On Non Local $p$-Laplacian with Right Hand Side Radon Measure

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Abstract: The aim of this paper is to investigate the following non local $p$-Laplacian problem with data a bounded Radon measure $\vartheta \in M_b(\Omega)$: $(-\Delta)^s_p u = \vartheta$ in $\Omega$, with vanishing conditions outside $\Omega$, and where $s \in (0,1), \frac{N}{p} < p \leq N$. An existence result is provided, and some sharp regularity has been investigated. More precisely, we prove by using some fractional isoperimetric inequalities the existence of weak solution $u$ such that: 1. If $\vartheta \in M_b(\Omega)$, then $u \in W^{s,p}_0(\Omega)$ for all $s < p$ and $q < \frac{N(\frac{1}{p} - 1)}{s}$. 2. If $\vartheta$ belongs to the Zygmund space $L \log^a L(\Omega), a > \frac{N - s}{s}$, then the limiting regularity $u \in W^{s,p}_0(\Omega)$ (for all $s < p$). 3. If $\vartheta \in L \log^a L(\Omega)$, and $\alpha = \frac{N - s}{s}$ with $p = N$, then we reach the maximal regularity with respect to $s$ and $N$, $u \in W^{s,N}_0(\Omega)$.

Keywords: fractional Sobolev space; non local problem; isoperimetric inequalities

1. Introduction

Over the past few years, there has been an increasing interest in fractional Laplacian operators, as well as nonlocal operators. In partial differential equations, fractional spaces, and their corresponding nonlocal equations, are undergoing a new phase of exploration in different topics, such as, finance [1], thin obstacle problem [2], stratified materials [3], crystal dislocation [4], water waves [5], semipermeable membranes and flame propagation [6], soft thin films [7], phase transitions [8], conservation laws [9], gradient potential theory [10], quasi-geostrophic flows [11], ultra-relativistic limits of quantum mechanics [12], multiple scattering [13], materials science [14], minimal surfaces [15] and singular set of minima of variational functionals [16]. See also [17] for other motivations. The singularity at infinity describes the nonlocal effect in the examples above.

In this paper, we will be interested with the study of the following fractional $p$-Laplacian problem with Radon measure data:

\[
\begin{cases}
(-\Delta)^s_p u = \vartheta & \text{in } \Omega \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega
\end{cases}
\]  

The problem involved in the last equation is a nonlocal integro-differential operators named by fractional $p$-laplacian operator $(-\Delta)^s_p$ defined as following:

\[
(-\Delta)^s_p u(x) = 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}} \, dy
\]

along all $u \in C^\infty_0(\mathbb{R}^N)$.

One typical feature of these operators is the nonlocality, in the sense that the value of $(-\Delta)^s_p u$ at any point $x \in \Omega$ depends not only on the values of $u$ on the whole $\Omega$, but actually on the whole $\mathbb{R}^N$.

The necessity to substitute fractional Laplacians for Laplacians stems from the need to represent anomalous diffusion. In probabilistic terms, it is equivalent to replacing the next neighbor interaction of random walks and their limit, the Brownian motion, by long-distance interaction. In a stochastic process, $u(x)$ represents the expected value of a random...
variable related to a process with a random jump far from the point $x$. A distinct difference from the classical case is that when one exits from $\Omega$, they are necessarily on $\partial \Omega$, since the Brownian motion is continuous. In view of the process’ jumping nature, it is possible to end up anywhere outside $\Omega$ at the exit time. Thus, the natural nonhomogeneous Dirichlet boundary condition consists of associating the values of $u$ with $\mathbb{R}^N \setminus \Omega$ instead of $\partial \Omega$. Then, the adequate functional framework to look for a solution is the space $W^{0,p}(\mathbb{R}^N)$ vanishing outside $\Omega$. It must be noted that, in a bounded domain, there is another way to give a formulation of the problem.

In [18], Karelsen et al. consider a duality method to prove existence and uniqueness of solutions to the following nonlocal problem:

$$(-\Delta)^s u = \mu \quad \text{in} \quad \mathbb{R}^N$$

with vanishing conditions at infinity. Where $\mu$ is a bounded Radon measure whose support is compactly contained in $\mathbb{R}^N$, $N \geq 2$, and $s \in (1/2, 1)$.

In [19], the authors treated the existence and regularity up to the boundary for the problem $(-\Delta)^s u = g$ in $\Omega$ and $u = 0$ in $\mathbb{R}^N \setminus \Omega$, for some $s \in (0, 1)$. Precisely, they proved, using a variant version of the Krylov boundary Harnack method in the fractional case, that under the hypothesis $g \in L^\infty(\Omega)$, $\Omega$ is a regular domain and $\delta(x) = \text{dist}(x, \partial \Omega)$, the solution $u$ of the problem satisfies $u \in C^\gamma(\mathbb{R}^N)$ and $\frac{\partial^\gamma u}{\partial \nu^\gamma} \in C^\alpha$ up to the boundary $\partial \Omega$ for some $\alpha \in (0, 1)$.

