Joule-Thomson expansion of hyperscaling violating black holes with spherical and hyperbolic horizons

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Abstract

We study the Joule-Thomson expansion of spherical and hyperbolic black holes with hyperscaling violating metric background. We compute the Joule-Thomson coefficient and inversion temperature for two horizons. Also, we investigate the effects of dynamical and hyperscaling violating exponents for two different horizons. Here, the minimum inversion and the critical temperatures for the spherical AdS case are discussed. Finally, by comparing two corresponding topologies with determined mass we show that the hyperbolic black hole has much higher inversion temperature than the spherical black hole. So, the corresponding results are shown by some interesting figures.

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1. Introduction

As we know the AdS/CFT correspondence play an important role for the describing of relation between strong and weakly coupled systems [1–3]. In strongly coupled systems as $N = 4$ a quantum phase transition occurs at a critical point, where the system reveals a scaling symmetry. On the other hand, some condensed matter systems at this critical point may exhibit an anisotropic scaling symmetry between space and time as $t \rightarrow \lambda^z t$, $x_i \rightarrow \lambda x_i$ [4,5], where $z$ is dynamical exponent. As a result of this anisotropic scaling, at a low temperature, the specific heat scales with respect to temperature as $c_V \sim T^{d/z}$. According to the holographic duality, the
gravitational theory lead us to scaling symmetry which is given by Lifshitz spacetimes [6]. On the other hand, we have Lifshitz scaling symmetries at quantum critical points and a dynamical critical exponent \( z \), and they are related to each other by different relations. In that case the corresponding metric background is hyperscaling violation [7,8]. The systems with hyperscaling violation metric background have a specific heat as \( c_V \sim T^{d-\theta/z} \). Here, \( \theta \) is hyperscaling violation exponent and also play an important role for the investigation of holographic in the above mentioned theory.

Here, we note that the hyperscaling violating geometry is a generalization of Lifshitz spacetime. The black hole solutions of hyperscaling violating spacetime with planar topology are investigated by [9–19]. Also, here we note that behavior of hyperscaling violation background asymptotically lead us to have spherical and hyperbolic solution [20]. In the pure Lifshitz case as \( \theta = 0, z > 1 \), there is not hyperbolic black hole solutions. In order to have the hyperbolic black holes solution, one can use the null energy conditions for the \( z > 1 \) at special values of \( \theta \) [20]. Generally, we say that The null energy conditions give us opportunity to have suitable parameter space. So, in such cases the hyperscaling violation exponent must be nonzero and equal to \( \theta = d(z - 1) \) which is summarized in Table 1. On the other hand, the thermal properties of black hole lead us to investigate the above mentioned metric background and also find the suitable value of \( \theta \).

Holographic correspondence is a powerful tool to description strongly coupled systems by gravity of weakly coupling [1–3]. In strongly coupled systems a quantum phase transition occurs at a critical point, where the system reveals a scaling symmetry. For instance some strongly condensed matter systems at this critical point may exhibit an anisotropic scaling symmetry between space and time, which is characterized by the dynamical exponent \( z: (t \rightarrow \lambda^z t, x_i \rightarrow \lambda x_i) \) [4,5]. As a result of this anisotropic scaling, at a low temperature, the specific heat scales with respect to temperature as \( c_V \sim T^{d/z} \) (where \( d \) is the dimensionality of space). According to the holographic duality, the gravitational theory which admits this scaling symmetry is given by Lifshitz spacetimes [6].

On the other hand, a quantum critical point is governed by other critical exponents, which are related to each other with various relations [7]. Hyperscaling relations are a class of these critical exponent relations which the dimensionality of space explicitly appears [8], and are valid only below the upper critical dimension. So, critical theories include above their upper critical dimension, violate the hyperscaling relations. In these theories, the specific heat scales as \( c_V \sim T^{(d-\theta)/z} \), where \( \theta \) is the hyperscaling violation exponent and obviously lowers the dimensionality of the theory. Such behavior is geometrically described by hyperscaling violating spacetime which is a generalization of the Lifshitz spacetime. The hyperscaling violating planar black holes have been found in [9–19] which are solutions to gravitational theories with higher order gravitational corrections or additional matter fields.

| Table 1 |
| Range of parameter space. |
| \( z \) | Hyperbolic \( k = -1 \) | Planar \( k = 0 \) | Spherical \( k = 1 \) |
| \( z < 1 \) | no solution | no solution | no solution |
| \( 1 \leq z < 2 \) | \( \theta = d(z - 1) \) | \( \theta \leq d(z - 1) \) | \( \theta \leq d(z - 1) \) |
| \( z \geq 2 \) | no solution | \( \theta < d \) | \( \theta < d \) |
structured in [20]. As we know the hyperbolic black hole solutions are absent for pure Lifshitz case ($\theta = 0, z > 1$). But, by using the null energy conditions one can find the hyperbolic black holes for $z > 1$, when the hyperscaling violation parameter is given by $\theta = d(z - 1)$ [20]. The null energy conditions create a limitation on the parameter space which is summarized in Table 1.

