On matrix elements of phase-angular momentum commutator in Hilbert space of arbitrary dimensions

Ramandeep S. Johal
Institut für Theoretische Physik,
Technische Universität Dresden,
01062 Dresden, Germany.
e-mail: rjohal@theory.phy.tu-dresden.de
Ph.: + 49 (0351) 463 35582
Fax: + 49 (0351) 463 37299

Abstract

We discuss correspondence between the predictions of quantum theories for rotation angle formulated in infinite and finite dimensional Hilbert spaces, taking as example, the calculation of matrix elements of phase-angular momentum commutator. A new derivation of the matrix elements is presented in infinite space, making use of a unitary transformation that maps from the state space of periodic functions to non-periodic functions, over which the spectrum of angular momentum operator is in general, fractional. The approach can be applied to finite dimensional Hilbert space also, for which identical matrix elements are obtained.
Many interesting quantum phenomena, such as in Josephson junctions [1], Aharonov-Bohm effect [2], fractional statistics in quantum hall effect [3] or superfluid thin films [4], are a consequence of the periodic or non-periodic nature of wave function under change in the phase variable associated with it. However, the idea of a hermitian phase operator, which may serve as possible observable in quantum mechanics, has a long history. Dirac [5] first of all attempted to obtain a phase operator for harmonic oscillator by defining a polar decomposition of annihilation operator, \( a = e^{i\phi_D} \sqrt{a^\dagger a} \). However, the exponential phase operator so defined is not unitary [6] and so does not yield a hermitian operator \( \hat{\phi}_D \). It is now generally accepted that a hermitian phase operator does not exist for harmonic oscillator. A related system is of a quantum rotor in plane. Here due to the angular momentum being unbounded, a hermitian operator \( \hat{\phi} \) may be defined, which has continuum of phase eigenvalues \( \phi \). The Hilbert space of the system is space of square integrable functions of polar angle \( \phi \) in the range \([0, 2\pi]\), with Lebesgue measure \( d\phi \). A differential realization for the third component of angular momentum \( \hat{L}_z = -i\partial/\partial\phi \), then implies the canonical commutator \([\hat{\phi}, \hat{L}_z] = i\).

However to deal with bounded operators [7] or finite spin systems [8], finite dimensional version of phase operator formalism have been proposed. A distinct feature of this formalism is that eigenvalues of phase operator are restricted to quantised values. Naturally then, differential realization for \( \hat{L}_z \) is not valid as it assumes a continuity of phase eigenvalues. A hermitian phase operator \( \hat{\phi}_l \) so defined, satisfies a different commutator with \( \hat{L}_z \)

\[
[\hat{\phi}_l, \hat{L}_z] = \frac{2\pi}{2l + 1} \sum_{m,m'=l} \frac{(m' - m)|m'\rangle\langle m|}{\exp[2\pi i(m - m')/(2l + 1)] - 1}, \quad (m \neq m') \quad (1)
\]

where \((2l + 1)\) is dimension of the space and Dirac notation for states is used. One such formalism that has gained attention in recent years, is the Pegg-Barnett (PB)
theory \[9\]. It emphasizes a specific limiting procedure: the physical results are to be obtained when the infinite dimensional limit is taken, after the calculation of expectation values for observables. For the angular momentum case, it means that only in the semi-classical limit (\(l \to \infty\)), can predictions of the theory match with physical results. Although the analogous formalism for phase of quantised electromagnetic field has been intensively studied and also debated in recent years \[11\], there have been relatively fewer studies elaborating the implications of the PB theory for rotation angle in quantum mechanics. An example is non-trivial notion of angular velocity in this formalism, where the usual Heisenberg equation for phase operator does not predict physically intuitive result. A solution was presented in \[10\] by recourse to the classical-quantum correspondence.

The infinite space formalism has its own share of subtleties. For example, the usual Heisenberg uncertainty relation between \(\hat{\phi}\) and \(\hat{L}_z\) does not apply \[12\]. One of the often discussed issues has been the seeming inconsistency in the calculation of matrix elements of canonical commutator, \((n, [\hat{\phi}, \hat{L}_z]m)\), over periodic wave functions \(u(\phi) \propto e^{im\phi}\), satisfying \(u(\phi + 2\pi) = u(\phi)\). It was thought that the problem lies in the operation \(\hat{\phi}u(\phi) = \phi u(\phi)\), which projects wave functions out of the space of periodic functions. So the use of a bare phase operator was rejected in favour of defining a suitable periodic extension of phase operator \[8, 13\] or using periodic functions of the phase operator \[14\]. However, if the adjoint of \(\hat{L}_z\) is taken carefully, then the problem vanishes \[15\] and well behaved matrix elements of the commutator are obtained

\[
(n, [\hat{\phi}, \hat{L}_z]m) = \frac{1}{2\pi} \int_0^{2\pi} d\phi \ e^{-im\phi}[\hat{\phi}, \hat{L}_z]e^{im\phi} = i\delta_{mn}. \tag{2}
\]

