Computation of Lickorish’s Three Manifold Invariants using Chern-Simons Theory

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It is well known that any three-manifold can be obtained by surgery on a framed link in \(S^3\). Lickorish gave an elementary proof for the existence of the three-manifold invariants of Witten using a framed link description of the manifold and the formalisation of the bracket polynomial as the Temperley-Lieb Algebra. Kaul determined three-manifold invariants from link polynomials in \(SU(2)\) Chern-Simons theory. Lickorish’s formula for the invariant involves computation of bracket polynomials of several cables of the link. We describe an easier way of obtaining the bracket polynomial of a cable using representation theory of composite braiding in \(SU(2)\) Chern-Simons theory. We prove that the cabling corresponds to taking tensor products of fundamental representations of \(SU(2)\). This enables us to verify that the two apparently distinct three-manifold invariants are equivalent for a specific relation of the polynomial variables.

January 1999

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1. Introduction

Classification of three dimensional manifolds has been a long standing problem. Witten [1] succeeded in giving an intrinsically three-dimensional definition for Jones’ type polynomial invariants \[2,3\] of links using a topological quantum field theory known as Chern-Simons theory. Witten’s approach gives rise to three-manifold invariants \(Z(M)\) (also called the partition function for the manifold \(M\)) via surgery on framed links.

The existing definitions of three-manifold invariants rely on two results:

I: The fundamental theorem of Lickorish [4] and Wallace [5] that any connected, closed, orientable three-manifold can be obtained by surgery on a framed link in \(S^3\).

II: The theorem due to Kirby [6] that two framed links determine the same three-manifold if and only if they are related by a sequence of diagram moves which are referred to as Kirby moves.

It follows from Kirby’s theorem that invariants of framed links which are unchanged under Kirby moves give an invariant of three-manifolds.

Computations of the Witten invariant are achieved by exploiting the connection between Chern-Simons field theory and a two dimensional conformal field theory known as Wess-Zumino conformal field theory. Though there are questions about measure in the functional integral formulation of Chern-Simons theory, the computed values of these invariants agree with the ones obtained from other mathematically rigorous approaches—viz., exactly solvable two dimensional statistical mechanical models [7], quantum groups [8-10], Temperley-Lieb algebra [14,12]. The interconnections between Chern-Simons theory, Wess-Zumino conformal field theory, solvable models and quantum groups have been summarised in Refs.[15,16].

Given a primitive 4\(r\)th root of unity, Lickorish [12,13] defines an invariant \(F_t(M)\) as a linear combination of bracket polynomials of cables of a framed link on \(S^3\) which under surgery gives the three-manifold \(M\). The cabling is necessary for the preservation of the invariant under Kirby moves. However, it introduces a large number of crossings in the diagram, and determining the bracket polynomials of link diagrams with several crossings is extremely cumbersome by the recursive method. Hence, the computation of Lickorish’s invariant is not easy.

Witten showed that the Jones’ and HOMFLY polynomials of links in \(S^3\) correspond to expectation values of Wilson loops carrying defining representation of the \(SU(2)\) and \(SU(N)\) gauge groups respectively [1]. This method has been generalised to arbitrary higher
dimensional representations of any compact semi-simple group resulting in a whole lot of new invariants of (framed) links in $S^3$\cite{15-19}. We refer to these field theoretic invariants as generalised invariants. Unlike Jones’, HOMFLY and bracket polynomials, the generalised invariants cannot be solved completely by the recursive method. Hence, a direct method of evaluating these was developed in Refs. \cite{15-19}.

By construction these generalised invariants depend on the framing chosen for the link. However, by fixing the framing to be standard, i.e., one in which the linking number of the link with its frame is zero, ambient isotopy invariants of links are obtained. In Refs. \cite{15-19}, the emphasis was on obtaining ambient isotopy invariants of links and hence computations were done in standard framing. In the present problem, we require a field theoretic presentation for the bracket polynomial of a link diagram. Bracket polynomials are regular isotopy invariants. So we choose the vertical or the black-board framing. We show how the Chern-Simons invariant $P_{1,1,\ldots,1}[D_L](q)$ in vertical framing for defining representation placed on any $n$-component link $L$ is related to the bracket polynomial $\langle D_L \rangle(A)$ provided the polynomial variables satisfy

$$q^{1/4} = -A. \quad (1.1)$$

The relation is proved by first establishing a connection between the field theoretic invariant $P_{1,1,\ldots,1}[D_L](q)$ and the Jones’ polynomial, and then using the well-known relationship between the Jones’ polynomial and the bracket polynomial (See Section 3 and Theorem 1).

Using the generalised regular isotopy invariants in $SU(2)$ Chern-Simons theory, Kaul \cite{16} has derived a three-manifold invariant $F_k[M]$. It is mentioned in \cite{16} that wherever computed, the Chern-Simons partition function $Z[M]$ turns out to be the same as his three-manifold invariant except for the normalisation. That is, $F_k(M) = Z[M]/Z[S^3]$. We expect Kaul’s invariant to be equivalent to Witten’s invariant for an arbitrary $M$. We show the equivalence by the following sequence of steps:

(i) We prove that the Lickorish’s and Kaul’s invariants are equivalent with the polynomial variables obeying (1.1) by finding an elegant and easier method of determining bracket polynomials of cables of link diagrams. (See Theorem 2.) This is achieved by using the techniques developed in Ref. \cite{20} -viz., representation theory of composite braids.

(ii) In \cite{13} the equivalence between Lickorish’s invariant and the Reshetikin-Turaev invariant was established. The Reshetikin-Turaev invariant is considered as a reformulation of Witten’s invariant using quantum groups (see \cite{9}, \cite{10}).
It follows that Kaul’s invariant is a reformulation of the Reshetikin-Turaev invariant in terms of the Chern-Simons generalised framed link invariants.

The plan of the paper is as follows. In Section 2, we describe Lickorish’s three-manifold invariant obtained using bracket polynomials. We present in Section 3, the techniques used in evaluating Chern-Simons regular isotopy invariants. We show detailed computations for the Hopf link and then generalise the method to prove Theorem 1. Then, we define the three-manifold invariant derived by Kaul from these generalised link invariants. In Section 4, we study the representation theory of parallel copies of braids. This is essential to compute directly the bracket polynomial for cables of link diagrams without going through the extremely tedious process of recursive evaluation. We show the details of our computation for the (2, 3)-cable of the Hopf link and generalise the techniques to arbitrary links. In the concluding section, we show that the three-manifold invariants obtained by Lickorish’s approach and the field theory approach are equivalent and thereby provide an easier method for computing Lickorish’s invariant.

2. Lickorish’s Three Manifold Invariant

We briefly present the salient features of Lickorish’s three-manifold invariant obtained from bracket polynomials.

