ON THE POSITIVITY OF HIGH-DEGREE SCHUR CLASSES OF AN AMPLE VECTOR BUNDLE

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Abstract. Let $X$ be a smooth projective variety of dimension $n$, and let $E$ be an ample vector bundle over $X$. We show that any non-zero Schur class of $E$, lying in the cohomology group of bidegree $(n-1, n-1)$, has a representative which is strictly positive in the sense of smooth forms. This conforms the prediction of Griffiths conjecture on the positive polynomials of Chern classes/forms of an ample vector bundle on the form level, and thus strengthens the celebrated positivity results of Fulton-Lazarsfeld for certain degrees.

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1. Introduction

1.1. Positive vector bundles. Let $X$ be a smooth projective variety of dimension $n$ defined over complex numbers. Let $E$ be a holomorphic vector bundle of rank $r$ over $X$, endowed with a smooth Hermitian metric $h$. If $D_{E,h}$ is the Chern connection of $(E,h)$, and $\Theta_{E,h} = D_{E,h}^2 \in \Lambda^{1,1}(X, \text{End}(E))$ its curvature, then the corresponding Chern forms $c_k(E,h)$ are computed formally as follows:

$$\det(\text{Id} + \frac{it}{2\pi} \Theta_{E,h}) = \sum_{k=0}^{r} c_k(E,h)t^k,$$

or equivalently,

$$c_k(E,h) = \text{trace}(\wedge^k \frac{i}{2\pi} \Theta_{E,h}).$$

The Chern form $c_k(E,h)$ is a globally defined real $d$-closed $(k,k)$ form on $X$. By the Chern-Weil theory, the $k$-th Chern class of $E$, denoted by $c_k(E)$, can be represented by the smooth form $c_k(E,h)$ and lies in $H^{k,k}(X, \mathbb{Z})$.

Let $\pi: \mathbb{P}(E) \to X$ be the projective bundle of hyperplanes of $E$, and $\mathcal{O}_E(1)$ the tautological line bundle on $\mathbb{P}(E)$:

$$0 \to S \to \pi^* E \to \mathcal{O}_E(1) \to 0.$$ 

Then $E$ is called ample if the line bundle $\mathcal{O}_E(1)$ is ample on $\mathbb{P}(E)$. Corresponding to the ampleness of $E$, there is a closely related differential-geometric positivity notion, which we describe below. Denote by $(e_1, ..., e_r)$ a local normal frame of $E$ over a coordinate patch $\Omega \subset X$ with a local holomorphic coordinates system $(z_1, ..., z_n)$. Then over $\Omega$ the curvature tensor $\Theta_{E,h}$ can be written as

$$\Theta_{E,h} = \sum_{\lambda, \mu=1}^{r} \sum_{j,k=1}^{n} c_{j,k\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e_{\lambda}^* \otimes e_{\mu}, \text{ satisfying } c_{j,k\lambda\mu} = \overline{c_{k,j\mu\lambda}}.$$ 

To $\Theta_{E,h}$ correspond a natural Hermitian form $\theta_{E,h}$ on $TX \otimes E$ defined by

$$\theta_{E,h} = \sum_{j,k,\lambda,\mu} c_{j,k\lambda\mu}(dz_j \otimes e_{\lambda}^*) \otimes (d\bar{z}_k \otimes e_{\mu}).$$
such that at a point $x \in \Omega$ we have
\[
\theta_{E,h}(u, u) = \sum_{j,k,\lambda,\mu} c_{j,k,\lambda,\mu} u_{j,\lambda} \overline{u_{k,\mu}}, \quad u \in T_x X \otimes E_x.
\]

The vector bundle $E$ endowed with the metric $h$ is called Griffiths-positive, if for any $\xi \in T_x X, \xi \neq 0$ and $s \in E_x, s \neq 0$,
\[
\theta_{E,h}(\xi \otimes s, \xi \otimes s) > 0.
\]

It is a celebrated and still widely open problem of Griffiths [Gri69] that the above algebraic-geometric ampleness and differential-geometric Griffiths-positivity are equivalent. One direction is clear. Endowing $O_E(1)$ with the induced metric from $h$, it is easy to see that:

$E$ being Griffiths-positive $\Rightarrow E$ being ample.

1.2. Griffiths conjecture on numerically positive polynomials. For an ample vector bundle $E$, its first Chern class $c_1(E) = c_1(\det E)$ has a representative which is a Kähler metric, i.e., a smooth strictly positive $(1,1)$ form. It is natural to ask whether this also holds for the $k$-th Chern classes $c_k(E)$. We will confirm this for $c_{n-1}(E)$ whenever it is non-zero, i.e., when $E$ is an ample vector bundle of rank $r \geq n - 1$.

Indeed, in the seminal paper [Gri69], Griffiths asked the following question, which we formulate as follows (see [Gul06] for a nice exposition).

**Question 1.1** (Griffiths). Let $P \in \mathbb{Q}[c_1, \ldots, c_r]$ be a homogeneous polynomial of weighted degree $k$, the variable $c_i$ being assigned weight $i$.

