Abstract. This is a continuation of [1] and [2]. We prove that if function \( f \) belongs to the class \( \Lambda_\omega \) \( \text{def} = \{ f : \omega_f(\delta) \leq \text{const} \omega(\delta) \} \) for an arbitrary modulus of continuity \( \omega \), then
\[
    s_j(f(A) - f(B)) \leq c \cdot \omega^*((1 + j)^{-\frac{1}{p}}\|A - B\|_{S^p}) \cdot \|f\|_{\Lambda_\omega}
\]
for arbitrary self-adjoint operators \( A, B \) and all \( 1 \leq j \leq l \), where \( \omega^*(x) \text{def} = x \int_x^\infty \omega(t) dt \) (\( x > 0 \)). The result is then generalized for contractions, maximal dissipative operators, normal operators and \( n \)-tuples of commuting self-adjoint operators.

CONTENTS

1. Introduction 1
2. Space \( \Lambda_\omega \) 2
3. Estimates on singular values of functions of perturbed self-adjoint and unitary operators 3
4. Estimates for other types of operators 8
References 9

1. Introduction

In this note we study the behavior of functions of operators under perturbations. We are going to find estimates for the singular values \( s_n(f(A) - f(B)) \), where both \( A \) and \( B \) are arbitrary self-adjoint or unitary operators. These results are based on the methods developed in [1] and [3] for estimates of operator norms \( \|f(A) - f(B)\| \), in these papers the authors proved if \( f \) belongs to the Hölder class \( \Lambda_\alpha(\mathbb{R}) \) with \( 0 < \alpha < 1 \), then \( \|f(A) - f(B)\| \leq \text{const} \|f\|_{\Lambda_\alpha} \|A - B\|^\alpha \) for all pairs of self-adjoint or unitary operators \( A \) and \( B \). The authors also generalized their results to the class \( \Lambda_\omega \), and obtained estimate \( \|f(A) - f(B)\| \leq \text{const} \|f\|_{\Lambda_\omega} \|A - B\| \).

In [2], it was shown that for functions \( f \) in the Hölder class \( \Lambda_\alpha(\mathbb{R}) \) with \( 0 < \alpha < 1 \) and if \( 1 < p < \infty \), the operator \( f(A) - f(B) \) belongs to \( S_{p/\alpha} \); whenever \( A \) and \( B \) are arbitrary self-adjoint operators such that \( A - B \in S_p \).
In particular, it was proved that if $0 < \alpha < 1$, then there exists a constant $c > 0$ such that for every $l \geq 0$, $p \in [1, \infty)$, $f \in \Lambda_\alpha(\mathbb{R})$, and for arbitrary self-adjoint operators $A$ and $B$ on Hilbert space with bounded $A - B$, the following inequality holds for every $j \leq l$:

$$s_j(f(A) - f(B)) \leq c \|f\|_{\Lambda_\alpha(\mathbb{R})}(1 + j)^{-\frac{\alpha}{p}} \|A - B\|_{S_p}^\alpha \text{(see (3.1) for definition).}$$

In section §3, we generalize this estimate to the class $\Lambda_\omega$ and also obtain some lower-bound estimates for rank one perturbations which also extend the results in [2]. In section §4, similar estimates are given without proofs in case of contractions, maximal dissipative operators, normal operators and $n$-tuples of commuting self-adjoint operators.

Necessary information on Space $\Lambda_\omega$ is given in section §2. We refer the reader to [1] for more detailed information.

2. Space $\Lambda_\omega$

Let $\omega$ be a modulus of continuity, i.e., $\omega$ is a nondecreasing continuous function on $[0, \infty)$ such that $\omega(0) = 0$, $\omega(x) > 0$ for $x > 0$, and $\omega(x + y) \leq \omega(x) + \omega(y), x, y \in [0, \infty)$.

We denote by $\Lambda_\omega(\mathbb{R})$ the space of functions on $\mathbb{R}$ such that

$$\|f\|_{\Lambda_\omega(\mathbb{R})} \overset{\text{def}}{=} \sup_{x \neq y} \frac{|f(x) - f(y)|}{\omega(|x - y|)}.$$

The space $\Lambda_\omega(\mathbb{T})$ on the unit circle can be defined in a similar way.

