On Feller and Strong Feller Properties and Exponential Ergodicity of Regime-Switching Jump Diffusion Processes with Countable Regimes

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Abstract

This work focuses on a class of regime-switching jump diffusion processes, in which the switching component has countably infinite many states or regimes. The existence and uniqueness of the underlying process are obtained by an interlacing procedure. Then the Feller and strong Feller properties of such processes are derived by the coupling method and an appropriate Radon-Nikodym derivative. Finally the paper studies exponential ergodicity of regime-switching jump-diffusion processes.

Key Words and Phrases. Jump-diffusion, switching, existence, uniqueness, Feller property, strong Feller property, exponential ergodicity.

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1 Introduction

Jump processes have become a key model in stochastic analysis over the recent years. On one hand this is due to an increasing need for modeling stochastic processes with jumps in areas ranging from physics and biology to finance and economics. On the other hand, there is a more and more profound understanding of theories and properties of jump processes. While a general framework is certainly provided by semimartingale theory, Lévy processes remain the basic building blocks. We refer the reader to Applebaum (2009) for extensive

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treatments of Lévy processes. Meanwhile, thanks to their ability in incorporating structural changes, regime-switching processes have attracted many interests lately. See, for example, Cloez and Hairer (2015), Mao and Yuan (2006), Shao and Xi (2014), Wang (2014), Xi (2008a), Xi and Yin (2011), Xi and Zhao (2006), Yin and Zhu (2010), Zhu (2011) and the references therein for investigations of such processes and their applications in areas such as inventory control, ecosystem modeling, manufacturing and production planning, financial engineering, risk theory, etc.

Motivated by the increasing need of modeling complex systems, in which both structural changes and small fluctuations as well as big spikes coexist and are intertwined, this paper aims to study regime-switching jump diffusion processes. Unlike some of the earlier work on regime-switching jump diffusion processes such as Xi (2009), Zhu et al. (2015), in which the switching component takes value in a finite state space, this paper allows the switching component to have an infinite countable state space. This is motivated by the formulation in the recent work Shao (2015), in which the switching component has an infinite countable state space. In the formulation of Shao (2015), starting from an arbitrary state, the switching component can only switch to a finite neighboring states (Assumption (A1)). Assumption (A1), together with other conditions, allows the author to derive the existence of a weak solution directly by invoking a result in Situ (2005). This paper does not require such a condition. Instead, certain Lyapunov type condition (condition (1.6)) is used. Note that Assumption (A1) of Shao (2015) certainly implies condition (1.6) but not necessarily the other way around. As a result of this relaxation, care is needed to establish the existence of a weak solution to the associated stochastic differential equations corresponding to the regime-switching jump diffusion processes. In this paper, we use an interlacing procedure together with exponential killing to construct a (possibly local) solution to the stochastic differential equations. Condition (1.6) together with the growth condition on the coefficients of the stochastic differential equations guarantee that the solution is actually global with no finite explosion time. Finally we establish the pathwise uniqueness result, which gives us the existence and uniqueness of a strong solution for the associated stochastic differential equations by virtue of Yamada and Watanabe’s result on weak and strong solutions.

Next we use the coupling method to derive the Feller property for regime-switching jump diffusion processes. The coupling method has been extensively used to study diffusion and jump diffusion processes; see, for example, Chen and Li (1989), Lindvall and Rogers (1986), Priola and Wang (2006), Wang (2010) and the references therein. Some earlier work of using the coupling method in the studies of regime-switching (jump) diffusions can be found in Xi (2008a,b), Xi and Zhao (2006). But in these papers, it is assumed that either the switching component is given by a continuous-time Markov chain, resulting the so-called Markovian
regime-switching diffusion processes, or the diffusion matrix is independent of the switching component. In this paper, we construct a coupling operator \( \tilde{\mathcal{A}} \) in (3.6), which can handle the general state-dependent regime-switching jump diffusions. The key idea is that, for the coupled process \((\tilde{X}, \tilde{\Lambda}, \tilde{Z}, \tilde{\Xi})\) generated by \( \tilde{\mathcal{A}} \) starting from \((x, k, z, k')\), one needs to carefully treat the first time when the switching components \( \tilde{\Lambda} \) and \( \tilde{\Xi} \) are different; see the proof of Theorem 3.3 for details.

For the investigation of strong Feller property, we use the idea developed in Xi (2009). More precisely, we first show that under certain conditions, the jump diffusion \( X^{(k)} \) of (4.2) has strong Feller property. Then we establish the strong Feller property for the auxiliary process \((V, \psi)\) constructed in equations (5.1)–(5.2). Next we use the Radon-Nikodym derivative \( M_T \) of (5.16) to derive the strong Feller property for the process \((X, \Lambda)\). In Section 6, as an application of the strong Feller property, we also obtain the exponential ergodicity for the regime-switching jump diffusion process \((X, \Lambda)\). In particular, when the coefficients of the associated stochastic differential equations are linearizable in a neighborhood of \( \infty \), we present some easily verifiable sufficient conditions for exponential ergodicity.

The rest of the paper is arranged as follows. Section 1.1 presents the precise formulation for regime-switching jump diffusion processes. The standing assumptions are also collected in Section 1.1. The existence and uniqueness results for the associated stochastic differential equations are presented in Section 2. Section 3 studies Feller property of regime-switching jump diffusion processes. Sections 4 and 5 establish strong Feller property for jump diffusion and regime-switching jump diffusion processes, respectively. Section 6 is devoted to exponential ergodicity of regime-switching jump diffusion process. Finally, concluding remarks are made in Section 7.

### 1.1 Formulation

Throughout the rest of this paper we let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})\) be a complete probability space with a filtration \( \{\mathcal{F}_t\}_{t\geq 0} \) satisfying the usual conditions (i.e., it is right continuous and \( \mathcal{F}_0 \) contains all \( \mathbb{P} \)-null sets). To formulate our model, let \( d \) be a positive integer, and put \( S := \{0, 1, 2, \ldots\} \), the totality of nonnegative integers. Let \((X, \Lambda)\) be a right continuous, strong Markov process with left-hand limits on \( \mathbb{R}^d \times S \). The first component \( X \) satisfies the following stochastic differential-integral equation

\[
dX(t) = \sigma(X(t), \Lambda(t))dB(t) + b(X(t), \Lambda(t))dt + \int_{U_0} c(X(t^-), \Lambda(t^-), u)\tilde{N}(dt, du) + \int_{U \setminus U_0} c(X(t^-), \Lambda(t^-), u)N(dt, du),
\]

where \( \sigma(x, k) \) is \( \mathbb{R}^{d \times d} \)-valued and \( b(x, k) \) and \( c(x, k, u) \) are \( \mathbb{R}^d \)-valued for \( x \in \mathbb{R}^d, k \in S \) and \( u \in U \), \((U, \mathcal{B}(U))\) is a measurable space, \( B(t) \) is an \( \mathbb{R}^d \)-valued Brownian motion, \( N(dt, du) \)
(corresponding to a random point process \( p(t) \)) is a Poisson random measure independent of \( B(t), \tilde{N}(dt, du) = N(dt, du) - \Pi(du) dt \) is the compensated Poisson random measure on \([0, \infty) \times U, \Pi(\cdot) \) is a deterministic \( \sigma \)-finite characteristic measure on the measurable space \((U, \mathcal{B}(U))\), and \( U_0 \) is a set in \( \mathcal{B}(U) \) such that \( \Pi(U \setminus U_0) < \infty \). The second component \( \Lambda \) is a discrete random process with an infinite state space \( S \) such that

\[
\mathbb{P}(\Lambda(t + \Delta) = l | \Lambda(t) = k, X(t) = x) = \begin{cases} 
q_{kl}(x) \Delta + o(\Delta), & \text{if } k \neq l, \\
1 + q_{kk}(x) \Delta + o(\Delta), & \text{if } k = l,
\end{cases} \tag{1.2}
\]

uniformly in \( \mathbb{R}^d \), provided \( \Delta \downarrow 0 \). As usual, we assume that for all \( x \in \mathbb{R}^d, q_{kl}(x) \geq 0 \) for \( k \neq l \) and \( \sum_{l \in S} q_{kl}(x) = 0 \) for all \( k \in S \). For \( x \in \mathbb{R}^d \) and \( \sigma = (\sigma_{ij}) \in \mathbb{R}^{d \times d} \), define

\[
|x| = \left( \sum_{i=1}^{d} |x_i|^2 \right)^{1/2}, \quad |\sigma| = \left( \sum_{i,j=1}^{d} |\sigma_{ij}|^2 \right)^{1/2}.
\]

Define a metric \( \lambda(\cdot, \cdot) \) on \( \mathbb{R}^d \times S \) as \( \lambda((x, m), (y, n)) = |x - y| + d(m, n) \), where \( d(\cdot, \cdot) \) is the discrete metric on \( S \) so \( d(m, n) = 1_{\{m \neq n\}} \). Let \( \mathcal{B}(\mathbb{R}^d \times S) \) be the Borel \( \sigma \)-algebra on \( \mathbb{R}^d \times S \). Then \( (\mathbb{R}^d \times S, \lambda(\cdot, \cdot), \mathcal{B}(\mathbb{R}^d \times S)) \) is a locally compact and separable metric space. For the existence and uniqueness of the strong Markov process \((X, \Lambda)\) satisfying the system (1.1) and (1.2), we make the following assumptions.

**Assumption 1.1.** Assume that \( c(x, k, u) \) is \( \mathcal{B}(\mathbb{R}^d \times S) \times \mathcal{B}(U) \) measurable, and that for some constant \( H > 0 \),

\[
|b(x, k)|^2 + |\sigma(x, k)|^2 + \int_U |c(x, k, u)|^2 \Pi(du) \leq H(1 + |x|^2), \tag{1.3}
\]

\[
|b(x, k) - b(y, k)|^2 + |\sigma(x, k) - \sigma(y, k)|^2 + \int_U |c(x, k, u) - c(y, k, u)|^2 \Pi(du) \leq H|x - y|^2, \tag{1.4}
\]

for all \( x, y \in \mathbb{R}^d \) and \( k \in S \).

**Assumption 1.2.** Assume that for all \((x, k) \in \mathbb{R}^d \times S, \) we have

\[
q_k(x) := -q_{kk}(x) = \sum_{l \in S \setminus \{k\}} q_{kl}(x) \leq H(k + 1), \tag{1.5}
\]

\[
\sum_{l \in S \setminus \{k\}} (f(l) - f(k)) q_{kl}(x) \leq H(1 + |x|^2 + f(k)), \tag{1.6}
\]

where the constant \( H > 0 \) is the same as in Assumption 1.1 without loss of generality, and the function \( f : S \rightarrow \mathbb{R}_+ \) is nondecreasing and satisfies \( f(m) \rightarrow \infty \) as \( m \rightarrow \infty \). In addition, assume there exists some \( \delta \in (0, 1] \) such that

\[
\sum_{l \in S \setminus \{k\}} |q_{kl}(x) - q_{kl}(y)| \leq H|x - y|^\delta \tag{1.7}
\]

for all \( k \in S \) and \( x, y \in \mathbb{R}^d \).
2 Existence and Uniqueness

In this section, we prove that there exists a unique strong solution to the system (1.1)–(1.2).

Theorem 2.1. Suppose that Assumptions 1.1 and 1.2 hold. Then for each \((x, k) \in \mathbb{R}^d \times S\), system (1.1) and (1.2) has a unique strong solution \((X(t), \Lambda(t))\) with \((X(0), \Lambda(0)) = (x, k)\).

The proof of this theorem is divided into three steps. In the first step, we construct a solution \((X, \Lambda)\) to (1.1) and (1.2) with \((X(0), \Lambda(0)) = (x, k)\) on the interval \([0, \tau_{\infty})\), where \(\tau_{\infty} \leq \infty\) is a stopping time to be defined in (2.10). After some preparatory work, we then show in the second step that \(\tau_{\infty} = \infty\) a.s. Finally we establish pathwise uniqueness for (1.1) and (1.2) in Step 3.

