Towards a Schubert Calculus for Maps from a Riemann Surface to a Grassmannian

by

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0. Introduction. The notion of a quantum cohomology theory for complex projective varieties was introduced by the physicist C. Vafa (cf. [13]), and has recently received a more mathematical treatment by Witten (cf. [14]) and others (see [8] and [10]). Given such a variety $X$ with polarization $O_X(1)$, the idea is to change the multiplicative structure of $H^*(X, \mathbb{C})$ by carrying out the intersection of cohomology classes not on $X$ itself, but rather after pulling them back to the scheme $\text{Mor}_d(C, X)$ parametrizing the morphisms of degree $d$ from a fixed Riemann surface $C$ to $X$. When the right number of cohomology classes are intersected on $\text{Mor}_d(C, X)$ to produce a top codimensional cohomology class on $\text{Mor}_d(C, X)$, then the degree of such a class is referred to as the “$k$ point correlation function” in the literature, but following [1], we will simply call it the Gromov invariant of the cohomology classes.

Although the objects in question, as described here, are all algebro-geometric, it is not readily apparent how to make sense of the Gromov invariants in terms of algebraic geometry. For example, the scheme $\text{Mor}_d(C, X)$ often does not have the expected dimension, is essentially never projective, and tends to have singularities. All these features evidently pose problems when one attempts to intersect cohomology classes.

The purpose of this paper is first to show that if $X = G = G(r, k)$, the Grassmannian of complex $r$-planes in $\mathbb{C}^k$, then for all Riemann surfaces and sufficiently large $d$, the Gromov invariants for the special Schubert cycles can be rigorously defined, and are realized as an intersection of chern classes on a projective scheme. (One probably ought to call this “quantum Schubert calculus”.) It is almost immediate when one defines the invariants in this way that they do not depend upon the choice of complex structure on the Riemann surface of genus $g$. Another advantage of realizing the Gromov invariants in

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this way comes from the general principle that intersection numbers tend to
be more easily computed when they are defined algebraically.

This principle was demonstrated by Daskalopoulos, Wentworth and the
author in [1], where it was shown that in case $r = 2$ and $C$ is an elliptic
curve, the Gromov invariants for Grassmannians agree with a remarkable
conjectural formula of Vafa and Intriligator ([6]), generalizing the classical
Schubert calculus on the Grassmannian.

The second aim of this paper is to rigorously prove an induction on the
genus. That is, an explicit relation is derived between the Gromov invariants
associated to Riemann surfaces of genus $g$ and $g - 1$. The induction, together
with the results of [1], enables us to calculate all the invariants when $r = 2,$
and again the result agrees with the generalized Schubert calculus formula.

The results of this paper mostly follow from a detailed analysis of the
Grothendieck quot scheme parametrizing quotients of a trivial vector bundle
on an algebraic curve $C$. These schemes, as was shown in [1], provide com-
 pactifications of the scheme of morphisms from $C$ to a Grassmann variety.
Moreover, the pull backs of the special Schubert cycles to $\text{Mor}_d(C, G)$ ex-
tend to chern classes on the quot scheme, which may therefore be intersected
in spite of the presence of singularities. It is shown that representat-ives of
the pulled-back cycles on $\text{Mor}_d(C, G)$ may be chosen in sufficiently general
position so that the actual intersection number is defined, and agrees with
the intersection of the corresponding Chern classes on the quot scheme. The
induction on the genus is obtained by considering the family of quot schemes
associated to a family of smooth curves degenerating to an irreducible curve
with one node.

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the preparation of this paper. I would also like to thank János Kollár and
Yongbin Ruan for their useful suggestions.
1. **Intersections on the Space of Maps.** If $C$ is a Riemann surface of genus $g$ and $X$ is a projective variety equipped with an ample line bundle $O_X(1)$, then the moduli space $\text{Mor}_d(C, X)$ parametrizing the morphisms $f : C \to X$ of degree $d$ is quasiprojective and admits a universal evaluation map:

$$ev : C \times \text{Mor}_d(C, X) \to X$$

Given a subvariety $Y \subset X$ and a point $p \in C$, one gets an induced subscheme of $\text{Mor}_d(C, X)$ by pulling back and intersecting:

$$W_d(p, Y) := ev^{-1}(Y) \cap (\{p\} \times \text{Mor}_d(C, X))$$

where $\{p\} \times \text{Mor}_d(C, X)$ is identified with $\text{Mor}_d(C, X)$.

If $Y_1, ..., Y_N \subset X$ are subvarieties of codimension $c_1, ..., c_N$, chosen so that $\sum_{i=1}^N c_i = \dim(\text{Mor}_d(C, X))$, then one expects the intersection: $\cap_{i=1}^N W(p_i, Y_i)$ to consist of distinct reduced points.

**Imprecise** Definition: The Gromov invariant associated to the subvarieties $Y_1, ..., Y_N$ above is the number of points in the intersection of the $W_d(p_i, Y_i) \subset \text{Mor}_d(C, X)$, assuming that $\text{Mor}_d(C, X)$ is of the expected dimension, and that there is no extra “intersection at infinity”.

The Gromov invariant ought not to depend upon the choice of distinct points $p_i \in C$. Moreover, it may be extended by linearity to a function of cycles $Z_1, ..., Z_N$ on $X$, and it ought in fact to be an invariant attached to the corresponding cohomology classes $[Z_i] \in H^c(X, C)$.

Our object in this section is to make rigorous sense of this definition in case $X$ is a Grassmann variety, and to prove at least the independence of the choice of points.