Barrios et al. [20] consider the following semilinear problem with the presence of a fractional Laplacian:

$$\begin{cases}
(-\Delta)^s u = \lambda \frac{f(x)}{u^p} + Mu^p & \text{in } \Omega,

u > 0 & \text{in } \mathbb{R}^N \setminus \Omega,

u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}$$

where $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$, $N > 2s$, $M \in (0, 1)$, $0 < s < 1$, $\gamma > 0$, $\lambda > 0$, $p > 1$ and $f$ is a non-negative function. Precisely, they proved that:

If $M = 0$, there exist a weak solution for all $\gamma > 0$ and $\lambda > 0$.

If $M = 0, f = 1$, there exists a threshold $\lambda^*$ such that there exist a weak solution for $0 < \lambda < \lambda^*$, and there does not for $\lambda > \lambda^*$.

Additionally, Barrios et al. in [21] proved the summability of the finite energy solution when the source belongs to some Lebesgue spaces $L^m(\Omega)$ for the problem

$$\begin{cases}
(-\Delta)^s u = f(x) & \text{in } \Omega,

u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}$$

with $0 < s < 1$, and $1 < p < +\infty, m > 1$.

Recently, Abdellaoui et al. [22] considered a more general problem under the form

$$\begin{cases}
(-\Delta)^s_{\beta,p} u = f(x, u) & \text{in } \Omega,

u = 0 & \text{in } \mathbb{R}^N \setminus \Omega.
\end{cases}$$

where $\Omega$ is a smooth bounded domain of $\mathbb{R}^N$ containing the origin and where

$$(-\Delta)^s_{\beta,p} u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{\beta + sp}} \frac{dy}{|x|^{N+sp}|y|^\beta}$$

With $0 \leq \beta < \frac{N - ps}{2}$, $1 < p < N$, $s \in (0, 1)$, $sp < N$ and $f \in L^1(\Omega)$. The main result was the existence of a weak solution in some adequate weighted fractional Sobolev space.

Our main goal in this paper is to study the existence and regularity results of the singular problem (1) when the source term is a bounded Radon measure. Indeed, the
results will present an extension and improvement of existing works in [18,22,23]. More precisely, we prove (by using some isoperimetric inequalities) in the fractional case the existence of weak solution \( u \in W^{1,q}_0(\Omega) \) for all \( s_1 < s \) and \( q < \frac{N(p-1)}{N-s} \) which present a generalization of [23] in the framework of fractional Sobolev spaces. Furthermore, we prove that if \( \vartheta \) belongs to the Zygmund space \( L\log^aL(\Omega) \), \( a > \frac{N-s}{N} \), then the limiting regularity \( u \in W^{1,q}_0(\Omega) \) for all \( s_1 < s \) is obtained and if \( \alpha = \frac{N-s}{N} \) with \( p = N \), then we reach the maximal regularity with respect to \( s \) and \( N \), \( u \in W^{1,N}_0(\Omega) \).

This paper is organized as follows. Section 2 presents some definitions and results concerning the Orlicz spaces, the fractional Sobolev spaces, and also some isoperimetric results. Section 3 focuses on the main result of the paper related to the existence, regularity and sharp regularity of solutions of the problem (1).

2. Preliminaries

In this section, for the reader’s convenience, we present various definitions and known results.

2.1. Orlicz Spaces

For the following spaces, one can see [24].

**Definition 1.** Let \( M : \mathbb{R}^+ \to \mathbb{R}^+ \) we say that \( M \) is an \( N \)-function, if it is continuous, convex and strictly positive for \( t > 0 \). We can assume without loss of generality that \( \frac{M(t)}{t} \to 0 \) as \( t \to 0 \) and \( \frac{M(t)}{t} \to \infty \) as \( t \to \infty \). The \( N \)-function conjugate to \( M \) is defined by

\[
\overline{M}(t) = \sup_{s > 0} \{ st - M(s) \}.
\]

**Definition 2.** Let \( \Omega \) be an open set in \( \mathbb{R}^N \). The Orlicz space \( L_M(\Omega) \) is defined by

\[
L_M(\Omega) := \left\{ u : \Omega \to \mathbb{R} \text{measurable} : \int_{\Omega} M\left( \frac{u(x)}{\lambda} \right) \, dx < +\infty \text{ for some } \lambda > 0 \right\}
\]

which is a Banach space under the norm

\[
\|u\|_{M,\Omega} = \inf \left\{ \lambda > 0, \int_{\Omega} M\left( \frac{u(x)}{\lambda} \right) \, dx \leq 1 \right\}.
\]

The closure of the set of bounded measurable functions with compact support in \( \overline{\Omega} \) in \( L_M(\Omega) \) will be denoted by \( E_M(\Omega) \).

The dual of \( E_M(\Omega) \) can be identified with \( L_{\overline{M}}(\Omega) \) by means of the pairing \( \int_{\Omega} uv \, dx \), and the dual norm of \( L_{\overline{M}}(\Omega) \) is equivalent to \( \|\cdot\|_{\overline{M},\Omega} \).

