TRIPLE PRODUCTS AND YANG–BAXTER EQUATION (I): OCTONIONIC AND QUATERNIONIC TRIPLE SYSTEMS

by

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Abstract

We can recast the Yang–Baxter equation as a triple product equation. Assuming the triple product to satisfy some algebraic relations, we can find new solutions of the Yang–Baxter equation. This program has been completed here for the simplest triple systems which we call octonionic and quaternionic. The solutions are of rational type.

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1. Introduction

The Yang–Baxter (Y–B) equation

\[
R^{b' a'}_{a_1 b_1} (\theta) R^{c' a_2}_{a' c_1} (\theta') R^{b_2 c}_{b' c'} (\theta'') = R^{c' b'}_{b_1 c_1} (\theta'') R^{a_1 c'}_{a' c'} (\theta') R^{b_2 a_2}_{b' a'} (\theta)
\]

(1.1a)

with

\[
\theta' = \theta + \theta''
\]

(1.1b)

appears in many subjects ranging from statistical physics\(^1\), exactly solvable one-dimensional field theory\(^1,2\), and, the braid group\(^3,4\) to the quantum group\(^1,5\). Here, and hereafter, all repeated indices are understood to be automatically summed over \(N\)-values, \(1, 2, \ldots, N\) for some integer \(N\). It is often more convenient to consider a \(N\)-dimensional vector space \(V\) over a field \(F\) (normally real or complex) with a basis vector \(e_1, e_2, \ldots, e_N\). We ordinarily introduce\(^1\) the general linear transformations \(R_{12}(\theta), R_{13}(\theta)\) and \(R_{23}(\theta)\) operating on the tensor product \(V \otimes V \otimes V\) by

\[
R_{12}(\theta) e_a \otimes e_b \otimes e_c = R^{b' a'}_{ab} (\theta) e_a' \otimes e_{b'} \otimes e_c
\]

(1.2a)

\[
R_{23}(\theta) e_a \otimes e_b \otimes e_c = R^{c' b'}_{bc} (\theta) e_a \otimes e_{b'} \otimes e_{c'}
\]

(1.2b)

\[
R_{13}(\theta) e_a \otimes e_b \otimes e_c = R^{c' a'}_{ac} (\theta) e_{a'} \otimes e_b \otimes e_{c'}
\]

(1.2c)

Then, the Y–B equation Eq. (1.1a) is rewritten as

\[
R_{12}(\theta) R_{13}(\theta') R_{23}(\theta'') = R_{23}(\theta'') R_{13}(\theta') R_{12}(\theta)
\]

(1.3)

which is convenient to discuss the underlying structure of the theory, such as the Hopf algebra. However, it is not easy in general to find explicit solutions of Eq. (1.3) except for some simple systems.

In this note, we will present a new way of solving the Y–B equation in terms of some triple product systems\(^6,7\). Here, we will first assume for the sake of simplicity that \(R_{cd}(\theta)\) satisfies the symmetric condition

\[
R_{da}(\theta) = R_{cd}(\theta)
\]

(1.4)
in what follows. The general case without assuming this will be discussed in the proceeding paper. Second, we suppose that the vector space $V$ possesses a non-degenerate bilinear form $< x|y > : V \otimes V \to F$, and set

$$ g_{ab} = < e_a|e_b > $$

(1.5)

with its inverse $g^{ab}$ so that

$$ g^{ab}g_{bc} = \delta^a_c. $$

(1.6)

Although we ultimately restrict ourselves to the case of $g_{ab} = g_{ba}$ in this note, we need not yet assume it. In terms of $g^{ab}$, we set

$$ e^a = g^{ab}e_b, \quad e_a = g_{ab}e^b $$

(1.7)

which satisfies

$$ < e^a|e_b > = \delta^a_b $$

(1.8)

as well as

$$ e_a < e^a|x > = < x|e_a > e^a = x. $$

(1.9)

We now introduce a $\theta$-dependent triple product

$$ [x, y, z]_\theta : V \otimes V \otimes V \to V $$

(1.10)

by

$$ [e^b, e_c, e_d]_\theta = R^{ab}_{cd}(\theta)e_a $$

(1.11)

or equivalently by

$$ < e^a|[e^b, e_c, e_d]_\theta > = R^{ab}_{cd}(\theta). $$

(1.12)

The symmetric condition Eq. (1.4) can be readily rewritten as

$$ < u|[z, x, y]_\theta > = < z|[u, y, x]_\theta > $$

(1.13)
while the Y–B equation Eq. (1.1a) becomes a triple product equation

\[
[v, [u, e_j, z]_{\theta''}, [e^j, x, y]_{\theta'}]_{\theta''} = [u, [v, e_j, x]_{\theta''}, [e^j, z, y]_{\theta'}]_{\theta}
\] (1.14)

in a basis–independent notation. Indeed, identifying \( x = e_{a_1}, y = e_{b_1}, z = e_{c_1}, u = e^{a_2}, \) and \( v = e^{c_2} \) together with Eq. (1.4), it will reproduce Eq. (1.1a). We note also that Eq. (1.14) is invariant under

\[ u \leftrightarrow v , \; x \leftrightarrow z , \; \theta \leftrightarrow \theta'' . \] (1.15)

Our ultimate task is to find solutions of the triple product equation Eq. (1.14) under the condition Eq. (1.13). To this end, we assume existence of \( \theta \)-independent triple (or ternary) product

\[
[x, y, z] : V \otimes V \otimes V \to V
\] (1.16)
satisfying the symmetry condition

\[
<u|[x, y, z]> = <z|[y, x, u]>, \tag{1.17}
\]

and seek a solution of Eq. (1.14) in a form of

\[
[z, x, y]_{\theta} = P(\theta)[x, y, z] + A(\theta)<x|y > z + B(\theta)<z|x > y + C(\theta)<z|y > x
\] (1.18)

for some functions \( P(\theta), A(\theta), B(\theta), \) and \( C(\theta) \) of \( \theta \) to be determined. Note first that we have changed the orders of variables between both sides of Eq. (1.18) and second that the condition Eq. (1.13) is automatically satisfied by Eq. (1.18), provided that we have

\[
<y|x> = \pm <x|y>
\] (1.19)

Inserting the expression Eq. (1.18) to both sides of Eq. (1.14), and assuming some suitable triple product relations for \( [x, y, z] \), it will give a rather complicated algebraic relations among \( P(\theta), A(\theta), B(\theta), \) and \( C(\theta) \), as we will see in sections 3 and 4 as well as in subsequent papers.
In this paper, we will consider perhaps the simplest cases for the triple product \([x, y, z]\) which we may call quaternionic and octonionic triple products, where \([x, y, z]\) is totally anti-symmetric in 3 variables \(x, y, z\). We will give its general discussions in section 2, and solve the corresponding Y–B equation in section 3 for \(N = 8\) and in section 4 for \(N = 4\). The case of \(N = 8\) reproduces the previously known result by de Vega and Nicolai\(^8\) who solved the Y–B equation in the traditional component-wise fashion. The more general triple systems will be discussed in the subsequent paper\(^9\). In the Appendix, we will explore some properties of octonionic triple systems, especially its relation to the standard octonion algebra.

