Reconstructing Minkowski Space-Time

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Abstract

Minkowski space is a physically important space-time for which the finding an adequate holographic description is an urgent problem. In this paper we develop further the proposal made in [1] for the description as a duality between Minkowski space-time and a Conformal Field Theory defined on the boundary of the light-cone. We focus on the gravitational aspects of the duality. Specifically, we identify the gravitational holographic data and provide the way Minkowski space-time (understood in more general context as a Ricci-flat space) is reconstructed from the data. In order to avoid the complexity of non-linear Einstein equations we consider linear perturbations and do the analysis for the perturbations. The analysis proceeds in two steps. We first reduce the problem in Minkowski space to an infinite set of field equations on de Sitter space one dimension lower. These equations are quite remarkable: they describe massless and massive gravitons in de Sitter space. In particular, the partially massless graviton appears naturally in this reduction. In the second step we solve the graviton field equations and identify the holographic boundary data. Finally, we consider the asymptotic form of the black hole space-time and identify the way the information about the mass of the static gravitational configuration is encoded in the holographic data.

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1 Introduction

In this paper we continue the study started in [1] of the duality between Minkowski space-time and a Conformal Field Theory defined on the boundary of the light-cone. That semiclassical gravity may know something about quantum field theories was first demonstrated by Brown and Henneaux in 1986 [2] who looked at the algebra of gravitational constraints generating the asymptotic symmetries of three-dimensional anti-de Sitter space-time and found that those constraints form conformal Virasoro algebra with calculable central charge. This was the first indication in the physical literature that asymptotically anti-de Sitter space encodes some non-trivial information about conformal symmetry and quantum anomalies in the space one dimension lower. At approximately same time mathematicians Fefferman and Graham [3] were interested in the purely mathematical problem of finding possible conformal invariants and discovered that all such invariants are naturally induced from the ordinary metric invariants near conformal boundary of the hyperbolic space one dimension higher. The relation between conformal symmetry and the Einstein spaces with negative cosmological constant thus was established. In their analysis they invented a technical tool of asymptotic expansion, the now famous “Fefferman-Graham expansion”, which later on proved to be very useful in the physical applications.

For almost a decade the two sides (physical and mathematical) of the story followed in parallel without making any close contact. This was until the holographic principle [4], [5] was formulated and a concrete realization of this principle, the AdS/CFT correspondence, was suggested [6], [7], [8]. According to the holographic idea the space-time physics of gravitationally interacting particles should be more economically described in terms of some theory living on the boundary (the so-called “holographic screen”). In the AdS/CFT correspondence the bulk space-time is anti-de Sitter space and the theory on the boundary is a quantum conformal field theory. It is possible to formulate precise bulk/boundary dictionary translating the (super)gravity phenomena in the bulk to that of CFT on the boundary and vice versa. This led to many interesting developments. Among others, it was understood that there is a deep relation between geometry of the negative constant curvature space-time and quantum properties of the conformal theories. In particular, the conformal anomalies can be calculated purely geometrically [9] by first expanding the bulk Einstein metric near the conformal boundary and then inserting the expansion back to the gravitational action. The Fefferman-Graham expansion thus made its new appearance and helped to reproduce the Brown-Henneaux central charge in a purely geometrical fashion. The asymptotic diffeomorphisms in anti-de Sitter space play an important role generating the conformal symmetry at the boundary [10] and imposing severe constraints on the possible form of the anomalies [11], [12]. Not only conformal anomalies but also the whole structure of the anomalous stress-tensor of quantum CFT might be possible to extract from the geometry of hyperbolic space [13], [14]. However, with the exception of the three-dimensional case [13] this is still an open problem. An important element in the holographic description is the way how the hologram should be decoded, i.e. how the bulk gravitational physics is restored from the boundary CFT data. In [15] the necessary holographic data were found to be the metric representing the conformal class on the boundary and the boundary CFT stress tensor. The ref.[15] then gives precise prescriptions for how the space-time metric can be reconstructed from these data.
The success of the AdS/CFT duality has motivated the attempts to extend the holographic description to other spaces. It was rather natural to generalize it first to de Sitter space, many elements of this new duality extend straightforwardly from the anti-de Sitter case while many new subtleties arise [16], [17], [18], [19]. One of them is the problem with unitarity since typical conformal weights arising in the duality with de Sitter space are complex. Another is the problem of formulating the S-matrix description in de Sitter space. Both problems are still open although some suggestions have been made [20], [21].

Minkowski space-time is another important space a holographic description of which should be understood. A number of ideas and proposals has been circulated in the literature [22]-[27]. In [1] it was suggested to associate the holographic picture with a choice of light-cone in Minkowski space. The part of the space-time which is out-side the light-cone is naturally foliated with de Sitter slices while the part which is inside is sliced with the Euclidean anti-de Sitter spaces. The only boundary of these slices is the boundary of the light-cone itself which is suggested to be the place where the holographic data should be collected. Formulating the holographic dictionary one can make use of the known prescriptions of the AdS/CFT and dS/CFT dualities applying these prescriptions to each separate slice and then summing over all slices. The details of this procedure have been worked out in [1]. In fact, this line of reasoning follows the inspirational paper [3] where the Euclidean hyperbolic space was considered in the context of the cone structure in flat space one dimension higher. The symmetry plays an important role in the identifying the way the holographic data should be presented. The Lorentz group of (d+2)-dimensional Minkowski space becomes the conformal group acting on d-sphere lying at (past or future) infinity of the light-cone. The data thus are expected to have a CFT representation. In this picture it is natural that the propagating near null infinity plane waves are dually described by an infinite set of the conformal operators living on the d-sphere. Moreover the quantum-mechanical S-matrix can be restored in terms of the correlation functions of operators on two d-spheres: at infinite past and infinite future on the light-cone.

In the present study we extend the picture suggested in [1] and apply it to the gravitational field itself. More specifically, we want to identify the minimal set of data which has to be specified at the boundary of the light-cone and which is sufficient for complete reconstruction of the bulk Minkowski space-time. We understand Minkowski space in a wide sense as a Ricci-flat space-time asymptotically approaching the standard flat space structure. It should be noted that there have been earlier attempts in the literature to proceed in a similar direction [28], [29], [30]. The main idea was to integrate the Einstein equations with zero cosmological constant starting with the boundary at spatial infinity and developing the series expansion in the radial direction in the similar fashion as Fefferman and Graham have taught us to do in the case of equations with negative cosmological constant. This program, however, does not work as nicely for asymptotically flat space as it did for asymptotically adS space. The recurrent relations between terms in the formal series are now differential rather than algebraic as in the adS case. It is not possible to resolve them and express the coefficients in the series in terms of some boundary data at the spatial infinity. The proposal made in [1] is to start at infinity of the light-cone and integrate the equations from there. One has to develop double expansion in this case: first expansion goes along the constant-curvature slice and the second is in the radial direction enumerating the slices. The relations between coefficients are now algebraic. They can be resolved and the necessary boundary data identified. In order to avoid the complexity of non-linear gravitational equations we consider the linear perturbations and
do the analysis for the perturbations. This certainly simplifies the problem and provides us with the important information on the non-linear case as well.

This paper is organized as follows. In section 2 we give some comments on the holographic reconstruction in general emphasizing the role of the causality. We use two-dimensional examples to illustrate our point. In section 3 we review the holographic proposal of [1] in the case of the scalar field. The way the duality works in two-dimensional Rindler space is briefly discussed. We turn to the gravitational case in section 4 and proceed in two steps. First, we reduce the Minkowski problem to an infinite set of gravitational equations on de Sitter space one dimension lower. These equations are quite remarkable: they describe massless and massive gravitons on de Sitter space. In particular the partially massless graviton appears naturally in this reduction. In the second step we solve each graviton field equation on de Sitter space and identify the boundary data. Decoding the hologram we then have to set the rules and translate the boundary data to the bulk gravitational physics. In section 5 we make a step in this direction and consider the asymptotic form of the black hole space-time and identify the holographic data which encode the information about the mass of the static gravitational configuration. We conclude in section 6.

2 Holographic reconstruction as a boundary value problem

We start with some general comments on the holographic reconstruction and show that it can be formulated as a (somewhat unusual) boundary value problem. More specifically it is the problem in which the boundary data are entirely specified on time-like or null-like boundary. To make this discussion concrete and simple we take a particular example of massless field in (1+1)-dimensional space-time. Let’s first consider the flat space-time with coordinates \((t, z)\), the field equation than takes the form

\[
-\partial^2_t \phi + \partial^2_z \phi = 0.
\]

Suppose we consider only a part of the space which lies at positive values of coordinate \(z\). The standard way to formulate the Cauchy problem in this case would be to specify 1) some initial conditions at \(t = 0\), \(\phi(t = 0, z)\) and \(\partial_t \phi(t = 0, z)\); and 2) the boundary conditions: \(\phi(t, z = 0)\) or \(\partial_z \phi(t, z = 0)\). The necessity to have two pieces of data, one on the “initial surface” \(t = 0\) and another on the boundary \(z = 0\), follows from simple causality argument: in order to determine the value of the function \(\phi(t, z)\) at a point \((t, z)\) we have to have data inside the past-directed light-cone with the tip at the point \((t, z)\). For small values of \(t\) the light-cone hits only some part of the “initial surface” and the data on that part is sufficient for the determining the value at the point \((t, z)\). But for large value of \(t\) the light-cone starts to hit the boundary at \(z = 0\) and the additional data should be specified there. Thus, two pieces of the data come out very naturally in this standard formulation.

In the holographic formulation we would like to have only one piece of data, namely data to be fixed on the time-like boundary \(z = 0\), and determine the field \(\phi(t, z)\) for all values of \(-\infty < t < +\infty\) and \(z > 0\) from just this data. Simple causality picture considered above in the case of standard formulation helps to visualize the problem in
this new formulation. We should again draw a light-cone with the tip at the point \((t, z)\) but now directed towards the boundary. The boundary data specified on the part of the boundary which lies inside of this light-cone should be sufficient for determining the field \(\phi\) at the point \((t, z)\). Another words, in order to reconstruct the field at the point \((t, z)\) from the data on the boundary \((z = 0)\) we should be able to communicate with that point by sending a signal and wait long enough to get the signal back from the point \((t, z)\). That’s qualitatively how this sort of reconstruction should work. Of course, in order to reconstruct the field for all points \((t, z)\) we should have at our disposal the all-time boundary data, i.e. for \(-\infty < t < +\infty\).

As for the question what kind of boundary data should be specified the case of equation (2.1) shows us that we have to specify a pair of functions

\[
\phi(t, z = 0) = \psi(t) \quad \text{and} \quad \partial_z \phi(t, z = 0) = \chi(t)
\]  

(2.2)
on the boundary \(z = 0\). This pair forms the holographic data on the boundary. This is typical situation when the holographic data comes in a pair (function itself and its normal derivative) and we will call this a “holographic pair”. The field equation (2.1) subject to the boundary conditions (2.2) can be easily solved and the solution reads

\[
\phi(t, z) = \frac{1}{2} [\psi(t + z) + \psi(t - z)] + \frac{1}{2} \int_{t-z}^{t+z} dt' \chi(t') \quad .
\]  

(2.3)
Thus in order to reconstruct the field at a point \((t, z)\) we have to know boundary value \(\phi(t, z = 0)\) of the field in two moments of time: \(t + z\) and \(t - z\), and the normal derivative \(\partial_z \phi(t, z = 0)\) on the boundary for all moments of time in-between.

We of course could try to find the solution as an expansion in distance from the boundary. The solution then would take the form of sum of two infinite series

\[
\phi(t, z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \psi_t^{(2n)}(t) + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n)!} \chi_t^{(2n)}(t) \quad ,
\]  

(2.4)
where within each series all coefficients are determined by the boundary function \(\psi(t)\), \(\psi_t^{(2n)}(t) = \partial_t^{(2n)} \psi(t)\) and the function \(\chi(t)\), \(\chi_t^{(2n+1)}(t) = \partial_t^{(2n+1)} \chi(t)\). Notice that the causal structure obvious in the complete solution (2.3) is now invisible in the expansion (2.4).

The expansion similar to (2.4) is the standard way to solve the holographic boundary value problem for a field in anti-de Sitter space. The boundary data then are associated with some CFT data on the boundary. Let’s for simplicity consider the two-dimensional anti-de Sitter space with metric

\[
ds^2 = \frac{d\rho^2}{4\rho^2} - \frac{1}{\rho} (1 - \frac{\rho}{4})^2 dt^2
\]

\[
= g(\rho)[-dt^2 + dz^2] \quad ,
\]  

(2.5)
where we introduced coordinate

\[
z(\rho) = \ln(\frac{2 + \sqrt{\rho}}{2 - \sqrt{\rho}})
\]

The anti-de Sitter space has time-like boundary located at \(\rho = 0\) or at \(z = 0\) in terms of the coordinate \(z\). The holographic boundary data thus should be specified there. The
metric (2.5) is conformal to two-dimensional flat space-time, actually to a part of it with \( z \geq 0 \). The massless scalar field in two dimensions is conformally invariant so that the solution to the boundary value problem again takes the form (2.3) where \( z \) should be replaced with \( z(\rho) \). The small \( \rho \) expansion then has two sort of terms

\[
\phi(t, \rho) = [\psi(t) + \sum_{k=1}^{\infty} F_k(t) \rho^k] + \rho^{1/2} [\chi(t) + \sum_{k=1}^{\infty} G_k(t) \rho^k] ,
\]

where \( \psi(t) \) is the boundary value of the field \( \phi \) and \( \chi(t) \) is the normal derivative of the field at the boundary of anti-de Sitter. The coefficients \( F_k(t) \) are completely determined by \( \psi(t) \) and its derivatives while \( G_k(t) \) are determined by \( \chi(t) \). Notice again that the causal structure present in solution (2.3) (with \( z = z(\rho) \)) is lost when we re-write it in the form of the \( \rho \)-expansion. In the adS/CFT correspondence the holographic pair \( (\psi(t), \chi(t)) \) has the following interpretation: \( \psi(t) \) is associated with the source which couples to a “dual” operator \( \mathcal{O}(t) \) while \( \chi(t) \) should be associated with the quantum expectation value of that operator, \( \chi(t) = \langle \mathcal{O}(t) \rangle \). The correlation function of the operators at different moments of time can be derived according to the standard adS/CFT prescription by taking the normal derivatives of the Green’s function

\[
D = -\frac{1}{2\pi} \ln \tanh \frac{\sigma}{2} ,
\]

where \( \sigma \) is the geodesic distance between two points on two-dimensional anti-de Sitter space. The 2-point function on the boundary of \( \text{adS}_2 \) then reads [32]

\[
\langle \mathcal{O}(t)\mathcal{O}(t') \rangle \sim \frac{1}{\sinh^2 \left( \frac{t-t'}{2} \right)} .
\]

The two-dimensional case is of course too simplistic for applying the adS/CFT dictionary in full since the “boundary” in this case is just a point cross the time. However, this is a good illustration since in higher dimensions the logic in identifying the holographic data and recovering the way of reconstructing the bulk physics from that data is essentially the same. In particular the \( \rho \)-expansion similar to (2.6) is the usual tool for analyzing the reconstruction of supergravity in the bulk from the CFT data on the boundary of anti-de Sitter. In the case of the gravitational field itself it is actually the only tool available due to extreme non-linearity of the gravitational field equations [9], [15]. We, however, want to emphasize the role of causality in the holographic reconstruction. This role is not obvious when the local series expansion is used.

