THE HAGEDORN-TYPE STRUCTURE OF THE GLUON PRESSURE WITHIN THE MASS GAP APPROACH TO QCD

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We have shown in detail that the low-temperature expansion for the gluon pressure has the Hagedorn-type structure. Its exponential spectrum of all the effective gluonic excitations are expressed in terms of the mass gap. It is this which is responsible for the large-scale dynamical structure of the QCD ground state. The gluon pressure properly scaled has a maximum at some characteristic temperature \( T = T_c = 266.5 \) MeV, separating the low- and high temperature regions. The gluon pressure is exponentially suppressed in the \( T \to 0 \) limit. In the \( T \to T_c \) limit it demonstrates an exponential rise in the number of dynamical degrees of freedom. This makes it possible to identify \( T_c \) with the Hagedorn-type transition temperature \( T_h \), i.e., to put \( T_h = T_c \) within our approach to QCD at finite temperature. The gluon pressure has a complicated dependence on the mass gap and temperature near \( T_c \) and up to approximately \( (4 - 5)T_c \). In the limit of very high temperatures \( T \to \infty \) its polynomial character is confirmed, containing the terms proportional to \( T^2 \) and \( T \), multiplied by the corresponding powers of the mass gap.

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I. INTRODUCTION

The properties of Quantum Chromodynamics (QCD) at finite temperature and density are subject to the intense investigations by lattice and analytic methods \cite{1,2} (and references therein). The effective potential approach for composite operators \cite{5} turned out to be effective and perspective analytical tool for the generalization of QCD to non-zero temperature and density. In the absence of external sources it is nothing but the vacuum energy density (VED), i.e., the pressure apart from the sign. This approach is nonperturbative (NP) from the very beginning, since it deals with the expansion of the corresponding skeleton vacuum loop diagrams in powers of the Planck constant, and thus allows one to calculate the VED from first principles. In accordance with this program we have extended \cite{6} to non-zero temperature \( T \) in \cite{7}. This made it possible to introduce the correctly defined temperature-dependent bag constant (bag pressure) as a function of the mass gap \( \Delta \). It is this which is responsible for the large-scale dynamical structure of the QCD ground state \cite{8} and coincides with the Jaffe-Witten (JW) mass gap \cite{9} by properties. The confining dynamics in the gluon matter (GM) is therefore nontrivially taken into account directly through the mass gap and via the temperature-dependent bag constant itself, but other NP effects due to the mass gap are also present. Being NP, the effective approach for composite operators, nevertheless, makes it possible to incorporate the thermal perturbation theory (PT) expansion in a self-consistent way. In our auxiliary work \cite{10} we have formulated and developed the analytic thermal PT which allows one to calculate the PT contributions in terms of the convergent series in integer powers of a small \( \alpha_s \). We have also explicitly derived the first PT correction of the \( \alpha_s \Delta^2 T^2 \)-order to the purely NP part of the gluon pressure. In this article we add the purely PT correction of the \( \alpha_s^3 \)-order to it. We call the sum of all the calculated terms as the gluon pressure, denoting it as \( P_g(T) \) in what follows. It will be understood as a main part of the full gluon pressure within our approach (see discussion and conclusions in sect. VIII).

From the very beginning here, we are investigating a system at non-zero temperature, which consists of \( SU(3) \) purely Yang-Mills (YM) gauge fields without quark degrees of freedom (i.e., at zero density). The primary aim of this article is to explicitly show the Hagedorn-type structure \cite{11} (and references therein) of the above-mentioned gluon pressure below some characteristic temperature \( T_c = 266.5 \) MeV, see Fig. 1. It has been missed in our previous investigations \cite{7,8,10}. Also here we have analytically and numerically calculated the \( \alpha_s^3 \)-order correction (mentioned above) to the gluon pressure \( P_g(T) \) for the first time.

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II. THE GLUON PRESSURE AT NON-ZERO TEMPERATURE

For the readers convenience in order to have a general picture at hand, we begin with short sects. II, III, and IV, in which we briefly describe our results obtained earlier in [3, 8, 10], especially taking into account that the book [8] is not freely available. In the imaginary-time formalism [12–14], all the four-dimensional integrals can be easily generalized to non-zero temperature $T$ according to the prescription

$$
\int \frac{dq_0}{(2\pi)^4} \rightarrow T \sum_{n=-\infty}^{+\infty}, \quad q^2 = q_0^2 + \omega_n^2 = \omega^2 + \omega_n^2, \quad \omega_n = 2n\pi T,
$$

(2.1)
i.e., each integral over $q_0$ of the loop momentum is to be replaced by the sum over the Matsubara frequencies labeled by $n$, which obviously assumes the replacement $q_0 \rightarrow \omega_n = 2n\pi T$ for bosons (gluons). Let us also remind that in all our publications as well as in this paper the signature is always Euclidean in order to avoid non-physical singularities at light-cone from the very beginning.

Introducing the temperature dependence into the gluon pressure [3, 8, 10], we obtain

$$
P_g(T) = P_{NP}^T(T) + P_M(T) = B_{YM}(T) + P_{YM}(T) + P_M(T),
$$

(2.2)

where the corresponding terms in frequency-momentum space are:

$$
B_{YM}(T) = \frac{8}{\pi^2} \int_0^{\omega_{eff}} d\omega \omega^2 T \sum_{n=-\infty}^{+\infty} \left[ \ln \left( 1 + 3\alpha_{INP}(\omega^2, \omega_n^2) \right) - \frac{3}{4} \alpha_{INP}(\omega^2, \omega_n^2) \right],
$$

(2.3)

$$
P_{YM}(T) = -\frac{8}{\pi^2} \int_0^{\omega_{eff}} d\omega \omega^2 T \sum_{n=-\infty}^{+\infty} \left[ \ln \left( 1 + \frac{3}{4} \alpha_{INP}(\omega^2, \omega_n^2) \right) - \frac{3}{4} \alpha_{INP}(\omega^2, \omega_n^2) \right],
$$

(2.4)

$$
P_M(T) = -\frac{8}{\pi^2} \int_{\Lambda_{YM}}^{\omega_{eff}} d\omega \omega^2 T \sum_{n=-\infty}^{+\infty} \left[ \ln \left( 1 + \frac{3}{4} \alpha_{INP}(\omega^2, \omega_n^2) \right) - \frac{3}{4} \alpha_{INP}(\omega^2, \omega_n^2) \right].
$$

(2.5)

In frequency-momentum space the intrinsically nonperturbative (INP) and PT effective charges become

$$
\alpha_{INP}(q^2) = \frac{\Delta^2}{q^2} = \alpha_{INP}(\omega^2, \omega_n^2) = \frac{\Delta^2}{\omega^2 + \omega_n^2},
$$

(2.6)

denoting

$$
\alpha_{PT}(q^2) = \frac{\alpha_s}{1 + \alpha_s b_0 \ln(q^2/\Lambda_{YM}^2)} = \alpha_{PT}(\omega^2, \omega_n^2) = \frac{\alpha_s}{1 + \alpha_s b_0 \ln(\omega^2 + \omega_n^2/\Lambda_{YM}^2)},
$$

(2.7)

respectively. The last term (2.5) is called mixed (M) since it depends on both effective charges. It is also convenient to introduce the following standard notations:

$$
T^{-1} = \beta, \quad \omega = \sqrt{q^2},
$$

(2.8)

where, evidently, in all the expressions $q^2$ is the square of the three-dimensional loop momentum, in complete agreement with the relations (2.1), and $\omega_{eff}$ is a scale separating the low- and high frequency-momentum regions.

In eq. (2.6) $\Delta^2$ is the mass gap, mentioned above, which is responsible for the large-scale dynamical structure of the QCD vacuum, and thus determines the scale of its NP dynamics. We have shown that confining effective charge (2.6), and hence its $\beta$-function, is a result of the summation of the skeleton (i.e., NP) loop diagrams, contributing to the full gluon self-energy in the $q^2 \rightarrow 0$ limit (the strong coupling regime for the effective charge). This summation has been performed within the corresponding equation of motion. It has been done without violating the $SU(3)$ color
gauge invariance of QCD \[8\] (and references therein). In more detail the derivation of the bag constant as a function of the mass gap and its generalization to non-zero temperature has been completed in \[6\] and \[7\], respectively.

