The solution to the q-KdV equation

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Abstract: Let KdV stand for the Nth Gelfand-Dickey reduction of the KP hierarchy. The purpose of this paper is to show that any KdV solution leads effectively to a solution of the q-approximation of KdV. Two different q-KdV approximations were proposed, first one by E. Frenkel [7] and a variation by Khesin, Lyubashenko and Roger [10]. We show there is a dictionary between the solutions of q-KP and the 1-Toda lattice equations, obeying some special requirement; this is based on an algebra isomorphism between difference operators and D-operators, where \( Df(x) = f(qx) \). Therefore every notion about the 1-Toda lattice can be transcribed into q-language.

Consider the q-difference operators \( D \) and \( D_q \), defined by

\[
Df(y) = f(qy) \quad \text{and} \quad D_q f(y) := \frac{f(qy) - f(y)}{(q - 1)y},
\]

and the q-pseudo-differential operators

\[
Q = D + u_0(x)D^0 + u_{-1}D^{-1} + \ldots \quad \text{and} \quad Q_q = D_q + v_0(x)D_q^0 + v_{-1}(x)D_q^{-1} + \ldots
\]
The following $q$-versions of KP were proposed by E. Frenkel and a variation by Khesin, Lyubashenko and Roger, for $n = 1, 2, ...$:

\[
\frac{\partial Q}{\partial t_n} = [(Q^n)_+, Q] \quad \text{(Frenkel system)} \tag{0.1}
\]

\[
\frac{\partial Q_q}{\partial t_n} = [(Q^n_q)_+, Q_q], \quad \text{(KLR system)}
\tag{0.2}
\]

where $(\ )_+$ and $(\ )_-$ refer to the $q$-differential and strictly $q$-pseudo-differential part of $(\ )$. The two systems are closely related, as will become clear from the isomorphism between $q$-operators and difference operators, explained below.

The purpose of this paper is to give a large class of solutions to both systems. 

The $\delta$-function $\delta(z) := \sum_{i \in \mathbb{Z}} z^i$; enjoys the property $f(\lambda, \mu)\delta(\lambda/\mu) = f(\lambda, \lambda)\delta(\lambda/\mu)$. Consider an appropriate space of functions $f(y)$ representable by “Fourier” series in the basis $\varphi_n(y) := \delta(q^{-n}x^{-1}y)$ for fixed $q \neq 1$,

\[
f(y) = \sum_{n} f_n \varphi_n(y);
\]

the operators $D$, defined by $Df(y) = f(qy)$, and multiplication by a function $a(y)$ act on the basis elements, as follows:

\[
D\varphi_n(y) = \varphi_{n-1}(y) \quad \text{and} \quad a(y)\varphi_n(y) = a(xq^n)\varphi_n(y).
\]

Therefore, the Fourier transform,

\[
f \mapsto \hat{f} = (\ldots, f_n, \ldots)_{n \in \mathbb{Z}},
\]

induces an algebra isomorphism, mapping $D$-operators onto a special class of $\Lambda$-operators in the shift $\Lambda := \left(\delta_{i, j-1}\right)_{i, j \in \mathbb{Z}}$, as follows:

\[
\sum_i a_i(y)D^i \mapsto \sum_i \text{diag}(\ldots, a_i(xq^n), \ldots)_{n \in \mathbb{Z}} \Lambda^i; \quad (0.3)
\]

conversely, any difference operator, depending on $x$, of the type $(0.3)$ i.e., annihilated by $D - Ad_{\Lambda}$, where $(Ad_{\Lambda})a = \Lambda a\Lambda^{-1}$, maps into a $D$-operator. This is the crucial basic isomorphism used throughout this paper.
To the shift $\Lambda$ and to a fixed diagonal matrix $\lambda = \text{diag}(\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots)$, we associate new operators
\[
\tilde{\Lambda} = -\lambda \Lambda \quad \text{and} \quad \tilde{\tilde{\Lambda}} = \tilde{\Lambda} + \lambda = -\lambda (\Lambda - 1).
\]
Observe that, under the isomorphism (0.3),
\[
D \mapsto -\Lambda, \quad (q-1)x \mapsto \tilde{\Lambda} \quad \text{and} \quad D_q \mapsto \tilde{\tilde{\Lambda}},
\]
upon setting $\lambda_n^{-1} = (1-q)xq^{n-1}$.

Defining the simple vertex operators
\[
X(t, z) := e^\sum_{i=1}^{\infty} t_i z^i e^{-\sum_{i=1}^{\infty} \frac{z^i}{i} \frac{\partial}{\partial t_i}}, \tag{0.4}
\]
we now make a statement concerning the so-called full one-Toda lattice; the latter describes deformations of of a bi-infinite matrix $L$, which is lower-triangular, except for a diagonal and a constant subdiagonal just above the main diagonal. The first formula (0.6) below gives a solution to the Frenkel system (Theorem 0.1), upon replacing $\tilde{\Lambda}$ by $\Lambda$, whereas the second formula (0.6) gives a solution to the KLR system (Theorem 0.2). Note, Theorem 0.1 is given for arbitrary $\lambda = (\ldots, \lambda_{-1}, \lambda_0, \lambda_1, \ldots)$. We shall need the well-known Hirota symbol for a polynomial $p$,

\[
p(\pm \tilde{\partial}) f \circ g := p \left( \pm \frac{\partial}{\partial y_1}, \pm \frac{1}{2} \frac{\partial}{\partial y_2}, \ldots \right) f(t+y)g(t-y) \bigg|_{y=0}.
\]

Note $A_+$ refers to the upper-triangular part of a matrix $A$, including the diagonal, and for $\alpha \in \mathbb{C}$, set $[\alpha] := (\alpha, \frac{1}{2} \alpha^2, \frac{1}{3} \alpha^3, \ldots) \in \mathbb{C}^\infty$.

