Shock Wave Polarizations and Optical Metrics in the Born and the Born-Infeld Electrodynamics

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We analyze the behavior of shock waves in nonlinear theories of electrodynamics. For this, by use of generalized Hadamard step functions of increasing order, the electromagnetic potential is developed in a series expansion near the shock wave front. This brings about a corresponding expansion of the respective electromagnetic field equations what allows for deriving relations that determine the jump coefficients in the expansion series of the potential. The solution of the first-order jump relations shows that, in contrast to linear Maxwell’s electrodynamics, in general the propagation of shock waves in nonlinear theories is governed by optical metrics and polarization conditions describing the propagation of two differently polarized waves (leading to a possible appearance of birefringence). In detail, shock waves are analyzed in the Born and Born-Infeld theories. The obtained results are compared to those ones found in literature. New results for the polarization of the two different waves are derived.

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I. INTRODUCTION

Foundation and study of nonlinear electrodynamic theories go back to ideas of Gustav Mie. The first gauge-invariant version of nonlinear electromagnetic field equations was presented by Born1 soon after, his ansatz was generalized by him and Infeld2,3. As an alternative to the linear field equations of Maxwell’s electrodynamics, the new equations should provide solutions that can be interpreted as classical models of electrons. Another type of nonlinear field equation was introduced by Heisenberg and Euler4. They showed that radiative quantum-electrodynamic effects semi-classically could be described by nonlinear correction terms to the linear classical equations. Nowadays, such theories also attract the interest because Born-Infeld-like actions arise as effective actions in superstring theory5,6.

Whatever the reason for the interest in nonlinear electrodynamics was, again and again it appeared papers on different aspects of such theories. In particular, this concerns the analysis of the propagation of waves. This goes back to the early 1950s and wins new attention not only of theorists, but - due to improved experimental technologies - also of experimental physicists. (For the theoretical analysis, see especially the paper by Obukhov and Rubilar7 and the literature cited therein.) The propagation of electromagnetic fields is also the topic of the present paper, whereby the Born and the Born-Infeld theories are in our focus.

There are mainly two methods used in such investigations, the method of geometric optics and, alternatively, the shock wave method which bases on Hadamard’s theory of discontinuities8. In the following, we shall use a shock wave method developed by Treder9.

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It differs from the method usually used by that (i) it starts from discontinuities in the electromagnetic potential, instead from those ones of the field strengths, and (ii) develops the field quantities in a series expansion in the neighborhood of the jump hypersurface, where, following Stellmacher\textsuperscript{10}, the jump relations of the different approximation steps are solved by contracting them with a suitably chosen tetrad field.

The paper is organized as follows\textsuperscript{11}. After providing fundamentals of nonlinear electrodynamics, describing the used shock-wave method in brief, and introducing the tetrad field, the electromagnetic potential is developed in a series expansion near the jump hypersurface. Then, by inserting this series in the field equations and performing the appropriate limes procedure, the jump relations can be calculated up to an arbitrary order (see Ref.\textsuperscript{12}). The first-order relations are solved by tetrad contraction. The solutions reproduce results of Obukhov and Rubilar\textsuperscript{7} (also discussed by Novello et al.\textsuperscript{13}) concerning the so-called optical metric (sometimes referred to as effective metric). Furthermore, as a new result we calculate the polarization angle of the first-order jump relations and the possible appearance of birefringence.

II. THE BORN AND THE BORN-INFELD ELECTRODYNAMICS

A. Lagrange Densities

In 1933, Born started the discussion on a new electromagnetic field theory\textsuperscript{11} for which the Lagrange density can be written in the form

\[ L_B = -\frac{b^2}{\mu_0} \left( \sqrt{1 + \frac{F}{b^2}} - 1 \right) \]  

using the International System of Units. This theory is known as the Born electrodynamics and was investigated in more detail by Born and Infeld\textsuperscript{2}. One year later, Born and Infeld published the foundations of the new field theory,\textsuperscript{3} the Born-Infeld electrodynamics, where they derived another Lagrange density for a free field,

\[ L_{BI} = -\frac{b^2}{\mu_0} \left( \sqrt{1 + \frac{F}{b^2} - \frac{G}{b^4}} - 1 \right). \]  

Born and Infeld introduced \( b \) as a new natural constant\textsuperscript{2} with the physical dimension of the electromagnetic field \( F_{\alpha\beta} \).