The PT effective charge \(\alpha_{\text{PT}}(q^2)\) (2.7) is the generalization to non-zero temperature of the renormalization group equation solution, the so-called sum of the main PT logarithms \[8, 15–17\] (its analog as a function of the variable \(T/T_c\) (see below) can be found, for example in \[8, 13, 18\]). Here \(\Lambda_{YM}^2 = 0.09 \text{ GeV}^2\) \[19\] is the asymptotic scale parameter for \(SU(3)\) YM fields, and \(b_0 = (11/4\pi)\) for these fields, while the strong fine-structure constant is \(\alpha_s \equiv \alpha_s(m_Z) = 0.1185(6)\) \[20\]. In eq. (2.7) \(q^2\) cannot go below \(\Lambda_{YM}^2\), i.e., \(\Lambda_{YM}^2 \leq q^2 \leq \infty\), which has already been symbolically shown in eq. (2.5).

It is worth reminding that the separation between effective charges (2.6) and (2.7), is not only exact but it is unique \[6, 8\].

The purely NP pressure \(P_{NP}^g(T) = B_{YM}(T) + P_{YM}(T)\) and the mixed pressure \(P_M(T)\), and hence the gluon pressure \(P_g(T)\) (2.2) itself, are normalized to zero when the interaction is formally switched off, i.e., letting \(\alpha_s = \Delta^2 = 0\). This means that the initial normalization condition of the free PT vacuum to zero also holds at non-zero temperature.

### III. \(P_{NP}^g(T)\) CONTRIBUTION

One of the attractive features of the confining effective charge (2.6) is that it allows an exact summation over the Matsubara frequencies in the purely NP pressure \(P_{NP}^g(T)\) given by the sum of the integrals (2.3) and (2.4). Collecting all the analytical results obtained in \[7, 8\], we can write

\[
P_{NP}^g(T) = B_{YM}(T) + P_{YM}(T) = \frac{6}{\pi^2} \Delta^2 P_1(T) + \frac{16}{\pi^2} T N(T). \tag{3.1}
\]

Here \(P_1(T)\) and \(N(T)\) are

\[
P_1(T) = \int_{\omega_{eff}}^{\infty} \frac{\omega}{e^{\beta \omega} - 1}, \tag{3.2}
\]

and

\[
N(T) = [P_2(T) + P_3(T) - P_4(T)], \tag{3.3}
\]

respectively, while

\[
P_2(T) = \int_{\omega_{eff}}^{\infty} \omega^2 \ln \left(1 - e^{-\beta \omega}\right),
\]

\[
P_3(T) = \int_0^{\omega_{eff}} \omega^2 \ln \left(1 - e^{-\beta \omega'}\right),
\]

\[
P_4(T) = \int_0^{\infty} \omega^2 \ln \left(1 - e^{-\beta \bar{\omega}}\right), \tag{3.4}
\]

and \(\omega'\) and \(\bar{\omega}\) are given by the relations

\[
\omega' = \sqrt{\omega^2 + 3\Delta^2} = \sqrt{\omega^2 + m_{eff}^2}, \quad \bar{\omega} = \sqrt{\omega^2 + \frac{3}{4} \Delta^2} = \sqrt{\omega^2 + \bar{m}_{eff}^2}. \tag{3.5}
\]

It is worth reminding that in the purely NP pressure (3.1) the bag pressure \(B_{YM}(T)\) (2.3) is responsible for the formation of the massive gluonic excitations \(\omega'\), while the YM part \(P_{YM}(T)\) (2.4) is responsible for the formation of the massive gluonic excitations \(\bar{\omega}\).

The so-called gluon mean number \[12\], also known as Bose-Einstein distribution, is

\[
N_g \equiv N_g(\beta, \omega) = \frac{1}{e^{\beta \omega} - 1}, \tag{3.6}
\]
where $\beta$ and $\omega$ are defined in eq. (2.8). It appears in the integrals (3.3)-(3.4) and describes the distribution and correlation of massless gluons in the medium. Replacing $\omega$ by $\bar{\omega}$ and $\omega'$ we can consider the corresponding gluon mean numbers as describing the distribution and correlation of the corresponding massive gluonic excitations in the medium, see integrals $P_3(T)$ and $P_4(T)$ in eqs. (3.4). They are of NP dynamical origin, since their corresponding masses are due to the mass gap, namely $m'_{eff} = \sqrt{3} \Delta$ and $\bar{m}_{eff} = (\sqrt{3}/2) \Delta$, see (3.5). All three different gluon mean numbers range continuously from zero to infinity [12]. We have the two different massless excitations, propagating in accordance with the integral (3.2) and the first of the integrals (3.4). However, they are not free, since in the PT $\Delta^2 = 0$ limit they vanish (the composition (3.3) becomes zero in this case). So the purely NP pressure $P^{NP}_T(T)$ (3.1) describes the four different effective gluonic excitations. The gluon mean numbers are closely related to the thermodynamic observables, especially to the pressure. Its exponential suppression in the $T \to 0$ limit and the polynomial structure in the $T \to \infty$ limit are determined by the corresponding asymptotics of the gluon mean numbers, see below.

Concluding, let us emphasize that the effective scale $\omega_{eff}$ is not an independent scale parameter. Due to extremization of the mass gap-dependent effective potential, from the stationary condition at zero temperature in [7] it follows that

$$\omega_{eff} = 1.48 \Delta, \quad \Delta = 0.6756 \text{ GeV}. \quad (3.7)$$

So it is expressed in terms of the initial fundamental and unique mass scale parameter in our approach - the mass gap $\Delta$ (for simplicity, its squared version $\Delta^2$ is conventionally called the mass gap as well throughout this paper). The introduction of $\omega_{eff}$ is also convenient from the technical point of view in order to simplify our expressions, which otherwise would be rather cumbersome (see below).

### IV. THERMAL PT

One of our primary goals in [10] was to develop the analytic formalism for the numerical calculation of the mixed term (2.5). It made it possible to calculate the PT contributions to the gluon pressure (2.2) in terms of the convergent series in integer powers of a small $\alpha_s$. For this goal, it is convenient to re-write the integral (2.5) as follows:

$$P_M(T) = -\frac{8}{\pi^2} \int_{\Lambda_{YM}}^{\infty} d\omega \omega^2 T \sum_{n=-\infty}^{+\infty} \left[ \ln[1 + x(\omega^2, \omega_n^2)] - \frac{3}{4} \alpha^{PT}(\omega^2, \omega_n^2) \right], \quad (4.1)$$

where

$$x(\omega^2, \omega_n^2) = \frac{3 \alpha^{PT}(\omega^2, \omega_n^2)}{4 + 3 \alpha^{PT}(\omega^2, \omega_n^2)} = \frac{3}{4} \frac{\omega^2 + \omega_n^2}{M(\omega^2, \omega_n^2)} \frac{\alpha_s}{1 + \alpha_s \ln z_n} \quad (4.2)$$

with the help of the expressions (2.6) and (2.7), and where

$$M(\omega^2, \omega_n^2) = \omega^2 + \omega_n^2, \quad \ln z_n \equiv \ln z(\omega^2, \omega_n^2) = b_0 \ln[\Lambda_{YM}^2/(\omega^2 + \omega_n^2)], \quad (4.3)$$

and $\bar{\omega}^2$ is given in eq. (3.5). Let us also note that in these notations $\alpha^{PT}(\omega^2, \omega_n^2)$ shown in eq. (2.7), becomes

$$\alpha^{PT}(\omega^2, \omega_n^2) \equiv \alpha(z_n) = \frac{\alpha_s}{1 + \alpha_s \ln z_n}. \quad (4.4)$$

