Twisted Virtual Bikeigebras and Twisted Virtual Handlebody-Knots

Sam Nelson*       Yuqi Zhao†

Abstract

We generalize unoriented handlebody-links to the twisted virtual case, obtaining Reidemeister moves for handlebody-links in ambient spaces of the form $\Sigma \times [0, 1]$ for $\Sigma$ a compact closed 2-manifold up to stable equivalence. We introduce a related algebraic structure known as twisted virtual bikeigebras whose axioms are motivated by the twisted virtual handlebody-link Reidemeister moves. We use twisted virtual bikeigebras to define $X$-colorability for twisted virtual handlebody-links and define an integer-valued invariant $\Phi^X$ of twisted virtual handlebody-links. We provide example computations of the new invariants and use them to distinguish some twisted virtual handlebody-links.

Keywords: Twisted virtual handlebody-links, twisted virtual bikeigebras, counting invariants

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1 Introduction

A spatial graph is a graph embedded in $\mathbb{R}^3$, which we may conceptualize as a “knotted graph.” A regular neighborhood of a trivalent spatial graph is a handlebody embedded in $\mathbb{R}^3$, known as a handlebody-knot. In [6] Reidemeister moves for handlebody-knots are described and in [7] genus-two handlebody-knots with up to six crossings are classified.

In [10], virtual knot theory was introduced, with a geometric interpretation of virtual knots as knotted curves in thickened orientable surfaces up to stabilization moves described in [3]. In [2], this geometric interpretation of virtual knots and link was extended to include knotted curves in thickened non-orientable surfaces known as twisted virtual links.

Bikei, also known as involutory biquandles, were considered in [1]; twisted biracks and their special case, twisted bikei, were considered in [4]. These algebraic structures define invariants of unoriented twisted virtual links by counting colorings of the semiarcs in a twisted virtual link diagram satisfying certain conditions at the crossings and twist bars.

In this paper we consider twisted virtual handlebody-knots, which are handlebodies embedded in 3-manifolds with boundary of the form $\Sigma \times [0, 1]$ for a possibly non-orientable surface $\Sigma$. We identify a list of Reidemeister moves for diagrams representing these objects and use these moves to define an algebraic structure we call twisted virtual bikeigebras, generalizing algebraic structures such as those in [4] [9] [11]. We then show that twisted virtual bikeigebras colorings of unoriented twisted virtual handlebody-link diagrams are preserved by our moves in the sense that for every coloring before a move, there is a unique corresponding coloring after the move; hence, the set of such colorings is an invariant of twisted handlebody-knots.

The paper is organized as follows. In Section 2 we introduce twisted virtual handlebody-knots. In Section 3 we introduce twisted virtual bikeigebras and use them to define computable invariants of twisted virtual handlebody-links with explicit examples. We end in Section 4 with some questions for future work.

* Email: Sam.Nelson@cmc.edu. Partially supported by Simons Foundation collaboration grant 316709
† Email: YZhao17@students.claremontmckenna.edu.
2 Twisted Virtual Handlebody-Knots

We begin with a few basic definitions.

A handlebody-knot or (handlebody-link if there is more than one component) is a handlebody of any genus embedded in $S^3$. In [6], handlebody-knots are shown to be equivalent via taking regular neighborhoods to equivalence classes of trivalent spatial graphs modulo a set of moves including the classical Reidemeister moves together with additional moves involving the interaction of trivalent vertices with classical crossings.

A twisted virtual link is an ambient isotopy class of simple closed curves in an ambient space of the form $\Sigma \times [0, 1]$, where $\Sigma$ is a compact surface, up to stabilization of $\Sigma$. If $\Sigma$ is orientable, we have a virtual link and if $\Sigma$ is non-orientable, we have a twisted virtual link. In [2, 3, 8, 10], virtual links and twisted virtual links were show to be equivalent to equivalence classes of diagrams including, in addition to the usual classical crossings, virtual crossings representing genus in $\Sigma$ and twist bars representing cross-caps in $\Sigma$. Starting with a twisted virtual link diagram $D$, we build an abstract surface $\Sigma$ by “skimming” from $D$ as shown:

We may think of our knot as the center line of a road with classical crossings representing intersections, virtual crossings representing bridges and twist bars representing the highway twisting upside down like a roller coaster. We cap off the resulting boundary circles with discs, possibly stabilizing with handles, to complete the abstract supporting surface $\Sigma$.

