Conductance of Two-Dimensional Imperfect Conductors: Does the Elastic Scattering Preclude from Localization at $T = 0$?

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The elastic electron-impurity scattering is proven analytically to prevent from interferential localization in 2D wires with more then one conducting channel. Unconventional diffusive regime is found in the length region where the electrons are usually considered as localized. Ohmic dependence of $T = 0$ conductance is predicted instead of exponential, with length-dependent diffusion coefficient.

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Electronic and classical-wave transport in random systems of various dimensionalities have been attracting much attention for scores of years. Numerous attempts in this field are concentrated around the problem of Anderson localization whose aspects build up to a great extent the understanding of metal-insulator transitions. Prospects of researches in this area are substantially determined by the claims of the one-parameter scaling theory of localization. Although universality of the one-parameter scaling was quite long challenged, attempts were made (and still persist) to improve the scaling approach for its relative convenience and simplicity. They were stimulated considerably by experimental findings of unexpectedly anomalous transport in dilute two-dimensional (2D) electron and hole systems. Also unconventional experimental results have stimulated the development of different approaches to the problem of quantum transport in disordered 2D systems (see, e.g., discussion in Ref. [7]), among which the most intriguing expectations are associated with the Coulomb interaction of carriers. Yet the transport theories with e-e interaction still cannot claim for the general acceptance because of substantial controversy in interpreting the role of the interaction within different domains of parameters corresponding to diffusive and localized regimes.

In spite of an ample variety of theoretical approaches to the problem of localization, some points in this field are still vague, and therefore attract intensive researches. The important obstacle is insufficient mathematical grounds for localization in 2D and 3D random systems, as opposed to rigorous results for 1D systems with arbitrary strength of disorder. Meanwhile, it is instructive to point out that elaboration of practical asymptotic methods for calculating the disorder-averaged many-particle characteristics (conductivity, density-density correlator, etc), leaving out the profound spectral analysis, was even more important for the theory of 1D random systems than development of the mathematical foundation. Arguments of a comparable standard in favour of localization, as well as against it, for 2D and 3D systems have not been found yet, except for some aspects of weak localization problem.

We consider a two-dimensional rectangular sample of the length $L$ in $x$ direction and the width $D$ in $y$ direction, where non-interacting electrons subject to static random potential are confined between the hard-wall lateral boundaries $y = \pm D/2$ while in the direction of current ($x$) we suppose the system open. The dimensionless conductance $g(L)$ (in units $e^2/\pi h$) is computed from the linear response theory, whence at zero temperature the formula follows

$$
g(L) = -\frac{4}{L^2} \int_0^L dx_1 dx_2 \sum_{n,m=1}^{\infty} \frac{\partial}{\partial x_1} G_{nn'}(x,x') \frac{\partial}{\partial x_2} G_{nn'}^*(x,x').$$

Here $G_{nn'}(x,x')$ is the retarded one-electron Green function in the coordinate-mode representation, i.e. Fourier-transformed over the transverse coordinate. This function obeys the equation

$$
\left[ \frac{\partial^2}{\partial x^2} + k_n^2 + i0 - V_n(x) \right] G_{nn'}(x,x') - \sum_{m=-1}^{\infty} U_{nm}(x) G_{mn'}(x,x') = \delta_{nn'} \delta(x - x'),
$$

where $k_n^2 = k_F^2 - (n\pi/D)^2$ is the longitudinal mode energy, $k_F$ is the Fermi wavenumber, $U_{nm}(x)$ is the mode matrix element of the “bulk” random potential $V(r)$ which is assumed to have zero mean and the binary correlator
\[ \langle V(r)V(r') \rangle = QW(r-r') , \]

where \( r = (x,y) \). The angular brackets in Eq. (3) stand for impurity averaging, the function \( W(r) \) is normalized to unity and has the correlation radius \( r_c \).

From the technical point of view it is important that the diagonal matrix element \( V_n(x) \equiv U_{nn}(x) \) is initially separated in Eq. (2) from off-diagonal elements, so that the matrix \( \| U_{nm} \| \) governs inter-mode transitions only. This enables to reduce strictly the problem of finding overall set of the functions \( G_{nn}(x,x') \) to the solution of a subset of purely one-dimensional closed equations for the diagonal mode functions \( G_{nn}(x,x') \). The exact “one-dimensionalization” procedure is sketched out below.

