Running vacuum model in non-flat universe

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Abstract

We investigate observational constraints on the running vacuum model (RVM) of $\Lambda = 3\nu(H^2 + K/a^2) + c_0$ in the spatially curved universe, where $\nu$ is the model parameter, $K$ corresponds to the spatial curvature constant, and $c_0$ is a constant defined by the boundary conditions. As $\dot{\Lambda} \neq 0$, there are energy exchanges between vacuum, matter and radiation in RVM. We study the “geometrical degeneracy” of RVM on the CMB power spectra. By fitting the cosmological data, we find that the values of $\chi^2$ in RVM and $\Lambda$CDM are similar to each other for the non-flat universe. Explicitly, we obtain the constraints of $\nu \leq O(10^{-4})$ (68% C.L.) and $|\Omega_K| \leq O(10^{-2})$ (95% C.L.) in our study. In addition, we show that the cosmological constraints of $\Sigma m_\nu = 0.416^{+0.311}_{-0.407}$ (RVM) and $\Sigma m_\nu = 0.497^{+0.335}_{-0.387}$ (LCDM) at 95% C.L. for the neutrino mass sum are relaxed in both models in the spatially curved universe.

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I. INTRODUCTION

Since the discovery that our universe has been expanding at an accelerated rate at recent time from the type Ia supernova data [1–3], many dark energy models have been proposed to explain these phenomena [4–7]. The simplest one is the ΛCDM model, in which Λ represents the cosmological constant term. However, ΛCDM encounters some difficulties, mainly the “fine tuning” [8, 9] and “coincidence” [10–12] problems.

The running vacuum model (RVM) [13, 14] has been introduced in order to solve the “coincidence problem”, where the cosmological constant term is assumed to be varying with the Hubble parameter $H$. This model links the existence of dark energy to the theoretical mechanism of the quantum field, which may trigger the primordial inflation scenario [15], and fit with the observational data better than ΛCDM [16]. In the literature, the spatially flat RVM has been extensively investigated [16–33].

However, the Planck Legacy 2018 analysis by Valentino, Melchiorri and Silk in Ref. [34] suggests that the universe is closed at 99 % C.L., which encourages us to study RVM in a non-flat universe. With the involvement of the non-zero spatial curvature, it is inevitable to encounter the degeneracies between curvature and other parameters. One of them is the famous “geometrical degeneracy” [35, 36] on CMB power spectra, caused by different sets of parameters that lead to same value of the angular diameter distance of the last scattering. On the other hand, when fitting with the observational data, the non-zero spatial curvature also broadens the constraints of the cosmological parameters [37].

In this work, we concentrate on the running cosmological constant in the non-flat universe, $\Lambda = 3\nu(H^2 + K/a^2) + c_0$, where $\nu$ and $c_0$ are the model parameters and $K$ corresponds to the spatial curvature constant. We first study the CMB power spectra in this non-flat RVM and discuss the degeneracy between $\nu$ and the density parameter of curvature, $\Omega_K$. We then constrain the cosmological parameters of both non-flat RVM and ΛCDM with the observational data by using the Markov chain Monte Carlo (MCMC) method, and compare the results with those in the flat universe. The effectiveness of RVM versus ΛCDM in the non-flat universe is also tested based on the minimal $\chi^2$ values.

This paper is organized as follows. In Sec. II, we introduce the non-flat running vacuum model and derive the background evolution equations. We compare the CMB power spectra of RVM in the non-flat universe along with the Planck 2018 data, and show the constraints
of the cosmological parameters in Sec. III. Our conclusions is presented in Sec. IV.

II. RVM IN CURVED UNIVERSE

We start with the Einstein field equation of RVM, given by

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + g_{\alpha\beta}\Lambda = \kappa^2T_{\alpha\beta},$$

(1)

where $\kappa^2 = 8\pi G$ is set to be 1 for simplicity, $R = g^{\alpha\beta}R_{\alpha\beta}$ represents the Ricci scalar, $T_{\alpha\beta}$ stands for the energy-momentum tensor for matter and radiation, and $\Lambda$ corresponds to the dynamical cosmological constant.