The authors of [20] also found thermodynamic quantities and studied extended thermodynamics of these black holes, extensively. Here, as a further step in this direction, we study the Joule-Thomson effect of the hyperscaling violating black holes with spherical and hyperbolic horizons.

The Joule-Thomson effect describes expansion of a gas from high pressure to low pressure during an isenthalpic process. Since the black hole mass is interpreted as enthalpy, such a process implies the black hole expansion at constant mass. Joule-Thomson coefficient $\mu$ measures the black hole temperature with respect to the pressure ($\mu = (\partial T/\partial P)_\mu$), so that $\mu > 0$ refers to cooling process and $\mu < 0$ to the heating one. $\mu = 0$ determines the inversion point ($T_i, P_i$), which discriminates the heating process from cooling process.

The first study on the Joule-Thomson expansion of the black holes was carried out by Ökçü and Aydiner in [21]. They investigated on expansion of the charged Ads black holes in constant mass and compared the result with van der Waals fluids. Subsequently, several papers were presented on this topic for different black holes [22–28].

In the case of hyperscaling violating black holes with non-trivial topologies there are questions that might be noteworthy. How does change the temperature of black hole during the constant mass expansion? How does inversion temperature depend on parameters $z$ and $\theta$? Where does occur the cooling/heating process? Hyperbolic black holes are known to have peculiar features, such as having finite temperature and non-vanishing entropy at their massless limit. Is there any unexpected behavior here of the hyperbolic black hole as compared to spherical case? In this paper we will answer the aforementioned questions. We can only study the spherical black holes for $z > 2$, because the hyperbolic black holes can only exist at $1 < z < 2$. We study both spherical and hyperbolic black holes for $1 \leq z < 2$ and compare the results together.

The outline of this paper is as follows: In section 2, we review the spherical and hyperbolic asymptotic hyperscaling violating black holes and their essential thermal quantities. In section 3, we study the Joule-Thomson effect for spherical and hyperbolic black holes, individually. Also here, we compare the results of two topologies together. Our conclusion will be presented in the last section.

2. A brief review of hyperscaling violating black holes with spherical and hyperbolic horizons

The Einstein-Maxwell-Dilaton action in (d+2)-dimensional spacetime is considered as [20]

$$S = -\frac{1}{16\pi G} \int d^{d+2}x \sqrt{-g} \left[ R - \frac{1}{2} (\nabla \mu \phi)^2 + V(\phi) - \frac{1}{4} X(\phi) F^2 - \frac{1}{4} Y(\phi) H^2 - \frac{1}{4} Z(\phi) K^2 \right],$$

(1)

where $F = dA$ is introduced to support the Lifshitz asymptotic of the geometry, $H = dB$ to support the non-trivial topology and $K = dC$ for solution with electric charge. Potential and dilaton couplings are considered as

$$V = V_0 e^{\lambda_0 \phi}, \quad X = X_0 e^{\lambda_1 \phi}, \quad Y = Y_0 e^{\lambda_2 \phi}, \quad Z = Z_0 e^{\lambda_3 \phi},$$

(2)

where $V_0, X_0, Y_0, Z_0$ and $\lambda_i$ are assumed as arbitrary constants. The metric is given by
\[ ds^2 = \left( \frac{r}{r_F} \right)^{-2\theta/d} \left( - \left( \frac{r}{r_F} \right)^2 f(r) dt^2 + \frac{l^2}{f(r)r^2} dr^2 + r^2 d\Omega_{k,d}^2 \right). \]  
\[ A = a(r)dt, \quad B = b(r)dt, \quad C = c(r)dt, \quad \phi = \phi(r), \]

where
\[ d\Omega_{k=1,d}^2 = d\chi_0^2 + \sin(\chi_0)^2 d\chi_1^2 + \cdots + \sin(\chi_{d-2})^2 d\chi_{d-1}^2, \]
\[ d\Omega_{k=0,d}^2 = \frac{d\chi_d^2}{l^2}, \quad d\Omega_{k=-1,d}^2 = d\chi_0^2 + \sinh(\chi_0)^2 d\Omega_{k=1,d}^2. \]

The blackening factor \( f(r) \) satisfies \( f(r) = 1 \) near the asymptotic boundary, and in this limit the Ansatz (3) is covariant under the following scale transformations
\[ t \rightarrow \lambda^2 t, \quad \Omega \rightarrow \lambda \Omega, \quad r \rightarrow \lambda^{-1} r, \quad ds \rightarrow \lambda^{0/d} ds. \]

It is clear that with a non-zero \( \theta \), the distance is not invariant under the scaling which implies violation of hyperscaling in the dual field theory.

