Evaluating Critical Exponents in the Optimized Perturbation Theory

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We use the optimized perturbation theory, or linear $\delta$ expansion, to evaluate the critical exponents in the critical $3d$ $O(N)$ invariant scalar field model. Regarding the implementation procedure, this is the first successful attempt to use the method in this type of evaluation. We present and discuss all the associated subtleties producing a prescription which can, in principle, be extended to higher orders in a consistent way. Numerically, our approach, taken at the lowest nontrivial order (second order) in the $\delta$ expansion produces a modest improvement in comparison to mean field values for the anomalous dimension $\eta$ and correlation length $\nu$ critical exponents. However, it nevertheless points to the right direction of the values obtained with other methods, like the $\epsilon$-expansion. We discuss the possibilities of improving over our lowest order results and on the convergence to the known values when extending the method to higher orders.

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I. INTRODUCTION

A distinct feature as we approach second order phase transition points is the emergence of critical phenomena and associated universality and scaling properties, as a result of diverging correlation lengths. Thanks to these characteristics we can relate many different systems just by simple general distinctions, like dimension and symmetry, with their universal behavior set by critical exponents, independent of the microscopic dynamics [1]. On the other hand, close to the critical temperature of transition we are faced with the problem of breakdown of perturbation methods in field theory due to the appearance of infrared divergences, as the correlation lengths diverge (or masses vanish), which then require the use of nonperturbative methods to study the physical system around the critical points and, ultimately, the phase transition process itself. The present work was motivated by the recent progress in dealing with field theory phase transitions in the context of the nonperturbative method of the optimized mass, or linear $\delta$ expansion (LDE) [2, 3, 4, 5]. Here, for the first time, this method is used in the derivation of the critical exponents of the three-dimensional $O(N)$ invariant $\lambda \phi^4$ model.

Critical exponents have been extensively studied at different levels of accuracy, particularly within the $\epsilon$-expansion and renormalization group (RG) methods (see, e.g. Ref. [6] for a throughout discussion of various methods of evaluating critical exponents in field theory). However, it is well known that these methods may not be applicable or appropriate for a number of important physical systems. Examples include, for instance, the reliability of the $\epsilon$-expansion in the study of the electroweak phase transition, where there may not be a visible fixed point in three dimensions when we perform an expansion around dimension $d = 4 - \epsilon$ [7]. There are also the important cases of multicritical phenomena in two dimensions with no obvious upper critical dimension or with no $O(N)$ invariants, in which case nonperturbative techniques like the $\epsilon$-expansion and large-$N$ approximation break down [8]. The use of renormalization group techniques have shown recently to produce impressive good results to critical exponents [9], however, there we commonly have to deal with a set of flow equations that may become very complicated to solve for complex systems.

Using the LDE one makes use of an interpolation of the original action by modifying the masses and coupling constants in terms of a fictitious parameter $\delta$ that at the end is taken to the unity, so as to recover the original model. At the same time an arbitrary mass parameter must be introduced to balance the dimensions of the interpolated theory. At the end it is fixed in such a way that nonperturbative information can be taken into account in quantities which have been perturbatively computed. Concerning the $O(N)$ invariant scalar field model in three dimensions, the convergence properties of the LDE have been studied recently in great detail, specifically at the critical point of phase

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transition in the large-\(N\) limit.\(^{4,5,10}\) Those results are also encouraging concerning the computation of critical exponents as performed in this paper. An advantage of the LDE method when compared to other nonperturbative methods is that we remain all the time within the familiar grounds of perturbation theory. The method then becomes of much simpler use than the traditional ones and we expect that the LDE may even further improve the accuracy of other nonperturbative methods when used in conjunction with them. We also expect the method to be suitable in the evaluation of the critical properties in the transition point in those cases where no clear expansion parameter exists or when the conventional techniques do not apply, as in the models discussed in the previous paragraph. The LDE has been successfully applied before to study field theory at finite temperature and phase transitions in those situations where conventional perturbation theory breaks down.\(^{2,3,11}\) In particular, in Ref.\(^{12}\), the LDE was applied to study the phase transition patterns in coupled scalar field models reproducing previous renormalization group applications to the problem and also predicting new critical points, showing the advantages of the method for applications regarding complex systems that may exhibit multicritical points.