**Theorem 0.1.** Given an integer $N \geq 2$, consider an arbitrary $\tau$-function for the KP equation such that $\partial \tau / \partial t_i = 0$ for $i = 1, 2, 3, \ldots$ (N-KdV hierarchy). For a fixed $\lambda$, $\nu$, $c \in \mathbb{C}^\infty$, the infinite sequence of $\tau$-functions
\[
\tau_n := e^{\nu_n \sum_{i=1}^{\infty} t_i N} X(t, \lambda_n) \ldots X(t, \lambda_1) \tau(c + t), \quad \tau_0 = \tau(c + t), \quad \text{for } n \geq 0; \tag{0.1}
\]

$\tau_n$ for $n < 0$ is defined later in (3.3).
satisfies the 1-Toda bilinear identity for all $t, t' \in \mathbb{C}^\infty$ and all $n > m$:

$$\oint_{z=\infty} \tau_n(t - [z^{-1}])\tau_{m+1}(t' + [z^{-1}])e^{\sum_1^\infty (t_i - t'_i)z^i}z^{n-m-1}dz = 0.$$  

The bi-infinite matrix (a full matrix below the main diagonal), where $p_\ell$ are the elementary Schur polynomials,

$$L = \sum_{\ell=0}^\infty \frac{p_\ell(\tilde{\partial})\tau_{n+2-\ell} \circ \tau_n}{\tau_{n+2-\ell}\tilde{\Lambda}^{1-\ell}}$$  \hspace{1cm} (0.5)

has the following properties:

(i) $L^N$ satisfies the 1-Toda lattice

$$\frac{\partial L^N}{\partial t_n} = [(L^n)_+, L^N], \quad n = 1, 2, ...$$

(ii) $L^N$ is upper triangular and admits the following expression in terms of $\tilde{\Lambda}$ and $\tilde{\Lambda}$:

$$L^N = \tilde{\Lambda}^N + \sum_{j=1}^N (\lambda_j + b_j)\tilde{\Lambda}^{p-1} + \left(\sum_{j=1}^{N-1} a_j + \sum_{1 \leq i \leq j \leq N-1} (\lambda_i + b_i)(\lambda_j + b_j)\right)\tilde{\Lambda}^{N-2}$$

$$+ ... + (\nu_{n+1} - \nu_n)_{n \in \mathbb{Z}} \tilde{\Lambda}^{0}$$

$$= \tilde{\Lambda}^N + \left(\sum_{j=1}^N b_j\right)\tilde{\Lambda}^{N-1}$$

$$+ \left(\sum_{j=1}^{N-1} a_j - \sum_{j=1}^{N-1} (b_N - b_i)\lambda_i + \sum_{1 \leq i \leq j \leq N-1} b_i b_j\right)\tilde{\Lambda}^{N-2} + ... \hspace{1cm} (0.6)$$

with

$$b_k = \frac{\partial}{\partial t_1} \log \frac{\tau(c + t - \sum_{i=k}^{k-1} [\lambda_i^{-1}])}{\tau(c + t - \sum_{i=1}^{k-1} [\lambda_i^{-1}])}, \quad a_k = \left(\frac{\partial}{\partial t_1}\right)^2 \log \tau \left(c + t - \sum_{i=1}^k [\lambda_i^{-1}]\right),$$

$$\hspace{1cm} (0.7)$$

in the expressions below, the coefficients of the $\tilde{\Lambda}$'s are diagonal matrices, whose 0th component is given by the expression appearing below; i.e., $\sum_{j=1}^N b_j$ stands for $\text{diag}(\sum_{j=1}^N b_j + n)_{n \in \mathbb{Z}}$.
for \( k \geq 1 \). These expressions for \( k \leq 0 \) will be given in (3.4) and (3.5).

