A STUDY OF GENERAL 2D, N=2 MATTER COUPLED TO SUPERGRAVITY IN SUPERSPACE

S. James Gates, Jr

Department of Physics
University of Maryland at College Park
College Park, MD 20742-4111, USA

and

M. T. Grisaru

and

M. E. Wehlau

Physics Department
Brandeis University
Waltham, MA 02254, USA

ABSTRACT

We study two-dimensional $N = 2$ supersymmetric actions describing general models of scalar and vector multiplets coupled to supergravity.

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1 Introduction

Aspects of 2D local supersymmetry theories have been studied most extensively in connection with realizations of local superconformal symmetry and its applications to string theory [1], conformal field theory [2], etc. But even beyond these applications where superconformal symmetry is of paramount importance, local supersymmetric actions have been observed to play an important role for integrable systems [3]. On one hand, we have the observation [4] that 2D, \( N = 2 \) superstrings possess very close relations to self-dual target space theories. On the other hand there is the well-known conjecture due to Atiyah that suggests a relation between self-dual Yang-Mills gauge theory and integrable models in less than four dimensions [5]. Some evidence has been presented [6] to indicate that the supersymmetric extension of this suggestion is also valid. It is in this spirit that one might believe that a general study of 2D locally supersymmetric Lagrangian field theories is worthwhile.

With this as our main motivation, we propose to answer the question, “What is the most general 2D, \( N = 2 \) action involving scalar and vector multiplets coupled to supergravity?” The analogous question for 4D, \( N = 1 \) supergravity coupled to supermatter was answered a very long time ago and plays a crucial role in determining the low-energy superstring theory effective action which in turn provides the starting point in many discussions of string-inspired GUT models. As we shall see, 2D, \( N = 2 \) theory brings about some new features in comparison to 4D, \( N = 1 \) theory. To some extent this is due to the special role of the 2D superconformal group. However, another feature is the fact that 2D, \( N = 2 \) models can realize another symmetry that presently has no known 4D field theoretic analogue, namely “mirror symmetry.” As far as 2D supersymmetric field theory is concerned, mirror symmetry has its origins in the simple fact that the fundamental and simplest representation of 2D, \( N = 2 \) supersymmetry, the scalar multiplet, comes in two different varieties: there are “chiral” multiplets [7] and there are “twisted chiral” multiplets [8].

It is possible to consider a transformation where a given theory involving chiral multiplets is mapped into a theory of twisted chiral multiplets and vice versa. This is the mirror map. Theories that are invariant under this map are said to possess mirror symmetry [9]. In addition to its applications in string theory, mirror symmetry is proving to be a useful tool in algebraic geometry and has topological implications too. It is possible that mirror symmetry plays a useful role in the study of some integrable systems as well. Although we will not study this question, this provides added motivation for us to emphasize the mirror-symmetric aspects of the models we consider.

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1 Although we take the name for this symmetry from its designation in string theory [9], the discovery of the twisted chiral multiplet preceeded the discovery of mirror symmetry.
2 The Mirror Versions of 2D, N = 2 Supermatter

There is a simple argument\(^2\) to show the existence of mirror versions of 2D, N = 2 scalar multiplets, namely chiral and twisted chiral multiplets. One of the consequences of supersymmetry is the existence of equal numbers of bosons and fermions within a supersymmetric multiplet. In two dimensions, N = 2 scalar multiplets must contain two spin-0 bosonic fields and two spin-1/2 fermionic fields. The two scalars can be presented as complex fields, say \(\phi\) and \(\bar{\phi}\), which are the leading components of superfields \(\Phi\) and \(\bar{\Phi}\). The spinor components can be defined via application of the superspace spinor derivatives \(D_+\), \(D_-\), \(D_\perp\), and \(D_-\) to either \(\Phi\) or \(\bar{\Phi}\). Thus, for possible spinor components we have,

\[
D_+ \Phi , \quad D_- \Phi , \quad D_\perp \Phi , \quad D_- \Phi .
\]

Since there are too many spinor states to describe the required two spin-1/2 degrees of freedom we are forced to set four of the derivatives to zero in order to obtain a minimal multiplet. If we set \(D_+ \Phi = D_- \Phi = 0\), we find a chiral multiplet. If we set \(D_\perp \Phi = D_- \Phi = 0\), we find a twisted chiral multiplet. Finally if we set \(D_+ \Phi = D_- \Phi = 0\), (breaking 2D chiral symmetry) we find a 2D, (2,0) righton multiplet. There are no other choices.\(^3\) Thus, simple counting reveals the presence of the twisted chiral multiplet as a possibility in any 2D supersymmetric theory. Of course, nonminimal representations are possible. For example, the four extra spinors above may be identified as auxiliary spinors. There are two ways of doing this, leading to the 2D, N = 2 complex and twisted complex linear scalar multiplets. We shall generically denote chiral superfields by \(\Phi\) and twisted chiral superfields by \(\mathcal{X}\), satisfying the constraints \(D_+ \Phi = D_- \Phi = 0\) and \(D_\perp \mathcal{X} = D_- \mathcal{X} = 0\) respectively, as well as corresponding conditions for their complex conjugates.

Having defined the two irreducible scalar multiplets through their respective differential constraints, the component fields are defined using the projection method. These are for the chiral multiplet

\[
\Phi| = \phi , \quad \bar{\Phi}| = \bar{\phi} \\
D_+ \Phi| = \psi_+ , \quad D_\perp \Phi| = \psi_+ \\
D_- \Phi| = \psi_- , \quad D_- \Phi| = \psi_-
\]

and for the twisted chiral multiplet

\[
\mathcal{X}| = \chi , \quad \bar{\mathcal{X}}| = \bar{\chi}
\]

\(^2\)This observation is due to T.Hubsch.

\(^3\)From this point of view, the existence of semi-chiral superfields which rely on nonlinear constraints is puzzling.
\[ D_+ \mathcal{X} = \eta_+ , \quad D_- \mathcal{X} = \eta_- \]
\[ D_+ \overline{\mathcal{X}} = \eta_+ , \quad D_- \overline{\mathcal{X}} = \eta_- \]
\[ \frac{i}{2} [D_+, D_-] \mathcal{X} = G , \quad \frac{i}{2} [D_+, D_-] \overline{\mathcal{X}} = \overline{G} . \]

Although we will not study them in detail, we discuss briefly the nonminimal scalar multiplets [7] described by complex linear superfields \( \Sigma \) and twisted complex linear superfields \( \Xi \), and defined through the second order differential equations,
\[ D_+ D_- \Sigma = 0 , \quad D_+ D_- \Xi = 0 . \] (2.4)
The components of the non-minimal multiplet \( \Sigma \) are given by
\[ B \equiv \Sigma | , \quad \zeta_+ \equiv D_+ \Sigma | , \quad \zeta_- \equiv D_- \Sigma | , \]
\[ \rho_+ \equiv D_+ \Sigma | , \quad \rho_- \equiv D_- \Sigma | , \quad H \equiv -i D_+ D_- \Sigma | , \]
\[ u \equiv -i D_+ D_- \Sigma | , \quad v \equiv -i D_+ D_- \Sigma | , \quad p_+ \equiv -i D_+ D_- \Sigma | , \]
\[ p_- \equiv -i D_+ D_- \Sigma | , \quad \beta_+ \equiv -i D_+ D_- \Sigma | , \quad \beta_- \equiv -i D_+ D_- \Sigma | , \] (2.5)
with the components of \( \Xi \) defined in a similar manner. We see that many more components exist for the non-minimal multiplet than for the chiral multiplet. However, by explicit evaluation of the free kinetic energy actions one can show that only \( B, \zeta_+ \) and \( \zeta_- \) are propagating degrees of freedom and this is exactly the same as for a chiral multiplet.

As in 4D, \( N = 1 \) superspace, there are several types of superspace Berezinian integrals associated with the D-terms and F-terms (and their conjugates). The D-term superinvariants are of the form
\[ S_D = \int d\theta^+ d\theta^- d\theta^+ d\theta^- \mathcal{L} , \] (2.6)
where \( \mathcal{L} \) is any general superfield. The F-term superinvariants are obtained from expressions of the form
\[ S_F = \int d\theta^+ d\theta^- \mathcal{L}_c , \] (2.7)
where \( \mathcal{L}_c \) is a chiral superfield. However, unlike 4D, \( N = 1 \) superspace, there is an additional possibility of forming superinvariants, namely
\[ S_{TF} = \int d\theta^+ d\theta^- \mathcal{L}_{tc} . \] (2.8)
We can call these “twisted F-terms” in analogy with the usual F-terms. The superfield \( \mathcal{L}_{tc} \) must be a twisted chiral superfield.

These actions can be used to construct in particular, as is well-known, nonlinear \( \sigma \)-models involving chiral and twisted chiral superfields. Furthermore, the existence
of linear multiplets gives rise to new possibilities for the manifest realization of $2D$, $N = 2$ nonlinear $\sigma$-models with torsion. The simplest linear kinetic term describing the free propagation of several such multiplets is given by

$$S = \int d^2x \int d^4\theta \left[ - \sum_i \Sigma^i + \Xi^I \Xi_I \right]. \quad (2.9)$$

More complicated actions that involve $\alpha(\Sigma^i \Sigma_i + \sum^I \sum^I)$ terms as well as $\beta(\Xi^I \Xi^I + \Xi_I \Xi_I)$ may also be used. For most values of the constant parameters $\alpha$ and $\beta$, these correspond to field redefinitions at the level of component fields. Clearly, it is possible to describe non-linear $\sigma$-models that also involve ordinary chiral and twisted chiral supermultiplets by considering a general action of the form

$$S = \int d^2x \int d^4\theta \hat{\Omega}(\Phi, X, \Sigma, \Xi, \bar{\Phi}, \bar{X}, \bar{\Sigma}, \bar{\Xi}), \quad (2.10)$$

where we regard each type of superfield as providing a coordinate to describe a subspace of a complex manifold. A remarkable point is that there are now new manifest $(2,2)$ supersymmetric ways to introduce torsion on the complex manifold. We note that the superfields $D_+ \Sigma$ and $D_- \Sigma$ are chiral superfields while $D_+ \Xi$ and $D_- \Xi$ are twisted chiral superfields. Therefore it is possible to introduce additional terms in the action of the form

$$S' = \int d^2x \int d^2\theta \left[ h_{JK}(\Phi)(D_+ \Sigma^J)(D_- \Sigma^K) + \text{h.c.} \right]$$

$$+ \int d^2x \int d\theta^+ d\theta^- \left[ k_i(\mathcal{X})(D_+ \Xi^i)(D_- \Xi^i) + \text{h.c.} \right], \quad (2.11)$$

which may be added freely to the non-linear $\sigma$-model term in $(2.10)$.

Let us specifically point out what is new about the sum of $(2.10)$ and $(2.11)$. For a suitable class of such models, constructed with two multiplets, we expect conformal invariance to be maintained at the quantum level. As a specific example, let us retain one chiral scalar and one non-minimal scalar multiplet. The target space is then a four-dimensional real manifold and in the usual manner of string theory possesses a metric, axion and dilaton. In our specific example, these three target space fields are determined by $\hat{\Omega}$ and $h_{JK}$. The new feature of this construction is that previously a single real potential determined all three target space fields. In these models, the target space fields are determined by both $\hat{\Omega}$ and $h_{JK}$ and the geometrical constraints of the older models may no longer apply. Thus we have additional $2D$, $N = 2$ superstrings beyond the three that were pointed out in 1989 [1]. These new $N = 2$ superstrings may lead to new classes of exact solutions for $4D$ gravity coupled to matter.

In a $N = 2$ string/conformal field theory context, the feature that distinguishes chiral and twisted chiral modes is the assignment of their $U(1)$ charges, and the
mirror map is defined as the isomorphism that interchanges these modes \[12\]. In a lagrangian field theory context we introduce a number of chiral multiplets (denoted by \(\Phi^i\)) and an equal number of twisted chiral multiplets (denoted by \(\mathcal{X}^i\)). An operator \(\mathcal{M}_m\), “the mirror map operator”, can be introduced through its action on these fields

\[
\mathcal{M}_m : \Phi^i = \mathcal{X}^i, \quad \mathcal{M}_m : \mathcal{X}^i = \Phi^i. \tag{2.12}
\]

The mirror map transformation within the context of supersymmetric 2D \(N = 2\) field theory is somewhat similar to the mapping of a scalar into a pseudo scalar.

Let \(\mathcal{L}(\Phi, \mathcal{X})\) denote a “D-term” superfield Lagrangian that depends on chiral and twisted chiral superfields. If the equation

\[
\mathcal{M}_m : \mathcal{L} = - \mathcal{L}, \tag{2.13}
\]

is satisfied, then we say the Lagrangian possesses “mirror symmetry.” Two examples of such Lagrangians are provided by,

\[
\mathcal{L}_1 = \bar{\Phi}\Phi - \bar{\mathcal{X}}\mathcal{X}, \quad \mathcal{L}_2 = \ln \left( \frac{\bar{\Phi}\Phi}{\bar{\mathcal{X}}\mathcal{X}} \right). \tag{2.14}
\]

The condition above will lead to an invariant action if \(\mathcal{M}_m\) acts on the spinorial coordinates of 2D, \(N = 2\) superspace according to the rules

\[
\mathcal{M}_m : \theta^+ = \theta^+, \quad \mathcal{M}_m : \theta^- = \theta^-, \quad \mathcal{M}_m : \theta^\dagger = \theta^\dagger, \quad \mathcal{M}_m : \theta^\ddagger = \theta^\ddagger. \tag{2.15}
\]

This equation acts as the fundamental superspace definition of the mirror transformation. Applying this definition of the mirror operator to the coordinates of the scalar superfield immediately leads to the results in (2.12). A lagrangian satisfying (2.13) will lead to an action

\[
S = \int d^2x d^4\theta \mathcal{L} \tag{2.16}
\]

invariant under mirror symmetry, since the sign change in the lagrangian is compensated by the sign change in the measure \(d\theta^+ d\theta^- d\theta^\dagger d\theta^\ddagger \to d\theta^+ d\theta^- d\theta^\dagger d\theta^\ddagger = -d\theta^+ d\theta^- d\theta^\dagger d\theta^\ddagger\).

The argument above applies to the invariance of a superfield Lagrangian that is integrated over the full superspace. It must be slightly modified to cover the case of the chiral integrals. To achieve invariance of superpotentials we must consider an action of the form

\[
\mathcal{S} = \int d^2x d\theta^+ d\theta^- U(\Phi^i) + \int d^2x d\theta^\dagger d\theta^\ddagger U(\mathcal{X}^i) + \text{h.c.}, \tag{2.17}
\]

with the same superpotential function \(U\).
3 The Mirror Versions of 2D, N = 2 SUSY YM

Although it was known some time ago that both chiral and twisted chiral matter multiplets existed [8], it was not realized until the work of Hull, Papadopoulos and Spence [13] and later Roˇcek and Verlinde [14], that the mirror transformation can be extended to 2D, N = 2 supersymmetric vector multiplets. This can be seen through the following simple argument: in all 2D theories wherein a gauge spin-1 field occurs only via its field strength, it is possible to perform a duality transformation to replace the field strength by an auxiliary scalar field. This is the usual Hodge duality transformation in the special case of D = 2. Consequently, one can regard a 2D supersymmetric vector multiplet as simply an off-shell 2D scalar multiplet where one of the usual auxiliary fields has been replaced by the field strength of a gauge spin-1 field. This argument applies directly to the two distinct scalar multiplets (chiral or twisted chiral). Replacing one of the auxiliary fields of the chiral matter multiplet by the field strength of a gauge spin-1 field yields the 2D, N = 2 “twisted vector multiplet”. Replacing one of the auxiliary fields of a twisted chiral matter multiplet by the field strength of a gauge spin-1 field yields the 2D, N = 2 “vector multiplet”.

