NONCONVENTIONAL MODERATE DEVIATIONS THEOREMS AND EXPONENTIAL CONCENTRATION INEQUALITIES

YEOR HAFOUTA

INSTITUTE OF MATHEMATICS
HEBREW UNIVERSITY
JERUSALEM, ISRAEL

Abstract. We obtain moderate deviations theorems and exponential (Bernstein type) concentration inequalities for “nonconventional” sums of the form

\[ S_N = \sum_{n=1}^{N} (F(\xi_{q_1}(n), \xi_{q_2}(n), ..., \xi_{q_{\ell}}(n)) - \bar{F}) \]

where most of the time we consider \( q_i(n) = i \), but our results also hold true for more general \( q_i(n) \)'s such as polynomials. Here \( \xi_n, n \geq 0 \) is a sufficiently fast mixing vector process with some stationarity conditions, \( F \) is a function satisfying certain regularity conditions and \( \bar{F} \) is a certain centralizing constant. When \( \xi_n, n \geq 0 \) are independent and identically distributed a large deviations theorem was obtained in [27] and one of the purposes of this paper is to obtain related results in the (weakly) dependent case. Several normal approximation type results will also be derived. In particular, two more proofs of the nonconventional central limit theorem are given and a Rosenthal type inequality is obtained. Our results hold true, for instance, when \( \xi_n = (\tau(f_i)_{i=1}^{\infty}) \) where \( \tau \) is a topologically mixing subshift of finite type, a Gibbs-Markov map, a hyperbolic diffeomorphism, a Young tower or an expanding transformation taken with a Gibbs invariant measure, as well as in the case when \( \xi_n, n \geq 0 \) forms a stationary and (stretched) exponentially fast \( \phi \)-mixing sequence, which, for instance, holds true when \( \xi_n = (f_i(T_n))_{i=1}^{\infty} \) where \( T_n \) is a Markov chain satisfying the Doeblin condition considered as a stationary process with respect to its invariant measure.

1. Introduction

Partially motivated by the research on nonconventional ergodic theorems (the term “nonconventional” comes from [13]), probabilistic limit theorems for sums of the form

\[ S_N = \sum_{n=1}^{N} F(\xi_{q_1}(n), \xi_{q_2}(n), ..., \xi_{q_{\ell}}(n)) \]

have become a well studied topic. Here \( \xi_n, n \geq 0 \) is a sufficiently fast mixing vector process with some stationarity properties and \( F \) is a function satisfying some regularity conditions. During the past decade many of the classical results such as the (functional) central limit theorem, Berry-Esseen type theorem, the local central limit theorem, Poissonian limit theorems and large deviations theorems were obtained for such sums (see [23], [26], [27], [19] and references therein). One of the most interesting choices of
$q_i$'s is the situation when $q_i(n) = in$ for any $i = 1, 2, \ldots, \ell$. This was the original motivation for the study of nonconventional sums and yields appropriate limit theorems for number of multiple recurrences to a given set by $\xi_k$'s at times forming arithmetic progressions of the type $n, 2n, \ldots, \ell n$.

The large deviation principle proved in [27] holds true in the case when $S_N = \sum_{n=1}^N F(\xi_n, \xi_{2n}, \ldots, \xi_{\ell n})$ only for independent and identically distributed $\xi_n$'s, while when the $q_i(n)$'s satisfy certain (faster than linear) growth conditions the results from there hold true also for certain Markov chains and dynamical systems. The main goal of this paper is to obtain related results when the $\xi_n$'s are weakly dependent and not necessarily generated by a Markov chain or a dynamical system. We will first obtain moderate deviation type theorems for such sums, namely, study the asymptotic behaviour as $N \to \infty$ of probabilities of the form

$$P\left(\frac{1}{N^\zeta} S_N \in \Gamma\right)$$

for arbitrary Borel measurable sets $\Gamma \subset \mathbb{R}$. Here $\frac{1}{2} < \zeta < 1$ depends on the amount of regularity of $F$ and on the growth of $\text{E} |\xi|^k$ as $k \to \infty$. Formally (see [9]), any choice of $\zeta$ is considered as large deviations type result, but under our conditions $\frac{1}{N^\zeta} S_N$ will satisfy the law of large numbers (see [24]) and so we will use the standard informal convention of referring to the case when $\zeta = 1$ as the large deviations case, while the case when $0 < \zeta < 1$ will be referred to as the moderate deviations case, where in our situation it is natural to require that $\frac{1}{2} < \zeta$ since $N^{-\frac{1}{2}} S_N$ satisfies the central limit theorem (see [20] and [17]). Exponential concentration inequalities (i.e. estimates of $P(S_N \geq x)$, $x > 0$) and Gaussian type estimates of the moments of $S_N$ will also be derived. All of the above results are obtained using the so-called method of cumulants (see [30]) and the local dependence structure of nonconventional sums introduced in [19]. The best exponential inequality obtained by this method yields estimates of the form

$$P(S_N \geq \varepsilon N) \leq e^{-c(\varepsilon N)^{\frac{1}{2}}}, \varepsilon > 0, N \geq c \varepsilon^{-\frac{1}{2}}$$

where $c > 0$ is some constant. Such estimates are not optimal since the power of $N$ is $\frac{1}{2}$ and not 1. In the case when $F$ is bounded we are able to improve these estimates. We first approximate $S_N$ in the $L^\infty$ norm by martingales with bounded differences and then apply the Hoeffding-Azuma inequality in order to obtain, in particular, estimates of the form

$$P(S_N \geq \varepsilon N) \leq e^{-c(\varepsilon N)^{\frac{1}{2}}}, \varepsilon > 0, N \geq 1$$

where $c(\varepsilon) > 0$ is some constant which depends on $\varepsilon$ but not on $N$. In the case when either $\xi_n$, $n \geq 0$ forms a sufficiently fast $\phi$-mixing process or it is generated by a topologically mixing subshift of finite type or a Young tower with exponential tails we can choose $c(\varepsilon) = c \varepsilon^2$ for some $c > 0$ which does not depend on $\varepsilon$ and $N$.

Our results hold true, for instance, when $\xi_n = T^n f$ where $f = (f_1, \ldots, f_d)$, $T$ is a topologically mixing subshift of finite type, a hyperbolic diffeomorphism (see [2]), a Young tower (see [31] and [32]), a Gibbs-Markov map considered in [1] or an expanding transformation taken with a Gibbs invariant measure, as well as in the case when $\xi_n = f(T_n)$, $f = (f_1, \ldots, f_d)$ where $T_n$ is a Markov chain satisfying the Doeblin condition considered as a stationary process with respect to its invariant measure. In fact, any stationary and exponentially fast $\phi$-mixing sequence $\{\xi_n\}$ can be considered. In the dynamical systems case each $f_i$ should be
either Hölder continuous or piecewise constant on elements of Markov partitions. As an application we can consider \( \xi_n = (\xi_n, \ldots, (\xi_n)_t) \), \( (\xi_n)_j = \mathbb{I}_{A_j}(T^n x) \) in the dynamical systems case and \( (\xi_n)_j = \mathbb{I}_{A_j}(T^n x) \) in the Markov chain case where \( \mathbb{I}_A \) is the indicator of a set \( A \). Let \( F = F(x_1, \ldots, x_t) \), \( x_j = (x^{(1)}_j, \ldots, x^{(\ell)}_j) \) be a bounded Hölder continuous function which identifies with the function \( G(x_1, \ldots, x_t) = x^{(1)}_1 \cdot x^{(2)}_2 \cdots x^{(\ell)}_\ell \) on the cube \( ([0, 1]^\ell)^\ell \). Let \( N(n) \) be the number of \( l \)'s between 0 and \( n \) for which \( T^{q(l)} x \in A_j \) for \( j = 0, 1, \ldots, \ell \) (or \( \Upsilon_{q(l)} \in A_j \) in the Markov chains case), where we set \( q_0 = 0 \), namely the number of \( \ell \)-tuples of return times to \( A_j \)'s (either by \( T^{q(l)} \) or by \( \Upsilon_{q(l)} \)). Then our results yield moderate deviation theorems and exponential concentration inequalities for the numbers \( N(n) \). In fact, in this case, and more generally for product functions of the form \( F(x_1, \ldots, x_t) = \prod_{l=1}^t q_l(x_l) \), our results hold true for (stretched) exponentially fast mixing \( \alpha \)-mixing processes. When \( f_j \)'s and \( q_l \)'s are Hölder continuous then our results also hold true for the (deterministic) distance expanding maps considered in [29], even though there are no underlying Markov partitions.

In general, the sum \( S_N \) is a nonlinear function of the random vector \( \{\xi_1, \xi_2, \ldots, \xi_{q(N)}\} \), and therefore our results can also be viewed as a part of the research on nonlinear large deviations theorems (see [4] and [44]). Moreover, in view of the large variety of dynamical systems that can be considered, our results can be viewed as a part of the research on concentration of measure for dynamical systems (see, for instance, [27]), as well.

2. Preliminaries and main results

Our setup consists of a \( \varphi \)-dimensional stochastic process \( \xi_n \), \( n \geq 0 \) on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and a family of sub-\( \sigma \)-algebras \( \mathcal{F}_{k,l} \), \( -\infty \leq k \leq l \leq \infty \) such that \( \mathcal{F}_{k,l} \subset \mathcal{F}_{k',l'} \subset \mathcal{F} \) if \( k' \leq k \) and \( l' \geq l \). We will impose restrictions on the mixing coefficients

\[
\phi(n) = \sup \{\phi(\mathcal{F}_{-\infty,k}, \mathcal{F}_{k+n,\infty}) : k \in \mathbb{Z}\}
\]

where we recall that for any two sub-\( \sigma \)-algebras \( \mathcal{G}, \mathcal{H} \subset \mathcal{F} \),

\[
\phi(\mathcal{G}, \mathcal{H}) = \sup \left\{ \left| \frac{P(A \cap B)}{P(A)} - P(B) \right| : A \in \mathcal{G}, B \in \mathcal{H}, P(A) > 0 \right\}
\]

In order to ensure some applications, in particular, to dynamical systems we will not assume that \( \xi_n \) is measurable with respect to \( \mathcal{F}_{n,n} \) but instead impose restrictions on the approximation rates

\[
\beta_q(r) = \sup_{k \geq 0} \|\xi_k - E[\xi_k | \mathcal{F}_{k-r,k+r}]\|_q
\]

where \( \|X\|_q := \|X\|_{L^q} \) for any \( 0 < q \leq \infty \) and a random variable \( X \).

We do not require stationarity of the process \( \xi_n \), \( n \geq 0 \), assuming only that the distribution of \( \xi_n \) does not depend on \( n \) and that the joint distribution of \((\xi_n, \xi_m)\) depends only on \( n - m \), which we write for further reference by

\[
\xi_n \sim \mu \quad \text{and} \quad (\xi_n, \xi_m) \sim \mu_{m-n}
\]

where \( Y \sim \mu \) means that \( Y \) has \( \mu \) for its distribution. In fact, some of our results hold true assuming only that \( \xi_n \sim \mu \) for any \( n \geq 0 \), and we will point out when the assumption about the distribution of \((\xi_n, \xi_m)\) is not needed.
Next, for any $\theta > 0$ set

$$\tau_\theta = \mathbb{E}|\xi_1|^\theta = \int |x|^\theta d\mu(x).$$

Let $F = F(x_1, \ldots, x_\ell)$, $x_j \in \mathbb{R}^\rho$ be a function on $(\mathbb{R}^\rho)^\ell$ such that for some $K \geq 1$, an integer $\ell \geq 0$, $\kappa \in (0, 1]$ and all $x_i, z_i \in \mathbb{R}^\rho$, $i = 1, \ldots, \ell$, we have

$$|F(x) - F(z)| \leq K[1 + \sum_{i=1}^\ell (|x_i| + |z_i|)]\sum_{i=1}^\ell |x_j - z_j|$$

and

$$|F(x)| \leq K[1 + \sum_{i=1}^\ell |x_i|]$$

where $x = (x_1, \ldots, x_\ell)$ and $z = (z_1, \ldots, z_\ell)$. In fact, if $\xi_n$ is measurable with respect to $\mathcal{F}_{n,n}$ then our results will follow with any Borel function $F$ satisfying (2.6) without imposing (2.5), since the latter is needed only for approximation of $\xi_n$ by conditional expectations $\mathbb{E}[\xi_n|\mathcal{F}_{n-r,n+r}]$ using (2.3). To simplify formulas we assume the centering condition

$$(2.7) \bar{F} := \int F(x_1, \ldots, x_\ell)d\mu(x_1)\ldots d\mu(x_\ell) = 0$$

which is not really a restriction since we can always replace $F$ by $F - \bar{F}$. Let $\ell \geq 1$ be an integer, set

$$S_N = \sum_{n=1}^N F(\xi_n, \xi_2n, \ldots, \xi_\ell n)$$

and $\bar{S}_N = S_N - \mathbb{E}S_N$. All the results presented here hold true in the situation when $q_i(n)$’s are polynomials with positive leading coefficients taking integer values on the integers, while some of the results hold true even for more general $q_i(n)$’s. This “nonlinear indexation” case requires some preparation, and so, for the sake of readability, we will discuss it only in Section 5.

