An Exact Calculation of the Energy Density of Cosmological Gravitational Waves

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Abstract

In this paper we calculate the Bogoliubov coefficients and the energy density of the stochastic gravitational wave background for a universe that undergoes inflation followed by radiation domination and matter domination, using a formalism that gives the Bogoliubov coefficients as continuous functions of time. By making a reasonable assumption for the equation of state during reheating, we obtain in a natural way the expected high frequency cutoff in the spectral energy density.

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1 Introduction

In the usual framework for the calculation of the stochastic gravitational wave background spectrum [1] it is assumed that the transition between two consecutive epochs in the evolution of the universe is sudden, requiring only the continuity of the metric and its first derivative at the transition point (when the unperturbed space-time manifold is the Friedmann-Lemaître-Robertson-Walker (FLRW) model, this reduces to the continuity of the scale factor and its first derivative). Matching the wave function for the perturbation and its first derivative at both sides of the transition, we obtain the Bogoliubov coefficients relating the initial and the final states. This sudden transition approximation gives correct results, except for very large frequencies where it gives an overproduction of gravitons that causes the energy density to diverge [2]. This overproduction can be traced back to the fact that in this approximation, the second derivative of the scale factor and therefore the scalar curvature is discontinuous [3]. To solve this problem, a cutoff is introduced for frequencies $f > f_t$ at the time the transition occurs, by invoking general adiabatic considerations [4], [5]. Loosely speaking, the modes with $f \gg f_t$ will not notice that the universe is expanding and hence will be exponentially suppressed; this suppression is well simulated by the cutoff. But for frequencies near $f_t$ the problem cannot be correctly solved by the use of a sharp cutoff.

Contrary to what happens with density perturbations where the high frequency part of the spectrum can be easily distorted, the high frequency part of the gravitational wave spectrum will remain unchanged during the evolution of the universe due to the fact that gravitational waves decouple very early. Thus, if our goal is the comparison of our spectra either with data from direct observations [6] or Cosmic Microwave Background Radiation measurements (CMBR) [7], [8], more attention should be given to the high frequency part of the gravitational wave background spectrum.

In this paper we calculate the exact Bogoliubov coefficients for the production of the gravitational wave background using a method developed in [9]. This method does not rely on the sudden transition approximation and can be used to obtain the Bogoliubov coefficients for the high frequency modes without any approximation. Making a reasonable assumption about the equation of state during reheating and using a scale factor that interpolates smoothly between a radiation dominated universe and a matter domi-
nated universe we obtain in a natural way the high frequency cutoff.

The outline of this paper is as follows: in section II we summarize our method for the calculation of the Bogoliubov coefficients; in section III we calculate the exact Bogoliubov coefficients for a universe that undergoes inflation, reheats and evolves through a radiation dominated epoch followed by a matter dominated epoch; we end the paper with our conclusions in section IV.

2 The method

For simplicity we assume that the unperturbed space-time manifold is the flat FLRW model. The perturbed metric can be expressed in the synchronous gauge as

\[ ds^2 = a^2(\tau) \left[ -d\tau^2 + (\delta_{ij} + h_{ij}(\tau, \mathbf{x})) \, dx^i dx^j \right] \] (1)

where \( \tau \) is the conformal time, \( a(\tau) \) is the scale factor and \( h_{ij}(\tau, \mathbf{x}) \) is the perturbation (latin indices run from 1 to 3). Expanding \( h_{ij} \) in plane waves we get

\[ h_{ij}(\tau, \mathbf{x}) = \sqrt{8\pi G} \sum_{p=1}^{2} \int \frac{d^3k}{(2\pi)^{3/2} a(\tau) \sqrt{2k}} \times \left[ a_p(k, \tau)\varepsilon_{ij}(k, p)e^{ik \cdot \mathbf{x}}\xi(k, \tau) + \text{herm. conj.} \right] \] (2)

where \( \mathbf{x} \) is the spatial coordinates three-vector, \( k \) is the comoving wave-number three-vector, \( k = |k| = \frac{2\pi a}{\lambda} = \omega a \), \( p \) runs over the two possible polarizations of the gravitational waves, \( \varepsilon_{ij}(k, p) \) is the polarization tensor, \( a_p(k, \tau) \) the annihilation operator and \( \xi(k, \tau) \) the mode function for the gravitational waves obeying the equation

\[ \xi'' + (k^2 - \frac{a''}{a})\xi = 0 \] (3)