Hyperbolic, planar and spherical topologies are characterized by the \( k = -1, 0, 1 \), respectively. The case \( k = 0 \) (planar topology) is not our interest in this article. Total scale of spatial geometry is specified by \( l \) which can be considered as a generalization of the Ads radius. The constants \( z \) and \( \theta \) are Lifshitz dynamical and hyperscaling violation exponents, respectively. Also, \( \chi_i \) are standard angles and the radius \( r_F \) is related to \( UV \) physics (see [20] for more details).

Ref. [20] derived hyperscaling violating black hole solutions as
\[ \phi = \phi_0 + \gamma \log r, \]
\[ F = -\rho_1 e^{-\lambda_1 \phi(r)} r^{-2\theta/d} d\theta z - 1 - d \gamma - 1 d t d r, \]
\[ H = -\rho_2 e^{-\lambda_2 \phi(r)} r^{-2\theta/d} d\theta z - 1 - d \gamma - 1 d t d r, \]
\[ K = -\rho_3 e^{-\lambda_3 \phi(r)} r^{-2\theta/d} d\theta z - 1 - d \gamma - 1 d t d r, \]
\[ f = 1 + k \frac{(d - 1)^2}{(d - \theta + z - 2)^2 r^2} - \frac{m}{r^{d-\theta+z}} + \frac{q^2}{r^{2(d-\theta+z-1)}}. \]

where \( \gamma \equiv \sqrt{(d - \theta)(z - 1 - \theta/d)}. \) The parameters are defined as follows
\[ \lambda_0 = \frac{2\theta}{\gamma d}, \quad \lambda_1 = - \frac{2(d - \theta + \theta/d)}{\gamma}, \quad \lambda_2 = - \frac{2(d - 1)(d - \theta)}{\gamma d}, \quad \lambda_3 = - \frac{\gamma}{d - \theta}. \]
\[ V_0 = (d - \theta + z - 1)(d - \theta + z) l^{-2} r_F^{-2\theta/d} e^{-\lambda_0 \phi_0}, \]
\[ \rho_1^2 = 2(z - 1)(d - \theta + z) X_0^{-2} r_F^{-2\theta/d} e^{\lambda_1 \phi_0}, \]
\[ \rho_2^2 = 2k \frac{(d - 1)(d - 1)(-\theta)}{d - \theta + z - 2} Y_0^{-2} (1 - z) r_F^{-2\theta/d} e^{\lambda_2 \phi_0}, \]
\[ \rho_3^2 = 2q^2 (d - \theta)(d - \theta + z - 2) Z_0^{-2} r_F^{-2\theta/d} e^{\lambda_3 \phi_0}. \]

The constants \( X_0, Y_0, Z_0 \) are positive parameters, which indicate the coupling strength at the gauge fields with gravity. To attain this solution, it is considered that \( d - \theta + z - 2 > 0 \) and \( \gamma \in \mathbb{R} \). In the blackening factor \( f(r) \), the mass \( m \) and the charge \( q \) parameters are related to the black hole mass \( M \) and charge \( Q \), respectively.

As mentioned before, the hyperbolic black holes exist for specific value \( \theta = d(z - 1) \). But we have \( \gamma = 0 \) for this value of \( \theta \), which causes \( \lambda_i \rightarrow \infty \) for \( i = \{0, 1, 2\} \) and \( \lambda_3 = 0 \). In this limit the solution seems singular. Hence, for a physical solution in this case, the authors of [20] redefined the scalar field...
\[ \phi(r) \rightarrow \gamma \tilde{\phi}(r) \]  

(17)

So, the action becomes

\[
S = -\frac{1}{16\pi G} \int d^{d+2}x \sqrt{-g} \left[ R - \gamma^2 \frac{1}{2} (\nabla \phi)^2 + V(\phi) - \frac{1}{4} X(\phi) F^2 - \frac{1}{4} Y(\phi) H^2 - \frac{1}{4} Z(\phi) K^2 \right].
\]

(18)

where

\[ V = V_0 e^{\tilde{\lambda}_0 \phi}, \quad X = X_0 e^{\tilde{\lambda}_1 \phi}, \quad Y = Y_0 e^{\tilde{\lambda}_2 \phi}, \quad Z = Z_0 e^{\tilde{\lambda}_3 \phi}. \]