The two irreducible vector multiplets are defined by the covariant derivatives
\[ \nabla_A = D_A + i\Gamma_A t + i\Gamma_A' t' \]
which satisfy the constraints
\[
\begin{align*}
\{\nabla_+ , \nabla_+ \} &= 0 , & \{\nabla_- , \nabla_- \} &= 0 \\
\{\nabla_+ , \nabla_- \} &= -i\mathcal{P} t' , & \{\nabla_+ , \nabla_\pm \} &= i\mathcal{W} t \\
\{\nabla_\pm , \nabla_\pm \} &= i\nabla_\pm , & \{\nabla_- , \nabla_\pm \} &= i\nabla_\pm \\
[\nabla_+ , \nabla_\pm \} &= 0 , & [\nabla_- , \nabla_\pm \} &= 0 \\
[\nabla_+ , \nabla_\mp \} &= - (\nabla_- \mathcal{W}) t + (\nabla_\pm \mathcal{P}) t' \\
[\nabla_- , \nabla_\pm \} &= (\nabla_+ \mathcal{W}) t + (\nabla_\pm \mathcal{P}) t' \\
[\nabla_\pm , \nabla_\pm \} &= i\mathcal{F} t + i\mathcal{F}' t' ,
\end{align*}
\] (3.1)
and the relations implied by the Bianchi identities. Here t and t’ are Lie algebra generators of two distinct U(1) groups (the generalization to other Lie algebras is simple). It is clear that under the mirror map we have for vector multiplets
\[ \mathcal{M}_m : \mathcal{P} = \mathcal{W} , \quad \mathcal{M}_m : \mathcal{W} = \mathcal{P} \] (3.2)

The action of the Lie algebra generators on covariantly chiral matter fields Φ or twisted chiral matter fields X is restricted by an integrability condition, (e.g. 0 = {\nabla_\pm , \nabla_\pm }Φ = -i\mathcal{P} t' Φ) so that
\[
\begin{align*}
[t , \Phi] &= iq\Phi , & [t' , X] &= iq'X \\
[t' , \Phi] &= 0 , & [t , X] &= 0 ,
\end{align*}
\] (3.3)
where \( q, q' \) are the \( U(1) \) charges of the multiplets. The components of the twisted vector multiplet are given by

\[
\begin{align*}
\mathcal{P} | &= P, \quad \overline{\mathcal{P}} | = \overline{P} \\
\nabla_+ \mathcal{P} | &= \rho_+ \quad \nabla_+ \overline{\mathcal{P}} | = \rho_+ \\
\nabla_- \mathcal{P} | &= \rho_- \quad \nabla_- \overline{\mathcal{P}} | = \rho_- \\
\frac{i}{2} [\nabla_+, \nabla_-] \mathcal{P} | &= H, \quad \frac{i}{2} [\nabla_+, \nabla_-] \overline{\mathcal{P}} | = \overline{H}
\end{align*}
\]

while those of the vector multiplet are given by

\[
\begin{align*}
\mathcal{W} | &= W, \quad \overline{\mathcal{W}} | = \overline{W} \\
\nabla_+ \mathcal{W} | &= \lambda_+ \quad \nabla_+ \overline{\mathcal{W}} | = \lambda_+ \\
\nabla_- \mathcal{W} | &= \lambda_- \quad \nabla_- \overline{\mathcal{W}} | = \lambda_- \\
\frac{i}{2} [\nabla_+, \nabla_-] \mathcal{W} | &= J, \quad \frac{i}{2} [\nabla_+, \nabla_-] \overline{\mathcal{W}} | = \overline{J}
\end{align*}
\]

The actual gauge fields occur through their respective field strengths inside the quantities \( H \) and \( J \) above, as seen by writing \( H \) and \( J \) in terms of their real and imaginary parts

\[
H = \frac{1}{2} [d' + iF(A')] \quad J = \frac{1}{2} [d + iF(A)]
\]

where \( F(A) = \partial_\bullet A_\bullet - \partial_\bullet A_\bullet \) and \( F(A') = \partial_\bullet A'_\bullet - \partial_\bullet A'_\bullet \).

Comparing the component projection equations for the scalar multiplets with those for the vector multiplets shows quite clearly that one of the auxiliary fields in the former is replaced by a spin-1 field strength in the latter.

In closing, it should be mentioned that by the duality argument presented above there are also more vector multiplets and twisted vector multiplets since both the chiral and twisted chiral matter superfields have two auxiliary fields. We can choose to replace both auxiliary fields by the field strengths of gauge spin-1 vectors. Thus, it is possible to have irreducible \( 2D, N = 2 \) multiplets that have two gauge fields in the same multiplet. Obviously, there exist also distinct vector multiplets obtained by applying a duality transformation to auxiliary spin-zero fields of complex linear or twisted complex linear superfields. (This same phenomenon occurs in the description of \( 2D, N = 2 \) supergravity. One can replace some or all of the auxiliary fields that occur in the off-shell theory by field strengths of gauge spin-1 vectors, leading to many distinct off-shell versions of \( 2D, N = 2 \) supergravity.)
4 The Mirror Versions of 2D, $N = 2$ Supergravity

In the previous sections we saw that matter scalar and vector multiplets realizing 2D, $N = 2$ supersymmetry come in mirror realizations. In the same manner, irreducible $N = 2$ supergravity comes in mirror versions, the so-called $U_V(1)$ and $U_A(1)$ versions \[15, 11, 16\]. Mirror symmetry can be realized in $N = 2$ supergravity by considering the reducible $U_V(1) \otimes U_A(1)$ theory \[15, 16\].

The mirror symmetric form of 2D, $U_V(1) \otimes U_A(1) N = 2$ supergravity is described in ref. \[15\] and further discussed in ref. \[16\]. Its tangent space symmetries are Lorentz, $U_V(1)$ and $U_A(1)$. Their respective generators (with a slight change of notation from the above references) are denoted by $\mathcal{M}$, $\mathcal{Y}$ and $\mathcal{Y}'$. Their action on spinors is

\[
[\mathcal{M}, \psi_\pm] = \pm \frac{1}{2} \psi_\pm, \quad [\mathcal{M}, \bar{\psi}_\pm] = \pm \frac{1}{2} \bar{\psi}_\pm
\]

\[
[\mathcal{Y}, \psi_\pm] = -i \frac{1}{2} \psi_\pm, \quad [\mathcal{Y}, \bar{\psi}_\pm] = +i \frac{1}{2} \bar{\psi}_\pm
\]

\[
[\mathcal{Y}', \psi_\pm] = \mp i \frac{1}{2} \psi_\pm, \quad [\mathcal{Y}', \bar{\psi}_\pm] = \pm i \frac{1}{2} \bar{\psi}_\pm.
\]

(4.1)

We also define the combinations

\[
M = \frac{1}{2}(\mathcal{M} + i\mathcal{Y}), \quad \bar{M} = \frac{1}{2}(\mathcal{M} - i\mathcal{Y})
\]

\[
N = \frac{1}{2}(\mathcal{M} + i\mathcal{Y}), \quad \bar{N} = \frac{1}{2}(\mathcal{M} - i\mathcal{Y}).
\]

(4.2)

The covariant derivatives are defined by

\[
\hat{\nabla}_A = E_A + \Phi_A \mathcal{M} + \Sigma_A \mathcal{Y} + \Sigma_A \mathcal{Y}'.
\]

(4.3)

The constraints which define the 2D, $N = 2 U_V(1) \otimes U_A(1)$ supergravity are given by

\[
\{\tilde{\nabla}_+, \tilde{\nabla}_+\} = 0, \quad \{\tilde{\nabla}_-, \tilde{\nabla}_-\} = 0, \quad \{\tilde{\nabla}_+, \tilde{\nabla}_\pm\} = -F_\mathcal{N},
\]

\[
\{\tilde{\nabla}_+, \tilde{\nabla}_-\} = -R_M,
\]

\[
\{\tilde{\nabla}_+, \tilde{\nabla}_+\} = i\tilde{\nabla}_+, \quad \{\tilde{\nabla}_-, \tilde{\nabla}_-\} = i\tilde{\nabla}_-.
\]

(4.4)

In writing this, we have used a “hatted” symbol for the $U_V(1) \otimes U_A(1)$ covariant derivatives to distinguish them from the irreducible derivatives below.

\[4\]This again suggests that it may actually be possible to find remnants of mirror symmetry in supersymmetric integrable systems.

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From these it follows that

\[
[\hat{\nabla}_+ , \hat{\nabla}_\pm ] = 0 , \quad [\hat{\nabla}_- , \hat{\nabla}_\mp ] = 0 , \\
\hat{\nabla}_\pm , \hat{\nabla}_\pm ] = 0 , \quad [\hat{\nabla}_\pm , \hat{\nabla}_\mp ] = 0 , \\
[\hat{\nabla}_+ , \hat{\nabla}_+ ] = -\frac{i}{2} R \hat{\nabla}_+ - i(\hat{\nabla}_+ R)\hat{\nabla}_- - \frac{i}{2} F \hat{\nabla}_- - i(\hat{\nabla}_- F)\hat{\nabla}_+ , \\
[\hat{\nabla}_- , \hat{\nabla}_- ] = \frac{i}{2} R \hat{\nabla}_- + i(\hat{\nabla}_- R)\hat{\nabla}_+ + \frac{i}{2} F \hat{\nabla}_+ + i(\hat{\nabla}_+ F)\hat{\nabla}_- , \\
[\hat{\nabla}_\pm , \hat{\nabla}_\pm ] = -\frac{i}{2} R \hat{\nabla}_\pm + i(\hat{\nabla}_\pm R)\hat{\nabla}_\pm - \frac{i}{2} F \hat{\nabla}_\pm + i(\hat{\nabla}_\pm F)\hat{\nabla}_\pm ,
\] (4.5)

and also

\[
[\hat{\nabla}_+, \hat{\nabla}_\pm ] = \frac{1}{2}(\hat{\nabla}_+ R)\hat{\nabla}_- + \frac{1}{2}(\hat{\nabla}_- R)\hat{\nabla}_+ - \frac{1}{2}(\hat{\nabla}_+ R)\hat{\nabla}_- - \frac{1}{2}(\hat{\nabla}_- R)\hat{\nabla}_+ \\
-\frac{1}{2} R R \hat{\nabla}_- - \frac{1}{2} R R M + (\hat{\nabla}_2 R) M - (\hat{\nabla}_2 R) \hat{\nabla}_M \\
+\frac{1}{2}(\hat{\nabla}_+ F)\hat{\nabla}_- + \frac{1}{2}(\hat{\nabla}_+ F)\hat{\nabla}_+ - \frac{1}{2}(\hat{\nabla}_+ F)\hat{\nabla}_- - \frac{1}{2}(\hat{\nabla}_+ F)\hat{\nabla}_+ \\
-\frac{1}{2} F \hat{\nabla}_- I + \frac{1}{2} F \hat{\nabla}_- + (\hat{\nabla}_+ \hat{\nabla}_2 F) N - (\hat{\nabla}_2 \hat{\nabla}_+ \hat{\nabla}_- F) \hat{\nabla}_+ .
\] (4.6)

The two distinct irreducible forms of 2D, \( N = 2 \) supergravity are obtained by restricting the gauge group so that either \( F = 0 \) for the \( U_A(1) \) version, or \( R = 0 \) for the \( U'V(1) \) version, so that the corresponding connections \( \Sigma_A \) or \( \Sigma'_A = \Omega_A - \Gamma_A \) are pure gauge and can be removed. Thus, the \( U_V(1) \) theory is described by

\[
\{\nabla_+ , \nabla_+ \} = 0 , \quad \{\nabla_- , \nabla_- \} = 0 , \quad \{\nabla_+ , \nabla_\mp \} = -\hat{\bar{F}} \hat{N} ,
\]

\[
\{\nabla_+ , \nabla_- \} = 0 ,
\]

\[
\{\nabla_+ , \nabla_\pm \} = i\nabla_\pm , \quad \{\nabla_- , \nabla_\pm \} = i\nabla_\mp ,
\] (4.7)

while the \( U_A(1) \) theory is described by

\[
\{\nabla_+ , \nabla_+ \} = 0 , \quad \{\nabla_- , \nabla_- \} = 0 , \quad \{\nabla_+ , \nabla_\mp \} = 0 ,
\]

\[
\{\nabla_+ , \nabla_- \} = -R \hat{M} ,
\]

\[
\{\nabla_+ , \nabla_\pm \} = i\nabla_\pm , \quad \{\nabla_- , \nabla_\pm \} = i\nabla_\mp ,
\] (4.8)

and their consequences. However, setting either \( \bar{F} \) or \( R \) to zero breaks mirror symmetry. Thus, in order to construct a local 2D, \( N = 2 \), mirror symmetric, supersymmetric theory, we must work with the reducible representation in (4.4).
Let us note that there are precedents for working with a reducible supergravity representation. The most important of these is the \(4D, N = 1\) limit of heterotic string theory. It is known that in this limit the supergravity multiplet appears along with a specific matter multiplet (the linear or tensor multiplet) that contains the axion, dilaton and dilatino. Thus, the low energy limit of the heterotic string is reducible with respect to supersymmetry. However, it forms an irreducible representation of some larger symmetry present in string theories. The reducible representation described by (4.4) is the analogue of the supergravity plus tensor multiplets seen in the low energy limit of superstring theory.

In [17] we have shown how to construct the covariant derivatives of the \(U_V(1) \otimes U_A(1)\) theory in terms of those of the \(U_A(1)\) theory and the prepotentials that solve the supergravity constraints. An equivalent procedure consists of starting with the supergravity covariant derivatives that solve the constraints of the \(U_A(1)\) covariant derivative in (4.8), and “entangle” them with a matter twisted vector multiplet. We define new covariant derivatives through

\[
\hat{\nabla}^+ = \exp\left[\frac{1}{2} (k + l) V \right] \left[ \nabla^+ + (\nabla^+ V) (- (k + m) M + i k Y + i k Y') \right], \\
\hat{\nabla}^- = \exp\left[\frac{1}{2} (k - l) V \right] \left[ \nabla^- + (\nabla^- V) ((k - m) M + i k Y + i k Y') \right]
\]

(4.9)

where \(V\) is an arbitrary real scalar superfield and \(k, l\) and \(m\) are real parameters. We note that since we are dealing with an \(N = 2\) theory, the scale transformation represented by the exponential factor on the right-hand-side of these equations does not lead to a rescaling of the vielbein determinant (as would be the case for other values of \(N\)), and hence

\[
\hat{E}^{-1} = E^{-1}.
\]

(4.10)

The commutator algebra of the covariant derivatives thus defined is isomorphic to the commutator algebra of the \(U_V(1) \otimes U_A(1)\) supergravity covariant derivative. In particular, the field strengths are given by

\[
\bar{F}_{U_V \otimes U_A} = 4k \exp[-kV][\nabla^+ \nabla V] , \quad \bar{R}_{U_V \otimes U_A} = \exp[-kV]\left[\bar{R}_{U_A} - 4k(\nabla^+ \nabla V)\right].
\]

(4.11)

An equivalent result is obtained if one starts from the irreducible \(U_V(1)\) supergravity covariant derivative and entangles properly with a (untwisted) vector multiplet.

5 Local Integration in 2D, \(N = 2\) Superspace

In this section we summarize the information concerning the local supermeasures for generalizing the global invariants in (2.3)-(2.8) in the presence of supergravity. We
are primarily interested in the projection formulae for obtaining component actions from the corresponding superspace actions.

In local supersymmetry, the superinvariant given by a full superspace integral involves the vielbein superdeterminant $E$:

$$ S_D = \int d\theta^+ d\theta^- d\theta^\dagger d\theta^\ddagger E^{-1} \mathcal{L} , $$

where $\mathcal{L}$ is a general superfield. This expression is valid for either of the minimal theories, or the non-minimal one, with the same $E$.

Corresponding to the global F-term invariants one finds local expressions of the form

$$ S_F = \int d\theta^+ d\theta^- \mathcal{E}^{-1} \mathcal{L}_C $$

and

$$ S_{TF} = \int d\theta^+ d\theta^- \tilde{\mathcal{E}}^{-1} \mathcal{L}_{TC} , $$

where $\mathcal{L}_C$ and $\mathcal{L}_{TC}$ are covariantly chiral and twisted chiral superfields and $\mathcal{E}^{-1}$ and $\tilde{\mathcal{E}}^{-1}$ are suitable measures. For $U_A(1)$ supergravity, which contains the chiral compensator $\sigma$ in addition to the (vector) prepotential $H^a$, the chiral measure is given explicitly by

$$ \mathcal{E}^{-1} = e^{-2\sigma} (1.e^\tilde{H}) $$

and a covariantly chiral $\mathcal{L}_C$ is given in terms of an ordinary chiral $\mathcal{L}_c$ by

$$ \mathcal{L}_C = e^H \mathcal{L}_c e^{-H} $$

or by $\mathcal{L}_C = \tilde{\nabla}^2 \mathcal{L}$ in terms of a general superfield. In the $U_V(1)$ theory similar results hold for the mirror-mapped quantities, i.e. twisted chiral measure, etc. (We are using the notation $H = iH^m \partial_m$ where in $\tilde{H}$ the derivative acts on everything to its left.)

