On the relation between Stokes multipliers and the T-Q systems of conformal field theory

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Abstract

The vacuum expectation values of the so-called $Q$-operators of certain integrable quantum field theories have recently been identified with spectral determinants of particular Schrödinger operators. In this paper we extend the correspondence to the $T$-operators, finding that their vacuum expectation values also have an interpretation as spectral determinants. As byproducts we give a simple proof of an earlier conjecture of ours, proved by another route by Suzuki, and generalise a problem in $P\bar{T}$ symmetric quantum mechanics studied by Bender and Boettcher. We also stress that the mapping between $Q$-operators and Schrödinger equations means that certain problems in integrable quantum field theory are related to the study of Regge poles in non-relativistic potential scattering.

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1 Introduction and review

An unexpectedly precise connection between certain two-dimensional quantum field theories and the quantum mechanics of anharmonic oscillators was uncovered in [1]. The quantum-mechanical side of this correspondence concerned the Schrödinger problem

\[ \hat{H} \psi(x) = \left( -\frac{d^2}{dx^2} + |x|^{2M} \right) \psi(x) = E \psi(x) \]  

(1.1)
on the real line \( x \in (-\infty, \infty) \), and the associated spectral determinant \( D_M(\lambda) \), defined for \( M > 1 \) as

\[ D_M(E) = D_M(0) \prod_{k=0}^{\infty} \left( 1 - \frac{E}{E_k} \right), \]  

(1.2)
with the product running over the set \( \{E_k\} \) of eigenvalues of \( \hat{H} \). The normalisation \( D_M(0) = \sin(\pi/(2M+2))^{-1} \) is automatic if \( D_M \) is defined as a zeta-regularized functional determinant (see [2]), and is also the natural choice for current purposes. However we have chosen to negate \( E \) as compared to the conventions of [1,2], so that the zeroes of \( D_M(E) \) as defined here coincide with the eigenvalues of \( \hat{H} \) rather than their negatives.

The first result of [1] was a mapping between the functional relation satisfied by \( D_2(E) \), discovered in [3], and another set of functional relations, known as a Y-system, which had previously arisen in the study of perturbed conformal field theories [4]. (The precise relation given in [3] had in fact also cropped up in the context of integrable lattice models, in [5].) One consequence was a novel expression for \( D_2(E) \) as the solution of a certain non-linear integral equation, a result that was fixed uniquely by the functional relations and known analyticity properties on the two sides of the correspondence. The move to higher integer values of \( M \) brings a considerable increase in the complexity of the functional relations. In [1] the spectral problems for \( M > 2 \) were related to certain further sets of functional relations, associated with the Y-systems, called T-systems, but this result rested on a combination of analytical and numerical evidence without being proved. This gap was filled by Suzuki in [6]; an alternative proof will be given below.

The other strand in [1] was a relation between the even and odd spectral subdeterminants for the problem (1.1) and the vacuum expectation values of objects known as Q-operators [7, 8], taken at certain values of the ‘Virasoro parameter’ \( p \). The subdeterminants are defined via a split of the eigenvalues according to the parity of their eigenfunctions, as

\[ D_M^\pm(E) = D_M^\pm(0) \prod_{k \text{ even/odd}} \left( 1 - \frac{E}{E_k} \right) \]  

(1.3)

(so that \( D_M(E) = D_M^+(E)D_M^-(E) \)), while Q-operators \( Q_+ \) and \( Q_- \) were introduced, in the form that we shall need, in [8]. They are continuum analogues of the Q-matrices of Baxter [9]. To state the result of [1] more precisely, we recall that in the construction of [8], the operators \( Q_\pm \) act on Virasoro modules labeled by \( p \), with central charge
\[ c = 1 - 6(\beta - \beta^{-1})^2, \beta \text{ being a further parameter.} \]

In each module the 'vacuum' (highest weight) state \(|p\rangle\) has conformal dimension \((p/\beta)^2 + (c-1)/24\), and is an eigenstate of the \(Q\)-operators. Setting

\[ A_{\pm}(\lambda, p) = \lambda^{\mp 2p/\beta^2} \langle p|Q_{\pm}(\lambda)|p\rangle , \]

the eigenvalues \(A_{\pm}(\lambda, p)\) are entire functions of the square of the spectral parameter \(\lambda\), and we have

\[ A_{\pm}(\lambda, \beta^2/4) = \alpha_{\pm}D_{M}^{\mp}(\lambda^2/\nu^2) \quad (1.5) \]

where \(M = \beta^{-2} - 1, \nu = \left(\frac{1}{2}\beta^2\right)^{1-\beta^2}\Gamma(1-\beta^2)^{-1}\), and the proportionality constants \(\alpha_{\pm}\) are rederived below.

In contrast to the first result, \(M\) does not have to be an integer for this to hold, and so the modulus signs in (1.1) are in general obligatory when the problem is posed on the full real line. However (as exploited in [9] for \(M\) half-integral) we can alternatively set up the problem on the half-line \((0, \infty)\) with the potential \(x^{2M}\), so long as the boundary condition at \(x = 0\) is chosen correctly. The even wavefunctions are picked out by the Neumann condition \(\psi'_{2k}(0) = 0\), and the odd by the Dirichlet condition \(\psi_{2k+1}(0) = 0\). These two spectral problems thus yield the even and odd spectral subdeterminants for the whole-line problem directly.

The problem of finding spectral determinants related to the \(Q\)-operators at general values of \(p\) was addressed in [10], where it was found that the problem (1.1) should be modified to

\[ \left(-\frac{d^2}{dx^2} + x^{2M} + \frac{l(l+1)}{x^2}\right)\psi(x) = E\psi(x) \quad (1.6) \]

on the half-line \((0, \infty)\). Imposing the two possible power-like behaviours at \(x = 0\), namely \(\psi(x) \propto x^{l+1}\) and \(\psi(x) \propto x^{-l}\), results in two different spectral problems, and their spectral determinants are proportional to the functions \(A_{+}(\lambda, (2l+1)\beta^2/4)\) and \(A_{-}(\lambda, (2l+1)\beta^2/4)\) respectively.

Much of this work rested on the so-called quantum Wronskian relation satisfied by the operators \(Q_+\) and \(Q_-\) [8]. But a key feature of [7,8] was another functional equation, relating the \(Q\)-operators to further operators \(T(\lambda)\), sometimes called quantum transfer matrices. Called the T-Q relation, it is

\[ T(\lambda)Q_{\pm}(\lambda) = Q_{\pm}(q^{-1}\lambda) + Q_{\pm}(q\lambda), \quad (1.7) \]

where

\[ q = e^{i\pi\beta^2}. \quad (1.8) \]

The vacua \(|p\rangle\) are also eigenstates of the \(T(\lambda)\), and if we set \(T(\lambda, p) = \langle p|T(\lambda)|p\rangle\) then the T-Q relation for these vacuum eigenvalues can be written as

\[ T(\lambda, p)A_{\pm}(\lambda, p) = e^{\mp 2\pi i p}A_{\pm}(q^{-1}\lambda, p) + e^{\pm 2\pi i p}A_{\pm}(q\lambda, p), \quad (1.9) \]

an equation that was also obtained in [11].

2
In this paper we point out that this relation also has a natural interpretation in the context of the Schrödinger equation, thus finding a rôle for the $T$ operators at general $\beta$ and $p$ in the ‘Schrödinger picture’. This leads to an alternative derivation of the results of [1,1], and a novel interpretation of the fusion relations and their truncations. It also has the bonus, for integer values of $M$, of providing a simple proof of the $T$-system conjecture alluded to above. All of this material is contained in §§2–6, while §7 applies the results to a problem in quantum mechanics, and §8 discusses duality properties. Finally, §9 contains our conclusions, and an appendix details the calculation of a certain asymptotic used in §2.

2 The general $T$-$Q$ relation

The results of [12] provide a convenient framework for our discussion, and we begin with a quick summary of some of this material.

Consider the differential equation

$$\left(-\frac{d^2}{dx^2} + P(x)\right)\psi(x) = 0$$

(2.1)

for general complex values of $x$. For our purposes we must set $P(x) = x^{2M} - E + l(l+1)x^{-2}$. In [12] $P(x)$ was taken to be a polynomial in $x$, restricting $2M$ to be a positive integer, and $l(l+1)$ to be zero. Allowing $2M$ to be a general real number larger than $-2$, or $l(l+1)$ to be nonzero, should not change the result quoted below in any essential way, though it will usually introduce a branch point in $\psi(x)$ at $x = 0$. Modulo this proviso, we have [12]:

- The equation (2.1) has a solution $y = y(x, E, l)$ such that
  (i) $y$ is an entire function of $(x, E)$ (though, for the reason just mentioned, $x$ must in general be considered to live on a suitable cover of the punctured complex plane).
  (ii) $y$ and $y' = dy/dx$ admit the asymptotic representations
        $$y \sim x^{-M/2} \exp\left(-\frac{1}{M+1}x^{M+1}\right)$$
        (2.2)
        $$y' \sim -x^{M/2} \exp\left(-\frac{1}{M+1}x^{M+1}\right)$$
        (2.3)
  as $x$ tends to infinity in any closed sector contained in the sector
        $$|\arg x| < \frac{3\pi}{2M + 2}. \quad (2.4)$$

Furthermore, the solution $y$ is uniquely characterised by this information. (Note though that the asymptotic (2.2) is only valid as given if $M > 1$. Extra terms appear for $M \leq 1$, consistent with the WKB result that the solution which decays as $x \to +\infty$ is asymptotically proportional to $P(x)^{-1/4} \exp(-\int^x P(t)^{1/2} dt)$.)

Let $S_k$ denote the sector

$$|\arg x - \frac{2k\pi}{2M+2}| < \frac{\pi}{2M+2}. \quad (2.5)$$
From (2.2) it follows that $y$ tends to zero as $x$ tends to infinity in $S_0$, and to infinity as $x$ tends to infinity in $S_{-1}$ and $S_1$. More technically, one says that $y$ is subdominant in $S_0$, and dominant in $S_{\pm 1}$. To find solutions subdominant in other sectors, consider $\tilde{y}(x) = y(ax, E, l)$ for any constant $a$. This function solves the equation

$$
\left(-\frac{d^2}{dx^2} + a^{2M+2}x^{2M} - a^2E + \frac{l(l+1)}{x^2}\right)\tilde{y}(x) = 0.
$$

Thus if $a^{2M+2} = 1$, $y(ax, a^{-2}E, l)$ is another solution to the original problem (2.1). Setting $\omega = \exp(\pi i/(M+1))$ we therefore have a set of solutions

$$
y_k \equiv y_k(x, E, l) = \omega^{k/2}y(\omega^{-k}x, \omega^{2k}E, l),
$$

with $y_k$ subdominant in $S_k$ and dominant in $S_{k\pm 1}$. (Our convention differs from [12] by the factor of $\omega^{k/2}$, which is included for later convenience.)

A consequence of these facts is that each pair $\{y_k, y_{k+1}\}$ provides a set of linearly independent solutions to the second-order equation (2.1), and any other solution can be expanded in terms of them. In particular,

$$
y_{k-1}(x, E, l) = C_k(E, l)y_k(x, E, l) + \tilde{C}_k(E, l)y_{k+1}(x, E, l).
$$

The functions $C_k$ and $\tilde{C}_k$ are called the Stokes multipliers for $y_{k-1}$ with respect to $y_k$ and $y_{k+1}$. It follows from (2.7) that $C_k(E, l) = C_{k-1}(\omega^2E, l)$ and $\tilde{C}_k(E, l) = \tilde{C}_{k-1}(\omega^2E, l)$. For brevity we will write $C_0$ and $\tilde{C}_0$ as $C$ and $\tilde{C}$ respectively. (Again, we differ slightly from the conventions of [12], where the abbreviations $C$ and $\tilde{C}$ were instead reserved for $C_1$ and $\tilde{C}_1$.)