The Lagrange densities (1) and (2) are functions of the field invariants

\[ F := \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta}, \]

\[ G := \frac{1}{4} F_{\alpha\beta}^{*} F^{\alpha\beta}, \]

where

\[ F^{\alpha\beta} := \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}. \]

is the dual field tensor, the contraction with the fully antisymmetric Levi-Civita tensor \( \epsilon^{\alpha\beta\gamma\delta} \) (\( \epsilon^{0123} = 1 \)). Indices are contracted with the Minkowski metric of flat space-time, \( \eta_{\alpha\beta} \), using the \( -2 \) signature, \( [\eta_{\alpha\beta}] = \text{diag}(1, -1, -1, -1) \).

B. Field Momenta

Using the Euler-Lagrange formalism with the Lagrange densities (2) and (1), one finds the vacuum field equations \( D^{\alpha\beta}_{\gamma} = 0 \) with the displacement field tensor (electromagnetic
field momentum)
\[ D_{\text{BI}}^{\alpha\beta} = \frac{1}{\mu_0} \frac{F^{\alpha\beta} - G^{\phi}_{\beta}^{\alpha\beta}}{\sqrt{1 + \frac{F^{\alpha\beta}}{\beta x}}}, \] (5)

and
\[ D_{\text{B}}^{\alpha\beta} = \frac{1}{\mu_0} \frac{F^{\alpha\beta}}{\sqrt{1 + \frac{F^{\alpha\beta}}{\beta x}}} \] (6)

respectively.

Both field theories have several mathematical advantages. For example, the inversion of the momenta (5) and (6) show the same mathematical structure when using the momentum scalars
\[ D := -\frac{\mu_0^2}{2} D_{\alpha\beta} D^{\alpha\beta}, \] (7a)
\[ E := \frac{\mu_0^2}{4} D_{\alpha\beta}^{\ast} D^{\alpha\beta}. \] (7b)

The inverted relations for the Born-Infeld electrodynamics are
\[ F^{\alpha\beta} = \frac{1}{\mu_0} \frac{D_{\text{BI}}^{\alpha\beta} + E_{\text{BI}} \delta^{\alpha\beta}}{\sqrt{1 + D_{\text{BI}}^{\phi} - E_{\text{BI}}^2}}. \] (8)

For the Born electrodynamics it reads
\[ F^{\alpha\beta} = \frac{1}{\mu_0} \frac{D_{\text{B}}^{\alpha\beta}}{\sqrt{1 + D_{\text{B}}^{\phi}}}. \] (9)

In case of the Born and the Born-Infeld electrodynamics, the field equations can be rewritten so that the square root disappears.

C. Rewritten Field Equations

In order to avoid a Taylor expansion of the square root in the shock wave formalism and to compare it to the linear field equations of Maxwell, they are multiplied with the third power of the square root. This mathematical reformulation does not change the results.

For the Born-Infeld field equations, one defines
\[ \phi_{\text{BI}}^{\beta} := \mu_0 \left( 1 + \frac{F}{\beta x} \right)^{-\frac{3}{2}} D_{\text{BI}}^{\alpha\beta}, \] (10)

and, analogously, for Born’s field equations
\[ \phi_{\text{B}}^{\beta} := \mu_0 \left( 1 + \frac{F}{\beta x} \right)^{-\frac{3}{2}} D_{\text{B}}^{\alpha\beta}. \] (11)

In more detail, the vanishing vector field \( \phi^{\beta} \) (with index BI for the Born-Infeld theory and index B for the Born electrodynamics) reads
\[ \phi_{\text{BI}}^{\beta} = F^{\beta}_{\alpha,\alpha} + \frac{1}{b^2} \left( F F^{\alpha\beta}_{\alpha,\alpha} - \frac{1}{2} F^{\alpha\beta} F_{\alpha} - \hat{F}^{\alpha\beta} G_{\alpha} \right) \\
+ \frac{1}{b^4} \left( -G^2 F^{\alpha\beta},_{\alpha} + \frac{1}{2} \hat{G} F^{\alpha\beta} F_{\alpha} + G F^{\alpha\beta} G_{\alpha} - \hat{F} F^{\alpha\beta} G_{\alpha} \right). \] (12)
and
\[ \phi_B^\beta = F^{\alpha\beta,\alpha} + \frac{1}{2\beta^2} F F^{\alpha\beta,\alpha} - \frac{1}{2\beta^2} F^{\alpha\beta} F_{\alpha}, \]
(13)
respectively. For the calculations, the homogeneous field equations \( \hat{F}^{\alpha\beta,\alpha} = 0 \) have been used which are also fulfilled for the Born and the Born-Infeld electrodynamics. The homogeneous field equations are mathematical identities when developed in a jump series.[12]

The first summand in equations (12) and (13) is the linear term known from Maxwell’s electrodynamics. The nonlinear terms in equation (13) are underlined to identify them in the jump series expansion (28) below.

