Interference Channel with Generalized Feedback (a.k.a. with source cooperation)
Part I: Achievable Region. *

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Abstract

An Interference Channel with Generalized Feedback (IFC-GF) is a model
for a wireless network where several source-destination pairs compete for
the same channel resources, and where the sources have the ability to sense
the current channel activity. The signal overheard from the channel provides
information about the activity of the other users, and thus furnishes the basis
for cooperation. In this two-part paper we study achievable strategies and
outer bounds for a general IFC-GF with two source-destination pairs. We
then evaluate the proposed regions for the Gaussian channel.

Part I: Achievable Region. We propose that the generalized feedback is
used to gain knowledge about the message sent by the other user and then
exploited in two ways: (a) to relay the messages that can be decoded at both
destinations—thus realizing the gains of beam-forming of a distributed multi-
antenna system—and (b) to hide the messages that can not be decoded at the
non-intended destination—thus leveraging the interference “pre-cancellation”
property of dirty-paper-type coding. We show that our achievable region
generalizes several known achievable regions for IFC-GF and that it reduces

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to known achievable regions for some of the channels subsumed by the IFC-GF model.

Part II: Outer Bounds. We strengthen the cut-set bound in two ways. We first derive two new sum-rate bounds by using Sato’s “receiver cooperation with worst possible correlation” idea: we enhance the channel by letting also the destinations cooperate, but—as in broadcast channels—we choose the correlation among the destination outputs that gives the tightest bound. We show that several bounds known in the literature—each derived with an ad-hoc technique—are in fact examples of this “receiver cooperation with worst possible correlation” idea. We then derive a sum-rate outer bound for a class of channels with a special interference structure. This class includes the Gaussian channel. When evaluating the proposed outer bound for the Gaussian channel, we observe that our bound is the tightest among existing bounds for certain range of parameters.

1 Introduction

The practical bottleneck of today’s communication networks is interference. The solution of commercial available networks is thus to avoid interference through division of the resources, i.e., time, spectrum, space and waveforms, among the competing users. This approach is appealing in practice because it results in a simple network architecture. It might also appear a good solution in theory since it is well known that in SISO (Single Input Single Output) uplink and downlink wireless channels with perfect centralized channel state information, it is sum-rate optimal to allocate all the available channel resources to the user who experiences the instantaneous highest channel gain [39]. Perfect orthogonalization of the users is however not possible in practice. The practical approach to deal with residual interference is to treat it as noise, thus completely neglecting its structure. Whenever interference is treated as noise, the system becomes rate-limited even when there is no power limitation at the transmitters. This negative view of interference was further reinforced by scaling law results of the early 00’s that showed that the total rate of the wireless network with $K$ users only scales as $\sqrt{K}$, thus yielding a vanishing per-user rate as the network grows [34]. It was also conjectured that, even with cooperation, the multiplexing gain of an interference network would be one irrespectively of the number of users, thus again the per-user rate would vanish as the number of users in the network grows [22].

On the other hand, it has been known since the mid 70’s that there exist channels where interference does not reduce capacity [4,41]. These channels are said
to have “very strong interference”, that is, the power imbalance between the use-
ful signal and the interfering signal at a receiver is so large that a given user can
first decode the interfering signal by treating its own signal as noise, then strip
the (now known) interference from its received signal, thus effectively having an
interference-free channel. This early example showed that interference should not
be treated as noise in general—in fact, as opposed to noise, it has a structure that
can be exploited—and that power imbalance must be leveraged upon in the system
design. With these observations in mind, much progress has been made recently
in understanding the ultimate performance limits of interference networks [1,14].
In the authors’ opinion, the major recent result has been to show that in an inter-
fering network with \( K \) users the sum-rate capacity scales as \( \frac{K}{2} \log(1+\text{SNR}) \). This
implies that each user can get half the rate it would get if it were alone on the net-
work, no matter how how many users are present in the network. The technique
that achieves this capacity scaling is referred to as interference alignment [1]. Un-
der the interference alignment paradigm, users do not avoid interference. Instead,
they make sure that the interference they collectively generate at a given receiver
is neatly confined in a specific signal subspace. This leaves the complement of
this subspace interference free. With interference alignment, a per-user rate of
\( \frac{1}{2} \log(1 + \text{SNR}) \) can be achieved irrespectively of the number of users.

In this work we consider networks of full-duplex nodes, where several source-
destination pairs share the same channel. The channel is assumed to be static so
that every node in the network has perfect knowledge of the channel state. Exten-
sions to networks of half-duplex nodes and/or of time-varying channels, such as
fading channels, are planned as part of future work. We focus on the case where
all nodes can listen to the channel activity. In particular, we are interested in the
case where the sources can “overhear” what the other sources are sending, as in
relay networks. The key observation is that interference due to simultaneous com-
munications effectively spreads “common information” around the network. This
information provides the basis for cooperation among otherwise uncoordinated
nodes.

1.1 Related Works

The signal a node can “overhear” form the channel is a form of feedback. To
distinguish this feedback information from the classical Shannon output feedback,
Willems referred to it as Generalized Feedback (GF) [48]. GF encompasses a wide
range of possible situations, examples are: noisy output feedback (with the non-
feedback case and the output feedback case at the two extremes); conferencing
encoders (where there are separate noise-free and interference-free links between
the sources, each of finite capacity); the case of GF with independent noises,
sometimes referred to as user cooperation. The motivation to study GF comes
from the work of Gaarder and Wolf [49] who showed that output feedback can
enlarge the capacity region of a MAC channel. Here, we consider the general
case of GF. Hence our results can be specialized to all the above situations. Our
approach—especially when in dealing with outer bounds—shows that all the above
cases can be dealt with in great generality. One of our contributions is to show a
unifying way of deriving some of the results available in the literature.

MAC-GF In [48], an achievable region for the MAC-GF (Multiple Access Chan-
nels with Generalized Feedback) was derived. The two main ingredients are regular block-Markov superposition coding and backward decoding. Block-Markov
coding, also known as Decode-and-Forward from the work by Cover and A. El
Gamal [10] in the context of relay channels, works as follows. Communication
proceeds over a frame of $N > 1$ slots. The source splits the message to be trans-
mittted in a given slot into two parts. The first part is decoded by the destination,
but treated as noise by the relay. The second part is decoded also by the relay, that
then retransmits it to the destination in the next slot. Because the source knows
what the relay is going to send in the next slot, it can “coordinate” with the re-
lay. In each slot, the destination receives the superposition of the new information
(sent by the source) and the repetition of part of the old information (forwarded
by the relay). The receiver waits until the whole frame has been received. Then
appropriately combines the information sent in consecutive slots to recover all the
transmitted messages.

Willems’s coding scheme for Gaussian channels was popularized by Sendonaris
et al. [42] under the name of user cooperation diversity in the context of cellular
networks. In [42] it was showed that cooperation between users achieves collective-
ly higher data rates or, alternatively, allows to reach the same data rates for
less transmission power. Since the publication of [42], the interest in coopera-
tive strategies has not ceased to increase (we do not attempt here to review all the
extensions of [42] for sake of space).

Although the MAC-GF has proved to be instrumental in understanding the
potential of user cooperation in networks, it is not as well suited for ad-hoc/peer-
to-peer networks, where the absence of coordination among users exacerbate the
problem of interference. Host-Madsen [21] first extended the Gaussian MAC-GF
model of [42] to the case of Gaussian IFC-GF. Before dwelling into the literature
of IFC-GF, we briefly revise the known results on IFC without feedback.

**IFC without feedback**  The capacity region of a general IFC without feedback is still unknown. The largest achievable region is due to Han and Kobayashi [19], whose “compact” expression appeared [6]. In IFCs without feedback, communications is as follows. Each transmitter splits its message in two parts: a *common message* and a *private message*. The two messages are superimposed and sent through the channel. Each receiver decodes its intended common and private messages, as well as the common message of the other user, by treating the other user’s private message as noise. The goal of this joint decoding is to reduce the interference level. The idea of information splitting was first proposed by Carleial [5], to whom many other early results on IFCs are due.

The Han-Kobayashi scheme is optimal in strong interference [4, 8, 9, 27, 41], and it is shown to be sum-rate optimal in mixed interference [33, 46], in very-weak interference [33, 47, 52], for the Z-IFC [40] (where only one receiver experiences interference), and for certain semi-deterministic channels [15, 44]. Moreover, a simple rate-splitting choice in the Han-Kobayashi scheme is optimal to within 1 bit for the Gaussian IFC [14].

In this work, we propose an achievable region that combines the idea of information splitting with that of Block Markov Coding & Backward Decoding.

**Early work on IFC-GF**  Host-Madsen [21] first studied inner and outer bounds for the sum-rate of IFC with both source and destination cooperation. He showed that in IFC networks with two SISO source-destination pairs, where the channel gains stay constant and the user’ powers increase, the multiplexing gain is one, with both types of cooperation (instead of two, which would be the case if cooperation were equivalent to a $2 \times 2$ virtual MIMO channel). In [21], several achievable strategies are proposed for the Gaussian channel only, each relatively simple and tailored to a specific sets of channel gains.

The work in [21] was extended to a general DMC (Discrete Memoryless Channel) IFC-GF in [24, 45]. These works proposed to add one extra level of information splitting to the original Han-Kobayashi scheme.

**IFC-GF: Cooperation on sending the common information**  In [45], we proposed to further split the common message in two parts: one part (referred to as *non-cooperative common information*) is as in the Han-Kobayashi scheme, while the other part (referred to as *cooperative common information*) is decoded at the
other source too. The sources use a block-Markov encoding scheme where in a given slot they “retransmit” the cooperative common information learnt in the previous slot. This cooperation strategy aims to realize the beam-forming gain of a distributed MISO channel.

A scheme similar to ours in [45] was independently proposed by Jiang et al. in [24] for the IFC with output feedback (i.e., not for a general GF setting). The difference lies in the way the cooperative common information is dealt with, both at the sources and at the destination. It is not clear which scheme achieves the largest achievable region (we will elaborate more on this point later on).

Recently, Prabhakaran and Viswanath [35] developed an outer bound for the sum-rate of a symmetric Gaussian IFC-GF with independent noises, inspired by the semi-determinisctic model of [44]; they showed that their upperbound is achievable within a constant gap of 18 bits when the cooperation link gains are smaller than the direct link gains; their achievable strategy is a simple form of our region in [45].

_cooperation on sending the common information is the essence of the first achievable region presented in Part I this work._

**IFC-GF: Cooperation on sending the private information** In [2], the authors proposed to further split the private message in two parts: one part (which we shall refer to as _non-cooperative private information_) is as in the Han-Kobayashi scheme, while the other part (which we shall refer to as _cooperative private information_) is decoded at the other source too. The sources use a block-Markov encoding scheme where in a given slot they “hide” the cooperative private information learnt in the previous slot to their intended receiver by using Gelfand-Pinsker coding [7,18]. An approach similar to [2] (commonly referred to as “dirty paper coding” for Gaussian channels [7]) was already used in [21] to characterize the high SNR sum-rate capacity of the Gaussian IFC-GF. In [21], it was first noted that in the high SNR regime, nulling the interference (i.e., an ancestor of interference alignment) is asymptotically optimal.

In [3], the ideas of [2] and of [45] were merged into a scheme where both the common and the private information are split into two parts. In [53] we proposed a different (more structured) coding strategy than in [3]. As opposed to [3], in [53] we did not use independent Gelfand-Pinsker binning codes to pre-cancel the effect of the cooperative private information. Instead, we proposed to superimpose several Gelfand-Pinsker binning codes (inspired by the work in [30] for cognitive IFCs) in such a way that the different binning stages are performed se-
quentially and conditionally on the previous ones. In addition, we also added a binning step similar to Marton’s achievable scheme for a general two-user broadcast channel [32]. The broadcast-type binning step is possible in IFC-GFs because each encoder knows part of the message sent by the other encoder, i.e., each transmitter is partially cognitive in the sense of [30]. A simple form of [33]’s region was shown to be optimal within a constant gap of 18 bits for the symmetric Gaussian IFC-GF with independent noises when the cooperation link gains are larger than the direct link gains.

The second achievable region presented in Part I of this work is a further enhancement of [33].

**IFC with degraded output feedback** The GF setting covers a wide range of situations. In particular, the case of degraded output feedback has been studied by a number of groups. Degraded output feedback refers to the case where the GF signal received at a source is a noisier version of the signal received at the intended destination. When the variance of the extra noise on the GF signal is zero, we have the so-called output feedback. Kramer in [25, 26] developed inner and outer bounds for the Gaussian channel with output feedback. Kramer and Gastpar [17], and more recently Tandon and Ulukus [37], derived an outer bound for the degraded output feedback case based on the dependance balance idea of Hekstra and Willems [20]. Suh and Tse [43] developed a novel outer bound for the Gaussian case with output feedback and showed it to be achievable to within 1.7 bits. In [43] it was also shown that the achievable region of [24] is optimal for the case of deterministic channels with output feedback.

As mentioned earlier, our achievable region is applicable to all forms of GF, thus also to the case of output feedback.

**Other channel models** The IFC-GF reduces to some well-known channel models. Under certain conditions, it reduces to a Broadcast Channel (BC) [32], or to MAC-GF [48], or to a Relay Channel (RC) [10], or to a Cognitive IFC (C-IFC) [38].

We shall also describe how our results encompass known results for these channels.
1.2 Summary of Contributions

In this first part of the paper we present our achievable regions. We first present a region where cooperation is on sending the common information only. The purpose is to highlight the key elements of our encoding and decoding scheme before proceeding to describe our more general scheme, where the sources cooperate in sending both the common and the private messages. Our contributions are as follows.

Cooperation to send the common information only:

1. We propose a structured way of superimposing the different codebooks that greatly simplifies the error analysis. Our codebook “nesting” is such that the “cloud center” codebook is the one all terminals will be decoding, i.e., the cooperative common codebook, to which we superimpose the non-cooperative common codebook (to be decoded by the two destinations but not by the other source) and finally we superimpose the non-cooperative private codebook (to be decoded at the intended receiver only). We shall see that although the destinations have to decode five messages, only 5 out of the possible $2^5 - 1 = 31$ error events matter.

2. We then perform Fourier Motzking elimination to obtain a region with only five types of rate bounds, as in the case of IFC without feedback. This was not immediately obvious since, with GF, each message is split into three parts, and not in two as for the case without feedback.

3. We also show that when cooperation is on sending the common information only, the achievable rate region cannot be enlarged if the sources are required to decode more information than they will eventually “relay” to the destinations.

4. We show how our region does reduce to the achievable regions of the channels subsumed by the IFC-GF model.

For the more general form of cooperation (to send both the common information and the private information):

1. We extend our previous achievable region so as to include cooperation on sending the private information too. Our achievable region uses superposition of several Gelfand-Pinsker binning codes. We propose a structured way of binning that reduces the number of rate constraints necessary to guarantee arbitrary small probability of error at the encoders.
Moreover, we propose to “jointly bin” the different codebooks, rather than perform several binning steps, one per codebook. Our proposed joint binning is similar in spirit to multiple description source coding.

2. The error analysis at the decoders could be very messy and lengthy with so many messages to decode. Our proposed method for error analysis leverages the way the codebooks are superimposed. The type of error analysis we use could be of interest in its own. In particular, with our encoding structure, we show that only 28 error events matter, out of the $2^8 - 1 = 255$ possible.

3. We show how our new region does reduce to the achievable regions of the channels subsumed by the IFC-GF model. Our region however does not give the largest possible achievable rate for the relay channel because it does not include Compress-and-Forward \cite{10}. Extensions of the current achievable region so as to include Compress-and-Forward are left for future work.

4. For the Gaussian channel, by means of a numerical example, we show that cooperation greatly increases the achievable rates with respect to the case without GF.

The rest of the paper is organized as follows. Section 2 introduces the channel model and the notation. Section 3 revises known results for IFC without feedback. Section 4 presents our achievable regions; in particular, subsection 4.1 describes the region where cooperation is on sending the common message only, and subsection 4.2 describes the region where cooperation is also on sending the private message. Section 5 evaluates the achievable region in subsection 4.1 for the Gaussian channel (the evaluation of the region in subsection 4.2 is postponed to Part-II of this paper where we will also compare it with an outer bound). Section 6 concludes Part I of this paper. All the proofs are in the Appendix.

## 2 Network Model and Definitions

Fig. 1 shows an IFC-GF with two source-destination pairs. It consists of a channel with two input alphabets ($\mathcal{X}_1, \mathcal{X}_2$), four output alphabets ($\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \mathcal{Y}_4$), and a

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\footnote{We however notice here that the performance loss due to Decode-and-Forward is less than within 1 bit for the Gaussian relay channel.}
transition probability $P_{Y_1 Y_2 Y_3 Y_4|X_1 X_2}$. We assume that all the alphabets are finite sets (the extension to continuous alphabets follows from standard argument [12]) and that the channel is memoryless. Source $u$, $u \in \{1, 2\}$, has a message $W_u$ for destination $u$. The messages $W_1$ and $W_2$ are independent and uniformly distributed over the set $\{1, \cdots, e^{n R_1}\} \times \{1, \cdots, e^{n R_2}\}$, where $n$ denotes the codeword length and $R_u$ the transmission rate for user $u$, $u \in \{1, 2\}$. At time $t$, $t \in \{1, \cdots, n\}$, source $u$ maps its message $W_u$ and its past channel observations $Y_{u t-1}$ into a channel input symbol

$$X_{u,t} = f_{u,t}^{(n)}(W_u, Y_{u t-1}^t) : \mathcal{X}_u \ni f_{u,t}^{(n)} : \{1, \cdots, e^{n R_u}\} \times \mathcal{Y}_{u t-1} \to \mathcal{X}_u.$$  

At time $n$, destination $u$ outputs the estimate of its intended message $\hat{W}_u$ based on all its channel observations $Y_{u n+1}^n$, i.e.,

$$\hat{W}_u = g^{(n)}_{u}(Y_{u n+2}^n) : \mathcal{Y}_{u n+2} \ni g^{(n)}_{u} : \{1, \cdots, e^{n R_u}\}.$$  

The capacity region is the closure of all rate pairs $(R_1, R_2)$ such that

$$\max_{u \in \{1, 2\}} \Pr[\hat{W}_u \neq W_u] \to 0 \text{ as } n \to \infty.$$  

### 2.1 Notation

The symbol $R_{xyz}$ is used to indicate the information rate “from source $x$ to destination $y$ with the help of $z$”, with $x \in \{1, 2\}$, $y \in \{0, 1, 2\}$ and $z \in \{c, n\}$. In
particular, $y = 0$ means that the message is decoded at both destinations (common message), $y = 1$ means that the message is decoded only at destination 1 (private message for user 1), and $y = 2$ means that the message is decoded only at destination 2 (private message for user 2); $z = c$ means that the message is sent cooperatively by both sources, and $z = n$ means that the message is sent non-cooperatively.

$T_\epsilon^{(n)}(P|S)$ indicates the set of length-$n$ sequences that are strongly $\epsilon$-typical with respect to the distribution $P$, conditioned on the sequences in $S$ [13].

3 IFC without feedback

In this section we briefly revise the best known inner and outer bound regions for an IFC without feedback. This will make the exposition of our coding strategy and outer bound easier. The channel transition probability of an IFC without feedback is

$$P_{Y_1 Y_2 Y_3 Y_4|X_1 X_2} = P_{Y_1 Y_2} P_{Y_3 Y_4|X_1 X_2},$$

that is, the GF signals $Y_1$ and $Y_2$ are independent of the inputs (for example $Y_1 = Y_2 = \emptyset$).

3.1 Inner Bound Region

The largest known achievable region for an IFC without feedback is due to Han and Kobayashi [19] and is as follows [6].

Class of Input Distributions: Consider a distribution from the class

$$P_{Q U_1 T_1 X_1 U_2 T_2 X_2 Y_3 Y_4} = P_{Q} P_{U_1 T_1 X_1 |Q} P_{U_2 T_2 X_2 |Q} P_{Y_3 Y_4|X_1 X_2}. \quad (1)$$

The channel transition probability $P_{Y_3 Y_4|X_1 X_2}$ is fixed, while the other factors in (1) can be varied.

Rate Splitting: The message $W_u \in \{1, \ldots, e^{nR_u}\}$, $u \in \{1, 2\}$, is split into two parts $(W_{u0n}, W_{uun})$: $W_{u0n} \in \{1, \ldots, e^{nR_{u0n}}\}$ is the common information decoded at both receivers while $W_{uun} \in \{1, \ldots, e^{nR_{uun}}\}$ is the private information decoded only at the intended receiver, with $R_u = R_{u0n} + R_{uun}$. Without feedback, all messages are sent non-cooperatively.