Proof of Theorem 2.1 (Step 1). Here we use the “interlacing procedure” as termed in Applebaum (2009) to demonstrate that under Assumptions 1.1 and 1.2, the system (1.1) and (1.2) has a (possibly local) weak solution \((X, \Lambda)\). To this end, let the complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), the \(d\)-dimensional standard Brownian motion \(B\), and the Poisson random measure \(N(\cdot, \cdot)\) on \([0, \infty) \times U\) be specified as in Section 1.1. In addition, let \(\{\xi_n\}\) be a sequence of independent mean 1 exponential random variables on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) that is independent of \(B\) and \(N\). Fix some \((x, k) \in \mathbb{R}^d \times S\) and consider the stochastic differential equation

\[
X^{(k)}(t) = x + \int_0^t \sigma(X^{(k)}(s), k)dB(s) + \int_0^t b(X^{(k)}(s), k)ds + \int_0^t \int_{U_0} c(X^{(k)}(s-), k, u)\tilde{N}(ds, du) + \int_0^t \int_{U \setminus U_0} c(X^{(k)}(s-), k, u)N(ds, du).
\]

(2.1)

In view of Theorem IV.9.1 of Ikeda and Watanabe (1989), such a solution exists and is pathwise unique thanks to (1.3) and (1.4) of Assumption 1.1. Let

\[
\tau_1 = \theta_1 := \inf \left\{ t \geq 0 : \int_0^t q_k(X^{(k)}(s))ds > \xi_1 \right\}.
\]

(2.2)

Then we have

\[
\mathbb{P}\{\tau_1 > t | \mathcal{F}_t\} = \mathbb{P}\left\{ \xi_1 \geq \int_0^t q_k(X^{(k)}(s))ds \bigg| \mathcal{F}_t \right\} = \exp \left\{ -\int_0^t q_k(X^{(k)}(s))ds \right\}.
\]

(2.3)

Thanks to (1.5) in Assumption 1.2, we have \(\mathbb{P}\{\tau_1 > t\} \geq e^{-H(k+1)t}\) and therefore \(\mathbb{P}(\tau_1 > 0) = 1\). We define a process \((X, \Lambda) \in \mathbb{R}^d \times S\) on \([0, \tau_1]\) as follows:

\[
X(t) = X^{(k)}(t) \text{ for all } t \in [0, \tau_1], \text{ and } \Lambda(t) = k \text{ for all } t \in [0, \tau_1).
\]
Moreover, we define $\Lambda(\tau_1) \in \mathcal{S}$ according to the probability distribution:

$$
\mathbb{P}\{\Lambda(\tau_1) = l | \mathcal{F}_{\tau_1}\} = \frac{q_{kl}(X(\tau_1-))}{q_k(X(\tau_1-))} \left(1 - \delta_{kl}\right) \mathbf{1}_{\{q_k(X(\tau_1-)) > 0\}} + \delta_{kl} \mathbf{1}_{\{q_k(X(\tau_1-)) = 0\}}.
$$

(2.4)

In general, having determined $(X, \Lambda)$ on $[0, \tau_n]$, we let

$$
\theta_{n+1} := \inf\left\{ t \geq 0 : \int_0^t q_{\Lambda(\tau_n)}(X^{(\Lambda(\tau_n))}(s))\,ds > \xi_{n+1}\right\},
$$

(2.5)

where

$$
X^{(\Lambda(\tau_n))}(t) := X(\tau_n) + \int_0^t \sigma(X^{(\Lambda(\tau_n))}(s), \Lambda(\tau_n))\,dB(s) + \int_0^t b(X^{(\Lambda(\tau_n))}(s), \Lambda(\tau_n))\,ds
$$

$$
+ \int_0^t \int_{\mathcal{U}} c(X^{(\Lambda(\tau_n))}(s-), \Lambda(\tau_n), u) \tilde{N}(ds, du)
$$

$$
+ \int_0^t \int_{\mathcal{U}\setminus U_0} c(X^{(\Lambda(\tau_n))}(s-), \Lambda(\tau_n), u) N(ds, du).
$$

As argued in (2.3), we have

$$
\mathbb{P}\{\theta_{n+1} > t | \mathcal{F}_{\tau_n+t}\} = \mathbb{P}\left\{\xi_{n+1} \geq \int_0^t q_{\Lambda(\tau_n)}(X^{(\Lambda(\tau_n))}(s))\,ds | \mathcal{F}_{\tau_n+t}\right\}
$$

$$
= \exp\left\{-\int_0^t q_{\Lambda(\tau_n)}(X^{(\Lambda(\tau_n))}(s))\,ds\right\}.
$$

(2.6)

Again, Assumption 1.2 implies that $\mathbb{P}\{\theta_{n+1} > 0\} = 1$. Then we let

$$
\tau_{n+1} := \tau_n + \theta_{n+1}
$$

(2.7)

and define $(X, \Lambda)$ on $[\tau_n, \tau_{n+1}]$ by

$$
X(t) = X^{(\Lambda(\tau_n))}(t - \tau_n) \text{ for } t \in [\tau_n, \tau_{n+1}], \quad \Lambda(t) = \Lambda(\tau_n) \text{ for } t \in [\tau_n, \tau_{n+1}],
$$

(2.8)

and

$$
\mathbb{P}\{\Lambda(\tau_{n+1}) = l | \mathcal{F}_{\tau_{n+1}}\} = \frac{q_{\Lambda(\tau_n), l}(X(\tau_{n+1}-))}{q_{\Lambda(\tau_n)}(X(\tau_{n+1}-))} \left(1 - \delta_{\Lambda(\tau_n), l}\right) \mathbf{1}_{\{q_{\Lambda(\tau_n)}(X(\tau_{n+1}-)) > 0\}} + \delta_{\Lambda(\tau_n), l} \mathbf{1}_{\{q_{\Lambda(\tau_n)}(X(\tau_{n+1}-)) = 0\}}.
$$

(2.9)

This “interlacing procedure” uniquely determines a strong Markov process $(X, \Lambda) \in \mathbb{R}^d \times \mathcal{S}$ for all $t \in [0, \tau_\infty)$, where

$$
\tau_\infty = \lim_{n \to \infty} \tau_n.
$$

(2.10)

Since the sequence $\tau_n$ is strictly increasing, the limit $\tau_\infty \leq \infty$ exists. Moreover it follows from (2.6)–(2.9) that the process $(X, \Lambda)$ satisfies (1.1) and (1.2) on $[0, \tau_\infty)$. \qed
Remark 2.2. Note that in general condition (1.5) alone can not guarantee that \( \tau_\infty = \infty \) a.s. To see this, let us consider a continuous-time Markov chain \( \Lambda \) with state space \( \mathbb{S} = \{0, 1, \ldots, \} \) and \( Q \)-matrix given by \( Q = (q_{kl}) \) such that \( -q_{kk} = q_{k,(k+1)^2} = k + 1 \) and \( q_{kl} = 0 \) for all \( l \in \mathbb{S} \setminus \{k, (k+1)^2\} \). For this example, (1.5) is satisfied.

Assume \( \Lambda(0) = 0 \), then \( \Lambda \) will stay in state 0 for an exponential amount of time with mean \( \frac{1}{2} \); it next switches to state 4, whose holding time is exponentially distributed with mean \( \frac{1}{5} \); and then switches to state 26, whose holding time is exponentially distributed with mean \( \frac{1}{27} \); and so on. It is then clear that \( \mathbb{P}(\tau_\infty < \infty) = 1 \). Of course, we can easily check that condition (1.6) can not be satisfied for this example.

Remark 2.3. However, if the upper bound \( H(k+1) \) in (1.5) of Assumption 1.2 is replaced by \( H \), then we have \( \tau_\infty = \infty \) a.s. and therefore the proof of Theorem 2.1 can be much simplified. Indeed, with the uniform upper bound, we have \( \mathbb{P}\{\theta_k > t\} \geq e^{-Ht} \) for all \( k \in \mathbb{N} \) and \( t > 0 \) and hence

\[
\mathbb{P}\{\tau_\infty = \infty\} \geq \mathbb{P}\{\{\theta_k > t\ \text{i.o.}\} = \mathbb{P}\left\{\bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \{\theta_k > t\}\right\} = \lim_{m \to \infty} \mathbb{P}\left\{\bigcup_{k=m}^{\infty} \{\theta_k > t\}\right\} \geq \limsup_{m \to \infty} \mathbb{P}\{\theta_m > t\} \geq e^{-Ht}.
\]

Letting \( t \downarrow 0 \) yields that \( \mathbb{P}\{\tau_\infty = \infty\} = 1 \). Thus the “interlacing procedure” directly leads to the existence of a solution \((X, \Lambda)\) to (1.1)–(1.2) for all \( t \in [0, \infty) \).

To proceed, we construct a family of disjoint intervals \( \{\Delta_{ij}(x) : i, j \in \mathbb{S}\} \) on the positive half real line as follows:

\[
\Delta_{01}(x) = [0, q_{01}(x)),
\Delta_{02}(x) = [q_{01}(x)), q_{01}(x) + q_{02}(x)),
\vdots
\Delta_{10}(x) = [q_0(x), q_0(x) + q_{10}(x)),
\Delta_{12}(x) = [q_0(x) + q_{10}(x)), q_0(x) + q_{10}(x) + q_{12}(x)),
\vdots
\Delta_{20}(x) = [q_0(x) + q_{1}(x), q_0(x) + q_{1}(x) + q_{20}(x)),
\vdots
\]
where for convenience of notations, we set $\Delta_{ij}(x) = \emptyset$ if $q_{ij}(x) = 0$, $i \neq j$. Note that for each $x \in \mathbb{R}^n$, \{\$\Delta_{ij}(x) : i, j \in S\$\} are disjoint intervals, and the length of the interval $\Delta_{ij}(x)$ is equal to $q_{ij}(x)$, which is bounded above by $Hi$ thanks to Assumption 1.2. We then define a function $h: \mathbb{R}^d \times S \times \mathbb{R}_+ \to \mathbb{R}$ by
\[
h(x, k, r) = \sum_{l \in S} (l - k)1_{\Delta_{kl}(x)}(r).
\] (2.12)
That is, for each $x \in \mathbb{R}^d$ and $k \in S$, we set $h(x, k, r) = l - k$ if $r \in \Delta_{kl}(x)$ for some $l \neq k$; otherwise $h(x, k, r) = 0$.

**Proposition 2.4.** Let Assumptions 1.1 and 1.2 hold. For any $f \in C_c^2(\mathbb{R}^d \times S)$, we have
\[
\mathbb{E}_{x,k}[f(X(t \wedge \tau_\infty), \Lambda(t \wedge \tau_\infty))] = f(x, k) + \mathbb{E}_{x,k}\left[\int_0^{t \wedge \tau_\infty} \mathcal{A}f(X(s), \Lambda(s))ds\right],
\] (2.13)
where
\[
\mathcal{A}f(x, k) := \mathcal{L}_k f(x, k) + Q(x)f(x, k),
\] (2.14)
with
\[
\mathcal{L}_k f(x, k) := \frac{1}{2} \text{tr}(a(x, k)\nabla^2 f(x, k)) + \langle b(x, k), \nabla f(x, k) \rangle
\] (2.15)
\[
+ \int_U \left(f(x + c(x, k, u), k) - f(x, k) - \langle \nabla f(x, k), c(x, k, u) \rangle\right)1_{\{u \in V_0\}} \Pi(du),
\]
\[
Q(x)f(x, k) := \sum_{j \in S} q_{kj}(x)[f(x, j) - f(x, k)] = \int_{[0, \infty)} \left[f(x, k + h(x, k, z)) - f(x, k)\right]m(dz).
\] (2.16)

**Proof.** Put $\lambda(t) := \int_0^t q_{\Lambda(s)}(X(s))ds$ and $n(t) := \max\{n \in \mathbb{N} : \xi_1 + \cdots + \xi_n \leq \lambda(t)\}$ for all $t \in [0, \tau_\infty)$, where \{\$\xi_n, n = 1, 2, \ldots$\} is a sequence of independent exponential random variables with mean 1. Then in view of (2.2), (2.3), (2.5), (2.6), and (2.7), the process \{\$n(t \wedge \tau_\infty), t \geq 0$\} is a counting process that counts the number of switches for the component $\Lambda$. We can regard $n(\cdot)$ as a nonhomogeneous Poisson process with random intensity function $q_{\Lambda(t)}(X(t)), t \in [0, \tau_\infty)$.

Now for any $s < t \in [0, \tau_\infty)$ and $A \in \mathcal{B}(S)$, let
\[
p((s, t] \times A) := \sum_{u \in (s, t]} 1_{\{\Lambda(u) \neq \Lambda(u-), \Lambda(u) \in A\}} \quad \text{and} \quad p(t, A) := p((0, t] \times A).
\]
Then we have $p(t \wedge \tau_\infty, S) = n(t \wedge \tau_\infty)$ and
\[
\Lambda(t \wedge \tau_\infty) = \Lambda(0) + \sum_{k=1}^{\infty} \left[\Lambda(\tau_k) - \Lambda(\tau_k-]\right]1_{\{\tau_k \leq t \wedge \tau_\infty\}}
\] (2.17)
\[
= \Lambda(0) + \int_0^{t \wedge \tau_\infty} \int_S [l - \Lambda(s-)] p(ds, dl).
\]
We can also define a Poisson random measure $N_1(\cdot, \cdot)$ on $[0, \infty) \times \mathbb{R}_+$ by
\[
N_1(t \wedge \tau_\infty, B) := \sum_{t \leq \tau_\infty} p(t \wedge \tau_\infty, l), \quad \text{for all } t \geq 0 \text{ and } B \in \mathcal{B}(\mathbb{R}_+).
\]

Observe that for any $(x, k) \in \mathbb{R}^d \times S$ and $l \in S \setminus \{k\}$, we have
\[
m\{r \in [0, \infty) : h(x, k, r) \neq 0\} = q_k(x) \quad \text{and} \quad m\{r \in [0, \infty) : h(x, k, r) = l - k\} = q_{kl}(x),
\]
where $m$ is the Lebesgue measure on $\mathbb{R}_+$. Therefore we can rewrite (2.9) and (2.17) as
\[
\Lambda(t \wedge \tau_\infty) = \Lambda(0) + \int_0^{t \wedge \tau_\infty} \int_{\mathbb{R}_+} h(X(s-), \Lambda(s-), r) N_1(ds, dr).
\]  

(2.18)

Then we can use the same argument as that in the proof of Lemma 3 on p. 105 of Skorokhod (1989) to show that for any $f \in C^2(\mathbb{R}^d \times S)$, we have
\[
f(X(t \wedge \tau_\infty), \Lambda(t \wedge \tau_\infty))
= f(x, k) + \int_0^{t \wedge \tau_\infty} A f(X(s), \Lambda(s)) ds + \int_0^{t \wedge \tau_\infty} \nabla f(X(s), \Lambda(s)) \cdot \sigma(X(s), \Lambda(s)) dB(s)
+ \int_0^{t \wedge \tau_\infty} \int_0^t \left[ f(X(s-), c(X(s-), \Lambda(s-), u), \Lambda(s-)) - f(X(s-), \Lambda(s-)) \right] \tilde{N}(ds, du)
+ \int_0^{t \wedge \tau_\infty} \int_{\mathbb{R}_+} \left[ f(X(s-), \Lambda(s-) + h(X(s-), \Lambda(s-), r)) - f(X(s-), \Lambda(s-)) \right] \tilde{N}_1(ds, dr),
\]
where $\tilde{N}_1(ds, dr) := N_1(ds, dr) - ds m(dr)$. In particular, (2.13) follows.  

