An implementation of the weighted essential non-oscillatory scheme for numerical approximations of the pressureless gas dynamics equations

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Abstract. In this study, the Weighted Essential Non-Oscillatory (WENO) scheme is implemented for numerical approximations of the pressureless gas dynamics equations. The Harten-Lax-van Leer-Contact (HLLC) approximate Riemann solver serves as a basic building block with the fifth order WENO scheme involving the first order Lax-Friedrichs scheme as a flux limiter for the numerical stability. In particular, the WENO scheme suppressed the oscillatory behavior around the delta shock waves and the flux limiter guaranteed the positivity of densities around the vacuum. Lastly, numerical tests for one-dimensional problems are presented for capturing the delta shock waves and vacuum.

1. Introduction

A wide variety of physical phenomena are governed by the dilute two-phase flows that a dispersed phase is transported within a continuous fluid phase. Practical interests for the flows includes a multiphase combustion, liquid water droplets in atmosphere and alumina particles in rocket engines, as well as pollutant particle dispersion or cosmology [1, 2, 3, 4]. Those particle flows with high Knudsen number [5, 6], where the Knudsen number ($Kn = \frac{\lambda}{L}$) is a dimensionless number defined as the ratio of the molecular mean free path length ($\lambda = \frac{h}{\sqrt{2\pi m}}$) to a representative physical length scale ($L$), result in the zero pressure (zero thermodynamic temperature, $T \rightarrow 0$) of gas dynamic equations (compressible Euler equations), called as the pressureless gas dynamic (PGD) equations [7], which have a non-distinct eigensystem. That is,

\begin{align*}
\rho_t + \nabla_x \cdot (\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \nabla_x \cdot (\rho \mathbf{u} \otimes \mathbf{u}) &= 0, 
\end{align*}

where, $\rho_t = \partial_t \rho$, $\mathbf{u} = \partial_t (\rho \mathbf{u})$, $t > 0$, $\mathbf{x} \in \mathbb{R}$, and $\rho(\mathbf{x}, t) \geq 0$ and $\mathbf{u}(\mathbf{x}, t)$ are in $\mathbb{R}$. $\rho$ and $\mathbf{u}$ denote the density and velocity vector of the liquid or particle, respectively. [11] represents the conservation of mass and momentum in the absence of pressure and has a non-diagonalizable Jacobian matrix with degenerate eigenvalues, \( \Lambda = [\lambda_1, \lambda_2]^T = [u, u]^T \), which may suffer a numerical difficulty to apply for the well-posed Godunov-type upwind schemes [8] that can provide the single-valued weak solutions, such as shock waves. Indeed, the main features of the PGD equations are occurrences of delta shock waves and vacuum. Thus, [11] may need special...
treatments, like relaxation models, satisfying the strictly hyperbolic conservation law \[9\], which has all real eigenvalues and a complete set of linearly independent eigenvectors.

The relaxation models has been studied by various techniques, representatively isothermal and isentropic Euler equations with a very small constant propagation speed of sound, \(c \to 0\). The models, however, have a fundamental issue that cannot maintain a same mathematical form with the PGD equations. Based on this finding, Jung \textit{et al.} \[2\] proposed a new type of the relaxation model for air-mixed droplet flows, where the convective part of droplet equations are governed by the PGD equations. That is,

\[
\rho_t + \nabla_x \cdot (\rho \mathbf{u}) = 0, \\
(\rho \mathbf{u})_t + \nabla_x \cdot (\rho \mathbf{u} \otimes \mathbf{u} + A \mathbf{I}) = \nabla_x \cdot (A \mathbf{I}),
\]

where, \(A = \rho gd\) and \(\mathbf{I}\) is an identity matrix. \(g\) and \(d\) denote a gravity and diameter of the droplet, respectively. \[2\] maintains the same mathematical form with the PGD equations and the left side of \[2\] satisfies the strictly hyperbolic conservation law with a distinct eigensystem, \(\Lambda = [\lambda_1, \lambda_2]^T = [u - \sqrt{gd}, u + \sqrt{gd}]^T\).

