D-homothetically fixed, weakly $(\kappa, \mu)$-structures on contact metric spaces

Philippe Rukimbira
Department of Mathematics and Statistics, Florida International University
Miami, Florida 33199, USA
E-MAIL: rukim@fiu.edu

Abstract

Contact metric $(\kappa, \mu)$-spaces are generalizations of Sasakian spaces. We introduce a weak $(\kappa, \mu)$ condition as a generalization of the K-contact one and show that many of the known results from generalized Sasakian geometry hold in the weaker generalized K-contact geometry setting. In particular, we prove existence of K-contact and $(\kappa, \mu = 2)$-structures under some conditions on the Boeckx invariant.

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1 Basic properties of contact metric structures

A contact form on a $2n + 1$-dimensional manifold $M$ is a one-form $\eta$ such that $\eta \wedge (d\alpha)^n$ is a volume form on $M$. Given a contact manifold $(M, \eta)$, there exist tensor fields $(\xi, \phi, g)$, where $g$ is a Riemannian metric and $\xi$ is a unit vector field, called the Reeb field of $\eta$ and $\phi$ is an endomorphism of the
tangent bundle of $M$ such that

(i) $\eta(\xi) = 1, \ \phi^2 = -Id + \eta \otimes \xi, \ \phi \xi = 0$

(ii) $d\eta = 2g(., \phi.)$

The data $(M, \eta, \xi, \phi, g)$ is called a contact metric structure; see ([3]) for more details.

Denoting by $\nabla$ the Levi-Civita connection of $g$, and by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

its curvature tensor, a contact metric structure $(M, \eta, \xi, \phi, g)$ is called Sasakian if the condition

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X$$

is satisfied for all tangent vectors $X$ and $Y$. A well known curvature characterization of the Sasakian condition is as follows:

**Proposition 1** A contact metric structure $(M, \eta, \xi, \phi, g)$ is Sasakian if and only if

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

for all tangent vectors $X$ and $Y$.

A condition weaker than the Sasakian one is the K-contact condition. A contact metric structure $(M, \eta, \xi, \phi, g)$ is called K-contact if the tensor field $h = \frac{1}{2}L_\xi \phi$ vanishes identically. Here, $L_\xi \phi$ stands for the Lie derivative of $\phi$ in the direction of $\xi$. The above K-contact condition is known to be equivalent
to the Reeb vector field $\xi$ being a $g$-infinitesimal isometry, or a Killing vector field. The tensor field $h$ is known to be symmetric and anticommutes with $\phi$.

An equally well known curvature characterization of K-contactness is as follows:

**Proposition 2** A contact metric structure $(M, \eta, \xi, \phi, g)$ is K-contact if and only if

$$R(X, \xi)\xi = X - \eta(X)\xi$$

for all tangent vectors $X$.

The notation "$l$" is common for the tensor

$$lX = R(X, \xi)\xi$$

**Proposition 3** On a contact metric structure $(M, \eta, \xi, \phi, g)$, the following identities hold:

$$\nabla_\xi h = \phi - h^2 \phi - \phi l$$  \hspace{1cm} (1)

$$\phi l \phi - l = 2(h^2 + \phi^2)$$  \hspace{1cm} (2)

$$L_\xi h = \nabla_\xi h + 2\phi h + 2\phi h^2$$  \hspace{1cm} (3)

**Proof** The first two identities appear in Blair’s book ([3]). We establish the
(L_\xi h)X = [\xi, hX] - h[\xi, X]
= \nabla_\xi(hX) - \nabla_{hX}\xi - h(\nabla_\xi X - \nabla_X\xi)
= (\nabla_\xi h)X + h\nabla_\xi X - [\phi hX - \phi h^2 X] - h\nabla_\xi X + h[-\phi X - \phi hX]
= (\nabla_\xi h)X + h\nabla_\xi X + \phi hX + \phi h^2 X - h\nabla_\xi X - h\phi X + \phi h^2 X
= (\nabla_\xi h)X + 2\phi hX + 2\phi h^2 X

\square

Given a contact metric structure \((M, \eta, \xi, \phi, g)\), its \(D_a\)-homothetic deformation is a new contact metric structure \((M, \tilde{\eta}, \tilde{\xi}, \tilde{\phi}, \tilde{g})\) given by a real number \(a > 0\) and

\[\tilde{\eta} = a\eta, \quad \tilde{\xi} = \frac{\xi}{a}, \quad \tilde{\phi} = \phi\]

\[\tilde{g} = ag + a(a - 1)\eta \otimes \eta\]

\(D\)-homothetic deformations preserve the K-contact and Sasakian conditions.