Suppose now that $G = G(r, k)$, the Grassmannian of $r$-planes in $V \cong \mathbb{C}^k$. Let

$$0 \to S \to V \otimes \mathcal{O}_G \to Q \to 0$$

be the universal exact sequence of vector bundles on $G$. The polarization $\mathcal{O}_G(1)$ is the determinant $\wedge^r(S^*)$, which determines the Plücker embedding.

It was shown in [1] that the dimension of $\text{Mor}_d(C, G)$ has the expected value $kd - r(k-r)(g-1)$ if $d >> 0$, and moreover that in this case $\text{Mor}_d(C, G)$ is irreducible and generically reduced.
We regard $G = G(r, k)$ as a homogeneous space for the group $GL(V)$. If $Y \subset G(r, k)$ is any subvariety, let $gY$ be the translate of $Y$ by the element $g \in GL(V)$.

**Lemma 1.1:** Suppose $Y \subset G(r, k)$ is an irreducible subvariety of codimension $c$, and suppose $Z \subset Mor_d(C, G)$ is an irreducible subscheme. Then for any $p \in C$ and a general translate $g$, the intersection $Z \cap W_d(p, gY)$ is either empty or has codimension $c$ in $Z$.

**Proof:** Let $ev_p$ be the restriction of $ev$ to $\{p\} \times Mor_d(C, G)$, and consider the reduced image $T = ev_p(Z) \subset G(r, k)$. Under a general translation, $gY$ intersects $T$ in codimension $c$ or the empty set. More generally, $gY$ intersects each locus in $T$ over which $ev_p$ has constant fiber dimension in codimension $c$ or the empty set, so $W_d(p, gY) \cap Z = ev_p^{-1}(gY) \cap Z$ is of codimension $c$ in $Z$ or else empty.

Suppose $Y_1, Y_2, ..., Y_N$ are irreducible subvarieties of $G(r, k)$ such that the codimension of $Y_i$ is $c_i$ and suppose that $p_1, ..., p_N$ are (not necessarily distinct) points of $C$.

**Corollary 1.2:** If $\sum_{i=1}^{N} c_i > \dim(Mor_d(C, G))$, then for general elements $g_1, ..., g_N \in GL(V)$, we have $W_d(p_1, g_1Y_1) \cap ... \cap W_d(p_N, g_NY_N) = \emptyset$.

**Proof:** Immediate from the Lemma.

Suppose in addition that $Mor_d(C, G)$ is irreducible and generically reduced. Then:

**Corollary 1.3:** If $\sum_{i=1}^{N} c_i = \dim(Mor_d(C, G))$, then for general $g_1, ..., g_N \in GL(V)$, the intersection $\cap_{i=1}^{N} W_d(p_i, g_iY_i)$ is finite, and is contained in the smooth locus.

**Proof:** Again, immediate from the Lemma.

As a consequence of Corollary 1.3, one can count the multiplicities at each of the finite points of the intersection $\cap_{i=1}^{N} W_d(p_i, g_iY_i)$, and thus a well-defined intersection number is obtained on $Mor_d(C, G)$.

One can now define the Gromov invariant associated to any subvarieties $Y_1, ..., Y_N$ as the intersection number in Corollary 1.3 for general choices of $p_1, ..., p_N \in C$ and $g_1, ..., g_N \in G$. However, for special choices of $Y_1, ..., Y_N$, the intersection numbers may be realized as intersections of Chern classes on a projective scheme.
**Definition:** If \( W \subset V^* \) has dimension \( n \leq r \), and \( Y \subset G(r, k) \) is the degeneracy locus of the map \( W \otimes \mathcal{O}_G \to S^* \), then following the literature (see e.g. [4]), we will call \( Y \) a special Schubert subvariety of \( G \).

The special subvarieties of the Grassmannian are straightforward generalizations of the hyperplanes in projective space. The special subvariety \( Y \subset G \) is always irreducible (though not necessarily smooth), and \( Y \) represents the \( r + 1 - n \)th chern class of \( S^* \) (see [4]). Any translate \( gY \) by an element of \( g \in GL(V) \) is simply the degeneracy locus associated to the translate \( gW \). Finally, the special subvarieties generate the cohomology ring of \( G \), as we will remind the reader in §3.

**Theorem 1.4:** If \( d >> 0 \) and \( Y_i, i = 1, \ldots, N \) are special subvarieties of \( G(r, k) \) of codimension \( c_i \) satisfying \( \sum_{i=1}^{N} c_i = dk - (g - 1)r(k - r) \), then the intersection number associated to \( \cap_{i=1}^{N} W_d(p_i, g_i Y_i) \) is independent of the choice of general \( g_i \in GL(V) \), and distinct points \( p_i \in C \).

The plan of the proof of Theorem 1.4 is as follows:

**Step 1:** To find a natural (irreducible, generically reduced) projective scheme compactifying the space of morphisms \( Mor_d(C, G) \), on which each subscheme \( W_d(p_i, g_i Y_i) \subset Mor_d(C, G) \) extends to a projective subscheme representing a chern class.

**Step 2:** To show that for general \( g_i \) and distinct \( p_i \), the intersection of the \( N \) extensions of the \( W_d(p_i, g_i Y_i) \) is contained in \( Mor_d(C, G) \), that is, there is no intersection at infinity. Thus the intersection number of the theorem is computable as an intersection of chern classes on the compactification. Changing \( g_i \) simply changes representatives of the chern classes, while changing the \( p_i \) deforms them, and the theorem follows.

