Lipschitz Adaptivity with Multiple Learning Rates in Online Learning

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Abstract
We aim to design adaptive online learning algorithms that take advantage of any special structure that might be present in the learning task at hand, with as little manual tuning by the user as possible. A fundamental obstacle that comes up in the design of such adaptive algorithms is to calibrate a so-called step-size or learning rate hyperparameter depending on variance, gradient norms, etc. A recent technique promises to overcome this difficulty by maintaining multiple learning rates in parallel. This technique has been applied in the MetaGrad algorithm for online convex optimization and the Squint algorithm for prediction with expert advice. However, in both cases the user still has to provide in advance a Lipschitz hyperparameter that bounds the norm of the gradients. Although this hyperparameter is typically not available in advance, tuning it correctly is crucial: if it is set too small, the methods may fail completely; but if it is taken too large, performance deteriorates significantly. In the present work we remove this Lipschitz hyperparameter by designing new versions of MetaGrad and Squint that adapt to its optimal value automatically. We achieve this by dynamically updating the set of active learning rates. For MetaGrad, we further improve the computational efficiency of handling constraints on the domain of prediction, and we remove the need to specify the number of rounds in advance.

1. Introduction
We consider online convex optimization (OCO) of a sequence of convex functions \( f_1, \ldots, f_T \) over a given bounded convex domain, which become available one by one over the course of \( T \) rounds (Shalev-Shwartz, 2011; Hazan, 2016). Typically \( f_t(w) = \text{LOSS}(w, x_t, y_t) \) represents the loss of predicting with parameters \( w \) on the \( t \)-th data point \((x_t, y_t)\) in a machine learning task. At the start of each round \( t \), a learner has to predict the best parameters \( w_t \) for the function \( f_t \) before finding out what \( f_t \) is, and the goal is to minimize the regret, which is the difference in the sum of function values between the learner’s predictions \( w_1, \ldots, w_T \) and the best fixed oracle parameters \( u \) that could have been chosen if all the functions had been given in advance. A special case of OCO is prediction with expert advice (Cesa-Bianchi and Lugosi, 2006), where the functions \( f_t(w) = w^\top \ell_t \) are convex combinations of the losses \( \ell_t = (\ell_{t,1}, \ldots, \ell_{t,K})^\top \) of \( K \) expert predictors and the domain is the probability simplex.

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Central results in these settings show that it is possible to control the regret with almost no prior knowledge at all about the functions. For instance, knowing only an upper bound $G$ on the $\ell_2$-norms of the gradients $g_t = \nabla f_t(w_t)$, the online gradient descent (OGD) algorithm guarantees $O(G\sqrt{T})$ regret by tuning its learning rate hyperparameter $\eta_t$ proportional to $1/(G\sqrt{T})$ (Zinkevich, 2003), and in the case of prediction with expert advice the Hedge algorithm achieves regret $O(L\sqrt{T\ln K})$ knowing only an upper bound $L$ on the range $\max_k \ell_{t,k} - \min_k \ell_{t,k}$ of the expert losses (Freund and Schapire, 1997). Here $G$ is the $\ell_2$-Lipschitz constant of the learning task\footnote{We slightly abuse terminology here, because the standard definition of a Lipschitz constant requires an upper bound on the gradient norms for any parameters $w$, not just for $w = w_t$, and may therefore be large.}, and $L/2$ is the $\ell_1$-Lipschitz constant over the probability simplex.

The above guarantees are tight if we make no further assumptions about the functions $f_t$ (Hazan, 2016; Cesa-Bianchi et al., 1997), but they can be significantly improved if the functions have additional special structure that makes the learning task easier. The literature on online learning explores multiple orthogonal dimensions in which tasks may be significantly easier in practice (see ‘related work’ below). Here we focus on the following regret guarantees that are known to exploit multiple types of easiness at the same time:

\[
\text{OCO:} \quad O\left(\sqrt{V_T^u d \log T}\right) \quad \text{for all } u, \quad \text{with } V_T^u = \sum_{t=1}^T ((w_t - u)\top g_t)^2, \quad (1)
\]

\[
\text{Experts:} \quad O\left(\sqrt{E_{\rho(k)}[V_T^k]} \mathrm{KL}(\rho||\pi)\right) \quad \text{for all } \rho, \quad \text{with } V_T^k = \sum_{t=1}^T ((w_t - e_k)\top \ell_t)^2, \quad (2)
\]

where $d$ is the number of parameters and $\mathrm{KL}(\rho||\pi) = \sum_{k=1}^K \rho(k) \ln \rho(k)/\pi(k)$ is the Kullback-Leibler divergence of a data-dependent distribution $\rho$ over experts from a fixed prior distribution $\pi$.

The OCO guarantee is achieved by the MetaGrad algorithm (Van Erven and Koolen, 2016), and implies regret that grows at most logarithmic in $T$ both in case the losses are curved (exp-concave, strongly convex) and in the stochastic case whenever the losses are independent, identically distributed samples with variance controlled by the Bernstein condition (Van Erven and Koolen, 2016; Koolen et al., 2016). The guarantee for the expert case is achieved by the Squint algorithm (Koolen and Van Erven, 2015; Koolen, 2015). It also exploits special structure along two dimensions simultaneously, because the $V_T^k$ term is much smaller than $L^2 T$ in many cases (Gaillard et al., 2014; Koolen et al., 2016) and the so-called quantile bound $\mathrm{KL}(\rho||\pi)$ is much smaller than the worst case $\ln K$ when multiple experts make good predictions (Chaudhuri et al., 2009; Chernov and Vovk, 2010). Squint and MetaGrad are both based on the same technique of tracking the empirical performance of multiple learning rates in parallel over a quadratic approximation of the original loss. A computational difference though is that Squint is able to do this by a continuous integral that can be evaluated in closed form, whereas MetaGrad uses a discrete grid of learning rates.

Unfortunately, to achieve (1) and (2), both MetaGrad and Squint need knowledge of the Lipschitz constant ($G$ or $L$, respectively). Overestimating $G$ or $L$ by a factor of $c > 1$ has the effect of reducing the effective amount of available data by the same factor $c$, but underestimating the Lipschitz constant is even worse because it can make the methods fail completely. In fact, the ability to adapt to $G$ has been credited (Ward et al., 2018) as one of the main reasons for the practical success of the AdaGrad algorithm (Duchi et al., 2011; McMahan and Streeter, 2010). Thus getting
the Lipschitz constant right makes the difference between having practical algorithms and having promising theoretical results.

For OCO, an important first step towards combining Lipschitz adaptivity to $G$ with regret bounds of the form (1) was taken by Cutkosky and Boahen (2017b), who aimed for (1) but had to settle for a weaker result with $G \sum_{t=1}^{T} \| g_t \|_2 \| w_t - u \|_2^2$ instead of $V_{T}^{m}$. Although not sufficient to adapt to the Bernstein condition, they do provide a series of stochastic examples where their bound already leads to fast $O(\ln^4 T)$ rates. For the expert setting, Wintenberger (2017) has made significant progress towards a version of (2) without the quantile bound improvement, but he is left with having to specify an initial guess $L_{\text{guess}}$ for $L$ that enters as $O(\ln \ln (L/L_{\text{guess}}))$ in his bound, which may yet be arbitrarily large when the initial guess is on the wrong scale.

**Main Contributions** Our main contributions are that we complete the process began by Cutkosky and Boahen (2017b) and Wintenberger (2017) by showing that it is indeed possible to achieve (1) and (2) without prior knowledge of $G$ or $L$. In fact, for the expert setting we are able to adapt to the tighter quantity $B \geq \max_k |(w_t - e_k)^\top t|$. We achieve these results by dynamically updating the set of active learning rates in MetaGrad and Squint depending on the observed Lipschitz constants. In both cases we encounter a similar tuning issue as Wintenberger (2017), but we avoid the need to specify any initial guess using a new restarting scheme, which restarts the algorithm when the observed Lipschitz constant increases too much. In addition to these main results, we remove the need to specify the number of rounds $T$ in advance for MetaGrad by adding learning rates as $T$ gets larger, and we improve the computational efficiency of how it handles constraints on the domain of prediction: by a minor extension of the black-box reduction for projections of Cutkosky and Orabona (2018), we incur only the computational cost of projecting on the domain of interest in Euclidean distance. This should be contrasted with the usual projections in time-varying Mahalanobis distance for second-order methods like MetaGrad.

**Related Work** If adapting to the Lipschitz constant were our only goal, a well-known way to achieve it for OCO would be to change the learning rate in OGD to $\eta_t \propto 1/\sqrt{\sum_{s \leq t} \| g_s \|_2^2}$, which leads to $O(\sqrt{\sum_{t \leq T} \| g_t \|_2}) = O(G\sqrt{T})$ regret. This is the approach taken by AdaGrad (for each dimension separately) (Duchi et al., 2011; McMahan and Streeter, 2010). In prediction with expert advice, Lipschitz adaptive methods are sometimes called scale-free and have previously been obtained by Cesa-Bianchi et al. (2007); De Rooij et al. (2014) with generalizations to OCO by Orabona and Pál (2015). In addition, the first two of these works obtain a data-dependent variance term that is different from $V_{T}^{k}$ in (2), but no quantile bounds are known for the former. Results for the latter have previously been obtained by Gaillard et al. (2014); Wintenberger (2014) without quantile bounds, and with a slightly weaker notion of variance by Luo and Schapire (2015). Quantile bounds without variance adaptivity were introduced by Chaudhuri et al. (2009); Chernov and Vovk (2010). These may be interpreted as measures of the complexity of the comparator $\rho$. The corresponding notion in OCO is to adapt to the norm of $u$, which has been achieved in various different ways, see for instance (McMahan and Abernethy, 2013; Cutkosky and Orabona, 2018). For curved functions, existing results achieve fast rates assuming that the degree of curvature is known (Hazan et al., 2007), measured online (Bartlett et al., 2007; Do et al., 2009) or entirely unknown (Van Erven and Koolen, 2016; Cutkosky and Orabona, 2018). Fast rates are also possible for slowly-varying linear functions and, more generally, optimistically predictable gradient sequences (Hazan and Kale, 2010; Chiang et al., 2012; Rakhlin and Sridharan, 2013).
We view our results as a step towards developing algorithms that automatically adapt to multiple relevant measures of difficulty at the same time. It is not a given that such combinations are always possible. For example, Cutkosky and Boahen (2017a) show that Lipschitz adaptivity and adapting to the comparator complexity in OCO, although both achievable independently, cannot both be realized at the same time (at least not without further assumptions). A general framework to study which notions of task difficulty do combine into achievable bounds is provided by Foster et al. (2015). Foster et al. (2017) characterize the achievability of general data-dependent regret bounds for domains that are balls in general Banach spaces.

