Recovery of small electromagnetic inhomogeneities from boundary measurements on part of the boundary

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Abstract

We consider for the Helmholtz equation the inverse problem of identifying locations and certain properties of the shapes of small dielectric inhomogeneities in a homogeneous background medium from boundary measurements on part of the boundary. Using as weights particular background solutions constructed by solving a minimization problem we develop an asymptotic (variational) method based on appropriate averaging of the partial boundary measurements.

Identification de petites inhomogénéités diélectriques à partir de mesures sur une partie du bord

Résumé

Nous considérons le problème d’identification de petites inhomogénéités diélectriques à partir de mesures (incomplètes) sur uniquement une partie du bord. Grâce à un résultat de densité, nous construisons une fonction dont la transformée de Fourier inverse permet de localiser les petites inhomogénéités.

Version française abrégée

Soit Ω un ouvert borné de R^d, d ≥ 2, de classe C^2. Supposons que Ω contient m inhomogénéités, \{z_j + \alpha B_j\}_{j=1}^m, où α est un petit paramètre, B_j ⊂ R^d est un ouvert borné et les points \{z_j\}_{j=1}^m vérifient les hypothèses (1). Soient la perméabilité magnétique μ_α et la permittivité électrique ε_α de forme (2).

Cette Note concerne le problème de reconstruction des points \{z_j\}_{j=1}^m et des tenseurs de polarisation M_j des domaines B_j, définis par (3), à partir de la mesure de la dérivée normale du champ électrique E_α, solution de l’équation de Helmholtz (4), sur une partie Γ_1 ⊂⊂ ∂Ω. Grâce à un résultat de densité, établi dans la Proposition 2.1, et à la formule asymptotique (4), nous réduisons ce problème inverse au calcul de la transformée de Fourier inverse de la fonction Λ_α, définie dans (5). La fonction test w_α, utilisée dans (5), peut être construite numériquement en résolvant le problème de minimisation (6).Cette fonction vérifie les propriétés d’approximation énoncées dans le Lemme 2.1. Cette Note généralise la méthode introduite dans (7) aux situations où on ne disposerait pas de la mesure de la dérivée normale du champ électrique E_α sur tout le bord ∂Ω.

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1 Introduction

Let $\Omega$ be a bounded $C^2$-domain in $\mathbb{R}^d$, $d \geq 2$ and $\nu$ be the outward unit normal to $\partial \Omega$. Assume that $\Omega$ contains a finite number of inhomogeneities, each of the form $z_j + \alpha B_j$, where $B_j \subset \mathbb{R}^d$ is a bounded, smooth domain containing the origin. The total collection of inhomogeneities is $\mathcal{B}_n = \bigcup_{j=1}^{m}(z_j + \alpha B_j)$. The points $z_j \in \Omega$, $j = 1, \ldots, m$, which determine the location of the inhomogeneities, are assumed to satisfy the following inequalities:

$$|z_j - z_l| \geq c_0 > 0, \forall j \neq l \quad \text{and} \quad \text{dist}(z_j, \partial \Omega) \geq c_0 > 0, \forall j. \tag{1}$$

This assumption implies $m \leq \frac{2d|\Omega|}{\pi \varepsilon_0^d}$. Assume that $\alpha > 0$, the common order of magnitude of the diameters of the inhomogeneities, is sufficiently small, that these inhomogeneities are disjoint, and that their distance to $\mathbb{R}^d \setminus \overline{\Omega}$ is larger than $c_0/2$. Let $\mu_0$ and $\varepsilon_0$ denote the permeability and the permittivity of the background medium, and assume that $\mu_0 > 0$ and $\varepsilon_0 > 0$ are positive constants. Let $\mu_j > 0$ and $\varepsilon_j > 0$ denote the permeability and the permittivity of the $j$-th inhomogeneity, $z_j + \alpha B_j$, these are also assumed to be positive constants. Introduce the piecewise-constant magnetic permeability

$$\mu_\alpha(x) = \begin{cases} \mu_0, & x \in \Omega \setminus \bar{\mathcal{B}}_\alpha, \\ \mu_j, & x \in z_j + \alpha B_j, j = 1 \ldots m. \end{cases} \tag{2}$$

If we allow the degenerate case $\alpha = 0$, then the function $\mu_0(x)$ equals the constant $\mu_0$. The piecewise constant electric permittivity, $\varepsilon_\alpha(x)$ is defined analogously.

