Self-Similar Polytropic Champagne Flows in H II Regions

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ABSTRACT
We explore large-scale hydrodynamics of H II regions for various self-similar shock flows of a polytropic gas cloud under self-gravity and with quasi-spherical symmetry. We formulate cloud dynamics by invoking specific entropy conservation along streamlines and obtain global self-similar “champagne flows” for a conventional polytropic gas with shocks as a subclass. Molecular cloud cores are ionized and heated to high temperatures after the onset of nuclear burning of a central protostar. We model subsequent evolutionary processes in several ways and construct possible self-similar shock flow solutions. We may neglect the mass and gravity of the central protostar. The ionization and heating of the surrounding medium drive outflows in the inner cloud core and a shock travels outwards, leading to the so-called “champagne phase” with an expanding outer cloud envelope. Complementarily, we also consider the expansion of a central cavity around the centre. As the inner cloud expands plausibly due to powerful stellar winds, a cavity (i.e., ‘void’ or ‘bubble’) can be created around the centre, and when the cavity becomes sufficiently large, one may neglect the gravity of the central protostar. We thus present self-similar shock solutions for “champagne flows” with an expanding central void. We compare our solutions with isothermal solutions and find that the generalization to the polytropic regime brings about significant differences of the gas dynamics, especially for cases of $n < 1$, where $n$ is a key scaling index in the self-similar transformation. We also compare our global polytropic self-similar solutions with numerical simulations on the expansion of H II regions. We further explore other possible dynamic evolutions of H II regions after the initiation of nuclear burning of the central protostar, for example asymptotic inflows or contractions far from the cloud centre and the ongoing infall around a central protostar. In particular, it is possible to use the downstream free-fall solution with shocks to describe the dynamic evolution of H II regions shortly after the nascence of the central protostar. We also give an analysis on the invariant form of self-similar polytropic flows by ignoring self-gravity.

Key words: H II regions — hydrodynamics — ISM: clouds — shock waves — star: formation — stars: winds, outflows

1 INTRODUCTION
The compact (scale sizes of $\sim 0.1 - 1$ pc) and ultracompact (UC; scale sizes $\lesssim 0.15$ pc) H II regions are associated with massive OB stars (e.g., Habing & Israel 1979). The UC H II region stage ($\lesssim 10^5$ yr) stands for a substantial part of the relatively short main-sequence lifetimes of OB stars. Several radio surveys (e.g., Wood & Churchwell 1989; Fish 1993; Kurtz, Churchwell & Wood 1994) observed expansions of luminous H II regions with shock signatures. “Champagne flow” models (e.g., Tenorio-Tagle 1979, Tenorio-Tagle, York & Bodenheimer 1979, York 1986) successfully explain the expansion of H II regions by considering a protostar formed in a cloud core, photonionizing and heating the cloud, as well as driving a shock that accelerates the ionized gas to expand rapidly. Observations tend to support “champagne flow” models, such as Lumsden & Hoare (1999) for UC H II regions G 29.96-0.02 and Barriault & Joncas (2007) for Compact H II region Sh 2-158. Champagne flows in clouds of larger scales have also been identified, such as Foster et al. (2007) for the dense Galactic H II region G84.9+0.5 and Maheswar et al. (2007) for the classical H II region S236 in the cluster of OB stars NGC 1893.

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are not completely known. We then approximate the energy equation by a general polytropic equation of state $p = \kappa(r, t) \rho^\gamma$, where $p$ is the thermal gas pressure, $\gamma$ is the polytropic index and the proportional coefficient $\kappa(r, t)$ (related to specific entropy) depends on radius $r$ and time $t$ in general. Setting $\kappa$ as a global constant, the equation of state simply becomes a conventional polytropic one. By adjusting $\gamma$, we may model various situations of H II regions in molecular clouds. For example, for $\gamma = 1$ and a constant $\kappa$, our solutions reduce to isothermal ones. Since “champagne flows” in a polytropic molecular cloud has not been studied, we would generalize the isothermal analyses of Shu et al. (2002) and of Tsai & Hsu (1995) to a polytropic description of self-similar “champagne flows”. We shall provide the basic formulation with the most general polytropic equation of state (i.e., specific entropy conservation along streamlines; Wang & Lou 2008) and present global “champagne flow” solutions with shocks for a conventional polytropic gas.

Shu et al. (2002) introduced the Bondi-Parker radius as a measure for the effective distance of the central gravity. The Bondi-Parker radius is defined by $r_{BP} = GM_*/(2a^2)$, where $M_*$ is the mass of the central gravity source and $a$ is the sound speed of the surrounding medium, which is a constant for an isothermal cloud. The mass originally residing within a radius $r_0$ is dumped into the star during the star formation. After the star formation, as the surrounding cloud becomes much hotter with a higher sound speed, the Bondi-Parker radius becomes much less than $r_0$. Therefore for the gas in $r > r_0$, the gravity of the central massive star may be neglected. This reasoning naturally leads to the possible existence of a cavity around the center of a molecular cloud, which we refer to as ‘void’ or ‘bubble’. At $t = 0$, the void boundary is at $r = r_0$. For an expanding cloud, the central void also expands. Indeed, a stellar wind drives also a principal shock and is capable of sweeping the surrounding ionized gas into an expanding shell. Wood & Churchwell (1989) identified central cavities in the shell or cometary UC H II regions, which are thought to be supported by stellar wind and radiation pressures in their survey. Lumsden & Hoare (1999) suggested a “champagne flow” surrounding a hot stellar wind bubble to interpret observations of G 29.96-0.02. Comerón (1997, 1999) numerically simulated the dynamic evolution of wind-driven H II regions with strong density gradients and found that features of classical champagne model are not substantially changed, except that the compression of the swept-up matter would, rapidly and particularly in densest cases, lead to the trapping of the IF and inhibit the champagne phase. Therefore, the dynamic evolution of void expansion represents an important physical aspect of “champagne flows”. Recently, Lou & Hu (2008, in preparation) explore self-similar solutions for voids in a more general context. In this paper, we construct self-similar solutions for “champagne flows” with central voids in self-similar expansion. The inclusion of a central void not only makes our model more realistic, but also allows us to take into account stellar wind bubbles.

We outline the model formulation of a general polytropic gas and present self-similar asymptotic solutions in section 2 and construct global solutions of “champagne flows” in section 3. Section 4 provides solutions of self-similar “champagne flows” with an expanding central void. In section 5, we discuss behaviours and astrophysical applications of these solutions for various settings.
of our novel solutions, and suggest other plausible forms of H II regions. Details of an invariant form of self-similar solutions in a conventional polytropic gas with the self-gravity ignored are summarized in Appendix A.

2 SELF-SIMILAR POLYTROPIC FLOWS

2.1 General Polytropic Formulation

Dynamic evolution of a quasi-spherical general polytropic gas under self-gravity can be described by nonlinear hydrodynamic equations in spherical polar coordinates \((r, \phi, \theta)\),

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 ru \right) = 0 ,
\]

\[
\frac{\partial M}{\partial t} + u \frac{\partial M}{\partial r} = 0 ,
\]

\[
\frac{\partial M}{\partial r} = 4\pi r^2 \rho ,
\]

\[
\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right) = -\frac{\partial p}{\partial r} - \frac{GM\rho}{r^2} ,
\]

\[
p = \kappa(r, t) \rho^n ,
\]

where \(\rho(r, t)\) is the mass density, \(u(r, t)\) is the bulk gas radial flow velocity, \(M(r, t)\) is the enclosed mass within \(r\) at time \(t\), \(p\) is the thermal pressure, \(G = 6.67 \times 10^{-8}\) dyne cm\(^2\) g\(^{-2}\) is the gravity constant. Equations (1), (2), (3) describe the mass conservation, equation (4) is the radial momentum equation and equation (5) is the general polytropic equation of state, in which \(\gamma\) is the polytropic index and the coefficient \(\kappa\) directly related to the ‘specific entropy’ depends on \(r\) and \(t\). For a conventional polytropic gas, \(\kappa\) is a global constant in space and time. More generally, we require the conservation of ‘specific entropy’ along streamlines, namely

\[
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} \right) \left( \ln \frac{p}{\rho^n} \right) = 0 .
\]

This set of equations is the same as those of Wang & Lou (2008) but without a completely random magnetic field.

To reduce the nonlinear partial differential equations (PDEs) to ordinary differential equations (ODEs) for self-similar flows, we introduce the following transformation

\[
r = k^{1/2} t^n x , \quad u = k^{1/2} t^{n-1} v , \quad \rho = \frac{\alpha}{4\pi G l^2} , \quad p = \frac{k^{2n-4}}{4\pi G} \beta , \quad M = \frac{k^{3/2} l^{3n-2} m}{(3n-2) G} ,
\]

where \(x\) is a dimensionless independent self-similar variable, \(k\) is a dimensional parameter related to the polytropic sound speed making \(x\) dimensionless, \(v(x)\), \(\alpha(x)\), \(\beta(x)\), \(m(x)\) are dimensionless reduced dependent variables of \(x\) only, and \(n\) is a key scaling index which controls the relation between \(r\) and \(x\) as well as various scalings of reduced dependent variables. We refer to \(v(x)\), \(\alpha(x)\), \(\beta(x)\), and \(m(x)\) as the reduced radial flow speed, mass density, thermal pressure, and enclosed mass, respectively. Transformation (7) is identical with that of Lou & Wang (2006).

