Research Article

Semilinear Evolution Problems with Ventcel-Type Conditions on Fractal Boundaries

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Received 30 April 2013; Accepted 17 October 2013; Published 22 January 2014

Academic Editor: William E. Fitzgibbon

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A semilinear parabolic transmission problem with Ventcel’s boundary conditions on a fractal interface \(S\) or the corresponding prefractional interface \(S\) is studied. Regularity results for the solution in both cases are proved. The asymptotic behaviour of the solutions of the approximating problems to the solution of limit fractal problem is analyzed.

1. Introduction

In this paper we study the parabolic semilinear second-order transmission problem which we formally state as

\[
\begin{cases}
  u_t (t, P) - \Delta u (t, P) = J (u (t, P)) & \text{in } [0, T] \times Q^1, \\
  -c_0 \Delta_L u (t, P) = \left[ \frac{\partial u (t, P)}{\partial n} \right] & \text{on } [0, T] \times L, \\
  u (t, P) = 0 & \text{on } [0, T] \times \partial Q, \\
  u^1 (t, P) = u^2 (t, P) & \text{on } [0, T] \times L, \\
  u (t, P) = 0 & \text{on } [0, T] \times \partial L, \\
  u (0, P) = \phi & \text{on } Q,
\end{cases}
\]

where \(Q\) is the bounded open set \((-1,1)^2 \times (0,1)\), and \(L\) is a “cylindrical” layer dividing the set \(Q\) into two subsets \(Q^1\) and \(Q^2\) (see Figure 2). When \(L\) is the Koch-type surface \(S = K \times I\), where \(K\) is the snowflake and \(I = [0,1]\) (see Section 2), \(E_L\) is the energy functional introduced in (12); when \(L\) is the prefractional surface \(S_h\), \(E_L\) is the energy functional \(E_{S_h}\) introduced in (24). \(J\) is a nonlinear function from a subset of \(L^2 (Q)\) into \(L^2 (Q)\). \(u^1\) denotes the restriction of \(u\) to \(Q^1\), and \([\partial u/\partial n] = \partial u^1/\partial n_1 + \partial u^2/\partial n_2\) denotes the jump of the normal derivatives across \(L\), to be intended in a suitable sense.

More precisely, we assume that \(J(u)\) is a nonlinear mapping from \(L^{2p} (Q)\) to \(L^2 (Q)\) for any fixed \(p > 1\), locally Lipschitz; that is, Lipschitz on bounded sets in \(L^{2p} (Q)\) with Lipschitz constant \(l(r)\) when restricted to \(B(0, r) \subset L^{2p} (Q)\), satisfying suitable growth conditions (see conditions (i) and (ii) in Section 4). Examples of this type of nonlinearity include, for example, \(J(u) = u |u|^{p-1}\), \(p > 1\) which occur in combustion theory (see [1]) and in the Navier-Stokes system (see [2]).

In the recent years there has been an increasing interest in the study of linear transmission problems across irregular layers of fractal type and the corresponding prefractional layers [3–7]. Problems of this type are also known in the literature as problems with Ventcel’s boundary conditions [8] or second-order transmission conditions. Fractal layers can provide new interesting settings in those model problems, in which the surface absorption of tension, electric conduction, or flow is the relevant effect. The literature on semilinear equations on smooth domains is extensive (see e.g., [9–13] and the recent review in [14]); the fractal case is more awkward (see e.g., [15–19]).

In our case one has to take into account that the diffusion phenomenon takes place both across the smooth domain.
2 International Journal of Partial Differential Equations

\( Q \) and the cylindrical layer \( L \); this fact has a counterpart in the structure of the energy functional \( E[u] \) and hence on problem \( \overline{P} \). In [18] the authors proved local existence and uniqueness results of the “mild” solution of an abstract evolution transmission problem across a prefractal or fractal interface (see (36) and (37)).

In this paper we give a strong interpretation of the abstract problem studied in [18]; namely, we prove that the solution of the abstract problem solves problem \( \overline{P} \) in a suitable sense (see Theorems 22 and 20).

The results on the strong interpretation in the prefractal case are deduced by proving regularity results for the solutions of elliptic problems in polyhedral domains. It turns out that the restriction \( u^i_h \) of the solution \( u_h \) to \( Q^i_h \) belongs to suitable weighted Sobolev spaces (see the proof of Theorem 22). This regularity result is important not only in itself but also in the numerical approximation procedure; to this regard, see [20]. Following this point of view, it is also important to study the asymptotic behaviour of the solutions of the prefractal problems.

The proof of the convergence of the solution of the prefractal problems to the one of the (limit) fractal problem relies on the convergence, in the Mosco’s sense, of the energy forms which, in turn, implies the convergence of semigroups in the strong operator topology of \( L^2(Q) \) (see Theorem 16). The plan of the paper is as follows. In Section 2 we describe the geometry of the problem; in Section 3 we introduce the Dirichlet energy forms and the associated semigroups and we recall the results on the convergence of the approximating energy forms (see [21] for details). In Section 4 we recall existence and uniqueness results for the local mild solution as well as global existence and regularity results. In Section 5 we prove that the solution of the abstract Cauchy problems \( (P) \) and \( (P_h) \) solves problem \( \overline{P} \) in the fractal and prefractal cases, respectively, (see Theorems 22 and 20). In Section 6 we prove the convergence of the solutions of the approximating problems to the solution of the limit fractal problem in a suitable functional space. In Appendices A and B, for the reader convenience, we introduce the functional spaces and traces involved.

2. Geometry of the Fractal Layers \( S \) and \( S_h \)

In the paper by \( |P - P_0| \) we denote the Euclidean distance in \( \mathbb{R}^d \) and the Euclidean balls by \( B(P_0, r) = \{ P \in \mathbb{R}^d : |P - P_0| < r \}, P_0 \in \mathbb{R}^d, r > 0 \). By the Koch snowflake \( F \), we will denote the union of three coplanar Koch curves (see [22]) \( K_1, K_2, \) and \( K_3 \) as shown in Figure 1. We assume that the junction points \( A_1, A_2, \) and \( A_3 \) are the vertices of a regular triangle with unit side length; that is, \( |A_1 - A_2| = |A_2 - A_3| = |A_3 - A_1| = 1 \).

In this section we briefly recall the essential notions on the geometry; for details see [18].

The Hausdorff dimension of the Koch snowflake is given by \( d_f = \log 4/\log 3 \). This fractal is no longer self-similar (and hence not nested).

One can define, in a natural way, a finite Borel measure \( \mu_F \) supported on \( F \) by

\[
\mu_F := \mu_1 + \mu_2 + \mu_3,
\]

where \( \mu_i \) denotes the normalized \( d_f \)-dimensional Hausdorff measure, restricted to \( K_i, i = 1, 2, 3 \).

