From Graph Isoperimetric Inequality to Network Connectivity – A New Approach

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Abstract

We present a new, novel approach to obtaining a network’s connectivity. More specifically, we show that there exists a relationship between a network’s graph isoperimetric properties and its conditional connectivity. A network’s connectivity is the minimum number of nodes, whose removal will cause the network disconnected. It is a basic and important measure for the network’s reliability, hence its overall robustness. Several conditional connectivities have been proposed in the past for the purpose of accurately reflecting various realistic network situations, with extra connectivity being one such conditional connectivity. In this paper, we will use isoperimetric properties of the hypercube network to obtain its extra connectivity. The result of the paper for the first time establishes a relationship between the age-old isoperimetric problem and network connectivity.

Index Terms

Conditional connectivity; Hypercube; Interconnection networks; Isoperimetric problem; Network reliability.

I. INTRODUCTION

With the increase of the number of processors in multiprocessor computer systems, the possibility of some nodes failing/malfunctioning increases as well. The overall system reliability

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is therefore a key issue in the design, implementation, and maintenance of large multiprocessor systems. There are two basic criteria in evaluating the reliability of multiprocessor systems. One is to determine if a certain structure can be embedded into the remaining healthy system. The other is to determine whether a fault-free communication path exists between any two fault-free nodes. We focus on the latter in this paper.

A multiprocessor system at the system-level can be modeled with an undirected graph $G(V, E)$. Each vertex (or node) in $V(G)$ represents a processor in the multiprocessor system, and each edge in $E(G)$ represents a communication link between two processors. A vertex cut $S$ (resp. an edge cut $S$) of a graph $G$ is a vertex subset $S \subseteq V(G)$ (resp. an edge subset $S \subseteq E(G)$) such that $G - S$ is disconnected. The connectivity (resp. edge connectivity) of a graph $G$ is the cardinality of a minimum vertex cut (resp. edge cut) of $G$. Connectivity (resp. edge connectivity) has been used as a traditional measure to evaluate the fault tolerance ability of multiprocessor systems. However, it has shown some deficiencies as a measure for fault tolerance. On one hand, as surveyed in [7], it cannot correctly reflect different situations of disconnected graphs when removing a vertex cut, which will render inaccuracy for some applications; on the other hand, for many interconnection networks, the probability that all vertices in a minimum vertex cut fail at the same time is quite small. So the classical definition of connectivity may have over-pessimistically underestimated many networks’ reliability [16].

Motivated by the above-mentioned shortcomings, Harary [20] introduced the concept of conditional connectivity by requiring that the disconnected components of $G - F$ have certain properties. Restricted connectivity, super connectivity and $h$-extra connectivity are examples of conditional connectivity, proposed by A.H.Esfahanian and S.L.Hakimi[17], D. Bauer et. al.[6], J. Fàbrega and M.A. Fiol[18], respectively. All these connectivities require some properties of the disconnected components, or have some restrictions on the faulty sets. Thus they are more refined measures of reliability for multiprocessor systems. The restricted connectivity, super connectivity and extra connectivity of many interconnection networks have been explored [2], [3], [4], [5], [10], [11], [29], [37], [44], [51], [15], [38], [39], [40], [31], [44], [28], [41], [42], [27], [33], [35], [19], [30], [32], [50], [51], [26], [48], [23], [44], [46], [24], [9], [49], [45], [47].

When faults occur in an interconnection network, it may become disconnected. But if the disconnected network has a very large component and the remaining small components have very few vertices in total, its performance will not degrade dramatically. This is much preferred than having a disconnected graph without any large components. This phenomenon
has been studied in [11], [12], [13], [14].

The hypercube [34] is a well-known interconnection network for multiprocessor computers. It possesses many attractive properties. The restricted connectivity, super connectivity, and extra connectivity of hypercubes have been studied in [9], [23], [44], [45], [51].

The age-old isoperimetric problem dates back to ancient literature, and in its original form is about finding the largest possible area with a given boundary length. In graph theory, an isoperimetric inequality is a lower bound for the size of the boundary in terms of the order of subgraph. The isoperimetric problem of many graphs have been studied [1], [22], [25], [36]. However to the best of our knowledge, the isoperimetric problem for graphs has never been related to network connectivity/reliability in the past.

In this paper, we will for the first time establish a relationship between network connectivity and graph isoperimetric problem. Opening a new direction in the study of interconnection networks, we will use the isoperimetric properties of the hypercube to study its reliability. More specifically, we will show that if the number of removed vertices is less than the \( h \)-minimum vertex boundary number of hypercube \( Q_n \), \( 1 \leq h \leq 3n - 6 \), then there must exist a large component, and the total number of vertices in the remaining small components is upper-bounded by a function of \( h \). We will then prove that when

- \( 1 \leq h \leq n - 3 \) and \( n \geq 5 \); or
- \( n + 2 \leq h \leq 2n - 4 \) and \( n \geq 7 \); or
- \( 2n + 1 \leq h \leq 3n - 6 \) and \( n \geq 9 \).

the hypercube \( Q_n \)'s \((h - 1)\)-extra connectivity is equal to its minimum \( h \)-vertex boundary number; when \( n - 2 \leq h \leq n + 1 \) and \( n \geq 5 \), \( Q_n \)'s \((h - 1)\)-extra connectivity is equal to its minimum \((n - 2)\)-vertex boundary number; when \( 2n - 3 \leq h \leq 2n \) and \( n \geq 7 \), \( Q_n \)'s \((h - 1)\)-extra connectivity is equal to its minimum \((2n - 3)\)-vertex boundary number.

The rest of this paper is organized as follows. Section II provides preliminaries, and introduces terminology and useful lemmas. In section III, we use the results on the isoperimetric problems for the hypercube to explore the structure of faulty hypercube, and determine its extra connectivity. Section IV summarizes the paper with concluding remarks.

## II. Preliminaries and Terminologies

For all terminologies and notations not defined in this section, we follow [8]. Let \( G = (V, E) \) be a simple undirected graph. For a vertex subset or a subgraph \( H \) of \( G \), the vertex
boundary $N_G(H)$ of $H$ is the set of vertices not in $V(H)$ joined to some vertices in $V(H)$. We use $C_G(H)$ to denote the set $N_G(H) \cup V(H)$. The vertex boundary number of $H$ is the number of vertices in $N_G(H)$, denoted by $b_v(H; G)$. The minimum $k$-boundary number of $G$ is defined as the minimum boundary number of all its subgraphs with order $k$, denoted by $b_v(k; G)$. Given a vertex subset $S$, we use $[S]$ to denote the induced subgraph of $S$ in $G$ and $G - S$ to denote the induced subgraph $[V(G) - S]$. The $cn$-number of $G$ is defined as the maximum number of common neighbors shared by a pair of different vertices in $G$, denoted by $cn(G)$\cite{43}.

