Niels Bohr wrote: "There is no quantum world. There is only an abstract quantum physical description. It is wrong to think that the task of physics is to find out how Nature is. Physics concerns what we can say about Nature." In an analogous way, von Weizsäcker suggested that the notion of the elementary alternative, the "Ur", should play a pivotal role when constructing physics. Both approaches suggest that the concept of information should play an essential role in the foundations of any scientific description of Nature. We show that if, in our description of Nature, we use one definite proposition per elementary constituent of Nature, some of the essential characteristics of quantum physics, such as the irreducible randomness of individual events, quantum complementary and quantum entanglement, arise in a natural way. Then quantum physics is an elementary theory of information.

Dedicated to Prof. C. F. von Weizsäcker at the occasion of his 90th birthday.

I. INTRODUCTION

All our description of objects is represented by propositions. The use of propositions is not a matter of our choice. In contrast, it is a necessity which is behind each of our attempts to learn something new about Nature and to communicate this knowledge with others. It is a necessity which we follow constantly and without any intention and it seems that there is no way to avoid it even if the phenomena to be described and to be understood are highly counterintuitive and distinct from both our everyday experience and the classical world view. One may even say that there is no need to avoid it. The reason is that the only way we are able to understand any phenomena in Nature, including quantum phenomena, is exclusively through the epistemological structure of classical physics and everyday experience. Bohr (1949) emphasized that "How far the [quantum] phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms. The argument is simply that by the word 'experiment' we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of the experimental arrangement and the result of observation must be expressed in unambiguous language with suitable application of the terminology of classical physics." von Weizsäcker (1974) emphasized that the understanding of new physical theories will also be given in language: "This verbalized language must be the language spoken by those physicists who do not know yet the theory we are telling them. The language used in order to explain a theory which we propose in mathematical form is the language which has been existed before the theory. On the other hand, it is not self-evident that this language has a clear meaning at all, because if it had a completely clear meaning probable the new theory would not be needed. Thus it may happen that by applying our new formalism to experience - an application made possible by our existing language - we may tacitly or explicitly change the rules of this very language."

Rigorously speaking a system is nothing else than a construct based on a complete list of propositions together with their truth values. The propositions from the list could be (1) "The velocity of the object is $v$" or (2) "The position of the object is $x$" and could be associated both to classical and to quantum objects. Yet, there is an important difference between the two cases. From the theorems of Bell (1964) and of Kochen and Specker (1967) we know that for a quantum system one cannot assert definite (noncontextual) truth values to all conceivable propositions simultaneously. For example, if the proposition (1) above is a definite proposition, then the proposition (2) must necessarily be completely indefinite and vice versa. The two propositions are mutually exclusive. This is a specific case of quantum complementarity.

Therefore, in an attempt to describe quantum phenomena we are unavoidably put in the following situation. On one hand the epistemological structure applied has to be inherited from the classical physics: the description of a quantum system has to be represented by the propositions which are used in the description of a classical system, and on the other hand, those propositions cannot be assigned to a quantum system simultaneously. Now, a natural question arises: How to join these two, seemingly inconsistent, requirements? We suggest to use the concept of "knowledge" or "information". Then even in situations where we cannot assert simultaneously definite truth-values to mutually exclusive propositions we can assert measures of information about their truth values. The structure of the theory
including the description of the time evolution can then be expressed in terms of measures of information\(^1\). To us this seems to be a change with the lowest possible "costs" in the epistemological structure of classical physics. And since some costs are unavoidable anyway we believe that the information-theoretical formulation of quantum physics leads to the "easiest" understanding of the theory.

From the point of view that the information content of a quantum system is fundamentally limited we will discuss precisely the empirical significance of the terms involved in formulating quantum theory, particularly the notion of a quantum state. However we are aware of the possibility that this might not carry the same degree of intuitive appeal for everyone. It is clear that it may be matter of taste whether one accepts the suggested concepts and principles as self-evident as we do or not. If not, then one may turn the reasoning around and, following our approach in (Brukner and Zeilinger, 1999; Brukner et al. 2001), argue for the validity of the statements given in the paper on the basis of known features of quantum physics.

The conceptual groundwork for the ideas presented here has been prepared most notably by von Bohr (1958), Weizsäcker (1958) and Wheeler (1983). In contrast to those other authors who look for deterministic mechanisms hidden behind the observed facts, these authors attempt to understand the structure of quantum theory as a necessity for extracting whatever meaning from the data of observations.

In recent years several different ideas were put forward suggesting that information can help us to learn more about the foundations of quantum physics. The foundations of quantum mechanics are interpreted in the light of quantum information (Fuchs, 2001; 2002, Caves et al., 2001a; 2001b). It was also suggested how to reduce quantum theory to few statements of physical significance by generalizing and extending classical probability theory (Hardy, 2001a; 2001b). In another approach it was shown how certain elements of the structure of quantum theory emerge from looking for invariants of probabilistic observations assuming that any newly gained information shall lead to more accurate knowledge of these invariants (Summhammer, 1988; 1994; 2000; 2001).

II. FINITENESS OF INFORMATION, UR, ELEMENTARY SYSTEM

One of the most distinct features of quantum physics with respect to classical physics is that prediction with certainty of individual outcomes is only possible for a very limited class of experiments. Such a prediction is equivalent to saying that the corresponding propositions have definite truth values. For all other (complementary) propositions the truth values are necessarily indefinite. We suggest this to be a consequence of the feature that\(^2\)

*The information content of a quantum system is finite.*

With this we mean that a quantum system cannot carry enough information to provide definite answers to all questions that could be asked experimentally. Then, by necessity the answer of the quantum system to some questions must contain an element of randomness. This kind of randomness must then be irreducible, that is, it cannot be reduced to "hidden" properties of the system. Otherwise the system would carry more information than what is available. Thus, without any additional physical structure assumed, we let the irreducible randomness of an individual event and complementarity, be a consequence of the finiteness of information.

How much information is available to a quantum system? If this information is limited than it is natural to assume that if we decompose a physical system, which may be represented by numerous propositions, into its constituents, each such constituent will be described by fewer propositions. This process of subdividing a system can go further until we reach a final limit when an individual system represents the truth value to one single proposition only. It is then suggestive to replace the above statement by a more precise one (Zeilinger, 1999):

*The most elementary system represents the truth value of one proposition.*

We call this the principle of quantization of information. One may consider the above statement as a definition of what is the most elementary system. Note that the truth value of a proposition can be represented by one bit of information with "true" being identified with the bit value "1" and "false" being identified with the bit value "0". Thus, the principle becomes simply:

*The most elementary system carries 1 bit of information.*

---

1 Heisenberg (1958) wrote: "The laws of nature which we formulate mathematically in quantum theory deal no longer with the particles themselves but with our knowledge of the elementary particles. ... The conception of objective reality ... evaporated into the ... mathematics that represents no longer the behavior of elementary particles but rather our knowledge of this behavior."

2 Feynman wrote: "It always bothers me that, according to the laws as we understand them today, it takes a computing machine an infinite number of logical operations to figure out what goes on in no matter how tiny a region of space and no matter how tiny a region of time, ... why should it take an infinite amount of logic to figure out what one tiny piece of space-time is going to do?" A closely related view was assumed by Landauer, who writes in his article "Information is Physical" (1991): "... the laws of physics are ... limited by the range of information processing available.".
FIG. 1 Spin measurement of a spin-1/2 particle. The particle passes through the Stern-Gerlach magnet oriented at the angle $\theta$, and then it hits one of the detector plates behind the Stern-Gerlach magnet. Depending on whether the upper or the lower detector plate is hit by a particle we call the outcome "yes" and "no", respectively.

We relate the notion of the most elementary system to that of the "Ur" introduced by von Weizsäcker. He was the first who introduced the concept of the most basic informational constituent of all objects ("Urs"). von Weizsäcker wrote (1974): "It is certainly possible to decide any large alternative step by step in binary alternatives. This may tempt us to describe all objects as composite systems composed from the most simple possible objects. The simplest possible object is an object with a two-dimensional Hilbert space, the 'ur'. The word 'ur' is introduced to have an abstract term for something which can be described by quantum theory and has a two-dimensional Hilbert space, and nothing more."

How much information is contained in more complex systems consisting of $N$ elementary systems? It is natural to assume that the information content of a complex system is proportional to the number of elementary constituents. The principle of quantization of information is thus generalized to (Zeilinger, 1999)

$N$ elementary systems represent the truth values of $N$ propositions, or equivalently,

$N$ elementary systems carry $N$ bits.

The finding compatible with this principle were reported (Donath and Svozil, 2002; Svozil 2002). Again one may consider the two statements as definitions of what is a composite system consisting of $N$ elementary systems. Note that the principle given above does not make any statement about how the information contained in $N$ propositions ($N$ bits) is distributed over the $N$ systems. It can be represented by the $N$ systems individually or, alternatively, it can be represented by $N$ systems jointly. The latter is the feature of quantum entanglement, discussed in more detail below, for which Schrödinger (1935) wrote: "If two separated bodies, each by itself known maximally, enter a situation in which they influence each other, and separate again, then there occurs regularly that which I have just called entanglement of our knowledge of the two bodies. The combined expectation-catalog consists initially of a logical sum of the individual catalogs; during the process it develops causally in accord with known law (there is no question whatever of measurement here). The knowledge remains maximal, but at its end, if the two bodies have again separated, it is not again split into a logical sum of knowledges about the individual bodies. What still remains of that may have becomes less than maximal, even very strongly so. -One notes the great difference over against the classical model theory, where of course from known initial states and with known interaction the individual end states would be exactly known."

The fundamental statements given above seem to suggest that binary (yes-no) alternatives are representatives of basic information units of all systems. Consider the case of $n$-fold (i.e. ternary, quaternary etc.) alternatives generalizing the binary ones. Obviously, any $n = 2^N$-fold alternative is decomposable into binary ones. Note that such an alternative can be realized in measurement of $N$ elementary systems. Yet, it is not obvious how to decompose or how to consider a general $n$-fold alternative with $n \neq 2^N$? An interesting possibility would be to find the factorization of number $n$ into its prime-number factors $p_1, p_2, \ldots$ and then to decompose the $n$-fold alternative into a sequential serial of $p_1$-fold alternatives, $p_2$-fold alternatives etc. Obviously such an approach would require to extend the notion of elementary system to all prime-number dimensional systems. Interestingly, as it will be shown later (see Sec. VI), only in these cases where the dimension of the quantum system is (a power of) a prime number, the total information content of the system can be defined unambiguously. This might suggest that the notion of the elementary system should indeed be extended to all prime-number dimensions. However, in this manuscript we will mainly restrict our analysis to binary decomposable alternatives.