In [17], we have established the existence of the measures $\mathcal{E}^{-1}, \tilde{\mathcal{E}}^{-1}$ in the $U_V(1) \otimes U_A(1)$ theory, and given an explicit formula for the former. However for our purpose, the existence of the full superspace measure $E^{-1}$ and of the chiral measure $\mathcal{E}^{-1}$, along with the projection formulae, will suffice.

Component actions are obtained from superspace actions by projection formulae. We consider here the $U_A(1)$ version of supergravity described by the vector prepotential $H^a$ and the chiral scalar prepotential $\sigma$. (In subsection 7.5 we will discuss generalizations for the $U_V(1) \otimes U_A(1)$ theory.) In order to go from a full superspace integral to a chiral one we use the formula (for a general $\mathcal{L}$)

$$ \int d^2xd^4\theta E^{-1} \mathcal{L} = \int d^2xd^2\theta \mathcal{E}^{-1} \tilde{\nabla}^2 \mathcal{L} \bigg|_{\theta = 0} . $$
This superspace integral can be reduced to a component action by means of the chiral density projection formula \[17\]

\[
\int d^2 x d^4 \theta E^{-1} \mathcal{L} = \int d^2 x e^{-\frac{1}{2} \nabla^2 \bar{\mathcal{L}}} \left[ \nabla^2 + i \psi^\dagger_\alpha \nabla_\alpha - i \psi^*_\alpha \nabla_\alpha + \left( -\frac{1}{2} \bar{B} - \psi^*_\alpha \psi^\dagger_\alpha + \psi^\dagger_\alpha \psi^*_\alpha \right) \right] \nabla^2 \mathcal{L} \tag{5.7}
\]

where \( \psi^\dagger_\alpha \) is the gravitino field and \( \bar{\mathcal{R}} = \bar{B} \).

Since, by construction, the expression on the right-hand-side of this equation is invariant, we can replace \( \nabla^2 \mathcal{L} \) by any covariantly chiral lagrangian \( \mathcal{L}_C \) and obtain the component projection formula for any F-type local invariant:

\[
\int d^2 x d^4 \theta \bar{\mathcal{E}}^{-1} \mathcal{L}_C = \int d^2 x e^{-\frac{1}{2} \nabla^2 \bar{\mathcal{L}}} \left[ \nabla^2 + i \psi^\dagger_\alpha \nabla_\alpha - i \psi^*_\alpha \nabla_\alpha + \left( -\frac{1}{2} \bar{B} - \psi^*_\alpha \psi^\dagger_\alpha + \psi^\dagger_\alpha \psi^*_\alpha \right) \right] \mathcal{L}_C \tag{5.8}
\]

By straightforward but non-trivial algebra \[17\], the general component expression can be rewritten in terms of twisted chiral projectors given by

\[
\int d^2 x d^4 \theta E^{-1} \mathcal{L} = \int d^2 x e^{-\frac{1}{2} \nabla^2 \bar{\mathcal{L}}} \left[ \nabla^2 + i \psi^\dagger_\alpha \nabla_\alpha - i \psi^*_\alpha \nabla_\alpha + \left( \psi^\dagger_\alpha \psi^\dagger_- + \psi^\dagger_- \psi^\dagger_+ \right) \right] \nabla^2 \mathcal{L} \tag{5.9}
\]

Although in the \( U_A(1) \) theory we do not know an explicit expression for \( \bar{\mathcal{E}}^{-1} \), it is reasonable to assume that replacing \( \nabla^2 \mathcal{L} \) by any covariantly twisted chiral lagrangian \( \mathcal{L}_{TC} \) would allow us to project to components a corresponding twisted F-term according to

\[
\int d^2 x d^4 \theta \bar{\mathcal{E}}^{-1} \mathcal{L}_{TC} = -\int d^2 x e^{-\frac{1}{2} \nabla^2 \bar{\mathcal{L}}} \left[ \nabla^2 + i \psi^\dagger_\alpha \nabla_\alpha - i \psi^*_\alpha \nabla_\alpha + \left( \psi^\dagger_\alpha \psi^\dagger_- + \psi^\dagger_- \psi^\dagger_+ \right) \right] \mathcal{L}_{TC} \tag{5.10}
\]

We are now in a position to construct component lagrangians for a variety of models, but let us conclude this section with the following remark:

In \( D = 4 \), it is known that the transition from global to local theory for a general Kähler action \( \int d^4 \theta K(\Phi, \bar{\Phi}) \), invariant under Kähler transformations \( K \rightarrow K + \Lambda_c + \bar{\Lambda}_a \) \( (D\Lambda_c = D\bar{\Lambda}_a = 0) \), requires replacing the global action by the local action

\[
\mathcal{S} = \int d^4 x d^4 \theta E^{-1} e^K \tag{5.11}
\]
simply because under the Kähler transformation $K \rightarrow K + \Lambda C + \bar{\Lambda} A$ ($\nabla \Lambda_C = \nabla \bar{\Lambda} A = 0$), the variation of $S = \int d^4x d^4\theta E^{-1}K$

$$\delta S = \int d^4\theta E^{-1}(\Lambda_C + \bar{\Lambda} A)$$

$$= \int d^2\theta \sigma^3(\nabla^2 + R)\bar{\Lambda} A + \int d^2\theta \sigma^3(\nabla^2 + R)\Lambda_C \ , \quad (5.12)$$
does not vanish (but in the exponential it can be cancelled by scale transformations of the compensator $\sigma$). However, in $D = 2$ the projection formula (5.6) to (anti-)chiral integrals does not involve the superfield strength $R$ and there is no need to exponentiate the Kähler potential.

6 General Coupling of Scalar and Vector Multiplets

In this section we consider the coupling of matter scalar and vector multiplets. To begin with we take a general function of covariantly chiral and twisted chiral (with respect to the $U(1)$ groups) superfields

$$S = \int d^2x d^4\theta K(\Phi, \overline{\Phi}, \mathcal{X}, \overline{\mathcal{X}}) \ , \quad (6.1)$$
in the presence of vector and twisted vector multiplets and evaluating at $\theta = 0$.

We take this down to components by replacing $d^4\theta$ by $\nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm}$, in this order. Here the covariant derivatives satisfy the constraints in (3.1). After distributing the four derivatives, but for ease of reading suppressing everywhere the derivatives of $K$ with respect to the superfields (factors of $K$ and its derivatives with respect to the fields can be easily reinstated by counting the number, and types of the fields which appear - i.e. a term such as $\nabla_{\pm} \Phi \nabla_{\pm} \bar{\Phi} \nabla_{\pm} \Phi \nabla_{\pm} \mathcal{X}$ should be multiplied by the factor $K_{\Phi, \overline{\Phi}, \mathcal{X}}$), we obtain:

$$S = \int d^2x\{(\nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \overline{\Phi} + \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \overline{\mathcal{X}})$$

$$- (\nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\Phi} + \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\mathcal{X}})(\nabla_{\pm} \Phi + \nabla_{\mp} \mathcal{X})$$

$$+ (\nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\Phi} + \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\mathcal{X}})(\nabla_{\pm} \Phi + \nabla_{\mp} \mathcal{X})$$

$$+ (\nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\Phi} + \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\mathcal{X}})(\nabla_{\pm} \Phi + \nabla_{\mp} \mathcal{X})$$

$$- (\nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\Phi} + \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\mathcal{X}})(\nabla_{\pm} \Phi + \nabla_{\mp} \mathcal{X})$$

$$- (\nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\Phi} + \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\mathcal{X}})(\nabla_{\pm} \Phi + \nabla_{\mp} \mathcal{X})$$

$$- (\nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\Phi} + \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\mathcal{X}})(\nabla_{\pm} \Phi + \nabla_{\mp} \mathcal{X})$$

$$+ (\nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\Phi} + \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\mathcal{X}})(\nabla_{\pm} \Phi + \nabla_{\mp} \mathcal{X})$$

$$+ (\nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\Phi} + \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \nabla_{\pm} \bar{\mathcal{X}})(\nabla_{\pm} \Phi + \nabla_{\mp} \mathcal{X})$$
\[
+ (\nabla_+ \nabla_\pm \Phi + \nabla_+ \nabla_- \Phi) (\nabla_- \nabla_+ \Phi + \nabla_- \nabla_\mp \Phi) \\
- (\nabla_- \Phi + \nabla_- \Phi) (\nabla_- \nabla_- \Phi + \nabla_- \nabla_\pm \Phi) \\
- (\nabla_- \Phi + \nabla_- \Phi) (\nabla_- \nabla_- \Phi + \nabla_- \nabla_\pm \Phi) (\nabla_\pm \Phi + \nabla_\mp \Phi) \\
- (\nabla_- \nabla_- \Phi + \nabla_- \nabla_- \Phi) (\nabla_- \nabla_- \Phi + \nabla_- \nabla_- \Phi) \\
+ (\nabla_- \Phi + \nabla_- \Phi) (\nabla_- \nabla_- \Phi + \nabla_- \nabla_\pm \Phi) (\nabla_- \Phi + \nabla_- \Phi) \\
- (\nabla_- \Phi + \nabla_- \Phi) (\nabla_- \nabla_- \Phi + \nabla_- \nabla_\pm \Phi) (\nabla_- \Phi + \nabla_- \Phi) \\
+ (\nabla_- \Phi + \nabla_- \Phi) (\nabla_- \Phi + \nabla_- \Phi) (\nabla_- \Phi + \nabla_- \Phi) \} .
\]

In the reduction to components, we use the (twisted) chirality conditions and the (anti-)commutators of (3.1) to write

\[
\begin{align*}
\nabla_+ \nabla_- \nabla_\pm \nabla_\mp \Phi &= \left\{ \frac{1}{2} (\nabla_+ \nabla_\pm + \nabla_- \nabla_\pm) - \frac{i}{4} q [\nabla_+, \nabla_-] \nabla \phi - \frac{i}{2} q [\nabla_\pm, \nabla_-] \nabla \phi \\
&+ q (\nabla_- \nabla_\phi) \nabla_- - q (\nabla_+ \nabla_\phi) \nabla_+ - q^2 \nabla \nabla \phi \right\} \\
&= \Box \phi + \frac{i}{2} q J \phi + \frac{i}{2} q J \phi + q \lambda_+ \psi_+ - q \lambda_+ \psi_- - q^2 \nabla \nabla \phi \\
\nabla_+ \nabla_- \nabla_\pm \Phi &= \left\{ \frac{1}{2} q' [\nabla_+ \nabla_\pm, \nabla_] \nabla \phi + q' (\nabla_- \nabla_\pm) \nabla_+ \phi \right\} \\
&= i q' H \chi + q' \rho_+ \eta_+ \\
\nabla_- \nabla_\pm \nabla_\pm \Phi &= \left\{ -i \nabla_\pm \nabla_\pm \Phi - q (\nabla_- \nabla_\phi) \nabla_\phi - q \nabla \nabla \phi \right\} \\
&= -i D_\phi \psi_+ - q \lambda_+ \phi - q \nabla \psi_+ \\
\nabla_- \nabla_\pm \nabla_\pm \Phi &= \left\{ -q' (\nabla_- \nabla_\pm) \nabla_+ \phi - q' \nabla \nabla_\phi \right\} \\
&= -q' \rho_- \chi - q' \nabla \nabla_\phi \\
\nabla_+ \nabla_- \nabla_\pm \Phi &= \left\{ i \nabla_\pm \nabla_- \Phi - q (\nabla_- \nabla_\phi) \nabla_\phi - q \nabla \nabla_- \Phi \right\} \\
&= i D_\phi \psi_- - q \lambda_- \phi - q \nabla \psi_- \\
\nabla_+ \nabla_- \nabla_\pm \phi &= \left\{ -q' (\nabla_- \nabla_\phi) \nabla_\phi \right\} \\
&= -q \rho_+ \chi - q' \nabla \nabla_- \phi \\
\nabla_+ \nabla_- \nabla_\mp \Phi &= \left\{ -q' (\nabla_- \nabla_\phi) \nabla_\phi \right\} \\
&= -q \rho_- \chi - q' \nabla \nabla_- \phi \\
\nabla_+ \nabla_- \nabla_\mp \phi &= \left\{ -q' (\nabla_- \nabla_\phi) \nabla_\phi \right\} \\
&= -q \rho_+ \chi - q' \nabla \nabla_- \phi \\
\nabla_+ \nabla_- \nabla_\pm \phi &= \left\{ -i \nabla_\pm \nabla_- \Phi - i (\nabla_- \nabla_\phi) \nabla_\phi \right\} \\
&= -i D_\phi \eta_- - q' \rho_- \chi - q' \nabla \eta_- \\
\nabla_\pm \nabla_- \phi &= -i F \\
\nabla_\pm \nabla_- \phi &= -i F 
\end{align*}
\]
\[ \nabla_+ \nabla_\phi = q \overline{W} \phi \]
\[ \nabla_- \nabla_\phi = -q W \overline{\phi} \]
\[ \nabla_- \nabla_\phi = i F \]
\[ \nabla_- \nabla_\phi = q W \phi \]
\[ \nabla_+ \nabla_\phi = -q \overline{W} \phi \]
\[ \nabla_- \nabla_\phi = i F \]
\[ \nabla_+ \nabla_- \phi = -q' \overline{P} \overline{\chi} \]
\[ \nabla_+ \nabla_- \phi = -q' P \chi \]
\[ \nabla_+ \nabla_- \chi = -i G \]
\[ \nabla_- \nabla_+ \chi = i G \]
\[ \nabla_- \nabla_+ \chi = q' P \overline{\chi} \]
\[ \nabla_- \nabla_+ \chi = q' P \overline{\chi} . \]

(6.3)

We obtain the “raw” component lagrangian

\[
\begin{align*}
&\left(\Box \phi + \frac{i}{2} q J \phi + \frac{i}{2} q \overline{J} \phi + q \lambda_+ \psi_+ - q \lambda_- \psi_- - q^2 W \overline{W} \phi + i q^2 H \chi - q' \rho_- \eta_+ \right) \\
&+ (i D_\pm \phi + q \lambda_+ \overline{\phi} + q W \psi_+ + q' \rho_- \chi)(\psi_+ + \eta_+) \\
&+ (i D_\pm \phi + q \lambda_- \overline{\phi} + q \overline{W} \psi_- + q' \rho_+ \chi - q' P \eta_+)(\psi_- + \eta_-) \\
&- (F - i q' P \chi)(F - i q' \overline{P} \phi) \\
&+ i(F - i q' P \chi)(\psi_+ + \eta_-)(\psi_+ + \eta_+) \\
&+ (q \lambda_+ \overline{\phi} + q' \rho_- \chi - i D_- \eta_+)(\psi_+ + \eta_+) \\
&+ (D_\pm \phi + D_- \chi)(D_\pm \overline{\phi} + D_- \overline{\chi}) \\
&+ i(D_\pm \phi + D_- \chi)(\psi_+ + \eta_+)(\psi_+ + \eta_+) \\
&- (q W \overline{\phi} - i G)(q W \overline{\phi} - i G) \\
&+ (\psi_+ + \eta_-)(q \lambda_+ \overline{\phi} + i D_+ \eta_- + q' \rho_+ \overline{\chi} + i q' \overline{P} \eta_+) \\
&+ (q W \overline{\phi} - i G)(\psi_+ + \eta_+) \\
&- (q W \overline{\phi} - i G)(\psi_+ + \eta_+)(\psi_- + \eta_-) \\
&+ i(\psi_+ + \eta_-)(D_\pm \overline{\phi} + D_- \overline{\chi})(\psi_- + \eta_-) \\
&+ i(\psi_+ + \eta_-)(\psi_+ + \eta_-)(F - i q' \overline{P} \chi) \\
&+ (\psi_+ + \eta_-)(\psi_+ + \eta_-)(\psi_- + \eta_-)(\psi_+ + \eta_+) \quad (6.4)
\end{align*}
\]

(Here the component derivatives \( D_\pm \) are covariant with respect to both vector and twisted vector multiplets.)