Consider solutions to the time-harmonic Maxwell’s equations with $\exp(-i\omega t)$ time dependence. Let $E_\alpha$ be the electric field in the presence of the inhomogeneities. It solves the Helmholtz equation

$$\nabla \cdot \left( \frac{1}{\mu_\alpha} \nabla E_\alpha \right) + \omega^2 \varepsilon_\alpha E_\alpha = 0 \quad \text{in} \ \Omega,$$

with the boundary condition $E_\alpha = f$ on $\partial \Omega$, where $\omega > 0$ is a given frequency. The electric field, $E_0$, in the absence of any inhomogeneities, satisfies the following equation:

$$\Delta E_0 + k^2 E_0 = 0 \quad \text{in} \ \Omega, \tag{3}$$

where $k^2 = \omega^2 \mu_0 \varepsilon_0$, with $E_0 = f$ on $\partial \Omega$. In order to insure well-posedness (also for the $\alpha$-dependent case for $\alpha$ sufficiently small) we shall assume that $k^2$ is not an eigenvalue for the operator $-\Delta$ in $L^2(\Omega)$ with the Dirichlet boundary conditions. It has been shown in [1] that the following asymptotic formula holds uniformly on $\partial \Omega$

$$\frac{\partial E_\alpha}{\partial \nu}(x) - \frac{\partial E_0}{\partial \nu}(x) - 2 \int_{\partial \Omega} \left( \frac{\partial E_\alpha}{\partial \nu} - \frac{\partial E_0}{\partial \nu} \right)(y) \frac{\partial G(x, y)}{\partial \nu} \, ds(y)$$

$$= 2\alpha^d \sum_{j=1}^{m} \left( 1 - \frac{\mu_0}{\mu_j} \right) \nabla_y \frac{\partial G(x, z_j)}{\partial \nu(x)} \cdot M_j (\frac{\mu_j}{\mu_0}) \nabla E_0(z_j)$$

$$- 2\alpha^d k^2 \sum_{j=1}^{m} \left( 1 - \frac{\varepsilon_0}{\varepsilon_j} \right) \frac{\partial G(x, z_j)}{\partial \nu(x)} |B_j| E_0(z_j) + o(\alpha^d), \tag{4}$$

where the remainder $o(\alpha^d)$ is independent of the set of points $\{z_j\}_{j=1}^{m}$ provided that [1] holds, $G(x, y)$ is a free space Green’s function for $\Delta + k^2$, and each $M_j$ is a $d \times d$, symmetric, positive definite matrix associated with the $j$-th inhomogeneity, called the polarizability tensor, which is given by

$$(M_j)_{ij} = |B_j| \delta_{ij} + \left( \frac{\mu_j}{\mu_0} - 1 \right) \int_{\partial B_j} \frac{\partial \phi_i^j}{\partial \nu_j} \, d\sigma_y, \tag{5}$$
where, for \( 1 \leq l' \leq d \), \( \phi_{l'}(y) \) is the unique function which satisfies

\[
\begin{cases}
\Delta \phi_{l'} = 0 & \text{in } B_j \text{ and } \mathbb{R}^d \setminus \overline{B_j}, \\
\frac{1}{\mu_0} \frac{\partial \phi_{l'}}{\partial v_j} - \frac{1}{\mu_j} \frac{\partial \phi_{l'}}{\partial v_j} = -\frac{1}{\mu_j} \mu_j \cdot e_{l'} & \text{on } \partial B_j,
\end{cases}
\]

with \( \phi_{l'} \) continuous across \( \partial B_j \) and \( \lim_{|y| \to \infty} \phi_{l'}(y) = 0 \). Here \( \{e_{l'}\}_{l'=1}^d \) is an orthonormal basis of \( \mathbb{R}^d \), \( \nu_j \) denotes the outward unit normal to \( \partial B_j \), superscripts \( - \) and \( + \) indicate the limiting values as the point approaches \( \partial B_j \) from outside \( B_j \), and from inside \( B_j \), respectively.