2. Octonionic and Quaternionic Triple Systems

In this section, we will present some mathematical analysis of triple product systems which we call octonionic and quaternionic systems, since they are needed for calculations of the next sections. More details of their mathematical structure will be found in the Appendix. Let \(V\) be the \(N\)-dimensional vector space with non-degenerate inner product \(<x|y>\) and with the \(\theta\)-independent triple (or ternary) product \([x, y, z]\) as in the previous section. We assume now that they obey the following axioms:

(i) \(<x|y> = <y|x>\),

(ii) \([x, y, z]\) is totally antisymmetric in \(x, y, z\);

(iii) \(<w|[x, y, z]\) is totally antisymmetric in \(x, y, z\) and \(w\);

(iv) \(<[x, y, z]|[u, v, w]\) 

\[ = \alpha \{ <x|u><y|v><z|w> + <y|u><z|v><x|w> + <z|u><x|v><y|w> - <x|w><y|v><z|u> - <y|w><z|v><x|u> - <z|w><x|v><y|u> \} \]

\[ + \beta \{ <x|u><y|[z, v, w]> + <y|u><z|[x, v, w]> + <z|u><x|[y, v, w]> + <x|v><y|[z, w, u]> \} \]

\[ + \gamma \{ <x|u><y|[z, v, w]> + <y|u><z|[x, v, w]> + <z|u><x|[y, v, w]> + <x|v><y|[z, w, u]> \} \]

\[ + \delta \{ <x|u><y|[z, v, w]> + <y|u><z|[x, v, w]> + <z|u><x|[y, v, w]> + <x|v><y|[z, w, u]> \} \]
for some constants $\alpha$ and $\beta$. In view of the non-degeneracy of $< w|x >$ as well as by Eq. (2.3), the relation Eq. (2.4) is equivalently rewritten as a triple product equation:

(iv)' $[[x, y, z], u, v]$

\[
\begin{align*}
&= \{\alpha[< y|v >< z|u ] - < y|u > < z|v >] - \beta < u|[v, y, z] > \}x \\
&+ \{\alpha[< z|v >< x|u ] - < z|u > < x|v >] - \beta < u|[v, z, x] > \}y \\
&+ \{\alpha[< x|v >< y|u ] - < x|u > < y|v >] - \beta < u|[v, x, y] > \}z \\
&- \beta < x|v > [u, y, z] + < y|v > [u, z, x] + < z|v > [u, x, y] \\
&+ < x|u > [v, z, y] + < y|u > [v, x, z] + < z|u > [v, y, x] \}
\end{align*}
\] (2.5)

Before going into further details, we will prove shortly that the solutions of Eqs. (2.1)–(2.5) are possible only for three cases of

(a) $N = 8$ with $\alpha = \beta^2$  \hspace{1cm} (2.6a)

(b) $N = 4$ with $\beta = 0$  \hspace{1cm} (2.6b)

(c) $[[x, y, z], u, v] = 0$ identically with $\alpha = \beta = 0$.  \hspace{1cm} (2.6c)

Since the last possibility (c) is uninteresting, we will not consider the case in this note. We call then two cases of $N = 8$ and $N = 4$ to be octonionic and quaternionic triple systems, respectively, by a reason to be explained in the Appendix. We will also show there that the underlying vector space $V$ for the case of $N = 8$ can be identified as the spinor representation space of the Lie algebra $\text{so}(7)$. Moreover, the reason for the validity of Eq. (2.4) will also be given.

Let $e_1, e_2, \ldots, e_N$ be a basis of $V$ normalized now by

\[
< e_a|e_b > = \delta_{ab} \quad , \tag{2.7}
\]

and introduce the structure constant $c_{abcd}$ of the triple algebra by

\[
[e_a, e_b, e_c] = c_{abcd}e^d \quad . \tag{2.8}
\]
Then, Eq. (2.3) implies that $c_{abcd}$ is totally antisymmetric in 4 indices $a, b, c,$ and $d$, while Eq. (2.4) is rewritten as

$$c_{abcd}c_{ijk} = \alpha \sum_P (-1)^P \delta_{ai} \delta_{bj} \delta_{ck}$$

$$+ \frac{1}{4} \beta \sum_P \sum_{P'} (-1)^P (-1)^{P'} \delta_{ai} c_{bcjk},$$

(2.9)

where the summations over $P$ and $P'$ are over $3!$ permutations of $a, b, c$ and of $i, j, k$, respectively. For $N = 8$ with normalization $\alpha = -\beta = 1$, Eq. (2.9) reproduces the result of de Wit and Nicolai\(^ {10} \) as well as that of Gürsey and Tze\(^ {11} \). The case of $N = 4$ is simpler. Let $\epsilon_{ijk}$ with $\epsilon_{1234} = 1$ to be the totally antisymmetric Levi-Civita symbol in 4–dimensional space. Then,

$$[\epsilon_i, \epsilon_j, \epsilon_k] = \epsilon_{ijk\ell} \epsilon^\ell$$

(2.10)

defines the quaternionic triple system with $\alpha = 1$ and $\beta = 0$, as we will show in the Appendix.

Now, we will prove the results of Eqs. (2.6). First, we calculate the expression

$$J_0 = [v, [u, e_j, z], [e^j, x, y]]$$

$$= -[v, [u, z, e_j], [x, y, e^j]]$$

(2.11)

in the following way. Setting

$$w = [u, e_j, z] = -[u, z, e_j]$$

(2.12)

we evaluate

$$J_0 = -[[x, y, e^j], w, v]$$

$$= \{- \{ \alpha [< y | v > < e^j | w > - < y | w > < e^j | v >] - \beta < w | [v, y, e^j] > \} x$$

$$- \{ \alpha [< e^j | v > < x | w > - < e^j | w > < x | v >] - \beta < w | [v, e^j, x] > \} y$$

$$- \{ \alpha [< x | v > < y | w > - < x | w > < y | v >] - \beta < w | [x, y, v] > \} e^j$$

$$+ \beta \{ < x | v > [w, y, e^j] + < y | v > [w, e^j, x] + < e^j | v > [w, x, y]$$

$$+ < x | w > [v, e^j, y] + < y | w > [v, x, e^j] + < e^j | w > [v, y, x] \}$$

(2.13)

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by replacing \( z \) and \( u \) by \( e^j \) and \( w \), respectively, in Eq. (2.5). We calculate for example

\[
[w, e^j, x] = -[[u, z, e^j], e^j, x]
\]

\[
= \alpha(N-2)\{< x|u > z - < x|z > u \} - \beta(N-4) [x, u, z]
\]

again from Eq. (2.5) together with Eq. (1.9). Also, we can reduce a term such as

\[
< w[e^j, x, y] > = < [u, e^j, z][e^j, x, y] >
\]

into a simpler form when we utilize Eq. (2.4). Inserting these results into (2.13) and noting Eq. (1.9), we find

\[
J_0 = -[v, [u, z, e^j], [x, y, e^j]]
\]

\[
= A_1 x + \overline{A_1} y + A_2 u + \overline{A_2} z
\]

\[
+ \{ (N-6)\beta^2 - \alpha \} \{< x|v > [y, u, z] - < y|v > [x, u, z] \}
\]

\[
+ \beta^2\{ - < x|u > [y, z, v] + < y|u > [x, z, v] + < x|z > [y, u, v]
\]

\[
- < y|z > [x, u, v] - < u|v > [x, y, z] + < z|v > [x, y, u] \}
\]

