functions, see [6, Lemma 2.37], and so it is natural to prove it here for the class $\text{BV}_0(W,\Omega)$, even though we will not need this result later.

**Proposition 3.17.** Let $W \subset \Omega \subset X$ be open sets and let $u \in \text{BV}(W)$ and $v, w \in \text{BV}_0(W,\Omega)$ such that $v \leq u \leq w$ in $\Omega$. Then $u \in \text{BV}_0(W,\Omega)$.

**Proof.** By observing that $u \in \text{BV}_0(W,\Omega)$ if and only if $u - v \in \text{BV}_0(W,\Omega)$, we can assume that $v \equiv 0$. Denote the zero extension of $u$ to $\Omega \setminus W$ by $u^0$. By Theorem 3.1, we find a sequence of nonnegative functions $(w_k) \subset \text{BV}(\Omega)$ with $\text{spt}_\Omega w_k \subset W$ and $w_k \to w$ in $\text{BV}(\Omega)$ (the nonnegativity can be achieved by truncating, if needed). Then $\varphi_k := \min\{w_k, u^0\} \in \text{BV}(W)$ by (2.8) and $\varphi_k \in \text{BV}_0(W,\Omega)$ by Lemma 3.14, for each $k \in \mathbb{N}$. Moreover, $\varphi_k \to u^0$ in $L^1(\Omega)$ and

$$\liminf_{k \to \infty} \|D\varphi_k\|(\Omega) = \liminf_{k \to \infty} \|D\varphi_k\|(W) \quad \text{by Lemma 3.14}$$

$$\leq \liminf_{k \to \infty} \|Dw_k\|(W) + \|Du_0\|(W) \quad \text{by (2.8)}$$

$$= \|Dw\|(W) + \|Du\|(W).$$

Thus by the lower semicontinuity of the total variation with respect to $L^1$-convergence, $u_0 \in \text{BV}(\Omega)$. Moreover, $u_0^\vee(x) \leq w(x) = 0$ for $\mathcal{H}$-a.e. $x \in \Omega \setminus W$, and obviously $u_0^\wedge(x) \geq 0$ for all $x \in \Omega \setminus W$, guaranteeing that $u_0^\wedge = u_0^\vee = 0$ $\mathcal{H}$-a.e. in $\Omega \setminus W$. \qed

### 4 Variational capacities

In this section we study variational capacities. Our approximation result will be based on the main result of this section, Theorem 4.5.

We begin by defining the variational 1-capacity and its Lipschitz and BV analogs.

**Definition 4.1.** Let $A \subset D \subset H \subset X$ be nonempty sets such that $H$ is $\mu$-measurable. We define the variational (Newton-Sobolev) 1-capacity by

$$\text{cap}_1(A, D, H) := \inf \int_H g_u \, d\mu,$$

where the infimum is taken over functions $u \in N^{1,1}_0(D, H)$ such that $u \geq 1$ on $A$.

We define the variational Lipschitz 1-capacity by

$$\text{cap}_{\text{lip},1}(A, D, H) := \inf \int_H g_u \, d\mu,$$

where the infimum is taken over functions $u \in N^{1,1}_0(D, H) \cap \text{Lip}_{\text{loc}}(H)$ such that $u \geq 1$ on $A$.

Finally, we define the variational BV-capacity by

$$\text{cap}_{\text{BV}}(A, D, H) := \inf \|Du\|(H),$$

where the infimum is taken over functions $u \in L^1(H)$ such that $u^\wedge = u^\vee = 0$ $\mathcal{H}$-a.e. on $H \setminus D$ and $u^\wedge \geq 1$ $\mathcal{H}$-a.e. on $A$. 

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If \( H = X \), we omit it from the notation. In each case, we say that the functions \( u \) over which we take the infimum are admissible (test) functions for the capacity in question.

Again, \( g_u \) always denotes the minimal 1-weak upper gradient of \( u \) (in \( H \)). Recall that we understand Newton-Sobolev functions to be defined at every point, but in the definition of \( \text{cap}_1(A, D, H) \) we can equivalently require \( u \geq 1 \) \( 1 \text{-q.e.} \) on \( A \), by (2.2). However, the same is not true for \( \text{cap}_{\text{lip},1}(A, D, H) \). In each definition, we see by truncation that it is enough to consider test functions \( 0 \leq u \leq 1 \), and then the conditions \( u \geq 1 \) and \( u^\wedge \geq 1 \) are replaced by \( u = 1 \) and \( u^\wedge = 1 \), respectively.

In the definition of the variational BV-capacity, it is implicitly understood that the test functions need to satisfy \( \|Du\|_1(\Omega) < \infty \) for some open \( \Omega \supset H \). Note that if \( H \) is itself open, then the infimum is taken over functions \( u \in \text{BV}_0(D, H) \) such that \( u^\wedge \geq 1 \) \( \mathcal{H}\text{-a.e.} \) on \( A \).

Using (2.10), (2.17), and Proposition 2.11, it is straightforward to see that for open \( \Omega \subset X \),

\[
\text{cap}_{\text{BV}}(A, D, \Omega) \leq \text{cap}_1(A, D, \Omega) \leq \text{cap}_{\text{lip},1}(A, D, \Omega). \tag{4.2}
\]

In [33, Theorem 4.23] it was shown that for a compact subset \( A \) of an open set \( D \), we have

\[
\text{cap}_{\text{BV}}(A, D) = \text{cap}_1(A, D) = \text{cap}_{\text{lip},1}(A, D).
\]

For more general \( A, D \), if there is no positive distance between \( A \) and \( X \setminus D \), then of course \( \text{cap}_{\text{lip},1}(A, D) = \infty \), while the other capacities may be finite. However, it is natural to expect the equality \( \text{cap}_{\text{BV}}(A, D) = \text{cap}_1(A, D) \) to hold more generally. We show this in Theorem 4.5 below.

**Lemma 4.3 ([30, Lemma 3.4]).** Let \( G \subset X \) and \( \varepsilon > 0 \). Then there exists an open set \( V \supset G \) with \( \text{Cap}_1(V) \leq C_1(\text{Cap}_1(G) + \varepsilon) \) and a function \( \eta \in N_0^{1,1}(V) \) with \( 0 \leq \eta \leq 1 \) on \( X \), \( \eta = 1 \) on \( G \), and \( \|\eta\|_{1^1,1(X)} \leq C_1(\text{Cap}_1(G) + \varepsilon) \), for some constant \( C_1 = C_1(C_d, C_F, \lambda) \geq 1 \).

