Landau-Lifshitz hierarchy and infinite dimensional Grassmann variety

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Abstract
The Landau-Lifshitz equation is an example of soliton equations with a zero-curvature representation defined on an elliptic curve. This equation can be embedded into an integrable hierarchy of evolution equations called the Landau-Lifshitz hierarchy. This paper elucidates its status in Sato, Segal and Wilson’s universal description of soliton equations in the language of an infinite dimensional Grassmann variety. To this end, a Grassmann variety is constructed from a vector space of \(2 \times 2\) matrices of Laurent series of the spectral parameter \(z\). A special base point \(W_0\), called “vacuum,” of this Grassmann variety is chosen. This vacuum is “dressed” by a Laurent series \(\phi(z)\) to become a point of the Grassmann variety that corresponds to a general solution of the Landau-Lifshitz hierarchy. The Landau-Lifshitz hierarchy is thereby mapped to a simple dynamical system on the set of these dressed vacua. A higher dimensional analogue of this hierarchy (an elliptic analogue of the Bogomolny hierarchy) is also presented.

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Running head: Landau-Lifshitz hierarchy and Grassmann variety
1 Introduction

The Landau-Lifshitz equation
\[ \partial_t S = S \times \partial^2_x S + S \times JS \]
is the equation of motion of classical spin fields \( S = (S_1, S_2, S_3) \) with totally anisotropic coupling constants \( J = \text{diag}(J_1, J_2, J_3), J_1 \neq J_2 \neq J_3 \neq J_1 \). This is an example of soliton equations whose zero-curvature representation is related to an elliptic curve. The building blocks \( A(z) \) and \( B(z) \) of its zero-curvature equation \[15, 2\]

\[ [\partial_x - A(z), \partial_t - B(z)] = 0 \]
are indeed \( 2 \times 2 \) matrices of elliptic functions of the spectral parameter \( z \).

It is now widely known, after the seminal work of Sato [13] and Segal and Wilson [14], that infinite dimensional Grassmann varieties provide a universal language for understanding soliton equations. According to their observation, many soliton equations can be translated to a simple dynamical system on a subset of an infinite dimensional “universal” Grassmann variety. This fundamental observation has been confirmed for a variety of cases, including higher dimensional generalizations as well [16]. Almost all of the cases thus examined, however, are equations with a rational zero-curvature representation, namely, those with rational matrices \( A(z) \) and \( B(z) \) of the spectral parameter \( z \). The status of soliton equations with an elliptic (and higher genus) spectral parameter in the language of Grassmann varieties still remains obscure. This is the problem that we address in this paper.

There are a few notable studies closely related to this issue. One is the work of Date et al. [3] on a free fermion formalism of the Landau-Lifshitz equation. Although one should, in principle, be able to translate such a free fermion formalism to a Grassmannian picture based on the vector space of creation/annihilation operators, this is actually not an easy task. The work of Carey et al. [1] is more close to our standpoint. They use an infinite dimensional Grassmann variety as a tool to analyze a factorization problem in a loop group, thereby solving the Landau-Lifshitz equation in much the same way as other soliton equations with a rational zero-curvature representation. From our point of view, however, their approach is yet unsatisfactory, because their usage of the infinite dimensional Grassmann variety fails to incorporate a fundamental geometric structure of the Landau-Lifshitz equation.

The geometric structure, also discussed by Carey et al., is hidden in the twisted double periodicity
\[ A(z + 2\omega_a) = \sigma_a A(z) \sigma_a, \quad B(z + 2\omega_a) = \sigma_a B(z) \sigma_a, \quad a = 1, 2, 3, \]
of the matrices $A(z)$ and $B(z)$. Here $z$ is the complex coordinate of the torus $\Gamma = \mathbb{C}/(2\omega_1 \mathbb{Z} + 2\omega_3 \mathbb{Z})$ that represents a nonsingular complex elliptic curve. $\omega_a$'s denote the half periods chosen to satisfy the linear relation $\omega_1 + \omega_2 + \omega_3 = 0$, and $\sigma_a$'s the Pauli matrices

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Geometrically, this twisted double periodicity is related to a nontrivial holomorphic sl(2, $\mathbb{C}$) bundle over the torus; the factorization method works because this bundle is rigid [5]. The same bundle is known to play a fundamental role in the elliptic Gaudin model and an associated conformal field theory as well [8].