We immediately have the following corollary from Proposition 2.4.

**Corollary 2.5.** Suppose Assumptions 1.1 and 1.2. Then the extended generator of the process $(X, \Lambda)$ is given by $A$ of (2.14) on the temporal interval $[0, \tau_\infty)$.

**Proof of Theorem 2.1 (Step 2).** Now we are ready to show that $\tau_\infty = \infty$ a.s. and hence the “interlacing procedure” presented in Step 1 actually determines a strong Markov process $(X, \Lambda) \in \mathbb{R}^d \times S$ for all $t \in [0, \infty)$. To this end, fix $(X(0), \Lambda(0)) = (x, k) \in \mathbb{R}^d \times S$ as in Step 1 and for any $m \geq k + 1$, we denote by $\tau_m := \inf\{t \geq 0 : \Lambda(t) \geq m\}$ the first exit time for the $\Lambda$ component from the finite set $\{0, 1, \ldots, m - 1\}$. Let $A^c := \{\omega \in \Omega : \tau_\infty > \tau_m \text{ for all } m \geq k + 1\}$ and $A := \{\omega \in \Omega : \tau_\infty \leq \tau_{m_0} \text{ for some } m_0 \geq k + 1\}$. Then we have
\[
\mathbb{P}\{\tau_\infty = \infty\} = \mathbb{P}\{\tau_\infty = \infty|A^c\} \mathbb{P}(A^c) + \mathbb{P}\{\tau_\infty = \infty|A\} \mathbb{P}(A).
\]  

(2.19)

Let us first show that $\mathbb{P}\{\tau_\infty = \infty|A\} = 1$. To this end, we note that on the event $A$, we have $\Lambda(\tau_n) \in \{0, 1, \ldots, m_0 - 1\}$ and hence by (1.5),
\[
q_{\Lambda(\tau_n)}(X^{(\Lambda(\tau_n))}(s)) \leq H(\Lambda(\tau_n) + 1) \leq H m_0, \quad \text{for all } n = 1, 2, \ldots \text{ and } s \geq 0.
\]
Then it follows from (2.6) that for all $n = 0, 1, \ldots$

\[
\mathbb{P}\{\theta_{n+1} > t | \mathcal{F}_{n+t}\} = \exp\left\{- \int_0^t q_{\Lambda(\tau)}(X(\Lambda(\tau)))(s) \, ds \right\} \geq 1_A \exp\left\{- \int_0^t q_{\Lambda(\tau)}(X(\Lambda(\tau)))(s) \, ds \right\} \geq e^{-H_{\text{mol}}t} 1_A.
\]

Taking expectations on both sides yields $\mathbb{P}(\theta_{n+1} > t) \geq e^{-H_{\text{mol}}t} \mathbb{P}(A)$ and hence $\mathbb{P}\{\theta_{n+1} > t | A\} \geq e^{-H_{\text{mol}}t}$. Thus, as argued in (2.11), we obtain that for any $t > 0$,

\[
\mathbb{P}\{\tau_\infty = \infty | A\} \geq \mathbb{P}\{\{\theta_n > t\} \text{ i.o.} | A\} \geq \limsup_{m \to \infty} \mathbb{P}\{\theta_m > t | A\} \geq e^{-H_{\text{mol}}t}.
\]

Letting $t \downarrow 0$ yields that $\mathbb{P}\{\tau_\infty = \infty | A\} = 1$.

If $\mathbb{P}(A) = 1$ or $\mathbb{P}(A^c) = 0$, then (2.19) implies that $\mathbb{P}\{\tau_\infty = \infty\} = 1$ and the proof is complete. Therefore it remains to consider the case when $\mathbb{P}(A^c) > 0$. Denote $\bar{\tau}_\infty := \lim_{m \to \infty} \bar{\tau}_m$. Note that $A^c = \{\tau_\infty \geq \bar{\tau}_\infty\}$. Thus $\mathbb{P}\{\tau_\infty = \infty | A^c\} \geq \mathbb{P}\{\bar{\tau}_\infty = \infty | A^c\}$ and hence (2.19) will hold true if we can show that

\[
\mathbb{P}\{\bar{\tau}_\infty = \infty | A^c\} = 1. \tag{2.20}
\]

Assume on the contrary that (2.20) was false, then there would exist a $T > 0$ such that

\[
\delta := \mathbb{P}\{\bar{\tau}_\infty \leq T, A^c\} > 0.
\]

Let $f : S \mapsto \mathbb{R}_+$ be as in Assumption 1.2. Then by virtue of the Dynkin formula (2.13), we have for any $m \geq k + 1$,

\[
f(k) = \mathbb{E}[e^{-H(T \wedge \tau_\infty \wedge \bar{\tau}_m)} f(\Lambda(T \wedge \tau_\infty \wedge \bar{\tau}_m))] \\
+ \mathbb{E}\left[ \int_0^{T \wedge \tau_\infty \wedge \bar{\tau}_m} e^{-Hs} \left( Hf(\Lambda(s)) - \sum_{l \in S} q_{\Lambda(s),l}(X(s))[f(l) - f(\Lambda(s))] \right) \, ds \right] \\
\geq \mathbb{E}[e^{-H(T \wedge \tau_\infty \wedge \bar{\tau}_m)} f(\Lambda(T \wedge \tau_\infty \wedge \bar{\tau}_m))] \\
+ \mathbb{E}\left[ \int_0^{T \wedge \tau_\infty \wedge \bar{\tau}_m} e^{-Hs}[Hf(\Lambda(s)) - H(1 + |X(s)|^2 + f(\Lambda(s)))] \, ds \right] \\
\geq \mathbb{E}[e^{-H(T \wedge \tau_\infty \wedge \bar{\tau}_m)} f(\Lambda(T \wedge \tau_\infty \wedge \bar{\tau}_m))],
\]

where the first inequality above follows from (1.6) in Assumption 1.2. Consequently we have

\[
e^{HT} f(k) \geq \mathbb{E}[f(\Lambda(T \wedge \tau_\infty \wedge \bar{\tau}_m))] \geq \mathbb{E}[f(\Lambda(\bar{\tau}_m))] 1_{\{\bar{\tau}_m \leq T \wedge \tau_\infty\}} \\
\geq f(m) \mathbb{P}\{\bar{\tau}_m \leq T \wedge \tau_\infty\} \geq f(m) \mathbb{P}\{\bar{\tau}_m \leq T \wedge \tau_\infty, A^c\} \tag{2.21}\\n\geq f(m) \mathbb{P}\{\bar{\tau}_\infty \leq T \wedge \tau_\infty, A^c\},
\]

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where the third inequality follows from the facts that \( \Lambda(\tilde{\tau}_m) \geq m \) and that \( f \) is nondecreasing, and the last inequality follows from the fact that \( \tilde{\tau}_m \uparrow \tilde{\tau}_\infty \). Recall that \( A^c = \{ \tau_\infty \geq \tilde{\tau}_\infty \} \). Thus

\[
P\{\tilde{\tau}_\infty \leq T \land \tau_\infty, A^c\} = P\{\tilde{\tau}_\infty \leq T \land \tau_\infty, \tilde{\tau}_\infty \leq \tau_\infty\} \\
\geq P\{\tilde{\tau}_\infty \leq T, \tilde{\tau}_\infty \leq \tau_\infty\} = P\{\tilde{\tau}_\infty \leq T, A^c\} = \delta > 0.
\]

Using this observation in (2.21) yields

\[
\infty > e^{HT} f(k) \geq f(m) \delta \to \infty \text{ as } m \to \infty,
\]

thanks to the fact that \( f(m) \to \infty \) as \( m \to \infty \). This is a contradiction. This establishes (2.20) and therefore completes the proof. \( \square \)

**Lemma 2.6.** Under Assumptions 1.1 and 1.2, the process \((X, \Lambda)\) has no finite explosion time with probability one; that is, \( P\{T_\infty = \infty\} = 1 \), where 

\[
T_\infty := \lim_{n \to \infty} T_n, \text{ and } T_n := \inf\{t \geq 0 : |X(t)| \lor \Lambda(t) \geq n\}.
\]

**Proof.** Consider the function \( V(x, k) := |x|^2 + f(k) \), where the function \( f : S \mapsto \mathbb{R}_+ \) is as in Assumption 1.2. Then we have from Assumptions 1.1 and 1.2 that

\[
\mathcal{A}V(x, k) = 2x \cdot b(x, k) + \frac{1}{2} \text{tr}(\sigma\sigma'(x, k)2I) + \sum_{l \in S} q_{kl}(x)[f(l) - f(k)] \\
+ \int_U |x + c(x, k, u)|^2 - |x|^2 - 2x \cdot c(x, k, u)\mathbf{1}_{U_0}(u)\Pi(du) \\
\leq 2|x|^2 + |b(x, k)|^2 + |\sigma(x, k)|^2 + H(1 + |x|^2 + k) \\
+ \int_U |c(x, k, u)|^2 \Pi(du) + \int_{U \setminus U_0} 2x \cdot c(x, k, u)\Pi(du) \\
\leq K(1 + |x|^2 + f(k)) = K(1 + V(x, k)),
\]

where \( K \) is a positive constant. Then the conclusion follows from Theorem 2.1 of Meyn and Tweedie (1993c). \( \square \)

**Proof of Theorem 2.1 (Step 3).** Finally we show that pathwise uniqueness for (1.1)–(1.2) holds. This, together with the existence result established in Steps 1 and 2, then implies that (1.1)–(1.2) has a unique strong solution \((X, \Lambda)\).

Suppose \((X, \Lambda)\) and \((\tilde{X}, \tilde{\Lambda})\) are two solutions to (1.1)–(1.2) starting from the same initial
condition \((x, k) \in \mathbb{R}^d \times S\). Then we have
\[
\tilde{X}(t) - X(t)
= \int_0^t \left[ b(\tilde{X}(s), \tilde{\Lambda}(s)) - b(X(s), \Lambda(s)) \right] ds + \int_0^t \left[ \sigma(\tilde{X}(s), \tilde{\Lambda}(s)) - \sigma(X(s), \Lambda(s)) \right] dW(s)
+ \int_{U_0} [c(\tilde{X}(s), \tilde{\Lambda}(s), z) - c(X(s), \Lambda(s), z)] \tilde{N}(ds, du)
+ \int_{U \setminus U_0} [c(\tilde{X}(s), \tilde{\Lambda}(s), z) - c(X(s), \Lambda(s), z)] N(ds, du),
\]
and
\[
\tilde{\Lambda}(t) - \Lambda(t) = \int_0^t \int_{\mathbb{R}^+} [h(\tilde{X}(s), \tilde{\Lambda}(s), z) - h(X(s), \Lambda(s), z)] N_1(ds, dz).
\]
Let \(\zeta := \inf\{ t \geq 0 : \Lambda(t) \neq \tilde{\Lambda}(t) \}\) be the first time when the discrete components differ from each other and define \(T_R := \inf\{ t \geq 0 : |\tilde{X}(t)| \vee |X(t)| \vee |\tilde{\Lambda}(t) \vee \Lambda(t) \geq R \}\) for \(R > 0\). Lemma 2.6 implies that \(T_R \to \infty\) a.s. as \(R \to \infty\). Note that \(\tilde{\Lambda}(s) = \Lambda(s)\) for all \(s < \zeta\). Detailed computations using (1.4) in Assumption 1.1 reveal that
\[
\mathbb{E}[|\tilde{X}(t \wedge \zeta \wedge T_R) - X(t \wedge \zeta \wedge T_R)|^2]
= \mathbb{E} \left[ \int_0^{t \wedge \zeta \wedge T_R} \left( 2(\tilde{X}(s) - X(s)) \cdot (b(\tilde{X}(s), \Lambda(s)) - b(X(s), \Lambda(s)))
+ |\sigma(\tilde{X}(s), \Lambda(s)) - \sigma(X(s), \Lambda(s))|^2
+ \int_{U_0} |c(\tilde{X}(s), \Lambda(s), u) - c(X(s), \Lambda(s), u)|^2 \Pi(du)
+ \int_{U_0} 2(\tilde{X}(s) - X(s)) \cdot (c(\tilde{X}(s), \Lambda(s), u) - c(X(s), \Lambda(s), u)) \Pi(du) \right) ds \right]
\leq K \mathbb{E} \left[ \int_0^{t \wedge \zeta \wedge T_R} |\tilde{X}(s) - X(s)|^2 ds \right]
= K \int_0^t \mathbb{E}[|\tilde{X}(s \wedge \zeta \wedge T_R) - X(s \wedge \zeta \wedge T_R)|^2] ds,
\]
where \(K\) is a positive constant. Applying Gronwall’s inequality, we see that
\[
\mathbb{E}[|\tilde{X}(t \wedge \zeta \wedge T_R) - X(t \wedge \zeta \wedge T_R)|^2] = 0
\]
for all \(R > 0\) and thus \(\mathbb{E}[|\tilde{X}(t \wedge \zeta) - X(t \wedge \zeta)|^2] = 0\), which, in turn, implies that
\[
\mathbb{E}[|\tilde{X}(t \wedge \zeta) - X(t \wedge \zeta)|] = 0 \text{ and } \mathbb{E}[|\tilde{X}(t \wedge \zeta) - X(t \wedge \zeta)|^\delta] = 0, \quad (2.22)
\]
where \(\delta \in (0, 1]\) is the Hölder constant in (1.7).
Note that \( \zeta \leq t \) if and only if \( \tilde{\Lambda}(t \wedge \tau) - \Lambda(t \wedge \tau) \neq 0 \). Therefore it follows that

\[
\mathbb{P}\{ \zeta \leq t \} = \mathbb{E}[1_{\{\tilde{\Lambda}(t \wedge \tau) - \Lambda(t \wedge \tau) \neq 0\}}]
\]

\[
= \mathbb{E}\left[ \int_0^{t \wedge \zeta} \left( 1\{\tilde{\Lambda}(s) - \Lambda(s) + h(\tilde{\Lambda}(s), \Lambda(s), z) - h(X(s), \Lambda(s), z) \neq 0\} - 1\{\tilde{\Lambda}(s) - \Lambda(s) \neq 0\}\right) m(\,dz\,)ds \right]
\]

\[
= \mathbb{E}\left[ \int_0^{t \wedge \zeta} 1\{h(\tilde{\Lambda}(s), \Lambda(s), z) - h(X(s), \Lambda(s), z) \neq 0\} m(\,dz\,)ds \right]
\]

\[
\leq \mathbb{E}\left[ \int_0^{t \wedge \zeta} \sum_{l \in S, l \neq \Lambda(s-)} |q_{\Lambda(s-), l}(\tilde{X}(s)) - q_{\Lambda(s-), l}(X(s))|ds \right]
\]

\[
\leq \kappa \mathbb{E}\left[ \int_0^{t \wedge \zeta} |\tilde{X}(s) - X(s)|^\delta ds \right] = \kappa \int_0^t \mathbb{E}[|\tilde{X}(s \wedge \zeta) - X(s \wedge \zeta)|^\delta]ds = 0,
\]

where the second inequality follows from (1.7). In particular, it follows that

\[
\mathbb{E}[1_{\{\tilde{\Lambda}(t) \neq \Lambda(t)\}}] = 0.
\]

Note also that \( \tilde{X}(t) - X(t) \) is integrable and hence it follows that \( \mathbb{E}[|\tilde{X}(t) - X(t)|1_{\{\zeta \leq t\}}] = 0 \).