Codeword Generation: Consider a distribution in (1). Pick uniformly at random a length-$n$ sequences $Q^n$ from the typical set $T_\epsilon^{(n)}(P_Q)$. For the codeword $Q^n = q^n$, 11
pick uniformly at random $e^{nR_{10n}}$ length-$n$ sequences $U^n_t(k)$, $k \in \{1, \ldots, e^{nR_{10n}}\}$, from the typical set $T^{(n)}_e(P_{U_t|Q}|q^n)$. For each codeword $Q^n = q^n$ and each codeword $U^n_t(k) = u^n_t(k)$, pick uniformly at random $e^{nR_{11n}}$ length-$n$ sequences $T^n_1(m, k), m \in \{1, \ldots, e^{nR_{11n}}\}$, from the typical set $T^{(n)}_e(P_{T_1|QU_t}|q^n, u^n_t(k))$. For each $Q^n = q^n$, $U^n_t(k) = u^n_t(k)$ and $T^n_1(m, k) = t^n_1(m, k)$, choose uniformly at random a sequence $X^n_t(m, k)$ from the typical set $T^{(n)}_e(P_{X_t|QU_tT_1}|q^n, u^n_t(k), t^n_1(m, k))$.

The generation of the codebooks at source 2 proceeds similarly.

**Encoding:** In order to send the messages $W_u = (W_{u0n}, W_{uun})$, source $u$, $u \in \{1, 2\}$, transmits $X^n_u(W_{u0n}, W_{uun})$.

**Decoding:** Destination 1 decodes from $Y^n_3$ the triplet $(W_{10n}, W_{20n}, W_{11n})$ by searching for a unique pair $(i_1, j_1), j_1 \in \{1, \ldots, e^{nR_{10n}}\}$ and $i_1 \in \{1, \ldots, e^{nR_{11n}}\}$, and some index $j_2, j_2 \in \{1, \ldots, e^{nR_{20n}}\}$, such that

$$(U^n_t(j_1), T^n_1(i_1, j_1), U^n_2(j_2), Y^n_3) \in T^{(n)}_e(P_{U_1T_1U_2Y_3|Q}|Q^n),$$

where

$$P^{(\text{dec1})}_{U_1T_1U_2Y_3|Q} = \frac{\sum_{X_1T_2X_2} P_{X_1T_1|Q}P_{U_1T_1X_1}P_{U_2T_2X_2}P_{Y_3|X_1X_2}}{P_Q}$$

$$= P_{U_1T_1|Q}P_{U_2|Q}\left(\sum_{X_1X_2} P_{X_1|QU_1T_1}P_{X_2|QU_2}P_{Y_3|X_1X_2}\right).$$

If no pair $(i_1, j_1)$ is found, or more than one pair is found, the receiver sets $(i_1, j_1) = (1, 1)$; in this case we say that an error at destination 1 has occurred.

Decoding at destination 2 proceeds similarly.

**Error Analysis:** The error analysis can be found in [6]. The probability of error at destination 1 can be driven to zero if the rates $R_u = R_{u0n} + R_{uun}, u \in \{1, 2\}$, are such that

$$R_{11n} \leq I(Y_3 \wedge T_1|U_1, U_2, Q) \quad (2a)$$

$$R_{20n} + R_{11n} \leq I(Y_3 \wedge T_1, U_2|U_1, Q) \quad (2b)$$

$$R_{10n} + R_{11n} \leq I(Y_3 \wedge T_1, U_1|U_2, Q) \quad (2c)$$

$$R_{20n} + R_{10n} + R_{11n} \leq I(Y_3 \wedge T_1, U_1, U_2|Q), \quad (2d)$$

and similarly, the probability of error at destination 2 can be driven to zero if

$$R_{22n} \leq I(Y_4 \wedge T_2|U_1, U_2, Q) \quad (3a)$$

$$R_{10n} + R_{22n} \leq I(Y_4 \wedge T_2, U_1|U_2, Q) \quad (3b)$$

$$R_{20n} + R_{22n} \leq I(Y_4 \wedge T_2, U_2|U_1, Q) \quad (3c)$$

$$R_{20n} + R_{10n} + R_{22n} \leq I(Y_4 \wedge T_2, U_1, U_2|Q). \quad (3d)$$
Achievable Region: The region given by the intersection of (2) and (3) can be compactly expressed after Fourier-Motzkin elimination as [6]:

**Theorem 3.1 ([6]).** For any distribution in (7) the following region is achievable:

\[
\begin{align*}
R_1 &\leq (2c), \quad (4a) \\
R_2 &\leq (3c), \quad (4b) \\
R_1 + R_2 &\leq \min\{(2d) + (3a), (2a) + (3d), (2b) + (3b)\}, \quad (4c) \\
2R_1 + R_2 &\leq (2d) + (2a) + (3b), \quad (4d) \\
R_1 + 2R_2 &\leq (2b) + (3a) + (3d), \quad (4e)
\end{align*}
\]

Without loss of generality, one can set \(T_1 = X_1\) and \(T_2 = X_2\) in (4), and, by Caratheodory’s theorem, choose the auxiliary random variables \((Q, U_1, U_2)\) from alphabets with cardinality \(|Q| \leq 7, |U_1| \leq |X_1| + 4\) and \(|U_2| \leq |X_2| + 4\).

**Remark 3.2.** It was remarked in [6] that after Fourier-Motzkin elimination, the following bounds also appear:

\[
\begin{align*}
R_1 &\leq (2a) + (3b), \\
R_2 &\leq (3a) + (2b),
\end{align*}
\]

however they can be shown to be redundant. Intuitively, the reason is as follows: if

\[
I(Y_3 \land T_1 | U_1, U_2, Q) + I(Y_4 \land T_2, U_1 | U_2, Q) = (2a) + (3b) < (2c) = I(Y_3 \land T_1, U_1 | U_2, Q),
\]

then decoding at destination 2 constrains the rate of source 1 too much; in this case, destination 2 should not be required to decode the common information \(U_1\). Indeed, it can be shown [6] (see also Appendix A) that the rate points for which \((2a) + (3b) < R_1\), but otherwise satisfy all other rate constraints in (4), are contained in the sub-region of (4) obtained by choosing \(U_1 = \emptyset\); thus the constraint \(R_1 \leq (2a) + (3b)\) can be removed without enlarging the achievable region. A similar reasoning holds for the constraint \(R_2 \leq (3a) + (2b)\).


3.2 Outer Bounds

As shown in the second part of this paper, the outer bound techniques of [28] for the Gaussian channel, and the bound in [44] for the semi-deterministic channel, give:

**Theorem 3.3** ([28] and [44]). For any \( P_{UX_1X_2} = P_U P_{X_1|U} P_{X_2|U} \), the following region is an outer bound for a general IFC without feedback:

\[
\begin{align*}
R_1 &\leq I(X_1; Y_3|X_2, U), \\
R_2 &\leq I(X_2; Y_4|X_1, U), \\
R_1 + R_2 &\leq I(X_1; Y_3|Y'_4, X_2, U) + I(X_1, X_2; Y_4|U), \\
R_1 + R_2 &\leq I(X_2; Y_4|Y'_3, X_1, U) + I(X_1, X_2; Y_3|U),
\end{align*}
\]

(5a) (5b) (5c) (5d)

where \( P_{Y_3,Y'_4|X_1,X_2} \) (and similarly for \( P_{Y'_3,Y_4|X_1,X_2} \)) is such that \( P_{Y'_4|X_1,X_2} = P_{Y_4|X_1,X_2} \), i.e., same marginal distribution, but otherwise \( P_{Y_3,Y'_4|X_1,X_2} \) has arbitrary joint distribution.

Furthermore, if the channel is semi-deterministic as defined in [44], that is, if there exist deterministic functions \( f_3, f_4, v_{32}, v_{41} \) and noise random variables \( Z_3, Z_4 \) such that

\[
\begin{align*}
Y_3 & = f_3(X_1, v_{32}(X_2, Z_3)) \text{ where } f_3 \text{ is invertible given } X_1, \\
Y_4 & = f_4(X_2, v_{41}(X_1, Z_4)) \text{ where } f_4 \text{ is invertible given } X_2,
\end{align*}
\]

then

\[
\begin{align*}
R_1 + R_2 &\leq H(Y_3|\tilde{V}_{41}, U) + H(Y_4|\tilde{V}_{32}, U) \\
&\quad - H(\tilde{V}_{41}|X_1, U) - H(\tilde{V}_{32}|X_2, U) \\
2R_1 + R_2 &\leq H(Y_3|\tilde{V}_{41}, X_2, U) + H(Y_3|U) + H(Y_4|\tilde{V}_{32}, U) \\
&\quad - H(\tilde{V}_{41}|X_1, U) - 2H(\tilde{V}_{32}|X_2, U), \\
R_1 + 2R_2 &\leq H(Y_4|\tilde{V}_{32}, X_1, U) + H(Y_4|U) + H(Y_3|\tilde{V}_{41}, U) \\
&\quad - 2H(\tilde{V}_{41}|X_1, U) - H(\tilde{V}_{32}|X_2, U).
\end{align*}
\]

(5e) (5f) (5g)

where \( \tilde{V}_{41} \) and \( \tilde{V}_{32} \) are independent copies of \( V_{41} = v_{41}(X_1, Z_4) \) and \( V_{32} = v_{32}(X_2, Z_3) \) conditioned on the input \( (X_1, X_2) \).

**Remark 3.4.** Other outer bounds exist for specific channels. For example, for the Gaussian channel in weak interference, Kramer [28] developed a bound based
on the idea of enhancing the IFC and make it equivalent to a degraded broadcast channels. Shang et al. [52], simultaneously and independently of [33, 47], showed that for the Gaussian channel in very weak interference, it is sum-rate optimal to treat the interference as noise. Outer bounds based on the idea of dependance balance appeared in [17,37]. Extensions of these bounding techniques to non-Gaussian channels and/or their evaluation is not straightforward and it will not be attempted in this paper.

4 Inner Bounds

In this section we derive our achievable region. We divide the section into two parts. In the first part (Section 4.1), we propose a scheme where the sources cooperate by “beam-forming” part of the common messages. In this scheme, all messages are superimposed and thus we refer to it as superposition-only achievable region. In the second part (Section 4.2), we propose a scheme where the sources also cooperate on sending part of the private messages by using binning, or dirty paper coding. We refer to this schemes as superposition & binning achievable region. The superposition & binning achievable region includes the superposition-only achievable region as special case. We present them both to guide the reader into the two possible cooperation mechanisms.

4.1 Superposition-only achievable region

Class of Input Distributions: Consider a distribution from the class

\[ P_{QV_1U_1T_1X_1V_2U_2T_2X_2Y_1Y_2Y_3Y_4} = \]

that is, conditioned on \( Q \), the random variables \((V_1, U_1, T_1, X_1)\) generated at source 1 are independent of the random variables \((V_2, U_2, T_2, X_2)\) generated at source 2. The channel transition probability \( P_{Y_1Y_2Y_3Y_4|X_1X_2} \) is fixed, while the other factors in the distribution in (6) can be varied.

Rate Splitting and Transmission Strategy: The message \( W_u \in \{1, ..., e^{nR_u}\}, u \in \{1, 2\} \), is divided into three parts \((W_{u0c}, W_{u0n}, W_{uun})\): \( W_{u0c} \in \{1, ..., e^{nR_{u0c}}\} \) is the part of the common message sent cooperatively by the sources, \( W_{u0n} \in \{1, ..., e^{nR_{u0n}}\} \) is the part of the common message sent by source \( u \) alone (i.e., non-cooperatively), and \( W_{uun} \in \{1, ..., e^{nR_{uun}}\} \) is the private private message sent non-cooperatively, with \( R_u = R_{u0c} + R_{u0n} + R_{uun} \).
We propose that the cooperative common message sent by a source is decoded at the other source thanks to the generalized feedback. Then, the sources send both cooperative common messages to the receivers as in a virtual MIMO channels, thus realizing the gain of beamforming. This is possible by using regular block Markov superposition encoding [11] at the sources and backward decoding [48] at the destinations. In particular, transmission occurs over a frame of N slots of n channel uses each. Source 1 in slot $b$, $b \in \{1, \cdots , N-1\}$, has an estimate $W_{10c,b-1}$ of the cooperative common message sent by source 2 in the previous slot. Similarly, source 2 in slot $b$ has an estimate $W_{10c,b-1}$ of the cooperative common message sent by source 1 in the previous slot. The random variable $Q$ in (6) conveys the two cooperative common messages ($W_{10c}, W_{20c}$) from the previous time slot to the destinations. In slot $b$, $b \in \{1, \cdots , N-1\}$, the random variables $V_u$, $U_u$, and $T_u$ in (6), $u \in \{1, 2\}$, convey the new cooperative common message $W_{u0c,b}$, the new non-cooperative common message $W_{u0n,b}$, and the new non-cooperative private message $W_{uan,b}$ respectively.

By setting $V_1 = V_2 = \emptyset$, i.e., no cooperative common messages, our proposed encoding schemes reduces to the Han and Kobayashi [19] for the IFC without feedback in Th. [3, 1]. In this case, $Q$ acts as a simple time-sharing random variable.

**Codebook Generation:** Pick uniformly at random $e^{n(R_{10c}+R_{20c})}$ length-$n$ sequences $Q^n([i,j])$, $i \in \{1, \cdots , e^{nR_{10c}}\}$ and $j \in \{1, \cdots , e^{nR_{20c}}\}$, from the typical set $T_{e}^{(n)}(P_Q)$. For each $Q^n([i,j]) = q^n([i,j])$, pick uniformly at random $e^{nR_{10c}}$ length-$n$ sequences $V_1^n(k, [i,j])$, $k \in \{1, \cdots , e^{nR_{10c}}\}$, from the typical set $T_{e}^{(n)}(P_{V_1|Q}[q^n([i,j]))$. For each $Q^n([i,j]) = q^n([i,j])$ and $V_1^n(k, [i,j]) = v_1^n(k, [i,j])$, pick uniformly at random $e^{nR_{11n}}$ length-$n$ sequences $U_1^n(\ell, k, [i,j])$, $\ell \in \{1, \cdots , e^{nR_{11n}}\}$, from the typical set $T_{e}^{(n)}(P_{V_1|Q}[q^n([i,j]), v_1^n(k, [i,j])]$. For each $Q^n([i,j]) = q^n([i,j])$, $V_1^n(k, [i,j]) = v_1^n(k, [i,j])$, and $U_1^n(\ell, k, [i,j]) = u_1^n(\ell, k, [i,j])$, pick uniformly at random $e^{nR_{11n}}$ length-$n$ sequences $T_1^n(m, \ell, k, [i,j])$, $m \in \{1, \cdots , e^{nR_{11n}}\}$, from the typical set $T_{e}^{(n)}(P_{V_1|Q}[q^n([i,j]), v_1^n(k, [i,j]), u_1^n(\ell, k, [i,j])]$. For each $Q^n([i,j]) = q^n([i,j])$, $V_1^n(k, [i,j]) = v_1^n(k, [i,j])$, $U_1^n(\ell, k, [i,j]) = u_1^n(\ell, k, [i,j])$, and $T_1^n(m, \ell, k, [i,j]) = t_1^n(m, \ell, k, [i,j])$, pick uniformly at random one sequence $X_1^n(m, \ell, k, [i,j])$ from the typical set

$T_{e}^{(n)}(P_{X_1|QV_1U_1T_1|[q^n([i,j]), v_1^n(k, [i,j]), u_1^n(\ell, k, [i,j]), t_1^n(m, \ell, k, [i,j])])$.

The generation of the codebooks at source 2 is similar.

**Encoding:** In slot $b$, $b \in \{1, \cdots , N\}$, given the new message $W_{u,b} = (W_{u0c,b}, W_{u0n,b}, W_{uan,b})$
for \( u \in \{1, 2\} \), the transmitted codewords are

\[
\begin{align*}
X_1^n(W_{11n,b}, W_{10n,b}, W_{10c,b}, [W_{10c,b-1}; W_{20c,b-1}]), \\
X_2^n(W_{22n,b}, W_{20n,b}, W_{20c,b}, [W_{10c,b-1}; W_{20c,b-1}]),
\end{align*}
\]

with the “boundary” conditions \( W_{u0c,0} = W_{u0c,N} = 1 \), \( u = \{1, 2\} \), i.e., on the first slot of the frame there is no cooperative common information from a previous slot to relay, and on the last slot of the frame there is no new cooperative common information to send because there will not be a future slot to relay it. With this scheme, user \( u \in \{1, 2\} \) transmits at an actual rate \( R_u' = (1 - 1/N)R_u0c + R_u0n + R_uun < R_u \) (because no new cooperative common information is sent on the last slot). The rate \( R_u' \) can be made arbitrarily close to \( R_u \) by taking the frame length \( N \) to be sufficiently large.

Fig. 2 visualizes the proposed superposition coding scheme: an arrow to a codebook/random variable indicates that the codebook is superimposed to all the codebooks that precede it, and codebooks linked by a vertical line are conditionally independent given everything that precedes them. For example, codebook \( U_1 \) is superimposed to \( Q \) and \( V_1 \), and it is conditionally independent of any codebook with index 2 when conditioned on \( Q \).
Cooperation: In slot $b$, $b \in \{1, \cdots, N-1\}$, we can assume that the users’ estimate of the previous cooperative common messages is exact, that is, $W'_{10c,b-1} = W_{10c,b-1}$ and $W'_{20c,b-1} = W_{20c,b-1}$, since the total average probability of error at the receivers can be upper bounded by the sum of the decoding error probabilities at each step, under the assumption that no error propagation from the previous steps has occurred [48]. Next we describe how source 2 cooperates with source 1. Source 1 proceeds similarly.

Source 2 at the end of slot $b$ decodes user 1’s new cooperative common message $W_{10c,b}$ carried by $V^n_1$ from its channel output $Y^n_{2,b}$, knowing everything that was generated at source 2 at the beginning of the slot. Formally, at the end of slot $b$, $b \in \{1, \cdots, N-1\}$, source 2 has received $Y^n_{2,b}$ and looks for the unique index $i \in \{1, \ldots, e^{nR_{10c}}\}$ such that

$$(V^n_1(i, \cdots), Y^n_{2,b}) \in T^n(\epsilon | P_{V^n_1 Y^n_2 | Q X^n_2}),$$

where the dots indicate known message indices, where all that is known at source 2 is represented by $X_2 = (Q, V_2, U_2, T_2, X_2)$, and where

$$P_{V^n_1Y^n_2 | Q X^n_2} = \frac{\sum_{U_1,T_1,X_1} P_Q P_{V_1 U_1 T_1 X_1} P_{V_2 U_2 T_2 X_2} | Q P_{Y_2} | X_1 X_2}{P_Q P_{V_2 U_2 T_2 X_2}} = P_{V_1 | Q} \left( \sum_{X_1} P_{X_1 | Q V_1} P_{Y_2 | X_1 X_2} \right).$$

If none or more than one index $i$ is found, then source 2 sets $i = 1$; in this case we say that an error has occurred at source 2.

Error Analysis: By standard arguments [45] (see Appendix B), the probability of error at source 2 can be made as small as desired if

$$R_{10c} \leq I(V_1 \wedge Y_2 | Q, V_2, U_2, T_2, X_2).$$

(7a)

Decoding: The destinations wait until the last slot of the frame (i.e. slot $N$) has been received and then perform backward decoding. In slot $b \in \{N, \cdots, 1\}$ destination 1 looks for the unique triplet $(i_1, j_1, m_1) \in \{1, \ldots, e^{nR_{10c}}\} \times \{1, \ldots, e^{nR_{10c}}\} \times \cdots$
\{1, \ldots, e^{nR_{11n}}\} and some pair \((i_2, j_2) \in \{1, \ldots, e^{nR_{20c}}\} \times \{1, \ldots, e^{nR_{20n}}\}\) such that

\[
\left( Q^n([i_1, i_2]), V^n_1(W'_{10c,b}, [i_1, i_2]), U^n_1(j_1, W'_{10c,b}, [i_1, i_2]), T^n_1(m_1, j_1, W'_{10c,b}, [i_1, i_2]),
\right.

\[
V^n_2(W'_{20c,b}, [i_1, i_2]), U^n_2(j_2, W'_{20c,b}, [i_1, i_2]),
\]

\[
Y^n_{3,b} \in T^n_e(P_{QV_1U_1T_1V_2U_2Y_3}^{(dec)})
\]

where the pair \((W'_{10c,b}, W'_{20c,b})\) was decoded in the previous step \((W'_{u0c,b} = 1\) by assumption, hence \(W'_{u0c,b} = 1\) too) and where

\[
P_{QV_1U_1T_1V_2U_2Y_3}^{(dec)} = \sum_{X_1, T_2, X_2} P_QP_{V_1U_1T_1|Q}P_{V_2U_2T_2X_2}|Q_P_{Y_3|X_1X_2}
\]

\[
= P_QP_{V_1U_1T_1|Q}P_{V_2U_2|Q} \left( \sum_{X_1, X_2} P_{X_1|QV_1U_1T_1}P_{X_2|QV_2U_2}P_{Y_3|X_1X_2} \right).
\]

In words, destination 1 decodes the old cooperative common messages in \(Q^n\), the current non-cooperative common messages in \((U^n_1, U^n_2)\), and the current non-cooperative private message in \(T^n_1\), from its channel output \(Y^n_{3,b}\). The current cooperative common messages in \((V^n_1, V^n_2)\) are known from the decoding of slot \(b + 1\). The current non-cooperative private message of user 2 in \(T^n_3\) is treated as noise.