Our strategy is, firstly, to construct an infinite dimensional Grassmann variety from a vector space $V$ of $2 \times 2$ matrices—rather than two dimensional vectors as used by Carey et al.—of Laurent series of $z$. Secondly, we choose a special base point $W_0$ of the Grassmann variety. This base point has to be chosen so that the structure of the aforementioned sl(2, $\mathbb{C}$) bundle is encoded therein. More precisely, $W_0$ is to be identified with the space of its holomorphic sections over the punctured torus $\Gamma \setminus \{P_0\}$, where $P_0$ is the point at $z = 0$. The Grassmann variety that fits this purpose turns out to be a slightly unusual one, which we call $\text{Gr}_{-4}$. The base point $W_0$, called “vacuum”, is then “dressed” by a Lauernt series $\phi(z)$ to become a point of the Grassmann variety that corresponds to a general solution of the Landau-Lifshitz equation (or of an integrable hierarchy of evolution equations referred to as the Landau-Lifshitz hierarchy [4, 1]). We thus construct a mapping to the set $\mathcal{M} \subset \text{Gr}_{-4}$ of “dressed vacua.” This mapping converts the Landau-Lifshitz equation (or hierarchy) to a simple dynamical system on $\mathcal{M}$.

This paper is organized as follows. Sections 2 and 3 are mostly a review of basic notions. Sections 2 is concerned with the factorization problem, and Section 3 the construction of the Landau-Lifshitz hierarchy. New results are presented in Sections 4, 5 and 6. In Section 4, the infinite dimensional Grassmann variety, the set of dressed vacua and the mapping from the Landau-Lifshitz hierarchy are introduced. In Section 5, the dynamical system on the space of dressed vacua is specified. Section 6 is a digression on a higher dimensional analogue of the Landau-Lifshitz hierarchy. We conclude this paper with Section 7.
2 Lie algebras and groups of Laurent series

In the usual setting of a Riemann-Hilbert problem on the torus $\Gamma$, one chooses a small circle $S^1 = \{z \mid |z| = a\}$ around the origin $z = 0$, and factorizes a (smooth or real analytic) map $g : S^1 \to \text{SL}(2, \mathbb{C})$ to the product $g_{\text{out}}(z)g_{\text{in}}(z)$ of two holomorphic maps defined in the outside and inside of the circle. For an algebraic interpretation, however, it is simpler to avoid fixing the circle and to reorganize the factorization as follows [11].

Let $\mathfrak{g}$ be the Lie algebra of Laurent series

$$X(z) = \sum_{n=-\infty}^{\infty} X_n z^n, \quad X_n \in \text{sl}(2, \mathbb{C}),$$

that converges in a neighborhood of $z = 0$ except at $z = 0$. This Lie algebra has a direct sum decomposition of the form

$$\mathfrak{g} = \mathfrak{g}_{\text{out}} \oplus \mathfrak{g}_{\text{in}},$$

(1)

where $\mathfrak{g}_{\text{out}}$ and $\mathfrak{g}_{\text{in}}$ denote the following subalgebras:

1. $\mathfrak{g}_{\text{in}}$ is the Lie subalgebra of all $X(z) \in \mathfrak{g}$ that are also holomorphic at $z = 0$; in other words,

$$\mathfrak{g}_{\text{in}} = \{X(z) \in \mathfrak{g} \mid X_n = 0 \text{ for } n < 0\}.$$  

(2)

2. $\mathfrak{g}_{\text{out}}$ consists of all $X(z) \in \mathfrak{g}$ that can be extended to a holomorphic mapping $X : \mathbb{C} \setminus (2\omega_1 \mathbb{Z} + 2\omega_3 \mathbb{Z}) \to \text{sl}(2, \mathbb{C})$ with singularity at each point of $2\omega_1 \mathbb{Z} + 2\omega_3 \mathbb{Z}$ and satisfy the twisted double periodicity condition

$$X(z + 2\omega_a) = \sigma_a X(z) \sigma_a \quad a = 1, 2, 3.$$  

(3)

Note that constant matrices are excluded from $\mathfrak{g}_{\text{out}}$, so that $\mathfrak{g}_{\text{out}} \cap \mathfrak{g}_{\text{in}} = \{0\}$.

