Abstract

In this paper, we prove lower bounds on the logical expressibility of optimization problems. There is a significant difference between the expressibilities of decision problems and optimization problems. This is similar to the difference in computation times for the two classes of problems; for example, a 2SAT Horn formula can be satisfied in polynomial time, whereas the optimization version in NP-hard. Grädel [E. 91] proved that all polynomially solvable decision problems can be expressed as universal ($\Pi_1^0$) Horn sentences. We show here that, on the other hand, optimization problems defy such a simple characterization, by demonstrating that even a simple $\Pi_0^0$ formula is unable to guarantee polynomial time solvability.

See Section 4 for Duality using Lagrangian Dual to characterize optimality conditions — that section also describes using a single call to a “decision machine” (a Turing machine that solves decision problems) to obtain optimal solutions.

1 Background

The reader is assumed to have some background in Finite Model Theory. If not, the book by Ebbinghaus and Flum [EF99] serves as a very good introduction. (See Section 2 for notation and definitions.)

In a recent paper, one of the authors [Man08] proved the following:

**Theorem 1.** Let $A$ be a structure (instance) defined over a signature $\sigma$. The value of an optimal solution to an instance $A$ of a maximization problem $Q$ can be represented by

$$opt_Q(A) = \max_S \{|w : (A, S) \models \forall x \, \eta(w, x, S)|\}$$

(1)

if $Q \in P_{\text{opt}}^\text{pb}$, where $x, A, S$ and $\eta$ are defined in Table 4. The Horn condition in the formula $\eta$ applies only to the second order predicates in $S$, not to first order predicates.
The converse of Theorem 1 can be stated as follows:

**Proposition 2.** If the optimal solution value to a maximization problem $Q$ can be represented as in (1), then $Q$ belongs to the class $P_{opt}^{pb}$.

The problem referred to, in Proposition 2, can be cast as an optimization problem as follows:

**Problem 3. Syntactic optimization.**

Given. (i) A structure $A$,
(ii) a sequence of second order variables $S = \{S_1, S_2, \cdots, S_p\}$ where each $S_i$ is of arity $r_i$ ($1 \leq i \leq p$). That is, each $S_i$ is of the form $S_i(z_1, z_2, \cdots, z_{r_i})$, where each $z_j$ can take any value in the domain of $A$; and
(iii) a quantifier-free conjunction of Horn clauses $\eta(w, x, S)$.

(As in Theorem 1, the Horn condition in the formula $\eta$ applies only to the second order predicates in $S$.)

To Do. For $1 \leq i \leq p$, assign truth values to each $S_i$, such that the number of tuples $w$ that satisfy $\forall x \eta(w, x, S)$ is maximized. The goal is to achieve the maximum value for $\text{opt}_Q(A)$ as given in (1).

(We hope the problem definition above is clear. If not, we hope the referees or the editor can help us in re-phrasing it.)

Due to difficulties in computing the optimal solution value for a general maximization problem in $P_{opt}^{pb}$, Bueno and Manyem [BM08] made the following conjecture:

**Conjecture 4.** The optimal value for an instance $A$ of an optimization problem, as measured in (1), cannot be computed in polynomial time by a deterministic Turing machine using syntactic techniques. We need optimization algorithms that exploit the particular problem structure.

Gate and Stewart [GS08] made an attempt to prove Conjecture 4. The decision version of MaxHorn2Sat (see Definition 9) is known to be NP-complete [JS87]. Gate and Stewart were able to show a polynomial time reduction from the decision version of MaxHorn2Sat to the decision version of Problem 3 thus proving that Problem 3 is NP-hard.

In other words, the authors in [GS08] essentially showed that just because the optimal solution value to an optimization problem can be expressed in the form in (1) does not mean that the problem is polynomially solvable, it may also be NP-hard.

Here we prove a stronger negative result. Notice that the first order part in (1) is in $\Pi_1$ Horn form (universal Horn). One would expect that if we simplify the expression from $\Pi_1$ Horn to $\Pi_0$ Horn (that is, a quantifier-free Horn formula), we should be able to guarantee polynomial time solvability.

Unfortunately this is not the case. We will show below that even a quantifier-free Horn expression is unable to guarantee polynomial time solvability. We show this by exhibiting such an expression for an NP-hard problem, MaxHorn2Sat defined above.
A structure defined over a signature $\sigma$; $A$ captures an instance of an optimization problem.

$\eta$ a quantifier-free first order (FO) formula, which is a conjunction of Horn clauses. (Recall that a Horn clause contains at most one positive literal.)

$x$ an $m$-tuple of FO variables.

$S$ a sequence of predicate symbols or second order (SO) variables; $S$ captures a solution to the optimization problem.

$P_{\text{opt}} (NP_{\text{opt}})$ $P$-optimization (NP-optimization) problems. See Definition 5 (6).

$P_{\text{opt}}^{\text{bb}} (NP_{\text{opt}}^{\text{bb}})$ Polynomially bound $P$-optimization (NP-optimization) problems. See Definition 7.