The measure \( \mu_F \) has the property that there exist two positive constants \( c_1 \) and \( c_2 \):

\[
c_1 r^d \leq \mu_F(B(P, r) \cap F) \leq c_2 r^d, \quad \forall P \in F,
\]

where \( d = d_f = \log 4/\log 3 \) and \( B(P, r) \) denotes the Euclidean ball in \( \mathbb{R}^2 \). As \( \mu_F \) is supported on \( F \), it is not ambiguous to write in (3) \( \mu_F(B(P, r)) \) in place of \( \mu_F(B(P, r) \cap F) \). In the terminology of Appendices A and B, we say that \( F \) is a \( d \)-set with \( d = d_f \).

Remark 1. The Koch snowflake can be also regarded as a fractal manifold (see [23] Section 2.2).

Let \( Q \) denote a bounded open set in \( \mathbb{R}^3 \); in our basic model, \( Q \) denotes the parallelepiped \( Q = (-1,1)^2 \times (0,1) \) and \( S \) denotes a “cylindrical” layer in \( Q \) of the type \( S = F \times I \), where \( I = [0,1] \) and \( F \) is the Koch snowflake. We assume that \( S \) is located in a median position inside \( Q \) and divides \( Q \) in two subsets \( Q_1 \) and \( Q_2 \) (see Figure 2).

We give a point \( P \in S \) the Cartesian coordinates \( P = (x, y) \), where \( x = (x_1, x_2) \) are the coordinates of the orthogonal projection of \( P \) on the plane containing \( F \) and \( y \) is the coordinate of the orthogonal projection of \( P \) on the \( y \)-line containing the interval \( I : P = (x, y) \), \( x = (x_1, x_2) \in F \), \( y \in I \).

One can define, in a natural way, a finite Borel measure \( m \) supported on \( S \) as the product measure

\[
dm = d\mu_F dy,
\]

where \( dy \) denotes the one-dimensional Lebesgue measure on \( I \). The measure \( m \) has the property that there exist two positive constants \( c_1 \) and \( c_2 \):

\[
c_1 r^d \leq m(B(P, r) \cap S) \leq c_2 r^d, \quad \forall P \in S,
\]

where \( d = d_f + 1 = \log 12/\log 3 \) and \( B(P, r) \) denotes the Euclidean ball in \( \mathbb{R}^3 \). As \( m \) is supported on \( S \), it is not ambiguous to write in (S) \( m(B(P, r)) \) in place of \( m(B(P, r) \cap S) \).
Thus $S$ turns out to be a $d$-set with $d = d_f + 1$ (see Appendices A and B).

By $S_h$, we denote the prefractal layer of the type $S_h = F_h \times I$, $h = 1, 2, \ldots, F_h$ is the piecewise linear prefractal approximation of $F$ at the step $h$. $S_h$ is a surface of polyhedral type. $S_h$ divides $Q$ in two subsets $Q_h^i$, $i = 1, 2$.

We give a point $P \in S_h$ the Cartesian coordinates $P = (x, y)$, where $x = (x_1, x_2)$ are the coordinates of the orthogonal projection of $P$ on the plane containing $F_h$ and $y$ is the coordinate of the orthogonal projection $P$ on the $y$-line containing the interval $I$.

3. Energy Forms and Semigroups Associated

3.1. The Energy Form $E$. In this section we introduce the energy functional on $S$. We first define the energy functional on the cross section $F$ by integrating its Lagrangian on $F$. For the concept of Lagrangian on fractals, that is, the notion of a measure-valued local energy, we refer to [24, 25]. Here for the sake of simplicity we only mention that the Lagrangian on $K$, $\mathcal{L}_K$, is a measure-valued map on $\mathcal{D}(F) \times \mathcal{D}(F)$ which is bilinear symmetric and positive ($\mathcal{L}_K[u]$ is a positive measure.) The measure-valued Lagrangian takes on the fractal $K$ the role of the Euclidean Lagrangian $d \mathcal{L}(u, v) = \nabla u \cdot \nabla v dx$. Note that in the case of the Koch curve, the Lagrangian is absolutely continuous with respect to the measure $\mu$; on the contrary, this is not true on most fractals (see [24]). In [23] the Lagrangian $\mathcal{L}_F$ on the snowflake $F$ has been defined by using its representation as a fractal manifold. Here we do not give details on the construction and definition of $\mathcal{L}_F$ and we refer to Section 4 in [23] for details; in particular in Definition 4.5 a Lagrangian measure $\mathcal{L}_F$ on $F$ and the corresponding energy form $\mathcal{E}_F$ as

$$\mathcal{E}_F(u, v) = \int_F d\mathcal{L}_F(u, v)$$

with domain $\mathcal{D}(F)$ have been introduced. The domain $\mathcal{D}(F)$, which is a Hilbert space with norm

$$\left( \|u\|_{\mathcal{D}(F, \mu_F)}^2 + \mathcal{E}_F(u, u) \right)^{1/2},$$

has been characterized in terms of the domains of the energy forms on $K$ (see [23] Theorem 4.6).

In the following, we will omit the subscript $F$, the Lagrangian measure will be simply denoted by $\mathcal{L}(u, v)$, and we will set $\mathcal{L}[u] = \mathcal{L}(u, u)$; an analogous notation will be adopted for the energies.

We define the energy forms $E_S$ on the fractal layer $S = F \times I$ by setting

$$E_S[u] = \sigma^1 \int_F \int_I \mathcal{L}_x[u] (dx) dy$$

$$+ \sigma^2 \int_F \int_I |\mathcal{D}_y u|^2 d\mu_F (dx),$$

where $\sigma^1$ and $\sigma^2$ are positive constants. Here $\mathcal{L}_x(\cdot, \cdot)(dx)$ denotes the measure-valued Lagrangian (of the energy form $\mathcal{E}_F$ of $F$ with domain $\mathcal{D}(F)$) now acting on $u(x, y)$ and $v(x, y)$ as function of $x \in F$ for a.e. $y \in I$; $\mu_F(dx)$ is the $d_f$-Hausdorff measure acting on each section $F$ of $S$ for a.e. $y \in I$ with $d_f = \log 4/\log 3$; $\mathcal{D}_y(\cdot)$ denotes the derivative in the $y$ direction.

