Modeling information systems based on the use of discrete and indiscrete laws of distribution and their approximation

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Abstract. Based on a comparative analysis of the basic numerical characteristics of continuous and discrete distribution laws, it is established that the approximation of discrete laws is continuous, and vice versa, is possible. And the highest accuracy of approximation will be provided if their mathematical expectations, variances and asymmetry coefficients are equated, and the joint coefficient of asymmetry and kurtosis for a continuous law is approximately equal to the joint coefficient of asymmetry and kurtosis for a discrete law. Also, the decomposition of the distribution laws of a discrete random variable on orthogonal polynomials Kravchuk, Meixner, Charlier, Chebyshev and Khan is obtained and practical recommendations are given on the application of the obtained expansions for approximation of distribution laws.

1. Introduction
To solve the problems of information systems synthesis, discrete distribution laws are often used as probabilistic models. At present, as the analysis of literature [1-3] shows, about 20 types of discrete distribution laws are known, while more than 1000 are known as indiscrete laws [4]. On the other hand, often in order to obtain simpler expressions and simplify the computations, one must make a transition from a discrete to an integral form of writing, and, accordingly, from discrete to indiscrete distribution laws. Such transitions, based on the expansion of counting functions with respect to orthonormal sample functions with further consideration of limit relations, often require cumbersome calculations and lead to significant errors.

In the well-known literature, for example, in [5], the attempts were repeatedly made to develop a similar approximation technique based on equating the first and the second central moments of discrete and continuous distribution laws, as well as their asymmetry $\eta_a$ and kurtosis $\eta_k$ coefficients. However, this approach did not always lead to the required accuracy of approximation and was often ineffective.

2. Mutual approximation of discrete and indiscrete laws of distribution when information systems modeling
Based on the comparative analysis of the basic numerical characteristics of indiscrete and discrete distribution laws, it is established that the approximation of discrete laws for indiscrete, and vice versa,
is possible if to equate their mathematical expectations, dissipations and asymmetry coefficients, and the joint coefficient of asymmetry and kurtosis \( K_{2i} \) for indiscrete laws, defined by expression

\[
K_{2i} = \frac{1.5 \mu_{3i}^2 + 6 \mu_{2i}^3}{\mu_i (\mu_{4i} + 3 \mu_{2i}^2)} = \frac{1.5 \eta_{a,i}^2 + 6}{\eta_{k,i} + 6},
\]

consider approximately equal to the coefficient \( K_{2d} \) for discrete laws

\[
K_{2d} = \frac{1.5 \mu_{3d}^2 + 6 \mu_{2d}^3 - 1.5 \mu_{2d}^2}{\mu_{2d} (\mu_{4d} + 3 \mu_{2d}^2 - \mu_{2d}^2)} = \frac{1.5 \eta_{a,d}^2 + 6 - 1.5}{\eta_{k,d} + 6 - 1/\mu_{2d}}.
\]

The indexes \( u \) and \( \phi \) indicate the ownership of the coefficients and parameters under consideration to the indiscrete and discrete distribution laws, respectively.

The main difference of this approach is that along with mathematical expectations, variances and asymmetry coefficients, it is suggested to equate not the kurtosis coefficients, as suggested earlier, but the joint coefficient of asymmetry and kurtosis \( K_{2i} \) and coefficient \( K_{2d} \), determined by expressions (1) and (2). Computational experiment showed that at the same time high accuracy of approximation of indiscrete distribution laws by discrete and vice versa is ensured. Moreover, as the comparative analysis of expressions (1) and (2) shows, the accuracy of approximation of the discrete distribution law increases with the increase of its dispersion.

Part of the results obtained in accordance with the developed technique for the case of approximation of discrete laws by indiscrete ones, including expressions for their parameters, are given in the table.