We continue with the class $\Lambda_\omega$ of functions on $\mathbb{T}$ first. Let $w$ be an infinitely differentiable function on $\mathbb{R}$ such that $w \geq 0$, $\text{supp } w \subset \left[\frac{1}{2}, 2\right]$, and $w(x) = 1 - w\left(\frac{x}{2}\right)$ for $x \in [1, 2]$. (2.1)

Define a $C^\infty$ function $v$ on $\mathbb{R}$ by

$$v(x) = 1 \text{ for } x \in [-1, 1] \text{ and } v(x) = w(|x|) \text{ if } |x| \geq 1. \quad (2.2)$$

Define trigonometric polynomials $W_n$, $W_n^\sharp$ and $V_n$ by

$$W_n(z) = \sum_{k \in \mathbb{Z}} w\left(\frac{k}{2n}\right)z^k, n \geq 1, \quad W_0(z) = z + 1 + z, \text{ and } W_n^\sharp(z) = \overline{W_n(z)}, n \geq 0$$

and

$$V_n(z) = \sum_{k \in \mathbb{Z}} w\left(\frac{k}{2n}\right)z^k, n \geq 1.$$

$V_n$ is called de la Vallée Poussin type kernel.

If $f$ is a distribution on $\mathbb{T}$, we define $f_n$, $n \geq 0$ by

$$f_n = f * W_n + f * W_n^\sharp, n \geq 1, \text{ and } f_0 = f * W_0,$$

then $f = \sum_{n \geq 0} f_n$ and $f - f * V_n = \sum_{k=n+1}^\infty f_n$. 

2
Now we proceed to the real line case. We use the same functions $w, v$ as in (2.1), (2.2), and define functions $W_n, W_n^\sharp$ and $V_n$ on $\mathbb{R}$ by

$$\mathcal{F}W_n(x) = w\left(\frac{x}{2^n}\right), \mathcal{F}W_n^\sharp(x) = \mathcal{F}W_n(-x), n \in \mathbb{Z}$$

and

$$\mathcal{F}V_n(x) = v\left(\frac{x}{2^n}\right), n \in \mathbb{Z},$$

where $\mathcal{F}$ is the Fourier transform:

$$(\mathcal{F}f)(t) = \int_{\mathbb{R}} f(x)e^{-ixt}dx, f \in L^1.$$ 

$V_n$ is also called de la Vallée Poussin type kernel.

If $f$ is a tempered distribution on $\mathbb{R}$, we define $f_n$ by

$$f_n = f \ast W_n + f \ast W_n^\sharp, n \in \mathbb{Z}.$$ 

We will use the same notation $\Lambda_\omega, W_n, W_n^\sharp$ and $V_n$ on $\mathbb{R}$ and on $\mathbb{T}$ in the following discussion.

In [1], it is proved that there exists a constant $c$ such that for an arbitrary modulus of continuity $\omega$ and for an arbitrary function $f$ in $\Lambda_\omega$, the following inequalities hold for all $n \in \mathbb{Z}$, in $\mathbb{R}$ case, or for all $n \geq 0$, in $\mathbb{T}$ case:

$$\|f - f \ast V_n\|_{L^\infty} \leq c \omega(2^{-n})\|f\|_{\Lambda_\omega} \quad (2.3)$$

$$\|f \ast W_n\|_{L^\infty} \leq c \omega(2^{-n})\|f\|_{\Lambda_\omega}, \|f \ast W_n^\sharp\|_{L^\infty} \leq c \omega(2^{-n})\|f\|_{\Lambda_\omega} \quad (2.4)$$

Let $\mathcal{S}'(\mathbb{R})$ be the space of all tempered distributions on $\mathbb{R}$. Denote by $\mathcal{S}'_+(\mathbb{R})$ the set of all $f \in \mathcal{S}'(\mathbb{R})$ such that $\text{supp } \mathcal{F}f \subset [0, \infty)$. Put $(\Lambda_\omega(\mathbb{R}))_+ \overset{\text{def}}{=} \Lambda_\omega(\mathbb{R}) \cap \mathcal{S}'_+(\mathbb{R})$ and $\mathbb{C}_+ \overset{\text{def}}{=} \{z \in \mathbb{C} : \text{Im } z > 0\}$. Then a function in $(\Lambda_\omega(\mathbb{R}))_+$ if and only if it has a (unique) continuous extension to the closed upper half-plane $\text{clos } \mathbb{C}_+$ that is analytic in the open upper half-plane $\mathbb{C}_+$ with at most a polynomial growth rate at infinity.

