On the Hamilton-Jacobi equation for second class constrained systems

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Abstract
We discuss a general procedure for arriving at the Hamilton-Jacobi equation of second-class constrained systems, and illustrate it in terms of a number of examples by explicitly obtaining the respective Hamilton principal function, and verifying that it leads to the correct solution to the Euler-Lagrange equations.

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1 Introduction

There has recently been much interest in a Hamilton-Jacobi (HJ) formulation for constrained systems [1]-[3]. Although there exists consensus in the formulation for purely first class systems, the case of second-class systems has to be dealt with separately.

For an unconstrained system described by a Lagrangian $L(q_i, \dot{q}_i)$, $i = 1, \ldots, n$, the HJ-equations read,

$$H'_0 \equiv p_0 + H_0(q_0, q_i, p_i) = 0 \quad (1)$$

where $H_0$ is the canonical Hamiltonian, and where, for later convenience, $p_0$ and $p_i$ are shorthand for the partial derivatives

$$p_0 = \frac{\partial S}{\partial q_0}, \quad p_i = \frac{\partial S}{\partial q_i}, \quad q_0 = t, \quad (2)$$

with $S$ the Hamilton principal function $S = S(t, q_i)$ to be obtained as a solution of eq. (1).

Consider now a constrained system in which not all canonical variables are independent. In that case, the Lagrangian $L(q_i, \dot{q}_i)$ is singular, so that the determinant of the Hessian matrix $H_{ij} = \frac{\partial^2 L}{\partial q_i \partial q_j}$ is zero and the accelerations of some variables $\dot{q}_i$ are not uniquely determined by the positions and the velocities at a given time.

Let $n - m_1$ be the rank of the Hessian. In that case only $n - m_1$ velocities $\dot{q}_a \ (a = m_1 + 1, \ldots, n)$ can be solved for as a function of the coordinates $q_i$, the momenta $p_a$ canonically conjugate to $q_a$, and the remaining velocities $\dot{q}_a$, to yield $\dot{q}_a = \dot{q}_a(q_i, q_0, p_b, \dot{q}_b)$. The remaining momenta $p_a \ (\alpha = 1, \ldots, m_1)$ can be shown to be functions of $q_i$ and $p_a$ only, and represent the primary constraints $\phi_\alpha(q, p) \equiv p_\alpha + H_\alpha(q, p_\alpha) = 0$, in the Dirac terminology [5]. We write them in a form appropriate for our purposes:

$$H'_\alpha \equiv p_\alpha + H_\alpha(q_0, q_i, p_a) = 0, \quad (\alpha = 1, \ldots m_1) \quad (3)$$

Adjoining these constraints to (1), we are led to consider the coupled set of differential equations

$$H'_0 \equiv \frac{\partial S}{\partial q_0} + H_0 \left( q_0, q_\alpha, q_a, \frac{\partial S}{\partial q_\alpha} \right) = 0, \quad (4)$$

$$H'_\alpha \equiv \frac{\partial S}{\partial q_\alpha} + H_\alpha \left( q_0, q_\alpha, q_a, \frac{\partial S}{\partial q_\alpha} \right) = 0, \quad \alpha = 1, \ldots, m_1, \quad (4)$$
where $H_0$ is the canonical Hamiltonian evaluated on the subspace of the primary constraints, which only depends on the variables $q_i$, $p_a$ (and possibly $q_0 = t$), as is well known. Continuing the above line of reasoning, the system of differential equations (4) should further be supplemented with the corresponding set of equations associated with possible secondary constraints $\varphi(q, p)\sigma = 0$. Let there be $m_2$ such constraints. Adjoining these to (4), we have in our generalized notation,

$$H'_\alpha(q_i, \frac{\partial S}{\partial q_i}) = 0, \quad i = 0, 1, \ldots, n, \quad \alpha = 0, 1, \ldots, m = m_1 + m_2. \quad (5)$$

Assuming that the secondary constraints do not represent constraints among the coordinates $q$ themselves, this coupled set of differential equations only admits a solution provided the $H'_\alpha$ are in weak involution in the sense

$$\{H'_\alpha, H'_\beta\}|_{p_i = \frac{\partial S}{\partial q_i}} = C_{\alpha \beta}^\gamma H'_\gamma(q_i, \frac{\partial S}{\partial q_i}), \quad (6)$$

where the (generalized) Poisson bracket is defined by

$$\{A, B\} = \sum_{i=0}^{m} \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i} \frac{\partial A}{\partial p_i} \right). \quad (7)$$

Hence, for second class systems, our Ansatz (5) for the HJ equations must be incorrect.