We would like to stress again that notions such as that a system "represents" the truth value of a proposition or that it "carries" one bit of information only implies a statement concerning what can be said about possible measurement results. For us a system is no more than a representative of a proposition.
III. MUTUALLY COMPLEMENTARY PROPOSITIONS

We consider an explicit example of an elementary system, the spin-1/2 particle, and the Stern-Gerlach experiment as depicted in Fig. 1 schematically. Depending on whether the upper or the lower detector plate is hit by a particle we call the outcome "yes" and "no" respectively, where "yes" and "no" represent the truth values of the proposition for the spin to be up along a chosen direction. The upper detector plate is hit with probability \( p \). If it is not hit the other detector plate will be hit with probability \( 1 - p \). Therefore we consider a binary alternative. Different experimental situations are specified by the orientation \( \theta \) of the magnet in the Stern-Gerlach apparatus as shown in Fig. 1.

Consider an elementary system specified by the true proposition "The spin along the \( z \)-axis is up" (or, equivalently, by the false proposition "The spin along the \(-z\)-axis is up"). This situation is described by the probabilities \( p(0) = 1 \) and \( p(\pi) = 0 \) for the "yes" outcome. Because a spin can carry one bit of information only, each proposition: "The spin along the direction tilted at an angle \( \theta \) (\( 0 < \theta < \pi \)) from the \( z \)-axes is up" has to be probabilistic (Fig. 2). How does the probability \( p(\theta) \) of a "yes" count depend upon the angle \( \theta \)?

We assume that the mapping of \( \theta \) to \( p(\theta) \) is analytic and monotonic. Then using the Cauchy theorem about continuous and monotonic functions one concludes that there has to be one and only one angle of orientation of the magnet in the Stern-Gerlach apparatus where the probabilities for a "yes" and for a "no" outcome are equal. Because of the symmetry of the problem this obviously has to be the angle \( \pi/2 \). For each direction \( \vec{n} \) in the \( x-y \) plane (the green circle on the sphere in Fig. 2 and 3) the proposition "The spin along the \( \vec{n} \)-axis is up" is completely indefinite, that is, we have absolutely no knowledge which outcome "yes" or "no" will be observed in a specific individual measurement. Note, however, that in principle this equal number of yes-no outcomes could also be achieved by an ensemble of systems each giving a definite result for each direction such that the same number of "yes" or "no" results is obtained. Yet again this would imply that an individual system carries enough information to permit assignment of definite truth values to all possible propositions, in contradiction to our basic principle.

Consider now the state of a spin-1/2 particle specified by the proposition "The spin along the \( x \)-axis is up (down)". In this case we have complete knowledge which outcome will be observed when the Stern-Gerlach magnet is oriented

---

3 If, in contrast, \( p(\theta) \) would only be sectionally analytic in \( \theta \) then there would be points of nonanalyticity separating two regions in which the function \( p(\theta) \) has different analytic forms. Thus the values of the function on a finite segment in the interior of a domain of analyticity would only determine, by the uniqueness theorem for analytic functions, the function up to the next point of nonanalyticity. Clearly, to describe such a system completely we would need catalogs both of functional values on finite segments in the interior of each domain of analyticity and of the positions of the points of nonanalyticity. Such a catalog would require large amount of information to describe the functional dependence and thus contradicts our desideratum of minimal information content of a quantum system.
FIG. 3 The formation of mutually complementary propositions associated with orthogonal spin components. If measurement along the $z$-axis (x-axis) [$y$-axis] gives a definite result, measurement along any direction in the $x$-$y$ plane, the green circle ($y$-$z$ plane, the yellow circle) or [$x$-$z$ plane, the red circle] will be maximally random, respectively. There are altogether three mutually complementary spin measurements represented by three intersection points of the green, yellow, and red circle.

along the $\pm x$-axis at the expense of the fact that we have absolutely no knowledge about the outcome for the orientation of the magnet along any direction in the $y$-$z$ plane (the yellow circle on the sphere in Fig. 3).

Finally, consider the state of a spin-1/2 particle specified by the proposition "The spin along the $y$-axis is up (down)". In that case we know precisely the outcome of the experiment when the Stern-Gerlach magnet is oriented along the $\pm y$-axis at the expense of complete uncertainty about the outcome when it is oriented along any direction in the $x$-$z$ plane (the red circle on the sphere in Fig. 3).

There are, therefore, altogether three mutually exclusive or complementary propositions (represented by three intersection points of the green, yellow and red circle on the sphere in Fig. 3): "The spin along direction $\vec{n}_1$ is up (down)", "The spin along direction $\vec{n}_2$ is up (down)" and "The spin along direction $\vec{n}_3$ is up (down)", where $\vec{n}_1$, $\vec{n}_2$ and $\vec{n}_3$ are mutually orthogonal directions. These are propositions with a property of mutually exclusiveness: the total knowledge of one proposition is only possible at the cost of total ignorance about the other two complementary ones. In other words precise knowledge of the outcome of one experiment implies that all possible outcomes of complementary ones are equally probable.

Why are there exactly 3 mutually complementary propositions for the elementary system and not, e.g., 2 or 4? We do not understand that fully. However the discussion above indicates that there is a strong link between the number (3) of mutually complementary propositions and the (three-)dimensionality of the ordinary space. We will come back once more to these question in the conclusions. But it is important to note that in any system with dichotomic (2-valued) observables there are always three complementary propositions even if these cannot be linked to the dimensionality of ordinary space.

IV. MEASURE OF INFORMATION IN A PROBABILISTIC EXPERIMENT

Consider a probabilistic experiment with $n$ possible outcomes. Suppose that the experimenter plans to perform $N$ trials of the experiment. All he knows before the trials are performed are the probabilities $p_1,...,p_i,...,p_n$ for all possible outcomes to occur: What kind of prediction can the experimenter make?

In general two cases are conceivable. The experimenter can ask: "What is the precise sequence of the $N$ outcomes?" or "What is the number of occurrences of the outcome $i$?". We will say that in answering the first question the experimenter makes a "deterministic" prediction and in answering the second one he makes a "probabilistic" prediction (following discussion in Summhammer, 2000). Obviously the deterministic prediction can only make sense if different outcomes follow from the intrinsically different individuals of the ensemble measured - the situation which we have in classical measurements. Then the precise sequence of outcomes reveal which property which individual member of

---

4 Summhammer (2000) wrote: "I want to discard a deterministic link. The reason is that the amount of records available to the observer to form a conception of the world is always finite, so that many different sets of laws can be invented to account for them. Pinning down any one of these sets as the laws of nature is then purely speculative. On the other hand we have the probabilistic view, which is successfully used to interpret quantum observations. It seems that in this view we assign a minimum of information content to observed data. To see this, imagine the $N$ trials of a probabilistic yes-no experiment, like tossing a coin, in which the outcome "yes" occurs $L$ times. If we want to tell somebody else the result it is sufficient to state the values of $N$ and of $L$. With the deterministic view, in which the precise sequence of outcomes is important, we would in general have to communicate many more details to enable the receiver to reconstruct this sequence."
the experimenter can only make probabilistic predictions. If the experimenter decides to perform  
Timpson 2001). Since outcomes of quantum-mechanical experiments are in general intrinsically probabilistic, there the  
measure (for discussion of these points see Brukner and Zeilinger, 2001; Hall, 2000; Brukner and Zeilinger 2000; Timpson 2001). Since outcomes of quantum-mechanical experiments are in general intrinsically probabilistic, there the experimenter can only make probabilistic predictions. If the experimenter decides to perform \( N \) future experimental trails, all he can guess is the number of occurrences of a specific outcome. Such a prediction will now be analyzed for the case of two possible outcomes "yes" and "no".

Because of the statistical fluctuations associated with any finite number of experimental trials, the number \( L \) of occurrences of the "yes" outcome in future repetitions of the experiment is not precisely predictable\(^5\). The random variable \( L \) is subject to a binomial distribution. Since it has a finite \( \sigma \) deviation, it fulfills Chebyshev’s inequality (Gnedenko, 1976):

\[
\text{Prob}\{ |L - pN| > k\sigma \} \leq \frac{1}{k^2},
\]

where the standard deviation \( \sigma \) is given by

\[
\sigma = \sqrt{p(1-p)N}.
\]

This inequality means that the probability that the number \( L \) will deviate from the product \( pN \) by more often than \( k \) deviations is less than or equal to \( 1/k^2 \). In the case of small \( \sigma \), large deviations of the number of occurrences of the "yes" outcome from the mean value \( pN \) are improbable. In this case the experimenter knows the future number of occurrences with a high certainty. Conversely, a large \( \sigma \) indicates that not all highly probable values of \( L \) lie near the mean \( pN \). In that case the experimenter knows much less about the future number of occurrences.