Outline We add Lipschitz adaptivity to Squint for the expert setting in Section 3. Then, in Section 4, we do the same for MetaGrad in the OCO setting. The developments are analogous at a high level but differ in the details for computational reasons. We highlight the differences along the way. Section 4 further describes how to avoid specifying \( T \) in advance for MetaGrad. Then, in Section 5, we add efficient projections for MetaGrad, and finally Section 6 concludes with a discussion of directions for future work.

2. Problem Setting and Notation

In OCO, a learner repeatedly chooses actions \( w_t \) from a closed convex set \( \mathcal{U} \subseteq \mathbb{R}^d \) during rounds \( t = 1, \ldots, T \), and suffers losses \( f_t(w_t) \), where \( f_t : \mathcal{U} \rightarrow \mathbb{R} \) is a convex function. The learner’s goal is to achieve small regret \( R_T^w = \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(u) \) with respect to any comparator action \( u \in \mathcal{U} \), which measures the difference between the cumulative loss of the learner and the cumulative loss they could have achieved by playing the oracle action \( u \) from the start. A special case of OCO is prediction with expert advice, where \( f_t(w) = w^T \ell_t \) for \( \ell_t \in \mathbb{R}^K \) and the domain \( U \) is the probability simplex \( \triangle_K = \{(w_1, \ldots, w_K) : w_i \geq 0, \sum_i w_i = 1\} \). In this context we will further write \( p \) instead of \( w \) for the parameters to emphasize that they represent a probability distribution. We further define \([K] = \{1, \ldots, K\}\).

3. Adaptive Second-order Quantile Method for Experts

In this section, we present an extension of the SQUINT algorithm that adapts automatically to the loss range in the setting of prediction with expert advice.

Throughout this section, we denote \( r_t^k := (\hat{p}_t - e_k, \ell_t) \) and \( v_t^k := (r_t^k)^2 \), where \( \hat{p}_t \in \triangle_K \) is the weight vector played by the algorithm at round \( t \) and \( \ell_t \) is the observed loss vector. The cumulative regret with respect to expert \( k \) is given by \( R_t^k := \sum_{s=1}^{t} r_s^k \). We use \( V_t^k := \sum_{s=1}^{t} v_s^k \) to denote the cumulative squared excess loss (which can be regarded as a measure of variance) of expert \( k \) at round \( t \). In the next subsection, we review the SQUINT algorithm.

3.1. The SQUINT Algorithm

We first describe the original SQUINT algorithm, as introduced by Koolen and Van Erven (2015). Let \( \pi \) and \( \gamma \) be prior distributions with supports on \([K]\) and \([0, \frac{1}{2}]\), respectively. Then SQUINT outputs predictions

\[
\begin{align*}
 p_{t+1} & \propto \mathbb{E}_{\pi(k)\gamma(n)} \left[ \eta e^{-\sum_{s=1}^{t} f_s(k, n)} e_k \right],
\end{align*}
\]  

(3)
where \( f_t(k, \eta) \) are quadratic surrogate losses defined by
\[
f_t(k, \eta) := -\eta \langle \hat{p}_t - e_k, \ell_t \rangle + \eta^2 \langle \hat{p}_t - e_k, \ell_t \rangle^2.
\] (4)

Koolen and Van Erven (2015) propose to use the improper prior \( \gamma(\eta) = \frac{1}{\eta} \) which does not integrate to a finite value over its domain, but because of the weighting by \( \eta \) in (3) the predictions \( \hat{p}_{t+1} \) are still well-defined. The benefit of the improper prior is that it allows calculating \( \hat{p}_{t+1} \) in closed form (Koolen and Van Erven, 2015). For any distribution \( \rho \in \triangle_K \), SQUINT achieves the following bound:
\[
R_T^\rho = O \left( \sqrt{V_T^\rho (\text{KL}(\rho|\pi) + \ln \ln T)} \right),
\]
(5)
where \( R_T^\rho = \mathbb{E}_{\rho(k)} \left[ R_T^k \right] \) and \( V_T^\rho = \mathbb{E}_{\rho(k)} \left[ V_T^k \right] \). This version of Squint assumes the loss range \( \max_k \ell_{t,k} - \min_k \ell_{t,k} \) is at most 1, and can fail otherwise. In the next subsection, we present an extension of SQUINT which does not need to know the Lipschitz constant.

### 3.2. Lipschitz Adaptable Squint

We first design a version of SQUINT, called SQUINT+C, that still requires an initial estimate \( B > 0 \) of the Lipschitz constant. The next section will be devoted to setting this parameter online. For now, we consider it fixed. In addition to this, the algorithm takes a prior distribution \( \pi \in \triangle_K \). In a sequence of rounds \( t = 1, 2, \ldots \) the algorithm predicts with \( \hat{p}_t \in \triangle_K \), and receives a loss vector \( \ell_t \in \mathbb{R}^K \). We denote the instantaneous regret of expert \( k \) in round \( t \) by \( r_t^k := \langle \hat{p}_t - e_k, \ell_t \rangle \). We denote the observed Lipschitz constant in round \( t \) at point \( \hat{p}_t \) by \( b_t := \max_k |r_t^k| \), and we denote its running maximum by \( B_t := B \vee \max_{s \leq t} b_s \), and we use the convention that \( B_0 = B \). In addition, we will also require a clipped version of the loss vector \( \ell_t = \frac{B_t}{b_t} \ell_t \), and we denote by \( \bar{r}_t^k := (\hat{p}_t - e_k, \bar{\ell}_t) \) the rescaled instantaneous regret. We will use that \( |\bar{r}_t^k| \leq B_{t-1} \). It suffices to control the regret for the clipped loss, because the cumulative difference is a negligible lower-order constant\(^2\):
\[
R_T^k - \bar{R}_T^k := \sum_{t=1}^T (r_t^k - \bar{r}_t^k) = \sum_{t=1}^T (B_t - B_{t-1}) \frac{r_t^k}{B_t} \leq B_T - B_0.
\]
(6)
This means we can focus on regret for \( \bar{\ell}_t \), for which the range bound \( |\bar{r}_t^k| \leq B_{t-1} \) is available ahead of each round \( t \). To motivate SQUINT+C, we define the potential function after \( T \) rounds by
\[
\Phi_T := \sum_k \pi_k \int_0^{2T \rho T - 1} \frac{e^{\eta \bar{r}_T^k - \eta^2 \bar{V}_T^k} - 1}{\eta} \, d\eta \quad \text{where} \quad \bar{R}_T^k := \sum_{t=1}^T \bar{r}_t^k \quad \text{and} \quad \bar{V}_T^k := \sum_{t=1}^T (\bar{r}_t^k)^2.
\]
(7)
We also define \( \Phi_0 = 0 \) (due to the integrand being zero), even though it involves the meaningless \( B_{-1} \) in the upper limit. The algorithm is now derived from the desire to keep this potential under control. As we will see in the analysis, this requirement uniquely forces the choice of weights
\[
\hat{p}_{T+1}^k \propto \pi_k \int_0^{2T \rho T} e^{\eta \bar{r}_T^k - \eta^2 \bar{V}_T^k} \, d\eta.
\]
(8)
\(^2\) We learned this technique from Ashok Cutkosky
Algorithm 1 Restarts to make SQUINT+C or METAGRAD+C scale-free

Require: ALG is either SQUINT+C or METAGRAD+C, taking as input parameter an initial scale \( B \)
1: Play \( w_1 \) until the first time \( t = \tau_1 \) that \( b_t \neq 0 \).
2: Run ALG with input \( B = B_{\tau_1} \) until the first time \( t = \tau_2 \) that \( \frac{B_t}{B_{\tau_1}} > \sum_{s=1}^{t} \frac{b_s}{B_s} \).
3: Set \( \tau_1 = \tau_2 \) and goto line 2.

Like the original SQUINT, we see that the weights \( p_{t+1} \) can be evaluated in closed form using Gaussian CDFs. The regret analysis consists of two parts. First, we show that the algorithm keeps the potential small.

Lemma 1 Given parameter \( B \geq 0 \), SQUINT+C ensures \( \Phi_T \leq \ln \frac{B_T - 1}{B} \).

The next step of the argument is to show that small potential is useful. The argument here follows Koolen and Van Erven (2015), specifically the version by Koolen (2015). We have

Lemma 2 Definition (7) implies that for any comparator distribution \( \rho \in \triangle_K \) the regret is at most

\[
R_T^\rho \leq \sqrt{2V_T^\rho} \left( 1 + \sqrt{2C_T^\rho} \right) + 5B_{T-1} \left( C_T^\rho + \ln 2 \right),
\]

where,

\[
C_T^\rho := KL(\rho\|\pi) + \ln \left( \Phi_T + \frac{1}{2} + \ln \left( 2 + \sum_{t=1}^{T-1} \frac{b_t}{B_t} \right) \right)
\]

Keeping only the dominant terms, this reads

\[
R_T^\rho = O \left( \sqrt{V_T^\rho} \left( KL(\rho\|\pi) + \ln \Phi_T + \ln T \right) \right).
\]

The significance of (6), Lemmas 1 and 2 is that we obtain a bound of the form

\[
R_T^\rho = O \left( \sqrt{\frac{V_T^\rho}{B_T} \left( KL(\rho\|\pi) + \ln \frac{T B_{T-1}}{B} \right)} + 5B_T \left( KL(\rho\|\pi) + \ln \frac{T B_{T-1}}{B} \right) \right).
\]

However, there does not seem to be any safe a-priori way to tune \( B = B_0 \). If we set it too small, the factor \( \ln \ln \frac{B_T - 1}{B} \) explodes. If we set it too large, the lower-order contribution \( B_{T-1} \geq B \) blows up. It does not appear possible to bypass this tuning dilemma within the current construction. Fortunately, we are able to resolve it using restarts. Algorithm 1, which applies to both SQUINT+C and METAGRAD+C (presented in the next section), monitors a condition of the sequences \( (b_t) \) and \( (B_t) \) to trigger restarts.