One way of circumventing the integrability problem would be to embed the model in question into a larger phase space, such as to become first class [8]. Then the integrability conditions (6) are satisfied and our Ansatz is self-consistent [6]. This is, however, not the point of view we wish to take. As we shall rather illustrate in terms of a number of examples, a direct way of proceeding in the case of systems with second class constraints is to perform a canonical change of variables in which the second-class constraints become part of the variables. If the Poisson brackets of constraints are given by constant matrices, one can always perform a transformation to variables, where the constraints themselves can be grouped into canonically conjugate

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3As we shall see, this condition is not satisfied for the “non-linear sigma model”.

4On the space of solutions to (4) this then implies strong involution [6].
pairs. Otherwise this may not be possible \[7\], and the embedding procedure would provide a way out of the dilemma.

Unlike the discussions found in the literature \[1, 9, 12\] it is the final objective of this paper to actually construct the Hamilton-principal function \(S\) for the examples we consider, and to obtain from there the solution of the corresponding Euler-Lagrange equations following standard methods.

Our strategy to be illustrated by a number of examples will be the following: We begin by seeking all constraints, following the Dirac algorithm. We then seek a canonical transformation from a set of phase space variables \(q_i, p_i\) to a new set \((Q, P) = (q^*_a, \chi_\alpha, p^*_a, \mathcal{P}_\alpha)\), in which the complete set of primary and secondary constraints are grouped into canonically conjugate pairs \((\chi_\alpha, \mathcal{P}_\alpha)\) satisfying \(\{\chi_\alpha, \mathcal{P}_\beta\} = \delta_{\alpha\beta}\). Since the number of second class constraints is even, this is always possible for linear constraints. \(^5\)

The remaining set of variables \((q^*_a, p^*_a)\), which in turn are canonically conjugate to each other, \(\{q^*_a, p^*_b\} = \delta_{ab}\), are chosen to commute with the set \((\chi_\alpha, \mathcal{P}_\alpha)\). Denoting by \(F_2(q; p^*, \mathcal{P}; t)\) the generating function for this canonical transformation, the new Hamiltonian will be given by \(\tilde{H}(q^*, p^*, \chi, \mathcal{P}; t) = H(q(q^*, p^*, \chi, \mathcal{P}), p(q^*, p^*, \chi, \mathcal{P}); t) + \frac{\partial F_2}{\partial t}\), where \(q(q^*, p^*, \chi, \mathcal{P})\) and \(p(q^*, p^*, \chi, \mathcal{P})\) are obtained by solving the coupled set of equations \(p_i = \frac{\partial F_2}{\partial q_i}, \chi_\alpha = \frac{\partial F_2}{\partial \mathcal{P}_\alpha}\) and \(q^*_a = \frac{\partial F_2}{\partial p^*_a}\). The extended action in terms of the new variables now reads, \(^6\)

\[
S_E = \int dt \{p^*_a \dot{q}^*_a + \mathcal{P}_\alpha \dot{\chi}_\alpha - \tilde{H}(q^*, p^*, \chi, \mathcal{P}; t) - \lambda_\alpha \chi_\alpha - \eta_\alpha \mathcal{P}_\alpha\}
\]

Since the constraints commute with \(q^*_a\) and \(p^*_a\), the equations of motion for these variables are determined by \(\tilde{H}\) with the constraints set equal to zero, strongly:

\[
\dot{q}^*_a = \{q^*_a, \tilde{H}(q^*, p^*, 0, 0; t)\}, \quad \dot{p}^*_a = \{p^*_a, \tilde{H}(q^*, p^*, 0, 0; t)\}
\]

while the equations for the remaining coordinates, the constraints, just reproduce the persistency equations, and represent a set of differential equations.