It is interesting that the boundary should not be necessarily time-like. The holographic boundary problem can be set for a null-like boundary. To illustrate this let’s once again exploit our two-dimensional example and consider arbitrary two-dimensional metric which always can be brought to a conformally flat form

\[
ds^2 = e^{\sigma(x_+, x_-)} dx_+ dx_- .
\]

Let’s consider the part of the space-time which lies in the conner \( x_- \geq 0, x_+ \leq 0 \), the boundary thus consists of two “null surfaces” \( x_+ = 0 (\mathcal{H}_+) \) and \( x_- = 0 (\mathcal{H}_+) \). As the boundary data we specify the value of the field function

\[
\phi(x_+, x_-)|_{\mathcal{H}_-} = \psi(x_+) \quad \text{and} \quad \phi(x_+, x_-)|_{\mathcal{H}_+} = \chi(x_-)
\]
on these null-surfaces. Since on the intersection of $\mathcal{H}_+$ and $\mathcal{H}_-$ the data should agree we have a constraint, $\psi(x_+ = 0) = \chi(x_- = 0) = \phi_H$. The solution of such formulated boundary value problem for the massless scalar field equation then takes a very simple form

$$\phi(x_+, x_-) = \psi(x_+) + \chi(x_-) - \phi_H.$$  \hfill (2.11)

Thus, in order to reconstruct the field at the point $(u, v)$ in the bulk we have to know the boundary data at three points on the null boundary: at point $x_+ = u$ on $\mathcal{H}_+$, at point $x_- = v$ on $\mathcal{H}_-$ and at the bifurcation point $H$ ($x_+ = x_- = 0$).

A natural example of the null-surface is horizon in black hole space-time or de Sitter space. That horizon can play the role of the holographic screen and there might be a dual CFT living on the horizon with bulk/boundary dictionary similar to the one in the case of adS/CFT correspondence was proposed in [34]. We refer the reader to that paper for further details. Another example when the null-surfaces are natural holographic screens is the Minkowski space-time and the null-screens are the light-cone and null-infinity. This is a possible way of looking at the Minkowski/CFT duality suggested in [1]. We discuss this briefly in the next section.

3 Holographic description in Minkowski space

The holographic construction suggested in [1] is associated with a choice of light-cone. The null-surface of a given light-cone $\mathcal{C}$ naturally splits Minkowski spacetime $\mathcal{M}_{d+2}$ on two regions: the region $\mathcal{A}$ lying inside light-cone $\mathcal{C}$ and the region $\mathcal{D}$ outside light-cone. The inside region $\mathcal{A}$ on the other hand splits on the part which is inside the future light-cone ($\mathcal{A}_+$) and the part which is inside the past light-cone ($\mathcal{A}_-$). Each region admits natural slicing with constant curvature hypersurfaces. Outside the light-cone it is the slicing with $(d+1)$-dimensional de Sitter spaces (which is positive constant curvature spacetime with Lorentz signature) while inside the light-cone one may choose the foliation with Euclidean anti-de Sitter hypersurfaces defined as positive constant curvature space with Euclidean signature. Enumerating the slices we choose the radial coordinate $r$ in the region $\mathcal{D}$ and the time-like coordinate $t$ in the region $\mathcal{A}$. The Minkowski metric then reads

\begin{align*}
\mathcal{D} : \quad & ds^2 = dr^2 + r^2(-dt^2 + \cosh^2 \tau d\omega^2(\theta)) \\ 
\mathcal{A} : \quad & ds^2 = -dt^2 + t^2(dy^2 + \sinh^2 y d\omega^2(\theta)) ,
\end{align*}

where $(\tau, \theta)$ and $(y, \theta)$ are the coordinates on a de Sitter and anti-de Sitter slice respectively, $d\omega^2(\theta)$ is metric on unit radius d-sphere with angle coordinates $\theta$.

Each slice in this foliation of Minkowski spacetime is a $(d+1)$-dimensional space which has some boundaries. In the anti-de Sitter case the boundary is the $d$-dimensional sphere $S_d$ lying at infinity of the space while boundaries of de Sitter space are two spheres $S_d^+$ and $S_d^-$ lying respectively in the future and in the past of de Sitter space. The considered foliation has a nice property that all slices have same boundaries as the light-cone, namely either $S_d^+$ or $S_d^-$. More precisely, all anti-de Sitter slices covering region $\mathcal{A}_-$ inside the light-cone have $S_d^-$ as a boundary while $S_d^+$ is the only boundary of slices covering region $\mathcal{A}_+$. Outside the light-cone, in the region $\mathcal{D}$, all de Sitter slices have same boundary $S_d^+$ in the future and $S_d^-$ in the past. This property motivated the suggestion made in [1] to
associate the holographic information on Minkowski space with these two $d$-dimensional spheres. The Lorentz group of $(d + 2)$-dimensional Minkowski space acts as a conformal group on the spheres $S^+_d$ or $S^-_d$. The holographic information thus is expected to have a conformal field theory description. Some of the details of this holographic description have been demonstrated in [1]. An important element of the description is the specification of necessary information to be stored on the holographic screens as well as the way how the hologram should be decoded, i.e. the rules of reconstruction the bulk Minkowski physics from the holographic data on the spheres.

This can be illustrated on the example of a massless scalar field\(^\dagger\) described by the field equation $\nabla^2 \phi = 0$. Let’s take for concreteness the region $D$ outside the light-cone. Then the field equation reads

$$\left( \partial^2_r + \frac{(d + 1)}{r} \partial_r + \frac{1}{r^2} \nabla^2_{ds} \right) \phi(r, \tau, \theta) = 0,$$

$$\nabla^2_{ds} = -\partial^2_r - d \tanh \tau \partial_\tau + \cosh^{-2} \tau \Delta_\theta,$$

(3.2)

where $\Delta_\theta$ is the Laplace operator on unit radius $d$-sphere. Solving equation (3.2) in the region $D$ we expand the field $\phi(r, \tau, \theta)$ in powers of radial coordinate $r$ so that the solution takes the form of Mellin transform

$$\phi(r, \tau, \theta) = \frac{1}{2\pi i} \int_{\frac{d}{2} - i\infty}^{\frac{d}{2} + i\infty} d\lambda r^{-\lambda} \phi_\lambda(\tau, \theta),$$

(3.3)

where the functions $\phi_\lambda(\tau, \theta)$ satisfy the massive wave equation on $(d + 1)$-dimensional de Sitter space,

$$(\nabla^2_{ds} - m^2_\lambda) \phi_\lambda(\tau, \theta) = 0, \quad m^2_\lambda = \lambda(d - \lambda).$$

(3.4)

The important question is the range for the spectral parameter $\lambda$. A typical configuration in the field theory is a plane wave for which the relevant spectral parameter is complex, $\lambda = \frac{d}{2} + i\alpha$, with $\alpha$ changing from minus to plus infinity. This explains the choice of the limits in the integral (3.3). Later on, the parameter $\lambda$ is identified with the conformal

\(^\dagger\)The consideration is naturally generalized for massive fields and higher spin field equations. The case of Dirac fermions was considered in [33].
weight of the dual operator. Mass term in (3.4) is real and positive, \( m^2 = \frac{d^2}{4} + \alpha^2 \), in this case. Notice that in general the field \( \phi(\tau, r, \theta) \) can be a superposition of propagating modes as well as solitonic configurations. The Coulomb-like potential would be an example of a configuration which requires inclusion in (3.3) of terms with real values of \( \lambda \). Indeed, in \( d + 2 \) space-time dimensions the Coulomb-like configuration

\[
\phi = \frac{Q}{r^{d-1}(\cosh \tau)^{d-1}}
\]

(3.5)
corresponds to \( \lambda = d - 1 \). We will see that similarly we have to include both complex and real \( \lambda \) in the gravitational case in order to describe both the gravitational waves and black holes. For description of the latter the purely real \( \lambda \) are appropriate. Notice also that in the case \( \lambda = \frac{d}{4} \) the two independent solutions to the radial differential equation (3.2) are \( r^{-d/2} \) and \( r^{-d/2} \ln r \).

The representation (3.3) is the first step in the holographic reduction: it reduces the quantum field in Minkowski space to a (infinite) set of massive fields living on de Sitter space of one dimension lower. As we have discussed in the beginning of this section each de Sitter slice has two boundaries, \( S_+^d \) and \( S_-^d \). The next step thus would be to relate each solution of the equation (3.4) to the boundary values of the functions \( \phi_\lambda(\tau, \theta) \). Since on the de Sitter space the boundary value problem is in fact the initial value problem the boundary data should be specified on the surface \( S_+^d \). The general solution of eq.(3.4) is expressed in terms of \( P- \) and \( Q-\)Legendre functions (see Appendix B)

\[
\phi_\lambda(\tau, \theta) = (\cosh \tau)^{\frac{1-d}{2}} [A(\Delta) P^{\frac{\sqrt{(d-1)^2 - 4\lambda}}{2}}_{\frac{(d-1)^2 - 4\lambda}{4}}(-i \sinh \tau) O_\lambda^> \theta (\theta) + B(\Delta) Q^{\frac{\sqrt{(d-1)^2 - 4\lambda}}{2}}_{\frac{(d-1)^2 - 4\lambda}{4}}(-i \sinh \tau) O_\lambda^< \theta (\theta)] ,
\]

(3.6)
where the “constants” \( A(\Delta) \) and \( B(\Delta) \) can be chosen in a way to remove the non-locality in the leading term when expression (3.6) is expanded in powers of \( e^{\tau} \).

The expansion in powers of \( e^{\tau} \) could be used as an alternative way to solve the equation (3.4). The coefficients in front of terms \( e^{\lambda \tau} \) and \( e^{(d-\lambda) \tau} \) in this expansion are not determined from the equation: these are the initial data to be specified on \( S_+^d \). In the form (3.6) of the solution these are the functions \( O_\lambda^> \theta (\theta) \) and \( O_\lambda^< \theta (\theta) \) which form the “boundary” data. Combining this expansion with the integral representation (3.3) we find that to the leading order near the sphere \( S_+^d \) the solution to the field equation (3.2) in Minkowski space reads

\[
\phi(r, \tau, \theta) = \frac{1}{2\pi i} \int_{\frac{d}{2} + i\infty}^{\frac{d}{2} - i\infty} d\lambda (r^{-\lambda} e^{(d-\lambda) \tau}) \left[ O_\lambda^< \theta (\theta) + \sum_{n=1}^{\infty} \varphi_{(n)\lambda}^< \theta e^{2n\tau} \right] + r^{-\lambda} e^{\lambda \tau} \left[ O_\lambda^> \theta (\theta) + \sum_{n=1}^{\infty} \varphi_{(n)\lambda}^> \theta e^{2n\tau} \right] ,
\]

(3.7)
where the higher order terms in the expansion are uniquely determined by \( O_\lambda^< \theta (\theta) \) and \( O_\lambda^> \theta (\theta) \). The coefficients \( O_\lambda^< \theta (\theta) \) and \( O_\lambda^> \theta (\theta) \) are thus the holographic data which are needed to be specified at the sphere \( S_+^d \) for the reconstruction of the scalar field everywhere in the region \( \mathcal{D} \). These functions can be also associated with the left- and right-moving waves. The sphere \( S_+^d \) lies in the intersection of two null-hypersurfaces: the past null infinity \( \mathcal{I}^- \) and the past light-cone \( \mathcal{C}_- \). Therefore, instead of two infinite sets of functions on

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the d-sphere we may consider just two functions defined on null-space of one dimension higher,

\[
\mathcal{O}^>(\xi, \theta) = \frac{1}{2\pi i} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\lambda \xi^{-\lambda} \mathcal{O}^>_{\lambda}(\theta)
\]

\[
\mathcal{O}^<(\eta, \theta) = \frac{1}{2\pi i} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\lambda \eta^{-\lambda} \mathcal{O}^<_{\lambda}(\theta)
\]

(3.8)

where \(\xi (\eta)\) is the affine parameter along the null infinity \(I^- (C_-)\).

Similar analysis can be done for the solution near the sphere \(S^+_d\) in the future of the de Sitter slices. A similar infinite set of functions could be specified there. In the quantum mechanical picture the data on the future sphere \(S^+_d\) can be associated with the quantum out-state while the data specified on \(S^-_d\) form the quantum in-state. In the CFT/Minkowski duality proposed in [1] each coefficient \(\mathcal{O}^>(\lambda)\) appearing in the expansion (3.7) (and in analogous expansion near \(S^+_d\)) is associated with quantum conformal operator of conformal dimension \(\lambda\). The correlation functions of the in- and out-operators are

\[
<0|_{\text{out}} \mathcal{O}^>(\lambda_1)_{\lambda_2} \mathcal{O}^>(\lambda_3)_{\lambda_4}|0> \sim \delta(\lambda_1 + \lambda_2 - d) \delta(\theta_1, \theta_2)
\]

\[
<0|_{\text{out}} \mathcal{O}^<(\lambda_1)_{\lambda_2} \mathcal{O}^<(\lambda_3)_{\lambda_4}|0> \sim \delta(\lambda_1 + \lambda_2 - d) \frac{1}{(1 + \cos \gamma(\theta_1, \theta_2))^\lambda}
\]

(3.9)

where \(\gamma(\theta, \theta')\) is the geodesic distance between two points on d-sphere. The S-matrix of (interacting in the bulk) field than can be reconstructed in terms of the correlation functions between the conformal operators living on \(S^+_d\) and \(S^-_d\), as was shown in [1].

Things are slightly different inside the light-cone, for instance in the region \(\mathcal{A}_+\) inside the past light-cone. One needs to specify only one set of functions, namely \(\mathcal{O}(\lambda)\) there. Or, equivalently, only a single function on the past null-infinity (more precisely, on that component of \(I^-\) which is inside the past light-cone) should be specified. This is quite obvious since \(I^-\) forms a Cauchy surface in the region \(\mathcal{A}_-\). In this region the solution to the Cauchy problem can be brought to a nice integral form [1]

\[
\phi(t, y, \theta) = \frac{1}{2\pi i} \int_{\frac{d}{2}-i\infty}^{\frac{d}{2}+i\infty} d\lambda (-t)^{-\lambda} \int_{S^-_d} d\mu(\theta') G(\lambda, y, \theta, \theta') \mathcal{O}(\theta')
\]

\[
G(\lambda, y, \theta, \theta') = \frac{g(\lambda)}{[\cosh y - \sinh y \cos \gamma(\theta, \theta')]^{\lambda}}
\]

(3.10)

where \(g(\lambda)\) is some normalization factor, \(d\mu(\theta)\) is the measure on d-sphere. \(G(\lambda, y, \theta, \theta')\) has the meaning of boundary-to-bulk propagator on \(AdS_{d+1}\).