Following [4], the compactification of \( Mor_d(C, G) \) we use is Grothendieck’s quot scheme. Namely, let \( Quot_d(C, G) \) be the projective scheme representing the functor:

\[
F(S) = \{ \text{flat families of quotients of } V \otimes \mathcal{O}_{C \times S} \text{ of relative Hilbert polynomial } d - (k - r)(g - 1) \text{ over } S. \}
\]

Let

\[
0 \to \mathcal{E} \to V \otimes \mathcal{O}_{C \times Quot} \to \mathcal{Q} \to 0
\]

be the universal exact sequence on \( C \times Quot_d(C, G) \). (See [4] for the details.)
While the universal quotient, $Q$, is a rather unpleasant sheaf, the kernel $E$ always locally free of rank $r$ and has degree $-d$ on the fibers over $Quot_d(C, G)$. So $Quot_d(C, G)$ may be thought of dually as parametrizing injective maps $E \to V \otimes O_C$ from vector bundles $E$ of degree $-d$ and rank $r$.

Of course, the morphisms to the Grassmannian are special cases of this where the map $E \to V \otimes O_C$ is injective on each fiber, and one easily sees that this determines $Mor_d(C, G)$ as an open subscheme of $Quot_d(C, G)$. It was shown in [1] moreover that for $d >> 0$, the quot scheme is irreducible and generically reduced, so $Mor_d(C, G)$ is dense.

Now suppose that $W \subset V^*$ is a plane of dimension $n$ and $Y \subset G(r, k)$ is the associated special subvariety. Given $p \in C$, we can define a corresponding degeneracy locus inside $Quot_d(C, G)$ as follows. Restrict the universal map $V^* \otimes O_{C \times Quot} \to E^*$ to $p \times Quot_d(C, G) \cong Quot_d(C, G)$. Let $E_p$ be the restriction of $E$ to $p \times Quot_d(C, G)$, and let $V_d(p, Y)$ be the degeneracy locus of the resulting map $W \otimes O_{Quot} \to E_p^*$.

It is immediate from the definitions that $V_d(p, Y) \cap Mor_d(C, G) = W_d(p, Y)$. Thus Step 1 is complete once we show that every component of $V_d(p, Y)$ has codimension $r+1-n$ in $Quot_d(C, G)$, since in that case $V_d(p, Y)$ will represent the $r+1-n$th chern class of $E_p^*$.

Let $Pl : G(r, k) \to P^n$ be the Plücker embedding of the Grassmannian (so $n = \binom{k}{r} - 1$). This induces an embedding of moduli spaces of morphisms $Pl : Mor_d(C, G) \hookrightarrow Mor_d(C, P^n)$, and this extends to a morphism of quot schemes $Pl : Quot_d(C, G) \to Quot_d(C, P^n)$ by sending an injective map $E \to O_C^k$ to the (injective) determinant map $\wedge^r E \to \wedge^r O_C^k$. One readily checks that this induces a map of quot functors, hence of the quot schemes.

The boundary $Quot_d(C, P^n) - Mor_d(C, P^n)$ decomposes as a disjoint union of locally closed subschemes $\sqcup_{0 < m \leq d} C_m \times Mor_{d-m}(C, P^n)$, where $C_m$ is the $m$th symmetric product of $C$. This is because any map $L \to O_C^k$ can be recovered from its divisor $D$ of zeroes and the induced map $L(D) \to O_D^k$.

As a result, the quot scheme decomposes as

$$Quot_d(C, G) = Mor_d(C, G) \sqcup_{0 < m \leq d} B_m$$

where $B_m = Pl^{-1}(C_m \times Mor_{d-m}(C, G))$. One checks that the fibers of the morphisms $Pl : B_m \to C_m \times Mor_{d-m}(C, G)$ all have dimension $(r-1)m$. In fact, the schemes $B_m$ are all fiber bundles over $Mor_{d-m}(C, G)$ with fiber over
Upon the choice of smooth curve of genus $g$, the intersection number of Theorem 1.4 does not depend on $g$. Proposition 1.5:

Let $Y$ be thought of as an invariant associated to the subvarieties $Y_1, ..., Y_N \subset G$, or equivalently to the chern classes of $S^*$. This proves Step 1, in particular.

But this setup also gives Step 2. Suppose that $W_d(p, g_i Y_i), i = 1, ..., N$ are chosen as in the Theorem, and consider $\cap_{i=1}^N V_d(p, g_i Y_i) \cap B_m$. In the analysis of the previous paragraph, for at most $m$ of the points $p_i$ may the intersection $V_d(p_i, g_i Y_i) \cap B_m$ lie in the scheme $Pl^{-1}(p_i + C_{m-1} \times Mor_{d-m}(C, G))$ because the points are distinct. The rest must lie in $Pl^{-1}(C_m \times W_{d-m}(p_i, g_i Y_i)).$ This means, after reordering the points, that the intersection $\cap_{i=1}^N V_d(p_i, g_i Y_i) \cap B_m$ is contained in $Pl^{-1}(C_m \times \cap_{i=1}^{N-m} W_{d-m}(p_i, g_i Y_i)).$

Suppose that $d_1$ is chosen so that $d > d_1$ implies that $\dim(Quot_d(C, G)) = kd - r(k - r)(g - 1)$. (Such a $d_1$ was proven to exist in [1]). Let $D = kd_1$. Then Step 2 follows from:

Claim: If $d > d_1 + D$ and $g_1 Y_1, ..., g_N Y_N$ are chosen as in the Theorem, then for all $0 < m \leq N$, $\cap_{i=1}^{N-m} W_{d-m}(p_i, g_i Y_i) = \emptyset$.