The spatially isotropic and homogeneous universe can be described by the Robertson-Walker metric:

$$ds^2 = -dt^2 + a^2(t) \left\{ \frac{dr^2}{1-Kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \right\},$$

(2)

where $a$ is the scale factor, while $K$ is a constant describe the spatial curvature with $K = 1, 0, -1$ corresponding to the closed, flat, and open universe, respectively. Then, the Friedmann equations can be expressed as

$$H^2 = \frac{1}{3}(\rho_m + \rho_r + \rho_\Lambda) - \frac{K}{a^2},$$

(3)

$$\dot{H} = -\frac{1}{2}(\rho_m + \rho_r + \rho_\Lambda + P_m + P_r + P_\Lambda) + \frac{K}{a^2},$$

(4)

where $\rho_{m,r,\Lambda}$ ($P_{m,r,\Lambda}$) are the energy densities (pressures) of matter, radiation and dark energy, respectively, and $H = da/(adt)$ represents the Hubble parameter. The corresponding equations of state in this model can be written as

$$w_{m,r,\Lambda} = \frac{P_{m,r,\Lambda}}{\rho_{m,r,\Lambda}} = 0, \frac{1}{3}, -1.$$

(5)

In addition, the density parameters are given by

$$\Omega_{m,r} = \frac{\rho_{m,r}}{3H^2},$$

(6)

$$\Omega_\Lambda = \frac{\Lambda}{3H^2},$$

(7)

$$\Omega_K = -\frac{K}{a^2H^2}.$$

(8)

In the non-flat universe, the running cosmological constant term is found to be

$$\Lambda = 3\nu H^2 + 3\nu \frac{K}{a^2} + c_0,$$

(9)
where $\nu$ is a non-negative model parameter to ensure that the energy density of dark energy is positive in the early universe, $c_0$ is given by $c_0 = -3\nu(H_0^2 + K) + \Lambda_0$ with $H_0$ and $\Lambda_0$ the present values of the Hubble parameter and cosmological constant, respectively. The model becomes to be $\Lambda$CDM when $\nu = 0$.

For the energy transformations from dark energy to matter and radiation, the modified continuity equations are given by

$$\dot{\rho}_{m,r} + 3H(1 + w_{m,r})\rho_{m,r} = Q_{m,r},$$

$$\dot{\rho}_\Lambda + 3H(1 + w_\Lambda)\rho_\Lambda = -Q,$$

with $Q_m + Q_r = Q$, where $Q_{m,r}$ can be written as

$$Q_{m,r} = -\frac{\dot{\rho}_\Lambda(\rho_{m,r} + P_{m,r})}{\rho_m + \rho_r + P_m + P_r} = 3\nu H(1 + w_{m,r})\rho_{m,r}.$$

By combining Eqs. (10)-(12), we obtain the energy densities as follows:

$$\rho_{m,r} = \rho_{m,r}^{(0)} a^{-3(1+w_{m,r})(1-\nu)},$$

where $\rho_{m,r}^{(0)}$ are the current values.

### III. NUMERICAL CALCULATIONS

To study the degeneracy between the cosmological parameters, we first modify the CAMB [38] program to generate theoretical CMB power spectra for both models of RVM and $\Lambda$CDM. The results are presented in Sec. IIIA. We then use the CosmoMC package [39], which is a Markov Chain Monte Carlo (MCMC) engine exploring the cosmological parameter space, to constrain RVM and $\Lambda$CDM from the observational data. For simplification, we take $\Omega_K$ afterward to represent the density parameter of curvature at the present time except those specifically indicated.

