A proof of Björner’s totally nonnegative conjecture

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Abstract

The McMullen Correspondence gives a linear dependence between M-sequences of length \( \left\lfloor \frac{d}{2} \right\rfloor + 1 \) and f-vectors of simplicial d-polytopes. Denote the transfer matrix between g and f by \( M_d \). Recently, Björner proved that any 2×2-minor of \( M_d \) is nonnegative and conjectured that the same would be true for arbitrary minors. In this paper we answer the question in the affirmative.

1 Introduction

Let \( P \) be a d-dimensional simplicial polytope, and let \( f_i \) denote the number of \( i \)-dimensional faces. The nonnegative integer vector

\[
\mathbf{f} = (f_0, \ldots, f_{d-1})
\]

is called the f-vector of \( P \). We assume the convention \( f_{-1} = f_d = 1 \). The Euler-Poincaré formula gives a first rough estimate on the dependence between the elements of \( f \),

\[
-f_{-1} + f_0 - f_1 + \ldots + (-1)^d f_d = 0.
\]

Define the numbers \( g_k = h_k - h_{k-1} \) for \( k = 0, \ldots, \left\lfloor \frac{d}{2} \right\rfloor \), where

\[
h_i = \sum_{j=0}^{i} (-1)^{i+j} \binom{d-j}{i-j} f_{j-1}, \quad i = 0, \ldots, d.
\]

The nonnegative vector \( \mathbf{g} = (g_0, \ldots, g_{\left\lfloor \frac{d}{2} \right\rfloor}) \) will be referred to as the g-vector of the simplicial d-polytope \( P \). The McMullen correspondence asserts that the map

\[
g \mapsto g \cdot M_d,
\]

where \( M_d \) is a nonnegative \( \left( \left\lfloor \frac{d}{2} \right\rfloor + 1 \right) \times d \) matrix given by

\[
m_{ij} = \binom{d+1-i}{d-j} - \binom{i}{d-j}, \quad i = 0, \ldots, \left\lfloor \frac{d}{2} \right\rfloor \quad \text{and} \quad j = 0, \ldots, d-1,
\]
is a bijection between $M$-sequences $g$ with $g_1 = n - d - 1$ and $f$-vectors in $\mathbb{N}_0^n$ of simplicial $d$-polytopes with $n = g_1 + d + 1$ vertices. Recall that a sequence $n_0, n_1, \ldots,$ of nonnegative integers is a $M$-sequence if $n_0 = 1$ and

$$\partial^k(n_k) \leq n_{k-1} \quad \text{for all } k > 1,$$

where $\partial^k$ is the “$k$-boundary”-operator (see e.g. [8], 262). A theorem by Macaulay [6] gives an algebro-combinatorial characterization of such sequences.

This paper is devoted to the analysis of the transfer matrices $M_d$ above. In dimensions $d = 1, 2,$ and $3$, they are given by

$$M_1 = \begin{pmatrix} 1 & 2 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 4 & 6 & 4 \\ 0 & 1 & 3 & 2 \end{pmatrix}.$$

Björner proved in [2] that every $2 \times 2$-minor in $M_d$ is nonnegative. In the same paper he conjectured that any minor in $M_d$ is nonnegative (such a matrix is called totally nonnegative). The conjecture has been verified up to dimension $d = 13$ by A. Hultman. In this paper we settle the conjecture in the affirmative for all dimensions.

## 2. Lattice paths and nonnegative minors

A path from $(x_1, y_1)$ to $(x_2, y_2)$ in $\mathbb{Z}^2$, where $x_1 \leq x_2$ and $y_1 \leq y_2$, is called a lattice path if only the steps $(1, 0)$ and $(0, 1)$ are allowed. The number of lattice paths from $(0, 0)$ to $(m, n)$ which do not touch the line $y = x + t$ are $\binom{m+n}{n} - \binom{m+n}{m-t}$ if $t > 0$ [1, 7]. This is sometimes referred to as the ballot numbers.