PNF Prenex Normal Form.

| Table 1: Notation |
|-------------------|
| $A$               | a structure defined over a signature $\sigma$; $A$ captures an instance of an optimization problem. |
| $\eta$            | a quantifier-free first order (FO) formula, which is a conjunction of Horn clauses. (Recall that a Horn clause contains at most one positive literal.) |
| $x$               | an $m$-tuple of FO variables. |
| $S$               | a sequence of predicate symbols or second order (SO) variables; $S$ captures a solution to the optimization problem. |
| $P_{\text{opt}} (NP_{\text{opt}})$ | $P$-optimization (NP-optimization) problems. See Definition 5 (6). |
| $P_{\text{opt}}^{\text{bb}} (NP_{\text{opt}}^{\text{bb}})$ | Polynomially bound $P$-optimization (NP-optimization) problems. See Definition 7. |
| PNF | Prenex Normal Form. |

2 Notation and Definitions

Definition 5. A $P$-optimization problem $Q$ is a set $Q = \{I_Q, F_Q, f_Q, opt_Q\}$, where

(i) $I_Q$ is a set of instances to $Q$,

(ii) $F_Q(I)$ is the set of feasible solutions to instance $I$,

(iii) $f_Q(I, S)$ is the objective function value to a solution $S \in F_Q(I)$ of an instance $I \in I_Q$. It is a function $f : \bigcup_{I \in I_Q} \{I\} \times F_Q(I) \rightarrow R^+$ (non-negative reals)\(^{1}\), computable in time polynomial in the size of the universe of $I$\(^{2}\).

(iv) For an instance $I \in I_Q$, $opt_Q(I)$ is either the minimum or maximum possible value that can be obtained for the objective function, taken over all feasible solutions in $F_Q(I)$.

\[ opt_Q(I) = \max_{S \in F_Q(I)} f_Q(I, S) \text{ (for P-maximization problems)}, \]
\[ opt_Q(I) = \min_{S \in F_Q(I)} f_Q(I, S) \text{ (for P-minimization problems)}, \]

(v) The following decision problem is in the class $P$: Given an instance $I$ and a non-negative constant $k$, is there a feasible solution $S \in F_Q(I)$, such that $f_Q(I, S) \geq k$ (for a P-maximization problem), or $f_Q(I, S) \leq k$ (in the case of a P-minimization problem)? And finally,

(vi) An optimal solution $S_{\text{opt}}(I)$ for a given instance $I$ can be computed in time polynomial in $|I|$, where $opt_Q(I) = f_Q(I, S_{\text{opt}}(I))$. (LET ME LEAVE THIS POINT HERE FOR THE TIME BEING.)

The set of all such $P$-optimization problems is the $P_{\text{opt}}$ class.

\(^{1}\)Of course, when it comes to computer representation, rational numbers will be used.

\(^{2}\)Strictly speaking, we should use $|I|$ here, where $|I|$ is the length of the representation of $I$. However, $|I|$ is polynomial in the size $A$ of the universe $A$, hence we can use $|I|$. 

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A similar definition, for NP-optimization problems, appeared in Panconesi and Ranjan (1993) [PR93]:

**Definition 6.** An NP-optimization problem is defined as follows. Points (i)-(iv) in Definition 5 above apply to NP-optimization problems, whereas (vi) does not. Point (v) is modified as follows:

(v) The following decision problem is in **NP**: Given an instance $I$ and a non-negative constant $k$, is there a feasible solution $S \in F_Q(I)$, such that $f_Q(I, S) \geq k$ (for an NP-maximization problem), or $f_Q(I, S) \leq k$ (in the case of an NP-minimization problem)?

The set of all such NP-optimization problems is the NP_{opt} class, and P_{opt} \subseteq NP_{opt}.

**Definition 7.** An optimization problem $Q$ is said to be polynomially bound if the value of an optimal solution to every instance $I$ of $Q$ is bound by a polynomial in the size of $I$. In other words, for every problem $Q$, there exists a polynomial $p_Q$, such that

$$\text{opt}_Q(I) \leq p_Q(|I|),$$

for every instance $I$ of $Q$. $P^{pb}_{opt}$ (NP^{pb}_{opt}) is the set of polynomially-bound P-optimization (NP-optimization) problems. Naturally, P^{pb}_{opt} \subseteq P_{opt} and NP^{pb}_{opt} \subseteq NP_{opt}.

Informally, an instance of MaxHorn2Sat consists of a formula in conjunctive normal form (CNF), where each clause is Horn, and each clause contains at most two literals. (Such a formula is also known as a quadratic Horn formula.) The problem is to maximize the number of satisfiable clauses.

**Definition 8.** An existential second-order (ESO) Horn expression is of the form $\exists S \psi$, where $\psi$ is a first order formula, and $S = (S_1, \cdots, S_p)$ is a sequence of predicate symbols not in the vocabulary of $\psi$. The formula $\psi$ can be written in $\Pi_1$ form as

$$\psi = \forall x_1 \forall x_2 \cdots \forall x_k \eta = \forall x \eta,$$

where $\eta$ is a conjunction of Horn clauses ($\eta$ is, of course, quantifier-free), and $x_i$ ($1 \leq i \leq k$) are first order variables. Each clause in $\eta$ contains at most one positive occurrence of any of the second order predicates $S_i$ ($1 \leq i \leq p$).

A general ESO formula is the same as an ESO Horn expression, except that $\eta$ can now be any quantifier-free first order formula.