In view of (0.6), the shift
\[
\Lambda : \; b_k \mapsto \Lambda b_k = b_{k+1} \quad \text{and} \quad a_k \mapsto \Lambda a_k = a_{k+1}
\]
corresponds to the following transformation,
\[
\Lambda : \; c \mapsto c - [\lambda_1^{-1}] \quad \text{and} \quad \lambda_i \mapsto \lambda_{i+1}.
\] (0.8)
Therefore, in order that \( L_N \) satisfies the form of the right hand side of (0.3), we must make \( c \) and \( \lambda_i \) depend on \( x \) and \( q \), such that the map \( \Lambda \) on \( a, b, \lambda \) corresponds to \( D \), in addition to the fact that all \( \lambda_i \) must tend to \( \infty \) simultaneously and \( c \) to \( (x, 0, 0, ... \) when \( q \) goes to 1. So, \( c \) and \( \lambda \) must satisfy:
\[
\left\{
\begin{array}{l}
Dc(x) = c(x) - [\lambda_1^{-1}] \\
D\lambda_n = \lambda_{n+1} \\
\lim_{q \to 1} \lambda_i = \infty \\
\lim_{q \to 1} c(x) = \bar{x} := (x, 0, 0, ...);
\end{array}
\right.
\] (0.9)
its only solution is given by:
\[
\lambda_n^{-1} = (1 - q)xq^{n-1} \quad \text{and} \quad c(x) = \left( \frac{(1 - q)x}{1 - q}, \frac{(1 - q)^2x^2}{2(1 - q^2)}, \frac{(1 - q)^3x^3}{3(1 - q^3)}, ... \right),
\] (0.10)
and thus \( D^n c(x) = c(x) - \sum_1^n [\lambda_i^{-1}] \). With this choice of \( \lambda_n \),
\[
\frac{1}{(q-1)x} D \mapsto \tilde{\Lambda} \quad \text{and} \quad D_q := \frac{D - 1}{(q-1)x} \mapsto \tilde{\Lambda}. \] (0.11)
In analogy with (0.4), we define the simple \( q \)-vertex operators:
\[
X_q(x, t, z) := e^{xz} X(t, z) \quad \text{and} \quad \tilde{X}_q(x, t, z) := (e^{xz})^{-1} X(-t, z).
\] (0.12)
in terms of (0.4) and the \( q \)-exponential \( e^z_q := e^{\sum_{k=1}^{\infty} \frac{(1-q)k^k}{k(1-q^k)}} \). Therefore under the isomorphism (0.3), Theorem 0.1 can be translated into \( q \)-language, to read:
Theorem 0.2. Any KdV \( \tau \)-function leads to a \( q \)-KdV \( \tau \)-function \( \tau(c(x) + t) \); the latter satisfies the bilinear relations below, for all \( x \in \mathbb{R}, \ t, t' \in \mathbb{C}^\infty \), and all \( n > m \), which tends to the standard KP-bilinear identity, when \( q \) goes to 1:

\[
\oint_{z=\infty} D^n (X_q(x,t,z) \tau(c(x) + t)) D^{n+1} (\hat{X}_q(x,t',z) \tau(c(x) + t')) \, dz = 0
\]

\[
\longrightarrow \oint_{z=\infty} X(t,z) \tau(\bar{x} + t) \ X(t',z) \tau(\bar{x} + t') \, dz = 0
\]

\[ (0.13) \]

Moreover, the \( q \)-differential operator \( Q^N_q \) has the form below and tends to the differential operator \( L^N \) of the KdV hierarchy, when \( q \) goes to 1:

\[
Q^N_q = D^N_q + \frac{\partial}{\partial t_1} \log \frac{\tau(D^N_c + t)}{\tau(c + t)} D_q^{N-1}
\]

\[
+ \left( \sum_{i=0}^{N-1} \frac{\partial^2}{\partial t_1^2} \log \tau(D^i c + t) \right)
\]

\[
- \sum_{i=0}^{N-2} \lambda_i+1 \left( \frac{\partial}{\partial t_1} \log \frac{\tau(D^N_c + t)}{\tau(D^{N-1}c + t)} - \frac{\partial}{\partial t_1} \log \frac{\tau(D^{i+1}c + t)}{\tau(D^i c + t)} \right)
\]

\[
+ \sum_{0 \leq i, j \leq N-2} \frac{\partial}{\partial t_1} \log \frac{\tau(D^{i+1}c + t)}{\tau(D^i c + t)} \frac{\partial}{\partial t_1} \log \frac{\tau(D^{j+1}c + t)}{\tau(D^j c + t)} D_q^{N-2} + ...
\]

\[ (0.14) \]

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1 The KP \( \tau \)-functions and Grassmannians

KP \( \tau \)-functions satisfy the differential Fay identity for all \( y, z \in \mathbb{C} \), in terms of the Wronskian \( \{ f, g \} := f'g - fg' \), as shown in \[11\] \[14\]:

\[ \{ \tau(t - [y^{-1}]), \tau(t - [z^{-1}]) \} \]
\[ (y - z)(\tau(t - [y^{-1}])\tau(t - [z^{-1}]) - \tau(t)\tau(t - [y^{-1}]) - \tau(t)\tau(t - [z^{-1}]) = 0. \tag{1.1} \]

In fact this identity characterizes the \( \tau \)-function, as shown in [13]. We shall need the following, shown in [1]:

**Proposition 1.1.** Consider \( \tau \)-functions \( \tau_1 \) and \( \tau_2 \), the corresponding wave functions

\[ \Psi_i = e^{\sum_{i \geq 1} t_i z^i} \tau_i(t - [z^{-1}]) = e^{\sum_{i \geq 1} t_i z^i} (1 + O(z^{-1})) \tag{1.2} \]

and the associated infinite-dimensional planes, as points in the Grassmannian \( \text{Gr} \),

\[ \tilde{W}_i = \text{span}\left\{ \left( \frac{\partial}{\partial t_1} \right)^k \Psi_i(t, z) \right\} \text{ for } k = 0, 1, 2, ... \]

then the following statements are equivalent

(i) \( z \tilde{W}_2 \subset \tilde{W}_1 \);
(ii) \( z\Psi_2(t, z) = \frac{\partial}{\partial t_1} \Psi_1(t, z) - \alpha \Psi_1(t, z) \), \( \text{for some function } \alpha = \alpha(t); \)
(iii)

\[ \{ \tau_1(t - [z^{-1}]), \tau_2(t) \} + z(\tau_1(t - [z^{-1}])\tau_2(t) - \tau_2(t - [z^{-1}])\tau_1(t)) = 0 \tag{1.3} \]

When (i), (ii) or (iii) holds, \( \alpha(t) \) is given by

\[ \alpha(t) = \frac{\partial}{\partial t_1} \log \frac{\tau_2}{\tau_1}. \tag{1.4} \]