Given a graph $G$, a vertex subset $S$ is called a vertex cut if $G - S$ is disconnected or trivial. The connectivity of $G$ is the cardinality of the smallest vertex cut, denoted by $\kappa(G)$. Given a graph $G$, a vertex cut $F$ is called a super vertex cut if each connected component of $G - F$ has at least 2 vertices. The super connectivity $\kappa_1(G)$ of $G$ is defined as the cardinality of the smallest super vertex cut of $G$, if exists. Given a graph $G$ and a non-negative integer $h$, a vertex cut $T$ of $G$ is called an $h$-extra vertex cut if each component of $G - T$ has at least $h + 1$ vertices. The $h$-extra connectivity $\kappa_h(G)$ of $G$ is defined as the cardinality of the smallest $h$-extra vertex cut of $G$, if exists. By the above definitions, the 1-extra connectivity is the super connectivity and may provide more accurate measures of the reliability of some interconnection networks.

The hypercubes are the most famous and widely studied class of interconnection networks. The vertices in an $n$-dimensional hypercube can be labelled by $n$-bit binary strings. Two vertices are adjacent if and only if they differ in exactly one bit position. Let $\oplus$ denotes the binary operation exclusive or. For a vertex $u = u_1u_2\cdots u_n$, let $u^i = u_1\cdots u_{i-1}u_i^0u_{i+1}\cdots u_n$ where $u_i^0 = u_i \oplus 1$. We call $u^i$ the $i$-th neighbor of $u$. Similarly, $u^{ij}$ the $j$-th neighbor of $u^i$ and $u^{ijk}$ the $k$-th neighbor of $u^{ij}$. For an edge $e = (u, u^i)$, we call $e$ an $i$-th edge. Thus an $n$-dimensional hypercube has $2^n$ vertices, each vertex has $n$ neighboring vertices.

Given an $i$, $1 \leq i \leq n$, let $S_i^0 = \{u_1u_2\cdots u_n | u_i = 0\}$, $S_i^1 = \{u_1u_2\cdots u_n | u_i = 1\}$. By the definition of hypercubes, the induced subgraph of $S_i^0$ and $S_i^1$ are both isomorphic to an $(n - 1)$-dimensional hypercube. Furthermore, there exists a perfect matching $M_i$ between $S_i^0$ and $S_i^1$ in $Q_n$. All $i$-th edges belong to $M_i$ and $M_i$ contains only $i$-th edges. We call this a decomposition of an $n$-dimensional hypercube along the $i$-th dimension, denoted by $Q_n = G(Q_{n-1}, Q_{n-1}; M_i)$. Given the decomposition, we call the edges in $M_i$ inter edges while other edges inner edges. Given the decomposition of an $n$-cube along the $i$-th dimension, we call the $i$-th neighbor of any vertex $u$ the pair vertex of $u$ in the decomposition.
By the definition of the hypercube, it’s easy to know that the connectivity of an $n$-dimensional hypercube $Q_n$ is $n$ \cite{21}.

**Lemma 2.1:** Given an $n$-dimensional hypercube $Q_n$, $\kappa(Q_n) = n$.

The $cn$-number of hypercubes are first explored in \cite{43}.

**Lemma 2.2:** \cite{43} Given an $n$-dimensional hypercube $Q_n$ with $n \geq 2$, any pair of different vertices in $V(Q_n)$ have exactly two common neighbors if they have any.

In 1966, Harper et. al. determined the minimum $m$-vertex boundary number of $Q_n$.

**Lemma 2.3:** \cite{22} Every integer $m$, $1 \leq m \leq 2^n - 1$, has a unique representation in the form

$$m = \sum_{i=r+1}^{n} \binom{n}{i} + m', \quad 0 < m' \leq \binom{n}{r},$$

$$m' = \sum_{j=s}^{r} \binom{m_j}{j}, \quad 1 \leq s \leq m_s < m_{s+1} < \ldots < m_r.$$

Then

$$b_v(m; Q_n) = \binom{n}{r} - m' + \sum_{j=s}^{r} \binom{m_j}{j-1}.$$

\section*{III. Minimum Boundary Number of Hypercubes}

By \textbf{lemma 2.3} each integer $m \leq 2^n - 1$ has a unique representation and the minimum $m$-vertex boundary number of $Q_n$ can be obtained by this representation. For an integer $m$ between 1 and $2^n - 1$, we use $r(m)$ (resp. $s(m)$) to denote the $r$ (resp. $s$) in the above unique representation of $m$.

By \textbf{Lemma 2.3} the following lemma can be obtained.

**Lemma 3.1:** Let $m$, $M$ be two different integers between 1 and $2^n - 1$, suppose the unique expression of $m$, $M$ are as follows:

$$m = \sum_{i=r_m+1}^{n} \binom{n}{i} + m', \quad 0 < m' \leq \binom{n}{r_m}; \quad M = \sum_{i=r_M+1}^{n} \binom{n}{i} + M', \quad 0 < M' \leq \binom{n}{r_M}$$

$$m' = \sum_{j=s_m}^{r_m} \binom{m_j}{j}, \quad 1 \leq s_m \leq m_s < m_{s+1} < \ldots < m_{r_m}$$

$$M' = \sum_{j=s_M}^{r_M} \binom{M_j}{j}, \quad 1 \leq s \leq M_s < M_{s+1} < \ldots < M_{r_M}$$

Then $m < M$ if and only if one of the following conditions holds:

(1) $r_m > r_M$. 

(2) \( r_m = r_m = r, m_j < M_j, m_k = M_k \) \( k = j + 1, \ldots, r \).

By Lemma 3.1 it’s easy to determine the explicit minimum \( m \)-boundary number of \( Q_n \).

As shown in the following corollary, which gives the explicit expression of the minimum \( m \)-boundary number of \( Q_n \) when \( 1 \leq m \leq 6n - 15 \).

**Corollary 3.2:**

\[
\begin{align*}
\text{Case 1:} & \\
& 1) \text{When } r(m) = n. \text{ By lemma 2.3, } m = m' = 1, s(m) = n, m_n = n. \text{ Thus } \nu_1(1; Q_n) = n. \\
& 2) \text{When } r(m) = n - 1, m = 1 + m', 0 < m' \leq n. \text{ So } 2 \leq m \leq n + 1.
\end{align*}
\]

When \( 2 \leq m \leq n \), since in the unique representation of \( m' = \sum_{j=s}^{n-1} (m_j) \), \( 1 \leq s \leq m_s < m_{s+1} < \ldots < m_{n-1}, m_{n-1} \geq n - 1 \). If \( m_{n-1} = n, m' \geq (m_{n-1}) = n \) which contradicts to \( m' \leq n - 1 \). So \( m_{n-1} = n - 1 \). Thus \( m_j = j \) for \( s \leq j \leq n - 1, m' = n - 1 - (s - 1), s = n - m' = n - m + 1 \). So \( \nu_v(m; Q_n) = (\binom{n}{s}) - (m-1) + ((n-1) + (n-2) + \ldots + (n-m+1) = -\frac{m^2}{2} + (n - \frac{1}{2})m + 1 \).

