A DECOMPOSITION OF THE GROUP ALGEBRA OF A HYPEROCTAHEDRAL GROUP

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Abstract. The descent algebra of a finite Coxeter group \( W \) is a subalgebra of the group algebra defined by Solomon. Descent algebras of symmetric groups have properties that are not shared by other Coxeter groups. For instance, the natural map from the descent algebra of a symmetric group to its character ring is a surjection with kernel equal the Jacobson radical. Thus, the descent algebra implicitly encodes information about the representations of the symmetric group, and a complete set of primitive idempotents in the character ring leads to a decomposition of the group algebra into a sum of right ideals indexed by partitions. Stanley asked whether this decomposition of the regular representation of a symmetric group could be realized as a sum of representations induced from linear characters of centralizers. This question was answered positively by Bergeron, Bergeron, and Garsia, using a connection with the free Lie algebra on \( n \) letters, and independently by Douglass, Pfeiffer, and Röhrle, who connected the decomposition with the configuration space of \( n \)-tuples of distinct complex numbers.

The Mantaci-Reutenauer algebra of a hyperoctahedral group is a subalgebra of the group algebra that contains the descent algebra. Bonnafé and Hohlweg showed that the natural map from the Mantaci-Reutenauer algebra to the character ring is a surjection with kernel equal the Jacobson radical. In 2008, Bonnafé asked whether the analog to Stanley’s question about the decomposition of the group algebra into a sum of induced linear characters holds. In this paper, we give a positive answer to Bonnafé’s question by explicitly constructing the required linear characters.

1. Introduction

Let \( W \) be a Coxeter group with a generating set of simple reflections \( S \). For a subset \( I \) of \( S \), \( W_I = \langle I \rangle \) is the standard parabolic subgroup of \( W \) generated by \( I \). Then \( W_I \) has a set of minimal length coset representatives \( X_I \), where length is the Coxeter length. Set \( x_I = \sum_{w \in X_I} w \). Then \( \{ x_I \mid I \subseteq S \} \) forms a basis for a subalgebra of the group algebra \( \mathbb{C}W \) called the descent algebra of \( W \) (see [14]).

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When $W = S_n$ is a symmetric group, the descent algebra has two particularly nice properties. First, the natural map from the descent algebra of $S_n$ to the character ring of $S_n$ is a surjection with kernel equal to the Jacobson radical, and second, a complete set of orthogonal idempotents of the descent algebra determines a decomposition of the group algebra of $S_n$ as a direct sum of induced representations of linear characters of centralizers. This decomposition of the group algebra has applications in surprisingly different contexts; see [1] for connections with the free Lie algebra, [10] for an application to Hochschild homology, and [7] and [5] for connections with the cohomology of the configuration space of $n$ distinct points in the plane. The first property, that the natural map from the descent algebra to the character ring is a surjection, only holds for symmetric groups. It is an open problem (the Lehrer-Solomon conjecture, see [2]) to determine whether the second property, that a complete set of orthogonal idempotents of the descent algebra determines a decomposition of the group algebra as a direct sum of induced representations of linear characters of centralizers, holds for a general Coxeter group $W$.

In this paper we consider hyperoctahedral groups and extensions of their descent algebras known as Mantaci-Reutenauer algebras. The Mantaci-Reutenauer algebra of a hyperoctahedral group $W_n$ has the property that the natural map to the character ring of $W_n$ is a surjection with kernel equal to the Jacobson radical [4]. The main result in this paper is that a complete set of orthogonal idempotents of the Mantaci-Reutenauer algebra of $W_n$ determines a decomposition of the group algebra of $W_n$ as a direct sum of induced representations of linear characters of centralizers. This answers a question raised by Bonnafé [3, §10]. It would be interesting to find a connection between the results in this paper and the corresponding conjectural result [2] for the descent algebra of $W_n$.

The idempotents in the Mantaci-Reutenauer algebra that play a central role in this paper were constructed by Vazirani [15], generalizing a construction given by Garsia and Reutenauer [9] for symmetric groups. Mantaci-Reutenauer algebras are defined for more general wreath products, and it seems likely that Vazirani’s construction, as well as the results in this paper, can be extended to the complex reflection groups $G(r, 1, n)$ for $r > 2$.

The rest of this paper is organized as follows. In the next section we set out some notation and state the main theorems. Proofs of the main results are then completed in subsequent sections.

2. A decomposition of $\mathbb{C}W_n$

To begin, we need some notation. First, throughout this paper, $n$ is a positive integer; for a positive integer $k$, set $[k] = \{1, \ldots, k\}$; for convenience the additive inverse of an integer will frequently be denoted by an overbar (so for example $\overline{3} = -3$ and $\overline{-3} = 3$); and for $J \subseteq \mathbb{Z}$, the notation $J = \{j_1 < j_2 < \cdots < j_k\}$ indicates that $J = \{j_1, j_2, \ldots, j_k\}$ and $j_1 < j_2 < \cdots < j_k$. 

2.1. Compositions and partitions. A composition of $n$ is a tuple of positive integers $p = (p_1, \ldots, p_k)$ such that $\sum_i p_i = n$. A partition of $n$ is a composition in which the entries of the tuple are nonincreasing. More generally, a signed composition of $n$ is a tuple of nonzero integers $p = (p_1, \ldots, p_k)$ such that $\sum_i |p_i| = n$. A signed partition of $n$ is a signed composition in which the positive entries of the tuple appear first, in nonincreasing order, followed by the negative entries, in order of nonincreasing absolute value. The entries of a signed composition $p$ are called parts of $p$. Define

- $C(n)$ to be the set of compositions of $n$,
- $P(n)$ to be the set of partitions of $n$,
- $SC(n)$ to be the set of signed compositions of $n$, and
- $SP(n)$ to be the set of signed partitions of $n$.

For $p = (p_1, p_2, \ldots, p_k) \in SC(n)$ set $\hat{p}_0 = 0$, and for $i \in [k]$ define

$$\hat{p}_i = \sum_{j=1}^{i} |p_j| \quad \text{and} \quad P_i = \{ \hat{p}_{i-1} + l \mid l = 1, \ldots, |p_i| \} = \{ \hat{p}_{i-1} + 1, \ldots, \hat{p}_i \}.$$ 

The subsets $P_i$ of $[n]$ will be referred to as “blocks” of $p$.

Signed partitions will frequently be written as $\lambda = (\lambda_1, \ldots, \lambda_a, \lambda_{a+1}, \ldots, \lambda_{a+b})$ to indicate that $\lambda_1, \ldots, \lambda_a$ are the positive parts of $\lambda$ and $\lambda_{a+1}, \ldots, \lambda_{a+b}$ are the negative parts. With the conventions above,

$$\hat{\lambda}_i = \sum_{j=1}^{i} |\lambda_j| \quad \text{and} \quad \Lambda_i = \{ \hat{\lambda}_{i-1} + 1, \hat{\lambda}_{i-1} + 2, \ldots, \hat{\lambda}_i \}.$$ 

For $p \in SC(n)$ let $\hat{\lambda}$ be the signed partition of $n$ formed by rearranging the parts of $p$.

Suppose $k$ is a positive integer and consider the set of signed compositions of $n$ with $k$ parts. The symmetric group $S_k$ acts on this set by permuting the parts of a signed composition, and the set of signed partitions of $n$ with $k$ parts forms a set of orbit representatives for this action. If $p$ has $k$ parts, let $\text{Stab}(p)$ be the stabilizer of $p$ in $S_k$.

For example, $p = (\overline{1}, 3, \overline{2}, 1, 3, \overline{1})$ is a signed composition of eleven with six parts, $\overline{\hat{p}} = (3, 3, 1, \overline{2}, \overline{1}, \overline{1})$, and $\text{Stab}(p)$ is isomorphic to the Klein four group.

If a signed composition $p$ is fixed, then $\xi_i$ will denote the sign of $p_i$, where the sign of a positive number is $+$ and the sign of a negative number is $-$. 

2.2. Hyperoctahedral groups. A signed permutation of $n$ is a permutation $w$ of the set $\{1, 2, \ldots, n\}$ such that $w(\overline{a}) = \overline{w(a)}$ for $a$ in $[n]$. Signed permutations of $n$ naturally form a group under composition, called the $n^{th}$ hyperoctahedral group and denoted by $W_n$. 

We identify $S_n$ with the subgroup of $W_n$ consisting of all signed permutations $w$ such that $w([n]) = [n]$. For a subset $P$ of $[n]$, let $S_P$ and $W_P$ denote the subgroups of $S_n$ and $W_n$, respectively, that fix $[n] \setminus P$ pointwise. Then $S_P = S_n \cap W_P \subseteq W_n$. Similarly, for an integer $m \leq n$, we identify $S_m$ with the subgroup $S_{[m]}$ of $S_n$, and $W_m$ with the subgroup $W_{[m]}$.

In this paper, $W$, $S$, and several other symbols can have four types of subscripts: a positive integer, usually $m$, $n$, or $|\lambda|$; a (signed) composition, usually $p$, $|p|$, or $|\lambda|$; a (signed) partition, usually $\lambda$; or a subset of $[n]$, usually $P_i$ or $\Lambda_i$. The meaning should always be clear from context.

Signed permutations may be represented in two row (or function) notation, one row notation, or cycle notation. The conventions we will use in this paper are most clearly demonstrated with an example. In two row, one row, and cycle notation, respectively,

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & \tau & 4 & 6 & 5 \end{pmatrix} = 23\tau4\overline{5} = (1 \ 2 \ 3)^{-} (4)(5 \ \overline{6})$$

is the signed permutation that maps 1 to 2, 2 to 3, 3 to $\tau$, and so on. Here the superscript $-$ in cycle notation denotes a negative cycle. Given $a_1, a_2, \ldots, a_r$, $(a_1 \ a_2 \ \ldots \ a_r)^-$ is called a negative $r$-cycle and denotes the signed permutation that maps $a_j$ to $a_{j+1}$ for $j \in [r-1]$ and maps $a_r$ to $\overline{a_1} = -a_1$. Note that a negative $r$-cycle has order $2r$ as an element of $W_n$.

Each signed permutation $w$ has a signed cycle type that is the signed partition $\lambda$ for which the positive parts of $\lambda$ are the lengths of the positive cycles in the cycle decomposition of $w$, and the negative parts of $\lambda$ are the lengths of the negative cycles in the cycle decomposition of $w$. For example, the signed cycle type of $w = (1 \ 2 \ 3)^{-} (4) (5 \ \overline{6})$ is the signed partition $(2, 1, 3)$. Note that two signed permutations in $W_n$ are conjugate if and only if they have the same signed cycle type.

For a positive integer $i$, let $s_i$ be the positive two-cycle $(i \ i + 1)$ that switches $i$ and $i + 1$, and let $t_i$ be the negative one-cycle $(i)^-$ that sends $i$ to $\tau$. Then \{ $t_1, s_1, s_2, \ldots, s_{n-1}$ \} is a set of Coxeter generators of $W_n$.

Finally, let $w_{0,n}$ be the “longest element” in $W_n$, so

$$w_{0,n} = t_1 \cdots t_n \quad \text{and} \quad w_{0,n}(a) = \overline{a}$$

for $a \in [n]$. Note that $\langle w_{0,n} \rangle$ is the center of $W_n$. Similarly, for $P \subseteq [n]$ define $w_{0,P} = \prod_{j \in P} t_j$ in $W_P$.

