The D0-brane metric in $N = 2$ sigma models

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Abstract

We investigate the physical metric seen by a D0-brane probe in the background geometry of an $N = 2$ sigma model. The metric is evaluated by calculating the Zamolodchikov metric for the disc two point function of the boundary operators corresponding to the displacement of the D0-brane boundary. At two loop order we show that the D0 metric receives an $R^2$ contribution.

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1. Introduction

D-branes have provided new insights into the meaning of quantum geometry. Used as local probes of string/M theory they have led to the concept of D-geometry, the particular geometry seen by a D-brane (see [1] [2] for reviews). In this article we investigate the D-brane metric, the metric seen by a D0 brane probe in curved space. We will work in the classical string limit, $g_s = 0$, and will focus on a background Kahler geometry provided by an $N = 2$ non-linear sigma model. The calculations will be performed as a perturbative expansion in $l_s/l_R$ where $l_s$ is the string length and $l_R$ is the typical curvature radius of the background geometry.

The sigma model metric must satisfy certain equations of motion to provide a conformally invariant theory. They can be written in terms of a powers series in $l_s^2$ with the powers of $l_s^{2n}$ arising from an $n$ loop calculation. At lowest order the metric must be Ricci flat. On general grounds it was known that for a Ricci flat metric the two loop contribution to the beta function must also be zero since, for a Kahler metric, the only allowed tensors of the correct order vanish for Ricci flat metrics. In particular terms such as $R^2$ (which occur for the bosonic theory) cannot be generated since they cannot arise from a Kahler potential. Specific calculations showed that this was indeed the case [3] and were eventually pushed out to four loop order [4]. The results being that up to three loops the beta function vanishes for Ricci flat metrics, but at four loops there is an $R^4$ contribution which is non-zero for Ricci flat metrics. Ricci flat metrics are thus a first approximation to the allowed background metrics of string theory. Starting from a Ricci flat metric one can perturbatively (in $l_s^2$) construct a finite metric satisfying the four loop beta function equation [4]. It is always possible to find finite globally well defined non-Ricci flat corrections to the Kahler potential whose one loop divergences cancel the divergences from higher loops.

There is nevertheless an ambiguity in the definition of the sigma model metric. At each order in $l_s^2$ counter-terms are added to cancel the divergences. Nothing in the renormalization procedure, however, determines the finite part of these counter-terms. They can in principle be any covariant tensor of the correct order constructed from the metric. These counterterms might themselves then lead to divergences, and can thus alter the beta function and hence the equation of motion that the sigma model metric must satisfy. There is nevertheless a physical metric seen by the string. Its equations of motion are determined by string theory scattering amplitudes. The calculations of [4] [5] showed that, at least to four loop order, the procedure of minimal subtraction led to beta function
equations of motion for the sigma model metric identical to those deduced from string scattering amplitudes. The finite counter term procedure of [5] showed how to construct a finite sigma model metric satisfying the string scattering amplitude equations of motion. This metric can be called the “physical” sigma model metric.

The question addressed in this paper is the relation between this physical sigma model metric and the metric seen by a D0 brane probe.

The low energy effective action for the motion of a D0-brane in curved space is of the form

$$S = \int dt g_{ij}^D \partial_i X^i \partial_j X^j.$$  \hspace{1cm} (1.1)

Since we are looking at the classical string limit, $g_s = 0$, we will be considering the disc amplitude. The metric $g^D$ seen by a D0-brane is a well defined physical metric, given in terms of the disc two point function for the boundary operators $\mathcal{O}_i = g_{ij} \partial_n X^j$ corresponding to shifting the D0-brane boundary[1] (The derivative $\partial_n$ is the derivative normal to the boundary). Specifically we have

$$< \mathcal{O}_i(x_1) \mathcal{O}_j(x_2) > = g_{ij}^D \frac{1}{2\pi(x_1 - x_2)^2}$$  \hspace{1cm} (1.2)

The $1/(x_1 - x_2)^2$ dependence is determined on dimensional grounds. The Zamolodchikov metric $g_{ij}^D$ is the metric on the moduli space for the D0-brane with the moduli space being the position of the D0-brane in the curved target space. A heuristic way to understand the connection between the metric $g_{ij}^D$ in (1.2) and the metric appearing in the low energy effective action is that the Zamolodchikov metric gives the normalization for the states created by the operators $\mathcal{O}_i$.

For the calculation of the Zamolodchikov metric we will take as our bulk CFT an $N = 2$ sigma model whose metric satisfies the beta function equations of motion. The sigma model metric will thus be written as a power series in $l_s^2$. To order $l_s^4$ the metric is Ricci flat. Although the calculations resemble those used to evaluate the sigma model beta function, there are important differences. For the sigma model beta function calculation it is only the divergent contributions that are important, whereas for the Zamolodchikov metric it is the finite terms that are physically relevant (the vanishing of the divergent terms is assured by the fact that metric satisfies the beta function equations of motion). It is thus not at all obvious that the Zamolodchikov metric will be identical to the sigma model metric. There are, for example, a priori no reasons why the metric should be a Kahler metric and hence no a priori reason why at two loop order there cannot be terms in the
metric of the form $R^2$ which do not vanish for Ricci flat metrics. Below the Zamolodchikov metric is calculated out to two loop order. It is found that there is such a contribution. This leads to a D0 brane metric that is neither Ricci flat (at order $l_s^4$) nor derivable from a Kahler potential. It is thus a physically different metric from that seen by the string.

