Unfolded Scalar Supermultiplet

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Abstract

Unfolded equations of motion for $\mathcal{N} = 1$, $D = 4$ scalar supermultiplet are presented. We show how the superspace formulation emerges from the unfolded formulation. To analyze supersymmetric unfolded equations we extend the $\sigma$-cohomology technics to the case with several operators $\sigma_-$. The role of higher $\sigma$-cohomology in the derivation of constraints is emphasized and illustrated by the example of the scalar supermultiplet.
A remarkable feature of the unfolded formulation of partial differential equations [1, 2] is that for the case of universal unfolded equations [3], which in fact includes all known examples, their form is insensitive to a particular space-time where the fields live. Since the full system of unfolded equations keeps the same form both in space and in superspace it is elementary to promote any supersymmetric unfolded system from, say, Minkowski space to superspace. This idea has been already applied to the analysis of higher-spin gauge theory in superspace [4] and generalized space-time [5]. Although the form of unfolded equations remains the same, its reduction to the standard field-theoretical formulation essentially depends on the structure of background space-time usually described by a flat connection of the symmetry group of the model at hand. The standard machinery that uncovers the conventional field-theoretical pattern of one or another unfolded system, answering the questions what are independent dynamical fields, field equations, gauge transformations etc, is the so-called $\sigma_-$ technics [6].

This property of the unfolded equations makes them convenient for the systematic derivation (rather than guessing) the manifestly invariant form of $G$-invariant equations in one or another $G$-invariant space. In particular, this approach was used in [4] for superspace reformulation of nonlinear supersymmetric higher-spin field equations and in [5] for the derivation
of the manifestly $Sp(8)$ invariant field equations for $4d$ massless fields reformulated in the $Sp(8)$ invariant ten-dimensional space-time which is (the big cell of) the Lagrangian Grassmannian $\mathcal{M}_4$ with local coordinates $X^{AB} = X^{BA}$ ($A, B = 0, \ldots, 4$ are $4D$ Majorana spinor indices). Although the unfolded equations have similar form both in the case of $4d$ Minkowski space and in the the ten-dimensional space $\mathcal{M}_4$, the patterns of dynamical fields and field equations look differently in the two cases: in four dimensions, the system contains an infinite set of fields of different spins, that satisfy massless field equations, while in $\mathcal{M}_4$ the same system is described by a single scalar hyperfield $C(X)$ that satisfies certain second-order differential field equations; similarly, all massless fermions are described by a single spinor hyperfield $C_A(X)$, that satisfies certain first-order differential equations. By construction, the two systems are equivalent, describing the same degrees of freedom as was explicitly checked in [7].

The aim of this paper is to apply the methods of unfolded dynamics [1, 2, 3] to the simplest supersymmetric model in superspace [8, 9, 10], namely free scalar supermultiplet in four space-time dimensions. On the one hand, this provides an illustration of how the unfolded dynamics approach can be used to derive superfield formulations of supersymmetric models. On the other hand, supersymmetric models suggest an interesting generalization of the $\sigma_-$ technics at least in two respects.

One is that usually the operator $\sigma_-$, which is the negative grade part of the covariant derivative in the unfolded field equations, is assumed to have definite grade $-1$. We show that in supersymmetric models it is more convenient to consider the situation with several $\sigma_-$ operators that carry different negative grades. The corresponding generalization of the $\sigma_-$--cohomology technics suggested in this paper is rather straightforward via application of the spectral sequence machinery operating with cohomologies of $\sigma_-$ operators of higher grades on the cohomologies of those with lower grades.

Another comment is on the role of higher $\sigma_-$--cohomologies for the derivation of consequences of the field equations. In particular, we show how one can use the field equations associated with higher $\sigma_-$--cohomologies to distinguish between fundamental field equations and their consequences.

Although application of the presented machinery to the simplest supersymmetric model may look a bit too heavy especially in the case of the massive model\footnote{In fact, the unfolded formulation of the massless case is simple enough while its massive deformation can be reached in two ways. One is to deform fiber space constraints that relate traces of higher grade fields to the lower grade ones. Leading to the simpler form of equations, the analysis of the resulting system needs independent $H(\sigma_-)$ computation compared to the massless case. In the scheme followed in this paper we keep the same tracelessness fiber space constraints, modifying instead unfolded equations which turn out to be a bit more involved. The benefit is that the $H(\sigma_-)$ analysis remains the same as in the massless case.}, this is to large extent because it contains complete information about the system under consideration, including the structure of on-shell representation of supersymmetry where the infinite set of zero-form fields is valued. Once obtained, the unfolded formulation answers many questions which may be hard to answer by other means such as, for example, the higher-spin extension of supersymmetry of the scalar supermultiplet [5].

The proposed approach can be useful for the analysis of generic (super)symmetric theories where the $\sigma_-$--cohomology analysis makes it possible to derive systematically analogues of the Dragon theorem [11]. It should be stressed that unfolded formulation contains all possible $G$-
invariant dual formulations of the same theory. A particular field-theoretical form depends on the choice of one or another $G$ invariant background geometry (i.e., one or another (super)space) and on the choice of the grading which distinguishes between dynamical and auxiliary fields in the system. This provides a powerful tool for the analysis and classification of $G$-invariant dynamical systems. In particular, this approach was used in [12] to classify all conformal invariant differential equations in $d$-dimensional Minkowski space-time. Once obtained, the unfolded formulation of a given supersymmetric system contains various its superfield formulations, allowing systematic investigation of all options. On the top of that unfolded formulation provides a powerful tool for the construction of the action and conserved currents along the lines of [13]. One of the most interesting problems to be explored within the unfolded formulation in the future is the manifestly supersymmetric off-shell action formulation of less trivial supersymmetric systems like $N = 4$ and/or $d = 10$ SYM theories.

Since the unfolded formulation proved to be most efficient for the description of nonlinear higher-spin theories [14], [15], [3], their superfield formulation should contain the description of the scalar supermultiplet in the form presented in this paper.

The rest of the paper is organized as follows. In Section 2 we review the unfolded dynamics approach. In Section 3 the extensions of the standard $\sigma$-technics suggested by the analysis of supersymmetric models are discussed. In Section 4 we recall the formulation of the flat superspace in the form of the flatness conditions for the SUSY algebra. In Section 5 we present the final result for the unfolded equations of a massless scalar supermultiplet with the emphasize in its general properties such as supersymmetry. The $\sigma$-cohomology analysis of unfolded equations of Section 5 is performed in Section 6. It is shown in particular that the standard $\sigma$-cohomology technics treats the two dynamical equations for the chiral superfield $C(z)$

$$D_\alpha C(z) = 0, \quad D^\alpha D_\alpha C(z) = 0$$

(1.1)

on the same footing with their consequence

$$\left(\sigma^a\right)^{\dot{\alpha} \alpha} D_a D_\alpha C(z) = 0.$$  

(1.2)

The origin of this peculiarity in the $\sigma$-cohomology language as well as its relation to the fact that superspace possesses nonzero torsion are discussed in Section 7. In Section 8 we derive unfolded equations for a massive scalar supermultiplet. Our notations and conventions are summarized in Appendix A. Some technicalities are collected in Appendix B.

## 2 Unfolded dynamics

Unfolded dynamics approach [11], [2] implies reformulation of equations of motion in the form of generalized zero curvature equations

$$R^\Omega (x) \overset{\text{def}}{=} dW^\Omega (x) + G^\Omega (W(x)) = 0,$$

(2.1)

where $d = dx^m \frac{\partial}{\partial x^m}$ is the exterior differential and

$$G^\Omega (W^T) \overset{\text{def}}{=} \sum_{n=1}^{\infty} f^\Omega_{\tau_1 \ldots \tau_n} W^{\tau_1} \ldots W^{\tau_n}$$
is built from exterior product (which is implicit in this paper) of differential forms $W^\Omega(x)$ and satisfies the compatibility condition

$$G^\Omega(W)\frac{\delta G^\Omega(W)}{\delta W^\Omega} \equiv 0.$$  \hspace{1cm} (2.2)

Here index $\Omega$ enumerates a set of differential forms. Let us note that (2.2) is the condition on the function $G^\Omega(W)$ to be satisfied identically for all $W^\Omega$.

For field-theoretical systems with infinite number of degrees of freedom, unfolding requires an infinite number of auxiliary fields subjected to an infinite set of equations most of which are algebraic constraints that express auxiliary fields via (derivatives of) dynamical fields.

The property (2.2) guarantees the generalized Bianchi identity

$$dR^\Omega = R^\Psi \frac{\delta G^\Omega}{\delta W^\Psi},$$

which tells us that the differential equations on $W^\Omega$

$$R^\Omega(W) = 0$$

are consistent with $d^2 = 0$.

Universal unfolded field equations are those where $W^\Omega$ can be treated as coordinates of some target superspace [3]. (Alternatively, one can say that the compatibility condition (2.2) holds independently on the number of values of the indices of differential forms, i.e., it is insensitive to the fact that $p$-forms with $p > d$ are zero in the $d$-dimensional space.) In this case, it is possible to differentiate freely over $W^\Omega$ and the equations (2.1) are manifestly invariant under the gauge transformation with a degree $p^\Omega - 1$ differential form gauge parameter $\varepsilon^\Omega(x)$ associated to any degree $p^\Omega > 0$ form $W^\Omega$

$$\delta W^\Omega = d\varepsilon^\Omega - \varepsilon^\Psi \frac{\delta L G^\Omega(W)}{\delta W^\Psi} \hspace{1cm} (2.4)$$

because

$$\delta R^\Omega = -R^\Psi \frac{\delta L}{\delta W^\Psi} \left( \varepsilon^\Psi \frac{\delta L G^\Omega(W)}{\delta W^\Psi} \right).$$

The gauge transformations of 0-forms $\delta C^{\Omega_0}$ only contain the gauge parameters $\varepsilon^{\Omega_1}$ associated to 1-forms $W^{\Omega_1}$ in (2.4)

$$\delta C^{\Omega_0} = -\varepsilon^{\Omega_1} \frac{\delta L G^{\Omega_0}(W)}{\delta W^{\Omega_1}} \hspace{1cm} (2.5)$$

In the case of universal unfolded equations, once the condition (2.2) is satisfied for some base manifold, it remains consistent for any larger (super)space. Another important fact is that in the topologically trivial situation all information about dynamical degrees of freedom described by an unfolded system is encoded by 0-forms $C^{\Omega_0}(x)$ at any given point $x_0$ of space-time (see [3] and references therein). Since this set of local data remains the same in any space, it follows that a universal unfolded system provides an equivalent description in a larger (super)spaces simply by adding additional coordinates corresponding to a larger (super)space. (In this consideration it is important that the original unfolded system constitutes
a subsystem of that extended to a larger (super)space.) In particular, unfolding provides
the systematic way for derivation of constraints in superfield formulations of supersymmetric
theories.

We use the following terminology. The fields, that neither can be expressed in terms of
derivatives of other fields, nor can be gauged away are called dynamical. Auxiliary fields
are expressed via derivatives of the dynamical fields. (As already mentioned, unfolding usually
requires an infinite set of auxiliary fields.) Differential conditions on the dynamical fields
imposed by the unfolded equations are called dynamical equations. Other equations are
either consequences of dynamical equations or constraints, which are the equations satisfied
identically when auxiliary fields are expressed in terms of dynamical ones. The standard
tool for the analysis of physical content of unfolded equations is provided by $\sigma$–cohomology
technics which will be discussed in the next section.

An important example of unfolded equations is

$$d\Omega_0 + \Omega_0 \Omega_0 = 0,$$

where $\Omega_0 = \Omega^a_0 T_a$ is a 1-form valued in some Lie algebra $g$ with a basis $T_a$. The consistency
condition (2.2) translates to the Jacoby identity for $g$. Eq. (2.6) implies that the connection
$\Omega_0$ is flat which is the standard way of the description of a $g$-invariant vacuum. This example
shows how $g$-invariant background fields appear in the unfolded equations. In the pertur-
bative analysis, $\Omega_0$ is assumed to be of the zeroth order because it contains the background
metric.

The transformation law (2.4) gives the usual gauge transformations of the connection $\Omega_0$
$$\delta \Omega_0(x) = d \varepsilon(x) + \Omega_0(x) \varepsilon(x) - \varepsilon(x) \Omega_0(x),$$
where $\varepsilon(x)$ is a 0-form valued in $g$. Given flat connection $\Omega_0(x)$ is invariant under the
transformations with the covariantly constant parameters, that satisfy

$$d \varepsilon(x) + \Omega_0(x) \varepsilon(x) - \varepsilon(x) \Omega_0(x) = 0.$$  (2.7)

This equation is formally consistent by virtue of (2.2). Locally, it reconstructs $\varepsilon(x)$ in terms
of its value $\varepsilon(x_0)$ at any given point $x_0$. Hence, the number of independent solutions of (2.7)
coincides with dim $g$. Solutions of the equations (2.7) describe the leftover global symmetry
$g$ of any solution of (2.6).