Theorem 3 Let SQUINT+L be the result of applying Algorithm 1 to SQUINT+C (as ALG). SQUINT+L guarantees, for any comparator \( \rho \in \triangle_K \),

\[
R_T^\rho \leq 2 \sqrt{\frac{V_T^\rho}{B_T} \left( 1 + \sqrt{2\Gamma_T^\rho} \right)} + 10B_T \left( \Gamma_T^\rho + \ln 2 \right) + 4B_T,
\]

where \( \Gamma_T^\rho := KL(\rho\|\pi) + \ln \left( \sum_{t=1}^{T-1} \frac{b_t}{B_t} \right) + \frac{1}{2} + \ln \left( 2 + \sum_{t=1}^{T-1} \frac{b_t}{B_t} \right) \).

Theorem 3 shows that the bound on the regret of SQUINT+L has a term of order \( O(\ln \ln \sum_{t=1}^{T-1} \frac{b_t}{B_t}) = O(\ln \ln T) \), which does not depend on the initial guess \( B_0 \) anymore.
4. Adaptive Method for Online Convex Optimization

We consider the Online Convex Optimization (OCO) setting where at each round \( t \), the learner predicts by playing \( \hat{u}_t \) in a closed convex set \( \mathcal{U} \subset \mathbb{R}^d \), then the environment announces a convex function \( \ell_t : \mathcal{U} \rightarrow [0, +\infty] \) and the learner suffers loss \( \ell_t(\hat{u}_t) \). The goal of the learner is to minimize the regret with respect to the single best action \( u \in \mathcal{U} \) in hindsight (after \( T \) rounds); that is, minimizing \( R^u_T := \sum_{t=1}^T \ell_t(\hat{u}_t) - \sum_{t=1}^T \ell_t(u) \) for the worst case \( u \in \mathcal{U} \). Since the losses are convex, it suffices to bound the sum of linearized losses \( \tilde{R}^u_T := \sum_{t=1}^T \langle \hat{u}_t - u, g_t \rangle \), where \( g_t := \nabla \ell_t(\hat{u}_t) \). We will assume that the set \( \mathcal{U} \) is bounded and let \( D \in [0, +\infty] \) be its diameter

\[
D := \sup_{u,v \in \mathcal{U}} \|u - v\|_2.
\]

Our main contribution in this section is to devise a simple modification of META\( \text{GRAD} \) — META\( \text{GRAD}+\text{C} \) — which, without prior knowledge of the maximum value of the gradient range \( G := \max_{t \leq T} \|\nabla \ell_t(\hat{u}_t)\| \), guarantees the following regret bound

\[
\forall u \in \mathcal{U}, \quad R^u_T \leq \tilde{R}^u_T = O \left( \sqrt{V^u_T d \ln T} + B T d \ln T \right),
\]

where \( V^u_T := \sum_{t=1}^T \langle \hat{u}_t - u, g_t \rangle^2 \). Consequently, this algorithm inherits the fast convergence results of standard \( \text{META\text{GRAD}} \) (Van Erven and Koolen, 2016). In particular, it was shown that due to the form of the bound in (13), META\( \text{GRAD} \) achieves a logarithmic regret when the sequence of losses are exp-concave (Van Erven and Koolen, 2016). Furthermore, when the sequence of gradient functions \( \nabla \ell_t \) are i.i.d distributed with common distribution \( P \) and satisfy the \((B, \beta)\)-Bernstein condition for \( B > 0 \) and \( \beta \in [0, 1] \) with respect to the risk minimizer \( u^* = \arg\min_{u \in \mathcal{U}} \mathbb{E}_{f \sim P}[f(u)] \), then \( \text{META\text{GRAD}} \) (and thus \( \text{META\text{GRAD}+C} \)) achieves the expected regret

\[
\mathbb{E} \left[ R^u_T \right] = O \left( (d \ln T)^{1+\frac{1}{B}} T^{\frac{1}{2-\beta}} + d \ln T \right).
\]

See (Koolen et al., 2016) for more detail.

4.1. The META\( \text{GRAD} \) Algorithm

The META\( \text{GRAD} \) algorithm runs several sub-algorithms at each round; namely, a set of slave algorithms, which learn the best action in \( \mathcal{U} \) given a learning rate \( \eta \) in some predefined grid \( G \), and the master algorithm, which learns the best learning rate. The goal of META\( \text{GRAD} \) is to maximize the sum of payoff functions \( \sum_{t=1}^T f_t(u, \eta) \) over all \( \eta \in G \) and \( u \in \mathcal{U} \) simultaneously, where

\[
f_t(u, \eta) := -\eta \langle \hat{u}_t - u, g_t \rangle + \eta^2 \langle \hat{u}_t - u, g_t \rangle^2, \quad t \in [T],
\]

and \( \hat{u}_t \) is the master prediction at round \( t \geq 1 \). Each slave algorithm takes as input a learning rate from a finite, exponentially-spaced grid \( G \) (with \( \lceil \log_2 \sqrt{T} \rceil \) points) within the interval \( \left[ \frac{1}{5DG}, \frac{1}{5DG} \right] \), where \( G \) is an upper bound on the norms of the gradients. In this case, the bound \( G \) must be known in advance. In what follows, we let \( M_t := \sum_{s=1}^t g_s g_s^\top \), for \( t \geq 0 \).

Slave predictions. Every slave \( \eta \in G \) starts with \( \hat{u}_1^\eta = 0 \). At the end of round \( t \geq 1 \), it receives the master prediction \( \hat{u}_t \) and updates the prediction in two steps

\[
\begin{align*}
u_{t+1}^\eta &:= \hat{u}_t^\eta - \eta \Sigma_{t+1}^\eta g_t \left( 1 + 2 \eta \langle \hat{u}_t^\eta - \hat{u}_t \rangle^\top g_t \right), \quad \text{where} \quad \Sigma_{t+1}^\eta := \left( \frac{1}{\eta^2} + 2 \eta^2 M_t \right)^{-1}, \\
\text{and} \quad \hat{u}_{t+1}^\eta &:= \arg\min_{u \in \mathcal{U}} \langle u_{t+1}^\eta - u \rangle^\top \left( \Sigma_{t+1}^\eta \right)^{-1} \left( u_{t+1}^\eta - u \right),
\end{align*}
\]

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**Master predictions** After receiving the slaves predictions, \((\hat{\mathbf{u}}^\eta_t)_{\eta \in G}\), the master algorithm aggregates them and outputs \(\hat{\mathbf{u}}_t \in \mathcal{U}\) according to:

\[
\hat{\mathbf{u}}_t := \frac{\sum_{\eta \in G} \eta w^\eta_t \hat{\mathbf{u}}^\eta_t}{\sum_{\eta \in G} \eta w^\eta_t}; \quad w^\eta_t := e^{- \sum_{s=1}^{t-1} f_s(\hat{\mathbf{u}}^\eta_s, \eta)}, \quad (17)
\]

As mentioned earlier, the META\textsc{Grad} algorithm requires the knowledge of the maximum value of the gradient range \(G\) and the horizon \(T\) in advance. These are needed to define the grid of the slave algorithms. In the analysis of META\textsc{Grad}, it is crucial for the \(u\)'s to be in the right interval in order to invoke a Gaussian exp-concavity result for the surrogate losses in (14) (see e.g. (Van Erven and Koolen, 2016, Lemma 10)). In the next subsection, we explore a natural extension of META\textsc{Grad} which does not require the knowledge of the gradient range or the horizon \(T\).

### 4.2. An Extension of META\textsc{Grad} for Unknown Gradient Range and Horizon

We present a natural extension of META\textsc{Grad}, called META\textsc{Grad}+C, which does not assume any knowledge on the gradient range or the horizon. Contrary to the original META\textsc{Grad} which requires knowledge of the horizon \(T\) to define the grid for the slaves, META\textsc{Grad}+C circumvents this by defining an infinite grid \(G\), in which, at any given round \(t \geq 1\), only a finite number of slaves (up to \(\log_2 t\) many) output a prediction (see Remark 4). Each slave \(\eta\) in this grid receives a prior weight \(\pi(\eta) \in [0,1]\), where \(\sum_{\eta \in G} \pi(\eta) = 1\). The expressions of \(G\) and \(\pi\) are given by

\[
G := \left\{ \eta_i := \frac{2^{-i}}{5B} : i \in \mathbb{N} \right\}; \quad \pi(\eta_i) := \frac{1}{(i+1)(i+2)}, \quad i \in \mathbb{N}, \quad (18)
\]

where \(B > 0\) is the input to META\textsc{Grad}+C.