2.3. Mantaci-Reutenauer algebras. Mantaci-Reutenauer algebras for hyperoctahedral groups were first defined by Mantaci and Reutenauer [12]. Subsequently, Bonnafé and Hohlweg [4] gave a construction in the spirit of Solomon’s construction of descent algebras described above. Theirs is the description of Mantaci-Reutenauer algebras used in this paper.
For a signed composition \( p = (p_1, p_2, \ldots, p_k) \) of \( n \) with blocks \( P_1, \ldots, P_k \), define \( W_p \) to be the subset of \( W_n \) consisting of all signed permutations \( w \) such that
\[
\text{sign}(P_i) \subseteq \pm P_i \text{ } \forall \text{ } i \in [k] \text{ and } w(P_i) \subseteq P_i \text{ if } p_i < 0.
\]

For example, if \( \lambda = (\lambda_1, \ldots, \lambda_a, \lambda_{a+1}, \ldots, \lambda_{a+b}) \in \mathcal{SP}(n) \), then
\[
W_\lambda = W_{\lambda_1} \cdots W_{\lambda_a} S_{\lambda_{a+1}} \cdots S_{\lambda_{a+b}} \cong W_{\lambda_1} \times \cdots \times W_{\lambda_a} \times S_{|\lambda_{a+1}|} \times \cdots \times S_{|\lambda_{a+b}|}.
\]

In analogy with the case of symmetric groups, the subgroups \( W_p \) for \( p \in \mathcal{SC}(n) \) are called signed Young subgroups. With respect to the length function determined by the Coxeter generating set \( \{t_1, s_1, \ldots, s_{n-1}\} \) of \( W_n \), every left coset of \( W_p \) in \( W_n \) contains a unique element of minimal length. Define \( X_p \) to be the set of these minimal length coset representatives and define
\[
x_p = \sum_{w \in X_p} w \in \mathbb{C}W_n.
\]

It turns out that \( \{ x_p \mid p \in \mathcal{SC}(n) \} \) is linearly independent and spans a subalgebra of \( \mathbb{C}W_n \). This subalgebra is the Mantaci-Reutenauer algebra of \( W_n \) (see [4, \S 2.3]). In this paper, the Mantaci-Reutenauer algebra of \( W_n \) is denoted by \( \Sigma(W_n) \). The reader should be aware that this is not in accordance with the notation in [9, \S 3], where \( \Sigma(W_n) \) denotes the descent algebra of \( W_n \) and \( \Sigma'(W_n) \) denotes the Mantaci-Reutenauer algebra of \( W_n \).

2.4. Idempotents in \( \Sigma(W_n) \). Our next task is to define a complete set of primitive, orthogonal idempotents in \( \Sigma(W_n) \), and hence a complete set of orthogonal idempotents in \( \mathbb{C}W_n \), that gives rise to a decomposition of the right regular representation of \( W_n \) as a direct sum of induced representations.

First, suppose \( m \) is a positive integer. Recall that if \( w \in S_m \) and \( i \in [m-1] \), then \( i \) is a descent of \( w \) if \( w(i) > w(i+1) \). Let \( D(w) \) denote the set of descents of \( w \) and for \( A \subseteq [m-1] \) define \( D_{\leq A} = \sum_{\ell \leq A} w \). It is shown in [13, Section 8.4] that
\[
r_m = \sum_{A \subseteq [m-1]} \frac{(-1)^{|A|}}{|A| + 1} D_{\leq A}
\]
is an idempotent in the group algebra \( \mathbb{C}S_m \). In fact, by [9, \S 3], \( r_m \) lies in the descent algebra of \( S_m \), and by Lemma 3.3 or [4, Theorem 3.7], \( r_m \) lies in the Mantaci-Reutenauer algebra of \( W_n \). We call the idempotent \( r_m \) the Reutenauer idempotent in \( \mathbb{C}S_m \).

Notice that if \( P = \{z_1 < \cdots < z_m\} \) is an ordered set of positive integers, then the Reutenauer idempotent \( r_P \) is unambiguously defined in the group algebra \( \mathbb{C}S_P \) by replacing the set \( [m] \) by \( P \) in the preceding paragraph.

Next, define
\[
\epsilon_m^\pm = (1/2)(\text{id} \pm w_{0,m}),
\]
where id denotes the identity permutation in $W_m$. Then $\epsilon^+_m$ and $\epsilon^-_m$ are idempotents in $\mathbb{C}W_m$, and it follows from [4, Example 3.5] that $\epsilon^\pm_m$ is in $\Sigma(W_m)$. Similarly, define $\epsilon^+_P = (1/2)(\text{id} \pm w_0, P)$ for $P \subseteq [n]$.

Finally, suppose $p = (p_1, \ldots, p_k)$ is a signed composition of $n$, and let $\xi_i$ denote the sign of $p_i$. Define a composition $|p|$ of $n$ by

$$|p| = (|p_1|, \ldots, |p_k|)$$

(note that this is non-standard notation!), and define

$$e_p = x_{|p|} \epsilon^\xi_1 r_{P_1} \cdots \epsilon^\xi_k r_{P_k},$$

where $x_{|p|}$ is the basis element of $\Sigma(W_n)$ corresponding to $|p|$ (note that $|p| \in \mathcal{SC}(n)$).

Because $\mathcal{SP}(n) \subseteq \mathcal{SC}(n)$, if $\lambda$ is a signed partition of $n$, then $e_{\lambda}$ is defined.

**Proposition 2.5.** The elements $e_p$, for $p$ a signed composition of $n$, coincide with the elements $I_p$ defined by Vazirani in [15, Chapter 3].

This proposition is proved in §3. The next corollary follows from the preceding proposition and [15, §3.7].

**Corollary 2.6.** For a signed composition $p$ of $n$, the element $e_p$ in $\Sigma(W_n)$ is a quasi-idempotent with $e_p^2 = |\text{Stab}(p)| e_p$. More generally, if $p$ and $q$ are signed compositions of $n$ with $\frac{p}{q}$, then $e_p e_q = |\text{Stab}(q)| e_q$.

2.7. Now suppose $\lambda = (\lambda_1, \ldots, \lambda_a, \lambda_{a+1}, \ldots, \lambda_{a+b})$ is a signed partition of $n$ and define

$$E_{\lambda} = \frac{1}{(a + b)!} \sum_{\frac{p}{q} = \lambda} e_p.$$  

It follows from Corollary 2.6 that $E_{\lambda}$ is an idempotent in $\mathbb{C}W_n$, and by Proposition 2.5, $E_{\lambda}$ coincides with $E_\lambda$ as defined by Vazirani in [15, Chapter 4], so the set $\{ E_{\lambda} \mid \lambda \in \mathcal{SP}(n) \}$ is a complete family of primitive, orthogonal idempotents in $\Sigma(W_n)$. Because the $E_{\lambda}$’s form a complete set of orthogonal idempotents in $\mathbb{C}W_n$, we have the direct sum decomposition

$$\mathbb{C}W_n \cong \bigoplus_{\lambda \in \mathcal{SP}(n)} E_{\lambda}\mathbb{C}W_n.$$  

2.8. For $i \in [a + b]$, define the positive $|\lambda_i|$-cycle

$$c_i = (\hat{\lambda}_{i-1} + 1 \ \hat{\lambda}_{i-1} + 2 \ \cdots \ \hat{\lambda}_i)$$

and the negative $|\lambda_i|$-cycle

$$d_i = \begin{cases} c_i w_0, \lambda_i & \text{if } \lambda_i \text{ is odd} \\ (\hat{\lambda}_{i-1} + 1 \ \hat{\lambda}_{i-1} + 2 \ \cdots \ \hat{\lambda}_i) & \text{if } \lambda_i \text{ is even.} \end{cases}$$
Note that $c_i$ and $d_i$ are supported on the block $\Lambda_i$ of $[n]$ and are defined for both the positive and negative parts of $\lambda$. Finally, define

$$w_\lambda = c_1 \cdots c_ad_{a+1} \cdots d_{a+b}.$$  

Then $w_\lambda$ is an element of $W_n$ with signed cycle type $\lambda$. Because of our sign conventions, in general $w_\lambda$ is not in the signed Young subgroup $W_\lambda$.

As in [11, §4.2], the centralizer in $W_n$ of $w_\lambda$ is generated by

$$\{ c_i, w_{0,\Lambda_i} \mid i \in [a] \} \Pi \{ d_i \mid i \in [a+b] \setminus [a] \} \Pi \{ y_i \mid \lambda_i = \lambda_{i+1}, \ i \in [a+b-1] \},$$

where for $i \in [a+b-1]$, $y_i$ is the permutation in $W_n$ defined by

$$y_i(l) = \begin{cases} l & \text{if } l \not\in \Lambda_i \cup \Lambda_{i+1} \\ l + |\lambda_i| & \text{if } l \in \Lambda_i \\ l - |\lambda_i| & \text{if } l \in \Lambda_{i+1}. \end{cases}$$

Then $y_i$ fixes $[n] \setminus (\Lambda_i \cup \Lambda_{i+1})$ pointwise and switches the blocks $\Lambda_i$ and $\Lambda_{i+1}$.

For example, if $\lambda = (2, 2, 1, 3, 2, 2)$, then

$$w_\lambda = c_1c_2c_3d_4d_5d_6 = (1 2)(3 4)(5 7)(8 10)(9 12)^{-1},$$

and $Z_{W_12}(w_\lambda) = \langle c_1, c_2, c_3, w_{0,\Lambda_1}, w_{0,\Lambda_2}, w_{0,\Lambda_3}, d_4, d_5, d_6, y_1, y_5 \rangle$, where $c_3 = \text{id}$, $w_{0,\Lambda_1} = t_1t_2$, $w_{0,\Lambda_2} = t_3t_4$, $w_{0,\Lambda_3} = t_3$,

$$y_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \cdots \\ 3 & 4 & 1 & 2 & 5 & \cdots \end{pmatrix} \quad \text{and} \quad y_5 = \begin{pmatrix} 1 & \cdots & 8 & 9 & 10 & 11 & 12 \\ 1 & \cdots & 8 & 11 & 12 & 9 & 10 \end{pmatrix}.$$

2.9. For a positive integer $m$, let $\omega_m$ be the primitive $m^{th}$ root of unity

$$\omega_m = e^{2\pi \sqrt{-1}/m}.$$

Also, for a group $G$ and an element $g$ in $G$ of order $|g| = m$, define an idempotent $\zeta_g$ in $\mathbb{C}G$ by

$$\zeta_g = \frac{1}{m} \sum_{j=1}^{m} \omega_m^{-j} g^j.$$  

If $m$ is odd, define also

$$\tilde{\zeta}_g = \frac{1}{m} \sum_{j=1}^{m} (\omega_m^{- (m+1)/2})^{-j} g^j.$$  

(The coefficient of $g^j$ has been chosen to simplify the formula for $\varphi_\lambda$ in Theorem 2.10.)

For $i \in [a+b]$, set

$$f_i = \begin{cases} \epsilon_i^+ \zeta_{c_i} & \text{if } i \in [a] \\ \epsilon_i^- \tilde{\zeta}_{c_i} & \text{if } i \in [a+b] \setminus [a] \text{ and } \lambda_i \text{ is odd} \\ \zeta_{d_i} & \text{if } i \in [a+b] \setminus [a] \text{ and } \lambda_i \text{ is even}, \end{cases}$$

where $\epsilon_i^+$ and $\epsilon_i^-$ are the minimum integers such that $\epsilon_i^+ \geq |\lambda_i|$ and $\epsilon_i^- \geq |\lambda_i|$, respectively.
and define $\tilde{e}_\lambda$ in $CW_n$ by

$$\tilde{e}_\lambda = x_{\lambda|}f_1 \cdots f_{a+b}.$$ 

We can now state the first main theorem.

**Theorem 2.10.** Suppose $\lambda$ is a signed partition of $n$.

1. The group $Z_{\lambda}(w)$ acts on $\tilde{e}_\lambda$ on the right as scalars. Let $\varphi_\lambda$ be the character afforded by the $CZ_{\lambda}(w)$-module $C\tilde{e}_\lambda$. Then $\varphi_\lambda$ is given by

$$\varphi_\lambda(w) = \begin{cases} 
\omega_{|c_i|} & \text{if } w = c_i \text{ for } i \in [a] \\
\omega_{|d_i|} & \text{if } w = d_i \text{ for } i \in [a+b] \setminus [a] \\
1 & \text{if } w = w_{0,\lambda_i} \text{ for } i \in [a], \text{ or if } w = y_i \text{ for } i \in [a+b-1] \\
& \text{with } \lambda_i = \lambda_{i+1}.
\end{cases}$$

2. There is an isomorphism of right $CW_n$-modules

$$E_\lambda CW_n \cong \text{Ind}_{Z_{\lambda}(w)}^{W_n}(C\tilde{e}_\lambda).$$

The theorem is proved in §5. A key ingredient in the proof is Proposition 4.1, where it is shown that if $\lambda_i$ is even, $C = \langle c_i, w_{0,\lambda_i} \rangle$ (the direct product of a cyclic group of order two and a cyclic group of order $|\lambda_i|$), and $D = \langle d_i \rangle$ (a cyclic group of order $2|\lambda_i|$), then $\text{Ind}_{C}^{W_{\lambda_i}}(C\tilde{e}_{\lambda_i} \zeta_{c_i}) = \text{Ind}_{D}^{W_{\lambda_i}}(C\zeta_{d_i})$.