To derive the full jump series expansion of the nonlinear electrodynamics, the shock wave formalism by Treder and Stellmacher will be used.

III. SHOCK WAVE FORMALISM

A. Series Expansion of the Electromagnetic Potential

The mathematical formulation of shock wave perturbations bases on the parameter \( \Sigma \) which specifies a position relative to the shock front in length units. The equation \( \Sigma = 0 \) defines the shock front as a hypersurface in space-time. The electromagnetic potential \( A_{\alpha} \) can be expanded with respect to the parameter \( \Sigma \). According to Treder[9], this reads
\[ A_{\alpha} = A^{-\alpha} + \sum_{m=1}^{\infty} \varphi_m h_m. \]
(14)

\( A^{-\alpha}(x^\mu) \) denotes the undisturbed field in front of the shock \( (\Sigma < 0) \) as well as its analytic continuation for \( \Sigma \geq 0 \). Fields marked with the minus symbol are referred to as background fields.

The letter \( m \) always stands for the first index in the perturbation summation so that the first discontinuities occur for the \( m \)-th derivative of the electromagnetic potential. However, \( m \) has to be greater than 1, because for \( m = 1 \) surface charges along \( \Sigma = 0 \) have to be discussed[14].

The expansion (14) is similar to a conventional Taylor series. The coefficients \( \varphi_m(x^\mu) \) are assumed to be analytic, since they do not depend on the expansion parameter \( \Sigma \), whereas \( h_m(\Sigma) \) represent general step functions discontinuous at \( \Sigma = 0 \) in their \( m \)-th derivative,
\[ h_m(\Sigma) := \begin{cases} 0 & \text{if } \Sigma < 0, \\ \frac{\Sigma^m}{m!} & \text{if } \Sigma \geq 0 \end{cases} \]
(15)

(see also FIG. 1). When truncating the expression (14) at a certain number of summands, the field perturbation can be approximated in a small region \( \Sigma > 0 \).

The jump coefficients of the electromagnetic potential are
\[ \varphi_m^{\alpha} = \left[ \frac{\partial^m A_{\alpha}}{\partial \Sigma^m} \right]_{\Sigma=0}, \]
(16)
where the jump brackets are defined as
\[ [J]_{\Sigma=0} := \lim_{\Sigma \to +0} (J) - \lim_{\Sigma \to -0} (J). \]
(17)

\( J \) stands for an arbitrary function[13].
B. Series Expansion of the Electromagnetic Field

The electromagnetic field tensor is defined as the exterior derivative of the electromagnetic potential $A_\alpha$,

$$F_{\alpha\beta} = F_{[\alpha\beta]} := A_{\beta,\alpha} - A_{\alpha,\beta}.$$  \hfill (18)

Similarly to the potential series expansion $\text{(14)}$, the field series expansion is given by

$$F_{\alpha\beta} = F^{-\alpha\beta} + \sum_{m=l'}^{\infty} f_{\alpha\beta} h^m,$$  \hfill (19)

where the coefficients $f_{\alpha\beta} h^m$ are functions of $\varphi_\alpha$ linking the equation $\text{(19)}$ to the derivatives of $A_\alpha$,

$$f_{m}^{\alpha\beta} = p_{\alpha} \varphi_{\beta} - p_{\beta} \varphi_{\alpha},$$  \hfill (20a)

$$f_{m}^{\alpha\beta} = \varphi_{\beta,\alpha} - \varphi_{\alpha,\beta} + p_{\alpha} \varphi_{\beta} + p_{\beta} \varphi_{\alpha},$$  \hfill (20b)

where $m \geq l$. The new start index is $l' = l - 1$ and the normal vector field

$$p_\alpha := \partial_\alpha \Sigma$$  \hfill (21)

appears in the coefficients $\text{(20)}$ due to the chain rule,

$$\partial_{\alpha} h^m = p_{\alpha} h^{m-1}.$$  \hfill (22)

$p_\alpha$ is perpendicular to the shock front $\Sigma = 0$ in every point.