We will obtain our main results under either

2.1. **Assumption.** $\iota = 0$ (i.e. $F$ is a bounded Hölder function) and there exist $a, d, \eta > 0$ so that

$$\phi(n) + \beta_\rho^\iota(n) \leq de^{-an^\eta}$$

for any $n \geq 1$,

or

2.2. **Assumption.** $\iota > 0$ and there exist $d, a, \eta, M, \zeta > 0$ so that

$$\phi(n) + \beta_\infty^\iota(n) \leq de^{-an^\eta}$$

for any $n \geq 1$, and for any $k \in \mathbb{N}$,

$$\tau_\rho^k = \mathbb{E}|\xi_1|^k = \int |x|^k d\mu(x) \leq M^k(k!)^\zeta.$$

Note that under either Assumption 2.1 or Assumption 2.2 there exists a constant $a_0$ so that $|\mathbb{E}S_N| \leq a_0K$ for any $N \geq 1$. In fact, this estimate holds true under weaker conditions, see the paragraph preceding Theorem 2.6.
Recall now (see [9]) that a sequence of probability measures \( \mu_N, N \geq 1 \) on a topological space \( \mathcal{X} \) is said to satisfy the large deviation principle (LDP) with speed \( s_N \to \infty \) and good rate function \( I(\cdot) \) if \( I \) is lower semicontinuous, the sets \( I^{-1}[0, \alpha], \alpha > 0 \) are compact and for any Borel measurable set \( \Gamma \subset \mathcal{X} \),

\[
\liminf_{N \to \infty} \frac{1}{s_N} \ln \mu_N(\Gamma) \geq - \inf_{x \in \Gamma^o} I(x)
\]

and

\[
\limsup_{N \to \infty} \frac{1}{s_N} \ln \mu_N(\Gamma) \leq - \inf_{x \in \Gamma^o} I(x)
\]

where \( \Gamma^o \) denotes the interior of a set \( \Gamma \) and \( \bar{\Gamma} \) denotes its closure. A sequence of random variables \( W_N, N \geq 1 \) is said to satisfy the LDP with speed \( s_N \) and good rate function \( I(\cdot) \) if the sequence \( \mathcal{L}(W_N), N \geq 1 \) of the laws of the \( W_N \)'s satisfies the appropriate LDP. We also recall the following terminological convention. When \( W_N, N \geq 1 \) satisfies the law of large numbers and \( s_N \) grows slower than linear in \( N \) the appropriate LDP is usually called a moderate deviation principle (MDP) and the case when \( s_N = N \) is referred to as the LDP.

Next, let \( a_N, N \geq 1 \) be a sequence of real numbers so that

\[
\lim_{N \to \infty} a_N = \infty \quad \text{and} \quad \lim_{N \to \infty} a_N N^{-\frac{1}{1+\gamma}} = 0
\]

where \( \gamma = \frac{1}{\eta} \) under Assumption 2.1 and \( \gamma = \frac{1}{\eta} + \zeta \) under Assumption 2.2. Our main result is the following

**2.3. Theorem.** Set \( v_N = \sqrt{\text{Var} S_N} \) and when \( v_N > 0 \) set \( Z_N = \frac{S_N}{v_N} \). Let \( \Phi \) be the standard normal distribution function.

(i) Suppose that Assumption 2.1 holds true and set \( \gamma = \frac{1}{\eta} \). Then the limit

\[
D^2 = \lim_{N \to \infty} \frac{1}{N} \mathbb{E} S_N^2
\]

exists and there exist constants \( c_1, c_2 > 0 \) which depend only on \( K, \ell, d, a, \eta \) and \( \kappa \) so that for any \( x > 0 \),

\[
P(\hat{S}_N \geq x) \leq \exp \left( - \frac{x^2}{2(c_1 + c_2 x N^{-\frac{1-\gamma}{2+\gamma}})^{1+\gamma}} \right).
\]

Moreover, when \( D^2 > 0 \) then there exist constants \( c_3, c_4, c_5 > 0 \) which depend only on \( \ell, K, \kappa, a, d \) and \( \eta \) so that for any \( N \geq c_3 \) we have \( v_N > 0 \) and for any \( 0 \leq x < c_4 N^{-\frac{1}{2+\gamma}} \),

\[
\left| \ln \frac{P(Z_N \geq x)}{1 - \Phi(x)} \right| \leq c_5 (1 + x^3) N^{-\frac{1-\gamma}{2+\gamma}} \quad \text{and}
\]

\[
\left| \ln \frac{P(Z_N \leq -x)}{\Phi(-x)} \right| \leq c_5 (1 + x^3) N^{-\frac{1-\gamma}{2+\gamma}}.
\]

Furthermore, the sequence \( (DN^{\frac{1}{2}} a_N)^{-1} S_N, N \geq 1 \) satisfies the MDP with the speed \( s_N = a_N^2 \) and the rate function \( I(x) = \frac{x^2}{a_N^2} \).

(ii) When Assumption 2.2 holds true all the results stated above hold true with \( \gamma = \frac{1}{\eta} + \zeta \) in place of \( \frac{1}{\eta} \) and constants \( c_i, 1 \leq i \leq 5 \) which depend only on \( K, \ell, d, a, \eta, M, \zeta, \kappa, \nu \) and \( \tau_i \).

The above theorem holds true also for certain nonlinear \( \beta_i(n) \)'s (e.g. polynomials), see Section 3. Note that when \( \beta_0(r_0) = 0 \) for some \( q \) and \( r_0 \) then Theorem 2.3 holds true for any Borel function \( F \) satisfying 2.6, namely, there is no need in 2.5 or in any other type of continuity. We also remark that 2.9 is obtained
using Lemma 2.3 in [30]. This lemma yields certain estimates close to the ones in [2.9] but for larger domain of $x$'s. For the sake of readability these results are not stated here.

Next, by taking $x = \varepsilon N$, $\varepsilon > 0$ in (2.8) (or in the corresponding estimate from Theorem [2.3(ii)]) and using that $|\mathbb{E}S_N| \leq a_0 K$ we obtain that

$$
(2.10) \quad \max \left( P(\bar{S}_N \geq \varepsilon N), P(S_N \geq \varepsilon N) \right) \leq e^{-c_7(\varepsilon N)^{1/\eta}}, \quad N \geq c_6\varepsilon^{-2 - \frac{1}{\eta}}
$$

where $c_6$ and $c_7$ are positive constants which do not depend on $N$ and $a$, and $\gamma$ equals either $\frac{1}{\eta}$ or $\frac{1}{\eta} + \zeta$, depending on the case. The power of $N$ in (2.10) is not optimal since it is smaller than 1. In order to obtain more accurate estimates on the tail probabilities we also prove the following

2.4. **Theorem.** Suppose that $F$ is a bounded Hölder function and that

$$
\varphi := \sum_{n=0}^{\infty} \phi(n) < \infty.
$$

Fix some $N \geq 1$ and $r \geq 0$ and set $\delta_1 := K(\varphi + r + 1)$ and $\delta_2 = KN\beta_0^\gamma (r) + \delta_1$. Then there exists a constant $B > 0$ which depends only on $\ell$ so that for any $\lambda > 0$,

$$
(2.11) \quad \mathbb{E}e^{\lambda S_N} \leq e^{B\lambda^2 N(\delta_1 + B\lambda\beta_2)}
$$

and therefore, for any $t > 0$,

$$
(2.12) \quad P(S_N \geq t + \delta_2) \leq e^{-\frac{t^2}{4B^2N^2\lambda^2}}.
$$

When $\beta_\infty(r_0) = 0$ for some $r_0 \geq 0$ then the above results hold true with $r = r_0$ for any bounded Borel function $F$, i.e. there is no need in any kind of continuity.

Theorem 2.4 holds true also when $q_i(n)$’s are polynomials with positive leading coefficients taking integer values on the integers, see Section 3. Note that the above theorem does not require that $(\xi_n, \xi_m) \sim \mu_{m-n}$ since it does not involve the limit $D^2$ (which does not necessarily exist without this assumption about the distribution of $(\xi_n, \xi_m)$). When $\beta_\infty(r_0) = 0$ for some $r_0 \geq 0$ then by taking $r = r_0$ the terms $\delta_1$ and $\delta_2$ are constants, and therefore we obtain optimal exponential concentration inequalities of the form

$$
P(S_N \geq \varepsilon N) \leq e^{-c_2 N}, \quad N \geq \frac{2\delta_2}{\varepsilon}
$$

where $c = \frac{\delta_1}{16a_0} > 0$ and $\varepsilon > 0$. When $\beta_\infty(r)$ convergence to 0 as $r \to \infty$ then for any $\varepsilon > 0$ we can take a sufficiently large $r_0 = r_0(\varepsilon)$ and obtain that there exists a constant $c(\varepsilon) > 0$ so that for any $N \geq 1$ and $t > 0$,

$$
P(S_N \geq t + 0.5\varepsilon N) \leq e^{-c(\varepsilon)\frac{t^2}{N}}
$$

and in particular

$$
(2.13) \quad P(S_N \geq \varepsilon N) \leq e^{-c_1(\varepsilon)N}
$$

for some constant $c_1(\varepsilon) > 0$ which depends on $\varepsilon$ but not on $N$. When some rate of decay of $\beta_\infty^\gamma (r)$ to 0 is known we can find an explicit $c(\varepsilon)$. For instance, when $\beta_\infty^\gamma (r) \leq de^{-ur}$, $d, u > 0$ for any $r \geq 0$, we can take $r_0$ of the form $r_0 = -c \ln \varepsilon$ and then the above estimate will hold true with $c(\varepsilon)$ having the form $c(\varepsilon) = q_0 |\ln \varepsilon|^{-1}$ for some constant $q_0$ which depends only on $\ell, d, u$, $\kappa$ and $K$. 


2.5. Remark. Let \((\mathcal{X},T)\) be a Young tower (see \([31]\) and \([32]\)) and \(\mu\) be an appropriate Gibbs measure. Consider \(\sigma\)-algebras \(\mathcal{F}_{n,m}\) are generated by an appropriate Markov partition. Then (see \([21]\)), the mixing coefficients \(\phi(n)\) decay in the same speed as the tails of the tower. Let \(h_1, ..., h_\rho\) be real valued functions on \(\mathcal{X}\) which are either constant on atoms of the partition or are Hölder continuous functions and let \(\xi_n = (h_1 \circ T^n, ..., h_\rho \circ T^n)\), \(n \geq 1\). Then, (2.13) holds true (with an appropriate \(c(\varepsilon)\)'s) assuming that the tails converge sufficiently fast to 0. Note that when \(h_1, ..., h_\rho\) are Hölder continuous functions, then the centralized sum \(\bar{S}_N\) can be written as a reverse martingale, and therefore (see \([4]\)), in these circumstances we obtain optimal exponential concentration inequality of the form

\[
P(\bar{S}_N \geq t) \leq e^{-c_2^{\frac{t^2}{N}}}, \quad N \geq 1, \quad t > 0
\]

where \(c\) is some constant. Plugging in \(t = \varepsilon N, \varepsilon > 0\) we derive that for any \(N \geq 1, \varepsilon > 0\),

\[
P(\bar{S}_N \geq \varepsilon N) \leq e^{-c_2 \varepsilon^2 N},
\]

namely we can take \(c(\varepsilon)\) of the form \(c(\varepsilon) = \varepsilon \varepsilon^2\) when \(S_N\) is replaced with \(\bar{S}_N\).