The next corollary follows immediately from 2.7(a) and the theorem.

**Corollary 2.11.** Let $\rho_n$ denote the regular character of $W_n$. Then

$$\rho_n = \bigoplus_{\lambda \in SP(n)} \text{Ind}_{Z_{\lambda}(w)}^{W_n}(\varphi_\lambda).$$

2.12. A question of Bonnafé. Let $\text{cf}_C(W_n)$ denote the algebra of $C$-valued class functions on $W_n$. Then $\text{cf}_C(W_n)$ is a split, semisimple, commutative $C$-algebra.

For $\lambda \in SP(n)$, let $\rho_\lambda$ be the characteristic function of the conjugacy class of $w_\lambda$. Then $\{ \rho_\lambda \mid \lambda \in SP(n) \}$ is the (unique) basis of $\text{cf}_C(W_n)$ consisting of primitive idempotents.

Next, let $1_{W_p}$ be the trivial character of $W_p$ and define

$$\theta_n: \Sigma(W_n) \to \text{cf}_C(W_n) \quad \text{by} \quad \theta_n(x_p) = \text{Ind}_{W_p}^{W_n}(1_{W_p})$$

and linearity. Bonnafé and Hohlweg [4, Theorem 3.7] have shown that $\theta_n$ is an algebra homomorphism with kernel equal to the Jacobson radical of $\Sigma(W_n)$. Therefore, if $\{ F_\lambda \mid \lambda \in SP(n) \}$ is a complete set of primitive, orthogonal idempotents in $\Sigma(W_n)$, then $\{ \theta_n(F_\lambda) \mid \lambda \in SP(n) \}$ is the set of primitive idempotents in $\text{cf}_C(W_n)$, and so there is a permutation of $SP(n)$, say $\lambda \mapsto \lambda^*$, so that $\theta_n(F_\lambda) = \rho_{\lambda^*}$ for all $\lambda \in SP(n)$.

Bonnafé [3, §10] asked whether it was possible to find a set of primitive idempotents $\{ F_\lambda \}$ such that $F_\lambda CW_n \cong \text{Ind}_{Z_{\lambda}(w)}^{W_n}(\eta_\lambda)$ for some linear character $\eta_\lambda$ of $Z_{\lambda}(w)$.

It follows from Theorem 2.10 that the idempotents $\{ E_\lambda \mid \lambda \in SP(n) \}$ constructed
by Vazirani give a positive answer to this question. The permutation \( \lambda \mapsto \lambda^* \) such that \( \theta_n(E_\lambda) = u_{\lambda^*} \) is given in the next theorem.

For \( p = (p_1, \ldots, p_k) \in SC(n) \) define \( p' \in SC(n) \) by

\[
P'_i = \begin{cases} 
p_i & \text{if } p_i \text{ is odd} \\
\overline{p_i} & \text{if } p_i \text{ is even}.
\end{cases}
\]

For example, if \( \lambda = (4, 3, 2, 2, 1, \overline{7}, \overline{5}, \overline{3}, \overline{3}, \overline{\overline{5}}, \overline{5}) \), then

\[
\lambda' = (\overline{7}, \overline{3}, \overline{\overline{5}}, \overline{1}, \overline{6}, \overline{\overline{5}}, 4, \overline{3}, \overline{3}, \overline{2}), \quad \text{and} \quad \overline{\lambda'} = (6, 4, 3, 2, 1, \overline{5}, \overline{4}, \overline{3}, \overline{3}, \overline{2}, \overline{\overline{5}}).
\]

**Theorem 2.13.** Suppose \( \lambda \) is a signed partition of \( n \). Then \( \theta_n(E_\lambda) = u_{\overline{\lambda'}} \).

This theorem is proved in §6.

3. Proof of Proposition 2.5

In this section, \( p = (p_1, \ldots, p_k) \) is a fixed signed composition of \( n \), and we show that the idempotent denoted by \( I_p \) in [15, Chapter 3] is equal to \( e_p = x_{\mid p} \xi^{e_1} P_1 r P_1 \cdots \xi^{e_k} r P_k \) (here \( \xi_i \) is the sign of \( p_i \)). In order to do so, we first reformulate the definition of the idempotents in \( CS_n \) denoted by \( I_p \) in [9] (for \( p \in C(n) \)). This requires the basis of \( \Sigma(W_n) \) used in [12] and [15].

3.1. **The Mantaci-Reutenauer basis of** \( \Sigma(W_n) \). Define a partial order on \( SC(n) \) by “signed refinement,” that is, for \( p, q \in SC(n) \), define \( p \leq q \) if \( q \) can be obtained from \( p \) by combining consecutive parts with the same sign. In this case, say that \( p \) is finer than \( q \). Here we are following the presentation in [15, Chapter 3], in which the partial order is reversed from that in [12].

For example, \( p = (1, 1, \overline{2}, \overline{1}, 2, 3, 2) \) is finer than \( q = (2, \overline{3}, 2, 3, 2) \), which in turn is finer than \( r = (2, \overline{3}, 7) \); moreover, \( r \) is maximal with respect to the partial order.

Next, for \( w \in W_n \), say that \( i \in [n - 1] \) is a descent of \( w \) if

- \( w(i) \) and \( w(i + 1) \) have the same sign and \( \left| w(i) \right| > \left| w(i + 1) \right| \), or
- \( w(i) \) and \( w(i + 1) \) have opposite signs.

Let \( D(w) \) denote the set of descents of \( w \). Notice that for \( w \in S_n \) this definition agrees with that in 2.4, but that in general \( D(w) \) is not the descent set of \( w \) with respect to a positive system of roots.

For example, in one row notation let \( w = w(1)w(2) \cdots w(n) = 2 \ 1 \ \overline{3} \ \overline{6} \ \overline{5} \ 4 \ 8 \ 7 \) in \( W_8 \).

Then \( D(w) = \{1, 2, 4, 5, 7\} \), where the descents at 2, 5, and 7 arise from sign changes. Note that the descents of \( w \) partition the set \([8]\) into six blocks:

\[
w = 2 \ | \ 1 \ | \ \overline{3} \ | \ \overline{6} \ | \ \overline{5} \ | \ 4 \ | \ 8 \ | \overline{7}.
\]

Finally, for \( w \in W_n \), the descent shape of \( w \), denoted by \( ds(w) \), is the signed composition \( p = (p_1, p_2, \ldots, p_k) \) such that
• $D(w) = \{ \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_{k-1} \}$, and
• the sign of $p_i$ is equal to the sign of $w(\hat{p}_i)$.

In other words, the descent shape of $w$ is found by using the descents of $w$ to break the set $[n]$ into blocks, and the sizes and signs of the blocks determine the parts of $ds(w)$. For example, with $w$ as above, $ds(w) = (1, 1, 2, 1, 2, 1)$.

Now for $p \in SC(n)$, define
\[
x_p^w = \sum_{ds(w) \geq p} w.
\]
Then $\{ x_p^w | p \in SC(n) \}$ is a basis of $\Sigma(W_n)$ (see [4, §2.8]).

3.2. Notice that for $w \in W_n$, the following statements are equivalent:
• $w \in S_n$,
• $ds(w) \in C(n)$, and
• $ds(w) \leq (n)$.

For a composition or signed composition $p$ of $n$, let $k(p)$ denote the number of parts of $p$. Also, let $\psi$ denote the bijection between (unsigned) compositions of $n$ and subsets of $[n-1]$ given by
\[
\psi(p) = \{ \hat{p}_1, \hat{p}_2, \ldots, \hat{p}_{k-1} \}
\]
when $p = (p_1, p_2, \ldots, p_k)$. The following statements follow immediately from the definitions:
• For $p \in C(n)$ and $w \in W_n$, $ds(w) = p$ if and only if $w \in S_n$ and $D(w) = \psi(p)$.
• For $p, q \in C(n)$, $p \leq q$ if and only if $\psi(q) \subseteq \psi(p)$.
• For $p \in C(n)$, $k(p) = |\psi(p)| + 1$.

It follows from the first two statements that for $p \in C(n)$,
\[
x_p^w = \sum_{ds(w) \geq p} w = \sum_{w \in S_n \atop D(w) \subseteq \psi(p)} w = D_{\subseteq \psi(p)}.
\]

Lemma 3.3. Let $m$ be a positive integer. Then the Reutenauer idempotent $r_m \in \mathbb{C}S_m$ may be expressed as
\[
r_m = \sum_{p \in C(m)} \frac{(-1)^{k(p)-1}}{k(p)} x_p^w.
\]

Proof. Using the assertions in 3.2 we have
\[
r_m = \sum_{A \subseteq [m-1]} (-1)^{|A|} D_{\subseteq A} = \sum_{p \in C(m)} \frac{(-1)^{|\psi(p)|}}{|\psi(p)| + 1} D_{\subseteq \psi(p)} = \sum_{p \in C(m)} \frac{(-1)^{k(p)-1}}{k(p)} x_p^w.
\]
\[\square\]
3.4. **Garsia-Reutenauer idempotents.** Suppose \( p = (p_1, \ldots, p_k) \) is a composition of \( n \). Garsia and Reutenauer \([9, (3.17)]\) define a quasi-idempotent \( I_p \in \mathbb{C}S_n \) by

\[
I_p = \sum_{J_1 + \cdots + J_k = [n], |J_i| = p_i} \rho[J_1] * \cdots * \rho[J_k],
\]

where (using the notation in [9])

- the sum is over all ordered set partitions \( J_1, \ldots, J_k \) of \( [n] \) such that \( |J_i| = p_i \) for \( i \in [k] \),
- if \( J = \{j_1 < \cdots < j_m\} \subseteq [n] \), then \( \rho[J] = w_J r_m \), where \( r_m \) is the Reutenauer idempotent in \( \mathbb{C}S_m \) (considered as a subalgebra of \( \mathbb{C}S_n \)) and \( w_J \) is the function from \( [m] \) to \( J \) given in two row notation by

\[
w_J = \begin{pmatrix} 1 & 2 & \cdots & m \\ j_1 & j_2 & \cdots & j_m \end{pmatrix},
\]

and

- \( * \) is the concatenation product.

In the second bullet point, the equality \( \rho[J] = w_J r_m \) uses the formulation of the Reutenauer idempotent in Lemma 3.3.

Consider a summand

\[
\rho[J_1] * \cdots * \rho[J_k] = w_{J_1} r_{p_1} * \cdots * w_{J_k} r_{p_k}.
\]

It is straightforward to check that

\[
w_{J_1} r_{p_1} * \cdots * w_{J_k} r_{p_k} = w_{(J_1, \ldots, J_k)} r_{p_1} \cdots r_{p_k},
\]

where as above the product on the right-hand side is the usual multiplication in the group algebra \( \mathbb{C}S_n \) and \( w_{(J_1, \ldots, J_k)} \) is the permutation given in two row notation by

\[
w_{(J_1, \ldots, J_k)} = \begin{pmatrix} P_1 & P_2 & \cdots & P_k \\ J_1 & J_2 & \cdots & J_k \end{pmatrix},
\]

with the convention that entries in \( P_i \) and \( J_i \) are written in increasing order.