**Example: (1+1)-dimensional Minkowski space-time.** As an illustration we consider a simple example of two-dimensional Minkowski space with metric

\[
ds^2 = -dX_0^2 + dX_1^2
\]

(3.11)

The light-cone is defined as \(X_0^2 - X_1^2 = 0\), the boundary “spheres” \(S^+\) and \(S^-\) are now null-dimensional. In two dimensions the region out-side the light-cone has two components: \(\mathcal{D}_L\) and and \(\mathcal{D}_R\) that are analogous to the region \(\mathcal{D}\) in the case of higher-dimensional
Minkowski space-time. The region out-side the light-cone can be foliated with hyperbolic curves which are one-dimensional analog of the de Sitter space-time. The foliation is most transparent in new coordinates \((\tau, r)\) defined as

\[
X_0 = r \sinh \tau, \quad X_1 = \pm r \cosh \tau, \tag{3.12}
\]

where \(+\) stands for region \(D_R\) and \(-\) for region \(D_L\). In this coordinates the metric takes the form

\[
ds^2 = -r^2 d\tau^2 + dr^2 \tag{3.13}
\]

which can be recognized as two-dimensional Rindler metric. Thus in two dimensions our general holographic construction naturally leads to Rindler space. The solution to the field equation takes the form (3.7) with \(d = 0\) inserted and all \(\varphi_{(n)\lambda}\) vanishing. In the present case there are two sets of operators living on the boundary \(S^+\) or \(S^-\) of the light-cone. These operators, \(\mathcal{O}^>(\omega)\) and \(\mathcal{O}^<(\omega)\), are associated with left- and right-moving modes there. Notice that there is no angle dependence since \(S^+\) and \(S^-\) are just points. Alternatively, we could consider operators \(\mathcal{O}^>(\xi)\) and \(\mathcal{O}^<(\eta)\) living on the past null-infinity and the light-cone and defined in (3.8). In this two-dimensional case the reconstruction of the bulk field in terms of the data on these null-surfaces is especially simple and is given by expression similar to (2.11). The correlation functions of the dual operators can be read off from the structure of the two-function on the Minkowski space when each of the functions is approaching one of the boundaries. Let’s restrict our consideration to a simple case of massless field. The Green’s function in this case takes the form

\[
D(X, X') = \frac{1}{4\pi} \ln s^2(X, X') , \tag{3.14}
\]

where the interval \(s^2(X, X')\) between two points in terms of the coordinates \((\tau, r)\) takes the form

\[
s^2 = r^2 + r'^2 - 2rr' \cosh(\tau - \tau') . \tag{3.15}
\]

The propagator (3.14) has a nice representation in terms of the coordinates \((\tau, r)\). In order to get it we first note that the following representation of the logarithmic function [31]

\[
\ln(1 - 2zx + x^2) = -1 - 2 \sum_{n=1}^{\infty} \frac{1}{n} T_n(z)x^n \tag{3.16}
\]

in terms of the Tchebycheff polynomials \(T_n(z) = \cos(n \arccos z)\). Using this representation and replacing the infinite sum in (3.16) with an integral we arrive at another representation for the propagator (3.14)

\[
D = \frac{1}{4\pi} \ln r^2 - \frac{1}{4\pi} - \frac{1}{4\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{d\lambda}{\lambda \sin \pi \lambda} \cosh \lambda(\tau - \tau')(\frac{r'}{r})^\lambda , \tag{3.17}
\]

where we have introduced small \(\epsilon\) in order to avoid the point \(\lambda = 0\) in the integral (3.17). This propagator is in fact a superposition of left- and right-moving modes that is easily seen after the substitution \(\lambda = i\omega\) into (3.17)

\[
D = \frac{1}{2\pi} \ln r - \frac{1}{4\pi} - \frac{1}{8\pi} \int_{-\infty-i\epsilon}^{+\infty-i\epsilon} d\omega K(\omega) \left( e^{i\omega(\tau - \tau')}(\frac{r'}{r})^{i\omega} + e^{-i\omega(\tau - \tau')}(\frac{r'}{r})^{i\omega} \right) , \tag{3.18}
\]
where
\[ K(\omega) = \frac{2}{\omega e^{2\pi \omega} - 1}. \]

From this we find that the correlation function between operators \( \mathcal{O}^> \) and \( \mathcal{O}^< \) reads
\[ \langle \mathcal{O}^>(\omega) \mathcal{O}^<(\omega') \rangle = \frac{2}{\omega e^{2\pi \omega} - 1} \delta(\omega + \omega') . \tag{3.19} \]

Similar expression is valid for “\( \mathcal{O}^> \)” and “\( \mathcal{O}^< \)” correlation functions. It is interesting to note that the function staying in the right hand side in (3.19) is exactly the thermal factor for the Hawking radiation. The latter is expected to appear in the Rindler space, the Rindler horizon should be in fact identified with the light-cone we have chosen. It is quite remarkable that the conformal operators living on the boundaries of the horizon (or the light-cone) carry certain information about the Hawking radiation. This observation is another piece of evidence in favor of the so-called “horizon holography” suggested in [34]. We however do not discuss this in the present paper.

4 Reconstruction of metric: linear perturbation analysis

The holographic construction being applied to the metric of space-time is a somewhat more complicated due to the dual nature of the space-time metric. It defines the dynamics of the gravitational field and also sets the background for other fields. Therefore we should consider a class of metrics which in some sense generalize Minkowski space-time. An appropriate class is that of Ricci flat metrics. The gravitational equations thus take the form
\[ R_{\mu \nu} = 0 . \tag{4.1} \]

These equations are essentially non-linear that makes the analysis more difficult. Additionally to the field equations (4.1) one has to specify the boundary conditions, i.e. the asymptotic conditions which should be satisfied by the metric for describing space-time asymptotically approaching the ordinary Minkowski space. The meaning of the words “asymptotically approaching” should be specified as well. A natural (from the perspective of the holographic picture associated with the boundary of the light-cone) condition formulated in [1] is that the space-time should approach the Minkowski structure at least in a vicinity of a d-sphere lying in the intersection of the null-infinity and the light-cone. A possible way to solve the gravitational equations (4.1) subject to this asymptotic condition is to expand the metric in powers of distance from the sphere (somewhat analogously to what has been done in the case of asymptotically anti-de Sitter space). In fact, in the present case a double expansion is needed: in powers of \( 1/r \) of the inverse radial coordinate enumerating the de Sitter slices close to the sphere \( S^d_\pm \) and in powers of \( e^\tau \) measuring the distance to the sphere along the de Sitter slice. In general this analysis is very complicated. To the leading order in \( r \) we however know that the solution to equation (4.1) takes the form [3]
\[ ds^2 = dr^2 + r^2 \left( g_{ij}(\tau, \theta) + O(1/r) \right) dx^i dx^j , \]
\[ R_{ij}[g] = d g_{ij}(\tau, \theta) , \tag{4.2} \]
where \( \{x^i\} = \{\tau, \theta\} \). So that to the leading order in \( r \) the Ricci flat space-time is foliated with constant positive curvature slices generalizing the structure (3.1) of the ordinary Minkowski space-time. This asymptotic structure is further modified by \( 1/r \) corrections necessarily present in (4.2). The large \( r \) expansion however leads to differential relations between coefficients in the expansion \([28],[29],[30]\). These relations can not be solved in general. On the other hand, expanding the metric \( g_{ij}(\tau, \theta) \) in powers of \( e^\tau \) we would get the standard asymptotic expansion near the boundary of Einstein space of positive constant curvature. This expansion is algebraic and is similar to the well-known expansion for the hyperbolic Einstein space. In most cases this expansion is an infinite series in \( e^\tau \). However, when the slice is 3-dimensional the expansion contains a finite number of terms as was shown in [13]. The boundary \( S^d_2 \) of the slice is two-dimensional and the conformal symmetry is infinite-dimensional in this case. Quite remarkably, it is the case when the asymptotically flat space-time is 4-dimensional. This might be an argument for looking more carefully at the physically interesting 4-dimensional case.

In this paper we take a different route. Instead of dealing with the nonlinearity of the equations (4.1) we look at linear perturbations of (4.1) around Minkowski space with metric (3.1). The holographic analysis then boils down to applying the holographic construction reviewed in the previous section to the equations giving the linear perturbations of (4.1). This is certainly a much simpler problem than solving the non-linear equations but it also tells us on the possible structure of the solution to the non-linear problem and on the necessary holographic data to be specified for the equation (4.1).

Starting the linear perturbation analysis we re-write the Minkowski metric (3.1) in the form

\[
ds^2 = dr^2 + r^2 \gamma_{ij}(x) dx^i dx^j ,
\]

where \( \gamma_{ij}(x) \) is metric on \((d+1)\)-dimensional de Sitter space,

\[
\gamma_{ij}(x) dx^i dx^j = -d\tau^2 + e^{A(\tau)} \beta_{ab}(\theta) \, d\theta^a \, d\theta^b ,
\]

\[
A(\tau) = 2 \ln \cosh \tau ,
\]

and \( \beta_{ab}(\theta) \) is the metric on unite radius d-sphere.

The equation for linear perturbations \( h_{\mu\nu}(r, x) \) takes the form

\[
\nabla^\alpha \nabla_\mu h_{\nu\alpha} + \nabla^\alpha \nabla_\nu h_{\mu\alpha} - \nabla^\alpha \nabla_\alpha h_{\mu\nu} - \nabla_\mu \nabla_\nu \hat{\text{Tr}} h = 0 ,
\]

where \( \hat{\text{Tr}} h \) and covariant derivative \( \nabla_\mu \) are defined with respect to the Minkowski metric (4.3). As in the case of scalar field we further proceed in two steps.

### 4.1 First step: reduction to field equations on de Sitter slice

First we fix the gauge, \( h_{rr} = h_{ri} = 0 \) so that the only non-vanishing components are \( h_{ij}(r, x) \). In particular, we have that \( \hat{\text{Tr}} h = r^{-2} \text{Tr} h \), \( \text{Tr} h = \gamma^{ij} h_{ij} \). The equations (4.5) then reduce to a set of equations

\[
r'^2 \text{Tr} h'' - 2r \text{Tr} h' + 2 \text{Tr} h = 0
\]

\[
r \partial_r(\nabla^j h_{ij} - \partial_i \text{Tr} h) - 2(\nabla^j h_{ij} - \partial_i \text{Tr} h) = 0
\]
\[ \nabla^k \nabla_j h_{jk} + \nabla^k \nabla_j h_{ik} - \nabla^k \nabla_k h_{ij} - \nabla_i \nabla_j \mathrm{Tr} h - 4 h_{ij} - r^2 \partial^2 h_{ij} + (3 - d) r \partial_r h_{ij} - \gamma_{ij} r^3 \partial_r (r^{-2} \mathrm{Tr} h) = 0. \]  \hfill (4.8)

The solution can be taken in the form

\[ h_{ij}(r, x) = \sum_{\lambda} r^{2-\lambda} \chi_{ij}^{(\lambda)}(x), \]  \hfill (4.9)

where the sum (or the integral if appropriate) is taken over all appropriate \( \lambda \). The analysis is in fact different for \( \lambda = 0, \lambda = 1 \) and \( \lambda \neq 0, 1 \). The equation (4.6) is satisfied automatically if \( \lambda = 0 \) or \( \lambda = 1 \) and imposes condition \( \text{Tr} \chi^{(\lambda)} = 0 \) when \( \lambda \neq 0, 1 \). The equation (4.7) is identically satisfied if \( \lambda = 0 \) and otherwise imposes condition

\[ \nabla^i \chi_{ij}^{(\lambda)} - \partial_i \text{Tr} \chi^{(\lambda)} = 0, \quad \lambda \neq 0. \]  \hfill (4.10)

The third equation, (4.8), takes the form

\[ \nabla^k \nabla_i \chi_{jk}^{(\lambda)} + \nabla^k \nabla_j \chi_{ik}^{(\lambda)} - \nabla^k \nabla_k \chi_{ij}^{(\lambda)} - \nabla_i \nabla_j \text{Tr} \chi^{(\lambda)} + (-2d + m_\lambda^2) \chi_{ij}^{(\lambda)} + \lambda \gamma_{ij} \text{Tr} \chi^{(\lambda)} = 0 \]  \hfill (4.11)

for any \( \lambda \), where the mass term \( m_\lambda^2 = \lambda (d - \lambda) \) is defined in the same way as for the scalar field (3.4). Notice, that in (4.9) all terms can be grouped in pairs, \( r^{2-\lambda} \chi_{ij}^{(\lambda)} \) and \( r^{2-(d-\lambda)} \chi_{ij}^{(d-\lambda)} \) corresponding to same mass term \( m^2 = \lambda (d - \lambda) \). These are two independent solutions to the second order differential equation.

Another case which requires a special treatment is when \( m^2 = \frac{d^2}{4} \) and there is only one \( \lambda = \frac{d}{2} \) which is related to this mass and hence only one radial function \( r^{2-\frac{d}{2}} \). Since the second order differential equations should have two independent solutions there must be another solution which is not of the form \( r^{2-\lambda} \). This second solution is \( r^{2-\frac{d}{2}} \ln r \) so that in this case we should search the solution to the gravitational equations in the form

\[ h_{ij} = r^{2-\frac{d}{2}} \left( \chi_{ij}^{(d/2)}(x) + \varphi_{ij}^{(d/2)}(x) \ln r \right). \]  \hfill (4.12)

Inserting this into equations (4.6), (4.7) and (4.8) we find that the fields \( \chi_{ij}^{(d/2)} \) and \( \varphi_{ij}^{(d/2)} \) decouple from each other in the gravitational equations and are thus independent functions. When \( d \neq 2 \) both tensors \( h_{ij}^{(d/2)}(x) \) and \( \varphi_{ij}^{(d/2)}(x) \) are transverse and traceless and satisfy equation (4.11) with \( m^2 = \frac{d^2}{4} \). When \( d = 2 \) the tensor \( \varphi_{ij}^{(1)} \) is still transverse and traceless while the trace of \( \chi_{ij}^{(d/2)} \) is not restricted.