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\(^5\)As shown in ref. \[7\] this is not always possible, if the constraints are nonlinear.

\(^6\)To formulate the general problem it is convenient to work with the extended action, where all of the constraints are implemented via Lagrange multipliers. Since all the constraints are supposed to be generated from the total Hamiltonian \(H_T\), involving only the primary constraints, via the Dirac self-consistent algorithm, the Lagrange multipliers associated with the secondary constraints will eventually turn out to be zero in a variational calculation.
in the new coordinates. With respect to the coordinates \((q^*, p^*)\) the problem has thus been reduced to that of an unconstrained system, and the usual Hamilton-Jacobi treatment applies. Note that the canonical change in phase space variables has now removed the inconsistency of the originally envisaged HJ formulation. The reason is that, unlike the Hamiltonian, the Hamilton Principal function in the new coordinates \((Q, P)\) cannot be obtained by a simple substitution \((q, p) \rightarrow (q(Q, P), p(Q, P))\).

In different terms: from the total action (8) we obtain for the HJ equation

\[
\tilde{H}(q^*, \frac{\partial S}{\partial q^*}, \chi, \frac{\partial S}{\partial \chi}, t) + \frac{\partial S}{\partial t} = 0
\]

(9)

together with the constraint equations,

\[
\chi_\alpha = 0, \quad P_\alpha = \frac{\partial S}{\partial \chi_\alpha} = 0.
\]

(10)

In general our problem thus reduces to solving the differential equations

\[
\hat{H}(q^*, \frac{\partial S}{\partial q^*}, t) + \frac{\partial S}{\partial t} = 0
\]

(11)

with

\[
\hat{H}(q^*, p^*) = \tilde{H}(q^*, p^*, \chi = P = 0).
\]

(12)

In the following section we illustrate these ideas in terms of a number of simple examples. We shall be rather detailed in the first example, in order to illustrate various aspects of the problem.

2 Second class systems and HJ formulation

In this section we shall consider several simple quantum mechanical models of purely second class systems and show how to properly deal with them in order to construct the corresponding Hamilton Principal function \(S\). We also verify that the formalism yields the solutions to the corresponding equations of motion.
Example 1

Consider the Lagrangean

\[ L = \frac{1}{2} \dot{x}^2 + \dot{x}y - \frac{1}{2}(x - y)^2 \]  

(13)

The corresponding Euler-Lagrange equations can be written in the form, \( \ddot{x} = 0 \), \( \dot{x} + x = y \). We have one primary constraint \( \phi := p_y = 0 \), and the canonical Hamiltonian on the primary surface reads,

\[ H_0 = \frac{1}{2}(p_x - y)^2 + \frac{1}{2}(x - y)^2. \]  

(14)

The usual Dirac algorithm yields a secondary constraint \( \varphi = 0 \),

\[ \{\phi, H_0\} = -2\varphi, \quad \varphi = y - \frac{1}{2}(p_x + x). \]  

(15)

There are no further constraints. Since \( \{\varphi, \phi\} = 1 \) these constraints are second class and canonically conjugate to each other.

If we were to proceed naively and make the replacements \( p_y = \frac{\partial S}{\partial y} \) in the primary constraint \( p_y = 0 \), then this would imply that \( S \) is independent of \( y \), which would be in direct conflict with the equation obtained by making the substitution \( p_x = \frac{\partial S}{\partial x} \) in the constraint \( \varphi = 0 \). This reflects the non-commutative structure of our second class system, as well as the violation of the integrability condition (6).

