NEIGHBORS

GÁBOR FEJES TÓTH, LÁSZLÓ FEJES TÓTH, AND WŁODZIMIERZ KUPERBERG

Abstract. Two members of a packing are neighbors if they have a common boundary point. A multitude of problems arises in connection with neighbors in a packing. The oldest one concerns a dispute between Newton and Gregory about the maximum number of neighbors a member can have in a packing of congruent balls. Other problems ask for the average number of neighbors or the maximum number of mutually neighboring members in a packing. The present work gives a survey of these problems.

Two members of a packing are neighbors if they have a common boundary point. In this chapter we survey results connected with the number of neighbors in a packing.

1. The Newton number of convex disks

Let \( N(S) \) denote the maximum of the number of neighbors that one member can have in a packing of congruent copies of \( S \). After L. Fejes Tóth [1969] we call \( N(S) \) the Newton number of \( S \). The name recalls the dispute between Newton and Gregory about the maximum number of congruent balls that can touch another one of the same size without overlapping with each other. A rigorous proof settling the dispute in favor of Newton was given by Schütte and van der Waerden [1953]. An alternative name for \( N(S) \) is the kissing number of \( S \).

Some experiments with regular polygonal disks point to the conjecture that the Newton number of a regular \( n \)-gon is: 
- 12 for \( n = 3 \);
- 8 for \( n = 4 \);
- and 6 for \( n \geq 5 \).

Indeed, with the exception of \( n = 5 \), this was proved by Böröczky [1971] (see also Youngs [1939] for the case of a square, Klamkin, Lewis and Liu [1994] for the cases \( n = 3, 4 \) and 6, and Zhao [1998] for the case \( n > 6 \)). The Newton number of the regular pentagon was proved to be six by Linhart [1973] and independently by Pankov and Dolmatov [1977, 1979] and Zhao and Xu [2002].

Schopp [1970] proved that the Newton number of any disk of constant width is at most 7. The Newton number of the Reuleaux triangle is equal to 7 (Figure 1). Kemnitz and Möller [1997] determined the Newton number of all rectangles. L. Fejes Tóth [1967] proved the inequality

\[
N(S) \leq (4 + 2\pi) \frac{D}{w} + 2 + \frac{w}{D}
\]
for a convex disk $S$ with diameter $D$ and width $w$. This estimate is exact in many cases. For example, for the isosceles triangle $\Delta$ with $\frac{w}{D} = \sin \frac{\pi}{19}$ we get $N(\Delta) \leq 64$. But a simple construction (Figure 2) shows that $N(\Delta) \geq 64$, hence $N(\Delta) = 64$.

Other bounds for the Newton number of a convex disk involving different parameters were given by Hortobágyi [1972, 1975, 1976b] and Wegner [1992]. Each of them considered a generalization of the Newton number produced by counting the number of congruent copies of a convex disk $K$ that can touch another convex disk $C$. Wegner’s result includes the computation of the Newton number of the $30^\circ$-$30^\circ$-$120^\circ$ triangle, which, as conjectured, turns out to be 21. Further problems about Newton numbers are treated in Harborth, Koch and Szabó [2001], Kemnitz, Möller and Wojzischke [2002], Kemnitz and Szabó [2001] and Kemnitz, Szabó and Ujváry-Menyhárt [2000].

2. The Hadwiger number of convex disks

The Hadwiger number is defined in a similar way as the Newton number, but considering only translated copies of $S$ instead of allowing all congruent ones. The notion was so named by L. Fejes Tóth [1970] because of the following result of Hadwiger [1957c]: If each of $k$ mutually non-overlapping translates of an $n$-dimensional convex body $K$ touches $K$, then $k \leq 3^n - 1$. The inequality is sharp, as equality occurs for the parallelepiped, and, as Groemer [1961a] proved, only for the parallelepiped (see also Grünbaum [1961] for the planar case). The Hadwiger number of a convex body $K$, denoted by $H(K)$, is alternately called the translative kissing number of $K$.

In the plane, how large can the Hadwiger number be for a non-convex Jordan region? While for each such region the number is finite, Cheong and Lee [2007] showed that, surprisingly, there exist Jordan regions with arbitrarily large Hadwiger number. For starlike regions, however, A. Bezdek [1997] showed that 75 is an upper bound and Lángi [2011b] lowered Bezdek’s bound to 35. For the Hadwiger number of centrally symmetric starlike regions Lángi [2009] established the stronger upper bound of 12. It is still unknown if the number can be greater than
8. LÁNGI [2011b] determined the Hadwiger number of a special class of regions. According to Lángi a pocket of $R$ is a connected component of $\text{conv} R \setminus R$. In their examples, Cheong and Lee achieve an arbitrarily large Hadwiger number only for a sequence of polygons whose number of pockets is not bounded above. Lángi determined the Hadwiger number of a region with one pocket; depending on certain properties of the region, the number is either 6 or 8.

Boju and Funar [1993, 2011] investigated the generalization of the Hadwiger number in which mutually non-overlapping, $\lambda$-homothetic copies of $K$ for some $\lambda > 0$ touch $K$ and they gave corresponding bounds.

3. Translates of a Jordan disk with a common point

An interesting related question was addressed by A. Bezdek, K. Kuperberg and W. Kuperberg [1995]: What is the maximum number of non-overlapping translates of a Jordan disk that can have a common point? They proved that the number is at most 4, and they characterized the disks for which 4 non-overlapping translates can have a common point. Earlier, K. Kuperberg and W. Kuperberg [1994] proved the same bound for starlike disks. In contrast, in space the corresponding number can be arbitrarily large even for starlike solids.

4. The number of touching pairs in finite packings

A problem of Erdős [1946] asks for the maximum number of occurrences of the minimum distance between $n$ points in the plane. Erdős proved the upper bound $3n - 6$ and pointed out that the example of the regular triangular lattice shows that the minimum distance can occur $3n - c\sqrt{n}$ times. Unaware of the work of Erdős, Reuter [1972] restated the problem as a conjecture in the language of circle packings: The maximum number of touching pairs among $n$ unit circles forming a packing in the plane is $[3n - \sqrt{12n} - 3]$. The conjecture was verified by Harborth [1974]. Kupitz [1994] gave a complete description of the extremal packings; they are all subsets of the triangular lattice. Brass [1990] extended Harborth’s result to packings consisting of translates of a convex disk different from a parallelogram and showed that the corresponding number for parallelograms is $[4n - \sqrt{28n} - 12]$. The papers by K. Bezdek [2002, 2012a] and K. Bezdek and Reid [2013] give bounds for the number of touching pairs in packings of congruent balls. The latter paper investigates the number of mutually touching triples and quadruples as well. Bowen [2000] studied contact numbers in circle packings in the hyperbolic plane. K. Bezdek, B. Szalkai and I. Szalkai [2016], K. Bezdek and Naszódi [2018a], K. Bezdek, Khan and Oliwa [2019] and Naszódi and Swanepeel [2022] investigated the number of touching pairs in totally separable packings. For a survey on contact numbers in packings see K. Bezdek and Khan [2018b].

5. $n$-neighbor packings

If every member of a packing has exactly $n$, or at least $n$ neighbors, we call it an $n$-neighbor or an $n^+$-neighbor packing, respectively. An easy construction shows that there exists a zero-density 5-neighbor packing of the plane with translates of a parallelogram. It turned out that this property characterizes parallelograms (see L. Fejes Tóth [1973a]). For a convex disk $K$ other than a parallelogram, Makai [1987] proved that every $5^+$-neighbor packing with translates of $K$ is of
density greater than or equal to $3/7$. Equality can occur only if $K$ is a triangle. For centrally symmetric convex disks the corresponding lower bound is $9/14$, attained only for affine regular hexagons. In [1987] Makai only sketched the proof for this last statement; details are given in [2018].

Concerning $6^+$-neighbor packings, the following is known: L. Fejes Tóth [1973a] showed that the density of a $6^+$-neighbor packing with translates of a convex disk is at least $1/2$, and Makai [1987] obtained the corresponding lower bound of $3/4$ for centrally symmetric disks. Each of these bounds is sharp, and, again, the extreme values are produced only by triangles and affine regular hexagons, respectively. Chvátal [1975] proved that the density of a $6^+$-neighbor packing with translated parallelograms is at least $11/15$.

According to L. Fejes Tóth [1973a] the density of a $5^+$-neighbour packing of congruent circular disks is at least $\sqrt{3} \pi / 7$. Concerning $5^+$ neighbor packing of non-congruent circles G. Fejes Tóth and L. Fejes Tóth [1991] proved that there is a constant $h_0 = 0.53329 \ldots$ such that any $5^+$ neighbor packing of circular disks whose homogeneity exceeds $h_0$ has positive density. The constant $h_0$ is the unique real root of the equation $8h^3 + 3h^2 - 2h - 1$. It cannot be replaced by a smaller one: A $5^+$-neighbor packing of circles of homogeneity $h_0$ and density 0 is shown in Figure 3.

