On $c_2$ invariants of some 4-regular Feynman graphs

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Abstract The obstruction for application of techniques like denominator reduction for the computation of the $c_2$ invariant of Feynman graphs in general is the absence of a 3-valent vertex. In this paper such a formula for a 4-valent vertex is derived. The formula allows us to compute the $c_2$ invariant of new graphs, for instance, some 4-regular graphs with small loop number.

Keywords Graph hypersurface · $c_2$ Invariant · Rational points · Feynman graph

Mathematics Subject Classification 14G05 · 81Q30

1 Introduction

It is known that the evaluation of certain Feynman periods in $\phi^4$ theory leads to interesting values, usually combinations of multiple zeta values, [1,9]. There is so far no good understanding of these numbers, and they are hard to compute, even numerically. The $c_2$ invariant can be regarded as a discrete analogue of the period. It is defined as the coefficient of the $q^2$ in the $q$-expansion of the number of $\mathbb{P}_q$-rational points of the graph hypersurface. This invariant respects many relations between the periods.

Starting with a (Feynman) graph $G$ the Feynman period is given by an integral of a differential form with double poles along the graph hypersurface $\mathcal{V}(\Psi_G)$ given by the graph polynomial
\[ \Psi_G := \sum_T \prod_{e \notin T} \alpha_e \in \mathbb{Z}[\alpha_1, \ldots, \alpha_{N_G}], \]  

where \( T \) runs over all spanning trees of \( G \), \( \alpha_i \) are formal variables associated to edges, and \( N_G \) is the number of edges. For any prime power \( q \) the number \( \#V(\Psi_G)(\mathbb{F}_q) \) of \( \mathbb{F}_q \)-rational points of \( V(\Psi_G) \) is divisible by \( q^2 \) for any \( G \) with at least 3 vertices. We define

\[ c_2(G)_q := \#V(\Psi_G)(\mathbb{F}_q)/q^2 \mod q. \]

A \( c_2 \) invariant depends on \( q \), but for simple (like denominator reducible) graphs it is just a constant. It can also be a constant outside primes of bad reduction [6,10], or even have a modular nature [3]. The naive way to compute \( c_2(G)_q \) for a prime power \( q \) is just brute force counting of all the rational points over \( \mathbb{F}_q \) and then take the coefficient of \( q^2 \). This can be done only for small \( q \) and small number of edges and does not compute the whole \( c_2 \). The much better idea is to try to eliminate the variables step by step and compute only \( c_2 \) itself, it can be done for many small and for several infinite series of graphs (like zigzag graphs \( ZZ_h \)) and this procedure is called denominator reduction. Even if the graph is not denominator reducible, it is possible to eliminate a big part of the variables first, decreasing the degree, and then try to compute the rest by other techniques, for example, analysing the underlying geometry, see K3-example in [3].

The construction of the Feynman period for a (primitive and log-divergent) graph in \( \phi^4 \) involves the following operation: one takes a 4-regular graph \( \Gamma \) and deletes one of the vertices together with the 4 incident edges (for getting rid of the symmetries), then for the resulting graph \( G \) one defines the Feynman period (in parametric space) using the graph polynomial (1). Later we write \( \widehat{G} = \Gamma \) and call it the completion of \( G \). Thus a (primitive log-divergent) Feynman graph in \( \phi^4 \) theory has 4 3-valent vertices, while the others are 4-valent. The existence of a 3-valent vertex is an important necessary condition for application of the denominator reduction process.

For a graph \( G \) with \( h_G := N_G - |V_G| + 1 \) loops and \( N_G \) edges, define \( \delta_G := 2h_G - N_G \). The graphs with \( \delta_G = 0 \) (for example graphs in \( \phi^4 \)) always have a vertex of degree \( \leq 3 \), so the denominator reduction can be applied for small graphs. For denominator reducible graphs the reduction gives us a chain of congruences \( \mod q \) going down to trivial case, so that \( c_2(G) = -1 \) for these graphs if no weight drop happens, or \( c_2(G) \) vanishes otherwise. We also know that the graphs with \( \delta_G < 0 \) always have \( c_2(G) = 0 \).

In the literature there is no one example of graphs with \( \delta_G \geq 1 \) such that the \( c_2 \) invariant is known. If \( G \) has \( \delta_G \geq 2 \), then \( G \) could have no 3-valent vertex, thus that denominator reduction technique cannot be applied. This happens for 4-regular graphs. Even with a 3-valent formula, the analogue of the generic step of denominator reduction for a graph with \( \delta_G > 0 \) will give additional terms on each step, provided that expected factorizations occur.

In this article we derive a formula for the \( c_2 \) invariant in the case of a 4-valent vertex (see Theorem 18) as a result of an organized elimination of 4 variables. Based on this we can apply a reduction similar to denominator reduction and compute the
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$c_2(G)_q$ for some small graphs, namely for the completions $G = \hat{ZZ}_h$ of graphs from the zigzag series for $h \leq 8$. It is not expected to compute many of $c_2$ invariants of 4-regular graphs even with small loop number since the log-divergent graphs in $\phi^4$ also stop to be denominator reducible at 8–9 loops, and since $\delta_G = 2$ is a priori worse for factorization of the appearing polynomials then $\delta_G = 0$. Nevertheless, our 4-valent formula and reduction procedure similar to denominator reduction can be used for elimination of big part of the variables in order to find some nice and understandable geometry behind, like it was done in [3].

There is another deep reason to study the $c_2$ invariant of 4-regular graphs, namely the relation to the completion conjecture (see [3]).

**Conjecture 1** Let $G_1$ and $G_2$ be two Feynman graphs in $\phi^4$ such that $\hat{G}_1 = \hat{G}_2$ (this means they come from the same 4-regular graph by deletion of two different vertices). Then $c_2(G_1) = c_2(G_2)$.

This conjecture is the one of the most interesting statements in the theory of the $c_2$ invariant and remains unproved. One of the approaches how to prove it was the following. The fact that $c_2$ invariants are the same for graphs with same completions could have something to do with the completion itself. Since $c_2$ is defined for the completion—4-regular graph—then there could possibly be a way to compare $c_2(G)$ and $c_2(\hat{G})$. The statement will then follow from non-symmetry of this relation on the vertex we remove. In this article we check this and see that the idea was too optimistic. The formula we get for $c_2$ is symmetric and does not help for proving the conjecture. It also produces no other similar relations for the sub-quotient graphs of 4-regular graphs.

Finally, applying our 4-valent formula for the $c_2$ invariant to the case of 4-regular graph $G = \hat{ZZ}_h$, we are able to compute (Proposition 21)

$$c_2(G)_q \equiv -h(h + 2) \mod q$$

for $h \leq 8$ and we conjecture that this holds for all $h$ (Conjecture 20).

**2 Preliminary results**

Let $G$ be a graph with the set of vertices $V = V(G)$ and the set of edges $E = E(G)$. Let $N = N_G := E(\hat{G})$ be the number of edges and let $h_G := N_G - |V| + 1$ be the loop number. We numerate edges $e_1, \ldots, e_N$ and associate with each edge $e_i$ the variable (Schwinger parameter) $\alpha_i$. For a given graph, we can define the following polynomial

$$\Psi_G := \sum_T \prod_{e \notin T} \alpha_e \in \mathbb{Z}[\alpha_1, \ldots, \alpha_{N_G}].$$

where $T$ runs over all spanning trees of $G$, the subgraphs that are trees and contain all the vertices. This polynomial is called the graph polynomial (or the first Symanzik polynomial) of $G$. 