We suggest to identify the experimenter’s uncertainty \( U \) with \( \sigma^2 \). Then it will be proportional to the number of trials. This important property guarantees that each individual performance of the experiment contributes the same amount of information, no matter how many times the experiment has already been performed. After each trial the experimenter’s uncertainty about the specific outcome therefore decreases by

\[
U = \frac{\sigma^2}{N} = p(1-p).
\]

This is the lack of information about a specific outcome with respect to a single future experimental trial. If, instead of two outcomes, we have \( n \) of them with the probabilities \( p_j \equiv (p_1, p_2, ..., p_n) \) for the individual occurrences, then we suggest to define the total lack of information regarding all \( n \) possible experimental outcomes as

\[
U(\vec{p}) = \sum_{j=1}^{n} U(p_j) = \sum_{j=1}^{n} p_j(1-p_j) = 1 - \sum_{j=1}^{n} p_j^2.
\]

\(^5\) Here, a very subtle and careful position was assumed by Weizsäcker (1974) who writes: "It is most important to see that this [the fact that probability is not a prediction of the precise value of the relative frequency] is not a particular weakness of the objective empirical use of the concept of probability, but a feature of the objective empirical use of any quantitative concept. If you predict that some physical quantity, say a temperature, will have a certain value when measured, this prediction also means its expectation value within a statistical ensemble of measurements. The same statement applies to the empirical quantity called relative frequency. But here are two differences which are connected to each other. The first difference: In other empirical quantities the dispersion of the distribution is in most cases an independent empirical property of the distribution and can be altered by more precise measurements of other devices; in probability the dispersion is derived from the theory itself and depends on the absolute number of cases. The second difference: In other empirical quantities the description of their statistical distributions is done by another theory than the one to which they individually belong, namely by the general theory of probability; in probability this discussion evidently belongs to the theory of this quantity, namely of probability itself. The second difference explains the first one."
FIG. 4 A set of three mutually complementary Stern-Gerlach arrangements labeled by a single experimental parameter $\theta$ which specifies the orientations of the Stern-Gerlach magnets in the three experiments. The three experimental arrangements are associated to the mutually complementary propositions: $P_1(\theta)$: "The spin along the $x$-axis is up", $P_2(\theta)$: "The spin is up along the direction tilted at angle $\theta$ from the $z$-axes" and $P_3(\theta)$: "The spin is up along the direction tilted at angle $\theta + 90^\circ$ from the $z$-axes".

The uncertainty is minimal if one probability is equal to one and it is maximal if all probabilities are equal.

This suggests that the knowledge, or information, with respect to a single future experimental trial an experimentalist possesses before the experiment is performed is somehow the complement of $U(\vec{p})$ and, furthermore, that it is a function of a sum of the squares of probabilities. A first ansatz therefore would be $I(\vec{p}) = 1 - U(\vec{p}) = \sum_{i=1}^{n} p_i^2$. Expressions of such a general type were studied in detail by Hardy, Littlewood and Pólya (1952). Notice that this expression can also be viewed as describing the length of the probability vector $\vec{p}$. Obviously, because of $\sum_i p_i = 1$, not all vectors in probability space are possible. Indeed, the minimum length of $\vec{p}$ is given when all $p_i$ are equal ($p_i = 1/n$). This corresponds to the situation of complete lack of information in an experiment about its future outcome. Therefore we suggest to normalize the measure of information in an individual quantum measurement as obtaining finally

$$I(\vec{p}) = N \sum_{i=1}^{n} \left( p_i - \frac{1}{n} \right)^2,$$

where $N$ is the normalization$^6$. Specifically, for a binary experiment the measure of information is given as

$$I(p_1, p_2) = 2 \left( p_1 - \frac{1}{2} \right)^2 + 2 \left( p_2 - \frac{1}{2} \right)^2 = (p_1 - p_2)^2.$$

It reaches its maximal value of 1 bit of information if one of the probabilities is one and it takes its minimal value of 0 bits of information if both probabilities are equal.

V. THE CATALOG OF KNOWLEDGE OF A QUANTUM SYSTEM

Consider again a stationary experimental arrangement with two detectors, where only one detector fires in each experimental trial. The first detector, say, fires (we call this the "yes" outcome) with probability $p_1$. If it is does not fire the other detector fires with probability $p_2 = 1 - p_1$ (the "no" outcome).

Note that the experimenter’s measure of information for the binary experiment as defined by Eq. (6) is invariant under permutation of the set of possible outcomes. In other words, it is a symmetrical function of $p_1$ and $p_2$.

---

$^6$ In (Brukner and Zeilinger, 1999) only those cases were considered where maximally $k$ bits of information can be encoded, i.e. $n = 2^k$. The normalization there is $N = 2^k k/(2^k - 1)$. Then $I(\vec{p})$ results in $k$ bits of information if one $p_i = 1$ and it results in 0 bits of information when all $p_i$ are equal.
A permutation of the set of possible outcomes can be achieved in two manners, which may be called "active" and "passive". In the passive point of view the permutation is obtained by a simple relabelling of the possible outcomes and the property of invariance is self evident because relabelling obviously does not make an experiment more predictable.

From the active point of view, one retains the same labeling, and the permutation of the set of outcomes refers to a real change of the experimental set-up. For a spin measurement this would be a re-orientation of the Stern-Gerlach magnet. In that case the property of invariance states that the measure of information is indifferent under certain real physical changes of the experimental situation. This requirement is more stringent and may be precisely formulated as an invariance of the measure of information under interchange of the following two physical situations: (a) the probability for "yes" is $p_1$ and for "no" is $p_2$; and (b) the probability for "yes" is $p_2$ and for "no" is $p_1$. Yet these are different experimental situations.

In order to remove this ambiguity in the description of the experiment one can assign probabilities for occurrences or different numbers or other distinct labels to possible outcomes, the particular scheme is of no further relevance. For example, one can use the statement "the probability for the outcome 'yes' is 0.6, and for the outcome 'no' is 0.4", or the statement "the probability for the outcome 'yes' is 0.4, and for the outcome 'no' is 0.6" to distinguish between the situations (a) and (b) given above. Note that in both cases the measure of information as defined by $I(\theta)$ is $I = 0.04$.

Here we will use a particular description which is based on the quantity

$$i = p_1 - p_2.$$  

Then, on one hand, the sign of $i$ differs between the two situations in (a) and (b), and on the other hand, the square of $i$ is equal to the measure of information $(I = i^2)$. Therefore $i$ represents an economic and complete description of the experimental situation (equivalent to the assignment of specific probabilities for the two results)\(^7\).

All the "quantum state" is meant to be is a representation of that catalog of our knowledge of the system that is necessary to arrive at the set of, in general probabilistic, predictions for all possible future observations of the system. Such a view was assumed by Schrödinger (1935) who wrote\(^8\): "Sie (die $\psi$-Funktion) ist jetzt das Instrument zur Voraussage der Wahrscheinlichkeit von Maßzahlen. In ihr ist die jeweils erreichte Summe theoretisch begründeter Zukunftserwartungen verkörpert, gleichsam wie in einem Katalog niedergelegt." The $\psi$ function is characterized by a set of complex numbers which are very remote from our everyday experience. Yet, if the origin of the structure of quantum theory is to be sought in a theory of observations, of observers, and of meaning, then we should focus our attention not on complex numbers, but rather on real-value quantities which are directly observable\(^9\). Interestingly, quantum theory allows descriptions of quantum state in terms of real numbers. An example for this is the description of density operators in terms of the real coefficients in the decomposition into generators of SU(N) algebra (basis of generalized Pauli matrices as used in, e.g., Schlienz and Mahler, 1998).

We will use a description of the state of an elementary system by a vector $\vec{i} = (i_1, i_2, i_3) = (p_1^+ - p_1^-, p_2^+ - p_2^-, p_3^+ - p_3^-)$, which is a catalog of knowledge about a set of three mutually complementary propositions and where, in the case of

\(^7\) We give another justification for introducing $i$. Our main goal in the next section will be to derive the functional dependence of probability $p_1(\theta)$ (recall $p_2(\theta) = 1 - p_1(\theta)$) on the value of the experimental parameter $\theta$. We will first derive the functional dependence $i(\theta)$ and from that of $p_1(\theta)$. Note that for this purpose one could not use $I(\theta)$ instead of $\vec{i}(\theta)$ because with any value $I(\theta)$ one can associate two physically non-equivalent situations (a) and (b) which correspond to different values of the probabilities.

\(^8\) Translated: "It (the $\psi$-function) is now the instrument for predicting the probability of measurement results. In it is embodied the respectively attained sum of theoretically grounded future expectations, somehow like laid down in a catalogue."

\(^9\) As Peres put it: "After all, quantum phenomena do not occur in a Hilbert space. They occur in a laboratory."
spin, $p^i_x$ is the probability to find the particle’s spin up along $x$ etc. It is assumed that the catalog $i$ is a complete description of the system in the sense that its knowledge is sufficient to determine the probabilities for the outcomes of all possible future measurements.

Denote by $\theta$ an arbitrary direction within the $y-z$ plane and oriented at an angle $\theta$ with respect to the $z$-axis. Now, for all $\theta$ the propositions: $P_1(\theta)$: "The spin is up along the direction $x$", $P_2(\theta)$: "The spin is up along the direction $\theta$", and $P_3(\theta)$: "The spin is up along the direction $\theta + 90^\circ$" are mutually complementary. The different lists of the three mutually complementary propositions are labeled by a single experimental parameter $\theta$ as given in Fig. 4. They correspond to different representations $i(\theta) = (i_1(\theta), i_2(\theta), i_3(\theta))$ of the catalog of our knowledge of the system as shown in Fig. 4.

VI. TOTAL INFORMATION CONTENT OF A QUANTUM SYSTEM

The finiteness of the information content of a quantum system comprises not just extreme cases of maximal knowledge of one proposition at the expense of complete ignorance of complementary ones but it also applies to intermediate cases. For example, it has been pointed out that in the interference experiments one can obtain some partial knowledge about the particle’s path and still observe an interference pattern of reduced contrast as compared to the ideal interference situation (Wootters and Zurek, 1979; Englert 1999). In other words the information content of the system can manifest itself as path information or as modulation of the interference pattern or partially in both to the extent defined by the finiteness of information (Brukner and Zeilinger, 2002). How to define then the total information content of a quantum system?

Bohr (1958) remarked that "... phenomena under different experimental conditions, must be termed complementary in the sense that each is well defined and that together they exhaust all definable knowledge about the object concerned". This suggests that the total information content of a quantum system is somehow contained in the full set of mutually complementary experiments. We define the total information (of 1 bit) of the elementary (or binary, or two-state) system as a sum of the individual measures of information over a complete set of three mutually complementary experiments

$$I_{\text{total}} = I_1 + I_2 + I_3 = 1. \quad (9)$$

How to define the total information content of more complex systems? In a $n$-dimensional Hilbert space, one needs $n^2 - 1$ real parameters to specify a general density matrix $\rho$, which must be hermitean and have $Tr(\rho) = 1$. Since measurements within a particular basis set can yield only $n-1$ independent probabilities (the sum of all probabilities for all possible outcomes in an individual experiment is one), one needs $n+1$ distinct basis sets to provide the required total number of $n^2 - 1$ independent probabilities. Ivanović (1981) showed that the required number $n + 1$ of unbiased basis sets indeed exists if $n$ is a prime number, and Wootters and Fields (1989) showed that it exists if $n$ is any power of a prime number. This suggests that the complete information represented by the density matrix is fully contained in a complete set of mutually complementary observables.

