FURTHER REFINEMENT OF SELF-NORMALIZED CRAMÉR-TYPE MODERATE DEVIATIONS*

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Abstract. In this paper, we study the self-normalized Cramér-type moderate deviations for centered independent random variables $X_1, X_2, \ldots$ with $0 < E|X_i|^3 < \infty$. The main results refine Theorems 1.1 and 1.2 of Wang \cite{Wang2011}, the Berry–Essen bound (2.11) and Corollaries 2.2 and 2.3 of Jing, et al. \cite{Jing2003} under stronger moment conditions.

Mathematics Subject Classification. 60F10, 62E20.

Received January 29, 2015. Revised September 6, 2016. Accepted May 12, 2017.

1. Introduction

Let $X_1, X_2, \ldots$ be independent random variables with $EX_i = 0$ and $0 < EX_i^2 < \infty$. Set

\[ S_n = \sum_{i=1}^{n} X_i, \quad V_n^2 = \sum_{i=1}^{n} X_i^2, \quad \text{and} \quad B_n^2 = \sum_{i=1}^{n} EX_i^2. \]

The last two decades have witnessed a significant development on the limit theorems in the self-normalized form $S_n/V_n$, including central limit theorem, weak invariance principle, law of the iterated logarithm, Berry–Essen inequality, large and moderate deviation probabilities. The last two in this list are the main approaches for estimating the error of the normal approximation of the self-normalized probabilities. One advantage of these self-normalized limit theorems is that they usually require less moment conditions than those for the corresponding regular limit theorems. An incomplete list of reference includes Griffin and Kuelbs \cite{Griffin1995, Griffin1996}, Giné, Götze and Mason \cite{Gine1991}, Shao \cite{Shao1997, Shao2000}, Wang and Jing \cite{Wang2001}, Csörgő, Szyszkowicz and Wang \cite{Csorgo1997}, Jing, Shao and Wang \cite{Jing2001}, Jing, Shao and Zhou \cite{Jing2002}, Robinson and Wang \cite{Robinson2003}, Wang \cite{Wang2004, Wang2005}, the survey papers of Shao \cite{Shao2004, Shao2006} and Shao and Wang \cite{Shao2007}. A systematic treatment of self-normalized limit theory is also collected in the book by de la Peña, Lai and Shao \cite{de-la-Pena1999}.

Keywords and phrases. Cramér-type moderate deviations, self-normalized sums, normal approximation.

* This paper is dedicated to the memory of Evarist Giné.

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The focus of this paper is on the self-normalized Cramér-type moderate deviations. Let \( b = x/B_n \) and \( \tau = B_n/\max\{1, x\} \) with \( x \geq 0 \). Set

\[
L_{kn} = B_n^{-k} \sum_{i=1}^{n} E X_i^k \quad \text{and} \quad \bar{L}_{kn} = B_n^{-k} \sum_{i=1}^{n} E X_i^k I(|bX_i| \leq 1), \quad k \geq 2,
\]

\[
\mathcal{L}_{kn} = B_n^{-k} \sum_{i=1}^{n} E|X_i|^k \quad \text{and} \quad \bar{\mathcal{L}}_{kn} = B_n^{-k} \sum_{i=1}^{n} E|X_i|^k I(b|X_i| \leq 1), \quad k \geq 2,
\]

\[
\Delta_{n,x}^{k,j} = \tau^{-j} \sum_{i=1}^{n} E|X_i|^j I(|X_i| \leq \tau) + \tau^{-k} \sum_{i=1}^{n} E|X_i|^k I(|X_i| \leq \tau), \quad k > j \geq 2.
\]

For the i.i.d. case, Shao [13] proved that as \( n \to \infty \),

\[
\frac{P(S_n/V_n \geq x)}{1 - \Phi(x)} \to 1, \quad \frac{P(S_n/V_n \leq -x)}{\Phi(-x)} \to 1
\]

holds uniformly for \( x \in [0, o(n^{5/(4+2d)})] \) under the conditions \( \mathbb{E}X_1 = 0 \) and \( \mathbb{E}|X_i|^{2+d} < \infty \) for \( 0 < d \leq 1 \). Here \( \Phi(x) \) is the distribution function of the standard normal random variables.

For independent random variables \( X_1, X_2, \ldots \) with \( \mathbb{E}X_i = 0, 0 < \mathbb{E}X_i^2 < \infty, x^2 \max_i \mathbb{E}X_i^2 \leq B_n^2 \), and \( \Delta_{n,x}^{2,3} \leq (1+x)^2/A \) where \( A \) is a constant sufficiently large, Jing, Shao and Wang [6] established a Cramér-type deviation result for self-normalized sums

\[
P(S_n/V_n \geq x) = e^{O(1)\Delta_{n,x}^{2,3}}.
\]

Wang [17] developed new techniques to refine the self-normalized Cramér-type deviation results and simplified the proof. Theorem 1.2 there states that if \( \mathbb{E}X_i = 0, 0 < \mathbb{E}|X_i|^3 < \infty, x\mathcal{L}_{3n} \leq 1/A, x^3 \max_i \mathbb{E}|X_i|^3 \leq B_n^{3/2}, \) and \( |c| \leq x/5 \), then

\[
P(S_n \geq xV_n + cB_n) = \tilde{\Psi}_x(\lambda_1, c)/(1 - \Phi(x + c))e^{O(1)\Delta_{n,x}^{3,4}}\{1 + O(1)(1+x)\mathcal{L}_{3n}\},
\]

where \( \tilde{\Psi}_x(t,c) = e^{m(t)+(x+c)^2/2-(x^2+2xc)} \) for \( m(t) = \sum_{i=1}^{n} \log \mathbb{E}e^{(2bX_i - b^2X_i^2)} \), and \( \lambda_1 \) is the solution of \( \lambda \) for \( m'(\lambda) = x^2 + 2cx \). His Theorem 1.1 states that if \( \mathbb{E}X_i = 0, 0 < \mathbb{E}X_i^4 < \infty \) and \( x \leq \mathcal{L}_{4n}^{-1/4} \), then

\[
P(S_n/V_n \geq x) = \tilde{\Psi}_x(\lambda_0, 0)/(1 - \Phi(x))\{1 + O(1)(1+x)\mathcal{L}_{3n} + O(1)(1+x^4)\mathcal{L}_{4n}\},
\]

where \( \lambda_0 \) is the solution of \( \lambda \) for \( m'(\lambda) = x^2 \).

In this paper, we further refine the proofs and results of Wang [17]. Section 2 gives the main results including a theorem and three corollaries. The theorem refines Theorems 1.1 and 1.2 of Wang [17] under stronger moment conditions. The corollaries refine the Berry–Essen bound (2.11) and Corollaries 2.2 and 2.3 of Jing, Shao and Wang [6] under stronger moment conditions. The proof of the theorem is also given in Section 2 using the propositions in Section 3. In Proposition 3.4, we obtain a formula for the probability \( P\{2bS_n - (\alpha_0 - \alpha_3x\bar{L}_{3n})b^2V_n^2 \geq (2 - \alpha_0)x^2 + \delta(x)\} \) where \( \alpha_0 \) and \( \alpha_3 \) are constants with different values in different situations, and \( \delta(x) \) is a function of \( x \). This generalizes the probability \( P\{2bS_n - b^2V_n^2 \geq x^2\} \) given by Jing, Shao and Wang [6] and the probability \( P\{2bS_n - b^2V_n^2 \geq x^2 + \delta(x)\} \) given by Wang [17]. This generalized probability is necessary to produce the desired accuracy in Theorem 2.1, in particular, the exponent \( -x^3L_{3n}/3 - x^4L_{4n}/12 \) in equations (2.5) and (2.6).
2. **Main Results**

**Theorem 2.1.** Let $X_1, X_2, \ldots$ be independent random variables with $EX_i = 0$ and $0 < E|X_i|^3 < \infty$. Assume that there exists a constant $1 < A < \infty$ sufficiently large such that for $x \geq 0$,

$$xL_{3n} \leq 1/A$$

(2.1)

and

$$x^2B_{n}^{-2} \max_{1 \leq i \leq n} EX_i^2 \leq 1/12.$$  

(2.2)

Then

$$P(S_n \geq xV_n) = \frac{x^3L_{3n}}{1 - \Phi(x)}\exp{-\frac{x^3L_{3n}}{3} - \frac{x^4L_{4n}}{12} + O(1)\Delta_{n,x}^{3,5}} \{1 + O(1)(1 + x)L_{3n}\}.$$  

(2.3)

Consequently,

$$P(S_n \geq xV_n) = \frac{P(S_n \geq xV_n)}{1 - \Phi(x)} = \frac{x^3L_{3n}}{3} + O(1)\Delta_{n,x}^{3,4} \{1 + O(1)(1 + x)L_{3n}\}.$$  

(2.4)

If $0 < EX_i^4 < \infty$, then

$$P(S_n \geq xV_n) = \frac{x^3L_{3n}}{1 - \Phi(x)}\exp{-\frac{x^3L_{3n}}{3} - \frac{x^4L_{4n}}{12} + O(1)\Delta_{n,x}^{4,5}} \{1 + O(1)(1 + x)L_{3n}\}.$$  

(2.5)

If $0 < E|X_i|^{4+\delta} < \infty$ with $0 < \delta \leq 1$, then for $0 \leq x \leq L_{(4+\delta)n}^{-1}$,

$$P(S_n \geq xV_n) = \exp{-\frac{x^3L_{3n}}{3} - \frac{x^4L_{4n}}{12}} \{1 + O(1)(1 + x)L_{3n} + O(1)(1 + x)^{4+\delta}L_{(4+\delta)n}\}.$$  

(2.6)

**Proof.** First we prove the lower bound. By Lemma 3.1(i), we have $x^3L_{3n}^3 \leq x^3 L_{5n} \leq \Delta_{n,x}^{3,5}/x^2$. Since $|1 - s|^{1-x} - 1 - s - s^2| \leq 2|s|^3$ for $|s| \leq 1/2$, then

$$\left|\frac{1}{1 - xL_{3n}/2} - 1 - \frac{x^3L_{3n}}{2} - \frac{x^3L_{3n}^2}{4}\right| \leq \frac{x^3L_{3n}}{4} \leq \frac{\Delta_{n,x}^{3,5}}{x^2}.$$  