Theorems 2.3 and 2.4 will follow from the following general results. The first one is

2.6. Theorem. Suppose that for some \(b > 2\) and \(m > 0\),

\[
\frac{1}{b} \geq \frac{t}{m} + 1, \max(\tau_m, \tau_b) < \infty
\]

and

\[
\Theta(b, \kappa) := \sum_{n=0}^{\infty} (n+1)\phi^{1-\frac{1}{b}}(n) + \sum_{n=0}^{\infty} (n+1)\beta^\kappa_n(n) < \infty.
\]

Then the limit \(D^2 = \lim_{N \to \infty} \frac{1}{N} \mathbb{E}S_N^2\) exists and there exists \(c_\ell > 0\) which depends only on \(\ell\) so that

\[
\mathbb{E}S_N^2 - D^2 N \leq c_\ell C_0 N^\frac{1}{2}
\]

for any \(N \in \mathbb{N}\), where \(C_0 = K^2(1 + \gamma_m^\ell)\Theta(b, \kappa)\). Moreover, \(D^2 > 0\) if and only if there exists no stationary in the wide sense process \(\{V_n : n \geq 1\}\) such that

\[
F(\xi^{(1)}_{\ell}, \xi^{(2)}_{\ell}, ..., \xi^{(\ell)}_{\ell}) = V_{n+1} - V_n, \quad P \text{ a.s.}
\]

for any \(n \in \mathbb{N}\), where \(\xi^{(i)}\), \(i = 1, ..., \ell\) are independent copies of \(\xi = \{\xi_n : n \geq 1\}\). When \(\nu = 0\) then the above results hold true without assuming (2.13) while when \(\beta_\infty(r) = 0\) for some \(r\) they hold true for Borel measurable \(F\)’s without assuming (2.9).

This theorem is a particular case of Theorem 1.3.4 in \([19]\) and Theorem 2.2 in \([18]\). In fact, it is a consequence of the arguments in \([26]\), \([25]\) and \([16]\) and is formulated here for readers’ convenience. We refer the readers’ to \([15]\) for conditions in the special case when \(\xi_n, n \geq 0\) forms a sufficiently fast mixing Markov chain. Remark that in the circumstance of Theorem 2.6 there exists a constant \(a_\epsilon\) which depends only on \(\ell\) so that \(|\mathbb{E}S_N| \leq a_\epsilon K C_0\) for any \(N \geq 1\). Indeed this is a consequence of (2.7) and Corollary 1.3.14 in \([19]\). Therefore, for any \(N \geq 1, \ell > 0\),

\[
|\text{Var}S_N - D^2 N| \leq C_1 N^\frac{1}{2}
\]

for some constant \(C_1\) which depends only on \(C_0, \ell\) and \(K\).
We recall next that the $k$-th cumulant of a random variable $W$ with finite moments of all orders is given by
\[ \Gamma_k(W) = \frac{1}{i^k} \frac{d^k}{dt^k} \left( \ln \mathbb{E} e^{itW} \right) \bigg|_{t=0}. \]
Note that $\Gamma_1(W) = \mathbb{E} W$, $\Gamma_2(W) = \text{Var} W$ and that $\Gamma_k(aW) = a^k \Gamma_k(W)$ for any $a \in \mathbb{R}$ and $k \geq 1$.

2.7. **Theorem.** Under Assumption 2.1 there exists a constant $c_0$ which depends only on $K, \ell, d, a, \eta$ and $\kappa$ so that for any $k \geq 3$,
\[ |\Gamma_k(\bar{S}_N)| \leq N(k!)^{1+\gamma_1}(c_0)^{k-2} \]
where $\gamma_1 = \frac{1}{\eta}$. When Assumption 2.2 holds true there exists a constant $c_0$ which depends only on $K, \ell, d, a, \eta, M, \zeta, \kappa$ and $\iota$ so that for any $k \geq 3$,
\[ |\Gamma_k(\bar{S}_N)| \leq N(k!)^{1+\gamma_2}(c_0)^{k-2} \]
where $\gamma_2 = \gamma_1 + \iota \zeta$.

Note that Theorem 2.7 holds true without assuming that $(\xi_n, \xi_m) \sim \mu_{m-n}$ since its proof does not require that the limit $D^2$ exists. When $(\xi_n, \xi_m) \sim \mu_{m-n}$ then $N^{-\frac{1}{2}} \bar{S}_N$ satisfies the CLT and so the term $N$ on the right hand side should not be alarming since theorem 2.7 implies that
\[ |\Gamma_k(N^{-\frac{1}{2}} \bar{S}_N)| \leq (k!)^{1+\gamma}(N^{-\frac{1}{2}}c_0)^{k-2} \]
for any $k \geq 3$, where $\gamma$ is either $\gamma_1$ or $\gamma_2$, depending on the case. After establishing Theorem 2.7 the moderate deviations theorems and exponential concentration inequalities stated in Theorem 2.3 follow from the so called method of cumulants (see [30] and [10]).

Theorem 2.4 will follow from the following result together with the Hoeffding-Azuma inequality.

2.8. **Theorem.** Suppose that $F$ is a bounded Hölder function and that
\[ \varphi := \sum_{n=0}^{\infty} \phi(n) < \infty. \]
Then there exists a constant $B > 0$ which depends only on $\ell$ so that for any $N \geq 1$ and $r \geq 0$ there is a martingale $M^{(N,r)}_n$, $n \geq 1$ whose differences are bounded by $\delta'_1 := BK(\varphi + r + 1)$ and
\[ \|S_N - M^{(N,r)}_N\|_{\infty} \leq \delta'_2 := BK N \beta^{\kappa}_{\infty}(r) + \delta'_1. \]
When $\beta_{\infty}(r_0) = 0$ for some $r_0 \geq 0$ then the above results hold true with $r = r_0$ for any bounded Borel function $F$.

2.1. **Product functions.** In the special case when $F$ has the form
\[ F(x_1, ..., x_\ell) = \prod_{i=1}^{\ell} f_i(x_i) \]
the results from Theorem 2.3 hold true under weaker assumptions, as described in what follows.

Recall first that the $\alpha$-mixing coefficients are given by
\[ \alpha(n) = \sup \{ \alpha(F_{-\infty,k}, F_{k+n,\infty}) : k \in \mathbb{Z} \} \]
where for any two sub-$\sigma$-algebras $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$,

\begin{equation}
\alpha(\mathcal{G}, \mathcal{H}) = \sup \{ |P(A \cap B) - P(A)P(B)| : A \in \mathcal{G}, B \in \mathcal{H} \}.
\end{equation}

Then (see [3]) $\alpha(n) \leq \frac{1}{n} \phi(n)$ for any $n \geq 0$, and so, assumptions involving $\alpha(n)$ are weaker than ones involving $\phi(n)$. We also recall that (see [11]) for any bounded functions $g_1, \ldots, g_l$, numbers $m_1 < n_1 < m_2 < n_2 < \ldots < m_L < n_L$ and $\mathcal{F}_{m_i,n_i}$-measurable random vectors $U_i$, $i = 1, 2, \ldots, L$,

\begin{equation}
|\mathbb{E} \prod_{i=1}^{L} g_i(U_i) - \mathbb{E} g(U)| \leq 8 \left( \prod_{i=1}^{L} \sup_{j} |g_j| \right) \sum_{i=2}^{L} \alpha(m_i - n_{i-1}).
\end{equation}

Relying on (2.20) we obtain in Section 3.4 that the results stated in Theorem 2.3 (i) hold true when $\mathcal{G}$'s are bounded. The situation of unbounded $\mathcal{G}$'s satisfying certain moment conditions is discussed there, as well.

Next, let $T : \Omega \to \Omega$ be a measurable and $P$-preserving map. We assume here that there exists a space $\mathcal{H}$ of real valued bounded functions on $\Omega$, a norm $\| \cdot \|_\mathcal{H}$ on $\mathcal{H}$, a constant $d$ and a sequence $c(m)$, $m \geq 1$, which converges to 0 as $m \to \infty$, so that for any $f, g \in \mathcal{H}$ and $n \geq 1$,

\begin{equation}
\text{Cov}_P(g, f \circ T^n) \leq d \|g\|_\mathcal{H} \sup |f| c(n).
\end{equation}

Usually $\Omega$ will be a topological space and $\mathcal{H}$ will be a space of Hölder continuous functions equipped with an appropriate norm. We also assume that the $f_i$'s are members of $\mathcal{H}$. Obtaining the MDP and exponential concentration inequalities under condition (2.21) is important when either there are no underlying Markov partitions or there is no effective estimate on the diameter of such partitions (so it is impossible to approximate effectively Hölder continuous functions by functions which are constant on elements of such partitions). For instance, (2.21) holds true with $c(n) = e^{-an}$, $a > 0$ in the (nonrandom) setup of [29], where $T$ is a locally distance expanding map and $\mathcal{H}$ is a space of (locally) Hölder continuous functions, while there are no underlying Markov partitions. Let $n_1 < n_L < \ldots < n_L$ and $g_1, \ldots, g_l \in \mathcal{H}$. By writing

$$
\prod_{i=1}^{L} g_i \circ T^{n_i} = (g_1 \cdot G \circ T^{n_1}) \circ T^{n_i}
$$

where $G = \prod_{i=2}^{L} g_i \circ T^{n_i-n_2}$ we obtain that

\begin{equation}
|\mathbb{E}_P \prod_{i=1}^{L} g_i \circ T^{n_i} - \prod_{i=1}^{L} \mathbb{E}_P g_i \circ T^{n_i}| \leq d M^L \sum_{i=2}^{L} c(n_i - n_{i-1})
\end{equation}

where $M = \max \{ \sup |g_i|, \|g_i\|_\mathcal{H} : i = 1, 2, \ldots, L \}$. Suppose next that

$$
\sum_{n=1}^{\infty} n c(n) < \infty.
$$

Using (2.22) in place of (2.20), we will prove in Section 3.4 that all the results stated in Theorems 2.3 (i) and 2.6 hold true with $\beta_c(n) \equiv 0$ and $c(n)$ in place of $\phi(n)$. 

3. Nonconventional moderate deviations and exponential inequalities via the method of cumulants

3.1. General estimates of cumulants. Let $V$ be a finite set and $\rho : V \times V \to [0, \infty)$ be so that $\rho(v,v) = 0$ and $\rho(u,v) = \rho(v,u)$ for any $u, v \in V$. For any $A, B \subseteq V$ set

$$\rho(A, B) = \min\{ \rho(a,b) : a \in A, b \in B \}.$$

Let $X_v, v \in V$ be a collection of centered random variables with finite moments of all orders, and for each $v \in V$ and $t \in (0, \infty]$ let $\varrho_{v,t} \in [0, \infty]$ be so that $\|X_v\|_t \leq \varrho_{v,t}$. Set $W = \sum_{v \in V} X_v$. The following result is (essentially) proved in [14] (see Theorem 1 there).