When \( m = n + 1, m' = m - \left( \binom{n}{n} \right) = n \). Since in the unique representation of \( m' = \sum_{j=s}^{n-1} (m_j) \), \( 1 \leq s \leq m_s < m_{s+1} < \ldots < m_{n-1}, m_{n-1} = n \) and \( s = n - 1 \). So \( \nu_v(n + 1; Q_n) = \binom{n}{n-1} - n + \left( \binom{n}{n-2} \right) = -\frac{(n+1)^2}{2} + (n - \frac{1}{2})m + 1 \).

So \( \nu_v(m; Q_n) = -\frac{m^2}{2} + (n - \frac{1}{2})m + 1 \) for \( 2 \leq m \leq n + 1 \).

3) When \( r(m) = n - 2, m = 1 + n + m' \), \( 0 < m' \leq \left( \binom{n}{n-2} \right) \). Thus \( n + 2 \leq m \leq \frac{(n+1)^2}{2} + 1 \). Since in the unique representation of \( m' = \sum_{j=s}^{n-2} (m_j) \), \( 1 \leq s \leq m_s < m_{s+1} < \ldots < m_{n-2} \), \( m_{n-2} \geq n - 2 \), then we will discuss according to the following cases:

Case 1: \( m_{n-2} = n - 2 \), then \( m_j = j \) for \( 1 \leq s \leq j \leq n - 2 \). Since \( m' = \sum_{j=s}^{n-2} (m_j) = n - 2 - s + 1 \) for \( 1 \leq s \leq j \leq n - 2 \), \( n + 2 \leq m \leq 2n - 1 \).

\( s = n - 2 + 1 - m' = 2n - m \). So \( \nu_v(m; Q_n) = \binom{n}{n-2} - (m - n - 1) + ((n-2) + (n-3) + \ldots + (2n-m) = -\frac{m^2}{2} + (2n - \frac{3}{2})m - n^2 - 2 \).

Case 2: \( m_{n-2} = n - 1 \) and \( m_{n-3} = n - 3 \), then \( m_j = j \) for \( s \leq j \leq n - 3 \). Since \( m' = \sum_{j=s}^{n-2} (m_j) = \binom{n-1}{n-3} + (n-3 - s + 1) \) for \( 1 \leq s \leq j \leq n - 2 \), \( n - 1 \leq m' \leq 2n - 4 \),
m = n + 1 + m′, 2n ≤ m ≤ 3n − 3, s = 2n − 3 − m′ = 3n − m − 2. So \( b_v(m; Q_n) = \binom{n}{n-2} - (m-n-1) + \sum_{j=s}^{n-1} \binom{n-1}{n-3} + ((n-2) + (n-4) + \cdots + (3n-m-2)) = -\frac{m^2}{2} + (3n - \frac{3}{2})m - 3n^2 + 4n + 2. \)

Case 3: for remained small components we have at most

\[\left(\binom{n}{n-2} - (m-n-1) + \sum_{j=s}^{n-1} \binom{n-1}{n-3} + ((n-2) + (n-4) + \cdots + (3n-m-2)) = -\frac{m^2}{2} + (3n - \frac{3}{2})m - 3n^2 + 4n + 2. \]

Case 4: for remained small components we have at most

\[\left(\binom{n}{n-2} - (m-n-1) + \sum_{j=s}^{n-1} \binom{n-1}{n-3} + ((n-2) + (n-4) + \cdots + (3n-m-2)) = -\frac{m^2}{2} + (3n - \frac{3}{2})m - 3n^2 + 4n + 2. \]

Case 5: for remained small components we have at most

\[\left(\binom{n}{n-2} - (m-n-1) + \sum_{j=s}^{n-1} \binom{n-1}{n-3} + ((n-2) + (n-4) + \cdots + (3n-m-2)) = -\frac{m^2}{2} + (3n - \frac{3}{2})m - 3n^2 + 4n + 2. \]

By Corollary 3.2 the following results are immediate.

**Lemma 3.3:** Given an n-dimensional hypercube \( Q_n \), then

\[ b_v(i * n - 1 - \frac{i* (i-1)}{2}; Q_n) = b_v(i * n - \frac{i* (i-1)}{2}; Q_n) = b_v(i * n - 2 - \frac{i* (i-1)}{2}; Q_n) + 1 = b_v(i * n + 1 - \frac{i* (i-1)}{2}; Q_n) + 1 \]

when \( 1 \leq i \leq 5 \),

\[ b_v(m; Q_n) < b_v(m + 1; Q_n) \]

for \( 1 \leq m \leq n - 2 \), \( n + 1 \leq m \leq 2n - 3 \), \( 2n \leq m \leq 3n - 5 \), \( 3n - 2 \leq m \leq 4n - 8 \), \( 4n - 5 \leq m \leq 5n - 12 \), \( 5n - 9 \leq m \leq 6n - 17 \).

According to the above Lemma, the following results can be obtained.

**Lemma 3.4:** Given an n-dimensional hypercube \( Q_n \) with \( n \geq 5 \), we have:

\[ b_v(h; Q_n) - b_v(h - 1; Q_{n-1}) = n - 1 \]

for \( 1 \leq h \leq n + 1 \). \[ b_v(h; Q_n) - b_v(h - 1; Q_{n-1}) = h - 2 \]

for \( n + 2 \leq h \leq 2n - 1 \). The following lemma shows that there exists a large component in \( Q_n - S \) when \( |S| < b_v(h; Q_n) \) for \( h \leq n - 2 \).

**Lemma 3.5:** Given an n-dimensional hypercube \( Q_n \) where \( n \geq 5 \), for any vertex subset \( S \) of \( Q_n \) with \( |S| < b_v(h; Q_n) \) for \( 1 \leq h \leq n - 2 \), \( Q_n - S \) has a large component and all the remaining small components have at most \( h - 1 \) vertices in total.
Proof: We use induction to prove this.

(i). When \( h = 1 \), \( b_v(1; Q_n) = n = \kappa(Q_n) \) by Lemma 2.2. Thus \( Q_n - S \) is connected when \(|S| < b_v(1; Q_n)\), the proposition holds.

