O(1) loop model with different boundary conditions
and symmetry classes of alternating-sign matrices

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Abstract

This work as an extension of our recent paper where we have found a numerical
evidence for the fact that the numbers of the states of the fully packed loop (FPL)
model with fixed link-patterns coincide with the components of the ground state
vector of the dense O(1) loop model for periodic boundary conditions and an even
number of sites. Here we give two new conjectures related to different boundary
conditions. Namely, we suggest that the numbers of the half-turn symmetric states
of the FPL model with fixed link-patterns coincide with the components of the
ground state vector of the dense O(1) loop model for periodic boundary conditions
and an odd number of sites and that the corresponding numbers of the vertically
symmetric states describe the case of the open boundary conditions and an even
number of sites.

In paper [1] we made some conjectures related to combinatorial propertie s of the
ground state vector of the XXZ spin chain for the asymmetry parameter \( \Delta = -1/2 \)
and an odd number of sites. In the subsequent paper [2] Batchelor, de Gier and Nienhuis
considered two variations of this model along with the corresponding dense O(n) loop
model at \( n = 1 \) and notably increased the number of models and related combinatorial
objects. Later we made some additional conjectures for the case of the XXZ spin chain
with twisted boundary conditions [3] and for the case of the dense O(1) loop model [4].
In the present paper we continue the consideration of the latter case.

Let us first review the results of papers [2, 4] and then give our new conjectures.

The state space of the dense O(1) loop model can be constructed as follows, see
paper [5] and references therein. For the open case one considers \( N \) vertices placed on a
line. Then for an even \( N \) one connects the vertices pairwise from the same side of the
line without intersections, see table 1 for \( N = 6 \). The state space in question is the vector
space of formal linear combinations with complex coefficients of the obtained pairings.
In the case of an odd \( N \) one vertex remains unpaired and this vertex divides the line into
two parts formed by non-intersecting pairings, see table 2 for \( N = 5 \). For the periodic case
we consider \( N \) vertices placed on a circle. Then for an even \( N \) we connect the vertices
pairwise inside the circle without intersections, see table 3 for \( N = 6 \). For an odd \( N \) one
vertex remains again unpaired, see table 4 for \( N = 7 \) where only the pairings which do
not differ by a rotation are presented. The state space of the model is again the vector
Table 1: Basis vectors and components of ground state vector of dense O(1) loop model with open boundary condition for $N = 6$

Table 2: Basis vectors and components of ground state vector of dense O(1) loop model with open boundary condition for $N = 5$

space of formal linear combination of the pairings with complex coefficients.

To construct the Hamiltonian of the dense O(1) loop model let us define the set of operators $h_i$, where the index $i$ runs from 1 to $N - 1$ for the open case, and from 1 to $N$ for the periodic case.

Consider first the case of an even $N$. For a fixed $i$ take a general basis vector. Let in this basis vector the $i$-th vertex is paired with the $(i+1)$-th one. In this case the operator $h_i$ left the basis vector unchanged. Otherwise, let the $i$-th vertex is paired with the $j$-th one and the $(i + 1)$-th vertex is paired with the $k$-th one. In this case the operator $h_i$ removes the two pairings under consideration and pairs the $i$-th vertex with the $(i+1)$-th one and the $j$-th vertex with the $k$-th one. In the case of an odd $N$ one of the vertices $i$-th and $(i + 1)$-th may be unpaired. Let it be the $i$-th vertex, and let the $(i+1)$-th vertex is paired with the $k$-th vertex. In this case the operator $h_i$ removes this pairing and pairs the $i$-th and $(i + 1)$-th vertices. Here the $k$-th vertex becomes unpaired. The case where the $(i + 1)$-th vertex is unpaired is treated similarly.
Table 3: Basis vectors and components of ground state vector of dense O(1) loop model with periodic boundary condition for $N = 6$ (up to rotations)

The Hamiltonian of the dense O(1) loop model is defined as the sum of the operators $h_i$ taken with the minus sign. Let $H$ be the matrix of the Hamiltonian. From the definition of the operators $h_i$ it follows that the sum of the matrix elements belonging to each column of the matrix $H$ is equal to $-(N - 1)$ for the open case, and it is equal to $-N$ for the periodic case. Therefore, this matrix has a left eigenvector with all components equal to 1. Thus, the Hamiltonian of the system has an eigenvector with the eigenvalue $-(N - 1)$ for the open case, and with the eigenvalue $-N$ for the periodic case. There is a strong evidence that this vector is the ground state vector of the model, see, for example, [4]. The components of the ground state vector for some partial cases are presented in tables 1–4. More examples can be found in paper [4].

The periodic case with an even number of sites is in a sense the simplest one. It was conjectured in paper [2] that under some appropriate normalization the sum of the components of the ground state vector for $N = 2n$ is equal to the number of $n \times n$ alternating sign matrices (ASMs), usually denoted by $A_n$. We supposed that the values of the ground state vector components correspond to some subclasses of the ASMs [4]. To define this correspondence one has to use the specific ‘reincarnation’ of the ASMs which we will discuss now.

