Tensor products of coherent configurations

Gang CHEN¹, Ilia PONOMARENKO¹,²,³

¹ School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China
² Steklov Institute of Mathematics at St. Petersburg, Russia
³ Sobolev Institute of Mathematics, Novosibirsk, Russia

© Higher Education Press 2022

Abstract A Cartesian decomposition of a coherent configuration $\mathcal{X}$ is defined as a special set of its parabolics that form a Cartesian decomposition of the underlying set. It turns out that every tensor decomposition of $\mathcal{X}$ comes from a certain Cartesian decomposition. It is proved that if the coherent configuration $\mathcal{X}$ is thick, then there is a unique maximal Cartesian decomposition of $\mathcal{X}$, i.e., there is exactly one internal tensor decomposition of $\mathcal{X}$ into indecomposable components. In particular, this implies an analog of the Krull–Schmidt theorem for the thick coherent configurations. A polynomial-time algorithm for finding the maximal Cartesian decomposition of a thick coherent configuration is constructed.

Keywords Coherent configuration, Cartesian decomposition, Krull–Schmidt theorem

MSC2020 20B99, 68R99

1 Introduction

A natural problem arising in the study of direct products in various categories is to find an analog of the Krull–Schmidt theorem which asserts that a group subject to certain conditions can be essentially uniquely written as a finite direct product of indecomposable groups. Despite a general solution proposed in [10], for many categories such an analog can be very nontrivial as, for example, for finite graphs, see [6]. A more subtle problem concerns the internal direct decompositions. Indeed, in this case, the uniqueness of the indecomposable factors up to isomorphism is replaced by the ‘absolute’ uniqueness of the set of indecomposable subobjects corresponding to the factors; a good example here is the famous Artin–Wedderburn theorem on semisimple rings. Finally, the third problem is of algorithmic nature, namely, it consists in constructing
an efficient algorithm finding a direct product decomposition of a given object
into indecomposable subobjects. Two examples here are algorithms for finding
the Wedderburn decomposition of associative semisimple algebras [5] and the
direct product decomposition of a finite permutation group [12]. In the present
paper, we examine these three problems in relation to coherent configurations.

The coherent configurations introduced by Higman [7] as a tool for studying
finite permutation groups, includes finite groups and association schemes as
full subcategories. An analog of the Krull-Schmidt theorem for commutative
coherent configurations was found in [4]. Later, it was generalized to infinite
case (with slightly weakened commutativity condition) in [14] and to
homogeneous (not necessarily commutative) coherent configurations in [13].
We did not find any result on coherent configurations related to the other two
problems mentioned above.

A coherent configuration on a finite set $\Omega$ can be thought as a finite arc-
colored directed complete graph with vertex set $\Omega$, and the color classes of
which satisfy certain combinatorial conditions (for exact definitions and related
material, see Section 2). As shown by Higman [8], the coherent configurations
on $\Omega$ are in one-to-one correspondence with coherent algebras on $\Omega$, which are
special semisimple subalgebras of the algebra of all complex matrices whose
rows and columns are indexed by the elements of $\Omega$. In the sense of this
 correspondence, the direct products of coherent configurations are exactly the
tensor product of the corresponding coherent algebras. That is why in theory
of coherent configurations we use the term ‘tensor product’ rather than ‘direct
product’.

A tensor decomposition of a coherent configuration $\mathcal{X}$ on $\Omega$ is a set of
coherent configurations $\mathcal{X}_i$ on $\Omega_i$, $i = 1, \ldots, m$, and a bijection
$$\pi: \Omega \to \Omega_1 \times \cdots \times \Omega_m,$$
which is an isomorphism from $\mathcal{X}$ to $\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_m$. Each tensor decomposition
of $\mathcal{X}$ defines a Cartesian decomposition of $\Omega$ in the sense of [1,11]. In Section 4,
we introduce the concept of Cartesian decomposition of a coherent configuration
and prove Theorems 5 and 6 that establish the following relationship between
tensor and Cartesian decompositions of coherent configurations.

**Theorem 1** Each tensor decomposition of a coherent configuration $\mathcal{X}$ defines
a uniquely determined Cartesian decomposition of $\mathcal{X}$. Conversely, every
Cartesian decomposition of $\mathcal{X}$ defines a tensor decomposition of $\mathcal{X}$.

The Cartesian decompositions of coherent configurations play the role of
‘internal’ tensor decompositions. A maximal Cartesian decomposition of $\mathcal{X}$
corresponds to a tensor decomposition of $\mathcal{X}$ in which every factor is indecomposable with respect to the tensor product. In general, $\mathcal{X}$ may
have several maximal Cartesian decompositions. For example, if the coherent
algebra corresponding to $\mathcal{X}$ is a full matrix algebra, then every Cartesian
decomposition of $\Omega$ is also a Cartesian decomposition of $\mathcal{X}$, and the number
of distinct maximal Cartesian decompositions of $\mathcal{X}$ can be exponential on $|\Omega|$. 
In this example, $\mathcal{X}$ has irreflexive basis relations of valency 1; a coherent configuration is said to be \textit{thick} if it has no such basis relations.

It is quite natural in group theory to consider groups with trivial centers, and this assumption corresponds to thick coherent configurations. Also, the readers should note that we do not use thickness when we prove general facts on tensor product and Cartesian decomposition.

\textbf{Theorem 2} Every thick coherent configuration $\mathcal{X}$ has exactly one maximal Cartesian decomposition.

To provide the uniqueness of the tensor decomposition, we make the thickness assumption (without it, the uniqueness does not necessarily hold). It is not clear whether an analog of the Krull-Schmidt theorem holds for all coherent configurations. However, as a consequence of Theorems 1 and 2, we have the following statement.

\textbf{Corollary 1} The analog of the Krull-Schmidt theorem holds for all thick coherent configurations.

Let $G$ be a finite group. The center of the group algebra of $G$ is a homogeneous coherent configuration $\mathcal{X}$ on $\Omega = G$. The valencies of the basis relations of $\mathcal{X}$ are exactly the cardinalities of the conjugacy classes of $G$. Therefore, $\mathcal{X}$ is thick if and only if the center of $G$ is trivial. Using Theorem 1, it is not hard to verify that the decompositions of $G$ into the internal direct product of normal subgroups are in one-to-one correspondence with Cartesian decompositions of $\mathcal{X}$. Thus, by Theorem 2, we obtain the following statement.

\textbf{Corollary 2} Let $G$ be a finite group with trivial center. Assume that $G = G_1 \cdots G_m$ and $G = H_1 \cdots H_k$ are two decompositions of $G$ into the internal direct product of indecomposable normal subgroups. Then the multisets $\{\{G_1, \ldots, G_m\}\}$ and $\{\{H_1, \ldots, H_k\}\}$ are equal.

In the last part of the paper, we construct an efficient algorithm finding the maximal tensor decomposition of a thick coherent configuration $\mathcal{X}$. The idea behind the algorithm is somewhat similar to that from [9]. Using a natural greedy algorithm (Algorithm A in Section 5), one can find a nontrivial Cartesian decomposition $P^*$ of the underlying set $\Omega$. The thickness condition implies that $P^*$ is a refinement of every Cartesian decomposition $P$ of $\mathcal{X}$ (Theorem 9). Since the number of all possible $P$ is at most $|\Omega|$, the maximal tensor decomposition of $\mathcal{X}$ can be found by exhaustive search of all $P$.

\textbf{Theorem 3} The maximal tensor decomposition of a thick coherent configuration of degree $n$ can be found in polynomial time in $n$.

This paper is organized as follows. The definitions and basic properties related with coherent configurations are given in Section 2. The basic facts concerning Cartesian decompositions of finite sets are discussed in Section 3. The goal of Section 4 is to introduce Cartesian decompositions of a coherent configuration and then to prove Theorems 5 and 6. In Sections 5 and 6, we prove Theorems 2 and 3, respectively.
2 Preliminaries

In our presentation of coherent configurations, we mainly follow the monograph [2], where all the details can be found.

2.1 Notation

Throughout the paper, \( \Omega \) denotes a finite set. For \( \Delta \subseteq \Omega \), the Cartesian product \( \Delta \times \Delta \) and its diagonal are denoted by \( 1_\Delta \) and \( 1_\Delta \), respectively. For a relation \( s \subseteq 1_\Omega \), we set

\[
s^* = \{ (\alpha, \beta) : (\beta, \alpha) \in s \}, \quad \alpha s = \{ \beta \in \Omega : (\alpha, \beta) \in s \}, \quad \forall \alpha \in \Omega,
\]

and define \( \langle s \rangle \) as the minimal equivalence relation on \( \Omega \), containing \( s \).

The left and right supports of \( s \) are the sets

\[
\Omega_-(s) = \{ \alpha \in \Omega : \alpha s \neq \emptyset \}, \quad \Omega_+(s) = \{ \alpha \in \Omega : \alpha s^* \neq \emptyset \},
\]

respectively.

The dot product (sometimes called the composition) of relations \( r \) and \( s \) on the same set \( \Omega \) is defined as follows:

\[
r \cdot s = \{ (\alpha, \beta) \in 1_\Omega : \alpha r \cap \beta s^* \neq \emptyset \}.
\]

Clearly, \( r \cdot s = \emptyset \) if and only if \( \Omega_+(r) \cap \Omega_-(s) = \emptyset \), and also \( (r \cdot s)^* = s^* \cdot r^* \). We say that \( r \) and \( s \) commute if \( r \cdot s = s \cdot r \).

An equivalence relation is said to be discrete if each class of it is a singleton. Given equivalence relations \( e \) and \( f \), we set

\[
e \land f = e \cap f, \quad e \lor f = \langle e \cup f \rangle.
\]

The following statement is a part of [1, Proposition 2.19].

**Lemma 1** Let \( e \) and \( f \) be commuting equivalence relations. Then \( e \cdot f = e \lor f \). Moreover, if \( e \land f = 1_\Omega \), then the intersection of any two classes, one of \( e \) and another one of \( f \), contained in the same class of \( e \cdot f \), is a singleton.

Let \( e \) be an equivalence relation on \( \Omega \) and \( \Omega/e \) the set of all classes of \( e \). Then there is a surjection

\[
\rho_e : \Omega \to \Omega/e,
\]

\[
\alpha \mapsto \alpha e.
\]

It induces a natural surjection from the binary relations on \( \Omega \) to those on \( \Omega/e \), which is also denoted by \( \rho_e \). The following lemma is straightforward.

