Stochastic functional differential equations driven by G-Brownian motion with monotone nonlinearity

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Abstract

By using the Picard iteration scheme, this article establishes the existence and uniqueness theory for solutions to stochastic functional differential equations driven by G-Brownian motion. Assuming the monotonicity conditions, the boundedness and existence-uniqueness results of solutions have been derived. The error estimation between Picard approximate solution $y^k(t)$ and exact solution $y(t)$ has been determined. The $L^2_G$ and exponential estimates have been obtained. The theory has been further generalized to weak monotonicity conditions. The existence, uniqueness and exponential estimate under the weak monotonicity conditions have been inaugurated.

Key words: Existence and uniqueness, boundedness, error estimation, $L^2_G$ and exponential estimates, stochastic functional differential equations, G-Brownian motion.

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1 Introduction

The existence and uniqueness theory for solutions to stochastic dynamical systems is always a significant theme and has received a huge attention, for instance see [1, 3, 5, 15, 20, 23, 24, 27]. In several evolution phenomena, the hereditary properties such as time-lag, time-delay or after-effect arise in the variables [2, 17, 21, 22, 31]. This naturally leads us to use stochastic functional differential equations which take into consideration the history of the system. Assuming the growth and Lipschitz conditions, Ren et al. [26] and Faizullah [13] gave the existence-uniqueness results for solutions to stochastic functional differential equations in the G-framework (G-SFDEs). The idea was generalized by Faizullah to G-SFDEs with non-Lipschitz conditions [9]. He further extended the theory to develop the $p$th moment estimates for the solutions to G-SFDEs [10, 11]. The existence-uniqueness theory for neutral stochastic functional differential equations driven by G-Brownian motion (G-NSFDEs) was developed by Faizullah [8] and Faizullah et al. [7]. The
exponential stability, the $p$th moment exponential estimate and stability with markovian switching for solutions to G-NSFDEs were respectively given by Zhu et al. [32], Faizullah et al. [6] and Li et al. [18]. However, to the best of our knowledge, no text is available on the existence, uniqueness, exponential estimate, error estimation for Picard approximate solution with monotone nonlinearity conditions for functional differential equation driven by G-Brownian motion. The aim of this article is to inaugurate the mentioned unavailable literature. In addition, the existence, uniqueness, $L^2_G$ and exponential estimates for solutions of G-SFDEs with weak nonlinear monotonicity conditions are studied. Let $\mathcal{C}_q((t,0];\mathbb{R}^n)$ be the collection of continuous functions from $(-\infty,0]$ to $\mathbb{R}^n$, then for a given number $q > 0$ we define the phase space with fading memory $C_q((t,0];\mathbb{R}^n)$ by

$$C_q((t,0];\mathbb{R}^n) = \{ \psi \in \mathcal{C}_q((t,0];\mathbb{R}^n) : \lim_{\vartheta \to -\infty} e^{q\vartheta} \psi(\vartheta) \text{ exists in } \mathbb{R}^n \}.$$  

This space is complete with norm $\| \psi \|_q = \sup_{-\infty < \vartheta \leq 0} e^{q\vartheta} |\psi(\vartheta)| < \infty$. The space $C_q((t,0];\mathbb{R}^n)$ is a Banach space of bounded and continuous functions and $C_{q_1} \subseteq C_{q_2}$ for any $0 < q_1 \leq q_2 < \infty$. Let $\mathcal{B}(C_q)$ be the $\sigma$-algebra generated by $C_q$ and $C_q^0 = \{ \psi \in C_q : \lim_{\vartheta \to -\infty} e^{q\vartheta} \psi(\vartheta) = 0 \}$. Denote by $L^2(C_q)$ (resp. $L^2(C_q^0)$) the space of all $\mathcal{F}$-measurable $C_q$-valued (resp. $C_q^0$-valued) stochastic processes $\psi$ such that $E[\| \psi \|^2] < \infty$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space, $\mathcal{B}(t)$ be an $n$-dimensional G-Brownian motion and $\mathcal{F}_t = \sigma\{ B(v) : 0 \leq v \leq t \}$ be the natural filtration. Assume that the filtration $\{ \mathcal{F}_t ; t \geq 0 \}$ assures the usual conditions. Let $\mathcal{P}$ be the set of all probability measures on $(C_q,\mathcal{B}(C_q))$ and $L_q(C_q)$ be the collection of all continuous bounded functionals. Let $f : [0,T] \times C_q((t,0];\mathbb{R}^n) \to \mathbb{R}^n$, $g : [0,T] \times C_q((t,0];\mathbb{R}^n) \to \mathbb{R}^{n \times m}$ and $h : [0,T] \times C_q((t,0];\mathbb{R}^n) \to \mathbb{R}^{n \times m}$ be Borel measurable. Consider the following stochastic functional differential equation driven by G-Brownian motion

$$dy(t) = f(t,y_t)dt + g(t,y_t)dB(t) + h(t,y_t)dB(t), \quad (1.1)$$

on $t \in [0,T]$ with given initial condition $\zeta(0) \in \mathbb{R}^n$ and $y_t = \{ y(t + \vartheta) : -\infty < \vartheta \leq 0 \}$. The coefficients $f$, $g$ and $h$ are given functions such that for all $y \in \mathbb{R}^n$, $f(\cdot,y), g(\cdot,y), h(\cdot,y) \in M^2_G((t,0];\mathbb{R}^n)$. For problem (1.1), the initial data is given as follows.

$$y_0 = \zeta = \{ \zeta(\vartheta) : -\infty < \vartheta \leq 0 \}, \quad (1.2)$$

is $\mathcal{F}_0$-measurable, $C_q((t,0];\mathbb{R}^n)$-value random variable such that $\zeta \in M^2_G((t,0];\mathbb{R}^n)$.

**Definition 1.1.** A stochastic process $y(t) \in \mathbb{R}^n$, $t \in (-\infty,T]$, is said to be a solution of the above equation (1.1) with the given initial data (1.2), if

1. For all $t \in [0,T]$, $y(t)$ is $\mathcal{F}_t$-adapted and continuous.
2. The coefficients $f(t,y_t) \in \mathcal{L}^1([0,T];\mathbb{R}^n)$, $g(t,y_t) \in \mathcal{L}^1([0,T];\mathbb{R}^{n \times m})$ and $h(t,y_t) \in \mathcal{L}^2([0,T];\mathbb{R}^{n \times m})$
3. For each $t \in [0,T]$, $y(t) = \zeta + \int_0^t f(v,y_v)dv + \int_0^t g(v,y_v)dB(v) + \int_0^t h(v,y_v)dB(v)$ q.s.

The rest of the paper is organized as follows. Section 2 is devoted to the basic notions and results required for the subsequent sections of this article. Section 3 presents the boundedness of solutions and contains the existence-uniqueness results with monotone nonlinearity conditions for G-SFDEs. The error estimation for Picard approximate solution $y^k(t)$ and exact solution $y(t)$ is determined in section 4. Section 5 gives the $L^2_G$ and exponential estimates for the unique solution of G-SFDEs. With weak monotonicity conditions, section 6 studies the existence and uniqueness while section 7 the $L^2_G$ and exponential estimates for G-SFDEs.
2 Preliminaries

Assume that $\mathcal{H}$ be a space of real valued functions defined on a given non-empty set $\Omega$ and let $(\Omega, \mathcal{H}, \mathbb{E})$ be a sublinear expectation space. Let $\Omega$ denotes the space of all $\mathbb{R}^n$-valued continuous paths $(w(t))_{t \geq 0}$ with $w(0) = 0$ equipped with the distance
\[
\rho(w^1, w^2) = \sum_{i=1}^{\infty} \frac{1}{2^i} \left( \max_{t \in [0,i]} |w^1(t) - w^2(t)| \land 1 \right).
\]
Let for any $w \in \Omega$ and $t \geq 0$, $B(t) = B(t, w) = w(t)$ be the canonical process. For any fixed $T \in [0, \infty)$, set
\[
L_{ip}(\Omega_T) = \left\{ \phi(B(t_1), B(t_2), ..., B(t_n)) : n \geq 1, t_1, t_2, ..., t_n \in [0, T], \phi \in C_{b,Lip}(\mathbb{R}^{nxm}) \right\},
\]
where $C_{b,Lip}(\mathbb{R}^{nxm})$ is a space of bounded Lipschitz functions and $L_{ip}(\Omega_t) \subseteq L_{ip}(\Omega_T)$ for $t \leq T$, $L_{ip}(\Omega) = \bigcup_{t=1}^{\infty} L_{ip}(\Omega_n)$. The completion of $L_{ip}(\Omega)$ under the Banach norm $\mathbb{E}[|.|^p]^{\frac{1}{p}}$, $p \geq 1$ is denoted by $L_{ip}^p(\Omega)$, where $L_{ip}^p(\Omega_t) \subseteq L_{ip}^p(\Omega_T) \subseteq L_{ip}^p(\Omega)$ for $0 \leq t \leq T < \infty$. Generated by the canonical process $\{B(t) : t \geq 0\}$, the filtration is given by $\mathcal{F}_t = \sigma \{B(s), 0 \leq s \leq t\}$, $\mathcal{F} = \{\mathcal{F}_t\}_{t \geq 0}$. Let $\pi_T = \{t_0, t_1, ..., t_N\}$, $0 \leq t_0 \leq t_1 \leq ... \leq t_N \leq \infty$ be a partition of $[0, T]$. For all $N \geq 1$, $0 = t_0 < t_1 < ... < t_N = T$ and $i = 0, 1, ..., N - 1$, define the space $M_{ip}^{p,0}([0, T])$, $p \geq 1$ of simple processes as
\[
M_{G}^{p,0}([0, T]) = \left\{ \eta_t(w) = \sum_{i=0}^{N-1} \xi_{t_i}(w) I_{[t_i,t_{i+1}]}(t) : \xi_{t_i}(w) \in L_{ip}^p(\Omega_{t_i}) \right\}.
\]
Let $M_{G}^{p}(0, T)$, $p \geq 1$ denotes the completion of $M_{ip}^{p,0}(0, T)$ with the norm given below
\[
||\eta|| = \left\{ \int_0^T \mathbb{E}[|\eta(v)|^p]dv \right\}^{1/p}.
\]