**Theorem 1.** (Hölder inequality in Orlicz spaces) Let \( M \) be an \( N \)-function and let \( u \in L_{M}(\Omega) \), \( v \in L_{\overline{M}}(\Omega) \). Then, we have

\[
\int_{\Omega} uv \, dx \leq 2\|u\|_{M}\|v\|_{\overline{M}}.
\]

2.2. Fractional Sobolev Spaces

In this paragraph, we introduce the fractional Sobolev space (see for instance [25]), which is the suitable space for the study of problem (1).

**Definition 3.** Let \( s \in (0,1) \) and \( p \in [1, +\infty) \). We define the fractional Sobolev space by

\[
W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{ps+N}} \, dxdy < +\infty \right\}
\]
endowed with the norm
\[
\|u\|_{W^{s,p}(\Omega)} = \left( \|u\|_{L^p(\Omega)}^p + [u]_{s,p}^p \right)^{\frac{1}{p}}
\]
where \([u]_{s,p}\) is the semi-norm of Gagliardo defined as
\[
[u]_{s,p} = \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{ps+N}} \, dx \, dy \right)^{\frac{1}{p}}.
\]

We define \(W^{s,p}_0(\Omega) = \{ u \in W^{s,p}(\Omega) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \}\). One can also define the space \(W^{s,p}(\Omega)\) as the closure of \(C^\infty_0(\Omega)\) with respect to the norm of \(W^{s,p}(\Omega)\).

**Theorem 2** (cf. [26]). Let \(\Omega\) be an open subset in \(\mathbb{R}^N\), \(s \in (0,1)\) and \(1 \leq p < +\infty\), we have
1. \(W^{s,p}(\Omega)\) is a Banach space.
2. If \(1 \leq p < +\infty\), then \(W^{s,p}(\Omega)\) is separable.
3. If \(1 < p < +\infty\), then \(W^{s,p}(\Omega)\) is reflexive.

**Theorem 3** (cf. [26]). Let \(\Omega\) be a bounded open subset in \(\mathbb{R}^N\) with \(C^{0,1}\) regularity, \(s \in (0,1)\) and \(1 \leq p < +\infty\), then \(W^{s,p}(\Omega)\) is continuously embedded in \(L^q(\Omega)\) for all \(q \in [1, p_s^*]\), where \(p_s^*\) is given by
\[
p_s^* = \frac{Np}{N - sp} \quad \text{if } sp < N \quad \text{and} \quad p_s^* = +\infty \quad \text{if } sp \geq N.
\]

**Theorem 4** (cf. [26]). Let \(\Omega\) be a bounded open subset in \(\mathbb{R}^N\) with \(C^{0,1}\) regularity, \(s \in (0,1)\) and \(1 \leq p < +\infty\). Then
1. if \(sp < N\), then the embedding \(W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)\) is compact for all \(q \in [1, p_s^*]\).
2. if \(sp = N\), then the embedding \(W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)\) is compact for all \(q \in [1, +\infty)\).
3. if \(sp > N\), then the embedding \(W^{s,p}(\Omega) \hookrightarrow L^q(\Omega)\) for all \(q < \infty\).

**Theorem 5** (cf. [26]). on \(W^{s,p}_0(\Omega)\), we have the equivalence between \(\|u\|_{W^{s,p}(\Omega)}\) and \([u]_{s,p}\).

**Theorem 6** (cf. [27]). Let \(\Omega\) be a bounded open subset in \(\mathbb{R}^N\) with \(C^{0,1}\) regularity, \(s \in (0,1)\) and \(1 \leq p < +\infty\). Then
\[
W^{s,N}_0(\Omega) \hookrightarrow L_A(\Omega), \quad A(t) = e^{\frac{Nt}{s}} - 1
\]
and the embedding \(W^{s,N}_0(\Omega) \hookrightarrow L_B(\Omega)\) is compact for all \(N\)-function \(B\), which grows essentially more slowly near infinity than \(A\), namely
\[
\lim_{t \to +\infty} \frac{B(\lambda t)}{e^{\frac{Nt}{s}}} = 0
\]
for every \(\lambda > 0\).

We define the fractional Orlicz–Sobolev spaces by the same way as \(W^{s,p}(\Omega)\) and \(W^{s,p}_0(\Omega)\).

2.3. Isoperimetric Inequalities

The following results concerning isoperimetric inequality for fractional perimeters can be found in [28].

