Relativistic formulation of curl force, relativistic Kapitza equation and trapping

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Abstract In this present communication, the relativistic formulation of the curl forces with saddle potentials has been performed. In particular, we formulated the relativistic version of the Kapitza equation. The dynamics and trapping phenomena of this equation have been studied both theoretically and numerically. The numerical results show interesting characteristics of the charged particles associated with the particle trapping and escaping in the relativistic domain. In addition, the relativistic generalization of the Kapitza equation associated with the monkey saddle has also been discussed.

Keywords Curl forces · Kapitza equation · Relativistic Lagrangian · Higher-order saddle potentials · Trapping and escaping

Mathematics Subject Classification 01A75 · 34A05 · 70J25 · 70H14

1 Introduction

In recent times, the understanding of the trapping of charged particles gets high attention to the researchers in connection with making an accelerated beam of charged particles with more control and with the possible applications toward astrophysics, radiation phenomenon, controlled fusion reactor, plasma stability characteristics and so on. The charged particles can have both relativistic and non-relativistic limits, and the trapping of those charged particles are necessary in understanding the stability of the system. There have been several works reported regarding this interesting exploration of the dynamics of the charged particles in the past few years. The approach of tackling the complex coupled dynamical equations of the system spans over phase space analysis, solitary wave approach, adiabatic mechanism, non-stationary Hamiltonian theory, curl force analysis, etc. Bégué et al. developed a semi-Lagrangian (2-D) fully relativistic Vlasov code for studying the trapped-particle dynamics in phase space and observed some coherent vortex structures which is a consequence of nonlinear saturation mechanism of relativistic modulational instability [1]. Lembege studied the acceleration of the relativistic charged particles, viz. electrons and ions via magnetosonic waves numerically. It has been shown under the influence of the nonlinear saturation the charged particles can gain unlimited acceleration. Nonlinear effects are shown to form the saturation mechanism and limit the amplitude.
below the level where a particle species can undergo unlimited acceleration. Around the trapping time scale soliton like wavelets can generate from the mail wave packet [2]. Mora studied the adiabatic theory of particle acceleration for the longitudinal plasma waves in the relativistic regime where the trapping conditions for the charged particles have been found [3]. Gonoskov et al. have demonstrated that in a sufficiently intense standing wave the charged particles are compressed and oscillate synchronously at the antinodes of the concerned electric field. This unusual behavior has been termed as the anomalous radiative trapping. This phenomenon opens up the possibilities to generate accelerated particle beams with a control [4]. Kostyukov et al. have shown that the electrons can be trapped in a plasma bubble and can be accelerated to very high energies via plasma wakefields. They have also shown the condition for trapping the high energetic electrons [5]. Zhang et al. have studied the characteristics of the relativistic plasma physics in supercritical fields [6]. Suk et al. have studied the acceleration and the trapping of plasma electrons in plasma wave fields via density transition [7]. Kalmykov et al. have shown the criteria for plasma bubble expansion and electron trapping via non-stationary Hamiltonian theory [8]. The prime motivation in conducting the present study comes due to the above mentioned observations and or the findings with the possible application toward various physical fields like relativistic plasma jets in astrophysical scenario, controlled thermonuclear fusion reactor, etc., as discussed later.

A Newtonian dynamics associated with a nonzero curl force has been a topic of interest to many of the researchers round the globe in the recent past. These forces are known as curl forces which has been introduced to the research community by Berry and Shukla [9,10]. These forces, in general, are velocity independent in order to satisfy a nonzero curl, viz. force \( F \) depends only on position \( r \) and is independent of velocity \( v \). The interesting fact is that these curl forces cannot be derivable from a scalar potential. The direct applications about the understanding of the system characteristics associated with these curl forces span to many compulsive fields like optics, laser physics, noncentral forces, anisotropic Kepler problem etc. [11–20]. The generalization of these curl forces has been made with the introduction of dissipation factor present in the system and gyroscopic terms. The application to this directs us to ion trapping, two level atomic state, gain-loss mechanisms, etc. The curl force preserves the volume in phase space domain, viz. \((r,v)\) without any attractors, and this can be visualized as follows. Let us take the dynamical equation for a unit mass as \( \ddot{r} = F(r) \) with a nonzero curl, i.e., \( \nabla \times F(r) \neq 0 \). Henceforth, one can observe that
\[
\nabla_r \cdot \dot{r} + \nabla_v \cdot \dot{v} = \nabla_r \cdot v + \nabla_v \cdot F(r) = 0, \quad \dot{v} = \dot{r}.
\]

