QUADRATIC TIME–VARYING SPECTRAL ESTIMATION
FOR UNDERSPREAD PROCESSES*

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Abstract: Time–varying spectral estimation is studied for nonstationary processes with restricted
time–frequency (TF) correlation. Explicit bias and variance expressions are given for quadratic
TF–invariant (QTFI) estimators of an expected real–valued QTFI representation based on a single
noisy observation. Unbiased theoretical estimators with globally minimal variance are derived and
approximately realized by a matched multi-window method.

1 INTRODUCTION

Time-frequency distribution are widely used to search for hidden structure in the signal. When
the signal consists of a small number of slowly varying sinusoids, the Wentzel-Kramer-Brillioun
representation reduces the signal to curves in the time-frequency plane [16]. We consider the case
of nonstationary stochastic processes with underlying time-frequency structure in the correlation
operator.

The evolutionary spectrum is one common representation of nonstationary processes. In
[15], we propose estimating the evolutionary spectrum by smoothing the log-spectrogram using a
data-adaptive kernel smoother in the time-frequency plane. The evolutionary spectrum has two
advantages: it is always positive and it converges to the spectrum as the ratio of the characteristic
time scale to the sampling rate becomes large. Its disadvantages are its lack of uniqueness and
its relatively poor time frequency resolutions.

We consider a different class of representations of nonstationary processes: quadratic Cohen’s
class spectra. These representations correspond to the expected value of Cohen’s class time-
frequency representations. An important member of this class is the Wigner-Ville spectrum.
This class of spectral representations possess useful operator properties and a reproducing kernel
Hilbert space structure.

In this article, we consider the estimation problem: how to estimate the Cohen’s class spectra.
This same problem has been considered by Sayeed and Jones as well. In [19], a complete knowledge
of the correlation operator is assumed. This assumption is appropriate for the signal classification
problem of recognizing one or more specific signals. Our approach assumes much weaker a priori
knowledge. We assume only that the signal is underspread which corresponds to being double
band limited in the ambiguity plane.

In Section 2, we review time-frequency representations of deterministic signals. Section 3
presents the analogous theory for time varying spectra. Section 4 defines and motivates unders-
spread processes.

Section 5 analyzes the bias and variance of a special class of quadratic estimators of Cohen’s
class spectra. Section 6 determines the minimum variance unbiased estimator of an underspread

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process. Section 7 describes a related estimation using multiple windows. Section 8 presents a biomedical example.

## 2 Quadratic Time-Frequency Distributions

Every real–valued quadratic time–frequency (TF) shift–invariant (QTFI) representation of a signal $x(t)$ can be represented as a quadratic form [1]. Comprehensive reviews of Cohen’s class are given in [2, 4]. We now cast Cohen’s class of time-frequency representations in an operator theoretic framework. Let $P$ be a self–adjoint Hilbert-Schmidt (H-S) “prototype” operator. We define the quadratic time-frequency shift invariant (QTFI) distribution

$$T_x(t, f) = \langle P^{(t,f)} x, x \rangle,$$

where $P^{(t,f)}$ is a TF shifted version of the self–adjoint prototype operator $P$. The choice of the kernel, $p(s, t)$, determines a particular representation in Cohen’s class. The TF–shifting of operators is defined as $P^{(t,f)} = S^{(t,f)}P S^{(t,f)}$, where $S^{(\tau, \nu)}$ is a unitary TF shift operator, acting as $(S^{(\tau, \nu)} x)(t) = x(t - \tau)e^{-j2\pi \nu t}$. The standard H-S inner product, $< R, P >$, is

$$< R, P > = \int \int \tau(t, s)p(s, t)dt ds,$$

where $r(t, s)$ and $p(s, t)$ are the respective kernels of the H.-S. operators $R$ and $P$. Throughout this article, we assume an infinite time domain and suppress replace $\int_\infty^{-\infty} dt$ with $\int dt$.