Proof: To prove that (i) \( \Rightarrow \) (ii), the inclusion \( z \tilde{W}_2 \subset \tilde{W}_1 \) implies \( z \tilde{W}_2^t \subset \tilde{W}_1^t \), where \( \tilde{W}_2^t = \tilde{W}_2 e^{-\sum t_i z^i}; \) it follows that

\[ z\psi_2(t, z) = z(1 + O(z^{-1})) \in W_1^t \]

must be a linear combination, involving the operator \( \nabla = \frac{\partial}{\partial x} + z \) and the wave functions \( \Psi_i = e^{\sum_{i \geq 1} t_i z^i} \psi_i; \)

\[ z\psi_2 = \nabla \psi_1 - \alpha(t) \psi_1, \text{ and thus } z\Psi_2 = \frac{\partial}{\partial t_1} \Psi_1 - \alpha(t) \Psi_1. \tag{1.5} \]
The expression (1.4) for $\alpha(t)$ follows from equating the $z^0$-coefficient in (1.5), upon using the $\tau$-function representation (1.2). To show that (ii) $\Rightarrow$ (i), note that

$$z\Psi_2 = \frac{\partial}{\partial t_1} \Psi_1 - \alpha_1 \Psi_1 \in W_1^0$$

and taking $z$-derivatives, we have

$$z \left( \frac{\partial}{\partial t_1} \right)^j \Psi_2 = \left( \frac{\partial}{\partial t_1} \right)^{j+1} \Psi_1 + \beta_1 \left( \frac{\partial}{\partial t_1} \right)^j \Psi_1 + \cdots + \beta_{j+1} \Psi_1,$$

for some $\beta_1, \ldots, \beta_{j+1}$ depending on $t$ only; this implies the inclusion (i). The equivalence (ii) $\iff$ (iii) follows from a straightforward computation using the $\tau$-function representation (1.2) of (ii) and the expression for $\alpha(t)$. □

2 The full one-Toda lattice

For details on this sketchy exposition, see [3]. The one-Toda lattice equations

$$\frac{\partial L}{\partial t_n} = [(L^n)_+, L], \quad (2.1)$$

are deformations of an infinite matrix

$$L = \sum_{-\infty < i \leq 0} a_i \tilde{\Lambda}^i + \tilde{\Lambda}, \text{with } \tilde{\Lambda} := \lambda \Lambda = \varepsilon \Lambda \varepsilon^{-1}, \quad (2.2)$$

for diagonal matrices $\lambda$ and $\varepsilon$, with non-zero entries, and diagonal matrices $a_i$, depending on $t = (t_1, t_2, \ldots)$. One introduces wave and adjoint wave vectors $\Psi(t, z)$ and $\Psi^*(t, z)$, satisfying

$$L \Psi = z \Psi \quad \text{and} \quad L^\top \Psi^* = z \Psi^*$$

and

$$\frac{\partial \Psi}{\partial t_n} = (L^n)_+ \Psi \quad \frac{\partial \Psi^*}{\partial t_n} = -((L^n)_+)\top \Psi^*. \quad (2.3)$$

The wave vectors $\Psi$ and $\Psi^*$ can be expressed in terms of one sequence of $\tau$-functions $\tau(n, t) := \tau_n(t_1, t_2, \ldots), \quad n \in \mathbb{Z}$, to wit:

$$\Psi(t, z) = \left( e^{\sum_{i=1}^{\infty} t_i z^i} \psi(t, z) \right)_{n \in \mathbb{Z}} = \left( \frac{\tau_n(t - [z^{-1}])}{\tau_n(t)} e^{\sum_{i=1}^{\infty} t_i z^i \varepsilon_n z^n} \right)_{n \in \mathbb{Z}}.$$
\( \Psi^*(t, z) = \left( e^{-\sum_{i=1}^{\infty} i z^i} \psi^*(t, z) \right)_{n \in \mathbb{Z}} = \left( \frac{\tau_{n+1}(t + [z^{-1}])}{\tau_{n+1}(t)} e^{-\sum_{i=1}^{\infty} i z^i \varepsilon_n^{-1} z^{-n}} \right)_{n \in \mathbb{Z}} \)

(2.4)

It follows that, in terms of \( \chi(z) := (z^n)_{n \in \mathbb{Z}} \):

\[ \Psi = e^{\sum_{i=1}^{\infty} i z^i} S \varepsilon \chi(z), \quad \text{with} \quad S = \sum_{\ell=0}^{\infty} p_n(\tilde{\partial}) \tau(t) \tilde{\Lambda}^{-n}, \]

\[ \Psi^* = e^{-\sum_{i=1}^{\infty} i z^i} (S^\top)^{-1} \varepsilon^{-1} \chi(z^{-1}), \quad \text{with} \quad S^{-1} = \sum_{\ell=0}^{\infty} \tilde{\Lambda}^{-n} \Lambda p_n(\tilde{\partial}) \tau(t) \]

and thus,

\[ L = S \tilde{\Lambda} S^{-1} \]

\[ = \sum_{\ell=0}^{\infty} \frac{p_n(\tilde{\partial}) \tau_{n+2-\ell} \circ \tau_n}{\tau_{n+2-\ell} \tau_n} \tilde{\Lambda}^{-1-\ell} \]

\[ = \tilde{\Lambda} + \left( \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n} \right)_{n \in \mathbb{Z}} \tilde{\Lambda}^0 + \left( \left( \frac{\partial}{\partial t_1} \right)^2 \log \tau_n \right)_{n \in \mathbb{Z}} \tilde{\Lambda}^{-1} + ... \]