Proof: For $m \leq D$, this follows immediately from Corollary 1.2 since

$$\sum_{i=1}^{N-m} \text{codim}_G(Y_i) \geq \dim(Quot_d(C, G)) - mr \geq \dim(Quot_{d-m}(C, G))$$

If $m > D$, only the last equality may fail to hold, since $Quot_{d-m}(C, G)$ may have larger than the expected dimension, but in this case,

$$\dim(Quot_d(C, G)) - mr > D - r(k - r)(g - 1) \geq \dim(Quot_{d_1}(C, G)) \geq \dim(Quot_{d-m}(C, G))$$

and again Corollary 1.2 gives the claim.

Thus, the intersection number for $\cap_{i=1}^N W_d(p_i, g_i Y_i)$ of Theorem 1.4 may be thought of as an invariant associated to the subvarieties $Y_1, ..., Y_N \subset G$, or equivalently to the chern classes of $S^*$ represented by the $Y_i$.

Proposition 1.5: The intersection number of Theorem 1.4 does not depend upon the choice of smooth curve of genus $g$. 

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Proof: In light of the irreducibility of the moduli space of smooth curves of genus $g$, we have only to prove the following. For each family $C$ of smooth curves over a connected base $B$, the intersection numbers in Theorem 1.4 are the same for all fibers $C(b)$.

Let $Q_d$ be the relative quot scheme over the base $B$, with fibers isomorphic to $\text{Quot}_d(C(b), G)$. The fact that the quot schemes are all of the expected dimension ($d$ is assumed to be large) implies that the map $Q_d \to B$ is a complete intersection morphism, hence the family of quot schemes is flat over $B$ (see e.g. [7]).

Let $U \hookrightarrow V \otimes O_{C \times B}Q_d$ be the universal subbundle. After restriction and base change, we can find a section $\sigma$ of $C$ near each point $b \in B$, and the restriction of $U^*$ to $\sigma \times_B Q_d$ gives a varying (flat) family of vector bundles over a neighborhood of $b$. The Proposition now follows from:

Lemma 1.6: If $f : \mathcal{X} \to \mathcal{Y}$ is a projective, flat morphism of relative dimension $n$ over an irreducible base, and if $E$ is a vector bundle of rank $r$ on $\mathcal{X}$, then for any polynomial $P(X_1, ..., X_r)$ of weighted degree $n$, the intersection number $\int_{\mathcal{X}_y} P(c_1(E_y), ..., c_r(E_y))$ is independent of the point $y \in \mathcal{Y}$.

Proof: Let $O_{\mathcal{X}}(1)$ be a relatively ample line bundle on $\mathcal{X}$. If $L_1, ..., L_k$ are line bundles on $\mathcal{X}$ and $\sum_{i=1}^k a_i = n$, then it follows from the invariance of the Hilbert polynomials: $\chi(\mathcal{X}_y, \otimes L_i^{a_i}(n))$ that the intersection numbers $c_1(L_1)^{a_1} ... c_1(L_k)^{a_k} [\mathcal{X}_y]$ are independent of $y \in \mathcal{Y}$. If we apply this to the appropriate intersections of pullbacks of $O_{\mathbb{P}(1)}$’s to fiber products of projective bundles $\mathbb{P}(E) \times ... \times \mathbb{P}(E)$ over $\mathcal{Y}$, then we get invariance of the intersections of segre classes, hence of the chern classes as well.

Suppose that $X_1, ..., X_r$ are weighted variables such that the weight of $X_i$ is $i$. If $P(X_1, ..., X_r)$ is a homogeneous polynomial of weighted degree $kd - r(k-r)(g-1)$, then for $d >> 0$:

Definition: The genus-$g$ Gromov invariant $N_d(P(X_1, ..., X_r), g)$ is the intersection number of Theorem 1.4, extended by linearity from the monomials, where $X_i$ are replaced by special subvarieties of codimension $i$.

Warning: The Gromov invariants do not respect intersections! Namely, as noted following Corollary 1.3, it is possible to attach a Gromov invariant to any subvarieties $Y_1, ..., Y_N$ of $G$. In general, if $Y_1$ and $Y_2$, for example,
intersect transversally, then the Gromov invariant for $Y_1 \cap Y_2, Y_3, ..., Y_N$ will not coincide with the Gromov invariant for $Y_1, ..., Y_N$.

**Remark:** It was shown in [1] that one can define a consistent Gromov invariant for all degrees by downward induction from the observation that

$$N_d(P(X_1, ..., X_r), g) = N_{d-r}(X^k P(X_1, ..., X_r), g)$$

for sufficiently large $d$.

**Proposition 1.7:** If $C_0$ is an irreducible curve with a single node $\nu$ and arithmetic genus $g$, then the Gromov invariants for $C_0$ and special subvarieties of $G$ agree with the genus-$g$ Gromov invariants.

**Proof:** This time, we use a family of curves $\mathcal{C} \to B$ smoothing the nodal curve $C_0$, and use the invariance property of Lemma 1.6 to conclude that if $\mathcal{E} \hookrightarrow V \otimes \mathcal{O}_{C_0 \times Quot}$ is the universal (torsion-free) subsheaf and $p \in C_0 - \nu$, then the intersection numbers of chern classes of the (locally free) sheaf $\mathcal{E}_p$ are the same as the corresponding intersection numbers in the smooth case.

The argument proceeds just as in the proofs of Theorem 1.4 and Proposition 1.5. That is, we need to show that:

1. For $d \gg 0$, the quot scheme $Quot_d(C_0, G)$ is irreducible and generically reduced of the expected dimension $dk - r(k - r)(g - 1)$.