**Lemma 1.** Let \(\Omega\) be an open subset in \(\mathbb{R}^N\). For all measurable sets \(D \subseteq \Omega\), we have
\[
|D|^{1-\frac{s}{N}} \leq 2CP_s(D, \Omega),
\]
where the perimeter of \(D\) is defined by
\[
P_s(D, \Omega) = \int_D \int_{\Omega \setminus D} \frac{1}{|x-y|^{N+s}} \, dx \, dy.
\]
When we take $D = \{ |u| > t \}$, we have the following equality, which we will use in the paper.

\[
P_t(|u| > t) = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |\chi_{|u|>t}(x) - \chi_{|u|>t}(y)| \cdot \frac{1}{|x-y|^{N+s}} dxdy
\]

\[
\int_{-\infty}^{+\infty} P_t(|u| > t)dt = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |u(x) - u(y)| \cdot \frac{1}{|x-y|^{N+s}} dxdy
\]

\[
|u(x) - u(y)| = \int_{-\infty}^{+\infty} |\chi_{|u|>t}(x) - \chi_{|u|>t}(y)| dt
\]

\[
\chi_{|u|>t}(x) - \chi_{|u|>t}(y) = \chi_{|u|>t}(x)\chi_{\mathbb{R}^N - \{ |u| > t \}}(y) + \chi_{|u|>t}(y)\chi_{\mathbb{R}^N - \{ |u| > t \}}(x).
\]

3. Main Results

In what follows, for simplicity we will use the following notations

\[D^s u = \frac{u(x) - u(y)}{|x-y|^s}, \quad dv = \frac{dxdy}{|x-y|^N} \quad \text{and} \quad \mu(t) = |\{ x \in \Omega : |u(x)| > t \}|.
\]

The following two lemmas are the fractional version of the well known results presented by Talenti (see [29] for more details).

Lemma 2. Let $s \in (0,1)$ and $u \in W^{1,1}_0(\Omega)$, we have

\[-\frac{d}{dt} \int_{\{ |u| > t \}} \int_{\Omega \setminus \{ |u| > t \}} |D^s u| dv \geq c\mu(t)^{1-\frac{s}{p}}.
\]

Proof. Let $(x,y) \in \{ |u| > t \} \times \Omega \setminus \{ |u| > t \}$, then

\[
|u(x) - u(y)| = \int_t^{+\infty} |\chi_{|u|>t}(x) - \chi_{|u|>t}(y)| dt
\]

and

\[
\int_{\{ |u| > t \}} \int_{\Omega \setminus \{ |u| > t \}} \frac{|u(x) - u(y)|}{|x-y|^{N+s}} dxdy = \int_t^{+\infty} P(|u| > t)
\]

Then,

\[
P(|u| > t) = -\frac{d}{dt} \int_{\{ |u| > t \}} \int_{\Omega \setminus \{ |u| > t \}} \frac{|u(x) - u(y)|}{|x-y|^{N+s}} dxdy \geq c\mu(t)^{1-\frac{s}{p}}
\]

\[\square\]

Lemma 3. Let $s \in (0,1)$ and $u \in W^{0,p}_0(\Omega), 1 < p < +\infty$ we have

\[-\mu'(t) \geq c\mu(t)^{1-\frac{s}{p}} C \left( -\frac{1}{c\mu(t)^{1-\frac{s}{p}}} \frac{d}{dt} \int_{\{ |u| > t \}} \int_{\Omega} |D^s u|^p dv \right), \quad \forall t > 0
\]

where $C(t) = t^{-\frac{1}{p-1}}$. 

**Proof.** Recall that $C$ is convex and decreasing. We have

$$
C \left[ \frac{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} |D^s u|^p dv}{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} |D^s u| dv} \right] \leq C \left[ \frac{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} |D^s u|^p dv}{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} |D^s u| dv} \right] \leq \frac{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} C(|D^s u|^{p-1})|D^s u| dv}{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} |D^s u| dv}
$$

Let us denote by $\Omega_+ = \{y \in \Omega : |x - y| \geq 1\}$, $\Omega_- = \{y \in \Omega : |x - y| \leq 1\}$, and $\delta(\Omega)$ designed the diameter of $\Omega$. Then,

$$
\int_{\Omega_+ \setminus \{u > t\}} C(|D^s u|^{p-1})|D^s u| \frac{dy}{|x - y|^N} = \int_{\Omega_+ \setminus \{u > t\}} \frac{dy}{|x - y|^N} \leq |\Omega|
$$

Choosing a real $\gamma$ large enough, we have

$$
\int_{\Omega_+ \setminus \{u > t\}} C(|D^s u|^{p-1})|D^s u| \frac{dy}{|x - y|^N} = \int_{\Omega_+ \setminus \{u > t\}} C \left[ \frac{|x - y|^\gamma}{|x - y|^N} \right] |D^s u| \frac{dy}{|x - y|^N} \leq \int_{\Omega_+ \setminus \{u > t\}} \frac{dy}{|x - y|^N} \leq |\Omega| \delta(\Omega)^{\gamma p' - N}.
$$

In both cases, we have

$$
C \left[ \frac{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} |D^s u|^p dv}{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} |D^s u| dv} \right] \leq \frac{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} |D^s u|^p dv}{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} |D^s u| dv} \leq \frac{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} C(-\mu(t + h) + \mu(t))}{\int_{\{t < |u| < t+h\}} \int_{\Omega \setminus \{u > t\}} |D^s u| dv}
$$