**Definition 2.1.** For $\eta_t \in M_{G}^{2,0}(0, T)$, the G-Itô’s integral $I(\eta)$ is defined by
\[
I(\eta) = \int_0^T \eta(v)dB^a(v) = \sum_{i=0}^{N-1} \xi_i \left( B^a(t_{i+1}) - B^a(t_i) \right).
\]
A mapping $I : M_{G}^{2,0}(0, T) \rightarrow L_{G}^2(\mathcal{F}_T)$ can be continuously extended to $I : M_{G}^{2}(0, T) \rightarrow L_{G}^2(\mathcal{F}_T)$ and for $\eta \in M_{G}^{2}(0, T)$ the G-Itô integral is still defined by
\[
\int_0^T \eta(v)dB^a(v) = I(\eta).
\]

**Definition 2.2.** The G-quadratic variation process $\{\langle B^a \rangle(t)\}_{t \geq 0}$ of G-Brownian motion is defined by
\[
\langle B^a \rangle(t) = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} \left( B^a(t_{i+1}) - B^a(t_i) \right)^2 = B^a(t)^2 - 2 \int_0^t B^a(v)dB^a(v),
\]
which is an increasing process with $\langle B^a \rangle(0) = 0$ and for any $0 \leq s \leq t$,
\[
\langle B^a \rangle(t) - \langle B^a \rangle(s) \leq \sigma_{aa^*}(t - s).
\]
Assume that $a, \hat{a} \in \mathbb{R}^n$ be two given vectors. Then the mutual variation process of $B^a$ and $B^{\hat{a}}$ is defined by $\langle B^a, B^{\hat{a}} \rangle = \frac{1}{4}[\langle B^a + B^{\hat{a}} \rangle(t) - \langle B^a - B^{\hat{a}} \rangle(t)]$. A mapping $H_{0,T} : M^0_G(0,T) \mapsto L^2_G(\mathcal{F}_T)$ is defined by

$$H_{0,T}(\eta) = \int_0^T \eta(v)d\langle B^a \rangle(v) = \sum_{i=0}^{N-1} \xi_i \left( \langle B^a \rangle(t_{i+1}) - \langle B^a \rangle(t_i) \right),$$

which can be continuously extended to $M^1_G(0,T)$ and for $\eta \in M^1_G(0,T)$ this is still defined by

$$\int_0^T \eta(v)d\langle B^a \rangle(v) = H_{0,T}(\eta).$$

The G-Itô integral and its quadratic variation process satisfy the following properties [25, 29].

**Proposition 2.3.**

1. $\hat{E}[\int_0^T \eta(v)dB(v)] = 0$, for all $\eta \in M^p_G(0,T)$.
2. $\hat{E}[\int_0^T \eta(v)dB(v)] = \hat{E}[\int_0^T \eta(x)\langle B, B \rangle(v)] \leq \sigma^2 \hat{E}[\int_0^T \eta^2(v)dv]$, for all $\eta \in M^2_G(0,T)$.
3. $\hat{E}[\int_0^T |\eta(v)|^p dv] \leq \int_0^T \hat{E}[|\eta(v)|^p] dv$, for all $\eta \in M^p_G(0,T)$.

The concept of G-capacity and lemma 2.6 can be found in [3].

**Definition 2.4.** Let $\mathcal{B}(\Omega)$ be a Borel $\sigma$-algebra of $\Omega$ and $\mathcal{P}$ be a collection of all probability measures on $(\Omega, \mathcal{B}(\Omega))$. Then the G-capacity denoted by $\hat{C}$ is defined as the following

$$\hat{C}(A) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{P}(A),$$

where set $A \in \mathcal{B}(\Omega)$.

**Definition 2.5.** A set $A \in \mathcal{B}(\Omega)$ is said to be polar if its capacity is zero i.e. $\hat{C}(A) = 0$ and a property holds quasi-surely (q.s) if it holds outside a polar set.

**Lemma 2.6.** Let $y \in L^p$ and $\hat{E}[|y|^p] < \infty$. Then for each $\alpha > 0$, the G-Markov inequality is defined by

$$\hat{C}(|y| > \alpha) \leq \frac{\hat{E}[|y|^p]}{\alpha}.$$

For the proof of the following lemmas 2.7 and 2.8 see [14].

**Lemma 2.7.** Let $p \geq 2$, $\eta \in M^2_G(0,T)$, $a \in \mathbb{R}^n$ and $y(t) = \int_0^t \eta(v)dB^a(v)$. Then there exists a continuous modification $\tilde{y}(t)$ of $y(t)$, that is, on some $\tilde{\Omega} \subset \Omega$ with $\hat{C}(\Omega^c) = 0$ and for all $t \in [0,T]$, $\hat{C}(|y(t) - \tilde{y}| \neq 0) = 0$ such that

$$\hat{E}\left[\sup_{s \leq v \leq t} |\tilde{y}(v) - \tilde{y}(s)|^p\right] \leq \hat{K} \sigma_a^p \hat{E}\left(\int_s^t |\eta(v)|^p dv\right)^{\frac{p}{2}},$$

where $0 < \hat{K} < \infty$ is a positive constant.
Lemma 2.8. Let $p \geq 1$, $\eta \in M^p(0, T)$ and $a, \hat{a} \in \mathbb{R}^n$, then there exists a continuous modification $\bar{y}^{a, \hat{a}}(t)$ of $y^{a, \hat{a}}(t) = \int_0^t \eta(v) d\langle B^a, B^{\hat{a}} \rangle(v)$ such that for $0 \leq s \leq v \leq t \leq T$,

$$
\hat{E} \left[ \sup_{0 \leq s \leq v \leq t} |\bar{y}^{a, \hat{a}}(v) - \bar{y}^{a, \hat{a}}(s)|^p \right] \leq \left( \frac{1}{4} \sigma_{a+\hat{a}}(a-\hat{a})^p \right)^p (t-s)^{p-1} \hat{E} \int_s^t |\eta(v)|^p dv,
$$

The following lemma can be found in [19].

Lemma 2.9. Let $a, b \in \mathbb{R}^n$ and $\hat{c} > 0$. Then

$$
|a + b|^2 \leq (1 + \hat{c})|a|^2 + (1 + \hat{c}^{-1})|b|^2.
$$

The following lemma is borrowed from [12].

Lemma 2.10. Let $p \geq 2$ and $\lambda < pq$. Then for any $\zeta \in C_q((\infty, 0]; \mathbb{R}^n)$,

$$
\hat{E} \|y_t\|_p^p \leq e^{-\lambda t} \hat{E} \|\zeta\|_p^p + \hat{E} \left[ \sup_{0 < v \leq t} |y(v)|^p \right].
$$

For the following lemma see [19, 28].

Lemma 2.11. Let $\kappa(.) : \mathbb{R}^+ \to \mathbb{R}^+$ be a concave non-decreasing continuous function satisfying $\kappa(0) = 0$ and $\kappa(y) > 0$ for $y > 0$. Assume that $\mu(t) \geq 0$ for all $0 \leq t \leq T < \infty$, satisfies

$$
\mu(t) \leq c + \int_0^t \varphi(s)\kappa(\mu(s)) ds,
$$

where $c$ is a positive real number and $\varphi : [0, T] \to \mathbb{R}^+$. Then the following properties hold.

(i) If $c = 0$, then $\mu(t) = 0$, $t \in [0, T]$.

(ii) If $c > 0$, we define $\omega(t) = \int_0^t \frac{1}{\mu(s)} ds$, for $t \in [0, T]$, then

$$
\mu(t) \leq \omega^{-1}(\omega(c) + \int_0^t \varphi(s)\omega) ds,
$$

where $\omega^{-1}$ is the inverse function of $\omega$.

3 The G-SFDEs with monotone nonlinearity

Consider equation (1.1) with the corresponding initial data (1.2). Let the coefficients $f$, $g$ and $h$ of (1.1) satisfy the following conditions.

(H1) For any $y \in C_q((\infty, 0]; \mathbb{R}^n)$, there exists a constant $K$ such that,

$$
2\langle y(0), f(t, y) \rangle + 2\langle y(0), g(t, y) \rangle + |h(t, y)|^2 \leq K(1 + \|y\|_q^2), \quad t \in [0, T]. \tag{3.1}
$$
Lemma 3.1. Let $y(t)$ be any solution of problem (1.1) with initial data (1.2) and $\hat{E}\|\eta\|_q^2 < \infty$. Assume that assumption $H_1$ holds, then

$$\hat{E}\left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \leq \hat{E}\|\eta\|_q^2 + C_1 e^{c_1 t},$$

where $C_1 = c_3 T + (2 + c_3 \lambda^{-1})\hat{E}\|\eta\|_q^2$, $c_3 = 2K(1 + 2c_1 + 2c_2^2)$, $c_1$ and $c_2$ are positive constants.