**Problem 9.** MaxHorn2Sat.

Given. A set of clauses $c_i$, $1 \leq i \leq n$. Each clause $c_i$ is one of the following: (i) a Boolean variable $x_j$, (ii) its negation, $\neg x_j$, (iii) $x_j \lor \neg x_k$, or (iv) $\neg x_j \lor \neg x_k$.

To Do. Assign truth values to the $x_i$’s such that the number of satisfied clauses is maximized.

**Definition 10.** $\Pi_n$ and $\Sigma_n$ formulae. These formulae have $n$ quantifier blocks at the beginning, followed by a quantifier-free formula. Each block contains only one type of quantifier (either existential or universal). Here is the difference between $\Pi_n$ and $\Sigma_n$: A $\Pi_n$ ($\Sigma_n$) formula begins with a universal (existential) block.
$\text{MAX } \Pi_0$ is the class of maximization problems whose optimal solution value to an instance $A$ of a Problem $Q$ can be represented as

$$\text{opt}_Q(A) = \max_S \{w : (A,S) \models \eta(w,S)\}$$

where $\eta$ is a quantifier-free formula.

### 3 A Syntactic Expression for MaxHorn2Sat

We need instances at two different levels. For example, suppose we are given a MaxHorn2Sat instance (formula) such as $M \equiv (z_1 \lor \neg z_2) \land (z_3) \land (\neg z_3 \lor \neg z_1)$.

The variables in this instance are $Z = \{z_1, z_2, z_3\}$, and a structure $B$ maps $Z$ to its universe $V = \{\text{TRUE}, \text{FALSE}\}$.

However, to represent the MaxHorn2Sat instance $M$ as in (1) or (10), the variables used will be $X = \{x, y\}$, and the universe of the structure $A$ would be $Z$. Diagrammatically,

$$X = \{x, y\} \rightarrow Z = \{z_1, z_2, z_3\} \rightarrow V = \{\text{TRUE}, \text{FALSE}\}. \quad (5)$$

$A$ maps (instantiates) $X$ to $Z$, and $B$ maps (instantiates) $Z$ to $V$.

The second order variables $S$ (to be used with $A$) consists of a single unary predicate $S$, that is $S = \{S\}$ where $S$ is of arity one. $S$ can be considered as a guess of the map $B$. In the above example, for a certain MaxHorn2Sat clause, if $A(x) = z_1$, $A(y) = z_3$, $S(x) = \text{FALSE}$ and $S(y) = \text{TRUE}$, then $S$ would have guessed that $B(z_1) = \text{FALSE}$ and $B(z_3) = \text{TRUE}$.

If variables $x$ and $y$ appear in a 2-literal MaxHorn2Sat clause, then the clause can assume one of the following forms (and represented in the signature of $A$ by the corresponding first order predicate on the right):

| $\neg x \lor \neg y$ | BothNeg$(x,y)$ |
| $\neg x \lor y$ | FirstNegSecondPos$(x,y)$, or simply $\text{FNSP}(x,y)$ |
| $x \lor \neg y$ | FirstPosSecondNeg$(x,y)$, or simply $\text{FPSN}(x,y)$ |

If a clause contains only one literal, insert a second literal and set it to FALSE.

### 3.1 Counting satisfying clauses

Our approach is similar to that of Kolaitis-Thakur 1994 [KT94], where they provide an expression for the optimal value for Max3Sat (optimization version).

We only count satisfying MaxHorn2Sat clauses for the objective function. That is, we count the number of tuples $(x, y)$ that satisfy $\phi$, where

$$\phi = \bigvee_{i=1}^{4} \phi_i = \phi_1 \lor \phi_2 \lor \phi_3 \lor \phi_4. \quad (6)$$

(The $\phi_i$’s are described below.)
3.2 Two-literal MaxHorn2Sat clauses

Two-literal MaxHorn2Sat clauses can be satisfied in one of the following ways:

\[ \phi_1 = \text{FPSN}(x, y) \land [S(x) \lor \neg S(y)]. \]
\[ \phi_2 = \text{FNSP}(x, y) \land [\neg S(x) \lor S(y)]. \]
\[ \phi_3 = \text{BothNeg}(x, y) \land [\neg S(x) \lor \neg S(y)]. \]

3.3 One-literal MaxHorn2Sat clauses

As mentioned earlier, convert one-literal clauses to two-literal clauses. (We do this, so that we can simply count the number of \((x, y)\) tuples that satisfy \(\phi\).

If the literal is positive, then create a special predicate called \(\text{BothPos}(x, y)\), as if the clause is \(x \lor y\); but of course, we set \(y\) to \(\text{FALSE}\), so \(y\) has no effect. So we use

\[ \phi_4 = \text{BothPos}(x, y) \land S(x). \]

If the literal is negative, then just use \(\phi_2\) above; no need for another \(\phi_i\). Since we set the second literal to \(\text{FALSE}\), \(S(y)\) will never hold. (For \(\phi_2\) to be true, \(S(x)\) must be false.)