Letting $h$ to zero, we obtain

$$
C \left[ \frac{d}{dt} \int_{\{u > t\}} \int_{\Omega \setminus \{u > t\}} |D^s u| dv \right] \leq \frac{d}{dt} \int_{\{u > t\}} \int_{\Omega \setminus \{u > t\}} |D^s u| dv \leq C\mu'(t)
$$

Using Lemma 7, we obtain the result. \qed

**Definition 4.** We say that a measurable function $u$ is a weak solution for the problem (1) if it satisfies:

$$
\begin{cases}
  u \in W^{1,1}_0(\Omega) \\
  \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |(u(x) - u(y))|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) |x - y|^{ps} dv = \int_{\Omega} \varphi dv
\end{cases}
$$

(5)

for all $\varphi \in C^0_0(\Omega)$.

In this definition, the double integral can be written in the domain

$$
Q = \mathbb{R}^N \times \mathbb{R}^N \setminus (\Omega^c \times \Omega^c).
$$

If we denote by $\Delta(x, y, u, \varphi) = |u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y)) |x - y|^{ps}$, then
\[ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Delta(x, y, u, \varphi) \, dv = \int_{\Omega} \int_{\Omega} \Delta(x, y, u, \varphi) \, dv + 2 \int_{\Omega} \int_{\mathbb{R}^N - \Omega} \Delta(x, y, u, \varphi) \, dv. \]

**Theorem 7.** Let \( \Omega \) be a bounded open subset in \( \mathbb{R}^N \) with \( C^{0,1} \) regularity. Suppose \( s \in (0, 1), 2 - \frac{2}{N} < p \leq N \) and \( \theta \) is a bounded Radon measure, the problem (1) has at least one weak solution \( u \in W^{s,p}_0(\Omega) \), for all \( s < s_1 \), \( q < \frac{N(p-1)}{N-s} \).

Moreover,
- if \( \theta \in L^p L^\alpha(\Omega) \), with \( \alpha > \frac{N-s}{N} \) then \( u \in W^{s_1,\frac{N(p-1)}{N-s}}_0(\Omega) \), for all \( s_1 < s \),
- if \( \alpha = \frac{N-s}{N} \) with \( p = N \), then \( u \in W^{s,N}_0(\Omega) \).

**Proof.** Approximation and a priori estimate:

Let \((f_n)\) be a sequence of smooth functions such that \((f_n)\) converges weakly to \( \varphi \) in the space of Radon measure \( M_b(\Omega) \) and \( \|f_n\|_1 \leq \|\varphi\|_{M_b(\Omega)} \).

Consider the approximate problem:

\[
\begin{align*}
(-\Delta)^s u_n &= f_n & \text{in } \Omega \\
u_n &= 0 & \text{in } \mathbb{R}^N \setminus \Omega
\end{align*}
\]  

(6)

The existence and uniqueness of \( u_n \in W^{s,p}_0(\Omega) \) is guaranteed by using a classical variational argument in the space \( W^{s,p}_0(\Omega) \).

Consider the following truncation function \( \varphi \) (see Figure 1) defined, for any \( t \) and \( h \) positive, by

\[
\varphi(\zeta) = \begin{cases} 
0 & \text{if } 0 \leq \zeta \leq \theta, \\
\frac{1}{h}(\zeta - t) & \text{if } \theta < \zeta < \theta + h, \\
1 & \text{if } \zeta \geq \theta + h, \\
-\varphi(-\zeta) & \text{if } \zeta < 0.
\end{cases}
\]

Figure 1. The function \( \varphi(\zeta) \).

By taking \( v = \varphi(u_n) \) as test function, we obtain

\[
\int_{\Omega} \int_{\Omega} \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(u_n(x)) - \varphi(u_n(y))) \, dv \leq \int_{\Omega} f_n \varphi(u_n) \, dx
\]

Since \( x \) and \( y \) play the same role let fixed \( x \in \{ t < |u_n| < t + h \} \).

Since \( \Omega = \{ t < |u_n| < t + h \} \cup \{|u_n| < t \} \cup \{|u_n| \geq t + h \} \).

If \( y \in \{ t < |u_n| < t + h \} \), then \( \varphi(u_n(x)) - \varphi(u_n(y)) = \frac{1}{h} (u_n(x) - u_n(y)) \).