The form $E_S$ is defined for $u \in \mathcal{D}(S)$, where $\mathcal{D}(S)$ is the closure in the intrinsic norm

$$\left\| u \right\|_{\mathcal{D}(S)} = \left( E_S[u] + \|u\|^2_{L^2(S, \mu)} \right)^{1/2}$$

of the set

$$C_0(S) \cap L^2((0, 1); \mathcal{D}(F)) \cap H^1_0((0, 1); L^2(F)),$$

where $L^2(F) = L^2(F, \mu_F(dx))$. In the following, we will also use the form $E_S(u, v)$ which is obtained from $E_S[u]$ by the polarization identity:

$$E_S(u, v) = \frac{1}{2} \left\{ E_S[u + v] - E_S[u] - E_S[v] \right\},$$

where $u, v \in \mathcal{D}(S)$.

**Proposition 2.** In the previous notations and assumptions, the form $E_S$ with domain $\mathcal{D}(S)$ is a regular Dirichlet form in $L^2(S, \mu)$ and the space $\mathcal{D}(S)$ is a Hilbert space under the intrinsic norm (9).
The proof can be carried on as in Proposition 3.1 of [26]. For the definition and properties of regular Dirichlet forms, we refer to [25]. We now define the Laplace operator on $S$. As $(E_S,\mathcal{D}(S))$ is a closed, bilinear form on $L^2(S,m)$, there exists (see Chapter 6, Theorem 2.1 in [27]) a unique self-adjoint, nonpositive operator $\Delta_S$ on $L^2(S,m)$—with domain $\mathcal{D}(\Delta_S) \subseteq \mathcal{D}(S)$ dense in $L^2(S,m)$—such that

$$E_S(u,v) = -\int_S (\Delta_S u) vdm, \quad u,v \in \mathcal{D}(\Delta_S).$$

(12)

Let $(\mathcal{D}(S))^\prime$ denote the dual of the space $\mathcal{D}(S)$. We now introduce the Laplace operator on the fractal $S$ as a variational operator from $\mathcal{D}(S) \to (\mathcal{D}(S))^\prime$ by

$$E_S(z,w) = -\langle \Delta_S z, w \rangle_{(\mathcal{D}(S))^\prime}$$

(13)

for $z \in \mathcal{D}(S)$ and for all $w \in \mathcal{D}(S)$, where $\langle \cdot, \cdot \rangle_{(\mathcal{D}(S))^\prime}$ is the duality pairing between $(\mathcal{D}(S))^\prime$ and $\mathcal{D}(S)$. We use the same symbol $\Delta_S$ to define the Laplace operator both as a self-adjoint operator in (12) and as a variational operator in (13). It will be clear from the context to which case we refer.

In the next, we will also use the spectral dimension $d$ of $S$. We find that if $r(\lambda)$ is the number of eigenvalues associated with $E_S$ smaller than $\lambda$, then $r(\lambda) \sim \lambda^{d/2}$. It can be shown that in our case $d = 2$ (see [28, 29]). We stress the fact that in the fractal case $\nu < d < D$, while in the Euclidean setting $\nu = d$.

Consider now the space of functions $u : Q \to R$ as

$$V(Q,S) = \{ u \in H^1_0(Q) : u|_S \in \mathcal{D}(S) \}.$$  

(14)

Here we denote by the symbol $f|_S$ the trace $\gamma_0 f$ of $f$ to $S$ (see Appendices A and B).

The space $V(Q,S)$ is nontrivial; see Proposition 3.3 of [4]. We now introduce the energy form

$$E[u] = \int_Q |Du|^2 dQ + \alpha E_S [u|_S]$$

(15)

defined on the domain $V(Q,S)$. Here and in the following, $dQ$ denotes the 3-dimensional Lebesgue measure and $E(u,v)$ denotes the corresponding bilinear form

$$E(u,v) = \int_Q DuDv dQ + \alpha E_S (u|_S, v|_S)$$

(16)

defined on $V(Q,S) \times V(Q,S)$.

As in Theorem 3.2 of [26], the following result can be proved.

**Proposition 3.** The form $E$ defined in (15) is a regular Dirichlet form in $L^2(Q)$ and the space $V(Q,S)$ is a Hilbert space equipped with the scalar product

$$(u,v)_{V(Q,S)} = E(u,v).$$

(17)

We denote by $\|u\|_{V(Q,S)}$ the norm in $V(Q,S)$, associated with (17), that is

$$\|u\|_{V(Q,S)} = \left( \alpha E_S [u|_S] + \int_Q |Du|^2 dQ \right)^{1/2}.$$  

(18)

As in Propositions (3.6) and (3.1) in [4], the following result can be proved.

**Proposition 4.** The space $\mathcal{D}(S)$ is embedded in $B^{2,2}_\beta$, $\beta = d/2$.

**Proposition 5.** The space $\mathcal{D}(S)$ is embedded in $B^{2,2}_\alpha$, $\alpha < 1$.

As $(E,V(Q,S))$ is a closed bilinear form on $L^2(Q)$ with domain $V(Q,S)$ dense in $L^2(Q)$, there exists (see Chapter 6 Theorem 2.1 in [27]) a unique self-adjoint nonpositive operator $A$ on $L^2(Q)$ with domain $\mathcal{D}(A) \subseteq V(Q,S)$ dense in $L^2(Q)$ such that

$$E(u,v) = -\int_Q Au v dQ, \quad u \in \mathcal{D}(A), \quad v \in V(Q,S).$$

(19)

Moreover in Theorem 13.1 of [25] it is proved that to each closed symmetric form $E$ a family of linear operators $\{G_a, \alpha > 0\}$ can be associated with the property

$$E(G_a u, v) + \alpha (G_a u, v) = (u,v),$$

(20)

taking $a = 2, 2$. This family is a strongly continuous resolvent with generator $A$, which also generates a strongly continuous semigroup $(T(t))_{t \geq 0}$.

For the reader's convenience, we recall here the main properties of the semigroup $(T(t))_{t \geq 0}$: the reader is referred to Proposition 3.5 in [21] for the proof.

**Proposition 6.** Let $(T(t))_{t \geq 0}$ be the semigroup generated by the operator $A$ associated with the energy form in (19). Then $(T(t))_{t \geq 0}$ is an analytic contraction positive preserving semigroup in $L^2(Q)$.

**Remark 7.** It is well known that the symmetric and contraction analytic semigroup $T(t)$ uniquely determines analytic semigroups on the space $L^p$, $1 \leq p < \infty$ (see Theorem 1.4.1 [30]) which we still denote by $T(t)$ and by $A_p$ its infinitesimal generator.

From Theorem 2.11 in [31], the following estimate on the decay of the heat semigroup holds.

**Proposition 8.** There exists a positive constant $M$ such that

$$\|T(t)\|_{L^2 \to L^\infty} \leq \begin{cases} Mr^{n/2}, & \text{for every } t \in (0,1] \\ Mr^{-\nu}, & \text{for every } t \in [1, \infty) \end{cases}.$$

(21)

One will consider the case $n = 3$ and $\nu = 2$; here $\nu$ is the spectral dimension of $S$.