If none or more than one triplet \((i_1, j_1, m_1)\) is found, then destination 1 sets \((i_1, j_1, m_1) = (1, 1, 1)\); in this case we say that an error has occurred at destination 1.

**Error Analysis:** By standard arguments \([45]\) (see Appendix \([C]\), the probability of error at destination 1 can be made as small as desired if

\[
R_{11n} \leq I(Y_3 \wedge T_1|Q, V_1, V_2, U_1, U_2) \quad (7b)
\]

\[
R_{11n} + R_{20n} \leq I(Y_3 \wedge T_1, U_2|Q, V_1, V_2, U_1) \quad (7c)
\]

\[
R_{11n} + R_{10n} \leq I(Y_3 \wedge T_1, U_1|Q, V_1, V_2, U_2) \quad (7d)
\]

\[
R_{11n} + R_{10n} + R_{20n} \leq I(Y_3 \wedge T_1, U_1, U_2|Q, V_1, V_2) \quad (7e)
\]

\[
R_{11n} + R_{10n} + R_{20n} + (R_{20c} + R_{10c}) \leq I(Y_3 \wedge T_1, U_1, U_2, Q, V_1, V_2). \quad (7f)
\]

Notice that \((7b) \leq \min\{(7c), (7d)\} \leq \max\{(7c), (7d)\} \leq (7e) \leq (7f)\).

By similar arguments, the probability of error at source 1 can be made as small as desired if

\[
R_{20c} \leq I(V_2 \wedge Y_1|Q, V_1, U_1, T_1, X_1), \quad (8a)
\]
and the probability of error at destination 2 can be made as small as desired if

\[ R_{22n} \leq I(Y_4 \wedge T_2|Q, V_2, V_1, U_2, U_1) \]  \(8b\)

\[ R_{22n} + R_{10n} \leq I(Y_4 \wedge T_2, U_1|Q, V_2, V_1, U_2) \]  \(8c\)

\[ R_{20n} + R_{22n} \leq I(Y_4 \wedge T_2, U_2|Q, V_2, V_1, U_1) \]  \(8d\)

\[ R_{20n} + R_{22n} + R_{10n} \leq I(Y_4 \wedge T_2, U_2, U_1|Q, V_1, V_2) \]  \(8e\)

\[ R_{22n} + R_{20n} + R_{10n} + (R_{20c} + R_{10c}) \leq I(Y_4 \wedge T_2, U_1, U_2, Q, V_1, V_2). \]  \(8f\)

**Achievable region:** The intersection of the region in (7) with the region in (8) can be compactly expressed after Fourier-Motzkin elimination as follows:

**Theorem 4.1.** For any distribution in (6), the following region is achievable:

\[ R_1 \leq (7f) \]  \(9a\)

\[ R_1 \leq (7a) + (7d) \]  \(9b\)

\[ R_2 \leq (8f) \]  \(9c\)

\[ R_2 \leq (8a) + (8d) \]  \(9d\)

\[ R_1 + R_2 \leq (7f) + (8b) \]  \(9e\)

\[ R_1 + R_2 \leq (7b) + (8f) \]  \(9f\)

\[ R_1 + R_2 \leq (7a) + (8a) + (7c) + (8b) \]  \(9g\)

\[ R_1 + R_2 \leq (7a) + (8a) + (7b) + (8e) \]  \(9h\)

\[ R_1 + R_2 \leq (7a) + (8a) + (7c) + (8c) \]  \(9i\)

\[ 2R_1 + R_2 \leq (7a) + (7b) + (7c) + (8c) \]  \(9j\)

\[ 2R_1 + R_2 \leq 2 \cdot (7a) + (8a) + (7b) + (7e) + (8e) \]  \(9k\)

\[ R_1 + 2R_2 \leq (8a) + (7c) + (8b) + (8f) \]  \(9l\)

\[ R_1 + 2R_2 \leq (7a) + 2 \cdot (8a) + (7c) + (8b) + (8e) \]  \(9m\)

*Without loss of generality, one can take* \(X_1 = T_1\) *and* \(X_2 = T_2\) *in* \(9\).*

We remark here that the proposed structured way of superimposing the codebooks greatly simplifies the error analysis. Our codebook “nesting”, in fact, is such that the “cloud center” codebook is the one all terminals will be decoding, i.e., the cooperative common codebook, to which we superimposed the non-cooperative common codebook (to be decoded by the two destinations but not by the other source) and finally we superimposed the non-cooperative private codebook (to be decoded at the intended receiver only). As a consequence, although a
destination has to decode five messages, only 5 out of the possible $2^5 - 1 = 31$
error events matter (see (7) for destination 1 and (8) for destination 2).

**Remark 4.2.** The following rate constraints also appear after Fourier-Motzkin
elimination:

\[
R_1 \leq (7a) + (7b) + (8c), \\
R_2 \leq (8a) + (8b) + (7c).
\]

(10) 
(11)

It can be shown (see Appendix A) that (10) and (11) can be removed without
enlarging the region in (9). The intuitive argument is as for the channel without
feedback in Remark 3.2.

The achievable region in (9) subsumes achievable regions for other multiuser
channels. For example:

1. By setting $(7a) = (8a) = 0$ in (9), we get the achievable region for an IFC
   **without feedback** in [4]. In fact, (9e) is redundant because of (9g); (9f)
is redundant because of (9h); (9i) is redundant because of (9k); and (9j)
is redundant because of (9m). With $(7a) = (8a) = 0$, the region in (9) is
unchanged if we set $V_1 = V_2 = Q$ (since the random variables $Q$, $V_1$ and $V_2$
always appear together).

2. By setting $Y_1 = Y_3$ and $Y_2 = Y_4$ we obtain an IFC **with output feedback**
as studied in [24,25,43]. In particular, the region in [24, eq.(18)-(29)] has
the same codebook structure of our Th. 4.1. However, they differ in the
encoding and decoding of the messages in $Q$. In our achievable region, the
sources repeat in $Q$ the whole past cooperative common messages. In [24],
the sources repeat in $Q$ a quantized version of the past cooperative common
message indices. In principle, thus the encoding in [24] is more general.
However, in our achievable region, the destinations decode $Q$ jointly with
all other messages. In [24], $Q$ is decoded first by treating all the rest as
noise, and then all the other messages are jointly decoded. It is thus not
clear a priori whether the rate-saving due to sending a quantized version of
$Q$ are wiped out by the low rate imposed by decoding $Q$ first. A formal
comparison among the two regions is however difficult, since the two could
appear different but be actually the same when the union over all possible
input distributions is taken.
By setting (7a) = ∞, and (8a) = 0 in (9), we obtained a **cognitive interference channel**, in which source 2 knows the message of source 1, while source is unaware of the messages sent by source 2. The cognitive channel has also been referred to as **IFC with degraded message set** [51] and as **IFC with unidirectional cooperation** [31].

With (7a) = ∞, and (8a) = 0 (and with $U_1 = V_1 = V_2 = Q$ without loss of generality) the achievable region in (9) reduces to:

$$
R_1 \leq (7f) \quad (12a)
$$

$$
R_2 \leq (8d) \quad (12b)
$$

$$
R_1 + R_2 \leq (7f) + (8b) \quad (12c)
$$

$$
R_1 + R_2 \leq (7b) + (8f) \quad (12d)
$$

$$
R_1 + 2R_2 \leq (7c) + (8b) + (8f) \quad (12e)
$$

The region in (12) is only a subset of the best known achievable region for a cognitive IFC by Rini et al [38] because the region in (9) does not use binning.

By setting (7a) = (8a) = ∞ (and with $U_1 = V_1 = U_2 = V_2 = Q$ without loss of generality) in (9), we obtain the achievable region with superposition-only for a **broadcast channel** (BC), namely

$$
R_1 \leq (7f) \quad (13a)
$$

$$
R_2 \leq (8f) \quad (13b)
$$

$$
R_1 + R_2 \leq (7f) + (8b) \quad (13c)
$$

$$
R_1 + R_2 \leq (7b) + (8f) \quad (13d)
$$

The region in (13) is only a subset of the largest known achievable region for a general BC by Marton [32] because the region in (9) does not use binning. The region in (13) is however optimal for the case of “more capable BC channels” [16] and for the case of “BC with degraded message set” [23].

By setting $Y_3 = Y_4 = Y$ (and thus $T_1 = T_2 = \emptyset$ without loss of generality) we obtain the achievable region for a **multiple access channel with GF** [48]. With $Y_3 = Y_4 = Y$ and $T_1 = T_2 = \emptyset$, we have that (7b) = (8b) = 0, (7d) = (8c), (7c) = (8d), (7e) = (8c), and (7f) = (8f), and thus the region
in (9) reduces to

\[ R_1 \leq (7a) + (7d) \]  \hspace{1cm} (14a)  
\[ R_2 \leq (8a) + (7c) \]  \hspace{1cm} (14b)  
\[ R_1 + R_2 \leq (7f) \]  \hspace{1cm} (14c)  
\[ R_1 + R_2 \leq (7a) + (8a) + (7c) \]  \hspace{1cm} (14d)  

as derived in [48].

6. The region in (14) is for a MAC-GF “without common message”. The case with common message, that is, a message \( W_0 \) available at both sources and to be decoded at both destinations, can be easily incorporated by having the codebook/random variable \( Q \) also carry the common message \( W_0 \). If the rate of the common message \( W_0 \) is \( R_0 \), the region in (14) must be modified as follows: the rate constraint (14c) becomes

\[ R_0 + R_1 + R_2 \leq (7f). \]

This trick (i.e., the random variable \( Q \) carries the common message) can be used whenever a common message has to be included in the IFC-GF setting. In particular, with a common message, our Th. 4.1 must be modified as follows: the left hand side of the inequalities in (7f) and (8f) must also include the rate of the common message.

7. Further setting \( R_2 = 0 \) in (14) gives

\[ R_1 \leq \min \{ (7a) + (7d), (7f) \}, \]

which is the achievable rate for a full-duplex relay channel with partial decode and forward. The above rate does not include the case of Compress-and-Forward [10].

8. Channels with conferencing encoders are also a special case of GF. A two-source conferencing model [48] assumes that there are two non-interferring, noise-free channels of finite capacity between the communicating nodes, one for each direction of communication. Let \( C_{ij} \) be the capacity of the conferencing channel from node \( j \) to node \( i \). Following [29] and with some abuse notation, the conferencing model is captured as follows. Let the inputs be \( X_1 = [F_1; X_1] \) and \( X_2 = [F_2; X_2] \), where \( F_1 \) and \( F_2 \) have alphabet
sizes $\log(C_{12})$ and $\log(C_{21})$, respectively. Further set $Y_1 = F_2$ and $Y_2 = F_1$ and define the channel transition probability to be

$$P_{Y_1,Y_2,Y_3,Y_4|F_1;X_1,[F_2;X_2]} = P_{Y_3,Y_4|X_1,X_2} 1\{Y_1=F_2\} 1\{Y_2=F_1\}. $$

In this model, the choice $V_1 = F_1$ and $V_2 = F_2$ gives (7a) = $C_{21}$ and (8a) = $C_{12}$ is capacity achieving [48].

**Remark 4.3.** In [35] it is show that the region in Th. 4.1 is sum-rate optimal to within 18 bits for Gaussian channels with independent noises and symmetric cooperation links when the gain of the cooperation links are smaller than the gain of the interfering links.

In the same work, it is show that when the gain of the cooperation links are larger than the gain of the interfering links, then the transmitters should decode more information from their received generalized feedback signal than they will actually use for cooperation. In our setting this amounts to: at the end of slot $b$, $b \in \{1, \ldots, N-1\}$, source 1 looks for a unique index $i \in \{1, \ldots, e^{nR_{20n}}\}$ and some index $j \in \{1, \ldots, e^{nR_{20c}}\}$ such that the sequences

$$(V_2^n(i, [\cdots]), U_2^n(j, i, [\cdots]), Y_1^n)$$

$$\in T_e(n)(P_{V_2,U_2,Y_1,Q,X_1}^{(enc)}|X^n_1),$$

where the dots indicate known message indices, where all that is known at source 2 is represented by

$$X_1 = (Q, V_1, U_1, T_1, X_1),$$

and where

$$P_{V_2,U_2,Y_1|Q,X_1}^{(enc)} = \sum_{T_2,X_2} P_{Q}P_{V_1,U_1,T_1,X_1|Q}P_{V_2,U_2,T_2,X_2|Q}P_{Y_1|X_1,X_2}$$

$$= P_{V_2,U_2|Q} \left( \sum_{X_2} P_{X_2|Q,V_2,U_2}P_{Y_1|X_1,X_2} \right).$$

Decoding is successful with high probability if

$$R_{20n} + R_{20c} \leq I(V_2, U_2 \land Y_1|Q, V_1, U_1, T_1, X_1). \quad (15)$$

Notice that the constraint in (15) allows $R_{20c} \leq I(V_2, U_2 \land Y_1|Q, V_1, U_1, T_1, X_1)$ while the constraint in (8a) only allowed for $R_{20c} \leq I(V_2 \land Y_1|Q, V_1, U_1, T_1, X_1)$. However, the constraint in (15) constrains $R_{20n}$ to satisfies $R_{20n} \leq I(V_2, U_2 \land
while the constraint in (8a) does not constrain \( R_{20c} \) at all. It is not clear at priori which cooperation strategy is better, i.e., whether (8a) or (15). We will show next—as intuition suggests—that the two are equivalent, that is, with superposition-only, the sources should relay to the destinations all the common information they have acquired through the generalized feedback.

**Corollary 4.4.** After FM elimination of the regions in (7) and (8) with (8a) replaced by (15), we get

\[
R_1 \leq \min\{ (7f), (7a) + (7d) \}
\]

\[
R_2 \leq \min\{ (8f), (15) + (8b) \}
\]

\[
R_1 + R_2 \leq \min\{ (7f) + (8b), (7a) + (15) + (7b) + (8c) \}
\]

\[
2R_1 + R_2 \leq (7a) + (7b) + (7f) + (8c).
\]

It can be easily verified that the region in (16) is the same as region (9) computed for \( P_{QV_1U_1T_1X_1V_2U_2T_2X_2} \) where \( U'_2 = \emptyset \) and \( V'_2 = (V_2, U_2) \) (i.e., in the case of no feedback this choice corresponds to sending only private information for user 2). Hence, requiring source 1 to decode more information than actually used for cooperation does not enlarge the achievable region in (9).

Along the same line of reasoning, one could also require that at the end of slot \( b, b \in \{1, \cdots, N - 1\} \), source 1 looks for a unique indices \( i \in \{1, \cdots, e^{nR_{20c}}\} \), and some pair of indices \( j \in \{1, \cdots, e^{nR_{20n}}\} \) and \( k \in \{1, \cdots, e^{nR_{22n}}\} \), such that the sequences

\[
(V^n_2(i, \cdots)), U^n_2(j, i, \cdots), T^n_2(k, j, i, \cdots), Y^n_{1,b})
\]

\[
\in T_e^{(n)}(P^{(\text{enc1})}_{V_2U_2T_2Y_1|QX_1} | X^n_1),
\]

where

\[
P^{(\text{enc1})}_{V_2U_2T_2Y_1|QX_1} = P_{V_2U_2T_2|Q} \left( \sum_{X_2} P_{X_2|QV_2U_2T_2} P_{Y_1|X_1X_2} \right).
\]

Decoding is successful if

\[
R_2 = R_{22n} + R_{20n} + R_{20c} \leq I(T_2, U_2, V_2 \land Y_1|Q, V_1, U_1, T_1, X_1).
\]

The constraint in (17) can be too restrictive for rate \( R_2 \) when the cooperation link is weak. Indeed, consider the extreme case where \( Y_1 \) is independent of everything else (i.e., unilateral cooperation), then the constraint in (17) implies \( R_2 = 0 \), which can be easily beaten by ignoring the generalized feedback.
4.2 Superposition & binning achievable region

Because the achievable region with superposition-only does not reduce to the largest known achievable region when the IFC-GF reduces to a cognitive channel or a broadcast channel, we now introduce binning in the superposition only achievable scheme. In this new achievable scheme, the sources also cooperate in sending part of the private messages.

Each message is divided into four parts: two common messages and two private messages. The sources cooperate in sending one part of the common messages and one part of the private messages. Communication again proceeds on a frame on \( N \) slots. In any given slot, the sources decode the cooperative messages from their GF signal; these message are then relayed in the next slot. Since the common messages are decoded at both destinations, the sources cooperate by forwarding the cooperative common messages to the destinations as in a virtual MIMO channel. On the other hand, the private messages are decoded at the intended destination only and treated as noise at the non-indended destination. In this case, a source treats the other source’s cooperative private message as “non-causally know interference” in the next slot. Cooperation is then in the form of repetition/forwarding for one destination and pre-coding/binning for the other destination. The details of our proposed superposition & binning scheme are as follows.

**Class of Input Distributions:** Consider a distribution from the class

\[
P_{QV_1U_1T_1S_1Z_1X_1V_2U_2T_2S_2Z_2X_2Y_1Y_2Y_3Y_4}
\]

that is, conditioned on \((Q,S_1,S_2)\), the random variables \((V_1, U_1, T_1, Z_1, X_1)\) generated at source 1 are independent of the random variables \((V_2, U_2, T_2, Z_2, X_2)\) generated at source 2. The channel transition probability \(P_{Y_1Y_2Y_3|X_1X_2}\) is fixed, while the other factors in the distribution in (18) can be varied.

The random variables \(S_u\) and \(Z_u\), \(u = \{1, 2\}\), which did not appear in (6), convey the previous and the new, respectively, cooperative private information in a block-Markov encoding scheme.

**Codebook Generation:** From the input distribution in (18), compute the marginals \(P_{V_1U_1T_1|Q}\) and \(P_{V_2U_2T_2|Q}\) (i.e., drop the dependance on \((S_1, S_2)\)), and construct codebooks \((Q^n, V_1^n, U_1^n, T_1^n, V_2^n, U_2^n, T_2^n)\) as described in Section 4.1.

From the input distribution in (18), compute the marginals \(P_{S_1|Q}\) and \(P_{S_2|Q}\). Conditioned on each \(Q^n([i, j]) = q^n([i, j])\), pick uniformly at random length-\(n\)
sequences \( S_1^n(k_1, [i, j]) \) from the typical set \( T^n_{\epsilon}(P_{S_1|Q} | q^n([i, j]) \)). We shall specify later the range of the index \( k \) in \( S_1^n(k_1, [i, j]) \). Similarly, generate a codebook \( S_2^n(k_2, [i, j]) \) by picking uniformly at random length-\( n \) sequences from the typical set \( T^n_{\epsilon}(P_{S_2|Q} | q^n([i, j]) \)). The range of the index \( k_2 \) will be specified later.

From the input distribution in (18), compute the the marginal \( P_{Z_1^1|QS_1V_1} \). For each triplet \( (Q^n([i, j]), S_1^n(k_1, [i, j]), V_1^n(\ell_1, [i, j])) = (q^n([i, j]), s_1^n(k_1, [i, j]), v_1^n(\ell_1, [i, j])) \), generate a codebook \( Z_1^n(m_1, \ell_1, k_1, [i, j]) \) by picking uniformly at random length-\( n \) sequences from the typical set \( T^n_{\epsilon}(P_{Z_1|QS_1V_1} | q^n([i, j]) \), \( s_1^n(k_1, [i, j]), v_1^n(\ell_1, [i, j])) \). The range of the indices \( m_1 \) will be specified later. Similarly, but with the role of the users swapped, construct a codebook \( Z_2^n(m_2, \ell_2, k_2, [i, j]) \).

