LINEAR ELLIPTIC SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS WITHOUT LANDESMAN-LAZER CONDITIONS

ALZAKI FADLALLAH

Abstract. The boundary value problem is examined for the system of elliptic equations of from \(-\Delta u + A(x)u = 0\) in \(\Omega\), where \(A(x)\) is positive semi-definite matrix on \(\mathbb{R}^{k \times k}\), and \(\frac{\partial u}{\partial \nu} + g(u) = h(x)\) on \(\partial \Omega\). It is assumed that \(g \in C(\mathbb{R}^k, \mathbb{R}^k)\) is a bounded function which may vanish at infinity. The proofs are based on Leray-Schauder degree methods.

1. Introduction

Let \(\mathbb{R}^k\) be real \(k\)-dimensional space, if \(w \in \mathbb{R}^k\), then \(|w|_E\) denotes the Euclidean norm of \(w\). Let \(\Omega \subset \mathbb{R}^N\), \(N \geq 2\) is a bounded domain with boundary \(\partial \Omega\) of class \(C^\infty\). Let \(g \in C^1(\mathbb{R}^k, \mathbb{R}^k), h \in C(\partial \Omega, \mathbb{R}^k)\), and the matrix

\[
A(x) = \begin{bmatrix}
a_1(x) & 0 & \cdots & 0 \\
0 & a_2(x) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_k(x)
\end{bmatrix}
\]

Verifies the following conditions:

(A1) The functions \(a_i : \Omega \to \mathbb{R}, a_i(x) \geq 0, \forall i = 1, \cdots, k \ x \in \Omega\) with strict inequality on a set of positive measure.

(A2) \(A(x)\) is positive semidefinite matrix on \(\mathbb{R}^{k \times k}\), almost everywhere \(x \in \Omega\), and \(A(x)\) is positive definite on a set of positive measure with \(a_{ij} \in L^p(\Omega) \ \forall \ i = 1, \cdots, k\) for \(p > \frac{N}{2}\) when \(N \geq 3\), and \(p > 1\) when \(N = 2\)

We will study the solvability of

\[
-\Delta u + A(x)u = 0 \quad \text{in } \Omega,
\]

\[
\frac{\partial u}{\partial \nu} + g(u) = h(x) \quad \text{on } \partial \Omega.
\]

The interest in this problem is the resonance case at the boundary with a bounded nonlinearity, we will assume that \(g\) a bounded function, and there is a constant \(R > 0\) such that

\[
|g(w(x))|_E \leq R \quad \forall \ w \in \mathbb{R}^k \ & x \in \partial \Omega
\]

Our assumptions allow that \(g\) is not only bounded, but also may be vanish at infinity i.e.,

\[
\lim_{|w|_E \to \infty} g(w) = 0 \in \mathbb{R}^k
\]

Condition \(1.3\) is not required by our assumptions, but allowing for it is the main result of this paper.
In case of the scalar equation $k = 1$ and $g$ doesn’t satisfy condition \ref{1.3} but satisfying the Landesman-Lazer condition

$$g_− < \bar{h} < g_+$$

where $\lim_{w \to -\infty} g(w) = g_−, \bar{h} = \frac{1}{|\partial \Omega|} \int_{\partial \Omega} h \, dx, \lim_{w \to \infty} g(w) = g_+$

And $A(x) = 0 \in \mathbb{R}^{k \times k}$ Then it is well know that there is a solution \ref{1.1}. The first results when the nonlinearity in the equation in scalar case was done by Landesman and Lazer [16] in 1970, Their work led to great interest and activity on boundary value problems at resonance which continues to this day. A particularly interesting extension of Landesman and Lazer’s work to systems was done by Nirenberg [18], [19] in case of system and the nonlinearity in the equation was done by Ortega and Ward [30], in the scalar case without Landesman-Lazer condition was done by Iannacci and Nkashama [13], Ortega and Sánchez [29], more completely the case for periodic solutions of the system of ordinary differential equations with bounded nonlinear $g$ satisfying Nirenberg’s condition. They studied periodic solutions

$$u'' + cu' + g(u) = p(t)$$

for $u \in \mathbb{R}^k$.