Except for an elementary system (see Sec. III) we cannot give the justification for the number of mutually complementary observations in the general case from our basic considerations. We take this number in the further discussion as given in the quantum theory.

Generalizing Eq. (3) we suggest to define the total information content of a $n$-dimensional quantum system as the sum of individual measures of information $I_i(\rho^j)$ over a complete set of $n+1$ mutually complementary measurements

$$I_{\text{total}} = \sum_{i=1}^{n+1} I_i(\rho^j) = N \sum_{i=1}^{n+1} \sum_{j=1}^n \left(p^j_i - \frac{1}{n}\right)^2. \quad (10)$$

Here $p^j_i = (p^j_1, ..., p^j_n)$ are the probabilities for the outcomes in the $i$-th measurement. In the case of a system composed of $N$ elementary systems and with appropriate normalization it results in just $N$ bits of information (for the system in a pure state).

The question whether or not one can find a complete set of mutually complementary observations in the general case of a Hilbert space of arbitrary dimensions is still open. If it should turn out to be the case, then the definition (10) can be applied to arbitrarily dimensional quantum systems. If, in contrast, such sets only exist if the dimension is the power of a prime number, then we suggest to take this seriously, as implying that the prime number alternatives are

---

10 The composite system consisting of $N$ elementary systems with dimension $n = 2^N$ of the Hilbert space is a special case.
Etienne Louis Malus (1775-1812), a French physicist, was almost entirely concerned with the study of light. He conducted experiments to verify Huygens’ theory of light and rewrote the theory in analytical form. His discovery of the polarization of light by reflection was published in 1809 and his theory of double refraction of light in crystals in 1810.

Quantum theory predicts \( p(\theta) = \cos^2(\theta/2) \) for the probability to find the spin up along the direction at an angle \( \theta \) with respect to the direction along which the system gives spin up with certainty. From what deeper foundation emerges this law in quantum mechanics, originally formulated by Malus\(^{11}\) for light? The most important contributions so far in that direction are those of Wootters (1981), Summhammer (1988, 1994) and Fivel (1994). In this section we argue that the most natural functional relation \( p(\theta) \) consistent with the principle of quantization of information is indeed the sinusoidal dependence of Malus.

We wish to specify a mapping of \( \theta \) onto \( \vec{\theta}(\theta) \). It is of importance to note that we can invent this mapping freely. The reason for this is that \( \theta \) will have functional relations to other physical parameters of the experiment. Then, the laws relating those parameters with the information vector \( \vec{\theta}(\theta) \) can be seen as laws about relations between those parameters and \( \theta \) plus a mapping of \( \theta \) onto \( \vec{\theta}(\theta) \). What basic assumptions should we follow to obtain the mapping from \( \theta \) to \( \vec{\theta}(\theta) \) most appropriate for quantum mechanics?

There are two basic assumptions. The first one is the assumption of the invariance of the total information content under the change of representation of the catalog of our knowledge of the system. Or, in other words, it is the assumption that total information content must be independent of the particular choice of mutually complementary propositions considered (see Fig. \( \text{III} \)). In the same spirit as choosing a coordinate system, one may choose any set of mutually complementary propositions to represent our knowledge of the system and the total information about the

---

\(^{11}\) Etienne Louis Malus (1775-1812), a French physicist, was almost entirely concerned with the study of light. He conducted experiments to verify Huygens’ theory of light and rewrote the theory in analytical form. His discovery of the polarization of light by reflection was published in 1809 and his theory of double refraction of light in crystals in 1810.
system must be invariant under that choice, i.e. for all $\theta$
\begin{equation}
I_{\text{total}} = I_1(\theta) + I_2(\theta) + I_3(\theta) = i^2_1(\theta) + i^2_2(\theta) + i^2_3(\theta) = 1.
\end{equation}
(11)

In fact, this property of invariance is the reason why we may use the phrase "the total information content of the system" without explicitly specifying a particular reference set of mutually complementary propositions.

We suggest that only mappings where neighboring values of $\theta$ correspond to neighboring values of $i(\theta)$ are natural. Thus, if we gradually change the orientation of the magnets in a set of Stern-Gerlach apparatus defining a complete set of mutually complementary observables a continuous change of the information vector will result. The property of invariance defined by Eq. (11) implies that with a gradual change of the experimental parameter from $\theta_0$ to $\theta_1$ the information vector rotates in the space of information

\[ \vec{t}(\theta_1) = \hat{R}(\theta_1 - \theta_0, \theta_0) \vec{t}(\theta_0), \]
(12)

such that the length of the information vector is conserved (Fig. 8). The rotation matrix depends on two independent variables $\theta_0$ and $\theta_1$; here specific arguments $\theta_1 - \theta_0$ and $\theta_0$ are chosen in the functional dependence for convenience. Equation (12) expresses our expectation that the transformation law is linear, that is, independent of the actual information vector transformed. $\hat{R}(\theta_1 - \theta_0, \theta_0)$ is an orthonormal matrix

\[ \hat{R}^{-1}(\theta_1 - \theta_0, \theta_0) = \hat{R}^T(\theta_1 - \theta_0, \theta_0). \]

Notice that transformation matrices do not build up a group in general because of the explicit dependence on both the initial and final parametric value.

The second basic assumption in the derivation of the Malus law in quantum physics is that no physical process a priori distinguishes one specific value of the physical parameter from others, that is, that the parametric $\theta$-axis is homogeneous. In our example with the orientation of Stern-Gerlach magnets as an experimental parameter, the homogeneity of the parametric axis becomes equivalent to the isotropy of the ordinary space. The homogeneity of the parametric axis precisely requires that if we transform physical situations of three complementary experiments together with the state of the system along the parametric axis for any real number $b$, we cannot observe any effect. Using a more formal language this means the following. Suppose two lists each with three mutually complementary experimental arrangements are associated with a specific parametric value $\theta_0$ and to some other value $\theta_0 + b$ ($-\infty < b < +\infty$) respectively. Furthermore, suppose the information vectors $\vec{t}(\theta_0)$ and $\vec{t}(\theta_0 + b)$ associated with the two lists are equal (i.e. all components of the two vectors are equal). The homogeneity of the parametric $\theta$-axis then requires that if we change the physical parameter in each experiment by an equal interval of $\theta - \theta_0$ in the two lists of complementary experiments, the resulting information vectors will be equivalent as shown in Fig. 8. Mathematically, if $\vec{t}(\theta_0) = \vec{t}(\theta_0 + b)$ for all $\theta_0$ implies $\hat{R}(\theta - \theta_0, \theta_0)\vec{t}(\theta_0) = \hat{R}(\theta - \theta_0, \theta_0 + b)\vec{t}(\theta_0 + b)$, then

\[ \hat{R}(\theta - \theta_0, \theta_0) = \hat{R}(\theta - \theta_0, \theta_0 + b). \]
(13)

The transformation matrix then depends only on the difference between the initial and final value of the experimental parameter, and not on the location of these values on the parametric $\theta$-axis.

The orthogonality condition leads to the following general form of the transformation matrix

\[ \hat{R}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f(\theta) & -g(\theta) \\ 0 & g(\theta) & f(\theta) \end{pmatrix}, \]
(14)

12 Precisely speaking, the invariance property only implies that the transformation law $\vec{t}(\theta_1) = \hat{R}(\theta_1, \theta_0)\vec{t}(\theta_0)$ is described by a general mapping $\hat{R}$ which preserves the length of the information vector. Now, consider the situation where with probability $w_A$ a system is prepared in state $\vec{t}_A(\theta_0)$ and with probability $w_B$ in state $\vec{t}_B(\theta_0)$. Then the information vector is given by $\vec{t}(\theta_0) = w_A\vec{t}_A(\theta_0) + w_B\vec{t}_B(\theta_0)$. This means that the state $\vec{t}(\theta_0)$, where $\vec{t}(\theta_0)$ can, just formally, be written as $w_A\vec{t}_A(\theta_0) + w_B\vec{t}_B(\theta_0)$, is equivalent to the state of the system which is with probability $w_A$ prepared in state $\vec{t}_A$ and with probability $w_B$ in state $\vec{t}_B$. Let us now suppose that the experimental parameter in each of the three mutually complementary experiments is changed from the value $\theta_0$ to $\theta_1$. The individual information vectors $\vec{t}_A(\theta_0)$ and $\vec{t}_B(\theta_0)$ evolve independently, resulting in $w_A\vec{t}_A(\theta_1, \theta_0, \vec{t}_A(\theta_0)) + w_B\vec{t}_B(\theta_1, \theta_0, \vec{t}_B(\theta_0))$ for the total information vector at $\theta_1$. This shows that the function $\hat{R}$ is linear: $\vec{t}(\theta_1, \theta_0, w_A\vec{t}_A(\theta_0) + w_B\vec{t}_B(\theta_0)) = w_A\hat{R}(\theta_1, \theta_0, \vec{t}_A(\theta_0)) + w_B\hat{R}(\theta_1, \theta_0, \vec{t}_B(\theta_0))$ for convex sums over $\vec{t}_A$ and $\vec{t}_B$. For an extension of the proof to arbitrary sums follow the idea from Appendix 1 of (Hardy, 2001a), which is there applied in a different context.

13 We give another line of reasoning, that is to require the same functional dependence of the transformation law for each initial value $\theta_0$ of the parameter. This can only be done with Eq. (4).
where we take $\theta_0 = 0$ for simplicity and $f(\theta)$ and $g(\theta)$ are not yet specified but assumed to be analytical functions satisfying

$$f^2(\theta) + g^2(\theta) = 1, f(0) = 1 \text{ and } g(0) = 0. \tag{15}$$

We further require that a change of the experimental parameter in a set of mutually complementary arrangements from $\theta_0$ to $\theta_1$ and subsequently from $\theta_1$ to $\theta_2$ must have the same physical effect as a direct change of the parameter from $\theta_0$ to $\theta_2$. The resulting transformation will then be independent, whether we apply two consecutive transformations $\hat{R}(\theta_1 - \theta_0)$ and $\hat{R}(\theta_2 - \theta_1)$ or a single transformation $\hat{R}(\theta_2 - \theta_0)$

$$\hat{R}(\theta_2 - \theta_0) = \hat{R}(\theta_2 - \theta_1)\hat{R}(\theta_1 - \theta_0). \tag{16}$$

This together with the property that for $\theta = \theta_0$ the transformation matrix equals the unity matrix (since there is no change of the physical situations of the complementary experiments one has $\hat{R}(0) = \hat{1}$) implies that transformation matrices build up the group of rotations $SO(3)$, a connected subgroup of the group of orthogonal matrices $O(3)$ which contains the identity transformation.