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The graph polynomial is homogeneous of degree \( h_G \) and is linear with respect to each variable. Let \( e_k \) be one of the edges. Then \( \Psi_G \) can be written in the following way called \textit{contraction–deletion formula}:

\[
\Psi_G = \Psi^k_G \alpha_1 + \Psi_{G,k}
\]

with \( \Psi^k_G \) and \( \Psi_{G,k} \) independent of \( \alpha_k \). It turns out that these two polynomials are again graph polynomials, namely \( \Psi^k_G = \Psi_{G\backslash k} \) and \( \Psi_{G,k} = \Psi_{G/k} \), where \( G \backslash k \) (respectively, \( G/k \)) is the graph obtained from \( G \) by deletion (respectively, contraction) of the edge \( e_k \). Equivalently, \( \Psi_G \) can be defined as the determinant of the matrix.

\[
\Psi_G = \det M_G, \quad M_G = \begin{pmatrix} \Delta(\alpha) & E_G \\ -E_G^T & 0 \end{pmatrix} \in \text{Mat}_{N+n,N+n}(\mathbb{Z}[[\alpha_i]_{i \in I_N}]),
\]

where \( \Delta(\alpha) \) is the diagonal matrix with entries \( \alpha_1, \ldots, \alpha_N \), and \( E_G \in \text{Mat}_{N,n}(\mathbb{Z}) \) is the incidence matrix after deletion of the last column, \( N = N_G, n = n_G \) (see [2], Section 2.2). The equivalence of the two definitions of \( \Psi_G \) is the content of the Matrix Tree Theorem.

We should enlarge the set of polynomials we work with.

Let \( I, J, K \subset E(G) \) be sets of edges with \( |I| = |J| \) and \( K \cap (I \cup J) = \emptyset \). Out of the matrix \( M_G \), one defines the \textit{Dodgson polynomials} \( \Psi_{G,K}^{I,J} \) by

\[
\Psi_{G,K}^{I,J} := \det M_G(I; J)_K,
\]

where \( M_G(I; J)_K \) obtained from \( M_G \) by removing rows indexed by \( I \) and columns indexed by \( J \), and by putting \( \alpha_t = 0 \) for all \( t \in K \). Such a polynomial \( \Psi_{G,K}^{I,J} \) is of degree \( h_G - |I| \) and depends on \( N_G - |I| - |K| \) variables. We usually write \( \Psi_{K}^{I,J} \) for \( \Psi_{G,K}^{I,J} \) and this is consistent with (5).

Similar to graph polynomials, the Dodgson polynomials satisfy many identities, see [2]. We give here those that will be used in the sequel.

For any Dodgson polynomial \( \Psi_{G,K}^{I,J} \) and any edge \( e_a \in E \setminus I \cup J \cup K \), the \textit{contraction–deletion formula} holds:

\[
\Psi_{G,K}^{I,J} = \pm \Psi_{G,K}^{I_a,J_b} \alpha_a \pm \Psi_{G,K}^{I,J_a}^{I_a,J_b} \]

where the signs depend on the choices made for the matrix \( M_G \). This formula agrees with (5) in the case of the graph polynomial \( (I = J) \).

Let \( I, J \subset E(G) \) be subsets of edges with \( I \cap J = \emptyset \) and let \( e_a, e_b, e_x \notin I \cup J \).

Then

\[
\Psi_{I_x,J_x}^{I_a,J_b} \Psi_{I_x,Ja}^{I_a,J_b} - \Psi_{I_x,Jb}^{I_a,J_x} \Psi_{Ia,Jx}^{Ia,Jb} = \gamma \Psi_{I,J}^{Ia,Jb} \Psi_{Ia,Jb}^{Ia,Jx}
\]

where \( \gamma \in \{1, -1\} \) can be understood combinatorially. This is called the \textit{first Dodgson identity}. 

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Consider again sets of (pairwise non-intersecting) edges \( I, J, U \) with \(|I| = |J|\), and assume \(|U| = h_G\). Define by \( E_G(U) \) the matrix obtained from the incidence matrix \( E_G \) by removing all the rows corresponding to \( U \). We know that \( \det(E_G(U)) = 0 \) or \( \pm 1 \), and nonzero exactly when \( U \) forms a spanning tree. One has the following equality for the Dodgson polynomials

\[
\Psi^I_J = \sum_{U \subseteq G \setminus (I \cup J), u \notin U} \prod_{u \in U} \alpha_u \det(E_G(U \cup I)) \det(E_G(U \cup J)),
\]

where \( U \) ranges over all subgraphs of \( G \setminus (I \cup J) \) which have the property that \( U \cup I \) and \( U \cup J \) are both spanning trees of \( G \), see Proposition 8, [5].

**Definition 2** Let \( P = P_1 \cup P_2 \cup \ldots \cup P_k \) be a set partition of a subset of vertices of \( G \). Define

\[
\Phi^P_G := \sum_{F} \prod_{e \notin F} \alpha_e,
\]

where the sum runs over spanning forests \( F = T_1 \cup \ldots \cup T_k \), and each tree \( T_i \) of \( F \) contains the vertices in \( P_i \) and no other vertices of \( P \), i.e. \( V(T_i) \supseteq P_i \) and \( V(T_i) \cup P_j = \emptyset \) for \( i \neq j \). These polynomials \( \Phi^P_G \) are called the spanning forest polynomials, see [5] for examples and more explanations.

There is an interpretation of the Dodgson polynomials in terms of the spanning forest polynomials, cf. Proposition 12, [5].

**Lemma 3** For two sets of edges \( I \) and \( J \) with \(|I| = |J|\) and \( I \cap J = \emptyset \), one has

\[
\Psi^I_J = \sum_k \gamma_k \Phi^P_k.
\]

Here the sum runs over all partitions of \( V(I \cup J) \) and \( \gamma_k \) are the coefficients in \( \{-1, 0, 1\} \) that can be also controlled (see Sect. 2, loc.cit.).

We need the Jacobi’s determinant formula

**Lemma 4** Let \( M = (a_{ij}) \) be an invertible \( n \times n \) matrix and let \( \text{adj}(M) = (A_{ij}) \) be the adjugate matrix of \( M \) (the transpose of the matrix of cofactors). Then for any \( k \), \( 1 \leq k \leq n \),

\[
\det(A_{ij})_{k \leq i, j \leq n} = \det(M)^{n-k-1} \det(a_{ij})_{1 \leq i, j \leq k}.
\]

The easy consequence is the following (see [2], Lemma 31)

**Lemma 5** Consider two sets of edges \( I \) and \( J \) with \(|I| = |J|\) and let \( S = \{s_1, \ldots, s_r\} \subset E_G \setminus (I \cup J) \) be some other subset of edges. If \( \Psi^{I_S} \) vanishes as a polynomial of \( \alpha_i \)’s, then

\[
\Psi^{I_s, J_s} = \sum_{k \neq t} \pm \Psi^{I_{s_t}, J_{s_k}}
\]
the signs depending on the order of rows in $M_G$.

The main theorem of the article considers the situation of a 4-valent vertex in a graph and a good way to eliminate first variables. Here we should discuss the case of a 3-valent vertex that is "classical" (see [2], p.17), but the technique is important for what comes latter.

