NONLINEAR GRAVITON AS A LIMIT OF
\( s_l(N;C) \) CHIRAL FIELDS AS \( N \to \infty \)

Maciej Przanowski

Institute of Physics, Technical University of Łódź,
Wólczańska 219. 93-005 Łódź, Poland
Department of Theoretical Physics, University of Łódź,
Pomorska 149/153. 90-236 Łódź, Poland

Sebastian Formański and Francisco J. Turrubiates

Institute of Physics, Technical University of Łódź,
Wólczańska 219. 93-005 Łódź, Poland

Abstract

An example of a sequence of the \( s_l(N;C) \) chiral fields, for \( N \geq 2 \), tending to the complex heavenly metric (nonlinear graviton) of the type \([4] \times [-]\) when \( N \to \infty \) is given.

Keywords: Self-dual gravity, chiral fields, infinite-dimensional Lie algebras.

1 Introduction

Ward, Park, Strachan and Husain were the first who observed the close relation between the principal chiral model and self-dual gravity. Then in this relation has been analysed from several points of view. Briefly speaking, self-dual gravity on a real 4-dimensional manifold endowed

\(^1\) e-mail: mprzan@ck-sg.p.lodz.pl
\(^2\) e-mail: sforman@ck-sg.p.lodz.pl
\(^\dagger\) Permanent address: Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, Apartado Postal 14-740, 07000 México D.F., México
e-mail: fturrub@fis.cinvestav.mx
with metric of signature (+ + − −) may be locally considered to be the principal chiral model on $V \subset \mathbb{R}^2$ for the real Poisson algebra of a 2-surface $\Sigma^2$. Assuming that $\Sigma^2$ is the 2-torus $T^2$ one can construct the isomorphism between the real Poisson algebra of $T^2$ and the $su(\infty)$ algebra. Similarly, the complex Poisson algebra of $T^2$ is isomorphic to the $sl(\infty; C)$ algebra.

Thus the natural question arises: "Can one construct a sequence of the $su(N)$ chiral fields, for $N = 2, 3, \ldots$, tending to a curved space in a limit". This question was first stated by Ward [1]. In previous works [6, 7] some approach to Ward’s question has been proposed.

The aim of the present paper is to give an example of a sequence of $sl(N, C)$ chiral fields for $N \geq 2$, tending to a curved heavenly space-time for $N \to \infty$.

Our paper is organized as follows. In Section 2 we recall the Husain-Park heavenly equation and its Moyal deformation. Section 3 is devoted to Ward’s question. Here we present an approach to this question given in the previous works [6, 7]. Finally, in Section 4 we give an example of the $sl(N; C)$ chiral fields on $\mathbb{R}^2$, or $\mathbb{C}^2$, for $N = 2, 3, \ldots$, tending to the complex heavenly metric on a real space time, or on the complex space-time, respectively, for $N \to \infty$. This heavenly metric (nonlinear graviton) appears to be of the type $[4] \times [-]$.

## 2 Husain-Park heavenly equation

V.Husain [4] has shown that the Ashtekar-Jacobson-Smolin equations describing the heavenly metric can be reduced to the following equation

$$\partial^2_x \Theta_0 + \partial^2_y \Theta_0 + \{\partial_x \Theta_0, \partial_y \Theta_0\}_P = 0 \quad (2.1)$$

where $\Theta_0 = \Theta_0(x, y, p, q)$ and $\{\cdot, \cdot\}_P$ stands for the Poisson bracket i.e.,

$$\{\partial_x \Theta_0, \partial_y \Theta_0\}_P = \partial_q(\partial_x \Theta_0)\partial_p(\partial_y \Theta_0) - \partial_q(\partial_y \Theta_0)\partial_p(\partial_x \Theta_0)$$

Then the heavenly metric $ds^2$ reads

$$ds^2 = dx(\partial_p \partial_x \Theta_0 dp + \partial_q \partial_x \Theta_0 dq) + dy(\partial_p \partial_y \Theta_0 dp + \partial_q \partial_y \Theta_0 dq)$$

$$- \frac{1}{\{\partial_x \Theta_0, \partial_y \Theta_0\}_P}[((\partial_p \partial_x \Theta_0 dp + \partial_q \partial_x \Theta_0 dq)^2 + (\partial_p \partial_y \Theta_0 dp + \partial_q \partial_y \Theta_0 dq)^2]^{1/2} \quad (2.2)$$

Observe that in the works [4, 5, 6] there are some mistakes in sign in formulae for $ds^2$. 