An \(n\)-component link \(L\) in \(S^3\) is a subset of \(S^3\) homeomorphic to the union of \(n\) disjoint circles. A framing \(f = (f_1, f_2, \ldots, f_n)\) on \(L\) is an assignment of an integer to each component of \(L\). A regular projection of a link in the plane is one with transverse double points as the only self-intersections. These double points are referred to as crossings. A link diagram is obtained from a regular projection by marking the under crossing arc at a crossing with a break to indicate that part of the curve dips below the plane. Regular isotopy is an equivalence relation on the set of link diagrams. It is generated by Reide-meister moves II and III \([21]\). A framed link \([L, f]\) can be represented by a diagram \(D_L\) in the plane such that the framing on each component of \(L\) equals the sum of crossing signs in the part of the diagram that represents that component. Given a link diagram, the framing it represents will be called the blackboard framing or the vertical framing.

The bracket polynomial as normalised in \([12]\) is a function

\[
\langle \ angle: \text{\{link diagrams in } (\mathbb{R}^2 \cup \infty) \text{ of unoriented links \}} \rightarrow \mathbb{Z}[A^{\pm 1}] 
\]

defined by the following three properties.
(i) $\langle \phi \rangle = 1$;
(ii) $\langle D_L \cup U \rangle = (-A^2 - A^{-2}) < D_L >$, where $U$ is a component with no crossings;
(iii) $\langle \chi \rangle = A\langle \chi \rangle + A^{-1}\langle \chi \rangle$ where this refers to three diagrams identical except at one
    crossing where they look as shown.

This is a regular isotopy invariant of link diagrams.

In order to define a three-manifold invariant using the bracket, Lickorish obtains an
expression which is invariant under Kirby moves on link diagrams. Before we can state
Lickorish’s result we need the following definition.

**Definition**: For a diagram $D_L$ representing an $n$ component framed link $[L, f]$, and a given
$n$-tuple of nonnegative integers $c = [c(1), c(2), \ldots, c(n)]$, a $c$-cable $c \ast D_L$ is defined as the
diagram obtained by replacing the $i$th component of $L$ in $D_L$ by $c(i)$ copies all parallel in
the plane.

As in [13], let $r$ be a fixed integer, $r \geq 3$, and let $C(n, r)$ denote the set of all functions
$c: \{1, 2, \ldots, n\} \to \{0, 1, \ldots, r - 2\}$. Let $A$ be a primitive $4r$-th root of unity. Let $G = G(A)$
be the Gauss sum $\sum_{l=1}^{4r} A^{l^2}$ and let $\bar{G}$ denote the complex conjugate of $G$.

**Proposition** [12] [13] : Let $M$ be a three-manifold obtained from $S^3$ by surgery on an
$n$-component framed link represented by a diagram $D_L$, and let $\sigma$ and $\nu$ be the signature
and the nullity of the linking matrix, respectively. Then

$$F_l(M) = \kappa \sum_{c \in C(n, r)} \lambda_{c(1)}\lambda_{c(2)}\ldots,\lambda_{c(n)}\langle \ast c D_L \rangle$$ (2.2)

is an invariant of the three-manifold with $\kappa$ and $\lambda_c$ given by

$$\kappa = (-1)^{r+1} A^6(\bar{G}/G),$$ (2.3)

$$\lambda_c = 2G^{-1} A^{r^2+3} \sum_{0 \leq 2j \leq r-2-c} (-1)^{c+j} \left( \frac{c+j}{j} \right) \left( A^{2(c+2j+1)} - A^{-2(c+2j+1)} \right).$$ (2.4)

In [12] only the existence and uniqueness of the $\lambda_c$ was shown without giving any
method of computation. The formulas (2.3) and (2.4) for $\kappa$ and $\lambda_c$, $0 \leq c \leq n$, were
obtained in [13]. The equivalence of Lickorish invariant to Reshetikin-Turaev invariant as
in the Kirby-Melvin [10] formulation is established in [13] for $A = -e^{\frac{2\pi i}{4r}}$.  

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In spite of having these formulas the computation of the invariant $F_l$ is quite difficult as one has to compute bracket polynomials of the c-cables which is very cumbersome. As mentioned in [12] if $r = 6$, and $D_L$ is a standard diagram of the trefoil with only 3 crossings, the computation of $<4\ast D_L>$ using the definition of the bracket as given above would involve $2^{48}$ operations. In the next two Sections, we will concentrate on the link invariants from Chern-Simons field theory with the motivation of finding an easier method of computing the bracket polynomials of cables.

3. SU(2) Chern-Simons Theory and Link Invariants

Now, we briefly present the methods employed in the direct evaluation of Chern-Simons link invariants [1,15-19].

The metric independent action $S$ of the SU(2) Chern-Simons theory on any three-manifold $M^3$ is given by

$$S = \frac{k}{4\pi} \int_{M^3} A \wedge dA + \frac{2}{3} A \wedge A \wedge A$$

(3.1)

where $A$ is a one-form (matrix-valued in the Lie algebra of a compact semi-simple Lie Group SU(2)). Explicitly, $A = A^a T^a dx^\mu$ where $T^a$ are the SU(2) generators and $k$ is the coupling constant.

The Wilson loop operators of any link $L$ embedded in $M^3$ is given by

$$W_{R_1 R_2 ... R_n} (L) = \prod_i W_{R_i} (C_i) = \prod_i \{ Tr_{R_i} P \exp \oint_{C_i} A_\mu dx^\mu \}$$

(3.2)

where $C_i$’s are the component knots of the link $L$ and $Tr_{R_i}$ refers to the trace over the SU(2) representation $R_i$ placed on the component $C_i$.

The link invariants are given by the expectation value of the Wilson loop operator:

$$P_{2R_1, 2R_2, ..., 2R_n} [D_L] = \langle W_{R_1, R_2, ..., R_n} (L) \rangle = \frac{\int [DA_\mu] \prod_i W_{R_i} (C_i) e^{iS}}{\int [DA_\mu] e^{iS}}$$

(3.3)

where $D_L$ denotes a diagram representing the framed link $L$ in vertical or black-board frame. This functional integral over the space of matrix valued one forms $A$ is evaluated by exploiting the connection between the Chern-Simons theory in three dimensional space with boundary and the SU(2)$_k$ Wess-Zumino conformal field theory on the two dimensional boundary [1]. The computation of these invariants has been considered in detail in Refs. [15-19]. We summarize this below.
We represent a link as the closure of a braid. The braid group $B_m$ consisting of $m$-strand braids is generated by $b_i, 1 \leq i \leq m - 1$, where $b_i$ represents a right-handed half-twist between the $i$-th strand and the $i + 1$-st strand. The inverse $b_i^{-1}$ corresponds to a left-handed half-twist. For a standard reference on braid groups see [22]. In order to illustrate the technique of direct evaluation of link invariant (3.3), we will take the example of the Hopf link. The details we present in this example are general enough for deriving invariants of links obtained from a four-strand braid either by plaiting or capping. In Theorem 1 we generalise the technique to arbitrary links.