Proof. Applying the G-Itô formula to $|y(t)|^2$, taking the G-expectation on both sides, using properties of G-Itô integral, lemma 2.7 and lemma 2.8, there exist $c_1 > 0$ and $c_2 > 0$ such that for any $t \in [0, T]$,

$$\hat{E}\left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \leq \hat{E}\|\eta\|_q^2 + \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle y(v), f(v, y_v) \rangle dv \right] + \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle y(v), h(v, y_v) \rangle dB(v) \right]$$

$$+ \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t [2\langle y(v), g(v, y_v) \rangle + |h(v, y_v)|^2] d\langle B, B \rangle(v) \right]$$

$$\leq \hat{E}\|\eta\|_q^2 + \hat{E}\int_0^t 2\langle y(v), f(v, y_v) \rangle dv + 2c_2 \hat{E}\left[ \int_0^t |\langle y(v), h(v, y_v) \rangle|^2 dv \right]^\frac{1}{2}$$

$$+ c_1 \hat{E}\int_0^t [2\langle y(v), g(v, y_v) \rangle + |h(v, y_v)|^2] dv$$

$$\leq \hat{E}\|\eta\|_q^2 + \hat{E}\int_0^t 2\langle y(v), f(v, y_v) \rangle dv + 2c_2 \hat{E}\int_0^t |h(v, y_v)|^2 dv + \frac{1}{2} \hat{E}\left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right]^2$$

$$+ c_1 \hat{E}\int_0^t [2\langle y(v), g(v, y_v) \rangle + |h(v, y_v)|^2] dv.$$

By using assumption $H_1$ and lemma 2.10, it follows

$$\hat{E}\left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \leq 2\hat{E}\|\eta\|_q^2 + 2(1 + 2c_1 + 2c_2^2)K\hat{E}\int_0^t (1 + |y(v)|^2) dv$$

$$\leq 2\hat{E}\|\eta\|_q^2 + 2(1 + 2c_1 + 2c_2^2)K(1 + \lambda^{-1}\hat{E}\|\eta\|_q^2)$$

$$+ 2(1 + 2c_1 + 2c_2^2)K\hat{E}\int_0^t \left( \sup_{0 \leq v \leq t} |y(v)|^2 \right) dv.$$

In virtue of the Gronwall inequality, we have

$$\hat{E}\left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \leq C_1 e^{c_3 t},$$

(3.3)
where \( C_1 = c_3 T + (2 + c_3 \lambda^{-1}) \hat{E} \| \zeta \|_q^2 \) and \( c_3 = 2K(1 + 2c_1 + 2c_2^2) \). Noticing that
\[
\hat{E} \left[ \sup_{-\infty < v \leq t} |y(v)|^2 \right] \leq \hat{E} \| \zeta \|_q^2 + \hat{E} \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right],
\]
we get
\[
\hat{E} \left[ \sup_{-\infty < v \leq t} |y(v)|^2 \right] \leq \hat{E} \| \zeta \|_q^2 + C_1 e^{ct}.
\]
By letting \( t = T \), the proof of the required assertion completes. \( \square \)

**Remark 3.2.** Lemma 3.1 states that the solution \( y(t) \) is bounded, in particular, \( y(t) \in M^2_G((-\infty, T]; \mathbb{R}^n) \).

Next under the assumptions \( H_1 \) and \( H_2 \), we prove the existence-uniqueness results for the G-SFDE (1.1) with the given initial data (1.2) in the phase space with fading memory \( C_q((-\infty, T]; \mathbb{R}^n) \).

First, we derive the uniqueness of solutions.

**Definition 3.3.** A solution \( y(t) \) of problem (1.1) with the initial data (1.2) is said to be unique if it is indistinguishable from any other solution \( z(t) \), that is,
\[
\hat{E} \left[ \sup_{-\infty < v \leq t} |z(v) - y(v)|^2 \right] = 0,
\]
quasi-surely.

**Theorem 3.4.** Let assumption \( H_2 \) holds. Then (1.1) has a unique solution, if exists.

**Proof.** Let (1.1) has two solutions say \( y(t) \) and \( z(t) \) with the same initial data. By virtue of lemma 3.1 \( y(t), z(t) \in M^2_G((-\infty, T]; \mathbb{R}^n) \). Define \( \Lambda(t) = z(t) - y(t), \hat{f}(t) = f(t, z_t) - f(t, y_t), \hat{g}(t) = g(t, z_t) - g(t, y_t) \) and \( \hat{h}(t) = h(t, z_t) - h(t, y_t) \). Applying the G-Itô formula to \( |\Lambda(t)|^2 \), taking the G-expectation on both sides, using properties of G-Itô integral, lemma 2.7 and lemma 2.8, there exist \( c_1 > 0 \) and \( c_2 > 0 \) such that for any \( t \in [0, T] \),
\[
\hat{E} \left[ \sup_{0 \leq v \leq t} |\Lambda(v)|^2 \right] \leq \hat{E} \left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle \Lambda(v), \hat{f}(v) \rangle dv \right] + \hat{E} \left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle \Lambda(v), \hat{g}(v) \rangle dB(v) \right] + \hat{E} \left[ \sup_{0 \leq v \leq t} \int_0^t |\hat{h}(v)|^2 dv \right] + c_1 \hat{E} \left[ \int_0^t |\Lambda(v)|^2 dv \right] + c_2 \hat{E} \left[ \int_0^t |\Lambda(v)|^2 dv \right] + 2c_2 \hat{E} \left[ \int_0^t |\hat{h}(v)|^2 dv \right]
\]
\[
\leq \hat{E} \int_0^t 2\langle \Lambda(v), \hat{f}(v) \rangle dv + 2c_2 \hat{E} \left[ \int_0^t |\hat{h}(v)|^2 dv \right] + c_1 \hat{E} \int_0^t 2\langle \Lambda(v), \hat{g}(v) \rangle dv + c_1 \hat{E} \int_0^t |\Lambda(v)|^2 dv + 2c_2 \hat{E} \left[ \sup_{0 \leq v \leq t} |\Lambda(v)|^2 \right],
\]
In view of assumption \( H_2 \), it follows
\[
\hat{E} \left[ \sup_{0 \leq v \leq t} |z(v) - y(v)|^2 \right] \leq 2(1 + 2c_1 + 2c_2^2) K \hat{E} \int_0^t \| z - y \|_q^2 dv
\]
Noticing that initial data of \( z(t) \) and \( y(t) \) is same, lemma \([2,10]\) yields

\[
\hat{E}\|z - y\|_q^2 \leq \hat{E}\left[ \sup_{0 \leq v \leq t} |z(v) - y(v)|^2 \right],
\]

which on substituting in the above last inequality gives

\[
\hat{E}\left[ \sup_{0 \leq v \leq t} |z(v) - y(v)|^2 \right] \leq 2(1 + 2c_1 + 2c_2^2)\hat{K}\int_0^t \hat{E}\left[ \sup_{0 \leq v \leq t} |z(v) - y(v)|^2 \right] dv.
\]

By using the Gronwall inequality, we derive

\[
\hat{E}\left[ \sup_{0 \leq v \leq t} |z(v) - y(v)|^2 \right] = 0,
\]

because the initial data of \( y(t) \) and \( z(t) \) is same, it follows

\[
\hat{E}\left[ \sup_{-\infty < v \leq t} |z(v) - y(v)|^2 \right] = 0.
\]

This shows that for \( t \in (-\infty, T] \), \( y(t) = z(t) \) quasi-surely. The proof of uniqueness is complete. \(\Box\)

To prove the existence of solutions we use the Picard iteration scheme. For \( t \in [0, T] \), define \( y^0(t) = \zeta(0) \) and \( y_0^0 = \zeta \). For each \( k = 1, 2, \ldots \), set \( y_0^k = \zeta \) and for \( t \in [0, T] \), define the Picard iterations,

\[
y^k(t) = \zeta(0) + \int_0^t f(v, y_v^{k-1})dv + \int_0^t g(v, y_v^{k-1})d\langle B, B \rangle(v) + \int_0^t h(v, y_v^{k-1})dB(v). \tag{3.4}
\]

It is obvious that \( y^0(t) \in M^2_G((-\infty, T]; \mathbb{R}^n) \) and by induction for each \( k = 1, 2, \ldots \), \( y^k(t) \in M^2_G((-\infty, T]; \mathbb{R}^n) \), which is derived in the following lemma \([3,5]\).

**Lemma 3.5.** Let assumption \( H_1 \) holds and \( \hat{E}\|\zeta\|_q^2 < \infty \). Then

\[
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] \leq C_2e^{c_2t}, \tag{3.5}
\]

where \( C_2 = c_3T + (2 + c_3(T + \lambda^{-1}))\hat{E}\|\zeta\|_q^2 \), \( c_3 = 2K(1 + 2c_1 + 2c_2^2) \) and \( c_1, c_2 \) are positive constants.