Let us now linearize the unfolded equation (2.1) around a vacuum flat connection $\Omega_0,
that solves (2.1). To this end we set

$$W = \Omega_0 + \mathcal{C},$$

where $\mathcal{C}$ are differential forms of various degrees, that are treated as small perturbations and,
therefore, contribute to equations linearly. Consider the subset of $\mathcal{C}$ constituted by forms $\mathcal{C}_p^i$
of some definite degree $p$, enumerated by index $i$. Usually, each field $\mathcal{C}_p^i$ with fixed $i$ is valued in
some representation of the Lorentz-like subalgebra $h \subset g$. In the linearized approximation,
one has to consider the part of $G^i$ bilinear in $\Omega_0$ and $\mathcal{C}_p^i$, that is $G^i = \Omega^a_0 (T_a)^{ij} \mathcal{C}_p^j$. In this case
the condition (2.2) implies that the matrices $(T_a)^{ij}$ form a representation of $g$ in a vector
space $V$ where the set of $\mathcal{C}_p^i$ with all $i$ is valued. The corresponding equation (2.1) is the
covariant constancy condition

$$D_{\Omega_0} \mathcal{C}_p^i = 0$$

with $D_{\Omega_0} \equiv d + \Omega_0$ being the covariant derivative in the $g$-module $V$. 5
3 σ_-–cohomology

3.1 General setup

A useful tool of the unfolding machinery is the identification of the dynamical content of unfolded equations with σ_-–cohomology groups [6] (see also [5, 3]). The aim is to work out the dynamical pattern of the linearized unfolded system of the form

\[ \mathcal{R} \overset{\text{def}}{=} (D + \sum \sigma)\mathcal{C} = 0, \]  

(3.1)

where \( \mathcal{C} \) are differential \( p \)-form fields, valued in \( V \), \( \mathcal{R} \) are generalized curvatures, \( D \) is a covariant derivative with respect to Lorenz-like subalgebra \( h \) of \( g \) and \( \sigma \) are operators, that act algebraically in the space-time sense (that is, they do not differentiate space-time coordinates). Some part of the equations (3.1) has the meaning of the algebraic constraints that express some auxiliary fields in terms of dynamical fields. The leftover equations in (3.1) may contain differential equations on the dynamical fields as well as their consequences.

Note that the decomposition of fields into auxiliary and dynamical is not necessarily unique. For example, in the system

\[ \frac{\partial}{\partial x} B(x) + A(x) = 0, \quad \frac{\partial}{\partial x} A(x) + B(x) = 0, \]

either \( A \) can be interpreted as an auxiliary field and \( B \) as a dynamical field or vice versa. The resulting systems are equivalent (dual) to each other.

In the σ_-–cohomology technics this ambiguity is controlled by a \( \mathbb{Z} \) grading \( \mathcal{G} \) that distinguishes between dynamical and auxiliary fields in such a way that auxiliary fields have higher \( \mathcal{G} \)-grade than the respective dynamical fields. The grading operator \( \mathcal{G} \) is required to be diagonalizable on the space of fields and to have spectrum bounded below. Also the \( \mathcal{G} \)-grade of the exterior differential \( d \) is required to be zero. Assuming that \( h \) is the \( \mathcal{G} \)-grade zero subalgebra of \( g \), the \( h \) covariant derivative \( D \) also has \( \mathcal{G} \)-grade zero. Dual formulations of the same theory are usually associated with different choices of the grading \( \mathcal{G} \). An example of this phenomenon in higher-spin theory is considered in [16].

Usually \( \mathcal{G} \) counts a number of tensor indices of tensor fields of \( h \), that constitute \( \mathcal{C} \). The σ_-–cohomology analysis applies once \( \sigma \) contains a part of negative \( \mathcal{G} \)-grade. In the case where the negative grade part of \( \sigma \) contains several operators we identify \( \sigma_- \) with that of the lowest grade. Then unfolded equations (3.1) acquire the form

\[ \mathcal{R} \overset{\text{def}}{=} (D + \sigma_- + \ldots)\mathcal{C} = 0, \]  

(3.2)

where \( \ldots \) denotes all those operators that do not contain space-time derivatives and have grade \( \mathcal{G} \) higher than \( \sigma_- \). The compatibility condition (2.2) for (3.2) is

\[ (D + \sigma_- + \ldots)^2 = 0. \]  

(3.3)

Decomposing this relation into different grades gives in particular that

\[ (\sigma_-)^2 = 0 \]  

(3.4)
since $\sigma_-$ carries the most negative grade.

For the unfolded equations (3.2), the Bianchi identities (2.3) and gauge transformations (2.4) are

$$T \overset{\text{def}}{=} (D + \sigma_- + \ldots)R = 0,$$

$$\delta C = (D + \sigma_- + \ldots)\varepsilon.$$ (3.5)

(3.6)

The $\sigma_-$-cohomology technics works as follows. Starting from the lowest grade we analyze Eq. (3.2). Fields, that are not annihilated by $\sigma_-$, can be expressed in terms of fields of lower grade by means of (3.2), hence they are auxiliary. The rest of fields, if cannot be gauged away by $\sigma_-\varepsilon$ in (3.6), are treated as dynamical. Hence, nontrivial dynamical fields are classified by $H_p(\sigma_-)$. Let us note, that the space of nontrivial dynamical 0-form fields associated to $H_0(\sigma_-)$ is always not empty because it at least contains the fields of the lowest grade.

Similarly, starting from the lowest grade, we impose equations (3.2) (it is equivalent to say, that we set associated $R$ to zero) and analyze Bianchi identities (3.5). By virtue of Bianchi identities (3.5), the part of $R$, that is not annihilated by $\sigma_-$, is zero as a consequence of equations (3.2) for lower grades, that is $R$ is $\sigma_-$ closed. On the other hand, the part of the equations $R = 0$ with $R \in \text{Im}(\sigma_-)$ is not dynamical because these just impose constraints, that express auxiliary fields in terms of (derivatives of) the fields of lower grade. We conclude that nontrivial differential equations contained in (3.2) are associated with $H_{p+1}(\sigma_-)$.

Let us note that the $\sigma_-$-cohomology analysis can be naturally extended to the bigraded or even multigraded cases where $G = \mathbb{Z} \times \mathbb{Z}$ or $\mathbb{Z}^n$. Then the $\sigma_-$ complex extends to a bicomplex or multicomplex. For example, the $\mathbb{Z} \times \mathbb{Z}$ bicomplex structure naturally appears in conformal supersymmetric theories [5] as well as in the analysis of partially massless [17] and massive [18] higher-spin fields. In the latter cases it counts the numbers of indices in the first and second rows of Young diagrams, associated to the tensors fields in these systems.

### 3.2 Bianchi identities and consequences of dynamical equations

Let $R_n$ be a part of $R$ of grade $n$ with respect to $G$. Suppose that by virtue of constraints and field equations $R_n=0 \ \forall n \leq n_0$. Then, as already mentioned in the analysis of the field equations, it follows that those components of the curvatures $R_{n_0+1}$ that do not belong to $\text{Ker}(\sigma_-)$ are also zero as a consequence of Bianchi identities. The part of Bianchi identities that is $\sigma_-$ exact relates the curvatures $R_{n_0+1}$ to derivatives of $R_n$. The part of the Bianchi identities, that does not involve the higher curvatures $R_{n_0+1}$ and hence may yield nontrivial identities for dynamical equations, therefore is in $H_{p+2}(\sigma_-)$.

The comment we wish to make in this paper is that the Bianchi identities in $H_{p+2}(\sigma_-)$ necessarily have the trivial form $0 = 0$ only when $\sigma$ does not contain parts of subleading grades. Otherwise, the Bianchi identities may give nontrivial consequences of the field equations that belong to $H_{p+1}(\sigma_-)$.

In this paper we show how this mechanism works in the the $\mathcal{N} = 1$, $D = 4$ scalar supermultiplet model. Similarly, in models where $\sigma_-$ does not carry a definite grade, higher cohomology groups responsible for Bianchi identities can lead to nontrivial consequences of field equations. In fact, as we hope to show in more detail elsewhere, this comment translates the Fierz-Pauli idea [19] of elimination of auxiliary fields

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2One of us (MV) acknowledges stimulating discussion of this phenomenon in a different context with Kostya Alkalaev.
in massive field theories into $\sigma-$cohomology language. Generally, non-trivial consequences of the field equations are characterized by $H_{p+2}(\sigma_-)$.

Let us explain this phenomenon in some more detail. Let the unfolded equations be of the form

$$R^n = DC^n + \sigma_- C^{n+1} + \sigma_+ \sum_{\varepsilon > 0} C^{n-\varepsilon} = 0,$$

where $C^n$ is a $p$-form field of grade $n$, $\sigma_+$ is a set of operators with positive grades and $\varepsilon > 0$. The $\sigma_-$ technics yields that the dynamical equations are

$$(DC^n + \sigma_+ \sum_{\varepsilon > 0} C^{n-\varepsilon})|_{H_{p+1}(\sigma_-)} = 0. \tag{3.8}$$

The term $\sigma_- C^{n+1}$ does not contribute to (3.8) because it is $\sigma_-$-exact. Fields $C$ are expressed in terms of lower grade dynamical fields by means of constraints. Since $\varepsilon > 0$, the number of derivatives in $C^n$, expressed in terms of dynamical fields, is greater than in $C^{n-\varepsilon}$. Since $C^n$ contributes to (3.8) with an additional derivative compared to $C^{n-\varepsilon}$, the number of derivatives in the first term on the l.h.s. of (3.8) is greater than the number of derivatives in the second one.

Various consequences of (3.8) result from its differentiations. The problem is to find such low-derivative consequences that contain a number of derivatives smaller than in the general case, i.e., those where the highest derivative part is zero. To achieve this, we should find an operator, that annihilates the first term in (3.8). Let us note, that the highest derivative term is exactly the l.h.s. of the dynamical equations of the unfolded system of the form

$$\tilde{R}^n = DC^n + \sigma_- C^{n+1} = 0, \tag{3.9}$$

which can be thought of as resulting from (3.7) in the limit where all mass parameters are set to zero. Hence, the problem of finding the low-derivative consequences of (3.8) is equivalent to the problem of constructing nontrivial identities for dynamical equations of the unfolded system (3.9). These are given by Bianchi identities for (3.9) projected to $H_{p+2}(\sigma_-)$

$$(D\tilde{R})|_{H_{p+2}(\sigma_-)} \equiv 0. \tag{3.10}$$

So, low-derivative consequences of (3.8) are given by

$$(DR)|_{H_{p+2}(\sigma_-)} = 0. \tag{3.11}$$

Suppose that grades of $H_p(\sigma_-)$, $H_{p+1}(\sigma_-)$ and $H_{p+2}(\sigma_-)$ are $n_f$, $n_e$ and $n_i$ respectively. Taking into account that $\sigma_-$ has grade $-1$, to express auxiliary field $C^{n_e}$ in terms of $C^{n_f}$ we should use $\sigma_-$-constraints $n_e-n_f$ times. This expresses $C^{n_e}$ in terms of $(n_e-n_f)$-th derivatives of $C^{n_f}$, which entails that the highest order term $(DC^{n_e})|_{H_{p+1}(\sigma_-)}$ in (3.8) contains $n_e-n_f+1$ derivatives. Eq. (3.11) has the form $(D \sum_{\varepsilon > 0} C^{n_i-\varepsilon})|_{H_{p+2}(\sigma_-)}$. So its highest derivative term contains $n_i - \min(\varepsilon) - n_f + 1$ derivatives. As a result, the number of derivatives in (3.11) is less than that in (3.8) if $n_e - n_i + \min(\varepsilon) > 0$. 

8
3.3 Several $\sigma$-operators

Usually, in the $\sigma$-analysis of unfolded equations it is assumed, that the negative grade part of $\sigma$ has grade $-1$. Hence, the standard $\sigma$–cohomology analysis considered in [6, 3, 20, 21, 22] only treats this particular case. However, as was mentioned already in [5] and will be explained in more detail in Section 6 in supersymmetric models this is not the case. Hence in this subsection we consider the peculiarities of the situation where $\sigma$ contains parts of several negative grades. In this case, usual $\sigma$–cohomology analysis naturally extends to the $\sigma$ spectral sequence analysis.