This is followed by some additional manipulations: integration by parts, and the use of relations that follow from the fact that \( K \) is neutral under the separate action.
of the two symmetries gauged by the vector multiplets, namely
\[ \Phi K_\Phi - \bar{\Phi} K_\Phi = 0 \]
\[ \chi K_\chi - \bar{\chi} K_\chi = 0 \quad . \quad (6.5) \]

From these two equations, it is possible to derive many more by applying various spinorial derivatives and evaluating at \( \theta = 0 \). Applying one derivative we obtain the following superspace identities:

\[ \nabla_+ \Phi = (\bar{\Phi} - \Phi)(\nabla_+ \Phi + \nabla_+ \chi) \]
\[ \nabla_- \Phi = (\bar{\Phi} - \Phi)(\nabla_- \Phi + \nabla_- \chi) \]
\[ \nabla_+ \bar{\Phi} = (\Phi - \bar{\Phi})(\nabla_+ \bar{\Phi} + \nabla_+ \bar{\chi}) \]
\[ \nabla_- \bar{\Phi} = (\Phi - \bar{\Phi})(\nabla_- \bar{\Phi} + \nabla_- \bar{\chi}) \]
\[ \nabla_+ \chi = (\bar{\chi} - \chi)(\nabla_+ \chi + \nabla_+ \bar{\chi}) \]
\[ \nabla_- \chi = (\chi - \bar{\chi})(\nabla_- \chi + \nabla_- \bar{\chi}) \]
\[ \nabla_+ \bar{\chi} = (\chi - \bar{\chi})(\nabla_+ \bar{\chi} + \nabla_+ \bar{\chi}) \]
\[ \nabla_- \bar{\chi} = (\bar{\chi} - \chi)(\nabla_- \bar{\chi} + \nabla_- \bar{\chi}) \quad . \quad (6.6) \]

We stress once again that the derivatives of \( K \) with respect to \( \Phi, \bar{\Phi}, \chi, \bar{\chi} \) are implicit; for example, the first equation reads

\[ K_\Phi \nabla_+ \Phi = K_{\Phi \Phi} \bar{\Phi} \nabla_+ \Phi - K_{\Phi \Phi} \bar{\Phi} \nabla_+ \Phi + K_{\Phi \chi} \bar{\Phi} \nabla_+ \chi - K_{\Phi \chi} \bar{\Phi} \nabla_+ \chi \quad . \quad (6.7) \]

The associated component expansions are:
\[ \psi_+ = (\bar{\phi} - \phi)(\psi_+ + \eta_+) \]
\[ \psi_- = (\bar{\phi} - \phi)(\psi_- + \eta_-) \]
\[ \psi_+ = (\phi - \bar{\phi})(\psi_+ + \eta_+) \]
\[ \psi_- = (\phi - \bar{\phi})(\psi_- + \eta_-) \]
\[ \eta_+ = (\bar{\chi} - \chi)(\psi_+ + \eta_+) \]
\[ \eta_- = (\chi - \bar{\chi})(\psi_- + \eta_-) \]
\[ \eta_+ = (\chi - \bar{\chi})(\psi_+ + \eta_+) \]
\[ \eta_- = (\bar{\chi} - \chi)(\psi_- + \eta_-) \quad . \quad (6.8) \]

Applying two derivatives gives us (in superspace) additional relations, of which we need the following:

\[ \nabla_- \nabla_+ \Phi = \nabla_+ \Phi(\nabla_- \Phi + \nabla_- \chi) + \nabla_- \Phi(\nabla_- \bar{\Phi} + \nabla_- \bar{\chi}) \]
\[ + (\Phi - \bar{\Phi})(\nabla_- \nabla_+ \Phi + \nabla_- \nabla_+ \chi) - (\nabla_+ \Phi + \nabla_+ \chi)(\nabla_- \Phi + \nabla_- \bar{\chi}) \]
\[ \nabla_- \nabla_+ \bar{\Phi} = \nabla_+ \bar{\Phi}(\nabla_- \Phi + \nabla_- \chi) + \nabla_- \bar{\Phi}(\nabla_- \Phi + \nabla_- \bar{\chi}) \]
\[ + (\bar{\Phi} - \Phi)(\nabla_- \nabla_+ \bar{\Phi} + \nabla_- \nabla_+ \bar{\chi}) - (\nabla_+ \Phi + \nabla_+ \chi)(\nabla_- \Phi + \nabla_- \bar{\chi}) \]
\[ \nabla_- \nabla_+ \chi = \nabla_+ \chi(\nabla_- \Phi + \nabla_- \chi) + \nabla_- \chi(\nabla_- \Phi + \nabla_- \bar{\chi}) \]
\[ + (\chi - \bar{\chi})(\nabla_- \nabla_+ \chi + \nabla_- \nabla_+ \bar{\chi}) - (\nabla_+ \Phi + \nabla_+ \chi)(\nabla_- \Phi + \nabla_- \bar{\chi}) \quad (6.9) \]
The respective component expansions are:

\[ -qW \tilde{\phi} = \psi_\parallel (\psi_- + \eta_-) + \psi_- (\psi_\parallel + \eta_\parallel) + (\phi - \tilde{\phi})[-qW + i\tilde{G} - (\psi_\parallel + \eta_\parallel)(\psi_- + \eta_-)] \]

\[ -q\tilde{W} \phi = \psi_\parallel (\psi_\parallel + \eta_\parallel) + \psi_- (\psi_\parallel + \eta_\parallel) + (\tilde{\phi} - \phi)[-q\tilde{W} \phi + i\tilde{G} - (\psi_\parallel + \eta_\parallel)(\psi_- + \eta_-)] \]

\[ i\tilde{G} = \eta_\parallel (\psi_- + \eta_-) + \eta_- (\psi_\parallel + \eta_\parallel) + (\chi - \tilde{\chi})[-qW \tilde{\phi} + i\tilde{G} - (\psi_\parallel + \eta_\parallel)(\psi_- + \eta_-)] \quad (6.10) \]

Utilizing (6.4) and (6.10), one obtains the following component lagrangian, which is symmetric with respect to all the fields (again factors of K should be reinstated as appropriate, so that, e.g., a term \( PF\psi_+\eta_- \) should be multiplied by \( K_{\phi\dot{\chi}} \)):

\[
S = \int d^2x \left\{ -\frac{1}{2} D_\mu \phi D_\mu \tilde{\phi} + \frac{1}{2} D_\mu \chi D_\mu \tilde{\chi} \right. \\
- \frac{1}{2} \epsilon_{\mu\nu} D_\mu \phi D_\nu \chi + \frac{1}{2} \epsilon_{\mu\nu} D_\mu \tilde{\chi} D_\nu \tilde{\phi} \\
- i\tilde{\psi}_\parallel \tilde{D}_\parallel \psi_+ + i\tilde{\eta}_\parallel \tilde{D}_\parallel \eta_+ - i\tilde{\psi}_- \tilde{D}_- \psi_- + i\tilde{\eta}_- \tilde{D}_- \eta_- \\
+ \frac{i}{2} (\psi_\parallel \psi_- + \eta_- \eta_+) \tilde{D}_\parallel (\tilde{\phi} - \phi - \chi + \tilde{\chi}) \\
+ i\tilde{\psi}_\parallel \eta_+ D_\parallel (\phi + \chi) + i\tilde{\eta}_\parallel \psi_+ D_\parallel (\tilde{\phi} + \chi) \\
- i\tilde{\psi}_- \eta_- D_- (\phi + \chi) + i\tilde{\eta}_- \psi_- D_- (\tilde{\phi} + \tilde{\chi}) \\
+ (\psi_- + \eta_-) (\psi_\parallel + \eta_\parallel)(\psi_- + \eta_-)(\psi_\parallel + \eta_\parallel) \\
+ FF + iF(\psi_- + \eta_-)(\psi_\parallel + \eta_\parallel) + iF(\psi_- + \eta_-)(\psi_\parallel + \eta_\parallel) \\
+ GG + iG(\psi_\parallel + \eta_\parallel)(\psi_- + \eta_-) - i\tilde{G}(\psi_- + \eta_-)(\psi_\parallel + \eta_\parallel) + q\phi[\lambda_- (\psi_\parallel + \eta_\parallel) - \lambda_\parallel (\psi_- + \eta_-)] \\
+ q\tilde{\phi}[\lambda_- (\psi_\parallel + \eta_\parallel) - \lambda_\parallel (\psi_- + \eta_-)] + qW[\psi_\parallel (\psi_- + \eta_\parallel) + \tilde{\phi}(\psi_- + \eta_-)(\psi_\parallel + \eta_\parallel)] \\
+ qW[\psi_- (\psi_\parallel + \eta_\parallel) - \phi(\psi_- + \eta_-)(\psi_\parallel + \eta_\parallel)] \\
+ q' \tilde{\chi}[\rho_- (\psi_\parallel + \eta_\parallel) - \rho_\parallel (\psi_- + \eta_-)] \\
+ q' \chi[\rho_- (\psi_\parallel + \eta_\parallel) - \rho_\parallel (\psi_- + \eta_-)] \\
+ q' \tilde{P}[-\eta_\parallel (\psi_- + \eta_-) + \chi(\psi_- + \eta_-)(\psi_\parallel + \eta_\parallel)] \\
+ q' \tilde{P}[-\eta_- (\psi_- + \eta_-) + \tilde{\chi}(\psi_- + \eta_-)(\psi_\parallel + \eta_\parallel)] \\
+ (q')^2 \tilde{P}\tilde{P}\tilde{\chi} + iq' \tilde{P}F\tilde{\chi} + iq' PF\chi + iq' F(\chi + \tilde{\chi}) \\
- q^2 W\tilde{W} \phi \tilde{\phi} + iqWG\tilde{\phi} + iqWG\phi + \frac{i}{4} q(J + \tilde{J})(\phi + \tilde{\phi}) \right\} . \quad (6.11) \]
Note that in the last two lines the terms $J + \bar{J}$ and $H + \bar{H}$ are proportional to the auxiliary fields of the vector multiplets.

In the above equations $\mathcal{D}$ is the component gauge covariant derivative, and

\[
\mathcal{D}_\mu \phi \mathcal{D}_\mu \bar{\phi} = \mathcal{D}_\mu \phi \mathcal{D}_\mu \bar{\phi} + \mathcal{D}_\mu \phi \mathcal{D}_\mu \bar{\phi}
\]

\[
\epsilon_{\mu
u} \mathcal{D}_\mu \phi \mathcal{D}_\nu \chi = \mathcal{D}_\mu \phi \mathcal{D}_\mu \chi - \mathcal{D}_\mu \phi \mathcal{D}_\mu \chi.
\] (6.12)

We consider now extensions of these results by including explicit dependence on the field strengths of vector and twisted vector multiplets. First we note that the action in (6.1) can be extended to include multiplets of chiral and twisted chiral matter superfields, $\Phi \rightarrow \{\Phi^i\}$ and $\mathcal{X} \rightarrow \{\mathcal{X}^I\}$ (where $i = 1, \ldots, m$ and $I = 1, \ldots, n$). The indices are just carried along passively through the calculations. We take advantage of this by noting that an action of the form

\[
S = \int d^2x \ d^4\theta \ K(\Phi, \bar{\Phi}, \mathcal{X}, \bar{\mathcal{X}}; \mathcal{P}, \bar{\mathcal{P}}, \mathcal{W}, \bar{\mathcal{W}})
\] (6.13)

looks exactly like the case with $m = 2$ and $n = 2$, if we replace $\Phi \rightarrow \{\Phi, \mathcal{P}\}$ and $\mathcal{X} \rightarrow \{\mathcal{X}, \mathcal{W}\}$. Arbitrary numbers of vector and twisted vector multiplets can be included by simply re-interpreting the value of the indices on $\Phi^i$ and $\mathcal{X}^I$. For example, the indices $i = 1, \ldots, p$ are associated with chiral multiplets while $i = p + 1, \ldots, m$ are associated with twisted vector multiplets. Similarly, the indices $I = 1, \ldots, q$ are associated with twisted chiral multiplets while $I = q + 1, \ldots, n$ are associated with vector multiplets.

As an example, we give the general coupling of an SU(2) vector multiplet to an SU(2) doublet of chiral scalars. We have $p = 2$, $m = 2$, $q = 0$ and $n = 3$. We begin with (6.13) and for simplicity write only the purely bosonic terms.

\[
S = \int d^2x \ d^4\theta \ K(\Phi, \bar{\Phi}, 0, 0; 0, \mathcal{W}, \bar{\mathcal{W}})\big|_{\text{bosonic fields}}
\]

\[
= \int d^2x \ \left\{ -\frac{1}{2} K_{\phi^i \bar{\phi}^j} \mathcal{D}_\mu \phi^i \mathcal{D}_\nu \phi^j + \frac{1}{2} K_{W^I \bar{W}^J} \mathcal{D}_\mu W^I \mathcal{D}_\nu \bar{W}^J \\
- \frac{1}{2} \epsilon_{\mu \nu} K_{\phi^i \bar{W}^J} \mathcal{D}_\mu \phi^i \mathcal{D}_\nu W^J + \frac{1}{2} \epsilon_{\mu \nu} K_{\bar{W}^I \bar{\phi}^j} \mathcal{D}_\mu \bar{W}^I \mathcal{D}_\nu \bar{\phi}^j \\
- K_{\phi^i \bar{\phi}^j} F^i \bar{F}^j + \frac{1}{4} K_{W^I \bar{W}^J} [F^I(A)F^J(A) + d^I d^J] \\
- g^2 K_{\phi^i \bar{\phi}^j} ([W, \phi])^i ([\bar{W}, \bar{\phi}])^j \\
+ \frac{i}{2} g K_{\phi^i \bar{W}^J} ([W, \bar{\phi}])^i [d^J + iF^J(A)] \\
+ \frac{i}{2} g K_{\phi^i \bar{\phi}^j} ([\bar{W}, \phi])^i [d^J + iF^J(A)] \\
+ \frac{i}{4} g K_{\phi^i ([J + \bar{J}], \phi)}^i + \frac{i}{4} g K_{\phi^i ([J + \bar{J}], \bar{\phi})^i} \right\}.
\] (6.14)

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In this expression, it is understood that when a quantity such as \( W \) appears without an index, we are referring to the corresponding Lie-algebraic operator (i.e. \( W \equiv W^1t_1 \)).

By use of the mirror map operator, we can easily obtain the mirror reflection, namely

\[
\int d^2x d^4\theta K(0,0,\mathcal{X},\mathcal{X}; \mathcal{P}, \mathcal{P}, 0, 0) = \mathcal{M}_m : \int d^2x d^4\theta K(\Phi, \Phi, 0, 0; 0, \mathcal{W}, \mathcal{W})
\]

where at the component level we make the obvious replacement of fields.