![Figure 3](image)

L. Fejes Tóth [1974] stated the following interesting conjecture: The homogeneity of a $6^+$-neighbor packing of circles is either 1 or $0$. The conjecture was confirmed by Bárány, Füredi and Pach [1984]. They proved the somewhat stronger statement that in a $6^+$-neighbor of circular disks either all disks are congruent or arbitrarily small disks occur. Their proof combines a geometric idea with a combinatorial one, each of interest on its own.

The angle at the boundary point $a$ of a convex disk $K$ is the measure of the smallest angular region with apex $a$ containing $K$. The minimum of the angle taken for all boundary points of $K$ is called the minimal angle of $K$. A further problem due to L. Fejes Tóth [1969c] asked for the numbers $n$ for which there is an $n$-neighbor packing consisting of convex disks with given minimal angle $\pi/h$. Linhart [1974b] proved that for such a packing $n \leq \max\{5, 2\lfloor h \rfloor - 1\}$ and showed that this bound is sharp for $\lfloor h \rfloor \leq 6$.

Let $t(K)$ denote the largest number $n$ for which there is a finite $n^+$-neighbor packing of translates of a convex disk $K$, and let $m(K)$ denote the minimum cardinality of such a packing. Talata [2002] proved that $t(K) = 4$ and $m(K) = 12$ if $K$ is a parallelogram, otherwise $t(K) = 3$ and $m(K) = 7$.

Wegner [1971] solved two problems posed by L. Fejes Tóth [1969c] concerning the existence of certain packings in which every member has the same number of neighbors. On one hand, he constructed for every $n \geq 3$ a stable $n$-neighbor packing
of congruent convex disks, and, on the other hand, he presented a 5-neighbor packing consisting of 32 congruent smooth convex disks (see Figure 4). Independently from Wegner, Linhart [1973b] also found a finite 5-neighbor packing of congruent smooth convex disks using a very similar idea.

Figure 4

6. Maximal packings

A packing of congruent copies of a convex disks is said to be a maximal packing if the number of neighbors of every disk equals its Newton number. The faces of each of the tilings \{6, 3\}, \{4, 4\} and \{3, 6\} form such a maximal packing.

Gács [1972] proved that there is an absolute constant \(N\), such that the number of neighbors in a maximal packing cannot exceed \(N\). In the tiling obtained by dissecting each face of \{3, 6\} into three congruent triangles (Figure 5) every cell has 21 neighbors. In view of the result of Wegner [1992] mentioned above, it follows that this tiling is a maximal packing. On the other hand, Linhart [1977a] showed that in no maximal packing of convex disks can the number of neighbors exceed 21.

Figure 5

Böröczky [1971] proved that the faces of every regular spherical tiling form a maximal packing. However at most 48 regular hyperbolic tilings are maximal (see Florian [1975b] and A. Florian and H. Florian [1975a, 1975b]). Linhart [1975] showed that the tiling \{7, 3\} is maximal.
Higher order neighbors

We say that in a packing of convex disks each disk is the “zero-th neighbor” of itself. A disk \( B \) is the \( k \)-th neighbor of the disk \( A \) if it is a neighbor of a \((k-1)\)-th neighbor of \( A \) other than a \( j \)-th neighbor of \( A \) with \( 0 \leq j < k - 1 \). The \( k \)-th Newton number \( N_k(D) \) of a convex disk \( D \) is the maximum of the total number of its \( j \)-th neighbors for \( j = 1, \ldots, k \). The densest lattice packing of circles yields the lower bound \( N_k(B^2) \geq 3k(k + 1) \). One could conjecture that, for small values of \( k \), say for \( k \leq 12 \), \( N_k(B^2) = 3k(k + 1) \) holds. This is obvious for \( k = 1 \) and was confirmed by L. Fejes Tóth and Heppes [1967] for \( k = 2 \). But L. Fejes Tóth [1969a] showed that \( N_{14}(B^2) \geq 636 \), which is greater than \( 3 \cdot 14 \cdot 15 = 630 \). He also determined the asymptotic behavior of \( N_k(B^2) \) for large \( k \), namely

\[
\lim_{k \to \infty} N_k(B^2)/k^2 = 2\pi/\sqrt{3}.
\]

Higher order Hadwiger numbers are defined and treated in L. Fejes Tóth [1975a] and L. Fejes Tóth and Heppes [1977].

The Newton number of balls

Alternates to Schütte and Van der Waerden’s proof of the equality \( N(B^3) = 12 \) were given by Leech [1956], Böröczky [2003], Anstreicher [2004], Musin [2006b], Glazyrin [2020b], and Maehara [2007b], the last one being perhaps the most elementary among them.

Flatley, Tarasov, Taylor and Theil [2013] proved that the maximum number of tangent pairs among twelve non-overlapping unit balls tangent to a thirteenth unit ball is 24, attained only in the case when the centers of the twelve balls are the vertices either of a cuboctahedron or of a twisted cuboctahedron. A twisted cuboctahedron is obtained by cutting a cuboctahedron into two parts by a plane containing 6 edges forming a regular hexagon, and rotating one part by an angle of \( \pi/3 \) around the axis through the center of the hexagon and perpendicular to its plane. R. Kusner, W. Kusner, Lagarias and Shlomson [2018] described the configuration space of 12 non-overlapping equal spheres of radius \( r \) touching a central unit sphere. They also gave a nice survey of the history of the twelve spheres problem and the Tammes problem.

The value of \( N(B^4) \) was determined by Musin [2003, 2008], while \( N(B^8) \) and \( N(B^{24}) \) were determined by Odlyzko and Sloane [1979] and, independently, by Levenstein [1979]. The cases \( n = 8 \) and \( n = 24 \) were resolved by means of the linear programming method, and the case \( n = 4 \) by its modification. The corresponding values are:

\[
N(B^4) = 24, \quad N(B^8) = 240, \quad \text{and} \quad N(B^{24}) = 196560.
\]

Each of these Newton numbers is realized in the unique densest lattice packing of balls in the corresponding dimension. Moreover, as shown by Bannai and Sloane [1981], each of the arrangements of balls that realize the Newton number for \( n = 8 \) and \( n = 24 \) is unique, which for \( n = 4 \) is still only conjectured. The Kabatjanski–Levenstein bound yields \( N(B^n) \leq 2^n.4014d+o(d) \), while the best known lower bound due to Jenssen, Joos and Perkins [2018] is \( N(B^n) \geq (1 + o(1))\sqrt{3\pi/8} \log(3\sqrt{2}/4) n^{3/2}(2\sqrt{3}/2)^n \). Dostert and Kolpakov [2020] gave upper bounds for the Newton number of balls in spherical and hyperbolic space.
Let $N^k_n$ be the $k$-th Newton number of the $n$-dimensional ball. We have $N^1_2 = 4$ and, as expected, $N^2_2 = 18$. L. Fejes Tóth and Heppes [1967] showed that $56 \leq N^3_2 \leq 63$ and $168 \leq N^4_2 \leq 232$. The lower bound $N^3_2 \geq 56$ comes from the enumeration of balls in the first and second neighborhood in the trapezorhombododecahedral packing. The outcome is $12 + 44 = 56$. Interestingly enough, the corresponding outcome for the rhombododecahedral packing amounts to just $12 + 42 = 54$. This could be of some significance to the still unanswered question why certain metals form the trapezo-rhombododecahedral structure in their crystals, while others form a rhombododecahedral structure. In the densest lattice packing of the 4-dimensional ball, found by Korkine and Zolotareff [1872], the corresponding number is $24 + 144 = 168$.

9. $n$-NEIGHBOR PACKING OF CONGRUENT BALLS

What is the chromatic number of a finite packing of congruent balls? For which numbers $n$ does there exist a finite $n$-neighbor packing of congruent balls? Two consecutive layers from the densest lattice packing of balls is a nine-neighbor packing of density 0. Does there exist a ten-neighbor packing of zero density? Motivated by these questions L. Fejes Tóth and Sachs [1976] stated the following conjectures.

“Conjecture A. The maximum number of points which can be placed on an open unit hemisphere with at least unit distance from one another is equal to eight.

Conjecture B. The maximum number of points which can be placed on a closed unit hemisphere with at least unit distance from one another is equal to nine.

Conjecture AB. If nine points lie on a closed unit hemisphere with at least unit distance from one another then six of them are on the boundary of the hemisphere.”

Since the condition that six points lie on the boundary uniquely determines the configuration, Conjecture AB implies the other two. It follows from Conjecture A that the chromatic number of a finite packing of unit balls is at most 9, while the highest known chromatic number of a finite packing of unit balls is 5 (see Maehara [2007a]). It also follows that no finite nine-neighbor packing of congruent balls exists. Conjecture B implies that the density of a ten-neighbor packing is positive.