For the special case of infinitesimally small variation of the experimental conditions, Eq. (16) reads

$$\hat{R}(\theta + d\theta) = \hat{R}(\theta)\hat{R}(d\theta). \tag{17}$$

Inserting the form (14) of the transformation matrix into the latter expression, one obtains

$$f(\theta + d\theta) = f(\theta)f(d\theta) - g(\theta)g(d\theta). \tag{18}$$

Using conditions (15), we transform Eq. (18) into the differential equation

$$\frac{df(\theta)}{d\theta} = -n\sqrt{1 - f^2(\theta)}, \tag{19}$$

where

$$n = -g'(0) \tag{20}$$

is a constant. The solution of the differential equation reads

$$f(\theta) = \cos n\theta, \tag{21}$$

where we integrate between 0 and $\theta$ using the condition $f(0) = 1$ from Eq. (15). This finally leads to

$$\hat{R}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos n\theta & -\sin n\theta \\ 0 & \sin n\theta & \cos n\theta \end{pmatrix}. \tag{22}$$

This result directly gives the familiar expression

$$p = \cos^2 \frac{n\theta}{2} \tag{23}$$
for probability in quantum theory.

Mathematically one could consider our result as a direct and immediate consequence of the theory of group representations; the cosine dependence follows from a particular representation of the rotation group. However from the physical perspective it is implied by fundamental assumptions: (1) the total information of the system is invariant under the change of the representation of the catalog of our knowledge about the system and (2) the parametric space is homogeneous. If (1) and (2) are satisfied, then the probability must vary as \( \cos^2 n\theta \), where \( n \) is a parameter not determined by the derivation. Quantum-mechanical probabilities are just of this form, with \( \theta \) for a relative polarization angle and with \( n = \frac{1}{2} \) for electrons and neutrinos, or with \( n = 1 \) for photons, or with \( n = 2 \) for gravitons. The same functional dependence \( \cos^2 \phi \) undergoes also the probability to find a particle in a specific output beam in the Mach-Zehnder type of interferometer with the phase shift \( \phi \) between two paths inside the interferometer.

In the discussion so far we considered a change of a single experimental parameter and the rotation of the information vector within one plane only. This can be generalized. Let us define the orientations of the three mutually orthogonal directions \( \vec{n}_1(\alpha, \beta, \gamma) \), \( \vec{n}_2(\alpha, \beta, \gamma) \) and \( \vec{n}_3(\alpha, \beta, \gamma) \) in ordinary space by the Euler angles \( 0 \leq \alpha < 2\pi \), \( 0 \leq \beta \leq \pi \), and \( 0 \leq \gamma < 2\pi \). Then the mutually complementary propositions which are associated to measurements along the three directions can be represented in terms of the Euler angles as \( P_1(\alpha, \beta, \gamma) \): "The spin along the direction \( \vec{n}_1(\alpha, \beta, \gamma) \) is up," \( P_2(\alpha, \beta, \gamma) \): "The spin along the direction \( \vec{n}_2(\alpha, \beta, \gamma) \) is up" and \( P_3(\alpha, \beta, \gamma) \): "The spin along the direction \( \vec{n}_3(\alpha, \beta, \gamma) \) is up".

Given a specific set of three orthogonal directions, all other sets of orthogonal directions can be obtained by rotating the reference set. Any general rotation for Euler’s angles \( \alpha, \beta, \gamma \) can be performed as a sequence of three rotations, the first around the \( z \)-axes by \( 0 \leq \gamma < 2\pi \), the second around the new \( y \)-axes by \( 0 \leq \beta < \pi \) and finally the third around the new \( z \)-axes by \( 0 \leq \alpha < 2\pi \).

A list of mutually complementary propositions associated to the spin measurements along directions obtained by the first rotation is \( P_1(0, 0, \gamma) \), \( P_2(0, 0, \gamma) \), and \( P_3(0, 0, \gamma) \). Following the argumentation given above one obtains

\[
\hat{R}(\gamma) = \begin{pmatrix}
\cos \gamma & -\sin \gamma & 0 \\
\sin \gamma & \cos \gamma & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

for the corresponding transformation matrix in the space of information\(^{14} \). If we fix the angle of the first rotation at \( \gamma_0 \) and consider only propositions \( P_1(0, \beta, \gamma_0) \), \( P_2(0, \beta, \gamma_0) \) and \( P_3(0, \beta, \gamma_0) \) about spins along directions obtained by

\(^{14}\) One should always keep in mind the difference between directions along which mutually complementary measurements are performed in ordinary space (such as the vertical direction and the direction at +45° along which a photon’s polarization is measured), or three spatially orthogonal directions along which complementary spin components of a spin-1/2 particle are measured) and directions associated with mutually complementary propositions (components of an information vector) in the space of information. The latter always constitute an orthogonal coordinate system. These again have to be distinguished from the orthogonal directions in Hilbert space which do not correspond to complementary measurements.
To this end we will follow Schrödinger’s (1935) view about entanglement: “Whenever one has a complete expectation-catalog - a psi-function - for two completely separated bodies, or, in better terms, for each of them singly, then one obviously has it also for the two bodies together, i.e., if one imagines that neither of them singly but rather the two of them together make up the object of interest, of our questions about the future. But the converse is not true. Maximal knowledge of a total system does not necessarily include total knowledge of all its parts, not even when these are fully separated from each other and at the moment are not influencing each other at all.”

the second rotation around the new $y$-axis for an angle $0 \leq \beta \leq \pi$, the corresponding transformation matrix reads

$$
\hat{R}(\beta) = \begin{pmatrix}
\cos \beta & 0 & \sin \beta \\
0 & 1 & 0 \\
-\sin \beta & 0 & \cos \beta
\end{pmatrix}.
$$

(25)

In the last step we fix both the angle $\gamma_0$ of the first rotation and the angle $\beta_0$ of the second rotation, and consider only sets of mutually complementary propositions $P_1(\alpha, \beta_0, \gamma_0)$, $P_2(\alpha, \beta_0, \gamma_0)$ and $P_3(\alpha, \beta_0, \gamma_0)$ about spins along directions obtained by the third rotation around the new $z$-axis for $0 \leq \alpha < 2\pi$. The corresponding transformation matrix is again of the form (24) with the angle $\alpha$.

Finally, the transformation matrix for a general rotation in the space of information is given as

$$
\hat{R}(\alpha, \beta, \gamma) = \hat{R}(\alpha)\hat{R}(\beta)\hat{R}(\gamma).
$$

(26)

While these relations have an obvious meaningful for spin they hold equally for any elementary system. Specifically they also hold for a two-path interferometer.

VIII. ENTANGLEMENT - MORE INFORMATION IN JOINT PROPERTIES THAN IN INDIVIDUALS

Entanglement is the feature which distinguishes quantum physics most succinctly from classical physics as quantitatively expressed by the violation of Bell’s inequalities (Bell 1964, Clauser et al., 1969). In 1964 John Bell obtained certain bounds (the Bell inequalities) on combinations of statistical correlations for measurements on two-particle systems if these correlations were to be understood within a realistic picture based on local properties of each individual particle. In such a picture the measurement results are determined by properties the particles carry prior to and independent of observation. In a local picture the results obtained at one location are independent of any measurements or actions performed at space-like separation. Quantum mechanics predicts violation of these constraints for certain statistical predictions for the composite (entangled) systems. By today, the predictions of quantum physics have been confirmed in many experiments (Freedman and Clauser 1972; Aspect et al., 1981; Weils et al., 1998; Pan et al., 2000)

In this section we will investigate how much information can be contained in the correlations between quantum systems in order to give an information-theoretic criterion of quantum entanglement. We suggest that a natural understanding of quantum entanglement results when one accepts that the information in a composite system can reside more in the correlations than in properties of individuals. The quantitative formulation of these ideas leads to a rather natural criterion of quantum entanglement.

The total information of a composite system can be distributed in various ways within the composite system. We will consider only that part of the total information of the system which is exclusively contained in correlations, or joint properties of its constituents. This is also the reason why now we will not consider complete sets of mutually complementary propositions for the composite system but just that subset of them which concerns joint properties of its constituents. As it is our final goal to compare that criterion with the one given by Bell-type inequalities where one considers correlations between spin measurements confined on each side within one plane we restrict our analysis to an $x$-$y$ plane locally defined for each subsystem.

As an explicit example of a composite systems a system consisting of two spin-1/2 particles will be considered. The propositions about their joint properties will be binary propositions, i.e. will be associated to experiments with two possible outcomes. The two outcomes will correspond to the proposition of the type: ”The spin of particle 1 along $x$ and the spin of particle 2 along $y$ are the same”, and to its negation ”The spin of particle 1 along $x$ and the spin of particle 2 along $y$ are different”. Therefore the measure of information Eq. (6) for binary experiments can be applied. If we denote the probabilities for the two outcomes by $p_{xy}^+$ and $p_{xy}^-$ respectively, then the information contained in proposition ”The spin of particle 1 along $x$ and the spin of particle 2 along $y$ are the same (different)” is given by

$$
I_{xy} = (p_{xy}^+ - p_{xy}^-)^2.
$$

(27)

---

15 To this end we will follow Schrödinger’s (1935) view about entanglement: ”Whenever one has a complete expectation-catalog - a maximum total knowledge - a psi-function - for two completely separated bodies, or, in better terms, for each of them singly, then one obviously has it also for the two bodies together, i.e., if one imagines that neither of them singly but rather the two of them together make up the object of interest, of our questions about the future. But the converse is not true. Maximal knowledge of a total system does not necessarily include total knowledge of all its parts, not even when these are fully separated from each other and at the moment are not influencing each other at all.”
We first consider a product state e.g. $|\psi\rangle = |+x\rangle_1 |-x\rangle_2$. This is the case of a composite system composed of two elementary systems carrying therefore $N = 2$ bits of information, i.e. representing the truth value of two propositions. Here the state $|\psi\rangle$ represents the two-bit combination true–false of the truth values of the propositions about the spin of each particle along the $x$-axis: (1) "The spin of particle 1 is up along $x$" and (2) "The spin of particle 2 is up along $x$". Instead of the second proposition describing the spin of particle 2, we could alternatively choose a proposition which describes the result of a joint observation: (3) "The two spins are the same along $x$." Then the state $|\psi\rangle$ represents the two-bit combination true–false of the truth values of the propositions (1) and (3).