3.4 The complete DNF formula

For convenience of writing, let us substitute

\[ A = \text{FPSN}(x, y), B = \text{FNSP}(x, y), C = \text{BothNeg}(x, y), D = \text{BothPos}(x, y), \]
\[ P = S(x), Q = S(y). \]

Then we can rewrite \(\phi_i\) \((1 \leq i \leq 4)\) as

\[ \phi_1 = (A \land P) \lor (A \land \neg Q), \quad \phi_2 = (B \land \neg P) \lor (B \land Q), \]
\[ \phi_3 = (C \land \neg P) \lor (C \land \neg Q), \quad \phi_4 = (D \land P). \]  

From (6), since one of the \(\phi_i\)'s should be satisfied for a MaxHorn2Sat clause to be counted towards the objective function,

\[ \phi \equiv \phi_1 \lor \phi_2 \lor \phi_3 \lor \phi_4 \equiv \]
\[ (A \land P) \lor (A \land \neg Q) \lor (B \land \neg P) \lor (B \land Q) \lor (C \land \neg P) \lor (C \land \neg Q) \lor (D \land P). \]

Write \(\phi = k_1 \lor k_2 \lor \cdots \lor k_7\), corresponding to each of the 7 “conjunct” clauses above in \(\phi\).

That is, \(k_1 = A \land P, k_2 = A \land \neg Q, \ldots, k_7 = D \land P\).

\((A \land P), \ldots, (D \land P)\) can be called “conjunct” clauses.
3.5 Converting DNF to CNF

Now $\phi$ is in DNF, so we should convert it to CNF. Call the CNF form as $\psi$ (or $\psi(x, y, S)$, to be more accurate). There are 7 clauses in $\phi$ with 2 literals each, so $\psi$ will have $2^7 = 128$ clauses, with 7 literals each — one literal from each of the 7 clauses in $\phi$. (See footnote \footnote{128 may be “large”, but still a finite number.}) From (8), we can write $\psi$ in lexicographic order as

$$\psi = (A \lor A \lor B \lor B \lor C \lor C \lor D) \land \cdots \land (P \lor \neg Q \lor \neg P \lor Q \lor \neg P \lor \neg Q \lor P). \quad (9)$$

We should ensure that each of the 128 disjunct clauses in $\psi$ is Horn, which is what we do next.

**Lemma 11.** Each of the 128 clauses in $\psi$ is Horn.

**Proof.** Note that the literals $A$, $B$, $C$ and $D$ are first order (part of the input), hence these do not affect the Horn condition; only the $P$’s and $Q$’s and their negations do.

If there is a 7-literal clause in $\psi$ containing literals $P$ and $\neg P$, it can be set to TRUE. Similarly for a clause containing $Q$ and $\neg Q$.

Anyway, we will run into trouble only if we have a clause $\psi_1$ in $\psi$, that (i) contains literals $P$ and $\neg P$, and (ii) contains neither $\neg P$ nor $\neg Q$. However, can such a clause evaluate to TRUE and hence can be “ignored”? Will such a clause obey the Horn condition? The answer turns out to be yes.

There are only three ways in which we can come across a “$P \lor Q$” within a 7-literal clause of $\psi$:

- Pick $P$ from $k_1$, $Q$ from $k_4$, and one of $\{A, B, C, D\}$ from the other clauses, to obtain $\psi_1 = P \lor A \lor B \lor Q \lor C \lor C \lor D$.
  
  This clause of $\psi$ contains $A$, $B$, $C$ and $D$ — the four types of clauses mentioned in (7), and one of them must occur; they are mutually disjoint and collectively exhaustive. So $A \lor B \lor C \lor D$ is (always) valid. So $\psi_1$ can be set to TRUE.

- Pick $P$ from $k_7$, $Q$ from $k_4$, and one of $\{A, B, C, D\}$ from the other clauses, to obtain $\psi_2 = A \lor A \lor B \lor Q \lor C \lor C \lor D$.
  
  This clause only contains $A$, $B$ and $C$, but not $D$. However, we know that $A \lor B \lor C \lor D$ is valid. If $A$, $B$ and $C$ are false, then $D$ will be true (the BothPos predicate) — this means, every clause in $\phi$ is false except $k_7$, which implies that $P$ is true. Hence $A \lor B \lor C \lor P$ is valid, which means $\psi_2$ can be set to TRUE.

- Pick $P$ from $k_1$ and $k_7$, $Q$ from $k_4$, and one of $\{A, B, C, D\}$ from the other clauses, to obtain $\psi_3 = P \lor A \lor B \lor Q \lor C \lor C \lor P$.
  
  Apply the same argument as for $\psi_2$. This sets $\psi_3$ to TRUE.

Hence each of the 128 clauses in $\psi$ is Horn. $\Box$
We need one more first order predicate to represent whether a certain \((x, y)\) combination actually occurs in the MaxHorn2Sat formula, and in which of the four \(\phi_i\) \((1 \leq i \leq 4)\) varieties it occurs. For each of the four \(\phi_i\) types, introduce a predicate \(L_i(x, y)\).

For instance, if \(L_2(z_2, z_3)\) is true, it means that \((\neg z_2 \lor z_3)\) is a clause in the given MaxHorn2Sat instance. Similarly, the truth of \(L_3(z_2, z_3)\) implies the existence of the clause \((\neg z_2 \lor \neg z_3)\).

Thus we need to append \([L_1(x, y) \lor L_2(x, y) \lor L_3(x, y)]\) as the 129th clause. However, these four are first order predicates, hence their truth values can be evaluated and substituted. This does not affect the Horn condition.

