On unimodality problems in Pascal’s triangle *

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Abstract

Many sequences of binomial coefficients share various unimodality properties. In this paper we consider the unimodality problem of a sequence of binomial coefficients located in a ray or a transversal of the Pascal triangle. Our results give in particular an affirmative answer to a conjecture of Belbachir et al which asserts that such a sequence of binomial coefficients must be unimodal. We also propose two more general conjectures.

1 Introduction

Let $a_0, a_1, a_2, \ldots$ be a sequence of nonnegative numbers. It is called unimodal if $a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots$ for some $m$ (such an integer $m$ is called a mode of the sequence). In particular, a monotone (increasing or decreasing) sequence is known as unimodal. The sequence is called concave (resp. convex) if for $i \geq 1$, $a_{i-1} + a_{i+1} \leq 2a_i$ (resp. $a_{i-1} + a_{i+1} \geq 2a_i$). The sequence is called log-concave (resp. log-convex) if for all $i \geq 1$, $a_{i-1}a_{i+1} \leq a_i^2$ (resp. $a_{i-1}a_{i+1} \geq a_i^2$). By the arithmetic-geometric mean inequality, the concavity implies the log-concavity (the log-convexity implies the convexity). For a sequence $\{a_i\}$ of positive numbers, it is log-concave (resp. log-convex) if and only if the sequence $\{a_{i+1}/a_i\}$ is decreasing (resp. increasing), and so the log-concavity implies the unimodality. The unimodality problems, including concavity (convexity) and log-concavity (log-convexity), arise naturally in many branches of mathematics. For details, see [3, 4, 13, 17, 18, 19, 21, 22] about the unimodality and log-concavity and [7, 10] about the log-convexity.

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Many sequences of binomial coefficients share various unimodality properties. For example, the sequence \( \binom{n}{k} \) is unimodal and log-concave in \( k \). On the other hand, the sequence \( \binom{n}{k} \) is increasing, log-concave and convex in \( n \) (see Comtet [5] for example). As usual, let \( \binom{n}{k} = 0 \) unless \( 0 \leq k \leq n \). Tanny and Zuker [14, 15] showed the unimodality and log-concavity of the binomial sequences \( \binom{n}{0}, \binom{n}{i} \) and \( \binom{n}{i} \). Very recently, Belbachir et al [1] showed the unimodality and log-concavity of the binomial sequence \( \binom{n}{0} \). They further proposed the following.

**Conjecture 1** ([1, Conjecture 1]). Let \( \binom{n}{k} \) be a fixed element of the Pascal triangle crossed by a ray. The sequence of binomial coefficients located along this ray is unimodal.

![Figure 1: a ray with \( d = 3 \) and \( \delta = 2 \).](image)

The object of this paper is to study the unimodality problem of a sequence of binomial coefficients located in a ray or a transversal of the Pascal triangle. Let \( \binom{n+i}{k+i} \) be such a sequence. Then \( \{n_i\}_{i \geq 0} \) and \( \{k_i\}_{i \geq 0} \) form two arithmetic sequences (see Figure 1). Clearly, we may assume that the common difference of \( \{n_i\}_{i \geq 0} \) is nonnegative (by changing the order of the sequence). For example, the sequence \( \binom{n+1}{i} \) coincides with the sequence \( \binom{n+1-i}{i} \) except for the order. On the other hand, the sequence \( \binom{n_i}{k_i} \) is the same as the sequence \( \binom{n_i}{n_i-k_i} \) by the symmetry of the binomial coefficients. So we may assume, without loss of generality, that the common difference of \( \{k_i\}_{i \geq 0} \) is nonnegative. Thus it suffices to consider the unimodality of the sequence \( \binom{n+id}{k+id} \) for nonnegative integers \( d \) and \( \delta \). The following is the main result of this paper, which in particular, gives an affirmative answer to Conjecture 1.

**Theorem 1.** Let \( n_0, k_0, d, \delta \) be four nonnegative integers and \( n_0 \geq k_0 \). Define the sequence

\[
C_i = \binom{n_0 + id}{k_0 + i\delta}, \quad i = 0, 1, 2, \ldots
\]
Then

(i) if $d = \delta > 0$ or $\delta = 0$, the sequence is increasing, convex and log-concave;

(ii) if $d < \delta$, the sequence is log-concave and therefore unimodal;

(iii) if $d > \delta > 0$, the sequence is increasing, convex, and asymptotically log-convex (i.e., there exists a nonnegative integer $m$ such that $C_m, C_{m+1}, C_{m+2}, \ldots$ is log-convex).