#### 4.2.1. Algorithm Description

**Preliminaries.** As in the previous subsection, we let \(\hat{\mathbf{u}}_t\) and \(\hat{\mathbf{u}}^\eta_t\) be the predictions of the master and slave \(\eta\), respectively, at round \(t \geq 1\) (we give their explicit expressions further below). Let \((b_t)\) and \((B_t)\) be the sequences in \(\mathbb{R}_{\geq 0}\) defined by

\[
b_t := D \|g_t\|_2; \quad B_t := B \vee \max_{s \in [t]} b_s, \quad t \in [T], \quad (19)
\]

where \(B\) is the input of META\textsc{Grad}+C, and we use the convention that \(B_0 = B\). Using the sequence \((B_t)\), we define the clipped gradients \(\bar{g}_t := \frac{B_{t-1}}{B_t} g_t\), and \(\forall \mathbf{u} \in \mathcal{U}, \forall t \geq 1, \forall \eta > 0\), we let

\[
\bar{r}_t^u := (\bar{\mathbf{u}}_t - \mathbf{u}, \bar{g}_t), \quad \bar{f}_t(\mathbf{u}, \eta) := -\eta \bar{r}_t^u + (\eta \bar{r}_t^u)^2, \quad \bar{M}_t := \sum_{s=1}^{t} \bar{g}_s \bar{g}_s^\top. \quad (20)
\]

For each slave \(\eta \in G\), we define the time \(s_\eta\) to be

\[
s_\eta := \min \left\{ t \geq 0 \left| \eta \geq \frac{1}{D \sum_{s=1}^{t} \|g_s\|_2 + B_t} \right. \right\}, \quad (21)
\]

and define the set \(\mathcal{A}_t\) of “active” slaves by

\[
\mathcal{A}_t := \{ \eta \in G_t : s_\eta < t \}, \quad \text{where} \quad G_t := G \cap \left[ 0, \frac{1}{s_{Bt-1}} \right], \quad t \geq 1. \quad (22)
\]
Slaves’ predictions. A slave \( \eta \in A_t \) issues its first prediction \( \hat{u}^\eta_t = 0 \) in round \( t = s_\eta + 1 \). From then on, it receives the master prediction \( \hat{u}_t \) as input and updates in two steps as

\[
\hat{u}^\eta_{t+1} := \hat{u}^\eta_t - \eta \Sigma^\eta_{t+1} g_t \left( 1 + 2\eta \left( \hat{u}^\eta_t - \hat{u}_t \right)^T \hat{g}_t \right), \quad \text{where} \quad \Sigma^\eta_{t+1} := \left( \frac{1}{\eta^2} + 2\eta^2 (M_t - M_\eta) \right)^{-1},
\]

and

\[
\hat{u}^\eta_{t+1} = \arg\min_{u \in \mathcal{U}} (u_{t+1} - u)^T (\Sigma^\eta_{t+1})^{-1} (u_{t+1} - u).
\]

Slaves that are outside the set \( A_t \) at round \( t \) are irrelevant to the algorithm. Note that restricting the slaves to the set \( \mathcal{G}_t \) is similar to clipping the upper integral range in the SQUINT+C case.

Master predictions. At each round \( t \geq 1 \), the master algorithm receives the slaves predictions \( \hat{u}^\eta_t \) and outputs the \( \hat{u}_t \):

\[
\hat{u}_t = \frac{\sum_{\eta \in A_t} \eta w_t^\eta \hat{u}^\eta_t}{\sum_{\eta \in A_t} \eta w_t^\eta}; \quad w_t^\eta := \pi(\eta) e^{-\sum_{s=s_\eta+1}^{t-1} \tilde{f}_s(\hat{u}^\eta_s, \eta)}, \quad t \geq 1.
\]

**Remark 4 (Number of active slaves)** At any round \( t \geq 1 \), the number of active slaves is at most \( \lfloor \log_2 t \rfloor \). In fact, if \( \eta \in A_t \), then by definition \( \eta \geq 1/(D \sum_{s=1}^{s_\eta} \|g_s\|_2 + B_\eta) \geq 1/(tB_{t-1}) \) (since \( s_\eta \leq t - 1 \)), and thus \( A_t \subset [1/(tB_{t-1}), 1/(5B_{t-1})] \). Since \( A_t \) is an exponentially-spaced grid with base 2, there are at most \( \lfloor \log_2 t \rfloor \) slaves in \( A_t \).

4.2.2. Analysis

To analyse the performance of \textsc{MetaGrad+C}, we consider the potential function

\[
\Phi_t := \pi(\mathcal{G}_t \setminus A_t) + \sum_{\eta \in A_t} \pi(\eta) e^{-\sum_{s=s_\eta+1}^{t-1} \tilde{f}_s(\hat{u}^\eta_s, \eta)}, \quad t \geq 0.
\]

For \( u \in \mathcal{U} \), we define the pseudo-regret \( \bar{R}^u_T := \sum_{t=1}^{T} (\hat{u}_t - u, g_t) \) and its clipped version \( \bar{R}^u_T := \sum_{t=1}^{T} (\hat{u}_t - u, \hat{g}_t) \). The following analogue to (6) relates these two regrets.

**Lemma 5** Let \( (b_t) \) and \( (B_t) \) be as in (19), respectively, then for all \( u \in \mathcal{U} \),

\[
\bar{R}^u_T \leq \bar{R}^u_T + B_T.
\]

Similarly to the SQUINT case, one can use the prod-bound to control the growth of this potential function as shown in the proof of the following lemma (see Appendix B):

**Lemma 6** \textsc{MetaGrad+C} guarantees that \( \Phi_T \leq \cdots \leq \Phi_0 = 1 \), for all \( T \in \mathbb{N} \).

We now give a bound on the clipped regret \( \bar{R}^u_T \) in terms of the clipped variance \( \bar{V}^u_T := \sum_{t=1}^{T} (\bar{R}^u_t)^2 \):

**Theorem 7** Given input \( B > 0 \), the clipped pseudo-regret for \textsc{MetaGrad+C} is bounded by

\[
\bar{R}^u_T \leq 3 \sqrt{\bar{V}^u_T C_T} + 15B TC_T \quad \text{for any} \ u \in \mathcal{U},
\]

where \( C_T := d \ln \left( 1 + \frac{2\sum_{t=1}^{T-1} b_t^2 + 2b_T^2}{25dB_T^{-1}} \right) + 2 \ln \left( \log_2 \sqrt{\sum_{t=1}^{T} b_t^2} + 3 \right) + 2 \text{ and } \log_2^+ = 0 \vee \log_2 \).

3. The predictions of the slaves outside \( A_t \) do not appear anywhere in the description or analysis of the algorithm. Alternatively, we may think of each slave \( \eta \) as operating with \( \eta_t = 0 \) in the first \( s_\eta \) rounds and with \( \eta_t = \eta \) afterwards. The presence of the factor \( \eta \) in (17) renders the master oblivious to inactive slaves.
Remark 8 We can relate the clipped pseudo-regret to the ordinary regret via \( R_T^u \leq \hat{R}_T^u \leq \bar{R}_T^u + B_T \) (see (26)) and on the right-hand side we can also use that \( V_T^u \leq V_T^u \).

An important thing to note from the result of Theorem 7 is that the ratio \( \sqrt{\sum_{t=1}^T b_t^2}/B \), could in principle be arbitrarily large if the input \( B \) is too small compared to the actual regret range. To resolve this issue, one can use the same restart trick as in the SQINT case:

**Theorem 9** Let MetaGrad+L be the result of applying Algorithm 1 to MetaGrad+C. Then the regret for MetaGrad+L is bounded by

\[
\hat{R}_T^u \leq 3\sqrt{V_T^u \Gamma_T} + 15B_T \Gamma_T + 4B_T \quad \text{for all } u \in U,
\]

where \( \Gamma_T := 2d \ln \left( \frac{2\eta^+}{\epsilon} + 2 \frac{2}{d} \sum_{t=1}^T \frac{b_t^2}{B_t} \right) + 4 \ln \left( \log^+ \sqrt{\sum_{t=1}^T (\sum_{i=1}^T b_i^2/B_i^2)^2 + 3} \right) + 4 = O(d \ln T) \).

In Theorem 9, we have replaced the ratio \( \sqrt{\sum_{t=1}^T b_t^2}/B \) appearing in the (clipped) pseudo-regret bound of MetaGrad+C by the term \( \sigma_T := \sqrt{\sum_{t=1}^T (\sum_{i=1}^T b_i^2/B_i)^2} \) which is always smaller than \( T^{3/2} \), but this is acceptable since \( \sigma_T \) appears inside a \( \ln \ln \). From the bound of Theorem 9 on can easily recover an ordinary regret bound, i.e. a bound on \( R_T^u, u \in U \) (see Remark 8).

5. Efficient Implementation Through a Reduction to the Sphere

Using MetaGrad+C (or MetaGrad), the computation of each vector \( \hat{u}_t^u \) requires a (Mahalanobis) projection step onto an arbitrary convex set \( U \). Numerically, this typically requires \( O(d^p) \) floating point operations (flops), for some \( p \in \mathbb{N} \) which depends on the topology of the set \( U \). Since \( p \) can be large in many applications, evaluating \( \hat{u}_t^u \) at each grid point \( u \) can become computationally prohibitive, especially when the number of grid points grows with \( T \) — in the case of MetaGrad+C there can be at most \( \lfloor \log_2 T \rfloor \) slaves at round \( T \geq 1 \) (see Remark 4).

5.1. An efficient implementation of MetaGrad+C on the ball

In this subsection, we assume that \( U \) is the ball of diameter \( D \), i.e. \( U = B_D := \{ u \in \mathbb{R}^d : \| u \|_2 \leq D/2 \} \).

In order to compute the slave prediction \( \hat{u}_t^u \), for \( t \geq 1 \) and \( u \in U \), the following quadratic program needs to be solved:

\[
\hat{u}_{t+1}^u = \arg\min_{u \in U} \left( u_{t+1}^u - u \right)^T \left( \sum_{t=1}^T \Sigma_{t+1}^u \right)^{-1} \left( u_{t+1}^u - u \right),
\]

where \( u_{t+1}^u \) (the unprojected prediction) and \( \Sigma_{t+1}^u \) (the co-variance matrix) are defined in (23). Since \( U \) is a ball, (23) can be solved efficiently using the result of following lemma:

**Lemma 10** Let \( t \geq 1, \eta \in \mathbb{R}^+, u \in U \), and \( v_{t+1}^u := \left( \frac{1}{\eta^2} + 2\eta^2 (\tilde{M}_t - \tilde{M}_s) \right) u_{t+1}^u \). Let \( Q_t \) be an orthogonal matrix which diagonalizes \( M_t \), and \( \Lambda_t := [\lambda_{t,i}^2]_{i=1}^d \) the diagonal matrix which satisfies \( Q_t M_t Q_t^T = \Lambda_t \). The solution of (29) is given by \( \hat{u}_{t+1}^u = u_{t+1}^u \), if \( u_{t+1}^u \in U \); and otherwise, \( \hat{u}_{t+1}^u = Q_t x_{t+1}^u = Q_t (x_{t+1}^u I + 2\eta^2 (\Lambda_t - \Lambda_s))^{-1} \), where \( x_{t+1}^u \) is the unique solution of

\[
\rho_t^u(x) := \sum_{i=1}^d \frac{\langle e_i, Q_t v_{t+1}^u \rangle^2}{(x + 2\eta^2 (\lambda_{t,i}^2 - \lambda_s^2))^2} = \frac{D^2}{4},
\]
The proof of the lemma is in Appendix C. Note that since the matrix $M_t$ is symmetric for all $t \geq 1$, the existence of the matrices $Q_t$ and $\Lambda_t$ in Lemma 10 is always guaranteed. Since $\rho_t^0$ in (30) is strictly convex and decreasing, one can use the Newton method to efficiently solve $\rho_t^0(x) = D^2/4$. Thus, since the computation of $Q_t v_{t+1}^0$ only involves matrix-vector products, Lemma 10 gives an efficient way of solving (29).