**Lemma 4.4 ([32, Lemma 3.8]).** Let \( \Omega \subset X \) be an open set and let \( u \in L^1_{\text{loc}}(\Omega) \) with \( \|Du\|_1(\Omega) < \infty \). Then for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that if \( A \subset \Omega \) with \( \text{Cap}_1(A) < \delta \), then \( \|Du\|_1(A) < \varepsilon \).

**Theorem 4.5.** Let \( D \subset \Omega' \subset X \) be 1-quasiopen sets and let \( A \subset D \) be 1-quasiclosed with respect to \( \Omega' \). Then \( \text{cap}_1(A, D, \Omega') \leq \text{cap}_{\text{BV}}(A, D, \Omega') \).

Note that if \( \Omega' \) is in fact open, then by (4.2) we have \( \text{cap}_1(A, D, \Omega') = \text{cap}_{\text{BV}}(A, D, \Omega') \).

**Proof.** We can assume that \( \text{cap}_{\text{BV}}(A, D, \Omega') < \infty \). Fix \( \varepsilon > 0 \). Take an open set \( \Omega \supset \Omega' \) and a function \( h \in L^1(\Omega') \) such that \( 0 \leq h \leq 1 \) on \( \Omega \), \( h^\wedge = 0 \) \( \mathcal{H}\text{-a.e.} \) on \( \Omega' \setminus D \), \( h^\wedge = 1 \) \( \mathcal{H}\text{-a.e.} \) on \( A \), and \( \|Dh\|_1(\Omega) < \text{cap}_{\text{BV}}(A, D, \Omega') + \varepsilon \). As \( \Omega' \) is 1-quasiopen we can assume that \( \mu(\Omega \setminus \Omega') < \infty \), and then \( h \in L^1(\Omega) \) and so \( h \in \text{BV}(\Omega) \). It follows from Proposition 2.15 that the super-level sets of \( h^\wedge \) are 1-quasiopen, and so using also Lemma 2.21, we conclude that the set

\[
U := (\{x \in \Omega : h^\wedge(x) > 1 - \varepsilon\} \cup A) \cap D
\]
is 1-quasiopen. Clearly $A \subset U \subset D$. Defining
\[ u := \min\{1, (1 - \varepsilon)^{-1}h\} \in BV(\Omega) \]
we have $u^\vee = 0$ $\mathcal{H}$-a.e. on $\Omega' \setminus D$, $u^\wedge = 1$ $\mathcal{H}$-a.e. on $U$, and $\|Du\|(\Omega) \leq (1 - \varepsilon)^{-1}(\text{cap}_{BV}(A, D, \Omega') + \varepsilon)$.

According to Lemma 4.4, there exists $\delta \in (0, \varepsilon)$ such that whenever $H \subset \Omega$ with Cap$_1(H) < \delta$, then $\|Du\|(H) < \varepsilon$. Since $A$ is 1-quasiclosed with respect to $\Omega'$, and $\Omega', D$, and $U$ are 1-quasiopen, we find an open set $G \subset \Omega$ such that Cap$_1(G) < \delta/C_1^2$, $\Omega' \cup G$ is open, $D \cup G$ is open, $U \cup G$ is open, and $A \setminus G$ is relatively closed (in the subspace topology of $\Omega'$, and then clearly also in that of $\Omega' \cup G$). By Lemma 4.3 we then find a set $V \supset G$ and a function $\eta \in N_0^1(V)$ such that Cap$_1(V) < \delta/C_1$, $0 \leq \eta \leq 1$ on $X$, $\eta = 1$ on $G$, and $\|\eta\|_{N_0^1(X)} < \delta/C_1$.

