A LARGE WIENER SAUSAGE FROM CRUMBS

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AMS 1991 Subject classification: Primary 60J45; Secondary 60J65, 31C15

Brownian motion, capacity, polar set, Wiener sausage

Abstract
Let $B(t)$ denote Brownian motion in $\mathbb{R}^d$. It is a classical fact that for any Borel set $A$ in $\mathbb{R}^d$, the volume $V_1(A)$ of the Wiener sausage $B[0,1] + A$ has nonzero expectation iff $A$ is nonpolar. We show that for any nonpolar $A$, the random variable $V_1(A)$ is unbounded.

1. Introduction

The impetus for this note was the following message, that was sent to one of us (Y. P.) by Harry Kesten:

"... First a question, though. It is not of major importance but has bugged me for a while in connection with some large deviation result for the Wiener sausage (with Yuji Hamana). Let $V_1(A)$ be the volume of the Wiener sausage at time 1, that is, $V_1(A) = \text{Vol}_d \left( \bigcup_{s \leq 1} B_s + A \right)$, where $B_s$ is $d$-dimensional Brownian motion, and $A$ is a $d$-dimensional..."
set of positive capacity. Is it true that the support of $V_1(A)$ is unbounded, i.e., is $P[V_1(A) > x] > 0$ for all $x$? This is easy if $A$ has a section of positive $(d-1)$-dimensional Lebesgue measure, but I cannot prove it in general. Do you have any idea?"

We were intrigued by this question, because it led us to ponder the source of the volume of the Wiener sausage when $A$ is a "small" set (e.g., a nonpolar set of zero Hausdorff dimension, in the plane). Is it due to the macroscopic movement of $B$ (in which case $V_1(A)$ would not be bounded) or to the microscopic fluctuations (in which case $V_1(A)$ might be bounded, like the quadratic variation)?

Our proof of the following theorem indicates that while the microscopic fluctuations of $B$ are necessary for the positivity of $V_1(A)$, the macroscopic behaviour of $B$ certainly affects the magnitude of $V_1(A)$.

**Theorem 1.** If the capacity $C(A)$ of $A \subseteq \mathbb{R}^d$ is positive, then $V_1(A)$ is not bounded.

The relevant capacity can be defined for $A \subseteq \mathbb{R}^d$ with $d \geq 3$, by

$$C(A) = \sup_{\mu} \frac{\mu(A)^2}{\mathcal{E}(\mu)},$$

where $\mathcal{E}(\mu) = \int \int c_d \frac{d\mu(x) \, d\mu(y)}{|x-y|^{d-2}}$, and the supremum is over measures supported on $A$. (the constant $c_d$ is unimportant for our purpose). A similar formula holds for $d = 2$ with a logarithmic kernel; in that case $C(A)$ is often called Robin’s constant, and it will be convenient to restrict attention to sets $A$ of diameter less than 1.

Denote by $\tau_A$ the hitting time of $A$ by Brownian motion. By Fubini’s theorem

$$\mathbf{E}[V_1(A)] = \int_{\mathbb{R}^d} \mathbf{P}_x[\tau_A \leq 1] \, dx.$$ 

It follows from the relation between potential theory and Brownian motion, that $\mathbf{E}[V_1(A)]$ is nonzero if and only if $A$ has positive capacity; see, e.g., [3], [2], or [4].

2. The recipe

For any kernel $K(x,y)$, the corresponding capacity is defined by $C_K(A) = \sup_{\mu} \frac{\mu(A)^2}{\mathcal{E}_K(\mu)}$ where $\mathcal{E}_K(\mu) = \int \int K(x,y) \, d\mu(x) \, d\mu(y)$ and the supremum is over measures on $A$. We assume that $K(x,x) = \infty$ for all $x$, and that for $0 < |x-y| < R_K$, the kernel $K$ is continuous and $K(x,y) > 0$.

The following lemma holds for all such kernels.