2 Weakly \((\kappa, \mu)\)-spaces

A direct calculation shows that under a \(D_a\) homothetic deformation, the curvature tensor transforms as follows:

\[a\tilde{R}(X,Y)\tilde{\xi} = R(X,Y)\xi - (a - 1)[(\nabla_X\phi)Y - (\nabla_Y\phi)X + \eta(X)(Y + hY)
- \eta(Y)(X + hX)] + (a - 1)^2[\eta(Y)X - \eta(X)Y]\]
Letting $Y = \xi$ and recalling $\nabla_{\xi} \phi = 0$, we get:

$$a\mathcal{R}(X, \xi)\xi = R(X, \xi)\xi - (a - 1)[(\nabla_X \phi)\xi + \eta(X)\xi - (X + hX)] + (a - 1)^2[X - \eta(X)\xi]$$

On any contact metric manifold, the following identity holds:

$$(\nabla_X \phi)\xi = -\phi \nabla_X \xi = -X + \eta(X) - hX.$$  

Taking into account of this identity, we see that the curvature tensor deforms as follows:

$$a^2\mathcal{R}(X, \xi)\xi = R(X, \xi)\xi + (a^2 - 1)(X - \eta(X)\xi) + 2(a - 1)hX$$

Equivalently, since $\xi = a\xi$ and $h = ah$,

$$\mathcal{R}(X, \xi)\xi = \frac{1}{a^2}R(X, \xi)\xi + \frac{a^2 - 1}{a^2}(X - \eta(X)\xi) + \frac{2a - 2}{a^2}hX \quad (4)$$

It follows from (4) that, under a $D_a$-homothetic deformation, the condition $R(X, \xi)\xi = 0$ transforms into

$$\mathcal{R}(X, \xi)\xi = \kappa(X - \eta(X)\xi) + \mu hX$$

where $\kappa = \frac{a^2 - 1}{a^2}$ and $\mu = \frac{2a - 2}{a^2}$.

As a generalization of both $R(X, \xi)\xi = 0$ and the K-contact condition, $R(X, \xi)\xi = X - \eta(X)\xi$, we consider

$$R(X, \xi)\xi = \kappa(X - \eta(X)\xi) + \mu hX.$$  

We call this the weak $(\kappa, \mu)$ condition. The same generalization was referred to as Jacobi $(\kappa, \mu)$-contact s manifold in [5]. Let us point out also that a strong
\((\kappa, \mu)\) condition \(R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)\)

has been introduced in [2]. Examples of weakly \((\kappa, \mu)\) spaces which are not strongly \((\kappa, \mu)\) are provided by the Darboux contact forms \(\eta = \frac{1}{2}(dz - \sum y^i dx^i)\)
on \(\mathbb{R}^{2n+1}\) with associated metric

\[
g = \frac{1}{4} \begin{pmatrix}
\delta_{ij} + y^i y^j + \delta_{ij} z^2 & \delta_{ij} z & -y^i \\
\delta_{ij} z & \delta_{ij} & 0 \\
-y^j & 0 & 1
\end{pmatrix}
\]

(see [3]).

Other examples of weakly \((\kappa, \mu)\)-spaces have been found on normal bundles of totally geodesic Legendre submanifolds in Sasakian manifolds (see [1]).

The two notions of \((\kappa, \mu)\)-spaces are D-homothetically invariant. It follows from identity (4) that, if \((M, \eta, \xi, \phi, g)\) is a (weak) \((\kappa, \mu)\) structure, then the \(D_a\)-homothetic deformation \((\tilde{\eta}, \tilde{\xi}, \tilde{\phi}, \tilde{g})\) is a (weak) \((\tilde{\kappa}, \tilde{\mu})\)-structure with:

\[
\tilde{\kappa} = \kappa + \frac{a^2 - 1}{a^2}, \quad \tilde{\mu} = \mu + \frac{2a - 2}{a}
\]

(5)

The tensor fields \(\phi\) and \(h\) on a weakly \((\kappa, \mu)\)-space are related by the identities in the following proposition.

**Proposition 4** On a weakly \((\kappa, \mu)\)-space \((M, \eta, \xi, \phi, g)\), the following identities hold:

\[
h^2 = (\kappa - 1)\phi^2, \quad \kappa \leq 1
\]

(6)

\[
\nabla_\xi h = -\mu \phi h
\]

(7)

\[
L_\xi h = (2 - \mu)\phi h + 2(1 - \kappa)\phi
\]

(8)
Proof Starting with identity (2), which is valid on any contact metric structure, one has, for any tangent vector $X$:

$$2h^2X + 2\phi^2X = \phi(\kappa\phi X + \mu h\phi X) - (\kappa(-\phi^2X) + \mu hX)$$

$$= \kappa\phi^2X + \mu\phi h\phi X + \kappa\phi^2X - \mu hX$$

$$= \kappa\phi^2X - \mu\phi^2X + \kappa\phi^2X - \mu hX$$

$$= 2\kappa\phi^2X$$

Hence, grouping terms

$$2h^2X = (2\kappa - 2)\phi^2X$$

So

$$h^2 = (\kappa - 1)\phi^2$$

But, since $h$ is symmetric, $h^2$ must be a non-negative operator, hence $\kappa \leq 1$, proving (6).

