Casimir forces on deformed fermionic chains

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We characterize Casimir forces for the Dirac vacuum on free-fermionic chains with smoothly varying hopping amplitudes, which correspond to (1+1)D curved space-times with a static metric in the continuum limit. The first-order energy potential for an obstacle on that lattice corresponds to the Newtonian potential associated to the metric. The finite-size corrections are described by a curved extension of the conformal field theory predictions, using the same central charge and an average value for the Fermi velocity, and including a suitable boundary term. We consider a variety of (1+1)D deformations: Minkowski, Rindler, anti-de Sitter (the so-called rainbow system) and sinusoidal metrics. Our predictions might be tested in quantum simulators using ultracold atoms in optical lattices.

Introduction.- The quantum vacuum on a static spacetime is nothing but the ground state (GS) of a certain Hamiltonian. Therefore, it is subject to quantum fluctuations which help minimize its energy. Yet, these fluctuations are clamped near the boundaries, giving rise to the celebrated Casimir effect [1], see [2] for experimental confirmations. Its relevance extends away from the quantum realm, with applications to thermal fluctuations in fluids [3]. Its initial description required two infinite parallel plates, giving rise to an attractive force between them. In fact, this attraction was rigorously proved for identical plates by Kenneth and Klich [4], yet they can become repulsive or even cancel out when the boundary conditions do not match [5]. The special features of fermionic 1D systems have been already considered [6, 7].

For fields subject to conformal invariance, the Casimir force is associated to the conformal anomaly, measured by the central charge in 2D conformal field theory (CFT), c. Using open boundaries on a system with size N, the energy of the ground state can be proved to be [8–11]

\[ E(N) = -\epsilon_0(N-1) - \epsilon_B - \frac{c_0 v_F^2}{24N} + O(N^{-2}), \]

with \( \epsilon_0 \) and \( \epsilon_B \) positive constants, and \( v_F \) standing for the Fermi velocity. In this case, conformal invariance is strong enough to yield an analytical expression for the Casimir forces in presence of arbitrarily shaped boundaries [12].

The peculiarities of Casimir forces in curved space-times have been considered by several authors [13], although the problem is already difficult for static space-times and weak gravitational fields [14]. It has been proposed that the effective index of refraction of space-time determines the deformed Casimir forces for weak static gravitational fields [15].

Even if our technological abilities do not allow us to access direct measurements of the Casimir effect in curved space-times, there are several strategies to develop quantum simulators using current technologies, such as ultracold atoms in optical lattices [16]. Concretely, it has been shown that the Dirac vacuum on certain static space-times can be characterized in such a quantum simulator [18], and an application has been devised to measure the Unruh radiation, including its non-trivial dimensional dependence [17, 19]. The key insight is the use of curved optical lattices, in which fermionic atoms are distributed on an optical lattice with inhomogeneous hopping amplitudes, thus simulating a position-dependence index of refraction or, in other terms, an optical metric.

Dirac vacua in such curved optical lattices present quite novel properties. When the background metric is negatively curved, i.e.: 1+1D anti-de Sitter (AdS), the entanglement entropy (EE) may violate maximally the area law [21], forming the so-called rainbow state [22–24]. Interestingly, the EE of blocks within the GS of a 1+1D system with conformal invariance is fixed by CFT arguments [25–28]. Such CFT arguments can be extended to a statically deformed 1+1D system, and the EE of the rainbow system was successfully predicted [29], along with other interesting magnitudes, such as the entanglement spectrum, entanglement contour and entanglement Hamiltonian [30, 31].

The aim of this letter is to extend the aforementioned field-theoretic predictions on curved backgrounds to characterize the Casimir force for the fermionic vacuum on curved optical lattices.

Model.- Let us consider an open fermionic chain with (even) N sites, whose Hilbert space is spanned by creation operators \( c_m^\dagger, m \in \{1, \cdots, N\} \) following standard anticommutation relations. We can define an inhomogeneous hopping Hamiltonian,

\[ H(J)_N = -\sum_{m=1}^{N-1} J_m c_m^\dagger c_{m+1} + h.c., \]

where \( J = \{J_m\}_{m=1}^{N-1} \) are the hopping amplitudes, \( J_m \in \mathbb{R}^+ \) referring to the link between sites \( m \) and \( m+1 \), see Fig. 1 (a). In order to obtain a physical intuition, let us remember that the set of \( \{J\} \) constitute a position-

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dependent Fermi velocity, i.e.: a signal takes a time of order $J_m^{-1}$ to travel between sites $m$ and $m + 1$.