One can use the well known weight functions

$$w_1(z) = \frac{\alpha \text{cn}(\alpha z)}{\text{sn}(\alpha z)}, \quad w_2(z) = \frac{\alpha \text{dn}(\alpha z)}{\text{sn}(\alpha z)}, \quad w_3(z) = \frac{\alpha}{\text{sn}(\alpha z)},$$

(4)

($\alpha = \sqrt{e_1 - e_3}$, $e_a = \wp(\omega_a)$, and $\text{sn}, \text{cn}, \text{dn}$ are Jacobi’s elliptic functions) in the zero-curvature representation [15, 2] to obtain a basis $\{\partial^n w_a(z)\sigma_a \mid n \geq 0, \ a = 1, 2, 3\}$ of $\mathfrak{g}_{\text{out}}$. The projection $(\cdot)_{\text{out}} : \mathfrak{g} \to \mathfrak{g}_{\text{out}}$ takes the simple form

$$\left(z^{-n-1}\sigma_a\right)_{\text{out}} = \frac{(-1)^n}{n!} \partial^n w_a(z)\sigma_a, \quad \left(z^n\sigma_a\right)_{\text{out}} = 0 \quad (n \geq 0)$$

(5)
in this basis.

The direct sum decomposition of the Lie algebra \( \mathfrak{g} \) induces the factorization of the associated Lie group \( G = \exp \mathfrak{g} \) to the subgroups \( G_{\text{out}} = \exp \mathfrak{g}_{\text{out}} \) and \( G_{\text{in}} = \exp \mathfrak{g}_{\text{in}} \), namely, any element \( g(z) \) of \( G \) near the identity element \( I \) can be uniquely factorized as

\[
g(z) = g_{\text{out}}(z)g_{\text{in}}(z), \quad g_{\text{out}}(z) \in G_{\text{out}}, \quad g_{\text{in}}(z) \in G_{\text{in}}.
\]  

(6)

3 Construction of Landau-Lifshitz hierarchy

The Landau-Lifshitz hierarchy is a multi-time nonlinear dynamical system on \( G_{\text{in}} \) derived from the linear dynamical system

\[
g(z) \mapsto g(z) \exp \left( - \sum_{n=1}^{\infty} t_n z^{-n} \sigma_3 \right)
\]  

(7)

on \( G \) by the factorization \([4]\). Here \( t = (t_1, t_2, \ldots) \) are the “time” variables of this system; the first variable \( t_1 \) will be identified with the spatial variable of the Landau-Lifshitz equation in \( 1 + 1 \) dimensions. The fundamental dynamical variable of this system is thus an \( SL(2, \mathbb{C}) \)-valued Laurent series of the form

\[
\phi(z) = \sum_{n=0}^{\infty} \phi_n z^n, \quad \det \phi(z) = 1,
\]

that converges in a neighborhood of \( z = 0 \). The time evolution \( \phi(0, z) \mapsto \phi(t, z) \) is achieved by the factorization

\[
\phi(0, z) \exp \left( - \sum_{n=1}^{\infty} t_n z^{-n} \sigma_3 \right) = \chi(t, z)^{-1} \phi(t, z),
\]  

(8)

where \( \chi(t, z) \) is an element of \( G_{\text{out}} \) that also depends on \( t \).

Equations of motion of \( \phi(t, z) \) can be derived by the following standard procedure. Rewrite the factorization relation as

\[
\chi(t, z) \phi(0, z) = \phi(t, z) \exp \left( \sum_{n=1}^{\infty} t_n z^{-n} \sigma_3 \right),
\]

differentiate both hand side by \( t_n \), and eliminate \( \phi(0, z) \) and the exponential matrix in the outcome by the factorization relation itself. This yields the relation

\[
\partial_{t_n} \chi(t, z) \cdot \chi(t, z)^{-1} = \partial_{t_n} \phi(t, z) \cdot \phi(t, z)^{-1} + \phi(t, z) z^{-n} \sigma_3 \phi(t, z)^{-1}.
\]  

(9)
Let $A_n(t, z)$ denote the $2 \times 2$ matrix defined by both hand sides of this relation. $A_n(t, z)$ thus has two expressions

$$A_n(t, z) = \partial_t \chi(t, z) \cdot \chi(t, z)^{-1}$$

and

$$A_n(t, z) = \partial_t \phi(t, z) \cdot \phi(t, z)^{-1} + \phi(t, z) z^{-n} \sigma_3 \phi(t, z)^{-1}.$$ 