Consider the Hopf link $H$, obtained from a four-strand braid (drawn as a closure of a two-strand braid), embedded in $S^3$ as shown in Fig.1. Let $j_1$ and $j_2$ denote the representations placed on the two component knots of $H$.

![Fig.1 Hopf Link](image)

Let us slice the three-manifold $S^3$ into two three dimensional balls as shown in Fig.2(a) and (b). The two dimensional $S^2$ boundaries of the three-balls are oppositely oriented and have four points of intersections with the braid, which we refer to as four punctures.
Now, exploiting the connection between Chern-Simons theory and Wess-Zumino conformal field theory, the functional integrals of these three-balls correspond to states in the space of four point correlator conformal blocks of the Wess-Zumino conformal field theory \[1\]. The dimensionality of this space is dependent on the representation of \( SU(2) \) placed on the strands and the number of punctures on the boundary. In the present example, the dimension of the space is

\[
\min(2j_1 + 1, 2j_2 + 1, k - 2j_1 + 1, k - 2j_2 + 1)
\]  

These states can be written in a suitable basis. Two such choices of bases (|\( \phi_s^{side} \rangle \)), and (|\( \phi_t^{cent} \rangle \)) are pictorially depicted in Fig.3(a),(b). Here \( s \in j_1 \otimes j_2 \) and \( t \in \min(j_1 \otimes j_1, j_2 \otimes j_2) \) where \( \otimes \) (also called tensor product notation) is defined as:

\[
\hat{j}_1 \otimes \hat{j}_2 = |\hat{j}_1 - \hat{j}_2| \oplus |\hat{j}_1 - \hat{j}_2| + 1 \oplus \ldots, \oplus \min(k - \hat{j}_1 - \hat{j}_2, \hat{j}_1 + \hat{j}_2)
\]  

with \( \oplus \) usually referred to as direct sum.
The basis $|\phi_{\text{side}}^s\rangle$ is chosen when the braiding is done in the side two parallel strands. In other words, it is the eigen basis corresponding to the generators $b_1$ and $b_3$:

$$b_1^2|\phi_{\text{side}}^s\rangle = b_3^2|\phi_{\text{side}}^s\rangle = (\lambda_{s,R}(j_1,j_2)^{(+)})^2|\phi_{\text{side}}^s\rangle \quad (3.6)$$

with the eigenvalues $\lambda_{s,R}^{(+)}(j_1,j_2)$ for the left-handed and right-handed half-twists in parallel strands (vertical framing) being:

$$\lambda_{s,R}^{(+)}(j_1,j_2) = \left(\lambda_{s,L}^{(+)}(j_1,j_2)\right)^{-1} = (-1)^{j_1+j_2-q}|j_1(j_1+1)+j_2(j_2+1)-s(s+1)|/2. \quad (3.7)$$

Similarly for braiding in the middle two anti-parallel strands $b_2$, we choose the basis $|\phi_{\text{cent}}^t\rangle$:

$$b_2^2|\phi_{\text{cent}}^t\rangle = (\lambda_{t,R}(j_1,j_2)^{(-)})^2|\phi_{\text{cent}}^t\rangle \quad (3.8)$$

with the eigenvalues in anti-parallel strands being:

$$\left(\lambda_{t,R}^{(-)}(j_1,j_2)\right)^{-1} = \lambda_{t,L}^{(-)}(j_1,j_2) = (-1)^{|j_1-j_2|}\left(q|j_1(j_1+1)+j_2(j_2+1)-t(t+1)|/2 . \right. \quad (3.9)$$

These two bases are related by a duality matrix

$$\left( a_{st} \begin{pmatrix} j_1 & j_2 \\ j_2 & j_1 \end{pmatrix} \right), \ s \in j_1 \otimes j_2, \ t \in \min(j_1 \otimes j_1, j_2 \otimes j_2),$$

defined as:

$$|\phi_{\text{side}}^s\rangle = a_{st} \begin{pmatrix} j_1 & j_2 \\ j_2 & j_1 \end{pmatrix} |\phi_{\text{cent}}^t\rangle \quad (3.10)$$

The matrix elements of the duality matrix are the $SU(2)$ quantum Racah coefficients which are known [18,8].
For example, the duality matrix for $j_1 = j_2 = \frac{1}{2}$ (defining/ fundamental representation of $SU(2)$) is a $2 \times 2$ matrix:

$$
\frac{1}{[2]} \begin{pmatrix}
-1 & \sqrt{3} \\
\sqrt{3} & 1
\end{pmatrix}
$$

where the number in square bracket refers to the quantum number defined as

$$
[n] = \frac{q^{\frac{s}{2}} - q^{-\frac{s}{2}}}{q^{\frac{s}{2}} - q^{-\frac{s}{2}}}
$$

with $q$ (also called deformation parameter in quantum algebra $SU(2)_q$) related to the coupling constant $k$ as $q = \exp(\frac{2i\pi}{k+1})$. We will see that the invariants are polynomials in $q$.

Now, let us determine the states corresponding to Figs. 2(a) and (b). Since the braiding is in the side two parallel strands, it is preferable to use $|\phi^{side}_s\rangle$ as basis states. Let $|\Psi_1\rangle$ be the state corresponding to Fig. 2(a). Clearly, we can write the state for Fig. 2(b) as

$$
|\Psi_2\rangle = b_1^2 |\Psi_1\rangle
$$

This state should be in the dual space as its $S^2$ boundary is oppositely oriented compared to the boundary in Fig. 2(a). Then the link invariant is given by

$$
P_{2j_1,2j_2}[D_H] = \langle \Psi_1 | b_1^2 | \Psi_1 \rangle
$$

For determining the polynomial, we will have to express the states as linear combination of the basis states $|\phi^{side}_s\rangle$. The coefficients in the linear combination are chosen such that

$$
\langle \Psi_1 | \Psi_1 \rangle = P_{2j_1,2j_2}[D_{U^2}] = [2j_1 + 1][2j_2 + 1],
$$

where $P_{2j_1,2j_2}[D_{U^2}]$ gives the polynomial of the unlink with 2 components.