(19)

In the limit \( \gamma \rightarrow 0 \), the charge associated to \( H(\rho_2) \), the kinetic term for the dilaton in the action and the parameter \( \lambda_3 \) will be vanished. By using these simplifications, the black hole solutions are now given by

\[
\tilde{\phi} = \tilde{\phi}_0 + \gamma \log r, \\
F = -\rho_1 e^{-\tilde{\lambda}_1 \phi(r)} r^{-d(2-z)-z+1} dt dr, \\
K = -\rho_3 r^{-d(2-z)-z+1} dt dr, \\
f = 1 + \frac{k}{(2-z)^2} \frac{l^2}{r^2} + \frac{m}{2(d(2-z)+z-1)} + \frac{q^2}{r^2(d(2-z)+z-1)}
\]

(20) (21) (22) (23)

where

\[
\tilde{\lambda}_0 = 2(z - 1), \quad \tilde{\lambda}_1 = -2[d(2-z)+z-1], \\
V_0 = [d(2-z)+z][d(2-z)+z-1] l^2 r_F^{-2(z-1)} e^{-\tilde{\lambda}_0 \phi_0}, \\
\rho_1^2 = 2(z-1)[d(2-z)+z] X_0^{-1} l^{-2} r_F^{2(z-1)} e^{\tilde{\lambda}_1 \phi_0}, \\
\rho_3^2 = 2q^2 d(d-1)(2-z)^2 Z_0^{-1} l^{-2} r_F^{2(z-1)}. \]

(24) (25) (26) (27)

So, we assume the second solution of the black hole for special case \( \theta = d(z - 1) \) throughout this paper. More precisely, for hyperbolic black hole and special case \( (\theta = d(z - 1)) \) of spherical black hole we will have \( \phi(r) \rightarrow \tilde{\phi}(r) \) and \( \lambda_i \rightarrow \tilde{\lambda}_i \).

The Hawking temperature, the entropy, the pressure, the black hole total electric charge and its associated potential are expressed as \([20]\)

\[
T = \frac{r_h^2}{4\pi l^2+1} \left[ (d - \theta + z) + \frac{l^2}{r_h^2} - \frac{(d - \theta + z - 2)q^2}{r_h^{2d(2\theta+z-1)}} \right].
\]

(28)

\[
S = \frac{\omega_{k,d}}{4G} r_h^{d-\theta} r_F^\theta.
\]

(29)

\[
P = \frac{1}{16\pi G} \frac{(d - \theta + z - 1)(d - \theta + z)}{l^2 r_F^{2d} e^{\tilde{\lambda}_0 \phi_0}}.
\]

(30)

\[
Q = \frac{\omega_{k,d}}{16\pi G} Z_0 \rho_3 l^{-1} r_F^{\theta-2d}. \]

(31)

\[
\Phi = \frac{q r_F^{\theta/d}}{l^2 e^{\tilde{\lambda}_3 \phi_0/2} r^{d-\theta+z-2}} \sqrt{\frac{2(d - \theta)}{Z_0(d - \theta + z - 2)}}.
\]

(32)
which $\omega_{k,d}$ denotes the volume of the space described by unit metric $d\Omega_{k,d}^2$. The mass and the volume of the black hole have separate definitions for spherical and hyperbolic topology. For spherical black hole we have

$$
M = \frac{\omega_{k=1,d}}{16\pi G} (d - \theta) m l^{(-z-1)} r_F^\theta.
$$

(33)

$$
V = \frac{\omega_{k=1,d}}{16\pi G} r_h^{d-\theta+z} l^{1-z} \theta^{+2\theta/d} \phi_0 e^{\lambda_0 \phi_0} \frac{1}{2(d - \theta + z - 1)(d - \theta + z)} \times \\
\left( (d(z + 1) - \theta) + \frac{(d - 1)^2 (d(z - 1) - \theta)^2}{(d - \theta + z - 2)^2 r_h^2} - \frac{(d(z - 1) - \theta) q^2}{r^2(d - \theta + z - 1)} \right).
$$

(34)

While for hyperbolic black hole these quantities are defined as follows

$$
M = \frac{\omega_{k=-1,d}}{16\pi G} \frac{d(2 - z)}{l^{1+z} r_F^{d(1-z)}} \times \\
\left[ m + \frac{2 l^2}{(2-z)^2(z + d(2-z))} \left( l \sqrt{\frac{d - 1}{(2-z)(z + d(2-z))}} \right)^{(d-1)(2-z)} \right].
$$

(35)

$$
V = \frac{\omega_{k=-1,d}}{16\pi G} \frac{l^{1-z} r_F^{(z-1)(d+2)} e^{\lambda_0 \phi_0} d}{(d(2-z) + z)(d(2-z) + z - 1)} \times \\
\left[ r_h^{d(2-z)+z} \left( \frac{l^2(d - 1)}{(2-z)(z + d(2-z))} \right)^{(d(2-z)+z)/2} \right].
$$

(36)

These thermodynamic quantities satisfy the first law of black hole thermodynamics

$$
dM = Tds + \Phi dQ + Vdp.
$$

(37)

By using these all definitions, we study the Joule-Thomson effect in the next section.

