On the Distribution of Range for Tree-Indexed Random Walks

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Abstract

We study tree-indexed random walks for spiders, trees with one vertex of degree greater than two. Our main result confirms a conjecture of Benjamini, Häggström, and Mossel for such graphs, namely that the distribution of the range for any such tree is dominated by that of a path on the same number of edges.

1 Introduction

In 2000, Benjamini, Häggström, and Mossel \(^1\) began the study of random graph homomorphisms into \(\mathbb{Z}\), alternatively known as graph-indexed random walks. For a graph \(G = (V, E)\) with distinguished vertex \(v_0 \in E\), the \(G\)-indexed walks are labelings of the following form:

\[ \mathcal{F}(G, v_0) := \{ f : V \to \mathbb{Z} \mid f(v_0) = 0, \{u, v\} \in E \implies |f(u) - f(v)| = 1 \}. \]

As defined, such walks only exist when \(G\) is bipartite, although other variants have been considered that are defined for all graphs. We will focus only on the original notion, however. Benjamini, Häggström, and Mossel let \(f\) be a \(G\)-indexed walk chosen uniformly at random from \(\mathcal{F}\), and study properties such as the expected distance between a fixed pair of vertices and the expected range. Note that both of these quantities are invariant when adding a constant to all labels in a labeling of \(G\), and as such are independent of the choice of \(v_0\). We can then ignore the information of the choice of \(v_0\) and simply refer to the space of labelings as \(\mathcal{F}(G)\).

When comparing walks on different graphs, intuition would suggest that graph-indexed random walks on paths would be likely to have the largest range, and that adding more edges to a graph would necessarily bring vertices closer together in expectation. This second statement is not always true: Benjamini et al. exhibit a graph \(G\) with two vertices \(u\) and \(v\), such that \(\mathbb{E}(|f(u) - f(v)|)\) actually increases upon adding an edge to \(G\). Despite this, they also show that \(\mathbb{E}(|f(u) - f(v)|)\) increases when \(G\) is pared down to any path from \(u\) to \(v\), as one would expect. Moreover, a stronger statement holds, namely that the distribution is stochastically dominated by that of a path:

**Theorem 1** (\(^1\), Theorem 2.8). Let \(G = (V, E)\) be a bipartite, connected, finite graph, let \(u, v \in V\) and let \(P\) be any path from \(u\) to \(v\) in \(G\). Then for all \(k\),

\[
\mathbb{P}_{f \in \mathcal{F}(G)} (|f(u) - f(v)| \geq k) \leq \mathbb{P}_{f \in \mathcal{F}(P)} (|f(u) - f(v)| \geq k).
\]

(1)

Stochastic domination is equivalent to stating that for any increasing \(g\), the expectation of \(g(|f(u) - f(v)|)\) is greater for a path than for any other graph. Taking \(g(x) = x\) yields the weaker result that the expected difference

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between the labels of $u$ and $v$ is larger for a path, but domination also implies that quantities such as the expected squared distance are larger for the path as well.

For the purposes of this paper we define the range of a graph labeling $\text{Range}(f) := \max_{u,v \in V} f(u) - f(v)$.

Benjamini, Häggström, and Mossel make the following two conjectures regarding the range of a graph-indexed random walk:

**Conjecture 2** ([1], Conjecture 2.10). Let $G$ be a simple connected graph on $n$ vertices and let $P$ be the path on $n$ vertices. Then:

- (Weak). $\mathbb{E}_{f \in \mathcal{F}(G)} \text{Range}(f) \leq \mathbb{E}_{f \in \mathcal{F}(P)} \text{Range}(f)$.
- (Strong). $\Pr_{f \in \mathcal{F}(G)}(\text{Range}(f) \geq k) \leq \Pr_{f \in \mathcal{F}(P)}(\text{Range}(f) \geq k)$, for all $k$.

In the literature, there has been some progress made on the weak conjecture, and no progress made on the strong conjecture. Wu, Xu, and Zhu [10] resolve the weak conjecture in the affirmative for trees, and Bok and Nešetřil [5] extend this work to confirm the weak conjecture for unicyclic graphs. Loebl, Nešetřil, and Reed [9] show that the expected range for any graph is bounded by some absolute constant multiple of the expected range of a path. The main result of our paper will be to resolve the strong conjecture in the affirmative for spiders, trees with (at most) one vertex with degree greater than 2:

**Theorem 3.** Let $T$ be a spider on $n$ vertices and $P$ be the path on $n$ vertices. Then for all $k$,

$$\Pr_{f \in \mathcal{F}(T)}(\text{Range}(f) \geq k) \leq \Pr_{f \in \mathcal{F}(P)}(\text{Range}(f) \geq k).$$