In case $c = 0$ was done by Mawhin [23]. In case the nonlinear terms vanish at infinity, as in \ref{1.3}, the Landesman-Lazer conditions fail. We would like to know what we can do in this case, and what conditions on a bounded nonlinearity that vanishes at infinity might replace that ones of the Landesman-Lazer type. Several authors have considered the case when the nonlinearity $g: \partial \Omega \times \mathbb{R} \to \mathbb{R}$ is a scalar function satisfies Carathéodory conditions i.e.;

i: $g(., u)$ is measurable on $\partial \Omega$, for each $u \in \mathbb{R}$,

ii: $g(x, .)$ is continuous on $\mathbb{R}$, for a.e. $x \in \partial \Omega$,

iii: for any constant $r > 0$, there exists a function $\gamma_r \in L^2(\partial \Omega)$, such that

$$|g(x, u)| \leq \gamma_r(x), \quad (1.4)$$

for a.e. $x \in \Omega$, and all $u \in \mathbb{R}$ with $|u| \leq r$,

Was done by Fadlallah [8] and the others have considered the case when the nonlinearity does not decay to zero very rapidly. For example in case the nonlinearity in the equation if $g = g(t)$ is a scalar function, the condition

$$\lim_{|t| \to \infty} tg(t) > 0 \quad (1.5)$$

and related ones were assumed in [1], [2], [3], [11], [12], [20], [21], [22], [31]. These papers all considered scalar problem, but also considered the Dirichlet (Neumann) problem at resonance (non-resonance) at higher eigenvalues (Steklov-eigenproblems). The work in some of these papers makes use of Leray-Schauder degree arguments, and the others using critical point theory both the growth restrictions like \ref{1.5} and Lipschitz conditions have been removed (see [21], [31]). In this paper we study systems of elliptic boundary value problems with nonlinear boundary conditions Neumann type and the nonlinearities at boundary vanishing at the infinity. We do not require the problem to be in variational from.
1.1. Assumptions.

G1: \( g \in C^1(\mathbb{R}^k, \mathbb{R}^k) \) and \( g \) is bounded with \( g(w) \neq 0 \) for \( |w|_E \) large. Let \( S^{k-1} \) be the unit sphere in \( \mathbb{R}^k \).

G2: We will assume that \( S^{k-1} \cap \partial \Omega \neq \emptyset \) and let \( S = S^{k-1} \cap \partial \Omega \).

G3: For each \( z \in S \) the \( \lim_{r \to \infty} \frac{g(rz)}{|g(rz)|_E} = \varphi(z) \) exists, and the limits is uniform for \( z \in S \).

G4: \( \text{deg}(\varphi) \neq 0 \)

1.2. Notations.

- Let \( \langle ., . \rangle_{L^2} \) denote the inner product in \( L^2 := L^2(\Omega, \mathbb{R}^k) \) where \( L^2 \) is Lebesgue space.
- Let \( \langle ., . \rangle_E \) denote the standard inner product in \( \mathbb{R}^k \) for \( u, v \in H^1 = H^1(\Omega, \mathbb{R}^k) \) where \( H^1 \) the Sobolev space.

We note that if follows from the assumptions G1-G4 that on large balls

\[
B(R) := \{ y : |y|_E \leq R \},
\]

the \( \text{deg}(g, B(R), 0) \neq 0 \) see [17], [24].

We modify the Lemma 1 and Theorem 1 [30] to fit our problem.

**Lemma 1.1.** Assume that G1 and G3 hold and \( C > 0 \) is a given constant. Then there is \( R > 0 \) such that

\[
\int_{\partial \Omega} g(u(x)) \, dx \neq 0,
\]

for each function \( u \in C(\partial \Omega, \mathbb{R}^k) \) (we can write \( u = \bar{u} + \tilde{u} \) where \( \bar{u} = \int_{\partial \Omega} u(x) \, dx \), and \( \bar{u}, \tilde{u} \) with \( |\bar{u}|_E \geq R \) and \( ||u - \bar{u}||_{L^\infty(\partial \Omega)} \leq C \).