### Table 1. Examples of approximation of discrete laws by indiscrete.

| № | Type of discrete law \( p(x) \) | Type of approximating indiscrete law \( pl(x) \) | Gaussian law |
|---|-----------------|-----------------|----------------|
| 1. | a) binomial law \( q = 0.5 \) | \( p(x) = \frac{N!}{x!(N-x)!} \left( \frac{q}{1-q} \right)^x (1-q)^N \) | \( pl(x) = \frac{1}{\sqrt{2\pi D}} \exp \left( -\frac{(x-m)^2}{2D} \right) \) |
|   | \( x = 0...N \) | | \( D = Nq(1-q) \), \( m = qN \) |
|   | b) binomial law \( q < 0.5 \) | \( p(x) = \frac{N!}{(N-x)!x!} \left( \frac{q}{1-q} \right)^x (1-q)^N \) | | \( \lambda = \frac{2}{1-2q} \), \( v = \frac{4Nq(1-q)}{(1-2q)^2} \), \( \mu = \frac{qN}{1-2q} \) |
|   | \( x = 0...N \) | | Gamma-distribution |
|   | c) binomial law \( q > 0.5 \) | \( p(x) = \frac{N!}{(N-x)!x!} \left( \frac{q}{1-q} \right)^x (1-q)^N \) | | \( \lambda = \frac{2}{1-2q} \), \( v = \frac{4Nq(1-q)}{(1-2q)^2} \), \( \mu = \frac{qN}{1-2q} \) |
|   | \( x = 0...N \) | | Gamma-distribution |
2. Poisson distribution
\[ p(x) = \frac{\alpha^x}{x!} \exp(-\alpha) \]
\[ p_l(x) = \frac{\lambda^x}{\Gamma(x)} \exp(-\lambda) \]
\[ \lambda = 2, \nu = 4\alpha, \mu = -\alpha \]

3. Negative binomial law
\[ p(x) = \frac{\Gamma(\alpha + x)}{\Gamma(\alpha) x!} q^x (1-q)^{\alpha} \]
\[ p_l(x) = \frac{\lambda^x}{\Gamma(x)} \exp(-\lambda) \]
\[ \lambda = 2 \frac{1-q}{q+1}, \nu = \frac{4aq}{(q+1)^2}, \mu = -\frac{aq}{q+1} \]

4. a) hypergeometric distribution
\[ (-1 < K < 1) \]
\[ p(x) = \frac{(c - N + 1)(N + 1 - x)(b + 2 - x)}{(b + c - N + 1)(c - N + 1)} x! \]
\[ x = 0...N \]
\[ K_s = \frac{(c-b)(b+c-2N)}{4\sqrt{bcN(b+c-N)}} \]
\[ v = b + c + 1, \alpha = \frac{(c-b)(b+c-2N)}{\sqrt{4bc - (b+c-2N)^2}}, \lambda = \frac{\sqrt{4bc - (b+c-2N)^2}}{4}, \mu = \frac{N}{2} - \frac{c-b}{4} \]

b) hypergeometric distribution
\[ (K_s = 1) \]
\[ p(x) = \frac{(c - N + 1)(N + 1 - x)(b + 2 - x)}{(b + c - N + 1)(c - N + 1)} x! \]
\[ x = 0...N \]
\[ v = b + c + 1, \lambda = \frac{bcN(b+c-N)}, \mu = \frac{m_1}{v-1}, m_1 = \frac{bN}{b+c} \]

c) hypergeometric distribution
\[ (K_s = -1) \]
\[ p(x) = \frac{(c - N + 1)(N + 1 - x)(b + 2 - x)}{(b + c - N + 1)(c - N + 1)} x! \]
\[ x = 0...N \]
\[ v = b + c + 1, \lambda = \frac{bcN(b+c-N)}, \mu = \frac{m_1}{v-1}, m_1 = \frac{bN}{b+c} \]

