Curvature-induced phase transition in
three-dimensional Thirring model

B. Geyer∗

Institute of Theoretical Physics and Center for Advanced Studies,
Leipzig University, Augustusplatz 10, D–04109 Leipzig, Germany

Yu. I. Shil’nov†

Department of Theoretical Physics, Faculty of Physics,
Kharkov State University,
Svobody Sq. 4, 310077, Kharkov, Ukraine

and

Center for Advanced Studies, Leipzig University,
Augustusplatz 10, D–04109, Leipzig, Germany

The effective potential of composite fermion fields in three dimensional
Thirring model in curved space–time is calculated in linear curvature ap-
proximation. The phase transition accompanied by the creation of non–
zero chiral invariant bifermionic vector–like condensate is shown to exist.
The type of this phase transition is discussed.

∗e-mail: geyer@ntz.uni-leipzig.d400.de
†e-mail: shilnov@itp.uni-leipzig.de. Permanent e-mail: shilnov@express.kharkov.ua.
As is well established the four-fermion models provide a useful tool for the description of low and intermediate energy physics of strong interactions [1-6]. They can be obtained from the full QCD by means of the integration over the gluon degrees of freedom. However, it requires the introduction of a dimensional scale, limiting the low energy region where the non-local character of fermion interactions due to exchange of gluons can be neglected.

Furthermore, the Nambu–Jona-Lasinio (NJL) model has recently been used to introduce a dynamical symmetry breaking mechanism which is very important for high energy physics (see, for example [7, 8] and literature, cited there).

The most popular and the simplest four-fermion model is the NJL model with the scalar and pseudoscalar types of fermion interaction only [1]. It has been frequently applied to obtain the dynamical symmetry breaking and to create bifermionic bound states (mesons) in a manner appropriate for the phenomenology of hadrons [3 - 5]. The effective potential and vacuum condensate of four-fermion models on the light-cone have been computed as well [9]. Simultaneously, much attention has been paid for the investigation of dynamical symmetry breaking in 3-dimensional four-fermion models [5, 6, 10]. There are different reasons for this. First, 3-dimensional theories are directly related to the high temperature limit of 4-dimensional ones [11, 12]. Second, it has very interesting non-trivial topological features [13, 14]. Third, 3-dimensional models are usually renormalizable in the large–N limit [5]. And, finally, there are some hints that they could be useful also in condensed matter physics [14].

It is quite natural to generalize the known results concerning four-fermion models for the case of curved spacetime. The phase structure of the NJL model in curved spacetime turned out to be nontrivial and the phase transition accompanied by the chiral symmetry breaking has been shown also to take place [15 - 17].

The extended NJL model, considered in [4 - 6], generalizes the results for the restricted one to the case when different kinds of fermion interactions are present in the effective Lagrangian. This means, strickly speaking, that the Thirring model (TM) [18], containing vector–like interactions of fermions, and the NJL model are combined to consider their effects simultaneously.
However the TM is very interesting also by itself. It has been investigated in detail from different points of view (see, for example, review [19]). Its renormalizability has been proved in the $d < 4$ dimensions [6]. Its phase structure, dynamical symmetry breaking and dynamical mass generation have been analysed [20] as well. Also the gauged TM has been recently investigated both in curved [19] and in flat [19, 21] spacetimes.

In the present paper we are dealing with the three–dimensional massless Thirring model in curved spacetime. Using the local momentum expansion [22] we obtain the linear curvature corrections for the effective potential of the composite auxiliary vector field. The creation of a non–zero chiral invariant vector-like bifermion condensate, taking place in flat spacetime, is shown to exist for the model with a finite cutoff above some critical value of the coupling constant (a similar phenomenon has been described in [23]). The positive spacetime curvature induces a second order phase transition making the vacuum expectation value of $< \bar{\psi} \gamma_{\mu} \psi >$ tend to zero. The phase structure of the model is discussed.