**Lemma 2** Let \( e \) be an equivalence relation on \( \Omega \) and \( \rho = \rho_e \). Then

1. \( \rho(1_\Omega) = 1_{\Omega/e} \), \( \rho(1_\Omega) = 1_{\Omega/e} \);
2. if \( e \) commutes with \( r, s \subseteq 1_\Omega \), then \( \rho(r \cdot s) = \rho(r) \cdot \rho(s) \);
if \( e \) commutes with equivalence relations \( f, g \subseteq 1_Ω \), then

\[
\rho(f \lor g) = \rho(f) \lor \rho(g), \quad \rho(f \land g) = \rho(f) \land \rho(g).
\]

Let \( m \geq 1 \), and let \( Ω = Ω_1 \times \cdots \times Ω_m \) be the Cartesian product. The tensor product of relations \( s_1 \subseteq 1_{Ω_1}, \ldots, s_m \subseteq 1_{Ω_m} \) is defined to be

\[
s_1 \otimes \cdots \otimes s_m = \{(α, β) \in 1_Ω : (α_i, β_i) \in s_i, i = 1, \ldots, m\},
\]
where \( α_i \) and \( β_i \) are the \( i \)-th components of the \( m \)-tuples \( α \) and \( β \), respectively. In the matrix language, this means that the adjacency matrix of the tensor product is equal to the Kronecker product of the adjacency matrices of the factors. A relationship between the dot and tensor products is given by the formula

\[
(s_1 \otimes \cdots \otimes s_m) \cdot (t_1 \otimes \cdots \otimes t_m) = (s_1 \cdot t_1) \otimes \cdots \otimes (s_m \cdot t_m),
\]
the proof of which is straightforward.

We define a partial order \( \preceq \) on the sets of relations on the same set by setting \( S \preceq T \) if every relation of \( T \) is contained in some relation of \( S \).

### 2.2 Coherent configurations

Let \( Ω \) be a finite set and \( S \) a partition of \( Ω^2 \); in particular, the elements of \( S \) are treated as binary relations on \( Ω \). A pair \( X = (Ω, S) \) is called a coherent configuration on \( Ω \) if the following conditions are satisfied:

1. (C1) \( 1_Ω \) is the union of some relations of \( S \);
2. (C2) \( s^* \in S \) for all \( s \in S \);
3. (C3) given \( r, s, t \in S \), the number \( c_{rs}^t = |αr \cap βs^*| \) does not depend on \( (α, β) \in t \).

The number \( |Ω| \) is called the degree of \( X \). We say that \( X \) is trivial if \( S \) consists of \( 1_Ω \) and its complement (unless \( Ω \) is a singleton) homogeneous if \( 1_Ω \in S \), and commutative if \( c_{rs}^t = c_{sr}^t \) for all \( r, s, t \).

An isomorphism from \( X \) to a coherent configuration \( X' = (Ω', S') \) is a bijection \( π : Ω \to Ω' \) such that for each \( s \in S \), the relation

\[
π(s) = \{(α^π, β^π) : (α, β) \in s\}
\]
belongs to \( S' \). In this case, we say that \( X \) and \( X' \) are isomorphic and write \( X \cong X' \).

Let \( G \) be a permutation group on \( Ω \). Denote by \( S \) the set of all orbits \( (α, β)^G \) of the induced action of \( G \) on \( Ω \times Ω \), \( α, β \in Ω \). Then the pair \( (Ω, S) \) is a coherent configuration; we say that it is the coherent configuration associated with \( G \).

### 2.3 Relations

The elements of \( S = S(X) \) are called basis relations of the coherent configuration \( X \). We say that \( s \in S \) is reflexive if \( s \subseteq 1_Ω \), and irreflexive...
otherwise. The unique basis relation containing the pair \((\alpha, \beta)\) is denoted by 
\[ r(\alpha, \beta). \]
Thus, \(r(\alpha, \beta) = r(\beta, \alpha)^*\).

A fiber of \(\mathcal{X}\) is a set \(\Delta \subseteq \Omega\) such that \(1_\Delta \in S\). By condition (C1), the set of all fibers form a partition of \(\Omega\). Moreover, the sets \(\Omega_-(s)\) and \(\Omega_+(s)\) are fibers of \(\mathcal{X}\) for all \(s \in S\). In particular, \(S\) is partitioned into the disjoint union of the sets
\[ S_{\Delta, \Gamma} = \{ s \in S \colon \Omega_-(s) = \Delta, \Omega_+(s) = \Gamma \}, \]
where \(\Delta\) and \(\Gamma\) run over the fibers of \(\mathcal{X}\).

Any union of basis relations is called a relation of \(\mathcal{X}\). The set of all of them is closed with respect to intersections, unions, and the dot product. Together with condition (C1), this implies that if \(s\) is a nonempty relation of \(\mathcal{X}\), then so are \(1_{\Omega_-(s)}\) and \(1_{\Omega_+(s)}\). Moreover, if \(s \in S\), then the latter relations are basis.

Let \(s \in S\) and \(t = 1_{\Omega_-(s)}\). The integer \(n_s = c_{ss}^t\) is positive and is equal to \(|\alpha s|\) for all \(\alpha \in \Omega_-(s)\); it is called the valency of \(s\). We say that \(s\) is thin if \(n_s = n_{s^*} = 1\); in particular, every reflexive basis relation is thin. An important property of a thin relation \(s \in S\) is that given \(r \in S\), each of the dot products \(r \cdot s\) and \(s \cdot r\) is either a basis relation or empty. A coherent configuration is said to be thick if it contains no irreflexive thin relations.

2.4 Parabolics and quotients

An equivalence relation which is a relation of the coherent configuration \(\mathcal{X}\) is called a parabolic of \(\mathcal{X}\). The set of all parabolics of \(\mathcal{X}\) is denoted by \(E = E(\mathcal{X})\). It is a lattice and the maximal and minimal elements of which are trivial parabolics \(1_\Omega\) and \(1_\Omega\), respectively; the meet \(\land\) and join \(\lor\) are exactly the operations defined in Subsection 2.1. Note that if parabolics \(e\) and \(f\) commute, then
\[ e \lor f = e \cdot f = f \cdot e. \]

We define a relation \(\perp\) on the set \(E\) as follows: \(e \perp f\) if for any two basis relations \(x \subseteq e\) and \(y \subseteq f\), we have
\[ x \cdot y \neq \emptyset \implies x \cdot y \in S. \tag{2.3} \]
This relation is not necessarily reflexive, but is symmetric. Indeed, if \(y \cdot x \neq \emptyset\), then \(x^* \cdot y^* \neq \emptyset\) and \(y \cdot x = (x^* \cdot y^*)^* \in S\). From the definition, it immediately follows that given parabolics \(e', f' \in E\), we have
\[ e' \subseteq e, f' \subseteq f, e \perp f \implies e' \perp f'. \tag{2.4} \]

**Lemma 3** If \(\mathcal{X}\) is thick and \(e, f \in E\) are such that \(e \perp f\), then \(e \land f = 1_\Omega\).

**Proof** Assume on the contrary that \(e \land f \neq 1_\Omega\). Then there is an irreflexive basis relation \(s \subseteq e \land f \subseteq e\). Since
\[ s^* \subseteq e \land f \subseteq f, \quad s \cdot s^* \neq \emptyset, \quad s^* \cdot s \neq \emptyset, \]
the condition \(e \perp f\) implies that \(s \cdot s^*, s^* \cdot s \in S\). Since
\[ 1_{\Omega_-(s)} \subseteq s \cdot s^*, \quad 1_{\Omega_+(s)} \subseteq s^* \cdot s, \]

we have \[ s \cdot s^* = 1_{\Omega_-}(s), \quad s^* \cdot s = 1_{\Omega_+}(s). \]
Consequently, \( n_s = n_{s^*} = 1 \), which is impossible, because \( \mathcal{X} \) is thick. \( \Box \)

We say that parabolics \( e, f \in E \) strongly commute if for any two basis relations \( x \subseteq e \) and \( y \subseteq f \), we have

\[ x \cdot f = f \cdot x, \quad e \cdot y = y \cdot e. \]

Again the relation ‘to be strongly commute’ is not reflexive but is symmetric. Clearly, if \( e \) and \( f \) strongly commute, then \( e \) and \( f \) commute. But the reverse statement is not true.

Let \( e \) be a parabolic of \( \mathcal{X} \). Denote by \( S_{\Omega/e} \) the set of all relations \( \rho_e(s), s \in S \). Then the pair

\[ \mathcal{X}_{\Omega/e} = (\Omega/e, S_{\Omega/e}) \]

is a coherent configuration; it is called the quotient of \( \mathcal{X} \) modulo \( e \). The mapping \( \rho_e \) induces a surjection from the relations (resp., parabolics) of \( \mathcal{X} \) to the relations (resp., parabolics) of \( \mathcal{X}_{\Omega/e} \).

### 2.5 Tensor product

Let \( \mathcal{X}_i = (\Omega_i, S_i) \) be a coherent configuration, \( i = 1, \ldots, m \). Denote by \( S_1 \otimes \cdots \otimes S_m \) the set of all tensor products \( s_1 \otimes \cdots \otimes s_m \), where \( s_i \in S_i \) for all \( i \). The pair

\[ \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_m := (\Omega_1 \times \cdots \times \Omega_m, S_1 \otimes \cdots \otimes S_m) \]

is a coherent configuration; it is called the tensor product of \( \mathcal{X}_1, \ldots, \mathcal{X}_m \). We say that the tensor product is nontrivial if each factor is of degree at least 2, and trivial otherwise. A coherent configuration is said to be indecomposable if it is not isomorphic to a nontrivial tensor product of coherent configurations.

### 3 Cartesian decompositions

#### 3.1 Atomic Cartesian decomposition of a set

In what follows, \( m \) is a positive integer and \( P = \{e_1, \ldots, e_m\} \) is a set of equivalence relations on \( \Omega \). The index set \( \{1, \ldots, m\} \) is denoted by \( M = M(P) \).

Assume that the relations \( e_i \in P \) commute pairwise. For every \( I \subseteq M \), the dot product \( P_I \) of all \( e_i, i \in I \), is again an equivalence relation (Lemma 1); when \( I \) is empty, \( P_I = 1_\Omega \). Moreover, if \( I, J \subseteq M \), then

\[ P_I \vee P_J = P_{I \cup J}. \] (3.1)

The \( P \)-complement of \( e = P_I \) is defined to be \( e' = P_{M \setminus I} \); so, \( e \vee e' = P_M \). Thus, \( L(P) = \{P_I : I \subseteq M\} \) is a (join) semilattice in which every element has a complement, and the minimal and maximal elements equal \( P_\emptyset \) and \( P_M \), respectively. Note that, in general, \( L(P) \) is not a lattice, see [1, Example 2.21].
The following statement is probably well known, but the authors are not aware of the publication where this fact was originally established.