We start in section 2 with a brief overview of Kahler geometry and the background field method. Calculations are greatly simplified by the use of the superfield formalism (see for example [7]). The fact that we have a boundary leads to slight modifications in the propagators and superderivative/propagator identities from the case without a boundary. Section 3 is devoted to determining these differences and setting up the Feynman rules. In section 4 we calculate the Zamolodchikov metric up to two loops.

2. The $N=2$ action and the background field method

The sigma model action, written in terms of chiral $\Phi^I(z, \theta, \bar{\theta})$ and antichiral $\bar{\Phi}^J(z, \theta, \bar{\theta})$ superfields, is given by

$$S = \int d^2 z \, d^4 \theta \, K(\Phi, \bar{\Phi}).$$

(2.1)

where $K(\Phi, \bar{\Phi})$ is the Kahler potential and the bosonic components of $\Phi$ and $\bar{\Phi}$ are coordinates on the Kahler manifold. The world sheet topology is that of the disc with boundary mapped to the real axis and the integral over $d^2 z$ defined over the upper half plane.

Calculations will be performed using the background field method. We start with a classical string world sheet. Since the world sheets we are considering have the topology of a disc they will classically collapse down to a single space time point. We write the fields as a constant part $\Phi_{cl}$, corresponding to this space time point plus a quantum part $\Phi(z, \theta)$:

$$\Phi_{total}(z, \theta) = \Phi_{cl} + \Phi(z, \theta).$$

(2.2)

The Kahler potential is then expanded as a power series in the quantum fields.

$$K(\Phi_{total}, \bar{\Phi}_{total}) = K_{IJ} \Phi^I \bar{\Phi}^J + \frac{1}{2!} K_{IKJ} \Phi^I \Phi^J \bar{\Phi}^K + \frac{1}{2!} K_{IJ\bar{K}} \Phi^I \bar{\Phi}^J \bar{\Phi}^K$$

$$+ \frac{1}{3!} K_{IKJL} \Phi^I \Phi^J \Phi^K \bar{\Phi}^L + \frac{1}{(2!)^2} K_{IKJK} \Phi^I \bar{\Phi}^J \Phi^K \bar{\Phi}^L + \frac{1}{3!} K_{IKJKL} \Phi^I \bar{\Phi}^J \Phi^K \bar{\Phi}^L + \cdots$$

(2.3)
where the coefficients $K_{I_1,I_2,\ldots,J_1,J_2,\ldots}$ are given by taking derivatives of the Kahler potential:

$$K_{I_1,I_2,\ldots,J_1,J_2,\ldots} = \left. \frac{\partial}{\partial \Phi_{I_1}} \frac{\partial}{\partial \Phi_{I_2}} \cdots \frac{\partial}{\partial \Phi_{J_1}} \frac{\partial}{\partial \Phi_{J_2}} \cdots K(\Phi, \bar{\Phi}) \right|_{\Phi = \phi_{cl}}. \quad (2.4)$$

We have dropped terms that involve only chiral or only antichiral fields in (2.3). The fact that the world sheet has collapsed to a single space time point means that the coefficients of the power series are constant and thus that in the action (2.1) such terms can be written as total derivatives.

Note that it is not possible to use normal coordinates since the field redefinitions necessary to transform to normal coordinates would in general mix chiral and antichiral fields. The individual coefficients are thus not covariant. As we will see below, however, the coefficients nevertheless combine together to give a covariant result for the Zamolodchikov metric.

Below we give the expressions for the Kahler metric, connection and curvature tensor. The Kahler metric $g_{IJ}$ is given by

$$g_{IJ} = K_{IJ}. \quad (2.5)$$

Its inverse we denote by $K^{IJ}$. The only non zero components of the connection are

$$\Gamma^I_{JK} = K^{IL} K_{LJK}, \quad \text{and} \quad \Gamma^I_{JK} = K^{IL} K_{LJK}. \quad (2.6)$$

The curvature tensor $R_{IJKL}$ takes the simple form

$$R_{IJKL} = K_{IJKL} - K_{IKLM} K_{JLN} K^{MN}. \quad (2.7)$$

3. $N = 2$ superfields and feynman rules in the presence of a boundary

In this section we fix our conventions for definitions of superfields and superderivatives, and derive the superfield propagator and superderivative/propagator identities in the presence of a boundary.

There are four fermionic variables $\theta^+, \theta^-, \bar{\theta}^+$ and $\bar{\theta}^-$, and a complex coordinate $z = x + iy$. Integration over $\theta$ and $z$ are given by

$$\int d^4 \theta = \int d\theta^+ d\theta^- d\bar{\theta}^+ d\bar{\theta}^- \quad \text{with} \quad \int d\theta \theta = 1 \quad \text{and} \quad \int d^2 z = \int d^2 x. \quad (3.1)$$
There are four superderivatives:

\[ D^+ = \frac{\partial}{\partial \theta^+} + \bar{\theta}^+ \partial \]
\[ \bar{D}^+ = \frac{\partial}{\partial \theta^+} + \theta^+ \bar{\partial} \]
\[ D^- = \frac{\partial}{\partial \theta^-} + \bar{\theta}^- \partial \]
\[ \bar{D}^- = \frac{\partial}{\partial \theta^-} + \theta^- \bar{\partial} \]

(3.2)

They satisfy

\[ \{D^+, \bar{D}^+\} = 2\bar{\partial}, \quad \{D^-, \bar{D}^-\} = 2\partial \quad \text{and} \quad [D^2, \bar{D}^2] = -4\bar{\partial}\partial, \]  

(3.3)

where we are using the conventions

\[ D^2 = D^+ D^- \quad \text{and} \quad \theta^2 = \theta^+ \theta^- \]  

(3.4)

and similarly for the barred superderivatives and \( \theta \)'s.

