General derivative Thomae formula for singular half-periods

J. Bernatska

Received: 1 February 2020 / Revised: 22 June 2020 / Accepted: 8 July 2020 / Published online: 16 July 2020
© Springer Nature B.V. 2020

Abstract
The paper develops second Thomae theorem in hyperelliptic case. The main formula, called general Thomae formula, provides expressions for values at zero of the lowest non-vanishing derivatives of theta functions with singular characteristics of arbitrary multiplicity in terms of branch points and period matrix. We call these values derivative theta constants. First and second Thomae formulas follow as particular cases. Some further results are derived. Matrices of second derivative theta constants (Hessian matrices of zero-values of theta functions with characteristics of multiplicity two) have rank three in any genus. Similar result about the structure of order three tensor of third derivative theta constants is obtained, and a conjecture regarding higher multiplicities is made. As a byproduct, a generalization of Bolza formulas are deduced.

Keywords Second and third derivative theta constants · Characteristic via partition · Bolza formula

Mathematics Subject Classification 14K25 · 32A15 · 32G15 · 14H15

1 Introduction
Thomae formulas are of great interest in many areas of mathematics and physics such as quantum field theory, string theory, theory of integrable systems, number theory, p-adic analysis, etc. This paper provides a development of the classical work of Thomae [1]. First and second Thomae formulas give a representation of theta constants with non-singular even and odd characteristics in terms of branch points and period matrix of a hyperelliptic Riemann surface. In the present paper, a similar representation of derivative theta constants with singular characteristics is found for
all possible multiplicities. This problem was not considered in mathematical literature since the time of Thomae.

Instead, generalizations of Thomae formulas to the case of \( \mathbb{Z}_N \)-curves, also called cyclic covers of \( \mathbb{CP}^1 \) or simply cyclic curves, were discovered. This was initiated by paper [2], where a generalization of first Thomae formula provided an expression for determinant of Dirac’s operator in terms of branch points of \( \mathbb{Z}_N \)-curve. This gave rise to a flow of publications. In [3] a rigorous proof of the mathematical result from [2] was given. Then, first Thomae formula was generalized to a special class of singular \( \mathbb{Z}_N \)-curve in [4], to general cyclic covers of \( \mathbb{CP}^1 \) in [5], to Abelian covers of \( \mathbb{CP}^1 \) in [6], and developed in other papers and a book of H. Farkas and Sh. Zemel Generalization of Thomae’s Formula for \( \mathbb{Z}_N \) Curves (2010) containing many examples. A detailed generalization of first Thomae formula for a trigonal cyclic curve with a specific choice of symplectic cohomology basis is given in [7]. The only generalization of second Thomae formula was obtained in [8] for trigonal cyclic curves.

This paper is organized as follows. Section 2 contains the minimal background, definitions and notation regarding theta and sigma functions, Thomae theorems and some auxiliary lemmas. Section 3 is devoted to the main theorem (Theorem 1), which generalizes second Thomae theorem, with a detailed proof. In Sect. 4 corollaries of the main theorem are presented with examples in genera 3, 4, 5 and 6. Corollaries 1–3 provide some further representations of first, second and third derivative theta constants. This allows to establish some essential results. In particular, the rank of matrices of second derivative theta constants (Hessian matrices) is three in any genus (Theorem 3). Also a generalization of Bolza formulas [9] for branch points and their symmetric functions is obtained. Section 5 contains a summary of our results and links to related problems posed in the literature.

2 Preliminaries

2.1 Hyperelliptic curve

A hyperelliptic curve \( C \) is described by its branch points \( \{ (e_j, 0) \} \), and defined by equation

\[
0 = f(x, y) = -y^2 + \prod_{j=1}^{2g+1} (x - e_j). \tag{1}
\]

The branch points are all distinct if a curve is non-degenerate. One more branch point is located at infinity which serves as a base point.

Homology basis is defined after H. Baker [10, p. 303]. One can imagine a continuous path through all branch points, which ends at infinity. The branch points are denoted by \( \{ e_j \}_{j=1}^{2g+1} \) along the path, infinity is denoted by \( e_{2g+2} \). Cuts are made between points \( e_{2k-1} \) and \( e_{2k} \) with \( k \) from 1 to \( g + 1 \). Canonical homology cycles are defined as follows. Each \( a_k \)-cycle encircles the cut \( (e_{2k-1}, e_{2k}) \), \( k = 1, \ldots, g \), and each \( b_k \)-cycle enters two cuts \( (e_{2k-1}, e_{2k}) \) and \( (e_{2g+1}, e_{2g+2}) \), see Fig. 1.
The standard cohomology basis is employed, it consists of first kind differentials $d\upsilon = (d\upsilon_1, d\upsilon_3, \ldots, d\upsilon_{2g-1})'$ and second kind differentials $d\tau = (d\tau_1, d\tau_3, \ldots, d\tau_{2g-1})'$ associated to the first kind differentials, see for example [10, p. 306],

$$d\upsilon_{2n-1} = \frac{x^{g-n}dx}{\partial_y f}, \quad n = 1, \ldots, g,$$

$$d\tau_{2n-1} = \frac{dx}{\partial_y f} \sum_{k=1}^{2n-1} k\lambda_{4n-2k+2}x^{g-n+k}, \quad n = 1, \ldots, g.$$  (2) (3)

Here, $\lambda_i$ denote coefficients of the curve (1) written in the form

$$f(x, y) = -y^2 + x^{2g+1} + \sum_{i=0}^{2g} \lambda_{4+2-2i}x^i.$$

The differentials are labeled by Satō weights in subscripts, for convenience. The weight shows an exponent of the leading term in expansion of the corresponding integral about infinity in parameter $\xi$, $x = \xi^{-2}$, namely $\text{wgt} \, \upsilon_{2n-1} = 2n - 1$, and $\text{wgt} \, \tau_{2n-1} = -(2n - 1)$.

Integrals of the differentials along the canonical homology cycles give first and second kind periods

$$\omega = (\omega_{ij}) = \left( \int_{a_j} d\upsilon_i \right), \quad \omega' = (\omega'_{ij}) = \left( \int_{b_j} d\upsilon_i \right),$$

$$\eta = (\eta_{ij}) = \left( \int_{a_j} d\tau_i \right), \quad \eta' = (\eta'_{ij}) = \left( \int_{b_j} d\tau_i \right).$$

The $g \times g$ matrices $\omega, \omega', \eta, \eta'$ form $2g \times 2g$ matrix

$$\Omega = \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix},$$

which is symplectic with respect to a complex structure $J$, $J^2 = -1_{2g}$, where $1_{2g}$ is the identity matrix of size $2g$, and $J' = -J$,

$$\Omega J \Omega' = 2\pi i J.$$  (4)
Relation (4) is known as the generalized Legendre relation, see for example [15, p. 16]. With complex structure matrix $J$ of the form

$$J = \begin{pmatrix} 0_g & 1_g \\ -1_g & 0_g \end{pmatrix},$$

where $1_g$ is the identity matrix of size $g$, and $0_g$ is $g \times g$ zero matrix. (4) produces relations

$$\omega \omega' = \omega' \omega, \quad \eta \eta' = \eta' \eta, \quad \eta \omega' = \omega' \eta - 2\pi i 1_g, \quad \omega' \eta = \eta' \omega - 2\pi i 1_g.$$

(5)

### 2.2 Theta and sigma functions

Each curve of the family $C$ has a Jacobian variety $\text{Jac}(C) = \mathbb{C}^g \setminus \mathcal{B}$, which is a quotient space of $\mathbb{C}^g$ by the lattice $\mathcal{B}$ of periods formed by columns of the matrix $(\omega, \omega')$. Let $u$ with coordinates $(u_1, u_3, \ldots, u_{2g-1})^t$ denote a point of Jacobian. These are variables of sigma function, which is defined here with the help of theta function.

Theta function is an entire function on $\mathbb{C}^g$ defined with respect to normalized periods $(1_g, \tau)$, where $\tau = \omega^{-1} \omega'$ is a symmetric matrix with positive imaginary part: $\tau' = \tau$, $\Im \tau > 0$, that is $\tau$ belongs to Siegel upper half-space. Normalized holomorphic differentials are

$$dv = \omega^{-1} du,$$

and similarly normalized coordinates of the Jacobian are defined: $v = \omega^{-1} u$, $v = (v_1, v_2, \ldots, v_g)^t$. This change of coordinates is essential for the relation (11) between theta and sigma functions. Riemann theta function is defined for $v \in \mathbb{C}^g$ as a Fourier series of the form

$$\theta(v; \tau) = \sum_{n \in \mathbb{Z}^g} \exp \left( i\pi n^t \tau n + 2i\pi n^t v \right).$$

(6)

Abel’s map $A$ maps the curve to its Jacobian

$$\text{Jac}(C) \ni A(P) = \int_{-\infty}^P dv, \quad P \in C.$$

(7)

Abel’s map of a positive divisor $D = \sum_{i=1}^n P_i$ on $C$ is defined by

$$A(D) = \sum_{i=1}^n \int_{-\infty}^{P_i} dv.$$

(8)
Each branch point \((e, 0)\) of a hyperelliptic curve (1) is identified with a half-period, see [20, Sect. 202, pp. 300–301]

\[
\mathcal{A}(e) = \int_{\infty}^{(e,0)} dv = \varepsilon/2 + \tau \varepsilon'/2, \quad \left[ \varepsilon'' \right] = [\varepsilon],
\]

(9)

where components of \(\varepsilon\) and \(\varepsilon'\) are 0 or 1. The integer \(2 \times g\)-matrix \([\varepsilon]\) is the characteristic of branch point \(e\). Characteristics are added by the rule \([\varepsilon] + [\delta] = ([\varepsilon] + [\delta]) \mod 2\). Theta function with characteristic \([\varepsilon]\) is given by the formula

\[
\theta[\varepsilon](v; \tau) = \exp \left( i\pi (\varepsilon''/2) \tau (\varepsilon'/2) + 2i\pi (v + \varepsilon/2) \varepsilon'/2 \right) \\
\times \theta(v + \varepsilon/2 + \tau \varepsilon'/2; \tau).
\]

(10)

A characteristic \([\varepsilon]\) is odd whenever \(\varepsilon \varepsilon' \mod 2 = 0\), and even whenever \(\varepsilon \varepsilon' \mod 2 = 1\). Theta function with characteristic has the same parity as its characteristic.

Another entire function, sigma function, is connected to theta function by relation

\[
\sigma(u) = \frac{1}{C} \exp \left( -\frac{1}{2} u' \kappa u \right) \theta[K] \left( \omega^{-1} u; \omega^{-1} \omega' \right),
\]

(11)

where \(\kappa = \eta \omega^{-1}\) is a symmetric matrix defined through period matrices \(\omega\) and \(\eta\), and \([K]\) denotes the characteristic of vector of Riemann constants. This formula arises as a definition of fundamental sigma functions in [11, p. 97] without constant \(C\). In [12] this constant was found by methods developed in [10, 11]. Also the constant is explicitly defined in [13, Eq. (3.29), p. 906], which is proved in [15, p. 33].

### 2.3 Characteristics in hyperelliptic case

The method of constructing characteristics in hyperelliptic case is adopted from [16, p. 1012]. It is based on the definition (9) of half-period characteristics. Let \([\varepsilon_k]\) be the characteristic of branch point \(e_k\). Evidently, \([\varepsilon_{2g+2}] = 0\). Guided by the picture of canonical homology cycles, one can find

\[
\mathcal{A}(e_{2g+1}) = \mathcal{A}(e_{2g+2}) + \sum_{k=1}^{g} \int_{e_{2k-1}}^{e_{2k}} dv \quad \left[ \varepsilon_{2g+1} \right] = \begin{bmatrix} 00...00 \end{bmatrix}_{11...11},
\]

\[
\mathcal{A}(e_{2g}) = \mathcal{A}(e_{2g+1}) - \int_{e_{2g}}^{e_{2g+1}} dv \quad \left[ \varepsilon_{2g} \right] = \begin{bmatrix} 00...01 \end{bmatrix}_{11...11},
\]

\[
\mathcal{A}(e_{2g-1}) = \mathcal{A}(e_{2g}) - \int_{e_{2g-1}}^{e_{2g}} dv \quad \left[ \varepsilon_{2g+1} \right] = \begin{bmatrix} 00...01 \end{bmatrix}_{11...10},
\]

for \(k\) from \(g - 1\) to 2

\[
\mathcal{A}(e_{2k}) = \mathcal{A}(e_{2k+1}) - \int_{e_{2k}}^{e_{2k+1}} dv \quad \left[ \varepsilon_{2k} \right] = \begin{bmatrix} 00...010...0 \end{bmatrix}_{11...10...0},
\]

\[
\mathcal{A}(e_{2k+1}) = \mathcal{A}(e_{2k}) - \int_{e_{2k+1}}^{e_{2k+2}} dv \quad \left[ \varepsilon_{2k+1} \right] = \begin{bmatrix} 00...010...0 \end{bmatrix}_{11...10...0}.
\]
\( \mathcal{A}(e_{2k-1}) = \mathcal{A}(e_{2k}) - \int_{e_{2k-1}}^{e_{2k}} dv \)

and finally

\( \mathcal{A}(e_2) = \mathcal{A}(e_3) - \int_{e_2}^{e_3} dv \)

\( \mathcal{A}(e_1) = \mathcal{A}(e_2) - \int_{e_1}^{e_2} dv \)

This set of characteristics is azygetic and serves as a fundamental system, see [17, pp. 181–184].