1.1 Remarks

The above definitions and conjectures are even more natural when restricted to trees. In the case of trees on $n$ vertices, there are always $2^{n-1}$ elements of $\mathcal{F}(T)$, and consequently the computations of probabilities are replaced by enumerations of sets. The case of a tree-indexed random walk had been studied before the introduction of $G$-indexed random walks, although this earlier work was concentrated on infinite trees (for example, [2, 3]). Regarding graph homomorphisms specifically, much of the literature so far has been asymptotic and hence does not provide the exact precision required to show domination of distributions (see [4, 7, 8]). In addition to the work mentioned above, Csikvári and Lin [6] study random graph homomorphisms from trees into paths, the number of which is counted (in our notation) by $F^k(T)$, a key quantity we work with in the body of this paper.

It is also worth remarking that while our main theorem only resolves the case of spiders, there is reason to hope that this will be a significant portion of the work in resolving the full strong range conjecture for trees. When Wu, Xu, and Zhu resolve the weak conjecture for trees, the bulk of the proof lies in considering a vertex with degree at least three such that two of the subtrees attached to this vertex are paths. They show that the expected range of $T$ is increased when this pair of paths is replaced by a single path of length equal to the sum of the original two. Inductively, they are then able to push the tree closer and closer to a path, increasing the range at each step. Our proof of the strong conjecture for spiders works in exactly the same way, singling out two “legs” of the spider to combine, and showing that the distribution of range changes in the correct way.

Despite this, we are quite confident that naive inductive arguments will not work. Let $T$ be the tree with 7 edges given by taking a path of 3 edges and appending a pair of leaves to both endpoints. This tree has two vertices of degree 3, but is not dominated by any other tree with seven edges other than the path of length seven. Consequently, no inductive argument that considers only one high-degree vertex at a time will be sufficient to handle this tree.
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2 Preliminaries

Definition 1. For a given tree $T$, let $F^k(T)$ be the number of labelings of $T$ with integers from 0 to $k$ such that adjacent vertices are labeled with consecutive integers. Such labelings will be referred to as “valid.”

Definition 2. For a given tree $T$, let $f^k(T)$ be the number of labelings of $T$ (with, say, integers from 0 to $k$) such that adjacent vertices are labeled with connected integers, up to equivalence by translation.

Remark. We may now restate the strong range conjecture as: $f^k(T) \geq f^k(P)$ for all $k$.

Proposition 4. $f^k(T) = F^k(T) - F^{k-1}(T)$.

Proof. Since every valid labeling bounded by $k - 1$ is also a valid labeling bounded by $k$, $F^k(T) - F^{k-1}(T)$ counts the number of valid labelings of $T$ bounded by $k$ that are not bounded by $k - 1$, i.e. those for which at least one vertex is labeled $k$. Every equivalence class of labelings with range at most $k$ will have exactly one member in this set; simply translate the labeling so the maximum label equals $k$.

Definition 3. For a tree $T$ with specified root, let $F^k_i(T)$ be the number of labelings of $T$ with labels in $\{0, \ldots, k\}$ such that the root is labeled $i$.

Let $P_a = \{p_0, p_1, \ldots, p_a\}$ be the path with $a$ edges rooted at its endpoint $p_0$. For paths, let $F^k_{i-j}(P_a)$ denote the number of valid labelings of $T$ such that the label of $p_0$ is $i$, and the label of $p_a$ is $j$. If, for example, $i < 0$, this quantity is simply 0. Similarly, if $i - j \neq a \mod 2$, this quantity will be zero as well.

Remark. By reflection, we have that $F^k_i(T) = F^{k-1}_{k-i}(T)$. For paths in particular, we can condition on whether $p_1 - p_0$ is positive or negative to obtain the recursive formula $F^k_i(P_a) = F^k_{i+1}(P_{a-1}) + F^k_{i-1}(P_{a-1})$. Similarly, given the first $a - 1$ labels, we have no more than 2 choices for the final label, so $F^k_i(P_a) \leq 2F^k_i(P_{a-1})$.

Lemma 5. $F^k_i(P_a) \geq F^k_j(P_a) \iff |i - \frac{k}{2}| \leq |j - \frac{k}{2}|$.

Proof. Note that for $k = 0$ or 1, the result is trivial. For $k \geq 2$, we proceed by induction on $a$. When $a = 0$, both quantities are 0. Assume now the result holds for $a - 1$. By reflection, it suffices to consider $i \leq a/2$.