Example 1. The twisted virtual knot diagram below represents a knot in a thickened $T^2 \# \mathbb{R}P^2$.

We are interested in the case of twisted virtual handlebody-knots and twisted virtual handlebody-links. Geometrically, these are embeddings of handlebodies into ambient spaces of the form $\Sigma \times [0, 1]$ regarded up to ambient isotopy of $\Sigma \times [0, 1]$ and stabilization of $\Sigma$. Considering the interaction of virtual crossings, twist bars and trivalent vertices, the reasoning in [6] and [2] yields two new additional moves as described below.

Remark 1. One can consider the special case of virtual handlebody-links without allowing twist bars; in this case, the analog of the Reidemeister IV move replacing the classical crossing with a virtual crossing is treated as a forbidden move.

Definition 1. Twisted virtual handlebody-knots are equivalence classes of unoriented trivalent spatial graph diagrams under the following rather lengthy set of moves. First, equivalence classes of diagrams containing only classical crossings under the moves

\[ r_1, r_1', r_{II}, r_{II}' \]
form classical knots and links. Including virtual crossings in our diagrams and allowing the moves

![Virtual Knot Diagrams](image)

yields virtual knots and links [10]. Allowing classical crossings and trivalent vertices together with moves rI, rII, rIII and

![Handlebody-Knot Diagrams](image)

yields handlebody-knots [6 7]. Allowing classical crossings, virtual crossings and twist bars with moves rI, rII, rIII, vI, vII, vIII and v and

![Twisted Virtual Knot Diagrams](image)

yields twisted virtual knots and links [2].

Since the interaction of trivalent vertices with classical crossings is considered in [6] and the interaction of twist bars and virtual crossings with classical crossings is considered in [2, 3, 8, 10], it remains only to consider the interaction of twist bars and virtual crossings with trivalent vertices. Thinking in terms of the skimming process, we have the following two pictures

![Twist Bar and Virtual Crossings Interaction](image)

which yield the moves tIV and tV below. Hence, allowing classical crossings, virtual crossings, trivalent vertices and twist bars with all previously listed moves, we need only two more moves, namely

![More Moves for Twisted Virtual Handlebody-Knots](image)

to define twisted virtual handlebody-knots.
3 Twisted Virtual Bikeigebra

Definition 2. Let \( X \) be a set. A twisted virtual bikeigebra structure on \( X \) consists of three binary operations \( x, y \mapsto x \circledast y, x \otimes y \) and one partially defined operation \( (x, y) \mapsto xy \) and an involution \( T : X \to X \) satisfying the following axioms:

(i) For all \( x \in X \), we have \( x \circledast x = x \otimes x \) \( (ri.i) \),

(ii) For all \( x, y \in X \) we have

\[
(x \circledast y) \circledast y = x \quad (rii.i) \\
(x \otimes y) \otimes y = x \quad (rii.ii)
\]

(iii) For all \( x, y, z \in X \), we have the classical and virtual exchange laws:

\[
(x \circledast y) \circledast (z \circledast y) = (x \circledast z) \circledast (y \circledast z) \quad (riii.i) \\
(x \otimes y) \otimes (z \otimes y) = (x \otimes z) \otimes (y \otimes z) \quad (riii.ii)
\]

(iv) For elements where the operation \( (x, y) \mapsto xy \) is defined, the operation is associative and we have

\[
xy = z \iff yz = x \iff zx = y, \quad xy = (y \otimes x)(x \circledast y) \quad (riv.i) \\
xy = (y \circledast x)(x \otimes y) \quad (riv.ii)
\]

and

\[
(xy) \circledast z = (x \circledast z)(y \circledast (z \otimes x)) \quad (rv.i) \\
(xy) \otimes z = (x \otimes z)(y \otimes (z \circledast x)) \quad (rv.ii) \\
(x \circledast y) \circledast z = (x \otimes y) \otimes (z \circledast x) \quad (t.i) \\
x \circledast (yz) = (x \circledast y) \otimes z \quad (t.ii)
\]