3. Estimates on singular values of functions of perturbed self-adjoint and unitary operators

Recall that if $T$ is a bounded linear operator on Hilbert space, then the singular values $s_j(T), j \geq 0$, are defined by

$$s_j(T) = \inf \{\|T - R\| : \text{rank } R \leq j\}.$$ 

For $l \geq 0$ and $p \geq 1$, we consider the norm $S^l_p$ (see [9]) defined by

$$\|T\|_{S^l_p} \overset{\text{def}}{=} \left(\sum_{j=0}^{l} (s_j(T))^p\right)^{\frac{1}{p}}. \quad (3.1)$$
It is shown in [13] and [2] that if \( f \) is an entire function of exponential type at most \( \sigma \) that is bounded on \( \mathbb{R} \), and \( A, B \) are self-adjoint operators with bounded \( A - B \), then

\[
\| f(A) - f(B) \| \leq \text{const} \| f \|_{L^\infty} \| A - B \|,
\]

and

\[
\| f(A) - f(B) \|_{S_p^\lambda} \leq \text{const} \| f \|_{L^\infty} \| A - B \|_{S_p^\lambda},
\]

(3.2)

(3.3)

For the proof and more details, see [1], [2], [8], [10], [11] and [13].

Given a modulus of continuity \( \omega \), define functions \( \omega^* \) and \( \omega^\# \) by

\[
\omega^*(x) = x \int_{x}^{\infty} \frac{\omega(t)}{t^2} dt, \quad x > 0
\]

and

\[
\omega^\#(x) = x \int_{x}^{\infty} \frac{\omega(t)}{t^2} dt + \int_{0}^{x} \frac{\omega(t)}{t} dt, \quad x > 0.
\]

In this paper, we assume that \( \omega^\# \) is finite valued whenever it is used.

For example, if we define \( \omega \) by

\[
\omega(x) = x^\alpha, \quad x > 0, \quad 0 < \alpha < 1,
\]

then \( \omega^\#(x) \leq \text{const} \omega(x) \).

It is well known (see [6], Ch.3, Theorem 13.30) that if \( \omega \) is a modulus of continuity, then the Hilbert transform maps \( \Lambda_\omega \) into itself if and only if \( \omega^\#(x) \leq \text{const} \omega(x) \).

**Theorem 3.1.** There exists a constant \( c > 0 \) such that for every modulus of continuity \( \omega \), every \( f \) in \( \Lambda_\omega(\mathbb{R}) \) and for arbitrary self-adjoint operators \( A \) and \( B \), the following inequality holds for all \( l \) and for all \( j, 1 \leq j \leq l \):

\[
s_j(f(A) - f(B)) \leq c \cdot \omega^*\left((1 + j)^{-\frac{1}{2}} \| A - B \|_{S_p^\lambda}\right) \cdot \| f \|_{\Lambda_\omega}.
\]

(3.4)

**Proof.** \( A \) and \( B \) can be taken as bounded operators (see [3], Lemma 4.4), then we may further assume \( f \) is bounded. Let \( R_N = \sum_{n=-\infty}^{N} (f_n(A) - f_n(B)) \), \( Q_N = (f - f * V_N)(A) - (f - f * V_N)(B) \). Here \( f_n \) and the de la Vallée Poussin type kernel \( V_N \) are defined as in [2]. Then \( f(A) - f(B) = R_N + Q_N \), with convergence in the uniform operator topology as shown in [1]. Note that for any integer \( m \in \mathbb{Z} \), functions \( f_m \) and \( f - f * V_m \) are entire functions of exponential type at most \( 2^{m+1} \). Thus it follows from (3.2), (3.3), (2.3), and (2.4) that

\[
\| Q_N \| \leq c \cdot \omega(2^{-N}) \cdot \| f \|_{\Lambda_\omega},
\]

where \( c \) is a constant independent of \( N \) and \( f \).
and

\[ \|R_N\|_{S_p^j} \leq \sum_{n=-\infty}^{N} \|f_n(A) - f_n(B)\|_{S_p^j} \]
\[ \leq c \cdot \sum_{n=-\infty}^{N} (2^n \cdot \|f_n\|_{L^\infty}) \cdot \|A - B\|_{S_p^j} \]
\[ \leq c \cdot 2^N \cdot \omega_s(2^{-N}) \cdot \|A - B\|_{S_p^j} \cdot \|f\|_{\Lambda_\omega} \text{(see [1])} \]