Now we can compute

\[
\mathbb{E}[|\tilde{X}(t) - X(t)|] = \mathbb{E}[|\tilde{X}(t) - X(t)|1_{\{\zeta > t\}}] + \mathbb{E}[|\tilde{X}(t) - X(t)|1_{\{\zeta \leq t\}}]
\]

\[
= \mathbb{E}[|\tilde{X}(t \wedge \zeta) - X(t \wedge \zeta)|1_{\{\zeta > t\}}] + \mathbb{E}[|\tilde{X}(t) - X(t)|1_{\{\zeta \leq t\}}]
\]

\[
\leq \mathbb{E}[|\tilde{X}(t \wedge \zeta) - X(t \wedge \zeta)|] + 0
\]

\[
= 0,
\]

Recall that \( \lambda((x, m), (y, n)) := |x - y| + 1_{\{m \neq n\}} \) is a metric on \( \mathbb{R}^d \times \mathbb{S} \). Hence we have shown that

\[
\mathbb{E}[\lambda((\tilde{X}(t), \tilde{\Lambda}(t)), (X(t), \Lambda(t)))] = 0 \text{ for all } t \geq 0.
\]

Thus \( \mathbb{P}\{ (\tilde{X}(t), \tilde{\Lambda}(t)) = (X(t), \Lambda(t)) \} = 1 \) for all \( t \geq 0 \). This, together with the fact that the sample paths of \( (X, \Lambda) \) are right continuous, implies the desired pathwise uniqueness result.

\( \square \)

We finish the section with some moment estimates for the solution \( (X, \Lambda) \) of (1.1)–(1.2).

**Proposition 2.7.** Suppose Assumptions 1.1 and 1.2. Then we have for any \( T \geq 0 \)

\[
\mathbb{E}_{x,k} \left[ \sup_{0 \leq t \leq T} |X(t)|^2 \right] \leq C_1,
\]

where \( C_1 = C_1(x, T, H) \) is a positive constant. Assume in addition that

\[
\sum_{l \neq k}(l - k)q_{kl}(x)^2 \leq H(1 + |x|^2 + k^2)
\]

\( \text{(2.24)} \)
for all \((x, k) \in \mathbb{R}^d \times S\). Then for any \(T \geq 0\), we have
\[
\mathbb{E}_{x, k} \left[ \sup_{0 \leq t \leq T} (|X(t)|^2 + \Lambda(t)^2) \right] \leq C_2,
\] (2.25)
where \(C_2 = C_2(x, k, T, H)\) is a positive constant.

**Proof.** We notice that the standard arguments using the linear growth condition \((1.3)\) in Assumption 1.1 and the BDG inequality (see, for example, the proof of Lemma 3.1 in Zhu et al. (2015)) allow us to derive
\[
\mathbb{E}_{x, k} \left[ \sup_{0 \leq t \leq T} |X(t)|^2 \right] \leq K_1 + K_2 \int_0^T \mathbb{E}_{x, k} \left[ \sup_{1 \leq u \leq s} |X(u)|^2 \right] ds,
\] (2.26)
where \(K_1, K_2\) are positive constants depending only on \(x, H\), and \(T\). Then (2.23) follows from Gronwall’s inequality.

It remains to establish (2.25) under the additional condition (2.24). Since
\[
\Lambda(t) = k + \int_0^t \int_{\mathbb{R}_+} h(X(s), \Lambda(s), r)N_1(ds, dr)
\]
\[
= k + \int_0^t \int_{\mathbb{R}_+} h(X(s), \Lambda(s), r)\tilde{N}_1(ds, dr) + \int_0^t \int_{\mathbb{R}_+} h(X(s), \Lambda(s), r)m(dr)ds,
\]
we can use the BDG and Hölder inequalities to compute
\[
\mathbb{E}_{x, k} \left[ \sup_{0 \leq t \leq T} \Lambda(t)^2 \right] \leq 3k^2 + 3\mathbb{E}_{x, k} \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \int_{\mathbb{R}_+} h(X(s), \Lambda(s), r)\tilde{N}_1(ds, dr) \right)^2 \right]
\]
\[
+ 3\mathbb{E}_{x, k} \left[ \sup_{0 \leq t \leq T} \left( \int_0^t \int_{\mathbb{R}_+} h(X(s), \Lambda(s), r)m(dr)ds \right)^2 \right]
\]
\[
\leq 3k^2 + 3\mathbb{E}_{x, k} \left[ \int_0^T \int_{\mathbb{R}_+} h^2(X(s), \Lambda(s), r)m(dr)ds \right]
\]
\[
+ 3\mathbb{E}_{x, k} \left[ \left( \int_0^T \sum_{l \in S, l \neq \Lambda(s)} (l - \Lambda(s))q_{\Lambda(s), l}(X(s))ds \right)^2 \right]
\]
\[
\leq 3k^2 + 3\mathbb{E}_{x, k} \left[ \int_0^T \sum_{l \in S, l \neq \Lambda(s)} (l - \Lambda(s))^2q_{\Lambda(s), l}(X(s))ds \right]
\]
\[
+ 3\mathbb{E}_{x, k} \left[ \int_0^T \sum_{l \in S, l \neq \Lambda(s)} (l - \Lambda(s))^2q_{\Lambda(s), l}(X(s))ds \right]
\]
\[
\leq 3k^2 + 3H(1 + T)\mathbb{E}_{x, k} \left[ \int_0^T (1 + |X(s)|^2 + \Lambda(s)^2)ds \right]
\]
\[
\leq 3k^2 + 3H(1 + T)\mathbb{E}_{x, k} \left[ \int_0^T \left[ 1 + \sup_{0 \leq u \leq s} (|X(u)|^2 + \Lambda(u)^2) \right] ds \right],
\] (2.27)
where we used (1.6) and (2.24) to derive the second last inequality. Then (2.25) follows from a combination of (2.26) and (2.27) and Gronwall’s inequality. \(\square\)
3 Feller Property

We make the following assumption throughout this section:

**Assumption 3.1.** Suppose that for all \( x, z \in \mathbb{R}^d \) and \( k \in \mathcal{S} \), we have

\[
\int_U |c(x, k, u) - c(z, k, u)| \Pi(du) \leq H |x - z|, \tag{3.1}
\]

and

\[
\sum_{l \in \mathcal{S} \setminus \{k\}} |q_{kl}(x) - q_{kl}(y)| \leq H |x - y|, \tag{3.2}
\]

where the constant \( H > 0 \) is the same as in Assumption 1.1 without loss of generality.

**Remark 3.2.** In (1.4) of Assumption 1.1, we assumed that

\[
\int_U |c(x, k, u) - c(y, k, u)|^2 \Pi(du) \leq H |x - y|^2
\]

for all \( x, y \in \mathbb{R}^d \) and \( k \in \mathcal{S} \). This condition in general does not necessarily imply (3.1). Consider for example \( U = (0, 1) \) and \( \Pi(du) = \frac{du}{1 + x^2} \) with some \( \alpha \in (0, 1) \). We can check directly that the function \( c(x, k, u) := xu^{2/\alpha} \) satisfies (1.4) but not (3.1).

The main result of this section is:

**Theorem 3.3.** Suppose that Assumptions 1.1, 1.2, and 3.1 hold. Then the process \((X, \Lambda)\) generated by the operator \( \mathcal{A} \) of (2.14) has Feller property.
which is a coupling of the jump part in the generator $\mathcal{L}_i$ defined in (2.15). Next we define the basic coupling (see, e.g., p. 11 on Chen (2004)) for the $q$-matrices $Q(x)$ and $Q(y)$. For any $f(x, i, z, j) \in C^2_c(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}^d \times \mathbb{S})$, we define

$$
\tilde{\Omega}_{\text{switching}} f(x, i, z, j) := \sum_{l \in \mathbb{S}} [q_{il}(x) - q_{jl}(z)]^+(f(x, l, z, j) - f(x, i, z, j)) \\
+ \sum_{l \in \mathbb{S}} [q_{jl}(z) - q_{id}(x)]^+(f(x, i, z, l) - f(x, i, z, j)) \\
+ \sum_{l \in \mathbb{S}} [q_{id}(x) \wedge q_{jl}(z)][f(x, l, z, l) - f(x, i, z, j)].
$$

It is easy to verify that $\tilde{Q}(x, z)$ defined in (3.5) is a coupling to $Q(x)$ defined in (2.16).

Finally, the coupling operator to $A$ of (2.14) can be written as

$$
\tilde{A} f(x, i, z, j) := [\tilde{\Omega}_{\text{diffusion}} + \tilde{\Omega}_{\text{jump}} + \tilde{\Omega}_{\text{switching}}] f(x, i, z, j).
$$

In fact, we can verify directly that for any $f(x, i, z, j) = g(x, i) \in C^2_c(\mathbb{R}^d \times \mathbb{S})$, we have $\tilde{A} f(x, i, z, j) = Ag(x, i)$.

As in the proof of Proposition 5.2.13 in Karatzas and Shreve (1991), we can construct a sequence $\{\psi_n(r)\}_{n=1}^\infty$ of twice continuously differentiable functions satisfying $|\psi_n''(r)| \leq 1$ and $\lim_{n \to \infty} \psi_n(r) = |r|$ for $r \in \mathbb{R}$, and $0 \leq \psi_n''(r) \leq 2n^{-1}H^{-1}r^{-2}$ for $r \neq 0$, where $H$ is as in (1.4). Furthermore, for every $r \in \mathbb{R}$, the sequence $\{\psi_n(r)\}_{n=1}^\infty$ is nondecreasing.

**Lemma 3.4.** For each $n \in \mathbb{N}$, let the function $\psi_n$ be defined as above and further define the function

$$
f_n(x, k, z, l) := \psi_n(|x - z|) + 1_{\{k \neq l\}}, \quad (x, k, z, l) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}^d \times \mathbb{S}.
$$

Then for all $(x, k, z, k) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}^d \times \mathbb{S}$ with $x \neq z$, we have

$$
\bar{A} f_n(x, k, z, k) \leq \frac{1}{n} + C|x - z|,
$$

in which $C = C(H)$ is a positive constant.

**Proof.** For any $x, z \in \mathbb{R}^d$ and $k, l \in \mathbb{S}$, set

$$
A(x, k, z, l) = a(x, k) + a(z, l) - 2\sigma(x, k)\sigma(z, l)',
$$

$$
\hat{B}(x, k, z, l) = \langle x - z, b(x, k) - b(z, l) \rangle,
$$

and

$$
\bar{A}(x, k, z, l) = \langle x - z, A(x, k, z, l)(x - z) \rangle / |x - z|^2.
$$
Then as in the proof of Theorem 3.1 in Chen and Li (1989), we can verify that

\[ 2 \tilde{\Omega} \text{diffusion}_n(x, k, z, l) = \psi_n''(|x - z|)A(x, k, z, l) + \frac{\psi_n'(|x - z|)}{|x - z|} \left[ \text{tr}(A(x, k, z, l)) - A(x, k, z, l) + 2\hat{B}(x, k, z, l) \right]. \]

Note that \( \text{tr}(A(x, k, z, k)) = \|\sigma(x, k) - \sigma(z, k)\|^2 \) and hence we obtain from (1.4) that

\[ \text{tr}A(x, k, z, k) + \hat{B}(x, k, z, k) \leq H|x - z|^2. \]

On the other hand, using (1.4) again,

\[ A(x, k, z, k) = \langle x - z, (\sigma(x, k) - \sigma(z, k))(\sigma(x, k) - \sigma(z, k))^T(x - z) \rangle \leq H|x - z|^2. \]

Thus it follows that

\[ \tilde{\Omega} \text{diffusion}_n(x, k, z, k) \leq \frac{1}{2}\psi_n''(|x - z|)H|x - z|^2 + \frac{3}{2}\psi_n'(|x - z|)H|x - z| \leq \frac{1}{n} + \frac{3}{2}H|x - z|, \tag{3.8} \]

where the last inequality follows from the construction of the function \( \psi_n \).