d) hypergeometric distribution
\[ (K_s > 1) \]
\[ p(x) = \frac{(c - N + 1)(N + 1 - x)(b + 2 - x)}{(b + c - N + 1)(c - N + 1)} x! \]
\[ x = 0...N \]
\[ v = b + c + 1, \alpha = \frac{(c-b)(b+c-2N)}{2\sqrt{(b+c-2N)^2-4bc}}, \lambda = \frac{\sqrt{(b+c-2N)^2-4bc}}{2}, \mu = \frac{m_1 - \alpha \lambda}{v-1} \]
For the inverse approximation of indiscrete laws by discrete as an approximating law it is convenient to use the generalized two-sided law of the distribution of a discrete RV

$$p(x) = \begin{cases} C, x = \mu, \\ (1 + \frac{a_1(x_m-x) - 0.5(a_1 + K_1)}{(x_m-x)(b_2(x_m-b_1) + b_0)})p(x-1), x > \mu, \\ (1 + \frac{a_1(m_1-x) - 0.5(a_1 - K_1)}{(m_1-x)(b_2(m_1-b_1) + b_0)})p(x+1), x < \mu \end{cases}$$

or

$$p(x) = \begin{cases} C, x = \mu, \\ C \prod_{k=x+\mu}^{x} (1 + \frac{a_1(k-x_m) - 0.5(a_1 + K_1)}{(k-x_m)(b_2(k-x_m) + b_0)})x > \mu, \\ C \prod_{k=x-\mu}^{-1} (1 + \frac{a_1(m_1-x) - 0.5(a_1 - K_1)}{(m_1-x)(b_2(m_1-b_1) + b_0)})x < \mu \end{cases}$$
where the parameters $a_1$, $a_0$, $b_2$, $b_0$ and $b_1$ are determined as

$$b_2 = 1 - K_2, \quad a_1 = 4K_2 - 5, \quad b_1 = 0.5K_1 + 1.5 - K_2, \quad b_0 = (2 - K_2)\mu_2, \quad a_0 = 0.5(K_1 - 4K_2 + 5). \quad (3)$$

In this case, the approximation reduces to calculating the coefficient $K_1 = \mu_3 / \mu_2$ and the joint coefficient of asymmetry and kurtosis $K_{2i}$ (1) for indiscrete law, the values of which, and also $\mu_3 = \mu_2$, are then substituted in (3) for determining the parameters of approximate law.

Thus, as a result of a comparative analysis of the basic numerical characteristics of discrete and indiscrete laws, it is established that the highest accuracy of approximation will be ensured if their mathematical expectations, variances and asymmetry coefficients are equated, and the joint coefficient of asymmetry and kurtosis $K_{2i}$ for an indiscrete law is approximately equal to the coefficient $K_{2d}$ for a discrete law.

### 3. Approximation of discrete distribution laws by orthogonal series when modeling information systems

To approximate the distribution laws of discrete random variable (RV), orthogonal discrete polynomials, defined on a finite or countable point system, are sometimes used [6,7]. For example, when approximating the distribution laws close to the Poisson law, Charlier polynomials are often used [1]. However, the basic properties of the discrete polynomials of Kravchuk, Meixner, Charlier, Chebyshev and Khan are not fully described in the literature, and their application is not shown for the representation of the distribution laws of discrete RV by orthogonal series.

Probabilities of possible values accepted by a discrete RV can be described by using the Charlier orthogonal series

$$p(x) = \frac{\lambda^x}{x!} \exp(-\lambda) \sum_{n=0}^{\infty} C_n S_n(x), \quad x = 0, 1, 2, \ldots, \quad (4)$$

where $S_n(x)$ - discrete Charlier polynomial.

Using the orthogonality conditions and the expression $h_n = \frac{n!}{\lambda^n}$ for constant $h_n$ find expansion coefficient $C_n$ in the formula (4):

$$C_n = \frac{2^n}{n!} \sum_{x=0}^{\infty} S_n(x)p(x) \quad (5)$$

From (5) accounting for $S_0(x) = 1$, $S_1(x) = 1 - \frac{x}{\lambda}$, $S_2(x) = 1 - \left(2 + \frac{1}{\lambda}\right) \frac{x}{\lambda} + \frac{x^2}{\lambda^2}$ it follows that

$$C_0 = 1; \quad C_1 = \lambda - m_1; \quad C_2 = 0.5[m_2 - (1 + 2\lambda)m_1 + \lambda^2] \quad (6)$$