**Lemma 1.** If a set $A \subseteq \mathbb{R}^d$ has $C_K(A) > 0$, then for any $L < \infty$ there exists $\epsilon > 0$ and subsets $A_1, A_2, \ldots, A_m$ of $A$ such that $\sum_{i=1}^{m} C_K(A_i) \geq L$, and the distance between $A_i$ and $A_j$ is at least $\epsilon$ for all $i \neq j$. ($m$ and $\epsilon$ depend on $A$ and $L$).

**Proof:** We can assume that $\text{diam}(A) < R_K$, for otherwise we can replace $A$ by a subset of positive capacity and diameter less than $R_K$.

Let $\mu$ be a measure supported on $A$ such that $\mu(A) = 1$ and $\mathcal{E}_K(\mu) < \infty$. 

By dominated convergence,
\[ \lim_{\delta \to 0} \int_{|x-y| \leq \delta} K(x, y) \, d\mu(x) \, d\mu(y) = 0. \]

Choose \( \delta \) so that this integral is less then \( 2^{-2d} L^{-1} \). Let \( \epsilon = \delta d^{-1/2} \) and let \( \mathcal{F} \) be a grid of cubes of side \( \epsilon \), i.e.,
\[ \mathcal{F} = \left\{ \prod_{i=1}^{d} [\ell_i, \ell_i + \epsilon) : (\ell_1, \ldots, \ell_d) \in \mathbb{Z}^d \right\}. \]

We can partition \( \mathcal{F} \) into \( 2^d \) subcollections \( \{ \mathcal{F}_v : v \in \{0,1\}^d \} \) according to the vector of parities of \( (\ell_1, \ldots, \ell_d) \). Then the distance between any two cubes in the same \( \mathcal{F}_v \) is at least \( \epsilon \). Since \( \mu \) is a probability measure, there exists \( v \in \{0,1\}^d \) such that
\[ \sum_{Q \in \mathcal{F}_v} \mu(Q) \geq 2^{-d}. \]

Let \( A_1, A_2, \ldots, A_m \) be all the nonempty sets among \( \{ A \cap Q : Q \in \mathcal{F}_v \} \). Since \( \mu \) is supported on \( A \), we can rewrite (1) as \( \sum_{i=1}^{m} \mu(A_i) \geq 2^{-d} \).

By Cauchy-Schwarz,
\[ \left( \sum_{i=1}^{m} e_i \right) \left( \sum_{i=1}^{m} \frac{\mu(A_i)^2}{e_i} \right) \geq \left( \sum_{i=1}^{m} \mu(A_i) \right)^2 \geq 2^{-2d}. \]

We have \( C_K(A_i) \geq \mu(A_i)^2 / e_i \), whence
\[ \sum_{i=1}^{m} C_K(A_i) \geq \sum_{i=1}^{m} \frac{\mu(A_i)^2}{e_i} \geq L, \]
by (2) and (3).

\textbf{Proof of Theorem 1:} Suppose that
\[ \text{esssup } V_1(A) = M < \infty. \]

Let \( V_t(A) \) denote the volume of the Wiener sausage \( B[0,t] + A \). From Spitzer [3] (see also [2] or [1]) it follows that \( \mathbb{E}[V_t(A)] > 2 \alpha_d C(A) \) for some absolute constant \( \alpha_d \). (If \( d = 2 \) we assume that \( \text{diam } A < 1 \).) We infer that \( \mathbb{E}[V_t(A)] > \alpha_d t C(A) \) for \( 0 < t < 1 \), by subadditivity of Lebesgue measure and monotonicity of \( V_t(A) \).