(2.7)

Hence

$$P(S_n \geq xV_n) = P\left\{2bs_n - (1 - xL_{3n}/2)b^2V_n^2 \geq \frac{x^2}{1 - xL_{3n}/2}\right\}$$

$$\geq P\left\{2bs_n - (1 - xL_{3n}/2)b^2V_n^2 \geq \frac{x^2}{1 - xL_{3n}/2}\right\}$$

$$\geq P\left\{2bs_n - (1 - xL_{3n}/2)b^2V_n^2 \geq x^2 + \frac{x^3L_{3n}}{2} + \frac{x^4L_{4n}^2}{4} + \Delta_{n,x}^{3,5}\right\}.$$
Next we prove the upper bound. First we consider $0 \leq x \leq 2$. Observe that $x^4L_{4n} \leq x^3L_{3n} \leq x^2/A \leq 1$ by condition (2.1). Since $|e^s - 1| \leq e^{s|s|}$, then
\[
\left| \exp \left( -\frac{1}{3}x^3 \tilde{L}_{3n} - \frac{1}{12}x^4 \tilde{L}_{4n} \right) - 1 \right| \leq \left( \frac{1}{3} + \frac{1}{12} \right) e^{1/3x^3 \tilde{L}_{3n}} \leq 3x \tilde{L}_{3n}.
\]
Hence by condition (2.1),
\[
\exp \left( -\frac{1}{3}x^3 \tilde{L}_{3n} - \frac{1}{12}x^4 \tilde{L}_{4n} \right) \{1 + 2A(1 + x)\tilde{L}_{3n}\} \geq (1 - 3x \tilde{L}_{3n})\{1 + 2A(1 + x)\tilde{L}_{3n}\} \geq 1 + A(1 + x)\tilde{L}_{3n}. \tag{2.8}
\]
By (2.16) of Wang [17], $|P(S_n \geq xV_n) - (1 - \Phi(x))| \leq A\tilde{L}_{3n}$. Then for $0 \leq x \leq 2$,
\[
\frac{P(S_n \geq xV_n)}{1 - \Phi(x)} \leq 1 + A(1 + x)\tilde{L}_{3n}. \tag{2.9}
\]
Combining (2.8) and (2.9), we obtain the upper bound. For $x \geq 2$,
\[
P(S_n \geq xV_n) = P \left( S_n \geq xV_n, \max_{1 \leq i \leq n} |X_i| > \tau \right) + P(\tilde{S}_n \geq x\tilde{V}_n)
\]
where $\tau = B_n/x$, $\tilde{S}_n = \sum_{i=1}^n X_i I(|X_i| \leq \tau)$ and $\tilde{V}^2_n = \sum_{i=1}^n X_i^2 I(|X_i| \leq \tau)$. Page 2181 of Jing, Shao and Wang [6] shows that
\[
P \left( S_n \geq xV_n, \max_{1 \leq i \leq n} |X_i| > \tau \right) \leq \sum_{i=1}^n P \left( S^{(i)}_n (x^2 - 1)^{1/2} V^{(i)}_n \right) P(|X_i| > \tau).
\]
Since $\sum_{i=1}^n P(|X_i| > \tau) \leq \Delta_{3, n, x}$, then the upper bound follows from Propositions 3.5 and 3.3. □

Note that equations (2.5) and (2.6) refine Theorems 1.2 and 1.1 (when $c = 0$) of Wang [17], respectively, under higher moment conditions.

**Corollary 2.2.** Suppose that $\max_{1 \leq i \leq n} E|X_i|^4 \leq C$ for some $C < \infty$ and $B^2_n \geq cn$ for some $c > 0$. If $\sum_{i=1}^n E|X_i|^3 = O(n^{\gamma})$ for $0 \leq \gamma \leq 1$, then
\[
P(S_n \geq xV_n) - (1 - \Phi(x)) = O \left( \frac{x^2}{n^{3/2 - \gamma}} + \frac{x^3}{n} + \frac{1}{\sqrt{n}} \right) e^{-x^2/2}
\]
for $0 \leq x \leq \min\{O(n^{1/2 - \gamma/3}), O(n^{1/4})\}$.

**Proof.** Let $y = -x^4L_{3n}/3 - x^4L_{4n}/12 + O(1)\Delta_{4, n, x}$ in (2.5). Note that $\Delta_{4.5, n, x} \leq (1 + x^4)L_{4n}$. Then $y$ is bounded for $x = \min\{O(n^{1/2 - \gamma/3}), O(n^{1/4})\}$. Since $|e^y - 1| \leq e^{y|y|}$, then $e^y = 1 + O(1)y$. Hence $P(S_n \geq xV_n) = (1 - \Phi(x))(1 + O(1)y + O(1)(1 + x)\tilde{L}_{3n})$ by (2.5). Note that $1 - \Phi(x) \leq 2e^{-x^2/2}/(1 + x)$ for $x \geq 0$. Then $x^4L_{3n}(1 - \Phi(x)) = O(x^2/n^{3/2 - \gamma})e^{-x^2/2}$ and $x^4L_{4n}(1 - \Phi(x)) = O(x^3/n)e^{-x^2/2}$. Moreover, $(1 + x)\tilde{L}_{3n}(1 - \Phi(x)) = O(1/\sqrt{n})e^{-x^2/2}$ because $E|X_i|^5 \leq (EX_i^4)^{3/4} \leq C^{3/4}$. Therefore, the corollary follows. □

Compared with the Berry–Esseen bound (2.11) by Jing, Shao and Wang [6], this corollary shows that the bound can be lowered and the range of $x$ can be extended under stronger moment conditions. For example, if $\max_{1 \leq i \leq n} E|X_i|^3 \leq C$ and $B^2_n \geq cn$, their result shows that $P(S_n > xV_n) - (1 - \Phi(x)) = O(1)(1 + x^2)e^{-x^2/2}/\sqrt{n}$ for $x = O(n^{1/6})$. However, if $\max_{1 \leq i \leq n} E|X_i|^4 \leq C$ and $B^2_n \geq cn$, the above corollary shows that $P(S_n > xV_n) - (1 - \Phi(x)) = O(1)(1 + x)e^{-x^2/2}/\sqrt{n}$ for $\sum_{i=1}^n E|X_i|^3 = O(n^{3/4})$ and $x = O(n^{1/4})$.  

Lemma 3.1. Let
\[ \bar{X} = \frac{S}{n} \]

\[
\frac{P(S_n \geq x V_n)}{1 - \Phi(x)} \to 1
\]

uniformly for \( 0 \leq x \leq \min\{o(n^{1/2 - \gamma/3}), o(n^{1/4})\} \).

This corollary extends the range of \( x \) for Corollary 2.2 of Jing, Shao and Wang [6] where \( 0 \leq x \leq O(n^{\delta/(4+2\delta)}) \) for \( 0 < \delta < 1 \).

Corollary 2.3. Suppose that \( \max_{1 \leq i \leq n} E|X_i|^4 \leq C \) for some \( C < \infty \) and \( B_n^2 \geq cn \) for some \( c > 0 \). If \( \sum_{i=1}^n E|X_i|^3 = O(n^\gamma) \) for \( 0 \leq \gamma \leq 1 \), then by (2.5),

\[
P(S_n \geq x V_n) - 1 \to 1
\]

Note that since \( X_1, X_2, \ldots \) are i.i.d. random variables, condition (2.1) becomes \( x \leq \sqrt{n}/A \), and (2.2) becomes \( x \leq \sqrt{n}/12 \).

3. Lemmas and Propositions

From now on, all \( |O(1)| \leq A \). First we establish some preliminary facts in the following lemma.

Lemma 3.1.

(i)

\[ \bar{L}_{4n}^2 \leq \bar{L}_{3n} L_{5n}, \quad \bar{L}_{3n} \bar{L}_{4n} \leq \bar{L}_{5n} \quad \text{and} \quad \bar{L}_{3n}^3 \leq \bar{L}_{5n}. \]

(ii)

\[ x^2 \bar{L}_{4n} \leq \max\{\bar{L}_{3n}, x^4 \bar{L}_{5n}\}. \]

Proof. (i) Let \( \bar{X}_i = X_i I(|bX_i| \leq 1) \). Then \( E\bar{X}_i^4 \leq (E|\bar{X}_i|^3)^{1/2}(E|\bar{X}_i|^5)^{1/2} \). Hence

\[
\left( \sum_{i=1}^n E\bar{X}_i^4 \right)^2 \leq \sum_{i=1}^n E|\bar{X}_i|^3 \sum_{i=1}^n E|\bar{X}_i|^5
\]

or
by the Cauchy–Schwarz inequality. Therefore,
\[ \mathcal{L}_{4n}^2 \leq \mathcal{L}_{3n} \mathcal{L}_{5n}. \] (3.1)

Similarly,
\[ \mathcal{L}_{3n}^2 \leq \mathcal{L}_{2n} \mathcal{L}_{4n} \leq \mathcal{L}_{4n}. \] (3.2)

By (3.1) and (3.2), \( \mathcal{L}_{4n} \mathcal{L}_{4n} \leq \mathcal{L}_{5n} \), which, together with (3.2), implies \( \mathcal{L}_{3n} \leq \mathcal{L}_{5n} \).

