SECOND ORDER ESTIMATES FOR CONVEX SOLUTIONS OF DEGENERATE $k$-HESSIAN EQUATIONS

HEMING JIAO AND ZHIZHANG WANG

Abstract. The $C^{1,1}$ estimate of the Dirichlet problem for degenerate $k$-Hessian equations with non-homogenous boundary conditions is an open problem, if the right hand side function $f$ is only assumed to satisfy $f^{1/(k-1)} \in C^{1,1}$. In this paper, we solve this problem for convex solutions defined in the strictly convex bounded domain.

Keywords: Degenerate $k$-Hessian equations; Second order estimates; convex solutions.

1. Introduction

Suppose $u$ is some function defined in a bounded domain $\Omega \subset \mathbb{R}^n$ and $\varphi$ is some given function defined on the boundary $\partial \Omega$. In this paper, we concern the Dirichlet problem of degenerate $k$-Hessian equations with non-homogenous boundary functions ($k \geq 2$)

\begin{equation}
\begin{aligned}
\sigma_k(\lambda(D^2u)) &= f \quad \text{in} \quad \Omega, \\
u &= \varphi \quad \text{on} \quad \partial \Omega,
\end{aligned}
\end{equation}

where $f \geq 0$ in $\Omega$ and $\sigma_k$ are the elementary symmetric functions

$$
\sigma_k(\lambda) = \sum_{i_1 < \ldots < i_k} \lambda_{i_1} \ldots \lambda_{i_k}, \quad k = 1, \ldots, n,
$$

and $\lambda(D^2u)$ is the eigenvalue vector of the hessian $D^2u$. The Poisson equation and Monge-Ampère equation fall into the form of (1.1) as $k = 1$ and $k = n$ respectively. We call a function $u \in C^2(\Omega)$ is $k$-convex if $\lambda(D^2u) \in \Gamma_k$ in $\Omega$, where $\Gamma_k$ is the Gårding’s cone

$$
\Gamma_k = \{ \lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0, j = 1, \ldots, k \}.
$$

A bounded domain $\Omega$ in $\mathbb{R}^n$ is called uniformly ($k-1$)-convex, if there is a positive constant $K$ such that for each $x \in \partial \Omega$,

$$(\kappa_1(x), \ldots, \kappa_{n-1}(x), K) \in \Gamma_k,$$

where $\kappa_1(x), \ldots, \kappa_{n-1}(x)$ are the principal curvatures of $\partial \Omega$ at $x$.

The central issue of the degenerate equations is the existence of $C^{1,1}$ solution, which major needs the a prior $C^{1,1}$ estimates of the solutions. Unlike the non degenerate equations, the establishment of $C^{1,1}$ estimate always requires some regularity of $f$ near the set of $f = 0$. In [8] and [9], Guan-Li have studied the degenerate

\begin{thebibliography}{9}

\bibitem{8} [8] Guan-Li have studied the degenerate

\bibitem{9} [9] Guan-Li have studied the degenerate

\end{thebibliography}

The first author is supported by the NSFC (Grant No. 11871243). Research of the second author is sponsored by Natural Science Foundation of Shanghai, No. 20JC1412400, 20ZR1406600 and supported by NSFC Grants No. 11871161, 12141105.

1
Weyl problem and degenerate Gauss curvature measure problem, which both are degenerate Monge-Ampère type equations. They found that the conditions
\[(1.2) \quad \Delta \left(\frac{f^{1/(n-1)}}{n-1}\right) \geq -A \text{ and } |D\left(\frac{f^{1/(n-1)}}{n-1}\right)| \leq A\]
for some constant $A$, are sufficient to get the global $C^{1,1}$ bound. Soon after, for the Dirichlet problem of the degenerate Monge-Ampère equations, Guan [7] found that the above conditions also are sufficient to get the $C^{1,1}$ boundary estimates, if we only consider the homogenous boundary problem. For the non homogenous boundary problem, Guan-Trudinger-Wang [13] established the $C^{1,1}$ boundary estimate, if one require
\[(1.3) \quad f^{1/(n-1)} \in C^{1,1}(\overline{\Omega}).\]
It is not difficult to see, (1.3) implies (1.2).

In view of the above results, a natural and interesting question is that can we establish the $C^{1,1}$ estimates for $k$-convex solutions of (1.1), including boundary estimates and global estimates, by using the condition
\[(1.4) \quad f^{1/(k-1)} \in C^{1,1}(\overline{\Omega}).\]
Note that if $k = n$, (1.4) is (1.3). This question is first proposed by Ivochkina-Trudinger-Wang [14], which is not solved until now.

Let’s review some related research of the above question. Write $\tilde{f} := f^{1/(k-1)}$. Dong [4] has considered the homogenous Dirichlet problem (1.1) with homogenous boundary condition $\varphi \equiv 0$. He established the $C^2$ estimates by (1.4) and
\[(1.5) \quad |D\tilde{f}| \leq C\tilde{f}^{1/2} \text{ on } \overline{\Omega}\]
for some positive constant $C$. In [15], by using (1.4), the authors obtain the $C^{1,1}$ estimate for the Dirichlet problem of degenerate $k$-curvature equations with homogenous boundary condition, which can be view as a generalization of [4]. If we require $f^{1/k} \in C^{1,1}(\overline{\Omega})$, which is a little stronger than (1.4), for non homogenous boundary problem, the $C^{1,1}$ regularity has been established by Krylov [16] [17] [18] [19], seeing an alternative proof by Ivochkina-Trudinger-Wang [14]. In [22], Wang gave an example, which shows that the condition (1.4) is optimal for the solutions of $k$-Hessian equations being in $C^{1,1}(\overline{\Omega})$. Therefore, condition (1.4) should be sharp. For more reference, the reader may see [3] [6] [21] and the reference therein.

The main results of this paper is to give the boundary estimate for convex solutions of (1.1) with non homogenous boundary functions, only using condition (1.4). Note that, for $k$-Hessian equations, the assumption of convex solutions has been used in [12] [11] [5]. Our result partially answers Ivochkina-Trudinger-Wang’s question.

**Theorem 1.1.** Suppose $\Omega$ is uniformly convex with $\partial \Omega \in C^{3,1}$, $\varphi \in C^{3,1}(\partial \Omega)$, $f > 0$ in $\Omega$ and (1.4). Then any convex solution $u \in C^3(\Omega) \cap C^2(\overline{\Omega})$ to the Dirichlet problem (1.1) satisfies the estimates
\[(1.6) \quad \max_{\partial \Omega} |D^2u| \leq C,\]
where the positive constant $C$ depends on $n$, $k$, $\Omega$, $|\varphi|_{C^{3,1}(\partial \Omega)}$ and $|f^{1/(k-1)}|_{C^{1,1}(\overline{\Omega})}$ but is independent of the lower bound $\inf_{\Omega} f$. 

Theorem 1.1 can be regarded as a generalization of the main results in [13]. The major difficulty to prove (1.6) is the estimates for double normal derivatives which we establish in two steps. One is Lemma 5.1 which generalizes Lemma 3.1 of [13]. We utilize the idea of Trudinger [20], then we can obtain the same lower bound of tangential-tangential derivatives only assuming that the solutions are \( k \)-convex for \( k \)-Hessian equations. Our method is completely different from [13].

The other one is an estimate of mixed tangential-normal derivatives in terms of tangential-tangential derivatives, namely, Lemma 6.1. Similar estimate was proved in [13] for Monge-Ampère equation, where the special structure of Monge-Ampère equation plays a key role to obtain the bound of the tangential-normal derivatives. Unfortunately, the \( k \)-Hessian equations with \( 2 \leq k \leq n - 1 \) do not possess such structure. Therefore, we need some new idea to reestablish these estimates without using the affine transformation, which is our novelty of Section 6 and Section 7.

Since the constant in (1.6) is independent of \( \inf_{\Omega} f \), by approximation and the \( C^1 \) estimates, global \( C^2 \) estimates in section 2, section 3, we obtain the following \( C^1, 1 \) estimates for degenerate \( k \)-Hessian equations.

**Theorem 1.2.** Suppose \( \Omega \) is uniformly convex with \( \partial \Omega \in C^{3,1}, \varphi \in C^{3,1}(\partial \Omega), f \geq 0 \) in \( \Omega \) and (1.4). Then any convex solution \( u \in C^4(\Omega) \cap C^2(\partial \Omega) \) to the Dirichlet problem (1.1) satisfies the estimates

\[
|u|_{C^{1,1}(\Omega)} \leq C,
\]

where the positive constant \( C \) depends on \( n, k, \Omega, |\varphi|_{C^{3,1}(\partial \Omega)} \) and \( |f^{1/(k-1)}|_{C^{1,1}(\Omega)} \).

The rest of the paper is organized as follows. In Section 2, we establish the \textit{a priori} \( C^1 \) estimates. Section 3 is devoted to the maximal principle for second order derivatives. We concern the boundary estimates of the pure tangential derivatives and mixed second order derivatives in Section 4. In Section 5, we prove a lower bound of the pure tangential derivatives on the boundary. In the last two sections, we obtain an upper bound of the tangential-normal derivatives in terms of the pure tangential derivatives for convex solutions.

## 2. \textit{C}^1 \textit{estimates}

In Section 2 to Section 5, we assume that \( u \) is the \( k \)-convex solution to (1.1). We establish the \( C^1 \) estimates for \( u \) in this section. By using the \((k-1)\)-convexity of the domain \( \Omega \), we can construct a subsolution \( \underline{u} \) to (1.1) as [2].

\[
\begin{aligned}
\sigma_k(\lambda(D^2 u)) &\geq f \quad \text{in } \Omega, \\
\underline{u} &= \varphi \quad \text{on } \partial \Omega.
\end{aligned}
\]

Let \( h \) be the harmonic function in \( \Omega \) with \( h = \varphi \) on \( \partial \Omega \). By the comparison principal, we have

\[
\underline{u} \leq u \leq h \quad \text{in } \Omega \text{ and } \underline{u} = u = h \quad \text{on } \partial \Omega
\]

Thus, we have

\[
\sup_{\Omega} |u| + \sup_{\partial \Omega} |Du| \leq C,
\]

where the constant \( C > 0 \) depends only on \( |u|_{C^1(\Omega)} \) and \( |h|_{C^1(\partial \Omega)} \). Define

\[
S_k[r] = \sigma_k(\lambda(r))
\]
for a symmetric matrix $r = \{r_{ij}\}$ with $\lambda(r) \in \Gamma_k$ and

$$
S_{ij}^k = \frac{\partial S_k[D^2u]}{\partial r_{ij}}.
$$

By the Newton-Maclaurin inequality, we have

$$
\sum_i S_{ii}^k = (n - k + 1)S_{k-1} \geq c_0 S_{k}^{-1/(k-1)} S_{1}^{1/(k-1)}
$$

for some positive constant $c_0$ depending only on $n$ and $k$.