Then

\[ s_j(f(A) - f(B)) \leq s_j(R_N) + \|Q_N\| \leq (1 + j)^{-1/p} \cdot \|R_N\|_{S_p^j} + \|Q_N\| \]
\[ \leq c \cdot [(1 + j)^{-\frac{1}{p} \cdot 2^N \cdot \omega_s(2^{-N})} \cdot \|A - B\|_{S_p^j} + \omega(2^{-N})] \cdot \|f\|_{\Lambda_\omega} \]

Take \( N \) such that \( 1 \leq (1 + j)^{-\frac{1}{p} \cdot 2^N \cdot \|A - B\|_{S_p^j} \leq 2 \) and use the fact that \( \omega(t) \leq \omega_s(t) \) for any \( t > 0 \), we get \((3.4)\). \( \square \)

**Theorem 3.2.** There exists a constant \( c > 0 \) such that for every modulus of continuity \( \omega \), every \( f \) in \( \Lambda_\omega(\mathbb{T}) \) and for arbitrary unitary operators \( U \) and \( V \), the following inequality holds for all \( l \) and for all \( j, 1 \leq j \leq l \):

\[ s_j(f(U) - f(V)) \leq c \cdot \omega_s((1 + j)^{-\frac{1}{p} \cdot \|U - V\|_{S_p^j}}) \cdot \|f\|_{\Lambda_\omega}. \tag{3.5} \]

**Proof.** If \( (1 + j)^{-\frac{1}{p} \cdot \|U - V\|_{S_p^j} \leq 2 \), the proof is similar to Theorem 3.1 with \( R_N = \sum_{n=0}^{N} (f_n(U) - f_n(U)) \); if \( (1 + j)^{-\frac{1}{p} \cdot \|U - V\|_{S_p^j} > 2 \), then

\[ s_j(f(U) - f(V)) \leq \|f(U) - f(V)\| \leq c \cdot \omega_s(\|U - V\|) \cdot \|f\|_{\Lambda_\omega} \leq c \cdot \omega_s(2) \cdot \|f\|_{\Lambda_\omega}. \]

\( \square \)

**Corollary 3.3.** Let \( \omega \) be a modulus of continuity such that

\[ \omega_s(x) \leq \text{const} \cdot \omega(x), \quad x \geq 0. \]

Then for an arbitrary function \( f \in \Lambda_\omega(\mathbb{R}) \) and for arbitrary self-adjoint operators \( A \) and \( B \), the following inequality holds for all \( l \) and for all \( j, 1 \leq j \leq l \):

\[ s_j(f(A) - f(B)) \leq \text{const} \cdot \omega((1 + j)^{-\frac{1}{p} \cdot \|A - B\|_{S_p^j}}) \cdot \|f\|_{\Lambda_\omega}. \]

Let \( H, \mathcal{H} \) be the Hankel operators defined in [2].

**Theorem 3.4.** Let \( \omega \) be a modulus of continuity on \( \mathbb{T} \). There exist unitary operators \( U, V \) and a real function \( h \) in \( \Lambda_{\omega_1}(\mathbb{T}) \) such that

\[ \text{rank}(U - V) = 1 \quad \text{and} \quad s_m(h(U) - h(V)) \geq \omega((1 + m)^{-1}). \]

[5]
Proof. Consider the operators $U$ and $V$ on space $L^2(\mathbb{T})$ with respect to the normalized Lebesgue measure on $\mathbb{T}$ defined by (see [2])

$$ Uf = \tilde{z}f \text{ and } Vf = \tilde{z}f - 2(f, 1)\tilde{z}, \quad f \in L^2. $$