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Equation (2.1) has been also found by Park [2]. (However, in his paper [2] Park was not aware that (2.1) defined the metric of a heavenly space-time). We call Eq. (2.1) the Husain-Park heavenly equation (H-P equation).

As it has been pointed out in the previous papers [5, 6, 7] it seems to be very fruitful to deal with the Moyal deformation of the H-P equation

\[
\partial_x^2 \Theta + \partial_y^2 \Theta + \{\partial_x \Theta, \partial_y \Theta\}_M = 0
\]

\[
\Theta = \Theta(h; x, y, p, q)
\]

(2.3)

where \(\{\cdot, \cdot\}_M\) stands for the Moyal bracket i.e.,

\[
\{f_1, f_2\}_M := f_1 \frac{2}{\hbar} \sin \left(\frac{\hbar}{2} \vec{P} \right) f_2
\]

\[
\vec{P} := \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q}
\]

(2.4)

\(\hbar\) is the deformation parameter.

It is evident that if \(f_1\) and \(f_2\) are independent of \(\hbar\) then

\[
\lim_{\hbar \to 0} \{f_1, f_2\}_M = \{f_1, f_2\}_P
\]

(2.5)

Let \(\Theta = \Theta(h; x, y, p, q)\) be an analytic in \(\hbar\) solution of Eq. (2.3) i.e.,

\[
\Theta = \sum_{n=0}^{\infty} \hbar^n \Theta_n \quad \Theta_n = \Theta(x, y, p, q)
\]

(2.6)

Consequently one quickly finds that the function \(\Theta_0\) satisfies the H-P equation (2.1).

In the present paper we deal with solutions of Eq. (2.3) on \(V \times T^2\) i.e., \((x, y) \in V\) and \((p, q) \in T^2\), where \(V \subset \mathbb{R}^2\) and \(T^2\) is the 2-torus of periods \(2\pi\) and \(2\pi\). Thus \(\Theta\) can be written in the following form

\[
\Theta = \sum_{(m_1, m_2) \in Z \times Z} \Theta(m_1, m_2)(\hbar; x, y) \exp[i(m_1 p + m_2 q)]
\]

(2.7)

It is an easy matter to show that from (2.4) one gets

\[
\{E_{(m_1, m_2)}, E_{(n_1, n_2)}\}_M = \frac{2}{\hbar} \sin \left(\frac{\hbar}{2} (m_1 n_2 - m_2 n_1) \right) E_{(m_1 + n_1, m_2 + n_2)}
\]

(2.8)

\((m_1, m_2), (n_1, n_2) \in Z \times Z\)

where

\[
E_{(m_1, m_2)} := \exp[i(m_1 p + m_2 q)]
\]

(2.9)
3 The $sl(N;C)$ chiral fields

Consider the $sl(N;C)$ principal chiral model. The field equations read now
\begin{align}
\partial_x A_y - \partial_y A_x + [A_x, A_y] &= 0 \quad (3.1) \\
\partial_x A_x + \partial_y A_y &= 0 \quad (3.2)
\end{align}
where $A_x = A_x(x, y)$ and $A_y = A_y(x, y)$ are $sl(N;C)$-valued functions on $V \subset \mathbb{R}^2$. From (3.2) one infers that there exists an $sl(N;C)$-valued function $\theta = \theta(x, y)$ such that
\begin{align}
A_x = -\partial_y \theta \quad \text{and} \quad A_y = \partial_x \theta \quad (3.3)
\end{align}
Inserting (3.3) into (3.1) we get
\begin{align}
\partial_x^2 \theta + \partial_y^2 \theta + [\partial_x \theta, \partial_y \theta] &= 0 \quad (3.4)
\end{align}
(Compare with [5, 6, 7]). In the classical field theory Eq. (3.4) can be equivalently considered to be the field equation of the $sl(N;C)$ principal chiral model.