Characteristic \([K]\) of the vector of Riemann constants \(K\) equals the sum of all odd characteristics of branch points, there are \(g\) such characteristics, see [20, Sect. 200, p. 297, Sect. 202, p. 301]. Actually,

\[ [K] = \sum_{k=1}^{g} [\varepsilon_{2k}] . \]

### 2.4 Characteristics and partitions

Let \( \mathcal{I} \cup \mathcal{J} \) be a partition of the set of indices of all branch points \( \{1, 2, \ldots, 2g + 2\} \), denote by \( \varepsilon(\mathcal{I}) = \sum_{i \in \mathcal{I}} \varepsilon_i \) the characteristic of

\[ \mathcal{A}(\mathcal{I}) = \sum_{i \in \mathcal{I}} \mathcal{A}(e_i) = \frac{1}{2} \varepsilon_\mathcal{I} + \frac{1}{2} \tau^{e_\mathcal{I}}. \]

Below a partition is often referred to by the part of less cardinality, denoted by \( \mathcal{I} \).

Characteristics of \(2g + 2\) branch points of (1) serve as a basis for constructing all \(2^{2g}\) half-period characteristics. According to [18, p. 13] and [20, Sect. 202, p. 301] all half-period characteristics are represented by partitions of \(2g + 2\) indices of the form \( \mathcal{I}_m \cup \mathcal{J}_m \) with \( \mathcal{I}_m = \{i_1, \ldots, i_{g+1-2m}\} \) and \( \mathcal{J}_m = \{j_1, \ldots, j_{g+1+2m}\} \), where \(m\) runs from 0 to \([(g+1)/2]\), and \([\cdot]\) means the integer part. Number \(m\) is called multiplicity. Index \(2g + 2\) corresponding to infinity is usually omitted in the sets, and inferred in the part with an incomplete number of indices.

Introduce also characteristic \( [\mathcal{I}_m] = [\varepsilon(\mathcal{I}_m)] + [K] \) of

\[ \sum_{i \in \mathcal{I}_m} \mathcal{A}(e_i) + K = \frac{1}{2} \delta_{\mathcal{I}_m} + \frac{1}{2} \tau \delta_{\mathcal{I}_m} , \]

which corresponds to a partition \( \mathcal{I}_m \cup \mathcal{J}_m \). Note that \( [\mathcal{J}_m] \) represents the same characteristic as \( [\mathcal{I}_m] \). Characteristics \([\mathcal{I}_m]\) of even multiplicity \(m\) are even, and of odd \(m\) are odd. According to Riemann theorem \( \theta(v + \mathcal{A}(\mathcal{I}_m) + K) \) vanishes to order \(m\) at
$v = 0$. Characteristics of multiplicity 0 are called *non-singular even characteristics*, there are $\left(\frac{2g+1}{g-1}\right)$ such characteristics. There exist $\left(\frac{2g+2}{g-1}\right)$ characteristics of multiplicity 1, which are called *non-singular odd*. All other characteristics are called *singular*. The number of characteristics of multiplicity $m > 1$ is $\left(\frac{2g+2}{g+1-2m}\right)$.

Characteristic $[K]$ corresponds to the partition $\{\} \cup \{1, 2, \ldots, 2g + 1\}$, which is always unique, and $\theta[K](v)$ vanishes to the maximal order $\left(\frac{g+1}{2}\right)$ at $v = 0$. In what follows, representation of characteristics in terms of partitions is preferable, because this makes clear which order of vanishing a theta function has at $v = 0$.

Let a collection of branch points $\{e_i \mid i \in \mathcal{I}\}$ correspond to a partition $\mathcal{I} \cup \mathcal{J}$. Then, $s_n(\mathcal{I})$ denotes an elementary symmetric polynomial of degree $n$ in $\{e_i \mid i \in \mathcal{I}\}$, and $\Delta(\mathcal{I})$ denotes the Vandermonde determinant in branch points from the collection

$$\Delta(\mathcal{I}) = \prod_{i,l \in \mathcal{I}} (e_i - e_l).$$

The Vandermonde determinant in all branch points of the curve is denoted by $\Delta$.

**Remark 1** Let branch points in all factors $(e_i - e_l)$ be ordered in such a way that $i > l$, we call this right ordering. This allows to avoid multiplier $\epsilon$, which arises in many relations. Such ordering was suggested by Baker [10, p. 346].

Theta function with characteristic $[\mathcal{I}]$ corresponds to sigma function at a half-period $A(\mathcal{I})$ as defined in [13,14], indeed

$$\sigma(\omega A(\mathcal{I})) = \frac{1}{C} \exp\left(-\frac{1}{2}A(\mathcal{I})' \omega' \eta A(\mathcal{I})\right)\theta[\epsilon_K](A(\mathcal{I}); \omega^{-1} \omega')$$

$$= \frac{1}{C} \exp\left(-\frac{1}{8}(\omega \epsilon + \omega' \epsilon')(\eta \epsilon + \eta' \epsilon')\right)\theta[\mathcal{I}](0; \omega^{-1} \omega')$$

(12)

where $\epsilon, \epsilon'$ denote $\epsilon(\mathcal{I}), \epsilon'(\mathcal{I})$, and Legendre relations (5) are used

$$\tau \omega' \eta = \omega^{-1} \omega' \omega' \eta = \omega'' \eta = \eta'' \omega - 2\pi i 1_g.$$

One can notice that exponential factor in (12) is $\exp\left(-\frac{1}{8} \tilde{\epsilon} \tilde{\epsilon}'\right)$ with the characteristic $[\epsilon]$ transformed by symplectic matrix $\Omega$

$$\begin{pmatrix} \tilde{\epsilon} \\ \tilde{\epsilon}' \end{pmatrix} = \Omega \begin{pmatrix} \epsilon \\ \epsilon' \end{pmatrix}.$$

### 2.5 Notation of theta constants

In what follows we use notation $\partial_{v_i}$ for derivative with respect to variable $v_i$, and omit argument $\tau$ of theta function, so

$$\partial_{v_i} \theta[\epsilon](v) = \frac{\partial}{\partial v_i} \theta[\epsilon](v; \tau),$$
\[ \frac{\partial^2}{\partial v_i \partial v_j} \theta[\varepsilon](v) = \frac{\partial^2}{\partial v_i \partial v_j} \theta[\varepsilon](v; \tau), \]

etc.

We also use the standard notation \( \theta[\varepsilon] \) for theta constant with characteristic \( [\varepsilon] \), or \( \theta[I_0] \) for one with the characteristic corresponding to partition \( I_0 \cup J_0 \). Thus

\[ \theta[I_0] = \theta[I_0](0; \tau), \]
\[ \partial_{v_i} \theta[I_1] = \left. \frac{\partial}{\partial v_i} \theta[I_1](v; \tau) \right|_{v=0}, \]
\[ \partial^2_{v_i, v_j} \theta[I_2] = \left. \frac{\partial^2}{\partial v_i \partial v_j} \theta[I_2](v; \tau) \right|_{v=0}, \]

etc.

In the literature we find values at \( v = 0 \) of theta functions with non-singular even characteristics, which are called theta constants, or theta constants of the first kind [16, p. 1011]. Ibid values at \( v = 0 \) of first derivatives of theta functions with non-singular odd characteristics are called theta constants of the second kind. However, there is no consensual term for the latter, and no term at all for the values of higher derivatives of theta functions at \( v = 0 \).

As mentioned above, \( \theta[I_m](v) \) vanishes to order \( m \) at \( v = 0 \). We are interested in values at \( v = 0 \) of the lowest non-vanishing derivatives of theta functions with characteristics of arbitrary multiplicity \( m \), that is \( \partial^m v \theta[I_m] \). Here symbol \( \partial^m v \) denotes a tensor of order \( m \) of partial \( m \)-th derivatives with respect to all combinations constructed from \( g \) components of \( v \). We call them derivative theta constants, or \( m \)-th derivative theta constants more precisely.

### 2.6 First Thomae formula

**First Thomae theorem** Let \( I_0 \cup J_0 \) with \( I_0 = \{ i_1, \ldots, i_g \} \) and \( J_0 = \{ j_1, \ldots, j_{g+1} \} \) be a partition of the set \( \{1, 2, \ldots, 2g + 1\} \) of indices of finite branch points, and \( [I_0] \) denotes the non-singular even characteristic corresponding to \( \mathcal{A}(I_0) + K \). Then

\[ \theta[I_0] = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(I_0)^{1/4} \Delta(J_0)^{1/4}. \tag{13} \]

where \( \epsilon \) satisfies \( \epsilon^8 = 1 \), and \( \Delta(I_0) \), \( \Delta(J_0) \) denote Vandermonde determinants built from \( \{ e_i \mid i \in I_0 \} \) and \( \{ e_j \mid j \in J_0 \} \).

For a proof see [18, Proposition 3.6, p. 46]. This form of first Thomae theorem is taken from [16, p. 1014], as well as second Thomae theorem [16, p. 1015] below.
2.7 Second Thomae formula

Second Thomae theorem Let $\mathcal{I}_1 \cup \mathcal{J}_1$ with $\mathcal{I}_1 = \{i_1, \ldots, i_{g-1}\}$ and $\mathcal{J}_1 = \{j_1, \ldots, j_{g+1}\}$ be a partition of the set $\{1, 2, \ldots, 2g + 1\}$ of indices of finite branch points, and $[\mathcal{I}_1]$ denote the non-singular odd characteristic corresponding to $A(\mathcal{I}_1) + K$. Then for any $n \in \{1, \ldots, g\}$

$$\frac{\partial}{\partial v_n} \theta[I_1](v) \bigg|_{v=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(I_1)^{1/4} \Delta(J_1)^{1/4} \sum_{j=1}^{g} (-1)^{j-1} s_{j-1}(I_1) \omega_{j,n}. \quad (14)$$

where $\epsilon$ satisfies $\epsilon^8 = 1$, and $\Delta(I_1)$, $\Delta(J_1)$ denote Vandermonde determinants built from $\{e_i \mid i \in \mathcal{I}_1\}$ and $\{e_j \mid j \in \mathcal{J}_1\}$, then $s_j(I)$ denotes the elementary symmetric polynomial of degree $j$ in $\{e_i \mid i \in \mathcal{I}\}$.

This result is nicely presented in a matrix form

$$\begin{pmatrix} \frac{\partial v_1}{\partial v_1} \\ \frac{\partial v_2}{\partial v_2} \\ \vdots \\ \frac{\partial v_g}{\partial v_g} \end{pmatrix} \theta[I_1](v) \bigg|_{v=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(I_1)^{1/4} \Delta(J_1)^{1/4} \begin{pmatrix} s_0(I_1) \\ -s_1(I_1) \\ \vdots \\ (-1)^{g-1} s_{g-1}(I_1) \end{pmatrix}. \quad (15)$$

In terms of non-normalized variables $u$

$$\begin{pmatrix} \frac{\partial u_1}{\partial u_1} \\ \frac{\partial u_3}{\partial u_3} \\ \vdots \\ \frac{\partial u_{2g-1}}{\partial u_{2g-1}} \end{pmatrix} \theta[I_1](\omega^{-1}u) \bigg|_{u=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(I_1)^{1/4} \Delta(J_1)^{1/4} \begin{pmatrix} s_0(I_1) \\ -s_1(I_1) \\ \vdots \\ (-1)^{g-1} s_{g-1}(I_1) \end{pmatrix}. \quad (16)$$

Second Thomae theorem is derived from Lemma 1, for more detail see the proof invented by V. Enolski, published in [19, pp. 92–95] (Enolski supervised K. Eilers’ research) and [8, pp. 2–4].
Here, the definition of elementary symmetric polynomials $s_l$ in $\{e_i \mid i \geq 0\}$ is recalled

$$\sum_{l \geq 0} t^l s_l = \prod_{i \geq 0} (1 + e_i t).$$

### 2.8 Auxiliary lemmas

The following Lemmas are essential in the proof of second Thomae formula. They are used below as well.