In this work we shall consider the same \(O(N)\) model studied in Refs.\(^{2,3,4,5}\). It is described by the \(O(N)\) invariant super-renormalizable action

\[
S_\phi = \int d^3x \left[ \frac{1}{2} \nabla \phi^2 + \frac{1}{2} r_0 \phi^2 + \frac{u}{4!} \phi^4 \right],
\]

where \(r_0 = r + A\), with \(A\) representing the mass renormalization counterterm needed to remove any ultraviolet divergence. The full propagator for this theory is given by

\[
G(p) = \left[ p^2 + m^2 + \Sigma_{\text{ren}}(p) - \Sigma_{\text{ren}}(0) \right]^{-1},
\]

where \(m^2 = r + \Sigma_{\text{ren}}(0)\) with \(\Sigma_{\text{ren}}\) representing the renormalized self-energies. At the critical point, \(m^2 = 0\), which implies the Hugenholtz-Pines theorem, \(r_c = -\Sigma_{\text{ren}}(0)\). Using the model described by Eq.\(^{1(1)}\) is advantageous since our results can readily be compared with other results that have been extensively obtained in connection with the \(\phi^4\) \(O(N)\) invariant scalar model. This is a particularly important model also for the reason that it can describe the critical properties (due to universality) of many different physical systems, such as polymers (\(N = 0\)), Ising models (\(N = 1\)), superfluid Helium, Bose-Einstein condensation of atomic atoms (\(N = 3\), etc. A similar attempt to compute critical exponents with the \(\delta\) expansion was performed by the authors of Ref.\(^{13}\). However, they made use of the nonlinear version of method which leads to a considerable more complicated perturbation series (see also\(^{14}\)), which quickly becomes cumbersome beyond leading order in \(\delta\) and their results were not so good for the same critical exponents evaluated here. On the other hand, the way the linear version is employed here avoids those difficulties and the results can in principle be extended to arbitrary orders as usual perturbative calculations. Our results for the correlation length (or mass) critical exponent \(\nu\) and the anomalous dimension exponent \(\eta\), already to the lowest nontrivial order (second order) in \(\delta\), though showing only modest improvements over the mean field values, are shifted away from these in the direction of the values known from high precision numerical results despite the simplicity of our calculations.

This paper is organized as follows. In Sec. II we briefly review the LDE method and present the interpolated version of the \(O(N)\) invariant scalar field action relevant to our study. In Sec. III, we carry out the formal evaluation of the critical exponents for the correlation length \(\nu\) and anomalous dimension \(\eta\). Our concluding remarks are presented in Sec. IV, where we also discuss the extension of our method to higher orders.

II. LDE AND THE INTERPOLATED EFFECTIVE SCALAR MODEL

Let us start our work by reviewing the application of the LDE method to our problem. The LDE was conceived to treat nonperturbative physics while staying within the familiar calculation framework provided by perturbation theory. In practice, this can be achieved as follows. Starting from an action \(S\) one performs the following interpolation

\[
S \to S_\delta = \delta S + (1 - \delta)S_0(M),
\]

where \(S_0\) is the soluble quadratic action, added by an (optimizable) mass term \(M\), and \(\delta\) is an arbitrary parameter. The above modification of the original action somewhat reminds the usual trick consisting of adding and subtracting a mass term to the original action. One can readily see that at \(\delta = 1\) the original theory is retrieved, so that \(\delta\) actually works just as a bookkeeping parameter. The important modification is encoded in the field dependent quadratic term
$S_0(M)$ that, for dimensional reasons, must include terms with mass dimensions $(M)$. In principle, one is free to choose these mass terms and within the Hartree approximation they are replaced by a direct (or tadpole) type of self-energy before one performs any calculation. In the LDE they are taken as being completely arbitrary mass parameters, which will be fixed at the very end of a particular evaluation by an optimization method. One then formally pretends that $\delta$ labels interactions so that $S_0$ is absorbed in the propagator whereas $\delta S_0$ is regarded as a quadratic interaction. So, one sees that the physical essence of the method is the traditional dressing of the propagator to be used in the evaluation of physical quantities very much like in the Hartree case. What is different between the two methods is that, within the LDE the propagator is completely arbitrary, whereas it is constrained to cope only with the so-called direct terms (i.e. tadpoles) within the Hartree approximation. So, within the latter approximation the relevant contributions are selected according to their topology from the start.