- If $i \leq a/2 - 1$, then for all $j < i$, we have $j + 1 < i + 1 \leq a/2$ and $j - 1 < i - 1 \leq a/2$. Inductively, we obtain
  
  $F^k_i(P_a) = F^k_{i+1}(P_{a-1}) + F^k_{i-1}(P_{a-1}) \geq F^k_j(P_{a-1}) + F^k_j(P_{a-1}) = F^k_j(P_a)$.

- Otherwise, if $a$ is even and $i = a/2$, then
  
  $F^k_i(P_a) = F^k_{a/2+1}(P_{a-1}) + F^k_{a/2-1}(P_{a-1}) = 2F^k_{a/2-1}(P_{a-1}) \geq F^k_{a/2-1}(P_a) \geq F^k_j(P_a)$.

- Else, if $a$ is odd and $i = a/2 - 1/2$, then
  
  $F^k_i(P_a) = F^k_{a/2+1/2}(P_{a-1}) + F^k_{a/2-3/2}(P_{a-1}) = F^k_{a/2-1/2}(P_{a-1}) + F^k_{a/2-3/2}(P_{a-1}) \geq F^k_{a/2-3/2}(P_a) \geq F^k_j(P_a)$.

\[ \square \]
Corollary 6. Let $T_{a_1, a_2, \ldots, a_i}$ be the spider with paths of length $a_1, a_2, \ldots, a_i$ emanating from a root. Then

$$F_i^k(T_{a_1, a_2, a_3, \ldots, a_i}) \geq F_j^k(T_{a_1, a_2, a_3, \ldots, a_i}) \iff \left| i - \frac{k}{2} \right| \leq \left| j - \frac{k}{2} \right|.$$  

Proof. By Lemma 5 if $\left| i - \frac{k}{2} \right| \leq \left| j - \frac{k}{2} \right|$ we have

$$F_i^k(T_{a_1, a_2, a_3, \ldots, a_i}) = \prod_{t=1}^l F_i^k(P_{a_t}) \geq \prod_{t=1}^l F_j^k(P_{a_t}) = F_j^k(T_{a_1, a_2, a_3, \ldots, a_i}).$$

\[\square\]

3 Main Results

Lemma 7. Let $T_{a_1, a_2, \ldots, a_i}$ be the spider with paths of length $a_1, a_2, \ldots, a_i$ emanating from a root. Then

$$F^k(T_{a_1, a_2, a_3, \ldots, a_i}) - F^k(T_{a_1 + a_2, a_3, \ldots, a_i}) = \sum_{0 \leq i < j \leq k} F_{i \rightarrow j}^k(P_{a_1}) (F_i^k(P_{a_2}) - F_j^k(P_{a_2})) (F_i^k(T_{a_3, \ldots, a_i}) - F_j^k(T_{a_3, \ldots, a_i})).$$

Proof. We have:

$$F^k(T_{a_1, a_2, \ldots, a_i}) = \sum_{i=0}^k F_i^k(P_{a_1}) \cdots F_i^k(P_{a_i})$$

$$= \sum_{i=0}^k \sum_{j=0}^k F_{i \rightarrow j}^k(P_{a_1}) F_i^k(P_{a_2}) F_i^k(T_{a_3, \ldots, a_i})$$

$$= \sum_{0 \leq i < j \leq k} F_{i \rightarrow j}^k(P_{a_1}) (F_i^k(P_{a_2}) F_i^k(T_{a_3, \ldots, a_i}) + F_j^k(P_{a_2}) F_j^k(T_{a_3, \ldots, a_i})).$$

and

$$F^k(T_{a_1 + a_2, \ldots, a_i}) = \sum_{i=0}^k F_i^k(P_{a_1 + a_2}) \cdots F_i^k(P_{a_i})$$

$$= \sum_{i=0}^k \sum_{j=0}^k F_{i \rightarrow j}^k(P_{a_1}) F_j^k(P_{a_2}) F_i^k(T_{a_3, \ldots, a_i})$$

$$= \sum_{0 \leq i < j \leq k} F_{i \rightarrow j}^k(P_{a_1}) (F_i^k(P_{a_2}) F_j^k(T_{a_3, \ldots, a_i}) + F_j^k(P_{a_2}) F_i^k(T_{a_3, \ldots, a_i})).$$

Subtracting these equations and factoring yields the desired expression. \[\square\]