For example, suppose \( p = (a, b) \), so \( P_1 = \{1, \ldots, a\} \) and \( P_2 = \{a + 1, \ldots, a + b\} \). Consider \( w_J x * w_K y \), with \( J = \{j_1 < \cdots < j_a\} \), \( K = \{k_1 < \cdots < k_b\} \), \( x \in S_a \), and \( y \in S_b \). Say \( x = x_1 x_2 \cdots x_a \) and \( y = y_1 y_2 \cdots y_b \) in one row notation. Using two row “block” notation for permutations, write

\[
w_J x = \begin{pmatrix} [a] \\ J \end{pmatrix} \begin{pmatrix} [a] \\ x \end{pmatrix} = \begin{pmatrix} [a] \\ J' \end{pmatrix} \quad \text{and} \quad w_K y = \begin{pmatrix} [b] \\ K \end{pmatrix} \begin{pmatrix} [b] \\ y \end{pmatrix} = \begin{pmatrix} [b] \\ K' \end{pmatrix},
\]

where now \( J' = j'_1 \cdots j'_a \) is obtained from \( J = j_1 \cdots j_a \) by permuting the entries, and similarly for \( K' \). Then

\[
w_J x * w_K y = \begin{pmatrix} [a] \\ J' \end{pmatrix} * \begin{pmatrix} [b] \\ K' \end{pmatrix}.
\]
\[
\begin{align*}
&= \left( \begin{array}{c} P_1 \\ J' \end{array} \right) \cdot \left( \begin{array}{c} P_2 \\ P_2 \\ a + K' \end{array} \right) \\
&= \left( \begin{array}{c} P_1 \\ J \end{array} \right) \cdot \left( \begin{array}{c} P_2 \\ P_2 \\ x \\ P_2 \\ a + K \end{array} \right) \cdot \left( \begin{array}{c} P_1 \\ P_2 \\ P_1 \\ a + y \end{array} \right) \\
&= \left( \begin{array}{c} P_1 \\ J \end{array} \right) \cdot \left( \begin{array}{c} P_2 \\ a + K \end{array} \right) \cdot \left( \begin{array}{c} P_1 \\ x \\ P_2 \\ a + y \end{array} \right) \\
&= w_{(J,K)} \left( \begin{array}{c} P_1 \\ P_2 \\ x \\ P_2 \\ P_1 \\ a + y \end{array} \right)
\end{align*}
\]

where \( a + K', a + K, \) and \( a + y \) denotes adding \( a \) to each entry of \( K', K, \) and \( y \), respectively.

It follows from the definitions (see [4, Remark 2.1]) that if \( w \in W_n \), then \( w \in X_p \) if and only if \( w(j) > 0 \) for \( j \in [n] \), and \( w|_{P_l} : P_l \rightarrow [n] \) is increasing for \( l \in [k] \). Thus \( w_{(J_1,\ldots,J_k)} \in X_p \) and in fact

\[
X_p = \{ w_{(J_1,\ldots,J_k)} \mid J_1 + \cdots + J_k = [n] \text{ and } \forall i \in [k], |J_i| = p_i \}.
\]

Putting the pieces together gives

\[
I_p = \sum_{J_1 + \cdots + J_k = [n] \atop |J_i| = p_i} \rho[J_1] \cdots \rho[J_k] = \sum_{w \in X_p} w r_{P_1} \cdots r_{P_k} = x_p r_{P_1} \cdots r_{P_k}.
\]

Garsia and Reutenauer [9, §3, §4] prove the remarkable fact that \( \{ I_p \mid p \in C(n) \} \) is a basis of the descent algebra of \( S_n \) consisting of quasi-idempotents.

3.5. Vazirani’s idempotents. Vazirani [15] extends the constructions of Garsia and Reutenauer to \( \Sigma(W_n) \).

Suppose \( m \) is a positive integer, and set

\[
I_{(m)}^\pm = e_m^\pm \cdot r_m \in CW_m.
\]

Now, given a signed composition \( p = (p_1,\ldots,p_k) \) of \( n \), in analogy with 3.4(a) define

\[
I_p = \sum_{J_1 + \cdots + J_k = [n] \atop |J_i| = |p_i|} I_{[J_1]}^{\xi_1} \cdots I_{[J_k]}^{\xi_k},
\]

where \( \xi_i \) is the sign of \( p_i \),

- the sum is over all ordered set partitions \( \{J_1,\ldots,J_k\} \) of \( [n] \) with \( |J_i| = |p_i| \) for \( i \in [k] \),
• if \( J = \{ j_1 < \cdots < j_m \} \subseteq [n] \), then
  \[
  I_{[J]}^\xi = w_J I_{[m]}^\xi = w_J \epsilon_m r_m,
  \]
  where \( w_J \) is as in 3.4 and \( \xi \in \{ +, - \} \), and
• * is the concatenation product.

Consider a summand
  \[
  I_{[J_1]}^{\xi_1} \cdots I_{[J_k]}^{\xi_k} = w_{J_1} \epsilon_{[p_1]} r_{[p_1]} \cdots w_{J_k} \epsilon_{[p_k]} r_{[p_k]},
  \]
Substituting \( \epsilon_{[p_i]} = (1/2)(\text{id} \pm w_{0,[p_i]}) \), expanding the right-hand side, using the computation of the concatenation product in 3.4(b), and then simplifying the expression using the definition of \( \epsilon_{[p_i]} \) again shows that
  \[
  w_{J_1} \epsilon_{[p_1]} r_{[p_1]} \cdots w_{J_k} \epsilon_{[p_k]} r_{[p_k]} = w_{(J_1,\ldots,J_k)} \epsilon_{P_1} r_{P_1} \cdots \epsilon_{P_k} r_{P_k},
  \]
where the product on the right-hand side is the usual multiplication in the group algebra and \( w_{(J_1,\ldots,J_k)} \) is the permutation in 3.4(c).

Putting the pieces together this time gives
  \[
  I_p = \sum_{J_1+\cdots+J_k=[n], |J_i|=|p_i|} I_{[J_1]}^{\xi_1} \cdots I_{[J_k]}^{\xi_k} = \sum_{w \in X_{[p_1]}} w \epsilon_{P_1} r_{P_1} \cdots \epsilon_{P_k} r_{P_k} = x_{[p]} \epsilon_{P_1} r_{P_1} \cdots \epsilon_{P_k} r_{P_k},
  \]
which is the assertion in the proposition.

4. Computation of some induced characters

In this section, \( m \) is a positive integer, and we consider the hyperoctahedral group \( W_m \). Let \( c \) be the positive \( m \)-cycle \( c = (1 \ 2 \ \cdots \ m) \), let \( d \) be the negative \( m \)-cycle \( d = (1 \ 2 \ \cdots \ m)^- \), and set \( w_0 = w_{0,m} \). In preparation for the proof of Theorem 2.10, we compare characters induced from \( \langle d \rangle \) and \( \langle c, w_0 \rangle \) to \( W_m \), when \( m \) is even. This section is devoted to the proof of Proposition 4.1, which asserts that when \( m \) is even, suitably chosen characters of \( \langle d \rangle \) and \( \langle c, w_0 \rangle \) induce to the same character of \( W_m \).

Recall that for a group \( G \) and an element \( g \in G \) of order \( m \), the idempotent \( \zeta_g \) in \( \mathbb{C}\langle g \rangle \) is defined by \( \zeta_g = \frac{1}{m} \sum_{i=1}^{m} \omega_m^{-i} g^i \). Throughout this section \( \epsilon = \frac{1}{2} (\text{id} - w_0) \). Then \( \epsilon \) and \( \zeta_c \) are commuting idempotents in \( \mathbb{C}\langle c, w_0 \rangle \). Let \( \chi_{\epsilon \zeta_c} \) be the character of \( \langle c, w_0 \rangle \) afforded by the right ideal \( \epsilon \zeta_c \mathbb{C}\langle c, w_0 \rangle \), and let \( \chi_{\zeta_d} \) be the character of \( \langle d \rangle \) afforded by the right ideal \( \zeta_d \mathbb{C}\langle d \rangle \).

**Proposition 4.1.** Suppose \( m \) is even. Then there is an isomorphism of right \( \mathbb{C}W_m \)-modules
  \[
  \epsilon \zeta_c \mathbb{C}W_m \cong \zeta_d \mathbb{C}W_m.
  \]
By [6, 11.21], $\text{Ind}_{\langle c, w_0 \rangle}^{W_m} (\chi_{\epsilon c})$ is the character of the representation of $W_m$ acting on the right ideal $\epsilon c \mathbb{C}W_m$, and $\text{Ind}_{\langle d \rangle}^{W_m} (\chi_{\zeta d})$ is the character of the representation of $W_m$ acting on the right ideal $\zeta d \mathbb{C}W_m$. Thus, to prove the proposition it is enough to show that

(a) $\text{Ind}_{\langle c, w_0 \rangle}^{W_m} (\chi_{\epsilon c}) = \text{Ind}_{\langle d \rangle}^{W_m} (\chi_{\zeta d})$.

We prove (a) by showing that both induced characters take the same values on all conjugacy classes. In order to do so we need some preliminary lemmas. The first lemma is due to Littlewood (see [6, Exercise 9.16] for the left-sided version).

**Lemma 4.2.** For a finite group $G$ and an idempotent $e = \sum_{g_1 \in G} \gamma_{g_1} g_1$ in $\mathbb{C}G$, the character $\chi_e$ of $G$ afforded by $e \mathbb{C}G$ is given by

$$\chi_e(g) = |Z_G(g)| \sum_{g_1 \in \text{ccl}(g)} \gamma_{g_1},$$

where $\text{ccl}(g)$ is the conjugacy class of $g$.

4.3. It follows from [6, 11.21] and the lemma that for $g$ in $W_m$,

$$\text{Ind}_{\langle c, w_0 \rangle}^{W_m} (\chi_{\epsilon c}) (g) = |Z_{W_m}(g)| \sum_{g_1 \in \text{ccl}(g)} \tilde{\gamma}_{g_1},$$

where

$$\tilde{\gamma}_{g_1} = \begin{cases} \frac{1}{2m} \omega_m^{-j} & \text{if } g_1 = c^j \text{ for some } j \in [m] \\ -\frac{1}{2m} \omega_m^{-j} & \text{if } g_1 = w_0 c^j \text{ for some } j \in [m] \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\text{Ind}_{\langle d \rangle}^{W_m} (\chi_{\zeta d}) (g) = |Z_{W_m}(g)| \sum_{g_1 \in \text{ccl}(g)} \gamma_{g_1},$$

where

$$\gamma_{g_1} = \begin{cases} \frac{1}{2m} \omega_{2m}^{-j} & \text{if } g_1 = d^j \text{ for some } j \in [2m] \\ 0 & \text{otherwise}. \end{cases}$$

To continue, we need to know the signed cycle types of the elements that appear in the subgroups $\langle c, w_0 \rangle$ and $\langle d \rangle$ of $W_m$. The proof of the next lemma is straightforward and is omitted.

**Lemma 4.4.** Suppose that $j \in [m]$. Set $\ell = \gcd(m, j)$, $a = m/\ell$, and $b = j/\ell$.

1. The cycle type of $c^j$ is $(a^\ell)$, i.e., $\ell$-many cycles of length $a$.
2. The cycle type of $w_0 c^j$ is

$$\begin{cases} (a^\ell) & \text{if } a \text{ is even} \\ (\bar{a}^\ell) & \text{if } a \text{ is odd}, \end{cases}$$

where $(\bar{a}^\ell)$ denotes $\ell$-many negative $a$-cycles.
3. The cycle type of $d^j$ is

$$
\begin{cases}
(a^\ell) & \text{if } b \text{ is even} \\
(\overline{a}) & \text{if } b \text{ is odd.}
\end{cases}
$$

4. The cycle type of $d^{m+j}$ is

$$
\begin{cases}
(a^\ell) & \text{if } a \text{ is even} \\
(a^\ell) & \text{if } a \text{ is odd and } b \text{ is odd} \\
(\overline{a}) & \text{if } a \text{ is odd and } b \text{ is even.}
\end{cases}
$$

In particular, the cycle types that occur in $\langle c, w_0 \rangle$ and $\langle d \rangle$ are the cycle types $(a^\ell)$ and $(\overline{a})$, where $\ell$ divides $m$ and $a = m/\ell$.