We now review the basic unitary TF representations of HS operators [6]. The generalized Weyl symbol is defined as

$$L_H^{(\alpha)}(t, f) \overset{\text{def}}{=} \int_{\tau} \int_{\nu} S_H^{(\alpha)}(\tau, \nu) e^{-j2\pi \nu t}d\tau d\nu,$$

where $|\alpha| \leq 1/2$. The Weyl correspondence is given by $\alpha = 0$, and the Kohn-Nirenberg correspondence (time–varying transfer function) by $\alpha = 1/2$ [20]. (When we suppress the superscript, this means validity for any $\alpha$.) The TF shifting of operators corresponds to a shift of the symbol, $L_{P^{(\tau, \nu)}}(t, f) = L_P(t - \tau, f - \nu)$, which shows that whenever $P$ is a TF localization operator that selects signals centered in the origin of the TF plane then $P^{(t,f)}$ localizes signal components centered around $(t, f)$. The generalized spreading function (GSF) of a linear operator [6] is

$$S_H^{(\alpha)}(\tau, \nu) \overset{\text{def}}{=} \int_{t} \int_{f} h(t + \left(\frac{1}{2} - \alpha\right)\tau, t - \left(\frac{1}{2} + \alpha\right)\tau) e^{-j2\pi \nu t}dt.$$

The GSF is the symplectic Fourier transform of the generalized Weyl symbol $L_H^{(\alpha)}(t, f)$:

$$S_H^{(\alpha)}(\tau, \nu) = \int_{t} \int_{f} L_H^{(\alpha)}(t, f)e^{-j2\pi (\nu t - \tau f)}dt df,$$

$$L_H^{(\alpha)}(t, f) = \int_{\tau} \int_{\nu} S_H^{(\alpha)}(\tau, \nu)e^{-j2\pi (-\nu t + \tau f)}d\tau d\nu.$$

When the Weyl symbol is smoothly varying in time and frequency, then the generalizing spreading function decays in $\tau$ and $\nu$. 2
3 TIME VARYING SPECTRUM

For a nonstationary process, a time–varying spectrum may be defined as the expectation of (1)

\[ P_x(t, f) \overset{\text{def}}{=} E \left\{ \langle P^{(t,f)}_x, x \rangle \right\}, \]  

(1)

A prominent example for \( P_x(t, f) \) is the Wigner–Ville spectrum [13]. Priestley’s evolutionary spectrum [14, 15] is a different, popular definition of a stochastic time–varying spectrum that cannot be brought into the form of (1).

We consider circular complex, zero–mean Gaussian processes with trace–class correlation kernel

\[ (R_x)(t, t') = r_x(t, t') = E \left\{ x(t)x^*(t') \right\}, \quad \text{tr} R_x < \infty. \]

The trace–class convention implies a HS inner product representation of \( P_x(t, f) \), alternatively written as the trace of the product operator:

\[ P_x(t, f) = (R_x, P^{(t,f)}_x) = \text{tr} \left\{ R_x P^{(t,f)}_x \right\}. \]  

(2)

The expected ambiguity function is defined as the GSF of the correlation operator [8]

\[ EA_x^{(\alpha)}(\tau, \nu) \overset{\text{def}}{=} S_{R_x}^{(\alpha)}(\tau, \nu) \]  

(3)

With the generalized Wigner–Ville spectrum, defined as

\[ EW_x^{(\alpha)}(t, f) \overset{\text{def}}{=} L_{R_x}^{(\alpha)}(t, f), \]

Eq. (2) carries over to a “nonstationary Wiener–Khintchine relation”:

\[ EW_x^{(\alpha)}(t, f) = \int_\tau \int_\nu EA_x^{(\alpha)}(\tau, \nu) e^{j2\pi(\nu t - \tau f)} d\tau d\nu, \]  

(4)

\[ EA_x^{(\alpha)}(\tau, \nu) = \int_t \int_f EW_x^{(\alpha)}(t, f) = e^{j2\pi(\tau f - \nu t)} dt df. \]  

(5)

These relationships are summarized in Table 1.

| \( L_{R_x}^{(\alpha)}(t, f) \) | \( \text{Weyl symbol of correlation} \) | \( \uparrow t \leftrightarrow \nu \) | \( \downarrow f \leftrightarrow \tau \) |
| \( S_{R_x}^{(\alpha)}(t, \nu) \) | \( \text{GSF of correlation} \) | \( \uparrow t \leftrightarrow \nu \) | \( \downarrow f \leftrightarrow \tau \) |
| \( EW_x^{(\alpha)}(t, f) \) | \( \text{Generalized W-V spectrum} \) | \( \uparrow t \leftrightarrow \nu \) | \( \downarrow f \leftrightarrow \tau \) |
| \( EA_x^{(\alpha)}(\tau, \nu) \) | \( \text{Expected ambiguity function} \) |