First, we introduce the auxiliary (trial) Green function \( G^{(V)}(x,x') \) obeying the equation

\[ \left( \frac{\partial^2}{\partial x^2} + k_n^2 + i0 - V_n(x) \right) G^{(V)}_n(x,x') = \delta(x-x') \] (4)

and Sommerfeld’s radiative conditions\(^2\) at the strip ends \( x = \pm L/2 \), which seem natural for an open system. Then, turning from Eq. (2) to the consequent integral equation

\[ G_{nn}(x,x') = G^{(V)}_n(x,x')\delta_{nn'} + \sum_{m=1}^{\infty} \int_L dt R_{nm}(x,t)G_{mn'}(t,x') \] (5)

with the kernel

\[ R_{nm}(x,t) = G^{(V)}_n(x,t)U_{nm}(t) , \]

one can express all the off-diagonal mode elements \( G_{mn} \) via the diagonal ones \( G_{nn} \) by means of the linear operator \( \hat{K} \),

\[ G_{mn}(x,x') = \int_L dt K_{mn}(x,t)G_{nn}(t,x') . \] (6)

The equation for the matrix elements \( K_{mn}(x,x') \) of \( \hat{K} \) results directly from Eq. (5).

\[ K_{mn}(x,x') = R_{mn}(x,x') + \sum_{k \neq n}^{\infty} \int_L dt R_{mk}(x,t)K_{kn}(t,x') . \] (7)

This equation belongs to a class of multi-channel Lippmann-Schwinger equations that are known to be extremely singular in general, in contrast to their single-channel counterparts\(^2\). However by choosing the trial Green function \( G^{(V)}_n \) as a zero approximation for \( G_{nn} \) and perturbing it by the inter-mode potentials \( U_{nm}(x) \) only, we manage to avoid the above mentioned singularity. Therefore the solution of Eq. (5) can be written in the form

\[ \hat{K} = \left( \mathbb{1} - \hat{R} \right)^{-1} \hat{R}\mathbb{P}_n , \] (8)

where \( \hat{R} \) is an operator acting in the mixed coordinate-mode space \((x,n)\) and specified by the matrix elements \( \langle \delta \rangle \). It is important that the indicated space contains all the waveguide modes except for the \( n \)-th mode itself. The projection operator \( \mathbb{P}_n \) makes the mode index of any operator that stands next to \( \mathbb{P}_n \) (both from the left and right) equal to \( n \).

From Eqs. (2), (4), (5) we obtain the exact closed one-dimensional equation for each diagonal function \( G_{nn}(x,x') \) separately,

\[ \left( \frac{\partial^2}{\partial x^2} + \kappa_n^2 + i0 - V_n(x) - \Delta T_n \right) G_{nn}(x,x') = \delta(x-x') , \] (9)

with \( \kappa_n^2 = k_n^2 - \langle \hat{T}_n \rangle, \Delta \hat{T}_n = T_n - \langle \hat{T}_n \rangle \). The operator \( \Delta \hat{T}_n \) acts on the variable \( x \) only since from Eq. (4) it follows that the operator \( \hat{T}_n \) is a two-dimensional T-matrix\(^2\) enveloped by the projective operators \( \mathbb{P}_n \),

\[ \hat{T}_n = \mathbb{P}_n \hat{U} \left( \mathbb{1} - \hat{R} \right)^{-1} \hat{R}\mathbb{P}_n = \mathbb{P}_n \hat{U} \left( \mathbb{1} - \hat{R} \right)^{-1} \mathbb{P}_n , \] (10)

\( \hat{U} \) is the intermode scattering operator in \((x,n)\) space, specified by matrix elements \( \langle x, k| \hat{U} |x', m \rangle = U_{km}(x)\delta(x-x') \).

Hereinafter, when analyzing Eq. (10), we regard a set of the renormalized energies \( \kappa_n^2 \) \( (n=1,2,\ldots) \) as representing
the new “unperturbed spectrum” of the system, instead of the primordial spectrum \( \{ k_n^2 \} \). The perturbation theory will then be developed making use of the appropriate zero-mean potentials \( V_n(x) \) and \( \Delta T_n \).