From interpolation theory results, it can be proved (see Section 3.1 in [18]) that

$$\|T(t)\|_{L^2 \to L^p} \leq \begin{cases} M^{(1/2)-(1/2)p}r^{-(n/4)(1-1/p)}, & \text{for every } t \in (0,1] \\ M^{(1/2)-(1/2)p}r^{-(\nu/4)(1-1/p)}, & \text{for every } t \in [1, \infty). \end{cases}$$

(22)
3.2. The Energy Forms $E_{S_h}$. By $Q$ we denote the parallelepiped as defined in Section 3 and by $S_h$ we denote the prefractal layer of the type $S_h = F_h \times I$, $h = 1, 2, \ldots$, $F_h$ is the prefractal approximation of $F$ at the step $h$ (see Section 2). $S_h$ divides $Q$ in two subsets $Q_i$, $i = 1, 2$.

We first construct the energy forms $E_{S_h}$ on the prefractal layers $S_h = F_h \times I$, $h \in \mathbb{N}$. By $\ell$ we denote the natural arc-length coordinate on each edge of $F_h$ and we introduce the coordinates $x_1 = x_1(\ell)$, $x_2 = x_2(\ell)$, and $y = y$ on every affine “face” $S_h^{(i)}$ of $S_h$. By $d\ell$ we denote the one-dimensional measure given by the arc-length as defined in Section 3 and by affine “face” $F_h$ approximations of functions in the form (see Appendix A and B). By Fubini theorem, we can write this

$$V(Q, S_h) \cup V(Q, S_h) = \{u \in H^1_0(Q) : u|_{S_h} \in H^1_0(S_h)\}; \quad (23)$$

We denote the corresponding bilinear form by $E_{S_h}(u, v)$. In the sequel we denote by the symbol $f|_{S_h}$ the trace $\gamma_0 f$ to $S_h$.

Consider now the space of functions $u : Q \rightarrow \mathbb{R}$ as

$$V(Q, S_h) = \{u \in H^1_0(Q) : u|_{S_h} \in H^1_0(S_h)\}; \quad (24)$$

It is not trivial as it contains $\partial Q$.

Consider now the energy form

$$E_{S_h}[u] = \sigma_1^h \int_I \left( \int_{F_h} |Du|^2 d\ell \right) dy + \sigma_2^h \int_I \left( \int_{F_h} |D_y u|^2 d\ell \right) dy,$$

(25)

defined on the domain $V(Q, S_h)$.

By $E_{S_h}(u, v)$ we will denote the corresponding bilinear form

$$E_{S_h}(u, v) = \int_{Q} Du Dv dQ + E_{S_h}[u|_{S_h}, v|_{S_h}], \quad (26)$$

defined on $V(Q, S_h) \times V(Q, S_h)$.

**Theorem 9.** The form $E_{S_h}$, defined in (26), with domain $V(Q, S_h)$ is a regular Dirichlet form in $L^2(Q)$ and the space $V(Q, S_h)$ is a Hilbert space equipped with the scalar product

$$(u, v)_{V(Q, S_h)} = E_{S_h}(u, v). \quad (27)$$

For the proof, see Theorem 4.1 in [4].

We denote by $\|u\|_{V(Q, S_h)}$ the corresponding energy norm in $V(Q, S_h)$; that is,

$$\|u\|_{V(Q, S_h)} = \left( \int_{Q} |Du|^2 dQ + E_{S_h}[u|_{S_h}] \right)^{1/2}. \quad (28)$$

Proceeding as in Section 3.1 we denote by $\{G_h^\alpha, \alpha > 0\}, A_h,$ and $\{T_h(t)\}_{t \geq 0}$ the resolvents, the generators, and the semigroups associated to $E_{S_h}$, for every $h \in \mathbb{N}$, respectively.

As in Proposition 6, the following result can be proved.

**Proposition 10.** Let $\{T_h(t)\}_{t \geq 0}$ be the semigroup generated by the operator $A_h$ associated with the energy form in (27). Then $\{T_h(t)\}_{t \geq 0}$ is an analytic contraction positive preserving semigroup in $L^2(Q)$.

By proceeding as in Remark 7, one can show that for every $h \in \mathbb{N}$ the symmetric and contraction analytic semigroup $T_h(t)$ uniquely determines analytic semigroups on the space $L^p$, $1 < p < \infty$ (see Theorem 1.4.1 [30]) which we still denote by $T_h(t)$ and by $A_p^h$ its infinitesimal generator.

The following estimate on the decay of the heat semigroup holds (see e.g., [32]).

**Proposition 11.** There exists a positive constant $\overline{M}$ such that

$$\|T_h(t)\|_{L^1 \rightarrow L^\infty} \leq \frac{\overline{M}^\nu t^{-n/2}}{t^{-\nu/2}}, \quad \text{for every } t \in [0, 1], \quad \text{and } \nu = \text{the Euclidean dimension of } S,$$

(29)

where $\overline{M}$ does not depend on $h$. One considers the cases $n = 3$ and $\nu = 2$; here $\nu$ is the Euclidean dimension of $S$.

As before by interpolation results it can be proved that

$$\|T_h(t)\|_{L^1 \rightarrow L^p} \leq \min \left\{ \frac{\overline{M}^\nu t^{-n/2}}{t^{-\nu/2}}, \frac{\overline{M}^{(1/2)(1-1/p)}}{t^{-(n/4)(1-1/p)}}, \frac{\overline{M}^{(1/2)(1-1/p)}}{t^{-(n/4)(1-1/p)}}, \quad \text{for every } t \in [1, \infty) \right\}. \quad (30)$$

3.3. The Convergence of Forms and Semigroups. We now recall the results proved in [21] on the convergence of the approximating energy forms $E_{S_h}$ to the fractal energy $E$. In this asymptotic behaviour, the factors $\sigma_1^h$ and $\sigma_2^h$ have a key role and can be regarded as a sort of renormalization factors of the approximating energies. These factors take into account the nonrectifiability of the curve $F$ and hence the irregularity of the surface $S$ and in particular the effect of the $D$-dimensional length intrinsic to the curve; for details, see [6]. The convergence of functional is here intended in the sense of the $M$-convergence which we define below.

3.3.1. The $M$-Convergence of Forms. We recall, for the sake of completeness, the definition of $M$-convergence of forms introduced by Mosco in [33].