From identity (1) combined with $lX = \kappa(X - \eta(X)\xi) + \mu hX$, we see that

$$(\nabla_\xi h)X = \phi X - h^2\phi X - \phi(\kappa(X - \eta(X)\xi) + \mu hX)$$

$$= \phi X - (\kappa - 1)\phi^3X - \kappa\phi X - \mu \phi hX$$

$$= (1 - \kappa + (\kappa - 1))\phi X - \mu \phi hX$$

$$= -\mu \phi hX$$

proving (7).
Next, combining identities (3), (7) and (6), one has:

\[
L_\xi h = \nabla_\xi h + 2\phi h + 2\phi h^2 \\
= -\mu \phi h + 2\phi h + 2\phi h^2 \\
= -\mu \phi h + 2\phi h + 2\phi(\kappa - 1)\phi^2 \\
= -\mu \phi h + 2\phi h - 2(\kappa - 1)\phi \\
= (2 - \mu)\phi h + 2(1 - \kappa)\phi
\]

proving (8). \(\blacksquare\)

Tangent bundle’s structure on a \((\kappa, \mu)\)-space is described by the following theorem:

**Theorem 1** Let \((M^{2n+1}, \eta, \xi, \phi, g)\) be a weakly \((\kappa, \mu)\), contact metric manifold. Then \(\kappa \leq 1\). If \(\kappa = 1\), then the structure is K-contact. If \(\kappa < 1\), then the tangent bundle \(TM\) decomposes into three mutually orthogonal distributions \(D(0), D(\lambda)\) and \(D(-\lambda)\), the eigenbundles determined by tensor \(h\)’s eigenspaces, where \(\lambda = \sqrt{1-\kappa}\).

**Proof** Clearly, \(\kappa = 1\) is exactly the K-contact condition.

Suppose \(\kappa < 1\). Since \(h\xi = 0\) and \(h\) is symmetric, it follows from identity (6), Proposition 4, \((h^2 = (\kappa - 1)\phi^2)\), that the restriction \(h|_D\) of \(h\) to the contact subbundle \(D\) has eigenvalues \(\lambda = \sqrt{1-\kappa}\) and \(-\lambda\). By \(D(\lambda), D(-\lambda)\) and \(D(0)\), we denote the corresponding eigendistributions. If \(X \in D(\lambda)\), then \(h\phi X = -\phi h X = -\lambda \phi X\). Thus \(\phi X \in D(-\lambda)\) which shows that the three distributions above are mutually orthogonal. \(\blacksquare\)

To shade some light on the difference between weak \((\kappa, \mu)\) and strong
(\(\kappa, \mu\))-spaces, we propose a weak, semi-symmetry condition. We say that a contact metric space \((M, \eta, \xi, \phi, g)\) is weakly semi-symmetric if \(R(X, \xi)R = 0\) for all tangent vectors \(X\) where \(R\) is the curvature operator.

We will prove the following:

**Theorem 2** Let \((M, \eta, \xi, \phi, g)\) be a weakly semi-symmetric, contact metric weakly \((\kappa, 0)\)-space. Then \((M, \eta, \xi, \phi, g)\) is a strongly \((\kappa, 0)\)-space.

**Proof**

The weakly semi-symmetric condition means that \((R(X, \xi)R)(Y, \xi)\xi = 0\) holds for any tangent vectors \(X\) and \(Y\). Extending \(Y\) into a local vector field, we have:

\[
0 = R(X, \xi)R(Y, \xi)\xi - R(R(X, \xi)Y, \xi)\xi - R(Y, R(X, \xi)\xi)\xi - \\
R(Y, \xi)R(X, \xi)\xi
\]

\[
= R(X, \xi)(\kappa(Y - \eta(Y))\xi) - \kappa(R(X, \xi)Y - \eta(R(X, \xi)Y)\xi) - \\
R(Y, \kappa(X - \eta(X))\xi - R(Y, \xi)(\kappa(X - \eta(X))\xi)
\]

\[
= \kappa R(X, \xi)Y - \kappa\eta(Y)R(X, \xi)\xi - \kappa R(X, \xi)Y + \kappa\eta(R(X, \xi)Y)\xi - \\
\kappa R(Y, X)\xi + \kappa\eta(R(Y, X)\xi - \kappa R(Y, \xi)X + \kappa\eta(X)R(Y, \xi)\xi
\]

\[
= -\kappa^2\eta(Y)X - \kappa g(R(X, \xi)\xi, Y)\xi - \kappa R(Y, X)\xi + \kappa^2\eta(X)Y - \\
\kappa R(Y, \xi)X + \kappa^2\eta(X)Y - \kappa^2\eta(X)\eta(Y)\xi
\]

\[
= -\kappa^2\eta(Y)X - \kappa g(\kappa(X - \eta(X)\xi), Y)\xi - \kappa R(Y, X)\xi + 2\kappa^2\eta(X)Y - \\
\kappa R(Y, \xi)X - \kappa^2\eta(X)\eta(Y)\xi
\]

\[
0 = -\kappa^2\eta(Y)X - \kappa^2g(X, Y)\xi - \kappa R(Y, X)\xi + 2\kappa^2\eta(X)Y - \kappa R(Y, \xi)X
\]
Equation

\[-\kappa^2 \eta(Y)X - \kappa^2 g(X, Y)\xi - \kappa R(Y, X)\xi + 2\kappa^2 \eta(X)Y - \kappa R(Y, \xi)X = 0 \quad (9)\]

is valid for any \(X\) and \(Y\). Exchanging \(X\) and \(Y\) leads to

\[-\kappa^2 \eta(X)Y - \kappa^2 g(Y, X)\xi - \kappa R(X, Y)\xi + 2\kappa^2 \eta(Y)X - \kappa R(X, \xi)Y = 0 \quad (10)\]

