P-(S.P) Submodules and C1(Extending) Modules

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Abstract. In this work, we study C1-(extending) module M through the sub-module of M which have two properties namely prime and semi-prime (P. (S.P)). Indeed these properties with another concept like fully invariant of any sub-modules explain the main objective of this study. Also heart submodule of a module M and the socle of M are studied with fully invariant property to obtain the same objective.

Keywords. C₁-module, Multiplication module, Cyclic module, Heart sub-module, Socle module.

1. INTRODUCTION
Throughout this paper all rings will be multiplication with identity and all modules will be unitary. “Let M be an R-module. Any submodule N of M is prime when if rM ⊆ N, r ∈ R, m ∈ M \ N, then M ⊆ N and N ⊆ M is said to be semi-prime if N ≠ M and whenever r ∈ R, m ∈ M, and r^n m ∈ N for n where n ∈ N⁺, then rm ∈ N’[2]. “Any module M is called multiplication if N = IM, such that N ⊆ M and I is an ideal of R and AnnR(M/N) is a prime ideal, then N is a prime submodule of M” [3]. Moreover; to find more information about multiplication and cyclic modules one can see ([4]). In [5], “can find fully invariant property in details. Let (S, ∘) be an algebraic structure which has an identity eS”. If x∈S has an inverse, then x is say be invertible for ∘. That is, x is invertible if and only if: ∃ y ∈ S: x ∘ y = eS = y ∘ x.

Recall that a module M is called D₁- module if for each sub-module N of M; there exists a direct summand of M is is a coessential submodule of N; equivalently; if N, K ≤ M and H ≤ N, then M is direct summand H and K, (N∩H)≪ M. So D₁-module is C₁(extend)ing module.

In this paper, we studied a famous module in algebra through other properties of any submodule of this module like prime (semi-prime) sub-module and so is duo submodule.

2. MAIN RESULTS
Several concepts have been used in this section for the purpose of reaching the main objective of the paper, for example, heart submodule of M and socle of M. We also will use the fully invariant property to achieve the same goal. Note that P ideal of a ring R is called maximal ideal if P ≠ R and J also is ideal of a ring R then P ⊈ J ⊈ R. So J=R.
(*) Assume that $K$ be a proper sub module of multiplication module $M$. So for every $m, n \in M$, if $m, n \subseteq K$, then $m \in K$ or $n \in K$.

**Theorem 2.1.** Let $M$ be a $D_1$-module over an integral domain $R$. If $M$ is simple(cyclic) and $f(N) \subseteq N$ and satisfy (*), then $M$ is prime-duo-$C_1$-module.

**Proof.** Suppose that $M$ is simple(cyclic) and generated by one element $M = (X) = RX = \{ rx: r \in R, x \in M \}$. Or $M = \{ yR: y \in M \}$.

So $M$ is a multiplication $R$-module. So any $N \leq M$ is a prime sub-module with fully in variant property ($N$ is duo sub-module) and $N = SM \ni S$ is prime ideal of a ring $R$ and $\text{Ann}_R(M) \subseteq S$ imply that $M$ is P-duo-$C_1$-module.

Recall that $(L :_M N) = \{ m \in M, n \in N \rightarrow mn \leq L \}$, this is called the residual of $L$ by $N$ in $M$ such that $M$ is a multiplication $R$-module and $L, N \leq M$. Also, $\text{Ann}_M(N) = (0;_M N)$ is annihilator of $N$ in $M$.

**Proposition 2.2.** Let $R$ be a semi-local ring. If $M$ is cyclic $D_1$-module such that for any $N_1, N_2 \leq M$ are a fully invariant, $N_2 \not\subseteq N_1$ and $(L :_M N_2) = N_1$, then $M$ is P-duo-$C_1$-module.

**Proof.** Since $R$ is a semi-local ring, then $R$ having just finitely many maximal ideals ($I$ is a maximal ideal of a ring $R$ if there are no other ideals existing between $I$ and $R$).

