Summary
Event–selected $C^r$ vector fields yield piecewise–differentiable flows [Burden et al., 2016], which possess a continuous and piecewise–linear Bouligand (or B–)derivative [Scholtes, 2012, Prop. 4.1.3]; here we provide an algorithm for computing this B–derivative. The number of “pieces” of the piecewise–linear B–derivative is factorial $(d!)$ in the dimension $(d)$ of the space, precluding a polynomial–time algorithm. We show how an exponential number $(2^d)$ of points can be used to represent the B–derivative as a piecewise–linear homeomorphism in such a way that evaluating the derivative reduces to linear algebra computations involving a matrix constructed from $d$ of these points.

Computing the Bouligand derivative of a class of piecewise–differentiable flows
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§ 1 Background

This brief note serves as an addendum to [Burden et al., 2016] that provides a compact representation of a nonclassical derivative operator, namely, the Bouligand (or B–)derivative of the piecewise–differentiable flow associated with a class of vector fields with discontinuous right–hand–sides, termed (in [Burden et al., 2016]) event–selected $C^r$. This B–derivative is a piecewise–linear map. In [Burden et al., 2016, Sec. 7] we provided the means to evaluate the B–derivative in a chosen direction (i.e. on a given tangent vector). In this note we construct a triangulation for the B–derivative, thus representing it as a piecewise–linear homeomorphism using the techniques from [Groff et al., 2003, Sec. 2].

Before proceeding, we informally recapitulate definitions and results from [Burden et al., 2016] to provide notational and conceptual context for what follows. Given an open subset $D \subset \mathbb{R}^d$ and order of differentiability $r \in \mathbb{N}$, the vector field $f : D \to TD$ is termed event–selected $C^r$ at $\rho \in D$ if there exists an open set $U \subset D$ containing $\rho$ and a collection of event functions $\{h_k\}_{k=1}^n \subset C^r(U, \mathbb{R})$ for $f$ [Burden et al., 2016, Def. 1] such that for all $b \in B = \{-1, +1\}^n$, with

$$D_b = \{x \in \mathbb{R}^d \mid \forall k \in \{1, \ldots, n\} : b_k(h_k(x) - h_k(\rho)) \geq 0\},$$

$f|_{\text{int } D_b}$ admits a $C^r$ extension $f_b : U \to TU$ [Burden et al., 2016, Def. 2]. Note that $f$ is allowed to be discontinuous along each event surface $E_k = h_k^{-1}(h_k(\rho))$, but is $C^r$ elsewhere.
If $f$ is event–selected $C^r$ at $\rho \in D$, then [Burden et al., 2016, Thm. 4] ensures there exists a piecewise–$C^r$ flow $\Phi : F \to D$ for $f$ defined on an open subset $F \subset \mathbb{R} \times D$ containing $(0, \rho)$. The notion of piecewise–differentiability used in [Burden et al., 2016] and in what follows is due to Robinson [Robinson, 1987]; for a highly–readable exposition of piecewise–$C^r$ flows and their properties, we refer the interested reader to [Scholtes, 2012]. In short, a continuous function is piecewise–$C^r$ (or $\mathcal{PC}^r$) if its graph is everywhere locally covered by the graphs of a finite number of $C^r$ functions (termed selection functions). Piecewise–$C^r$ functions always possess a continuous first–order approximation termed a Bouligand (or $B$–)derivative [Scholtes, 2012, Prop. 4.1.3]; due to the local finiteness of selection functions, the $B$–derivative of $\mathcal{PC}^r$ functions is piecewise–linear. The remainder of this note will be devoted to constructing a triangulation for the piecewise–linear $B$–derivative of the $\mathcal{PC}^r$ flow $\Phi$.

\section{Normal forms}

Let $f : U \to TU$ be an event–selected $C^r$ vector field at $\rho \in \mathbb{R}^n$ with respect to $h : U \to \mathbb{R}$ where $U$ is a neighborhood of $\rho$. By assumption, $f$ is $C^r$ everywhere except (perhaps) $h^{-1}(h(\rho)) = \{x \in U \mid \exists k : h_k(x) = h_k(\rho)\}$, the “transition surfaces”. Also by assumption, $h_k(\rho)$ is a regular value for event function $h_k$ and $Dh_k \cdot f > \varepsilon$ everywhere in $U$ for some $\varepsilon > 0$. If $n \neq d$, the system can be embedded via the technique in [Burden et al., 2016, Remark 4] in a higher–dimensional system where the dimension of the state space equals the number of event surfaces, and the matrix $Dh(\rho)$ is invertible.