C. Tetrad Fields

The tetrad which is used to formulate shock wave solutions and to contract the jump relations consists of four vector fields, the normal vector field $p_\alpha$, the complex spacelike vector field $\omega_\alpha = \text{Re}(\omega_\alpha) + i \text{Im}(\omega_\alpha)$, its conjugate $\bar{\omega}_\alpha$, and a timelike vector field $d_\alpha$. $\omega_\alpha$ was chosen complex to interpret the fields perpendicular to the propagation direction similar to optics with an amplitude and polarization angle.
In contrast to the tetrad fields introduced by Stellmacher\cite{19} and Treder\cite{18}, the scalar
\[ p := p_\alpha p^\alpha = \eta^{\alpha\beta} p_\alpha p_\beta \] (23)
need not be always zero. However, the four tetrad fields still fulfill the conditions
\[ p_\alpha \omega^\alpha = p_\alpha \bar{\omega}^\alpha := 0, \quad p_\alpha d^\alpha := 1, \] (24a)
\[ \omega_\alpha \omega^\alpha = \bar{\omega}_\alpha \bar{\omega}^\alpha := 0, \quad \omega_\alpha \bar{\omega}^\alpha := -\frac{1}{2}, \] (24b)
\[ \omega_\alpha d^\alpha = \bar{\omega}_\alpha d^\alpha := 0, \quad d_\alpha d^\alpha := 1. \] (24c)

For a better physical interpretation of the parts of the electromagnetic fields pointing in \( d_\alpha \)-direction, one may choose \( d_\alpha \) so that \( p_\alpha d^\alpha = 0 \) when \( p_\alpha \) is not lightlike \( (p \neq 0) \). However, to avoid the discussion of two different cases \( p \neq 0 \) (wave front traveling slower than the speed of light \( c \)) and \( p = 0 \) (wave front traveling with the maximum speed \( c \)), the above relations \( \text{(24)} \) will be used according to Treder’s tetrad\cite{18}.

For some calculations, it is useful to express the Minkowskian metric in terms of the tetrad fields,
\[ \eta^{\alpha\beta} = -p_1^{-1} d_\alpha d_\beta + \frac{1}{1-p} (d_\alpha p_\beta + p_\alpha d_\beta - p_\alpha p_\beta) - 2 (\omega_\alpha \bar{\omega}_\beta + \bar{\omega}_\alpha \omega_\beta). \] (25)

With the parameters shock excitation \( J \), polarization angle \( \kappa \), temporal amplitude \( \lambda \), and normal amplitude \( a \) the general solution of the \( m \)-th-order coefficient \( (m \geq l) \) is a superposition of the four vector fields,
\[ \varphi_\alpha = \sqrt{J} \left( e^{i\kappa \omega_\alpha} + e^{-i\kappa \bar{\omega}_\alpha} \right) + \lambda d_\alpha + a p_\alpha + \partial_\alpha a. \] (26)

The terms \( a p_\alpha + \partial_\alpha a \) can be removed by a discontinuous gauge transformation\cite{18} along \( \Sigma = 0 \). The gradient field \( \partial_\alpha a \) \( \forall m > l \) simplifies the transformation and can be composed of the four vector fields itself\cite{21}. Therefore, the gauged solutions (symbolized with a prime) are a superposition of the three remaining space-time directions \( \omega_\alpha, \bar{\omega}_\alpha, \) and \( d_\alpha \),
\[ \varphi'_\alpha = \sqrt{J} \left( e^{i\kappa \omega_\alpha} + e^{-i\kappa \bar{\omega}_\alpha} \right) + \lambda d_\alpha. \] (27)

When looking for the shock wave solutions, one has to find the expressions for the free parameters.

IV. JUMP CONDITIONS

A. Series Expansion and General Jump Conditions

The series expansion of the field equations \( \phi^\beta = 0 \) is expressed with respect to the electromagnetic field \( \text{(19)} \). Evaluating every term of Born’s field equations \( \text{(13)} \) and combining it to one jump series yields
\[ 0 = \phi_B^\beta = F^{-\alpha\beta,\alpha} + \frac{1}{b^2} F^{-\alpha\beta,\alpha} - \frac{1}{2b^2} F^{-\alpha\beta} F^{-,\alpha} \]
\[ + \sum_{m=l'}^{\infty} \left( f_m^{\alpha\beta,\alpha} + \frac{1}{b^2} F_f^{\alpha\beta,\alpha} + \frac{1}{b^2} F^{-\alpha\beta,\alpha} F^{-,\gamma\delta} f_m^{\gamma\delta} \right). \]
The full jump series expansion of the Born-Infeld field equations $\phi_{\text{BI}} = 0$ is given in Ref.\textsuperscript{12}. The series expansion of the field equations $\phi = 0$ of any nonlinear electrodynamic theory follows the same mathematical formalism. However, analytic functions of the field invariants have to be developed in a Taylor series so that every Taylor term can be expanded in a jump series.