The above mentioned restrictions determine the state $|\Psi_1\rangle$ (see [17]) as:

$$
|\Psi_1\rangle = \sum_{s = |j_1 - j_2|}^{\min(j_1 + j_2, k - j_1 - j_2)} \sqrt{[2s + 1]} |\phi^{side}_s\rangle
$$

Substituting it in eqn. (3.14) and using the braiding eigenvalue (3.7), we obtain

$$
P_{2j_1,2j_2}[D_H] = \sum_{s = |j_1 - j_2|}^{\min(j_1 + j_2, k - j_1 - j_2)} [2s + 1] (\lambda^{(+)}_{s,R}(j_1, j_2))^2.
$$

\[3\] We work in the unknot polynomial normalisation $P_{2j}[D_U] = [2j + 1]$ with the representation $j$ placed on unknot. The square bracket denotes the quantum number (3.12).
For \( j_1 = j_2 = 1/2 \), we get the following polynomial.

\[
P_{1,1}[D_H] = q^{3/2} + (q + 1 + q^{-1})q^{-1/2} = q^{3/2} + q^{1/2} + q^{-1/2} + q^{-3/2}.
\] (3.17)

The bracket polynomial for Hopf link, represented by diagram \( D_H \), obtained by the recursive method is

\[
\langle D_H \rangle = A^6 + A^2 + A^{-2} + A^{-6} = P_{1,1}[D_H] \mid_{q^{1/4} = -A}.
\] (3.18)

The bracket polynomial \( \langle D_L \rangle (A) \) and Jones’ polynomial \( V[L](q) \) for any \( n \)-component link are related as given below (see, for instance, [13] which uses a different normalisation\(^4\))

\[
(-1)^n V[L](q) \mid_{q^{1/4} = -A} = (-A)^{3\omega} \langle D_L \rangle (A),
\] (3.19)

where \( \omega \) is the writhe or the sum of crossing signs\(^5\) in the diagram \( D_L \).

The Jones’ polynomial is obtained by placing the defining representation \( j_1 = j_2 = 1/2 \) on the component knots in standard framing. The standard framing braiding eigenvalue for a right-handed half-twist in parallel strands \( \tilde{\lambda}_{r,R}^{(+)}(j_1, j_2) \) is related to the corresponding vertical framing eigenvalue as

\[
\tilde{\lambda}_{r,R}^{(+)}(1/2, 1/2) = q^{3/2} \lambda_{r,R}^{(+)}(1/2, 1/2).
\] (3.20)

This equation determines the frame correction factor between the ambient isotopy invariant and the regular isotopy invariant. Using (3.16), it is clear that for the Hopf link the Jones’ polynomial \( V[H] \) is related to the invariant in vertical framing as

\[
V[H] = \sum_{s=0}^{\min(1,k-1)} [2s + 1](\tilde{\lambda}_{s,R}^{(+)}(j_1, j_2))^2 = q^{3/2} P_{1,1}[D_H].
\] (3.21)

For the diagram in Figure 1, \( \omega = 2 \), and the number of components \( n = 2 \). Combining (3.19) and (3.21) we get \( P_{1,1}[D_H] \mid_{q^{1/4} = -A} = \langle D_H \rangle \) which confirms eqn. (3.18).

The method elaborated above for a specific example can be generalised for any \( n \)-component link obtained as a closure of an \( m \)-strand braid \((n \leq m)\). We will briefly outline the steps below.

\(^4\) We work in the unknot normalisation : \( V[U] = (q^{1/2} + q^{-1/2}), \langle D_U \rangle = -(A^2 + A^{-2}) \).

\(^5\) In standard literature \( \omega \) in the exponent may appear with a negative sign. This is a matter of replacing the variable \( q \) with \( q^{-1} \).
**Theorem 1**: For a diagram $D_L$ of an $n$-component link the bracket polynomial and the invariant in vertical framing are related as

$$\langle D_L \rangle \mid_{A=-q^\frac{1}{4}} = (-1)^n P_{1,1,\ldots,1}[D_L]. \quad (3.22)$$

**Proof:**

Let us take an arbitrary braid word in the braid group $B_{2m}$ denoted by the black box $P$ as shown in Fig. 4. Here, the dotted lines denote the closure and the dashed line represents slicing of $S^3$ into two pieces with oppositely oriented $S^2$ boundaries. Note that the last $m$ strands are trivial, so this can also be considered the closure of an $m$-braid.

**FIG. 4**

**FIG. 5**
The states on the $2m$ punctured surface corresponding to the two three-balls can be expanded in a suitable basis (see Fig. 5):

$$|\Psi_1\rangle = \sum_{t_0, t_1, \ldots, t_{2m-4}} A_{j_1, \ldots, j_{2m}, t_0, t_1, \ldots, t_{2m-4}} [P] |\phi_{t_0, t_1, \ldots, t_{2m-4}}\rangle$$

$$\langle \Psi_2 | = \sum_{t_0, t_1, \ldots, t_{2m-4}} B_{j_1, j_2, \ldots, j_{2m}, t_0, t_1, \ldots, t_{2m-4}} \langle \phi_{t_0, t_1, \ldots, t_{2m-4}} |$$

(3.23)

where the summation variables $t_i \in t_{i-1} \otimes j_{i+2}$ (3.5) with $t_{-1} = j_1$ and $t_{2m-3} = j_{2m}$.

The element $A$ in the summation depends on the braid word $P$ and $B$ is chosen such that $\langle \Psi_2 | \Psi_2 \rangle$ gives $\prod_{i=1}^{m} [2j_i + 1]$.

There are various ways to obtain a link by closing a braid. Some such ways are discussed in [18]. The closure as shown in Fig. 4 will demand that

$$(j_{m+1}, j_{m+2}, \ldots, j_{2m}) = (j_m, j_{m-1}, \ldots, j_1).$$

Additional restrictions on $j_i$ are dictated by the braid word $P$ and the $n$-component link. The link invariant is well-defined only if all strands which correspond to the same component of the link are marked with the same $j_i$. (i.e.), at most $n$ of the $j_i$'s can be different.

With these inputs, any $n$-component link invariant will be

$$P_{2j_1, 2j_2, \ldots, 2j_n} = \langle \Psi_2 | \Psi_1 \rangle$$

(3.24)

In order to compare this invariant with the Jones’ polynomial, we choose $j_1 = j_2 = \ldots = j_n = 1/2$ and as a consequence of the frame correction factor (3.20) between ambient isotopy (standard framing) and regular isotopy (vertical framing) we have

$$V[L] = (q^{\frac{3}{4}})^{\omega} P_{1,1,\ldots,1}[D_L].$$

(3.25)

Hence from eqn. (3.19) we have the result (3.22) (Proved).

The generalised link polynomials (3.24) of some of the knots up to eight crossings and two component links have been tabulated in Appendix II of [18]. We have to use vertical framing braiding eigenvalues (3.7), (3.9) instead of the standard framing eigenvalues.