(2.14)

after some calculations where we have set

\[
A_1 = -\{ (N-6)\beta^2 - \alpha \} < y|[z, u, v] >
\]

\[
+ \alpha\beta(N-4)\{ < y|u > < z|v > - < y|z > < v|u > \} \ , \quad (2.15a)
\]

\[
\overline{A_1} = \{ (N-6)\beta^2 - \alpha \} < x|[u, v, z] >
\]

\[
- \alpha\beta(N-4)\{ < x|u > < z|v > - < x|z > < v|u > \} \ , \quad (2.15b)
\]

\[
A_2 = \beta^2 < y|[x, z, v] >
\]

\[
+ \alpha\beta(N-4)\{ < x|v > < y|z > - < y|v > < x|z > \} \ , \quad (2.15c)
\]

\[
\overline{A_2} = -\beta^2 < y|[x, u, v] >
\]

\[
- \alpha\beta(N-4)\{ < x|v > < y|u > - < y|v > < x|u > \} \ . \quad (2.15d)
\]

However, the left side of Eq. (2.14) is antisymmetric for the exchange of \( x \leftrightarrow u \) and \( y \leftrightarrow z \), so that the right side of Eq. (2.14) must share the same property. This fact can be easily
seen to lead to the validity of either

\[(N - 7)\beta^2 = \alpha \quad (2.16)\]

or

\[<x|v > [y, u, z] - <y|v > [x, u, z] + <u|v > [x, y, z] - < z|v > [x, y, u]
\]
\[= < y|[z, u, v] > x - < x|[z, u, v] > y
\]
\[+ < y|[x, z, v] > u - < y|[x, u, v] > z . \quad (2.17)\]

For the second case of Eq. (2.17), we set further \(u = e_j\) and \(v = e_j\) and sum over \(j = 1, 2, \ldots, N\) to find

\[(N - 4)[x, y, z] = 0 \quad . \quad (2.17)'\]

Therefore, excluding the trivial case of \([x, y, z] = 0\) identically, the second condition Eq. (2.17) is possible only for \(N = 4\).

Next, we calculate another expression

\[J_1 = [[x, y, z], e_j, [u, v, e^j]]\]

in the following two different ways. First, we set \(w = [u, v, e^j]\) and evaluate

\[J_1 = [[x, y, z], e_j, w]\]

by repeated uses of Eq. (2.5). Alternately, we can compute \(J_1\) in a similar way by setting \(\overline{w} = [x, y, z]\) and hence

\[J_1 = [\overline{w}, e_j, [u, v, e^j]] = [[u, v, e^j], \overline{w}, e_j] . \]

Comparing both expressions, we obtain

\[2(\beta^2 - \alpha)\{< y|[u, v, z] > x + < z|[u, v, x] > y + < x|[u, v, y] > z\}
\]
\[+ [6\beta^2 - (N - 2)\alpha]\{< v|[x, y, z] > u - < u|[x, y, z] > v\}
\]
\[= (N - 8)\beta^2\{< x|u > [y, z, v] + < y|u > [z, x, v] + < z|u > [x, y, v]
\]
\[= < x|v > [z, y, u] - < y|v > [x, z, u] - < z|v > [y, x, u]\} = 0 \quad . \quad (2.18)\]
Setting $u = e_j$ and $v = e^j$, and summing over $j$, Eq. (2.18) leads to

$$-6\beta^2(N - 8)[x, y, z] = 0$$

so that we have $N = 8$ or $\beta = 0$, provided that $[x, y, z]$ is not identically zero. Combining this fact with Eqs. (2.16) and (2.17)', we find the desired results Eqs. (2.6). Note that for $N = 8$ with $\alpha = \beta^2$, Eq. (2.18) is identically satisfied, while, for $N = 4$ with $\beta = 0$, it gives an identity

$$<y|[u, v, z] > x + < z|[u, v, x] > y + < x|[u, v, y] > z$$

$$+ < v|[x, y, z] > u - < u|[x, y, z] > v = 0.$$  \hspace{1cm} (2.19)

The relation Eq. (2.19) is automatically satisfied for $N = 4$, since its left side is totally antisymmetric in 5 variables $x, y, z, u,$ and $v$. Relations Eqs. (2.17) and (2.19) will be utilized in section 4.

For the case of $N = 8$ with $\alpha = \beta^2$, Eq. (2.14) is rewritten as

$$J_0 = [v, [u, e_j, z], [e^j, x, y]]$$

$$= \{4\alpha\beta[< y|u >> z|v > - < y|z >> v|u >] - \beta^2 < y|[z, u, v] >\}x$$

$$- \{4\alpha\beta[< x|u >> z|v > - < x|z >> v|u >] - \beta^2 < x|[u, v, z] >\}y$$

$$+ \{4\alpha\beta[< x|v >> y|z > - < y|v >> x|z >] + \beta^2 < y|[x, z, v] >\}u$$

$$- \{4\alpha\beta[< x|v >> y|u > - < y|v >> x|u >] + \beta^2 < y|[x, u, v] >\}z$$

$$+ \beta^2 \{< x|v > [y, u, z] - < y|v > [x, u, z] - < x|u > [y, z, v] + < y|u > [x, z, v]$$

$$+ < x|z > [y, u, v] - < y|z > [x, u, v] - < u|v > [x, z, v] + < z|v > [x, y, u]\}$$

which will be used in the calculation of the next section.

We can similarly compute

$$[[x, y, z], t, [u, v, w]]$$

in two different ways. In this way, we obtain a very complicated identity which is however not given here.
Further discussions of the triple product as well as its relation to the octonion and quaternion algebras will be given in the Appendix.

3. Solution for $N = 8$

Here, let $[x, y, z]$ be the octonionic triple product as has been defined in the previous section. We seek a solution of the Y–B equation Eq. (1.14) in a form given by Eq. (1.18) for 4–unknown functions $P(\theta)$, $A(\theta)$, $B(\theta)$, and $C(\theta)$ to be determined. For simplicity, we set

$$P = P(\theta), \quad P' = P(\theta'), \quad P'' = P(\theta''), \quad (3.1)$$

and similarly for $A(\theta)$, $B(\theta)$, and $C(\theta)$. Inserting the expression Eq. (1.18) into Eq. (1.14), each side of Eq. (1.14) contains 64 terms, among whom the most complicated term is the one proportional to $P''P'P$, i.e.

$$(3.2)$$

However, for $N = 8$, this can be computed easily from Eq. (2.20). At any rate, after some straightforward calculations, we find the following result:

$$0 = [v, [u, e_j, z], [e_j, x, y]] - (u \leftrightarrow v, x \leftrightarrow z)$$