Let $\sigma^{\prime}$ be the next to minimal negative grade, i.e.,

$$\sigma = \sigma_- + \sigma^{\prime} + \ldots,$$

(3.12)

where ... denote operators of higher grades. Fields, equations, gauge symmetries, Bianchi identities, etc, resulting from the $\sigma$–cohomology analysis can be further analyzed using the operator $\sigma^{\prime}$. From (3.3) it follows that

$$\{\sigma_-, \sigma^{\prime}\} = 0,$$

(3.13)

$$(\sigma^{\prime})^2 + \{\sigma_-, \Sigma^{\prime}\} = 0,$$

(3.14)

where $\Sigma^{\prime}$ is the part of the operator (3.2) of the appropriate grade, which can in particular contain $D$. From (3.13) it follows that $\sigma^{\prime}$ maps $H(\sigma_-)$ to $H(\sigma_-)$. In addition, from (3.14) it follows that $(\sigma^{\prime})^2 = 0$ when restricted to $H(\sigma_-)$.

As a result, for fields and curvatures that belong to $H(\sigma_-)$, the analysis goes along the same lines as for $\sigma_-$, namely, to express $\mathcal{C} \in H_p(\sigma_-)$ in terms of derivatives of lower grade fields one can use the $\sigma^{\prime}$-constraints that belong to $H_{p+1}(\sigma_-)$. Analogously, the Bianchi identities, that remain unused for expression of curvatures in terms of lower grade ones in $\sigma$–cohomology analysis belong to $H_{p+2}(\sigma_-)$. These can be used to find the $\sigma^{\prime}$-constraints between eqs. (3.2) with $\mathcal{R} \in H_{p+1}(\sigma_-)$.

Let $\tilde{\sigma}^{\prime}$ be the restriction of $\sigma^{\prime}$ to $H(\sigma_-)$. In this terms, the dynamical pattern of unfolded equations is encoded by $H(\tilde{\sigma}^{\prime})$ alternatively denoted as $H(\sigma^{\prime}|\sigma_-)$. Let us stress that in the analysis of the action of $\sigma^{\prime}$ restricted to $H(\sigma_-)$ one should factor out all $\sigma_-$-exact terms, i.e., $\mathcal{C}$ such that $\sigma^{\prime} \mathcal{C} \in \text{Im}(\sigma_-)$ belongs to $\text{Ker}(\tilde{\sigma}^{\prime})$.

Analogously, one proceeds in the case where $\sigma$ contains any number of different negative grade parts, repeating the analysis of cohomologies on cohomologies for all algebraic operators of increasing negative grades. The more cohomologies are computed, the more relations between seemingly independent dynamical equations, Bianchi identities, etc are extracted. Eventually, the pattern of the equations is governed by the cohomology $H^p(\sigma^{\prime}_{\ldots} \ldots |\sigma^{\prime}_{\ldots} |\sigma_-|\sigma_-)$. Mathematically, this translates to the spectral sequence computation. Note that analogous spectral sequences are familiar in computation of BRST cohomology (see, e.g., [23]).

Application of this analysis to higher cohomologies $H^p(\sigma^{\prime}_{\ldots} |\sigma^{\prime}_{\ldots} |\sigma_-|\sigma_-)$, associated to consequences of Bianchi identities for unfolded equations that contain $\sigma_+$ type algebraic operators of positive grades as discussed in the previous subsection, may help to control nontrivial consequences of dynamical equations containing lower-derivative terms. In Section 7 this phenomenon will be illustrated by the $\sigma_-$ analysis of scalar supermultiplet.
4 Supersymmetric vacuum

To start unfolding we first introduce gauge fields of supergravity [24] (for review see, e.g. [25]) resulting from gauging the SUSY algebra that has nonzero commutators (for notations see Appendix A)

\[
[M_{\mu\nu}, M_{\rho\sigma}] = -(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}),
\]

\[
[P_{\mu}, M_{\mu\nu}] = \eta_{\mu\nu}P_{\nu} - \eta_{\nu\mu}P_{\mu},
\]

\[
\{Q_\alpha, Q_\beta\} = -2i(\sigma^\mu)_{\alpha\beta}P_\mu,
\]

\[
[M_{\mu\nu}, Q_\alpha] = \frac{i}{2}(\sigma_{\mu\nu})_{\alpha}^\beta Q_\beta,
\]

\[
[M_{\mu\nu}, \bar{Q}^{\dot{\alpha}}] = \frac{i}{2}(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} \bar{Q}^{\dot{\beta}},
\]

where

\[
(\sigma_{\mu\nu})_{\alpha}^\beta = \frac{1}{2}((\sigma_{\mu})_{\alpha}^{\dot{\alpha}}(\bar{\sigma}_{\nu})_{\dot{\alpha}}^{\beta} - (\sigma_{\nu})_{\alpha}^{\dot{\alpha}}(\bar{\sigma}_{\mu})_{\dot{\alpha}}^{\beta}),
\]

\[
(\bar{\sigma}_{\mu\nu})_{\dot{\alpha}}^{\dot{\beta}} = \frac{1}{2}((\bar{\sigma}_{\mu})_{\dot{\alpha}}^{\dot{\alpha}}(\sigma_{\nu})_{\alpha}^{\beta} - (\bar{\sigma}_{\nu})_{\dot{\alpha}}^{\dot{\alpha}}(\sigma_{\mu})_{\alpha}^{\beta}).
\]

These include vierbein 1-form \( e^a = e^m a dx^m \), spin connection 1-form \( \omega^{a,b} = \omega^m_{\ a,b} dx^m \) and gravitino 1-form \( \phi^a = \phi^m_a dx^m \). Carrying spinorial index, the gravitino 1-form \( \phi^a \) is Grassmann odd. Taking into account that \( \phi^a \) is a degree 1 differential form, we have the following commutation relations for \( \phi^a \) (recall that we use the exterior products of differential forms discarding the wedge product symbol)

\[
[\phi^a, \bar{\phi}^{\dot{a}}] = 0, \quad \{\phi^a, e^a\} = 0, \quad \{\phi^a, \omega^{a,b}\} = 0.
\]

Gauge fields \( e^a, \omega^{a,b} \) and \( \phi^\gamma \) combine into a 1-form connection \( \Omega \) valued in the SUSY algebra

\[
\Omega \equiv e^a P_a + \frac{1}{2} \omega^{a,b} M_{ab} + \phi^a Q_\alpha + \bar{\phi}^{\dot{a}} \bar{Q}^{\dot{\alpha}}.
\]

A globally supersymmetric background corresponding to the most symmetric solution of supergravity is described by a flat connection \( \Omega \) that satisfies

\[
R \equiv d\Omega + \Omega \Omega = 0.
\]

The decomposition of the curvature \( R \) into components associated to particular generators

\[
R \equiv S^a P_a + \frac{1}{2} R^{a,b} M_{ab} + S_\alpha Q_\alpha + \bar{S}_{\dot{a}} \bar{Q}^{\dot{\alpha}}
\]

gives the standard results

\[
S^a = D^L e^a + 2i \phi^a \bar{\phi}^{\dot{a}} (\sigma^a)_{\alpha\dot{\alpha}} = de^a + \omega^{a,b} e_b + 2i \phi^a \bar{\phi}^{\dot{a}} (\sigma^a)_{\alpha\dot{\alpha}} = 0,
\]

\[
R^{a,b} = d\omega^{a,b} + \omega^{a,c} \omega_{c}^{\ b} = 0,
\]

\[
S^\alpha = D^L \phi^\alpha = d\phi^\alpha + \frac{i}{4} \omega^{a,b} \phi^\beta (\sigma_{ab})^\alpha_\beta = 0.
\]
\[ S_{\dot{a}} = D^{L} \dot{\phi}_{\dot{a}} = d\dot{\phi}_{\dot{a}} + \frac{i}{4} \omega^{a,b}_{\dot{a}} \dot{\phi}_{\dot{b}}(\bar{\sigma}_{ab})\dot{\alpha} = 0, \]  

(4.6)

where \( D^{L} \) is the Lorentz covariant derivative. As is well-known, (4.3) implies that the torsion \( T^{a} = D^{L} e^{a} \) is nonzero

\[ T^{a} = -2i \phi^{a} \dot{\phi}(\sigma^{a})_{\alpha\dot{a}}. \]  

(4.7)

As explained in Section 2, Eq. (4.2) has unfolded form. Global SUSY transformations identify with those gauge (super)transformations that leave invariant the background gauge connections in (4.3)-(4.6), satisfying the equations

\[ \delta e^{a} = d\varepsilon^{a} - \varepsilon^{a,b} e_{b} + \varepsilon b^{a} - 2i(e^{a} \tilde{\phi}(\sigma^{a})_{\alpha\dot{a}} + \varepsilon^{a} \phi(\sigma^{a})_{\alpha\dot{a}}) = 0; \]

\[ \delta \omega^{a,b} = d\varepsilon^{a,b} + \omega^{a,c} \varepsilon_{c} - \varepsilon^{a,c} \omega_{c} = 0; \]

\[ \delta \phi^{a} = d\varepsilon^{a} + \frac{i}{4} e^{a,b,c} \varepsilon_{c} - \varepsilon^{a,c} \omega_{c} = 0; \]

\[ \delta \tilde{\phi}^{a} = d\varepsilon_{\dot{a}} + \frac{i}{4} e^{a,b,c} \varepsilon_{c} - \varepsilon^{a,c} \omega_{c} = 0. \]  

(4.8)

Note that the flatness condition (4.2) guarantees the compatibility of the equations (4.8) which therefore reconstruct the dependence of the symmetry parameters on space-time coordinates in terms of their values at any point of space-time.

As explained in Section 2 to uplift the unfolded system (which is obviously universal in the case of vacuum equations) to superspace it suffices to add supercoordinates, \( x^{m} \rightarrow z^{A} = (x^{n}, \theta^{\mu}) \), extending appropriately the indices of differential forms:

\[ e_{m}^{a}(x) dx^{m} \rightarrow E_{M}^{a}(z) dz^{M}, \quad \omega_{m}^{a,b}(x) dx^{m} \rightarrow \Omega_{M}^{a,b}(z) dz^{M}, \]

\[ \phi_{m}^{a}(x) dx^{m} \rightarrow E_{M}^{a}(z) dz^{M}, \quad \tilde{\phi}_{m}^{a}(x) dx^{m} \rightarrow E_{M}^{a}(z) dz^{M}. \]

Now the zero curvature equations

\[ D^{L} E^{a} + 2i E^{a} E^{\dot{a}}(\sigma^{a})_{\alpha\dot{a}} = dE^{a} + \Omega^{a,b} E_{b} + 2i E^{a} E^{\dot{a}}(\sigma^{a})_{\alpha\dot{a}} = 0, \]  

(4.9)

\[ d\Omega^{a,b} + \Omega^{a,c} \Omega_{c}^{b} = 0, \]  

(4.10)

\[ D^{L} E^{\alpha} = dE^{\alpha} + \frac{i}{4} \Omega^{a,b} E^{\beta}(\sigma_{ab})_{\beta}^{\alpha} = 0, \]  

(4.11)

\[ D^{L} E_{\dot{a}} = dE_{\dot{a}} + \frac{i}{4} \Omega^{a,b} E^{\dot{\beta}}(\bar{\sigma}_{ab})_{\dot{\beta}}^{\dot{a}} = 0 \]  

(4.12)

describe flat superspace. Fields \( E^{a} \), \( E^{\alpha} \) and \( E_{\dot{a}} \) can be combined into supervierbein \( E^{A} = dz^{M} E_{M}^{A} \). A particular solution of (4.9)-(4.12), which extends Cartesian coordinates to superspace is

\[ E^{a} = dx^{m} \delta^{a}_{m} + d\theta^{\mu}(i \tilde{\theta}^{\mu}(\sigma^{a})_{\mu\dot{a}}) + d\bar{\theta}_{\dot{a}}(i \theta^{\mu}(\sigma^{a})_{\mu\dot{a}}), \]

\[ E^{a} = d\theta^{\mu} \delta^{a}_{\mu}, \quad E_{\dot{a}} = d\bar{\theta}_{\dot{a}} \delta^{a}_{\dot{a}}, \quad \Omega^{a,b} = 0. \]  

(4.13)

The Lorentz covariant derivative in the superspace can be rewritten in the form

\[ D^{L} = E^{a} D_{a} + E^{\alpha} D_{\alpha} + E_{\dot{a}} \tilde{D}_{\dot{a}}. \]  

(4.14)

In coordinates (4.13) they are

\[ D_{a} = \partial_{a}, \quad D_{\alpha} = \partial_{\alpha} - i(\sigma^{b})_{\alpha\dot{a}} \tilde{\theta}^{\dot{a}} \partial_{\dot{b}}, \quad \tilde{D}_{\dot{a}} = \partial_{\dot{a}} - i \theta^{a}(\sigma^{b})_{\alpha\dot{a}} \partial_{b}. \]

From \( (D^{L})^{2} = 0 \) it follows that

\[ \{ D_{a}, \tilde{D}_{\dot{a}} \} = 2i(\sigma^{a})_{\alpha\dot{a}} D_{a}. \]  

(4.15)
5 Free massless scalar supermultiplet unfolded

In this Section unfolded equations of motion for $\mathcal{N} = 1, D = 4$ free massless scalar supermultiplet are presented. Here we give the final result leaving details of its derivation to Section 8 where the more complicated massive case will be considered.