The Stokes multipliers can be expressed in terms of Wronskians. Recall that the Wronskian $W[f, g]$ of two functions $f(x)$ and $g(x)$ is defined as

$$
W[f, g] = fg' - f'g.
$$

If $f$ and $g$ both solve a Schrödinger equation such as (2.1), then $W[f, g]$ is independent of $x$; furthermore, it vanishes if and only if $f$ and $g$ are linearly dependent. Taking the Wronskian of (2.3) at $k = 0$ with $y_1$ and $y_0$ shows that

$$
C = \frac{W_{-1,1}}{W_{0,1}}, \quad \tilde{C} = -\frac{W_{-1,0}}{W_{0,1}},
$$

where we used the abbreviation $W_{k_1, k_2}$ for $W[y_{k_1}, y_{k_2}]$. These Wronskians are entire functions of $E$ and $l$. Since $y_0$ and $y_1$ are independent, $W_{0,1}$ never vanishes, and $C$ and $\tilde{C}$ are also entire.

In fact, all of the $\tilde{C}_k$ are identically equal to $-1$ [12]. This follows from (2.10) and the relations $W_{k_1+1, k_2+1}(E, l) = W_{k_1, k_2}(\omega^2E, l)$ and $W_{0,1}(E, l) = 2i$. (The second of these is found by evaluating $W_{0,1}$ as $x$ tends to infinity in the sectors $S_0$ or $S_1$, where the asymptotic behaviours of $y_0$ and $y_1$ and their derivatives are determined by (2.2).
and \((2.3)\). Since \(y_1(x, E, l) = y_1(x^*, E^*, l^*)^*\), it also follows from \((2.10)\) that \(C(E, l)\) is real whenever \(E\) and \(l\) are real.

The basic Stokes relation \((2.8)\) at \(k = 0\) is therefore

\[
C(E, l)y_0(x, E, l) = y_{-1}(x, E, l) + y_1(x, E, l)
\]

(2.11)

with

\[
C(E, l) = \frac{1}{2l}W_{-1,1}(E, l).
\]

(2.12)

If \((2.11)\) is rewritten in terms of \(y\) it becomes

\[
C(E, l)y(x, E, l) = \omega^{-1/2}y(\omega x, \omega^{-2}E, l) + \omega^{1/2}y(\omega^{-1}x, \omega^2E, l)
\]

(2.13)

With \(x\) formally set to zero, this has exactly the form of \((1.9)\) for \(A_+ + (\lambda, p)\), albeit at the specific value \(\beta_2/4\) of \(p\) for which \(e^{2\pi ip} = \omega_1/2\). (It also matches the T-Q relation for \(A_-\) at \(p = -\beta_2/4\), but since \(A_-(\lambda, p) = A_+(\lambda, -p)\) \(\text{[8]}\) this is not an independent result.)

However this tactic only works when \(l(l+1) = 0\). Otherwise, the resulting equation in \(E\) is either trivial or meaningless: if \(l(l+1) < 0\), that is \(-1 < l < 0\), then \(y(x=0, E, l)\) is identically zero, while if \(l(l+1) > 0\) then \(y(x=0, E, l)\) is almost everywhere infinite.

The problem arises because any solution to \((2.1)\) is a linear combination of one solution, \(\psi^+\), behaving near \(x = 0\) as \(x^{l+1}\), and one, \(\psi^-\), behaving there as \(x^{-l}\). Both of these are zero at \(x = 0\) if \(l(l+1) < 0\), whilst one or other is infinite if \(l(l+1) > 0\). However, rather than considering the functions \(y(x, E, l)\) at \(x=0\) directly, we can take a hint from the result of \([\text{10}]\) and project onto either \(\psi^+\) or \(\psi^-\). We choose to fix \(\psi^+\) by the \(x \to 0\) asymptotic

\[
\psi^+(x, E, l) \sim x^{l+1} + O(x^{l+3}).
\]

(2.14)

Since \(\psi^-\), the other solution, behaves as \(x^{-l}\), this only determines \(\psi^+\) uniquely if \(\Re l > -3/2\). If necessary, \(\psi^+\) can be defined outside this domain by analytic continuation. In particular, since \(l\) only appears in \((2.1)\) in the combination \(l(l+1)\), we can continue from \(l\) to \(-1-l\) and define \(\psi^-\) as

\[
\psi^-(x, E, l) \equiv \psi^+(x, E, -1-l).
\]

(2.15)

This procedure does bring some subtleties, to which we shall return \([\text{3}]\) below, but they do not affect the arguments of this section. In discussions of the radial Schrödinger equation (see, for example, chapter 2 of \([\text{3}]\) or chapter 4 of \([\text{4}]\) \(\psi^+(x, E, l)\) for \(\Re l > -1/2\) is sometimes called the regular solution.

In analogy to \((2.7)\), we define ‘shifted’ solutions \(\psi^\pm_k\):

\[
\psi^\pm_k \equiv \psi^\pm_k(x, E, l) = \omega_k^{\pm/2} \psi^\pm(\omega^{-k}x, \omega^{2k}E, l).
\]

(2.16)

These also solve the original problem \((2.1)\). By considering the \(x \to 0\) limit it is easily seen that

\[
\psi^\pm_k(x, E, l) = \omega^{\pm k(l+1/2)} \psi^\pm(x, E, l).
\]

(2.17)
We also have \( W[y_{k_1+1}, \psi_{k_2}^-](E, l) = W[y_{k_1}, \psi_{k_2}^-](\omega^2 E, l) \), so
\[
W[y_k, \psi^\pm](E, l) = \omega^{\pm k(l+1/2)} W[y_k, \psi^\pm](E, l) = \omega^{\pm k(l+1/2)} W[y, \psi^\pm](\omega^2 E, l). \tag{2.18}
\]
We can now take the Wronskian of both sides of (2.11) with \( \psi^\pm \). Defining
\[
D^\pm(E, l) \equiv W[y(x, E, l), \psi^\pm(x, E, l)] \tag{2.19}
\]
and using equation (2.18), the Stokes relation (2.11) becomes
\[
C(E, l) D^\pm(E, l) = \omega^{\pm(1/2+l)} D^\mp(\omega^{-2} E, l) + \omega^{\pm(1/2+l)} D^\mp(\omega^2 E, l) \tag{2.20}
\]
and (1.9) has indeed been matched, provided \( \omega \) is equal \( q \) (that is, \( e^{i\pi/(M+1)} = e^{i\pi\beta^2} \), and \( \omega^{l+1/2} \) is equal to \( e^{2\pi ip} \). These are the same relations between \( M \) and \( \beta \), and \( l \) and \( p \), as obtained in [1, 10], found here by an alternative route.

To establish the precise relation between the functions appearing in equations (1.9) and (2.20), we can use the fact that, when combined with certain analyticity properties, T-Q relations of this kind are extremely restrictive [7, 8, 15]. Since \( D^+(E, l) = D^-(E, -l-1) \), we need only consider \( D^-(E, l) \). In addition to (2.20), we have

(i) \( C \) and \( D^- \) are entire functions of \( E \);

(ii) If \( l \) is real and larger than \(-1/2\), then the zeroes of \( D^- \) all lie on the positive real axis of the complex-\( E \) plane;

(iii) If \(-1-M/2 < l < M/2\), then the zeroes of \( C \) all lie away from the positive real axis of the complex-\( E \) plane;

(iv) If \( M > 1 \) then \( D^- \) has the large-\( E \) asymptotic
\[
\log D^-(E, l) \sim \frac{a_0}{2} (-E)^\mu \quad , \quad |E| \to \infty \quad , \quad |\arg(-E)| < \pi \quad \tag{2.21}
\]
where \( \mu = (M+1)/2M \) and
\[
a_0 = 2 \int_0^\infty [(t^2M + 1)^{1/2} - tM] dt = -\frac{1}{\sqrt{\pi}} \Gamma(-\frac{1}{2} - \frac{1}{2M}) \Gamma(1 + \frac{1}{2M}); \tag{2.22}
\]

(v) If \( E = 0 \) then
\[
D^-(0, l) = \frac{1}{\sqrt{\pi}} \Gamma(1 + \frac{2M+1}{2M+2}) (2M+2)^{\frac{2M+1}{2M+2} + \frac{1}{2}}. \tag{2.23}
\]
Property (i) follows from the definition (2.19) of \( D^- \) as a Wronskian, given that the functions involved are themselves entire functions of \( E \). Property (ii) is also straightforward, since a zero of \( D^-(E, l) \) signals the existence of an eigenfunction for (1.6) at that value of \( E \), decaying as \( x^{l+1} \) as \( x \to 0 \), and exponentially as \( x \to +\infty \). The self-adjoint
nature of this problem for $l > -1/2$ then ensures the reality of these zeroes. For $l > 0$, the potential is everywhere positive and multiplying (1.6) by $\psi^*(x)$ and integrating from 0 to $\infty$ shows that all of the eigenvalues $E$ must also all be positive. For $-1/2 < l < 0$ the centrifugal term in the potential, $l(l+1)/x^2$, is negative but the same style of argument can be applied to the transformed equation (A.3), with the conclusion that the eigenvalues are again all positive. Property (iii) is more delicate, and further discussion will be postponed until § 3, where a partial result will be established. Finally, property (iv) follows from a WKB analysis, which is outlined in appendix A, and property (v) from a mapping of the problem at $E = 0$ into an exactly-solvable case, given in § 3 below.

We now claim that for $M > 1$ and $-1/2 < l < M/2$, the T-Q relation (2.20) and properties (i)–(v) characterise the functions $C(E,l)$ and $D^-(E,l)$ uniquely. Furthermore, with the identifications

$$\beta^2 = \frac{1}{M+1}, \quad p = \frac{2l+1}{4M+4} \quad (2.24)$$

the same T-Q relation and the same properties (i)–(v) hold for the functions $T(\lambda,p)$ and $A_+(\lambda,p)$ of § 3, save for $\lambda^2$ replacing $E$ in (i), the asymptotic in (iv) becoming

$$\log A_+(\lambda,p) \sim (M+1)\Gamma(\frac{1}{2\nu}) 2^{2\mu} a_0 (-\lambda^2)\mu, \quad |\lambda^2| \to \infty, \quad |\arg(-\lambda^2)| < \pi, \quad (2.25)$$

and $A_+(0,p)$ being equal to one rather than the value given in (v). The asymptotics can be made to agree by setting

$$\lambda = \nu E^{1/2}, \quad \nu = (2M+2)^{-1/2}\mu \Gamma(\frac{1}{2\nu})^{-1} \quad (2.26)$$

while (v) is fixed by multiplying $A_+(\lambda,p)$ by $D^-(0,l)$. By the claimed uniqueness, we therefore have

$$A_+(\lambda,p)\bigg|_{\beta^2} = \alpha^+ D^\pm\left(\left(\frac{\lambda}{\nu}\right)^2, \frac{2p}{\beta^2-\frac{1}{2}}\right)\bigg|_{M=\beta^{-2}-1} \quad (2.27)$$

where $\alpha^+ = D^\pm(0,2p/\beta^2-1/2)^{-1}$, and

$$T(\lambda,p)\bigg|_{\beta^2} = C\left(\left(\frac{\lambda}{\nu}\right)^2, \frac{2p}{\beta^2-\frac{1}{2}}\right)\bigg|_{M=\beta^{-2}-1} \quad (2.28)$$

Continuation from $l$ to $-l-1$ and the identity $A_-(\lambda,p) = A_+(\lambda,-p)$ were used to deduce the relation between $A_-$ and $D^+$ in (2.27), while the result (2.12) was used to express $C(E,l)$ in terms of $W_{-1,l}(E,l)$ in (2.28). As will seen in the next section, the expression in terms of the Wronskian appears to be the most general.