By performing self-similar transformation (7) in equations (1)–(6) and introducing parameter \(q \equiv 2(n + \gamma - 2)/(3n - 2)\), we obtain two integral relations

\[
m = \alpha x^2 (\pi x - v) ,
\]

\[
\beta = \alpha^* m^q ,
\]

where there is no loss of generality to set the proportional coefficient (i.e., an integration constant) equal to unity in integral (9) for \(q \neq 2/3\) or \(\gamma \neq 4/3\) (Wang & Lou 2008). The special case of \(\gamma = 4/3\) corresponds to a relativistically hot gas as studied by Goldreich & Weber (1980) and Lou & Cao (2008). By setting \(q = 0\), the general polytropic formulation reduces to the conventional polytropic case of a global constant \(\kappa\) (e.g., Suto & Silk 1988; Lou & Gao 2006; Lou & Wang 2006; Lou, Jiang & Jin 2008) with \(n + \gamma = 2\). According to expression \(M(r, t)\) for the enclosed mass in transformation (7), we require \(3n - 2 > 0\) and \(\pi x - v > 0\) to ensure a positive enclosed mass. The inequality \(3n - 2 > 0\) will reappear later for a class of asymptotic solutions at large \(x\). By equation (5), we emphasize that for \(\pi x - v > 0\) at a certain \(x^*\), the enclosed mass within \(x^*\) becomes zero; we refer to this as a central void and \(x^*\) is the independent similarity variable marking the void boundary which expands with time \(t\) in a self-similar manner.

Combining all reduced equations above, we readily derive two coupled nonlinear ODEs for \(\alpha'\) and \(v'\) as

\[
\mathcal{X}(x, \alpha, v) = \mathcal{A}(x, \alpha, v) , \quad \mathcal{X}(x, \alpha, v) v' = \mathcal{V}(x, \alpha, v) ,
\]

where functionalities \(\mathcal{X}, \mathcal{A}\) and \(\mathcal{V}\) are defined by

\[
\mathcal{X}(x, \alpha, v) \equiv \left[ 2 - n + (3n - 2)q/2 \right] \alpha^{1-n+3nq/2} x^{2q}(\pi x - v)^q - (\pi x - v)^2 ,
\]

\[
\mathcal{A}(x, \alpha, v) \equiv 2\pi \alpha \left[ q \alpha^{1-n+3nq/2} x^{2q}(\pi x - v)^q - (\pi x - v)^2 \right]
\]

\[
+ (\pi x - v)^2 \left[ n - (n - 1)v + \frac{\alpha}{4\pi G} \right]
\]

\[
+ q \alpha^{1-n+3nq/2} x^{2q-1}(\pi x - v)^q(3n x - 2v) ,
\]

\[
\mathcal{V}(x, \alpha, v) \equiv 2\pi \alpha \left[ 2 - n + \frac{\alpha}{4\pi G} \right] \alpha^{1-n+3nq/2} x^{2q}
\]

\[
\times (\pi x - v)^q - (\pi x - v)^2 \left[ n - (n - 1)v + \frac{\alpha}{4\pi G} \right]
\]

\[
+ q \alpha^{1-n+3nq/2} x^{2q-1}(\pi x - v)^q(3n x - 2v) .
\]

For a conventional polytropic gas of constant \(\kappa\) with \(n + \gamma = 2\), we simply set \(q = 0\) in equation (11) to derive

\[
\frac{\alpha'}{\alpha^2} = \frac{(n - 1)v + \frac{\alpha}{4\pi G}}{\alpha(\pi x - v)^2 - \gamma \alpha^2} ,
\]

\[
v' = \frac{(n - 1)\alpha v(\pi x - v) + \frac{\alpha}{4\pi G}}{\alpha(\pi x - v)^2 - \gamma \alpha^2} .
\]

Up to this point, our basic self-similar hydrodynamic formulation is the same as that of Lou & Wang (2006) and of Wang & Lou (2008) without a random magnetic field.

For energy conservation, we define the energy density \(\epsilon\) and the energy flux density \(\mathcal{F}\) as follows,

\[
\epsilon = \frac{\alpha m^2}{2} - \frac{GM\rho}{r} + \frac{i}{2} p ,
\]

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where \( \epsilon \) is the energy density, \( J \) is the energy flux density and \( i \) is the degree of freedom of an individual gas particle. The three terms in expressions (14) and (15) correspond to densities of the kinetic energy, the gravitational energy and the internal energy, respectively. With equations (11) – (15) and a globally constant \( \kappa \), we derive

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 J) = \mathcal{P} \equiv u \frac{\partial p}{\partial r} \left[ \frac{i}{2} (\gamma - 1) - 1 \right]
\]

(16)

for energy conservation, where \( \mathcal{P} \) represents the net energy input. If the gas expands adiabatically or \( \gamma = (i + 2)/i \), then \( \mathcal{P} = 0 \). Whether the gas locally gains or loses energy depends not only on the difference between \( \gamma \) and \( (i + 2)/i \), but also on the signs of \( \partial p/\partial r \) and \( u \).

### 2.2 Self-Similar Solutions

An exact globally static solution, known as the singular polytropic sphere (SPS) takes the following form of

\[
v = 0, \quad \alpha = \left[ \frac{n^2 - q}{2(2 - n)(3n - 2)} \right]^{-1/(n - 3q/2)} x^{1-2/n}.
\]

(17)

This is a straightforward generalization of the singular isothermal sphere (SIS; e.g., Shu 1977) and of the SPS for a conventional polytropic gas with \( q = 0 \) (Lou & Wang 2006; Lou & Gao 2006). For a general SPS here, the mass density profile scales as \( \rho \propto r^{-2/n} \), independent of \( q \) parameter.

For large \( x \), the asymptotic flow behaviour is

\[
v = \left[ -\frac{nA}{(3n - 2)} + 2(2 - n)n^{-1} A^{1-n+3nq/2} \right] x^{1-2/n} + B x^{1-1/n} + \cdots,
\]

(18)

\[
\alpha = Ax^{-2/n} + \cdots,
\]

(19)

where \( A > 0 \) and \( B \) are two constant parameters. As \( x \to +\infty \), the mass density profile scales the same way as (17) above, independent of \( q \) parameter to the leading order. Since \( x \to +\infty \) means \( t \to 0^+ \), the asymptotic solution at large \( x \) is thus equivalent to initial conditions of the system at a finite \( r \). Coefficients \( A \) and \( B \) are referred to as the mass and velocity parameters, respectively. A global solution with asymptotic behaviour (14) and (15) represents a fluid whose density profile scales initially similar to that of a SPS. Thus, by varying the scaling index \( n \) or the polytropic index \( \gamma \), we are able to model the initial density profile with an index \( l = -2/n \). Furthermore, as a physical requirement for plausible similarity solutions of our polytropic flow, both \( v(x) \) and \( \alpha(x) \) should remain finite or tend to zero at large \( x \). Hence, in cases of \( 2/3 < n < 1 \), corresponding to \( 2 < l < 3 \), \( A \) and \( B \) are fairly arbitrary, while in cases of \( 1 \leq n < 2 \) (general polytropic), corresponding to \( 1 < l < 2 \), \( B \) should vanish for a finite radial flow velocity at large \( x \).

In the regime of \( x \to 0^+ \), there exists a free-fall asymptotic solution for which the gravity is virtually the only force in action, and the radial velocity and the mass density profile both diverge in the limit of \( x \to 0^+ \). To the leading order, the free-fall asymptotic solution takes the form of

\[
\alpha(x) = \left[ \frac{3n - 2}{2x^3} \right]^{1/2}, \quad v(x) = -\left[ \frac{2m(0)}{3n - 2} \right]^{1/2},
\]

(21)

where constant \( m(0) \) represents an increasing central point mass. Such solutions were first found by Shu (1977) in the isothermal case, and were generalized to the conventional polytropic case (Cheng 1978; Lou & Gao 2006; Lou & Wang 2006) and to the general polytropic case with a random magnetic field by Wang & Lou (2008). The asymptotic form does not depend on \( q \). The validity of such solutions requires \( n > 2/3 \) and \( \gamma < 5/3 \); the last inequality appears as a result of comparing various terms in series expansions.

Another exact global solution, known as the Einstein-de Sitter (EdS) solution, exists in two cases of \( q = 0 \) and \( q = 2/3 \). The EdS solution for a conventional polytropic gas of \( q = 0 \) reads as

\[
v = \frac{2}{3} x, \quad \alpha = \frac{2}{3}, \quad m = \frac{2(n - 2/3)}{3} x^3,
\]

(22)

Lou & Wang (2006); for \( q = 2/3 \) and thus \( \gamma = 4/3 \), this global EdS solution without and with a random magnetic field takes a slightly different form (Lou & Cao 2008). With shocks, EdS solutions can be used to construct polytropic “champagne flows” with various upstream dynamics.

From now on, we focus on the conventional polytropic case of \( q = 0 \) with \( n + \gamma = 2 \). To simulate “champagne flows” in H II regions in star-forming clouds, we solve coupled nonlinear ODEs (12) and (13) subject to the inner boundary conditions at \( x = 0 \), namely,

\[
\alpha = \alpha_0, \quad v = 0,
\]

(23)

where \( \alpha_0 \) is a constant. A series expansion yields the LP type of asymptotic solutions at small \( x \) in the form of

\[
v = \frac{2}{3} x - \frac{x_0}{15\gamma} \left( \frac{\alpha_0 - 2}{3} \right) \left( n - \frac{2}{3} \right) x^3 + \cdots,
\]

(24)

\[
\alpha = \alpha_0 - \frac{\alpha_0^{(2-\gamma)}}{6\gamma} \left( \frac{\alpha_0 - 2}{3} \right) x^2 + \cdots
\]

(25)

The isothermal counterpart of this polytropic series solution was obtained earlier (Larson 1969; Penston 1969). Hunter (1977; Shu et al. 2002) with \( n = 1 \) and \( \gamma = 1 \). Such LP type solutions may be utilized to construct champagne flows, if we ignore the central protostar as an approximation and assume the surrounding gas to be initially static (e.g., Shu et al. 2002). Physically, as a result of the gravity of the central protostar, the gas infall towards the very central region may not stop, even when the protostar starts to shine and the ‘champagne phase expansion’ has occurred in the outer cloud envelope due to the photoionization and ultraviolet (UV) heating. Thus with central free falls (20) and (21) as the downstream side of a shock, we can also construct global solutions for the dynamics of H II regions surrounding a nascent central massive protostar which involves free-fall materials.