This paper is organized as follows. In the next section, we prove Theorem 1. In Section 3, we present a combinatorial proof of the log-concavity in Theorem 1 (ii). In Section 4, we show more precise results about the asymptotically log-convexity for certain particular sequences of binomial coefficients in Theorem 1 (iii). Finally in Section 5, we propose some open problems and conjectures.

Throughout this paper we will denote by $[x]$ and $[x]$ the largest integer $\leq x$ and the smallest integer $\geq x$ respectively.

2 The proof of Theorem 1

The following result is folklore and we include a proof of it for completeness.

**Lemma 1.** If a sequence $\{a_i\}_{i \geq 0}$ of positive numbers is unimodal (resp. increasing, decreasing, concave, convex, log-concave, log-convex), then so is its subsequence $\{a_{n_0+id}\}_{i \geq 0}$ for arbitrary fixed nonnegative integers $n_0$ and $d$.

**Proof.** We only consider the log-concavity case since the others are similar. Let $\{a_i\}_{i \geq 0}$ be a log-concave sequence of positive numbers. Then the sequence $\{a_{i-1}/a_i\}_{i \geq 0}$ is increasing. Hence $a_{j-1}/a_j \leq a_k/a_{k+1}$ for $1 \leq j \leq k$, i.e., $a_{j-1}a_{k+1} \leq a_ja_k$. Thus

$$a_n a_{n+d} \leq a_{n-d+1} a_{n+d-1} \leq a_{n-d+2} a_{n+d-2} \leq \cdots \leq a_{n-1} a_{n+1} \leq a_n^2,$$

which implies that the sequence $\{a_{n_0+id}\}_{i \geq 0}$ is log-concave. $\square$

The proof of Theorem 1. (i) If $\delta = 0$, then $C_i = \binom{n_i+id}{k_i}$. The sequence $\binom{i}{k_i}$ is increasing, convex and log-concave in $i$, so is the sequence $C_i$ by Lemma 1. The case $d = \delta$ is similar since $C_i = \binom{n_i+id}{n_i-k_i}$.

(ii) To show the log-concavity of $\{C_i\}$ when $d < \delta$, it suffices to show that

$$\binom{n+d}{k+\delta} \binom{n-d}{k-\delta} \leq \binom{n}{k}^2$$

for $n \geq k$. Write

$$\binom{n+d}{k+\delta} \binom{n-d}{k-\delta} = \frac{(n+d)!(n-d)!}{(n-k+d-\delta)!(k+\delta)!(n-k-\delta-d)!(k-\delta)!}$$

$$= \left(\frac{n+d}{n-k}\right) \left(\frac{n-d}{n-k}\right) \frac{\binom{n-k}{\delta-d}}{\binom{k+\delta}{\delta-d}}.$$

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Now \((n-k) \leq (n-k+\delta-d)\), \((k-d) \leq (k+\delta-d)\) and \((n+d)/(n-k) \leq (n)\) by (i). Hence
\[
\binom{n+d}{k+\delta} \binom{n-d}{k-\delta} \leq \left( \binom{n}{n-k} \right)^2 = \left( \frac{n}{k} \right)^2,
\]
as required.

(iii) Assume that \(d > \delta > 0\). By Vandermonde’s convolution formula, we have
\[
\binom{n+d}{k+\delta} = \sum_{r+s=k+\delta} \binom{n}{r} \binom{d}{s} \geq \left( \binom{n}{k} \right)^2 \geq \left( \frac{n}{k} \right)^2,
\]
which implies that \((n+d)/(k+\delta)\) and \((n+d)/(k-\delta)\) are increasing and convex. Hence the sequence \(\{C_i\}\) is increasing and convex.