Putting things together, we find that depending on \( \lambda \) the equations reduce to one of the following form

\( \lambda = 0 \):

\[ \nabla^j \chi_{ij}^{(0)} \] and \( \text{Tr} \chi^{(0)} \) are arbitrary;

\[ \nabla^k \nabla_i \chi_{jk}^{(0)} + \nabla^k \nabla_j \chi_{ik}^{(0)} - \nabla^k \nabla_k \chi_{ij}^{(0)} - \nabla_i \nabla_j \text{Tr} \chi^{(0)} - 2d \chi_{ij}^{(0)} = 0 \]  \hfill (4.13)

\( \lambda = 1 \):

\[ \nabla^j \chi_{ij}^{(1)} - \partial_i \text{Tr} \chi^{(1)} = 0 \]  \hfill (4.14)
\[\nabla^k \nabla_i \chi^{(1)}_{jk} + \nabla^k \nabla_j \chi^{(1)}_{ik} - \nabla^k \nabla_k \chi^{(1)}_{ij} - \nabla_i \nabla_j \text{Tr} \chi^{(1)} - (d+1)\chi^{(1)}_{ij} + \gamma_{ij} \text{Tr} \chi^{(1)} = 0 \quad (4.15)\]

\(\lambda \neq 0, 1:\)

\[\nabla^j \chi^{(\lambda)}_{ij} = 0 \quad \text{and} \quad \text{Tr} \chi^{(\lambda)} = 0 \quad , \quad (4.16)\]

\[-\nabla^k \nabla_k \chi^{(\lambda)}_{ij} + (2 + m^2_\lambda) \chi^{(\lambda)}_{ij} = 0 \quad . \quad (4.17)\]

\(\lambda = d/2:\) the solution takes the form (4.12) where depending on value \(d\) we have one of the following possibilities

\[d \neq 2 \quad \text{both} \quad \chi^{(d/2)}_{ij} \quad \text{and} \quad \varphi^{(d/2)}_{ij} \quad \text{satisfy equations (4.16) and (4.17)} \quad \text{with} \quad m^2 = d^2/4.\]

\[d = 2 \quad \text{tensor} \quad \varphi^{(d/2)}_{ij} \quad \text{satisfies equations (4.16) and (4.17)} \quad \text{with} \quad m^2 = 1 \quad \text{while the tensor} \quad \chi^{(d/2)}_{ij} \quad \text{satisfies equations (4.14) and (4.15)} \quad \text{and thus should be identified with} \quad \chi^{(1)}_{ij} \quad \text{when} \quad d = 2.\]

These are equations for \(\chi^{(\lambda)}_{ij}(x)\) considered as some symmetric tensor fields on de Sitter spacetime. It is not difficult to recognize that eq.(4.13) is in fact equation for the massless graviton on de Sitter space. In particular, it is invariant under the usual gauge transformations,

\[\chi^{(0)}_{ij} \rightarrow \chi^{(0)}_{ij} + \nabla_i \xi_j + \nabla_j \xi_i \quad . \quad (4.18)\]

The \(\lambda = 0\) perturbations thus describe deformations of Einstein space with positive constant curvature. This is of course consistent with the asymptotic analysis (4.2). The equations (4.17) for linear perturbations characterized by \(\lambda \neq 0, 1\) on the other hand describe the massive graviton on \((d+1)\)-dimensional de Sitter space. No gauge symmetry is present in this case.

The case \(\lambda = 1\) is a bit trickier. Both equations (4.14) and (4.15) are invariant under gauge transformations

\[\chi^{(1)}_{ij} \rightarrow \chi^{(1)}_{ij} + \nabla_i \nabla_j \xi + \gamma_{ij} \xi \quad (4.19)\]

generated by some scalar function \(\xi(x)\). This signals that equation (4.15) is some field equation which is already known in the literature. Indeed, it is the equation which describes the spin-two partially massless field\(^1\). In the context of adS/CFT and dS/CFT dualities it was considered in [35], [36] and [37]. Note that equation (4.14) arises as a constraint from the field equation (4.15). It is interesting to note that the gauge transformation (4.19) has a natural origin from the Minkowski space perspective. The \((d + 2)\)-dimensional diffeomorphism preserving the form (4.3) of Minkowski metric takes the following form, as was found in [1],

\[\xi^r = \alpha(x) \quad , \quad \xi^i = \frac{1}{r} \gamma^{ij}(x) \partial_j \alpha(x) \quad , \quad (4.20)\]

\(^1\)I thank K. Skenderis for pointing this out to me.
where $\alpha(x)$ is arbitrary function, the linear perturbations $h_{ij}(r, x)$ of Minkowski metric then changes as follows
\[ \delta_{\alpha} h_{ij} = 2r (\nabla_i \nabla_j \alpha + \gamma_{ij} \alpha) \ . \] (4.21)

So that this diffeomorphism acts only on the $\lambda = 1$ component in $h_{ij}(r, x)$ the transformation law for which is identical to (4.19) (after identifying $\xi = 2\alpha$).

Thus, in this first step the gravitational equations on Minkowski space-time reduce to a set of massless and massive graviton field equations on de Sitter space one dimension lower. This is in strict similarity with the scalar field case.

4.2 Second step: solving field equations on de Sitter slice

In the next step we want to solve the equations (4.13), (4.15) and (4.17) for all values of $\lambda$. One way to do it is to develop an expansion in powers of $e^\tau$ starting from the boundary $S_d^-$ on the de Sitter slice. One first takes the linear perturbation in the form
\[ \chi_{ij}^{(\lambda)}(x) = \sum_{\kappa} e^{-(\sigma_{ij} - \kappa)\tau} \chi_{ij}^{(\lambda, \kappa)}(\theta) , \] where $\sigma_{ab} = 2$, $\sigma_{\tau a} = 1$, $\sigma_{\tau \tau} = -2$ and $\chi_{ij}^{(\lambda, \kappa)}(\theta)$ is set of functions on $d$-sphere, and look for certain values of $\kappa$ for which coefficients $\chi_{ij}^{(\lambda, \kappa)}(\theta)$ are not completely determined by previous terms in the expansion, they then form the boundary data to be specified on $S_d^-$. In all cases it is found that appropriate values are
\[ \kappa = \lambda \quad \text{or} \quad \kappa = d - \lambda \ . \]

In fact we can do better than just an expansion - we can solve the gravitational field equations exactly. The details again depend on the value of $\lambda$.

4.2.1 $\lambda = 0$: Massless graviton in de Sitter space

The equation for the perturbations in this case is equation for massless graviton on de Sitter space which has usual gauge freedom (4.18). We choose this freedom to further fix the gauge. Specifically we impose conditions: $\chi_{\tau \tau} = 0$ and $\chi_{\tau a} = 0$, $a = 1, 2, \ldots, d$. So that the only non-vanishing components are $\chi_{ab}(\tau, \theta)$. Then we have that $Tr \chi = e^{-A(\tau)} tr \chi$, $tr \chi = \beta^{ab} \chi_{ab}$. The equations (4.13) then take the form
\[ e^A \partial^2_{\tau} [e^{-A} tr \chi] - A^2 tr \chi + A' \partial_{\tau} tr \chi = 0 \] (4.22)
\[ \partial_{\tau} [\nabla^b \chi_{ba} - \partial_a tr \chi] - A'(\tau) [\nabla^b \chi_{ba} - \partial_a tr \chi] = 0 \] (4.23)
\[ e^{-A} [\nabla^c \nabla_a \chi_{bc} + \nabla^c \nabla_b \chi_{ac} - \nabla^c \nabla_c \chi_{ab} - \nabla_a \nabla_b tr \chi] + \partial^2_{\tau} \chi_{ab} + \chi_{ab}[A^2 - 2d] + \frac{1}{2} \beta_{ab}(A' tr h' - A^2 tr h) = 0 \ , \] (4.24)

where covariant derivative $\nabla_a$ is with respect to metric $\beta_{ab}(\theta)$ on sphere $S_d^-$. Recall that in (4.22), (4.23), (4.24) we have that
\[ e^{A(\tau)} = \cosh^2 \tau \ . \]

§In the sub-sections 4.2.1 and 4.2.2 we drop the subscript $\lambda$ in the components $\chi_{ij}$. We hope this should not cause confusion since value of $\lambda$ is explicitly indicated in the heading of each sub-section.
Equations (4.22) and (4.23) are solved immediately and we find that

\[ \text{tr} \chi = B(\theta) \cosh \tau \sinh \tau + D(\theta) \cosh^2 \tau \]
\[ \nabla^b \chi_{ab} - \nabla_a \text{tr} \chi = C_a(\theta)e^{A(\tau)} , \] (4.25)

where the integration constants \( D(\theta) \) and \( B(\theta) \) are some functions on sphere \( S^{-d}_d \) and \( C_a(\theta) \) is arbitrary vector field on \( S^{-d}_d \). As a consequence of (4.25) we have

\[ \text{tr} \chi' = A' \text{tr} \chi + B \]
\[ \text{tr} \chi'' = (A'' + A'^2) \text{tr} \chi + A' B \] (4.26)

Taking trace of equation (4.24) with respect to metric \( \beta_{ab}(\theta) \) and using (4.25) and (4.26) we find the relation between “integration constants” \( C_a(\theta) \) and \( D(\theta) \):

\[ D(\theta) = \frac{1}{d-1} \nabla^a C_a(\theta) . \] (4.27)

The metric \( \beta_{ab} \) on sphere \( S^{-d}_d \) is maximally symmetric one for which the curvature tensor reads

\[ R_{cabd} = \beta_{cb}(\theta) \beta_{ad} - \beta_{cd}(\theta) \beta_{ba} \]
\[ R_{ab} = (d-1) \beta_{ab} . \] (4.28)

Commuting the covariant derivatives with the help of identity

\[ \nabla_c \nabla_a \chi^c_{ab} - \nabla_a \nabla_c \chi^c_{ab} = d \chi_{ab} - \beta_{ab} \text{tr} \chi \]
we find that equation (4.24) (after substituting (4.25)) can be written in the form

\[ \chi''_{ab} + \frac{(d-4)}{\coth \tau} \chi'_{ab} + \frac{(4-2d)}{\coth^2 \tau} \chi_{ab} - \frac{1}{\cosh^2 \tau} \nabla^2 \chi_{ab} - \frac{1}{\coth \tau} B_{ab} + F_{ab} = 0 , \]
(4.30)

where we define

\[ B_{ab} = \beta_{ab} B - \nabla_a \nabla_b B \]
\[ F_{ab} = \nabla_a C_b + \nabla_b C_a + \nabla_a \nabla_b D - 2 \beta_{ab} D , \]
(4.31)

and \( D \) is defined in (4.27).

As usual the general solution to the inhomogeneous differential equation (4.30) is sum of general solution of homogeneous \( (B_{ab} = F_{ab} = 0) \) equation and a particular solution of the inhomogeneous equation, i.e.

\[ \chi_{ab} = \chi_{ab}^{(\text{hom})} + \chi_{ab}^{(\text{inh})} . \] (4.32)

A solution of the inhomogeneous equation can be easily found and it takes the form

\[ \chi_{ab}^{(\text{inh})} = \chi_{ab}^{(F)} \cosh^2 \tau + \chi_{ab}^{(B)} \sinh \tau \cosh \tau , \] (4.33)

where

\[ \chi_{ab}^{(F)} = \frac{1}{\nabla^2 - 2} F_{ab} , \]
\[ \chi_{ab}^{(B)} = \frac{1}{d - \nabla^2} B_{ab} . \] (4.34)
Since \( \text{tr} \mathcal{F} = (\nabla^2 - 2)D \) and \( \text{tr} \mathcal{B} = (d - \nabla^2)B \) we have that
\[
\text{tr} \chi^{(\text{inh})} = D \cosh^2 \tau + B \sinh \tau \cosh \tau
\]
and hence (taking into account (4.25)) the homogeneous part in (4.32) should be traceless,
\[
\text{tr} \chi^{(\text{hom})} = 0.
\]
Similarly we can analyze the divergence of (4.33). For that we need to know the commutation relation of covariant derivative \( \nabla \) and Laplace type operator \( \nabla^2 = \nabla^a \nabla_a \) as acting on symmetric tensor \( \chi_{ab} \). Useful relation for these purposes is the following
\[
\nabla_b \nabla^2 \chi^{ab} - \nabla_a \nabla^2 \text{tr} \chi = \nabla^2 (\nabla_b \chi^{ab} - \nabla_a \text{tr} \chi) + (d + 1) \nabla_b \chi^{ab} + (d - 3) \nabla_a \text{tr} \chi.
\]
(4.35)
Using this relation (and after some algebra) we find that (4.34) satisfy
\[
(1 - \nabla^2)(\nabla_b \chi^{(B)}_{ab} - \partial_a \text{tr} \chi^{(B)}) = 0
\]
\[
(\nabla^2 + d - 1)(\nabla_b \chi^{(F)}_{ab} - \partial_a \text{tr} \chi^{(F)}) = (\nabla^2 + d - 1)C_a.
\]
(4.36)
Resolving these equations and ignoring the homogeneous part we find that
\[
(\nabla_b \chi^{(B)}_{ab} - \partial_a \text{tr} \chi^{(B)}) = 0
\]
\[
(\nabla_b \chi^{(F)}_{ab} - \partial_a \text{tr} \chi^{(F)}) = C_a(\theta).
\]
(4.37)
Thus the homogeneous part in (4.32) should be transverse and traceless,
\[
\nabla_b \chi^{(\text{hom})}_{ab} = 0, \quad \text{tr} \chi^{(\text{hom})} = 0.
\]
(4.38)
Its exact form can be easily found
\[
\chi^{(\text{hom})}_{ab} = (\cosh \tau)^{\frac{5 - d}{2}} [A_0(\nabla^2)P_{d-1}^{\sqrt{9 - 2d + d^2 - 4\nabla^2}}(\sqrt{1 - \cosh^2 \tau})f_{ab}(\theta)
\]
\[
+ B_0(\nabla^2)Q_{d-1}^{\sqrt{9 - 2d + d^2 - 4\nabla^2}}(\sqrt{1 - \cosh^2 \tau})\psi_{ab}(\theta)]
\]
(4.39)
where \( P_{\nu}(z) \) and \( Q_{\nu}(z) \) are Legendre functions and \( f_{ab}(\theta) \) and \( \psi_{ab}(\theta) \) are any transverse-traceless tensors,
\[
\text{tr} f = \text{tr} \psi = 0, \quad \nabla^a f_{ab} = \nabla^a \psi_{ab} = 0.
\]
(4.40)
These conditions guarantee that the homogeneous part (4.39) of the solution is transverse and traceless. Although the tracelessness is quite obvious the transverseness should be verified. Indeed, using the identity (4.35) for a traceless tensor we find that
\[
\nabla^b \mathcal{P}(\nabla^2)\chi_{ab} = \mathcal{P}(\nabla^2 + d + 1)\nabla^b \chi_{ab}
\]
(4.41)
is valid for any function \( \mathcal{P}(\nabla^2) \) of the Laplace operator \( \nabla^2 \). Applying this relation to (4.39) we find that condition \( \nabla^b \chi^{(\text{hom})}_{ab} = 0 \) is equivalent to conditions \( \nabla^b f_{ab} = \nabla^b \psi_{ab} = 0 \).

The eq.(4.32) where the inhomogeneous part is given by (4.33)-(4.34) and the homogeneous part has the form (4.39) is the general exact solution to the set of gravitational equations (4.22)-(4.24). As it stands expression (4.39) is highly non-local since it contains
a very complicated function of operator $\nabla^2$. However, in the expansion in powers of $e^\tau$ (when $\tau \to -\infty$) few first terms are local.