With these regular isotopy polynomial invariants, a three-manifold invariant which respects Kirby’s theorem has been constructed in [16]; the formula being

$$F_k(M) = \alpha^{-\sigma[L]} \sum_{c \in C(n, k+2)} \mu_c(1) \mu_c(2) \ldots, \mu_c(n) P_{c(1), c(2), \ldots, c(n)}[D_L],$$

(3.26)
where
\[ \mu_c = \frac{1}{2i} \sqrt{\frac{2}{k+2}} \left( q^{\frac{c+1}{2}} - q^{-\frac{(c+1)}{2}} \right), \quad \alpha = (q^\frac{1}{2})^{\frac{3k}{4}} \]
and \( \sigma \) denotes the signature of the linking matrix in a framed link representation of the manifold \( M \).

It is not a priori clear whether this formula gives the same invariant as the Lickorish invariant (2.2) with the polynomial variables related as in eqn. (1.1). In order to verify the equality, we need to find a method of determining brackets of cables in terms of the invariant \( P_{c(1),c(2),...,c(n)} \).

In the next section, we will present the representation theory of composite braiding which will prove useful to directly compute the invariants of cables of link diagrams.

4. Composite Braiding and c-cable Link invariants

The representation theory of composite braiding involves determination of braiding eigenvalues, eigen basis and the duality matrix. One such representation in standard framing was presented in [19] in an attempt to distinguish a class of knots called mutants. In this paper, we have a slightly different composite braiding. The braiding eigenvalues and eigen basis are derived in a similar fashion as in Ref. [19].

**Definition:** A \( c \)-Composite of a given braid is obtained by replacing every strand by \( c \)-strands and the generator \( b_i \) by a composite braiding \( B_i^{(r)} \)

\[ B_i^{(c)} = b(ci, ci + c - 1)b(ci - 1, ci + r - 2) \ldots b(ci - c + 1, ci) \]

where \( b(i, j) = b_i b_{i+1} \ldots b_j \).

For convenience we will call the original braid an **elementary braid** and the new one the **composite braid**. When we are dealing with a link it is possible to replace different components by a different number of parallel copies. In order to handle that case we have to consider mixed composite braids which we describe below.

**Definition:** Let \( c = (c_1, c_2, \ldots, c_n) \). An \( c \)-composite braid of an elementary braid is obtained by replacing the \( i \)-th strand by \( c_i \)-strands and the generator \( b_i \) by \( B_i^{(c)} \)

\[ B_i^{(c)} = b(\sum_{j=1}^{i} c_j - 1, \sum_{j=1}^{i+1} c_j - 1)b(\sum_{j=1}^{i} c_j - 1, \sum_{j=1}^{i+1} c_j - 2) \ldots b(\sum_{j=1}^{i} c_j - c_i + 1, \sum_{j=1}^{i+1} c_j - c_i) \]

\[ \text{[3.24]} \]

\[ \text{[4.1]} \]

\[ \text{[4.2]} \]

\[ \text{[6] The presence or absence of the nullity \( \nu \) of the linking matrix for the framed link appears to be a matter of normalisation as \( \nu \) is unchanged under both the Kirby moves.} \]
Clearly, for \( c_1 = c_2 = \ldots = c_n = c \), \( B_i^{(e)} \) is the same as \( B_i^{(e)} \) (4.1).

We shall again take the Hopf link from Fig.1 as an example and work out the details for the \((2, 3)\)-cable. Let us denote the resulting diagram as \( (2, 3) \ast D_H \) and the corresponding field theory invariant in vertical framing as \( P_{\{1\},\{1\}}[(2, 3) \ast D_H] \). This notation implies that \( j_1 \) is the representation of \( SU(2) \) placed on all the elementary strands constituting the \( r_1 = 2 \) bunch of strands and \( j_2 \) on all the elementary strands in the \( r_2 = 3 \) bunch.

We are interested in determining the invariant \( P_{\{1\},\{1\}}[(2, 3) \ast D_H] \) so that using (3.22), we get the bracket polynomial \( \langle (2, 3) \ast D_H \rangle \).

We present the steps, analogous to the ones in Section 3, to evaluate \( P_{\{1\},\{1\}}[(2, 3) \ast D_H] \) which is obtained by gluing the two three-balls as shown in Fig. 6(a) and (b). Clearly, the boundary is a ten-punctured surface as against the earlier elementary case considered in Section 3. So, the functional integrals on these three-balls corresponds to states in the space of ten-point correlator conformal blocks in the Wess-Zumino conformal field theory. The basis is so chosen that it is the eigen basis of the composite braiding operator \( B_1^{(2, 3)} \).

Using the elementary braiding eigenvalues (3.7), duality matrix (3.11), and some of the properties of the duality matrix which are given in Appendix I of [18], it can be shown that the eigen basis for \((2, 3)\)-mixed braiding in the side strands is \( |\phi_{(l_1,(n_1,l_2),m),((n_2,l_3),l_4,m)}^{\text{side}} \rangle \) with eigenvalue \( \lambda_{m,R}^{(+)}(l_1, l_2) \) (3.7) (i.e.),

\[
B_1^{(3,2)} B_1^{(2,3)} |\phi_{(l_1,(n_1,l_2),m),((n_2,l_3),l_4,m)}^{\text{side}} \rangle = [\lambda_{m,R}^{(+)}(l_1, l_2)]^2 |\phi_{(l_1,(n_1,l_2),m),((n_2,l_3),l_4,m)}^{\text{side}} \rangle \quad (4.3)
\]

The derivation of composite basis states and eigenvalues is along a similar direction as elaborated in Appendix of [20] for a different 2-composite braiding.
Similarly, for composite braiding \((3,3)\) in the middle two strands, we choose the basis \(|\phi_{l_1,(n_1,l_2),(n_2,l_3),n,l_4}^{\text{cent}}\rangle\). These basis states are pictorially depicted in Fig. 7(a) and (b).

Fig. 7

For \(j_1 = j_2 = \frac{1}{2}\),

\[ l_1, n_1, n_2, l_4 \in \left(\frac{1}{2} \otimes \frac{1}{2}\right) ; l_2 \in (n_1 \otimes \frac{1}{2}) \text{ and } l_3 \in (n_2 \otimes \frac{1}{2}) ; m \in \min(l_1 \otimes l_1, l_2 \otimes l_2) ; n \in (l_1 \otimes l_2). \]

The two bases in Fig. 7(a) and (b) are related by the duality matrix

\[ |\phi_{\text{side}}^{l_1,(n_1,l_2),m}((n_2,l_3),l_4,m)\rangle = a_{mn} \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix} |\phi_{\text{cent}}^{l_1,(n_1,l_2),(n_2,l_3),n,l_4}\rangle \]

(4.4)

With these ingredients, it is clear that the state for Fig. 6(a) is:

\[ |\Psi^{(2,3)}_1\rangle = \sum_{l_2 \in n_1 \otimes \frac{1}{2}} \sum_{l_1,n_1} \sum_{m=|l_1-l_2|} \sqrt{2m + 1} |\phi_{\text{side}}^{l_1,(n_1,l_2),m}((n_2,l_3),l_4,m)\rangle \cdot (4.5) \]

The restriction \(l_1 = l_4, n_1 = n_2, l_2 = l_3\) in the basis states is obtained as a consequence of closure of the braid to obtain the link. The coefficients in the linear combination are obtained from the fact that \(\langle \Psi^{(2,3)}_1 |\Psi^{(2,3)}_1\rangle = P_{1,1,1,1}[D_{U^5}] = [2]^{5}\) where \(D_{U^5}\) is the unlink with five components.