It remains to show that the sequence \(\{C_i\}\) is asymptotically log-convex. Denote
\[
\Delta(i) := \left( \binom{n_0 + (i+1)d}{k_0 + (i+1)\delta} \binom{n_0 + (i-1)d}{k_0 + (i-1)\delta} - \binom{n_0 + id}{k_0 + id} \right)^2.
\]
Then we need to show that \(\Delta(i)\) is positive for all sufficiently large \(i\). Write
\[
\Delta(i) = \frac{(n_0 + id)! [n_0 + (i-1)d]!}{(k_0 + i\delta)! [k_0 + (i+1)\delta]! [n_0 - k_0 + i(d - \delta)]! [n_0 - k_0 + (i+1)(d - \delta)]!}
\times \left\{ \prod_{j=1}^{d} (n_0 + id + j) \prod_{j=1}^{d-\delta} [n_0 - k_0 + (i-1)(d - \delta) + j] \prod_{j=1}^{\delta} [k_0 + (i-1)\delta + j] \right\}
\quad - \left\{ \prod_{j=1}^{d} (n_0 + (i-1)d + j) \prod_{j=1}^{d-\delta} [n_0 - k_0 + i(d - \delta) + j] \prod_{j=1}^{\delta} (k_0 + i\delta + j) \right\}
\quad \times \frac{(n_0 + id)! [n_0 + (i-1)d]! d^\delta (d - \delta)^{(d-\delta)}}{(k_0 + i\delta)! [k_0 + (i+1)\delta]! [n_0 - k_0 + i(d - \delta)]! [n_0 - k_0 + (i+1)(d - \delta)]!} P(i),
\]
where
\[
P(i) = \prod_{j=1}^{d} \left( i + \frac{n_0 + j}{d} \right)^{d-\delta} \prod_{j=1}^{d} \left( i + \frac{n_0 - k_0 - d + \delta + j}{d - \delta} \right)^{\delta} \prod_{j=1}^{d} \left( i + \frac{k_0 - \delta + j}{\delta} \right)
\quad - \prod_{j=1}^{d} \left( i + \frac{n_0 - d + j}{d} \right)^{d-\delta} \prod_{j=1}^{d} \left( i + \frac{n_0 - k_0 + j}{d - \delta} \right)^{\delta} \prod_{j=1}^{d} \left( i + \frac{k_0 + j}{\delta} \right).
\]
Then it suffices to show that \(P(i)\) is positive for sufficiently large \(i\). Clearly, \(P(i)\) can be viewed as a polynomial in \(i\). So it suffices to show that the leading coefficient of \(P(i)\) is positive.

Note that \(P(i)\) is the difference of two monic polynomials of degree \(2d\). Hence its degree is less than \(2d\). Denote
\[
P(i) = a_{2d-1}i^{2d-1} + a_{2d-2}i^{2d-2} + \cdots.
\]
By Vieta’s formula, we have

\[
a_{2d-1} = - \left( \sum_{j=1}^{d} \frac{n_0 + j}{d} + \sum_{j=1}^{d-\delta} \frac{n_0 - k_0 - d + \delta + j}{d - \delta} + \sum_{j=1}^{\delta} \frac{k_0 - \delta + j}{\delta} \right) 
+ \left( \sum_{j=1}^{d} \frac{n_0 - d + j}{d} + \sum_{j=1}^{d-\delta} \frac{n_0 - k_0 + j}{d - \delta} + \sum_{j=1}^{\delta} \frac{k_0 + j}{\delta} \right)
\]

\[
= \sum_{j=1}^{d} \left( \frac{n_0 - d + j}{d} - \frac{n_0 + j}{d} \right) 
+ \sum_{j=1}^{d-\delta} \left( \frac{n_0 - k_0 + j}{d - \delta} - \frac{n_0 - k_0 - d + \delta + j}{d - \delta} \right) 
+ \sum_{j=1}^{\delta} \left( \frac{k_0 + j}{\delta} - \frac{k_0 - \delta + j}{\delta} \right)
\]

\[
= \sum_{j=1}^{d} (-1) + \sum_{j=1}^{d-\delta} 1 + \sum_{j=1}^{\delta} 1
= -d + (d - \delta) + \delta
= 0.
\]