(ii). Suppose the proposition holds for \( h - 1 \) where \( h \geq 2 \), in the following we use contradiction to prove that it also holds for \( h \). Suppose not, let \( C_1, C_2, \ldots, C_m \) be all the components of \( Q_n - S \) and \(|V(C_1)| \leq |V(C_2)| \leq \cdots \leq |V(C_m)|\), then \( m \geq 2 \) and \( \sum_{i=1}^{m-1} |V(C_i)| \geq h \). Let \( B = \bigcup_{i=1}^{m-1} V(C_i) \), \( \forall u = u_1u_2 \cdots u_n, v = v_1v_2 \cdots v_n, u, v \in B \), \( u \neq v \), then \( \exists i \) s.t. \( u_i \neq v_i \), let \( Q_n = G(Q_{n-1}^0; Q_{n-1}^1; M_i) \) be a decomposition of \( Q_n \) along the \( i \)-th dimension, then \( u, v \) don’t belong to the same subcube.

Let \( S_i = S \cap V(Q_{n-1}^i) \), \( i = 0, 1 \), then \(|S_0| + |S_1| = |S| < b_v(h; Q_n)\). By Lemma 3.4 \( b_v(h; Q_n) - b_v(h - 1; Q_{n-1}) = n - 1 \). It can be verified that \( n - 1 \leq b_v(h - 1; Q_{n-1}) \) for \( n \geq 2 \). So at most one of \(|S_0|\) and \(|S_1|\) can be not less than \( b_v(h - 1; Q_{n-1})\).

Case 1. \(|S_0| < b_v(h - 1; Q_{n-1})\) and \(|S_1| < b_v(h - 1; Q_{n-1})\).

By the induction hypothesis, \( Q_{n-1}^0 - S_0 \) (resp. \( Q_{n-1}^1 - S_1 \)) has a large component \( C_0 \) (resp. \( C_1 \)), and all the small components have at most \( 2v - 2 \) vertices in total. Since there exists a perfect matching \( M \) between \( Q_{n-1}^0 \) and \( Q_{n-1}^1 \) and has at least \( 2n - 2 - |S| - 2(h - 2) > 0 \) edges between \( C_0 \) and \( C_1 \) in \( Q_n - S \), thus \( C_0 \) and \( C_1 \) are connect to each other in \( Q_n - S \). So there exist a large component \( C \) in \( Q_n - S \), and all the small components have at most \( 2(h - 2) \leq 2(n - 4) = 2n - 8 \) vertices in total. Let \( A \) denotes the union of all the vertex sets of the small components in \( Q_n - S \), if \(|A| \leq h - 1 \), then we are done. Suppose not, then \( h \leq |A| \leq 2n - 8 \). Since \( N_{Q_n}(A) \subset S \), \(|S| \geq |N_{Q_n}(A)| \geq b_v(|A|; Q_n)\). By Lemma 3.3 \( b_v(h; Q_n) < b_v(l; Q_n) \) for \( n \geq 5 \), \( h \leq n - 2 \) and \( h < l \leq 2n - 8 \). So \(|S| \geq b_v(|A|; Q_n) \geq b_v(h; Q_n)\), this is a contradiction to \(|S| < b_v(h; Q_n)\).

Case 2. \(|S_0| \geq b_v(h - 1; Q_{n-1})\) or \(|S_1| \geq b_v(h - 1; Q_{n-1})\).

Without loss of generality, suppose \(|S_1| \geq b_v(h - 1; Q_{n-1})\), then \(|S_0| = |S| - |S_1| \leq n - 2\), \( Q_{n-1}^0 - S_0 \) is connected. There exists a large component in \( Q_n - S \) which contains all vertices in \( Q_{n-1}^0 - S_0 \), so there is no vertex of the small components in \( Q_n^0 - S_0 \), which contradicts to the decomposition of \( Q_n \).

Lemma 3.6: Let \( S \) be a vertex set in \( Q_n \) with \( n \geq 5 \) and \( b_v(n - 2; Q_n) \leq |S| < b_v(n - 1; Q_n) \), then \( Q_n - S \) has a large component and all the remaining small components have at most \( n + 1 \) vertices in total.

Proof: Let \( S_i = S \cap V(Q_{n-1}^i) \), \( i = 0, 1 \). According to Corollary 2.4, \( b_v(n - 2; Q_n) = n^2 - n \), \( b_v(n - 1; Q_n) = \frac{n^2 - n}{2} + 1 \), so \(|S| = \frac{n^2 - n}{2} \). Since \( b_v(n - 3; Q_{n-1}) + b_v(1; Q_{n-1}) = b_v(n - 2; Q_n) \),
it can be verified that \( b_v(1; Q_{n-1}) \leq b_v(n-3; Q_{n-1}) \) for \( n \geq 4 \), so at most one of \( |S_0| \) and \( |S_1| \) can be not less than \( b_v(n-3; Q_{n-1}) \).

Case 1. \( |S_0| < b_v(n-3; Q_{n-1}) \) and \( |S_1| < b_v(n-3; Q_{n-1}) \).

By Lemma 3.5, \( Q_{n-1}^0 - S_0 \) (resp. \( Q_{n-1}^1 - S_1 \)) has a large component \( C_0 \) (resp. \( C_1 \)), and all the small components have at most \( n-4 \) vertices in total. And similarly as Lemma 3.5, \( C_0 \) and \( C_1 \) can be proved to be connected to each other in \( Q_n - S \). So there exists a large component \( C \) in \( Q_n - S \). Let \( A \) denote the union of all the vertex sets of the small components in \( Q_n - S \), if \( A \leq n + 1 \), then we are done. Suppose not, we have \( n + 2 \leq |A| \leq 2n - 8 \). Similarly, \( |S| \geq |N_{Q_n}(A)| \geq b_v(|A|; Q_n) \geq b_v(n + 2; Q_n) > \frac{n^2-n}{2} \) by Lemma 3.3 which contradicts to \( |S| = \frac{n^2-n}{2} \).

Case 2. \( |S_0| \geq b_v(n-3; Q_{n-1}) \) or \( |S_1| \geq b_v(n-3; Q_{n-1}) \).

Without loss of generality, we assume \( |S_1| \geq b_v(n-3; Q_{n-1}) \), then \( |S_0| < n - 1 = b_v(1; Q_{n-1}) \), so \( Q_{n-1}^0 - S_0 \) is connected. So there is no vertex of the small components in \( Q_{n-1}^0 - S_0 \), which contradicts to our construction of the decomposition of \( Q_n \).

**Lemma 3.7:** Let \( S \) be a vertex set in \( Q_n \), \( b_v(n-1; Q_n) \leq |S| \leq b_v(h; Q_n) \) for \( n \geq 7 \) and \( n + 2 \leq h \leq 2n - 3 \), then \( Q_n - S \) has a large component and all the small components have at most \( h - 1 \) vertices in total.