**Proof.** Applying the G-Itô formula to \( |y^k(t)|^2 \), taking the G-expectation on both sides, using properties of G-Itô integral, lemma \([2,7]\) and lemma \([2,8]\), there exist \( c_1 > 0 \) and \( c_2 > 0 \) such that for any \( t \in [0, T] \)

\[
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] \leq \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle y^k(v), f(v, y_v^{k-1}) \rangle dv \right] + \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle y^k(v), h(v, y_v^{k-1}) \rangle dB(v) \right]
\]

\[
+ \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t |2\langle y^k(v), g(v, y_v^{k-1}) \rangle + |h(v, y_v^{k-1})|^2 d\langle B, B \rangle(v) \right]
\]

\[
\leq \hat{E}\int_0^t 2\langle y^k(v), f(v, y_v^{k-1}) \rangle dv + 2c_2\hat{E}\int_0^t |h(v, y_v^{k-1})|^2 dv + \frac{1}{2}\hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right]
\]

\[
+ c_1\hat{E}\int_0^t |2\langle y^k(v), g(v, y_v^{k-1}) \rangle + |h(v, y_v^{k-1})|^2| dv.
\]

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By using assumption $H_1$ and lemma 2.10, it follows

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] \leq 2\hat{E}\|\zeta\|^2_q + 2(1 + 2c_1 + 4c_2^2)K\hat{E}\int_0^t (1 + \|y^{k-1}\|^2_q) dv
\leq 2\hat{E}\|\zeta\|^2_q + 2(1 + 2c_1 + 2c_2^2)K(T + \lambda^{-1}\hat{E}\|\zeta\|^2_q)
+ 2(1 + 2c_1 + 2c_2^2)K\hat{E}\int_0^t \left[ \sup_{0 \leq v \leq t} |y^{k-1}(v)|^2 \right] dv.
$$

Noticing that

$$
\max_{1 \leq k \leq n} \hat{E}\left[ \sup_{0 \leq v \leq t} |y^{k-1}(v)|^2 \right] \leq \max \left\{ \hat{E}\|\zeta\|^2_q, \max_{1 \leq k \leq n} \hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] \right\}
\leq \hat{E}\|\zeta\|^2_q + \max_{1 \leq k \leq n} \hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right],
$$

we derive

$$
\max_{1 \leq k \leq n} \hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] \leq 2\hat{E}\|\zeta\|^2_q + 2K(1 + 2c_1 + 2c_2^2)T + (T + \lambda^{-1})\hat{E}\|\zeta\|^2_q
+ 2K(1 + 2c_1 + 2c_2^2)\int_0^t \max_{1 \leq k \leq n} \hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] dv.
$$

In virtue of the Gronwall inequality and noting that $n$ is arbitrary, we have

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] \leq 2\left[ \hat{E}\|\zeta\|^2_q + K(1 + 2c_1 + 2c_2^2)T + (T + \lambda^{-1})\hat{E}\|\zeta\|^2_q \right] e^{2K(1 + 2c_1 + 2c_2^2)t}.
$$

Finally, by using the fact

$$
\hat{E}\left[ \sup_{-\infty < v \leq t} |y^k(v)|^2 \right] \leq \hat{E}\|\zeta\|^2_q + \hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right],
$$

and letting $t = T$, the proof of assertion (3.5) completes.

\begin{proof}
Consider the Picard iteration sequence $\{y^k(t); t \geq 0\}$ defined by (3.4). Then from (3.4), we have

$$
y^1(t) = \zeta(0) + \int_0^t f(v, y_v^0) dv + \int_0^t g(v, y_v^0) d\langle B, B \rangle(v) + \int_0^t h(v, y_v^0) dB(v).
$$

Applying the G-Itô formula to $|y^1(t)|^2$ and using similar arguments as earlier, it follows

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^1(v)|^2 \right] \leq \hat{E}\|\zeta\|^2_q + \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle y^1(v), f(v, y_v^0) \rangle dv \right] + \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle y^1(v), h(v, y_v^0) \rangle dB(v) \right]
+ \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t [2\langle y^1(v), g(v, y_v^0) \rangle + |h(v, y_v^0)|^2 d\langle B, B \rangle(v)] \right]
\leq \hat{E}\|\zeta\|^2_q + \hat{E}\int_0^t 2\langle y^1(v), f(v, y_v^0) \rangle dv + 2c_1\hat{E}\int_0^t |h(v, y_v^0)|^2 dv + \frac{1}{2} \hat{E}\left[ \sup_{0 \leq v \leq t} |y^1(v)|^2 \right]
+ c_1\hat{E}\int_0^t [2\langle y^1(v), g(v, y_v^0) \rangle + |h(v, y_v^0)|^2] dv.
$$
\end{proof}
By assumption $H_2$, we have

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^1(v)|^2 \right] \leq 2\hat{E}\|\zeta\|_q^2 + 2K(1 + 2c_1 + 2c_2^2)\hat{E}\int_0^t (1 + \|y^0_v\|^2) dv \\
\leq 2\hat{E}\|\zeta\|_q^2 + 2K(1 + 2c_1 + 2c_2^2)T + 2K(1 + 2c_1 + 2c_2^2)\hat{E}\int_0^t [e^{-\lambda t}\|\zeta\|_q^2 + \|\zeta\|_q^2] dv \\
\leq 2K(1 + 2c_1 + 2c_2^2)T + 2[1 + K(1 + 2c_1 + 2c_2^2)(T + \lambda^{-1})]\hat{E}\|\zeta\|_q^2.
$$

By using lemma 2.9 it follows

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^2(v) - y^0(v)|^2 \right] \leq (1 + \hat{c})\hat{E}\left[ \sup_{0 \leq v \leq t} |y^1(v)|^2 \right] + (1 + \hat{c}^{-1})\hat{E}\|\zeta\|_q^2 \leq L,
$$

where $L = (1 + \hat{c})|c_3T + [2 + c_3(T + \lambda^{-1})]\hat{E}\|\zeta\|_q^2 + (1 + \hat{c}^{-1})\hat{E}\|\zeta\|_q^2$. In a similar fashion as above, from 3.4 we obtain

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^3(v) - y^1(v)|^2 \right] \leq \hat{E}\int_0^t 2(y^2(v) - y^1(v), f(v, y^1_v) - f(v, y^0_v)) dv \\
+ 2c_2^2\hat{E}\int_0^t |h(v, y^1_v) - h(v, y^0_v)|^2 dv + \frac{1}{2}\hat{E}\left[ \sup_{0 \leq v \leq t} |y^2(v) - y^1(v)|^2 \right] \\
+ c_1\hat{E}\int_0^t [2(y^2(v) - y^1(v), g(v, y^1_v) - g(v, y^0_v)) + |h(v, y^1_v) - h(v, y^0_v)|^2] dv.
$$

By using assumption $H_2$ and lemma 2.10 it follows

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^3(v) - y^1(v)|^2 \right] \leq 2(1 + 2c_1 + 2c_2^2)K\hat{E}\int_0^t \|y^1_v - y^0_v\|^2 dv \\
\leq 2(1 + 2c_1 + 2c_2^2)K\int_0^t \hat{E}\left[ \sup_{0 \leq v \leq t} |y^1(v) - y^0(v)|^2 \right] dv \\
\leq 2(1 + 2c_1 + 2c_2^2)KLt.
$$

Similarly, we derive

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^3(v) - y^1(v)|^2 \right] \leq 2(1 + 2c_1 + 2c_2^2)K\hat{E}\int_0^t \|y^2_v - y^1_v\|^2 dv \\
\leq 2(1 + 2c_1 + 2c_2^2)K\int_0^t \hat{E}\left[ \sup_{0 \leq v \leq t} |y^2(v) - y^1(v)|^2 \right] dv \\
\leq L\frac{1}{2!}[2(1 + 2c_1 + 2c_2^2)K]^2t^2.
$$

Continuing this procedure, we get

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^3(v) - y^3(v)|^2 \right] 2(1 + 2c_1 + 2c_2^2)K\hat{E}\int_0^t \|y^2_v - y^1_v\|^2 dv \\
\leq 2(1 + 2c_1 + 2c_2^2)K\int_0^t \hat{E}\left[ \sup_{0 \leq v \leq t} |y^3(v) - y^2(v)|^2 \right] dv \\
\leq L\frac{1}{3!}[2(1 + 2c_1 + 2c_2^2)K]^3t^3.
$$
By using lemma 2.10, we obtain which on substituting in the above last inequality gives lemma 2.6 we derive integer $H$

For $k = 0$, it has been proved above. Next suppose that (3.6) holds for some $k \geq 0$, we have to show that it holds for $k + 1$. Define $\Lambda^{k+2,k+1}(t) = y^{k+2}(t) - y^{k+1}(t)$, $\hat{f}^{k+1,k}(t) = f(y^{k+1}) - f(y^k)$, $\hat{g}^{k+1,k}(t) = g(y^{k+1}) - g(y^k)$ and $\hat{h}^{k+1,k}(t) = h(y^{k+1}) - h(y^k)$. By using the G-Itô formula, lemma 2.7 and lemma 2.8 for any $t \in [0,T]$, we obtain

$$
\mathbb{E}\left[\sup_{0 \leq v \leq t} |\Lambda^{k+2,k+1}(v)|^2 \right] \\
\leq \mathbb{E}\left[\sup_{0 \leq v \leq t} \int_0^t 2\langle \Lambda^{k+2,k+1}(v), \hat{f}^{k+1,k}(v) \rangle dv \right] + \mathbb{E}\left[\sup_{0 \leq v \leq t} \int_0^t 2\langle \Lambda^{k+2,k+1}(v), \hat{g}^{k+1,k}(v) \rangle dB(v) \right] \\
+ \mathbb{E}\left[\sup_{0 \leq v \leq t} \int_0^t |\hat{h}^{k+1,k}(v)|^2 d\langle B, B \rangle(v) \right] \\
\leq \mathbb{E}\int_0^t 2\langle \Lambda^{k+2,k+1}(v), \hat{f}^{k+1,k}(v) \rangle dv + 2c_2^2 \mathbb{E}\int_0^t |\hat{h}^{k+1,k}(v)|^2 dv + \frac{1}{2} \mathbb{E}\left[\sup_{0 \leq v \leq t} |\Lambda^{k+2,k+1}(v)|^2 \right] \\
+ c_4 \mathbb{E}\int_0^t 2\langle \Lambda^{k+2,k+1}(v), \hat{g}^{k+1,k}(v) \rangle + |\hat{h}^{k+1,k}(v)|^2 dv.
$$

In view of assumption $H_2$, it follows

$$
\mathbb{E}\left[\sup_{0 \leq v \leq t} |y^{k+2}(v) - y^{k+1}(v)|^2 \right] \leq 2(1 + 2c_1 + 2c_2^2)K\mathbb{E}\int_0^t \|y^{k+1} - y^k\|^2 dv.
$$