We can take LP type or EdS solutions as the downstream side of a shock and model classical champagne flows for a conventional polytropic gas. We shall come to the possible scenario of inner free-fall solutions with an outer champagne flow.

With the restriction \( n + \gamma = 2 \) for a conventional polytropic gas, the initial density profile with the scaling index \( n \) is directly linked to the polytropic index \( \gamma \), depending on the energy exchange process in the gas (see eq (16)). For a
general polytropic gas with \( q \neq 0 \) in contrast, the scaling index \( n \) and polytropic index \( \gamma \) can be independently specified; in particular, we can have \( \gamma > 1 \) and \( n \geq 1 \) (this is impossible for a conventional polytropic gas). On the other hand, the initial mass density profile is affected by the star formation or other energetic processes before \( t = 0 \). Hence for \( l = -2/n \), we postulate that the energy exchange process remains largely unchanged after the protostar formation at \( t = 0 \). This is plausible because the ionization front travels relatively fast to large distances in a cloud during the "champagne flow" phase. Our general polytropic model allows inequality \( 2/3 < n < 2 \), corresponding to \( 1 < l < 3 \), which covers so far the entire range of initial mass density profiles of H II regions. This range of mass density distribution has been obtained from radio observations of cloud fragments and isolated dark clouds (e.g., Arquilla & Goldsmith [1985; Myers [1985]).

Franco et al. (1990) reveals that the initial mass density profile index \( 3/2 < l < 3 \) (i.e., \( 2/3 < n < 4/3 \) in a polytropic model) leads to "champagne flows" in clouds with weak shocks; and \( l > 3 \) corresponds to a "champagne flow" with strong and accelerating shocks. We then require parameter \( n \) within the range \( 2/3 < n < 4/3 \), as we assume the cloud is 'density bounded'. With such range of parameter, we provide solutions of "champagne flow" with shocks. We emphasize in particular that there is one more degree of freedom to choose the velocity parameter \( B \) in asymptotic solution (18) and (19) in polytropic cases with \( 2 < l < 3 \), which covers so far the entire range of initial mass density profiles of H II regions.

Finally, we need to include shocks at proper places in LP type solutions, EdS solutions or free-fall solutions to match with appropriate asymptotic solutions at large \( x \) to determine relevant coefficients. The shock jump conditions between the downstream and upstream variables are determined by the mass conservation, the radial momentum conservation and the energy conservation. With these three equations for shock conditions, one can determine upstream self-similar variables \( (x_u, \alpha_u, v_u) \) uniquely from downstream self-similar variables \( (x_d, \alpha_d, v_d) \) or vice versa. The two subscripts \( d \) and \( u \) here denote the downstream and upstream variables, respectively. Detailed formulation and procedure of self-similar shocks can be found in Section 5 of Lou & Wang (2006). All solutions in this paper are obtained by solving coupled nonlinear ODEs (12) and (13) for a conventional polytropic gas with \( n + \gamma = 2 \).

### 3 POLYTROPIC CHAMPAGNE FLOWS

In cases of \( n < 1 \), there is a range of shock positions (or speeds) for a specified downstream solution with a fixed density at the centre \( \alpha_0 \) at \( x = 0 \), corresponding to different asymptotic flow behaviours at large \( x \) on the upstream side. We now discuss two situations: first, for cases with a fixed value of \( \alpha_0 \), we adjust shock positions for a specified LP type solution and observe the relation between shock positions and asymptotic flow behaviours at large \( x \). Secondly, for cases with a fixed shock position, we alter the value of \( \alpha_0 \) and examine the variation of upstream conditions. It is expected to set certain limits on relevant parameters for polytropic "champagne flows" to exist. As a series of examples, we choose the scaling index \( n = 0.9 \). Numerical explorations have also been performed for cases of \( n = 0.7 \) and \( n = 0.8 \) and the results are qualitatively similar.

#### 3.1 Cases with a Fixed \( \alpha_0 \) Value

With a fixed value of \( \alpha_0 \) for \( x \to 0^+ \), one can uniquely determine a LP type solution by a standard numerical integration. Such a LP type solution will encounter the sonic critical curve at a certain \( x_{\text{max}} \) uniquely corresponding to \( \alpha_0 \). It is natural to consider possible hydrodynamic shock positions \( x_{\text{sh}} < x_{\text{max}} \) on the downstream side of shock front. One such example of \( n = 0.9 \) and \( \alpha_0 = 1 \) is shown in Figure [1] with relevant parameters summarized in Table 1. To model "champagne flows" with outward expansions at large radii in molecular clouds, we provide the following analysis. In principle, there are two other conditions giving rise to two minima of \( x_{\text{sh}} \) in order to obtain an outflow in the upstream side of a shock. First, our extensive numerical explorations reveal that there exists a \( x_{\text{min}1} \) and for \( x_{\text{sh}} < x_{\text{min}1} \), the upstream shock position \( x_{\text{sh}} \) becomes complex by the shock conditions. This only happens when we attempt to obtain upstream variables from downstream variables. For a real \( x_{\text{sh}} \), downstream variables \( (x_d, \alpha_d, v_d) \) should satisfy

\[
(1 - \gamma)\alpha_d^{\gamma-1} + 2(\alpha - v_d)^2 > 0 .
\]  

(26)

For \( \gamma < 1 \) (unphysical) and \( \gamma = 1 \), inequality (26) is readily satisfied; for \( \gamma > 1 \) or \( n < 1 \), this condition does not always hold. Algebraic manipulations give the downstream Mach number in the shock reference framework \( M_d \) as

\[
M_d^2 = \frac{(n \alpha_d - v_d)^2}{\gamma \alpha_d^{\gamma-1}}.
\]  

(27)

Inequality (26) imposed on a subsonic downstream Mach number is \( 1 > M_d^2 > (\gamma - 1)/(2\gamma) \). The downstream Mach number and the upstream Mach number is related by

\[
M_d^2 = \frac{2 + (\gamma - 1)M_u^2}{2\gamma M_u^2 - (\gamma - 1)}.
\]  

(28)

This relation was provided by Lou & Cao (2008) for a relativistically hot gas and is proven valid in our model consideration. With the possible range of upstream Mach number \( 1 < M_u^2 < \infty \), we then have the limit on \( M_d \) shown above. As we integrate LP or EdS solutions from \( x = 0 \) with \( \alpha = \alpha_0 \) and \( v = 0 \), solutions do not satisfy inequality (26) when \( x \) remains sufficiently small and \( x_{\text{min}1} \) is the minimum value of \( x \) satisfying (26). This value of \( x_{\text{min}1} \) is uniquely determined by the \( \alpha_0 \) value. Therefore, for a fixed LP type solution around small \( x \), polytropic shocks can be constructed with a downstream shock in the range of \( x_{\text{min}1} < x_{\text{sh}} < x_{\text{max}} \), and across such a shock, the LP type solution at small \( x \) can be matched with different asymptotic flows at large \( x \).
Systematic numerical explorations reveal that the upstream velocity increases monotonically with the increase of \( x_{sd} \), as shown by the variation trend of the \( B \) parameter (see Table I). There is thus another critical value imposed on \( x_{sd} \), denoted by \( x_{min2} \). For \( x_{sd} > x_{min2} \), the upstream solution matches to an asymptotic solution at large \( x \) in the form of (15) and (19) with \( B > 0 \), referred to as an outflow. Complementarily with \( x_{sd} < x_{min2} \), the upstream solution matches to an asymptotic solution with \( B < 0 \), referred to as an inflow. As \( B \) varies continuously and monotonically with \( x_{sd} \), for \( x_{sd} = x_{min2} \) the upstream solution corresponds to an asymptotic solution with \( B = 0 \), which describes a breeze or a contraction in association with “champagne flows”. According to asymptotic expressions (18) and (19) with \( q = 0 \), the breeze or contraction correspond to slow outward or inward flows. To obtain a breeze, we need a mass parameter \( A < A_s \approx \left[ \frac{n^2}{2\gamma(3n-2)} \right]^{-1/n} \) (29).

For the specific case of \( A = A_s \), either a breeze or a contraction reduces to SPS solution (17). With \( n < 1 \), there are three possibilities in general. First, \( x_{min1} > x_{min2} \), for the allowed range to construct shocks, the upstream solutions always correspond to outflows. Secondly, \( x_{min1} < x_{min2} \) for the allowed range to construct shocks, it is possible to obtain outflows, inflows and breezes or contractions for the upstream side. Thirdly, if any of \( x_{min1} \) or \( x_{min2} \) exceeds \( x_{max} \), a global champagne flow is not allowed. For the isothermal case of \( n = 1 \), we may set \( B \) equal to zero; asymptotic breezes or contractions on the upstream side are allowed, and the only allowed value of the downstream shock position is \( x_{sd} = x_{min2} \) [Shu et al. (2002)] indicated that the isothermal shock position is uniquely determined by the value of \( \alpha_0 \), which is consistent with our more general analysis here. The unique shock position found by Shu et al. (2002) corresponds to \( x_{min2} \) above.