We start with the unfolded equations for scalar supermultiplet component fields in Minkowski space. Then we modify the equations to manifestly supersymmetric form. Finally, we add supercoordinates to uplift the system to superspace.

Minkowski space is described by (4.6) with $\phi = \bar{\phi} = 0$
\begin{equation}
T^a = D^L e^a = 0, \quad R^{a,b} = d\omega^{a,b} + \omega^{a,c}\omega_c{}^b = 0.
\end{equation}

Sometimes we will use Cartesian coordinates in the sequel
\begin{equation}
e_m{}^a = \delta_m{}^a, \quad \omega_m{}^{a,b} = 0, \quad D^L = d.
\end{equation}

Unfolded equations for a massless scalar field $C$ in Minkowski space are [2, 6]
\begin{equation}
R^{a(k)} \overset{def}{=} D^L C^{a(k)} + e_b C^{a(k)b} = 0,
\end{equation}
where $C^{a(k)} \in \mathbf{U}_0^k$, that is $C^{a(k)}$ are complex 0-forms valued in symmetric traceless rank-$k$ Lorentz tensors. Introducing notation
\begin{equation}
(\sigma_1 C)^{a(k)} \overset{def}{=} e_b C^{a(k)b},
\end{equation}
(5.3) can be rewritten in the form
\begin{equation}
(D^L + \sigma_1)C = 0.
\end{equation}

In these terms, the compatibility condition (2.2) requires $(D + \sigma_1)^2 = 0$ which is true by virtue of (5.1). Note that $(\sigma_1)^2 = 0$ since it contains the antisymmetrization of two symmetric indices when acting on $C^{a(k)}$.

Let us show following [6] that (5.3) indeed describes a massless spin 0 particle. In Cartesian coordinates (5.2) the first two equations of (5.3) can be rewritten in the form
\begin{equation}
\partial_a C + C_a = 0,
\end{equation}
\begin{equation}
\partial_b C_a + C_{ab} = 0.
\end{equation}

Substituting $C_a$ expressed in terms of $C$ from (5.5) into (5.6) and taking trace we have
\begin{equation}
\partial^a \partial_a C = 0.
\end{equation}

Other equations in (5.3) do not impose further conditions on $C$ just expressing auxiliary fields $C^{a(k)}$ in terms of the dynamical field $C$.

Similarly, the equations
\begin{equation}
r_\alpha{}^{a(k)} \overset{def}{=} D^L \chi_\alpha a^{a(k)} + e_b \chi_\alpha a^{a(k)b} = 0
\end{equation}
for $0$-forms $\chi_\alpha^{a(k)} \in V_0^k$, that is $\chi_\alpha^{a(k)}$ are complex $0$-forms valued in symmetric traceless rank-$k$ tensor-spinors subjected to the condition
\begin{equation}
(\bar{\sigma}_b)^{\dot{\alpha}}_{\alpha} \chi_\alpha^{a(k-1)b} = 0, 
\end{equation}
which will be referred to as $\sigma$-transversality condition, describe massless spin $1/2$ field in Minkowski space $\mathbb{M}^4$.

The system of a free scalar and fermion in four dimension is supersymmetric. However, within the above formulation supersymmetry is not manifest. To make it manifest it suffices to extend the system (5.3) and (5.7) to the case where the connections $\phi$ and $\bar{\phi}$ associated with supertransformations are introduced into the equations. Clearly, they mix bosons and fermions. An elementary analysis then shows that the appropriate modification is
\begin{align}
R^{a(k)} &= D^L C^{a(k)} + e_b C^{a(k)b} - \sqrt{2} \bar{\phi}^\alpha \chi_\alpha^{a(k)} = 0, \\
r^{a(k)} &= D^L \chi_\alpha^{a(k)} + e_b \chi_\alpha^{a(k)b} - \sqrt{2} i \bar{\phi}^{\dot{\alpha}} \sigma_b^{\dot{\alpha}} \bar{\chi}_b^{a(k)b} = 0.
\end{align}

The formal consistency of this system relies on the flatness conditions (4.2) and the identity
\begin{equation}
(\sigma_b)^{\beta\dot{\alpha}} \chi_\alpha^{a(k)b} = (\sigma_b)^{\dot{\alpha}\dot{\alpha}} \chi_b^{a(k)b},
\end{equation}
which is the consequence of the $\sigma$-transversity of $\chi$ along with the fact that spinorial indices take just two values, expressed by the formula (A.1).

Application of (2.5) to (5.9), (5.10) gives the gauge transformation rules of fields $C^{a(k)}$ and $\chi_\alpha^{a(k)}$
\begin{equation}
\delta C^{a(k)} = \sqrt{2} \xi^\alpha \chi_\alpha^{a(k)}, \quad \delta \chi_\alpha^{a(k)} = \sqrt{2} i \bar{\xi}^{\dot{\alpha}} (\sigma_b)^{\dot{\alpha}\dot{\alpha}} C^{a(k)b},
\end{equation}
where $\xi^\alpha$ is a gauge parameter associated to $\phi^\alpha$. In the Cartesian coordinates $e_m^a = \delta^a_m$, $D^L = d$, $\phi = \bar{\phi} = 0$ one easily finds from (5.9) that $C_a = -\partial_a C$. Then, Eq. (5.11) yields
\begin{equation}
\delta C = \sqrt{2} \xi^\alpha \chi_\alpha, \quad \delta \chi_\alpha = -\sqrt{2} i \bar{\xi}^{\dot{\alpha}} (\sigma_b)^{\dot{\alpha}\dot{\alpha}} \partial^b C,
\end{equation}
which is the standard supertransformation [10, 9] with the parameter $\xi^\alpha$. Thus, in accordance with the general consideration of Section 2 that the equations (5.9), (5.10) are compatible with background (4.6) guarantees supersymmetry of the model.

The unfolded formulation of massless scalar supermultiplet in superspace is now reached easily by adding spinorial supercoordinates into the same system of unfolded equations (5.9), (5.10)
\begin{align}
R^{a(k)} &= D^L C^{a(k)} + e_b C^{a(k)b} - \sqrt{2} E^\alpha \chi_\alpha^{a(k)} = 0, \\
r^{a(k)} &= D^L \chi_\alpha^{a(k)} + e_b \chi_\alpha^{a(k)b} - \sqrt{2} i E^\dot{\alpha} (\sigma_b)^{\dot{\alpha}\dot{\alpha}} C^{a(k)b} = 0.
\end{align}

To show that these equations indeed describe massless scalar supermultiplet in superspace one should sort out independent dynamical superfields as well as their field equations. This is achieved via the $\sigma$-cohomology analysis.

## 6 Dynamical content of unfolded equations for scalar supermultiplet

### 6.1 Irreducible subspaces

To perform $\sigma$-cohomology analysis it is useful to characterize the pattern of the spaces, where fields and curvatures are valued, in terms of irreducible representations of the diagonal Lorentz group, that acts both on fiber indices and those of differential forms.
Spaces $\mathbf{U}_k^0$ and $\mathbf{V}_k^0$, where the fields $C$ and $\chi$ are valued, are irreducible under Lorentz transformations for fixed $k$. This is not the case, however, for the spaces $\mathbf{U}_k^1$ and $\mathbf{V}_k^1$, where curvatures $R$ and $r$ are valued. The differential form index can be transformed to a tangent index by the supervierbein $E_M^A$. For $R_M^{\alpha(k)} \in \mathbf{U}_k^1$ and $r_M^{\alpha(k)} \in \mathbf{V}_k^1$ we have

$$R_M^{\alpha(k)} = E_M R_A^{\alpha(k)}, \quad r_M^{\alpha(k)} = E_M r_A^{\alpha(k)}. \quad (6.1)$$

Vertical lines in $R_A^{\alpha(k)}$ and $r_A^{\alpha(k)}$ separate the fiber indices from those resulting from the base ones. Decomposing the respective reducible representation of the Lorentz group into irreducible components we obtain the following decomposition of the 1-forms into irreducible parts:

$$R = (\pi_\downarrow + \pi_\uparrow + \pi^\times)t + (\pi_\swarrow + \pi_\nearrow + \pi_\nwarrow + \pi_\searrow)\tau, \quad (6.2)$$

$$r = (\pi_\downarrow + \pi_\uparrow + \pi_\rightarrow + \pi_\leftarrow)\tau + (\pi_\rightarrow + \pi_\leftarrow + \pi_\nwarrow + \pi_\searrow)t, \quad (6.3)$$

where $t$ denotes 0-forms without spinor indices and $\tau$ denotes 0-forms with one spinor index, while the operators $\pi$ are defined up to overall factors by\(^3\)

$$\begin{align*}
\pi_\downarrow : \quad & \mathbf{U}_p^k \to \mathbf{U}_{p+1}^{k-1}, \quad \mathbf{V}_p^k \to \mathbf{V}_{p+1}^{k-1},
\pi_\uparrow : \quad & \mathbf{U}_p^k \to \mathbf{U}_{p+1}^{k+1}, \quad \mathbf{V}_p^k \to \mathbf{V}_{p+1}^{k+1},
\pi^\times : \quad & \mathbf{U}_p^{k,1} \to \mathbf{U}_{p+1}^k, \quad \mathbf{V}_p^{k,1} \to \mathbf{V}_{p+1}^k,
\pi_\swarrow : \quad & \mathbf{V}_p^k \to \mathbf{U}_{p+1}^k, \quad \mathbf{U}_p^k \to \mathbf{V}_{p+1}^{k-1},
\pi_\nearrow : \quad & \mathbf{V}_p^k \to \mathbf{U}_{p+1}^k, \quad \mathbf{U}_p^k \to \mathbf{V}_{p+1}^{k+1},
\pi_\nwarrow : \quad & \mathbf{V}_p^k \to \mathbf{U}_{p+1}^k, \quad \mathbf{U}_p^k \to \mathbf{V}_{p+1}^{k+1},
\pi_\searrow : \quad & \mathbf{V}_p^{k,1} \to \mathbf{V}_{p+1}^{k-1}, \quad \mathbf{U}_p^{k,1} \to \mathbf{V}_{p+1}^k,
\pi_\leftarrow : \quad & \mathbf{V}_p^k \to \mathbf{U}_{p+1}^k, \quad \mathbf{U}_p^k \to \mathbf{V}_{p+1}^k,
\pi_\rightarrow : \quad & \mathbf{V}_p^k \to \mathbf{U}_{p+1}^k, \quad \mathbf{U}_p^k \to \mathbf{V}_{p+1}^{k+1}.
\end{align*}$$

Explicit expressions for $\pi$ with appropriately fixed overall factors are given in Appendix B.

It is convenient to endow the spaces of (spinor-)tensors with the \textit{vertical} $\mathcal{Z}$ grading $\mathcal{G}$ equal to the number $n$ of vector indices for bosons and to $n + \frac{1}{2}$ for fermions and \textit{horizontal} $\mathcal{Z}$ grading $\mathcal{H}$ with three nonzero homogeneous spaces taking “left”, “center” or “right” values for spin-tensors with dotted indices, tensors and spin-tensors with undotted indices, respectively. In these terms, the labels of the $\pi$-operators acquire the simple meaning. In particular, the operators that increase or decrease $\mathcal{G}$-grade are endowed with arrows pointing up or down, respectively, and similarly for the horizontal grading. For example, an operator that maps $\mathbf{U}_p^k$ to $\mathbf{V}_{p+1}^k$ increases $\mathcal{G}$-grade by 1/2 and maps a tensor to a spin-tensor with undotted index. Hence it is denoted $\pi_\rightarrow$.