**Example 6** Let $G$ be a graph with a 3-valent vertex incident to the edges $e_1$, $e_2$ and $e_3$. Deletion of these 3 edges leads to disconnectedness of the vertex, thus $\Psi^{123} = 0$. So we are in the settings of the previous lemma. Other equality to mention is $\Psi^{12} = \Psi^{23} = \Psi^{13}$. This holds since the deletion of any 2 of the 3 edges and contraction of the third one leads to the same sub-quotient graph. The Jacobi determinant formula (13) implies

$$\det\begin{pmatrix}
\psi^1 & \psi^{1,2} & \psi^{1,3} \\
\psi^{2,1} & \psi^2 & \psi^{2,3} \\
\psi^{3,1} & \psi^{3,2} & \psi^3
\end{pmatrix} = 0.$$  

(15)

In the spirit of (14), the first row of the matrix gives $\psi^1 = \psi^{1,2} - \psi^{1,3}$. We rewrite this in terms of the 3 variables:

$$\Psi^{12} \alpha_2 + \Psi^{13} \alpha_3 + \Psi^{23} = \Psi^{13,23} \alpha_3 + \Psi^{1,2} + \Psi^{12,3} \alpha_2 - \Psi^{1,3}. \quad (16)$$

Define

$$f_0 := \psi_k^{ij}, \quad f_k := \psi_k^{i,j}, \quad f_{123} := \psi_{123} \quad \text{for } \{i, j, k\} = \{1, 2, 3\}. \quad (17)$$

Equation above (and similar after permutation of the 3 edges) implies

$$f_0 = \psi^i_{j,k} = \psi^{ik,jk} \quad \text{and} \quad \psi_{jk}^i = f_j + f_k \quad (18)$$

for all $\{i, j, k\} = \{1, 2, 3\}$. It follows that the graph polynomial for a graph with a 3-valent vertex has the form

$$\Psi_G = f_0 (\alpha_1 \alpha_2 + \alpha_2 \alpha_3 + \alpha_1 \alpha_3) + (f_1 + f_2) \alpha_3 + (f_1 + f_3) \alpha_2 + (f_2 + f_3) \alpha_1 + f_{123}, \quad (19)$$

together with the identity

$$f_0 f_{123} = f_1 f_2 + f_2 f_3 + f_1 f_3. \quad (20)$$

### 3 Point-counting functions

For a prime power $q$ and for an affine variety $Y$ defined over $\mathbb{Z}$, we define by $[Y]_q := \# \tilde{Y}(\mathbb{F}_q)$ the number of $\mathbb{F}_q$-rational points of $Y$ after extension of scalars to $\mathbb{F}_q$. Roughly speaking, this means that $[f]_q$ is a number of solutions of $f = 0$ in $\mathbb{F}_q^n$ after taking
the coefficients mod $q$ for an affine hypersurface given by $f \in \mathbb{Z}[x_1, \ldots, x_n]$. Here and later, we use the shortcut $[f, \ldots, f_n]_q$ for $[\mathcal{V}(f_1, \ldots, f_n)]_q$. We think of $[\cdot]_q$ as a function of $q$. Sometimes (but not in general), this function is a polynomial of $q$.

The very basic situation one always meet while computing of the point-counting function is the case of a linear polynomial or system of two linear polynomials.

**Lemma 7** Let $f^1, f_1, g^1, g_1, h \in \mathbb{Z}[\alpha_2, \ldots, \alpha_n]$ be polynomials. Then, considering the varieties on the right hand side of the coming formulas to be in $\mathbb{A}^{n-1}$ and the varieties on the left to be in $\mathbb{A}^n$, for every $q$,

1. for $f = f^1\alpha_1 + f_1$, one has
   \[ [f, h]_q = [h]_q - [f^1, h]_q + q[f^1, f_1, h]_q. \tag{21} \]
   and, in particular,
   \[ [f]_q = q^{n-1} - [f^1]_q + q[f^1, f_1]_q. \tag{22} \]

2. for $f = f^1\alpha_1 + f_1$ and $g = g^1\alpha_1 + g_1$, one has
   \[ [f, g, h]_q = q[f^1, f_1, g^1, g_1, h]_q + [f^1 g_1 - g^1 f_1, h]_q - [f^1, g^1, h]_q. \tag{23} \]
   and
   \[ [f, g]_q = q[f^1, f_1, g^1, g_1]_q + [f^1 g_1 - g^1 f_1]_q - [f^1, g^1]_q. \tag{24} \]

**Proof** See, for example, Lemma 3.1 in [7]. \qed

**Definition 8** For 2 polynomials linear in one of the variables, $f = f^k\alpha_k + f_k$ and $g = g^k\alpha_k + g_k$ we use the following notation for their resultant
\[ [f, g]_k := \pm(f^k g_k - f_k g^k). \tag{25} \]

**Lemma 9** Let $G$ be a graph with at least 3 vertices and let $q$ be a fixed prime power. Then for the number of rational points on the graph hypersurface $\mathcal{V}(\Psi_G)$ over $\mathbb{F}_q$ the following holds:
\[ [\Psi_G]_q \equiv 0 \mod q^2 \tag{26} \]
and
\[ [\Psi^1_G, \Psi_{G,1}]_q \equiv 0 \mod q \tag{27} \]
for any edge $e_1$ and any $q$.

**Proof** See, for example, Theorem 2.9 in [10] and Proposition-Definition 18 in [3]. \qed

This allows us to introduce the main object of our study.
Definition 10  Let $G$ be a graph with at least 3 vertices. Then
\[
c_2(G)_q := \frac{[\Psi_G]_q}{q^2} \mod q.
\] (28)

We will intensively use the following vanishing argument proved by Katz in [8].

Theorem 11 (Chevalley–Warning vanishing) Let $f_1, \ldots, f_k \subseteq \mathbb{Z}[x_1, \ldots, x_n]$ be polynomials and assume that the degrees $d_i := \deg f_i$ satisfy $\sum d_i < n$. Then, for the number of $\mathbb{F}_q$-rational points of the variety $V(f_1, \ldots, f_n)$ given by the intersection of the hyperplanes $V(f_i)$ in $\mathbb{A}^n$, the following congruence holds
\[
[f_1, \ldots, f_k]_q \equiv 0 \mod q.
\] (29)

For a polynomial $f \in \mathbb{Z}[x_1, \ldots, x_N]$ of degree $d$, define $\delta(f) := 2d - N$. In the case of a graph polynomial $f = \Psi_G$ the equality $\delta(f) = 0$ corresponds to $G$ being log-divergent. If $G$ is a 4-regular graph, then $\delta(\Psi_G) = 2$. The positivity of $\delta$ is the obstruction to the vanishing of the big part of the summands in the reduction procedures of iterative elimination of the variables like denominator reduction. What we will try to do instead is to keep track of all the summands.

From the graph polynomial we can always eliminate the first 2 variables.