1. Assume that $(E, h)$ is an arbitrary Griffiths-positive bundle, characterize the polynomial $P$ such that the cohomology class $P(c_1(E), \ldots, c_r(E))$ is numerically positive, that is, for any irreducible subvariety $V$ of dimension $k$ in $X$,
\[
\int_V P(c_1(E), \ldots, c_r(E)) > 0.
\]

2. Assume that $(E, h)$ is an arbitrary Griffiths-positive bundle, characterize the polynomial $P$ such that the $(k,k)$ form $P(c_1(E,h), \ldots, c_r(E,h))$ is positive.

**Remark 1.2.** The first part of Griffiths question is on the cohomology level, while the second part is on the form level which is a priori stronger than the first part.

It is conjectured explicitly in [Gri69, Conjecture 0.7] that the desired polynomials in Question 1.1 (1) are given by Griffiths-positive polynomials. For the definition of Griffiths-positive polynomials, see [Gri69, Definition 5.9]. The numerical positivity of Chern classes for ample vector bundles is also conjectured in [Har66].

Let $\Lambda(k, r)$ be the set of all partitions of $k$ by non-negative integers $\leq r$. Associated to each partition $\lambda \in \Lambda(k, r)$, there is a Schur polynomial $s_\lambda \in \mathbb{Q}[c_1, \ldots, c_r]$ of weighted degree $k$ defined as the determinant:
\[
s_\lambda(c_1, \ldots, c_r) = \det[c_{\lambda_j-j+l}]_{j,l \leq k}.
\]

The space of homogeneous polynomials $P \in \mathbb{Q}[c_1, \ldots, c_r]$ of weighted degree $k$ is spanned by Schur polynomials, one can write such $P$ uniquely as a linear combination of the $s_\lambda$:
\[
P = \sum_{\lambda \in \Lambda(k, r)} a_\lambda(P)s_\lambda.
\]

On the cohomology level, not only for Griffiths-positive vector bundles but also for ample vector bundles, Question 1.1 (1) was answered completely in the celebrated work of Fulton-Lazarsfeld [FL83]. This extends the previous works of Kleiman [Kle69] for surfaces, Bloch-Gieseker [BG71] for Chern classes, Gieseker [Gie71] for monomials of Chern classes and Usui-Tango [UT77] for ample and globally generated bundles.

**Theorem 1.3** (Fulton-Lazarsfeld). A weighted homogeneous polynomial $P$ is numerically positive for ample vector bundles of rank $r$ if and only if
\[
P \neq 0 \text{ and } a_\lambda(P) \geq 0 \text{ for all } \lambda \in \Lambda(k, r).
\]

Equivalently, the Schur polynomials $s_\lambda, \lambda \in \Lambda(k, r)$ span the cone of numerically positive polynomials.
Moreover, it is proved that a non-zero weighted homogeneous polynomial \( P \in \mathbb{Q}[c_1, ..., c_r] \) is Griffiths-positive if and only if \( P \) is a non-trivial non-negative linear combination of Schur polynomials (see [FL83, Appendix A]).

It should be noted that Fulton-Lazarsfeld’s positivity results are extended to nef vector bundles on compact Kähler manifolds in [DPS94].

Regarding Question 1.1 (2), Griffiths conjectured [Gri69, Page 247] that such polynomials are also given by Griffiths-positive polynomials. Combining with [FL83, Appendix A], Griffiths conjecture can be formulated as follows:

**Griffiths Conjecture.** Let \( P \in \mathbb{Q}[c_1, ..., c_r] \) be a weighted homogeneous polynomial, then the forms \( P(c_1(E, h), ..., c_r(E, h)) \) are positive for any Griffiths-positive vector bundles \( (E, h) \) over any smooth projective variety \( X \) if and only if \( P \) is a non-negative linear combination of Schur polynomials.

For the above conjecture, Griffiths commented that “this will require a better understanding of the algebraic properties of the curvature form \( \Theta \)”, and verified that for a Griffiths-positive vector bundle \( (E, h) \) of rank 2, the second Chern form \( c_2(E, h) \) is positive. See also [Gul12, Div16, Pin18, Li20] for some related results and progress. In the general case, the conjecture is still widely open.

1.3. The main result. Motivated by the well known Nakai-Moishezon criterion, we are interested in the relation between the ampleness, numerical positivity and pointwise positivity of Chern classes for a vector bundle. We propose the following conjecture which is in the intermediate of Fulton-Lazarsfeld’s positivity theorem and Griffiths conjecture.

**Conjecture 1.4.** Let \( X \) be a smooth projective variety of dimension \( n \), and \( E \) an ample (or Griffiths-positive) vector bundle of rank \( r \) over \( X \). Let \( s_\lambda(c_1, ..., c_r) \) be a Schur polynomial, then the class \( s_\lambda(c_1(E), ..., c_r(E)) \) has a smooth positive representative.

By [Ful76], even on \( \mathbb{P}^2 \) the existence of positive representatives does not imply the ampleness of \( E \).

**Remark 1.5.** It is unclear whether for every smooth representative \( \Phi \) of \( s_\lambda(c_1(E), ..., c_r(E)) \) there is a smooth Hermitian metric \( h \) on \( E \) such that \( \Phi = s_\lambda(c_1(E, h), ..., c_r(E, h)) \). Nevertheless, see [Pin18] for some results on a surface.

In this note, we confirm Conjecture 1.4 when the weighted degree \( |\lambda| = n - 1 \) by showing that the Schur class \( s_\lambda \) has very strong pointwise positivity.