(ii) If \( x^2 \mathcal{L}_{4n} \leq x^4 \mathcal{L}_{5n} \), then \( x^2 \mathcal{L}_{4n} \leq \max\{ \mathcal{L}_{3n}, x^4 \mathcal{L}_{5n} \} \). If \( x^2 \mathcal{L}_{4n} \geq x^4 \mathcal{L}_{5n} \), then \( \mathcal{L}_{2n} \leq \mathcal{L}_{3n} \mathcal{L}_{4n} \) by (3.1) and hence \( x^2 \mathcal{L}_{4n} \leq \mathcal{L}_{3n} \leq \max\{ \mathcal{L}_{3n}, x^4 \mathcal{L}_{5n} \} \).

\[ \square \]

Let \( b = x/B_n \) with \( x > 0 \). Define the function
\[ \xi(y) = 2by - (\alpha_0 - \alpha_3 x L_{3n})b^2 y^2 \]
where \( \alpha_0 \) and \( \alpha_3 \) are constants such that \( 1/3 \leq \alpha_0 - \alpha_3 x L_{3n} \leq 7/3 \). Here \( 1/3 \) and \( 7/3 \) are selected from the estimates of \( L_2 \) and \( L_3 \) in Proposition 3.5. In other situations, we only need \( \alpha_0 = 1 \) and \( \alpha_3 = 0 \) or \( 1/2 \), and thus \( \alpha_0 - \alpha_3 x L_{3n} \) is close to 1 by condition (2.1).

Denote that \( X_{i,(1)} = X_i \) and \( X_{i,(2)} = X_i I(\|bX_i\| \leq 1) \) for \( i \geq 1 \). For \( \lambda > 0 \), let \( Z_{\lambda,i,(k)} \) be independent random variables with \( Z_{\lambda,i,(k)} \) having the distribution function
\[ P(Z_{\lambda,i,(k)} \leq u) = \frac{\int_{-\infty}^{u} e^{\lambda \xi(v)} dP(X_{i,(k)} \leq v)}{E e^{\lambda \xi(X_{i,(k)})}} \] (3.3)
where \( k = 1 \) or \( 2 \). Then for any function \( f \),
\[ Ef(Z_{\lambda,i,(k)}) = \frac{\int_{-\infty}^{\infty} f(u) e^{\lambda \xi(u)} dP(X_{i,(k)} \leq u)}{E e^{\lambda \xi(X_{i,(k)})}}. \] (3.4)

In particular,
\[ Ef(Z_{\lambda,i,(k)}) = \frac{\int_{-\infty}^{\infty} \xi(u) e^{\lambda \xi(u)} dP(X_{i,(k)} \leq u)}{E e^{\lambda \xi(X_{i,(k)})}} = \frac{d \left( \log E e^{\lambda \xi(X_{i,(k)})} \right)}{d\lambda} \] (3.5)
and
\[ \text{Var}(\xi(Z_{\lambda,i,(k)})) = \frac{d^2 \left( \log E e^{\lambda \xi(X_{i,(k)})} \right)}{d\lambda^2}. \] (3.6)

**Lemma 3.2.** Let \( b = x/B_n \) with \( x > 0 \), \( X_{i,(1)} = X_i \) and \( X_{i,(2)} = X_i I(\|bX_i\| \leq 1) \). Under conditions (2.1) and (2.2), for \( 7/16 \leq \lambda \leq 9/16 \) and \( k = 1 \) or 2,
\[ \sum_{i=1}^{n} \log E e^{\lambda \xi(X_{i,(k)})} = \left( 2\lambda^2 - \lambda_0 \right) x^2 + \left( \frac{4\lambda^3}{3} - 2\lambda^2 \alpha_0 + \lambda \alpha_3 \right) x^3 L_{3n} \]
\[ + \left( \frac{2\lambda^4}{3} - 2\lambda^3 \alpha_0 + \frac{\lambda^2 \alpha^2_0}{2} \right) x^4 L_{4n} + 2\lambda^2 \alpha_3 x^4 L_{3n}^2 \]
\[ - \frac{(2\lambda^2 - \lambda_0)^2}{2} x^4 \sum_{i=1}^{n} \left( E X_i^2 \right)^2 + \frac{O(1)}{B_n^4} \Delta^{3,5}_{n,x}. \] (3.7)
Consequently,

\[
\sum_{i=1}^{n} E \xi(Z_{\lambda,i,(k)}) = (4 \lambda - \alpha_0)x^2 + (4 \lambda^2 - 4 \alpha_0 + \alpha_3)x^3 \bar{L}_3 + \\
\left( \frac{8 \lambda^3}{3} - 6 \lambda^2 \alpha_0 + \lambda \alpha_0^2 \right)x^4 \bar{L}_4 + 4 \lambda \alpha_3 x^4 \bar{E}_3 + \\
(-8 \lambda^3 + 6 \lambda^2 \alpha_0 - \lambda \alpha_0^2)x^4 \bar{E}_3 + 4 \lambda \alpha_3 x^4 \bar{E}_3 + O(1) \Delta_{n,x}^{3.5} \tag{3.8}
\]

and

\[
\sum_{i=1}^{n} \text{Var}(\xi(Z_{\lambda,i,(k)})) = 4x^2 + O(1)x^3 \bar{L}_3. \tag{3.9}
\]

Moreover, for \( m \geq 2 \) and \( |O(1)| \leq A^{1/5} \),

\[
EZ_{\lambda,i,(2)}^m = E \bar{X}_i^m + 2 \lambda b E \bar{X}_i^{m+1} + O(1)b^2 E|\bar{X}_i|^{m+2}. \tag{3.10}
\]

Proof. We follow some proofs in Lemmas 6.1 and 6.2 of Jing, Shao and Wang [6]. Let \( \gamma = 2 \lambda \) and \( \theta = \lambda(\alpha_0 - \alpha_3 x \bar{L}_3) \). Then \( \lambda \xi(X_i) = \gamma b X_i - \theta b^2 X_i^2 \). Hence

\[
E(e^{\lambda \xi(X_i)}) - 1)I(|bX_i| > 1) \leq e^{\gamma^2/(4\theta)}b E|X_i|I(|bX_i| > 1). \tag{3.11}
\]

Since \( |e^s - 1 - s - s^2/2 - s^3/6 - s^4/24| \leq |s|^5 e^{s/5})/120 \) for \( s \in R \), then

\[
E(e^{\lambda \xi(X_i)}) - 1)I(|bX_i| \leq 1) = E(\gamma b \bar{X}_i - \theta b^2 \bar{X}_i^2) + \frac{1}{2} E(\gamma b X_i - \theta b^2 X_i^2)^2 + \frac{1}{6} E(\gamma b X_i - \theta b^2 X_i^2)^3 + \\
+ \frac{1}{24} E(\gamma b X_i - \theta b^2 X_i^2)^4 + O(1) \frac{e^{\gamma^2/(4\theta)}}{120} E|\gamma b \bar{X}_i - \theta b^2 \bar{X}_i|^5
\]

\[
= \gamma b E\bar{X}_i + \left( \frac{\gamma^2}{2} - \theta \right)b^2 E\bar{X}_i^2 + \left( \frac{\gamma^3}{6} - \theta \right)b^3 E\bar{X}_i^3 + \\
+ \left( \frac{\gamma^4}{24} - \frac{\gamma^2 \theta}{2} + \frac{\theta^2}{2} \right)b^4 E\bar{X}_i^4 + O'_{\gamma, \theta}b^5 E|\bar{X}_i|^5, \tag{3.12}
\]

where

\[
|O'_{\gamma, \theta}| \leq \left\{ \frac{1}{6}(\gamma + \theta)^3 + \frac{1}{24}(\gamma + \theta)^4 + \frac{e^{\gamma^2/(4\theta)}}{120}(\gamma + \theta)^5 \right\}. \tag{3.13}
\]

By (3.11) and (3.12),

\[
E e^{\lambda \xi(X_i)} = 1 + \gamma b E\bar{X}_i + \left( \frac{\gamma^2}{2} - \theta \right)b^2 E\bar{X}_i^2 + \left( \frac{\gamma^3}{6} - \theta \right)b^3 E\bar{X}_i^3 + \\
+ \left( \frac{\gamma^4}{24} - \frac{\gamma^2 \theta}{2} + \frac{\theta^2}{2} \right)b^4 E\bar{X}_i^4 + O(1) e^{\gamma^2/(4\theta)}b E|X_i|I(|bX_i| > 1) + O'_{\gamma, \theta}b^5 E|\bar{X}_i|^5, \tag{3.14}
\]
where $|O(1)| \leq 1$. We want to show that $|e^{\lambda X_{i,(k)}} - 1| \leq 11/12$. Since $7/16 \leq \lambda \leq 9/16$ and $1/3 \leq \alpha_0 - \alpha_3x \bar{L}_{3n} \leq 7/3$, then $7/8 \leq \gamma \leq 9/8$ and $7/48 \leq \theta \leq 63/48$. Hence by (3.13),

$$
\left| \frac{\gamma^2}{2} - \theta \right| + \left| \frac{\gamma^3}{6} - \gamma \cdot \theta \right| + \left| \frac{\gamma^4}{24} - \frac{\gamma^2 \theta}{2} + \frac{\theta^2}{2} \right| + |O'_{\gamma, \theta}|
$$

$$
\leq \left| \frac{(7/8)^2}{2} - 63/48 \right| + \left| \frac{(7/8)^3}{6} - (9/8)(63/48) \right| + \left| (9/8)^4/24 - (7/8)^2(7/48)/2 + (63/48)^2/2 \right|

+ \left| (9/8 + 63/48)^3/6 + (9/8 + 63/48)^4/24 + e^{27/16}(9/8 + 63/48)^5/120 \right|

\leq 0.93 + 1.37 + 0.88 + 7.77 = 10.95.
$$

Note that $\gamma_b \hat{X}_i = -\gamma b eX_i I(|bX_i| > 1)$. Since $\gamma^2/(4\theta) = \lambda/(\alpha_0 - \alpha_3x \bar{L}_{3n}) \leq 27/16$, then

$$
\left| \gamma + e^{\gamma^2/(2\theta)}|b\hat{E}|X_i|I(|bX_i| > 1) \right| \leq (9/8 + e^{27/16})b^2 E \hat{X}_i^2 I(|bX_i| > 1)