(3.3)

where $G_\mu (\mu = 1, 2, \ldots, 12)$ are cubic polynomials of $P$, $A$, $B$, and $C$ to be given below, and $\hat{G}_\mu$ is the function which can be obtained from $G_\mu$ by interchanging $\theta \leftrightarrow \theta''$. Their
explicit forms are found to be

\[ G_1 = 2\beta'^2 P''P' P - 2\beta'' P' B' P - \beta'' P' B - \beta' P'' P - P'' A'B - B'' A'P \, , \]
\[ G_2 = -G_1 \, , \]
\[ G_3 = 0 \, , \]
\[ G_4 = \beta^2 P'' P' P + 2\beta P'' B' P + \beta'' P' B + \beta B'' P' P \\
- C'' P'C - B'' C' P + C'' C' P - P'' C'B + P'' C'C + 4\beta P'' C' P \, , \]
\[ G_5 = -\beta^2 P'' P' P + \beta'' B' P + 4\beta A'' P' P - \beta' P'' P' B \\
+ 2\beta B'' P' P - A'' B' P + A'' P' C - A'' A' P + A'' B' P - P'' A'C \, , \]
\[ G_6 = 2\beta^2 P'' P' P + \beta'' B' P - \beta'' P' P - P'' P' C + 2\beta P'' P' B + B'' P'C \, , \]
\[ G_7 = 4\alpha\beta P'' P' P - \alpha P'' B' P + \alpha P'' P' B + \alpha B'' P' P \\
- 6\alpha P'' C' P + B'' C'C + C'' C'B - C'' B'C \, , \]
\[ G_8 = -4\alpha \beta P'' P' P + \alpha P'' B' P - 6\alpha A'' P' P - \alpha P'' P' B \\
- \alpha B'' P' P + B'' A'C - A'' A' P + A'' A'C \, , \]
\[ G_9 = 4\alpha \beta P'' P' P - \alpha P'' B' P + \alpha P'' P' B + \alpha B'' P' P - A'' A' A - A'' B' A \\
- 8A'' C'A - B'' C'A - C'' C'A - A'' C'B - A'' C'C + C'' A'C \, , \]
\[ G_{10} = \hat{G}_8 \, , \]
\[ G_{11} = -G_6 \, , \]
\[ G_{12} = -\hat{G}_5 \, . \]

Therefore, the Y–B equation is satisfied, if we have

\[ G_1 = G_4 = G_5 = G_6 = G_7 = G_8 = G_9 = G_{10} = G_{11} = G_{12} = 0 \, . \]

The conditions Eq. (3.5) agree with those found by de Vega and Nicolai\(^8\), if we set \( \alpha = -\beta = 1 \), and \( \theta \rightarrow \theta'' \rightarrow \theta' \rightarrow \theta \) with \( P(\theta) = -c(\theta) \), \( A(\theta) = b(\theta) \), and \( B(\theta) = a(\theta) \). For simplicity, we normalize \( \alpha \) and \( \beta \) by

\[ \alpha = -\beta = 1 \, . \]
Then, the solution of Eq. (3.5) can be found by first considering $G_1 = 0$ which can be rewritten as

$$\left(\frac{B''}{P''} + \frac{B}{P} \right) \left( \frac{A'}{P'} - 1 \right) = 2 + 2 \frac{B'}{P'} .$$  \hspace{1cm} (3.7)

However, since we must have $\theta' = \theta + \theta''$, the solution must be of form

$$\frac{B}{P} = \frac{B(\theta)}{P(\theta)} = \lambda_0 + b\theta$$  \hspace{1cm} (3.8)

for some constants $\lambda_0$ and $b$, so that

$$\frac{A}{P} = \frac{A(\theta)}{P(\theta)} = 1 + \frac{2(1 + \lambda_0 + b\theta)}{2\lambda_0 + b\theta} .$$

We can then examine the rest of Eqs. (3.5) to find the final solution:

$$\frac{A(\theta)}{P(\theta)} = \frac{18 - 3b\theta}{10 - b\theta} ,$$  \hspace{1cm} (3.9a)

$$\frac{B(\theta)}{P(\theta)} = b\theta - 5 ,$$  \hspace{1cm} (3.9b)

$$\frac{C(\theta)}{P(\theta)} = \frac{12 - 3b\theta}{b\theta} ,$$  \hspace{1cm} (3.9c)

where $b$ is an arbitrary constant and $P(\theta)$ is undetermined. With the choice of normalization $b = -3$ and $C(\theta) = \frac{1}{\theta}$, it reproduces the result of reference 8,

$$P(\theta) = -\frac{1}{3\theta + 4} ,$$  \hspace{1cm} (3.10a)

$$A(\theta) = -\frac{9(\theta + 2)}{(3\theta + 4)(3\theta + 10)} ,$$  \hspace{1cm} (3.10b)

$$B(\theta) = \frac{3\theta + 5}{3\theta + 4} ,$$  \hspace{1cm} (3.10c)

$$C(\theta) = \frac{1}{\theta} .$$  \hspace{1cm} (3.10d)

Although we have assumed $P(\theta) \neq 0$ in our derivation, we can find a solution for $P(\theta) = 0$ by first setting $P(\theta) = b/12$ in Eqs. (3.9) and then letting $b \to 0$. In this way, we obtain a rather trivial classical Y–B solution

$$P(\theta) = A(\theta) = B(\theta) = 0 ,$$

$$C(\theta) = \frac{1}{\theta} .$$  \hspace{1cm} (3.11)
The more general solution for $P(\theta) = 0$ will be given in the proceeding paper.

4. Solution for $N = 4$

As we will explain shortly, the case of $N = 4$ requires a separate discussion. Surprisingly, this case is slightly more complicated than the previous one of $N = 8$, although it is simpler for calculations of triple products. For instance, since we have $\beta = 0$, Eq. (2.14) reduces to a simpler equation

\[
\begin{align*}
[v, [u, e_j, z], [e^j, x, y]] &= \alpha < y|[z, u, v] > x - \alpha < x|[u, v, z] > y \\
&- \alpha \{ < x|v > [y, u, z] - < y|v > [x, u, z]\}.
\end{align*}
\]

(4.1)

However, we have, now, various constraints to be taken into account. For any 6 variables $x, y, z, u, v,$ and $w$, we first note the validity of the identity

\[
< w|x > < y|[z, u, v] > + < w|y > < z|[u, v, x] > + < w|z > < u|[v, x, y] > + < w|u > < v|[x, y, z] > + < w|v > < x|[y, z, u] > = 0
\]

(4.2)

for $N = 4$, since the left side of Eq. (4.2) is totally antisymmetric in 5 variables $x, y, z, u$ and $v$. In view of the non-degeneracy of the inner product $< w|x >$, Eq. (4.2) immediately give

\[
< y|[z, u, v] > x + < z|[u, v, x] > y + < u|[v, x, y] > z \\
+ < v|[x, y, z] > u + < x|[y, z, u] > v = 0
\]