**Theorem 4** Let $P = \{e_i: i \in M\}$ be a set of pairwise commuting equivalence relations on $\Omega$. Assume $e_i \wedge e'_i = 1_\Omega$ for all $i \in M$. Then, for all $I, J \subseteq M$,

$$P_I \wedge P_J = P_{I \cap J}.$$  

In particular, $L(P)$ is a lattice isomorphic to the Boolean lattice $2^M$.

**Proof** First, let $I \cap J = \emptyset$. In this case, $P_J \subseteq P_{M \setminus I}$, and we may assume $J = M \setminus I$, and also $|I| \leq |J|$. The proof goes by induction on $|I|$. When $|I| = 1$, the statement follows from the theorem assumption.

Let $|I| \geq 2$. Assume on the contrary that the equivalence relation $f := P_I \cap P_{M \setminus I}$ is not discrete. Then $P_I$ is not discrete; let $i \in I$ be such that $e = e_i$ is not discrete. Put $\bar{e} = P_I \setminus \{i\}$ and $g = P_{M \setminus I}$; in particular, $f = (e \lor \bar{e}) \land g$.

Clearly, $f \subseteq e \lor \bar{e}$ and $f \subseteq g$. We claim that

$$\bar{e} \subsetneq f \lor \bar{e} \subsetneq e \lor \bar{e}, \quad (f \lor \bar{e}) \land e = 1_\Omega. \quad (3.2)$$

Indeed, if $\bar{e} = f \lor \bar{e}$, then $f \subseteq \bar{e}$, and hence, $f \subseteq \bar{e} \land g$. However, $\bar{e} = P_I \setminus \{i\}$ and then by induction, $\bar{e} \land g = 1_\Omega$ implying $f = 1_\Omega$, a contradiction. This proves that $\bar{e} \subsetneq f \lor \bar{e}$. Next, by the theorem assumption, we have $e \land e' = 1_\Omega$, and hence,

$$(f \lor \bar{e}) \land e \subseteq (g \lor \bar{e}) \land e = e' \land e = 1_\Omega.$$  

This proves the second formula in (3.2). Finally, if $f \lor \bar{e} = e \lor \bar{e}$, then

$$(f \lor \bar{e}) \land e = (e \lor \bar{e}) \land e = e \neq 1_\Omega$$

in contrast to what we just proved. Thus, the diagram of the partial order between the defined equivalence relations is as in Fig. 1.

![Fig. 1 Diagram of partial order from Theorem 4](image-url)
The first formula in (3.2) implies that there are two distinct classes \( \Delta_1, \Delta_2 \) of \( \overline{e} \), contained in the same class \( \Delta \) of \( f \lor \overline{e} \). Since \( \overline{e} \) and \( e \) commute and \( \Delta \) is contained in a class of \( e \lor \overline{e} \), there are points \( \delta_1 \in \Delta_1 \) and \( \delta_2 \in \Delta_2 \) such that \( (\delta_1, \delta_2) \in e \) (Lemma 1). Since \( \delta_1, \delta_2 \in \Delta_1 \cup \Delta_2 \subseteq \Delta \), this shows that \( (\delta_1, \delta_2) \in f \lor \overline{e} \). Thus, \( (f \lor \overline{e}) \land e \neq 1_\Omega \), in contrast to the second formula in (3.2). This completes the proof for the case \( I \cap J = \emptyset \).

Now, we turn to the case \( I \cap J \neq \emptyset \). Without loss of generality, we may assume that \( |M| > 1 \). Let \( i \in I \cap J \) and put
\[
\rho = \rho_{e_i}, \quad \overline{P} = \{\rho(e_k) : k \in \overline{M}\},
\]
where for all \( K \subseteq M \), we set \( \overline{K} = K \setminus \{i\} \). By Lemma 2(2), the relations of \( \overline{P} \) pairwise commute. Furthermore, given \( \overline{K} \subseteq \overline{M} \), we have
\[
\rho(P_K) = \overline{P_K}, \tag{3.3}
\]
Indeed, the left-hand side obviously contains the right-hand side. On the other hand, if this inclusion is strict, then there exists \( l \notin K \cup \{i\} \) such that \( \rho(e_l) \subseteq \rho(P_K) \). Then
\[
eq e_l \subseteq e_l \cdot e_i \subseteq P_{K \cup \{i\}} \subseteq e'_i,
\]
and hence, \( e_l \subseteq e_i \land e'_i \), in contrast to the theorem assumption.

Thus, the set \( \overline{P} \) satisfies the theorem assumption. Using Lemma 2(3), formula (3.3), and induction on \( |P| \), we obtain
\[
\rho(P_I \land P_J) = \rho(P_I) \land \rho(P_J) = \overline{P_I} \cap \overline{P_J} = \overline{P_{I \cap J}}.
\]
Since \( i \in I \cap J \), the \( \rho \)-preimage of the left-hand side equals \( P_I \cap P_J \), and \( \overline{I} \cap \overline{J} = \overline{I \cap J} \). By formula (3.3), this implies that
\[
P_I \land P_J = \rho^{-1}(\rho(P_I \land P_J)) = \rho^{-1}(\overline{P_{I \cap J}}) = \rho^{-1}(\rho(P_{I \cap J})) = P_{I \cap J},
\]
as required. \( \square \)

**Corollary 3** In the condition of Theorem 4, assume that \( P_M = 1_\Omega \). Let \( I \subseteq M \), and let \( \Delta'_i \) be a class of \( e'_i \) for each \( i \in I \). Then the intersection of \( \Delta'_i, i \in I \), is a class of \( P_{M \setminus I} \).

**Proof** For \( |I| = 1 \), the statement is trivial. Let \( |I| > 1 \), \( i \in I \), and \( J = I \setminus \{i\} \). By induction, the intersection \( \Gamma \) of all classes \( \Delta'_j, j \in J \), is a class of \( P_{M \setminus J} \). We need to verify that \( \Delta'_i \cap \Gamma \) is a class of \( P_{M \setminus I} \). However, either \( \Delta'_i \cap \Gamma = \emptyset \) or \( \Delta'_i \cap \Gamma \) is a class of the equivalence relation
\[
eq e'_i \land P_{M \setminus J} = P_{M \setminus \{i\}} \land P_{M \setminus J} = P_{M \setminus I},
\]
where the last equality follows from Theorem 4. It remains to note that the former case is impossible. Indeed, the equivalence relations \( e'_i \) and \( P_{M \setminus J} \) commute and
\[
eq e'_i \lor P_{M \setminus J} \supseteq P_{M \setminus \{i\}} \lor P_{\{i\}} = P_M = 1_\Omega.
\]
Thus, each class of $e'_i$ intersects each class of $P_M\setminus J$ (Lemma 1).

From Corollary 3, for $I = M$, it follows that if $P_M = 1_\Omega$ and $|M| > 1$, then \{\(e'_i: i \in M\)\} is a Cartesian decomposition of $\Omega$ in the sense of \[1, \text{Section 3.1}\]. In particular, there is a well defined bijection
\[
\pi_P: \Omega/e'_1 \times \cdots \times \Omega/e'_m \to \Omega,
\]
\[
(\Delta'_1, \ldots, \Delta'_m) \mapsto \delta,
\] (3.4)
where $\delta$ is a unique point of the intersection $\Delta'_1 \cap \cdots \cap \Delta'_m$ (Corollary 3). It immediately follows that
\[
\rho_{e'_i}(\pi_P(\Delta'_1, \ldots, \Delta'_m)) = \Delta'_i, \quad i \in M,
\] (3.5)
and
\[
|\Omega| = \prod_{i \in M} |\Omega/e'_i|.
\] (3.6)

Moreover, Theorem 4 shows that $L(P)$ is the Cartesian lattice defined by this decomposition (again in the sense of \[1, \text{Section 3.1}\]), and $P$ consists of the atoms of this lattice.

**Definition 1** A nonempty set $P$ of nondiscrete equivalence relations on $\Omega$ is called an atomic Cartesian decomposition of $\Omega$, if they pairwise commute and for every $e \in P$,
\[
e \land e' = 1_\Omega, \quad e \lor e' = 1_\Omega.
\] (3.7)
The atomic Cartesian decomposition is trivial if $|P| = 1$.

The mapping $P \mapsto P' = \{e': e \in P\}$ is a poset anti-isomorphism between the atomic and ordinary Cartesian decompositions of the same set $\Omega$. An advantage to use atomic Cartesian decomposition comes from the fact that condition (3.7) is sometimes more easier to check (at least from algorithmic point of view) than the conclusion of Corollary 3 for $I = M$.

### 3.2 Standard atomic Cartesian decomposition

Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$, where $|\Omega_i| > 1$ for all $i \in M$. Denote by $e_i$ the equivalence relation on $\Omega$ defined by the equalities at all coordinates except for the $i$th one, i.e.,
\[
e_i = 1_{\Omega_1} \otimes \cdots \otimes 1_{\Omega_{i-1}} \otimes 1_{\Omega_i} \otimes 1_{\Omega_{i+1}} \otimes \cdots \otimes 1_{\Omega_m}, \quad i \in M.
\] These equivalence relations are obviously pairwise commuting. Put $P = \{e_i: i \in M\}$. Then the $P$-complement of $e_i$ is given by formula
\[
e'_i = 1_{\Omega_1} \otimes \cdots \otimes 1_{\Omega_{i-1}} \otimes 1_{\Omega_i} \otimes 1_{\Omega_{i+1}} \otimes \cdots \otimes 1_{\Omega_m}, \quad i \in M.
\] It easily follows that $P$ is an atomic Cartesian decomposition of $\Omega$; it is said to be a standard one.
Lemma 4 Let $P$ be an arbitrary atomic Cartesian decomposition and $\pi = \pi_P$ the bijection (3.4). Then $\pi^{-1}(P)$ is the standard atomic Cartesian decomposition of the $\text{Dom}(\pi)$.