Within the LDE one calculates in powers of $\delta$ as if it was a small parameter. In this respect the LDE resembles the large-$N$ calculation since both methods use a bookkeeping parameter which is not a physical parameter like the original coupling constants and within each method one performs the calculations formally working as if $N \to \infty$ or $\delta \to 0$, respectively. Finally, in both cases the bookkeeping parameters are set to their original values at the end which, in our case, means $\delta = 1$. However, quantities evaluated at any finite LDE order from the dressed propagator will depend explicitly on $M$, unless one could perform a calculation to all orders. Up to this stage the results remain strictly perturbative and very similar to the ones which would be obtained via a true perturbative calculation. It is now that the freedom in fixing $M$ generates nonperturbative results. Since $M$ does not belong to the original theory one may require that a physical quantity $\Phi^{(k)}$ calculated perturbatively to order-$\delta^k$ be evaluated at the point where it is less sensitive to this parameter. This criterion, known as the Principle of Minimal Sensitivity (PMS), translates into the variational relation

$$\frac{d\Phi^{(k)}}{dM} \bigg|_{M,\delta=1} = 0 \ . \tag{2.2}$$

The optimum value $\overline{M}$ which satisfies Eq. (2.2) must be a function of the original parameters including the couplings, which generates the nonperturbative results.

Following the considerations above, in order to interpolate the original theory described by Eq. (1.1), one may use

$$S_0 = \frac{1}{2} \left[ \nabla \phi^2 + r \phi^2 + M^2 \phi^2 \right] \ , \tag{2.3}$$

as in Refs. [3]. Then, the interpolated action reads

$$S_\delta = \int d^3x \left[ \frac{1}{2} \nabla \phi^2 + \frac{1}{2} (r + M^2) \phi^2 - \frac{\delta}{2} M^2 \phi^2 + \frac{\delta u}{4!} (\phi^2)^2 + \frac{\delta}{2} A_\delta \phi^2 \right] , \tag{2.4}$$

where $A_\delta$ represents the renormalization mass counterterm for the interpolated theory, which depends on the parameters $M$ and $\delta$. It is important to note that by introducing only extra mass terms in the original theory the LDE does not alter the polynomial structure and, hence, the renormalizability of a quantum field theory. In practice, the original counterterms change in an almost trivial way so as to absorb the new $M$ and $\delta$ dependence (for details, see for instance [3]). The compatibility of the LDE with the renormalization program has been shown in the framework of the O($N$) scalar field theory at finite temperatures, in Ref. [11], showing that it consistently takes into account anomalous dimensions in the critical regime.

For the interpolated theory, the full propagator $G^{(\delta)}(p)$, can be written as

$$G^{(\delta)}(p) = \left[ p^2 + m^2 + (1-\delta)M^2 + \Sigma^{(\delta)}_{\text{ren}}(p) - \Sigma^{(\delta)}_{\text{ren}}(0) \right]^{-1} , \tag{2.5}$$

where $\Sigma^{(\delta)}_{\text{ren}}$ are the (renormalized) momentum dependent self-energies evaluated in powers of $\delta$ and $m^2 = r + \Sigma^{(\delta)}_{\text{ren}}(0)$. At the same time, the bare propagator is given by

$$G^{(0)}(p) = \left[ p^2 + \Omega^2 \right]^{-1} , \tag{2.6}$$

where $\Omega^2 = m^2 + M^2$. It is interesting to note that at the critical point $(m = 0)$ the propagator given by Eq. (2.6) does not generate any infrared divergences since it is automatically regulated by the LDE mass parameter $M$ whereas
the equivalent bare propagator of the original theory is massless. In fact, this is the main problem concerning any eventual perturbative evaluation of quantities at the critical point, an example being the problem of calculations of critical temperature shifts, \( \Delta T \). In our evaluation of the renormalized quantities and fixed points we shall use the following Feynman rules for the vertices: \( \delta M^2 \), \( -\delta A_4 \) for the quadratic interaction and \( -\delta u \) for the quartic one. Those rules are used to evaluate, perturbatively in \( \delta \), all the relevant physical quantities. After that, as discussed in Sec. II, the nonperturbative results are produced by means of the PMS condition, Eq. (2.2).