Evidently, for pure product states at most one proposition with definite truth-value can be made about joint properties because one proposition has to be used up to define a property of one of the two subsystems. In other words 1 bit of information defines the correlations. In our example where $|\psi\rangle = |+x\rangle_1 |-x\rangle_2$ the correlations are fully represented by the correlations between spin $x$-measurements on the two sides, therefore

$$I_{xx} = 1.$$  \hspace{1cm} (28)

We denote the states with property (28) as classically composed states.

Obviously, the choice of directions $x$ and $y$ within each of the planes of measurements on the two sides is arbitrary. It is physically not acceptable that the total information contained in correlations between spin measurements confined on each side within $x$-$y$ planes depends on this choice. We therefore require that the total information contained in the correlations must be invariant upon the choice of general $x$ and $y$ measurement directions within the $x$-$y$ planes on each side. Only with this requirement the statement "the total information contained in the correlations between measurements within the $x$-$y$ planes" can have a meaning independent of the specific set of mutually complementary measurements considered. This invariance property can only be guaranteed with our measure of information (27).

We define the total information contained in the correlations as the sum over the individual measures of information about a complete set of mutually complementary observations within the planes $x$-$y$ on the two sides. The total information contained in the correlations is thus defined as the sum

$$I_{\text{corr}} = I_{xx} + I_{xy} + I_{yx} + I_{yy} \hspace{1cm} (29)$$

of the partial measures of information contained in the set of complementary observations within the $x$-$y$-planes. These observations are mutually complementary for product states and the set is complete as there exists no further complementary observation within the chosen $x$-$y$-planes. By this we mean that for any product state a complete knowledge contained in any proposition from the set: "The two spins are equal along $x$", "The spin of particle 1 along $x$ and the spin of particle 2 along $y$ are the same", "The spin of particle 1 along $y$ and the spin of particle 2 along $x$ are the same" and "The two spins are equal along $y$" excludes any knowledge about other three propositions.

In general there can also be some amount of information contained in the correlations for measurements involving $z$ direction, for example, for measurement directions within the $x$-$z$ planes on the two sides. Obviously if the general $x$ and $y$ directions are chosen to include directions outside of the old $x$-$y$ planes measure of information $I_{\text{corr}}$ cannot be assumed to remain an invariant. The maximal value of $I_{\text{corr}}$ can then be obtained by an optimization over all possible two-dimensional planes of measurements on both sides.

Consider now a maximally entangled Bell state, e.g.

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|+x\rangle_1 |-x\rangle_2 - | -x\rangle_1 |+x\rangle_2)$$

$$= \frac{1}{\sqrt{2}}(|+y\rangle_1 |-y\rangle_2 - | -y\rangle_1 |+y\rangle_2). \hspace{1cm} (30)$$

The two propositions here both are statements about results of joint observations (Zeilinger 1997), namely (1′) "The two spins are equal along $x$" and (2′) "The two spins are equal along $y$". Now the state represents the two-bit combination false–false of these propositions. Note that here the 2 bits of information are all carried by the 2 elementary systems in a joint way, with no individual elementary system carrying any information on its own. In other words, as the two available bits of information are already exhausted in defining joint properties, no further possibility exists to also encode information in individuals. Therefore

$$I_{\text{Bell}} = 2. \hspace{1cm} (31)$$

Note that in our example of Bell-state $|\psi^-\rangle$, $I_{xx} = I_{yy} = 1$ and $I_{xy} = I_{yx} = 0$. Also, note that the truth value for another proposition, namely, "The two spins are equal along $z$" must follow immediately from the truth values of the propositions (1′) and (2′), as only 2 bits of information are available. Interestingly this is also a direct consequence of the formalism of quantum mechanics as the joint eigenstate of $\sigma_x^1 \sigma_x^2$ and of $\sigma_y^1 \sigma_y^2$ is also an eigenstate of $\sigma_z^1 \sigma_z^2 = - (\sigma_z^1 \sigma_z^2)(\sigma_y^1 \sigma_y^2)$. 


In contrast to product states we suggest entanglement of two elementary systems to be defined in general such that *more than one bit* (of the two available ones) is used to define joint properties, i.e.

\[ I_{\text{entgl}} > 1 \] (32)

for at least one choice of the planes of measurements for the two elementary systems (or, equivalently, for that choice of the planes of measurements for which \( I_{\text{corr}} \) reaches its maximal value). Most importantly this simple information-theoretic criterion of entanglement can be shown to be equivalent to a *necessary and sufficient* condition (Horodeccy family, 1995) for a violation of a Bell-type inequality for two-elementary systems. A generalization of our information-theoretic criterion for entanglement to \( N \) elementary systems and its relation to the criteria for violation of Bell’s inequalities can be found in (Brukner et al., 2001).

**IX. TIME EVOLUTION OF THE CATALOG OF KNOWLEDGE**

Any assignment of properties to an object is always a consequence of some observation. Using information obtained in previous observations we wish to make predictions about the future. Again our predictions might be formulated as, in general probabilistic, predictions about future properties of a system. Clearly, these predictions can be verified or falsified by performing measurements and checking whether the experimental results agree with our predictions. It is then important to connect past observations with future observations. Or, more precisely, to make, based on past observations, specific statements about possible results of future observations.

In quantum mechanics this connection between past observation and future observation exactly is achieved by the quantum-mechanical Liouville equation (for pure states it reduces to the Schrödinger equation)

\[
i\hbar \frac{d\hat{\rho}(t)}{dt} = [\hat{H}(t), \hat{\rho}(t)].\] (33)

The initial state \( \hat{\rho}(t_0) \) represents all our information as obtained by earlier observation. Using the quantum-mechanical Liouville equation we can derive a time evolved final state \( \hat{\rho}(t) \) at some future time \( t \) which gives us predictions for any possible observation of the system at that time. In this section the dynamics of an elementary system is formulated as a time evolution of the catalog of our knowledge of the system. This is specified by the evolution of the information vector in the space of information. The Liouville equation will then be derived from the differential equation describing the motion of the information vector in the information space.

We will consider now the time evolution of an elementary system with no information exchange with an environment. Suppose that the state of the system at some initial time \( t_0 \) is represented by the catalog \( \vec{i}(t_0) = (i_1(t_0), i_2(t_0), i_3(t_0)) \) of our knowledge. Now let the system evolve in time. Because there is no information exchange with an environment during the evolution, the total information of the system at some later time \( t \) must still be the same as at the initial time. This may be seen as an ultimate constant of the evolution of the system motion independent of the strength, time dependence or any other characteristic of the "external field" of the system. Therefore

\[ I_{\text{total}}(t) = \sum_{n=1}^{3} i_n^2(t) = \sum_{n=1}^{3} i_n^2(t_0) = I_{\text{total}}(t_0). \] (34)

Mathematically, the conservation of the total information is equivalent to the conservation of the length of the information vector during its motion in the information space. This means that time evolution of an isolated quantum system is just a rotation of the information vector in the space of information (see footnote 12)

\[
\vec{i}(t) = \hat{R}(t, t_0)\vec{i}(t_0),
\] (35)

where again \( \hat{R}(t, t_0) \) is a rotation matrix

\[
\hat{R}^{-1}(t, t_0) = \hat{R}^T(t, t_0)
\]

and \( \hat{R}^T(t, t_0) \) is its transposed matrix.

---

16 If there is information exchange between the system and the environment we cannot formulate system’s evolution law independently of the environment, but we have to consider it as a subsystem of a larger system that contains both the system and the environment where again the total information is conserved.
The derivative of Eq. (35) with respect to time is
\[
\frac{d\vec{u}(t)}{dt} = \frac{d\hat{R}(t, t_0)}{dt} \vec{u}(t_0) = \hat{K}(t, t_0) \vec{u}(t),
\]  
(36)

where \( \hat{K}(t, t_0) = \frac{d\hat{R}(t, t_0)}{dt} \hat{R}^T(t, t_0) \). We will now show that the operator \( \hat{K}(t, t_0) \) is antisymmetric. We find
\[
\hat{K}^T(t) = \hat{R}(t) \frac{d\hat{R}^T(t)}{dt} = \hat{R}(t) \lim_{\Delta t \to 0} \frac{\hat{R}^T(t + \Delta t) - \hat{R}^T(t)}{\Delta t} = \hat{R}(t) \lim_{\Delta t \to 0} \frac{\hat{R}^T(t) - \hat{R}(t + \Delta t)}{\Delta t} \hat{R}^T(t + \Delta t)
\]
\[
= \lim_{\Delta t \to 0} \frac{\hat{R}(t) - \hat{R}(t + \Delta t)}{\Delta t} \hat{R}^T(t) = -\hat{K}(t),
\]

where the initial time \( t_0 \) is identified with the time 0.

It is a well-known result of vector analysis that with every antisymmetric operator \( \hat{K} \) one may uniquely associate the "vector of rotation" \( \vec{u} \) by the relation \( \hat{K}\vec{y} = \vec{u} \times \vec{y} \) for all \( \vec{y} \),
\[
\hat{K}\vec{y} = \vec{u} \times \vec{y} \quad \text{for all \( \vec{y} \)}
\]  
(37)

where "\( \times \)" denotes vector product. Using this result we now rewrite Eq. (36) as
\[
\frac{d\vec{u}(t)}{dt} = \vec{u}(t, t_0) \times \vec{u}(t).
\]  
(38)

Mathematically, this equation describes the rotation of the information vector around the axis \( \vec{u}(t, t_0) \) which itself changes in the course of time. Physically, this is the formulation of the dynamical law for the evolution of the catalog of our knowledge.