In order to circumvent this difficulty, we make a canonical transformation to a new set of variables \( (q^*, p^*) \) and \( (\chi, P) \), in which one of the canonically conjugate pairs are chosen to be the constraints themselves:

\[ \chi = \varphi, \quad P = \phi, \]

\[ q^* = x - \frac{1}{2}p_y, \quad p^* = p_x + \frac{1}{2}p_y. \]  

(16)

We evidently have

\[ \{\chi, P\} = 1, \quad \{q^*, p^*\} = 1. \]  

(17)

Furthermore, \((q^*, p^*)\) and \((\chi, P)\) “commute”. Our change of variables is thus canonical. Indeed one readily checks that

\[ p_x \ddot{x} + p_y \ddot{y} = p^* \ddot{q}^* + P \ddot{\chi} + \frac{1}{2} \frac{d}{dt}(p^* P - \frac{1}{2} P^2). \]
Furthermore, the corresponding generating functional \( F_2(x, y; p^*, \mathcal{P}) \) for the transformation (16) is readily constructed:

\[
F_2 = (p^* - \frac{1}{2}\mathcal{P})x + \mathcal{P}y - \frac{1}{2}\mathcal{P}p^* + \frac{1}{8}\mathcal{P}^2.
\]

The inverse transformation to (16) reads

\[
\begin{align*}
x &= q^* + \frac{1}{2}\mathcal{P}, & p_x &= p^* - \frac{1}{2}\mathcal{P}, \\
y &= \frac{1}{2}(q^* + p^*) + \chi, & p_y &= \mathcal{P}.
\end{align*}
\] (18)

In terms of the new variables,

\[
\hat{H} = \frac{1}{2} \left[ \frac{1}{2}(p^* - q^*) - \left(\frac{1}{2}\mathcal{P} + \chi\right) \right]^2 + \frac{1}{2} \left[ \frac{1}{2}(q^* - p^*) - \left(\frac{1}{2}\mathcal{P} - \chi\right) \right]^2
\] (19)

One checks,

\[
\begin{align*}
\dot{q}^* &= \frac{1}{2}(p^* - q^*) - \frac{1}{2}\mathcal{P} \\
\dot{p}^* &= \frac{1}{2}(p^* - q^*) - \frac{1}{2}\mathcal{P} \\
\dot{\mathcal{P}} &= -2\chi \\
\dot{\chi} &= \eta - \frac{1}{2}(p^* - q^*) + \frac{1}{2}\mathcal{P}
\end{align*}
\] (20)

The 3’rd equation reproduces the secondary constraint, and the 4’th equation fixes the parameter \( \eta \) as expected.

The above shows, that as a consequence of having the conjugate pairs in strong involution, we can simply set the constraints in \( \tilde{H}_0 \) equal to zero and proceed with the Hamiltonian

\[
\hat{H}_0 = \frac{1}{4}(p^* - q^*)^2.
\] (21)

Hence we are left with only one HJ equation,

\[
\frac{\partial S}{\partial t} + \frac{1}{4}(\frac{\partial S}{\partial q^*} - q^*)^2 = 0
\] (22)
showing that $S$ will be just a function of $q^*$. The HJ equation associated
with the primary constraint $P = 0$ tells us that $S$ is independent of the new
coordinate $\chi$: $P = \frac{\partial S}{\partial \chi} = 0$.

Since the Hamiltonian does not depend explicitely on time, we must have
$$\frac{\partial S}{\partial t} = -\alpha^2 = \text{constant}$$
so that equation (22) has the solution
$$S(q^*, \alpha, t) = \pm 2\alpha q^* + \frac{1}{2}q^{*2} - \alpha^2 t.$$ (23)

Notice that $S$ depends only on one integration constant, since when regarded
as the generating function taking us to new momenta $p^*$ and $P$, one of the
momenta ($P$) is just the primary constraint $P = \phi = 0$. There is another
constant arising from the equation
$$\beta = \frac{\partial S}{\partial \alpha},$$
which implies from (23),
$$\beta = \pm 2q^* - 2\alpha t.$$ Together with (we choose the plus sign in (23), without loss of generality)
$$p^* = \frac{\partial S}{\partial q^*} = 2\alpha + q^*$$
we now solve the problem in the standard way. Taking account of the con-
straints $\chi_\alpha = P_\alpha = 0$ in (18), we find
$$x = \frac{1}{2}\beta + \alpha t, \quad y = (\alpha + \frac{1}{2}\beta) + \alpha t$$
which evidently is a solution of the Euler-Lagrange equations associated with
(13).