Finally, for each set of codewords \( (Q^n, S_1^n, S_2^n, V_1^n, V_2^n, U_1^n, T_1^n, Z_1^n) = (q^n, s_1^n, s_2^n, v_1^n, v_2^n, u_1^n, t_1^n, z_1^n) \) (we omit here the list of indices for sake of space) pick uniformly at random one sequence \( X_1^n \) from the typical set \( T^n_{\epsilon}(P_{X_1|QS_1S_2V_1U_1T_1Z_1} | q^n, s_1^n, s_2^n, v_1^n, v_2^n, u_1^n, t_1^n, z_1^n) \). Similarly, but with the role of the users swapped, construct a codebook \( X_2^n \).

Fig. 3 visualizes the proposed codebook generation (the convention is the same used for Fig. 2). The class of input distributions in (18) however allows for codebooks as depicted in Fig. 4. With binning we will force a codebook generated as in Fig. 3 to look like a codebook generated as in Fig. 4.

In order to complete the generalization of the scheme in Section 4.1 we must
Figure 4: A visual representation of the possible codebooks for the superposition & binning achievable scheme.

Consider each index $W_{xyz}$ (with the exception of the pair $[W_{10c,b-1}, W_{20c,b-1}]$ in $Q^n$) as a pair of indices $[W_{xyz}, B_{xyz}]$, where $W_{xyz} \in \{1, \ldots, e^{nR_{xyz}}\}$ represents a message index, and $B_{xyz} \in \{1, \ldots, e^{nR_{xyz}}\}$ represents a “bin index”. Notice that each bin index $B_{xyz}$ has the same subscript of the corresponding message index $W_{xyz}$ and has rate $R_{xyz}'$ (i.e., notice the prime in the superscript). The only exception is for $S_1$ and $S_2$ where the rate of the bin index is indicated with a double prime as a superscript, that is, $R_{11c}'$ and $R_{22c}''$, so as not to confuse them with $R_{11c}'$ in $Z_1$ and $R_{22c}''$ in $Z_2$.

**Encoding:** We can assume correct decoding of the message indices at the sources’ side at the end of slot $b - 1$ (no error propagation), i.e.,

- $W_{10c,b-1}' = W_{10c,b-1}''$ (carried by $V_1^n$ and to be repeated in $Q^n$),
- $W_{11c,b-1}'' = W_{11c,b-1}''$ (carried by $Z_1^n$ and to be repeated in $S_1^n$),
- $W_{20c,b-1}'' = W_{20c,b-1}''$ (carried by $V_2^n$ and to be repeated in $Q^n$),
- $W_{21c,b-1}'' = W_{21c,b-1}''$ (carried by $Z_2^n$ and to be repeated in $S_1^n$),

since the total average probability of error can be upper bounded by the sum of the decoding error probabilities at each step, under the assumption that no error propagation from the previous steps has occurred [6, 54].

The encoding process at the beginning of slot $b$ consists of the following binning and superposition steps, whose purpose is to allow the most general possible class of input distributions:

- **Binning codebooks $S_1^n$ and $S_2^n$ against each other.** At the beginning of slot $b$, given the past messages $(W_{10c,b-1}, W_{20c,b-1}, W_{11c,b-1}, W_{22c,b-1})$, source 1
tries to find a pair \((b_{11c,b-1}, b_{22c,b-1})\) such that

\[
\begin{align*}
&S_1^n([W_{11c,b-1}, b_{11c,b-1}], [\cdots]), \\
&S_2^n([W_{22c,b-1}, b_{22c,b-1}], [\cdots])
\end{align*}
\]

\[\in T^n_\epsilon(P_{S_1S_2}Q^n|Q^n),\]

where the dots are in place of the known messages \([W_{10c,b-1}, W_{20c,b-1}]\) from the previous slot. If more than one pair is found, then source 1 chooses one at random. If the search fails, source 1 sets \((b_{11c,b-1}, b_{22c,b-1}) = (1, 1)\); in this case we say that an error has occurred at source 1.

Error analysis. The codewords \((Q^n, S_1^n, S_2^n)\) were sampled in an i.i.d. fashion from the distribution \(P_{Q_1Q_2} = P_Q P_{S_1} P_{S_2Q}.\) The encoding process “forces” them to actually look as if they were sampled in an i.i.d. fashion from the distribution \(P_Q P_{S_1} P_{S_2Q}.\) For that to be feasible with high probability we must have (see Appendix D)

\[
R_{11c}'' + R_{22c}'' \geq I(S_1 \land S_2|Q). \tag{19}
\]

This encoding step is the same as the encoding in Marton’s achievable region for a general two-user broadcast channel with common message [32].

This first encoding step is run in parallel at both sources, so that the two sources have the same set of past cooperative messages \((W_{10c,b-1}, W_{20c,b-1}, W_{11c,b-1}, W_{22c,b-1})\), and of bin indices \((B_{11c,b-1}, B_{22c,b-1})\). This “common knowledge” furnishes the basis for cooperation in slot \(b\), where

\[
S_2^n([W_{22c,b-1}, B_{22c,b-1}], [W_{10c,b-1}, W_{20c,b-1}])
\]

can be treated as “non-causally known interference” at source 1, and

\[
S_1^n([W_{11c,b-1}, B_{11c,b-1}], [W_{10c,b-1}, W_{20c,b-1}])
\]

can be treated as “non-causally known interference” at source 2.

- **Joint conditional binning:** Given the new messages \((W_{10c,b}, W_{10n,b}, W_{11n,b})\), source 1 tries to find a set of bin indices \((b_{10c,b}, b_{10n,b}, b_{11n,b})\) such that

\[
\begin{align*}
&V_1^n([W_{10c,b}, b_{10c,b}], [\cdots]), \\
&U_1^n([W_{10n,b}, b_{10n,b}], [W_{10c,b}, b_{10c,b}], [\cdots]), \\
&T_1^n([W_{11n,b}, b_{11n,b}], [W_{10n,b}, b_{10n,b}], [W_{10c,b}, b_{10c,b}], [\cdots])
\end{align*}
\]

\[\in T^n_\epsilon(P_{V_1U_1T_1}Q^n|Q^n, S_1^n, S_2^n),\]
where the dots are in place of the known previous message $[W_{1c,b-1}, W_{2b-1}]$. If more than one triplet is found, then source 1 chooses one at random. If the search fails, source 1 sets $(b_{10c,b}, b_{10n,b}, b_{11n,b}) = (1, 1, 1)$; in this case we say that an error has occurred at source 1.

This encoding step is a generalization of the “sequential binning” idea introduced in [30]. Here, instead of doing several sequential binning steps, we bin all the codewords at once.

Error analysis. The triplet $(Q^n, S^n_1, S^n_2)$ found in the previous encoding step, appears jointly typical according to $P_{QS_1S_2}$. The set of codewords $(Q^n, V^n_1, U^n_1, T^n_1)$ is jointly typical according to $P_{QS_1U_1T_1}$ by construction. However, codewords $(V^n_1, U^n_1, T^n_1)$ and $(S^n_1, S^n_2)$ were generated independently conditioned on $Q^n$. The purpose of this binning step is to make the whole set $(Q^n, S^n_1, S^n_2, V^n_1, U^n_1, T^n_1)$ look jointly typical according to $P_{QS_1S_2V_1U_1T_1}$. By standard arguments, similar to those used in Multiple Description Coding [36] (see Appendix E), the joint binning step is successful with arbitrary high probability if

\[
R'_{10c} \geq I(V_1 \land S_1, S_2|Q) \quad (20a)
\]
\[
R'_{10n} + R'_{10c} \geq I(U_1, V_1 \land S_1, S_2|Q) \quad (20b)
\]
\[
R'_{11n} + R'_{10n} + R'_{10c} \geq I(V_1, U_1, T_1 \land S_1, S_2|Q). \quad (20c)
\]

The rate constraint in (20a) can be understood as follows. After the first (successful) binning step, the codewords $(Q^n, S^n_1, S^n_2, V^n_1)$ look as if they were sampled from the distribution $P_QP_{S_1S_2}Q_{P_{V_1|Q}}$. The encoding step requires them to look as if they were sampled from the distribution $P_QP_{S_1S_2}Q_{P_{V_1|Q S_1S_2}}$; hence we need to be able to search among an exponential (in $n$) number of codewords $V^n_1$, whose exponent must be at least $H(V_1|Q) - H(V_1|Q S_1S_2) = I(V_1 \land S_1, S_2|Q)$. The rate constraints in (20b) and in (20c) have a similar interpretation.

- **Final binning step:** Given the new messages $W_{11c,b}$, source 1 tries to find a bin index $b_{11c,b}$ such that

\[
\left( Q^n(\cdots), S^n_1(\cdots), S^n_2(\cdots), V^n_1(\cdots), U^n_1(\cdots), T^n_1(\cdots), \\
Z^n_1([W_{11c,b}, b_{11c,b}], \cdots) \right) 
\in T^n_c(P_{Z_1|QS_1S_2V_1U_1T_1}Q^n, S^n_1, S^n_2, V^n_1, U^n_1, T^n_1),
\]

30
where the dots are in place of the know messages and bin indices. If more than one index is found, then source 1 chooses one at random. If the search fails, source 1 sets $b_{11,c,b} = 1$; in this case we say that an error has occurred at source 1.

Error analysis. The set $(Q^n, S^n_1, S^n_2, V^n_1, U^n_1, T^n_1)$ found from the previous encoding steps, appears jointly typical according to $P_{QS_1 S_2 V_1 U_1 T_1}$. By construction, $(Q^n, S^n_1, V^n_1, Z^n_1)$ are jointly typical according to $P_{QS_1 V_1 Z_1}$, that is, conditioned on $(Q, S_1, V_1)$. $Z_1$ is independent of $(S_2, U_1, T_1)$. The purpose of this binning step is to make the whole set jointly typical according to $P_{Q S_1 S_2 V_1 U_1 T_1 Z_1}$. By standard arguments (see Appendix E), this binning step is successful with arbitrary high probability if

$$R_{11c}' \geq I(Z_1 \wedge S_2, U_1, T_1 | Q, S_1, V_1).$$  \hfill (20d)

• Finally, source 1 sends a codeword $X^n_1$ that is jointly typical with all the sequences found in the previous binning steps.

Encoding at source 2 proceeds similarly.

**Cooperation:** If the encoding steps are successful at both sources, all the transmitted and received signals are jointly typical according to the distribution in (18).

At the end of slot $b$, source 1 knows all the messages generated at source 1 at the beginning of the slot, including $(Q^n, S^n_2)$ (because we can assume successfully decoding of all past cooperative messages). Source 1 then searches for a unique pair of codeword indices $(i, j)$ and some bin index $(b_i, b_j)$ such that

$$\left( V^n_2([j, b_j], \cdots), Z^n_2([i, b_i], \cdots), Y^n_{1i,j} \right) 
\in T^n_{\epsilon} (P_{V_2 Z_2 Y_1 | X_1} | X^n_1),$$

where all that is known at source 1 is represented by

$$X_1 = (Q, S_1, S_2, Z_1, V_1, U_1, T_1, X_1),$$

where the dots indicate known old cooperative messages and related bin indices, and where

$$P_{V_2 Z_2 Y_1 | X_1}^{(enc)} = \frac{\sum_{U_2, T_2, X_2} P_{Q S_1 S_2 V_1 U_1 T_1 Z_1 X_1} (Q S_1 S_2 P_{V_2 U_2 T_2 Z_2 X_2|Q S_1 S_2} P_{V_1|X_1 X_2})}{P_{Q S_1 S_2 V_1 U_1 T_1 Z_1 X_1}}$$

$$= P_{V_2 Z_2|Q S_1 S_2} \left( \sum_{X_2} P_{X_2|Q S_1 S_2 V_2 Z_2} P_{Y_1|X_1 X_2} \right)$$

31
If the search fails, source 1 sets \((i, j) = (1, 1);\) in this case we say that an error has occurred at source 1.

Error analysis. From the point of view of source 1, this decoding step is equivalent to decoding the two codewords \(V_n^2([W_{20c,b}, B_{20c,b}], \ldots)\) and \(Z_n^2([W_{22c,b}, B_{22c,b}], \ldots)\) in a MAC-like channel with output \(Y_{11,b}\) and state \(X_{11}\) known at the receiver only, similar in spirit to [50]. Decoding at source 1 is depicted in Fig. 5 where the variables circled in blue are to be decoded, and the ones circled in red are treated as noise (the rest is known). By standard arguments (see Appendix F), decoding is correct with high probability if

\[
R_{Z_2} \leq I(Z_2 \wedge Y_1|X_1, V_2) + I(Z_2 \wedge S_1|Q, S_2, V_2) \tag{21a}
\]

\[
R_{V_2} + R_{Z_2} \leq I(V_2, Z_2 \wedge Y_1|X_1) + I(Z_2 \wedge S_1|Q, S_2, V_2) + I(V_2 \wedge S_1, S_2|Q), \tag{21b}
\]

with \(R_{Z_2} \equiv (R_{22c} + R'_{22c})\) and \(R_{V_2} \equiv (R_{20c} + R'_{20c}).\) Intuitively, the constraints in (21) are a consequence of the following observations. Conditioned on \((Q, S_1, S_2),\) a wrong \(V_2\) looks sampled from \(P_{V_2|Q S_1 S_2}\) but it was actually sampled from \(P_{V_2|Q};\) this accounts for the term \(I(V_2 \wedge S_1, S_2|Q).\) Similarly, conditioned on \((Q, S_1, S_2),\) a wrong \(Z_2\) looks sampled from \(P_{Z_2|Q S_1 S_2 V_2}\) but it was actually sampled from \(P_{Z_2|Q S_2 V_2};\) this accounts for the term \(I(Z_2 \wedge S_1|Q, S_2, V_2).\)
The inequalities in (21) generalize the one in (7a), and reduce to (7a) when $S_1 = Z_1 = S_2 = Z_2 = Q$.

Decoding: The receivers wait until the last slot has been received, and then proceed to decode by using backward decoding. We can assume that when decoding the information sent in slot $b$, the decoding of the information sent in the slots $b+1, \ldots, N$ was successful [6,48]. When decoding slot $b$, destination 1 knows the current cooperative messages $(W_{10c,b}, W_{20c,b}, W_{11c,b})$ carried by $(V_1, Z_1, V_2)$, and tries to decode the previous cooperative common messages $(W_{10c,b-1}, W_{20c,b-1})$ in $Q$, the previous cooperative private message $W_{11c,b-1}$ in $S_1$, and the current non-cooperative messages $W_{10n,b}$ in $U_1$, $W_{20n,b}$ in $U_2$, and $W_{11n,b}$ in $T_1$. Decoding at destination 1 is depicted in Fig. 6 where the variables circled in blue are those whose message index is known (but the bin index is not), and the ones circled in red are treated as noise; the rest is to be decoded.

Formally, destination 1 tries to find a unique set of message indices $(q_1, q_2, s_1, u_1, u_2, t_1)$ and some bin indices $(b_{v_1}, b_{v_2}, b_{z_1}, b_{u_1}, b_{u_2}, b_{t_1})$ such that

\[
\left( Q^n([q_1, q_2]), S^n_1([s_1, b_{s_1}], [q_1, q_2]), V^n_1([1, b_{v_1}], [q_1, q_2]), Z^n_1([1, b_{z_1}], [s_1, b_{s_1}], [1, b_{v_1}], [q_1, q_2]), U^n_1([u_1, b_{u_1}], [1, b_{v_1}], [q_1, q_2]), T^n_1([t_1, b_{t_1}], [u_1, b_{u_1}], [1, b_{v_1}], [q_1, q_2]), V^n_2([1, b_{v_2}], [q_1, q_2]), U^n_2([u_2, b_{u_2}], [1, b_{v_2}], [q_1, q_2]), Y^n_{3,b} \right) \in T^n_{\text{dec}}(P^{(\text{dec})}_{Q^nS_1V^nU^nT^nZ^nV^nU^nY^n3})
\]

where

\[
P^{(\text{dec})}_{Q^nS_1V^nU^nT^nZ^nV^nU^nY^n3} = \sum_{S_2,\tilde{X},T_2,\tilde{Z},U_2} P_{Q^nS_1S_2} P_{V^nU^nT^nZ^nX^n1|Q^nS_1S_2} P_{V^nU^nT^nZ^nX^n2|Q^nS_1S_2} P_{Y^n3|X^n1X^n2}
\]

\[
= P_{Q^nS_1} P_{V^nU^nT^nZ^n|Q^nS_1} P_{V^nU^nT^nZ^n|Q^nS_1} \left( \sum_{S_2,\tilde{X},T_2,\tilde{Z},U_2} \frac{P_{X^n1S_2|Q^nS_1V^nU^nT^nZ^n1} P_{X^n2S_2|Q^nS_1V^nU^nT^nZ^n2} P_{Y^n3|X^n1X^n2}}{P_{S_2|Q^nS_1}} \right),
\]

and where, without loss of generality, we have assumed that the known messages have index one. Notice that the above expression for $P^{(\text{dec})}_{Q^nS_1V^nU^nT^nZ^nV^nU^nY^n3}$ does not
imply that \((V_1, U_1, T_1, Z_1)\) and \((V_2, U_2)\) are independent conditioned on \((QS_1)\), i.e.,

\[
P_{QS_1 V_1 U_1 T_1 Z_1 V_2 U_2}^{(dec1)} = \sum_{Y_3} P_{QS_1 V_1 U_1 T_1 Z_1 V_2 U_2 Y_3}^{(dec1)}
= P_{QS_1} P_{V_1 U_1 T_1 Z_1|QS_1} P_{V_2 U_2|QS_1} \sum_{S_2} \frac{P_{S_2|QS_1 V_1 U_1 T_1 Z_1} P_{S_2|QS_1 V_2 U_2}}{P_{S_2|QS_1}}.
\]

The error analysis can be found in Appendix G. The probability of error at destination 1 can be made as small as desired if the following rate constraints are
satisfied:

\[ R_{V_1} + R_{V_2} + R_{U_1} + R_{T_1} + R_{U_2} + R_{Z_1} \leq E_0^{(1)} \]  \hspace{1cm} (22a)

\[ R_{U_1} + R_{T_1} + R_{U_2} + R_{Z_1} \leq \min \{ E_1^{(1)}, E_2^{(1)}, E_4^{(1)}, E_5^{(1)} \} \]  \hspace{1cm} (22b)

\[ R_{U_1} + R_{T_1} + R_{Z_1} \leq \min \{ E_3^{(1)}, E_6^{(1)} \} \]  \hspace{1cm} (22c)

\[ R_{T_1} + R_{U_2} + R_{Z_1} \leq \min \{ E_7^{(1)}, E_8^{(1)} \} \]  \hspace{1cm} (22d)

\[ R_{T_1} + R_{Z_1} \leq E_9^{(1)} \]  \hspace{1cm} (22e)

\[ R_{U_2} + R_{Z_1} \leq \min \{ E_{10}^{(1)}, E_{11}^{(1)} \} \]  \hspace{1cm} (22f)

\[ R_{Z_1} \leq E_{12}^{(1)} \]  \hspace{1cm} (22g)

\[ R_{U_1} + R_{T_1} + R_{U_2} \leq \min \{ E_{13}^{(1)}, E_{14}^{(1)}, E_{16}^{(1)}, E_{17}^{(1)}, E_{22}^{(1)}, E_{23}^{(1)} \} \]  \hspace{1cm} (22h)

\[ R_{U_1} + R_{T_1} \leq \min \{ E_{15}^{(1)}, E_{18}^{(1)}, E_{24}^{(1)} \} \]  \hspace{1cm} (22i)

\[ R_{T_1} + R_{U_2} \leq \min \{ E_{19}^{(1)}, E_{20}^{(1)}, E_{25}^{(1)}, E_{26}^{(1)} \} \]  \hspace{1cm} (22j)

\[ R_{T_1} \leq \min \{ E_{21}^{(1)}, E_{27}^{(1)} \} \]  \hspace{1cm} (22k)

where the rates \( R_\star, \star \in \{ Q, V_1, U_1, T_1, S_1, Z_1, V_2, U_2, T_2, S_2, Z_2 \} \), and the quantities \( E_\ell^{(1)}, \ell \in \{ 0, \ldots, 27 \} \), are defined in Appendix G. Similarly, the rate constraints at destination 2 are as in (22) but with the role of the users swapped.