Let us consider in which case the expansion in the Charlier series the distribution law of discrete RV can be restricted just to the first term, which coincides with the Poisson law. We use the coefficients $K_1$ and $K_2$, which are determined by the expression

$$K_1 = \frac{\mu_3}{m_1 + \mu_2}, \quad K_2 = 1 - \frac{\mu_3/\mu_2 + 1/2 \mu_2/m_1}{2(m_1 - 1 + \mu_2/m_1)}.$$

For the Poisson law $K_1 = 0.5$ and $K_2 = 1$. Then in the expansion (2) $C_0 = 1, C_1 = C_2 = C_3 = 0$. Parameter value $\lambda$ is found from (3), accounting that $C_1 = 0$. Herewith we get $\lambda = m_1$.

We give an example of the expansion in the Charlier series the distribution law of discrete RV. Often, it is used the sum distribution of two independent discrete RVs, one of which is distributed according to Poisson law and the other according to the negative binomial law [8, 9].
\[ p(x) = (1-q)^x \exp(-\lambda) \frac{\lambda^x}{x!} \; _2F_0(-x, \alpha, -q/\lambda). \]  

(7)

The Charlier expansion for the distribution (7) has the following form:

\[ p(x) = \frac{\lambda^x}{x!} \exp(-\lambda) \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} \left( \frac{-q}{1-q} \right)^n S_n(x), \quad x = 0, 1, 2, \ldots \]  

(8)

Using Charlier polynomials, we can also get expressions for the distribution law of two Poisson dependent random variables

\[ p(x, y) = p(x)p(y) \sum_{n=0}^{\infty} \frac{(r\lambda)^n}{n!} S_n(x)S_n(y) = 
\]

\[ = p(x)p(y)(1-r)^{x+y} \exp(r\lambda) \; _2F_0\left(-x, -y; \frac{r}{(1-r)^2 \lambda}\right) \]

(9)

and for conditional probability of Poisson random variable

\[ p(x/y) = p(x,y)/p(y) = \frac{\lambda^x}{x!} (1-r)^{x+y} \exp(r\lambda - \lambda) \; _2F_0\left(-x, -y; \frac{r}{(1-r)^2 \lambda}\right), \]

(10)

where \( r \) - correlation coefficient, characterizing the degree of linear connection between random variables.

Probabilities of possible values accepted by a discrete RV can also be described with the help of the orthogonal Kravchuk series

\[ p(x) = \frac{p^x}{(1-p)^{x-N}} \sum_{n=0}^{N} C_n K_n(x), x = 0, 1, 2, \ldots, N; \]

(11)

where \( K_n(x) \) - Kravchuk discrete polynomial.

With the help of orthogonality conditions and the expression \( h_n = \frac{n!(1-p)^n(N-n)!}{p^n N!} \) for constant \( h_n \) find the expansion coefficient \( C_n \) in the formula (11):

\[ C_n = \binom{N}{n} \frac{p^n}{(1-p)^{n}} \sum_{x=0}^{N} K_n(x)p(x). \]

(12)

From (12) considering that \( K_0(x) = 1; \; K_1(x) = 1 - \frac{x}{pN}; \; K_2(x) = 1 - \frac{2x}{pN} + \frac{x(x-1)}{p^2 N(N-1)} \) it follows that:

\[ C_0 = 1; \quad C_1 = \frac{1}{(1-p)}(pN - m_1); \quad C_2 = \frac{0.5}{(1-p)^2} \left[m_2 - (1 + 2p(N-1))m_1 + p^2N(N-1)\right] \]

(13)