**Lemma 1** Let $D = \sum_{i=1}^{g} P_i$ be a divisor of $g$ finite points $\{P_i = (x_i, y_i)\}$ of a hyperelliptic curve of genus $g$, and $[\varepsilon_k]$ denote the characteristic of a branch point $e_k$. Then

$$\frac{\theta[\varepsilon_k](v(D) + K)^2}{\theta(v(D) + K)^2} = \frac{\epsilon}{\sqrt{\partial_x f(e_k, 0)}} \prod_{i=1}^{g} (e_k - x_i),$$

where $\epsilon^4 = 1$, and $K$ denotes the Riemann constant vector related to the base point at infinity.

Here and below $v(D)$ denotes the argument of theta functions which is considered as a function of a divisor $D$.

A proof can be found in [19, pp. 92–95] or [8, pp. 2–4].

**Lemma 2** The Jacobian matrix $\frac{\partial (x_1, x_2, \ldots, x_g)}{\partial (v_1, v_2, \ldots, v_g)}$ of Abel’s map $A : C^g \mapsto \text{Jac}(C)$ has the following entries

$$\frac{\partial x_p}{\partial v_n} = -2y_p \sum_{j=1}^{g} (-1)^{j-1} s_{j-1}^{(p)} \omega_{jn} \prod_{l=1}^{g} (x_p - x_l),$$

where $p$ and $n$ run from 1 to $g$, and $x_p$ represents a points $P_p = (x_p, y_p)$ of the curve.

**Proof** Introduce

$$P_p(x) = \prod_{l \neq p}^{g} \frac{x - x_l}{x_p - x_l} = \frac{\sum_{j=1}^{g} (-1)^{j-1} s_{j-1}^{(p)} x^{g-j}}{\prod_{l \neq p}^{g} (x_p - x_l)}, \quad p = 1, \ldots, g,$$

where $s_{j}^{(p)}$ denotes the elementary symmetric polynomial of order $l$ built from elements $\{x_1, \ldots, x_g\} \setminus \{x_p\}$. Evidently,

$$P_p(x_k) = \delta_{pk}.$$
Taking into account that $v = \omega^{-1} u$, where $u$ is defined by (2), we find the inverse to Jacobian matrix 
\[
\frac{\partial(x_1, \ldots, x_g)}{\partial(u_1, \ldots, u_{2g-1})} = J
\]
in non-normalized variables
\[
J^{-1} = \begin{pmatrix}
x_1^{g-1} & x_2^{g-1} & \cdots & x_1^{g-1} \\
-2y_1 & -2y_2 & \cdots & -2y_g \\
\vdots & \vdots & \ddots & \vdots \\
-2y_1 & -2y_2 & \cdots & -2y_g \\
-2y_1 & -2y_2 & \cdots & -2y_g
\end{pmatrix}.
\]

This implies
\[
\sum_{j=1}^{g} J_{pj} x_k^{g-j} = \delta_{pk}.
\]

Comparing the last equation with (18), we obtain
\[
J_{pj} = -2y_p (-1)^{j-1} s_{j-1}^{(p)} \prod_{l \neq p}^{g} (x_p - x_l).
\]

Thus,
\[
\frac{\partial x_p}{\partial v_n} = \sum_{j=1}^{g} J_{pj} \omega_{jn} = -2y_p \sum_{j=1}^{g} (-1)^{j-1} s_{j-1}^{(p)} \omega_{jn} \prod_{l \neq p}^{g} (x_p - x_l).
\]

\[\square\]

2.9 Verification

All formulas and relations given in the paper are verified by direct computation of left and right hand sides. Curves with real branch points are used. Period matrices $\omega, \omega', \eta, \eta'$ are computed explicitly, as well as matrices $\tau$ and $\varkappa$ for each curve. Hyperelliptic curves of genera 3, 4, 5, and 6, and theta functions with characteristics up to multiplicity 3 are taken.

3 Main theorem

**Theorem 1** (General Thomae formula) Let $\mathcal{I}_m \cup \mathcal{J}_m$ with $\mathcal{I}_m = \{i_1, \ldots, i_{g+1-2m}\}$ and $\mathcal{J}_m = \{j_1, \ldots, j_{g+1+2m}\}$ be a partition of the set $\{1, 2, \ldots, 2g + 2\}$ of indices of branch points of hyperelliptic curve, and $[\mathcal{I}_m]$ denote the singular characteristic of multiplicity $m$ corresponding to $\mathcal{A}(\mathcal{I}_m) + K$. Then for any $n_1, \ldots, n_m \in \{1, \ldots, g\}$
and any set $\mathcal{K} \subset \mathcal{J}_m$ of cardinality $\ell = 2m - 1$ or $2m$ the following relation holds

$$
\frac{\partial}{\partial v_{n_1}} \cdots \frac{\partial}{\partial v_{n_m}} \theta(\mathcal{I}_m)(v)\bigg|_{v=0} = \epsilon \left(\frac{\det \omega}{\pi^g}\right)^{1/2} \Delta(\mathcal{I}_m)^{1/4} \Delta(\mathcal{J}_m)^{1/4}
\times \sum_{p_1, \ldots, p_m \in \mathcal{K} \atop \text{all different}} \prod_{i=1}^m \sum_{j=1}^g (-1)^{j-1} s_{j-1}(\mathcal{I}_m \cup \mathcal{K}(p_i)) \omega_{j n_i},
$$

(19)

where $\mathcal{K}(p_i) = \mathcal{K}\setminus\{p_i\}$, $\epsilon$ satisfies $\epsilon^8 = 1$, then $\Delta(\mathcal{I}_m)$, $\Delta(\mathcal{J}_m)$ denote Vandermonde determinants built from $\{e_i \mid i \in \mathcal{I}_m\}$ and $\{e_j \mid j \in \mathcal{J}_m\}$, and $s_j(\mathcal{I})$ denotes the elementary symmetric polynomial of degree $j$ in $\{e_i \mid i \in \mathcal{I}\}$. The relation does not depend on the choice of $\mathcal{K}$.

**Remark 2** Formula (19), in particular, gives first Thomae formula when $m = 0$, and second Thomae formula when $m = 1$ and $\ell = 2m - 1$. That is why it is called general. However, the proof given below covers only the cases of $\ell \geq 2$.

**Proof of Theorem 1** We start with the known result from [20, Sect. 209, p. 312] with factor $\sqrt{\Delta(\mathcal{K})}$ obtained in [10, Sect. VII, pp. 348–350].

**Theta quotient theorem 1** Let $\mathcal{D} = \sum_{k=1}^g P_k$ be a non-special divisor of $g$ finite points $\{P_k = (x_k, y_k)\}$ of a hyperelliptic curve of genus $g$, and $\mathcal{K}$ be a subset of $\{1, 2, \ldots, 2g + 1\}$ of cardinality $\ell$, and $[e_\kappa]$ denote the characteristic of a branch point $e_\kappa$. Then

$$
\theta\left[\sum_{\kappa \in \mathcal{K}} e_\kappa\right](v(\mathcal{D}) + K) \theta(v(\mathcal{D}) + K)^{\ell-1}
\times \det \left| \begin{array}{cccc}
 y_k x_k^l & y_k x_k^{l-1} & \cdots & y_k x_k^{n-1} \\
 \phi_\kappa(x_k) & \phi_\kappa(x_k) & \cdots & \phi_\kappa(x_k) \\
 \cdots & \cdots & \cdots & \cdots \\
 x_k^n & x_k & \cdots & 1 \\
\end{array} \right|_{k=1}^g,
$$

(20)

where $\epsilon^4 = 1$, then $l = [\ell/2] - 1$ and $n = g - 1 - [\ell/2]$ for $\ell \geq 2$, here $[\cdot]$ denotes the integer part, and $\phi_\kappa(x) = \prod_{\kappa \in \mathcal{K}} (x - e_\kappa)$.

As mentioned, formula (20) works for $\ell \geq 2$. If $\ell = 1$, which is the case of second Thomae formula, one should assign $n = g$, and entries with $y_k$ are absent. So (20) turns into a trivial identity, and Lemma 1 is used instead.

**Theta quotient theorem 2** Under the assumptions of Theta quotient Theorem 1 the following relation holds

$$
\frac{\theta_\kappa(v(\mathcal{D}) + K)}{\theta(v(\mathcal{D}) + K)} = \frac{\epsilon}{\left(\prod_{\kappa \in \mathcal{K}^\ast}(e_\kappa - e_j)\right)^{1/4}} \frac{\Phi_\kappa[x_1, \ldots, x_g]}{\Delta[x_1, \ldots, x_g]},
$$

(21)

where $[\epsilon(\mathcal{K})] = \sum_{\kappa \in \mathcal{K}} [e_\kappa]$, and $\mathcal{K}^\ast$ denotes the complement set to $\mathcal{K}$ that is $\mathcal{K} \cup \mathcal{K}^\ast$ serves as a partition of all $2g + 1$ indices of finite branch points, and the following

@ Springer
General derivative Thomae formula for singular half-periods

Let \( \Delta[x_1, \ldots, x_g] = \det \hat{V}, \quad \hat{V} = \left\| x_k^{g-1}, x_k^{g-2}, \ldots, x_k, 1 \right\|_{k=1}^g, \)

(22a)

\[ \phi_{K}[x_1, \ldots, x_g] = \det \hat{\phi}_{K}, \]

(22b)

\[ \hat{\phi}_{K} = \| x_k^1 \sqrt{\phi_{K}(x_k)}, x_k^{g-1} \sqrt{\phi_{K}(x_k)}, \ldots, \sqrt{\phi_{K}(x_k)} \|_{k=1}^g, \]

(22c)

Here \( \phi_{K}(x) = \prod_{j \in K^*} (x - e_j). \)

**Proof** Combining Theta quotient Theorem 1 and Lemma 1 we find

\[
\frac{\theta[e(K)](v(D) + K)}{\theta(v(D) + K)} = \frac{\epsilon \sqrt{\Delta(K)}}{\left( \prod_{\kappa \in K} \partial_x f(e_\kappa, 0) \right)^{1/4}} \\
\times \sqrt{\frac{\prod_{k=1}^g \phi(x_k)}{\Delta[x_1, \ldots, x_g]}} \det \| y_k x_k^1 \sqrt{\phi_{K}(x_k)}, y_k x_k^{g-1} \sqrt{\phi_{K}(x_k)}, \ldots, y_k, x_k^n, x_k^{n-1}, \ldots, 1 \|_{k=1}^g.
\]

Then, we take into account that

\[
\sqrt{\Delta(K)} = \frac{1}{\left( \prod_{\kappa \in K} \partial_x f(e_\kappa, 0) \right)^{1/4}} \left( \prod_{j \in K^*} (e_\kappa - e_j) \right)^{1/4},
\]

and

\[
y_k = \left( \prod_{i=1}^{2g+1} (x_k - e_i) \right)^{1/2} = \sqrt{\phi_{K}(x_k) \phi_{K}(x_k)}.\]

\[ \square \]

Derivation of (21) leads to

\[
\frac{\partial}{\partial v_n} \frac{\theta[e(K)](v(D) + K)}{\theta(v(D) + K)} = \frac{\epsilon}{\left( \prod_{\kappa \in K} (e_i - e_j) \right)^{1/4}} \\
\times \sum_{p=1}^g \frac{\partial x_p}{\partial v_n} \left( \frac{\Phi_{K}^{(p)}}{\Delta} - \Phi_{K} \frac{\Delta^{(p)}}{\Delta^2} \right),
\]

(23)

where \( \Phi_{K} \) and \( \Delta \) are constructed with the set of points \( \{x_1, \ldots, x_g\} \) of divisor \( D \), and the notation \( \Phi_{K}^{(p)} = \partial \Phi_{K}/\partial x_p \), and \( \Delta^{(p)} = \partial \Delta/\partial x_p \) is adopted. Since each row of \( \hat{\Phi}_{K} \) depends on a particular point \( x_k \), derivative of \( \det \hat{\Phi}_{K} \) with respect to \( x_p \) is the
determinant of matrix $\hat{\Phi}_K^{(p)}$ with the same entries as in $\hat{\Phi}_K$ except for $p$-th row which is replaced by its derivative. Namely, the $p$-th row has the form

$$
(\hat{\Phi}_K^{(p)})_p = \frac{1}{2y_p} \left( x_p^1 \hat{\Phi}_K^{(p)}(x_p) \sqrt{\phi_K(x_p)} + 2y_p x_p^{l-1} \sqrt{\phi_K}(x_p), \right.
$$

$$
\ldots, \hat{\Phi}_K^{(p)}(x_p) \sqrt{\phi_K(x_p)}, x_p^n \hat{\Phi}_K^{(p)}(x_p) \sqrt{\phi_K^{(p)}(x_p)} + 2y_p x_p^{n-1} \sqrt{\phi_K}(x_p),
$$

$$
\ldots, \phi_K^{(p)}(x_p) \sqrt{\phi_K(x_p)} \right).
$$

(24)

The same is true for $\hat{V}$.