The spaces $\mathbf{U}_p^{k,1}$ and $\mathbf{V}_p^{k,1}$, that appear in the decomposition into irreducible parts \((6.2), (6.3)\) are spanned by tensors of the symmetry of the two-row Young diagram with one cell

\(^3\)Notations are collected in Appendix A
in the second row. The π-operators acting on $U_{p}^{k,1}$ and $V_{p}^{k,1}$, that annihilate a cell in the second row of a Young diagram, are endowed with the label $\chi$.

Eqs. (5.12)-(5.13) can be rewritten in the form:

$$R = D L C + \sigma_{↓} C + \sigma_{\downarrow} \chi = 0,$$
(6.4)

$$r = D L \chi + \sigma_{↓} \chi + \sigma_{\downarrow} C = 0,$$
(6.5)

where

$$(\sigma_{\downarrow} C)^{a(k)} \overset{\text{def}}{=} E_{b} C^{a(k)b}, \quad (\sigma_{\downarrow} \chi)^{a(k)} \overset{\text{def}}{=} -\sqrt{2} E^{a_{\alpha}} C^{a(k),b},$$

$$(\sigma_{\downarrow} \chi)^{a(k)} \overset{\text{def}}{=} E_{b} C^{a(k)b}, \quad (\sigma_{\downarrow} C)^{a(k)} \overset{\text{def}}{=} -\sqrt{2} i E^{a_{\alpha}} C^{a(k)b}. \quad (6.6)$$

The operators $\sigma$ are introduced similarly to $\pi$ but the overall factors in $\sigma$ are fixed by the compatibility conditions (2.2) which have the form

$$(\sigma_{↓})^{2} C = 0, \quad (\sigma_{↓})^{2} \chi = 0,$$

$$\{\sigma_{↓}, \sigma_{\downarrow}\} C = 0, \quad \{\sigma_{↓}, \sigma_{\downarrow}\} \chi = 0,$$

$$(\{D L, \sigma_{↓}\} + \sigma_{\downarrow} \sigma_{\downarrow}) C = 0, \quad (\{D L, \sigma_{↓}\} + \sigma_{\downarrow} \sigma_{\downarrow}) \chi = 0,$$

$$\{D L, \sigma_{\downarrow}\} C = 0, \quad \{D L, \sigma_{\downarrow}\} \chi = 0.$$

The $\sigma$-operators possess the following $G$-grades

$$[G, \sigma_{↓}] = -\sigma_{↓}, \quad [G, \sigma_{\downarrow}] = -\frac{1}{2} \sigma_{\downarrow}, \quad [G, \sigma_{\downarrow}] = -\frac{1}{2} \sigma_{\downarrow}.$$

In accordance with their grades, $\sigma_{↓}$ will also be denoted as $\sigma_{↓}^{1}$ and operators $\sigma_{\downarrow}$ and $\sigma_{\downarrow}$ will be combined to $\sigma_{↓}^{1/2}$. To carry out the analysis sketched in Subsection 3.3, we first compute cohomology of $\sigma_{↓}^{1}$ and then restrict $\sigma_{↓}^{1/2}$ to $H(\sigma_{↓}^{1})$ to obtain $\sigma_{↓}^{1/2}$. The dynamical content of the system is encoded by $H(\sigma_{↓}^{1/2})$.

**6.2 $H(\sigma_{↓}^{1})$.**

Since only the spaces $U_{0}^{0}$ and $V_{0}^{0}$ are annihilated by $\sigma_{↓}^{1}$ and $\text{Im}(\sigma_{↓}^{1})$ is empty,

$$H_{0}(\sigma_{↓}^{1}) \cong U_{0}^{0} \oplus V_{0}^{0}. \quad (6.7)$$

Analogously, all the 1-forms of the lowest grade

$$U_{1}^{0} = \pi_{↓} U_{0}^{1} \oplus \pi_{\downarrow} V_{0}^{0} \oplus \pi_{\downarrow} V_{0}^{0}$$

and next to the lowest grade

$$V_{1}^{0} = \pi_{↓} V_{0}^{1} \oplus \pi_{\downarrow} V_{0}^{0} \oplus \pi_{\downarrow} U_{0}^{1} \oplus \pi_{\downarrow} U_{0}^{1}$$

spaces are annihilated by $\sigma_{↓}^{1}$. In addition, one can check, that the projection of $U_{1}^{0}$ to $\pi_{↓} U_{0}^{0}$ also belongs to $\text{Ker}(\sigma_{↓}^{1})$. The subspaces $\pi_{↓} U_{1}^{0}$ in (6.8) and $\pi_{↓} V_{0}^{0}$ in (6.9) belong to $\text{Im}(\sigma_{↓}^{1})$. Factoring them out, we obtain

$$H_{1}(\sigma_{↓}^{1}) \cong \pi_{\downarrow} V_{0}^{0} \oplus \pi_{\downarrow} U_{0}^{0} \oplus \pi_{\downarrow} V_{0}^{0} \oplus \pi_{\downarrow} U_{0}^{1} \oplus \pi_{\downarrow} U_{0}^{1} \oplus \pi_{\downarrow} U_{0}^{1} \oplus \pi_{\downarrow} U_{0}^{0}. \quad (6.10)$$
6.3 $\mathcal{H}(\tilde{\sigma}^{1/2})$.

Since $\tilde{\sigma}^{1/2}$ is a restriction of $\sigma^{1/2}$ to $\mathcal{H}(\sigma^1)$, $C \in \text{Ker}(\tilde{\sigma}^{1/2})$ means that $\sigma^{1/2}C$ vanishes up to $\sigma^1$-exact terms, that is $\sigma^{1/2}C \in \text{Im}(\sigma^1)$. Since $\sigma^{1/2}U^0_0 = 0$ and $\sigma^{1/2}V^0_0 \notin \text{Im}(\sigma^1)$, we conclude, that

$$\mathcal{H}_0(\tilde{\sigma}^{1/2}) \cong U^0_0.$$  \hfill (6.11)

Hence, after all the constraints are taken into account, the only dynamical field is $C \in U^0_0$.

To analyze $\mathcal{H}_1(\tilde{\sigma}^{1/2})$, first, we drop the term $\pi \varpi V^0_0$ in (6.10), which is $\tilde{\sigma}^{1/2}$-exact. Note that the term $\pi \varpi U^0_0$ is $\sigma^{1/2}$-exact but not $\sigma^{1/2}$-exact on $\mathcal{H}(\sigma^1)$, because $U^0_0 \notin \mathcal{H}(\sigma^1)$.

Analogously to the case of 0-forms, for 1-forms we keep those of the rest terms in (6.10), that satisfy $\sigma^{1/2}R \in \text{Im}(\sigma^1)$. The result is

$$\mathcal{H}_1(\tilde{\sigma}^{1/2}) \cong \pi \varpi \dot{V}^0_0 \oplus \pi \varpi \dot{V}^0_0 \oplus \pi \varpi U^0_0.$$  \hfill (6.12)

Let us note, that although $\sigma \varpi \pi \sigma U^0_0$ has the form $E_\alpha E^{\dot{\alpha}}(\sigma_a)_{a\dot{a}}t$ for any $t \in U^0_0$, it does not belong to $\text{Im}(\sigma^1)$, because, being proportional to $\sigma$-matrices, $E^{\dot{\alpha}}(\sigma_a)_{a\dot{a}}t$ does not satisfy the $\sigma$-transversality condition (5.8), hence not belonging to $V^1_0$ or any other space where the curvatures are valued.

6.4 Dynamical interpretation

Let us summarize the results of the $\sigma_-$-cohomology analysis. The only dynamical superfield is $C(z)$. Other fields are auxiliary and express in terms of its derivatives. For example, the projection of $R(z) = 0$ to $\pi \varpi V^0_0$ expresses $\chi_\alpha(z)$ in terms of $C(z)$:

$$\chi_\alpha(z) = \frac{1}{\sqrt{2}} D_\alpha C(z).$$  \hfill (6.13)

Equating to zero curvature projections, that belong to (6.12), and taking into account (6.13) we get superfield equations

$$R(z)\big|_{\pi \varpi \dot{V}^0_0} = 0 \quad \Rightarrow \quad \bar{D}_\alpha C(z) = 0,$$  \hfill (6.14)

$$r(z)\big|_{\pi \varpi \dot{V}^0_0} = 0 \quad \Rightarrow \quad D^\alpha D_\alpha C(z) = 0,$$  \hfill (6.15)

$$r(z)\big|_{\pi \varpi \dot{V}^0_0} = 0 \quad \Rightarrow \quad (\bar{\sigma}_a)^{\dot{\alpha}\alpha} D_{\alpha} D_{\dot{\alpha}} C(z) = 0,$$  \hfill (6.16)

where vertical lines, that carry labels associated to some spaces, indicate the projections to these spaces.

Obviously, (6.16) can be obtained by the application of $\bar{D}_\alpha$ to the both sides of (6.15), taking into account (6.14) and (4.15). This means that the $\sigma_-$-cohomology analysis performed so far does not lead to the minimal set of independent dynamical equations contained in the unfolded system (5.12), (5.13). As we show in the next section, in accordance with the consideration of Subsection 3.2, the missed information on the dependence of the system of equations (6.14)-(6.16) follows from the analysis of higher $\sigma_-$-cohomology.
7 Extra equations from higher $\sigma_-$–cohomology

Let us show that Bianchi identities relate Eq. (6.16) to (6.14), (6.15) and explain why the analysis of Subsection 3.1 and extended analysis of Subsection 3.3 do not catch it. Namely, along the lines of Subsection 3.2 it will be shown that, resulting from Bianchi identities, Eq. (6.16) belongs to the projection of $D^L r_\alpha$ to $H_2(\tilde{\sigma}_{1/2}^\perp)$. The left hand side of the Bianchi identity

$$D^L r_\alpha + E_b r_\alpha^b - \sqrt{2i} E^\hat{a} (\sigma_b)_{\alpha\hat{a}} R^\hat{a} = 0 \quad (7.1)$$

can be decomposed into irreducible parts analogously to the case of 0- and 1-forms of the previous section. For the second term this gives

$$E_b r_\alpha^b = (\pi_4 r)^\alpha = (\pi_4 (\pi_+ + \pi_\rightarrow + \pi_\leftarrow + \pi_\uparrow) r_2 + \pi_5 (\pi_\rightarrow + \pi_\downarrow + \pi_\uparrow + \pi_\downarrow) t_2)^\alpha \quad (7.2)$$

(here we use the decomposition (6.3)), while for the third term gives

$$E^\hat{a} (\sigma_b)_{\alpha\hat{a}} R^\hat{a} = (\pi_\downarrow R)^\alpha = (\pi_\downarrow (\pi_+ + \pi_\rightarrow + \pi_\downarrow + \pi_\uparrow) r_3 + \pi_\downarrow (\pi_\rightarrow + \pi_\downarrow + \pi_\uparrow + \pi_\downarrow) t_3)^\alpha \quad (7.3)$$

(here we use the decomposition (6.2)).

To bring the first term

$$D^L r_\alpha = (E^b D_b + E^\beta D_\beta + E^\phi \bar{D}^\phi)((\pi_4 + \pi_\rightarrow) r_1 + (\pi_\rightarrow + \pi_\downarrow + \pi_\uparrow) t_1)^\alpha \quad (7.4)$$

to the desired form, $D^a$, $D^\alpha$ and $\bar{D}_\hat{a}$ should be commuted to the operators $\pi$ in (7.4). Most conveniently this can be done by commuting $D^L$ to $\pi$ according to (4.9). For example, let us show how this works for the term $(\pi_\rightarrow t_1)^\alpha$:

$$D^L E^b (\sigma_b)_{\alpha\hat{a}} \tau_1^\hat{a} = -2iE^\beta E^\phi (\sigma_b)_{\beta\hat{a}} (\sigma_b)_{\alpha\hat{a}} \tau_1^\hat{a} - E^b (\sigma_b)_{\alpha\hat{a}} (E^c D_c + E^\gamma D_\gamma + E^\phi \bar{D}^\phi) \tau_1^\hat{a} \quad (7.5)$$

The first term on the r.h.s. of (7.5) results from the action of $D^L$ on $E^b$, which brings factor $-2iE^\beta E^\phi (\sigma_b)_{\beta\hat{a}}$ due to nonzero torsion (4.7) while the second term results from permutation of 1-forms $D^L$ and $E^b$ bringing a minus sign.