**Theorem A.** Let \( X \) be a smooth projective variety of dimension \( n \), and let \( E \) be an ample vector bundle of rank \( r \) on \( X \). Then for any Schur polynomial \( s_\lambda(c_1, ..., c_r) \) with \( |\lambda| = n - 1 \), whenever it is non-zero the cohomology class \( s_\lambda(c_1(E), ..., c_r(E)) \) has a smooth representative which is strictly positive in the sense of \((n-1, n-1)\) forms.

A \( d \)-closed strictly positive \((n-1, n-1)\) form on \( X \) is also called a balanced metric in differential-geometric category.

**Example 1.6.** Let \( \lambda \in \Lambda(k, r) \) be a partition of \( k \). We list some simple Schur polynomials. By definition, we have:

\[
\begin{align*}
  s_\lambda(c_1(E), ..., c_r(E)) &= \begin{cases} 
  c_k(E), & \text{when } \lambda = (k, 0, ..., 0); \\
  \text{Segre class } s_k(E), & \text{when } \lambda = (1, 1, ..., 1, 0, ..., 0); \\
  c_j(E) \cdot c_{k-j}(E) - c_{j-1}(E) \cdot c_{k+1-j}(E), & \text{when } \lambda = (k-j, j, 0, ..., 0).
  \end{cases}
\end{align*}
\]

Since a numerically positive polynomial is a non-negative linear combination of Schur polynomials, Theorem A strengthens the positivity results of [FL83] for \((n-1, n-1)\) classes pointwisely.

**Example 1.7.** It is proved in [FL83] that the product of numerically positive polynomials is again numerically positive. In particular, any monomial \( c_l = c_1^{l_1} \cdot c_2^{l_2} \cdot c_r^{l_r} \) and

\[
c_k^k - c_k = \sum_{j=1}^{k} c_k^{k-j} (c_1 c_{j-1} - c_j)
\]
are numerically positive for ample vector bundles\(^1\). Applying Theorem A implies that whenever non-zero the classes \(c_1(E)\) with \(\sum_{j=1}^n j i_j = n - 1\), \(c_1^{n-1}(E) - c_{n-1}(E)\) have smooth representative which are strictly positive.

**Remark 1.8.** It is possible that \(s_\lambda(c_1(E),...,c_r(E))\), \(|\lambda| = n - 1\) has a smooth representative which is strictly positive in the sense of \((n - 1, n - 1)\) forms even if \(E\) is not ample. For example, let \(F\) be an ample vector bundle of rank \(r \geq n - 1\) and let \(E = F \oplus \mathcal{O}\), where \(\mathcal{O}\) is the trivial line bundle. Then \(c_{n-1}(E) = c_{n-1}(F)\), but \(E\) is not ample.

For general \(k\), by restricting \(E\) to a subvariety of dimension \(k + 1\), we get:

**Corollary A.** Let \(X\) be a smooth projective variety of dimension \(n\), and let \(E\) be an ample vector bundle of rank \(r\) on \(X\). Let \(V \subset X\) be an irreducible smooth subvariety of dimension \(k + 1\), given by the complete intersection of very ample divisors \(H_1,...,H_{n-k-1}.\) Then for any Schur polynomial \(s_\lambda(c_1,...,c_r)\) with \(|\lambda| = k\), whenever it is non-zero the cohomology class \(s_\lambda(c_1(E),...,c_r(E))\) : \([V]\) has a smooth representative which is strictly positive in the sense of \((n - 1, n - 1)\) forms.

**Remark 1.9.** Theorem A and Corollary A also hold for \(\mathbb{R}\)-twisted ample vector bundles.

In Section 2 we present some notions and basic results that are needed in the proof of our main result. In Section 3, we give the proof of the main result.

## 2. Preliminaries

### 2.1. \(\mathbb{R}\)-twisted vector bundles

In the proof of the main result, we need to apply a perturbation argument by using the notion of \(\mathbb{R}\)-twisted vector bundles. We give a very brief review on \(\mathbb{R}\)-twisted vector bundles. The reader can find more facts in [Laz04, Section 6.2] and the references therein.

**Definition 2.1.** A \(\mathbb{R}\)-twisted vector bundle \(E(\delta)\) on \(X\) is an ordered pair consisting of a vector bundle \(E\) on \(X\), defined up to isomorphism, and a cohomology class \(\delta \in H^{1,1}(X, \mathbb{R})\). The rank of \(E(\delta)\) is the rank of \(E\).

A \(\mathbb{R}\)-twisted vector bundle \(E(\delta)\) is understood as a formal object, yet when \(\delta = c_1(L)\) for some line bundle, \(E(\delta)\) can be considered as \(E \otimes L\).

Let \(\pi : \mathbb{P}(E) \to X\) be the projective bundle of hyperplanes of \(E\), and \(\mathcal{O}_E(1)\) the tautological line bundle:

\[
0 \to S \to \pi^*E \to \mathcal{O}_E(1) \to 0,
\]

where \(S_{(x,[l])} = l^{-1}(0) \subset E_x, l \in E^*_x\). If \(L\) is a holomorphic line bundle on \(X\), then \(\mathbb{P}(E \otimes L) \cong \mathbb{P}(E)\) by an isomorphism under which \(\mathcal{O}_E(1) \otimes \pi^*L\) corresponds to \(\mathcal{O}_{\mathbb{P}(E)}(1) \otimes \pi^*L\) on \(\mathbb{P}(E)\). This motivates the following:

**Definition 2.2.** A \(\mathbb{R}\)-twisted vector bundle \(E(\delta)\) on \(X\) is called ample (resp. nef) if \(\eta_E + \pi^*\delta\) is a Kähler (resp. nef) class on \(\mathbb{P}(E)\), where \(\eta_E = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))\).

