Roughness of Crack Interfaces in Two-Dimensional Beam Lattices

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Abstract

The roughness of crack interfaces is reported in quasistatic fracture, using an elastic network of beams with random breaking thresholds. For strong disorders we obtain $\zeta = 0.86(3)$ for the roughness exponent, a result which is very different from the minimum energy surface exponent, i.e., $\zeta = \frac{2}{3}$. A cross-over to lower values is observed as the disorder is reduced, the exponent in these cases being strongly dependent on the disorder.

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Phenomena associated with fracture are a central theme in materials science, with great importance in a wide range of technological applications. In recent years it has also been the subject of much attention in the statistical physics community, resulting in methods which model materials in terms of disordered rather than continuous media [1]. This approach has drawn attention to certain features apparently sharing a common basis with other, seemingly unrelated, problems showing critical behavior. For instance, the scaling of interfaces which characterize deposition and growth processes, or propagation in substrates with a random structure, have been found to obey non-trivial laws [2]. This is also the case of certain equilibrium phenomena, where an interface is obtained as a result of interactions with a surrounding random medium. In the directed polymer problem, for instance, $\zeta = \frac{2}{3}$ is obtained for the roughness exponent of the minimum energy path in a two-dimensional embedding medium [3].

Furthermore, in numerical simulations with the random fuse model [4], the roughness exponent of the interface which characterizes electrical breakdown of a conducting network is found to be $\zeta = 0.70(7)$. This is close to the value for the minimum energy path, $\zeta = \frac{2}{3}$ [3], and there have been speculations that they indeed are equal [3]. However, in three dimensions the fuse model exponent seems higher than the minimum energy surface exponent [3]. The two dimensional fuse network result [4] also agrees with experimental results in two dimensions [8]. However, experimental results in three dimensions suggest a much higher value, $\zeta = 0.8$, than the fuse model gives, $\zeta = 0.62(5)$ [6].

A question related to whether or not brittle fracture falls within this class of problems, however, is how well the random fuse model actually describes fracture processes. In the fuse model each element has a single degree of freedom, i.e., the voltage difference between neighboring nodes. The interface obtained, which is characteristic of electrical breakdown rather than a physical crack, nevertheless provides valuable information on the interplay between quenched disorder and the current distribution. As opposed to vector fracture, where the elastic elements each have three degrees of freedom, the random fuse model is thus referred to as describing scalar fracture. It is due to the analogy between Ohm’s law and Hooke’s law that the electrical problem has been regarded as similar to its elastic counterpart.

In this Letter, we report the results of computer simulations using the elastic beam model [10,11] which has previously been used to study the scaling properties of forces and displacements in brittle fracture. Furthermore, we address the universality issue by using two different types of distribution with a wide range of disorders.

The beam model in two dimensions may be defined as a regular square lattice of size $L \times L$, where the spacing is unity, and each node in the horizontal and vertical in-plane directions is connected to its nearest neighbours by elastic beams. A given beam is then soldered to other beams in such a way that, upon subsequent displacement of neighbouring nodes, the angle between beams remains the same as in the original underlying square lattice. The three possible degrees of freedom, i.e., translations in the horizontal and vertical directions and rotations about the axis perpendicular to the plane, thus allow for bending moments as well as axial elongation and compression. The beam is also imagined as having a certain thickness, providing shear elasticity.

The forces between neighbouring nodes may be derived by considering a concentrated end load on an elastic beam with no end restraints [12]. We define, for notational convenience,
\[ p_j = \frac{1}{2}[1 - (-1)^j], \quad q_j = \prod_{n=0}^{j-1} (-1)^n, \quad \]  

which entails an anti-clockwise labeling beginning with the beam to the right of \( i \). With \( \delta z = z_j - z_i \) denoting the displacements, we obtain at node \( i \), due to the beam which connects \( i \) with \( j \),

\[ M_i^{(j)} = \frac{1}{\beta + \frac{\gamma}{12}} \left\{ \frac{\beta}{\gamma} \delta \theta + \frac{q_j}{2} T_i^{(j)} - \frac{\theta_i}{3} - \frac{\theta_j}{6} \right\}, \]

\[ V_i^{(j)} = \frac{1}{\beta + \frac{\gamma}{12}} \left[ T_i^{(j)} - \frac{q_j}{2} (\theta_i + \theta_j) \right], \]

\[ S_i^{(j)} = \frac{1}{\alpha} [T_i^{(j+1)}], \quad (2) \]

where \( T_i^{(j)} = \delta x + p_j (\delta y - \delta x) \), for the contributions in moment, shear and strain, respectively.