**Proof.** By the way of contradiction. Assume that for some \( C > 0 \) there is exist a sequence of functions \( \{u_n\}_{n=1}^{\infty} \subset C(\overline{\Omega}, \mathbb{R}^k) \), with

\[
|\bar{u}_n|_E \to \infty, \quad ||u_n - \bar{u}_n||_{L^\infty(\partial \Omega)} \leq C
\]

and

\[
\int_{\partial \Omega} g(u_n(x)) \, dx = 0 \quad (1.6)
\]

We constructed a subsequence of \( u_n \) one can assume that \( \bar{z}_n = \frac{\bar{u}_n}{||\bar{u}_n||_E} \) converges to some point \( z \in S \). The uniform bound on \( u_n - \bar{u}_n \) implies that also \( \frac{\bar{u}_n}{||\bar{u}_n||_E} \) converges to \( z \) and this convergence is uniform with respect to \( x \in \overline{\Omega} \). It follows from the assumption G3 that

\[
\lim_{n \to \infty} \frac{g(u_n(x))}{|g(u_n(x))|_E} = \varphi(z)
\]

uniformly in \( \overline{\Omega} \). Since \( \varphi(z) \) is in the unit sphere one can find an integer \( n_0 \) such that if \( n \geq n_0 \) and \( x \in \overline{\Omega} \), then

\[
\frac{g(u_n(x))}{|g(u_n(x))|_E} \cdot \varphi(z) = \frac{g(u_n(x))}{|g(u_n(x))|_E} \cdot \varphi(z) \geq 1
\]

Define

\[
\gamma_n(x) = |g(u_n(x))|_E.
\]
By G1 clearly \( \gamma_n > 0 \) everywhere. For \( n \geq n_0 \)

\[
\langle \int_{\partial \Omega} g(u_n(x)) \, dx, \varphi(z) \rangle_E = \int_{\partial \Omega} \langle g(u_n(x)), \varphi(z) \rangle_E \, dx
\]

\[
= \int_{\partial \Omega} \gamma_n(x) \frac{g(u_n(x))}{\gamma_n(x)} \varphi(z) \rangle_E \, dx \geq \frac{1}{4} \int_{\partial \Omega} \gamma_n(x) \, dx > 0
\]

Therefore \( \int_{\partial \Omega} g(u_n(x)) \, dx > 0 \). Now we have contradiction with (1.6)

The proof completely of the lemma. \( \square \)

2. Main Result

Let

\[ Qu = Nu \tag{2.1} \]

be a semilinear elliptic boundary value problem. Suppose \( N \) is continuous and bounded (i.e., \( |N| \leq C \) for all \( u \)). If \( Q \) has a compact inverse \( Q^{-1} \) then by Leray-Schauder theory (2.1) has a solution. On the other hand if \( Q \) is not invertible the existence of a solution depends on the behavior of \( N \) and its interaction with the null space of \( Q \) see [24].

Theorem 2.1. Assume That \( g \in C^1(\mathbb{R}^k, \mathbb{R}^k) \) satisfies G1, G3, and G4. If \( h \in C(\partial \Omega, \mathbb{R}^k) \), satisfies \( \bar{h} = 0 \) then (1.1) has at least one solution.

Proof. Define

\[ Dom(L) := \{ u \in H^1(\Omega) : -\Delta u + A(x)u = 0 \} \]

Define an operator \( L \) on \( L^2 = L^2(\Omega, \mathbb{R}^k) \) for \( u \in Dom(L) \) and each \( v \in H^1(\Omega) \) by

\[ Lu = \frac{\partial u}{\partial \nu} \quad \forall \; u \in Dom(L) \]

we use the embedding theorem see [5] since you know that \( H^1(\Omega) \hookrightarrow L^2(\Omega) \) and the trace theorem \( H^1 \rightarrow L^2(\partial \Omega) \), thus \( L : Dom(L) \subset L(\partial \Omega) \rightarrow L^2(\partial \Omega) \) then the equation

\[ < Lu, v > = \langle h, v \rangle_{L^2(\partial \Omega)} \quad \forall \; v \in H^1(\partial \Omega) \]

if and only if

\[ Lu = h. \]

The latter equation is solvable if and only if

\[ Ph := \frac{1}{|\partial \Omega|} \int_{\partial \Omega} h = 0 \]

