Neutrino Oscillations in the Vacuum

Hosam Mohammed\textsuperscript{1,2}*, Jarah Evslin\textsuperscript{1,2}† and Emilio Ciuffoli\textsuperscript{1}‡

1) Institute of Modern Physics, NanChangLu 509, Lanzhou 730000, China
2) University of the Chinese Academy of Sciences, YuQuanLu 19A, Beijing 100049, China

Abstract

The initial configuration of a neutrino experiment consists of wave packets, created by the experimenter. Unmeasured final states are essentially delocalized, but it is conventional to nonetheless treat them as localized wave packets as they are in a sense measured by the environment. In this note we consider an experiment in the vacuum, and so there is no environment and the unmeasured final states are not localized. We introduce a simplified but consistent model of neutrino interactions and numerically evolve the system over a finite time using the Hamiltonian to obtain the oscillation probability. All particles involved in neutrino production or detection, localized or not, are treated and evolved consistently in quantum field theory, keeping all entanglements.

1 Introduction

In the standard, wave packet treatment of neutrino oscillations \cite{1, 2}, neutrino wave packets of each mass eigenstate are created simultaneously at the same position. They then separate with time leading to decoherence. On the other hand, in Ref. \cite{3}, the authors argue that the neutrinos are measured in a flavor eigenstate at a fixed moment in space and time, and so the mass eigenstates must have been emitted separately and then coalesced. Clearly at most one of these descriptions is realized in a given experiment, and one would like to know which. Similarly in Ref. \cite{4} the Daya Bay collaboration analyzed their data using the oscillation probability found in Ref. \cite{5}, which they note is distinct from that of Ref. \cite{2}. The difference, as is described in Ref. \cite{6}, arises from a difference in the functional form of the wave packets considered. It is not known which functional form is correct.

The reason for these ambiguities is as follows. The neutrino wave packet is only an approximation to the true state of the quantum field theory. As has been stressed in Ref. \cite{7},

\*hosam@impcas.ac.cn
†jarah@impcas.ac.cn
‡emilio@impcas.ac.cn
the true state is actually an entangled state of the neutrino with various particles involved in its production which are in turn entangled with the environment. Unmeasured particles, such as the muon that is emitted in pion decay in accelerator neutrino experiments, interact with the environment. Interaction with the environment is similar to a measurement in that it leads to distinct final states, indeed distinct superselection sectors, which must be summed incoherently [9]. The environment is complicated, and so can only be treated explicitly in the simplest of cases, such as [10]. Therefore the usual approach is to not include the environment explicitly, but rather to include this effective measurement by modifying the states of the particles directly involved in neutrino production and measurement.

There are at least two distinct prescriptions for such an effective treatment of environmental interactions. First, in the external wave packet approach [11], measurement (of a position) by the environment is in a sense included [12, 13] by projecting the unmeasured particles onto localized wave packets. On the other hand, in Ref. [8] the measurement of the neutrino production time by the environment is included by restricting this time to a fixed window.

Such effective descriptions lead to two basic problems. First of all, within the external wave packet or the time window approach, there are unmeasured parameters that need to be fixed. Namely, one must fix the wave packet size or the size of the time window. Estimates of these parameters in the literature differ by many orders of magnitude [14, 15, 16, 17] as they are generally based on ill-defined concepts, such as the mean time between interactions. Of course this time is equal to zero, a correct treatment should somehow incorporate the phase introduced by each interaction [11]. One may also worry about the wave packet shape [18], for example whether it must be covariant [5]. Second, if there are two such different ways of incorporating environmental interactions, one may wonder if either is correct.

Our goal in this line of research is to attempt to derive the environmental effects directly from the microphysics of the environmental interactions. We want to create a dictionary which describes how various environmental interactions affect neutrino oscillations. We suspect that they can be characterized by an effective resolution with which the neutrino baseline is determined and also an effective energy resolution. In the external wave packet model, these two resolutions are related by the kinematics of the interaction. For example, if all particles involved in neutrino production and detection move very slowly, then the corresponding wave packets will have a large temporal overlap as compared with their spatial overlap and so the neutrino energy will be fixed more precisely than its baseline. On the other hand, in the time window approach, the energy resolution can be tuned independently of the position resolution by changing the size of the time window. We expect that in a fully microscopic treatment of environmental interactions, it will be possible to interpolate...
between these two cases depending on the environmental interactions chosen. Thus we expect
to find a richer set of possibilities than can be realized in either model.

To systematically investigate the effects of various environmental interactions, we must
first define a baseline model in which environmental interactions are not present. That is
the goal of the present paper. In such a model, unmeasured particles must not be localized
into wave packets. Also all particles must be treated consistently in quantum field theory.
Therefore all wave packets are treated dynamically and all entanglements are kept. Fur-
thermore, as the importance of finite time effects is not known, we do not calculate the S
matrix and we do not use the interaction picture. Instead we restrict our study to finite
time, starting with an initial condition at time 0 and calculating probability densities for
evolution to various states at time $t$. We provide the most robust possible treatment: we
simply evolve the state at time 0 to time $t$ using the Hamiltonian. Thus we do not need to
impose that either the source or the detector is stationary.

This program was initiated in Ref. [19] where we introduced a model of neutrino produc-
tion. This alone was not sufficient to compare even qualitatively with neutrino oscillation
phenomenology, as detection was not incorporated into the model. In the current note
we complete the model of [19] by including neutrino detection. As this model is not the
electroweak model, quantitative predictions for neutrino oscillations will not be possible.
However the qualitative questions that we are trying to answer are independent of the de-
tails of the model. We feel that a robust treatment of a simplified model is more useful than
a truncated treatment of the correct model.

We define the model in Sec. 2. Then in Sec. 3 we use the Hamiltonian evolution of
the state to derive a closed form expression for the probability of detecting a neutrino as
a function of position. In Sec. 4 we fix the parameters of the detector so as to maximize
its detection efficiency and, for a fixed set of parameters, we numerically evaluate the ap-
pearance and disappearance channel oscillation probabilities as a function of the baseline $x$
between the neutrino source and detector. Finally in Sec. 5 we develop two semianalytic
approximations which roughly reproduce our results. Sec. 6 describes other approaches to
neutrino decoherence in the literature.