3.1. Theorem. Let $0 < \delta \leq \infty$. Suppose that for any $k \geq 1$, $b > 0$ and a finite collection $A_j$, $j \in J$ of (nonempty) subsets of $V$ so that $\min_{i \neq j} \rho(A_i, A_j) \geq b$ and $r := \sum_{j \in J} |A_j| \leq k$ we have

$$(3.1) \quad |\mathbb{E} \prod_{j \in J} \prod_{i \in A_j} X_i - \mathbb{E} \prod_{j \in J} X_i| \leq (r - 1) \left( \prod_{j \in J} \prod_{i \in A_j} \varrho_{i,(1 + \delta)k} \right) \gamma_{\delta}(b, k)$$

where $\gamma_{\delta}(b, r)$ is some nonnegative number which depends only on $\delta, b$ and $r$, and $|J|$ stands for the cardinality of a finite set $\Delta$. Then for any $k \geq 2$ and $s > 0$, $0 < t < \infty$,

$$|\Gamma_k(W)| \leq k^k \left( 2^k C(k)(L_s(k))^{k-1} + R_s(\delta, k) \right)$$

where for any $0 < t < \infty$,

$$L_s(t) = \sup \left\{ \sum_{u \in V, \rho(u,v) \leq s} \varrho_{u,t} : v \in V \right\}, \quad C(t) = \sum_{v \in V} \varrho_{v,t},$$

$$R_s(\delta, k) = \sum_{m \geq s+1} \left( L_m((1 + \delta)k) \right)^{k-1} C((1 + \delta)k) \lambda(\gamma_{\delta}(m, k), k),$$

$$\tilde{\gamma}_{\delta}(m, k) = \max\{\frac{\gamma_{\delta}(m, r)}{r} : 1 \leq r \leq k\}$$

and 

$$\lambda(\varepsilon, k) = k! \sum_{r=1}^{\left\lfloor \frac{k}{2} \right\rfloor} \frac{\varepsilon^r(3r+1)k-2r}{r(k-2r)!}.$$

The difference in the formulations of Theorem 1 in [14] and Theorem 3.1 is that the result from [14] relies on a certain local mixing condition instead of (3.1). But in proof from there the author obtains (3.1) with $\varrho_{v,t} = \|X_v\|_t$ and appropriate $\gamma_{\delta}(b, k)$ relying on that mixing condition, and so Theorem 3.1 is proved exactly as in [14]. We reformulated this theorem in order to include the case when $\beta_{\delta}(r) \neq 0$ for any $r$ and the second situation considered in Section 2.4.

Note that by Stirling’s approximation there exists a constant $C > 0$ so that $k^k \leq C e^{k^k}k!$ for any $k \geq 1$. Remark also that when condition (3.1) holds true only in the case when $|J| = 2$, then using induction this implies that (3.1) holds true with $k \gamma_{\delta}(b, k)$ instead of $\gamma_{\delta}(b, k)$, for collections of more than two sets. Compare this with (22), (12) and (8) in the case when $V = \{1, ..., n\}$ and $\rho(x, y) = |x - y|$.

Next, the following result is a consequence of Theorem 3.1.

3.2. Corollary. Suppose, in addition to the assumptions of Theorem 3.1, that there exist $c_0 \geq 1$ and $u_0 \geq 0$ so that

$$|\{u \in V : \rho(u, v) \leq s\}| \leq c_0 s^{u_0}. $$
for any \( v \in V \) and \( s \geq 1 \). Assume also that \( \tilde{\gamma}_\delta(m, k) \leq d e^{-a m^\eta} \) for some \( a, \eta > 0 \), \( d \geq 1 \) and all \( k, m \geq 1 \). Then there exists a constant \( c \) which depends only on \( c_0, a, u_0 \) and \( \eta \) so that for any \( k \geq 2 \),

\[
|\Gamma_k(W)| \leq d^k |V| e^k(k!)^{1+\frac{\eta}{2}} (M_k^k + M_{k(1+\delta)}^k)
\]

where for any \( q > 0 \),

\[
M_q = \max\{g_{v, q} : v \in V\} \quad \text{and} \quad M_q = (M_q)^k.
\]

When the \( X_v \)'s are bounded and (3.3) holds true with \( \delta = \infty \) we can always take \( g_{v, t} = g_{v, \infty}, t > 0 \) and then for any \( k \geq 2 \),

\[
|\Gamma_k(W)| \leq 2d^k |V| M_k^k e^k(k!)^{1+\frac{\eta}{2}}.
\]

When \( \delta < \infty \) and there exist \( \theta \geq 0 \) and \( M > 0 \) so that

\[
(g_{v, k})^k \leq M^k(k!)^\theta
\]

for any \( v \in V \) and \( k \geq 1 \), then for any \( k \geq 2 \),

\[
|\Gamma_k(W)| \leq 3C \frac{1}{\delta} d^k |V| e^k(1+\delta)^k M_k^k(k!)^{1+\frac{\eta}{2}+\theta}
\]

where \( C \) is some absolute constant.

The proof of this corollary is elementary but for readers’ convenience we will give all the details.

**Proof.** Let \( k \geq 2 \) and \( m \geq s \geq k^{\frac{1}{4}} \). Set \( \varepsilon = \varepsilon_m = e^{-a m^\eta} \). Then \( \tilde{\gamma}_\delta(m, k) \leq d \varepsilon \) and so

\[
\lambda(\tilde{\gamma}_\delta(m, k), k) \leq d^k k!4^k \sum_{r=1}^{[\frac{k}{2}]} \varepsilon r^{k-2r-1} \leq d^k k!4^k \sum_{r=1}^{[\frac{k}{2}]-1} \varepsilon r^{k-2r} + d^k k!4^k \varepsilon^{[\frac{k}{2}]}.\]

Observe that \( k!4^k \varepsilon^{[\frac{k}{2}]} \leq H \varepsilon \) for some constant \( H \) which depends only on \( a \) and \( \eta \), where we used that \( m^\eta \geq k \). Moreover, by Stirling’s approximation there exists an absolute constant \( C > 0 \) so that for any \( 1 \leq r \leq [\frac{k}{2}] - 1 \),

\[
\frac{1}{(k-2r)!} \leq C \varepsilon e^{k-2r} (k-2r)^{k-2r}.
\]

Therefore,

\[
\lambda(\tilde{\gamma}_\delta(m, k), k) \leq C k! (4de)^k \sum_{r=1}^{[\frac{k}{2}]-1} \varepsilon r \left( \frac{r}{k-2r} \right)^{k-2r} + d^k H \varepsilon.
\]

Consider next the function \( g_m = g_{m, k} : [1, \frac{k}{2}] - 1 \rightarrow \mathbb{R} \) given by

\[
g_m(r) = e^r \left( \frac{r}{k-2r} \right)^{k-2r} = e^r \ln e^{-r} (k-2r) \ln(\frac{k}{2}-2).
\]

Then,

\[
g_m'(r) = (\ln \varepsilon + 2 \ln \frac{k}{r-2} + \frac{k}{r}) g_m(r).
\]

If \( g_m'(r_0) = 0 \) for some \( r_0 \in [1, \frac{k}{2}] - 1 \) then

\[
ak \leq am^\eta = - \ln \varepsilon = 2 \ln \frac{k}{r_0-2} + \frac{k}{r_0} \leq 3 \frac{k}{r_0}
\]
and so \( r_0 \leq \frac{3}{a} \) := \( q \). Hence,\
\[
\max_{r \in [1, \frac{k}{2} - 1]} g_m(r) \leq \max \left( g_m(1), g_m\left( \frac{k}{2} - 1 \right), \ max_{w \in [1, q] \setminus \{1, \frac{k}{2} - 1\}} g_m(w) \right)
\]
where \( q_k = \min \left( \frac{k}{2} - 1, q \right) \) and we set \( \max \emptyset = -\infty \). Observe now that\
\[
g_m(1) = \frac{\varepsilon}{(k - 2)^{k - 2}} \leq \frac{k^2 \varepsilon}{k!} \leq 3^k(k!)^{-1} \varepsilon.
\]
Since \( m^n \geq k \) we also have\
\[
g_m\left( \frac{k}{2} - 1 \right) \leq k^2 \varepsilon^{\frac{k}{2} - 1} \leq \varepsilon k^2 e^{-ak\left( \frac{k}{2} - 2 \right)} \leq c_1 \varepsilon(k!)^{-1}
\]
where \( c_1 \) is a constant which depends only on \( a \). When \( k \leq 2(q + 1) \) we can trivially write\
\[
\max_{w \in [1, q_k]} g_m(w) \leq \varepsilon(\psi_0)^k(k!)^{-1}
\]
for some constant \( \psi_0 \) which depends only on \( a \). On the other hand, when \( k > 2(q + 1) \) then using that the function \( x \rightarrow x^{-x} \) is strictly decreasing on \([1, \infty)\) and then Stirling’s approximation we derive that\
\[
\max_{w \in [1, q_k]} g_m(w) = \max_{w \in [1, q]} g_m(w) \leq \varepsilon(q + 1)^k(k - [2q] - 1)^{-(k - [2q] - 1)} \leq \varepsilon^k(k!)^{-1}
\]
where \( \psi \) is a constant which depends only on \( a \), and we also used the inequality \( k! \leq (k - l)!k^l \leq (k - l)3^l \), \( 1 \leq l \leq k \). We conclude from the above estimates that there exists a constant \( R = R(a, \eta) \) which depends only on \( a \) and \( \eta \) so that for any \( 1 \leq r \leq \frac{k}{2} - 1 \),\
\[
g_m(r) = \varepsilon^r \left( \frac{r}{k - 2r} \right)^{k - 2r} \leq \varepsilon R^k(k!)^{-1}
\]
which together with (3.7) yields\
\[
\lambda(\gamma_\delta(m, k), k) \leq d^k R_0^k \varepsilon = d^k R_0^k e^{-am^n}
\]
where \( R_0 = R_0(a, \eta) \geq 1 \) is another constant.

Next, using (3.2), (3.8) and the definitions of \( C(t) \) and \( L_s(t) \) we obtain that\
\[
R_\delta(\delta, k) \leq d^k \left( 1 + H \right) R_0^k \left( M(1 + \delta)k \right)^k |V| \sum_{m \geq s + 1} m^{u_0(k - 1)} e^{-am^n}
\]
where \( L_s(t), C(t), R_\delta(\delta, k) \) are defined in Theorem 3.1. Set \( j_0 = j_0(k, \eta) = \left( \frac{(k - 1)u_0 + 2}{\eta} \right) + 1 \). Then\
\[
m^{u_0(k - 1)} e^{-am^n} \leq m^{u_0(k - 1)} j_0^1(\delta m^n)^{-j_0} \leq j_0 a^{-j_0} m^{-2}.
\]
By Stirling’s approximation there exists a constant \( Q \) which depends only on \( \eta \) and \( u_0 \) so that \( j_0 \leq Q^k(k!)^{\frac{\eta}{u_0}} \) and therefore,
\[
\sum_{m \geq s + 1} m^{u_0(k - 1)} e^{-am^n} \leq j_0! \sum_{m \geq s + 1} \frac{1}{m^2} \leq \frac{1}{s} j_0! \leq \frac{1}{s} \left( Q_1(k!)^{\frac{u_0}{\eta}} \right)
\]
where \( Q_1 \) is a constant which depends only on \( \eta, a \) and \( u_0 \). Taking \( s = k^\frac{1}{2} \) the estimate (3.3) follows by Theorem 3.1 the definition of \( L_s(m) \), Stirling’s approximation and (3.2). By Stirling’s approximation \(((1 + \delta)k)! \leq C(k!)^{1+\delta}(1+\delta)(1+\delta)k\) and (3.6) follows now by (3.3) and the inequality \((1+\delta)^{\frac{1}{1+\delta}} \leq e \). \( \square \)
3.2. **Proof of Theorem 2.7.** Fix some \( N \geq 1 \) and set \( V = V_N = \{1, 2, ..., N\} \). For any \( n, m \in V \) set
\[
\rho(n, m) = \rho_\ell(n, m) = \min_{1 \leq i, j \leq \ell} |in - jm|.
\]
Then for any \( \Delta_1, \Delta_2 \subset V \),
\[
\rho(\Delta_1, \Delta_2) = \inf\{|x - y| : x \in T_1, y \in T_2\} := \text{dist}(T_1, T_2)
\]
where \( T_i = \{jt : t \in \Delta_i, 1 \leq j \leq \ell\} \), \( i = 1, 2 \). Moreover, for any \( s \geq 1 \) and \( 1 \leq n \leq N \),
\[
A_s(n, N) := \{m \in V : \rho(m, n) \leq s\} = \bigcup_{1 \leq i, j \leq \ell} \left[ \frac{in - s}{j}, \frac{in + s}{j} \right]
\]
and so
\[
|A_s(n, N)| \leq 3\ell^2 s.
\]
Therefore (3.2) holds true in our situation with \( u = 1 \). For each \( n \in V \) put
\[
\Theta_n = (\xi_1, \xi_2, ..., \xi_{\ell n})
\]
and
\[
X_n = F(\Theta_n) - EF(\Theta_n).
\]
Then \( \bar{S}_N = \sum_{n \in V} X_n \). We will verify that the remaining assumptions of Corollary 3.2 hold true with the above \( X_n \)'s. First, for each \( r \geq 0 \) and \( n \geq 1 \), set \( \xi_{n,r} = \mathbb{E}[\xi_n | F_{n-r, n+r}] \), \( \Theta_{n,r} = (\xi_{n,r}, \xi_{n+r,1}, ..., \xi_{n+r, \ell n}) \) and
\[
X_{n,r} = F(\Theta_{n,r}) - EF(\Theta_{n,r}).
\]
Set \( \rho_\infty = 2K(1 + \ell) \) and \( \rho_t = 2K(1 + \ell t^\ell) \), \( 0 < t < \infty \). When \( \ell = 0 \) then by (2.6) for any \( n \geq 1 \) and \( r \geq 0 \),
\[
\max(\|X_n\|_\infty, \|X_{n,r}\|_\infty) \leq 2K(1 + \ell) = \rho_\infty
\]
while when \( \ell > 0 \) we derive similarly that for any \( 0 < t < \infty \), \( n \geq 1 \) and \( r \geq 0 \),
\[
\max(\|X_n\|_t, \|X_{n,r}\|_t) \leq 2K(1 + \ell t^\ell) = \rho_t
\]
where we also used the contraction of conditional expectations. Note that \( \rho_{t_1} \leq \rho_{t_2} \) whenever \( 0 < t_1 \leq t_2 < \infty \). In our future applications of Corollary 3.2 we will always take \( \rho_{v,\infty} = \rho_\infty \) and \( \rho_{v,t} = \rho_t \) for \( 0 < t < \infty \).