(v) For all \( x, y \in X \),

\[
T(x \circledast y) = T(x) \circledast y \quad (tii.i) \\
x \circledast y = x \otimes T(y) \quad (tii.ii) \\
T(T(x)T(y)) = (y \otimes x)(x \circledast y) \quad (tii.iii) \\
T(T(x) \circledast T(y)) = [(x \circledast y) \otimes (y \otimes x)] \circledast [(y \otimes x) \circledast (x \circledast y)] \quad (tii.iii)
\]

Example 2. Let \( X \) be a set and define trivial bikei and virtual bikei operations on \( X \), so we have for all \( x, y \in X \)

\[
x \circledast y = x \otimes y = x \circledast y = x.
\]

Then the conditions of axioms (i), (ii) and (iii) are satisfied automatically, as are conditions (rv.i-iv) and (tv.i-ii); the rest of axiom (iv) requires that the partially defined multiplication on \( X \) be associative and commutative when defined, with

\[
xy = x \iff yz = x \iff zx = y.
\]

Writing \( xy \) additively, this condition says

\[
x + y = z \iff y + z = x \iff z + x = y;
\]

we can satisfy this by making \( X \) a \( \mathbb{Z}_2 \)-module so that all three equations become simply

\[
x + y + z = 0.
\]
For example, the Klein 4-group \( X = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) is a virtual bikeigebra with \( x \bar{\circ} y = x \circledast y = x \otimes y = x \) and \( xy = x + y \). Then any involution \( T : X \to X \) satisfies (tii.iii) and (tii.ii) automatically, and the final condition (tiv.iii) requires that
\[
T(T(x) + T(y)) = y + x
\]
or equivalently
\[
T(x) + T(y) = T(x + y),
\]
so the twist maps admitted by \( X \) are the automorphisms of \( X \).

**Example 3.** An *Alexander bikei* is a module \( X \) over \( \mathbb{Z}[t^{\pm 1}, s^{\pm 1}] / (s^2 - 1, t^2 - 1, (1 - s)(s - t)) \) with bikei operations
\[
\begin{align*}
  x \ast y &= tx + (s - t)y, \\
  x \bar{\circ} y &= sx.
\end{align*}
\]
Then if \( X \) is a twisted virtual bikeigebra, axiom (rv.iv) requires
\[
sx = x \bar{\circ} (yz) = (x \bar{\circ} y) \bar{\circ} z = s^2 x = x
\]
so we must have \( s = 1 \); then axiom (rv.iii) requires
\[
tx + (1 - t)yz = x \ast (yz) = (x \ast y) \ast z = t^2 x + t(1 - t)y + (1 - t)z = x + t(1 - t)y + (1 - t)z
\]
so we must also have \( t = 1 \); hence any twisted virtual bikeigebra with Alexander bikei operations must have trivial bikei operations
\[
\begin{align*}
  x \ast y &= 1x + (1 - 1)y = x, \\
  x \bar{\circ} y &= 1x = x.
\end{align*}
\]

More generally, we can specify a twisted virtual bikeigebra with a block matrix encoding the operation tables of the three operations \( \ast, \bar{\circ}, \otimes \), one partially defined operation \( (x, y) \mapsto xy \), and one involution \( T \).