3. Joule-Thomson expansion

In thermodynamics, the Joule-Thomson expansion is described by expansion of a gas through a tube separated into two parts by a porous valve. The tube is insulated during the process, so the enthalpy remains constant. The change in the gas temperature during the expansion can be determined by the Joule-Thomson coefficient $\mu$ which is given by [21]

$$
\mu = \left( \frac{\partial T}{\partial P} \right)_H.
$$

(38)

It is easy to see that with $\mu > 0$ ($\mu < 0$) the gas experiences a cooling (heating) process which by passing the inversion point ([$T_i$, $P_i$] outcome of $\mu = 0$) this process will change to heating (cooling).

The real gas is described by the van der Waals equation which is a generalization of ideal gas equation and it includes the size of molecules and interaction between them. By applying this equation to the Joule-Thomson effect, it is found that in isenthalpic expansion of the real gas, there are two upper and lower inversion curves (inversion points in $(T - P)$ plane) which cooling process takes place inside of them. While in black hole case, there is only one inversion
curve that corresponds to the lower curve and cooling process occurs above it [21]. Note that such a process implies to expansion of the black hole with constant mass, because the black hole mass is interpreted as enthalpy.

The Joule-Thomson coefficient is derived in [21,24] by using the first law of black hole thermodynamics and some straightforward calculations as

\[
\mu = \left( \frac{\partial T}{\partial P} \right)_H = \frac{1}{C_p} \left[ T \left( \frac{\partial V}{\partial T} \right)_{P,Q} - V \right].
\]

(39)

In this section by using the above equation, we study the effects of dynamical and hyperscaling violation exponents (\(\gamma\) and \(\theta\) respectively) on this process in two spherical and hyperbolic topologies of the horizon and then we will compare the results of each topology.

3.1. Spherical horizon

Using chain derivatives and definition of specific heat, we can rewrite Eq. (39) as

\[
\mu = \frac{1}{T} \left[ T \left( \frac{\partial V}{\partial r} \right)_{P,Q} - V \left( \frac{\partial T}{\partial r} \right)_{P,Q} \right].
\]

(40)

Substituting Eqs. (28), (29) and (34) into Eq. (40) and using given phrase for pressure Eq. (30), one can obtain

\[
\mu = \frac{r_h^{-(d-\theta-\gamma-1)} l^{-2(z+1)}}{32\pi^2 (d-\theta) PT} \left( [(d(z+1) - \theta)(d-\theta + z)(d-\theta)] r_h^{d-\theta+z-1} 
+ \frac{(d-1)^2 l^2}{(d-\theta + z - 2)^2} [(d(z - 1) - \theta)(d-\theta + z)(d-\theta - 2)] r_h^{d-\theta+z-3} 
+ (d(z+1) - \theta)(d-\theta + z - 2)(d-\theta + 2)] r_h^{d-\theta+z-5} 
+ \frac{(d-1)^4 l^4}{(d-\theta + z - 2)^3} [(d(z - 1) - \theta)(d-\theta)] r_h^{d-\theta+z+1} 
+ q^2 [(d + \theta + 2(z - 1))(d(z-1) - \theta)(d-\theta + z) 
- (3(d-\theta) + 2(z - 1))(d-\theta + z - 2)(d(z+1) - \theta)] r_h^{-(d-\theta+z-1)} 
- 2(d-1)^2 l^2 q^2 [(d(z - 1) - \theta)(d-\theta)] r_h^{-(d-\theta+z+1)} 
+ q^4 [(d(z-1) - \theta)(d-\theta + z - 2)(d-\theta)] r_h^{-(d-\theta+z-1)} \right).
\]

(41)

By combining Eqs. (30) and (31), we can find a useful expression for \(q^2\) as

\[
q^2 = \frac{\gamma P}{\gamma_q} \frac{Q^2}{P},
\]

(42)

where we define

\[
\gamma_P = \frac{1}{16\pi G} (d-\theta + z - 1)(d-\theta + z) r_F^{-2\theta/d} e^{-\lambda_0 \phi_0},
\]

\[
\gamma_q = \left( \frac{\alpha_{k,d}}{16\pi G} \right)^2 2 Z_0 (d-\theta + z - 2)(d-\theta) r_F^{2\theta-2\theta/d} e^{\lambda_3 \phi_0}.
\]