Fix \( L > 6M / \alpha_d \), and let \( A_1, \ldots, A_m \) be the subsets of \( A \) given by the lemma. A Wiener sausage on \( A \) contains the union of Wiener sausages on the \( A_i \), and the sum of their volumes is expected to be large. If we can arrange for the intersections to be small, then \( V_1(A) \) will be large as well.
Consider the event
\[ H_n = \left\{ \max_{0 \leq s \leq \frac{n}{2n}} |B_s| < \frac{\epsilon}{2} \right\}. \]

By Brownian scaling and standard estimates for the maximum of Brownian motion,
\[ P[H_n^c] \leq 4d \exp\left(-\frac{2n^2}{4d}\right). \]

Choose \( n \) large enough so that the right-hand side is less than \( \frac{1}{nm} \). For each \( i \), we have
\[ \mathbb{E}[V_1(A_i) | H_n^c] \leq M \] by (4), so
\[ \mathbb{E}[V_1(A_i) | H_n] \geq \mathbb{E}[V_1(A_i)] - M P[H_n^c] \geq \frac{\alpha_d}{2n} C(A_i) - \frac{M}{mn}. \] (5)

For \( 0 \leq j < n \), denote by \( G_j \) the event that
\[ \max_{\frac{2j}{2n} \leq s \leq \frac{2j+1}{2n}} |B_s - B_{\frac{2j}{2n}}| < \epsilon/2 \]
and the first coordinate of the increment \( B_{\frac{2j+2}{2n}} - B_{\frac{2j+1}{2n}} \) is greater than the diam(\( A \)) + 2\( \epsilon \).

Define \( G = \cap_{j=0}^{n-1} G_j \). We will see that the expectation of \( V_1(A) \) given \( G \) is large.

On the event \( G \), for each fixed \( j \), the \( m \) sausages \( \{B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A_i\}_{i=1}^m \) are pairwise disjoint due to the separation of the \( A_i \) and the localization of \( B \) in the time interval \( [\frac{2j}{2n}, \frac{2j+1}{2n}] \). Therefore,
\[ \mathbb{E}\left[ \text{Vol}(B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A) \mid G \right] \geq \sum_{i=1}^{m} \left( \frac{\alpha_d}{2n} C(A_i) - \frac{M}{mn} \right) \geq \frac{\alpha_d L}{2n} - \frac{M}{n} > \frac{2M}{n}. \]

Also, on \( G \), the sausages on the odd intervals, \( B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A \) for \( 0 \leq j < n \), are pairwise disjoint due to the large increments of \( B \) (in the first coordinate) on the even intervals. We conclude that
\[ \mathbb{E}[V_1(A) \mid G] \geq \sum_{j=0}^{n-1} \mathbb{E}\left[ \text{Vol}(B[\frac{2j}{2n}, \frac{2j+1}{2n}] + A) \mid G \right] > 2M. \] (6)

This contradicts the assumption (4) and completes the proof. \( \square \)

Questions:
- Can the event \( G \) that we conditioned on at the end of the preceding proof, be replaced by a simpler event involving just the endpoint of the Brownian path?
- In particular, does every nonpolar \( A \subset \mathbb{R}^d \) satisfy
\[ \lim_{R \to \infty} \mathbb{E}\left[ V_1(A) \mid |B(1)| > R \right] = \infty? \]
- Can one estimate precisely the tail probabilities \( P[V_1(A) > v] \) for specific nonpolar fractal sets \( A \) and large \( v \), e.g., when \( d = 2 \) and \( A \) is the middle-third Cantor set on the \( x \)-axis?

Acknowledgments. We are grateful to Harry Kesten for suggesting the problem, and for a correction to an earlier version of this note. We thank Dimitris Gatzouras and Yimin Xiao for helpful comments. Research of Peres was partially supported by
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NSF grant #DMS-9803597 and by the Landau Center for Mathematical Analysis at the Hebrew University.

References

[1] I. Benjamini, R. Pemantle and Y. Peres, Martin capacity for Markov chains. *Ann. Probab.* 23 (1995), 1332–1346.
[2] K. Itô and H. P. McKean (1974), *Diffusion Processes and Their Sample Paths*, Springer-Verlag.
[3] F. Spitzer (1964), Electrostatic capacity, heat flow, and Brownian motion. *Z. Wahrschein. Verw. Gebiete* 3, 110–121.
[4] A. S. Sznitman (1998), *Brownian motion, Obstacles and Random Media*. Springer Monographs in Mathematics. Springer-Verlag, Berlin.