(4.3)

which is equivalent to Eq. (2.19). Moreover, if we note

\[
< y|[z, u, v] > = - < z|[y, u, v] > \\
< v|[x, y, z] > = - < z|[x, y, v] > \\
< x|[y, z, u] > = - < z|[y, x, u] >
\]

then Eq. (4.2) leads also to another identity

\[
- < w|x > [y, u, v] + < w|y > [u, v, x] + < u|[v, x, y] > w \\
- < w|u > [x, y, v] - < w|v > [y, x, u] = 0
\]
Rewriting $w$ by $z$, and noting Eq. (4.3), we find then

\[
< z | y > [u, v, x] \\
= < z | x > [y, u, v] + < z | u > [x, y, v] + < z | v > [y, x, u] - < u | [v, x, y] > z \\
= < z | x > [y, u, v] + < z | u > [x, y, v] + < z | v > [y, x, u]
\] (4.4)

\[
- < z | [y, u, v] > x + < z | [u, v, x] > y - < z | [x, y, v] > u + < z | [x, y, u] > v
\]

which reproduces Eq. (2.17) when we let $z \leftrightarrow v$. Interchanging $z \leftrightarrow x$, or $z \leftrightarrow u$, or $z \leftrightarrow v$ etc., we can therefore eliminate now $< z | y > [u, v, x]$, $< x | y > [u, v, z]$, $< u | y > [z, v, x]$ and $< v | y > [u, z, x]$ as linear combinations of terms of form $< a | b > [c, d, y]$, $< y | [a, b, c] > d$ and $< z | [u, v, x] > y$ for $a, b, c, d$ being some permutations of $x, z, u$ and $v$. Except for this fact, the calculation for $N = 4$ proceeds just as in the previous case of $N = 8$ to lead to

\[
0 = [v, [u, e_{j}, z]_{\theta'}, [e^{j}, x, y]_{\theta}]_{\theta''} - (u \leftrightarrow v, x \leftrightarrow z, \theta \leftrightarrow \theta'')
\]

\[
= F_{1} < u | v > [z, x, y] + F_{2} < x | z > [y, u, v] + F_{3} < x | u > [v, z, y]
- \hat{F}_{3} < z | v > [u, x, y] + F_{4} < x | v > [y, z, u] - \hat{F}_{4} < z | u > [y, x, v]
+ F_{5} < y | [z, u, v] > x - \hat{F}_{5} < y | [x, u, v] > z + F_{6} < y | [z, x, v] > u
- \hat{F}_{6} < y | [z, x, u] > v + F_{7} < y | z > < u | v > x - \hat{F}_{7} < y | x > < u | v > z
+ F_{8} < z | u > < y | v > x - \hat{F}_{8} < z | v > < y | u > z + F_{9} < x | z > < y | v > u
- \hat{F}_{9} < x | z > < y | u > v + F_{10} < y | z > < x | v > u - \hat{F}_{10} < y | x > < z | u > v
\] (4.5)

where $\hat{F}_{\mu}(\mu = 1, 2, \ldots, 10)$ is the function obtained from $F_{\mu}$ by interchanging $\theta \leftrightarrow \theta''$ as
before. Explicit expressions of \( F_{\mu} \)'s are found after some calculations to be

\[
F_1 = 2\alpha P'' P' P - C'' P' B - B'' P' C + P'' A' B + B'' A' P + P'' B'C + C'' B' P ,
\]

\[
F_2 = -B'' P' A + C'' P' A - A'' P' B + A'' P' C - P'' A' B - B'' A' P - P'' A' A
\]
\[
+ P'' B' A - C'' A' P - A'' A' P + A'' B' P - P'' A' C ,
\]

\[
F_3 = -\alpha P'' P' P + B'' P' A - C'' P' A + B'' P' C + P'' A' A - P'' B' A + C'' A' P - P'' B' C ,
\]

\[
F_4 = C'' P' C - B'' P' A + C'' P' A + C'' P' B + B'' C' P - C'' C' P + P'' C' B
\]
\[
- P'' C' C - P'' A' A + P'' B' A - C'' A' P - C'' B' P ,
\]

\[
F_5 = -F_3 ,
\]

\[
F_6 = F_3 ,
\]

\[
F_7 = \alpha P'' B' P - 2\alpha A'' P' P - \alpha P'' P' B - \alpha B'' P' C - A'' A' B - A'' B' C ,
\]

\[
F_8 = -\alpha P'' B' P - 2\alpha P'' C' B + \alpha P'' B' P + \alpha B'' P' B + B'' C' C + C'' C' B - C'' B' C ,
\]

\[
F_9 = \hat{F}_7 ,
\]

\[
F_{10} = -\alpha P'' B' P + \alpha P'' P' B + \alpha B'' P' P - A'' A' A - A'' B' A - 4A'' C' A
\]
\[
- B'' C' A - C'' C' A - A'' C' B - A'' C' C + C'' A' C .
\]

Therefore, the Y–B equation can be satisfied if we have

\[ F_1 = F_2 = F_3 = F_4 = F_7 = F_8 = F_{10} = 0 . \quad (4.7) \]

In contrast to the previous case of \( N = 8 \), we could not, however, succeed in obtaining the most general solution of Eq. (4.7). Consider first the simplest equation \( F_1 = 0 \) which can be rewritten as

\[ 2\alpha + \frac{C''}{P''} \left( \frac{B'}{P'} - \frac{B}{P} \right) + \frac{C}{P} \left( \frac{B'}{P'} - \frac{B''}{P''} \right) + \frac{A'}{P'} \left( \frac{B}{P} + \frac{B''}{P''} \right) = 0 . \quad (4.8) \]

Since \( \theta' = \theta + \theta'' \), this gives a functional equation whose general solutions are difficult to be found in contrast to Eq. (3.7). However, we can find some solutions of Eq. (4.8) as follows. In analogy to Eq. (3.8), we seek a solution of form

\[ \frac{B(\theta)}{P(\theta)} = a + b\theta \quad (4.9) \]
for some constants $a$ and $b$. Then, Eq. (4.8) can be solved to yield

$$\frac{C(\theta)}{P(\theta)} = a' + \frac{d}{b\theta}, \quad (4.10a)$$

$$\frac{A(\theta)}{P(\theta)} = -a' - \frac{2(\alpha + d - aa')}{b\theta + 2a} \quad (4.10b)$$

for some other constants $a'$ and $d$. Inserting these into the rest of equations $F_2 = F_3 = F_4 = F_7 = F_8 = F_{10} = 0$, we can verify that they are satisfied also, provided that we have

$$a' = a, \quad d = a^2 - \alpha.$$

Therefore, the desired solution is given by

$$\frac{A(\theta)}{P(\theta)} = -a, \quad \frac{B(\theta)}{P(\theta)} = a + b\theta, \quad \frac{C(\theta)}{P(\theta)} = a + \frac{a^2 - \alpha}{b\theta} \quad (4.11)$$

where $a$ and $b$ are arbitrary constants. It is interesting to observe that Eq. (4.11) will be singular–free at $\theta = 0$, if we choose the arbitrary constant $a$ to satisfy $a^2 = \alpha$. Its possible relevance for the knot theory will be discussed elsewhere.