As noted before, integrability conditions forbid minimal couplings of twisted vector multiplets to chiral scalar multiplets or vector multiplets to twisted chiral scalar multiplets. Nonminimal couplings are possible through Pauli-moment type terms. This possibility occurs because vector multiplet field strengths are chiral or twisted chiral and can therefore be used in superpotential actions. The superpotentials of (2.17) can be generalized to include the respective vector multiplets.

\[
S_c = \int d^2xd^2\theta U(\Phi, \mathcal{P}) + h.c.
= \int d^2xe^{-1}\left\{ U_{\phi\phi}\psi_+\psi_- + U_{\phi P}(\psi_+\rho_- + \rho_+\psi_-) + U_{P P}\rho_+\rho_- \\
- iU_{\phi}F + -\frac{i}{2}U_{P}[d' + iF(A')] + h.c.\right\},
\]

and

\[
S_{tc} = \int d^2x d\theta^+ d\theta^- \tilde{U}(\mathcal{X}, \mathcal{W}) + h.c.
= \int d^2xe^{-1}\left\{ \tilde{U}_{\chi\chi}\eta_+\eta_- + \tilde{U}_{\chi W}(\eta_+\lambda_- + \lambda_+\eta_-) + \tilde{U}_{WW}\lambda_+\lambda_- \\
- i\tilde{U}_{\chi}G - \frac{i}{2}\tilde{U}_{W}[d + iF(A)] + h.c.\right\}.
\]

7 Lagrangians for Matter Multiplets Coupled to Supergravity

In this section we consider matter-supergravity systems, and their component actions. We concentrate primarily on the irreducible \( U_A(1) \) version of supergravity, but we present in subsection 7.5 results involving the nonminimal \( U_V(1) \otimes U_A(1) \) theory.
7.1 Chiral Multiplet

In this subsection we obtain the component lagrangian for the kinetic term of a covariantly chiral superfield (defined by $\nabla_\alpha \Phi = \nabla_\alpha \bar{\Phi} = 0$) coupled to the $U_A(1)$ version of $(2,2)$ supergravity. The chiral projection formula (5.8)

$$S_{\Phi\bar{\Phi}} = \int d^2xd^4\theta E^{-1}\Phi\bar{\Phi}$$

$$= \int d^2xe^{-1}[\nabla^2 + i\psi_\pm \nabla_+ - i\psi_+^\dagger \nabla_- + (-\frac{1}{2}B - \psi_-^\dagger \psi_+ + \psi_+^\dagger \psi_-)^2]\nabla^2(\Phi\bar{\Phi})]$$

$$= \int d^2xe^{-1}[((\nabla^2 \Phi)\bar{\Phi}) + (\nabla_+ \nabla^2 \Phi)(\nabla_- \Phi)] - (\nabla_- \nabla^2 \Phi)(\nabla_+ \Phi)]$$

$$(\nabla_+ \nabla^2 \Phi)| + i\psi_+^\dagger \nabla_- (\nabla^2 \Phi)|\bar{\Phi}] + (-\frac{1}{2}B - \psi_-^\dagger \psi_+ + \psi_+^\dagger \psi_-)(\nabla^2 \Phi)|\bar{\Phi}] .$$

(7.1)

We list the component expansions for the quantities appearing above:

$$\nabla_+ \nabla^2 \Phi| = i\mathbf{D}_\Phi \psi_\pm - \psi_\pm \mathbf{D}_\pm \bar{\Phi} - \psi_-^\dagger \psi_+^\dagger \psi_+^\dagger \psi_- + \psi_+^\dagger \psi_-^\dagger \psi_+^\dagger \psi_- + \psi_+^\dagger \psi_-^\dagger \bar{\Phi}$$

$$\nabla_- \nabla^2 \Phi| = -i\mathbf{D}_\pm \psi_\pm - \psi_\pm \mathbf{D}_\pm \bar{\Phi} + \psi_-^\dagger \psi_+^\dagger \psi_+^\dagger \psi_- + \psi_+^\dagger \psi_-^\dagger \psi_+^\dagger \psi_- + \psi_+^\dagger \psi_-^\dagger \bar{\Phi}$$

$$\nabla_+ \nabla^2 \Phi| = \nabla_+ \nabla_\Phi + \frac{1}{2}(\nabla_+ R) \nabla_\Phi + \frac{1}{2}R \nabla^2 \bar{\Phi}$$

$$= \nabla_+ \nabla_\Phi - \frac{i}{2}D_\mp \bar{R} \nabla_\Phi + i\psi_-^\dagger \psi_+^\dagger \psi_+^\dagger \psi_- + i\psi_-^\dagger \psi_-^\dagger \psi_+^\dagger \psi_- + i\psi_-^\dagger \psi_-^\dagger \psi_+^\dagger \psi_- \nabla_\Phi + \frac{1}{2}R \nabla^2 \bar{\Phi}$$

$$\nabla_- \nabla_\Phi = \mathbf{D}_\Phi \mathbf{D}_\pm + \psi_\pm \mathbf{D}_\pm \nabla_\Phi - \frac{i}{2} \psi_\mp \bar{R} \mathbf{D}_\pm \nabla_\Phi$$

$$+ \psi_-^\dagger \psi_\dagger \nabla_\mp + i\psi_-^\dagger \psi_\dagger \mathbf{D}_\mp \nabla_\Phi - \psi_-^\dagger \nabla_\mp + \psi_-^\dagger \nabla_\mp + \psi_-^\dagger \nabla_\mp + \psi_-^\dagger \nabla_\mp$$

$$= \mathbf{D}_\Phi \mathbf{D}_\pm \bar{\Phi} + \psi_-^\dagger \psi_\dagger \psi_\dagger \psi_- + \psi_-^\dagger \psi_-^\dagger \psi_\dagger \psi_- + \psi_-^\dagger \psi_-^\dagger \psi_\dagger \psi_-$$

$$- \frac{i}{2} \psi_-^\dagger \psi_-^\dagger \psi_\dagger \psi_- + i\psi_-^\dagger \psi_-^\dagger \psi_-^\dagger \psi_- + i\psi_-^\dagger \psi_-^\dagger \psi_-^\dagger \psi_- + i\psi_-^\dagger \psi_-^\dagger \psi_-^\dagger \psi_-$$

$$- i\psi_-^\dagger \psi_-^\dagger \bar{\Phi} + i\psi_-^\dagger \psi_-^\dagger \bar{\Phi}$$

(7.2)

We have used the identities that appear in the Appendix, as well as the component identifications

$$\Phi| = \phi, \quad \bar{\Phi}| = \bar{\phi}$$

$$\nabla_+ \Phi| = \psi_+, \quad \nabla_\Phi| = \psi_+$$

$$\nabla_- \Phi| = \psi_-, \quad \nabla_\Phi| = \psi_-$$

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\[
\frac{i}{2}[\nabla_+, \nabla_-] \Phi = F , \quad \frac{i}{2}[\nabla_+^*, \nabla_-^*] \Phi = \bar{F} .
\] (7.3)

Once we have the complete component expansion, we group together similar terms (component bosonic, fermionic, and spinorial-bosonic). For each set, once one separates out the contribution from the extra gravitino pieces in the connection as in (A.24), the results are symmetric in barred/unbarred quantities. To illustrate how this works, we collect together all the bosonic terms, and get

\[
- \bar{F} F + (D_\Phi D_\Phi) \phi + i(\psi_+^* \psi_-^* + \psi_-^* \psi_+^*) (D_\Phi \bar{\phi}) \phi .
\] (7.4)

We separate out the gravitino terms in \( D_\Phi \) when it acts on \( D_\Phi \bar{\phi} \), as follows:

\[
D_\Phi(D_\Phi \bar{\phi}) = [e* + \omega* M + \gamma* M](D_\Phi \bar{\phi})
= [e* - \frac{1}{2}(\omega + \gamma)*]D_\Phi \bar{\phi}
= [e* - \phi*]D_\Phi \bar{\phi}
= [e* - \omega* - i(\psi_+^* \psi_-^* + \psi_-^* \psi_+^*) (D_\Phi \bar{\phi})]
= D_\Phi(D_\Phi \bar{\phi}) - i(\psi_+^* \psi_-^* + \psi_-^* \psi_+^*) (D_\Phi \bar{\phi}) ,
\] (7.5)

where \( D_\Phi \) is the ordinary gravitational covariant derivative.

Substituting into the bosonic terms above, we find that the gravitino pieces cancel. Furthermore, when \( D \) acts on a scalar field, one can replace \( D_\Phi \bar{\phi} \) by \( \partial_\Phi \bar{\phi} \), and inside the \( d^2x e^{-1} \) integral we can integrate by parts, leaving only \( -\bar{F} F - \partial_\Phi \bar{\phi} \partial_\Phi \phi \).

For the fermionic terms, the gravitino pieces from the connection combine with similar terms that were produced explicitly by the component projections. We also separate out the \( U_A(1) \) connection \( V'_a \) explicitly, as in the following example:

\[
D_\Phi \psi_\perp = [e* + \gamma* M] \psi_\perp
= [e* - \frac{1}{2} \gamma*] \psi_\perp
= [e* - \frac{1}{2} \phi* - \frac{i}{2} V'_a] \psi_\perp
= [e* - \frac{1}{2} \phi* - \frac{i}{2} V'_a - \frac{i}{2} (\psi_+^* \psi_-^* + \psi_-^* \psi_+^*)] \psi_\perp
= (D_\Phi - \frac{i}{2} V'_a) \psi_\perp - \frac{i}{2} (\psi_+^* \psi_-^* + \psi_-^* \psi_+^*) \psi_\perp .
\] (7.6)

Summing all the terms, the final result for the component lagrangian for the kinetic term of the chiral multiplet coupled to the \( U_A(1) \) version of (2,2) supergravity is

\[
S_{\Phi} = \int d^2 x d^3 \theta E^{-1} \Phi \Phi
\]
\[ S_\mathcal{X} = - \int d^2x e^{-1} \{ -FF - \partial_\mathbf{\tilde{\phi}} \partial_\mathbf{\phi} + i(D_\mathbf{\tilde{\phi}} \psi_-)\psi_- + i(D_\mathbf{\phi} \psi_+)\psi_+ + \frac{1}{2} V'_\mathbf{\tilde{\phi}} \psi_-\psi_- - \frac{1}{2} V'_\mathbf{\phi} \psi_+\psi_+ \] 
\[ - [\partial_\mathbf{\tilde{\phi}} \mathbf{\phi} + \frac{1}{2} \psi'_\mathbf{\tilde{\phi}} \psi_\alpha] \psi_\alpha \psi_- + [\partial_\mathbf{\phi} \mathbf{\tilde{\phi}} + \frac{1}{2} \psi'_\mathbf{\phi} \psi_\alpha] \psi_- \psi_- \] 
\[ - [\partial_\mathbf{\tilde{\phi}} \mathbf{\phi} + \frac{1}{2} \psi'_\mathbf{\tilde{\phi}} \psi_\alpha] \psi_\alpha \psi_+ - [\partial_\mathbf{\phi} \mathbf{\tilde{\phi}} + \frac{1}{2} \psi'_\mathbf{\phi} \psi_\alpha] \psi_+ \psi_+ \} \] 
(7.7)

which agrees with the expression first derived in ref. [18].

### 7.2 Twisted Chiral Multiplet

To obtain the component lagrangian for the kinetic term of a twisted chiral multiplet, we start with the same chiral projection formula as above, except that now \( \mathcal{X} \) is subject to the twisted chirality conditions \( \nabla_\downarrow \mathcal{X} = \nabla_\downarrow \bar{\mathcal{X}} = \nabla_\uparrow \mathcal{X} = \nabla_\uparrow \bar{\mathcal{X}} = 0 \).

\[ S_\mathcal{X} = - \int d^2xd^4\theta E^{-1} \bar{\mathcal{X}} \mathcal{X} \] 
\[ = - \int d^2xe^{-1}[\nabla^2 + i\psi^\dagger \nabla_\downarrow - i\psi^\dagger \nabla_\downarrow - (-\frac{1}{2} \bar{B} - \psi^\dagger \psi^\dagger + \psi^\dagger \psi^\dagger)\nabla^2(\bar{\mathcal{X}}\mathcal{X})] \] 
\[ + (\nabla_{\mathbf{\tilde{\phi}}} \bar{\mathcal{X}})(\nabla_\downarrow \mathcal{X}) + i(\nabla_{\mathbf{\tilde{\phi}}} \bar{\mathcal{X}})(\nabla_\downarrow \mathcal{X}) \] 
\[ + i\psi^\dagger [i(\nabla_{\mathbf{\tilde{\phi}}} \bar{\mathcal{X}})(\nabla_\downarrow \mathcal{X})] - (\nabla_{\mathbf{\tilde{\phi}}} \bar{\mathcal{X}})(\nabla_\downarrow \mathcal{X})] \] 
\[ - i\psi^\dagger [-i(\nabla_{\mathbf{\tilde{\phi}}} \bar{\mathcal{X}})(\nabla_\downarrow \mathcal{X})] + (\nabla_{\mathbf{\tilde{\phi}}} \bar{\mathcal{X}})(\nabla_\downarrow \mathcal{X})] \] 
\[ + (-\frac{1}{2} \bar{B} - \psi^\dagger \psi^\dagger + \psi^\dagger \psi^\dagger)(\nabla_{\mathbf{\tilde{\phi}}} \bar{\mathcal{X}})(\nabla_\downarrow \mathcal{X})] \} \] 
(7.8)

The components of the twisted chiral multiplet are defined by covariant projection:

\[ \mathcal{X}| = \chi \quad , \quad \bar{\mathcal{X}}| = \bar{\chi} \] 
\[ \nabla_\downarrow \mathcal{X}| = \eta_\downarrow \quad , \quad \nabla_\uparrow \mathcal{X}| = \eta_\uparrow \] 
\[ \nabla_\downarrow \bar{\mathcal{X}}| = \eta_\downarrow \quad , \quad \nabla_\uparrow \bar{\mathcal{X}}| = \eta_\uparrow \] 
\[ \frac{i}{2} [\nabla_\downarrow, \nabla_\downarrow] \mathcal{X}| = G \quad , \quad \frac{i}{2} [\nabla_\uparrow, \nabla_\downarrow] \bar{\mathcal{X}}| = \bar{G} \] 
(7.9)

In this case, the required component derivative expansions are:

\[ \nabla_{\mathbf{\tilde{\phi}}} \nabla_\downarrow \mathcal{X}| = D_{\mathbf{\tilde{\phi}}} \eta_\downarrow - i\psi^\dagger \bar{G} + i\psi^\dagger (D_{\mathbf{\tilde{\phi}}} \bar{\chi} + \psi^\dagger \eta_\downarrow + \psi^\dagger \eta_\downarrow) \] 
\[ \nabla_\uparrow \nabla_\downarrow \mathcal{X}| = -\frac{i}{2} \bar{B} \eta_\downarrow + D_{\mathbf{\tilde{\phi}}} \eta_\downarrow + i\psi^\dagger (D_{\mathbf{\tilde{\phi}}} \bar{\chi} + \psi^\dagger \eta_\downarrow + \psi^\dagger \eta_\downarrow) + i\psi^\dagger \bar{G} \] 
\[ \nabla_{\mathbf{\tilde{\phi}}} \bar{\mathcal{X}}| = D_{\mathbf{\tilde{\phi}}} \bar{\chi} + \psi^\dagger \bar{\eta}_\downarrow + \psi^\dagger \bar{\eta}_\downarrow \] 
\[ \nabla_\downarrow \mathcal{X}| = D_\downarrow \chi + \psi^\dagger \eta_\downarrow + \psi^\dagger \eta_\downarrow \] 
(7.10)
We follow the same procedure as in the case of the chiral multiplet. Collecting all the terms in the sum and separating out the gravitino pieces as before, we obtain

\[ S_{\bar{X}X} = -\int d^2xd^4\theta E^{-1}\bar{X}X \]

\[ = \int d^2x e^{-1}\{-\bar{G}G - \partial_{-\bar{X}}\partial_{+\chi} + i(D_{-\eta_-})\eta_- + i(D_{+\eta_+})\eta_+ \]

\[ + \frac{1}{2}V'_{+\eta_-}\eta_- - \frac{1}{2}V'_{-\eta_+}\eta_+ \]

\[ - [\partial_{+\chi} + \frac{1}{2}(\psi_+^\eta_- + \psi_-^\eta_+)]\psi_+^\eta_+ \]

\[ - [\partial_{-\chi} + \frac{1}{2}(\psi_+^\eta_+ + \psi_-^\eta_-)]\psi_+^\eta_- \]

\[ - [\partial_{+\bar{X}} + \frac{1}{2}(\psi_+^\eta_- + \psi_-^\eta_+)]\psi_+^\eta_+ \]

\[ - [\partial_{-\bar{X}} + \frac{1}{2}(\psi_+^\eta_+ + \psi_-^\eta_-)]\psi_+^\eta_- \] (7.11)

as the component lagrangian for the kinetic term of the twisted chiral multiplet coupled to the \(U_A(1)\) version of (2,2) supergravity.