Let us consider in which case in the expansion (11) we can restrict ourselves just to the first term, which coincides with the binomial law. For the binomial law \( 0 < K_1 < 0.5 \) and \( K_2 = 1 \). Then in the
expansion $C_0 = 1, C_1 = C_2 = C_3 = 0$. The parameter values $p$ and $N$ must be capable of requiring the conditions: $C_1 = C_2 = 0$. Equating the right hand sides of expressions (13) for the coefficients $C_i$ and $C_2$ to zero, we obtain a system of equations, in the result of its solution we will get:

$$N = \frac{m_2}{m_1 - \mu_2}; \quad p = \frac{m_1}{N}. \quad (14)$$

Using the Kravchuk polynomials, one can obtain an expression for the distribution law of two dependent random variables

$$p(x, y) = p(x)p(y)\sum_{n=0}^{N} \binom{N}{n} \left(\frac{r}{1-r}\right)^n K_n(x)K_n(y) =$$

$$= p(x)p(y)\left(\frac{(1-r)(1-p)}{1-p(1-r)}\right)^{x+y-N} 2F_1\left(-x, -y; -N; \frac{-r}{(1-r)^2 p(1-p)}\right) \quad (15)$$

each of which is distributed according to the binomial law. Herewith the conditional probability of a random variable

$$p(x/y) = \frac{N!p^y(1-r)^{x+y}(1-p)^y}{x!(N-x)![(1-p(1-r)]^{x+y-N}} 2F_1\left(-x, -y; -N; \frac{-r}{(1-r)^2 p(1-p)}\right) \quad (16)$$

Approximation of the distribution laws of discrete RV can also be performed with the help of the orthogonal Meixner series

$$p(x) = \frac{(\alpha x)^{\alpha}}{\alpha^x} q^{\alpha}(1-q)^{x\alpha} \sum_{n=0}^{\infty} C_n M_n(x), x = 0, 1, 2, \ldots, \quad (17)$$

where $M_n(x)$ - discrete Meixner polynomial.

Considering the orthogonality conditions and the expression $h(x) = \frac{n!}{(\alpha)_n q^n}$ for constant $h_n$ find the coefficients of expansion $C_n$ in the formula (14):

$$C_n = \frac{(\alpha)_n q^n}{n!} \sum_{x=0}^{\infty} M_n(x)p(x). \quad (18)$$

From (18) considering $M_0(x) = 1, M_1(x) = 1 - \frac{x(1-q)}{\alpha q}, M_2(x) = 1 - \frac{2x(1-q)}{\alpha q} + \frac{x(x-1)(1-q)^2}{\alpha(\alpha + 1)q^2}$ it is followed that:

$$C_0 = 1; \quad C_1 = \alpha q - m_1(1-q);$$

$$C_2 = 0.5m_2(1-q)^2 - m_1(1-q)[q(\alpha + 1) + 0.5(1-q)] + 0.5\alpha(\alpha + 1)q^2. \quad (19)$$

Let us consider in which case in the expansion (17) we can restrict ourselves just to the first term, which coincides with the negative binomial law. For a negative binomial law $0.5 < K_1 < 1$ and $K_2 = 1$. Then in the expansion (18) $C_0 = 1, C_1 = C_2 = C_3 = 0$. Parameters values $\alpha$ and $q$ must meet the conditions: $C_1 = C_2 = 0$. Equating the right hand sides of expressions (19) for coefficients $C_1$ and $C_2$ to zero, we obtain a system of equations, in the result of its solution we will get:
\[ q = 1 - \frac{m_1}{\mu_2}; \quad \alpha = \frac{m_1(1 - q)}{q} = \frac{m_1^2}{\mu_2 - m_1}. \quad (20) \]

We give an example of the expansion in the Meixner series the distribution law of a discrete RV. The Laguerre distribution [8-10] is widely used:

\[ p(x) = \exp(-b)(1 - q)^x q^x L_{n-1}^x\left(-\frac{b(1 - q)}{1}\right), \quad (21) \]

where \( a > 0, \ b \geq 0, \ 0 < q < 1 \) – distribution parameters.