The natural operations on vector bundles extend to the \(\mathbb{R}\)-twisted situation, here we just list a few.

**Definition 2.3.** The tensor of two \(\mathbb{R}\)-twisted vector bundles is defined by

\[
E_1(\delta_1) \otimes E_2(\delta_2) = (E_1 \otimes E_2)(\delta_1 + \delta_2),
\]

in particular, if \(E_2\) is a trivial line bundle then \(E_1(\delta_1) \otimes E_2(\delta_2) = E_1(\delta_1 + \delta_2)\).

The direct sum of two \(\mathbb{R}\)-twisted vector bundles with the same twisting class is defined by

\[
E_1(\delta) \oplus E_2(\delta) = (E_1 \oplus E_2)(\delta).
\]

A quotient of \(\mathbb{R}\)-twisted vector bundle \(E(\delta)\) is a \(\mathbb{R}\)-twisted vector bundle \(Q(\delta)\), where \(Q\) is a quotient of \(E\). Subbundles of \(E(\delta)\) are defined similarly.

Let \(f : Y \to X\) be a morphism between two projective varieties, then we define the pullback of a \(\mathbb{R}\)-twisted vector bundle \(E(\delta)\) on \(X\) to be the \(\mathbb{R}\)-twisted vector bundle \(f^*(E(\delta)) = (f^*E)(f^*\delta)\) on \(Y\).

\(^1\)The classes \(c_r\) and \(c_1^r - c_k\) (and its variant) are studied in [Gie71] and [DPS94, Li120] respectively.
Remark 2.4. From the above discussions, if $E(\delta)$ is ample, then for $\varepsilon \in H^{1,1}(X, \mathbb{R})$ small enough (by endowing the space $H^{1,1}(X, \mathbb{R})$ with some norm), the bundle $E(\delta + \varepsilon)$ is still ample. The pullback of a nef $\mathbb{R}$-twisted vector bundle by a morphism is nef, and the pullback of an ample $\mathbb{R}$-twisted vector bundle by a finite morphism is ample.

Remark 2.5. The positivity results of [FL83, BG71, DPS94] also hold in the $\mathbb{R}$-twisted setting.

2.2. Schur polynomials and Schur classes. Let $\Lambda(n, r)$ be the set of all partitions of $n$ by non-negative integers $\leq r$, i.e., any element $\lambda \in \Lambda(n, r)$ is a sequence

$$r \geq \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$$

satisfying $|\lambda| = \sum_{i=1}^{n} \lambda_i = n$. Associated to each partition $\lambda \in \Lambda(n, r)$, there is a Schur polynomial $s_\lambda \in \mathbb{Q}[c_1, \ldots, c_r]$ of weighted degree $n$ defined as the following determinant:

$$s_\lambda(c_1, \ldots, c_r) = \det[c_\lambda - j + k|_{1 \leq j, k \leq n} = \begin{vmatrix} c_{\lambda_1} & c_{\lambda_1 + 1} & \cdots & c_{\lambda_1 + n - 1} \\ c_{\lambda_2 - 1} & c_{\lambda_2} & \cdots & c_{\lambda_2 + n - 2} \\ \vdots & \vdots & \ddots & \vdots \\ c_{\lambda_n - n + 1} & c_{\lambda_n - n + 2} & \cdots & c_{\lambda_n} \end{vmatrix},$$

where we use the convention $c_0 = 1, c_i = 0$ whenever $i \notin [0, r]$. These polynomials form a basis for the vector space of all homogeneous polynomials of weighted degree $n$ in $r$ variables, where the variable $c_i$ is of weighted degree $i$. Geometrically, Schur polynomials appear in describing the cohomology of Grassmannians (see e.g. [Ful98, Chapter 14]).

Example 2.6. We list some simple Schur polynomials.

1. For small $n$, the Schur polynomials with $\lambda \in \Lambda(n, n)$, are given by

$$n = 1 : s_{(1)} = c_1,$$
$$n = 2 : s_{(2,0)} = c_2, s_{(1,1)} = c_1^2 - c_2,$$
$$n = 3 : s_{(3,0,0)} = c_3, s_{(2,1,0)} = c_1c_2 - c_3, s_{(1,1,1)} = c_1^3 - 2c_1c_2 + c_3.$$

2. For general $n$, we have

$$s_\lambda(c_1, \ldots, c_r) = \begin{cases} c_n, & \text{when } \lambda = (n, 0, \ldots, 0); \\ c_{j_n - j} c_{j-1} c_{n+1-j}, & \text{when } \lambda = (n - j, j, 0, \ldots, 0). \end{cases}$$

3. For $\lambda = (1, 1, \ldots, 0, 0, \ldots, 0)$, $s_\lambda(c_1, \ldots, c_r)$ is the Segre polynomial.

Definition 2.7. Let $X$ be a smooth projective variety of dimension $n$, and $E$ a holomorphic vector bundle of rank $r$ on $X$. Let $c_1(E), \ldots, c_r(E)$ be the Chern classes of $E$, then the Schur classes of $E$ are defined by

$$s_\lambda(E) = s_\lambda(c_1(E), \ldots, c_r(E)) \in H^{1,1}(X, \mathbb{R}).$$

The definition of Chern classes and Schur classes also make sense for $\mathbb{R}$-twisted bundles (see [Laz04, Chapter 8]).