Subtracting equation (9) from equation (10), we obtain:

\[3\kappa^2 (\eta(Y)X - \eta(X)Y) + \kappa (R(Y, X)\xi - R(X, Y)\xi) + \kappa (R(Y, \xi)X - R(X, \xi)Y) = 0 \quad (11)\]

By the first Bianchi Identity, \(R(Y, \xi)X - R(X, \xi)Y = R(Y, X)\xi\) holds. Incorporating this identity into (11), we obtain the following:

\[3\kappa^2 (\eta(Y)X - \eta(X)Y) + 3\kappa R(Y, X)\xi = 0 \quad (12)\]

which implies the strong \((\kappa, 0)\) condition

\[R(X, Y)\xi = \kappa (\eta(Y)X - \eta(X)Y)\]

\[\square\]

Theorem 2 applies to the case \(\kappa = 1\) and has the following interesting corollary:

**Corollary 1** A weakly semi-symmetric K-contact manifold is Sasakian.

**Proof** In the \(\kappa = 1\) case, the strong \((\kappa, \mu)\) condition is exactly the Sasakian condition. \(\square\)

Weakly \((\kappa, \mu)\)-structures with \(\kappa = 1\), (the K-contact ones), are D-homothetically fixed. A non-K-contact weakly \((k, \mu)\) structure cannot be D-homothetically fixed.
deformed into a K-contact one. Weakly \((k, \mu)\)-structures with \(\mu = 2\) are also D-homothetically fixed. As a consequence, a weakly \((k, \mu)\) structure with \(\mu \neq 2\) cannot be deformed into one with \(\mu = 2\) neither.

Existence of these homothetically fixed \((\kappa, \mu)\) structures depends on an invariant that was first introduced by Boeckx for strongly \((\kappa, \mu)\) structures in ([4]).

3 The Boeckx invariant

The Boeckx invariant, \(I_M\) of a weakly contact \((\kappa, \mu)\) space is defined by

\[
I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1 - k}} = \frac{1 - \frac{\mu}{2}}{\lambda}.
\]

\(I_M\) is a D-homothetic invariant. Any two D-homothetically related weakly \((\kappa, \mu)\)-structures have the same Boeckx invariant.

The following lemma is crucial in proving existence of D-homothetically fixed \((\kappa, \mu)\)-structures.

**Lemma 1** Let \((M, \alpha, \xi, \phi, g)\) be a non-K-contact, weakly \((\kappa, \mu)\) space.

(i) If \(I_M > 1\), then \(2 - \mu - \sqrt{1 - k} > 0\) and \(2 - \mu + \sqrt{1 - k} > 0\).

(ii) If \(I_M < -1\), then \(2 - \mu + \sqrt{1 - k} < 0\) and \(2 - \mu - \sqrt{1 - k} < 0\).

(iii) \(|I_M| < 1\) if and only if \(0 < 2\lambda + 2 - \mu\) and \(0 < 2\lambda + \mu - 2\)
Proof  (i). Suppose $I_M > 1$. Then $1 - \frac{\mu}{2} > \sqrt{1 - \kappa}$ and $\mu < 2$.

\[
1 - \frac{\mu}{2} > \sqrt{1 - \kappa} \Rightarrow 2 - \mu > 2\sqrt{1 - \kappa}
\]

\[
\Rightarrow 2 - \mu > \sqrt{1 - \kappa} \text{ and } 2 - \mu > -\sqrt{1 - \kappa}
\]

\[
\Rightarrow 2 - \mu - \sqrt{1 - \kappa} > 0 \text{ and } 2 - \mu + \sqrt{1 - \kappa} > 0
\]

(ii). Suppose $I_M < -1$. Then $1 - \frac{\mu}{2} < -\sqrt{1 - \kappa}$ and $\mu > 2$.

\[
1 - \frac{\mu}{2} < -\sqrt{1 - \kappa} \Rightarrow 2 - \mu < -2\sqrt{1 - \kappa} \text{ and } 2 - \mu < \sqrt{1 - \kappa}
\]

\[
\Rightarrow 2 - \mu < -\sqrt{1 - \kappa} \text{ and } 2 - \mu < \sqrt{1 - \kappa}
\]

\[
\Rightarrow 2 - \mu + \sqrt{1 - \kappa} < 0 \text{ and } 2 - \mu - \sqrt{1 - \kappa} < 0
\]

\[
\square
\]

(iii). $|I_M| < 1$ if and only if $-1 < \frac{1 - \frac{\mu}{2}}{\lambda} < 1$. Equivalently

\[
-1 < \frac{2 - \mu}{2\lambda} < 1 \text{ and } -1 < \frac{\mu - 2}{2\lambda} < 1
\]

Thus

\[
-2\lambda < 2 - \mu < 2\lambda \text{ and } -2\lambda < \mu - 2 < 2\lambda
\]

Or,

\[
0 < 2\lambda + 2 - \mu \text{ and } 0 < 2\lambda + \mu - 2
\]

\[
\square
\]
4 D-homothetically fixed structures on weakly \((\kappa, \mu)\)-spaces

4.1 K-contact structures on weakly \((\kappa, \mu)\) spaces

We have pointed out that D-homothetic deformations of non K-contact weakly \((\kappa, \mu)\) structures remain non K-contact. However, on \((\kappa, \mu)\)-spaces with large Boeckx invariant, K-contact structures coexist with \((\kappa, \mu)\) structures.