For $f \in C(\mathbb{T})$, we have

$$ ((f(U) - f(V))z^j, z^k) = -2 \begin{cases} \hat{f}(j - k), & \text{if } j \geq 0, k < 0; \\ \hat{f}(j - k), & \text{if } j < 0, k \geq 0; \\ 0, & \text{otherwise}. \end{cases} $$

Define function $g$ by

$$ g(\zeta) = \sum_{n=1}^{\infty} \omega(4^{-n})(\zeta^{4^n} + \zeta^{4^n}), \quad \zeta \in \mathbb{T}. $$

Then we have

$$ \|g \ast W_n\|_{L^\infty} \leq \text{const } \omega(2^{-n}), \quad \|g \ast W_n^T\|_{L^\infty} \leq \text{const } \omega(2^{-n}), \quad n \geq 0. $$

Let $\xi, \eta$ be two arbitrarily different fixed points on $\mathbb{T}$, choose $N \geq 0$ such that $\frac{1}{2} \leq \frac{2^{-N}}{|\xi - \eta|} \leq 1$, then

$$ |g(\xi) - g(\eta)| \leq \sum_{n=0}^{N} |g_n(\xi) - g_n(\eta)| + |(g - g \ast V_N)(\xi) - (g - g \ast V_N)(\eta)| $$

$$ \leq \sum_{n=0}^{N} |g_n(\xi) - g_n(\eta)| + 2 \sum_{n=N+1}^{\infty} \|g_n\|_{L^\infty} $$

$$ \leq \text{const } \sum_{n=0}^{N} 2^n \|g_n\|_{L^\infty} |\xi - \eta| + 2 \sum_{n=N+1}^{\infty} \|g_n\|_{L^\infty} $$

$$ \leq \text{const } \sum_{n=0}^{N} 2^n \omega(2^{-n}) |\xi - \eta| + \text{const } \sum_{n=N+1}^{\infty} \omega(2^{-n}) $$

$$ \leq \text{const } \omega_s(|\xi - \eta|) + \text{const } \int_{0}^{2^{-N}} \frac{\omega(t)}{t} dt $$

$$ \leq \text{const } \omega_s(|\xi - \eta|). $$

Consider the matrix $\Gamma_g = \{\hat{g}(-j - k)\}_{j \geq 1, k \geq 0} = \{\hat{g}(j + k)\}_{j \geq 1, k \geq 0}$. Let $n \geq 1$. Define matrix $T_n = \{\hat{g}(j + k + 4^n - 1)\}_{0 \leq j, k \leq 3 \cdot 4^n - 1}$, then

$$ T_n = \begin{bmatrix} \omega(4^{-n}) \\ \cdots \\ \omega(4^{-n}) \end{bmatrix}. $$

If $R$ is any matrix with the same size of $T_n$ such that rank($R$) $< 3 \cdot 4^n - 1$, then $\|T_n - R\| \geq \omega(4^{-n})$. It follows that $s_j(T_n) \geq \omega(4^{-n})$ for $j < 3 \cdot 4^n - 1$. 


For each $T_n$, there is some orthogonal projection $P_n$ such that $T_n = P_n \Gamma_g P_n$, hence $s_j(\Gamma_g) \geq s_j(T_n) \geq \omega(4^{-n})$ for all $n$ and for all $j, j < 3 \cdot 4^n - 1$. Thus for all $j \geq 0$, we have

$$s_j(\Gamma_g) \geq \omega\left(\frac{3}{16} \cdot (j + 1)^{-1}\right) \geq \frac{3}{32} \cdot \omega\left((j + 1)^{-1}\right).$$

To complete the proof, it suffices to take $h = \frac{32}{7} g$. 

**Corollary 3.5.** Let $\omega$ be a modulus of continuity such that

$$\omega_k(x) \leq \text{const} \cdot \omega(x), \quad 0 \leq x \leq 2.$$ 

There exist unitary operators $U, V$ and a real function $h$ in $\Lambda_\omega(T)$ such that

$$\text{rank}(U - V) = 1 \quad \text{and} \quad s_m(h(U) - h(V)) \geq \omega\left((1 + m)^{-1}\right).$$

**Theorem 3.6.** Let $\omega$ be a modulus of continuity on $\mathbb{R}$ and $f$ be a continuous function on $\mathbb{R}$. If for all unitary operators $U$ and $V$, we have

$$s_n(f(U) - f(V)) \leq \text{const} \cdot \omega\left((1 + n)^{-\frac{1}{2}}\|U - V\|_p\right), \quad \text{for all } n \geq 0,$$

then $f \in \Lambda_\omega(\mathbb{R})$.