Note that \(l^2\) is related to the pressure \(P\) as
Using Eqs. (42) and (43) and applying $\mu = 0$, we obtained the following relation at the inversion point for spherical black hole

$$P_i^2 \left( (d(z + 1) - \theta)(d - \theta + z)(d - \theta) r_{h,i}^{2(d-\theta+z-1)} \right)$$

$$+ P_i \left( \frac{\gamma P}{(d - \theta + z - 2)^2} [ (d(z - 1) - \theta)(d - \theta + z)(d - \theta - 2) + (d(z + 1) - \theta)(d - \theta + z - 2)(d - \theta + 2)] r_{h,i}^{4(d-\theta+z-1) - 2} + \frac{\gamma P Q^2}{\gamma_q} \times$$

$$[(d + \theta + 2(z - 1))(d(z - 1) - \theta)(d - \theta + z) - (3(d - \theta) + 2(z - 1))(d - \theta + z - 2) (d(z + 1) - \theta)] r_{h,i}^{2(d-\theta+z-1)} \right)$$

$$+ \frac{\gamma P (d - 1)^2}{(d - \theta + z - 2)^3} r_{h,i}^{4(d-\theta+z-1) - 4} - \frac{2 Q^2 (d - 1)^2}{\gamma_q (d - \theta + z - 2)} r_{h,i}^{2(d-\theta+z-1) - 2} + \frac{Q^4}{\gamma_q^2} (d - \theta + z - 2) = 0.$$  

(44)

Where $P_i$ and $r_{h,i}$ represent the inversion pressure and corresponding horizon radius, respectively. On the other hand, by using Eqs. (28), (42) and (43), we can write the inversion temperature as

$$T_i = \frac{r_{h,i}^2}{4\pi} \left( P_i \right)^{(z-1)/2} \left[ (d - \theta + z) P_i + \frac{(d - 1)^2}{(d - \theta + z - 2) r_{h,i}^3} - \frac{(d - \theta + z - 2) Q^2}{\gamma_q r_{h,i}^{2(d-\theta+z-1)}} \right].$$

(45)

The first point we can see here is that the minimum inversion temperature which can be obtained by demanding $P_i = 0$, can only be nonzero when $z = 1$

$$T_i^{min} = \frac{(d - 1)^2}{4\pi (d - \theta - 1) r_{h,i}^2} - \frac{(d - 1) Q^2}{4\pi \gamma_q r_{h,i}^{2(d-\theta-1)}}. \quad (46)$$

At this state ($z = 1$, $P_i = 0$), the positive root of Eq. (44) can be expressed as

$$r_{h,i} = \begin{cases} \left( \frac{Q^2}{\gamma_q} \left( \frac{d-\theta - 1}{d-1} \right)^2 \right)^{1/(2(d-\theta) - 2)} & \theta \neq d(z - 1), \\ \left( \frac{3d Q^2}{\gamma_q (d+2)} \right)^{1/(2(d-1))} & \theta = d(z - 1). \end{cases} \quad (47)$$

The inversion temperature can be found by substituting Eq. (47) into (46)

$$T_i^{min} = \begin{cases} 0 & \theta \neq d(z - 1), \\ \frac{(d-1)^2}{6\pi d} \left( \frac{\gamma_q (d+2)}{3d Q^2} \right)^{1/(2(d-1))} & \theta = d(z - 1). \end{cases} \quad (48)$$

Therefore, the minimum inversion temperature has nonzero value only at Ads case ($z = 1$, $\theta = d(z - 1)$).
By using Eqs. (28) and (30), one can write the equation of state \( P = P(V, T) \), in Ads case as
\[
P = \frac{dT}{4r_h} + \frac{d(d-1)}{16\pi} \left( \frac{Q^2}{\gamma_q r_h^{2d}} - \frac{1}{r_h^2} \right), \quad r_h = \left( \frac{(d+1)V}{\omega_k=1,d} \right)^{1/(d+1)}
\]
(49)

Applying the critical points condition \( \frac{\partial P}{\partial r_h} = \frac{\partial^2 P}{\partial^2 r_h} = 0 \), we find the critical temperature
\[
T_c = \frac{(d-1)^2}{(2d-1)\pi} \left( \frac{\gamma_q}{dQ^2(2d-1)} \right)^{1/2(d-1)}.
\]
(50)

Which yields following ratio
\[
\frac{T_{i_{\min}}}{T_c} = \frac{(2d-1)}{6d} \left( \frac{(2d-1)(d+2)}{3} \right)^{1/2(d-1)}.
\]
(51)

This ratio decreases gradually with increasing \( d \) as obtained in [24] and is in agreement with the result of [21] for charged Ads black hole at \( d = 2 \) \( \left( \frac{T_{i_{\min}}}{T_c} = \frac{1}{2} \right) \).