For numerical approximations of \[2\], well-established Godunov-type upwind schemes can be employed to simulate the delta shock waves and vacuum. In this study, the Harten-Lax-van Leer-Contact (HLCC) scheme \[10\], which enables a positivity of density derived by the depth positivity condition, serves as the basic building block to \[2\], and the Weighted Essential Non-Oscillatory (WENO) \[11\] scheme are employed for the monotonicity and high-order accuracy (5th order). Based on author’s observations, the Monotonic Upwind Scheme for Conservation Laws (MUSCL)-type second-order scheme with van Albada’s limiter \[12, 13\] showed a local pick around the delta shock waves that the monotonicity is not preserved. A major difference between the MUSCL-type scheme with Van Albada limiter and the WENO scheme can be taken into account for the order of accuracy and monotonicity of numerical solutions. A key idea in WENO schemes is a linear combination of reconstruction to obtain a higher order approximation that achieves non-oscillatory property near discontinuities to avoid spurious oscillatory. Hence, the WENO scheme may be proper to suppress the oscillatory around the delta shock waves. Further, the WENO scheme involves a simple flux limiter for a purpose of the numerical stability. The first order Lax-Friedriches scheme is employed for the simple flux limiter \[14\]. Lastly, the current approaches are compared with the MUSCL-type second order method for one-dimensional test problems.

2. Numerical methods

The finite volume method with the Godunov-type upwind scheme \[15\] is employed as a computational framework for \[2\]. For the sake of simplicity, let us consider a two-dimensional \(x\)-split conservative vector form of \[2\],

\[
\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}(\mathbf{U})_x, \tag{3}
\]

and \(\mathbf{U} = [\rho, \rho u, \rho \psi]^T, \mathbf{F}(\mathbf{U}) = [\rho u, \rho u^2 + A, \rho \psi]^T, \mathbf{S}(\mathbf{U}) = [0, A, 0]^T\), where \(A = \rho gd\). A tangential velocity component \(\psi\) represents the concentration of a pollutant or other passive scalar. When the cell average of interval \(I_i = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]\) is defined as \(\mathbf{U}_i = \frac{1}{\Delta x_i} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathbf{U}(x, t) dx\), where \(\Delta x_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}\), the explicit finite volume conservative formula in the interval \(I_i\) can be derived:

\[
\frac{d\mathbf{U}_i(t)}{dt} + \frac{1}{\Delta x_i} (\mathbf{F}_{i+\frac{1}{2}} - \mathbf{F}_{i-\frac{1}{2}}) = \frac{1}{\Delta x_i} (\mathbf{S}_{i+\frac{1}{2}} - \mathbf{S}_{i-\frac{1}{2}}), \tag{4}
\]

where \(\mathbf{F}_{i+\frac{1}{2}} = \mathbf{F}(\mathbf{U}_{i+\frac{1}{2}}), \mathbf{S}_{i+\frac{1}{2}} = \mathbf{S}(\mathbf{U}_{i+\frac{1}{2}}), \) and \(\mathbf{U}_{i+\frac{1}{2}} = \mathbf{U}(x_{i+\frac{1}{2}}, t)\). The numerical fluxes \(\mathbf{F}_{i+\frac{1}{2}}\) and \(\mathbf{S}_{i+\frac{1}{2}}\), and the associated wave speed estimates can be derived using the Godunov-type upwind scheme.
2.1. HLLC approximate Riemann solver
Let us recall the left-hand side of (3) for the general initial value problem,

\[ \mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0, \quad \text{and} \quad \mathbf{U}(x,0) = \begin{cases} \mathbf{U}_L & \text{if } x < 0, \\ \mathbf{U}_R & \text{if } x > 0. \end{cases} \]  

(5)

For (5), the HLLC solver and WENO scheme are employed. The HLLC solver (10) restores the missing Rarefaction wave, which is relevant to preserve a positivity of density, for the middle wave speed. Also, for problems comprising both shocks and complex smooth solution structure, the WENO scheme (11) provides higher accuracy in smooth regions and essentially non-oscillatory transition for solution discontinuities. The Godunov inter-cell numerical flux \( \mathbf{F}_{i+1/2} \) can be derived for the HLLC as

\[
\mathbf{F}_{i+\frac{1}{2}} = \begin{cases} 
\mathbf{F}_L & \text{if } 0 \leq S_L, \\
\mathbf{F}_{*L} & \text{if } S_L \leq 0 \leq S_*, \\
\mathbf{F}_{*R} & \text{if } S_* \leq 0 \leq S_R, \\
\mathbf{F}_R & \text{if } S_R \leq 0,
\end{cases}
\]  