**Theorem 3** Let \((M, \eta, \xi, \phi, g)\) be a non-K-contact, weakly \((k, \mu)\)-space whose Boeckx invariant \(I_M\) satisfies \(|I_M| > 1\). Then, \(M\) admits a K-contact structure \((M, \eta, \xi, \tilde{\phi}, \tilde{g})\) compatible with the contact form \(\eta\).

**Proof** We define tensor fields \(\tilde{\phi}\) and \(\tilde{g}\) by

\[
\tilde{\phi} = \frac{\epsilon}{(1 - k) \sqrt{(2 - \mu)^2 - 4(1 - k)}} (L_\xi h \circ h)
\]

\[
\tilde{g} = \frac{1}{2} d\eta(\cdot, \cdot, \tilde{\phi}) + \eta \otimes \eta
\]

where

\[
\epsilon = \begin{cases} 
+1 & \text{if } I_M > 0 \\
-1 & \text{if } I_M < 0
\end{cases}
\]

From the formula \(h^2 = -(1 - k)\phi^2\) and \(L_\xi h = (2 - \mu)\phi h + 2(1 - k)\phi\) in\n
Proposition 4, we obtain

\[
(L_\xi h \circ h)^2 = (2 - \mu)^2(1 - k)^2\phi^2 - 4(1 - k)^2\phi^2 h^2
\]

\[
= (1 - k)^2((2 - \mu)^2 - 4(1 - k))\phi^2
\]

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That is:

\[(L\xi h \circ h)^2 = \lambda^4 \alpha (-Id + \eta \otimes \xi)\]

where \(\lambda = \sqrt{1-k}\) and \(\alpha = (2 - \mu)^2 - 4(1 - k)\). One sees that, if \(\alpha > 0\), then \(\overline{\phi} = \frac{\xi}{\lambda^2 \sqrt{\alpha}} (L\xi h \circ h)\) defines an almost complex structure on the contact subbundle. Notice also that \(\alpha > 0\) is equivalent to \(|I_M| > 1\).

We will show that \(\overline{\phi}\) is \(\xi\) invariant. For that, it suffices to show that the Lie derivative of \(L\xi h \circ h\) vanishes in the \(\xi\) direction.

\[
L\xi (L\xi h \circ h) = L\xi((2 - \mu)(1 - k)\phi + 2(1 - k)\phi h)
\]
\[
= 2(2 - \mu)(1 - k)h + 4(1 - k)h^2 + 2(1 - k)[(2 - \mu)\phi^2 h + 2(1 - k)\phi^2]
\]
\[
= 2(2 - \mu)(1 - k)h + 4(1 - k)h^2 + 2(1 - k)(2 - \mu)\phi^2 h + 4(1 - k)^2 \phi^2
\]
\[
= -4(1 - k)^2 \phi^2 + 4(1 - k)^2 \phi^2 = 0
\]

Next, we will show that \(\overline{\gamma} = -\frac{1}{2} d\eta(\cdot, \overline{\phi}) + \eta \otimes \eta\) is an adapted Riemannian metric for the structure tensors \((\eta, \overline{\xi}, \overline{\phi})\). That is \(\overline{\gamma}\) is a bilinear, symmetric, positive definite tensor with

\[
d\eta = 2\overline{\gamma}(\cdot, \overline{\phi}).
\]

From the definition of \(\overline{\gamma}\), we have, for arbitrary tangent vectors \(X\) and \(Y\):
\[ g(X, Y) = -\frac{1}{2} d\eta(X \bar{\phi} Y) + \eta(X) \eta(Y) \]
\[ = -\frac{1}{2} d\eta(X, (1 - \kappa)(2 - \mu)\phi + (1 - \kappa)2\phi h) Y) + \eta(X) \eta(Y) \]
\[ = \frac{2 - \mu}{\sqrt{4(1 - \kappa) - (2 - \mu)^2}} g(X, Y) + \frac{2}{\sqrt{4(1 - \kappa) - (2 - \mu)^2}} g(X, hY) + \frac{1}{\sqrt{4(1 - \kappa) - (2 - \mu)^2}} \eta(X) \eta(Y) \]
\[ = \frac{2 - \mu}{\sqrt{4(1 - \kappa) - (2 - \mu)^2}} g(Y, X) + \frac{2}{\sqrt{4(1 - \kappa) - (2 - \mu)^2}} g(Y, hX) + \frac{1}{\sqrt{4(1 - \kappa) - (2 - \mu)^2}} \eta(Y) \eta(X) \]
\[ = \bar{g}(Y, X) \]

proving symmetry of \( \bar{g} \). We used \( h \)'s symmetry in the step before the last.