**Proof:** We use induction to prove it.

(i). When \( h = n + 2 \). Since \( b_v(n + 2; Q_n) - b_v(n - 2; Q_{n-1}) = 2n - 5 \) and \( 2n - 5 \leq b_v(n - 2; Q_{n-1}) \), then at most one of \( S_0 \) and \( S_1 \) can be not less than \( b_v(n - 2; Q_{n-1}) \).

Case 1.1. \( |S_0| < b_v(n-2; Q_{n-1}) \) and \( |S_1| < b_v(n-2; Q_{n-1}) \).

By Lemma 3.5 and Lemma 3.6 \( Q_{n-1}^0 - S_0 \) (resp. \( Q_{n-1}^1 - S_1 \)) has a large component and all small components have at most \( n \) vertices in total. Similarly as the proof of Lemma 3.6 the two large components of \( Q_{n-1}^0 - S_0 \) and \( Q_{n-1}^1 - S_1 \) are connected in \( Q_n - S \). Let \( A \) denote the union of the vertex sets of all the small components in \( Q_n - S \), then \( |A| \leq n + 1 \). If \( |A| \leq n + 1 \), we are done. Suppose not, then \( n + 2 \leq |A| \leq 2n \). According to Lemma 3.3 we have \( b_v(n + 2; Q_n) \leq b_v(l; Q_n) \) for \( n + 2 \leq l \leq 2n \). So \( |S| \geq |N_{Q_n}(A)| \geq b_v(|A|; Q_n) \geq b_v(n + 2; Q_n) \), this is a contradiction to \( |S| < b_v(n + 2; Q_n) \).

Case 1.2. \( |S_0| \geq b_v(n-2; Q_{n-1}) \) or \( |S_1| \geq b_v(n-2; Q_{n-1}) \).

Without loss of generality, we assume \( |S_0| \geq b_v(n-2; Q_{n-1}) \), then \( |S_1| \leq 2n - 6 < b_v(2; Q_{n-1}) = 2n - 4 \), so \( Q_{n-1}^1 - S_1 \) has a large component and the small components have at most 1 vertex in total. By the construction of the decomposition, all small components have at least 1 vertex in \( Q_{n-1}^1 - S_1 \), so the small components have exactly 1 vertex in \( Q_{n-1}^1 - S_1 \),
we use \( u \) to denote it. Since there exists a perfect matching between \( Q_{n-1}^0 \) and \( Q_{n-1}^1 \) in \( Q_n \), the vertices of \( Q_{n-1}^0 - S_0 \) whose pair vertex is not in \( S_1 \cup \{ u \} \) must be in the large component of \( Q_n - S \). Let \( A \) denote the union of all the vertex sets of the small components in \( Q_n - S \), then \( |A| \leq |S_1| + 2 \leq 2n - 4 \). If \( |A| \leq n + 1 \), we are done. Suppose not, we assume \( |A| \geq n + 2 \). Then \( 2n - 5 \geq |A| \geq n + 2 \). Similar to the proof of Case 1. of Lemma 3.5, a contradiction can be obtained.

(ii). Suppose the proposition holds for \( h - 1 \) where \( h \geq n + 3 \), in the following we will prove it also holds for \( h \). Suppose not, let \( C_1, C_2, \ldots, C_m \) be all the components in \( Q_n - S \); that is, \( m \geq 2 \) and \( \sum_{i=1}^{n-1} |V(C_i)| \geq h \). Let \( B = \bigcup_{i=1}^{m-1} V(C_i) \), \( u, v \) be two different vertices in \( B \). Suppose they differ in the \( i \)-th dimension, let \( Q_n = G(Q_{n-1}^0, Q_{n-1}^1; M) \) be a decomposition of \( Q_n \) along the \( i \)-th dimension. Then \( u \) and \( v \) don't belong to the same subcube.

Since \( b_v(h - 1; Q_{n-1}) + (h - 2) = b_v(h; Q_n) \), it can be verified that \( h - 2 \leq b_v(h - 1; Q_{n-1}) \) when \( n \geq 7 \) and \( h \geq n + 3 \). So at most one of \( |S_0| \) and \( |S_1| \) can be not less than \( b_v(h - 1; Q_{n-1}) \).

Case 2.1. \( |S_0| < b_v(h - 1; Q_{n-1}) \) and \( |S_1| < b_v(h - 1; Q_{n-1}) \).

By the induction hypothesis, \( Q_{n-1}^0 - S_0 \)(resp. \( Q_{n-1}^1 - S_1 \)) has a large component \( C_0 \)(resp. \( C_1 \)) and all the small components have at most \( h - 2 \) vertices in total. Similar as the proof of Lemma 3.6, \( C_0 \) and \( C_1 \) are connected to each other in \( Q_n - S \). So there exists a large component in \( Q_n - S \). Let \( A \) be the union of the vertex sets of all the small components in \( Q_n - S \), then \( |A| \leq 2(h - 2) \). If \( |A| \leq h - 1 \), then we are done. Suppose not, we have \( h \leq |A| \leq 2(h - 2) \leq 2(2n - 5) = 4n - 10 \). Since \( N_{Q_n}(A) \subset S \), \( |S| \geq |N_{Q_n}(A)| \geq b_v(|A|; Q_n) \). According to the Lemma 3.3, we have \( b_v(h; Q_n) \leq b_v(l; Q_n) \) for \( n \geq 7 \) and \( h < l \leq 4n - 10 \). So \( |S| \geq b_v(|A|; Q_n) \geq b_v(h; Q_n) \), this is a contradiction to \( |S| < b_v(h; Q_n) \).

Case 2.2. \( |S_0| \geq b_v(h - 1; Q_{n-1}) \) or \( |S_1| \geq b_v(h - 1; Q_{n-1}) \).

Without loss of generality, we suppose \( |S_0| \geq b_v(h - 1; Q_{n-1}) \). So \( |S_1| \leq (h - 2) \leq 2n - 5 < b_v(2; Q_{n-1}) = 2n - 4 \). Similarly as the proof of Case 1.2, a contradiction can be obtained, so the result holds.

Lemma 3.8: Let \( S \) be a vertex set in \( Q_n \) with \( n \geq 7 \) and \( b_v(2n - 3; Q_n) \leq |S| < b_v(2n + 1; Q_n) \), then \( Q_n - S \) has a large component and all the remaining small components have at most \( 2n \) vertices in total.