**Proof** The set $\text{Dom}(\pi)$ consists of the tuples $\Delta' = (\Delta'_1, \ldots, \Delta'_m)$, where $\Delta'_i$ is a class of $e'_i, i \in M$. The definitions of $e_i$ and $\pi$ imply that

$$\pi^{-1}(e_i) = \{(\Delta', \Gamma'): \Delta'_j = \Gamma'_j, \forall j \in M \setminus \{i\}\}$$

is exactly the equivalence relation on $\text{Dom}(\pi)$ defined by the equalities at all coordinates except for the $i$th one. $\square$

Let $I = \{i_1, \ldots, i_k\}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq m$. The projection of $\Omega$ with respect to $I$ is defined to be the surjection

$$\text{pr}_I: \Omega \to \Omega_{i_1} \times \cdots \times \Omega_{i_k},$$

$$(\alpha_1, \ldots, \alpha_m) \mapsto (\alpha_{i_1}, \ldots, \alpha_{i_k}).$$

It is easily seen that $\text{pr}_I(P) := \{\text{pr}_I(e_{i_1}), \ldots, \text{pr}_I(e_{i_k})\}$ is the standard atomic Cartesian decomposition of $\text{pr}_I(\Omega) = \Omega_{i_1} \times \cdots \times \Omega_{i_k}$.

### 3.3 Properties of atomic Cartesian decompositions

The atomic Cartesian decompositions on $\Omega$ are partially ordered with respect to the relation $\preceq$. The following lemma shows that nontrivial upper ideals of this partially ordered set admit a good description.

Lemma 5 Let $P$ be an atomic Cartesian decomposition of $\Omega$. Then

1. for every partition $\mathcal{R}$ of $M$, the set $P^\mathcal{R} = \{P_I: I \in \mathcal{R}\}$ is an atomic Cartesian decomposition of $\Omega$;
2. if $R \preceq P$ is an atomic Cartesian decomposition of $\Omega$, then $R = P^\mathcal{R}$ for some partition $\mathcal{R}$ of $M$.

**Proof** Statement (1) immediately follows from formula (3.1) and Theorem 4. To prove statement (2), let $f \in R$ and

$$M(f) = \{i \in M: e_i \subseteq f\}.$$ 

Since $R \preceq P$, every element of $P$ is contained in either $f$, or contained in the $R$-complement $f'$ of $f$. Thus,

$$P_{M(f)} \subseteq f, \quad P_{M \setminus M(f)} \subseteq f'.$$  \hspace{1cm} (3.8)

In particular, the nonempty sets $M(f), f \in R$, are pairwise disjoint and form a partition $\mathcal{R}$ of $M$. It suffices to verify that $P_{M(f)} = f$ for all $f \in R$.

By statement (1) applied to $P$ and $\mathcal{R}$, each of the sets $\{P_{M(f)}, P_{M \setminus M(f)}\}$ and $\{f, f'\}$ is an atomic Cartesian decomposition of $\Omega$. By formula (3.6), this yields

$$|\Omega/P_{M(f)}| |\Omega/P_{M \setminus M(f)}| = |\Omega| = |\Omega/f||\Omega/f'|.$$
On the other hand, by formula (3.8), we have

\[ |\Omega/P_M(f)| \geq |\Omega/f|, \quad |\Omega/P_M\setminus(f)| \geq |\Omega/f'|. \]

These two formulas together imply that

\[ |\Omega/P_M(f)| = |\Omega/f|, \quad |\Omega/P_M\setminus(f)| = |\Omega/f'|, \]

which, in particular, means that \( P_M(f) = f \), as required.

Let \( P \) and \( Q \) be atomic Cartesian decompositions of \( \Omega \). In general, the set

\[ P \cap Q = \{ e \land f : e \in P, f \in Q, e \land f \neq 1_\Omega \} \]

is not an atomic Cartesian decomposition. The following statement shows that this never happens if there is an atomic Cartesian decomposition of \( \Omega \), which is a common refinement of \( P \) and \( Q \).

**Lemma 6** Let \( Q \) and \( R \) be atomic Cartesian decompositions of \( \Omega \). Assume that there is an atomic Cartesian decomposition \( P \) of \( \Omega \) such that \( Q \cap R \preceq P \). Then \( Q \cap R \) is also an atomic Cartesian decomposition of \( \Omega \).

**Proof** We have

\[ Q \preceq Q \cap R \preceq P, \quad R \preceq Q \cap R \preceq P. \]

By Lemma 5 (2), this implies that there exist partitions \( \mathcal{Q}, \mathcal{R} \) of \( M \) such that

\[ Q = P_{\mathcal{Q}}, \quad R = P_{\mathcal{R}}. \]

For any \( f \in Q \) and any \( g \in R \), there are \( I \in \mathcal{Q} \) and \( J \in \mathcal{R} \) for which \( f = P_I \) and \( g = P_J \). By Theorem 4, we have

\[ f \land g = P_I \land P_J = P_{I \cap J}. \]

The nonempty sets \( I \cap J \) form a partition \( \mathcal{P} \) of the set \( M \). Consequently, \( Q \cap R = P_{\mathcal{P}} \) and we are done by Lemma 5 (1). \( \square \)

### 4 Cartesian decompositions and tensor products

Unless otherwise stated, in what follows, we always assume that \( \mathcal{X} = (\Omega, S) \) is a coherent configuration and \( E = E(\mathcal{X}) \).

**Definition 2** An (atomic) Cartesian decomposition \( P \) of \( \Omega \) is called the (atomic) Cartesian decomposition of \( \mathcal{X} \), if \( P \subseteq E \) and each \( e \in P \) strongly commutes with \( e' \), and also \( e \perp e' \).

Clearly, the poset anti-isomorphism between the atomic and ordinary Cartesian decompositions of a set \( \Omega \) induces a poset anti-isomorphism between the atomic and ordinary Cartesian decompositions of \( \mathcal{X} \). Thus, in the sequel, we deal with atomic Cartesian decompositions only.
Let \( P \) be an atomic Cartesian decomposition of \( \mathcal{X} \). Clearly, the set \( \{e, e'\} \) is also an atomic Cartesian decomposition of \( \mathcal{X} \) for all \( e \in P \). Furthermore, for any \( I \subseteq M \), the parabolic \( P_{M \setminus I} \) commutes with all relations of \( P \). By Lemma 2, this implies that \( \rho_e(P) = \{ \rho_e(f) : f \in P \} \) is an atomic Cartesian decomposition of the coherent configuration \( \mathcal{X}_{\Omega/e} \), where \( e = P_{M \setminus I} \).

**Lemma 7** Let \( P \) be an atomic Cartesian decomposition of \( \mathcal{X} \), and let \( \Delta, \Gamma \) be fibers of \( \mathcal{X} \). Then

1. \( \Delta = \Gamma \) if and only if \( \rho_e(\Delta) = \rho_e(\Gamma) \) for all \( e \in P \);
2. the restriction of \( \rho_e' \) to \( \{ s \in S_{\Delta, \Gamma} : s \subseteq e \} \) is injective for all \( e \in P \).

**Proof** (1) Let \( \pi = \pi_P \). Recall that \( \{e, e'\} \) is a Cartesian decomposition of \( \Omega \). We claim that

\[
\rho_e(\Delta) \times \rho_e'(\Delta) = \pi^{-1}(\Delta).
\]

Indeed, the inclusion \( \supseteq \) is obvious. To prove the reverse inclusion, let \( \Lambda \in \rho_e(\Delta) \) and \( \Lambda' \in \rho_e'(\Delta) \), i.e., there are points \( \alpha, \alpha' \in \Delta \) such that \( \rho_e(\alpha) \in \Lambda \) and \( \rho_e'(\alpha') \in \Lambda' \). By the definition of \( \pi \), \( \pi(\Lambda, \Lambda') \) is equal to the unique point \( \beta \) of the set \( \Lambda \cap \Lambda' \). We need to verify that \( \beta \in \Delta \).

Note that \( r(\alpha, \beta) \subseteq e \), because both \( \alpha \) and \( \beta \) belong to the class \( \Lambda \) of \( e \), and similarly, \( r(\beta, \alpha') \subseteq e' \). Since \( e \perp e' \), we have

\[
r(\alpha, \alpha') = r(\alpha, \beta) \cdot r(\beta, \alpha').
\]

Furthermore, \( e \) and \( e' \) strongly commute. Consequently, there exists a basis relation \( s' \subseteq e' \) such that

\[
r(\alpha, \alpha') = s' \cdot r(\alpha, \beta).
\]

It follows that \( \beta \in \Omega_+(s) \) with \( s = r(\alpha, \alpha') \). It remains to note that \( \Omega_+(s) = \Delta \), because \( \alpha' \in \Delta \). The claim is proved.

Now, from the claim, it easily follows that for every fiber \( \Lambda \) of \( \mathcal{X} \), we have

\[
\pi(\Lambda) = \rho_{e_1}(\Lambda) \times \cdots \times \rho_{e_m}(\Lambda).
\]

Therefore, if \( \pi_{e_i}(\Delta) = \pi_{e_i}(\Gamma) \) for all \( i \in M \), then \( \pi(\Delta) = \pi(\Gamma) \). Since \( \pi \) is a bijection, we obtain \( \Delta = \Gamma \).

(2) Let \( s, t \in S_{\Delta, \Gamma} \) be such that \( \rho_e(s) = \rho_e(t) \). Then there exist classes \( \Lambda', \Lambda'' \) of \( e' \) and points

\[
\alpha_s, \alpha_t \in \Lambda' \cap \Delta, \quad \beta_s, \beta_t \in \Lambda'' \cap \Gamma,
\]

for which \( (\alpha_s, \beta_s) \in s \) and \( (\alpha_t, \beta_t) \in t \). Assume that \( r(\alpha_s, \beta_s), r(\alpha_t, \beta_t) \subseteq e \). Since \( e \perp e' \) and \( e, e' \) strongly commute, we have

\[
r(\alpha_t, \alpha_s) \cdot r(\alpha_s, \beta_s) = r(\alpha_s, \beta_s) \cdot s' = s \cdot s'.
\]
for some $s' \in S$ contained in $e'$. It follows that the set $\alpha_t r(\alpha_s, \beta_s) = \alpha_t s$ intersects the class $\Lambda''$. Thus, we may assume that $\alpha_s = \alpha_t =: \alpha$. Now,

$$(\alpha, \beta_s) \in s \subseteq e, \quad (\alpha, \beta_t) \in t \subseteq e.$$ 

Therefore, $(\beta_s, \beta_t) \in e$. On the other hand, $(\beta_s, \beta_t) \in e'$ by formula (4.1). Since $e \wedge e' = 1_{\Omega}$, this implies that $\beta_s = \beta_t =: \beta$. Thus, $s$ and $t$ contain a common pair $(\alpha, \beta)$ and hence $s = t$. \hfill \Box

The following two theorems establish a one-to-one correspondence between the tensor decompositions and atomic Cartesian decompositions of $\mathcal{X}$.