By using lemma 2.10 we obtain

$$
\mathbb{E}\|y^{k+1} - y^k\|^2 \leq \mathbb{E}\left[\sup_{0 \leq v \leq t} |y^{k+1}(v) - y^k(v)|^2 \right],
$$

which on substituting in the above last inequality gives

$$
\mathbb{E}\left[\sup_{0 \leq v \leq t} |y^{k+2}(v) - y^{k+1}(v)|^2 \right] \leq 2(1 + 2c_1 + 2c_2^2)K\mathbb{E}\int_0^t \mathbb{E}\left[\sup_{0 \leq v \leq t} |y^{k+1}(v) - y^k(v)|^2 \right] dv \\
\leq 2(1 + 2c_1 + 2c_2^2)K\int_0^t \frac{L[Mt]^k}{k!} dv = \frac{L[Mt]^{k+1}}{(k+1)!}.
$$

This implies that (3.6) holds for $k + 1$. Thus by induction (3.6) holds for all $k \geq 0$. Next by using lemma 2.4 we derive

$$
\mathcal{C}\left[\sup_{0 \leq s \leq T} |y^{k+1}(t) - y^k(t)|^2 \right] \leq \frac{1}{2k} \mathbb{E}\left[\sup_{0 \leq t \leq T} |y^{k+1}(t) - y^k(t)|^2 \right] \leq \frac{L[Mt]^k}{k!}
$$

Since $\sum_{k=0}^{\infty} \frac{L[2Mt]^k}{k!} < \infty$, the Borel-Cantelli lemma gives that for almost all $w$ there exists a positive integer $k_0 = k_0(w)$ such that

$$
\sup_{0 \leq s \leq T} |y^{k+1}(t) - y^k(t)|^2 \leq \frac{1}{2k}, \text{ as } k \geq k_0.
$$
It implies that q.s. the partial sums
\[ y^0(t) + \sum_{i=0}^{k-1} [y^{k+1}(t) - y^k(t)] = y^k(t), \]
are convergent uniformly on \( t \in (-\infty, T] \). Denote the limit by \( y(t) \). Then the sequence \( \{y^k(t)\}_t \geq 0 \) converges uniformly to \( y(t) \) on \( t \in (-\infty, T] \). Clearly, \( y(t) \) is continuous and \( \mathcal{F}_t \)-adapted because \( \{y^k(t)\}_t \geq 0 \) is continuous and \( \mathcal{F}_t \)-adapted. Also, from (3.6), we can see that \( \{y^k(t) : k \geq 1\} \) is a Cauchy sequence in \( L^2_G \). Hence \( y^k(t) \) converges to \( y(t) \) in \( L^2_G \); that is,
\[ \hat{E}|y^k(t) - y(t)|^2 \to 0, \text{ as } k \to \infty. \]

Taking limits \( k \to \infty \), from (3.5) in lemma 3.5 we obtain
\[ \hat{E}\left[ \sup_{0 \leq s \leq t} |y(s)|^2 \right] \leq C_2 e^{c_3 T}. \]

To show that the sequence of solution maps \( \{y^k : n \geq 1\} \) is convergent in \( L^2_G \), by using lemma 2.10 we get
\[ \hat{E}|y^k(t) - y(t)|^2 \leq e^{-\lambda t} \hat{E}|\zeta - \xi|^2 + \hat{E}\left[ \sup_{0 < s \leq t} |y^k(s) - y(s)|^2 \right]. \]

Since \( y^k(t) \) and \( y(t) \) have the same initial data and \( y^n(t) \) is convergent to \( y(t) \), we therefore have
\[ \hat{E}|y^n(t) - y(t)|^2 \to 0 \text{ as } n \to \infty. \]

This implies that the sequence \( \{y^n\}_n \geq 0 \) converges to \( y \) in \( L^2_G \) and we have
\[
\begin{align*}
\int_0^t g(v, y^n_v) dv & \to \int_0^t g(v, y_v) dv, \text{ in } L^2_G, \\
\int_0^t h(v, y^n_v) d(B, B)(v) & \to \int_0^t h(v, y_v) d(B, B)(v), \text{ in } L^2_G, \\
\int_0^t h(v, y^n_v) dB(v) & \to \int_0^t h(v, y_v) dB(v), \text{ in } L^2_G.
\end{align*}
\]

For \( t \in [0, T] \), by taking limits \( n \to \infty \) in (3.4) we derive
\[
\lim_{k \to \infty} y^k(t) = \zeta(0) + \int_0^t \lim_{k \to \infty} f(v, y_v^{k-1}) dv + \int_0^t \lim_{k \to \infty} g(v, y_v^{k-1}) d(B, B)(v) + \int_0^t \lim_{k \to \infty} h(v, y_v^{k-1}) dB(v),
\]
which yields that
\[ y(t) = \zeta(0) + \int_0^t f(v, y_v) dv + \int_0^t g(v, y_v) d(B, B)(v) + \int_0^t h(v, y_v) dB(v), \]
t \in [0, T]. This shows that \( y(t) \) is the solution of (1.1). The proof of existence is complete. \( \square \)
4 Error Estimation

The procedure of the proof of above existence results demonstrates how to construct the Picard sequence \( \{y^k(t); t \geq 0\} \) and gain the accurate solution \( y(t) \). We now show the estimate of error for Picard approximate solution \( y^k(t) \) and exact solution \( y(t) \).

**Theorem 4.1.** Let \( y(t) \) be the unique solution of problem (1.1) with initial data (1.2) and \( y^k(t) \) be defined by (3.4). Assume that the assumptions \( H_1 \) and \( H_2 \) hold. Then for all \( k \geq 1 \),

\[
\mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v) - y(v)|^2 \right] \leq \frac{L[Mt]^k}{k!} e^{MT},
\]

where \( L = (1 + \hat{c})[c_3 T + [2 + c_3(T + \lambda^{-1})] \mathbb{E} \|\zeta\|_q^2] + (1 + \hat{c}^{-1}) \mathbb{E} \|\zeta\|_q^2 \) and \( M = 2(1 + 2c_1 + 2c_2^2) \hat{K} \).

**Proof.** We define \( \Lambda(t) = y^k(t) - y(t), \hat{f}(t) = f(t, y^k) - f(t, y), \hat{g}(t) = g(t, y^k) - g(t, y) \) and \( \hat{h}(t) = h(t, y^k) - h(t, y) \). Then in a similar fashion as earlier we obtain

\[
\mathbb{E} \left[ \sup_{0 \leq v \leq t} |\Lambda(v)|^2 \right] \leq \mathbb{E} \int_0^t 2 \langle \Lambda(v), \hat{f}(v) \rangle dv + 2c_2^2 \mathbb{E} \int_0^t |\hat{h}(v)|^2 dv + \frac{1}{2} \mathbb{E} \left[ \sup_{0 \leq v \leq t} |\Lambda(v)|^2 \right]
\]

\[
+ c_1 \mathbb{E} \int_0^t [2 \langle \Lambda(v), \hat{g}(v) \rangle + |\hat{h}(v)|^2] dv.
\]

In view of assumption \( H_2 \), it follows

\[
\mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v) - y(v)|^2 \right] \leq 2(1 + 2c_1 + 2c_2^2) \hat{K} \mathbb{E} \int_0^t |y^k - y|^2 dv
\]

\[
\leq 2(1 + 2c_1 + 2c_2^2) \hat{K} \int_0^t \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v) - y(v)|^2 \right] dv
\]

\[
\leq M \int_0^t \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v) - y^{k-1}(v)|^2 \right] dv + M \int_0^t \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^{k-1}(v) - y(v)|^2 \right] dv
\]

By using (3.4), we derive

\[
\mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v) - y(v)|^2 \right] \leq \frac{L[Mt]^k}{k!} + M \int_0^t \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^{k-1}(v) - y(v)|^2 \right] dv
\]

Finally, the Gronwall inequality yields,

\[
\mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v) - y(v)|^2 \right] \leq \frac{L[Mt]^k}{k!} e^{MT},
\]

for \( t \in [0, T] \). By letting \( t = T \), we get the desired expression. The proof stands completed. \( \square \)

Consider, the following stochastic differential equation driven by G-Brownian motion \( \{B(t); t \geq 0\} \)

\[
dy(t) = f(t, y(t))dt + g(t, y(t))dB(t) + h(t, y(t))dB(t),
\]

(4.1)
Lemma 5.1. Let $y(t)$ be a unique solution of problem (1.1) under the hypothesis $H_1$ and $H_2$. By straightforward calculations in a similar procedure used in lemma 3.1, we obtain it follows

$$\lim_{t \to \infty} \sup_{0 \leq v \leq t} |y(v)|^2 \leq C_3 e^{\alpha t},$$

where $C_3 = c_3 T + (3 + c_3 \lambda^{-1}) \hat{E}||\zeta||^2_q + c_3 = 2(1 + 2c_1 + 4c_2)K$ and $c_1$, $c_2$ are positive constants.

Proof. By straightforward calculations in a similar procedure used in lemma 3.1, we obtain

$$\hat{E} \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \leq c_3 T + (2 + c_3 \lambda^{-1}) \hat{E}||\zeta||^2_q + c_3 \hat{E} \int_0^t \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] dv. \quad (5.1)$$

Noticing that

$$\hat{E} \left[ \sup_{-\infty < v \leq t} |y(v)|^2 \right] \leq \hat{E}||\zeta||^2_q + \hat{E} \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right],$$

it follows

$$\hat{E} \left[ \sup_{-\infty < v \leq t} |y(v)|^2 \right] \leq \hat{E}||\zeta||^2_q + c_3 T + (2 + c_3 \lambda^{-1}) \hat{E}||\zeta||^2_q + c_3 \hat{E} \int_0^t \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] dv$$

$$\leq c_3 T + (3 + c_3 \lambda^{-1}) \hat{E}||\zeta||^2_q + c_3 \hat{E} \int_0^t \left[ \sup_{-\infty < v \leq t} |y(v)|^2 \right] dv$$

Finally, by using the Gronwall inequality, it follows

$$\hat{E} \left[ \sup_{-\infty < v \leq t} |y(v)|^2 \right] \leq (3 + c_3 \lambda^{-1}) \hat{E}||\zeta||^2_q e^{\alpha t}.$$ 

The proof is complete.