III. CRITICAL EXponents EVALUATION up to ORDER-\( \delta^2 \)

We now proceed to the evaluation of the critical exponents for the correlation length and anomalous dimension, which follow from the usual scaling relations at the critical point. Following the standard conventions and definitions given, e.g., in Parisi’s textbook \[16\] one writes the scaling relations as

\[
m^2 \propto (r - \delta r_c)^2 \Rightarrow Z_2 \equiv \frac{d}{dm^2} \sim Z_1 m^{2C_2} \Rightarrow C_2 = \frac{1}{2\nu} - 1 ,
\]

and

\[
Z_1 \equiv \left\{ \left[ \frac{G^{(\delta)}(p)}{dG^{(\delta)}(p)} \right]^2 \right\}_{p^2=0} \propto m^{2C_1} \Rightarrow C_1 = \frac{\eta}{2} ,
\]

where \( Z_1 \) is the field renormalization function and \( \lambda \) is the effective renormalized coupling, defined in terms of the one-particle irreducible four-point function \( \Gamma_4 (0) \) as

\[
\lambda = Z_1^2 \frac{\Gamma_4 (0)}{m} .
\]

In order to compute the critical exponents defined in Eqs. (3.1) and (3.2), we need to compute the \( \beta \)-function,

\[
\beta = m^2 \frac{\partial}{\partial m^2} \lambda = - \frac{g}{2} \frac{\partial}{\partial g} \lambda ,
\]

from which the fixed points are obtained. Note that \( g = u/m \) is taken as a dimensionless coupling. The fixed points are then obtained, as usual, from the solutions of \( \beta (\lambda_c) = 0 \).

From the relations (3.1) and (3.2) we can also define

\[
C_1 (\lambda) \equiv - \frac{g}{2} \frac{\partial}{\partial g} \ln[Z_1 (\lambda)] ,
\]

and

\[
C_2 (\lambda) \equiv - \frac{g}{2} \frac{\partial}{\partial g} \ln[Z_2 (\lambda)/Z_1 (\lambda)] ,
\]

which, when evaluated at the fixed point, will determine the constants \( C_1 = C_1 (\lambda_c) \) and \( C_2 = C_2 (\lambda_c) \) defining the critical exponents in (3.1) and (3.2).

Having introduced the quantities given above, let us now use the LDE to evaluate \( \Gamma_4 (0) \), \( Z_1 \) and the self-energy \( \Sigma^{(\delta)} \) expanded in powers of \( \delta \). Note that at first order in \( \delta \) the results are trivial, since it corresponds just to the lowest order one-loop perturbative expansion for which \( Z_1 = Z_2 = 1 \). In this case, using the definitions (3.3) and (3.6), one obtains \( C_1 = C_2 = 0 \) which lead to the well known mean-field critical exponents \( \eta = 0 \) and \( \nu = 1/2 \). The next order results in \( \delta \) will however already lead to results departing from the mean-field ones. For consistency we carry the calculations of all quantities to the first non-trivial order, \( \delta^2 \). Also, all our calculations are performed
using dimensional regularization and the modified minimal subtraction (\( \overline{\text{MS}} \)) renormalization scheme which amounts to replace the the momentum integrals by

\[
\int \frac{d^3p}{(2\pi)^3} \to \int p = \left( \frac{e^\gamma \mu_s^2}{4\pi} \right)^\epsilon \int \frac{d^{2\omega}p}{(2\pi)^{2\omega}},
\]

where \( 2\omega = 3 - 2\epsilon \) while \( \mu_s \) is an arbitrary mass scale and \( \gamma_E \approx 0.5772 \) is the Euler-Mascheroni constant.