Note that $u \in BV_0(D \cup G, \Omega' \cup G)$. By Proposition 3.16 we find functions $(v_i) \subset \text{Lip}_{loc}(\Omega' \cup G)$ such that $0 \leq v_i \leq 1$, spt$_{\Omega' \cup G} v_i \subset D \cup G$, $v_i \to u$ in $L^1(\Omega' \cup G)$, and
\[ \lim_{i \to \infty} \int_{\Omega' \cup G} g_{v_i} \, d\mu = \|Du\|(\Omega' \cup G). \]
Since $A \setminus G$ is a relatively closed (in the subspace topology of $\Omega' \cup G$) subset of the open set $U \cup G$, by Lemma 3.13 we also find a function $\rho \in \text{Lip}_{loc}(\Omega' \cup G)$ such that $0 \leq \rho \leq 1$, $\rho = 1$ on $A \setminus G$, and spt$_{\Omega' \cup G} \rho \subset U \cup G$. Take open sets $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega' \cup G$ such that $\Omega' \cup G = \bigcup_{j=1}^{\infty} \Omega_j$, and $\Omega_0 := \emptyset$. Denote by $L_j$ the Lipschitz constant of $\rho$ in $\Omega_j$. By Lemma 3.2, from the functions $v_i$ we can construct a new function $v \in \text{Lip}_{loc}(\Omega' \cup G)$ such that $0 \leq v \leq 1$, spt$_{\Omega' \cup G} v \subset D \cup G$ (this follows from (3.3)),
\[ \|v - u\|_{L^1(\Omega_j \setminus \Omega_{j-1})} < 2^{-j} \varepsilon L_j^{-1} \quad \text{for all } j \in \mathbb{N}, \tag{4.6} \]
and
\[ \int_{\Omega' \cup G} g_v \, d\mu \leq \|Du\|(\Omega' \cup G) + \varepsilon. \tag{4.7} \]
Then define
\[ w := \rho(1 - \eta) + (1 - \rho)(1 - \eta)v. \]
Note that $0 \leq w \leq 1$, $w = 1$ on $A \setminus V$, and spt$_{\Omega' \cup G} w \subset D$, so that $w$ is admissible for cap$_1(A \setminus V, D, \Omega')$. By the Leibniz rule [6, Theorem 2.15, Lemma 2.18], we have in $\Omega' \cup G$
\[
g_w \leq \rho g_{\eta} + (1 - \rho)g_{(1 - \eta)v} + g_{\rho}(1 - \eta)(1 - v) \\
\leq \rho g_{\eta} + (1 - \rho)(g_{\eta}v + g_{\rho}(1 - \eta)) + g_{\rho}(1 - \eta)(1 - v) \\
\leq g_{\eta} + g_{\eta} + g_{\epsilon} + g_{\rho}(1 - \eta)(1 - v),
\]
since $\rho$ and $v$ take values between 0 and 1. Since $u = 1$ a.e. on $U$ and $g_{\rho} = 0$ outside
We have
\[
\int_{\Omega \cup G} g_\rho(1-\eta)(1-v) \, d\mu \leq \int_{\Omega \setminus G} g_\rho(1-v) \, d\mu \\
= \int_{\Omega \setminus G} g_\rho(u-v) \, d\mu \\
\leq \sum_{j=1}^{\infty} \int_{\Omega_j \setminus \Omega_{j-1}} g_\rho |u-v| \, d\mu \\
\leq \sum_{j=1}^{\infty} L_j \|u-v\|_{L^1(\Omega_j \setminus \Omega_{j-1})} \\
< \varepsilon \quad \text{by (4.6)}.
\]
Thus
\[
\int_{\Omega'} g_w \, d\mu \leq 2 \int_{\Omega'} g_\eta \, d\mu + \int_{\Omega'} g_v \, d\mu + \varepsilon \leq 2\delta/C_1 + \|Du\|_{(\Omega' \cup G)} + 2\varepsilon
\]
by (4.7). Thus we have
\[
\text{cap}_1(A \setminus V, D, \Omega') \leq \int_{\Omega'} g_w \, d\mu \leq \|Du\|_{(\Omega' \cup G)} + 2\delta/C_1 + 2\varepsilon
\]
(4.8)
\[
\leq \|Du\|_{\Omega} + 4\varepsilon
\]
\[
\leq (1-\varepsilon)^{-1} (\text{cap}_{BV}(A, D, \Omega') + \varepsilon) + 4\varepsilon.
\]
Moreover, by using Lemma 4.3 again, we find a set $W \supset V$ and a function $\xi \in \mathcal{N}^{1,1}_0(W)$ such that $\text{Cap}_1(W) < \delta$, $0 \leq \xi \leq 1$ on $X$, $\xi = 1$ on $V$, and $\|\xi\|_{\mathcal{N}^{1,1}(X)} < \delta$.
Since $\xi$ is 1-finely continuous 1-q.e. by [29, Corollary 5.4], we have $\xi = 1$ 1-q.e. on $V$ (the 1-fine closure of $V$). By Proposition 2.11 and (2.3) we have $\xi_1 = 1$ $\mathcal{H}$-a.e. on $\overline{V}$, and now clearly $(\xi u)^{\wedge} = 1$ $\mathcal{H}$-a.e. on $A \cap \overline{V}$. Clearly also $(\xi u)^\vee = (\xi u)^{\wedge} = 0$ $\mathcal{H}$-a.e. on $\Omega \setminus D$, and $\xi u \in L^1(\Omega)$. Thus $\xi u$ is admissible for $\text{cap}_{BV}(A \cap \overline{V}, D, \Omega')$.
By the Leibniz rule, see [24, Proposition 4.2], we get for constant $C = C(C_d, C_P, \lambda)$
\[
\text{cap}_{BV}(A \cap \overline{V}, D, \Omega') \\
\leq \|D(\xi u)\|_{\Omega}
\leq C \left( \int_{\Omega} |\xi|^\vee \|Du\| + \int_{\Omega} u^{\wedge} \|D\xi\| \right)
\]
\[
\leq C \left( \int_{\Omega} \xi \|Du\| + \int_{\Omega} ug_\xi \, d\mu \right) \quad \text{by Proposition 2.11, (2.7), and (2.10)}
\]
\[
\leq C \left( \|Du\|_{\Omega \cap W} + \int_{\Omega} g_\xi \, d\mu \right)
\leq C(\varepsilon + \delta);
\]
recall that $\|Du\|_{(W \cap \Omega)} < \varepsilon$ since $\text{Cap}_1(W) < \delta$. Combining the above with (4.8),
we conclude that for any $\varepsilon > 0$ there is an open set $V \subset X$ such that $\text{Cap}_1(V) < \varepsilon$,
\[
\text{cap}_1(A \setminus V, D, \Omega') \leq \text{cap}_{BV}(A, D, \Omega') + \varepsilon, \quad \text{and} \quad \text{cap}_{BV}(A \cap \overline{V}, D, \Omega') < \varepsilon.
\]
Fix a new $\varepsilon > 0$. Note that $\overline{V}$ is 1-finely closed and thus 1-quasiclosed by Theorem 2.20, and thus $A \cap \overline{V}$ is 1-quasiclosed with respect to $\Omega$. Thus we can repeat the above procedure with $A$ replaced by $A \cap \overline{V}$. Denote $V = V_1$. Inductively, for each $i \in \mathbb{N}$ we find a set $V_i$ such that $\mathrm{Cap}_1(V_i) < 1/i$.

\[
\mathrm{cap}_1 \left( A \cap \bigcap_{j=1}^{i-1} \overline{V_j} \setminus V_i, D, \Omega' \right) \leq \mathrm{cap}_{\mathrm{BV}} \left( A \cap \bigcap_{j=1}^{i-1} \overline{V_j}, D, \Omega' \right) + 2^{-i} \varepsilon, \tag{4.9}
\]

and

\[
\mathrm{cap}_{\mathrm{BV}} \left( A \cap \bigcap_{j=1}^{i} \overline{V_j}, D, \Omega' \right) \leq 2^{-i} \varepsilon. \tag{4.10}
\]

For each $k \in \mathbb{N}$, clearly

\[
A \setminus \bigcup_{i=1}^{k} \left( A \cap \bigcap_{j=1}^{i-1} \overline{V_j} \setminus V_i \right) \subset V_k,
\]

with $\mathrm{Cap}_1(V_k) < 1/k$, and so

\[
A \setminus \bigcup_{i=1}^{\infty} \left( A \cap \bigcap_{j=1}^{i-1} \overline{V_j} \setminus V_i \right)
\]

is a set of 1-capacity zero. Thus by the subadditivity of $\mathrm{cap}_1$ (see [8, Theorem 3.4])

\[
\mathrm{cap}_1(A, D, \Omega') \leq \sum_{i=1}^{\infty} \mathrm{cap}_1 \left( A \cap \bigcap_{j=1}^{i-1} \overline{V_j} \setminus V_i, D, \Omega' \right) \\
\leq \sum_{i=1}^{\infty} \left( \mathrm{cap}_{\mathrm{BV}} \left( A \cap \bigcap_{j=1}^{i-1} \overline{V_j}, D, \Omega' \right) + 2^{-i} \varepsilon \right) \text{ by (4.9)} \\
\leq \mathrm{cap}_{\mathrm{BV}}(A, D, \Omega') + 2^{-1} \varepsilon + \sum_{i=2}^{\infty} (2^{-i+1} \varepsilon + 2^{-i} \varepsilon) \text{ by (4.10)} \\
= \mathrm{cap}_{\mathrm{BV}}(A, D, \Omega') + 2\varepsilon.
\]