One might recognize Eq. (38) as a description of the state evolution in terms of the Bloch vector. Based on the known features of the quantum formalism we will now argue for the validity of Eq. (38). Suppose that the quantum state of the system is described by the density matrix \( \hat{\rho} \). We decompose the density matrix into the unity operator and the generators of SU(2) algebra (Pauli matrices)
\[
\hat{\rho}(t) = \frac{1}{2} \mathbb{1} + \frac{1}{2} \sum_{j=1}^{3} i_j(t) \hat{\sigma}_j,
\]  
(39)

where \( \hat{\sigma}_j \) is spin operator for the direction \( j = x, y, z \). Note that the quantity \( i_j \) for the spin along the direction \( j \) is equal to the expectation value of spin along this direction, i.e. \( i_j(t) = \text{Tr}(\hat{\rho}(t)\hat{\sigma}_j) \).

If we take a derivative of Eq. (39) in time we obtain
\[
i\hbar \frac{d\hat{\rho}(t)}{dt} = \frac{1}{2} \sum_{j=1}^{3} i_j(t) \frac{d\hat{\sigma}_j}{dt}.
\]  
(40)

Inserting Eq. (38) on the right-hand side we find
\[
i\hbar \frac{d\hat{\rho}(t)}{dt} = \frac{i}{2} \sum_{i,j,k=1}^{3} \epsilon_{ijk} u_i(t) i_j \hat{\sigma}_k.
\]  
(41)

\[\text{17}\] The operator \( \hat{K} \) is represented by an antisymmetric matrix
\[
\hat{K} = \begin{pmatrix}
0 & -k_{21} & -k_{31} \\
k_{21} & 0 & -k_{32} \\
k_{31} & k_{32} & 0
\end{pmatrix}.
\]

From there we read out the components of the vector of rotation \( \vec{u} \) as
\[
u_1 = k_{32}, u_2 = -k_{31}, u_3 = k_{21}.
\]
FIG. 9 One complete rotation of the information vector after a time elapse of the de-Broglie wave-period.

Since the Pauli matrices satisfy $[\hat{\sigma}_i, \hat{\sigma}_j] = 2i \sum_{k=1}^{3} \epsilon_{ijk} \hat{\sigma}_k$, we proceed with

$$i\hbar \frac{d\rho(t)}{dt} = \frac{1}{4} \sum_{i,j=1}^{3} u_i(t) i_j (\hat{\sigma}_i \hat{\sigma}_j - \hat{\sigma}_j \hat{\sigma}_i).$$

(42)

Introducing the operator $\hat{H}(t)$ such that

$$u_i(t) := \text{Tr}(\hat{H}(t)\hat{\sigma}_i),$$

(43)

we finally obtain the quantum-mechanical Liouville equation

$$i\hbar \frac{d\rho(t)}{dt} = [\hat{H}(t), \rho(t)].$$

(44)

For the special case of a conservative system, the evolution of a quantum state in time is constrained by a higher constant of motion, namely our information about the energy of the system, apart from the ultimate one of the total information content of the system. In the space of information this corresponds to the rotation of the information vector around a fixed axis that is associated to our knowledge of energy of the system$^{18}$. This is only possible if the axis $\vec{u}$ in Eq. (38) is a fixed axis in time around which the information vector rotates. This further implies the existence of a minimal interval of time the information vector needs to make one complete rotation in the space of information (Fig. 9). After this time interval the values $i$ for all propositions about the system take the same value. This time interval is known as the deBroglie wave-period.

Obviously the result given above is just a very first and the simplest step toward an information-theoretical formulation of the quantum-mechanical evolution in time. Mathematically, it is an immediate consequence of the (nearly) isomorphism between SU(2) group of unitary rotations in a two-dimensional Hilbert space and SO(3) group of rotations in the real three-dimensional Euclidian space. Obviously one needs to consider more complex systems and to find position- and momentum-representations of the catalog of our knowledge in order to give an information-theoretical formulation of the Schrödinger equation.

X. MEASUREMENT - THE UPDATE OF INFORMATION

In this section, it will be argued that identifying the quantum state of a system with the catalog of our knowledge of the system leads to the resolution of many of the seemingly paradoxical features of quantum mechanics connected to the so-called measurement problem.

In a quantum measurement, we find the system to be in one of the eigenstates of the observable defined by the measurement apparatus. A specific example is the case when we are considering a wave packet as being composed of a superposition of plane waves. Such a wave packet is more or less well-localized, but we can always perform a position

$^{18}$ Note that we consider elementary systems, that is, systems with two possible energy values. By information about the energy of the system we mean our knowledge about which of the two values will be observed in an appropriately designed experiment.
measurement on a wave packet which is better localized than the dimension of the packet itself. This, sometimes called "reduction of the wave packet" or "collapse of the wave function", can only be seen as a "measurement paradox" if one views this change of the quantum state as a real physical process. In the extreme case it is often even related to an instant collapse of some physical wave in space.

There is no basis for any such assumption. In contrast, there is never a paradox if we realize that the wave function is just an encoded mathematical representation of our knowledge of the system. When the state of a quantum system has a non-zero value at some position in space at some particular time, it does not mean that the system is physically present at that point, but only that our knowledge (or lack of knowledge) of the system allows the particle the possibility of being present at that point at that instant.

What can be more natural than to change the representation of our knowledge if we gain new knowledge from a measurement performed on the system? When a measurement is performed, our knowledge of the system changes, and therefore its representation, the quantum state, also changes. In agreement with the new knowledge, it instantaneously changes all its components, even those which describe our knowledge in the regions of space quite distant from the site of the measurement. Then no need whatsoever arises to allude to notions like superluminal or instantaneous transmission of information.

Schrödinger (1935) wrote\(^\text{19}\): "Bei jeder Messung ist man genötigt, der \(\psi\)-Funktion (=dem Voraussagenkatalog) eine eigenartige, etwas plötzliche Veränderung zuzuschreiben, die von der gefundenen Maßzahl abhängt und sich nicht vorhersehen läßt; woraus allein schon deutlich ist, daß diese zweite Art von Veränderung der \(\psi\)-Funktion mit ihrem regelmässigen Abrollen \(zweischen\) zwei Messungen nicht das mindeste zu tun hat. Die abrupte Veränderung durch die Messung ... ist der interessanteste Punkt der ganzen Theorie. Es ist genau \emph{der} Punkt, \emph{der} den Bruch mit dem naiven Realismus verlangt. Aus \emph{diesem} Grund kann man die \(\psi\)-Funktion \emph{nicht} direkt an die Stelle des Modells oder des Realdings setzen. Und zwar nicht etwa weil man einem Realding oder einem Modell nicht abrupte unvorhergesehene Änderung zuzumuten dürfte, sondern weil vom realistischen Standpunkt die Beobachtung ein Naturvorgang ist wie jeder andere und nicht per se eine Unterbrechung des regelmässigen Naturlaufs hervorrufen darf".

A closely related position was assumed also by Heisenberg, who wrote in a letter to Remninger dated February 2, 1960: "The act of recording, on the other hand, which leads to the reduction of the state, is not a physical, but rather, so to say, a mathematical process. With the sudden change of our knowledge also the mathematical presentation of our knowledge undergoes of course a sudden change.\(^\text{19}\)", as translated by Jammer (1974).

We will now bring the role of the observer in a quantum measurement to the center of our discussion. In classical physics we can assume that an observation reveals some property already existing in the outside world. For example, if we look at the moon, we just find out where it is and it is certainly safe to assume that the property of the moon to be there is independent of whether anyone looks or not. The situation is drastically different in quantum mechanics and it is just the very attitude of the Copenhagen interpretation giving a fundamental role to observation which is a major intellectual step forward over this naive classical realism. With the only exception of the system being in an eigenstate of the measured observable, a quantum measurement changes the system into one of the possible new states defined by the measurement apparatus in a fundamentally unpredictable way, and thus cannot be claimed to reveal a property existing before the measurement is performed. The reason for this is again the fact that a quantum system cannot, not even in principle, carry enough information to specify observation-independent properties corresponding to all possible measurements. In the measurement the state therefore must appear to be changed in accord with the new information, if any, \emph{acquired} about the system together with unavoidable and irrecoverable loss of complementary information. Unlike a classical measurement, a quantum measurement thus does not just add (if any) some knowledge, it changes our knowledge in agreement with a fundamental finiteness of the total information content of the system\(^\text{20}\).

We as observers have a significant role in the measurement process, because we can decide by choosing the measuring device which attribute will be realized in the actual measurement\(^\text{21}\). Since the information content of the system is

\(^{19}\) Translated: "For each measurement one is required to ascribe to the \(\psi\)-function (=the prediction catalog) a characteristic, quite sudden change, which \textit{depends on the measurement result obtained}, and so \textit{cannot be foreseen}; from which alone it is already quite clear that this second kind of change of the \(\psi\)-function has nothing whatever in common with its orderly development \textit{between} two measurements. The abrupt change by measurement ... is the most interesting point of the entire theory. It is precisely the point that demands the break with naive realism. For \textit{this} reason one \textit{cannot} put the \(\psi\)-function directly in place of the model or of the physical thing. And indeed not because one might never dare impute abrupt unforeseen changes to a physical thing or to a model, but because in the realism point of view observation is a natural process like any other and cannot per se bring about an interruption of the orderly flow of natural events."

\(^{20}\) Wheeler (1989) stated that "... yes or no that is recorded constitutes an unsplittable bit of information".

\(^{21}\) Wheeler explicates this by example of the well-known case of a quasar, of which we can see two pictures through the gravity lens action of a galaxy that lies between the quasar and ourselves. By choosing which instrument to use for observing the light coming from that quasar, we can decide here and now whether the quantum phenomenon in which the photons take part is interference of amplitudes passing on both sides of the galaxy or whether we determine the path the photon took on one or the other side of the galaxy.
limited, by choosing which measurement device to use we not only decide what particular knowledge will be gained, but simultaneously what complementary knowledge will be lost after the measurement is performed. Here, a very subtle position was assumed by Pauli (1955) who writes: "The gain of knowledge by means of an observation has as a necessary and natural consequence, the loss of some other knowledge. The observer has however the free choice, corresponding to two mutually exclusive experimental arrangements, of determining what particular knowledge is gained and what other knowledge is lost (complementary pairs of opposites). Therefore every irrevocable interference by an observation about a system alters its state, and creates a new phenomenon in Bohr’s sense."