Let $I_0$ be a set of $g$ indices of finite branch points, and divisor $D$ consists of branch points $\{(e_i, 0)\}_{i \in I_0}$, namely: $v(D) = A(I_0)$. At the same time let $K \subset I_0$, that is $\{[e_k] | k \in K\}$ are characteristics of branch points from $D$. We also denote $I_0 \setminus K = I_m$. With these assumptions the left hand side of (21) equals

$$
\frac{\theta[\epsilon(K)](A(I_0) + K)}{\theta(A(I_0) + K)} = \frac{\theta[\epsilon(I_m) + \epsilon_K](0)}{\theta[\epsilon(I_0) + \epsilon_K](0)}
$$

and the numerator vanishes with order $m = [(\ell + 1)/2]$.

In what follows we suppose $D = \sum_{i \in I_m} (e_i, 0)$, and the points are arranged in the order $K, I_m$. Because $\phi_K(e_k) = 0$ for all $k \in K$, and $\phi_K^{*}(e_i) = 0$ for all $i \in I_m$ since $I_m \subset K^*$, matrix $\hat{\Phi}_K$ consists of four blocks, the right upper block of size $\ell \times (g - \ell)$ and the left lower block of size $(g - \ell) \times [\ell/2]$ are zero. In more detail,

$$
\hat{\Phi}_K = \left( \begin{array}{cc}
B_K & 0 \\
0 & B_K^* \\
\end{array} \right)^{\ell\times(g-\ell)}
$$

where

$$
B_K = \left( \begin{array}{cccc}
e_{k_1}^1 \sqrt{\phi_K(e_{k_1})} & e_{k_1}^{l-1} \sqrt{\phi_K^{*}(e_{k_1})} & \cdots & \sqrt{\phi_K^{*}(e_{k_1})} \\
e_{k_2}^1 \sqrt{\phi_K(e_{k_2})} & e_{k_2}^{l-1} \sqrt{\phi_K^{*}(e_{k_2})} & \cdots & \sqrt{\phi_K^{*}(e_{k_2})} \\
\vdots & \vdots & \ddots & \vdots \\
e_{\ell}^1 \sqrt{\phi_K(e_{\ell})} & e_{\ell}^{l-1} \sqrt{\phi_K^{*}(e_{\ell})} & \cdots & \sqrt{\phi_K^{*}(e_{\ell})} \\
\end{array} \right),
$$

$$
B_K^* = \left( \begin{array}{cccc}
e_{t_1}^n \sqrt{\phi_K^{*}(e_{t_1})} & e_{t_1}^{n-1} \sqrt{\phi_K(e_{t_1})} & \cdots & \sqrt{\phi_K(e_{t_1})} \\
e_{t_2}^n \sqrt{\phi_K^{*}(e_{t_2})} & e_{t_2}^{n-1} \sqrt{\phi_K(e_{t_2})} & \cdots & \sqrt{\phi_K(e_{t_2})} \\
\vdots & \vdots & \ddots & \vdots \\
e_{t_{g-\ell}}^n \sqrt{\phi_K^{*}(e_{t_{g-\ell}})} & e_{t_{g-\ell}}^{n-1} \sqrt{\phi_K(e_{t_{g-\ell}})} & \cdots & \sqrt{\phi_K(e_{t_{g-\ell}})} \\
\end{array} \right)
$$

with $K = \{k_1, k_2, \ldots, k_\ell\}$, and $I_m = \{t_1, t_2, \ldots, t_{g-\ell}\}$. Recall that $n + \ell + 2 = g$. Evidently, $\det \hat{\Phi}_K = 0$ if $\ell \geq 1$. 
Next, analyse derivative \( \partial \det \hat{\Phi}_K/\partial x_p \), which we denote by \( \Phi_K^{(p)} \). Taking into account (24) we find for \( \kappa \in K \) and \( \iota \in I_m \)

\[
\lim_{x_p \to e_p} y_p \left( \hat{\Phi}_K^{(p)} \right)_p = \frac{1}{2} \left( 0, 0, \dotsc, 0, e_k^{n-1}, \dotsc, 1 \right) \Phi_K(e_k) \sqrt{\Phi_K(e_k)},
\]

\[
\lim_{x_p \to e_p} y_p \left( \hat{\Phi}_K^{(p)} \right)_p = \frac{1}{2} \left( e^{\iota}, e^{\iota-1}, \dotsc, 1, 0, \dotsc, 0 \right) \Phi_K(e_{\iota}) \sqrt{\Phi_K(e_{\iota})},
\]

that is row \( (\hat{\Phi}_K^{(k)})_k \) makes a contribution to block \( B^*_K \), and row \( (\hat{\Phi}_K^{(\iota)})_\iota \) makes a contribution to block \( B_K \). The term \( y_p \) at branch point \( (e_p, 0) \) vanishes, however we keep it as far as it cancels with the same term in \( \partial e_p / \partial v_n \). It is easy to observe that

- \( \det \hat{\Phi}_K^{(i)} = 0 \) for \( \iota \in I_m \) since block matrix \( \hat{\Phi}_K^{(i)} \) is non-diagonal,
- \( \det \hat{\Phi}_K^{(j,j)} = 0 \) and \( \det \hat{\Phi}_K^{(j,j,\dotsc)} = 0 \) for any index \( j \);
- \( \det \hat{\Phi}_K^{(p_1,\dotsc,p_m)} = 0 \) when all \( p_1, \dotsc, p_m \) are different and \( m < m \),
- \( \det \hat{\Phi}_K^{(p_1,\dotsc,p_m)} \neq 0 \) when all \( p_1, \dotsc, p_m \) are different and the condition holds \( \iota = m + [\iota/2] \), that is \( \iota = 2m - 1 \) or \( \iota = 2m \).

Thus, \( \det \hat{\Phi}_K^{(p_1,\dotsc,p_m)} \) does not vanish when blocks \( B_K \) and \( B^*_K \) are both square. For the sake of simplicity we keep the notation \( B_K \), \( B^*_K \) for diagonal non-vanishing blocks of \( \hat{\Phi}_K^{(p_1,\dotsc,p_m)} \). At \( \iota = 2m - 1 \), block \( B^*_K \) is of size \( (g - m + 1) \) and \( B_K \) of size \( (m - 1) \). At \( \iota = 2m \), block \( B^*_K \) is of size \( (g - m) \) and \( B_K \) has size \( m \). In the both cases

\[
\det \hat{\Phi}_K^{(p_1,\dotsc,p_m)} = \Delta[K\setminus\{p_1,\dotsc,p_m\}] \Delta[I_m \cup \{p_1, \dotsc, p_m\}] \times \prod_{\kappa \in K} \sqrt{\Phi_K^*(e_\kappa)} \prod_{\iota \in I_m} \sqrt{\Phi_K(e_\iota)} \prod_{i=1}^m \frac{\phi_{K}^{\prime}(e_{p_i})}{2y_{p_i}}.
\]

Firstly, we consider the case of \( m = 2 \). Taking the second derivative of (21) we find

\[
\frac{\partial}{\partial v_{n_1}} \frac{\partial}{\partial v_{n_2}} \theta[\epsilon(K)](v(D) + K) = \frac{\epsilon}{\left( \prod_{\kappa \in K^*} (e_\kappa - e_j) \right)^{1/4}} \times \sum_p \sum_q \frac{\partial x_p}{\partial v_{n_1}} \frac{\partial x_q}{\partial v_{n_2}} \left( \frac{\Phi_K^{(p,q)}}{\Delta} - \Phi_K \frac{\Delta^{(p,q)}}{\Delta^2} \right) - \Phi_K \frac{\Delta^{(q)}}{\Delta^2} - \Phi_K \frac{\Delta^{(p)}}{\Delta^2} + 2\Phi_K \frac{\Delta^{(p)}}{\Delta^3}.
\]

On the right hand side the only non-vanishing term is \( \Phi_K^{(p,q)}/\Delta \) with \( p \neq q \). Thus, assigning all points of divisor \( D \) to branch points \( \{e_{2l}\}_{l=1}^g \) which form \( K = v(D) \), we
write down
\[
\frac{\partial v_{n_1} \partial v_{n_2} \theta[J(K)](v(D) + K)}{\theta(v(D) + K)} \bigg|_{v(D) = A(I_0)} = \frac{\epsilon}{(\prod_{j \in K^*} (e_k - e_j))^{1/4}} \sum_{p \neq q, p, q \in K} \frac{\partial e_p}{\partial v_{n_1}} \frac{\partial e_q}{\partial v_{n_2}} \Phi_{K(p, q)}^{(p, q)} \Delta[I_0].
\]

Using Lemma 2 we obtain
\[
\text{RHS} = \frac{\epsilon (\prod_{k \in K} \Phi_{K}^*(e_k))^{1/4}}{(\prod_{i \in I_m} \phi_K(e_i))^{1/2}} \sum_{p \neq q, p, q \in K} \left( \sum_{j=1}^{g} (-1)^{j-1} s_j^p \omega j \right) \\
\times \left( \sum_{j=1}^{g} (-1)^{j-1} s_j^q \omega j \right) \frac{\Delta[K \setminus \{p, q\}] \Delta[K]}{\Delta[K]}. \tag{25}
\]

Finally, with \( I_2 = I_0 \setminus K \) and \( J_2 = J_0 \cup K \), applying first Thomae theorem we come to
\[
\frac{\partial}{\partial v_{n_1}} \frac{\partial}{\partial v_{n_2}} \theta[I_2](v) \bigg|_{v=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(I_2)^{1/4} \Delta(J_2)^{1/4} \\
\times \sum_{p \neq q, p, q \in K} \frac{\left( \sum_{j=1}^{g} (-1)^{j-1} s_j^p \omega j \right) \left( \sum_{j=1}^{g} (-1)^{j-1} s_j^q \omega j \right)}{\prod_{k \in K \setminus \{p, q\}} (e_p - e_k)(e_q - e_k)}. \tag{26}
\]

where \( I_0^{(k)} = I_0 \setminus \{k\} \). The formula works for \( t = 3 \) and \( t = 4 \), where \( t \) is the cardinality of \( K \).