A chiral field \( \Phi \) satisfies \( \bar{D}\Phi(z, \theta) = 0 \) and an antichiral field \( \bar{\Phi} \) satisfies \( D\bar{\Phi}(z, \theta) = 0 \). Their \( \theta \) component expansions are

\[ \Phi(z, \theta) = \left[ 1 + (\theta^- \bar{\theta}^- \partial + \theta^+ \bar{\theta}^+ \bar{\partial}) - \theta^2 \bar{\theta}^2 \partial \bar{\partial} \right] X(z) \]
\[ + \theta^+ \left[ 1 + \theta^- \bar{\theta}^- \partial \right] \Psi_+(z) + \theta^- \left[ 1 + \theta^+ \bar{\theta}^+ \bar{\partial} \right] \Psi_-(z) \]
\[ + \theta^2 F(z) \]  

(3.5)

\[ \bar{\Phi}(z, \theta) = \left[ 1 - (\theta^- \bar{\theta}^- \partial + \theta^+ \bar{\theta}^+ \bar{\partial}) - \theta^2 \bar{\theta}^2 \partial \bar{\partial} \right] X^*(z) \]
\[ + \bar{\theta}^+ \left[ 1 - \theta^- \bar{\theta}^- \partial \right] \bar{\Psi}_+(z) + \bar{\theta}^- \left[ 1 - \theta^+ \bar{\theta}^+ \bar{\partial} \right] \bar{\Psi}_-(z) \]
\[ + \bar{\theta}^2 F^*(z) \]  

(3.6)

We derive the propagator in flat space for a single chiral and antichiral field. The propagator of the curved space action (2.1) is then given by introducing indices \( I, \bar{J} \) for the chiral and antichiral fields and prefactoring the propagator we find below by the inverse metric \( K^{IJ} \).

The flat space action for a single chiral and antichiral field is

\[ S = \int d^2 z \, d^4 \theta \, \Phi \bar{\Phi} \]
\[ = \int d^2 z \left[ X(-4\partial\bar{\partial})X^* + 2\Psi_+ \partial \bar{\Psi}_+^* + 2\Psi_- \bar{\partial} \Psi_-^* + |F|^2 \right] \]  

(3.7)

\[ = \int d^2 x \left[ X(-\nabla)X^* + \Psi_+(\partial_x - i\partial_y)\Psi_+^* + \Psi_-(\partial_x + i\partial_y)\Psi_-^* + |F|^2 \right] \]
The boundary conditions on the fields are

\[ X(\bar{z}, z) = -X(z, \bar{z}) \quad \text{and} \quad \Psi_{-}(z, \bar{z}) = -\Psi_{+}(\bar{z}, z) \quad (3.8) \]

This leads to the bosonic propagator

\[ \langle X(z_1, \bar{z}_1)X^{*}(z_2, \bar{z}_2) \rangle = -\frac{1}{2\pi} \left( \ln |z_1 - z_2| - \ln |z_1 - \bar{z}_2| \right), \quad (3.9) \]

satisfying

\[ -4\partial_1 \bar{\partial}_1 \langle X(z_1, \bar{z}_1)X^{*}(z_2, \bar{z}_2) \rangle = -\nabla \langle X(z_1, \bar{z}_1)X^{*}(z_2, \bar{z}_2) \rangle > = \delta^2(z_1 - z_2) - \delta^2(z_1 - \bar{z}_2), \quad (3.10) \]

and fermionic propagators

\[
\begin{align*}
\langle \Psi_{+}(z)\Psi_{+}(z') \rangle &= -\frac{1}{2\pi} \frac{1}{z - z'} \\
\langle \Psi_{-}(z)\Psi_{-}(z') \rangle &= -\frac{1}{2\pi} \frac{1}{\bar{z} - \bar{z}'} \\
\langle \Psi_{+}(z)\Psi_{-}(z') \rangle &= +\frac{1}{2\pi} \frac{1}{\bar{z} - \bar{z}'} \\
\langle \Psi_{-}(z)\Psi_{+}(z') \rangle &= +\frac{1}{2\pi} \frac{1}{z - z'} \\
\end{align*}
\]

(3.11)

The superfield propagators can be built up from the component propagators \([3.10]\) and \([3.11]\). One finds (see the appendix for some useful identities for the superderivatives).

\[
\begin{align*}
\langle \Phi(z_1, \theta_1)\bar{\Phi}(z_2, \theta_2) \rangle &= -\frac{1}{2\pi} D_1^2 \bar{D}_2^2 [(\theta_1 - \theta_2)^4 \ln |z_1 - z_2| + (\theta_1 - \bar{\theta}_2)^4 \ln |z_1 - \bar{z}_2|] \\
&= -\frac{1}{2\pi} D_1^2 \bar{D}_1^2 [(\theta_1 - \theta_2)^4 \ln |z_1 - z_2| - (\theta_1 - \bar{\theta}_2)^4 \ln |z_1 - \bar{z}_2|] \\
&= -\frac{1}{2\pi} \bar{D}_2^2 \bar{D}_2^2 [(\theta_1 - \theta_2)^4 \ln |z_1 - z_2| - (\theta_1 - \bar{\theta}_2)^4 \ln |z_1 - \bar{z}_2|] \\
\end{align*}
\]

(3.12)

where \(\bar{\theta}\) means the + and \(-\) components have been interchanged.