Note that if we interchange \{\(\{-\}\) and \{\(\{-\}\) in the (7.11) above (and change the \(U(1)\) charge of the left-moving spinors), we get exactly the result (7.7) for the ordinary chiral multiplet. We also point out that both (7.7) and (7.11) are valid for the \(U_A(1)\) version of supergravity, but that to get the corresponding expressions for the \(U_V(1)\) version, one just interchanges \{-\} and \{-\}.

### 7.3 Potentials and Scalar-Vector Models

To begin, let us calculate the component form of the potential terms in (2.17) but in the presence of a \(U_A(1)\) supergravity background. We apply the chiral and twisted chiral density projectors of (5.8) and (5.9) to \(U(\Phi)\) and \(\tilde{U}(\chi)\) respectively and obtain the component actions

\[ S_c = \int d^2xe^{-1}\{ U''(\phi)\psi_+\psi_- + U'(\phi)[-iF + i\psi_+^\eta_-\psi_+ - i\psi_-^\eta_+\psi_-] \]

\[ - U(\phi)[\frac{1}{2}\tilde{B} + \psi_+^\eta_-\psi_- - \psi_-^\eta_+\psi_+] + h.c.} \] , (7.12)

for the chiral case and

\[ S_{tc} = -\int d^2xe^{-1}\{ \tilde{U}''(\chi)\eta_-\eta_+ + \tilde{U}'(\chi)[iG + i\psi_+^\eta_-\eta_- - i\psi_-^\eta_+\eta_+] \]

\[ + \tilde{U}(\chi)[\psi_+^\eta_-\psi_- + \psi_-^\eta_+\psi_+] + h.c.} \] , (7.13)

for the twisted chiral case.
We can discuss now local actions for the chiral and twisted chiral multiplets with potential terms, given by the sum of (7.7), (7.11), (7.12) and (7.13). In the following, we are assuming that the potentials have the dimensions of mass. We assume flat kinetic terms for the two types of multiplets: $\int d^2 xd^2 \theta \overline{E}^{-1} \overline{\Phi}$ for the chiral multiplet and $- \int d^2 xd^2 \theta \overline{E}^{-1} \overline{\mathcal{X}} \mathcal{X}$ the twisted chiral multiplet. We obtain

\[ S = - \int d^2 x \, e^{-1} \{ \partial_x \overline{\phi} \partial_x \phi + \partial_x \overline{\chi} \partial_x \chi + \mathcal{F} \mathcal{F} + \overline{G} G \]

\[ + i(D_x \eta_+ \eta_-) + i(D_\eta_+ \eta_-) + i \partial_x \phi + \frac{1}{2} V_\phi \phi + \frac{1}{2} V_\chi \chi - \frac{1}{2} V_\psi \psi \psi_+ \psi_- \]

\[ - \partial_x \phi + \frac{1}{2} \psi_\alpha \psi_\alpha \psi_+ \psi_- \]

\[ - \partial_x \phi + \frac{1}{2} \psi_\alpha \psi_\alpha \psi_+ \psi_- \]

\[ - \partial_x \chi + \frac{1}{2} \psi_\alpha \psi_\alpha \psi_+ \psi_- \]

\[ - \partial_x \chi + \frac{1}{2} \psi_\alpha \psi_\alpha \psi_+ \psi_- \]

\[ - \partial_x \chi + \frac{1}{2} \psi_\alpha \psi_\alpha \psi_+ \psi_- \]

\[ - \partial_x \chi + \frac{1}{2} \psi_\alpha \psi_\alpha \psi_+ \psi_- \]

\[ - \partial_x \chi + \frac{1}{2} \psi_\alpha \psi_\alpha \psi_+ \psi_- \]

\[ + \left[ U''(\phi) \psi_+ \psi_- + U'(\phi) (-i F + i \psi_\pm \psi_\mp - i \psi_\mp \psi_+ \psi_-) \right] \]

\[ - U'(\phi) \left( - \frac{1}{2} \mathcal{B} + \psi_\pm \psi_\mp - \psi_\mp \psi_\pm \right) \]

\[ + \overline{U''}(\chi) \eta_+ \eta_- - \overline{U'}(\chi) \left( i G + i \psi_\pm \eta_+ \psi_- - i \psi_- \psi_+ \right) \]

\[ - \overline{U}(\chi) \left( \psi_+ \psi_- + \psi_- \psi_+ \right) + \text{h.c.} \} \right\} \quad (7.14) \]

This expression can be generalized trivially to any number of chiral and twisted chiral multiplets. In particular, if we set the potential terms to zero and consider the case of just two multiplets, one chiral and one twisted chiral, two chiral, or two twisted chiral, the action above describes the mirror symmetric, or the two types of nonsymmetric, $N = 2$ superstrings, whose possible existence was mentioned in some time ago.

Outside the context of $N = 2$ superstrings, we can think of the potential terms above as massive perturbations of $N = 2$ superstring theory. We note that the potentials introduce mass for the spinors in the two multiplets in very different ways. The spinor in the chiral multiplet has a Majorana-type mass term while the spinor in the twisted chiral multiplet has a Dirac-type mass term.

\[ \text{Recently, we have found evidence that the action for the } N = 4 \text{ superstring possesses this same type of ambiguity. There appear to be } four \text{ distinct } N = 4 \text{ superstring actions and six distinct mirror map operators.} \]
If we set all the fermions above to zero we obtain

\[- \int d^2 x \ e^{-1} \left\{ \partial_\pm \bar{\phi} \partial_\pm \phi + \partial_\pm \bar{\chi} \partial_\pm \chi + \bar{F} F + \bar{G} G + \left[ iU'(\phi) F + \frac{1}{2} U(\phi) B + i\bar{U}'(\chi) G + \text{h.c.} \right] \right\} \]  

(7.15)

After eliminating the $F$ and $G$ auxiliary fields by their equations of motion, potentials are generated for the scalar fields. However, the presence of the supergravity auxiliary field $B$ introduces an asymmetry in the constraints on the potentials. The $B$ auxiliary field imposes the condition $\bar{U}(\phi) = 0$. The trace of the zweibein also imposes the conditions $U'(\phi) = 0$ and $\bar{U}'(\chi) = 0$. If we think of these conditions as defining hypersurfaces in the $\phi$-space and in the $\chi$-space, the two classes of surfaces are different, since one can have a constant solution $\bar{U} = \bar{U}_0$ whose effect is to produce a type of mass term for the gravitino. This particular deformation of the mirror symmetric $N = 2$ superstring may possess some interesting properties.

We next consider scalar-vector matter systems coupled to $2D$ supergravity. This can be done by assuming that in the generalization of the action (7.14) to one with several scalar multiplets $\Phi^i, X^I$, some of them are replaced by vector multiplets $P, W$. The main effect of this is that, aside from the replacement of physical components (e.g. matter spinor into gauge spinor, etc.), one matter auxiliary field is replaced by a component (supercovariantized) gauge field strength (similar to the discussion in section 3). For example, substituting a twisted vector multiplet $P$ for the chiral multiplet $\Phi'$ in the superpotential $U(\Phi, \bar{\Phi'})$ leads to the component expression

\[
S_c = \int d^2 x d^2 \theta U(\Phi, P) + \text{h.c.}
\]

(7.16)

and in a similar manner,

\[
S_{tc} = \int d^2 x d^4 \theta \bar{U}(X, W) + \text{h.c.}
\]

(7.17)

Here we should mention an important difference between a vector multiplet and a twisted vector multiplet, when coupled to $U_A(1)$ supergravity. The component
spin-1 field strength of the twisted vector multiplet (in addition to containing supercovariantized derivatives of the component gauge field) is uniformly modified in the Lagrangian above according to the replacement rule (which follows from the commutator algebra of (3.1) in the presence of supergravity)

\[ F(A') \rightarrow F(A') - \frac{1}{2} [B \psi + B \overline{\psi}] \]  

(7.18)

This replacement rule does not occur for a vector multiplet. The appearance of the supergravity auxiliary field has the following effect. From the kinetic action \( \int d^2 \sigma d^4 \theta \bar{\mathcal{P}} \mathcal{P} \) one generates the component terms \( \frac{1}{4} [F(A') - \frac{1}{2} (B \psi + B \overline{\psi})] \). As has been noted in the \( N = 1 \) case \[19\], in the absence of a potential for the chiral matter superfields, the supergravity auxiliary field’s equation of motion forces trivial dynamics for the twisted vector multiplet.

### 7.4 Models with Gauged \( U_V(1) \) or \( U_A(1) \)

In a (2,2) string-type theory the matter multiplets are neutral under the tangent space \( U_A(1) \) or \( U_V(1) \) transformations. One way in which a superconformal (2,2) model may be deformed is for the \( U_V(1) \) or \( U_A(1) \) charge to be realized in a non-trivial manner on the scalar multiplets coupled to supergravity. We investigate here the resulting structure.

We fix the form of the supergravity multiplet to correspond to the \( U_A(1) \) version of supergravity. For the matter scalar superfields, we choose some number \( N_C \) of chiral multiplets and some number \( N_T \) of twisted chiral multiplets. For the sake of simplicity, we choose only flat kinetic terms. However, we want the matter multiplets to carry a non-trivial realization of the charge that is gauged by the vector in the supergravity multiplet. Since we are using the \( U_A(1) \) form of supergravity, the integrability condition requires the chiral multiplets to be neutral under the \( U_A(1) \) charge generator while the twisted chiral multiplet transforms non-trivially:

\[
[\mathcal{Y}', \Phi] = 0 , \quad [\mathcal{Y}', \mathcal{X}] = i Q' \mathcal{X} ,
\]  

(7.19)

where \( Q' \) is an arbitrary real \( N_T \times N_T \) diagonal matrix. Since the chiral fields do not carry the \( U_A(1) \) charge, their coupling to supergravity is unchanged and their lagrangian given by (7.11). For the twisted chiral superfields we have first

\[
- \int d^2 \sigma d^4 \theta E^{-1} \overline{\mathcal{X}} \mathcal{X} = - \int d^2 \sigma d^4 \theta E^{-1} \left[ (\nabla \overline{\mathcal{X}})(\nabla \mathcal{X}) - \frac{1}{2} R(\overline{\mathcal{X}} Q' \mathcal{X}) \right] . \]  

(7.20)

\(^6\)For the \( N = 2 \) superstring we only have the choices \( N_C = 2, N_T = 0, N_C = 1, N_T = 1, \) and \( N_C = 0, N_T = 2 \). We may call these the \( C^2, CT \) and \( T^2 \) \( N = 2 \) superstrings, respectively.
Applying the chiral projection formula in (5.8), we obtain
\[ S_Q' = \int d^2x e^{-1} \left\{ \frac{1}{2} [ \nabla_+ R - i \psi_0^+ \overline{R}] \overline{Q'} \eta_+ - \frac{1}{2} [ \nabla_+ R - i \psi_0^+ \overline{R}] \eta_- Q' \chi \ight. \\
- \frac{1}{2} [ \nabla_- R - i \psi_0^- \overline{R}] \eta_+ Q' \chi + \left. \frac{1}{2} [ \nabla_- R - i \psi_0^- \overline{R}] \eta_- Q' \chi \right\} + \frac{1}{2} R(\eta_- Q' \eta_+ + \eta_+ Q' \eta_-) \] \quad (7.21)

where
\[ \Delta = [ \nabla_+ \nabla_- R + i \psi_0^+ \nabla_- R - i \psi_0^- \nabla_+ R - R(\frac{1}{2} \overline{R} + \psi_0^+ \psi_0^+ - \psi_0^- \psi_0^-) ] \] \quad (7.22)

It is clear that (super)conformal invariance is broken because this result depends explicitly upon the supergravity auxiliary field \( B \). This breaking is seen most explicitly if we retain only the bosonic terms in \( S_Q' \) and note that the term involving \( \Delta \) takes the form
\[ \frac{1}{2} [ \overline{Q}' \chi ] \Delta \rightarrow \frac{1}{2} R[\overline{Q}' \chi] \] \quad (7.23)

where \( R \) is the 2D curvature scalar. In 4D, terms of this form lead to an “improvement” of the spin-0 energy-momentum tensor. For a special nonvanishing choice of \( Q' \) the 4D coupling is conformal. In 2D, precisely the opposite happens. For all nonvanishing \( Q' \) such a term “degrades” the conformal coupling of a free spin-0 field coupling to 2D gravity.

In fact, this is an illustration of a point not generally recognized. We state this result in the form of a theorem:

**In 2D, if the spin-0 field of a matter supermultiplet carries a non-trivial realization of an internal symmetry charge that is gauged by a spin-1 field in the superconformal multiplet, the action for the spin-0 field is neither conformally nor superconformally invariant.**

If \( Q' \) is small, \( S_Q' \) may be regarded as a massless perturbation of the \( CT \) or \( T^2 N = 2 \) superstring.

Before we leave the result in (7.21), it is worth mentioning one of its consequences for the pure gauge formulation of \( N = 2 U_A(1) \) supergravity. There is a form of \( U_A(1) \) supergravity where the usual auxiliary scalar fields are replaced by the supercovariantized field strengths of a complex central charge generator \( Z \). This form of 2D, \( N = 2 \) supergravity may be regarded as the result of a dimensional reduction of one of the original off-shell forms of 4D, \( N = 1 \) supergravity [20]. The commutator algebra of this formulation is given by
\[ [\nabla_+ , \nabla_+] = 0 \quad , \quad [\nabla_-, \nabla_-] = 0 \quad , \quad [\nabla_+ , \nabla_-] = 0 \quad , \]
\[ [\nabla_+ , \nabla_-] = 2Z - \bar{R}M , \quad [\nabla_+ , \nabla_+] = -2\bar{Z} + RM , \]
\[ [\nabla_+ , \nabla_] = i
\]
\[ [\nabla_- , \nabla_+] = i\nabla_+ , \quad [\nabla_- , \nabla_-] = i\nabla_- , \quad (7.24) \]

where \( Z \) annihilates all of the supergravity superfields, \( [Z, \nabla_A] = 0 \). All higher dimension commutators remain unchanged with the exception of
\[ [\nabla_+, \nabla] = \frac{1}{2}(\nabla_+ R)\nabla_- + \frac{1}{2}(\nabla_- R)\nabla_+ - \frac{1}{2}(\nabla_+ \bar{R})\nabla_- - \frac{1}{2}(\nabla_- \bar{R})\nabla_+ \]
\[ - \frac{1}{2}R\bar{R}M - \frac{1}{2}R\bar{R}M + (\nabla^2 R)M - (\bar{\nabla}^2 \bar{R})\bar{M} \]
\[ + RZ + \bar{R}\bar{Z} \quad . \quad (7.25) \]

The main point of these equations is that this modified form of 2D, \( N = 2 \) supergravity at the component level amounts to the substitution for the auxiliary field
\[ B \rightarrow F(U + iV) = \partial_+(U + iV)\nabla_- - \partial_- (U + iV)\nabla_+ - 2[\psi_+^\dagger \bar{\psi}_-^\dagger - \psi_-^\dagger \bar{\psi}_+^\dagger] \quad , \quad (7.26) \]
in terms of new real gauge fields \( (U_+, U_-) \) and \( (V_+, V_-) \). The result of (7.21) shows that in the presence of a matter multiplet that carries the \( U_A(1) \) charge, the gauge fields \( U_a \) and \( V_a \) appear in the action with the standard form of their kinetic terms multiplied by matter scalars.
\[ \frac{1}{4}R\bar{R}[\bar{\mathcal{X}}(Q')^2 \mathcal{X}] \rightarrow \frac{1}{4}[F^2(U) + F^2(V)] [\bar{\mathcal{X}}(Q')^2 \mathcal{X}] \quad . \quad (7.27) \]

The supergravity auxiliary fields of this formulation become physical degrees of freedom with the vacuum value of the operator \([\bar{\mathcal{X}}(Q')^2 \mathcal{X}]\) setting the scale of the central charge coupling constants. The replacement of the auxiliary fields by supercovariant field strengths occurs uniformly in this modified form of \( U_A(1) \) supergravity. In particular, it occurs in the chiral density projector. Under this circumstance, the quantity \( \int d^2xd\theta^+ d\theta^{-} \mathcal{E}^{-1} \) becomes a topological quantity proportional to a \( U(1) \otimes U(1) \) index. (In the standard form of the chiral projector, after the replacement of the complex auxiliary field by the complex supercovariant spin-1 field strength, the quadratic gravitini terms in the non-derivative part of the density projector cancel.)