Although, in general, the dynamics associated with the curl forces are non-dissipative. Therefore, in the presence of dissipation the phase space volume is not preserved under the curl force, i.e., \( \nabla \times F(r) \neq 0 \) and \( \nabla_r \cdot v + \nabla_v \cdot F(r) \neq 0 \). A dynamical motion associated with saddle potentials generates curl forces. In general, a particle remains stable to its equilibrium position when the surface rotates sufficiently fast around its vertical axis. To visualize this phenomenon, we take a saddle force field for unit mass with components \( (F_x = x, F_y = -y) \). We now apply a counterclockwise rotation with angular velocity \( 2\omega \) to each of those vectors. This implementation converts the system dynamics from time-independent to a time-dependent one. It is worth mentioning about the Earnshaw’s theorem now. This theorem states that we cannot trap any charged particle with the help of a static potential. The basic reason behind this is static potential satisfies the Laplace’s law and henceforth it lacks to provide a potential minimal to the system under consideration. Therefore, to trap a charged particle we must apply a rotation around a suitable axis to the saddle potential with suitable angular frequency and that was the notable idea of Wolfgang Paul [21]. The basic idea for stabilization was incorporated by means of ‘vibrating’ electrostatic field which is analogous to the dynamics associated with the so called Stephenson–Kapitza [22,23] pendulum. In contrary to this, magnetic fields have been incorporated for charged particle’s trapping in case of the Penning trap [24–26]. Both radial and axial trapping can be possible via this Penning trap where a strong axial homogeneous magnetic field used in confining the charged particles radially and quadrupole electric field for axial confinement.

In recent times, the extensive study regarding the stabilization and or other geometric properties of these exceedingly interesting curl forces and the dynamical characteristics of nonlinear Hamiltonian curl forces have been reported [27–29]. The dynamics of the curl forces are related to the theory of Kapitza-Merkin non-conservative positional forces. The linearized dynam-
ics of a rotating shaft (with position coordinates \( x \) and \( y \)) formulated by Kapitza [30] is given by
\[
\ddot{x} + ay + bx = 0, \quad \ddot{y} - ax + by = 0.
\] (1.1)

The corresponding characteristics equation reflects that if we add a nonzero non-conservative curl force (i.e., \( a \neq 0 \)) to a stable system associated with a stable potential, then it becomes unstable. This is associated with the Merkin's result [31,32], which states that "the introduction of non-conservative linear forces into a system with a stable potential and with equal frequencies destroys the stability regardless of the form of nonlinear terms." It is worth mentioning that the positional force, i.e., the terms \( ay \) and \(-ax \), is proportional to \( \omega^2 \), where \( \omega \) is the rotation rate of the shaft. It is also possible to derive Eq. (1.1) associated with a saddle potential via Euler–Lagrange method. It is very straightforward to check that the Lagrangian
\[
L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - \frac{1}{2}a(x^2 - y^2) - bxy,
\] (1.2)
yields Eq. (1.1). A symmetric saddle surface can be described by \( \bar{U} = a(x^2 - y^2) \), where \( a \) is a geometrical parameter that specifies the curvature of the saddle. It is note worthy to mention that an ion trap potential is a rotating saddle surface on which a ball can be trapped. The ponderomotive potential of an ion trap is that of a saddle in 2D, but the time evolution of the potential is one that flaps up and down. It must be noted that saddle point on a curved surface refers to that point at which the curvatures in two mutually perpendicular planes are associated with opposite signs. In case of flapping saddle, the system stability has been restored with the corresponding flapping at the saddle points, whereas in case of rotating saddle the stability criterion can be achieved by giving a controlled rotation with respect to an axis passing through the saddle point. The potential of the modified rotating system is
\[
\bar{U}(x, y, t) = \frac{1}{2}(x^2 - y^2)\cos(2\omega t) - xy\sin(2\omega t),
\]
where \( \omega \) is the angular drive frequency of the spinning saddle. The saddle-shaped potential oscillates or ‘flaps’ with the driving frequency. The time-dependent potential of a saddle spinning at angular frequency \( \Omega \) is similar enough to the flapping potential. It turns out that the motion in a rotating saddle trap can be described by trigonometric functions which are described the Mathieu equations. We show the generalization of this picture in Sect. 3. In our previous paper [20], we studied the integrable modulation problem of the parametric Kapitza equation using curl force formalism. The relativistic extension of the nonlinear dynamical equations are not a straightforward job due to the intrinsic extra nonlinearity triggered by the presence of the Lorentz factor [33–35]. The relativistic harmonic oscillator is a topic of immense interest, although widely discussed, still contains elements of interest. Here, the Newtonian kinetic energy is replaced by its special relativistic counterpart. This version of the relativistic harmonic oscillator model has recently been probed experimentally also [36]. Recently, Haas [37] generalized Ermakov systems toward the special relativity domain. Guha et al. formulated a relativistic generalization of non-central force [38] which is a prototypical example of relativistic curl force. The reduction of this equation yields the relativistic generalization of the Emden–Fowler equation. In fact, relativistic nonlinear dynamics is an open area of research and deserves more attention and intense studies.