Lemma 12  Let $G$ be graph with the 2 edges $e_1$ and $e_2$. Then
\[
[\Psi_G]_q = q^{N_G-1} - [\Psi^1_1]_q + q^2[\Psi^1_2, \Psi^2_1, \Psi^2_2, \Psi^1_2]_q + q[\Psi^1_1, \Psi^2_1]_q - q[\Psi^1_2, \Psi^2_1]_q.
\] (30)

Proof We use (22) for $f := \Psi$ and (24) for the pair $(\Psi^1, \Psi^1_1)$
\[
[\Psi]_q = q^{N_G-1} - [\Psi^1_1]_q + q[\Psi^1_1, \Psi^1_1]_q = q^{N_G-1} - [\Psi^1]_q + q^2[\Psi^1_2, \Psi^2_1, \Psi^2_2, \Psi^1_2]_q + q[\Psi^1_2, \Psi^2_1]_q + q[\Psi^1_2, \Psi^2_1]_q - q[\Psi^1_2, \Psi^2_1]_q.
\] (31)

and then the first Dodgson identity (9)
\[
\Psi^1_2 \Psi^2_1 - \Psi^1_2 \Psi^2_1 = (\Psi^1_2)^2.
\] (32)

The statement follows. \qed

Lemma 13  Let $G$ be a graph with $|V| \geq 4$ and with a 3-valent vertex, say, incident to the edges $e_1, e_2$ and $e_3$. Then
\[
c_2(G) \equiv [f_0, f_3]_q \mod q.
\] (33)

Proof We are going to use (30). Since $G$ has a 3-valent vertex, the graph polynomial $\Psi_G$ has the form (19). In these terms one computes
\[
[\Psi^1_2, \Psi^2_1, \Psi^2_1, \Psi^1_2]_q = [f_0, f_1 + f_3, f_2 + f_3, (f_1 + f_2)x_3 + f_123]_q.
\] (34)
Using (20), one derives that the vanishing \( f_1 + f_3 = 0 \) on \( V(f_0) \) implies \( f_1 = f_3 = 0 \). So the term above vanishes mod \( q \), since the defining polynomials become independent of \( \alpha_3 \).

We also have

\[
[Psi^{1,2}]_q = [f_0\alpha_3 + f_3]_q = q^{N-3} - [f_0]_q + q[f_0, f_3]_q
\]  
(35)

and since \( f_0 \) is again a graph polynomial for \( G := G\setminus \{3\} \), it implies \( q^2[[f_0]_q \) and

\[
[Psi^{1,2}]_q \equiv q[f_0, f_3]_q \mod q^2.
\]  
(36)

In the case of a log-divergent \( G, c_2(G\setminus \{3\}) = 0 \). From the equation

\[
[Psi^2]_q = q^{N-2} - [Psi^{12}]_q + q[Psi^{12}, Psi^2]_q
\]  
(37)

and the fact that the existence of a vertex of valency \( \leq 2 \) implies the vanishing of the \( c_2 \) invariant, we derive \( q^2[[Psi^{12}, Psi^2]_q \) (and additionally also \( q^3[[Psi^1]_q \). Now (30) yields

\[
[Psi]_q \equiv [f_0, f_3]_q q^2 \mod q^3.
\]  
(38)

In other words, for any graph \( G \) with \( |V| \geq 3 \) and a 3-valent vertex,

\[
c_2(G) \equiv [f_0, f_3]_q \mod q.
\]  
(39)

Remark 14 A different way to obtain the formula is by using the structure of an \( A_2 \)-fibration, then the similar statement holds also in the Grothendieck ring of varieties \( K_0(Var_k) \) for the corresponding \( c_2(G) \), see Lemma 24 in [3].

There is a good idea to compute the \( c_2 \) invariant of a log-divergent graph (i.e. \( N_G = 2h_G \)) by eliminating the variables step by step starting with (33) called denominator reduction.

For a log-divergent graph \( G \) (with \( e_1, e_2 \) and \( e_3 \) incident to a 3-valent vertex), the denominator reduction algorithm with respect to some order of the edges \( e_1, \ldots, e_n \) is a sequence of polynomials \( D_k \in \mathbb{Z}[\alpha_{k+1}, \ldots, \alpha_{N-1}] \) for all \( 3 \leq k \leq n \) with \( D_3 := f_0 f_3 \) and with the following rules:

1. if \( D_{k-1} \) is defined and is factorizable into factors linear in \( \alpha_k \)

\[
D_{k-1} = (f^k \alpha_k + f_k)(g^k \alpha_k + g_k).
\]  
(40)

then the next \( D_k \) is defined by

\[
D_k := \pm (f^k g_k - f_k g^k).
\]  
(41)
2. if \( D_{k-1} = (f)^{2}h \) with \( f \) linear in \( \alpha_k \) and \( h \) independent of \( \alpha_k \) and \( k \leq N_G - 2 \), then \( D_k = 0 \), the algorithm stops and we say that \( G \) has a weight drop.

3. if \( D_k \) is defined for all \( k, 3 \leq k \leq N_G - 1 \), then \( G \) is called denominator reducible for this order of edges.

A graph is called **denominator reducible** if there exist an order of edges for which it is denominator reducible in the above sense.

One can start with \( D_0 := \Psi_G \), but the first 3 (initial) steps are special, we have 2 initial weight drops at the first and third reduction step, this is where the divisibility of \( [\Psi_G]_q \) by \( q^2 \) comes from. Generically, a graph fails to be denominator reducible already at step five: \( D_5 \) always exists but is usually not factorizable. Nevertheless, all small graphs and several infinite series of graphs are denominator reducible. The starting point for the reduction is \( D_3 = f_0 f_3 \) because of formula (33) for edges \( e_1, e_2 \) and \( e_3 \) forming a 3-valent vertex, but it was shown by Brown that one can start with any other 3 edges. It can happen that a graph is denominator reducible for a particular order of edges but non-reducible for some other order. For more explanations see [5] and [3].

In the case \( \delta_G < 0 \), we apply Theorem 11 to \( [f_0, f_3]_q \) and formula (33) implies \( c_2(G) = 0 \), thus there is no need for application of any reduction process.

In the case \( \delta_G > 0 \), the denominator reducibility of a graph for certain order (ignoring the possible non-existence of a 3-valent vertex) in the sense above does not imply that we can compute the \( c_2 \) invariant. Indeed, in the reduction process we apply formula (24), and, in addition to the middle summand, the rightmost summand is also nonzero (mod \( q \)) for \( \delta_G > 0 \), so we get additional summands. If a graph is denominator reducible in the sense that for some order of edges all the \( D_k \) are defined, then this does not in general imply that the polynomials in the reduction of additional summands are factorizable for the same order of edges. An example for that is \( \tilde{P}_{6,2} \) (from the table in [9]) when one starts with the 4-valent formula (50) from the next section, even forgetting the summands \( c_2(G \setminus i) \) and working only with \([D_4(G)]_q\).

### 4 4-Valent vertex formula

Now consider the situation of a graph \( G \) with a 4-valent vertex incident to the edges, say, \( e_1, \ldots, e_4 \). We are going to think of \( \Psi_G \) as polynomial in the 4 corresponding variables \( \alpha_1, \ldots, \alpha_4 \). We can easily eliminate the first 2 variables using Lemma 12 and get

\[
[\Psi_G]_q = q^{N_G - 1} - [\Psi^1]_q + q^2 [\Psi^{12}, \Psi^2, \Psi^1, \Psi_{12}]_q \\
+ q [\Psi^{12}]_q - q [\Psi^{12}, \Psi^2]_q.
\] (42)

For further reduction using this formula, we firstly need to understand the summand \([\Psi^{12}_G]_q\).
Lemma 15 Let \( G \) be as above. Then
\[
[\Psi^{1,2}]_q \equiv q \big(- [\Psi^{13,24}, \Psi^{14,23}]_q - c_2(G \setminus 3) - c_2(G \setminus 4)\big) \mod q^2.
\] (43)