\subsection{Piecewise–constant sampled systems}

Let $f : D \to \mathbb{R}^n$ be an event–selected $\mathcal{PC}^r$ with respect to $h : U \to \mathbb{R}^n$ at $\rho$. From the previous section, we assume without loss of generality that $U$ is $n$–dimensional.

We refer to the zero level sets of the components of $h$, $E_k := \{x \in U \mid h_k(x) = 0\}$ as local sections [Burden et al., 2016, Def. 1]. Since $f$ is $\mathcal{PC}^r$, zero is a regular value of each of the $h_k(\cdot)$ functions, and thus $E_k$ are embedded codimension–1 submanifolds.

Let $b \in B = \{-1,+1\}^n$ be a corner of the hypercube $\{-1,+1\}^n$. Define $F_b := \lim_{\alpha \to 0^+} f(h^{-1}(\alpha b))$ the corner value of $f$ at the corner $b$. Note that by construction, all coordinates of $F_b$ are positive and larger than $\varepsilon$. Extend (by slight abuse of notation) to $F(x) := F_b(\text{sign}(Dh(\rho) \cdot (x - \rho)))$. The flow $\hat{\Phi}(\cdot)$ of $F(\cdot)$ near $\rho$, has transition manifolds which are affine subspaces of co-dimension 1, tangent to the transition manifolds of the original system at $\rho$. Furthermore, [Burden et al., 2016, Eqn. (63)] shows that the sampled system’s flow provides a first–order approximation for that of the original system at $\rho$.

\section{The structure of corner limit flows}

We recall from [Burden et al., 2016, Thm. 7] that the time–to–impact any local section of an event–selected $C^r$ vector field is a piecewise–$C^r$ function. For each surface index $k \in \{1, \ldots, n\}$, let $\tau_k : V_k \to \mathbb{R}$ denote the time–to–impact map for the event surface $E_k = h_k^{-1}(\mathbb{R})$ defined over a neighborhood $V_k$ containing $\rho$. Letting $\tau = (\tau_1, \ldots, \tau_n) : V \to \mathbb{R}^n$ denote the composite function defined over $V = \cap_{k=1}^n V_k$,

it follows that for any point $x \in V$ and index $k \in \{1, \ldots, n\}$, $\tau_k(x) = \tau_k(x)$ is the time required for $x$ to flow to surface $E_k$, i.e. $h_k(\Phi(\tau_k(x), x)) = 0$ for the flow $\Phi : F \to D$ of the vector field $f$. Furthermore, it is clear that $\tau$ is injective and its image is an open set, whence Brouwer’s Open Mapping Theorem [Brouwer, 1911; Hatcher, 2002] implies $\tau$ is a homeomorphism onto its image.

\textbf{Lemma 3.1.} Time–to–impact maps ($\tau$ in the preceding paragraph) have the following properties:

1. they are $C^r$ on any submanifold that encounters events in the same order;
2. they take the zero level sets of the event functions ($h_k$) to the standard arrangement;
3. they take the vector field ($f$) to the constant vector field $1$ (referred to as the diagonal flow hereon);
4. for the diagonal flow, the time–to–impact is $x \mapsto -x$;
5. for sampled systems, the time–to–impact map is piecewise–linear.

\textbf{Proof.} Properties 1–4 are direct; property 5 follows from [Burden et al., 2016, Remark 4]
From lemma 3.1 we conclude that the flow $\Phi^t(\cdot)$ in a neighborhood of $\rho$ is $PC^r$ conjugate through $\tau(\cdot)$ to the diagonal flow. By the definition of $\tau$, an event $h_k(\cdot)$ occurs in the original coordinates at $x$ if, and only if, $e_1^T\tau(x) = 0$.

It remains to analyze the structure of the diagonal flow.

3.1. The Standard Cone — The cone span of a set of vectors $X \subseteq \mathbb{R}^n$ is given by $\text{Cone} X := \{ y \in \mathbb{R}^n \mid \sum_{i=1}^m \alpha_i k_i, \alpha_i \in \mathbb{R}^+ \}$. In this section we show that the diagonal flow in $\mathbb{R}^n$ comprises $n$ identical cones, whose form we make computationally explicit. The interior of each of these cones consists of all the points whose transitions happen in a specific order, as specified by a permutation $\sigma \in S^n$.