**Achievable region:** The achievable region with superposition \& binning as a function of \( R_1 \) and \( R_2 \) only, can be obtained by applying the Fourier-Motzkin elimination procedure to the intersection of (19), and (20), (21), (22), and the regions corresponding to (20), (21), (22) but with the role of the users swapped. However, the rates constraints are expressed as the minimum of several quantities that we do not report it here for sake of space. We note that when the binning rates are taken to satisfy the constraints in (20) and in (21) with equality for both users, the achievable region has five types of rate bounds as for the non-feedback case, i.e., for \( R_1 \), for \( R_2 \), for \( R_1 + R_2 \), for \( 2R_1 + R_2 \), and for \( R_1 + 2R_2 \).

As for the case of superposition-only, the error analysis for the case of binning \& superposition is greatly simplified by the structured way we performed superposition. In particular, only 28 error events matter out of the \( 2^8 - 1 = 255 \) possible that result from the joint decoding of 8 messages.

**Remark 4.5.** In [53], we constructed the codebooks as depicted in Fig. 7. The difference with respect to the encoding proposed in this paper is that \( Z_1 \) was superimposed to \( S_2 \) too (which carries the private message from source 2–sent
cooperatively by source 1 too— but not decoded at destination 1), and $Z_2$ was superimposed to $S_1$ too. Thus, with the encoding of [53], $Z_1$ could not be decoded at destination 1, since its decoding implies the decoding of all messages to which $Z_1$ is superimposed, thus also $S_2$; but $S_2$ is a private message that must not be decoded at destination 1. Similarly, $Z_2$ could be decoded at destination 2.

With the encoding proposed in [53], the rate constraint in (20d) should be replaced by

$$R'_{11c} \geq I(Z_1 \land U_1, T_1 | Q, S_1, S_2, V_1),$$

and the rate constraints in (21) should be replaced by

$$R_{Z_2} \leq I(Z_2 \land Y_1 | X_1, V_2)$$

$$R_{V_2} + R_{Z_2} \leq I(V_2, Z_2 \land Y_1 | X_1), + I(V_2 \land S_1, S_2 | Q)$$

where the term $I(Z_2 \land S_1 | Q, S_2, V_2)$ in (21) does not appear in (24) since $Z_2$ is superimposed to $S_1$ by construction (thus it already has the desired joint distribution). In other words, the encoding of [53] is less stringent in terms of “binning rates constraints” but it is more stringent in terms on “decoding rate constraints”. However, these two effects do compensate one another. Indeed, from (24) and (23)
we have:

\[ R_{22c} \leq I(Z_2 \land Y_1 | X_1, V_2) - R'_{22c} \]

\[ \leq I(Z_2 \land Y_1 | X_1, V_2) - I(Z_2 \land U_2, T_2 | Q, S_1, S_2, V_2) \]

\[ R_{20c} + R_{22c} \leq I(V_2, Z_2 \land Y_1 | X_1) + I(V_2 \land S_1, S_2 | Q) - R'_{20c} - R'_{22c} \]

\[ \leq I(V_2, Z_2 \land Y_1 | X_1) - I(Z_2 \land U_2, T_2 | Q, S_1, S_2, V_2) \]

which is the same we obtain from (27) and (20d).

The two schemes are not the same. With the encoding proposed in this paper, the codeword in Z₁ and Z₂ can be decoded at the corresponding destination. This improves on the performance since now the Z’s do not longer act as noises at the intended destination.

We showed in the previous section that the achievable region with superposition-only in (9) did not reduced to the known achievable regions for some channels subsumed by the IFC-GF model. With superposition & binning we have:

1. When the IFC-GF channel reduces to a broadcast channel, our region with superposition & binning with only (Q, S₁, S₂) reduces to Marton’s inner bound for a general broadcast channel [32].

2. When the IFC-GF channel reduces to a cognitive channel, our region with superposition & binning achieves a subset of the largest know achievable region for a cognitive channel [38].

The reason why our superposition & binning region does not comprises the region in [38] is as follows. Assume source 1 is the cognitive user and source 2 is the primary user. Since the primary user is unaware of the message of the cognitive user, we need to set S₁ = V₁ = Z₁ = ∅. Since there is not part of the primary user’s message that is not available at the cognitive user, we must set Z₂ = V₂ = U₂ = T₂ = ∅. With these choices, our encoding scheme only uses (Q, U₁, T₁, S₂) and is equivalent to the scheme in [38] with U₂pb = ∅ (U₂pb in [38] carries part of the message of the primary user that is only sent by the cognitive user; this feature is not present in our encoding scheme.)

3. When the IFC-GF channel reduces to a relay channel, our region with superposition & binning does not seem to include the compress-and-forward relaying strategy.
5 Example: the Gaussian IFC-GF

In this section we provide an evaluation of our superposition-only achievable region for the Gaussian channel. We will provide an evaluation of our superposition & binning achievable region, as well as, a comparison with outer bounds, in the second part of this paper.

We assume full duplex communication, perfect channel state information at all terminals, and an average power constraint on the inputs. A Gaussian channel in standard form has outputs:

\[ Y_c = h_{c1}X_1 + h_{c2}X_2 + N_c, \quad N_c \sim \mathcal{N}(0,1), \ c \in \{1, \ldots, 4\} \]

subject to \( \mathbb{E}[|X_u|^2] \leq P_u, \ u \in \{1, 2\} \). We assume that the additive noises on the different channels are independent (the case of correlated noises will be discussed in the second part of this paper). Without loss of generality we assume that the direct link channel gains \( h_{31} \) and \( h_{42} \) are real-valued since the destinations can compensate for the phase of the intended signal. Similarly, we assume that the cooperation link channel gains \( h_{21} \) and \( h_{12} \) are also real-valued. As opposed to the case without GF, the phase of the interfering link channel gains \( h_{32} \) and \( h_{41} \) matter because of transmitter cooperation.

In the following we are going to consider jointly Gaussian inputs only when evaluating the achievable region in Th. 4.1. Without loss of generality, let \( Q \sim \mathcal{N}(0,1) \) and let

\[
V_u = \alpha_u Q + X_{u0c} \\
U_u = V_u + X_{u0n} \\
T_u = X_u = U_u + X_{uun},
\]

for \( u \in \{1, 2\} \), where \( (\alpha_1, \alpha_2) \in \mathbb{C}^2 \) and \( X_m \sim \mathcal{N}(0, \sigma_m^2) \) for \( m \in \{10c, 10n, 11n, 20c, 20n, 22n\} \) are independent random variables whose variances satisfy

\[
0 \leq |\alpha_u|^2 + \sigma_{u0c}^2 + \sigma_{u0n}^2 + \sigma_{uun}^2 \leq P_u, \quad u \in \{1, 2\}.
\]
The right hand side of the rate bounds in region (7) are then

\[(7a) \log \left(1 + \frac{|h_{21}|^2 \sigma_{10c}^2}{1 + |h_{21}|^2 \sigma_{10m}^2 + \sigma_{11n}^2}\right)\]

\[(7b) \log \left(1 + \frac{|h_{31}|^2 \sigma_{11n}^2}{1 + |h_{32}|^2 \sigma_{22n}^2}\right)\]

\[(7c) \log \left(1 + \frac{|h_{31}|^2 \sigma_{10n}^2 + |h_{32}|^2 \sigma_{20n}^2}{1 + |h_{32}|^2 \sigma_{22n}^2}\right)\]

\[(7d) \log \left(1 + \frac{|h_{31}|^2 \sigma_{10m}^2 + \sigma_{11n}^2}{1 + |h_{32}|^2 \sigma_{22n}^2}\right)\]

\[(7e) \log \left(1 + \frac{|h_{31}|^2 \sigma_{20n}^2 + \sigma_{11n}^2 + |h_{32}|^2 \sigma_{20n}^2}{1 + |h_{32}|^2 \sigma_{22n}^2}\right)\]

\[(7f) \log \left(1 + \frac{|h_{31}|^2 \sigma_{10c}^2 + \sigma_{10m}^2 + \sigma_{11n}^2 + |h_{32}|^2 \sigma_{20c}^2 + \sigma_{20n}^2}{1 + |h_{32}|^2 \sigma_{22n}^2}\right)\]

and similarly for the other user. Notice that, conditioning on \(Q\) removes the dependency between \(X_1\) and \(X_2\). The cooperation gain shows in eq. (7f) (where instead of \(|h_{31} \alpha_1| + |h_{32} \alpha_2|^2\) one has \(|h_{31} \alpha_1 + h_{32} \alpha_2|^2\)) and in equation (7a). The private message \(X_{22n}(X_{11n})\) acts as noise for destination 1 (destination 2).

For the purpose of numerical example, we consider a symmetric network where, in a two dimensional Cartesian place with \(x \geq 0\) and \(y \geq 0\), source 1 is in position \((-x/2, -y/2)\), source 2 is in position \((-x/2, +y/2)\), destination 1 is in position \((+x/2, -y/2)\), and destination 2 is in position \((+x/2, +y/2)\). Both users have the same power constraint \(P\) and the channel gains are inversely proportional to the distance between the terminals, that is, \(h_{21} = h_{12} = h_{34} = h_{43} = 1/y\), \(h_{31} = h_{13} = h_{24} = h_{42} = 1/x\) and \(h_{41} = h_{14} = h_{23} = h_{32} = 1/\sqrt{x^2 + y^2}\).

Fig. 8 shows the performance of our superposition-only scheme and compares it with the case without generalized feedback, for the case \(P_1 = P_2 = 6\) and \(x = 2\) and \(y = 1\). It can be seen that great rate improvements are possible thanks to cooperation, when compared to the case without GF. For example, the maximum rate for a given user improves from 1.3 bit/sec/Hz to 1.9 bit/sec/Hz (1.9=1.3×(1+0.46), i.e., 46% improvement), while the sum-rate improves from 1.7 bit/sec/Hz to 2.2 bit/sec/Hz (2.2=1.7×(1+0.29), i.e., 29% improvement).
Figure 8: Performance comparison among standard IFC and IFC-GF.

6 Conclusion

In this paper we presented an achievable region for a general IFC-GF. We built on the idea of message splitting into common and private messages and proposed a coding scheme where the sources cooperate to send part of the two messages. The cooperation in sending the common message aims to realizing the gains of beam-forming, as in a distributed multi-antenna system, while the cooperation in sending the private message aims to leverage the interference “pre-cancellation” property of dirty-paper-type coding. We showed that our achievable region generalizes several known achievable regions for IFC-GF and that it reduces to known achievable regions for some of the channels subsumed by the IFC-GF model. Numerical results show that cooperation can improve the achievable rate of all the involved sources. In the second part of this paper, we will derive an outer bound for a general IFC-GF and use to assess the performance of our proposed achievable strategy.

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A Proof of the redundancy of two single-rate constraints

For a fixed distribution $P_{QVUR}$ consider another distribution $P'_{Q'V'UR}$ with

$$V_1' = U_1' = \emptyset, \quad T_1' = (T_1, U_1), \quad Q' = (Q, V_1),$$

that is, source 1 does not send any common information. With $P'_{Q'V'UR}$, the achievable region in (9) reduces to:

$$R_1 \leq (7d)' \quad (25a)$$
$$R_2 \leq (8f)' \quad (25b)$$
$$R_2 \leq (8a)' + (8d)' \quad (25c)$$
$$R_1 + R_2 \leq (7a)' + (7f)' \quad (25d)$$
$$R_1 + R_2 \leq (8a)' + (7e)' + (8b)' \quad (25e)$$

where a prime as a superscript indicates that the mutual information in the corresponding equation must be computed for the distribution $P'_{Q'V'UR}$ (rather than for the distribution $P_{QVUR}$). Notice that all the mutual informations in (7) are larger under $P'_{Q'V'UR}$ than under $P_{QVUR}$ and satisfy

$$0 \leq (7b)' = (7d)' = (7d)$$
$$\leq (7c)' = (7e)' = (7e)$$
$$\leq (7f)' = (7f)$$
$$(7a)' = 0,$$

while the mutual informations in (8) under $P'_{Q'V'UR}$ satisfy

$$0 \leq (8b)' = (8d)' = I(Y_4 \wedge T_2 | Q, V_1, V_2, U_2)$$
$$\leq (8c)' = (8e)' = I(Y_4 \wedge T_2, U_2 | Q, V_1, V_2)$$
$$\leq (8f)' = I(Y_4 \wedge T_2, U_2, Q, V_1, V_2)$$
$$(8a)' = (8a) = I(V_2 \wedge Y_1 | Q, T_1, U_1, V_1, X_1).$$

Consider the region in (9) (under $P_{QVUR}$). If

$$\min\{ (7f), (7a) + (7d) \} \geq \min\{ (7f), (7a) + (7d) \}$$
then the rate constraint in (10) is redundant and can be omitted from the region in (9). We will now show that the rate constraint in (10) can always be omitted from the region in (9) without enlarging the achievable region. We will do so by showing that the rate points for which the rate constraint in (10) is violated, that is,

\[
\begin{align*}
\text{(7a)} + \text{(7b)} + \text{(8c)} < R_1 \leq \min &\{ \text{(7f)} , \text{(7a)} + \text{(7d)} \}\end{align*}
\]  

(26)

holds together with all the rate constraint in (9), are contained in the region in (25) (under \(P_{QV_1U_1T_1X_1V_2U_2T_2X_2} \)). The region in (25) is a special case of the region in (9) for which the rate constraint in (10) is redundant. This shows that the region in (9) is indeed achievable.

Assume now that (26) holds together with all the rate constraint in (9) (under \(P_{QV_1U_1T_1X_1V_2U_2T_2X_2} \)). We will show that that (26) and (9) imply (25). We have:

\[
\begin{align*}
R_2 &\leq \text{(7b)} + \text{(8f)} - \left( \text{(7a)} + \text{(7b)} + \text{(8c)} \right) \\
&= I(Y_4; U_2, Q, V_1, V_2) - I(Y_2; V_1|X_2) \\
&\leq I(Y_4; T_2, U_2, Q, V_1, V_2) - 0 = \text{(8f)}' \\
\end{align*}
\]

and

\[
\begin{align*}
R_2 &\leq \text{(7a)} + \text{(8a)} + \text{(7b)} + \text{(8c)} - \left( \text{(7a)} + \text{(7b)} + \text{(8c)} \right) \\
&= \text{(8a)} + I(Y_4; U_2|Q, V_1, V_2) \\
&\leq \text{(8a)} + I(Y_4; T_2, U_2|Q, V_1, V_2) = \text{(8a)} + \text{(8d)}' \\
\end{align*}
\]

and

\[
\begin{align*}
R_1 + R_2 &\leq \text{(7a)} + \text{(7b)} + \text{(8f)} + \text{(8c)} - \left( \text{(7a)} + \text{(7b)} + \text{(8c)} \right) \\
&= \text{(7f)} \\
&\leq \text{(7f)} + \text{(8c)}' \\
\end{align*}
\]

47
and
\[ R_1 + R_2 \leq 2 \left( 7a + 8a + 7b + 7c + 8c\right) - \left( 7a + 7b + 8c\right) \]
\[ = \left( 7a + 8a + 7c\right) \]
\[ \leq \left( 7a + 8a + 7c + 8b\right)'. \]

This concludes the proof.

B Proof of (7a) and of (8a)

At the end of slot \(b\), \(b \in \{1, \cdots, N-1\}\), transmitter 1 has received \(Y^n_{1,b}\) and looks for the unique index \(i \in \{1, \cdots, e^{nR_{20c}}\}\) such that the sequences
\[ (V^n_2(i, \cdots), Y^n_{1,b}) \]
\[ \in T^{(n)}_e(P_{V_2|Q}P_{Y_1|Q,X_1,Y_2|Q,X_1^n,X_2^n}), \]
where the dots indicate known message indices, and where all that is known at transmitter 1 is represented by
\[ X^n_1 = (Q^n, V^n_1, U^n_1, T^n_1, X^n_1). \]

If none or more than one index \(i \in \{1, \cdots, e^{nR_{20c}}\}\) are found, then transmitter 1 sets \(i = 1\).

Assume message \(i = 1\) was sent. An error occurs when transmitter 1 declares \(i \neq 1\) in (27), which occurs with probability
\[
\Pr[i \neq 1|1\text{ sent}] \\
= \Pr[\bigcup_{i > 1}(V^n_2(i, \cdots), Y^n_{1,b}) \in T^{(n)}_e(P_{V_2|Q}P_{Y_1|Q,X_1,Y_2|Q,X_1^n})] \\
\leq \sum_{i=2} e^{nR_{20c}+H(V_2|Q)+H(Y_1|Q,X_1,Y_2)-H(V_2|Q)-H(Y_1|Q,X_1)+O(\epsilon)} \\
= e^{n[R_{20c}-I(Y_1 \wedge V_2|Q,X_1)+O(\epsilon)]}
\]

This concludes the proof.
because \( V_2 \) and \( Y_1 \) are independent conditioned on \( Q \) for any \( i \neq 1 \). The error probability goes to zero if (8a) holds, where \( O(\epsilon) \) denotes a function of \( \epsilon \) that goes to zero when \( \epsilon \to 0 \).

Similarly, we can prove (15) and (17) as follows. We only give the proof for (15). Assume \( i = 1 \) was sent. An error occur if the search for

\[
(V_2^n(i, \ldots, j, i, \ldots), Y_1^n) 
\in T_{\epsilon}^{(n)}(P_{V_2,U_2|Q}P_{Y_1|Q,X_1,V_2,U_2|Q^n,X_1^n}),
\]

results in an \( i \neq 1 \). Thus an error occurs with probability

\[
\Pr[i \neq 1|1 \text{ sent}] 
= \Pr[\cup_{i,j \geq 1} (V_2^n(i, \ldots, j, i, \ldots), Y_1^n(j, i, \ldots, j, i, \ldots, Y_1^n) \in T_{\epsilon}^{(n)}(P_{V_2,U_2|Q}P_{Y_1|Q,X_1,V_2,U_2|Q^n,X_1^n})] 
\leq \sum_{i=2}^{n} \sum_{j=1}^{n} \left| T_{\epsilon}^{(n)}(P_{V_2,U_2|Q}P_{Y_1|Q,X_1,V_2,U_2|Q^n,X_1^n}) \right| P_{V_2,U_2|Q}P_{Y_1|Q,X_1,V_2,U_2|Q^n,X_1^n} 
\leq e^{n[R_{20c}+R_{20n}+H(V_2,U_2|Q)+H(Q,X_1,V_2,U_2) - H(V_2,U_2|Q) - H(Y_1|Q,X_1)+O(\epsilon)]} 
= e^{n[R_{20c}+R_{20n}-I(Y_1,V_2,U_2|Q,X_1)+O(\epsilon)]}
\]

because \( (V_2, U_2) \) and \( Y_1 \) are independent conditioned on \( Q \) for any \( i \neq 1 \). Hence, the probability of error vanishes as the block-length increases if (15) holds.

### C Proof of (7) and (8)

For the error analysis, we consider only the probability of error in each block as the total average probability of error can be upper bounded by the sum of the decoding error probabilities at each step, under the assumption that no error propagation from the previous steps has occurred [654].

Receiver 1 looks for the unique triplet \((q_1, u_1, t_1)\) and some pair \((q_2, u_2)\) such that the sequences

\[
(Q^n([q_1, q_2]), V_1^n(W_{10c,b}, [q_1, q_2]), U_1^n(u_1, W_{10c,b}, [q_1, q_2]), T_1^n(t_1, u_1, W_{10c,b}, [q_1, q_2]),
V_2^n(W_{20c,b}, [q_1, q_2]), U_2^n(u_2, W_{20c,b}, [q_1, q_2]), Y_3^n) 
\in T_{\epsilon}^{(n)}(P_{Q}P_{V_1,U_1,T_1|Q}P_{V_2,U_2|Q}P_{Y_3|Q,V_1,U_1,T_1,V_2,U_2}),
\]

(29)

where \((W_{10c,b}, W_{20c,b})\) were decoded exactly in the previous step. Let \(E_{q_1,q_2,u_1,u_2,t_1}\) be the event that the sequences in (29) are strongly jointly typical. Assume that
\((q_1 q_2 u_1 u_2 t_1) = (11111)\) was sent. The total probability of error at receiver 1 can be bounded as

\[
\begin{align*}
P_{e,1}^{(n)} &= \Pr[E_{11111}^c \cup (q_1 \neq 1, q_2 u_1 u_2 t_1) \cup E_{q_1 q_2 u_1 u_2 t_1}] \\
&\leq \Pr[E_{11111}^c] \\
&+ \sum_{q_2 > 1, u_2 > 1} \Pr[\cup (q_1, q_2) \neq (1, 1) E_{q_1 q_2 u_1 u_2 t_1}] + \Pr[\cup (q_1, u_1) \neq (1, 1) E_{q_1 u_1 t_1}] \\
&+ \sum_{u_2 > 1} \Pr[\cup (q_1, u_1) \neq (1, 1) E_{q_1 u_1 t_1}] + \sum_{q_2 > 1} \Pr[\cup (q_1, u_1) \neq (1, 1) E_{q_1 q_2 u_1 t_1}]
\end{align*}
\]

where all probabilities are conditioned on the event that \((q_1 q_2 u_1 u_2 t_1) = (11111)\) was sent.