2. For $d \gg 0$, the degeneracy loci $V_d(p_i, g_i Y_i)$ represent chern classes of $\mathcal{E}_p$, and the intersections $\cap_{i=1}^N V_d(p_i, g_i Y_i)$ and $\cap_{i=1}^N W_d(p_i, g_i Y_i)$ are the same.

We need to use the following result on the structure of torsion-free sheaves on $C_0$, which may be found, for example, in [11]. Namely, if $E$ is a torsion-free sheaf, then the cokernel, $T_\nu$, of the double-dual exact sequence:

$$0 \to E \to E^{**} \to T_\nu \to 0$$

is a sheaf (necessarily of rank $< r$) supported on the node $\nu$.

In other words, the torsion-free sheaves of rank $r$ and degree $-d$ are identified with the locally free sheaves $F$ of rank $r$ and degree $-d + m$, together with a quotient of the fiber $F(\nu)$ of rank $m \leq r$.

This implies that the quot scheme $Quot_d(C_0, G)$ decomposes as a disjoint union of locally closed subschemes:

$$Quot_d(C_0, G) = Mor_d(C_0, G) \sqcup_{m=1}^d B_m \sqcup_{m=1}^r A_m$$
where $P \colon B_m \to \text{Mor}_{d-m}(C_0, G)$ is smooth, of relative dimension $mr$ as before, and the new pieces $A_m$ are $G(m, r)$-bundles over $\text{Quot}_{d-m}(C_0, G) := \text{Mor}_{d-m}(C_0, G) \sqcup_{\bigcup_{i=1}^{d-m} B_i}$, the subscheme of $\text{Quot}_{d-m}(C_0, G)$ parametrizing locally-free subsheaves $E \hookrightarrow V \otimes O_{C_0}$.

The same argument as in the proof of Theorem 1.4 will give (1) and (2) once we show:

Claim: $\text{Mor}_d(C_0, G)$ is of the expected dimension for $d >> 0$.

Proof: Let $f : \tilde{C}_0 \to C_0$ be the normalization of $C_0$. It suffices to show that the image of the obvious embedding: $\iota : \text{Mor}_d(C_0, G) \hookrightarrow \text{Mor}_d(\tilde{C}_0, G)$ has codimension $r(k - r)$ for $d >> 0$.

The codimension of the locus $\{E \to V \otimes O_{\tilde{C}_0} | H^1(\tilde{C}_0, E^*(-p - q)) \neq 0\}$ in $\text{Mor}_d(\tilde{C}_0, G)$ grows with $d$. This follows, for example, from the proof of Theorem 4.28 in [1]. In particular, for large $d$, it exceeds $r(k - r)$, and thus this locus may be ignored.

On the other hand, if $H^1(\tilde{C}_0, E^*(-p - q)) = 0$, then the natural map $\text{Hom}(E, V \otimes O_{\tilde{C}_0}) \to \text{Hom}(E_p, V) \oplus \text{Hom}(E_q, V)$ is surjective, and the claim follows by pulling back the diagonal from the product of Grassmannians.

2. Induction on the Genus. The universal exact sequence implies that the Chern classes of the universal quotient bundle $Q$ on $G$ are all expressible in terms of $c_1(S^*), ..., c_r(S^*)$, and the Chern classes of the tangent bundle $TG = S^* \otimes Q$ are therefore expressed in terms of $c_1(S^*), ..., c_r(S^*)$ by the standard formulas for the Chern classes of a tensor product. (See §3 for more details.) In particular, we get in this way an “euler” polynomial $e(X_1, ..., X_r)$ of weighted degree $r(k - r)$, with the property that when evaluated at the Chern classes of $S^*$, $e$ produces the euler class, that is, the top Chern class of $TG$. The point of this section is to prove the following induction formula:

**Theorem 2.1:** The Gromov invariants for genus $g$ and $g - 1$ are related by:

$$N_d(P(X_1, ..., X_r, g) = N_d(e(X_1, ..., X_r)P(X_1, ..., X_r), g - 1)$$

The idea of the proof is as follows. We have already seen in Proposition 1.7 that the intersection $\cap_{i=1}^{N} W_d(p_i, g_i Y_i)$ for an irreducible curve $C_0$ with one node and arithmetic genus $g$ computes the Gromov invariant for genus $g$. On the other hand, the scheme $\text{Mor}_d(C_0, G)$ embeds in $\text{Mor}_d(\tilde{C}_0, G)$ as
the subscheme parametrizing morphisms which send $p$ and $q$ to the same point. If the other points $p_i \in C_0$ are identified with their preimages in $\tilde{C}_0$, then the schemes $W_d(p_i, g_i Y_i) \subset \text{Mor}_d(C_0, G)$ are just the intersections of the corresponding subschemes in $\text{Mor}_d(\tilde{C}_0, G)$ with the image of $\text{Mor}_d(C_0, G)$.

The proof of Theorem 2.1 consists in showing that $\text{Mor}_d(C_0, G)$ extends to a subscheme of $\text{Quot}_d(\tilde{C}_0, G)$ which represents the pullback from $G$ of the euler polynomial in $c_i(S^*)$, and that there are no “intersections at infinity” when this scheme is intersected with the $V_d(p_i, g_i Y_i)$.

The first problem we encounter is the fact that $\text{ev}_p^*(Q)$ does not extend as a vector bundle to the quot scheme.

Definitions: (a) Let $G^* = G(k - r, k)$ be the Grassmannian of $k - r$-dimensional subspaces of $V^*$ and let $F \hookrightarrow V^* \otimes \mathcal{O}$ be the universal subbundle on $C \times \text{Quot}_d(C, G^*)$.