Corollary 4.2. Under the hypothesis $H_1$ and $H_2$, problem (1.1) admits a unique solution $y(t) \in M^2_{\hat{E}}([0, T]; \mathbb{R}^n)$.

5 The exponential estimate

First, we assume that under hypothesis $H_1$ and $H_2$ problem (1.1) with the given initial data (1.2) admits a unique solution on $[0, t]$. Then we derive the $L^2_{\hat{E}}$ and exponential estimates as follows.

Lemma 5.1. Let $y(t)$ be a unique solution of (1.1) on $t \in [0, \infty)$ and $\hat{E}||\zeta||^2_q < \infty$. Then under the hypothesis $H_1$ for all $t \geq 0$,

$$\hat{E} \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \leq C_3 e^{\alpha t},$$

where $C_3 = c_3 T + (3 + c_3 \lambda^{-1}) \hat{E}||\zeta||^2_q$, $c_3 = 2(1 + 2c_1 + 2c_2)K$ and $c_1$, $c_2$ are positive constants.

Theorem 5.2. Let $y(t)$ be a unique solution of (1.1) on $t \in [0, \infty)$ and $\hat{E}||\zeta||^2_q < \infty$. Then for all $t \geq 0$,

$$\lim_{t \to \infty} \sup_{0 \leq v \leq t} \frac{1}{t} \log |y(t)| \leq \alpha,$$

where $\alpha = K(1 + 2c_1 + 4c_2)$. 

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Proof. From (5.1), we derive
\[
\hat{E} \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \leq C_1 e^{c_3 t},
\] (5.2)
where \( C_1 = c_3 T + (2 + c_3 \lambda^{-1}) \hat{E} \| \zeta \|_q^2 \). By virtue of the above result (5.2), for each \( m = 1, 2, 3, \ldots \), we have
\[
\hat{E} \left[ \sup_{m-1 \leq t \leq m} |y(t)|^2 \right] \leq C_1 e^{c_3 m}.
\]
For any \( \epsilon > 0 \), by using lemma 2.6 we get
\[
\hat{C} \left\{ w : \sup_{m-1 \leq t \leq m} |y(t)|^2 > e^{(c_3 + \epsilon)m} \right\} \leq \frac{\hat{E} \left[ \sup_{m-1 \leq t \leq m} |y(t)|^2 \right]}{e^{(c_3 + \epsilon)m}} \leq \frac{C_1 e^{c_3 m}}{e^{(c_3 + \epsilon)m}} = C_1 e^{-\epsilon m}.
\]
But for almost all \( w \in \Omega \), the Borel-Cantelli lemma yields that there exists a random integer \( m_0 = m_0(w) \) so that
\[
\sup_{m-1 \leq t \leq m} |y(t)|^2 \leq e^{(c_3 + \epsilon)m}, \text{ as } m \geq m_0,
\]
which implies
\[
\lim_{t \to \infty} \sup_t \frac{1}{t} \log|y(t)| \leq \frac{2K(1 + 2c_1 + 2c_2^2) + \epsilon}{2} = K(1 + 2c_1 + 2c_2^2) + \frac{\epsilon}{2},
\]
but \( \epsilon \) is arbitrary and the above result reduces to
\[
\lim_{t \to \infty} \sup_t \frac{1}{t} \log|y(t)| \leq \alpha,
\]
where \( \alpha = K(1 + 2c_1 + 2c_2^2) \). The proof is complete. \( \square \)

6 Existence and uniqueness with weak monotonicity

In this section, we assume that the coefficients of problem (1.1) with the initial data (1.2) satisfy the following weak nonlinear monotonicity conditions.

(A_1) There exists a non-decreasing and concave function \( \kappa(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \) with \( \kappa(0) = 0 \), \( \kappa(z) > 0 \) for \( z > 0 \) and \( \int_0^\infty \frac{dz}{\kappa(z)} = \infty \) such that for any \( y, z \in C_q((-\infty, 0]; \mathbb{R}^n) \),
\[
2 \langle z(0) - y(0), f(z, t) - f(y, t) \rangle + 2 \langle z(0) - y(0), g(z, t) - g(y, t) \rangle
\]
\[
\vee |h(z, t) - h(y, t)|^2 \leq \kappa(\|z - y\|_q^2), \quad t \in [0, T],
\]
where for all \( z \geq 0 \) and \( a, b \in \mathbb{R}^+ \), \( \kappa(z) \leq a + bz \).

(A_2) For any \( t \in [0, T] \) and \( f(t, 0), g(t, 0), h(t, 0) \in L^2 \), there exists a positive constant \( \bar{K} \) such that
\[
|f(t, 0)|^2 \vee |g(t, 0)|^2 \vee |h(t, 0)|^2 \leq \bar{K}.
\]
We now prove two useful lemmas. They will be used in the upcoming existence-uniqueness results.

**Lemma 6.1.** Let assumptions $A_1$ and $A_2$ hold and $\hat{E}\|\zeta\|_q^2 < \infty$. Then

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] \leq C,
$$

where $C = C_4 e^{\hat{C}_4 T}$, $C_4 = (1 + \hat{c}b\lambda^{-1} + \hat{c}T)\hat{E}\|\zeta\|_q^2 + \hat{c}(\bar{K} + a)T$, $\hat{C}_4 = (2 + 2c_1 + \hat{c}b)$, $\hat{c} = 2(1 + 2c_1 + 2c_2^2)$ and $c_1, c_2$ are positive constants.

**Proof.** Consider the Picard iteration sequence $\{y^k(t); t \geq 0\}$ defined by (3.4). By using the G-Itô formula, lemma (2.7) and lemma (2.8) for any $t \in [0, T]$, we derive

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right]
\leq \hat{E}\|\zeta\|_q^2 + \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle y^k(v), f(v, g(v, y^k_{v-1})) \rangle dv \right] + \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle y^k(v), h(v, y^k_{v-1}) \rangle dB(v) \right]
+ \hat{E}\left[ \sup_{0 \leq v \leq t} \int_0^t |2\langle y^k(v), g(v, y^k_{v-1}) \rangle + |h(v, y^k_{v-1})|^2| dB, B(v) \right]
\leq \hat{E}\|\zeta\|_q^2 + \hat{E}\int_0^t 2\langle y^k(v), f(v, g(v, y^k_{v-1})) \rangle dv + 2c_2^2 \hat{E}\int_0^t |h(v, y^k_{v-1})|^2 dv + \frac{1}{2} \hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right]
+ c_1 \hat{E}\int_0^t |2\langle y^k(v), g(v, y^k_{v-1}) \rangle + |h(v, y^k_{v-1})|^2| dv.
$$

By using assumptions $A_1$ and $A_2$, it follows

$$
\hat{E}\left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right]
\leq \hat{E}\|\zeta\|_q^2 + 2\hat{E}\int_0^t [\kappa(\|y^{k-1}\|_q^2) + 2|y^k(v)||f(v, 0)||dv + 4c_2^2 \hat{E}\int_0^t [\kappa(\|y^{k-1}\|_q^2) + |h(v, 0)|^2] dv
+ 2c_1 \hat{E}\int_0^t [\kappa(\|y^{k-1}\|_q^2) + 2|y^k(v)||g(v, 0)| + \kappa(\|y^{k-1}\|_q^2) + |h(v, 0)|^2] dv
\leq \hat{E}\|\zeta\|_q^2 + 2\hat{E}\int_0^t [\kappa(\|y^{k-1}\|_q^2) + |y^k(v)|^2 + |f(v, 0)|^2] dv + 4c_2^2 \hat{E}\int_0^t [\kappa(\|y^{k-1}\|_q^2) + |h(v, 0)|^2] dv
+ 2c_1 \hat{E}\int_0^t [\kappa(\|y^{k-1}\|_q^2) + |y^k(v)|^2 + |g(v, 0)|^2 + \kappa(\|y^{k-1}\|_q^2) + |h(v, 0)|^2] dv
\leq \hat{E}\|\zeta\|_q^2 + \hat{c}KT + \hat{c}aT + 2(1 + c_1) \hat{E}\int_0^t |y^k(v)|^2 dv
+ \hat{c}b \hat{E}\int_0^t |\|y^{k-1}\|_q^2| dv,
$$

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where \( \hat{c} = 2(1 + 2c_2^2 + 2c_4) \). By virtue of Lemma 2.10, we have

\[
\mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] \leq \mathbb{E} \|\zeta\|_q^2 + \hat{c}(\hat{K} + a)T + 2(1 + c_1)\mathbb{E} \int_0^t |y^k(v)|^2 dv
\]

\[
+ \hat{c}b\mathbb{E} \int_0^t \|\|\zeta\|_q e^{-\lambda v} + \sup_{0 \leq v \leq t} |y^{k-1}(v)|^2|dv
\]

\[
\leq \mathbb{E} \|\zeta\|_q^2 + \hat{c}(\hat{K} + a)T + 2(1 + c_1)\int_0^t \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] dv
\]

\[
+ \hat{c}b\lambda^{-1}\mathbb{E} \|\zeta\|_q^2 + \hat{c}b \int_0^t \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^{k-1}(v)|^2 \right] dv.
\]