2 The Model

We consider a model in (1+1)-dimensions consisting of six real scalar fields. At the source
there is a heavy and light field $\phi_{SH}$ and $\phi_{SL}$ and at the detector a heavy and light field $\phi_{DH}$
and $\phi_{DL}$. These have masses $M_{SH}$, $M_{SL}$, $M_{DH}$ and $M_{DL}$ respectively. There are also two
| Field | Description |
|-------|-------------|
| $\phi_{SL}$ | The light source field |
| $\phi_{SH}$ | The heavy source field |
| $\phi_{DL}$ | The light detector field |
| $\phi_{DH}$ | The heavy detector field |
| $\psi_1$ | The light neutrino field |
| $\psi_2$ | The heavy neutrino field |

| Symbol | Description |
|--------|-------------|
| $M_{\alpha I}$ | The mass of $\phi_{\alpha I}$ |
| $m_i$ | The mass of $\psi_i$ |
| $E_{\alpha I}$ | The on-shell energy of $\phi_{\alpha I}$ |
| $e_i$ | The on-shell energy of $\psi_i$ |
| $\mathcal{E}_0$ | Total on-shell energy before neutrino emission |
| $\mathcal{E}_{1i}$ | Total on-shell energy between emission and absorption of $\psi_i$ (Process A) |
| $\hat{\mathcal{E}}_{1i}$ | Total on-shell energy between emission and absorption of $\psi_i$ (Process B) |
| $\mathcal{E}_2$ | Total on-shell energy after neutrino absorption |
| $F$ | A function of the $\mathcal{E}$'s which oscillates rapidly if far off-shell |
| $A(k, l)$ | Amplitude for a final state with source (detector) momentum $l$ ($k$) |
| $P(k, l)$ | Probability density for a final state with source (detector) momentum $l$ ($k$) |
| $P(k)$ | Probability density for a final state with detector momentum $k$ |

Table 1: Summary of Notation
scalar fields with mass eigenstates $\psi_1$ and $\psi_2$ which will play the role of the neutrino. Their masses will be denoted $m_1$ and $m_2$. We will refer to these scalars as neutrinos.

The evolution of these fields is described by the standard free massive scalar Hamiltonian $H_0$ plus an interaction term

$$H_I = \int dx \cdot \mathcal{H}_I(x) ; \quad \mathcal{H}_I(x) = \sum_{\alpha = \{S,D\}} \phi_{\alpha H}(x) \phi_{\alpha L}(x) (\psi_1(x) + \psi_2(x)).$$

Here the sum $\psi_+ = \psi_1 + \psi_2$ plays the role of a flavor eigenstate and $\mathcal{H}_I(x)$ denotes the standard normal ordering of the creation and annihilation operators defined below. We remind the interested reader that in 1+1 dimensions, normal-ordering of such theories renders them finite and so no renormalization is necessary. The state $\psi_- = \psi_1 - \psi_2$ plays the role of the other flavor eigenstate. Thus the decay of the heavy source particle creates $\psi_+$ which later oscillates to $\psi_-$ and back. We will consider the disappearance channel in which $\psi_+$ is measured and more briefly the appearance channel, in which $\psi_-$ is measured.

Note that our toy model neutrinos are created in two body decays, such as the decay $\pi^+ \rightarrow \mu^+ + \nu_\mu$. However the kinematics of the decay will not be essential, and we are in fact motivated by reactor neutrinos. In this case $\phi_{DH}$ represents a nucleus which $\beta$ decays to $\psi_{DL}$ plus an electron antineutrino and an electron. The electron is not present in our model. The electron antineutrino corresponds to $\psi_+$. Imagine that there were only two flavors of neutrino. Then $\psi_1$ and $\psi_2$ would be the two mass eigenstates. The other flavor eigenstate, for example the muon antineutrino, would be $\psi_-$. Similarly we identify $\phi_{DL}$ with a free proton which detects electron antineutrinos via inverse $\beta$ decay, producing $\phi_{DH}$ which represents the neutron. The positron is not included in our model.

One may object that $\psi_-$ cannot be produced and so our mass matrix appears to be nonunitary. In the real world a $\beta$ decay may produce a muon antineutrino but it would be hopelessly off-shell because a muon must be produced simultaneously. There is no muon in our model, but if desired we could mock up this effect in our model by including another coupling of $\phi_{SH}$ to $\psi_-$ and also a new very heavy $\phi_{SVH}$ which plays the role of the muon, and so guarantees that any $\psi_-$ produced by $\phi_{SH}$ decay will be hopelessly off-shell. The same may be done with the detector particles. In this way, our model may be given an ultraviolet completion in terms of a unitary mass matrix. However we will not be interested in an ultraviolet completion of our model in this paper.
The canonical scalar fields will be decomposed in the Schrödinger picture as 

$$\phi_{\alpha I}(x) = \int \frac{dp}{2\pi} \frac{1}{\sqrt{2E_{\alpha I}(p)}} (A_{\alpha I,-p} + A_{\alpha I,p}^\dagger) e^{-ipx}, \quad E_{\alpha I}(p) = \sqrt{M_{\alpha I}^2 + p^2}$$

$$\Pi_{\alpha I}(x) = -i \int \frac{dp}{2\pi} \frac{E_{\alpha I}(p)}{2} (A_{\alpha I,-p} - A_{\alpha I,p}^\dagger) e^{-ipx}$$

$$\psi_{i}(x) = \int \frac{dp}{2\pi} \frac{1}{\sqrt{2e_i(p)}} (a_{i,-p} + a_{i,p}^\dagger) e^{-ipx}, \quad e_i(p) = \sqrt{m_i^2 + p^2}$$

$$\pi_{i}(x) = -i \int \frac{dp}{2\pi} \frac{e_i(p)}{2} (a_{i,-p} - a_{i,p}^\dagger) e^{-ipx}.$$  (2.2)

We remind the reader that in the Schrödinger picture even interacting fields admit such a decomposition with $A$ and $A^\dagger$ and also $a$ and $a^\dagger$ satisfying the Heisenberg algebra. Here and throughout the paper Greek indices run over $S$ and $D$, capital Roman indices over $H$ and $L$ and lower case Roman indices over 1 and 2.

We will be interested in states that have precisely one source particle, one detector particle and zero or one neutrinos. These will be denoted respectively by

$$|I, p_2; J, p_1\rangle = A_{DI,p_2}^\dagger A_{SJ,p_1}^\dagger |\Omega\rangle, \quad |I, p_2; J, p_1; i, q\rangle = a_{i,q}^\dagger |I, p_2; J, p_1\rangle$$  (2.3)

where the free particle ground state $|\Omega\rangle$ is annihilated by all $a$ and $A$ operators. To avoid clutter in the formulas, we will not normalize the states or probabilities.