#### A. CMB power spectra of the models

There is “geometrical degeneracy” between curvature and other parameters for CMB power spectra. To see this effect, we compare CMB power spectra of RVM and $\Lambda$CDM with different $\nu$ and $\Omega_K$ along with the observational data from Planck 2018 [40]. From the previous studies in the literature [16, 31, 32] with $0 \leq \nu \leq O(10^{-3})$ in RVM for the
flat universe and the result of $-0.007 \geq \Omega_K \geq -0.095$ at 99% C.L. in Ref. [34], we choose $0 \leq \nu < 0.01$ and $0 \geq \Omega_K \geq -0.01$ to see the degeneracy between $\nu$ and $\Omega_K$ on CMB power spectra. Furthermore, the $\Lambda$CDM model is recovered when $\nu = 0$ and $\Omega_K = 0$ in Eq. 9.

In Fig. 1, we present the CMB power spectra for the TT, EE and TE modes from the CAMB package. It can be seen that $0 \leq \nu \leq O(10^{-3})$ (solid lines) and $0 \geq \Omega_K \geq -O(10^{-2})$ (dashed lines) fit well with the data from Planck 2018. The residues with respect to $\Lambda$CDM are plotted in Fig. 2. We find that the geometrical degeneracy with $(\nu, \Omega_K) = (0.001, 0)$ (green solid line) and $(0.0, -0.01)$ (purple dashed line) has the similar results on CMB power spectra. However, only $\nu$ can cause strong suppressions on the TT mode spectra when $\nu > 0$. In addition, the effects of $\nu$ and $K$ show additive property on CMB power spectra (red dash-dotted line).

FIG. 1. Power spectra of CMB TT, EE and TE for RVM and $\Lambda$CDM in the flat and non-flat universe along with the Planck 2018 data.
FIG. 2. Residuals of $\Delta D^\ell_{TT}$, $\Delta D^\ell_{EE}$ and $\Delta D^\ell_{TE}$ in RVM with respect to $\Lambda$CDM for CMB power spectra, respectively, along with the observational data from Planck 2018.

TABLE I. data points of $f\sigma_8$

| $z$   | $f\sigma_8$    | Ref. | $z$   | $f\sigma_8$    | Ref. | $z$   | $f\sigma_8$    | Ref. |
|-------|----------------|------|-------|----------------|------|-------|----------------|------|
| 1     | 0.482 ± 0.116  | [49] | 10    | 0.488 ± 0.06  | [57] | 19    | 0.440 ± 0.05  | [52, 60] |
| 2     | 0.470 ± 0.08   | [50] | 11    | 0.444 ± 0.038 | [58] | 20    | 0.394 ± 0.062 | [58]  |
| 3     | 0.38 ± 0.04    | [51] | 12    | 0.452 ± 0.057 | [55] | 21    | 0.407 ± 0.055 | [56]  |
| 4     | 0.490 ± 0.18   | [52, 53] | 13 | 0.427 ± 0.043 | [56] | 22    | 0.351 ± 0.058 | [59]  |
| 5     | 0.437 ± 0.072  | [54] | 14    | 0.413 ± 0.080 | [54] | 23    | 0.42 ± 0.07   | [51]  |
| 6     | 0.457 ± 0.052  | [55] | 15    | 0.45 ± 0.04   | [51] | 24    | 0.51 ± 0.06   | [52, 61] |
| 7     | 0.390 ± 0.063  | [54] | 16    | 0.419 ± 0.041 | [56] | 25    | 0.49 ± 0.15   | [62]  |
| 8     | 0.433 ± 0.067  | [56] | 17    | 0.430 ± 0.054 | [55] | 26    | 0.423 ± 0.055 | [63]  |
| 9     | 0.43 ± 0.04    | [51] | 18    | 0.460 ± 0.038 | [59] | 27    | 0.36 ± 0.04   | [64]  |
TABLE II. Priors for cosmological parameters with the non-flat RVM of $\Lambda = 3\nu(H^2 + K/a^2) + c_0$

| Parameter                        | Prior                        |
|---------------------------------|------------------------------|
| RVM parameter $\nu$             | $0.0 \leq \nu \leq 3.0 \times 10^{-4}$ |
| Curvature parameter $\Omega_K$  | $-0.25 \leq \Omega_K \leq 0.2$ |
| Baryon density                  | $0.5 \leq 100\Omega_b h^2 \leq 10$ |
| CDM density                     | $0.1 \leq 100\Omega_c h^2 \leq 99$ |
| Optical depth                   | $0.01 \leq \tau \leq 0.8$      |
| Neutrino mass sum               | $0 \leq \Sigma m_\nu \leq 2$ eV |
| Angular diameter distance       | $0.5 \leq 100\theta_{MC} \leq 10$ |
| Scalar power spectrum amplitude | $2 \leq \ln(10^{10} A_s) \leq 4$ |
| Spectral index                  | $0.8 \leq n_s \leq 1.2$        |