Next, when (2.7) holds true and \( \ell > 0 \) then by Stirling’s approximation there exists an absolute constant \( C > 1 \) so that for any \( k \geq 1 \),
\[
\tau_{ik}^k = \mathbb{E}[\xi_i^{k}] \leq M^{k}((k!)!)^\zeta \leq C^{(\ell+1)}Q^{k}(k!)^\zeta \leq (C^{(\ell+1)}Q)^{k}(k!)^\zeta
\]
where \( Q = \ell^\ell M^\ell \geq 1 \). Therefore, the collection of numbers \( \rho_{v,k} = \rho_k \) satisfies (3.5) with \( 4KQ^{(\ell+1)}Q \) in place of \( M \) and with \( \theta = \zeta \).

Now we will verify condition (3.1). We will need first the following general result. Let \( U_i, i = 1, 2, ..., L \) be \( d_i \)-dimensional random vectors defined on the probability space \( (\Omega, \mathcal{F}, P) \) from Section 1 and \( \{C_j : 1 \leq j \leq s\} \) be a partition of \( \{1, 2, ..., L\} \). Consider the random vectors \( U(C_j) = \{U_i : i \in C_j\} \), \( j = 1, ..., s \), and let
\[
U^{(j)}(C_j) = \{U_i^{(j)} : i \in C_j\}, \ j = 1, ..., s
\]
be independent copies of the \( U(C_j)'s \). For each \( 1 \leq i \leq L \) let \( a_i \in \{1, ..., s\} \) be the unique index such that \( i \in C_{a_i} \), and for any bounded Borel function \( H : \mathbb{R}^{d_1 + d_2 + ... + d_L} \rightarrow \mathbb{R} \) set
\[
\mathcal{D}(H) = \left| \mathbb{E}H(U_1, U_2, ..., U_L) - \mathbb{E}H(U_1^{(a_1)}, U_2^{(a_2)}, ..., U_L^{(a_L)}) \right|.
\]
The following result is proved in Corollary 1.3.11 in [19] (see also Corollary 3.3 in [18]),

3.3. **Lemma.** Suppose that each \( U_i \) is \( \mathcal{F}_{m_i,n_i} \)-measurable, where \( n_{i-1} < m_i \leq n_i < m_{i+1}, \ i = 1, \ldots, L, \ n_0 = -\infty \) and \( m_{L+1} = \infty \). Then, for any bounded Borel function \( H : \mathbb{R}^{d_1+d_2+\ldots+d_L} \to \mathbb{R}, \)

\[
\mathcal{D}(H) \leq 4 \sup |H| \sum_{i=2}^{L} \phi(m_i - n_{i-1})
\]

where \( \sup |H| \) is the supremum of \( |H| \). In particular, when \( s = 2 \) then

\[
\alpha(\sigma\{U(C_1)\}, \sigma\{U(C_2)\}) \leq 4 \sum_{i=2}^{L} \phi(m_i - n_{i-1})
\]

where \( \sigma\{X\} \) stands for the \( \sigma \)-algebra generated by a random variable \( X \).

Next, in order to show that (3.1) holds true we first notice that for any set of pairs \((a_i, b_i), \ i = 1, 2, \ldots, m, \)

\[
\prod_{i=1}^{m} a_i - \prod_{i=1}^{m} b_i = \sum_{1 \leq i < j \leq m} a_j(a_i - b_i) \prod_{i<j} b_j.
\]

Let \( n_1, \ldots, n_m \in V \) and \( q \geq 0 \). When \( \iota = 0 \) using (3.17), (3.11) and (2.5) we obtain that for each \( 1 \leq i \leq m, \)

\[
|m(\varrho_{\infty})^{m-1} \max\{E|X_{n_i} - X_{n_i,q}|, 1 \leq i \leq m\} \leq m(\varrho_{\infty})^{m} \ell^\infty_{b,q}(q) |
\]

When \( \iota > 0 \) then by the contraction of conditional expectations for any \( 1 \leq i \leq m, \)

\[
\|X_{n_i} - X_{n_i,q}\| \leq K\|1 + \sum_{j=1}^{\iota} (\xi_{jn_i,q} + |\xi_{jn_i,q}|)\| + \iota \sum_{j=1}^{\iota} \|\xi_{jn_i,q}\| \leq K(1 + 2\ell^\infty_{b,q}).
\]

where \( q_m \) is defined in (3.12) Therefore by (3.17), (3.12), (2.5) and the H"older inequality,

\[
\|E\prod_{i=1}^{m} X_{n_i} - E\prod_{i=1}^{m} X_{n_i,q}\| \leq m(\varrho_{m})^{m} \ell_{b,q}(q).
\]

Now, let \( k, b \geq 1 \) and a finite collection \( A_j, j \in J \) of nonempty subsets of \( V \) be so that \( r := \sum_{j \in J} |A_j| \leq k \) and \( \rho(A_j, A_{j'}) \geq b \) whenever \( i \neq j \). Set \( q_b = \left[ \frac{b}{3} \right] \). When \( \iota = 0 \) set \( \delta = \infty \) and

\[
\gamma(b,r) = \gamma_{\infty}(b,r) = 128\ell r (\varrho q_b + \beta_{\infty}^\infty(q_b))
\]

while when \( \iota > 0 \) set \( \delta = 1 \) and

\[
\gamma(b,r) = \gamma_1(b,r) = 128\ell r (\varrho q_b + \beta_{\infty}^\infty(q_b)).
\]

We claim that in both cases (3.1) holds true with \( \varrho_{\iota,\delta} = \varrho_{\iota} \) defined in (3.12) and (3.11) and the above \( \delta \) and \( \gamma_{\delta}(b,r) \) (depending on the case). Indeed, when \( \iota = 0 \) and \( \delta = \infty \) set \( \gamma_0(b,r) = 32\ell r \varrho(q_b) \), while when \( \iota > 0 \) and \( \delta = 1 \) we set \( \gamma_{\delta}(b,r) = \)
In order to prove this claim we first assert that in both cases,
\begin{equation}
\alpha \left( \prod_{i \in J} \prod_{a_i \in A_{a_i}} X_{i,a_i} - \prod_{i \in J} \prod_{a_i \in A_{a_i}} X_{i,a_i} \right) \leq (r - 1) \left( \prod_{i \in J} \prod_{a_i \in A_{a_i}} \|X_{i,a_i}\| \|1 + \delta\|^k \right) \gamma_0(b, k).
\end{equation}

It is clear that (3.1) with these \( A_j \)'s, \( b \) and \( k \) follow from either (3.18) and (3.20) or (3.19) and (3.20), depending on the case, where when \( r \geq 2 \) we use that \( r \leq 2(r - 1) \). In order to obtain (3.20) we need first the following. Let \( \Delta_1 \) and \( \Delta_2 \) be so that \( \rho(\Delta_1, \Delta_2) \geq b \) and set \( d_1 = |\Delta_1| + |\Delta_2| \) and \( T_i = \{ j x : x \in \Delta_i, 1 \leq j \leq \ell \}, i = 1, 2 \). Then by (3.9) we have \( \text{dist}(T_1, T_2) = \rho(\Delta_1, \Delta_2) \geq b \) and so we can write
\[ T := T_1 \cup T_2 = \bigcup_{i = 1}^L C_i \]
where \( L \leq \ell d_1 \), \( c_i + b \leq c_{i+1} \) for any \( c_i \in C_i \) and \( c_{i+1} \in C_{i+1}, \ i = 1, 2, ..., L - 1 \) and each one of the \( C_i \) is either a subset of \( T_1 \) or a subset of \( T_2 \). Applying (3.16) with the random vectors \( U_i = \{ \xi_{j,a} : j \in C_i \}, i = 1, 2, ..., L \) and the partition of \( \{1, 2, ..., L\} \) into the sets \( C_1 = \{ 1 \leq i \leq L : C_i \subset T_1 \} \) and \( C_2 = \{ 1 \leq i \leq L : C_i \subset T_2 \} \) we obtain that
\begin{equation}
\alpha(\sigma\{ X_{i,a} : i \in \Delta_1 \}, \sigma\{ X_{j,a} : j \in \Delta_2 \}) \leq 4d_1 \phi(qb).
\end{equation}

Recall next that (see Corollary A.2 in [20]) for any two sub-\( \sigma \)-algebras \( \mathcal{G}, \mathcal{H} \subset \mathcal{F}, \)
\begin{equation}
\text{Cov}(\eta_1, \eta_2) \leq 8\|\eta_1\|_u \|\eta_2\|_v \alpha(\mathcal{G}, \mathcal{H})^{1 - \frac{1}{u} - \frac{1}{v}},
\end{equation}
whenever \( h_1 \) is \( \mathcal{G} \)-measurable, \( h_2 \) is \( \mathcal{H} \)-measurable and \( 1 < u, v \leq \infty \) satisfy that \( \frac{1}{u} + \frac{1}{v} < 1 \) (where we set \( \frac{1}{\infty} = 0 \)). The estimate (3.20) follows now exactly as in the paragraph preceding equality (10) in [14], relying on (3.22) and on (3.21), in place of the mixing conditions from [14]. Indeed, writing \( J = \{ 1, 2, ..., J \}, \) setting \( \Delta_1 = A_1 \) and \( \Delta_2 = \bigcup_{1 \leq i \leq J} A_i \), and applying (3.22) with \( u = \frac{1 + \delta}{|\Delta_1|} \) and \( v = \frac{1 + \delta}{|\Delta_2|} \) we obtain that
\begin{equation}
\text{Cov}\left( \prod_{i \in \Delta_1} X_{i,q}, \prod_{i \in \Delta_2} X_{i,q} \right) \leq 8\|\prod_{i \in \Delta_1} X_{i,q}\|_u \|\prod_{i \in \Delta_2} X_{i,q}\|_v \alpha^{1 - \frac{1}{u} - \frac{1}{v}},
\end{equation}
where \( \alpha = \alpha(\sigma\{ X_{i,a} : i \in \Delta_1 \}, \sigma\{ X_{j,a} : j \in \Delta_2 \}) \) and we also used that \( \alpha \leq 1 \) and
\begin{align*}
\frac{1}{u} + \frac{1}{v} &= \frac{|\Delta_1 \cup \Delta_2|}{k(1 + \delta)} \leq \frac{r}{k(1 + \delta)} = \frac{1}{1 + \delta}.
\end{align*}

Using the Hölder inequality to estimate the norms on the right hand side of (3.23) and then repeating the above arguments with \( J_i = \{ i, i + 1, ..., J \}, i = 2, 3, ..., J \) in place of \( J \) we obtain (3.20), taking into account that \( J = |J| \leq \sum_{i \in J} |A_i| \). Using either (3.11) or (3.12) we conclude that all the conditions of Corollary 5.2 are satisfied under either Assumption 2.1 or Assumption 2.2 and the proof of Theorem 2.7 is complete.