In the present communication, the relativistic analog of the Kapitza equation has been investigated mainly to bridge a connection between the saddle potentials and charged particle trapping in the relativistic regime. The basic theme of the present work is based on the generalization of a curl force associated with rotating saddles which directs us the charged particle trapping just like in the case of normal trapping of charged particles inside a magnetic mirror in plasma. It has been found that in the relativistic domain, the charged particles often escape from the saddle, whereas in some cases for the generalized Kapitza equation particles can be trapped in some specific parametric domain. Inclusion of the relativistic nature to the particle dynamics makes the dynamical equation so clumsy that we have to solve the dynamical equations via numerical simulation and we have plotted the respective phase-space plots in order to observe the nature of the trapping and or escaping. We believe that the present theoretical findings may direct us about the charged particles trapping in a more precise way with direct applications toward plasma fusion reactors, charged particle acceleration in a controlled way, relativistic plasma jets in astrophysical events, radiation belts of trapped charged particles around some of the planets [39–42], etc.

The paper is organized as follows: In Sect. 2, we briefly recapitulate the construction of the linear curl forces according to Berry and Shukla. The introduction of Kapitza equation and dynamics related to the
higher saddles have also been placed there. Section 3 is mainly dedicated to the relativistic extension of the curl forces and Kapitza equation where we discuss the trapping phenomena with associated phase plots. Lastly, in Sect. 4, we summarize our findings in the relativistic domain with the comparison to the non-relativistic results along with the possible application toward realistic scenarios.

### 2 Preliminaries: linear curl forces and Hamiltonization

In this section, the recapitulation of the construction of linear curl forces according to Berry and Shukla has been placed. The central fact about curl force theory is that only a small subset of all curl forces are Hamiltonian cases and they find nice applications also. But it is the non-Hamiltonian curl forces that are really distinctive and exhibit various new features of dynamics. Let us summarize the classical theory of linear curl forces[9,10,17]. We recall from Ref. [9] that the form of the Hamiltonian in the case of curl forces is given

\[ H = \frac{1}{2} (\alpha p_x^2 + \beta p_x p_y + \gamma p_y^2) + U(x, y) \]  

(2.1)

where the potential is defined by

\[ U(x, y) = \frac{1}{2} g x^2 + b x y + \frac{1}{2} c y^2. \]  

(2.2)

The first set of Hamiltonian equations then gives

\[ \dot{x} = \alpha p_x + \beta p_y, \quad \dot{y} = \beta p_x + \gamma p_y. \]  

(2.3)

We can then get the forces from the second set of Hamiltonian equations as

\[ a_x = \dot{x} = \alpha \dot{p}_x + \beta \dot{p}_y \]
\[ = - \alpha \frac{\partial U(x, y)}{\partial x} - \beta \frac{\partial U(x, y)}{\partial y}, \]  

(2.4)

\[ a_y = \dot{y} = \beta \dot{p}_x + \gamma \dot{p}_y \]
\[ = - \beta \frac{\partial U(x, y)}{\partial x} - \gamma \frac{\partial U(x, y)}{\partial y}. \]  

(2.5)

The curl is

\[ \nabla \times F = (\alpha - \gamma) \frac{\partial^2 U(x, y)}{\partial x \partial y} \mathbf{i} \]
\[ + \beta \left( \frac{\partial^2 U(x, y)}{\partial x^2} - \frac{\partial^2 U(x, y)}{\partial y^2} \right) \mathbf{j}, \]  

(2.6)

which yields \( \nabla \times F = (\alpha - \gamma) b \mathbf{i} + \beta (c - a) \mathbf{j} \). To ensure that \( \nabla \times F \neq 0 \), it is necessary to take \( \beta = 0 \) and \( \alpha = 1 = -\gamma \). Also the another choice we have is \( \beta \neq 0 \) and \( c = -a \). Henceforth, the Hamiltonian turns out to be

\[ H = \frac{1}{2} (p_x^2 - p_y^2) + \frac{1}{2} (x^2 - y^2) + b x y, \]  

(2.7)

which is a Hamiltonian of the Kapitza equation. Next we discuss about the rotating saddle in connection with the Kapitza equation and higher saddles.

#### 2.1 Rotating saddle and Kapitza equation related to higher-saddle

Consider the motion of a particle in the rotating saddle potential in the plane

\[ U(x, t) = U_0(R^{-1}x), \quad U_0 = \frac{1}{2} (x^2 - y^2), \]
\[ x = (x, y); \]  

(2.8)

where

\[ R = R(\Omega t) = \begin{pmatrix} \cos (\Omega t) & \sin (\Omega t) \\ -\sin (\Omega t) & \cos (\Omega t) \end{pmatrix}. \]

The Lagrangian of the rotating saddle potential is given by

\[ L_{\text{rotsaddle}} = \frac{1}{2} (\dot{x}^2 - \dot{y}^2) - \frac{\Lambda}{2} \left( (x^2 - y^2) \cos (2\Omega t) + 2xy \sin(2\Omega t) \right), \]  

(2.9)

where \( \Lambda = \frac{2mgh_0}{I_0} \), is proportional to the tension, viz. force on the particle per length. The corresponding equations of motion are as follows:

\[ \ddot{x} + \Lambda x \cos (2\Omega t) + \Lambda y \sin (2\Omega t) = 0, \]
\[ \ddot{y} + \Lambda y \cos (2\Omega t) - \Lambda x \sin (2\Omega t) = 0. \]  