With arbitrary \( m \), and \( t = 2m - 1 \) or \( 2m \), we find the \( m \)-th derivative of (21), keeping only the non-vanishing term
\[
\frac{\partial v_{n_1} \cdots \partial v_{n_m} \theta[\sum_{k \in K} \epsilon_k](v(D) + K)}{\theta(v(D) + K)} \bigg|_{v(D) = K} = \frac{\epsilon}{(\prod_{j \in K^*} (e_k - e_j))^{1/4}} \sum_{p_1, \ldots, p_m \in K \text{ all different}} \frac{\partial e_{p_1}}{\partial v_{n_1}} \cdots \frac{\partial e_{p_m}}{\partial v_{n_m}} \Phi_K^{(p_1, \ldots, p_m)} \Delta[I_0], \tag{26}
\]

where
\[
\text{RHS} = \frac{\epsilon \prod_{k \in K} \sqrt{\phi_K^*(e_k)} \prod_{i \in I_m} \sqrt{\phi_K(e_i)}}{(\prod_{j \in K^*} (e_k - e_j))^{1/4} \Delta[I_0]}
\]
\[ \times \sum_{p_1, \ldots, p_m \in \mathcal{K} \text{ all different}} \sum_{j=1}^{g} (-1)^{j-1}s_{j-1}^{(p_1)} \omega_{jn_1} \prod_{i \in \mathcal{I}_m} (e_{p_1} - e_i) \prod_{\kappa \in \mathcal{K} \setminus \{p_1, \ldots, p_m\}} (e_{p_1} - e_\kappa) \]

\[ \times \Delta[\mathcal{K}\setminus\{p_1, \ldots, p_m\}] \Delta[\mathcal{I}_m \cup \{p_1, \ldots, p_m\}] \]

\[ = \epsilon \left( \prod_{\kappa \in \mathcal{K}} \phi_{K}^*(e_\kappa) \right)^{1/4} \left( \prod_{i \in \mathcal{I}_m} \phi_{K}(e_i) \right)^{1/2} \sum_{p_1, \ldots, p_m \in \mathcal{K} \text{ all different}} \sum_{j=1}^{g} (-1)^{j-1}s_{j-1}^{(p_1)} \omega_{jn_1} \prod_{\kappa \in \mathcal{K} \setminus \{p_1, \ldots, p_m\}} (e_{p_1} - e_\kappa) \]

Taking into account

\[ \prod_{\kappa \in \mathcal{K}} \phi_{K}^*(e_\kappa) = \pm \prod_{i \in \mathcal{I}_m} \phi_{K}(e_i) \prod_{\kappa \in \mathcal{K}, j \in \mathcal{J}_0} (e_\kappa - e_j) \]

and applying first Thomae formula we come to (19). This completes the proof. \( \square \)

### 4 Corollaries and applications

As the simplest development of general Thomae formula we simplify the sums of products of symmetric polynomials in (19).

Firstly, we consider theta functions with non-singular odd characteristics obtained by dropping two indices from \( \mathcal{I}_0 \) (corresponding to a non-singular even characteristic). Let me emphasize that there exist two types of first derivative theta constants: whose characteristics are constructed from \( g - 1 \), and from \( g - 2 \) indices. The former corresponds to \( \mathcal{I}_0 \) with one index dropped, we denote this set by \( [I_1] \). And the latter corresponds to \( \mathcal{I}_0 \) with two dropped indices, the corresponding characteristic is denoted by \( [I_1^\infty] \), here symbol \( \infty \) indicates that the index of infinity belongs to this part of partition. Similar situation happens with characteristics of arbitrary multiplicity \( m \), they can be obtained by dropping \( 2m - 1 \) or \( 2m \) indices. So we have two types of derivative theta constants of each order \( m \).

We start with first derivative theta constants corresponding to \( [I_1^\infty] \) and find a modification of formulas (14)–(16) in this case (Corollary 1). This result seems to be known, since it is inferred in some formulas and statements from [13,14]. Then, we examine second derivative theta constants, which are naturally arranged in matrices (Hessians). A matrix representation with simple structure is obtained (Corollary 2), and examples in genera from 3 to 5 are given. The representation leads to an essential result (Theorem 3) that the rank of matrices of second derivative theta constants is three in any genus.

Finally, third derivative theta constants are considered. Simplification of sums of products of symmetric polynomials is made in Corollary 3, and examples of genera...
and 6 are given. A conjecture about the simplification for higher derivative theta constants is proposed.

As a byproduct expressions for branch points and their symmetric functions, which generalize Bolza formulas, are figured out.

4.1 First derivative theta constants

Note that second Thomae formula (14) works only for a partition \( I_1 \) consisting of \( g - 1 \) indices of finite branch points. If \( I_1 \) contains \( g - 2 \) indices and the omitted index of infinity, here we denote it by \( I_1^\infty \), its characteristic \([I_1^\infty]\) is also non-singular odd, though (14) requires a modification.

**Corollary 1** (Second Thomae theorem with infinity) Let \( I_1^\infty \cup J_1 \) with \( I_1^\infty = \{i_1, \ldots, i_{g-2}\} \) and \( J_1 = \{j_1, \ldots, j_{g+3}\} \) be a partition of the set of \( 2g + 1 \) indices of finite branch points, and \([I_1^\infty]\) denote the non-singular odd characteristic corresponding to \( A(I_1^\infty) + K \). Then

\[
\frac{\partial}{\partial v_n} \theta[I_1^\infty](v) \Big|_{v=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(I_1^\infty)^{1/4} \Delta(J_1)^{1/4} \sum_{j=2}^g (-1)^{j-2} s_{j-2}(I_1^\infty) \omega_{jn},
\]

(27)

where \( \epsilon \) satisfies \( \epsilon^8 = 1 \), and \( \Delta(I_1^\infty) \), \( \Delta(J_1) \) denote Vandermonde determinants built from \( \{e_i \mid i \in I_1^\infty\} \) and \( \{e_j \mid j \in J_1\} \), then \( s_j(I) \) denotes the elementary symmetric polynomial of degree \( j \) in \( \{e_i \mid i \in I\} \).

**Proof** This is the case of \( k = 2 \) and \( m = 1 \). Let \( I_0 = \{i_1, \ldots, i_g\}, J_0 = \{j_1, \ldots, j_{g+1}\}, \) and \( K = \{\kappa_1, \kappa_2\} \). Then \( I_1^\infty = I_0 \setminus K \), and \( J_1 = J_0 \cup K \). Here, we start with (23) which reads as

\[
\frac{\partial}{\partial v_n} \theta[I_0](v) \Big|_{v(D) = A(I_0)} = \epsilon \left( \prod_{\kappa \in K} (e_i - e_{\kappa}) \right)^{1/4} \sum_p \frac{\partial e_p}{\partial v_n} \Phi_K^{(p)} \Delta.
\]

With the help of Lemma 2 and first Thomae theorem one finds

\[
\text{RHS} = \epsilon \left( \prod_{j \in J_0} (e_{\kappa_1} - e_j)(e_{\kappa_2} - e_j) \right)^{1/4} \left( \prod_{i \in I_1^\infty} (e_i - e_{\kappa_1})(e_i - e_{\kappa_2}) \right)^{1/4} \times \left( \sum_{j=1}^g (-1)^{j-1} s_{j-1}(I_0^{(\kappa_1)}) \omega_{jn} \right) + \sum_{j=1}^g (-1)^{j-1} s_{j-1}(I_0^{(\kappa_2)}) \omega_{jn} \right)
\]

\[\square\] Springer
\[= \epsilon \left( \prod_{j \in J_0} (e_{\kappa_1} - e_j) (e_{\kappa_2} - e_j) \right)^{1/4} \left( \prod_{t \in I_1(\infty)} (e_t - e_{\kappa_1}) (e_t - e_{\kappa_2}) \right)^{1/4} \sum_{j=2}^{g} (-1)^{j-2} s_{j-2} (I_1(\infty)) \omega_{jn}. \]

where \( I_0(\kappa) = I_0 \setminus \{\kappa\} \). This leads to (27). \( \square \)

In a matrix form (27) looks as follows

\[
\begin{pmatrix}
\frac{\partial v_1}{\partial v_2} \\
\vdots \\
\frac{\partial v_g}{\partial v_g}
\end{pmatrix} \theta[I_1^\infty](v) \big|_{v=0} = \epsilon \left( \frac{\det \omega}{\pi g} \right)^{1/2} \Delta(I_1^\infty)^{1/4} \Delta(J_1)^{1/4} \omega^t \begin{pmatrix}
0 \\
s_0(I_1^\infty) \\
\vdots \\
(-1)^{g-2} s_{g-2} (I_1^\infty)
\end{pmatrix}. \quad (28)
\]

In terms of non-normalized variables \( u \)

\[
\begin{pmatrix}
\frac{\partial u_1}{\partial u_3} \\
\vdots \\
\frac{\partial u_{2g-1}}{\partial u_{2g-1}}
\end{pmatrix} \theta[I_1^\infty](\omega^{-1} u) \big|_{u=0} = \epsilon \left( \frac{\det \omega}{\pi g} \right)^{1/2} \Delta(I_1^\infty)^{1/4} \Delta(J_1)^{1/4} \omega^t \begin{pmatrix}
0 \\
s_0(I_1^\infty) \\
\vdots \\
(-1)^{g-2} s_{g-2} (I_1^\infty)
\end{pmatrix}. \quad (29)
\]

This result is used in [13, Proposition 4.3, p. 911] to obtain a generalization of Bolza formulas.

**Remark 3** We introduce a constant related to a genus \( g \) curve with a fixed basis of homologies and holomorphic differentials

\[C_g = \left( \frac{\det \omega}{\pi g} \right)^{1/2} \Delta^{1/4}, \quad (30)\]

where \( \Delta \) denotes the Vandermonde determinant in all branch points of the curve. This constant arises in formula (11) connecting theta and sigma functions. From first Thomae theorem it follows

\[C_g = \epsilon \theta[I_0] \left( \prod_{\kappa \in I_0} \prod_{j \in J_0} (e_{\kappa} - e_j) \right)^{1/4}. \]
and \( C_g \) is independent of the partition \( I_0 \cup J_0 \). With the right ordering we get rid of \( \epsilon \), here \( \epsilon = -1 \) as follows from computation.

**Example 1** In genus 2 case formula (29) reads as

\[
\left( \frac{\partial u_1}{\partial u_3} \right) \theta[\{\}] = -C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

where \( C_2 \) is defined by (30) with the right ordering of branch points. Note that partition \( \{\} \cup \{1, 2, 3, 4, 5\} \) in genus 2 corresponds to characteristic \( [K] \) of the vector of Riemann constants. Formula (31) can be used to determine constant \( C_2 \).

**Remark 4** This coincides with the result of [12], devoted to genus 2 case. The fact that \( \partial u_3 \theta[\{\}](\omega^{-1}u) \) gives constant \( C_2 \) was proven with the help of addition theorem by methods of [11]. The other part of the main result of [12] connects constant \( C_2 \) to the product of all theta constants (with non-singular even characteristics), which easily extends to higher genera.

**Theorem 2** For hyperelliptic curve of arbitrary genus \( g \) the following relation holds

\[
\prod_{\text{all } I_0} \theta[I_0] = \left( \frac{\det \omega}{\pi^g} \right)^\frac{1}{2} (2g+1)^{\frac{1}{2}} \Delta^\frac{1}{4} (2g-1)^{\frac{1}{4}} (2g+1)^{\frac{1}{4}}.
\]

The proof follows directly from first Thomae formula.