It remains to discuss the uniqueness property. Observe first that the asymptotic (2.21) applies as $|E| \to \infty$ along any ray with $\arg E \neq 0$, since the WKB problem only has turning-points if $E$ is on the positive real axis. Since the growth of $\log D^-(E,l)$
in this remaining direction is no greater we conclude that the order of $D^-(E, l)$ as a function of $E$ is equal to $(M+1)/2M$. (The order of an entire function $f(z)$, if finite, is the lower bound of all positive numbers $A$ such that $|f(z)| = O(e^{Az})$ as $|z| = r \to \infty$. See, for example, chapter 8 of \[16\].) For $M > 1$, $(M+1)/2M$ is less than 1, and Hadamard’s factorization theorem implies that $D^-$ can be written as an infinite product over its set of zeroes $\{E_n\}$ of the following form:

$$D^-(E, l) = D^-(0, l) \prod_{n=0}^{\infty} \left(1 - \frac{E}{E_n}\right). \quad (2.29)$$

As an aside, we note that this is the final ingredient needed to prove that $D^-$, and hence $A_+$, is proportional to a spectral determinant: from the discussion of property (ii) above, the zeroes of $D^-$ coincide with those of the relevant spectral determinant, and the residual ambiguity of an entire function with no zeroes is reduced to an overall constant by (2.29). (In fact, the normalisation of $D^-$ given by (2.29) coincides with an earlier ‘natural’ normalisation for the spectral determinant when $M$ is an integer and $l$ is equal to 0 or 1 \[9\], so it is reasonable to say that $D^-$ is actually equal to the spectral determinant.)

If $M \leq 1$ the order is larger, permitting a more complicated prefactor to the product in (2.29), and furthermore the asymptotic density of zeroes necessitates modifications to the terms in the product to ensure their convergence (again, see \[16\]). For the most part we have been excluding such cases from our discussion, but we remark that there is evidence that many of our results continue to hold even after the region $M > 1$ (called the ‘semiclassical domain’ in \[8\]) is left. Some of this evidence was given in \[1\]; some more can be found in the next section, and in \[5\].

Reverting to $M > 1$, the product representation (2.29) allows the steps described in \[8,17\] to be repeated to show that $D^-$ is determined by a nonlinear integral equation of the type introduced in \[15,17\]. Introduce the function

$$d(E, l) = \omega^{2l+1} \frac{D^-(\omega^2 E, l)}{D^-(\omega^{-2} E, l)}. \quad (2.30)$$

From (2.20) and property (i), the points at which $d = -1$ are exactly the zeroes of $C$ and $D^-$. By properties (ii) and (iii), those on the positive real axis are the zeroes of $D^-$, those on the negative real axis the zeroes of $C$. (The need for property (iii) was not mentioned in \[8\], but if it fails equation (2.32) below has to be modified, becoming the massless version of one of the equations obtained in \[18\].) The large-$E$ behaviour of $d(E, l)$ follows from (2.21): \[\]

\[
\log d(E, l) \sim \begin{cases} 
-\frac{1}{2} ib_0(1-e^{-i\pi/M})(E)^\mu & \frac{2\pi}{M+1} < \arg(E) < 2\pi - \frac{2\pi}{M+1} \\
-\frac{1}{2} ib_0(E)^\mu & -\frac{2\pi}{M+1} < \arg(E) < \frac{2\pi}{M+1} \\
-\frac{1}{2} ib_0(1-e^{i\pi/M})(E)^\mu & -2\pi + \frac{2\pi}{M+1} < \arg(E) < -\frac{2\pi}{M+1}
\end{cases} \quad (2.31)
\]

*this asymptotic is only correct for $M > 1$. The situation for $M < 1$ is discussed in appendix A
where \(b_0 = 2 \cos\left(\frac{\pi}{2}M\right)a_0\). (By \((E)^\mu\) we imply \(\epsilon^{\mu \arg(E)}|E|^\mu\). Thus the first and third asymptotics coincide, as indeed they must since \(d\) is a single-valued function of \(E\). In the language of \([17]\), they correspond to the ‘second determination’.)

Now trade \(E\) for the new variable \(\theta = \mu \log(\nu^2 E)\). Then the arguments of \([8]\) can be followed to show that the function 
\[
f(\theta, l) = \log d(\nu^{-\frac{2}{\mu}}e^{\theta/\mu}, l)
\]
solves
\[
f(\theta, l) = i\pi \left(l + \frac{1}{2}\right) - \frac{i}{2} b_0 \nu^{-2\mu} e^{\theta} + \int_{\mathcal{C}_1} \varphi(\theta - \theta') \log(1 + e^{f(\theta', l)}) d\theta'
- \int_{\mathcal{C}_2} \varphi(\theta - \theta') \log(1 + e^{-f(\theta', l)}) d\theta'
\]
where the contours \(\mathcal{C}_1\) and \(\mathcal{C}_2\) run from \(-\infty\) to \(+\infty\), just below and just above the real \(\theta\)-axis, and
\[
\varphi(\theta) = \int_{-\infty}^{\infty} \frac{e^{ik\theta} \sinh \frac{\pi}{4}(\xi - 1)k}{2 \cosh \frac{\pi}{2}k \sinh \frac{\pi}{2} \xi k 2\pi} dk , \quad \xi = \frac{1}{M}.
\]
As mentioned above, such nonlinear integral equations first arose in \([15, 17]\). We must now assume that the solution to this equation is unique. In this (‘massless’) context such an assumption seems reasonable, though we did not attempt a complete proof. (Note though that for the closely-related TBA integral equations, such a uniqueness result can fail at nonzero complex values of the mass scale, as can be seen from figure 1 of \([19]\).)

A knowledge of \(f\) fixes \(D^-\) up to an overall constant, given the general information on the locations of the zeroes of \(D^-\) and \(C\) contained in properties (i) and (ii) above. In turn, this constant is determined by property (v), and so \(D^-\) and hence \(C\) have been determined. While a more direct approach to the main uniqueness claim would perhaps be more satisfactory, the fact, already stressed in \([1]\), that spectral problems can be solved with the aid of a nonlinear integral equation is of independent interest, and will be put to use in \(\S\) 7 below.

3 The harmonic oscillator and the free-fermion point

The case \(M = 1\) falls just outside the ‘semiclassical domain’ \(M > 1, \beta^2 < 1/2\) that we have been discussing. However, it corresponds to the three-dimensional harmonic oscillator, for which the exact wave-functions are known. As explained in, for example, \([20]\), the general solution to the radial Schrödinger equation for the harmonic oscillator at angular momentum \(l\) can be written in terms of the confluent hypergeometric functions \(M(a, b, z)\) and \(U(a, b, z)\) (the notation is as in \([21]\)). The correctly-normalised solution, subdominant in the sector \(S_0\), turns out to be
\[
y(x, E, l) = x^{l+1} e^{-x^2/2} U\left(\frac{1}{2}(l+\frac{3}{2}) - E, l+\frac{3}{2}, x^2\right)
\]
and has the asymptotic behaviour
\[
y(x, E, l) \sim x^{-1/2+E/2} [1 + O(x^{-2})] \exp(-\frac{1}{2}x^2) .
\]
(Notice the extra factor of $x^{E/2}$ here compared to the earlier formula (2.2), reflecting the fact that the semiclassical domain has been left.) Taking from (21) the analytic continuation formula

$$U(a, b, z e^{2\pi in}) = (1 - e^{-2\pi ibn}) \frac{\Gamma(1-b)}{\Gamma(1+a-b)} M(a, b, z) + e^{-2\pi ibn} U(a, b, z)$$

(3.3)

and the Wronskian

$$W[U(a, b, z), M(a, b, z)] = \frac{\Gamma(b)}{\Gamma(a)} z^{-b} e^z,$$

(3.4)

the function $\frac{1}{2i} W_{-1,1}(E, l)$ can be calculated explicitly. The result is

$$\left| \frac{1}{2i} W_{-1,1}(E, l) \right|_{M=1} = \frac{2\pi}{\Gamma \left( \frac{3}{4} + \frac{l}{2} + \frac{E}{4} \right) \Gamma \left( \frac{1}{4} - \frac{l}{2} + \frac{E}{4} \right)}.$$

(3.5)

At $M=1$ we have $\nu = 1/\sqrt{4\pi}$, $E = 4\pi\lambda^2$, and $l = 4p-1/2$, and so from (2.28) we find

$$T(\lambda, p) \big|_{\beta^2=1/2} = \frac{2\pi}{\Gamma \left( \frac{1}{4} + 2p + \pi\lambda^2 \right) \Gamma \left( \frac{1}{4} - 2p + \pi\lambda^2 \right)}.$$

(3.6)

This agrees with the result obtained by the authors of [8] for the vacuum expectation value of the operator $T$ at $\beta^2 = 1/2$, the so-called free-fermion point, for the particular value 0 of their non-universal renormalisation constant $C$.

One subtlety has been glossed over here. The modified asymptotic (3.2) changes the value of $W_{0,1}(E, l)$ from $2i$ to $2i e^{\pi i E/4}$, so that $\tilde{C}(E, l)$ is no longer equal to $-1$, but rather $-e^{-\pi i E/2}$. Hence (2.11) becomes

$$C(E, l) y_0(x, E, l) = y_{-1}(x, E, l) + e^{-\pi i E/2} y_1(x, E, l)$$

(3.7)

and the would-be T-Q relation (2.20) must be replaced by

$$e^{\pi i E/4} C(E, l) D^+(E, l) = \omega^{\pi i(l+1/2)+E/2} D^+(\omega^{-2E} E, l) + \omega^{\pi i(l+1/2)-E/2} D^+(\omega^2 E, l)$$

(3.8)

with $\omega = e^{\pi i/2}$. This precisely matches the ‘renormalised’ T-Q relation obtained in [8] for the free-fermion point, so long as $T(\lambda, p)$ is identified with $e^{\pi i E/4} C(E, l)$ at this point rather than $C(E, l)$. But when this $T$ is reexpressed in terms of Wronskians, the new prefactor is exactly cancelled by the modification to $W_{0,1}(E, l)$, with the result that the general relation (2.28) between $T$ and $W_{-1,1}$ survives unscathed. Considering the asymptotics discussed in appendix A, we expect similar modifications to the T-Q relation to occur at $M = 1/(2m-1)$, $\beta^2 = 1 - 1/2m$, for any positive integer $m$.