(2.10)

#### 2.1.1 Higher saddle and generalized Kapitza equation

In this part, the generalization of the Kapitza equation with rotation in connection with higher saddles has been placed. The rotating shaft equation formulated by Kapitza using higher-order saddles can be generalized as follows. Equation (1.1) can be manufactured from the Euler–Lagrange equation using simple saddle potential \( g_1(x, y) = x^2 - y^2 \) and the rotated version
of the same surface, $g^R_1(x, y) = xy$. In other words, the other one is obtained from the rotation of the saddle by 90 degrees. Let $z = x + iy$, then $x^2 - y^2 = Re(z^2)$ and $2xy = Im(z^2)$. We follow this guideline to construct equations of rotating shaft associated with monkey saddle from $z^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$. The monkey saddle belongs to a specific class of the saddle surfaces where it has been assumed that the saddle allows a monkey to sit with spreading two legs in two depressions and also there is space for the tail in the other direction. This can be visualized as a monkey as a rider of that saddle and henceforth the name “monkey saddle” has been assigned to those type of saddles. The respective monkey saddle surface yields

$$g_2(x, y) = x^3 - 3xy^2 = Re(z^3)$$

and its rotated version

$$g^R_2(x, y) = 3x^2y - y^3 = Im(z^3).$$

The generalized rotating shaft pair of equations associated with degree three potential or monkey saddle potential is given by

$$U_3 = \frac{1}{3}(k_1g_2(x, y) + k_2g^R_2(x, y)) = \frac{1}{3}(k_1(x^3 - 3xy^2) + k_2(3x^2y - y^3)).$$

The corresponding equations of motion are given by

$$\ddot{x} + k_1(x^2 - y^2) + 2k_2xy = 0,$$

$$\ddot{y} + 2k_1xy - k_2(x^2 - y^2) = 0.$$  

(2.13)

3 Relativistic curl force

We are now in a position to extend the curl forces and Kapitza equation in the relativistic domain. In this section, the formalism has been placed along with the discussion of particle trapping with associated phase plots. The relativistic Lagrangian in the presence of scalar potential fields is given by

$$L = -mc^2\sqrt{1 - \frac{|v|^2}{c^2}} + \frac{2U}{mc^2}$$

$$= -mc^2\sqrt{1 - \frac{|v|^2}{c^2}}\sqrt{1 + \frac{2U}{mc^2}(1 - \frac{|v|^2}{c^2})^{-1}}$$

$$= -mc^2\gamma^{-1}\sqrt{1 + \frac{2U}{mc^2}\gamma^2}, \quad \gamma = (1 - \frac{|v|^2}{c^2})^{-\frac{1}{2}}.$$  

Expanding the expression within square-root binomially up to second order gives us

$$\hat{L} = -mc^2\gamma^{-1}(1 + \frac{U}{mc^2}\gamma^2) = -mc^2\gamma^{-1} - U\gamma.$$  

(3.2)

Although in Goldstein’s book the semi-relativistic Lagrangian ($L_{sr}$) is given by

$$L_{sr} = mc^2\gamma^{-1} - U.$$  

(3.3)

If we assume

$$\frac{|v|}{c} = \alpha << 1, \Rightarrow U\gamma \sim U + \frac{U}{2}\alpha^2,$$

for practical purpose; we can drop the second term for small value of $\alpha$ which immediately yields the semi-relativistic Lagrangian given in Goldstein [43]. Let us illustrate it for planar relativistic harmonic oscillator. We can express the Lagrangian for a unit mass particle using polar coordinate as

$$L = -c^2\sqrt{1 - \frac{\dot{r}^2 + r^2\dot{\theta}^2}{c^2}} - \frac{1}{2}\omega(t)^2r^2.$$  

(3.4)

We obtain the conserved angular momentum using the property of the cyclicity of $\theta$ coordinate

$$J = \gamma r^2\dot{\theta}, \quad \gamma = \frac{1}{\sqrt{1 - \frac{\dot{\theta}^2 + r^2\dot{\theta}^2}{c^2}}}.$$  

(3.5)

The radial equation yields

$$\frac{d}{dt}(\gamma\dot{r}) - \gamma r\dot{\theta}^2 + \omega^2r = 0, \quad \text{or}$$

$$\frac{d}{dt}(\gamma\dot{r}) - \frac{J^2}{\gamma r^3} + \omega^2r = 0.$$  

(3.6)

We can remove the $\dot{\theta}$ term using angular momentum, and we obtain

$$\dot{\theta} = \frac{J}{\gamma_1\sqrt{1 + \frac{\dot{r}^2}{c^2}}} = \frac{1}{\gamma_1\sqrt{1 - \frac{\dot{\theta}^2}{c^2}}}.$$  

(3.7)
In our previous paper [38], we focus on non-central curl force problem; hence, we divert our attention to azimuthal force and give a relativistic formulation of non-central forces.