Table 1

As an example, the real–valued generalized Wigner–Ville spectrum can be written as

\[ \text{Re} \left\{ EW_x^{(\alpha)}(t, f) \right\} = E \left\{ \langle P^{(t,f)}_x(\alpha), x \rangle \right\}, \]

where the \( \alpha \)–dependent prototype operator is given by:

\[ S_{P(\alpha)}^{(0)}(\tau, \nu) = \cos(2\pi \nu \tau \alpha). \]  

(6)
Since both the Weyl symbol and the spreading function are unitary representations of HS operators we can rewrite the general time–varying spectrum,
\[ \langle R_x, P^{(t,f)} \rangle = \langle EW_x, L_{P(t,f)} \rangle = \langle EA_x, S_{P(t,f)} \rangle . \]

Note furthermore that the GSF of the TF shifted prototype operator is just a modulated version of the GSF of the original version:
\[ S_{P(t,f)}(\tau, \nu) = S_P(\tau, \nu)e^{j2\pi(\nu t - \tau f)} , \]
thus in particular \( |S_{P(t,f)}(\tau, \nu)| = |S_P(\tau, \nu)|. \)

4 UNDERSPREAD PROCESSES

The bias-variance analysis of Sec. 5 is valid for any circular Gaussian process with a trace class covariance. We now restrict our consideration to the case where the process’ expected ambiguity function \( EA_x^{(a)}(\tau, \nu) \) is zero outside a rectangle in the ambiguity plane. Our requirement that the expected ambiguity is double band-limited implies that the Weyl symbol is smooth in time and frequency.

We denote the maximum temporal correlation width \( \tau_{\text{max}} \) and the maximum spectral correlation width \( \nu_{\text{max}} \); i.e., we assume that the expected ambiguity function satisfies
\[ EA_x^{(a)}(\tau, \nu) = EA_x^{(a)}(\tau, \nu)\chi_x(\tau, \nu), \tag{1} \]
where \( \chi_x(\tau, \nu) \) is the 0/1–valued indicator function of a centered rectangle with area \( s_x = 4\tau_{\text{max}}\nu_{\text{max}}. \) According to the recently introduced terminology we call a process with \( s_x < 1 \) underspread and in the converse case overspread [7]. For asymptotics we assume that the underspread parameter is very small: \( s_x \ll 1. \) The underspread parameter, \( s_x, \) corresponds to the expansion parameter \( 1/(\tau \lambda f) \), which is used in the analysis of evolutionary spectra [15].

As to the relevance and realizability of the underspread processes we note that practically important linear time–varying (LTV) systems, as e.g. the mobile radio channel or underwater acoustic channel [21], are characterized by an (at least in good approximation) restricted spreading function (this is the field where the underspread/overspread terminology was originally introduced). Now, we apply stationary white noise \( n(t) \) with \( E\{n(t)n^*(t')\} = \delta(t - t') \) to an underspread LTV system \( H \) characterized by
\[ S_H^{(a)}(\tau, \nu) = S_H^{(a)}(\tau, \nu)\chi_H(\tau, \nu), \]
where \( \chi_H(\tau, \nu) \) covers a centered rectangle with halfwidths \( \tau_{\text{max},H} \) and \( \nu_{\text{max},H} \). Then the output process \( x(t) = (Hn)(t) \) is nonstationary with correlation
\[ R_x = HH^{\dagger}. \]

Applying the triangle inequality to the spreading function of the product operator [11] gives
\[ |EA_x(\tau, \nu)| \leq |S_H(\tau, \nu)| * |S_H(\tau, \nu)|, \]
where the ** denotes double convolution. The output process is thus underspread with \( \tau_{\text{max},x} = 2\tau_{\text{max},H} \) and \( \nu_{\text{max},x} = 2\nu_{\text{max},H}. \) Hence, we have shown that underspread processes are realizable and relevant.