Prefactors characteristic of the material and its dimensions in Eq. (2) depend on

\[ \alpha = \frac{1}{EA}, \quad \beta = \frac{1}{GA}, \quad \gamma = \frac{1}{EI}, \quad (3) \]

where \( E \) is Young’s modulus, \( A \) and \( I \) the area of the beam section and its moment of inertia about the centroidal axis, respectively, and \( G \) the shear modulus.

For the sum of forces and moments on each node, we then have

\[ \Sigma_{ix} = S_i^{(1)} + V_i^{(2)} + S_i^{(3)} + V_i^{(4)}, \]

\[ \Sigma_{iy} = V_i^{(1)} + S_i^{(2)} + V_i^{(3)} + S_i^{(4)}, \]

\[ \Sigma_{i\theta} = \sum_{j=1}^{4} M_i^{(j)}, \quad (4) \]

the lattice being in equilibrium when, at any point in the fracture, \( \Sigma_{ix} = \Sigma_{iy} = \Sigma_{i\theta} = 0 \).

Such a configuration is realized when the elastic energy, i.e.,

\[ \mathcal{E} = \frac{1}{2} \sum_i \sum_{j=1}^{2} \left\{ [S_i^{(j)}]^2 + [V_i^{(j)}]^2 + [M_i^{(j)}]^2 \right\}, \quad (5) \]

is at its minimum. This minimum we obtain via relaxation, using the conjugate gradient method with a tolerance in the residual error of \( \epsilon = 10^{-12} \).

For a brittle material we assume that each beam is linearly elastic up to the breaking threshold. Using \( t_S \) and \( t_M \) for the strain and bending thresholds respectively, the breaking criterion \[ \square \], inspired from Tresca’s formula, is given by

\[ \left( \frac{S}{t_S} \right)^2 + \frac{|M|}{t_M} \geq 1, \quad (6) \]

where \( |M| = \max(|M_i|, |M_j|) \) is the largest of the bending moments at the two beam ends \( i \) and \( j \).
The fracture process is initiated by imposing on the lattice an external vertical displacement of unit magnitude, i.e., a displacement which at the top row corresponds to one beam in length. In its initial state, the lattice now consists of horizontally undeformed beams and beams which in the vertical direction are stretched lengthwise by an amount $1/L$. With an extra row at the top there are $L(L - 1)$ inner nodes, for which any neighbouring beam may be broken, and $L$ nodes each at the top and bottom, the positions of which are held fixed. This defines the vertical boundary conditions.

As for the horizontal direction, previous results obtained with the random fuse model have relied on the use of periodic boundary conditions. This is a good strategy to avoid edge-effects, especially in a situation where numerical resources are limited to small system sizes. However, when considering fracture in a periodic system, the topology is essentially that of a plane intersecting a cylinder. We thus need to address the problem of how the trace of a sine curve affects results obtained for the roughness. To avoid this, we instead use open boundary conditions, i.e., we adopt the procedure used in Ref. [13] of subtracting the average vertical drift of the crack as it traverses the width of the lattice.

The first beam to break is that for which the sum of the two ratios is largest, this being the vertically oriented beam which has the lowest value of $t_S$.

If all threshold values are approximately the same, the next beam to break will be one of the nearest lateral neighbours since these now carry a larger load than other beams in the lattice. The case of no disorder is thus one in which the crack propagates horizontally from the initial damage, taking the shortest possible path to break the lattice apart. This results in a smooth interface.