The background information on the ASMs and their different combinatorial forms can be found in the recent review by Propp [6] and in references therein. The most important for us is the bijection of the ASMs and the states of the fully packed loop (FPL) model which can be described as follows. Following paper [6] define the ‘generalized tic-tac-toe’ graph as the graph formed by $n$ horizontal lines and $n$ vertical lines meeting $n^2$ intersections of degree 4, with $4n$ vertices of degree 1 at the boundary. Then we number the vertices of degree 1. We start with the left top vertex and number clockwise every other vertex. Now consider subgraphs of the underlying tic-tac-toe graph such that each of the $n^2$ internal vertices lies on exactly two of the selected edges and each numbered external vertex lies on a selected edge, while each unnumbered external vertex does not lie on a selected edge (see, for example, figure [4]). These subgraphs are the states of the FPL.
Table 4: Basis vectors and components of ground state vector of dense O(1) loop model with periodic boundary condition for $N = 7$

| Basis Vectors | Components |
|---------------|------------|
| 1 2 3 4 5 6 7 | 1 1 1 1 1 1 1 |
| 6 5 4 3 2 1 7 | 1 1 1 1 1 1 1 |
| 1 2 3 4 5 6 7 | 1 1 1 1 1 1 1 |

Figure 1: One of the possible states of the FPL model for $n = 7$

Figure 2: The pairing-pattern corresponding to figure 1

They are in bijective correspondence with the ASMs. In particular, the number of such states is equal to the number of ASMs. Each state of the FPL model defines a so-called pairing-pattern describing the pairings of the external vertices. We depict such a pattern as a circle with $2n$ vertices placed on it and connected pairwise inside the circle without intersection, see figure 2. Having in mind the relation to ASMs we denote the number of the states of the FPL model corresponding to the pairing-pattern $\pi$ by $A_n(\pi)$. Denote the set of all possible pairing-patterns by $\Pi_n$. It is evident that we can identify any pairing-pattern with the corresponding basis vector of the dense O(1) loop model. Then the main conjecture of our paper [4] can be formulated as

**Conjecture 1** For the case of an even $N = 2n$ and the periodic boundary conditions the vector

$$\Psi = \sum_{\pi \in \Pi_n} \pi A_n(\pi)$$

is the ground state vector of the dense O(1) loop model.

Proceed now to the periodic case with an odd $N = 2n + 1$. In paper [2] a formula
for the sum of the components of the ground state vector was conjectured for this case. It appears that this formula actually gives the number of \((2n + 1) \times (2n + 1)\) half-turn symmetric ASMs, which we denote by \(A_{2n+1}^{HT}\). Information about various symmetry classes of alternating sign matrices can be found in papers by Robbins [7] and Kuperberg [8] and in references therein.

It can be shown that the bijection of the ASMs and the states of the FPL model sends a half-turn symmetric ASM into a half-turn symmetric state and vice versa. Here by a half-turn symmetric state we mean a state whose picture rotated on 180° over its center coincides with itself, see figure 3 for an example. The corresponding pairing-pattern also

\[
\text{Figure 3: One of the possible half-turn symmetric states of the FPL model for } n = 7
\]

has the half-turn symmetry. Denote the set of such pairing-patterns by \(\Pi_{2n+1}^{HT}\), and the number of the half-turn symmetric ASMs corresponding to the pairing-pattern \(\pi \in \Pi_{2n+1}^{HT}\) by \(A_{2n+1}^{HT}(\pi)\).

It is possible to establish a bijection of the set \(\Pi_{2n+1}^{HT}\) and the basis of the dense O(1) loop model for the case under consideration. To this end we use a special numbering of the vertices, which should be clear from consideration of figure 3. The bijection in question is realised via identifying \(i\)-th vertex with the \(i'\)-th one, see figure 4. Identifying the states of the dense O(1) loop model with the half-turn symmetric pairing-patterns we formulate our first new conjecture as follows.

**Conjecture 2** For the case of an odd \(N = 2n + 1\) and the periodic boundary conditions the vector

\[
\Psi = \sum_{\pi \in \Pi_{2n+1}^{HT}} \pi A_{2n+1}^{HT}(\pi)
\]

is the ground state vector of the dense O(1) loop model.

Now consider the open case for an even \(N = 2n\). It was conjectured in paper [3] that in this case the sum of the components of the ground state vector coincides with the number of \((2n + 1) \times (2n + 1)\) vertically symmetric ASMs, which we denote by \(A_{2n+1}^{V}\).

Similarly to the previous case one sees that the bijection of the ASMs and the states of the FPL model sends a vertically symmetric ASM into a vertically symmetric state and vice versa, where we define a vertically symmetric state of the FPL model in an evident way, see figure 5 for an example. Note that we use a special choice of numbered vertices.
and the numbering which is more appropriate for the case under consideration.

The pairing-pattern corresponding to a vertically symmetric state of the FPL model also has such a symmetry. We denote the set of such pairing-patterns by $\Pi_{2n+1}^V$, and the number of the vertically symmetric ASMs corresponding to the pairing-pattern $\pi \in \Pi_{2n+1}^V$ by $A_{2n+1}^V(\pi)$.

Now we establish a bijection of $\Pi_{2n+1}^{HT}$ and the basis of the dense $O(1)$ loop model identifying the $i$-th vertex with the $i'$-th one and removing the pairing of the 0-th vertex and the 0'-th one, see figure 6. Identifying the states of the dense $O(1)$ loop model with the vertical symmetric pairing-patterns we formulate our second new conjecture as follows.

**Conjecture 3** For the case of an even $N = 2n$ and the open boundary conditions the vector

$$\Psi = \sum_{\pi \in \Pi_{2n+1}^V} \pi A_{2n+1}^V(\pi)$$

is the ground state vector of the dense $O(1)$ loop model.

We have not succeeded to formulate a similar conjecture for the open case with an odd $N$.

Our results partially overlap the results of the recent paper by Pearce, Rittenberg, and de Gier [9]. Namely, our conjecture 3 actually coincides with the conjecture that the components of the ground state vector for the open case and an even $N$ are given by the numbers of states of the FPL model on the corresponding pyramid grid domain with specified pairing-patterns.

**Acknowledgments** The work was supported in part by the Russian Foundation for Basic Research under grant #01–01–00201 and the INTAS under grant #00–00561.

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