From all the arguments above including Lemma 11 we conclude as follows:

**Theorem 12.** Let the structure \(A\) represent an instance of MaxHorn2Sat defined in Problem 9. Then the value of an optimal solution to \(A\) can be represented by

\[
O(A) = \max_S |\{(x, y) : (A, S) \models \psi(x, y, S)\}|,
\]

where \(\psi(x, y, S)\) is defined in (9).

Note that \(\psi\) above is quantifier free (\(\Pi_0\) or \(\Sigma_0\) form). This means

**Corollary to Theorem 12 and Discussion:** Since it is known that MaxHorn2Sat is NP-hard, even a \(\Pi_0\) Horn expression does not guarantee polynomial time solvability for maximization problems.

In [Man08], it was shown that the MaxFlow\(_{PB}\) problem (the MaxFlow problem with unit weight edges) cannot be represented in Horn \(\Pi_0\) or Horn \(\Sigma_1\) form first order form; it needs a Horn \(\Pi_1\) sentence. The optimal solution to this problem can be obtained in polynomial time using MaxFlow algorithms.

Hence it is a surprise that while a polynomially solvable problem, MaxFlow\(_{PB}\), has a Horn \(\Pi_1\) lower bound, an NP-hard problem, MaxHorn2Sat, can be expressed by a quantifier-free Horn sentence.

A similar anamoly was observed by Panconesi and Ranjan (1993) [PR93]: While the class MAX \(\Pi_0\) (defined below) can express NP-hard problems such as Max3Sat, it is unable to express polynomially solvable problems such as Maximum Matching! This suggests that

**Conjecture 13.** Quantifier alternation does not provide a precise characterization of computation time. A hierarchy in quantifier alternation does not translate to one in computation time. We need to look at other characteristics of logical formulae such as the number of variables, or a combination of these.

### 4 Optimality Conditions: Duality to the Rescue

**Recognizing (Verifying) Optimality.** In general, the question, *Given a solution \(T\) to an instance \(A\) of an optimization problem \(Q\), is it an optimal solution?* is as hard to answer as determining an optimal solution, necessitating a \(\Sigma_2\) second order sentence as in (11) below. However, under certain conditions, such as when the duality gap is zero, optimal solutions can be recognized more efficiently, and can be expressed in existential second order (ESO, or second order \(\Sigma_1\)) logic.
**Duality Gap** is the difference between the optimal solution values for the primal and dual problems. For problems such as LP and MaxFlow-MinCut, the duality gap has been shown to be zero; that is, they possess the strong duality property. However, for other problems such as Integer Programming, there is no known dual problem that guarantees strong duality; hence expressions such as (17) and (20) cannot be derived, at least until a dual that guarantees strong duality is discovered.

The above question can also be phrased as a classical decision problem (for maximization): *Given a solution T for instance A with solution value f(T), is there another solution S such that f(S) > f(T)?*

An optimal solution T to an instance A of an optimization problem Q can easily be represented as the best among all feasible solutions S:

\[ \exists T \forall S \phi \land [f(A, T) \geq f(A, S)], \tag{11} \]

where \( \phi \) represents satisfaction of the constraints to A, and f is the objective function referred to, in Definitions 5 and 6. The formula \( \phi \) captures the constraints, such as \( g(x) = b \) and \( h(x) \leq c \) in (12) below.

[Note that the above formula represents an optimal solution to a maximization problem; we can write a similar formula for minimization; simply change the last condition to \( f(A, T) \leq f(A, S) \).]

Recall that a maximization problem \( P_1 \) in the \( \mathbb{R}^n \) Euclidean space can be represented as follows [BSS06]:

\[
\begin{align*}
\text{Maximize} & \quad f_1(x) : \mathbb{R}^n \to \mathbb{R}, \\
(P_1) & \quad \text{subject to} \quad g(x) = b, \quad h(x) \leq c, \\
& \quad \text{where} \quad x \in \mathbb{R}^n, b \in \mathbb{R}^{m_1}, \text{ and } c \in \mathbb{R}^{m_2}. 
\end{align*}
\tag{12}
\]

For several optimization problems, an optimal solution can be recognized when a feasible solution obeys certain optimality conditions. In such cases, it is unnecessary to represent an optimal solution T as in (11). The Duality concept in optimization can play an important role here.

Let \( u \in \mathbb{R}^{m_1} \) and \( v \in \mathbb{R}^{m_2} \) be two vectors of variables with \( v \geq 0 \). Given a primal problem \( P_1 \) as in (12), its Lagrangian dual problem \( P_2 \) can be represented as (see [BSS06]):

\[
\begin{align*}
\text{Minimize} & \quad \theta(u, v) \\
(P_2) & \quad \text{subject to} \quad v \geq 0, \\
& \quad \text{where} \quad \theta(u, v) = \inf_{x \in \mathbb{R}^n} \{ f(x) + \sum_{i=1}^{m_1} u_i g_i(x) + \sum_{j=1}^{m_2} v_j h_j(x) \}. 
\end{align*}
\tag{13}
\]

Furthermore, \( g_i(x) = b_i \) \( [h_j(x) \leq c_j] \) is the \( i^{th} \) equality \( [j^{th} \text{ inequality}] \) constraint.