Assume hereon that $y = 1$ is the diagonal flow, conjugate through $\tau(\cdot)$ to $\dot{x} = f(x)$, i.e. $D\tau(x) \cdot f(x) = 1$ everywhere. For any $y \in \tau(U)$, let $\sigma_y \in S^n$ be a permutation that sorts the elements of $y$. This permutation can be represented by a permutation matrix $Z_\sigma$ such that $z := Z_\sigma y$ satisfies:

$$\forall 0 < i < j \leq \dim U : (i < j) \rightarrow (e_i^T z \leq e_j^T z)$$

We define the group of permutation matrices $G := \{ Z_\sigma \mid \sigma \in S^n \}$; this is a representation of $S^n$ which is valid over all fields.

We define the sets

$$K_\sigma := \{ Z_\sigma^{-1} z \mid z \in \mathbb{R}^n, z_1 \leq z_2 \leq \ldots \leq z_{n-1} \leq z_n \}$$

and denote by $K$ the set associated with the identity permutation.

By construction, $K_\sigma$ are exactly the points whose impact times are (weakly) in the order specified by $\sigma$. If impact times are different, i.e. strongly in the order specified by $\sigma$, the inequalities in 2 are strong, and the corresponding point is an interior point of $K_\sigma$.

3.2. Constructing $K$ as Cone Span — In this section we will use addition and scalar multiplication in a setwise sense, i.e. $AB$ is the set of all products of elements of $A$ and of $B$, $A + B$ is the Minkowski sum — the set of all possible sums comprising an element of $A$ and an element of $B$.

We define the vectors that form the columns of a lower triangular matrix $M^U$ with elements zero to be $s_k := \frac{n}{n+1-k} \sum_{i=k}^n e_k$, as follows:

$$M^U := \begin{bmatrix} 1 & \cdots & \cdots & 1 & 1 \\ 0 & n/(n-1) & n/(n-1) & \cdots & n/(n-1) \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \cdots & \cdots & n/2 \\ 0 & \cdots & \cdots & 0 & n \end{bmatrix} \quad \Delta^U := \frac{1}{n} \begin{bmatrix} n & -(n-1) & 0 & \cdots & 0 \\ 0 & (n-1) & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & -2 & 0 \\ 0 & \cdots & \ddots & 2 & -1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

Note that $1 \cdot s_k = n$ for all $k$, $s_1 = 1$.

$K$ is a cone — all positive multiples of $z$ will satisfy the same inequality as $z$ in 2, i.e. $\mathbb{R}^+ K = K$. $K$ is also invariant under the diagonal flow, i.e. $K = \mathbb{R}^1 + K$. Using the $s_k$ vectors we note that $K = \mathbb{R}s_n + \sum_{k=1}^{n-1} \mathbb{R}^+ s_k$.

Define $D_b$ for $b \in \mathbb{R}^n$ as $D_b = \{ x \in \mathbb{R}^n \mid \sigma x = \sigma b \}$. For the diagonal flow, the points of a set $D_b$ are all points that have the same events as for $b$ happen to them in the past (in whatever order), the same events will happen to them as for $b$ (in whatever order) in the future, and the same set of events are currently happening to them as are happening to $b$.

Let us define the anti-diagonal as $A_0 := \{ p \in \mathbb{R}^n \mid \dot{p} = 0 \}$, and similarly define two affine subspaces parallel to $A_0$ by $A_\pm := A_0 \pm 1$.

Lemma 3.2. The following set-wise equality holds

$$A_+ \cap D_1 \cap K = \text{conv}\{s_1, \ldots, s_n\}$$

and describes a simplex of dimension $n - 1$. 

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Proof. We use some facts from convexity: (1) for a convex set $A$, when $B \subseteq A$ then $(\text{conv } B) \subseteq A$; (2) for convex $A, B$ the intersection $A \cap B$ is also convex; (3) convex closure preserves linear constraints: if points $B$ meet a constraint $\forall b \in B : w \cdot b = c$, then so does their convex closure $\forall x \in \text{conv } b : w \cdot x = c$. Note that the sets $A_+$, $K$, and $D_1$ are convex.

First we show the inclusion of the RHS in the LHS of 4. For all corners $s_k, s_k \in D_1$ because they have non-negative coordinates; they satisfy $1 \cdot s_k = 1$; and $s_k \in K$ because the coordinates of $s_k$ are non-decreasing. From the convexity properties mentioned, the inclusion of the RHS in the LHS follows.