Using the identity

\[
\sum_{1 \leq i < j \leq n} x_i x_j = \frac{1}{2} \left[ \left( \sum_{i=1}^{n} x_i \right)^2 - \sum_{i=1}^{n} x_i^2 \right],
\]

we obtain again by Vieta’s formula

\[
a_{2d-2} = \frac{1}{2} \left[ \left( \sum_{j=1}^{d} \frac{n_0 + j}{d} \right)^2 - \left( \sum_{j=1}^{d-\delta} \frac{n_0 - k_0 - d + \delta + j}{d - \delta} \right)^2 - \left( \sum_{j=1}^{\delta} \frac{k_0 - \delta + j}{\delta} \right)^2 \right]
- \left( \sum_{j=1}^{d} \frac{n_0 - d + j}{d} \right)^2 - \left( \sum_{j=1}^{d-\delta} \frac{n_0 - k_0 + j}{d - \delta} \right)^2 - \left( \sum_{j=1}^{\delta} \frac{k_0 + j}{\delta} \right)^2
\]

\[
- \frac{1}{2} \left[ \left( \sum_{j=1}^{d} \frac{n_0 - d + j}{d} \right)^2 - \left( \sum_{j=1}^{d-\delta} \frac{n_0 - k_0 + j}{d - \delta} \right)^2 - \left( \sum_{j=1}^{\delta} \frac{k_0 + j}{\delta} \right)^2 \right].
\]
But $a_{2d-1} = 0$ implies
\[
\left( \sum_{j=1}^{d} \frac{n_0 + j}{d} + \sum_{j=1}^{d-\delta} \frac{n_0 - k_0 - d + \delta + j}{d - \delta} + \sum_{j=1}^{\delta} \frac{k_0 - \delta + j}{\delta} \right)^2
\]
\[
= \left( \sum_{j=1}^{d} \frac{n_0 - d + j}{d} + \sum_{j=1}^{d-\delta} \frac{n_0 - k_0 + j}{d - \delta} + \sum_{j=1}^{\delta} \frac{k_0 + j}{\delta} \right)^2,
\]
so we have
\[
a_{2d-2} = \frac{1}{2} \sum_{j=1}^{d} \left[ \left( \frac{n_0 - d + j}{d} \right)^2 - \left( \frac{n_0 + j}{d} \right)^2 \right]
+ \frac{1}{2} \sum_{j=1}^{d-\delta} \left[ \left( \frac{n_0 - k_0 + j}{d - \delta} \right)^2 - \left( \frac{n_0 - k_0 - d + \delta + j}{d - \delta} \right)^2 \right]
+ \frac{1}{2} \sum_{j=1}^{\delta} \left[ \left( \frac{k_0 + j}{\delta} \right)^2 - \left( \frac{k_0 - \delta + j}{\delta} \right)^2 \right]
\]
\[
= -\frac{1}{2} \sum_{j=1}^{d} \frac{2n_0 - d + 2j}{d} + \frac{1}{2} \sum_{j=1}^{d-\delta} \frac{2(n_0 - k_0) - (d - \delta) + 2j}{d - \delta} + \frac{1}{2} \sum_{j=1}^{\delta} \frac{2k_0 - \delta + 2j}{\delta}
\]
\[
= -\frac{1}{2} (2n_0 + 1) + \frac{1}{2} (2n_0 - 2k_0 + 1) + \frac{1}{2} (2k_0 + 1)
\]
\[
= \frac{1}{2}.
\]
Thus $P(i)$ is a polynomial of degree $2d - 2$ with positive leading coefficient, as desired. This completes the proof of the theorem.

\[\square\]

3 Combinatorial proof of the log-concavity

In Section 2 we have investigated the unimodality of sequences of binomial coefficients by an algebraic approach. It is natural to ask for a combinatorial interpretation. Lattice path techniques have been shown to be useful in solving the unimodality problem. As an example, we present a combinatorial proof of Theorem 1 (ii) following Bóna and Sagan’s technique in [2].

Let $\mathbb{Z}^2 = \{(x, y) : x, y \in \mathbb{Z}\}$ denote the two-dimensional integer lattice. A lattice path is a sequence $P_1, P_2, \ldots, P_\ell$ of lattice points on $\mathbb{Z}^2$. A southeastern lattice path is a lattice path in which each step goes one unit to the south or to the east. Denote by $P(n, k)$ the set of southeastern lattice paths from the point $(0, n - k)$ to the point $(k, 0)$. Clearly, the number of such paths is the binomial coefficient $\binom{n}{k}$.