2. We start from the action of the three–dimensional Thirring model in curved spacetime [19, 20]:

\[
S = \int d^3x \sqrt{-g} \{ i \bar{\psi} \gamma^\mu(x) D_\mu \psi - \frac{\lambda}{2N} (\bar{\psi} \gamma_a \psi)(\bar{\psi} \gamma^a \psi) \},
\]

where the covariant derivative $D_\mu$ is given by $D_\mu = \partial_\mu + \frac{1}{2} \omega^{ab}_\mu \sigma_{ab}$, the local Dirac matrices $\gamma_\mu(x)$ are expressed through the usual flat ones $\gamma_a$ and tetrads $e^a_{\mu}$, $\gamma^\mu(x) = \gamma^a e^\mu_a(x)$, $\sigma_{ab} = \frac{1}{4}[\gamma_a, \gamma_b]$, the spin–connection has the form

\[
\omega^{ab}_\mu = \frac{1}{2} e^{av}(\partial_\mu e^b_v - \partial_v e^b_\mu) + \frac{1}{4} e^{av} e^{bp} e_{cm}(\partial_p e^c_v - \partial_v e^c_p) - \frac{1}{2} e^{bv}(\partial_\mu e^a_v - \partial_v e^a_\mu) - \frac{1}{4} e^{bv} e^{ap} e_{cm}(\partial_p e^c_v - \partial_v e^c_p),
\]

and $N$ is the number of bispinor fields $\psi_n$.

Introducing the chiral invariant auxiliary field

\[
A^a = -\frac{\lambda}{N} (\bar{\psi} \gamma^a \psi)
\]

\[\text{It should be noted that in the interaction term of (1) we don’t care for the difference between Greek and Latin indices, corresponding to the curved and tangent flat spacetimes. This term can be written using the definition } A_a = e^a_\mu A_\mu \text{ in the same form but with the Greek indices instead of Latin ones.} \]
we have the following expression for the action (1):

\[ S = \int d^3x \sqrt{-g} \{ i \bar{\psi} \gamma^\mu D_\mu \psi + \frac{N}{2\lambda} A^a A_a + A_a (\bar{\psi} \gamma^a \psi) \}. \] (3)

By a functional integration the effective potential is obtained in leading order of large–\(N\) expansion according to:

\[ \frac{1}{N} \Gamma_{\text{eff}} = -\frac{1}{2\lambda} \int d^3x \sqrt{-g} A^a A_a - i \ln \det \{ i \gamma^\mu (x) D_\mu + \gamma_a A^a \}. \] (4)

Defining now the effective potential for the constant auxiliary field \(A_a\) as

\[ V_{\text{eff}} = -\Gamma_{\text{eff}} / \left( N \int d^3x \sqrt{-g} \right) \]

one gets

\[ V_{\text{eff}} = -\frac{1}{2\lambda} A_a A^a + i Sp \ln <x|[i \gamma^\mu (x) D_\mu + \gamma_a A^a]|x>. \]

Now we introduce the special Green function (GF) obeying the following equation:

\[ (i \gamma^\mu D_\mu + \gamma_a A^a - s)_x G(x, x'; s) = \delta(x - x'). \] (5)

Then, neglecting an infinite constant, we find

\[ \ln <x|[i \gamma^\mu D_\mu + \gamma_a A^a]|x> = \int_0^\infty G(x, x; s) ds, \]

and for the effective potential we obtain:

\[ V_{\text{eff}} = -\frac{1}{2\lambda} A_a A^a + i Sp \int_0^\infty G(x, x; s) ds. \]

The most convenient way to calculate the linear curvature corrections is to use the local momentum expansion, writing the spacetime quantities contained in the equation (5) by the help of Riemannian normal coordinates up to linear curvature accuracy in the following form [22]:

\[ g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} R_{\rho\sigma\mu\nu} y^\rho y^\sigma \quad \text{with} \quad y = x - x', \]

\[ e^\mu_a(x) = \delta^\mu_a + \frac{1}{6} R^\mu_{\rho\sigma a} y^\rho y^\sigma, \quad \omega^{ab}_\mu \sigma_{ab} = \frac{1}{2} R^{ab}_\mu \lambda y^\lambda \sigma_{ab}. \]

Substituting these expressions into (5) we obtain the equation for the Green function \(G(x, x'; s)\):

\[ \left[ i \gamma^a (\delta^\mu_a + \frac{1}{6} R^\mu_{\rho\sigma a} y^\rho y^\sigma) (\partial_\mu + \frac{1}{4} R_{bc\mu\lambda} y^\lambda \sigma^{bc}) + A_a \gamma^a - s \right] G(x, x', s) = \delta(x - x') \]