However, within the finite dimensional space, different matrix elements are ob-
For instance, given the commutator (1), the diagonal elements vanish
\[ \langle m | [\hat{\phi}_l, \hat{L}_z] | m \rangle = 0. \] (3)
This result is exact for all angular momentum eigenstates \( |m\rangle \) where \(-l \leq m \leq l\). Thus taking \( l \to \infty \), does not yield the result in infinite space. In this paper, we try to address the question of correspondence between results of these matrix elements in the finite and infinite dimensional spaces. We adopt a different approach for the calculation of these elements and argue that results in finite space can also match with those in the infinite space.

The difficulty discussed above for infinite space springs from the fact that \( \hat{L}_z \) is not self-adjoint on the space of nonperiodic functions of the kind \( \phi u(\phi) \) which turn up during the calculation. Still, this does not exclude the use of other non-periodic functions (if they serve any useful purpose) over which angular momentum operator is self-adjoint. Note that using the differential realisation and by partial integration, we obtain
\[
(u_2, \hat{L}_z u_1) = (\hat{L}_z u_2, u_1) - \frac{i}{2\pi} [u_2^*(2\pi)u_1(2\pi) - u_2^*(0)u_1(0)]. \] (4)
The surface term vanishes or alternately, \( \hat{L}_z \) becomes hermitian if its eigenfunctions satisfy \( u_s(2\pi) = e^{i2\pi s}u_s(0) \), where \( s \) is positive real parameter. We denote the angular momentum operator by \( \hat{L}_z(s) \), whose domain \( (D) \) is the above mentioned space of non-periodic functions \( u_s(\phi) \). It is clear that \( D[\hat{L}_z] \neq D[\hat{L}_z(s)] \) for non-integer values of parameter \( s \), and that both operators have same differential realisation \(-i\partial/\partial\phi\). In other words, both satisfy the same canonical commutator with \( \hat{\phi} \) and the matrix elements of this commutator can be correctly evaluated over their respective domains along the lines presented in [15]. In literature, usually hermiticity is guaranteed for \( \hat{L}_z \) by restricting its state space to periodic functions (\( s \) is zero or integer valued). But it is clear that periodicity of wave
functions is required only to keep the spectrum of $\hat{L}_z$ integer valued. Theoretical possibilities such as generalized statistics do exist, where spectrum of $\hat{L}_z$ is in general, fractional [3, 4]. In the following, we make use of the freedom of parameter $s$ to derive the matrix elements of canonical commutator.

We have seen that $D[\hat{L}_z(s)]$ consists of wave functions $u_s(\phi) \propto e^{i(m+s)\phi}$. These can be generated from the usual periodic functions $u(\phi)$, by a unitary transformation: $e^{is\hat{\phi}}u(\phi) = u_s(\phi)$. Also we have $\hat{L}_z(s)u_s(\phi) = (m + s)u_s(\phi)$, in analogy with $\hat{L}_zu(\phi) = mu(\phi)$. Now consider the term

$$\hat{R} = e^{is\hat{\phi}}\hat{L}_z - \hat{L}_z(s)e^{is\hat{\phi}}. \quad (5)$$

Evaluating $\lim_{s \to 0} \frac{1}{is}(n, \hat{R}m)$, we can easily verify that (2) is satisfied. Note that the need for taking adjoint of angular momentum operator is dispensed with, since $\hat{L}_z$ and $\hat{L}_z(s)$ have to act in their respective domains only.

It is the unitary transformation $e^{is\hat{\phi}}$ which is of present interest to us. For instance, it can be used to define a generalized dynamics [16] for current-biased Josephson junction. The parameter $s$ in this case, takes a physical interpretation and represents an external magnetic flux linked to the junction. This system is equivalent to the system of a charged particle on a ring with magnetic flux through the centre of the ring. Thus the limit $s \to 0$ in our calculation, represents the limit in which magnetic flux at the centre of ring goes to zero.