3.1 Relativistic Kapitza equation

We now discuss about the Kapitza equation in the relativistic domain here. The relativistic Lagrangian of the curl force is defined as

\[ L = -\frac{c^2}{\Gamma} - U(x, y), \quad \text{with} \quad \Gamma = (1 - \frac{x^2}{c^2} + \frac{y^2}{c^2})^{-\frac{1}{2}}. \]

Here, the saddle potential is assumed as

\[ U(x, y) = \frac{1}{2} k(x^2 - y^2) + bxy. \]  

**Lemma 3.1** The Euler–Lagrange equation of the relativistic Lagrangian \( L = -\frac{c^2}{\Gamma} - \frac{1}{2} k(x^2 - y^2) - bxy \) yields

\[ \ddot{x} \left( 1 + \frac{\dot{x}^2}{c^2} \right) - \frac{\dot{x} \dot{y}}{c^2} = -\frac{kx}{\Gamma^3} - \frac{by}{\Gamma^3}, \quad (3.10) \]

\[ \ddot{y} \left( 1 - \frac{\dot{x}^2}{c^2} \right) + \frac{\dot{x} \dot{y}}{c^2} \dot{x} = -\frac{ky}{\Gamma^3} + \frac{bx}{\Gamma^3}, \quad (3.11) \]

where \( \Gamma = (1 - \frac{\dot{x}^2}{c^2} + \frac{\dot{y}^2}{c^2})^{-\frac{1}{2}}. \)

**Proof** It is straightforward to check that

\[ \frac{\partial L}{\partial \dot{x}} = \dot{x} \left( 1 - \frac{\dot{x}^2}{c^2} + \frac{\dot{y}^2}{c^2} \right)^{-\frac{1}{2}}, \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = \left[ \dot{x} \left( 1 + \frac{\dot{x}^2}{c^2} \right) - \frac{\dot{x} \dot{y}}{c^2} \right] \left( 1 - \frac{\dot{x}^2}{c^2} + \frac{\dot{y}^2}{c^2} \right)^{-\frac{3}{2}}, \]

\[ \frac{\partial L}{\partial \dot{y}} = -\dot{y} \left( 1 - \frac{\dot{x}^2}{c^2} + \frac{\dot{y}^2}{c^2} \right)^{-\frac{1}{2}}, \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) = \left[ -\dot{y} \left( 1 - \frac{\dot{x}^2}{c^2} \right) - \frac{\dot{x} \dot{y}}{c^2} \right] \left( 1 - \frac{\dot{x}^2}{c^2} + \frac{\dot{y}^2}{c^2} \right)^{-\frac{3}{2}}. \]

Then, using the derivative of \( \frac{\partial L}{\partial \dot{x}} \) and \( \frac{\partial L}{\partial \dot{y}} \) we obtain our result from the Euler–Lagrange equation. We must note here that both Eqs. (3.10) and (3.11) contain \( \ddot{x} \) and \( \ddot{y} \) terms. The equations of motion are therefore obtained by separating these terms.

**Proposition 3.1** The Euler–Lagrange equation for the relativistic Kapitza Lagrangian yields the relativistic Kapitza equation

\[ \ddot{x} = \frac{\dot{x} \dot{y}}{c^2} \left( -ky + bx \right) - \frac{1}{\Gamma c^2} \left( kx + by \right), \quad (3.12) \]

\[ \ddot{y} = \frac{\dot{x} \dot{y}}{c^2} \left( kx + by \right) + \frac{1}{\Gamma c^2} \left( -ky + bx \right), \quad (3.13) \]

where

\[ \Gamma_x = (1 - \frac{\dot{x}^2}{c^2})^{-\frac{1}{2}}, \quad \Gamma_y = (1 + \frac{\dot{y}^2}{c^2})^{-\frac{1}{2}}. \]

**Proof** Multiplying (3.10) by \( (1 - \frac{\dot{x}^2}{c^2}) \), we obtain

\[ \ddot{x} \left( 1 + \frac{\dot{y}^2}{c^2} \right) \left( 1 - \frac{\dot{x}^2}{c^2} \right) - \ddot{y} \left( 1 - \frac{\dot{x}^2}{c^2} \right) \left( 1 - \frac{\dot{x}^2}{c^2} \right) = -\frac{kx (1 - \frac{\dot{x}^2}{c^2}) + by (1 - \frac{\dot{x}^2}{c^2})}{\Gamma c^2} \]

\[ \left( \ddot{x} \right) \Gamma_x = \frac{\dot{x} \dot{y}^3}{\Gamma} \left[ -ky + bx \right] - \frac{1}{\Gamma_x \Gamma y} \left( kx + by \right). \]

Similarly, we obtain the second equation via the multiplication of Eq. (3.11) with \( (1 + \frac{\dot{y}^2}{c^2}) \).

**Corollary 3.1** Equations (3.12) and (3.13) for non-relativistic limit reduce to the Kapitza equation

\[ \ddot{x} + kx + by = 0, \quad \ddot{y} - ky + bx = 0. \]  

The relativistic Kapitza equation can also be formulated using Hamiltonian method.