**Example 4.** The set \( X = \{1, 2, 3, 4\} \) has twisted virtual bikeigebra structures including
\[
\begin{array}{cccc|cccc|cccc|cccc}
  & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 2 \\
  \hline
  \ast & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\
  \bar{\circ} & 2 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 2 & 2 & 4 & 3 & 2 & 1 \\
  \otimes & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 1 & 2 & 3 & 4 \\
  \cdot & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

which we can write more compactly as a block matrix
\[
\begin{bmatrix}
  2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 3 & 4 & 1 & 2 & 2 \\
  1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 4 & 3 & 2 & 1 & 1 \\
  3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 1 & 2 & 3 & 4 & 3 \\
  4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 2 & 1 & 4 & 3 & 4 \\
\end{bmatrix}.
\]

The same bikei can have several different virtual structures, partial products and twist maps; for example, we also have twisted virtual bikeigebra
\[
\begin{bmatrix}
  2 & 2 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
  1 & 1 & 2 & 2 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
  3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
  4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{bmatrix},
\]

where \(-\) indicates an undefined product.
The twisted virtual bikeigebra axioms are motivated by the Reidemeister moves for unoriented twisted virtual handlebody-knots with the coloring rules

\[
\begin{align*}
L_1 & \\
\left[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\right].
\end{align*}
\]

For instance, the move tV gives us axiom (tv.i):

Analogously to other algebraic structures such as groups, quandles, biquandles etc. [5], we can define computable integer-valued invariants of twisted virtual handlebody-links by counting colorings of a diagram \( D \) of a twisted virtual handlebody-link by elements of a twisted virtual bikeigebra \( X \) satisfying the coloring rule above. By construction, we have the following theorem:

**Theorem 1.** The sets of colorings of twisted virtual handlebody-link diagram by a twisted virtual bikeigebra before and after each Reidemeister move are in one-to-one correspondence.

This motivates the following definition:

**Definition 3.** Let \( D \) be a twisted virtual handlebody-link diagram and \( X \) a twisted virtual bikeigebra. Then the cardinality of the set \( C(D, X) \) of colorings of \( D \) by \( X \) is denoted \( \Phi^Z_X(D) = |C(D, X)| \).

**Theorem 1** implies the following corollary:

**Corollary 2.** If \( D \) and \( D' \) are twisted virtual handlebody-links related by twisted virtual handlebody Reidemeister moves and \( X \) is a twisted virtual bikeigebra, then \( \Phi^Z_X(D) = \Phi^Z_X(D') \). In particular, \( \Phi^Z_X \) is an invariant of twisted virtual handlebody-links.

**Definition 4.** Let \( K \) be a twisted virtual handlebody-link and \( X \) a twisted virtual bikeigebra. If \( \Phi^Z_X(K) \neq 0 \), we say \( K \) is \( X \)-colorable.

**Remark 2.** We note that \( X \)-colorability is a two-valued invariant ("Yeah, bikeigebra-colorable by \( X \)" or "No, not bikeigebra-colorable by \( X \)"") analogous to the usual \( p \)-colorability of knots, with the distinction that a twisted virtual handlebody-link \( L \) is \( X \)-colorable is there is any \( X \)-coloring of \( L \); we do not need a notion of trivial vs. non-trivial colorings.

**Example 5.** Let us compute the sets of colorings of the twisted virtual handlebody-knot \( L_1 \) below by the twisted virtual bikeigebra \( X \) on the set \( \{1, 2\} \) given by the matrix

\[
\begin{align*}
\left[
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 \\
\end{array}
\right].
\end{align*}
\]
The system of coloring equations is
\[
\begin{align*}
  x_3 &= T(x_1) \circ x_1 \\
  x_4 &= x_1 \circ T(x_2) \\
  x_5 &= x_1 \triangledown x_3 \\
  x_5 &= x_2 \triangledown x_4 \\
  x_6 &= x_3 \triangledown x_1 \\
  x_5 &= x_4 \triangledown x_2 \\
  x_7 &= x_1 \triangledown x_6 \\
  x_8 &= x_2 \circ x_7 \\
  x_9 &= x_6 \triangledown x_1 \\
  x_9 &= x_7 \circ x_2
\end{align*}
\]

We can then verify for each of the four possible colorings that the system is not satisfied:

\[
\begin{array}{c|c|c|c}
  x_1 & x_2 & (i) & (ii) \\
  \hline
  1 & 1 & 1 = 1 & 1 \neq 1 \\
  1 & 2 & 1 \neq 2 & 1 \neq 2 \\
  2 & 1 & 2 \neq 1 & 1 = 1 \\
  2 & 2 & 2 \neq 2 & 1 \neq 2
\end{array}
\]

so the twisted virtual handlebody-knot $L_1$ is not $X$-colorable.