We have demonstrated \( \Sigma_1 \) second order expressibility using Lagrangian duality in this paper. However, other types of duality may be used, such as Fenchel duality or the geometric duality or the canonical duality, as long as they provide a zero duality gap, and optimality conditions that can be verified efficiently (say, in polynomial time).
4.1 Computational Models

Turing machine (TM) based computational models for solving an optimization problem $Q$ come in two flavours:

**Model 1.** The input consists of a problem instance such as in (12). If the instance has a feasible solution, the output is a string representing an optimal solution; otherwise, the TM crashes. Corresponding to classes P and NP in the world of decision problems, the classes here are $FP$ and $FNP$ respectively.

**Model 2.** In addition to a problem instance such as in (12), the input consists of a parameter $K$, which is a bound on the optimal solution value. The TM is a “decision” machine, that is, one whose output is simply a yes or a no; call this machine as $M_1$. The method then to solve $Q$ by a Turing machine, say $M_2$, is to do a binary search on solution values, calling $M_1$ a logarithmic ($\log V$) number of times, where $V$ is the optimal solution value. Thus we make a *weakly polynomial* number of calls to $M_1$. Each call to $M_1$ involves answering a question such as: “Is there a feasible solution $S$ satisfying the constraints, such that the objective function value $f(A, S)$ is greater than or equal to $K$?”, for a maximization problem.

If the problem answered by $M_1$ is in the NP class, then the complexity of solving $Q$ is $P^{NP}$, since $M_2$ makes a polynomial number of calls to the oracle $M_1$ (and $M_1$ solves a problem in NP).

Similarly, if the problem answered by $M_1$ is in the class P, then the complexity of solving $Q$ as $P^P$, which is simply P (although strictly speaking, this is weakly polynomial due to the $\log V$ number of calls).

The method used in Model 2, binary search, has been recognised/adopted for solving optimization problems since the discovery of the class NP. It involves making a polynomial number of calls to a “decision TM” (a TM that solves decision problems).

However, we show in this section that for pairs of problems with a duality gap of zero, a single call to a decision TM is sufficient. If the machine answers yes, then the primal and the dual problems have optimal solutions; otherwise, neither problem has an optimal solution (one of the problems will be infeasible and the other one will have an unbounded optimal solution). This is demonstrated by second order $\Sigma_1$ sentences such as (17) and (20), which implies, as per Fagin’s result below, that such a machine produces a yes/no answer in NP time.

**Theorem 14.** [Fag74] A decision problem can be logically expressed in ESO if and only if it is in NP.

The following theorem is the deterministic counterpart of Fagin’s result. It characterizes P as the class of decision problems definable by ESO universal Horn formulae.

**Theorem 15.** (Grädel [E. 91]) For any ESO Horn expression as defined in Definition 8, the corresponding decision problem is a member of P.

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4For a graph problem, an algorithm is strongly polynomial if the running time is a polynomial in the number of vertices and/or edges; it becomes weakly polynomial if the running time is a polynomial in the logarithm of edge weights. In Linear Programming, this translates to the number of variables/constraints versus the data in the coefficient matrix $A$ and the right side vector $b$.  

10
The converse is also true — if a problem \( \mathcal{P} \) is a member of \( \mathcal{P} \), then it can be expressed in ESO Horn form — but only if a successor relation is allowed to be included in the vocabulary of the first-order formula \( \psi \).

The polynomial time computability in the first part of Theorem 15 is due to the fact that the first order part of formulae representing decision problems can be reduced to propositional Horn formulae, which can be solved in time linear in the number of predicates (which are unknown second order).

4.2 Linear Programming

For example, in the case of Linear Programming (LP), using Lagrangian duality, the primal and dual problems \( P_3 \) and \( P_4 \) respectively, can be stated as follows:

\[
(P_3) \quad \text{Maximize} \quad f_1(\mathbf{x}) = c^T \mathbf{x}, \quad \text{subject to} \quad \mathbf{A}\mathbf{x} \leq \mathbf{b}, \quad \mathbf{x} \geq \mathbf{0},
\]

where \( \mathbf{x}, \mathbf{c} \in \mathbb{R}^n \),

\[
(P_4) \quad \text{Minimize} \quad f_2(\mathbf{y}) = \mathbf{b}^T \mathbf{y}, \quad \text{subject to} \quad \mathbf{A}^T \mathbf{y} \geq \mathbf{c}, \quad \mathbf{y} \geq \mathbf{0},
\]

after the usual process [Had62] of converting unrestricted variables (if any) to non-negative variables, and equality constraints (if any) to inequality constraints, in the primal problem. Here, \( y_i \) (\( x_j \)) is the \( i^{th} \) dual (\( j^{th} \) primal) variable corresponding to the \( i^{th} \) primal (\( j^{th} \) dual) constraint. When the primal and dual problems have feasible solutions, then they both have optimal solutions \( \mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*) \) and \( \mathbf{y}^* = (y_1^*, y_2^*, \ldots, y_m^*) \) such that the two objective functions are equal: \( c^T \mathbf{x}^* = \mathbf{b}^T \mathbf{y}^* \). (Almost every book on LP should explain this result. See for example, [Had62].)