4.5. With the computation of the cycle types and Lemma 4.2 in hand we can compute $\text{Ind}_{W_m}^{\langle c, w_0 \rangle} (\chi_{\epsilon \zeta c})$ and $\text{Ind}_{W_m}^{\langle d \rangle} (\chi_{\zeta d})$. The proof of Proposition 4.1 follows from the next lemma.

For $\lambda \in SP(m)$, denote the value of a character $\chi$ on the conjugacy class with cycle type $\lambda$ by $(\chi)_{\lambda}$.

**Lemma 4.6.** Suppose $\lambda \in SP(m)$. Then

$$
\left(\text{Ind}_{\langle c, w_0 \rangle}^{W_m} (\chi_{\epsilon \zeta c})\right)_{\lambda} = \left(\text{Ind}_{\langle d \rangle}^{W_m} (\chi_{\zeta d})\right)_{\lambda}
$$

$$
= \begin{cases}
\alpha^{\ell-1}(\ell - 1)! 2^{\ell-1} \mu(a) & \text{if } m = a\ell, a \text{ is odd, and } \lambda = (a^\ell) \\
-a^{\ell-1}(\ell - 1)! 2^{\ell-1} \mu(a) & \text{if } m = a\ell, a \text{ is odd, and } \lambda = (\overline{a}) \\
0 & \text{otherwise,}
\end{cases}
$$

where $\mu$ is the Möbius function.

**Proof.** The strategy of the proof is to compute the values of both induced characters and then observe that they are equal. We give the details for the character $\text{Ind}_{\langle d \rangle}^{W_m} (\chi_{\zeta d})$. The computation of the character values of $\text{Ind}_{\langle c, w_0 \rangle}^{W_m} (\chi_{\epsilon \zeta c})$ is similar (and easier), and is omitted.

Now consider $\left(\text{Ind}_{\langle d \rangle}^{W_m} (\chi_{\zeta d})\right)_{\lambda}$. It follows from Lemma 4.4 that if $\lambda$ is not equal to $(a^\ell)$ or $(\overline{a})$ for some factorization $m = a\ell$, then $\left(\text{Ind}_{\langle d \rangle}^{W_m} (\chi_{\zeta d})\right)_{\lambda} = 0$.

Suppose $m = a\ell$ and $\lambda = (a^\ell)$ or $(\overline{a})$. There are two cases depending on whether $a$ is even or odd.

First suppose that $a$ is even. By Lemma 4.4, no elements in $\langle d \rangle$ have signed cycle type $(a^\ell)$, and the elements with signed cycle type $(\overline{a})$ are the elements $d^j$ and $d^{m+j}$ with $j \in [m]$ and gcd$(m, j) = \ell$. Using the notation in 4.3, for each such $j$ we can pair the elements $d^j$ and $d^{m+j}$ to obtain

$$
\gamma_d^j + \gamma_{d^{m+j}} = (\omega_{2m}^{-j} + \omega_{2m}^{-(m+j)})/2m = (\omega_{2m}^{-j} - \omega_{2m}^{-j})/2m = 0.
$$
Therefore, \( \text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \) vanishes on the conjugacy classes of \( W_m \) with signed cycle type \((a^\ell)\) and \((\overline{a}^\ell)\) when \( a \) is odd.

Now suppose that \( m = a\ell \) and \( a \) is odd. There are four subcases.

First, the only element in \( \langle d \rangle \) with signed cycle type \( \lambda = (1^m) \) is the identity, and

\[
\left( \text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_{(1^m)} = |W_m|/2m = 2^{m-1}(m-1)! = a^{\ell-1}(\ell - 1)!2^{\ell-1}\mu(a),
\]

because \( \ell = m \) and \( \mu(1) = 1 \). Similarly, the only element in \( \langle d \rangle \) with signed cycle type \( \lambda = (\overline{1}^m) \) is \( w_0 = d\overline{m} \), and

\[
\left( \text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_{(\overline{1}^m)} = |W_m|/2m = -2^{m-1}(m-1)! = -a^{\ell-1}(\ell - 1)!2^{\ell-1}\mu(a).
\]

Now suppose that \( a > 1 \) and \( \lambda = (a^\ell) \). Set

\[
A = \{ j \in [m-1] \mid \gcd(m, j) = \ell \text{ and } j/\ell \text{ is even} \}.
\]

Because \( d^j \) and \( (d^j)^{-1} = d^{2m-j} \) are conjugate in \( W_m \), the set of elements in \( \langle d \rangle \) with signed cycle type \( (a^\ell) \) is \( \{ d^j, d^{2m-j} \mid j \in A \} \). Note that because \( m \) is even and \( a \) is odd, \( \ell = m/a \) is even, and therefore if \( j \in A \), then \( \ell \) divides \( j \) and so \( j \) is even. Thus, using the fact that \( c^\ell \) has signed cycle type \((a^\ell)\) we have

\[
\left( \text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_{(a^\ell)} = |Z_{W_m}(c^\ell)| \sum_{j \in A} (\omega_{2m}^{-j} + \omega_{2m}^{-(2m-j)})/2m
\]

\[
= a^{\ell-1}(\ell - 1)!2^{\ell-1} \sum_{j \in A} (\omega_{2m}^{-2(j/2)} + \omega_{2m}^{-(2m-2(j/2))})
\]

\[
= a^{\ell-1}(\ell - 1)!2^{\ell-1} \sum_{k \in A/2} (\omega_m^{-k} + \omega_m^{-(m-k)}).
\]

Observe that with our assumptions on \( m \) and \( a \), if \( k = j/2 \in A/2 \), then \( \gcd(m, k) = \gcd(m, j) = \ell \). Moreover, \( \gcd(m, k) = \gcd(m, m-k) \), so \( \gcd(m, m-k) = \ell \). Therefore,

\[
\{ k, m-k \mid k \in A/2 \} = \{ h \in [m] \mid \gcd(m, h) = \ell \},
\]

and so letting \( k' = h/\ell \), and using that \( m/\ell = a \), we have

\[
\{ h \in [m] \mid \gcd(m, h) = \ell \} = \{ \ell k' \mid k' \in [a] \text{ and } \gcd(a, k') = 1 \}.
\]

Thus

\[
\sum_{k \in A/2} (\omega_m^{-k} + \omega_m^{-(m-k)}) = \sum_{k \in [m] \atop \gcd(m, h) = \ell} \omega_m^{-h} = \sum_{k' \in [a] \atop \gcd(a, k') = 1} \omega_m^{-\ell k'} = \sum_{k' \in [a] \atop \gcd(a, k') = 1} \omega_a^{-k'} = \mu(a),
\]

where the last equality holds because the sum is over all primitive \( a \)th roots of unity. Substituting \( b \) in \( a \) gives

\[
\left( \text{Ind}_{\langle d \rangle}^{W_m}(\chi_{\zeta_d}) \right)_{(a^\ell)} = a^{\ell-1}(\ell - 1)!2^{\ell-1}\mu(a).
\]
Finally suppose that \( a > 1 \) and \( \lambda = (\pi^a) \). Set

\[ B = \{ j \in [m - 1] \mid \gcd(m, j) = \ell \text{ and } j/\ell \text{ is odd} \}. \]

Then the set of elements in \( \langle d \rangle \) with signed cycle type \( (\pi^\ell) \) is \( \{ d^j, d^{2m-j} \mid j \in B \} \), and because \( w_0 c^\ell \) has signed cycle type \( (\pi^\ell) \), we have

\[
\left( \text{Ind}_{(d)}^{W_m}(\chi_{\zeta_d}) \right)(\pi^\ell) = |Z_{W_m}(w_0 c^\ell)| \sum_{j \in B} (\omega_{2m}^{-j} + \omega_{2m}^{-(2m-j)})/2m
= a^{\ell-1}(\ell - 1)! 2^{\ell-1} \sum_{j \in B} (\omega_{2m}^{-j} + \omega_{2m}^{-(2m-j)}).
\]

Again, if \( j \in B \), then \( j \) is even. Hence \( m + j \) and \( m - j \) are both even, so

\[
\omega_{2m}^{-j} + \omega_{2m}^{-(2m-j)} = \omega_{2m}^m(\omega_{2m}^{-j} + \omega_{2m}^{m+j}) = -(\omega_{2m}^{-j} + \omega_{2m}^{m+j}) = -(\omega_{m}^{-j} + \omega_{m}^{m+j}).
\]

We now show that if \( j \in B \), then \( \ell \) divides \( m + j \)/2 and \( m - j \)/2, from which it follows that \( \gcd(m, (m \pm j)/2) = \ell \), and hence that

\[
\{ (m \pm j)/2 \mid j \in B \} = \{ h \in [m] \mid \gcd(m, h) = \ell \}.
\]

Say \( m = 2^{k_m} q_m, \ell = 2^{k_\ell} q_\ell \), and \( j = 2^{k_j} q_j \), where \( q_m, q_\ell, \) and \( q_j \) are all odd. Since \( \ell | m \) and \( \ell | j \), and \( m/\ell = a \) and \( j/\ell \) are odd, it must be that \( k_m = k_\ell \) and \( k_j = k_\ell \). Set \( k = k_\ell = k_j = k_m \). Then \( m \pm j = 2^k (q_m \pm q_j) \). Because \( q_m \) and \( q_j \) are odd, \( q_m \pm q_j \) is even. Hence \( 2^k \) divides \( (m \pm j)/2 \). Moreover, \( q_\ell \) divides \( (q_m \pm q_j)/2 \) because \( q_m, q_\ell, \) and \( q_j \) are all odd. Hence \( 2^k q_\ell = \ell \) divides \( (m \pm j)/2 \).

Now, arguing as in (b) we have

\[
\sum_{j \in B} (\omega_{2m}^{-j} + \omega_{2m}^{-(2m-j)}) = -\sum_{j \in B} (\omega_{m}^{-j} + \omega_{m}^{m+j}) = -\sum_{h \in [m]} \omega_{-m}^h = -\mu(a),
\]

and so \( \left( \text{Ind}_{(d)}^{W_m}(\chi_{\zeta_d}) \right)(\pi^\ell) = -a^{\ell-1}(\ell - 1)! 2^{\ell-1}\mu(a) \). This completes the proof of the lemma. \( \square \)

5. Proof of Theorem 2.10

Throughout this section,

\[ \lambda = (\lambda_1, \ldots, \lambda_a, \lambda_{a+1}, \ldots, \lambda_{a+b}) \]

is a signed partition of \( n \) and we use the notation and conventions introduced in 2.1.
5.1. To show that $Z_{W_n}(w_\lambda)$ acts on $\tilde{e}_\lambda$ on the right as scalars we compute the action of the generators of $Z_{W_n}(w_\lambda)$. Recall from 2.8 that $Z_{W_n}(w_\lambda)$ is generated by
\[
\{ c_i, w_{0,\lambda}, | i \in [a]\} \Pi \{ d_i \mid i \in [a+b] \setminus [a] \} \Pi \{ y_i \mid \lambda_i = \lambda_{i+1}, i \in [a+b-1] \},
\]
and from 2.9 that $\tilde{e}_\lambda = x_{|\lambda|} f_1 \cdots f_{a+b}$, where
\[
f_i = \begin{cases} 
\epsilon^+_\lambda \zeta_{c_i} & \text{if } i \in [a] \\
\epsilon^-_\lambda \zeta_{c_i} & \text{if } i \in [a+b] \setminus [a] \text{ and } \lambda_i \text{ is odd} \\
\zeta_{d_i} & \text{if } i \in [a+b] \setminus [a] \text{ and } \lambda_i \text{ is even.}
\end{cases}
\]
In the computations below we repeatedly use the fact that $W_{\lambda_i}$ and $W_{\lambda_j}$ commute elementwise whenever $i \neq j$.