where the \( ' \) denotes the derivative with respect to the conformal time. The annihilation and creation operators can be expressed in terms of the initial time annihilation and creation operators \( A_p(k) \) and \( A^\dagger_p(k) \) through the Bogoliubov coefficients \( \alpha(k, \tau) \) and \( \beta(k, \tau) \)

\[ a(k, \tau) = \alpha(k, \tau)A(k) + \beta^*(k, \tau)A^\dagger(k) \] (4)
(we have dropped the polarization subscripts for clarity) where the $\alpha$ and $\beta$ must satisfy

$$|\alpha|^2 - |\beta|^2 = 1. \quad (5)$$

In ref. [9] it has been shown that the Bogoliubov coefficients obey the set of coupled ODE’s

$$\alpha'(\tau) = \frac{i}{2k} \left( \alpha(\tau) + \beta(\tau)e^{2ik(\tau - \tau_0)} \right) \frac{a''(\tau)}{a(\tau)} \quad (6)$$

$$\beta'(\tau) = -\frac{i}{2k} \left( \beta(\tau) + \alpha(\tau)e^{-2ik(\tau - \tau_0)} \right) \frac{a''(\tau)}{a(\tau)} \quad (7)$$

where $\tau_0$ is an arbitrary constant. These equations are equivalent to Parker’s integral equations [4] for $\alpha$ and $\beta$ provided we make for his $W(k, \tau)$ the ansatz

$$W(k, \tau) = k. \quad (8)$$

Substituting

$$\alpha = \frac{1}{2}(X + Y)e^{ik(\tau - \tau_0)} \quad (8)$$

$$\beta = \frac{1}{2}(X - Y)e^{-ik(\tau - \tau_0)} \quad (9)$$

with $X \equiv X(k, \tau)$ and $Y \equiv Y(k, \tau)$ our system takes the form

$$X'' + (k^2 - \frac{a''}{a})X = 0 \quad (10)$$

$$Y = \frac{i}{k}X'. \quad (11)$$

When the scale factor has the power-law form, the solution of equation (10) can be expressed in terms of the Hankel functions [10]. Once we have solved equation (10) for the entire period under investigation, $\alpha$ and $\beta$ can be easily calculated from equations (11), (8) and (9). To obtain the final Bogoliubov coefficients we must now perform one more Bogoliubov transformation to the mode functions appropriate to the present state of the universe as explained in ref. [9]. Denoting the readily calculable Bogoliubov coefficients corresponding to the final transformation by $\alpha_{ft}$ and $\beta_{ft}$, the final Bogoliubov coefficients $\alpha_F$ and $\beta_F$ will then be given by

$$\alpha_F = \alpha \alpha_{ft} + \beta \beta_{ft}^* \quad (12)$$

$$\beta_F = \beta \alpha_{ft}^* + \alpha \beta_{ft}. \quad (13)$$
With the formalism described above no sudden transition approximation is needed and the high frequency part of the graviton spectrum can be obtained exactly, provided we can make some reasonable assumption about the behaviour of the scale factor during the transitions between two consecutive epochs.

3 The Bogoliubov coefficients

We will consider a Universe that undergoes inflation, reheating and enters an epoch where dynamics is governed by a mixture of radiation and dust. The scale factor has the form

\[ a(\tau) = \begin{cases} 
\frac{1}{H(\tau_1 - \tau)} & \tau \leq \tau_I \\
\frac{a_R(\tau)}{\tau_I} & \tau_I < \tau \leq \tau_R \\
a_{eq}(\tau^2 + 2\tau) & \tau > \tau_R 
\end{cases} \]  

(14)

where \( a_R(\tau) \) is an yet undetermined function of \( \tau \) that will smoothly interpolate between inflation and the next epoch, \( a_{eq} \) is the scale factor at the time of equality between the densities of radiation and dust \( \tau_{eq} \) and \( H \) (the Hubble constant during inflation) and \( \tau_1 \) will be determined later. The solution is normalized in such a way that \( \tau_{eq} = \sqrt{2} - 1 \). The scale factor for \( \tau > \tau_R \) is a solution of the Friedmann equation for a universe filled with a non-interacting mixture of radiation and dust. This assumption provides a smooth transition from a radiation dominated universe to a dust dominated universe and is the simplest possible improvement over the usual treatment, certainly giving a better approximation near \( \tau_{eq} \); for \( \tau \ll \tau_{eq} \) the scale factor reduces to the one for a radiation dominated universe \( a \propto \tau \) and for \( \tau \gg \tau_{eq} \) we recover the scale factor for a dust dominated universe \( a \propto \tau^2 \).