Conjecture B was confirmed by G. Fejes Tóth [1981]. Alternative proofs based on the difficult result that the Newton number of the 3-ball is 12 were given by Sachs [1986] and by A. Bezdek and K. Bezdek [1988]. Finally, Kertész [1994] proved Conjecture AB. The analogous problem was investigated in higher dimensions as well. Let $B(n)$ be the maximum number of points which can be placed on a closed unit hemisphere in $E^n$ with at least unit distance from one another. Equivalently, $B(n)$ is the maximum number of non-overlapping unit balls that can touch another unit ball at points of a closed hemisphere. Corresponding to this, $B(n)$ is called the one-sided kissing number of $B^n$. Szabó [1991] proved that $B(4) \leq 20$. Based on an extension of the Delsarte method Musin [2006a] lowered this bound to the sharp value 18. Bachoc and Vallentin [2009] proved that $B(8) = 183$ and Dostert, de Laat and Moustrou [2020] proved uniqueness of the arrangement. Clearly, a finite $k^+$-packing of congruent balls in $E^n$ exists only for $k < B(n)$. It appears, however, that the maximum number $k$ for which there exists a finite $k^+$-packing of congruent balls in $E^n$ is considerably less than $B(n)$. Alon [1997] gave an explicit construction of a finite $2\sqrt{n}$-packing of unit balls.

Unit balls centered at the vertices of a triangle, tetrahedron, octahedron, and dodecahedron form a two-, three-, four-, and five-neighbor packing, respectively.
It is easy to see that for \( n \leq 4 \) these are the arrangements of minimal cardinality. G. Fejes Tóth and Harborth [1981] showed that a five-neighbor packing of congruent balls has at least 12 members, and reported a six-neighbor packing consisting of 240 congruent balls constructed by Gerd Wegner (see Figure 6). It is an open question whether a finite seven-neighbor or eight-neighbor packing of congruent balls exists.

It is natural to ask for the minimum density of an \( n \)-neighbor packing of congruent balls for \( n = 10, 11 \) and 12. For \( n = 10 \) and 11 our knowledge is limited. Even the question whether an 11-neighbor packing of congruent balls exists is open. The case \( n = 12 \) is of special interest. L. Fejes Tóth [1969b, 1989] conjectured that if in a packing with congruent balls each ball is touched by exactly twelve other balls, then the packing must consist of parallel hexagonal layers. This long-standing conjecture was verified by Hales [2012b, 2013]. Böröczky and Szabó [2015] gave an alternative proof based on the result of Musin and Tarasov [2012] about the densest packing of 13 spherical caps.

Since the smallest ball in a finite ball-packing has at most twelve neighbors, there is no finite \( n \)-neighbor packing for \( n \geq 12 \). On the other hand, the average number of neighbors in a finite packing of balls can be greater than 12. G. Kuperberg and Schramm [1994] constructed a finite packing of balls in which the average number of neighbors is \( 666/53 = 12.566 \ldots \). A packing of balls with a slightly greater average number of neighbors of \( 7656/607 = 12.612 \ldots \) was given by Eppstein, G. Kuperberg and Ziegler [2003]. G. Kuperberg and Schramm [1994] proved the upper bound \( 8 + 4\sqrt{3} = 14.928 \ldots \) for the average number of neighbors in a packing of balls, which was improved to 13.955 by Glazyrin [2020] and to 13.606 by Dostert, Kolpakov and Oliveira Filho [2020]. The latter two papers also give upper bounds for the average number of neighbors in ball packings in higher dimensions.
K. BEZDEK, CONNELLY and KERTÉSZ [1987] investigated packings of circles of radius $r$ on the sphere and proved that there are positive numbers $\varepsilon$ and $r_0$ such that for $r \leq r_0$ the average number of neighbors in such packings is at most $5 - \varepsilon$.

10. RESULTS ABOUT CONVEX BODIES

TALATA [2002] investigated packings of translates of cylinders in $E^3$ and showed that the maximum $n$ for which there exists a finite $n^+$-neighbor packing of translates of a cylinder $C$ is 10 if $C$ is not a parallelepiped and 13 if $C$ is a parallelepiped. He constructed a $10^+$-neighbor packing of 172 translates of a cylinder if the base is not a parallelogram and a $13^+$-neighbor packing of 382 translates of a parallelepiped.

L. FEJES TÓTH and SAUER [1977] proved that if in a packing of translates of an $n$-dimensional cube, for each cube the total number of its $j$-th neighbors for $0 \leq j \leq k$ is more than $(k + 1)(2k + 1)^{n-1}$, then the packing has positive density. The result is sharp, as shown by the example of two consecutive layers in the grid of cubes in which every cube has a total number of $(k + 1)(2k + 1)^{n-1}$ $j$-th neighbors with $0 \leq j \leq k$. K. BEZDEK and BRASS [2003] generalized the case $k = 1$ of this result for packings by translates of an arbitrary $n$-dimensional convex body.

ZONG [1996] constructed a lattice packing of tetrahedra in which every tetrahedron touches 18 others, conjectured that for a tetrahedron $T H(T) \leq 18$, and proved that $H(T) \leq 19$. Since in a lattice packing each member has an even number of neighbors, it follows that the number of neighbors of a member in a lattice packing of tetrahedra cannot exceed 18. TALATA [1998a] gave a simple alternative proof for this. Later, TALATA [1999a] succeeded in proving Zong’s conjecture about the Hadwiger number of tetrahedra. Moreover, he showed that the packing of 18 translates of a tetrahedron touching a nineteenth one is unique. He also gave a description of all possible packings of 17 translates of a tetrahedron touching an eighteen one. He applied this result for the determination of the minimum and maximum densities of $17^+$-neighbor translative packings of tetrahedra. The Hadwiger number of the octahedron is 18. This was proved independently by ROBINS and SALOWE [1995], TALATA [1999a] and LARMAN and ZONG [1996]. The latter author showed that the Hadwiger number of the rhombic dodecahedron is also 18.

GRÜNBAUM [1961] proved that the Hadwiger number of a convex disk is always attained in a lattice packing. ZONG [1994] showed that this statement does not hold in any dimension $n \geq 3$, namely it fails for a cube truncated at some of its vertices. In the same article, Grünbaum conjectured that the Hadwiger number of every convex body is always even. Disproving the conjecture, JOÓS [2008a] constructed a 3-dimensional convex body whose Hadwiger number is 15.

TALATA [1998a] proved that there is an absolute constant $c > 0$ such that $H(K) \geq 2^cn$ for every $n$-dimensional convex body $K$. In [2000b] TALATA gave the bound $H(S_n) \geq 1.13488^{1-\alpha(1)n}$ for the $n$-dimensional simplex $S_n$, and for strictly convex bodies $K$ he gave in [2005] the following explicit bound: $H(K) \geq \frac{16}{3}\gamma^{(n-1)/2}$. ROBINS and SALOWE [1995], SWANEPOEL [1999], LARMAN and ZONG [1999] and XU [2007] gave lower bounds for the Hadwiger number of superballs.

ZONG [1997] studied the Hadwiger number of Cartesian products of convex bodies $K$ and $L$, and he proved that if $\text{dim} L \leq 2$, then $H(K \times L) = (H(K) + 1)(H(L) + 1) - 1$. TALATA [2005] showed that if the dimension $L$ is higher than 2 the equality does not always hold.
It is clear that the density of a saturated packing of congruent copies of a convex body with a large average number of neighbors cannot be arbitrarily small. Groemer [1961] proved that the density of a saturated packing of translates of a convex body in $E^n$ with average number of neighbors $\mu$ is at least $\frac{1}{\pi - \mu}$.

11. Mutually Touching Translates of a Convex Body

A problem of Erdős [1948] asked for the greatest cardinality of a set of points in $E^n$ with the property that no angle determined by three points is greater than 90°. Another problem posed by Klee asked for the largest antipodal set in $E^n$. Two points of a set are antipodal if there are two parallel supporting hyperplanes of the set, each containing one of the points while the whole set is contained in the closed slab bounded by the supporting planes. A set is antipodal if every pair of its points are antipodal.

Danzer and Grünbaum [1962] proved that the two problems are equivalent and confirmed the conjecture of Erdős that the answer is $2^n$. Moreover, they showed that these problems are also equivalent to the problem of finding the largest family of mutually touching translates of a convex body in $E^n$. The touching number $t(K)$ of a convex body $K$ in $E^n$ is the maximum number of pairwise touching translates of $K$. Thus, the theorem of Danzer and Grünbaum states that $t(K) \leq 2^n$ with equality attained only for a parallelotope.

Károly Bezdek and János Pach (see [2005, p. 98, Conjecture 13]) conjectured that if $C$ is a centrally symmetric convex body, then even a family of pairwise touching homothetic copies of $C$ has at most $2^n$ members. Naszódi [2004] proved the upper bound $2^{n+1}$ without assuming symmetry of $C$ and replacing “homothetic” with “positively-homothetic.” For centrally symmetric bodies Naszódi’s bound was lowered by Lángi and Naszódi [2009] to $3 \times 2^{n-1}$. Földvári [2020] proved that the maximum number of pairwise touching positive homothetic copies of a convex disk is 4.

There is a fourth equivalent formulation of the problem of Erdős and Klee. A subset $S$ of a metric space $M$ is said to be equilateral if all pairs of points in $S$ have the same distance. The maximum number of elements in an equilateral set of $M$ is denoted by $e(M)$. If $K$ is the unit ball of a Minkowski space $M$, then the set of mutually touching translates of $K$ corresponds to a set of equilateral points in $M$, and vice versa. Thus, for an $n$-dimensional Minkowski space $M$, $e(M) \leq 2^n$, as was noted by Petty [1971] and P. S. Soltan [1975].