Proof Using contraction–deletion formula (8)
\[
[\Psi^{1,2}]_q = [\Psi^{13,23} \alpha_3 + \Psi_3^{1,2}]_q = q^{N-3} - [\Psi^{13,23}]_q + q[\Psi^{13,23}, \Psi_3^{1,2}]_q
\]
\[
= q^{N-3} - [\Psi^{13,23}]_q + q^2[\Psi^{13,23}, \Psi_4^{13,24}, \Psi_3^{14,24}, \Psi_3^{14,24}]_q
\]
\[
+ q[\Psi^{13,23}, \Psi_3^{13,24}]_q - q[\Psi^{13,23}, \Psi_3^{14,24}]_q = \Psi^{13,24} = \Psi^{14,23}.
\] (44)

The most relevant polynomial above is
\[
\Psi^{13,23} \Psi_3^{13,24} - \Psi_4^{13,23} \Psi_3^{14,24} = \Psi^{13,24} \Psi^{14,23}.
\] (45)

here we use the Dodgson identity (9). The summand \([\Psi^{13,23}, \Psi_3^{14,24}]_q\) has the form \([f_0, f_3]_q\) for the graph \(G \setminus 4\) with a 3-valent vertex, so it supports the \(c_2\) invariant. Similarly to (36), \([\Psi^{13,23}]_q\) gives the summand \(c_2(G \setminus 3)\). Putting everything together, one gets
\[
[\Psi^{1,2}]_q \equiv q \big([\Psi^{13,24}]_q + [\Psi^{14,23}]_q - [\Psi^{13,24}, \Psi^{14,23}]_q
\]
\[
- c_2(G \setminus 3) - c_2(G \setminus 4)\big) \mod q^2.
\] (46)

The polynomials \(\Psi^{13,24}\) and \(\Psi^{14,23}\) are both of degree \(h_G - 2\) and depend on \(N_G - 4\) variables. Since \(G\) has at least 4 vertices, \(|V_G| - 1 = N_G - h_G \geq 3\), Chevalley–Warning implies \([\Psi^{13,24}]_q \cong [\Psi^{14,23}]_q \cong 0 \mod q\). \(\square\)

Proposition 16 Let \( G \) be a graph with a 4-valent vertex with the 4 incident edges denoted by \(e_1, \ldots, e_4\). Then for any prime power \(q\),
\[
[\Psi[G]]_q \equiv q^2 \big([\Psi^{12}, \Psi_2^1, \Psi_1^2, \Psi_{12}]_q - \sum_{i=1}^{4} c_2(G \setminus i) - [\Psi^{13,24}, \Psi^{14,23}]_q\big) \mod q^3.
\] (47)

Proof By Lemma 12, the elimination of the first 2 variables gives
\[
[\Psi[G]]_q = q^{N_G - 1} - [\Psi^1]_q + q^2 [\Psi^{12}, \Psi_2^1, \Psi_1^2, \Psi_{12}]_q + q[\Psi^{1,2}]_q - q[\Psi^{12}, \Psi_1^2]_q.
\] (48)

Both the summands \([\Psi^1]_q\) and \([\Psi^{12}, \Psi_1^2]_q\) give us \(c_2\) invariants of the corresponding graphs with 3-valent vertices (use that \(G \setminus 12\) has a 2-valent vertex in (37)). Together with Lemma 15, this implies the statement. \(\square\)
The first summand on the right hand side of (47), \([\Psi^{12}, \Psi^1_2, \Psi^2_1, \Psi_{12}]_q\), is given by the point-counting function of the intersection of 4 hypersurfaces and can a priori be very complicated. Nevertheless, we can prove that it vanishes \(\mod q\).

**Proposition 17** Let \(G\) be a graph with a 4 valent vertex with incident edges \(e_1, \ldots, e_4\) (and with at least 5 vertices). Then, in the notation above, for any \(q\):

\[
[\Psi^{12}, \Psi^1_2, \Psi^2_1, \Psi_{12}]_q \equiv 0 \mod q.
\]  

(49)

The proof is moved to a separate section.

We finally state the main result of the paper, the 4-valent formula for computation of the \(c_2\) invariant.

**Theorem 18** Let \(G\) be a graph with \(|V_G| \geq 5\) with a 4-valent vertex with incident edges \(e_1, \ldots, e_4\). Then, for any \(q\):

\[
[\Psi_G]_q \equiv -q^2 \left( [\Psi^{13,24}, \Psi^{14,23}]_q + \sum_{i=1}^{4} c_2(G\setminus i) \right) \mod q^3.
\]  

(50)

Proof Since \([\Psi^{12}, \Psi^1_2, \Psi^2_1, \Psi_{12}]_q \equiv 0 \mod q\) by the previous theorem, the statement is the immediate consequence of Formula (47).

\(\square\)

## 5 Proof of Proposition 17

The idea and the technique of the proof is very similar to Theorem 4.1, [7]. We rewrite \([\Psi^{12}, \Psi^1_2, \Psi^2_1, \Psi_{12}]_q\) as a sum several summands, each of which is a point-counting function of an intersection of at most 2 hypersurfaces, as it is usual for the denominator reduction technique, and then we study all the cancellations. To do this we look closer to \(\Psi_G\), use special notation for some of the appearing graph polynomials and other Dodgson polynomials and we will intensively use the Dodgson identities.

In this section we think of \(q\) to be fixed and we omit the index \(q\) in the point-counting function and will write \([\Psi_G]\) and \([f, g]\) instead of \([\Psi_G]_q\) and \([f, g]_q\), this will make our formulas more readable.

Since deletion of all of the first 4 edges disconnects \(G\), one has \(\Psi^{1234} = 0\). Similarly to Example 6, the Jacobi identity gives us the vanishing of the corresponding \(4 \times 4\) matrix. The first row implies

\[
\Psi^1 = \Psi^{1,2} - \Psi^{1,3} + \Psi^{1,4}.
\]  

(51)

Expanding the polynomials in \(\alpha_2, \alpha_3\) and \(\alpha_4\), we obtain \(\Psi^{123} = \Psi^{123,24}, \Psi^{12} = \Psi_4^{12,23} - \Psi_3^{12,24}, \Psi_1^{234} = \Psi_{34}^{1,2} - \Psi_{24}^{1,3} + \Psi_{23}^{1,4}\). Let’s define

\[
a := \Psi_i^{ijk}, \quad c^{i,j} = (-1)^{i-j-1} \Psi_{ki}^{i,j}, \quad b^i_j := (-1)^{ki} \Psi_j^{ki,ii},
\]  

(52)
where \{i, j, k, t\} = \{1, 2, 3, 4\}, and \(r_b = (k-t)\) if \((k-i)(t-i) > 0\), and \(r_b = (k-t-1)\) otherwise. Managing other rows of the matrix similar to (51), we derive

\[
\psi_{ijk,ijt} = a = \psi_{ij}^{ijk},
\]

\[
\psi_{ij}^k = b_i^k + b_j^k,
\]

\[
\psi_{ijk,ijkl} = c_i^j + c_i^k + c_{i,t}
\]

for all \{i, j, k, t\} = \{1, 2, 3, 4\}. The Dodgson identities for the polynomials above imply

\[
(b_i^j)^2 \equiv \psi_{ik,ij} \mod \alpha \subset \mathbb{Z}[\alpha].
\]

We need to prove the following identity.