**Proposition 2.1.** Assume that

$$
|Df(x)| \leq Af^{1-\frac{1}{k-1}}(x)
$$

holds for some positive constant $A$ and any $x \in \Omega$. Then there exists positive constant $C$ depending only on $n$, $k$, $A$ and $\Omega$ such that

$$
\sup_{\Omega} |Du| \leq C(1 + \sup_{\partial \Omega} |Du|).
$$

**Proof.** We may assume $0 \notin \overline{\Omega}$ and $|x|^2 \geq \delta_0 > 0$ for all $x \in \overline{\Omega}$. Suppose

$$
W := \max_{x \in \overline{\Omega}, \xi \in \mathbb{S}^n} \{u_\xi + B|x|^2\}
$$

is attained at an interior point $x_0 \in \Omega$ and $\xi_0 \in \mathbb{S}^n$, where $B$ is a positive constant sufficiently large to be determined. We may assume $\xi_0$ is in the direction $x_1$ by rotating the coordinates. We have, at $x_0$,

$$
\sum_{l=1}^n u_{ll}^2 = 4B^2 \sum_l x_l^2 = 4B^2|x|^2 \geq 4B^2\delta_0.
$$

By (2.2) and (2.3), at $x_0$, we have

$$
0 \geq S_{ij}^k(u_1 + B|x|^2)_{ij} = f_1 + 2B \sum_i S_{ii}^k
$$

$$
\geq -Af^{1-1/(k-1)} + 2c_0 BS_{1}^{1/(k-1)} S_{k}^{-1/(k-1)}
$$

$$
= -Af^{1-1/(k-1)} + 2c_0 B(\Delta u)^{1/(k-1)} f^{1-1/(k-1)}.
$$

Since $\lambda(D^2u) \in \Gamma_k \subset \Gamma_2$, we have

$$
0 < 2\sigma_2(\lambda(D^2u)) = 2 \sum_{1 \leq i < j \leq n} (u_{ii}u_{jj} - u_{ij}^2)
$$

$$
= (\Delta u)^2 - \sum_i u_{ii}^2 - \sum_{i \neq j} u_{ij}^2.
$$

We have, by (2.6),

$$
(\Delta u)^2 \geq \sum_i u_{ii}^2 \geq 4B^2\delta_0.
$$

Thus, in view of (2.7), we derive

$$
0 \geq -Af^{1-1/(k-1)} + 2c_0 B(4B^2\delta_0)^{1/(k-1)} f^{1-1/(k-1)} > 0
$$

provided $B$ is sufficiently large which is a contradiction. Then we conclude that $W$ is attained on the boundary when choosing $B$ sufficiently large and (2.5) holds. □

The $C^1$ estimate has been established in terms of (2.2) and (2.5).
3. THE MAXIMAL PRINCIPLE FOR SECOND ORDER DERIVATIVES

In this section, we prove

**Proposition 3.1.** Suppose (2.4) and

\[ \inf \{ \Delta f_{x^r} (x) \} \geq -\frac{A}{k-1} \]

hold for some positive constant \( A \) and any \( x \in \Omega \). Then there exists a positive constant \( C \) depending on \( n, k, A \) and \( \Omega \) such that

\[ \sup_{\Omega} |D^2 u| \leq C(1 + \sup_{\partial \Omega} |D^2 u|). \]

To prove Theorem 3.1, we need the following lemma which is Lemma 3.2 in [10].

**Lemma 3.2.** Let \( \alpha = \frac{1}{k-1} \). If \( \lambda (D^2 u) \in \Gamma_k \), then

\[ \sum_i S_k^{pq,rs} u_{pqi} u_{rsi} \leq -S_k \sum_h \left[ \frac{(S_k)_h}{S_k} - \frac{(S_1)_h}{S_1} \right] \left[ (\alpha - 1) \frac{(S_k)_h}{S_k} - (\alpha + 1) \frac{(S_1)_h}{S_1} \right], \]

where

\[ S_k^{pq,rs} = \frac{\partial^2 S_k [D^2 u]}{\partial p_q \partial r_s}. \]

Lemma 3.2 comes from the concavity of \( \left( \frac{S_k}{S_1} \right)^{1/(k-1)} \) in \( \Gamma_k \).

**Proof of Proposition 3.1.** We consider the test function

\[ H = \Delta u + \frac{B}{2} |x|^2, \]

where \( B \) is a positive undetermined constant. First, by differentiating the equation (1.1), we have

\[ S_k^{ij} (\Delta u)_{ij} + \sum_i S_k^{pq,rs} u_{pqi} u_{rsi} = \Delta f, \]

which implies

\[ S_k^{ij} H_{ij} = -\sum_i S_k^{pq,rs} u_{pqi} u_{rsi} + B(n - k + 1) S_{k-1} + \Delta f. \]

Suppose \( H \) attains its maximum at an interior point \( x_0 \in \Omega \). Write \( \alpha := \frac{1}{k-1} \). We have, at \( x_0 \),

\[ 0 = H_i = (\Delta u)_i + B x_i \]

\[ = \sum_j S_k^{ij} (\Delta u)_{ij} + \sum_i S_k^{pq,rs} u_{pqi} u_{rsi} + B(n - k + 1) S_{k-1} + \Delta f. \]
for each $1 \leq i \leq n$ and therefore, by $[3.3]$, $[2.3]$, $[3.1]$ and $[2.4]$, we get

\[ 0 \geq - \sum_{i} S^p_{k} u_{pq} u_{rs} + B(n - k + 1) S_{k-1} + \Delta f \]

\[ \geq S_k \sum_{h} \left( \frac{(S_k)_{h}}{S_k} - \frac{(S_1)_h}{S_1} \right) \left( (\alpha - 1) \frac{(S_k)_{h}}{S_k} - (\alpha + 1) \frac{(S_1)_h}{S_1} \right) \]

\[ + (c_0 BS^1_1 - A) f^{1-\alpha} + (1-\alpha) \frac{\| \nabla f \|^2}{f} \]

\[ (3.7) \]

\[ = (\alpha - 1) \frac{\| \nabla f \|^2}{f} - 2\alpha \frac{\nabla \Delta u}{\Delta u} \cdot \nabla f + (1+\alpha) f \frac{\| \nabla \Delta u \|^2}{(\Delta u)^2} \]

\[ + (c_0 B(\Delta u)^{\alpha} - A) f^{1-\alpha} + (1-\alpha) \frac{\| \nabla f \|^2}{f} \]

\[ = \frac{2\alpha B x \cdot \nabla f}{\Delta u} + (1+\alpha) f \frac{B^2 |x|^2}{(\Delta u)^2} + (c_0 B(\Delta u)^{\alpha} - A) f^{1-\alpha} \]

\[ \geq \left( c_0 B(\Delta u)^{\alpha} - A - \frac{CB}{\Delta u} \right) f^{1-\alpha}, \]

where $C$ is some constant only depending on $\Omega, n, k$ and $A$. Thus, for sufficient large $B$ and $\Delta u$, we have the desired estimates. \qed

If there is no degenerate point on $\partial \Omega$, we have the usual boundary estimate as $[2]$. Therefore, using Proposition $2.1$ and Proposition $3.1$, we can prove

**Theorem 3.3.** Suppose $\Omega$ is $(k-1)$-convex with $\partial \Omega \in C^{3,1}$, $\varphi \in C^{3,1}(\overline{\Omega})$, $f \geq 0$ in $\Omega$, $f > 0$ on $\partial \Omega$ and $[2.4]$, $[3.1]$ hold for some constant $A > 0$. Then the estimate $[1.7]$ holds for any $k$-convex $C^{4,1}$ solution $u$ of $[1.1]$. 

4. **Estimates for mixed tangential-normal derivatives**

In this and the following sections, we derive the second order boundary estimates. For any point $x_0 \in \partial \Omega$, we may assume that $x_0$ is the origin and that the positive $x_n$-axis is in the interior normal direction to $\partial \Omega$ at the origin. Suppose near the origin, the boundary $\partial \Omega$ is given by

\[ x_n = \rho(x') = \frac{1}{2} \sum_{\alpha, \beta < n} B_{\alpha \beta} x_{\alpha} x_{\beta} + O(|x'|^3), \]

where $x' = (x_1, \ldots, x_{n-1})$ and $B_{\alpha \beta}$ is the second fundamental form of $\partial \Omega$ at $x_0$. Differentiating the boundary condition $u = \varphi$ on $\partial \Omega$ twice, we can find a constant $C$ depending on $|\varphi|_{C^{2}(\partial \Omega)}$ and $|u|_{C^{1}(\Omega)}$ such that

\[ |u_{\alpha \beta}(0)| \leq C \text{ for } \alpha, \beta \leq n - 1. \]

Next, we establish the estimate

\[ |u_{\alpha n}(0)| \leq C \text{ for } \alpha \leq n - 1. \]

For $x \in \partial \Omega$ near the origin, let

\[ T_{\alpha} = \partial_{\alpha} + \sum_{\beta < n} B_{\alpha \beta} (x_{\beta} \partial_{\alpha} - x_{n} \partial_{\beta}), \text{ for } \alpha < n, \]

\[ T_{n} = \partial_{n} \text{ and } \omega_{\delta} = \{ x \in \Omega : \rho(x') < x_n < \rho(x') + \delta^2, |x'| < \delta \}. \]

We have

\[ S_{k}^{ij}(T_{\alpha} u)_{ij} = T_{\alpha} f. \]
It follows that
\[ |S_{ij}^{ij}(T_\alpha(u - \varphi))_{ij}| \leq C \left( S_{k-1} + f^{1-1/(k-1)} \right) \]
and
\[ |T_\alpha(u - \varphi)| \leq C|x'|^2 \text{ on } \partial \Omega \cap \partial \delta \text{ for } \alpha < n \]
when \( \delta \) is sufficiently small since \( u = \varphi \) on \( \partial \Omega \). Because \( \Omega \) is uniformly \((k-1)\)-convex, there exist positive constants \( \theta \) and \( K \) such that
\[ (\kappa_1 - 2\theta, \ldots, \kappa_{n-1} - 2\theta, 2K) \in \Gamma_k. \]
Define
\[ \Psi = \rho(x') - x_n - \theta|x'|^2 + Kx_n^2. \]
Note that the boundary \( \partial \omega_\delta \) consists three parts: \( \partial \omega_\delta = \partial_1 \omega_\delta \cup \partial_2 \omega_\delta \cup \partial_3 \omega_\delta \), where \( \partial_1 \omega_\delta, \partial_2 \omega_\delta \) are defined by \( \{x_n = \rho\} \cap \partial \delta \), \( \{x_n = \rho + \delta^2\} \cap \partial \delta \) respectively, and \( \partial_3 \omega_\delta \) is defined by \( \{|x'| = \delta\} \cap \partial \delta \).

We see that when \( \delta \) is sufficiently small (depending on \( \theta \) and \( K \)), \( \Psi \leq 0 \) in \( \omega_\delta \) and furthermore,
\[ \Psi \leq -\frac{\theta}{2}|x'|^2, \text{ on } \partial_1 \omega_\delta \]
\[ \Psi \leq -\frac{\delta^2}{2}, \text{ on } \partial_2 \omega_\delta \]
\[ \Psi \leq -\frac{\theta \delta^2}{2}, \text{ on } \partial_3 \omega_\delta. \]

We derive from (4.7) that \( \lambda(D^2 \Psi) \in \Gamma_k \) and
\[ S_{ij}^{ij} \Psi_{ij} \geq \eta_0 S_{k-1} \text{ on } \partial \delta \]
for some uniform constant \( \eta_0 > 0 \) by further requiring \( \delta \) small.

We claim that there exist uniform positive constants \( A \) and \( \delta \) such that \( A\Psi \pm T_\alpha(u - \varphi) \leq 0 \) on \( \partial \Omega \). It is easy to get (4.8) from the claim since \( A\Psi(0) \pm T_\alpha(u - \varphi)(0) = 0 \).