For transparency, we also calculate in Dirac notation. The states $\{|m+s\rangle\}$ form a complete orthonormal basis like the standard angular momentum basis $\{|m\rangle\}$. The periodic phase eigenstate can be written as

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} e^{-im\phi}|m\rangle, \quad (6)$$

or equivalently

$$|\phi\rangle = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} e^{-i(m+s)\phi}|m+s\rangle. \quad (7)$$
Relevant operators are defined by the following representation

\[
\hat{L}_z|m\rangle = m|m\rangle, \quad (8)
\]

\[
\hat{L}_z(s)|m+s\rangle = (m+s)|m+s\rangle, \quad (9)
\]

\[
\hat{\phi}|\phi\rangle = \phi|\phi\rangle. \quad (10)
\]

The representations (8) and (9) imply that the following unitary transformation exists

\[
e^{is\hat{\phi}} = \sum_{m=-\infty}^{+\infty} |m+s\rangle\langle m|. \quad (11)
\]

The unitarity of this operator follows simply from completeness of states. For \( s = 0 \), we get resolution of unity over standard angular momentum states. Note again that due to the angular momentum being unbounded, the system of states is invariant under any finite integer shift \( s \). Due to this invariance, we can restrict the range of \( s \) as \( 0 \leq s \leq 1 \). Particularly, for \( s = 1 \), we have the canonical unitary phase operator which shifts an angular momentum state by unity. For this case, the unitary operator may be said to shift states within the same orthonormal basis, while for other values of \( s \), (11) clearly takes from one orthonormal basis to another.

Next we consider the matrix elements when dimension of angular momentum space is finite. From the \( \hat{R} \) term (3) defined in infinite space, we were able to derive the matrix elements of canonical commutator \([\hat{\phi}, \hat{L}_z]\). Thus it is interesting to extend this calculation to finite space also and see if we can recover, say (3) or not. We take \( \hat{R} = e^{is\hat{\phi}_l} \hat{L}_z - \hat{L}_z(s)e^{is\hat{\phi}_l} \), and calculate the matrix elements \( \langle n|\hat{R}|k\rangle \), where now all operators act in \((2l+1)\) dimensional space. Analogous to (3) and (4), we can define the phase states in finite space in terms of either the \{\(|m\rangle\}_{-l,...,l}\} states or \{\(|m-1+s\rangle\}_{-l,...,l}\} states, as follows

\[
|\phi_l\rangle = \frac{1}{\sqrt{2l+1}} \sum_{m=-l}^{+l} e^{-im\phi_l}|m\rangle,
\]
\[
\frac{1}{\sqrt{2l+1}} \sum_{m=-l}^{-l+1} e^{-i(m-1+s)\phi_l} |m-1+s\rangle.
\] (12)

These are eigenstates of phase operator \(\hat{\phi}_l|\phi_l\rangle = \phi_l|\phi_l\rangle\), which form a conjugate \((2l+1)\)-dimensional orthonormal basis, if the phase eigenvalues are quantised as \(\phi_l = \frac{2\pi n}{2l+1}\), where \(n = 0, 1, ..., 2l\), for phase operator defined in \([0, 2\pi]\) range. Apart from \(\hat{L}_z|m\rangle = m|m\rangle\), we also have

\[
\hat{L}_z(s)|m-1+s\rangle = (m-1+s)|m-1+s\rangle.
\] (13)

From (12), we have the following unitary transformation

\[
e^{is\hat{\phi}_l} = \sum_{m=-l}^{l-1} |m+s\rangle \langle m| + |l-1+s\rangle \langle l|,
\] (14)

which may be looked as a generalized phase operator producing arbitrary shift in the angular momentum states. Its operation on angular momentum states has been shown pictorially in the Figure. Note that for \(s = 1\), we have the standard phase operator formalism in finite space [8, 9].

Before we present results for matrix elements, a remark is in place. For infinite dimensional case, we can take the \(s \to 0\) limit at the operator level, for example, in (11). Thus one recovers the canonical commutator \([\hat{\phi}_l, \hat{L}_z]\) from the quantity \(\frac{R}{is}\), in the \(s \to 0\) limit. However, this is not correct within finite space theory, as can be seen from (14); taking \(s \to 0\) limit does not give the identity operator. The reason for this difference is that in order to satisfy unitarity, the operator (14) produces a shift of \(s\) in all \(|m\rangle\) states, except for \(m = l\) case, for which the shift is \(\sigma = -2l-1+s\). This discontinuity in the shift property of unitary phase operator is also present in the \(s = 1\) case of standard formalism [9]. Note that we should investigate the limit in which magnitude of the shift produced goes to zero. In the infinite dimensional case, shifts in all states were of equal magnitude. As an implication, separate limits are to be taken depending on whether in the
calculation of matrix elements, the state $|l\rangle$ is involved or not. Thus the said
limit is taken only after the matrix elements are calculated.