Again using the identities in the appendix one can show that the propagators satisfy

\[
\begin{align*}
D_2^2 \langle \Phi(z_1, \theta_1)\Phi(z_2, \theta_2) \rangle &= D_2^2 [(\theta_1 - \theta_2)^4 \delta^2(z_1 - z_2) - (\theta_1 - \bar{\theta}_2)^4 \delta^2(z_1 - \bar{z}_2)] \\
\bar{D}_2^2 \langle \Phi(z_1, \theta_1)\Phi(z_2, \theta_2) \rangle &= \bar{D}_2^2 [(\theta_1 - \theta_2)^4 \delta^2(z_1 - z_2) - (\theta_1 - \bar{\theta}_2)^4 \delta^2(z_1 - \bar{z}_2)] \\
\end{align*}
\]

(3.13)

It is not so obvious that these identities would still hold true in the presence of a boundary, (indeed for the \(N = 1\) superfield formalism the analogous identities no longer hold when
there is a boundary). The fact that they do hold means that in evaluating Feynman
diagrams one can manipulate superderivatives and collapse propagators just as one does
for the case without boundary.

Finally we give the expressions for the propagators connected to the boundary operator
\( \mathcal{O}(x) = \partial_y X(z)|_{y=0} \):

\[
\langle \mathcal{O}(x_0) \Phi(z_1, \theta_1) \rangle = -\frac{1}{\pi} \bar{D}_1^2 \bar{\theta}_1^2 \partial_{y_1} \ln |x_0 - z_1|,
\]

\[
\langle \mathcal{O}^*(x_0) \Phi(z_1, \theta_1) \rangle = -\frac{1}{\pi} D_1^2 \theta_1^2 \partial_{y_1} \ln |x_0 - z_1|,
\]

and for the tadpole propagator which starts and finishes at the same point

\[
\langle \Phi(z, \theta) \bar{\Phi}(z, \theta) \rangle = -\frac{1}{2\pi} \left[ \ln |0| - \ln |2y| - i(\bar{\theta}^- \theta^+ - \bar{\bar{\theta}}^+ \theta^-) \frac{1}{2y} + \theta^4 \left[ \frac{1}{4y^2} - \pi \delta(0) \delta(2y) \right] \right].
\]

The tadpole propagator satisfies

\[
D^2 \langle \Phi(z, \theta) \bar{\Phi}(z, \theta) \rangle^n = \bar{D}^2 \langle \Phi(z, \theta) \bar{\Phi}(z, \theta) \rangle^n = 0,
\]

where \( n \) is any positive integer.

3.1. Feynman rules

The propagators given above were for a single chiral anti-chiral field pair in flat space.
The curved space propagator is given by including chiral and anti-chiral indices and an
inverse metric \( K^{IJ} \). Since the superderivatives act at opposite ends of the propagator it
is a standard convention to include the superderivatives on the vertices rather than the
propagators. In other words diagrammatically we have for the propagators and vertices.

\[
\begin{align*}
\begin{array}{c}
\text{I} \\
\downarrow \\
\ell \\
\end{array}
\begin{array}{c}
\text{J} \\
\downarrow \\
\bar{z}_2 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
K^{IJ} = \frac{1}{2\pi} \left[ \delta(\theta_1 - \theta_2) \ln |z_1 - z_2| + \delta(\bar{\theta}_1 - \bar{\theta}_2) \ln |z_1 - \bar{z}_2| \right],
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{I} \\
\downarrow \\
\bar{K} \\
\end{array}
\begin{array}{c}
\text{J} \\
\downarrow \\
\ell \\
\end{array}
\end{align*}
\]

The dots after the the \( D^2 \) and \( \bar{D}^2 \) for the vertices mean that the superderivatives act on
the propagators attached to the vertices. For compactness of diagrammatic notation a
solid bar on the leg of a vertex denotes a \( \bar{D}^2 \) whereas for legs without a bar we associate
a \( D^2 \).
There are three diagrammatic rules that are easily derived using the Feynman rules (3.17) and the identities (3.13). Firstly each time one has a vertex with a single chiral field and all other fields anti-chiral one can integrate by parts two superderivatives off the antichiral legs onto the chiral leg. By the identity (3.13) this collapses a propagator.

\[
\begin{align*}
\frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial t^2} = 0
\end{align*}
\]

(3.18)

An analogous identity obviously applies when there is a single chiral leg and all the other legs are chiral.

A second observation helps to reduce the number of diagrams that contribute to the Zamolodchikov metric. For non-zero diagrams the internal legs of the vertices connected to the boundary must have at least one field of opposite chirality to the boundary operator. Diagrams in which all legs are of the same chirality as that of the boundary operator give zero since one can integrate by parts superderivatives from the internal legs onto the external leg, collapsing a propagator. The internal legs of the vertex then have their endpoints on the boundary where the propagators are zero. °

\[
\begin{align*}
\frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial t^2} = 0
\end{align*}
\]

(3.19)

Finally there is an identity involving the tadpole propagator which can be stated graphically as

\[
\begin{align*}
\frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial x^2} \frac{\partial^2}{\partial t^2} = 0
\end{align*}
\]

The vertex to which the tadpoles are attached has all other legs of the same chirality. One can thus integrate by parts a \( \frac{\partial^2}{\partial x^2} \) (or \( \bar{\partial}^2 \)) off one of the legs and onto the tadpoles, leading to zero by (3.16).