If \( y \in \{|u_n| < t \} \), then \( \varphi(u_n(x)) - \varphi(u_n(y)) = \frac{1}{h} (u_n(x) - t) \) and we deduce that:

\[
\frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(\varphi(u_n(x)) - \varphi(u_n(y))) \geq \frac{1}{h} |u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))(u_n(y) - u_n(x)) \geq \frac{1}{h} |u_n(x) - u_n(y)|^p.
\]
If \( y \in \{|u_n| \geq t + h\} \), then \( \varphi(u_n(x)) - \varphi(u_n(y)) = \frac{1}{h}(u_n(x) - (t + h)) \). As above, we deduce that
\[
|u_n(x) - u_n(y)|^p - (u_n(x) - u_n(y))(\varphi(u_n(x)) - \varphi(u_n(y)) \\
\geq \frac{1}{h}|u_n(x) - u_n(y)|^p - (u_n(x) - u_n(y))(u_n(y) - u_n(x)) \\
= \frac{1}{h}|u_n(x) - u_n(y)|^p.
\]

Then, for all \( h > 0 \), we have
\[
\frac{1}{h} \int_{|u_n|<t} \int \omega \bigg| \frac{D^p u_n}{|x-y|^p} \bigg| dv \\
\leq \int \int \omega \bigg| \frac{u_n(x) - u_n(y)}{|x-y|^p} \bigg| \omega \bigg| \frac{D^p u_n}{|x-y|^p} \bigg| dv \\
\leq \int \int \omega \varphi(u_n) dx
\]
and we deduce easily that
\[
-\frac{d}{dt} \int_{|u_n|>t} \int \omega \bigg| \frac{D^p u_n}{|x-y|^p} \bigg| dv \leq \int_{|u_n|>t} \omega \varphi(u_n) dx.
\]

By using Lemma 13, it’s easy to deduce that
\[
\left[ \frac{1}{\mu(t)} \frac{d}{dt} \int_{|u_n|>t} \int \omega \bigg| \frac{D^p u_n}{|x-y|^p} \bigg| dv \right]^{1/p} \leq \left[ -\frac{1}{\mu(t)^{1-s/N}} \frac{d}{dt} \int_{|u_n|>t} \int \omega \bigg| \frac{D^p u_n}{|x-y|^p} \bigg| dv \right]^{1/(p-1)}
\]
Let \( M \) be an N-function and \( K \) a convex function such that \( K(M(t)) = t^q, 1 < q < p \). Let \( 0 < s_1 < s < 1 \) and let \( \varepsilon > 0 \), which will be chosen later.

\[
K \left[ \int_{|u_n|<t} \int \omega \bigg| \frac{D^p u_n}{|x-y|^p} \bigg| dv \right] \leq \int_{|u_n|<t} \int \omega \bigg| \frac{D^p u_n}{|x-y|^p} \bigg| dv
\]

Claim: \( K \left[ \int \omega \bigg| \frac{D^p u_n}{|x-y|^p} \bigg| dv \right] \leq C \int \omega \bigg| \frac{D^p u_n}{|x-y|^p} \bigg| dv. \)

Let us denote by \( \Omega_+ = \{ y \in \Omega : |x-y| \geq 1 \} \), \( \Omega_- = \{ y \in \Omega : |x-y| \leq 1 \} \), \( d\lambda_\varepsilon = \frac{dy}{|x-y|^N} \) and \( \delta(\Omega) \) design the diameter of \( \Omega \).

We start by proving that \( \lambda_\varepsilon(\Omega) \leq C_1(\varepsilon,|\Omega|). \)

Indeed, let \( \delta > 0 \), then
\[
\int \omega \bigg| \frac{D^p u_n}{|x-y|^p} \bigg| dv \leq \int_{B_{\delta}(0)} \frac{dy}{|x-y|^N} + \int_{\Omega \setminus B_{\delta}(0)} \frac{dy}{|x-y|^N} \\
\leq \int_{B_{\delta}(0)} \frac{dy}{|x-y|^N} + \int_{\Omega \setminus B_{\delta}(0)} \frac{dy}{|x-y|^N} \\
\leq \int_{B_{\delta}(0)} \frac{dy}{|x-y|^N} + \frac{\delta}{\delta^N} := C_1(\varepsilon,|\Omega|).
\]

Case 1: If \( \lambda_\varepsilon(\Omega) \leq 1, \)
By using Jensen inequality in the first inequality and the convexity in the second one, we have

\[
\int_{\Omega} K \left[ \frac{M(|D^{s_1} u_n|)}{|x-y|^\varepsilon} \right] \frac{dy}{|x-y|^{N-\varepsilon}} \leq \frac{1}{\lambda_{\varepsilon}^{\prime}(\Omega)} \int_{\Omega} K \left[ \frac{M(|D^{s_1} u_n|)}{|x-y|^\varepsilon} \right] \frac{dy}{|x-y|^{N-\varepsilon}}.
\]

On one hand, we have

\[
\int_{\Omega} K \left[ \frac{M(|D^{s_1} u_n|)}{|x-y|^\varepsilon} \right] \frac{dy}{|x-y|^{N-\varepsilon}} \leq \int_{\Omega} \frac{1}{|x-y|^\varepsilon} K(M(|D^{s_1} u_n|)) \frac{dy}{|x-y|^{N-\varepsilon}} = \int_{\Omega} \frac{1}{|x-y|^\varepsilon} K(M(|D^{s_1} u_n|)) \frac{dy}{|x-y|^{N-\varepsilon}} = \int_{\Omega} K(M(|D^{s_1} u_n|)) \frac{dy}{|x-y|^N} \leq \delta(\Omega)^q \int_{\Omega} K(M(|D^{s_1} u_n|)) \frac{dy}{|x-y|^N}.
\]