The first-order jump conditions are calculated with the help of the jump brackets \textsuperscript{[17]},

$$0 = \left[ \frac{\partial^{l-2}\phi^\beta}{\partial \Sigma^{l-2}} \right]_{\Sigma=0},$$

(29)

Higher-order jump conditions, with the index $m > l$ follow from similar equations,

$$0 = \left[ \frac{\partial^{m-2}\phi^\beta}{\partial \Sigma^{m-2}} \right]_{\Sigma=0}.$$  

(30)

We discuss the first-order jump conditions in the following.

\textbf{B. First-Order Jump Conditions}

The field coefficients $f_{\alpha\beta}$ in the first-order jump conditions are replaced according to equation (20a) what yields

$$0 = B^{-}\alpha p_{\alpha} \left( p^\alpha \phi'_{l} - p^\beta \phi'_{l} \right) + B^{-}\beta \phi'_{l}$$

(31)

with the symmetric contraction

$$B^{-}\beta \phi := p_{\alpha} p_{\gamma} B^{-}\alpha\beta\gamma\delta.$$  

(32)

For the first-order jump conditions (29) it is useful to define two background tensors. Their expressions are picked from the jump series expansion. The series expansion (28) of the Born electrodynamics gives

$$B_{\text{B}} := 1 + \frac{1}{b^2} F^{-}.$$  

(33)
When analyzing the series expansion of the Born-Infeld electrodynamics, one defines
\[ a_B^- := 1 + \frac{1}{b^2} F^- - \frac{1}{b^4} (G^-)^2 \]  
(34)

and
\[ b_B^- := a_B^- - 2 \frac{L_F}{b^2} F^- F^- - \frac{1}{b^4} (G^-)^2 \]  
(35)

The four first-order jump conditions for any other nonlinear electrodynamic theory derived from a Lagrange density always has the form (31). The general first-order background field tensors read
\[ a_B^- = -2 \mathcal{L}_F \big|_\perp, \]
\[ b_B^- := -4 \mathcal{L}_{FF} \big|_\perp F^- \alpha \gamma \delta - \mathcal{L}_{GG} \big|_\perp \hat{F}^- \alpha \gamma \delta - 2 \mathcal{L}_{FG} \big|_\perp \left( F^- \alpha \gamma \delta + \hat{F}^- \alpha \gamma \delta \right). \]  
(36)

The derivatives of the Lagrange density (\( \mathcal{L}_F, \mathcal{L}_{FF}, \mathcal{L}_{FG}, \) and \( \mathcal{L}_{GG} \)) are evaluated with the background fields.

The background tensor \( b_B^- \alpha \beta \gamma \delta \) is symmetric when exchanging the first and last index pair,
\[ b_B^- \alpha \beta \gamma \delta = b_B^- \gamma \delta \alpha \beta. \]  
(37)

Due to the antisymmetry of indices \( \alpha \beta \) and \( \gamma \delta \), respectively, the contracted background field tensor \( B^- \alpha \beta \) is orthogonal to the normal direction,
\[ p_{\alpha} B^- \alpha \beta = p_{\alpha} B^- \beta \alpha = 0. \]  
(38)

Furthermore, the first-order jump conditions (29) are a homogeneous, linear system of equations
\[ N^\alpha \beta \dot{\varphi}_{\dot{\lambda}}^\beta = 0 \]  
(39)

with the coefficient matrix
\[ N^\alpha \beta := a^- \left( p\eta^{\alpha \beta} - p^\alpha p^\beta \right) + b^- \alpha \beta. \]  
(40)

Calculations of the matrix rank of \( N^\alpha \beta \) give the degrees of freedom of the solutions \( \dot{\varphi}_{\dot{\lambda}}^\beta \) under certain conditions yet to be identified.