**Example 2: QM $\sigma$ - model**

Consider now the quantum mechanical version of the “sigma model” as
given by the Lagrangean
$$L = \frac{\dot{q}^2}{2} + \lambda(q^2 - 1)$$ (24)
implying the primary constraint \( p_\lambda = 0 \). The canonical Hamiltonian on the primary surface reads,

\[
H_0 = \frac{\vec{p}^2}{2} - \lambda(\vec{q}^2 - 1).
\]

The complete set of primary and secondary constraints can be written in the form

\[
\phi = p_\lambda, \quad \varphi = \lambda + \frac{\vec{p}^2}{2}, \quad \varphi_1 = \vec{q}^2 - 1, \quad \varphi_2 = \frac{\vec{p} \cdot \vec{q}}{2\vec{q}^2}.
\]

The Poisson brackets of the constraints show that they can be grouped into mutually commuting, canonically conjugate pairs:

\[
\{ \varphi, \phi \} = 1, \quad \{ \varphi_1, \varphi_2 \} = 1.
\]

Let us consider for simplicity the case of two dimensions. One may be tempted to make a transformation from \((\lambda, p_\lambda)\) to \((\varphi, \phi)\) and \((q_1, q_2, p_1, p_2)\) to \((\varphi_1, \varphi_2, \theta, J)\), where \((\theta, J)\) are the canonically conjugate variables,

\[
\theta = \arctan(q_2/q_1), \quad J = \epsilon_{ij} q_i p_j,
\]

with \( J \) evidently playing the role of the “third” component of angular momentum. The second set of variables commute with the first set, except for \( \theta \), which does not commute with \( \varphi \): \( \{ \varphi, \theta \} = J/\vec{q}^2 \). Hence this transformation is not canonical.

In order to remove this problem we eliminate \( \lambda \) and \( p_\lambda \) using the constraints \( \varphi = 0 \) and \( \phi = p_\lambda = 0 \). \(^7\) The resulting Hamiltonian is \( H_0 = \frac{1}{2} \vec{p}^2 \vec{q}^2 \).

Noting that

\[
J^2 = \vec{q}^2 \vec{p}^2 - (\vec{q} \cdot \vec{p})^2 = \vec{p}^2(1 + \varphi_1) - 4\varphi_2^2(\varphi_1 + 1)^2
\]

we then obtain for \( \hat{H}_0 \),

\[
\hat{H}_0 = \frac{1}{2} J^2
\]

We are thus left to solve the equation

\[
\frac{\partial S}{\partial t} + \frac{1}{2} \left( \frac{\partial S}{\partial \theta} \right)^2 = 0
\]

\(^7\) It is easy to see, that this is a legitimate procedure since \( \lambda \) plays the role of a Lagrange multiplier in the Lagrangean. More precisely, implementing strongly the constraint \( \varphi = 0 \) in the Hamiltonian (25) from the outset, leads to an equivalent set of equations of motion.
which has the solution (an irrelevant additive constant has been omitted)

\[ S(\theta, \alpha, t) = -\frac{\alpha^2}{2} t + \alpha \theta. \]  

(30)

In order to obtain the solution to the Euler-Lagrange equations we recall that we still have the equation \( \frac{\partial S}{\partial \alpha} = \beta \), which gives the solution in terms of two integrations constants:

\[ \theta = \alpha t + \beta \]  

(31)

One again verifies that this provides the solution of the Euler-Lagrange equations.

**Example 3: Multidimensional rotator**

Consider the Lagrangean [8, 11],

\[ L = \frac{1}{2} \dot{\vec{q}}^2 + \lambda \vec{q} \cdot \vec{q}. \]

There is one primary constraint \( \phi = p_\lambda = 0 \) and the canonical Hamiltonian evaluated on the primary surface reads,

\[ H_0 = \frac{1}{2} (\vec{p} - \lambda \vec{q})^2. \]

(32)

The model effectively describes the motion on a \( n - 1 \) dimensional sphere without specification of the radius of the sphere. Note that unlike in the case of the \( \sigma \)-model, the momentum conjugate to \( \vec{q} \) is no longer the mechanical momentum: \( \vec{p} = \dot{\vec{q}} + \lambda \vec{q} \). There is only one secondary constraint

\[ \varphi = \lambda - \frac{\vec{p} \cdot \vec{q}}{\vec{q}^2}, \]

