Extremality of Bi-invariant Metrics on Lie Groups and Homogeneous Spaces

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Abstract

Gromov asked if the bi-invariant metric on an $n$ dimensional compact Lie group is extremal compared to any other metrics. In this note, we prove that the bi-invariant metric on an $n$ dimensional compact connected semi-simple Lie group $G$ is extremal in the sense of Gromov when compared to the left invariant metrics. In fact the same result holds for a compact connected homogeneous Riemannian manifold $G/H$ with the Lie algebra of $G$ having trivial center.

Key words: extremal metric, Lie group, homogeneous spaces, scalar curvature

1 Introduction

In [3], Gromov asks: are bi-invariant metrics on compact Lie groups extremal (This is already problematic for $SO(5)$)? Here a metric $g$ on a Riemannian manifold $M$ is extremal in the sense of Gromov (not to be confused with Calabi’s extremal metrics in Kähler geometry) if any metric $g'$ on $M$ with $g' \geq g$ must have smaller scalar curvature $R(g')$ at some point of the manifold. More precisely, $g' \geq g$ and $R(g') \geq R(g)$ implies that $g' = g$.

The first result of this type is [6] in which Llarull showed that the standard metric on $S^n$ is extremal. The work gives a positive answer to an earlier question of Gromov, which is motivated by Gromov-Lawson’s famous work on the non-existence of positive scalar curvature metrics on the torus [4], later extended to more general class of manifolds, namely the enlargeable manifolds. In the same spirit, Llarull in fact proved that a metric on a compact manifold admitting a $(1, \Lambda^2)$-contracting map to $S^n$ is extremal. Min-Oo [9] proved that hermitian symmetric spaces of compact type are extremal. The extremality of complex and quaternionic projective spaces is established by Kramer [5]. Later, in 2002, S. Goette and U. Semmelmann prove that compact symmetric spaces of type $G/K, rk(G) - rk(K) \leq 1$ are extremal [2]. Then Mario Listing improves Goette-Semmelmann’s result in [7], by weakening the extremality condition.

Note that a Lie group with a bi-invariant metric is a symmetric space, but not of the types considered above. In this short note, we present a partial positive answer to Gromov’s question. Namely we show that the bi-invariant metric on an $n$ dimensional compact connected semi-simple Lie group $G$ is extremal when compared to the left invariant metrics. In fact, as pointed out by Wolfgang Ziller, the same result holds for any compact homogeneous spaces without torus factor.

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According to [8], if a connected Lie group admits a bi-invariant metric, it is isomorphic to the product of a compact Lie group with an abelian one. Being semi-simple rules out the abelian factor. On the other hand, we have the famous result of Gromov-Lawson [4] and Schoen-Yau [12, 11] which implies that the only metrics of nonnegative scalar curvatures on the torus are the flat ones.

Acknowledgment: We are deeply grateful to Wolfgang Ziller who pointed the more general result for the homogeneous space as well as bringing the work [13] to our attention, which considerably simplifies our previous computation as well as generalizes to the more general case of homogeneous spaces. We thank Wolfgang for many helpful discussions. Thanks are also due to Professor Yurii Nikonorov for similar remarks and useful comments.

2 Preliminaries

Given a Riemannian manifold \((M, g)\), we denote by \(R_g\) the scalar curvature of \(g\). We recall Gromov's notion of extremal metrics.

**Definition 1.** A \(g\) on \(M\) is extremal (in the sense of Gromov), if any metric \(g_1\) on \(M\) satisfying \(g_1 \geq g\) and \(R_{g_1} \geq R_g\) must be \(g\) itself, \(g_1 = g\).

For a Lie group \(G\), and \(a \in G\), we denote by \(L_a : G \to G; L_a b = a \cdot b\), and \(R_a : G \to G; R_a b = b \cdot a\) the left action and the right action, respectively. Conjugation by \(a\) induces the adjoint action on the Lie algebra \(\mathfrak{g}\), \(Ad(a) : \mathfrak{g} \to \mathfrak{g}\). If a metric on \(G\) is both left invariant and right invariant, then it is called bi-invariant. If \(G\) is compact, bi-invariant metrics always exist.