To complete the one-dimensionalization we express the conductance \( I \) through the diagonal Green functions \( G_{nn} \) and the trial functions \( G_m^{(V)} \) (both one-dimensional!). In this paper we focus on the case of \textit{weak electron-impurity scattering} specified by the inequalities

\[
k_F^{-1}, r_c \ll \ell ,
\]

with \( \ell = 2k_F/Q \) denoting a \textit{semiclassical} mean free path evaluated for \( \delta \)-correlated 2D random potential, i.e. \( W(r) = \delta(r) \) in Eq. (3). The conditions (12) allow to expand the operator \( \hat{K} \), Eq. (4), to lowest order in the inter-mode operator \( \hat{R} \), what in turn enables to replace the exact operator \( \hat{T}_n \) from Eq. (11) by its approximate value

\[
\hat{T}_n \approx P_n \hat{U} G_m^{(V)} \hat{u} P_n ,
\]

with the operator \( G_m^{(V)} \) defined by matrix elements \( \langle x, k \rangle \hat{G}^{(V)} | x', m \rangle = \delta_{km} G_m^{(V)} (x, x') \). Applying then Eqs. (7), (8), and (13) to Eq. (11) we arrive at the following expression for the impurity-averaged conductance,

\[
\langle g(L) \rangle = -\frac{4}{L^2} \sum_{n,m=1}^{\infty} \int_L dx \ dx' \left[ \left( \partial G_{nn}(x, x') / \partial x \right) \partial G_{nn}^*(x, x') \right] \\
+ \frac{Q}{d} \sum_{n,m=1}^{\infty} \int_L dy \left[ (G_m^*)^s (x, y) \partial / \partial y G_m^{(V)} (x, y) \right] \left[ (G_{nn})^s (x', y) \partial / \partial y' G_{nn}(x', x') \right] . \tag{14}
\]

At this point it is useful to discuss the spectral properties of the quantum-mechanical system governed by the equation (10). First, the term (13) \( \Delta k_n^2 = -i/\tau_n^{(x)} \) which modifies the initial spectrum \( \{ k_n^2 \} \) can be readily calculated. In the limit (12) an explicit form of the function \( W(r) \) is not so important, and we obtain from Eq. (13)

\[
\Delta k_n^2 = \frac{Q}{D} \sum_{m=1}^{N_c} \int_\mathcal{P} \frac{d_0 W(q + k_m)}{2\pi k_m^2 - q^2} , \tag{15a}
\]

\[
\frac{1}{\tau_n^{(x)}} = \frac{Q}{D} \sum_{m=1}^{N_c} \frac{1}{4k_m} \left[ \tilde{W}(k_m - k_n) + \tilde{W}(k_n + k_m) \right] . \tag{15b}
\]

In Eq. (15a) the symbol \( \mathcal{P} \) stands for principal value, \( \tilde{W}(q) \) is the Fourier transform of the correlation function \( W(r - r') \) over \( y, y' \) and \( x - x' \). The summation in Eq. (15a) is restricted by the number \( N_c = \lfloor k_F D / \pi \rfloor \) of \textit{conducting channels} (extended waveguide modes), because only for \( n \leq N_c \) the disorder-averaged diagonal in \( n \) Green matrix \( \langle \hat{G}^{(V)} \rangle \) in Eq. (13) is essentially complex. The real addition (15a) to the primordial mode energy \( k_n^2 \) is small under conditions Eq. (12), so that it can be omitted. At the same time, the “level broadening” \( 1/\tau_n^{(x)} \), which can be interpreted as the inverse phase-breaking time for the \( n \)-th mode state, relates to the mean level spacing as \( N_c / k_F \ell \) and thus cannot be omitted in the framework of the weak scattering approximation in general. Just the addition (15b) to \( k_n^2 \) is of crucial importance for the further analysis.