Now we can return to property (iii) from the last section, which states that for $l \in (-M/2, M/2)$, there are no zeroes of $C$ on the positive-real $E$ axis. We will only discuss $M \geq 1$. Then a WKB analysis as in [22] can be used to show that at large $|E|$ all zeroes must lie near the negative-real axis, and hence satisfy property (iii). (In fact, it seems clear that for $M \geq 1$ they lie on the negative axis, but this is harder to
prove.) There remains the possibility that some zeroes at smaller values of \( |E| \) lie on the positive-real axis, and to exclude this at least for \( M = 1 \) we use the fact that the zero positions at \( M = 1 \) are determined by (3.5), and are

\[
E = -4n - 2 \pm (2l + 1), \quad n = 0, 1, \ldots \quad (3.9)
\]

Thus for \( M = 1 \) and \(-3/2 < l < 1/2\) the zeroes are bounded away from the positive-real axis. By uniform continuity in compact domains, this will also be true for the low-lying zeroes if \( M \) is sufficiently close to 1. (Note that continuity alone does not exclude the possibility of a zero at large \( |E| \) violating (iii) for \( M \) arbitrarily near to 1, which is why we also had to invoke the WKB argument.) In fact, it appears that all of the other values of \( l \) is recovered:

the possibility of a zero at large \( |E| \) violating (iii) for \( M \) arbitrarily near to 1, which

Thus for \( M = 1 \) and \(-3/2 < l < 1/2\) the zeroes are bounded away from the positive-real axis. By uniform continuity in compact domains, this will also be true for the low-lying zeroes if \( M \) is sufficiently close to 1. (Note that continuity alone does not exclude the possibility of a zero at large \( |E| \) violating (iii) for \( M \) arbitrarily near to 1, which is why we also had to invoke the WKB argument.) In fact, it appears that all of the zeroes of \( C \) actually lie on the negative-real axis for \( M \geq 1 \) and \( l \in (-1 - M/2, M/2) \). At \( l = 0 \), this is the main conjecture of [22]; some numerical evidence for the claim at other values of \( l \) is given in [8]. However, the given range for \( l \) is certainly maximal:

from the formula

\[
C(0, l) = 2 \cos \frac{2l+1}{2M+2} \pi, \quad (3.10)
\]

a consequence of the T-Q relation (2.20), it follows that \( C(E, l) \) has a zero at \( E = 0 \) when \( l = -1 - M/2 \) and \( l = M/2 \).

Finally in this section we calculate \( D^-(0, l)|_M \). At \( M = 1 \), (3.1) can be used with the relation \( U(a, 2a, x^2) = \frac{1}{\sqrt{\pi}} x^{1-2a} e^{x^2/2} K_{a-1/2}(x^2/2) \), where \( K_a \) is a Bessel function of second kind, to see that

\[
y(x, 0, l)|_{M=1} = \frac{1}{\sqrt{\pi}} x^{1/2} K_{1/2 + 1/4}(x^2/2). \quad (3.11)
\]

For \( \nu > 0 \) and \( z \to 0 \), \( K_{\nu}(z) \sim \frac{1}{2} \Gamma(\nu) \left(\frac{z}{2}\right)^{-\nu} [21] \), so for \( l > -1/2 \) we have

\[
y(x, 0, l)|_{M=1} \sim \frac{1}{\sqrt{\pi}} 2^{l-1/2} \Gamma\left(\frac{l+1}{2}\right) x^{-l} \quad \text{as } x \to 0, \quad (3.12)
\]

and hence, using (3.1), \( D^-(0, l)|_{M=1} = \frac{1}{\sqrt{\pi}} 2^{l+3/2} \Gamma(1 + \frac{l}{2} + \frac{1}{4}) \).

The general-\( M \) case can be recovered by using the fact that at \( E = 0 \) a variable change relates a solution at arbitrary \( M \) and \( l \) to a solution at \( M = 1 \), but with \( l \) replaced by \( l^* = l'(l, M) \). Considering that the normalisation at large \( x \) must be in agreement with (2.2), the relevant transformation is

\[
y(x, 0, l)|_{M} = \left( \frac{2}{M+1} \right)^{1/4} x^{-(M+1)/4} y\left( \frac{2}{M+1} \right)^{1/2} x^{(M+1)/2}, 0, \frac{2l+1}{M+1} - \frac{1}{2} \right)|_{M=1} \quad (3.13)
\]

and repeating the steps already described for \( M = 1 \), the result quoted in the last section is recovered:

\[
D^-(0, l)|_M = (\alpha^{-})^{-1} = \frac{1}{\sqrt{\pi}} \Gamma(1 + \frac{2l+1}{2M+2}) (2M+2)^{\frac{2l+1}{2M+2} + \frac{1}{2}}. \quad (3.14)
\]

To find \( D^+(0, l) \), we continue from \( l \) to \(-1 - l\):
\[ D^+(0,l)\rvert_M = (\alpha^+)^{-1} = \frac{1}{\sqrt{\pi}} \Gamma(1 - \frac{2l+1}{2M+2}) (2M + 2)^{-\frac{2l+1}{2M+2} + \frac{1}{2}}. \] (3.15)

Notice that
\[ D^-(0,l)\rvert_M D^+(0,l)\rvert_M = (\alpha - \alpha^+)^{-1} = (2l+1)/\sin \frac{2l+1}{2M+2} \pi. \] (3.16)

When \( l=0 \), \( D^- \) and \( D^+ \) correspond to half-line problems with Dirichlet and Neumann boundary conditions at the origin respectively, and therefore yield the odd and even spectral determinants for the whole-line problem (1.1). (This is why the ± signs were swapped in our definitions between \( D^\pm \) and \( A_\mp \).) Their product is the full spectral determinant, and the result (3.16) at \( l=0 \) matches the formula for \( D_M(0) \) given in [23,24].

### 4 Fusion relations and monodromy

The T-Q relation is not the only functional equation which arises in the context of the T- and Q-operators (see, for example, [8,11,25–28]). Further relations are conveniently expressed using the ‘fused’ T-operators \( T_j \), which can be built up by a process known as fusion from the basic operator \( T \). Introducing a half-integer valued index \( j = 0, 1/2, 1, \ldots \), the first set of fusion relations, sometimes called a T-system, reads as follows:

\[ T_j(q^{-1/2}\lambda)T_j(q^{1/2}\lambda) = 1 + T_{j+1/2}(\lambda)T_{j-1/2}(\lambda) \] (4.1)

where \( T_0(\lambda) \equiv 1 \) and \( T_{1/2}(\lambda) \equiv T(\lambda) \). The fused T’s can also be obtained from

\[ T(\lambda)T_j(q^{j+1/2}\lambda) = T_{j-1/2}(q^{j+1}\lambda) + T_{j+1/2}(q^j\lambda) \] (4.2)

or

\[ T(\lambda)T_j(q^{-j-1/2}\lambda) = T_{j-1/2}(q^{-j-1}\lambda) + T_{j+1/2}(q^{-j}\lambda). \] (4.3)

The vacuum states \(|p\rangle \) are also eigenstates of these fused T-operators. In this section we shall show that the vacuum expectation values \( T_j(\lambda) \equiv \langle p|T_j(\lambda)|p\rangle \) arise naturally in the context of the Schrödinger equation (2.1), leading to a reinterpretation of the fusion relations and their truncations in terms of the behaviour of solutions to this equation under analytic continuation.

As remarked earlier, each pair of functions \( \{y_m, y_{m+1}\} \) provides a basis for the space of solutions to (2.1). So far, we have only examined the expansion of \( y_{k-1} \) in the basis \( \{y_k, y_{k+1}\} \), but it is natural to ask about other possibilities. To this end, we extend the definition (2.3) of \( C_k \) and \( \tilde{C}_k \) by setting

\[ y_{k-1} = C_k^{(m)} y_{k+m-1} + \tilde{C}_k^{(m)} y_{k+m} \] (4.4)

(so that \( C_k^{(1)} = C_k \) and \( \tilde{C}_k^{(1)} = \tilde{C}_k = -1 \)). The change from the \( \{y_{k+m-1}, y_{k+m}\} \) basis to the \( \{y_{k-1}, y_k\} \) basis is then effected by a \( 2 \times 2 \) matrix \( C_k^{(m)} \) as

\[
\begin{pmatrix}
  y_{k-1} \\
  y_k
\end{pmatrix} = C_k^{(m)} \begin{pmatrix}
  y_{k+m-1} \\
  y_{k+m}
\end{pmatrix}, \quad C_k^{(m)} = \begin{pmatrix}
  C_k^{(m)} & \tilde{C}_k^{(m)} \\
  C_k^{(m-1)} & \tilde{C}_k^{(m-1)}
\end{pmatrix}.
\] (4.5)
This matrix depends on $E$ and $l$, but not $x$. The following properties are immediate:

$$C_k^{(m)}(E, l) = C_{k-1}^{(m)}(\omega^2 E, l), \quad (4.6)$$

$$C_k^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_k^{(1)} = \begin{pmatrix} C_k & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.7)$$

Further relations reflect the fact that the change from the basis $\{y_{k+m+n-1}, y_{k+m+n}\}$ to $\{y_{k+m-1}, y_{k+m}\}$, followed by the change from $\{y_{k+m-1}, y_{k+m}\}$ to $\{y_k, y_k\}$, has the same effect as accomplishing the overall change in one go:

$$C_k^{(m)} C_{k+m}^{(n)} = C_k^{(m+n)}. \quad (4.8)$$

(These express the consistency of the analytic continuations, and can be thought of as monodromy relations.) Consider first the case $m = 1$. We have

$$\begin{pmatrix} C_k & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} C_{k+1}^{(n)} \\ C_{k+2}^{(n-1)} \end{pmatrix} = \begin{pmatrix} C_k^{(n+1)} \\ C_{k+1}^{(n)} \end{pmatrix}, \quad (4.9)$$

which gives two non-trivial relations:

$$C_k C_{k+1}^{(n)} - C_{k+2}^{(n-1)} = C_k^{(n+1)} \quad (4.10)$$

and

$$C_k \tilde{C}_{k+1}^{(n)} - \tilde{C}_{k+2}^{(n-1)} = \tilde{C}_k^{(n+1)}. \quad (4.11)$$

In addition, we have the initial conditions

$$C_k^{(0)} = 1, \quad C_k^{(1)} = C_k \quad (4.12)$$

$$\tilde{C}_k^{(0)} = 0, \quad \tilde{C}_k^{(1)} = -1 \quad (4.13)$$

The $n = 1$ case of (4.11) shows that $\tilde{C}_k^{(2)} = -C_k = -C_k^{(1)}$; and then the more general equality

$$\tilde{C}_k^{(n)} = -C_k^{(n-1)} \quad (4.14)$$

follows on comparing (4.11) with (4.10). If we now set

$$C^{(n)}(E) = C_0^{(n)}(\omega^{-n+1}E), \quad (4.15)$$

then (4.10) is equivalent to

$$C(E)C^{(n)}(\omega^{n+1}E) = C^{(n-1)}(\omega^{n+2}E) + C^{(n+1)}(\omega^n E), \quad (4.16)$$

and this matches the fusion relation (4.2). Since $C^{(0)}(E) = 1 = T_0(E)$ and, from the last section, $C^{(1)}(E) = C(E) = T_{1/2}(\nu E^{1/2})$, this establishes the basic equality

$$C^{(n)}(E) = T_{n/2}(\nu E^{1/2}). \quad (4.17)$$
It is easy to check that the fusion relation (4.3) emerges in a similar manner from (4.8) at \( n = 1 \). To recover the T-system (4.1), one more piece of information is needed. Taking Wronskians in (4.4) yields

\[
C_k^{(m)} = \frac{1}{2i} W_{k-1,k+m} , \quad \tilde{C}_k^{(m)} = -\frac{1}{2i} W_{k-1,k+m-1} .
\]  

(4.18)

An immediate consequence is the recovery of (4.14), but we also obtain

\[
C_k^{(m)} = -C_{k+m+1}^{(-m-2)} .
\]  

(4.19)

Using this result, the \( n = -m \) case of (4.8), namely \( C_k^{(m)} C_{k+m}^{(-m)} = 1 \), implies that

\[
C_k^{(m-1)} (\omega^{-1} E) C_k^{(m-1)} (\omega E) - C_k^{(m)} (E) C_k^{(m-2)} (E) = 1 .
\]  

(4.20)

Given the identification (4.21), this is the T-system (4.1), evaluated on the vacuum state \(|p\rangle\). Finally, the formula (4.18) allows the function \( T_{n/2}^{(n)} \) to be expressed alternatively in terms of a Wronskian:

\[
T_{n/2}^{(n)} (\nu E^{1/2}) = C^{(n)} (E) = \frac{1}{2i} W_{-1,n} (\omega^{-n+1} E) .
\]  

(4.21)

This will be relevant in §7 below.