Also, if we wish, we can find a solution for $P(\theta) = 0$ by first letting $P(\theta) = \frac{1}{a} F(\theta)$ and $b = ak$ and then taking the limit $a \to \infty$ for fixed values of $\alpha, k$, and $F(\theta)$. In this way, we obtain a solution of form

$$P(\theta) = 0, \quad A(\theta) = -F(\theta), \quad B(\theta) = (1 + k\theta)F(\theta), \quad C(\theta) = (1 + \frac{1}{k\theta})F(\theta) \quad (4.12)$$

for an arbitrary function $F(\theta)$ and for arbitrary constant $k$. This solution corresponds essentially to that of the so(4) model of Zamolodchikov’s12).
5. Concluding Remarks

We have seen in this note that the triple products can be useful for solving the Yang–Baxter equation. We may, of course, note contrarily that we rewrite $R_{cd}^{ab}(\theta)$ first as

$$R_{cd}^{ab}(\theta) = P(\theta)C_{cd}^{ba} + A(\theta)g_{cd}g^{ab} + B(\theta)\delta_c^a\delta_d^b + C(\theta)\delta_c^a\delta_d^b$$  \hspace{1cm} (5.1)

from Eqs. (1.5), (1.8), (1.12), (1.18) and (2.8), and then determine $P(\theta)$, $A(\theta)$, $B(\theta)$, and $C(\theta)$ directly by inserting Eq. (5.1) into Eq. (1.1a), as has been done in ref. 8. However, the use of the triple product makes the calculation far easier. Moreover, the direct method becomes hardly manageable for more complicated triple systems to be discussed in the subsequent papers.

The triple system discussed in this note is actually a special case of more general class of system which we call orthogonal triple (or ternary) system. Setting

$$\lambda = -3\beta$$  \hspace{1cm} (5.2)

and introducing new triple product $xyz$ by

$$xyz = [x, y, z] + \lambda < y|z > x - \lambda < z|x > y$$  \hspace{1cm} (5.3)

we can easily see that Eqs. (2.1)–(2.5) will lead to the validity of

(i) $< x|y >= < y|x >$  \hspace{1cm} (5.4)
(ii) $yxz + xyz = 0$  \hspace{1cm} (5.5)
(iii) $xyz + xzy = 2\lambda < y|z > x - \lambda < x|y > z - \lambda < z|x > y$  \hspace{1cm} (5.6)
(iv) $uv(xyz) = (uvx)yz + x(uvy)z + xy(uvz)$  \hspace{1cm} (5.7)
(v) $< uvx|y >= - < x|uvy >$  \hspace{1cm} (5.8)

The triple system satisfying Eqs. (5.4)–(5.8) is a supersymmetric analogue of the symplectic triple system$^{13}$ and may be called the orthogonal triple system. Solutions of the Y–B equation in terms of orthogonal or symplectic triple product systems will be discussed in the proceeding paper$^9$. Then, the utilization of mathematical structures of ternary algebras for these cases becomes essential to find solutions.
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Appendix: Properties of Octonionic Triple System

We first prove existence of non-trivial solutions of triple systems satisfying Eqs(2.1)–(2.4) by following the method given elsewhere\textsuperscript{14}). Let $L$ be a simple or semi-simple Lie algebra over the complex number field $F$, and let $V$ be a irreducible module of $L$. The tensor product $V \otimes V$ can always be decomposed into a direct sum of the symmetric and antisymmetric components, $(V \otimes V)_S$ and $(V \otimes V)_A$, respectively as

$$V \otimes V = (V \otimes V)_S \oplus (V \otimes V)_A \ . \quad (A.1)$$

Analogously, the tensor product $V \otimes V \otimes V$ can be decomposed into a direct sum of various components of various Young tableau\textsuperscript{15}). As usual, we specify totally antisymmetric, symmetric and mixed tableaus by the symbols $[1^3]$, $[3]$, and $[2,1]$ with $V = [1]$, respectively and so on. Suppose that the irreducible module $V$ obeys conditions

$$\text{Dim } \text{Hom}((V \otimes V)_S \to F) = 1 \ , \quad (A.2)$$

$$\text{Dim } \text{Hom}([1^3] \to V) = 1 \ , \quad (A.3)$$

$$\text{Dim } \text{Hom}([1^4] \to F) = 1 \ , \quad (A.4)$$

and

$$\text{Dim } \text{Hom}\{([1^3] \otimes [1^3])_S \to F\} \leq 2 \ . \quad (A.5)$$

Here, $\text{Dim } W$ implies the dimension of a vector space $W$, and $\text{Hom}(W_1 \to W_2)$ designates the vector space of all homorphisms between $L$-modules $W_1$ and $W_2$, which commute with actions of $L$. 
As we will explain shortly, the validity of Eqs. (A.2)–(A.5) implies that of Eqs. (2.1)–(2.4). Consider first Eq. (A.2) which implies existence of symmetric bi-linear form $< x | y >$. Moreover, its invariance under the action of $L$ assures its non-degeneracy, since $V$ is assumed to be irreducible. Next, totally antisymmetric triple product $[x, y, z]$ exists in $V$ in view of Eq. (A.3). Further, Eq. (2.3) follows from Eq. (A.4) as we explained in ref. 14. Finally, the validity of Eq. (A.5) leads to that of Eq. (2.4) for some constants $\alpha$ and $\beta$ by the same reasoning given there. Therefore, if we can find a simple Lie algebra $L$ and its irreducible module $V$ satisfying conditions Eqs. (A.2)–(A.5), then we have constructed the triple product $[x, y, z]$ in $V$ obeying Eqs. (2.1)–(2.4). Let $L$ be the Lie algebra $B_3$ which is the same as the Lie algebra so(7) of the SO(7) group. Choose $V$ to be its eight-dimensional spinor representation $V = \{ \Lambda_1 \}$ so that

$$N = \text{Dim } V = 8 \quad . \quad (A.6)$$

It is easy to verify the validity of Eqs. (A.2)–(A.5) for this case, when we note for example

$$[1^3] = V \oplus \{ \Lambda_1 + \Lambda_3 \} \quad ,$$

or

$$56 = 8 + 48$$

where $\Lambda_1$, $\Lambda_2$, and $\Lambda_3$ are fundamental weights of $B_3$, and $\{ \Lambda_1 + \Lambda_3 \}$ designates the 48-dimensional irreducible module of $B_3$ with the highest weight $\Lambda_1 + \Lambda_3$. Also, we calculate

$$[1^4] = \{ 0 \} \oplus \{ \Lambda_1 \} \oplus \{ 2\Lambda_1 \} \oplus \{ 2\Lambda_3 \} \quad \text{or} \quad 70 = 1 + 7 + 27 + 35.$$  

Similarly, the same argument applies for the case of $L = \text{so}(4)$ with $V$ being its 4-dimensional irreducible vector representation, although $\text{so}(4)$ is not simple but semi-simple.