Using (4.5), the state corresponding to Fig. 6(b) will be:

\[ |\Psi^{(2,3)}_2\rangle = B^{(3,2)}_1 B^{(2,3)}_1 |\Psi^{(2,3)}_1\rangle \]

\[ = \sum_{l_1,l_2,n_1} \sum_{m=|l_1-l_2|} \sqrt{2m + 1} \lambda^{(+)}_{m,R}(l_1,l_2) |\phi_{\text{side}}^{l_1,(n_1,l_2),m}((n_2,l_3),l_4,m)\rangle \cdot (4.6) \]

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The summation over \( n_1 \) can be suppressed but we should remember that \( l_2 \in (\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2}) \).

Now that we have determined the states for Fig. 6(a) and (b), the invariant for the (2, 3)-cable of Hopf link can be rewritten in terms of elementary Hopf link invariants (3.16):

\[
P_{\{1\},\{1\}}[(2, 3) \ast D_H] = \langle \Psi_{\{1\}}^{(2, 3)} | \Psi_{\{2\}}^{(2, 3)} \rangle = \sum_{l_1, l_2} P_{2l_1, 2l_2}[H] ,
\]

where \( l_1 = 0, 1 \) and \( l_2 = 1/2, 1/2, 3/2 \) for \( k \geq 3 \). The explicit form of the polynomial is:

\[
P_{\{1\},\{1\}}[(2, 3) \ast D_H] = 2P_{0, 1}[D_H] + P_{0, 3}[D_H] + 2P_{2, 1}[D_H] + P_{2, 3}[D_H]
\]

\[
= 2(q^{\frac{1}{2}} + q^{\frac{1}{2}}) + (q^{\frac{1}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{3}{2}})
\]

\[
+ 2\left(q^2(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) + q^{-1}(q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{3}{2}})\right)
\]

\[
+ \left(q^5(q^{\frac{1}{2}} + q^{-\frac{1}{2}}) + q^2(q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-\frac{3}{2}})\right)
\]

\[
+ q^{-3}(q^{\frac{3}{2}} + q^{\frac{1}{2}} + q^{\frac{1}{2}} + q^{-\frac{1}{2}} + q^{-3} + q^{-\frac{3}{2}})
\]

\[
= q^{\frac{1}{2}} + q^{\frac{3}{2}} + q^{\frac{1}{2}} + 3q^{\frac{3}{2}} + 4q^{\frac{1}{2}} + 6q^{\frac{1}{2}}
\]

\[
6q^{-\frac{1}{2}} + 4q^{-\frac{3}{2}} + 3q^{-\frac{1}{2}} + 9q^{-\frac{3}{2}} + q^{-1} + q^{-\frac{1}{2}}
\]

The following relation is easy to check by computing the bracket polynomial using the recursive method:

\[
P_{\{1\},\{1\}}[(2, 3) \ast D_H]|_{(q^{\frac{1}{2}} = -A)} = -\langle (2, 3) \ast D_H \rangle .
\]

This is expected from Theorem 1 for the five component link.

Now we generalise the technique used here for cables of arbitrary link diagrams. A bold face lower case letter will indicate an \( n \)-tuple of numbers, for ex., \( c = (c_1, c_2, \ldots, c_n) \). Using Theorem 1 we obtain:

**Theorem 2:** The bracket polynomial of a \( c \)-cable of the diagram of an \( n \)-component link can be expressed in terms of elementary link invariants in vertical framing (3.24); the exact relation being:

\[
\langle c \ast D_{L_n} \rangle = (-1)^{c_1 + c_2 + \ldots + c_n} \left( \sum_{l_1, l_2, \ldots, l_n} P_{2l_1, 2l_2, \ldots, 2l_n}[D_{L_n}] \right) |_{(q^{\frac{1}{2}} = -A)}
\]

where \( l_i \) takes values in

\[
l_i \in \left( \frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \right) \ldots \frac{1}{2} ,
\]

\[
\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \ldots \frac{1}{2} ,
\]

\[
(\frac{1}{2} \otimes \frac{1}{2} \otimes \frac{1}{2} \ldots \frac{1}{2}).
\]

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Proof: Consider an \( c \)-cable of an \( n \)-component link \( L \). Suppose that \( L \) is obtained from closure of an elementary \( m \)-braid. Let us replace each strand corresponding to the \( i \)-th component of the link by \( c_i \) parallel strands, \( 1 \leq i \leq n \). This gives a mixed composite braid say \( (c'_1, c'_2, \ldots, c'_{2m}) \) where as a set \( \{c'_1, c'_2, \ldots, c'_{2m}\} \) is the same as \( \{c_1, c_2, \ldots, c_n\} \).

We will place defining representation \( j = 1/2 \) on all the elementary strands. Closure of composite braids forces,

\[
(c'_{m+1}, \ldots, c'_{2m}) = (c'_m, c'_{m-1}, \ldots, c'_1).
\]

Again writing the states for the mixed composite braiding in a suitable basis (See Fig. 8):

\[
|\Psi^{(c_1, c_2, \ldots, c_n)}\rangle = \sum_{l_1, l_2, \ldots, l_{2m}} \sum_{t_0, t_1, \ldots, t_{2m-4}} A_{l_1, l_2, \ldots, l_{2m}, t_0, t_1, \ldots, t_{2m-4}} \left| \tilde{\phi}_{l_1, l_2, t_0, t_1, \ldots, t_{2m-4}} \right|
\]

\[
\langle\Psi^{(c_1, c_2, \ldots, c_n)}| = \sum_{l_1, l_2, \ldots, l_{2m}} \sum_{t_0, t_1, \ldots, t_{2m-4}} B_{l_1, l_2, \ldots, l_{2m}, t_0, t_1, \ldots, t_{2m-4}} \langle \tilde{\phi}_{l_1, l_2, t_0, t_1, \ldots, t_{2m-4}} |\]

where \( t_i \in (t_{i-1} \otimes l_{i-2}) \) with \( t_{-1} = l_2, t_{2m-3} = l_{2m} \) and \( l_i \)'s as in (4.11). The closure of the braid demands:

\[
(l_{m+1}, \ldots, l_{2m}) = \mathcal{P}(l_1, l_2, \ldots, l_m).
\]

The constraint of closing the braid to give an \( n \)-component link, \( n \leq m \) requires that at most \( n \) of the \( l_i \)'s are distinct, and the rest are determined by the link under consideration.