(6)

where \( \mathbf{F}_{*K} = \mathbf{F}_K + S_K (\mathbf{U}_{*K} - \mathbf{U}_K) \), and a subscript \( K \) denotes the left and right states. The states \( \mathbf{U}_{*K} \) are given by \( \mathbf{U}_{*K} = \rho_K \left( \frac{S_K - u_K}{S_K - S_*} \right) [1, S_*, \psi_K]^T \). Left and right wave speeds, \( S_L \) and \( S_R \), are given as \( S_L = u_L - a_L q_L \) and \( S_R = u_R + a_R q_R \), respectively. Here, \( a_L = a_R = \sqrt{g d} \).

Then, \( q_K \) is defined by

\[
q_K = \begin{cases} 
\frac{\sqrt{\rho_s}}{\rho_K} & \text{if } \rho_s > \rho_K, \\
1 & \text{if } \rho_s \leq \rho_K.
\end{cases}
\]  

(7)

The middle wave \( S_* \) in (6) can be obtained by assuming \( \rho_{*L} = \rho_{*R} \) in the exact Riemann solver,

\[
S_* = \frac{S_{LR}}{S_{RL}} u_R - S_{RPL} (u_L - S_L) \rho_R (u_R - S_R) - \rho_L (u_L - S_L).
\]  

(8)

The third component of the flux in (3) can be expressed in term of the first component, that is, \( \mathbf{F}^3 = \mathbf{F}^1 \psi \), using the generalized Riemann invariant, \( \psi_{*L} \neq \psi_{*R} \) and the Rankine-Hugoniot conditions, \( \psi_1 = \psi_R \) and \( \psi_s = \psi_L \). An expression for the third component based on the velocity in a star region is given as

\[
\mathbf{F}_{i+\frac{1}{2}} = \begin{cases} 
\mathbf{F}^{1*}_{i+\frac{1}{2}} \psi_L & \text{if } u_s \geq 0, \\
\mathbf{F}^{1*}_{i+\frac{1}{2}} \psi_R & \text{if } u_s < 0.
\end{cases}
\]  

(9)

In the (7) and (9), two unknowns, \( \rho_*, \) and \( u_* \), that should be defined to compute the fluxes as a closed form in the HLLC solver are given as

\[
\rho_* = \frac{1}{2} (\rho_L + \rho_R) - \frac{1}{4} (u_R - u_L) \frac{\rho_L + \rho_R}{a_L + a_R}, \quad u_* = \frac{1}{2} (u_L + u_R) - \frac{1}{4} (\rho_R - \rho_L) \frac{a_L + a_R}{\rho_L + \rho_R}.
\]  

(10)

2.2. Weighted Essentially Non-Oscillatory scheme for high order accuracy
The Weighted Essentially Non-Oscillatory (WENO) schemes use the idea of adaptive stencils in the reconstruction procedure based on the local smoothness of the numerical solution to automatically achieve high order accuracy and non-oscillatory property near discontinuities. In this study, the fifth order WENO reconstruction is employed and briefly reviewed. The fifth
order WENO reconstruction for a scalar function \( u(x) \) defines the extrapolated value of the left side as \( u^L_{i+1/2} = \sum_{n=1}^{3} u_n v_n \), then, \( v_k \) is given as

\[
v_0 = \frac{(-u_{i+2} + 5u_{i+1} + 2u_i)}{6}, \quad v_1 = \frac{(-u_{i-1} + 5u_i + 2u_{i+1})}{6}, \quad v_2 = \frac{(2u_{i-2} - 7u_{i-1} + 11u_i)}{6},
\]

where nonlinear WENO weights are given as \( w_k = \alpha_k / \sum_{n=0}^{2} \alpha_n \), where a subscript, \( k = 0, 1, 2 \). The coefficients \( \alpha_k \) are

\[
\alpha_0 = \frac{3}{10(10^{-6} + IS_0)^2}, \quad \alpha_1 = \frac{3}{5(10^{-6} + IS_1)^2}, \quad \alpha_2 = \frac{1}{10(10^{-6} + IS_2)^2}.
\]