In particular, let $\lambda$ be a partition, for each $0 \leq i \leq |\lambda|$ the “derived” Schur classes (see [RT19]) $s^{(i)}_\lambda(E)$ are defined by requiring that

$$s_\lambda(E(\delta)) = \sum_{i=0}^{\frac{|\lambda|}{|\lambda|}} s^{(i)}_\lambda(E) \cdot \delta^i.$$

Then $s^{(i)}_\lambda(E) \in H^{1,1}(X, \mathbb{R})$, with $s^{(0)}_\lambda(E) = s_\lambda(E)$. (1) can be understood as follows. Let $x_1, \ldots, x_r$ be the Chern roots of $E$, then $x_1 + \delta, \ldots, x_r + \delta$ are the Chern roots of $E(\delta)$. The Schur class is a polynomial of Chern roots, thus one can write

$$s_\lambda(x_1 + \delta, \ldots, x_r + \delta) = \sum_{i=0}^{\frac{|\lambda|}{|\lambda|}} s^{(i)}_\lambda(x_1, \ldots, x_r) \delta^i.$$
where \( s^{(i)}_{\lambda}(x_1, \ldots, x_r) \) is a symmetric polynomial of degree \(|\lambda| - i\), giving the class \( s^{(i)}_{\lambda}(E) \). This is motivated by the identity
\[
s_{\lambda}(E(\delta)) = s_{\lambda}(E \otimes L),
\]
when \( \delta = c_1(L) \) for some line bundle \( L \).

One also has:
\[
s^{(i)}_{\lambda}(E(\delta)) = \sum_{k=i}^{|\lambda|} \binom{k}{i} s^{(k)}_{\lambda}(E) \cdot \delta^{k-i}.
\]

**Remark 2.8.** The classes \( s_{\lambda}(E), s^{(i)}_{\lambda}(E) \) and their twisted variants have functorial properties under pullbacks, just as Chern classes.

### 2.3. Positivity of smooth forms.

We recall several positivity notions on \((p, p)\) forms. The standard reference is [Dem12]. Let \((z_1, \ldots, z_n)\) be the Euclidean coordinates on \(\mathbb{C}^n\), and \(\Lambda^{p,q}(\mathbb{C}^n)\) the space of \((p, q)\) forms on \(\mathbb{C}^n\) with constant coefficients. Denote the volume form of \(\mathbb{C}^n\) by
\[
d \text{vol}_{\mathbb{C}^n} = idz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge idz_n \wedge d\bar{z}_n.
\]

**Definition 2.9.** A \((p, p)\) form \(u \in \Lambda^{p,p}(\mathbb{C}^n)\) is said to be positive if for any \(\alpha_j \in \Lambda^{1,0}(\mathbb{C}^n)\), \(1 \leq j \leq n-p\), the \((n, n)\)-form
\[
\sum_{s=1}^{m} a_s i \alpha_{s,1} \wedge \bar{\alpha}_{s,1} \wedge \ldots \wedge i \alpha_{s,p} \wedge \bar{\alpha}_{s,p}
\]
diffs with \(d \text{vol}_{\mathbb{C}^n}\) by a non-negative multiplier. A \((p, p)\) form \(u \in \Lambda^{p,p}(\mathbb{C}^n)\) is said to be strongly positive if \(u\) is a non-negative combination
\[
u = \sum_{s=1}^{m} a_s i \alpha_{s,1} \wedge \bar{\alpha}_{s,1} \wedge \ldots \wedge i \alpha_{s,p} \wedge \bar{\alpha}_{s,p},
\]
where \(\alpha_{s,j} \in \Lambda^{1,0}(\mathbb{C}^n)\) and \(a_s \geq 0\).

The set of positive \((p, p)\) forms is a closed convex cone in \(\Lambda^{p,p}(\mathbb{C}^n)\), similarly for the set of strongly positive \((n-p, n-p)\) forms. These two cones are dual to each other via the pairing between \(\Lambda^{p,p}(\mathbb{C}^n)\) and \(\Lambda^{n-p,n-p}(\mathbb{C}^n)\). It is easy to see that strongly positive forms must be positive.

**Remark 2.10.** For forms of bidegrees \((0, 0), (1, 1), (n-1, n-1), (n, n)\), the above two positivity notions are equivalent. Furthermore, an \((1, 1)\) form
\[
u = \sum_{j,k} u_{jk} dz_j \wedge d\bar{z}_k
\]
is positive or strongly positive if and only if the Hermitian matrix \([u_{jk}]_{1 \leq j,k \leq n}\) is semi-positive. Denote \(d\bar{z}_j \wedge d\bar{z}_k\) to be the \((n-1, n-1)\)-form such that
\[
idz_j \wedge d\bar{z}_k \wedge dz_j \wedge d\bar{z}_k = d \text{vol}_{\mathbb{C}^n}.
\]

Then an \((n-1, n-1)\) form
\[
\sum_{j,k} u_{jk} d\bar{z}_j \wedge d\bar{z}_k
\]
is positive or strongly positive if and only if the Hermitian matrix \([u_{jk}]_{1 \leq j,k \leq n}\) is semi-positive.