For $1 \leq p < \infty$ and $n \geq 1$ let $l_p^n$ and $l_\infty^n$ denote $R^n$ endowed with the norm $\|(x_1, \ldots, x_n)\| = (\sum_{i=1}^n |x_i|^p)^{1/p}$ and $\|(x_1, \ldots, x_n)\| = \max_{1 \leq i \leq n} |x_i|$, respectively. In $l_1^n$ the standard basis vectors and their negatives form an equilateral set of $2n$ points, and the set of standard basis vectors together with an appropriate multiple of the all 1 vectors shows that $e(l_1^n) \geq n + 1$. R. B. Kusner (see Guy [1983]) conjectured that both examples are extremal, that is $e(l_2^n) = 2n$ and $e(l_p^n) = n + 1$ for $1 < p < \infty$. The conjecture concerning $l_3^n$ was confirmed for $n = 3$ by Bandelt, Chepoi and Laurent [1998] and for $n = 4$ by Koolen, Laurent and Schrijver [2000]. Thus, the touching number of an octahedron is 6, and the touching number of a cross-polytope in $E^4$ is 8. Besides the Euclidean case $l_2^n$ settled by C. Smith [2001], Kusner’s conjecture was confirmed for $l_4^n$ by Swanepoel [2004a] who also proved that the conjecture is false for all $1 < p < 2$ and sufficiently large $n$, depending on $p$. However, it follows by continuity that, for fixed $n$, if $p$ is close to
2 or 4 then \( e(l^n_2) = n + 1 \) and \( e(l^n_4) = n + 1 \). C. Smith [2001] and Swanepeol [2014] gave explicit bounds for \( p \): If \( |p - 2| < \frac{2 \log(1 + 2/n)}{\log(n + 2)} \) then \( e(l^n_2) = n + 1 \) and if \( |p - 4| < \frac{4 \log(1 + 2/n)}{\log(n + 2)} \) then \( e(l^n_4) = n + 1 \). C. Smith [2013] proved that there exists a constant \( c_p \) such that \( e(l^n_p) \leq c_p n^{(p+1)/(p-1)} \). Extending Smith’s method, Alon and Pudlák [1992a] proved \( e(l^n_p) \leq c_p n^{(2p+1)/(2p-1)} \), and for odd integers \( p \geq 1 \) established the bound \( e(l^n_p) \leq c_p n \log n \).

Petty conjectured that \( e(M) \geq n + 1 \) for every \( n \) dimensional Minkowski space. In other terms the conjecture states, on one hand, that in every \( n \)-dimensional Minkowski space there exists a full-dimensional regular simplex and, on the other hand, that every convex body \( K \) in \( \mathbb{E}^n \) admits a packing of \( n + 1 \) mutually touching translates of \( K \). Petty proved the bound \( e(M) \geq \min \{ 4, n + 1 \} \), confirming the conjecture for \( n = 3 \). Alternative proofs for the 3-dimensional case were given by Kobos [2013] and Väisälä [2012]. This case is well understood: Grünbaum [1963a] proved that \( e(M) \leq 5 \) if the unit ball of \( M \) is strictly convex, and Schürmann and Swanepeol [2006] proved that \( e(M) \leq 6 \) if the unit ball is smooth. The latter authors gave an example of a smooth space \( M \) with \( e(M) = 6 \) and also characterized the 3-dimensional Minkowski spaces that admit equilateral sets of 6 and 7 points. The case \( n = 4 \) of Petty’s conjecture was confirmed by Makeev [2005]. For \( n \geq 5 \) the conjecture is still open, except for special classes of spaces.

The Banach-Mazur distance between two \( n \)-dimensional Minkowski spaces is defined as \( d(X, Y) = \inf \| T \| \| T^{-1} \| \), where the infimum is taken over all linear, invertible operators \( T \) from \( X \) to \( Y \). Let \( M \) be an \( n \)-dimensional Minkowski space. Brass [1990] and Dekster [2000a] proved that if \( d(M, l^n_2) \leq 1 + 1/n \), then \( e(M) \geq n + 1 \). Swanepeol and Villa [2008] verified Petty’s conjecture also for the case that \( d(M, l^n_\infty) \leq 3/2 \), and Averkov [2010] proved that even \( d(M, l^n_\infty) \leq 2 \) guarantees \( e(M) \geq n + 1 \). From these results Swanepeol and Villa [2008] derived the lower bound \( e(M) \geq c \sqrt{\log n} \) using the theorem of Alon and Milman [1983] stating that for every \( \varepsilon > 0 \) there exists a constant \( c(\varepsilon) \) such that any \( n \)-dimensional Minkowski space contains a subspace of dimension at least \( c(\varepsilon) \sqrt{\log n} \) whose Banach-Mazur distance to either \( l^n_2 \) or \( l^n_\infty \) is at most \( \varepsilon \).

González Merino [2020] verified Petty’s conjecture for spaces whose unit ball satisfies certain intersection properties and Kobos [2014] proved it if the unit ball is symmetric in each of the hyperplanes \( x_i = x_j \). Kobos also proved that the conjecture holds for any \( n - 1 \)-dimensional subspace of \( l^n_\infty \). Frankl [2020] gave the following extension of this result: To every integer \( k \geq 2 \) there is a bound \( N(k) \) such that for \( n > N(k) \) any \( k \)-dimensional subspace of \( l^n_\infty \) contains a set of \( k + 1 \) equidistant points. As a corollary she obtained that if a centrally symmetric polytope in \( \mathbb{E}^n \) has at most \( \frac{4}{3}n - \frac{1 + \sqrt{1 + 16n^2}}{6} \) opposite pairs of facets, then there are \( n + 1 \) mutually translates of it. Unfortunately, the difference body of the simplex has more faces, so this does not give a lower bound for the touching number of simplices. Koolen, Laurent and Schrijver [2001] gave examples of \( n + 2 \) mutually touching \( n \)-dimensional simplices. Moreover, Lemmens and Parsons [2015] proved that for \( n \geq 5 \) and \( n \equiv 1 \) (mod 4) the touching number of the \( n \)-dimensional simplex is at least \( n + 3 \).

Further reading about equilateral sets can be found in the papers by Swanepeol [2004d, 2018].
12. Mutually touching cylinders

Littlewood ([1968, Problem 7 on p. 20]) asked the following question: “Is it possible in 3-space for seven infinite circular cylinders of unit radius each to touch all the others?” There are several examples of six cylinders mutually touching each other. All known examples are flexible with one degree of freedom. Bozoki, Lee and Rónyai [2015] answered Littlewood’s question in the affirmative. Fixing the angle of two cylinders the position of the remaining five cylinders can be described by 20 parameters satisfying 20 multivariate polynomial equations. Fixing the position of the initial cylinders perpendicularly two essentially different approximate solutions were found by numerical methods. Having found the approximate solutions, it was proved by Smale’s \(\alpha\)-theory (see Smale [1986]), as well as by interval-arithmetic computations, that the system of equations does indeed have real solution in the neighborhood of the approximate solutions. Further numerical investigation indicated that by fixing the angle of the initial two cylinders between any angle \(0 < \phi < \pi\) gives a one-parametric class of solutions (see the demonstration by Scherer [2014]). The two arrangements found by fixing the angle perpendicularly are special elements of this class. Figure 7 shows seven mutually touching cylinders.

Unaware of Littlewood’s problem Pikhitsa [2004] studied the same question motivated by applications in physics. He also described a configuration of seven mutually touching infinite cylinders numerically, without giving a mathematically rigorous proof for its existence. Pikhitsa and Choi [2014] gave numerical evidence for the existence of nine mutually touching incongruent infinite cylinders.

W. Kuperberg presented a seemingly convincing physical model of eight mutually touching congruent cylinders and asked whether the cylinders really are mutually touching? Ambrus and A. Bezdek [2008] showed that in this model there are two cylinders that do not touch. The question whether there are eight mutually touching congruent cylinders remains unanswered.

Figure 7

A. Bezdek [2005a, 2005b] gave two different proofs of the statement that the number of mutually touching congruent infinite cylinders is bounded. The first proof uses Ramsey’s theorem, and the argument given there shows that the number
of cylinders is bounded even if we allow incongruent cylinders. In the second paper he proved that no more than 24 congruent infinite cylinders can mutually touch. Pikhitsa and Pikhitsa [2017] also proved that the number of mutually touching infinite cylinders of arbitrary base is bounded. Moreover, in [2019] they claimed that no more than 10 infinite cylinders can mutually touch, and gave numerical evidence for the existence of 10 mutually touching elliptic cylinders.