It remains to demonstrate the inclusion of the LHS of 4 in the RHS. Let $z$ be a point in the intersection in the LHS. From $z \in A_+ \cap D_1$ it follows that $z \cdot 1 = n$, and $\forall k : z_k \geq 0$. Given $z = z\Delta^U M^U$ we have that for $a := z\Delta^U$ we need to show $\sum_{k=1}^{n} a_k$ is one, and $a_k \geq 0$.

By examining the columns of $\Delta^U$ we can see that $a_1 = z_1$, and for $k > 1$ we have $a_k = \frac{n+1-k}{n} (z_k - z_{k-1})$.

From $z \in K$, we know $z_k \geq z_{k-1}$, giving us that $a_k \geq 0$. Note that $1$ is (by direct examination) a right eigenvector of $\Delta^U$ with eigenvalue $n^{-1}$, giving

$$\sum_{k=1}^{n} a_k = z\Delta^U \cdot 1 = \frac{1}{n} z \cdot 1 = 1$$

proving the inclusion of the LHS in the RHS, and thus the desired equality.

This allows us to state a more general theorem

**Theorem 3.3.**

$$A_+ \cap D_1 \cap K_\sigma = \text{conv}\{Z_\sigma s_1, \ldots, Z_\sigma s_n\}$$

**Proof.** Convex closure commutes with all linear maps, in particular the $Z_\sigma$ maps that give $K_\sigma = Z_\sigma K$. We obtain $K_\sigma = \text{conv}\{Z_\sigma s_k\}_{k=1}^{n}$. Furthermore $Z_\sigma A_+ = A_+$ and $Z_\sigma D_1 = D_1$, allowing us to conclude the desired result.

By construction $\{K_\sigma\}_{\sigma \in S_m}$ are a cover of $\mathbb{R}^n$. We conclude that by knowing how the corners in 6 map through the flow we may deduce the values at all other points by barycentric (convex) interpolation.

### 3.2.1 The sampling points $\{Z_\sigma s_k\}_{k=1}^{n}$ — The set of simplex corner points $P := \{Z_\sigma s_k\}_{k=1}^{n}$ is smaller than may appear at first. This is due to the fact that the cardinality $\#\{Z_\sigma s_k\}_{\sigma \in S_m} = \binom{n}{m}$ — there are only as many of them as there are ways of choosing $k$ of the $n$ coordinates to be zero. Overall we obtain $\#\mathcal{P} = 2^n$, the number of ways to select which entries will be zero out of $n$ possibilities. Thus by flowing forward all $2^n$ points of $\mathcal{P}$ from $A_-$ until they impact $A_+$, we obtain all the necessary information for computing all $n!$ affine transformations that comprise the first order approximation of the flow near the origin.

### 3.3. Bypassing the need for the map $\tau(\cdot)$ — The trajectories going through points of $\mathcal{P}$ other than $1$ have a unique property: each of them has only two times at which events will occur. For $Z_\sigma s_k$, a set of events occurs simultaneously at time 0, and all other events occurs simultaneously at time $\frac{n-1}{n+1-k}$.

We will now show how to compute $\tau^{-1}(\mathcal{P})$ for corner limit flows. Let $p \in \mathcal{P}$ such that $p = Z_\sigma s_k$, and $p_i \neq 0$ if and only if $i \in S \subseteq \{1, \ldots, n\}$ (i.e. $S$ is the support of $p$). Let $b \in B$ such that $b_i = -1$ for $i \in S$ and $b_i = 1$ otherwise (for $i \in S$). In the diagonal flow, $D_b$ is the set of points for which the events of $S$ have already happened, and the remaining events $S$ are yet to occur.

Let us construct a trajectory which at time 0 goes through $q(0) := \tau^{-1} p$. The state $q(0)$ is on the event surfaces of $h_i$ for $i \in S$, i.e. $\forall i \in S : \nabla h_i(p) \cdot q(0) = 0$. For any positive time $t_+ > 0$, we have $q(t_+) = q(0) + F(1)t_+$. The remaining events for this trajectory occurred at time $-t := \frac{n}{n+1-k}$, and in the time interval $(-t, 0)$ the trajectory was moving under the influence of $F(b)$. Thus $q(-t) = q(0) - t F(b)$ is on the remaining event surfaces, i.e. $\forall i \in S : \nabla h_i(p) \cdot q(0) = t \nabla h_i(p) \cdot F(b)$. For time $t_+ < -t$ the trajectory moved under the influence of $F(-1)$, i.e. $q(t-) = q(0) - t F(b) + (t_+ + t) F(-1)$, completing the description of the trajectory.
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Figure 1: A corner limit flow [left] is mapped by \( \tau(\cdot) \) into impact time coordinates [right]. In impact times coordinates, we define triangulation \( \mathcal{X}_+ \) which comprises simplices that go through known transition sequences (id and \( \sigma \) in this 2D case; black lines). The differential we seek is given by the piecewise linear homeomorphism produced by carrying this triangulation through the flow [left] from the initial locations \( q_k(-T) \) (green) to the final locations \( q_k(0) \).