If the $\{J_m\}$ are smooth enough, we can take the continuum limit by assuming $J_m = J(x_m)$ for a certain smooth function $J(x)$, with $x_m = m\Delta x$, $\Delta x \to 0^+$ and $N \to \infty$ with $L = \Delta x N = \text{const}$. We can prove that Hamiltonian (2) corresponds in that limit to a Dirac fermion on a static metric \cite{18,19,30,31}.

$$ds^2 = -J^2(x)dt^2 + dx^2,$$

i.e. a space-time metric with a position dependent speed of light or, in other terms, a modulated index of refraction. Yet, we will define our hopping amplitudes as $J_m = J(m\Delta x)$, with $\Delta x = 1$.

Some interesting metrics fall into this category. If $J(x) = J_0$ is a constant, we recover Minkowski space-time, although on a finite spatial interval. The Rindler metric, which is the space-time structure perceived by an observer moving with constant acceleration $a$ in a Minkowski metric, is described by

$$J(x) = J_0 + ax.$$

Notice that it presents an horizon at $x_h = -J_0/a$, where the local speed of light vanishes. Information can not cross this point, thus separating space-time into two Rindler wedges. We will consider some other choices for the hopping amplitudes, such as the sine metric,

$$J(x) = J_0 + A\sin(kx),$$

or an hyperbolic metric given by

$$J(x) = J_0 \exp(-h|x|),$$

for $h \geq 0$, with $h = 0$ corresponding to the Minkowski case. This metric has constant negative curvature except at the center, thus resembling an anti-de Sitter (a$dS$) space, and has considered recently because its vacuum presents volumetric entanglement \cite{23,24}. Unless otherwise stated, we will always assume $J_0 = 1$.

Metric (3) is, evidently, conformally equivalent to the Minkowski metric. Defining $\tilde{x}(x)$ such that $d\tilde{x}/dx = J(x)^{-1}$, we have to $d\tilde{s}^2 = J^2(x)(-dt^2 + dx^2)$. This deformation is illustrated in Fig. 1 (b): sites get closer when the hopping amplitude associated to their link is large, giving rise to an homogeneous effective hopping amplitude. In the continuum limit, $\Delta x \to 0$, $J_0\Delta x \to \text{const}.$, it has been proved \cite{24,29,30} that the Hamiltonian can be written as

$$H \approx -i \int_0^L d\tilde{x} \left[ \hat{\psi}_L^\dagger \frac{\partial}{\partial \tilde{x}} \hat{\psi}_L - \hat{\psi}_L^\dagger \frac{\partial}{\partial \tilde{x}} \hat{\psi}_R \right].$$

The exact diagonalization of Hamiltonian (2) is a straightforward procedure which only involves the solution of the associated single-body problem. Let us define the hopping matrix, $T_{ij} = T_{ji} = -J_0\delta_{ij+1}$, such that $H(J)_N = -\sum_{i,j} T_{ij} \hat{c}_i^\dagger \hat{c}_j$, then we can diagonalize the hopping matrix, $T_{ij} = \sum_k U_{i,k} \epsilon_k U_{j,k}$, where $\epsilon_k$ are the single-body energies and the columns of $U_{i,k}$ represent the single-body modes. The GS of Hamiltonian (2) can be written as $|\Psi\rangle = \prod_{n=1}^{N/2} |0\rangle$, where $|0\rangle$ is the Fock vacuum and $b_k^\dagger = \sum_k U_{i,k} c_i^\dagger$.

The system presents particle-hole symmetry, $\epsilon_k = -\epsilon_{N+1-k}$, with $U_{i,k} = (-1)^i U_{i,N+1-k}$. At half-filling the local density is always homogeneous, $\langle c_i^\dagger c_i \rangle = 1/2$ for all $n$, independently of the metric. For the Minkowski metric, $\langle c_i^\dagger c_{i+n} \rangle = \sum_{k=1}^{N/2} U_{i,k} U_{i+n,k} \approx c_0/2 = 1/\pi$ plus a correction term presenting parity oscillations.

The Casimir potential resembles a gravitational potential.- Let us consider a free fermionic chain of $N$ sites on a deformed metric, following Eq. (2), and let $E_0$ be its GS energy. A classical particle between sites $p$ and $p + 1$, which acts like an obstacle which inhibits the local hopping by a factor $\gamma \ll 1$, $J_p \to \gamma J_p$. Let us now evaluate the excess energy of the deformed GS as a function of $p$, $V(p) = E_p(p) - E_0$, which acts as a potential energy function for the obstacle. The results are shown in Fig. 2 where we plot $V(p)$ for four different situations: Minkowski, Rindler, rainbow and sine metric, using $N = 100$ and both $\gamma = 0.01$ and $\gamma = 0.75$. As $\gamma$ approaches the trivial case is recovered, i.e. the potential energy is equivalent to $E_0$.