Collecting all the results obtained in [8, 10], where it has been explicitly shown that variable $x(\omega^2, \omega_n^2)$ always is very small, we are able to present the mixed part of the gluon pressure (4.1) as a sum of the two terms, namely

$$P_M(T) = P^s_{NP}(T) + P^s_{PT}(T) \quad (4.5)$$

where

$$P^s_{NP}(T) = \sum_{k=1}^{\infty} \alpha_s^k P_k(\Delta^2; T) \quad (4.6)$$
with
\[ P_k(\Delta^2; T) = \frac{9}{2\pi^2} \Delta^2 \int_{\Lambda_{YM}}^\infty d\omega \ \omega^2 T \sum_{n=-\infty}^{+\infty} \left[ \frac{1}{M(\omega^2, \omega_n^2)} (-1)^{k-1} \ln^{k-1} z_n \right], \] (4.7)

while \( P_{PT}^s(T) \) describes the pure PT contribution to the gluon pressure \( P_g(T) \), namely
\[ P_{PT}^s(T) = -\frac{9}{2\pi^2} \alpha_s^2 \int_{\Lambda_{YM}}^\infty d\omega \ \omega^2 T \sum_{m=-\infty}^{+\infty} \left[ \sum_{m=0}^{\infty} \left( \frac{3}{4} \right)^m \frac{\alpha_s^m}{m+2} \sum_{k=0}^{\infty} c_k (m+2) \alpha_s^k \ln^k z_n \right], \] (4.8)
and the coefficients \( c_k(m+2) \) are defined as follows:
\[ c_0(m) = 1, \quad c_p(m) = \frac{1}{p} \sum_{k=1}^{p} (km - p + k)(-1)^k c_{p-k}, \quad p \geq 1. \] (4.9)

Here \( P_{NP}^s(T) \) (4.6) describes the \( \Delta^2 \)-dependent contribution, beginning with the \( \alpha_s \)-order term. In fact, the whole expansion (4.6) is the correction in integer powers of \( \alpha_s \) to the purely NP pressure \( P_{NP}^g(T) \) (3.1). Let us also note that in the expression (4.5) is not taken into account the \( \alpha_s^3 \)-order independent correction to the \( P_{NP}^g(T) \) term. It also depends on the mass gap \( \Delta^2 \), but numerically it is very small in comparison with \( P_{NP}^s(T) \). For this reason it is omitted from the consideration, as it was pointed out above. It is worth emphasizing that all the series which arise from the initial term (4.1) in integer powers of a small \( x(\omega^2, \omega_n^2) \) (or, equivalently, \( \alpha_s \)) are convergent [8, 10]. Thus, the approximation of the PT effective charge by the summation of the main PT logarithms (2.7) is fully sufficient to calculate all the PT corrections to leading orders in powers of a small \( \alpha_s \).

Concluding this section, let us note that in order to clarify and simplify notations of [8, 8, 10] we change notation \( P_{NP}(T) \) there to \( P_{NP}^s(T) \) here. Also \( P_{PT}(T) \) to \( P_{M}(T) \), \( P_{PT}(\Delta^2; T) \) to \( P_{NP}^s(T) \) and \( P_{PT}(T) \) to \( P_{PT}^s(T) \), while retaining the same notation for \( P_g(T) \).

V. THE GLUON PRESSURE \( P_g(T) \)

Taking into account the above-mentioned remarks and eq. (4.5), the gluon pressure (2.2) then becomes
\[ P_g(T) = P_{NP}^s(T) + P_{NP}^a(T) + P_{PT}^s(T). \] (5.1)
In the integral (4.7) for \( k = 1 \) the summation over the Matsubara frequencies can be performed analytically, i.e., exactly [8, 8, 10]. Absolutely in the same way, we can perform the summation over the Matsubara frequencies in the integral (4.8) at \( k = 1 \), while the summation over \( m \) can be done exactly as well (note that at \( k = 0 \) this integral is zero, for some details of calculation see appendix A). So finally for the gluon pressure \( P_g(T) \) (5.1), one obtains
\[ P_g(T) = P_{NP}(T) + P_{PT}(T) = P_{NP}^s(T) + P_{NP}^a(T) + P_{PT}^s(T), \] (5.2)
on account of eqs. (3.1)-(3.4) and where
\[ P_{NP}^s(T) = \alpha_s \times \frac{9}{2\pi^2} \Delta^2 \int_{\Lambda_{YM}}^\infty d\omega \ \omega^2 \frac{1}{\omega} \frac{1}{e^{\beta \omega} - 1} \] (5.3)
and
\[ P_{PT}^s(T) = \left( \frac{9b_0}{\pi^2} \frac{\alpha_s^3}{1 + (3/4)\alpha_s} \right) \frac{1}{T} \int_{\Lambda_{YM}}^\infty d\omega \ \omega^2 \ln (1 - e^{-\beta \omega}), \] (5.4)
where, obviously, we retain the same notations for the integrals (4.6) and (4.8) at \( k = 1 \), for convenience. It is instructive to explicitly show that the last integral, being purely PT, can be expressed as the so-called SB-type term as follows:
FIG. 1: The gluon pressure (5.2) scaled (i.e., divided) by $T^4/3$ is shown as a function of $T/T_c$ (solid curve). It has a maximum at $T = T_c = 266.5$ MeV (vertical solid line). The horizontal dashed line is the general Stefan-Boltzmann (SB) constant $3P_{SB}(T)/T^4 = (24/45)\pi^2$. One can conclude that NP effects due to the mass gap are still important approximately up to $5T_c$.

\[P^g_{PT}(T) = q_0 q(T) P_{SB}(T), \tag{5.5}\]

where

\[q(T) = T^{-3} \int_{N_{YM}}^{\infty} d\omega \omega^2 \ln \left(1 - e^{-\beta \omega}\right), \tag{5.6}\]

while

\[q_0 = \left(\frac{9b_0}{\pi^2} \frac{\alpha_s^3}{1 + (3/4)\alpha_s}\right)\left(\frac{45}{8\pi^2}\right), \tag{5.7}\]

and this constant is numerically very small $q_0 \sim 0.0007$, indeed.

It is worth noting that the NP term (5.3) describes the same massive gluonic excitation $\bar{\omega}$ (3.5), but its propagation, however, suppressed by the $\alpha_s$-order. In the free PT $\alpha_s = \Delta^2 = 0$ limit, the above-defined composition $N(T)$ becomes zero as well as $P^g_{NP}(T)$ itself, as it follows from eqs. (3.1)-(3.4). Thus the gluon pressure $P_g(T)$ (5.2), satisfies to the normalization condition of the free PT vacuum to zero, as underlined above. Numerically calculated the gluon pressure (5.2) is shown in fig. 1. It has a maximum at some “characteristic” temperature $T = T_c = 266.5$ MeV. Let us now analytically investigate the low-temperature (below $T_c$) behavior of the gluon pressure (5.2) in more detail. It will make it possible to explicitly show the Hagedorn-type nature of the corresponding expansion in this region. At the same time, its high-temperature (above $T_c$) behavior suffices to briefly discuss.

VI. LOW-TEMPERATURE EXPANSION. THE HAGEDORN-TYPE STRUCTURE

In order to investigate the behavior of the gluon pressure (5.2) in the low-temperature region ($T \leq T_c$) it is convenient to present it as follows:

\[P_g(T) = \frac{6}{\pi^2} \Delta^2 P_1(T) + \frac{16}{\pi^2} T N(T) + P^s_{NP}(T) + P^s_{PT}(T), \tag{6.1}\]

using eqs. (3.1)-(3.4) and integrals (5.3) and (5.4). Let us note that in the integrals (3.2), (3.4) and (5.3), (5.4) the variable $e^{-\beta \omega}$ with the replacements $\omega \to \omega'$, $\bar{\omega}$ is always small in this region, especially in the ($T \to 0$, $\beta = T^{-1} \to \infty$)
limit. So one can expand the corresponding mean numbers (3.6) in the form of the corresponding Taylor series as follows:

\[ N_g \equiv N_g(\beta, \omega) = \frac{1}{e^{\beta \omega} - 1} = e^{-\beta \omega}(1 - e^{-\beta \omega})^{-1} = \sum_{n=1}^{\infty} e^{-n \beta \omega} \] (6.2)

and

\[ \ln (1 - e^{-\beta \omega}) = -\sum_{n=1}^{\infty} \frac{1}{n} e^{-n \beta \omega} \] (6.3)

with the above-mentioned replacements (here \( n \) is different from \( n \) in eqs. (2.3)-(2.5)). After substitution of these series into the corresponding integrals, such obtained terms can be explicitly integrated termwise, since the Taylor series (6.2) and (6.3) are convergent in this temperature region and integrals calculated in this section are not divergent.