The probability of the event \(E_{11111}^c\) vanishes because the transmitted codewords are jointly typical with the received sequence with high probability.

Although it seems we need to consider \(7 \times 4 = 28\) different error events, all events with \((q_1 q_2) \neq (1, 1)\), i.e., \(Q^n\) wrong, are such that the estimated codewords are independent of the actual transmitted ones, and hence of the output. Hence, the probabilities that either \(q_1 \neq 1\) or \(q_2 \neq 1\) are dominated by

\[
\sum_{i > 1, j > 1, k \geq 1, l \geq 1, m \geq 1} \Pr[E_{ijklm}]
\]

for which we have

\[
\begin{align*}
\Pr[E_{ijklm}] &\leq T^n_1 \left(P_{Q^n} P_{V_1, U_1, T_1} P_{V_2, U_2} P_{Y_3} | V_1, U_1, T_1, V_2, U_2, Q^n \right) \\
&\leq e^{-n \left(H(Q) + H(V_1, U_1, T_1, V_2, U_2) + H(Y_3) + H(U_1, T_1, V_2, U_2, Q^n)\right)} \\
&\leq e^{-n \left[H(V_1, U_1, T_1, V_2, U_2) + H(Y_3) + O(c)\right]}
\end{align*}
\]

Hence, the probability in (30) vanishes as \(n \to \infty\) if (71) holds.

Besides the events with \((q_1, q_2) \neq (1, 1)\), the other probability affecting \(P_{e,1}^{(n)}\) are as follows. From now on \((q_1, q_2) = (1, 1)\), i.e., \(Q^n\) correct. Recall that when \(Q^n\) is correct, also \(V^n_1\) and \(V^n_2\) are correct.

The events with \((u_1 > 1, t_1 = 1)\) and \((u_1 > 1, t_1 > 1)\), i.e., \(U^n_1\) wrong, have the same probability since, once \(U^n_1\) is wrong, the received signal is independent of the chosen \((U^n_1(u_1, \cdots), T^n_1(t_1, u_1, \cdots))\) conditioned on the transmitted/correct
$$(Q^n, V^n_1)$$. This is so because, even if $t_1 = 1$, the chosen $T^n_1$ is not the transmitted one because superimposed to a wrong $U^n_1$. Hence, if $u_2 = 1$, i.e., $U^n_2$ correct, we have

$$\sum_{k>1, m \geq 1} \Pr[E_{11k1m}]$$

$$\leq \sum_{k>1, m > 1} \left| T^n_e \left( P_Q P_{V_1, U_1, T_1} | Q P_{V_2, U_2} | Q P_{V_3} | V_1, U_1, T_1, V_2, U_2, Q \right) \right|$$

$$P_{Q^n} P_{V_1^n, U_1^n, T_1^n} | Q^n P_{V_2^n, U_2^n} | Q^n P_{V_3^n} | Q^n, V^n_1, V^n_2, U^n_2$$

$$\leq \sum_{k>1, m > 1} e^n(+H(Q) + H(V_1, U_1, T_1, X_1 | Q) + H(V_2, U_2 | Q) + H(Y_3 | Q, V_1, U_1, T_1, V_2, U_2))$$

$$= e^n(R_{10n} + R_{11n} - I(Y_3 \wedge U_1, T_1 | Q, V_1, U_2) + O(\epsilon))$$

which gives (7d); while if $u_2 > 1$, i.e., $U^n_2$ wrong, we have

$$\sum_{\ell>1, k>1, m \geq 1} \Pr[E_{11k\ell m}]$$

$$\leq \sum_{\ell>1, k>1, m > 1} \left| T^n_e \left( P_Q P_{V_1, U_1, T_1} | Q P_{V_2, U_2} | Q P_{V_3} | V_1, U_1, T_1, V_2, U_2, Q \right) \right|$$

$$P_{Q^n} P_{V_1^n, U_1^n, T_1^n} | Q^n P_{V_2^n, U_2^n} | Q^n P_{Y_3^n} | Q^n, V^n_1, V^n_2, U^n_2$$

$$\leq \sum_{\ell>1, k>1, m > 1} e^n(+H(Q) + H(V_1, U_1, T_1, X_1 | Q) + H(V_2, U_2 | Q) + H(Y_3 | Q, V_1, U_1, T_1, V_2, U_2))$$

$$= e^n(R_{10n} + R_{11n} - I(Y_3 \wedge T_1, U_1, U_2 | Q, V_1, V_2) + O(\epsilon))$$

which gives (7e).

From now on $(q_1, q_2, u_1) = (1, 1, 1)$, i.e., $Q^n$ and $U^n_1$ correct. If $u_2 = 1$, i.e.,
which gives (7c), while if $u_2 > 1$, i.e., $U_2^n$ wrong, and $t_1 > 1$, i.e., $T_1^n$ wrong, we have

$$\sum_{m > 1} \Pr[E_{1111m}]$$

$$\leq \sum_{m > 1} |T^n_\ell (P_Q P_{V_1, U_1, T_1} | Q P_{V_2, U_2} | Q P_{Y_3} | V_1, U_1, T_1, V_2, U_2, Q)|$$

$$P_Q^n P_{V_1^n, U_1^n, T_1^n | Q^n} P_{V_2^n, U_2^n | Q^n} P_{Y_3^n | Q^n, V_1^n, V_2^n, U_1^n, U_2^n}$$

$$\leq \sum_{\ell > 1, m > 1} e^{-n(H(Q) + H(V_1, U_1, T_1, X_1 | Q) + H(V_2, U_2 | Q) + H(Y_3 | Q, V_1, U_1, T_1, V_2, U_2))}$$

$$e^{-n(\epsilon - H(Q) - H(V_1, U_1, T_1, X_1 | Q) - H(V_2, U_2 | Q) - H(Y_3 | Q, V_1, U_1, T_1, V_2, U_2))} + O(\epsilon))$$

which gives (7b).

The developed analysis can be summarized as in Table 1. In Table 1 the symbols “1”, “0” and “⋆” have the following meaning. A “1” indicates that the message index is in error. A “0” indicates that the message index is correct. A “⋆” indicates that it does not matter whether the message index is in error; this is so because of superposition coding; in this case in fact, the codeword selected by the decoder—even though with the correct message index—is superimposed to a wrong codeword and it is thus independent of the received signal. In case of a “⋆”, the factorization
of the joint probability needed for the evaluation of the probability of error is as for the case where the message is wrong; this implies that the error event that gives the most stringent rate bound is that for which the message is wrong (i.e., as far as error bounds are concerned, a “⋆” is equivalent to a “1”). The second to last column in Table 1 counts how many error events are included in the corresponding row (i.e., each “⋆” corresponds to two possible cases), and the last column lists the elements of the set $C$, where $C$ is the set of correctly decoded codewords.

### D Proof of (19)

The probability that encoder 1 fails to find a good pair of indices $B_{11c,b−1} = i$, $B_{22c,b−1} = j$ is

\[
\Pr[\bigcup_{i,j \geq 1} \left( S_1^n([W_{11c,b−1}, i], [W_{10c,b−1}, W_{20c,b−1}]),
S_2^n([W_{22c,b−1}, j], [W_{10c,b−1}, W_{20c,b−1}]) \notin T^n(P_{Q_n S_1 S_2 | Q^n}) \right) = (1 - p)^{e^n(R'_{11c}+R'_{22c})} \leq \exp(-e^n(R'_{11c}+R'_{22c})p) \leq \exp(-e^n(R'_{11c}+R'_{22c}-I(S_1 \wedge S_2 | Q) - O(\varepsilon)))
\]
which goes to zero as $n \to \infty$ if (19) holds, where

$$p = \Pr\left[ \left( S_1^n([W_{11c,b-1}, i], [W_{10c,b-1}, W_{20c,b-1}]), S_2^n([W_{22c,b-1}, j], [W_{10c,b-1}, W_{20c,b-1}]) \right) \in T^n_\epsilon (P_{QS_1} S_2^n(Q^n)) \right]$$

$$\leq \left| T^n_\epsilon (P_{QS_1} S_2^n(Q^n)) \right| |P_{S_1^n} P_{S_2^n}| Q^n$$

$$\leq e^n (H(S_1^n|Q^n) - H(S_2|Q^n) - O(\epsilon))$$

$$= e^n (-I(S_1 \wedge S_2|Q^n) - O(\epsilon)),$$

where $O(\epsilon) \to 0$ as $\epsilon \to 0$.

### E  Proof of (20)

Encoding fails if for all set of indices $(B_{10c,b}, B_{10n,b}, B_{11n,b})$ can be found no triplet $(V_1^n, U_1^n, T_1^n)$ can be found to be jointly typical with $(Q^n, S_1^n, S_2^n)$. The probability of this event can be bounded as

$$\Pr[ e^{nR_{10c}} e^{nR_{10n}} e^{nR_{11n}} \cap \bigcap_{B_{10c}=1} \bigcap_{B_{10n}=1} \bigcap_{B_{11n}=1} V_1^n([W_{10c,b}, B_{10c}], \ldots), U_1^n([W_{10n,b}, B_{10n}], [W_{10c,b}, B_{10c}], \ldots), T_1^n([W_{11n,b}, B_{11n}], [W_{10n,b}, B_{10n}], [W_{10c,b}, B_{10c}], \ldots)) \notin T^n_\epsilon (P_{Q,S_1,S_2,V_1,U_1,T_1}|Q^n, S_1^n, S_2^n)]$$

$$= \Pr[K = 0] \leq \frac{\text{Var}[K]}{E^2[K]},$$

where

$$K = \sum_{B_{10c}=1} e^{nR_{10c}} e^{nR_{10n}} e^{nR_{11n}} K_{B_{10c},B_{10n},B_{11n}}$$
for

\[ K_{B_10c, B_{10n}, B_{11n}} = 1 \{ 
\begin{align*}
  &V^n([W_{10c,b}, B_{10c}], \cdots), \\
  &U^n([W_{10n,b}, B_{10n}], [W_{10c,b}, B_{10c}], \cdots), \\
  &T^n([W_{11n,b}, B_{11n}], [W_{10n,b}, B_{10n}], [W_{10c,b}, B_{10c}], \cdots)
\end{align*}
\] \]

\[ \in T^n(\Pr[Q,S_1,S_2,V_1,T_1|Q^n,S^n_1,S^n_2]) \]

and \(1_{\{A\}}\) is the indicator function that equals one whenever the condition in \(A\) is true.

The mean of the random variable \(K\) is easily lower bounded as

\[
\mathbb{E}[K] = \sum_{B_{10c}=1}^{e^{nR'_{10c}}} \sum_{B_{10n}=1}^{e^{nR'_{10n}}} \sum_{B_{11n}=1}^{e^{nR'_{11n}}} \Pr[ \\
\begin{align*}
  &V^n([W_{10c,b}, B_{10c}], \cdots), \\
  &U^n([W_{10n,b}, B_{10n}], [W_{10c,b}, B_{10c}], \cdots), \\
  &T^n([W_{11n,b}, B_{11n}], [W_{10n,b}, B_{10n}], [W_{10c,b}, B_{10c}], \cdots)
\end{align*}
\] \[
\in T^n(\Pr[Q,S_1,S_2,V_1,T_1|Q^n,S^n_1,S^n_2]) \]

\[
= \sum_{B_{10c}=1}^{e^{nR'_{10c}}} \sum_{B_{10n}=1}^{e^{nR'_{10n}}} \sum_{B_{11n}=1}^{e^{nR'_{11n}}} |T^n(\Pr[Q,S_1,S_2,V_1,T_1|Q^n,S^n_1,S^n_2])|P^n_V |U^n |T^n |Q^n
\]

\[
\geq \exp\{n[R'_{10c} + R'_{10n} + R'_{11n} + H(V_1, U_1, T_1|Q,S_1,S_2) - H(V_1, U_1, T_1|Q) - O(\epsilon)]\}
\]

\[
= \exp\{n[R'_{10c} + R'_{10n} + R'_{11n} - I(S_1,S_2 \land V_1, U_1, T_1|Q) - O(\epsilon)]\},
\]

and upper bounded as

\[
\mathbb{E}[K] \leq \exp\{n[R'_{10c} + R'_{10n} + R'_{11n} - I(S_1,S_2 \land V_1, U_1, T_1|Q) + O(\epsilon)]\}.
\]

The variance of \(K\) can be computed as

\[
\text{Var}[K] = \sum_{B_{10c}=1}^{e^{nR'_{10c}}} \sum_{B_{10n}=1}^{e^{nR'_{10n}}} \sum_{B_{11n}=1}^{e^{nR'_{11n}}} \sum_{B'_{10c}=1}^{e^{nR'_{10c}}} \sum_{B'_{10n}=1}^{e^{nR'_{10n}}} \sum_{B'_{11n}=1}^{e^{nR'_{11n}}} \\
\left( \Pr[K_{B_{10c},B_{10n},B_{11n}} = 1, K_{B'_{10c},B'_{10n},B'_{11n}} = 1] \\
- \Pr[K_{B_{10c},B_{10n},B_{11n}} = 1] \Pr[K_{B'_{10c},B'_{10n},B'_{11n}} = 1] \right).
\]
When $B_{10c} \neq B_{10c}'$, the random variables $K_{B_{10c}, B_{10n}, B_{11n}}$ and $K_{B_{10c}', B_{10n}', B_{11n}}$ are independent by construction, hence they do not contribute to the summation. When $B_{10c} = B_{10c}'$, we upper-bound $\text{Var}[K]$ by neglecting the non-negative term $\Pr[K_{B_{10c}, B_{10n}, B_{11n}} = 1]\Pr[K_{B_{10c}', B_{10n}', B_{11n}} = 1]$. Hence we have

$$\text{Var}[K] \leq \sum_{B_{10c}=B_{10c}'} \sum_{B_{10n}=B_{10n}'} \sum_{B_{11n}=B_{11n}'} \Pr[K_{B_{10c}, B_{10n}, B_{11n}} = 1] \Pr[K_{B_{10c}', B_{10n}', B_{11n}} = 1] \leq e^{-nA}.$$ 

We now evaluate $A$ and $B$. We have, neglecting the terms that go to zero as $\epsilon \to 0$,

$$e^{-nB} = \Pr[(U_1^n, T_1^n) \in T_\epsilon^n(P_{Q,S_1,S_2,V_1}, U_1, T_1 | Q^n, S_1^n, S_2^n, V_1^n)] \leq e^{-nH(U_1, T_1 | Q, S_1, S_2, V_1) - nH(U, T_1 | Q, V_1)} \leq e^{-nI(U_1, T_1 \wedge S_1, S_2 | V_1)},$$

and

$$e^{-nA} = \Pr[T_1^n \in T_\epsilon^n(P_{Q,S_1,S_2,V_1}, U_1, T_1 | Q^n, S_1^n, S_2^n, V_1^n, U_1^n)] \leq e^{-nH(T_1 | Q, S_1, S_2, U_1) - nH(T_1 | Q, V_1, U_1)} \leq e^{-nI(T_1 \wedge S_1, S_2 | Q, V_1, U_1)}.$$

After having evaluated $A$ and $B$, we have that

$$\frac{\text{Var}[K]}{\mathbb{E}^2[K]} \leq 1 + e^{n[R_{11n}^n - A]} + e^{n[R_{10n}^n + R_{11n}^n - B]} \leq e^{n[R_{10n}^n + R_{11n}^n - I(S_1S_2 \wedge V_1U_1T_1 | Q)]}$$

goess to zero as $n \to \infty$ if (20) holds.

**F Proof of (21)**

Let

$$E_{i,j,b_j} = (V_2^n([i, j], \ldots), Z_2^n([i, b_i] \ldots), Y_{1,b_i}^n) \in T_\epsilon^n(P_{V_2Z_2 | QS_1S_2} P_{Y_1 | QS_1S_2V_2}, |X_1^n),$$

56
where all that is known at transmitter 1 is represented by

\[ X^n = (Q^n, S^n_1, S^n_2, Z^n_1, V^n_1, U^n_1, T^n_1, X^n_1). \]

Assume that \((i, j, b_i, b_j) = (1111)\) was sent. The probability that the estimate of 
\((j, i) = (W_{20c, b}, W_{22c, b})\) is wrong is bounded by

\[
P_{e_{enc1}}^{(n)} = \Pr[E_{1111}^c \cup (i,j) \neq (1111), \forall (b_i, b_j) E_{ijk}]
\leq \Pr[E_{1111}^c]
+ \sum_{i>1,j>1} \Pr[\cup_{\forall (b_i, b_j)} E_{ijb_i, b_j}]
+ \sum_{i>1} \Pr[\cup_{\forall (b_i)} E_{i1b_i, 1}]
+ \sum_{j>1} \Pr[\cup_{\forall (b_j)} E_{1j1b_j, 1}]
\]

where all probabilities are conditioned on \((i, j, b_i, b_j) = (1111)\) being sent.

The probability of \(E_{1111}^c\) is vanishing as \(n \to \infty\) because the transmitted code-
words are jointly typical with the received signal with high probability.

For the other three terms we have: if \(i > 1, j > 1\), i.e., when \(V^n_2\) is wrong then
whether \(Z^n_2\) is correct or wrong does not change the distribution to use in the error
events and the most stringent error bound is for the case where both are wrong; thus
we have

\[
\sum_{i>1,j>1} \Pr[\cup_{\forall (b_i, b_j)} E_{ijb_i, b_j}]
\leq \sum_{i>1,j>1} |\mathcal{T}_e^n (P_{V_2, Z_2|Q, S_1, S_2, P_{Y_1|Q, S_1, S_2, V_2, Z_2, X_1}|X^n_1)|P_{V_2|Q}P_{Z_2|Q, S_2, V_2}P_{Y_1|Q, S_1, S_2, X_1}
\leq e^{n[R_{22c}+R_{20c}+R_{22c}'+R_{20c}']}
\]

\[
e^{n[H(V_2|Q, S_1, S_2)+H(Z_2|Q, S_1, S_2, V_2)+H(Y_1|Q, S_1, S_2, V_2)|S_1, S_2, X_1)]}
\leq e^{n[R_{22c}+R_{20c}+R_{22c}'+R_{20c}' - I(V_2 \wedge S_1, S_2|Q) - I(Z_2 \wedge S_1|Q, S_2, V_2) - I(Y_1 \wedge V_2, Z_2|Q, S_1, S_2, X_1)]}
\]

since by construction \(V_2\) and \(Z_2\) are independent conditioned on \(Q\) and \(Z_2\) is
superimposed to \((S_2, V_2)\). This probability can be driven to zero if \((21b)\) holds.
Finally, if \( i > 1, j = 1 \), i.e., \( V^o_2 \) correct and \( Z^o_2 \) wrong, we have

\[
\sum_{i>1} \Pr[\bigcup_{b_i \geq 1} E_{i1b_i}]
\leq \sum_{i>1, b_i \geq 1} |T^e_i (P_{V_2Z_2|QS_1S_2}P_{Y_1|QS_1S_2V_2Z_2X_1} | X^u_i) | P_{V_2|QS_1S_2}P_{Z_2|QS_2V_2}P_{Y_1|QS_1S_2V_2X_1}
\leq e^n[R_{22c}+R'_{22c}]
\leq e^n[H(V_2|QS_1S_2)+H(Z_2|QS_1S_2V_2)+H(Y_1|QS_1S_2V_2X_1)]
\leq e^{-n[H(V_2|QS_1S_2)+H(Z_2|QS_2V_2)+H(Y_1|QS_1S_2V_2X_1)]}
\leq e^n[R_{22c}+R'_{22c}-I(Z_2|QS_2V_2)-I(Y_1|Z_2|QS_1S_2V_2X_1)]
\]

since now the \( V_2 \) has the marginal imposed by the binning step during the encoding process. This probability can be driven to zero if (21a) holds.