(33)

which has been chosen to be canonically conjugate to \( \phi \): \( \{ \varphi, \phi \} = 1 \).

i) We first consider this model in two dimensions. A coordinate change \( (\lambda, p_\lambda) \to (\varphi, \phi) \) and \( (q_1, p_1, q_2, p_2) \to (r, p_r, \theta, J) \) would again lead to the second set not commuting with the first, so that \( \lambda \) should first be eliminated using the constraint \( \varphi = 0 \). Having done this we may replace the canonical Hamiltonian (32) by (see footnote 7)

\[ \hat{H}_0 = \frac{1}{2} \left( \vec{p} - \frac{(\vec{p} \cdot \vec{q})}{\vec{q}^2} \vec{q} \right)^2 = \frac{J^2}{2r^2} \]

(34)
where \( r = \sqrt{q^2} \), \( J \) is given in (28), and \( \{ \theta, J \} = 1 \). Notice that \( r \) just enters as a parameter. Replacing \( J \) by \( \left( \frac{\partial S}{\partial \theta} \right) \), we obtain from (34) the HJ equation,

\[
\frac{\partial S}{\partial t} + \frac{1}{2r^2} \left( \frac{\partial S}{\partial \theta} \right)^2 = 0 ,
\]

which just differs from (29) by \( r \) being arbitrary, and not equal to one. Correspondingly (30) is replaced by

\[
S(\theta, \alpha, t) = -\frac{\alpha^2}{2} t + r\alpha \theta .
\]

The rest proceeds as in the case of model 2.

ii) We now reconsider the problem in \( n \)-dimensions. Solving again the constraint \( \varphi = 0 \) for \( \lambda \), the HJ equation (35) is replaced by

\[
\frac{\partial S}{\partial t} + \frac{1}{2} \left( \frac{\partial S}{\partial q_a} \right)^2 - \frac{1}{2} \left( \frac{\partial S}{\partial q_a} \right) \left( \frac{\partial S}{\partial q_b} \right) \frac{q_a q_b}{q^2} = 0 .
\]

We make the Ansatz,

\[
S = f(\vec{n} \cdot \vec{q})
\]

with \( \vec{n} \) a unit normal vector parametrized by \( n-1 \) constants. The HJ equation for \( f \) then reads,

\[
\frac{1}{2} \left( 1 - \frac{(\vec{n} \cdot \vec{q})^2}{q^2} \right) f'(\vec{n} \cdot \vec{q}) = \frac{\alpha^2}{2} .
\]

Setting \( x = \vec{n} \cdot \vec{q} \) and \( r^2 = \vec{q}^2 \), we then find

\[
f'(x) = \pm \frac{\alpha}{\sqrt{1 - \frac{x^2}{r^2}}}
\]

so that upon integration in \( x \) the Hamilton principal function takes the form

\[
S = \alpha r \tan^{-1} \frac{\vec{n} \cdot \vec{q}}{\sqrt{r^2 - (\vec{n} \cdot \vec{q})^2}} - \frac{\alpha^2}{2} t + \text{const}
\]

The Hamilton principal function contains \( n \) independent constants, which we take to be \( \alpha \) and \( n_1, n_2, \ldots n_{n-1} \), while the normalization of \( \vec{n} \) implies for the

\[^8\text{Note that the coordinates } q_a \text{ play the role of } q^*_a \text{ in section 1.}\]
The $n$'th component, $n_n = \sqrt{1 - \sum_{a=1}^{n-1} n_a^2}$. Differentiating the Hamilton principal function with respect to these constants (new momenta in the corresponding generating functional) yields the $n$ time-independent new coordinates:

$$
\beta = \frac{\partial S}{\partial \alpha}, \quad \beta_a = \frac{\partial S}{\partial n_a}, \quad a = 1, 2, \ldots n - 1.
$$