These states are eigenstates of the bare Hamiltonian $H_0$. We will need the following

$$H_0|L, p_2; H, p_1\rangle = \mathcal{E}_0(p_1, p_2) |L, p_2; H, p_1\rangle$$

$$H_0|L, p_2; L, p_1; i, q\rangle = \mathcal{E}_{1i}(p_1, p_2, q) |L, p_2; L, p_1; i, q\rangle$$

$$H_0|H, p_2; H, p_1; i, q\rangle = \tilde{\mathcal{E}}_{1i}(p_1, p_2, q) |H, p_2; H, p_1; i, q\rangle$$

$$H_0|H, p_2; L, p_1\rangle = \mathcal{E}_2(p_1, p_2) |H, p_2; L, p_1\rangle$$  (2.4)

where the eigenvalues are

$$\mathcal{E}_0(p_1, p_2) = E_{SH}(p_1) + E_{DL}(p_2), \quad \mathcal{E}_{1i}(p_1, p_2, q) = E_{SL}(p_1) + E_{DL}(p_2) + e_i(q)$$

$$\tilde{\mathcal{E}}_{1i}(p_1, p_2, q) = E_{SH}(p_1) + E_{DH}(p_2) + e_i(q), \quad \mathcal{E}_2(p_1, p_2) = E_{SL}(p_1) + E_{DH}(p_2).$$  (2.5)
We will also need the action of the interaction Hamiltonian $H_I$ on some of the states

$$H_I|L, p_2; H, p_1\rangle \supset \sum_{i=1}^2 \int \frac{dq}{2\pi} \left( \frac{|L, p_2; L, p_1 - q; i, q\rangle}{\sqrt{8E_{SL}(p_1 - q)E_{SH}(p_1)e_i(q)}} + \frac{|H, p_2 - q; H, p_1; i, q\rangle}{\sqrt{8E_{DL}(p_2)E_{DH}(p_2 + q)e_i(q)}} \right)$$

$$\mathcal{P}H_I|L, p_2; L, p_1 - q; i, q\rangle = \frac{|H, p_2 + q; L, p_1 - q\rangle}{\sqrt{8E_{DL}(p_2)E_{DH}(p_2 + q)e_i(q)}}$$

$$\mathcal{P}H_I|H, p_2 - q; H, p_1; i, q\rangle = \frac{|H, p_2 + q; L, p_1 - q\rangle}{\sqrt{8E_{SL}(p_1 - q)E_{SH}(p_1)e_i(q)}}$$

\(\mathcal{P}\) projects onto the subspace of the Fock space with one light source particle, one heavy detector particle and no other particles. In the first equation, we only considered the terms in the Fock space with precisely two source particles and a neutrino, although actually there will be two more terms in the Fock space with four source particles and a neutrino. In applications to neutrino physics, such terms could correspond for example to processes in which a virtual nucleon or meson is produced along with the neutrino. At current reactor and accelerator experiments, such processes are strongly suppressed and so we do not consider these terms here.

3 Analytic Results

We will consider the following experiment. We begin with a heavy source particle and a light detector particle. We measure the heavy detector particle at fixed time $t$. We will work to leading nonvanishing order in $H_I$. The leading order contribution comes at order $H_I^2$ at which two processes contribute, shown in Fig. 1. In process A, the heavy source particle decays to a light source particle and a neutrino. The neutrino is then absorbed by the light detector particle, yielding the desired heavy detector particle. In process B, the light detector particle emits a virtual neutrino and becomes a heavy detector particle. The virtual neutrino is absorbed by the heavy source particle which becomes light. In process B, the neutrino is necessarily well off-shell and so its contribution will always be small for a large separation between the source and the detector, but we keep it for completeness.

When is this leading order justified? Clearly it is not possible to begin with $\phi_{SH}$ and $\phi_{DL}$ and finish with $\phi_{SL}$ and $\phi_{DH}$ with less than two interactions. However it is possible with more. Our approximation is justified when the two-interaction amplitude dominates over the sum of the multi-interaction amplitudes. What could ruin such an approximation? If the kinetic energy of the neutrino $E$ were much greater than the mass of the $\phi$ particles, then one would imagine that there would be many loop corrections of order $E^2/M_\phi^2$ which could be large. We will always choose initial conditions that avoid this regime, as the creation
Figure 1: At leading order, contributions to the amplitude come from the same Feynman diagram which encodes two physical processes. In process A (left) the source emits a neutrino which is absorbed by the detector. In process B (right) a neutrino travels from the detector to the source. The neutrino in process B is always off-shell.

of virtual nucleons in a neutrino beam is not relevant at reactor neutrino experiments. Furthermore if one considers a time interval such that the probability of $\phi_{SH}$ decay is of order unity, then higher order corrections reproduce the reduction in decay probability with time as the source is depleted. Therefore the equations below cannot be trusted at time scales of order or greater than the half-life of $\phi_{SH}$. In the case of reactor experiments, this means that our approximations will not describe the reactor fuel evolution, but can safely be applied to analyze the data on any given day. This is sufficient for our purposes, as we are interested in the decoherence involved in individual events which last a small fraction of a second. One day is much longer than any relevant time scale in this problem. Needless to say, loop corrections to electroweak processes are rarely relevant in neutrino physics, but the above caveat is important for interpreting the large $t$ semianalytic approximation of Subsec. 5.1.

The relevant amplitude for this process is

$$A(k, l, p_1, p_2) = \langle H, k; L, l | e^{-iHt} | L, p_2; H, p_1 \rangle. \quad (3.1)$$

As the final state in this leading order process always consists of a heavy detector particle and a light source particle, we may insert the projection operator $\mathcal{P}$ from Eq. (2.6)

$$A_i(k, l, p_1, p_2) = \langle H, k; L, l | \mathcal{P} e^{-iHt} | L, p_2; H, p_1 \rangle. \quad (3.2)$$

We are thus interested in those terms in

$$\mathcal{P} e^{-iHt} | L, p_2; H, p_1 \rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} c_n, \quad c_n = \mathcal{P} (H_0 + H_I)^n | L, p_2; H, p_1 \rangle \quad (3.3)$$
which have precisely two powers of $H_I$. These are the leading terms in $c_n$ in perturbation theory. In other words, if we chose to multiply $H_I$ by a parameter $g$ and expand in $g$, then the leading contribution to the amplitude would be of order $g^2$ and would be given by these terms. We will drop higher order terms and, since all amplitudes will be proportional to $g^2$, we will simply drop the $g^2$ to avoid clutter.