### G Proof of (22)

In slot \( b, b = N, N-1, ..., 1 \), destination 1 tries to find a unique set of message indices \( (q_1, q_2, s_1, u_1, u_2, t_1) \) and some bin indices \( (b_{v_1}, b_{v_2}, b_{s_1}, b_{a_1}, b_{u_2}, b_{t_1}) \) such that

\[
\left( Q^n([q_1, q_2]),
S^n_1([s_1, b_{s_1}], [q_1, q_2]),
V^n_1([1, b_{v_1}], [q_1, q_2]),
Z^n([1, b_{v_2}], [s_1, b_{s_1}], [1, b_{v_1}], [q_1, q_2]),
U^n_1([u_1, b_{u_1}], [1, b_{v_1}], [q_1, q_2]),
T^n_1([t_1, b_{t_1}], [u_1, b_{u_1}], [1, b_{v_1}], [q_1, q_2]),
V^n_2([1, b_{v_2}], [q_1, q_2]),
U^n_2([u_2, b_{u_2}], [1, b_{v_2}], [q_1, q_2]),
Y^n_{3,b} \right) \in \mathcal{T}_{e}^{(n)} (P_{QS_1V_2U_1T_1Z_1V_2U_2Y_3}^{(dec)})
\]

where

\[
P_{QS_1V_2U_1T_1Z_1V_2U_2Y_3}^{(dec)}
= P_{QS_1}P_{V_2U_1T_1Z_1|QS_1}P_{V_2U_2|QS_1} \left( \sum_{S_2, X_1, X_2} P_{X_1S_2|QS_1V_2U_1T_1Z_1} \frac{P_{X_2S_2|QS_1V_2U_2}P_{Y_3|X_1X_2}}{P_{S_2|QS_1}P_{Y_3|X_1X_2}} \right).
\]
Table 2: Error events at destination 1.

| $E^{[1]}_1$ | $E^{[1]}_2$ | $E^{[1]}_3$ | $E^{[1]}_4$ | $E^{[1]}_5$ | $E^{[1]}_6$ | $E^{[1]}_7$ | $E^{[1]}_8$ | $N$ | $C$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | * | * | * | * | * | * | * | $2^7$ | $\emptyset$ |
| $E^{[2]}_1$ | 0 | 1 | * | * | 1 | * | 1 | 4 | $Q$ |
| $E^{[2]}_2$ | 0 | 1 | * | * | 1 | * | 0 | 2 | $Q, V_2$ |
| $E^{[2]}_3$ | 0 | 1 | * | * | 1 | * | 0 | 2 | $Q, V_2, U_2$ |
| $E^{[3]}_1$ | 0 | 0 | 1 | * | 1 | * | 1 | 2 | $Q, V_1$ |
| $E^{[3]}_2$ | 0 | 0 | 1 | * | 1 | * | 0 | 2 | $Q, V_2, V_1$ |
| $E^{[3]}_3$ | 0 | 0 | 1 | * | 1 | * | 0 | 2 | $Q, V_2, U_2, V_1$ |
| $E^{[4]}_1$ | 0 | 0 | 0 | 1 | 1 | * | 1 | 2 | $Q, V_1, U_1$ |
| $E^{[4]}_2$ | 0 | 0 | 0 | 1 | 1 | * | 0 | 2 | $Q, V_2, V_1, U_1$ |
| $E^{[4]}_3$ | 0 | 0 | 0 | 1 | 1 | * | 0 | 2 | $Q, V_2, U_2, V_1, U_1$ |
| $E^{[5]}_1$ | 0 | 0 | 0 | 0 | 1 | * | 1 | 2 | $Q, V_1, U_1, T_1$ |
| $E^{[5]}_2$ | 0 | 0 | 0 | 0 | 1 | * | 0 | 2 | $Q, V_2, V_1, U_1, T_1$ |
| $E^{[5]}_3$ | 0 | 0 | 0 | 0 | 1 | * | 0 | 2 | $Q, V_2, U_2, V_1, U_1, T_1$ |
| $E^{[6]}_1$ | 0 | 1 | * | * | 0 | * | 1 | 4 | $Q, S_1$ |
| $E^{[6]}_2$ | 0 | 1 | * | * | 0 | * | 0 | 2 | $Q, S_1, V_2$ |
| $E^{[6]}_3$ | 0 | 1 | * | * | 0 | * | 0 | 2 | $Q, S_1, V_2, U_2$ |
| $E^{[7]}_1$ | 0 | 0 | 1 | * | 0 | 1 | 1 | 2 | $Q, S_1, U_1$ |
| $E^{[7]}_2$ | 0 | 0 | 1 | * | 0 | 1 | 0 | 2 | $Q, S_1, V_2, V_1$ |
| $E^{[7]}_3$ | 0 | 0 | 1 | * | 0 | 1 | 0 | 2 | $Q, S_1, V_2, U_2, V_1$ |
| $E^{[8]}_1$ | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 2 | $Q, S_1, U_1, T_1$ |
| $E^{[8]}_2$ | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 2 | $Q, S_1, V_2, V_1, U_1$ |
| $E^{[8]}_3$ | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 2 | $Q, S_1, V_2, U_2, V_1, U_1$ |
| $E^{[9]}_1$ | 0 | 0 | 1 | * | 0 | 0 | 1 | 2 | $Q, S_1, Z_1, V_1$ |
| $E^{[9]}_2$ | 0 | 0 | 1 | * | 0 | 0 | 0 | 2 | $Q, S_1, Z_1, V_2, V_1$ |
| $E^{[9]}_3$ | 0 | 0 | 1 | * | 0 | 0 | 0 | 2 | $Q, S_1, Z_1, V_2, U_2, V_1$ |
| $E^{[10]}_1$ | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 2 | $Q, S_1, Z_1, U_1$ |
| $E^{[10]}_2$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | $Q, S_1, Z_1, V_2, U_1$ |
| $E^{[10]}_3$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | $Q, S_1, Z_1, V_2, U_1, U_1$ |
| OK | 0 | 0 | 0 | 0 | 0 | 0 | * | 2 | all correct |

Notice that, given $(Q, S_1)$ the input variables for source 1 are not independent of the input variables for source 2.

The possible error events are listed in Table 2. In Table 2 the symbols “1”, “0” and “*” have the following meaning. A “1” indicates that either the message index or the bin index are in error. A “0” indicates that both the message index and the bin index are correct. A “*” indicates that it does not matter whether the message index is in error; this is so because of superposition coding; in this case in fact, the codeword selected by the decoder—even though with the correct message index—is superimposed to a wrong codeword and it is thus independent of the received signal. In case of a “*”, the factorization of the joint probability needed for the evaluation of the probability of error is as for the case where the message is wrong; this implies that the error event that gives the most stringent rate bound is that for which the message is wrong (i.e., as far as error bounds are concerned, a “*” is equivalent to a “1”). The second to last column in Table 2
counts how many error events are included in the corresponding row (i.e., each “⋆” corresponds to two possible cases).

There are several groups of error events in Table 2. For event $E_{0}^{(1)}$: $Q$ is wrong, and hence all the decoded codewords are independent of the received signal. For events from $E_{1}^{(1)}$ to $E_{12}^{(1)}$: $S_{1}$ is wrong, and thus also $Z_{1}$ is wrong (because superimposed to $S_{1}$). For events from $E_{13}^{(1)}$ to $E_{21}^{(1)}$: $S_{1}$ is correct but $Z_{1}$ is wrong. For events from $E_{22}^{(1)}$ to $E_{27}^{(1)}$: both $S_{1}$ and $Z_{1}$ are correct. Notice that, because of the way codebooks are superimposed, out of the possible $2^{8} - 1 = 255$ error events, only 28 events matter. A way to understand the error events listed in Table 2 is as follows. From destination 1’s perspective, given $Q$, there are three “super-codebooks” to decode. Conditioned on $Q$, each “super-codebook” is the superposition of one or more codebooks; each “super-codebook” is represented as a separate line in Fig. 9.

Decoding proceeds as for a multiple access channel. In particular, we need to consider all possible combinations of events that consist of jointly decoding a set of messages from the first column of Fig. 10 and a set of messages from the second column of Fig. 10 and a set of messages from the third column of Fig. 10 (even though not all combinations are actual errors for destination 1). In considering such “joint-decoding events”, the messages that do not appear in the “set of jointly decoded messages” must be considered as correctly decoded and stripped from the received signal, as in a standard multiple access channel.

The last column in Table 2 is used as follows. Let $X$ be the set of transmitted
Figure 10: Destination 1 must consider all possible combinations of events that consist of jointly decoding a set of messages from the first column and a set of messages from the second column and a set of messages from the third column (even though not all combinations are actual errors for destination 1).

tabular}{c c c c}
0 & 0 & 0 & \\
T \_1 & 0 & 0 & \\
T \_1 \ U \_1 & Z \_1 & U \_2 & \\
T \_1 \ U \_1 \ V \_1 & Z \_1 \ S \_1 & U \_2 \ V \_2 & \\
\end{tabular}

codewords (we do not write here the superscript \( n \) that indicates the block-length in order to have a lighter notation), and \( X' \) be the set of decoded codewords. Let \( \mathcal{C} \) be the subset of the correctly decoded message indices such that \( X(\mathcal{C}) = X'(\mathcal{C}) \) (recall that with superposition coding, the decoder might select a codeword \( X' \) that is different from the transmitted codeword \( X \) but with same message index; this happens when an error is committed on one of the “base layer” codewords). The last column of Table 2 lists the elements of \( \mathcal{C} \), i.e., \( \mathcal{C} \) contains the codewords that have a “0” in the corresponding row. The sets \( \mathcal{C} \) are important for the factorization of the joint probability needed for the evaluation of the probability of error. The error analysis proceeds as follows. The joint distribution of the decoded codewords and the received signal is

\[
\sum_{X(\mathcal{C})X(\mathcal{C}')} P_{X'X'(\mathcal{C})X(\mathcal{C})Y} = \sum_{X(\mathcal{C})} P_{X(\mathcal{C})} P_{X(\mathcal{C})|X(\mathcal{C})} P_{X'(\mathcal{C})|X(\mathcal{C})} P_{X(\mathcal{C})X(\mathcal{C})} P_{Y|X(\mathcal{C})X(\mathcal{C})} = P_{X'(\mathcal{C})|X(\mathcal{C})} P_{X(\mathcal{C})Y},
\]

where \( P \) is a distribution from the set of possible input distributions in \((18)\) and \( P^{(g)} \) is computed from \( P \) as described in the codebook generation paragraph in Section 4.2 that is,

\[
P^{(g)}_{QV_1U_1T_1S_1Z_1V_2U_2} = P_Q P_{S_1|Q} P_{V_1U_1T_1|Q} P_{Z_1|Q} P_{S_1V_1} P_{V_2U_2|Q},
\]

where all the factors of \( P^{(g)} \) are obtained from the corresponding marginalization
of $P$. In the following, we shall drop the prime superscript to distinguish between the wrongly decoded codewords and the transmitted codewords. We will add a superscript “(g)” to the symbol for entropy to indicate that the entropy must be evaluated by using the distribution $P^{(g)}$; the symbol for entropy without any superscript indicates that the entropy must be evaluated by using the distribution $P$.

Destination 1 searches for codewords that are joint typicality with the received signal according to $P_{X(\mathcal{C})X(\mathcal{C}^c)Y}$; however, assuming that the messages in $\mathcal{C}$ are correctly decoded and those in $\mathcal{C}^c$ are wrongly decoded (this is the case that gives the most stringent error bound), the actual joint distribution is $P_{X(\mathcal{C})X(\mathcal{C}^c)Y}^{(g)}$. The probability of the error for the messages in $\mathcal{C}^c$ (neglecting the terms that will eventually be taken to go to zero) is:

\[
\Pr[\text{error } \mathcal{C}^c] = \sum_{x \in T_{n}((P_{X(\mathcal{C})X(\mathcal{C}^c)Y}|X(\mathcal{C})))} P_{X(\mathcal{C})X(\mathcal{C}^c)Y}^{(g)} P_{Y|X(\mathcal{C})}^{}
\]

\[
\leq \exp \left( n \left[ R(\mathcal{C}^c) + H(X(\mathcal{C}^c)|X(\mathcal{C})) + H(Y|X(\mathcal{C})X(\mathcal{C}^c)) - H(\mathcal{C}^c)(X(\mathcal{C})) + H(X(\mathcal{C}^c), X(\mathcal{C})) - H(\mathcal{C}^c)(X(\mathcal{C})), X(\mathcal{C})) \right] \right)
\]

\[
= \exp \left( n \left[ R(\mathcal{C}^c) - I(Y \land X(\mathcal{C}^c)|X(\mathcal{C})) + H(X(\mathcal{C}^c), X(\mathcal{C})) - H^{}(\mathcal{C}^c)(X(\mathcal{C})) \right] \right)
\]

where $R(\mathcal{C}^c)$ is the sum of the rates corresponding to the wrongly decoded messages that are indexed by $\mathcal{C}^c$. In the following, for any two distributions $P$ and $Q$, the notation $\mathbb{E}[\log(P/Q)]$ stands for the Kullback-Leibler divergence $D(P||Q)$. Let

\[
\Delta^{(1)} \triangleq H^{}(X(\mathcal{C}^c), X(\mathcal{C})) - H^{}(X(\mathcal{C}^c), X(\mathcal{C})) = \mathbb{E} \left[ \log \frac{P_{X(\mathcal{C}^c)\cup\mathcal{C}}}{P_{X(\mathcal{C}^c)\cup\mathcal{C}}} \right]
\]

\[
= \mathbb{E} \left[ \log \frac{P_{QV_1U_1T_1S_1Z_1V_2U_2}}{P_{Q}P_{V_1U_1T_1}|Q} \right]
\]

\[
= \mathbb{E} \left[ \log \frac{P_{S_1|QV_1U_1T_1}P_{Z_1|QV_1U_1T_1S_1}P_{V_2U_2|QV_1U_1T_1S_1Z_1}}{P_{S_1|Q}P_{Z_1|Q}P_{V_2U_2|Q}} \right]
\]

\[
= I(S_1 \land V_1 U_1 T_1|Q) + I(Z_1 \land U_1 T_1|Q S_1 V_1) + I(V_2 U_2 \land V_1 U_1 T_1 S_1 Z_1|Q)
\]
Finally, \( \Pr[\text{error } C^c] \to 0 \) as \( n \to \infty \) if

\[
R(C^c) \leq I(Y \wedge X(C^c)|X(C)) + \Delta^{(1)} - \mathbb{E} \left[ \log \frac{P_{X(C)}}{P^{(y)}_{X(C)}} \right].
\]

We now evaluate \( \Delta^{(1)} \) for all possible error events in Table 2. For \( \mathcal{E}_{0}^{(1)} \): \( Q \) is wrong and hence--because of superposition encoding--the most stringent error event is when the messages carried by \( Q \) and all the messages superimposed to \( Q \) are wrong. In this case \( C = \emptyset \) and thus \( \Delta^{(1)}_{0} = 0 \). It can be also easily verified that

\[
\Delta^{(1)}_{0} = \Delta^{(1)}_{\{Q\}} = \Delta^{(1)}_{\{Q,V_2\}} = \Delta^{(1)}_{\{Q,V_2,U_2\}} = \Delta^{(1)}_{\{Q,V_1\}} = \Delta^{(1)}_{\{Q,V_1,U_1\}} = \Delta^{(1)}_{\{Q,V_1,U_1,T_1\}} = 0,
\]

for \( \mathcal{E}_{0}^{(1)} \), \( \mathcal{E}_{1}^{(1)} \), \( \mathcal{E}_{2}^{(1)} \), \( \mathcal{E}_{3}^{(1)} \), \( \mathcal{E}_{4}^{(1)} \), \( \mathcal{E}_{7}^{(1)} \), \( \mathcal{E}_{10}^{(1)} \), \( \mathcal{E}_{13}^{(1)} \).
because in these cases \( P_{X(c)} = P_{X(c)}^{(g)} \). Then we have:

\[
\mathcal{E}_{5}^{(1)} : \Delta_{Q,V_{1},V_{2}}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QV_{2}}}{P_{Q}P_{V_{1}|Q}P_{V_{2}|Q}} \right] = I(V_{1} \wedge V_{2}|Q),
\]

\[
\mathcal{E}_{6}^{(1)} : \Delta_{Q,V_{1},V_{2},U_{2}}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QV_{2}U_{2}}}{P_{Q}P_{V_{1}|Q}P_{V_{2}|Q}P_{U_{2}|Q}} \right] = I(V_{1} \wedge V_{2}, U_{2}|Q),
\]

\[
\mathcal{E}_{7}^{(1)} : \Delta_{Q,V_{1},U_{1},V_{2}}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QV_{2}U_{1}V_{2}}}{P_{Q}P_{V_{1}|Q}P_{V_{2}|Q}P_{U_{1}|Q}P_{U_{2}|Q}} \right] = I(V_{1}, U_{1} \wedge V_{2}|Q),
\]

\[
\mathcal{E}_{8}^{(1)} : \Delta_{Q,V_{1},U_{1},V_{2},U_{2}}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QV_{2}U_{1}V_{1}U_{2}}}{P_{Q}P_{V_{1}|Q}P_{V_{2}|Q}P_{U_{1}|Q}P_{U_{2}|Q}} \right] = I(V_{1}, U_{1} \wedge V_{2}, U_{2}|Q),
\]

\[
\mathcal{E}_{9}^{(1)} : \Delta_{Q,V_{1},U_{1},T_{1},V_{2}}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QV_{2}U_{1}T_{1}V_{1}V_{2}}}{P_{Q}P_{V_{1}|Q}P_{V_{2}|Q}P_{U_{1}|Q}P_{U_{2}|Q}P_{T_{1}|Q}P_{T_{2}|Q}} \right] = I(V_{1}, U_{1} \wedge V_{2}, T_{1}|Q),
\]

\[
\mathcal{E}_{10}^{(1)} : \Delta_{Q,V_{1},U_{1},T_{1},V_{2},U_{2}}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QV_{2}U_{1}T_{1}V_{1}V_{2}U_{2}}}{P_{Q}P_{V_{1}|Q}P_{V_{2}|Q}P_{U_{1}|Q}P_{U_{2}|Q}P_{T_{1}|Q}P_{T_{2}|Q}P_{U_{2}|Q}} \right] = I(V_{1}, U_{1} \wedge V_{2}, T_{1}|Q),
\]

\[
\mathcal{E}_{11}^{(1)} : \Delta_{Q,S_{1},V_{2}}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_{1}V_{2}}}{P_{Q}P_{S_{1}|Q}P_{V_{2}|Q}} \right] = I(S_{1} \wedge V_{2}|Q),
\]

\[
\mathcal{E}_{12}^{(1)} : \Delta_{Q,S_{1},V_{2},U_{2}}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_{1}V_{2}U_{2}}}{P_{Q}P_{S_{1}|Q}P_{V_{2}|Q}P_{U_{2}|Q}} \right] = I(S_{1} \wedge V_{2}, U_{2}|Q),
\]

\[
\mathcal{E}_{13}^{(1)} : \Delta_{Q,S_{1},V_{2},U_{2}}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_{1}V_{2}}}{P_{Q}P_{S_{1}|Q}P_{V_{2}|Q}} \right] = I(S_{1} \wedge V_{2}, U_{2}|Q),
\]

\[
\mathcal{E}_{14}^{(1)} : \Delta_{Q,S_{1},V_{2}}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_{1}V_{2}}}{P_{Q}P_{S_{1}|Q}P_{V_{2}|Q}} \right] = I(S_{1} \wedge V_{2}, U_{2}|Q),
\]

\[
\mathcal{E}_{15}^{(1)} : \Delta_{Q,S_{1},V_{2}}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_{1}V_{2}}}{P_{Q}P_{S_{1}|Q}P_{V_{2}|Q}} \right] = I(S_{1} \wedge V_{2}, U_{2}|Q),
\]
and

\[ \mathcal{E}_{16}^{(1)} : \Delta_{Q,S_1,V_1}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_1V_1}}{P_Q P_{S_1|Q} P_{V_1|Q}} \right] = I(S_1 \wedge V_1|Q), \]

\[ \mathcal{E}_{17}^{(1)} : \Delta_{Q,S_1,V_1,V_2}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_1V_1V_2}}{P_Q P_{S_1|Q} P_{V_1|Q} P_{V_2|Q}} \right] = \mathbb{E} \left[ \log \frac{P_{V_1|Q} P_{V_2|Q} P_{QS_1V_1}}{P_{V_1|Q} P_{V_2|Q}} \right] = I(S_1 \wedge V_1|Q) + I(S_1, V_1 \wedge V_2|Q), \]