The first expression shows that $A_n(t, z)$ takes values in $\mathfrak{g}_{\text{out}}$. This implies that $A_n(t, z)$ should be equal to the projection, down to $\mathfrak{g}_{\text{out}}$, of the right hand side of the second expression. Since the first term $\partial_t \phi(t, z) \cdot \phi(t, z)^{-1}$ on the right hand side disappears by the projection, one obtains the formula

$$A_n(t, z) = \left(\phi(t, z) z^{-n} \sigma_3 \phi(t, z)^{-1}\right)_{\text{out}}. \quad (10)$$

One can insert this formula into the linear equations

$$\partial_t \phi(t, z) = A_n(t, z) \phi(t, z) - \phi(t, z) z^{-n} \sigma_3 \quad (11)$$

(which follow from one of the foregoing expressions of $A_n(t, z)$) to obtain the nonlinear equations

$$\partial_t \phi(t, z) = -\left(\phi(t, z) z^{-n} \sigma_3 \phi(t, z)^{-1}\right)_{\text{in}} \phi(t, z), \quad (12)$$

where $(\cdot)_{\text{in}}$ stands for the projection $\mathfrak{g} \to \mathfrak{g}_{\text{in}}$. This is the final form of equations of motion of $\phi(t, z)$.

The zero-curvature equations

$$[\partial_t m - A_m(t, z), \partial_t n - A_n(t, z)] = 0 \quad (13)$$

follow from the auxiliary linear system

$$(\partial_t n - A_n(t, z)) \chi(t, z) = 0 \quad (14)$$

as the Frobenius integrability condition. Also note that

$$\psi(t, z) = \phi(t, z) \exp\left(\sum_{n=1}^{\infty} t_n z^{-n} \sigma_3\right) \quad (15)$$

satisfies a linear system of the same form. The zero-curvature equation for $m = 1$ and $n = 2$ amounts to the Landau-Lifshitz equation; $t_1$ and $t_2$ are identified with the spatial
coordinate $x$ and the time variable $t$ therein. The spin variables can be read off, in the matrix form $S = \sum_{a=1}^{3} S_a \sigma_a$, from the Laurent expansion

$$A_1(z) = Sz^{-1} + O(1)$$

at $z = 0$ or, equivalently,

$$S = \phi(t, 0)\sigma_3\phi(t, 0)^{-1}.$$ (17)

### 4 Infinite dimensional Grassmann variety

The construction of an infinite dimensional Grassmann variety starts with the choice of an infinite dimensional vector space $V$. Two options are available here, namely, Sato’s algebraic or complex analytic construction based on a vector space of Laurent series [13] and Segal and Wilson’s functional analytic construction based on the Hilbert space of square-integrable functions on a circle [14]. Of course, the latter will be a natural choice for the present setting.

Let $V$ be the vector space of Laurent series

$$X(z) = \sum_{n=-\infty}^{\infty} X_n z^n, \quad X_n \in \text{gl}(2, \mathbb{C})$$

that converges in a neighborhood of $z = 0$ except at $z = 0$. Note that the coefficients are now taken from $\text{gl}(2, \mathbb{C})$ i.e., complex $2 \times 2$ matrices without any algebraic constraints. This vector space is a matrix analogue of $V^{\text{ana}(\infty)}$ in Sato’s list of models [13]; as noted therein, one can introduce a natural linear topology in this vector space. The following infinite dimensional Grassmann variety $\text{Gr}_{-4}$ of closed vector subspaces $W \subset V$ turns out to be relevant to the Landau-Lifshitz hierarchy:

$$\text{Gr}_{-4} = \{W \subset V \mid \dim \text{Ker}(W \to V/V_+) = \dim \text{Coker}(W \to V/V_+) - 4 < \infty\}. \quad (18)$$

Here $V_+$ denotes the subspace of all elements $X(z)$ of $V$ that are holomorphic and vanish at $z = 0$, i.e.,

$$V_+ = \{X(z) \in V \mid X_n = 0 \text{ for } n \leq 0\}. \quad (19)$$

The map $W \to V/V_+$ is the composition of the inclusion $W \hookrightarrow V$ and the canonical projection $V \to V/V_+$.