The leading order terms are

$$c_n = \sum_{j=0}^{n-2} \sum_{k=0}^{n-j-2} \mathcal{P} H_0^j H_I^k H_I H_0^{n-j-k-2} |L, p_2; H, p_1).$$

Substituting in Eqs. (2.4), (2.5) and (2.6) one finds

$$c_n = \sum_{j=0}^{n-2} \sum_{k=0}^{n-j-2} \mathcal{E}_0^{n-j-k-2}(p_1, p_2) \int \frac{dq}{2\pi} \mathcal{E}_2^j(p_1 - q, p_2 + q)$$

$$\times 2 \sum_{i=1}^{2} \left( \frac{\mathcal{E}_i^k(p_1 - q, p_2, q) + \mathcal{E}_i^k(p_1, p_2 + q, -q)}{8e_i(q)\sqrt{E_{SL}(p_1 - q)E_{SH}(p_1)E_{DL}(p_2)E_{DH}(p_2 + q)}} \right) |H, p_2 + q; L, p_1 - q).$$

Using the standard series expansion of the exponential and geometric series one arrives at the identity

$$\sum_{n=2}^{\infty} \sum_{j=0}^{n-2} \sum_{k=0}^{n-j-2} \frac{(-i)^n}{n!} \mathcal{E}_0^{n-j-k-2}\mathcal{E}_2^j\mathcal{E}_1^k = 1 \frac{E_0 - E_1}{E_0 - E_2} - \frac{E_0 - E_1}{E_1 - E_2}.$$ (3.6)

Let us define the shorthand notation

$$F(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) = \frac{1}{\mathcal{E}_0 - \mathcal{E}_1} \left( \frac{e^{-it\mathcal{E}_1} - e^{-it\mathcal{E}_2}}{\mathcal{E}_0 - \mathcal{E}_2} - \frac{e^{-it\mathcal{E}_1} - e^{-it\mathcal{E}_2}}{\mathcal{E}_1 - \mathcal{E}_2} \right).$$ (3.7)

As a consistency check, note that this may be rewritten as an integral of $e^{-iEt}$ over the times $t_1$ and $t_2$ of the neutrino emission and absorption

$$F(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) = (-i)^2 \int_0^{t_1} dt_2 \int_0^{t_2} dt_1 e^{-it_1\mathcal{E}_0} e^{-it_2\mathcal{E}_1} e^{-i(t_2-t_1)\mathcal{E}_2}. (3.8)$$

In the wave packet approaches of Refs. [2, 13], the neutrino emission time $t_1$ is fixed to be close to 0 by imposing that the unobserved source particles, like $\phi_{SL}$, are in localized wave packets. In our approach, on the other hand, $t_1$ can assume any value in the range [0, $t_2$]. This is not imposed by hand, but is a result of our calculation of the Schrodinger picture evolution.
Inserting (3.5) into (3.3) and performing the $j$, $k$ and $n$ sums using the identity (3.6) we find the evolved state at time $t$

$$\mathcal{P} e^{-iHt} |L, p_2; H, p_1\rangle = \sum_{i=1}^{2} \int \frac{dq}{2\pi} \left( F(E_0(p_1, p_2), E_{1i}(p_1 - q, p_2, q), E_2(p_1 - q, p_2 + q)) 
+ F(E_0(p_1, p_2), E_{1i}(p_1, p_2 + q, -q), E_2(p_1 - q, p_2 + q)) \right)$$

$$\times \frac{|H, p_2 + q; L, p_1 - q\rangle}{8e_i(q)\sqrt{E_{SL}(p_1 - q)E_{SH}(p_1)E_{DL}(p_2)E_{DH}(p_2 + q)}}. \quad (3.9)$$

The matrix elements are therefore

$$A(k, l, p_1, p_2) = \sum_{i=1}^{2} A_i(k, l, p_1, p_2)$$

$$A_i(k, l, p_1, p_2) = \int \frac{dq}{2\pi} \left( F(E_0(l + q, k - q), E_{1i}(l, k - q, q), E_2(l, k)) 
+ F(E_0(l + q, k - q), E_{1i}(l + q, k, q), E_2(l, k)) \right)$$

$$\times \frac{\delta(p_2 + q - k)\delta(p_1 - q - l)}{8e_i(q)\sqrt{E_{SL}(l)E_{SH}(l + q)E_{DL}(k - q)E_{DH}(k)}}. \quad (3.10)$$

This is the amplitude for an initial momentum eigenstate, which is an infinitely extended plane wave. To observe spatial neutrino oscillations, we must begin with localized states. In particular, we would like the source and detector to be separated by a distance $x$. Thus we will begin with the initial state

$$|0\rangle = \int dp_1 e^{-\frac{p_1^2}{2\sigma_1^2}} \int dp_2 e^{-\frac{p_2^2}{2\sigma_2^2}} e^{-ixp_2} |L, p_2; H, p_1\rangle \quad (3.11)$$

where $\sigma_1$ and $\sigma_2$ are the uncertainties in the momenta of the initial source and detector particles. These are in principle controlled and known by the experimenter, for example if the source is a radioactive particle in an optical trap.

Therefore our initial states are described by wave packets. Does this mean that we are using a wave packet model in the sense of Refs. [2, 13]? Our claim is that we are not. In those papers, the neutrino itself was described by a localized wave packet. This is because it was assumed that $\phi_{SL}$ was measured, and the overlap between the trajectories of $\phi_{SL}$ and $\phi_{SH}$ was used to fix the neutrino production region. Thus the neutrino was produced during a fixed, microscopic time. On the other hand, in our approach $t_1$ is not fixed, rather it is integrated over a macroscopic time interval because our $\phi_{SL}$ is not measured. As the neutrino production time is not fixed to a small interval, the neutrino wave function is not localized.
Therefore while our initial conditions for the source and detector are spatially-localized wave packets, our neutrino is not described by a spatially-localized wave packet and so our model is quite different from the wave packet models of Refs. [2, 13]. We believe that our choice, in which the initial conditions are fixed and $\phi_{SL}$ is not measured, is appropriate for most if not all neutrino oscillation experiments today.

The evolution of this state is characterized by the amplitude

$$A(k, l) = \langle H, k; L, l | e^{-iHt} | 0 \rangle = \int dp_1 e^{-\frac{p_1^2}{2\sigma_1^2}} \int dp_2 e^{-\frac{p_2^2}{2\sigma_2^2}} e^{-ixp_2} A(k, l, p_1, p_2)$$

$$= \sum_{i=1}^{2} \int \frac{dq}{2\pi} \left( F(E_0(l + q, k - q), E_0(l, k - q, q), E_2(l, k)) + F(E_0(l + q, k - q), \tilde{E}_1(l + q, k, q), E_2(l, k)) \right)$$

$$\times e^{-\frac{(l+q)^2}{2\sigma_1^2}} e^{-\frac{(k-q)^2}{2\sigma_2^2}} e^{-ix(k-q)} \frac{1}{8e_1(q) \sqrt{E_{SL}(l)E_{SH}(l+q)E_{DL}(k-q)E_{DH}(k)}}.$$  (3.12)

The corresponding unnormalized probability density for a final state consisting of a heavy detector particle of momentum $k$ and a light source particle of momentum $l$ is

$$P(k, l) = |A(k, l)|^2.$$  (3.13)

The source particle is not observed, and so we are instead interested in the probability density to observe a detector particle with momentum $k$

$$P(k) = \int dl P(k, l).$$  (3.14)

Note that one may also define an appearance channel probability by multiplying the amplitude $A_2$ in Eq. (3.10) by $-1$, which would correspond to a sign flip in the $\phi_{DH}\phi_{DL}\psi_2$ interaction in $H_I$. These unnormalized probability densities are proportional to the probability that the neutrino is detected at any time $t_2 < t$, and so in this sense it is proportional to the time-integrated neutrino signal.