**Proof.** Let $\zeta, \eta \in \mathbb{T}$, we can select commuting unitary operators $U$ and $V$ such that $s_0(U - V) = s_1(U - V) = \ldots = s_n(U - V) = |\zeta - \eta|$ and $s_k(U - V) = 0$, $k \geq n + 1$. Then $s_n(f(U) - f(V)) = |f(\zeta) - f(\eta)|, \|U - V\|_p = (1 + n)^{\frac{1}{2}} \cdot |\zeta - \eta|$. 

**Theorem 3.7.** Let $\omega$ be a modulus of continuity on $\mathbb{R}$ and $f$ be a continuous function on $\mathbb{R}$. If for all self-adjoint operators $A$ and $B$, we have

$$s_n(f(A) - f(B)) \leq \text{const} \cdot \omega\left((1 + n)^{-\frac{1}{2}}\|A - B\|_p\right), \quad \text{for all } n \geq 0,$$

then $f \in \Lambda_\omega(\mathbb{R})$.

**Proof.** Similar to Theorem 3.6.

**Theorem 3.8.** Let $\omega$ be a modulus of continuity over $\mathbb{R}$. There exist self-adjoint operators $A, B$, and a real function $f$ in $\Lambda_\omega(\mathbb{R})$ such that

$$\text{rank}(A - B) = 1 \quad \text{and} \quad s_m(f(A) - f(B)) \geq \omega((1 + m)^{-1}), \quad \text{for all } m \geq 0.$$

**Proof.** WLOG, we assume $\omega(t) = \omega(2)$, for all $t \geq 2$, that is, $\omega$ can be regarded as a modulus of continuity on $\mathbb{T}$.

We then choose a function (see [2], Lemma 9.6) $\rho \in C^\infty(\mathbb{T})$ such that $\rho(\zeta) + \rho(i\zeta) = 1, \rho(\zeta) = \rho(\bar{\zeta})$ for all $\zeta \in \mathbb{T}$, and $\rho$ vanishes in a neighborhood of the set $\{-1, 1\}$.

Note that $\rho \in \Lambda_\omega(\mathbb{T})$, since $\omega(st) \geq \frac{1}{2} \omega(t)$, for all $t \geq 0$ and $s, 0 < s < 1$.

Define function $g_1$ by

$$g_1(\zeta) = \sum_{n=1}^{\infty} \omega(4^{-n})(\zeta 4^n + \bar{\zeta} 4^n), \quad \zeta \in \mathbb{T}.$$
Then \( g_1 \in \Lambda_{\omega_4}(\mathbb{T}) \). If \( g_0 \defeq C\rho g_1 \) for a sufficient large number \( C \), then \( g_0 \in \Lambda_{\omega_4}(\mathbb{T}) \), vanishes in a neighborhood of the set \( \{-1, 1\} \) and \( g_0(\zeta) = g_0(\bar{\zeta}) \) for all \( \zeta \in \mathbb{T} \), and \( s_m(H_{g_0}) \geq \omega((1 + m)^{-1}) \) for all \( m \geq 0 \).

Define \( \varphi(x) = (x^2 + 1)^{-1} \) (as in \cite{2}, Theorem 9.9), then there exists a compactly supported real bounded function \( f \) such that \( f(\varphi(x)) = g_0(\frac{x^2 - 1}{x^2 + 1}) \) and a simple calculation shows that \( f \) belongs to \( \Lambda_{\omega_4}(\mathbb{R}) \). Denote \( L_2^2(\mathbb{R}) \) the subspace of even functions in \( L^2(\mathbb{R}) \). Consider operators \( A \) and \( B \) on \( L_2^2(\mathbb{R}) \) defined by \( A(g) = \mathbf{H}^{-1}M_{\varphi}H(g) \) and \( B(g) = \varphi g \), here \( \mathbf{H} \) is the Hilbert transform defined on \( L_2^2(\mathbb{R}) \) (see \cite{2}) and \( M_{\varphi} \) is the multiplication by \( \varphi \). Then \( \text{rank}(A - B) = 1 \), and we have

\[
s_m(f(B) - f(A)) \geq \sqrt{2}s_m(H_{f_{\varphi}}) = \sqrt{2}s_m(H_{g_0}) \geq \sqrt{2}\omega((1 + m)^{-1}). \tag{4}
\]