The expansion in the Meixner series for distribution (21) has the following form:

\[ p(x) = \frac{(a)}{x!} q^x (1 - q)^x \sum_{n=0}^\infty \frac{b^n}{n!} M_n(x), \quad x = 0, 1, 2, \ldots. \]

Using the Meixner polynomials, one can obtain an expression for the distribution law of two dependent random variables

\[ p(x, y) = p(x)p(y) \sum_{n=0}^\infty \frac{(a)_n (rq)_n}{n!} M_n(x)M_n(y) = \]

\[ = p(x)p(y) \frac{(1 - r)^{x+y}}{(1 - qr)^{x+y}} \, _2F_1\left(-x, y; a; \frac{r(1-q)^2}{(1-r)^2q}\right), \quad (22) \]

each of which is distributed by negative binomial law. Herewith the conditional probability of a random variable

\[ p(x|y) = \frac{(a)_x q^x (1 - q)^x (1 - r)^{x+y}}{x!(1 - qr)^{x+y}} \, _2F_1\left(-x - y, a; \frac{r(1-q)^2}{(1-r)^2q}\right). \quad (23) \]

Probabilities of possible values accepted by a discrete RV can be described with the help of the orthogonal Chebyshev series

\[ p(x) = \frac{1}{N + 1} \sum_{n=0}^\infty C_n T_n(x), \quad x = 0, 1, 2, \ldots, N, \quad (24) \]

where \( T_n(x) \) – discrete Chebyshev polynomial.

Using the orthogonality condition and the expression \( h_n = \frac{(N + n + 1)(N - n)!}{(2n + 1)N!(N + 1)!} \) for constant \( h_n \), find the coefficients of expansion \( C_n \) in the formula (24):

\[ C_n = \frac{(2n + 1)N!(N + 1)!}{(N + n + 1)!(N - n)!} \sum_{x=0}^N T_n(x)p(x). \quad (25) \]

From (22) with regard to \( h_n = \frac{(c+1-N)_n (N-x+1)_x}{(b+c+1-N)_n (c+1-N)_x} \), \( x = 0, 1, 2, \ldots, N \), when \( b=1 \)

and \( c=1 \) it is followed that:

\[ C_0 = 1; \quad C_1 = \frac{3}{(N + 2)} (N - 2m_1); \]
Using the Chebyshev polynomials one can obtain an expression for the distribution law of two dependent random variables

\[ p(x, y) = \frac{1}{(N + 1)^2} \sum_{n=0}^{N} \frac{(-N)_{n} (-r)^{n} (2n + 1)}{(N + 2)_{n}} T_{n}(x)T_{n}(y). \]  

(26)

Each of which is distributed by discrete uniform law.

Here with the conditional probability of random value

\[ p(x/. y) = \frac{1}{(N + 1)} \sum_{n=0}^{N} \frac{(-N)_{n} (-r)^{n} (2n + 1)}{(N + 2)_{n}} T_{n}(x)T_{n}(y). \]

(27)

Probabilities of possible values accepted by a discrete RV can also be described by an orthogonal series using the Hahn polynomials of the first kind

\[ p(x) = \frac{(c)_{N}}{(2n + b + c - 1)(b + c)_{n} (N + b + c)_{n} (N - n)!} x! \sum_{n=0}^{N} C_{n} P_{n}(x), \quad x = 0, 1, 2, ..., N, \]

(28)

where \( P_{n}(x) \) – discrete Hahn polynomials of the first kind.

Using the orthogonality condition and expression

\[ h_{n} = \frac{(n + b + c - 1)(N + b + c)_{n} (N - n)!}{(2n + b + c - 1)(N + b + c)_{n} N!} \]

For constant \( h_{n} \), find the coefficients of expansion \( C_{n} \) in the formula (24):

\[ C_{n} = \binom{N}{n} \frac{(2n + b + c - 1)(b + c)_{n} (b + c - 1) N!}{(n + b + c - 1)(N + b + c)_{n} (N - n)!} \sum_{n=0}^{N} P_{n}(x) p(x). \]

(29)