\leq 6.54b^2 E \hat{X}_i^2 I(|bX_i| > 1).
$$

From the above, $|e^{\lambda X_{i,(k)}} - 1| \leq 10.95b^2 E \hat{X}_i^2 \leq 11/12$ by condition (2.2). Note that $|\log(1 + y) - y - y^2/2| \leq 12^3|y|^3/3$ for $|y| \leq 11/12$. Since $\gamma = 2\lambda$ and $\theta = \lambda(\alpha_0 - \alpha_3x \bar{L}_{3n})$, then by (3.14) and Lemma 3.1(i),

$$
\log E^{\lambda X_{i,(k)}} = \log \left(1 + \left\{ e^{\lambda X_{i,(k)}} - 1 \right\} \right)
$$

$$
= \left\{ 2\lambda^2 - \lambda(\alpha_0 - \alpha_3x \bar{L}_{3n}) \right\} b^2 E \hat{X}_i^2 + \left\{ \frac{4\lambda^3}{3} - 2\lambda^2(\alpha_0 - \alpha_3x \bar{L}_{3n}) \right\} b^3 E \hat{X}_i^3
$$

$$
+ \left\{ \frac{2\lambda^4}{3} - 2\lambda^3(\alpha_0 - \alpha_3x \bar{L}_{3n}) + \frac{\lambda^2(\alpha_0 - \alpha_3x \bar{L}_{3n})^2}{2} \right\} b^4 E \hat{X}_i^4
$$

$$
- \frac{1}{2} \left\{ 2\lambda^2 - \lambda(\alpha_0 - \alpha_3x \bar{L}_{3n}) \right\}^2 b^4 (E\hat{X}_i^2)^2 + O(1)b^3 E|X_i|^3 I(|bX_i| > 1)
$$

$$
+ O(1)b^5 E|\hat{X}_i|^5.
$$

Hence (3.7) follows from Lemma 3.1(i). Taking derivative with respect to $\lambda$ on both sides of (3.7) and by (3.5), we obtain (3.8). Taking derivative with respect to $\lambda$ on both sides of (3.8) and by (3.6), we obtain (3.9). Similar to (3.14),

$$
E \hat{X}_i^m e^{\lambda X_{i,(2)}} = E \hat{X}_i^m + \gamma b E \hat{X}_i^m + O(1)b^2 E|\hat{X}_i|^{m+2}
$$

(3.15)

for $m \geq 2$. Since $|(1 + y)^{-1} - 1| \leq 12^2|y|$ for $|y| \leq 11/12$ and $\gamma = 2\lambda$, then by (3.4), (3.14) and (3.15),

$$
EZ_{\lambda,t,(2)} = \frac{E \hat{X}_i^m e^{\lambda X_{i,(2)}}}{e^{\lambda X_{i,(2)}}} = E \hat{X}_i^m + 2\lambda b E \hat{X}_i^m + O(1)b^2 E|\hat{X}_i|^{m+2}.
$$

\[ Q.E.D. \]

**Lemma 3.3.** Let $x > 0$, $|\delta(x)| \leq x^2/5$ and $k = 1$ or 2. Under conditions (2.1) and (2.2), for $\sum_{i=1}^n E \xi(Z_{\lambda,i,(k)})$ in (3.8), the equation

$$
\sum_{i=1}^n E \xi(Z_{\lambda,i,(k)}) = (2 - \alpha_0)x^2 + \delta(x)
$$

(3.16)
has a unique solution of \( \lambda \), denoted as \( \lambda_\delta \), with \( 7/16 \leq \lambda_\delta \leq 9/16 \). If \( \delta(x) = \beta_3 x^3 \bar{L}_{3n} + \beta_4 x^4 \bar{L}_{4n} + \beta_5 x^4 \bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3.5} \) with \( |\beta_\delta| \leq \sqrt{A} \), then

\[
\lambda_\delta = \frac{1}{2} + \left( \frac{\beta_3}{4} + \frac{\alpha_0 - \alpha_3}{2} - \frac{1}{4} \right) x \bar{L}_{3n} + \left( \frac{\beta_4}{4} + \frac{3\alpha_0}{8} - \frac{\alpha_3}{4} - \frac{1}{12} \right) x^2 \bar{L}_{4n} + \left( \frac{\beta_5}{4} - \frac{\alpha_3}{2} + (\alpha_0 - 1) \left( 4 \alpha_0 \frac{\alpha_3}{4} - \frac{1}{4} \right) \right) x^2 \bar{L}_{3n}^2 + \left( \frac{1}{4} - \frac{3\alpha_0}{8} + 2 \right) x^2 \frac{\sum_{i=1}^{n}(EX_i^2)^2}{B_i^4} + \frac{O(1)\Delta_{n,x}^{3.5}}{x^2}.
\]

(3.17)

Proof. Since the left-hand side of (3.9) is equal to \( d(\sum_{i=1}^{n} E\xi(Z_{\lambda_i,(k)}))/d\lambda \) and the right-hand side is positive, then \( \sum_{i=1}^{n} E\xi(Z_{\lambda_i}) \) increases strictly as \( \lambda \) increases. By (3.8), condition (2.1) and \( |\delta(x)| \leq x^2/5 \),

\[
\sum_{i=1}^{n} E\xi(Z_{7/16,i,(k)}) \leq (2 - \alpha_0)x^2 + \delta(x) \leq \sum_{i=1}^{n} E\xi(Z_{9/16,i,(k)}).
\]

Hence (3.16) has a unique solution \( \lambda_\delta \) and \( 7/16 \leq \lambda_\delta \leq 9/16 \). By (3.8) and (3.16),

\[
\lambda_\delta = \frac{1}{2} + \left( \frac{\beta_3}{4} - \lambda_\delta \alpha_0 + \frac{\alpha_3}{4} \right) x \bar{L}_{3n} + \left( \frac{\beta_4}{4} + \frac{3\alpha_0}{8} - \lambda_\delta \alpha_3 \right) x^2 \bar{L}_{4n} + \left( 2\lambda_\delta^3 - \frac{3\lambda_\delta^2 \alpha_0}{2} + \lambda_\delta \alpha_0 \frac{\alpha_3}{4} \right) \frac{x^2 \sum_{i=1}^{n}(EX_i^2)^2}{B_i^4} + \frac{O(1)\Delta_{n,x}^{3.5}}{x^2}.
\]

(3.18)

If \( \delta(x) = \beta_3 x^3 \bar{L}_{3n} + \beta_4 x^4 \bar{L}_{4n} + \beta_5 x^4 \bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3.5} \), then

\[
\lambda_\delta = \frac{1}{2} + \left( \frac{\beta_3}{4} - \lambda_\delta \alpha_0 + \frac{\alpha_3}{4} \right) x \bar{L}_{3n} + \left( \frac{\beta_4}{4} + \frac{3\alpha_0}{8} - \lambda_\delta \alpha_3 \right) x^2 \bar{L}_{4n} + \left( \frac{\beta_5}{4} - \lambda_\delta \alpha_3 \right) x^2 \bar{L}_{3n}^2 + \frac{O(1)\Delta_{n,x}^{3.5}}{x^2}.
\]

By Lemma 3.1(i) and (3.18),

\[
\lambda_\delta^2 x \bar{L}_{3n} = \frac{1}{4} x \bar{L}_{3n} + \left( \frac{\beta_3}{4} - \frac{1}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4} \right) x^2 \bar{L}_{3n}^2 + \frac{O(1)\Delta_{n,x}^{3.5}}{x^2}
\]

and

\[
\lambda_\delta \alpha_0 x \bar{L}_{3n} = \frac{\alpha_0}{2} x \bar{L}_{3n} + \alpha_0 \left( \frac{\beta_3}{4} - \frac{1}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4} \right) x^2 \bar{L}_{3n}^2 + \frac{O(1)\Delta_{n,x}^{3.5}}{x^2}.
\]

Hence the second term on the right-hand side of (3.18)

\[
\left( \frac{\beta_3}{4} - \lambda_\delta^2 \alpha_0 - \frac{\alpha_3}{4} \right) x \bar{L}_{3n} = \left( \frac{\beta_4}{4} - \frac{1}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4} \right) x \bar{L}_{3n} + (\alpha_0 - 1) \left( \frac{\beta_4}{4} - \frac{1}{4} + \frac{\alpha_0}{2} - \frac{\alpha_3}{4} \right) x^2 \bar{L}_{3n}^2 + \frac{O(1)\Delta_{n,x}^{3.5}}{x^2}.
\]

For the other terms in (3.18) involving \( x^2 \bar{L}_{4n} \), \( x^2 \bar{L}_{3n}^2 \) and \( x^2 B_{-1}^4 \sum_{i=1}^{n}(EX_i^2)^2 \), by Lemma 3.1(i), \( \lambda_\delta \) can be replaced by 1/2 with a difference of \( O(1)\Delta_{n,x}^{3.5}/x^2 \). Therefore (3.17) follows. \( \square \)
Proposition 3.4. Let \( S_{n,(k)} = \sum_{i=1}^{n} X_{i,(k)} \) and \( V_{n,(k)}^2 = \sum_{i=1}^{n} X_{i,(k)}^2 \), where \( X_{i,(1)} = X_i \) and \( X_{i,(2)} = X_i I(|bX_i| \leq 1) \). Under conditions (2.1) and (2.2), for \( x > 0 \), \( |\delta(x)| \leq x^2/5 \) and \( k = 1 \) or 2,

\[
P\left\{2bS_{n,(k)} - (\alpha_0 - \alpha_3 x L_{3n}) b^2 V_{n,(k)}^2 \geq (2 - \alpha_0)x^2 + \delta(x)\right\}
= \exp\left(\frac{x^2}{2} + \sum_{i=1}^{n} \log \text{exp}_{\lambda_{i}\xi(X_{i,(k)})} - \lambda_{3}(2 - \alpha_0)x^2 - \lambda_{4}\delta(x)\right) (1 - \Phi(x)) \times (1 + O(1)x L_{3n}).
\] (3.19)