4. Estimates for other types of operators

The following estimates are given without proofs in case of contractions, maximal dissipative operators, normal operators and \( n \)-tuples of commuting self-adjoint operators.

**Theorem 4.1.** There exists a constant \( c > 0 \) such that for every modulus of continuity \( \omega \), every \( f \) in \( (\Lambda_{\omega}(\mathbb{R}))_+ \) and for arbitrary contractions \( T \) and \( R \) on Hilbert space, the following inequality holds for all \( l \) and for all \( j \), \( 1 \leq j \leq l \):

\[
s_j(f(T) - f(R)) \leq c\omega_{\ast}(\mathbb{R})(1 + j)^{-\frac{1}{2}}\|T - R\|S_j), \|f\|_{\Lambda_{\omega}}.
\]

To prove this result, the following result is important (see \cite{1}, \cite{2} and \cite{12}):

There exists a constant \( c \) such that for arbitrary trigonometric polynomial \( f \) of degree \( n \) and for arbitrary contractions \( T \) and \( R \) on Hilbert space,

\[
\|f(T) - f(R)\|_{S_p} \leq c n\|f\|_{L_\infty}(\mathbb{R})\|T - R\|_{S_p}.
\]

Denote \( \mathcal{F} \) the Fourier transform on \( L_1(\mathbb{R}^n) \), \( n \geq 1 \) by:

\[
(\mathcal{F}f)(t) = \int_{\mathbb{R}^n} f(x)e^{-i(x,t)dx}, \text{where}
\]

\[
(x = (x_1, \ldots, x_n), t = (t_1, \ldots, t_n), (x,t) \defeq x_1t_1 + \ldots + x_nt_n.
\]

**Theorem 4.2.** There exists a constant \( c > 0 \) such that for every modulus of continuity \( \omega \), every \( f \) in \( (\Lambda_{\omega}(\mathbb{R}))_+ \) and for arbitrary maximal dissipative operators \( L \) and \( M \) with bounded difference, the following inequality holds for all \( l \) and for all \( j \), \( 1 \leq j \leq l \):

\[
s_j(f(L) - f(M)) \leq c\omega_{\ast}(\mathbb{R})(1 + j)^{-\frac{1}{2}}\|L - M\|_{S_j^p}, \|f\|_{\Lambda_{\omega}}.
\]
To prove this result, the following result is important (see [4]):

There exists a constant $c > 0$ such that for every function $f$ in $H^\infty(\mathbb{C}_+)$ with $\text{supp } \mathcal{F}f \subset [0, \sigma]$, $\sigma > 0$, and for arbitrary maximal dissipative operators $L$ and $M$ with bounded difference,

$$\|f(L) - f(M)\|_{S_p} \leq c\sigma\|f\|_{L^\infty}\|L - M\|_{S_p}.$$  

**Theorem 4.3.** There exists a constant $c > 0$ such that for every modulus of continuity $\omega$, every $f$ in $\Lambda_\omega(\mathbb{R}^2)$ and for arbitrary normal operators $N_1$ and $N_2$, the following inequality holds for all $l$ and for all $j$, $1 \leq j \leq l$:

$$s_j(f(N_1) - f(N_2)) \leq c\omega((1 + j)^{-\frac{1}{p}}\|N_1 - N_2\|_{S_p})\|f\|_{\Lambda_\omega}.$$  

To prove this result, the following result is important (see [5]):