To understand the relation between self-dual gravity and $sl(N;C)$ principal chiral model we use some results of a distinguished paper by Fairlie, Fletcher and Zachos [8]. In particular in [8] the basis of the $su(N)$ algebra has been considered which plays a crucial role in our construction. To define this basis consider $N \times N$ matrices
\begin{align}
S := \sqrt{\omega} \left( \begin{array}{cccc}
1 & 0 & 0 & \ldots & 0 \\
0 & \omega & 0 & \ldots & 0 \\
0 & 0 & \omega^2 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & \omega^{N-1}
\end{array} \right) \\
\omega := \exp\left(\frac{2\pi i}{N}\right), \quad \sqrt{\omega} = \exp\left(\frac{\pi i}{N}\right)
\end{align}

\begin{align}
T := \left( \begin{array}{cccc}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 1 \\
-1 & 0 & 0 & \ldots & 0
\end{array} \right) \quad (3.5)
\end{align}
\[ S^N = T^N = -1, \quad T \cdot S = \omega S \cdot T \]

Then one defines
\[
L_{(m_1, m_2)} := iN \omega \frac{m_1 m_2}{2\pi} S^{m_1} T^{m_2}; \quad (m_1, m_2) \in Z \times Z \quad (3.6)
\]

It is an easy matter to show that
\[
L_{(m_1, m_2)} + N(r_1, r_2) = (-1)^{(m_1+1)r_2 + (m_2+1)r_1 + N r_1 r_2} L_{(m_1, m_2)} \quad (3.7)
\]

and
\[
[L_{(m_1, m_2)}, L_{(n_1, n_2)}] = \frac{N}{\pi} \sin\left[\frac{\pi}{N} (m_1 n_2 - m_2 n_1)\right] L_{(m_1 + n_1, m_2 + n_2)} \quad (3.8)
\]

As it has been proved in \[3\] \(N^2 - 1\) matrices \(L_{(\mu_1, \mu_2)}\), \(0 \leq \mu_1 \leq N - 1, 0 \leq \mu_2 \leq N - 1\) and \((\mu_1, \mu_2) \neq (0, 0)\), span the \(su(N)\) algebra (and of course the \(sl(N; C)\) algebra). In what follows we assume that the Greek indices \((\mu_1, \mu_2), (\nu_1, \nu_2), \ldots, \) etc. satisfy the above written conditions.

Comparing (3.8) and (3.9) with (3.8), employing also (3.7) one finds that the linear extension of the mapping

\[
E_{(\mu_1, \mu_2)} + N(r_1, r_2) \mapsto L_{(\mu_1, \mu_2)} + N(r_1, r_2) = (-1)^{(\mu_1+1)r_2 + (\mu_2+1)r_1 + N r_1 r_2} L_{(\mu_1, \mu_2)}
\]

\[
E_{N(r_1, r_2)} \quad \mapsto \quad 0
\]

is a Lie algebra homomorphism of the complex Moyal bracket algebra on \(T^2\) with \(\hbar = \frac{2\pi}{N}\) onto the Lie algebra \(sl(N; C)\). Denote this homomorphism by \(\Psi\). Let \(\Theta = \Theta(\hbar; x, y, p, q)\) be some solution of the Moyal deformation of the H-P equation (2.3) on \(V \times T^2\). Then we write \(\Theta\) in the form (2.7) and we put \(\hbar = \frac{2\pi}{N}\). Define

\[
\theta = \theta(N; x, y) := \Psi(\Theta(\frac{2\pi}{N}; x, y, p, q))
\]

\[
= \sum_{(\mu_1, \mu_2)} \sum_{(r_1, r_2) \in Z \times Z} (-1)^{(\mu_1+1)r_2 + (\mu_2+1)r_1 + N r_1 r_2} \Theta_{(\mu_1, \mu_2)} + N(r_1, r_2) \left( \frac{2\pi}{N}; x, y \right) L_{(\mu_1, \mu_2)}
\]