° Note that there is a normal derivative from the boundary operator acting on one of the propagators (rendering it non-zero) but since there is more than one propagator with its endpoint on the boundary the total result is zero.
4. Zero, one and two loop contributions to the Zamolodchikov metric

The tree level contribution to the Zamolodchikov metric is just given by the propagator. In other words we have

\[ g_{IJ}^{(0)} = 2\pi(x_1 - x_2)^2 < O_I(x_1)\bar{O}_J(x_2) >_{\text{tree level}} = K_{IJ} \]  

(4.1)

To simplify the presentation all Feynman diagrams in this section include the prefactor \(2\pi(x_1 - x_2)^2\). The Feynman diagram consisting of a single propagator connecting the two boundary operators is thus, by definition, equal to one.

Using the diagrammatic rule (3.19) there are only two diagrams that contribute at one loop :

\[ g_{IJ}^{(1)} = -K_{IIJJ}K_{JJ} + K_{IIJK}K_{JJK}K_{JK} \]

(4.2)

It is then easy to use the diagrammatic rule (3.18) to collapse the bottom propagator of the first diagram so that it has an identical form to the first.

\[ g_{II}^{(1)} = [-K_{IIJJ}K_{JJ} + K_{IIJK}K_{JJK}K_{JK}] = -R_{II} \]

(4.3)

= 0

The tensors \(K_{..}\) have combined to give the Ricci tensor which is zero at this order in \(l_s^2\).

We now turn to the two loop diagrams. They involve vertices of order three, four, five and six. Using the diagrammatic identity (3.18) however all three vertices can have one of their propagators collapsed, as can the four vertices with three legs of same chirality. One thus finds that all diagrams collapse down to one of four distinct types. We have

\[ g_{II}^{(2)} = g_{II}^{(2a)} + g_{II}^{(2b)} + g_{II}^{(2c)} + g_{II}^{(2d)} \]  

(4.4)

The first type of contribution, \(g_{II}^{(2a)}\) consists of diagrams that collapse down to a double tadpole :

\[ g_{II}^{(2a)} = g_{II}^{(2a1)} + g_{II}^{(2a2)} + g_{II}^{(2a3)} + g_{II}^{(2a4)} \]  

(4.5)

where

\[ g_{II}^{(2a1)} = -\frac{1}{2}K_{IIJK}K_{JK}K_{KK} \]  

(4.6)
The factors of $1/2$ are symmetry factors coming from symmetry under interchange of two propagators. Using the diagrammatic rule (3.18) all diagrams can be seen to reduce down to the double tadpole structure. The non-covariant tensors $K_{\cdots}$ then combine to give the covariant result

$$g^{(2a2)}_{II} = \left[ \frac{1}{2} K_{IJKL} K_{IJKL} + K_{IJL} K_{IJKL} + \frac{1}{2} K_{IJKL} K_{JKL} \right] K_{J}^{J} K_{K}^{K} K_{L}^{L} K_{L}^{L},$$

(4.7)

$$g^{(2a3)}_{II} = - \left[ K_{IJKL} K_{IJLM} K_{JLM} + \frac{1}{2} K_{IJKL} K_{IJLM} K_{JLM} + \frac{1}{2} K_{IJKL} K_{IJLM} K_{JLM} 
+ K_{IJKL} K_{IJLM} K_{JLM} + \frac{1}{2} K_{IJKL} K_{IJLM} K_{JLM} \right] K_{J}^{J} K_{K}^{K} K_{L}^{L} K_{M}^{M},$$

(4.8)

and

$$g^{(2a4)}_{II} = \left[ K_{IJKL} K_{IJMN} K_{KM} K_{MN} + K_{IJKL} K_{IJMN} K_{KM} K_{MN} + \frac{1}{2} K_{IJKL} K_{IJMN} K_{KM} K_{MN} \right] K_{J}^{J} K_{K}^{K} K_{L}^{L} K_{M}^{M} K_{N}^{N},$$

(4.9)

The second type of contribution consists of all diagrams that collapse down to a contraction

$$g^{(2a)}_{II} = - \frac{1}{2} \left[ \nabla_{I} \nabla_{J} R + R_{IJKL} R_{IJ}^{JKL} + R_{IJK} R_{JK}^{IJK} \right]$$

(4.10)
between two four vertices, each with two chiral and two antichiral indices

\[ g_{IJ}^{(2b)} = \frac{1}{2} K_{KL} I_{JK} K K_{LL} - \left[ \frac{1}{2} K_{LM} I_{JK} K K_{KM} \right] K_{KL} K + \frac{1}{2} K_{LK} I_{JK} K K_{LM} K K_{KL} K M \bar{M} + \frac{1}{2} K_{LM} I_{JK} K K_{LM} K K_{KM} K N \bar{N} \]

\[ = \frac{1}{2} R_{IJ} R_{IJK} K_{JK} K \]

(4.11)