The conventional scenario for “champagne flows” would require the entire fluid to expand outward. Numerical simulations on “champagne flows” (e.g., Tenorio-Tagle et al. 1984) assume that at \( t = 0^+ \) the central star is formed and the surrounding cloud is initially at rest. For \( t > 0^+ \), the fluid is photoionized and heated by the ultraviolet radiation from the central star and expands. In this scenario, we would require an expanding upstream fluid in order to model “champagne flows”. However, since solutions with an asymptotic inflow or contraction as upstream part may also exist, the outer part of HI regions can also have inward velocities. In fact, during the star formation even some time after the star formation, the surrounding cloud may continue to collapse towards the centre (e.g., Fatuzzo et al. 2004). With the core nuclear burning of the central protostar, the surrounding gas is ionized and heated, and the inner part of the fluid starts to expand, while the outer part continues to fall inwards. Those solutions of the LP type as the downstream side of a shock and an asymptotic inflow or contraction for the upstream side correspond to this scenario just described; and such global solutions are referred to as Inner Shock Expansions in a Collapsing Envelope (ISECE).

For the case of \( n = 0.9 \) and \( \alpha_0 = 1 \), we have determined \( x_{min1} = 0.95 \), \( x_{min2} = 2.287 \) and \( x_{max} = 3 \); the shock range \( 2.287 < x_{sd} < 3 \) gives sensible classical polytropic “champagne flow” solutions and \( 0.95 < x_{sd} < 2.287 \) gives the ISECE solutions as shown in Figure [I]. For \( n = 0.9 \), we have \( A_s = 2.042 \); with \( x_{sd} = x_{min2} \) and \( A = 1.536 < A_s \), the asymptotic solution is a breeze and should be considered as a classical champagne flow. Given other parameters the same, the situation of \( A > A_s \) would give rise to an asymptotic contraction. The shock location and shock speed can be determined once \( x_{sd} \) is specified. The dimensionless shock position in the self-similar variable represents the shock strength and velocity in dimensional form. The shock velocity reads \( dr_s/dt = nk_1^{1/2}x_{sd}t^{-\alpha} \). The outgoing shock slightly decelerates (for \( n \) slightly less than 1) and the shock velocity is proportional to \( x_{sd} \).

As shown in Figure [I] and Table I with increasing \( x_{sd} \), the upstream variables at the shock front \( v_s \) and \( \alpha_0 \) increase, and the two parameters \( A \) and \( B \) of asymptotic solutions (18) and (19) also increase. This is a fairly common feature, observed in all other cases that we have studied numerically. Different shock positions match with different asymptotic solutions at large \( x \). Once \( A \) and \( B \) are specified, the shock position \( x_{sd} \) is uniquely determined. As \( A \) is the mass parameter and \( B \) is the velocity parameter, \( x_{sd} \), proportional to the shock velocity and strength, is determined not only by initial mass density but also by the initial motion. This differs from the isothermal case when \( B = 0 \) with the mass parameter \( A \) determining the shock behaviour. We expect a faster and stronger shock with a higher initial speed.

We further identify two sub-types of such ISECE solutions: (i) the upstream side has an outward velocity near the shock and an asymptotic inward velocity far from the centre (e.g., solution with \( x_{sd} = 2.2 \) in Figure [I]; (ii) the upstream side has an inward velocity everywhere (e.g., solution with

![Figure 1. The reduced mass density \( \alpha(x) \) (top) and the reduced radial flow velocity \( v(x) \) (bottom) for global “champagne flow” solutions in cases with \( n = 0.9 \) (thus \( \gamma = 1.1 \)) and \( \alpha_0 = 1 \). In both panels the dashed curve represents the sonic critical curve and in the bottom panel the dotted line is \( v = 0 \). The downstream solution is connected with various upstream solutions with light solid curves. The downstream solution is integrated numerically from \( x \to 0^+ \) with a LP type solution. In both panels, the upstream solutions from top to bottom correspond to \( x_{sd} = 3, 2.923, 2.773, 2.623, 2.473, 2.287, 2.2 \) and 2 respectively. We note that the upstream solution with \( x_{sd} = 2.287 \) corresponds to a breeze. Relevant parameters are summarized in Table I.](image-url)
The variation of LP type solutions with different $x_{sd}$ is expected to occur in certain H II regions (e.g., Shen & Lou 2004; Fatuzzo et al. (2004)).

### 3.2 Cases of a Fixed Dimensionless Shock Position

The variation of LP type solutions with different $x_{sd}$ is shown in Figure 2. For $\alpha_0 > 2/3$, $\alpha$ decreases with increasing $x$ while for $\alpha_0 < 2/3$, $\alpha$ increases with increasing $x$. With a larger $\alpha_0$, the LP type solution encounters the sonic critical curve earlier; therefore $\alpha_0$ cannot be too large, otherwise the LP type solution encounters the sonic critical curve before reaching the preset shock position $x_{sd}$. One case of $n = 0.9$ and $x_{sd} = 3$ is shown in Figure 2 with parameters given in Table 2. Here, we have $x_{sd} < 2.5$.

As shown in Figure 2 and Table 2 with the increase of $\alpha_0$, the upstream conditions at the shock front $u_0$ and the mass parameter $A$ of the asymptotic solution increases, but $v_0$ and $B$ do not have a steady trend. As $\alpha_0$ increases from $10^{-5}$ to 0.04, $v_0$ and $B$ decrease, while as $\alpha_0$ increases from 0.04 to 2.5, $v_0$ and $B$ increase (see Table 2). This shows that $v_0$ and $B$ are correlated. The minimum values of $v_0$ and $B$ are associated with $\alpha_0 = 0.04$. Here we show that with a prefixed downstream shock position $x_{sd}$, it is possible that neither breeze nor contraction solutions are allowed. With a $x_{sd}$ as large as 3, all the asymptotic upstream solutions correspond to outflows. In the isothermal analysis of Shu et al. (2002), the mass parameter $A$ tends to zero with $\alpha_0 \rightarrow 0^+$. In polytropic cases, we obtain similar results. According to series expansions (21) and (22), if setting $\alpha_0 = 0$ exactly, the integration gives a trivial solution of $\alpha = 0$.

Based on numerical explorations for $\alpha_0 \rightarrow 0^+$ in isothermal cases, Shu et al. (2002) argued that the self-gravity may be neglected for cases with small central density, and developed another self-similar transformation, viz., the so-called invariant form, in order to model the initial mass density profile other than that of a SIS (i.e., $\rho \propto r^{-2}$). The initial mass density profile $\rho \propto r^{-l}$, where index $l$ does not necessarily equal to 2, can be described by the invariant form when the self-gravity is ignored. We perform a similar reduction with the invariant self-similar transformation for a conventional polytropic gas without the self-gravity (see Appendix A), and show that with $n \neq 1$ (i.e., non-isothermal cases), the power index $l$ must be equal to $2/n$ for a self-
similar form. This mass density profile with a scaling index \( l = 2/n \) is the same as that in a SPS and in asymptotic solutions at large \( x \). In summary, the freedom of choosing \( l \) in the invariant form disappears for non-isothermal cases. From another perspective, since the index \( n \) ranges in \( 2/3 < n < 4/3 \), self-similar polytropic champagne flows can model the initial mass density profile with \( 3/2 < l < 3 \). In other words, the objective to model the initial density profile other than \( l = 2 \) is naturally fulfilled, without the necessity of dropping the self-gravity. The clear advantage of our polytropic approach is that the self-gravity is included in the model consideration. Therefore, to apply our polytropic solutions, \( \alpha_0 \to 0^+ \) is no longer required. From transformation (7), parameter \( \alpha_0 \) is tightly linked with the central mass density \( \rho_0 \) and timescale \( t \), and should have different values in different situations. Therefore polytropic champagne flow solutions are adaptable to a much wider range of astrophysical cloud systems. Moreover, the \( \alpha_0 \to 0^+ \) cases in our polytropic framework can be approximated by a central “void” as discussed in the following section. With a central “void”, we are able to neglect the gravity of the central region where the density is sufficiently low and still consider the self-gravity of the outer more dense gas medium.

In the isothermal case of Shu et al. (2002), the solution in which the outer part is a static SIS represents a limiting solution defining the maximum value of \( \alpha_0 \). In the polytropic cases with \( n = 0.9 \), we can also identify such a limit by requiring \( B = 0 \) for the upstream asymptotic solutions, such that for a preset \( \alpha_0 \) the downstream shock position \( x_{sd} \) is uniquely determined. A family of such “champagne flow” solutions with asymptotic upstream breezes or contractions is shown by solid curves in Figure 3 with parameters summarized in Table 3. With a gradual increase of \( \alpha_0 \), the upstream side varies from outward breezes, to a SPS, and to an inward contraction; this trend leads to a maximum \( \alpha_0 \) if we define an outward breeze for classical “champagne flows”. Here, the critical value \( \alpha_0 = 3.13 \) for an upstream SPS corresponds to the upstream SIS limit in Shu et al. (2002) (i.e., \( \alpha_0 = 7.9 \) in the isothermal case). Naturally, this critical value depends on the choice of \( n \). For \( \alpha_0 > 3.13 \), an upstream asymptotic contraction or an ISECE solution appears (dashed curve in Fig. 3). Thus, isothermal champagne flows form a special family with \( B = 0 \) and \( n = 1 \), referred to as breeze champagne flows. With \( 2/3 < n < 1 \) or \( 2 < l < 3 \), there are many more physically possible champagne flows, for which the upstream side corresponds to asymptotic outflows at large \( x \). The physical meaning of this special solution whose upstream side is the outer part of a static SPS is clear: the outer envelope of gas is initially in a hydrostatic equilibrium, and the expanding shock created by the UV photoionization travels into the static envelope; on the downstream side of this expanding shock, the gas is heated to high temperatures. As the static SPS relies on the scaling parameter \( n \) or the polytropic index \( \gamma \), we expect one single solution for a preset \( n \). From Table 3 we see clearly that with the increase of \( \alpha_0 \), the downstream shock position to obtain a breeze in the upstream, or \( x_{min} \) decreases. Meanwhile, numerical explorations suggest that \( x_{min} \) increases with increasing \( \alpha_0 \). Hence we expect for a sufficiently large \( \alpha_0 \), \( x_{min2} \) would become less than \( x_{min1} \) to forbid ISECE solutions.