### A. Evaluating the optimized self-energy.

As already emphasized, all relevant physical quantities evaluated perturbatively with the LDE will depend on the arbitrary quantity \( \Omega^2 = m^2 + M^2 \), which is present in the LDE bare propagator, Eq. (2.6). Nonperturbative results can be generated by optimizing the renormalized self-energy, \( \Sigma_{\text{ren}}(0) \), which then generates the optimum \( \overline{M} \). Having mass dimensions, this quantity can be expressed as a function of the original \( m \) and \( u \) parameters which, as we shall see, will allow us to express the optimum quantity \( \Omega \) in terms of \( m \) and the dimensionless coupling \( g = u/m \). The choice of \( \Sigma_{\text{ren}}(0) \) as the physical quantity to be optimized is well justified since this is an important quantity in the study of critical phenomena signaling phase transitions via the Hugenholtz-Pines theorem. It also fixes the effective scale determining all critical exponents through the scaling relations at the critical point. To order-\( \delta^2 \) the self-energy has been explicitly evaluated in Ref. [3]. When \( m \neq 0 \) it is given by

\[
\Sigma_{\text{ren}}^{(2)}(0) = -\frac{\delta u \Omega}{8\pi} \left( \frac{N + 2}{3} \right) + \frac{\delta^2 u M^2}{16\pi^2} \left( \frac{N + 2}{3} \right) + \frac{\delta^2 u^2}{128\pi^2} \left( \frac{N + 2}{3} \right)^2
\]

\[
-\frac{\delta^2 u^2}{(8\pi)^2} \frac{N + 2}{18} \left[ 4 \ln \left( \frac{\mu_s^2}{2m^2} \right) + 2 + 4 \ln \left( \frac{2}{3} \right) \right] + O(\delta^3).
\]

We then set \( \delta = 1 \) solving the PMS equation

\[
\frac{d\Sigma_{\text{ren}}^{(2)}(0)}{dM} \bigg|_{\text{M}} = 0,
\]

(3.9)

to obtain the \((N\text{-independent})\) roots

\[
\overline{M}_0 = 0
\]

\[
\overline{M}_\pm^2 = \frac{2}{(12\pi)^2} \left\{ u^2 \pm u \left[ (12\pi m)^2 + u^2 \right]^{1/2} \right\}.
\]

(3.10)

The solutions \( \overline{M}_0 \) and \( \overline{M}_- \), at the critical point \( m = 0 \), are trivial ones, while \( \overline{M}_+ (m \to 0) = u/(6\pi) \) remains nonzero even at the critical point and it is then able to effectively lead to nonperturbative results as shown below, apart from agreeing with our previous results [3]. This nontrivial root (omitting the index “+” from now on) can be written in terms of the dimensionless coupling, \( g = u/m \), as

\[
\overline{M}_+^2 = \frac{m^2}{(12\pi)^2} \left\{ g^2 + 12\pi g \left[ 1 + \frac{g^2}{(12\pi)^2} \right]^{1/2} \right\},
\]

(3.11)

which is, manifestly, a nonperturbative quantity. Finally, the optimized \( \overline{\Omega}^2 = m^2 + \overline{M}_+^2 \) can be written as

\[
\overline{\Omega}^2 = m^2 \left\{ 1 + \frac{1}{(72\pi)^2} \left[ g^2 + 12\pi g F(g) \right] \right\},
\]

(3.12)

\(^1\) See Ref. [3] for details concerning the \( m = 0 \) case.
where we have defined

\[ F(g) = \left[ 1 + \frac{g^2}{(12\pi)^2} \right]^{1/2}. \tag{3.13} \]

So, in the following, our strategy will be to evaluate all relevant quantities with the LDE to order-\(\delta^2\) and then perform the substitution \(\Omega \to \overline{\Omega}\) where the latter quantity is given by Eq. (3.12).