For the third type of contribution the diagrams collapse down to the another possible contraction of two four vertices.

\[ g_{IJ}^{(2c)} = K_{JK} K_{LM} K_{KL} K_{JK} K K_{LL} - \left[ K_{LM} I_{JK} K K_{LM} \right] K_{KL} K + K_{LM} I_{JK} K K_{LM} K K_{KL} K M \bar{M} + \frac{1}{2} K_{LM} I_{JK} K K_{LM} K K_{KM} K N \bar{N} \]

\[ = R_{IJ} R_{IJ} K \]

(4.12)

Finally there is the contribution consisting of two one loop diagrams

\[ g_{IJ}^{(2d)} = K_{JK} K_{LM} K_{LM} K_{KL} K_{IJ} K K_{LL} - \left[ K_{LM} I_{JK} K K_{LM} \right] K_{KL} K + K_{LM} I_{JK} K K_{LM} K K_{KL} K M \bar{M} + \frac{1}{2} K_{LM} I_{JK} K K_{LM} K K_{KM} K N \bar{N} \]

\[ = R_{IJ} R_{IJ} \]

(4.13)

If one was confident that the result would be covariant one could specialise to Kahler potentials where all three vertices are zero, calculate using only the four vertices and the
six vertex and find directly the covariant results of (4.10) (4.11) (4.12) (4.13) It can be shown [4] that the divergent part will always be covariant. The proof of [4] does not however apply to the finite part of the boundary two point function.

For Ricci flat metrics the first and third terms in (4.13) are zero as are the contributions (4.12) and (4.13). For Ricci flat metrics we thus have

\[
\frac{1}{2} R_{IJKL} R_{\bar{J}\bar{K}\bar{L}} \left[ \frac{1}{2} \left( x^I \right)^2 + \left( x^\bar{J} \right)^2 \right] = \frac{1}{2} R_{IJKL} R_{\bar{J}\bar{K}\bar{L}} \left( x^I \right)^2 \quad (4.14)
\]

Note that the second term cannot be further reduced using (3.18). One can still integrate by parts the superderivatives to collapse an internal propagator and end up with double tadpole structure of the first term (thus canceling the divergence). In so doing however one will also end up with (in addition to the double tadpole structure) terms involving superderivatives acting on the boundary propagators. It is these terms (which are finite) which give the contribution to the Zamolodchikov metric. The whole contribution to the Zamolodchikov metric comes from the finite part of the second term. We thus see that the Zamolodchikov metric potentially receives a contribution proportional to

\[ R_{IJKL} R_{\bar{J}\bar{K}\bar{L}} \]

All that remains to do is to calculate the precise coefficient that goes with this term.

4.1. Calculation of coefficient of \( R^2 \) term

Below we indicate explicitly the superderivatives for the double four vertex term.

\[
D_1^2 \quad D_2^2 \quad \bar{D}_1^2 \quad \bar{D}_2^2
\]

\[
D_1^2 \quad D_2^2
\]

\[
\bar{D}_1^2 \quad \bar{D}_2^2
\]

\[
x_0 \quad z_1 \quad z_2 \quad x_3
\]

To reproduce the tadpole structure one can integrate by parts the \( \bar{D}_1^2 \) superderivatives off the bottom propagator. This will generate several types of contribution. There will be terms with superderivatives acting on the left hand external propagator. There will also be a contribution in which both \( \bar{D}_1^2 \) act on the top propagator. This is equivalent (via identity (3.13)) to \( \bar{D}_1^2 \) acting on a collapsed propagator. Integrating the superderivatives back off the collapsed propagator leaves one with the double tadpole structure and in addition further terms in which superderivatives act on the external propagator.* Alternatively

* Note that if one was calculating the beta function all terms with superderivatives acting on the external legs would be dropped since, by power counting arguments, they give rise to finite contributions.
and more simply one can manipulate directly the expression for the top propagator and express it as a collapsed propagator + other terms. The identity

\[ D_1^2 D_1^2 = D_1^2 D_1^2 - 4 \partial_1 \bar{\partial}_1 + 2(\bar{\partial}_1 D_1^- D_1^- + \partial_1 \bar{D}_1^+ \bar{D}_1^+), \] (4.16)

along with the expressions (3.12) for the propagators means that we can rewrite the diagram as follows

\[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram.png} \\
= \includegraphics[width=0.3\textwidth]{diagram.png} + \includegraphics[width=0.3\textwidth]{diagram.png} + \includegraphics[width=0.3\textwidth]{diagram.png}
\end{array} \] (4.17)

where the three diagrams on the right hand side of (4.17) correspond, respectively to the three terms on the right hand side of (4.16). In particular the dotted line of the top propagator of the third diagram comes from the third term of (4.16).