Now we prove the claim. We first note that \( A\Psi \pm T_\alpha(u - \varphi) \leq 0 \) on \( \partial_\Omega \) by (1.7) when \( A \) is sufficiently large. So we may suppose
\[ W := \max_{\partial \Omega}(A\Psi \pm T_\alpha(u - \varphi)) \]

is attained at an interior point \( x_0 \in \omega_\delta \) and \( A \) is sufficiently large to be chosen.

We may assume, at \( x_0 \), \( A\Psi \pm T_\alpha(u - \varphi) \geq 0 \) for otherwise we are done. Now we consider two cases: (i) \( \Delta u(x_0) \leq 1 \) and (ii) \( \Delta u(x_0) > 1 \).

**Case (i).** As in the gradient estimates, we see \( |u_i(x_0)| \leq C_0 \Delta u(x_0) \leq C_0 \) for some uniform positive constant \( C_0 \) and each \( 1 \leq i, j \leq n \). We note that, at \( x_0 \),
\[ 0 = (A\Psi \pm T_\alpha(u - \varphi))_n \]
\[ = -A + 2AKx_n \]
\[ \pm \left[ (u - \varphi)_{\alpha n} + \sum_{\beta < n} B_{\alpha \beta}(x_\beta(u - \varphi)_{mn} - x_n(u - \varphi)_{\beta n}) - \sum_{\beta < n} B_{\alpha \beta}(u - \varphi)_{\beta n} \right] \]
\[ < 0, \]
if we choose $\delta$ is sufficiently small and $A$ is sufficiently large using the bound of $|D^2u|$ and $|Du|$ at $x_0$. Then we have a contradiction.

**Case (ii).** By (2.3) and (4.3), we see at $x_0$ where $W$ is attained,

$$0 \geq S_k^i(A\Psi \pm T_\alpha(u - \varphi))_{ij} \geq A\eta_0 S_{k-1} - C(S_{k-1} + f^{1-1/(k-1)})$$

$$\geq \frac{A}{2} \eta_0 S_{k-1} + \frac{A}{2} \eta_0 c_0 f^{1-1/(k-1)} - C(S_{k-1} + f^{1-1/(k-1)}) > 0$$

provided $A$ is sufficiently large, which is a contradiction. The claim follows and (4.3) is proved.

For the homogenous problem, combining (4.2), (4.3) with the estimates for double normal derivatives in [1], the conditions (2.4) and (3.1) are sufficient to obtain the existence of $C^{1,1}$ solutions, which is a slight different from Dong’s theorem [4].

**Theorem 4.1.** Suppose $\Omega$ is uniformly $(k - 1)$-convex with $\partial \Omega \in C^{3,1}$, $f \geq 0$ and there exists a positive constant $A$ such that (2.3) and (3.1) hold. Then the equation (1.1) with $\varphi \equiv 0$ has a unique $k$-convex solution $u \in C^{1,1}(\Omega)$ satisfying the estimates (1.7), where the constant $C$ depends on $\Omega$, $A$, $(\sup_{\partial \Omega} f)^{-1}$ and modulus continuity of $f$ in $\Omega$.

5. An Inequality on the Boundary

Suppose $W$ is a $(0, 2)$ tensor on $\overline{\Omega}$, namely $W \in C^2(T^*\overline{\Omega} \otimes T^*\overline{\Omega})$, where $T^*\overline{\Omega}$ is the co-tangent bundle of $\overline{\Omega}$. Let $W'$ be the projection of $W|_{\partial \Omega}$ in the bundle $T^*\partial \Omega \otimes T^*\partial \Omega$, where $T^*\partial \Omega$ is the co-tangent bundle of $\partial \Omega$. $\lambda'(W')$ denotes the eigenvalue vector of $W'$ with respect to the induced metric on $\partial \Omega$. In this section, we prove

**Lemma 5.1.** There exists a uniform positive constant $\delta_0$ such that

$$\sigma_{k-1}(\lambda'((D^2u)'')) \geq \delta_0 f$$

on $\partial \Omega$.

**Proof.** We use an idea being closed to [20]. Define $G(r) = \sigma_{k-1}^{1/(k-1)}(\lambda'(r))$ for a $(n - 1) \times (n - 1)$ matrix $r$ with its eigenvalues $\lambda'(r) \in \Gamma_{k-1}$. Let

$$m := \inf_{\partial \Omega} \frac{G((D^2u)')}{f},$$

where $\tilde{f} = f^{1/(k-1)}$. It suffices to prove $m \geq \delta_0$ for some positive constant $\delta_0$. Suppose $m$ is attained at $x_0 \in \partial \Omega$. We may assume that $x_0$ is the origin and the positive $x_n$-axis is in the interior normal direction to $\partial \Omega$ at the origin as before. Choose local orthonormal frames $\{e_1, \ldots, e_n\}$ around $x_0$ such that $e_n$ is the interior normal to $\partial \Omega$. $\nabla$ denotes the standard connection of $\mathbb{R}^n$. Write $e_i(x) = e_i^0(x)\partial_j$ for $i = 1, \ldots, n$, where $\partial_1, \ldots, \partial_n$ is the rectangular coordinate system. Thus, we have

$$\nabla_i u := \nabla_{e_i} u = e_i^0 \partial_j u = e_i^0 u_j$$

and

$$\nabla_{ij} u := \nabla_{e_i} \nabla_{e_j} u = e_i^k e_j^l \partial_k \partial_l u = e_i^k e_j^l u_{kl}.$$ 

We may also assume that $e_i^0(x_0) = \delta_{ij}$ for $1 \leq i, j \leq n$ and $\{\nabla_{\alpha \beta} u(x_0)\}_{1 \leq \alpha, \beta \leq n-1}$ is diagonal. Let $\overline{u}$ be a $k$-convex subsolution satisfying (2.3). Since $u - \overline{u} = 0$ on $\partial \Omega$, we find

$$\nabla_{\alpha \beta} \overline{u} = \nabla_{\alpha \beta} \overline{u}^0 - \nabla_{\alpha \beta}(u - \overline{u})\sigma_{\alpha \beta}, 1 \leq \alpha, \beta \leq n - 1$$

(5.2)
on $\partial \Omega$ near $x_0$, where $\sigma_{\alpha \beta} = \langle De_\alpha e_\beta, e_n \rangle$ is the second fundamental form of $\partial \Omega$. Let 
\[
G_0^{\alpha \beta} = \frac{\partial G}{\partial r_{\alpha \beta}}(\nabla_{\alpha \beta} u(x_0)), 1 \leq \alpha, \beta \leq n - 1.
\]

By the concavity of $G$, we have
\[
G_0^{\alpha \beta} (r_{\alpha \beta} - \nabla_{\alpha \beta} u(x_0)) \geq G(r) - G(\nabla_{\alpha \beta} u(x_0))
\]
for any matrix $r$ satisfying $\lambda'(r) \in \Gamma_{k-1}$. It follows from (5.2), (5.3) and the definition of $m$ that
\[
-G_0^{\alpha \beta} \sigma_{\alpha \beta} \nabla_n (u - \underline{u}) = G_0^{\alpha \beta} (\nabla_{\alpha \beta} u - \nabla_{\alpha \beta} \underline{u}) + G_0^{\alpha \beta} \nabla_{\alpha \beta} u(x_0)
\]
\[
\geq G(\nabla_{\alpha \beta} u) - G(\nabla_{\alpha \beta} u(x_0)) + G_0^{\alpha \beta} \nabla_{\alpha \beta} u(x_0) - G_0^{\alpha \beta} \nabla_{\alpha \beta} \underline{u}
\]
\[
\geq m \tilde{f} - G_0^{\alpha \beta} \nabla_{\alpha \beta} \underline{u}
\]
on $\partial \Omega$ near $x_0$. Since $\Omega$ is uniformly $(k - 1)$-convex and $\sigma_{\alpha \beta}$ is the second fundamental form of $\partial \Omega$, we have
\[
\lambda'(\{\sigma_{\alpha \beta} - \theta \delta_{\alpha \beta}\}) \in \Gamma_{k-1}
on\]
on $\partial \Omega$ near $x_0$ for some positive constant $\theta$ depending only on the geometry of $\partial \Omega$. Thus, by the concavity of $G$, we get
\[
G_0^{\alpha \beta} \sigma_{\alpha \beta} \nabla_{\alpha \beta} u(x_0) \geq G(\nabla_{\alpha \beta} u) - G(\nabla_{\alpha \beta} u(x_0)) + G_0^{\alpha \beta} \nabla_{\alpha \beta} u(x_0) - G_0^{\alpha \beta} \nabla_{\alpha \beta} \underline{u}
\]
\[
\geq m \tilde{f} - G_0^{\alpha \beta} \nabla_{\alpha \beta} \underline{u}
\]
for some positive constant $\gamma$. Let
\[
\Omega_\delta := \{ x \in \Omega : |x - x_0| < \delta \}
\]
for $\delta$ sufficiently small and $\eta := G_0^{\alpha \beta} \sigma_{\alpha \beta}$. Define the following barrier in $\Omega_\delta$, 
\[
\tilde{\Phi} = -\nabla_n (u - \underline{u}) + \frac{1}{\eta} \left( G_0^{\alpha \beta} \nabla_{\alpha \beta} \underline{u} - m \tilde{f} \right).
\]
It follows that $\tilde{\Phi} \geq 0$ on $\partial \Omega \cap \partial \Omega_\delta$ and $\tilde{\Phi}(x_0) = 0$. Since $e_j^i = \delta_{ij}$ at $x_0 = 0$, we have
\[
|e_{\alpha}^\alpha (u - \underline{u})_\alpha| \equiv | \sum_{j \neq \alpha} e_j^\alpha (u - \underline{u})_j | \leq C \sum_{j \neq \alpha} |e_j^\alpha| \leq C_0 |x|
\]
on $\partial \Omega \cap \partial \Omega_\delta$ for $1 \leq \alpha \leq n - 1$ and the constant $C_0$ depends only on the bound of $|D(u - \underline{u})|$ and $|De_j^\alpha|$ for $j \neq \alpha$. We may also assume that $\inf_{\Omega_\delta} e_{\alpha}^\alpha e_{\alpha}^\alpha \geq c_0 > 0$ for any $1 \leq \alpha \leq n - 1$ and some positive constant $c_0$ by choosing $\delta$ sufficiently small.
By (5.6) and that $e^i_j = \delta_{ij}$ at $x_0 = 0$ again, we have
\[
\nabla_n (u - \underline{u}) = e^\alpha_n (u - \underline{u}) + \sum_{\alpha=1}^{n-1} e^\alpha_n (u - \underline{u})_\alpha
\]
\[
= e^\alpha_n (u - \underline{u}) + \sum_{\alpha=1}^{n-1} e^\alpha_n e^\alpha_n (u - \underline{u})_\alpha \geq e^n_n (u - \underline{u})_n - C C_0 |x|^2
\]
on $\partial \Omega \cap \partial \Omega_\delta$, where the constant $C$ depends only on $\inf_{\partial \Omega} e^\alpha_n$ and the bound of $|D e^n_n|$, $1 \leq \alpha \leq n-1$. Since $\Phi \geq 0$ on $\partial \Omega \cap \partial \Omega_\delta$ and $\Phi(x_0) = 0$, we have
\[
\Phi := - (u - \underline{u})_n + \frac{1}{\eta e^n_n} (C^\alpha_0 \nabla_{\alpha \beta} \underline{u} - m \tilde{f}) + M |x|^2 \geq 0
\]
on $\partial \Omega \cap \partial \Omega_\delta$ and $\Phi(x_0) = 0$, where $M = C C_0 / \eta_0$. Let $\Psi$ be the function defined in (4.6). Note that $\Psi$ satisfies (4.7) and (4.8). Consider the function $B \Psi - \Phi$ on $\overline{\omega}$, where $B$ is a positive sufficiently large constant to be chosen later.