B. Global fitting

In order to constrain the cosmological parameters of RVM and $\Lambda$CDM in the non-flat universe, we use the CosmoMC package with a MCMC engine to explore the parameter space with the combinations of the observational data sets, which include the CMB temperature fluctuation from Planck 2015 with TT, EE, low-$l$ polarization from SMICA [41–43], BAO data from 6dF Galaxy Survey [44] and BOSS [45], supernova(SN) data from the JLA compilation [46], the weak lensing (WL) data from CFHTLenS [47] and direct large scale structure (LSS) formation data, and the datapoints of $f\sigma_8$ listed in Table I. The priors of parameters are given in Table II. Due to the tension between the geometry data (SNIa, BAO etc.) and growth data (WL, $f\sigma_8$) [48], we choose the two combinations of CMB+BAO+SN and CMB+BAO+SN+WL+$f\sigma_8$ in our fits. To calculate the best fitted values of $\chi^2$, we use that

$$\chi^2_c = \sum_{i=1}^{n} \frac{(T_c(z_i) - O_c(z_i))^2}{E^i_c}, \quad (14)$$

where $c$ denotes the type of the data, $n$ is the number of the data in each data set, $T_c$ represents the theoretical value derived form CAMB at the redshift $z_i$, and $O_c(E_c)$ corresponds to the observational value (covariance).

The global fitting results of RVM and $\Lambda$CDM in the non-flat universe are plotted in
FIG. 3. One and two-dimensional distributions of $\Omega_b h^2$, $\Omega_c h^2$, $\tau$, $\Omega_K$, $\sum m_\nu$, $10^4 \nu$, $H_0$, $\sigma_8$ for RVM and $\Lambda$CDM in the non-flat universe with the combined data of CMB+BAO+SN, where the contour lines represent 68% and 95% C.L., respectively.

Figs. 3 and 4, while those listed in Table III correspond to the cosmological parameters and $\nu$, given at 95% and 68% C.L., respectively. Our results show that $\nu \lesssim 1.65 \times 10^{-4}$ at 68% C.L. in the non-flat universe of RVM for the data set of CMB+BAO+SN+WL+$f\sigma_8$, which is similar to the previous result of $1.54 \times 10^{-4}$ at 68% C.L. in RVM for the flat universe [31]. For the density parameter of curvature, $|\Omega_K|$, we find that it is constrained to be $\leq O(10^{-2})$ in both RVM and $\Lambda$CDM. We obtain that $\chi^2_{RV M} = 2039.18(2089.66)$ and $\chi^2_{\Lambda CD M} = 2038.45(2089.66)$ with $\chi^2_{RV M} - \chi^2_{\Lambda CD M} = 0.7(0.0)$ when fitting with the data set of CMB+BAO+SN (CMB+BAO+SN+WL+$f\sigma_8$), indicating that our results in RVM are consistent with those in $\Lambda$CDM for the non-flat universe.

In addition, the tension of $\sigma_8$ is reduced in RVM compared with $\Lambda$CDM. On the other hand, the best-fit values of the neutrino mass sum, $\sum m_\nu$, in the non-flat universe are much larger than those in the flat universe [16]. This is caused by the degeneracy of $\Omega_K$ and
FIG. 4. One and two-dimensional distributions of $\Omega_b h^2$, $\Omega_c h^2$, $\tau$, $\Omega_K$, $\sum m_\nu$, $10^4 \nu$, $H_0$, $\sigma_8$ for RVM and $\Lambda$CDM in the non-flat universe with the combined data of CMB+BAO+SN+WL+$f\sigma_8$, where the contour lines represent 68% and 95% C.L., respectively.