The smoothness indicators \( IS_k \) are

\[
IS_0 = \frac{13}{12} (u_i - 2u_{i+1} + u_{i+2})^2 + \frac{1}{4} (3u_i - 4u_{i+1} + u_{i+2})^2, \\
IS_1 = \frac{13}{12} (u_{i-1} - 2u_i + u_{i+1})^2 + \frac{1}{4} (u_{i-1} - u_{i+1})^2, \\
IS_2 = \frac{13}{12} (u_{i-2} - 2u_{i-1} + u_i)^2 + \frac{1}{4} (u_{i-2} - 4u_{i-1} + 3u_i)^2.
\]

The extrapolated value of the right side \( u^R_{i+1/2} \) is obtained by symmetry.

2.3. Flux limiter for the WENO scheme

Direct usage of high order WENO scheme may result in the appearance of negative densities that can lead to an ill-posed system, which may cause blow-up of the numerical solution. Therefore, a limiter to keep the positivity of the density may be essential to guarantee a robustness of numerical solutions. In this study, a whole procedure of limiter for positive density follows Hu et al. work as the positivity-preserving flux limiter [14], which is based on the first-order Lax-Friedrichs (LF) scheme. The left side of (16) can be rewritten as a convex combination

\[
U_{i+1}^{n+1} = \frac{1}{2} \left( U_i^n + 2\lambda F_{i+1/2} \right) + \frac{1}{2} \left( U_i^n - 2\lambda F_{i-1/2} \right) = \frac{1}{2} U_i^n - \frac{1}{2} U_i^n,
\]

where \( U^\pm_i = U_i^n \pm 2\lambda F_{i+1/2} \) and \( \lambda = \frac{\Delta t}{\Delta x} = \frac{CFL}{(u|u|)_{\text{max}}} \). A sufficient condition for positivity of density is fulfilled by \( \rho(U^\pm_i) > 0 \). The first-order LF scheme has positivity property \( \rho(U_i^\text{LF,\pm}) = \rho(U_i^n \pm 2\lambda F_{i+1/2}) > 0 \), under an additional CFL condition, \( \text{CFL} \leq \frac{1}{2} \). A straightforward way to ensure positivity is to limit the magnitude of interface flux by utilizing the locally Lipschitz continuous and constants. The limitation of magnitude of interface flux can be enforced by the flux limiter. The architecture of the flux limiter is described as below:

1. For all \( i \), initialize \( \theta_{i+1/2}^+ = \theta_{i+1/2}^- = 1 \).
2. If \( \rho(U^+_i) < \epsilon \), solve \( \theta_{i+1/2}^+ \) from \( \left( 1 + \theta_{i+1/2}^+ \right) \rho(U^\text{LF,}^+_i) + \theta_{i+1/2}^+ \rho(U^+_i) = \epsilon \).
3. If \( \rho(U^-_{i+1}) < \epsilon \), solve \( \theta_{i+1/2}^- \) from \( \left( 1 - \theta_{i+1/2}^- \right) \rho(U^\text{LF,}^-_{i+1}) + \theta_{i+1/2}^- \rho(U^-_{i+1}) = \epsilon \).
4. Set \( \theta_{i+1/2} = \min\left( \theta_{i+1/2}^+, \theta_{i+1/2}^- \right) \), \( F^*_{i+1/2} = (1 - \theta_{i+1/2}) F_{i+1/2}^\text{LF(1st)} + \theta_{i+1/2} F_{i+1/2}^\text{HLLC(WENO)} \).

Here, \( \epsilon = \min(\epsilon^0, \rho_{\text{min}}^0) \), where \( \epsilon^0 = 10^{-13} \) and \( \rho_{\text{min}} \) is the minimum density in the initial condition. \( F^*_{i+1/2} \) is the limited flux, and \( 0 \leq \theta^\pm_{i+1/2} \leq 1 \) are the limiting factors corresponding to the two neighboring cells. Finally, the HLLC solver with the WENO scheme involving the first order LF limiter are summarized as

\[
U^n_{i+1} = U^n_i - \lambda \left( F^*_{i+1/2} + F^*_{i-1/2} \right).
\]
2.4. Finite volume formulation and treatment of source term