Introducing disorder in the threshold values, material strength is no longer uniformly distributed throughout the lattice and consequently the crack will not necessarily develop from the initial damage point. Instead microcracks and voids form wherever the stress concentration most exceeds the local strength, i.e., wherever Eq. (6) dictates that the next beam should be broken. Towards the end of the process some of these merge into a macroscopic crack, forming a sinuous, or rough, interface which is characteristic of the disorder in the system.

Hence we have a highly correlated process in which the quenched disorder and the non-uniform stress distribution combine to determine where the next break will occur while, simultaneously, the stress distribution itself continually changes as the damage spreads.

To study this, we generate a random number $r$ on the unit interval $[0, 1]$ and let this represent the cumulative threshold distribution. Assigning the threshold values according to

$$t_F = r^D,$$  \hfill (7)

the threshold distribution approaches that of no disorder when $|D| \to 0$. In the fuse model [7,14], several types of distribution have been used for the threshold values. Although at present we restrict ourselves to Eq. (7), the two cases $D < 0$ and $D > 0$ represent widely different distributions, i.e., for $D > 0$ the distribution is a power law with a tail which extends towards weak beams whereas for $D < 0$ the tail of the distribution extends towards strong beams. In both cases we use a wide range of disorders between $|D| = \frac{1}{12}$ and $|D| = 4$.

The roughness is now obtained for a large number of lattices, each of size $L$, the thresholds being re-cast according to Eq. (7) each time a new sample is broken. Generally the number
of samples depend on $L$ as well as, to a lesser degree, on the disorder $D$. Presently lattices of all sizes from $L = 4$ up to $L = 20$ were studied, with sample sizes ranging from $N = 250000$ in the the former case to about $N = 1000$ in the latter case. For the larger systems we studied, typical sample sample sizes are shown in Table [1].

Fig. [1] shows a log-log plot of $W$ as a function of $L$ for a range of disorders with $D > 0$. For all $L$, the interface is seen to become more rough with increasing disorder. Each curve also has a characteristic crossover, beyond which the asymptotic relationship is that of a straight line, i.e., where $W \sim L^\zeta$. This feature is seen to be disorder dependent, with the onset of asymptotic behaviour being deferred to larger $L$ as $D$ increases. At some point, the crossover becomes difficult to define before it again reduces to a point well within the range of the system sizes presently studied. Closely associated with this behaviour is an even more striking feature, i.e., the dependency of $\zeta$ upon the disorder. Specifically, with the enumeration scheme used for the disorders in Fig. [1], we obtain (f) $\zeta = 0.16$ for $D = 0.25$ as opposed to (k) $\zeta = 0.55$ for $D = 0.09$, between which $\zeta$ increases monotonously as the disorder decreases, i.e., (g) $\zeta = 0.23$, (h) $\zeta = 0.32$, (i) $\zeta = 0.42$ and (j) $\zeta = 0.51$. A very pronounced transition is observed between (d) $D = 0.5$ and (f) $D = 0.25$, the exponents for $D > 0.5$ apparently having a constant value of $\zeta \sim 0.86$, that is, we obtain (a) $\zeta = 0.87$, (b) $\zeta = 0.86$ and (c) $\zeta = 0.86$.

Although values obtained for $\zeta$ with $D < 0$, shown in Fig. [2], are seen to be different from those obtained with the same $|D|$ when $D > 0$, the qualitative features in this case are the same. Again there is a pronounced transition in $\zeta$ from (a) $\zeta = 0.87$ and (b) $\zeta = 0.86$ to (e) $\zeta = 0.41$, between which the exponent is difficult to determine. With a further decrease in disorder the exponent increases up towards (k) $\zeta = 0.60$, the intermediate values shown in Fig. [2] being (f) $\zeta = 0.48$, (g) $\zeta = 0.51$, (h) $\zeta = 0.55$, (i) $\zeta = 0.57$ and (j) $\zeta = 0.60$.