**Lemma 19** Let \(G\) be a graph as above. In terms of polynomials defined above,

\[
\psi_{12,34} = b_4^2 - b_3^2 = b_4^2 - b_3^1.
\]

**Proof** We will have in mind the following picture with labelled vertices and edges, see Fig. 1.

Using (10), one writes

\[
\psi_{G_{12,34}} = \sum_{U \subset G \setminus \{12\}} \prod_{u \notin U} \alpha_u \det(E_G(U \cup \{12\})) \det(E_G(U \cup \{34\})),
\]

where \(U\) ranges over all subgraphs of \(G' := G \setminus \{1234\}\) which have the property that \(U \cup \{12\}\) and \(U \cup \{34\}\) are both spanning trees of \(G\). One easily sees that the subgraph \(U\) should necessarily consist of two trees and the vertices \(v_1\) and \(v_2\) (resp., \(v_3\) and \(v_4\)) are on different trees. Hence, we have only 2 possibilities and in terms of (12) one obtains

\[
\psi_{G_{12,34}} = \Phi_{G'}^{13,24} - \Phi_{G'}^{23,14}.
\]

**Fig. 1** \(G\) with a 4-valent vertex
Some readers may think that $\Phi_{G'}^{13,14}$ vanishes by the intuition coming from Fig. 1, but this is true only for planar graphs. Analysing similarly $\Psi_4^{12,13}$ and $\Psi_4^{12,23}$, one finds

$$
\begin{align*}
    b_4^1 &= \Phi_{G'}^{23,14} + \Phi_{G'}^{123,4} \\
    b_4^2 &= \Phi_{G'}^{13,14} + \Phi_{G'}^{123,4}
\end{align*}
$$

(58)

The first statement follows from (57) and (58). The second statement trivially follows from this using the middle equation of (53).

The statement holds for all permutation of indexes also modulo the correct signs, but here we only need that form stated above.

We are going to use a formula for elimination of one of the variables from the system of many polynomials linear in that variable, see Proposition 29, [4]. Let $f_1, \ldots, f_k \in \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$ with $f_i = f_i^1\alpha + f_i,1$ for $\alpha = \alpha_1$. Then

$$
[f_1, \ldots, f_n] = [f_1^\alpha, f_1, \alpha, \ldots, f_n^\alpha, f_n, \alpha]q
+ \left(\sum_{k=1}^{n-2} ([f_1^\alpha, f_1, \alpha, \ldots, f_k^\alpha, f_k, \alpha, f_{k+1}, f_{k+2}, \alpha, \ldots, f_n, \alpha] - [f_1^\alpha, f_1, \alpha, \ldots, f_n^\alpha])
\right)
$$

(59)

Here $[f, g]_\alpha = \pm (f_i^i g_i - f_i g_i^i)$ is the resultant with respect to $\alpha_i$ for polynomials $f$ and $g$ linear in $\alpha$. We also write $[f, g]_i$ for $[f, g]_{\alpha_i}$ sometimes. We apply Formula (59) to the right hand side of (49) for the variable $\alpha = \alpha_3$:

$$
\begin{align*}
    f_a &= \Psi_4^{12}, \quad f_b = \Psi_4^1, \quad f_c = \Psi_4^2, \quad f_d = \Psi_{12}.
\end{align*}
$$

(60)

One obtains

$$
\begin{align*}
    [\Psi_4^{12}, \Psi_4^1, \Psi_4^2, \Psi_{12}] &= [f_a, f_b, f_c, f_d] \\
    &= [f_a^3, f_b^3, f_b^3, f_c^3, f_c^3, f_d^3, f_d^3]q \\
    &\quad + (A + B + C) - ([f_a^3, f_b^3, f_c^3, f_d^3] + [f_a^3, f_a^3])
\end{align*}
$$

(61)

where

$$
\begin{align*}
    A &= [f_a, f_b, f_c, f_d], \\
    B &= [f_a^3, f_a^3, f_b, f_c, f_d], \\
    C &= [f_a^3, f_a, f_b^3, f_b, f_c, f_d].
\end{align*}
$$

(62)

Each of the three summands in the last brackets of (61) is divisible by $q$. Indeed, the variety $\mathcal{V}(f_a^3, f_b^3, f_c^3, f_d^3) \subset \mathbb{A}^{N-2}$ is the cone over the variety defined by the
same equations but in $\mathbb{A}^{N-3}$ (no $\alpha_3$), thus $q[[f_a^3, f_b^3, f_c^3, f_d^3]]$. Now $[f_a^3, f_a^3] = [\Psi_{G'}^{123}, \Psi_{G'}^{12}] = [\Psi_{G'}^{3}, \Psi_{G'}^{3}]$ for $G' = G \setminus 12$, so $q[[f_a^3, f_a^3]]$ by Lemma 9. For the last summand $[f_a^3, f_a^3, f_b^3, f_b^3] = [\Psi_{G'}^{123}, \Psi_{G'}^{12}, \Psi_{G'}^{13}, \Psi_{G'}^{123}]$ we are going to use the formula for a 3-valent vertex (19) for the graph $G' := G \setminus 1$ with edges $e_2, e_3, e_4$ forming the 3-valent vertex. In the notation with $f_i$'s but with indices $i = 2, 3, 4$, we have

$$[\Psi_{G'}^{23}, \Psi_{G'}^{2}, \Psi_{G'}^{3}, \Psi_{G'}^{23}] = [f_0, f_0 \alpha_3 + (f_3 + f_4), g_0 \alpha_3 + (f_2 + f_3), (f_2 + f_4) \alpha_3 + f_234]
= [f_0, f_3 + f_4, f_2 + f_3, (f_2 + f_4) \alpha_3 + f_234].$$

The connecting identity (20) is of the form $f_0 f_234 = f_2(f_3 + f_4) + f_3f_4$, thus again the vanishing of $f_3 + f_4$ on $V(f_0)$ implies the vanishing of both summands $f_3$ and $f_4$. Analogously,

$$[f_0, f_2 + f_3] = [f_0, f_2, f_3].$$

It follows now that all the terms in the brackets (63) become independent of $\alpha_3$. As a consequence, it gives us a cone over a variety in $\mathbb{A}^{N-3}$, and thus, the number of points is divisible by $q$.

Summarizing, we derive the following congruence from (61):

$$[\Psi_{G'}^{12}, \Psi_{G'}^{1}, \Psi_{G'}^{12}] \equiv (A + B + C) \mod q$$

with $A, B, C$ given by (62). Now we will work with these 3 summands separately and then will show that they sum up to $0 \mod q$. For simplicity, we list here the involved polynomials:

$$[f_a, f_b]_3 = \Psi_{123}^{23} \Psi_{2}^{13} - \Psi_{3}^{12} \Psi_{2}^{13} = (\Psi_{12}^{12,13})^2 = (a \alpha + b_4^1)^2,$n
$$[f_a, f_c]_3 = \Psi_{123}^{12} \Psi_{13}^{1} - \Psi_{3}^{12} \Psi_{13}^{1} = (\Psi_{12}^{12,12})^2 = (a \alpha + b_4^2)^2,$n
$$[f_c, f_d]_3 = \Psi_{123}^{23} \Psi_{123} - \Psi_{13}^{2} \Psi_{123} = (\Psi_{12}^{2,3})^2,$n
$$[f_b, f_d]_3 = \Psi_{123}^{13} \Psi_{123} - \Psi_{12}^{3} \Psi_{123} = (\Psi_{12}^{1,3})^2,$n
$$[f_b, f_c]_3 = \Psi_{123}^{13} \Psi_{13}^{1} - \Psi_{12}^{23} \Psi_{123}^{1},$$n
$$[f_a, f_d]_3 = \Psi_{123}^{12} \Psi_{123} - \Psi_{3}^{12} \Psi_{123}^{3}.$$