Recall that, to show the log-concavity of $C_i = \binom{n_0 + id}{k_0 + i\delta}$ where $n_0 \geq k_0$ and $d < \delta$, it suffices to show $\binom{n+d}{k+d} \binom{n-d}{k-d} \leq \binom{n}{k}^2$ for $n \geq k$. Here we do this by constructing an injection
\[
\phi : P(n + d, k + \delta) \times P(n - d, k - \delta) \longrightarrow P(n, k) \times P(n, k).
\]
Consider a path pair \((p, q) \in P(n + d, k + \delta) \times P(n - d, k - \delta)\). Then \(p\) and \(q\) must intersect. Let \(I_1\) be the first intersection. For two points \(P(a, b)\) and \(Q(a, c)\) with the same \(x\)-coordinate, define their vertical distance to be \(d_v(P, Q) = b - c\). Then the vertical distance from a point of \(p\) to a point of \(q\) starts at \(2(\delta - d)\) for their initial points and ends at 0 for their intersection \(I_1\). Thus there must be a pair of points \(P \in p\) and \(Q \in q\) before \(I_1\) with \(d_v(P, Q) = \delta - d\). Let \((P_1, Q_1)\) be the first such pair of points. Similarly, after the last intersection \(I_2\) there must be a last pair of points \(P_2 \in p\) and \(Q_2 \in q\) with the horizontal distance \(d_h(P_2, Q_2) = -\delta\) (the definition of \(d_h\) is analogous to that of \(d_v\)). Now \(p\) is divided by two points \(P_1, P_2\) into three subpaths \(p_1, p_2, p_3\) and \(q\) is divided by \(Q_1, Q_2\) into three subpaths \(q_1, q_2, q_3\). Let \(p'_1\) be obtained by moving \(p_1\) down to \(Q_1\) south \(\delta - d\) units and \(p'_3\) be obtained by moving \(p_3\) right to \(Q_2\) east \(\delta\) units. Then we obtain a southeastern lattice path \(p'_1q_2p'_3\) in \(P(n, k)\). We can similarly obtain the second southeastern lattice path \(q'_1p_2q'_3\) in \(P(n, k)\), where \(q'_1\) is moved north \(\delta - d\) units and \(q'_3\) is moved west \(\delta\) units. Define \(\phi(p, q) = (p'_1q_2p'_3, q'_1p_2q'_3)\). It is not difficult to verify that \(\phi\) is the required injective. We omit the proof for brevity.

![Figure 2: the constructing of \(\phi\).](image)

4 Asymptotic behavior of the log-convexity

Theorem 1 (iii) tells us that the sequence \(C_i = \binom{n_0 + id}{k_0 + id}\) is asymptotically log-convex when \(d > \delta > 0\). We can say more for a certain particular sequence of binomial coefficients. For example, it is easy to verify that the central binomial coefficients \(\binom{2i}{i}\) is log-convex for \(i \geq 0\) (see Liu and Wang [10] for a proof). In this section we give two generalizations of this result. The first one is that every sequence of binomial coefficients located along a ray with origin \(\binom{0}{0}\) is log-convex.

Proposition 1. Let \(d\) and \(\delta\) be two positive integers and \(d > \delta > 0\). Then the sequence \(\left\{\binom{id}{id}\right\}_{i \geq 0}\) is log-convex.

Before showing Proposition 1, we first demonstrate two simple but useful facts.

Let \(\alpha = (a_1, a_2, \ldots, a_n)\) and \(\beta = (b_1, b_2, \ldots, b_n)\) be two \(n\)-tuples of real numbers. We say that \(\alpha\) alternates left of \(\beta\), denoted by \(\alpha \preceq \beta\), if

\[a_1^* \leq b_1^* \leq a_2^* \leq b_2^* \cdots \leq a_n^* \leq b_n^*,\]

where \(a_j^*\) and \(b_j^*\) are the \(j\)th smallest elements of \(\alpha\) and \(\beta\), respectively.
**Fact 1** Let \( f(x) \) be a nondecreasing function. If \((a_1, a_2, \ldots, a_n) \preceq (b_1, b_2, \ldots, b_n)\), then \( \prod_{i=1}^{n} f(a_i) \preceq \prod_{i=1}^{n} f(b_i) \).

