Proof of Jacobi identity in generalized quantum dynamics

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**ABSTRACT**

We prove that the Jacobi identity for the generalized Poisson bracket is satisfied in the generalization of Heisenberg picture quantum mechanics recently proposed by one of us (SLA). The identity holds for any combination of fermionic and bosonic fields, and requires no assumptions about their mutual commutativity.
1. Introduction

In a recent paper, one of us (SLA) proposed a new quantum dynamics by generalizing the fundamental field equations from c-number to unitary or bi-unitary operator gauge invariance [1]. The approach was motivated by an investigation of quaternionic quantum mechanics and field theory [2], and more generally, gives a symplectic dynamics for general non-commutative degrees of freedom, provided only that multiplication is associative and there exists a trace permitting cyclic permutation of the non-commuting variables.

An important ingredient missing from the formalism of [1] was the proof of the Jacobi identity for the generalized Poisson bracket, of arbitrary polynomial trace functionals of fermionic and bosonic quantum variables. The Jacobi identity is necessary for the correct incorporation of symmetries. Specifically, if $A$ and $B$ are conserved symmetry generators, then the Jacobi identity implies that their generalized Poisson bracket is also a conserved symmetry generator. More generally, because of the Jacobi identity, we expect total trace symmetry generators, such as the Poincaré generators, to obey a Lie algebra under the generalized Poisson bracket operation, which is isomorphic to the Lie algebra under commutation obeyed by the corresponding abstract generators of symmetries of the total trace Lagrangian.

In this paper we prove that the Jacobi identity is indeed satisfied for the formalism of [1]. To keep the paper self-contained, we define below the ingredients of the formalism necessary for our proof. The interested reader should consult [1] and [2] for further details.

2. The proof

One starts by defining a Hilbert space $V_H$ (based either on complex number or quaternionic scalars) which is the direct sum of a bosonic space $V_H^+$ and a fermionic space $V_H^-$. Next, following Witten [3], one defines an operator $(-1)^F$ with eigenvalue $+1$ for states in $V_H^+$ and $-1$ for states in $V_H^-$. Finally, one needs a trace operation $\text{Tr} \, O$ for a general operator $O$, defined by

$$\text{Tr} \, O = \text{Re} \, Tr \, (-1)^F O = \text{Re} \sum_n \langle n | (-1)^F O | n \rangle. \quad (1)$$

It is easy to show that the trace $\text{Tr}$ is non-vanishing only for operators $O$ which commute with $(-1)^F$.

Let $\{q_\nu(t)\}$ be a set of time-dependent quantum variables, which act as operators on the underlying
Hilbert space, with each individual $q_r$ of either bosonic or fermionic type, defined respectively as commuting, or anti–commuting with $(-1)^F$. No other \textit{a priori} assumptions about commutativity of the $q_r$ are made. The Lagrangian $L[q_r, \dot{q}_r]$ is then defined as the trace of a polynomial function of $\{q_r(t)\}$ and its time derivative $\{\dot{q}_r(t)\}$, or as a suitable limit of such functions. The action $S$ is defined as the time integral of $L$, and generalizations of the Euler–Lagrange equations follow from the requirement that $\delta S = 0$ for arbitrary (same–type) variations of the operators. Derivatives of $L$ with respect to $q_r$ and $\dot{q}_r$ are defined by writing the variation of $L$, for infinitesimal variations in the $\{q_r\}$, in the form,

$$\delta L = \text{Tr} \sum_r \left( \frac{\delta L}{\delta q_r} \delta q_r + \frac{\delta L}{\delta \dot{q}_r} \delta \dot{q}_r \right),$$

(2)

where cyclic permutations of operators inside $\text{Tr}$ have been used to order $\delta q_r$ and $\delta \dot{q}_r$ to the right. The momentum $p_r$ conjugate to $q_r$ is defined by

$$\frac{\delta L}{\delta \dot{q}_r} = p_r,$$

(3)

and the Hamiltonian $H$ is given by

$$H = \text{Tr} \sum_r p_r \dot{q}_r - L.$$  

(4)

The generalized Poisson bracket of two trace functionals $A[q_r, p_r]$ and $B[q_r, p_r]$ is then defined as

$$\{A, B\} \equiv \text{Tr} \sum_r \epsilon_r \left( \frac{\delta A}{\delta q_r} \frac{\delta B}{\delta p_r} - \frac{\delta B}{\delta q_r} \frac{\delta A}{\delta p_r} \right),$$

(5)

with $\epsilon_r = +1$ if $q_r$ is bosonic, and $\epsilon_r = -1$ if $q_r$ is fermionic.