**B. Evaluating the critical coupling \(g_c\)**

Let us start by evaluating the \(\beta\) function whose roots define \(g_c\). The relevant Green’s function is the four-point one, which to order-\(\delta^2\) reads

\[ \Gamma_4(0) = \delta u \frac{3}{2} \delta^2 u^2 \left( \frac{N + 8}{9} \right) \int \frac{d^3q}{(2\pi)^3} \frac{1}{(q^2 + \Omega^2)^2} = \delta u \frac{3\delta^2 u^2}{16\pi \Omega} \left( \frac{N + 8}{9} \right) \], \tag{3.14} \]

where the first term on the RHS is the tree vertex and the second term is the one-loop correction. In addition to the four-point function, according to Eq. (3.3), one needs the field renormalization function in order to define the effective renormalized coupling. To evaluate \(Z_1\), as given by Eq. (3.2), we need the full propagator \(G^{(2)}(p)\) which, to order-\(\delta^2\), is

\[ G^{(2)}(p) = [p^2 + m^2 + \Sigma^{(2)}_{\text{ren}}(p) - \Sigma^{(2)}_{\text{ren}}(0)]^{-1}, \tag{3.15} \]

where \(\Sigma^{(2)}_{\text{ren}}(p) - \Sigma^{(2)}_{\text{ren}}(0)\) is given by \(\Sigma^{(2)}_{\text{ren}}(p) - \Sigma^{(2)}_{\text{ren}}(0) = \frac{(N + 2)\delta^2 u^2}{18(4\pi)^2} \left[ \frac{3\Omega}{p} \arctan \left( \frac{p}{3\Omega} \right) - \frac{1}{2} \ln \left( \frac{p^2 + 9\Omega^2}{9\Omega^2} \right) \right] \). \tag{3.16} \]

Substituting Eqs. (3.16) and (3.15) in (3.2), one gets

\[ Z_1 = \left\{ 1 + \frac{(N + 2)\delta^2 u^2}{18(4\pi)^2 54\Omega^2} \right\}^{-1} \], \tag{3.17} \]

which, after expanding to order-\(\delta^2\), becomes

\[ Z_1 = 1 - \frac{(N + 2)\delta^2 u^2}{18(4\pi)^2 54\Omega^2} \]. \tag{3.18} \]

From Eqs. (3.3), (3.14) and (3.18) we then obtain the renormalized coupling to order \(\delta^2\)

\[ \lambda = \frac{\delta u}{m} - \frac{3\delta^2 u^2}{16\pi m\Omega} \left( \frac{N + 8}{9} \right) + O(\delta^3) \], \tag{3.19} \]

from which the \(\beta\)-function will follow via Eq. (3.4). But before performing the derivatives with respect to \(g\) (or \(m\)) let us recall that this quantity also appears in \(\overline{\Omega}\). Then, substituting the optimized quantity Eq. (3.12) in (3.10) and setting \(\delta = 1\) we obtain the optimized renormalized coupling given by

\[ \overline{\lambda}(g) = g - \frac{g^2(N + 8)}{48\pi} \left\{ 1 + \frac{1}{72\pi^2} \left[ g^2 + 12\pi g F(g) \right] \right\}^{-1/2} \]. \tag{3.20} \]
which is, of course, a nonperturbative quantity. Now, by using the definition for the $\beta$-function, which from Eq. (3.20) can be expressed as

$$\beta = -\frac{g}{2} \frac{\partial}{\partial g} \ln Z,$$

(3.21)

one finds the fixed points, $g_c$, as given by the solutions of $\beta(g_c) = 0$. For a fixed value of $N \neq 0$ we always find three roots, a trivial one ($g_c = 0$), a purely imaginary one (which leads to unacceptable scaling relations at the critical point) and a positive real root. This last one can easily be found numerically, and is given, e.g., for a few cases, by $g_c(N = 1) = 15.3524$, $g_c(N = 2) = 12.2968$, $g_c(N = 3) = 10.3692$. The nontrivial real positive roots are the ones that will be used in the subsequent determination of the critical exponents.