**Proposition 3.2** The (effective) Hamiltonian of the relativistic Kapitza equation is

\[ H = \frac{p_x^2}{2\Gamma} - \frac{p_y^2}{2\Gamma} + \frac{1}{2} k(x^2 - y^2) + bxy, \]

and corresponding Hamiltonian equations are given by

\[ \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{\Gamma}, \quad \dot{p}_x = -\frac{\partial H}{\partial x} = -kx - by, \]

\[ \dot{y} = \frac{\partial H}{\partial p_y} = -\frac{p_y}{\Gamma}, \quad \dot{p}_y = -\frac{\partial H}{\partial y} = ky - bx. \]

One can clearly see that \( p_x = \dot{x} \Gamma, \quad p_y = -\dot{y} \Gamma \) and the rest follows from lemma 3.1. It can be easily shown that the relativistic Kapitza equation follows from Eqs. (3.17) and (3.18).
It is clear that for the non-relativistic limit the Hamiltonian equation yields the Kapitza equation. One must note that non-relativistic Kapitza equation has two integrals of motion, the Hamiltonian $H = p_x^2 - p_y^2 + \frac{1}{2}k(x^2 - y^2) + bxy$ and the Fradkin tensor $I = pxpy + \frac{1}{2}b(x^2 - y^2) = kxy$. Unfortunately, for the relativistic case we have just one conserved quantity due to the presence of the Lorentz factor.

3.2 Relativistic rotation of saddle

Introduction of the rotation to the higher-order saddles changes the perspective of the system dynamics which can be understood by the following. The Lagrangian of the rotating saddle potential is given by

$$L_{\text{rel saddle}} = -\frac{c^2}{\Gamma} - \frac{\Lambda}{2} \left( x^2 - y^2 \cos(2\Gamma_\omega t) + 2xy \sin(2\Gamma_\omega t) \right).$$

where the constant coefficients $b$ and $k$ are replaced by the sinusoidal terms, the corresponding equations of motion are

$$\dot{x} = \frac{\dot{y}}{\Gamma c^2} \Lambda \left( -\cos(2\Gamma_\omega t)y + \sin(2\Gamma_\omega t)x \right) - \frac{1}{\Gamma \Gamma_x^2} \Lambda \left( \cos(2\Gamma_\omega t)x + \sin(2\Gamma_\omega t)y \right),$$

$$\dot{y} = \frac{\dot{x}}{\Gamma c^2} \Lambda \left( \cos(2\Gamma_\omega t)x + \sin(2\Gamma_\omega t)y \right) + \frac{1}{\Gamma \Gamma_y^2} \Lambda \left( -\cos(2\Gamma_\omega t)y + \sin(2\Gamma_\omega t)x \right),$$

where

$$\Gamma_x = \left( 1 - \left( \frac{\dot{x}^2}{c^2} \right)^{-\frac{1}{2}} \right), \quad \Gamma_y = \left( 1 + \left( \frac{\dot{y}^2}{c^2} \right)^{-\frac{1}{2}} \right).$$

The phase space plots for the rotating saddle are depicted in Fig. (1) for Eqs. (3.20) and (3.21) in both relativistic and non-relativistic cases. In the non-relativistic scenario, it can be seen from Fig. (1) that the trajectory is analogous to usual limit cycle under one
specific saddle point (SP) and spanning over smaller to larger amplitudes. The particle can be trapped inside those periodic boundaries for shorter times but not completely as the amplitude grows with time. Figure (1) corresponds to heteroclinic orbit where it moves from one SP to another over time directs to the escaping of particles. In the relativistic case, Fig. (1) both corresponds to heteroclinic orbit and particle escaping. It must be noted that Fig. (1) reflects the escaping nature of the particle due to initial condition which has been imposed in the simulation. As we have set \( \dot{y} \neq 0 \) at the start therefore the rotating saddle is unable to hold and or confine the particle which falls to relativistic domain. Over the time, due to the this initial thrust, the rotation of the saddle also provides the energy to the particle which results to an increase in the amplitudes of the phase space and directs particle escaping. Therefore, we are in a position to safely say that the course of a particle over time is dependent on the time rate change of the respective coordinates, viz. the velocity components along various direction. The dependency is undoubtedly a very complex one as we can see that the respective velocity components are coupled with each other and with the specific parametric range the flapping and trapping of a particle are greatly affected by this. The detail exposure of this dependency can generate a lot of interest which motivates us to work further in this direction. One must note that, in the non-relativistic limit Eqs. (3.20) and (3.21) reduce to

\[
\begin{align*}
\ddot{x} + \Lambda x \cos(2\omega t) - \Lambda y \sin(2\omega t) &= 0, \\
\ddot{y} + \Lambda y \cos(2\omega t) - \Lambda x \sin(2\omega t) &= 0.
\end{align*}
\]