If \( \delta(x) = \beta_3 x^3 \bar{L}_{3n} + \beta_4 x^4 \bar{L}_{4n} + \beta_5 x^4 \bar{L}_{5n}^2 + O(1)\Delta_{n.x}^3 \) with \( |\beta_j| \leq \sqrt{\lambda} \), then

\[
\frac{x^2}{2} + \sum_{i=1}^{n} \log \text{exp}_{\lambda_{i}\xi(X_{i,(k)})} - \lambda_{3}(2 - \alpha_0)x^2 - \lambda_{4}\delta(x)
= \left(\frac{\alpha_3 - \beta_3}{2} + \frac{1 - 3\alpha_0}{6}\right)x^3 \bar{L}_{3n} + \left((-\frac{\beta_3}{2} + \frac{(\alpha_0 - 1)^2}{8} - \frac{1}{12}\right)x^4 \bar{L}_{4n}
+ \left(\frac{\alpha_3 - \beta_5}{2} - \frac{(\beta_3 - \alpha_3 + 2\alpha_0 - 1)^2}{8}\right)x^4 \bar{L}_{5n}^2
- \frac{(1 - \alpha_0)^2 x^4 \sum_{i=1}^{n}(E X_i^2)^2}{B_{n}^2} + O(1)\Delta_{n.x}^3.
\] (3.20)

Proof. By the conjugate method (e.g., (4.9) of Petrov [8] or (2.11) of Petrov [9]) together with (3.3) and (3.16),

\[
P\left\{2bS_{n,(k)} - (\alpha_0 + \alpha_3 x L_{3n}) b^2 V_{n,(k)}^2 \geq (2 - \alpha_0)x^2 + \delta(x)\right\}
= \int \cdots \int \sum_{i=1}^{n} \log \text{exp}_{\lambda_{i}\xi(X_{i,(k)})} - \lambda_{3}\sum_{i=1}^{n}\xi(X_{i,(k)})
\times \int \sum_{i=1}^{n} \log \text{exp}_{\lambda_{i}\xi(X_{i,(k)})} - \lambda_{3}(2 - \alpha_0)x^2 - \lambda_{4}\delta(x)\right) \int \prod_{i=1}^{n} dP(X_{i,(k)} \leq x_i)
\times \int \left(\sum_{i=1}^{n}(\xi(Z_{\lambda_3,i,(k)}) - E\xi(Z_{\lambda_3,i,(k)})) \geq 0\right)\right\}.
\] (3.21)

Let \( W_i = \xi(Z_{\lambda_3,i,(k)}) \) and \( S = \sum_{i=1}^{n}(W_i - EW_i) \). By (3.9), we have

\[
B_{n}^2 := \sum_{i=1}^{n}(W_i - EW_i)^2 = \sum_{i=1}^{n}\text{Var}(\xi(Z_{\lambda_3,i,(k)})) = 4x^2 + O(1)x^3 \bar{L}_{3n}.
\]

Then by the proof of Proposition 2.2 of Wang [17] with \( \theta = 0 \),

\[
E\left\{e^{-\lambda_3 \sum_{i=1}^{n}(\xi(Z_{\lambda_3,i,(k)}) - E\xi(Z_{\lambda_3,i,(k)}))} \times \int \left(\sum_{i=1}^{n}(\xi(Z_{\lambda_3,i,(k)}) - E\xi(Z_{\lambda_3,i,(k)})) \geq 0\right)\right\}
= Ee^{-\lambda_3 S} I(S \geq 0)
= e^{x^2/2}(1 - \Phi(x))(1 + O(1)x L_{3n}).
\] (3.22)
Hence (3.19) follows from (3.21) and (3.22). By (3.7),
\[
\sum_{i=1}^{n} \log E e^{\delta x i(X_{i}(k))} - \lambda_{i}(2 - \alpha_{0})x^{2} = (2\lambda_{i}^{2} - 2\lambda_{i})x^{2} + \left(\frac{4\lambda_{i}^{3}}{3} - 2\lambda_{i}^{2}\alpha_{0} + \lambda_{i}\alpha_{3}\right)x^{3}\bar{L}_{3n} + \\
\left(\frac{2\lambda_{i}^{3}}{3} - 2\lambda_{i}^{2}\alpha_{0} + \frac{\lambda_{i}^{2}\alpha_{0}^{2}}{2}\right)x^{4}\bar{L}_{4n} + 2\lambda_{i}^{3}\alpha_{3}x^{4}\bar{L}_{3n}^{2} - \left(\frac{2\lambda_{i}^{3} - \lambda_{i}\alpha_{0}}{2}\right)x^{4}\sum_{i=1}^{n}(E X_{i}^{2})^{2} + O(1)\Delta_{n,x}^{3.5}. \quad (3.23)
\]
From the expression of \(\lambda_{i}\) in (3.17) and by Lemma 3.1(i), we have
\[
(2\lambda_{i}^{2} - 2\lambda_{i})x^{2} = \frac{x^{2}}{2} + 2\left(\frac{\beta_{i}^{2}}{4} + \frac{\alpha_{0}}{2} - \frac{\alpha_{3}}{4} - \frac{1}{4}\right)x^{4}\bar{L}_{3n}^{2} + O(1)\Delta_{n,x}^{3.5} \quad (3.24)
\]
and
\[
\left(\frac{4\lambda_{i}^{3}}{3} - 2\lambda_{i}^{2}\alpha_{0} + \lambda_{i}\alpha_{3}\right)x^{3}\bar{L}_{3n} = \left(\frac{1}{6} - \frac{\alpha_{0}}{2} + \frac{\alpha_{3}}{2}\right)x^{3}\bar{L}_{3n} + (1 - 2\alpha_{0} + \alpha_{3})\left(\frac{\beta_{i}^{2}}{4} + \frac{\alpha_{0}}{2} - \frac{\alpha_{3}}{4} - \frac{1}{4}\right)x^{4}\bar{L}_{3n} + O(1)\Delta_{n,x}^{3.5}. \quad (3.25)
\]
Applying (3.24) and (3.25) to (3.23) and by Lemma 3.1(i),
\[
\sum_{i=1}^{n} \log E e^{\delta x i(X_{i}(k))} - \lambda_{i}(2 - \alpha_{0})x^{2} = -\frac{x^{2}}{2} + \left(\frac{1}{6} - \frac{\alpha_{0}}{2} + \frac{\alpha_{3}}{2}\right)x^{3}\bar{L}_{3n} + \left(\frac{1}{24} - \frac{\alpha_{0}}{4} + \frac{\alpha_{3}^{2}}{8}\right)x^{4}\bar{L}_{4n} + \\
2\left(\frac{\beta_{i}^{2}}{4} + \frac{\alpha_{0}}{2} - \frac{\alpha_{3}}{4} - \frac{1}{4}\right)x^{4}\bar{L}_{3n}^{2} + (1 - 2\alpha_{0} + \alpha_{3})\left(\frac{\beta_{i}^{2}}{4} + \frac{\alpha_{0}}{2} - \frac{\alpha_{3}}{4} - \frac{1}{4}\right)x^{4}\bar{L}_{3n}^{2} + O(1)\Delta_{n,x}^{3.5}. \quad (3.26)
\]
Since \(\delta(x) = \beta_{i}x^{3}\bar{L}_{3n} + \beta_{4}x^{4}\bar{L}_{4n} + \beta_{5}x^{4}\bar{L}_{3n}^{2} + O(1)\Delta_{n,x}^{3.5}\), then
\[
-\lambda_{i}\delta(x) = -\frac{\beta_{i}^{2}}{2}x^{3}\bar{L}_{3n} - \frac{\beta_{i}^{4}}{2}x^{4}\bar{L}_{4n} - \left\{\frac{\beta_{i}}{2} + \beta_{3}\left(\frac{\beta_{i}}{4} + \frac{\alpha_{0}}{2} - \frac{\alpha_{3}}{4} - \frac{1}{4}\right)\right\}x^{4}\bar{L}_{3n}^{2} + O(1)\Delta_{n,x}^{3.5}. \quad (3.27)
\]
Combining (3.26) and (3.27), we obtain (3.20). \(\square\)

**Proposition 3.5.** Under conditions (2.1) and (2.2), for \(x \geq 2\),
\[
P(S_{n} \geq x V_{n}) \leq A \exp \left\{-\frac{1}{3}x^{3}\bar{L}_{3n} - \frac{1}{12}x^{4}\bar{L}_{4n} + A\Delta_{n,x}^{3.5}\right\} \times (1 - \Phi(x))(1 + AxL_{3n}). \quad (3.28)
\]
Consequently, for \(S_{n}^{(i)} = S_{n} - X_{i}\) and \(V_{n}^{(i)} = (V_{n}^{2} - X_{i}^{2})^{1/2}\),
\[
P \left(S_{n}^{(i)} \geq (x^{2} - 1)^{1/2}V_{n}^{(i)}\right) \leq A \exp \left\{-\frac{1}{3}x^{3}\bar{L}_{3n} - \frac{1}{12}x^{4}\bar{L}_{4n} + A\Delta_{n,x}^{3.5}\right\} \times (1 - \Phi(x))(1 + AxL_{3n}). \quad (3.29)
\]
Proof. Let $P(S_n \geq xV_n) = I_1 + I_2 + I_3 + I_4$ where

\[ I_1 = P\left\{ S_n \geq xV_n, \ |V_n/B_n - 1| \leq |x\bar{L}_{3n}/2 + x^2\bar{L}_{4n}| \right\}, \]
\[ I_2 = P\left\{ S_n \geq xV_n, \ 0 \leq V_n/B_n \leq 1 - |x\bar{L}_{3n}/2 - x^2\bar{L}_{4n}| \right\}, \]
\[ I_3 = P\left\{ S_n \geq xV_n, \ 1 + |x\bar{L}_{3n}/2 + x^2\bar{L}_{4n} < V_n/B_n \leq 3 \right\}, \]
\[ I_4 = P\left\{ S_n \geq xV_n, \ V_n/B_n > 3 \right\}. \]

Observe that for any real valued set $B$ and any constants $\gamma > 0$ and $\theta > 0$,

\[ P\left\{ S_n \geq xV_n, \ \frac{V_n}{B_n} \in B \right\} \leq P\left\{ \gamma bS_n - \theta b^2 V_n^2 \geq \inf_{V_n/B_n \in B} \left( \gamma x\frac{V_n}{B_n} - \theta x^2\frac{V_n^2}{B_n^2} \right) \right\}. \quad (3.30) \]