C. Evaluating the critical exponents

Now, the evaluation of the critical exponent $\eta$ is fairly easy. From Eqs. (3.2) and (3.5) we have that

$$\eta = -g \frac{\partial}{\partial g} \ln \left[ Z_1(g) \right] \bigg|_{g=g_c},$$

(3.22)

where $Z_1$ represents the optimized field renormalization function. This quantity can be obtained directly from Eq. (3.18) with the replacement $\Omega \rightarrow \Omega$. Then, setting $\delta = 1$, one obtains

$$Z_1 = 1 - \frac{g^2(N + 2)}{18(4\pi)^2} \left\{ 1 + \frac{1}{72\pi^2} \left[ g^2 + 12\pi gF(g) \right] \right\}^{-1}.$$

(3.23)

From Eq. (3.22) and the previous results for the fixed points, we then obtain the results (for some representative values of $N$) $\eta(N = 1) = 0.0026$, $\eta(N = 2) = 0.0029$, $\eta(N = 3) = 0.0030$. These values are contrasted to the results from other methods in Table I. Results for other values of $N$ can also be easily obtained from the previous equations. We now turn to the calculation of the critical exponent $\nu$ to order-$\delta^2$. According to our prescription, to evaluate this quantity one computes the optimized $Z_2$ which is given by

$$Z_2 = \frac{\partial \sigma}{\partial \ln \tau},$$

(3.24)

where $\tau = m^2 - \Sigma_{\text{ren}}(2)(0)$. The optimized self-energy at order $\delta^2$, $\Sigma_{\text{ren}}(2)(0)$, can be trivially obtained from Eq. (3.13) by performing the substitutions $M \rightarrow \bar{M}$ and $\Omega \rightarrow \bar{\Omega}$ as given by Eqs. (3.11) and (3.12). The critical exponent $\nu$ then follows from Eqs. (3.6) and (3.1), or

$$\nu = \left\{ 2 - g \frac{\partial}{\partial g} \ln \left[ \frac{Z_2(g)}{Z_1(g)} \right] \right\}^{-1} \bigg|_{g=g_c},$$

(3.25)

which, at $\delta = 1$ and for a few representative values of $N$, yields the results $\nu(N = 1) = 0.5287$, $\nu(N = 2) = 0.5362$, $\nu(N = 3) = 0.5422$. These results, together with those for $\eta$, are shown in Table I so that they can be compared with the results obtained using different approximations.

The $\epsilon$-expansion results for the critical exponents can be found e.g. in Ref. [6]. In Parisi’s book [16], one can find, for instance the results for the critical exponents for one and two-loop perturbation theory (PT). In Ref. [17], it is used a variational perturbation theory (VPT) to continue the renormalization constants of three-dimensional $\phi^4$ theories to the regime of strong bare coupling. The authors of Ref. [18] have used the logarithmic $\delta$-expansion (DE-log) to obtain the exponents $\eta$ and $\nu$. All the results from these different methods are shown in Table I together with ours.

IV. CONCLUSIONS

In this paper, the LDE has been applied for the first time in the evaluation of critical exponents. An early attempt to use this type of method was carried out by Gandhi and McKane [15] who used a variant of the method, known as the
logarithmic $\delta$ expansion. However, the authors did not implement the method correctly, since in their implementation they have not made use of any optimization scheme, just plain expansion in $\delta$, which led to many difficulties associated to the fact that their fixed point coupling $g_c$ behaves like $1/\delta$.

Here, our main concern was to show how to circumvent the difficulties found in Ref. [13]. Our approach to the problem is based on the central idea of first obtaining optimized values for the arbitrary parameter $M$ by applying the optimization procedure PMS to the renormalized self-energy at a given order in $\delta$, $\Sigma_{\text{ren}}^{(\delta)}(0)$, which then generates an optimum value for $M$ in terms of the original parameters of the theory ($m$ and $g = u/m$). Then, the full propagator and four-point functions are evaluated to the same perturbative order with the LDE and their optimum values obtained by performing the substitution $M \rightarrow M(m, g)$. Our analytical results show explicitly that the $M$ dependence on $g$ is nonperturbative (all orders are present). The relevant optimized quantities $\tilde{Z}_1$, $\tilde{Z}_2$ and the $\beta$ function are obtained upon using derivatives with respect to $g$. This whole procedure then generates the optimized values for the fixed point, $g_c$, as well as for the critical exponents $\eta$ and $\nu$ (note that, in contrast to the approach used in [13], the optimization procedure is already performed prior to computing the $\beta$-function and the fixed points). In principle this prescription can be extended to any perturbative order in $\delta$. The only difficulties of extending our results to higher orders are technical ones, associated to the evaluation of multi-loop diagrams containing massive propagators.