Now if \( h \in L^\infty(\partial \Omega, \mathbb{R}^k) \) and \( Ph = 0 \) then each solution \( u \in H^1(\Omega) \) is Hölder continuous, so \( u \in C^\gamma(\bar{\Omega}, \mathbb{R}^k) \) for some \( \gamma \in (0,1) \). Since we know that there is constant \( r_1 \) such that

\[ \|u\|_\gamma \leq r_1 \left( \|u\|_{L^2(\partial \Omega)} + \|h\|_{L^\infty(\partial \Omega)} \right) \]

When \( Ph = 0 \) there is a unique solution \( Kh = \bar{u} \in H^1(\Omega) \) with \( P\bar{u} = 0 \) to

\[ Lu = h, \]

and if \( h \in C(\partial \Omega) = C(\partial \Omega, \mathbb{R}^k) \) then

\[ \|Kh\|_\gamma \leq r_1 \left( \|Kh\|_{L^2(\partial \Omega)} + \|h\|_{L^\infty(\partial \Omega)} \right) \leq r_2 \|h\|_{C(\partial \Omega)} \]
and $K$ maps $C(\partial \Omega)$ into itself take compact set to compact set i.e.; compactly.

Let $Q$ be the restriction of $L$ to $L^{-1}(C(\partial \Omega)) = KC(\partial \Omega) + \mathbb{R}^k$. We define $N : C(\partial \Omega) \to C(\partial \Omega)$ define by

$$N(w)(x) := h(x) - g(w(x)) \quad \forall w \in C(\partial \Omega)$$

is continuous. Now (1.1) can be written as

$$Qu = Nu$$

and $\ker Q = \exists P, \exists Q = \ker P$. The linear map $Q$ is a Fredholm map (see [22]) and $N$ is $Q$–compact (see [24]). Now we define the Homotopy equation as follows

Let $\lambda \in [0, 1]$ such that

$$Qu = \lambda Nu.$$  \hspace{1cm} (2.2)

The a priori estimates (i.e.; the possible solutions of [22] are uniformly bounded in $C(\partial \Omega)$) Now we show that the possible solutions of [22] are uniformly bounded in $C(\partial \Omega)$ independent of $\lambda \in [0, 1]$ Since we know that $u = \bar{u} + \bar{v}$ where $\bar{u} = Pu$. Then

$$||\bar{u}||_{L^\gamma} = ||\lambda KNu||_{L^\gamma} \leq r_2||Nu||_{C(\partial \Omega)} \leq R_1$$

Where $R_1$ is a constant ($g$ is a bounded function). It remains to show that $\bar{u} \in \mathbb{R}^k$ is bounded, independent of $\lambda \in [0, 1]$. By the way of contradiction assume is not the case (i.e.; $\bar{u}$ unbounded). Then there are sequence $\{\lambda_n\} \subset [0, 1]$, and $\{u_n\} \subset \text{Dom}(Q)$ with $||\bar{u}_n||_{L^\gamma} \leq R_1$,

$$Qu_n = \lambda_n Nu_n \quad \text{and} \quad ||u_n||_{E} \to \infty$$

We get that

$$PNu_n = PN(\bar{u}_n + \bar{u}_n) = -\int_{\partial \Omega} g(\bar{u}_n(x) + \bar{u}_n(x)) \, dx = 0$$

Now $u_n = \bar{u}_n + \bar{u}_n$ so $||u_n - \bar{u}_n||_{L^\infty(\partial \Omega)} = ||\bar{u}_n||_{L^\infty(\partial \Omega)} \leq R_1$ and $||\bar{u}_n||_{L^\infty(\partial \Omega)} \to \infty$. It follows from Lemma [13] that for all sufficiently large $n$

$$\int_{\partial \Omega} g(u_n(x)) \, dx \neq 0$$

We have reached a contradiction, and hence all possible solutions of [22] are uniformly bounded in $C(\partial \Omega)$ independent of $\lambda \in [0, 1]$