Now, we return to Eq. (45), to understand the behavior of inversion curves. This equation depends except for the inversion pressure on the black hole horizon and we need to find horizon radius to attain an explicit relation between the inversion temperature and pressure. But it is impossible to solve Eq. (44), analytically. So, we obtain numerical root \((r_h,i)\) of this equation and substitute it into Eq. (45) to plot inversion curves in \((T-P)\) plane. From inspection of Fig. 1 generally it can be seen that with increasing \( \theta \) and \( z \), the inversion temperature increases for given pressure.\(^1\) However, we see an exception in Fig. 1(d). Although two lower panels have the same parameter values, but have the different inversion curves behavior. These sub-figures show that with increasing \( z \), for \( z \geq 2 \), the inversion temperature decreases for low pressure while it increases for high pressure. Fig. 1(b) illustrate the inversion curves for special case \( \theta = d(z-1) \), which will be useful for compare the results with the hyperbolic topology of the horizon.

Isenthalpic curves in \((T-P)\) plane can be obtained by eliminating the event horizon \( r_h \) between blackening factor \( f(r) \) and Hawking temperature \( T_h \). Not that the mass \( m \) and charge \( q \) parameters should be replaced by black hole mass \( M \) and charge \( Q \), respectively. Fig. 2 shows an instance of the isenthalpic curves and corresponding inversion curve in \((T-P)\) and \((P-r)\) planes. Intersection points between inversion curve and isenthalpic curves indicate the inversion points where the cooling-heating transition occurs. It is easy to see that higher inversion pressure corresponds to the higher mass of the black hole. Also, comparison of the Fig. 2(a) and Fig. 2(b) shows that if the black hole has a maximum pressure at the beginning, the black hole heats up with the initial expansion, from the maximum pressure to the inversion pressure and with further expansion from the inversion pressure to the minimum pressure, the black hole will cool down.\(^2\)

3.2. Hyperbolic horizon

Like the spherical case, we begin by calculating the Joule-Thomson coefficient. By using Eqs. (28), (29), (30) and (36), following relation is obtained

\(^1\) We choose parameters in that range to compare the result with limiting case of \( Ads_4 \) \((z = 1, \theta = 0, d = 2)\) which is studied before [21].

\(^2\) Not that the Fig. 2(b) illustrates horizon radius of the black hole with respect to the pressure, and the smaller radius for given pressure implies to inner horizon which is not our consideration.
Fig. 1. The inversion curves for spherical black hole. The parameters are set as (a) $z=1$ (b) $\theta = d(z - 1)$ (c, d) $\theta = -1$, with $d = Q = 2$ and $\phi_0 = Z_0 = r_F = G = 1$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

\[
\mu = \frac{r_h^{z-d(2-z)+1}(d(2-z)+z-1)^2}{16\pi^2(d-1)(2-z)^2P T} \left\{ d(2-z)[(d(2-z)+z-1)^2 - 1]r_h^{d(2-z)+z-1} \\
- l^2(d-1)^2(d(2-z)+2)r_h^{d(2-z)+z-3} \\
- q^2(d-1)^2(2-z)^2(3d(2-z)+2(z-1))r_h^{-(d(2-z)+z-1)} \\
+ \left( \frac{d-1}{(d(2-z)+z)(2-z)} \right)^{d(2-z)+z} \left( (d(2-z)+z-1)^2 - 1 \right) z r_h^{-1} \\
+ (2-z)(d-2)^2(1^2 r_h^{-3} + q^2(2d-1)(2-z)^2 r_h^{-2(d(2-z)+z-1)-1}) \right\}
\]

And the vanishing Joule-Thomson coefficient leads us to the following equation
Fig. 2. The isenthalpic curves and corresponding inversion curve in $T - P$ and $P - r$ planes for spherical Ads black hole with fixed parameters: $d = Q = 2$ and $\phi_0 = Z_0 = r_F = G = 1$.