There is also the adjoint action \(\text{ad}(X) : \mathfrak{g} \to \mathfrak{g}\), \(X \in \mathfrak{g}\), induced by the adjoint action of \(G\). In fact,

\[
\text{ad}(X)Y = [X, Y], \quad X, Y \in \mathfrak{g}.
\]

Now left invariant metrics on \(G\) are in one-to-one correspondence with inner products on its Lie algebra \(\mathfrak{g}\).

**Theorem 1** ([8] Lemma 7.2). In the case of a connected group \(G\), a left invariant metric is actually bi-invariant if and only if the linear transformation \(\text{ad}(X)\) is skew-adjoint for every \(X\) in the Lie algebra \(\mathfrak{g}\) of \(G\).

**Definition 2.** A Lie group \(G\) is semi-simple if its Lie algebra \(\mathfrak{g}\) is semi-simple, i.e., its Killing form \(K(X, Y) = \text{Tr}(\text{ad}(X)\text{ad}(Y)), \quad X, Y \in \mathfrak{g}\) is nondegenerate.

Clearly, if \(\mathfrak{g}\) is semi-simple, it has a trivial center.

If \(M\) is a compact connected Riemannian homogeneous space, then \(M = G/H\) for a compact connected Lie group \(G\) and a closed subgroup \(H\). The left action of \(G\) extends to \(G/H\) and \(G\) acts effectively on \(M = G/H\).

Let \(\mathfrak{h} \subset \mathfrak{g}\) be the Lie algebra of \(H\). Denote by \(Ad_G\) the adjoint action of \(G\) on \(\mathfrak{g}\) and \(Ad_H\) its restriction to \(H\). Since \(Ad_H\) preserves \(\mathfrak{h}\), it induces an action on \(\mathfrak{g}/\mathfrak{h}\). A metric \(g\) on \(M = G/H\) is called \(G\)-invariant if it is invariant under the left action of \(G\). \(G\)-invariant metrics on \(G/H\) are naturally identified with inner products on \(\mathfrak{g}/\mathfrak{h}\) which are invariant under the \(Ad_H\) action, see, e.g., [1]. In particular, a bi-invariant metric on \(G\) gives rise to a \(G\)-invariant metric on \(G/H\), which we will continue to call it a bi-invariant metric on \(G/H\).

**Definition 3.** We say a metric \(g\) on \(G/H\) is extremal compared to the \(G\) invariant metrics if for any \(G\) invariant metric \(g_1, g_1 \geq g\) and \(R_{g_1} \geq R_g\) imply that \(g_1 = g\).
3 The Theorem and its Proof

Our main result is

**Theorem 2.** Let \( M = G/H \) be an \( n \) dimensional compact Riemannian homogeneous space, with \( G \) a compact connected Lie group. Assume that the Lie algebra \( \mathfrak{g} \) of \( G \) has trivial center. Then any bi-invariant metric \( g_0 \) on \( G/H \) is extremal when compared to the \( G \) invariant metrics.

Since the Lie algebra of a semi-simple Lie group has trivial center, we have

**Corollary 3.** If \( G \) is a compact connected Lie group whose Lie algebra \( \mathfrak{g} \) has trivial center, then any bi-invariant metric on \( G \) is extremal when compared to the left invariant metrics. In particular, this holds for any compact connected semi-simple Lie group.

Our proof relies crucially on a simple elegant formula for the scalar curvature for \( G \)-invariant metrics, as well as another lemma, in \([13]\). We first recall this formula and the setup.