Our goal is to identify the locations \( \{z_j\}_{j=1}^m \) and the polarizability tensors \( \{M_j\}_{j=1}^m \) of the small inhomogeneities \( \mathcal{B}_a \) from the boundary measurements of \( \frac{\partial \mathcal{E}_a}{\partial v} \) on a given part \( \Gamma_1 \subset \subset \partial \Omega \). This inverse problem is more complicated from the mathematical point of view and more interesting in applications than the one solved in [1], because in many applications one cannot get measurements on the whole boundary. As in [1], we want to reduce this reconstruction problem to the calculation of an inverse Fourier transform. Our work is a generalization of [1] to the case of the reconstruction of small inhomogeneities from the measurements on a part of the boundary.

In [4] a different method is proposed for finding small inhomogeneities from the scattering data. In [5] it is proved that the singular (delta-type) potentials are uniquely determined by the scattering data. In [6] a numerical realization of the method proposed in [10] is given. In [7] a method for finding small inhomogeneities from tomographic data is given. In [8] analytic formulas are derived for electric and magnetic polarizability tensors for bodies of arbitrary shapes. In [9] a number of multi-dimensional inverse scattering problems are studied and some of the methods developed were related to our approach.

## 2 Identification procedure

Before describing our identification procedure, let us introduce the sets \( N(\Omega) = \{ v : v \in H^1(\Omega) \cap H^2(\Omega), \Delta v + k^2 v = 0 \text{ in } \Omega \} \) and \( \mathcal{N}(\Omega) = \{ v : v \in H^1(\Omega) \cap H^2(\Omega), \Delta v + k^2 v = 0 \text{ in } \Omega, v = 0 \text{ on } \Gamma_2 \} \), where \( \Gamma_2 = \partial \Omega \setminus \Gamma_1 \), where \( \Gamma_1 \) is an open in \( \partial \Omega \) subset.

The general approach we use to recover the locations and the polarizability tensors of the small inhomogeneities is to integrate the solution \( \mathcal{E}_a \) against special test functions in the set \( \mathcal{N}(\Omega) \). Let \( v \) be any function in \( \mathcal{N}(\Omega) \). As in [10], the following estimate can be derived from [10]:

\[
\int_{\Gamma_1} \frac{\partial \mathcal{E}_a}{\partial v} v \ ds - \int_{\partial \Omega} \frac{\partial v}{\partial v} \mathcal{E}_a \ ds = \alpha^d \sum_{j=1}^m (1 - \mu_j) \mu_0 \nabla \mathcal{E}_0(z_j) \cdot M_j \mu_0 \nabla v(z_j)
\]

\[
\quad \quad \quad + \alpha^d k^2 \sum_{j=1}^m (1 - \frac{\varepsilon_j}{\varepsilon_0}) |B_j| \mathcal{E}_0(z_j) v(z_j) + o(\alpha^d),
\]

where \( |B_j| \) stands for the volume of the set \( B_j \). We want to make suitable choices for the test functions \( v \) in \( \mathcal{N}(\Omega) \) and the boundary condition \( \mathcal{E}_a |_{\partial \Omega} \) in order to get simple equations for the unknown parameters, namely, for the points \( \{z_j\}_{j=1}^m \) and matrices \( \{M_j\}_{j=1}^m \). Similar idea was used and the associated numerical experiments have been successfully conducted in the case of the conductivity problem [10] with boundary measurements on all of \( \partial \Omega \).