As far as our numerical values are concerned, they already at nontrivial lowest order show improvement, though modest, over the mean field theory ones. In a sense, the physical quantities evaluated by us with the LDE to order-$\delta^2$ contain only two-loop diagrams so it comes as no surprise the fact that our lowest order results are not as good as the ones obtained by resumming higher order corrections. One expects, as shown by the many LDE applications, that by considering higher order contributions one can quickly obtain better results from a calculation which has the advantage of being completely perturbative regarding the evaluation of Feynman diagrams. In this context, from our previous experience [3, 4, 5] in extending to higher order the LDE to obtain the critical temperature shift for an homogeneous Bose gas, which is a problem that share many similarities to the one studied here, we can advance a number of important issues that must be handled. First is the slow convergence behavior observed in [10] by the direct extension of the LDE method studied here. Since both the calculation of the temperature shift for Bose condensation (for $N = 2$) and the exponents computed here make use of the same critical model, it is reasonable to expect that the same convergence behavior will show up in higher order calculations of the critical exponents within the LDE method. The general explanation and throughout understanding of the slow convergence behavior for the Bose condensate critical temperature shift was given in Ref. [10]. Also, in the same reference, it was discussed how appropriate resummation techniques can speed convergence. The same resummation techniques used there could also

| Method            | $\nu$ (N=1) | $\eta$ (N=1) | $\nu$ (N=2) | $\eta$ (N=2) | $\nu$ (N=3) | $\eta$ (N=3) |
|-------------------|-------------|--------------|-------------|--------------|-------------|--------------|
| LDE $- O(\delta^2)$ | 0.5287      | 0.0026       | 0.5362      | 0.0029       | 0.5422      | 0.0030       |
| $\epsilon$ expansion | 0.626       | 0.037        | 0.655       | 0.039        | 0.679       | 0.039        |
| PT one-loop       | 0.6         | 0            | 0.625       | 0            | 0.647       | 0            |
| PT two-loops      | 0.63 ± 0.015 | 0.031 ± 0.004| 0.63        | 0.030        | 0.670       | 0.032        |
| VPT               |              |              | 0.706       | 0.032        |              |              |
| DE $- \log$       | 0.53        | 0.0013       | --          | --           | --          | --           |

TABLE I: Numerical results for $\nu$ and $\eta$. |
be used here in order to improve convergence. A second important issue that commonly arises when extending the LDE beyond second order is the appearance of imaginary solutions upon the use of the optimization, Eq. (2.2). In previous references we have shown that only the real part of those solutions make physical sense and indeed they lead to correct results, despite some obvious embarrassment of having to deal with those imaginary solutions. To circumvent this problem an alternative optimization has been proposed recently \[18\], which make use of additional parameters from third order onwards. In practice this can be achieved by replacing, in Eq. (2.1), terms like 
\[M^* = (1 - \delta)M^2\] by terms like 
\[M^* = \left[1 - a_1 \delta - (1 - a_1)\delta^2\right]^{1/2} M, \text{ at third order}\]
\[M^* = \left[1 - a_1 \delta - (1 - a_1)\delta^2 + a_2^2(1 - \delta)\right]^{1/2} M, \text{ at fourth order} \]
and so on, where the \(a_j, j = 1, \ldots, n - 2\) are additional parameters added at order \(n\) in the LDE and to be determined with \(M\) by generalizing the optimization condition Eq. (2.2) to a system of equations.

As shown in \[18\] this optimization procedure generates, in the Bose-Einstein condensation case, only real values for the LDE interpolating parameters with the added bonus of fast convergence already at lowest orders. As far the LDE application of Ref. \[18\] to the Bose-Einstein condensation problem is concerned, it has produced some of the most precise and stable analytical predictions for the critical temperature shift. Those results thus also give an indication of the applicability of the use of the LDE in analogous models close and at the critical point. We are currently working on the use of this modified optimization to the problem studied in this paper and we will report the results elsewhere.

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