The first diagram of (4.17) is zero by (3.19), the second leads to cancellation of the double tadpole in (4.14) leaving the final diagram. Writing out the final diagram explicitly we have

\[ \begin{array}{c}
\includegraphics[width=0.5\textwidth]{final_diagram.png} \\
= 4\pi |x_0 - x_3|^2 \int d^2 z_1 d^4 \theta_1 \int d^2 z_2 d^4 \theta_2 [(D_1^2 P_{10})(D_2^2 P_{23})(D_1^2 D_1^2 P_{12})]^2 \\
(\bar{\partial}_1 D_1^- D_1^- + \partial_1 \bar{D}_1^+ \bar{D}_1^+) P_{12}],
\end{array} \] (4.18)

where

\begin{align*}
P_{12} &= -\frac{1}{2\pi} [\delta^4(\theta_1 - \theta_2) \ln |z_1 - z_2| - \delta^4(\theta_1 - \bar{\theta}_2) \ln |z_1 - \bar{z}_2|] \\
P_{10} &= -\frac{1}{\pi} \theta_1^2 \partial_{y_1} \ln |z_1 - x_0| \\
P_{23} &= -\frac{1}{\pi} \bar{\theta}_2^2 \partial_{y_1} \ln |z_2 - x_3|
\end{align*} (4.19)

As discussed for the zero loop contribution we include in the definition of the Feynman diagrams a prefactor \(2\pi|x_0 - x_3|^2\). The integrals over \(z_1\) and \(z_2\) in (4.18) are over the upper half complex plane. By interchanging \(z_1\) with \(\bar{z}_1\), \(\theta_1^+\) with \(\bar{\theta}_1^-\) and \(\bar{\theta}_1^-\) with \(\bar{\theta}_1^+\) the integral over \(z_1\) can be completed into an integral over the whole complex \(z_1\) plane. After using the identities

\begin{align*}
\bar{\partial}_1 D_1^- \delta^4(\theta_1 - \theta_2) f(z_1 - z_2) &= \bar{\partial}_2 D_2^- \delta^4(\theta_1 - \theta_2) f(z_1 - z_2) \\
\bar{\partial}_1 D_1^- \delta^4(\theta_1 - \bar{\theta}_2) f(z_1 - \bar{z}_2) &= \bar{\partial}_2 D_2^+ \delta^4(\theta_1 - \bar{\theta}_2) f(z_1 - \bar{z}_2), \quad (4.20)
\end{align*}

13
where $f(z)$ is an arbitrary function of $z$ and its complex conjugate, the $z_2$ integral can similarly be completed into an integral over the whole complex $z_2$ plane. We thus arrive at the integral

$$
\int \cdots = |x_0 - x_3|^2 \int d^2 z_1 d^4 \theta_1 \int d^2 z_2 d^4 \theta_2 \left[ (\bar{D}_1 D_1^2 P_{10})(\bar{D}_2 D_2^2 P_{23}) 
- (D_1^2 D_1^2 P_{12})^2 \delta(\theta_1 - \theta_2)^4 \frac{1}{\bar{z}_1 - \bar{z}_2} \right],
$$

(4.21)

where the integrals are now over the whole complex plane. Performing the integrations over the fermionic parameters we find (see appendix)

$$
\int \cdots = \frac{1}{2\pi^4} \int d^2 z_1 d^2 z_2 \left[ \frac{1}{\bar{z}_1^2} - \frac{1}{\bar{z}_1^2} \right] \frac{1}{z_2 \bar{z}_2 - z_2 \bar{z}_2 + 1} \frac{1}{z_2 \bar{z}_2 - z_2 \bar{z}_2 + 1} \ln |z_1 - z_2 + 1| \quad (4.22)
$$

Evaluating the double integral over the complex plane one finds (see appendix)

$$
\int \cdots = \frac{1}{6} \quad (4.23)
$$

Up to two loop order the D0 metric is thus given by :

$$
g_{\bar{I}I}^D = g_{\bar{I}I}^\sigma + \frac{l_s^4}{12} R_{IJK\bar{L}} R_{\bar{I}jK\bar{L}} + \mathcal{O}(l_s^6). \quad (4.24)
$$

5. Conclusions

The conclusion of this paper is that there is a non-trivial contribution to the Zamolodchikov metric at order $l_s^4$. A D-brane in a weakly curved background thus experiences a different metric from that seen by the string.

6. Acknowledgements

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7. Appendix

In this appendix we list the identities necessary to prove the results of section 3. We also give a few technical details on the calculation of the coefficient of the $R^2$ term.
7.1. Useful identities

\[
D_1^2(\theta_1 - \theta_2)^2 = -\exp[\bar{\theta}_1^- (\theta_1^- - \theta_2^-) \partial + \bar{\theta}_1^+ (\theta_1^+ - \theta_2^+) \bar{\partial}]
\]
\[
D_2^2(\bar{\theta}_1 - \bar{\theta}_2)^2 = -\exp[\theta_1^- (\theta_1^- - \theta_2^-) \partial + \theta_1^+ (\theta_1^+ - \theta_2^+) \bar{\partial}]
\]
\[
D_2^2 D_1^2(\theta_1 - \theta_2)^4 = \exp[\bar{\theta}_1^- \theta_1^- \partial_1 + \theta_1^+ \theta_1^+ \bar{\partial}_1 - \theta_2^- \theta_2^- \partial_2 - \theta_2^+ \theta_2^+ \bar{\partial}_2 - \bar{\theta}_1^+ \theta_1^+ (\partial_1 - \partial_2) - \theta_1^- \theta_2^- (\partial_1 - \partial_2)]
\]
\[
D_2^2 D_1^2 (\bar{\theta}_1 - \bar{\theta}_2)^4 = -\exp[\bar{\theta}_1^- \theta_1^- \partial_1 + \theta_1^+ \theta_1^+ \bar{\partial}_1 - \theta_2^- \theta_2^- \partial_2 - \theta_2^+ \theta_2^+ \bar{\partial}_2 - \bar{\theta}_1^+ \theta_1^+ (\partial_1 - \partial_2) - \theta_1^- \theta_2^- (\partial_1 - \partial_2)]
\]
\[
D_1^2(\theta_1 - \theta_2)^4 f(z_1 - z_2) = D_2^2(\theta_1 - \theta_2)^4 f(z_1 - z_2)
\]
\[
D_1^2(\bar{\theta}_1 - \bar{\theta}_2)^4 f(z_1 - \bar{z}_2) = -D_2^2(\theta_1 - \bar{\theta}_2)^4 f(z_1 - \bar{z}_2)
\]

where \(f(z)\) is an arbitrary function of \(z\) and \(\bar{z}\) and \(\bar{\theta}\) means the + and − components have been interchanged.