In view of the “nonstationary Wiener–Khintchine relation” (4), the overspread/underspread classification may be interpreted as a smoothness condition for the time–varying spectrum of the process. Applying the sampling theorem on the symbol level leads to a discrete Weyl–Heisenberg expansion of the correlation operator [11]:
\[ R_x = \sum_l \sum_m EW_x^{(a)}(lT, mF)P^{(lT, mF)}(\alpha) \]
valid for a sampling grid with
\[ T \leq \frac{1}{2 \nu_{\text{max}}} \quad \text{and} \quad F \leq \frac{1}{2 \tau_{\text{max}}} \]
and the prototype operator defined by
\[ S_{P(a)}^{(\alpha)}(\tau, \nu) = \chi_x(\tau, \nu). \tag{2} \]

The critical spread \( s_x = 1 \) corresponds to the Nyquist sampling density \( TF = 1 \). Hence, considering bandlimited processes, for \( s_x = 1 \) the rate of innovation in the process second order statistics is equal to the sampling rate of the realization [8]. However, a robust estimation procedure maps a time series with \( N \) samples on a model with less than \( N \) coefficients such that the critical spread is a threshold for robust estimation of the generalized Wigner–Ville spectrum. It is furthermore remarkable that the evolutionary spectrum of an underspread process is 2D bandlimited in exactly the same manner as the generalized Wigner–Ville spectrum [11].

It should be noted that one can view the stationarity assumption underlying any time-invariant spectrum estimation as a limit case of (1) since the expected ambiguity function of a wide-sense stationary process is characterized by ideal concentration on the \( \tau \)-axis:
\[ E A_x(\tau, \nu) = r_x(\tau) \delta(\nu), \]
where \( r_x(\tau) \) is the autocorrelation function.

The Wiener–Khintchine relation requires strict band-limiting the ambiguity plane. The remainder of our analysis requires only a concentration in the ambiguity plane with characteristic spread, \( s_x \ll 1 \), but not complete band limitation.

5 REPRODUCING KERNEL HILBERT SPACE

We now show that time-frequency distributions are a reproducing kernel Hilbert spaces (RKHS) [5] using the Wigner-Ville kernel. A RKHS is Hilbert space \( \mathcal{H} \) of complex valued functions, defined on a set \( \mathcal{S} \), that has a reproducing kernel \( K(s, t) \) defined on \( \mathcal{S} \times \mathcal{S} \) with two properties: (i) for each \( t \), the function \( K(s, t) \) lies in \( \mathcal{H} \) and (ii) for each \( x \in \mathcal{H} \) and each \( t \in \mathcal{S} \) one has the reproducing property:
\[ x(t) = \langle x, K(., t) \rangle = \int K^*(t', t)x(t')dt'. \]

In our case, the Hilbert space, \( \mathcal{H} \), is the set of Weyl symbols of underspread operators \( \mathbf{H} \) satisfying a given spreading constraint (1). The reproducing kernel is given by the Weyl symbol of the prototype operator:
\[ K(t', f', t, f) = L_{P(t,f)}(t', f'). \]

This is in fact a reproducing kernel as (i) for each \( (t, f) P^{(t,f)} \) remains underspread since
\[ |S_{P^{(t,f)}}(\tau, \nu)| = |S_{P}(\tau, \nu)|, \]
and (ii) one has the reproducing formula as follows:
\[ L_{H}(t, f) = \langle L_{H}, L_{P^{(t,f)}} \rangle = \int \int L_{H}(t', f')L_{P}(t'-t, f'-f)dt' df' \tag{1} \]

WERNER: DO YOU MEAN 1 for the Wigner Ville kernel or for the kernel in2 or WHAT?
We now consider QTFI estimators of the time varying spectrum of the signal process $x(t)$ when it is contaminated with noise. We are given a single noisy observation, $y(t)$ of the signal process $x(t)$:

$$y(t) = x(t) + n(t) \text{ with } \mathbb{E}\{n(t)n^*(t')\} = \sigma_n^2 \delta(t-t'),$$

where $n(t)$ is statistically independent, zero–mean, circular complex Gaussian white noise. To estimate $P_x(t, f)$ we use a generally different QTFI representation of the observation:

$$\hat{P}_x(t, f) = \langle \hat{\mathbf{P}}^{(t,f)} y, y \rangle.$$

We define the “bias operator” $\tilde{\mathbf{P}}$ as $\tilde{\mathbf{P}} \defeq \hat{\mathbf{P}} - \mathbf{P}$. The QTFI estimator is consistent with classical, “non–parametric” time–invariant spectrum estimation where the predominant class of estimators [22, 18] can be basically written as a frequency parameterized quadratic form:

$$\hat{S}_x(f) = \langle \hat{\mathbf{P}}^{(0,f)} y, y \rangle. \quad (1)$$

7 BIAS AND VARIANCE ANALYSIS

With the statistical independence of signal and noise and using (2) we have the following expectation of the estimate:

$$\mathbb{E}\{\hat{P}_x(t, f)\} = \mathbb{E}\{\langle \hat{\mathbf{P}}^{(t,f)} x, x \rangle\} + \sigma_n^2 \text{tr} \mathbf{P},$$

such that the bias is given by

$$B(t, f) \defeq \mathbb{E}\{\hat{P}_x(t, f)\} - P_x(t, f) = \text{tr} \{\hat{\mathbf{P}}^{(t,f)} \mathbf{R}_x\} + \sigma_n^2 \text{tr} \hat{\mathbf{P}}.$$ 

Using the Schwarz inequality for operator inner products and triangle inequality, we immediately get a tight bound for the maximum bias:

$$|B(t, f)| \leq \|\hat{\mathbf{P}}\| \|\mathbf{R}_x\| + \sigma_n^2 |\text{tr} \hat{\mathbf{P}}|,$$

where the operator norm is the HS norm. We assume knowledge of the noise level $\sigma_n^2$ such that we can trivially correct the TF–independent bias term:

$$\hat{P}_x^*(t, f) = \hat{P}_x(t, f) - \sigma_n^2 \text{tr} \hat{\mathbf{P}},$$

where $\hat{P}_x^*(t, f)$ denotes the corrected estimate.

The variance,

$$V(t, f) \defeq \mathbb{E}\{\hat{P}_x^2(t, f)\} - \left(\mathbb{E}\{\hat{P}_x(t, f)\}\right)^2,$$

is evaluated using of Isserlis’ fourth order moment formula (for the special case of circular complex variables), $\mathbb{E}\{x(t_1)x^*(t_2)x(t_3)x^*(t_4)\} = r_x(t_1, t_2)r_x(t_3, t_4) + r_x(t_1, t_4)r_x(t_3, t_2)$, one has:

$$V(t,f) = \text{tr} \left\{ \left(\hat{\mathbf{P}}^{(t,f)} \mathbf{R}_x\right)^2 \right\} + 2\sigma_n^2 \text{tr} \left\{ \left(\hat{\mathbf{P}}^{(t,f)}\right)^2 \mathbf{R}_x \right\} + \sigma_n^4 \|\hat{\mathbf{P}}\|^2.$$

The Schwarz inequality for the operator inner product leads to a bound on the maximum variance, 

$$V_{\text{max}} \leq \|\hat{\mathbf{P}}\|^2 \left(\|\mathbf{R}_x\| + \sigma_n^2\right)^2,$$

proportional to the HS norm of the prototype operator $\hat{\mathbf{P}}$. 

6 QTFI ESTIMATION
Global Mean Square Error. The bias and variance results are complicated TF–dependent expressions. Due to our restriction to QTFI estimators we need TF–invariant, thus global indicators for the estimator performance. After correcting for the TF independent bias term, \( B_0 \) defined as:

\[
B_0 \overset{\text{def}}{=} \sigma_n^2 \text{tr} \hat{P},
\]

we characterize the global square bias as follows:

\[
B^2_{\text{tot}} \overset{\text{def}}{=} \int \int (B(t,f) - \sigma_n^2 \text{tr} \hat{P})^2 dt df = \langle |S_P|^2, |EA_x|^2 \rangle.
\]

(4)

Just as for the bias we give a global characterization of the variance. The TF independent term is given by:

\[
V_0 = \sigma_n^4 \| \hat{P} \|^2.
\]

We define a total variance as the integral over the TF dependent variance terms, one has:

\[
V_{\text{tot}} \overset{\text{def}}{=} \int \int (V(t,f) - V_0) dt df = \| \hat{P} \|^2 \left( \text{tr} R_x^2 + 2 \sigma_n^2 \text{tr} R_x \right).
\]

(5)

Equations (4) and (5) are derived in the appendix.