**Implementation on the ball.** At round $t \geq 1$, the implementation of $\text{META\textsc{Grad}+C}$ on the ball $B_D$ keeps in memory the orthogonal matrix $Q_{t-1}$ which diagonalizes $M_{t-1}$. In this case, since $M_t = M_{t-1} + g_t g_t^\top$ it is possible to compute the new matrices $Q_t$ and $\Lambda_t$ in $O(d^2)$ flops (Stor et al., 2015). Note that this operation only needs to be performed once — the diagonalization does not depend on $\eta$. Therefore, computing $Q_t v_{t+1}^0$ (and thus $\hat{u}_{t+1}^0$) can be performed in only $O(d^2)$ flops. Thus, aside from the matrix-vector products, the time complexity involved in computing $\hat{u}_{t+1}^0$ for a given $\eta \in A_t$ is of the same order as that involved in solving $\rho_t^0(x) = D^2/4$.

5.2. A Reduction to the ball

In this subsection, we make use of a recent technique by Cutkosky and Orabona (2018) that reduces constrained optimization problems to unconstrained ones, to reduce any OCO problem on an arbitrary bounded convex set $\mathcal{U} \subset \mathbb{R}^d$ to an OCO problem on a ball, where we can apply $\text{META\textsc{Grad}+C}$ efficiently. Let $D$ be the diameter of $\mathcal{U} \subset \mathbb{R}^d$ as in (12), so that the ball $B_D$ of radius $D/2$ encloses $\mathcal{U}$. For $u \in \mathcal{U}$, we denote $d_{\mathcal{U}}(u) = \min_{w \in \mathcal{U}}\|u - w\|_2$ the distance function from the set $\mathcal{U}$, and we define $\Pi_{\mathcal{U}}(u) := \{w \in \mathcal{U} : \|w - u\|_2 = d_{\mathcal{U}}(u)\}$.

Algorithm 2 reduces the OCO problem on the set $\mathcal{U}$ to one on the ball $B_D$, where $\text{META\textsc{Grad}+C}$ is used as a black-box to solve it efficiently. As a result, Algorithm 2 (including its $\text{META\textsc{Grad}+C}$ subroutine) only performs a single Euclidean projection (as opposed to the projection in Mahalanobis distance as in (29)) onto the set $\mathcal{U}$, which is applied to the output of $\text{META\textsc{Grad}+C}$ — the $\text{META\textsc{Grad}+C}$ subroutine only performs projections onto the ball $B_D$, which can be done efficiently as described in the previous subsection.

Let $R_T^u := \sum_{t=1}^T \langle \hat{w}_t - u, \hat{g}_t \rangle$ and $V_T^u := \sum_{t=1}^T \langle \hat{w}_t - u, \hat{g}_t \rangle^2$ be the pseudo-regret and variance of Algorithm 2. The following theorem, whose proof is in Appendix C, shows how the regret guarantee of $\text{META\textsc{Grad}+C}$ readily transfers to Algorithm 2:

| Algorithm 2 | Fast implementation of $\text{META\textsc{Grad}+C}$ on any bounded convex set $\mathcal{U}$ via reduction to the ball. |
|-------------|--------------------------------------------------------------------------------------------------|
| **Require:** | A bounded convex set $\mathcal{U} \subset \mathbb{R}^d$ with diameter $D$, and a fast $\text{META\textsc{Grad}+C}$ implementation on the ball $B_D$, $\text{META\textsc{Grad}+C}$, taking input $B$. |
| **for** $t = 1$ to $T$ **do** | Get $\hat{u}_t$ from $\text{META\textsc{Grad}+C}$  
Play $\hat{w}_t = \Pi_{\mathcal{U}}(\hat{u}_t)$, receive $\hat{g}_t = \nabla \ell_t(\hat{w}_t)$  
Set $g_t \in \frac{1}{2} (\hat{g}_t + \|\hat{g}_t\| \partial d_{\mathcal{U}}(\hat{u}_t))$  
Send $g_t$ to $\text{META\textsc{Grad}+C}$ |
| **end for** | |

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**Theorem 11** Algorithm 2, which uses MetaGrad+C as a black-box, guarantees:

\[
\sum_{t=1}^{T} (\ell_t(\hat{w}_t) - \ell_t(u)) \leq \hat{R}_T^u \leq 3\sqrt{V^u_T \Gamma_T} + 24B_T \Gamma_T + B_T, \quad \text{for } u \in \mathcal{U}, \quad (31)
\]

where \( \Gamma_T := d \ln \left( \frac{27}{25} + \frac{2}{25} B_T \right) + 2 \ln \left( \log_2 \sqrt{\sum_{t=1}^{T-1} b_t^2} + 3 \right) + 2 = O(d \ln T), \) and

\[
b_t := D \|g_t\|_2; \quad B_t := B \vee \max_{s \in [t]} b_s, \quad t \in [T]. \quad (32)
\]

Note that Algorithm 2, guarantees the same type of regret as MetaGrad+C, and thus can also adapt to exp-concavity of the losses \((\ell_t)\) and the Bernstein condition.

**6. Conclusion**

We present algorithms that adapt to the Lipschitz constant of the loss for OCO and experts. Stepping back, we see that an interesting combination of problem complexity dimensions can be adapted to, with hardly any overhead in either regret or computation. The main question for future work is to obtain a better understanding of the landscape of interactions between measures of problem complexity and their algorithmic reflection.

One surprising conclusion from our work, which provides a curious contrast with incompatibility of Lipschitz adaptivity with comparator complexity adaptivity in general OCO (Cutkosky and Boahen, 2017a), is the following observation. Our results for the expert setting, which we phrased for a finite set of \( K \) experts, in fact generalise unmodified to priors with infinite support. Considering a countable set of experts, we find a scenario where the comparator complexity \( \text{KL}(\rho\|\pi) \) is unbounded, yet our Squint strategy adapts to the Lipschitz constant of the loss without inflating the regret compared to an a-priori known complexity by more than a constant.

A final very interesting question is when it is possible to exploit scenarios with large ranges that occur only very infrequently. An example of this is found in statistical learning with heavy-tailed loss distributions. Martingale methods for such scenarios that are related to our potential functions suggest that it may be necessary to replace the “surrogate” negative quadratic term \( f_t(u, \eta) \) that our algorithms include in the exponent by another function appropriate for the specific distribution (Howard et al., 2018, Table 3). It is not currently clear what individual sequence analogues can be obtained.

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Appendix A. Proofs of Section 3

Proof of Lemma 1 We proceed by induction on \(T\). By definition \(\Phi_0 = 0\). For \(T \geq 0\), the definition (7) gives

\[
\Phi_{T+1} = \sum_k \pi_k \int_0^{\frac{1}{B_T}} \frac{e^{\eta \bar{R}_T^k - \eta^2 \bar{V}_T^k}}{\eta} \left( e^{\eta \bar{R}_T^k - \eta^2 (\bar{R}_T^k)^2} - 1 \right) \, d\eta + \sum_k \pi_k \int_0^{\frac{1}{B_T}} \frac{e^{\eta \bar{R}_T^k - \eta^2 \bar{V}_T^k}}{\eta} - 1 \, d\eta.
\]

To control the first term \(Q_1\), we apply the “prod bound” \(e^{x-x^2} \leq 1+x\) for \(x \geq -1/2\) to \(x = \eta \bar{r}_T^k\), which we may do as \(\eta \bar{r}_T^{k+1} \geq -\frac{1}{2B_T}B_T\). Linearity and the definition of the weights (8) further yield

\[
Q_1 \leq \sum_k \pi_k \int_0^{\frac{1}{B_T}} \frac{e^{\eta \bar{R}_T^k - \eta^2 \bar{V}_T^k}}{\eta} \left( \eta \bar{r}_T^k \right) \, d\eta = \left( \sum_k \pi_k \int_0^{\frac{1}{B_T}} e^{\eta \bar{R}_T^k - \eta^2 \bar{V}_T^k} \left( \bar{p}_{T+1} - e_k \right) \, d\eta, \bar{e}_{T+1} \right) = 0.
\]

To control the second term \(Q_2\), we extend the range of the integral to find

\[
Q_2 \leq \sum_k \pi_k \int_0^{\frac{1}{B_T}} \frac{e^{\eta \bar{R}_T^k - \eta^2 \bar{V}_T^k}}{\eta} - 1 \, d\eta + \ln \frac{B_T}{B_{T-1}} = \Phi_T + \ln \frac{B_T}{B_{T-1}}.
\]

Proof of Lemma 2 For any \(\epsilon \in [0, \frac{1}{2B_{T-1}}]\), we may split the potential (7) as follows

\[
\Phi_T = \sum_k \pi_k \int_0^{\epsilon} \frac{e^{\eta \bar{R}_T^k - \eta^2 \bar{V}_T^k}}{\eta} - 1 \, d\eta + \sum_k \pi_k \int_{\epsilon}^{\frac{1}{B_{T-1}}} \frac{e^{\eta \bar{R}_T^k - \eta^2 \bar{V}_T^k}}{\eta} - 1 \, d\eta.
\]