Proof: Let \( S_i = S \bigcap V(Q_{n-i}^i), i = 0, 1 \). According to Corollary 2.4, \( b_v(2n - 2; Q_n) = 3 - 3n + n^2, b_v(2n + 1; Q_n) = -2 - 2n + n^2 \), so \( 3 - 3n + n^2 \leq |S| \leq -3 - 2n + n^2 \). Since \( 2b_v(2n - 5; Q_{n-1}) > |S| \) when \( n \geq 6 \), so at most one of \( |S_0| \) and \( |S_1| \) can be not less than
By Lemma 3.3, \( Q_{n-1}^0 - S_0 \) (resp. \( Q_{n-1}^1 - S_1 \)) has a large component \( C_0 \) (resp. \( C_1 \)), and all the small components have at most \( 2n - 6 \) vertices in total. And similarly as the proof of Lemma 3.5, \( C_0 \) and \( C_1 \) can be proved to be connected to each other in \( Q_n - S \). So there exists a large component \( C \) in \( Q_n - S \). Let \( A \) denotes the union of all the vertex sets of the small components in \( Q_n - S \), then \( |A| \leq 4n - 12 \). If \( |A| \leq 2n \), then we are done. Suppose not, then \( 2n + 1 \leq |A| < 4n - 12 \). Since \( N_{Q_n}(A) \subset S \), \( |S| \geq |N_{Q_n}(A)| \). But by Lemma 3.3, \( b_v(m, Q_n) \geq b_v(2n + 1, Q_n) \) when \( 2n + 1 \leq m \leq 4n - 12 \). Thus \( |S| \geq |N_{Q_n}(A)| \geq b_v(|A|, Q_n) \geq b_v(2n + 1, Q_n) > |S|\), a contradiction. Thus the small components have at most \( 2n \) vertices in total.

**Case 1.** \( |S_0| \leq b_v(2n - 5; Q_{n-1}) \) and \( |S_1| < b_v(2n - 5; Q_{n-1}) \).

**Case 2.** \( |S_0| \geq b_v(2n - 5; Q_{n-1}) \) or \( |S_1| \geq b_v(2n - 5; Q_{n-1}) \).

Without loss of generality, we assume \( |S_1| \geq b_v(2n - 5; Q_{n-1}) \), then \( |S_0| \leq -2 - 2n + n^2 - b_v(2n - 5; Q_{n-1}) = 3n - 9 < b_v(3; Q_{n-1}) \). So there exists a large component in \( Q_{n-1}^0 - S_0 \) and the small components have at most \( 2n \) vertices in total. Since there exits a perfect matching between \( Q_{n-1}^0 \) and \( Q_{n-1}^1 \) in \( Q_n \). At most \( |S_0| + 2 \) vertices in \( Q_{n-1}^1 - S_1 \) are not connected to the large component in \( Q_{n-1}^0 - S_0 \). So there exists a large component in \( Q_n - S \) and the remaining small components have at most \((|S_0| + 2) + 2 \leq 3n - 5 \) vertices in total. Similar as the proof of case 1, we can prove that the small components have at most \( 2n \) vertices in total.

**Lemma 3.9:** Let \( S \) be a vertex set in \( Q_n \), \( b_v(2n + 1; Q_n) \leq |S| < b_v(h; Q_n) \) for \( n \geq 9 \) and \( 2n + 2 \leq h \leq 3n - 6 \), then \( Q_n - S \) has a large component and all the small components have at most \( h - 1 \) vertices in total.

**Proof:** We use induction to prove this.

(1) When \( h = 2n + 2 \), Let \( S_1 = S \cap V(Q_{n-1}^i) \), \( i = 0, 1 \). According to Corollary 2.4, \( b_v(2n + 1; Q_n) = -2 - 2n + n^2 \) and \( b_v(2n + 2; Q_n) = -7 - n + n^2 \). So \( -2 - 2n + n^2 \leq |S| \leq -8 - n + n^2 \). Since \( 2b_v(2n - 1; Q_{n-1}) > b_v(2n + 2; Q_n) \) when \( n \geq 6 \), so at most one of \( |S_0| \) and \( |S_1| \) can be not less than \( b_v(2n - 1; Q_{n-1}) \).

**Case 1.1.** \( |S_0| \leq b_v(2n - 1; Q_{n-1}) \) and \( |S_1| < b_v(2n - 1; Q_{n-1}) \).

By Lemma 3.8, \( Q_{n-1}^0 - S_0 \) (resp. \( Q_{n-1}^1 - S_1 \)) has a large component \( C_0 \) (resp. \( C_1 \)), and all the small components have at most \( 2n - 2 \) vertices in total. And similarly as the proof of Lemma 3.5, \( C_0 \) and \( C_1 \) can be proved to be connected to each other in \( Q_n - S \). So there exists a large component \( C \) in \( Q_n - S \). Let \( A \) denotes the union of all the vertex sets...
of the small components in $Q_n - S$, then $|A| \leq 4n - 4$. If $|A| \leq 2n + 1$, then we are done. Suppose not, then $2n + 2 \leq |A| < 4n - 4$. Since $N_{Q_n}(A) \subset S$, $|S| \geq |N_{Q_n}(A)|$.

But by Lemma 3.3, $b_v(m, Q_n) \geq b_v(2n + 2, Q_n)$ when $2n + 2 \leq m \leq 4n - 4$. Thus $|S| \geq |N_{Q_n}(A)| \geq b_v(|A|, Q_n) \geq b_v(2n + 1, Q_n) > |S|$, a contradiction. Thus the small components have at most $2n + 1$ vertices in total.

**Case 1.2.** $|S_0| \geq b_v(2n - 1; Q_{n-1})$ or $|S_1| \geq b_v(2n - 1; Q_{n-1})$.

Without loss of generality, we assume $|S_1| \geq b_v(2n - 1; Q_{n-1})$, then $|S_0| \leq -8n + n^2 - b_v(2n - 1; Q_{n-1}) = 3n - 9 < b_v(3; Q_{n-1})$. So there exists a large component in $Q_{n-1}^0 - S_0$ and the small components have at most 2 vertices in total. Since there exists a perfect matching between $Q_{n-1}^0$ and $Q_{n-1}^1$ in $Q_n$. At most $|S_0| + 2$ vertices in $Q_{n-1}^0 - S_1$ are not connected to the large component in $Q_{n-1}^0 - S_0$. So there exists a large component in $Q_n - S$ and the remaining small components have at most $(|S_0| + 2) + 2 \leq 3n - 5$ vertices in total. Similar as the proof of case 1, we can prove the small components have at most $2n + 1$ vertices in total.

(II). Suppose the proposition holds for $h - 1$ where $3n - 5 \geq h \geq 2n + 3$. In the following we will prove it also holds for $h$. Suppose not, let $C_1, C_2, \cdots, C_m$ be all the components in $Q_n - S$ with $|V(C_1)| \leq |V(C_2)| \leq \cdots \leq V(|C_m|)$; that is, $m \geq 2$ and $\sum_{i=1}^{m-1} |V(C_i)| \geq h$. Let $B = \bigcup_{i=1}^{m-1} V(C_i)$, $u, v$ be two different vertices in $B$. Suppose they differ in the $i$-th dimension, let $Q_n = G(Q_{n-1}^0, Q_{n-1}^1; M)$ be a decomposition of $Q_n$ along the $i$-th dimension. Then $u$ and $v$ don’t belong to the same subcube.