Let $t = xV_n/B_n$. Then $f_x(t) := 2xt - t^2$ has minimum at $a$ for $t \in [a, x]$ and has minimum at $b$ for $t \in [x, b]$. Hence by (3.30), Lemma 3.1(i) and Proposition 3.4 with $\alpha_0 = 1$, $\alpha_3 = 0$, $\beta_3 = 0$, $\beta_4 = 0$ and $\beta_5 = -1/4$,

\[ I_1 \leq P\left\{ 2bS_n - b^2 V_n^2 \geq 2x^2 \left( 1 - \frac{|x\bar{L}_{3n}/2 - x^2\bar{L}_{4n}|}{2} \right) \right\} \]
\[ + P\left\{ 2bS_n - b^2 V_n^2 \geq 2x^2 \left( 1 + \frac{|x\bar{L}_{3n}/2 + x^2\bar{L}_{4n}|}{2} \right) \right\} \]
\[ = 2P\left\{ 2bS_n - b^2 V_n^2 \geq x^2 - \frac{1}{4} x^4\bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5} \right\} \]
\[ \leq 2 \exp \left\{ -\frac{1}{3} x^3\bar{L}_{3n} - \frac{1}{12} x^4\bar{L}_{4n} + A\Delta_{n,x}^{3,5} \right\} (1 - \Phi(x))(1 + Ax\bar{L}_{3n}). \]

To estimate $I_2$, note that $f_x(t) := 2xt - t^2/3$ has minimum at $a$ for $t \in [0, a]$ with $a \geq 6x/7$. Then by (3.30), Lemma 3.1(i) and Proposition 3.4 with $\alpha_0 = 7/3$, $\alpha_3 = 0$, $\beta_4 = 8/3$, $\beta_5 = -7/12$, and $\beta_3 = 4/3$ if $x^3\bar{L}_{3n} \geq 0$ and $\beta_3 = -4/3$ if $x^3\bar{L}_{3n} < 0$,

\[ I_2 \leq P\left\{ 2bS_n - \frac{7}{3} b^2 V_n^2 \geq 2x^2 \left( 1 - \frac{|x\bar{L}_{3n}/2 - x^2\bar{L}_{4n}|}{2} \right) \right\} \]
\[ = P\left\{ 2bS_n - \frac{7}{3} b^2 V_n^2 \geq -\frac{x^2}{3} + \frac{4}{3} x^3|\bar{L}_{3n}| + \frac{8}{3} x^4\bar{L}_{4n} - \frac{7}{12} x^4\bar{L}_{3n}^2 + O(1)\Delta_{n,x}^{3,5} \right\} \]
\[ \leq \exp \left\{ -\frac{1}{3} x^3\bar{L}_{3n} - \frac{1}{12} x^4\bar{L}_{4n} + A\Delta_{n,x}^{3,5} \right\} (1 - \Phi(x))(1 + Ax\bar{L}_{3n}) \]

To estimate $I_3$, note that $f_x(t) := 2xt - t^2/3$ has minimum at $a$ for $t \in [a, 3x]$. Then by (3.30), Lemma 3.1(i) and Proposition 3.4 with $\alpha_0 = 1/3$, $\alpha_3 = 0$, $\beta_4 = 4/3$, $\beta_5 = -1/12$, and $\beta_3 = 2/3$ if $x^3\bar{L}_{3n} \geq 0$ and $\beta_3 = -2/3$ if $x^3\bar{L}_{3n} < 0$,

\[ I_3 \leq P\left\{ 2bS_n - \frac{1}{3} b^2 V_n^2 \geq 2x^2 \left( 1 + \frac{|x\bar{L}_{3n}/2 + x^2\bar{L}_{4n}|}{2} \right) \right\} \]
\[ \leq \exp \left\{ -\frac{1}{3} x^3\bar{L}_{3n} - \frac{1}{12} x^4\bar{L}_{4n} + A\Delta_{n,x}^{3,5} \right\} (1 - \Phi(x))(1 + Ax\bar{L}_{3n}), \]
where we also use the fact that $\bar{L}_{3n}^2 \leq \bar{L}_{4n}$.

Finally by the proof of Lemma 8.1 of Jing, Shao and Wang [6], for $x \geq 2$,
\[
I_4 \leq 2e^{-x^2} \leq A \exp \left\{ -\frac{1}{3} x^3 \bar{L}_{3n} - \frac{1}{12} x^4 \bar{L}_{4n} + A \Delta_{n,x}^{3,5} \right\} (1 - \phi(x)).
\]
Therefore, (3.28) follows from all of the above. \hfill \Box

For $x \geq 2$, let $\tau = B_n/x$ and set
\[
\bar{X}_i = X_i I(|X_i| \leq \tau), \quad \bar{Z}_{\lambda^3,i} = Z_{\lambda^3,i}(2), \quad \bar{S}_n = \sum_{i=1}^n \bar{X}_i, \quad \bar{V}_n^2 = \sum_{i=1}^n \bar{X}_i^2,
\]
\[
\bar{V}_{n,\delta}^2 = \sum_{i=1}^n \bar{Z}_{\lambda^3,i}^2, \quad u_i = \bar{Z}_{\lambda^3,i}^2 - E\bar{Z}_{\lambda^3,i}^2, \quad \nabla_n = \frac{x^2}{B_n^2} \left( \sum_{i=1}^n u_i \right)^2.
\]

**Proposition 3.6.** Under conditions (2.1) and (2.2), for $x \geq 2$,
\[
P(\bar{S}_n \geq x\bar{V}_n) \leq \exp \left\{ -\frac{1}{3} x^3 \bar{L}_{3n} - \frac{1}{12} x^4 \bar{L}_{4n} + A \Delta_{n,x}^{3,5} \right\} (1 - \phi(x))(1 + Ax\bar{L}_{3n}).
\]

**Proof.** Let $\xi(y) = 2by - (1 - x\bar{L}_{3n}/2)b^2 y^2$ and
\[
\delta(x) = \frac{x^3 \bar{L}_{3n}}{2} + \frac{x^4 \bar{L}_{3n}^2}{4} - 2A \Delta_{n,x}^{3,5}.
\]
By the conjugate method similar to (3.21),
\[
P(\bar{S}_n \geq x\bar{V}_n) = \int \cdots \int I \left\{ \sum_{i=1}^n x_i \geq x \left( \sum_{i=1}^n x_i^2 \right)^{1/2} \right\} \prod_{i=1}^n dP(\bar{X}_i \leq x_i)
\]
\[
= \int \cdots \int e^{\sum_i \log \text{E}e^{\lambda^3 \xi(x_i)} - \lambda^3 \sum_i \xi(x_i)} I(s_n \geq xv_n) \prod_{i=1}^n dP(\bar{Z}_{\lambda^3,i} \leq x_i),
\]
where $s_n = \sum_{i=1}^n x_i$ and $v_n^2 = \sum_{i=1}^n x_i^2$. We can write
\[
v_n = \frac{B_n}{1 - x\bar{L}_{3n}/2} (1 + y_n)^{1/2},
\]
where
\[
y_n = \frac{(1 - x\bar{L}_{3n}/2)^2 v_n^2 - B_n^2}{B_n^2}.
\]
Since $\bar{L}_{3n}^2 \leq \bar{L}_{4n}$, then $\lambda_3 = 1/2 + O(1) x \bar{L}_{3n} + O(1) x^{-2} \Delta_{n,x}^{3,5}$ by (3.17). Hence by (3.10) and (3.31),
\[
E\bar{V}_{n,\delta}^2 = \sum_{i=1}^n E\bar{X}_i^2 + \frac{x}{B_n} \sum_{i=1}^n E\bar{X}_i^3 + O(1) \frac{x}{B_n^2} \sum_{i=1}^n E\bar{X}_i^4 + O(1) x^{-2} \Delta_{n,x}^{3,5} B_n^2
\]
\[
= B_n^2 + x\bar{L}_{3n}B_n^2 + O(1) x^2 \bar{L}_{4n}B_n^2 + O(1) x^{-2} \Delta_{n,x}^{3,5} B_n^2.
\]
Since \((1 - x\bar{L}_{3n}/2)^2 = 1 - x\bar{L}_{3n} + x^2\bar{L}_{3n}^2/4\). Then
\[
(1 - x\bar{L}_{3n}/2)^2EV_{n,\delta}^2 = B_n^2 + O(1)x^2\bar{L}_{4n}B_n^2 + O(1)x^{-2}\Delta_{n,x}^{3,5}B_n^2. \tag{3.36}
\]
Applying (3.36) to (3.35), we have
\[
y_n = \frac{(1 - x\bar{L}_{3n}/2)^2(v_n^2 - EV_{n,\delta}^2)}{B_n^2} + \frac{(1 - x\bar{L}_{3n}/2)^2EV_{n,\delta}^2 - B_n^2}{B_n^2}
= \frac{(1 - x\bar{L}_{3n}/2)^2(v_n^2 - EV_{n,\delta}^2)}{B_n^2} + O(1)x^2\bar{L}_{4n} + O(1)x^{-2}\Delta_{n,x}^{3,5}. \tag{3.37}
\]
Note that \(x\bar{L}_{3n}^2 \leq x\bar{L}_{3n}\bar{L}_{5n} \leq \bar{L}_{5n}/A\) by Lemma 3.1(i). Then
\[
y_n^2 \leq \frac{2(1 - x\bar{L}_{3n}/2)^4(v_n^2 - EV_{n,\delta}^2)^2}{B_n^4} + Ax^{-2}\Delta_{n,x}^{3,5}. \tag{3.38}
\]
Observe that \((1 + y)^{1/2} \geq 1 + y/2 - y^2/2\) for any \(y \geq -1\) and \((1 + y)^{1/2} \geq 1 + y/2 - y^2/m\) for \(y \geq m/2\) with \(m > 0\). By (3.37), \(B_n^{-2}(v_n^2 - EV_{n,\delta}^2) \geq -3/2\) because \(y_n \geq -1\). Moreover, \(B_n^{-2}(v_n^2 - EV_{n,\delta}^2) \geq m\) implies that \(y_n \geq m/2\) for \(m \geq 2\). Hence
\[
(1 + y_n)^{1/2} \geq 1 + \frac{y_n}{2} - \theta y_n^2, \tag{3.39}
\]
where
\[
\theta = \begin{cases} 
1/2 & \text{if } B_n^{-2}|v_n^2 - EV_{n,\delta}^2| \leq 2 \\
1/m & \text{if } m < B_n^{-2}(v_n^2 - EV_{n,\delta}^2) \leq m + 1 \text{ for } m \geq 2.
\end{cases}
\]
Since \(4(1 - x\bar{L}_{3n}/2)^3 \leq 5\), then combining (3.34), (3.35), (3.38) and (3.39), we have
\[
I(s_n \geq xv_n) \leq I \left\{ 2bs_n \geq \frac{2x^2}{1 - x\bar{L}_{3n}/2} (1 + y_n)^{1/2} \right\}
\leq I \left\{ 2bs_n - (1 - x\bar{L}_{3n}/2)b^2v_n^2 + \frac{5\theta x^2(v_n^2 - EV_{n,\delta}^2)^2}{B_n^4} \geq \frac{x^2}{1 - x\bar{L}_{3n}/2} - A\Delta_{n,x}^{3,5} \right\}. \tag{3.40}
\]
From (2.7) and (3.32),
\[
\frac{x^2}{1 - x\bar{L}_{3n}/2} - A\Delta_{n,x}^{3,5} \geq x^2 + \delta(x). \tag{3.41}
\]
By (3.33), (3.40), (3.41) and the definition of \(\nabla_n\) in (3.31),
\[
P(s_n \geq x\bar{V}_n) \leq Ee^{\sum_{i=1}^n \lambda_i x_i} I \left\{ \sum_{i=1}^n \xi(\bar{Z}_{\lambda_i,i}) + 5\theta \nabla_n \geq x^2 + \delta(x) \right\}. \tag{3.42}
\]
Let
\[
T_n = \sum_{i=1}^n (\xi(\bar{Z}_{\lambda_i,i}) - E\xi(\bar{Z}_{\lambda_i,i})) + 5\theta \nabla_n.
\]
Note that $\sum_{i=1}^{n} E\xi(\tilde{Z}_{\lambda,i}) = x^2 + \delta(x)$ in (3.16). Hence