**Theorem 5** Let $\mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_m$, where $\mathcal{X}_i$ is a coherent configuration on $\Omega_i$, $i = 1, \ldots, m$. Then the standard atomic Cartesian decomposition $P = \{e_1, \ldots, e_m\}$ of $\Omega = \Omega_1 \times \cdots \times \Omega_m$ is an atomic Cartesian decomposition of $\mathcal{X}$. Moreover,

$$\mathcal{X}_i \cong \mathcal{X}_{\Omega/e_i}, \quad i = 1, \ldots, m.$$ 

**Proof** Clearly, $P \subseteq E$. Next, $\mathcal{X} = \mathcal{X}_i \otimes \mathcal{X}_i'$ for all $i$, where

$$\mathcal{X}_i' = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_{i-1} \otimes \mathcal{X}_{i+1} \otimes \cdots \otimes \mathcal{X}_m$$

is a coherent configuration on

$$\Omega_i' = \Omega_1 \times \cdots \times \Omega_{i-1} \times \Omega_{i+1} \times \cdots \times \Omega_m.$$ 

Let $x \subseteq e_i$ and $x' \subseteq e'_i$ be arbitrary basis relations of $\mathcal{X}$. There exist sets $\Delta_i \subseteq \Omega_i$ and $\Delta_i' \subseteq \Omega_i'$ (which are, in fact, fibers of $\mathcal{X}_i$ and $\mathcal{X}_i'$, respectively), and relations $s_i \in S(\mathcal{X}_i)$ and $s_i' \in S(\mathcal{X}_i')$ such that

$$x = s_i \otimes 1_{\Delta_i}, \quad x' = 1_{\Delta_i} \otimes s_i'.$$

Assume that $x \cdot x' \neq \emptyset$. Then

$$\Delta_i = \Omega_+(s_i), \quad \Delta_i' = \Omega_-(s_i').$$

By formula (2.2), we obtain

$$x \cdot x' = (s_i \otimes 1_{\Delta_i'}) \cdot (1_{\Delta_i} \otimes s_i') = (s_i \cdot 1_{\Delta_i}) \otimes (1_{\Delta_i'} \cdot s_i') = s_i \otimes s_i'.$$

Now, the relation on the right-hand side belongs to $S$. Thus, $e_i \perp e'_i$. Next, $x'' = 1_{\Omega_-(s_i)} \otimes s_i'$ is a basis relation of $\mathcal{X}$, contained in $e'_i$. Moreover, again by formula (2.2), we have

$$x \cdot x' = s_i \otimes s_i' = (1_{\Omega_-(s_i)} \otimes s_i') \cdot (s_i \otimes 1_{\Delta_i'}) = x'' \cdot x.$$ 

Hence, $x \cdot e'_i = e'_i \cdot x$. Similarly, one can verify that $e_i \cdot y = y \cdot e_i$. Therefore, $e_i$ and $e'_i$ strongly commute. Thus, $P$ is an atomic Cartesian decomposition of $\mathcal{X}$.
The statement about the isomorphism follows from [2, Theorem 3.2.9] applied to the tensor product $\mathcal{X}' = \mathcal{X}_i \otimes \mathcal{X}_i'$.

The reverse statement to Theorem 5 is given below.

**Theorem 6**  Let $\mathcal{X}$ be a coherent configuration on $\Omega$, $P$ an atomic Cartesian decomposition of $\mathcal{X}$, $\pi_P$ the bijection (3.4), and

$$\mathcal{X}' = \bigotimes_{e \in P} \mathcal{X}_{\Omega/e \prime}.$$  

Then $\pi_P$ is an isomorphism from $\mathcal{X}'$ to $\mathcal{X}$, and $P = \pi_P(P')$, where $P'$ is the standard atomic Cartesian decomposition of $\mathcal{X}'$.

**Proof**  Let $P = \{e_1, \ldots, e_m\}$ and $\pi = \pi_P$. By Lemma 4, we have $P = \pi(P')$; in particular, $P' = \{f_1, \ldots, f_m\}$, where $f_i = \pi(e_i)$, $i \in M$. To prove that $\pi$ is an isomorphism from $\mathcal{X}'$ to $\mathcal{X}$, we verify that $\pi(s') \in S$ for all basis relations $s'$ of $\mathcal{X}'$. Let

$$s' = s'_1 \otimes \cdots \otimes s'_m,$$

where $s'_i \in S(\mathcal{X}_i)$, $i \in M$.

First, assume that $s'$ is reflexive. In this case, $\pi(s') \subseteq 1_\Omega$, and $s'_i = 1_{\Delta'_i}$ for all $i \in M$, where $\Delta'_i$ is a fiber of the coherent configuration $\mathcal{X}_{\Omega/e'_i}$. By formula (3.5),

$$\rho_{e'_i}(\pi(\Delta'_1, \ldots, \Delta'_m)) = \Delta'_i, \quad i \in M.$$  

It follows that $\pi(\Delta'_1, \ldots, \Delta'_m)$ is equal to the intersection $\Delta$ of the $\rho_{e'_i}$-preimages of the sets $\Delta'_i$, $i \in M$. Each of these preimages is a union of some fibers of $\mathcal{X}$. Moreover, $\Delta$ cannot contain two distinct fibers of $\mathcal{X}$ by Lemma 7 (1). Thus, $\Delta$ is a fiber of $\mathcal{X}$ and $\pi(s') = 1_\Delta \in S$. In fact, we proved that $\pi$ takes the fibers of $\mathcal{X}'$ to the fibers of $\mathcal{X}$.

Now, let $s'$ be irreflexive. We use induction on the number $k$ of all $i \in M$ for which the relation $s'_i$ is irreflexive. Let $k = 1$. Then $s' \subseteq f_i$ for exactly one $i \in M$. Therefore, $\pi(s') \subseteq \pi(f_i) = e_i$. Since also $\pi(s') \subseteq \rho_{e'_i}^{-1}(s'_i)$, we obtain

$$\pi(s') \subseteq e_i \cap \rho_{e'_i}^{-1}(s'_i).$$  

The reverse inclusion follows from the definition of $\pi$. Thus, $\pi(s')$ equals the right-hand side of the above formula. On the other hand, $s' \in S'_{\Delta', \Gamma'}$, where $S' = S(\mathcal{X}')$ and $\Delta', \Gamma'$ are fibers of $\mathcal{X}'$. From the above paragraph, it follows that $\Delta = \pi(\Delta')$ and $\Gamma = \pi(\Gamma')$ are fibers of $\mathcal{X}$. Thus, $\pi(s')$ is a union of some relations belonging to the set $\{s \in S_{\Delta, \Gamma}: s \subseteq e_i\}$. From Lemma 7 (2), it follows that there is a unique $s$ such that $\rho_{e'_i}(s) = \pi(s')$. Thus, $\pi(s') = s \in S$.

Let $k > 1$. Take any $i \in \{1, \ldots, m\}$ for which $s'_i$ is irreflexive; to simplify the notation, let $i = 1$. Put

$$x = s'_1 \otimes 1_{\Omega_-(s''_i)}, \quad y = 1_{\Omega_+(s'_1)} \otimes s''.$$
where \( s'' = s_2 \otimes \cdots \otimes s_m \). Then \( s' = x \cdot y \) by formula (2.2). Furthermore, \( \pi(x), \pi(y) \in S \) by induction. Thus,
\[
\pi(s') = \pi(x \cdot y) = \pi(x) \cdot \pi(y).
\]
On the other hand, \( x \subseteq f_1 \) and \( y \subseteq f'_1 \). Consequently, \( \pi(x) \subseteq e_1 \) and \( \pi(y) \subseteq e'_1 \). Since \( e_1 \perp e'_1 \), we have \( \pi(s') = \pi(x) \cdot \pi(y) \in S \), as required. \( \square \)

**Corollary 4** Let \( P = \{e_1, \ldots, e_m\} \) be an atomic Cartesian decomposition of a coherent configuration \( \mathcal{X} \). Assume that \( e \) and \( f \) are parabolics of \( \mathcal{X} \), and also \( e \subseteq e_i \) and \( f \subseteq e_j \) for some \( 1 \leq i \neq j \leq m \). Then \( e \) and \( f \) strongly commute.

**Proof** By Theorem 6, we may assume that \( \mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_m \) and \( P \) is the standard atomic Cartesian decomposition of \( \Omega = \Omega_1 \otimes \cdots \otimes \Omega_m \), where \( \mathcal{X}_i \) is a coherent configuration on \( \Omega_i \), \( i = 1, \ldots, m \). Then the rest of the proof is as that of Theorem 5. \( \square \)

**Theorem 7** Let \( \mathcal{X} = \mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_m \) and \( P \) the standard atomic Cartesian decomposition of \( \mathcal{X} \). If the coherent configuration \( \mathcal{X}_i \) is indecomposable, \( i = 1, \ldots, m \), then \( P \) is a \( \preceq \)-maximal atomic Cartesian decomposition of \( \mathcal{X} \).

**Proof** Assume on the contrary that \( P \) is not \( \preceq \)-maximal. Then there exists an atomic Cartesian decomposition \( R \neq P \) of \( \mathcal{X} \), such that \( P \preceq R \). By Lemma 5 (2), there exists a partition \( R \) of the set \( M \) for which \( P = R^F \). Since \( R \neq P \), one can find a non-singleton set \( I \subseteq \{1, \ldots, |R|\} \) such that \( R_I = e_i \in P \) for some \( i \in M \). Then \( pr_I(R) \) is a nontrivial atomic Cartesian decomposition of \( \mathcal{X}_i \). By Theorem 6, this implies that \( \mathcal{X}_i \) is decomposable, a contradiction. \( \square \)

## 5 Uniqueness of maximal Cartesian decomposition

### 5.1 Irredundant relations

Let \( \mathcal{X} = (\Omega, S) \) be a coherent configuration. A relation \( s \in S \) is said to be redundant if there exist irreflexive relations \( x, y \in S \) such that
\[
s = x \cdot y, \quad \langle x \rangle \land \langle y \rangle = 1_\Omega;
\]
otherwise, \( s \) is said to be irredundant. Thus, every redundant relation is the dot product of at least two irreflexive basis relations. One can also see that \( s^* \) is redundant if and only if so is \( s \).

**Lemma 8** If \( s \in S \) satisfies conditions (5.1) for some \( x, y \in S \), then \( n_s = n_x n_y \).

**Proof** We claim that \( c_{xy}^s = 1 \). Indeed, otherwise, \( c_{xy}^s \geq 2 \) and
\[
|\alpha x \cap \beta y^*| = c_{xy}^s \geq 2,
\]
where \( \alpha \) and \( \beta \) are points such that \( r(\alpha, \beta) = s \). For any two distinct points \( \gamma \) and \( \delta \) of \( \alpha x \cap \beta y^* \), we have
\[
r(\gamma, \delta) \subseteq (r(\gamma, \alpha) \cdot r(\alpha, \delta)) \cap (r(\gamma, \beta) \cdot r(\beta, \delta)) = (x^* \cdot x) \cap (y^* \cdot y) \subseteq \langle x \rangle \cap \langle y \rangle = 1_\Omega.
\]
Consequently, $\gamma = \delta$, a contradiction. The claim is proved.