The monodromy relations used so far have all been ‘local’, in that they can be built up from continuations of the functions \( y_k \) from one sector \( S_n \) to its neighbours \( S_{n \pm 1} \). But there is also the possibility to continue all the way round the origin a number of times. If the cover of the punctured \( x \)-plane on which the \( y_k \) live closes in a suitable sense, this leads to new relations, and these turn out to correspond to the truncations of fusion hierarchies in the integrable quantum field theory.

As an illustration, we will discuss the simplest class of examples, which arise when \( 2M \) is an integer and \( l(l+1) \) is zero. Then all solutions to (2.1), and in particular the \( y_k(x,E,l) \), are single valued on the once-punctured \( x \)-plane, and the sector \( S_{n+2M+2} \) coincides with the sector \( S_n \). Both \( y_n \) and \( y_{n+2M+2} \) are subdominant in this sector, and so they must be proportional to each other. To find the precise relationship, consider their behaviour as \( |x| \to \infty \) in \( S_n \), setting \( x = \rho e^{n\pi i/(M+1)} \) and letting \( \rho \to \infty \). From (2.2) and (2.7),

\[
y_n \sim \omega^{n/2} \rho^{-M/2} \exp\left(-\frac{1}{M+1} \rho^{M+1}\right) , \quad y_{n+2M+2} \sim -\omega^{n/2} \rho^{-M/2} \exp\left(-\frac{1}{M+1} \rho^{M+1}\right)
\]  

(4.22)

and so \( y_{n+2M+2}(x,E,l) = -y_n(x,E,l) \). From (4.21) we immediately deduce

\[
C^{(2M)} (E) = 1 \quad , \quad C^{(2M+1)} (E) = 0
\]  

(4.23)

and hence the relation (4.20) can be rewritten as

\[
C^{(m)} (\omega^{-1} E) C^{(m)} (\omega E) = 1 + \prod_{n=1}^{2M-1} \left(C^{(n)} (E)\right)^{l_{nm}}
\]  

(4.24)
where \( l_{am} \) is the incidence matrix of the \( A_{2M-1} \) Dynkin diagram. This is the simplest example of truncation, and ultimately \([23]\) leads, via the \( Y \) functions discussed briefly in \( \S \) below, to alternative sets of integral equations of the ‘TBA’ type \([29]\). For integer values of \( M \), these equations were applied in \([1]\) to the computation of the energy levels of anharmonic oscillators with analytic potentials \( x^{2M} \).

At arbitrary rational values of \( M+1 \), a similar collapse of the set of solutions \( \{y_k\}_{k=-\infty}^{\infty} \) occurs, but this time on some finite cover of the punctured \( x \)-plane. Again, a truncation of the fusion hierarchy results. For nonzero values of \( l(l+1) \) the situation is a little more subtle, as there is a multivaluedness in the solutions induced by the singularity at \( x = 0 \). To handle this behaviour the alternative basis \( \{\psi^-, \psi^+\} \) is more appropriate. We leave further discussion of this point to future work, but we expect that all of the more general truncations of the fusion hierarchy \([4,20]\) will eventually find a geometrical interpretation in terms of the behaviours of solutions to the basic differential equation \([2.1]\).

5 The (fused) quantum Wronskians

The next task is to relate the fused Stokes multipliers \( C^{(m)} \) to the functions \( D^\mp \) discussed in \([2]\). Recall that the Wronskian is bilinear in the space of differentiable functions, so that given four arbitrary functions \( f(x), g(x), h(x) \) and \( l(x) \), and arbitrary constants \( \alpha, \beta, \gamma, \delta \), we have

\[
W[\alpha f + \beta g, \gamma h + \delta l] = \alpha \gamma W[f, h] + \alpha \delta W[f, l] + \beta \gamma W[g, h] + \beta \delta W[g, l] .
\]  

(5.1)

For almost all \( l \) (exceptions will be discussed below), the functions \( \{\psi^-, \psi^+\} \) introduced in \([2.14] \) and \([2.15] \) provide an alternative basis for the space of solutions to the differential equation \([2.1] \). In particular, using the results \( W[y, \psi^\pm] = D^\mp \) and \( W[\psi^-, \psi^+] = 2l+1 \), we have

\[
(2l+1)y(x, E, l) = D^{-}(E, l)\psi^-(x, E, l) - D^{+}(E, l)\psi^+(x, E, l) ,
\]  

(5.2)

More generally, the shifted solutions defined by \([2.7] \) and \([2.16] \) satisfy

\[
(2l+1)y_k(x, E, l) = D^{-}(\omega^{2k}E, l)\psi^-(x, E, l) - D^{+}(\omega^{2k}E, l)\psi^+(x, E, l) .
\]  

(5.3)

Taking the Wronskian \([5.3] \) at \( k = -1 \) with the same equation at \( k = n \), shifting \( E \) to \( \omega^{1-n}E \) and then using the formula \([4.21] \) for \( C^{(n)}(E) \), property \( (5.1) \) and the results

\[
W[\psi^+_k, \psi^+_p] = W[\psi^-_k, \psi^-_p] = 0 ,
\]

\[
W[\psi^-_p, \psi^+_k] = (2l+1)\omega^{(k-p)(l+1/2)}
\]  

(5.4)

(valid at arbitrary ‘shifts’ \( p \) and \( k \)), we find

\[
(4l+2)i C^{(n)}(E) = \omega^{(n+1)(l+1/2)}D^{-}(\omega^{n+1}E, l)D^{+}(\omega^{-n-1}E, l) - \omega^{-(n+1)(l+1/2)}D^{-}(\omega^{-n-1}E, l)D^{+}(\omega^{n+1}E, l) .
\]  

(5.5)
In the context of integrable quantum field theory, a corresponding set of relations was given in [3]:

\[
2i \sin(2\pi p)T_j(\lambda) = q^{(4j+2)p/\beta^2}A_+(q^{j+1/2}\lambda, p)A_-(q^{-j-1/2}\lambda, p) - q^{-(4j+2)p/\beta^2}A_+(q^{-j-1/2}\lambda, p)A_-(q^{j+1/2}\lambda, p)
\]  

(5.6)

where \(j = 0, 1/2, 1, \ldots\). With the identifications (2.27) and (2.28), and using the result (3.16), the two sets of relations agree if, as before, \(q = \omega\) and \(2p/\beta^2 = l+1/2\). At \(j = 0\), (5.6) is the ‘quantum Wronskian’ relation of [3], while the \(n = 0\) case of (5.3) was first found (for \(M\) an integer and \(l = 0\)) in [3]. The match between these two was a key ingredient in [1, 10].

The ‘T-system conjecture’ of [1], proved in [6], is a simple corollary of this result. If \(l = 0\) and \(M\) is an integer, then (5.3) at \(n = M\) becomes

\[
C^{(M)}(E) = D^+(-E)D^-(E) = D_M(-E)
\]  

(5.7)

the second equality following because at \(l = 0\), \(D^+\) and \(D^-\) are the even and odd spectral subdeterminants respectively for the full-line problem (1.4). Since \(C^{(M)}(E) = T_{M/2}(\nu E^{1/2})\), this establishes the conjectured relation between \(D_M\) and the vacuum expectation value \(T_{M/2}\). (Note though the small differences in notation from [1]: as well as the negation of \(E\) already mentioned in the introduction, a half-integer \(j\) has been used to index the \(T\) operators in this paper, in line with the conventions of [8, 27, 28], while in [1] the index was integer-valued. Thus in [1] the correspondence was with \(T_M\) rather than \(T_{M/2}\).)

Next we return to the fact that the pair of functions \(\{\psi^-, \psi^+\}\) does not always furnish a basis for the space of solutions to the differential equation (2.3). The most obvious counterexample to such a claim is the point \(l = -1/2\), at which \(\psi^- = \psi^+\). More generally, since \(\psi^-\) is initially defined by analytic continuation in \(l\), there may be points at which poles are encountered. If these poles are removed by multiplying by a regularising factor, the resulting solution \(\tilde{\psi}^-\) may fail to be independent of \(\psi^+\) at those values of \(l\) where previously there had been divergences. This is related to the remark in [3] that at certain values of \(p\) the functions \(A_+(\lambda, p)\) and \(A_-(\lambda, p)\) may coincide. In the context of the radial Schrödinger equation, a discussion of the issue can be found in, for example, chapter 4 of [14]. The most direct way to locate the ‘problem’ values of \(l\) is probably the iterative construction of \(\psi^-\) and \(\psi^+\) given in [30]. For the potential \(x^{2M}\), for which the function \(U(x)\) of [20] is equal to \(x^{2M} - E\), poles \(\psi^+(x, E, l)\) occur at \(l + 1/2 = -m_1 - (M+1)m_2\), with \(m_1\) and \(m_2\) non-negative integers. Hence \(\{\psi^-, \psi^+\}\) fails as a basis at the points

\[
l + 1/2 = \pm (m_1 + (M+1)m_2), \quad m_1, m_2 \geq 0.
\]  

(5.8)

For integer values of \(2M\), this is just a standard phenomenon in the Frobenius method, which predicts that one of the pair \(\{\psi^-, \psi^+\}\) may have a logarithmic component whenever the two solutions to the indicial equation \(\alpha(\alpha-1) = l(l+1)\) differ by an integer, or an even integer when \(2M\) is even.
Rather than giving a complete treatment, we will focus here on the \( l = -1/2 \) case. This corresponds to the ground state of the untwisted sine-Gordon model, and so is particularly interesting from the field-theoretic point of view. The emergence of logarithmic corrections can be understood by taking an appropriate limit \( l \to -1/2 \). Using (5.2), valid away from the points (5.8), set \( l = -1/2 + \varepsilon \) with \( \varepsilon \) tending to zero. Near \( x = 0 \) we have

\[
y(x, E, -\frac{1}{2} + \varepsilon) \sim D^{-}(E, -\frac{1}{2})' x^{1/2} \frac{x^{\varepsilon} + x^{-\varepsilon}}{2} - D^{-}(E, -\frac{1}{2}) x^{1/2} \frac{x^{\varepsilon} - x^{-\varepsilon}}{2\varepsilon} \tag{5.9}
\]

where the primes denote differentiation with respect to \( l \) and we used the facts that \( D^{-}(E, l) \) and \( D^{-}(E, l)' \) are nonsingular at \( l = -1/2 \) to expand \( D^{-}(E, -1/2 + \varepsilon) \sim D^{-}(E, -1/2) + D^{-}(E, -1/2)' \varepsilon \), Expanding \( x^{\pm \varepsilon} = \exp(\pm \varepsilon \log x) \) to first order in \( \varepsilon \), (5.9) becomes

\[
y(x, E, -\frac{1}{2}) \sim D^{-}(E, -\frac{1}{2})' x^{1/2} - D^{-}(E, -\frac{1}{2}) x^{1/2} \log x. \tag{5.10}
\]