13. Cylinders touching a ball

W. Kuperberg [1990, 2014] asked for the maximum number of unit-radius infinite cylinders touching a unit-radius ball. He conjectured that the number in question is six, which can be realized in several different ways. Heppes and Szabó [1991] gave two different proofs of the upper bound 8 on the number of cylinders. They also discussed the same problem for higher dimensions, and for other radii of the touching cylinders. Brass and Wenk [2000] computed the portion of area cut out by a cylinder touching the unit ball from a concentric sphere of radius \( \sqrt{4.7} \), which came out to be greater than 1/8, showing that the number of touching cylinders is at most 7. While the question whether 7 mutually disjoint infinite cylinders of unit radius can touch the unit ball remains open, it turned out that 6 cylinders of radius \( r > 1 \) can touch it. The first such example with radius \( r = 1.049659 \) was given by Firsching [2015] (see also Firsching [2014]) by a numerical exploration of the corresponding 18-dimensional configuration manifold. Ogievetsky and Shlosman [2019a, 2019b, 2019c, 2021a, 2021b] devoted a series of papers to the study of the configuration space of cylinders touching a ball. They found a packing of 6 infinite cylinders of radius \( r = \frac{1}{8}(3 + \sqrt{33}) \approx 1.093070331 \) touching the unit ball, and believe that this value of the radius is the maximum.

Starostin [2006] investigated tubes touching a ball or another tube.

14. Neighbors in lattice packings

Concerning the number of neighbors in densest lattice packings Swinnerton-Dyer [1953] proved that for every convex body \( K \) in \( E^n \) there is a lattice packing of \( K \) in which every member touches at least \( n(n+1) \) others. M. J. Smith [1975] extended the result of Swinnerton-Dyer to compact sets \( S \) for which \( S - S \) has non-empty interior. On the other hand, Gruber [1986] proved that, in the sense of Baire category, typical convex bodies have at most \( 2n^2 \) neighbors in their densest lattice packings. The ball is not typical: Vlăduţ [2019] constructed for a sequence of dimensions \( n \) lattice ball packings in \( E^n \) in which the balls have \( 2^{0.038n^2 + o(n)} \) neighbors, and in [2021] he constructed sequences of lattice packings of superballs in \( E^n \) with an exponential number of neighbors. The difference between the Hadwiger number and the maximum number \( H_L(K) \) of neighbors of \( K \) in a lattice packing of \( K \) can be large. Talata [1998b] proved that for every \( n \geq 3 \) there exists an \( n \)-dimensional convex body \( K \) such that \( H(K) - H_L(K) \geq 2^{n-1} \).

Groemer [1968c] studied the number of neighbors in connected lattice packings and proved that in a thinnest connected lattice packing of an \( n \)-dimensional convex body, each body has at least \( 2n \) and at most \( 2(2^n - 1) \) neighbors.
References

Alon, N. [1997] Packings with large minimum kissing numbers. Discrete Math. 175 (1997) no. 1-3, 249–251. MR1475852, DOI 10.1016/S0012-365X(97)00071-X

Alon, N. and Milman, V. D. [1983] Embedding of \( l^k_\infty \) in finite-dimensional Banach spaces. Israel J. Math. 45 (1983) 265–280. MR720303, DOI 10.1007/BF02804012

Alon, N. and Pudlák, P. [1992a] Equilateral sets in \( l^n_p \). Geom. Funct. Anal. 13 (2003) no. 3, 467–482. MR1995795, https://doi.org/10.1007/s00039-003-0418-7

Ambrus, G. and Bezdek, A. [2008] On the number of mutually touching cylinders. Is it 8? European J. Combin. 29 (2008) no. 8, 1803–1807. MR2463157

Anstreicher, K. M. [2004] The thirteen spheres: a new proof. Discrete Comput. Geom. 31 (2004) no. 4, 613–625. MR2053501, DOI 10.1007/s00454-003-0819-2

Averkov, G. [2010] On nearly equilateral simplices and nearly \( l^\infty \) spaces. Can. Math. Bull. 53 (2010) no. 3, 394–397. MR2682535, DOI 10.4153/CMB-2010-055-1

Bachoc, Ch. and Vallentin, F. [2009] Semidefinite programming, multivariate orthogonal polynomials, and codes in spherical caps. European J. Combin. 30 (2009) no. 3, 625–637. MR2494437, DOI 10.1016/j.ejc.2008.07.017

Bandelt, H. J.; Chepoi, V. and Laurent, M. [1998] Embedding into rectilinear spaces. Discrete Comput. Geom. 19 (1998) 595–604.

Bannai, E. and Sloane, N. J. A. [1981] Uniqueness of certain spherical codes. Canad. J. Math. 33 (1981) no. 2, 437–449. MR0617634, DOI 10.4153/CJM-1981-038-7

Bárány, I.; Füredi, Z. and Pach, J. [1984] Discrete convex functions and proof of the six circle conjecture of Fejes Tóth. Canad. J. Math. 36 (1984) no. 3, 569–576. MR0752985, DOI 10.4153/CJM-1984-035-1

Bezdek, A. [1997] On the Hadwiger number of a starlike disk. Intuitive geometry (Budapest, 1995) 237—245, Bolyai Soc. Math. Stud., 6, János Bolyai Math. Soc., Budapest, 1997. MR1470761

[2005a] A Ramsey-type bound on the number of mutually touching cylinders. Rev. Roumaine Math. Pures Appl. 50 (2005) no. 5-6, 455–460. MR2204125

[2005b] On the number of mutually touching cylinders. Combinatorial and computational geometry, 121–127, Math. Sci. Res. Inst. Publ., 52, Cambridge Univ. Press, Cambridge, 2005. MR2178316

Bezdek, A. and Bezdek K. [1988] A note on the ten-neighbour packings of equal balls. Beiträge Algebra Geom. 27 (1988) 49–53. MR0984401

Bezdek, A.; Kuperberg, K. and Kuperberg, W. [1995] Mutually contiguous translates of a plane disk. Duke Math. J. 78 (1995) no. 1, 19–31. MR1328750,DOI 10.1215/S0012-7094-95-07802-8

Bezdek, K. [2002] On the maximum number of touching pairs in a finite packing of translates of a convex body. J. Combin. Theory Ser. A 98 (2002) no. 1, 192–200. MR1897933, DOI 10.1006/jcta.2001.3204

[2012a] Contact numbers for congruent sphere packings in Euclidean 3-space. Discrete Comput. Geom. 48 (2012) no. 2, 298–309. MR2946449, DOI 10.1007/s00454-012-9405-9

Bezdek, K. and Brass, P. [2003] On \( k^{+} \)-neighbour packings and one-sided Hadwiger configurations. Beiträge Algebra Geom. 44 (2003) no. 2, 493–498. MR2017050
Bezdek, K. and Connelly, R. and Kertész, G.
[1987] On the average number of neighbors in a spherical packing of congruent circles. Intuitive geometry (Siófok, 1985), 37–52, Colloq. Math. Soc. János Bolyai, 48, North-Holland, Amsterdam, 1987. MR910699

Bezdek, K. and Khan, M. A.
[2018a] Contact numbers for sphere packings. New trends in intuitive geometry, 25–47, Bolyai Soc. Math. Stud., 27, János Bolyai Math. Soc., Budapest, 2018. MR3889255, DOI 10.1007/978-3-662-57413-3_2

Bezdek, K.; Khan, M. A. and Oliwa, M.
[2019] On contact graphs of totally separable domains. Aequationes Math. 93 (2019) no. 4, 757–780. MR3984326, DOI 10.1007/s00010-018-0617-9

Bezdek, K. and Naszódi, M.
[2018b] On contact graphs of totally separable packings in low dimensions. Adv. in Appl. Math. 101 (2018) 266–280. MR3857559, DOI 10.1016/j.aam.2018.08.003

Bezdek, K. and Reid, S.
[2013] Contact graphs of unit sphere packings revisited. J. Geom. 104 (2013) no. 1, 57–83. MR3047448, DOI 10.1007/s00022-013-0156-4

Boju, V. and Funar, L.
[1993] Generalized Hadwiger numbers for symmetric ovals. Proc. Amer. Math. Soc. 119 (1993) no. 3, 931–934. MR1176065, DOI 10.1090/S0002-9939-1993-1176065-5
[2011] Asymptotics of generalized Hadwiger numbers. Studia Sci. Math. Hungar. 48 (2011) no. 1, 44–74. MR2868176, DOI 10.1556/SScMath.2009.1151

Böröczky, K.
[1971] Über die Newtonsche Zahl regulärer Vielecke. Period. Math. Hungar. 1 (1971) no. 2, 113–119. MR0287440
[2003] The Newton-Gregory problem revisited. in: A. Bezdek (Ed.) Discrete Geometry, Monogr. Textbooks Pure Appl. Math., 253, Marcel Dekker, New York, 2003, pp. 103–110. MR2034712

Böröczky, K. and Szabó, L.
[2015] 12-neighbour packings of unit balls in $E^3$. Acta Math. Hungar. 146 (2015) no. 2, 421–448. MR3369506, DOI 10.1007/s10474-015-0527-4

Bowen, L.
[2000] Circle packing in the hyperbolic plane. Math. Phys. Electron. J. 6 (2000) Paper 6, 10 pp. MR1797777, DOI 10.1142/9789812778741_0012