On the other hand, the twisted virtual handlebody-knot below is $X$-colorable in two ways:

and hence $L_1$ is distinguished from $L_2$ by $X$-colorability.

**Example 6.** Let $X$ be the twisted virtual bikeiegebra on the set $\{1, 2, 3, 4\}$ given by the operation matrix
\[
\begin{bmatrix}
  1 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & - & - & - & 2 \\
  2 & 2 & 2 & 1 & 2 & 1 & 2 & - & 2 & - & - & 1 \\
  3 & 3 & 4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 3 & - & - & - & - & - & 4 \\
  4 & 4 & 3 & 3 & 4 & 4 & 3 & 3 & 3 & 3 & 3 & - & - & - & - & - & 3
\end{bmatrix}
\]
Then the twisted virtual handlebody-link $L_3$ below is not $X$-colorable,

![Diagram of $L_3$]

but switching the position of the twist bar yields a twisted virtual handlebody-link $L_4$ with two $X$-colorings.

![Diagram of $L_4$]

When two twisted virtual handlebody-links are both colorable by the same twisted virtual bikeigebra, we can use the number of colorings to distinguish the links.

**Example 7.** Consider the two twisted virtual handlebody-knots $L$ and $L'$ below, both of which are handlebody-knots of genus 2 represented by diagrams with four classical crossings, two virtual crossings and one twist bar. They are distinguished by their counting invariants with respect to the twisted virtual bikeigebra $X$ given by the operation matrix

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \\
2 & 6 & 6 & 2 & 2 & 6 & 6 & 2 & 2 & 6 & 6 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & 2 \\
3 & 3 & 7 & 7 & 3 & 3 & 7 & 7 & 3 & 3 & 7 & 7 & 3 & 3 & 7 & 7 & 3 & 4 & 1 & 2 & 7 & 8 & 5 & 6 & 4 \\
4 & 8 & 4 & 4 & 8 & 4 & 4 & 8 & 4 & 4 & 8 & 4 & 4 & 8 & 4 & 4 & 8 & 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 & 3 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 & 4 \\
6 & 2 & 6 & 6 & 6 & 6 & 2 & 6 & 2 & 6 & 2 & 6 & 2 & 6 & 2 & 6 & 2 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 \\
7 & 7 & 3 & 3 & 7 & 7 & 3 & 7 & 3 & 3 & 7 & 7 & 3 & 7 & 3 & 3 & 7 & 7 & 3 & 7 & 8 & 5 & 6 & 3 & 4 & 1 & 2 \\
8 & 8 & 4 & 8 & 4 & 8 & 4 & 8 & 4 & 8 & 4 & 8 & 4 & 8 & 4 & 8 & 4 & 8 & 4 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1
\end{bmatrix}
$$

with 40 and 64 colorings respectively as computed by our **Python** code.

![Diagram of $L_4$]

$\Phi_X^L(L) = 40$

$\Phi_X^L(L') = 64$
4 Questions

We end with some questions for future research.

We have only looked at the unoriented case of twisted virtual handlebody-links and their associated algebraic structure; the oriented case should define a generalized algebraic structure we could call twisted virtual biqualgebras. What are the appropriate axioms and examples of such structures?

We are very interested in examples of families of twisted virtual bikeigebra structures defined in terms of groups, modules, matrix algebras, and any other algebraic structures whose additional properties can be used to simplify coloring calculations.

As with all counting invariants, we ask what enhancements of the twisted virtual bikeigebra counting invariant are possible. Cocycle invariants, structure enhancements, twisted virtual bikeigebra module enhancements and many more possibilities remain to be investigated.

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Department of Mathematical Sciences
Claremont McKenna College
850 Columbia Ave.
Claremont, CA 91711