4 Numerical Results

4.1 Fixing Parameters

Imagine that the initial configuration is on-shell, for example with $p_1 = p_2 = 0$. As distance scales are macroscopic at neutrino experiments, we are interested in regimes where also the
final state and even the intermediate neutrino are also on-shell. This yields two conditions. In the expression (3.10) for the amplitude, the neutrino momentum $q$ is integrated over. Thus we effectively have one free parameter which needs to satisfy two conditions. There is no solution for a generic choice of masses.

This is one reason that we begin with the initial state $|0\rangle$. Now $p_1$ and $p_2$ in principle run over all values, including those for which all particles are on-shell. However, $\sigma_1$ and $\sigma_2$ combine to form the energy resolution of our measurement of neutrino oscillations and so if we wish to observe oscillations these should be small. Therefore in general the on-shell contribution to the amplitude will be exponentially suppressed, unless the central value $p_1 = p_2 = 0$ of the initial momentum distributions is close to a value which allows both the neutrino and the final states to be on-shell.

To remove this exponential suppression, we will choose the mass $M_{DL}$ so that for $p_1 = p_2 = 0$ and $m_i = 0$ all particles will be nearly on-shell for some value of $q$. Such a tuning of $M_{DL}$ is very physical, it means that the detector is chosen to have an energy splitting which is close to the expected neutrino energy minus the recoil energy, which maximizes the probability that the neutrino will be detected. Note that $M_{DL} = M_{HL}$ does not satisfy this condition as a result of the recoil energy.

Let us fix $p_1 = p_2 = 0$, $M_{DH} = M_{SH}$, $M_{SL} = M_{DH}(1 - \epsilon)$, $m_i = 0$. (4.1)

As process B can never be on-shell, we only consider process A.

At the first vertex, a heavy source particle at rest decays to a light source particle and a neutrino with momentum $q$. If all particles are on-shell, the initial and final energies at this vertex are

$$E^1_i = M_{SH}, \quad E^1_j = q + \sqrt{M_{SH}^2(1 - \epsilon)^2 + q^2}. \quad (4.2)$$

Energy conservation yields

$$q \sim M_{SH} \left( \epsilon - \frac{\epsilon^2}{2} \right) \quad (4.3)$$

plus terms of order $\epsilon^3$. Recall that $q$ is integrated over, and so the on-shell condition for these particles can always be satisfied for some $q$ in the range of integration.

At the second vertex, the neutrino is absorbed by a light detector particle at rest, which becomes a heavy detector particle. If all particles are on-shell, the initial and final energies at this vertex are

$$E^2_i = M_{DL} + q, \quad E^2_j = \sqrt{M_{SH}^2 + q^2}. \quad (4.4)$$
Energy conservation together with (4.3) then yields

\[ M_{DL} = M_{SH} \left(1 - \epsilon + \epsilon^2\right). \] (4.5)

As we would like a large on-shell contribution to our probabilities, we choose this mass for the light detector particle. Equivalently, the detector mass splitting is chosen to optimize the chance that it absorbs a neutrino.

### 4.2 Numerical Results

We choose source and detector masses as in (4.1) and (4.5) with

\[ M_{SH} = 10, \quad \epsilon = 0.1. \] (4.6)

We also choose neutrino masses

\[ m_1 = 0, \quad m_2 = 0.1 \] (4.7)

and initial wave packet momentum spreads

\[ \sigma_1 = \sigma_2 = 0.1. \] (4.8)

We vary the separation \(x\) between the source and the detector. The detector measures the detector heavy particle momentum \(k\), and so we are free to decide which value of \(k\) interests us. We fix

\[ k = 1 \] (4.9)

as this is close to the on-shell value. With these choices, the standard neutrino oscillation formula suggests oscillations with a wavelength of

\[ \lambda = 4\pi \frac{|q|}{M_1^2 - M_2^2} \sim 400\pi. \] (4.10)

The oscillation amplitudes and probabilities at time \(t = 10^4\) and varying baseline \(x\) are shown in Figs. 2 and 3 respectively. As can be seen on the left hand side of Fig. 2, near the on-shell value of \(l\) the absolute value of the one flavor amplitude shrinks linearly with \((t - x)\). As a result, the two flavor amplitude exhibits oscillations whose amplitude shrinks linearly with \((t - x)\). This is because the neutrino production begins at \(t = 0\) and so neutrinos arrive at a distance \(x\) between \(t = x\) and \(t = 10^4\). This time window shrinks linearly with \(x\). Thus the unnormalized probability \(P(k)\), which is essentially the integrated neutrino flux, also shrinks linearly with \(x\). On the other hand, in the wave packet approximation one might
Figure 2: Left: The one-flavor disappearance channel amplitudes $|A_1(1,l)|$ at $l = -0.942$ to $-0.956$ in steps of $-0.002$ in black, red, blue, green, brown, purple, grey and yellow respectively at time $t = 10^4$. Right: The sum $|A_1(1,l) + A_2(1,l)|$, which exhibits two-flavor oscillations. In both cases, the highest curves are closest to the on-shell values.

Figure 3: Left: The disappearance channel probability density $P(1,l)$ at $l = -0.942$ to $-0.956$ in steps of $-0.002$ in black, red, blue, green, brown, purple, grey and yellow respectively at $t = 10^4$. Right: The disappearance (black) and appearance (red) oscillation probabilities $P(1)$ at time $t = 10^4$. 


expect instead a single bump at a value of $x$ equal to the time since production times the velocity. The fact that no bump exists demonstrates that our production is not localized in time, as it would be in the wave packet approximation, but rather occurs continuously from time 0 to $t$ as can be seen in Eq. (3.8). The oscillation wavelength is compatible with the standard formula (4.10).

5 Semianalytic Estimates

5.1 Infinite Time

The amplitudes simplify considerably if the time $t$ is infinite. Of course our leading tree-level calculation is not valid if $t$ is of order or greater than the half life of $\phi_{SH}$, and so we will arrive at some nonsensical answers in this subsection such as infinite decay probabilities. These will be remedied in the following subsection when we consider finite time.