The important point is that the algebraic operator $E^\beta E^\phi (\sigma_b)_{\beta\hat{a}} (\sigma_b)_{\alpha\hat{a}}$, that appears in (7.5), is not explicitly present in (6.1), (6.3) as an algebraic $\sigma$-operator. Analogously to operators $\sigma^1_-$ and $\sigma^2_-$, it can be used to express the component $\tau_1^\hat{a}$ of curvature $r_\alpha$ in terms of other curvatures by (7.1). A simple calculation shows that contraction of undotted spinor indices in the part of (7.1) proportional to $E^a E^\hat{a}$ gives

$$4i\tau_1^\hat{a} + 2\bar{D}^\hat{a} t_1 - (\bar{\sigma}_a)^{\hat{a}\alpha} D_\alpha t_1^a - 2\sqrt{2i} \tau_3^\hat{a} = 0. \quad (7.6)$$

This can be used to express $\tau_1^\hat{a}$, which is just the l.h.s. of (6.16), in terms of other curvatures. It is easy to see, that (7.6) means that the projection of (7.1) to $\pi_\downarrow \pi_\rightarrow \hat{\nu}^0_\alpha$ vanishes.

To see how the resulting relation (7.6) can be derived from the analysis of Bianchi identities in terms of higher $\sigma_-$–cohomology we observe that

$$\pi_\downarrow \pi_\rightarrow \hat{\nu}^0_\alpha \in H_2(\tilde{\sigma}_{1/2}^\perp), \quad (7.7)$$
which means, that this part of Bianchi identity \((\ref{7.1})\) has not been used yet to express curvatures in terms of lower grade ones in \(\sigma^-\) and \(\tilde{\sigma}^{1/2}\)-cohomology analysis. Indeed, 
\[ \sigma^- \pi \pi \pi \dot{V}_0 \pi = 0 \quad \text{and} \quad \pi \pi \pi \pi \dot{V}_0 \pi \notin \text{Im}(\sigma^-), \] 
so \(\pi \pi \pi \pi \dot{V}_0 \pi \in H_2(\sigma^-)\). Moreover, \(\sigma^{1/2} \pi \pi \pi \pi \dot{V}_0 \pi = 0\) and \(\pi \pi \pi \pi \dot{V}_0 \pi = i/\sqrt{2} \sigma^{1/2} \pi \pi \pi \dot{V}_0 \pi \), but \(\pi \pi \pi \pi \dot{V}_0 \pi \notin H_1(\sigma^-)\), that proves \((7.7)\).

This example provides an illustration of a general phenomenon that higher \(\sigma^-\)-cohomology may encode nontrivial relations between field equations resulting from the naive \(\sigma^-\) analysis.

### 8 Massive scalar supermultiplet

In this Section we derive unfolded equations of motion for \(\mathcal{N} = 1, D = 4\) scalar supermultiplet of any mass and carry out their \(\sigma^-\)-cohomology analysis. We use the same set of fields as in the massless case and consider the most general form of unfolded equations. Then, imposing compatibility conditions \((\ref{2.2})\) and fixing some field redefinition ambiguity, we determine all the terms in equations which will contain an arbitrary parameter of mass.

Let us introduce additional operators, that act in the following way (see Appendix B):

- \(\pi^- : U^k_p \rightarrow \dot{V}_{p+1}^k\),
- \(\pi^\wedge : U^k_p \rightarrow \dot{V}_{p+1}^k\),
- \(\pi^\wedge \wedge : V^k_p \rightarrow \dot{V}_{p+1}^k\),
- \(\pi^\wedge : \dot{V}_p^k \rightarrow \dot{V}_{p+1}^{k-1}\),
- \(\pi^\wedge \wedge : \dot{V}_p^k \rightarrow \dot{V}_{p+1}^{k+1}\).

The general form of equations is

\[
R = DC + \sigma_k C + \sigma^\wedge \wedge + \sigma^\wedge \chi + \sigma_\wedge \chi + \sigma_\wedge \wedge \chi = 0, \quad (8.1)
\]

\[r = D\chi + \sigma_k \chi + \sigma^\wedge \chi + \sigma_\wedge \chi + \sigma^\wedge \chi + \sigma^\wedge \chi + \sigma^\wedge \chi = 0, \quad (8.2)
\]

and conjugated equations

\[
R^+ = D\bar{C} + \sigma_k \bar{C} + \sigma^\wedge \bar{C} + \sigma^\wedge \bar{C} + \sigma^\wedge \bar{C} + \sigma^\wedge \bar{C} + \sigma^\wedge \bar{C} = 0, \quad (8.3)
\]

\[r = D\bar{\chi} + \sigma_k \bar{\chi} + \sigma^\wedge \bar{\chi} + \sigma_\wedge \bar{\chi} + \sigma^\wedge \bar{\chi} + \sigma^\wedge \bar{\chi} + \sigma^\wedge \bar{\chi} = 0. \quad (8.4)
\]

Operators \(\sigma\) are proportional to the corresponding operators \(\pi\) with the complex overall factors \(U, T, V\) and \(W\) introduced as follows (to avoid writing all indices explicitly, we denote \(C^{a(k)}\) \(\in U^k_p\) as \(C(k)\), \(\chi^{a(k)} \in V^k_p\) as \(\chi(k)\) and \(\bar{\chi}^{\bar{a}(k)} \in V^k_p\) as \(\bar{\chi}(k)\)):

\[
\sigma_k C(k) = U_k(k) \pi_k C(k), \quad \sigma_\wedge C(k) = U_\wedge(k) \pi_\wedge C(k), \quad \sigma^\wedge C(k) = U^\wedge(k) \pi^\wedge C(k),
\]

\[
\sigma_\wedge C(k) = U_\wedge(k) \pi_\wedge C(k), \quad \sigma^\wedge C(k) = U^\wedge(k) \pi^\wedge C(k), \quad \sigma_\wedge C(k) = U_\wedge(k) \pi_\wedge C(k),
\]

\[
\sigma^\wedge \bar{C}(k) = U^\wedge(k) \pi^\wedge \bar{C}(k), \quad \sigma_\wedge \bar{C}(k) = U_\wedge(k) \pi_\wedge \bar{C}(k), \quad \sigma^\wedge \bar{C}(k) = U^\wedge(k) \pi^\wedge \bar{C}(k),
\]

\[
\sigma_\wedge \bar{C}(k) = U_\wedge(k) \pi_\wedge \bar{C}(k), \quad \sigma^\wedge \bar{C}(k) = U^\wedge(k) \pi^\wedge \bar{C}(k), \quad \sigma_\wedge \bar{C}(k) = U_\wedge(k) \pi_\wedge \bar{C}(k),
\]

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\( \sigma_\perp \chi(k) = V_\perp(k) \pi_\perp \chi(k) \), \( \sigma_\parallel \chi(k) = V_\parallel(k) \pi_\parallel \chi(k) \), \( \sigma_\perp \chi(k) = V_\perp(k) \pi_\parallel \chi(k) \), \( \sigma_\parallel \chi(k) = V_\parallel(k) \pi_\perp \chi(k) \).

\( \sigma_\perp \bar{\chi}(k) = W_\perp(k) \pi_\perp \bar{\chi}(k) \), \( \sigma_\parallel \bar{\chi}(k) = W_\parallel(k) \pi_\parallel \bar{\chi}(k) \), \( \sigma_\perp \bar{\chi}(k) = W_\perp(k) \pi_\parallel \bar{\chi}(k) \), \( \sigma_\parallel \bar{\chi}(k) = W_\parallel(k) \pi_\perp \bar{\chi}(k) \).

\( \sigma_\perp \chi(k) = V_\perp(k) \pi_\perp \chi(k) \), \( \sigma_\parallel \chi(k) = V_\parallel(k) \pi_\parallel \chi(k) \), \( \sigma_\perp \bar{\chi}(k) = W_\perp(k) \pi_\perp \bar{\chi}(k) \), \( \sigma_\parallel \bar{\chi}(k) = W_\parallel(k) \pi_\parallel \bar{\chi}(k) \).

\( \sigma_\perp \bar{\chi}(k) = W_\perp(k) \pi_\perp \bar{\chi}(k) \), \( \sigma_\parallel \bar{\chi}(k) = W_\parallel(k) \pi_\perp \bar{\chi}(k) \), \( \sigma_\perp \bar{\chi}(k) = W_\perp(k) \pi_\parallel \bar{\chi}(k) \), \( \sigma_\parallel \bar{\chi}(k) = W_\parallel(k) \pi_\perp \bar{\chi}(k) \).

\( \sigma_\perp \chi(k) = V_\perp(k) \pi_\perp \chi(k) \), \( \sigma_\parallel \chi(k) = V_\parallel(k) \pi_\parallel \chi(k) \), \( \sigma_\perp \bar{\chi}(k) = W_\perp(k) \pi_\perp \bar{\chi}(k) \), \( \sigma_\parallel \bar{\chi}(k) = W_\parallel(k) \pi_\parallel \bar{\chi}(k) \).

(8.5)

Let us note, that it is possible to add terms of the form \( \pi_\parallel \bar{C} \) and \( \pi_\parallel \bar{C} \) to (8.1). These terms are built from vierbeine, and survive in Minkowski case with \( \phi = 0 \). Since such terms are absent in the unfolded equations for spin-0 field in Minkowski space, we require them to be absent in (8.1). In fact, such terms can be removed by a field redefinition.

Using that (8.3) and (8.4) are conjugated to (8.1) and (8.2), we have:

\[ U_\perp^*(k) = T_\perp(k), \quad U_\parallel^*(k) = T_\parallel(k), \quad V_\perp^*(k) = W_\perp'(k), \]
\[ -V_\parallel^*(k) = W'_\parallel(k), \quad W_\perp^*(k) = V_\perp'(k), \quad -W_\parallel^*(k) = V_\parallel'(k), \]
\[ V_\perp^*(k) = W_\perp(k), \quad V_\parallel^*(k) = W_\parallel(k), \quad -W_\perp^*(k) = V_\perp(k), \quad -U_\parallel^*(k) = T_\parallel(k), \]
\[ U_\perp^*(k) = T_\perp(k), \quad -T_\parallel^*(k) = U_\perp(k), \quad T_\parallel^*(k) = U_\parallel(k). \]

(8.6)

The compatibility conditions (2.2) impose constraints on these coefficients (8.5) given in Appendix B.

Eqs. (8.1)- (8.4) have a freedom in field rescaling

\[ C(n) = X(n) \bar{C}(n), \quad \chi(n) = Y(n) \bar{\chi}(n), \]

(8.7)

which induces the following redefinition of the coefficients \( U_\perp \) and \( V_\perp \)

\[ \tilde{U}_\perp(n) = \frac{X(n)}{X(n-1)} U_\perp(n), \quad \tilde{V}_\perp(n) = \frac{Y(n)}{Y(n-1)} V_\perp(n). \]

It is convenient to fix

\[ U_\perp(n) = 1, \quad V_\perp(n) = 1. \]

(8.8)

The remaining part of the rescaling ambiguity is given by (8.7) with \( X(n) = X(0) \) and \( Y(n) = Y(0) \). Resolving the compatibility conditions in the scaling (8.8) we express all the coefficients

\[ U_\perp(k) = V_\perp(k) = T_\perp(k) = W_\perp(k) = 1, \]
\[ W_\perp(k) = -B_\perp \frac{1}{k+2}, \quad V_\perp(k) = B_\perp \frac{1}{k+2}, \]
\[ V_\parallel(k) = W_\parallel(k) = (-2B_\perp B_\parallel)^* \left( \frac{k+1}{k+2} \right)^2, \]
\[ U_\parallel(k) = T_\parallel(k) = (-2B_\perp B_\parallel)^* \frac{k+1}{k+2}, \]
\[ U_\perp(k) = A_\perp, \quad U_\parallel(k) = -2B_\perp A_\perp, \quad T_\perp(k) = -2B_\perp A_\perp, \quad T_\parallel(k) = -2B_\perp A_\perp, \]
\[ U_\perp(k) = A_\perp, \quad U_\parallel(k) = 2B_\perp A_\perp, \quad T_\perp(k) = -A_\perp, \quad T_\parallel(k) = 2B_\perp A_\perp. \]
\[ V'_{\langle}(k) = B_{\langle}, \quad V_{\langle}(k) = C_{\langle} B_{\langle} \frac{k + 1}{k + 2}, \quad W'_{\langle}(k) = B_{\langle}^*, \quad W_{\langle}(k) = -C_{\langle}^* B_{\langle}^* \frac{k + 1}{k + 2}, \]

\[ V'_{\rangle}(k) = C^*_{\rangle}, \quad V_{\rangle}(k) = B_{\rangle}^* B_{\rangle} \frac{k + 1}{k + 2}, \quad W'_{\rangle}(k) = C_{\rangle}, \quad W_{\rangle}(k) = -B_{\rangle} B_{\rangle}^* \frac{k + 1}{k + 2}, \] (8.9)
in terms of parameters \( B_{\langle}, A_{\langle}, A_{\rangle}, B_{\rangle}, C_{\rangle} \), subjected to the conditions

\[ C_{\langle} A_{\langle} = iD_1, \] (8.10)

\[ B_{\langle} A_{\rangle} = iD_2, \] (8.11)

\[ A_{\rangle} C_{\rangle} = (A_{\langle} B_{\rangle})^*, \] (8.12)

\[ D_1 + D_2 = 2, \] (8.13)

where \( D_1 \) and \( D_2 \) are some real parameters. From (8.10)-(8.13) it follows that

\[ D_1 D_2 \leq 0. \] (8.14)