Letting $\varepsilon \to 0$, we get the result. \qed

5 The approximation result

In this section we prove our approximation result, Theorem 1.1. Recall that this theorem states that we can approximate a given BV function by SBV functions in the strict sense, pointwise uniformly, and without adding significant jumps. First we note that if we were to drop one of the last two conditions, the proof would be straightforward. Again we will always denote by $\Omega$ a nonempty open set.
Example 5.1. Let $u \in \text{BV}(\Omega)$. From Lemma 3.2 (essentially, from the definition of the total variation) we obtain a sequence $(u_i) \subset \text{SBV}(\Omega)$ (in fact, $(u_i) \subset \text{Lip}_{\text{loc}}(\Omega)$) such that $u_i \to u$ strictly and $\mathcal{H}(S_{u_i} \setminus S_u) = 0$, because in fact $S_{u_i} = \emptyset$. Usually, however, the $u_i$’s do not converge to $u$ uniformly, and this is in fact impossible for example when $u$ is a function on the real line with a nonempty jump set. Nonetheless, when $u$ is the Cantor ternary function on the unit interval and the $u_i$’s are the usual Lipschitz functions used in its construction (see e.g. [4, Example 1.67]), then also $u_i \to u$ uniformly.

On the other hand, assuming for simplicity that $\Omega$ is bounded and $u$ is nonnegative, if we define approximations

$$u_i := \frac{1}{\beta} \sum_{j=1}^{\infty} \chi_{\{u > t_{i,j}\}}$$

with $\|Du_i\|(\Omega) \leq \frac{1}{\beta} \sum_{j=1}^{\infty} P(\{u > t_{i,j}\}, \Omega)$,

then by the coarea formula (2.6) we can see that with a suitable choice of the numbers $t_{i,j} \geq 0$, we get $u_i \to u$ strictly, and uniformly. However, now the jump sets $S_{u_i}$ are usually very large.

To prove the approximation result, we first consider a case where the function only has small jumps.

Proposition 5.2. Let $\Omega' \subset \Omega$ such that $\Omega'$ is 1-quasiopen and $\Omega$ is open, and let $u \in \text{BV}(\Omega)$ and $\beta > 0$ such that $u^\vee - u^\wedge < \beta$ in $\Omega'$. Then for every $\varepsilon > 0$ there exists $v \in N^{1,1}(\Omega')$ such that $\|v - u\|_{L^\infty(\Omega')} \leq 4\beta$ and

$$\int_{\Omega'} g_v \, d\mu \leq \|Du\|(\Omega') + \varepsilon.$$

Proof. First assume that $u \geq 0$. Fix $\varepsilon > 0$. For each $i \in \mathbb{N}$, let

$$A_i := \{x \in \Omega' : u^\vee(x) \geq (i + 1)\beta\} \quad \text{and} \quad D_i := \{x \in \Omega' : u^\wedge(x) > (i - 2)\beta\}.$$

By Proposition 2.15, each $A_i$ is 1-quasiclosed with respect to $\Omega'$. It is straightforward to check that the intersection of two 1-quasiopen sets is 1-quasiopen, and so each $D_i$ is 1-quasiopen (with respect to $\Omega'$). Moreover, for all $i \in \mathbb{N}$,

$$A_i \subset \{x \in \Omega' : u^\wedge(x) \geq i\beta\} \subset \{x \in \Omega' : u^\vee(x) > (i - 1)\beta\} \subset D_i.$$

Fix $i \in \mathbb{N}$. Let

$$u_i := \frac{1}{\beta} \min\{\beta, (u - (i - 1)\beta)_+\} \in \text{BV}(\Omega).$$

Then

$$\text{cap}_{\text{BV}}(A_i, D_i, \Omega') \leq \text{cap}_{\text{BV}}(\{u^\wedge \geq i\beta\} \cap \Omega', \{u^\vee > (i - 1)\beta\} \cap \Omega', \Omega')$$

$$\leq \|Du_i\|(\Omega')$$

$$= \frac{1}{\beta} \int_{(i-1)\beta}^{i\beta} P(\{u > t\}, \Omega') \, dt$$

$$= \frac{1}{\beta} \int_{(i-1)\beta}^{i\beta} P(\{u > t\}, \Omega') \, dt.$$
by the coarea formula (2.6), which also applies to 1-quasiopen sets, see [30, Proposition 3.8]. By Theorem 4.5 we find a function \( v_i \in N_{0,1}(D_i, \Omega') \) such that \( v_i = 1 \) on \( A_i \) and

\[
\int_{\Omega'} g_{v_i} \, d\mu < \text{cap}_{\text{BV}}(A_i, D_i, \Omega') + 2^{-i} \varepsilon \beta \leq \frac{1}{\beta} \int_{(i-1)\beta}^{i\beta} P\{u > t\}, \Omega'\} dt + 2^{-i} \varepsilon .
\]

Now define

\[
v := \beta \sum_{i=3}^{\infty} v_i .
\]

It is easy to check that \( u^\vee - 4 \beta \leq v \leq u^\wedge \) on \( \Omega' \), so that \( \|v - u\|_{L^\infty(\Omega')} \leq 4 \beta \) and also \( v \in L^1(\Omega') \). Since \( g_v \leq \beta \sum_{i=3}^{\infty} g_{v_i} \) (see e.g. [6, Lemma 1.52]),

\[
\int_{\Omega'} g_v \, d\mu \leq \beta \sum_{i=1}^{\infty} \int_{\Omega'} g_{v_i} \, d\mu \\
\leq \beta \sum_{i=1}^{\infty} \left( \frac{1}{\beta} \int_{(i-1)\beta}^{i\beta} P\{u > t\}, \Omega'\} dt + \frac{2^{-i} \varepsilon}{\beta} \right) \\
= \int_{0}^{\infty} P\{u > t\}, \Omega'\} dt + \varepsilon \\
= \|D u\|(\Omega') + \varepsilon .
\]

This completes the proof in the case \( u \geq 0 \).