Let us describe our inversion method. Take \( \eta \) to be a vector in \( \mathbb{R}^d \), \( \eta^+ \) a unit vector in \( \mathbb{R}^d \) which is orthogonal to \( \eta \), and \( \gamma \) a complex number. Then \( e^{i(\eta^+ + \gamma^+) \cdot x} \) is a solution to the Helmholtz equation in \( \mathbb{R}^d \) if and only if \( \gamma^2 = k^2 - |\eta|^2 \) and in this case \( e^{i(\eta - \gamma \eta^+) \cdot x} \) is also a solution to the Helmholtz equation in \( \mathbb{R}^d \). For simplicity, let us consider the case where all the \( B_j \) are balls. In this case all the matrices \( M_j \) are multiples of the identity matrix which makes our analysis simpler.
If \( \frac{\partial E_\alpha}{\partial \nu} \) is known on the whole boundary \( \partial \Omega \) then taking \( E_\alpha = e^{i(\eta - \gamma \eta^+) \cdot x} \) on \( \partial \Omega \) we know from Lemma 2.1 that

\[
\int_{\partial \Omega} \frac{\partial E_\alpha}{\partial \nu}(y)e^{i(\eta - \gamma \eta^+) \cdot y} \, ds(y) - \int_{\partial \Omega} \frac{\partial}{\partial \nu}e^{i(\eta - \gamma \eta^+) \cdot y}E_\alpha(y) \, ds(y) = \alpha^2 \sum_{j=1}^m e^{2i\eta \cdot x} \left( 1 - \frac{\mu_j^2}{\mu_0^2} \right) M_j \left( \frac{\mu_j^2}{\mu_0^2} \right)(2|\eta|^2 - k^2) + k^2(1 - \frac{\eta^2}{\mu_0^2})|B_j| + o(\alpha^d).
\]

The main difficulty in generalizing this approach to the case when \( \frac{\partial E_\alpha}{\partial \nu} \) is known only on a part \( \Gamma_1 \subset \partial \Omega \) is to construct a function \( w_\alpha(x) \) in \( \tilde{N}(\Omega) \), that is asymptotically \( e^{i(\eta - \gamma \eta^+) \cdot x} \) as \( \alpha \) approaches 0. The following lemma holds.

**Lemma 2.1** Let \( \Omega' \subset \subset \Omega \) be a \( C^2 \)-domain. Let \( \eta \in \mathbb{R}^d \) and \( \eta^\perp \) be a unit vector in \( \mathbb{R}^d \) that is orthogonal to \( \eta \). There exists \( w_\alpha \in \tilde{N}(\Omega) \) such that

\[
w_\alpha(x) = e^{i(\eta - \gamma \eta^+) \cdot x} + o(\alpha^d) \quad \text{and} \quad \nabla w_\alpha(x) = i(\eta - \gamma \eta^+)e^{i(\eta - \gamma \eta^+) \cdot x} + o(\alpha^d),
\]

uniformly in \( \Omega' \).

This lemma is an immediate corollary of the following general density result.

**Proposition 2.1** The set \( \tilde{N}(\Omega) \) is dense, in the \( L^2(\Omega') \) norm, in the set \( N(\Omega) \).

**Proof.** Assume the contrary and let \( v \in N(\Omega) \) be an element which cannot be approximated in \( L^2(\Omega') \) by the functions from \( \tilde{N}(\Omega) \) with a prescribed accuracy. Then there is an element in \( N(\Omega) \), which we denote again \( v \), such that \( \int_{\Omega'} v w dx = 0, \forall w \in \tilde{N}(\Omega) \). Let \( G_0 \) be the Dirichlet Green’s function in \( \Omega \):

\[
\begin{cases}
\Delta G_0 + k^2 G_0 = \delta_y(x) & \text{in } \Omega, \\
G_0 = 0 \text{ on } \partial \Omega.
\end{cases}
\]