Let $\bar{B}(0, r) = \{x : |x|_E \leq r\}$ denote the ball in $C(\partial \Omega, \mathbb{R}^k)$ Now you can apply Leray-Schauder degree theorem see ([17], [24]), the only thing left to show is that

$$\text{deg}(PN, \bar{B}(0, r) \cap \ker Q, 0) \neq 0,$$

for large $r > 0$. So $\text{deg}(PN, \bar{B}(0, r) \cap \ker Q, 0) = \text{deg}(g, \bar{B}_r, 0)$, where $\bar{B}_r$ is the ball in $\mathbb{R}^k$ of radius $r$. Since for $|x|_E$ large, and $\text{deg}(\phi) \neq 0$ we have that $\text{deg}(g, \bar{B}_r, 0) \neq 0$ for large $r$. Therefore $\text{deg}(PN, \bar{B}(0, r) \cap \ker Q, 0) \neq 0$ By Leray-Schauder degree theorem equation (2.2) has a solution when $\lambda = 1$. Therefore equation (1.1) has at least one solution. This proves the theorem. \hfill \Box

We will give one Example

**Example 2.1.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ is a bounded domain with boundary $\partial \Omega$ of class $C^\infty$. Let

$$-\Delta u + A(x)u = 0 \quad \text{in} \ \Omega,$$

$$\frac{\partial u}{\partial \nu} + \frac{u}{1 + |u|^3} = h(x) \quad \text{on} \ \partial \Omega$$

(2.3)
where $A(x)$ is positive semidefinite matrix on $\mathbb{R}^{2 \times 2}$, and Where $u = (u_1, u_2) \in \mathbb{R}^2$ and $h$ real valued function and continuous on $\partial \Omega$, and $\int_{\partial \Omega} h(x) \, dx = 0$ and $g(u) = \lim_{u \to \infty} \frac{u}{1 + |u|^2}$

$$\lim_{u \to \infty} g(u) = \lim_{|u| \to \infty} \frac{u}{1 + |u|^2} = 0$$

g(u) vanishes at infinity, clearly $g \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ and bounded with $g(u) \neq 0$, for $|u|_E$ large. Therefore $g$ satisfies G1.

$$\frac{g(ru_1, ru_2)}{|g(ru_1, ru_2)|} = \frac{g(ru)}{|g(ru)|} = \frac{ru}{1 + |ru|^2} = \frac{u}{|u|^2} = u$$

For all $u$ in $\mathcal{S}$ and $r > 0$. Therefore G3 holds.

And $\varphi(u) = u$ so that $\deg(\varphi) \neq 0$. Therefore G4 holds. By theorem 2.4, 2.3 has at least one solution.

REFERENCES

[1] A. Ambrosetti and G. Mancini, Existence and multiplicity results for nonlinear elliptic problems with linear part at resonance the case of the simple eigenvalue, J. Differential Equations 28 (1978), 229–245
[2] A. Ambrosetti and G. Mancini, Theorems of existence and multiplicity for nonlinear elliptic problems with noninvertible linear part, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 5 (1978), 15–28
[3] G. Auchmuty. Steklov eigenproblems and the representation of solutions of elliptic boundary value problems. Numer. Func. Analysis Optim. 25 (2004), 321–348.
[4] Ambrosetti, A. (Antonio) ., A primer of nonlinear analysis, Cambridge, University Press, 1993.
[5] Brezis, H. Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, 2011
[6] Brown, Robert F., A topological introduction to nonlinear analysis, Boston, Birkhauser, 1993.
[7] A. Castro. Semilinear equations with discrete spectrum. Contemp. Math. 347 (2004), 1–16.
[8] A. Fadlallah Linear second order elliptic partial differential equations with Nonlinear Boundary Conditions at Resonance Without Landesman-Lazer Conditions, http://arxiv.org/pdf/1410.7315v2 Submitted on 27 Oct 2014.
[9] L.C. Evans, Partial Differential Equations, Amer. Math. Soc., Providence, RI, 1998.
[10] DE Figueiredo D.G, Semilinear elliptic at resonance; higher eigenvalues and unbounded nonlinearities , in Recent Advances in differential Educational (Edited by Conti), pp. 89-99, Academic Press, London 1981.
[11] S. Fucik and M. Krbeč, Boundary value problems with bounded nonlinearity and general null-space of the linear part, Math. Z. 155 (1977), 129–138
[12] P. Hess, A remark on the preceding paper of Fucik and Krbec, Mat. Z. 155 (1977), 139–141
[13] R.Iannacci, M. N. Nkashama; nonlinear two point boundary value problems at resonance without Landesman-lazer condition. Journal of Proceeding of the american mathematical society Vol.106, NO.4, pp 943-952, August 1989.
[14] R.Iannacci, M. N. Nkashama; Unbounded Perturbations of forced second order ordinary differential equations at resonance. Journal of Differential Equations Vol.69, NO.4, pp 289-309, 1987.
[15] R.Iannacci, M. N. Nkashama; nonlinear boundary value problems at resonance. Journal of Nonlinear analysis, theory, methods,85 applications Vol.11, NO.4, pp 455-473,1987. Printed in Great Britain.
[16] E. Landesman and A. Lazer: Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1970), 609-623.
[17] Lloyd, N.G ., Degree theory, Cambridge, University Press, 1978.
[18] L. Nirenberg, An application of generalized degree to a class of nonlinear problems. Troisième Colloque sur l’Analyse Fonctionnelle (Liège, 1976), pp. 57–74. Vander, Louain 1971.
[19] L. Nirenberg, Generalized degree and nonlinear problems. Contributions to nonlinear functional analysis (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1971), pp. 1–9. Academic Press, New York, 1971.