We see that \( y(x, E, -1/2) \) has an \( x^{1/2} \log x \) component at small \( x \), as expected from the Frobenius method. The basis \( \{ \psi^-, \psi^+ \} \) should therefore be replaced at \( l = -1/2 \), and a suitable choice is the pair \( \{ \chi^-, \chi^+ \} \):

\[
\begin{align*}
\chi^+(x, E) &\sim x^{1/2} + O(x^{5/2}) , \\
\chi^-(x, E) &\sim x^{1/2} \log x + O(x^{5/2}).
\end{align*} \tag{5.11}
\]

The ‘Jost functions’ (2.19) become

\[
\tilde{D}^\mp(E) = W[y(x, E, -\frac{1}{2}), \chi^\pm(x, E)], \tag{5.12}
\]

so that \( \tilde{D}^-(E) = D^-(E, -\frac{1}{2}) \), and \( \tilde{D}^+(E) = D^-(E, -\frac{1}{2})' \). It is interesting to see the effect that these changes have on the basic functional relations. We set \( \chi^\pm_k(x, E) = \omega^{k/2} \chi^\pm(\omega^{-k}x, \omega^{2k}E) \) and use the Wronskians

\[
\begin{align*}
W[\chi^+_p, \chi^-_q] &= 0 , \\
W[\chi^-_p, \chi^-_q] &= (q-p) \log \omega , \\
W[\chi^+_p, \chi^-_q] &= 1 ,
\end{align*} \tag{5.13}
\]

to find, with \( \tilde{D}^\mp_k \equiv \tilde{D}^\mp(\omega^{2k}E) \),

\[
W[y_p, y_q] = (q-p) \tilde{D}^-_p \tilde{D}^-_q \log \omega + (\tilde{D}^-_p \tilde{D}^+_q - \tilde{D}^-_q \tilde{D}^+_p) \tag{5.14}
\]

and feeding this into the formula (4.21) for \( C^{(n)} \) yields

\[
C^{(n)}(E) = \frac{\pi(n+1)}{2M+2} \tilde{D}^-(\omega^{n+1}E) \tilde{D}^-(\omega^{-n-1}E) \tag{5.15}
\]

\[
+ \frac{1}{2i} \left( \tilde{D}^-(\omega^{-n-1}E) \tilde{D}^+(\omega^{n+1}E) - \tilde{D}^-(\omega^{n+1}E) \tilde{D}^+(\omega^{-n-1}E) \right).
\]

17
Note also, from (3.14) and (3.15), that
\[
\tilde{D}^-(0) = \sqrt{\frac{2M + 2}{\pi}}, \quad \tilde{D}^+(0) = \frac{2}{\sqrt{\pi}} (2 + 2M)^{-1/2} (\log(2 + 2M) - \gamma_E) \tag{5.16}
\]
where \( \gamma_E \equiv -\Gamma'(1) = 0.57721 \ldots \) is the Euler-Mascheroni constant. Finally from (2.11), (5.10) and (5.11) we obtain
\[
C(\mathcal{E}) \tilde{D}^-(\mathcal{E}) = \tilde{D}^- (\omega - 2\mathcal{E}) + \tilde{D}^- (\omega^2 \mathcal{E})
\]
\[
C(\mathcal{E}) \tilde{D}^+(\mathcal{E}) = \tilde{D}^+ (\omega - 2\mathcal{E}) + \tilde{D}^+ (\omega^2 \mathcal{E}) + \left( \tilde{D}^- (\omega - 2\mathcal{E}) - \tilde{D}^- (\omega^2 \mathcal{E}) \right) \log \omega. \tag{5.17}
\]
Thus at \( l = -1/2 \) one of the two T-Q relations has to be changed. Similarly, comparing (5.15) with the fused quantum Wronskian (5.5) shows that the relationship between \( C(n) \) and \( D_{\mathcal{E},l} \) undergoes a nontrivial modification. The fact that for the ground state energy of the untwisted sine-Gordon model things need to be slightly modified was already observed, from a completely different angle, in [31]. There, the general solution of the sine-Gordon Y-system was described in terms of a single quasi-periodic function \( h(\mathcal{E}) \), satisfying \( h(\mathcal{E}^2 + 2M + 2) = p + q h(\mathcal{E}) \), where \( p \) and \( q \) can, for the purposes of [31] and the example investigated here, be taken to be constants. In order to have the correct UV-limit for the untwisted sine-Gordon model (corresponding to \( l = -1/2 \)), a non-vanishing \( p \) was required, leading to a \( \log(\mathcal{E}) \) component in \( h(\mathcal{E}) \) (see (6.14) below). In a slightly different language, this turns out to be what we have derived here. A match with the results of [31] is the topic of the next section, where, as simple byproduct of the mapping, a curious property of the spectral determinants is pointed out.

## 6 Y-systems and dilogarithm identities

As mentioned in [4], there is a second set of functional relations, the so-called Y-system, closely related to the T-system discussed in the previous sections. The relation between these two systems is
\[
Y_n(\mathcal{E}) = C^{n+1}(\mathcal{E}) C^{n-1}(\mathcal{E}) \tag{6.1}
\]
and the Y’s fulfil the relation
\[
Y_n(\omega \mathcal{E}) Y_n(\omega^{-1} \mathcal{E}) = (1 + Y_{n+1}(\mathcal{E}))(1 + Y_{n-1}(\mathcal{E})) \tag{6.2}
\]
For \( M \) integer or half-integer and \( l = 0 \), this system truncates \( Y_0(\mathcal{E}) = Y_{2M}(\mathcal{E}) = 0 \), and it coincides with the \( A_{2M-1} \)-related Y-system discussed in [4].

On the other hand, the Y-functions are related to the solutions of TBA equations. In this framework they encode finite-size effects in integrable quantum field theories, and, through the consideration of ultraviolet limits, lead to certain remarkable sum rules for the Rogers dilogarithm function involving the stationary \( (\mathcal{E} = 0) \) solutions of the system (6.2). For \( M \) integer and \( l = 0 \), for example, the relevant sum rule is
\[
\frac{6}{\pi^2} \sum_{n=1}^{2M-1} L \left( \frac{1}{1 + Y_n(0)} \right) = \frac{2M-1}{M+1} = c_{UV} \tag{6.3}
\]

18
where $c_{UV}$ is the central charge of the $Z_{2M}$ parafermionic conformal field theory, $L(x)$ is the Rogers dilogarithm

$$L(x) = -\frac{1}{2} \int_0^x dy \left[ \log\left(\frac{y}{1-y}\right) + \log\left(\frac{1-y}{y}\right) \right],$$

and the values of the constants $Y_n(0)$ involved in (6.3) are

$$Y_n(0) = \frac{\sin\left(\pi \frac{n}{2M+2}\right) \sin\left(\pi \frac{n+2}{2M+2}\right)}{\sin^2\left(\frac{\pi}{2M+2}\right)}.$$ (6.5)

With some additional complications, sum rules similar to (6.3) can be written for any rational $M$ and arbitrary $l$. In [33] a generalisation of (6.3) involving the $E$-dependent $Y$-functions was proposed. For an arbitrary solution $Y_n(E)$ to (6.2) (but again with $M$ an integer and $l = 0$) the result is

$$\frac{6}{\pi^2} \sum_{n=1}^{2M-1} \sum_{k=0}^{2M+1} \frac{1}{Y_n(\lambda^k E)} = 2(2M-1).$$ (6.6)

Dilogarithms also appear in certain volume calculations in three-dimensional manifolds (see for example [34]), and related to this idea is the fact [35] that the general solution to (6.2) can be expressed using cross-ratios

$$(a, b, c, d) = \frac{(a-c)(b-d)}{(a-d)(b-c)}.$$ (6.7)

Surprisingly, the points involved in the cross-ratio can be expressed in terms of a single quasi-periodic function of $E$: if $h(E)$ satisfies

$$h(E\omega^{2M+2}) \equiv p + qh(E)$$ (6.8)

then

$$Y_n(E) = -(h(\omega^n E), h(\omega^{-n-2} E), h(\omega^{-n} E), h(\omega^{n+2} E))$$ (6.9)

solves (6.2).

Now we would like to show that the function $h(E)$ is a natural object in the context of the Schrödinger equation. Since $C^{(0)}(E) = 1$ for all $E$, we can rewrite $C^{(n+1)}(E)$ as $C^{(n+1)}(E)/C^{(0)}(\omega^{n+1} E)$ and, for generic values of $l$, expand out numerator and denominator using (5.5) to find

$$C^{(n+1)}(E) = \omega^{(l+1/2)(n+1)} \frac{D^+(\omega^{-n-2} E, l)}{D^+(\omega^n E, l)} \frac{(k(\omega^{n+2} E) - k(\omega^{-n-2} E))}{(k(\omega^{n+2} E) - k(\omega^n E))}$$ (6.10)

where

$$k(E) = E^{l+1/2} D^-(E, l)/D^+(E, l).$$ (6.11)
Similarly,
\[ C^{(n-1)}(E) = \omega^{-(l+1/2)(n+1)} \frac{D^+(\omega^n E, l)}{D^-(\omega^{-n-2} E, l)} \frac{(k(\omega^n E) - k(\omega^{-n} E))}{(k(\omega^{-n} E) - k(\omega^{-n-2} E))} \]  
(6.12)
and so, from (6.1),
\[ Y_n(E) = -\frac{k(\omega^{n+2} E) - k(\omega^{-n-2} E)}{k(\omega^{n+2} E) - k(\omega^n E)} \frac{k(\omega^{-n} E) - k(\omega^n E)}{k(\omega^{-n} E) - k(\omega^{-n-2} E)}. \]  
(6.13)

This was for generic values of \( l \): the story changes at those points where \( \{\psi^-, \psi^+\} \) fails to be a basis. The first example is \( l = -1/2 \), where (5.5) is replaced by (5.15). It turns out that the final result (6.13) continues to hold, so long as the definition (6.11) is swapped for
\[ k(E) = \log E + 2 \frac{D^+(E)}{D^-(E)}. \]  
(6.14)
Comparing (6.13) with (6.9), the function \( k(E) \) as defined in (6.11) gives a realisation of \( k(E) \) for \( p = 0 \) and \( q = e^{2\pi i(l+1/2)} \), while the definition (6.14) coincides with the case \( q = 1, p = 2\pi i \). Note also that when \( l = 0 \), \( k(E) \) is essentially the (reciprocal of) the alternating, or skew, spectral determinant discussed in [23, 24]. It is interesting that the result (6.6) (and its generalisations) appears to furnish a novel form of sum rule for the spectral determinants of Schrödinger equations.

### 7 The T operators, spectral determinants, and a generalisation of a problem considered by Bender and Boettcher

The result
\[ T_{n/2}(\nu E^{1/2}) = C^{(n)}(E) = \frac{1}{2i} W_{-1,n}(\omega^{-n+1} E) \]  
(7.1)
from §4 implies that the vacuum expectation values of the T-operators also have an interpretation as spectral determinants. The right hand side of (7.1) vanishes if and only if \( E \) is such that the functions \( y_{-1} \) and \( y_{l} \) are linearly dependent, which in turn is true if and only if there is a nontrivial solution to (2.1) which decays to zero as \( x \) tends to infinity in the sectors \( S_{-1} \) and \( S_{n} \). This is an eigenvalue problem, and the argument just given shows that the zeroes of \( C^{(n)} \) coincide with the zeroes of its spectral determinant. From (5.3) and (2.21), the order of \( C^{(n)} \) is less than 1 for \( M > 1 \), so the discussion around (2.29) for \( D^- \) applies equally to \( C^{(n)} \), and shows that this function is actually equal to the spectral determinant, with a normalisation which again turns out to coincide with the natural one for \( 2M \) an integer and \( l \) equal to 0 or 1. (Incidentally, for \( M \) integer and \( l \) zero this gives us another easy proof of the T-system conjecture, obtained more indirectly via the subdeterminants \( D^\pm \) in [8] above.) General eigenproblems of this sort, where the boundary conditions are specified as \( |x| \to \infty \) in two sectors in the complex \( x \) plane, are discussed in chapter 6 of [12] and also in [22, 36]. They are the other natural
set of spectral problems associated with (2.1), and it is pleasing that they correspond so neatly with the fused T-operators.