We will now show that in this case the $t_1$ and $t_2$ integrations from Eq. (3.8) each yield delta functions for momenta imposing that the interactions preserve the on-shell energy, which combined with the two delta functions from Eq. (3.10) which imposed momentum conservation at the two vertices yield a total of four delta functions for the four momenta $p_1, p_2, q, k$ and $l$. This means that only one momentum is independent and it allows all integrals to be trivially performed using the delta functions.

Let us begin with the $t \to \infty$ limit of Eq. (3.8)

$$F(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) = -\lim_{t \to \infty} \int_0^t dt_2 \int_0^{t_2} dt_1 e^{-it_1 \mathcal{E}_0} e^{-it(t_2-t_1) \mathcal{E}_1} e^{-it(t-t_2) \mathcal{E}_2} = -(2\pi)^2 \lim_{t \to \infty} e^{-it\mathcal{E}_2} \delta(\mathcal{E}_0 - \mathcal{E}_1) \delta(\mathcal{E}_1 - \mathcal{E}_2). \quad (5.1)$$

Recall that the functions $\mathcal{E}_i$ are the total energy that the system would have before, between or after the interactions were all particles on-shell, or equivalently $\mathcal{E}_i$ are the eigenvalues of $H_0$. Of course the true energy, which is the eigenvalue of the total Hamiltonian $H$, is conserved for each $H$ eigenstate but these eigenstates are very complicated. Therefore the fact that the $\mathcal{E}_i$ must be equal in Eq. (5.1) is a consequence of the fact that the states are on-shell in the infinite $t$ limit.

Process B can never be on-shell, and so in the infinite time limit only process A contributes
to \(\mathcal{A}\) in Eq. \((3.12)\)

\[
\mathcal{A}(k, l) = -2\pi \lim_{t \to \infty} e^{-it\mathcal{E}_2(l,k)} \sum_{i=1}^{2} \int dq \delta(\mathcal{E}_0(l + q, k - q) - \mathcal{E}_{1i}(l, k - q, q)) \times \frac{\delta(\mathcal{E}_{1i}(l, k - q, q) - \mathcal{E}_2(l, k)) e^{-\frac{(l+q)^2}{2\sigma_i^2}} e^{-\frac{(k-q)^2}{2\sigma_i^2}} e^{-ix(k-q)}}{8\sigma_i(q)\sqrt{E_{SL}(l)E_{SH}(l + q)E_{DL}(k - q)E_{DH}(k)}}. \tag{5.2}
\]

The two delta functions can be numerically solved to yield the values of \(q\) and \(l\) for which this process is on-shell for a given value of \(k\). In general there are two solutions, however with the choice of parameters in Sec. 4 one of these solutions is strongly suppressed by the exponential factors. This suppression is caused by the small values of the momentum spreads \(\sigma_i\) which are necessary for oscillations to be observed. Let the remaining solutions be \(q_i\) and \(l_i\), which implicitly depend on \(k\).

In summary, the delta functions may be simplified by defining a function \(f_i(k)\) such that

\[
\delta(\mathcal{E}_0(l + q, k - q) - \mathcal{E}_{1i}(l, k - q, q))\delta(\mathcal{E}_{1i}(l, k - q, q) - \mathcal{E}_2(l, k)) = f_i(k)\delta(q - q_i)\delta(l - l_i). \tag{5.3}
\]

The function \(f_i(k)\) is given by the usual inverse Jacobian formula. Then the \(q\) integral may be performed, leaving

\[
\mathcal{A}(k, l) = -2\pi \lim_{t \to \infty} \sum_{i=1}^{2} f_i(k)\delta(l - l_i) \frac{e^{-\frac{(l+q_i)^2}{2\sigma_i^2}} e^{-\frac{(k-q_i)^2}{2\sigma_i^2}} e^{-ix(k-q_i)} e^{-it\mathcal{E}_2(l_i,k)}}{8\sigma_i(q_i)\sqrt{E_{SL}(l_i)E_{SH}(l_i + q_i)E_{DL}(k - q_i)E_{DH}(k)}}. \tag{5.4}
\]

So far our amplitude is exact, at leading order in perturbation theory in the infinite time limit. In practice, in neutrino oscillation experiments the various kinematic factors are, to within experimental error, equal for each relevant active neutrino mass eigenstate. Therefore up to an over all constant, only the phase

\[
\mathcal{A}(k, l) \sim \sum_{i=1}^{2} \delta(l - l_i)e^{ixq_i}e^{-it\mathcal{E}_2(l_i, k)} \tag{5.5}
\]

is relevant for neutrino oscillations. Here and from now on the infinite time limit will be implicit. This yields an unnormalized probability density of

\[
P(k, l) \sim \left| \sum_{i=1}^{2} \delta^2(l - l_i)e^{ixq_i}e^{-it\mathcal{E}_2(l_i, k)} \right|^2 = 4\delta^2(l - l_i)\cos^2\left(\frac{l}{2}(q_2 - q_1) - \frac{t}{2}(\mathcal{E}_2(l_2, k) - \mathcal{E}_2(l_1, k))\right). \tag{5.6}
\]

The integration over the unobserved momentum \(l\) in Eq. \((3.14)\) uses one delta function but leaves one \(\delta(0)\) using the approximation that \(l_i\) is independent of \(i\). This infinity is to be
expected, as this probability density is proportional to the number of neutrinos measured after an infinite time with a constant neutrino flux, which of course is infinite. Thus the observed neutrino flux at momentum $k$ will be proportional to

$$P(k) \sim 4\delta(0)\cos^2\left(\frac{x}{2}(q_2 - q_1) - \frac{t}{2}(E_2(l_2, k) - E_2(l_1, k))\right). \quad (5.7)$$

The time $t$ is infinite and so the phase has an infinite shift, but at fixed time nonetheless the $x$ wavelength is well-defined, it is

$$\Delta x = \frac{2\pi}{q_2 - q_1}. \quad (5.8)$$

In the case of the parameters of Sec. 4 the on-shell conditions are easily solved numerically to yield

$$q_1 \sim 0.9497, \quad q_2 \sim 0.9444, \quad \Delta x \sim 1183. \quad (5.9)$$

This wavelength is again consistent with Fig. 3. Of course the overall shrinking of the oscillation amplitude with $x$ is not present in the infinite time case, as that effect resulted from the limited time for the neutrino beam to reach large $x$. Instead the probability (5.7) is periodic.