For LP’s, the complementary slackness conditions below are known to be necessary and sufficient conditions for the existence of an optimal primal solution \( \mathbf{S} \) and an optimal dual solution \( \mathbf{T} \):

\[
y_i^*(b_i - A_i \mathbf{x}^*) = 0, \quad y_i^* \geq 0, \quad b_i - A_i \mathbf{x}^* \geq 0, \quad i \in \{1, 2, \ldots, m\} \tag{15}
\]

\[
x_j^*(c_j - A_j^T \mathbf{y}^*) = 0, \quad x_j^* \geq 0, \quad c_j - A_j^T \mathbf{y}^* \geq 0, \quad j \in \{1, 2, \ldots, n\} \tag{16}
\]

where \( A_i \) is the \( i^{th} \) row of \( \mathbf{A} \), \( A_j^T \) is the \( j^{th} \) column of \( \mathbf{A} \), \( (b_i - A_i \mathbf{x}^*) = 0 \) is derived from the \( i^{th} \) primal constraint, and \( (c_j - A_j^T \mathbf{y}^*) = 0 \) is derived from the \( j^{th} \) dual constraint.

Thus the existence of \( \mathbf{S} \) and \( \mathbf{T} \) can be expressed as

\[
\exists \mathbf{S} \exists \mathbf{T} \left[ \forall i \psi_1(i) \land \forall j \psi_2(j) \land \phi_p(\mathbf{S}) \land \phi_d(\mathbf{T}) \right], \tag{17}
\]

where \( \psi_1(i) \) [\( \psi_2(j) \)] logically captures the \( i^{th} \) \( j^{th} \) constraint in [15] [16] respectively. Also, \( \phi_p \) and \( \phi_d \) model the primal and dual constraints in [14] respectively.

We are not concerned about the first order part of the above expression, \([\forall i \psi_1(i) \land \forall j \psi_2(j)] \land \phi_p \land \phi_d \). What is of interest to us is that the existence of optimal solutions for the primal and dual problems can be expressed in ESO, existential second order logic; a \( \Sigma_2 \) second order sentence as in [11] is unnecessary.

Applying Theorem 14 it follows that recognition of an optimal solution, for certain problems such as LP, is in the computational class NP.
(One could argue that the existence of a feasible solution\footnote{A word of caution — \textbf{Feasible solutions}, a difference in terminology: Fagin and Grädel\cite{Fagin1991} have syntactically characterized \textit{feasible} solutions for classes NP and P respectively. However the “feasibility” captured by an ESO expression, as described by Fagin and Grädel, also includes an upper (lower) bound on the objective function of a minimization (maximization) problem, such as $f_1(x) \geq K$ where $K$ is a constant — not just satisfaction of the constraints such as $Ax \leq b$, $x \geq 0$ in (14). In this paper, we differ from this view; when we talk about feasibility, we only refer to satisfaction of constraints such as $Ax \leq b$, $x \geq 0$.} for an optimization problem, satisfying constraints such as $Ax \leq b$, $x \geq 0$, implies the existence of an optimal solution.)

4.3 Polynomially Solvable Problems

But what if the primal and dual problems are polynomially solvable? Can this be reflected in expressions such as (17)? The answer turns out to be yes. Recall from Theorem (15) that to express polynomial solvability, the first order part of (17) needs to be a universal Horn formula.

From our assumption about the polynomial solvability of the primal and the dual problems and from Theorem (15), it follows that $\phi_p$ and $\phi_d$ can be expressed as universal Horn formulae.

As for the complementary slackness conditions (15) and (16), we only need to express $y^*_i (b_i - A_i x^*) = 0$ and $x^*_j (c_j - A^T_j y^*) = 0$, since the other conditions have been expressed in $\phi_p$ and $\phi_d$.

$y^*_i (b_i - A_i x^*) = 0$ can be expressed as $\psi_1(i) \equiv Y(i) \lor B_A (i, X)$, where $Y(i)$ is a predicate which is true iff $y^*_i = 0$, and $B_A (i, X)$ is a predicate which is true iff $b_i - A_i x^* = 0$. The formula $\psi_1(i)$ is not Horn. However, since $y^*_i = 0$ and $b_i - A_i x^* = 0$ do not occur anywhere else in (17), we can negate the predicates and modify $\psi_1(i)$.

As in Theorem 1, the Horn condition in the formula $\eta$ applies only to the second order predicates in $S$ and $T$. In this case, it applies to predicates that involve unknowns such as $x_j$ and $y_i$.

Let $Y_{\text{notEq}}(i)$ be true iff $y^*_i \neq 0$, and $B_{\text{AnotEq}}(i, X)$ be true iff $b_i - A_i x^* \neq 0$. Using these, one can rewrite $\psi_1(i)$ as

$$\psi_1(i) \equiv \neg Y_{\text{notEq}}(i) \lor \neg B_{\text{AnotEq}}(i, X), \quad (18)$$

which is a Horn formula. Similarly, the formula $\psi_2(j)$ in (17) can be expressed in Horn form:

$$\psi_2(j) \equiv \neg X_{\text{notEq}}(j) \lor \neg C_{\text{AnotEq}}(j, Y). \quad (19)$$

Now that we know that all four subformulae in the first order part of (17) can be expressed in universal Horn form, we can conclude that the formula in (17) fully obeys the conditions of Theorem [15] that is, ESO logic with the first order part being a universal Horn formulae (that is, the quantifier-free part is a conjunction of Horn clauses). Hence we can state that