XI. CONCLUSIONS

The laws we discover about Nature do not already exist as "Laws of Nature" in the outside world. Rather "Laws of Nature" are necessities of the mind for any possibility to make sense whatsoever out of the data of experience. This epistemological structure is a necessity behind the form of all laws an observer can discover. As von Weizsäcker has put it, and Heisenberg quoted in (1958) paper: "Nature is earlier than man, but man is earlier than natural science."

An observer is inescapably suspended in the situation of obtaining the data from observation, formatting concepts of Nature therefrom, and predicting the data of future observations. In observing she/he is able to distinguish only a finite number of results at each interval of time (compare Summhammer, 2000; 2001). Therefore the experience of the ultimate experimenter is a stream of ("yes" or "no") answers to the questions posed to Nature. Any concept of an existing reality is then a mental construction based on these answers. Of course this does not imply that reality is no more than a pure subjective human construct. From our observations we are able to build up objects with a set of properties that do not change under variations of modes of observation or description. These are "invariants" with respect to these variations. Predictions based on any such specific invariants may then be checked by anyone, and as a result we may arrive at an intersubjective agreement about the model, thus lending a sense of independent reality to the mentally constructed objects.

In quantum experiments an observer may decide to measure a different set of complementary variables, thus gaining certainty about one or more variable at the expense of losing certainty about the other(s). Thus the measure of information in an individual experiment is not an invariant but depends on the specific experimental context. However the total uncertainty, or equivalently, the total information, is invariant under such transformation from one complete set of complementary variables to another. In classical physics a property of a system is a primary concept prior to and independent of observation and information is a secondary concept which measures our ignorance about properties of the system. In contrast in quantum physics the notion of the total information of the system emerges as a primary concept, independent of the particular complete set of complementary experimental procedures the observer might choose, and a property becomes a secondary concept, a specific representation of the information of the system that is created spontaneously in the measurement itself. Bohr (1934) wrote that "... a subsequent measurement to a certain degree deprives the information given by a previous measurement of its significance for predicting the future course of phenomena. Obviously, these facts not only set a limit to the extent of the information obtainable by measurement, but they also set a limit to the meaning which we may attribute to such information."

Theorems like those of Bell (1964) and Greenberger-Horne-Zeilinger (1990) state that randomness of an individual quantum event cannot be derived from local causes (local hidden variables). Quantum physics is not able to "explain why (specific) events happen" as pointed out by Bell (1990). It is beyond the scope of quantum physics to answer the question why events happen at all (that is, why the detectors clicks at all). Yet, if events happen, then they must happen randomly. The reason is the finiteness of the information. Any detailed description of the reality that would be able to give an unambiguous answer to Bell’s question, that is, any description that would be able to arrive at an accurate and detailed prediction of the particular process resulting in a particular event, will necessarily include the definition of a number of "hidden" properties of the system which would carry information as to which specific result will be observed for all possible future measurements. Therefore no answer can be given to Bell’s question, because otherwise, quantum system would carry more information that it is in principle available.

It turns out that the lowest symmetry common for all elementary systems is the invariance of their total information content with respect to a rotation in a three-dimensional space. The three dimensionality of the information space is a consequence of the minimal number (3) of mutually exclusive experimental questions we may pose to an elementary system. This seems to justify the use of three-dimensional space as "the" space of the inferred world. Such a view was first suggested by v. Weizsäcker (1974): "It [quantum theory of the simple alternative] contains a two-dimensional complex vector space with a unitary metric, a two-dimensional Hilbert space. This theory has a group of transformations which is surprisingly near-isomorphic with a group of rotations in the real three-dimensional Euclidian space. This has been known for a very long time. I propose to take this isomorphism seriously as being the real reason why ordinary space is three-dimensional."

We end with another quote of v. Weizsäcker (1974): "But I feel these consideration make it plausible that quantum
theory is not just one out of a thousand equally possible theories, and the one which happens to please God so much that he chose to create a world in which it would be true. I rather think, if we had understood quantum theory just a little bit better than we understand it so far it would turn out to be a fairly good approximation towards the formulation of a theory which contains nothing but the rules under which we speak about future events if we can speak about them in an empirically testable way at all."

It has not escape our attention that our considerations presented here may be viewed as providing the necessary justification for this point of view.

Acknowledgements

We acknowledge discussions with Terry Rudolph, Christoph Simon, Johann Summhammer and Marek Žukowski. This work is supported by the Austrian FWF project F1506, and by the QIPC program of the EU.

XII. REFERENCES

Aspect, A., P. Grangier, and G. Roger, 1981, Phys. Rev. Lett. 47, 460-463.
Bell, J. S., 1964, Physics 1, 195-200; reprinted Bell, J. S., 1987, Speakable and Unspeakable in Quantum Mechanics (Cambridge Univ. Press).
Bell, J. S., 1990, Physics World (August 1990).
Bohr N., 1949, in Albert Einstein: Philosopher-Scientist, edited by P.A. Schillp (The Library of Living Philosophers Evanston, IL) 200. A copy can be found at the web site [http://www.emr.hibu.no/lars/eng/schlipp/Default.html].
Bohr, N., 1958, Atomic Physics and Human Knowledge (Wiley, New York).
Brukner, Č., and A. Zeilinger, 1999, Phys. Rev. Lett. 83, 3354-3357.
Brukner, Č. and A. Zeilinger, 2000, e-print quant-ph/0008091.
Brukner, Č., M. Žukowski, and A. Zeilinger, 2001, e-print quant-ph/0106119.
Brukner, Č. and A. Zeilinger, 2001, Phys. Rev. A 63, 022113 1-10.
Brukner, Č. and A. Zeilinger, 2002, Phil. Trans. R. Soc. Lond. A 360 (2002) 1061.
Caves, C. M., C. A. Fuchs, R. Schack, 2001a, e-print quant-ph/0104088.
Caves, C. M., C. A. Fuchs, R. Schack 2001b, Phys. Rev. A 022305.
Clauser, J., M. Horne, A. Shimony, and R. Holt, 1969, Phys. Rev. Lett. 23, 880-884.
Donath, N., and K. Svozil, 2002, Phys. Rev. A 66, 044302.
Freedman, S. J., and J. S. Clauser, 1972, Phys. Rev. Lett. 28, 938-941.
Englert B. G., 1996, Phys. Rev. Lett. 77, 2154.
Fivel, D.I., 1994, Phys. Rev. A 59, 2108.
Fuchs, C. A., 2001, e-print quant-ph/0105015.
Fuchs, C. A., 2001, e-print quant-ph/0101012.
Fuchs, C. A., 2002, e-print quant-ph/0205039.
Fuchs, C. A., 2001, e-print quant-ph/0106166.
Fuchs, C. A., 2002, e-print quant-ph/0105035.
Guedenko, B. V., 1976, The Theory of Probability (Mir Publishers, Moscow).
Greenberger, D. M., M. Horne, A. Shimony, and A. Zeilinger, 1990, Am. J. Phys. 58, 1131-1143.
Hall, M. J. W., 2000, e-print quant-ph/0007110.
Hardy G., J. E. Littlewood and G. Pólya, 1952 Inequalities (Cambridge University Press).
Hardy, L., 2001, e-print quant-ph/0105039.
Hardy, L., 2001b, e-print quant-ph/0111068.
Heisenberg, W., 1958, Daedalus 87, 95.
Horodecki, R., P. Horodecki, and M. Horodecki, 1995, Phys. Lett. A 200, 340-344.
Ivanović I., 1981, J. Phys. A 14, 3241.
Jammer, M., 1974, The Philosophy of Quantum Mechanics, (J. Wiley & Sons, New York).
Kochen, S. and E. P. Specker, 1967, J. Math. and Mech. 17, 59.
Landauer R., 1991 May, Physics Today, 23.
Pan, J. W., D. Bouwmeester, H. Weinfurter, and A. Zeilinger, 2000, Nature 403, 515-518.
Pauli W., 1955, in Writings on Philosophy and Physics edited by C. P. Enz and K. von Meyenn, translated by Robert Schlapp (Springer Verlag, Berlin).
Schlienz J. and G. Mahler, 1995, Phys. Rev. A 52, 4396-4404.
Shannon, C. E., 1948, Bell Syst. Tech. J. 27, 379. A copy can be found at [http://cm.bell-labs.com/cm/ms/what/shannonday/paper.html].
Schrödinger, E., 1935, Naturwissenschaften 23, 807-812; 823-828; 844-849. Translation published in Proc. Am. Phil. Soc. 124, p. 323-338 and in Quantum Theory and Measurement edited by J. A. Wheeler and W. H. Zurek (Princeton
University Press, Princeton), p. 152-167. A copy can be found at (www.emr.hibu.no/lars/eng/cat).

Summhammer, J., 1988, Found. Phys. Lett. 1, 123.

Summhammer, J., 1994, Int. J. Theor. Phys. 33, 171.

Summhammer, J., 2000, e-print quant-ph/0008009, to appear in "The Third Millenium" edited by Cristian Calude.

Summhammer, J. 2001, e-print quant-ph/0102099.

Svozil, K., 2002, Phys. Rev. A 66, 044306.

Timpson, C. G., 2001, e-print quant-ph/0112178.

don Weizsäcker, C. F., 1958, Aufbau der Physik (Carl Hanser, München).

don Weizsäcker, C. F., 1974, in Quantum Theory and the Structures of Time and Space, edited by L. Castell, M. Drieschner, C. F. von Weizsäcker (Hanser, München, 1975). Papers presented at a conference held in Feldafing, July 1974.

Weihs, G., T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, 1998, Phys. Rev. Lett. 81, 5039-5043.

Wheeler J. A., 1983, Law without Law in Quantum Theory and Measurement edited by J. A. Wheeler and W. H. Zurek (Princeton University Press, Princeton) 182.

Wheeler J. A., 1989, Proc. 3rd Int. Symp. Foundations of Quantum Mechanics, Tokyo, 354.

Wootters, W. K., 1981, Phys. Rev D 23, 357.

Wootters W. K. and B. D. Fields, 1989, Ann. Phys. 191, 363.

Wootters, W. K., and W. H. Zurek, 1979, Phys. Rev. D 19, 473.

Zeilinger, A., 1997, Phil. Trans. Roy. Soc. Lond. 1733, 2401-2404.

Zeilinger, A., 1999, Found. Phys. 29, 631-643.