Using Hahn polynomials of the first kind, one can obtain an expression for the distribution law of two dependent random variables

\[ p(x, y) = p(x)p(y) \sum_{n=0}^{N} \binom{N}{n} \frac{(b)_{n} (b + c)_{n} (2n + b + c - 1) r^{n}}{(N + b + c)_{n} (n + b + c - 1) N!} P_{n}(x) P_{n}(y). \]

(30)

Each of which has a distribution

\[ p(x) = \frac{(c)_{N}}{(b + c)_{N} (N + c - x)!} (b + c)_{N} (N + c - x), \quad x = 0, 1, 2, ..., N. \]

Here with the conditional probability of a random variable

\[ p(x/. y) = p(x) \sum_{n=0}^{N} \binom{N}{n} \frac{(b)_{n} (b + c)_{n} (2n + b + c - 1) r^{n}}{(N + b + c)_{n} (n + b + c - 1) N!} P_{n}(x) P_{n}(y), \]

(31)

where

\[ p(x) = \frac{(c)_{N}}{(b + c)_{N} (N + c - x)!} (b + c)_{N} (N + c - x), \quad x = 0, 1, 2, ..., N. \]

(32)

Probabilities of possible values accepted by a discrete RV can also be described by an orthogonal series with the use of Hahn polynomials of the second kind.
\[ p(x) = \frac{(1+c-N)_{n}(N-x+1)_{x}(b-x+1)_{x}}{(b+c+1-N)_{n}(1+c-N)_{x}x!} \sum_{n=0}^{N} C_{n} R_{n}(x) , \]  

where \( x = 0,1,2,...,N \); \( R_{n}(x) \) – discrete Hahn polynomials of the second kind.

Using the orthogonality condition and the expression

\[ h_{n} = \frac{(1+b+c)(c+1-n)_{n}(b+c+1-N-n)_{n}(N-n)!}{(1+b+c-2n)(2+b+c-n)_{n}(b+1-n)_{n}N!} \]

For constant \( h_{n} \), find the expansion coefficients \( C_{n} \) in the formula (30)

\[ C_{n} = \left( \frac{N}{n} \right) \frac{(1+b+c-2n)(c+1)_{n}(b+c+1-N-n)_{n}}{(1+b+c-n)(1+b+c)_{n}(1+b+c-n)_{n}} \sum_{n=0}^{N} R_{n}(x)p(x) . \]  

Using the Hahn polynomial of the second kind, one can obtain an expression for the distribution law of two dependent random variables

\[ p(x, y) = p(x)p(y) \sum_{n=0}^{N} \left( \frac{N}{n} \right) \frac{(c+1)_{n}(b+c+1-N-n)_{n}(1+b+c-2n)r_{n}^{n}}{(1+b+c-n)(1+b+c)_{n}(1+b+c-n)_{n}} R_{n}(x)R_{n}(y) , \]  

each of which is distributed by the law

\[ p(x) = \frac{(1+c-N)_{n}(N+1-x)_{x}(b+1-x)_{x}}{(1+b+c-N)_{x}(1+c-N)_{x}x!} , \quad x = 0,1,2,...,N . \]

Herewith the conditional probability of random value

\[ p(x \mid y) = p(x) \sum_{n=0}^{N} \left( \frac{N}{n} \right) \frac{(c+1)_{n}(b+1-N-n)_{n}(1+b+c-2n)r_{n}^{n}}{(1+b+c-n)(1+b+c)_{n}(1+b+c-n)_{n}} R_{n}(x)R_{n}(y) , \]  

where \( p(x) = \frac{(1+c-N)_{n}(N+1-x)_{x}(b+1-x)_{x}}{(1+b+c-N)_{x}(1+c-N)_{x}x!} , \quad x = 0,1,2,...,N . \)

Thus, the expanding of the distribution laws of a discrete RV on orthogonal polynomials of Kravchuk, Meixner, Charlier, Chebyshev, and Khan are obtained. Practical recommendations on the application of the obtained expansions to approximate the distribution laws of a discrete RV are given.

References

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