Next we show that for some positive constant \( K \), we have

\[ \tilde{\Omega} \text{jump}_n(x, k, z, k) \leq K|x - z|. \tag{3.9} \]

In fact, since \( |\psi_n'| \leq 1 \), we can use (3.1) to compute

\[ \int_{U_0^c} \left[ \psi_n(|x + c(x, k, u) - z - c(z, k, u)|) - \psi_n(|x - z|) \right] \Pi(du) \leq \int_{U_0^c} |c(x, k, u) - c(z, k, u)| \Pi(du) \leq H|x - z|. \]

On the other hand, note that \( D_z\psi_n(|x - z|) = -D_x\psi_n(|x - z|) \). Thus it follows that

\[ \int_{U_0} \left[ \psi_n(|x + c(x, k, u) - z - c(z, k, u)|) - \psi_n(|x - z|) \right. \]

\[ - \langle D_x\psi_n(|x - z|), c(x, k, u) \rangle - \langle D_z\psi_n(|x - z|), c(z, k, u) \rangle \Pi(du) \]

\[ = \int_{U_0} \left[ \psi_n(|x - z + c(x, k, u) - c(z, k, u)|) - \psi_n(|x - z|) \right. \]

\[ - \langle D_x\psi_n(|x - z|), c(x, k, u) - c(z, k, u) \rangle \Pi(du) \]

\[ \leq 2 \int_{U_0} |c(x, k, u) - c(z, k, u)| \Pi(du) \leq 2H|x - z|, \]
where we used (3.1) to obtain the last inequality. Combining the above two displayed equations gives (3.9).

Finally we estimate $\tilde{\Omega}_{\text{switching}} f_n(x, k, z, l)$. Clearly we have $\tilde{\Omega}_{\text{switching}} f_n(x, k, z, l) \leq 0$ when $k \neq l$. When $k = l$, we have from (3.2) that

$$
\tilde{\Omega}_{\text{switching}} f_n(x, k, z, k) = \sum_{i \in S}[q_{ki}(x) - q_{ki}(z)]^+ (1_{\{i \neq k\}} - 1_{\{k \neq k\}})
+ \sum_{i \in S}[q_{ki}(z) - q_{ki}(x)]^+ (1_{\{i \neq k\}} - 1_{\{k \neq k\}}) + 0
\leq \sum_{i \in S,i \neq k} |q_{ki}(x) - q_{ki}(z)|
\leq H|x - z|.
$$

(3.10)

Now plug (3.8), (3.9), and (3.10) into (3.6) yields (3.7). This completes the proof. □

**Proof of Theorem 3.3.** Denote by $\{P(t, x, k, A) : t \geq 0, (x, k) \in \mathbb{R}^d \times S, A \in \mathcal{B}(\mathbb{R}^d \times S)\}$ the transition probability family of the process $(X, \Lambda)$. Since $\mathbb{S}$ has a discrete topology, we need only to show that for each $t \geq 0$ and $k \in \mathbb{S}$, $P(t, x, k, \cdot)$ converges weakly to $P(t, z, k, \cdot)$ as $x - z \to 0$. By virtue of Theorem 5.6 in Chen (2004), it suffices to prove that

$$
W(P(t, x, k, \cdot), P(t, z, k, \cdot)) \to 0 \text{ as } x \to z,
$$

(3.11)

where $W(\cdot, \cdot)$ denotes the Wasserstein metric between two probability measures.

Let $(\tilde{X}(t), \tilde{\Lambda}(t), \tilde{Z}(t), \tilde{\Omega}(t))$ denote the coupling process corresponding to the coupling operator $\tilde{A}$ defined in (3.6). Assume that $(\tilde{X}(0), \tilde{\Lambda}(0), \tilde{Z}(0), \tilde{\Omega}(0)) = (x, k, z, k) \in \mathbb{R}^d \times S \times \mathbb{R}^d \times S$ with $x \neq z$. Define $\zeta := \inf \{t \geq 0 : \tilde{\Lambda}(t) \neq \tilde{\Omega}(t)\}$. Note that $\mathbb{P}\{\zeta > 0\} = 1$. In addition, similarly to the proof of Theorem 2.3 in Chen and Li (1989), set

$$
T_R := \inf \{t \geq 0 : |\tilde{X}(t)|^2 + |\tilde{Z}(t)|^2 + \tilde{\Lambda}(t) + \tilde{\Omega}(t) > R\}.
$$

Now we apply Itô’s formula to the process $f_n(\tilde{X}(\cdot), \tilde{\Lambda}(\cdot), \tilde{Z}(\cdot), \tilde{\Omega}(\cdot))$ to obtain

$$
\mathbb{E}[f_n(\tilde{X}(t \wedge T_R \wedge \zeta), \tilde{\Lambda}(t \wedge T_R \wedge \zeta), \tilde{Z}(t \wedge T_R \wedge \zeta), \tilde{\Omega}(t \wedge T_R \wedge \zeta))]
= f_n(x, k, z, k)
+ \mathbb{E}\left[\int_0^{t \wedge T_R \wedge \zeta} \tilde{\mathbb{A}} f_n(\tilde{X}(s), \tilde{\Lambda}(s), \tilde{Z}(s), \tilde{\Omega}(s))ds\right]
\leq \psi_n(|x - z|) + \frac{t}{n} + C\mathbb{E}\left[\int_0^{t \wedge T_R \wedge \zeta} |\tilde{X}(s) - \tilde{Z}(s)|ds\right],
$$

(3.12)

where the last step follows from the observation that $\tilde{\Lambda}(s) = \tilde{\Omega}(s)$ for all $s \in [0, t \wedge T_R \wedge \zeta)$ and the estimate in (3.7). Since $f_n(x, k, z, l) = \psi_n(|x - z|) + 1_{\{k \neq l\}} \geq \psi_n(|x - z|)$, we have
from (3.12) that

\[ E[\psi_n(\Xi(t \wedge T_R \wedge \zeta) - \tilde{Z}(t \wedge T_R \wedge \zeta))] \]
\[ \leq \psi_n(|x - z|) + \frac{t}{n} + CE\left[ \int_0^{t \wedge T_R \wedge \zeta} |\tilde{X}(s) - \tilde{Z}(s)|ds \right]. \]

Recall that \( \psi_n(|x|) \uparrow |x| \) as \( n \to \infty \). Therefore, passing to the limit as \( n \to \infty \) on both sides of the above equation, it follows from the Monotone Convergence Theorem that

\[ E[\Xi(t \wedge T_R \wedge \zeta) - \tilde{Z}(t \wedge T_R \wedge \zeta)] \]
\[ \leq |x - z| + CE\left[ \int_0^{t \wedge T_R \wedge \zeta} |\tilde{X}(s) - \tilde{Z}(s)|ds \right] \]
\[ = |x - z| + CE\left[ \int_0^t |\tilde{X}(s \wedge T_R \wedge \zeta) - \tilde{Z}(s \wedge T_R \wedge \zeta)|ds \right]. \]

Then an application of Gronwall’s inequality leads to

\[ E[|\Xi(t \wedge T_R \wedge \zeta) - \tilde{Z}(t \wedge T_R \wedge \zeta)|] \leq |x - z| \exp(Ct). \]

Now passing to the limit as \( R \uparrow \infty \), we conclude that

\[ E[|\Xi(t \wedge \zeta) - \tilde{Z}(t \wedge \zeta)|] \leq |x - z| \exp(Ct). \]  

(3.13)

Observe that \( \zeta \leq t \) if and only if \( \tilde{X}(t \wedge \zeta) \neq \tilde{Z}(t \wedge \zeta) \). Put \( f(x, k, z, l) := 1_{\{k \neq l\}} \) and apply Itô’s formula to the process \( f(\tilde{X}(t), \tilde{\Lambda}(t), \tilde{Z}(t), \tilde{\Sigma}(t)) \):

\[ \mathbb{P}\{\zeta \leq t\} = \mathbb{E}[1_{\{\tilde{X}(t \wedge \zeta) \neq \tilde{Z}(t \wedge \zeta)\}}] = \mathbb{E}[f(\tilde{X}(t \wedge \zeta), \tilde{\Lambda}(t \wedge \zeta), \tilde{Z}(t \wedge \zeta), \tilde{\Sigma}(t \wedge \zeta))] \]
\[ = \mathbb{E}\left[ \int_0^{t \wedge \zeta} \tilde{A}f(\tilde{X}(s), \tilde{\Lambda}(s), \tilde{Z}(s), \tilde{\Sigma}(s))ds \right] \]
\[ \leq HE\left[ \int_0^{t \wedge \zeta} |\tilde{X}(s) - \tilde{Z}(s)|ds \right] \]
\[ = H \int_0^t \mathbb{E}[|\tilde{X}(s \wedge \zeta) - \tilde{Z}(s \wedge \zeta)|]ds \]
\[ \leq K|x - z|e^{Ct}, \]  

(3.14)

where \( K = K(H, \Pi(U_0^c)) \) is a positive constant, the first inequality above follows from (3.10) and the last step follows from (3.13).

The standard argument using Assumptions 1.1 and 1.2 reveals that \( \mathbb{E}[\sup_{0 \leq s \leq t} |\tilde{X}(s)|^2 + |\tilde{Z}(s)|^2] \leq K(1 + |x|^2 + |z|^2) \), where \( K = K(t, H, \Pi(U_0^c)) \) is a positive constant. Then it follows from the Hölder inequality and (3.14) that

\[ \mathbb{E}[|\tilde{X}(t) - \tilde{Z}(t) - \tilde{X}(t \wedge \zeta) + \tilde{Z}(t \wedge \zeta)|1_{\{\zeta \leq t\}}] \leq K(1 + |x|^2 + |z|^2)^{\frac{1}{2}}|x - z|^{\frac{1}{2}}, \]  

(3.15)
where in the above, \( K \) is a positive constant depending only on \( t, H, \) and \( \Pi(U_0^c) \). Finally, we combine (3.13) and (3.15) to obtain
\[
\begin{align*}
\mathbb{E}[|\tilde{X}(t) - \tilde{Z}(t)|] & \leq \mathbb{E}[|\tilde{X}(t \wedge \zeta) - \tilde{Z}(t \wedge \zeta)|] + \mathbb{E}[|\tilde{X}(t) - \tilde{Z}(t) - \tilde{X}(t \wedge \zeta) + \tilde{Z}(t \wedge \zeta)||\zeta \leq t]\] (3.16)
\leq K|x - z| + K(1 + |x|^2 + |z|^2)^{1/2}|x - z|^{1/2}.
\end{align*}
\]

Observe that if \( \tilde{\Lambda}(t) \neq \tilde{\Xi}(t) \) then \( \zeta \leq t \). Thus thanks to (3.14), we also have
\[
\mathbb{E}[1_{\{\tilde{\Lambda}(t) \neq \tilde{\Xi}(t)\}}] \leq P\{\zeta \leq t\} \leq K|x - z|e^{Ct}.
\] (3.17)

Now let \( f \in C_b(\mathbb{R}^d \times S) \), then we have
\[
\begin{align*}
\mathbb{E}[|f(\tilde{X}(t), \tilde{\Lambda}(t)) - f(\tilde{Z}(t), \tilde{\Xi}(t))|] & \leq \mathbb{E}[|f(\tilde{X}(t), \tilde{\Lambda}(t)) - f(\tilde{Z}(t), \tilde{\Lambda}(t))|] + \mathbb{E}[|f(\tilde{Z}(t), \tilde{\Lambda}(t)) - f(\tilde{Z}(t), \tilde{\Xi}(t))|].
\end{align*}
\] (3.18)

Both terms on the right-hand side of (3.18) converge to 0 as \( x \to z \) thanks to (3.16), (3.17), the continuity of \( f \), and the bounded convergence theorem. This implies (3.11) and therefore completes the proof. \( \square \)

4 Strong Feller Property: Jump Diffusions

In order to prove the strong Feller property, we further make the following assumption:

**Assumption 4.1.** Assume that the characteristic measure \( \Pi(\cdot) \) is finite (i.e., \( U_0 \equiv \emptyset \)) and that for each \( k \in S \), the diffusion \( X^{(k),0} \) satisfying
\[
dX^{(k),0}(t) = b(X^{(k),0}(t), k)dt + \sigma(X^{(k),0}(t), k)dB(t),
\] (4.1)
has the strong Feller property and has a transition probability density with respect to the Lebesgue measure.

**Remark 4.2.** For a given \( k \in S \), a sufficient condition for \( X^{(k),0} \) to have the strong Feller property and to have a transition probability density is that the Fisk-Stratonovich type generator of \( X^{(k),0} \) is hypoelliptic (see, for example, Ichihara and Kunita (1974), Kliemann (1987) for details). In particular, if the diffusion matrix of \( X^{(k),0} \) is uniformly positive, then the diffusion process \( X^{(k),0} \) must have the strong Feller property and must have a transition probability density (see the last paragraph of Section 2 in Kliemann (1983) or Section 8 of Chapter V in Ikeda and Watanabe (1989)).
For later use, we now introduce a family of jump diffusions under Assumption 4.1. For each \( k \in \mathbb{S} \), let the single jump diffusion \( X^{(k)} \) satisfy the following stochastic differential-integral equation:

\[
dX^{(k)}(t) = b(X^{(k)}(t), k)dt + \sigma(X^{(k)}(t), k)dB(t) + \int_U c(X^{(k)}(t-), k, u)N(dt, du).
\]

(4.2)

**Lemma 4.3.** Suppose that Assumption 4.1 holds. For each given \( k \in \mathbb{S} \), the jump-diffusion process \( X^{(k)} \) has the strong Feller property with a transition probability density with respect to the Lebesgue measure.