The first salient feature of Fig. 2 is that the potential energy $V(p)$ resembles the hopping function $J(x)$. We are thus led to conjecture: a classical particle moving on a static metric in (1+1)D would be dragged by a force similar to the gravitational pull. Make use of Hellmann-Feynman’s theorem, we see that $V(p) \approx -2J_p \langle c_p^\dagger c_{p+1} \rangle$. Thus, our conjecture implies that the local correlators in the deformed vacuum are homogeneous. In fact, we will make the further claim that the local correlators are rigid, i.e. $\langle c_p^\dagger c_{p+1} \rangle \approx c_0/2$ even in the deformed case. This claim has been checked independently in Fig. 3, where the local correlators are shown for different deformations. Indeed, their average values are still very
close to \( c_0 = 2 / \pi \), with relevant deformations only in the parity oscillations. The exact vacuum energy can be written as \( E_0 = -2 \sum_p J_p \langle c_p^\dagger c_{p+1} \rangle \). Based on the correlator rigidity conjecture, we can write \( E_0 \approx -c_0 S_N \), where \( S_N \equiv \sum_{p=1}^{N-1} J_p \). The validity of this approximation can be checked in the bottom panel of Fig. 3 for the same four different deformations.

A heuristic argument to understand this correlator rigidity may be as follows. For fermionic fields in Minkowski space-time we have \( \langle \psi(x) \psi(x+\Delta x) \rangle \sim \Delta x^{-1} \). After a deformation, \( \Delta x \to \Delta \tilde{x} = \Delta x / J(x) \). Yet, the fields transform also as \( \psi(x) \to J^{1/2} \psi(x) \), and the local correlator remains invariant.

**Finite-size corrections.** In CFT, the GS of a finite open chain of \( N \) sites with Hamiltonian (2) in Minkowski space-time with \( J_0 = 1 \) is given by Cardy’s expression [8–11],

\[
E_N^{\text{Mink}} = -c_0 (N-1) - \epsilon_B - \frac{c \pi v_F}{24 N} + O(N^{-2}),
\]  

with \( c_0 = 2 / \pi \) and \( \epsilon_B = 4 / \pi - 1 \) non-universal constants, \( v_F = 2 \) stands for the Fermi velocity and \( c = 1 \) is the central charge of the associated CFT. Notice that only the last term is universal, since its form is fixed by conformal invariance [8,11]. Our main target is to generalize expression (8) to the case of deformed backgrounds. We will propose an extension based on physical grounds, term by term.

- The term \( c_0 (N-1) \) stands for the bulk energy, which should be replaced by \( S_N c_0 \), i.e. the sum of the \( N - 1 \) first hopping amplitudes, multiplied by the local correlator term.
- The boundary term, \( \epsilon_B \) should be proportional to the terminal hoppings, thus generalizing to \( \epsilon_B (J_1 + J_{N-1}) / 2 \).
- The conformal correction is universal. Thus, it will deform naturally, changing \( N^{-1} \) into \( L_N^{-1} \), where \( L_N \) is the effective length in deformed coordinates, given by \( L_N = \sum_{i=1}^{N-1} J_i^{-1} \).

Yet, the inverse of the deformed length \( L_N^{-1} \) can be given an interesting physical interpretation. Indeed, it is easy to recognize \( (N-1) L_N^{-1} \) as the harmonic average of the local speeds of light, which can be understood as an effective Fermi velocity, \( \bar{v}_F \). Yet, for small deformations, the harmonic average is similar (yet, lower than) the arithmetical average. Thus, for the sake of simplicity, we approximate \( \bar{v}_F \approx 2 S_N / (N-1) \). Thus, the full proposal for a weakly deformed \((1+1)D \) lattice should be

\[
E_N \approx -c_0 S_N - \frac{\epsilon_B}{2} (J_1 + J_{N-1}) - \frac{\pi S_N}{12 N^2}.
\]  

Obtaining a numerical confirmation of this expression is a subtle task. We can consider an alternative observable: the Casimir force measured by a local observable.
near the boundary. Because of the strong parity fluctuations, we will only consider even-sized chains. Thus, our definition for the force will be

\[ F_N = \frac{E_N - E_{N-2}}{J_{N-1} + J_{N-2},} \tag{10} \]

because energies should be divided by the local time-rescaling factor. Some elementary manipulations lead to a conjectured behavior for the energy difference which, upon dividing by \( J_{N-1} + J_{N-2} \) and assuming smoothly varying hopping amplitudes, yields

\[ F_N \approx -c_0 - \frac{\epsilon_B}{2} \left( \frac{J_N'}{J_N} \right) - \frac{\pi}{12N^2} \frac{\pi S_N}{6J_NN^3}, \tag{11} \]