3.3. **Proof of Theorem 2.3** First, (2.3) follows from Theorem 2.7 and Lemma 2.3 in [20]. Next, suppose that \( D^2 > 0 \). Then (2.4) follows by Lemma 6.2 in [10] (which is a consequence of Lemma 2.3 in [20]). Finally, let \( a_N, N \geq 1 \) be a sequence of real numbers so that
\[
\lim_{N \to \infty} a_N = \infty \quad \text{and} \quad \lim_{N \to \infty} a_N N^{-\frac{1}{1 + \delta}} = 0.
\]
where $\gamma = \gamma_1 = \frac{1}{\eta}$ under Assumption 2.1 and $\gamma = \gamma_2 = \gamma_1 + i\zeta$ under Assumption 2.2. The variances $v_N$ grow linearly fast in $N$ and therefore by Theorem 2.7 and Theorem 1.1 in [10] the sequence $(a_N)^{-1}Z_N$, $N \geq 1$ satisfies the MDP with the speed $s_N = a_N^2$ and the rate function $I(x) = \frac{1}{2}x^2$. Since $v_N/N$ converges to $D^2 > 0$ as $N \to \infty$, $\|S_N\|$ is bounded in $N$ and $I$ is continuous we derive that $(DN^\frac{1}{2}a_N)^{-1}S_N$, $N \geq 1$ satisfies the MDP stated in Theorem 2.3 and the proof of Theorem 2.3 is complete.

3.4. **Product functions case.** Consider the situation when $F$ has the form

$$F(x_1, ..., x_L) = \prod_{i=1}^{\ell} f_i(x_i).$$

We will describe here shortly how to prove Theorem 2.3 in the situations discussed at the end of Section 2.

3.4.1. **$\alpha$-mixing case.** First, in the notations of Lemma 3.3 we obtain that (3.15) holds true for functions of the form $H(u) = \prod_{i=1}^{L} g_i(u_i)$ when all of the $g_i$’s are bounded, where $\phi(m_i - n_{i-1})$ is replaced by $4\alpha(m_i - n_{i-1})$ for $i = 2, 3, ..., L$. Indeed, setting

$$u^{(c_j)} = \{u_i : i \in C_j\} \text{ and } G_j(u^{(c_j)}) = \prod_{i \in C_j} g_i(u_i), j = 1, 2, ..., s,$$

we derive from (2.20), exactly as in the proof of Corollary 1.3.11 in [19] (or Corollary 3.3 in [18]), that

$$\tag{3.24} \left| \mathbb{E}H(U_1, ..., U_L) - \sum_{j=1}^{s} \mathbb{E}G_j(U^{(c_j)}) \right| \leq 16 \left( \prod_{j=1}^{L} \sup_{|g_j|} \sum_{i=2}^{L} \alpha(m_i - n_{i-1}) \right).$$

Note that the derivation of (3.24) is indeed possible since (2.20) holds true for arbitrary bounded $g_i$’s, appropriate $U_i$’s and partitions $C_i$’s. Relying on (3.24) we can approximate the left-hand side of (3.1) and therefore the results stated in Theorem 2.3 (i) hold true with $\alpha(n)$ in place of $\phi(n)$.

We remark that (2.20) follows, in fact, by a repetitive application of (3.22) with $u = v = \infty$. Applying (3.22) with finite $u$’s and $v$’s we obtain similar estimates when the $g_i(U_i)$’s are not bounded but only satisfy certain moment conditions, where the product $\prod_{j=1}^{L} \sup_{|g_j|}$ is replaced with an appropriate product of the form $\prod_{i=1}^{\xi} \|g_i(U_i)\|_q$ and $\alpha(m_i - n_{i-1})$ is replaced with $(\alpha(m_i - n_{i-1}))^\zeta$ for an appropriate $0 < \zeta < 1$. We refer the readers’ to the proof of (3.21) for the exact details. Relying on this “unbounded version” of (3.24), we can approximate the left-hand side of (3.1) and obtain results similar to the ones stated in Theorem 2.3 (ii) but with $\alpha(n)$ in place of $\phi(n)$.

3.4.2. **Decay of correlations case.** Let $T, H$ and $c(m), m \geq 1$ be as described at the end of Section 2. Let $n_1 < n_2 < ... < n_L$ and $g_1, ..., g_L \in H$. In the notations of Lemma 3.3 using (2.22) and the $T$-invariance of $P$, we obtain similarly to the above $\alpha$-mixing case that

$$\tag{3.25} \left| \mathbb{E}_P \prod_{i=1}^{L} g_i \circ T^{n_i} - \prod_{j=1}^{S} \mathbb{E}_P \prod_{i \in C_j} g_i \circ T^{n_i} \right| \leq 2dM^L \sum_{t=2}^{L} c(n_t - n_{t-1})$$

where $\gamma = \gamma_1 = \frac{1}{\eta}$ under Assumption 2.1 and $\gamma = \gamma_2 = \gamma_1 + i\zeta$ under Assumption 2.2. The variances $v_N$ grow linearly fast in $N$ and therefore by Theorem 2.7 and Theorem 1.1 in [10] the sequence $(a_N)^{-1}Z_N$, $N \geq 1$ satisfies the MDP with the speed $s_N = a_N^2$ and the rate function $I(x) = \frac{1}{2}x^2$. Since $v_N/N$ converges to $D^2 > 0$ as $N \to \infty$, $\|S_N\|$ is bounded in $N$ and $I$ is continuous we derive that $(DN^\frac{1}{2}a_N)^{-1}S_N$, $N \geq 1$ satisfies the MDP stated in Theorem 2.3 and the proof of Theorem 2.3 is complete.
where $M = \max\{\sup |g_i|, \|g_i\|_H : i = 1, 2, ..., L\}$). Note that when $\sum_{n=1}^{\infty} nc(n) < \infty$ then all the results stated in Theorem 2.6 are proved similarly to [26], [16] and [25] relying on [3.25] instead of the mixing assumptions from there. The inequality (3.25) also yields appropriate estimates of the left-hand side of (3.1), and we conclude that that all the results stated in Theorem 2.6 (i) hold true with $\beta_c(n) \equiv 0$ and $c(n)$ in place of $\phi(n)$.

4. Exponential inequalities via martingale approximation-proof of Theorems 2.8 and 2.4

In this section we adapt the martingale approximation technique from [26] and approximate $S_N$ in the $L^\infty$ norm by martingales with bounded differences. As in [26] we first write

$$F(x_1, ..., x_{\ell}) = \sum_{i=1}^{\ell} F_i(x_1, ..., x_i)$$

where

$$F_i(x_1, ..., x_{\ell}) = F(x_1, ..., x_{\ell}) - \int F(x_1, ..., x_{\ell-1}, z) d\mu(z)$$

and for $i = 1, 2, ..., \ell - 1$,

$$F_i(x_1, ..., x_i) = \int F(x_1, ..., x_i, z_1, ..., z_\ell) d\mu(z_1) ... d\mu(z_\ell) - \int F(x_1, ..., x_{i-1}, z_1, ..., z_\ell) d\mu(z_1) ... d\mu(z_\ell).$$

Then for each $1 \leq i \leq \ell$,

$$\int F_i(y_1, ..., y_{i-1}, z) d\mu(z) = 0, \forall y_1, ..., y_{i-1}$$

where for $i = 1$ we used that $F = 0$.

Next, recall that (see [3], Ch. 4) for any two sub-$\sigma$-algebras $G, H \subset F$,

$$2\phi(G, H) = \sup\{|\mathbb{E}[g|G] - \mathbb{E}g|_\infty : g \in L^\infty(\Omega, H, P), \|g\|_\infty \leq 1\}$$

where $\phi(G, H)$ is defined by (2.2). The following result is a version of Corollary 3.6 in [26] and Lemma 1.3.10 in [19] (see also Lemma 3.2 in [18]). It does not seem to be new but for readers' convenience and completeness we will prove it here.

4.1. Lemma. Let $G, H \subset F$ be two sub-$\sigma$-algebras of $F$ and $d \in \mathbb{N}$. Let $f(\cdot, \omega) : \mathbb{R}^d \to \mathbb{R}$ be a random function so that $f(x, \omega)$ is $H$-measurable for any fixed $x \in \mathbb{R}^d$ and $P$-a.s. for any $x, y \in \mathbb{R}^d$, 

$$|f(x, \omega)| \leq C \text{ and } |f(x, \omega) - f(y, \omega)| \leq C|x - y|^\kappa$$

where $C > 0$ and $\kappa \in (0, 1]$ are constants which do not depend on $x, y$ and $\omega$. Set $f(x, \omega) = \mathbb{E}[f(x, \cdot)|G](\omega)$ and $f(x) = \int f(x, \omega) dP(\omega) = \int f(x, \omega) dP(\omega)$. Then there exists a measurable set $\Omega' \subset \Omega$ so that $P(\Omega') = 1$, $f(x, \omega)$ is defined for all $\omega \in \Omega'$ and $x \in \mathbb{R}^d$ and

$$\sup_{x \in \mathbb{R}^d} |\tilde{f}(x, \omega) - \bar{f}(x)| \leq 2C\phi(G, H), P - a.s.$$ 

In particular, for any $\mathbb{R}^d$-valued random variable $X$,

$$|\tilde{f}(X, \omega) - \bar{f}(X)| \leq 2C\phi(G, H), P - a.s.$$
Proof. Let $\mathcal{A} = \{A_i : i \in I\}$ be a countable partition of $\mathbb{R}^d$ and denote its diameter by $\text{diam}\mathcal{A}$. For each $i \in I$ let $I_{A_i}$ be the indicator function of $A_i$ and choose some $a_i \in A_i$. Then by (4.3), $P$-a.s. for any $x \in \mathbb{R}^d$ we have
\[
|f(x, \omega) - \sum_{i \in I} I_{A_i}(x) f(a_i, \omega)| \leq C(\text{diam}\mathcal{A})^\kappa.
\]
Taking conditional expectations with respect to $\mathcal{G}$ and then the limit as $\text{diam}\mathcal{A} \to 0$ we obtain the existence of $\Omega'$ as in the statement of the lemma. Fixing $\mathcal{A}$ and taking again conditional expectations with respect to $\mathcal{G}$ we derive that
\[
\sup_{x \in \mathbb{R}^d} |\tilde{f}(x, \omega) - \sum_{i \in I} I_{A_i}(x) \tilde{f}(a_i, \omega)| \leq C(\text{diam}\mathcal{A})^\kappa, \quad P - \text{a.s.}.
\]
Similarly, we obtain by taking expectations that
\[
\sup_{x \in \mathbb{R}^d} |\tilde{f}(x) - \sum_{i \in I} I_{A_i}(x) \tilde{f}(a_i)| \leq C(\text{diam}\mathcal{A})^\kappa.
\]
Using (4.2) and (4.3) we deduce that for each $i$,
\[
|\tilde{f}(a_i, \omega) - \tilde{f}(a_i)| \leq 2\|f(a_i, \cdot)\|_{\infty} \phi(\mathcal{G}, \mathcal{H}) \leq 2C \phi(\mathcal{G}, \mathcal{H}), \quad P - \text{a.s.}
\]
and therefore, $P$-a.s.,
\[
\sup_{x \in \mathbb{R}^d} |\tilde{f}(x, \omega) - \tilde{f}(x)| \leq 2C \phi(\mathcal{G}, \mathcal{H}) + 2C(\text{diam}\mathcal{A})^\kappa.
\]
Taking the limit as $\text{diam}\mathcal{A} \to 0$ we obtain (4.4). \hfill $\square$

Next, consider the random functions $F_{i,n,r}$ given by
\[
F_{i,n,r}(x_1, \ldots, x_{i-1}, \omega) = \mathbb{E}[F_i(x_1, \ldots, x_{i-1}, 1_{n}) | \mathcal{F}_{n-r, n+r}](\omega).
\]
Note that in view of the uniform continuity of $F$ these are indeed random functions, i.e. all the random variables $F_{i,n,r}(x_1, \ldots, x_{i-1}, \cdot)$, $x_1, \ldots, x_{i-1} \in \mathbb{R}^\kappa$ can be defined on a measurable set $\Omega'$ so that $P(\Omega') = 1$. Set
\[
(4.6) \quad Y_{i,n} = F(\xi_n, \xi_{2n}, \ldots, \xi_{in}) \quad \text{and} \quad Y_{i,m} = 0 \quad \text{if} \quad m \notin \{in : n \in \mathbb{N}\} \quad \text{and} \quad Y_{i,n,m} = F_{i,n,r}(\xi_n, \xi_{2n}, \ldots, \xi_{(i-1)n}, \omega) \quad \text{and} \quad Y_{i,m,n} = 0 \quad \text{if} \quad m \notin \{in : n \in \mathbb{N}\}
\]
where we recall that $\xi_{m,n} = \mathbb{E}[\xi_m | \mathcal{F}_{m, r+m}]$ for any $m \geq 1$.