Next, since $s \in S$ and $s = x \cdot y$, we have $c^t_{xy} = 0$ for all $t \neq s$. By the well-known identity for the intersection numbers (see, e.g., [2, formula (2.1.8)]), this implies that

$$n_s = n_s c^s_{xy} = \sum_{t \in S} n_t c^t_{xy} = n_x n_y,$$

as required. □

Given $s \in S$, put $d(s) = n_s n_s^*$. Clearly, $d(s) = 1$ if and only if the relation $s$ is thin. The invariant $d(s)$ is especially useful if the relation $s$ is redundant.

**Corollary 5** Under the condition of Lemma 8, we have $d(s) = d(x)d(y)$.

**Proof** By Lemma 8, $n_s = n_x n_y$ and similarly, $n_s^* = n_x^* n_y^*$. Thus,

$$d(s) = n_s n_s^* = (n_x n_y) (n_x^* n_y^*) = (n_x n_x^*) (n_y n_y^*) = d(x)d(y),$$

as required. □

Not every coherent configuration has at least one irredundant relation: take a coherent configuration associated with regular elementary abelian group of rank at least 2. However, as the following lemma shows, every thick coherent configuration has irredundant relations.

**Lemma 9** Let $\mathcal{X}$ be a thick coherent configuration, and let $d = \min_s d(s)$, where $s$ runs over the irreflexive basis relations of $\mathcal{X}$. Then every $s \in S$ with $d(s) \leq d$ is irredundant.

**Proof** Let $s \in S$ be such that $d(s) \leq d$. Assume on the contrary that $s$ is redundant. Then there are irreflexive relations $x, y \in S$ such that formula (5.1) holds. By Corollary 5, we have

$$d \geq d(s) = d(x)d(y) \geq d^2,$$

whence $d = 1$. Consequently, $d(x) = d(y) = 1$. Thus, the relations $x$ and $y$ are thin. Since the coherent configuration $\mathcal{X}$ is thick, they are reflexive, a contradiction. □

**Theorem 8** Let $\mathcal{X}$ be a thick coherent configuration and $s \in S$. Then there exist irredundant relations $s_1, \ldots, s_k \in S$ such that $s = s_1 \cdots s_k$.

**Proof** By Lemma 9, we may assume $d(s) > d$. If $s$ is irredundant, then we are done with $k = 1$. Otherwise, formula (5.1) holds for some irreflexive $x, y \in S$. Then $d(s) = d(x)d(y)$ by Corollary 5. Since $\mathcal{X}$ is thick and $x, y$ are irreflexive, we have $d(x) > 1$ and $d(y) > 1$. Therefore, $d(x) < d(s)$ and $d(y) < s$. Now, the statement follows by induction. □

### 5.2 Atomic Cartesian decomposition $P^*$

Let $\mathcal{X}$ be a coherent configuration. We define a special set $P^* \subseteq E$ as the output of the following procedure.
Algorithm A

Step 1 Set $Q = \{s : s \in S \text{ is irredundant}\}$. 

Step 2 While there are distinct $e, f \in Q$ that are not strongly commute or $e \not\perp f$, set 
$$Q := (Q \setminus \{e, f\}) \cup \{e \lor f\}.$$ 

Step 3 Output $P^* := Q \setminus \{1_\Omega\}$. 

Clearly, $P^* = \emptyset$ if and only if $\mathcal{X}$ contains no irredundant basis relations. Note that, in general, $P^*$ may depend on the choice of $e$ and $f$ at each iteration of Step 2. In any case, the cardinality of the set $Q$ defined in Step 1 is at most $|S|$, and is reduced by one at each iteration of Step 2. Thus, after at most $|S|$ iterations, Algorithm A arrives at Step 3.

The lemma below immediately follows from Lemma 9 and the description of Algorithm A.

Lemma 10 Let $\mathcal{X}$ be a thick coherent configuration. Then $P^*$ is a nonempty subset of $E$, satisfying the following conditions:

(P0) $1_\Omega \notin P^*$;
(P1) each irredundant relation of $\mathcal{X}$ is contained in some parabolic from $P^*$;
(P2) the parabolics from $P^*$ are pairwise strongly commute;
(P3) $e \perp f$ for all distinct $e, f \in P^*$.

In some special cases (e.g., if $\mathcal{X}$ is the scheme of conjugacy classes of an arbitrary group, see [2, Example 2.4.3]), $P^*$ is an atomic Cartesian decomposition of $\mathcal{X}$. It is not clear whether this is always true (if it would be so, then the proof of the main theorems can essentially be simplified): indeed, condition (P3) is much weaker than that in Definition 2. However, the following theorem implies, in particular, that $P^*$ is an atomic Cartesian decomposition of $\Omega$.

Theorem 9 Let $\mathcal{X}$ be a thick coherent configuration on $\Omega$, and let $P^* \subseteq E$ be the output of Algorithm A. Then

(i) $P^*$ is an atomic Cartesian decomposition of $\Omega$;
(ii) if $P$ is an atomic Cartesian decomposition of $\mathcal{X}$, then $P \preceq P^*$.

Proof (i) By statements (P0) and (P2) of Lemma 10, we need to verify formula (3.7) only. The first part of it follows from the lemma below for $Q = P^*$.

Lemma 11 Let $Q$ be an arbitrary set of pairwise commuting parabolics of the coherent configuration $\mathcal{X}$. Assume $e \perp f$ for all distinct $e, f \in Q$. Then $e \land e' = 1_\Omega$ for all $e \in Q$, where $e'$ is the $Q$-complement of $e$.

Proof Without loss of generality, we may assume that $|Q| \geq 2$. The proof involves the induction on the cardinality of $Q$. For $|Q| = 2$, the statement follows from Lemma 3. Now, let $|Q| \geq 3$ and $e \in Q$. 

Assume on the contrary that there is an irreflexive basis relation $s \subseteq e \land e'$. Then $d(s) > 1$ by the thickness of $\mathcal{B}$. Without loss of generality, we may assume $n_s \geq 2$; otherwise, we replace $s$ with $s^*$.

Let $f \in Q \setminus \{e\}$ and $g$ the $Q$-complement of $e \lor f$; in particular, $e' = f \lor g$. By the induction applied to $Q \setminus \{f\}$, we have

$$e \land g = 1_\Omega. \quad (5.2)$$

Moreover, since $s \subseteq e' = f \lor g = f \cdot g$, there are basis relations $x \subseteq f$ and $y \subseteq g$ such that $s \subseteq x \cdot y$. It follows that $y \subseteq x^* \cdot s$. Since also $x^* \subseteq f$, $s \subseteq e$, and $f \perp e$, we have

$$y = x^* \cdot s. \quad (5.3)$$

Take arbitrary $\alpha \in \Omega_-(s)$ and any two different points $\beta, \gamma \in \alpha s$ (recall that $n_s \geq 2$). Equality (5.3) yields $\Omega_+(x^*) = \Omega_-(s)$, and hence, there is a point $\delta \in \alpha x$ such that the chosen points form the configurations in Fig. 2.

![Fig. 2 Configuration of points $\alpha, \beta, \gamma, \delta$](image)

In particular, by (5.2), we obtain

$$r(\beta, \gamma) \subseteq (s^* \cdot s) \cap (y^* \cdot y) \subseteq e \land g = 1_\Omega,$$

which follows that $\beta = \gamma$, a contradiction. \hfill $\square$

To complete the proof of (i), assume on the contrary that $e \lor e' \neq 1_\Omega$. Then there is $s \in S$ such that $s \cap (e \lor e') = \emptyset$. Note that $s$ is redundant, because by condition (P1), every irredundant relation is contained in either $e$ or $e'$. By Theorem 8, $s = x_1 \cdots x_k$ for some $k \geq 2$ and irredundant $x_1, \ldots, x_k \in S$. Again by condition (P1), $x_i \subseteq e \lor e'$ for all $i$. Thus,

$$s = x_1 \cdots x_k \subseteq (e \lor e') \cdots (e \lor e') = e \lor e',$$

a contradiction.

(ii) Let $P = \{e_1, \ldots, e_m\}$ be an atomic Cartesian decomposition of $\mathcal{B}$. It suffices to verify that if $Q \subseteq E$ is obtained at a certain step of Algorithm A, then $P \preceq Q$, i.e., every $e \in Q$ is contained in some $e_i \in P$.

Let $Q$ be constructed at Step 1. Assume on the contrary that $e$ is contained in no $e_i \in P$. Denote by $J$ a maximal subset of $M$ such that $e \not\subseteq P_J$. Then
It follows that $e_i \subseteq P_J$ for every $i \in M \setminus J$. On the other hand, $e = \langle s \rangle$ for some irredundant $s \in S$. By formula (2.4), we have $e_i \perp P_J$, because $P_J \subseteq e_i'$ and $e_i \perp e_i'$. Consequently,

$$s = x \cdot y, \quad x \subseteq e_i, \ y \subseteq P_J.$$ 

Moreover, none of $x, y$ is reflexive: if $x$ is reflexive, then $s = y \subseteq P_J$ and hence $e = \langle s \rangle \subseteq P_J$, in contrast to the choice of $J$, whereas if $y$ is reflexive, then $s = x \subseteq e_i$ and hence $e = \langle s \rangle \subseteq e_i$, in contrast to the assumption. Finally,

$$\langle x \rangle \land \langle y \rangle \subseteq e_i \land P_J \subseteq e_i \land e_i' = 1_\Omega.$$ 

Consequently, the relation $s$ is redundant, a contradiction.

Now, let $Q$ be obtained at a certain iteration of Step 2, not just after Step 1. Denote by $R$ the set obtained at the previous iteration. Then

$$Q = (R \setminus \{f, g\}) \cup \{f \lor g\}$$

for some distinct $f, g \in R$ which are not strongly commute or $f \nmid g$. By induction, we may assume that $P \preceq R$. Consequently, $e_i \supseteq f$ and $e_j \supseteq g$ for some $i, j \in M$.