Let us consider the terms individually. The first, \( c_0 = 2/\pi \), is simply associated to the bulk energy. The second is a boundary force, which is absent from the homogeneous case, and will take a leading role in some cases. For very weak deformations, \( J_N \approx J_0 + \delta J_N \) is a small deformation, we can assume that \( S_N \approx NJ_N \), so we obtain

\[ F_N \approx -c_0 - \frac{\epsilon_B}{2} \left( \frac{J_N'}{J_N} \right) + \frac{\pi}{12N^2}. \tag{12} \]

The validity of expression (12) can be checked in Fig. 4. In all cases, the black continuous line is the theoretical prediction, Eq. (12). The top panel shows the forces \( F_N + c_0 \) as a function of \( N \) for Rindler metrics of different sizes, varying both \( J_0 \) and the acceleration \( a \). We have included the Minkowski case, which corresponds to \( J_0 = 1 \) and \( a = 0 \), as one of the limits. We notice that \( F_N + c_0 \) can be both positive and negative, depending on the values of \( J_0 \) and the acceleration \( a \). This behavior is explained through our expression (12): the boundary term scales like \( N^{-1} \) and it is always negative. Meanwhile, the universal conformal term scales like \( N^{-2} \) and is always positive. Thus, the prevalence of one or the other explains the global behavior, but for \( N \gg 1 \) the boundary term is always dominant. This trade-off can be visualized in the inset, where we plot the absolute value \( |F_N + c_0| \) as a function of \( N \) in log-log scale. For Minkowski, \( J_0 = 1 \) and \( a = 0 \), the \( 1/N^2 \) behavior extends for all sizes, but as soon as \( a > 0 \) we observe a small-\( N \) behavior like \( N^{-2} \) which performs a crossover into the dominant \( N^{-1} \) term beyond a finite size which scales as \((J_0/a)^{1/2}\).

The central panel of Fig. 4 shows the case of the Casimir forces in the rainbow state, for which the boundary term is constant: \( J_N'/J_N = -h \) for all \( N \). Thus, the behavior of \( F_N + c_0 \) corresponds merely to the CFT term, Eq. (8) with a constant additive correction. This behavior is further clarified when this constant is removed, and we observe the nearly perfect collapse of all the forces in the inset of Fig. 4 (center).

![Figure 4](image)

**Figure 4.** Casimir forces, \( F_N + c_0 \), for different metrics. Top: Rindler metric. Inset, log-log plot of \( |F_N + c_0| \) as a function of \( N \), in log-log scale. Notice most small systems are dominated by the CFT correction, while for larger sizes the boundary term \( N^{-1} \) dominates. Center: Rainbow metric, we observe that \( F + c_0 \) tends to \( \epsilon_B h \). Inset: log-log plot of \( F_N + c_0 - \epsilon_B h \). Bottom: Sinusoidal metric (top) and modulated frequency metric (bottom).

We have also considered is the sinusoidal metric, Eq. (5), where the boundary term dominates the force for large \( N \), while the CFT term dominates for low \( N \), as we can see in the bottom panel of Fig. 4. There, we can observe the behavior of the hoppings (in pale pink), along with the forces and their fit to expression (12). In-
 Indeed, the force behaves like the derivative of the hopping function. In order to highlight this behavior, we have considered yet another metric, given by

\[ J_N = 1 + A \sin(kN^2), \]  

(13)
i.e. a modulated frequency sinusoidal. The results are shown in the bottom panel of Fig. 4, showing again an excellent agreement between the theory and the numerical experiments.

**Conclusions.** We have derived an expression for the ground-state energy of the discretized version of the Dirac equation in a deformed (1+1)D medium, which corresponds to the vacuum state in static curved metrics. We can model a classical particle navigating through the system depressing a local hopping, and then it can be readily checked that the classical particle moves approximately in a potential which corresponds to the classical gravitational potential associated with the metric. The quantum corrections to this semi-classical result can be obtained by suitably deforming the predictions of conformal field theory (CFT). Indeed, we have checked that the finite-size corrections are dominated by two terms: a boundary term related to the derivative of the local hopping amplitude at the edge of the system, and a naturally deformed version of the CFT force, where the central charge is preserved. The correction can be interpreted in two complementary ways. Either the Fermi velocity is substituted by the average value of the hopping terms, or the system size is transformed by its deformed value.

It is relevant to ask whether our results extend to other field theories, both interacting, such as Heisenberg, or non-interacting, such as the Ising model in a transverse field. Even more challenging will be to extend these results to (2+1)D field theories and to consider non-static metrics, where the dynamical effects will be relevant, linking them to the dynamical Casimir effect [32]. Even if the energy is not defined in those cases, a force can still be found acting on classical particles. It is also interesting to consider chains under strong inhomogeneity or randomness [33–36]. We intend also to develop protocols in order to obtain these results in the laboratory, employing ultra-cold atoms in optical lattices, where similar curved-metric problems have been addressed in the past, such as the measurement of the Unruh effect.

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