Bozóki, S.; Lee, Tsung-Lin and Rónyai, L.
[2015] Seven mutually touching infinite cylinders. Comput. Geom. 48 (2015) no. 2, 87–93. MR3260249

Brass, P.
[1990] On equilateral simplices in normed spaces. Beiträge Algebra Geom. 40 (1990) no. 2, 303–307. MR1720106
[1996] Erdős distance problems in normed spaces. Comput. Geom. 6 (1996) no. 4, 195–214. MR1392310

Brass, P.; Moser, W. and Pach, J.
[2005] Research problems in discrete geometry. Springer, New York, 2005. xii+499 pp. ISBN: 978-0387-23815-8; 0-387-23815-8. MR2163782

Brass, P. and Wenk, C.
[2000] On the number of cylinders touching a ball. Geom. Dedicata 81 (2000) no. 1-3, 281–284. MR1772209

Cheong, O. and Lee, M.
[2007] The Hadwiger number of Jordan regions is unbounded. Discrete Comput. Geom. 37 (2007) no. 4, 497–501. MR2321737

Chvátal, V.
1975] On a conjecture of Fejes Tóth. Period. Math. Hungar. 6 (1975) no. 4, 357–362. MR0400065, DOI 10.1007/BF02017932

Danzer, L., and Grünbaum, B.
1962 Über zwei Probleme bezüglich konvexer Körper von P. Erdős und von V. L. Klee. Math. Z. 79 (1962) 95–99. MR0138040

Dekster, B. V.
2000a Simplexes with prescribed edge lengths in Minkowski and Banach spaces. Acta Math. Hungar. 86 (2000) no. 4, 343–358. MR1756257, DOI 10.1023/A:1006727810727

Dostert, M., and Kolpakov, A.
2020 Kissing number in non-Euclidean spaces. arXiv:2003.05547v1 [math.CO] 11 Mar 2020.

Dostert, M.; Kolpakov, A., and Oliveira Filho, F. M.
2020 Semidefinite programming bounds for the average kissing number. arXiv:2003.11832v1 [math.MG] 26 Mar 2020

Dostert, M; de Laat, D., and Moustrou P.
2020 Exact semidefinite programming bounds for packing problems. arXiv:2001.00256v1 [math.OC] 1 Jan 2020

Eppstein, D., Kuperber, G. and Ziegler, G.
2003 Fat 4-polytopes and fatter 3-spheres. Discrete geometry, 239–265, Monogr. Textbooks Pure Appl. Math., 253, Dekker, New York, 2003. MR2034720

Erdős, P.
1946 On sets of distances of n points. Amer. Math. Monthly 53 (1946) 248–250. MR0015796, DOI 10.1080/00029890.1946.11991674
1948 Problem 4306. Amer. Math. Monthly 55 (1948) 431. DOI 110.1080/00029890.1948.11999271

Fejes Tóth, G.
1981 Ten-neighbour packing of equal balls. Periodica Math. Hung. 12 (1981) 125–127. MR0603405

Fejes Tóth, G. and Fejes Tóth, L.
1991 Remarks on 5-neighbour packings and coverings with circles. Applied geometry and discrete mathematics, 275–288, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 4, Amer. Math. Soc., Providence, RI, 1991. MR1116354

Fejes Tóth, G. and Harborth, H.
1981 Kugelpackungen mit vorgegebenen Nachbarnzahlen. Studia Sci. Math. Hung. 22 (1981) 79–82. MR0613895

Fejes Tóth, L.
1967d On the number of equal discs that can touch another of the same kind. Studia Sci. Math. Hungar. 2 (1967) 363–367. MR0221388
1969a Über die Nachbarschaft eines Kreises in einer Kreispackung. Studia Sci. Math. Hungar. 4 (1969) 93–97. MR0253163
1969b Remarks on a theorem of R. M. Robinson. Studia Sci. Math. Hungar. 4 (1969) 441–445. MR0254744
1969c Scheibenpackungen konstanter Nachbarszahl. Acta Math. Acad. Sci. Hungar. 20 (1969) 375–381. MR0257887
1970 Über eine affininvariante Maßzahl bei Eipolyedern. Studia Sci. Math. Hungar. 5 (1970) 173–180. MR0268776
1973a Five-neighbour packing of convex discs. Period. Math. Hungar. 4 (1973) 221–229. MR0345006
1975a On Hadwiger numbers and Newton numbers of a convex body. Studia Sci. Math. Hungar. 10 (1975) no. 1–2, 111–115. MR0440469

1977 Research problem No. 21. Periodica Math. Hungar. 8 (1977) 103–104. MR1553600 DML, DOI 10.1007/BF02018053
1989 Research problem no. 44. Period. Math. Hungar. 20 (1989) no. 1, 89–91. MR1553649, DOI 10.1007/BF01849507

Fejes Tóth, L. and Heppes, A.
[1967] A variant of the problem of the thirteen spheres. Canad. J. Math. 19 (1967) 1092–1100. MR0216371

[1977] A remark on the Hadwiger numbers of a convex disc. Studia Sci. Math. Hungar. 12 (1977) no. 3-4, 409–412. MR0607095

FEJES TÓTH, L. and SACHS, H.
[1976] Research problem no. 17. Periodica Math. Hungar. 7 (1976) no. 1, 87–89. MR1553596 DML, DOI 10.1007/BF02019998

FEJES TÓTH, L. and SAUER, N.
[1977] Thinnest packing of cubes with a given number of neighbours. Canad. Math. Bull. 20 (1977) 501-507. MR0478017

FIRSCHING, M.
[2014] How many unit cylinders can touch a unit ball? MathOverflow, 2014. https://mathoverflow.net/q/162003.
[2015] Optimization methods in discrete geometry. PhD dissertation, Freie Universität Berlin, 2015. DOI 10.17169/refubium-11646

FLATLEY, L.; TARASOV, A. TAYLOR, M. and THEIL, F.
[2013] Packing twelve spherical caps to maximize tangencies. J. Comput. Appl. Math. 254 (2013) 220–225. MR3061078

FLORIAN, A.
[1975b] Reguläre hyperbolische Mosaiken und Newtonsche Zahlen. Period. Math. Hungar. 6 (1975) 97–101. MR0370371

FLORIAN, A. and FLORIAN, H.
[1975a] Reguläre hyperbolische Mosaiken und Newtonsche Zahlen. II. Period. Math. Hungar. 6 (1975) no. 2, 179–183. MR0380632
[1975b] Reguläre hyperbolische Mosaiken und Newtonsche Zahlen. III. Österreich. Akad. Wiss. Math.-Naturwiss. Kl. S.-B 184 (1975) no. 1-4, 29–40. MR0420440

FÖLDVÁRI, V.
[2020] Bounds on convex bodies in pairwise intersecting Minkowski arrangement of order $\mu$. J. Geom. (2020) 111:27 DOI 10.1007/s00022-020-00538-3
[2020] Large equilateral sets in subspaces of $l^\infty$ of small codimension. arXiv:2005.04256v1 [math.CO] 8 May 2020

GÁCS, P.
[1972] Packing of convex sets in the plane with a great number of neighbours. Acta Math. Acad. Sci. Hungar. 23 (1972) no. 3, 383–388. MR0328782, DOI 0.1007/BF01896958

GLAZYRIN, A.
[2020] Contact graphs of ball packings. Journal of Combinatorial Theory, Series B 145 (2020) 323–340. MR4131384, DOI 10.1016/j.jctb.2020.05.007
[2020b] A short solution of the kissing number problem in dimension three. arXiv:2012.15058v1 [math.MG] 30 Dec 2020

GONZÁLEZ MERINO, B.
[2020] On large equilateral point-sets in normed spaces. Arch. Math. (Basel) 114 (2020) 553–559. MR4088554, DOI 10.1007/s00013-019-01417-3

GROEMBER, H.
[1961a] Abschätzungen für die Anzahl der konvexen Körper, die einen konvexen Körper berühren. Monatsh. Math. 65 (1961) 74–81. MR0124819, DOI 10.1007/BF01322659
[1961c] Eine Ungleichung für die Dichte von Lagerungen konvexer Körper. Arch. Math. (Basel) 12 (1961) 477–480. MR0140006, DOI 10.1007/BF01650594
[1968b] Einige Bemerkungen über zusammenhängende Lagerungen. Monatsh. Math. 72 (1968) 212–216. MR0227862, DOI 10.1007/BF01362545

GRUBER, P. M.
[1986] Typical convex bodies have surprisingly few neighbours in densest lattice packings. Studia Sci. Math. Hungar. 21 (1986) no. 1-2, 163–173. MR0898853

GRÜNBAUM, B.
GÁBOR FEJES TÓTH, LÁSZLÓ FEJES TÓTH, AND WŁODZIMIERZ KUPERBERG

[1961] On a conjecture of H. Hadwiger. Pacific J. Math. 11 (1961) 215–219. MR0138044, DOI 10.2140/pjm.1961.11.215

[1963a] Strictly antipodal sets. Israel J. Math. 1 (1963) 5–10. MR0159263, DOI 10.1007/BF02759795

GUY, R. K.