7.2. Fermionic integrals for \(R^2\) contribution

To evaluate the fermionic integrals of equation (4.21), one first trivially integrates over \(\theta_2\), the delta function setting all occurrences of \(\theta_2\) equal to \(\theta_1\). One then uses the following expressions for the propagators (which follow from (4.19) and (7.1)) :

\[
\bar{D}_1^- D_1^2 P_{10} = \frac{1}{\pi} \partial_{y_1} D_1^+ \theta_1^2 \frac{1}{z_1 - x_0},
\]
\[
D_2^- \bar{D}_2^2 P_{23} = \frac{1}{\pi} \partial_{y_2} \bar{D}_2^+ \bar{\theta}_2^2 \frac{1}{z_2 - x_3},
\]
\[
\bar{D}_1^2 D_1^2 P_{12}|_{\theta_1 = \theta_2} = -\frac{1}{2\pi} \left[ \ln |z_1 - z_2| - \ln |z_1 - \bar{z}_2| 
\right.
\]
\[
\left. + \bar{\theta}_1^+ \theta_1^- \frac{1}{z_1 - \bar{z}_2} + \bar{\theta}_1^- \theta_1^+ \frac{1}{\bar{z}_1 - z_2} 
\right.
\]
\[
\left. - (\bar{\theta}_1^+ \theta_1^- + \bar{\theta}_1^- \theta_1^+) \frac{1}{2} \left[ \frac{1}{z_1 - \bar{z}_2} + \frac{1}{\bar{z}_1 - z_2} \right] 
\right.
\]
\[
\left. - \theta_1^+ \left[ \frac{1}{(z_1 - z_2)^2} + \frac{1}{(\bar{z}_1 - \bar{z}_2)^2} + \pi \delta^2(z_1 - \bar{z}_2) \right] \right].
\]
is collapsed and either the \( z_1 \) or \( z_2 \) vertex is pulled back to the boundary, where the \((D_2 P_1)\) term gives zero. For the second type of contribution just two \( \theta \)'s come from the external propagators \((\theta^- \theta^-)\), the remaining two coming from the \((D_2 P_1)\) term. This leads to (1.22).

### 7.3. Double integration over the complex plane

The double integral over the complex plane (4.22) is complicated by the fact that it mixes holomorphic and antiholomorphic variables. Below we describe briefly how the integral can be performed analytically and give an intermediate result which allows the integral to be checked by numerical integration.

Changing integration variables from \( z_1, z_2 \) to \( z^\pm = (z_1 \pm z_2)/2 \) the integral (4.22) can be written in the form

\[
\int \frac{d^2z^-}{2\pi} \left[ I_1(z^-) - I_2(z^-) \right] \frac{1}{2z^- + 1} \ln |2z^- + 1|, 
\]

with

\[
I_1(z^-) = \int d^2z^+ \frac{1}{(z^+ + z^-)^2} \frac{1}{z^+ - z^- + z^-},
\]

\[
I_2(z^-) = \int d^2z^+ \frac{1}{(\bar{z}^+ + \bar{z}^-)^2} \frac{1}{\bar{z}^+ - \bar{z}^- + \bar{z}^- + 1}. 
\]

Performing the integrals over \( z^+ \) one finds the results

\[
I_1(z^-) = -\frac{i\pi}{2z^- - 1} \tan^{-1}\left(\frac{y^-}{x^- + 1/2}\right),
\]

\[
I_2(z^-) = \frac{\pi}{4} \left[ \frac{2}{(2x^- + 1/2)^3} \left[ \ln |2z^-| - \ln |2z^- + 1| \right] - \frac{1}{(2x^- + 1/2)^2} \left[ \frac{1}{z^-} + \frac{1}{\bar{z}^-} \right] - \frac{1}{2(2x^- + 1/2) \left[ \frac{1}{(z^-)^2} + \frac{1}{(\bar{z}^-)^2} \right] } \right].
\]

The inverse tangent in \( I_1(z^-) \) takes on values between \(-\pi/2\) and \(\pi/2\). Note that the expression for \( I_2(z^-) \) is non-singular at \( x^- = -1/4 \) (contrary to first appearances). The singularities of the individual terms cancel among themselves leaving \( I_2(z^-) \) finite at \( x^- = -1/4 \).

Integration over \( z^- \) yields:

\[
\int d^2z^- I_1(z^-) \frac{1}{2z^- + 1} \ln |2z^- + 1| = \frac{\pi^4}{8},
\]

\[
\int d^2z^- I_2(z^-) \frac{1}{2z^- + 1} \ln |2z^- + 1| = \frac{\pi^4}{24},
\]

Which on substituting back into (7.3) leads to equation (1.23).

16
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