The behaviour of $\zeta$ as a function of the disorder $D$ is shown in Fig. [3]. Here, estimates for $\zeta$ which are difficult to define are also included. Hence, corresponding to the open circles in Fig. [3] we use (d) $\zeta = 0.89$, based on the data for $L = 27$ to $L = 100$ and (e) $\zeta = 0.31$, based on the four uppermost data points. In Fig. [2] the corresponding estimates are (c) $\zeta = 0.88$, based on data for $L = 19$ up to $L = 63$, and (d) $\zeta = 0.43$, again based on the four uppermost data points. As $|D| \to 0$, the interface becomes sufficiently smooth to frequently avoid detection by the course-graining of the lattice. Hence an accurate estimate for $\zeta$ now depends on the relative occurrence of those samples which are unusually rough for the given disorder, implying an excessively large amount of samples for each $L$. To obtain an estimate nonetheless, we note that the two sets of exponents for the six lowest values of $|D|$ corresponding to $D < 0$ and $D > 0$, respectively, each very nearly lie on a straight line. The intersection between the two lines is $D \approx 0.04$, with a limiting value of $\zeta \approx 0.65$ for the exponent. Although this is very close to the two-thirds value frequently referred to in connection with scalar fracture, the result obtained for $D = 0.08$ does not significantly alter the $D = 0.1$ result, which is $\zeta = 0.60$. Hence, the lines may taper off at this value, the limit $|D| \to 0$ possibly representing a Laplacian random walk [17] whereby crack advancement is governed by local conditions surrounding the crack tip.

Recently the role of propagating stress waves during brittle fracture has been investigated [10]. In our model the elastic wave emitted from a breaking beam would then result in stresses exceeding those due to the elastic deformations only, the stress enhancement being especially important in the case of an imminent burst of failures. Although this feature is
not included in the present quasistatic approach, the comparison with experimental results for \( \zeta \) in two dimensions should remain valid, i.e., Poirier et al. obtained \( \zeta = 0.73 \pm 0.07 \) by considering a two dimensional stacking of parallel collapsible cylinders (drinking straws) while Kertesz et al. and Engøy et al. obtained \( \zeta \approx 0.73 \) and \( \zeta = 0.68 \pm 0.04 \) by studying tear lines in (wet) paper and fractures in thin wood plates, respectively, none of which should generate stress waves significant enough to modify the result.

To summarize, the main feature of our results is the dependency of \( \zeta \) upon the disorder, apparently contradicting a universal value. Whereas values obtained at low disorders vary considerably, however, the more or less constant \( \zeta \) obtained at moderate and strong disorders nevertheless seems to be consistent with a universal value of \( \zeta \sim 0.86 \). While thus being similar to the experimental results in three dimensions, our results are different from other two dimensional results.
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TABLE I. Typical number $N$ of lattices generated for the various system sizes $L$.

| $L$ | $N$ | $L$ | $N$ | $L$ | $N$ | $L$ | $N$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
| 23  | 800 | 40  | 300 | 80  | 150 | 160 | 50  |
| 27  | 600 | 50  | 250 | 100 | 100 | 200 | 30  |
| 32  | 500 | 63  | 200 | 125 | 75  | 250 | 20  |
FIGURES

FIG. 1. The average roughness $W$ as a function of lattice size $L$, for disorders with $D > 0$. The enumeration scheme is (a) top to (k) bottom, with $|D|$ equal to (a) 4, (b) 2, (c) 1, (d) 0.5, (e) 0.33, (f) 0.25, (g) 0.20, (h) 0.17, (i) 0.14, (j) 0.10 and (k) 0.09.

FIG. 2. The average roughness $W$ as a function of lattice size $L$, for disorders with $D < 0$. The enumeration scheme is (a) top to (k) bottom, with $|D|$ equal to (a) 4, (b) 2, (c) 1.5, (d) 1, (e) 0.50, (f) 0.33, (g) 0.25, (h) 0.20, (i) 0.14, (j) 0.10 and (k) 0.08.

FIG. 3. The roughness exponent $\zeta$ as a function of the disorder $D$, with thresholds chosen according to Eq. (7). Labels and symbols refer to the enumeration scheme used in Fig. 1 and Fig. 2, with (⋆) referring to the extrapolated value for $D \approx 0$, i.e., $\zeta = 0.65$. 

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