We first show that $\tilde{e}_\lambda \cdot c_i = \omega_{|c_i|} \tilde{e}_\lambda$ for $i \in [a]$: This equality is clear because $f_i = \epsilon^+_\lambda \zeta_{c_i}$ and so
\[
f_i \cdot c_i = \epsilon^+_\lambda \zeta_{c_i} \cdot c_i = \omega_{|c_i|} \epsilon^+_\lambda \zeta_{c_i} = \omega_{|c_i|} f_i.
\]
Thus
\[
\tilde{e}_\lambda \cdot c_i = x_{|\lambda|} f_1 \cdots (f_{c_i}) \cdots f_{a+b} = \omega_{|c_i|} x_{|\lambda|} f_1 \cdots f_{a+b} = \omega_{|c_i|} \tilde{e}_\lambda.
\]

Next, we show that $\tilde{e}_\lambda \cdot w_{0,\lambda_i} = \tilde{e}_\lambda$ for $i \in [a]$: In this case $f_i = \epsilon^+_\lambda \zeta_{c_i} = (1/2)(\text{id} + w_{0,\lambda_i}) \zeta_{c_i}$. Therefore,
\[
\epsilon^+_\lambda \zeta_{c_i} \cdot w_{0,\lambda_i} = (\epsilon^+_\lambda \cdot w_{0,\lambda_i}) \cdot \zeta_{c_i} = (1/2)(w_{0,\lambda_i} + \text{id}) \cdot \zeta_{c_i} = \epsilon^+_\lambda \zeta_{c_i},
\]
and the result follows.

Now, we show that $\tilde{e}_\lambda \cdot d_i = \omega_{|d_i|} \tilde{e}_\lambda$ for $i \in [a+b] \setminus [a]$: There are two cases depending on whether $\lambda_i$ is odd or even. If $\lambda_i$ is odd, then $d_i = c_i w_{0,\lambda_i}$ and $f_i = \epsilon^+_\lambda \zeta_{c_i} = (1/2)(\text{id} - w_{0,\lambda_i}) \zeta_{c_i}$. Therefore
\[
f_i \cdot d_i = (1/2)(\text{id} - w_{0,\lambda_i}) \zeta_{c_i} \cdot c_i w_{0,\lambda_i} = \left((1/2)(\text{id} - w_{0,\lambda_i}) \cdot w_{0,\lambda_i}\right) \zeta_{c_i} = \left(-\omega_{|\lambda_i|+1}/2 \epsilon^-_\lambda \zeta_{c_i} = \omega_{|d_i|} f_i.
\]
On the other hand, if $\lambda_i$ is even, then $f_i = \zeta_{d_i}$ and the result follows as in the computation of $\tilde{e}_\lambda \cdot c_i$.

Last, we show that $\tilde{e}_\lambda \cdot y_i = \tilde{e}_\lambda$ for $i \in [a+b-1]$ with $\lambda_i = \lambda_{i+1}$: Recall that $y_i|\lambda_i$ is the identity for $l \neq i, i+1$ and that the restriction of $y_i$ to $\Lambda_i$ defines the unique order preserving bijection between $\Lambda_i$ and $\Lambda_{i+1}$. In particular, $y_i$ is an involution in $S_n$ such that $y_i c_i y_i = c_i + 1$, $y_i w_{0,\lambda_i} y_i = w_{0,\lambda_i+1}$, and $y_i d_i y_i = d_i + 1$. It is straightforward to check that $y_i f_i y_i = f_{i+1}$, and so
\[
\tilde{e}_\lambda \cdot y_i = x_{|\lambda|} f_1 \cdots f_i f_{i+1} y_i \cdots f_{a+b} = x_{|\lambda|} f_1 \cdots y_i f_{i+1} y_i \cdots f_{a+b} = x_{|\lambda|} y_i f_1 \cdots f_{a+b}.
\]
To complete the computation we need to show that $x_{|\lambda|} y_i = x_{|\lambda|}$. To see this, recall that $x_{|\lambda|} = \sum_{w \in X_{|\lambda|}} w$. Thus, it suffices to show that $X_{|\lambda|} y_i = X_{|\lambda|}$. As in 3.4, if
w ∈ W_n, then w ∈ X_{|\lambda|} if and only if w(j) > 0 for j ∈ [n] and w|_{|\lambda|}: \Lambda_l → [n] is increasing for l ∈ [a + b]. It is easy to see that if w ∈ X_{|\lambda|}, then wy_l(j) > 0 for j ∈ [n] and wy_l|_{|\lambda|}: \Lambda_l → [n] is increasing for l ∈ [a + b], so wy_l ∈ X_{|\lambda|}. Therefore X_{|\lambda|}y_l = X_{|\lambda|}, and so x|_{|\lambda|}y_l = x|_{|\lambda|}.

5.2. To complete the proof of Theorem 2.10, it remains to show that there is an isomorphism of right $\mathbb{C}W_n$-modules

$$E_\lambda \mathbb{C}W_n \cong \text{Ind}_{Z_{W_n}(\omega)}^{W_n}(\mathbb{C}e_\lambda).$$

Recall the idempotents $E_\lambda$ and the quasi-idempotents $e_p$ from 2.4. Notice that $e_\lambda$ is one of the summands in the definition of $E_\lambda$. The strategy is to show that

(a) $$E_\lambda \mathbb{C}W_n = e_\lambda \mathbb{C}W_n \cong e_\lambda \mathbb{C}W_n \cong \text{Ind}_{Z_{W_n}(\omega)}^{W_n}(\mathbb{C}e_\lambda).$$

Lemma 5.3. With the preceding notation,

$$E_\lambda \mathbb{C}W_n = e_\lambda \mathbb{C}W_n.$$

Proof. It follows from Corollary 2.6 that $E_\lambda e_\lambda = e_\lambda$ and $e_\lambda E_\lambda = |\text{Stab}(\lambda)| E_\lambda$, so

$$e_\lambda \mathbb{C}W_n = E_\lambda e_\lambda \mathbb{C}W_n \subseteq E_\lambda \mathbb{C}W_n, \quad \text{and} \quad E_\lambda \mathbb{C}W_n = e_\lambda E_\lambda \mathbb{C}W_n \subseteq e_\lambda \mathbb{C}W_n.$$

The next lemma is well-known and easy to prove.

Lemma 5.4. Suppose $e$ and $f$ are idempotents in a ring $A$.

(1) If $ef = f$ and $fe = e$, then $eA = fA$.

(2) If $ef = e$ and $fe = f$, then $eA \cong fA$ as right ideals.

Lemma 5.5. There is an isomorphism of right ideals

$$e_\lambda \mathbb{C}W_n \cong e_\lambda \mathbb{C}W_n.$$

Proof. In this proof we use the theory of Lie idempotents (see [13, §8.4]).

Suppose first that $m$ is any positive integer. Then by [13, Theorem 8.16, Theorem 8.17], the Reutenauer idempotent $r_m$ and the Klyachko idempotent $\kappa_m$ are both Lie idempotents in the group algebra $\mathbb{C}S_m$. Say $r_m = \sum_{x \in S_m} a_xx$ and $\kappa_m = \sum_{x \in S_m} \alpha_x x$.

With the notation of Lemma 4.2,

$$\chi_{r_m}(w) = |Z_{S_m}(w)| \sum_{x \in \text{ccl}(w)} a_x \quad \text{and} \quad \chi_\kappa(w) = |Z_{S_m}(w)| \sum_{x \in \text{ccl}(w)} \alpha_x$$

for $w \in W$. Garsia [8, Proposition 5.1] has shown that $\sum_{x \in \text{ccl}(w)} a_x = \sum_{x \in \text{ccl}(w)} \alpha_x$, which implies that $\chi_{r_m} = \chi_\kappa$, and hence that $r_m \mathbb{C}S_m \cong \kappa_m \mathbb{C}S_m$. In addition, Reutenauer [13, Lemma 8.19] has shown that if $c$ is the $m$-cycle $(1 \ 2 \ \cdots \ m)$, $\omega$ is any primitive $m^{th}$ root of unity (not necessarily $\omega_m$), and $\zeta_c^l = \frac{1}{m} \sum_{j=1}^m \omega^{-jc}$, then $\zeta_c^l \kappa = \kappa$, and $\kappa \zeta_c^l = \zeta_c^l$. Thus by Lemma 5.4 $\kappa_m \mathbb{C}S_m = \zeta_c^l \mathbb{C}S_m$, which implies that

(a) $$r_m \mathbb{C}S_m \cong \zeta_c^l \mathbb{C}S_m.$$
Suppose \( i \in [a+b] \) and consider the right ideals \( \epsilon_{\lambda_i}^+ r_{\lambda_i} \mathcal{C}W_{\lambda_i} \) and \( f_i \mathcal{C}W_{\lambda_i} \) in \( \mathcal{C}W_{\lambda_i} \). If \( i \in [a] \), then \( f_i = \epsilon_{\lambda_i}^+ \zeta_{c_i} \), and it follows from (a) that \( \epsilon_{\lambda_i}^+ r_{\lambda_i} \mathcal{C}W_{\lambda_i} \cong \epsilon_{\lambda_i}^+ \zeta_{c_i} \mathcal{C}W_{\lambda_i} = f_i \mathcal{C}W_{\lambda_i} \). If \( i \in [a+b] \setminus [a] \) and \( \lambda_i \) is odd, then \( f_i = \epsilon_{\lambda_i}^+ \zeta_{c_i} \), and it again follows from (a) that \( \epsilon_{\lambda_i}^+ r_{\lambda_i} \mathcal{C}W_{\lambda_i} \cong \epsilon_{\lambda_i}^+ \zeta_{c_i} \mathcal{C}W_{\lambda_i} = f_i \mathcal{C}W_{\lambda_i} \). Finally, if \( i \in [a+b] \setminus [a] \) and \( \lambda_i \) is even, then \( f_i = \zeta_{d_i} \), and it follows from (a) and Proposition 4.1 that \( \epsilon_{\lambda_i}^+ r_{\lambda_i} \mathcal{C}W_{\lambda_i} \cong \epsilon_{\lambda_i}^+ \zeta_{c_i} \mathcal{C}W_{\lambda_i} = f_i \mathcal{C}W_{\lambda_i} \). Thus
\[
\epsilon_{\lambda_i}^+ r_{\lambda_i} \mathcal{C}W_{\lambda_i} \cong f_i \mathcal{C}W_{\lambda_i}
\]
for all \( i \in [a+b] \).

To complete the proof we use (b) to compute
\[
e_{\lambda} \mathcal{C}W_n = x_{[a]} \epsilon_{\lambda_1}^+ r_{\lambda_1} \cdots \epsilon_{\lambda_{a+b}}^+ r_{\lambda_{a+b}} \mathcal{C}W_n
= x_{[a]} (\epsilon_{\lambda_1}^+ r_{\lambda_1} \mathcal{C}W_{\lambda_1}) \cdots (\epsilon_{\lambda_{a+b}}^+ r_{\lambda_{a+b}} \mathcal{C}W_{\lambda_{a+b}}) \cdot \mathcal{C}W_n
\cong x_{[a]} (f_1 \mathcal{C}W_{\lambda_1}) \cdots (f_{a+b} \mathcal{C}W_{\lambda_{a+b}}) \cdot \mathcal{C}W_n
= e_{\lambda} \mathcal{C}W_n.
\]

The last isomorphism in 5.2(a) follows from the next lemma.

**Lemma 5.6.** The multiplication map \( \mathbb{C}e_{\lambda} \otimes_{\mathbb{C}W_n(w_\lambda)} \mathcal{C}W_n \to e_{\lambda} \mathcal{C}W_n \) is an isomorphism of right \( \mathcal{C}W_n \)-modules.