[20] N. Mavinga and M. N. Nkashama. Steklov–Neumann eigenproblems and nonlinear elliptic equations with nonlinear boundary conditions. J. Diff. Eqns 248 (2010), 1212–1229.

[21] N. Mavinga and M. N. Nkashama. Nonresonance on the boundary and strong solutions of elliptic equations with nonlinear boundary conditions. Journal of Applied Functional Analysis, Vol. 7, No. 3, 243–257, 2011.

[22] N. Mavinga, M. N. Nkashama; Nonresonance on the boundary and strong solutions of elliptic equations with nonlinear boundary conditions. Journal of Applied Functional Analysis, Vol. 7, No. 3, 248–257, copyright 2012 Eudoxus Press, LLC.

[23] J. Mawhin, Forced second order conservative systems with periodic nonlinearity, Ann. Inst. Henri-Poincaré Anal. Non Linéaire 6 suppl. (1989), 415–434.

[24] J. Mawhin; Topological degree methods in nonlinear boundary-value problems, in NSFCBMS Regional Conference Series in Mathematics, American Mathematical Society, Providence, RI, 1979.

[25] J. Mawhin J.R. Ward, and M. Willem, Necessary and sufficient conditions for the solvability of a nonlinear two-point boundary value, Proc. Amer. Math. Soc. 93 (1985), 667–674

[26] J. Mawhin and K. Schmitt. Corrigendum: upper and lower solutions and semilinear second order elliptic equations with non-linear boundary conditions. Proc. R. Soc. Edinb. A 100 (1985), 361.

[27] J. Mawhin; Topological degree and boundary-value problems for nonlinear differential equations, in: P.M. Fitzpatrick, M. Martelli, J. Mawhin, R. Nussbaum (Eds.), Topological Methods for Ordinary Differential Equations, Lecture Notes in Mathematics, vol. 1537, Springer, New York/Berlin, 1991.

[28] J. Mawhin, Landesman-Lazer conditions for boundary value problems: A nonlinear version of resonance, Bol. de la Sociedad Española de Mat. Aplicada 16 (2000), 45–65.

[29] R. Ortega and L.A. Sánchez, Periodic solutions of forced oscillators with several degrees of freedom, Bull. London Math. Soc. 34 (2002) 308–318

[30] R. Ortega, and J.R. Ward, A Semilinear elliptic system with vanishing nonlinearities, proceedings of the fourth international conference on dynamical systems and differential equations May 24 – 27, 2002, Wilmington, NC

[31] T. Runst, A unified approach to solvability conditions for nonlinear second-order elliptic equations at resonance, Bull. London Math. Soc. 31 (1999) 385–394.

ALZAKI,M.M. FADALLAH

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALABAMA AT BIRMINGHAM

BIRMINGHAM, ALABAMA 35294-1170, USA

E-mail address: zakima99@math.uab.edu