The relativistic generalized Kapitza equation corresponding monkey saddle is the extension of the monkey saddle equation,

\[
\begin{align*}
\ddot{x} &= \frac{\dot{x}\dot{y}}{\Gamma c^2} \left( -2k_1 xy + k_2 (x^2 - y^2) \right) \\
&\quad - \frac{1}{\Gamma \Gamma_1^2} \left( k_1 (x^2 - y^2) + 2k_2 xy \right), \\
\ddot{y} &= \frac{\dot{x}\dot{y}}{\Gamma c^2} \left( k_1 (x^2 - y^2) + 2k_2 xy \right) \\
&\quad + \frac{1}{\Gamma \Gamma_1^2} \left( -2k_1 xy + k_2 (x^2 - y^2) \right).
\end{align*}
\]

The phase space plots for the relativistic rotating monkey saddle are depicted in Fig. (2) for Eqs. (3.23) and (3.24). Figure (2) shows the analogous nature of a heteroclinic trajectory which remove the possibility of particle trapping for both relativistic and non-relativistic domain. In contrary to this, the nature of the plots depicted in Fig. (2) is analogous to limit cycle trajectories under one SP for both relativistic and non-relativistic cases. As one can see that there are periodic boundaries which are changing very slowly over the time scale and closely spaced, henceforth particle can be trapped over the shorter time scale. Please note that the relativistic rotating monkey saddle equation can be obtained by replacing \( k_1 = \Lambda \alpha_1 \cos(2\omega t) \) and \( k_2 = \Lambda \alpha_2 \sin(2\omega t) \).

3.3 Flapping and spinning of relativistic saddle potential and trap

Suppose we allow the potential \( U = g\left( \frac{1}{2} (x^2 - y^2) \right) \) to vary sinusoidally with time. this becomes

\[
U(x, y, t) = A_{RF} \cos(\omega t) g\left( \frac{1}{2} (x^2 - y^2) \right),
\]

where \( A_{RF} \) is the amplitude of the AC component of the electric field applied to create the trap, and \( \omega \) is the frequency. The function \( g\left( \frac{1}{2} (x^2 - y^2) \right) \) can be expressed in general as \( g\left( \frac{1}{2} (x^2 - y^2) \right) = \sum_n \frac{1}{2} (x^2 - y^2)^n \) (also used in the numerical simulation). This can be viewed as a nonlinear generalization of ‘flapping saddle.’ Latter occurs in a quadrupole potential in which extreme oscillate between peaks and valleys. Thus, Newton’s law yields the following pair of equations

\[
\begin{align*}
\ddot{x} &= -A_{RF} \cos(\omega t) x g\left( \frac{1}{2} (x^2 - y^2) \right), \\
\ddot{y} &= -A_{RF} y \sin(\omega t) g\left( \frac{1}{2} (x^2 - y^2) \right).
\end{align*}
\]

The linear case coincides with the form of the Mathieu differential equations. Our motivation is to study this new set of equations. In general it is difficult to create flapping saddle on which the ball can move. So we bang on the time-dependent spinning saddle at angular frequency \( \omega \), which is similar to the flapping potential

\[
\tilde{U}(x, y, t) = A \left( g(x^2 - y^2) \cos(2\omega t) + 2f(xy) \sin(2\omega t) \right),
\]

where \( A \) is some constant. We now apply again the relativistic Lagrangian as

\[
L_{relflap} = -\frac{c^2}{\Gamma} - \tilde{U}(x, y, t),
\]
The phase space diagrams (with normalized parameters) via Eqs. (3.23) and (3.24) with $\Lambda = 0.1$, $\omega = 0.5$ and $\alpha_1 = \alpha_2 = 1$.

The top two figures are for non-relativistic cases with $v_x = 0 = v_y$, and the bottom two are for the relativistic cases with $v_x = 0.5c = v_y$. The initial conditions are $x(0) = 0$, $\dot{x}(0) = 0$, $y(0) = 0$, $\dot{y}(0) = 0.001$. The normalization scheme is same as mentioned in Fig. (1) along with $\alpha_1 \rightarrow \alpha_1 c/\omega c$ and $\alpha_2 \rightarrow \alpha_2 c/\omega c$.

If we take $\tau = \omega t$ and $x' = \frac{dx}{d\tau}$, we obtain

$$\ddot{U}(x, y, \tau) = A(g(x^2 - y^2) \cos(2\omega t) + 2f(xy) \sin(2\omega t)), \tag{3.28}$$

to get the relativistic coupled differential equations as

$$\ddot{x} = \frac{x'y'}{\Gamma c^2} \Lambda\left(-yg'(x^2 - y^2) \cos(2\Gamma \tau) + xf'(xy) \sin(2\Gamma \tau)\right)$$

$$- \frac{1}{\Gamma \Gamma_y} (xg'(x^2 - y^2) \cos(2\Gamma \tau) + yf'(xy) \sin(2\Gamma \tau)), \tag{3.29}$$

$$\ddot{y} = \frac{x'y'}{\Gamma c^2} \Lambda(xg'(x^2 - y^2) \cos(2\Gamma \tau) + yf'(xy) \sin(2\Gamma \tau))$$

$$+ \frac{1}{\Gamma \Gamma_y} \left(-yg'(x^2 - y^2) \cos(2\Gamma \tau) + xf'(xy) \sin(2\Gamma \tau)\right). \tag{3.30}$$