In the general case, we find a function \( w_1 \in N^{1,1}(\Omega') \) corresponding to \( u_+ \) and a function \( w_2 \in N^{1,1}(\Omega') \) corresponding to \( u_- \). Then for \( v := w_1 - w_2 \in N^{1,1}(\Omega') \) we have \( \|v - u\|_{L^\infty(\Omega')} \leq 4 \beta \) and

\[
\int_{\Omega'} g_v \, d\mu \leq \int_{\Omega'} g_{w_1} \, d\mu + \int_{\Omega'} g_{w_2} \, d\mu \leq \|D u_+\|(\Omega') + \|D u_-\|(\Omega') + 2 \varepsilon \\
= \|D u\|(\Omega') + 2 \varepsilon ,
\]

where the last inequality follows from the coarea formula.

\[\square\]

Now we consider the more general case where \( u \) may also have large jumps.

**Proposition 5.3.** Let \( u \in \text{BV}(\Omega) \) and let \( \varepsilon > 0 \). Then there exists \( v \in \text{BV}(\Omega) \) such that \( \|v - u\|_{L^\infty(\Omega)} \leq \varepsilon , \|D v\|(\Omega) \leq \|D u\|(\Omega) + \varepsilon , \|D v\|^c(\Omega) = 0 \), and \( \mathcal{H}(S_v \setminus S_u) = 0 \).

**Proof.** Fix \( 0 < \delta < \min\{1, \varepsilon\}/4 \) to be chosen later. Let \( S := \{x \in \Omega : u^\vee - u^\wedge \geq \delta\} \).

By Proposition 2.15, \( \Omega \setminus S \) is a 1-quasiopen set. Apply Lemma 5.2 to find a function \( v \in N^{1,1}(\Omega \setminus S) \) such that \( \|v - u\|_{L^\infty(\Omega \setminus S)} \leq 4 \delta \) and

\[
\int_{\Omega \setminus S} g_v \, d\mu \leq \|D u\|(\Omega \setminus S) + \varepsilon . \tag{5.4}
\]

By the decomposition (2.12) it is clear that \( \mathcal{H}(S) < \infty \), from which it easily follows that \( \mu(S) = 0 \). Thus we have in fact \( \|v - u\|_{L^\infty(\Omega)} \leq \varepsilon \) and \( v \in L^1(\Omega) \), as desired.
Now we estimate $\|Du\|((\Omega))$. Take a sequence $(u_i) \subset N^{1,1}(\Omega)$ (from Lemma 3.2) such that $u_i \to u$ in $L^1(\Omega)$ and $\int_\Omega g_{u_i} \, d\mu \to \|Du\|((\Omega))$. Then $v - u_i \to v - u$ in $L^1(\Omega)$. Letting $w_i := \min\{1, \max\{-1, v - u_i\}\}$, we have $w_i \to v - u$ in $L^1(\Omega)$. Let $i \in \mathbb{N}$ be fixed. We find a covering $\{B_j = B(x_j, r_j)\}_{j=1}^\infty$ such that $r_j \leq 1/i$ for all $j$, $S \subset \bigcup_{j=1}^\infty B_j$, and

$$\sum_{j=1}^\infty \frac{\mu(B(x_j, r_j))}{r_j} < \mathcal{H}(S) + 1/i.$$  

(5.5)

Then pick $1/r_j$-Lipschitz functions $\eta_j$ such that $0 \leq \eta_j \leq 1$ on $X$, $\eta_j = 1$ on $B(x_j, r_j)$, and $\eta_j = 0$ outside $B(x_j, 2r_j)$. Define $\rho_i := \sup_{j \in \mathbb{N}} \eta_j$. Consider the function

$$h_i := (1 - \rho_i)w_i.$$

Let $g \in L^1(\Omega \setminus S)$ be a 1-weak upper gradient of $w_i$ in $\Omega \setminus S$; for example $g_e + g_u$ will do. By [6, Corollary 2.21] we know that $\chi_{2B_j}/r_j$ is a 1-weak upper gradient of $\eta_j$ (in $X$), and then $\sum_{j=1}^\infty \frac{\chi_{2B_j}}{r_j}$ is a 1-weak upper gradient of $\rho_i$ (in $X$) by e.g. [6, Lemma 1.52]. We show that

$$g_i := g + \sum_{j=1}^\infty \frac{\chi_{2B_j}}{r_j}$$

is a 1-weak upper gradient of $h_i$ in $\Omega$. By the Leibniz rule [6, Theorem 2.15], $g_i$ is a 1-weak upper gradient of $h_i$ in $\Omega \setminus S$; recall that $\|w_i\|_{L^\infty(\Omega)} \leq 1$. Take a curve $\gamma$ in $\Omega$ such that the upper gradient inequality is satisfied by $h_i$ and $g_i$ on all subcurves of $\gamma$ in $\Omega \setminus S$; this is true for 1-a.e. curve in $\Omega$ by [6, Lemma 1.34]. Moreover, the fact that $\Omega \setminus S$ is 1-quasiopen implies by [37, Remark 3.5] that it is also 1-path open, meaning that for 1-a.e. curve $\gamma$, the set $\gamma^{-1}(\Omega \setminus S)$ is a relatively open subset of $[0, \ell_\gamma]$. Thus we can assume that $\gamma^{-1}(S)$ is a compact subset of the relatively open set $\gamma^{-1}(\bigcup_{j=1}^\infty B_j)$. Thus $\gamma$ can be split into a finite number of subcurves each of which lies either entirely in $\bigcup_{j=1}^\infty B_j$, or entirely in $\Omega \setminus S$. If $\gamma_1$ is a subcurve lying entirely in $\bigcup_{j=1}^\infty B_j$,

$$|h_i(\gamma_1(0)) - h_i(\gamma_1(\ell_{\gamma_1}))| = |0 - 0| = 0,$$

so the upper gradient inequality is satisfied. If $\gamma_2$ is a subcurve lying entirely in $\Omega \setminus S$, then

$$|h_i(\gamma_2(0)) - h_i(\gamma_2(\ell_{\gamma_2}))| \leq \int_{\gamma_2} g_i \, ds$$

by our choice of $\gamma$. Summing over the subcurves, we obtain

$$|h_i(\gamma(0)) - h_i(\gamma(\ell_\gamma))| \leq \int_{\gamma} g_i \, ds.$$

Thus $g_i$ is a 1-weak upper gradient of $h_i$ in $\Omega$. By (5.5) we have

$$\|g_i\|_{L^1(\Omega)} \leq \|g\|_{L^1(\Omega \setminus S)} + C_d(\mathcal{H}(S) + 1/i).$$  