The next task is to choose a suitable base point, to be called “vacuum,” in this Grassmann variety; this base point should correspond to the “vacuum solution” $\phi(t, z) =
$I$ of the Landau-Lifshitz hierarchy. A correct choice is the following vector subspace $W_0 \subset V$:

$$W_0 = \{ X(z) \in V \mid X(z) \text{ can be extended to a holomorphic mapping}$$

$$X : \mathbb{C} \setminus (2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}) \to \text{gl}(2, \mathbb{C}) \text{ with the twisted double periodicity}$$

$$X(z + 2\omega_a) = \sigma_a X(z)\sigma_a \ (a = 1, 2, 3)\} \quad (20)$$

This resembles the definition of $g_{out}$; the difference is, firstly, that $X(z)$ now takes values in $\text{gl}(2, \mathbb{C})$ rather than $\text{sl}(2, \mathbb{C})$, and secondly, that $X(z)$ can be a constant matrix. The following lemma implies that $W_0$ is indeed an element of $\text{Gr}_{-4}$.

**Lemma 1** The linear mapping $W_0 \to V/V_+$ has the following properties:

1. $\text{Ker}(W_0 \to V/V_+) = \{0\}$.
2. $\text{Im}(W_0 \to V/V_+) \oplus \text{sl}(2, \mathbb{C}) \oplus \mathbb{C}z^{-1}I = V/V_+$.

Here $\text{sl}(2, \mathbb{C})$ and $\mathbb{C}z^{-1}I$ are both understood to be embedded in $V/V_+$.

**Proof.** Notice that $\text{Ker}(W_0 \to V/V_+) = W_0 \cap V_+$. If an element $X(z)$ of $W_0$ belongs to $V_+$, it is a matrix-valued holomorphic function defined everywhere on $\mathbb{C}$ and has a zero at $z = 0$. The twisted double periodicity implies that $X(z)$ is bounded. By Liouville’s theorem, such a function should be identically zero. To confirm the statement on $\text{Im}(W_0 \to V/V_+)$, notice that $W_0$ contains $I$, $\partial_z^n w_a(z)\sigma_a$ and $\partial_z^n \wp(z)I$ for $n \geq 0$ and $a = 1, 2, 3$. The latter two have the Laurent expansion

$$\partial_z^n w_a(z)\sigma_a = (-1)^n n! z^{-n-1} \sigma_a + O(z) \quad (21)$$

and

$$\partial_z^n \wp(z)I = (-1)^n (n + 1)! z^{-n-2}I + O(z) \quad (22)$$

at $z = 0$. One can thus see that the image of $W_0$ in $V/V_+$ contains $I$, $z^{-n-1} \sigma_a$ and $z^{-n-2}I$ for $n \geq 0$ among the standard basis $\{z^{-n}\sigma_a, z^{-n}I \mid n \geq 0, \ a = 1, 2, 3\}$ of $V/V_+$. What is missing are $\sigma_1, \sigma_2, \sigma_3$ and $z^{-1}I$ that span the four dimensional subspace $\text{sl}(2, \mathbb{C}) \oplus \mathbb{C}z^{-1}I$ of $V/V_+$. On the other hand, it is an easy exercise of complex analysis to show that there is no element of $W_0$ that behaves as $\sigma_a + O(z)$, $a = 1, 2, 3$, or $z^{-1}I + O(z)$ at $z = 0$. One can thus confirm that the image of $W_0$ in $V/V_+$ is complementary to $\text{sl}(2, \mathbb{C}) \oplus \mathbb{C}z^{-1}I$. $\square$
Stated differently, the lemma says that the composition of the inclusion $W_0 \hookrightarrow V$ and the canonical projection $V \rightarrow V/(V_+ \oplus \text{sl}(2, \mathbb{C}) \oplus \mathbb{C}z^{-1}I)$ is an isomorphism

$$W_0 \cong V/(V_+ \oplus \text{sl}(2, \mathbb{C}) \oplus \mathbb{C}z^{-1}I), \quad (23)$$

where $\text{sl}(2, \mathbb{C})$ and $\mathbb{C}z^{-1}I$ are now understood to be a subspace of $V$. This condition satisfied by $W_0$ is an open condition, namely,

$$\text{Gr}^o_{-4} = \{ W \in \text{Gr}_{-4} \mid W \cong V/(V_+ \oplus \text{sl}(2, \mathbb{C}) \oplus \mathbb{C}z^{-1}I) \} \quad (24)$$

is an open subset of $\text{Gr}_{-4}$, in fact, the open cell (or “big cell”) in a cell decomposition of this Grassmann variety. Therefore, if a general solution $\phi(t,z)$ of the Landau-Lifshitz hierarchy is a small deformation of the vacuum solution $\phi(t,z) = I$, the “dressed vacuum”

$$W(t) = W_0 \phi(t,z) \quad (25)$$

remains in this open subset. The dynamics of the Landau-Lifshitz hierarchy is thus encoded to the motion of this dressed vacuum.