Noticing that

\[
\max_{1 \leq k \leq n} \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^{k-1}(v)|^2 \right] \leq \max \left\{ \mathbb{E} \|\zeta\|_q^2, \max_{1 \leq k \leq n} \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] \right\}
\]

\[
\leq \mathbb{E} \|\zeta\|_q^2 + \max_{1 \leq k \leq n} \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right],
\]

we derive

\[
\max_{1 \leq k \leq n} \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] \leq (1 + \hat{c}b\lambda^{-1} + \hat{c}bT)\mathbb{E} \|\zeta\|_q^2 + \hat{c}(\hat{K} + a)T
\]

\[
+ (2 + 2c_1 + \hat{c}b) \int_0^t \max_{1 \leq k \leq n} \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] dv.
\]

By the Gronwall inequality, it follows

\[
\max_{1 \leq k \leq n} \mathbb{E} \left[ \sup_{0 \leq v \leq t} |y^k(v)|^2 \right] \leq C_4 e^{C_4t},
\]

where \( C_4 = (1 + \hat{c}b\lambda^{-1} + \hat{c}bT)\mathbb{E} \|\zeta\|_q^2 + \hat{c}(\hat{K} + a)T \) and \( \hat{C}_4 = (2 + 2c_1 + \hat{c}b) \). But \( n \) is arbitrary and letting \( t = T \), we get the desired expression. The proof is complete. \( \square \)

**Lemma 6.2.** Let assumption \( A_1 \) holds. Then for \( t \in [0, T] \),

\[
\mathbb{E} \left[ \sup_{0 \leq s \leq t} |y^{k+m}(s) - y^k(s)|^2 \right] \leq \hat{c} \int_0^t \kappa \left( \mathbb{E} \left[ \sup_{0 \leq s \leq t} |y^{k+m-1}(s) - y^{k-1}(s)|^2 \right] \right) ds \leq \hat{C} t,
\]

where \( \hat{C} = \hat{c} \kappa(4C) \), \( \hat{c} = 2(1 + 2c_1 + 2c_2^2) \) and \( c_1, c_2 \) are positive constants.

**Proof.** Define \( \Lambda^{k+m,k}(t) = y^{k+m}(t) - y^k(t) \), \( \hat{f}^{k+m-1,k-1}(t) = f(t, y^{k+m-1}_t) - f(t, y^{k-1}_t) \), \( \hat{g}^{k+m-1,k-1}(t) = g(t, y^{k+m-1}_t) - g(t, y^{k-1}_t) \) and \( \hat{h}^{k+m-1,k-1}(t) = h(t, y^{k+m-1}_t) - h(t, y^{k-1}_t) \). By using the G-Itô formula,
Lemma 6.3. Let assumptions $A_1$ and $A_2$ hold. Let $y(t)$ be a unique solution of problem (1.1) with initial data (1.2). Then $y(t)$ is bounded, in particular, $y(t) \in M^2_G((−\infty,T];\mathbb{R}^n)$.

We omit the proof of the above lemma. It can be proved in a similar way like lemma 7.1. To show the existence of solution we set that for $t \in [0,T]$,

$$\mu_1(t) = \tilde{C} t,$$

(6.1)

and define a recursive function as follows. For every $k, m \geq 1$,

$$\mu_{k+1}(t) = \tilde{k} \int_0^t \kappa(\mu_k(s)) ds,$$

$$\mu_{k,m}(t) = E\left[ \sup_{-\tau \leq q \leq s} |y^{k+m}(q) - y^k(q)|^2 \right].$$

(6.2)

Choose $T_1 \in [0,T]$ such that for all $t \in [0,T_1]$,

$$\tilde{k}\kappa(\tilde{C} t) \leq \tilde{C}.$$  

(6.3)

Theorem 6.4. Let assumptions $A_1$ and $A_2$ hold. Then equation (1.1) with the corresponding initial data (1.2) admits a unique solution $y(t) \in M^2_G((−\infty,T];\mathbb{R}^n)$. 

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Proof. We claim that for all \( k \geq 1 \) and any \( m \geq 1 \) there exists a positive \( T_1 \in [0, T] \) such that

\[
0 \leq \mu_{k,m}(t) \leq \mu_k(t) \leq \mu_{k-1}(t) \leq \ldots \leq \mu_1(t),
\]

for all \( t \in [0, T] \). We use mathematical induction to prove that the inequality (6.4) holds for all \( k \geq 1 \). By virtue of lemma (7.1) and definition of function \( \mu(.) \) it follows

\[
\mu_{1,m}(t) = E\left[ \sup_{-\tau \leq q \leq s} |y^{1+m}(q) - y^1(q)|^2 \right] \leq \hat{C}t = \mu_1(t).
\]

\[
\mu_{2,m}(t) = E\left[ \sup_{-\tau \leq q \leq s} |y^{2+m}(q) - y^2(q)|^2 \right]
\leq \hat{k} \int_{t_0}^t \kappa(E\left[ \sup_{-\tau \leq q \leq s} |y^{1+m}(q) - y^1(q)|^2 \right])ds
\leq \hat{k} \int_{t_0}^t \kappa(\mu_1(s))ds = \mu_2(t).
\]

By using (6.3), we derive

\[
\mu_2(t) = \hat{k} \int_{t_0}^t \kappa(\mu_1(s))ds = \int_{t_0}^t \hat{k}\kappa(\hat{C}t)ds \leq \hat{C}t = \mu_1(t).
\]

Hence for every \( t \in [0, T_1] \), we get that \( \mu_{2,m}(t) \leq \mu_2(t) \leq \mu_1(t) \). Suppose that (6.4) holds for some \( k \geq 1 \). Then we have to show that the inequality (6.4) holds for \( k+1 \), as follows

\[
\mu_{n+1,m}(t) = E\left[ \sup_{-\tau \leq q \leq s} |y^{k+m+1}(q) - y^{k+1}(q)|^2 \right]
\leq \hat{k} \int_{t_0}^t \kappa(E\left[ \sup_{-\tau \leq q \leq s} |y^{k+m}(q) - y^k(q)|^2 \right])ds
= \hat{k} \int_{t_0}^t \kappa(\mu_{k,m}(s))ds \leq \hat{k} \int_{t_0}^t \kappa(\mu_k(s))ds = \mu_{k+1}(t).
\]

And

\[
\mu_{k+1}(t) = \hat{k} \int_{t_0}^t \kappa(\mu_k(s))ds \leq \hat{k} \int_{t_0}^t \kappa(\mu_{k-1}(s))ds = \mu_k(s).
\]

Hence for all \( t \in [0, T_1] \), \( \mu_{k+1,m}(t) \leq \mu_{k+1}(t) \leq \mu_k(s) \), that is, the expression (6.4) holds for \( k+1 \).

We observe that for \( k \geq 1 \), \( \mu_k(t) \) is continuous and decreasing on \( t \in [0, T_1] \). By using the dominated convergence theorem, we define the function \( \mu(t) \) by

\[
\mu(t) = \lim_{n \to \infty} \mu_n(t) = \lim_{n \to \infty} \hat{k} \int_{t_0}^t \kappa(\mu_{n-1}(s))ds = \hat{k} \int_{t_0}^t \kappa(\mu(s))ds, \quad 0 \leq t \leq T_1.
\]

So,

\[
\mu(t) \leq \mu(0) + \hat{k} \int_{t_0}^t \kappa(\mu(s))ds.
\]
Hence for every $0 \leq t \leq T_1$, lemma (3.4) gives that $\mu(t) = 0$. For all $t \in [0, T_1]$, (3.4) follows that $\mu_{k,m}(s) \leq \mu_k(s) \to 0$ as $k \to \infty$, which gives $\mathbb{E}|g^{k+m}(t) - g^k(t)|^2 \to 0$ as $k \to \infty$. Then the completeness of $L^2_G$ implies that for all $t \in [0, T_1]$,

$$f(t, y^k_t) \to f(t, y_t), g(t, y^k_t) \to g(t, y_t), h(t, y^k_t) \to h(t, y_t) \text{ in } L^2_G \text{ as } k \to \infty.$$  

Hence for all $t \in [0, T_1]$,

$$\lim_{k \to \infty} g^k(t) = \zeta(0) + \lim_{k \to \infty} \int_0^t f(s, y^k_{s-})dv + \lim_{k \to \infty} \int_0^t g(s, y^k_{s-})d(B(s), B(t)) + \lim_{k \to \infty} \int_0^t h(s, y^k_{s-})dB(s),$$

which implies

$$y(t) = \zeta(0) + \int_0^t f(v, y_v)dv + \int_0^t g(v, y_v)d(B(s), B(t)) + \int_0^t h(v, y_v)dB(s),$$

that is, equation (3.4) with the corresponding given initial data (1.2) admit a unique solution $z(t)$ on $t \in [t_0, T_1]$. By iteration, we get that equation (1.1) admits a solution on $t \in [0, T]$. The proof of existence is complete. To show the uniqueness, let equation (1.1) admits two solutions $y(t)$ and $z(t)$. Define $\Lambda(t) = z(t) - y(t)$, $\hat{f}(t) = f(t, z_t) - f(t, y_t)$, $\hat{g}(t) = g(t, z_t) - g(t, y_t)$ and $\hat{h}(t) = h(t, z_t) - h(t, y_t)$. Using the G-Itô formula, lemma (2.7) and lemma (2.8) we derive

$$\mathbb{E}\left[ \sup_{0 \leq v \leq t} |\Lambda(v)|^2 \right] \leq \mathbb{E}\left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle \Lambda(v), \hat{f}(v) \rangle dv \right] + \mathbb{E}\left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle \Lambda(v), \hat{h}(v) \rangle dB(v) \right]$$

$$+ \mathbb{E}\left[ \sup_{0 \leq v \leq t} \int_0^t [2\langle \Lambda(v), \hat{g}(v) \rangle + |\hat{h}(v)|^2]d(B(s), B(t)) \right]$$

$$\leq \mathbb{E}\int_0^t 2\langle \Lambda(v), \hat{f}(v) \rangle dv + 2c_2\mathbb{E}\left[ \int_0^t |\langle \Lambda(v), \hat{h}(v) \rangle|^2 dv \right]^{\frac{1}{2}}$$

$$+ c_1\mathbb{E}\int_0^t [2\langle \Lambda(v), \hat{g}(v) \rangle + |\hat{h}(v)|^2]dv$$

$$\leq \mathbb{E}\int_0^t 2\langle \Lambda(v), \hat{f}(v) \rangle dv + 2c_2\mathbb{E}\int_0^t |\hat{h}(v)|^2 dv + \frac{1}{2} \mathbb{E}\left[ \sup_{0 \leq v \leq t} |\Lambda(v)|^2 \right]$$

$$+ c_1\mathbb{E}\int_0^t [2\langle \Lambda(v), \hat{g}(v) \rangle + |\hat{h}(v)|^2]dv.$$