For convenience, let us introduce \(\bar{b}_t := \max_k |\bar{r}_T^k| = \frac{B_{t-1}}{B_t} b_t\) and abbreviate \(\bar{S}_T := \sum_{t=1}^T \bar{b}_t\). To bound the left term \(Q_1\) from below, we use \(e^x - 1 \geq x\). Then combined with \(\bar{R}_T^k \geq -\bar{S}_T\) and \(\bar{V}_T^k \leq \sum_{t=1}^{T-1} \bar{b}_t^2 \leq B_{T-1} \bar{S}_T\) we find

\[
Q_1 \geq \sum_k \pi_k \int_0^{\epsilon} \bar{R}_T^k - \eta \bar{V}_T^k \, d\eta \geq - \left( \epsilon + \frac{\epsilon^2}{2B_{T-1}} \right) \bar{S}_T.
\]
For the right term \( Q_2 \), we use KL duality to find
\[
Q_2 = \sum_k \pi_k \int_{\epsilon}^{\epsilon + \epsilon \cdot 2^{B_{T-1}}} \frac{e^{\eta R^\rho_k - \eta^2 V_k^\rho}}{\eta} \, d\eta + \ln (2B_{T-1}\epsilon)
\]
\[
\geq e^{-\text{KL}(\rho||\pi)} \int_{\epsilon}^{\epsilon + \epsilon \cdot 2^{B_{T-1}}} \frac{e^{\eta R^\rho_k - \eta^2 V_k^\rho}}{\eta} \, d\eta + \ln (2B_{T-1}\epsilon)
\]
Way pick the admissible \( \epsilon = \frac{1}{2(\bar{S}_T + B_{T-1})} \) for which \( \left( \epsilon + \frac{\epsilon^2}{2} B_{T-1} \right) \bar{S}_T \leq \frac{1}{2} \), and find
\[
\Phi_T \geq e^{-\text{KL}(\rho||\pi)} \int_{\epsilon}^{\epsilon + \epsilon \cdot 2^{B_{T-1}}} \frac{e^{\eta R^\rho_k - \eta^2 V_k^\rho}}{\eta} \, d\eta - \frac{1}{2} - \ln \left( 1 + \frac{\bar{S}_T}{B_{T-1}} \right)
\]
which we may reorganise to
\[
Q_3 := \ln \int_{\frac{1}{2(\bar{S}_T + B_{T-1})}}^{\epsilon + \epsilon \cdot 2^{B_{T-1}}} \frac{e^{\eta R^\rho_k - \eta^2 V_k^\rho}}{\eta} \, d\eta \leq \text{KL}(\rho||\pi) + \ln \left( \Phi_T + \frac{1}{2} + \ln \left( 1 + \frac{\bar{S}_T}{B_{T-1}} \right) \right).
\]
The argument to bound the integral in \( Q_3 \) splits in 3 cases. Let us abbreviate \( R \equiv R^\rho_k \) and \( V \equiv V^\rho_k \).

1. First the important case, where \( [\hat{\eta} - \frac{1}{\sqrt{4V}}, \hat{\eta}] \subseteq \left[ \frac{1}{2(\bar{S}_T + B_{T-1})}, \frac{1}{2B_{T-1}} \right] \). Then
\[
Q_3 \geq \ln \int_{\hat{\eta} - \frac{1}{\sqrt{4V}}}^{\hat{\eta}} \frac{e^{\eta R - \eta^2 V}}{\eta} \, d\eta \geq \ln \int_{\hat{\eta} - \frac{1}{\sqrt{4V}}}^{\hat{\eta}} \frac{e^{(\hat{\eta} - \frac{1}{\sqrt{4V}}) R - (\hat{\eta} - \frac{1}{\sqrt{4V}})^2 V}}{\eta} \, d\eta
\]
\[
= (\hat{\eta} - \frac{1}{\sqrt{2V}}) R - (\hat{\eta} - \frac{1}{\sqrt{2V}})^2 V + \ln \frac{\hat{\eta}}{\eta} - \frac{1}{\sqrt{2V}} \geq \frac{1}{2} \left( \frac{R}{\sqrt{2V}} - 1 \right)^2
\]
where the last inequality uses \( \ln \frac{1}{1 - x} \geq 1 - x \) for \( x \geq 1 \), which can be easily verified by a one-dimensional plot. We conclude
\[
R \leq \sqrt{2V} \left( 1 + \sqrt{2 \text{KL}(\rho||\pi) + 2 \ln \left( \Phi_T + \frac{1}{2} + \ln \left( 1 + \frac{\bar{S}_T}{B_{T-1}} \right) \right)} \right).
\]

2. Then in the case where \( \hat{\eta} - \frac{1}{\sqrt{2V}} < \frac{1}{S_T} \), we have
\[
R < \sqrt{2V} + \frac{2V}{S_T} \leq \sqrt{2V} + 2B_{T-1}
\]
and we are done again.
3. We come to the final case where \( \hat{\eta} > \frac{1}{2B_{T-1}} \), meaning that \( R > \frac{\sqrt{V}}{2B_{T-1}} \). Here we use that for any \( u \in \left[ \frac{1}{2(S_T + B_{T-1})}, \frac{1}{2B_{T-1}} \right] \)

\[
Q_3 \geq \ln \int_u^{\frac{1}{2B_{T-1}}} \frac{e^{uR-u^2V}}{\eta} \, d\eta \geq uR(1-uB_{T-1}) + \ln \ln \frac{1}{2uB_{T-1}}
\]

and hence

\[
R \leq \frac{Q_3 - \ln \ln \frac{1}{2uB_{T-1}}}{u(1-uB_{T-1})}.
\]

Picking the feasible \( u = \frac{5-\sqrt{5}}{10B_{T-1}} \) and using \(-\ln \ln \frac{5}{5-\sqrt{5}} \leq \ln 2 \) results in

\[
R \leq 5B_{T-1} \left( \text{KL} (\rho||\pi) + \ln \left( \Phi_T + \frac{1}{2} + \ln \left( 1 + \frac{\bar{S}_T}{B_{T-1}} \right) \right) + \ln 2 \right).
\]

Finally, using the fact that

\[
\frac{\bar{S}_T}{B_{T-1}} = \frac{1}{B_{T-1}} \sum_{t=1}^{T} \frac{B_{t-1}}{B_t} b_t \leq 1 + \sum_{t=1}^{T-1} \frac{b_t}{B_t}
\]

concludes the proof.

Proof of Theorem 3 The idea of the proof is to analyse the rounds in three parts, as shown in Figure 1. For comparator \( \rho \in \Delta_K, B > 0 \) and \( \tau_1, \tau_2 \in \mathbb{N} \) such that \( \tau_1 < \tau_2 \), we define the regret \( R^\rho_{(\tau_1, \tau_2)} \) and variance \( V^\rho_{(\tau_1, \tau_2)} \) of SQUINT+C started at round \( \tau_1 + 1 \) (with input \( B_{\tau_1} \)) and terminated after round \( \tau_2 \) by

\[
R^\rho_{(\tau_1, \tau_2)} := \sum_{t=\tau_1+1}^{\tau_2} \mathbb{E}_\rho(k) \left[ r^k_t \right], \quad V^\rho_{(\tau_1, \tau_2)} := \sum_{t=\tau_1+1}^{\tau_2} \mathbb{E}_\rho(k) \left[ (r^k_t)^2 \right].
\]
We also define

\[ \Gamma^\rho_{(\tau_1, \tau_2]} := KL (\rho \| \pi) + \ln \left( \ln \sum_{t=1}^{\tau_2-1} b_t \frac{1}{B_t} + \frac{1}{2} + \ln \left( 2 + \sum_{t=\tau_1+1}^{\tau_2-1} b_t \frac{1}{B_t} \right) \right). \] (34)

If we assume that \( \frac{B_{\tau_2-1}}{B_{\tau_1}} \leq \sum_{t=1}^{\tau_2-1} b_t \frac{1}{B_t} \) (this corresponds to the case when the restart condition in line 2 of Algorithm 1 is not triggered at round \( \tau_2 - 1 \)), then from Lemma 1 the potential function \( \Phi_{\tau_2} \), can be bounded by

\[ \Phi_{\tau_2} \leq \ln \frac{B_{\tau_2-1}}{B_{\tau_1}} \leq \sum_{t=1}^{\tau_2-1} \frac{b_t}{B_t}. \] (35)

Using this together with Lemma 2 and (6), we get the following bound on the regret for SQUINT+L:

\[ R^\rho_{(\tau_1, \tau_2]} \leq 2\sqrt{V^\rho_{(\tau_1, \tau_2)]}} \left( 1 + \sqrt{2\Gamma^\rho_{(\tau_1, \tau_2)]}} \right) + 5B_{\tau_2} \left( \Gamma^\rho_{(\tau_1, \tau_2]} + \ln 2 \right) + B_{\tau_2}. \] (36)

Now assume without loss of generality that \( b_1 \neq 0 \). Then the regret of SQUINT+L in round one is bounded by \( B_1 \leq B_T \), and SQUINT+C is started for the first time in round \( t = 2 \) with input \( B = B_1 \).

Now suppose first that the restart condition in line 2 of Algorithm 1 is never triggered, which means that \( \frac{B_t}{B_1} \leq \sum_{s=1}^{t} \frac{b_s}{B_s} \) for all rounds \( t = 2, \ldots, T \). Then for any comparator distribution \( \rho \in \Delta_K \), the result follows from Lemma 2 and the fact that \( V^\rho_{[1:T]} \leq V^\rho_T \) and the fact that \( \Gamma^\rho_{[1:T]} \leq \Gamma^\rho_T \).