\[
(3.10)
\]
As $\Psi$ is a Lie algebra homomorphism the $sl(N;C)$-valued function

$$\theta = \theta(N; x, y) = \sum_{(\mu_1, \mu_2)} \theta_{(\mu_1, \mu_2)}(N; x, y)L_{(\mu_1, \mu_2)}$$

$$\theta_{(\mu_1, \mu_2)}(N; x, y) := \sum_{(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z}} (-1)^{(\mu_1+1) r_2 + (\mu_2+1) r_1 + N r_1 r_2} \Theta_{(\mu_1, \mu_2) + N (r_1, r_2)} \left(\frac{2\pi}{N}; x, y\right)$$  \hspace{1cm} (3.11)

$$0 \leq \mu_1 \leq N - 1 \ , \ 0 \leq \mu_2 \leq N - 1 \ \text{and} \ \ (\mu_1, \mu_2) \neq (0, 0)$$

is a solution of the $sl(N;C)$ chiral field equation (3.4). (Compare with [7]).

Assume that our solution $\Theta = \Theta(h; x, y, p, q)$ is analytic in $h$. Consequently, from the results of Sec.2. one infers that

$$\Theta_0 = \Theta_0(0; x, y, p, q)$$  \hspace{1cm} (3.12)

fulfils the H-P equation (2.1).

Now if we put $h = \frac{2\pi}{N}$ then $h \rightarrow 0$ corresponds to $N \rightarrow \infty$. Assume that the metric $ds^2$ given by (2.2) for $\Theta_0$ defined by (3.12) describes the curved manifold. Thus one arrives at the sequence of $sl(N;C)$ chiral fields for $N = 2, 3, \ldots$ tending to a curved heavenly space-time.

## 4 Example

We look for the solution of the Moyal deformation of the H-P equation (2.3) on $V \times T^2$, $V \subset \mathbb{R}^2$, of the following form

$$\partial_x \Theta = \exp(iq)$$

$$\partial_y \Theta = \sum_{m=0}^{\infty} \eta(1, m)(h; y) \exp[i(p + mq)]$$  \hspace{1cm} (4.1)

Substituting (4.1) into (2.3) employing also (2.8) one gets the system of differential equations

$$\frac{d\eta(1, 0)}{dy} = 0 ,$$

$$\frac{d\eta(1, m)}{dy} - \frac{2}{h} \sin\left(\frac{h}{2}\right) \eta(1, m-1) = 0 \ \text{for} \ \ m \geq 1 \hspace{1cm} (4.2)$$

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The general solution of the system (4.2) reads
\[ \eta_{(1,m)} = \sum_{j=0}^{m} \frac{a_{m-j} 2^{j} \sin(\frac{h}{2} y)}{j!}, \tag{4.3} \]
where \( a_0, a_1, \ldots \) are complex constants (which may depend on \( h \)).

Inserting (4.3) into (4.1) we find that the general \( \partial_{y} \Theta \) of the form (4.1) is any linear combination of the function \( \exp(iq) \exp[\frac{2}{h} \sin(\frac{h}{2}) y \exp(iq)] \) and all its derivatives with respect to \( y \).

For further considerations we take the solution of Eq. (2.3) to be
\[ \partial_{x} \Theta = \exp(iq) \]
\[ \partial_{y} \Theta = \exp(ip) \exp[2 \frac{h}{2} \sin(\frac{h}{2}) y \exp(iq)] \]
\[ = \sum_{m=0}^{\infty} \frac{1}{m!} \left[ \frac{2}{h} \sin(\frac{h}{2}) y \right]^{m} \exp[i(p + mq)] \tag{4.4} \]

Hence
\[ \Theta = x \exp(iq) + \sum_{m=0}^{\infty} \left( \frac{2}{h} \sin(\frac{h}{2}) \right)^{m} y^{m+1} \exp[i(p + mq)] \tag{4.5} \]