We extend the form $E$ defined in (15) and $E_{S_h}$ defined in (26) on the whole space $L^2(Q)$ by defining

$$E[u] = +\infty \quad \text{for every } u \in \frac{L^2(Q)}{V(Q, S)}; \quad (31)$$

$$E_{S_h}[u] = +\infty \quad \text{for every } u \in \frac{L^2(Q)}{V(Q, S)}.$$

(32)
Definition 12. A sequence of form \( \{E^{(h)}\} \) \( M \)-converges to a form \( E \) in \( L^2(Q) \) if

(a) for every \( \{v_h\} \) converging weakly to \( u \) in \( L^2(Q) \)
\[
\liminf_{h \to \infty} E^{(h)}[v_h] \geq E[u];
\]

(33)

(b) for every \( u \in L^2(Q) \) there exists \( \{w_h\} \) converging strongly to \( u \) in \( L^2(Q) \) such that
\[
\limsup_{h \to \infty} E^{(h)}[w_h] \leq E[u].
\]

(34)

Definition 13. The sequence of forms \( \{E^{(h)}\} \) is asymptotically compact in \( L^2(Q) \) if every sequence \( \{u_h\} \) with
\[
\liminf_{h \to \infty} E^{(h)}[u_h] + \int_Q |u_h|^2 dQ < \infty
\]

(35)

has a subsequence strongly convergent in \( L^2(Q) \).

Proposition 14. The sequence of forms \( \{26\} \) is asymptotically compact in \( L^2(Q) \).

Remark 15. We point out that, as the sequence of forms \( \{26\} \) is asymptotically compact in \( L^2(Q) \), \( M \)-convergence is equivalent to the \( \Gamma \)-convergence (see Lemma 2.3.2 in [34]) and thus we can take in (a) \( v_h \) strongly converging to \( u \) in \( L^2(Q) \).

Theorem 16. Let \( \sigma^1_h = \sigma_{10}(3^d - 1)^h \) and \( \sigma^2_h = \sigma_{20}(3^d - 1)^h \); then the sequence of forms \( \{E^{(h)}\} \) defined in (26) \( M \)-converges in the space \( L^2(Q) \) to the form \( E \) defined in (15). The sequence of semigroups \( \{T_h(t)\} \) associated with the form \( E^{(h)} \) converges to the semigroup \( T(t) \) associated with the form \( E \) in the strong operator topology of \( L^2(Q) \) uniformly on every interval \([0, t_1]\).

4. Evolution Problems: Existence and Convergence of the Solutions

In this Section we recall the results on existence and uniqueness of the solution of the abstract problems \( (P) \) and \( (P_h) \) (see below) and the asymptotic behaviour of the solutions of the abstract problems. In Section 5 we will show that the solutions of the abstract problems solve \( (P) \) in both cases. We refer the reader to [18].

We consider the abstract Cauchy problems as

\[
(P) \quad \frac{d u(t)}{d t} = A u(t) + J(u(t)), \quad 0 \leq t \leq T
\]

(36)

and for every \( h \in \mathbb{N} \)

\[
(P_h) \quad \frac{d u_h(t)}{d t} = A_h u_h(t) + J(u_h(t)), \quad 0 \leq t \leq T
\]

(37)

where \( A : \mathcal{D}(A) \subset L^2(Q) \to L^2(Q) \) and \( A_h : \mathcal{D}(A_h) \subset L^2(Q) \to L^2(Q) \) are the generators associated, respectively, to the energy form \( E \) and the energy forms \( E^{(h)} \) introduced in (15) and (26), \( T \) is a fixed positive real number, and \( \phi \) and \( \phi_h \) are given functions in \( L^2(Q) \). We assume that \( J \) is a mapping from \( L^{2p}(Q) \to L^2(Q) \), \( p > 1 \) locally Lipschitz, that is, Lipschitz on bounded sets in \( L^{2p}(Q) \); we let \( l(r) \) denote the Lipschitz constant of \( J \):

\[
\|J(u) - J(v)\|_{L^2(Q)} \leq l(r) \|u - v\|_{L^{2p}(Q)},
\]

(38)

where \( \|u\|_{L^{2p}(Q)} \leq r \), \( \|v\|_{L^{2p}(Q)} \leq r \). We also assume that \( J(0) = 0 \). This assumption is not necessary in all that follows, but it simplifies the calculations (see [11]). In order to prove the local existence theorem, we make the following assumptions on the growth of \( l(r) \) when \( r \to \infty \).

We set for brevity \( a := (n/4)(1 - (1/p)) \); we note that \( 0 < a < 1 \), for \( n \leq 4 \), and \( p > 1 \).

(i) There exists \( 0 < b < a \) such that \( l(r) = o(r^{(1-a)/b}) \), \( r \to \infty \).

(ii) Consider

\[
\int_0^\infty l(r) r^{-1/4} dr < \infty
\]

(39)

for every \( r > 0 \).

We note that (ii) implies (i) for all \( 0 < b < a \) since \( l(r) \) is nondecreasing and

\[
\int_r^{2r} l(s) s^{-1/4} ds \geq rl(r)(2r)^{-1/4}.
\]

(40)

Thus \( l(r)r^{-1(1-a)/4} \) is bounded as \( r \to \infty \) which implies (i) for \( 0 < b < a \).

In Theorem 5.1 of [18], the following local existence theorem has been proved.

Theorem 17. Let condition (i) hold. Let \( K > 0 \) be sufficiently small if \( \phi \in L^2(Q) \) and

\[
\limsup_{t \to 0} \|e^{t T(t)} \phi\|_{L^2(Q)} < K.
\]

(41)

There is a \( T > 0 \) and \( a \)

\[
u \in C([0, T]; L^2(Q))
\]

(42)

with \( u(0) = \phi \) satisfying

(1) \( u \in C((0, T]; L^{2p}(Q)) \), and \( \|e^{t \phi} u(t)\|_{L^{2p}} < 2K \);

(2) for every \( t \in [0, T] \),

\[
u(t) = T(t) \phi + \int_0^t T(t - s) J(u(s)) ds
\]

(43)

with the integral being both an \( L^2 \)-valued and \( L^{2p} \)-valued Bochner integral;
(3) If \( v : (0, T_1] \to L^{2p} \) is strongly measurable with \( T_1 \leq T \), \( \| v(t) \|_{L^{2p}} \leq 2K \) and also satisfies (43), then \( u(t) = v(t) \), for every \( t \in (0, T_1] \).

Let condition (ii) hold; there exist a \( T > 0 \) and a unique \( u(t) \in C([0, T]; L^2(Q)) \) with \( u(0) = \phi \) satisfying

\[
(1) \; u \in C((0, T]; L^2(Q))
\]

\[
\lim_{t \to 0} \sup \| t^a u(t) \|_{L^{2p}} < \infty; \quad (44)
\]

(2) For every \( t \in [0, T] \), \( u(t) \) satisfies (43) with the integral being both an \( L^2 \)-valued and \( L^2p \)-valued Bochner integral;

(3) If \( v : (0, T_1] \to L^{2p} \) is strongly measurable with \( T_1 \leq T \), \( \| v(t) \|_{L^{2p}} \) bounded and also satisfies (43), then \( u(t) = v(t) \), for every \( t \in (0, T_1] \).