Because of (4.3) and $\tilde{f} \in C^{1,1}(\overline{\Omega})$, we have
\[
S^k_{ij} \Phi_{ij} \leq C \left( S_{k-1} + f^{1-1/(k-1)} \right) \text{ on } \overline{\omega}_\delta.
\]
As in Section 4, we can take sufficiently large $B$ to satisfy $B \Psi - \Phi \leq 0$ on $\partial \omega_k$. Suppose the maximum of $B \Psi - \Phi$ on $\overline{\omega}$ is achieved at an interior point $x_1 \in \omega_k$. We see, at $x_1$,
\[
(B \Psi - \Phi)_n = -B + 2 B K x_n - \Phi_n = 0,
\]
which implies that $u_{nn}(x_1) \geq B / 2$ provided $B$ is sufficiently large and $\delta$ is sufficiently small. Therefore, we get $\Delta u(x_1) \geq B / 2$. Using (2.3) again, we have, at $x_1$,
\[
0 \geq S^k_{ij} (B \Psi - \Phi)_{ij} > 0
\]
if $B$ is sufficiently large, which is a contradiction. Therefore, $B \Psi - \Phi \leq 0$ on $\overline{\omega}_\delta$. It follows that $\Phi_n(x_0) \geq B \Psi_n(x_0) \geq -C$. Thus, we get
\[
\frac{u_{nn}(x_0)}{C(1 + m)},
\]
where $C$ depends on $|u|_{C^{1,1}(\overline{\Omega})}$, $\sup_{\partial \Omega} |(D^2 u)'|$, $|\underline{u}|_{C^{2,1}(\overline{\Omega})}$ and $|f|_{C^{1,1}(\overline{\Omega})}$. Without loss of generality, we may assume $m \leq 1$. Combining with $\Delta u \geq 0$, we then obtain a bound
\[
\frac{u_{nn}(x_0)}{C} \leq C.
\]
Recall that $\nabla_{\alpha \beta} u(x_0) = u_{\alpha \alpha}(x_0) \delta_{\alpha \beta}$ for $1 \leq \alpha, \beta \leq n - 1$. Let $b_{\alpha} = u_{\alpha \alpha}(x_0)$ for $1 \leq \alpha \leq n - 1$ and $b = (b_1, \ldots, b_{n-1})$. By the equation (1.1), we see, at $x_0$,
\[
\frac{u_{n n} \sigma_{k-1} (b) - \sum_{\alpha=1}^{n-1} u_{\alpha \alpha}^2 \sigma_{k-2; \alpha} (b)}{\sigma_{k-1} (b)} = \sigma_{k} (b) = f.
\]
Since we have
\[
\frac{\sigma_k (b)}{\sigma_{k-1} (b)} = \frac{\sum_{\alpha=1}^{n-1} b_{\alpha} \sigma_{k-1; \alpha} (b)}{k \sigma_{k-1} (b)} = \frac{\sum_{\alpha=1}^{n-1} b_{\alpha} (\sigma_{k-1} (b) - b_{\alpha} \sigma_{k-2; \alpha} (b))}{k \sigma_{k-1} (b)} = \frac{\sum_{\alpha=1}^{n-1} b_{\alpha}}{k} - \frac{\sum_{\alpha=1}^{n-1} b_{\alpha}^2 \sigma_{k-2; \alpha} (b)}{k \sigma_{k-1} (b)} \leq \frac{\sum_{\alpha=1}^{n-1} b_{\alpha}}{k} \leq C,
\]
then we get
\[
\frac{f(x_0)}{\sigma_{k-1}(b)} \leq u_{nn}(x_0) + \frac{\sigma_k(b)}{\sigma_{k-1}(b)} \leq C.
\]
We are done. \(\square\)

6. Estimates for convex solutions, the case \(k = 2\)

From now on, we assume that \(u \in C^3(\Omega) \cap C^2(\overline{\Omega})\) is a convex solution of (1.1) and that \(\Omega\) is uniformly convex. We consider an arbitrary point \(x_0 \in \partial\Omega\). As in Section 4, we assume \(x_0\) is the origin, the positive \(x_n\)-axis is in the interior normal direction to \(\partial\Omega\) at the origin, and near the origin, the boundary \(\partial\Omega\) is given by (4.1).

In this and the following sections we always use the notation \(x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\) to denote the projection of \(x = (x_1, \ldots, x_n) \in \mathbb{R}^n\) in \(\mathbb{R}^{n-1}\) by its first \(n-1\) coordinates.

We may assume \(\{u_{\alpha\beta}(0)\}_{1 \leq \alpha, \beta \leq n-1}\) is diagonal and
\[
\{u_{\alpha\beta}(0)\}_{1 \leq \alpha, \beta \leq n-1} = \text{diag}\{b_1, \ldots, b_{n-1}\}.
\]
We may further assume \(b_1 \leq \cdots \leq b_{n-1}\).

Let \(\nabla_{\zeta\eta}\) denote the second order covariant derivative with respect to the standard connection \(\nabla\) for any local vector fields \(\zeta, \eta\) in \(\Omega\) near the origin. In this and the next section, we shall prove

**Lemma 6.1.** Let \(\nu\) denote the unit interior normal to \(\partial\Omega\) and \(\tau_\beta = \partial_\beta + \rho_\beta \partial_n\) be the tangential vector field of \(\partial\Omega\) for \(\beta \leq n-1\) near the origin. For \(\alpha = n-(k-1), \ldots, n-1\), there exist two positive constants \(\theta_\alpha\) and \(C\) depending only on \(\Omega\), \(f\in C^3(\overline{\Omega})\) and \(f^{1/(k-1)}\in C^1(\overline{\Omega})\) such that if any point \(x = (x_1, \ldots, x_n) \in \partial\Omega\) satisfies \(|x_\beta| \leq \theta_\alpha b_\alpha\) for any \(1 \leq \beta \leq \alpha\) and \(|x_\beta| \leq \theta_\alpha b_\beta\) for any \(\alpha + 1 \leq \beta \leq n-1\), we have the estimates
\[
|\nabla_{\nu\tau_\beta} u(x)| \leq C \sqrt{b_\alpha}
\]
for all \(\beta = 1, \ldots, \alpha\).

Let \(\omega_1 := \{x \in \Omega : x_n < \epsilon_1 b_{n-1}\}\) with a positive constant \(\epsilon_1\) sufficiently small to be chosen later. In this section, we prove
\[
|\nabla_{\nu\xi} u| \leq C \sqrt{b_{n-1}} \quad \text{on } \partial\omega_1 \cap \partial\Omega,
\]
for any convex solution \(u\) of (1.1), where \(\xi\) is any unit tangent vector of \(\partial\Omega\). Note that (6.2) implies Lemma 6.1 for \(\alpha = n-1\) and that Lemma 6.1 follows immediately for the case \(k = 2\).

Since \(f^{1/(k-1)} \in C^1,1(\overline{\Omega})\), we have
\[
|\hat{f}_\alpha(0)| \leq C \sqrt{\hat{f}(0)} \quad \text{for } \alpha \leq n-1,
\]
where \(\hat{f} := f^{1/(k-1)}\) and the constant \(C\) depends only on \(\hat{f}\in C^1,1(\overline{\Omega})\). By (6.1), (6.3) and Taylor’s expansion of \(\hat{f}\), we have
\[
\hat{f}(x) \leq \hat{f}(0) + \sum_{i=1}^n \hat{f}_i(0)x_i + C|x|^2 \leq C \left(\sigma_{k-1}^{1/(k-1)}(b) + \sigma_{k-1}^{1/2}(k-1)(b)|x'| + x_n + |x|^2\right)
\]
in $\Omega$ near the origin, where $x' = (x_1, \cdots, x_{n-1})$. It follows that

$$f(x) \leq C \left( \sigma_{k-1}(b) + \sqrt{\sigma_{k-1}(b)} |x'|^{k-1} + x_n^{k-1} + |x|^2(k-1) \right)$$

in $\Omega$ near the origin. Define $\omega := \{ x \in \Omega : x_n < 2b_{n-1} \}$. Therefore, we have

$$|x'| \leq C b_{n-1}, \quad \text{for } x \in \Omega$$

by (6.11). In the following, we always suppose $b_{n-1}$ is sufficiently small, otherwise by (6.10), we have got (6.2). Thus, by (4.1) and (6.4) we get

$$f \leq C b_{n-1}^{k-1} \text{ in } \omega.$$ 

Subtracting $Du(0) \cdot x + u(0)$, we may assume $\inf_{\Omega} u = u(0) = 0$ and $Du(0) = 0$. By Taylor’s expansion, we have

$$\varphi(x', \rho(x')) = \frac{1}{2} \sum_{\alpha \leq n-1} b_{i\alpha} x_{\alpha}^2 + \frac{1}{6} \sum_{\alpha, \beta, \gamma \leq n-1} \varphi_{\alpha\beta\gamma}(0) x_{\alpha} x_{\beta} x_{\gamma} + O(|x'|^4),$$

near the origin, where $\varphi_{\alpha\beta\gamma}(x') = \frac{\partial^3 \varphi(x', \rho(x'))}{\partial x_{\alpha} \partial x_{\beta} \partial x_{\gamma}}$. Since $u = \varphi \geq 0$ on $\partial \Omega$ and (6.3), we find

$$|\varphi_{\alpha\beta\gamma}(0)| \leq C \sqrt{b_{n-1}}.$$ 

We extend the boundary function $\varphi$ to be the right hand side of (6.7) near the origin in $\Omega$ and still denote it by $\varphi$. By the convexity of $u$, the maximum value of $u$ on $\{ x_n = 2b_{n-1} \}$ achieves on $\partial \Omega$. Therefore, again by convexity, we have

$$|u - \varphi| \leq \sup_{\partial \Omega \setminus \omega} (|u| + |\varphi|) \leq C b_n^2 \text{ on } \omega.$$ 

Let $d(x) := \text{dist}(x, \partial \Omega)$ and

$$v := -d + \frac{1}{8b_{n-1}} d^2.$$ 

We have $d \leq x_n$ near the origin so that $v \leq 0$ in $\omega$. Moreover, at $x = 0$, we have

$$D^2 v \geq \text{diag} \left\{ \frac{\kappa_1}{2}, \cdots, \frac{\kappa_{n-1}}{2}, \frac{1}{4b_{n-1}} \right\},$$

where $\kappa_1, \cdots, \kappa_{n-1}$ are the principal curvatures of $\partial \Omega$ at origin. Thus $v$ is convex near origin and

$$\sigma_k(D^2 v) \geq \frac{\delta_0}{b_{n-1}} \text{ in } \omega$$

for some positive constant $\delta_0$ because $\Omega$ is uniformly convex. Let

$$F(r) = \sigma_k^{1/k}(\lambda(r)) \text{ with } \lambda(r) \in \Gamma_k$$

where $r$ is a symmetric matrix and $\lambda(r)$ is the eigenvalue vector of $\lambda$, and

$$F^{ij} = \frac{\partial F(D^2 u)}{\partial u_{ij}}.$$ 

By using (6.10), $D^2 v - \delta_1 I$ is a positive definite matrix for some sufficiently small constant $\delta_1$, where $I$ is the identical matrix. By the concavity of $F$ and the uniformly convexity of $\Omega$, we find

$$F^{ij} v_{ij} \geq \delta_1 \left( \frac{1}{b_{n-1}^{1/k}} + \sum_{i=1}^{n} F^{ii} \right) \text{ in } \omega.$$
Let $\omega_0 := \{x \in \Omega : x_n < \epsilon b_{n-1}\}$, where $\epsilon$ is a positive constant sufficiently small to be determined. For any fixed $y = (y_1, \cdots, y_n) \in \partial \omega_0 \cap \partial \Omega$, we have

\begin{equation}
(6.12) \quad x_n - y_n \geq (2 - \epsilon) b_{n-1} \quad \text{on} \quad \partial \omega \cap \{x_n = 2b_{n-1}\}.
\end{equation}