So from ([4] Lemma3) $M$ is a cyclic $R$-module and hence $M$ is a multiplication $R$-module. But $(L :_M N_2) = N_1$ and has fully invariant. So $N_1$ is a $P$-sub module of $M$. Thus $M$ is a $P$-sub module.

Recall that if any sub module $N$ of $M$ is a $P$-sub-module, this means $M$ is prime module. Therefore we can introduce the following result:

**Corollary 2.3.** Let $M$ be a $D_1$-cyclic $R$-module If $M$ is a prime module and has fully invariant property, then $M$ is a P-duo-$C_1$-module.

**Proof.** The proof is very easy, because prime module gives every sub module of $M$ is prime with same way in with fully property, we obtain $M$ is P-duo-$C_1$-module.

Now we need to introduce two concepts namely heart sub-modules of $M$ ($H(M)$) and the socle of the module $M$ $\text{Soc}(M)$. $H(M)$ means intersection of all nonzero sub modules of $M$ and $H(M)$ is a minimal sub-module contained in non-zero sub-module when $H(M) \neq 0$.

Recall that if $M$ is any $R$-module, then the **socle** of $M$ can defined by

$$\text{Soc}(M) = \sum \{ N \leq M: N \text{ is simple} \}$$

On the other hand, the **socle** of $M$ is the largest submodule of $M$ generated by simple modules, or it is the largest semisimple submodule of $M$. Also, if $M_K$ be a module and $N \leq M$, then $N$ is called $h$-closed submodule of $M$ provided that $H(M/N) = 0$. 


Theorem 2.4. Let $M$ be a $C_1$-module. If $M$ has $h$-closed sub-module and for each $r \in R$, $r$ or $1 - r$ invertible element and $N_1 \nsubseteq N_2$; $\text{Ann}(N_2) = N_1 \nsubseteq N_1, N_2 \subseteq M$. Then $M$ is a $P$-dual-$C_1$-module.

Proof. Suppose that $I_1$ and $I_2$ are maximal ideal of $R$ s.t. $I_1 \nsubseteq I_2$. Hence $I_1 + I_2 = R$. So $1 = i_1 + i_2 \in I_1 \cap I_2$. Hence $I_1 = I_2$.

Since $i_1 \in I_1$; $i_1$ is not invertible element and hence $1 - i_1$ is invertible element. This implies $i_2$ is invertible element ($i_2 = 1 - i_1$) and this contradiction.

Since $i_2 \in I_2$ implies that $i_2$ is not invertible element. Then $R$ is local ring (R semi-local ring). Hence $M$ is cyclic $R$-module and then is multiplication $R$-module. Since $M$ have $h$-closed sub-module, then $H(M)$ is fully invariant. We have $N_1 \nsubseteq N_2$ and $\text{Ann}(N_2) = N_1$, therefore $N_1$ is a $P$-submodule. Thus $M$ is $P$-dual-$C_1$-module.

Proposition 2.5. Let $R$ be a PID. Let $M$ be a $D_1$-$R$-module If $M$ has $h$-closed sub module and $I$ is prime ideal of $R$; then $M$ is a $P$-dual-$C_1$-module.

Proof. Clear. If $I$ prime ideal of $R$ gives $I$ maximal ideal. Same proof of Theorem 2.4 we obtain the required.

Theorem 2.6. Let $M$ be multiplication $D_1$-$R$-module. If any sub module $N$ of $M$ is a fully invariant in $M$ $N_1, N_2$ subset of $N$, $N_1 \cap N_2$ subset of $N$ for all $N_1, N_2$ are sub-modules of $M$. So $M$ is $S$-$P$-dual-$C_1$-module.

Corollary 2.7. Every Semi Prime-sub-modules of a multiplication $C_1$-module is $\cap \{p$, $h$ - closed $\}$ sub modules and so $M$ is $P$-dual-$C_1$-module.

Proof. We know $h$-closed sub-modules means intersection of all sub-module. But $h$-closed ($H(M)$) is fully invariant in $M$ with prim property imply $M$ is $S$-$P$-dual-$C_1$-module.