The following result is proved exactly as in the proof of Proposition 5.8 in [26] using Lemma 4.11 and the inequality $|F| \leq K(1 + \ell)$ instead of Corollary 3.6 (ii) and the moment assumptions from there.

4.2. Corollary. Suppose that $\varphi := \sum_{n=0}^\infty \phi(n) < \infty$. Then there exists a constant $B > 0$ which depends only on $\ell$ so that for any $l \geq 0$ and $r \geq 0$,
\[
\sum_{n=l}^\infty \|\mathbb{E}[Y_{i,n,r} | \mathcal{F}_{-\infty, l+r}]\|_{\infty} \leq BK(r + 1 + \varphi).
\]

Now we introduced the martingales constructed in [16] relying on ideas originated in [26]. For any $1 \leq i \leq \ell$, $n \geq 0$ and $r \geq 0$ set $R_{i,n,r} = \sum_{s \geq n+1} \mathbb{E}[Y_{s,n,r} | \mathcal{F}_{-\infty, n+r}]$ and
\[
W_{i,n,r} = Y_{i,n,r} + R_{i,n,r} - R_{i,n-1,r}.
\]
Then when $i$ and $r$ are fixed $W_{i,n,r}$, $n \geq 1$ is a martingale difference with respect to the filtration $\{\mathcal{F}_{-\infty, n+r} : n \geq 1\}$ and by Corollary 4.2
\[
(4.7) \quad \|R_{i,n,r}\|_{\infty} \leq 2BK(\varphi + r + 1)
\]
and therefore there exists a constant $B_1 > 0$ which depends only on $\ell$ so that
\[ \|W_{i,n,r}\|_\infty \leq B_1 K (\varphi + r + 1). \]
Set $W^{(N)}_{i,n,r} = \mathbb{I}_{\{n \leq iN\}} W_{i,n,r},$
\[ W^{(N)}_{n,r} = \sum_{i=1}^{\ell} W^{(N)}_{i,n,r}, \]
\[ M^{(N)}_{i,n,r} = \sum_{m=1}^{n} W^{(N)}_{i,m,r} \quad \text{and} \quad M^{(N)}_{n,r} = \sum_{i=1}^{\ell} M^{(N)}_{i,n,r}. \]
Then when $r$ and $N$ are fixed $M^{(N)}_{N,r}$, $n \geq 1$ is a martingale (with respect to the above filtration) whose differences are bounded by $\ell B_1 K (\varphi + r + 1)$. We estimate now the $L^\infty$-norm
\[ \|S_N - M^{(N,r)}_{N,\ell}\|_\infty. \]

We first write
\[ S_N - M^{(N,r)}_{N,\ell} = \sum_{i=1}^{\ell} \sum_{n=1}^{N} (Y_{i,n} - Y_{i,n,r}) + \sum_{i=1}^{\ell} (R_{i,N\ell,r} - R_{i,0,r}) \]
where we used (4.1). By replacing $\xi_{jn}$ with $\xi_{jn,r}$, $j = 1, 2, ..., i$ in the definitions of $Y_{i,n}$ and $Y_{i,n,r}$, using the Hölder continuity of $F$ and that $\xi_{\ell n,r}$ is $F_{\ell n-r,\ell n+r}$-measurable we obtain that
\[ |Y_{i,n} - Y_{i,n,r}| \leq K B_2^2 \beta_{\infty}^{\ell n} (r), \quad P - \text{a.s.} \]
for any $1 \leq i \leq \ell$, $n \in \mathbb{N}$ and $r \geq 0$, where $B_2 = B_2(\ell)$ is some constant which depends only on $\ell$. Combining this with (4.7) we obtain that
\[ (4.8) \quad \|S_N - M^{(N,r)}_{N,\ell}\|_\infty \leq B_3 K (N \beta_{\infty}^{\ell n} (r) + \varphi + r + 1) := \delta_2^\prime \]
where $B_3 = B_3(\ell)$ is another constant, and the proof of Theorem 2.3 is complete. In order to prove Theorem 2.4 we first apply the Hoeffding-Azuma inequality (see, for instance, page 33 in [28]) and obtain that for any random variable $Z$, $t_0 > 0$ and $\lambda > 0$ we have
\[ P(Z \geq t_0) \leq e^{-\lambda t_0^2}. \]
Taking $Z = S_N, t_0 = t + \delta_2^\prime$, using (2.11) and then optimizing by taking $\lambda = \frac{-t}{2N \delta_2^\prime}$ we obtain (2.12), and the proof of Theorem 2.4 is complete.

5. **Nonlinear indexes**

Let $q_1, ..., q_\ell$ be functions which map $\mathbb{N}$ to $\mathbb{N}$, are strictly increasing on some ray $[R, \infty)$ and are ordered so that
\[ q_1(n) < q_2(n) < ... < q_\ell(n) \]
for any sufficiently large $n$. For any $N \in \mathbb{N}$ consider the random variable

$$S_N = \sum_{n=1}^{N} (F(\xi_{q_i(n)}, \xi_{q_{2i}(n)}, \ldots, \xi_{q_{\ell}(n)}) - \bar{F})$$

where $\bar{F}$ is given by (2.7). We further assume that the difference $q_i(n) - q_{i-1}(n)$ tends to $\infty$ as $n \to \infty$ for any $i = 1, 2, \ldots, \ell$, where $q_0 \equiv 0$, though the situation when some of these differences are nonnegative constants can be considered, as well (see Section 3 in [17]). Next, for any $n, m \in \mathbb{N}$ set

$$\hat{\rho}(n, m) = \hat{\rho}(n, m) = \min_{1 \leq i, j \leq \ell} (|q_i(n) - q_j(m)|).$$

We will rely on the following

5.1. Assumption. There exists $Q \geq 1$ so that for any $1 \leq j \leq \ell$ and $a, b \geq q_j(R)$,

$$|q_j^{-1}(a) - q_j^{-1}(b)| \leq Q(1 + |a - b|)$$

where $q_j^{-1}$ is the inverse of $q_j$ to the ray $[R, \infty)$.

Set $\tilde{A}_s(n, N) = \{1 \leq m \leq N : \hat{\rho}(n, m) \leq s\}$. When (5.2) holds true then for any $1 \leq n \leq N$ and $s \geq 1$,

$$|\tilde{A}_s(n, N)| \leq Q\ell^2(1 + 2s) \leq 3\ell^2Qs$$

which means that (5.2) holds true with $c_0 = 3Q\ell^2$ and $u_0 = 1$. Condition (5.2) holds true, for instance, when all $q_j$’s have the form $q_j(x) = [p_j(x)]$ where each $p_j$ is a strictly increasing function whose inverse $p_j^{-1}$ has bounded derivative on some ray $[K, \infty)$. For example we can take $p_j$’s to be a polynomial with positive leading coefficient, exponential function etc.

We conclude that under Assumption (5.1) the results stated in Theorem 2.7 hold true. Therefore, (2.8) holds true and all the results stated in Theorem 2.3 hold true when $D^2$ exists and it is positive. The limit $D^2$ exists when $q_i$’s satisfy the conditions from [20] or, as in [17], when they are polynomials taking integer values on the integers. See [16] and [17] for conditions equivalent to $D^2 > 0$. Note also that for such $q_i$’s Theorem 2.8 holds true, as well, since the martingale approximation method was applied in [26] and [16] successfully, and so the arguments from Section 3 can be repeated.

5.2. Remark. Let $q(n), n \geq 1$ be a strictly increasing sequence of natural numbers, and consider the process $\tilde{\xi}_n, n \geq 1$ given by $\tilde{\xi}_n = \xi_{q(n)}$. Set $\tilde{F}_{m,n} = F_{q(m),q(n)}$ and let $\tilde{\phi}(n)$ and $\tilde{\beta}_q(n)$ be defined similarly to $\phi(n)$ and $\beta_q(n)$ but with the $\tilde{F}_{m,n}$’s in place of the $F_{m,n}$’s. Then $\tilde{\beta}_q(n) \leq \beta_q(q(n))$ and $\tilde{\phi}(n) \leq \phi(j(n))$, where

$$j(n) = \inf_{m \geq 1} (q(m + n) - q(m)).$$

When $q(n), j(n) \geq cn^l$ for some $l \geq 2$ and $c > 0$ then the mixing and approximation coefficients $\tilde{\phi}(n)$ and $\tilde{\beta}_q(n)$ converge to 0 faster than $\phi(n)$ and $\beta_q(n)$, and by writing $s = q(s') \geq c(s')^l$ we can take $u_0 = \frac{1}{s'}$ in (5.2). Repeating the arguments from the proof of Theorem 2.7 we obtain similar estimates of $|\Gamma_k(S_N)|$, but with $\gamma_1' = \frac{1}{m} < \gamma_1$ in place of $\gamma_1 = \frac{1}{s'}$. The assumption that the distribution of $(\xi_n, \xi_m)$ depends only on $n - m$ was only needed in order for $D^2$ to exist and for obtaining convergence rate towards it. Therefore, (2.8) and the corresponding estimate from Theorem 2.3 (ii) hold true with $\xi_{q(n)}$ and $\gamma_1'$ in place of $\xi_n$ and $\gamma_1$, respectively. If we know
that the limit $D^2$ exists (after this replacement) then all the other results stated in Theorem 2.3 also hold true with $\frac{1}{p^2}$ in place of $\frac{1}{p}$.

Consider, for instance, the case when $q_i$’s are polynomials and $q(n) = n^l$ for some $l \geq 2$, namely, nonconventional sums of the form

$$\tilde{S}_N = \sum_{n=1}^{N} F(\xi_{n_1}, \xi_{n_2}, ..., \xi_{n_l})$$

(5.3)

when all $p_i$’s are polynomials. Then the limit $D^2$ exists (see [17]) and so all the results described above hold true.

6. ADDITIONAL RESULTS

6.1. The CLT and Berry-Esseen type estimates. We recall first the following result (see Corollary 2.1 in [30]),

6.1. Lemma. Let $W$ be a random variable. Suppose that there exist $\gamma \geq 0$ and $\Delta > 0$ so that for any $k \geq 3$,

$$|\Gamma_k(W)| \leq (k!)^{1+\gamma}\Delta^{-(k-2)}.$$

Let $\Phi$ be the standard normal distribution. Then,

$$\sup_{x \in \mathbb{R}} |P(W \leq x) - \Phi(x)| \leq c_\gamma \Delta^{\frac{1}{1+2\gamma}},$$

where $c_\gamma = \frac{1}{6} \left( \frac{\sqrt{2\pi}}{\sqrt{\eta}} \right)^{\frac{1}{1+2\gamma}}$.