[1983] Unsolved problems: An olla-podrida of open problems, often oddly posed. Amer. Math. Monthly 90 (1983) 196–200.

HADWIGER, H.

[1957c] Über Treffanzahlen bei translationsgleichen Eikörpern. Arch. Math. (Basel) 8 (1957) 212–213. MR0091490, DOI 10.1007/BF01899995

HALES, T. C.

[2012b] A proof of Fejes Tóth’s conjecture on sphere packings with kissing number twelve. arXiv:1209.6043v1 [math.MG] 26 Sep 2012.

[2013] The strong dodecahedral conjecture and Fejes Tóth’s conjecture on sphere packings with kissing number twelve. Discrete geometry and optimization, 121–132, Fields Inst. Commun., 69, Springer, New York, 2013. MR3156780

HARBORTH, H.

[1974] Lösung zu Problem 664A. Elem. Math. 29 (1974) 14–15.

[2001] Newton numbers for overlapping circular discs. Studia Sci. Math. Hungar. 37 (2001) no. 1-2, 119–130. MR1834326

[2002] Regular sphere packings. Arch. Math. (Basel) 78 (2002) no. 1, 81–89. MR1887319,DOI 10.1007/s00013-002-8219-z

HEPPES, A. and SZABÓ, L.

[1991] On the number of cylinders touching a ball. Geom. Dedicata 40 (1991) no. 1, 111–116. MR1130481, DOI 10.1007/BF00181656

HORMOBÁGYI, I.

[1972] The Newton number of convex plane regions. (Hungarian) Mat. Lapok 23 (1972) 313–317. MR0362043

[1975] Über die auf Scheibenklassen bezügliche Newtonsche Zahl der konvexen Scheiben. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 18 (1975) 123–127. MR0425775

[1976b] Über die Anzahl der Scheiben konstanter Breite, die eine gegebene Scheibe konstanter Breite berühren. Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 19 (1976) 45–48 (1977). MR0513961

HORVÁTH, H.

[2018] On kissing numbers and spherical codes in high dimensions. Adv. Math. 335 (2018) 307–321. MR3836667,DOI 10.1016/j.aim.2018.07.001

JOÓS, A.

[2008a] On a convex body with odd Hadwiger number. Acta Math. Hungar. 119 (2008) no. 4, 307–321. MR2429292,DOI 10.1007/s10474-008-6032-2

KEMNITZ, A. and MÖLLER, M.

[1997] On the Newton number of rectangles. Intuitive geometry (Budapest, 1995) Bolyai Soc. Math. Stud., 6, János Bolyai Math. Soc., Budapest, 1997, 373–381. MR1470775

[2002] Bounds for the Newton number in the plane. Results Math. 41 (2002) no. 1-2, 128–139. MR1887273, DOI 10.1007/BF03322760

KEMNITZ, A. and SZABÓ, L.

[2001] Relative Newton numbers of regular polygons with equal side lengths. Studia Sci. Math. Hungar. 37 (2001) no. 3-4, 343–354. MR1874689

[2000] Protecting regular polygons. Beiträge Algebra Geom. 41 (2000) no. 2, 391–399. MR1801429

KERTÉSZ, G.

[1994] Nine points on the hemisphere. Intuitive geometry (Szeged, 1991) 189–196, Colloq. Math. Soc. János Bolyai, 63, North-Holland, Amsterdam, 1994. MR1383625
Klamkin, M. S.; Lewis, T. and Liu, A.  [1995] The kissing number of the square. Math. Mag. 68 (1995) no. 2, 128–133. MR1573082

Koros, T.  [2013] An alternative proof of Petty’s theorem on equilateral sets. Ann. Pol. Math. 109 (2013) 165–175. MR3103122, DOI 10.4064/ap109-2-5

[2014] Equilateral dimension of certain classes of normed spaces. Numer. Funct. Anal. Optim. 35 (2014) no. 10, 1340–1358. MR3233155, DOI 10.1080/01630563.2014.930482

Koothen, J.; Laurent, M. and Schrijver, A.  [2000] Equilateral dimension of the rectilinear space. Designs, Codes and Crypt. 21 (2000) 19–164. MR1801196, DOI 10.1023/A:1008391712305

Korkin, A. and Zolotareff, G.  [1872] Sur les formes quadratiques positives quaternaires. Math. Ann. 5 (1872) 581–583. MR1509795 DML, DOI 10.1007/BF01442912

Kuperberg, G. and Schramm, O.  [1994] Average kissing numbers for non-congruent sphere packings. Math. Res. Lett. 1 (1994) no. 3, 339–344. MR1302648, DOI 10.4310/MRL.1994.v1.n3.a5

Kuperberg, K. and Kuperberg, W.  [1994] Translates of a starlike plane region with a common point. Intuitive geometry (Szeged, 1991) 205–126, Colloq. Math. Soc. János Bolyai, 63, North-Holland, Amsterdam, 1994. MR1383627

Kuperberg, W.  [1990] Problem 3.3. DIMACS report on Workshop on Polytopes and Convex Sets, Rutgers University 1990.

[2014] How many unit cylinders can touch a unit ball? MathOverflow, 2014. https://mathoverflow.net/q/156008

Kupitz, Y. S.  [1994] On the maximal number of appearances of the minimal distance among n points in the plane. Intuitive geometry (Szeged, 1991), 217–244, Colloq. Math. Soc. János Bolyai, 63, North-Holland, Amsterdam, 1994. MR1383628

Kusner, R.; Kusner, W.; Lagarias, J. C. and Shlosman, S.  [2018] Configuration spaces of equal spheres touching a given sphere: the twelve spheres problem. New trends in intuitive geometry, 219–277, Bolyai Soc. Math. Stud., 27, János Bolyai Math. Soc., Budapest, 2018. MR3889263, DOI 10.1007/978-3-662-57413-3_10

Lángi, Zs.  [2009] On the Hadwiger numbers of centrally symmetric starlike disks. Beiträge Algebra Geom. 50 (2009) no. 1, 249–257. MR2499792

[2011b] On the Hadwiger numbers of starlike disks. European J. Combin. 32 (2011) no. 8, 1203–1212. MR2838099, DOI 10.1016/j.ejc.2011.05.004

Lángi, Zs. and Naszódi, M.  [2009] On the Bezdek-Pach conjecture for centrally symmetric convex bodies. Canad. Math. Bull. 52 (2009) no. 3, 407–415. MR2547807, DOI 10.4153/CMB-2009-044-8

Larman, D. G. and Zong, C.  [1999] On the kissing number of some special convex bodies. Discrete Comput. Geom. 21 (1999) 233–242. MR1668102, DOI 10.1007/PL00009418

Leech, J.  [1956] The problem of the thirteen spheres. Math. Gaz. 40 (1956) 22–23. MR0076369. MR0076369, DOI 10.2307/3610264

Lemmens, B. and Parsons, C.  [2015] On the number of pairwise touching simplices. Involve 8 (2015) no. 3, 513–520. MR3356091

Levenštejn, V. I.  [1979] Bounds for packings in n-dimensional Euclidean space. (Russian) Dokl. Akad. Nauk SSSR 245 (1979) no. 6, 1299–1303. English translation in Soviet Math. Dokl. 20 (1979) 417–421. MR0529659

Linhart, J.
References

[2019b] Extremal cylinder configurations II: Configuration O_6. Experimental Mathematics (2019) DOI 10.1080/10586458.2019.1641768

[2019c] Platonic compounds of cylinders. arXiv:1904.02043v1 [math.MG] 3 Apr 2019

[2021a] Extremal cylinder configurations I: Configuration C_m. Discrete Comput. Geom. 66 (2021) no. 1, 140–164. MR4270638, DOI 10.1007/s00454-020-00244-6

[2021b] The six cylinders problem: D_3-symmetry approach. Discrete Comput. Geom. 65 (2021) no. 2, 385–404. MR4212969, DOI 10.1007/s00454-019-00064-3

PANKOV, P. S. and DOLMATOV, S. L.

[1977] Use of a computer to find the number of regular pentagons that can simultaneously touch a given one. (Russian) Numerical methods and questions on organization of computations. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 70 (1977) 169–177, 293. Translation in: Journal of Soviet Mathematics 23 (1983) no. 1, 2004–2011. MR0500522, DOI 10.1007/BF01093281

[1979] Substantiable evaluations by electronic computers and their application to one problem in combinatorial geometry. Information Processing Letters 8 (1979) no. 4, 202–203, Zbl 0408.68082, DOI 10.1016/0020-0190(79)90024-3

PETTY, C. M.

[1971] Equilateral sets in Minkowski spaces. Proc. Amer. Math. Soc. 29 (1971) 369–374. MR0275294, DOI 10.1090/S0002-9939-1971-0275294-8

PIKHITSA, P. V.

[2004] Regular network of contacting cylinders with implications for materials with negative poisson ratios. Phys. Rev. Lett. 93 015505 (2004)

PIKHITSA, P. V. and CHOI, M.