(b) Let $\Gamma \subset \text{Quot}_d(C, G) \times \text{Quot}_d(C, G^*)$ be the closure of the image of $\text{Mor}_d(C, G) = \text{Mor}_d(C, G^*)$ via the two embeddings. Let the two projections from $\Gamma$ be named $\pi : \Gamma \to \text{Quot}_d(C, G)$ and $\pi^* : \Gamma \to \text{Quot}_d(C, G^*)$.

(c) Let $Z_{p,q}$ be the zero scheme in $\Gamma$ of the canonical map $E_p \to F_q^*$ (which factors through $V \otimes \mathcal{O}_{C \times \Gamma}$).

Note that $\text{Mor}_d(C_0, G)$ is identified with an open subscheme of $Z_{p,q}$ via $\pi^{-1}$.

Let $M(X_1, ..., X_r)$ be a monomial of weighted degree $kd - r(k - r)(g - 1)$ with corresponding special subvarieties $Y_1, ..., Y_N$. Let $c_i = c_i(E_p^*)$ for some point $r \in \tilde{C}_0$. The theorem follows immediately from:

Lemma 2.2: (a) The following degrees are the same:

$$M(c_1, ..., c_r)[Z_{p,q}] = e(c_1, ..., c_r)M(c_1, ..., c_r).[\text{Quot}_d(\tilde{C}_0, G)]$$

(b) For general $p_1, ..., p_N \in \tilde{C}_0$ and $g_1, ..., g_N \in GL(V)$,

$$Z_{p,q} \cap_{i=1}^N V_d(p_i, g_i Y_i) = \text{Mor}_d(C_0, G) \cap_{i=1}^N W_d(p_i, g_i Y_i)$$

Proof of (a): Since $Z_{p,q}$ is the zero locus of a section $\sigma \in \text{Hom}(E_p, F_q^*)$, it follows that if $Z_{p,q}$ is irreducible, of codimension $r(k - r)$, then $Z_{p,q}$ represents the top chern class of $E_p^* \otimes F_q^*$. Moreover, since $E_p^* \otimes F_q^*$ coincides with the
pullback $ev_p^* S^* \otimes ev_q^* Q$ from $G$ over the open subscheme $\text{Mor}_d(\tilde{C}_0, G)$, any difference between $Z_{p,q}$ and the pull-back of the euler polynomial would be concentrated in higher codimension than $r(k - r)$, hence can be ignored.

The same argument as in the proof of Proposition 1.7 shows that $Z_{p,q}$ intersects $\text{Mor}_d(\tilde{C}_0, G)$ in codimension $r(k - r)$. As with Step 1 in the proof of Theorem 1.4, we next need to analyze the intersection of $Z_{p,q}$ with the boundary subschemes $\pi^{-1}B_m = \pi^{-1}Pl^{-1}(C_m \times \text{Mor}_{d-m})$ for $1 \leq m \leq d$. A new argument is required only for the intersections of $Z_{p,q}$ with the open subschemes of $\text{Hom}(\text{Hom}(\tilde{C}_0, G))$ parametrizing maps $E \hookrightarrow V \otimes \mathcal{O}_C$ which have rank $r - a$ at $p$ and $r - b$ at $q$.

If $F$ is a vector bundle on $\tilde{C}_0$ of degree $-d + a + b$ with $H^1(F^*(-p-q)) = 0$, then the codimension in

$$\{s : F \hookrightarrow V \otimes \mathcal{O}|rk(s_p) = rk(s_q) = r\}$$

of the set of such $s$ so that $Z_{p,q} \cap \pi^{-1}Pl^{-1}(s) \neq \emptyset$ is the same as the codimension in $G(r, k) \times G(r, k)$ of $\{(\Lambda, \Lambda')|\dim(\Lambda \cap \Lambda') \geq r - a - b\}$, which is $(k - r - a - b)(r - a - b)$. From this it follows that the codimension of $Z_{p,q} \cap \pi^{-1}Pl^{-1}(ap + bq \times \text{Mor}_{d-a-b})$ in $\Gamma$ is $r(k - r) + a^2 + b^2$.

This argument readily adapts to show that all other intersections with boundary components also have codimension larger than $r(k - r)$, and part (a) follows.

**Proof of (b):** As in the claim in the proof of Theorem 1.4, we may assume that all $\text{Mor}_{d-m}(\tilde{C}_0, G)$ are of dimension $dk - r(k - r)(g - 1) - mk$. But then by the analysis of the previous paragraph, the locus in $ap + bq \times \text{Mor}_{d-a-b}(\tilde{C}_0, G)$ over which $Z_{p,q}$ intersects nontrivially has codimension $r(k - r) - (a + b)k + (a + b)^2$, and as in the proof of Step 2, this fact, together with Corollary 1.2, implies part (b).

3. **Quantum Schubert Calculus.** Both the Chow ring and the cohomology ring of $G(r, k)$ are isomorphic to $\mathbb{C}[X_1, \ldots, X_r]/I$, where $I$ is the ideal generated by the coefficients of $t^{k-r+1}, \ldots, t^k$ in the formal-power-series inverse of

$$P_t := 1 + X_1 t + X_2 t^2 + \ldots + X_r t^r$$

The isomorphisms may be realized by identifying $X_i$ with the $i$th chern class of $S^*$. In particular, the $X_i$ are represented by the special Schubert subvarieties.
If $X_i$ is assigned weight $i$ and if $P(X_1, ..., X_r)$ is a polynomial of weighted degree $r(k-r) = \dim(G(r, k))$, then the computation of $\int_G P(c_1(S^*), ..., c_r(S^*))$ is an intersection of special subvarieties, and is a special case of the Schubert calculus on the Grassmannian. The classical Pieri formulas compute these integers (see [3] or [4]), but a recent formula due to Vafa and Intriligator ([6]) generalizes readily to the quantum case.