Let us begin with pointing out in advance that all exactly calculated integrals, discussed below can be found in \[21, 22\]. So the integral \( P_1(T) \) defined in eq. (3.2) becomes

\[ P_1(T) = \int_{\omega_{\text{eff}}}^{\infty} d\omega \omega N_g(\beta, \omega) = \int_{\omega_{\text{eff}}}^{\infty} d\omega \omega \sum_{n=1}^{\infty} e^{-n \beta \omega} = \sum_{n=1}^{\infty} \int_{\omega_{\text{eff}}}^{\infty} d\omega \omega e^{-n \beta \omega}. \] (6.4)

The almost trivial integration yields

\[ P_1(T) = \sum_{n=1}^{\infty} \left( \frac{1}{n^2} T^2 + \frac{1}{n} \omega_{\text{eff}} T \right) e^{-n \omega_{\text{eff}}}. \] (6.5)

The integral \( P_2(T) \) defined in eqs. (3.4) can be considered in the same way after the substitution of the expansion (6.3), so it becomes

\[ P_2(T) = \int_{\omega_{\text{eff}}}^{\infty} d\omega \omega^2 \ln (1 - e^{-\beta \omega}) = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{\omega_{\text{eff}}}^{\infty} d\omega \omega^2 e^{-n \beta \omega}, \] (6.6)

and exactly integrating it, one obtains

\[ P_2(T) = -\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{2}{n^3} T^3 + \frac{2}{n^2} \omega_{\text{eff}} T^2 + \frac{1}{n} \omega_{\text{eff}}^2 T \right) e^{-n \omega_{\text{eff}}}. \] (6.7)

The integral \( P_3(T) \) defined in eqs. (3.4) after the substitution of the expansion (6.3) with the replacement \( \omega \to \omega' \) looks like

\[ P_3(T) = \int_{0}^{\omega_{\text{eff}}} d\omega \omega^2 \ln (1 - e^{-\beta \omega'}) = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{\omega_{\text{eff}}} d\omega \omega^2 e^{-n \beta \omega'}. \] (6.8)

Replacing the variable \( \omega \) by the variable \( \omega' \) in accordance with the relation (3.5), this integral becomes

\[ P_3(T) = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{a}^{\omega_{\text{eff}}} d\omega' \omega' \sqrt{\omega'^2 - a^2} e^{-n \beta \omega'}, \] (6.9)

where

\[ \omega'_{\text{eff}} = \sqrt{(\omega_{\text{eff}}^2 + a^2)}, \quad a = \sqrt{3} \Delta. \] (6.10)
Noting further that the variable \( x = a^2/\omega^2 \leq 1 \), we can formally expand

\[
\sqrt{\omega^2 - a^2} = \omega'(1-x)^{1/2} = \omega' \left[ 1 - \frac{1}{2} \frac{a^2}{\omega^2} + \sum_{k=2}^{\infty} \frac{1/2}{k} (-x)^k \right],
\]

(6.11)

then from the last integral it follows

\[
P_3(T) = - \sum_{n=1}^{\infty} \frac{1}{n} \int_a^{\omega_{eff}'} \omega' \omega^2 e^{-n\beta\omega'} + \frac{3}{2} \Delta^2 \sum_{n=1}^{\infty} \frac{1}{n} \int_a^{\omega_{eff}'} \omega' e^{-n\beta\omega'} - \sum_{n=1}^{\infty} \frac{1}{n} P_3^{(n)}(T),
\]

(6.12)

where

\[
P_3^{(n)}(T) = - \int_a^{\omega_{eff}'} \omega' \omega^2 e^{-n\beta\omega'} \sum_{k=2}^{\infty} \frac{1/2}{k} (-x)^k.
\]

(6.13)

Let us consider the last integral (6.13) in more detail. Since the series over \( k \) are convergent in the interval of integration and the functions depending on \( k \) are integrable in this interval, these series may be integrated termwise [18], that is,

\[
P_3^{(n)}(T) = - \sum_{k=2}^{\infty} \frac{1/2}{k} (-a^2)^k \int_a^{\omega_{eff}'} \omega' e^{-n\beta\omega'}.
\]

(6.14)

Integrating it, one obtains

\[
P_3^{(n)}(T) = - \sum_{k=2}^{\infty} \frac{1/2}{k} (-a^2)^k \left[ N_3^{(n,k)}(T, \omega') \right]_{a}^{\omega_{eff}'}
\]

(6.15)

and \( \left[ N_3^{(n,k)}(T, \omega') \right]_{a}^{\omega_{eff}'} \) denotes the result of the integration over \( \omega' \) in eq. (6.14) in the interval \([a, \omega_{eff}']\), while the function \( N_3^{(n,k)}(T, \omega') \) itself is

\[
N_3^{(n,k)}(T, \omega') = -e^{-n\beta\omega'} \sum_{m=1}^{2k-3} \frac{(-n\beta)^{m-1}(\omega')^{m+2-2k}}{(2k-3)(2k-4)...(2k-2-m)} + (-n\beta)^{2k-3} \frac{(2k-3)!}{(2k-3)!} \text{Ei}(-n\beta\omega'), \quad k = 2, 3, 4, ...
\]

(6.16)

The series for the exponential integral function \( \text{Ei}(-n\beta\omega') \) is [18]

\[
\text{Ei}(-n\beta\omega') = e^{-n\beta\omega'} \sum_{l=1}^{p} (-1)^l \frac{l-1!}{(n\beta\omega')^l} + R_p,
\]

(6.17)

where the relative error in the expansion (6.17) should satisfy \(| R_p | < p!/(n\beta\omega')^{p+1} \) for real numbers. If one chooses \( p = 2k - 4 \) in the previous equation and correspondingly adjusting the relative error \( R_p \), it is easy to show that both terms in eq. (6.16) for \( N_3^{(n,k)}(T, \omega') \) cancel each other termwise for any \( k \geq 2 \), and thus

\[
N_3^{(n,k)}(T, \omega') = 0, \quad k = 2, 3, 4, ...
\]

(6.18)

which leads to

\[
P_3^{(n)}(T) = 0
\]

(6.19)
via eq. (6.15). Equivalently, we can choose \( p = 2k - 3 \) and neglecting \( R_p \), then both terms in eq. (6.16) for \( N_{3(n,k)}^{(n)}(T, \omega') \) again will cancel each other termwise for any \( k \geq 2 \), by neglecting the term of the same order as \( R_p \) in the first sum of eq. (6.16). Going back to eq. (6.12) and easily integrating the first two terms, and taking into account the previous result, one comes to the following expression, namely

\[
P_3(T) = \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{2}{n^3} T^3 + \frac{2}{n^2} \omega_{eff} T^2 + \frac{1}{n} \omega_{eff}^2 T \right) e^{-n \frac{T}{\omega_{eff}}} - \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{2}{n^3} T^3 + \frac{2}{n^2} aT^2 + \frac{1}{n} a^2 T \right) e^{-n \frac{T}{a}} - \frac{1}{2} a^2 T \sum_{n=1}^{\infty} \frac{1}{n^2} \left( e^{-n \frac{T}{a}} - e^{-n \frac{T}{a}} \right).
\]  

(6.20)

The integral \( P_4(T) \) defined in eqs. (3.4) after the substitution of the expansion (6.3) with the replacement \( \omega \to \tilde{\omega} \) looks like

\[
P_4(T) = \int_{0}^{\infty} d\omega \, \omega^2 \ln \left( 1 - e^{-\beta \omega} \right) = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{\infty} d\omega \, \omega^2 e^{-n \beta \omega},
\]  