$$P(\tilde{S}_n \geq x\tilde{V}_n) \leq \exp\left\{\sum_{i=1}^{n} \log E e^{\lambda_3 \xi(\tilde{X}_i)} - \lambda_3 x^2 - \lambda_3 \delta(x)\right\} \times E e^{-\lambda_3 \sum_{i=1}^{n} (\xi(\tilde{Z}_{\lambda,i}) - E\xi(\tilde{Z}_{\lambda,i})) I(T_n \geq 0).$$

(3.43)

Since $|e^x - 1| \leq e^{0\sqrt{x}|s|}$, then

$$E e^{-\lambda_3 \sum_{i=1}^{n} (\xi(\tilde{Z}_{\lambda,i}) - E\xi(\tilde{Z}_{\lambda,i}))} I(T_n \geq 0) = E e^{5\delta_3 \sqrt{n} - \lambda_3 T_n} I(T_n \geq 0) \leq E e^{-\lambda_3 T_n} I(T_n \geq 0) + 5\theta \lambda E \sqrt{n} e^{5\theta \lambda_3 \sqrt{n} - \lambda_3 T_n} I(T_n \geq 0) \leq E e^{-\lambda_3 T_n} I(T_n \geq 0) + 5\theta \lambda E \sqrt{n} e^{5\theta \lambda_3 \sqrt{n}}$$

(3.44)

To estimate $J_1$, let $B_n^2 = \sum_{i=1}^{n} (K_i - E K_i)^2$ where $K_i = \xi(\tilde{Z}_{\lambda,i}) + 5\theta_2 x^2 B_n^{-4} u_i^2$. Then $B_n^2 = 4x^2 + O(x^3 L_{3n})$ by (3.9) and (3.10). Following the proof of Proposition 2.2 of Wang [17], we have

$$J_1 = e^{x^2/2}(1 - \Phi(x))(1 + AxL_{3n}).$$

(3.45)

By Lemma 3.7 below and Lemma 3.1(ii),

$$J_2 \leq Ax^2 L_{4n} \leq A(\tilde{L}_{3n} + x^{-1} A^{3,5}_{n,x}).$$

(3.46)

Note that $(2\pi)^{-1/2}(x^{-1} - x^{-3})e^{-x^2/2} \leq (1 - \Phi(x))$ for $x \geq 2$. Then by (3.44)-(3.46),

$$E e^{-\lambda_3 \sum_{i=1}^{n} \xi(\tilde{Z}_{\lambda,i}(2)) - E\xi(\tilde{Z}_{\lambda,i}(2))} I(T_n \geq 0) \leq e^{x^2/2(1 - \Phi(x))(1 + AxL_{3n} + A\Delta^{3,5}_{n,x})} \leq e^{x^2/2 + A\Delta^{3,5}_{n,x}(1 - \Phi(x))(1 + AxL_{3n}).}$$

(3.47)

By (3.20) in Proposition 3.4 and the $\delta(x)$ in (3.32),

$$\frac{x^2}{2} + \sum_{i=1}^{n} \log E e^{\lambda_3 \xi(\tilde{X}_i)} - \lambda_3 x^2 - \lambda_3 \delta(x) \leq -\frac{1}{3} x^3 \tilde{L}_{4n} - \frac{1}{12} x^4 L_{4n} + A\Delta^{3,5}_{n,x}. $$

(3.48)

Therefore, the proposition follows from (3.43), (3.47) and (3.48).

$\square$

**Lemma 3.7.** Suppose that $\theta = 1/2$ if $B_n^{-2}|\sum_{i=1}^{n} u_i| \leq 2$ and $\theta = 1/m$ if $m < B_n^{-2}(\sum_{i=1}^{n} u_i) \leq m + 1$ for $m \geq 2$. Then for $x \geq 2$ and $\sqrt{n}$ defined in (3.31),

$$E\sqrt{n} e^{5\theta\sqrt{n}} \leq Ax^2 \tilde{L}_{4n}. $$

**Proof.** By the definition of $\sqrt{n}$ in (3.31),

$$E\sqrt{n} e^{5\theta\sqrt{n}} = \frac{x^2}{B_n^4} \sum_{i=1}^{n} B_i^2 e^{5\theta\sqrt{n}} + \frac{x^2}{B_n^4} \sum_{1 \leq i,j \leq n, i \neq j} E u_i u_j e^{5\theta\sqrt{n}}. $$

(3.49)

First, we want to show that for $d = \{1, \ldots, n\}\backslash\{i\}$ or $d = \{1, \ldots, n\}\backslash\{i, j\}$,

$$E e^{10\theta x^2 B_n^{-4}(\sum_{k \in d} u_k)^2} \leq A.$$ 

(3.50)
Since \( B_n^{-2} |u_k| \leq 1/x^2 \) and since \( B_n^{-2} (\sum_{k=1}^{n} u_k) \geq -3/2 \) as mentioned after (3.38), then \( B_n^{-2} (\sum_{k \in d} u_k) \geq -3/2 - 2/x^2 \geq -2 \). Hence

\[
E e^{10\theta x^2 B_n^{-4} (\sum_{k \in d} u_k)^2} = E e^{10\theta x^2 B_n^{-4} (\sum_{k \in d} u_k)^2} I \left( xB_n^{-2} \sum_{k \in d} u_k \leq 2 \right) + E e^{10\theta x^2 B_n^{-4} (\sum_{k \in d} u_k)^2} I \left( 2 < xB_n^{-2} \sum_{k \in d} u_k \leq 2x \right) + \sum_{m=2}^{\infty} E e^{10\theta x^2 B_n^{-4} (\sum_{k \in d} u_k)^2} I \left( mx < xB_n^{-2} \sum_{k \in d} u_k \leq (m+1)x \right)
\]

\[
:= K_1 + K_2 + K_3. \tag{3.51}
\]

Since \( \theta \leq 1/2 \), then

\[
K_1 \leq e^{20}. \tag{3.52}
\]

Let \([2x]\) be the largest integer smaller than or equal to \( x \). Then

\[
K_2 \leq \sum_{l=2}^{[2x]} E e^{10\theta x^2 B_n^{-4} (\sum_{k \in d} u_k)^2} I \left( l < xB_n^{-2} \sum_{k \in d} u_k \leq l + 1 \right) \leq \sum_{l=2}^{[2x]} e^{5(l+1)^2} P \left( xB_n^{-2} \sum_{k \in d} u_k > l \right). \tag{3.53}
\]

Since \( |e^s - 1 - s| \leq e^{0.5 s^2}/2 \) and \( Eu_k = 0 \), then for \( t > 0 \),

\[
P \left( xB_n^{-2} \sum_{k \in d} u_k > l \right) \leq e^{-tl} \prod_{k \in d} E e^{txB_n^{-2} u_k} + e^{-tl} \prod_{k \in d} E e^{-txB_n^{-2} u_k} \leq 2e^{-tl} \prod_{k \in d} \left( 1 + \frac{1}{2} e^{txB_n^{-2} |u_k|} \right) x^2 B_n^{-4} E u_k^2. \tag{3.54}
\]

Let \( t = (\log A^{1/2})/l \). Since \( xB_n^{-2} |u_k| \leq 1/x \), then \( e^{txB_n^{-2} |u_k|} \leq A^{1/2} \) for \( l \leq 2x \). Note that \( Eu_k^2 \leq E Z_{A,k,i}^4 \leq A^{1/4} E X_k^4 \) by (3.10) and \( x^2 \mathcal{L}_{4 \sigma} \leq 1/A \). Then

\[
P \left( xB_n^{-2} \sum_{k \in d} u_k > l \right) \leq 2e^{-tl} e^{A^{1/2} l^2} A^{1/4} x^2 \mathcal{L}_{4 \sigma} \leq 2e^{-(\log A)l^2/4+t^2}. \tag{3.55}
\]

Applying (3.54) to (3.53), we have

\[
K_2 \leq \sum_{l=2}^{[2x]} e^{-tl} \leq A. \tag{3.55}
\]

For \( K_3 \), if \( mx < xB_n^{-2} \sum_{k \in d} u_k \leq (m+1)x \), then \( m - 2/x^2 < B_n^{-2} \sum_{k=1}^{n} u_k \leq m + 1 + 2/x^2 \). By the assumptions of the lemma, \( \theta \leq 1/(m-1) \) for \( mx < xB_n^{-2} \sum_{k \in d} u_k \leq (m+1)x \) with \( m \geq 2 \). Hence

\[
K_3 \leq \sum_{m=2}^{\infty} e^{10x^2 (m+1)^2/(m-1)} P \left( B_n^{-2} \sum_{k \in d} u_k > m \right).
\]
Let \( t = (\log A^{1/2}) x^2 \). Then
\[
P\left( B_n^{-2} \sum_{k \in d} u_k > m \right) \leq e^{-tm} \prod_{k \in d} \left( 1 + \frac{1}{2} e^{tB_n^{-2}|u_k| t^2 B_n^{-4} E u_k^2} \right) \leq e^{-(\log A) m x^2 / 2 + x^2}.
\]

Hence
\[
K_3 \leq A \sum_{m=2}^{\infty} e^{-mx^2} \leq A. \tag{3.56}
\]

Therefore, (3.50) follows from (3.51), (3.52), (3.55) and (3.56).