Observe that any of the global variance constants; i.e., the maximum variance \( V_{\text{max}} \), the TF–independent variance term \( V_0 \), and the total variance \( V_{\text{tot}} \) are proportional to the HS norm of the prototype operator:

\[
V_0, V_{\text{max}}, V_{\text{tot}} \propto \| \hat{P} \|^2.
\]

(6)

8 ESTIMATOR OPTIMIZATION

Classical spectrum estimation produces smooth spectra since — due to the absence of a model — smoothing is the actual tool for variance reduction. The proposed estimators usually are the result of mean–squared error considerations. In the present work, we deviate from this point of view in a pragmatic way: we restrict ourselves to underspread processes whose true spectra are itself smooth (in the sense of 2D bandlimitation) such that there exist a whole class of unbiased estimators. While such a modelling ingredient may be questionable for time–invariant spectrum analysis we feel that it is necessary for time–varying spectral estimation. The reason lies in the often overlooked point that frequency parametrization is matched to any stationary process (the Fourier transform diagonalizes the correlation operator) while TF parametrization is not matched to a general nonstationary process. From the point of view of operator diagonalization it is the class of underspread processes where TF–parametrization is appropriate [11].

Unbiased estimation without further assumption on the signal process \( x(t) \) requires a vanishing “bias operator”, i.e., \( P = \hat{P} \). In the case of the generalized Wigner–Ville spectrum, the prototype operator (cf. (6)) is not HS since

\[
\| \hat{P} \|^2 = \int \int |S_P(\tau,\nu)|^2 d\tau d\nu,
\]

so that one can exclude finite–variance unbiased estimation of the generalized Wigner–Ville spectrum without a priori knowledge on the process. This is well–known [13].

Based upon the known support of \( EA_x(\tau,\nu) \) one has a large class of nontrivial unbiased estimators (with nonvanishing “bias operator”,)

\[
S_{P_{UB}}^{(\alpha)}(\tau,\nu) = \begin{cases} 
S_P^{(\alpha)}(\tau,\nu), & \text{where } EA_x(\tau,\nu) \neq 0 \\
\text{arbitrary}, & \text{where } EA_x(\tau,\nu) = 0 
\end{cases}.
\]

We interpret minimum variance in the sense of the combined consideration of the global variance constants \( V_0, V_{\text{max}}, V_{\text{tot}} \). Due to (6) one has to select the unbiased estimator with minimum HS
norm prototype operator. Using (1) this turns out to be trivial: the minimum–variance unbiased (MVUB) QTFI estimator is obtained by setting the spreading function of the prototype operator zero wherever possible:

\[
S_{\hat{P}_{MVUB}}^{(\alpha)}(\tau, \nu) = \begin{cases} 
S_P^{(\alpha)}(\tau, \nu), & \text{where } EA_x(\tau, \nu) \neq 0 \\
0, & \text{where } EA_x(\tau, \nu) = 0 
\end{cases}
\]

When \(\chi_x(\tau, \nu)\) is the smallest indicator function containing the support of \(EA_x(\tau, \nu)\), then the MVUB QTFI estimator can be written as:

\[
S_{\hat{P}_{MVUB}}^{(\alpha)}(\tau, \nu) = S_P^{(\alpha)}(\tau, \nu)\chi_x(\tau, \nu).
\]

In particular, for the \(\alpha\)–parametrized real–valued generalized Wigner–Ville spectrum one has:

\[
S_{\hat{P}_{MVUB}}^{(\alpha)}(\tau, \nu) = \cos(2\pi \tau \nu)\chi_x(\tau, \nu).
\]

This estimator is optimal among all QTFI estimators thus in the sense of global variance minimization. The estimate is locally stable since it minimizes a bound on the maximum variance \((V_{max})\) and it is unbiased for arbitrary time and frequency, but it deviates from the local TF–dependent MVUB estimate.