4 SIMILARITY POLYTROPIC “CHAMPAGNE FLOWS” WITH CENTRAL VOIDS

We now establish and analyze a new class of “champagne flow” solutions with voids surrounding the centre. We extend solutions from \( x = 0 \) to the dimensionless self-similar expanding boundary \( x^* \) of a void, inside of which there is no mass, i.e., \( m^* = m(x^*) = 0 \). In our notations, superscript “*” attached to variables indicates variables on the void boundary \( x^* \). By expression (3), we have \( v^* = nx^* \). The void boundary conditions are

\[
\alpha = \alpha^*, \quad v = nx^* \quad \text{at} \quad x = x^*.
\]

A Taylor series expansion to the first order around the void boundary \( x = x^* \) of ODEs (12) and (13) yields

\[
v(x) = nx^* + 2(1 - n)(x - x^*) + \cdots, \quad (31)
\]

\[
\alpha(x) = \alpha^* + \frac{n(1 - n)}{\gamma} x^*(x - x^*) + \cdots. \quad (32)
\]

Series expansions (31) and (32) are conspicuously different from series expansions (24) and (25). The proportional coefficient of the asymptotic velocity is no longer \( 2/3 \) but depends on the scaling index \( n \). By translating the origin, the dynamic flow behaves differently. For \( n < 2 \), solutions given by expression (31) are locally above the line \( nx^* - v = 0 \), indicating a positive enclosed mass. Numerical integrations reveal that the solutions are always above the line \( nx^* - v = 0 \) thereafter. Here, we model a gas flow with a central void in the presence of self-gravity and thermal pressure, directly relevant to galactic subsystems such as H II regions.

One can readily obtain the downstream portion of “champagne flow” solutions by numerically integrating coupled nonlinear ODEs (12) and (13) from the void boundary \( x^* \) with asymptotic expansions (31) and (32). All such void

| \( A \) | \( B \) | \( x_{sd} \) | \( \alpha_{sd} \) | \( v_{sd} \) | \( x_{su} \) | \( \alpha_{su} \) | \( v_{su} \) |
|-------|-------|-----------|-------------|-----------|-----------|-------------|-----------|
| 0.9902 | -0.7699 | 2         | 0.8416      | 1.3019    | 2.0693    | 0.2905      | -0.3012   |
| 1.3501 | -0.2267 | 2         | 0.8172      | 1.4266    | 2.2446    | 0.2404      | 1.0044    |
| 1.5357 | 0       | 2.2869    | 0.8068      | 1.4805    | 2.3234    | 0.2586      | 0.2599    |
| 2.0109 | 0.4767  | 2.4732    | 0.7848      | 1.5956    | 2.4964    | 0.2993      | 0.5789    |
| 2.474  | 0.8577  | 2.6232    | 0.7678      | 1.688     | 2.6387    | 0.3336      | 0.8172    |
| 3.0224 | 1.2416  | 2.7731    | 0.7516      | 1.7803    | 2.7832    | 0.369       | 1.0425    |
| 3.6696 | 1.6323  | 2.9231    | 0.7363      | 1.8728    | 2.9292    | 0.4054      | 1.2568    |
| 4.0455 | 1.8367  | 3         | 0.729       | 1.9204    | 3.0047    | 0.4243      | 1.3629    |

Table 1. Data Parameters of Global Polytropic “Champagne Flow” Solutions in Cases with \( n = 0.9 \) and \( \alpha_0 = 1 \).
solutions encounter the sonic critical curve, and the lower the value of $\alpha^*$ is, the later (or at larger $x$) the void solution encounters the sonic critical curve. In order to match with asymptotic solutions of finite density and velocity at large $x$, void solutions must either cross the critical curve smoothly or connect to another branch of solutions via shocks. To model “champagne flows”, we need to construct shocks to obtain global solutions. Global void solutions that cross the sonic critical curve smoothly will be discussed in Lou & Hu (2008, in preparation). We now present the solutions for large and small $\alpha^*$, respectively. A family of semi-complete “champagne flow” solutions with $n = 0.9$, void boundary $x^* = 1$ and $\alpha^* = 5$ is constructed by varying the self-similar shock position as shown in Figure 4. Complementarily, another family of solutions with $\alpha^* = 10^{-4}$ is also constructed and shown in Figure 5. Both outflow, inflow and breeze or contraction as the upstream side are presented. Relevant parameters of these void “champagne flow” solutions are summarized in Table 4.

Similar to no void cases, $\alpha$ decreases with increasing $x$ for large $\alpha^*$, so the density on the void boundary is a local maximum (see Fig. 4); while for small $\alpha^*$, $\alpha$ increases with increasing $x$, so the density maximum is at the downstream side of the shock. The latter corresponds to a shell-like structure in self-similar expansion (see Fig. 5).

With downstream void solutions and upstream outflow and breeze solutions connected by shocks, we establish semi-complete polytropic “champagne flow” solutions with central voids. Similar to LP type solutions with shocks, there are also one maximum $x_{\text{max}}$ and two minimum limits $x_{\text{min}1}$ and $x_{\text{min}2}$ imposed on the downstream shock position $x_{sd}$ in order to obtain “champagne flow” solutions. Systematic numerical explorations for cases of $n = 0.7$, $n = 0.8$ and $n = 0.9$ show that in general $x_{\text{min}2} < x_{sd} < x_{\text{min}1}$, a void solution can be matched with an asymptotic inflow to produce ISECE solutions, while for $x_{sd} > x_{\text{min}2}$, a central void solution can be matched with an asymptotic outflow to produce “champagne flow” solutions. For $x_{sd} = x_{\text{min}2}$, the upstream corresponds to a breeze or a contraction with $B = 0$. The analysis here parallels that in the previous section for cases without central voids; in particular, the parameters $x_{\text{min}1}$ and $x_{\text{min}2}$ are determined not only by $n$ and $\alpha^*$, but also by the expanding void boundary $x^*$. Numerical explorations suggest that for a certain $n$, with the increase of $\alpha^*$, $x_{\text{min}1}$ increases and $x_{\text{min}2}$ decreases. Hence for a sufficiently large $\alpha^*$, we expect $x_{\text{min}2} < x_{\text{min}1}$ for which ISECE solutions are not allowed. This is consistent with polytropic cases without central voids.

### Table 2. Data Parameters of Global Polytropic “Champagne Flow” Solutions in Cases with $n = 0.9$ and $x_{sd} = 3$.

| $\alpha_0$ | $A$     | $B$     | $x_{sd}$ | $\alpha_d$ | $v_d$   | $x_{su}$ | $\alpha_u$ | $v_u$   |
|-----------|---------|---------|----------|------------|---------|----------|------------|---------|
| 0.00001   | 0.0002532 | 1.145   | 3        | 0.0001     | 2.3469  | 3.0609   | 0.000030184 | 1.5294 |
| 0.001     | 0.01243 | 0.7205  | 3        | 0.0049     | 2.2821  | 3.0603   | 0.001372   | 1.2353 |
| 0.01      | 0.0912  | 0.4713  | 3        | 0.0358     | 2.2449  | 3.0746   | 0.009752   | 1.0536 |
| 0.04      | 0.3045  | 0.3899  | 3        | 0.1154     | 2.2135  | 3.0715   | 0.03192    | 0.9641 |
| 0.5       | 2.4053  | 1.1453  | 3        | 0.6048     | 2.0444  | 3.0197   | 0.2556     | 1.1567 |
| 1         | 4.0455  | 1.8367  | 3        | 0.729      | 1.9204  | 3.0047   | 0.4244     | 1.3629 |
| 1.5       | 5.5009  | 2.4063  | 3        | 0.7512     | 1.8166  | 3.008    | 0.5556     | 1.5059 |
| 2         | 6.9587  | 2.9308  | 3        | 0.7414     | 1.7274  | 3        | 0.6606     | 1.6083 |
| 2.5       | 8.5684  | 3.4633  | 3        | 0.7588     | 1.562   | 2.8444   | 0.7086     | 1.4913 |

### Table 3. Parameters of Global Polytropic “Champagne Flow” Solutions with $n = 0.9$ and Upstream Breeze or Contraction ($B = 0$).

| $\alpha_u$ | $A$     | $B$     | $x_{sd}$ | $\alpha_d$ | $v_d$   | $x_{su}$ | $\alpha_u$ | $v_u$   |
|-----------|---------|---------|----------|------------|---------|----------|------------|---------|
| 0.2209    | 0.7550  | 0       | 2.6250   | 0.3571     | 1.8473  | 2.6873   | 0.0990     | 0.5153 |
| 2/3       | 1.3240  | 2       | 2.3986   | 0.6667     | 1.5991  | 2.4419   | 0.2045     | 0.3407 |
| 1         | 1.5357  | 2       | 2.2869   | 0.8068     | 1.4805  | 2.3234   | 0.2586     | 0.2599 |
| 2         | 1.8643  | 2       | 2.0733   | 1.0783     | 1.2563  | 2.0996   | 0.3734     | 0.1069 |
| 3.13      | 2.0420  | 1       | 1.9264   | 1.2773     | 1.1036  | 1.9475   | 0.4643     | 0      |
| 5         | 2.1924  | 0       | 1.7706   | 1.5091     | 0.9420  | 1.7873   | 0.5746     | −0.1163 |