4
Performing the expansion according to the degree of the spacetime curvature, \( G = G_0 + G_1 + \ldots \), where \( G_n \sim R^n \), we receive the following chain of equations:

\[
(i \partial + \mathcal{A} - s)G_0(x, x', s) = \delta(x - x'),
\]

\[
(i \partial + \mathcal{A} - s)G_1(x, x', s) + i \left[ \frac{1}{6} R_{\rho \sigma \lambda} y^\rho y^\sigma y^\lambda \partial_\mu + \frac{1}{4} R_{\rho \sigma \tau} y^\lambda \gamma_\sigma \gamma_\tau \right] G_0(x, x', s) = 0.
\]

Here and below we can forget about the difference between Greek and Latin indices because it lies beyond of linear curvature approximation.

Equations (6) and (7) can be solved directly in the momentum representation:

\[
G_0(k) = \frac{q + s}{q^2 - s^2},
\]

\[
G_1(k) = - \frac{1}{12 (q^2 - s^2)^2} + \frac{2}{3} R^{\mu \nu} k_\mu k_\nu (q + s) - \frac{4}{3} R^{\mu \rho \sigma \nu} k_\mu \gamma_\nu A_\rho q_\sigma,
\]

\[
- \frac{1}{2} R^{\mu \nu \rho \lambda} \gamma_\rho \sigma_\mu \gamma_\lambda \frac{k_\mu (q + s) k_\nu (\gamma_\rho A_\sigma + \gamma_\sigma A_\rho)}{(q^2 - s^2)^2} - \frac{1}{3} R^{\mu \rho \sigma \nu} (q + s) k_\mu \gamma_\nu (\gamma_\rho A_\sigma + \gamma_\sigma A_\rho)
\]

\[
+ \frac{1}{3} R^{\mu \nu} \frac{\gamma_\mu A_\nu}{(q^2 - s^2)^2} + \frac{1}{3} R^{\mu \nu} A_\mu (A_\nu + 4 k_\nu) (q + s)
\]

where \( q_\mu = k_\mu + A_\mu \).

Let us now, for simplicity, assume that our spacetime has constant curvature such that

\[
R_{\mu \sigma \kappa \lambda} = \frac{1}{6} R \left( \eta_{\mu \kappa} \eta_{\sigma \lambda} - \eta_{\mu \lambda} \eta_{\kappa \sigma} \right), \quad R_{\mu \nu} = \frac{1}{3} R \eta_{\mu \nu}.
\]

Then, calculating the trace over the spinor indices (the dimension of the fermion representation is supposed to be equal to four) and making a Wick rotation, one gets:

\[
V_{\text{eff}} = \frac{A^2}{2 \lambda} - 2 \int_0^\infty ds^2 \int \frac{d^3 k}{(2\pi)^3} \left[ \frac{1}{(k + A)^2 + s^2} + \frac{3}{(k + A)^2 + s^2} \right] + \frac{R}{36} \left( \frac{k^2 - A^2 - 3s^2}{((k + A)^2 + s^2)^2} + 2 \frac{k^2 - A^2 - 3s^2}{((k + A)^2 + s^2)^3} \right),
\]

where \( A^2 = A_\alpha A^\alpha \) is the Euclidean square. Integrating over the angles (in momentum space), we finally get the expression for the effective potential.
in the model under consideration:

\[ V_{eff} = \frac{A^2}{2\lambda} - \frac{1}{2\pi^2} \left\{ \frac{3\Lambda^4 + 6A^2\Lambda^2 - A^4}{12A} \ln \left| \frac{\Lambda + A}{\Lambda - A} \right| \right. \]

\[ + \left. \frac{2}{3} \Lambda^3 \ln \left| \frac{\Lambda^2 - A^2}{\Lambda^2} \right| + \frac{1}{6} \Lambda A^2 - \frac{R}{36} A \ln \left| \frac{\Lambda + A}{\Lambda - A} \right| \right\}, \tag{10} \]

where the parameter \( \Lambda \) is the upper cut off in the above momentum integral.