(2.5)

With each component of the wave vector \( \Psi \), we associate a sequence of infinite-dimensional planes in the Grassmannian \( Gr \)

\[ W_n = \text{span}_\mathbb{C} \left\{ \left( \frac{\partial}{\partial t_1} \right)^k \Psi_n(t, z), \quad k = 0, 1, 2, ... \right\} \]

\[ = e^{\sum_{i=1}^{\infty} i z^i} \text{span}_\mathbb{C} \left\{ \left( \frac{\partial}{\partial t_1} + z \right)^k \psi_n(t, z), \quad k = 0, 1, 2, ... \right\} \] \hspace{1cm} (2.6)

and planes

\[ W^*_n = \text{span}_\mathbb{C} \left\{ \left( \frac{\partial}{\partial t_1} \right)^k \Psi_n^*(t, z), \quad k = 0, 1, 2, ... \right\} \]

which are orthogonal to \( W_n \) by the residue pairing

\[ \oint_{z = \infty} f(z) g(z) \frac{dz}{2\pi iz}. \] \hspace{1cm} (2.7)

Note that the plane \( z^{-n}W_n \) has so-called virtual genus zero, in the terminology of [12]; in particular, this plane contains an element of order \( 1 + O(z^{-1}) \).

The following statement is mainly contained in [3]:
Proposition 2.1. The following five statements are equivalent

(i) The 1-Toda lattice equations (2.1)

(ii) \( \Psi \) and \( \Psi^* \), with the proper asymptotic behaviour, given by (2.4), satisfy the bilinear identities for all \( t, t' \in C^\infty \)

\[
\oint_{z=\infty} \Psi_n(t,z)\Psi_{m}^*(t',z) \frac{dz}{2\pi iz} = 0, \quad \text{for all } n > m; \tag{2.8}
\]

(iii) the \( \tau \)-vector satisfies the following bilinear identities for all \( n > m \) and \( t, t' \in C^\infty \):  

\[
\oint_{z=\infty} \tau_n(t-[z^{-1}])\tau_{m+1}(t'+[z^{-1}])e^{\sum_1^{\infty} (t_i-t'_i)z^i} z^{n-m-1}dz = 0; \tag{2.9}
\]

(iv) The components \( \tau_n \) of a \( \tau \)-vector correspond to a flag of planes in \( Gr \),

\[ \supset W_{n-1} \supset W_n \supset W_{n+1} \supset \ldots \tag{2.10} \]

(v) A sequence of KP-\( \tau \)-functions \( \tau_n \) satisfying the equations

\[
\{\tau_n(t-[z^{-1}]), \tau_{n+1}(t)\} + z(\tau_n(t-[z^{-1}])\tau_{n+1}(t) - \tau_{n+1}(t-[z^{-1}])\tau_n(t)) = 0 \tag{2.11}
\]

Proof: The proof that (i) is equivalent to (ii) follows from the methods in [4, 14]. That (ii) is equivalent to (iii) follows from the representation (2.4) of wave functions in terms of \( \tau \)-functions. Finally, we sketch the proof that (ii) is equivalent to (iv). The inclusion in (iv) implies that \( W_n \), given by (2.6), is also given by

\[ W_n = \text{span}_C \{\Psi_n(t,z), \Psi_{n+1}(t,z), \ldots\}; \]

Since each \( \tau_n \) is a \( \tau \)-function, we have that

\[
\oint_{z=\infty} \Psi_n(t,z)\Psi_{n-1}^*(t',z) \frac{dz}{2\pi iz} = 0,
\]

implying that, for each \( n \in \mathbb{Z} \), \( \Psi_{n-1}^*(t,z) \in W_n^* \). Moreover the inclusions \( \ldots \supset W_n \supset W_{n+1} \supset \ldots \) imply, by orthogonality, the inclusions \( \ldots \subset W_n^* \subset W_{n+1}^* \subset \ldots \), and thus

\[ W_n^* = \{\Psi_{n-1}^*(t,z), \Psi_{n-2}^*(t,z), \ldots\}. \]
Since 
\[ W_n \subset W_m = (W_m^*)^\ast, \quad \text{all } n \geq m, \]
we have the orthogonality \( W_n \perp W_m^\ast \) by the residue pairing (2.7) for all \( n \geq m \), i.e.,
\[ \oint_{z=\infty} \Psi_n(t, z) \Psi_{m-1}^\ast(t', z) \frac{dz}{2\pi i z} = 0, \quad \text{all } n \geq m. \]
Note (ii) implies \( W_m^\ast \subset W_n^\ast, \quad n > m \), hence \( W_n \subset W_m, \quad n > m \), yielding (iv). That (iv) \( \iff \) (v) follows from proposition 1.1, by setting \( \tau_1 := \tau_n \) and \( \tau_2 = \tau_{n+1} \). Then (v) is equivalent to the inclusion property
\[ z(z^{-n-1}W_{n+1}) \subset (z^{-n}W_n), \quad \text{i.e. } W_{n+1} \subset W_n, \]
thus ending the proof of proposition 2.1.