**Proof.** For a given \( k \in \mathbb{S} \), let us denote by \( P^{(k)}(t, x, A) \) the transition probability for the process \( X^{(k)} \), and by \( P^{(k),0}(t, x, A) \) the transition probability for the process \( X^{(k),0} \). Following the proofs of (Skorokhod, 1989, Theorem 14 in Chapter I) and (Li et al., 2002, Lemma 2.3) with some elementary analysis, for any given \( t > 0 \), \( x \in \mathbb{R}^d \) and \( A \in \mathcal{B}(\mathbb{R}^d) \), we obtain the relation

\[
P^{(k)}(t, x, A) = \exp\{-t\Pi(U)\}P^{(k),0}(t, x, A) + \int_0^t \int_U \exp\{-s\Pi(U)\}P^{(k),0}(s, x, dy_1)\Pi(du_1)ds,
\]

(4.3)

From this we have

\[
P^{(k)}(t - s_1, y_1 + c(y_1, k, u_1), A)
= \exp\{-(t - s_1)\Pi(U)\}P^{(k),0}(t - s_1, y_1 + c(y_1, k, u_1), A)
+ \int_{t-s_1}^t \int_U \exp\{-s_2\Pi(U)\}P^{(k),0}(s_2, y_1 + c(y_1, k, u_1), dy_2)
\times \Pi(du_2)ds_2 P^{(k)}(t - s_1 - s_2, y_2 + c(y_2, k, u_2), A).
\]

(4.4)

Using (4.3) again we further have

\[
P^{(k)}(t - s_1 - s_2, y_2 + c(y_2, k, u_2), A)
= \exp\{-(t - s_1 - s_2)\Pi(U)\}P^{(k),0}(t - s_1 - s_2, y_2 + c(y_2, k, u_2), A)
+ \int_{t-s_1-s_2}^t \int_U \exp\{-s_3\Pi(U)\}
\times P^{(k),0}(s_3, y_2 + c(y_2, k, u_2), dy_3)\Pi(du_3)ds_3
\times P^{(k)}(t - s_1 - s_2 - s_3, y_3 + c(y_3, k, u_3), A).
\]

(4.5)

Using (4.3) countably many times, we conclude that for any given \( t > 0 \), \( x \in \mathbb{R}^d \) and \( A \in \mathcal{B}(\mathbb{R}^d) \),

\[
P^{(k)}(t, x, A) = \text{a series}.
\]

(4.6)

For this series, from (4.3)–(4.5) we derive that the first term (in which the process has no jump on \([0, t]\)) is

\[
\exp\{-t\Pi(U)\}P^{(k),0}(t, x, A),
\]

(4.7)
the second term (in which the process has just one jump on \([0, t]) is
\[
\exp\{-t\Pi(U)\} \int_0^t \int_U \int_U P^{(k),0}(s_1, x, dy_1)\Pi(du_1)ds_1 \\
\times P^{(k),0}(t - s_1, y_1 + c(y_1, k, u_1), A),
\]
(4.8)
the third term (in which the process has just two jumps on \([0, t]) is
\[
\exp\{-t\Pi(U)\} \int_0^t \int_U \int_U \int_U P^{(k),0}(s_1, x, dy_1)\Pi(du_1)ds_1 \\
\times P^{(k),0}(s_2, y_1 + c(y_1, k, u_1), dy_2)\Pi(du_2)ds_2 \\
\times P^{(k),0}(t - s_1 - s_2, y_2 + c(y_2, k, u_2), A),
\]
(4.9)
and moreover, the general term (in which the process has just \(n\) jumps on \([0, t]) is
\[
\exp\{-t\Pi(U)\} \int_0^t \int_U \int_U \int_U \int_U \cdots \int_U P^{(k),0}(s_1, x, dy_1)\Pi(du_1)ds_1 \\
\times P^{(k),0}(s_2, y_1 + c(y_1, k, u_1), dy_2)\Pi(du_2)ds_2 \cdots \\
\times P^{(k),0}(s_n, y_{n-1} + c(y_{n-1}, k, u_{n-1}), dy_n)\Pi(du_n)ds_n \\
\times P^{(k),0}(t - s_1 - \cdots - s_n, y_n + c(y_n, k, u_n), A).
\]
(4.10)
In general, it is easy to see that the \(n\)th term does not exceed
\[
\frac{(t\Pi(U))^{n-1}}{(n-1)!} \exp\{-t\Pi(U)\}.
\]
Hence it follows that the series in (4.6) converges uniformly with respect to \(x\) over \(\mathbb{R}^d\).

It is easy to prove that for any given \(t > 0\) and \(A \in \mathcal{B}(\mathbb{R}^d)\), each term of the series in (4.6) is lower semicontinuous with respect to \(x\) by the strong Feller property of \(X^{(k),0}\) (see Assumption 4.1). Therefore, it follows that for any given \(t > 0\) and \(A \in \mathcal{B}(\mathbb{R}^d)\), \(P^{(k)}(t, x, A)\) is also lower semicontinuous with respect to \(x\). As a result, \(X^{(k)}\) has the strong Feller property by Proposition 6.1.1 in Meyn and Tweedie (1993a). Finally, from (4.6), \(X^{(k)}\) has a transition probability density with respect to the Lebesgue measure since \(X^{(k),0}\) does so under Assumption 4.1. The proof is complete. \(\square\)

**Remark 4.4.** From (4.6) we can also see that if transition probability density of \(\tilde{X}^{(k),0}\) is positive, so is that of \(\tilde{X}^{(k)}\).

## 5 Strong Feller Property: Regime-Switching Jump Diffusions

In order to prove the strong Feller property for \((X, \Lambda)\), we further make the following assumption.
Assumption 5.1. There exists a positive integer $\kappa$ such that $q_{kl}(x) = 0$ for all $k, l \in S$ with $|k - l| \geq \kappa + 1$.

Now let us establish the strong Feller property for the regime-switching jump diffusion $(X, \Lambda)$.

Theorem 5.2. Suppose that Assumptions 1.1, 1.2, 3.1, 4.1, and 5.1 hold. Then $(X, \Lambda)$ has the strong Feller property.

To proceed, we first consider the strong Feller property for a special type of switching jump-diffusion $(V, \psi)$. Let the first component $V$ satisfy
\[
    dV(t) = b(V(t), \psi(t))dt + \sigma(V(t), \psi(t))dB(t) + \int_U c(V(t-), \psi(t-), u)N(dt, du),
\]
and the second component $\psi$ that is independent of the Brownian motion $B(\cdot)$ and Poisson random measure $N(\cdot, \cdot)$, be a time-homogeneous Markov chain with state space $S$ satisfying
\[
    \mathbb{P}\{\psi(t + \Delta) = l | \psi(t) = k\} = \begin{cases} 
        \hat{q}_{kl}\Delta + o(\Delta), & \text{if } k \neq l, \\
        1 + \hat{q}_{kk}\Delta + o(\Delta), & \text{if } k = l
    \end{cases}
\]
provided $\Delta \downarrow 0$, where $\hat{Q} = (\hat{q}_{kl})$ is a conservative Q-matrix such that (i) all the diagonal elements are equal to $-2\kappa$, (ii) there are exactly $2\kappa$ off diagonal elements being 1 that are as symmetric and adjacent to the diagonal entry as possible, and (iii) all other elements are zero. To be precise,
\[
    \hat{q}_{kl} = \begin{cases} 
        -2\kappa & \text{if } k = l = 0, 1, 2, \ldots, \\
        1 & \text{if } k = 0, 1, 2, \ldots, \kappa - 1, \text{ and } l = 0, 1, 2, \ldots, 2\kappa \text{ with } l \neq k, \\
        1 & \text{if } k = \kappa + 1, \kappa + 2, \ldots, \text{ and } |l - k| \leq \kappa, \\
        0 & \text{otherwise}.
    \end{cases}
\]

For example, when $\kappa = 1$,
\[
    \hat{Q} = (\hat{q}_{kl}) = \begin{pmatrix} 
        -2 & 1 & 1 & 0 & 0 & 0 & \cdots \\
        1 & -2 & 1 & 0 & 0 & 0 & \cdots \\
        0 & 1 & -2 & 1 & 0 & 0 & \cdots \\
        0 & 0 & 1 & -2 & 1 & 0 & \cdots \\
        \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
    \end{pmatrix}.
\]

Obviously, if the $-2$, 1 and 1 on the first row of this matrix were replaced by $-1$, 1 and 0, then this matrix would be a very simple birth-death matrix. In the sequel, we sometimes emphasize the process $(V(t), \psi(t))$ with initial condition $(V(0), \psi(0)) = (x, k)$ by $(V^{(x,k)}(t), \psi^{(k)}(t))$. 
Moreover, denote by \( \Gamma(t, (x, k), \cdot) \) the transition probability of \((V, \psi)\). For subsequent use, let us fix a probability measure \( \mu(\cdot) \) that is equivalent to the product measure on \( \mathbb{R}^d \times S \) of the Lebesgue measure on \( \mathbb{R}^d \) and the counting measure on \( S \). For example, \( \mu(\cdot) \) could be taken as the product measure of the Gaussian probability measure on \( \mathbb{R}^d \) and the Poisson probability measure on \( S \).

**Lemma 5.3.** Suppose that Assumptions 1.1, 4.1, and 5.1 hold. Then \((V, \psi)\) has the strong Feller property and the transition probability \( \Gamma(t, (x, k), \cdot) \) of \((V, \psi)\) has density \( \gamma(t, (x, k), \cdot) \) with respect to \( \mu(\cdot) \).

**Proof.** Denote by the \( v_1 \) the stopping time defined by \( v_1 = \inf\{s > 0 : \psi(t) \neq \psi(0)\} \). When \( \psi(0) = k \), \((v_1, \psi(v_1))\) on \([0, \infty) \times S_k\) with respect to the product of the Lebesgue measure and the counting measure has the probability density \( \exp(-2\kappa s)1_{\delta_k}(l) \), where \( \delta_k := \{l \in S : \bar{q}_{kl} = 1\} \) is a finite subset of \( S \). For any given \( t > 0 \), \( x \in \mathbb{R}^d \), \( k, l \in S \) and \( A \in \mathcal{B}(\mathbb{R}^d) \), we have the relation

\[
\Gamma(t, (x, k), A \times \{l\}) = \delta_{kl} \exp\{-2\kappa t\} P^{(k)}(t, x, A) + \int_0^t \sum_{l_1 \in S_k} \int \exp\{-2\kappa s_1\} P^{(k)}(s_1, x, dy_1) \Gamma(t-s_1, (y_1, l_1), A \times \{l\}) ds_1, \tag{5.4}
\]

where \( \delta_{kl} \) is the Kronecker symbol in \( k, l \), which equals 1 if \( k = l \) and is 0 if \( k \neq l \). From this we have

\[
\Gamma(t-s_1, (y_1, l_1), A \times \{l\}) = \delta_{l_1l} \exp\{-2\kappa (t-s_1)\} P^{(l_1)}(t-s_1, y_1, A) + \int_0^{t-s_1} \sum_{l_2 \in S_{l_1}} \int \exp\{-2\kappa s_2\} P^{(l_1)}(s_2, y_1, dy_2) \times \Gamma(t-s_1-s_2, (y_2, l_2), A \times \{l\}) ds_2. \tag{5.5}
\]

Using (5.4) countably many times, as in the proof of Lemma 4.3, we conclude that for any given \( t > 0 \), \( x \in \mathbb{R}^d \) and \( A \in \mathcal{B}(\mathbb{R}^d) \),

\[
\Gamma(t, (x, k), A \times \{l\}) = \text{a series}. \tag{5.6}
\]

For this series, as in the proof of Lemma 4.3, we derive that the first term (in which \( \psi \) has no jump on \([0, t]\)) is

\[
\delta_{kl} \exp\{-2\kappa t\} P^{(k)}(t, x, A), \tag{5.7}
\]

the second term (in which \( \psi \) has just one jump on \([0, t]\)) is

\[
\exp\{-2\kappa t\} \int_0^t \sum_{l_1 \in S_k, l_1 \neq l} \int P^{(k)}(s_1, x, dy_1) P^{(l_1)}(t-s_1, y_1, A) ds_1, \tag{5.8}
\]
and the third term (in which $\psi$ has just two jumps on $[0,t]$) is
\[
\exp\{-2\kappa t\} \int_0^t \int_0^{t-s_1} \sum_{l_1 \in S, l_2 \in S} \int \int P^{(k)}(s_1, x, dy_1) \\
\times P^{(l_1)}(s_2, y_1, dy_2) P^{(l_2)}(t-s_1-s_2, y_2, A) ds_2 ds_1.
\] (5.9)

Similar to the proof of Lemma of 4.3, we can easily verify that the $n$th term of the series in (5.6) is bounded above by $\frac{(2\kappa)^{n-1}}{(n-1)!} \exp\{-2\kappa t\}$. Thus it is uniformly convergent with respect to $x \in \mathbb{R}^d$. Noting that $S$ is an infinitely countable set with a discrete metric, and using similar arguments as those in the proof of Lemma 4.3, we derive Lemma 5.3. \(\square\)

**Lemma 5.4.** Suppose that Assumptions 1.1, 4.1, and 5.1 hold. Then for all $T > 0$, $\delta > 0$ and $k \in S$, we have
\[
\mathbb{P}\left\{ \sup_{0 \leq t \leq T} |V^{(x,k)}(t) - V^{(y,k)}(t)| \geq \delta \right\} \to 0
\] (5.10)
as $|x - y| \to 0$.

*Proof.* This lemma is just (Xi, 2009, Lemma 4.1). \(\square\)

**Lemma 5.5.** For any bounded and measurable function $f$ on $\mathbb{R}^d \times S$ and any positive number $\delta > 0$, there exists a compact subset $D \subset \mathbb{R}^d$ such that $\mu(D^c \times S) < \delta$ and $f|_{D \times S}$, the function $f$ restricted to $D \times S$, is uniformly continuous.