Evidently, the function \( \Theta = \Theta(h; x, y, p, q) \) defined by (4.3) is analytic in \( h \) and also we can put \( V = R^{2} \). Then
\[ \Theta_{0} = \Theta(0; x, y, p, q) = x \exp(iq) + \sum_{m=0}^{\infty} \frac{1}{(m+1)!} y^{m+1} \exp[i(p + mq)] \]
\[ = x \exp(iq) + \exp[i(p-q)]\{\exp[y \exp(iq)] - 1\} \tag{4.6} \]
is a solution of the Husain-Park heavenly equation (2.1). Now we put in (4.4) \( h = \frac{2\pi}{N} \). Then from (3.3), \( (1.11) \), (1.1) and (4.4) one gets
\[ A_{x} = - \sum_{\mu=0}^{N-1} \{ \sum_{r=0}^{\infty} \eta_{(1,\mu+Nr)} \left( \frac{2\pi}{N} y \right) \} L_{(1,\mu)} \]
\[ = - \sum_{\mu=0}^{N-1} \{ \sum_{r=0}^{\infty} \frac{1}{(\mu+Nr)!} \left| \frac{N}{\pi} \sin\left( \frac{\pi}{N} \right) y \right|^{\mu+Nr} \} L_{(1,\mu)} \tag{4.7} \]

As for \( \mu < N \)
\[ \sum_{r=0}^{\infty} \frac{1}{(\mu+Nr)!} \left| \frac{N}{\pi} \sin\left( \frac{\pi}{N} \right) y \right|^{\mu+Nr} = \frac{1}{N} \sum_{k=1}^{N} \omega^{(N-\mu)k} \exp[\omega k \frac{N}{\pi} \sin\left( \frac{\pi}{N} \right) y] \]
\[ \tag{4.8} \]
where \( \omega = \exp \frac{2\pi i}{N} \) we obtain finally

\[
A_x = -N^{-1} \sum_{\mu=0}^{N-1} \{ \frac{1}{N} \sum_{k=1}^{N} \omega^{(N-\mu)k} \exp[\omega^k N^2 \sin(\pi N)] y] \} L_{(1,\mu)}
\]

\[
A_y = L_{(0,1)}
\]

(4.9)

Straightforward calculations show that the \( \mathfrak{sl}(N;\mathbb{C}) \)-valued functions \( A_x = A_x(x,y) \) and \( A_y = A_y(x,y) \), indeed, fulfill the principal chiral model field equations (3.1) and (3.2) for every \( N \geq 2 \). Thus we get the sequence of the \( \mathfrak{sl}(N;\mathbb{C}) \) chiral fields tending for \( N \to \infty \) to the heavenly space-time of the metric defined by (2.2) with the function \( \Theta_0 \) given by (4.6). This metric can be written in the following null tetrad form

\[
ds^2 = e^1 \otimes e^2 + e^2 \otimes e^1 + e^3 \otimes e^4 + e^4 \otimes e^3
\]

\[
e^1 = \Omega[i\gamma - \exp(-iq)dp - ydq],
\]

\[
e^2 = \Omega[\exp(-iq)dp + ydq]
\]

\[
e^3 = \Omega dq,
\]

\[
e^4 = \Omega \exp[-(ip + y \exp(iq))]\{idx - \exp[-(ip + y \exp(iq))]dq\}
\]

\[
\Omega = \frac{1}{\sqrt{2}} \exp\left\{ \frac{1}{2} \left[ i(p + q) + y \exp(iq) \right] \right\}
\]

(4.10)

Then using the Cartan structure equations [9, 10] one finds that the only nonzero null tetrad component of the Riemann curvature tensor reads

\[
\frac{1}{2} C^{(1)} = R_{3131} = -2\Omega^{-2} \left[ \frac{1}{4} \Omega^{-4} \exp(4iq) + 1 \right]
\]

(4.11)

Consequently our heavenly space-time is of the type \([4] \times [-]\). (see [8]).

Note that if \( x, y, p \) and \( q \) are real coordinates then (4.10) defines the complex heavenly metric on \( \mathbb{R}^2 \times T^2 \). Assuming \( x, y, p \) and \( q \) to be complex we get the complex heavenly space-time (nonlinear graviton) of the holomorphic metric (4.10).

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