Contractions of equation (29) with the four tetrad fields \( p_{\alpha}, d_{\alpha}, \omega_{\alpha}, \) and \( \bar{\omega}_{\alpha} \) give conditions for the temporal parameter \( \lambda_{\dot{\lambda}} \), for the parameter \( p \), and the shock wave polarization angle \( \kappa_{\dot{\lambda}} \).
C. First-Order Tetrad Contractions

Because the first-order jump conditions do not differ for any electrodynamics when formulated in the form (31), this section presents the general contractions before the Born and the Born-Infeld electrodynamics are investigated in detail.

Similar to linear electrodynamics, the \( p_\beta \) contraction of the first-order jump conditions vanishes identically with the help of the equation (38),

\[
p_\beta \left[ \frac{\partial^2 \phi}{\partial \Sigma^{\gamma^2}} \right]_{\Sigma=0} = 0. \quad (41)
\]

After contracting with the timelike vector field \( d_\beta \),

\[
0 = d_\beta \left[ \frac{\partial^2 \phi}{\partial \Sigma^{\gamma^2}} \right]_{\Sigma=0} \quad (42)
\]

and replacing the coefficients by the gauged general solution (27), one obtains

\[
0 = -\left( a B^- (1 - p) - d_\beta d_\delta B^- \right) \lambda + \sqrt{J} \left( e^{i \kappa} d_\beta \omega + e^{-i \kappa} d_\beta \bar{\omega} \right) b B^- \beta \delta. \quad (43)
\]

The temporal parameter has to be

\[
\lambda = \frac{\sqrt{J}}{a B^- (1 - p) - d_\delta d_\gamma b B^- \gamma \delta}. \quad (44)
\]

It depends on the shock excitation \( J \) and the polarization angle \( \kappa \). In absence of background fields, \( F^- \alpha \beta = 0 \), or in the linear limit \( b \to \infty \) the first jump conditions are identical to Maxwell’s theory with \( \lambda = 0 \).

The general condition for the parameter \( p \) follows from the contraction with the spacelike vector fields \( \omega_\beta \). The complex result splits into one real and one imaginary condition which may also constrain the polarization angle \( \kappa \) for nonlinear theories. The first-order parameters \( J \) and \( \kappa \) fulfill differential equation which follow from the second-order jump relations. However, this work focuses on the physical results of the first order.

The \( \omega_\alpha \) contraction can be split into the real part

\[
0 = a_b \left( B^- d_\delta \right) \sqrt{J} \left( e^{i \omega_\beta} + e^{-i \omega_\beta} \right) \left[ \frac{\partial^2 \phi}{\partial \Sigma^{\gamma^2}} \right]_{\Sigma=0} = -a B^- \left( B^- d_\delta \right) p + \left( a_b \left( B^- d_\delta \right) \right) J + \left( a_b \left( B^- d_\delta \right) \right) J \cos 2 \kappa + \left( a_b \left( B^- d_\delta \right) \right) J \sin 2 \kappa \cos 2 \kappa. \quad (45a)
\]

and the imaginary part

\[
0 = a_b \left( B^- d_\delta \right) \sqrt{J} \left( e^{i \omega_\beta} - e^{-i \omega_\beta} \right) \left[ \frac{\partial^2 \phi}{\partial \Sigma^{\gamma^2}} \right]_{\Sigma=0} = i \left( a_b \left( B^- d_\delta \right) \right) J \sin 2 \kappa - i \left( a_b \left( B^- d_\delta \right) \right) J \cos 2 \kappa. \quad (45b)
\]

The symbols used in these equations are

\[
b \left( B^- \right) := \left( \omega_\alpha \omega_\beta + \bar{\omega}_\alpha \bar{\omega}_\beta \right) b B^- \alpha \beta, \quad (46a)
\]
\[ \begin{align*}
B^b_{I \omega} &:= i(\omega_\alpha \omega_\beta - \bar{\omega}_\alpha \bar{\omega}_\beta) B^{-\alpha \beta}, \\
B^b_{-} &:= 2\omega_\alpha \bar{\omega}_\beta B^{-\alpha \beta}, \\
B^b_{d} &:= d_\alpha d_\beta B^{-\alpha \beta}, \\
B^b_{R \omega d} &:= (\omega_\alpha \omega_\gamma + \bar{\omega}_\alpha \bar{\omega}_\gamma) d_\beta d_\delta B^{-\alpha \beta B^{-\gamma \delta}}, \\
B^b_{-} &:= 2\omega_\alpha d_\beta \bar{\omega}_\gamma d_\delta B^{-\alpha \beta B^{-\gamma \delta}}, \\
B^b_{d} &:= B^{-} (1 - p) - d_\alpha d_\beta B^{-\alpha \beta}. 
\end{align*} \] (46b, 46c, 46d)

If the background fields vanish, all terms in equation \( \textbf{45a} \) except \( \bar{B}^- p J \) disappear so that the result \( p = p_\alpha p^\alpha = 0 \) is identical to the linear, lightlike condition from Maxwell’s electrodynamics. The equation \( \textbf{45b} \) is trivially satisfied in absence of background fields.