There are two ways of explaining the introduction of this parameter. The first one is the standard renormalization cut off. Then, after renormalization of the coupling constant \( \lambda \) in the limit \( \Lambda \to \infty \),

\[ \frac{1}{\lambda_R} = \frac{1}{\lambda} - \frac{2\Lambda}{3\pi^2}, \tag{11} \]

we have

\[ V_{eff} = \frac{A^2}{2\lambda_R} \tag{12} \]

and, hence, the Thirring model is the trivial one in this approximation.

However, in general, four-fermion models may be considered as low energy effective theories being derived from a more complete version of quantum field theory (QCD, for example). In this case the parameter \( \Lambda \) can be treated as a natural characteristic scale limiting the range, where our low energy approximation is valid, and then the Thirring model would describe some phenomenological effects of elementary particles physics.

Keeping in mind these reasoning we can save \( \Lambda \) finite and obtain:

\[ V_{eff}''(0) = \frac{1}{\lambda} - \frac{2\Lambda}{3\pi^2} + \frac{R}{18\Lambda \pi^2}, \quad V_{eff}'''(0) = 0. \tag{13} \]

The creation of a non-zero vacuum expectation value of the vector–like bifermionic condensate \( A^a \sim \langle \bar{\psi} \gamma^a \psi \rangle \) takes place if \( V_{eff}''(0) < 0 \).

For \( R = 0 \) the critical value of coupling constant \( \lambda \) is given by

\[ \lambda_c = \frac{3\pi^2}{2\Lambda}. \tag{14} \]

Therefore, if \( \lambda > \lambda_c \), a nonvanishing condensate \( A_a \neq 0 \) appears. However, positive \( R \) makes \( A_a \) tend to zero if

\[ R > 8\Lambda^2 \left( 1 - \frac{\lambda_c}{\lambda} \right). \tag{15} \]
Furthermore, the equation of the critical line dividing the $R - \lambda$ plane into two regions, characterized by zero and non-zero vacuum expectation value of $\langle \bar{\psi}\gamma_a\psi \rangle$, is determined by:

$$\frac{R}{8\Lambda^2} + \frac{\lambda_c}{\lambda} = 1.$$  \hspace{1cm} (16)

Because of condition (15) we have the ordinary picture of second order curvature-induced phase transition where $A^2$ is the order parameter. It is clearly illustrated by Figs. 1 - 3.

Fig. 1 shows the dependence of $v(a) = [V(a) - V(0)]/\Lambda^3$ on $a = A/\Lambda$ for $R = 0$. The curves 1, 2, 3, 4 correspond to the following values of $g = \frac{\lambda\Lambda}{\Lambda} = 12.5, 15.25, 16.5, 17$.

Fig. 2 represents the behaviour of the function $v(a)$ (curves 1, 2, 3, 4) for different values of the curvature $r = R/\Lambda^2 = 0.8, 0.5, 0.3, 0$ for $g = 17$.

Fig. 3 contains the three–dimensional plot of effective potential $v(a, r)$ as a function of $a$ and $r$ simultaneously for $g = 17$ as well.

3. The authors would like to thank S. D. Odintsov for pointing out the problem and for helpful discussions. Yu. I. Sh. is very much indebted to Deutcher Akademischer Austauschdienst for financial support of his visit to Leipzig University, where this work has been done. He also expresses his deep gratitude to A. Letwin and R. Patov for their kind support.

References

[1] Y. Nambu, G. Jona- Lasinio, \textit{Phys. Rev.} \textbf{122}, 345 (1961); \textbf{124}, 246 (1961).

[2] D. J. Gross, A. Neveu, \textit{Phys. Rev.} \textbf{D10}, 3235 (1974). M. K. Volkov, \textit{Ann. Phys.} \textbf{157}, 282 (1984).

[3] M. Bando, T. Kugo, K. Yamawaki, \textit{Phys. Rept.} \textbf{164}, 217 (1988).

[4] M. K. Volkov, \textit{Ann. Phys.} \textbf{157}, 282 (1984). R. Alkofer, H. Reinhart \textit{Chiral Quark Dynamics} (Springer, 1995). J. Bijnens, \textit{Phys. Rept.} \textbf{265}, 369 (1996).