Let \( g_0 \) be a bi-invariant metric on \( G \); still denote by \( g_0 \) the induced metric on \( G/H \). Let \( g = h + m \) be an \( \text{Ad}_H \)-invariant decomposition orthogonal with respect to \( g_0 \). Then \( G \)-invariant metrics on \( G/H \) are identified with \( \text{Ad}_H \)-invariant inner products on \( m \).

Let \( \langle X, Y \rangle_0 \) be the \( \text{Ad}_H \)-invariant inner product on \( m \) corresponding to the bi-invariant metric \( g_0 \) and \( \langle X, Y \rangle \) the \( \text{Ad}_H \)-invariant inner product on \( m \) for a \( G \)-invariant metric \( g \). Then

\[
\langle X, Y \rangle = \langle S(X), Y \rangle_0
\]

for \( X, Y \in m \) and \( S \) a self-adjoint operator on \((m, \langle \cdot, \cdot \rangle_0)\) commuting with the \( \text{Ad}_H \)-action.

It follows then that there is an \( \text{Ad}_H \)-invariant decomposition

\[
m = m_1 + \cdots + m_s
\]

orthogonal with respect to \( g_0 \), such that

\[
\langle X, Y \rangle = \lambda_1 \langle X, Y \rangle_0 |_{m_1} + \cdots + \lambda_s \langle X, Y \rangle_0 |_{m_s}, \quad \lambda_i > 0.
\]

Here \( m_i \) is \( \text{Ad}_H \)-irreducible space coming from further decomposition of the eigenspace of \( S \) with eigenvalue \( \lambda_i \), \( i = 1, \cdots, s \). That is, the metric \( g \) is a so-called “diagonal” metric in \([12]\).

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Let \( B \) be the negative of the Killing form \( B(X, Y) = -K(X, Y) \). Then \( B(X, X) \geq 0 \), with equality iff \( X \) is central. The following formula is from \([13]\) , equation (1.3).

**Lemma 4** (Wang-Ziller). Let \( g \) be an \( G \)-invariant metric on \( G/H \) and the decomposition 1 as before. Let \( b_i \geq 0 \) be defined by \( B(X, Y) = b_i \langle X, Y \rangle_0 \), \( X, Y \in m_i \). Then the scalar curvature of \( g \) is given by

\[
R_g = \frac{1}{2} \sum_{i=1}^s b_i \frac{\lambda_i}{\lambda_i^2} - \frac{1}{4} \sum_{i,j,k=1}^s A_{ij}^k \frac{\lambda_i}{\lambda_i \lambda_j}.
\]

(2)
The following lemma from [13] relates $b_i d_i$ to the structural constants.

**Lemma 5** (Wang-Ziller). One has

$$\sum_{i,j,k} A^k_{ij} = b_i d_i - 2c_i d_i,$$

where $c_i \geq 0$ is defined in terms of the Casimir operator via $C_{m|m_i} = c_i Id$.

**Proof of theorem 2.** Since $\{E_n\}$ is an orthonormal basis for $\langle X, Y \rangle_0$, and $\langle X, Y \rangle_0$ is bi-invariant, $C_{\alpha\beta}$ is skew-symmetric in all three indices by Theorem 1. Hence $(A^k_{ij})$ is symmetric in all three indices. By the first lemma above,

$$R_g = \frac{1}{2} \sum_{i=1}^s b_i d_i - \frac{1}{4} \sum_{i,j,k=1}^s A^k_{ij} \frac{\lambda_k}{\lambda_i \lambda_j}$$

and

$$R_{g_0} = \frac{1}{2} \sum_{i=1}^s b_i d_i - \frac{1}{4} \sum_{i,j,k=1}^s A^k_{ij}.$$

Now the extremal conditions $\langle X, Y \rangle \geq \langle X, Y \rangle_0$ and $R_g \geq R_{g_0}$ yield $\lambda_i \geq 1(i = 1, \cdots, s)$ as well as $R_g - R_{g_0} \geq 0$. Then

$$0 \leq R_g - R_{g_0} \leq \frac{1}{2} \sum_{i=1}^s b_i d_i - \frac{1}{4} \sum_{i,j,k=1}^s A^k_{ij} \left( \frac{\lambda_k}{\lambda_i \lambda_j} - 1 \right)$$

$$= \sum_{i=1}^s c_i d_i (1 - \lambda_i) - \frac{1}{4} \sum_{i,j,k=1}^s A^k_{ij} \left[ \frac{\lambda_k}{\lambda_i \lambda_j} - 1 + 2 \frac{\lambda_i - 1}{\lambda_i} \right]$$

using the second lemma.