**Proof.** The mapping is obviously \( \mathcal{C}W_n \)-linear and surjective, so
\[
(a) \quad \dim (e_{\lambda} \mathcal{C}W_n) \leq \dim \left( \mathbb{C}e_{\lambda} \otimes_{\mathbb{C}W_n(w_\lambda)} \mathcal{C}W_n \right) = |W_n|/|Z_{W_n}(w_\lambda)|.
\]

Now, using the decomposition 2.7(a) and the isomorphism \( E_\mu \mathcal{C}W_n \cong e_\mu \mathcal{C}W_n \) from Lemma 5.3 and Lemma 5.5, we have
\[
\dim \mathcal{C}W_n = \sum_{\mu \in \mathcal{SP}(n)} \dim E_\mu \mathcal{C}W_n = \sum_{\mu \in \mathcal{SP}(n)} \dim e_\mu \mathcal{C}W_n
\leq \sum_{\mu \in \mathcal{SP}(n)} |W_n|/|Z_{W_n}(w_\mu)| = |W_n|,
\]
and so it follows from (a) that \( \dim e_\mu \mathcal{C}W_n = |W_n|/|Z_{W_n}(w_\mu)| \) for \( \mu \in \mathcal{SP}(n) \). Therefore \( \dim e_{\lambda} \mathcal{C}W_n = \dim \left( \mathbb{C}e_{\lambda} \otimes_{\mathbb{C}W_n(w_\lambda)} \mathcal{C}W_n \right) \), and it follows that the multiplication map in the statement of the lemma is an isomorphism as claimed. \( \square \)

6. **Computing \( \theta_n(E_\lambda) \)**

Recall from 2.12 that \( \text{cf}_C(W_n) \) denotes the algebra of \( \mathbb{C} \)-valued class functions on \( W_n \), and that for \( \mu \in \mathcal{SP}(n) \), \( u_\mu \) is the characteristic function for \( \text{ccl}(w_\mu) \). Recall also the surjective algebra homomorphism \( \theta_n : \Sigma(W_n) \to \text{cf}_C(W_n) \) with kernel equal to the Jacobson radical of \( \Sigma(W_n) \). In this section we prove Theorem 2.13:
Suppose $\lambda \in \mathcal{SP}(n)$. Then $\theta_n(E_\lambda) = u_{\lambda'}$, where $\lambda'$ is the signed composition of $n$ defined by

$$
\lambda'_i = \begin{cases} 
\lambda_i & \text{if } \lambda_i \text{ is odd} \\
-x_i & \text{if } \lambda_i \text{ is even.}
\end{cases}
$$

The proof requires several preliminary results. To begin, as observed in 2.12, $\{\theta_n(E_\mu) \mid \mu \in \mathcal{SP}(n)\} = \{u_\mu \mid \mu \in \mathcal{SP}(n)\}$ is the basis of $\text{cf}_C(W_n)$ formed by the primitive idempotents because $\theta_n$ identifies $\text{cf}_C(W_n)$ with the semisimple quotient of $\Sigma(W_n)$ by its Jacobson radical and $\text{cf}_C(W_n)$ is a commutative algebra. Thus, $\theta_n(E_\lambda) = u_\mu$ for some $\mu$. The first reduction is to replace $E_\lambda$ by $e_\lambda$.

**Lemma 6.1.** With the preceding notation, $\theta_n(E_\lambda) = |\text{Stab}(\lambda)|^{-1}\theta_n(e_\lambda)$.

**Proof.** It follows from the definition of $E_\lambda$ in 2.7, and Corollary 2.6, that $e_\lambda = E_\lambda e_\lambda$ and $e_\lambda E_\lambda = |\text{Stab}(\lambda)| E_\lambda$. Therefore,

$$
\theta_n(e_\lambda) = \theta_n(E_\lambda e_\lambda) = \theta_n(E_\lambda) \theta_n(e_\lambda) = \theta_n(e_\lambda) \theta_n(E_\lambda) = \theta_n(e_\lambda E_\lambda) = |\text{Stab}(\lambda)| \theta_n(E_\lambda).
$$

6.2. If $\chi \in \text{cf}_C(W_n)$, then $\chi = \sum_{\mu \in \mathcal{SP}(n)} |Z_{W_n}(w_\mu)| \langle \chi, u_\mu \rangle_{W_n} \cdot u_\mu$, where $\langle \cdot, \cdot \rangle_{W_n}$ is the usual inner product on $\text{cf}_C(W_n)$, so to prove the theorem it is enough to compute $\langle \theta_n(e_\lambda), u_\mu \rangle_{W_n}$ for $\mu \in \mathcal{SP}(n)$. More generally, in Proposition 6.7 we give an explicit formula for $\langle \theta_n(e_\mu), u_\mu \rangle_{W_n}$. The first step is to give a formula for $\theta_m(\epsilon_m r_m)$.

**Proposition 6.3.** Let $m$ be a positive integer. Then

(a) $$\theta_m(r_m) = u_{(m)} + u_{(\overline{m})},$$

and

(b) $$\theta_m(\epsilon_m r_m) = \begin{cases} u_{(m)} & \text{if } m \text{ is odd} \\
u_{(\overline{m})} & \text{if } m \text{ is even} \end{cases} \quad \text{and} \quad \theta_m(\epsilon_m r_m) = \begin{cases} u_{(m)} & \text{if } m \text{ is even} \\
u_{(\overline{m})} & \text{if } m \text{ is odd.} \end{cases}$$

**Proof.** Recall that $r_m$ lies in the descent algebra of $S_m$ and that there is a surjective algebra homomorphism $\theta_{S_m}$ from the descent algebra of $S_m$ to the algebra of class functions $\text{cf}_C(S_m)$. It follows from the results in [9, §3] that $\theta_{S_m}(r_m) = u_{S_m}^{S_m}$, where $u_{S_m}^{S_m}$ is the characteristic function of the conjugacy class of $m$-cycles in $S_m$.

By [4, (3.4)], $\theta_m(r_m)$ is the lift to $\text{cf}_C(W_m)$ of $u_{S_m}^{S_m}$, so if $\pi: W_m \to S_m$ is the projection with kernel equal to $T = \langle t_1, \ldots, t_m \rangle$, then $\theta_m(r_m) = u_{S_m}^{S_m} \circ \pi$. For $w \in S_m$ and $t \in T$, $\pi(wt) = w$, so $u_{S_m}^{S_m} \circ \pi(wt) = 0$ unless $w$ is an $m$-cycle, in which case $u_{S_m}^{S_m} \circ \pi(wt) = 1$. On the other hand, if $w$ is an $m$-cycle and $t \in T$, then $wt$ has signed cycle type $(m)$ or $(\overline{m})$. It follows that $\theta_m(r_m) = u_{(m)} + u_{(\overline{m})}$. 
Now, let $\varepsilon_m$ denote the sign character of $W_m$. By [4, Example 3.5], $\theta_m(u_{0,m}) = \varepsilon_m$. Then, using (a) and the fact that $\theta_m$ is an algebra homomorphism, we have that
\[
\theta_m(\varepsilon^\pm_m r_m) = (1/2)(\text{id} \pm \varepsilon_m)(u(m) + u(\overline{m})) = (1/2)(u(m) + u(\overline{m}) \pm \varepsilon_m u(m) \pm \varepsilon_m u(\overline{m})).
\]

One checks that
\[
\varepsilon_m u(m) = \begin{cases} u(m) & \text{if } m \text{ is odd} \\ -u(m) & \text{if } m \text{ is even} \end{cases}
\]
and
\[
\varepsilon_m u(\overline{m}) = \begin{cases} u(\overline{m}) & \text{if } m \text{ is even} \\ -u(\overline{m}) & \text{if } m \text{ is odd}. \end{cases}
\]
Hence, if $m$ is odd, then
\[
\theta_m(\varepsilon^+_m r_m) = (1/2)(u(m) + u(\overline{m}) \pm u(m) \mp u(\overline{m})) = \begin{cases} u(m) & \text{for } \varepsilon^+_m \\ u(\overline{m}) & \text{for } \varepsilon^-_m, \end{cases}
\]
and if $m$ is even, then
\[
\theta_m(\varepsilon^-_m r_m) = (1/2)(u(m) + u(\overline{m}) \mp u(m) \pm u(\overline{m})) = \begin{cases} u(m) & \text{for } \varepsilon^-_m \\ u(\overline{m}) & \text{for } \varepsilon^+_m. \end{cases}
\]

The formulas in (b) follow from (c) and (d). \hfill \Box

6.4. In this subsection and the next, $p = (p_1, \ldots, p_k)$ denotes a fixed signed composition of $n$. Our goal is to compute $\langle \theta_\mu(e_p), u_\mu \rangle_{W_n}$ for $\mu \in \mathcal{SP}(n)$. Recall that $e_p = x_{[|p|]} \varepsilon_{P_1}^1 r_{P_1} \cdots \varepsilon_{P_k}^k r_{P_k}$, where $\varepsilon_i$ is the sign of $p_i$,
\[
\varepsilon_{P_1}^1 r_{P_1} \cdots \varepsilon_{P_k}^k r_{P_k} \in \mathcal{C}W_{[p]}, \quad W_{[p]} = W_{P_1} \cdots W_{P_k} \cong W_{|p_1|} \times \cdots \times W_{|p_k|},
\]
and $W_{|p_i|} \cong W_{P_i} \subseteq W_n$.

For $i \in [k]$ define an isomorphism $f_i : W_{|p_i|} \cong W_{P_i}$ by
\[
(f_i(w))(l) = \begin{cases} \hat{P}_{i-1} + w(l - \hat{P}_{i-1}) & \text{if } l \in P_i, \ w(l - \hat{P}_{i-1}) > 0 \\ \hat{P}_{i-1} + w(l - \hat{P}_{i-1}) & \text{if } l \in P_i, \ w(l - \hat{P}_{i-1}) < 0 \\ l & \text{otherwise} \end{cases}
\]
\[
\text{for } w \in W_{|p_i|} \text{ and } l \in [p_i]. \text{ Then } f_i(w) \text{ is the identity on } [n] \setminus P_i, \text{ and the restriction of } f_i(w) \text{ to } P_i \text{ is the translation of } w \text{ from a map } [p_i] \to \pm [p_i] \text{ to a map } P_i \to \pm P_i.
\]

The embeddings $f_1, \ldots, f_k$ define a group isomorphism
\[
f = f_1 \times \cdots \times f_k : W_{|p_1|} \times \cdots \times W_{|p_k|} \cong W_{[p]},
\]
and an algebra isomorphism (also denoted by $f$)
\[
\text{(a)} \quad f : \mathcal{C}W_{[p_1]} \otimes \cdots \otimes \mathcal{C}W_{[p_k]} \cong \mathcal{C}W_{[p]}.
\]
Because $W_{[p]}$ is the internal product $W_{P_1} \cdots W_{P_k}$, the isomorphism in (a) restricts to an isomorphism (still denoted by $f$)
\[
f : \text{cf}_C(W_{[p_1]}) \otimes \cdots \otimes \text{cf}_C(W_{[p_k]}) \cong \text{cf}_C(W_{[p]}).
\]
Notice that
\[ f(\eta_1 \otimes \cdots \otimes \eta_k) = \eta_1 f_1^{-1} \otimes \cdots \otimes \eta_k f_k^{-1} \in \cf(C(W_{[\ell]})), \]
where \( (\phi_1 \otimes \cdots \otimes \phi_k)(v_1 \cdots v_k) = \phi_1(v_1) \cdots \phi_k(v_k) \) for \( i \in [k], \eta_i \in \cf(C(W_{[\ell]})), \phi_i \in \cf(C(W_{P_i})), \) and \( v_i \in W_{P_i}. \)

6.5. For \( q \in SC(n) \) with \( W_q \subseteq W_{[\ell]}, \) set \( X_q^{[\ell]} = X_q \cap W_{[\ell]} \), and let \( x_q^{[\ell]} = \sum_{w \in X_q^{[\ell]}} w. \)

Bonnafé and Hohlweg [4, Section 3.1] define
\[ \Sigma'(W_{[\ell]}) = \text{span}\{x_q^{[\ell]} \mid q \in SC(n), W_q \subseteq W_{[\ell]} \}. \]

They show that \( \{x_q^{[\ell]} \mid q \in SC(n), W_q \subseteq W_{[\ell]} \} \) is a basis of \( \Sigma'(W_{[\ell]}), \) that \( \Sigma'(W_{[\ell]}) \) is a subalgebra of \( \mathbb{C}W_n, \) and that there is an algebra homomorphism
\[ \theta_{[\ell]} : \Sigma'(W_{[\ell]}) \to \cf(C(W_{[\ell]})) \]
with the same properties as \( \theta_n. \) (More generally, Bonnafé and Hohlweg consider subalgebras \( \Sigma'(W_p) \) and homomorphisms \( \theta_p, \) and \( \Sigma(W_n) \) and \( \theta_n \) are defined as the special case when \( p = (n). \))

**Lemma 6.6.** With the preceding notation,
\[ f(\Sigma(W_{[p_1]}) \otimes \cdots \otimes \Sigma(W_{[p_k]})) = \Sigma'(W_{[\ell]}), \]
and the diagram
\[
\begin{array}{ccc}
\Sigma(W_{[p_1]}) \otimes \cdots \otimes \Sigma(W_{[p_k]}) & \xrightarrow{f} & \Sigma'(W_{[\ell]}) \\
\text{cf}_C(W_{[p_1]}) \otimes \cdots \otimes \text{cf}_C(W_{[p_k]}) & \xrightarrow{f} & \text{cf}_C(W_{[\ell]}) \\
(a) & & (a)
\end{array}
\]
commutes.