**Fact 2** Let \( x_1, x_2, y_1, y_2 \) be four positive numbers and \( \frac{a_1}{y_1} \leq \frac{a_2}{y_2} \). Then \( \frac{a_1}{y_1} \leq \frac{a_1+a_2}{y_1+y_2} \leq \frac{a_2}{y_2} \).

**Proof of Proposition 1.** By Lemma 1, we may assume, without loss of generality, that \( d \) and \( \delta \) are coprime. We need to show that

\[
\Delta(i) := \left( \frac{(i+1)d}{(i+1)\delta} \right) \left( \frac{(i-1)d}{(i-1)\delta} \right) - \left( \frac{id}{i\delta} \right)^2 \geq 0
\]

for all \( i \geq 1 \). Write

\[
\Delta(i) = \frac{(id)!(i-1)d!d\delta(d-\delta)^{(d-\delta)}}{(i\delta)!(i+1)d!(i(d-\delta))!(i+1)(d-\delta))} Q(i),
\]

where

\[
Q(i) = \prod_{j=1}^{\delta} \left( 1 - \frac{1}{i + \frac{j}{d}} \right)^{d-\delta} \prod_{j=1}^{\delta} \left( 1 - \frac{1}{i + \frac{j}{d-\delta}} \right)^{d-\delta} \prod_{j=1}^{d} \left( 1 - \frac{1}{i + \frac{j}{d}} \right).
\]

Then we only need to show that \( Q(i) \geq 0 \) for \( i \geq 1 \). We do this by showing

\[
\left( \frac{1}{d}, \ldots, \frac{d-1}{d}, \frac{1}{d} \right) \preceq \left( \frac{1}{\delta d}, \ldots, \frac{\delta-1}{\delta}, \frac{\delta}{d-\delta}, \ldots, \frac{d-\delta-1}{d-\delta}, \frac{d-\delta}{d-\delta} \right),
\]

or equivalently,

\[
\left( \frac{1}{d}, \ldots, \frac{d-1}{d} \right) \preceq \left( \frac{1}{\delta}, \ldots, \frac{\delta-1}{\delta}, \frac{\delta}{d-\delta}, \ldots, \frac{d-\delta-1}{d-\delta}, \frac{d-\delta}{d-\delta} \right).
\]

Note that \( (d, \delta) = 1 \) implies all fractions \( \left\{ \frac{j}{d} \right\}_{j=1}^{d-1} \) and \( \left\{ \frac{j}{d-\delta} \right\}_{j=1}^{d-\delta-1} \) are different. Hence it suffices to show that every term of \( \left\{ \frac{j}{d} \right\}_{j=1}^{d-1} \) \( \bigcup \left\{ \frac{j}{d-\delta} \right\}_{j=1}^{d-\delta-1} \) is precisely in one of \( d-2 \) open intervals \( \left( \frac{k}{d}, \frac{k+1}{d} \right) \), where \( k = 1, \ldots, d-2 \). Indeed, neither two terms of \( \left\{ \frac{j}{d} \right\}_{j=1}^{d-1} \) nor two terms of \( \left\{ \frac{j}{d-\delta} \right\}_{j=1}^{d-\delta-1} \) are in the same interval since their difference is larger than \( \frac{1}{d} \). On the other hand, if \( \frac{j}{d} \) and \( \frac{j'}{d-\delta} \) are in a certain interval \( \left( \frac{k}{d}, \frac{k+1}{d} \right) \), then so is \( \frac{j+j'}{d} \) by Fact 2, which is impossible. Thus there exists precisely one term of \( \left\{ \frac{j}{d} \right\}_{j=1}^{d-1} \) \( \bigcup \left\{ \frac{j}{d-\delta} \right\}_{j=1}^{d-\delta-1} \) in every open interval \( \left( \frac{k}{d}, \frac{k+1}{d} \right) \), as desired. This completes our proof. \( \square \)

For the second generalization of the log-convexity of the central binomial coefficients, we consider sequences of binomial coefficients located along a vertical ray with origin \( \binom{n_0}{0} \) in the Pascal triangle.