For \( \bar{g} \)'s positive definiteness, first observe that \( \bar{g}(\xi, \xi) = 1 > 0 \). Then for any non-zero tangent vector \( X \) in the contact bundle \( D \), using the definition of \( \bar{\phi} \) in (13), the formula for \( L_\xi h \) from identity (8) in Proposition 4, we have:

\[ \bar{g}(X, X) = -\frac{1}{2} d\eta(X, \bar{\phi} X) \]
\[ = -\frac{\epsilon(2 - \mu)}{2\sqrt{(2 - \mu)^2 - 4(1 - \kappa)}} d\eta(X, \phi X) - \frac{\epsilon}{\sqrt{(2 - \mu)^2 - (4(1 - \kappa))}} d\eta(X, \phi h X) \]
\[ = \frac{\epsilon(2 - \mu)}{\sqrt{(2 - \mu)^2 - 4(1 - \kappa)}} g(X, X) + \frac{2\epsilon}{\sqrt{(2 - \mu)^2 - 4(1 - \kappa)}} g(X, hX) \]
\[ = \bar{g}(X, X) \]

If \( X \in D(\lambda) \), then

\[ \bar{g}(X, X) = \frac{\epsilon}{\sqrt{(2 - \mu)^2 - 4(1 - \kappa)}} ((2 - \mu) + 2\sqrt{1 - \kappa}) g(X, X)). \]

By Lemma 1, (i), (ii), the inequality

\[ \epsilon((2 - \mu) - 2\sqrt{1 - \kappa}) > 0 \]
holds when $|I_M| > 1$. Therefore $\overline{g}(X, X) > 0$.

In the same way, if $X \in D(-\lambda)$, then

$$\overline{g}(X, X) = \frac{\epsilon}{\sqrt{(2 - \mu)^2 - 4(1 - \kappa)}}((2 - \mu) - 2\sqrt{1 - \kappa}g(X, X))$$

which is also $> 0$ by Lemma 1, (i) and (ii). This concludes the proof of $\overline{g}$'s positivity.

We easily verify that $\overline{g}$ is an adapted metric.

$$2\overline{g}(X, \overline{\phi}Y) = -d\eta(X, \overline{\phi}^{2}Y)$$

$$= -d\eta(X, -Y + \eta(Y)\xi)$$

$$= d\eta(X, Y).$$

\[ \square \]

Remark: As a consequence of Theorem 3, contact forms on compact, weakly $(\kappa, \mu)$-spaces with $|I_M| > 1$ admit associated K-contact structures, hence verify Weinstein’s Conjecture about the existence of closed Reeb orbits. (see[8]).

On weakly $(\kappa, \mu)$-spaces with small Boeckx invariant, it turns out that $(\kappa, 2)$ structures coexist with $(\kappa, \mu \neq 2)$ structures. This will be established in the next subsection.

4.2 Contact metric weakly $(\kappa, 2)$-spaces

Given a non-K-contact, weakly $(\kappa, \mu)$-space $(M, \eta, \xi, \phi, g)$, we define the D-homothetic invariant tensor field

$$\tilde{\phi} = \frac{1}{\sqrt{1 - k}} h$$
Lemma 2 Denoting by
\[ \tilde{h} = \frac{1}{2} L_\xi \hat{\phi} = \frac{1}{2\sqrt{1 - k}} L_\xi h, \]
the following identities are satisfied:
\[ \tilde{h} = \frac{1}{2\sqrt{1 - k}} ((2 - \mu)\phi h + 2(1 - k)\phi) \quad (14) \]
\[ \tilde{h}^2 = ((1 - k) - (1 - \frac{\mu}{2})^2)\phi^2 \quad (15) \]

Proof From the third identity in Proposition 3, combined with identity (8), Proposition 4, we get
\[ 2(\sqrt{1 - k})\tilde{h} = L_\xi h = (2 - \mu)\phi h + 2(1 - k)\phi \]

So
\[ \tilde{h} = \frac{1}{2\sqrt{1 - k}} (2 - \mu)\phi h + 2(1 - k)\phi \]
which is (14).

The proof of (15) is a straightforward calculation. □

Remark: If \( |I_M| < 1 \), then \( 1 - k - (1 - \frac{\mu}{2})^2 > 0 \). Therefore, identity (15) suggests that \( \tilde{h} \) can be used to define a complex structure on the contact subbundle.

Define the tensor field \( \phi_1 \) by:
\[ \phi_1 = \frac{1}{\sqrt{1 - k - (1 - \frac{\mu}{2})^2}} \tilde{h} = \frac{1}{\sqrt{1 - k - (1 - \frac{\mu}{2})^2}} \frac{1}{2\sqrt{1 - k}} ((2 - \mu)\phi h + 2(1 - k)\phi) \]

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Proposition 5  The tensor field $\phi_1$ satisfies

\[ \phi_1^2 = -I + \eta \otimes \xi \]

\[ h_1 = \frac{1}{2} L_\xi \phi_1 = (\sqrt{1 - I_3^2} h) \quad (16) \]

Proof  The identity $\phi_1^2 = \phi^2 = -I + \eta \otimes \xi$ follows from Lemma 2, (15).