Since $c_i \geq 0$, each summand in the first summation is less than or equal to zero, with equality iff either $c_i = 0$ or $\lambda_i = 1$. For the second summation, we use the symmetry to rewrite it as

$$-\frac{1}{12} \sum_{i,j,k} A^k_{ij} \left[ \frac{\lambda_k}{\lambda_i \lambda_j} + 2 \frac{\lambda_i - 1}{\lambda_i} + \frac{\lambda_i}{\lambda_k \lambda_j} + 2 \frac{\lambda_i - 1}{\lambda_k} + \frac{\lambda_j}{\lambda_i \lambda_k} + 2 \frac{\lambda_j - 1}{\lambda_j} - 3 \right]$$

$$= -\frac{1}{12} \sum_{i,j,k} A^k_{ij} \frac{\lambda^2_i + \lambda^2_j + \lambda^2_k - 2\lambda_i \lambda_j - 2\lambda_i \lambda_k - 2\lambda_k \lambda_j + 3\lambda_i \lambda_j \lambda_k}{\lambda_i \lambda_j \lambda_k}.$$

For distinct $i, j, k$ we consider the order of $\lambda_i, \lambda_j, \lambda_k$. Without loss of generality we can assume that $\lambda_k \geq \lambda_j \geq \lambda_i \geq 1$. Then the summand in the sum above can be re-organized as

$$\lambda^2_i + \lambda^2_j + \lambda^2_k - 2\lambda_i \lambda_j - 2\lambda_i \lambda_k - 2\lambda_k \lambda_j + 3\lambda_i \lambda_j \lambda_k$$

$$= (\lambda_i - \lambda_j)^2 + (\lambda_k - \lambda_j)^2 + \lambda_j \lambda_k (\lambda_i - 1) + \lambda_j (\lambda_k - \lambda_j) + 2\lambda_i \lambda_k (\lambda_j - 1)$$

$$\geq 0$$

with equality iff $\lambda_k = \lambda_j = \lambda_i = 1$. But then all the inequalities become equalities, which implies that, either $c_i = 0$ or $\lambda_i = 1$, and either $A^k_{ij} = 0$ or $\lambda_k = \lambda_j = \lambda_i = 1$. If for some $i$, $\lambda_i > 1$, then $c_i = 0$, and $A^k_{ij} = 0$ for all $j, k$. By the second lemma again, we have $b_i = 0$. Hence $m_i$ is in the center of $g$, see [13], p181. In other words, if $g$ has trivial center, we must have $\lambda_i = 1$ for $i = 1, \cdots, n$. The result follows. □
Remark 6. In fact, from the proof, we see that if a bi-invariant metric $g_0$ on $G/H$ is not extremal, then $G/H$ must have a torus factor. Indeed, let $\mathfrak{z} \subset \mathfrak{g}$ be the center. If for some $i$, $\lambda_i > 1$, then $\mathfrak{m}_i \subset \mathfrak{z}$. Decompose $\mathfrak{g} = \mathfrak{z} + \mathfrak{g}'$ and $\mathfrak{z} = \mathfrak{m}_i + \mathfrak{k}$. Then $\mathfrak{h} \subset \mathfrak{k} + \mathfrak{g}'$. It follows then that $G/H = T^d \times (K \times G')/H$. We thank Wolfgang Ziller for pointing this out to us.

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