The purpose of this paper is to prove that for the formalism defined above, the Jacobi identity is satisfied. That is, if $A[q_r, p_r]$, $B[q_r, p_r]$, and $C[q_r, p_r]$ are arbitrary polynomial trace functionals of the operator arguments $\{q_r\}$, $\{p_r\}$, then,

$$[A, B, C] \equiv \{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0.$$  

(6)

We first attempted to check or find a counter–example to Eq. (6) by generating many computer examples, and using a computer algorithm based on the manipulation of strings of integer labels to test the Jacobi
identity. After many successful verifications on the computer, we found the analytic proof which we present below.

For ease of exposition, we will use a more compact notation. Derivatives with respect to \( q_r \) or \( p_r \) of a trace functional \( A \) will be denoted by \( A_r \) and \( A^r \) respectively. The operation \( \text{Tr} \) will be implied by the parentheses ( ) and, for the time being, it will be assumed that repeated indices are summed.

In this notation, the Poisson bracket is given by,

\[
\{A, B\} = \epsilon_r (A_r B^r - B_r A^r) .
\]

(7)

It is useful to illustrate with an example how derivatives are computed. Consider the case where we have two kinds of field variables \( q_1, p_1 \) and \( q_2, p_2 \). Given the trace functional \( A = (q_1 p_1 q_2 p_2 q_1) \), its derivative with respect to \( q_1 \) is denoted by \( A_1 \) and is given by

\[
A_1 = q_1 p_1 q_2 p_2 + \epsilon_1 \epsilon_2 p_2 q_1 p_1 q_2 + \epsilon_1 p_1 q_2 q_1 p_2 q_1 .
\]

(8)

The three terms result from the three possible \( q_1 \) factors to differentiate, and the \( \epsilon \)'s come from cyclically permuting the factors to bring the particular \( q_1 \) which is to be differentiated to the right.

The first term on the right hand side of Eq. (6), expanded out in this notation, is

\[
\{A, \{B, C\}\} = \{A, \epsilon_r (B_r C^r - C_r B^r)\} ,
\]

(9a)

which can be expanded further to

\[
\{A, \{B, C\}\} = \epsilon_r \epsilon_s \left( A_s (B_r C^r)^s - A_s (C_r B^r)^s - (B_r C^r)_s A^s + (C_r B^r)_s A^s \right) .
\]

(9b)

Cyclic permutations of \( A, B, \) and \( C \) give the other two terms in Eq. (6). Thus, the left–hand side of Eq. (6) is

\[
[A, B, C] = \epsilon_r \epsilon_s \left[ (A_s (B_r C^r)^s - A_s (C_r B^r)^s - (B_r C^r)_s A^s + (C_r B^r)_s A^s) + (B_s (C_r A^r)^s - B_s (A_r C^r)^s - (C_r A^r)_s B^s + (A_r C^r)_s B^s) + (C_s (A_r B^r)^s - C_s (B_r A^r)^s - (A_r B^r)_s C^s + (B_r A^r)_s C^s) \right] .
\]

(10)
Let us first consider how the terms in Eq. (10) cancel in the classical, c-number case. A similar cancellation mechanism will also apply in the more general quantum operator case. For c-numbers, the trace operation is trivial, derivatives of functionals commute, and one can apply the product rule to expand the terms. For instance,

\[(B_r C^r)^s = B_r^s C^r + B_r C^r s.\] (11)

Note that \(B_r^s\) means that the \(q_r\) derivative is applied before the \(p_s\) derivative. \(B_r^s\) would mean that the same derivatives are applied in the opposite order. This distinction is meaningless for c-number fields, where derivatives commute, but it is crucial for non-commutative operators \(\{q_r\}\) and \(\{p_r\}\).

Equation (11) implies that each term in Eq. (10) will generate two terms. These terms cancel in pairs. For example, in the first term in Eq. (10), consider the derivative with respect to \(p_s\) applied to \(B_r\). This generates the term \(+ A_s B_r^s C^r\). This cancels against the term \(- A_r B_r^s C^r\) obtained by applying the derivative with respect to \(p_s\) on \(B_r\) in the eleventh term (the dummy indices \(r\) and \(s\) need to be interchanged for the terms to be the same). The other half of the eleventh term will in turn be cancelled by a part of the eighth term, and so on. After twelve such double terms have been computed, we come back to the beginning and all terms have been cancelled.