5 Dynamical system in Grassmann variety

The consideration in the following is limited to a small deformation of the vacuum solution and small values of $t$. The subspace $W(t) = W_0 \phi(t,z)$ of $V$ thereby remains in the open cell $\text{Gr}^o_{-4}$ of $\text{Gr}_{-4}$.

The dynamics of $W(t)$ turns out to take a simple form. This can be deduced from the factorization relation (8). A clue is the following.

**Lemma 2** $W_0 \chi(t,z) = W_0$.

**Proof.** By construction, $\chi(t,z)$ itself is an element of $W_0$. $W_0$ is closed under multiplication of two elements, because the analytical properties in the definition of $W_0$ are preserved under multiplication. Consequently, $W_0 \chi(t,z) \subseteq W_0$. On the other hand, $\chi(t,z)$ is invertible, and the inverse matrix $\chi(t,z)^{-1}$ has the same analytical properties as $\chi(t,z)$. Consequently, $\chi(t,z)^{-1}$ belongs to $W_0$, so that $W_0 \chi(t,z)^{-1} \subseteq W_0$. This implies that the equality holds. □

If one rewrites (8) as

$$\phi(t,z) = \chi(t,z)\phi(0,z) \exp\left(-\sum_{n=1}^{\infty} t_n z^{-n} \sigma_3\right)$$
and apply this expression of \( \phi(t,z) \) to the definition of \( W(t) \), the outcome is that

\[
W(t) = W_0 \chi(t,z) \phi(0,z) \exp \left( - \sum_{n=1}^{\infty} t_n z^{-n} \sigma_3 \right)
\]

\[
= W_0 \phi(0,z) \exp \left( - \sum_{n=1}^{\infty} t_n z^{-n} \sigma_3 \right)
\]

\[
= W(0) \exp \left( - \sum_{n=1}^{\infty} t_n z^{-n} \sigma_3 \right).
\]

Note that the foregoing lemma has been used between the first and second lines of this calculation. Thus the following result has been deduced.

**Theorem 1** The Landau-Lifshitz hierarchy can be mapped, by the correspondence \( \phi(t,z) \mapsto W(t) = W_0 \phi(t,z) \), to a dynamical system on the subset

\[
\mathcal{M} = \{ W \in \text{Gr}_{-4}^0 \mid W = W_0 \phi(z), \ \phi(z) \in G_{\text{un}} \}
\]

of \( \text{Gr}_{-4} \) with the exponential flows

\[
W(t) = W(0) \exp \left( - \sum_{n=1}^{\infty} t_n z^{-n} \sigma_3 \right). \tag{27}
\]

It will be instructive to derive, conversely, the factorization relation \( \phi(t,z) \) from the dynamical system in Grassmann variety. Suppose that one is given an element \( \phi(z) \) of \( G_{\text{un}} \) such that \( W(0) = W_0 \phi(z) \) belongs to the open cell \( \text{Gr}_{-4}^\circ \). The point \( W(t) \) of the trajectory of the exponential flows \( \exp \) remains in the same open cell as far as \( t \) is small. This implies that

\[
\dim \text{Im}(W(t) \to V/V_+) \cap \text{gl}(2, \mathbb{C}) = 1, \tag{28}
\]

so that \( \text{Im}(W(t) \to V/V_+) \cap \text{gl}(2, \mathbb{C}) \) contains a nonzero matrix \( \phi_0(t) \), which is close to the leading term of \( \phi(z) \) if \( t \) is sufficiently small. One can choose \( \phi_0(t) \) to be unimodular, i.e., \( \det \phi_0(t) = 1 \). Thus \( W(t) \) turns out to contain an element of the form

\[
\phi(t,z) = \sum_{n=0}^{\infty} \phi_n(t) z^n, \quad \det \phi_0(t) = 1.
\]