We emphasize that the level broadening (15b) implies the presence of other extended modes with \( m \neq n \) in the conductor. For extremely narrow strips with \( N_c = 1 \) the imaginary term is not present in the renormalized mode spectrum as the sum (15b) contains no terms in this case. Then the system should exhibit true one-dimensional properties. Specifically, the electrons can be transferred within two regimes only, \textit{ballistic} and \textit{localized}, and the conductance of such a wire is decreased \textit{exponentially} with the length \( L \) exceeding the \textit{localization length} \( \xi_1 = 4 L_b \sim \ell \), the quadruple Born backscattering length [1].

With increasing the conductor width, as soon as the wire ceases to be single-mode (\( N_c \geq 2 \)), the situation changes drastically. The \( n \)-th mode spectrum is modified jointly by both the potentials \( V_n(x) \) and \( \Delta T_n \), and acquires the level broadening (15b). We thus come to the necessity of analyzing the condition of (one-dimensional!) localization in \textit{lossy} media, though \textit{no inelastic scattering} was initially involved in the problem. The appropriate comprehensive theory is beyond the scope of the short article and will be given in more extensive publication [2].

Here we emphasize that in studying spectral properties of a system governed by Eq. (10) one should clearly distinguish between the \textit{direct} intramode scattering due to the \textit{local} potential \( V_n(x) \) and \textit{indirect} intramode scattering
due to the *operator* potential $\hat{T}_n$. The intramode potential $V_n(x)$ gives rise to the coherent localization effect, just as in the case with $N_c = 1$, Ref. [23]. This potential causes local (in $x$) elastic $n \to n$ transitions, so that the effect is purely interferential. Meanwhile, from Eqs. (11), (13) it follows that the potential $\hat{T}_n$ can also be associated with the $n \to n$ scattering, but via all the other modes, i.e. except for the $n$-th mode. Pictorially this can be thought of as diffusion in the mode space with returning to the initial mode.

It is justified thus to regard the operator potential $\hat{T}_n$ just as governing the *intermode* scattering within the effectively “single-mode” problem [10]. This scattering can lead both to the coherent localization (due to the potential $\Delta \hat{T}_n$) and to the uncertainty of a mode state due to the term $i/\tau_n^{(\omega)}$ arising from strong complexity of the trial functions $G^{(V)}_{m}(x,x')$ at $m \leq N_c$. This duality of the intermode scattering, especially the appearance of the “phase breaking” term $i/\tau_n^{(\omega)}$ in spite of the scattering due to the potential $\hat{T}_n$ being effectively *intramode* (i.e., at first sight elastic), has a clear physical explanation. It certainly results from probabilistic nature of electron transitions through intermediate mode states $m \neq n$ (intrinsic to the potential $\hat{T}_n$) with the mode energies different from $k_F^2$. This *hidden inelasticity* is exactly the reason for strong complexification of the quasi-particle spectrum (13).

In the final stage we discuss the role of the intermode scattering in the whole range of the conductor length by estimating the Born scattering rate $1/\tau_n^{(T)}$ which determines the fundamental frequency of the states presumably localized by the 1D random potential $\Delta \hat{T}_n$. Estimation of the operator norm $\|\Delta \hat{T}_n\|^2$ with the use of Eq. (13) yields

$$\frac{\tau_n^{(c)}}{\tau_n^{(T)}} \sim \frac{1}{\cos^2 \vartheta_n} \min \left(1, \frac{L/D}{k_F \ell} \right),$$

where $\vartheta_n$ is a “sliding angle” of the mode $n$ with respect to the $x$-axis, $|\sin \vartheta_n| = n\pi/k_F D$. The level broadening for an $n$-th mode exceeds the level spacing provided the wire is not extremely stretched along the $x$-axis, i.e. if the length $L$ does not fall into the interval

$$L \gg Dk_F \ell \sim N_c \ell .$$

Yet even within this interval the level spacing $1/\tau_n^{(T)}$ due to the potential $\Delta \hat{T}_n$ cannot exceed the level broadening $1/\tau_n^{(\omega)}$. Consequently it is useless to seek the traditional interferential localization at any length of the multi-mode ($N_c \geq 2$) conductor.