Since $b_v(h - 1; Q_{n-1}) + (2h - 3n + 1) = b_v(h; Q_n)$, it can be verified that $2h - 3n + 1 \leq b_v(h - 1; Q_{n-1})$ when $n \geq 7$ and $h \geq 2n + 3$. So at most one of $|S_0|$ and $|S_1|$ can be not less than $b_v(h - 1; Q_{n-1})$.

**Case 2.1.** $|S_0| < b_v(h - 1; Q_{n-1})$ and $|S_1| < b_v(h - 1; Q_{n-1})$.

By the induction hypothesis, $Q_{n-1}^0 - S_0$ (resp. $Q_{n-1}^1 - S_1$) has a large component $C_0$ (resp. $C_1$) and all the small components have at most $h - 2$ vertices in total. Similar as the proof of Lemma 3.6, $C_0$ and $C_1$ are connected to each other in $Q_n - S$. So there exists a large component in $Q_n - S$. Let $A$ be the union of the vertex sets of all the small components in $Q_n - S$, then $|A| \leq 2(h - 2)$. If $|A| \leq h - 1$, then we are done. Suppose not, we have $h \leq |A| \leq 2(h - 2) \leq 2(3n - 8) = 6n - 16$. Since $N_{Q_n}(A) \subset S$, $|S| \geq |N_{Q_n}(A)| \geq b_v(|A|; Q_n)$.

According to the Lemma 3.3, we have $b_v(h; Q_n) < b_v(l; Q_n)$ for $2n + 2 \leq h \leq 3n - 6$ and $h < l \leq 6n - 16$. So $|S| \geq b_v(|A|; Q_n) > b_v(h; Q_n)$, this is a contradiction to $|S| < b_v(h; Q_n)$.

**Case 2.2.** $|S_0| \geq b_v(h - 1; Q_{n-1})$ or $|S_1| \geq b_v(h - 1; Q_{n-1})$. 

Without loss of generality, we suppose \(|S_0| \geq b_v(h - 1; Q_{n-1})\). So \(|S_1| \leq 2h - 3n + 1 \leq 3n - 11 < b_v(3; Q_{n-1}) = 3n - 8\). Similarly as the proof of Case 1.2, a contradiction can be obtained, so the result holds.

Let \(n, h \in \mathbb{N}^+, 1 \leq h \leq 3n - 6\), we define \(f(h)\) as follows:

\[
f(h) = \begin{cases} 
  h - 1, & 1 \leq h \leq n - 2 \\
  n + 1, & h = n - 1, n \\
  n, & h = n + 1 \\
  h - 1, & n + 2 \leq h \leq 2n - 3 \\
  2n, & h = 2n - 2, 2n - 1, 2n + 1 \\
  2n - 4, & h = 2n \\
  h - 1, & 2n + 2 \leq h \leq 3n - 6 
\end{cases}
\]

By Lemma 3.5, Lemma 3.6, Lemma 3.7, Lemma 3.8 and Lemma 3.9, the following Theorem can be obtained:

**Theorem 3.10:** Given an \(n\)-dimensional hypercube \(Q_n\), for any vertex subset \(S\) of \(Q_n\) with \(|S| < b_v(h; Q_n)\) with \(1 \leq h \leq 3n - 6\), then there exists a large component in \(Q_n - S\) and all the remaining small components have at most \(f(h)\) vertices in total.

The following theorem shows the relationship between the hypercube’s extra connectivity and its minimum boundary number.

**Theorem 3.11:** Let \(Q_n\) be an \(n\)-dimensional hypercube and \(1 \leq h \leq 3n - 6\), the \(h-1\)-extra connectivity of \(Q_n\) are as follows:

\[
\kappa_{h-1}(Q_n) = \begin{cases} 
  b_v(h, Q_n), & 1 \leq h \leq n - 3, n \geq 5 \\
  b_v(n - 2, Q_n), & n - 2 \leq h \leq n + 1, n \geq 5 \\
  b_v(h, Q_n), & n + 2 \leq h \leq 2n - 4, n \geq 7 \\
  b_v(2n - 3, Q_n), & 2n - 3 \leq h \leq 2n, n \geq 7 \\
  b_v(h, Q_n), & 2n + 1 \leq h \leq 3n - 6, n \geq 9 
\end{cases}
\]

**Proof:** 1) Let \(Q_n\) be a \(n\) dimensional Hypercube with \(n \geq 5\). When \(h \leq n - 2\), by Lemma 3.5 if \(|S| < b_v(h; Q_n)\), there exists a large component in \(Q_n - S\) and all the small components have at most \(h - 1\) vertices in total. Thus \(\kappa_{h-1}(Q_n) \geq b_v(h; Q_n)\).

Let \(u = 0^n\), \(S_h = \{u, u^1, \cdots u^{h-1}\}\), when \(1 \leq h \leq n + 1\). By Lemma 2.2 and Lemma 3.2 \(|N_{Q_n}(S_h)| = (n-h+1) + (n-1)(h-1) - \binom{h-1}{2} = b_v(h, Q_n)\). It’s clear that \([S_h]\) is isomorphic to \(K_{1,h-1}\). When \(1 \leq h \leq n - 3\), by Lemma 3.3 \(b_v(h, Q_n) < b_v(h + 1, Q_n)\). Thus according
to Lemma 3.3, there exists a large component in \( Q_n - N_{Q_n}(S_h) \) and the small components have at most \( h \) vertices in total. Obviously \([S_n]\) is a connected component in \( Q_n - N_{Q_n}(S_h) \) with \( h \) vertices. So \( Q_n - C_{Q_n}(S_h) \) is the large connected component in \( Q_n - N_{Q_n}(S_h) \). \( N_{Q_n}(S_h) \) is an \((h-1)\)-extra vertex cut of \( Q_n \). Thus \( \kappa_{h-1}(Q_n) \leq |N_{Q_n}(S_h)| = b_v(h; Q_n) \) when \( 1 \leq h \leq n - 3 \).

2) Let \( u = 0^n, S_n = \{u, u^1, \ldots u^n\} \). By Lemma 3.3, \( b_v(n-2; Q_n) + 1 = b_v(n+1; Q_n) + 1 = b_v(n-1; Q_n) = b_v(n; Q_n) \), so \( |N_{Q_n}(S_n)| = b_v(n+1; Q_n) < b_v(n-1; Q_n) \). According to Lemma 3.6, when \( n \geq 5 \), \( Q_n - N_{Q_n}(S_h) \) has a large connected component and all the small components have at most \( n + 1 \) vertices in total. It’s clear that \([S_n]\) is a connected component in \( Q_n - N_{Q_n}(S_h) \) with \( n + 1 \) vertices. So \( Q_n - C_{Q_n}(S_h) \) is connected. Thus \( N_{Q_n}(S_n) \) is an \( n \)-extra vertex cut, \( \kappa_n(Q_n) \leq |N_{Q_n}(S_n)| = b_v(n-2; Q_n) \).