These two cases discussed above exhaust all possibilities for the system satisfying Eqs. (2.1)–(2.5) in some sense to be specified below. This fact is also intimately connected with their relation with octonion and quaternion algebras, as we will show shortly. First, if
\( \alpha = 0 \), then we must have \( \beta = 0 \) by the result of section 2 so that we assume \( \alpha \neq 0 \). Then, normalizing the triple product and/or inner product suitably, we may assume hereafter

\[
\alpha = 1 \quad .
\] (A.8)

Second, since the inner product \( \langle x|y \rangle \) is non-degenerate, there exists an element \( e \in V \) satisfying \( \langle e|e \rangle \neq 0 \). Normalizing it suitably, we can set

\[
\langle e|e \rangle = 1 \quad .
\] (A.9)

For any arbitrary but fixed element \( e \in V \) satisfying Eq. (A.9), we can now introduce a bi-linear product

\[
x \cdot y : V \otimes V \to V \quad (A.10)
\]

by

\[
x \cdot y = [x, y, e] + \langle x|e \rangle y + \langle y|e \rangle x - \langle x|y \rangle e \quad .
\] (A.11)

Here, we used the symbol \( x \cdot y \) rather than the customary \( xy \) in order to avoid possible confusions with the triple product \( xyz \) defined by Eq. (5.3). Moreover, we define the conjugate of \( x \) as usual by

\[
\overline{x} = 2 \langle x|e \rangle e - x \quad .
\] (A.12)

It is easy to verify from Eqs. (2.1)–(2.5) with \( \alpha = 1 \) that this bi-linear product satisfies the composition law

\[
\langle x \cdot y|x \cdot y \rangle = \langle x|y \rangle \quad .
\] (A.13)

as well as

\[
x \cdot e = e \cdot x = x \quad ,
\]

\[
x \cdot \overline{x} = \overline{x} \cdot x = \langle x|x \rangle e \quad ,
\]

\[
x \cdot (\overline{x} \cdot y) = (x \cdot \overline{x}) \cdot y = \langle x|x \rangle y \quad ,
\]

\[
(y \cdot x) \cdot \overline{x} = y \cdot (x \cdot \overline{x}) = \langle x|x \rangle y \quad ,
\]

\[
\langle x \cdot y|z \rangle = \langle \overline{x} \cdot z|y \rangle \quad .
\]
In other words, it defines a quadratic alternative composition algebra\(^{16}\), so that possible dimensions of \(V\) are limited to be \(N = 1, 2, 4\) or \(8\) by the Hurwitz’s theorem\(^{16}\). Moreover, since two cases of \(N = 1\), and \(2\) lead to the trivial case \([x, y, z] = 0\) identically, this reproduces the result of section 2. Two cases of \(N = 4\) and \(8\) correspond to the quaternion and octonion algebras. First consider the quaternion case of \(N = 4\). Let \(e(=e_4), e_1, e_2,\) and \(e_3\) be the standard quaternionic basis with normalization

\[
\langle e_\mu|e_\nu \rangle = \delta_{\mu\nu}
\]

for \(\mu, \nu = 1, 2, 3, 4\). Then, we can readily show the validity of

\[
\langle e_\alpha| [e_\mu, e_\nu, e_\lambda] \rangle = \epsilon_{\alpha\mu\nu\lambda}
\]

where \(\epsilon_{\alpha\mu\nu\lambda}\) with \(\epsilon_{1234} = 1\) is the totally antisymmetric Levi-Civita symbol in the 4-dimensional space. This, then, uniquely determines the triple product for \(N = 4\) by Eq. (2.10).

For \(N = 8\), Eq. (2.6a) with Eq. (A.8) determines \(\beta\) to be \(\pm 1\). Changing the sign of the triple product, if necessary, we can then assume \(\beta = -1\). Setting \(z = v = e\) in Eq. (2.5), and then changing \(u\) into \(z\), it gives

\[
(x \cdot y) \cdot z = [x, y, z] + 2 < y|e > x \cdot z - < y|z > x - < x|y > z + < x|z > y . \tag{A.15}
\]

Antisymmetrizing Eq. (A.15) together with Eq. (A.14), we can rewrite Eq. (A.15) into a more symmetrical form of

\[
[x, y, z] = \frac{1}{2} \{(x, y, z) + \langle x|e > [y, z] + \langle y|e > [z, x]
+ \langle z|e > [x, y] - < z|[x, y] > e\} \tag{A.16}
\]

where \((x, y, z)\) and \([x, y]\) are associator and commutator, respectively, defined by

\[
(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z) \tag{A.17}
\]

\[
[x, y] = x \cdot y - y \cdot x .
\]
We may mention the fact that in derivation of Eq. (A.16), we used the identity\(^{17}\)
\[
[[x, y], z] + [[y, z], x] + [[z, x], y]
\]
\[
= (x, y, z) + (y, z, x) + (z, x, y) - (z, y, x) - (y, x, z) - (x, z, y)
\]
which is equal to \(6(x, y, z)\) for alternative algebra. Since the triple product \([x, y, z]\) can be determined uniquely in terms of the octonion algebras by Eq. (A.15) or (A.16), we may say that the eight dimensional triple system is also unique. These are the reasons also why we called our triple products defined by Eqs. (2.1)–(2.4) to be quaternionic and octonionic ternary systems, respectively, for \(N = 4\) and 8. In spite of relationship between the octonion algebra and the octonionic triple system, the triple system enjoys a larger symmetry so(7) by construction in comparison to \(G_2\) of the octonion. The reason is, of course, due to the introduction of the fixed privileged element \(e\), which breaks\(^{18}\) the symmetry from the so(7) to \(G_2\). Its relation to so(8) will be discussed in the proceeding paper\(^9\).

In this connection, we remark that Eq. (A.16) can be generalizeable for any orthogonal triple system defined in section 5 by introducing the antisymmetric product \([x, y]\) by
\[
\frac{1}{2} [x, y] = [x, y, e] = xye + \lambda < x|e > y - \lambda < y|e > x
\]
for any privileged element \(e \in V\) satisfying \(< e|e > = 1\). Then, the relations Eqs. (5.4)–(5.8) can be used to prove the validity of the identity
\[
4\lambda[x, y, z] = [[x, y], z] + [[y, z], x] + [[z, x], y]
\]
\[
+ 2\lambda\{< e|x > [y, z] + < e|y > [z, x]
\]
\[
+ < e|z > [x, y] - < z|[x, y] > e\}
\]
by setting \(v = z = e\) in Eq. (5.7) and then rewriting \(u\) by \(z\). If \(\lambda \neq 0\), it implies that the triple product \([x, y, z]\) can be expressed in terms of the bi-linear product \([x, y]\). Indeed, for the octonion algebra, we have \(\lambda = -3\beta = 3\) and Eq. (A.20) will immediately reproduce Eq. (A.16). However for the case \(\lambda = 0\) as in the quaternionic triple system, Eq. (A.20) gives only
\[
[[x, y], z] + [[y, z], x] + [[z, x], y] = 0
\]
so that the bi-linear product \([x, y]\) must be automatically a Lie algebra.