(6.21)

and replacing the variable \( \omega \) by the variable \( \tilde{\omega} \) in accordance with the relation (3.5), this integral becomes

\[
P_4(T) = -\sum_{n=1}^{\infty} \frac{1}{n} \int_{(a/2)}^{\infty} d\tilde{\omega} \, \tilde{\omega} \sqrt{(\tilde{\omega}^2 - (a/2)^2)} \, e^{-n \beta \tilde{\omega}}.
\]  

(6.22)

Comparing eq. (6.9) with this eq. (6.22), one can conclude that the last one is the first one by putting formally \( \omega_{eff} = \infty \) and replacing \( a \to a/2 \). Doing so in the expansion (6.20), for integral (6.22) one finally obtains

\[
P_4(T) = -\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{2}{n^3} T^3 + \frac{a}{n^2} T^2 + \frac{a^2}{4n} T \right) e^{-n \frac{T}{2a}} + \frac{1}{8} a^2 T \sum_{n=1}^{\infty} \frac{1}{n^2} e^{-n \frac{T}{2a}}.
\]  

(6.23)

Let us now consider eq. (5.3), which after the substitution of the expansion (6.2) with the replacement \( \omega \to \tilde{\omega} \) becomes

\[
P_{np}(T) = \frac{9\alpha}{2\pi^2} \Delta^2 \int_{\Lambda_{YM}} d\omega \, \omega^2 \left( \frac{1}{\omega} \frac{1}{e^{\beta \omega} - 1} \right) = \frac{9\alpha}{2\pi^2} \Delta^2 \sum_{n=1}^{\infty} \int_{\Lambda_{YM}} d\omega \, \omega^2 \frac{1}{\omega} e^{-n \beta \tilde{\omega}},
\]  

(6.24)

and \( \tilde{\omega} \) is given by the relation (3.5). Replacing the variable \( \omega \) by the variable \( \tilde{\omega} \), one obtains

\[
P_{np}(T) = \frac{9\alpha}{2\pi^2} \Delta^2 \sum_{n=1}^{\infty} \int_{\tilde{\omega}_{eff}}^{\infty} d\tilde{\omega} \, \sqrt{(\tilde{\omega}^2 - (a/2)^2)} \, e^{-n \beta \tilde{\omega}}, \quad \tilde{\omega}_{eff} = \sqrt{\Lambda_{YM}^2 + (a/2)^2},
\]  

(6.25)

and for \( a \) see eq. (6.10). Noting that the variable \( z = a^2/4\tilde{\omega}^2 < 1 \) in this case, we can use the expansion like (6.11), taking into account only the substitution \( a \to a/2 \), in order to obtain

\[
P_{np}(T) = \frac{9\alpha}{2\pi^2} \Delta^2 \sum_{n=1}^{\infty} \left[ \int_{\tilde{\omega}_{eff}}^{\infty} d\tilde{\omega} \, \tilde{\omega} e^{-n \beta \tilde{\omega}} - \frac{1}{8} a^2 \int_{\tilde{\omega}_{eff}}^{\infty} d\tilde{\omega} \, \frac{e^{-n \beta \tilde{\omega}}}{\tilde{\omega}} + P_{np}^{(n)}(T) \right].
\]  

(6.26)

Due to the same formalism which has been used previously in order to get the result (6.19), one can conclude that \( P_{np}^{(n)}(T) = 0 \) as well. Easily integrating the first two terms, one comes to the following expansion

\[
P_{np}(T) = \frac{9\alpha}{2\pi^2} \Delta^2 \sum_{n=1}^{\infty} \left( \frac{1}{n^2} T^2 + \frac{1}{n} \tilde{\omega}_{eff}^2 T \right) e^{-n \frac{\tilde{\omega}_{eff}^2}{T}} + \frac{1}{8} a^2 Ei(-n \frac{\tilde{\omega}_{eff}^2}{T}),
\]  

(6.27)
where the corresponding exponential integral function is defined by eq. (6.17) and

$$\tilde{\omega}_{eff} = \sqrt{\Lambda_{YM}^2 + (3/4)\Delta^2}. \quad (6.28)$$

The integral $P_{PT}^g(T)$ defined in eq. (5.4) can be considered in the same way after the substitution of the expansion (6.3), so it becomes

$$P_{PT}^g(T) = -\left(\frac{9b_0}{\pi^2} \frac{\alpha_s^3}{1 + (3/4)\alpha_s} \right) T^2 \sum_{n=1}^{\infty} \frac{1}{n} \int_{\Lambda_{YM}}^{\infty} d\omega \omega^2 e^{-n\beta_0\omega}, \quad (6.29)$$

and exactly integrating it, one obtains

$$P_{PT}^g(T) = -\left(\frac{9b_0}{\pi^2} \frac{\alpha_s^3}{1 + (3/4)\alpha_s} \right) \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{2}{n^2} T^4 + \frac{2}{n} \Lambda_{YM} T^3 + \Lambda_{YM}^2 T^2 \right) e^{-n(\Lambda_{YM}/T)}. \quad (6.30)$$

Collecting all our results of the corresponding integrations and after some re-arrangement of the terms, as well as introducing the explicit dependence on the mass gap with the help of the relations (3.7) and (6.10), for the gluon pressure (5.2) one finally obtains

$$P_g(T) = P_{NP}(T) + P_{PT}^g(T) = P_{NP}^g(T) + P_{NP}^g(T) + P_{PT}^g(T)$$

$$= \frac{6}{\pi^2} \Delta^2 T^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \left( 1 + 1.48n \frac{\Delta}{T} \right) e^{-1.48n(\Delta/T)} - 4 \left( e^{-2.28n(\Delta/T)} - e^{-\sqrt{3}n(\Delta/T)} \right) - e^{-\left(\sqrt{3}/2\right)n(\Delta/T)} \right]$$

$$+ \frac{16}{\pi^2} T^4 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \left( \frac{2}{n^2} + \frac{4.56 \Delta}{n T} + 5.19 \frac{\Delta^2}{T^2} \right) e^{-2.28n(\Delta/T)} - \left( \frac{2}{n^2} + \frac{2.96 \Delta}{n T} + 2.19 \frac{\Delta^2}{T^2} \right) e^{-1.48n(\Delta/T)} \right]$$

$$- \frac{16}{\pi^2} T^4 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \left( \frac{2}{n^2} + \frac{2\sqrt{3} \Delta}{n T} + 3 \frac{\Delta^2}{T^2} \right) e^{-\sqrt{3}n(\Delta/T)} - \left( \frac{2}{n^2} + \frac{3 \Delta}{n T} + \frac{3 \Delta^2}{4 T^2} \right) e^{-\left(\sqrt{3}/2\right)n(\Delta/T)} \right]$$

$$+ \frac{9}{2\pi^2} \alpha_s \Delta^2 T^2 \sum_{n=1}^{\infty} \frac{1}{n^2} \left[ \left( 1 + \frac{n\tilde{\omega}_{eff}}{T} \right) e^{-n\tilde{\omega}_{eff}/T} + \frac{3}{8} n^2 \frac{\Delta^2}{T^2} \text{Ei}(-n\tilde{\omega}_{eff}/T) \right]$$

$$- q_0 P_{SB}(T) \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{2}{n^2} + \frac{2 \Lambda_{YM}}{n T} + \frac{\Lambda_{YM}^2}{T^2} \right) e^{-n(\Lambda_{YM}/T)}, \quad T \leq T_c. \quad (6.31)$$

where $\tilde{\omega}_{eff}$ is given in (6.28) and the last pure PT contribution (6.30) is written down as the SB-type term in the limit of low temperatures, (see eqs. (5.5-5.7)). The expression (6.31) is nothing else but the Hagedorn-type expansion for the gluon pressure in the low-temperature region. Its effective gluonic excitations are mainly expressed in terms of the mass gap. It is dynamically generated by the strong self-interaction of massless gluon modes, and thus is responsible for all the NP effects in the YM ground-state at any temperature [3]. All these effective gluonic excitations are of the NP origin. They vanish from GM spectrum in the PT $\Delta^2 = 0$ limit, apart from the last term in the expansion (6.31). It is not only exponential suppressed, but it is strongly suppressed, at least, by the $\alpha_s^2$-order as well, see eq. (6.30). This means that it can not penetrate deep into the low-temperature region in comparison with all other NP terms (as it should be expected). However, close to $T_c$ numerically it can be compared with the NP terms in this expansion. That is the reason why it has to be carefully taken into account, especially when we will go to the full gluon pressure. Other interesting features of the expansion (6.31) are: a non-analytical dependence on the mass gap $\Delta^2$ in some terms $\sim \Delta^2(\Delta^2)^{1/2}T \sim \Delta^4T$ and $\sim (\Delta^2)^{1/2}T^3 \sim \Delta^3T$. The PT correction of the $\alpha_s$-order depends on the mass gap squared analytically. The presence of terms $\sim T^4$, being, nevertheless, of the NP origin, since the overall coefficient in front of them becomes zero in the PT $\Delta^2 = 0$ limit (as underlined above). This is in agreement with the initial normalization condition of the free PT vacuum to zero.