On the other hand, using the convexity of \(M\), we have

\[
\int_{\Omega} K \left[ \frac{M(|D^{s_1} u_n|)}{|x-y|^\varepsilon} \right] \frac{dy}{|x-y|^{N-\varepsilon}} \leq \int_{\Omega} K \left[ \frac{M(D^{s_1} u_n)}{|x-y|^\varepsilon} \right] \frac{dy}{|x-y|^{N-\varepsilon}} = \int_{\Omega} K \left[ \frac{M(D^{s_1} u_n)}{|x-y|^\varepsilon} \right] \frac{dy}{|x-y|^{N-\varepsilon}} \leq \delta(\Omega)^{s_1} \int_{\Omega} |D^{s_1} u_n|^q \frac{dy}{|x-y|^N}.
\]

Combining the last two estimations, we get

\[
\int_{\Omega} K \left[ \frac{M(|D^{s_1} u_n|)}{|x-y|^\varepsilon} \right] \frac{dy}{|x-y|^{N-\varepsilon}} \leq (\delta(\Omega)^{s_1} + \delta(\Omega)^q) \int_{\Omega} |D^{s_1} u_n|^q \frac{dy}{|x-y|^N}.
\]

**Case 2: If \(\lambda_{\varepsilon}^{\prime}(\Omega) > 1\)**

Using the convexity of \(M\), we have

\[
K \left[ \int_{\Omega} \frac{M(|D^{s_1} u_n|)}{|x-y|^\varepsilon} dy \right] \leq \frac{1}{\lambda_{\varepsilon}^{\prime}(\Omega)} \int_{\Omega} K \left[ \frac{M(|D^{s_1} u_n|)}{|x-y|^\varepsilon} \right] \frac{dy}{|x-y|^{N-\varepsilon}}
\]

To estimate the integral in the right side over \(\Omega_+\) and \(\Omega_-\), we adopt the same way as in the above case to obtain

\[
\int_{\Omega} K \left[ \frac{M(|D^{s_1} u_n|)}{|x-y|^\varepsilon} \right] \frac{dxdy}{|x-y|^{N-\varepsilon}} \leq C_1^{s_1} \int_{\Omega} |D^{s_1} u_n|^q \frac{dxdy}{|x-y|^N}.
\]

Choose \(\varepsilon\) such that \(s_1 + \varepsilon = s\), we then deduce

\[
K \left[ \int_{\Omega} \frac{M(|D^{s_1} u_n|)}{|x-y|^\varepsilon} dy \right] \leq C \int_{\Omega} \frac{|D^{s} u_n|^q}{|x-y|^{N}} dy
\]

which proves the claim.

Combining the inequalities (7) and (8), we obtain

\[
K \left[ \int_{\Omega} M(|D^{s_1} u_n|) dv \right] \leq \frac{1}{\mu(t+h) + \mu(t)} \int_{\Omega} |D^{s} u_n|^q dv.
\]
Letting $h$ to zero, one has

$$K \left[ \frac{1}{\mu'(t)} \frac{d}{dt} \int_{\{u_n\geq t\}} \int_{\Omega} M(|D^{s_1} u_n|) dv \right] \leq \frac{1}{\mu'(t)} \frac{d}{dt} \int_{\{u_n\geq t\}} \int_{\Omega} |D^s u_n|^q dv.$$

Then

$$-\frac{d}{dt} \int_{\{u_n\geq t\}} \int_{\Omega} M(|D^{s_1} u_n|) dv \leq \int_0^{+\infty} (-\mu'(t)) M \left[ \frac{1}{\mu(t)^{1-s/N}} \frac{d}{dt} \int_{\{u_n\geq t\}} \int_{\Omega} |D^{s_1} u_n|^q dv \right]^{1/(p-1)}$$

If $M(t) = t^q, \mu \in M_b(\Omega)$ the last integral converges if $1 < q < \frac{N(p-1)}{N-s}$ and $(u_n)$ is bounded in $W_0^{s,q}(\Omega).$

If $M(t) = t^{N(p-1)/N}, \alpha > \frac{N-s}{N}, \theta \in L \log \alpha L(\Omega),$ we have

$$\int_0^{+\infty} (-\mu'(t)) M \left[ \frac{1}{\mu(t)^{1-s/N}} \int_{\{u_n\geq t\}} \int_{\Omega} |f_n| dx \right]^{1/(p-1)}$$

Then, $(u_n)$ is bounded in $W_0^{s,L}(\Omega)$ and up to a subsequence denoted $(u_n),$ there exists a function $u \in W_0^{s,L}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{s,L}(\Omega)$ and strongly in $L_M(\Omega)$ and a.e. in $\Omega.$

Passaging to the limit when $\theta \in M_b(\Omega), \ 2 - s/N < p < N.$

Let $\varphi \in C_0^\infty(\Omega),$ we start by denoting $U_n(x,y) = u_n(x) - u_n(y), U(x) = u(x) - u(y),$ and $\Phi(x,y) = \varphi(x) - \varphi(y),$ then

$$|U_n|^{p-2} U_n \rightharpoonup |U|^{p-2} U \ \text{a.e in } \Omega \times \Omega.$$

By the dominated convergence theorem and since $\varphi \in W^{s,l}(\Omega)$ for all $\delta \in \mathbb{R}^+$ and for all $1 < l < +\infty$, we obtain

$$\int_\Omega \frac{|U|^{p-2} U \Phi}{|x-y|^{N}} dv = \int_\Omega \varphi d\theta.$$
which means that \( u \) is a weak solution of (1).