*Proof.* This lemma can be derived from the Lusin Theorem (see, for example (Cohn, 1980, Theorem 7.4.3)). \(\square\)

**Lemma 5.6.** Suppose that Assumptions 1.1, 4.1, and 5.1 hold. For any given $t > 0$ and bounded measurable function $f$ on $\mathbb{R}^d \times S$, we have that
\[
f(V^{(x,k)}(t), \psi^{(k)}(t)) \to f(V^{(y,k)}(t), \psi^{(k)}(t))
\] in probability (5.11)
as $|x - y| \to 0$.

*Proof.* It follows from Lemma 5.3 that for any $(x, k) \in \mathbb{R}^d \times S$, any $A \in \mathcal{B}(\mathbb{R}^d)$ and $l \in S$,
\[
\mathbb{P}\{ (V^{(x,k)}(t), \psi^{(k)}(t)) \in A \times \{l\} \} = \int_A \gamma(t, (x, k), (y, l)) \mu(dy \times \{l\}).
\] (5.12)

By the strong Feller property proved in Lemma 5.3, for any sequence $\{x_n\}$ satisfying $x_n \to x$ and for any $g(y, l) \in L^\infty(\mu)$, we have
\[
\sum_{l \in S} \int g(y, l) \gamma(t, (x_n, k), (y, l)) \mu(dy \times \{l\}) \to \sum_{l \in S} \int g(y, l) \gamma(t, (x, k), (y, l)) \mu(dy \times \{l\})
\]
as \( n \to \infty \). Namely, when \( n \to \infty \), \( \gamma(t, (x_n, k), \cdot) \) converges weakly to \( \gamma(t, (x, k), \cdot) \) in \( L_1(\mu) \). Thus, by the Dunford-Pettis theorem, we obtain that the family \( \{\gamma(t, (x_n, k), \cdot) : n \geq 1\} \) is uniformly integrable in \( L_1(\mu) \). Hence for any given \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( A \in \mathcal{B}(\mathbb{R}^d) \), if \( \mu(A \times S) < \delta \), then for all \( n \geq 1 \),

\[
\mathbb{P}(\{V(x_n, k)(t), \psi(k)(t) \in A \times S\}) = \sum_{l \in S} \int_A \gamma(t, (x_n, k), (y, l)) \mu(dy \times \{l\}) < \varepsilon, \tag{5.13}
\]

and

\[
\mathbb{P}(\{V(x, k)(t), \psi(k)(t) \in A \times S\}) = \sum_{l \in S} \int_A \gamma(t, (x, k), (y, l)) \mu(dy \times \{l\}) < \varepsilon. \tag{5.14}
\]

By Lemma 5.5, we find a compact subset \( D \subseteq \mathbb{R}^d \) such that \( \mu(D^c \times S) < \delta \) and \( f|_{D \times S} \) is uniformly continuous. Namely, for any given \( \eta > 0 \), there exists \( \delta_1 > 0 \) such that for all \( (x, k), (x', k) \in D \times S \), if \( |x - x'| < \delta_1 \), then \( |f(x, k) - f(x', k)| < \eta \) for all \( k \in S \). Therefore, from (5.13) and (5.14), we arrive at

\[
\mathbb{P}(\{|f(V(x_n, k)(t), \psi(k)(t)) - f(V(x, k)(t), \psi(k)(t))| > \eta\}
\leq \mathbb{P}(\{|V(x_n, k)(t) - V(x, k)(t)| > \delta_1\}
+ \mathbb{P}(\{(V(x_n, k)(t), \psi(k)(t)) \notin D \times S\} + \mathbb{P}(\{(V(x, k)(t), \psi(k)(t)) \notin D \times S\}
\leq \mathbb{P}(\{|V(x_n, k)(t) - V(x, k)(t)| > \delta_1\} + 2\varepsilon. \tag{5.15}
\]

Meanwhile, by Lemma 5.4, \( \mathbb{P}(\{|V(x_n, k)(t) - V(x, k)(t)| > \delta_1\}) \to 0 \) as \( n \to \infty \). Inserting this into (5.15) and noting that \( \varepsilon \) and \( \eta \) are arbitrary, (5.11) holds. This completes the proof.

\[\square\]

In order to transfer the strong Feller property from \((V, \psi)\) to \((X, \Lambda)\), we need to make a comparison between these two processes. Let \( \{v_m\} \) be the sequence of stopping times defined by

\[v_0 = 0, \quad v_{m+1} = \inf\{s > v_m : \psi(t) \neq \psi(v_m)\} \quad \text{for} \quad m \geq 0.\]

Define \( n(t) = \max\{m : v_m \leq t\} \), which is the number of switches (i.e., jumps) of \( \psi \) up to time \( t \). Set \( D := D([0, \infty), \mathbb{R}^d \times S) \) and denote by \( \mathcal{D} \) the usual \( \sigma \)-field of \( D \). Likewise, for any \( T > 0 \), set \( D_T := D([0, T], \mathbb{R}^d \times S) \) and denote by \( \mathcal{D}_T \) the usual \( \sigma \)-field of \( D_T \). Moreover, denote by \( \mu_1(\cdot) \) the probability distribution induced by \((X, \Lambda)\) and \( \mu_2(\cdot) \) the probability distribution induced by \((V, \psi)\) in the path space \((D, \mathcal{D})\), respectively. Denote by \( \mu_1^T(\cdot) \) the restriction of \( \mu_1(\cdot) \) and \( \mu_2^T(\cdot) \) the restriction of \( \mu_2(\cdot) \) to \((D_T, \mathcal{D}_T)\), respectively. For any given \( T > 0 \), from (Xi, 2009, Lemma 4.2), we know that \( \mu_1^T(\cdot) \) is absolutely continuous with respect
to $\mu_2^T(\cdot)$ and the corresponding Radon-Nikodym derivative has the following form.

$$M_T(V(\cdot), \psi(\cdot)) := \frac{d\mu_T^T}{d\mu_2^T}(V(\cdot), \psi(\cdot)) = \prod_{i=0}^{\infty} q_{\psi(v_i)\psi(v_{i+1})}(V(v_{i+1})) \exp\left(-\sum_{i=0}^{n(T)} \int_{v_i}^{v_{i+1}} [q_{\psi(v_i)}(V(s)) - 2\kappa] \, ds\right),$$

(5.16)

where $q_k(x) = \sum_{l \neq k} q_{kl}(x)$.

**Remark 5.7.** Note that the Radon-Nikodym derivative defined in (5.16) is similar to the likelihood ratio martingale defined in Chow and Teicher (1977) and Rogers and Williams (2000).

We restate (Xi, 2009, Lemmas 4.3 and 4.4) as the following two lemmas respectively.

**Lemma 5.8.** For all $T > 0$, we have that

$$\mathbb{E}\left[|M_T(V^{(x,k)}(\cdot), \psi^{(k)}(\cdot)) - M_T(V^{(y,k)}(\cdot), \psi^{(k)}(\cdot))|\right] \to 0$$

(5.17)

as $|x - y| \to 0$.

**Lemma 5.9.** For all $T > 0$ and $(x,k) \in \mathbb{R}^d \times \mathbb{S}$, $M_T(V^{(x,k)}(\cdot), \psi^{(k)}(\cdot))$ is integrable.

Now we are ready to prove the main result of this section.

**Proof of Theorem 5.2.** To prove the desired strong Feller property, it is enough to prove that for any $t > 0$ and any bounded measurable function $f$ on $\mathbb{R}^d \times \mathbb{S}$, $\mathbb{E}[f(X^{(x,k)}(t), \Lambda^{(x,k)}(t))]$ is bounded continuous in both $x$ and $k$. Since $\mathbb{S}$ has a discrete metric, it is sufficient to prove that

$$\left|\mathbb{E}[f(X^{(x,k)}(t), \Lambda^{(x,k)}(t))] - \mathbb{E}[f(X^{(y,k)}(t), \Lambda^{(y,k)}(t))]\right| \to 0$$

(5.18)

as $|x - y| \to 0$. Indeed, by (5.16), for all $(x, k) \in \mathbb{R}^d \times \mathbb{S}$,

$$\mathbb{E}[f(X^{(x,k)}(t), \Lambda^{(x,k)}(t))] = \mathbb{E}[f(V^{(x,k)}(t), \psi^{(k)}(t)) \cdot M_t(V^{(x,k)}(\cdot), \psi^{(k)}(\cdot))].$$

(5.19)

Similarly to the proof of Proposition 1.2 in Wu (2001), for any given $\varepsilon > 0$, using (5.19), we have

$$\left|\mathbb{E}[f(X^{(x,k)}(t), \Lambda^{(x,k)}(t))] - \mathbb{E}[f(X^{(y,k)}(t), \Lambda^{(y,k)}(t))]\right| \leq \mathbb{E}\left[|f(V^{(x,k)}(t), \psi^{(k)}(t)) \cdot M_t(V^{(x,k)}(\cdot), \psi^{(k)}(\cdot)) - f(V^{(y,k)}(t), \psi^{(k)}(t)) \cdot M_t(V^{(y,k)}(\cdot), \psi^{(k)}(\cdot))|\right]$$

$$\leq \|f\| \cdot \mathbb{E}\left[|M_t(V^{(x,k)}(\cdot), \psi^{(k)}(\cdot)) - M_t(V^{(y,k)}(\cdot), \psi^{(k)}(\cdot))|\right]$$

$$\leq 2\|f\| \cdot \mathbb{E}\left[M_t(V^{(y,k)}(\cdot), \psi^{(k)}(\cdot)) I_{\{|f(V^{(x,k)}(t), \psi^{(k)}(t)) - f(V^{(y,k)}(t), \psi^{(k)}(t))| \geq \varepsilon\}}\right]$$

$$+ \varepsilon \cdot \mathbb{E}\left[M_t(V^{(y,k)}(\cdot), \psi^{(k)}(\cdot))\right]$$

$$= (I) + (II) + (III),$$

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where \( \| f \| := \sup \{|f(x,k)| : (x,k) \in \mathbb{R}^d \times S\} \). From Lemma 5.8 term (I) in (5.20) tends to zero as \(|x - y| \to 0\). From Lemmas 5.6 and 5.9, we derive that term (II) in (5.20) also tends to zero as \(|x - y| \to 0\). Meanwhile, term (III) in (5.20) can be arbitrarily small since the multiplier \( \varepsilon \) is arbitrary and \( M_t(V^{(y,k)}(\cdot), \psi^{(k)}(\cdot)) \) is integrable by Lemma 5.9. The proof is completed.  \( \square \)

6 Exponential Ergodicity

In this section, we follow Xi (2009) and investigate the exponential ergodicity for the process \((X, \Lambda)\). To this end, let us first recall some relevant terminologies. As in Meyn and Tweedie (1993b), the process \((X, \Lambda)\) is called bounded in probability on average if for each \((x,k) \in \mathbb{R}^d \times S\) and each \(\varepsilon > 0\), there exists a compact subset \(C \subset \mathbb{R}^d\) and a finite subset \(N \subset S\) such that

\[
\liminf_{t \to \infty} \frac{1}{t} \int_0^t P(s, (x,k), C \times N)ds \geq 1 - \varepsilon.
\]

We now introduce a Foster-Lyapunov drift condition as follows. For some \(\alpha, \beta > 0\), \(f(x,k) \geq 1\), a compact subset \(C \subset \mathbb{R}^d\) and a finite subset \(N \subset S\), and a nonnegative function \(V(\cdot, \cdot) \in C^2(\mathbb{R}^d \times S)\),

\[
\mathcal{A}V(x,k) \leq -\alpha f(x,k) + \beta 1_{C \times N}(x,k), \quad (x,k) \in \mathbb{R}^d \times S,
\]

where \(\mathcal{A}\) is the operator defined in (2.14).

**Proposition 6.1.** Suppose (6.1) and Assumptions 1.1, 1.2, 3.1, 4.1, and 5.1 hold. Then the process \((X, \Lambda)\) is bounded in probability on average and possesses an invariant probability \(\pi\).

**Proof.** By Theorem 5.2, the process \((X, \Lambda)\) is strong Feller and hence a \(T\)-process in the terminology of Meyn and Tweedie (1993c). In addition, Proposition 2.1 indicates that \((X, \Lambda)\) is non-explosive. Therefore Theorem 4.7 of Meyn and Tweedie (1993c) implies that \((X, \Lambda)\) is bounded in probability on average. The assertion that \((X, \Lambda)\) possesses an invariant probability \(\pi\) is a direct consequence of (Meyn and Tweedie, 1993c, Theorem 4.5).  \( \square \)

For any positive function \(f : \mathbb{R}^d \times S \mapsto [1, \infty)\) and any signed measure \(\nu\) defined on \(\mathcal{B}(\mathbb{R}^d \times S)\), we write

\[
\|\nu\|_f := \sup \{|\nu(g)| : g \in \mathcal{B}(\mathbb{R}^d \times S) \text{ satisfying } |g| \leq f\},
\]

where \(\nu(g) := \sum_{l \in S} \int_{\mathbb{R}^d} g(x,l)\nu(dx,l)\) is the integral of the function \(g\) with respect to the measure \(\nu\). Note that the usual total variation norm \(\|\nu\|\) is just \(\|\nu\|_f\) in the special case when \(f \equiv 1\). For a function \(\infty > f \geq 1\) on \(\mathbb{R}^d \times S\), the process \((X, \Lambda)\) is said to \(f\)-exponentially
ergodic if there exists a probability measure \( \pi(\cdot) \), a constant \( \theta \in (0, 1) \) and a finite-valued function \( \Theta(x, k) \) such that
\[
\| P(t, (x, k), \cdot) - \pi(\cdot) \|_f \leq \Theta(x, k) \theta^t
\]
for all \( t \geq 0 \) and all \( (x, k) \in \mathbb{R}^d \times \mathbb{S} \).

We need the following assumption:

**Assumption 6.2.** For any distinct \( k, l \in \mathbb{S} \), there exist \( k_0, k_1, \ldots, k_r \) in \( \mathbb{S} \) with \( k_i \neq k_{i+1} \), \( k_0 = k \) and \( k_r = l \) such that the set \( \{ x \in \mathbb{R}^d : q_{k, k_{i+1}}(x) > 0 \} \) has positive Lebesgue measure for \( i = 0, 1, \ldots, r - 1 \).