Let $Q_t = 1 + \sum_{m=1}^{\infty} y_m t^m$ be the formal-power-series inverse of $P_t$. Then the polynomials $y_m(X_1, ..., X_r)$ can all be computed from the expansion

$$\log(P_t) = \sum_{n=0}^{\infty} W_n(X_1, ..., X_r)t^n.$$ 

Namely, differentiating both sides with respect to $X_i$ and equating powers of $t^n$, one finds that $y_{n-i} = \frac{\partial W_n}{\partial X_i}$ for $1 \leq i \leq r$.

In particular, (plus or minus) the coefficients $y_{k-r+1}, ..., y_{k}$, which generate all of the relations in the cohomology ring, are the partial derivatives of a single polynomial $W := (-1)^k W_{k+1}(X_1, ..., X_r)$. It is immediate that $W$ has weighted degree $k + 1$. Indeed, if $q_1, ..., q_r$ are the chern roots of $P_t$, that is $q_1, ..., q_r$ are formal variables satisfying $P_t = \prod_{i=1}^{r}(1 + q_i t)$, then in terms of these chern roots, we may write

$$W = \sum_{i=1}^{r} \frac{q_i^{k+1}}{k+1}.$$ 

If we define the perturbed polynomial

$$\tilde{W} := W + X_1,$$

then the zeroes of the $X_i$-gradient $\nabla_X \tilde{W}$ are distinct and reduced. Indeed, the zeroes of the $q_i$-gradient $\nabla_q \tilde{W}$ are reduced, and occur when each $q_i$ is a $k$th root of $-1$. The change of variables coming from the elementary symmetric polynomials $X_i = \sigma_i(q_1, ..., q_r)$ implies that the zeroes of $\nabla_X \tilde{W}$ are reduced and that there are $\binom{k}{r}$ of them (one for each unordered $r$-tuple of distinct $k$th roots of $-1$).

We let

$$h(X_1, ..., X_r) = \det \left( \frac{\partial^2 \tilde{W}}{\partial X_i \partial X_j} \right)$$

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be the Hessian polynomial. Then Vafa and Intriligator tell us:

**Proposition 3.1:** The Schubert calculus for special subvarieties of $G(r, k)$ is computed by the following formula:

$$(*) \quad \int_{G} P(c_1(S^*), \ldots, c_r(S^*)) = (-1)^{\binom{r}{2}} \sum_{\nabla W = 0} P h^{-1}$$

where $P(X_1, \ldots, X_r)$ is a homogeneous polynomial of weighted degree $r(k - r)$.

**Proof:** Since both the left and right-hand sides of $(*)$ are linear in $P$, and $H^{r(k-r)}(G, \mathbb{C})$ is one-dimensional, it suffices to show:

1. $(*)$ holds and is nonzero for one choice of $P$.
2. If $P \in I$, where $I$ is the ideal of relations among the $c_i(S^*)$, then the right-hand side evaluates to zero.

The evaluation of $h(X_1, \ldots, X_r)$ (or $h^{-1}(X_1, \ldots, X_r)$) on the zeroes of $\nabla W$ may be written in terms of the $q_i$’s as follows. The determinant of the Jacobian matrix $J$ associated to the elementary symmetric functions $\sigma_i(q_1, \ldots, q_r)$ is the Vandermonde determinant $\prod_{i<j}(q_i - q_j)$, so $\nabla W = (\nabla W)(J^{-1})$, and when evaluated at the critical points $\nabla W = 0$, we have

$$h(X_1, \ldots, X_r) = \det \left( \frac{\partial^2 W}{\partial q_i \partial q_j} \right) \det(J^{-2}) = \frac{k^r \prod_{i=1}^{r} q_i^{k-1}}{(\prod_{i<j}(q_i - q_j))^2}$$

We verify (1) for the choice of $P(X_1, \ldots, X_r) = X_r^{k-r}$. The Schubert cycle $Y \subset G(r, k)$ consisting of the zero locus of a section $O_G \to S^*$ represents $c_r(S^*)$, and $k - r$ generally chosen sections yield Schubert cycles intersecting in a single point, corresponding to the plane $W \subset V^*$ spanned by the sections. Thus, $\int_G c_r^{k-r}(S^*) = 1$, that is, this choice of $P$ gives the volume form on the Grassmannian.

We compute the right-hand-side using the $q$ variables. Namely,

$$\sum_{\nabla W = 0} X_r^{k-r} h^{-1} = \sum_{\nabla W = 0} \prod_{i=1}^{r} q_i^{k-r} \frac{(\prod_{i<j}(q_i - q_j))^2}{k^r \prod_{i=1}^{r} q_i^{k-1}}$$

$$= \frac{1}{r! k^r} \sum_{(-q_i)^k = -1} \prod_{i=1}^{r} q_i^{1-r} (\prod_{i<j}(q_i - q_j))^2$$

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Only the term $q_i^{r-1} \cdots q_r^{r-1}$ in $(\prod_{i<j} (q_i - q_j))^2$ contributes nontrivially to the sum. This term appears with coefficient $(-1)^{r(r-1)/2} (r!)$, yielding $(-1)^{r(r-1)/2}$ as the sum, as desired.