Next, we would like to mention a fact that we can construct another type of a triple product from the octonion algebra, following the works of Allison\(^{19}\) and Kamiya\(^{20}\). Defining the left and right multiplication operators \(L_x\) and \(R_y\), respectively, by

\[
L_x y = x \cdot y , \quad R_x y = y \cdot x ,
\]

we know\(^{16}\) that

\[
D_{x,y} = [R_x, R_y] + [L_x, L_y] + [L_x, R_y]
\]

is its derivation, i.e., it satisfies

\[
D_{x,y}(u \cdot v) = (D_{x,y}u) \cdot v + u \cdot (D_{x,y}v) .
\]

We, now, define a new triple product \(x * y * z\) by

\[
x * y * z = -\frac{1}{4} D_{x,y} z
\]

\[
= \frac{1}{4} \{ y \cdot (x \cdot z) - (z \cdot y) \cdot x + (z \cdot x + x \cdot z) \cdot y - x \cdot (y \cdot z + z \cdot y) \} .
\]

Since \(D_{x,y}\) is a derivation of any alternative algebra, we have also

\[
D_{u,v}(x * y * z) = (D_{u,v} x) * y * z + x * (D_{u,v} y) * z + x * y * (D_{u,v} z)
\]

because of Eqs. (A.24) and (A.25). In other words, we have

\[
u * v * (x * y * z)
\]

\[
= (u * v * x) * y * z + x * (u * v * y) * z + x * y * (u * v * z)
\]

Using Eqs. (A.14) and (A.18), we can rewrite Eq. (A.25) to be

\[
x * y * z = \frac{3}{4} (x, y, z) - \frac{1}{4} [[x, y], z]
\]

\[
= \frac{1}{4} (x, y, z) + \{ < y|z > - < y|e >< z|e >\} x
\]

\[
- \{ < x|z > - < x|e >< z|e >\} y
\]

\[
- \{ < x|e >< y|z > - < y|e >< x|z >\} e
\]
from which we find

(i) \( x \ast y \ast z + y \ast x \ast z = 0 \)

(ii) \( x \ast y \ast z + x \ast z \ast y \)

\[
= \{2 < y|z> - 2 < y|e>< z|e>\}x \\
- \{< x|z> - < x|e>< z|e>\}y \\
- \{< x|y> - < x|e>< y|e>\}z \\
+ \{< y|e>< x|z> + < z|e>< x|y> - 2 < y|z>< x|e>\}e .
\]

\[(A.28)\]

The relationship between \( x \ast y \ast z \) and \( xyz \) given in section 5 is not straightforward. First, for \( N = 4 \), we find for instance the Lie–triple condition

\[
[x, y, z] = \frac{1}{3} \{x \ast y \ast z + y \ast z \ast x + z \ast x \ast y\} = 0
\]

while the quaternionic triple product \([x, y, z]\) does not satisfy the corresponding relation.

For \( N = 8 \), on the other side, we obtain

\[
2x \ast y \ast z - xyz \\
= \{< x|z> + 2 < x|e>< z|e>\}y - \{< y|z> + 2 < y|e>< z|e>\}x \\
- 2\{< x|e>< y|z> - < y|e>< x|z>\}e - \frac{1}{2} \{< x|e>[y, z] \\
+ < y|e>[z, x] + < z|e>[x, y] - < z|[x, y]> e\} .
\]

\[(A.29)\]

From Eqs. (A.26) and (A.28), we see that the new triple product \( x \ast y \ast z \) satisfies almost the same relations as Eqs. (5.4)–(5.7). We can make it to be the orthogonal ternary system, if we restrict ourselves to the sub-space

\[
V_0 = \{x| < x|e> = 0 , \ x \in V\}
\]

\[(A.30)\]

so that \( N_0 = \text{Dim } V_0 = 3 \) or 7 according to \( N = 4 \) or 8. Since we can readily verify

\[
< e|x \ast y \ast z > = 0
\]

\[(A.31)\]

whenever we have \( x, y, z \in V_0 \), the product \( x \ast y \ast z \) defines a triple product in \( V_0 \). Moreover, Eqs. (A.26) and (A.28) show now that it satisfies all axioms of orthogonal ternary systems.
with \( \lambda \) now being 1 for \( V_0 \). However for \( N_0 = 3 \), it is trivial in a sense that we have

\[
x \ast y \ast z = < y | z > x - < x | z > y .
\]

For \( N_0 = 7 \), the space \( V_0 \) corresponds to the 7-dimensional simple exceptional Malcev algebra\(^{17}\), so that we call the present system as the Malcev triple system. Although we have introduced \( x \ast y \ast z \) in terms of quadratic alternative algebra, we could have started directly with the known derivations\(^{21}\) of any Malcev algebra. However we will find the same result for \( N_0 = 7 \) as is expected.

We can characterize the orthogonal triple system just as we have done for both quaternionic and octonionic triple system by Eqs. (A.2)–(A.5). Suppose that we now have

\[
\text{Dim Hom } ((V \otimes V)_S \to F) = 1 ,
\]

\[
\text{Dim Hom } ([1^3] \to V) = 1 ,
\]

\[
\text{Dim Hom } ([1^4] \to F) = 1 ,
\]

just as Eqs. (A.2)–(A.4), but we now assume

\[
\text{Dim Hom } ([1^4] \otimes [1^2] \to F) \leq 1 \quad (A.33)
\]

instead of Eq. (A.5). Then, following the method given in ref. 14, the totally antisymmetric ternary product \([x, y, z]\) can be shown to satisfy

\[
[u, v, [x, y, z]] - [[u, v, x], y, z] - [x, [u, v, y], z] - [x, y, [u, v, z]] = \lambda\{< u|[x, y, z] > v - < v|[x, y, z] > u + < x|v > [u, y, z] + < y|v > [x, u, z] + < z|v > [x, y, u] - < x|u > [v, y, z] - < y|u > [x, v, z] - < z|u > [x, y, v]\}
\]

for a constant \( \lambda \). Setting

\[
xyz = [x, y, z] + \lambda\{< y|z > x - < z|x > y\} , \quad (A.35)
\]

it can be shown to satisfy the axioms Eqs. (5.3)–(5.8) of the orthogonal triple system. If we choose \( V = \{A_2\} \) to be the 7-dimensional irreducible module of the exceptional Lie
algebra $G_2$, then we can verify the validity of Eqs. (A.32) and (A.33) so that it reproduces the same triple product $x \ast y \ast z$, where we have used for example
\[
[1^2] = \{A_2\} \oplus \{A_1\} \quad \text{or} \quad 21 = 7 + 14, \tag{A.36}
\]
\[
[1^3] = [1^4] = \{0\} \oplus \{A_2\} \oplus \{2A_1\} \quad \text{or} \quad 35 = 1 + 7 + 27.
\]
Moreover, we can verify
\[
\text{Dim Hom } ([2, 1] \rightarrow V) = 1 \tag{A.37}
\]
since we find
\[
[2, 1] = \{A_2\} \oplus \{A_1\} \oplus \{2A_2\} \oplus \{A_1 + A_2\}
\]
or
\[
112 = 7 + 14 + 27 + 64. \tag{A.38}
\]
Conditions Eqs. (A.32), (A.33) and (A.37) also hold valid for $V$ being 8-dimensional spinor representation $\{\Lambda_3\}$ of $so(7) = B_3$, as we expect. The condition Eq. (A.37) will turn out to be crucial for the calculation of the proceeding paper for both cases.

In ending this note, we simply remark that there exists\(^2\) another entirely different class of triple systems which are nevertheless intimately related also to the octonion algebra. However, we will not go into detail.
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