Lemma 8. Let $i < j \leq k$ such that $\frac{i+j}{2} \leq \frac{k}{2}$. Then:

$$0 \leq F_j^k(P_a) - F_i^k(P_a) \leq F_j^{k+1}(P_a) - F_i^{k+1}(P_a).$$

Proof. Positivity follows directly from Lemma 5. For the second inequality we proceed in a similar manner to the proof of Lemma 5 by induction on $a$, with special cases when $\frac{i+j}{2} = \frac{k}{2}$ and $\frac{i+j}{2} = \frac{k-1}{2}$. When $a = 0$ the result is trivial. For our first special case, if $\frac{i+k}{2} = \frac{k}{2}$, then $|i - \frac{k}{2}| = |j - \frac{k}{2}| = \frac{k}{2}$, and so the left-hand side is zero by Lemma 5 whereas the right-hand side is non-negative by the same lemma.

For the remaining cases we have $\frac{i+k}{2} < \frac{k}{2}$. Let $f_j^{k+1}(P_a) := F_j^{k+1}(P_a) - F_j^k(P_a)$. We can rewrite the desired inequality as:

$$f_j^{k+1}(P_a) \geq f_i^{k+1}(P_a).$$
Combinatorially, one can show that $f_{j}^{k+1}(P_{a})$ counts the number of paths starting at $j$ of length $a$ such that at least one vertex is labeled $k + 1$. Consequently, the following recursive formula holds for $j < k + 1$:

$$f_{j}^{k+1}(P_{a}) = f_{j+1}^{k+1}(P_{a-1}) + f_{j-1}^{k+1}(P_{a-1}).$$

The next special case we consider is when $\frac{i + j}{2} = \frac{k - 1}{2}$. Then $\frac{i + j + 1}{2} = \frac{k}{2}$, so from our first special case $F(j) \geq f(j + 1)$. Consequently we reduce to the case $j = i + 1$. If $\frac{i + j + 1}{2} < \frac{k - 1}{2}$, this will be handled below. Otherwise, if $\frac{i + j + 1}{2} = \frac{k - 1}{2}$, then $i = \frac{k}{2} - 1$. Consequently,

$$f_{j}^{k+1}(P_{a}) = f_{\frac{k}{2}}^{k+1}(P_{a}) = f_{\frac{k}{2}+1}^{k+1}(P_{a-1}) + f_{\frac{k}{2}-1}^{k+1}(P_{a-1}) \geq 2f_{\frac{k}{2}-1}^{k+1}(P_{a-1}) \geq f_{\frac{k}{2}-1}^{k+1}(P_{a}) = f_{i}^{k+1}(P_{a}).$$

Here, the first inequality follows from the $\frac{i + j}{2} = \frac{k}{2}$ case, and the second follows from the remark after Definition 3.

We now cover the remaining cases inductively. Assume the statement holds for paths of length $a - 1$. From our arguments above, it remains to verify the inequality when $\frac{i + j}{2} \leq \frac{k}{2} - 1$. Note that as a consequence of this we have either $i < 0$ (in which case the statement is trivial) or $j \leq k - 2$. Inductively, $f_{j+1}^{k+1}(P_{a-1}) \geq f_{i+1}^{k+1}(P_{a-1})$, and $f_{j-1}^{k+1}(P_{a-1}) \geq f_{i-1}^{k+1}(P_{a-1})$, as long as $i + 1 < j + 1 \leq k$ and $\frac{i + j + 2}{2} \leq \frac{k}{2}$. Both of these conditions are satisfied by our hypotheses, and adding these two inequalities produces our desired statement.

**Lemma 9.** Let $T_{a_{1},a_{2},...,a_{l}}$ be the spider with paths of length $a_{1}, a_{2}, \ldots, a_{l}$ emanating from a root, and let $i < j \leq k$ such that $\frac{i + j}{2} \leq \frac{k}{2}$. Then:

$$0 \leq F_{j}^{k}(T_{a_{1},...,a_{l}}) - F_{i}^{k}(T_{a_{1},...,a_{l}}) \leq F_{j}^{k+1}(T_{a_{1},...,a_{l}}) - F_{i}^{k+1}(T_{a_{1},...,a_{l}}).$$