The remaining scaling ambiguity acts on these parameters as follows

\[ \tilde{B}_{\langle} = B_{\langle} \frac{Y(0)}{X(0)}, \quad \tilde{C}_{\langle} = C_{\langle} \frac{Y^*(0)}{X(0)}, \quad \tilde{A}_{\langle} = A_{\langle} \frac{X(0)}{Y(0)}, \quad \tilde{A}_{\rangle} = A_{\rangle} \frac{X(0)}{Y^*(0)}. \]

First, we fix \( B_{\langle} = -\sqrt{|D_2|} \). Then we can fix the phase of \( Y(0) \) in such a way that \( C_{\langle} \) is negative and real. Then the scaling symmetry is fixed up to (8.7) with

\[ X(n) = Y(n) = X(0), \quad X(0) \in \mathbb{R} \] (8.15)

and the solution of (8.10)-(8.13) is

\[ B_{\langle} = -\sqrt{|D_2|}, \quad A_{\langle} = -i \cdot \text{sign}(D_2) \sqrt{|D_2|}, \]

\[ C_{\langle} = -\sqrt{|D_1|}, \quad A_{\langle} = -i \cdot \text{sign}(D_1) \sqrt{|D_1|}. \] (8.16)

The curvatures (8.1), (8.2) are invariant under another type of symmetry. Namely, \( C \) can be replaced by appropriately normalized linear combination of \( C \) and \( \bar{C} \) (the normalization is necessary to preserve (8.16)). One can show, that

\[ \tilde{C}(k) = \frac{1}{\sqrt{2}} \left( \text{sign}(D_2) \sqrt{|D_2|} C(k) + \text{sign}(D_1) \sqrt{|D_1|} \bar{C}(k) \right) \]

redefines any \( D_1 \) and \( D_2 \) in such a way that

\[ D_1 = 0, \quad D_2 = 2. \] (8.17)

This fixes all the coefficients. The final result is given by Eqs. (8.9), (8.16) and (8.17).

Although the unfolded equations for a massive supermultiplet are different from the equations (5.12) and (5.13) for the massless case, this difference does not affect the operators \( \sigma_- \) which remain the same. Hence, \( \sigma_- \)-cohomology analysis yields the same results.
The dynamical field is $C(z)$. Constraint (6.13) and the first dynamical equation (6.14) remain the same. The second dynamical equation is given by the projection of

$$r_\alpha = D\chi_\alpha + E_b\chi^b - \frac{1}{2}B_{\tau}B^{*\tau}E^b(\sigma_b)_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} - \sqrt{2}iE^{\dot{\alpha}}(\sigma_\alpha)_{\alpha\dot{\alpha}}C^b + 2\sqrt{2}iB_{\tau}^*E_\alpha\bar{C} = 0$$

to $\pi/\mathbb{U}_0^0$ and yields

$$0 = D^\alpha D_\alpha C - 8iB_{\tau}^*\bar{C}.$$  \hspace{1cm} (8.18)\hspace{1cm}$$

Comparison of (8.18) with the standard formula gives

$$B_{\tau} = \frac{i}{2}m.$$  \hspace{1cm} (8.19)\hspace{1cm}$$

The consistency of unfolded equations (8.1)-(8.4) does not impose any constraints on $B_{\tau}$ which remains arbitrary. Formulas (6.14) and (8.18) yield the standard superfield description of the scalar supermultiplet [9]:

$$D_\alpha C(z) = 0, \quad D^\alpha D_\alpha C(z) - 4mC(z) = 0.$$  \hspace{1cm} \hspace{1cm}$$

9 Conclusion

In this paper we derive and analyze unfolded equations for the simplest supersymmetric model in four space-time dimensions associated to a scalar supermultiplet. The analysis of these equations and, more generally, other supersymmetric models, requires extension of the standard $\sigma$–cohomology technics to the case of unfolded equations, that contain two and more negative grade algebraic operators. This extension is presented in this paper for a general unfolded system. Also we explain how to associate the equations that result from Bianchi identities to higher $\sigma$–cohomology, which is in particular necessary to extract the full information on dynamical equations in superspace.

Since the reformulation of supersymmetric systems in the unfolded form makes it possible to elaborate its superfield pattern in a systematic way, it would be interesting to extend the results of this paper to the variety of other supersymmetric models (both on-shell and off-shell) in various dimensions. It should be noted that, in accordance with the general discussion of Section 2, the form of linearized unfolded equations can be systematically derived by choosing appropriate modules of SUSY algebra to avoid a complicated brut force analysis like that of Section 8 that was possible to use because of simplicity of the model. Also, let us mention that for off-shell unfolded supersymmetric systems one can look for manifestly supersymmetric actions along the lines of [13].

The results of this paper may have a wide area of applicability beyond supersymmetric models. In particular, as we will explain in more detail elsewhere, the proposed association of higher $\sigma$–cohomology with consequences of dynamical equations provides an interesting interpretation of the old Fierz-Pauli program [19] in the massive HS models whose unfolded form was recently given in [18].
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Appendix A: Notations

In this paper we deal with 4-dimensional space parametrized by coordinates $x^m$, ($m \in \{0, 1, 2, 3\}$). The vierbein field $e_m^a$ relates base indices (i.e. indices of differential forms) denoted by Latin letters $m, n, \ldots$ from the middle of the alphabet and fiber indices (i.e. indices of tensors in local basis) denoted by Latin letters $a, b, \ldots$ from the beginning of the alphabet. In the fiber space we use the mostly minus Minkowski metric $\eta_{ab} = \text{diag}(1, -1, -1, -1)$. In the process of unfolding we introduce auxiliary fields, which are traceless symmetric tensors on their fiber indices. Sometimes we do not write symmetrized indices explicitly, just writing one of them and indicating the number of indices in brackets (e.g., $a(k)$ instead of $(a_1 \ldots a_k)$).

The fiber spinor indices are from the beginning of Greek alphabet $\alpha, \dot{\alpha}, \beta, \dot{\beta} \ldots$ ($\alpha \in \{0, 1\}$), while base spinor indices $\mu, \dot{\mu}, \nu, \dot{\nu}, \ldots$ needed for differential forms in superspace are from the middle of Greek alphabet. Fiber spinor indices are raised/lowered by antisymmetric forms

$$
\varepsilon^{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
$$

$$
\xi_\alpha = \varepsilon_{\alpha\beta} \xi^\beta, \quad \xi^\alpha = \varepsilon^{\alpha\beta} \xi_\beta, \quad \bar{\chi}_\dot{\alpha} = \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\chi}^\beta, \quad \bar{\chi}^\dot{\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\chi}_\beta.
$$

We use the following convention for $\sigma$-matrices

$$
(\sigma^0)_{\alpha\dot{\alpha}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\sigma^1)_{\alpha\dot{\alpha}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (\sigma^2)_{\alpha\dot{\alpha}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad (\sigma^3)_{\alpha\dot{\alpha}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

The following useful relations are used

$$
T_{\dot{\alpha}\dot{\beta}\gamma\ldots} - T_{\alpha\gamma\ldots} = \varepsilon_{\alpha\beta} T_{\beta\gamma\ldots}, \quad (A.1)
$$

$$
(\sigma^m \bar{\sigma}^n + \sigma^n \bar{\sigma}^m)_{\alpha\beta} = 2\eta^{mn} \delta_{\alpha\beta}, \quad (\bar{\sigma}^m \sigma^n + \bar{\sigma}^n \sigma^m)_{\dot{\alpha}\dot{\beta}} = 2\eta^{mn} \delta_{\dot{\alpha}\dot{\beta}}, \quad (A.2)
$$

$$
(\sigma^m \bar{\sigma}^n)_{\alpha\beta} \delta_{\beta}^\alpha = 2\eta^{mn}, \quad (\bar{\sigma}^m \sigma^n)_{\dot{\alpha}\dot{\beta}} \delta_{\dot{\beta}}^\dot{\alpha} = 2\delta_{\dot{\alpha}\dot{\beta}} \delta_{\alpha\beta}, \quad (A.3)
$$

$$
(\sigma^m)_{\alpha\dot{\alpha}} (\bar{\sigma}_m)_{\beta\dot{\beta}} = 2\delta_{\alpha\beta} \delta_{\dot{\alpha}\dot{\beta}}, \quad (A.4)
$$

$$
\sigma^a \bar{\sigma}^b \sigma^c + \sigma^c \bar{\sigma}^b \sigma^a = -2(\eta^{ac} \sigma^b - \eta^{bc} \sigma^a - \eta^{ab} \sigma^c), \quad (A.5)
$$

$$
\bar{\sigma}^a \bar{\sigma}^b \bar{\sigma}^c + \bar{\sigma}^c \bar{\sigma}^b \bar{\sigma}^a = -2(\eta^{ac} \bar{\sigma}^b - \eta^{bc} \bar{\sigma}^a - \eta^{ab} \bar{\sigma}^c). \quad (A.6)
$$

Bosonic coordinates $x^m$ combined with fermionic coordinates $\theta^\mu, \bar{\theta}^\dot{\mu}$ constitute superspace coordinates $z^M \sim (x^m, \theta^\mu, \bar{\theta}^\dot{\mu})$. Base superspace indices are denoted by upper case letters from the middle of Latin alphabet, while fiber superspace indices are from the beginning of the alphabet. We use the standard convention for commutation rules of supercoordinates, exterior product and exterior differentiation in superspace [9], allowing to keep the standard rules of differentiation and multiplication of differential forms in superspace.

We also use notations for the following linear spaces over $\mathbb{C}$:
• \( U_p^k \) is a space of differential \( p \)-forms \( t^{a(k)} \) such that \( t_b^{ba(k-2)} = 0 \);

• \( V_p^k \) is a space of differential \( p \)-forms \( \tau^a_{\alpha} \) such that \( \tau_{ab}^{ba(k-2)} = 0 \), \((\bar{\sigma}_b)^\alpha\tau_{\alpha}^{ba(k-1)} = 0\);

• \( \dot{V}_p^k \) is a space of differential \( p \)-forms \( \dot{\tau}^a_{\alpha} \) such that \( \dot{\tau}^a_{b}^{ba(k-2)} = 0 \), \((\bar{\sigma}_b)^\alpha\dot{\tau}_{\alpha}^{ba(k-1)} = 0\);

• \( U_p^{k,1} \) is a space of differential \( p \)-forms \( t^{a(k),b} \), \( k \geq 1 \) such that \( t^{a(k),a} = 0 \), \( t^{ca(k-2),b} = 0 \) with other traces vanishing as a consequence of the first two conditions;

• \( V_p^{k,1} \) is a space of differential \( p \)-forms \( \tau^{a(k),b} \), \( k \geq 1 \) such that \( \tau^{a(k),a} = 0 \), \( \tau^{ca(k-2),b} = 0 \), \((\bar{\sigma}_c)^\alpha\tau_{\alpha}^{ca(k-1),b} = 0 \) with other traces and \( \sigma \)-longitudinal projections vanishing as a consequence of the first three conditions.