**Remark 5** In genus 2 Bolza formulas [9, Eq. (6)], or in more detail in Bolza’s dissertation (Göttingen, 1886), see page 15,

\[
e_i = -\frac{\partial u_3 \theta[\{i\}](\omega^{-1}u)}{\partial u_1 \theta[\{i\}](\omega^{-1}u)} \bigg|_{u=0}
\]

can be obtained directly from second Thomae formula in the form (16), which reads as

\[
\left( \frac{\partial u_1}{\partial u_3} \right) \theta[\{i\}](\omega^{-1}u) \bigg|_{u=0} = \frac{-C_2}{(\prod_{j \neq i}^{2g+1} (e_j - e_i))^{1/4}} \left( \frac{1}{-e_i} \right).
\]

**Example 2** In genus 3 generalization of Bolza formulas is obtained from (29). Namely, cf. [13, Eq. (6.26)] and [14, Eq. (6.24)]

\[
e_i = -\frac{\partial u_5 \theta[\{i\}](\omega^{-1}u)}{\partial u_3 \theta[\{i\}](\omega^{-1}u)} \bigg|_{u=0}.
\]

And second Thomae formula (16) implies the following, cf. [13, Eq. (6.25)], [14, Eq. (6.23)],

\[
e_i + e_k = -\frac{\partial u_3 \theta[\{i, k\}](\omega^{-1}u)}{\partial u_1 \theta[\{i, k\}](\omega^{-1}u)} \bigg|_{u=0}.
\]
\[ e_i e_k = \frac{\partial u_5 \theta[[i, k]](\omega^{-1}u)}{\partial u_1 \theta[[i, k]](\omega^{-1}u)} \bigg|_{u=0}. \]

**Example 3** In genus 4 formulas for symmetric polynomials in two branch points follow from (29)

\[ e_i + e_k = -\frac{\partial u_5 \theta[[i, k]](\omega^{-1}u)}{\partial u_3 \theta[[i, k]](\omega^{-1}u)} \bigg|_{u=0}, \quad e_i e_k = \frac{\partial u_7 \theta[[i, k]](\omega^{-1}u)}{\partial u_3 \theta[[i, k]](\omega^{-1}u)} \bigg|_{u=0}. \]

In [13, Proposition 4.3, p. 911] a generalization of Bolza formulas coming from (16) and (29) was found in terms of sigma function at half-periods. In our notation this generalization has the form

\[ s_j(I_1) = (-1)^j \frac{\partial u_{2j+1} \theta[I_1](\omega^{-1}u)}{\partial u_1 \theta[I_1](\omega^{-1}u)} \bigg|_{u=0}, \quad j = 1, \ldots, g - 1; \]

\[ s_j(I_1^\infty) = (-1)^j \frac{\partial u_{2j+3} \theta[I_1](\omega^{-1}u)}{\partial u_1 \theta[I_1](\omega^{-1}u)} \bigg|_{u=0}, \quad j = 1, \ldots, g - 2. \]

**Remark 6** With the help of first derivative theta constants, one can find expressions for symmetric polynomials in \( g - 1 \) or \( g - 2 \) branch points, these correspond to partitions \( I_1 \) and \( I_1^\infty \). Expressions for separate branch points arise for genera 2 and 3 only. For higher genera one should use higher derivative theta constants.

### 4.2 Second derivative theta constants

Next we consider characteristics of multiplicity 2, which arise when 3 or 4 indices drop from \( I_0 \). Again \( I_0 \cup J_0 \) is a partition of \( 2g + 1 \) indices of finite branch points with \( I_0 = \{i_1, \ldots, i_g\} \) and \( J_0 = \{j_1, \ldots, j_{g+1}\} \). Let \( K \) denote the set of indices which drop, then \( I_2 = I_0 \setminus K \), and \( J_2 = J_0 \cup K \).

**Corollary 2** Let \( I_2 \cup J_2 \) with \( I_2 = \{i_1, \ldots, i_{g-\ell}\} \) and \( J_2 = \{j_1, \ldots, j_{g+1+\ell}\} \), where \( \ell = 3 \) or 4, be a partition of the set of \( 2g + 1 \) indices of finite branch points, such that singular characteristic \( [I_2] \), corresponding to \( A(I_2) + K \), has multiplicity 2. Let \( \Delta(I_2) \) and \( \Delta(J_2) \) be Vandermonde determinants built from \( \{e_i \mid i \in I_2\} \) and \( \{e_j \mid j \in J_2\} \). Then

\[ \frac{\partial}{\partial v_{n_1}} \frac{\partial}{\partial v_{n_2}} \theta[I_2](v) \bigg|_{v=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(I_2)^{1/4} \Delta(J_2)^{1/4} \]

\[ \times \sum_{i,j=1}^{g} (-1)^{i+j} \left( 2s_{i-\ell+1}(I_2)s_{j-\ell+1}(I_2) - s_i(I_2)s_{j-\ell+2}(I_2) + s_{i-\ell+2}(I_2)s_{j-\ell}(I_2) \right) \omega_{i,n_1} \omega_{j,n_2}. \]
where \( \epsilon \) satisfies \( \epsilon^8 = 1 \), and elementary symmetric functions \( s_1(I_2) \) are replaced by zero when \( l < 0 \).

In matrix form

\[
\partial^2_v \theta[I_2](v)\big|_{v=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(I_2)^{1/4} \Delta(J_2)^{1/4} \omega^t \hat{S}[I_2] \omega, \tag{35}
\]

where \( \partial^2_v \) denotes the operator of second derivatives, whose entries are \( \partial_{v_{i_1}} \partial_{v_{i_2}} \), and \( \hat{S}[I_2] \) is a \( g \times g \) matrix with entries

\[
(\hat{S}[I_2])_{i,j} = (-1)^{i+j} \left( 2s_{i-t+1}(I_2)s_{j-t+1}(I_2) - s_{i-t+2}(I_2)s_{j-t}(I_2) - s_{i-t}(I_2)s_{j-t+2}(I_2) \right). \tag{36}
\]

With non-normalized variables \( u \)

\[
\partial^2_u \theta[I_2](\omega^{-1}u)\big|_{u=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(I_2)^{1/4} \Delta(J_2)^{1/4} \hat{S}[I_2]. \tag{37}
\]

**Proof of Corollary 2** Recall that multiplicity \( m = 2 \) arises when \( \ell = 3 \) or 4. First consider the case of three dropped indices with \( K = \{ \kappa_1, \kappa_2, \kappa_3 \} \). Starting from (25), by straightforward computation we find

\[
\sum_{\substack{p \neq q \\atop p,q \in K}} \frac{\left( \sum_{j=1}^{g} (-1)^{j-1} s_{j-1}(I_0^{(p)}) \omega_{jn_1} \right) \left( \sum_{j=1}^{g} (-1)^{j-1} s_{j-1}(I_0^{(q)}) \omega_{jn_2} \right)}{\prod_{\kappa \in K \setminus \{p,q\}} (e_p - e_\kappa)(e_q - e_\kappa)}
\]

\[
= \sum_{i,j=1}^{g} (-1)^{i+j} \left( 2s_{i-2}(I_2)s_{j-2}(I_2) - s_{i-1}(I_2)s_{j-3}(I_2) - s_{i-3}(I_2)s_{j-1}(I_2) \right) \omega_{in_1} \omega_{jn_2},
\]

where \( s_1(I_2) \) is replaced by zero when \( l < 0 \).

In the case of four dropped indices with \( K = \{ \kappa_1, \kappa_2, \kappa_3, \kappa_4 \} \) similar computation leads to the following

\[
\sum_{\substack{p \neq q \\atop p,q \in K}} \frac{\left( \sum_{j=1}^{g} (-1)^{j-1} s_{j-1}(I_0^{(p)}) \omega_{jn_1} \right) \left( \sum_{j=1}^{g} (-1)^{j-1} s_{j-1}(I_0^{(q)}) \omega_{jn_2} \right)}{\prod_{\kappa \in K \setminus \{p,q\}} (e_p - e_\kappa)(e_q - e_\kappa)}
\]

\[
= \sum_{i,j=2}^{g} (-1)^{i+j} \left( 2s_{i-3}(I_2)s_{j-3}(I_2) - s_{i-2}(I_2)s_{j-4}(I_2) - s_{i-4}(I_2)s_{j-2}(I_2) \right) \omega_{in_1} \omega_{jn_2}.
\]
These two expressions can be written as (34).

**Example 4** In genus 3 case \( \mathcal{J}_2 = \{ \} \), and

\[
\hat{S}[\{\}] = \begin{pmatrix}
0 & 0 & -1 \\
0 & 2 & 0 \\
-1 & 0 & 0
\end{pmatrix}.
\]

(38)

Let \( \mathcal{I}_0 = \{ i_1, i_2, i_3 \} \), then \( \mathcal{J}_0 \) is the complement to \( \mathcal{I}_0 \) in the set of 7 indices of finite branch points, \( \mathcal{K} \) coincides with \( \mathcal{I}_0 \). So (35) gives

\[
\partial^2 \theta[\{\}](v)|_{v=0} = -C_3 \omega^I \hat{S}[\{\}] \omega.
\]

(39)

where \( C_3 \) is defined by (30). Representation (37) in terms of non-normalized variables \( u \) allows to determine constant \( C_3 \), namely

\[
C_3 = \partial^2_{u_1, u_5} \theta[\{\}](\omega^{-1}u)|_{u=0} = -\frac{1}{2} \partial^2_{u_3, u_5} \theta[\{\}](\omega^{-1}u)|_{u=0}.
\]

**Example 5** In genus 4 there is the unique partition \( \mathcal{I}_2^{(\infty)} = \{ \} \) with

\[
\hat{S}[\{\}] = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 2 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}.
\]

The relation similar to (39) holds with constant \( C_4 \), defined by (30). Again constant \( C_4 \) can be found from the equalities

\[
C_4 = \partial^2_{u_3, u_7} \theta[\{\}](\omega^{-1}u)|_{u=0} = -\frac{1}{2} \partial^2_{u_3, u_5} \theta[\{\}](\omega^{-1}u)|_{u=0}.
\]

**Example 6** In genus 4 there are \( 2g + 1 = 9 \) singular even characteristics \( \mathcal{I}_2 = \{ \iota \} \) with

\[
\hat{S}[\{\iota\}] = \begin{pmatrix}
0 & 0 & -1 & e^i \\
0 & 2 & -e^i & -e^{2i} \\
-1 & -e^i & 2e^{2i} & 0 \\
e^i & -e^{2i} & 0 & 0
\end{pmatrix}.
\]

Let \( \mathcal{I}_0 = \{ i_1, i_2, i_3, \iota \} \), and \( \mathcal{J}_0 \) be the complement to \( \mathcal{I}_0 \) in the set of 9 indices of finite branch points, here \( \mathcal{K} = \{ i_1, i_2, i_3 \} \). Then, (37) gives (with right ordering)

\[
\partial^2 \theta[\{\iota\}](\omega^{-1}u)|_{u=0} = \frac{-C_4}{(\prod_{j \neq \iota}^{2g+1} (e_i - e_j))^{1/4}} \hat{S}[\{\iota\}].
\]
This immediately implies a generalization of Bolza formulas to genus 4

\[
e_t = - \frac{\partial^2}{\partial u_1, u_7} \theta[\{i\}](\omega^{-1}u) \bigg|_{u=0} = \frac{\partial^2}{\partial u_3, u_5} \theta[\{i\}](\omega^{-1}u) \bigg|_{u=0} = -2 \frac{\partial^2}{\partial u_3, u_5} \theta[\{i\}](\omega^{-1}u) \bigg|_{u=0} = -\frac{2}{\partial^2 u_3, u_5} \theta[\{i\}](\omega^{-1}u) \bigg|_{u=0} = -\frac{1}{\partial^2 u_3, u_5} \theta[\{i\}](\omega^{-1}u) \bigg|_{u=0} = -\frac{1}{\partial^2 u_3, u_5} \theta[\{i\}](\omega^{-1}u) \bigg|_{u=0}.
\]

(40)

See Example 2 for genus 3 case, and Example 3 for symmetric functions in two branch points in genus 4.

**Remark 7** Another generalization of Bolza formulas, in terms of sigma function, the reader could find in [13, Proposition 4.2, p. 911]. In the present paper, the generalization involves lower derivatives, actually the lowest non-vanishing derivatives of theta function with characteristic \([\{i\}]\) at \(v = 0\).