### 5 ANALYSIS AND DISCUSSION

#### 5.1 Comparison with Numerical Simulations

To adapt our self-similar solutions for modelling an astrophysical cloud system, we need to first specify the parameter $k$ related to the sound speed squared. By varying $k$, one can model clouds of different scales using a single self-similar solution. Parameter $k$ is determined by the thermodynamic parameters, including thermal pressure $p$, mass density $\rho$ and temperature $T$. A useful relation for a conventional polytropic gas derived from transformation (7) is

$$k = \frac{p}{\rho^\gamma (4\pi G)^{\frac{1}{2}}} = \frac{k_B T}{\mu p^{\gamma-1} (4\pi G)^{\frac{1}{2}}} ,$$

where $\mu$ is the mean molecular mass of gas particles. For a typical value provided by the classification of Habing & Israel (1979), the UC H II regions have an electron number density $n_e > 3000$ cm$^{-3}$ (corresponding to a mass...
Table 4. Polytropic “Champagne Flow” Solutions with a Central Void inside $x^* = 1$ in the Case of $n = 0.9$

| $\alpha^*$ | $A$      | $B$      | $x_{sd}$ | $\alpha_d$ | $v_d$      | $x_u$      | $\alpha_u$ | $v_u$      |
|------------|----------|----------|-----------|------------|------------|------------|------------|------------|
| $10^{-4}$  | 0.000106 | $-0.3813$| 1.8       | $1.4734 \times 10^{-4}$| 1.3322     | 1.9058     | $2.7895 \times 10^{-5}$| 0.1058     |
| $10^{-4}$  | 0.000191 | 0        | 2.068     | $1.8377 \times 10^{-4}$| 1.5386     | 2.1442     | $4.2673 \times 10^{-5}$| 0.4893     |
| $10^{-4}$  | 0.0014   | 1.0833   | 3.088     | $0.000517$ | 2.398      | 3.152      | 0.00015    | 1.501      |
| $10^{-4}$  | 0.0991   | 3.5315   | 5.701     | 0.0114     | 4.701      | 5.839      | 0.0032     | 3.669      |
| 5          | 2.2512   | $-1.0512$| 1.7       | 2.8906     | 1.0956     | 1.8424     | 0.4646     | $-1.2708$  |
| 5          | 2.9388   | 0        | 1.818     | 2.4555     | 1.1047     | 1.8777     | 0.5978     | $-0.5632$  |
| 5          | 5.2894   | 1.7273   | 2         | 1.8447     | 1.0816     | 2.0101     | 0.8364     | 0.2166     |

Figure 3. “Champagne flow” solution with a LP type downstream and the upstream part as a breeze or contraction for $n = 0.9$ (i.e., $\gamma = 1.1$). The top panel shows the density and the bottom panel shows the velocity. In both panels the dotted curve is the sonic critical curve. The solutions are integrated from $\alpha_0 = 0.2209$, 2/3, 1, 2, 3.13 and 5 (from bottom to top in the top panel and from top to bottom in the bottom panel), and the downstream shock positions $x_{sd}$ are carefully chosen to let the upstream solution correspond to the asymptotic solutions with $B = 0$. Relevant parameters are summarized in Table 3. For the solution with $\alpha_0 = 3.13$, the corresponding upstream is a SPS. For $\alpha_0 > 3.13$, the upstream contracts (dashed curve), while for $\alpha_0 < 3.13$, the upstream is a breeze. For a breeze “champagne flow”, we require $\alpha_0 \geq 3.13$ (solid curves).

Figure 4. The reduced mass density $\alpha(x)$ (top) and the reduced radial flow velocity $v(x)$ (bottom) for semi-complete “champagne flow” solutions with a central void inside $x^* = 1$ in the case of $n = 0.9$ and density on the void boundary $\alpha^* = 5$. In both panels, the dashed curve stands for the sonic critical curve. In the bottom panel the dotted line stands for the line $v = 0$. The solid curve on the upper left of the sonic critical curve is the downstream void solution and the solid curves on the lower right of the sonic critical curve are the corresponding upstream solutions with the downstream shock position $x_{sd} = 1.7$ (inflow), 1.818 (contraction) and 2 (outflow). In this case, $x_{min2} \approx 1.818$ and $x_{min1} = 1.408$. Numerical data about these solutions are tabulated in Table 3.

value of the polytropic index $\gamma$ does influence very much the resulting $k$ and thus $k$ should be evaluated specifically. For nearly isothermal cases, with relation (33), we estimate $k$ for UC and Compact H II regions to be $k \sim 10^{11} \sim 10^{12}$ cgs unit. One should be aware that $k$ and thus $k$ vary with the gas temperature and density in a cloud. Here, we presume a constant $k$ to convert self-similar variables to real space variables as a first approximation.

We now compare our self-similar solutions of quasi-spherical symmetry with previous numerical simulations. Tenorio-Tagle et al. (1986) performed a numerical study for a similar scenario as ours, i.e. a nascent central massive protostar ionizes and heats the ambient neutral gas and then leads to “champagne flows”. In their simulation, the radia-
In the bottom panel, the dotted line stands for the line both panels, the dashed curve represents the sonic critical curve.

In the gravitational field at a uniform temperature all parameters into our self-similar form of solutions, we first ex-
changed. The residual gas is initially in a hydrostatic equilibrium
pectedly choose $n = 0.9$ (hence the parameter $\gamma = 2 - n = 1.1$) to model the initial density profile $l = -2/n = -2.22$. Note that parameter $n$ for the “champagne flow” solution is slightly different from the value we obtain from the fitting. With $\rho_0$ and $T_0$ specified in the simulation, we estimate $k_d = 3.6 \times 10^{15}$ cgs unit with expression (53). We still have the freedom to require the shock traveling to $r_s = 2.51 \times 10^{21}$ cm at $t = 1.3 \times 10^9$ yr, giving $x_{sd} = 1.86$. With these parameters, we model “champagne flows” in diffuse H II regions with radius $r$ up to $10^{21}$ cm (i.e., $\sim 300$ pc). The full solution is shown in Figure 7. The timescale of $1.3 \times 10^7$ yr is regarded as the duration of the formation phase and the initial time for a “champagne flow”, and the timescale $5.1 \times 10^6$ yr is regarded as the lifetime of a “champagne flow”.

The orders of magnitude of all variables are consistent with typical values; e.g., the expansion velocity is several tens km s$^{-1}$ and the temperature is about $\sim 10^4$ K. The enclosed mass at $r = 10^{21}$ cm ($\sim 300$ pc) is about 850$M_{\odot}$, consistent with the value of $\sim 800M_{\odot}$ given by the numerical simulation and with the typical value for diffuse H II regions. The enclosed mass does not vary with time $t$, confirming the cut-off radius chosen at $r = 10^{21}$ cm. As time evolves, the central number density decreases from $10^{-0.5}$ to $10^{-3.5}$ cm$^{-3}$.
and the central thermal pressure decreases accordingly from $10^{-12}$ to $10^{-15}$ dyne cm$^{-2}$.

We can also compare variable profiles with the case $F$ of Tenorio-Tagle et al. (1986). The velocity profiles are very similar and we clearly see an expanding shock. As time evolves, the shock strength becomes weaker. In both numerical simulation and our model analysis, we observe a density peak on the downstream side of the shock and a significant temperature gradient on the upstream side, which cannot be accounted for by previous isothermal solutions. The upstream density and pressure profiles are also similar; however, the downstream density, pressure and temperature profiles (near centre) are somewhat different. In Tenorio-Tagle et al. (1986), temperature, pressure and density are initially very uniform behind the champagne shock

but at the end of the calculations show large inward gradients. In Figure 7, our model also produces quasi-uniform temperature, pressure and density behind the shock at the beginning ($t = 1.3 \times 10^5$ yr), but we do not observe large inward gradients at the end. This difference is primarily due to the different physical assumptions adopted in the simulation and our self-similar solutions. Tenorio-Tagle et al. (1986) treated the gas dynamics in protogalactic clusters and neglected the gas self-gravity as the gas mass is only about 0.1 percent that of the stars. As shown above, our self-similar solutions neglect the gravity of the central massive protostar but include the self-gravity effect. Another important factor that introduces such difference is that in the simulation both forward champagne shock and reverse rarefaction wave are taken into account. Our self-similar model can accommodate forward moving shock, so we only have the principle outgoing champagne shock.

We calculate the total energy $E_{\text{total}}$ defined as the energy of the gas under consideration in an infinite space. The total energy at time $t$ can be expressed as

$$E_{\text{total}} = E_K + E_G + E_I$$

$$= \int_{r_{\text{out}}}^{r_{\text{in}}} \left( \frac{3}{2} \rho u^2 - \frac{GM_0}{r} + \frac{7}{2} \right) 4\pi r^2 dr$$

$$= \frac{3}{5/2} \frac{5}{3} \frac{n-4}{2} \int_{x_{\text{in}}}^{x_{\text{out}}} \left[ \alpha \nu^2 x^2 - \frac{2}{(3\alpha-2)} \alpha x^3 (n x - v) \right] + \frac{i\alpha x^2}{x} dx ,$$

(34)

where $E_K$, $E_G$ and $E_I$ are the kinetic, gravitational and internal energies of the gas, respectively, and $r_{\text{in}}$, $r_{\text{out}}$, $x_{\text{in}}$, $x_{\text{out}}$ are the inner and outer boundaries of the gas under consideration, $i$ is the degree of freedom of the gas particle presumed to be 3. Note that one fixed $r_{\text{out}}$ at different times corresponds to different values of $x_{\text{out}}$. In this solution, $E_{\text{total}} = 5.5 \times 10^{48}$ erg at $t = 1.3 \times 10^5$ yr, and $E_{\text{total}} = 8.1 \times 10^{48}$ erg at $t = 5.1 \times 10^6$ yr, so there is net energy input. Especially, we see the kinetic energy $E_K = 6.4 \times 10^{47}$ erg at $t = 1.3 \times 10^5$ yr, indicating a small fraction of the total energy, and $E_K = 3.9 \times 10^{48}$ erg at $t = 5.1 \times 10^6$ yr, indicating a fairly large fraction of the total energy. The increase of kinetic energy shows clearly the development of a champagne flow. The gravitational energy is of order $10^{44}$ erg, much less than the kinetic energy throughout this duration. This confirms that the gas is not bounded and must have an outflow. We are also able to consider qualitatively the local energy exchange throughout the self-gravitating gas with relation (16). In Figure 7, $\partial p/\partial r$ is positive on the downstream side and negative on the upstream side. For $\gamma = 1.1$, the downstream and upstream sides locally loses and gains energy, respectively. In summary, the profiles on the order of magnitudes, and the time evolution of our self-similar solution in modelling this case are grossly consistent with the numerical simulation result of Tenorio-Tagle et al. (1986), which lends support to our polytropic self-similar “champagne flow” solution as a gross description of dynamics of H II regions.