Equations (15) and source term in (3) are recalled for an integral form of the equations

\[ \frac{\partial}{\partial t} \int_{\Omega} U d\Omega + \int_{\partial S} F^* dS = \int_{\Omega} S d\Omega, \]  

where \( \Omega \) and \( S \) are a volume and interface area of the cell. Then, we have a following discretized form as

\[ U^n_{i+1} = U^n_i - \frac{1}{\Omega_i} \left\{ \sum_{k=1}^{n} F^*_k \Delta S_k - gd \sum_{k=1}^{n} \rho_k c \Delta S_k \right\}. \]  

In the source term of (17), \( \rho_k \) at the interface is given by a basic idea of the HLLC scheme as

\[ \rho_k = \begin{cases} 
\rho_L & \text{if } 0 \leq S_L, \\
\rho_{*L} & \text{if } S_L \leq 0 \leq S_s, \\
\rho_{*R} & \text{if } S_s \leq 0 \leq S_R, \\
\rho_R & \text{if } S_R \leq 0. 
\end{cases} \]  

2.5. Temporal discretization

(16) and (17) can be rewritten as

\[ U^n_{i+1} = U^n_i - \lambda (Q^n_{i+1/2} - Q^n_{i-1/2}), \]  

where \( Q^n_{i+1/2} = F^n_{i+1/2} - S^n_{i+1/2}, \lambda = \Delta t/\Delta x, \) and \( \Delta t = \frac{CFL \Delta x}{(|u|+a)_{\text{max}}} \). The following strong stability preserving (SSP) third-order Runge-Kutta time discretization [18] is used:

\[ \begin{align*}
U^{n+1/3} &= U^n + \Delta t R(U^n), \\
U^{n+2/3} &= \frac{3}{4} U^n + \frac{1}{4} U^{n+1/3} + \frac{1}{4} \Delta t R(U^{n+1/3}), \\
U^{n+1} &= \frac{1}{3} U^n + \frac{2}{3} U^{n+2/3} + \frac{2}{3} \Delta t R(U^{n+2/3}).
\end{align*} \]  

3. Numerical tests

Typical one-dimensional test cases [16, 17] are employed to evaluate the current strategy for positivity-preserving high order numerical approaches. Special emphasis are placed on the suppression of local oscillatory around delta shock waves (a role of the WENO scheme) and the preservation of positivity of density (a role of first order LF flux limiter) around the vacuum. All the computations are carried out with a CFL number of 0.4 (the SSP third-order Runge-Kutta time discretization scheme requires the CFL condition as CFL < 0.5.) and \( \Delta x = 0.01 \) at which a grid dependency of the MUSCL and WENO schemes was clearly shown. The coefficients of \( g \) and \( d \) in (16) are set up with a constant value of \( g = d = 1 \) for the sake of simplicity of numerical treatments.

3.1. A delta-function problem

The first test case is devoted to the delta-function problem. The initial condition and the corresponding exact solution at the time \( t = 0.5 \) are

\[ (\rho_0, u_0) = \begin{cases} 
(1, 1) & \text{if } x < 0, \\
(0.25, 0) & \text{if } x > 0,
\end{cases} \]  

\[ (\rho_{0.5}, u_{0.5}) = \begin{cases} 
(1, 1) & \text{if } x < 2t/3, \\
(0.25, 0) & \text{if } x > 2t/3,
\end{cases} \]  

\[ (\rho_{1.0}, u_{1.0}) = \begin{cases} 
(0.25, 0) & \text{if } x < 4t/3, \\
(1, 1) & \text{if } x > 4t/3.
\end{cases} \]
where $\rho_0$ and $u_0$ are $\rho(x)$ and $u(x)$ at $t = 0$. At $x = 2t/3$, a delta-shock is immediately developed. Numerical and exact solutions are shown in fig. [1]. Although both schemes trace the exact solutions and capture the delta shock waves, the fifth order WENO scheme with the first order LF flux limiter shows a higher value of the maximum density than the MUSCL-type second order scheme with Van Albada limiter.

\[ 0 \leq x \leq 1 \]

**Figure 1.** Numerical (red line: WENO with LF flux limiter, blue line: MUSCL with van Albada limiter) and exact (black line) solutions on the initial condition, (21). Other parameters are taken to be $\Delta x = 0.01, t = 0.5, g = d = 1$ and CFL=0.4.