**Proposition 2.** Let \( n_0 \geq 0 \) and \( V_i(n_0) = \binom{n_0+2i}{i} \). Then \( V_0(n_0), V_1(n_0), \ldots, V_m(n_0) \) is log-concave and \( V_{m-1}(n_0), V_m(n_0), V_{m+1}(n_0), \ldots \) is log-convex, where \( m = n_0^2 - \left\lceil \frac{n_0}{2} \right\rceil \).
Proof. The sequence $V_i(0) = \binom{2i}{i}$ is just the central binomial coefficients and therefore log-convex for $i \geq 0$. It implies that the sequence $V_i(1) = \binom{i+2}{i}$ is log-convex for $i \geq 0$ since $V_i(1) = \frac{1}{2}V_{i+1}(0)$. Now let $n_0 \geq 2$ and define $f(i) = V_{i+1}(n_0)/V_i(n_0)$ for $i \geq 0$. Then, to show the statement, it suffices to show that

$$f(0) > f(1) > \cdots > f(m-1) \quad \text{and} \quad f(m-1) < f(m) < f(m+1) < \cdots$$

(1)

for $m = n_0^2 - \lceil \frac{n_0^2}{2} \rceil$.

By the definition we have

$$f(i) = \frac{\binom{n_0^2(i+1)}{i+1}}{\binom{n_0^2+2i}{i}} = \frac{(n_0 + 2i + 1)(n_0 + 2i + 2)}{(i + 1)(n_0 + i + 1)}.$$  

(2)

The derivative of $f(i)$ with respect to $i$ is

$$f'(i) = \frac{2i^2 - 2(n_0 - 2)(n_0 + 1)i - (n_0 + 1)(n_0^2 - 2)}{(i + 1)^2(n_0 + i + 1)^2}.$$  

The numerator of $f'(i)$ has the unique positive zero

$$r = \frac{2(n_0 - 2)(n_0 + 1) + \sqrt{4(n_0 - 2)^2(n_0 + 1)^2 + 8(n_0 + 1)(n_0^2 - 2)}}{2} = \frac{(n_0 - 2)(n_0 + 1)}{2} + \frac{n_0 \sqrt{n_0^2 - 1}}{2}.$$  

It implies that $f'(i) < 0$ for $0 \leq i < r$ and $f'(i) > 0$ for $i > r$. Thus we have

$$f(0) > f(1) > \cdots > f(\lfloor r \rfloor) \quad \text{and} \quad f(\lfloor r \rfloor) < f(\lfloor r \rfloor + 1) < f(\lfloor r \rfloor + 2) < \cdots.$$  

(3)

It remains to compare the values of $f(\lfloor r \rfloor)$ and $f(\lceil r \rceil)$. Note that

$$\frac{n_0^2 - n_0 \sqrt{n_0^2 - 1}}{2} = -\frac{n_0}{2(n_0 + \sqrt{n_0^2 - 1})} < \frac{1}{2}.$$  

Hence

$$\left\lfloor \frac{n_0 \sqrt{n_0^2 - 1}}{2} \right\rfloor = \begin{cases} \frac{n_0^2}{2}, & \text{if } n_0 \text{ is even;} \\ \frac{n_0^2 + 1}{2}, & \text{if } n_0 \text{ is odd}, \end{cases}$$

and so

$$\lfloor r \rfloor = \frac{(n_0 - 2)(n_0 + 1)}{2} + \left\lfloor \frac{n_0 \sqrt{n_0^2 - 1}}{2} \right\rfloor = \begin{cases} n_0^2 - \frac{n_0}{2} - 1, & \text{if } n_0 \text{ is even;} \\ n_0^2 - \frac{n_0 + 1}{2}, & \text{if } n_0 \text{ is odd}. \end{cases}$$

If $n_0$ is even, then by (2) we have

$$f(\lfloor r \rfloor) = \frac{16n_0^5 - 8}{4n_0^5 - 1} = 4 - \frac{4}{4n_0^5 - 1}$$
and
\[ f([r]) = f([r] - 1) = \frac{16n_0^4 - 40n_0^2 + 16}{4n_0^4 - 9n_0^2 + 4} = 4 - \frac{4(n_0^2 - 2)}{4n_0^4 - 9n_0^2 + 4}. \]
Thus \( f([r]) > f([r]) \) since 
\[ f([r]) - f([r]) = \frac{8}{(4n_0^2 - 1)(4n_0^2 - 9n_0^2 + 4)} > 0. \] Also, \( [r] = m - 1 \).
Combining (3) we obtain (1).