[5] B. Rosenstein, B. J. Warr, S. H. Park, \textit{Phys. Rept.} \textbf{205}, 59 (1991).
[6] S. Hands, A. Kocic, J. B. Kogut, *Ann. Phys.* **224**, 29 (1993).

[7] E. Fahri, R. Jackiw, *Dynamical symmetry breaking* (World Scientific, Singapore, 1981). *Proceedings of the Workshop on Dynamical Symmetry Breaking*, ed. T. Muta, K. Yamawaki (Nagoya, 1990). *Proceedings of the International Workshop on Electroweak Symmetry Breaking*, ed. W. A. Bardeen, J. Kodaira, T. Muta (World Scientific, Singapore, 1991). P. I. Fomin, V. P. Gusynin, V. A. Miransky, Yu. A. Sitenko, *Riv. Nuov. Cim.* **6**, 1 (1983).

[8] S. Khebnikov, R. D. Peccei, *Phys. Rev.* **D48**, 361 (1993).

[9] S.-I. Kojima, N. Sakai, T. Sakai, *Progr. Theor. Phys.* **95**, 621 (1996).

[10] G. W. Semenoff, L. C. R. Wijewardhana, *Phys. Rev.* **D45**, 1342 (1992).

[11] D. J. Gross, R. Pisarski, L. Yaffe, *Rev. Mod. Phys.* **53**, 43 (1981).

[12] A. D. Linde *Particle Physics and Inflationary Cosmology* (Harwood Acad., London, 1990).

[13] S. Deser, R. Jackiw, S. Templeton, *Ann. Phys.* **140**, 372 (1982).

[14] R. Jackiw, *Phys. Rev.* **D29**, 2375 (1984). I. V. Krive, A. S. Rozhavskii, *Sov. Phys. Usp.* **30**, 370 (1987). E. Fradkin *Field Theories of Condensed Matter Systems* (Addison-Wesley, Reading, 1991).

[15] T. Muta, S. D. Odintsov, *Mod. Phys. Lett.* **A6**, 3641 (1991). T. Inagaki, T. Muta, S. D. Odintsov, *Mod. Phys. Lett.* **A8**, 2117 (1993). E. Elizalde, S. Leseduarte, S. D. Odintsov, *Phys. Rev.* **D49**, 5551 (1994). C. T. Hill, D. S. Salopek, *Ann. Phys.* **213**, 21 (1992).

[16] E. Elizalde, S. D. Odintsov, *Phys. Rev.* **D51**, 5990 (1995).

[17] E. Elizalde, S. D. Odintsov, Yu. I. Shil’nov, *Mod. Phys. Lett.* **A9**, 913 (1994). E. Elizalde, S. Leseduarte, S. D. Odintsov, Yu. I. Shil’nov, *Phys. Rev.* **D53**, 1917 (1996). T. Inagaki, S. Mukaigawa, T. Muta, *Phys. Rev.* **D52**, 4267 (1995). B. Geyer, S. D. Odintsov, *Phys. Lett.* **B 376** 260 (1996), *Phys. Rev.* **D 53** 7321 (1996).

[18] W. Thirring, *Ann. Phys.* **3**, 91 (1958).
[19] I. Sachs, A. Wipf, *Ann. Phys.* **249**, 380 (1996).

[20] M. Gomes, M. S. Mendes, R. F. Ribeiro, A. J. de Silva, *Phys. Rev.* **D43**, 3516 (1991). D. K. Hong, S. H. Park, *Phys. Rev.*, **D49**, 5507 (1994)

[21] T. Itoh, Y. Kim, M. Sugiura, K. Yamawaki, *Progr. Theor. Phys.*, **93**, 417 (1995). K.-I. Kondo, *Progr. Theor. Phys.*, **94**, 899 (1995); *Nucl. Phys.* **B450**, 251 (1995). K. Ikekami, K.-I. Kondo, A. Nakamura, *Progr. Theor. Phys.*, **95**, 203 (1995).

[22] T. S. Bunch, L. Parker, *Phys. Rev.* **D20**, 2449 (1979).

[23] I. L. Buchbinder, S. D. Odintsov, *Europhys. Lett.* **4**, 147 (1987).