[2014] Seven, eight, and nine mutually touching infinitely long straight round cylinders: Entanglement in Euclidean space. arXiv:1312.6207v2 [math.MG] 27 Mar 2014

PIKHITSA, P. V. and PIKHITSA, S.

[2017] Symmetry, topology and the maximum number of mutually pairwise-touching infinite cylinders: configuration classification. R. Soc. Open Sci. 4 (2017) no. 1, January, 160729, 15 pp. MR3621393

[2019] Mutually touching infinite cylinders in the 3D world of lines. Sib. Élektron. Mat. Izv. 16 (2019) 96–120. MR3912027, DOI 10.33048/semi.2019.16.005

REUTER, G.

[1972] Problem 664A, Elem. Math. 27 (1972) 19.

ROBINS, G. and SALOWE, J. S.

[1995] Low-degree minimum spanning trees. Discrete Comput. Geom. 14 (1995) no. 2, 151–165. MR1331924, DOI 10.1007/BF02570700

SACHS, H.

[1986] No more than nine unit balls can touch a closed unit hemisphere. Studia Sci. Math. Hung. 21 (1986) 203–206. MR0898858

SCHERER, K.

[2014] Seven touching cylinders puzzle. http://demonstrations.wolfram.com/SevenTouchingCylindersPuzzle/ Wolfram Demonstrations Project, Published: October 15 2014.

SCHOPP, J.

[1970] Über die Newtonsche Zahl einer Scheibe konstanter Breite. Studia Sci. Math. Hung. 5 (1970) 475–478. MR0285983

SCHÜRMANN, A. and SWANEPOEL, K. J.

[2006] Three-dimensional antipodal and norm-equilateral sets. Pac. J. Math. 228 (2006) no. 2, 349–370. MR2274525, DOI 10.2140/pjm.2006.228.349

SCHÜTTE, K. and VAN DER WAERDEN, B. L.

[1953] Das Problem der dreizehn Kugeln. Math. Ann. 125, (1953) 325–334. MR0053537, DOI 10.1007/BF01343127

SMALE, S.
Newton’s method estimates from data at one point. in Ewing, R. E., Gross, K. I. and Martin, C. F. eds. The Merging of Disciplines: New Directions in Pure, Applied, and Computational Mathematics, 185–196, Springer, New York 1986. MR0870648, DOI 10.1007/978-1-4612-4984-9_13

Smith, C.
[2001] Equilateral or 1-distance sets and Kusner’s conjecture. https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.4755&rep=rep1&type=pdf
[2013] Equilateral sets in $l^d_p$. Thirty essays on geometric graph theory, Springer, New York, 2013. 483–488. MR3205169

Smith, M. J.
[1975] Packing translates of a compact set in Euclidean space. Bull. London Math. Soc. 7 (1975) 129–131. MR0375094, DOI 10.1112/blms/7.2.129

Soltan, P. S.
[1975] Analogues of regular simplexes in normed spaces. Dokl. Akad. Nauk SSSR 222 (1975) no. 6, 1303–1305 (Russian). Translation in Soviet Math. Dokl. 16 (1975) no. 3, 787–789. MR383246

Starostin, E. L.
[2006] On the number of tubes touching a sphere or a tube. Geom. Dedicata 117 (2006) 47–64. MR2231158, DOI 10.1007/s10711-005-9010-7

Swanepoel, K. J.
[1999] New lower bounds for the Hadwiger numbers of $l_p^n$ balls for $p < 2$. Appl. Math. Lett. 12 (1999) no. 5, 57–60. MR1750139, DOI 10.1016/S0893-9659(99)00057-9
[2004a] A problem of Kusner on equilateral sets. Arch. Math. (Basel) 83 (2004) no. 2, 164–170. MR2104945, DOI 10.1007/s00013-003-4840-8
[2004b] Equilateral sets in finite-dimensional normed spaces. Seminar of Mathematical Analysis, 195–237, Colecc. Abierta, 71, Univ. Sevilla Secr. Publ., Seville, 2004. MR2117069
[2014] Equilateral sets and a Schütte theorem for the 4-norm. Canad. Math. Bull. 57 (2014) no. 3, 640–647. MR3239128, DOI 10.4153/CMB-2013-031-0
[2018] Combinatorial distance geometry in normed spaces. New trends in intuitive geometry, 407–458, Bolyai Soc. Math. Stud., 27, János Bolyai Math. Soc., Budapest, 2018. MR3889270, DOI 10.1007/978-3-662-57413-3_17

Swanepoel, K. J. and Villa, R.
[2008] A lower bound for the equilateral number of normed spaces. Proc. Amer. Math. Soc. 136 (2008) no. 1, 127–131. MR2350397, DOI 10.1090/S0002-9939-07-08916-2

Swinnerton-Dyer, H. P. F.
[1953] Extremal lattices of convex bodies. Proc. Cambridge Philos. Soc. 49 (1953) 161–162. MR0051880, DOI 10.1017/S0305004100028188

Szabó, L.
[1991] 21-neighbour packing of equal balls in the 4-dimensional Euclidean space. Geom. Dedicata 38 (1991) no. 2, 193–197. MR1104344, DOI 10.1007/BF00181218

Talata, I.
[1998a] Exponential lower bound for the translatative kissing numbers of $d$-dimensional convex bodies. Dedicated to the memory of Paul Erdős. Discrete Comput. Geom. 19 (1998) no. 3, Special Issue, 447–455. MR1615129, DOI 10.1007/PL00009362
[1998b] On a lemma of Minkowski. Period. Math. Hungar. 36 (1998) no. 2-3, 199–207. MR1694585, DOI 10.1023/A:1004689911234
[1999a] The translatative kissing number of tetrahedra is 18. Discrete Comput. Geom. 22 (1999) no. 2, 231–248. MR1698544, DOI 10.1007/s004930070026
[2000a] A lower bound for the translatative kissing numbers of simplices. Combinatorica 20 (2000) no. 2, 281–293. MR1767027, DOI 10.1007/s004930070026
[1999b] On extensive subsets of convex bodies. Period. Math. Hung. 38 (1999) no. 3, 231–246. MR1756241, DOI 10.1023/a:1004814826820
[2002] On minimum kissing numbers of finite translatative packings of a convex body. Beiträge Algebra Geom. 43 (2002) no. 2, 501–511. MR1957754
[2005] On Hadwiger numbers of direct products of convex bodies. Combinatorial and computational geometry, 517–528, Math. Sci. Res. Inst. Publ., 52, Cambridge Univ. Press, Cambridge, 2005. MR2178337

Xu, L.

[2007] A note on the kissing numbers of superballs. Discrete Comput. Geom. 37 (2007) no. 3, 485–491. MR2301531, DOI 10.1007/s00454-006-1256-9

Youngs, J. W. T.

[1939] A lemma on squares. Amer. Math. Monthly 46 (1939) 20–22. MR1524467, DOI 10.1080/00029890.1939.11990885

Väisälä, J.

[2012] Regular simplices in three-dimensional normed spaces. Beiträge. Algebra Geom. 53 (2012) no. 2, 569–570. MR2971762 DOI 10.1007/s13366-012-0098-2

Vlăduţ, S.

[2019] Lattices with exponentially large kissing numbers. Moscow Journal of Combinatorics and Number Theory 8 (2019) no. 2, 163–177. MR3959884, DOI 10.2140/msc.2019.8.163

[2021] On the lattice Hadwiger number of superballs and some other bodies. Discrete Comput. Geom. 66 (2021) no. 3, 1105–1112. MR4310606, DOI 10.1007/s00454-020-00261-5

Wegner, G.

[1971] Bewegungsstabile Packungen konstanter Nachbarnzahl. Studia Sci. Math. Hungar. 6 (1971) 431–438. MR0295216

[1992] Relative Newton numbers. Monatsh. Math. 114 (1992) no. 2, 149–160. MR1191821, DOI 10.1007/BF01535582

Zhao, Likuan

[1998] The kissing number of the regular polygon. Discrete Math. 188 (1998) 293–296. MR1630435, DOI 10.1016/S0012-365X(98)00027-2

Zhao, Likuan and Xu, Junqin

[2002] The kissing number of the regular pentagon. Discrete Math. 252 (2002) no. 1-3, 293–298. MR1906662, DOI 10.1016/S0012-365X(01)00460-5

Zong, C.

[1994] An example concerning the translative kissing number of a convex body. Discrete Comput. Geom. 12 (1994) no. 2, 183–188. MR1283886, DOI 10.1007/BF02574374

[1996] The kissing numbers of tetrahedra. Discrete Comput. Geom. 15 (1996) no. 3, 239–252. MR1380392, DOI 10.1007/BF02711493

[1997] The translative kissing number of the Cartesian product of two convex bodies, one of which is two-dimensional. Geom. Dedicata 65 (1997) no. 2, 135–145. MR1451968, DOI 10.1023/A:1004968518946

Alfréd Rényi Institute of Mathematics, Reáltanoda u. 13-15., H-1053, Budapest, Hungary

Email address: gfejes@renyi.hu

Department of Mathematics & Statistics, Auburn University, Auburn, AL36849-5310, USA

Email address: kuperwl@auburn.edu