Alternatively, suppose there is at least one restart. Then let \( 1 \leq \tau_1 < \tau_2 < T \) be such that \( (\tau_1, \tau_2] \) and \( (\tau_2, T] \) are the two intervals over which the last two runs of SQUINT+C occurred. We invoke Theorem 2 separately for both these intervals to bound

\[ R^\rho_{(\tau_1, T]} \leq \sqrt{2V^\rho_{(\tau_1, \tau_2]} \left( 1 + \sqrt{2\Gamma^\rho_{(\tau_1, \tau_2)} + 5B_{\tau_2} \left( \Gamma^\rho_{(\tau_1, \tau_2]} + \ln 2 \right) + B_{\tau_2} \right) + \sqrt{2V^\rho_{(\tau_2, T]} \left( 1 + \sqrt{2\Gamma^\rho_{(\tau_2, T)} + 5B_T \left( \Gamma^\rho_{(\tau_2, T]} + \ln 2 \right) + B_T \right) \right) \leq 2\sqrt{V^\rho_{(\tau_1, T]}} \left( 1 + \sqrt{2\Gamma^\rho_{(\tau_1, T)} + 10B_T \left( \Gamma^\rho_{[1:T]} + \ln 2 \right) + 2B_T \right), \] (37)

where in (37) we used the fact that \( \sqrt{x} + \sqrt{y} \leq \sqrt{2x + 2y} \).

If there is exactly one restart, then this implies the desired result. If there are multiple restarts, then the proof is completed by bounding the contribution to the regret of all rounds \( 2, \ldots, \tau_1 \) by

\[ R^\mu_{[1, \tau_1]} \leq \sum_{t=2}^{\tau_1} b_t \leq B_{\tau_1} \sum_{t=1}^{\tau_1} \frac{b_t}{B_t} \leq B_{\tau_1} \sum_{t=1}^{\tau_2} \frac{b_t}{B_t} < B_{\tau_2} \leq B_T, \]

where the second to last inequality holds because there was a restart at \( t = \tau_2 \). Finally, by bounding the instantaneous regret from the first round by \( B_T \), we obtain the desired result.
Appendix B. Proofs of Section 4

Proof of Lemma 5 Let \( u \in \mathcal{U} \), and denote \( r^u_t \) := \( \langle \bar{u}_t - u, g_t \rangle \) and \( \bar{r}^u_t \) := \( \langle \bar{u}_t - u, \bar{g}_t \rangle \). We have

\[
\bar{R}^u_T - \bar{R}^u_T := \sum_{t=1}^T (r^u_t - \bar{r}^u_t) = \sum_{t=1}^T (B_t - B_{t-1}) \frac{\bar{r}^u_t}{B_t} \leq B_T - B_0.
\]

where to get to the last inequality we used Cauchy Schwarz inequality and the fact that \( \mathcal{U} \) has diameter \( D \), which imply that \( |r^u_t| \leq B_t \). \( \blacksquare \)

Proof of Lemma 6 Let \( t \geq 1 \). To simplify notation, we denote \( \bar{r}^u_t := \langle \bar{u}_t - \hat{u}_k^u, g_s \rangle \), for \( u \in \mathcal{U} \) and \( s \in \mathbb{N} \). By appealing to the prod-bound, we have

\[
\Phi_{t+1} = \pi(\mathcal{G}_{t+1} \setminus \mathcal{A}_{t+1}) + \sum_{\eta \in \mathcal{A}_{t+1}} w^\eta_{t+1} \left( e^{\eta w^0_{t+1} - \eta (\bar{r}^\eta_{t+1})^2} - 1 \right) + \sum_{\eta \in \mathcal{A}_{t+1}} w^\eta_{t+1}
\]

Now by (24)

\[
\sum_{\eta \in \mathcal{A}_{t+1}} w^\eta_{t+1} \bar{r}^\eta_{t+1} = \sum_{\eta \in \mathcal{A}_{t+1}} \eta w^\eta_{t+1} (\bar{u}_t + \bar{u}_t) \bar{g}_t = 0.
\]

Moreover, by definition of \( G_t \) and \( A_t \),

\[
\pi(\mathcal{G}_{t+1} \setminus \mathcal{A}_{t+1}) + \sum_{\eta \in \mathcal{A}_{t+1}} w^\eta_{t+1} = \pi(\{ \eta \in G_{t+1} : s_\eta > t \}) + \sum_{\eta \in G_{t+1} : s_\eta \leq t} w^\eta_{t+1}
\]

\[
\leq \pi(\{ \eta \in G_t : s_\eta > t \}) + \sum_{\eta \in G_t : s_\eta \leq t} w^\eta_{t+1} = \pi(\{ \eta \in G_t : s_\eta \geq t \}) + \sum_{\eta \in G_t : s_\eta < t} w^\eta_{t+1}
\]

\[
= \pi(\mathcal{G}_t \setminus \mathcal{A}_t) + \sum_{\eta \in \mathcal{A}_t} w^\eta_{t+1} = \Phi_t.
\]

Where we used that \( w^\eta_{s_\eta + 1} = \pi(\eta) \). Finally, as \( A_0 = \emptyset \) and \( G_0 = G \), we find \( \Phi_0 = \pi(G) = 1 \). \( \blacksquare \)

Proof of Theorem 7 Throughout this proof we will deal with slaves \( \eta \in G_T \setminus \mathcal{A}_T \) that are provisioned but not active yet by time \( T \), and we will interpret their \( s_\eta = T \) for uniform treatment, even though technically all we know from (21) is that \( s_\eta \geq T \). First due to Lemma 6, we have \( \Phi_T \leq 1 \), where \( \Phi_T \) is the potential defined in (25). Taking logarithms and rearranging, we find

\[
\forall \eta \in G_T, \quad -\sum_{t=s_\eta + 1}^T \bar{f}_t(\bar{u}_t^\eta, \eta) \leq -\ln \pi(\eta).
\]

On the other hand, every slave \( \eta \in \mathcal{G}_T \) guarantees the following regret for the rounds \( t = s_\eta + 1, \ldots, T \) (see Van Erven and Koolen 2016, Lemma 5):

\[
\sum_{t=s_\eta + 1}^T \left( \bar{f}_t(\bar{u}_t^\eta, \eta) - \bar{f}_t(u, \eta) \right) \leq \ln \det \left(I + 2\eta^2 D^2 (\bar{M}_T - \bar{M}_{s_\eta}) \right) + \frac{\|w\|^2}{2D^2},
\]

\[
\leq d \ln \left(1 + \frac{2D^2}{25DB_{T-1}} \text{tr} \bar{M}_T \right) + \frac{\|w\|^2}{2D^2},
\]

\[
= 19.
\]
where in (40) we used concavity of \( \ln \det \), \( \mathbf{M}_{s_{\eta}} \succeq 0 \) and the fact that \( \eta \in \mathcal{G}_T \subset \left[0, \frac{1}{5B_{T-1}}\right] \). We then invert the wakeup condition (21) at time \( s_{\eta} - 1 \) to infer

\[
- \sum_{t=1}^{s_{\eta}} \hat{f}_t(u, \eta) \leq \eta \sum_{t=1}^{s_{\eta}} \hat{r}_t^u \leq \frac{\sum_{t=1}^{s_{\eta}-1} \hat{r}_t^u + \hat{r}_{s_{\eta}}^u}{D \sum_{t=1}^{s_{\eta}-1} \| g_t \|_2 + B_{s_{\eta}-1}} \leq 1. 
\]

(41)

Combining the bounds (39), (40), and (41), then dividing through by \( \eta \), gives:

\[
\forall \eta \in \mathcal{G}_T, \quad \bar{R}^u_T \leq \eta \bar{V}^u_T + \frac{1}{\eta} C_T(\eta),
\]

(42)

where \( C_T(\eta) := d \ln \left(1 + \frac{2D^2}{25dB_{T-1}^2} \text{tr} \mathbf{M}_T\right) - \ln \pi(\eta) + 2 \).

Let \( C_T \) be as in the theorem statement and let \( \eta_* \) be the estimator defined by \( \eta_* := \sqrt{C_T/\bar{V}^u_T} \). Suppose that \( \eta_* \leq \frac{1}{5B_{T-1}} \). By construction of the grid, there exists \( \hat{\eta} \in \mathcal{G}_T \) such that \( \hat{\eta} \in [\eta_* / 2, \eta_*] \).

On the other hand, the estimator \( \eta_* \) can be lower bounded by \( 1/\sqrt{\bar{V}^u_T} \) since \( C_T \geq 1 \). From this \( \eta_* \) and the fact that there exists \( i \in \mathbb{N} \) such that \( 2^{-i}/(5B_0) = \hat{\eta} \in [\eta_* / 2, \eta_*], \) we have \( 2^{-i}/(5B_0) \geq \frac{1}{2\sqrt{\bar{V}^u_T}} \). This implies that the prior weight on \( \hat{\eta} \) satisfies

\[
\frac{1}{\pi(\hat{\eta})} = (i + 1)(i + 2) \leq \left( \log_2 \frac{2\sqrt{\bar{V}^u_T}}{5B_0} + 1 \right) \left( \log_2 \frac{2\sqrt{\bar{V}^u_T}}{5B_0} + 2 \right) \leq \left( \log_2 \frac{2\sqrt{\bar{V}^u_T}}{5B_0} + 3 \right)^2.
\]

(43)

Note also that from the fact that \( 1/\sqrt{\bar{V}^u_T} \leq \eta_* \leq 1/(5B_{T-1}) \leq 1/(5B_0) \), we have \( \sqrt{\bar{V}^u_T}/B_0 \geq 2 \). This fact combined with (43), implies that \( C_T(\hat{\eta}) \leq C_T \), where \( C_T \) is as in the statement of the theorem. Plugging \( \hat{\eta} \) into (42) and using the fact that \( \hat{\eta} \in [\eta_* / 2, \eta_*] \), gives

\[
\bar{R}^u_T \leq \eta \bar{V}^u_T + \frac{1}{\eta} C_T(\hat{\eta}) \leq \eta \bar{V}^u_T + \frac{2}{\eta_*} C_T = 3\sqrt{\bar{V}^u_T C_T}.
\]

(44)

Now suppose that \( \eta_* > \frac{1}{5B_{T-1}} \). Let \( \hat{\eta} := \max \mathcal{G}_T \geq \frac{1}{10B_{T-1}} \), where the last inequality follows by construction of \( \mathcal{G}_T \). Note that in this case \( \frac{1}{\pi(\hat{\eta})} \leq (\log_2 \frac{2B_{T-1}}{B_0} + 1)(\log_2 \frac{2B_{T-1}}{B_0} + 2) \), and thus, we still have \( C_T(\hat{\eta}) \leq C_T \). Plugging \( \hat{\eta} \) into (42) and using the assumption on \( \eta_* \), we obtain

\[
\bar{R}^u_T \leq \eta \bar{V}^u_T + \frac{1}{\eta} C_T(\hat{\eta}) \leq \eta \bar{V}^u_T + \frac{1}{\eta} C_T \leq 15B_T C_T.
\]

(45)

By combining (44) and (45), we get the desired result.