As an application, we consider and generalise a problem treated by Bender and Boettcher in [22]. These authors discussed the spectrum of the Hamiltonian $p^2 - (ix)^N$ with $N$ real, corresponding to the eigenvalue equation

$$-\psi''(x) - (ix)^N \psi(x) = E\psi(x). \quad (7.2)$$

For quantised energy levels the boundary conditions must be chosen appropriately. The standard requirement is for $\psi(x)$ to tend to zero as $|x| \to \infty$ on the real axis, but for arbitrary $N$ the problem needs to be continued into the complex $x$-plane. The real $x$-axis is replaced by some contour, and the boundary conditions are imposed at the two ends of this contour. The regions where $\psi(x)$ can vanish exponentially as $|x| \to \infty$ are wedges, bounded by what Bender and Boettcher referred to as Stokes lines, with the vanishing being most rapid at the centers of the wedges, called by them anti-Stokes lines. (Beware that many authors use exactly the opposite terminology – see, for example, [37].) There are thus a variety of eigenvalue problems associated with (7.2), depending on the choice of asymptotic directions for the contour relative to the wedges. The problem studied in [22] was the analytic continuation of the usual harmonic oscillator, so the wedges were picked to allow the quantisation contour to run along the real axis when $N = 2$. This selects the wedges centred on the directions

$$\theta_{\text{left}} = -\pi + \frac{N - 2}{N + 2} \frac{\pi}{2} \quad \text{and} \quad \theta_{\text{right}} = -\frac{N - 2}{N + 2} \frac{\pi}{2}, \quad (7.3)$$

each of opening angle $\Delta = 2\pi / (N+2)$. Bender and Boettcher studied the problem both numerically and analytically, and found strong evidence for an entirely real spectrum for $N \geq 2$, with a transition at $N = 2$ below which infinitely-many eigenvalues become complex. This they associated with a spontaneous breaking of $\mathcal{PT}$ symmetry in the quantum-mechanical problem.

If we set $\hat{\psi}(x) = \psi(x/i)$, (7.2) becomes

$$-\hat{\psi}''(x) + x^N \hat{\psi}(x) = -E\hat{\psi}(x). \quad (7.4)$$

and matches our original Schrödinger equation (2.1) with $M = N/2$, $l = 0$, and $E$ replaced by $-E$. Furthermore, it is easily checked that the wedges defined by (7.3) become the sectors $\mathcal{S}_{-1}$ and $\mathcal{S}_1$. Referring back to the discussion following (7.1), this means that the eigenvalues of the problem posed by (7.2) and (7.3) occur at the negated zeroes of $C^{(1)}(E)$, and are thus encoded in the pattern of zeroes of the vacuum expectation value of the fundamental quantum transfer matrix $T(\lambda) \equiv T_{1/2}(\lambda)$.

In turn, these can be found via the non-linear integral equation (2.32). The results are displayed in figure 1, and agree perfectly with the results of [22], which were obtained by a direct treatment of the differential equation in the complex plane. Note that the transition at $N = 2$ corresponds to the point at which the associated sine-Gordon model moves from the attractive to the repulsive regime, and that the match with the results
of Bender and Boettcher continues to hold even after this point has been passed, and
the semiclassical domain $N > 2$ has been left.

From the point of view of equation (2.32), the restriction to $l = 0$ is unnecessary.
Once it is relaxed, we are led to the following generalisation of the Bender-Boettcher
problem:

$$- \psi''(x) - (ix)^N \psi(x) + l(l+1)x^{-2} = E \psi(x).$$

(7.5)

This seemingly-innocent modification to the Hamiltonian turns out to have a remark-
able effect on its spectrum. In figure 2 we show some results from an initial numerical
investigation. A few features merit immediate comment. At $N = 2$, the exact eigenval-
ues can always be found by negating formula (3.9). The behaviour for $N < 2$ depends
significantly on the sign of $l$. The situation at $l = -0.025$ is shown in figure 2a. It
might appear similar to that at $l = 0$, but the connectivity of the states on the plots is
exactly reversed: the ground state is now joined to the first excited state, the second
to the third, and so on. As $l$ increases, the level joined to the ground state flicks up
through the rest of the spectrum, leaving a restored connectivity pattern in its wake.
The mechanism should be clear enough from figures 2b to 2d. By the time $l$ reaches
zero the earlier situation has been recovered, with the ground state no longer connected
to one of the excited states, but rather diverging to $+\infty$ along the line $N = 1$. Then for
$l > 0$ the ground state eigenvalue ventures for the first time into the domain $N \in (0, 1]$.  

Figure 1: Eigenvalues of the Hamiltonian $p^2 - (ix)^N$ plotted as a function of $N$, found via the
solution of the nonlinear integral equation (2.32) at $l = 0$.  

22
Figure 2: Eigenvalues of the Hamiltonian $p^2 - (ix)^N + l(l+1)x^{-2}$ found via the nonlinear integral equation (2.32), plotted as a function of $N$ at various nonzero values of $l$. 

2a) $l = -0.025$  
2b) $l = -0.015$  
2c) $l = -0.0015$  
2d) $l = -0.001$  
2e) $l = 0.001$  
2f) $l = 0.05$
(To make sense of the eigenvalue problem in this regime requires its consideration on a cover of the once-punctured complex plane, with the sectors $S_{-1}$ and $S_1$ overlapping on different sheets.) Figures 2e and 2f show this behaviour. Observe from figure 2f that the ground state eigenvalue is zero at $l = 0.05$, $N = 0.2$. More generally, (3.10) implies that the problem has a zero eigenvalue when $N = 4l$. This means that the steeply-climbing line on figure 2e should also return to zero, at $N = 0.004$. While our program currently breaks down at such small values of $N$, plots made at intermediate values of $l$ (between 0.001 and 0.05) offer clear support for this scenario.

For the values of $l$ pictured, the spectrum is entirely real for $N \geq 2$, thus generalising the reality property observed by Bender and Boettcher for $l = 0$.

In a subsequent paper [36] further eigenvalue problems were discussed, based on Hamiltonians of the form $p^2 + x^{2K}(ix)^\varepsilon$, with the two ‘quantisation wedges’ chosen so as to be centred on the positive and negative real axes when $\varepsilon = 0$. In these conventions, the earlier problem corresponds to $K = 1$, $\varepsilon = N - 2$. It is straightforwardly checked that, just as $T_{1/2}$ or $C^{(1)}$ encodes the spectrum of the problem for $K = 1$, so the functions $T_{K/2}$ or $C^{(K)}$ encode the spectra of the more general problems. This means that these problems, treated separately in [36], are in fact intimately linked together, since their spectral determinants participate in the fusion hierarchy, or T-system, discussed in §4 above. On the other hand, the numerical and analytical work in [36] immediately gives us a rather detailed picture of the behaviour of the zeroes of the general fused T operators. Worth noting in this regard is the fact that the first transition to complex eigenvalues for the Schrödinger problem, that is to complex zero positions for the fused T operator, always occurs at $\varepsilon = 0$. In the earlier notation of this paper, this point corresponds to $2M = N = 2K$, and for $K > 1$, it lies well inside the semiclassical domain $M > 1$, $\beta^2 = 1/(M+1) < 1/2$.

As this paper was being finished, a further article by Bender et al appeared [38]. There a form of ‘classical limit’ ($\beta^2 \to 0$) is discussed. More precisely, the limit is $M \to \infty$ with $l = 0$ and $E \equiv E/M^2$ held fixed. (This differs from the limit treated in appendix B of [38] where $M \to \infty$ with $l$ and $E$ is held fixed, and the functions $A_\pm$ and $T_j$, $j = O(M)$, are treated. Taking the limit with $E/M^2$ fixed instead of $E$ is appropriate for capturing the behaviour of the functions $T_j$ with $j$ remaining finite.) For the initial problem, corresponding to $K = 1$ and the function $T_{1/2}$ or $C^{(1)}$, Bender et al found

$$E_k(M) \sim (k + \frac{1}{2})^2 M^2 , \quad k = 0, 1 \ldots , \quad M \to \infty . \quad (7.6)$$

(Note that in [38] $M$ is instead used to label the different eigenproblems, the rôle played by $K$ here.) The result (7.6) suggests that the limiting form for the spectral determinant $C^{(1)}(\mathcal{E}, l)$ at $l = 0$ is $2 \cos(\pi \sqrt{-\mathcal{E}})$. (The minus sign appears for the same reason as before, namely the variable change in going from (7.2) to (7.4).) Assuming that the fusion hierarchy (4.20) holds in this limit, from it we deduce

$$C^{(K)}(\mathcal{E}, 0) \sim \frac{\sin((K+1) \pi \sqrt{-\mathcal{E}})}{\sin(\pi \sqrt{-\mathcal{E}})}. \quad (7.7)$$
This correctly reproduces the spectra for the higher problems that in [38] had to be obtained by a series of independent calculations. Finally, we remark that the angular momentum term $l(l+1)x^{-2}$ drops out of (7.5) in the limit considered in [38], and so the results (7.6) and (7.7) should hold unchanged for $l \neq 0$.

8 Singular potentials and duality

Up to now the parameter $M$ has been restricted to the range $(0, \infty)$, but simple transformation properties of the Schrödinger equation allow a wider range of $M$ to be accommodated. The idea is to define a mapping between the eigenvalue problem for $M > 0$ and the problem with $-1 < M < 0$. Our discussion here is not meant to be particularly original, essentially following [10], but we wish to underline the fact that the relevance of integral equations to Schrödinger problems, stressed in [1], actually applies to wider range of problems than might initially be thought. The first step is to make Langer’s [39] variable change

$$x = e^z , \quad y(x, E, l) = e^{z/2}\psi(z, E, l). \quad (8.1)$$

Then

$$\left(-\frac{d^2}{dz^2} + e^{2z(M+1)} - E e^{2z} \right) \psi(z, E, l) = \kappa^2 \psi(z, E, l) \quad (8.2)$$

with $\kappa = -i(l+\frac{1}{2})$. Now we interchange the role of the two exponential terms, with the transformation

$$z \to \frac{1}{M+1} z + \log \frac{M+1}{\sqrt{E}} \quad (8.3)$$

to obtain

$$\left(-\frac{d^2}{dz^2} - e^{2z/(M+1)} + \tilde{E} e^{2z} \right) \psi(z, E, l) = \tilde{\kappa}^2 \psi(z, E, l), \quad (8.4)$$

where $\tilde{\kappa} = \kappa/(M+1), \tilde{l} = (l-M/2)/(M+1)$ and

$$\tilde{E} = \frac{-1}{(M+1)^2} E^{-M-1}. \quad (8.5)$$

It is easy to check that for $M = 1$ [35] gives an exact mapping between the energy levels of the harmonic oscillator at angular momentum $\tilde{l}$ and those of the Coulomb potential at angular momentum $\tilde{l} = l/2 - 1/4$. Indeed it is straightforward to see that eigenfunctions of the original problem are transformed into eigenfunctions for the ‘dual’ problem: reversing the original variable change, we have that

$$\tilde{y}(x, \tilde{E}, \tilde{l}) = (M+1)^{-1/2} E^{1/4} x^{\frac{M}{M+1}} y((M+1)E^{-1/2} x^{1/(M+1)}, E, l) \quad (8.6)$$

solves

$$\left(-\frac{d^2}{dx^2} - x^{2M} + \frac{\tilde{l} (\tilde{l}+1)}{x^2} - \tilde{E} \right) \tilde{y}(x, \tilde{E}, \tilde{l}) = 0 , \quad (8.7)$$

25
with $\tilde{M} = (M+1)^{-1} - 1$. Furthermore, if $y$ decays as $x \to +\infty$ then so does $\tilde{y}$, and if $y \sim x^{l+1}$ near $x = 0$ then $\tilde{y} \sim x^{l+1}$. Hence eigenfunctions are mapped to eigenfunctions, and the relation (8.3) is true at the level of the eigenvalues too. A curious consequence of this transformation, applied to the problem with $M = 1/2$ and $l = 0$, is that the solution of the $x^{-2/3}$ potential with $\tilde{l} = -1/6$ can be written in terms of the Airy function.