The “constants” $A_0(\nabla^2)$ and $B_0(\nabla^2)$ in (4.39) can be chosen in a way that expansion in powers of $e^\tau$, or equivalently in powers of new variable $\rho = 4e^{2\tau}$, takes the form

$$\chi_{ab}^{(\text{hom})} = \frac{1}{\rho} \left( f_{ab}^{(0)}(\theta) + O(\rho) \right) + \rho^{d/2} \left( \psi_{ab}^{(d)}(\theta) + O(\rho) \right).$$

Combining this with the analogous expansion for the complete solution (4.32)

$$\chi_{ab} = \frac{1}{\rho} \left( \chi_{ab}^{(0)}(\theta) + O(\rho) \right) + \rho^{d/2} \left( \chi_{ab}^{(d)}(\theta) + O(\rho) \right),$$

where $\chi_{ab}^{(0)}$ has the meaning of deformation of the metric on sphere $S_d^-$ and $\chi_{ab}^{(d)}$ is related to the stress tensor of the dual CFT living on the sphere $S_d^-$, we find that

$$\chi_{ab}^{(0)} = f_{ab}^{(0)}(\theta) + \frac{1}{\nabla^2 - 2} F_{ab} + \frac{1}{\nabla^2 - d} B_{ab},$$
$$\chi_{ab}^{(d)} = \psi_{ab}^{(d)}(\theta).$$

Thus the so far undetermined “integration constants” $C_a(\theta)$ and $B(\theta)$ in (4.25) can be related to the trace and divergence of the deformation $\chi_{ab}^{(0)}$

$$C_a = \nabla^b \chi_{ab}^{(0)} - \partial_a \text{tr} \chi^{(0)},$$
$$B = \frac{1}{(d-1)}(\nabla^a \nabla^b \chi_{ab}^{(0)} - \nabla^2 \text{tr} \chi^{(0)}) - \text{tr} \chi^{(0)}.$$  

The first equation in (4.42) can be viewed as a way to represent arbitrary symmetric tensor $\chi_{ab}^{(0)}$ in terms of its trace, divergence and the transverse-traceless part.

**d=2 case is special.** In this case there takes place the following

**Statement:** If $\chi_{ab}$ is any tensor on two-dimensional manifold such that $\text{tr} \chi = 0$ and $\nabla^b \chi_{ab} = 0$ then

$$\nabla^2 \chi_{ab} = R \chi_{ab},$$

where $R$ is the scalar curvature of the manifold.

This statement can be verified by brut-force calculation. For sphere we have $R = 2$ so that any transverse-traceless tensor in two dimensions is an eigen-function of Laplace operator $\nabla^2$ with the eigen-value 2. This means that we can make a substitution $\nabla^2 = 2$ everywhere in (4.39). The Legendre functions then become trigonometric functions and the homogeneous part of the solution reads

$$\chi_{ab}^{(\text{hom})} = f_{ab} \cosh^2 \tau + \psi_{ab} \cosh \tau \sinh \tau.$$  

Combining this with the inhomogeneous part (4.33) we find that the total solution (4.32) in two dimensions being expressed in terms of variable $\rho$ has only few terms. This is of course consistent with the more general result obtained in [13] that in $d+1=3$ the solution to Einstein equations with nonzero cosmological constant has $\rho$-expansion which terminates on the first three terms. Here we have proven this for the perturbations. The
proof however is rather non-trivial since it was not quite clear from the expression (4.39) how the complicated Legendre functions may reduce to just few exponential terms even after we put \( d=2 \) in (4.39). The above statement was crucial for the demonstration of the consistency.

Summarizing this subsection, the arbitrary symmetric tensor \( \chi^{(0)}_{ab}(\theta) \) describing deformation of the metric structure on \( d \)-sphere and the transverse-traceless tensor \( \psi_{ab}(\theta) \) related to the stress tensor of the dual CFT are the holographic data to be specified on \( d \)-sphere \( S^d \) which completely determine the \((d+2)\)-dimensional Ricci flat metric in the sector \( \lambda = 0 \).

In the holographic pair \( (\chi^{(0)}_{ab}, \psi_{ab}) \) the function \( \chi^{(0)}_{ab} \) represents a source which on boundary \( S^d \) couples to a dual operator represented by \( \psi_{ab}(\theta) \). The coupling then is as follows

\[
\int_{S^d} \chi^{(0)}_{ab} \psi_{ab} .
\]

(4.45)

The gauge invariance (4.18) is usual coordinate invariance on the \( d \)-sphere

\[
\delta_{\xi} \chi^{(0)}_{ab} = \nabla_a \xi_b + \nabla_b \xi_a ,
\]

where \( \xi \) is a vector on \( S^d \), under which (4.45) is supposed to be invariant. This imposes constraint \( \nabla^a \psi_{ab} = 0 \) on the dual operator that also motivates its interpretation as a stress-tensor. This condition is what we also get by solving the massless graviton field equation (see (4.40)).

4.2.2 \( \lambda = 1 \): Partially massless graviton in de Sitter space

The equations for perturbations in this case are collected in Appendix B. As was discussed above these equations describe a partially massless graviton field in de Sitter space of dimension \( d+1 \). This equation has gauge symmetry (4.19). In order to fix the gauge-independent degrees of freedom we may want to impose some gauge conditions. A possible condition to impose is

\[
\text{Tr} \chi = -\chi_{\tau\tau} + e^A \text{tr} \chi = 0 .
\]

It is the gauge suggested in [35]. Another possible way to impose gauge fixing constraint is to demand that

\[
\chi_{\tau\tau} = 0 .
\]

(4.46)

Looking at the transformation for the component \( \chi_{\tau\tau} \),

\[
\delta_{\xi} \chi_{\tau\tau} = \partial^2 \xi - \xi ,
\]

(4.47)

we find that condition (4.46) restricts the gauge parameter \( \xi(\theta, \tau) \) to take the form

\[
\xi = \xi_0(\theta)e^{-\tau} + \xi_2(\theta)e^\tau .
\]

(4.48)

In this sub-section we prefer to use condition (4.46) and will see that field equations are considerably simplified in this gauge. As we can see from (4.48), the condition (4.46) does not fix the components \( \xi^{(0)} \) and \( \xi^{(2)} \) of the gauge parameter so that there still remains some fiducial gauge invariance. In fact this invariance is important and plays the role similar
to the asymptotic conformal symmetry in the case $\lambda = 0$. On $(\tau a)$- and $(a b)$-components of the perturbation the gauge transformation with parameter taking the form (4.48) acts as follows

$$
\delta_\xi \chi_{\tau a} = \frac{1}{\cosh \tau} (\partial_a \xi_0 - \partial_a \xi_2)
$$

$$
\delta_\xi \chi_{a b} = e^{-\tau} [\nabla_a \nabla_b \xi_0 + \frac{1}{2} \beta_{a b} (\xi_0 + \xi_2)] + e^{\tau} [\nabla_a \nabla_b \xi_2 + \frac{1}{2} \beta_{a b} (\xi_2 + \xi_0)] .
$$

(4.49)

In the gauge (4.46) the equations of Appendix C are simplified and can be solved explicitly. Substituting equation (C.1) into (C.3) and recalling that $e^A = \cosh^2 \tau$ we find that $\text{tr} \chi = \beta_{a b} \chi_{a b}$ satisfies a simple differential equation

$$
\text{tr} \chi'' - \text{tr} \chi = 0 ,
$$

(4.50)

the general solution is

$$
\text{tr} \chi = \alpha(\theta) \cosh \tau + \gamma(\theta) \sinh \tau ,
$$

(4.51)

where $\alpha(\theta)$ and $\gamma(\theta)$ are some integration constants. Substituting this back to equations (C.1) and (C.2) we get that

$$
\nabla^a \chi_{a \tau} = \frac{\gamma(\theta)}{\cosh \tau}
$$

(4.52)

and

$$
(\nabla^b \chi_{ba} - \partial_{\tau} \text{tr} \chi) = \cosh^2 \tau (\partial_{\tau} \chi_{\tau a} + d \frac{\sinh \tau}{\cosh \tau} \chi_{\tau a}) .
$$

(4.53)

Taking one more divergence of eq.(4.53) we get

$$
\nabla^a \nabla^b \chi_{a b} - \nabla^2 \text{tr} \chi = (d - 1) \gamma(\theta) \sinh \tau .
$$

(4.54)

The equations (4.51) and (4.54) tell us that in the expansion of the perturbation $\chi_{a b}$ in powers of $e^\tau$ all terms, except the first three terms, are traceless and partially conserved. The partial conservation is very important (see [35]) in the theory of partially massless graviton field and for its relation to a conformal field theory on the boundary. As was discussed in [35] the partial conservation is directly related to the gauge symmetry generated by (4.48). We discuss this point later in the paper. Here we just note that the functions $\alpha(\theta)$ and $\gamma(\theta)$ transform as

$$
\delta_\xi \alpha(\theta) = (\nabla^2 + d)(\xi_0 + \xi_2)
$$

$$
\delta_\xi \gamma(\theta) = \nabla^2 (\xi_2 - \xi_0) .
$$

(4.55)

These functions are the only variables which transform non-trivially under the gauge transformation (4.19).

Next equation to be solved is (C.4). Substituting the gauge condition (4.46), equation (C.2) and explicit expression for $A(\tau)$ we find that this equation takes a simpler form

$$
\chi''_{\tau a} + d \frac{\sinh \tau}{\cosh \tau} \chi'_{\tau a} + [d - 1 - \frac{\nabla^2 - 1}{\cosh^2 \tau}] \chi_{\tau a} + \frac{\partial_a \gamma}{\cosh^3 \tau} = 0 .
$$

(4.56)

\*In paper [35] only the part of transformations which is due to $\xi_0$ was considered.
This equation can be solved explicitly and general solution is a sum of a particular solution of the inhomogeneous equation and general solution of the homogeneous equation,

$$\chi_{\tau a} = \chi^{(\text{inh})}_{\tau a} + \chi^{(\text{hom})}_{\tau a},$$  \hspace{1cm} (4.57)

where we find that

$$\chi^{(\text{inh})}_{\tau a} = \frac{1}{\cosh\tau} \frac{1}{(\nabla^2 - d + 1)} \partial_a \gamma$$  \hspace{1cm} (4.58)

and

$$\chi^{(\text{hom})}_{\tau a} = \left(\cosh\tau\right)^{\frac{1+d}{2}} \left[A(\nabla^2) P_{d-3}^{\frac{\sqrt{5-4d+d^2+4\nabla^2}}{2}} (-i\sinh\tau) J_a(\theta)\right] + B(\nabla^2) Q_{d-3}^{\frac{\sqrt{5-4d+d^2+4\nabla^2}}{2}} (-i\sinh\tau) I_a(\theta)), \hspace{1cm} (4.59)

This solution should satisfy equation (4.52) and thus we get some conditions on the so far arbitrary “constants” $J_a(\theta)$ and $I_a(\theta)$. Using identity (A.5) we show that

$$\nabla^a \chi^{(\text{inh})}_{\tau a} = \frac{\gamma(\theta)}{\cosh\tau}.$$

So that the homogeneous part of the solution should be covariantly conserved, $\nabla^a \chi^{(\text{hom})}_{\tau a} = 0$. Using identity (A.6) we find that the latter condition imposes constraints

$$\nabla^a J_a(\theta) = \nabla^a I_a(\theta) = 0,$$

i.e. $J_a$ and $I_a$ are arbitrary covariantly conserved vectors on $S_d^-$. Choosing $A(\nabla^2)$ and $B(\nabla^2)$ appropriately we find that (4.57) has asymptotic expansion

$$\chi_{\tau a} = 2e^\tau (\chi^{(0)}_{\tau a} + O(e^\tau)) + e^{\tau(d-1)} (I_a(\theta) + O(e^\tau)),$$

where the rest terms in the expansion are completely determined by these two terms and we have that

$$\chi^{(0)}_{\tau a} = J_a + \frac{1}{\nabla^2 - d + 1} \partial_a \gamma, \hspace{1cm} \nabla^a \chi^{(0)}_{\tau a} = \gamma(\theta) \hspace{1cm} (4.63)$$

that is a way to present a vector in terms of its divergence ($\gamma$) divergence-free part ($J_a$). Thus two vectors: arbitrary vector $\chi^{(0)}_{\tau a}$ and divergence-free vector $I_a$ form the first holographic pair at the level $\lambda = 1$.

The only equation left is the equation (C.5) on the components $\chi_{ab}$ of the perturbation. After all substitutions made this equation reads

$$\chi''_{ab} + (d-4) \frac{\sinh\tau}{\cosh\tau} \chi'_{ab} + [3 - d - (\frac{4 - 2d + \nabla^2}{\cosh^2\tau})] \chi_{ab} + 2 \frac{\sinh\tau}{\cosh\tau} [\nabla_a \chi_{\tau a} + \nabla_b \chi_{\tau b}] + \frac{1}{\cosh\tau} \nabla_a \nabla_b \gamma - 2 \beta_{ab} \gamma = 0$$

(4.64)

It is again an inhomogeneous equation, the terms staying in the second line in (4.64) play the role of the source for the differential operator staying in the first line. The solution is again of the familiar form

$$\chi_{ab} = \chi^{(\text{inh})}_{ab} + \chi^{(\text{hom})}_{ab}.$$  \hspace{1cm} (4.65)
where the homogeneous part takes the form

\[
\chi_{ab}^{(\text{hom})} = (\cosh \tau)^{\frac{5-d}{2}} [A_1(\nabla^2)P_{d-3}^2 \sqrt{-2d^2 - 4\nabla^2}] (-i \sinh \tau) k_{ab}(\theta) + B_1(\nabla^2)Q_{d-3}^2 \sqrt{-2d^2 - 4\nabla^2} (-i \sinh \tau) p_{ab}(\theta) \ 
\]  

(4.66)

Several terms contribute to the inhomogeneous part

\[
\chi_{ab}^{(\text{inh})} = \chi_{ab}^{(\alpha)} + \chi_{ab}^{(\gamma)} + \chi_{ab}^{(J)} + \chi_{ab}^{(I)} ,
\]  

(4.67)

where

\[
\chi_{ab}^{(\alpha)} = \frac{1}{\nabla^2 - d}(\nabla_a \nabla_b - \beta_{ab}) \alpha(\theta) \cosh \tau \ 
\]  

(4.68)

\[
\chi_{ab}^{(\gamma)} = \frac{2}{\nabla^2 - 2d + 4} \frac{1}{2} \nabla_a \nabla_b - \beta_{ab} + \nabla_a \frac{1}{\nabla^2 - d + 1} \nabla_b \nabla_b - \beta_{ab} \gamma(\theta) \sinh \tau \ 
\]  

(4.69)

\[
\chi_{ab}^{(J)} = \mathcal{F}^{(J)}(\nabla^2, \tau)(\nabla_a J_b + \nabla_b J_a) \ 
\]

\[
\chi_{ab}^{(I)} = \mathcal{F}^{(I)}(\nabla^2, \tau)(\nabla_a I_b + \nabla_b I_a) ,
\]  