**Example 7** In genus 5 there exist \(2g + 1 = 11\) singular even characteristics of multiplicity 2 corresponding to partitions \(\mathcal{I}_2^\infty = \{i\}\) with matrices

\[
\hat{S}[\{i\}] = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & e_t \\
0 & 0 & 2 & -e_t & -e_t^2 \\
0 & -1 & -e_t & 2e_t^2 & 0 \\
e_t & -e_t & 2 & 0 & 0
\end{pmatrix},
\]

and \(\binom{2g+1}{2} = 55\) characteristics of multiplicity 2 corresponding to partitions \(\mathcal{I}_2 = \{i_1, i_2\}\) with

\[
\hat{S}[\{i_1, i_2\}] = \begin{pmatrix}
0 & 0 & -1 & -e_{i_1} - e_{i_2} \\
0 & 2 & e_{i_1} + e_{i_2} & -e_{i_1} - e_{i_2} & 2(e_{i_1} + e_{i_2} + e_{i_2}^2) \\
e_{i_1} + e_{i_2} & -e_{i_1} - e_{i_2} & 2 & e_{i_1} e_{i_2} (e_{i_1} + e_{i_2}) & -e_{i_1}^2 e_{i_2} \\
e_{i_1} e_{i_2} & e_{i_1} e_{i_2} (e_{i_1} + e_{i_2}) & -e_{i_1} e_{i_2} & e_{i_1} e_{i_2} (e_{i_1} + e_{i_2}) & -e_{i_1}^2 e_{i_2} \\
e_{i_1} e_{i_2} (e_{i_1} + e_{i_2}) & e_{i_1} e_{i_2} (e_{i_1} + e_{i_2}) & -e_{i_1}^2 e_{i_2} & 0 & 0
\end{pmatrix}.
\]
General derivative Thomae formula for singular half-periods

Thus, a generalization of Bolza formulas to genus 5 can be obtained

\[
e_i = - \frac{\partial_i^2 \theta(t) (\omega^{-1}u)}{\partial u^2 \theta(t)} \bigg|_{u=0} = - \frac{\partial_{u^2} \theta(t) (\omega^{-1}u)}{\partial u^2 \theta(t)} \bigg|_{u=0}
\]

and formulas for symmetric functions in two branch points

\[
e_{12} = - \frac{\partial_{u^2} \theta(t) (\omega^{-1}u)}{\partial u^2 \theta(t)} \bigg|_{u=0} = 2 \frac{\partial_{u^2} \theta(t) (\omega^{-1}u)}{\partial u^2 \theta(t)} \bigg|_{u=0}
\]

(41)

(42)
\[
\begin{align*}
\frac{2\partial^2_{u_5,u_9}\theta[i](\omega^{-1}u)}{(\partial^2_{u_5,u_5}+2\partial^2_{u_3,u_7})\theta[i](\omega^{-1}u)}\bigg|_{u=0} &= -\frac{\partial^2_{u_7,u_7}\theta[i](\omega^{-1}u)}{(\partial^2_{u_5,u_5}+2\partial^2_{u_3,u_7})\theta[i](\omega^{-1}u)}\bigg|_{u=0}. \\
\end{align*}
\]

(43)

In genus \( g \) with the help of second derivative theta constants one can compute symmetric polynomials in \( g-4 \) and \( g-3 \) branch points, these are the possible cardinalities of \( I_2 \), and we use the notation \( \ell = 3 \) or 4. The first equalities from (40), (41), (42), and (43) are combined as

\[
s_j(I_2) = (-1)^j \frac{\partial^2_{u_2e-5,u_2e+2j-1}\theta[I_2](\omega^{-1}u)}{\partial^2_{u_2e-5,u_2e-1}\theta[I_2](\omega^{-1}u)}\bigg|_{u=0},
\]

which holds in arbitrary genus \( g \).

**Remark 8** Note that matrix \( \hat{S}[I_2] \) is non-degenerate only in genus 3. In higher genera \( \hat{S}[I_2] \) has rank 3.

**Theorem 3** For hyperelliptic curves of genera \( g \geq 3 \), when characteristics of multiplicity 2 exist, rank of every matrix of second derivative theta constants equals three, that is

\[
\text{rank} \left( \partial^2_v \theta[I_2] \right) = 3.
\]

Therefore, \( \det \left( \partial^2_v \theta[I_2] \right) = 0 \) in genera \( g > 3 \).

**Proof** Corollary 2 gives a decomposition of Hessian matrix \( \partial^2_v \theta[I_2] \) as the product of non-degenerate matrices \( \omega \) and \( g \times g \) matrix \( \hat{S}[I_2] \). The latter, as easily seen from (36), is composed of columns spanned by three vectors. Therefore, rank of \( \partial^2_v \theta[I_2] \) equals 3. \( \square \)

It could be observed from (36) that matrix \( \hat{S}(I_2) \) belongs to the second tensor power \( S_3 \otimes S_3 \) of the vector space \( S_3 \) spanned by three vectors \( s_0, s_1, s_2 \) such that \( s_d = (s_{j-e+d}(I_2))_{j=1}^g \), recall that \( \ell = 3 \) or 4. And tensor rank of \( \hat{S}(I_2) \) is three, since it is spanned of three basis elements: \( s_1 \otimes s_1, s_0 \otimes s_2, s_2 \otimes s_0 \).

### 4.3 Third derivative theta constants

Here, we consider characteristics of multiplicity 3, obtained by dropping 5 or 6 indices from \( I_0 \). Again \( K \) denotes the set of dropped indices, \( I_3 = I_0 \setminus K \), and \( J_3 = J_0 \cup K \).

**Corollary 3** Let \( I_3 \cup J_3 \) with \( I_3 = \{i_1, \ldots, i_{g-\ell}\} \) and \( J_3 = \{j_1, \ldots, j_{g+1+\ell}\} \), where \( \ell = 5 \) or 6, be a partition of the set of indices of \( 2g+1 \) finite branch points, such that singular characteristic \( [I_3] \), corresponding to \( A(I_3) + K \), has multiplicity 3.
Let $\Delta(\mathcal{I}_3)$ and $\Delta(\mathcal{J}_3)$ be Vandermonde determinants built from $\{e_i \mid i \in \mathcal{I}_3\}$ and $\{e_j \mid j \in \mathcal{J}_3\}$. Then

$$\frac{\partial}{\partial v_1} \frac{\partial}{\partial v_2} \frac{\partial}{\partial v_3} \theta[\mathcal{I}_3](v) \bigg|_{v=0} = \epsilon \left( \frac{\det \omega}{\pi g} \right)^{1/2} \Delta(\mathcal{I}_3)^{1/4} \Delta(\mathcal{J}_3)^{1/4} \times \sum_{j_1,j_2,j_3=1}^{g} (\hat{S}[\mathcal{I}_3])_{j_1,j_2,j_3} \omega_{j_1n_1} \omega_{j_2n_2} \omega_{j_3n_3},$$

(45)

with tensor of order 3

$$(\hat{S}[\mathcal{I}_3])_{j_1,j_2,j_3} = (-1)^{j_1+j_2+j_3-3\epsilon} \left( 6s_{j_1-\epsilon+2s_{j_2-\epsilon+2s_{j_3-\epsilon+2}} 
-2\{s_{j_1-\epsilon+3s_{j_2-\epsilon+2s_{j_3-\epsilon+1}} \} + 2\{s_{j_1-\epsilon+3s_{j_2-\epsilon+2s_{j_3-\epsilon+3}} \} 
+2\{s_{j_1-\epsilon+1s_{j_2-\epsilon+1s_{j_3-\epsilon+4}} \} - \{s_{j_1-\epsilon+3s_{j_2-\epsilon+2s_{j_3-\epsilon+4}} \} \right).$$

(46)

where $\{s_{j_1-\epsilon+d},s_{j_2-\epsilon+d},s_{j_3-\epsilon+d}\}$ denotes the sum over all permutations of $\{d_1,d_2,d_3\}$, elementary symmetric functions $s_k$ are constructed in $\{e_i \mid i \in \mathcal{I}_3\}$, and replaced by zero when $l < 0$, and $\epsilon$ satisfies $\epsilon^8 = 1$.

In a tensor form with non-normalized variables $u$ (45) reads as

$$\frac{\partial^3}{\partial u^3} \theta[\mathcal{I}_3](\omega^{-1}u) \bigg|_{u=0} = \epsilon \left( \frac{\det \omega}{\pi g} \right)^{1/2} \Delta(\mathcal{I}_3)^{1/4} \Delta(\mathcal{J}_3)^{1/4} \hat{S}[\mathcal{I}_3].$$

(47)

To extend Corollary 3 to higher multiplicities the following Conjecture could be helpful.

**Conjecture 1** Let $\mathcal{I}_m \cup \mathcal{J}_m$ be a partition of the set of indices of $2g+1$ finite branch points with $\mathcal{I}_m = \{i_1, \ldots, i_{g-\epsilon}\}$ and $\mathcal{J}_m = \{j_1, \ldots, j_{g+1+\epsilon}\}$, where $\epsilon = 2m-1$ or $2m$, and characteristic $[\mathcal{I}_m]$ of multiplicity $m$ corresponds to $A(\mathcal{I}_m) + K$. In the expansion of

$$(\hat{S}[\mathcal{I}_m])_{j_1,\ldots,j_m} = \sum_{p_1,\ldots,p_m \in \mathcal{K}} I^{m-1} s_{j_1-\epsilon+1(j_{p_1}+1)}(\mathcal{I}_m \cup \mathcal{K}(p_i)) \prod_{k \in \mathcal{K} \setminus \{p_1,\ldots,p_m\}} (\epsilon p_i - \epsilon_k)$$

over $\prod_{i=1}^{m} s_{j_i-\epsilon+d}(\mathcal{I}_m)$ only terms of order $|d| = \sum_{i=1}^{m} d_i = m(m-1)$ do not vanish. This comes from the homogeneous degree of the ratio on the right hand side, which is $\sum_{i} j_i - m(\epsilon - m + 1)$.

**Example 8** In genus 5 the highest multiplicity of characteristics is 3, this characteristic is unique and corresponds to partition with $\mathcal{I}_3 = \{}$. Then, $\hat{S}[\{\} \}$ is a constant symmetric
tensor of order 3, whose non-vanishing entries are the following (with all permutations of indices)

\[(\hat{S}([]))_{1,3,5} = -1, \quad (\hat{S}([]))_{1,4,4} = (\hat{S}([]))_{2,2,5} = 2, \]
\[(\hat{S}([]))_{2,3,4} = -2, \quad (\hat{S}([]))_{3,3,3} = 6.\]

Again introducing constant \(C_5\) given by (30) into (47), we obtain

\[\partial^3_u \theta(\omega^{-1}u)\bigg|_{u=0} = -C_5\hat{S}([]), \quad (48)\]

so \(C_5\) is directly computed through

\[C_5 = \partial^3_{u_1,u_3,u_5} \theta(\omega^{-1}u)\bigg|_{u=0} = -\frac{1}{2} \partial^3_{u_1,u_7,u_9} \theta(\omega^{-1}u)\bigg|_{u=0} = -\frac{1}{6} \partial^3_{u_3,u_5,u_9} \theta(\omega^{-1}u)\bigg|_{u=0}.
\]

**Proposition 1** For a hyperelliptic curve of arbitrary genus \(g\) with period matrix \(\omega\) the constant defined by (30) and arising as a normalizing factor in the relation between sigma and theta functions (11) is determined by a directional derivative of theta function with the characteristic \([[\xi]] = [K]\) of maximal multiplicity as follows

\[C_g = \partial^[(g+1)/2]_{u_{2(g \mod 2)+1} \ldots u_{2g-7,u_{2g-3}}} \theta(\omega^{-1}u)\bigg|_{u=0}.
\]

**Example 9** In genus 6 the constant order 3 symmetric tensor \(\hat{S}([])\) has the following non-vanishing entries, cf. genus 5 case,

\[(\hat{S}([]))_{2,4,6} = -1, \quad (\hat{S}([]))_{2,5,5} = (\hat{S}([]))_{3,3,6} = 2, \]
\[(\hat{S}([]))_{3,4,5} = -2, \quad (\hat{S}([]))_{4,4,4} = 6.\]

There are \(2g + 1 = 13\) characteristics \([\mathcal{Z}_3] = [[\iota]]\) with symmetric tensor \(\hat{S}([[\iota]])\) such that

\[(\hat{S}([[\iota]]))_{1,3,5} = -1, \quad (\hat{S}([[\iota]]))_{1,4,4} = (\hat{S}([[\iota]]))_{2,2,5} = 2, \]
\[(\hat{S}([[\iota]]))_{2,3,4} = -2, \quad (\hat{S}([[\iota]]))_{3,3,3} = 6, \]
\[(\hat{S}([[\iota]]))_{1,3,6} = - (\hat{S}([[\iota]]))_{1,4,5} = -\frac{1}{2} (\hat{S}([[\iota]]))_{2,2,6} = (\hat{S}([[\iota]]))_{2,3,5}
\]
\[= \frac{1}{2} (\hat{S}([[\iota]]))_{2,4,4} = - \frac{1}{2} (\hat{S}([[\iota]]))_{3,3,4} = \iota, \]
\[(\hat{S}([[\iota]]))_{2,3,6} = - (\hat{S}([[\iota]]))_{1,4,6} = \frac{1}{2} (\hat{S}([[\iota]]))_{1,5,5} = - (\hat{S}([[\iota]]))_{2,4,5}.
\]

\(\exists\) Springer
\[
= -\frac{1}{2} (\hat{S}([t]))_{3,3,5} = \frac{1}{2} (\hat{S}([t]))_{3,4,4} = e_i^2,
\]
\[
(\hat{S}([t]))_{2,4,6} = -\frac{1}{2} (\hat{S}([t]))_{2,5,5} = -\frac{1}{2} (\hat{S}([t]))_{3,3,6}
\]
\[
= \frac{1}{2} (\hat{S}([t]))_{3,4,5} = -\frac{1}{6} (\hat{S}([t]))_{4,4,4} = e_i^3.
\]

There are plenty of possibilities to compute \( e_i \), one of them is
\[
e_i = -\frac{\partial^3}{\partial u_1, u_5, u_1} \frac{\theta[I_3](\omega^{-1}u)}{\theta[I_3](\omega^{-1}u)} \bigg|_{u=0}.
\]

The above observations (44) and (49) are generalized in

**Proposition 2** Let \( I_m \) be a set of \( g - \ell \) indices, and \( m = [(\ell + 1)/2] \). Elementary symmetric polynomials in branch points \( \{e_i \mid i \in I_m\} \) of genus \( g \) hyperelliptic curve with period matrix \( \omega \) are defined by
\[
s_j(I_m) = (-1)^j \frac{\partial^m}{\partial u_{2g-2}^{m-4(m-1)} \cdots u_{2g-5} u_{2g-2j-1}} \frac{\theta[I_m](\omega^{-1}u)}{\theta[I_m](\omega^{-1}u)} \bigg|_{u=0}.
\]

In particular,
\[
e_i = -\frac{\partial^{[g/2]}_{u_{2g-2} \mod 2+1, \ldots, u_{2g-5}, u_{2g-1}} \theta([t])(\omega^{-1}u)}{\partial^{[g/2]}_{u_{2g-2} \mod 2+1, \ldots, u_{2g-5}, u_{2g-3}} \theta([t])(\omega^{-1}u)} \bigg|_{u=0}.
\]

This extends Bolza formulas to an arbitrary genus hyperelliptic curve.