(5.6)
Since \(-1 \leq w_i \leq 1\),
\[
\|h_i - w_i\|_{L^1(\Omega)} = \|\rho_i w_i\|_{L^1(\Omega)} \leq \|\rho_i\|_{L^1(\Omega)} \leq \frac{1}{i} \sum_{j=1}^{\infty} \frac{\mu(2B_j)}{r_j} \leq \frac{C_d}{i}(\mathcal{H}(S) + 1/i).
\]
Recall that \(w_i \to v - u\) in \(L^1(\Omega)\). Thus also \(h_i \to v - u\) in \(L^1(\Omega)\), and so by (5.6)
\[
\|D(v - u)(\Omega)\| \leq \lim \inf_{i \to \infty} \|g_i\|_{L^1(\Omega)} \leq \|g\|_{L^1(\Omega \setminus S)} + C_d \mathcal{H}(S) < \infty.
\]
Thus also \(\|Dv\|(\Omega) < \infty\) (recall (2.9)). By the decomposition (2.12) and the discussion after it, we find that only the jump part of \(\|D(v - u)\|\) can charge \(S\), and then from the fact that \(\|v - u\|_{L^\infty(\Omega)} \leq 4\delta\) we get
\[
\|D(v - u)(\Omega)\| \leq C_d \int_S ((v - u)^\vee - (v - u)^\wedge) \, d\mathcal{H} \leq 8C_d \delta \mathcal{H}(S). \tag{5.7}
\]
By another application of the decomposition (2.12),
\[
\infty > \|Du\|(S_u) \geq \alpha \int_{S_u} (u^\vee - u^\wedge) \, d\mathcal{H} = \alpha \int_0^\infty \mathcal{H}\{(u^\vee - u^\wedge > t)\} \, dt
\]
by Cavalieri’s principle. Since the function \(t \mapsto \mathcal{H}\{(u^\vee - u^\wedge > t)\}\) is thus integrable, necessarily
\[
\lim \inf_{t \to 0} t \mathcal{H}\{(u^\vee - u^\wedge > t)\} = 0.
\]
Thus by choosing a suitable small \(\delta\), we can ensure that \(\delta \mathcal{H}(S) < \varepsilon/(8C_d)\). Hence (5.7) gives \(\|D(v - u)(\Omega)\| \leq \varepsilon\) and so
\[
\|Dv\|(S) \leq \|Du\|(S) + \varepsilon.
\]
Thus we get (note that 1-quasiopen sets can be seen to be \(\|Du\|\)-measurable by Lemma 4.4)
\[
\|Dv\|(\Omega) = \|Dv\|(\Omega \setminus S) + \|Dv\|(S)
\leq \int_{\Omega \setminus S} g_v \, d\mu + \|Dv\|(S) \quad \text{by Theorem 2.16}
\leq \|Du\|(\Omega \setminus S) + \varepsilon + \|Du\|(S) + \varepsilon \quad \text{by (5.4)}
= \|Du\|(\Omega) + 2\varepsilon,
\]
as desired. Note that for any \(A \subset \Omega \setminus S\) with \(\mu(A) = 0\), by Theorem 2.16 we have for any open \(W\) with \(A \subset W \subset \Omega\) that
\[
\|Dv\|(A) \leq \|Dv\|(W \setminus S) \leq \int_{W \setminus S} g_v \, d\mu,
\]
which becomes arbitrarily small by choosing \(\mu(W)\) small. We conclude that \(\|Dv\|\cap(\Omega \setminus S) = 0\), and thus \(\|Dv\|\cap(\Omega) = 0\) since \(\mathcal{H}(S) < \infty\).

By Theorem 2.14, \(v\) is 1-quasicontinuous on \(\Omega \setminus S\), so by [28, Theorem 5.1] it is also 1-finely continuous 1-q.e. on \(\Omega \setminus S\), and so by (2.19) clearly \(v^\wedge = v^\vee\) 1-q.e. on \(\Omega \setminus S\). Hence \(\mathcal{H}(S_v \setminus S) = 0\) and so \(\mathcal{H}(S_v \setminus S_u) = 0\). \(\square\)
To obtain the strongest possible result, we will apply the above proposition only in a small open subset of $\Omega$ where the Cantor part of $\|Du\|$ is concentrated. For this, we will need the following extension lemma.

**Lemma 5.8.** Let $W \subset \Omega \subset X$ be open sets, let $u \in BV(W)$, and suppose that

$$\lim_{W \ni y \to x} |u|^\vee(y) = 0$$

for all $x \in \partial W$. Then $u \in BV_0(W, \Omega)$.

**Proof.** First assume that $\text{spt}_X u \subset W$. Then $u \in BV_0(W, \Omega)$ with $\|Du\|(\Omega) = \|Du\|(\Omega)$ by Lemma 3.14.

In the general case, note that for the functions

$$u_\delta := (u - \delta)_+ - (u + \delta)_-, \quad \delta > 0,$$

we have $\text{spt}_X u_\delta \subset W$. Thus, understanding $u$ to be zero extended to $\Omega \setminus W$, we have $u_\delta \to u$ in $L^1(\Omega)$ and then

$$\|Du\|(\Omega) \leq \liminf_{i \to \infty} \|Du_\delta\|(\Omega) = \liminf_{i \to \infty} \|Du_i\|(W) \leq \|Du\|(W),$$

so that $u \in BV(\Omega)$. By (5.9), clearly $u^\wedge = u^\vee = 0$ on $\Omega \setminus W$, and so $u \in BV_0(W, \Omega).$

Now we prove our main approximation result, which we first give in the following form.