(4.70)

and function \( \mathcal{F}^{(J)}(\nabla^2, \tau) \) (\( \mathcal{F}^{(I)}(\nabla^2, \tau) \)) is a solution to differential equation

\[
\mathcal{F}'' + (d - 4) \frac{\sinh \tau}{\cosh \tau} \mathcal{F}' + (3 - d - \frac{4 - 2d + \nabla^2}{\cosh^2 \tau}) \mathcal{F} + 2 \frac{\sinh \tau}{\cosh \tau} \Phi = 0 \ 
\]  

(4.71)

with

\[
\Phi^{(J)}(\nabla^2, \tau) = (\cosh \tau)^{\frac{1-d}{2}} A(\nabla^2 - d - 1) P_{d-3}^2 \sqrt{-2d^2 - 4\nabla^2} (-i \sinh \tau) \ 
\]

and

\[
\Phi^{(I)}(\nabla^2, \tau) = (\cosh \tau)^{\frac{1-d}{2}} B(\nabla^2 - d - 1) Q_{d-3}^2 \sqrt{-2d^2 - 4\nabla^2} (-i \sinh \tau) .
\]  

Eq.(A.5) was used in deriving \( \Phi^{(I)} \) and \( \Phi^{(J)} \) from (4.59). We do not have a closed-form expression for functions \( \mathcal{F}^{(J)} \) and \( \mathcal{F}^{(I)} \) but the expansion is readily available

\[
\mathcal{F}^{(J)}(\nabla^2, \tau) = e^{\tau} \left[ \frac{1}{4 - d} + O(e^{\tau}) \right] 
\]

\[
\mathcal{F}^{(I)}(\nabla^2, \tau) = e^{\tau(d-1)} \frac{1}{d} + O(e^{\tau}) ,
\]  

(4.72)

where we keep only the leading terms. The dependence on \( \nabla^2 \) appears in the subleading terms. Two cases are special: \( d=2 \) and \( d=4 \). The expansion then should be modified

\[
\mathcal{F}^{(J, I)}(\nabla^2, \tau) = e^{\tau} \tau^2 [1 + O(e^{\tau})] , \quad d = 2 
\]

\[
\mathcal{F}^{(J)}(\nabla^2, \tau) = e^{\tau} \tau [1 + O(e^{\tau})] , \quad d = 4 .
\]  

(4.73)
In terms of variable $\rho$ it involves a logarithm, $\rho \ln \rho$ and $\rho^{1/2} \ln \rho$ respectively. As (4.72) and (4.73) indicate $\chi^{(J)}_{ab}$ and $\chi^{(I)}_{ab}$ contribute in a way that $J_a$ and $I_a$ show up in the subleading terms of the total solution (4.65).

Using identities from Appendix A we can now show that
\[\nabla^a \nabla^b \chi^{(\alpha)}_{ab} - \nabla^2 \text{tr} \chi^{(\alpha)} = 0\]
\[\nabla^a \nabla^b \chi^{(J)}_{ab} = \nabla^a \nabla^b \chi^{(I)}_{ab} = 0\]
\[\nabla^a \nabla^b \chi^{(\gamma)}_{ab} - (\nabla^2 + d - 1) \text{tr} \chi^{(\gamma)} = 0 \quad . \tag{4.74}\]

It indicates that the non-conservation in equation (4.54) is entirely due to the term $\chi^{(\gamma)}_{ab}$. Similarly for the trace we have that
\[\text{tr} \chi^{(\alpha)} = \alpha(\theta) \cosh \tau, \quad \text{tr} \chi^{(\gamma)} = \gamma(\theta) \sinh \tau, \quad \text{tr} \chi^{(I)} = 0 \quad . \tag{4.75}\]

Combining (4.74) and (4.75) with (4.51) and (4.54) we conclude that the homogeneous part of the solution should be traceless and partially conserved, $\text{tr} \chi^{(\text{hom})} = \nabla^a \nabla^b \chi^{(\text{hom})}_{ab} = 0$. This gives conditions
\[\text{tr} k = \text{tr} p = 0 \quad \text{and} \quad \nabla^a \nabla^b k_{ab} = \nabla^a \nabla^b p_{ab} = 0 \tag{4.76}\]

for the integration constants $k_{ab}(\theta)$ and $p_{ab}(\theta)$ in (4.66).

Choosing appropriately $A_1(\nabla^2)$ and $B_1(\nabla^2)$ we find that (4.65) has expansion
\[\chi_{ab} = \frac{1}{2} e^{-\tau}(\chi^{(0)}_{ab}(\theta) + O(e^\tau)) + e^{(d-3)\tau}(p_{ab}(\theta) + O(e^\tau)) \quad , \tag{4.77}\]

where the rest terms in the expansion are determined by these two terms and by the functions $J_a$ and $I_a$ which appear in the terms starting with $e^\tau$ and $e^{(d-1)\tau}$ respectively. The leading term $\chi^{(0)}_{ab}$ in (4.77) has the meaning of boundary value of the perturbation, we have that
\[\chi^{(0)}_{ab} = \frac{1}{\nabla^2 - d}(\nabla_a \nabla_b - \beta_{ab})\alpha + \frac{2}{\nabla^2 - 2d + 4}(\frac{1}{2} \nabla_a \nabla_b - \beta_{ab} + \nabla_a \nabla_b + \nabla_b \nabla_a)\gamma(\theta) + k_{ab} \quad . \tag{4.78}\]

Thus the functions $\alpha$ and $\gamma$ can be related to the trace and the partial non-conservation of the tensor $\chi^{(0)}_{ab}$
\[\alpha(\theta) = 2 \text{tr} \chi^{(0)} + \frac{1}{d - 1}[\nabla^a \nabla^b \chi^{(0)}_{ab} - \nabla^2 \text{tr} \chi^{(0)}] \]
\[\gamma(\theta) = - \frac{1}{d - 1}[\nabla^a \nabla^b \chi^{(0)}_{ab} - \nabla^2 \text{tr} \chi^{(0)}] \quad . \tag{4.79}\]

so that (4.78) is just a way to represent any symmetric tensor in terms of its trace, partial non-conservation and a traceless and partially conserved part.

We did not use yet the equation (4.53). This equation imposes certain relations between the so far independent functions $\chi^{(0)}_{ab}$, $p_{ab}$, $\chi^{(0)}_{\tau a}$ and $I_a$ appearing in the expansions (4.77) and (4.62). Substituting these expansions in the equation (4.53) and comparing terms at the same order of $e^\tau$ on both sides we find the relations
\[\nabla^b \chi^{(0)}_{ab} - \partial_a \text{tr} \chi^{(0)} = (1 - d)\chi^{(0)}_{\tau a} \tag{4.80}\]

and

$$\nabla^b p_{ab} = -\frac{1}{4} I_a . \quad (4.81)$$

Together with (4.57) equations (4.65)-(4.69) give us exact and complete solution to the partially massless graviton field equations on de Sitter space. We are now in the position to determine the holographic field equations on the boundary \( (S^d_d) \) of de Sitter space. Additionally to the pair \((\chi_{(0)}^{(0)}, I_a)\) the two functions \((\chi_{ab}^{(0)}, p_{ab})\) form another holographic pair at the level \(\lambda = 1\). This data is subject to constraints (4.61), (4.76) and (4.80) and (4.81). Notice that the partial conservation is a consequence of relations (4.80) and (4.81) and of condition (4.61). This completes the holographic data at this level.

In the dS/CFT duality in each pair \((\chi_{(0)}^{(0)}, I_a)\) and \((\chi_{ab}^{(0)}, p_{ab})\) the first function should be considered as a source which couples on the boundary \((S^d_d)\) to quantum operator associated with the second function of the pair. The couplings thus take the form

$$\int_{S^d_d} \chi_{(0)}^{(0)} I^a \quad \text{and} \quad \int_{S^d_d} \chi_{ab}^{(0)} p^{ab} .$$

The gauge invariance (4.49) for the source

$$\delta \xi \chi_{(0)}^{(0)} = 2 \partial_a (\xi_2 - \xi_0) \delta \xi \chi_{ab}^{(0)} = (\nabla_a \nabla_b \xi_0 + \frac{1}{2} \beta_{ab} \xi_0) + \frac{1}{2} \beta_{ab} \xi_2$$

then implies that the dual operators should satisfy certain constraints: vector \(I_a(\theta)\) should have vanishing divergence and tensor \(p_{ab}(\theta)\) should be traceless and partially conserved. This is exactly what we see from our solution (see (4.61) and (4.76)). Concluding this sub-section we want to stress that the description dual to the partially massless graviton in de Sitter space does not just contain a tensor operator \(p_{ab}(\theta)\) which is traceless and partially conserved, as it was suggested in [35]. It should contain also a divergence-free vector operator \(I_a(\theta)\) related to the operator \(p_{ab}(\theta)\) according to (4.81).

4.2.3 \(\lambda \neq 0,1\): Massive graviton in de Sitter space

The process of solving the field equations in this case goes pretty much in a similar fashion as before. One of the field equations (or rather constraints) is that the perturbation should be traceless (see second equation in (4.16)). This equation allows to express the component \(\chi_{(0)}^{(0)}\) of the perturbation in terms of the trace of components \(\chi_{ab}^{(0)}\) in the following way

$$\chi_{(0)}^{(0)} = e^{-A(r)} \text{tr} \chi^{(0)} . \quad (4.82)$$

The first equation in (4.16) then gives a pair of equations (where (4.82) has been taken into account)

$$\nabla^a \chi_{(\lambda)}^{(a)} = \partial_r \text{tr} \chi^{(\lambda)} + \frac{(d-1)}{2} A' \text{tr} \chi^{(\lambda)}$$

and

$$\nabla^b \chi_{(\lambda)}^{(b)} = e^A \left( \frac{d}{2} A' \chi_{(\lambda)}^{(a)} + \partial_r \chi_{(\lambda)}^{(a)} \right) . \quad (4.84)$$
The field equations (4.17) are collected in Appendix D. Notice that the \((\tau \tau)\) component of (4.17) is an equation on the trace \(\text{tr} \chi^{(\lambda)}\). With the help of (4.83) this equation takes the form
\[
\partial^2_\tau \text{tr} \chi^{(\lambda)} + \frac{d}{\coth \tau} \partial_\tau \text{tr} \chi^{(\lambda)} + \left[ \lambda (d - \lambda) - \frac{\nabla^2}{\cosh^2 \tau} \right] \text{tr} \chi^{(\lambda)} = 0 \tag{4.85}
\]
and in fact is identical to the scalar field equation (3.4) on de Sitter space considered in section 2. The solution takes the form similar to (3.6)
\[
\text{tr} \chi^{(\lambda)} = (\cosh \tau)^{1-d \over 2} \left[ A^{(0)}_\lambda (\nabla^2) P_{d-1 \over 2}^{(d-1)^2 - 4 \lambda^2} \right] (-i \sinh \tau) f^{(\lambda)}(\theta) + B^{(0)}_\lambda (\nabla^2) Q_{d-1 \over 2}^{(d-1)^2 - 4 \lambda^2} (-i \sinh \tau) g^{(\lambda)}(\theta) \right] , \tag{4.86}
\]
where \(A^{(0)}_\lambda (\nabla^2)\) and \(B^{(0)}_\lambda (\nabla^2)\) are chosen in a way that the asymptotic behavior of (4.86) take the form
\[
\text{tr} \chi^{(\lambda)} = e^{\tau (d-\lambda)} \left( f^{(\lambda)}(\theta) + O(e^\tau) \right) + e^{\tau \lambda} \left( g^{(\lambda)}(\theta) + O(e^\tau) \right) . \tag{4.87}
\]
Respectively using (4.82) we find the asymptotic expansion for the \((\tau \tau)\) components of the perturbation,
\[
\chi^{(\lambda)}_{\tau \tau} = 4 e^{2 \tau} \left( e^{\tau (d-\lambda)} \left( f^{(\lambda)}(\theta) + O(e^\tau) \right) + e^{\tau \lambda} \left( g^{(\lambda)}(\theta) + O(e^\tau) \right) \right) . \tag{4.88}
\]
The \((\tau a)\) component (D.2) of equation (4.17) takes a similar form of inhomogeneous differential equation
\[
\partial^2_\tau \chi^{(\lambda)}_{\tau a} + \frac{d}{\coth \tau} \partial_\tau \chi^{(\lambda)}_{\tau a} + \left[ \lambda (d - \lambda) - \frac{(\nabla^2 - 1)}{\cosh^2 \tau} \right] \chi^{(\lambda)}_{\tau a} + 2 \frac{\sinh \tau}{\cosh^3 \tau} \partial_a \text{tr} \chi^{(\lambda)} = 0 . \tag{4.89}
\]
The solution takes the form of sum of two terms
\[
\chi_{\tau a} = \chi^{(\text{hom})}_{\tau a} + \chi^{(\text{inh})}_{\tau a} , \tag{4.90}
\]
where
\[
\chi^{(\text{hom})}_{\tau a} = (\cosh \tau)^{1-d \over 2} \left[ A^{(1)}_\lambda (\nabla^2) P_{d-1 \over 2}^{(d-1)^2 - 4 \lambda^2} \right] (-i \sinh \tau) J^{(\lambda)}_a(\theta) + B^{(1)}_\lambda (\nabla^2) Q_{d-1 \over 2}^{(d-1)^2 - 4 \lambda^2} (-i \sinh \tau) I^{(\lambda)}_a(\theta) \right] , \tag{4.91}
\]
and \(I^{(\lambda)}_a(\theta)\) and \(J^{(\lambda)}_a(\theta)\) are (at this point) arbitrary vectors on d-sphere. The term \(\chi^{(\text{inh})}_{\tau a}\) in (4.90) is due to the inhomogeneity in (4.89) caused by the term depending on \(\text{tr} \chi^{(\lambda)}\). Although a closed-form expression for \(\chi^{(\text{inh})}_{\tau a}\) may be possible to find what we really need is an expansion of the solution in powers of \(e^\tau\),
\[
\chi^{(\text{inh})}_{\tau a}(\tau, \theta) = 4 e^{2 \tau} \left( e^{\tau (d-\lambda)} \left( \frac{\partial_a f^{(\lambda)}(\theta)}{2 + d - 2 \lambda} + O(e^\tau) \right) + e^{\tau \lambda} \left( \frac{\partial_a g^{(\lambda)}(\theta)}{2 \lambda + 2 - d} + O(e^\tau) \right) \right) . \tag{4.92}
\]
These terms are subleading with respect to those coming from the expansion of (4.91) so that for the total solution (4.90) we have that
\[
\chi^{(\lambda)}_{\tau a} = e^{\tau (d-\lambda)} \left( I^{(\lambda)}_a(\theta) + O(e^\tau) \right) + e^{\tau \lambda} \left( J^{(\lambda)}_a(\theta) + O(e^\tau) \right) . \tag{4.93}
\]
The equation (D.3)

\[ \chi''_{ab} + \frac{(d-4)}{\cosh \tau} \chi'_{ab} + \left[ \lambda (d-\lambda) + 4 - 2d - \left( \frac{4 - 2d + \nabla^2}{\cosh^2 \tau} \right) \right] \chi_{ab} \]

\[ + 2 \frac{\sinh \tau}{\cosh \tau} [\nabla_a \chi_{ra} + \nabla_b \chi_{ra}] - 2 \frac{\sinh^2 \tau}{\cosh^2 \tau} \beta_{ab} \text{ tr } \chi = 0 \quad (4.94) \]

has solution in the form

\[ \chi^{(\lambda)}_{ab} = \chi^{(\text{hom})}_{ab} + \chi^{(\text{inh})}_{ab}, \quad (4.95) \]

where the homogeneous part is

\[ \chi^{(\text{hom})}_{ab} = (\cosh \tau)^{\frac{\lambda - d}{2}} \left[ A^{(2)}_\lambda (\nabla^2) P_{\lambda - \frac{1}{2}}^{-1} - i \sinh \tau \right] f^{(\lambda)}_{ab}(\theta) \]

\[ + B^{(2)}_\lambda (\nabla^2) Q_{\lambda - \frac{1}{2}}^{-1} - i \sinh \tau \psi^{(\lambda)}_{ab}(\theta) \quad (4.96) \]