If we take $\tau = \omega t$ and $x' = \frac{dx}{d\tau}$, we obtain

$$\dddot{x} = \frac{x'y'}{\Gamma c^2} \Lambda\left(-yg'(x^2 - y^2) \cos(2\Gamma \tau) + xf'(xy) \sin(2\Gamma \tau)\right)$$

$$- \frac{1}{\omega^2 \Gamma \Gamma_y} (xg'(x^2 - y^2) \cos(2\Gamma \tau) + yf'(xy) \sin(2\Gamma \tau)), \tag{3.31}$$

$$\dddot{y} = \frac{x'y'}{\Gamma c^2} \Lambda(xg'(x^2 - y^2) \cos(2\Gamma \tau) + yf'(xy) \sin(2\Gamma \tau))$$

$$+ \frac{1}{\omega^2 \Gamma \Gamma_y} \left(-yg'(x^2 - y^2) \cos(2\Gamma \tau) + xf'(xy) \sin(2\Gamma \tau)\right). \tag{3.32}$$

The phase space plots for the relativistic flapping saddle are depicted in Fig. (3) for Eqs. (3.29) and (3.30). Like in the previous case of relativistic rotating monkey saddle here also we see that both trapping in shorter
Fig. 3 Phase space diagrams (with normalized parameters) via Eqs. (3.29) and (3.30) with $\Lambda = 0.1$, $\omega = 0.5$ and $\beta_1 = \beta_2 = \beta_3 = \beta_4 = 1$. The top two figures are for non-relativistic cases with $v_x = 0 = v_y$, and the bottom two are for the relativistic cases with $v_x = 0.5c = v_y$. The initial conditions are $x(0) = 0$, $\dot{x}(0) = 0.0$, $y(0) = 0.0$, $\dot{y}(0) = 0.001$. The normalization scheme is same as mentioned in Fig. 1 along with $\beta_1 \rightarrow \beta_1 \omega_1^2/c^2$, $\beta_2 \rightarrow \beta_2 \omega_2^2/c^4$, $\beta_3 \rightarrow \beta_3 \omega_3^2/c^2$ and $\beta_4 \rightarrow \beta_4 \omega_4^2/c^4$. It must be noted here that we have chosen $g(x^2 - y^2) = 1 + \beta_1 (x^2 - y^2) + \beta_2 (x^2 - y^2)^2$ and $f(xy) = 1 + \beta_3 (xy) + \beta_4 (xy)^2$. 

4 Summary and conclusion

The relativistic analog of the Kapitza equation has been investigated, and associated trapping phenomena of the charged particles have been discussed. We have presented an example of a relativistic curl force and the generalization of the same associated with saddles potentials formulated by Berry and Shukla. In case of relativistic rotating saddle, the obtained phase space plots correspond to some heteroclinic orbits suggesting particle escaping. Thereafter, with the inclusion of the monkey saddle to the generalized Kapitza equation in relativistic domain, the obtained phase plots also show some heteroclinic trajectories with a possibility of particle trapping in specific parametric domain in the shorter time scale. In addition to this, in case of relativistic flapping and spinning saddle, a possibility of trapping and escaping of the charged particles has also been observed. At last, we have discussed about the relativistic generalization of the Kapitza equation associated with the monkey saddle. The theoretical outcomes along with the numerical results may direct and or give an idea about the possible mechanism of particle trapping with finite curl forces in the relativistic domain with some other saddle potentials and or external fields. In the earlier observations [19,20], we see that there is a possibility of trapping the charged particles under the curl force dynamics with saddle potentials completely, whereas, in this work we have observed that the trapping is not complete in the relativistic domain. As the results in the relativistic domain direct us of incomplete trapping in a shorter time scale, it becomes a field of relative interest to further investigate this with the choice of different saddle potentials and to apply the outcomes toward astrophysical scenarios like relativistic jets, in making stable fusion reactors, etc. In the present time, when there is a global concern of obtain-
ing sustainable energy beyond the natural resources, pivotal attention has been shifted toward making controlled fusion reactors. In those kind of reactors charged particles need to be trapped well short of the chamber’s wall in order to make the mission is a successful one without damaging the whole configuration. The concerned charged particles are often belong to relativistic regime. Therefore, understanding the dynamics, especially the trapping of charged particles both in relativistic and non-relativistic regimes, is very crucial. The novelty of the present results comes via the implementation of a theoretical formalism to trap the charged particles via the flapping compared to the earlier findings and or observations where the trapping has been obtained via the techniques of magnetic mirrors, resonance in a magnetic field and electromagnetic wave [44,45], etc. Lastly, we believe that every little bit of novel understanding and or knowledge regarding the trapping of high velocity charged particles contributes to the noble cause of the human race in the modern era along with exploring the universe.

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Author contributions All the authors contributed equally to this work.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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