### 5.2 Finite Time

In the finite time $t$ case, the on-shell condition is violated via a correction to the energy of order $1/t$. This energy shift is much too small to be observed, but it is responsible for the space and time dependence of the unoscillated integrated flux. The finite time case can be treated using a saddle-point approximation. Eq. (3.8) for $F$ is similar to the $e^{-iEt}$ used in the standard wave-packet approach [2, 13] except that here the neutrino is created at $t_1$ and absorbed at $t_2$ which are integrated over all possible values between 0 and $t$. In the wave-packet approach, on the other hand, the creation time and absorption time are each fixed to within a small range. As the time $t$ will be taken to be large, the states are still nearly on-shell and so we may linearly expand the various energies

$$E_0 = \epsilon_0 + v_0(q - q_i), \quad E_{1i} = \epsilon_{1i} + v_{1i}(q - q_i), \quad E_2 = \epsilon_2 + v_2(q - q_i) \quad (5.10)$$

where $q_i$ again are the on-shell values of $q$. As the source and detector particles are very heavy, we will ignore their velocities and so set

$$v_0 = v_2 = 0. \quad (5.11)$$
Including these velocities would be straightforward. The important velocity is that of the neutrino
\[ v_{1i} = \frac{\partial \mathcal{E}_{1i}(l, k - q, q)}{\partial q} \bigg|_{q_i} \sim \frac{q_i}{\sqrt{m_i^2 + q_i^2}}. \] (5.12)

Finally, we will use the approximation
\[ \epsilon = \epsilon_0 = \epsilon_{1i} = \epsilon_2 \] (5.13)

which holds when all particles are nearly on-shell.

Again, only process A can be close enough to being on-shell to be relevant when \( t \) is large. \( F \) in Eq. (3.8) is then
\[ F(E_0, E_{1i}, E_2) \sim -e^{-i\epsilon t} \int_0^t dt_2 \int_0^{t_2} dt_1 e^{-i(q_2 - q_1)v_{1i}(t_2 - t_1)} \] (5.14)

The \( q \)-dependent phases in Eq. (3.12) are then
\[ A(k, l) \sim \sum_i e^{iq\cdot x} \int_0^t dt_2 \int_0^{t_2} dt_1 \int dq e^{i(q - q_1)(x - v_{1i}(t_2 - t_1))} \]
\[ = 2\pi \sum_i e^{iq\cdot x} \int_0^t dt_2 \int_0^{t_2} dt_1 \delta(x - v_{1i}(t_2 - t_1)) \]
\[ = 2\pi \sum_i \frac{e^{iq\cdot x}}{v_{1i}} \int_0^t dt_2 = 2\pi \sum_i \frac{e^{iq\cdot x}}{v_{1i}} \left( t - \frac{x}{v_{1i}} \right) \]
\[ \sim \sum_i e^{iq\cdot x}(t - x) \] (5.15)

where the last line is the ultrarelativistic limit. We have kept only the \( q \)-dependence of the phases and in particular we have dropped the \( l \) dependence, although we kept the \( l \) argument on the left-hand side of the equation for consistency of our notation. Again summing and squaring one finds
\[ P(k, l) \sim (t - x)^2 \cos^2 \left( \frac{x}{2}(q_2 - q_1) \right). \] (5.16)

This is plotted in Fig. 4 with the \( y \)-axis rescaled to match the on-shell curves of the left panel of Fig. 3. Good agreement is found despite the crude approximations here, where our on-shell approximation effectively fixes \( l \). In the right panel of Fig. 3, after \( l \) has been integrated, the \((t - x)\) behavior is linear and not quadratic, as a result of the \( l \)-dependence of \( P(k, l) \) which has been omitted in the approximation Eq. (5.15), which kept only the \( q \)-dependent phases.

More precisely, since the allowed distance of \( l \) from the on-shell value of \( l_i \) is inversely proportional to \((t - x)\), as is clear from the uncertainty principle, the \( l_i \) integral is suppressed
Figure 4: The disappearance channel probability density $P(1,l)$ as approximated in Eq. (5.16).

by a factor of $(t - x)$. This suppression converts the quadratic $(t - x)$-dependence of $P(k,l)$ into the linear dependence of $P(k)$. This is also evident in the left panel of Fig. 3 as the fractional distance between the curves is largest on the left, where $(t - x)$ is largest, reflecting that the $l$ integration has narrower support at large $(t - x)$.

6 Comparison with the Literature

There is already a vast literature on neutrino decoherence. In this section we try to place our work in the context of the wider literature. The literature largely uses three approaches. The first approach uses quantum mechanical models. The second uses quantum field theory, but in a wave packet formulation. The third uses quantum field theory directly, without imposing the existence of wave packets.

A general discussion and comparison of the first two approaches appears in Ref. [20]. Here it is explained that wave packets emerge in quantum field theory via a restriction on the region of space time in which the neutrino was produced. Neutrinos are produced in that paper via a 2-body decay, as in our work. Therefore localized wave packets arise whenever the other particle produced during the decay, corresponding to $\phi_{SL}$ in our paper, is detected. The detection of this particle places it in a fixed wave packet given by the function $f_{P_f}$. Such a detection of course never occurs for reactor neutrinos and is quite rare in general. The case in which $\phi_{SL}$ is not measured is also briefly and qualitatively discussed in Subsec. 4.1 of Ref. [20], however it is assumed that its momentum is known and so it is treated as a delocalized plane wave. As it is delocalized, the neutrino production point is not known and so presumably wave packets do not arise.

This is similar to our approach, however we do not assume that the momentum of $\phi_{SL}$
is known precisely. On the contrary, the momentum distribution of our \( \phi_{SL} \) is simply determined by the evolution equations with fixed initial conditions for \( \phi_{SH} \). This results in entanglement between \( \phi_{SL} \) and our neutrino, which we break by integrating the probability over all possible final states of \( \phi_{SL} \) as these are not measured. It is essential that we integrate the *probability* and not the amplitude because distinct \( \phi_{SL} \) final states are distinct final states and thus must be summed incoherently.

The canonical reference on the second approach is Ref. [2], which contains a very general treatment, for example allowing the neutrinos to be unstable. It is again assumed that the particles involved in the production are measured. As a result, as in Ref. [20], the amplitudes in their Eqs. (27) and (28) are folded into a fixed wave function for these particles, called the overlap function. The overlap function is completely arbitrary, but its size, together with the initial size of the source particle, fix the allowed space time region in which the neutrino is produced and so the size of the neutrino wave packet. This of course is all in sharp contrast with our approach, in which only the initial condition is known and \( \phi_{SL} \) is in no way measured or artificially restricted. Therefore, although we do assign parameters \( \sigma \) to our initial conditions, which we feel can be fixed and known by the experimenter, the allowed neutrino production region is macroscopic and so does not fit the general characterization of wave packet models in Ref. [20].

Ref. [13] considers a special case, that relevant to neutrino oscillations, of that considered in Ref. [2]. The treatment is equivalent. However there is one additional comment which gets to the heart of the issue. The author writes that the overlap function for \( \phi_{SL} \) may result from a measurement of \( \phi_{SL} \) or else from “interactions ... with the surrounding medium.” The first possibility is rarely or never realized in neutrino oscillation experiments. And so the claim is that interactions with the medium are equivalent to a measurement of \( \phi_{SL} \). The goal of our program is precisely to test this assertion.