Its expansion (when \( \tau \to -\infty \)) is

\[ \chi^{(\text{hom})}_{ab} = e^{-2\tau} \left( e^{\tau(d-\lambda)}(\psi^{(\lambda)}_{ab}(\theta) + O(e^\tau)) + e^{\tau \lambda}(f^{(\lambda)}_{ab}(\theta) + O(e^\tau)) \right) \quad (4.97) \]

The expansion of the inhomogeneous part in (4.95) can be obtained by inserting the expansions for tr \( \chi^{(\lambda)} \) (4.87) and \( \chi^{(\lambda)}_{\tau a} \) (4.93) into the equation (4.94) and taking the leading part in the equation. We then obtain

\[ \chi^{(\text{inh})}_{ab} = \frac{1}{2\lambda - d + 2} H^{(\lambda)}_{ab} e^{\lambda \tau} + \frac{1}{d + 2 - 2\lambda} G^{(\lambda)}_{ab} e^{(d-\lambda)\tau}, \quad (4.98) \]

where we keep only the leading terms and defined

\[ H^{(\lambda)}_{ab} = \nabla_a f^{(\lambda)}_b + \nabla_b f^{(\lambda)}_a + \beta_{ab} g^{(\lambda)} \]

\[ G^{(\lambda)}_{ab} = \nabla_a I^{(\lambda)}_b + \nabla_b I^{(\lambda)}_a + \beta_{ab} f^{(\lambda)} \]

Obviously (4.98) is subleading with respect to (4.97). Thus to the leading order the \((ab)\) components of the perturbation behave as

\[ \chi^{(\lambda)}_{ab} = e^{-2\tau}(e^{\tau(d-\lambda)}\psi^{(\lambda)}_{ab}(\theta) + e^{\tau \lambda}(f^{(\lambda)}_{ab}(\theta)) \quad (4.99) \]

At this point we see that the complete solution to the set of equations (4.82), (D.1)-(D.3) is characterized by the set of tensors \( f^{(\lambda)}_{ab}, \psi^{(\lambda)}_{ab}, J^{(\lambda)}_a, I^{(\lambda)}_a, g^{(\lambda)} \) and \( f^{(\lambda)} \) defined on the sphere \( S^d \). Now we have to take into account the equations (4.83) and (4.84). This will impose certain relations between these functions. Indeed, substituting expansions (4.87), (4.93) and (4.97) into (4.83) and (4.84) and looking at the leading order we get the relations

\[ \nabla^a J^{(\lambda)}_a = (\lambda - d + 1) g^{(\lambda)} \quad \text{and} \quad \nabla^a f^{(\lambda)}_a = (1 - \lambda) f^{(\lambda)} \quad (4.100) \]

and

\[ \nabla^b f^{(\lambda)}_{ab} = \frac{\lambda - d}{4} J^{(\lambda)}_a \quad \text{and} \quad \nabla^b \psi^{(\lambda)}_{ab} = -\frac{\lambda}{4} I^{(\lambda)}_a. \quad (4.101) \]
More relations come from the consistency condition that trace of (4.95) should coincide with (4.86). Comparing the leading terms in the expansions (4.87) and (4.99) we find that \(\psi^{(\lambda)}_{ab}\) and \(f^{(\lambda)}_{ab}\) should be traceless,

\[
\text{tr} \psi^{(\lambda)} = \text{tr} f^{(\lambda)} = 0 .
\]

This also means that the whole homogeneous part (4.96) should be traceless so that the non-vanishing trace \(\text{tr} \chi^{(\lambda)}\) (4.86) is entirely due to the inhomogeneous part in (4.95). To the leading order it can be checked directly (using relations (4.100) and expansions (4.98) and (4.87)).

Several remarks are in order. First, it should be noted that all holographic data satisfying the relations (4.100) and (4.101) can be grouped in pairs: the one which corresponds to \(\lambda\) and another one which corresponds to \((d - \lambda)\). Interestingly it can be extended to include the cases \(\lambda = 0\) and \(\lambda = 1\) which are not of the massive graviton case. Then, the pair to the massless graviton \(\lambda = 0\) is the massive graviton with \(\lambda = d\) and the pair to partially massless graviton \(\lambda = 1\) is the massive graviton with \(\lambda = d - 1\). Of course, these pairs are just two independent solutions to the second order radial differential equations discussed in section 4.1. Second, a somewhat degenerate case is \(\lambda = \frac{d}{2}\). Then the two independent asymptotic solutions considered in this sub-section should be \(e^{\tau \frac{d}{2}}\) and \(e^{\tau \frac{d}{2}} \tau\).

We do not consider in detail this case. Finally, we note that the asymptotic form (4.99) multiplied by the radial function \(r^{2-\lambda}\) with complex \(\lambda = \frac{d}{2} + i\alpha\) is exactly what one would have expected for the representation of plane gravitational waves and is similar to the representation (3.7) for the plane waves in the case of the scalar field.

### 4.3 Summary

Let us summarize our rather long analysis. The metric perturbation over Minkowski space has been represented in the form

\[
h_{ij}(r, \tau, \theta) = \sum_{(\lambda)} r^{2-\lambda} \chi^{(\lambda)}_{ij}(\tau, \theta) + r^{2-\frac{d}{2}} \left( \chi^{(d/2)}_{ij}(x) + \varphi^{(d/2)}_{ij}(x) \ln r \right) ,
\]

where sum over \(\lambda\) may contain also integral as it happens when \(\lambda = \frac{d}{2} + i\alpha\) with continuous \(\alpha\). Also, the case of \(\lambda = \frac{d}{2}\) is special and involves a logarithm in the representation for the perturbation and is written in the above expression explicitly.

\(\lambda = 0\): components \(\chi_{\tau\tau}\) and \(\chi_{\tau a}\) vanish by the gauge conditions; the asymptotic behavior for \((ab)\) components is

\[
\chi_{ab} = e^{-2\tau} \left[ (\chi_{ab}^{(0)} + \ldots) + (e^{(d-1)\tau} \psi_{ab} + \ldots) \right] ,
\]

where \(\chi_{ab}^{(0)}(\theta)\) is arbitrary and has the meaning of deformation of the metric structure on d-sphere; \(\psi_{ab}(\theta)\) should satisfy conditions

\[
\text{tr} \psi = 0 \quad \text{and} \quad \nabla^b \psi_{ab} = 0 .
\]

\(\lambda = 1\): component \(\chi_{\tau\tau} = 0\) by gauge fixing; the asymptotic behavior of the non-vanishing components

\[
\chi_{\tau a} = 2e^{\tau} (\chi_{\tau a}^{(0)}(\theta) + \ldots) + e^{\tau(d-1)} (I_a(\theta) + \ldots)
\]

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\[ \chi_{ab} = \frac{1}{2} e^{-\tau} \left[ (\chi_{ab}^{(2)}(\theta) + \ldots) + e^{(d-3)\tau}(p_{ab}(\theta) + \ldots) \right], \]

where the following conditions should be satisfied

\[ \nabla^a I_a = 0 \quad \text{and} \quad \text{tr} \, p = 0, \quad \nabla^a \nabla^b p_{ab} = 0 \]

as well as relations

\[ \nabla^b \chi_{ab}(0) - \partial_t \text{tr} \chi(0) = (1 - d) \chi_{\tau a}(0) \quad \text{and} \quad \nabla^b p_{ab} = -\frac{1}{4} I_a. \]

\[ \lambda \neq 0,1: \text{all components are non-vanishing, the asymptotic behavior is as follows} \]

\[ \chi^{(\lambda)}_{\tau\tau} = 4 e^{2\tau} [ e^{\tau(d-\lambda)}(f^{(\lambda)}(\theta) + \ldots) + e^{\tau\lambda}(g^{(\lambda)}(\theta) + \ldots) ] \]

\[ \chi^{(\lambda)}_{\tau a} = e^{\tau(d-\lambda)}[(f^{(\lambda)}_a(\theta) + \ldots) + e^{\tau\lambda}(J^{(\lambda)}_a(\theta) + \ldots)] \]

\[ \chi^{(\lambda)}_{ab} = e^{-2\tau} e^{\tau(d-\lambda)}(\psi^{(\lambda)}_{ab}(\theta) + \ldots) + e^{\tau\lambda}(f^{(\lambda)}_{ab}(\theta) + \ldots) \]

with constraints and relations

\[ \text{tr} \, \psi^{(\lambda)} = \text{tr} \, f^{(\lambda)} = 0 \]

\[ \nabla^a J^{(\lambda)}_a = (\lambda - d + 1) g^{(\lambda)} \quad \text{and} \quad \nabla^a I^{(\lambda)}_a = (1 - \lambda) f^{(\lambda)} \]

\[ \nabla^b f^{(\lambda)}_{ab} = (\frac{\lambda - d}{4}) f^{(\lambda)}_a \quad \text{and} \quad \nabla^b \psi^{(\lambda)}_{ab} = -\frac{\lambda}{4} f^{(\lambda)}_a. \]

\[ \lambda = \frac{d}{2}: \text{If } d \neq 2 \text{ both functions } \chi^{(d/2)}_{ij} \quad \text{and} \quad \varphi^{(d/2)}_{ij} \text{ have the same expansion as in the case } \lambda \neq 0,1 \text{ considered above; in the case } d = 2 \text{ the function } \chi^{(d/2)}_{ij} \text{ should be identified with the } \lambda = 1 \text{ perturbation.} \]

The coefficients in the above expansions (subject to the above mentioned constraints) form the holographic data on the sphere \( S^d_a \) which should be sufficient for the complete reconstruction of the \((d+2)\)-dimensional Ricci-flat metric. The uncovered holographic data should have interpretation in terms of the conformal field theory living on sphere \( S^d_a \) as well as from the point of view of the \((d+2)\)-dimensional gravitational physics. As for the CFT interpretation the tensors \( f^{(\lambda)}_{ab}(\theta) \) and \( \psi^{(\lambda)}_{ab}(\theta) \) are naturally interpreted as an infinite set of the stress-tensors corresponding to the infinite set of the conformal operators on \( d \)-sphere that represent the matter degrees of freedom. That the stress-tensors are not conserved indicates that the dual conformal theory couples to a set of sources (represented by operators \( J^{(\lambda)}_a \) and \( I^{(\lambda)}_a \)). In this case the conservation is replaced by the Ward identity (see [15] for some discussion of this). On the other hand, the operators \( J^{(\lambda)}_a \) and \( I^{(\lambda)}_a \) are not conserved as well due to coupling to the operators \( g^{(\lambda)} \) and \( f^{(\lambda)} \). It would be interesting to make more precise the relation between these operators and the infinite set of operators \( (\mathcal{O}_\lambda^\Delta, \mathcal{O}_\lambda^\Sigma) \) representing the matter fields. For that one would have to analyze the coupled gravity-matter system.

It seems natural to suggest that the holographic data should encode information about the mass and rotation of the asymptotically flat gravitational configuration. In particular, we expect that \( I^{(\lambda)}_a \) and \( J^{(\lambda)}_a \) should carry information about the angular momentum. Also, the data should contain information about the energy flow coming through the null-infinity. The latter seems to be encoded in the data corresponding to \( \lambda = \frac{d}{2} + i\alpha \). The details however need to be further understood. In the next section we solve a somewhat simpler problem and analyze where in the holographic data on \( S^d_a \) it is stored the information about mass of the static gravitational configuration.
5 Asymptotic form of the black hole metric

The holographic data which we revealed in the previous section should encounter for all relevant information about the gravitational physics in asymptotically flat space-time. In particular it should encode the energy balance between the mass of the gravitational configuration and the energy flow coming through the null-infinity. Thus, we expect that the Bondi mass can be appropriately re-defined in terms of the described holographic data. This is a problem for future investigation. Here we solve a simpler problem of encoding the information about the mass of a static configuration. We want to see which particular term in the \( \lambda \)-representation of the asymptotic metric contains information about the mass. We start with the known metric of (non-rotating) black hole and then bring it to the form which is more appropriate for our asymptotic analysis. The standard form of the metric is

\[
ds^2 = -g(\rho)dt^2 + g^{-1}(\rho)d\rho^2 + \rho^2 d\Omega_{S^d}^2,
\]

where the metric function \( g(\rho) \) depends on the space-time dimension. When the space-time dimension is \( d+2=4 \) the metric (5.1) is the Schwarzschild solution,

\[
g(\rho) = 1 - \frac{2m}{\rho}.
\]

In higher dimensions the metric is known as Myers-Perry metric [38],

\[
g(\rho) = 1 - \frac{2m}{\rho^{d-1}}.
\]

Parameter \( m \) is the mass in the case \( d=2 \) and is related to the mass when \( d > 2 \). The metric (5.1) should be brought to the form

\[
ds^2 = dr^2 + r^2(-F(r, \tau)d\tau^2 + R(r, \tau)d\omega_{S_d}^2)
\]

in terms of new coordinates \((r, \tau)\). When \( r \) goes to infinity the function \( F(r, \tau) \) should approach 1 while the function \( R(r, \tau) = \cosh^2 \tau \) in this limit so that the standard form of the flat space-time metric is restored. In this limit the relation between coordinates \((\rho, t)\) and \((r, \tau)\) is \( \rho = r \cosh \tau, \quad t = r \sinh \tau \).

That the metric (5.1) is non-flat manifests in modifying these relations by subleading terms. Respectively, the subleading terms appear in the \( r \)-expansion of the functions \( R(r, \tau) \) and \( F(r, \tau) \). It is rather straightforward although quite tedious to obtain this expansion. Below we present the result.