The weight of a path is the product of the weights of its arcs. From any subset $A$ of $\mathbb{Z}^2$ one can construct a planar acyclic directed graph with vertex set $A$ and arcs of types $(x, y) \to (x + 1, y)$ and $(x, y) \to (x, y + 1)$.

**Theorem 2.1** The matrix $M_d$ is totally nonnegative for all $d$.

**Proof:** For integers $n \geq 2$ define the graphs $T_n$ with vertex set

$$\{(x, y) \in \mathbb{Z}^2 \mid x \leq \lfloor n/2 \rfloor - 1, y - x \leq \lfloor n/2 \rfloor, \text{ and } x + y \geq \lfloor n/2 \rfloor - 1\}.$$

The weight of horizontal arcs in $T_n$ is 1, and the weight of any vertical arc $(x, y) \to (x, y + 1)$ is $w_y$. The graphs $T_8$ and $T_9$ are depicted in figures 1 and 2.

For all $0 \leq i \leq \lfloor n/2 \rfloor - 1$ and $0 \leq j < n$, let the sum of the weights of the directed paths from $([n/2] - 1 - i, i)$ to $([n/2] - 1, j)$ be $W(n, i, j)$.

If $i > j$ then $W(n, i, j) = 0$ and otherwise it is

$$w_i w_{i+1} \cdots w_{j-1} \left( \binom{j}{i} - \binom{j}{n-i} \right)$$

by the ballot numbers.
Figure 1: The weighted planar graph $T_8$
Figure 2: The weighted planar graph $T_9$
Now define the weights of the vertical arcs as \( w_i = \binom{n}{i+1}/\binom{n}{i} \). Note that all arc weights are positive real numbers. For \( i \leq j \) we get that

\[
W(n, i, j) = \binom{n}{i+1} \ldots \binom{n}{i+1} \binom{n}{j-i-1} \ldots \binom{n}{j-i-1}
\]

\[
\frac{\binom{n}{j-i-1} \ldots \binom{n}{j-i-1}}{\binom{n}{j-i-1} \ldots \binom{n}{j-i-1}} = \binom{n}{j} \binom{n}{i} - \binom{n}{j} \binom{n}{i}.
\]

The values of \( W(n, i, j) \) are tabulated in the \([n/2] \times n \) matrix \( W_n \):

\[
\begin{pmatrix}
\binom{n}{0} - \frac{0}{0} & \binom{n}{0} - \frac{0}{0} & \ldots & \binom{n}{0} - \frac{0}{0} \\
0 & \binom{n}{1} - \frac{1}{1} & \ldots & \binom{n}{1} - \frac{1}{1} \\
0 & 0 & \ldots & \binom{n}{2} - \frac{2}{2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \binom{n}{[n/2]+1} - \frac{[n/2]+1}{[n/2]+1} \\
\end{pmatrix}
\]

By removing the leftmost column of \( W_n \), we get \( M_{n-1} \). Thus it is sufficient to prove that \( W_n \) is totally nonnegative to conclude the same about \( M_{n-1} \).

Fomin and Zelevinsky wrote a nice survey on testing total positivity and related questions \[4]. We need a result by Lindström \[5\], and Gessel and Viennot \[6\].

**Lemma 2.2** If a weighted acyclic directed planar graph has nonnegative real weights, then its weight matrix is totally nonnegative.

The weight matrix \( X \) of a weighted acyclic directed planar graph \( G \), given a set \( I \) of sinks and \( J \) of sources of \( G \), is a matrix with the rows indexed by \( I \) and the columns indexed by \( J \). On the position \( i, j \) of \( X \), where \( i \in I \) and \( j \in J \), is the sum of the weights of all paths from the source \( i \) to the sink \( j \).

The graph \( T_n \), with its \([n/2]\) sources and \( n \) sinks described earlier gives the weight matrix \( W_n \). All weight are nonnegative, and hence \( W_n \) and its submatrix \( M_{n-1} \) are totally nonnegative. \(\Box\)

**References**

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