Next, for any $x = (x_1, \cdots, x_n) \in \partial \omega \cap \partial \Omega$, we have

\begin{equation*}
x_n - y_n = \sum_{\beta=1}^{n-1} \rho_\beta(y')(x_\beta - y_\beta) + O(|x' - y'|^2)
\end{equation*}

\begin{equation*}
\geq \sum_{\beta=1}^{n-1} \rho_\beta(y')(x_\beta - y_\beta) - \kappa |x' - y'|^2
\end{equation*}

by Taylor’s expansion of $\rho$ at $y'$, where $\kappa$ is a positive constant depending only on the principal curvatures of $\partial \Omega$. Let

\begin{equation*}
L(x) := \sum_{\beta=1}^{n-1} \rho_\beta(y')(x_\beta - y_\beta).
\end{equation*}

It follows that

\begin{equation}
(6.13) \quad w(x) := x_n - y_n - L(x) + \kappa |x' - y'|^2 \geq 0 \quad \text{on} \quad \partial \omega \cap \partial \Omega.
\end{equation}

Note that $\rho_\beta(0) = 0$ for each $1 \leq \beta \leq n - 1$. We find, by (4.1),

\begin{equation*}
|L(x)| \leq |D\rho(y')| |x' - y'| \leq C|y'| |x' - y'| \leq C\sqrt{\epsilon} b_{n-1}
\end{equation*}

for any $x \in \mathcal{W}$. In particular, by (6.12), we have

\begin{equation}
(6.14) \quad w(x) \geq (2 - \epsilon - C\sqrt{\epsilon}) b_{n-1} \geq b_{n-1}
\end{equation}

on $\partial \omega \cap \{x_n = 2b_{n-1}\}$ by fixing $\epsilon$ small enough. Thus, by (6.9), (6.13) and (6.14), we have

\begin{equation}
(6.15) \quad |u - \varphi| \leq C b_{n-1} w \quad \text{on} \quad \partial \omega.
\end{equation}

From (6.6), (6.15), (6.11) and using Lemma 5.1, we derive

\begin{equation}
(6.16) \quad F_{ij} (b_{n-1} (A_1 v - A_2 w) \pm (u - \varphi))_{ij} \geq 0 \quad \text{in} \quad \omega
\end{equation}

\begin{equation*}
b_{n-1} (A_1 v - A_2 w) \pm (u - \varphi) \leq 0 \quad \text{on} \quad \partial \omega
\end{equation*}

by choosing $A_1 \gg A_2 \gg 1$. It follows from the maximal principle that

\begin{equation}
(6.17) \quad |u_\nu(y)| \leq C b_{n-1},
\end{equation}

for $y \in \partial \Omega \cap \partial \omega$. By (4.1), (6.7), (6.17) and the convexity of $u$, we have

\begin{equation}
(6.18) \quad \sup_{\omega_0} |u_\alpha| \leq \sup_{\partial \omega_0 \cap \partial \Omega} |\varphi_\alpha - u \rho_\alpha| \leq C b_{n-1}^{3/2}, \quad \text{for each} \quad 1 \leq \alpha \leq n - 1.
\end{equation}

Let $\epsilon_1 = \epsilon/2$. For any $x = (x', \epsilon_1 b_{n-1}) \in \partial \omega_1 \cap \{x_n = \epsilon_1 b_{n-1}\}$, by the convexity of $u$, (6.9) and (6.18),

\begin{equation}
(6.19) \quad - \epsilon_1 b_{n-1} u_\alpha(x) \leq u(0) - u(x) + \sum_{\alpha=1}^{n-1} x_\alpha u_\alpha(x) \leq C b_{n-1}^2.
\end{equation}

It follows that

\begin{equation}
(6.20) \quad u_n(x) \geq - \frac{C}{\epsilon_1} b_{n-1}.
\end{equation}
On the other hand, we fix a point \( y = (y', \epsilon b_{n-1}) \in \partial \omega_0 \cap \partial \Omega \). By the convexity of \( u \), (6.9) and (6.18), we find

\[
(6.21) \quad \epsilon_1 b_{n-1} u_n(x) \leq u(y) - u(x) + \sum_{\alpha=1}^{n-1} (y_\alpha - x_\alpha) u_\alpha(x) \leq C b_{n-1}^2
\]

and we obtain

\[
(6.22) \quad u_n(x) \leq \frac{C}{\epsilon_1} b_{n-1}.
\]

Combining (6.18), (6.20) with (6.22), we get

\[
|Du| \leq C b_{n-1} \quad \text{on} \quad \partial \omega_1.
\]

By the convexity of \( u \) again, we have

\[
(6.23) \quad |Du| \leq C b_{n-1} \quad \text{in} \quad \omega_1.
\]

In the following, we always denote \( \nabla_\alpha = \nabla_{\tau_\alpha} \). Note that \( |\rho_\alpha| \leq C |x'| \leq C b_{n-1}^{1/2} \) on \( \omega_1 \). By (6.18) and (6.23), we see \( |\nabla_\alpha u| \leq C b_{n-1}^{3/2} \) on \( \omega_1 \). It follows that

\[
(6.24) \quad |\nabla_\alpha (u - \varphi)| \leq C \sqrt{b_{n-1}} w \quad \text{on} \quad \partial \omega_1
\]

by using (6.14). Since \( \tau_\alpha \) is a tangential vector field of \( \partial \Omega \) near the origin, we have

\[
(6.25) \quad |\nabla_\alpha f| \leq C \sqrt{f} \quad \text{near the origin.}
\]

The proof of (6.25) can be found in Lemma 3.1 of [1]. Differentiating the equation \( F(D^2 u) = f^{1/k} \) and using Lemma 5.1, we get

\[
(6.26) \quad |F^{ij}(\nabla_\alpha u)_{ij}| \leq |\nabla_\alpha f^{1/k}| + C \sum_{i,j=1}^{n} F^{ij} u_{ij} + C |Du| \sum_{i=1}^{n} F^{ii}
\]

\[
\leq C f^{1/k^{2-\frac{1}{k}}} + C b_{n-1} \sum_{i=1}^{n} F^{ii}
\]

\[
\leq C b_{n-1}^{1/2-1/k} + C b_{n-1} \sum_{i=1}^{n} F^{ii}
\]

in \( \omega_1 \). By (6.8), we have

\[
(6.27) \quad |F^{ij}(\nabla_\alpha \varphi)_{ij}| \leq C \sqrt{b_{n-1}} \sum_{i=1}^{n} F^{ii} \quad \text{in} \quad \omega_1.
\]

Combining (6.21), (6.26), (6.27) and (6.11), we can choose positive constants \( A_1 \gg A_2 \gg 1 \) again such that

\[
F^{ij} \left( \sqrt{b_{n-1}}(A_1 v - A_2 w) \pm \nabla_\alpha (u - \varphi) \right)_{ij} \geq 0 \quad \text{in} \quad \omega_1
\]

\[
A \sqrt{b_{n-1}}(A_1 v - A_2 w) \pm \nabla_\alpha (u - \varphi) \leq 0 \quad \text{on} \quad \partial \omega_1
\]

By the maximal principle again we get

\[
\sqrt{b_{n-1}}(A_1 v - A_2 w) \pm \nabla_\alpha (u - \varphi) \leq 0 \quad \text{in} \quad \omega_1
\]

and

\[
(6.28) \quad |\nu(\nabla_\alpha w)(y)| \leq C \sqrt{b_{n-1}} \quad \text{for} \; \alpha \leq n - 1,
\]
DEGENERATE HESSIAN EQUATIONS

for any \( y \in \partial \Omega \cap \partial \omega_1 \), where \( \nu \) is the unit interior normal of \( \partial \Omega \) at \( y \). Therefore, in view of (6.23), we get (6.2).

Now we consider the case \( k = 2 \). We have, at the origin,

\[
u_{nn} \sigma_1(b) - \sum_{\alpha \leq n-1} u^{2}_{\alpha} + \sigma_2(b) = f.
\]

Thus, by (5.1) and (6.2), we obtain

\[(6.29) \quad \nu_{nn}(0) \leq C.\]

7. Estimates for convex solutions, the case \( k \geq 3 \)

In this section, we continue to prove Lemma 6.1 for the case \( k \geq 3 \).

As \([14]\), we will prove Lemma 6.1 by induction. Our induction hypothesis is that for some given \( \alpha = n - (k - 1), \ldots, n - 2 \) and any index \( \alpha + 1 \leq \beta \leq \gamma \) and \( |x_\beta| \leq \theta_\gamma b_\beta \) for any \( 1 \leq \beta \leq \gamma \) and \( |x_\gamma| \leq \theta_\gamma b_\gamma \) for any \( \gamma + 1 \leq \beta \leq n - 1 \), we have the estimates

\[(7.1) \quad |\nabla_{\nu x_\beta} u(x', \rho(x'))| \leq C \sqrt{b_\gamma} \]

for \( 1 \leq \beta \leq \gamma \).

From (6.2), we see that (7.1) holds for \( \alpha = n - 2 \). In the following, we will prove there exist positive constants \( \theta_\alpha \) and \( C \) depending only on \( \Omega \), \( |\varphi|_{C^{1,1}(\Omega)} \) and \( |\tilde{f}|_{C^{1,1}(\Omega)} \) such that if any point \( (x', \rho(x')) \in \partial \Omega \) satisfies \( |x_\beta| \leq \theta_\gamma b_\gamma \) for any \( 1 \leq \beta \leq \gamma \) and \( |x_\gamma| \leq \theta_\gamma b_\gamma \sqrt{b_\beta} \) for any \( \gamma + 1 \leq \beta \leq n - 1 \), we have

\[(7.2) \quad |\nabla_{\nu x_\beta} u(x', \rho(x'))| \leq C b_\alpha \]

for \( 1 \leq \beta \leq \alpha \).