**Passing to the limit when** \( \alpha > \frac{N-s}{N} \), \( \theta \in L Log^a L(\Omega) \).

The passage to the limit can be obtained by the same process.

We treat now the case \( \theta \in L Log^a L(\Omega) \), \( \alpha = \frac{N-s}{N} \), and \( p = N \).

Consider the following problem:

\[
\begin{align*}
\left\{ \begin{array}{ll}
(\Delta)^{\frac{1}{p}} u_n + \frac{1}{n} (\Delta)^{\frac{1}{N}} u_n &= f_n & \text{in } \Omega \\
u_n &= 0 & \text{in } \mathbb{R}^N \setminus \Omega
\end{array} \right.
\]

(9)

Let us use \( v = u_n \) as test function, we obtain

\[
\int_\Omega \int_\Omega |D^s u_n|^p dv + \frac{1}{n} \int_\Omega \int_\Omega |D^s u_n|^N dv \leq \int_\Omega f_n u_n dx 
\]

\[
\leq \|f_n\|_{LLog^a L(\Omega)} \|u_n\|_{L_a(\Omega)} 
\]

\[
\leq \|f\|_{LLog^a L(\Omega)} \|u_n\|_{W^{s,N}_0(\Omega)} \text{ (see Theorem 10)}
\]

where \( A(t) = e^{t \frac{N}{N-s}} - 1 \).

Then,

\[
\frac{1}{n} \left( \int_\Omega \int_\Omega |D^s u_n|^N dv \right)^{\frac{N-1}{N}} \leq C.
\]

Let \( \psi \in C_c^\infty(\Omega) \), \( U_n(x,y) = u_n(x) - u_n(y) \), and \( \Phi(x,y) = \psi(x) - \psi(y) \) then

\[
\int_\Omega \int_\Omega \frac{|U_n(x,y)|^{N-2} U_n(x,y) \Phi(x,y)}{|x-y|^{Ns}} dv + \frac{1}{n} \int_\Omega \int_\Omega \frac{|U_n(x,y)|^{N-2} U_n(x,y) \Phi(x,y)}{|x-y|^{Ns}} dv 
\]

\[
\leq \int_\Omega f_n \psi dx \leq C \|\psi\|_{W^{s,N}_0(\Omega)}.
\]

Let us remark that \( \|\Psi\|_{W^{s,N}_0(\Omega)} = \|\Phi\|_{L^N(\Omega \times \Omega ; \frac{d\nu}{|x-y|^{Ns}})} \).

Additionally, we deduce by using Hölder inequality that

\[
\frac{1}{n} \left( \int_\Omega \int_\Omega \frac{|U_n(x,y)|^{N-2} U_n(x,y) \Phi(x,y)}{|x-y|^{Ns}} dv \right)^{\frac{N-1}{N}} 
\]

\[
\leq C \|\Phi\|_{L^N(\Omega \times \Omega ; \frac{d\nu}{|x-y|^{Ns}})}.
\]

Then,

\[
\int_\Omega \int_\Omega \frac{|U_n(x,y)|^{p-2} U_n(x,y) \Phi(x,y)}{|x-y|^{Ns}} dv \leq C \|\Phi\|_{L^N(\Omega \times \Omega ; \frac{d\nu}{|x-y|^{Ns}})}
\]

which means that \( |U_n|^{N-1} \) is bounded in \( L^N(\Omega \times \Omega ; \frac{d\nu}{|x-y|^{Ns}}) \). Then,

\[
\int_\Omega \int_\Omega |D^s u_n|^N dv \leq C.
\]

Then, \( (u_n) \) is bounded in \( W^{s,N}_0(\Omega) \) and up to a subsequence denoted \( (u_n) \), there exists a function \( u \in W^{s,N}_0(\Omega) \) such that \( u_n \to u \) weakly in \( W^{s,N}_0(\Omega) \) and a.e. in \( \Omega \).

**Passing to the limit when** \( \theta \in L Log^a L(\Omega) \), \( \alpha = \frac{N-s}{N} \), \( p = N \) and \( \alpha > \frac{N-s}{N} \) for \( p < N \).
For passing to the limit, we adopt the same way as above. □

4. Conclusions

In this paper, we have presented existence and regularity results for the nonlocal $p-$Laplacian. Indeed, the results presented are an extension and improvement of existing work in [18,22,23].

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