Incorporating the above mentioned conditions and also using (3.24), we get the composite invariant to be:

\[
P_{\{1, \{1, \ldots, 1\}|} = \langle \Psi^{(c_1, c_2, \ldots, c_n)}| = \sum_{l_1, l_2, \ldots, l_n} P_{2l_1, 2l_2, \ldots, 2l_n} |D_{L_n}\rangle. \quad (4.13)
\]
It is evident that the number of components in the $c$-cable of the $n$-component link is $c_1 + c_2 + \ldots c_n$. Hence Theorem 1 gives the result (4.10) (Proved).

We expand all the tensor products in (4.11) using (3.5) to get some useful relations for proving the equality of the two apparently distinct three-manifold invariants defined by Lickorish and Kaul. The closed form expression for the tensor product for $c_i \leq k$ turns out to be

$$l_i \in \bigoplus_{s \geq 0} a_s j_s - \bigoplus_{t \geq 0} a_t j_t$$

(4.14)

where representations $j_s$ and $\hat{j}_t$ are given by $2j_s = c_i - 2s$ and $2\hat{j}_t = 2t - c_i - 2$ and $a_s = \left(\frac{c_i - 1}{s}\right)$ are the constants.

We use this tensor product expansion to rewrite the bracket of an $c$-cable in the following Corollary.

**Corollary 1:** With $q^{\frac{1}{4}} = -A$, the bracket polynomial of the $c$-cable of an $n$-component link diagram $D$, where $c = (c_1, c_2, \ldots, c_n)$, is given by

$$\langle c \ast D \rangle = (-1)^{c_1 + c_2 + \ldots + c_n} \left\{ \sum_{(s_1, s_2, \ldots, s_n)} (\Pi_{i=1}^n A_{c_i, s_i}) P_{s_1, s_2, \ldots, s_n}[D] \right\}$$

(4.15)

where the $\{s_i\}$ are subjected to

$$r_i - s_i \text{ even }, \ 0 \leq s_i \leq c_i,$$

and

$$A_{c_i, s_i} = \left\{ \begin{array}{ll} \left(\frac{c_i - 1}{2}\right) - \left(\frac{c_i - 1}{2}\right), & \text{if } 0 \leq s_i \leq c_i - 4, \\
\left(\frac{c_i - 1}{2}\right), & \text{if } c_i - 3 \leq s_i \leq c_i. \end{array} \right.$$  

(4.16)

The inverse of Corollary 1 expressing elementary invariants $P_{l_1, l_2, \ldots, l_n}[D]$ in terms of the composite invariants $\langle c \ast D \rangle$ will be useful in proving the equivalence between Lickorish and Kaul’s three-manifold invariants. Hence inverting (4.15) we obtain the following:

**Theorem 3:** For a diagram $D$ of an $n$-component link, the invariant $P_{l_1, l_2, \ldots, l_n}[D]$ can be expressed in terms of bracket polynomials as follows.

$$P_{l_1, l_2, \ldots, l_n}[D] = \sum_{j} \left\{ \Pi_{i=1}^n (-1)^{l_i - j_i} \left(\frac{l_i - j_i}{j_i}\right)^t \langle 1 \ast 2j \rangle D \right\}$$

(4.17)
Proof: We use Corollary 1 to rewrite the RHS of the above equation. This changes (4.17) to:
\[ P_{l_1,l_2,\ldots,l_n}[D] = \sum_{j_0 \leq j_1 \leq l_i} \sum_s \prod_{i=1}^n \left\{ (-1)^{j_i} \left( \frac{l_i - j_i}{j_i} \right) A_{l_i - 2j_i,s_i} \right\} P_{s_1,s_2,\ldots,s_n}[D], \quad (4.18) \]
where \( s = (s_1, s_2, \ldots, s_n) \) is such that \( l_i - s_i \) is even and \( 0 \leq s_i \leq l_i - 2j_i \). Let us make a change of variable \( l_i - s_i = 2s'_i \), write each sum as a multiple sum, and interchange the summations to rewrite the statement we need to prove as:
\[ P_{l_1,l_2,\ldots,l_n}[D] = \sum \sum_{s_1',s_2',\ldots,s_n'} \prod_{i=1}^n \left\{ (-1)^{j_i} \left( \frac{l_i - j_i}{j_i} \right) A_{l_i - 2j_i,l_i - 2s'_i} \right\} P_{l_1 - 2s'_1,l_2 - 2s'_2,\ldots,l_n - 2s'_n}[D]. \quad (4.19) \]
We will work with these sums one at a time. Let
\[ S(s'_1) = \sum_{j_1=0}^{s'_1} (-1)^{j_1} \left( \frac{l_1 - j_1}{j_1} \right) A_{l_1 - 2j_1,l_1 - 2s'_1} P_{l_1 - 2s'_1,\ldots,l_n - 2s'_n}[D]. \]
Then the first sum in (4.19) equals
\[ \sum_{s'_1 = 0}^{[l_1/2]} S(s'_1), \]
where \([l_1/2]\) denotes the greatest integer less than or equal to \( l_1/2 \). We split this into two sums \( \sum_{s'_1 = 0}^{\min\{[l_1/2],1\}} \) and \( \sum_{s'_1 = 2}^{[l_1/2]} \) and use eqn.(4.16) to substitute for the \( A_{l_i - 2j_i,s_i} \). It is easy to see that:
\[ \sum_{s'_1 = 0}^{\min\{[l_1/2],1\}} S(s'_1) = P_{l_1,l_2 - 2s'_2,\ldots,l_n - 2s'_n}[D]. \quad (4.20) \]
For \( 2 \leq s'_1 \leq l_1/2 \), using (4.16) we have:
\[ S(s'_1) = \sum_{j_1 = s'_1 - 1}^{s'_1} (-1)^{j_1} \left( \frac{l_1 - j_1}{j_1} \right) \left( \frac{l_1 - 2j_1 - 1}{s'_1 - j_1} \right) P_{l_1 - 2s'_1,\ldots,l_n - 2s'_n}[D] \]
\[ + (l_1 - 2s'_1 + 1) \times \]
\[ \sum_{j_1 = 0}^{s'_1 - 2} (-1)^{j_1} \left( \frac{l_1 - j_1}{j_1} \right) \left( \frac{l_1 - j_1 - 1}{s'_1 - j_1} \right) \ldots \left( \frac{l_1 - j_1 - s'_1 + 2}{s'_1 - j_1} \right) P_{l_1 - 2s'_1,\ldots,l_n - 2s'_n}[D]. \quad (4.21) \]
We claim that this equals zero. The theorem will follow by treating the rest of the sums in eqn. (4.19) similarly.