Let

\[ \omega := \{ x \in \Omega : |x_\beta| < \delta b_\alpha \sqrt{b_\beta}, \beta = \alpha + 1, \ldots, n - 1, x_n < \delta^2 b_\alpha \} , \]

where \( \delta \) is a positive sufficiently small constant independent of \( b_\alpha \) to be determined later. By (6.1) and that \( u \geq 0 \) in \( \Omega \), we have

\[(7.3) \quad |\varphi_{\mu, \beta, \gamma}(0)| \leq \begin{cases} 
C \sqrt{b_\mu b_\beta b_\gamma} b_\alpha, & \text{if } \mu, \beta, \gamma > \alpha \\
C \sqrt{b_\beta b_\gamma} \sqrt{b_\alpha}, & \text{if } \mu \leq \alpha; \beta, \gamma > \alpha \\
C \sqrt{b_\gamma}, & \text{if } \mu, \beta \leq \alpha; \gamma > \alpha \\
C \sqrt{b_\alpha}, & \text{if } \mu, \beta, \gamma \leq \alpha 
\end{cases} \]

provided \( b_\alpha \) is sufficiently small. By (6.4) and the definition of \( \omega \), we have

\[(7.4) \quad f(x) \leq C b_{n-1} \cdots b_{\alpha+1} b_{\alpha+k+\alpha-n} \text{ in } \omega. \]

We then note that

\[(7.5) \quad u \leq C b_{\alpha}^2 \text{ in } \omega \]

by the convexity of \( u \).
Let
\[ u(x) = \frac{\sigma}{2} \left( b_0 |\hat{x}|^2 + \sum_{\beta = \alpha + 1}^{n-1} b_\beta x_\beta^2 \right) + \frac{1}{2} K x_n^2 - K^2 b_\alpha x_n, \]

where \( \hat{x} := (x_1, \ldots, x_n) \), \( \sigma \) and \( K \) are sufficiently small and sufficiently large positive constants to be chosen later. We then have
\[ \sigma_k (\lambda(D^2 u)) \geq \sigma^{k-1} K b_{n-1} \cdots b_{n+1} b_\alpha^{k+\alpha-n}. \]

Next, we prove
\[ \frac{u}{2} \leq u \]
on \( \partial \omega \) by choosing suitable \( \delta, \sigma \) and \( K \). Note that
\[ \partial \omega = \partial_1 \omega \cup \partial_2 \omega \cup \partial_3 \omega, \]
where \( \partial_1 \omega := \partial \omega \cap \partial \Omega, \partial_2 \omega := \partial \omega \cap \{ x_n = \delta^2 b_\alpha \} \) and \( \partial_3 \omega := \partial \omega \cap \{ |x_\beta| = \frac{\delta b_\alpha}{\sqrt{b_\beta}} \} \) for some \( \beta = \alpha + 1, \ldots, n-1 \).

We first consider any point \( x \in \partial_1 \omega \). By (6.7) and (7.3),
\[ u(x) \geq \left( \frac{1}{2} - C \delta \right) \sum_{\beta = \alpha + 1}^{n-1} b_\beta x_\beta^2 - C b_\alpha |\hat{x}|^2 \geq \frac{1}{4} \sum_{\beta = \alpha + 1}^{n-1} b_\beta x_\beta^2 - C b_\alpha |\hat{x}|^2 \]
provided \( \delta \) is sufficiently small. By the uniformly convexity of \( \Omega \), we have
\[ \frac{u}{2} \leq \frac{\sigma}{2} \sum_{\beta = \alpha + 1}^{n-1} b_\beta x_\beta^2 - b_\alpha \left( \frac{1}{2} K x_n - \frac{\sigma}{2} |\hat{x}|^2 \right) \leq \frac{\sigma}{2} \sum_{\beta = \alpha + 1}^{n-1} b_\beta x_\beta^2 - \frac{K^2}{4} b_\alpha |\hat{x}|^2 \]
if \( \sigma \) is sufficiently small. Thus, (7.7) holds on \( \partial_1 \omega \) provided \( \sigma \) is sufficiently small and \( K \) is sufficiently large.

Next, for \( x \in \partial_2 \omega \)
\[ u(x) \leq C \delta^2 b_\alpha^2 \sigma - \frac{K^2 \delta^2 b_\alpha^2}{2} \leq \frac{1}{2} u(x) \]
provided \( \sigma \) is sufficiently small or \( K \) is sufficiently large.

For \( \partial_3 \omega \), as [13], we only consider the piece \( \partial_3' \omega := \partial \omega \cap \left\{ x_{n-1} = \frac{\delta b_\alpha}{\sqrt{b_{n-1}}} \right\} \) and other cases can be handled similarly. We first prove that
\[ \frac{u}{4} \geq \frac{\sigma}{2} \delta^2 b_\alpha^2 \text{ on } \partial_3' \omega \cap \{ x_n < \epsilon_0 \delta^2 b_\alpha \} \]
provided \( \epsilon_0 \) is sufficiently small. On \( \partial_3' \omega \cap \{ x_n \geq \epsilon_0 \delta^2 b_\alpha \} \), (7.7) holds as on \( \partial_2 \omega \). Therefore, we only consider the set \( \partial_3' \omega \cap \{ x_n < \epsilon_0 \delta^2 b_\alpha \} \). Fix a point \( x = (\hat{x}, \tilde{x}, x_n) \in \partial_3' \omega \cap \{ x_n < \epsilon_0 \delta^2 b_\alpha \} \), where \( \hat{x} = (x_1, \ldots, x_n) \) and \( \tilde{x} = (x_{n+1}, \ldots, x_n) \). We consider two cases.

**Case 1.** Suppose \( \hat{x} = 0 \). Let \( x_0 = (0, \tilde{x}, \rho(0, \tilde{x})) \in \partial \Omega \). We may assume \( \delta \ll \min_{\alpha+1 \leq n \leq n-1} \theta_{\gamma} \). Let \( \hat{\nu} := \hat{\nu}(x') = (-D \rho(x'), 1) \in \mathbb{R}^n \) for \( x' \in \mathbb{R}^n \) near the origin. Note that \( |u_{\gamma}| = |\varphi_{\gamma} - \rho_{\gamma} u_n| \leq \sqrt{b_\alpha} \) on \( \partial \omega \cap \partial \Omega \) for \( 1 \leq \gamma \leq n-1 \).
Therefore, \( \nu = \frac{\partial}{\partial \nu} \) and for any point \( y = (0, \tilde{y}, y_n) \in \partial \omega \cap \partial \Omega \), by our induction hypothesis, we have
\[
|u_{\nu(y)}(y)| \leq |u_{\nu(0)}(0)|
\]
(7.9)
\[
+ \sum_{\beta = \alpha + 1}^{n-1} \sup_{\xi \in \partial \omega \cap \partial \Omega} \left( |\nabla_{\beta} u(\xi)| + \sum_{\gamma = 1}^{n-1} |p_{\gamma} u(\xi)| \right) |y_{\beta}| \leq Cb_{\alpha}
\]
by (7.11) and that \( Du(0) = 0 \). It follows that \( |u_{\nu(y)}| \leq Cb_{\alpha} \). Therefore \( |Du(y)| \leq Cb_{\alpha} \) for each \( 1 \leq \gamma \leq n-1 \). Thus, by the convexity of \( u \), \( x_n < \epsilon_0 b_{\alpha} \) and \( \bar{x} = 0 \), we have
\[
u(x) \geq u(x_0) - Cb_{\alpha} |x - x_0| = u(x_0) - Cb_{\alpha} (x_n - \rho(0, \bar{x})) \geq u(x_0) - C\epsilon_0 b_{\alpha}^2.
\]
(7.10)
By (7.3) and that \( x_{n-1} = \frac{b_{\alpha}}{\sqrt{b_{n-1}}} \), we have
\[
u(x_0) = \varphi(x_0) = \frac{1}{2} \sum_{\beta = \alpha + 1}^{n-1} b_{\beta} x_{\beta}^2 + \frac{1}{6} \sum_{\xi, \beta, \gamma \geq \alpha + 1} \varphi_{\xi, \beta, \gamma}(0) x_{\xi} x_{\beta} x_{\gamma} + O(|\bar{x}|^4)
\]
(7.11)
\[
\geq \frac{1}{2} \delta^2 b_{\alpha}^2 - C\delta^3 b_{\alpha}^2 \geq \frac{7}{16} \delta^2 b_{\alpha}^2
\]
provided \( \delta \) is sufficiently small. Combining with (7.10), we have
\[
u(x) \geq \frac{3}{8} \delta^2 b_{\alpha}^2
\]
(7.12)
when \( \epsilon_0 \) is sufficiently small.

**Case 2.** Suppose \( \bar{x} \neq 0 \). We may assume \( x \notin \partial \Omega \), otherwise (7.12) can be derived similar as (7.11). Let \( x_0 = (0, \bar{x}, x_n^0) = (0, \bar{x}, \rho(0, \bar{x})) \in \partial \Omega \) and \( P \) be the 2-dimensional plane spanned by \( x \) and the straight line through \( x_0 \) and parallel to the \( x_n \)-axis. Note that \( P \subset \{ x_{n-1} = \frac{b_{\alpha}}{\sqrt{b_{n-1}}} \} \). Let \( \gamma \) be the intersection of \( \partial \Omega \) and \( P \). It is clear that \( x^* = (0, \bar{x}, x_n + \varepsilon_1) \in P \), where \( \varepsilon_1 := - \sum_{\beta = 1}^{\alpha} x_{\beta} \rho_{\beta}(0, \bar{x}) \). Note that
\[
x_n^* - x_n^0 = x_n - \sum_{\beta = 1}^{\alpha} x_{\beta} \rho_{\beta}(0, \bar{x}) - \rho(0, \bar{x}) > \rho(\bar{x}, \bar{x}) - \sum_{\beta = 1}^{\alpha} x_{\beta} \rho_{\beta}(0, \bar{x}) - \rho(0, \bar{x}) \geq 0
\]
by the convexity of \( \rho \). Suppose
\[
\varepsilon_1 = \frac{(\bar{x}, 0, -\varepsilon_1)}{\sqrt{|\bar{x}|^2 + \varepsilon_1^2}}, \quad \varepsilon_2 = \frac{(\varepsilon_1 \bar{x}, 0, |\bar{x}|^2)}{|\bar{x}| \sqrt{|\bar{x}|^2 + \varepsilon_1^2}}.
\]
It is easy to see that \( \varepsilon_1 \) and \( \varepsilon_2 \) are the unit tangent vector and unit normal vector of the curve \( \gamma \) at \( x_0 \) respectively. By using the coordinate system \( \{ x_0; \varepsilon_1, \varepsilon_2 \} \), the curve \( \gamma \) can be written by
\[
\xi_2 = \kappa_0 \xi_1^2 + O(|\xi_1|^3) \text{ as } \xi_1 \to 0
\]
(7.13)
for some positive constant \( \kappa_0 \) depending only on \( \partial \Omega \). Thus, the straight line from \( x \) to \( x^* \) meets \( \gamma \) at a point \( \bar{x} \) satisfying
\[
\epsilon_0 |x - x^*| \leq |\bar{x} - x^*| \leq |\bar{x} - x|
\]
(7.14)
for some $0 < c_0 < 1$ depending only on $\partial \Omega$.

Let $\bar{x} = \xi_1 e_1 + \xi_2 e_2$. Suppose $\cos \theta = \langle e_2, E_n \rangle$, where $E_n = (0, \ldots, 0, 1)$ is the direction of the $x_n$-axis. Thus, we find

$$\xi_2 = (x_n^* - x_0^*) \cos \theta$$

and $|\xi_1| \leq C \sqrt{\xi_2}$

for some $C$ depending only on $\kappa_0$ by (7.13), since $|\xi_1| \leq C\epsilon_0 \delta^2 b_\alpha$ and $x_n^* - x_0^* \leq C\epsilon_0 \delta^2 b_\alpha$ which can be sufficiently small. Next, we see

$$E_n = \frac{-\epsilon_1}{\sqrt{\xi_1^2 + |\bar{x}|^2}} e_1 + \frac{|\bar{x}|}{\sqrt{\xi_1^2 + |\bar{x}|^2}} e_2.$$ 

It follows that, using the definition of $\epsilon_1$,

$$\bar{x}_n - x_0^* = \frac{-\epsilon_1 \xi_1 + |\bar{x}| \xi_2}{\sqrt{\xi_1^2 + |\bar{x}|^2}} \leq |D\rho(0, \bar{x})||\xi_1| + |\xi_2| \leq C\epsilon_0 \delta^2 b_\alpha$$

and therefore we get

$$\bar{x}_n \leq C\epsilon_0 \delta^2 b_\alpha + x_0^* \leq C\epsilon_0 \delta^2 b_\alpha.$$

Thus, $\bar{x} \in \omega$ if $\epsilon_0$ is sufficiently small.