\[ P_i^{(d(2-z)+z+2)/2} d(2-z)[(d(2-z) + z - 1)^2 - 1]^{3(d(2-z)+z-1)+1} \]
\[ - \gamma p P_i^{(d(2-z)+z)/2} (d-1)^2 \times \]
\[ (d(2-z) + 2)r_{h,i}^{3(d(2-z)+z-1)-1} + \frac{Q^2}{Y_q} (2-z)^2 (3d(2-z) + 2(z-1)) r_{h,i}^{d(2-z)+z} + \]
\[ \left( \frac{\gamma p (d-1)}{(d(2-z)+z)(2-z)} \right)^{(d(2-z)+z)/2} \left( P[(d(2-z) + z - 1)^2 - 1] z r_{h,i}^{2(d(2-z)+z-1)} + \right. \]
\[ \left. \gamma p (2-z)(d-1)^2 (r_{h,i}^{2(d(2-z)+z-2)} + \frac{Q^2}{Y_q} (2d-1)(2-z)^2) \right) = 0 \]

The inversion temperature with respect to the inversion pressure $P_i$ and corresponding $r_{h,i}$ can be expressed as

\[ T_i = \frac{d(2-z) + z}{4\pi Y_p^{(z+1)/2} P_i^{(z+1)/2} r_{h,i}^{z+1/2}} - \frac{d - 1}{4\pi (2-z) Y_p^{(z-1)/2} P_i^{(z-1)/2} r_{h,i}^{z-2}} \]
\[ - \frac{(d-1)(2-z) Q^2}{Y_q Y_p^{(z-1)/2}} P_i^{(z-1)/2} r_{h,i} 6z - 2(d(2-z) + z - 1) \]

(54)

As the spherical case there only exist nonzero minimum inversion temperature when $z=1$, but Eq. (53) has no real root by setting the inversion pressure to zero. So, the inversion temperature is undefined at $P_i = 0$ for Ads case. However, the numerical values show that the inversion temperature tends to zero when $P_i \to 0$ and $z = 1$.

To obtain the inversion temperature $T_i$ as a function of inversion pressure $P_i$, we substituted the numerical solution of Eq. (53) into Eq. (54). Fig. 3 shows that like the spherical black hole, the inversion temperature increases with increasing $z$. But for given $P_i$, the inversion temperature $T_i$ is more lower than in the spherical case.
Fig. 3. The inversion curves for hyperbolic Ads black hole with fixed parameters: \( d = Q = 2 \) and \( \phi_0 = Z_0 = r_F = G = 1 \).

Fig. 4. The isenthalpic curves and corresponding inversion curve in \( T - P \) and \( P - r \) planes for hyperbolic Ads black hole with fixed parameters: \( d = Q = 2 \) and \( \phi_0 = Z_0 = r_F = G = 1 \).

Nevertheless the isenthalpic curves illustrated in Fig. 4\(^3\) show an important difference between hyperbolic and spherical black holes structure that changes the result. The maximum pressure that a hyperbolic black hole can withstand is much higher than the spherical black hole. Therefore, for a specified mass, we see a higher inversion temperature (or maximum temperature) in the hyperbolic black hole. More precisely, when the black hole has a hyperbolic horizon, its isenthalpic expansion will have a higher inversion temperature than when it has a spherical horizon. In addition, the comparison of Fig. 4(b) and Fig. 2(b) shows that the higher inversion pressure \( P_i \) and consequently the higher inversion temperature \( T_i \) of the hyperbolic black hole tends to

\(^3\) Needless to say, both subfigures are plotted by described methods in the previous subsection.
zero at smaller horizon radius, with respect to spherical black hole. This means that the hyperbolic black hole requires less expansion to cool than the spherical black hole, despite the higher maximum temperature.

4. Conclusions

In this work, we studied the Joule-Thomson expansion of hyperscaling violating black holes with spherical and hyperbolic topologies. We computed the Joule-Thomson coefficient $\mu$ in each case and found the inversion temperature by applying $\mu = 0$. It is demonstrated that the minimum inversion temperature has nonzero value only at spherical Ads case ($z = 1, \theta = 0$). At this state, we found that the ratio between minimum inversion and critical temperatures decreases with increasing $d$ as explained in [24] and for $d = 2$ this ratio reaches to the obtained value in [21] for charged Ads black hole.

In the spherical case, the inversion temperature increases with increasing both of dynamical and hyperscaling violation exponents ($z$ and $\theta$ respectively). But for $z \geq 2$ with increasing $z$, the inversion temperature decreases at low inversion pressure and increases at higher inversion pressure. A notable point here is that more massive black holes have higher inversion pressure. So we can conclude massive (small) black holes will have higher (lower) inversion temperature, with increasing $z$. In hyperbolic black hole, we showed that for the given inversion pressure the inversion temperature is more lower than the spherical black hole. But the hyperbolic black hole can withstand more pressure, consequently it has a higher inversion temperature with respect to the spherical black hole. In future, it will be interesting to consider some asymptotically behavior of hyperscaling violating black holes in form of black fold or black brane and apply the Joule-Thomson expansion.

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