For \( m = n - 2 \), \( t \mapsto t + [\alpha], \quad t' \mapsto t - [\alpha] \), the bilinear identity (2.8) yields
\[
0 = \left. \frac{\tau_n(t + [\alpha])\tau_{n-1}(t - [\alpha])}{\tau_n(t)\tau_{n-1}(t)} \oint_{z=\infty} \Psi_n(t + [\alpha], z)\Psi_{n-2}^\ast(t - [\alpha], z) \frac{dz}{z} \right. \\
= \frac{1}{\tau_n\tau_{n-1}} \sum_{j \geq 0} \alpha^j \left( p_{j+2}(\tilde{\partial}) - \frac{\partial}{\partial t_{j+2}} \right) \tau_n \circ \tau_{n-1} \\
= \sum_{j \geq 0} \alpha^j \left( (L^{j+2})_{n-1,n-1} - \frac{\partial}{\partial t_{j+2}} \log \frac{\tau_n}{\tau_{n-1}} \right),
\]
from which the following useful formula follows:
\[
\frac{\partial}{\partial t_k} \log \frac{\tau_{n+1}}{\tau_n} = (L^k)_{nn}.
\tag{2.12}
\]

3 Proof of Theorems 0.1 and 0.2

At first, we exhibit particular solutions to equation (2.11), explained in [4].

Lemma 3.1. Particular solutions to equation
\[
\{\tau_1(t - [z^{-1}]), \tau_2(t)\} + z(\tau_1(t - [z^{-1}])\tau_2(t) - \tau_2(t - [z^{-1}])\tau_1(t)) = 0
\]
are given, for arbitrary \( \lambda \in \mathbb{C}^* \), by pairs \((\tau_1, \tau_2)\), defined by:

\[
\tau_2(t) = X(t, \lambda)\tau_1(t) = e^{\sum t_i \lambda_i} \tau_1(t - [\lambda^{-1}]), \tag{3.1}
\]
or

\[
\tau_1(t) = X(-t, \lambda)\tau_2(t) = e^{-\sum t_i \lambda_i} \tau_2(t + [\lambda^{-1}]). \tag{3.2}
\]

Proof: Using

\[ e^{-\sum_{t=1}^{\infty} \frac{1}{n!} (\xi)^n} = 1 - \frac{\lambda}{z}, \]

it suffices to check that \(\tau_2(t)\) satisfies the above equation (2.11)

\[
e^{-\sum t_i \lambda_i} \left\{ \{\tau_1(t - [z^{-1}]), \tau_2(t)\} + z(\tau_1(t - [z^{-1}])\tau_2(t) - \tau_2(t - [z^{-1}])\tau_1(t)\right\} \]

\[
= e^{-\sum t_i \lambda_i} \left\{ \{\tau_1(t - [z^{-1}]), e^{\sum t_i \lambda_i} \right\} \tau_1(t - [\lambda^{-1}])
\]

\[
+ z(\tau_1(t - [z^{-1}])\tau_1(t - [\lambda^{-1}]) - (1 - \frac{\lambda}{z})\tau_1(t)\tau_1(t - [z^{-1}] - [\lambda^{-1}])
\]

\[= 0, \]

using the differential Fay identity (1.1) for the \(\tau\)-function \(\tau_1\); a similar proof works for the second solution, given by (3.2). \(\blacksquare\)

Proof of Theorems 0.1 and 0.2: From an arbitrary KdV \(\tau\)-function, construct, for \(\lambda, c, \nu \in \mathbb{C}^\infty\), the following sequence of \(\tau\)-functions, for \(n \geq 0\), as announced in Theorem 0.1:

\[
\tau_0(t) = e^{\nu_n \sum t_i N} \tau(c + t)
\]

\[
\tau_n = e^{\nu_n \sum t_i N} X(t, \lambda_n)\ldots X(t, \lambda_1)\tau(c + t)
\]

\[
= \frac{\Delta(\lambda_1, \ldots, \lambda_n)}{\prod_1^n \lambda_i^{-1}} e^{\nu_n \sum t_i N} \prod_{k=1}^n e^{\sum_{i=1}^n t_i \lambda_i k} \tau(c + t - \sum_{i=1}^n [\lambda_i^{-1}]),
\]

\[
\tau_{-n} = e^{\nu_{-n} \sum t_i N} X(-t, \lambda_{-n+1})\ldots X(-t, \lambda_0)\tau(c + t)
\]

\[
= \frac{\Delta(\lambda_0, \ldots, \lambda_{-n+1})}{\prod_1^n \lambda_{-i+1}^{-1}} e^{\nu_{-n} \sum_{i=1}^n t_i N} \prod_{k=1}^n e^{-\sum_{i=1}^n t_i \lambda_{-i+1}^{-1} k} \tau(c + t + \sum_{i=1}^n [\lambda_{-i+1}^{-1}])
\]

(3.3)
and so, each \( \tau_n \) is defined inductively by

\[
\tau_{n+1} = e^{(\nu_{n+1} - \nu_n) \sum_{i=1}^{\infty} t_i N} X(t, \lambda_{n+1}) \tau_n;
\]

thus by Lemma 3.1, the functions \( \tau_{n+1} \) and \( \tau_n \) are a solution of equation (v) of proposition 2.1. Therefore, by the same proposition 2.1, the \( \tau_n \)'s form a \( \tau \)-vector of the 1-Toda lattice. Since each \( \tau_n \), except for the exponential factor \( \exp(\nu_n \sum_{i=1}^{\infty} t_i N) \), has the property that \( \partial \tau_n / \partial t_i N = 0 \) for \( i = 1, 2, \ldots \), we have that

\[
z^N W_n \subset W_n;
\]

in particular, the representation

\[
W_n = \text{span}\{ \Psi_n(t, z), \Psi_{n+1}(t, z), \ldots \},
\]

which follows from the inclusion \( \cdots \subset W_n \supset W_{n+1} \supset \cdots \), implies that, since \( L \Psi = z \Psi \),

\[
z^N \Psi_k = \sum_{j \geq k} a_j \Psi_j = (L^N \Psi)_k,
\]

and thus \( L^N \) is upper-triangular. Multiplying \( \tau_n \) with the exponential factor \( \exp(\nu_n \sum_{i=1}^{\infty} t_i N) \), does not modify the wave function \( \Psi_n \), except for a factor, which is a function of \( z^N \) only.