The coefficient of $\alpha_2$ in the expansion of the first Dodgson identity $\Psi_{1}^{12} \Psi_{13}^{3} - \Psi_{13}^{12} \Psi_{13} = (\Psi_{1,2}^{1,3})^2$ in $\alpha_2$ gives

$$\Psi_{3}^{12} \Psi_{12}^{3} + \Psi_{23}^{1} \Psi_{13}^{23} - \Psi_{123}^{12} \Psi_{123} - \Psi_{13}^{12} \Psi_{13}^{2} = -2 \Psi_{12}^{12,23} \Psi_{1,2}^{1,3}.\quad(67)$$
Similarly, for the expansion in $\alpha_1$ of the Dodgson identity for the pair of edges $e_2$ and $e_3$ implies
\[ \psi_{12}^3 \psi_{12}^3 + \psi_{13}^2 \psi_{2}^1 - \psi_{123}^1 \psi_{23}^1 + \psi_{123}^1 \psi_{23}^1 = 2\psi_{12,13}^1 \psi_{1}^{2,3}. \]  
(68)

The sum of the two equalities above reads
\[ \psi_{12}^3 \psi_{12}^3 - \psi_{123}^1 \psi_{23}^1 = \psi_{12,13}^1 \psi_{1}^{2,3} - \psi_{12,23}^1 \psi_{2}^{1,3}. \]  
(69)

It follows that $[f_a, f_d]_3 \in \mathbb{Z}[\alpha]$ is in the ideal generated by $\psi_{12,13}^1$ and $\psi_{12,23}^1$. One computes
\[ A = [[f_a, f_b], [f_a, f_c], [f_a, f_d]] = [\psi_{12,13}^1, \psi_{12,23}^1] = [a\alpha_4 + b_4^1, a\alpha_4 + b_4^1, b_4^1 - b_4^1]. \]  
(70)

Similarly to the Plücker identity, Lemma 27 in [2], one obtains:
\[ \det M_G([1, 2], [3, 4]) - \det M_G([1, 2], [1, 3]) + \det M_G([1, 2], [2, 3]) = 0. \]  
(71)

Expansion in $\alpha_4$ gives
\[ \psi_{12,34}^1 = b_4^2 - b_4^1. \]  
(72)

After elimination of $\alpha_4$ by (21), equalities (70) and (72) imply
\[ A \equiv [\psi_{12,34}^1] - [a, \psi_{12,34}^1] \mod q. \]  
(73)

Now we are going to compute $B$:
\[ B = [f_a^3, f_a f_b, f_a f_c, f_b f_d] = [a, \psi_{34}^{12}, [f_b, f_c]_3, \psi_{2}^{1,3}]. \]  
(74)

We use again equalities (67) and (68) but now subtracting instead of adding. We immediately get
\[ [f_b, f_c]_3 = [\psi_{13}^2 \psi_{13}^2 - \psi_{23}^1 \psi_{23}^1] = \psi_{12,13}^1 \psi_{1}^{2,3} + \psi_{12,23}^1 \psi_{2}^{1,3}. \]  
(75)

It follows that
\[ B = [a, \psi_{34}^{12}, \psi_{2}^{1,3}, \psi_{12,13}^1 \psi_{1}^{2,3}] = [a, \psi_{34}^{12}, \psi_{2}^{1,3}, (a\alpha_4 + b_4^1)\psi_{1}^{2,3}] = [a, \psi_{34}^{12}, b_4^2\alpha_4 + \psi_{24}^{1,3}, b_4^1 \psi_{1}^{2,3}]. \]  
(76)

The last term of the last brackets disappears, and this follows from (54): $b_4^1$ vanishes on $V(a, \psi_{34}^{12})$. By (21), eliminating $\alpha_4$, one now computes
\[ B \equiv [a, \psi_{34}^{12}] - [a, \psi_{34}^{12}, b_2^1] \mod q. \]  
(77)
Similarly, and, by \( (54) \),
\[
\frac{\psi_{14}}{\psi_{13}} = \frac{\psi_{12}}{\psi_{13}} = \frac{\psi_{24}}{\psi_{23}} = \frac{\psi_{24}}{\psi_{23}} \equiv 0. 
\]

We claim that \( \psi_{14} \) lies in the ideal generated by \( a, \psi_{12}, \psi_{13} \). Indeed, \( \psi_{14} = b_1^2 + b_3^1 \) and, by \( (54) \), \( b_1^2 \) vanishes on \( \mathcal{V}(a, \psi_{13}) \) while \( b_3^1 \) vanishes on \( \mathcal{V}(a, \psi_{12}) \). Thus only the last polynomial in \( (78) \) depends on \( \alpha_4 \). One computes
\[
\psi_{13} = b_3^4 \psi_{12} - b_2^2 \psi_{24} = b_3^4 \psi_{12} - b_2^2 \psi_{24}. 
\]

Each of the appearing polynomials depends on \( \alpha_3 \). A consideration of the constant coefficient gives
\[
\psi_{13} = b_3^4 \psi_{12} - b_2^2 \psi_{24}. 
\]

Consider the variety \( Z = \mathcal{V}(a, \psi_{12}, \psi_{13}) \subset A^{N_G-4} \) and let \( Y = Z \setminus Z \cap \mathcal{V}(b_1^4) \). Since the vanishing of \( \psi_{12} \) implies \( b_1^3 = 0 \) and the vanishing of \( \psi_{24} \) implies \( b_2^1 = 0 \) on \( \mathcal{V}(a) \) by \( (54) \), one gets also \( \psi_{14} = b_1^2 + b_3^1 = 0 \) on \( \mathcal{V}(a) \). Hence, again by \( (54) \), \( b_4^2 \) vanishes on \( Z \). Equation \( (81) \) now implies \( \psi_{13} = 0 \) on \( Z \). Since \( \psi_{13} = b_1^4 + b_3^2 \) and \( b_4^2 = 0 \) while \( b_4^2 \neq 0 \) on \( Y \), one derives \( Y \cap \mathcal{V}(\psi_{13}) \equiv Y \). Thus
\[
C = [\mathcal{V}(a, \psi_{12}, \psi_{13}, \psi_{14}) \setminus \mathcal{V}(\psi_{13})]. 
\]

For \( a, \psi_{12}, \psi_{13} \), one uses the equality \( (b_1^4)^2 \equiv \psi_{14} \psi_{12}^3 \mod a \) in \( (54) \) and gets
\[
[a, \psi_{12}, b_1^4] = [a, \psi_{12}, b_{13}^2, \psi_{12}^3] \\
[a, \psi_{12}, \psi_{13}, \psi_{24}^2, \psi_{12}^3] \\
[a, \psi_{12}, \psi_{13}, \psi_{24}^3, \psi_{12}^3] \\
[a, \psi_{12}, \psi_{13}, \psi_{24}^4, \psi_{12}^3]. 
\]