$\Sigma m_\nu$ on the distance parameters [65], as non-zero $\Omega_K$ lead to the relaxation of $\Sigma m_\nu$ in the spatially curved universe. We remark that the constraints on $\Sigma m_\nu$ are further relaxed for the data set of CMB+BAO+SN+WL+$f\sigma_8$ data. Explicitly, we have that $\Sigma m_\nu = 0.416_{-0.407}^{+0.311}$ and $0.497_{-0.387}^{+0.335}$ at 95% C.L., resulting in the non-zero lower bounds of $\Sigma m_\nu \geq 0.009$ and 0.110 eV at 95% C.L., for RVM and $\Lambda$CDM, respectively, in spatially curved universe.

IV. CONCLUSIONS

We have studied the model with the running cosmological constant of $\Lambda = 3\nu (H^2 + K/a^2) + c_0$ in the spatially curved universe. We have compared our results for several sets of $\nu$ and $\Omega_K$ with the Planck 2018 data in the CMB power spectra. We have found that $\nu$ and $\Omega_K$ have similar effects on the CMB power spectra, but only non-zero values of $\nu$ would
TABLE III. Fitting results for RVM and ΛCDM in the non-flat universe, where the cosmological parameters and ν are given at 95% and 68% C.L., respectively.

| Parameter | CMB+BAO+SN | CMB+BAO+SN | CMB+BAO+SN |
|-----------|------------|------------|------------|
| Model     | RVM        | ΛCDM       | RVM        | ΛCDM       |
| 100Ω_bh^2 | 2.22 ± 0.05| 2.24 ± 0.05| 2.21 ± 0.05| 2.22 ± 0.05|
| 100Ω_ch^2 | 11.8 ± 0.4 | 11.8 ± 0.4 | 11.8 ± 0.4 | 11.8 ± 0.4 |
| 100τ      | 11.6^{+5.5}_{-5.7} | 12.3^{+5.3}_{-5.9} | 9.0^{+6.4}_{-7.1} | 10.8^{+6.1}_{-7.2} |
| 10^3Ω_K   | 1.68^{+7.75}_{-6.97} | 0.32^{+6.99}_{-6.39} | 7.02^{+7.80}_{-9.04} | 7.03^{+6.84}_{-7.87} |
| Σm_ν      | < 0.434    | < 0.395    | 0.416^{+0.311}_{-0.407} | 0.497^{+0.335}_{-0.387} |
| 10^4ν     | < 1.50     | < 1.65     | < 1.65     | < 1.65     |
| H_0       | 67.6^{+1.5}_{-1.4} | 67.8^{+1.4}_{-1.5} | 67.6^{+1.5}_{-1.4} | 67.7^{+1.4}_{-1.5} |
| σ_8       | 0.836^{+0.054}_{-0.062} | 0.843^{+0.052}_{-0.059} | 0.759 ± 0.039 | 0.756^{+0.040}_{-0.037} |
| χ^2_{best-fit} | 2039.18 | 2038.45 | 2089.66 | 2089.66 |

lead to large suppressions in the CMB TT mode spectra. In the two combinations of the observational data, we have constrained that ν ≤ O(10^{-4}) together with |Ω_K| ≤ O(10^{-2}). Notably, the constraints of ν in the non-flat universe are similar to those in the flat universe. From the best fitted values of χ^2, we have shown that RVM is in consistent with ΛCDM. However, the tension of σ_8 can be a little bit alleviated in the non-flat RVM. Moreover, the constraints of Σm_ν are relaxed due to the curvature parameter. When fitting with the date set of CMB+BAO+SN+fσ_8, we have obtained the non-zero lower bounds of Σm_ν ≥ 0.110 and 0.009 eV at 95% C.L. in the non-flat ΛCDM and RVM, respectively, indicating that the involvement of a non-zero Ω_K would provide viable constraints on the absolute neutrino masses in cosmological models.

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