Corollary 2.8. Let $M$ be a $D_1$-$R$-module. If $N \subseteq M$ is a fully invariant and $\text{Ann}(M) = 0$ over $R$ is a field, then $M$ is a $P$-dual-$C_1$-module.

Proof. First, we claim $R$ is not field. So $0 \neq N \subseteq M$ is a prime. Hence $R$ is adomain. So $T(M) = 0$ (M is a Torsion free $R$-module) over $R$. Note that when $R$ is not field this means $M = R x$ ($M$ is not simple module); such that $0 \neq R x \subseteq M$. If $x \neq 0$ and is not invertible element of $R$. So $R x m$ is a prime and $x M \subseteq R x m$; contradiction (because $M = R x$) or $x \in R x m$, then $R x = R$ and this contradiction. Hence $N$ is a prime sub-module of $M$. We have $N$ is a fully invariant (sub module).Thus $M$ is a $P$-dual-$C_1$-module.

Corollary 2.9. Let $M$ be a $D_1$-$R$-module. If the following statements are true:
1- $N = IM$; I proper ideal of $R$ and $N \subseteq M$;
2- $\text{Ann}(M) = 0$;
3- $M = X R$;
4- $N$ is duo submodule, then $M$ is $P$-dual-$C_1$-modul.

Lemma 2.10. $H(M)$ is a fully invariant submodules.
Proof. From definition of $H(M)$, we get $H(M)$ is a submodule of a module $M$. Take any homomorphism $g$ in $\text{End}(M)$, $(g: m \mapsto m)$. We need to prove that $g(H(M)) \subseteq H(M)$. If $H(M)=0$, then $H(M)$ is invariant submodule. Now let $H(M)\neq 0$. Therefore $H(M)$ is simple and hence $H(M)=\text{Soc}(H(M))$. So $G(H(M))=g(\text{Soc}(H(M))) \subseteq \text{Soc}(H(M))=H(M)$.

Then $H(M)$ is also fully invariant.

**Theorem 2.11.** Let $M$ be a $D_1$-modul. If $M$ satisfy the following statements:
1- $M\setminus N$ is a multiplicatively closed $N \subseteq M$;
2-$N=IM$;
3-$H(M)=\text{Soc}(M)$; then $M$ is $P$-duo-$C_1$-module.

**Proof.** From condition (2) $M$ is a multiplication module. Let $m, n \in M\setminus N$. From condition (1) $(mn \cap N \neq \emptyset)$ (see [1]). So $N$ is a proper prime submodule of $M$. We have $H(M)$ is fully invariant submodule (Lemma12) with condition (3), $\text{Soc}(M)$ is also fully invariant submodule ($N$ is a du submodule). Since $M$ is a $D_1$-module, then it is $C_1$-module. Thus $M$ is $P$-duo-$C_1$-module.

**Theorem 2.12.** Let $M$ be a $D_1$(extending) module and satisfy the following conditions.
1- $M$ semisimple module.
2- $M$ is Cyclicmodule.
3- Let $P \leq M$. For every $m, n \in M$, if $mn$ subset of $P$, then $m \in P$ or $n \in P$.
4- $M$ closed-duomodule. Then $M$ is $P$-duo-$C_1$(extending) module.

**Proof.** Assume that $N$ be any sub-module of a semisimple module $M$. By assumption $N$ is a directsummand of $M$; hence, it is a closed sub-module. We have $M$ is a closed-duo-module, then $N$ is isfully invariant. Since $M$ is cyclic module, then $M$ is a multiplication module with condition (3), we obtain $N$ is prime submodule, and we obtained the result.

3. Conclusion
In this study we proved many facts about $C_1$(extending)module. Every S.P-sub-modules of a multiplication $C_1$(Extending) module is $\cap \{p - h - \text{closed}\}$ sub-modules and so $M$ is $P$-duo-$C_1$(extending) module. Also, we investigated a fact which say if $\text{Soc}(M)=H(M)$ with prime sub-module this imply $M$ is a $P$-duo-$C_1$(extending) module.

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