**Theorem 6.3.** Suppose Assumptions 1.1, 1.2, 3.1, 4.1, 5.1, and 6.2 hold. In addition, assume that there exist positive numbers \( \alpha, \gamma \) and a nonnegative function \( V \in C^2(\mathbb{R}^d \times \mathbb{S}) \) satisfying

\[
(i) \quad V(x, k) \to \infty \text{ as } |x| \vee k \to \infty,
\]

\[
(ii) \quad AV(x, k) \leq -\alpha V(x, k) + \gamma, \quad (x, k) \in \mathbb{R}^d \times \mathbb{S}.
\]

Then the process \((X, \Lambda)\) is \( f \)-exponentially ergodic with \( f(x, k) = V(x, k) + 1 \).

**Proof.** Note that the existence of \( V \) satisfying (i) and (ii) in the statement of the theorem trivially leads to (6.1), and hence, together with the other assumptions of the theorem, the conclusions of Proposition 6.1. We next show that the process \((X, \Lambda)\) is irreducible in the sense that for any \( t > 0, \ (x, k) \in \mathbb{R}^d \times \mathbb{S}, \ A \in \mathcal{B}(\mathbb{R}^d) \) with positive Lebesgue measure, and \( l \in \mathbb{S} \), we have \( P(t, (x, k), A \times \{l\}) > 0 \). To this end, for each \( k \in \mathbb{S} \), we kill the Lévy process \( X^{(k)} \) of (4.2) with killing rate \( q_k(\cdot) \). Denote by \( P^{(k)}(t, x, \cdot) \) the transition probability of the killed process. Then we have

\[
P(t, (x, k), A \times \{l\}) = \delta_{kl} P^{(k)}(t, x, A) + \sum_{m=1}^{+\infty} \int \cdots \int \sum_{0 \leq l_1 < l_2 < \cdots < l_m \leq t} P^{(l_m)}(t_{m} - t_{m-1}, y_{m}) P^{(l_{m-1})}(t_{m-1} - t_{m-2}, y_{m-1}) \cdots P^{(l_2)}(t_2 - t_1, y_2) \delta_{kl_1} \delta_{kl_2} \delta_{kl_m} dy_1 dy_2 \cdots dy_m,
\]

where \( \delta_{kl} \) is the Kronecker symbol in \( k, l \), which equals 1 if \( k = l \) and 0 if \( k \neq l \). As argued in Xi (2009), Assumption 4.1 guarantees that each term of the form \( P^{(l)}(s, x, A) \) with \( l \in \mathbb{S}, s > 0 \) and \( x \in \mathbb{R}^d \) is positive; this, together with Assumption 6.2, implies that \( P(t, (x, k), A \times \{l\}) > 0 \).
Using the same argument as that in the proof of Theorem 6.3 of Xi (2009), we can show that all compact subsets of $\mathbb{R}^d \times \mathbb{S}$ are petite for the skeleton $\{X(nh), \Lambda(nh)\}, n \geq 0$. Then the desired $f$-exponential ergodicity follows from Theorem 6.1 in Meyn and Tweedie (1993c). □

**Example 6.4.** In this example, we consider a coupled one-dimensional Ornstein-Uhlenbeck process

$$dX(t) = \alpha(\Lambda(t))X(t)dt + \sigma(\Lambda(t))dB(t) + \int_{\mathbb{R}\setminus\{0\}} \beta(\Lambda(t-))zN(dt, dz), \quad (6.4)$$

where for each $k \in \mathbb{S} = \{0, 1, 2, \ldots\}$, $\alpha_k := \alpha(k)$, $\beta_k := \beta(k)$, and $\sigma_k := \sigma(k)$ are real numbers to be determined later, $B$ is a standard one-dimensional Brownian motion, and $N(dt, dz)$ is Poisson random measure with characteristic measure $\nu(dz) = \frac{1}{2}e^{-|z|}dz$. Suppose the switching component $\Lambda$ is generated by the $q$-matrix

$$Q(x) = \begin{pmatrix}
-q_{01}(x) & q_{01}(x) & 0 & 0 & 0 & 0 & \ldots \\
-q_{10}(x) & -q_{10}(x) + q_{12}(x) & q_{12}(x) & 0 & 0 & 0 & \ldots \\
0 & -q_{21}(x) + q_{23}(x) & q_{23}(x) & 0 & 0 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\end{pmatrix}, \quad (6.5)$$

where $q_{k,k-1}(x)$ and $q_{k,k+1}(x)$ are positive and Lipschitz continuous functions. Obviously, Assumptions 1.1, 1.2, 3.1, 4.1, 5.1, and 6.2 hold.

Let us consider the functions $V(x, k) = (k + 1)x^2$ for $(x, k) \in \mathbb{R} \times \mathbb{S}$. Then we have

$$\mathcal{A}V(x, 0) = [2\alpha_0 - q_{01}(x) + 2q_{01}(x)]x^2 + \sigma_0^2 + \int_{\mathbb{R}\setminus\{0\}} [(x + \beta_0 z)^2 - x^2] \nu(dz)$$

$$= [2\alpha_0 + q_{01}(x)]x^2 + \sigma_0^2 + 4\beta_0^2,$$

and for $k = 1, 2, 3, \ldots$,

$$\mathcal{A}V(x, k) = [2(k + 1)\alpha_k + kq_{k,k-1}(x) - (k + 1)(q_{k,k-1}(x) + q_{k,k+1}(x)) + (k + 2)q_{k,k+1}(x)]x^2 + (k + 1)\sigma_k^2 + \int_{\mathbb{R}^0} [(k + 1)(x + \beta_k z)^2 - (k + 1)x^2] \nu(dz)$$

$$= [2(k + 1)\alpha_k - q_{k,k-1}(x) + q_{k,k+1}(x)]x^2 + (k + 1)\sigma_k^2 + 4(k + 1)\beta_k^2.$$

Now suppose there exist positive constants $K_1$ and $K_2$ so that the following conditions are satisfied:

(a) $2\alpha_0 + q_{01}(x) \leq -K_1 < 0$,

(b) for each $k \in \mathbb{S}$, we have $\sigma_k > 0$, and $(k + 1)\sigma_k^2 + 4(k + 1)\beta_k^2 \leq K_2 < \infty$,
(c) for all \( k \in \mathbb{S} \setminus \{0\} \), we have 
\[
2(k+1)\alpha_k - q_{k,k-1}(x) + q_{k,k+1}(x) \leq -K_1(k+1) < 0.
\]

Then it follows that for all \((x, k) \in \mathbb{R} \times \mathbb{S} \), we have 
\[
\mathcal{A}V(x, k) \leq -K_1(k+1)x^2 + K_2 = -K_1V(x, k) + K_2.
\]

This verifies conditions (i) and (ii) of Theorem 6.3. Hence we conclude that the process \( X \) of (6.4) is \( f \)-exponentially ergodic.

Note that we can choose \( \alpha_k, \beta_k, \sigma_k \) and \( Q(x) \) so that: (i) \( X^{(0)} \) is exponentially ergodic, (ii) \( X^{(k)} \) is transient for \( k = 1, 2, \ldots \), but (iii) the process \((X, \Lambda)\) of (6.4) is \( f \)-exponentially ergodic.

To proceed, we assume in the rest of the section that

**Assumption 6.5.** For each \( i \in \mathbb{S} \), there exist \( b(i), \sigma_j(i) \in \mathbb{R}^{d \times d}, j = 1, 2, \ldots, d \), such that as \( |x| \to \infty \),
\[
\frac{b(x, i)}{|x|} = b(i)\frac{x}{|x|} + o(1),
\]
\[
\frac{\sigma(x, i)}{|x|} = (\sigma_1(i)x, \sigma_2(i)x, \ldots, \sigma_d(i)x)\frac{1}{|x|} + o(1),
\]
where \( o(1) \to 0 \) as \( |x| \to \infty \).

**Proposition 6.6.** Suppose Assumptions 1.1, 1.2, 3.1, 4.1, 5.1, 6.2, and 6.5 hold. Assume that for each \( i \in \mathbb{S} \) and some \( p \in (0, 2) \) such that as \( |x| \to \infty \), we have
\[
\int_{U} \left( \frac{|x + c(x, i, u)|^p}{|x|^p} - 1 \right) \Pi(du) \leq \tilde{c}_i < \infty.
\]

Denote \( \mu_i := \lambda_{\max}(\frac{b(i)+b(i)'}{2} + \sum_{j=1}^{d} \sigma_j(i)\sigma_j(i)' + \tilde{c}_i) \) for each \( i \in \mathbb{S} \). Suppose there exist \( \alpha > 0 \) and \( g_i > 0, i \in \mathbb{S} \) such that \( g_i \to \infty \) as \( i \to \infty \) and when \( x \) is sufficiently large,
\[
\sum_{j \in \mathbb{S}} q_{ij}(x)g_j + p(\alpha + \mu_i)g_i \leq 0 \quad \text{for all } i \in \mathbb{S},
\]
where \( p \in (0, 2) \) is as in (6.7). Then \((X, \Lambda)\) is \( f \)-exponential ergodic.

**Proof.** Let \( p \in (0, 2), \alpha > 0 \) and \( g_i, i \in \mathbb{S} \) be as in the statement of the proposition. Let the function \( V(x, i) \in \mathcal{C}^2(\mathbb{R}^d \times \mathbb{S}) \) and \( V(x, i) = g_i|x|^p \) when \((x, i) \in (\mathbb{R}^d \setminus \{y : |y| \leq 1\}) \times \mathbb{S} \). It is readily seen that for each \( i \in \mathbb{S} \), \( V(\cdot, i) \) is continuous, nonnegative, and converges to \( \infty \) as \( |x| \vee i \to \infty \). Detailed calculations reveal that for \( x \neq 0 \), we have
\[
D(|x|^p) = p|x|^{p-2}x,
\]
\[
D^2(|x|^p) = p[|x|^{p-2}I + (p-2)|x|^{p-4}xx'].
\]

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Meanwhile, it follows from (6.6) that when $|x| \to \infty$

$$a(x,i) = \sigma(x,i)\sigma'(x,i) = \sum_{j=1}^{d} \sigma_j(i) xx' \sigma'_j(i) + o(|x|^2).$$

Then for all $(x,i) \in \mathbb{R}^d \times \mathbb{S}$ with $|x|$ sufficiently large, detailed computations using Assumption 6.5 reveal that

$$\mathcal{A}V(x,i) = pg_i|x|^p \left( \frac{x'b(i)x}{|x|^2} + \frac{\sum_{j=1}^{d} x' \sigma_j(i) \sigma'_j(i)x}{|x|^2} + (p-2) \frac{(x' \sigma_j(i)' x)^2}{|x|^4} \right)$$

$$+ \int_{\mathcal{U}} \left( \frac{|x + c(x,i,u)|^p}{|x|^p} - 1 \right) \Pi(du) + \sum_{j \in \mathbb{S}} q_{ij}(x) \frac{g_j}{pg_i} + o(1).$$

Notice that

$$\frac{x'b(i)x}{|x|^2} + \frac{\sum_{j=1}^{d} x' \sigma_j(i) \sigma'_j(i)x}{|x|^2} \leq \lambda_{\max} \left( \frac{b(i) + b(i)'}{2} + \sum_{j=1}^{d} \sigma_j(i) \sigma'_j(i) \right).$$

Also since $0 < p < 2$, we have $(p-2) \frac{(x' \sigma_j(i)' x)^2}{|x|^4} \leq 0$. Therefore for $|x|$ sufficiently large, it follows from (6.7) and (6.8) that

$$\mathcal{A}V(x,i) \leq pV(x,i) \left( \lambda_{\max} \left( \frac{b(i) + b(i)'}{2} + \sum_{j=1}^{d} \sigma_j(i) \sigma'_j(i) \right) + \tilde{c}_i + \sum_{j \in \mathbb{S}} q_{ij}(x) \frac{g_j}{pg_i} + o(1) \right)$$

$$= pV(x,i) \left( \mu_i + \sum_{j \in \mathbb{S}} q_{ij}(x) \frac{g_j}{pg_i} + o(1) \right) \leq pV(x,i)[-\alpha + o(1)].$$

In particular, we can choose $R > 0$ sufficiently large so that

$$\mathcal{A}V(x,i) \leq -\frac{\alpha}{2} pV(x,i), \text{ for all } (x,i) \in \mathbb{R}^d \times \mathbb{S} \text{ with } |x| \geq R.$$ 

Next we choose $\gamma > 0$ sufficiently large so that

$$\mathcal{A}V(x,i) \leq -\frac{\alpha}{2} pV(x,i) + \gamma, \text{ for all } (x,i) \in \mathbb{R}^d \times \mathbb{S}.$$ 

This verifies condition (ii) of Theorem 6.3. Therefore the desired assertion on $f$-exponential ergodicity follows. \(\square\)

7 Concluding Remarks

Motivated by the increasing need of modeling complex systems, this paper is devoted to the investigation of a class of regime-switching jump diffusions with countable regimes. By using
an interlacing procedure together with an exponential killing technique, this paper was able to establish the existence and uniqueness of a strong solution to the associated stochastic differential equations under more general formulation than those in the literature. The paper next used coupling method and an appropriate Radon-Nikodym derivative to derive Feller and strong Feller properties and exponential ergodicity for such processes.

A number of other problems deserve further investigation. In particular, in view of Yamada and Watanabe’s work on the uniqueness of solutions of stochastic differential equations (Yamada and Watanabe (1971)), one may naturally ask whether the Lipschitz condition can be relaxed. Also of interest is to consider the problem of successful couplings for regime-switching jump diffusions.

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