In general, \( J \neq 0 \), so that the nonlinear system of equations \( \textbf{45} \) gives conditions for \( p \) and conditions for the polarization angle \( \kappa \).

If the coefficients of the trigonometric functions in the second equation \( \textbf{45b} \) do not vanish, it can be solved for the polarization angle. The nonlinear shock wave polarization condition is

\[ \tan 2\kappa_l = \frac{\frac{\alpha-b}{b} B^\alpha_{d} B^\beta_{-} + \frac{\beta}{b} B^\beta_{R \omega d}}{\frac{\alpha-b}{b} B^\alpha_{d} B^\beta_{R \omega} + \frac{\beta}{b} B^\beta_{R \omega d}}, \] \] (47)

When the trigonometric functions in the equation \( \textbf{45a} \) are replaced by equation \( \textbf{47} \), a quadratic equation for the scalar \( p \) is obtained,

\[ 0 = B^- B^- d p - \left( \frac{\alpha-b}{b} B^\alpha_{d} B^\beta_{-} + \frac{\beta}{b} B^\beta_{R \omega d} \right) \pm \sqrt{\left( \frac{\alpha-b}{b} B^\alpha_{d} B^\beta_{-} + \frac{\beta}{b} B^\beta_{R \omega d} \right)^2 + \left( \frac{\alpha-b}{b} B^\alpha_{d} B^\beta_{R \omega} + \frac{\beta}{b} B^\beta_{R \omega d} \right)^2}. \] \] (48)

The two solutions can be formulated with optical metrics,

\[ g_{\text{opt.(1,2)}} \alpha^\beta p_\alpha p^\beta = 0. \] \] (49)

The two first-order solutions differ in their polarization angle \( \textbf{47} \) because they depend on the vector field \( p_\alpha \). This is called birefringence in vacuum. If \( g_{\text{opt.(1)}} \alpha^\beta = g_{\text{opt.(2)}} \alpha^\beta \), this effect does not appear.

**D. Optical Metric and Shock Wave Polarization of the Born Electrodynamics**

Since the Born electrodynamics \( \textbf{1} \) has a background tensor \( B^{-\alpha \beta \gamma \delta} \propto F^{-\alpha \beta} F^{-\gamma \delta} \), the contracted background tensor \( \textbf{32} \) shows the symmetry

\[ B^{-\alpha \beta \gamma \delta} = B^{-\alpha \delta \beta} B^{-\gamma \delta}. \] \] (50)
Contractions with different combinations of the timelike and spacelike tetrads yield three identities,

\[
\begin{align*}
B_c^b B_d^b &= B_{cd}^b, \\
B_{Re}^b B_d^b &= B_{Rw}^b, \\
B_{Le}^b B_d^b &= B_{Lw}^b.
\end{align*}
\] (51)

These three equations and the phase condition (47) yield

\[
\tan 2\kappa_l = \frac{b B_l^\omega B_d^\omega}{b B_{Rw}^\omega} = \left( \frac{\text{Re}(\omega_\alpha)p_\beta F^{\alpha \beta}}{\text{Im}(\omega_\gamma)p_\delta F^{\gamma \delta}} \right)^2.
\] (54)

The right-hand side with the squared bracket is the result for any nonlinear theory \( L(F) \), including the Born electrodynamics.

With the above symmetry, the quadratic equation (48) becomes

\[
a B^p - \frac{p}{1 - p} b B_d^p + (1 \pm 1) b B_d^p = 0.
\] (55)

The equation with the upper sign is fulfilled for the parameter \( p(1) = 0 \) (known from Maxwell’s electrodynamics), since, in general

\[
a B^p - \frac{1}{1 - p} b B_d^p \neq 0.
\]

Therefore, the first optical metric is identical to the Minkowskian metric,

\[
g_{\text{opt},(1)}^{\alpha \beta} = \eta^{\alpha \beta}.
\] (56)

When contracting the second background tensor with the Minkowskian metric given in equation (25), the second equation with the minus sign can be rewritten to

\[
\left( a B^\gamma + b B_{\gamma \delta} \right) p_\alpha p_\beta = 0.
\] (57)

The symmetric, second-order tensor in the brackets is the second optical metric. With the background tensors of the Born electrodynamics, one finds

\[
g_{\text{opt,B,(2)}}^{\alpha \beta} = \left( 1 + \frac{1}{b^2} F^{-} \right) \eta^{\alpha \beta} + \frac{1}{b^2} F^{-\alpha \gamma} F^{- \gamma \beta}.
\] (58)

The symmetry (50) also occurs for field theories with the Lagrange density \( L(F) \) or \( L(G) \).