**Definition 2.11.** For an \((1, 1)\) or \((n-1, n-1)\) form \(u\), we call it strictly positive if the Hermitian matrix \([u_{jk}]_{1 \leq j,k \leq n}\) is positive definite.

It is clear that these positivity notions can be also formulated on a complex manifold \(X\) by requiring the corresponding positivity at every point of \(X\). Using the duality between forms and currents, one can also define positivity for currents (see [Dem12, Chapter 3]). For example, on a compact complex manifold a \((k, k)\) current is positive (resp. strongly positive) if it takes non-negative values on all smooth strongly positive (resp. positive) \((n-k, n-k)\) forms.
3. Proof of the main result

In this section, we give the proof of the main result. As mentioned in the introduction, the main result holds in the $\mathbb{R}$-twisted setting, we are going to prove:

**Theorem 3.1.** Let $X$ be a smooth projective variety of dimension $n$, and let $E$ be a $\mathbb{R}$-twisted ample vector bundle of rank $r$ on $X$. Then for any Schur polynomial $s_\lambda(c_1,\ldots,c_r)$ with $|\lambda| = n-1$, whenever it is non-zero the cohomology class $s_\lambda(E)$ has a smooth representative which is strictly positive in the sense of $(n-1,n-1)$ forms.

**Proof.** Since on a surface the Schur class of bidegree $(1,1)$ is just the first Chern class, without loss of generality, in the following we always assume that $n \geq 3$.

We need to prove the existence of a strictly positive representative in an $(n-1,n-1)$ class. The idea is to apply the following very useful positivity criterion by using duality between the pseudo-effective and the movable cone [WN19, Corollary A] (see also [FX14, Appendix], [Tom10], [BDPP13]).

**Lemma 3.2.** Let $X$ be a smooth projective variety of dimension $n$ and $\alpha \in H^{n-1,n-1}(X,\mathbb{R})$, then $\alpha$ has a smooth representative which is strictly positive if and only if for any non-zero pseudo-effective (1,1) class $L \in H^{1,1}(X,\mathbb{R})$, the intersection number $\alpha \cdot L > 0$. Moreover, for $\alpha \in H^{n-1,n-1}(X,\mathbb{Q})$, one only needs to check the positivity condition $\alpha \cdot L > 0$ for $L$ rational.

Recall that $L \in H^{1,1}(X,\mathbb{R})$ is called pseudo-effective if it has a representative which is positive in the sense of $(1,1)$ currents.

Applying Lemma 3.2 in our setting, it is sufficient to check that:

$$s_\lambda(E) \cdot L > 0$$

for any non-zero pseudo-effective class $L \in H^{1,1}(X,\mathbb{R})$. If $L$ is the class of a hypersurface, then it follows from [FL83]; if $L$ is a Kähler class, then the non-negativity follows from [DPS94, Theorem 2.5].

We reduce the proof to the above two cases. To this end, we apply the divisorial Zariski decomposition due to [Bou04] (see also [Nak04]):

**Lemma 3.3.** Let $X$ be a smooth projective variety (or more generally, a compact Kähler manifold) of dimension $n$, let $L \in H^{1,1}(X,\mathbb{R})$ be a pseudo-effective class. Then there exists a decomposition

$$L = P(L) + N(L),$$

such that $P(L)$ is movable and $N(L)$ is effective.

More precisely, $N(L) = \sum_{i=1}^{m} a_i[D_i]$ for some $a_i \geq 0$ and some prime divisors $D_i$, and $P(L)$ can be written as

$$P(L) = \lim_{m} (\pi_m)_* \tilde{\omega}_m,$$

where $\pi_m : X_m \to X$ is a modification and $\tilde{\omega}_m$ is a Kähler class on the projective manifold $X_m$.

Now we fix a non-zero pseudo-effective class $L \in H^{1,1}(X,\mathbb{R})$, and apply the divisorial Zariski decomposition to $L$.

If $N(L) = \sum a_j[D_j] \neq 0$, i.e., in the summands there exists a prime divisor $D_{j_0}$ such that $a_{j_0} > 0$, then by [FL83],

$$s_\lambda(E) \cdot [D_{j_0}] = s_\lambda(E|_{D_{j_0}}) > 0.$$

If $N(L) = 0$, i.e., $L = P(L)$ is a non-zero movable class, then there is a sequence of modifications $\pi_m : X_m \to X$ and Kähler classes $\tilde{\omega}_m$ on $X_m$ such that

$$L = \lim_{m} (\pi_m)_* \tilde{\omega}_m.$$

Thus,

$$s_\lambda(E) \cdot L = \lim_m s_\lambda(E) \cdot (\pi_m)_* \tilde{\omega}_m = \lim_m s_\lambda(\pi_m)_* E \cdot \tilde{\omega}_m \geq 0$$

where the last inequality holds by [DPS94, Theorem 2.5], since $\pi_m^* E$ is nef on $X_m$.

In particular, we have proved:

**Lemma 3.4.** For $E$ a $\mathbb{R}$-twisted nef vector bundle, $s_\lambda(E) \cdot L \geq 0$ for any non-zero movable class $L$. 