We still have to do some assumptions about the scale factor during reheating. A reasonable way to proceed is to assume an independent equation of state for each of the components of the fluid filling the universe and integrating the Friedmann and energy density conservation equations to obtain \( a_R(\tau) \). A simple but not unique choice for the equation of state is

\[ p_i(\tau) = \lambda_i(\tau) \rho_i(\tau) \quad (i = 1, 2) \]  

(15)
where the $\lambda_i$ must satisfy the limiting values
\begin{equation}
\begin{aligned}
\lambda_1(\tau_I) &= \lambda_2(\tau_I) = -1 \\
\lambda_1(\tau_R) &= \frac{1}{3} \\
\lambda_2(\tau_R) &= 0
\end{aligned}
\end{equation}
(16)
as appropriate to the inflationary and radiation+dust stages respectively.

Of course, our choice of equation of state does not intend to be more than illustrative of the method we are using. From the present values of $\rho_{\text{rad}}$ and $\rho_{\text{dust}}$ we obtain $\rho_i(\tau_R)$
\begin{equation}
\begin{aligned}
\rho_1(\tau_R) &= \rho_{\text{rad}}(\tau_R) = \rho_{\text{rad}}(\tau_p) \left( \frac{a(\tau_p)}{a(\tau_R)} \right)^4 \\
\rho_2(\tau_R) &= \rho_{\text{dust}}(\tau_R) = \rho_{\text{dust}}(\tau_p) \left( \frac{a(\tau_p)}{a(\tau_R)} \right)^3
\end{aligned}
\end{equation}
(17)
(18)
where $\tau_p$ is the present conformal time.

Denoting by $f(\tau)$ the function
\begin{equation}
f(\tau) = \begin{cases}
0 & \tau \leq 0 \\
\exp(-c_1/\tau) & \tau > 0
\end{cases}
\end{equation}
(19)
we choose $\lambda_i$ as
\begin{equation}
\begin{aligned}
\lambda_1 &= \frac{1}{3} \frac{f(\tau - \tau_I)}{f(\tau - \tau_I) + f(\tau_R - \tau)} - \frac{f(\tau_R - \tau)}{f(\tau - \tau_I) + f(\tau_R - \tau)} \\
\lambda_2 &= \frac{f(\tau_R - \tau)}{f(\tau - \tau_I) + f(\tau_R - \tau)}
\end{aligned}
\end{equation}
(20)
(21)
where the constant $c_1$, can be used to control the rate at which the transition occurs. It can be easily seen that these $\lambda$’s obey conditions (16). As fig. 1 shows, these are very smooth functions.

We can finally write down the equations that determine $a_R$, $\rho_1$ and $\rho_2$
\begin{equation}
\left( \frac{a'}{a^2} \right)^2 = \frac{8\pi G}{3c^2} (\rho_1 + \rho_2)
\end{equation}
(22)
\begin{equation}
\rho_i' = -3 \frac{a'}{a} \rho_i (1 + \lambda_i) \quad (i = 1, 2)
\end{equation}
(23)
with the $\lambda_i$ given by (20) and (21). These equations can be easily solved numerically. If we want our model to be consistent with the present day observed universe then we must evolve back in time the present condition of the universe to obtain the state of the universe at $\tau_R$ and then we integrate equations (22) and (23) backwards in time up to $\tau_I$. This procedure will give us a scale factor that is compatible with the observed Universe. Once the numerical integration is done we can easily find $H$ and $\tau_1$ appearing in eq. (14)

$$H = \frac{a_R^2(\tau_I)}{a'_R(\tau_I)}; \quad \tau_1 = \frac{a_R(\tau_I)}{a'_R(\tau_I)} + \tau_I$$

(24)

In our calculations we used for the redshifts at $\tau_{eq}$, $\tau_R$ and $\tau_I$ the values $1 + Z_{eq} = 2.38 \times 10^4$, $1 + Z_R = 3 \times 10^{28}$ and $1 + Z_I = 1.6 \times 10^{29}$.