\textbf{Theorem 16. For a pair of primal and dual Linear Programming problems as in (14), and hence obeying strong duality, the existence of optimal solutions for the primal and the dual can be expressed in ESO logic with the first order part being a universal Horn formula, and the optimal solutions can be computed in polynomial time (a) syntactically, and (b) by a single call to a decision Turing machine (which returns yes/no answers).}

But does strong duality imply polynomial time solvability? This is the subject of another manuscript \cite{MS09}.\footnote{A word of caution — \textbf{Feasible solutions}, a difference in terminology: Fagin and Grädel\cite{Fagin1991} have syntactically characterized \textit{feasible} solutions for classes NP and P respectively. However the “feasibility” captured by an ESO expression, as described by Fagin and Grädel, also includes an upper (lower) bound on the objective function of a minimization (maximization) problem, such as $f_1(x) \geq K$ where $K$ is a constant — not just satisfaction of the constraints such as $Ax \leq b$, $x \geq 0$ in (14). In this paper, we differ from this view; when we talk about feasibility, we only refer to satisfaction of constraints such as $Ax \leq b$, $x \geq 0$.}
4.4 MaxFlow MinCut

(Still some work to be done here; The first order part is not universal Horn, but I think it should be possible to make it universal Horn.)

The MaxFlow-MinCut Theorem is another example where Lagrangian duality plays an important role in characterizing optimal solutions. The MaxFlow and MinCut problems are dual to each other. At optimality, the values of the two optimal solutions coincide. An optimal solution to MaxFlow can be syntactically recognized by an “optimality condition”, rather than a comparison of the objective function value with those of all other feasible solutions.

Definition 17. The MaxFlow problem:
Given. We are given a network \( G = (V, E) \) with 2 special vertices \( s, t \in V \), \( E \) is a set of directed edges, and each edge \( (i, j) \in E \) has a capacity \( C_{ij} > 0 \).
To Do. Determine the maximum amount of flow that can be sent from \( s \) to \( t \) such that in each edge \( (i, j) \in E \), the flow \( f(i, j) \) is at most its capacity \( C_{ij} \). That is, \( 0 \leq f(i, j) \leq C_{ij}, \forall (i, j) \in E \).

An S-T Cut is a non-empty subset \( U \) of \( V \) such that \( S \in U \) and \( T \in \bar{U} \), where \( \bar{U} = V - U \).

Definition 18. The MinCut problem:
Given. Same as the MaxFlow problem.
To Do. Of all the \( S-T \) cuts in \( G \), find a least cut; that is, a cut with the least capacity.

The optimality condition for the MaxFlow problem is that there exists a least \( S-T \) cut, \( U \), such that

- (forward direction) For every edge \( (i, j) \) in the edge set \( E \) such that \( i \in U \) and \( j \in \bar{U} \), the flow in \( (i, j) \), \( f(i, j) \), is equal to its capacity \( C_{ij} \);
- (backward direction) For every edge \( (i, j) \in E \) such that \( i \in \bar{U} \) and \( j \in U \), \( f(i, j) = 0 \); and
- The maximum flow, that is, the optimal solution value for the MaxFlow problem, is equal to \( C(U) \), the capacity of the cut \( U \).

This condition can be syntactically characterized as

\[
\exists U \exists F \forall i \forall j \ U(S) \land \neg U(T) \\
\land [E(i, j) \land U(i) \land \neg U(j) \rightarrow F(i, j, C_{ij})] \\
\land [E(i, j) \land \neg U(i) \land U(j) \rightarrow F(i, j, 0)] \land \psi, \text{ where}
\]

\( U \) and \( F \) are second order predicates;
\( E(i, j) \) is a first order relation which is true whenever \( (i, j) \) is an edge in the input graph;
$U(i)$ is true when $i \in$ vertex set $U$;

$F(i, j, v)$ is true when the flow in the edge $(i, j)$ equals $v$; and

$\psi$ models the flow conservation constraint at all nodes.

Once more, with the help of previously proven optimality conditions, we have been able to characterize the primal optimal solution $F$ and the dual optimal solution $U$, in existential second order logic (ESO).

Similarly in Convex Programming, the Karush-Kuhn-Tucker conditions provide sufficient conditions for the optimality of a feasible solution.

5 Zero Duality Gap

**Note.** Here, we only express the existence of optimal solutions. We do not compute optimal solutions (we do not provide a method to compute them). Grädel [E. 91] provided an expression for the existence of a feasible solution for polynomially solvable problems. What we provide here is an improvement on his result, for problems that obey strong duality.

We observe that expressions such as those in (17) and (20) are possible only if there is no duality gap, that is, when the duality gap is zero. The primal optimality condition implies dual feasibility and vice versa.

(ARE THE FORMULAE IN (17) and (20) Horn? If so, using the “propositional Horn formula in linear time” property, can you get something algorithmic out of it?)

To our knowledge, all known problem-pairs with a zero duality gap (also known as strong duality) are polynomially solvable. The decision versions of all such optimization problems can be shown to be in the complexity class NP $\cap$ CoNP [MS09].

Problems in NP $\cap$ CoNP can be expressed in both ESO and USO (universal second order logic), since USO precisely characterizes problems in USO.

Does strong duality imply polynomial time solvability? This is the subject of another manuscript [MS09].

(How does all this relate to the POLYNOMIAL HIERARCHY? Saddle Points? (Linear) complementarity problems?)

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