In order to prove the claim first note that

\[
\sum_{j_1=0}^{s'_1} \frac{1}{j_1!(s'_1 - j_1)!} (-1)^{j_1} x^{j_1} = \frac{(1 - x)^{s'_1}}{s'_1!}.
\]

Using this we see :

\[
\begin{align*}
\left( \sum_{j_1=0}^{s'_1} \frac{(-1)^{j_1}(l_1 - j_1)(l_1 - j_1 - 1) \ldots (l_1 - j_1 - s'_1 + 2)}{j_1!(s'_1 - j_1)!} \right) \\
\frac{d^{s'_1-1}}{ds'_1} \left[ \left( \sum_{j_1=0}^{s'_1} \frac{(-1)^{j_1} x^{j_1}}{j_1!(s'_1 - j_1)!} \right) x^{s'_1-l_1-2} \right] \\
= \text{Lt}_{x \to 1} \left( -1 \right)^{s'_1-1} \frac{d^{s'_1-1}}{ds'_1} \left\{ (1 - x)^{s'_1} x^{s'_1-l_1-2} \right\} = 0.
\end{align*}
\]

It follows that

\[
S(s'_1) = \sum_{j_1=s'_1-1}^{s'_1} \left[ (-1)^{j_1} \left\{ \left( \frac{l_1 - j_1}{j_1} \right) \left( \frac{l_1 - 2j_1 - 1}{s'_1 - j_1} \right) - \frac{(l_1 - 2s'_1 + 1) \times (l_1 - j_1)(l_1 - j_1 - 1) \ldots (l_1 - j_1 - s'_1 + 2)}{j_1!(s'_1 - j)!} \right\} \right]
\]

\[
P_{l_1-2s'_1,...,l_n-2s'_n} [D]
\]

A simple arithmetic shows that the expression in the RHS of (4.23) is 0. (Proved).

Now that we have given a direct method of determining the bracket polynomials of cables of link diagrams, we will show, in the next section, that the three-manifold invariants obtained from regular isotopy field theoretic invariants (3.26) are the same as (2.2) for the polynomial variables satisfying eqn. (1.1).

5. Conclusions

We have given a field theoretic presentation for bracket polynomials in terms of framed link invariants in SU(2) Chern-Simons theory with the polynomial variable obeying (1.1). Then, using representation theory of composite braids, we obtained a direct method of
evaluating bracket polynomials of cables of link diagrams. This enables us to show that
the three-manifold invariant obtained by Lickorish using the formalisation of the bracket
polynomial as Temperley-Lieb algebra and the invariant obtained by Kaul using generalised
link invariants from Chern-Simons theory are equal upto normalisation. However the
normalising factor depends on the choice of \( A \), the \( 4r \)-th root of unity.

In the discussion below \( \left( \begin{array}{c} a \\ b \end{array} \right) \) denotes the quadratic symbol for relatively prime integers \( a \) and \( b \), defined as [23]:

\[
\left( \begin{array}{c} a \\ b \end{array} \right) = \begin{cases} +1, & \text{if } a \equiv x^2 \text{ mod } b, \text{ for some integer } x, \\ -1, & \text{otherwise}. \end{cases}
\]  

(5.1)

**Theorem 4:** Let \( A = e^{\omega \pi i / 2r} \), where \( n \) is a positive integer relatively prime to \( 4r \) with \( r \) related to the coupling constant in field theory as \( k = r - 2 \). Lickorish’s invariant \( F_l \) obtained from the formalisation of the bracket polynomial as Temperley-Lieb algebra and
Kaul’s invariant \( F_k \) obtained using generalised link invariants from Chern-Simons theory,
for the polynomial variables obeying \((1.1)\), are related as:

\[
F_k(M) = \epsilon \kappa^{n/2} F_l(M), \text{ where}
\]

\[
\epsilon = \left( \begin{array}{c} r \\ n \end{array} \right) e^{(n-1)(r+1)\pi i / 2} = \pm 1.
\]  

(5.2)

(5.3)

**Proof:** It is an easy exercise in algebraic number theory (see [23]) to show that the
Gauss (quadratic) sum \( G = G(e^{\omega \pi i / 2r}) \) used in the definitions \((2.3)\) and \((2.4)\) is as given below.

\[
G = 2\sqrt{2r} \left( \begin{array}{c} r \\ n \end{array} \right) e^{\omega n \pi i / 2}.
\]  

(5.4)

Clearly, \( \frac{G}{G} = e^{-\omega \pi i / 2} \). Simplifying \((2.3)\) and using \((3.27)\) it is easy to see that

\[
\kappa = \alpha^{-2}.
\]  

(5.5)

Similarly using \((2.4)\) and \((3.27)\) we see that

\[
\lambda_l = \left( \begin{array}{c} r \\ n \end{array} \right) e^{(n-1)(r+1)\pi i / 2} \alpha^{-1} \sum_{j=0}^{L_2-2} (-1)^{l+j} \binom{l+j}{j} \mu_{l+2j}.
\]  

(5.6)
We use Theorem 3 to write $F_k$ in terms of brackets of cables of the diagram $D$ which represents the framed link associated with the manifold and compare the coefficients of $\langle c \ast D \rangle$. We see that

$$F_k(M) = \epsilon^n \kappa^{\frac{\nu}{2}} F_l(M).$$

Note that $\epsilon = \pm 1$, and $n$ is odd. The result follows. (Proved.)

It was shown [13] that with $A = -e^{\frac{i\pi}{2}}$ Lickorish’s invariant equals the Reshitikin Turaev invariant upto normalisation and a change of variable. So from Theorem 4, it follows that Kaul’s invariant defined using the generalised link invariants in vertical framing is a reformulation of the Reshetikin-Turaev invariant. The Kauffman-Lins invariant defined in Chapter 12 of [14] gives another normalization of the Witten-Reshetikin-Turaev invariant following Lickorish’s Temperley-Lieb algebra approach.

We shall rewrite the inferred result in the following corollary.

**Corollary 2:** The relationship between the Kauffman-Lins invariant $Z(M)$, which is Witten-Reshetikin-Turaev invariant upto a normalisation, and Kaul’s invariant $F_k$ is

$$F_k(M) = Z(M)/Z(S^3), \text{ where } Z(S^3) = \mu_0 = \sqrt{\frac{2}{r}} \sin\left(\frac{\pi}{r}\right). \quad (5.7)$$

We have shown by an indirect procedure that Kaul’s three-manifold invariant equals Witten’s partition function. It would be very interesting to see whether there is a direct method of deducing the above result.

**Acknowledgments:** We would like to thank Ashoke Sen for his valuable suggestions. We are also grateful to T.R. Govindarajan, R.K. Kaul, C. Livingston, and J. Prajapat for their comments.
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