The claims of the Theorem is proved by a contraction mapping argument on suitable spaces of continuous functions with values in Banach space.

By exploiting the analyticity of the semigroup \( T(t) \) both on \( L^2(Q) \) and \( L^2p(Q) \), the following regularity result for the maximal solution holds (see Theorem 5.3 [18]).

Theorem 18. Under the assumptions of Theorem 17, one has that the solution \( u(t) \) can be continuously extended to a maximal interval \( (0, T_\phi) \) as a solution of (43), until \( \| u(t) \|_{L^{2p}(Q)} \to \infty \) as \( t \to T_\phi \), and it is a classical solution; that is,

\[
\begin{align*}
\text{if } v : (0, T_1] \to L^2(Q) & \text{ strongly measurable with } T_1 \leq T, \quad \| v(t) \|_{L^2(Q)} \leq 2K, \\
\text{then } u(t) = v(t) & \text{ for every } t \in (0, T_1].
\end{align*}
\]

\[ u \in C \left( [0, T_\phi]; L^2(Q) \right) \cap C \left( (0, T_\phi); \mathscr{D}(A) \right) \quad (45) \]

and satisfies

\[
\frac{du}{dt} (t) = Au(t) + J(u(t)), \quad \text{for every } t \in (0, T_\phi). \quad (46)
\]

For every fixed \( h \in \mathbb{N} \), the claims of Theorems 17 and 18 hold for problem \( (P_h) \) with the obvious changes.

We now recall the convergence results of the sequence of the approximating solutions \( \{u_h\} \) when \( h \) goes to infinity (see Theorem 6.2 in [18]).

Theorem 19. Let \( u \) and \( u_h \) be the mild solutions of problems \( (P) \) and \( (P_h) \); let \( \alpha^1 \) and \( \alpha^2 \) be as in Theorem 16. In the notations and assumptions of Theorem 17, one has the following:

(a) let assumption (i) hold; let \( \phi_h \) and \( \phi \) belong to \( L^q(Q) \) with \( q = 2p/(n + 4pb) \) and \( \phi_h \to \phi \) in \( L^q(Q) \); then

\[
\begin{align*}
\text{if } v : (0, T_1] \to L^2(Q) & \text{ strongly measurable with } T_1 \leq T, \quad \| v(t) \|_{L^2(Q)} \leq 2K, \\
\text{then } u_h \to u & \text{ in } C \left( [0, T]; L^2(Q) \right), \quad (47)
\end{align*}
\]

(b) if assumption (ii) holds and \( \phi_h \to \phi \) in \( L^2(Q) \), then

\[
\begin{align*}
\text{if } v : (0, T_1] \to L^2(Q) & \text{ strongly measurable with } T_1 \leq T, \quad \| v(t) \|_{L^2(Q)} \leq 2K, \\
\text{then } u_h \to u & \text{ in } C \left( [0, T]; L^2(Q) \right), \quad (48)
\end{align*}
\]

with \( a = n/4(1 - 1/p) \).

5. Strong Formulation of the Transmission Problems

5.1. The Fractal Layer

Theorem 20. Let \( u \) be the solution of problem \( (P) \). Then one has, for every fixed \( t \in (0, T] \),

\[
\begin{align*}
& \text{if } v : (0, T_1] \to L^2(Q) \text{ strongly measurable with } T_1 \leq T, \quad \| v(t) \|_{L^2(Q)} \leq 2K, \\
& \text{then } u(t) = v(t) \text{ for every } t \in (0, T_1].
\end{align*}
\]

Theorem 21. Under assumption (i) hold, let \( \phi \) belong to \( L^q(Q) \) with \( q = 2p/(n + 4pb) \) and \( \phi_h \to \phi \) in \( L^q(Q) \); then

\[
\begin{align*}
\text{if } v : (0, T_1] \to L^2(Q) & \text{ strongly measurable with } T_1 \leq T, \quad \| v(t) \|_{L^2(Q)} \leq 2K, \\
\text{then } u_h \to u & \text{ in } C \left( [0, T]; L^2(Q) \right), \quad (47)
\end{align*}
\]

(b) if assumption (ii) holds and \( \phi_h \to \phi \) in \( L^2(Q) \), then

\[
\begin{align*}
\text{if } v : (0, T_1] \to L^2(Q) & \text{ strongly measurable with } T_1 \leq T, \quad \| v(t) \|_{L^2(Q)} \leq 2K, \\
\text{then } u_h \to u & \text{ in } C \left( [0, T]; L^2(Q) \right), \quad (48)
\end{align*}
\]

with \( a = n/4(1 - 1/p) \).

Theorem 22. Let \( u \) be the solution of problem \( (P) \). Then one has, for every fixed \( t \in (0, T] \),

\[
\begin{align*}
& \text{if } v : (0, T_1] \to L^2(Q) \text{ strongly measurable with } T_1 \leq T, \quad \| v(t) \|_{L^2(Q)} \leq 2K, \\
& \text{then } u(t) = v(t) \text{ for every } t \in (0, T_1].
\end{align*}
\]

The claim of the Theorem is proved by a contraction mapping argument on suitable spaces of continuous functions with values in Banach space.

By exploiting the analyticity of the semigroup \( T(t) \) both on \( L^2(Q) \) and \( L^2p(Q) \), the following regularity result for the maximal solution holds (see Theorem 5.3 [18]).
we obtain $\Delta u(t, P) = u_t(t, P) - J(u(t, P))$ and since the right-hand side belongs to $C((0, T]; L^2(Q))$ we deduce that $\Delta u(t, P) \in C((0, T]; L^2(Q))$; hence $u(t, \cdot) \in C((0, T]; V(Q)))$, where

$$V(Q) = \{u \in H^1_0(Q); \Delta u \in L^2(Q)\}; \quad (53)$$

here the Laplacian is intended in the distributional sense. By proceeding as in (3.26) of [4], we prove that, for every fixed $t$, the normal derivative $\partial u/i/\partial n_i$ is in the dual $(B_{\beta,0}^{2,2}(S))'$ of the space $(B_{\beta,0}^{2,2}(S))$, where $\beta = d/2$ and

$$\left\langle \frac{\partial u}{\partial n_i}, v|_S \right\rangle_{(B_{\beta,0}^{2,2}(S))',(B_{\beta,0}^{2,2}(S))} = \int_Q Du(t, P) Dv(P) \, dQ + \int_{\partial Q} v(P) \Delta u(t, P) \, dQ \quad (54)$$

for every $t \in (0, T]$ and every $v \in H^1_0(Q)$. By proceeding as in Section 6.1 of [21], we can prove that $\partial u/i/\partial n_i \in C((0, T]; (B_{\beta,0}^{2,2}(S))')$.