**Theorem 5.10.** Let $u \in BV(\Omega)$ and let $\varepsilon > 0$. Then there exists $w \in SBV(\Omega)$ and an open set $W \subset \Omega$ such that $w \geq u$ on $\Omega$, $w = u$ on $\Omega \setminus W$, $\|w - u\|_{L^1(\Omega)} < \varepsilon$, $\|w - u\|_{L^\infty(\Omega)} < \varepsilon$, $\mathcal{H}(S_w \setminus S_u) = 0$, $\mu(W) < \varepsilon$, $\|Du\|(W) < \|Du\|_{c}(\Omega) + \varepsilon$, $\|D(w - u)\|(\Omega \setminus W) = 0$, $\|Du\|(W) < \|Du\|_{c}(\Omega) + \varepsilon$, and

$$\lim_{W \ni y \to x} |w - u|^\vee(y) = 0 \quad \text{for all } x \in \partial W.$$ 

(5.11)

**Proof.** By the decomposition (2.12), we find a Borel set $A \subset \Omega$ such that $\mu(A) = 0$ and $\|Du\|_{c}(A) = \|Du\|(A) = \|Du\|_{c}(\Omega)$. Take an open set $W \subset \Omega$ such that $W \supset A$ and $\|Du\|(W) < \|Du\|_{c}(\Omega) + \varepsilon$, and $\mu(W) < \varepsilon$. By Proposition 5.3 we find a sequence $(u_i) \subset BV(W)$ such that $\|u_i - u\|_{L^\infty(W)} \to 0$,

$$\lim_{i \to \infty} \|Du_i\|(W) = \|Du\|(W),$$

$\|Du_i\|_{c}(W) = 0$, and $\mathcal{H}(W \cap S_{u_i} \setminus S_u) = 0$. Then by Lemma 3.6 we find a function $v \in BV(W)$ such that $v \geq u$ on $W$, $\|v - u\|_{L^1(W)} < \varepsilon$, $\|v - u\|_{L^\infty(W)} < \varepsilon$,

$$\|Dv\|(W) < \|Du\|_{c}(\Omega) + \varepsilon,$$

$$\lim_{W \ni y \to x} |v - u|^\vee(y) = 0 \quad \text{for all } x \in \partial W,$$

(5.12)
\[ \|Dv\|^c(W) = 0, \text{ and } \mathcal{H}(W \cap S_v \setminus S_u) = 0. \] Let
\[
w := \begin{cases} 
    u & \text{on } \Omega \setminus W, \\
    v & \text{on } W.
\end{cases}
\]

Clearly \(w \geq u\), \(\|w - u\|_{L^1(\Omega)} < \varepsilon\), and \(\|w - u\|_{L^\infty(\Omega)} < \varepsilon\). By Lemma 5.8 and (5.12), \(w - u \in \text{BV}(\Omega)\) and then \(w \in \text{BV}(\Omega)\). Equation (5.12) gives (5.11). By (5.11), \(\partial^*\{w - u > t\} \setminus W = \emptyset\) for all \(t \neq 0\). Thus by the coarea formula (2.6) and (2.5),
\[
\|D(w - u)\|(\Omega \setminus W) = \int_{-\infty}^\infty P(\{w - u > t\}, \Omega \setminus W) \, dt \\
\leq C_d \int_{-\infty}^\infty \mathcal{H}(\partial^*\{w - u > t\} \setminus W) \, dt \\
= 0. 
\tag{5.13}
\]

Also
\[
\|Dw\|(W) = \|Dv\|(W) < \|Du\|(W) + \varepsilon
\]
and
\[
\|Dw\|^c(\Omega) = \|Dw\|^c(W) + \|Dw\|^c(\Omega \setminus W) \\
= \|Dv\|^c(W) + \|Du\|^c(\Omega \setminus W) \quad \text{by (5.13)} \\
= 0. 
\]

Equation (5.11) also implies that \(w^\wedge = u^\wedge\) and \(w^\vee = u^\vee\) on \(\Omega \setminus W\), so that \(S_w \setminus W = S_u \setminus W\). We have \(\mathcal{H}(W \cap S_v \setminus S_u) = 0\) and so also \(\mathcal{H}(W \cap S_w \setminus S_u) = 0\). We conclude that \(\mathcal{H}(S_w \setminus S_u) = 0\).

Next we show the sharpness of the condition \(\|D(w - u)\|(\Omega) < 2\|Du\|^c(\Omega) + \varepsilon\); in particular this demonstrates the fact that it is generally impossible to approximate \(\text{BV}\) functions by \(\text{SBV}\) functions in the \(\text{BV}\) norm.

**Example 5.14.** Let \(u \in \text{BV}(\Omega)\) and let \((u_i) \subset \text{SBV}(\Omega)\) such that \(u_i \to u\) in \(L^1(\Omega)\). We show that necessarily
\[
\liminf_{i \to \infty} \|D(u_i - u)\|(\Omega) \geq 2\|Du\|^c(\Omega). 
\]

Fix \(\varepsilon > 0\). We find a Borel set \(F \subset \Omega\) such that \(\mu(F) = 0\) and \(\|Du\|^c(F) = \|Du\|^c(\Omega)\). We also find an open set \(W \subset \Omega\) such that \(W \supset F\) and \(\|D(u_i)\|(W) < \|Du\|(F) + \varepsilon\). Let \(S := \bigcup_{i=1}^\infty S_{u_i}\) and \(H := F \setminus S\). Since \(S\) is \(\sigma\)-finite with respect to \(\mathcal{H}\), \(\|Du\|^c(S) = 0\). Then
\[
\|D(u - u_i)\|(H) \geq \|Du\|(H) - \|Du_i\|(H) \\
= \|Du\|(F) - \|Du_i\|(H) \\
= \|Du\|^c(\Omega) - 0
\]
for all $i \in \mathbb{N}$. Moreover,

$$
\| D(u - u_i) \|(W \setminus H) \geq \| Du_i \|(W \setminus H) - \| Du \|(W \setminus H)
$$

$$
\geq \| Du_i \|(W \setminus H) - \varepsilon
$$

$$
= \| Du_i \|(W) - \varepsilon
$$

for all $i \in \mathbb{N}$, and thus

$$
\liminf_{i \to \infty} \| D(u - u_i) \|(W \setminus H) \geq \liminf_{i \to \infty} \| Du_i \|(W) - \varepsilon
$$

$$
\geq \| Du \|(W) - \varepsilon \quad \text{since } u_i \to u \text{ in } L^1(\Omega)
$$

$$
\geq \| Du \|^c(\Omega) - \varepsilon.
$$

In total, we get

$$
\liminf_{i \to \infty} \| D(u - u_i) \|(\Omega) \geq \liminf_{i \to \infty} \| D(u - u_i) \|(W \setminus H) + \liminf_{i \to \infty} \| D(u - u_i) \|(H)
$$

$$
\geq 2 \| Du \|^c(\Omega) - \varepsilon,
$$

and so we have the result.

Now we get the following corollary, which in particular implies Theorem 1.1.