**Remark 9** From (46) one could observe that order 3 tensor \( \hat{S}(I_3) \) belongs to the third tensor power \( S_5 \otimes S_3 \) of the vector space \( S_5 \) spanned by five vectors \( s_0, s_1, s_2, s_3, s_4 \) such that \( s_d = (s_{j-d} \otimes I_3)_{j=1}^g \), here \( \ell = 5 \) or 6. And tensor rank of \( \hat{S}(I_3) \) is 19, since it is spanned of 19 basis elements, which are

\[
\begin{align*}
& s_0 \otimes s_2 \otimes s_4, & s_0 \otimes s_4 \otimes s_2, & s_2 \otimes s_0 \otimes s_4, \\
& s_4 \otimes s_0 \otimes s_2, & s_2 \otimes s_4 \otimes s_0, & s_4 \otimes s_2 \otimes s_0, \\
& s_1 \otimes s_2 \otimes s_3, & s_1 \otimes s_3 \otimes s_2, & s_2 \otimes s_1 \otimes s_3, \\
& s_3 \otimes s_1 \otimes s_2, & s_2 \otimes s_3 \otimes s_1, & s_3 \otimes s_2 \otimes s_1, \\
& s_0 \otimes s_3 \otimes s_3, & s_3 \otimes s_0 \otimes s_3, & s_3 \otimes s_3 \otimes s_0, \\
& s_1 \otimes s_3 \otimes s_4, & s_1 \otimes s_4 \otimes s_1, & s_4 \otimes s_1 \otimes s_1, \\
& s_2 \otimes s_2 \otimes s_2.
\end{align*}
\]

Tensor products in the basis are composed in such a way that cumulative weight (in subscripts) is 6. So one could find the basis which spans \( \hat{S}(I_3) \) from partitions of 6 of length 3 formed from \{0, 1, 2, 3, 4\}.  

\[\text{Springer}\]
**Conjecture 2** With a characteristic $[\mathcal{I}_m]$ of multiplicity $m$ corresponding to a partition $\mathcal{I}_m \cup \mathcal{J}_m$ with $\mathcal{I}_m = \{i_1, \ldots, i_{g-\ell}\}$ and $\mathcal{J}_m = \{j_1, \ldots, j_{g+1+\ell}\}$, where $\ell = 2m - 1$ or $2m$, of indices of $2g + 1$ finite branch points the following holds

$$
\partial_u^m \theta[\mathcal{I}_m](\omega^{-1}u) \bigg|_{u=0} = \epsilon \left( \frac{\det \omega}{\pi^g} \right)^{1/2} \Delta(\mathcal{I}_m)^{1/4} \Delta(\mathcal{J}_m)^{1/4} \hat{S}[\mathcal{I}_m],
$$

where $u$ are non-normalized variables, and order $m$ tensor $\hat{S}[\mathcal{I}_m]$ belongs to the $m$-th tensor power $S^{\otimes m}_{2m-1}$ of the vector space $S_{2m-1}$ spanned by $2m - 1$ vectors $s_0$, $s_1, \ldots, s_{2m-2}$ such that $s_d = (s_j - \ell + d(I_m)) g_j = 1$. The basis spanning $\hat{S}(\mathcal{I}_m)$ could be found from partitions of $m(m-1)$ of length $m$ formed from numbers $\{0, 1, \ldots, 2m-2\}$.

---

**5 Conclusion and discussion**

The main Theorem 1 extends the result of second Thomae theorem to derivative theta constants of arbitrary order, namely it gives an expression in terms of period matrix $\omega$ and symmetric functions of branch points for the lowest non-vanishing derivative at $v = 0$ of theta function with characteristic of arbitrary multiplicity $m$ in hyperelliptic case. Formula (19), which is called general Thomae formula, provides a natural generalization of second Thomae formula, and includes, as particular cases, second and first Thomae formulas.

This result gives wide scope for producing further representations of derivative theta constants, and relations between theta functions. In the present paper, the simplest implication is derived, which comes from simplification of sums of products of symmetric polynomials in (19). Nevertheless, this leads to an essential result.

In the case of multiplicity 2 the lowest non-vanishing derivative of theta function is second, they are naturally arranged in matrices of second derivative theta constants (Hessians). All of them have a representation of the form $\omega^{\prime \prime} \hat{S}[\mathcal{I}_2] \omega$, and symmetric $g \times g$ matrix $\hat{S}[\mathcal{I}_2]$ consists of symmetric polynomials in branch points with indices from $\mathcal{I}_2$ (Corollary 2). An essential result is the following: the rank of $\hat{S}[\mathcal{I}_2]$ is three in any genus (Theorem 3). In connection to this result I refer to the main theorem (Theorem 10) from [21], which is formulated somewhat incorrect. Perhaps, the authors meant that Hessian is singular (not zero), that is its determinant is zero, since in the proof they assert that the rank of the Hessian is three. With this correction Theorem 10 tells about the unique in genus 4 theta function with characteristic $[\{\}]$, which is even and vanishes at $v = 0$. Theorem 3 from the present paper confirms that in hyperelliptic case the rank of the Hessian of this theta function at $v = 0$ is three, so the determinant of the Hessian vanishes. Moreover, Theorem 3 also extends this result to all hyperelliptic curves of higher genera, thus gives the answer to the question posed at the end of [21] for this class of curves.

Similar representation is obtained in the case of multiplicity 3 (Corollary 3), where third derivative theta constants arranged in $g \times g \times g$ tensors are expressed as a product of three matrices $\omega$ and an order 3 symmetric tensor $\hat{S}[\mathcal{I}_3]$ whose entries are symmetric.
polynomials in branch points with indices from $I_3$. Analysis of the structure of tensor $\hat{S}_3[Z_3]$ allows to make Conjectures 1 and 2 about the case of arbitrary multiplicity.

Also progress is made comparing to [12]. The result of [12] is extended to an arbitrary genus hyperelliptic curve, namely (i) the constant (30) related to the curve, which also serves as a normalizing factor in the relation between sigma and theta functions (11), is expressed in terms of directional derivative of theta function with the characteristic of maximal multiplicity (Proposition 1), and (ii) the product of all theta constants is computed through the Vandermonde determinant in all branch points and $\det \omega$ (Theorem 2).

One more result is a generalization of Bolza formulas, see Examples 6, 7, and 9 for the cases of genera 4, 5 and 6, and Proposition 2. This confirms the result of [13, Propositions 4.2, p. 911] and provides a representation in lower derivatives than the given in [13, Propositions 4.3, p. 911]. Moreover, Corollaries 2, 3, and similar representations for higher multiplicities allow to find a complete list of expressions which generalize Bolza formulas for separate branch points and symmetric polynomials of any number of them.

Acknowledgements The problem of obtaining general Thomae formula was posed by Y. Kopeliovich, who also encouraged the author to work on it. The author is grateful to Y. Kopeliovich and V. Enolski for fruitful discussion.

References

1. Thomae, J.: Beitrag zur Bestimmung $\theta(0, 0, \ldots, 0)$ durch die Klassenmoduln algebraischer Funktionen. J. Reine Angew. Math. 71, 201–222 (1870)
2. Bershadski, M., Radul, A.: Fermionic fields on $\mathbb{Z}_N$-curves. Commun. Math. Phys. 116, 689–700 (1988)
3. Nakayashiki, A.: On the Thomae formula for $\mathbb{Z}_N$ curves. Publ. Res. Inst. Math. Sci. 33(6), 987–1015 (1997)
4. Enolski, V., Grava, T.: Thomae type formulae for singular $\mathbb{Z}_N$ curves. Lett. Math. Phys. 76(2–3), 187–214 (2006)
5. Ya, K.: Thomae formula for general cyclic covers of $\mathbb{C}P^1$. Lett. Math. Phys. 94(3), 313–333 (2010)
6. Kopeliovich, Y., Zemel, S. (2019) Thomae formula for Abelian covers of $\mathbb{C}P^1$. Trans. Am. Math. Soc., arXiv:1612.09104
7. Matsumoto, K., Terasoma, T.: Degenerations of triple covering and Thomae’s formula (2010). arXiv:1001.4950
8. Enolski, V., Kopeliovich, Y., Zemel, S.: Thomae’s derivative formulae for Trigonal curves. Lett. Math. Phys. (submitted). arXiv:1810.06031
9. Bolza, O.: Ueber die Reduction hyperelliptischer Integrale erster Ordnung und erster Gattung auf elliptische durch eine Transformation vierten Grades. Math. Ann. 28(3), 447–456 (1887)
10. Baker, H.F.: On the hyperelliptic sigma functions. Am. J. Math. 20(4), 301–384 (1898)
11. Baker, H.F.: Multiply Periodic Functions. Cambridge University Press, Cambridge (1907)
12. Grant, D.: A generalization of Jacobi’s derivative formula to dimension two. J. Reine Angew. Math. 392, 125–136 (1988)
13. Enolski, V., Hackmann, E., Kagramanova, V., Kunz, J., Lämmerzahl, C.: Inversion of hyperelliptic integrals of arbitrary genus with application to particle motion in general relativity. J. Geom. Phys. 61, 899–921 (2011)
14. Enolski, V., Hartmann, B., Kagramanova, V., Kunz, J., Lämmerzahl, C., Sirimachan, P.: J. Math. Phys. 53, 012504 (2012)
15. Buchstaber, V.M., Enolski, V.Z., Leykin, D.V.: Multi-dimensional sigma-functions, p. 267 (2012). arXiv:1208.0990
16. Enolski, V.Z., Richter, P.H.: Periods of hyperelliptic integrals expressed in terms of $\theta$-constants by means of Thomae formulae. Philos. Trans. Lond. Math. Soc. A 366, 1005–1024 (2008)
17. Rauch, H.E., Farkas, H.M.: Theta Functions with Applications to Riemann Surfaces, p. 232. The Williams & Wilkins Company, Baltimore (1974)
18. Fay, J.D.: Theta Functions on Riemann Surfaces. Lectures Notes in Mathematics, vol. 352. Springer, Berlin (1973)
19. Eilers, K.: Rosenhain–Thomae formulae for higher genera hyperelliptic curves. J. Nonlinear Math Phys. 25(1), 86–105 (2018)
20. Baker, H.F.: Abel’s Theorem and the Allied Theory of Theta Functions. Cambridge University Press, Cambridge (1897). Reprinted in 1995
21. Grushevsky, S., Manni, R.S.: Jacobians with a vanishing theta-null in genus 4. Isr. J. Math. 164(1), 303–315 (2008). arXiv:math/0605160

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.