By (6.7) and $\bar{x} \in \omega$, we have

$$u(\bar{x}) = \frac{1}{2} \sum_{\beta=1}^{n-1} b_\beta \bar{x}_\beta^2 + \frac{1}{6} \sum_{\xi, \beta, \gamma} \varphi_{\xi \beta \gamma}(0) \bar{x}_\xi \bar{x}_\beta \bar{x}_\gamma + O(|\bar{x}|^4)$$

(7.15)

$$\leq u(x_0) + \frac{1}{2} \sum_{\beta=1}^{n} b_\beta \bar{x}_\beta^2 + C\delta^3 b_\alpha^2.$$ 

By (6.11), we have

$$\frac{1}{2} \sum_{\beta=1}^{n} b_\beta \bar{x}_\beta^2 \leq C b_\alpha \bar{x}_n + b_\alpha O(|\bar{x}|^3) \leq C b_\alpha \bar{x}_n + C\delta^3 b_\alpha^2.$$ 

Since $|\xi_1| \leq C\epsilon_0 \delta^2 b_\alpha$, using the same argument as Case 1, (7.10) and (7.12) hold for $x = x^*$ if $\epsilon_0$ is small enough, namely

$$u(x^*) \geq u(x_0) - C\epsilon_0 \delta^2 b_\alpha^2, \quad u(x^*) \geq \frac{3}{8} \delta^2 b_\alpha^2.$$ 

Combining with (7.15), (7.10), we get

$$u(\bar{x}) \leq u(x^*) + C\epsilon_0 \delta^2 b_\alpha^2 + C b_\alpha \bar{x}_n + C\delta^3 b_\alpha^2$$

(7.16)

$$\leq u(x^*) + C(\epsilon_0 + \delta) \delta^2 b_\alpha^2 \leq \left(1 + \frac{c_0}{3}\right) u(x^*)$$

provided $\epsilon_0$ and $\delta$ are sufficiently small. Therefore, by (7.14) and the convexity of $u$, we have

$$u(x) \geq u(x^*) + \frac{1}{c_0} \min\{0, u(x^*) - u(\bar{x})\} \geq \frac{2}{3} u(x^*) \geq \frac{1}{4} \delta^2 b_\alpha^2$$

and (7.8) is proved. Note that

$$u(x) \leq C\sigma \delta^2 b_\alpha^2 - \frac{K^2}{2} b_\alpha x_n \text{ in } \omega.$$ 

Thus, (7.7) holds when $\sigma$ is sufficiently small and $K$ is sufficiently large. Then (7.7) is valid on $\partial \omega$. 
By (7.4) and (7.6), we can further fix $K$ sufficiently large such that
\[
\sigma_k(\lambda(D^2u)) \geq \sup_\omega f \quad \text{in } \omega.
\]
We have constructed a lower barrier $u$ vanishing at the origin with $u(0) = -K^2b_\alpha$.

Now we construct a lower barrier for an arbitrary point $x_0 = (x_0^1, \ldots, x_0^n) \in \partial \Omega \cap \partial \omega$, where
\[
\omega_\epsilon := \{ x \in \Omega : |x_\beta| < \epsilon \delta \frac{b_\alpha}{\sqrt{b_\beta}}, \beta = \alpha + 1, \ldots, n-1, x_n < \epsilon^2 \delta^2 b_\alpha \}
\]
and $\epsilon$ is a positive constant sufficiently small to be determined later. For $x \in \omega$, write $y = x - x_0$ and define
\[
u(x) = u(y) = \frac{\sigma}{2} \left( b_\alpha |\hat{y}|^2 + \sum_{\beta = \alpha+1}^{n-1} b_\beta y_\beta^2 \right) + \frac{1}{2} Ky_n^2 - K^2 b_\alpha y_n
\]
as above, where
\[
\bar{y}_n := y_n - \sum_{\beta = 1}^{n-1} \rho_\beta(x_0^\beta) y_\beta.
\]
Define $l(y')$ to be the linear function
\[
l(y') := u(x_0) + \sum_{\beta = 1}^{n-1} \nabla_\beta u(x_0) y_\beta.
\]
Now we prove $u \leq u - l(y')$ on $\partial \omega$ by choosing suitable positive constants $\sigma$, $K$ and $\epsilon$.

First we consider $x \in \partial_1 \omega$ as before. Let $\tilde{b_\beta} := \max\{b_\alpha, b_\beta\}$ for $1 \leq \beta \leq n-1$. Thus, by (6.7) and (7.3), we have
\[
\frac{\partial^2 \varphi}{\partial x_\beta^2}(x', \rho(x')) \geq b_\beta - C\delta \tilde{b_\beta}, \beta = 1, \ldots, n-1
\]
and
\[
\left| \frac{\partial^2 \varphi}{\partial x_\beta \partial x_\mu}(x', \rho(x')) \right| \leq C\delta \sqrt{b_\beta b_\mu}, \beta, \mu = 1, \ldots, n-1; \beta \neq \mu
\]
for any $(x', \rho(x')) \in \partial \omega$. Hence by Taylor’s expansion of $u(x', \rho(x')) = \varphi(x', \rho(x'))$ at $x_0$, we obtain
\[
u(x) \geq l(y') + \frac{1}{4} \sum_{\beta = 1}^{n-1} b_\beta y_\beta^2 - Cb_\alpha |\hat{y}|^2
\]
provided $\delta$ is sufficiently small. By Taylor’s expansion of $\rho$ at $x'_0$ and the uniform convexity of $\Omega$, we get
\[
\bar{y}_n = \rho(x) - \rho(x_0) \geq \sum_{\beta = 1}^{n-1} \rho_\beta(x_0^\beta) y_\beta + \kappa |y'|^2,
\]
where $\kappa$ is a positive constant depending only on the principal curvatures of $\partial \Omega$. It follows that
\[
\bar{y}_n \geq \kappa |y'|^2 \quad \text{on } \partial_1 \omega.
\]
Next, since $|\rho_\beta(x'_0)| \leq C|x'_0| \leq C\epsilon \sqrt{b_\alpha}$ for any $\beta = 1, \ldots, n-1$, we have

$$ y_n^2 = \left( \tilde{y}_n + \sum_{\beta=1}^{n-1} \rho_\beta(x'_0)y_\beta \right)^2 \leq 2y_n^2 + C\epsilon^2 \delta^2 b_\alpha |y'|^2. $$

(7.22)

Therefore, by (7.21), (7.22) and $|\tilde{y}_n| \leq Cb_\alpha$, we have

$$ u(x) \leq \frac{\sigma}{2} \sum_{\beta=\alpha+1}^{n-1} b_\beta y_\beta^2 - b_\alpha \left( \frac{1}{2}K^2 \tilde{y}_n - \frac{\alpha}{2} |\tilde{y}|^2 - C\epsilon \delta^2 |y'|^2 \right) $$

(7.23)

provided $\sigma$ is sufficiently small and $K$ is sufficiently large. Combing (7.20) and (7.23) we obtain $u(x) - l(y') \geq u(x)$ if $K$ is large enough.

For $x \in \partial \omega$, we note that $\tilde{y}_n \geq (1 - \epsilon^2 - C\epsilon) \delta^2 b_\alpha$ since $x_0 \in \partial \Omega \cap \partial \omega$. By (7.8) and (7.22) again, we find

$$ |l(y')| \leq C(\epsilon^2 + \epsilon) \delta^2 b_\alpha^2. $$

(7.24)

It follows that

$$ u(x) + l(y') \leq C\sigma \delta^2 b_\alpha^2 + CK \delta^2 b_\alpha^2 - K^2(1 - \epsilon^2 - C\epsilon) \delta^2 b_\alpha^2 + C(\epsilon^2 + \epsilon) \delta^2 b_\alpha^2 $$

$$ < 0 \leq u(x) $$

if $K$ is sufficiently large and $\epsilon$ is sufficiently small.

Now we consider $x \in \partial \omega$. We only need to consider the case $x \in \partial \omega$ as before. First we note that if $x_n \geq \epsilon_0 \delta^2 b_\alpha$, where $\epsilon_0$ is the constant defined by (7.8),

$$ \tilde{y}_n \geq (\epsilon_0 - \epsilon^2 - C\epsilon) \delta^2 b_\alpha. $$

Thus, we have $u(x) + l(y') < 0 < u(x)$ if $\epsilon \ll \epsilon_0$ and $K$ is large enough as in the case $x \in \partial \omega$.

Now we consider $x \in \partial \omega \cap \{x_n < \epsilon_0 \delta^2 b_\alpha\}$. By (7.8) and (7.24), we have

$$ u(x) - l(y') \geq \left( \frac{1}{4} - C(\epsilon^2 + \epsilon^2) \right) \delta^2 b_\alpha^2 \geq \frac{1}{8} \delta^2 b_\alpha^2 $$

provided $\epsilon$ is sufficiently small. By (7.22), we have

$$ u(x) \leq C(\sigma + \epsilon) \delta^2 b_\alpha^2 - \frac{K^2}{2} b_\alpha \tilde{y}_n. $$

(7.23)

It follows that $u(x) \leq u(x) - l(y')$ if $\sigma$ and $\epsilon$ are small enough and $K$ is large enough. We can further fix $K$ sufficiently large such that (7.18) holds.

To construct an upper barrier for $x_0 \in \partial \Omega \cap \partial \omega$, we define

$$ \bar{u}(x) = \bar{u}(y) := M \left( \frac{1}{2} b_\alpha |\tilde{y}|^2 + \frac{1}{2} \sum_{\beta=\alpha+1}^{n-1} b_\beta y_\beta^2 + b_\alpha \tilde{y}_n \right), $$

where $M$ is a sufficiently large constant to be determined later. By (7.23), we can fix $\epsilon$ sufficiently small and $M$ sufficiently large such that

$$ u - l(y') \leq \bar{u} \text{ on } \omega. $$

Furthermore, we find

$$ \det(D^2 \bar{u}) = 0 \text{ in } \omega. $$
By the convexity of $u$ and the maximal principle, we have

$$u - l(y') \leq u \text{ in } \omega.$$ 

Thus, we have

(7.25) \[ |u_{\nu}| \leqCb_{\alpha} \text{ on } \partial_{1}\omega_{\epsilon}. \]

Similar to (6.18), we find, by (4.1), (6.7), (7.25) and the convexity of $u$,

(7.26) \[ \sup_{\omega} |u_{\beta}| \leq \sup_{\partial_{2}\omega_{\epsilon}} |\varphi_{\beta} - u_{\alpha} \rho_{\beta}| \leq Cb_{\alpha}^{3/2}, \text{ for } 1 \leq \beta \leq \alpha, \]

(7.26) \[ \sup_{\omega} |u_{\beta}| \leq \sup_{\partial_{2}\omega_{\epsilon}} |\varphi_{\beta} - u_{\alpha} \rho_{\beta}| \leq C \sqrt{b_{\beta}b_{\alpha}}, \text{ for } \alpha + 1 \leq \beta \leq n - 1. \]