On the other hand, as an element of

\[
W(t) = W(0) \exp \left( - \sum_{n=1}^{\infty} t_n z^{-n} \sigma_3 \right) = W_0 \phi(z) \exp \left( - \sum_{n=1}^{\infty} t_n z^{-n} \sigma_3 \right),
\]

\[
= W_0 \chi(t,z) \phi(0,z) \exp \left( - \sum_{n=1}^{\infty} t_n z^{-n} \sigma_3 \right).
\]
\( \phi(t, z) \) can be written as

\[
\phi(t, z) = \chi(t, z) \phi(z) \exp\left(-\sum_{n=1}^{\infty} t_n z^{-n} \sigma_3 \right)
\]

with a \( t \)-dependent element \( \chi(t, z) \) of \( W_0 \). Now examine the relation

\[
\det \phi(t, z) = \det \chi(t, z)
\]

that follows from the last equality. Since \( \chi(t, z) \) is an element of \( W_0 \), \( \det \chi(t, z) \) becomes a doubly periodic holomorphic function of \( z \) on \( \mathbb{C} \setminus (2\omega_1 \mathbb{Z} + 2\omega_3 \mathbb{Z}) \). On the other hand, one knows that \( \det \phi(t, z) = 1 + O(z) \) as \( z \to 0 \), which implies that \( \det \chi(t, z) \) is also holomorphic at each point of \( 2\omega_1 \mathbb{Z} + 2\omega_3 \mathbb{Z} \). Hence, by Liouville’s theorem, \( \det \chi(t, z) \) turns out to be a constant. Letting \( z \to 0 \), one finds that \( \det \phi(t, z) = \det \chi(t, z) = 1 \), so that

\[
\chi(t, z) \in G_{\text{out}}, \quad \phi(t, z) \in G_{\text{in}}.
\]

One can thus see that the exponential flows \((27)\) on the Grassmann variety solves the factorization problem \((8)\).

### 6 Elliptic analogue of Bogomolny hierarchy

It is nowadays well known [9] that many 1 + 1 dimensional soliton equations can be derived, via the Bogomolny equation in three dimensions, from the (anti)self-dual Yang-Mills equation in four dimensions. Speaking differently, the latter may be thought of as a higher dimensional analogue of soliton equations.

The problem addressed here is to construct a similar higher dimensional analogue of the Landau-Lifshitz hierarchy. Although the work of Carey et al. [11] briefly mentions a formulation of on such a higher dimensional analogue, this issue deserves to be investigated in more detail. For simplicity, the following consideration is limited to a higher dimensional analogue of the Bogomolny type; it is straightforward to generalize these results to equations of the self-duality type.

A clue for constructing such a higher dimensional analogue is the fact that the subspace \( W_0 \subset V \) is invariant, \( W_0 f(z) \subseteq W_0 \), under the multiplication by any elliptic function \( f(z) \) with a pole at \( z = 0 \) and holomorphic elsewhere. In the context of the Landau-Lifshitz hierarchy, this implies that the one-parameter flow generated by \( f(z)I \) on \( \mathcal{M} \) is trivial:

\[
W \exp(tf(z)I) = W \quad (W \in \mathcal{M}). \tag{29}
\]
Actually, as demonstrated in the usual cases (KdV, nonlinear Schrödinger, etc.) [6, 7], nontrivial higher dimensional flows stem from those trivial flows (or symmetries) of the soliton equations.

Let \( f_n(z) \), \( n = 1, 2, \ldots \), be a set of elliptic functions with a pole at \( z = 0 \) and holomorphic elsewhere, e.g., \( f_n(z) = \partial_z^{n-1} \varphi(z) \), and \( s = (s_1, s_2, \ldots) \) a set of corresponding “time” variables (though some of those may rather be understood as spatial variables in the context of the Bogomolny equation). One needs yet another spatial variable \( y \); a fundamental dynamical variable of the higher dimensional hierarchy is the Laurent series

\[
\phi(y, z) = \sum_{n=0}^{\infty} \phi_n(y) z^n \in G_{\text{in}}
\]

that depends on \( y \). In other words, the dynamical variable of this system is a \( G_{\text{in}} \)-valued field on the \( y \) space.

The time evolution \( \phi(0, y, z) \mapsto \phi(s, y, z) \) of this system is defined by the factorization relation

\[
\phi(0, y + \sum_{n=1}^{\infty} s_n f_n(z), z) = \chi(s, y, z)^{-1} \phi(s, y, z),
\]

\[\chi(s, y, z) \in G_{\text{out}}, \quad \phi(s, y, z) \in G_{\text{in}}. \tag{30}\]

In other words, this is the projection, down to \( G_{\text{in}} \), of the flows

\[
g(y, z) \mapsto g(y + \sum_{n=1}^{\infty} s_n f_n(z), z) = \exp\left(\sum_{n=1}^{\infty} s_n f_n(z) \partial_y\right) g(y, z) \tag{31}\]

generated by an exponential operator on the space of \( G \)-valued fields with one spatial variable \( y \).