Therefore, we conclude that the matrix \( L \), defined by (2.5), from the sequence of \( \tau \)-functions (3.3),

\[
L = \tilde{\Lambda} + \left( \frac{\partial}{\partial t_1} \log \frac{\tau_{n+1}}{\tau_n} \right)_{n \in \mathbb{Z}} + \left( \left( \frac{\partial}{\partial t_1} \right)^2 \log \tau_n \right)_{n \in \mathbb{Z}} \tilde{\Lambda}^{-1} + \ldots
\]

\[
= \tilde{\Lambda} + (\lambda_{n+1} + b_{n+1})_{n \in \mathbb{Z}} \tilde{\Lambda}^0 + (a_n)_{n \in \mathbb{Z}} \tilde{\Lambda}^{-1} + \ldots,
\]

satisfies the 1-Toda lattice equations, where

\[
b_{n+1} = \frac{\partial}{\partial t_1} \log \frac{\tau(c + t - \sum_{i=1}^{n+1} [\lambda_i^{-1}])}{\tau(c + t - \sum_{i=1}^{n} [\lambda_i^{-1}])}
\]

for \( n \geq 1 \)

\[
= \frac{\partial}{\partial t_1} \log \frac{\tau(c + t - [\lambda_1^{-1}])}{\tau(c + t)}, \quad \text{for } n = 0,
\]

\[
= \frac{\partial}{\partial t_1} \log \frac{\tau(c + t + \sum_{i=0}^{n+1} [\lambda_i^{-1}])(1 - \delta_{-1, n})}{\tau(c + t + \sum_{i=0}^{n+1} [\lambda_i^{-1}])}, \quad \text{for } n \leq -1. (3.4)
\]
\[ a_n = \frac{\partial^2}{\partial t^2} \log (\tau(c + t - \sum_{1}^{n}[\lambda_i^{-1}])) \text{ for } n \geq 1 \]
\[ = \frac{\partial^2}{\partial t^2} \log (\tau(c + t)) \text{ for } n = 0 \]
\[ = \frac{\partial^2}{\partial t^2} \log (\tau(c + t + \sum_{0}^{n+1}[\lambda_i^{-1}])) \text{ for } n \leq -1, \quad (3.5) \]

confirming (0.7). Using the fact that, in view of (2.12), the diagonal terms of \( L^N \) are given by
\[ \frac{\partial}{\partial t} \log \left( \frac{\tau_{n+1}}{\tau_n} \right) = \nu_{n+1} - \nu_n, \]
and the fact that
\[ \tilde{\Lambda}^n = (\tilde{\Lambda} + \lambda)^n = \tilde{\Lambda}^n + \left( \sum_{1}^{n} \lambda_i \right) \tilde{\Lambda}^{n-1} + \left( \sum_{1 \leq i \leq j \leq n-1} \lambda_i \lambda_j \right) \tilde{\Lambda}^{n-2} + ..., \]
one finds that the upper-triangular matrix \( L^N \) has the following expression:
\[ L^N = \tilde{\Lambda}^N + \sum_{1}^{N} (\lambda_j + b_j) \tilde{\Lambda}^{N-1} + \left( \sum_{0}^{N-1} a_j + \sum_{1 \leq i \leq j \leq N-1} (\lambda_i + b_i)(\lambda_j + b_j) \right) \tilde{\Lambda}^{N-2} + ... \]
\[ = \tilde{\Lambda}^N + \left( \sum_{1}^{N} b_j \right) \tilde{\Lambda}^{N-1} + \left( \sum_{0}^{N-1} a_j - \sum_{1 \leq i \leq j \leq N-1} (b_N - b_i) \lambda_i + \sum_{1 \leq i < j \leq N-1} b_ib_j \right) \tilde{\Lambda}^{N-2} + ... \quad (3.6) \]
in terms of \( b_k \) and \( a_k \) defined in (0.7), thus proving Theorem 0.1.

To prove Theorem 0.2, note at first:
\[ \prod_{k=m+2}^{n} (-\lambda_k)^i \prod_{k=m+2}^{n} e^{-\sum_{i=1}^{n} \frac{1}{z} (\lambda_k)} = \frac{z^{n-m-1}}{\prod_{k=m+2}^{n} (\lambda_k)} \prod_{k=m+2}^{n} \left( 1 - \frac{\lambda_k}{z} \right) \]
\[ = \prod_{k=m+2}^{n} \left( 1 - \frac{z}{\lambda_k} \right) \]
\[ \prod_{k=m+2}^{n} e^{-\sum_{i=1}^{\infty} \frac{1}{i} \left( \frac{z}{k} \right)^i} = \frac{e^{x z q^n}}{e^{x z q^{m+1}}} = D^n e^{x z} D^{m+1} \left( e^{x z} \right)^{-1}. \]