Mean–Squared Error. The theoretical MVUB estimator serves well as a starting point for obtaining practical estimators with good mean–squared error performance. The mean squared error is given by \(E(t, f) = V(t, f) + B^2(t, f)\). For any process that satisfies the spreading constraint (1) one can formally redefine the estimation target via the prototype operator of any unbiased estimator:

\[
P_x(t, f) = \text{tr}\left\{R_x \hat{P}(t, f)\right\} = \text{tr}\left\{R_x \hat{P}_{UB}(t, f)\right\},
\]

so that one can obtain a useful bound on the integrated mean–squared error

\[
E_{tot} < \|\hat{P}\|\|R_x\|^2 + \|\hat{P}\|^2\left(\|R_x\|^2 + 2\sigma_n^2\text{tr}R_x\right), \tag{2}
\]

with \(\hat{P} = \hat{P} - \hat{P}_{MVUB}\). This bound is based on (4), (5) and \(<|S_P^2|, |EA_x^2| < \|\hat{P}\|^2\text{tr}R_x\).

9 MATCHED MULTI-WINDOW ESTIMATOR

The eigenfunction decomposition of the prototype operator \(\hat{P}\) shows that \(\hat{P}^{(t,f)}\) is a weighted sum of rank one projections. Equivalently, any QTFI representation can be written as a weighted sum of spectrograms with orthonormal windows [20]. For practicality, we require our estimator to be based on a finite–rank prototype operator with finite–length eigenfunctions. The MVUB estimator of Sec. 6 does not satisfy these requirements. Thus, we choose the finite–rank, time–limited estimator which minimizes the upper bound on the integrated mean–squared error as given by (2). When we impose the additional requirement that the prototype operator be projection type with normalized trace, \(\hat{P}\) has the representation:

\[
\hat{P}_N = \frac{1}{N} \sum_{k=1}^{N} \gamma_k \otimes \gamma_k
\]

where \(\gamma_k \otimes \gamma_k\) denotes the rank–one projection on the orthonormal window functions \(\gamma_k(t)\) and \(N\) is the rank. In this case, \(\|\hat{P}_N\|^2 = 1/N\), and the optimization of (2) reduces to minimizing
\[ \| \hat{P}_{MVUB} - \hat{P}_N \|^2 \] subject to orthonormality constraints on the \( \gamma_k \). We define the matched multi-window estimator as the quadratic form based on a prototype operator that minimizes \( E_{tot} \) subject to (1). The optimization is performed in a two-step procedure: we optimize the windows subject to a fixed rank and then we optimize the rank. For practicality, we impose that the \( \gamma_k \) have support on \([-T/2, T/2] \). To impose this time localization on the optimization of \( \hat{P}_N \), we define \( T \) as the projection onto the centered interval and require \( \hat{P}_N = T \hat{P}_N T \). Minimizing (2) yields the optimal windows equation:

\[ T \hat{P}_{MVUB} T \gamma_{k, opt} = \lambda_k \gamma_{k, opt} \tag{2} \]

The optimum window set is independent of \( N \). For the specific case where \( \hat{P}_{MVUB} \) is an ideal bandpass (which may be considered as a theoretically optimal estimator for stationary processes) (2) yields the time-limited and optimally bandlimited prolate spheroidal wave functions consistent with [22, 18].

A more realistic and simpler family of tapers are the discrete sinusoidal tapers, \( \{ v^{(k)} \} \), where

\[ v^{(k)}_n = \sqrt{\frac{2}{N+1}} \sin \frac{\pi k n}{N+1}, \quad \text{and} \quad N \text{ is the number of points } [17]. \]

The resulting sinusoidal multi-taper spectral estimate is

\[ \hat{S}(t, f) = \frac{1}{K(N+1)} \sum_{j=1}^{K} (\gamma(t, f + j/2N+2) - \gamma(t, f - j/2N+2))^2, \]

where \( \gamma(t, f) \) is the local Fourier transform centered at time \( t \) with length \( N \). \( S(t, f) \) is the instantaneous spectral density, and \( K \) is the number of tapers. The sinusoidal tapers are asymptotically optimal when the bias error is local.