From Proposition 4 and proceeding as in Section 6 of [3], it can be proved that the transmission condition

$$-c_0 \Delta_S u|_S = \left[ \frac{\partial u}{\partial n} \right] \quad \text{holds in} \ (\mathcal{D}(S))'. \quad (55)$$

That is, for every $t \in (0, T)$,

$$-c_0 \langle \Delta_S u|_S, z \rangle_{(\mathcal{D}(S))' \times (\mathcal{D}(S))} = \left[ \left\langle \frac{\partial u}{\partial n}, z \right\rangle_{(\mathcal{D}(S))' \times (\mathcal{D}(S))} \right] \quad (56)$$

As a consequence of Theorem 20, the solution of problem (P) is the solution of the following transmission problem. For every $t \in (0, T)$,

\begin{align*}
\text{(j)} & \quad u_i^1 - \Delta u_i^1 = f(u_i^1) \quad \text{in} \ L^2(Q_i), \quad i = 1, 2, \\
\text{(jj)} & \quad -c_0 \Delta_S u = \left[ \frac{\partial u}{\partial n} \right] \quad \text{in} \ (\mathcal{D}(S))', \\
\text{(jjj)} & \quad u = 0 \quad \text{in} \ H^{1/2}(\partial Q), \\
\text{(jv)} & \quad u^1 = u^2 \quad \text{in} \ B_{d+1/2}^{2,2}(S), \\
\text{(v)} & \quad u = 0 \quad \text{in} \ B_{d-1/2}^{2,2}(\partial S). \quad (57) \quad (58) \quad (59) \quad (60) \quad (61)
\end{align*}

Remark 21. Actually from Proposition 6, one deduces that equalities (jv) and (v), respectively, hold in $B_{\alpha/2}^{2,2}(S)$ and in $B_{\alpha-1/2}(\partial S)$ with $\alpha < 1$.

5.2. The Prefractal Layer

Theorem 22. Let $u_h$ be the solution of problem $(P_h)$. Then one has, for every fixed $t \in (0, T]$,

$$(u_h)_i^1(t, P) - \Delta u_h(t, P) = J(u_h(t, P)) \quad \text{for} \ P \in Q_i^h, \ \text{a.e.} \ i = 1, 2,$$

$$\frac{\partial u_h^i}{\partial n} \in L^2(S_h), \quad i = 1, 2,$$

$$-\Delta_S u_h|_{S_h} = \left[ \frac{\partial u_h}{\partial n} \right], \quad \text{in} \ L^2(S_h), \quad (62)$$

$$u(t, P) = 0 \quad \text{for} \ P \in \partial Q,$$

$$u_h^1 = u_h^2 \quad \text{in} \ H^1_0(S_h),$$

$$u_h(0, P) = \phi(P) \quad \text{in} \ L^2(Q).$$

where $u_h^i$ is the restriction of $u_h$ to $Q_i^h$, $[\partial u_h^i/\partial n] = (\partial u_h^1/\partial n_1) + (\partial u_h^2/\partial n_2)$ is the jump of the normal derivatives across $S_h, n_i,$ $i = 1, 2$, is the inward normal vector, and $\Delta_S = \sigma_1^1 D_x^2 + \sigma_2^2 D_y^2$ is the piecewise tangential Laplacian associated to the Dirichlet form $E_{S_h}$. Moreover $\partial u_h^i/\partial n_i \in C((0, T]; L^2(S_h))$.

Proof. The first equality in (62) easily follows by proceeding as in Theorem 20. From this, it follows that, for every $t \in (0, T]$,

$$u_h(t, P) \in V(Q_h^i) = \{u \in H^1_0(Q_i^i); \Delta u_h^i \in L^2(Q_i^h)\}. \quad (63)$$

For every fixed $t \in (0, T]$, let $u_i^h$ denote the restriction of the solution $u_h$ to $Q_i^h$. By usual duality arguments (see Appendix 4 in [35]), the normal derivatives $\partial u_i^h/\partial n_i$, $i = 1, 2$, belong to the dual space of $H^{1/2}_0(S_h)$. By proceeding as in Section 6.2 of [21], it is possible to prove that $\partial u_h^i/\partial n_i \in C((0, T]; (H^{1/2}_0(S_h))')$.

Then, by the Green formula for Lipschitz domains, one can prove that

$$\left\langle -\Delta_S u|_{S_h}, z \right\rangle_{(H^{3/2}_0(S_h))' \times (H^{3/2}_0(S_h))} = \left\langle \left[ \frac{\partial u_h}{\partial n} \right], z \right\rangle_{(H^{1/2}_0(S_h))' \times (H^{1/2}_0(S_h))}. \quad (64)$$

That is, the transmission condition

$$-\Delta_S u|_{S_h} = \left[ \frac{\partial u_h}{\partial n} \right] \quad (65)$$

holds in the dual of $H^{3/2}_0(S_h)$ (see Proposition 2.2 in [5] for details). In order to prove that $\partial u_h^i/\partial n_i \in L^2(S_h)$, we proceed as in Section 4.2 of [4]. Let us consider, for each fixed
\( t \in (0, T] \), the weak solutions \( \bar{u}_h^i \) and \( \bar{w}_h^i \) in \( H^1(Q_h) \) of the following auxiliary problems:

\[
\begin{align*}
\Delta \bar{w}_h^i &= 0 \quad \text{in } Q_h^i, \\
\bar{w}_h^i &= u_{h} \quad \text{on } \partial Q_h^i, \\
-\Delta u_{h}^i &= -(u_{h}^i)_t + J(u_{h}^i) \quad \text{in } Q_h^i, \\
u_{h}^i &= 0 \quad \text{on } \partial Q_h^i.
\end{align*}
\]  

(66)

The regularity of \( u_{h}^i \) follows from the regularity of \( u_{h}^i \) and \( \bar{w}_h^i \) since

\[ u_{h}^i = u_{h}^i + \bar{w}_h^i. \]  

(68)

From a regularity result of Jerison and Kenig (see Theorems 2 and 3 of [36]), we deduce that

\[ \frac{\partial \bar{w}_h^i}{\partial n_i} \in L^2(S_h) \]  

(69)

and \( \| \frac{\partial \bar{w}_h^i}{\partial n_i} \| \leq C(h) \| u_{h}^i \|_{L^2(S_h)} \leq C(h) \| u_h \|_{L^2(D(A_h))}. \)

As to the solution \( w_{h}^i \) of (67), we preliminarily observe that the right-hand side in the first equation of (67) belongs to \( L^2(Q_h) \). From Proposition 4.5 in [4], it follows that

\[ w_{h}^i \in H^s(Q_h^i), \quad i = 1, 2, \]  