Combining the above discussions, we only need to prove (2) when $L$ is a non-zero movable class. In the following, we assume that $L$ is a non-zero movable class.

Similar to [BG71] (see also [Laz04, Chapter 8]), we use the ampleness of $E$ and a perturbation argument to prove that $s_\lambda(E) \cdot L > 0$.

Fix a very ample line bundle $H$ such that $\omega = c_1(H)$ on $X$. Since $E$ is ample, for $c > 0$ sufficiently small and $t \in [0, c]$, the $\mathbb{R}$-twisted bundle $E(-t\omega)$ is also ample, thus by Lemma 3.4

$$s_\lambda(E(-t\omega)) \cdot L \geq 0, \text{ for any } t \in [0, c].$$

Recall that the Schur classes $s_\lambda(E(-t\omega))$ and $s_\lambda(E)$ are related by the following identity (see (1)):

$$s_\lambda(E(-t\omega)) = s_\lambda(E) - ts_\lambda^{(1)}(E) \cdot \omega + O(t^2),$$

where $s_\lambda^{(1)}(E) \in H^{n-2,n-2}(X, \mathbb{R})$, and $O(t^2)$ is the term of order $t^2$. Therefore,

$$s_\lambda(E) \cdot L = s_\lambda(E(-t\omega)) \cdot L + ts_\lambda^{(1)}(E) \cdot \omega \cdot L + O(t^2).$$

To finish the proof, we only need to check that $s_\lambda^{(1)}(E) \cdot \omega \cdot L > 0$.

The idea is to apply the Hodge index theorem due to Ross-Toma [RT19]. Roughly speaking, we first verify that

$$s_\lambda^{(1)}(E) \cdot \omega \cdot L \geq 0 \text{ and } s_\lambda^{(1)}(E) \cdot L \cdot L \geq 0.$$  

Assuming that $s_\lambda^{(1)}(E) \cdot \omega \cdot L = 0$, i.e., $L$ is primitive with respect to $(s_\lambda^{(1)}(E), \omega)$, then by the Hodge index theorem,

$$s_\lambda^{(1)}(E) \cdot L \cdot L \leq 0$$

with equality holds only if $L = 0$. Combining with $s_\lambda^{(1)}(E) \cdot L \cdot L \geq 0$ we get $L = 0$, which contradicts with our assumption $L \neq 0$.

**Lemma 3.5.** The class $s_\lambda^{(1)}(E) \in H^{n-2,n-2}(X, \mathbb{R})$ satisfies Hodge index theorem, and (3) holds.

The proof of Lemma 3.5 mainly follows from [RT19].

Fix a complex vector space $V$ of dimension $\dim V = r + n$, and let $V = X \times V$ be the trivial vector bundle on $X$. Let $F = V \otimes E$, and denote the rank of $F$ by $f + 1 = (r + n)r$.

Let $p : P(F) \to X$ be the projective bundle of lines of $F$, and $U$ the universal quotient bundle on $P(F)$:

$$0 \to \mathcal{O}_{P(F)}(-1) \to p^*F \to U \to 0.$$

The bundle $U$ is of rank $f$, and it is nef when $F$ is nef. By [RT19, Proposition 5.2] (though it is stated for certain degrees in their setting, it also holds for general degrees),

$$s_\lambda^{(1)}(E) \cdot \beta = p_*c_{f-1}(U|_C) \cdot \beta$$

for any $\beta \in H^{1,1}(X, \mathbb{R})$, where $C \subset P(F)$ is an irreducible subvariety of dimension $f + 1$ and locally a product over $X$ (see [RT19] or [Ful98]). Therefore, by the projection formula

$$s_\lambda^{(1)}(E) \cdot \beta \cdot \beta' = c_{f-1}(U|_C) \cdot p^*\beta \cdot p^*\beta',$$

where $\beta, \beta' \in H^{1,1}(X, \mathbb{R})$.

The bundle $E$ being ample implies that $F$ is ample. By [RT19, Theorem 4.1], the quadratic form

$$Q(\beta, \beta') : = c_{f-1}(U|_C) \cdot p^*\beta \cdot p^*\beta'$$

$$= s_\lambda^{(1)}(E) \cdot \beta \cdot \beta'$$

satisfies Hodge index theorem, that is, it has signature $(1, h^{1,1}(X) - 1)$.

As for (3), we first claim that $s_\lambda^{(1)}(E) \cdot \omega \cdot L \geq 0$. Recall that $L = \lim_{m} (\pi_m)_*\omega_m$, thus

$$s_\lambda^{(1)}(E) \cdot \omega \cdot L = \lim_{m} s_\lambda^{(1)}(\pi_m^*E) \cdot \pi_m^*\omega \cdot \omega_m.$$
Applying the above argument and (4) to the nef bundle $\pi_m^*E$ on $X_m$ shows that every term
\[
s^{(1)}_\lambda(\pi_m^*E) \cdot \pi_m^*\omega \cdot \hat{\omega}_m = c_{f-1}(U_m|_{C_m}) \cdot p_m^*(\pi_m^*\omega) \cdot p_m^*\hat{\omega}_m \geq 0,
\]
where the second inequality follows from [DPS94, Corollary 2.2] and $\omega$ being an ample divisor class. Here we denote the corresponding objects by the subscript $m$. Thus the limit is also non-negative, finishing the proof of our claim.