Knowing $a(\tau)$ for the entire period under consideration, we are now prepared to calculate the Bogoliubov coefficients. The initial conditions for $\alpha$ and $\beta$ are

$$|\alpha(\tau = -\infty)| = 1; \quad |\beta(\tau = -\infty)| = 0$$

(25)

which translate for the $X$ and $Y$ variables into

$$|X(\tau = -\infty)| = |Y(\tau = -\infty)| = 1$$

(26)

Equation (14) can be integrated analytically up to $\tau = \tau_I$. The solution that satisfies (25) gives at $\tau_I$

$$X(\tau_I) = e^{-ik(\tau_I - \tau_1)} \left(1 + \frac{i}{k(\tau_1 - \tau_I)}\right)$$

(27)

$$Y(\tau_I) = e^{-ik(\tau_I - \tau_1)} \left(1 + \frac{i}{k(\tau_1 - \tau_I)} - \frac{1}{k^2(\tau_1 - \tau_I)^2}\right)$$

(28)

At this point we must proceed with the integration numerically using (27) and (28) as initial conditions. After recovering $\alpha$ and $\beta$ from (8) and (9) we must perform the final Bogoliubov transformation (12) and (13). To do this, we notice first that in spite of our final state being one of mixed radiation and dust, as $\tau_p \gg \tau_{eq}$ the contribution of the radiation component is negligible (we have seen at the beginning of section III that for $\tau \gg \tau_{eq}$ we recover the scale factor for a matter dominated universe $a \propto \tau^2$) and we can consider that the final state in our model is one of matter domination. As explained fully in ref. [9] the final Bogoliubov coefficients must be those
appropriate to the present day matter era mode functions and this requires a final Bogoliubov transformation to the matter era mode functions with the corresponding Bogoliubov coefficients $\alpha_{ft}$ and $\beta_{ft}$ in equations (12) and (13) given by [1], [2], [9]

$$\alpha_{ft} = \frac{e^{ik\tau_0} \left( 1 + \frac{i}{k\tau_p} - \frac{1}{2(k\tau_p)^2} \right)}{(29)}$$

$$\beta_{ft} = \frac{-e^{-ik(2\tau_p - \tau_0)} 2(k\tau_p)^2}{(30)}.$$

Figures 2 shows our final results expressed as the energy density $P(\omega) = dE/d\omega = \hbar \omega^3 |\beta|^2/\pi^2 c^3$. It is clear from this figure that with the approach developed in this paper it is possible to obtain, in a natural way the high frequency cutoff.

For the sake of comparison we also calculate the Bogoliubov coefficients in the sudden transition approximation for a universe that undergoes inflation, radiation domination and matter domination. Assuming that in this case the transition between inflation and radiation occurs at $\tau_{ir}$ and denoting the Bogoliubov coefficients associated with the transition from inflation to radiation domination and from radiation to matter domination respectively with the subscripts $r$ and $d$ we have

$$\alpha_r = \frac{e^{2ik\tau_I} \left( 1 + \frac{i}{k\tau_I} - \frac{1}{2(k\tau_I)^2} \right)}{(31)}$$

$$\beta_r = \frac{1}{2(k\tau_I)^2}$$

$$\alpha_d = \frac{e^{ik\tau_{eq}} \left( 1 + \frac{i}{2k\tau_{eq}} - \frac{1}{8(k\tau_{eq})^2} \right)}{(33)}$$

$$\beta_d = -e^{-3ik\tau_{eq}} \frac{1}{8(k\tau_{eq})^2}$$

The final Bogoliubov coefficients are then given by [1], [2]

$$\alpha = \begin{cases} \alpha_r & k_{min} < k \leq k_r \\ \alpha_r \alpha_d + \beta_r \beta_d^* & k_r < k \leq k_{cut} \end{cases}$$

$$\beta = \begin{cases} \beta_r & k_{min} < k \leq k_r \\ \beta_r \alpha_d^* + \alpha_r \beta_d & k_r < k \leq k_{cut} \end{cases}$$
with

\[k_{\text{min}} = 2 \pi a(\tau_p) H(\tau_p)\]  \hspace{1cm} (37)

\[k_r = 2 \pi a(\tau_{eq}) H(\tau_{eq})\]  \hspace{1cm} (38)

To fix the value of \(k_{\text{cut}}\) we notice that, according to general adiabatic arguments, the characteristic frequency for the cutoff is given by \(\omega_{\text{cut}} = 2 \pi \Delta t^{-1}\) where \(\Delta t\) is the (physical) time scale for the transition, usually taken to be \(H(\tau_{ir})^{-1}\) \[10\], \[11\]. However, as these authors point, the exact value of \(\Delta t\) depends on the details of the transition. In our model \(\Delta \tau\) is known and we have \(\Delta t \approx a(\tau_{ir}) \Delta \tau\) which gives

\[k_{\text{cut}} = \frac{2\pi}{\Delta \tau} .\]  \hspace{1cm} (39)

At this point we still have to determine \(\tau_{ir}\). Because in our model we have introduced one intermediate epoch between inflation and radiation there is, when comparing the two models, an ambiguity in the choice of \(\tau_{ir}\). To overcome this ambiguity we choose \(\tau_{ir}\) in such a way that the number of gravitons at the beginning of the radiation dominated era is the same in the two models.