It is instructive to show this expansion as a function of the variable $T_c/T$. It suffices to do this by introducing the corresponding number of the exponents, using the numerical values of the mass gap, characteristic temperature and $\Lambda_{YM}$, respectively, namely $\Delta = 0.6756$ GeV, $T_c = 0.2665$ GeV and $\Lambda_{YM} = 0.3$ GeV, and taking into account the relation (6.28). Such kind of the expansion looks like
So close to $T_c$ this expansion shows an exponential rise in the number of dynamical degrees of freedom in the $T \to T_c$ limit, explicitly seen in fig. 1. In the opposite $T \to 0$ limit the gluon pressure is exponentially suppressed. The maximum of temperature at which the Hagedorn-type expansion (6.32) is valid is $T_c$, then it makes sense to identify $T_c$ with the Hagedorn-type transition temperature $T_h$, i.e., to put $T_h = T_c$ within our approach (and see discussion in sect. VIII as well). It is worth underlying once more that the Hagedorn-type expansion (6.31) or, equivalently, (6.32) is nothing else but the gluon pressure (5.2) in the low-temperature region $T \leq T_h = T_c$.

Concluding this part, let us stress that the Hagedorn-type pressure (6.31), and hence (6.32), is closely related to the asymptotic of the gluon mean number (6.2) in the low-temperature region $T \leq T_c = T_h$, and on account of the replacements $\omega \to \omega', \tilde{\omega}$ in it. It is even possible to say that the Hagedorn-type structure of the expansion (6.31) is determined by them in this temperature interval within the mass gap approach to QCD at finite temperature. In other words, it was not introduced by hand, but it was due to the corresponding asymptotics of the gluon mean numbers and the structure of the gluon pressure as a function of the mass gap.

**VII. HIGH-TEMPERATURE EXPANSION. THE POLYNOMIAL STRUCTURE**

In order to investigate the behavior of the gluon pressure (5.2) in the high-temperature region ($T \geq T_c$), it is convenient to re-write it as follows:

$$P_g(T) = P_{NP}(T) + P_{PT}(T) = P_{NP}^0(T) + P_{NP}^s(T) + P_{PT}(T),$$

(7.1)

where

$$P_{NP}^0(T) = \Delta^2 T^2 - \frac{6}{\pi^2} \Delta^2 P_1(T) + \frac{16}{\pi^2} TN(T) + P_{NP}^s(T) + P_{PT}(T).$$

It is easy to show that the expressions (6.1) and (7.1) are the same, because of the relations $P_1(T) = (\pi^2/6) T^2 - P_1(T)$, $\int_0^\infty d\omega/\omega e^{\beta \omega} - 1 = (\pi^2/6) T^2$, where the integral $P_1(T)$ is explicitly given in eq. (3.2). At moderately high temperatures up to approximately a few $T_c$ the exact functional dependence on the mass gap $\Delta^2$ and temperature $T$ of the gluon pressure (7.1) remains rather complicated. From fig. 1 it follows that the NP effects due to the mass gap are still important up to rather high temperature, estimated as $(4 - 5)T_c$. The gluon pressure has a polynomial character in integer powers of $T$ up to $T^2$ at very high temperatures only (see below). As mentioned above, it is related to the corresponding asymptotic of the gluon mean number (3.6). In the high-temperature limit $T \to \infty$ ($\beta = T^{-1} \to 0$), the gluon mean number $N_g(\beta, \omega)$ can be reproduced by the corresponding series in powers of $(\beta \omega)$ if the variable $\omega$ is restricted, namely

$$N_g(\beta, \omega) = \frac{1}{e^{\beta \omega} - 1} = (\beta \omega)^{-1} [1 - \frac{1}{2} (\beta \omega) + O(\beta^2)], \quad \beta \to 0,$$

(7.3)
with the corresponding replacements $\omega \to \omega', \bar{\omega}$. It is worth noting in advance that in what follows for our purpose it is sufficient to keep only the positive powers of $T$ in the evaluation of the high-temperature expansion for the gluon pressure (7.1). Let us also remind that calculated finally terms not depending on temperature (if any) should be omitted, by definition (12). Omitting all these tedious derivations, which can be explicitly found in [8, 10], the high-temperature expansion for the gluon pressure (7.1) up to the leading and next-to-leading orders, is as follows:

$$P_g(T) = P_{NP}(T) + P^s_{PT}(T) = P^s_{NP}(T) + P^s_{PT}(T)$$

$$\sim \frac{12}{\pi^2} \Delta^3 \omega_{eff} T + \frac{8}{3\pi^2} \omega_{eff}^3 T \ln \left( \frac{\omega_{eff}}{\omega_{eff}} \right)^2$$

$$+ \frac{2\sqrt{3}}{\pi^2} \Delta^3 T \arctan \left( \frac{2\omega_{eff}}{\sqrt{3}\Delta} \right) - \frac{16\sqrt{3}}{\pi^2} \Delta^3 T \arctan \left( \frac{\omega_{eff}}{\sqrt{3}\Delta} \right)$$

$$+ \frac{9}{2\pi^2} \bar{\omega}_{eff} \Delta^2 \left[ \frac{\pi^2}{6} T^2 - T \left( \frac{\Delta M}{\sqrt{3} \Delta} \arctan \left( \frac{2\Delta M}{\sqrt{3} \Delta} \right) \right) \right]$$

$$- g_0 P_{SB}(T) \left[ \frac{\pi^4}{45} + \frac{1}{3} \frac{\Delta^3 M}{T^2} \left( \ln \frac{\Delta M}{T} - \frac{1}{3} \right) \right], \quad T \to \infty,$$

(7.4)

where $\omega_{eff} = \sqrt{\omega_{eff}^2 + (3/4) \Delta^2}$, while $\omega_{eff}$ and $\omega_{eff}'$ are shown in (3.7) and (6.10), respectively. Here it suffices to express the gluon pressure in terms of the above-mentioned effective $\omega_{eff}'$s and the mass gap itself. The last line of this expansion is written down as the SB-type term in the limit of high temperatures, see eqs. (5.5-5.7).

A non-analytical dependence on the mass gap occurs in terms $\sim (\Delta^2)^{3/2} T \sim \Delta^3 T$, though $\Delta^2$ is not an expansion parameter like $\alpha_s$ is in hot PT QCD, where a non-analytical dependence on $\alpha_s$ has been discovered (see, for example [29] and references therein). The term $\sim T^2$ has been first introduced in the phenomenological equation of state (EoS) (24) and widely discussed in [7, 8, 25–32]. On the contrary, in our approach both terms $\sim T^2$ and $\sim T$ have not been introduced by hand. They naturally appear on a general ground as a result of the explicit presence of the mass gap from the very beginning in our EoS (7.1).

It is interesting to note that the mass scale parameter in the leading NP term $\sim T^2$ in the expansion (7.4) is $(9/2\pi^2) \times (\pi^2/6) \Delta^2 = (3/4) \Delta^2 = \bar{m}_{eff}^2$. Its numerical value is $\bar{m}_{eff} = 585$ MeV. The scale of the NP dynamics investigated in [24] is $M = 596$ MeV at almost the same $T_c$ as ours, namely $T_c = 270$ MeV. It may or may not be a coincidence, but these numbers are very close to each other, though obtained by different approaches.