**Proof.** Suppose that \( q^i \) is a signed composition of \( [p_i] \) for \( i \in [k] \) and that \( q \) is the concatenation of \( q^1, \ldots, q^k. \) Then \( q \in SC(n) \) and \( X_q \subseteq W_{[\ell]} \). Straightforward computations using [4, Remark 2.1] and the definitions show that
\[ f_i(X_{q^i}) = X_{q_i}^{[\ell]}, \text{ where } q^i = ([p_1], \ldots, [p_{i-1}], q^i, [p_{i+1}], \ldots, [p_k]) \in SC(n), \]
and that
\[ X_{q_1}^{[\ell]} \cdots X_{q_k}^{[\ell]} = X_q^{[\ell]}. \]

It follows from (b) and (c) that
\[ f(x_{q_1} \otimes \cdots \otimes x_{q_k}) = x_{q_1}^{[\ell]} \cdots x_{q_k}^{[\ell]} = x_q^{[\ell]}. \]

One checks that the rule \( (q^1, \ldots, q^k) \mapsto q \) defines a bijection
\[ SC([p_1]) \times \cdots \times SC([p_k]) \leftrightarrow \{ q \in SC(n) \mid W_q \subseteq W_{[\ell]} \}, \]
and so \( f \) maps the basis
\[
\{ x_q^i \otimes \cdots \otimes x_q^k \mid \forall i \in [k], \ q^i \in \mathcal{SC}(|p_i|) \}
\]

of \( \Sigma(W_{|p_1|}) \otimes \cdots \otimes \Sigma(W_{|p_k|}) \)
to the basis
\[
\{ x_q^{|p|} \mid q \in \mathcal{SC}(n), \ W_q \subseteq W_{|p|} \}
\]

of \( \Sigma'(|p|) \).

Therefore
\[
f(\Sigma(W_{|p_1|}) \otimes \cdots \otimes \Sigma(W_{|p_k|})) = \Sigma'(|p|)
\]
as claimed.

Finally,
\[
\theta_{|p|}(f(x_q^1 \otimes \cdots \otimes x_q^k)) = \theta_{|p|}(x_q^{|p|})
\]

\[
= \text{Ind}_{W^{|p|}}(1_{W_q})
\]

\[
= \text{Ind}_{f_1(W^{|p|})}W_{|p|}^1 f_2 \cdots \text{Ind}_{f_k(W^{|p|})}W_{|p|}^k
\]

\[
= \text{Ind}_{W^{|p|}}(1_{W_q}) f_1^{-1} \cdots \text{Ind}_{W^{|p|}}(1_{W_q}) f_k^{-1}
\]

\[
f(\text{Ind}_{W^{|p|}}(1_{W_q}) \otimes \cdots \otimes \text{Ind}_{W^{|p|}}(1_{W_q}))
\]

\[
= f(\theta_{|p|}(x_q^1) \otimes \cdots \otimes \theta_{|p|}(x_q^k))
\]

\[
= f((\theta_{|p|} \otimes \cdots \otimes \theta_{|p|})(x_q^1 \otimes \cdots \otimes x_q^k))
\]

and it follows that (a) commutes. \( \square \)

We can now give a formula for \( \langle \theta_n(e_p), u_\mu \rangle_{W_n} \).

**Proposition 6.7.** Suppose \( p = (p_1, \ldots, p_k) \) is a signed composition of \( n \) and \( \mu \) is a signed partition of \( n \). Then
\[
\langle \theta_n(e_p), u_\mu \rangle_{W_n} = \begin{cases} 2^{-k|p_1| \cdots |p_k|} & \text{if } \mu = \frac{p}{p'} \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** Consider the diagram
\[
\begin{array}{ccc}
\Sigma(W_{|p_1|}) \otimes \cdots \otimes \Sigma(W_{|p_k|}) & \xrightarrow{f} & \Sigma'(W_{|p|}) \xrightarrow{x_{|p|}^*} \Sigma(W_{n}) \\
\theta_{|p_1|} \otimes \cdots \otimes \theta_{|p_k|} & \equiv & \theta_{|p|} \\
\text{cf}_C(W_{|p_1|}) \otimes \cdots \otimes \text{cf}_C(W_{|p_k|}) & \xrightarrow{f} & \text{cf}_C(W_{|p|}) \xrightarrow{\text{Ind}_{W_{|p|}}^{W_n}} \text{cf}_C(W_{n}),
\end{array}
\]

where \( x_{|p|}^* \) denotes left multiplication by \( x_{|p|} \). It was shown in Lemma 6.6 that the left square commutes and it is shown in [4, Section 3.2] that the right square commutes, so (a) is a commutative diagram.

Using the commutativity of the right square we have
\[
\theta_n(e_p) = \theta_n(x_{|p|}^{\xi_1} r_{p_1} \cdots \xi_k r_{p_k}) = \text{Ind}_{W_{|p|}}^{W_n}(\theta_{|p|}(\epsilon_{p_1} r_{p_1} \cdots \epsilon_{p_k} r_{p_k})
\]

(b) \( \theta_n(e_p) = \theta_n(x_{|p|}^{\xi_1} r_{p_1} \cdots \xi_k r_{p_k}) = \text{Ind}_{W_{|p|}}^{W_n}(\theta_{|p|}(\epsilon_{p_1} r_{p_1} \cdots \epsilon_{p_k} r_{p_k})) \),
and using the commutativity of the left square we have

\[
\theta_{[p]}(\epsilon^1_{P_1} r_{P_1} \cdots \epsilon^k_{P_k} r_{P_k}) = \theta_{[p]}(f(\epsilon^1_{[p]} r_{[p]} \otimes \cdots \otimes \epsilon^k_{[p]} r_{[p]}))
\]

\[
= f\left(\theta_{[p]}(\epsilon^1_{[p]} r_{[p]} \otimes \cdots \otimes \epsilon^k_{[p]} r_{[p]})\right)
\]

\[
= \theta_{[p]}(\epsilon^1_{[p]} r_{[p]} f^{-1}_1 \boxtimes \cdots \boxtimes \theta_{[p]}(\epsilon^k_{[p]} r_{[p]})) f^{-1}_k.
\]

Using (b), Frobenius reciprocity, and (c) gives

\[
\langle \theta_n(e_p), u_\mu \rangle_w = \langle \text{Ind}_{W_{[p]}}^{W_n}(\theta_{[p]}(\epsilon^1_{P_1} r_{P_1} \cdots \epsilon^k_{P_k} r_{P_k})), u_\mu \rangle_w
\]

\[
= \langle \theta_{[p]}(\epsilon^1_{[p]} r_{[p]} \otimes \cdots \otimes \epsilon^k_{[p]} r_{[p]})), u_\mu |_{W_{[p]}} \rangle_{W_{[p]}}
\]

\[
= \langle \theta_{[p]}(\epsilon^1_{[p]} r_{[p]} f^{-1}_1 \boxtimes \cdots \boxtimes \theta_{[p]}(\epsilon^k_{[p]} r_{[p]})) f^{-1}_k, u_\mu |_{W_{[p]}} \rangle_{W_{[p]}}.
\]

One checks that \(u_\mu |_{W_{[p]}} \neq 0\) if and only if for \(i \in [k]\) there are signed partitions \(\mu^i\) of \([p_i]\) such that if \(q \in SC(n)\) is the concatenation of \(\mu^1, \ldots, \mu^k\), then \(\mu = \hat{q}\). Suppose that this is the case. Then \(u_\mu |_{W_{[p]}} = u_\mu^1 f^{-1}_1 \boxtimes \cdots \boxtimes u_\mu^k f^{-1}_k\). Thus

\[
\langle \theta_{[p]}(\epsilon^1_{[p]} r_{[p]} f^{-1}_1 \boxtimes \cdots \boxtimes \theta_{[p]}(\epsilon^k_{[p]} r_{[p]})) f^{-1}_k, u_\mu |_{W_{[p]}} \rangle_{W_{[p]}}
\]

\[
= \langle \theta_{[p]}(\epsilon^1_{[p]} r_{[p]} f^{-1}_1 \boxtimes \cdots \boxtimes \theta_{[p]}(\epsilon^k_{[p]} r_{[p]})) f^{-1}_k, u_\mu |_{W_{[p]}} \rangle_{W_{[p]}}.
\]

\[
= \langle \theta_{[p]}(\epsilon^1_{[p]} r_{[p]} f^{-1}_1, u_\mu^1 f^{-1}_1) W_{[p]} \cdots \langle \theta_{[p]}(\epsilon^k_{[p]} r_{[p]} f^{-1}_k, u_\mu^k f^{-1}_k) W_{[p]}
\]

and so by Proposition 6.3,

\[
\langle \theta_n(e_p), u_\mu \rangle_w = \langle \theta_{[p]}(\epsilon^1_{[p]} r_{[p]} f^{-1}_1, u_\mu^1) W_{[p]} \cdots \langle \theta_{[p]}(\epsilon^k_{[p]} r_{[p]} f^{-1}_k, u_\mu^k) W_{[p]}
\]

\[
= \prod_{p_i > 0} \langle u(p_i), u_\mu^i \rangle_{W_{[p_i]}} \cdot \prod_{p_i > 0} \langle u(p_i), u_\mu^i \rangle_{W_{[p_i]}}
\]

\[
\cdot \prod_{p_i < 0} \langle u(p_i), u_\mu^i \rangle_{W_{[p_i]}} \cdot \prod_{p_i < 0} \langle u(p_i), u_\mu^i \rangle_{W_{[p_i]}}
\]

\[
= \prod_{p_i \text{ odd}} \langle u(p_i), u_\mu^i \rangle_{W_{[p_i]}} \cdot \prod_{p_i \text{ even}} \langle u(p_i), u_\mu^i \rangle_{W_{[p_i]}}
\]

\[
= \begin{cases} 
2^{-k} |p_1 \cdots p_k|^{-1} & \text{if } \mu = \hat{q} \\
0 & \text{otherwise}
\end{cases}
\]
6.8. Now suppose \( \lambda = (\lambda_1, \ldots, \lambda_a, \lambda_{a+1}, \ldots, \lambda_{a+b}) \) is a signed partition of \( n \). Then \( |\text{Stab}(\lambda)| = |\text{Stab}(\lambda')| \) and with the notation in 2.8, the subgroup of \( \mathbb{Z}_W(n)(w_\lambda) \) generated by \( \{ y_i \mid \lambda_i = \lambda_{i+1} \} \) is isomorphic to \( \text{Stab}(\lambda) \). Thus

\[
|Z_{W_n}(w_{\lambda'})| = |\text{Stab}(\lambda)| 2^{a+b} \lambda_1 \cdots \lambda_{a+b}
\]

and so by Lemma 6.1 and Proposition 6.7,

\[
\theta(E_\lambda) = |\text{Stab}(\lambda)|^{-1} \theta_n(e_\lambda) = |\text{Stab}(\lambda)|^{-1} \sum_{\mu \in \mathcal{SP}(n)} |Z_{W_n}(w_\mu)| \langle \theta_n(e_\lambda), u_\mu \rangle_{W_n} \cdot u_\mu
\]

\[
= |\text{Stab}(\lambda)|^{-1} |Z_{W_n}(w_{\lambda'})| 2^{-a-b} \lambda_1 \cdots \lambda_{a+b}^{-1} \cdot u_{\lambda'} = u_{\lambda'},
\]

as claimed.

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