Figure 3 shows the comparison of the results obtained with our method and the one using the sudden transition approximation. We can see in Fig. 3a that the method discussed in this paper predicts a larger \(P\) for the low frequency gravitons than is the case with the usual method of calculation. This can be easily explained by the fact that during the radiation dominated epoch, and if we assume only radiation to be present, there is no graviton production due to the well known conformal invariance of the equations \[4\], \[12\], \[13\]. In our model the invariance is broken because we also have dust mixed with the radiation and we thus have an additional contribution to \(P\). No matter how small the contribution of the dust component becomes, the conformal invariance will always be broken, as can be seen by the fact that \(a'' = a_{eq} \neq 0\) in conformal time. We should also notice (fig. 3b) that the point where the high frequency suppression starts to take place, depends on the time it takes for the transition between inflation and radiation domination, as expected from adiabatic arguments \[4\], \[3\]. The dots in Fig. 3b represent the energy density obtained with the sudden transition approximation and \(k_{\text{cut}}\) given by (39) with \(\Delta \tau = 6 \times 10^{-26}\), corresponding to the width of the transition in our model with \(c_1 = 11\). Comparing the two curves, we can
see that, for frequencies near the suppression, the energy density obtained by our exact calculation is one order of magnitude below the result obtained with the sudden transition approximation.

4 Conclusions

We have shown in this paper how the high frequency cutoff in the energy density of the stochastic gravitational wave background could be obtained in a natural way. Although our results are not surprising, the calculation done in this paper is exact, even in the high frequency region of the spectrum. We showed that for frequencies slightly below the high frequency cutoff, the sudden transition approximation predicts an overproduction of gravitational waves. Differences to previous calculations were also obtained in the region of frequencies corresponding to the transition between the radiation dominated and the matter dominated epochs.

After this work was completed we came across a preprint by Koranda and Allen [14] where the epoch following inflation is also treated as a mixture of radiation and dust.

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FIGURE CAPTIONS

Figure 1 The $\lambda$’s given by equations (20) (a) and (21) (b) as functions of the conformal time $\tau$ during the transition from inflation to the radiation+dust era. We can see that the width of the transition can be controlled with the parameter $c_1$. Larger values of $c_1$ correspond to faster transitions. We used $\tau_I = 3.9585 \times 10^{-27}$ and $\tau_R = 3.9585 \times 10^{-25}$ in all the figures.

Figure 2 The energy density of the gravitational waves produced during the expansion of the universe obtained with the method developed in this paper. The steeper part of the curve ($f < 10^{-16} s^{-1}$) corresponds to gravitons produced during the radiation+dust epoch and the region with constant slope is due to graviton production during inflation. The sharp suppression is due to the finite width of the transition from inflation to radiation domination. This figure was obtained with $c_1 = 11$.

Figure 3 For low frequencies, ($f < 10^{-16} s^{-1}$) the contribution for $P$ of the gravitons produced during the radiation+dust epoch is larger in our model than in the sudden transition approximation. Due to the presence of a small contribution of dust in the radiation dominated era, the conformal invariance of the equations is broken giving rise to more graviton production (a). As it was expected, the point where the high frequency suppression appears depends on the time it takes for the transition from inflation to radiation domination. In our model the width of the transition can be varied with $c_1$ (b). The dots represent the results obtained with the sudden transition approximation; we introduced a high frequency cutoff $k_{cut} = 2\pi/\Delta \tau$ with $\Delta \tau = 6. \times 10^{-26}$ corresponding to the width of the transition for the case $c_1 = 11$. (See Fig. 1).
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http://arxiv.org/ps/gr-qc/9410033v1
$\lambda_2$ vs $\tau \times 10^{27}$

$c_1 = 11.$
$c_1 = 3.$
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http://arxiv.org/ps/gr-qc/9410033v1
Log10 f (s^-1)

Log10 P (g cm^-1 s^-1)

Smooth Transition

Sudden Transition
Log10 f (s\(^{-1}\))

Log10 P (g cm\(^{-1}\) s\(^{-1}\))

c1=3.
c1=5.
c1=11.
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