If \( n_0 \) is odd, then
\[ f([r]) = 4 - \frac{4(n_0^2 + 1)}{4n_0^4 + 3n_0^2 + 1} \]
and
\[ f([r]) = 4 - \frac{4(n_0^2 - 1)}{4n_0^4 - 5n_0^2 + 1}. \]

It is easy to verify that \( f([r]) < f([r]) \). Also, \( [r] = [r] - 1 = m - 1 \). Thus (1) follows.
This completes our proof. \( \square \)

5 Concluding remarks and open problems

In this paper we show that the sequence \( C_i = \binom{n_0 + id}{k_0 + id} \) is unimodal when \( d < \delta \). A further problem is to find out the value of \( i \) for which \( C_i \) is a maximum. Tanny and Zuker [14, 15, 16] considered such a problem for the sequence \( \binom{n_0 - id}{i} \). For example, it is shown that

\[ \text{the sequence } \binom{n_0 - i}{i} \text{ attains the maximum when } i = \left\lfloor \frac{5n_0 + 7 - \sqrt{5n_0^2 + 10n_0 + 9}}{10} \right\rfloor. \]

Let \( r(n_0, d) \) be the least integer at which \( \binom{n_0 - id}{i} \) attains its maximum. They investigated the asymptotic behavior of \( r(n_0, d) \) for \( d \to \infty \) and concluded with a variety of unsolved problems concerning the numbers \( r(n_0, d) \). An interesting problem is to consider analogue for the general binomial sequence \( C_i = \binom{n_0 + id}{k_0 + id} \) when \( d < \delta \). It often occurs that unimodality of a sequence is known, yet to determine the exact number and location of modes is a much more difficult task.

A finite sequence of positive numbers \( a_0, a_1, \ldots, a_n \) is called a Pólya frequency sequence if its generating function \( P(x) = \sum_{i=0}^{n} a_i x^i \) has only real zeros. By the Newton’s inequality, if \( a_0, a_1, \ldots, a_n \) is a Pólya frequency sequence, then
\[ a_i^2 \geq a_{i-1} a_{i+1} \left( 1 + \frac{1}{i} \right) \left( 1 + \frac{1}{n-i} \right) \]
for \( 1 \leq i \leq n - 1 \), and the sequence is therefore log-concave and unimodal with at most two modes (see Hardy, Littlewood and Pólya [9, p. 104]). Darroch [6] further showed that each mode \( m \) of the sequence \( a_0, a_1, \ldots, a_n \) satisfies
\[ \left\lfloor \frac{P'(1)}{P(1)} \right\rfloor \leq m \leq \left\lceil \frac{P'(1)}{P(1)} \right\rceil. \]

We refer the reader to [3, 4, 8, 11, 12, 13, 20] for more information.

For example, the binomial coefficients \( \binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n} \) is a Pólya frequency sequence with the unique mode \( n/2 \) for even \( n \) and two modes \( (n \pm 1)/2 \) for odd \( n \). On the other
hand, the sequence \( \binom{n}{0}, \binom{n-1}{1}, \binom{n-2}{2}, \ldots, \binom{n/2}{n/2} \) is a Pólya frequency sequence since its generating function is precisely the matching polynomial of a path on \( n \) vertices. Hence we make the more general conjecture that every sequence of binomial coefficients located in a transversal of the Pascal triangle is a Pólya frequency sequence.

**Conjecture 2.** Let \( C_i = \binom{n_0+i\delta}{k_0+i\delta} \) where \( n_0 \geq k_0 \) and \( \delta > d > 0 \). Then the finite sequence \( \{C_i\} \) is a Pólya frequency sequence.

In Proposition 2 we have shown that the sequence \( V_i(n_0) = \binom{n_0+2i}{i} \) is first log-concave and then log-convex. It is possible that an arbitrary sequence of binomial coefficients located along a ray in the Pascal triangle has the same property as the sequence \( V_i(n_0) \). We leave this as a conjecture to end this paper.

**Conjecture 3.** Let \( C_i = \binom{n_0+i\delta}{k_0+i\delta} \) where \( n_0 \geq k_0 \) and \( d > \delta > 0 \). Then there is a nonnegative integer \( m \) such that \( C_0, C_1, \ldots, C_m \) is log-concave and \( C_{m+1}, C_m, C_{m+1}, \ldots \) is log-convex.

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