There exists a constant $c > 0$ such that for every bounded continuous function $f$ on $\mathbb{R}^2$ with

$$\text{supp } \mathcal{F}f \subset \{\zeta \in \mathbb{C} : |\zeta| \leq \sigma\}, \ \sigma > 0,$$

and for arbitrary normal operators $N_1$ and $N_2$,

$$\|(f(N_1) - f(N_2))\|_{S_p} \leq c\sigma\|f\|_{L^\infty}\|N_1 - N_2\|_{S_p}.$$  

**Theorem 4.4.** Let $n$ be a positive integer and $p \geq 1$. There exists a positive number $c_n$ such that for every modulus of continuity $\omega$, every $f$ in $\Lambda_\omega(\mathbb{R}^n)$ and for arbitrary $n$-tuples of commuting self-adjoint operators $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$, the following inequality holds for all $l$ and for all $j$, $1 \leq j \leq l$:

$$s_j(f(A_1, \ldots, A_n) - f(B_1, \ldots, B_n)) \leq c_n\max_{1 \leq j \leq n}\omega((1 + j)^{-\frac{1}{p}}\|A_j - B_j\|_{S_p})\|f\|_{\Lambda_\omega}.$$  

To prove this result, the following result is important (see [7]):

There exists a constant $c_n > 0$ such that for every bounded continuous function $f$ on $\mathbb{R}^n$ with

$$\text{supp } \mathcal{F}f \subset \{\xi \in \mathbb{R}^n : |\xi| \leq \sigma\}, \ \sigma > 0,$$

and for arbitrary $n$-tuples of commuting self-adjoint operators $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$,

$$\|(f(A_1, \ldots, A_n) - f(B_1, \ldots, B_n))\|_{S_p} \leq c_n\sigma\|f\|_{L^\infty}\max_{1 \leq j \leq n}\|A_j - B_j\|_{S_p}.$$  

**References**

1. A.B. Aleksandrov, V.V. Peller, Operator Hölder-Zygmund functions, Adv. Math. 224 (2010), 910–966.
2. A.B. Aleksandrov, V.V. Peller, Functions of operator under perturbation of class $S_p$, J. Func. Anal. 258 (2010), 3675–3724.
3. A.B. Aleksandrov, V.V. Peller, Functions of perturbed unbounded self-adjoint operators. Operator Bernstein type inequalities, Indiana Univ. Math. J. 59:4 (2010), 1451–1490.
4. A.B. Aleksandrov, V.V. Peller, Functions of perturbed dissipative operators, Algebra i Analiz 23 (2011), 9–51; translation in St. Petersburg Math. J. 23 (2012), 209–238.
5. A.B. Aleksandrov, V.V. Peller, D. Potapov and F. Sukochev, Functions of normal operators under perturbations, Adv. Math. 226 (2011), 5216–5251.
6. A. Zygmund, Trigonometric series, 2nd ed. Vols. I, II. Cambridge University Press, New York, 1959.
7. F.L. Nazarov, V.V. Peller, Functions of n-tuples of commuting self-adjoint operators, J. Funct.Anal. 266 (2014), 5398–5428.
8. M.S. Birman, M.Z. Solomyak, Double Stieltjes operator integrals. III, Problems of Math. Phys., Leningrad. Univ. 6 (1973), 27–53 (Russian).
9. M.S. Birman, M.Z. Solomyak, Spectral theory of selfadjoint operators in Hilbert spaces, Mathematics and its Applications (Soviet Series), D. Reidel Publishing Co., Dordrecht, 1987.
10. V.V. Peller, Hankel operators of class $S_p$ and their applications (rational approximation, Gaussian processes, the problem of majorizing operators), Mat. Sbornik, 113 (1980), 538–581 (Russian).
    English Transl. in Math. USSR Sbornik, 41 (1982), 443-479.
11. V.V. Peller, Hankel operators in theory of perturbations of unitary and self-adjoint operators, Funktsional. Anal. i Prilozhen. 19:2 (1985), 37–51 (Russian).
    English Transl. in Funct. Anal. Appl. 19 (1985), 111-123.
12. V.V. Peller, For which $f$ does $A - B \in S_p$ imply that $f(A) - f(B) \in S_p$?, Operator Theory, Birkhauser, 24 (1987), 289–294.
13. V.V. Peller, Hankel operators in the perturbation theory of unbounded self-adjoint operators. Analysis and partial differential equations, 529-544, Lecture Notes in Pure and Appl. Math., 122, Dekker, New York, 1990.