Next we claim that $s^{(1)}_\lambda(E) \cdot L \cdot L \geq 0$. The proof is similar to the first claim. Note that
\[
s^{(1)}_\lambda(E) \cdot L = \lim_{m} s^{(1)}_\lambda(\pi_m^*E) \cdot \pi_m^*(\pi_m^*\hat{\omega}_m) \cdot \hat{\omega}_m.
\]
By Siu decomposition [Siu74], we have\(^2\)
\[
\pi_m^*(\pi_m^*\hat{\omega}_m) = \hat{\omega}_m + [D]
\]
for some effective $\mathbb{R}$-divisor $D$. Therefore,
\[
s^{(1)}_\lambda(\pi_m^*E) \cdot \pi_m^*(\pi_m^*\hat{\omega}_m) \cdot \hat{\omega}_m = s^{(1)}_\lambda(\pi_m^*E) \cdot [D] \cdot \hat{\omega}_m + s^{(1)}_\lambda(\pi_m^*E) \cdot \hat{\omega}_m \cdot \hat{\omega}_m.
\]
Repeating the argument for $\pi_m^*E$ on $X_m$ as the first claim and applying [DPS94] show that every summand is non-negative. This finishes the proof of the second claim.

By using [RT19, Theorem 4.3] or [FL83],
\[
Q(\omega, \omega) = c_{f-1}(U|C) \cdot p^*\omega^2 > 0.
\]
Therefore, if $Q(\omega, L) = s^{(1)}_\lambda(E) \cdot \omega \cdot L = 0$, then by Lemma 3.5 $Q(L, L) \leq 0$ with the equality holds if and only if $L = 0$. By Lemma 3.5 again, $Q(L, L) \geq 0$, thus $s^{(1)}_\lambda(E) \cdot \omega \cdot L = 0$ yields $L = 0$, contradicting with $L \neq 0$.

Therefore, $s^{(1)}_\lambda(E) \cdot \omega \cdot L > 0$. As $L$ is arbitrary, this finishes the proof of the theorem.

\[\square\]

**Remark 3.6.** When $s_\lambda(E) = c_{n-1}(E)$ is the $(n-1)$-th Chern class of $E$ and the rank $r \geq n-1$, the estimate $c_{n-1}(E) \cdot L > 0$ can be alternatively proved by an inductive argument.

Recall that in our perturbation argument, we assume that $\omega = c_1(H)$ for a very ample line bundle $H$. Use the same notation $H$ to denote a very general smooth hypersurface in the linear system of $H$. Then by the identity
\[
(5) \quad c_k(E(\delta)) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E)\delta^{k-i}, \quad 0 \leq k \leq r,
\]
we have
\[
c_{n-1}(E) \cdot L = c_{n-1}(E(-t\omega)) \cdot L + t(r-n+2)c_{n-2}(E) \cdot \omega \cdot L + O(t^2)
\[
= c_{n-1}(E(-t\omega)) \cdot L + t(r-n+2)c_{n-2}(E|_H) \cdot L|_H + O(t^2).
\]
By the weak Lefschetz theorem, the restriction $i^* : H^2(X, \mathbb{R}) \to H^2(H, \mathbb{R})$ is injective whenever $2 \leq n-1$. Our assumption $n \geq 3$ shows that the weak Lefschetz theorem holds. Thus the class $L|_H$ is non-zero. As $H$ is very general, it has a representative which is a non-zero positive $(1,1)$ current on $H$.

By induction, $c_{n-2}(E|_H)$ has a representative which is a smooth strictly positive $(n-2, n-2)$ form on the hypersurface $H$. This yields that
\[
c_{n-2}(E) \cdot \omega \cdot L = c_{n-2}(E|_H) \cdot L|_H > 0,
\]
finishing the proof for $c_{n-1}(E)$.

\(^2\)This is the transcendental analogy of Negativity Lemma in algebraic geometry.
As for an inductive argument to the general Schur class \( s_\lambda(E) \), it is not clear to us if \( s_\lambda^{(1)}(E) \) is numerically positive, or equivalently, a non-negative linear combination of Schur polynomials of weighted degree \((n-2)\).

Corollary A follows from applying Theorem A to the subvariety \( V \) and the weak Lefschetz theorem.

**Proof of Corollary A.** We can assume that \( n \geq 3 \). By the proof of Theorem A, we only need to check the positivity of intersection numbers with non-zero pseudo-effective \((1,1)\) classes.

Since \( V \) is the complete intersection of very ample divisors and we only care about intersection numbers, we can assume that \( V \) is the intersection of very general elements of the linear systems of the \( H_i \). Given a non-zero pseudo-effective \((1,1)\) class \( L \), by weak Lefschetz theorem its restriction \( L|_V \) is non-zero. As \( V \) is very general, it is also pseudo-effective by \cite{Dem92}. Thus applying Theorem A on \( V \) shows that

\[
s_\lambda(E) \cdot [V] \cdot L = s_\lambda(E|_V) \cdot L|_V > 0.
\]

This finishes the proof of Corollary A. \( \square \)

**Remark 3.7.** For a nef vector bundle \( E \) over a compact Kähler manifold \( X \) of dimension \( n \), \cite{DPS94} and the proof of Theorem A imply that the Schur class \( s_\lambda(E) \) with \(|\lambda| = n - 1\) is always in the dual cone of the pseudo-effective cone of \((1,1)\) classes.

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