Similarly,
\[
[a, \psi_{12}, \psi_{24}, b_1^4] = [a, \psi_{12}, \psi_{24}, \psi_{13}, \psi_{12}^2] \\
[a, \psi_{12}, \psi_{24}, \psi_{13}, \psi_{12}^3] \\
[a, \psi_{12}, \psi_{24}, \psi_{13}, \psi_{12}^4] \\
[a, \psi_{12}, \psi_{24}, \psi_{13}, \psi_{12}^5]. 
\]
The last summands of (83) and (84) coincide. Indeed, \( \Psi_{34}^{12} = 0 = \Psi_{23}^{14} \) on \( \mathcal{V}(a) \) imply \( \Psi_{24}^{13} = 0 \) since \( e_2, e_3, e_4 \) form a 3-valent vertex in \( G \setminus 1 \), and also \( \Psi_{23}^{14} = 0 = \Psi_{12}^{14} \) imply \( \Psi_{13}^{24} = 0 \) for the 3-valent vertex formed by \( e_1, e_2, e_3 \) in \( G \setminus 4 \). Proceeding similarly, one derives

\[
[a, \Psi_{34}^{12}, \Psi_{24}^{13}, \Psi_{13}^{24}, \Psi_{12}^{34}] = [a, \Psi_{34}^{12}, \Psi_{24}^{13}, \Psi_{12}^{34}] = [a, \Psi_{34}^{12}, \Psi_{24}^{13}, \Psi_{13}^{24}] = [a, \Psi_{34}^{12}, \Psi_{24}^{13}, \Psi_{12}^{34}]. \tag{85}
\]

Hence,

\[
B + C \equiv [a, \Psi_{34}^{12}] + [a, \Psi_{34}^{12}, \Psi_{24}^{13}] - [a, \Psi_{34}^{12}, \Psi_{12}^{34}] = [a, \Psi_{34}^{12}, \Psi_{24}^{13}] \mod q. \tag{86}
\]

The first summand on the right hand side is divisible by \( q \) by Lemma 9 applied to \( [\Psi_{G'}^{13}, \Psi_{G'}^{14}, \Psi_{G'}^{12}] \) for \( G' = G \setminus 1, 2, 3 \). Similarly, the first two summands on the right hand side of the equality

\[
[a, \Psi_{34}^{12}, \Psi_{24}^{13}] = [a, \Psi_{34}^{12}] + [a, \Psi_{34}^{12}, \Psi_{24}^{13}] - [a, \Psi_{34}^{12}, \Psi_{24}^{13}] \tag{87}
\]

are divisible by \( q \). Using the equality (54), one gets

\[
[a, \Psi_{34}^{12}, \Psi_{24}^{13}] \equiv -[a, \Psi_{34}^{12}, \Psi_{24}^{13}] \equiv -[a, b_4^1] \mod q. \tag{88}
\]

The same thing can be done with \( [a, \Psi_{34}^{12}, \Psi_{12}^{34}] \) in (86). One can also do the step (87) for \( [a, \Psi_{34}^{12}, \Psi_{12}^{34}] \). The congruence (86) now implies

\[
B + C \equiv [a, b_3^1] - [a, b_4^1] + [a, \Psi_{34}^{12}, \Psi_{12}^{34}] \mod q. \tag{89}
\]

By (65) and (73), we finally get the desired formula

\[
[\Psi_{12}, \Psi_{12}^1, \Psi_{12}^2, \Psi_{12}] \equiv A + B + C \equiv [\Psi_{12}^{12}, \Psi_{12}^{14}] - [a, \Psi_{12}, \Psi_{12}^{14}] + [a, b_3^1] - [a, b_4^1] + [a, \Psi_{34}^{12}, \Psi_{12}^{34}] \mod q. \tag{90}
\]

One computes \( [a, b_3^1] = [\Psi_{23}^{23,34}, \Psi_{24}^{34,4}] \equiv c_2(G') \mod q \) for \( G' = G \setminus 1 \), since the additional (see (38)) summand \( c_2(G \setminus 3) \) vanishes \( \mod q \) by the existence of a 2-valent vertex. The same holds for \( [a, b_4^1] \), so these terms sum up to \( 0 \mod q \). One also has \( q||[\Psi_{12}, \Psi_{12}^{12}] \) by Chevalley–Warning since \( |\mathcal{V}_G| \geq 5 \). Hence,

\[
[\Psi_{12}, \Psi_{12}^1, \Psi_{12}^2, \Psi_{12}] \equiv [a, \Psi_{34}^{12}, \Psi_{12}^{34}] - [a, \Psi_{12}, \Psi_{12}^{14}] \mod q. \tag{91}
\]

Let’s consider a modification of our graph shown in Fig. 2.
In a graph $G$ with edges $e_1, \ldots, e_4$ forming a 4-valent vertex, we remove this 4 edges (and the vertex) and add two other edges $e_s$ and $e_t$. We denote the resulting graph by $G'$. One easily sees that

$$
\Psi_{G,34}^{12} = \Psi_{G',t}^{s'} \quad \text{and} \quad \Psi_{G,12}^{34} = \Psi_{G',s}^t.
$$

(92)

Using the first Dodgson identity for $I = \{s\}, J = \{t\}$, one gets

$$
\mathcal{V}(a, \Psi_{G,34}^{12} \Psi_{G,12}^{34}) \cong \mathcal{V}(\Psi_{G',t}^{s'}, \Psi_{G',s}^t \Psi_{G',t}^{s'}) \cong \mathcal{V}(\Psi_{G'}^{s,t}, \Psi_{G'}^{s,t}).
$$

(93)

Now it remains to show that $\Psi^{s,t}$ and $\Psi^{12,34}$ coincide.

One can again apply Formula (56) for both $\Psi^{s,t}$ and $\Psi^{12,34}$. It is easy to see that for each $U$ such that $U \cup \{1, 2\}$ (resp., $U \cup \{3, 4\}$) is a spanning tree for $G$, one also has $U \cup \{s\}$ (resp., $U \cup \{t\}$) is a spanning tree for $G'$ and vice versa. The coefficients in the formula above also coincide, so we get

$$
\Psi_{G}^{12,34} = \Psi_{G'}^{s,t}.
$$

(94)

As a consequence,

$$
[a, \Psi_{G'}^{s,t}] = [a, \Psi_{G}^{12,34}]
$$

(95)

By (91), we finally obtain

$$
[\Psi_{12}^1, \Psi_{12}^2, \Psi_{12}^3, \Psi_{12}^4] \equiv 0 \mod q,
$$

(96)

as desired. This finishes the proof of Proposition 17.

### 6 Applications

For an application of the 4-valent vertex formula (50), we can restrict to 4-regular graphs. These graphs do not have a 3-valent vertex, and thus, Formula (33) cannot help for computation of $c_2(G)$.

The most famous and simple infinite series of primitive log-divergent graphs in $\phi^4$ theory is $\mathbb{Z} \mathbb{Z}_h$ (see, for example, [3], Section 5.6), and the most simple nontrivial
4-regular graphs are their completions. They should behave nicely from the point of view of the $c_2$ invariant. We conjecture the following.

**Conjecture 20** Let $\hat{ZZ}_h$ be a zigzag graph with $h = h_G \geq 3$ loops and let $\hat{ZZ}_h$ be its completion. Then, for any $q$,

$$c_2(\hat{ZZ}_h)_q \equiv -h_G(h_G + 2) \mod q. \quad (97)$$

We have checked the conjecture for small loop number:

**Proposition 21** The statement of Conjecture 20 is true for first six graphs $\hat{ZZ}_h$, $3 \leq h \leq 8$.

**Proof** One starts from formula (50) and to each of 5 summands one applies the denominator reduction type argument by iterative use of formulas (21) and (23) together with Chevalley–Warning vanishing and factorization of the appearing polynomials into products of linear factors. □

There should be an analytic way to prove Conjecture 20 for all $\hat{ZZ}_h$ using the recurrence relations between Dodgson polynomials appearing in the reduction process.

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