Now observing that
\[
\langle n|e^{is\hat{\varphi}_l}\hat{L}_z|k\rangle = k \sum_{m=-l}^{l-1} \langle n|m+s\rangle \delta_{mk} + k\langle n|l-1+s\rangle \delta_{lk},
\]
and
\[
\langle n|\hat{L}_z(s)e^{is\hat{\varphi}_l}|k\rangle = \sum_{m=-l}^{l-1} (m+s)\langle n|m+s\rangle \delta_{mk} + (-l-1+s)\langle n|l-1+s\rangle \delta_{lk}
\]
we can calculate $\langle n|\hat{R}|k\rangle$. To evaluate proper limit, we distinguish two cases:

(i) If for some $m$ in the range $-l \leq m \leq l-1$, $k = m$, then we obtain
\[
\langle n|\hat{R}|k\rangle = -s\langle n|k+s\rangle.
\]
Thus we get
\[
\lim_{s \to 0} \frac{1}{is} \langle n|\hat{R}|k\rangle = i\delta_{nk}.
\]
(ii) If $k = l$, then
\[
\langle n|\hat{R}|l\rangle = -\sigma\langle n|l+\sigma\rangle,
\]
where parameter $\sigma$ has been defined above. Therefore considering the limit when
the shift $\sigma$ approaches zero
\[
\lim_{\sigma \to 0} \frac{1}{i\sigma} \langle n|\hat{R}|l\rangle = i\delta_{nl}.
\]
Thus from our result for infinite dimensional space, we might expect that limiting
behaviour of $\langle n|\hat{R}|k\rangle$ in finite space, may yield the matrix elements such as (15).
However, we see that the obtained matrix elements (18) and (20) exactly cor-
respond with the elements (2), provided the limits in which the shift parameter
goes to zero are carefully taken.

Concluding, we have presented a new derivation for matrix elements of canonical
commutator between phase and angular momentum operators. We have
utilised the more general state space of non-periodic eigenfunctions over which
the spectrum of angular momentum can be fractional. The matrix elements are
obtained in the limiting case \((s \to 0)\), when integer valued angular momenta are
realised. We remark that the limit \(s \to 0\) has been used here only as calculational
tool; we do not imply that matrix elements of the canonical commutator are well
defined only for \(s \to 0\) case. In particular, we do not question the relevance
of canonical commutator \([\hat{\phi}, \hat{L}_z(s)] = i\) for circumstances where \(s > 0\) may be
significant \[3, 4\]. We have extended the technique to finite dimensional space
also, where it is required to introduce a generalization \[14\] of unitary phase op-
erator such that standard case \[9\] of \(s = 1\) is also recovered. It is argued that
similar matrix elements as of infinite space can be obtained. Thus we have here
another example of the correspondence between predictions of quantum theories
for rotation angle in infinite and finite dimensional spaces.

**Acknowledgements**

The author is grateful to Alexander von Humboldt Foundation, Germany for
financial support.
References

[1] A. Barone and G. Paterno, Physics and Applications of the Josephson Effect (Wiley, New York, 1982).

[2] M. Peshkin and A. Tonamura, The Aharonov-Bohm Effect (Springer, New York, 1989).

[3] B. Halperin, Phys. Rev. Lett. 52 (1984) 1583;
D. Arovas, J.R. Schrieffer and F. Wilczek, Phys. Rev. Lett. 53 (1984) 722.

[4] J.M. Leinaas and J. Myrheim, Phys. Rev. B 37 (1988) 9286.

[5] P.A.M. Dirac, Proc. R. Soc. London, Ser. A 114 (1927) 243.

[6] L. Susskind and J. Glogower, Physics 1 (1964) 49.

[7] T. S. Santhanam, Phys. Lett. A 56 (1976) 345.

[8] I. Goldhirsch, J. Phys. A: Math. Gen. 13 (1980) 3479.

[9] S. M. Barnett and D. T. Pegg, Phys. Rev. A 41 (1990) 3427.

[10] R.S. Johal, Phys. Lett. A 263 (1999) 62.

[11] R. Lynch, Phys. Rep. 256 (1995) 367 and references therein.

[12] K. Kraus, Z. Phys. 188 (1965) 374; D. Judge, Phys. Lett. 5 (1963) 189.

[13] D. Judge and J.T. Lewis, Phys. Lett. 5 (1963) 190.

[14] P. Carruthers and M.M. Nieto, Rev. Mod. Phys. 40 (1968) 411.

[15] D. Loss and K. Mullen, J. Phys. A: Math. Gen. 25 (1992) L235.

[16] R.S. Johal, Mod. Phys. Lett. B 14 (2000) 961.
Figure 1: The states $|m\rangle$ and $|m-1+s\rangle$ of the $(2l+1)$-dimensional bases can be related by the unitary transformation $e^{is\hat{\phi}}$ of (14), as shown by arrows. For $s = 1$, the unitary operator of [8, 9] is obtained which shifts the states by unity within basis $|m\rangle$. 