**Appendix B: Technicalities**

The detailed expressions for cell operators introduced in Section 6 are:

\[
(\pi_\downarrow t)^{a(k-1)} = E_b t^{a(k-1)b},
\]

\[
(\pi_\downarrow \tau)^{a(k-1)} = E_b \tau^{a(k-1)b},
\]

\[
(\pi_\uparrow t)^{a(k+1)} = E^a t^{a(k)} - \frac{k}{2(k+1)} E_b t^{a(k-1)b} \eta^{aa},
\]

\[
(\pi_\uparrow \tau)^{a(k+1)} = E^a \tau^{a(k)} - \frac{k}{2(k+1)} E_b \tau^{a(k-1)b} \eta^{aa},
\]

\[
(\pi_\leftarrow t)^{a(k)} = E_a t^{a(k)} - \frac{k}{2(k+1)} E_b (\sigma_b)_{\alpha\alpha} \tau^{a(k-1)b},
\]

\[
(\pi_\rightarrow t)^{a(k)} = E_a t^{a(k)} - \frac{k}{2(k+1)} E_b (\sigma_a)_{\alpha\alpha} \tau^{a(k-1)b},
\]

\[
(\pi_\wedge \tau)^{a(k)} = E_a \tau^{a(k)},
\]

\[
(\pi_\vee \tau)^{a(k)} = E_a \tau^{a(k)},
\]

\[
(\pi_\wedge t)^{a(k-1)} = E_b (\sigma_b)_{\alpha\alpha} t^{a(k-1)b},
\]

\[
(\pi_\vee t)^{a(k-1)} = E_b (\sigma_a)_{\alpha\alpha} t^{a(k-1)b},
\]

\[
(\pi_\times t)^{a(k-1)} = E_b (\sigma_a)_{\alpha\alpha} t^{a(k-1)b},
\]

\[
(\pi_\times t)^{a(k)} = E_b (\sigma_a)_{\alpha\alpha} t^{a(k-1)b}.
\]
This is the full list of operators such that their image belongs to $\mathbf{U}^k_p$ and $\mathbf{V}^k_p$, $p \geq 1$. In Section 8, we use operators, that act to $\dot{\mathbf{V}}^k_p$ and have the form

$$
(\pi_\dot{\mathbf{V}}^k_p t)^{\dot{a}a(k-1)} = \phi_a(\tilde{\sigma}_m)^{\dot{a}a} t^{a(k-1)m},
$$

$$
(\pi_{\dot{\mathbf{V}}^k_p} t)^{\dot{a}a(k)} = \frac{k}{2(k+1)} \tilde{\phi}^{\dot{a}a} t^{a(k)} - \frac{k}{2(k+1)} \tilde{\phi}^{\dot{a}a} (\tilde{\sigma}^a)_{\alpha \beta} t^{a(k-1)b},
$$

$$
(\pi_{\dot{\mathbf{V}}^k_p} \tau)^{\dot{a}a(k)} = e_m (\tilde{\sigma}_m)^{\dot{a}a} \tau_a^{(k-1)m} = \frac{k}{k+1} e_m (\tilde{\sigma}^a)_{\alpha \beta} t^{a(k-1)m},
$$

$$
(\pi_{\dot{\mathbf{V}}}^k t)^{\dot{a}a(k+1)} = e_m (\tilde{\sigma}_m)^{\dot{a}a} (\sigma^a)^{\alpha} \tau_{\alpha}^{(k-1)m},
$$

The compatibility conditions for (8.1)-(8.4) imply

$$
\{D, \sigma_\uparrow \} C + \{\sigma_\downarrow, \sigma_\uparrow \} C = 0 \quad \Rightarrow \quad -2i \dot{U}_\downarrow(k) + V_\downarrow(k-1) U_\downarrow(k) + W_\downarrow(k-1) U_\downarrow(k) = 0,
$$

$$
\{D, \sigma_\uparrow \} C + \{\sigma_\downarrow, \sigma_\downarrow \} C = 0 \quad \Rightarrow \quad -2i \dot{U}_\uparrow(k) + V_\uparrow(k-1) U_\uparrow(k) + W_\uparrow(k-1) U_\uparrow(k) = 0,
$$

$$
\{\sigma_\uparrow, \sigma_\downarrow \} C + \{\sigma_\downarrow, \sigma_\downarrow \} C + \{\sigma_\downarrow, \sigma_\uparrow \} C = 0 \quad \Rightarrow \quad U_\uparrow(k+1) U_\downarrow(k) \frac{k + 2}{k + 1} = U_\uparrow(k-1) U_\downarrow(k),
$$

$$
W_\uparrow(k-1) U_\downarrow(k) = \frac{k}{2(k+1)} V_\uparrow(k) U_\uparrow(k), \quad V_\downarrow(k-1) U_\downarrow(k) = \frac{k}{2(k+1)} W_\downarrow(k) U_\downarrow(k),
$$

$$
\{\sigma_\uparrow, \sigma_\downarrow \} C = 0 \quad \Rightarrow \quad \dot{U}_\downarrow(k-1) U_\downarrow(k) = U_\downarrow(k-1) U_\downarrow(k),
$$

$$
\{\sigma_\uparrow, \sigma_\downarrow \} C = 0 \quad \Rightarrow \quad V_\uparrow(k) U_\uparrow(k) \frac{k + 2}{k + 1} = U_\uparrow(k+1) U_\uparrow(k),
$$

$$
\{\sigma_\uparrow, \sigma_\downarrow \} + \sigma_\downarrow \sigma_\uparrow \} C = 0 \quad \Rightarrow \quad V_\downarrow(k) U_\downarrow(k) = U_\downarrow(k-1) U_\downarrow(k),
$$

$$
V_\uparrow(k-1) U_\downarrow(k) \frac{k^2(k+2)}{(k+1)^2} = W_\downarrow(k-1) U_\downarrow(k),
$$

$$
\{\sigma_\uparrow, \sigma_\downarrow \} C = 0 \quad \Rightarrow \quad \dot{U}_\downarrow(k-1) U_\downarrow(k) = U_\downarrow(k-1) U_\downarrow(k),
$$

$$
\{\sigma_\uparrow, \sigma_\downarrow \} C = 0 \quad \Rightarrow \quad V_\uparrow(k) U_\downarrow(k) \frac{k + 2}{k + 1} = U_\downarrow(k+1) U_\downarrow(k),
$$

$$
\{\sigma_\uparrow, \sigma_\downarrow \} + \sigma_\downarrow \sigma_\downarrow \} C = 0 \quad \Rightarrow \quad V_\downarrow(k) U_\downarrow(k) = U_\downarrow(k-1) U_\downarrow(k),
$$

$$
V_\uparrow(k-1) U_\downarrow(k) \frac{k^2(k+2)}{(k+1)^2} = V_\downarrow(k-1) U_\downarrow(k),
$$

$$
\{\sigma_\uparrow, \sigma_\downarrow \} C = 0 \quad \Rightarrow \quad U_\downarrow(k+1) U_\uparrow(k) = (k+1) V_\downarrow(k) U_\uparrow(k),
$$

$$
V_\downarrow(k-1) U_\uparrow(k) = V_\downarrow(k) U_\uparrow(k) \frac{k^2(k+2)}{(k+1)^2},
$$

$$
V_\uparrow(k-1) U_\downarrow(k) = V_\downarrow(k) U_\uparrow(k) \frac{k^2(k+2)}{(k+1)^2},
$$

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\[
\{\sigma_i, \sigma'_i\} \chi = 0 \Rightarrow U_\downarrow(k) V_\downarrow(k) = V_\downarrow(k - 1) V_\downarrow(k),
\]
\[
\{\sigma_i, \sigma'_i\} \chi = 0 \Rightarrow T_\downarrow(k) V'_\downarrow(k) = V'_\downarrow(k - 1) V_\downarrow(k),
\]
\[
\{\sigma^\uparrow, \sigma^\uparrow\} \chi = 0 \Rightarrow U_\uparrow(k + 1) V_\downarrow(k) = \frac{k + 2}{k + 1} V_\downarrow(k + 1) V_\uparrow(k),
\]
\[
\{\sigma^\uparrow, \sigma^\downarrow\} \chi = 0 \Rightarrow T_\uparrow(k + 1) V_\downarrow(k) = \frac{k + 2}{k + 1} V_\downarrow(k + 1) V_\uparrow(k),
\]
\[
(\{\sigma_i, \sigma'_i\} + \sigma^\downarrow \sigma^\downarrow) \chi = 0 \Rightarrow U_\downarrow(k + 1) V_\downarrow(k) \frac{1}{k + 1} = W_\downarrow(k)V_\downarrow(k),
\]
\[
U_\downarrow(k + 1) V_\downarrow(k) \frac{k(k + 2)}{(k + 1)^2} = V_\downarrow(k - 1) V_\downarrow(k),
\]
\[
(\{\sigma_i, \sigma'_i\} + \sigma^\downarrow \sigma^\uparrow) \chi = 0 \Rightarrow T_\downarrow(k + 1) V'_\downarrow(k) \frac{1}{k + 1} = W'_\downarrow(k)V_\downarrow(k),
\]
\[
T_\downarrow(k + 1) V'_\downarrow(k) \frac{k(k + 2)}{(k + 1)^2} = V'_\downarrow(k - 1) V_\downarrow(k),
\]
\[
(\{\sigma_i, \sigma'_i\} + \sigma^\uparrow \sigma^\downarrow) \chi = 0 \Rightarrow V'_\downarrow(k + 1) V_\uparrow(k) \frac{1}{2(k + 1)} = W'_\uparrow(k)V_\downarrow(k),
\]
\[
U_\uparrow(k) V'_\downarrow(k) = \frac{k + 2}{k + 1} V'_\downarrow(k + 1) V_\uparrow(k),
\]
\[
(\{\sigma_i, \sigma'_i\} + \sigma^\uparrow \sigma^\downarrow) \chi = 0 \Rightarrow V'_\uparrow(k + 1) V_\uparrow(k) \frac{1}{2(k + 1)} = W'_\uparrow(k)V_\downarrow(k),
\]
\[
T_\uparrow(k) V'_\uparrow(k) = \frac{k + 2}{k + 1} V'_\uparrow(k + 1) V_\uparrow(k),
\]
\[
(\{D, \sigma_i\} + \sigma^\downarrow \sigma^\downarrow + \sigma^\downarrow \sigma'_i) \chi = 0 \Rightarrow -2iV_\downarrow(k) + U_\downarrow(k)V_\downarrow(k) + T_\downarrow(k)V'_\downarrow(k) = 0,
\]
\[
(\{D, \sigma_i\} + \sigma^\uparrow \sigma^\downarrow + \sigma'_i \sigma'_i) \chi = 0 \Rightarrow -2iV_\uparrow(k) + \frac{k + 1}{k + 2} (U_\uparrow(k + 1) V_\downarrow(k) + T_\uparrow(k + 1) V'_\downarrow(k)) = 0,
\]
\[
(\{\sigma_i, \sigma^\downarrow\} + \sigma^\uparrow \sigma^\downarrow + \sigma^\downarrow \sigma'_i + \sigma'_i \sigma'_i) \chi = 0 \Rightarrow
\]
\[
U_\downarrow(k + 1) V_\downarrow(k) + T_\downarrow(k + 1) V'_\downarrow(k) = 0, \quad
U_\uparrow(k) V'_\downarrow(k) + T_\uparrow(k) V'_\downarrow(k) = 0,
\]
\[
W_\downarrow(k) V_\downarrow(k) = \frac{1}{2(k + 1)^2} V_\downarrow(k + 1) V_\downarrow(k), \quad V_\downarrow(k + 1) V_\uparrow(k) = \frac{(k + 1)^4}{k^2(k + 2)^2} V_\downarrow(k - 1) V_\downarrow(k),
\]
\[
(\{D, \sigma^\downarrow\} + \{\sigma^\uparrow, \sigma^\downarrow\} + \{\sigma'_i, \sigma'_i\}) \chi = 0 \Rightarrow
\]
\[
-4iV_\downarrow(k) - 2U_\downarrow(k + 1) V_\downarrow(k) + U_\downarrow(k)V_\downarrow(k) - 2T_\downarrow(k + 1) V'_\downarrow(k) + T_\downarrow(k) V'_\downarrow(k) = 0,
\]
\[
(\{\sigma_i, \sigma^\downarrow\} + \sigma^\uparrow \sigma^\downarrow + \sigma'_i \sigma'_i) \chi = 0 \Rightarrow V_\downarrow(k - 1) V_\downarrow(k) = W_\downarrow(k)V_\downarrow(k) \frac{k + 2}{k + 1},
\]
\[
U_\downarrow(k) V'_\downarrow(k) + T_\downarrow(k) V'_\downarrow(k) = 0,
\]
\[
(\{\sigma_i, \sigma^\downarrow\} + \sigma^\uparrow \sigma^\downarrow + \sigma'_i \sigma'_i) \chi = 0 \Rightarrow W_\downarrow(k) V_\downarrow(k) = V_\downarrow(k + 1) V_\downarrow(k) \frac{k + 3}{k + 2},
\]
\[
U_\downarrow(k + 1) V_\downarrow(k) + T_\downarrow(k + 1) V'_\downarrow(k) = 0.
\]
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