From the first equation and (38) we easily obtain

$$
\vec{n} \cdot \vec{q} = r \cos \beta + \alpha t \equiv r \cos \Omega(t).
$$

Using this, we obtain from the second equation in (39)

$$
q_a = \frac{\beta_a}{\alpha} \sin \Omega(t) + \frac{n_a q_n}{n_n}, \quad a = 1, 2, \ldots , n - 1
$$

Multiplying this equation by $n_a$, summing from $a = 1$ to $a = n - 1$ and using the normalization of $\vec{n}$ we obtain from (41),

$$
\vec{n} \cdot \vec{q} = \frac{1}{\alpha} \sum_{a=1}^{n-1} n_a \beta_a \sin \Omega(t) + \frac{q_n}{n_n} = r \cos \Omega(t)
$$

This allows us to solve for $q_a$ as

$$
q_a = n_a \left( r \cos \Omega(t) - \frac{1}{\alpha} \sum_{a=1}^{n-1} n_a \beta_a \sin \Omega(t) \right).
$$

Substituting back in the equation for $q_a$ we obtain

$$
q_a = \frac{1}{\alpha} \left( \beta_a - n_a \sum_{a=1}^{n-1} n_a \beta_a \right) \sin \Omega(t) + n_a r \cos \Omega(t), \quad a = 1, 2, \ldots , n - 1.
$$

Introducing the $n$-dimensional vector $\vec{\beta} = (\beta_1, \beta_2, \ldots , \beta_{n-1}, 0)$, these results can be written compactly as

$$
\vec{q} = \frac{1}{\alpha} \left( \vec{\beta} - (\vec{n} \cdot \vec{\beta}) \vec{n} \right) \sin \Omega(t) + r \vec{n} \cos \Omega(t)
$$

Substituting the above result into our original condition, $r^2 = \vec{q}^2$, leads to

$$
r = \sqrt{\frac{\vec{\beta}^2 - (\vec{n} \cdot \vec{\beta})^2}{\alpha^2}}
$$
Thus the radius of motion is fixed if the new time independent coordinates are specified. More conventionally one would consider the radius of motion as a given initial condition; then the above condition eliminates one of the $\beta_a$ as an independent constant. Indeed, from (42) we see that the component of $\vec{\beta}$ orthogonal to $\vec{n}$ is fixed to be $\alpha r$. In this case one therefore considers $\alpha, r$ and the remaining $n - 2$ constants, which determine the component of $\vec{\beta}$ parallel to $\vec{n}$ as independent constants.

**Example 4: Landau model in the zero mass limit**

The Lagrangean of a spinless charged particle moving on a two dimensional plane in a constant background magnetic field $B$ perpendicular to the 12-plane, and a harmonic oscillator potential, can be written in the form

$$L_{\text{Landau}} = \frac{m}{2} \dot{q}^2 + \frac{B}{2} \vec{q} \times \dot{\vec{q}} - \frac{k}{2} \vec{q}^2.$$  

In the zero-mass limit the Lagrangian of the model reduces to

$$L = \frac{B}{2} \vec{q} \times \dot{\vec{q}} - \frac{k}{2} \vec{q}^2$$  

(43)

There are only two primary constraints

$$\phi_i = \frac{1}{\sqrt{B}} p_i + \frac{\sqrt{B}}{2} \epsilon_{ij} q_j$$

with non-vanishing Poisson brackets

$$\{ \phi_i, \phi_j \} = \epsilon_{ij}$$

We perform the following change of variables from $q_i, p_i$ to $\phi_1, \phi_2, q, p$:  

$$\begin{align*}
\phi_1 &= \frac{1}{\sqrt{B}} p_1 + \frac{\sqrt{B}}{2} q_2, \\
\phi_2 &= \frac{1}{\sqrt{B}} p_2 - \frac{\sqrt{B}}{2} q_1, \\
q &= \frac{1}{2} q_1 + \frac{1}{B} p_2, \\
p &= p_1 - \frac{B}{2} q_2
\end{align*}$$  

(44)

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9In the notation of section 1, $q, p$ now play the role of $q^*, p^*$, and $\phi_1 = \chi, \phi_2 = \mathcal{P}.$