\[ \mathcal{E}_{18}^{(1)} : \Delta_{Q,S_1,V_1,V_2}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_1V_1V_2}}{P_Q P_{S_1|Q} P_{V_1|Q} P_{V_2|Q}} \right] = I(S_1 \wedge V_1|Q) + I(S_1, V_1 \wedge V_2, U_2|Q), \]

and

\[ \mathcal{E}_{19}^{(1)} : \Delta_{Q,S_1,V_1,U_1}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_1V_1U_1}}{P_Q P_{S_1|Q} P_{V_1|Q}} \right] = I(S_1 \wedge V_1, U_1|Q), \]

\[ \mathcal{E}_{20}^{(1)} : \Delta_{Q,S_1,V_1,U_1,V_2}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_1V_1U_1V_2}}{P_Q P_{S_1|Q} P_{V_1|Q} P_{V_2|Q}} \right] = I(S_1 \wedge V_1, U_1|Q) + I(S_1, V_1, U_1 \wedge V_2|Q), \]

\[ \mathcal{E}_{21}^{(1)} : \Delta_{Q,S_1,V_1,U_1,V_2}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_1V_1U_1V_2}}{P_Q P_{S_1|Q} P_{V_1|Q} P_{V_2|Q}} \right] = I(S_1 \wedge V_1, U_1|Q) + I(S_1, V_1, U_1 \wedge V_2, U_2|Q), \]

and

\[ \mathcal{E}_{22}^{(1)} : \Delta_{Q,S_1,Z_1,V_1}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_1Z_1V_1}}{P_Q P_{S_1|Q} P_{V_1|Q} P_{Z_1|Q} P_{S_1V_1}} \right] = \mathbb{E} \left[ \log \frac{P_{V_1|Q} P_{S_1}}{P_{V_1|Q}} \right] = I(S_1 \wedge V_1|Q), \]

\[ \mathcal{E}_{23}^{(1)} : \Delta_{Q,S_1,Z_1,U_1,V_2}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_1Z_1V_1V_2}}{P_Q P_{S_1|Q} P_{V_1|Q} P_{Z_1|Q} P_{S_1V_1} P_{V_2|Q}} \right] = \mathbb{E} \left[ \log \frac{P_{V_1|Q} P_{S_1} P_{V_2|Q} P_{QS_1V_1}}{P_{V_1|Q} P_{V_2|Q}} \right] = I(S_1 \wedge V_1|Q) + I(V_2 \wedge S_1, Z_1, V_1|Q), \]

\[ \mathcal{E}_{24}^{(1)} : \Delta_{Q,S_1,Z_1,U_2,V_1}^{(1)} = \mathbb{E} \left[ \log \frac{P_{QS_1Z_1V_1U_2}}{P_Q P_{S_1|Q} P_{V_1|Q} P_{Z_1|Q} P_{S_1V_1} P_{U_2|Q}} \right] = I(S_1 \wedge V_1|Q) + I(V_2, U_2 \wedge S_1, Z_1, V_1|Q), \]
and

$$E^{(1)}_{25}: \Delta^{(1)}_{Q,S_1, z_1, v_1, u_1} = \mathbb{E} \left[ \log \frac{P_{Q|S_1} P_{S_1} P_{V_1|S_1} P_{Z_1|S_1, V_1}}{P_{V_1|Q} P_{U_1|Q}^{v_1}} \right] = \mathbb{E} \left[ \log \frac{P_{V_1|Q} P_{U_1|Q}^{v_1} P_{S_1} P_{V_1|S_1}}{P_{Q} P_{S_1} P_{V_1|S_1} P_{Z_1|S_1, V_1}} \right]$$

$$= I(S_1 \wedge V_1|Q) + I(U_1 \wedge S_1, Z_1|Q, V_1),$$

$$E^{(1)}_{26}: \Delta^{(1)}_{Q,S_1, z_1, v_1, u_1} = \mathbb{E} \left[ \log \frac{P_{Q|S_1} P_{S_1} P_{V_1|S_1} P_{Z_1|S_1, V_1} P_{V_2|Q}}{P_{V_1|Q} P_{U_1|Q}^{v_1}} \right]$$

$$= I(S_1 \wedge V_1|Q) + I(U_1 \wedge S_1, Z_1|Q, V_1) + I(V_2 \wedge S_1, V_1, Z_1, U_1|Q),$$

$$E^{(1)}_{27}: \Delta^{(1)}_{Q,S_1, z_1, v_1, u_1, u_1} = \mathbb{E} \left[ \log \frac{P_{Q|S_1} P_{S_1} P_{V_1|S_1} P_{Z_1|S_1, V_1} P_{V_2|Q}}{P_{V_1|Q} P_{U_1|Q}^{v_1}} \right]$$

$$= I(S_1 \wedge V_1|Q) + I(U_1 \wedge S_1, Z_1|Q, V_1) + I(V_2, U_2 \wedge S_1, V_1, Z_1, U_1|Q).$$

Let

$$R_Q = R_{10c} + R_{20c}$$

$$R_{V_1} = R_{10c}, \quad R_{V_1} = R_{10c} + R_{10c}$$

$$R_{V_2} = R_{20c}, \quad R_{V_2} = R_{20c} + R_{20c}$$

$$R_{U_1} = R_{10n} + R_{10m}$$

$$R_{T_1} = R_{11n} + R_{11n}$$

$$R_{S_1} = R_{11c}, \quad R_{S_1} = R_{11c} + R_{11c}$$

$$R_{Z_1} = R_{11c}, \quad R_{Z_1} = R_{11c} + R_{11c}$$

$$R_{U_2} = R_{20n} + R_{20m}$$

$$R_{T_2} = R_{22n} + R_{22n}$$

$$R_{S_2} = R_{22c}, \quad R_{S_2} = R_{22c} + R_{22c}$$

$$R_{Z_2} = R_{22c}, \quad R_{Z_2} = R_{22c} + R_{22c},$$

and hence

$$R_{V_1} + R_{V_2} = R_Q + (R_{V_1} + R_{V_2})$$

$$R_{Z_1} = R_{S_1} - R_{S_1} + R_{Z_1}$$

$$R_{Z_2} = R_{S_2} - R_{S_2} + R_{Z_2},$$

Recall that we defined

$$\Delta^{(1)} = I(S_1 \wedge V_1, U_1, T_1|Q) + I(Z_1 \wedge U_1, T_1|Q, S_1, V_1) + I(V_2, U_2 \wedge V_1, U_1, T_1, Z_1|Q, S_1).$$
In addition to the rate constraints from cooperation among the sources (see (21)), i.e.,

\[ R_{Z_1} \leq C_1^{(1)} = I(Z_1 \land Y_2|X_2, V_1) + I(Z_1 \land S_2|Q, S_1, V_1) \]

\[ R_{V_1} + R_{Z_1} \leq C_2^{(1)} = I(V_1, Z_1 \land Y_2|X_2) + I(Z_1 \land S_2|Q, S_1, V_1) + I(V_1 \land S_1, S_2|Q) \]

\[ R_{Z_2} \leq C_1^{(2)} = I(Z_2 \land Y_1|X_1, V_2) + I(Z_2 \land S_1|Q, S_2, V_2) \]

\[ R_{V_2} + R_{Z_2} \leq C_2^{(2)} = I(V_2, Z_2 \land Y_1|X_1) + I(Z_2 \land S_1|Q, S_2, V_2) + I(V_2 \land S_1, S_2|Q) \]

we have the following rate constraints arising from decoding at destination 1:

\[ \mathcal{E}_0^{(1)} : R_{U_1} + R_{T_1} + \frac{[R_{S_1} + R'_{Z_1} - R'_{S_1}]}{R_{Z_1}} + R_{U_2} \left[ R_Q + R'_{V_1} + R'_{V_2} \right] = \]

\[ = R_{U_1} + R_{T_1} + R_{Z_1} + R_{U_2} + R_{V_1} + R_{V_2} \leq E_0^{(1)} = I(Y_3 \land Q, V_1, U_1, T_1, S_1, Z_1, V_2, U_2) - (R'_{S_1}) + \Delta^{(1)} \]

\[ \mathcal{E}_1^{(1)} : R_{U_1} + R_{T_1} + R_{Z_1} + R_{U_2} \leq E_1^{(1)} = I(Y_3 \land Q, V_1, U_1, T_1, S_1, Z_1, V_2, U_2|Q) - (R'_{V_1} + R'_{S_1} + R'_{V_2}) + \Delta^{(1)} \]

\[ \mathcal{E}_2^{(1)} : R_{U_1} + R_{T_1} + R_{Z_1} + R_{U_2} \leq E_2^{(1)} = I(Y_3 \land Q, V_1, U_1, T_1, S_1, Z_1, U_2|Q, V_2) - (R'_{V_1} + R'_{S_1}) + \Delta^{(1)} \]

\[ \mathcal{E}_3^{(1)} : R_{U_1} + R_{T_1} + R_{Z_1} \leq E_3^{(1)} = I(Y_3 \land Q, V_1, U_1, T_1, S_1, Z_1|Q, V_2, U_2) - (R'_{V_1} + R'_{S_1}) + \Delta^{(1)} \]

\[ \mathcal{E}_4^{(1)} : R_{U_1} + R_{T_1} + R_{Z_1} + R_{U_2} \leq E_4^{(1)} = I(Y_3 \land U_1, T_1, S_1, Z_1, V_2, U_2|Q, V_1) - (R'_{S_1} + R'_{V_2}) + \Delta^{(1)} \]

\[ \mathcal{E}_5^{(1)} : R_{U_1} + R_{T_1} + R_{Z_1} + R_{U_2} \leq E_5^{(1)} = I(Y_3 \land U_1, T_1, S_1, Z_1, U_2|Q, V_1, V_2) - (R'_{S_1}) + \Delta^{(1)} - I(V_1 \land V_2|Q) \]

\[ \mathcal{E}_6^{(1)} : R_{U_1} + R_{T_1} + R_{Z_1} \leq E_6^{(1)} = I(Y_3 \land U_1, T_1, S_1, Z_1|Q, V_1, V_2, U_2) - (R'_{S_1}) + \Delta^{(1)} - I(V_1 \land V_2, U_2|Q) \]
\( \mathcal{E}_7^{(1)} : R_{T_1} + R_{Z_1} + R_{U_2} \leq E_7^{(1)} = I(Y_3 \land T_1, S_1, Z_1, V_2, U_2|Q, V_1, U_1) - (R_{S_1} + R_{V_2}) + \Delta^{(1)} \)

\( \mathcal{E}_8^{(1)} : R_{T_1} + R_{Z_1} + R_{U_2} \leq E_8^{(1)} = I(Y_3 \land T_1, S_1, Z_1, U_2|Q, V_1, U_1, V_2) - (R_{S_1}) + \Delta^{(1)} - I(V_1, U_1 \land V_2|Q) \)

\( \mathcal{E}_9^{(1)} : R_{T_1} + R_{Z_1} \leq E_9^{(1)} = I(Y_3 \land T_1, S_1, Z_1|Q, V_1, U_1, V_2, U_2) - (R_{S_1}) + \Delta^{(1)} - I(V_1, U_1 \land V_2, U_2|Q) \)

\( \mathcal{E}_{10}^{(1)} : R_{Z_1} + R_{U_2} \leq E_{10}^{(1)} = I(Y_3 \land S_1, Z_1, V_2, U_2|Q, V_1, U_1, T_1) - (R_{S_1} + R_{V_2}) + \Delta^{(1)} \)

\( \mathcal{E}_{11}^{(1)} : R_{Z_1} + R_{U_2} \leq E_{11}^{(1)} = I(Y_3 \land S_1, Z_1, U_2|Q, V_1, U_1, T_1, V_2) - (R_{S_1}) + \Delta^{(1)} - I(V_1, U_1, T_1 \land V_2|Q) \)

\( \mathcal{E}_{12}^{(1)} : R_{Z_1} \leq E_{12}^{(1)} = I(Y_3 \land S_1, Z_1|Q, V_1, U_1, T_1, V_2, U_2) - (R_{S_1}) + \Delta^{(1)} - I(V_1, U_1, T_1 \land V_2, U_2|Q) \)

\( \mathcal{E}_{13}^{(1)} : R_{U_1} + R_{T_1} + R_{U_2} \leq E_{13}^{(1)} = I(Y_3 \land V_1, U_1, T_1, Z_1, V_2, U_2|Q, S_1) - (R_{V_1} + R_{Z_1} + R_{V_2}) + \Delta^{(1)} \)

\( \mathcal{E}_{14}^{(1)} : R_{U_1} + R_{T_1} + R_{U_2} \leq E_{14}^{(1)} = I(Y_3 \land V_1, U_1, T_1, Z_1, U_2|Q, S_1, V_2) - (R_{V_1} + R_{Z_1}) + \Delta^{(1)} - I(S_1 \land V_2|Q) \)

\( \mathcal{E}_{15}^{(1)} : R_{U_1} + R_{T_1} \leq E_{15}^{(1)} = I(Y_3 \land V_1, U_1, T_1, Z_1|Q, S_1, V_2, U_2) - (R_{V_1} + R_{Z_1}) + \Delta^{(1)} - I(S_1 \land V_2, U_2|Q) \)

\( \mathcal{E}_{16}^{(1)} : R_{U_1} + R_{T_1} + R_{U_2} \leq E_{16}^{(1)} = I(Y_3 \land U_1, T_1, Z_1, V_2, U_2|Q, S_1, V_1) - (R_{Z_1} + R_{V_2}) + \Delta^{(1)} - I(S_1 \land V_1|Q) \)

\( \mathcal{E}_{17}^{(1)} : R_{U_1} + R_{T_1} + R_{U_2} \leq E_{17}^{(1)} = I(Y_3 \land U_1, T_1, Z_1, U_2|Q, S_1, V_1, V_2) - (R_{Z_1}) + \Delta^{(1)} - I(S_1 \land V_1|Q) - I(S_1, V_1 \land V_2|Q) \)

\( \mathcal{E}_{18}^{(1)} : R_{U_1} + R_{T_1} \leq E_{18}^{(1)} = I(Y_3 \land U_1, T_1, Z_1|Q, S_1, V_1, V_2, U_2) - (R_{Z_1}) + \Delta^{(1)} - I(S_1 \land V_1|Q) - I(S_1, V_1 \land V_2, U_2|Q) \)
$\mathcal{E}_{19}^{(1)} : R_{T_1} + R_{U_2} \leq E_{19}^{(1)} = I(Y_3 \land T_1, Z_1, V_2, U_2|Q, S_1, V_1, U_1) - (R'_{Z_1} + R'_{V_2})$
+ $\Delta^{(1)} - I(S_1 \land V_1, U_1|Q)$,

$\mathcal{E}_{20}^{(1)} : R_{T_1} + R_{U_2} \leq E_{20}^{(1)} = I(Y_3 \land T_1, Z_1, U_2|Q, S_1, V_1, U_1, V_2) - (R'_{Z_1})$
+ $\Delta^{(1)} - I(S_1 \land V_1, U_1|Q) - I(S_1, V_1, U_1 \land V_2|Q)$,

$\mathcal{E}_{21}^{(1)} : R_{T_1} \leq E_{21}^{(1)} = I(Y_3 \land T_1, Z_1|Q, S_1, V_1, U_1, V_2) - (R'_{Z_1})$
+ $\Delta^{(1)} - I(S_1 \land V_1, U_1|Q) - I(S_1, V_1, U_1 \land V_2, U_2|Q)$,

$\mathcal{E}_{22}^{(1)} : R_{U_1} + R_{T_1} + R_{U_2} \leq E_{22}^{(1)} = I(Y_3 \land U_1, T_1, V_2, U_2|Q, S_1, Z_1, V_1) - (R'_{V_2})$
+ $\Delta^{(1)} - I(S_1 \land V_1|Q)$,

$\mathcal{E}_{23}^{(1)} : R_{U_1} + R_{T_1} + R_{U_2} \leq E_{23}^{(1)} = I(Y_3 \land U_1, T_1, U_2|Q, S_1, Z_1, V_1, V_2)$
+ $\Delta^{(1)} - I(S_1 \land V_1|Q) - I(S_1, Z_1, V_1 \land V_2|Q)$,

$\mathcal{E}_{24}^{(1)} : R_{U_1} + R_{T_1} \leq E_{24}^{(1)} = I(Y_3 \land U_1, T_1|Q, S_1, Z_1, V_1, V_2, U_2)$
+ $\Delta^{(1)} - I(S_1 \land V_1|Q) - I(S_1, Z_1, V_1 \land V_2, U_2|Q)$,

$\mathcal{E}_{25}^{(1)} : R_{T_1} + R_{U_2} \leq E_{25}^{(1)} = I(Y_3 \land T_1, V_2, U_2|Q, S_1, Z_1, V_1, U_1) - (R'_{V_2})$
+ $\Delta^{(1)} - I(S_1 \land V_1|Q) - I(S_1, Z_1 \land U_1|Q)$,

$\mathcal{E}_{26}^{(1)} : R_{T_1} + R_{U_2} \leq E_{26}^{(1)} = I(Y_3 \land T_1, U_2|Q, S_1, Z_1, V_1, U_1, V_2)$
+ $\Delta^{(1)} - I(S_1 \land V_1|Q) - I(S_1, Z_1 \land U_1|Q) - I(S_1, Z_1, V_1, U_1 \land V_2|Q)$,

$\mathcal{E}_{27}^{(1)} : R_{T_1} \leq E_{27}^{(1)} = I(Y_3 \land T_1|Q, S_1, Z_1, V_1, U_1, V_2, U_2)$
+ $\Delta^{(1)} - I(S_1 \land V_1|Q) - I(S_1, Z_1 \land U_1|Q) - I(S_1, Z_1, V_1, U_1 \land V_2, U_2|Q)$.

All the above constraints combined give the region in (22).

Subsets of the above achievable region with fewer rate constraints can be obtained as follows:

- If $R'_{V_1} = R'_{V_2} = 0$, that is, $V_1$ and $V_2$ are not binned against the known interference, then $V_1$ and $V_2$ are correct whenever $Q$ is correct. In this case, 16 of the 31 error events listed in Table 2 are impossible (all those for which the bin index in either $V_1$ or $V_2$ is wrong).

- If $R'_{Z_1} = 0$ (similar observation can be made if $R'_{Z_2} = 0$), that is, $Z_1$ is not binned against the known interference, then $Z_1$ is correct whenever $Q$
and $V_1$ are correct. In this case, the 9 error events from $E_{13}^{(1)}$ to $E_{21}^{(1)}$ listed in Table 2 are impossible and the achievable region becomes

$$R_{V_1} + R_{V_2} + R_{U_1} + R_{T_1} + R_{U_2} \leq E_0^{(1)} \quad (32a)$$

$$R_{U_1} + R_{T_1} + R_{U_2} \leq \min \{ E_1^{(1)}, E_2^{(1)}, E_4^{(1)}, E_5^{(1)}, E_{13}^{(1)}, E_{14}^{(1)}, E_{16}^{(1)}, E_{17}^{(1)}, E_{22}^{(1)}, E_{23}^{(1)} \} \quad (32b)$$

$$R_{U_1} + R_{T_1} \leq \min \{ E_3^{(1)}, E_6^{(1)}, E_{15}^{(1)}, E_{18}^{(1)}, E_{24}^{(1)} \} \quad (32c)$$

$$R_{T_1} + R_{U_2} \leq \min \{ E_7^{(1)}, E_8^{(1)}, E_{19}^{(1)}, E_{20}^{(1)}, E_{25}^{(1)}, E_{26}^{(1)} \} \quad (32d)$$

$$R_{T_1} \leq \min \{ E_9^{(1)}, E_{21}^{(1)}, E_{27}^{(1)} \}, \quad (32e)$$

$$R_{U_2} \leq \min \{ E_{10}^{(1)}, E_{11}^{(1)} \} \quad (32f)$$

$$R_{Z_1} \leq E_{12}^{(1)} \quad (32g)$$

with only five rate constraints, as for the case of superposition only. Notice that the rate bound on $R_{U_2}$ can be removed since an error on $U_2$ alone is not an error from the point of view of source 1.

• Instead of joint decoding of all the messages, one can perform a two-step decoding as follows. First step: decode $Q$ and $S_1$ jointly, and then strip them from the received signal. This the first decoding step is successful if

$$R_{S_1} \leq I(Y_3 \land S_1 | Q)$$

$$R_Q + R_{S_1} \leq I(Y_3 \land S_1, Q).$$

Second step: jointly decode all the other messages. For this second step, one only needs to consider the error events from $E_{13}^{(1)}$ to $E_{27}^{(1)}$. 

70