Now we estimate the first term on the right-hand side of (3.49). Since \( \theta \leq 1/2 \), then for each \( i \) and \( d = \{1, \ldots, n\} \setminus \{i\} \),
\[
E u_i^2 e^{5 \theta \gamma_n} \leq E u_i^2 e^{10x^2 B_n^{-4} u_i^2 + 10x^2 B_n^{-4}(\sum_{k \in d} u_k)^2} = E e^{10x^2 B_n^{-4}(\sum_{k \in d} u_k)^2} E u_i^2 e^{10x^2 B_n^{-4} u_i^2}. \tag{3.57}
\]

Since \( x^2 B_n^{-2} u_i \leq 1 \), then applying (3.50) to (3.57), we have
\[
\frac{x^2}{B_n^2} \sum_{i=1}^{n} E u_i^2 e^{5 \theta \gamma_n} \leq \frac{A x^2}{B_n^4} \sum_{i=1}^{n} E u_i^2 \leq A x^2 \tilde{C}_n. \tag{3.58}
\]

Next we estimate the second term on the right-hand side of (3.49). For each \( i \) and \( j \) with \( d = \{1, \ldots, n\} \setminus \{i, j\} \),
\[
E u_i u_j e^{5 \theta \gamma_n} = E u_i u_j e^{5 \theta x^2 B_n^{-4}(u_i + u_j)^2 + 10 \theta x^2 B_n^{-4}(u_i + u_j) \sum_{k \in d} u_k + 5 \theta x^2 B_n^{-4}(\sum_{k \in d} u_k)^2}.
\]

Since \( |e^s - 1 - s| \leq e^{\|s\|_s^2 / 2} \) for \( s = 10 \theta x^2 B_n^{-4}(u_i + u_j) \sum_{k \in d} u_k \) and since \( 10 \theta x^2 B_n^{-4}|(u_i + u_j) \sum_{k \in d} u_k| \leq 10 B_n^{-2}|\sum_{k \in d} u_k| \), then
\[
\frac{x^2}{B_n^2} \sum_{i \neq j} E u_i u_j e^{5 \theta \gamma_n} \leq \frac{x^2}{B_n^2} \sum_{i \neq j} E u_i u_j e^{5 \theta x^2 B_n^{-4}(u_i + u_j)^2} e^{5 \theta x^2 B_n^{-4}(\sum_{k \in d} u_k)^2} + \frac{10 \theta x^4}{B_n^2} \sum_{i \neq j} E \left( u_i u_j (u_i + u_j) e^{5 \theta x^2 B_n^{-4}(u_i + u_j)^2} E \left( \sum_{k \in d} u_k \right) e^{5 \theta x^2 B_n^{-4}(\sum_{k \in d} u_k)^2} \right)
\]
\[
+ \frac{50 \theta^2 x^6}{B_n^2} \sum_{i \neq j} E |u_i u_j|(u_i + u_j)^2 e^{5 \theta x^2 B_n^{-4}(u_i + u_j)^2} \times \left( \sum_{k \in d} u_k \right)^2 e^{10 B_n^{-2} \sum_{k \in d} u_k + 5 \theta x^2 B_n^{-4}(\sum_{k \in d} u_k)^2}
\]
\[
:= L_1 + L_2 + L_3. \tag{3.59}
\]

For \( L_1 \), since \( |e^s - 1| \leq e^{\|s\|_s} |s| \), \( E u_i u_j = 0 \) and \( \theta \leq 1/2 \), then
\[
\left| E \left( u_i u_j e^{5 \theta x^2 B_n^{-4}(u_i + u_j)^2} u_k, k \in d \right) \right| \leq A x^2 B_n^{-4} E |u_i u_j|(u_i + u_j)^2.
\]
Then by (3.50),
\[
L_1 = \frac{x^2}{B^4_n} \sum_{i \neq j} E \left\{ e^{5\theta x^2 B^{-4}_n (\sum_{k \in d} u_k)^2} E \left( u_i u_j e^{5\theta x^2 B^{-4}_n (u_i + u_j)^2} \left| u_k, k \in d \right\} \right. \right.
\leq \frac{A x^4}{B^8_n} \sum_{i \neq j} E |u_i u_j| (u_i + u_j)^2 \leq A x^2 \bar{L}_4 n. \tag{3.60}
\]

For \( L_2 \), since \(|e^s - 1| \leq e^{0/s}|s| \) and \( E u_i u_j (u_i + u_j) = 0 \), then
\[
\left| E \left( u_i u_j (u_i + u_j) e^{5\theta x^2 B^{-4}_n (u_i + u_j)^2} \right| u_k, k \in d \right\} \leq A x^2 B^{-4}_n E |u_i u_j| u_i + u_j |^3. \tag{3.61}
\]

By Hölder’s inequality and (3.50),
\[
\frac{x}{B^2_n} \sum_{k \in d} u_k e^{5\theta x^2 B^{-4}_n (\sum_{k \in d} u_k)^2} \leq \left\{ \frac{x^2}{B^4_n} \left( \sum_{k \in d} u_k \right)^2 \right\}^{1/2} \left\{ E^{10 \theta x^2 B^{-4}_n (\sum_{k \in d} u_k)^2} \right\}^{1/2} \leq A. \tag{3.62}
\]

By (3.61) and (3.62),
\[
L_2 = \frac{10 \theta x^4}{B^8_n} \sum_{i \neq j} E \left\{ \left( \sum_{k \in d} u_k \right) e^{5\theta x^2 B^{-4}_n (\sum_{k \in d} u_k)^2} \right. \cdot \left. E \left( u_i u_j (u_i + u_j) e^{5\theta x^2 B^{-4}_n (u_i + u_j)^2} \right| u_k, k \in d \right\} \right.
\leq \frac{A x^5}{B^{16}_n} \sum_{i \neq j} E |u_i u_j| u_i + u_j |^3 \leq A x^2 \bar{L}_4 n. \tag{3.63}
\]

For \( L_3 \),
\[
E \left( |u_i u_j| (u_i + u_j)^2 e^{5\theta x^2 B^{-4}_n (u_i + u_j)^2} \right| u_k, k \in d \right\} \leq AE (|u_i u_j| (u_i^2 + u_j^2)). \tag{3.64}
\]

Applying Hölder’s inequality twice and by (3.50),
\[
\frac{x^2}{B^4_n} \left( \sum_{k \in d} u_k \right)^2 e^{20 B^{-2}_n |\sum_{k \in d} u_k|^2 + 5\theta x^2 B^{-4}_n (\sum_{k \in d} u_k)^2}
\leq \frac{x^4}{B^4_n} \left\{ E \left( \sum_{k \in d} u_k \right)^4 e^{40 B^{-2}_n |\sum_{k \in d} u_k|^2} \right\}^{1/2} \left\{ E^{10 \theta x^2 B^{-4}_n (\sum_{k \in d} u_k)^2} \right\}^{1/2}
\leq A \left\{ \frac{x^8}{B^{16}_n} E \left( \sum_{k \in d} u_k \right)^8 \right\}^{1/4} \left\{ E e^{80 B^{-2}_n |\sum_{k \in d} u_k|^2} \right\}^{1/4}. \tag{3.65}
\]

Since \(|e^s - 1| \leq e^{0/s}s^2/2\) and \( E u_k = 0 \), then
\[
E e^{80 B^{-2}_n |\sum_{k \in d} u_k|^2} \leq E e^{80 B^{-2}_n |\sum_{k \in d} u_k|^2} + E e^{-80 B^{-2}_n |\sum_{k \in d} u_k|^2}
\leq 2 \prod_{k \in d} (1 + A B^{-4}_n E u_k^2) \leq 2 e A B^{-4}_n \bar{L}_4 \leq 2 e. \tag{3.66}
\]
It is easy to check that \( x^8 B_n^{-16} E\left(\sum_{k \in d} u_k\right)^8 \leq A x^2 \tilde{L}_{4n} \leq 1 \). Then from the above, we have
\[
\frac{x^2}{B_n^4} E \left(\sum_{k \in d} u_k\right)^2 e^{20 B_n^{-2} \sum_{k \in d} u_k |+5x^2 B_n^{-4} (\sum_{k \in d} u_k)^2} \leq A. \tag{3.65}
\]
By (3.64) and (3.65),
\[
L_3 \leq \frac{A x^4}{B_n^8} \sum_{i \neq j} E(|u_i u_j| (u_i^2 + u_j^2)) \leq A x^2 \tilde{L}_{4n}. \tag{3.66}
\]
Combining (3.59), (3.60), (3.63) and (3.66), we obtain
\[
\frac{x^2}{B_n^4} \sum_{i \neq j} E u_i u_j e^{5 \theta^2 n} \leq A x^2 \tilde{L}_{4n}. \tag{3.67}
\]
Therefore, the lemma follows from (3.49), (3.58) and (3.67). \(\square\)

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