**Proof of Theorem 9** Assume without loss of generality that \( b_1 \neq 0 \). Then the regret of METAGRAD+L in round one is bounded by \( B_1 \leq B_T \), and METAGRAD+C is started for the first time in round \( t = 2 \) with parameter \( B = B_1 \).

Let \( V^u_{(1:T)} \) and \( C_{(1:T)} \) represent the quantities denoted by \( V^u_T \) and \( C_T \) in Theorem 7 but measured on rounds \( 2, \ldots, T \). Now suppose first that the restart condition in line 2 of Algorithm 1 is never triggered, which means that

\[
\frac{B_t}{B_1} \leq \sum_{s=1}^{t} \frac{b_s}{B_s}, \quad \text{for all rounds } t = 2, \ldots, T
\]

(46)
Then the result follows from Theorem 7 and

\[ V_{(1:T)}^u \leq V_T^m, \]

\[ C_{(1:T)} = d \ln \left( \frac{27}{25} + \frac{2}{25d} \frac{\sum_{t=2}^{T-1} b_t^2}{B_{T-1}^2} \right) + 2 \ln \left( \log_2 \sqrt{\frac{\sum_{t=2}^{T} b_t^2}{B_1}} + 3 \right) + 2, \]

\[ \leq d \ln \left( \frac{27}{25} + \frac{2}{25d} \frac{\sum_{t=2}^{T-1} b_t^2}{B_{T-1}^2} \right) + 2 \ln \left( \log_2 \sqrt{\frac{\sum_{t=2}^{T} \left( \sum_{s=1}^{t} b_s \right)^2}{B_s}} + 3 \right) + 2, \]

\[ \leq \Gamma_T, \]

where in (48), we used (46).

Alternatively, suppose there is at least one restart. Then let \( \tau_1 < \tau_2 < T \) be such that \( (\tau_1, \tau_2] \) and \( (\tau_2, T] \) are the two intervals over which the last two runs of METAGRAD+C occurred. We invoke Theorem 7 separately for both these intervals to bound

\[ R_{(\tau_1,T]}^u \leq 3 \sqrt{V_{(\tau_1,\tau_2]}^u C_{(\tau_1,\tau_2]} + 15 B_T C_{(\tau_1,\tau_2]} + B_{\tau_2}} \]

\[ + 3 \sqrt{V_{(\tau_2,\tau]}^u C_{(\tau_2,\tau]} + 15 B_T C_{(\tau_2,\tau]} + B_T} \]

\[ \leq 3 \sqrt{V_{(\tau_1,\tau_2]}^u \Gamma_T/2 + 3 \sqrt{V_{(\tau_2,\tau]}^u \Gamma_T/2 + 15 B_T \Gamma_T + 2B_T}}, \]

where a subscript \( (\tau_1, \tau_2] \) indicates a quantity measured only on rounds \( \tau_1 + 1, \ldots, \tau_2 \) and the last inequality uses \( \sqrt{x} + \sqrt{y} \leq \sqrt{2x + 2y}. \)

If there is exactly one restart, then this implies the desired result. If there are multiple restarts, then the proof is completed by bounding the contribution to the regret of all rounds \( 2, \ldots, \tau_1 \) by

\[ R_{(1,\tau_1]}^u \leq \sum_{t=2}^{\tau_1} b_t < B_{\tau_1} \sum_{t=2}^{\tau_1} \frac{b_t}{B_t} \leq B_{\tau_1} \sum_{t=1}^{\tau_2} \frac{b_t}{B_t} < B_{\tau_2} \leq B_T, \]

where the second to last inequality holds because there was a restart at \( t = \tau_2. \) Finally, by bounding the instantaneous regret from the first round by \( B_T, \) we obtain the desired result. ■

Appendix C. Proofs of Section 5

Proof of Lemma 10 We use the Lagrangian multiplier to solve (23). To this end let

\[ L(u, \mu) := (u_{t+1}^\eta - u)^T (\Sigma_{t+1}^\eta)^{-1} (u_{t+1}^\eta - u) + \mu (u^T u - D^2). \]

Setting \( \frac{\partial L}{\partial u} = 0 \) implies that \( 2 (\Sigma_{t+1}^\eta)^{-1} (u - u_{t+1}^\eta) + 2\mu u = 0. \) After rearranging, this becomes

\[ u = ((\mu + \frac{1}{D^2}) I + 2\eta^2 (M_t - \bar{M}_{\eta}))^{-1} (\Sigma_{t+1}^\eta)^{-1} u_t^\eta, \]

\[ = Q_{t}^\eta (xI + 2\eta^2 (A_t - \bar{A}_{\eta}))^{-1} Q_{t}^\eta u_{t+1}^\eta, \]

\[ Q_{t}^\eta, \]

\[ \leq \Gamma_T, \]
where we set $x := \mu + \frac{1}{T}$. The result follows by observing that $u^T u = D^2/4 \iff \rho_t^n(x) = D^2/4$.

\[ \text{Proof of Theorem 11} \]
Let $\hat{R}_T^n := \sum_{t=1}^T (\hat{u}_t - u, \hat{g}_t)$ and $\hat{V}_T^n := \sum_{t=1}^T (\hat{u}_t - u, \hat{g}_t)^2$ be the pseudo-regret and variance of Algorithm 2. From Theorem 7, the bound on the pseudo-regret $\hat{R}_T^n = \sum_{t=1}^T \langle \hat{u}_t - u, g_t \rangle$ with respect to $\hat{V}_T^n = \sum_{t=1}^T (\hat{u}_t - u, g_t)^2$ for the MetaGrad+C subroutine in Algorithm 2, can be written as

\[
\forall u \in U, \forall \eta > 0, \quad \eta \hat{R}_T^n - \eta^2 \hat{V}_T^n \leq \frac{\eta}{2} \Gamma_T + 15\eta B_T \left( \Gamma_T + \frac{1}{15} \right),
\]

where $\Gamma_T^n := d \ln \left( \frac{2\eta}{25} + \frac{2\sum_{t=1}^{T-1} \eta^2}{25dB_{t-1}} \right) + 2 \ln \left( \log_2 \frac{\sqrt{\sum_{t=1}^T \eta^2}}{\beta} + 3 \right) + 2$.

As in the proof of (Cutkosky and Orabona, 2018, Proposition 1), we have

\[
\langle \hat{w}_t - u, g_t \rangle \leq 2\hat{\ell}_t(\hat{u}_t) - 2\hat{\ell}_t(u),
\]

where $\hat{w}_t = \Pi_t(\hat{u}_t)$ is the prediction of Algorithm 2 at round $t$ and $\hat{\ell}_t$ is the function defined by $\hat{\ell}_t(u) := \frac{1}{2} \left( \langle g_t, u \rangle + \|g_t\|^2 \right)$. By the convexity of $\hat{\ell}_t$ and the fact that $g_t \in \partial \hat{\ell}_t(\hat{u}_t)$, we have

\[
\langle \hat{u}_t - u, g_t \rangle \geq \hat{\ell}_t(u) - \hat{\ell}_t(u) \geq \frac{1}{2} \langle \hat{w}_t - u, g_t \rangle,
\]

where the right-most inequality follows from (52). Since the function $x \mapsto x - x^2$ is strictly increasing on $]-\infty, \frac{1}{2}]$, (53) implies

\[
\frac{\eta}{2} \langle \hat{w}_t - u, g_t \rangle - \frac{\eta^2}{4} \langle \hat{w}_t - u, g_t \rangle^2 \leq \eta \langle \hat{u}_t - u, g_t \rangle - \eta^2 \langle \hat{u}_t - u, g_t \rangle,
\]

for all $\eta \in \left[ 0, \frac{1}{2B_T} \right]$. Summing over $t = 1..T$ and using the bound (51), we get

\[
\forall \eta \in \left[ 0, \frac{1}{2B_T} \right], \quad \hat{R}_T^n - \frac{\eta}{2} \hat{V}_T^n \leq \hat{R}_T^n - \hat{V}_T^n \leq \frac{9}{4\eta} \Gamma_T + 15B_T \left( \Gamma_T + \frac{1}{15} \right),
\]

which leads to, for all $\eta \in \left[ 0, \frac{1}{2B_T} \right]$,

\[
\hat{R}_T^n \leq \frac{\eta}{2} \hat{V}_T^n + \frac{9}{4\eta} \Gamma_T + 15B_T \left( \Gamma_T + \frac{1}{15} \right),
\]

The $\eta$ which minimizes the right-hand side of (42) is given by $\eta_* := \sqrt{\frac{9B_T}{2\hat{V}_T^n}}$. We consider two cases; suppose first that $\hat{\eta} \leq \frac{1}{2B_T}$. Then, by setting $\eta = \eta_*$, we have

\[
\frac{\eta}{2} \hat{V}_T^n + \frac{9}{4\eta} \Gamma_T = 3\sqrt{\hat{V}_T^n \Gamma_T}.
\]

Now suppose that $\eta_* > \frac{1}{2B_T}$. Then for $\eta = \frac{1}{2B_T}$ we have

\[
\frac{\eta}{2} \hat{V}_T^n + \frac{9}{4\eta} \Gamma_T \leq 9B_T \Gamma_T.
\]

Combining (55)-(57) yields the desired result. \[\blacksquare\]