Let $e$ be an arbitrary equivalence relation of $Q$. If $e \neq f \lor g$, then $e \in R$, and hence, $e$ is contained in some element of $P$, as required. Let $e = f \lor g$. If, in addition, $i = j$, then $e = e_i \in P$. Now, let $i \neq j$. Then $e_j \subseteq e_j'$. Since $e_i \perp e_i'$, formula (2.4) shows that $f \perp g$. By the choice of $f$ and $g$, this means that $f$ and $g$ are not strongly commute. However, this contradicts Corollary 4. \qed

The following sufficient condition for a thick coherent configuration to be indecomposable is an immediate consequence of Theorem 9 (2).

**Corollary 6**  A thick coherent configuration $\mathcal{X}$ is indecomposable if $|P^*| = 1$.

### 5.3 Proofs of Theorem 2 and Corollary 1

**Proof of Theorem 2**  Because of the poset anti-isomorphism between the Cartesian and atomic Cartesian decompositions of $\mathcal{X}$, we prove the statement for the latter ones. Clearly, $\mathcal{X}$ has at least one atomic Cartesian decomposition, the trivial one. Assume that $P$ and $Q$ are $\preceq$-maximal atomic Cartesian decompositions of $\mathcal{X}$. By Theorem 9, the set $P^*$ is an atomic Cartesian decomposition of $\Omega$, and

$$P \cap Q \preceq P^*. \quad (5.4)$$

Consequently, $P \cap Q$ is also an atomic Cartesian decomposition of $\Omega$ (Lemma 6). It remains to verify the claim below, because then $P = P \cap Q = Q$ by the maximality of $P \preceq P \cap Q$ and $Q \preceq P \cap Q$.

**Claim**  $P \cap Q$ is an atomic Cartesian decomposition of $\mathcal{X}$.

**Proof**  Let $e \in P \cap Q$ and $e'$ the $(P \cap Q)$-complement of $e$. There are unique $f \in P$ and $g \in Q$ such that $e = f \land g$. Denote by $f'$ the $P$-complement of $f$. 

$$f = x \cdot y, \quad x \subseteq e_i, \ y \subseteq P_J.$$ 

Moreover, none of $x, y$ is reflexive: if $x$ is reflexive, then $s = y \subseteq P_J$ and hence $e = \langle s \rangle \subseteq P_J$, in contrast to the choice of $J$, whereas if $y$ is reflexive, then $s = x \subseteq e_i$ and hence $e = \langle s \rangle \subseteq e_i$, in contrast to the assumption. Finally,

$$\langle x \rangle \land \langle y \rangle \subseteq e_i \land P_J \subseteq e_i \land e_i' = 1_\Omega.$$ 

Consequently, the relation $s$ is redundant, a contradiction.

Now, let $Q$ be obtained at a certain iteration of Step 2, not just after Step 1. Denote by $R$ the set obtained at the previous iteration. Then

$$Q = (R \setminus \{f, g\}) \cup \{f \lor g\}$$

for some distinct $f, g \in R$ which are not strongly commute or $f \nmid g$. By induction, we may assume that $P \preceq R$. Consequently, $e_i \supseteq f$ and $e_j \supseteq g$ for some $i, j \in M$.

Let $e$ be an arbitrary equivalence relation of $Q$. If $e \neq f \lor g$, then $e \in R$, and hence, $e$ is contained in some element of $P$, as required. Let $e = f \lor g$. If, in addition, $i = j$, then $e = e_i \in P$. Now, let $i \neq j$. Then $e_j \subseteq e_j'$. Since $e_i \perp e_i'$, formula (2.4) shows that $f \perp g$. By the choice of $f$ and $g$, this means that $f$ and $g$ are not strongly commute. However, this contradicts Corollary 4. \qed

The following sufficient condition for a thick coherent configuration to be indecomposable is an immediate consequence of Theorem 9 (2).

**Corollary 6**  A thick coherent configuration $\mathcal{X}$ is indecomposable if $|P^*| = 1$.

### 5.3 Proofs of Theorem 2 and Corollary 1

**Proof of Theorem 2**  Because of the poset anti-isomorphism between the Cartesian and atomic Cartesian decompositions of $\mathcal{X}$, we prove the statement for the latter ones. Clearly, $\mathcal{X}$ has at least one atomic Cartesian decomposition, the trivial one. Assume that $P$ and $Q$ are $\preceq$-maximal atomic Cartesian decompositions of $\mathcal{X}$. By Theorem 9, the set $P^*$ is an atomic Cartesian decomposition of $\Omega$, and

$$P \cap Q \preceq P^*. \quad (5.4)$$

Consequently, $P \cap Q$ is also an atomic Cartesian decomposition of $\Omega$ (Lemma 6). It remains to verify the claim below, because then $P = P \cap Q = Q$ by the maximality of $P \preceq P \cap Q$ and $Q \preceq P \cap Q$.

**Claim**  $P \cap Q$ is an atomic Cartesian decomposition of $\mathcal{X}$.

**Proof**  Let $e \in P \cap Q$ and $e'$ the $(P \cap Q)$-complement of $e$. There are unique $f \in P$ and $g \in Q$ such that $e = f \land g$. Denote by $f'$ the $P$-complement of $f$. 

$$f = x \cdot y, \quad x \subseteq e_i, \ y \subseteq P_J.$$ 

Moreover, none of $x, y$ is reflexive: if $x$ is reflexive, then $s = y \subseteq P_J$ and hence $e = \langle s \rangle \subseteq P_J$, in contrast to the choice of $J$, whereas if $y$ is reflexive, then $s = x \subseteq e_i$ and hence $e = \langle s \rangle \subseteq e_i$, in contrast to the assumption. Finally,

$$\langle x \rangle \land \langle y \rangle \subseteq e_i \land P_J \subseteq e_i \land e_i' = 1_\Omega.$$ 

Consequently, the relation $s$ is redundant, a contradiction.

Now, let $Q$ be obtained at a certain iteration of Step 2, not just after Step 1. Denote by $R$ the set obtained at the previous iteration. Then

$$Q = (R \setminus \{f, g\}) \cup \{f \lor g\}$$

for some distinct $f, g \in R$ which are not strongly commute or $f \nmid g$. By induction, we may assume that $P \preceq R$. Consequently, $e_i \supseteq f$ and $e_j \supseteq g$ for some $i, j \in M$.

Let $e$ be an arbitrary equivalence relation of $Q$. If $e \neq f \lor g$, then $e \in R$, and hence, $e$ is contained in some element of $P$, as required. Let $e = f \lor g$. If, in addition, $i = j$, then $e = e_i \in P$. Now, let $i \neq j$. Then $e_j \subseteq e_j'$. Since $e_i \perp e_i'$, formula (2.4) shows that $f \perp g$. By the choice of $f$ and $g$, this means that $f$ and $g$ are not strongly commute. However, this contradicts Corollary 4. \qed
Since \( P \cap Q \) is an atomic Cartesian decomposition of \( \Omega \), Lemma 5 (2) implies that \( f, f', g \in L(P \cap Q) \) and also \( f = e \cdot \overline{f} \) for some \( \overline{f} \in L(P \cap Q) \). Thus,

\[
e' = \overline{f} \cdot f'.
\] (5.5)

Let \( x, y \in S \) be arbitrary relations such that \( x \subseteq e \) and \( y \subseteq e' \). Note that \( \overline{f} \perp f' \), because \( \overline{f} \subseteq f \) and \( f \perp f' \). Since the parabolics \( \overline{f}, f' \in L(P \cap Q) \) commute and in view of (5.5), this implies that there exist \( \overline{y}, y' \in S \) such that \( \overline{y} \subseteq \overline{f}, y' \subseteq f' \), and

\[
y = \overline{y} \cdot y'.
\]

In order to verify \( e \perp e' \), we prove that \( x \cdot y \in S \) provided that \( x \cdot y \neq \emptyset \). We have

\[
x \subseteq e \subseteq g, \quad \overline{y} \subseteq \overline{f} \subseteq g'.
\]

Since \( g \perp g' \), this implies that \( x \cdot \overline{y} \in S \) (note that \( x \cdot \overline{y} \neq \emptyset \), for otherwise, \( x \cdot y = x \cdot \overline{y} \cdot y' = \emptyset \)). Taking into account that \( x \cdot \overline{y} \subseteq f, y' \subseteq f' \), and \( f \perp f' \), we obtain

\[
x \cdot y = x \cdot (\overline{y} \cdot y') = (x \cdot \overline{y}) \cdot y' \in S,
\]

as required.

Let us verify that \( e \) and \( e' \) strongly commute. Note that \( x \) is contained in both \( g \) and \( f \). Since \( g \) strongly commutes with \( g' \supseteq \overline{f} \) and \( f \) strongly commutes with \( f' \supseteq y' \), there are basis relations \( z \subseteq g' \) and \( z' \subseteq f' \) such that

\[
x \cdot y = x \cdot (\overline{y} \cdot y') = (x \cdot \overline{y}) \cdot y' = z \cdot x \cdot y' = z' \cdot x.
\]

Take an arbitrary \( z \in S \) contained in \( z \cdot z' \subseteq g' \cdot f' \subseteq e' \) and such that \( z \cdot x = \emptyset \) if \( x \cdot y = \emptyset \), and \( z \cdot x \) intersects \( x \cdot y \) otherwise. In the former case, \( z \cdot x = x \cdot y \).

In the latter case, we note that \( e' \perp e \) by the above, and hence, \( x \cdot y, z \cdot x \in S \).

Thus, \( z \cdot x = x \cdot y \), and hence, \( x \cdot e' = e' \cdot x \). Similarly, one can verify that \( e \cdot y = y \cdot e \).

\[ \square \]

**Proof of Corollary 1**

Let \( \mathcal{X} \) be a thick coherent configuration. Without loss of generality, we may assume that \( \mathcal{X} \) is decomposable. Suppose that we are given coherent configurations

\[
\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_l \cong \mathcal{X}, \quad \mathcal{Y}_1 \otimes \cdots \otimes \mathcal{Y}_m \cong \mathcal{X},
\]

the factors of which are indecomposable of degree greater than 1. The assumption implies that \( l, m > 1 \). By Theorem 5, there are atomic Cartesian decompositions \( P = \{e_1, \ldots, e_l\} \) and \( Q = \{f_1, \ldots, f_m\} \) of the coherent configuration \( \mathcal{X} \), such that

\[
\mathcal{X}_i \cong \mathcal{X}_{\Omega/e_i}, \quad \mathcal{Y}_j \cong \mathcal{X}_{\Omega/f_j}, \quad i = 1, \ldots, l, \quad j = 1, \ldots, m.
\] (5.6)

Moreover, these decompositions are \( \preceq \)-maximal by Theorem 7. By Theorem 2, this yields \( P = Q \). In particular, \( l = m \), and, after a suitable enumeration, \( e_i = f_i \) for \( i = 1, \ldots, l \). In view of (5.6), this implies that \( \mathcal{X}_i \cong \mathcal{Y}_i \) for all \( i \). \( \square \)
6 Finding maximal Cartesian decomposition

In this section, we prove Theorem 3 by constructing a polynomial-time algorithm to find the maximal Cartesian decomposition of a thick coherent configuration \( \mathcal{X} \). At the input of the algorithm, \( \mathcal{X} \) is given by a list of basis relations and each of which is considered as a directed graph. In this representation, for any two relations \( r \) and \( s \) of \( \mathcal{X} \), one can efficiently find \( \langle r \rangle \), \( r \cap s \), \( r \cup s \), and \( r \cdot s \), and check whether \( r \) and \( s \) commute; the corresponding algorithms are standard and can be implemented with the help of basic procedures described in, e.g., [3, Chap. VI].