To illustrate the above statement we find the average conductance (14) for different lengths in the relatively easy case $N_c \gg 1$. The exact mode function $G_{nn}$ can be obtained from the equation

$$G_{nn}(x,x') = G_{nn}^{(0)}(x,x') + (\hat{G}_{nn}^{(0)} \Delta \hat{T}_n \hat{G}_{nn}) (x,x'),$$

which stems directly from Eq. (10), where the “unperturbed” function $G_{nn}^{(0)}(x,x')$ obeys the equation (10) with $\Delta \hat{T}_n = 0$. With the estimate (10) taken into account one can solve Eq. (13) perturbatively in $\Delta \hat{T}_n$. In doing so the addition to the conductance emerges that is similar to the second term $\langle g^{(2)}(L) \rangle$ on the r.h.s. of Eq. (14), but proportional to higher degree of the small interaction strength $Q$. The potential $\Delta \hat{T}_n$ can thus be removed from Eq. (10) and the intermode scattering taken into account through the dephasing rate $1/\tau_n^{(\omega)}$ and the term $\langle g^{(2)}(L) \rangle$.

The potential $V_n(x)$, though different from $\Delta \hat{T}_n$ by its physical meaning, can as well be removed from Eq. (13) because of the relative smallness of its norm, $\langle \|V_n\|^2 / \|\Delta \hat{T}_n\|^2 \rangle \sim N_c^{-1}$. Then the Green function $G_{nn}$ can be replaced in Eq. (14), to the main approximation in $N_c^{-1} \ll 1$, by its “unperturbed” expression

$$G_{nn}^{(0)}(x,x') = \frac{1}{2ik_n} \exp \left\{ [ik_n - 1/(\ell \cos \vartheta_n)] \vartheta_n - x' \right\} ,$$

which nonetheless includes the most of inter-mode-scattering effects.

As to the functions $G_{mn}^{(V)}(x,y)$ in Eq. (14), at $L \ll N_c \ell$ we can put the potential $V_n(x) \equiv 0$ since the $m$-th mode localization length found from Eq. (11) with the use of the method of Ref. [23] is $\xi_m = \frac{16\pi}{k_F} N_c \ell \cos^2 \vartheta_m / W_{x}(2k_m) \sim N_c \ell$. In this case the second term in Eq. (14) turns out to be $-1/8$ of the first one, i.e. not parametrically small. Yet in the limit (17) all the functions $G_{mn}^{(V)}(x,y)$ are localized, and therefore the second term in Eq. (14) is negligibly small.

Basing on the above arguments we arrive at the following asymptotic expressions for the conductance (14), disregarding weak-localization corrections governed by the intra-mode potentials $V_n(x)$,

$$\begin{align*}
&i) \quad L < \ell : \quad \langle g(L) \rangle \approx N_c ; \\
&ii) \quad \ell \ll L \ll N_c \ell : \quad \langle g(L) \rangle \approx \frac{2}{3} N_c \ell / L \gg 1 ; \\
&iii) \quad N_c \ell \ll L : \quad \langle g(L) \rangle \approx \frac{2}{3} N_c \ell / L \ll 1 .
\end{align*}$$

(20)
The result given in Eq. (20) allows to distinguish three regimes of charge transport in multi-mode conductors, none of them localized in the anticipated sense. Regime (i) corresponds to entirely ballistic transport, both from semiclassical and quantum standpoints. In regimes (ii) and (iii) the semiclassical motion should be regarded as diffusive. The difference between them is that in regime (ii) all the mode states could be considered extended in the absence of the intermode scattering, whereas in regime (iii) they all would be localized due to the potentials $V_n(x)$. In both diffusive regimes (ii) and (iii) the conductance exhibits purely ohmic (inversely proportional to $L$) behaviour, but with different (classic) diffusion coefficients. Note that just in regime (iii), when all the trial states would be localized if the inter-mode scattering was disregarded, the result is exactly reproduced given by the classical kinetic theory. No exponential decay of the conductance appears at any length and width of the system provided $N_c \geq 2$.

To conclude, the $T = 0$ conductance of a 2D finite-size disordered metal strip was calculated. The interferential localization was shown to manifest itself strongly only for single-mode, i.e. purely 1D, conductors. In commonly examined square-shaped multi-mode samples the electron transport is diffusive as long as $L \gg \ell$, the semiclassical mean free path. For any extended (propagating) mode in a multi-mode strip all the other extended modes can be thought of as an effective phase-breaking reservoir destroying quantum interference and hence the exponential localization.

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