### 7.5 Classical Mirror Symmetric Models

The existence of the mirror map in 2D, \( N = 2 \) models is a simple consequence of the presence of a \( U(1) \) charge in the supersymmetry algebra. It leads to a pairing of nonlinear \( \sigma \)-models represented by the action
\[ S_\sigma = \int d^2x d^2\theta d^2\bar{\theta} K(\Phi, \bar{\Phi}; \mathcal{X}, \bar{\mathcal{X}}) \]
\[ + \left[ \int d^2x \, d^2\theta \, U(\Phi) + \text{h.c.} \right] \]
\[ + \left[ \int d^2x \, d\theta^+ \, d\theta^- \, \hat{U}(X) + \text{h.c.} \right]. \quad (7.28) \]

and the action
\[ S_\tilde{g} = \int d^2x \, d^2\theta \, d^2\bar{\theta} \left[ -K(X, \overline{X}; \Phi, \overline{\Phi}) \right] \]
\[ + \left[ \int d^2x \, d\theta^+ \, d\theta^- \, U(X) + \text{h.c.} \right] \]
\[ + \left[ \int d^2x \, d^2\theta \, \hat{U}(\Phi) + \text{h.c.} \right]. \quad (7.29) \]

with \( K \) real and \( U \) and \( \hat{U} \) holomorphic. It is generally believed that this pairing is related to the pairing of mirror manifolds in Calabi-Yau compactifications, but to the best of our knowledge the precise connection has not been established.

In the remainder of this subsection we concentrate on the class of models which possess mirror symmetry, i.e. are actually invariant under the map that interchanges \( \theta^- \) and \( \theta^+ \). For rigid \( N = 2 \) models the issue is very simple. As discussed in section 2, the lagrangians corresponding to \( D \)-terms must be odd under the interchange of chiral and twisted chiral scalar or vector multiplets, while \( F \)-terms and twisted \( F \)-terms must go into each other under the map. We discuss now the coupling of mirror-symmetric rigid models to supergravity. Obviously, in order to maintain the symmetry, we must couple to the \( U_V(1) \otimes U_A(1) \) version.

We note that using the reducible version of supergravity is perfectly natural if one regards the \( 2D, \, N = 2 \) theory as arising from compactification of the heterotic string. It is well known that the \( 4D, \, N = 1 \) field theory limit of the heterotic string consists of a \( 4D, \, N = 1 \) supergravity multiplet plus a \( 4D, \, N = 1 \) tensor multiplet (or linear multiplet) coupled to matter. Under further reduction to two dimensions the supergravity multiplet becomes a \( 2D, \, N = 2 \) supergravity multiplet (plus matter vector multiplets) and the tensor multiplet becomes a \( 2D, \, N = 2 \) tensor multiplet which is equivalent to a \( 2D \) vector multiplet. If we have started with the minimal version of \( 4D \) supergravity, the reduction will lead to the \( 2D, \, U_A(1) \) theory, while the reduction of the tensor multiplet leads to a twisted vector multiplet. Therefore, a heterotic string compactified so that its target space possesses a realization of \( 2D, \, N = 2 \) supersymmetry naturally leads to the appearance of the \( U_V(1) \otimes U_A(1) \) theory coupled to matter.

The coupling to the \( U_V(1) \otimes U_A(1) \) theory involves two issues. The first one concerns the existence of appropriate local measures, the second one the derivation and form of the projection formulae. There is no difficulty in generalizing global \( D \)-terms to local ones and in particular, as we have noted earlier, the vielbein determinant is the same as in the \( U_A(1) \) theory, \( \hat{E}^{-1} = E^{-1} \). Furthermore, in \[17\] we established the existence of a chiral measure by constructing it in terms of the chiral
measure of the $U_A(1)$ theory, and inferred the existence of a twisted chiral measure. Equivalently, the relation between the measures can be deduced from (4.9) which allows us to relate $\hat{\nabla}^2$ and $\bar{\nabla}^2$.

The derivation of the chiral projection formula (5.8) follows precisely that for the $U_A(1)$ theory. Similarly, the derivation of the twisted chiral projection formula (5.9) goes through as in [17], with due regard to the presence of the supergravity field strength $\tilde{F}$ in the anticommutator $\{\hat{\nabla}_+, \hat{\nabla}_-\}$. In particular, we find now complete symmetry between the projection formulae (5.6) and (5.9), with the component auxiliary field $\bar{R}$ entering in (5.9) in the same manner that the component auxiliary field $\tilde{R}| = B$ enters in (5.6).

We come to the conclusion that most of the results of the previous sections can be immediately generalized to the present situation. Obviously, since the extended supergravity theory contains more component fields (the components of the entangled vector multiplet), we may expect these components to appear in the final form of results which previously involved just the ordinary supergravity component fields, namely $B$, the gravitino, and the component curvature.

8 Quantum Considerations

We now examine at the quantum level the coupling of matter superfields to the $U_V(1) \otimes U_A(1)$ version of supergravity and determine the induced action. As usual it is sufficient to compute the one-loop contribution of chiral and twisted chiral superfields to the self-energy of the prepotential $H^a$ and covariantize the result.

We collect together the expressions that we need for various quantities in the $U_V(1) \otimes U_A(1)$ supergravity theory at the linearized level, in terms of the prepotentials $H^a$ and $S$ [16]. For the field strengths we have

$$\begin{align*}
\bar{R} &= -D_+\Gamma_- - D_-\Gamma_+ \\
\tilde{F} &= 2iD_+\Sigma_- + 2iD_-\Sigma_+ 
\end{align*}$$

and for the connections,

$$\begin{align*}
\Gamma_- &= -2D_-(S + \bar{S}) - D_+A_-^+ \\
\Gamma_+ &= 2D_+(S + \bar{S}) + D_-A_-^+ \\
\Sigma_+ &= -2iD_+\bar{S} - iD_-A_-^- \\
\Sigma_- &= 2iD_-S + iD_+A_-^+ .
\end{align*}$$

To first order the $A$’s are given by

$$\begin{align*}
A_-^+ &= i\hat{C}_+ - 2D_+D_-H^* \\
A_-^- &= i\hat{C}_- = -2D_+D_-H^*.
\end{align*}$$
\[ A_+^\dagger = i \hat{C}_+ = -2D_+H^\dagger \quad , \quad A_-^\dagger = i \hat{C}_- = -2D_-H^\dagger \quad . \quad (8.3) \]

Upon substitution we obtain
\[
\begin{align*}
\bar{R} &= 4D_+D_-(S + \bar{S}) \\
R &= -4D_+D_-(S + \bar{S}) \\
F &= 4D_+D_-(S + \bar{S} + D_+D_+H^\dagger - D_-D_-H^\dagger) \\
&= 4D_+D_-(S + \bar{S} + i\partial_\mp H^\dagger - i\partial_\pm H^\dagger) \\
\bar{F} &= -4D_+D_-(S + \bar{S} - D_+D_+H^\dagger + D_-D_-H^\dagger) \\
&= -4D_+D_-(S + \bar{S} - i\partial_\mp H^\dagger + i\partial_\pm H^\dagger) \quad . \quad (8.4)
\end{align*}
\]

We begin with the kinetic action for covariantly chiral and twisted chiral superfields, \( S = \int d^6z E E^{-1}(\Phi \Phi - \bar{X}X) \). (In this section ordinary chiral and twisted chiral superfields are denoted by \( \phi \) and \( \chi \), respectively.) We have \( \Phi = e^{iH_\cdot \partial} \phi \), but a closed expression for \( X \) in terms of \( \chi \) is not known. However in [L7], we derived an expression relating a covariantly twisted chiral superfield to an ordinary one at the linearized level, which is sufficient for our present one-loop calculation,
\[
\begin{align*}
\bar{X} &= 1 - 2D_+H^\dagger D_+ + 2D_-H^\dagger D_+ + iH^\dagger \partial_\mp - iH^\dagger \partial_\pm \chi \\
X &= 1 + 2D_+H^\dagger D_+ - 2D_-H^\dagger D_+ - iH^\dagger \partial_\mp + iH^\dagger \partial_\pm \bar{\chi} \quad . \quad (8.5)
\end{align*}
\]

We also expand the inverse of the superdeterminant to first order in \( H^a \)
\[
E^{-1} = 1 - [D_+, D_+]H^\dagger - [D_-, D_-]H^\dagger \quad . \quad (8.6)
\]

We obtain the action from which we can compute the one-loop contribution,
\[
S = \int d^6z \left[ - \bar{\phi} \phi - 2H^\dagger D_+ \bar{\phi} D_+ \phi - 2H^\dagger D_- \bar{\phi} D_- \phi + \cdots \right] \\
- \left[ \bar{\chi} \chi + 2H^\dagger D_+ \bar{\chi} D_+ \chi - 2H^\dagger D_- \bar{\chi} D_- \chi + \cdots \right] \quad . \quad (8.7)
\]

For the propagators, we have
\[
< \bar{\phi}(z)\phi(z') > = -\frac{D^2 D^2}{\partial_\mp \partial_\pm} \delta^{(2)}(z - z')\delta^{(4)}(\theta_z - \theta_{z'}) \quad , \quad (8.8)
\]

and
\[
< \bar{\chi}(z)\chi(z') > = -\frac{D_+ D_- \partial_\pm \partial_\mp}{\partial_\mp \partial_\pm} \delta^{(2)}(z - z')\delta^{(4)}(\theta_z - \theta_{z'}) \quad . \quad (8.9)
\]

We find that the contribution to the \( H^a \) self-energy at one-loop is
\[
- \frac{8}{\pi} \left[ D_+ D_- \partial_\pm H^\dagger \frac{1}{\Box} D_+ D_- \partial_\pm H^\dagger + D_+ D_- \partial_\pm H^\dagger \frac{1}{\Box} D_+ D_- \partial_\pm H^\dagger \right] \quad . \quad (8.10)
\]
In order to covariantize the result, we add in extra local cross-terms which give
\[ -\frac{8}{\pi} [D_+D_- (\partial_\Psi H^\Psi - \partial_\Xi H^\Xi) \frac{1}{\Box} D_- D_+ (\partial_\Psi H^\Psi - \partial_\Xi H^\Xi)] \],
(8.11)
and using (8.4) we rewrite the result in covariant form, representing the induced action
\[ S_{\text{ind}} = \frac{1}{4\pi} \int d^2 \theta \int d^4 \theta \left[ \tilde{R}^1_1 R - \tilde{F}^1_1 F \right] \],
(8.12)
which is manifestly mirror symmetric.

As formulated in this paper, the mirror transformation is a classical discrete transformation defined on rigid (2,2) supersymmetric off-shell field theories and the coordinates of rigid (2,2) superspace. The classical mirror symmetric theories are those that are invariant under this discrete transformation. By coupling to the \( U_{\text{V}}(1) \otimes U_{\text{A}}(1) \) supergravity we realize mirror symmetry within the context of a locally supersymmetric theory. The calculation above shows that this discrete symmetry can be preserved after quantization.

## 9 Concluding remarks

We have seen that the mirror transformation is an intrinsic geometric feature of 2D, \( N = 2 \) superspace theories. It has been conjectured that this is the origin of mirror manifolds as seen in the (2,2) Calabi-Yau compactification of the heterotic string, but the precise connection is unclear, as is the role of duality as an attempt to relate theories that are connected by the mirror map operator. Stated in its simplest form, is there a first order formulation such that eliminating some of its variables yields (7.28), while a different elimination yields (7.29)? A second such question is whether there exists a relation between the strong coupling limit of (7.28) and the weak coupling limit of (7.29).

We have presented in this paper a number of tools and results concerning the coupling of 2D, \( N = 2 \) matter multiplets to supergravity. Although we have not studied in detail linear complex multiplets, it is clear that our results can be easily generalized to them. We would be led to consider therefore general models of the form
\[ S = \int d^2 x d^4 \theta \, E^{-1} \tilde{\Omega} \\
+ \left[ \int d^2 x d^2 \theta \, \tilde{\mathcal{E}}^{-1} \left[ h_{1J} (\nabla_+ \Sigma^1)(\nabla_+ \Sigma^J) + U \right] + \text{h.c.} \right] \\
+ \left[ \int d^2 x d\theta^+ d\theta^+ \, \tilde{\mathcal{E}}^{-1} \left[ k_{ij} (\nabla_+ \Xi^i)(\nabla_+ \Xi^j) + \tilde{U} \right] + \text{h.c.} \right]. \] (9.1)

The functions that govern the form of this general action include the real Kähler-like potential \( \tilde{\Omega} \), holomorphic target space second-rank \( \text{not necessarily skew symmetric} \)
tensors $h_{IJ}$ and $k_{ij}$ and holomorphic target space scalars $P$ and $\hat{P}$. The Kähler-like potential may depend on any scalar superfield whose physical degrees of freedom consist of spins 0, 1/2 including $2D$, $N = 2$ vector multiplets. These vector multiplets are described in terms of superspace covariant derivatives that have a $\mathcal{G}_c \otimes \mathcal{G}'_{tc}$ structure, where $\mathcal{G}_c$ acts on chiral and complex linear superfields and $\mathcal{G}'_{tc}$ acts on twisted chiral and twisted complex linear superfields.

The existence of the holomorphic second-rank tensors $h_{IJ}$ and $k_{ij}$ has not been generally recognized previously. The former is restricted to depend solely on chiral superfields and the latter solely on twisted chiral superfields. This is the same as for the holomorphic target space scalars $U$ and $\hat{U}$, respectively. The off-shell supergravity fields that appear in the action may be chosen to be either the $U_V(1)$, $U_A(1)$ (minimal irreducible) or $U_V(1) \otimes U_A(1)$ theories. The rigid version of this action is expected to provide the most general off-shell (2,2) compactification of the heterotic string. To every action of this form there is naturally associated a mirror reflected theory. Thus, (2,2) supersymmetric theories necessarily come in pairs with the mirror map transformation providing the isomorphism between members of the pairs. Although we have not investigated the issue, it may be possible extend the notion of the mirror map to act from the space of chiral superfields to (twisted) complex linear superfields. Finally, we have every reason to believe that many of our results can be extended to $2D$, $N = 4$ theories.

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A Appendix

We restrict ourselves to the minimal $U_A(1)$ theory by setting $\Sigma_A = 0$. In defining components we follow the philosophy and methods described in *Superspace*, [7]. The expressions for the covariant derivatives evaluated at $\theta = 0$ are:

\begin{align*}
\nabla_a | &= \partial_a \\
\nabla_a | &= D_a + \psi_a ^\alpha \nabla_\alpha | + \psi_{\dot{a}} ^\alpha \nabla_{\dot{\alpha}} | \\
&= D_a + \psi_a ^\alpha \partial_\alpha + \psi_{\dot{a}} ^\alpha \partial_{\dot{\alpha}} \ . \tag{A.1}
\end{align*}

In the $U_A(1)$ theory we define $D_a$ as the fully covariant gravitational derivative with a Lorentz connection that includes, in addition to the ordinary connection, extra terms that are bilinear in the gravitini. Specifically, $D_a$ is defined to be

\begin{align*}
D_a &= e_a + \varphi_a M + V'_a \gamma' \\
&= e_a + \omega_a M + \gamma_a \overline{M} \tag{A.2}
\end{align*}

where $\omega_a$ is the gravitational plus $U(1)$ component connection, including gravitini.

We also introduce the ordinary gravitational covariant derivative (without $U(1)$ connection or gravitino terms), denoted $D_a$

\begin{equation}
D_a = e_a + \omega_a M \tag{A.3}
\end{equation}

and in particular define $\partial_\pm \equiv e_\pm = e_\pm ^m \partial_m$, $\partial_\mp \equiv e_\mp = e_\mp ^m \partial_m$.