**Proof.** As before, positivity follows directly from Corollary 3. For the second inequality, we proceed by induction on $l$. When $l = 1$, this is just Lemma 8. Otherwise, assume this is true for spiders with $l - 1$ legs and rewrite the desired inequality as

$$f_{j}^{k+1}(T_{a_{1},...,a_{l}}) \geq f_{i}^{k+1}(T_{a_{1},...,a_{l}}).$$

Combinatorially, we have that $f_{j}^{k+1}(T_{a_{1},...,a_{l}})$ counts the number of trees of the given form with root labeled $j$, such that at least one vertex is labeled $k + 1$. Thus we have either a vertex along $P_{a_{1}}$ labeled $k + 1$, a vertex along one of the remaining paths labeled $k + 1$, or both:

$$f_{j}^{k+1}(T_{a_{1},...,a_{l}}) = f_{j}^{k+1}(P_{a_{1}})F_{j}^{k}(T_{a_{2},...,a_{l}}) + F_{j}^{k}(P_{a_{1}})f_{j}^{k+1}(T_{a_{2},...,a_{l}}) + f_{j}^{k+1}(P_{a_{1}})f_{j}^{k+1}(T_{a_{2},...,a_{l}}).$$

Applying Lemma 5, Corollary 6, Lemma 8 and the induction hypothesis as appropriate, we see each term becomes smaller when $j$ is replaced by $i$, which completes the proof.

**Corollary 10.** Let $T = T_{a_{1},a_{2},...,a_{l}}$ be the spider with paths of length $a_{1}, a_{2}, \ldots, a_{l}$ emanating from a root, and let $i < j \leq k$ such that $\frac{i + j}{2} \geq \frac{k}{2}$. Then:

$$0 \leq F_{j}^{k}(T) - F_{i}^{k}(T) \leq F_{i+1}^{k+1}(T) - F_{j+1}^{k+1}(T).$$

**Proof.** Positivity once again follows directly from Lemma 5. For the second inequality, we note that Lemma 5 implies $F_{i}^{k}(T) = F_{k-i}^{k}(T)$. We see that the pair $(k - i, k - j)$ satisfies the hypotheses of Lemma 8 which yields:

$$F_{i}^{k}(T) - F_{j}^{k}(T) = F_{k-i}^{k}(T) - F_{k-j}^{k}(T) \leq F_{k-i}^{k+1}(T) - F_{k-j}^{k+1}(T) = F_{i+1}^{k+1}(T) - F_{j+1}^{k+1}(T).$$

We now prove our main result.
Proof of Theorem 3. Since $f^k(T) = F^k(T) - F^{k-1}(T)$ for any tree $T$, it suffices to show that

$$F^k(T_{a_1,a_2,a_3,...,a_i}) - F^k(T_{a_1+a_2,a_3,...,a_i}) \leq F^{k+1}(T_{a_1,a_2,a_3,...,a_i}) - F^{k+1}(T_{a_1+a_2,a_3,...,a_i}).$$

From Lemma 7 we have

$$F^k(T_{a_1,a_2,a_3,...,a_i}) - F^k(T_{a_1+a_2,a_3,...,a_i}) = \sum_{0 \leq i < j \leq k} F^k_{i\rightarrow j}(P_{a_1}) (F^k(P_{a_2}) - F^k_j(P_{a_2})) (F^k(T_{a_3,...,a_i}) - F^k(T_{a_3,...,a_i})).$$

We now look at what happens to each term when we increase $k$ to $k+1$. When $\frac{i+j}{2} \leq \frac{k}{2}$, we compare to the $(i, j)$ summand for $k + 1$. We have $F^k_{i\rightarrow j}(P_{a_1}) \leq F^{k+1}_{i\rightarrow j}(P_{a_1})$. By Lemma 9 we have that both $(F^k(P_{a_2}) - F^k_j(P_{a_2}))$ and $(F^k(T_{a_3,...,a_i}) - F^k(T_{a_3,...,a_i}))$ are negative, and decrease when $k$ is replaced by $k + 1$. Consequently the summand is positive and increases. On the other hand, when $\frac{i+j}{2} > \frac{k}{2}$, we compare to the $(i + 1, j + 1)$ summand for $k + 1$. We have $F^k_{i\rightarrow j}(P_{a_1}) = F^k_{(k-i)\rightarrow (k-j)}(P_{a_1}) \leq F^{k+1}_{(k-i)\rightarrow (k-j)}(P_{a_1}) = F^{k+1}_{i+1\rightarrow j+1}(P_{a_1})$. From Corollary 10 we have that both $(F^k(P_{a_2}) - F^k_j(P_{a_2}))$ and $(F^k(T_{a_3,...,a_i}) - F^k(T_{a_3,...,a_i}))$ are positive, and increase when $k$ is replaced by $k + 1$ and $(i, j)$ by $(i + 1, j + 1)$. Consequently the summand is again positive, and increases when moving from $k$ to $k + 1$. □

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