As for identity (16), we proceed as follows:

\[
h_1 = \frac{1}{2}(L_\xi \phi_1) = \frac{1}{\sqrt{1 - \kappa(1 - \kappa - \frac{1}{2})^2}} L_\xi((2 - \mu)\phi h + 2(1 - \kappa)\phi
\]

\[
= \frac{1}{\sqrt{1 - \kappa(1 - \kappa - \frac{1}{2})^2}} [(2 - \mu)((L_\xi \phi) h + \phi L_\xi h) + 2(1 - \kappa) L_\xi \phi]
\]

\[
= \frac{1}{\sqrt{1 - \kappa(1 - \kappa - \frac{1}{2})^2}} [(2 - \mu)(2h^2 + \phi((\mu - 2)h\phi + 2(1 - \kappa)\phi) + 4(1 - \kappa)h)
\]

\[
= \frac{1}{\sqrt{1 - \kappa(1 - \kappa - \frac{1}{2})^2}} [2(2 - \mu)h^2 - (2 - \mu)^2 \phi \phi + 2(2 - \mu)(1 - \kappa)\phi^2 + 4(1 - \kappa)h]
\]

\[
= \frac{1}{\sqrt{1 - \kappa(1 - \kappa - \frac{1}{2})^2}} [2(2 - \mu)(\kappa - 1)\phi^2 - (2 - \mu)^2 h + 2(2 - \mu)(1 - \kappa)\phi^2 + 4(1 - \kappa)h]
\]

\[
= \frac{1}{\sqrt{1 - \kappa(1 - \kappa - \frac{1}{2})^2}} [4(1 - \kappa) - (2 - \mu)^2]h
\]

\[
= \sqrt{\frac{4(1 - \kappa - (2 - \mu)^2}{4(1 - \kappa))} h = (\sqrt{1 - I_3^2}) h
\]

\[ \square \]

As pointed out earlier, when a D-homothetic deformation is applied to a weakly $(\kappa, \mu)$ structure with $\mu = 2$, the $\mu$ value remains the same as is seen from one of formulas (5):

\[ \mu = \frac{\mu + 2a - 2}{a} \]
As a consequence, weakly \((\kappa, 2)\) structures cannot be obtained through D-homothetic deformations. In the case \(|I_M| < 1\), we prove the following theorem:

**Theorem 4** Let \((M, \eta, \xi, \phi, g)\) be a non-K-contact, weakly \((\kappa, \mu)\)-space with Boeckx invariant \(I_M\) satisfying \(|I_M| < 1\). Then, there is a weakly \((\kappa_1, \mu_1)\) structure \((M, \eta, \xi, \phi_1, g_1)\) where \(\mu_1 = 2\) and \(\kappa_1 = \kappa + (1 - \frac{\mu}{2})^2\).

**Proof** Define \(g_1\) by

\[
g_1(X, Y) = -\frac{1}{2}d\eta(X, \phi_1 Y) + \eta(X)\eta(Y).
\]

We will show that \(g_1\) is a Riemannian metric adapted to \(\phi_1\) and \(\eta\), i.e.

\[
d\eta = 2g_1(., \phi_1)
\]
For any tangent vectors $X$ and $Y$,

\[
g_1(X, Y) = -\frac{1}{2} \frac{1}{\sqrt{1 - \kappa - (1 - \mu)^2}} \eta(X, \tilde{h}Y) + \eta(X)\eta(Y)
\]

\[
= -\frac{1}{2\sqrt{1 - \kappa - (1 - \mu)^2}} d\eta(X, (2 - \mu)\phi h Y) + d\eta(X, 2(1 - \kappa)\phi Y) + \eta(X)\eta(Y)
\]

\[
= -\frac{1}{2\sqrt{1 - \kappa - (1 - \mu)^2}} (2g(X, (2 - \mu)\phi^2 h Y) + 2d\eta(X, 2(1 - \kappa)\phi^2 Y) + \eta(X)\eta(Y))
\]

\[
= -\frac{1}{2\sqrt{1 - \kappa - (1 - \mu)^2}} (2g((2 - \mu)\phi h X, Y) + 2d\eta(2(1 - \kappa)\phi^2 X, Y) + \eta(Y)\eta(X))
\]

\[
= -\frac{1}{2} \frac{1}{\sqrt{1 - \kappa - (1 - \mu)^2}} d\eta(Y, \tilde{h}X) + \eta(Y)\eta(X)
\]

\[
g_1(Y, X)
\]

proving that $g_1$ is a symmetric tensor. To prove positivity of $g_1$, first observe that $g_1(\xi, \xi) = 1 > 0$. Next, for any $X$ in the contact distribution,

\[
g_1(X, X) = -\frac{1}{2} d\eta(X, \frac{1}{2\sqrt{1 - \kappa - (1 - \mu)^2}} ((2 - \mu)\phi h X + 2(1 - \kappa)\phi X)
\]

\[
= -\frac{(2 - \mu)}{4\sqrt{1 - \kappa - (1 - \mu)^2}} d\eta(X, \phi h X) - \frac{(1 - \kappa)}{2\sqrt{1 - \kappa - (1 - \mu)^2}} d\eta(X, \phi X)
\]

\[
= \frac{1}{2\sqrt{1 - \kappa - (1 - \mu)^2}} ((2 - \mu)g(X, h X) + 2(1 - \kappa)g(X, X))
\]

\[
g_1(X, X) = \frac{1}{2\sqrt{1 - \kappa - (1 - \mu)^2}} ((2 - \mu)g(X, h X) + 2(1 - \kappa)g(X, X))
\]