We are motivated in part by Refs. [4, 5, 6]. As these papers are written by overlapping authors and share essentially the same approach, let us discuss the first. The amplitude (6) depends on the arbitrary parameters defined in Eq (5), which are not integrated over. There is integration over the momenta of \( \phi_{SL} \), but using the arbitrary weights from Eq (5) which are equivalent to fixing the outgoing state \( \phi_{SL} \) as in the previous papers. In Ref. [6] the author arrives at a very different formula for wave packet spreading than that obtained in Ref. [2]. The difference is caused by a different choice of overlap functions, describing the state \( \phi_{SL} \). In our approach no such ambiguity occurs as we do not fix the wave function of \( \phi_{SL} \).

The rest of our motivation comes from Ref. [8]. The third paragraph of Section II contains
a very direct and interesting discussion of this issue: “The interactions ... may be turned ‘on’ and ‘off.’ ... a real (microscopic) source or detector will be in an environment which is ‘noisy,’ so that the coherent emission or absorption of a neutrino gets cut off after some time due to the interactions of the source or detector with its surrounding environment.” In other words, they restrict the neutrino production time by hand as a proxy for interactions with the environment. This restriction leads to the revival of neutrino oscillations. Our goal is to test whether or not such a cut is a reliable proxy for environmental interactions, by directly including environmental interactions and checking to see whether oscillations can be revived.

This revival claim was tested in quantum field theory in Ref. [12]. Here again it was assumed that all particles involved in production are measured. This assumption is incorporated as usual by integrating the amplitude over a Gaussian kernel corresponding to the measured wave function for \( \phi_{SL} \).

What about the third approach?

The approach of Ref. [7] is similar to ours. They aim to construct a consistent treatment of neutrino oscillations in quantum field theory. They explain that spin and three-body kinematics are not essential to understand oscillations, and so like us they consider a scalar model in which neutrinos are created in a two-body decay. The full entanglement with the parent particles is kept but then is traced out to calculate probabilities, which is equivalent to the integration of probabilities in our paper. They use the stationary phase approximation to extract the scaling of their probabilities. This is sufficient for their goal of trying to derive the standard neutrino oscillation formula. However we believe that our direct numerical evaluation of probabilities is more robust, and thus better suited to our goal of understanding decoherence.

Ref. [3] also contains a consistent approach to neutrino oscillations in quantum field theory. It uses observations only of quantities which are observed. The authors in particular do not restrict the space time region in which the neutrino is created. They conclude that, on the contrary to the usual wave packet description in which neutrinos of various mass eigenstates are created simultaneously at roughly the same position and then separate, instead the heavier eigenstate is created before the light and they coalesce at the detector. They conclude that the usual wave packet picture is wrong. However environmental interactions are not included, and we suspect that, as argued in Ref. [8], these would spoil the coherence between neutrinos created at sufficiently distinct times and so affect their conclusions. It is the goal of our project to test this suspicion.

In addition a number of papers have tried to estimate the wave packet size for various
kinds of neutrino experiment. The most common approach, used in Refs. \[14, 15, 16\] is simply to assert that the time between interactions of the source is the coherence time and so the wave packet size should be this time multiplied by some velocity. Different papers generally found very different wave packet sizes. Needless to say, the time between interactions depends in general on the infrared cutoff, and without that cutoff is generally equal to zero as interactions occur continuously. The first calculation of the wave packet size, in Ref. \[1\], included a phase shift from each interaction and assumed decoherence results when the shift is of order $2\pi$ radians. The IR cutoff dependence thus disappears and this is well-defined.

All of these estimates are in contradiction with the observation in Ref. \[19\] that only the difference between interactions of the source before and after neutrino production causes decoherence. For the solar neutrinos of interest in Ref. \[1\] the difference is of order the interaction itself and so this is not a problem. However, in the case of the $\beta$ decays relevant for reactor neutrinos, there is little difference between the parent and daughter source particle and so this difference may be one or two orders of magnitude smaller than the interaction itself. Even more importantly, the arguments of Ref. \[19\] suggest that the interatomic forces which are usually used are too small to create noticeable decoherence at reactor experiments, but strong interactions inside of nuclei may be sufficient. These would yield very different wave packet sizes from existing estimates, perhaps within the range of interest for JUNO. It is the final goal of our line of research to find a sensible method for estimating the wave packet size.

7 Conclusions

In Ref. \[19\] we constructed a toy model of neutrino production, considering the full entanglement of the quantum states. This entanglement exists in the full quantum field theory description, and so, as has been emphasized in Ref. \[7\], should be considered in any calculation. However the toy model of \[19\] could not be connected, even qualitatively, to observations because it did not include neutrino measurement.

In the current note that shortcoming has been remedied. We have modified the model of \[19\] to include neutrino detection and we have tested that it produces the standard oscillation formula.

There are many controversies in neutrino physics, for example regarding the observability of neutrino oscillation decoherence \[8, 17\] and also whether neutrino mass eigenstates are created at the same point and then separate or are created at different points and then
coalesce [3], which are not specific to any model of electroweak interactions. They depend only on the basic structure of the entanglement of the various wave packets. However, arguments in the literature often rely upon wave packet models, where these entanglements are not considered. In particular, interactions with the environment are generally included by simply projecting the state [13]. Our toy model manifests entanglement and yet is simple enough that the relevant probabilities can be expressed in closed form, albeit as integrals. Entanglement with the environment can be added as in Ref. [19]. Therefore we believe that our model is now ready to be applied to resolving these controversies, and more generally to understanding just when the wave packet approximation is and is not applicable.

Is our probability (3.14) equivalent to that obtained using quantum field theory with wave packets? In that case, the neutrino flavor eigenstate is created in a localized space-time region, the overlap of the wave packets of the initial and final production particles. In the present case, we have seen in Eq. (3.8) that our amplitude may be written as an integral over the production and detection times $t_1$ and $t_2$. These integrals extend over the entire, macroscopic interval from time 0 to $t$. As a result the probability $P$, being the square of the amplitude, will contain cross-terms arising from macroscopically separated production times $t_1$. In this sense, both the standard description in which the mass eigenstates are produced simultaneously and the paradigm of Ref. [3] are included in the amplitude. This story will become more interesting in the sequel, when we couple to environmental interactions, as these will spoil the coherence in cross-terms with sufficiently different values of $t_1$. In that case, neutrinos with sufficiently different values of $t_1$ will not contribute to the same final states and so will not be added coherently. Thus each coherent component of the amplitude will be integrated over only a small region of $t_1$, as in the external wave packet picture and the time window prescription.

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