\( d = 2 \):

\[
F(r, \tau) = 1 + \frac{4m}{\cosh^3 \tau} \ln \frac{r}{\tau} + \frac{2}{r} f(\tau)
\]

\[
R(r, \tau) = \cosh^2 \tau - \frac{2m}{\cosh \tau} \ln \frac{r}{\tau} + \frac{2}{r} a(\tau) \cosh \tau
\]

\( d > 2 \):

\[
F(r, \tau) = \left(1 + \frac{f(\tau)}{r}\right)^2 - \left(\frac{d}{d-2}\right)^2 \frac{2m}{\cosh^{d+1} \tau} \frac{1}{r^{d-1}}
\]

\[
R(r, \tau) = \left(\cosh \tau + \frac{a(\tau)}{r}\right)^2 + \frac{2m}{(d-2) \cosh^{d-1} \tau} \frac{1}{r^{d-1}}
\]
We skip the subleading terms in (5.5) and (5.6). In both cases the function \( f(\tau) \) is arbitrary and \( a(\tau) \) is defined as
\[
a(\tau) = \int f(\tau) \sinh \tau d\tau.
\]
As we have already seen, the \( d = 2 \) case (Minkowski space-time has physical dimension 4) is in many respects special. Equation (5.5) indicates another peculiarity of \( d = 2 \).

Quite surprisingly, the \( r \)-expansion of the Schwarzschild metric starts with the term with logarithm, \( r^{-1} \ln r \). We see also that both in (5.5) and (5.6) the mass makes its first appearance at the level \( \lambda = d - 1 \). To make connection with our notations let us re-write the expansions (5.5) and (5.6) in the form
\[
\begin{align*}
\phi_{ij}^{(1)}(1) & = 4m \cosh^3 \tau , \\
\chi_{ij}^{(1)}(1) & = -2m \cosh \tau \beta_{ab}(\theta) \\
\chi_{ij}^{(d-1)}(1) & = -\left(\frac{d}{d-2}\right) \frac{2m}{\cosh^{d+1} \tau} , \\
\chi_{ab}^{(d-1)} & = \frac{2m}{(d-2) \cosh^{d-1} \tau} \beta_{ab}(\theta)
\end{align*}
\]
and in both cases the components of \( \chi_{ij}^{(1)} \) are
\[
\begin{align*}
\phi_{\tau\tau}^{(1)}(1) & = 2f(\tau) , \\
\chi_{ab}^{(1)} & = 2a(\tau) \cosh \tau \beta_{ab}(\theta).
\end{align*}
\]
Comparing the expressions (5.8) with our analysis we see that components (5.8) are what we called the inhomogeneous part (4.98) of the perturbation while the homogeneous part vanishes identically in (5.8). Notice also that the \( \lambda = d - 1 \) solution to the radial differential equations corresponds to mass term \( m^2 = d - 1 \). The second independent solution is the one with \( \lambda = 1 \). In the case \( d=2 \) the two independent solutions which correspond to the mass term \( m^2 = 1 \) are \( r \ln r \) and \( r \). We see that in the expansion (5.6) there appear both independent solutions corresponding to \( m^2 = d - 1 \): the \( \lambda = d - 1 \) term (\( r \ln r \) term in \( d=2 \) case) contains information about the mass of gravitational configuration while the \( \lambda = 1 \) term contains an arbitrary function of \( \tau \). Freedom in the choice of this function is apparently a manifestation of the gauge symmetry (4.19) appearing exactly at the level \( \lambda = 1 \). Comparing (5.8) with our analysis in section 4.2.3 we can single out the primary element in the holographic data which contains the information about the mass. We find that it is the function \( g^{(\lambda=1)}(\theta) \) which is proportional to the mass \( m \) and brings the dependence on \( m \) into all metric components. Generically, \( g^{(\lambda)}(\theta) \) (defined in (4.87)) is a source for the vector \( J_a^{(\lambda)} \) via equation (4.100). But in the case \( \lambda = 1 \) the coefficient in front of the source vanishes and \( J_a^{(\lambda)} \) is divergence-free. On the other hand, the current \( J_a^{(\lambda)} \) plays the role of the source for \( f_{ab}^{(\lambda)} \) (4.101) and since \( J_a^{(\lambda)} \) is conserved it means that \( f_{ab}^{(\lambda)} \) is partially conserved. (It is also traceless by (4.102).) This latter property singles out the value \( \lambda = d - 1 \). Of course, all \( J_a^{(\lambda)} \) and \( f_{ab}^{(\lambda)} \) identically vanish in the solution (5.5) and (5.6). But they would be non-trivial in the general case of dynamical situation when there is flow of energy coming through the past null-infinity. It is certainly interesting to analyze how information about the flow is encoded in the holographic data.
6 Conclusion

Minkowski space-time can be reconstructed from some data specified on the boundary of light-cone. This works in a fashion similar to the known holographic reconstruction of asymptotically anti-de Sitter space from the boundary data. In the latter case the holographic pair consists on metric representing the conformal class on the boundary and the stress-tensor of the boundary theory. In the present case since infinitely many adS and dS slices end at the boundary of the light-cone we should expect that infinitely many stress-tensors need to be specified there. Also, since the data should represent the (d+2)-dimensional physics which does not confine to the boundary only, these stress-tensors are not expected to be conserved. Rather they should satisfy the Ward identities as required by the coordinate invariance. Indeed, what we have found is the chains of holographic operators that can be represented as follows

\[ f^{(\lambda)}_{ab} \rightarrow J^{(\lambda)}_a \rightarrow g^{(\lambda)} \]

and

\[ \psi^{(\lambda)}_{ab} \rightarrow I^{(\lambda)}_a \rightarrow f^{(\lambda)} \].

The label \( \lambda \) parameterizes the infinite family of such operators. Another expectation is that in the dual theory the central charge (coming naively from the radius of each (a)dS slice) should be continuous function perhaps parameterized by \( \lambda \) so that each stress-tensor in the infinite family should have its own central charge. However, we do not see this in our analysis since all tensors \( f^{(\lambda)}_{ab} \) and \( \psi^{(\lambda)}_{ab} \) are traceless. Of course, these tensors represent only the linearized part of the stress-tensors in the full non-linear problem. But that they are traceless in the linear order may indicate that the central charge is well-defined in the dual theory independently of \( \lambda \). This should be further investigated. From the gravitational perspective the above sets of operators represent the bulk gravitational dynamics and should describe the mass, angular momentum and the energy flow. The latter can be due to gravitational waves and is likely to be represented by the holographic operators with complex \( \lambda \). We finish with a remark that our construction may be a nice starting point for the quantization of asymptotically flat gravitational field since the set of the holographic data is obviously even-dimensional and seems to be well-suited for introduction of the symplectic structure.

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Appendix

A Some useful identities in $dS_d$ and $S_d$

In this Appendix we collect some useful commutation relations of covariant derivative and Laplace type operator $\nabla^2 = \nabla^a \nabla_a$ on d-dimensional de Sitter space. These identities are valid in any signature. In the case of Euclidean signature the space is d-dimensional sphere. Both spaces are maximally symmetric so that the Riemann curvature can be expressed in terms of metric $\beta_{ab}$.

\[
R_{abcd} = \beta_{ab} \beta_{cd} - \beta_{ad} \beta_{cb} \\
R_{ab} = (d - 1) \beta_{ab} .
\]

The commutation of covariant derivatives on such spaces is significantly simplified. The useful relations are

\[
\nabla_a \nabla^2 \phi = \nabla^2 \nabla_a \phi - (d - 1) \nabla_a \phi \quad (A.2)
\]

for scalar field $\phi$,

\[
\nabla_a \nabla^2 A_b = (\nabla^2 - d + 1) \nabla_a A_b + 2 \beta_{ab} \nabla^c A_c - 2 \nabla_b A_a \quad (A.3)
\]

for vector field $A_a$. Contracting the indices in (A.3) we find that

\[
\nabla_a \nabla^2 A_a = (\nabla^2 - d + 1) \nabla^a A_a .
\]

Taking the symmetrization of equation (A.3) and assuming that $\nabla^a A_a = 0$ we get another useful identity for vector

\[
\nabla_a \nabla^2 A_b + \nabla_b \nabla^2 A_a = (\nabla^2 - d - 1)(\nabla_a A_b + \nabla_b A_a) . \quad (A.5)
\]

For symmetric tensor $h_{ab}$ we get the identity

\[
\nabla_b \nabla^2 h_a^b = (\nabla^2 + d + 1)(\nabla_b h_a^b) - 2 \nabla_a \text{tr} h . \quad (A.6)
\]

B Legendre functions: differential equation and asymptotic behavior

Solution to the differential equation

\[
y''(\tau) + \frac{1 - 2a}{\coth \tau} y'(\tau) - (b - \frac{c}{\cosh^2 \tau}) y(\tau) = 0 ,
\]

where $a, b, c$ are some constants, is a combination of $P$- and $Q$-Legendre functions:

\[
y(\tau) = (\cosh(\tau))^a \left(C_1 P_{\nu}^{\frac{a+c}{2}+\frac{1}{2}}(-i \sinh \tau) + C_2 Q_{\nu}^{\frac{a+c}{2}+\frac{1}{2}}(-i \sinh \tau)\right) , \quad (B.2)
\]

where $C_1$ and $C_2$ are arbitrary integration constants.

For large $\xi$ the Legendre functions asymptotically behave as follows [31]

\[
P_{\nu}^{\mu}(\xi) = \frac{(2\xi)^\nu}{\sqrt{\pi}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu - \mu + 1)} \left(1 + O\left(\frac{\ln \xi}{\xi}\right)\right) , \quad \text{Re} \ \nu > -\frac{1}{2}
\]

\[
Q_{\nu}^{\mu}(\xi) = \frac{\sqrt{\pi}}{(2\xi)^{\nu+1}} \frac{\Gamma(\nu + \mu + 1)}{\Gamma(\nu + \frac{3}{2})} \left(1 + O\left(\frac{\ln \xi}{\xi}\right)\right) \quad (B.3)
\]
\[\text{C} \quad \lambda = 1 \text{ equations on de Sitter space } dS_{d+1}\]

Keeping all components of \(\chi_{ij}\), i.e. \(\chi_{\tau\tau}, \chi_{\tau a}\) and \(\chi_{ab}\), the equation (4.14) splits on two equations

\[
\frac{d}{2} A' \chi_{\tau\tau} = e^{-A} \left( \nabla^a \chi_{a\tau} + \frac{1}{2} A' \text{tr} \chi - \partial_{\tau} \text{tr} \chi \right) \quad (C.1)
\]

\[
\partial_a \chi_{\tau\tau} - \partial_{\tau} \chi_{a\tau} - \frac{d}{2} A' \chi_{a\tau} = e^{-A} (\partial_a \text{tr} \chi - \nabla^b \chi_{ba}) \quad , \quad (C.2)
\]

where \(\text{tr} \chi = \beta^{ab} \chi_{ab}\) and \(\nabla^a\) is with respect to metric \(\beta_{ab}\) on d-sphere.

The other group of equations comes from (4.15). For \((\tau\tau), (\tau a)\) and \((ab)\) components of (4.15) we have that

\[
\frac{d}{2} A' \partial_{\tau} \chi_{\tau\tau} - \frac{d}{2} (A^2 - 2) \chi_{\tau\tau}
+ e^{-A} \left( \text{tr} \chi - \frac{A^2}{2} \text{tr} \chi + e^A \partial^2_{\tau} (e^{-A} \text{tr} \chi) - \nabla^c \nabla_c \chi_{\tau\tau} + 2 A' \nabla^b \chi_{\tau b} \right) = 0 \quad (C.3)
\]

\[
\partial^2_{\tau} \chi_{a\tau} + A' \left( \frac{d - 2}{2} \partial_{\tau} \chi_{a\tau} + \frac{(d + 1)}{2} (2 - A^2) - \frac{1}{2} A'' \chi_{a\tau} + \frac{3}{2} A' \partial_a \chi_{\tau\tau} - \partial_a \partial_{\tau} \chi_{\tau\tau} \right)
+ \partial_a \partial_{\tau} (e^{-A} \text{tr} \chi) - \frac{A'}{2} \partial_a (e^{-A} \text{tr} \chi) + A' e^{-A} \nabla^b \chi_{ab} - e^{-A} \nabla^c \nabla_c \chi_{a\tau} = 0 \quad (C.4)
\]

\[
\partial^2_{\tau} \chi_{ab} + \frac{(d - 4)}{2} A' \partial_{\tau} \chi_{ab} + \chi_{ab} [(d + 1) - \frac{A^2}{2} (d - 1) - A'']
+ \beta_{ab} e^A [- \frac{A^2}{2} \chi_{\tau\tau} + \frac{A'}{2} \partial_{\tau} \chi_{\tau\tau} + \chi_{\tau\tau}]
+ \beta_{ab} [- \frac{A'}{2} e^A \partial_{\tau} (e^{-A} \text{tr} \chi) - \text{tr} \chi] + A' (\nabla_a \chi_{b\tau} + \nabla_b \chi_{a\tau})
- e^{-A} \nabla^c \nabla_c \chi_{ab} - \nabla_a \nabla_b \chi_{\tau\tau} + e^{-A} \nabla_a \nabla_b \text{tr} \chi = 0 \quad (C.5)
\]

\[\text{D} \quad \lambda \neq 0, 1 \text{ equations on de Sitter space } dS_{d+1}\]

For components \((\tau\tau), (a\tau)\) and \((ab)\) of equation (4.17) we find respectively

\[
e^A \partial^2_{\tau} (e^{-A} \text{tr} \chi^{(\lambda)}) - e^{-A} \nabla^2 \text{tr} \chi^{(\lambda)} + 2 A' \nabla^a \chi_{a\tau}^{(\lambda)}
+ \frac{d}{2} A' \partial_{\tau} \text{tr} \chi^{(\lambda)} + (2 + \lambda (d - \lambda) - (d + \frac{1}{2}) A^2) \text{tr} \chi^{(\lambda)} = 0 \quad (D.1)
\]

\[
\partial^2_{\tau} \chi_{a\tau}^{(\lambda)} + (\frac{d}{2} - 1) A' \partial_{\tau} \chi_{a\tau}^{(\lambda)} + (2 + \lambda (d - \lambda) - \frac{d + 1}{2} A^2 - \frac{1}{2} A'') \chi_{a\tau}^{(\lambda)}
- e^{-A} \nabla^2 \chi_{a\tau}^{(\lambda)} + A' \partial_{a\tau} \chi_{\tau\tau} + A' e^{-A} \nabla^b \chi_{ab}^{(\lambda)} = 0 \quad (D.2)
\]

\[
\partial^2_{\tau} \chi_{ab}^{(\lambda)} + (\frac{d}{2} - 2) A' \partial_{\tau} \chi_{ab}^{(\lambda)} - e^{-A} \nabla^2 \chi_{ab}^{(\lambda)} + (2 + \lambda (d - \lambda) - A'' - \frac{(d - 1)}{2} A^2)
+ A' (\nabla_a \chi_{b\tau}^{(\lambda)} + \nabla_b \chi_{a\tau}^{(\lambda)}) - \frac{1}{2} A^2 \beta_{ab} \text{tr} \chi^{(\lambda)} = 0 \quad . \quad (D.3)
\]
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