For \( L(G) \) theories, however, the background tensor \( B^\gamma \) equals zero what is nonphysical because the second optical metric would vanish in absence of background fields (for further arguments, see Ref. 7).

The result (58) was published by Novello et al.\cite{13} as the only solution. The birefringence of the Born theory, including the first solution (56), was also found by Obukhov et al.\cite{12}

Furthermore, the shock wave polarization condition (54) has to be fulfilled for both solutions, individually. Two different solutions for the vector field \( p_\alpha(1,2) = \Sigma_\alpha(1,2) \) are defined by the optical metrics (56) and (57). The vector fields are used in the numerator and denominator contractions of equation (54) yielding two different polarizations \( \kappa_l(1,2) \).
Considering a shock wave with \( [p_\alpha] = [1, \sqrt{1 - p_{(1,2)}}, 0, 0] \) and \( [\omega_\alpha] = [0, 0, 1/2, 1/2] \), the polarization condition reads (in the International System of Units)

\[
\tan 2\kappa_{l(1,2)} = \left( \frac{1}{2} E_y^+ + \sqrt{1 - p_{(1,2)} B_y^-} \right)^2.
\]

(59)

In an experimental setup with stationary, electric background fields \( E_{y}^- \) and \( E_{z}^- \) and stationary, magnetic background fields \( B_{y}^- \) and \( B_{z}^- \), two different polarizations appear. If either both electric or both magnetic background fields are zero, both polarizations are identical.

The polarization condition (54) is supported by the fact that the rank of the coefficient matrix (40) is reduced to

\[
\text{rank} \left[ N^{\alpha\beta}_l \right] = 2.
\]

(60)

with the help of the optical metric (56) or (58), respectively. Therefore, the gauged first-order coefficient (27) has one free parameter, the shock excitation \( J_{l} \).

E. Optical Metric and Shock Wave Polarization of the Born-Infeld Electrodynamics

Both optical metrics coincide for the Born-Infeld electrodynamics and have the same mathematical form as the second solution in the Born theory,

\[
g_{\text{opt.BI},(1,2)}^{\alpha\beta} = \left( 1 + \frac{1}{b^2} F^- \right) \eta^{\alpha\beta} + \frac{1}{b^2} F^- \alpha\gamma F^- \gamma. \]

(61)

These optical metrics reduce the rank of the coefficient matrix (40) to one,

\[
\text{rank} \left[ N_{BI}^{\alpha\beta}_l \right] = 1.
\]

(62)

Therefore, the gauged first-order solution has two free parameters. Not only the shock excitation \( J_{l} \) but also the shock polarization angle \( \kappa_{l} \) are free parameters. When evaluating the general equations from above, the polarization condition (47) becomes not applicable.

V. SUMMARY

This work uses the shock wave method which bases on the publications of Treder and Stellmacher. Here, the tetrad fields from these publications were generalized to become usable in nonlinear electrodynamics. It turns out that, in general, nonlinear theories of electrodynamics show birefringence in vacuum as it has been demonstrated with other methods, too. As a new result, the general condition (47) for the first-order shock polarization is calculated.

As an example, the Born electrodynamics is investigated for which the shock wave has two polarization modes traveling along different directions. One of them is identical to Maxwell’s theory of light in vacuum. The other polarization has a shock-wave front \( \Sigma_{l} \) with a normal vector field \( p_{\alpha} = \Sigma_{\alpha} \) characterized by the equation (49) with the optical metric (58).

The Born-Infeld electrodynamics, however, is an exceptional case and does not show the effect of birefringence. As in the linear theory of Maxwell, the shock waves with arbitrary polarizations share one optical metric. One significant difference between the Born-Infeld and Maxwell’s electrodynamics is caused by different optical metrics. The optical metric of the Born-Infeld electrodynamics is given by equation (61).

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