(17)
If $X \in D(\lambda)$, then (17) becomes
\[
g_1(X, X) = \frac{1}{\sqrt{1 - \kappa - (1 - \frac{2}{2})^2}}(2\sqrt{1 - \kappa} + (2 - \mu))g(X, X) > 0
\]
The last inequality follows from Lemma 1, (iii). If $X \in D(-\lambda)$, then (17) becomes
\[
g_1(X, X) = \frac{1}{\sqrt{1 - \kappa - (1 - \frac{2}{2})^2}}(2\sqrt{1 - \kappa} - (2 - \mu))g(X, X) > 0
\]
also following from Lemma 1, (iii).

We now prove that $g_1$ is an adapted metric. Directly from the definition of $g_1$,
\[
2g_1(X, \phi_1 Y) = -d\eta(X, \phi_1^2 Y)
= d\eta(X, Y)
\]

Finally, we show that the structure $(M, \eta, \xi, \phi_1, g_1)$ is a weakly $(\kappa_1, 2)$-structure. By Proposition 5, (16), the positive eigenvalue of $h_1$ is
\[
\lambda_1 = \sqrt{1 - I_M^2 \lambda} = \sqrt{(1 - \kappa)(1 - I_M^2)} = \sqrt{1 - \kappa - (1 - \frac{2}{2})^2}.
\]
Since $(\eta, \xi, \phi_1, g_1)$ is a contact metric structure, identity (1), Proposition 3 holds.
\[
\nabla_\xi h_1 = \phi_1 - \phi_1 l_1 - \phi_1 h_1^2.
\]
For any tangent vector field $X$, one has
\[
\phi_1 X - \phi_1 l_1 X - \phi_1 h_1^2 X = (\nabla_\xi h_1)X
\]
\[
\phi_1 X - \phi_1 l_1 X - \lambda_1^2 \phi_1 X = \nabla_\xi (h_1 X) - h_1 \nabla_\xi X
\]
\[
= \nabla_{h_1 X} \xi + [\xi, h_1 X] - h_1 (\nabla_X \xi + [\xi, X])
\]
\[
= -\phi_1 h_1 X - \phi_1 h_1^2 X + (L_\xi h_1) X + h_1 [\xi, X]
\]
\[
- h_1 (-\phi_1 X - \phi_1 h_1 X + [\xi, X])
\]
\[
\phi_1 X - \phi_1 l_1 X - \lambda_1^2 \phi_1 X = -2\phi_1 h_1 X + (L_\xi h_1) X
\]

Applying \( \phi_1 \) on both sides of the above identity, one has
\[
\phi_1^2 X + l_1 X - \lambda_1^2 \phi_1^2 X = 2h_1 X - 2\lambda_1^2 \phi_1^2 X + \phi_1 (L_\xi h_1) X
\]

Solving for the tensor field \( l_1 \) gives
\[
l_1 X = 2h_1 X - (1 + \lambda_1^2) \phi_1^2 X + (\phi_1 L_\xi h_1) X
\] (18)

From Proposition 5, we know \( L_\xi h_1 \equiv \sqrt{1 - \hat{I}_M^2} L_\xi h \) and \( L_\xi h = (\mu - 2) h \phi + 2(1 - \kappa) \phi \) from Proposition 4. Also \( \phi_1 \equiv \frac{1}{2\sqrt{(1 - \kappa)(1 - \kappa - (1 - \mu)^2)}} (L_\xi h) \). A direct calculation shows that
\[
\phi_1 L_\xi h_1 = \frac{\sqrt{1 - \hat{I}_M^2}}{2\sqrt{(1 - \kappa)(1 - \kappa - (1 - \mu)^2)}} (L_\xi h)^2
\]
\[
= \frac{\sqrt{1 - \hat{I}_M^2}}{2\sqrt{(1 - \kappa)(1 - \kappa - (1 - \mu)^2)}} (1 - \kappa)[4(1 - \kappa) - (\mu - 2)^2] \phi^2
\]
\[
= \frac{1}{2} [4(1 - \kappa) - (\mu - 2)^2] \phi^2
\]

Reporting this in identity (18), we get:
\[
l_1 X = 2h_1 X - (1 + \lambda_1^2) \phi_1^2 X + \frac{1}{2} [4(1 - \kappa) - (2 - \mu)^2] \phi^2 X
\]
\[
= 2h_1 X - (1 + 1 - \kappa - (1 - \frac{\mu}{2})^2)(-X + \eta(X) \xi + 2(1 - \kappa) - \frac{(2 - \mu)^2}{2}(-X + \eta(X) \xi
\]
\[
= 2h_1 X + (\kappa + \frac{(2 - \mu)^2}{4})(X - \eta(X) \xi)
\]

Which is the \((\kappa_1, \mu_1)\) condition with \( \mu_1 = 2 \) and \( \kappa_1 = \kappa + \frac{(2 - \mu)^2}{4} \). □
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