As (6.19) and (6.21), for any $x = (x', 2\delta^{2}b_{\alpha}) \in \partial_{2}\omega_{\epsilon}$, we have

$$-\epsilon^{2}\delta^{3}b_{\alpha}u_{n}(x) \leq u(0) - u(x) + \sum_{\beta=1}^{n-1} x_{\beta}u_{\beta}(x) \leq Cb_{\alpha}^{2}$$

and

$$\delta^{3}(1 - \epsilon^{2})b_{\alpha}u_{n}(x) \leq u(y) - u(x) \leq Cb_{\alpha}^{2}, \text{ with } y = (x', \delta^{2}b_{\alpha}) \in \partial_{2}\omega_{\epsilon}.$$

by (7.3) and (7.26). It follows that $|u_{n}| \leq Cb_{\alpha}$ on $\partial_{2}\omega_{\epsilon}$. Therefore, by the convexity of $u$ again, we can also get a bound $|u_{n}| \leq Cb_{\alpha}$ on $\partial_{3}\omega_{\epsilon}$. We then obtain

(7.27) \[ |D\varphi| \leq Cb_{\alpha} \text{ on } \omega_{\epsilon}. \]

To proceed we consider $\nabla_{\beta}(u - \varphi)$ for $1 \leq \beta \leq \alpha$. By (7.3), (4.1), (7.26) and (7.27), we have

(7.28) \[ |\nabla_{\beta}(u - \varphi)| \leq Cb_{\alpha}^{3/2} \text{ on } \omega_{\epsilon}. \]

By (7.3) again, we find

(7.29) \[ |F^{ij}(\nabla_{\beta}\varphi)_{ij}| \leq C \sqrt{b_{\alpha}} \left( \sum_{i=1}^{\alpha} F^{ii} + \sum_{i=\alpha+1}^{n-1} \frac{b_{i}}{b_{\alpha}} F^{ii} \right) \text{ in } \omega_{\epsilon}. \]

Let

$$m := b_{n-1} \cdots b_{\alpha+1} b_{\alpha}^{k+\alpha-n}.$$ 

We find

$$m^{-1/2(k-1)} \leq (b_{\alpha}^{k-1})^{-1/2(k-1)} = b_{\alpha}^{-1/2}.$$ 

By (6.25), (7.4) and (7.27), we have

(7.30) \[ |F^{ij}(\nabla_{\beta}u)_{ij}| \leq |\nabla_{\beta} f^{1/k}| + C \left( f^{1/k} + b_{\alpha} \sum_{i=1}^{n} F^{ii} \right) \]

\[ \leq Cm^{1/k} m^{-1/2(k-1)} + Cb_{\alpha} \sum_{i=1}^{n} F^{ii} \]

\[ \leq Cb_{\alpha}^{-1/2} m^{1/k} + Cb_{\alpha} \sum_{i=1}^{n} F^{ii}. \]
in \(\omega\). Now we consider an arbitrary point \(x_0 = (x_0^1, \ldots, x_0^n) \in \partial \Omega \cap \partial \omega\), with \(\epsilon_1 \ll \epsilon\) to be determined later. Let \(\underline{u}\) be the function defined in (7.19). By the concavity of \(F\), we have

\[
(7.31) \quad \frac{1}{\sqrt{b_\alpha}} F^{ij}(u - u)_{ij} \geq c_0 \left( \sqrt{b_\alpha} \sum_{i=1}^{n-1} F^{ii} + \sqrt{b_\alpha} \sum_{i=\alpha+1}^{n-1} b_i F^{ii} + F^{nn} + \frac{m^{1/k}}{\sqrt{b_\alpha}} \right)
\]

in \(\omega\) for some positive constant \(c_0\) which may depend on \(\sigma\) and \(K\). Let

\[
v = \frac{1}{\sqrt{b_\alpha}} \sum_{\beta=\alpha+1}^{n-1} b_\beta y_\beta + \sqrt{b_\alpha} \bar{y}_n.
\]

Thus, for any \(x \in \partial_2 \omega\), we have

\[
v(x) \geq \sqrt{b_\alpha} \bar{y}_n \geq \sqrt{b_\alpha} (\epsilon^2 - \epsilon_1^2 - C \epsilon_1) \delta^2 b_\alpha \geq \frac{\epsilon^2}{2} \delta^2 b_\alpha^{3/2}
\]

if we let \(\epsilon_1\) be small enough such that \(\epsilon_1^2 + C \epsilon_1 \leq \epsilon/2\). Next, for any \(x \in \partial_3 \omega\), which means \(|x_{\beta_0}| = \epsilon b_\alpha / \sqrt{b_{\beta_0}}\) for some \(\alpha + 1 \leq \beta_0 \leq n - 1\), we find

\[
v(x) \geq \frac{1}{\sqrt{b_\alpha}} \sum_{\beta=\alpha+1}^{n-1} b_\beta y_\beta^2 \geq \frac{1}{\sqrt{b_\alpha}} b_{\beta_0} y_{\beta_0}^2 \geq \frac{b_{\beta_0}}{\sqrt{b_\alpha}} \left( \frac{1}{2} x_{\beta_0}^2 - (x_0^0)^2 \right)
\]

\[
\geq \left( \frac{\epsilon}{2} - \epsilon_1^2 \right) \delta^2 b_\alpha^{3/2} \geq \frac{\epsilon}{2} \delta^2 b_\alpha^{3/2}
\]

provided \(\epsilon_1^2 \leq \epsilon/2\). Combining the above two inequalities and (7.28), we have

\[
|\nabla_\beta (u - \varphi)| \leq C v \text{ on } \partial \omega.
\]

Thus, by (7.29), (7.30) and (7.31), there exist positive constants \(A \gg B \gg 1\) such that

\[
F^{ij} \left( \frac{A}{\sqrt{b_\alpha}} (u - u) - B v \pm \nabla_\beta (u - \varphi) \right)_{ij} \geq 0 \text{ in } \omega
\]

\[
\frac{A}{\sqrt{b_\alpha}} (u - u) - B v \pm \nabla_\beta (u - \varphi) \leq 0 \text{ on } \partial \omega.
\]

By the maximal principle we have

\[
\frac{A}{\sqrt{b_\alpha}} (u - u) - B v \pm \nabla_\beta (u - \varphi) \leq 0 \text{ in } \omega
\]

and

\[
|v(\nabla_\beta u)(x_0)| \leq C \sqrt{b_\alpha}, \text{ for } 1 \leq \beta \leq \alpha.
\]

We thus have proved

\[
|\nabla_{n\tau_\beta} u| \leq C \sqrt{b_\alpha} \text{ on } \partial \omega_1 \cap \partial \Omega, \text{ for } 1 \leq \beta \leq \alpha.
\]

By induction, (6.1) is proved.

At the origin, we have

\[
(7.32) \quad \sigma_{k-1}(b) u_{nn}(0) - \sum_{\alpha \leq n-1} u_{\alpha\alpha}(0) \sigma_{k-2;\alpha}(b) + \sigma_k(b) = f(0).
\]
Note that, by (6.1),
\[
\sum_{\alpha \leq n-1} u_{n\alpha}^2 \sigma_{k-2\alpha}(b) \leq n - 1 \sigma_k^2(b) + \sum_{\alpha = n-k+1}^{n-1} u_{n\alpha}^2 b_{n-k+1} \cdots b_{n-1} b_{n-k+1} \cdots b_{n-1} + C \sum_{\alpha = n-k+1}^{n-1} u_{n\alpha} b_{n-k+1} \cdots b_{n-1} = C.
\]
(7.33)

Combining (5.1), (7.32) and (7.33), we obtain
\[
u_{nn}(0) \leq C.
\]

Acknowledgement. The authors wish to thank Professor Bo Guan for many helpful discussions.

References

[1] Z. Blocki, Regularity of the degenerate Monge-Ampère equation on compact Kähler manifolds, Math. Z. 244 (2003), 153-161.
[2] L. A. Caffarelli, L. Nirenberg and J. Spruck, Dirichlet problem for nonlinear second order elliptic equations III, Functions of the eigenvalues of the Hessian, Acta Math. 155 (1985), 261-301.
[3] L. A. Caffarelli, L. Nirenberg and J. Spruck, The Dirichlet problem for degenerate Monge-Ampère equations, Rev. Mat. Iberoamericana 2 (1986), 19-27.
[4] H. Dong, Hessian equations with elementary symmetric functions, Comm. Partial Diff. Eqns 31 (2006), 1005-1025.
[5] M. McGonagle, C. Song and Y. Yuan, Hessian estimates for convex solutions to quadratic Hessian equation, Ann. Inst. H. Poincaré Anal. Non Linéaire 36 (2019), no. 2, 451-454.
[6] B. Guan, The Dirichlet problem for Monge-Ampère equations in non-convex domains and spacelike hypersurfaces of constant Gauss curvature, Trans. Amer. Math. Soc. 350 (1998) 4955-4971.
[7] P. Guan, C² a priori estimate for degenerate Monge-Ampère equations, Duke Math. J. 86 (1997), 323-346.
[8] P. Guan and Y. Y. Li, The Weyl problem with nonnegative Gauss curvature, J. Differential Geom. 39 (1993), 331-342.
[9] P. Guan and Y. Y. Li, C¹,₁ estimates for solutions of a problem of Alexandrov, Comm. Pure Appl. Math. 50 (1997), 789-811.
[10] P. Guan, J. Li and Y.Y. Li, Hypersurfaces of prescribed curvature measure, Duke Math. J. 161 (2012), 1927-1942.
[11] P. Guan and G. Qiu, Interior C² regularity of convex solutions to prescribing scalar curvature equations, Duke Math. J. 168 (2019), no. 9, 1641-1663.
[12] P. Guan, C. Ren, Z. Wang, Global C² estimates for convex solutions of curvature equations, Comm. Pure Appl. Math. 68 (2015) 1287–1325.
[13] P. Guan, N. S. Trudinger and X.-J. Wang, On the Dirichlet problems for degenerate Monge-Ampère equations, Acta. Math. 182 (1999), 87-104.
[14] N. M. Ivochkina, N. S. Trudinger and X.-J. Wang, The Dirichlet problem for degenerate Hessian equations, Comm. Partial Diff. Eqs. 29 (2004), 219-235.
[15] H. Jiao, Z. Wang, The Dirichlet problem for degenerate curvature equations, J. Funct. Anal. 283 (2022), No. 109485.
[16] N. V. Krylov, Weak interior second order derivative estimates for degenerate nonlinear elliptic equations, Diff. Int. Eqsns. 7 (1994), 133-156.
[17] N. V. Krylov, Barriers for derivatives of solutions of nonlinear elliptic equations on a surface in Euclidean space, Comm. Partial Diff. Eqs. 19 (1994), 1909-1944.
[18] N. V. Krylov, A theorem on the degenerate elliptic Bellman equations in bounded domains, Diff. Int. Eqsns. 8 (1995), 961-980.
[19] N. V. Krylov, *On the general notion of fully nonlinear second-order elliptic equations*, Trans. Amer. Math. Soc. 347 (1995), 857-895.

[20] N. S. Trudinger, *On the Dirichlet problem for Hessian equations*, Acta Math. 175 (1995), 151-164.

[21] Q. Wang and C. Xu, *$C^{1,1}$ solution of the Dirichlet problem for degenerate $k$-Hessian equations*, Nonlinear Analysis 104 (2014), 133-146.

[22] X.-J. Wang, *Some counterexamples to the regularity of Monge-Ampère equations*, Proc. Amer. Math. Soc. 123 (1995), 841-845.