The component connections are defined by projection to be:

\begin{align*}
\Phi_A | &= \varphi_A \\
\Sigma'_A | &= V'_A \\
\Sigma_A | &= V_A \tag{A.4}
\end{align*}

and

\begin{align*}
\Omega_A | &= \omega_A \\
\Gamma_A | &= \gamma_A \ . \tag{A.5}
\end{align*}

Note that

\begin{equation}
\varphi_A = \frac{1}{2}(\omega + \gamma)_A \ \text{and} \ V'_A = \frac{i}{2}(\omega - \gamma)_A \ . \tag{A.6}
\end{equation}

We also need expressions for objects such as $\nabla_\alpha \nabla_\beta |$, for example. The $\nabla_\alpha$ component of $\nabla_\beta$ is given by (c.f. *Superspace* sec. 5.6.b)

\begin{equation}
\nabla_\alpha \nabla_\beta | = \frac{1}{2} \{ \nabla_\alpha, \nabla_\beta \} | \tag{A.7}
\end{equation}

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while the $\nabla_\alpha$ component of $\nabla_b$ is

$$
\nabla_\alpha \nabla_b = [\nabla_\alpha, \nabla_b] + \nabla_b \nabla_\alpha
= [\nabla_\alpha, \nabla_b] + D_b \nabla_\alpha + \psi^\beta_\alpha \partial_\beta \nabla_\alpha + \psi^\beta_\alpha \partial_\beta \nabla_\alpha .
$$

(A.8)

From (A.7) we obtain the following relations

$$
\nabla_+ \nabla_+ = \frac{1}{2} \{\nabla_+, \nabla_+\} = 0
\nabla_- \nabla_- = \frac{1}{2} \{\nabla_-, \nabla_-\} = 0
\nabla_+ \nabla_+ = \frac{1}{2} \{\nabla_+, \nabla_+\} = \frac{i}{2} \nabla_+
\nabla_- \nabla_- = \frac{1}{2} \{\nabla_-, \nabla_-\} = \frac{i}{2} \nabla=
\nabla_+ \nabla_- = \frac{1}{2} \{\nabla_+, \nabla_-\} = -\frac{1}{2} R |\nabla|
\nabla_+ \nabla_- = \frac{1}{2} \{\nabla_+, \nabla_-\} = 0 ,
\nabla_- \nabla_+ = \frac{1}{2} \{\nabla_-, \nabla_+\} = 0 ,
$$

(A.9)

and from (A.8), we get the series of identities that appears below.

$$
\nabla_+ \nabla_+ = [\nabla_+, \nabla_\#] + \nabla_\# \nabla_+
= \nabla_\# \nabla_- + \nabla_- \nabla_+ + \psi^\beta_\# \nabla_- \nabla_+ + \psi^\beta_- \nabla_+ \nabla_+
= \nabla_\# \nabla_- - \frac{1}{2} \psi^\beta_\# R |\nabla_\#| + \frac{i}{2} \psi^\beta_\# (D_\# + \psi^\beta_- \nabla_\alpha + \psi^\beta_\alpha \nabla_\#)
\nabla_- \nabla_- = [\nabla_-, \nabla_-] + \nabla_- \nabla_- + \psi^\beta_- \nabla_- \nabla_- + \psi^\beta_- \nabla_- \nabla_- + \frac{i}{2} \nabla_- |\nabla_-| - i(\nabla_- R) M|
= \nabla_- \nabla_- - \frac{1}{2} \psi^\beta_- R |\nabla_-| + \frac{i}{2} \psi^\beta_- (D_- + \psi^\beta_- \nabla_\alpha + \psi^\beta_- \nabla_\#) - \frac{i}{2} \nabla_- |\nabla_-| - i(\nabla_- R) M|
\nabla_\# \nabla_\# = \nabla_\# \nabla_\# + \frac{i}{2} \psi^\beta_\# R |\nabla_\#| + \frac{i}{2} \psi^\beta_\# (D_\# + \psi^\beta_- \nabla_\alpha + \psi^\beta_- \nabla_\#)
\nabla_\# \nabla_- = \nabla_\# \nabla_- + \frac{i}{2} \psi^\beta_- R |\nabla_-| + \frac{i}{2} \psi^\beta_- (D_- + \psi^\beta_- \nabla_\alpha + \psi^\beta_- \nabla_\#) - \frac{i}{2} \nabla_- |\nabla_-| - i(\nabla_- R) M|
\nabla_+ \nabla_\# = \nabla_+ \nabla_\# - \frac{i}{2} \psi^\alpha_- R |\nabla_\#| + \frac{i}{2} \psi^\alpha_- (D_\# + \psi^\alpha_- \nabla_\alpha + \psi^\alpha_- \nabla_\#)
\nabla_\# \nabla_\# = \nabla_\# \nabla_\# + \frac{i}{2} \psi^\alpha_- R |\nabla_\#| + \frac{i}{2} \psi^\alpha_- (D_\# + \psi^\alpha_- \nabla_\alpha + \psi^\alpha_- \nabla_\#)
\nabla_\# \nabla_- = \nabla_\# \nabla_- + \frac{i}{2} \psi^\beta_- R |\nabla_-| + \frac{i}{2} \psi^\beta_- (D_- + \psi^\beta_- \nabla_\alpha + \psi^\beta_- \nabla_\#)
$$

(A.10)
where we have used the commutation relations

\[
\begin{align*}
[\nabla_+, \nabla_+] &= 0 , \quad [\nabla_+, \nabla_\pm] = 0 \\
[\nabla_-, \nabla_\pm] &= 0 , \quad [\nabla_-, \nabla_\mp] = 0 \\
[\nabla_+, \nabla_\pm] &= -\frac{i}{2}\mathcal{R}\nabla_\pm - i(\nabla_\mp \mathcal{R})\mathcal{M} \\
[\nabla_+, \nabla_\mp] &= \frac{i}{2}\mathcal{R}\nabla_- + i(\nabla_- \mathcal{R})M \\
[\nabla_-, \nabla_+] &= \frac{i}{2}\mathcal{R}\nabla_\pm - i(\nabla_\mp \mathcal{R})\mathcal{M} \\
[\nabla_-, \nabla_\mp] &= -\frac{i}{2}\mathcal{R}\nabla_+ + i(\nabla_+ \mathcal{R})M
\end{align*}
\]  

(A.11)

and also

\[
[\nabla_\pm, \nabla_\mp] = \frac{1}{2}(\nabla_+ \mathcal{R})\nabla_- + \frac{1}{2}(\nabla_- \mathcal{R})\nabla_+ - \frac{1}{2}(\nabla_\pm \mathcal{R})\nabla_\mp - \frac{1}{2}(\nabla_\mp \mathcal{R})\nabla_\pm \\
&\quad -\frac{1}{2}\mathcal{R}\mathcal{R}\mathcal{M} - \frac{1}{2}\mathcal{R}\mathcal{R}M + (\nabla_\mp \mathcal{R})M - (\nabla_\pm \mathcal{R})\mathcal{M} .
\]  

(A.12)

We also find the \( \nabla_\pm \) component of \( \nabla_\mp \) to be:

\[
\nabla_\pm \nabla_\mp = (\mathbf{D}_\pm + \psi_\pm \nabla_\mp + \psi_\pm \nabla_\alpha)\mathcal{R} \nabla_\mp + \psi_\pm \nabla_\mp + \psi_\pm \nabla_\alpha \nabla_\mp + \psi_\pm \nabla_\alpha \nabla_\mp .
\]  

(A.13)

Using this we can show that

\[
[\nabla_\pm, \nabla_\mp] = [\nabla_\pm, \nabla_\mp] + \psi_\pm \nabla_\alpha \nabla_\mp + \psi_\pm \nabla_\alpha \nabla_\mp - \psi_\pm \nabla_\alpha \nabla_\mp - \psi_\pm \nabla_\alpha \nabla_\mp ,
\]  

(A.14)

where the first term can be expanded as

\[
[\nabla_\pm, \nabla_\pm] = [\mathbf{D}_\pm + \psi_\pm \nabla_\alpha + \psi_\pm \nabla_\alpha, \mathbf{D}_\pm + \psi_\pm \nabla_\alpha + \psi_\pm \nabla_\alpha] \\
= [\mathbf{D}_\pm, \mathbf{D}_\pm] + [\mathbf{D}_\pm (\psi_\pm \nabla_\alpha) + \mathbf{D}_\pm (\psi_\pm \nabla_\alpha)] \\
= [\mathbf{D}_\pm, \mathbf{D}_\pm] + \mathbf{D}_\pm (\psi_\pm \nabla_\alpha) + \mathbf{D}_\pm (\psi_\pm \nabla_\alpha) \\
&\quad + \frac{1}{2}\omega (\psi_\pm \nabla_\pm \nabla_\mp - \psi_\pm \nabla_\mp \nabla_\pm) + \frac{1}{2}\gamma (\psi_\pm \nabla_\pm \nabla_\mp - \psi_\pm \nabla_\mp \nabla_\pm) .
\]  

(A.15)

Substituting this into \( [\nabla_\pm, \nabla_\mp] \), we get

\[
[\nabla_\pm, \nabla_\mp] = [\mathbf{D}_\pm, \mathbf{D}_\mp] + \mathbf{D}_\pm (\psi_\pm \nabla_\alpha) + \mathbf{D}_\pm (\psi_\pm \nabla_\alpha) \\
+ \psi_\pm \nabla_\alpha \nabla_\mp + \psi_\pm \nabla_\alpha \nabla_\mp - \psi_\pm \nabla_\alpha \nabla_\mp - \psi_\pm \nabla_\alpha \nabla_\mp \\
= [\mathbf{D}_\pm, \mathbf{D}_\mp] + \mathbf{D}_\pm (\psi_\pm \nabla_\alpha) + \mathbf{D}_\pm (\psi_\pm \nabla_\alpha) \\
+ \psi_\pm \nabla_\pm \nabla_\mp + \frac{1}{2}\psi_\pm (\mathbf{D}_\pm + \psi_\pm \nabla_\alpha + \psi_\pm \nabla_\alpha) - \frac{1}{2}\psi_\pm \mathcal{R}\mathcal{M} - \frac{1}{2}\psi_\pm \mathcal{R}M - i\mathcal{R} \nabla_\mp - i(\nabla_\mp \mathcal{R})\mathcal{M} \\
+ \psi_\pm \mathbf{D}_\pm \nabla_\mp - \frac{1}{2}\psi_\pm \mathcal{R}M + \frac{1}{2}\psi_\pm (\mathbf{D}_\pm + \psi_\pm \nabla_\alpha + \psi_\pm \nabla_\alpha) \\
+ 6 \text{ other similar terms} .
\]  

(A.16)
If we compare this now with (A.12) (which is true in general, not just at $\theta = 0$), we can obtain expressions for the various derivatives of $R$ and $\bar{R}$, evaluated at $\theta = 0$. By comparing the coefficients of $\nabla_{\pm} = \partial_{\pm}$ on both sides, for example, we get:

$$\frac{1}{2} \nabla_- R = D_{\ast}\psi^+ + \frac{i}{2} \psi^+ \bar{R} + i\psi^+ \psi^+ \psi^+ + i\psi^+ \psi^+ \psi^+ + i\psi^+ \psi^+ \psi^+ .$$  \hspace{1cm} (A.17)

In the same fashion we obtain:

$$\frac{1}{2} \nabla_+ R = D_{\ast}\psi^- + \frac{i}{2} \psi^- \bar{R} + i\psi^- \psi^- \psi^- + i\psi^- \psi^- \psi^- + i\psi^- \psi^- \psi^- .$$

$$-\frac{1}{2} \nabla_- \bar{R} = D_{\ast}\psi^- - \frac{i}{2} \psi^- \bar{R} + i\psi^- \psi^- \psi^- + i\psi^- \psi^- \psi^- + i\psi^- \psi^- \psi^- .$$

$$-\frac{1}{2} \nabla_+ \bar{R} = D_{\ast}\psi^+ - \frac{i}{2} \psi^+ \bar{R} + i\psi^+ \psi^+ \psi^+ + i\psi^+ \psi^+ \psi^+ + i\psi^+ \psi^+ \psi^+ .$$

$$(-\frac{1}{2} \bar{R} R + \nabla^2 R) = [D_{\ast}, D_{\ast}] M - i\psi^- \nabla_+ R + i\psi^+ \nabla_- R - \psi^- \psi^- R - \psi^+ \psi^+ R .$$

$$(-\frac{1}{2} \bar{R} R - \nabla^2 R) = [D_{\ast}, D_{\ast}] M - i\psi^+ \nabla_+ R + i\psi^- \nabla_- R - \psi^+ \psi^+ R - \psi^- \psi^- R .$$  \hspace{1cm} (A.18)

We can get explicit information about the connections from looking at the commutator of two $D$’s. In particular

$$[D_{\ast}, D_{\ast}] = [e_{\ast}, e_{\ast}] + e_{\ast} [\omega_{\ast}] M + e_{\ast} [\gamma_{\ast}] M$$

$$-\frac{1}{2} \omega_{\ast} (e_{\ast}) - \frac{1}{2} \gamma_{\ast} (e_{\ast}) - \omega_{\ast} e_{\ast} M$$

$$+ \gamma_{\ast} e_{\ast} M - \frac{1}{2} \gamma_{\ast} (e_{\ast}) M - \frac{1}{2} \omega_{\ast} (e_{\ast}) M .$$  \hspace{1cm} (A.19)

The anholonomy coefficients are defined as usual by $[e_{\ast}, e_{\ast}] = C_{\ast} a e_{a}$. Denoting the “component” torsion by $t_{\pm}^a$ where

$$[D_{\ast}, D_{\ast}] = t_{\pm}^a D_a + r_{\pm} \bar{M} ,$$  \hspace{1cm} (A.20)

and comparing (A.20) with (A.19), we find the torsions

$$t_{\pm}^\pm = C_{\pm} a - \frac{1}{2} (\omega + \gamma)_{\ast}$$

$$t_{\pm}^\mp = C_{\pm} a - \frac{1}{2} (\omega + \gamma)_{\ast} .$$  \hspace{1cm} (A.21)

and the curvatures

$$r_{\pm} = e_{\ast} [\omega_{\ast}] - C_{\pm} a \omega_{\ast} - C_{\pm} \omega_{\ast}$$

$$\bar{r}_{\pm} = e_{\ast} [\gamma_{\ast}] - C_{\pm} a \gamma_{\ast} - C_{\pm} \gamma_{\ast} .$$  \hspace{1cm} (A.22)
The full torsions are again defined in the standard way, \(\{\nabla_A, \nabla_B\} = T^C_{\ AB} \nabla_C + R^M_{\ AB} M + R^M_{\ AB} M\). Therefore, from (A.14) at \(\theta = 0\) and using (A.1), we get the full torsions to be

\[
\begin{align*}
T^\#_{\ -} &= t^\#_{\ -} + i(\psi_\#^+ \psi_-^+ + \psi_\#^+ \psi_-^+) \\
T^\#_{\ -} &= t^\#_{\ -} + i(\psi_-^+ \psi_-^+ + \psi_-^+ \psi_-^+) \\
T^\#_{\ -} &= D_\#\psi_\#^+ + \frac{i}{2} \psi_\#^+ R \psi_\#^+ + \psi_\#^+ \psi_\#^+ \psi_\#^+ + \psi_\#^+ \psi_\#^+ \psi_\#^+ + \psi_\#^+ \psi_\#^+ \psi_\#^+ + \psi_\#^+ \psi_\#^+ \psi_\#^+ . \quad (A.23)
\end{align*}
\]

It is obvious from (A.12) that \(T^\#_{\ -} = 0\). Combining this with (A.21) and (A.23), we obtain

\[
\begin{align*}
\tilde{\varphi} \# &= \frac{1}{2}(\omega + \gamma) \# \\
&= C^\#_{\ -} + i(\psi_\#^+ \psi_-^+ + \psi_\#^+ \psi_-^+) \\
&= \omega \# + i(\psi_-^+ \psi_-^+ + \psi_-^+ \psi_-^+) \\
\tilde{\varphi} &= \frac{1}{2}(\omega + \gamma) \\
&= C^\#_{\ -} + i(\psi_-^+ \psi_-^+ + \psi_-^+ \psi_-^+) \\
&= \omega_- + i(\psi_-^+ \psi_-^+ + \psi_-^+ \psi_-^+) \quad (A.24)
\end{align*}
\]

for the full component Lorentz connection, written in terms of the ordinary component gravitational connection, \(\omega_a\), plus the gravitini terms mentioned previously.
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