10 FREE PARAMETER OPTIMIZATION

For a strongly underspread process \( s_x \ll 1 \), \( S_P(\tau, \nu) \) is approximately constant in the support of \( EA_x(\tau, \nu) \). Using the optimal window functions of (2), we approximate \( \hat{P}_{MVUB} \) with \( 1/s_x \) such rank one projections. In this case, \( \| \hat{P}_{MVUB} - \hat{P}_N \|^2 \) reduces to \((s_x - 1/N)\). Optimizing (2) with respect to \( N \) for moderate noise level yields

\[ N_{opt} \approx \frac{1}{s_x}, \quad \text{for} \quad \frac{\sigma_n^2}{\text{tr} R_x} < \frac{1 - s_x}{2}. \]

WERNER: YOUR ESTIMATE of \( s_x \) and \( T \)...

11 CONCLUSIONS

We have studied time-varying spectral estimation via quadratic TF-invariant estimators. For circular complex Gaussian signal and noise processes we have presented explicit (local and global) bias and variance results. For the specific case of an underspread process the design of matched multi-window estimators has been based on approximating a theoretical MVUB estimator.

The theoretical MVUB estimator as derived in Section 6 is a specific case of the recently proposed optimum kernel design for Wigner–Ville spectrum estimation [19]. We emphasize that [19] requires a complete knowledge of a second order statistic what makes this approach purely theoretical while our proposed estimator uses a more realistic, incomplete a priori knowledge of the process statistics.

For Cohen’s class time varying spectra. Using the reproducing kernel Hilbert space formalism, we derive expressions for the leading order bias and variance. Underspread processes are band limited in the ambiguity plane and smooth in the time frequency domain. For underspread processes, we give unbiased minimum variance estimators.

APPENDIX: PROOFS
We now derive (4) which equates the integral square bias with the inner product of the squared GSF of the “bias operator” and the process’ expected ambiguity function:

\[
\begin{align*}
\int_t \int_f \text{tr}^2 \left\{ \mathbf{P}^{(t,f)} \mathbf{R}_x \right\} dt df &= \int_t \int_f \left| \langle S_{\mathbf{P}^{(t,f)}} (t,f), EA_x \rangle \right|^2 dt df = \\
\int_t \int_f \int_{\tau_1 \nu_1} \int_{\tau_2 \nu_2} S_{\mathbf{P}}(\tau_1, \nu_1) \cdot EA_x(\tau_1, \nu_1) \cdot S_{\mathbf{P}}^*(\tau_2, \nu_2) \cdot EA_x^*(\tau_2, \nu_2) \\
& \quad \cdot e^{-j2\pi [(\nu_1-\nu_2)\tau_1-(\tau_1-\tau_2)f]} dt df d\tau_1 d\nu_1 d\tau_2 d\nu_2 = \\
& = \int_\tau \int_\nu \left| S_{\mathbf{P}}(\tau, \nu) \right|^2 |EA_x(\tau, \nu)|^2 d\tau d\nu = \langle \left| S_{\mathbf{P}} \right|^2, |EA_x|^2 \rangle.
\end{align*}
\]

Derivation of (5):

\[
\mathbf{P}^{(t,f)} \mathbf{P}^{(t,f)} = \mathbf{S}^{(t,f)} \mathbf{P}^2 \mathbf{S}^{(t,f)} = \left( \mathbf{P}^2 \right)^{(t,f)},
\]

together with

\[
\int_t \int_f \mathbf{P}^{(t,f)} dt df = \text{tr} \{ \mathbf{P} \} \mathbf{I}
\]

(which follows directly from the trace formula of the Weyl correspondence [6]); as well as

\[
\left( \mathbf{P}^{(t,f)} \mathbf{R} \right) (s, s') = \int_{s''} p(s-t, s''-t) e^{j2\pi f(s-s'')} r(s'', s') ds'' ,
\]

whence

\[
\begin{align*}
\int_t \int_f \text{tr} \left\{ \left( \mathbf{P}^{(t,f)} \mathbf{R} \right)^2 \right\} dt df &= \int_t \int_f \int_{s_1, s_2} p(s-t, s_1-t) r(s_1, s') \\
& \quad \cdot p^*(s-t, s_2-t) r^*(s_2, s') e^{j2\pi f(s_1-s_2)} dt df ds ds' ds_1 ds_2 \\
& = \int_t \int_f \int_{s_1, s_2} |p(s-t, s_1-t)|^2 |r(s_1, s')|^2 dt df ds ds' ds_1 \\
& = \| \mathbf{P} \|^2 \| \mathbf{R} \|^2.
\end{align*}
\]

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