**Corollary 5.15.** Let $u \in BV(\Omega)$. Then there exists a sequence $(u_i) \subset SBV(\Omega)$ such that

- $u_i \to u$ in $L^1(\Omega)$ and $\| Du_i \|(\Omega) \to \| Du \|(\Omega)$,
- $\lim_{i \to \infty} \| D(u_i - u) \|(\Omega) = 2 \| Du \|^c(\Omega)$,
- $\limsup_{i \to \infty} \| Du \|((|u_i - u|^\vee \neq 0)) \leq \| Du \|^c(\Omega)$, $\lim_{i \to \infty} \mu((|u_i - u|^\vee \neq 0)) = 0$,
- $u_i \geq u$ and $u_i \to u$ uniformly in $\Omega$,
- $\mathcal{H}(S_{u_i} \setminus S_u) = 0$ for all $i \in \mathbb{N}$, and
- $\lim_{\Omega \ni y \to x} |u_i - u|^\vee(y) = 0$ for all $x \in \partial \Omega$.

**Proof.** This follows almost directly from Theorem 5.10 and Example 5.14. The third condition follows from (5.11) and the estimates $\mu(W) < \varepsilon$ and $\| Du \|(W) < \| Du \|^c(\Omega) + \varepsilon$ given in Theorem 5.10. The last condition also follows from (5.11). \qed

Note that the first condition says that the $u_i$’s converge to $u$ strictly, the second condition describes closeness in the BV norm, and the third condition implies that

$$
\limsup_{i \to \infty} \| Du \|((u_i^\wedge \neq u^\wedge) \cup \{u_i^\vee \neq u^\vee\}) \leq \| Du \|^c(\Omega),
$$

so it describes approximation in the Lusin sense. The last condition expresses the fact that $u_i$ and $u$ have the same “boundary values”.

In closing, let us consider a few new capacities defined similarly as in Definition 4.1.
**Definition 5.16.** Let \( A \subset D \subset \Omega \subset X \) be nonempty sets with \( \Omega \) open. We define the variational SBV-capacity by

\[
\text{cap}_{\text{SBV}}(A, D, \Omega) := \inf \|Du\|(\Omega),
\]

where the infimum is taken over functions \( u \in \text{BV}_0(D, \Omega) \cap \text{SBV}(\Omega) \) such that \( u^\wedge \geq 1 \) \( \mathcal{H} \)-a.e. on \( A \).

We define the variational diffuse BV-capacity by

\[
\text{cap}_{\text{DBV}}(A, D, \Omega) := \inf \|Du\|(\Omega),
\]

where the infimum is taken over functions \( u \in \text{BV}_0(D, \Omega) \) such that \( \mathcal{H}(S_u) = 0 \) and \( u^\wedge \geq 1 \) \( \mathcal{H} \)-a.e. on \( A \).

One can also replace \( \Omega \) by a more general set, but we choose to consider the above simpler case here.

**Corollary 5.17.** We have

\[
\text{cap}_{\text{SBV}}(A, D, \Omega) = \text{cap}_{\text{BV}}(A, D, \Omega) \quad \text{and} \quad \text{cap}_{\text{DBV}}(A, D, \Omega) = \text{cap}_1(A, D, \Omega).
\]

**Proof.** To prove the first equality, we can assume that \( \text{cap}_{\text{BV}}(A, D, \Omega) < \infty \). Let \( 0 < \varepsilon < 1/2 \). Take a function \( u \) that is admissible for \( \text{cap}_{\text{BV}}(A, D, \Omega) \) such that \( \|Du\|(\Omega) < \text{cap}_{\text{BV}}(A, D, \Omega) + \varepsilon \). By Corollary 5.15 we find a function \( w \in \text{SBV}(\Omega) \) such that \( \|Dw\|(\Omega) < \|Du\|(\Omega) + \varepsilon \) and \( \|w - u\|_{L^\infty(\Omega)} < \varepsilon \). Then \( v := (w - \varepsilon)_+/(1 - 2\varepsilon) \) is admissible for \( \text{cap}_{\text{SBV}}(A, D, \Omega) \) and so

\[
\text{cap}_{\text{SBV}}(A, D, \Omega) \leq \|Dv\|(\Omega) \leq \frac{\|Du\|(\Omega) + \varepsilon}{1 - 2\varepsilon} \leq \frac{\text{cap}_{\text{BV}}(A, D, \Omega) + 2\varepsilon}{1 - 2\varepsilon}.
\]

Letting \( \varepsilon \to 0 \), the first inequality follows.

To prove the second equality, we can assume that \( \text{cap}_{\text{DBV}}(A, D, \Omega) < \infty \). Let \( 0 < \varepsilon < 1/2 \). Take a function \( u \) that is admissible for \( \text{cap}_{\text{DBV}}(A, D, \Omega) \) such that \( \|Du\|(\Omega) < \text{cap}_{\text{DBV}}(A, D, \Omega) + \varepsilon \). Apply Proposition 5.2 with the choice \( \Omega' = \Omega \setminus S_u \) to find a function \( w \in N^{1,1}(\Omega') \) such that \( \int_{\Omega'} g_w \, d\mu < \|Du\|(\Omega) + \varepsilon \) and \( \|w - u\|_{L^\infty(\Omega')} < \varepsilon \). Since \( \mathcal{H}(S_u) = 0 \) and thus \( \text{Cap}_1(S_u) = 0 \), we have in fact \( w \in N^{1,1}(\Omega) \) with \( \int_{\Omega} g_w \, d\mu < \|Du\|(\Omega) + \varepsilon \), see [6, Proposition 1.48]. Then \( v := (w - \varepsilon)_+/(1 - 2\varepsilon) \) is admissible for \( \text{cap}_1(A, D, \Omega) \) and so

\[
\text{cap}_1(A, D, \Omega) \leq \int_{\Omega} g_w \, d\mu \leq \frac{\|Du\|(\Omega) + \varepsilon}{1 - 2\varepsilon} \leq \frac{\text{cap}_{\text{DBV}}(A, D, \Omega) + 2\varepsilon}{1 - 2\varepsilon}.
\]

Letting \( \varepsilon \to 0 \), the second inequality follows.

Note that for the first equality we did not actually need the full strength of our approximation result; recall Example 5.1. However, with our result it is also possible to handle much more general energies than simply \( \|Du\|(\Omega) \), given for example by convex functionals of linear growth, or involving terms such as \( \mathcal{H}(S_u) \) (like for example in the Mumford-Shah functional). Generally, the implication is that the absolutely continuous and jump parts help to optimize energy — in particular, it is possible to have \( \text{cap}_{\text{BV}}(A, D) < \text{cap}_1(A, D) \), see [33, Example 4.27] — but the Cantor part does not.
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