However, it is not a coincidence at all that our lowest NP scale $\bar{m}_{eff} = (\sqrt{3}/2) \Delta = 585$ MeV is very close to the gluon mass $\bar{m}_g = 571$ MeV in a model with dimensional-2 gluon condensate [33 34] since their and our equations for the gluon self-energy coincides to the leading order in the deep infrared gluon momentum limit (the so-called nonperturbative iteration solution). In fact, what they call dimensional-2 gluon condensate is the mass gap squared for the gluon self-energy coincides to the leading order in the deep infrared gluon momentum limit (the so-called nonperturbative iteration solution). It is noteworthy in advance that in what follows for our purpose (8) (apart from some non-important numerical factor, of course).

The appearance of the NP massive gluonic excitation $\bar{m}_{eff}^2 = (3/4) \Delta^2$ in this expansion (though suppressed as it should be, see remarks below) is clear evidence of the importance of the NP effects up to a few $T_c$, as it has been underlined above, see fig. 1. However, it is worth to emphasize in advance that in our formalism (8) the effective gluonic excitations can be treated as massive single gluons, which effective masses are generated by the strong self-interaction of massless gluon modes and are expressed in terms of the mass gap or, equivalently, in terms of the effective $\omega_{eff}'$s, mentioned above. They play an important role in the whole temperature region (see sect. V). In the Hagedorn-type expansion (6.31), and hence in (6.32), and in eq. (7.4) they explicitly appear by the substitutions $\Delta \to (1/\sqrt{3}) \bar{m}_{eff}$ and $\Delta \to (2/\sqrt{3}) \bar{m}_{eff}$, as it follows from eqs. (3.5). They are suppressed in different ways in the limit of very high temperatures, only, as it can be concluded from the expansion (7.4).

A few important issues concerning the high-temperature asymptotic of the gluon pressure (7.4) are to be discussed in more detail. The corresponding expansion for the composition (16/\pi^2) \times T N_1(T) = (16/\pi^2) T [P_2(T) - P_4(T)], which enters the composition $N(T)$ in (7.1), is as follows:

$$\frac{16}{\pi^2} T N_1(T) \sim -2P_{SB}(T) + 2P_{SB}(T) - \Delta^2 T^2 + \frac{6}{\pi^2} \Delta^2 \omega_{eff} T - \frac{16}{\pi^2} T P^{(2)}_4(T)$$

$$\sim -\Delta^2 T^2 + \frac{6}{\pi^2} \Delta^2 \omega_{eff} T - \frac{16}{\pi^2} T P^{(2)}_4(T), \quad T \to \infty,$$

(7.5)

where the expression for the integral $P^{(2)}_4(T)$ is not important for present discussion. So one can conclude that at high temperatures the exact cancelation of the $P_{SB}(T)$ terms occurs within this composition. On the other hand,
substituting it into eq. (7.1) the cancelation of the $\Delta^2 T^2$ term occurs within the pressure $P_g(T)$ itself. Let us emphasize once more that the SB term disappears from the gluon pressure (7.1) above $T_c$ due to the normalization of the free PT vacuum to zero from the very beginning. The cancellation of the truly NP terms $\Delta^2 T^2$ simply shows that exact $T^2$ behavior cannot start just from $T_c$ due to the rather complicated dependence of the gluon pressure on the mass gap and temperature in the moderately high temperature interval (approximately up to $5T_c$, see fig. 1). It would be very surprised if a pure NP contribution were survived in the limit of very high temperature, while for its PT correction it would be expected/possible. In other words, the $\Delta^2 T^2$ behavior of $P_g(T)$ in (7.1) is replaced by $\sim \alpha_s \Delta^2 T^2$ behavior in (7.4) only in this limit. At the same time, the second purely NP term $\sim T$ is suppressed in comparison with the first term in the high temperature limit, indeed.

Nevertheless, the approximate $\sim T^2$ behavior (i.e., not suppressed by the $\sim \alpha_s$-order) up to the rather high temperature of such thermodynamic quantity as the trace anomaly or, equivalently, the interaction measure $I(T) = \epsilon(T) - 3P(T)$ is likely to appear, since it depends on the derivative of the pressure. It is very sensitive to the truly NP effects (and thus is free from all the types of the purely PT contributions). On the other hand, the so-called SB-type term, which is strongly suppressed even in this region, will play no any role after inclusion of the SB term $P_{SB}(T)$ in order to get the full gluon pressure, which will be discussed in sect. VIII below in some details.

Concluding, in a more compact form the previous expansion (7.4) looks like

$$P_g(T) = \alpha_s (3/4) \Delta^2 T^2 + [B_3 \Delta^3 + GeV^3]T + ..., \quad T \to \infty,$$

where the expressions for both constants $B_3$ and $GeV^3$ (which becomes zero in the PT $\Delta^2 = 0$ limit) can be easily restored from the expansion (7.4), if necessary. The SB-type term (explicitly shown in the last line of the expansion (7.4) for the sake of completeness only) has been omitted in this expansion due to the remarks made above.

VIII. DISCUSSION AND CONCLUSIONS

The gluon pressure (5.1) or, equivalently, (5.2) has a few remarkable features. First of all, below $T_c$ it is exponentially suppressed in the $T \to 0$ limit, see expansions (6.31) and (6.32) in this limit. Its the most important feature is that at low temperatures $T \leq T_c$ it is nothing else but the Hagedorn-type exponential series (6.32) for the effective gluonic excitations, which are expressed in terms of the mass gap, generated in its turn by the strong self-interaction of massless gluon modes. It is the only one which determines the NP dynamics in the GM within our approach [8]. Nevertheless, it plays a crucial role in the structure of the gluon pressure (5.2) in the whole temperature range, as it can be clearly seen throughout this investigation. We call our expansion (6.32) the Hagedorn-type since it has exponential increasing spectrum valid only up to $T_c$. The scale of the exponential increase determines the value of the Hagedorn temperature [11, 35, 36]. Just this happens in the expansion (6.32) in the $T \to T_c$ limit. This means that maximum of temperature at which the Hagedorn structure is valid is $T_c$, so one has to identify $T_c$ with the Hagedorn-type transition temperature $T_h$. Indeed, there is no other choice than to put $T_h = T_c$ in the mass gap approach to QCD at non-zero temperature (it can be also true for finite density, but this requires a separate investigation elsewhere). In other words, the mass gap approach makes/implies the Hagedorn-type exponential series to be necessarily arisen in hot QCD, see also discussion in [37] (and references therein).

It is instructive to point out that our dynamical degrees of freedom are different from those which appear in the Hagedorn pressure of the glueball gas model associated with a sum over a number of single noninteracting, relativistic particle species of the corresponding masses (low-lying glueballs) [11, 37, 38]. However, it was unable to correctly describe the corresponding thermodynamical lattice data below $T_c$ [30, 38]. Only adding the closed bosonic string contribution [39], modelling the high-lying glueballs [40–43] exponential spectrum, success has been achieved [30, 44]. Also the $SU(3)$ lattice entropy density has been nicely reproduced down to $0.7T_c$ by taking into account the string-type configurations of gluon fields in this joint approach (glueball gas model plus bosonic string) [38].

The gluon pressure (5.1) has a maximum at some characteristic temperature $T = T_c = 266.5$ MeV, see fig. 1, at which its exponential rise in the $T \to T_c$ limit is changed to fall off at $T \geq T_c$. The relevant degrees of freedom below $T_c$ are the effective gluonic excitations of dynamical origin, expressed in terms of the mass gap (some of these NP excitations can be treated as massive quasi-particles - bosonic species). They saturate the dynamical content of the GM below $T_c$, see eq. (6.32). Above $T_c$ the relevant degrees of freedom become weakly-interacting but still NP objects, which pressure is dominated by the $\sim \alpha_s T^2 \Delta^2$-order term, see eq. (7.4). In other words, in the limit of very high temperatures survives only lowest massive excitation $m_{eff}^2 = (3/4) \Delta^2$, while all others are suppressed in this limit, see eq. (7.6). Such drastic changes in the character of the transition at $T_c$ indicates that it might be understood as the temperature of the phase transition from the GM to the gluon plasma (GP). However, the nature of the transition at $T_c$ can be clearly established only by investigating the derivatives, or, equivalently, gradients of
the full gluon pressure (mentioned above and discussed below). To describe the dynamical content of GP below and above $T_c$ in detail is more convenient in their framework as well.