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with the inverse transformation

\[ q_1 = q - \frac{1}{\sqrt{B}} \phi_2, \quad q_2 = \frac{1}{B} (\sqrt{B} \phi_1 - p), \]
\[ p_1 = \frac{1}{2} (\sqrt{B} \phi_1 + p), \quad p_2 = \frac{1}{2} (Bq + \sqrt{B} \phi_2). \] (45)

In terms of the new coordinates the canonical Hamiltonian evaluated on the constraint surface reads,

\[ \hat{H}_0 = k^2 (q_2^2 + \frac{1}{B^2} p_2^2) \] (46)

Hence we have for the HJ equation

\[ \frac{\partial S}{\partial t} + \frac{k}{2B^2} \left( B^2 q^2 + \left( \frac{\partial S}{\partial q} \right) \right) = 0, \] (47)

having the solution

\[ S(q, \alpha, t) = -\alpha t + B \int q' \left\{ \frac{2 \alpha}{k} - q'^2 \right\} - \text{const.} \] (48)

From the equation \( \beta = \frac{\partial S}{\partial \alpha} \) we then obtain in the usual way the solution

\[ q(t) = \sqrt{\frac{2 \alpha}{k}} \cos(\omega t + b), \]

with \( \omega = k/B \) and \( b = \beta k/B \).

### 3 Conclusion

A *naive* extension of Hamilton-Jacobi (HJ) theory to second class constrained systems is condemned to failure right from the outset since it leads to differential equations which are in direct conflict. Mathematically this conflict is expressed by the violation of the integrability condition (6).
It is useful to reformulate these integrability conditions in terms of the commutator algebra of operators. To this end we begin by introducing for each $H'_{\alpha}$ a linear differential operator $X_{\alpha}$ defined by [6, 9, 10]

$$X_{\alpha}f = \{H'_{\alpha}, f\},$$

(49)

where the Poisson bracket is the generalized Poisson bracket defined in (7), and $f$ an arbitrary functions of $q_i$ and $p_i$. If $H'_{\alpha}$ does not possess an explicit time dependence, we recover the usual definition of the Poisson bracket. Making use of the Jacobi identity for Poisson brackets, we obtain from Eq.(49) the following relation between the commutator of the operators $X_{\alpha}$, and the Poisson algebra of the constraints:

$$[X_{\alpha}, X_{\beta}]f = \{\{H'_{\alpha}, H'_{\beta}\}, f\}.$$  

(50)

for arbitrary $f$. It is now clear that the commutator algebra of the operators $X_{\alpha}$ will not close unless the Poisson algebra of the $H'_{\alpha}$ closes.

One way in which this problem has been approached is to iteratively extend the set of operators $H'_{\alpha}$ until the algebra closes. Equivalently, new operators in (50) are iteratively introduced, enhancing the original set $X_{\alpha}$ to a larger set $X_{\bar{\alpha}}$ of operators until the closure relation

$$[X_{\bar{\alpha}}, X_{\bar{\beta}}] = c_{\bar{\alpha}\bar{\beta}} X_{\bar{\gamma}}.$$  

(51)

is achieved [12]. The corresponding system of partial differential equations will then be integrable. However, there is no general reason why the solution for the Hamilton principal function $S$ for this extended first-class algebra should provide the solution for the problem in question.

As an alternative approach to formulate a consistent set of HJ equations, one can consider [8] the embedding of the second class system into a first class characterization by a strongly involutive algebra, following the BFT [4] construction. This was not the point of view to be taken in this paper. It was rather our objective to confront ourselves directly with the problem of non-integrability, and to point out a procedure for coping with this problem. The procedure consisted in making a canonical transformation to a new set of variables in which the second class constraints become part of the variables, grouped in canonically conjugate pairs. As we illustrated by a number of examples, the new Hamiltonian is then obtained by setting the constraints
strongly equal to zero, and the HJ equations associated with the constraints are no longer in conflict with each other, nor with this new Hamiltonian. The solution for the coordinates obtained in this way were shown to coincide with the solutions of the corresponding Lagrange equations of motion.

If the constraints are nonlinear, one is in general not able to perform a canonical transformation, turning these constraints into canonical pairs [7], and our analysis as presented here is no longer valid. Investigations in this directions are presently under way.

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