In view of assumption $A_1$ and lemma (7.1) it follows

$$\mathbb{E}\left[ \sup_{0 \leq v \leq t} |z(v) - y(v)|^2 \right] \leq 2(1 + 2c_1 + 2c_2^2)\mathbb{E}\int_0^t \kappa(|z - y|^2) dv$$

$$\leq 2(1 + 2c_1 + 2c_2^2) \int_0^t \kappa\left( \mathbb{E}\left[ \sup_{0 \leq v \leq t} |z(v) - y(v)|^2 \right] \right) dv.$$  

Consequently lemma (2.11) gives

$$\mathbb{E}\left[ \sup_{0 \leq v \leq t} |z(v) - y(v)|^2 \right] = 0,$$

that is, for $t \in [0, T]$, $z(t) = y(t)$. Therefore we have $z(t) = y(t)$ holds quasi-surely for all $t \in (-\infty, T]$. The uniqueness has been proved. \qed

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7 The $L^2_G$ and exponential estimates with weak monotonicity

Let under the assumptions $A_1$ and $A_2$ equation (1.1) with the initial data (1.2) has a unique solution $y(t)$ on $t \in [0, \infty)$. We now find the $L^2_G$ estimate and then the exponential estimate as follows.

**Lemma 7.1.** Let assumptions $A_1$ and $A_2$ hold and $\hat{E} \| \zeta \|^2_q < \infty$. Then for all $t \geq 0$,

$$\hat{E} \left[ \sup_{|y(v)| \leq t} |y(v)|^2 \right] \leq C_5 e^{C_4 t},$$

where $C_5 = (2 + \hat{c}b\lambda^{-1})\hat{E} \| \zeta \|^2_q + \hat{c}(\hat{K} + a)T$ and $\hat{C}_4 = (2 + 2c_1 + \hat{b}c_2), \hat{c} = 2(1 + 2c_1 + 2c_2^2)$ and $c_1, c_2$ are positive constants.

**Proof.** Applying the G-Itô formula to $|y(t)|^2$ and using lemmas 2.7 and 2.8 for any $t \in [0, T]$, we derive

$$\hat{E} \left[ \frac{\sup_{0 \leq v \leq t} |y(v)|^2}{\sup_{|y(v)| \leq t}} \right] \leq \hat{E} \| \zeta \|^2_q + \hat{E} \left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle y(v), f(v, y_v) \rangle dv \right] + \hat{E} \left[ \sup_{0 \leq v \leq t} \int_0^t 2\langle y(v), h(v, y_v) \rangle dB(v) \right] + \hat{E} \left[ \frac{\sup_{0 \leq v \leq t} |y(v)|^2}{\sup_{|y(v)| \leq t}} \right] \leq \hat{E} \| \zeta \|^2_q + \hat{E} \int_0^t \left[ 2\langle y(v), f(v, y_v) \rangle + |h(v, y_v)|^2 \right] dv + 4\hat{E} \int_0^t |f(v, y_v)|^2 dv + \frac{1}{2} \hat{E} \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right]

$$

By using assumptions $A_1$ and $A_2$, it follows

$$\hat{E} \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \leq \hat{E} \| \zeta \|^2_q + 2\hat{E} \int_0^t \left[ \kappa(||y||^2_q) + |y(v)|^2 + |f(v, 0)|^2 \right] dv + 4\hat{E} \int_0^t \left[ \kappa(||y||^2_q) + |h(v, 0)|^2 \right] dv + 2c_1 \hat{E} \int_0^t \left[ \kappa(||y||^2_q) + |y(v)|^2 + |g(0)|^2 + \kappa(||y||^2_q) + |h(v, 0)|^2 \right] dv,$$

which by straightforward calculations, yileds

$$\hat{E} \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \leq \hat{E} \| \zeta \|^2_q + \hat{c}K + \hat{c}a + 2(1 + c_1)\hat{E} \int_0^t |y(v)|^2 dv + \hat{c}b\hat{E} \int_0^t ||y||^2_q dv,$$

where $\hat{c} = 2(1 + 2c_1 + 2c_2^2)$. By virtue of lemma 2.10 we have

$$\hat{E} \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \leq \hat{E} \| \zeta \|^2_q + \hat{c}(\hat{K} + a)T + 2(1 + c_1)\hat{E} \int_0^t |y(v)|^2 dv + \hat{c}b\hat{E} \int_0^t ||y||^2_q e^{-\lambda v} \sup_{0 \leq v \leq t} |y(v)|^2 dv$$

$$\leq \hat{E} \| \zeta \|^2_q + \hat{c}(\hat{K} + a)T + \hat{c}b\lambda^{-1}\hat{E} \| \zeta \|^2_q + (2 + 2c_1 + \hat{b}c_2)\hat{E} \int_0^t \sup_{0 \leq v \leq t} |y(v)|^2 dv.$$
Noting that, \( \hat{E} \left[ \sup_{-\infty < v \leq t} |y(v)|^2 \right] \leq \hat{E} \| \zeta \|_q^2 + \hat{E} \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \), it follows

\[
\hat{E} \left[ \sup_{-\infty < v \leq t} |y(v)|^2 \right] \leq (2 + \hat{c}b \hat{\lambda}^{-1})\hat{E} \| \zeta \|_q^2 + \hat{c}(\hat{K} + a)T + (2 + 2c_1 + \hat{b}c) \int_0^t \hat{E} \left[ \sup_{-\infty < v \leq t} |y(v)|^2 \right] dv.
\]

By using the Grownwall inequality, we get

\[
\hat{E} \left[ \sup_{-\infty < v \leq t} |y(v)|^2 \right] \leq C_5 e^{\hat{C}_4 t},
\]

where \( C_5 = (2 + \hat{c}b \hat{\lambda}^{-1})\hat{E} \| \zeta \|_q^2 + \hat{c}(\hat{K} + a)T \) and \( \hat{C}_4 = 2 + 2c_1 + \hat{b}c \). The proof is complete.

**Theorem 7.2.** Let assumptions \( A_1 \) and \( A_2 \) hold. Then

\[
\lim_{t \to \infty} \sup \frac{1}{t} \log |y(t)| \leq \beta,
\]

where \( \beta = 1 + c_1 + b(1 + 2c_1 + 2c_2^2) \).

**Proof.** By using the Grownwall inequality from (7.1), it follows

\[
\hat{E} \left[ \sup_{0 \leq v \leq t} |y(v)|^2 \right] \leq C_6 e^{\hat{C}_4 t},
\]

where \( C_6 = (1 + \hat{c}b \hat{\lambda}^{-1})\hat{E} \| \zeta \|_q^2 + \hat{c}(\hat{K} + a)T \) and \( \hat{C}_4 = 2 + 2c_1 + \hat{b}c \). By virtue of the above result (7.2), for each \( m = 1, 2, 3, \ldots \), we have

\[
\hat{E} \left[ \sup_{m-1 \leq t \leq m} |y(t)|^2 \right] \leq C_6 e^{\hat{C}_4 m}.
\]

For any \( \epsilon > 0 \), by using lemma 2.6 we get

\[
\tilde{C} \left\{ w : \sup_{m-1 \leq t \leq m} |y(t)|^2 > e^{(\hat{C}_4 + \epsilon)m} \right\} \leq \frac{\hat{E} \left[ \sup_{m-1 \leq t \leq m} |y(t)|^2 \right]}{e^{(\hat{C}_4 + \epsilon)m}} \leq \frac{C_6 e^{\hat{C}_4 m}}{e^{(\hat{C}_4 + \epsilon)m}} = C_6 e^{-\epsilon m}.
\]

Then by similar arguments used in theorem 5.2, we obtain

\[
\lim_{t \to \infty} \sup \frac{1}{t} \log |y(t)| \leq \frac{\hat{C}_4 + \epsilon}{2} = 1 + c_1 + b(1 + 2c_1 + 2c_2^2) + \frac{\epsilon}{2},
\]

but \( \epsilon \) is arbitrary and the above result reduces to

\[
\lim_{t \to \infty} \sup \frac{1}{t} \log |y(t)| \leq \beta,
\]

where \( \beta = 1 + c_1 + b(1 + 2c_1 + 2c_2^2) \). The proof is complete.

We derive the results in the phase space with fading memory \( C_q((\infty, 0); \mathbb{R}^n) \). However, all the results of this article also hold in the space \( BC((\infty, 0); \mathbb{R}^n) \) defined in [6, 19, 26].

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