Lemma 12  Let \( \mathcal{X} \) be a coherent configuration of degree \( n \). Then in polynomial time in \( n \), one can test

(i) whether a given relation \( s \in S \) is irredundant;
(ii) whether given parabolics \( e, f \in E \) strongly commute, or whether \( e \perp f \).

Proof  To test (i), one need to verify that \( s = x \cdot y \) for no irreflexive \( x, y \in S \) such that \( \langle x \rangle \cap \langle y \rangle = 1 \Omega \). By the above remarks, this can be done efficiently for a fixed pair \( x \) and \( y \). Since there is at most \( |S|^2 \leq n^4 \) of such pairs, we are done. Testing (ii) requires verifying equalities of the form \( r \cdot s = s' \cdot r' \) for fixed \( r, s \) and all \( r', s' \in S \), or relations (2.3); both can be done efficiently by the above remarks. \( \square \)

Corollary 7  In the condition of Lemma 12, the set \( P^* \) defined by Algorithm A can be found in polynomial time in \( n \).

A decomposability certificate of \( \mathcal{X} \) is an atomic Cartesian decomposition \( P \) of \( \mathcal{X} \) such that \( |P| \leq 2 \) and \( |P| = 2 \) if and only if \( \mathcal{X} \) is decomposable. The construction of such a certificate is sufficient to find the \( \preceq \)-maximal atomic Cartesian decomposition of \( \mathcal{X} \). The input coherent configuration in the following algorithm is assumed to be thick; for nonthick coherent configurations the algorithm is not always correct.

Algorithm B

Input: a thick coherent configuration \( \mathcal{X} \) on \( \Omega \).

Output: a decomposability certificate \( P \) of \( \mathcal{X} \).

Step 1 Find the set \( P^* \subseteq E \) with the help of Algorithm A; put \( M^* = M(P^*) \).

Step 2 For each proper subset \( I \subset M^* \), test whether \( P := \{P^*_I, P^*_{M^* \setminus I}\} \) is an atomic Cartesian decomposition of \( \mathcal{X} \), and if so, then output \( P \).

Step 3 Output \( P := \{1_\Omega\} \).

Lemma 13  Algorithm B constructs a decomposability certificate \( P \) correctly and in time polynomial in \( n \).

Proof  Let us prove the correctness. If \( P \) is constructed at Step 2, then \( |P| = 2 \) and \( \mathcal{X} \) is decomposable by Theorem 6. Let \( P \) be constructed at Step 3. Assume on the contrary that there is a nontrivial atomic Cartesian
decomposition $R$ of $\mathcal{X}$. By Theorem 9, the set $P^*$ found at Step 1 is an atomic Cartesian decomposition of $\Omega$ and $R \preceq P^*$. By Lemma 5 (2) applied to $P = P^*$, there is a partition $\mathcal{R}$ of $M^*$ such that

$$R = (P^*)^{\mathcal{R}}.$$ 

Now, let $I \in \mathcal{R}$. Then the above equality implies that $P^*_I \in R$. Consequently, the set $P$ defined at Step 2 is an atomic Cartesian decomposition of $\mathcal{X}$. This means that Algorithm B stops at Step 2, a contradiction.

Let us estimate the running time of Algorithm B on an input of degree $n$. We note that Step 1 can be implemented in polynomial time by Corollary 7. At Step 2, one needs to inspect all proper subsets of $M^*$. Since $P^*$ is an atomic Cartesian decomposition of $\Omega$, formula (3.6) implies that

$$n = |\Omega| = \prod_{i \in M^*} |\Omega/e'_i| \geq 2^{|M^*|}.$$ 

Assuming $n > 1$, we have $|M^*| \leq \log n$ and hence, the number of proper subsets of $M^*$ is at most $n$. Moreover, testing that for a given $I \subseteq M^*$ the set $\{P^*_I, P^*_{M^* \setminus I}\}$ is an atomic Cartesian decomposition of $\mathcal{X}$ can be done in polynomial time by Lemma 12 (ii). This proves that Step 2 and then the entire algorithm is polynomial-time.

**Proof of Theorem 3** Recall that a tensor decomposition of coherent configuration $\mathcal{X}$ on $\Omega$ into indecomposable components is a set $C$ of indecomposable coherent configurations $\mathcal{X}_i$ on $\Omega_i$, $i = 1, \ldots, m$, and an isomorphism $\pi: \Omega \rightarrow \Omega_1 \times \cdots \times \Omega_m$ from $\mathcal{X}$ to $\mathcal{X}_1 \otimes \cdots \otimes \mathcal{X}_m$.

**Algorithm C**

**Input:** a thick coherent configuration $\mathcal{X}$ on $\Omega$.

**Output:** a tensor decomposition $(C, \pi)$ of $\mathcal{X}$ into indecomposable components.

**Step 1** Find a decomposability certificate $P$ of $\mathcal{X}$ with the help of Algorithm B.

**Step 2** If $|P| = 1$, then output $C = \{\mathcal{X}\}$ and $\pi = \text{id}_\Omega$. Otherwise, let $P = \{e_1, e_2\}$, $\mathcal{X}_1 = \mathcal{X}_{\Omega/e_1}$, and $\mathcal{X}_2 = \mathcal{X}_{\Omega/e_2}$.

**Step 3** For $i = 1, 2$, recursively find a tensor decomposition $(C_i, \pi_i)$ of $\mathcal{X}_i$ into indecomposable components: $C_i = \{\mathcal{X}_{i,1}, \ldots, \mathcal{X}_{i,m_i}\}$ and $\pi_i$ is an isomorphism from $\mathcal{X}_i$ to $\mathcal{X}_{i,1} \otimes \cdots \otimes \mathcal{X}_{i,m_i}$.

**Step 4** Set $\pi'$ to be the isomorphism from $\mathcal{X}_1 \otimes \mathcal{X}_2$ to $\mathcal{X}_{1,1} \otimes \cdots \otimes \mathcal{X}_{1,m_1} \otimes \mathcal{X}_{2,1} \otimes \cdots \otimes \mathcal{X}_{2,m_2}$, induced by the isomorphisms $\pi_1$ and $\pi_2$.

**Step 5** Output $C = C_1 \cup C_2$ and the composition $\pi$ of $\pi_P$ with $\pi'$.

The correctness of the output at Step 2 follows from Lemma 13. Furthermore, the recursive calls at Step 3 are correct. Indeed, if one of the coherent configurations $\mathcal{X}_1$, $\mathcal{X}_2$ is not thick, say $\mathcal{X}_1$ contains an irreflexive
thin basis relation \( s_1 \), then \( \mathcal{X} \) is not thick: \( S \) has the irreflexive thin relation \( \pi_p^{-1}(s_1 \otimes s_2) \), where \( s_2 \) is an arbitrary reflexive relation of \( \mathcal{X}_2 \). Thus, the coherent configurations \( X_{i,j} \) found at Step 3 are indecomposable by induction. It remains to note that the bijection \( \pi \) constructed at Step 5 is a composition of two isomorphisms (the fact that \( \pi_p \) is an isomorphism follows from Theorem 6).

To prove that Algorithm C is polynomial-time, it suffices to bound from above the number \( c(n) \) of the recursive calls at Step 3 by a polynomial in \( n \): indeed, each step except for Step 1 can obviously be implemented efficiently, whereas for Step 1, this is true by Lemma 13. Finally, at Step 3, we have \( |\Omega| = |\Omega_1 \times \Omega_2| \) and hence \( n = n_1 n_2 \), where \( n_1 = |\Omega_1| \) and \( n_2 = |\Omega_2| \). Thus,

\[
c(n) = 2 + c(n_1) + c(n_2).
\]

This immediately implies that \( c(n) = \log n \), as required. \( \square \)

Acknowledgements The authors are grateful to Professor M. Muzychuk for numerous fruitful discussion on the topic of the paper. The first author was supported by the National Natural Science Foundation of China (Grant No. 11971189).

References

1. Bailey R A, Cameron P J, Praeger C E, Schneider C. The geometry of diagonal groups. arXiv: 2007.10726
2. Chen G, Ponomarenko I. Coherent Configurations. Wuhan: Central China Normal Univ Press, 2019; a draft is available at www.pDMI.ras.ru/\~inp/ccNOTES.pdf
3. Cormen T H, Leiserson C E, Rivest R L, Stein C. Introduction to Algorithms. 3rd ed. Cambridge: The MIT Press, 2009
4. Ferguson P A, Turull A. Algebraic decomposition of commutative association schemes. J Algebra, 1985, 96: 211–229
5. Friedl K, Ronyai L. Polynomial time solutions of some problems of computational algebra. In: Proc of the Seventeenth ACM STOC. 1985, 153–162
6. Hammack R, Imrich W, Klavzar S. Handbook of Product Graphs. Boca Raton: CRC Press, 2011
7. Higman D G. Coherent configurations. I. Rend Semin Mat Univ Padova, 1971, 44: 1–25
8. Higman D G. Coherent algebras. Linear Algebra Appl, 1987, 93: 209–239
9. Kayal N, Nezhmetdinov T. Factoring groups efficiently. Lecture Notes in Comput Sci, 2009, 5555: 585–596
10. Krause H. Krull–Schmidt categories and projective covers. Expo Math, 2015, 33: 535–549
11. Praeger C E, Schneider C. Permutation Groups and Cartesian Decompositions. Cambridge: Cambridge Univ Press, 2018
12. Wilson J B. Existence, algorithms, and asymptotics of direct product decompositions, I. Groups Complex Cryptol, 2012, 4(1): 1–39
13. Xu B. Direct products of association schemes and tensor products of table algebras. Algebra Colloq, 2013, 20(3): 475–494
14. Zieschang P -H. Theory of Association Schemes. Berlin: Springer, 2005