The mapping to singular potentials, decaying (albeit algebraically) at infinity, highlights the fact that the motion of the zeroes of the $Q$-operators as a function of the ‘twist’ $p$ corresponds to the motion of Regge poles for the radial Schrödinger equation as the angular momentum $l$ is continuously varied \cite{13, 14, 40}. Quite apart from the intrinsic curiosity of this fact, it means that the general results of, for example, chapters 8 and 13 of \cite{14} can be applied to deduce features of the motion of the zeroes of the $Q$-operators as $p$ and $\beta^2$, or equivalently $l$ and $E$, are varied. Observe also that, from the point of view of integrable field theory, the duality $M \to \tilde{M}$ sends $\beta^2$ to $1/\beta^2$. As $M$ passes through 0 into the ‘dual’ regime $-1 < M < 0$, $\beta^2$ increases through 1 and the associated perturbation of a $c = 1$ conformal field theory becomes irrelevant. The full range $-1 < M < \infty$ is mapped onto $0 < \beta^2 < \infty$. For $M < -1$, the nature of the Schrödinger problem \cite{2, 1} changes fundamentally, as the singularity at the origin ceases to be regular. Formally $M < -1$ corresponds to purely imaginary values of $\beta$, and it is tempting to suppose that there should be a relation with the non-compact sinh-Gordon model. However at this stage this is pure speculation.

9 Conclusions

In this paper we have further explored the correspondence between certain integrable quantum field theories and Schrödinger equations. The two natural classes of spectral problems for these equations have been associated with the two types of operators which arise in the integrable quantum field theories, namely the $T$ and $Q$-operators. The neatness of this fit convinces us that the matching is no accident, and that it promises to lead to a fruitful interaction between these previously distinct fields. A new angle on the theory of integrable models seems to be opening up, and this alone should justify some further work on the many unresolved issues which remain.

On a technical level, while we have tried to give most of the details of our calculations, our discussion has not been completely rigorous and some aspects will probably repay a more careful treatment. In addition, the treatment of this paper should be completed with a full discussion of the ‘quantum’ (non-semiclassical) domain $M < 1$. This is particularly important because the region includes the unitary minimal models. One example, the ‘Airy case’ ($M = 1/2$, $l = 0$), has already been treated in \cite{4}, and its consistency with integrable field theory predictions has been confirmed both numerically \cite{1} and analytically \cite{11}.

The literature on the spectral theory of the (radial) Schrödinger equation is extensive, much of it dating back to Regge pole theory, and we can hope to apply further
results to the study of integrable quantum field theories. In the other direction, the
realisation that some of the functional relations uncovered by Sibuya, Voros and
others in connection with ordinary differential equations also arise in integrable models,
and can be solved by means of nonlinear integral equations, promises to be very useful.
It should be remarked that functional relations can be found for more general potentials
than the $x^{2M} + l(l+1)x^{-2}$ cases used above (see [12, 24, 42]). It would be interesting
to know if these also have a rôle to play in the theory of integrable models. A relation
between certain functional relations and differential equations was recently discussed
in [13], though the approach is rather different from that adopted here.

The match that we have found connects spectral determinants of the Schrödinger
equation with the vacuum expectation values of the $T$ and $Q$-operators. However
this leaves to one side all of the other expectation values, which emerge when excited
states in the integrable quantum field theories are discussed. They obey the same sets of
functional relations as the vacuum expectation values [8, 25, 27, 28], and a major challenge
is to find an interpretation for them within the context of Schrödinger equations. One
way to access excited states in integrable quantum field theories is by a process of
analytic continuation in a suitable parameter [19]. However if we remain with the
massless models discussed in this paper, this idea is unlikely to help since the functions
$\psi^\pm$ and $y$ are single-valued in the one remaining parameter, namely $l$ (see [14, 30]). If the
correspondence could be extended to cover massive models, then the story would change
since these models are already known to exhibit a pattern of branch points joining the
ground state to excited states (examples can be found in [19, 44]). Whether massive
models can be incorporated into the differential equation framework is another question
that for the moment remains completely open, and any progress on this issue would be
extremely interesting.

Acknowledgements – We are grateful to Armen Allahverdyan, Clare Dunning, Paul
Fendley, Davide Fioravanti, Ferdinando Gliozzi, Bernard Nienhuis and Paul Pearce for
useful discussions. The work was supported in part by a TMR grant of the European
Commission, reference ERBFMRXCT960012. PED thanks the EPSRC for an Ad-
vanced Fellowship, and RT thanks the Universiteit van Amsterdam for a post-doctoral
fellowship.

Appendix A WKB details

In this appendix we sketch how to obtain (2.21). The equation describes the asymptotic behaviour of $\log D^-(E,l) = W[y(x,E,l), \psi^+(x,E,l)]$. Note first that $D^-$ can be evaluated as

$$D^-(E,l) = \lim_{x \to 0} \left( (2l+1)x^{l} y(x,E,l) \right). \quad (A.1)$$

For convenience we set $E = -a$, so that the problem under discussion becomes

$$\left( -\frac{d^2}{dx^2} + x^{2M} + \frac{l(l+1)}{x^2} + a \right) y = 0 \quad (A.2)$$
and we are interested in the behaviour as $|a| \to +\infty$. The WKB approximation cannot be applied directly to this equation, because of the nature of its singularity at $x = 0$. The remedy found in [33] is to make the variable change $x = e^z$, $y(x) = e^{z/2} \phi(z)$. Writing $\lambda = l + 1/2$ the equation becomes

$$\left(- \frac{d^2}{dz^2} + R(z, a, \lambda)\right) \phi = 0$$  \hspace{1cm} (A.3)

with $R(z, a, \lambda) = e^{(2M+2)z} + ae^{2z} + \lambda^2$. The WKB approximation for $\phi$ is

$$\frac{A}{R(z, a, \lambda)^{1/4}} \exp \left[ \int_{z_0}^{z} \sqrt{R(u, a, \lambda)} \, du \right] + \frac{B}{R(z, a, \lambda)^{1/4}} \exp \left[ - \int_{z_0}^{z} \sqrt{R(u, a, \lambda)} \, du \right].$$  \hspace{1cm} (A.4)

In contrast to the WKB treatment of the initial equation, this approximation is good for all real $z$ (and hence for $x$ all the way down to zero) since $R^{-3/4}(R^{-1/4})''$ tends to zero uniformly in $z$ as $|a| \to +\infty$, so long as arg $a \neq \pi$ (for a discussion, see [43]).

The solution that we hope to approximate is subdominant as $x \to +\infty$, which requires $A = 0$. The value of $B$ must be chosen so as to match the large-$x$ asymptotic (2.2). A little thought shows that the correctly-normalised approximate solution can be written as

$$\phi(z, a, \lambda) \sim \frac{1}{R(z, a, \lambda)^{1/4}} \exp \left[ \int_{z}^{\infty} \left( \sqrt{R(u, a, \lambda)} - e^{(M+1)u} \right) du - \frac{1}{M+1} e^{(M+1)z} \right].$$  \hspace{1cm} (A.5)

To estimate (A.1) we need to analyse this quantity as $z \to -\infty$. An integration by parts extracts a term $-\lambda z$ in this limit, after which $\lambda$ can be dropped from the remaining integral, giving a leading $a$-dependence

$$\phi(z, a, \lambda) \sim \lambda^{-1/2} \exp \left[ -\lambda z + \int_{\infty}^{\infty} \left( \sqrt{e^{(2M+2)u} + ae^{2u} - e^{(M+1)u}} \right) du \right] = \lambda^{-1/2} \exp \left[ -\lambda z + a^{(M+1)/2M} \int_{0}^{\infty} \left( \sqrt{t^{2M+1} - t^M} \right) dt \right].$$  \hspace{1cm} (A.6)

Reverting to the original variables and substituting into (A.1), it is easily checked that the result quoted in the main text is recovered.

If $M < 1$, the above treatment fails, because the integral in (A.5) diverges. The remedy is to introduce further regularising terms, by expanding $\sqrt{R(u, a, \lambda)}$ to higher order. For example, if $1/3 < M < 1$ the argument of the exponential in (A.3) should be replaced by

$$\left[ \int_{z}^{\infty} \left( \sqrt{R(u, a, \lambda)} - e^{(M+1)u} - \frac{a}{2} e^{(1-M)u} \right) du - \frac{1}{M+1} e^{(M+1)z} - \frac{a}{2-2M} e^{(1-M)z} \right]$$  \hspace{1cm} (A.7)

and (A.6) becomes

$$\phi(z, a, \lambda) \sim \lambda^{-1/2} \exp \left[ -\lambda z + a^{(M+1)/2M} \int_{0}^{\infty} \left( \sqrt{t^{2M+1} - t^M - \frac{1}{2} t^{-M}} \right) dt \right].$$  \hspace{1cm} (A.8)
However, for \(1/3 < M < 1\) we have

\[
2 \int_0^\infty \left[ (t^{2M} + 1)^{1/2} - t^M - \frac{1}{2} t^{-M} \right] dt = -\frac{1}{\sqrt{\pi}} \Gamma\left(-\frac{1}{2} - \frac{1}{2M}\right) \Gamma\left(1 + \frac{1}{2M}\right),
\]

and so the result (2.22) is preserved – we have simply continued it around the pole at \(M = 1\). Despite the fact that the \(|E| \to \infty\) asymptotic of \(D^-\) is unchanged, the move to \(M < 1\) does have an effect on the limiting behaviour of the auxiliary function

\[d(E,l) \equiv \omega^{2l+1} D^-(\omega^2 E, l) / D^-(\omega^{-2} E, l)\]

used in §2. The correct choice of branches when applying the asymptotic (2.21) at the shifted arguments \(\omega^{\pm 2} E\) changes, with the result that the asymptotic log \(d(E,l) \sim -\frac{1}{2} i b_0(E)^\mu\) now holds in the revised sector

\[-\frac{2\pi M}{M+1} < \arg(E) < \frac{2\pi M}{M+1}\].

Outside this sector, the leading contributions cancel and so all that can be concluded is that the asymptotic is subleading, though from the nonlinear integral equation (2.32) we expect that it will be \(O(1)\), and in fact equal to \((2l+1)\pi i\).

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