To verify (2), it is enough by linearity to show that if $P = \left( \frac{\partial W}{\partial X_i} \right) N$, then

$$\sum \nabla W = 0$$

But $\frac{\partial W}{\partial X_i} = \frac{\partial W}{\partial X_i} + 1$, it follows that if $P = \left( \frac{\partial W}{\partial X_i} \right) \cdot N$, then

$$\sum \nabla W = 0$$

and when written in terms of the $q_i$'s, this sum is easily seen to be zero.

**Corollary 3.2:** The euler class $c_{r(k-r)}(TG)$ and the (modified) hessian polynomial $\left( -1 \right)^{\binom{r}{2}} h(c_1(S^*), ..., c_r(S^*))$ determine the same element of $H^*(G)$.

**Proof:** We have determined that the number of zeroes of $\nabla W$ is $\left( \binom{k}{r} \right)$. On the other hand, (23.2.1) of [3] states that the Poincaré series of $G(r; k)$ is:

$$(1 - t^2) ... (1 - t^{2k})/(1 - t^2) ... (1 - t^{2r})(1 - t^2)...(1 - t^{2(k-r)})$$

Thus it follows from Proposition 2.1 that:

$$-1^{\binom{r}{2}} \int_G h(c_1(S^*), ..., c_r(S^*)) = \binom{k}{r} = \int_G c_{r(k-r)}(TG)$$

This says that the euler class, as a polynomial in the chern classes of $S^*$, and the modified hessian are the same modulo the ideal generated by $\frac{\partial W}{\partial X_1}, ..., \frac{\partial W}{\partial X_r}$. But the following improvement will show that they are in fact the same in the quantum cohomology ring.

**Proposition 3.3:** The euler polynomial and $\left( -1 \right)^{\binom{r}{2}} h$ are the same at the zeroes of $\nabla W$.

**Proof:** The splitting principle together with the tensor product formula imply that

$$e(X_1, ..., X_r) = c_{r(k-r)}(S^* \otimes Q) = \prod_{i=1}^r \left( q_i^{k-r} + q_i^{k-r-1} c_1(Q) + ... + c_{k-r}(Q) \right)$$

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Thus it follows from Proposition 2.1 that:

$$-1^{\binom{r}{2}} \int_G h(c_1(S^*), ..., c_r(S^*)) = \binom{k}{r} = \int_G c_{r(k-r)}(TG)$$

This says that the euler class, as a polynomial in the chern classes of $S^*$, and the modified hessian are the same modulo the ideal generated by $\frac{\partial W}{\partial X_1}, ..., \frac{\partial W}{\partial X_r}$. But the following improvement will show that they are in fact the same in the quantum cohomology ring.

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When evaluated at the points where $\nabla \tilde{W} = 0$, we have:

\[
\prod_{j=1}^{1}(1-q_j t) = \frac{1}{c_1(S)} = 1 + c_1(Q)t + \cdots + c_{k-r}(Q)t^{k-r} + 0 + \cdots + 0 - t^k - c_1(Q)t^{k+1} - \ldots
\]

\[
= (1 + c_1(Q)t + \cdots + c_{k-r}(Q)t^{k-r})(1 + t^k)^{-1}
\]

If we take residues of both sides at $t = q_i^{-1}$, where $q_i$ is a kth root of $-1$, we get:

\[
\prod_{j \neq i}(1-q_j q_i^{-1}) = \frac{1}{k}(1 + c_1(Q)q_i^{-1} + \cdots + c_{k-r}(Q)q_i^{-(k-r)})
\]

Multiplying by $k q_i^{k-r}$ and taking the product over all $i$, the right hand side gives the euler polynomial by (\ref{euler polynomial}), but the left hand side gives $(-1)^{(\frac{k}{2})}$ times the formula for the hessian in terms of the $q_i$’s derived earlier.

Putting together the results of this paper with \cite{1}, we get the following formula, first conjectured by Vafa and Intriligator in \cite{6}:

**Theorem 3.4:** The Gromov invariants for $G = G(2, k)$ are computed by the formula:

\[
N_d(P(X_1, X_2), g) = (-1)^{g-1} \sum_{\nabla W = 0} P(X_1, X_2)h(X_1, X_2)^{g-1}
\]

**Proof:** As already mentioned, the main result of \cite{1} shows that the Theorem holds for curves of genus one, and the arguments therein also imply the genus zero case. For higher genera, we use Theorem 2.1, Proposition 3.3 and the result for genus one to get:

\[
N_d(P, g) = N_d(P e^{g-1}, 1) = \sum_{\nabla W = 0} P e^{g-1} = (-1)^{g-1} \sum_{\nabla W = 0} P h^{g-1}
\]

as desired.

The full conjecture of Vafa and Intriligator states that the analogous formula computes the genus-$g$ Gromov invariants on any Grassmannian. Namely,
Conjecture: If $P(X_1,\ldots,X_r)$ is of weighted degree $kd - r(k - r)(g - 1)$, then

$$N_d(P(X_1,\ldots,X_r),g) = (-1)^{(g-1)(\binom{r}{2})} \sum_{\nabla W = 0} P h^{g-1}$$

In light of Theorem 2.1 and Proposition 3.3, it would suffice to verify this conjecture in the genus zero case to get the result for all genera. Since the genus zero quot scheme is smooth and indeed has been described quite explicitly by Strømme in [12], it may be possible to verify this directly. Indeed, it was recently shown in [9] by direct methods that the degree of the genus zero quot scheme, namely, $N_d(X_1^{kd+r(k-r)},0)$, agrees with the conjectured value.

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