Define \( \tilde{H}^{3/2}(\Gamma_1) = \{ p \in H^{3/2}(\Gamma_1) \text{ such that there exists } \tilde{p} \in H^{3/2}(\partial \Omega), \tilde{p}|_{\Gamma_2} = 0, \tilde{p}|_{\Gamma_1} = p \} \). Since any \( w \in \tilde{N}(\Omega) \) can be represented as follows

\[
w(x) = \int_{\Gamma_1} \frac{\partial G_0(x, y)}{\partial \nu(y)} p(y) \, ds(y), x \in \Omega,
\]

where \( p \in \tilde{H}^{3/2}(\Gamma_1) \) is arbitrary, we have

\[
\int_{\Omega'} v(y) \frac{\partial G_0(x, y)}{\partial \nu(x)} \, dy = 0, \forall x \in \Gamma_1.
\]

Define \( u(x) := \int_{\Omega'} v(y)G_0(x, y) \, dy \). Then \( u|_{\partial \Omega} = 0, \frac{\partial u}{\partial \nu}|_{\Gamma_1} = 0 \), and

\[
\Delta u + k^2 u = \begin{cases} v & \text{in } \Omega', \\
0 & \text{in } \Omega \setminus \Omega'.
\end{cases}
\]

By the uniqueness of the solution to the Cauchy problem for the Helmholtz equation, it follows that \( u = 0 \) in \( \Omega \setminus \Omega' \), and so \( u = \frac{\partial u}{\partial \nu} = 0 \) on \( \partial \Omega' \). Since \( \Delta u + k^2 u = v \) in \( \Omega' \), it follows by multiplying this equation by \( v \) and integrating by parts over \( \partial \Omega' \) that \( \int_{\Omega'} v^2 dx = 0 \). Thus, \( v = 0 \) in \( \Omega' \), and, by the unique continuation, \( v = 0 \) in \( \Omega \), which proves Proposition 2.1. In the above argument we assumed that \( u \) and \( v \) are real valued. This can be done without loss of generality since \( k^2 > 0 \).

From Proposition 2.1 it follows that the function \( w_\alpha \) in Lemma 2.1 can be constructed (numerically) by solving the minimization problem:

\[
\min_{p \in \tilde{H}^{3/2}(\Gamma_1)} \| \int_{\Gamma_1} \frac{\partial G_0(x, y)}{\partial \nu(y)} p(y) \, ds(y) - e^{i(\eta - \gamma \eta^+) \cdot x} \|_{L^2(\Omega')}.
\]
We can take
\[ w_\alpha(x) = \int_{\Gamma_1} \frac{\partial G_0(x, y)}{\partial \nu(y)} p_\alpha(y) \, ds(y), \]
where \( p_\alpha \) is any function in \( \tilde{H}^{1/2}(\Gamma_1) \) such that
\[ \| \int_{\Gamma_1} \frac{\partial G_0(x, y)}{\partial \nu(y)} p_\alpha(y) \, ds(y) - e^{i(\eta - \gamma \eta^\perp) \cdot x} \|_{L^2(\Omega')} = o(\alpha^d). \]  

(8)

Since \((\Delta + k^2)(w_\alpha - e^{i(\eta - \gamma \eta^\perp) \cdot x}) = 0\) in \( \Omega \), by the standard elliptic interior estimates [1], we obtain from (8) that \( \| w_\alpha - e^{i(\eta - \gamma \eta^\perp) \cdot x} \|_{H^s(\Omega')} \) is of order \( o(\alpha^d) \) for any \( s \) and any \( C^2 \)-domain \( \Omega'' \subset \subset \Omega' \). Lemma 2.1 follows then immediately from the Sobolev imbedding theorems.