The gluon pressure’s (7.1) fall off just after $T_c$ is not a simple polynomial-type one, see fig. 1. It is due to its rather complicated dependence on the temperature and mass gap in the region of high temperatures up to approximately $(4 - 5)T_c$. So NP effects are still important within our approach in this temperature interval, i.e., the GP can be considered as still remaining strongly interacting medium in this region. Only in the limit of very high temperature $T \rightarrow \infty$ it can be considered as weakly interacting medium, and the gluon pressure has a corresponding polynomial-type character, eq. (7.6). Possessing so nice features, the gluon pressure (7.1) at first sight seems to have one unpleasant “defect”. From fig. 1 it clearly follows that it will never reach the general SB constant/limit at very high temperatures.

VII. The gluon pressure (5.1) may change its exponential regime below $T_c$ only in the close neighborhood of $T_c$ in order for its full counterpart to reach the requested SB limit at high temperatures. The SB term cannot be added to eq. (5.1), even multiplied by the corresponding \(\Theta((T/T_c) - 1)\)-function. In this case the full pressure will get a jump at $T = T_c$, which is not acceptable. So some other term(s) multiplyed by the corresponding \(\Theta((T_c/T) - 1)\)-function, should be added as well in order to ensure a smooth transition across $T_c$ for the full gluon pressure. These problems make the inclusion of the SB term into the gluon EoS (5.1) highly non-trivial in order to transform it into the full gluon EoS. Only after its inclusion into eq. (5.1) in a self-consistent way, such obtained full equation can be called the GP pressure or the GP EoS, and denoted as $P_{GP}(T)$ (we call the dynamical content of GP below $T_c$ as GM).

In this connection, one thing has to be made perfectly clear. The gluon pressure (5.1) will remain an important part of the full GP pressure. It is this which will determine the low-temperature dynamical structure of the full pressure and even play a significant role in it rather far away from $T_c$. Let us emphasize once more that the gluon pressure $P_g(T)$ (though determined in the whole temperature range), but being the NP part of the full pressure, is not obliged and cannot reach SB limit at very high temperature. It is the full pressure $P_{GP}(T)$ which is obliged to approach this thermodynamical limit, and should be a continuously growing function of temperature at any point of its domain from zero to infinity. Thus the above-discussed unpleasant “defect” is not a real defect at all: on the contrary, the gluon pressure (5.1) has a correct thermodynamic limit at very high temperatures (7.6). The NP effects cannot indeed survive in the regime of very high temperatures, which is governed by the SB pressure $P_{SB} = (8/45)\pi^2T^4$ of non-interacting massless particles (an ideal gas limit of gluons).

The gluon pressure (5.1) satisfies to all the required thermodynamical limits. It is exponentially suppressed in the $T \rightarrow 0$ limit and has an exponential rise in the $T \rightarrow T_c$ limit. This exponential increase in the number of dynamical degrees of freedom or, equivalently, an exponential grows in the density of states in this limit is clearly seen in fig. 1 (the same picture occurs in the string theory at the Hagedorn transition \[42\]). Due to its Hagedorn’s-type nature (6.32) at $T \leq T_c$ the gluon pressure (5.1) describes the GM as a dense states of the various effective (both massive and massless) gluonic excitations of the dynamical origin, which are expressed in terms of the mass gap. As pointed out above, they are not glueballs, i.e., bound-states of two or three gluons, but rather single massive gluons, treated as strongly interacted quasi-particles – bosonic species. If in the above-mentioned bosonic string model glueballs can be interpreted as “rings of glue” \[44\] \[46\], then our degrees of freedom can be possibly interpreted as “chains of glue”. There is no doubts that QCD vacuum at zero and non-zero temperature and density contains a lot of degrees of freedom of various dynamical nature, which can be interpreted in many different ways. The gluon pressure (5.1) necessarily has a Hagedorn-type structure at $T \leq T_c = T_h$ and demonstrates rather complicated dependence on the mass gap and temperature up to approximately $(4 - 5)T_c$. In the limit of very high temperature $T \rightarrow \infty$ it has a polynomial behavior consistent with the SB limit. That is why it can serve as a basic equation for its transformation into the full GP EoS.

In the forthcoming paper we will present a general formalism how to transform the gluon pressure (5.1) into the full GP EoS in a self-consistent way (i.e., not destroying the Hagedorn-type structure below $T_c$, providing the smooth transition across $T_c$ and approaching to the SB limit above $T_c$ from below). For this we will need the lattice data \[27\] \[30\] \[44\] for the pressure on either sides of $T_c$ but close to it only. Some preliminary attempts in this direction have been already done in \[8\] \[33\]. Completing this program in much more satisfactory way, we will be able to analytically describe YM $SU(3)$ lattice thermodynamics \[27\] \[30\] \[38\] \[44\] \[48\] \[52\], and thus to compare it with other analytical approaches and models. Concluding, a few messages we would like to emphasize and convey are:

a). The Hagedorn-type structure of the pressure is of crucial importance to correctly understand and describe the GM dynamical content at low temperatures within any approach or model.

b). It necessary arise within the mass gap approach to QCD at finite temperature.

c). It is valid up to $T_c$ only, which implies to identify it with the Hagedorn-type temperature, i.e., to put $T_c = T_h$ within our approach.

d). All the dynamical degrees of freedom can be expressed in terms of the mass gap alone in this picture.
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Appendix A: The purely PT contribution

The pure PT contribution $P_{PT}^s(T)$ to the gluon pressure $P_g(T)$ is given in Eq. (4.7), namely

$$P_{PT}^s(T) = -\frac{9}{2\pi^2}\alpha_s^2 \int_{\Lambda_{YM}}^\infty d\omega \omega^2 T \sum_{m=-\infty}^{+\infty} \left( -\frac{3}{4} \right)^m \frac{\alpha_s^m}{m+2} \sum_{k=0}^{\infty} c_k(m+2)\alpha_s^k \ln k z_n ,$$

(A1)

where the coefficients $c_k(m+2)$ are defined in (4.9). Since at $k=0$ the sum over $n$ does not depend on it, the summation over $n$ yields zero. This means that first non-zero contribution appears at $k=1$ and the integral (A1) becomes

$$P_{PT}^s(T) = -\frac{9}{2\pi^2}\alpha_s^3 \int_{\Lambda_{YM}}^\infty d\omega \omega^2 T \ln z_n \left[ \sum_{m=0}^{\infty} \left( -\frac{3}{4} \right)^m \frac{\alpha_s^m}{m+2} c_1(m+2) \right] .$$

(A2)

From eq. (4.9) it follows that $c_1(m+2) = -(m+2)$, then the summation over $m$ can done exactly as follows:

$$\sum_{m=0}^{\infty} \left( -\frac{3}{4} \right)^m \frac{\alpha_s^m}{m+2} c_1(m+2) = -\frac{1}{1 + (3/4)\alpha_s} .$$

(A3)

The summation over $n$, on account of the relations (2.1) and (4.3) looks like

$$\sum_{n=-\infty}^{+\infty} T \ln z_n = b_0 + \sum_{n=-\infty}^{+\infty} T \ln \left[ 1 + \left( \frac{2\pi n T}{\omega} \right)^2 \right] = 2Tb_0 \ln(1 - e^{-\beta\omega}) ,$$

(A4)

where, for simplicity, we omit details of the exact summation of the thermal PT logarithms, which can be found in [7, 8] by using the formulae from [21, 22]. Substituting results of all the summations done, one finally obtains the expression given in eq. (5.4), namely

$$P_{PT}^s(T) = \left( \frac{9b_0}{\pi^2} \frac{\alpha_s^3}{1 + (3/4)\alpha_s} \right) T \int_{\Lambda_{YM}}^\infty d\omega \omega^2 \ln \left( 1 - e^{-\beta\omega} \right) , \quad \beta = T^{-1} .$$

(A5)

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