One can derive equations of motion of \( \phi(s, y, z) \) in much the same way as the previous case. Firstly, note that the differential operator \( \partial_{s_n} - f_n(z) \partial_y \) annihilates the left hand side of the foregoing factorization relation. Applying this operator to both hand sides of the factorization relation and doing some algebra, one obtains the relation

\[
(-\partial_{s_n} + f_n(z) \partial_y) \chi(s, y, z) \cdot \chi(s, y, z)^{-1} = (-\partial_{s_n} + f_n(z) \partial_y) \phi(s, y, z) \cdot \phi(s, y, z)^{-1}, \tag{32}\]

where both hand sides are multiplied by \(-1\) for convenience. Let \( B_n(s, y, z) \) denote the matrix defined by both hand sides of this relation. \( B_n(s, y, z) \) thus has two expressions

\[
B_n(s, y, z) = -\partial_{s_n} \chi(s, y, z) \cdot \chi(s, y, z)^{-1} + f_n(z) \partial_y \chi(s, y, z) \cdot \chi(s, y, z)^{-1}.
\]
and

\[ B_n(s, y, z) = -\partial_s \phi(s, y, z) \cdot \phi(s, y, z)^{-1} + f_n(z) \partial_y \phi(s, y, z) \cdot \phi(s, y, z)^{-1} \]

Notice here that \( \partial_s \chi(s, y, z) \cdot \chi(s, y, z)^{-1} \) and \( \partial_y \chi(s, y, z) \cdot \chi(s, y, z)^{-1} \), which appear in the first expression, take values in \( g_{\text{out}} \). Moreover, thanks to the analytic property of \( f_n(z) \), the product of \( f_n(z) \) and the second term remains in \( g_{\text{out}} \). Consequently, \( B_n(s, y, z) \in g_{\text{out}} \).

This implies that \( B_n(s, y, z) \) should be equal to the projection, down to \( g_{\text{out}} \), of the right hand side of the second expression of \( B_n(s, y, z) \). Since the first term \( \partial_s \phi(s, y, z) \cdot \phi(s, y, z)^{-1} \) on the right hand side disappears by the projection, one obtains the formula

\[ B_n(s, y, z) = \left( f_n(z) \partial_y \phi(s, y, z) \cdot \phi(s, y, z)^{-1} \right)_{\text{out}}. \] (33)

One can insert this formula into the linear equations

\[ (\partial_s - f_n(z) \partial_y + B_n(s, y, z)) \phi(s, y, z) = 0. \] (34)

(which follow from one of the foregoing expressions of \( B_n(s, y, z) \)) to obtain a system of nonlinear equations for \( \phi(s, y, z) \). They give equations of motion of \( \phi(s, y, z) \). Moreover, the zero-curvature equations

\[ [\partial_s - f_m(z) \partial_y + B_m(s, y, z), \partial_s - f_n(z) \partial_y + B_n(s, y, z)] = 0, \] (35)

can be derived from the same linear equations or, equivalently, from

\[ (\partial_s - f_n(z) \partial_y + B_n(s, y, z)) \chi(s, y, z) = 0. \] (36)

These equations comprise an elliptic analogue of the Bogomolny hierarchy.

## 7 Conclusion

We have elucidated the status of the Landau-Lifshitz hierarchy in the language of an infinite dimensional Grassmann variety. The main result (Section 5) shows that the hierarchy can be mapped to a simple dynamical system on the set \( \mathcal{M} \) of dressed vacua in the Grassmann variety \( \text{Gr}_{-4} \). The construction of the set of dressed vacua (in particular, the choice of the base point \( W_0 \)) is closely related to the holomorphic sl(2, \( \mathbb{C} \)) bundle hidden in the structure of the zero-curvature equations. Thus the Landau-Lifshitz hierarchy turns out to fall into the universal framework of Sato [13] and Segal and Wilson [14]. It is straightforward to generalize these results to the case of sl(\( N, \mathbb{C} \)).
Let us lastly stress an unusual nature of this case. Namely, we had to define the Grassmann variety itself in a slightly unusual way. For most soliton equations, the relevant Grassmann variety consists of vector subspaces \( W \subset V \) such that the linear map \( W \to V/V_+ \) is a Fredholm map of index zero; in our case, the index is equal to \(-4\). This is of course due to the special structure of \( W_0 \).

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