Now, if we choose \( E_\alpha = e^{i(\eta + \gamma \eta^\perp) \cdot x} \) on \( \partial \Omega \) and \( v = w_\alpha \) in \( \Omega \) then, since the points \( \{ z_j \}_{j=1}^m \) are away from the boundary \( \partial \Omega \), it follows from (6) and Lemma 2.2 that the following asymptotic expansion holds:
\[
\Lambda_\alpha(\eta) = \int_{\Gamma_1} \frac{\partial E_\alpha}{\partial \nu} w_\alpha \, ds - \int_{\partial \Omega} \frac{\partial w_\alpha}{\partial \nu} E_\alpha \, ds = o(d^m \sum_{j=1}^m e^{2i\eta \cdot z_j} \left( 1 - \frac{\mu_j}{\mu_0} \right) M_j \left( \frac{\beta_j}{\mu_0} \right) (2|\eta|^2 - k^2) + k^2 (1 - \frac{\xi_j}{\xi_0}) |B_j| \right) + o(\alpha^d). \]  

(9)

Recall that \( e^{2i\eta \cdot z_j} \) (up to a multiplicative constant) is the Fourier transform of the Dirac function \( \delta_{-2z_j} \) (a point mass located at \( -2z_j \)). Multiplication by powers of \( \eta \) in the Fourier space corresponds to differentiation of the Dirac function. Therefore, the function \( \Lambda_\alpha(\eta) \) is the inverse Fourier transform of a distribution supported at the points \( z_j \). A numerical Fourier inversion of a sample of \( \Lambda_\alpha(\eta) \) will yield the points \( z_j \) with a small error as \( \alpha \to 0 \). It is natural to use a fast Fourier transform for this inversion. One can estimate the number of the sampling points needed for an accurate discrete Fourier inversion using the Shannon’s theorem [3], page 18. This number is of order \( \left( \frac{h}{\delta} \right)^3 \). One needs this amount of sampled values of \( \eta \) to reconstruct, with resolution \( \delta \), a collection of inhomogeneities that lie inside a square of side \( h \). Once the points \( \{ z_j \}_{j=1}^m \) are found, one may find \( \{ M_j \}_{j=1}^m \) by solving appropriate linear system arising from the asymptotic formula (6). If \( B_j \) are general domains, our calculations become more complicated, and eventually we have to deal with pseudo-differential operators (independent of the space variable \( x \)) applied to the same Dirac functions. In view of the asymptotic results derived in [2] a similar approach may be applied to the full Maxwell equations in the presence of small dielectric inhomogeneities.

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References

[1] H. Ammari, S. Moskow, and M. Vogelius, **Boundary integral formulas for the reconstruction of electromagnetic imperfections of small diameter**, to appear in ESAIM: Cont. Opt. Calc. Var.

[2] H. Ammari, M. Vogelius, and D. Volkov, **Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of imperfections of small diameter II. The full Maxwell equations**, to appear in J. Math. Pures Appl.

[3] I. Daubechies, **Ten Lectures on Wavelets**, SIAM, Philadelphia, 1992.
[4] L. Desbat and A. G. Ramm, *Finding small objects from tomographic data*, Inverse Problems 13 (1997), 1239-1246.

[5] F. Gesztesy and A. G. Ramm, *An inverse problem for point inhomogeneities*, Methods of Functional Analysis and Topology 6 (2000), 1-12.

[6] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1984.

[7] S. Gutman and A. G. Ramm, *Application of the hybrid stochastic-deterministic minimization method to a surface data inverse scattering problem*, in: Operator Theory and its Applications, editors A. G. Ramm, P. N. Shivakumar, and A. V. Strauss, Amer. Math. Soc., Fields Institute Communications vol. 25, Providence, RI, 2000, 293-304.

[8] A. G. Ramm, *Iterative methods for calculating static fields and wave scattering by small bodies*, Springer-Verlag, New York 1992.

[9] A. G. Ramm, *Multidimensional inverse scattering problems*, Longman/Wiley, New York, 1992, 1-385.

[10] A. G. Ramm, *Finding small inhomogeneities from surface scattering data*, J. Inverse Ill-Posed Problems 8 (2000), 205-210.

[11] M. Vogelius and D. Volkov, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of inhomogeneities*, Math. Model. Numer. Anal. 34 (2000), 723-748.