The function \( \tau_n \), defined in Theorem 0.1, satisfies the bilinear identity of Theorem 0.1; therefore, using (3.3) and the above in the computation of \( \tau_n(t - [z^{-1}]) \), the following relations hold, up to a multiplicative factor depending on \( \lambda \) and \( \nu \):

\[
\alpha(\lambda, \nu) \int_{z=\infty} \tau_n(t - [z^{-1}]) \tau_{m+1}(t' + [z^{-1}]) e^{\sum_{i=1}^{\infty} (t_i - t_i') z_i} z^{n-m} \frac{dz}{z} = \int_{z=\infty} \tau(c(x) + t - [z^{-1}]) - \sum_{i=1}^{n} (\lambda_i - 1) \tau_{N} c(x) + t' + \sum_{i=1}^{m+1} (\lambda_i - 1) \]

\[
\prod_{k=m+2}^{n} \left( 1 - \frac{z}{\lambda_k} \right) e^{\sum_{i=1}^{\infty} (t_i - t_i') z_i} \frac{dz}{z} = \int_{z=\infty} D^n (X_q(x, t, z) \tau(c(x) + t)) \ D^{m+1} \left( \tilde{X}_q(x, t', z) \tau(c(x) + t') \right) dz = 0.
\]

When \( q \rightarrow 1 \), the second expression above tends to the standard KP-bilinear equation, upon using (0.10). Moreover, one checks by induction, using the first three terms in the expression for \( L \) and (2.12), that \( (L^N)_+ \) for \( N = 1, 2, 3, ... \) has the \( q \)-form (0.3). Also, note that \( a_k \) and \( b_k \) can be expressed in terms of the \( D \)-operator, using (0.7); to wit:

\[
b_k = \frac{\partial}{\partial t_1} \log \frac{\tau(D^k c + t)}{\tau(D^{k-1} c + t)}, \quad a_k = \left( \frac{\partial}{\partial t_1} \right)^2 \log \tau(D^k c + t).
\]

So, the expression for \( Q_N^q \) in Theorem 0.2 follows at once from (3.4). The fact that

\[-\lambda_1 \frac{\partial}{\partial t_1} \log \frac{\tau(D^{i+1} c + t)}{\tau(D^i c + t)} \rightarrow \frac{\partial^2}{\partial x^2} \log \tau(\bar{x} + t)\]

implies that all terms in (0.10) vanish in the limit \( q \rightarrow 1 \), except for the term \( \sum_{i=0}^{N-1} \frac{\partial^2}{\partial t_1} \log \tau(D^i c + t) \); so we have that

\[
\lim_{q \rightarrow 1} Q_N^q = \left( \frac{\partial}{\partial x} \right)^N + N \frac{\partial^2}{\partial x^2} \log \tau(\bar{x} + t) \left( \frac{\partial}{\partial x} \right)^{N-2} + ..., \]

thus ending the proof of theorem 0.2.
4 Examples and vertex operators

The isomorphism (0.3) enables one to translate every 1-Toda statement, having the form (0.3) into a $D$ or $D_q$ statement. Also every $\tau$-function of the KdV hierarchy leads automatically to a solution of $q$-KdV. For instance, by replacing $t \mapsto c(x) + t$ in the Schur polynomials, one finds $q$-Schur polynomials. The latter were obtained by Haine and Iliev [9] by using the $q$-Darboux transforms; the latter had been studied by Horozov and coworkers in [5, 6].

The $n$-soliton solution to the KdV (for $N = 2$) (for this formulation, see [4]),

$$\tau(t) = \det \left( \delta_{i,j} - \frac{a_j}{y_i + y_j} e^{-\sum_{k, \text{odd}} \tau_k (y_i^k + y_j^k)} \right)_{1 \leq i,j \leq n},$$

leads to a $q$-soliton by the shift $t \mapsto c(x) + t$, with $c(x)$ as in (0.8), namely

$$\tau(x,t) = \det \left( \delta_{ij} - \frac{a_j}{y_i + y_j} e^{-\sum_{k=1}^{\infty} \tau_k (y_i^k + y_j^k)} \right)_{1 \leq i,j \leq n}.$$

Moreover the vertex operator for the 1-Toda lattice is a reduction of the 2-Toda lattice vertex operator (see [2]), given by

$$X(t,y,z) = -\chi^*(z) X(-t,z) X(t,y) \chi(y) = \frac{z}{y - z} e^{\sum_{i=1}^{\infty} t_i (y^i - z^i)} e^{-\sum_{i=1}^{\infty} (y^i - z^i) \frac{1}{\omega^i} \frac{\partial}{\partial t^i} \left( \frac{y^n}{z^n} \right)}_{n \in \mathbb{Z}} ;$$

in particular, if $\tau$ is a 1-Toda vector, then $a\tau + bX(t,y,z)\tau$ is a 1-Toda vector as well. Using the dictionary, this leads to $q$-vertex operators

$$X_q(x,t;y,z) = e_q^{xy} (e_q^{xz})^{-1} e^{\sum_{i=1}^{\infty} t_i (y^i - z^i)} e^{-\sum_{i=1}^{\infty} (y^i - z^i) \frac{1}{\omega^i} \frac{\partial}{\partial t^i}} \quad \text{for } q\text{-KP},$$

and, for any $N$th root $\omega$ of 1,

$$X_q(x,t;z) = e_q^{x\omega z} (e_q^{xz})^{-1} e^{\sum_{i=1}^{\infty} t_i z^i (\omega^i - 1)} e^{-\sum_{i=1}^{\infty} z^{-i} (\omega^{-i} - 1) \frac{1}{\omega^i} \frac{\partial}{\partial t^i}} \quad \text{for } q\text{-KdV},$$

having the typical vertex operator properties.
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