Recent studies further suggest that the inclusion of stellar winds is also important and even necessary sometimes in understanding the large-scale dynamics of H II regions. Córondo (1997) found that a shocked stellar wind in the central region produces important morphological differences.

Figure 7. Self-similar “champagne flow” solutions for radius up to $10^{21}$ cm ($\sim 300$ pc) at time $t = 1.3 \times 10^5$ yr (solid curves) and $t = 5.1 \times 10^6$ yr (dashed curves). From top to bottom, the panels show number density, velocity, pressure, enclosed mass and the central thermal pressure decreases accordingly from $10^{-12}$ to $10^{-15}$ dyne cm$^{-2}$.
as compared to windless cases. Moreover, Comerón (1997) suggested that the spatial scale of an H II region undergoing “champagne flow” is systematically larger and the gas flow is generally faster as driven by a central stellar wind. Arthur & Hoare (2006) provided two-dimensional cylindrical radiative-hydrodynamic simulations of cometary H II regions using champagne flow models, by taking into account of strong stellar winds from the central ionizing star. In these simulations, the hydrodynamics and radiative transfer are coupled through an energy equation whose source term depends on the photoionization heating and radiative cooling rates; while with our polytropic approach, complicated energetic processes are relegated to the choice of \( \gamma \). Arthur & Hoare (2006) studied the gas dynamics up to a radius of 0.03 pc central void at a time of ∼ 200 yr after the triggering of UV ionizing photons and powerful stellar winds; a stellar wind bubble around the central mass point, and with dimensions in self-similar transformation (7), \( \gamma \) is different. In dimensionless form, \( \mathcal{M}(t) \) stands for a central mass accretion rate. For \( \mathcal{M}(t) \) scales as the central mass accretion rate. For \( \mathcal{M}(t) = 0.5 \times 10^{-7} \), the entire global solution corresponds to a free-fall solution of the form (4). The free-fall solution crosses the sonic critical curve smoothly at \( x = 0.3237 \), and can also be connected to the upstream solutions via shocks at various locations. The free-fall solution crosses the line \( v = 0 \) at \( x_{\text{stg}} = 0.74 \). This stagnation radius expands with time in a self-similar manner; inside the stagnation radius the gas falls inwards, while outside the stagnation radius the gas expands outwards. Therefore if \( x_{\text{stg}} > x_{\text{sh}}, \) the entire global solution corresponds to an inflow (solution 4 of Figure 9). This situation describes an accretion shock during a protostar formation phase. If \( x_{\text{sh}} > x_{\text{sh}}, \) the outer part of the downstream side is an outflow. This describes the scenario that the shock sweeps up that as time evolves, the void boundary expands, meanwhile the density and pressure in the vicinity of void boundary decrease by several orders of magnitude, consistent with the simulation. However, we see a very high density near the “champagne flow” shock on the downstream side, and as time evolves, the density profile becomes more and more smooth. For our self-similar “champagne flow” model with central expanding voids, the velocity can rise up to several hundred km s\(^{-1}\). In Figure 8 we also see clearly that the case is non-isothermal, and on the downstream side of the shock the temperature is the highest as expected. We note that at \( t = 800 \text{ yrs}, \) the shock is at ∼ 0.3 pc, beyond the scale of UC or compact H II regions. In reality, a champagne shock is so fast that even at a short timescale of ∼ 800 yrs the shock is well in the surrounding diffuse interstellar medium (ISM).

Compared with numerical simulations, the advantage of our semi-analytical self-similar approach is clear. We can generate self-similar shock solutions to model different H II regions by varying a few parameters. The self-similar processes and shock solutions of this paper describe the basic hydrodynamics of polytropic “champagne flows” and serve as test cases for bench marking numerical simulations.

### 5.2 Asymptotic Free-Fall Solutions around a Central Protostar

So far we have constructed “champagne flow” solutions with LT type asymptotic solutions on the downstream side as \( x \to 0^+ \), because such asymptotic solution satisfies boundary condition (23). Complementarily, free-fall asymptotic solutions (20) and (21) at \( x \to 0^+ \) represent gas inflow and collapse during the protostar formation phase; and the surrounding gas and the infall momentum associated with the star formation process may be sustained for a while during the evolution after the onset of stellar nuclear burning and UV photoionization of the surrounding gas. Cochran & Ostriker (1977) have investigated consequences of the birth of a massive star within a dense cloud with a free-fall density profile, and found that the radiation pressure from the star sweeps up grains from the infalling gas to form a dust shell which bounds the H II region. Here, we utilize such free-fall solutions as the downstream side and construct global solutions with shocks to model possible dynamic evolutions of H II regions surrounding a nascent protostar in nuclear burning. We present such solutions in Figures 9 and 11 where parameter \( m(0) \) for free-fall solutions is different. In dimensionless form, \( m(0) \) stands for a central mass point, and with dimensions in self-similar transformation (4) \( M(0, t) \propto t^{3n-3} m(0) \); therefore \( m(0) \) scales as the central mass accretion rate. For \( m(0) = 0.546 \) (Figure 9), the free-fall solution crosses the sonic critical curve smoothly at \( x = 0.3237 \), and can also be connected to the upstream solutions via shocks at various locations. The free-fall solution crosses the line \( v = 0 \) at \( x_{\text{stg}} = 0.74 \). This stagnation radius expands with time in a self-similar manner; inside the stagnation radius the gas falls inwards, while outside the stagnation radius the gas expands outwards. Therefore if \( x_{\text{sh}} < x_{\text{stg}}, \) the entire global solution corresponds to an inflow (solution 4 of Figure 9). This situation describes an accretion shock during a protostar formation phase. If \( x_{\text{sh}} > x_{\text{sh}}, \) the outer part of the downstream side is an outflow. This describes the scenario that the shock sweeps up...
the gas and turns the gas from infall to expansion on the downstream side near the shock front. Similar to the situation with downstream LP type solutions, there exists one specific $x_{sl}$, from which the upstream solution is a breeze with $B = 0$. In this case, $x_{sl} = 1.7747$ gives an upstream breeze (solution 2 of Figure 9). Thus for $x_{sl} < 1.7747$, the upstream side corresponds to an asymptotic inflow far from the centre (solution 3 of Figure 9) and for $x_{sl} > 1.7747$, the upstream side corresponds to an asymptotic outflow (wind) far from the centre (solution 1 of Figure 9). Another example shown in Figure 10 has $m(0) = 4.638$. This free-fall solution does not cross the sonic critical curve smoothly and can be connected with upstream solutions via shocks. In an analogous manner, we show the possibility to obtain an outflow (solution 1 of Figure 11), breeze (solution 2 of Figure 11) and inflow (solution 3 of Figure 11) for the upstream side. Solutions 1 and 2 of Figure 9 and solution 1 of Figure 11 have asymptotic outflow or breeze on the upstream side and in the outer part of the downstream side, which is very similar to champagne flow solutions obtained with a downstream LP type, with different behaviours in central regions. With a free-fall asymptotic solution, the gravity of the central massive star is not neglected and the gas immediately surrounding the massive protostar still undergoes infall when the outer envelope starts to expand. Hence, solutions with free-fall centre are plausibly suitable to describe the early stage of “champagne flows”. In general, with central free falls on the downstream side, we can also obtain asymptotic outflow, inflow, breeze and contraction for the upstream side, by varying the downstream shock position $x_{sl}$ in a proper range.

Central free-fall solutions describe the core collapse phase in star formation. For this purpose, Wang & Lou (2008) explored such solutions with free-fall inner core and an inflow or outflow in the outer envelope, for general polytropic cases in their Figure 2. The inner and outer portions are connected by magnetohydrodynamic (MHD) shocks. Such shocks are interpreted as accretion shocks, typically found in a star formation process, or around accreting black holes. Here we specifically emphasize that such shocks may also arise by the UV photoionization of ambient medium surrounding a nascent protostar. Under certain situations, the UV flux from the burning star might not be intense and rapid enough to turn the surrounding gas from infall to expansion by ionization and heating. Meanwhile the ionization front (IF) creates a weak shock traveling outwards, and the upstream side may have an outward velocity. Another possibility is that the gravity of the central star is so large that the gas immediately surrounding the star keeps falling towards the protostar, but the outer part of the downstream side and the corresponding upstream side expand. In summary, with different initial conditions of gas and different physical
conditions of a burning protostar, the radiative influence of the nascent protostar on the dynamic evolution of the surrounding gas may give rise to various self-similar solutions, including the classical champagne flow solutions, the ISECE solutions and the inner free-fall with outer inflow/outflows or contraction/breeze solutions. The solutions constructed in this paper (classical champagne flows) is suitable for situations that the gas is initially static and the protostar ionizes the entire gas immediately, and then the gas begins to expand in a “champagne phase” with an outgoing shock.