Note that the vector whose \( S \) coordinates are of value \( t \), and whose \( \bar{S} \) coordinates are \( 0 \) is

\[
s(b) := \frac{(b + 1)n}{2n + 2 - 1 \cdot (b + 1)} = Z_\sigma s_k
\]

(7)

Thus we obtain a simplified equation

\[
Dh(\rho)q(0) = \text{diag}(s(b)) \cdot Dh(\rho) \cdot F(b)
\]

(8)

which allows \( q(0) \) to be solved for in the original (fully non-linear) coordinates for all \( b \in \mathcal{B} \), provided the corner limits \( F_b \) and event Jacobian \( Dh \) can be computed.

Note also that \( Dh \) is never inverted, allowing this computation to be used with event surfaces that are not transverse.

3.4. Computing the derivative of the flow — Denote the \( q(\cdot) \) of 8 and the previous section by \( q_0(\cdot) \) to highlight its dependence on \( b \). In addition, define \( q_0(0) = \rho \) and for \( t_+ > 0 \) take \( q_0(t_+) = F(1)t_+ \) and \( q_0(-t_-) = -F(-1)t_- \) – the trajectory of the origin.

We will now define a set of simplices surrounding the point \( q_0(1) \). The vertices of these simplices will be

\[
\mathcal{V} := \{q_0(0), q_0(1), q_0(2)\} \cup \{q_0(0)\}_{b \in \mathcal{B}}
\]

(9)

For every permutation \( \sigma \in S_n \) we will have a \( n - 1 \) dimensional face

\[
Y_\sigma := \text{conv} \left( \{q_0(1)\} \cup \{q_0(0) \mid b_{\sigma(1)} = -1, b_{\sigma(k+1)} = 1, 0 < k < n \} \right)
\]

(10)

These faces cover a neighborhood of \( q_0(1) \) in the affine subspace \( \tau^{-1}(\mathcal{A}_+) \).

For each face \( Y_\sigma \) we will define two simplices \( X_- \) and \( X_+ \) by \( X_- = \text{conv} \left( \{q_0(0)\} \cup Y_\sigma \right) \) and \( X_+ = \text{conv} \left( \{q_0(2)\} \cup Y_\sigma \right) \). Let \( \mathcal{X}_+ := \{X_- \cup X_+ \}_{\sigma \in S_n} \) be the set of all these simplices. By construction: (1) \( \mathcal{X}_+ \) covers a neighborhood of \( q_0(1) \); (2) All points of each simplex \( X_{\pm} \in \mathcal{X}_+ \) experience events in the same order – the order designated by \( \sigma \).
At any time $-T < -n$ no events have occurred for any of trajectories $q_b(\cdot)$, i.e. $\forall b \in B : h(q_b(-T)) < 0$ and furthermore $h(q_b(2-T)) < 0$. We can conclude this because the event times are known for each $q_b(\cdot)$ and take on the values $0$ and $\frac{n}{n+1-k}$ for $1 \leq k \leq n$.

Let $\mathcal{X}_- := \{\text{conv}\{q_k(-T)\} | \text{conv}\{q_k(0)\} \in \mathcal{X}_+\}$, i.e. $\mathcal{X}_-$ are the simplices created by taking the convex closure of the corners of simplices of $\mathcal{X}_+$ after carrying them back to time $-T$. All that remains is to construct the piecewise linear homeomorphism mapping $\mathcal{X}_-$ to $\mathcal{X}_+$ using the techniques of [Groff et al., 2003, Sec. 2]. In the parlance of that reference, our $q_k(0)$ are the set of $q$ points; our $q_k(-T)$ points are the set of $p$ points; the simplicial complex structure $\Sigma$ is given by equation 10. Figure 1 shows a example 2-dimensional case.

§ 4 Acknowledgement

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