Let \( S \) be an \((n-3)\)-extra vertex cut of \( Q_n \), then each component of \( Q_n - S \) has at least \( n - 2 \) vertices. By Lemma 3.3, \( |S| \geq b_v(n-2; Q_n) \). Thus \( \kappa_{n-3}(Q_n) \geq b_v(n-2; Q_n) \).

So we have \( \kappa_{n-3}(Q_n) \geq b_v(n-2; Q_n) \geq \kappa_{n}(Q_n) \geq \kappa_{n-3}(Q_n) \). The inequalities are all equalities. So \( \kappa_{h-1}(Q_n) = b_v(n-2; Q_n) \) for \( n - 2 \leq h \leq n + 1 \).

3) By Lemma 3.7, \( \kappa_{h-1}(Q_n) \geq b_v(h; Q_n) \) when \( n \geq 7 \), \( n + 2 \leq h \leq 2n - 3 \). In the following paragraph, we will prove that \( \kappa_{h-1}(Q_n) \leq b_v(h; Q_n) \) when \( n + 2 \leq h \leq 2n - 4 \).

Let \( u = 0^n \), Let \( S_h = \{u, u^1, \ldots u^n, u^{12}, u^{13}, \ldots u^{1(h-n)}\} \), when \( n + 2 \leq h \leq 2n \). Then it’s easy to verify that \( |N_{Q_n}(S_h)| = b_v(h, Q_n) \). It’s clear that \([S_n]\) is connected. When \( n + 2 \leq h \leq 2n - 4 \), by Lemma 3.3, \( b_v(h; Q_n) < b_v(h + 1; Q_n) \). Thus by Lemma 3.6 there exists a large component in \( Q_n - N_{Q_n}(S_h) \) and the small components have at most \( h \) vertices in total. Obviously \([S_n]\) is a connected component in \( Q_n - N_{Q_n}(S_h) \) with \( h \) vertices, so \( Q_n - C_{Q_n}(S_h) \) is the large connected component in \( Q_n - N_{Q_n}(S_h) \). So \( N_{Q_n}(S_h) \) is an \( h - 1 \)-extra vertex cut of \( Q_n \). Thus \( \kappa_{h-1}(Q_n) \leq b_v(h; Q_n) \) when \( n + 2 \leq h \leq 2n - 4 \).

4) Let \( u = 0^n, S_{2n-1} = \{u, u^1, \ldots u^n, u^{12}, u^{13}, \ldots u^{1n}\} \). By Lemma 3.3, \( b_v(2n-3; Q_n) + 1 = b_v(2n; Q_n) + 1 = b_v(2n - 2; Q_n) = b_v(2n - 1; Q_n) \). Since \( |N_{Q_n}(S_{2n-1})| = b_v(2n; Q_n) < b_v(2n + 1; Q_n) \), according to Lemma 3.8, \( Q_n - N_{Q_n}(S_{2n-1}) \) has a large connected component and all the small components have at most \( 2n \) vertices in total. It’s clear that \([S_{2n-1}]\) is a connected component in \( Q_n - N_{Q_n}(S_{2n-1}) \) with \( 2n \) vertices. So \( Q_n - C_{Q_n}(S_{2n-1}) \) is connected. Thus \( N_{Q_n}(S_{2n-1}) \) is an \((2n - 1)\)-extra vertex cut. Thus \( \kappa_{2n-1}(Q_n) \leq |N_{Q_n}(S_{2n-1})| = b_v(2n; Q_n) \).

Let \( S \) be a minimum \((2n-4)\)-extra vertex cut of \( Q_n \), then each component of \( Q_n - S \) has at least \( 2n - 3 \) vertices. By Lemma 3.8, \( |S| \geq b_v(2n - 3; Q_n) \). Thus \( \kappa_{2n-4}(Q_n) \geq b_v(2n - 3; Q_n) \).
So we have $b_v(2n - 3; Q_n) \leq \kappa_{2n-4}(Q_n) \leq \kappa_{2n-3}(Q_n) \leq \kappa_{2n-2}(Q_n) \leq \kappa_{2n-1}(Q_n) \leq b_v(2n; Q_n)$. Since $b_v(2n - 3; Q_n) = b_v(2n; Q_n)$, the inequalities are all equalities. So $\kappa_{h-1}(Q_n) = b_v(2n - 3; Q_n)$ for $2n - 3 \leq h \leq 2n$.

5) By Lemma 3.9, $\kappa_{h-1}(Q_n) \geq b_v(h; Q_n)$ when $n \geq 9$, $2n + 1 \leq h \leq 3n - 6$.

For an integer $2n + 1 \leq h \leq 3n - 6$, let $u = 0^n$ and $k = h + 1 - 2n$. Let $S_h = \{u, u^1, \ldots u^n, u^{12}, \ldots u^{1n}, u^{2n}, u^{3n}, \ldots u^{kn}\}$ with $2 \leq k \leq n - 1$, when $2n + 1 \leq h \leq 3n - 2$.

By Lemma 2.2 and Corollary 3.2, it’s easy to verify that $|N_{Q_n}(S_h)| = b_v(h, Q_n)$. It’s clear that $[S_h]$ is connected. When $2n + 1 \leq h \leq 3n - 6$, by Lemma 3.3, $b_v(h, Q_n) < b_v(h+1, Q_n)$. Thus by Lemma 3.9 there exists a large component in $Q_n - N_{Q_n}(S_h)$ and the small components have at most $h$ vertices in total. Obviously $[S_h]$ is a connected component in $Q_n - N_{Q_n}(S_h)$ with $h$ vertices, so $Q_n - C_{Q_n}(S_h)$ is the large connected component in $Q_n - N_{Q_n}(S_h)$. So $N_{Q_n}(S_h)$ is an $(h-1)$-extra vertex cut of $Q_n$. Thus $\kappa_{h-1}(Q_n) \leq b_v(h; Q_n)$ when $2n + 1 \leq h \leq 3n - 6$.

IV. Conclusion

We have proposed a new approach to finding a network’s conditional connectivity based on its isoperimetric properties. Using the vertex isoperimetric results, we have analyzed a faulty hypercube $Q_n$’s structure, and determined its $h$-extra connectivity for $1 \leq h \leq 3n - 6$.

Our work in this paper is the first attempt to establish a link between the two fields, i.e. between graph isoperimetric problems and connectivity/reliability of interconnection networks. We have shown that the results and methods in the former can be applied in the study of the latter. We anticipate that the established link will help getting more insights and expanding toolkits for the research of interconnection networks.

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