6 CONCLUSIONS
We present newly established self-similar polytropic shock solutions with and without central voids to model “champagne flows” in H II regions featuring various asymptotic dynamic behaviours. As a substantial generalization of the isothermal model of Tsai & Hsu (1995) and Shu et al. (2002), we found similarities and differences in self-similar polytropic processes. Our general polytropic “champagne flow” model allows a much larger freedom to choose the polytropic index $\gamma \geq 1$ for $2/3 < n < 2$; for a conventional polytropic gas as a subclass of examples, the power-law index $l$ of the initial mass density profile $\rho \propto r^{-l}$ is linked to $\gamma$ by $l = 2/n = 2/\gamma$. Together, our model is adaptable to a wide range of initial mass density profile with $l < 3$ for H II regions. For conventional polytropic cases of $1 < \gamma < 4/3$ (i.e., $2/3 < n < 1$ and $2 < l < 3$), we have more freedom for convergent initial conditions. In this
fashion, our conventional polytropic shock flow solutions are
determined not only by the initial mass density profile (i.e.,
mass parameter $A$), but also by the motion at the very early
stage (i.e., velocity parameter $B$). The dimensionless shock
positions or the dimensional shock speed and strength are
determined by the initial conditions related to $A$ and $B$ and
the central density at $\alpha_0$. Our self-similar shock flow solutions
give a plausible description for the “champagne flow”
phase for the dynamics of H II regions. We conclude that
general polytropic “champagne flows” with the initial den-
sity power-law index $1 < l < 3$ may evolve in a self-similar
manner.

We have established novel “champagne flow” shock solu-
tions with an expanding void surrounding the centre to
model a certain cloud core whose inner part has fallen into
a nascent protostar. We observe that the evolution of central
void boundary plays an important role in determining the
asymptotic solution to approach and the general behaviour
of solutions as well. With even one more free parameter, the
“champagne flow” shock solutions with central voids can
model the dynamics of H II regions more realistically, in-
cluding the effect of central stellar wind bubbles.

We have further explored possibilities of asymptotic in-
flows or contractions far from the cloud centre. In addition,
we also establish global shock solutions with the asym-
ptotic free-fall solution approaching the centre. In general,
by varying dimensionless shock position, we connect the down-
stream side, with either LP type solutions, EdS solutions or
free-fall solutions, to upstream solutions which eventually
merge into asymptotic outflow, breeze, contraction and
inflow. Within the theoretical framework of the self-similar
polytropic fluid, global shock solutions with different be-
haviours correspond to different forms of hydrodynamic evo-
lation of H II regions after the nascence of a central mas-
setic protostar. Apparently, even within the framework of self-similarity, dynamic evolution of polytropic H II regions
depends on the initial and boundary conditions of molecu-
lar clouds. Numerical simulations are needed to probe and
connect various self-similar evolution phases.

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APPENDIX A: AN INVARIANT FORM

Transformation (2) leads to an initial density profile $\rho \propto r^{-2/n}$ as the outer part of SPS. To model an initial mass
density profile other than a SIS (e.g., Galli et al. 1999) in
the self-similarity framework, Shu et al. (2002) developed an
invariant self-similar transformation for an isothermal gas by
neglecting the self-gravity. We extend below this invariant
transformation to a conventional polytropic gas.

The independent similarity variable is $x = r/(k^{1/2}t^{n})$
and the reduced radial velocity $v(x)$ still reads as $v = u/(k^{1/2}t^{n-1})$. But the mass density $\rho$ is generalized to

$$\rho(r,t) = \frac{D}{r^\gamma} R(x), \quad (A1)$$

where $R(x)$ is a new reduced density dependent on $x$ only,
$t$ is a scaling index and $D$ is a constant parameter to be
determined. With the conventional polytropic relation $p = \kappa \rho^\gamma$ for constant $\kappa$ and $\gamma$, the thermal pressure $p$ is

$$p(r,t) = \kappa \frac{D^\gamma}{r^\gamma} R^\gamma. \quad (A2)$$

We now determine the relation between $k$ and $\kappa$ from the
radial momentum equation without self-gravity
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} \quad (A3)
\]
The left-hand side (LHS) of equation (A3) relates to the sound scaling parameter \( k \) and the right-hand side (RHS) relates to the specific entropy coefficient \( \kappa \). We substitute self-similar transformation (A1) and (A2) into both sides of the radial momentum equation (A3) to obtain
\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} = k^{1/2} \rho^{n-2} [(n-1)u + (v-nx)u'], \quad (A4)
\]
\[
-\frac{1}{\rho} \frac{\partial \rho}{\partial r} = -\kappa D^{-1} \gamma (k^{-1/2} R^{-n})^{(\gamma-1)/2} x^{-\gamma-1} \times \left( -\frac{1}{x} R^{\gamma-1} + R' R^{\gamma-2} \right), \quad (A5)
\]
where the superscript \('\) over \( v(x) \) and \( R(x) \) indicates the first derivative in \( x \). By expressions (A3) and (A5) and in order to remove the explicit \( t \) dependence in the self-similar form of equation (A3), we require
\[
-n[(\gamma-1)l+1] = n - 2. \quad (A6)
\]
Together with \( n + \gamma = 2 \) for a conventional polytropic gas, we obtain \( l = 2/n \) for \( n \neq 1 \). For \( n = 1 \), there is no constraint on \( l \). For a conventional polytropic gas, the possible initial mass density profile described by this invariant self-similar transformation is not arbitrary, but is preset by the scaling index \( n \) or by the polytropic index \( \gamma \). Here \( D \) provides a measure for the magnitude of density profile and \( \kappa \) is the constant in the polytropic equation of state; both parameters are determined physically. For simplicity of the formulation, we may choose the sound parameter \( k \) so that
\[
\kappa = k^{1/n} D^{n-1}. \quad (A7)
\]
This relation is verified by a dimensional analysis. As a result, these parameters disappear in the following ODEs,
\[
[(nx - v)^2 - \gamma x^{2-1} R^{\gamma-1}] R' = \frac{R}{x} \left( (l - 2)(v - nx) + (n - 1)x \right) - \gamma lx^{1-l} R^{\gamma}, \quad (A8)
\]
\[
[(nx - v)^2 - \gamma x^{2-1} R^{\gamma-1}] u' = 2\gamma x^{1-l} R^{\gamma} (v - x) + (n - 1)v(nx - v). \quad (A9)
\]
For \( n \neq 1 \) and \( l = 2/n \), coupled equations (A8) and (A9) describe non-isothermal self-similar flows of a conventional polytropic gas. For \( n = 1 \) and \( \gamma = 1 \), the value of \( l \) is arbitrary; for \( l = 2 \), these two coupled nonlinear ODEs reduce to the isothermal formulation of Shu et al. (2002) with two decoupled ODEs for \( l = 2 \). For \( n = 1 \) and \( \gamma = 1 \) with \( l \neq 2 \), we need to use the two decoupled ODEs (32) and (33) in Section 4 of Shu et al. (2002). In contrast to the isothermal case, the nonlinear ODEs here remain coupled for \( \gamma \neq 1 \). In summary, with our self-similar transformation, hydrodynamics for a spherical gas with ignorable gravity can be expressed by a set of two coupled nonlinear ODEs (A8) and (A9).

In cases with \( \gamma \neq 1 \) and for \( x \to +\infty \), the asymptotic boundary conditions are \( v \to 0 \) and \( R \to 1 \) for \( n < 1 \). The asymptotic solution at large \( x \) yields
\[
v = H x^{1-1/n} + \frac{2\gamma}{n} x^{1-2/n} + \cdots, \quad R = 1 + 3H \left( \frac{1}{n} - 1 \right) x^{-1/n} + \cdots, \quad x \to +\infty, \quad (A10)
\]
where \( H \) is an arbitrary constant but should be chosen for a positive mass density everywhere. The condition \( l = 2/n \) has already been taken into account in this solution. This form of asymptotic solution at large \( x \) differs from equation (34) of Shu et al. (2002), because for non-isothermal cases the leading terms in nonlinear ODEs (A8) and (A9) are different; however, the second term on the RHS of the expression of \( v \) reduces to the \( v \) expression in equation (34) of Shu et al. (2002) for \( n = 1 \) and \( \gamma = 1 \). The form (A10) is similar to the form of (18) and (19) in the overall scaling. The integration constant \( H \) in the invariant form is an analogy of the ‘velocity parameter’ \( B \), while the ‘mass parameter’ \( A \) is already absorbed into the parameter \( D \) in the invariant form. To make this asymptotic solution applicable, we should require \( H = 0 \) in cases of \( n < 1 \) and thus \( \gamma > 1 \). The boundary conditions at \( x \to +\infty \) correspond to the initial conditions of the fluid. Therefore, the initial density scales as \( r^{-1} \). More precisely, this new self-similar transformation requires the initial density profile to scale as \( r^{-2/n} \), identical with the scaling we obtain in the main text using self-similar transformation (7). For \( x \to 0^+ \), the asymptotic solution reads
\[
v \sim 2x/3, \quad R \sim I x^l, \quad \text{as} \quad x \to 0^+, \quad (A11)
\]
where \( I > 0 \) is an arbitrary parameter for a positive mass density. This \( v \) expression of asymptotic solution (A11) is independent of \( l \), different from the \( v \) expression (35) in Shu et al. (2002), but it is similar to asymptotic solution (24) in the main text; the \( R \) expression of our asymptotic solution is similar to the \( R \) expression (35) of Shu et al. (2002), and the parameter \( I \) here is proportional to the central reduced mass density \( \alpha_0 \). In the isothermal case with \( n = 1 \) and \( \gamma = 1 \), relation (A10) is satisfied automatically and parameter \( l \) is thus arbitrary. Here lies a fundamental difference between the conventional polytropic case and the isothermal case.