### 3.2. A vacuum problem

The second test case is devoted to the vacuum problem, which takes into account for the positivity-preserving properties. The initial condition and the corresponding exact solution at time $t = 0.5$ are

\[
(\rho_0, u_0) = \begin{cases} 
(0.5, -0.5) & \text{if } x < 0, \\
(0.5, 0.4) & \text{if } x > 0,
\end{cases}
\]

\[
(\rho_{0.5}, u_{0.5}) = \begin{cases} 
(0.5, -0.5) & \text{if } x < -0.25, \\
(0, \text{undefined}) & \text{if } -0.25 < x < 0.2, \\
(0.5, 0.4) & \text{if } 0.2 < x,
\end{cases}
\]

In vacuum problem induced by two opposite rarefaction waves, a positivity-preserving property play [16] a key role to the numerical scheme. Numerical and exact solutions are shown in fig. [2]. Although both schemes trace the exact solutions and guarantee the positivity of density, the fifth order WENO scheme with the first order LF flux limiter is closer to the exact solution than the MUSCL-type second order scheme with Van Albada limiter. In particular, the first order LF flux limiter does not destroy the high order accuracy as proven by Hu *et al.* [14]. Also, the velocity keeps the bound without any undershot or overshot.
3.3. A problem involving the delta shock wave and vacuum

The third test case is designed to create a vacuum and a mass accumulation. The initial condition and the corresponding exact solution at time \( t = 0.5 \) are

\[
\begin{align*}
(\rho_0, u_0) &= \begin{cases} 
(0.5, -0.5) & \text{if } x < -0.5, \\
(0.5, 0.4) & \text{if } -0.5 < x < 0, \\
(0.5, 0.4 - x) & \text{if } 0 < x < 0.8, \\
(0.5, -0.4) & \text{if } x > 0.8,
\end{cases} \\
(\rho_{0.5}, u_{0.5}) &= \begin{cases} 
(0.5, -0.5) & \text{if } x < -0.5 - 0.5t, \\
(0, \text{undefined}) & \text{if } -0.5 - 0.5t < x < -0.5 + 0.4t, \\
(0.5, 0.4) & \text{if } -0.5 + 0.4t < x < 0.4t, \\
\left(\frac{0.5}{1-t}, \frac{0.4-x}{1-t}\right) & \text{if } 0.4t < x < 0.8 - 0.4t, \\
(0.5, -0.4) & \text{if } x > 0.8 - 0.4t.
\end{cases}
\end{align*}
\]  

(23)

This test is a problem combining previous tests of [21] and [22]. Figure 3 shows the numerical and exact solutions. Both schemes can capture the vacuum and mass accumulation. In case of the MUSCL-type second order scheme with van Albada’s limiter, local oscillations around the delta shock waves are observed in the density profile. Whereas, the fifth order WENO scheme with the first order LF flux limiter shows a strong monotonicity suppressing the local oscillatory.

4. Conclusions

In this study, the relaxation model was based on a simple technique adding an artificial term into left- and right-hand sides of the PGD equations as a purpose of mathematically equivalent form with the PGD equations. The model satisfies the strictly hyperbolic conservation law that the well-posed approximate Riemann solver can be employed. For the numerical methods, the fifth order WENO scheme with the first order LF flux limiter was implemented to preserve
Figure 3. Numerical (red line: WENO with LF flux limiter, blue line: MUSCL with van Albada limiter) solutions on the initial condition, (23). Other parameters are taken to be $\Delta x = 0.01, t = 0.5, g = d = 1$ and $CFL=0.4$.

the positivity of density in the vacuum and suppress the local oscillatory around delta shock waves. Numerical experiments showed that the WENO scheme with the LF limiter are in a good agreement with exact solutions of test problems. Even though the first order LF flux limiter was imposed to the WENO scheme, a high-order accuracy of the WENO scheme was well preserved in comparisons with the MUSCL-type second order scheme with a van Albada limiter. Also, the monotonicity of the WENO scheme was clearly shown at the local oscillatory around the delta shock waves. In the future, the present work will be extended to the multiphase flow involving the delta function and vacuum in wide range of applications.

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