The order in which these cancellations occur is as follows,

\[
\begin{align*}
&\leftrightarrow (A_s (B_r C^r)^s) \leftrightarrow ((A_s B^r)_r C^r) \leftrightarrow ((A_r C^r)_s B^r) \leftrightarrow (A_r (C_s B^r)^r) \leftrightarrow (C_s (A_r B^r)^s) \leftrightarrow ((C_s A^r)_r B^r) \leftrightarrow ((C_r B^r)_s A^r) \leftrightarrow (C_r (B_s A^r)^r) \leftrightarrow (B_s (C_r A^r)^s) \leftrightarrow ((B_s C^r)_r A^r) \leftrightarrow ((B_r A^r)_s C^r) \leftrightarrow (B_r (A_s C^r)^r) \leftrightarrow
\end{align*}
\]

where we have used the fact that \(r\) and \(s\) are dummy indices and have interchanged them in some of the terms, and where the lower right of Eq. (12) links back to the upper left. By Eq. (11), each entry in Eq. (12) generates two terms; one of these cancels against a term from the entry to the immediate left in the chain, and the other cancels against a term from the entry to the immediate right.

We will now proceed to show that in the general case, the cancellations occur in a similar way. However, the absence of both commutativity and the product rule makes the proof a little less trivial. For the rest of this discussion, we will also not assume the summation convention, which means that repeated indices
are not summed henceforth, so that we are dealing with summands which appear, summed over $r$ and $s$, in the Jacobi identity. Also, we will assume that $A, B, C$ are monomials in $\{q_r\}$ and $\{p_r\}$. The proof for the general case of polynomial functionals follows from expanding out the generalized Poisson brackets in Eq. (6) in terms of monomials. Finally, recall that there is an implied trace $\text{Tr}$ for each pair of parentheses $(\ )$ in Eq. (10). This means that we can cyclically permute the factors within a parenthesis, if we include a factor $\epsilon_r$ every time a $q_r$ or $p_r$ is moved from the front of a trace to the back. Thus, in our shorthand notation, $(q_r\mathcal{O}) = \epsilon_r(\mathcal{O}q_r)$.

When one computes the derivative of some monomial with respect to $q_r$ (say), each particular occurrence of $q_r$ generates one term in the result. Consider the expression,

$$(B, C^r)^s,$$  \hspace{1cm} (13)

which appears in the first term of Eq. (12). In this expression, there are three derivatives, and a choice is made of which occurrence of $q_r$, $p_r$, and $p_s$ to differentiate in the appropriate terms. Each set of choices will produce a particular monomial term in the result. If $q_r$ appears $N(B, q_r)$ times in the monomial $B$, and $p_r$ appears $N(C, p_r)$ times in $C$, and so on, then the number of terms produced by Eq. (13) is at most $N(B, q_r)N(C, p_r)(N(B, p_s) + N(C, p_s))$.

We will show that in Eq. (10), each such monomial term in the result, for fixed $r$, $s$ (i.e., for a fixed choice of $q_r, p_r, q_s, p_s$), will cancel with its counterpart in the order defined by Eq. (12). Consider the case where the $p_s$ derivative is applied to $B$ in the first entry and the $q_r$ derivative is applied to $B$ in the second entry of Eq. (12). For these to give non-vanishing contributions, $B$ must contain at least one instance of both $q_r$ and $p_s$. Therefore the most general form for $B$ is

$$B = (\alpha q_r, \beta p_s),$$  \hspace{1cm} (14)

where $\alpha$ and $\beta$ are arbitrary monomials (and could possibly contain $q_r$ and $p_s$). The displayed $q_r$ and $p_s$ are the particular instances of these coordinates in $B$ upon which the derivatives will act.

We have

$$(A_s((B, C^r)^s) = (A_s((\alpha q_r, \beta p_s), C^r)^s))$$
\begin{align*}
= \epsilon_\alpha \epsilon_r (A_s (\beta p_s \alpha C^r)^s) \\
= \epsilon_\alpha \epsilon_r \epsilon_\beta \epsilon_s (A_s \alpha C^r \beta), \quad (15)
\end{align*}

and
\begin{align*}
((A_s B^s)_r C^r) &= ((A_s (\alpha q_r \beta p_s)^s)_r C^r) \\
&= ((A_s \alpha q_r \beta)_r C^r) \\
&= \epsilon_\beta (\beta A_s \alpha C^r) \\
&= (A_s \alpha C^r \beta). \quad (16)
\end{align*}

If \( B \) is not identically zero, making the equality of Eqs. (15) and (16) trivial, it must have an even number of fermion factors. Therefore, \( \epsilon_\alpha \epsilon_r \epsilon_\beta \epsilon_s = 1 \), and so the right hand sides of Eqs. (15) and (16) are always the same. Finally, these same cancellations can be shown to occur for every summand term in Eq. (10) in the order indicated by Eq. (12), and apply both to the summands with \( r \neq s \) and to those with \( r = s \), including the parts of the summands with \( r = s \) in which there are two derivatives with respect to the same variable \( q_r \) (or \( p_r \)). This proves that the Jacobi identity is true for arbitrary bosonic and fermionic quantum field operator variables \( \{q_r\} \) and \( \{p_r\} \).

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