Note that when $|\Gamma_k(W)| \leq C(k!)^{1+\gamma}\Delta^{-(k-2)}$, $k \geq 3$ for some constant $C \geq 1$ then the conditions of Lemma 6.1 are satisfied with $\Delta C^{-1}$ in place of $\Delta$. This lemma together with the cumulants’ estimates obtained in Theorem 2.7 yields convergence rates in the nonconventional CLT for $S_N \approx \sqrt{N}$ which (when $\eta = 1$) are at best of order $N^{-\frac{1}{2}}$, since in our circumstances $\Delta$ is of order $N^{\frac{1}{2}}$ and $\gamma \geq 1$, where in the case when $F$ is bounded we can take $\gamma = 1$. The rate $N^{-\frac{1}{2}}$ is better than the ones obtained in [16], which is important since the rates obtained in [18] and [19] do not apply to the cases considered in Section 5. Note that, in fact, we obtain here for the first time the CLT under condition (2.21) when $F$ has the form (2.27).

6.2. Remark. Consider the case discussed in Remark 5.2 when all $q_i$’s have the form $q_i(n) = p_i(n^l)$ for some polynomials $p_1, ..., p_l$ and an integer $l \geq 2$, namely the sums $N^{-\frac{1}{2}} \tilde{S}_N$ where $\tilde{S}_N$ is defined in (5.3). Then under Assumption 2.1 we obtain (when $D^2 > 0$) closer to optimal rates. Indeed, in these circumstances Theorem 2.7 holds true with $\gamma' = \frac{1}{\eta}$ in place of $\gamma = \frac{1}{\eta}$ and so, using the equality $\Gamma_k(aW) = a^k \Gamma_k(W)$, $a \in \mathbb{R}$, we can apply now Lemma 6.1 with $\gamma = \gamma'$ and $\Delta$ of the form $\Delta = c \sqrt{N}$ and obtain rates of order $N^{-\frac{1}{2}+\gamma'(a^{2\gamma'-1})}$, which are better than $N^{-\frac{1}{2}}$ when $\eta = 1$.

6.2. Moment estimates of Gaussian type. Theorem 2.7 also implies the following

6.3. Theorem. Suppose that the conditions of Theorem 2.7 hold true. Let $Z$ be a random variable which is distributed according to the standard normal law. Then for any $p \geq 1$,

$$|E(\tilde{S}_N)^p - (\text{Var} \tilde{S}_N)^{\frac{1}{2}} E Z^p| \leq N(p!)^{1+\gamma} c^p (c_{0,1})^p$$
where $c_{0,1} = \max(1,c_0), \gamma = \gamma_1$ when Assumption 2.1 holds true, $\gamma = \gamma_2$ when Assumption 2.2 holds true and $c_0, \gamma_1$ and $\gamma_2$ are specified in Theorem 2.7.

Proof. The arguments below are based on the proof of Theorem 3 in [12]. By formula (1.53) in [30], for any $p \geq 1$ and $N \in \mathbb{N}$,

$$\mathbb{E}(\bar{S}_N)^p = \sum_{1 \leq u \leq \frac{p}{2}} \frac{1}{u!} \sum_{k_1 + k_2 + \ldots + k_u = p} \frac{p!}{k_1! \ldots k_u!} \Gamma_{k_1} (\bar{S}_N) \cdots \Gamma_{k_u} (\bar{S}_N).$$

Let $1 \leq u \leq \frac{p}{2}$. When $k_i = 1$ for some $1 \leq i \leq u$ then $\Gamma_{k_i} (\bar{S}_N) = \mathbb{E} S_N = 0$ and so the corresponding summand $\frac{p!}{k_1! \ldots k_u!} \Gamma_{k_1} (\bar{S}_N) \cdots \Gamma_{k_u} (\bar{S}_N)$ vanishes. When $p$ is even and $u = \frac{p}{2}$ then the unique non-vanishing summand corresponds to the choice of $k_i = 2$, $i = 1, 2, \ldots, u$ and it equals $(\text{Var} S_N)^{\frac{p}{2}} \mathbb{E} Z^p$. When $p$ is odd then $\mathbb{E} Z^p = 0$, and therefore for any $p \geq 1$,

$$|\mathbb{E}(\bar{S}_N)^p - (\text{Var} S_N)^{\frac{p}{2}} \mathbb{E} Z^p| \leq \sum_{1 \leq u \leq \frac{p}{2}} \frac{1}{u!} \sum_{k_1 + k_2 + \ldots + k_u = p} \frac{p!}{k_1! \ldots k_u!} \left| \frac{\Gamma_{k_1} (\bar{S}_N) \cdots \Gamma_{k_u} (\bar{S}_N)}{k_1! \ldots k_u!} \right|.$$

Applying the Hölder inequality to Euler’s $\Gamma$ function we obtain that $(k!)^p \leq (p!)^k$ for any integers $k$ and $p$ so that $1 \leq k \leq p$. Using Theorem 2.7 we derive that

$$\left| \frac{\Gamma_{k_1} (\bar{S}_N) \cdots \Gamma_{k_u} (\bar{S}_N)}{k_1! \ldots k_u!} \right| \leq N c_{0,1}^u \prod_{i=1}^u (k_i)! \sum_{i=1}^u k_i - 2u \left( \prod_{i=1}^u (k_i)! \right)^{\gamma} \leq (c_{0,1})^p (p!)^{1+\gamma}$$

for any $1 \leq k_1, \ldots, k_u$ so that $\sum_{i=1}^u k_i = p$, where $\gamma$ is described in the statement of Theorem 6.3. Thus,

$$|\mathbb{E}(\bar{S}_N)^p - (\text{Var} S_N)^{\frac{p}{2}} \mathbb{E} Z^p| \leq (c_{0,1})^p (p!)^{1+\gamma} \sum_{1 \leq u \leq \frac{p}{2}} \frac{\mathcal{N}(u,p)}{u!}$$

where

$$\mathcal{N}(u,p) := |\{2 \leq k_1, \ldots, k_u \leq p : \sum_{i=1}^u k_i = p\}| \leq \sum_{1 \leq k_1, \ldots, k_u \leq p : \sum_{i=1}^u k_i = p} \left\lfloor \frac{p}{u!} \right\rfloor.
$$

We conclude from the above estimates that for any integer $p \geq 1$,

$$|\mathbb{E}(\bar{S}_N)^p - (\text{Var} S_N)^{\frac{p}{2}} \mathbb{E} Z^p| \leq N(c_{0,1})^p (p!)^{1+\gamma} \sum_{1 \leq u \leq \frac{p}{2}} \frac{p^u}{(u!)^2} \leq N(c_{0,1})^p (p!)^{1+\gamma} e^p$$

and the proof of Theorem 6.3 is complete. \hfill \Box

We note that this theorem yields an appropriate Rosenthal type inequality for the nonconventional sums $\bar{S}_N$ and that, in fact, makes the method of moments effective for them, which provides an additional proof of the nonconventional central limit theorem. See Remarks 4 and 5 in [12], where we also use (2.16) (which implies that $N^{-1} \text{Var} S_N$ converges to $D^2$ as $N \to \infty$).
Nonconventional moderate deviations theorems and exponential concentration inequalities

References

[1] J. Aaronson, M. Denker, Local Limit Theorems for Gibbs-Markov Maps, Stoch. Dyn. 1 (2001), 193-237.
[2] R. Bowen, Equilibrium states and the ergodic theory of Anosov diffeomorphisms, Lecture Notes in Mathematics, volume 470, Springer Verlag, 1975.
[3] R.C. Bradley, Introduction to Strong Mixing Conditions, Volume 1, Kendrick Press, Heber City, 2007.
[4] S. Chatterjee, An introduction to large deviations for random graphs, Bull. Amer. Math. Soc. 53 (2016) 617-642.
[5] S. Chatterjee, Nonlinear large deviations, Adv. Math. 299 (2016) 396-450.
[6] J.R. Chazottes and S. Gouëzel, Optimal concentration inequalities for dynamical systems, Comm. Math. Phys. 316 (2012), 843-889.
[7] J.R. Chazottes, Fluctuations of observables in dynamical systems: from limit theorems to concentration inequalities, Nonlinear dynamics new directions, Vol. 11, Nonlinear Syst. Complex., pages 4785. Springer, 2015.
[8] D. Dedecker, P. Doukhan, G. Lang, J.R. Len, S. Louhichi, C. Prieur, Weak Dependence: With Examples and Applications, Lecture Notes in Statistics, vol 190, Springer-Verlag, Berlin (2007).
[9] A. Dembo and O. Zeitouni, Large Deviations Techniques and Applications, 2nd edn. Applications of Mathematics, vol. 38. Springer, New York (1998).
[10] H. Döring and P. Eichelsbacher, Moderate deviations via cumulants, J. Theor. Probab. 26 (2013), 360-385.
[11] P. Doukhan, Mixing: Properties and Examples, Lecture Notes in Statistics, Vol. 85, Springer, Berlin (1994).
[12] P. Doukhan and M. Neumann, Probability and moment inequalities for sums of weakly dependent random variables, with applications, Stochastic Process. Appl. 117 (2007), 878-903.
[13] H. Furstenberg, Nonconventional ergodic averages, Proc. Symp. Pure Math. 50 (1990), 43-56.
[14] A.B. Grochakov, Upper estimates for semi invariants of the sum of multi-indexed random variables, Discrete Math. Appl. 5 (1995), 317-331.
[15] Y. Hafouta and Yu. Kifer, A nonconventional local limit theorem, J. Theor. Probab. 29 (2016), 1524-1553.
[16] Y. Hafouta and Yu. Kifer, Berry-Esseen type estimates for nonconventional sums, Stoch. Proc. Appl. 126 (2016), 2430-2464.
[17] Y. Hafouta and Yu. Kifer, Nonconventional polynomial CLT, Stochastics, 89 (2017), 550-591.
[18] Y. Hafouta, Stein’s method for nonconventional sums, arXiv:1704.01094.
[19] Y. Hafouta and Yu. Kifer, Nonconventional limit theorems and random dynamics, World Scientific, Singapore, 2018.
[20] P.G. Hall and C.C. Hyde, Martingale central limit theory and its application, Academic Press, New York, 1980.
[21] N.T.A. Haydn and F. Yang, Local escape rates for σ-mixing dynamical systems, arXiv preprint 1806.07148.
[22] R.S. Kallabis and M.H. Neumann, An exponential inequality under weak dependence, Bernoulli 12 (2006) 333-350.
[23] Yu. Kifer, Nonconventional limit theorems , Probab. Th. Rel. Fields, 148 (2010), 71-106.
[24] Yu. Kifer, A nonconventional strong law of large numbers and fractal dimensions of some multiple recurrence sets, Stoch. Dynam. 12 (2012), 1150023.
[25] Yu. Kifer, Strong approximations for nonconventional sums and almost sure limit theorems, Stochastic Process. Appl. 123 (2013), 2286-2302.
[26] Yu. Kifer and S.R.S Varadhan, Nonconventional limit theorems in discrete and continuous time via martingales, Ann. Probab. 42 (2014), 649-688.
[27] Yu. Kifer and S.R.S. Varadhan, Nonconventional large deviations theorems, Probab. Th. Rel. Fields 158 (2014), 197-224.

[28] V.D. Milman and G. Schechtman, Asymptotic theory of finite-dimensional normed spaces, Lecture Notes in Mathematics, Vol. 1200, Berlin: Springer-Verlag, 1986, (With an appendix by M. Gromov).

[29] V. Mayer, B. Skorulski and M. Urbański, Distance expanding random mappings, thermodynamical formalism, Gibbs measures and fractal geometry, Lecture Notes in Mathematics, vol. 2036 (2011), Springer.

[30] L. Saulis and V.A. Statulevicius, Limit Theorems for Large Deviations, Kluwer Academic, Dordrecht, Boston, 1991.

[31] L.S. Young, Statistical properties of dynamical systems with some hyperbolicity, Ann. Math. 7 (1998) 585-650.

[32] L.S. Young, Recurrence time and rate of mixing, Israel J. Math. 110 (1999) 153-88.

Institute of Mathematics, The Hebrew University, Jerusalem 91904, Israel
E-mail address: yeor.hafouta@mail.huji.ac.il