(70)

where \( 1 < s_1 < 8/5 \) and \( 1 < s_2 < 7/4 \); hence

\[
D^\alpha \bar{w}_h^1 \in H^{(3/5) - \varepsilon}(Q_h^i), \quad D^\alpha \bar{w}_h^2 \in H^{(3/4) - \varepsilon}(Q_h^i),
\]

\[ |\alpha| = 1 \]

(71)

for every \( \varepsilon > 0 \); then by trace results (see Proposition A.1), we obtain, for \( i = 1, 2, \)

\[
\frac{\partial u_{h}^i}{\partial n_i} \in L^2(S_h) \]

(72)

and \( \| \frac{\partial u_{h}^i}{\partial n_i} \|_{L^2(S_h)} \leq C(h) \| - (u_{h}^i)_t + J(u_{h}^i) \|_{L^2(Q_h)}. \) It follows from (67), (68), and (69) that \( \bar{u}_{h}^i, \bar{w}_h^i \in L^2(S_h), \) and hence the jump belongs to \( L^2(S_h). \) As \( H^1_{0,L}(S_h) \) is dense in \( L^2(S_h) \) (see e.g., [37]), we deduce that the transmission condition (64) actually holds in the \( L^2 \)-sense and in particular \( \Delta_{S_h} u_{h}^i \in L^2(S_h). \) The proof that \( \frac{\partial u_{h}^i}{\partial n_i} \in C((0, T); L^2(S_h)) \) easily follows from (69), (72), and the fact that \( u_{h}^i, A_i u_{h}^i, J(u_{h}^i), \) and \( (u_{h}^i)_t \) belong to \( C((0, T], L^2(Q_h)) \).

From Theorem 22, it follows that the solution of problem \( (P_h) \) is the solution of the following transmission problem. For every \( t \in (0, T], \)

\[
\begin{align*}
(\text{j}) & \quad u_{h}^i = \Delta u_{h}^i + J(u_{h}^i) \quad \text{in } L^2(Q_h^i), \quad i = 1, 2, \\
(\text{jj}) & \quad -\Delta_{S_h} u = \frac{\partial u}{\partial n} \quad \text{in } L^2(S_h),
\end{align*}
\]  

(73)

(74)

6. Convergence Results

Now we are interested in the behavior of the sequence \( \{u_h\} \) when \( h \) goes to \( 0. \)

Theorem 23. Let \( u \) and \( u_{h} \) be the solutions of problems \( (P) \) and \( (P_h) \) according to Theorem 19. Let \( \sigma_{h}^1 \) and \( \sigma_{h}^2 \) be as in Theorem 16. For every fixed positive \( \varepsilon, \) one has

(i) \( J(u_{h}) \) converges to \( J(u) \) in \( L^2([\varepsilon, T] \times Q); \)

(ii) \( [u_{h} / \partial t] \) weakly converges to \( u / \partial t \) in \( L^2([\varepsilon, T] \times Q); \)

(iii) \( \{A_i u_{h}^i\} \) weakly converges to \( A u \) in \( L^2([\varepsilon, T] \times Q); \)

(iv) \( \{u_{h}\} \) converges to \( u \) in \( L^2([\varepsilon, T]; H_0^1(Q)). \)

Proof. We prove condition (i), that is,

\[ \left\| J(u(t, P)) - J(u_{h}(t, P)) \right\|_{L^2([\varepsilon, T] \times Q)} \to 0. \]  

(75)

From (38), we have

\[
\begin{align*}
\int_\varepsilon^T \left\| J(u(t, P)) - J(u_{h}(t, P)) \right\|_{L^2(Q)}^2 dt &\leq \int_\varepsilon^T \beta(R) \left\| u(t) - u_{h}(t) \right\|_{L^2(Q)}^2 dt \\
&\leq (T - \varepsilon) \sup_{t \in [\varepsilon, T]} \beta(R) \left\| u(t) - u_{h}(t) \right\|_{L^2(Q)}^2.
\end{align*}
\]  

(76)

(77)

From Theorem 19 (a), we have

\[ \sup_{t \in [\varepsilon, T]} \left\| u(t) - u_{h}(t) \right\|_{L^2(Q)} \to 0. \]  

(78)

And hence, for every fixed \( \varepsilon > 0, \)

\[ \sup_{t \in [\varepsilon, T]} \left\| u(t) - u_{h}(t) \right\|_{L^2(Q)} \to 0. \]  

(79)

(80)

(81)

This concludes the proof of condition (i).

We now prove condition (ii). From the local Lipschitz continuity of \( J(u) \) and the Hölder continuity of \( u_{h} \) in \( (\varepsilon, T) \) into \( L^{1+\sigma}, \) one can prove that \( \| J(u_{h}) \|_{C^{1+\sigma}([\varepsilon, T]; L^{1+\sigma}(Q))} \) is bounded by a constant which does not depend on \( h; \) actually the constants depend only on the constants of the semigroups which in turn
do not depend on \(h\). From this, together with Theorem 18, we have that there exists a constant \(c\) independent of \(h\) such that
\[
\|u_h\|_{C^1([\epsilon, T]; L^2(Q))} + \|u_h\|_{C^0([\epsilon, T]; L^2(A))} \\
\leq c\left\|J(u_h)\right\|_{C^0([\epsilon, T]; L^2(Q))} \\
\leq c l(r)\|u_h\|_{C^0([\epsilon, T], L^2(\Omega))}.
\]
Thus in particular it holds \(\sup_{t \in [\epsilon, T]} \|u_h\|_{L^2(\Omega)} \leq c\); thus, for every fixed \(t \in [\epsilon, T]\), \(\|u_h(t)\|_{L^2(\Omega)} \leq c\).

From (82), it follows that for each \(h\), \(du_h/dt\) belongs to \(L^2([\epsilon, T] \times \Omega)\) and \(\|du_h/dt\|_{L^2([\epsilon, T] \times \Omega)} \leq c\).

From the boundedness of the sequence \(\{du_h/dt\}\) in \(L^2([\epsilon, T] \times \Omega)\), it follows that there exists a subsequence, which we denote with \(\{du_{h_n}/dt\}\) and a function \(v \in L^2([\epsilon, T] \times \Omega)\) such that \(\{du_{h_n}/dt\}\) weakly converges to \(v\) in \(L^2([\epsilon, T] \times \Omega)\) as \(h\) goes to \(\infty\).

In order to prove (ii), it is enough to prove that \(v = du/dt\). Since \(C^1([\epsilon, T] \times \Omega)\) is dense in \(L^2([\epsilon, T] \times \Omega)\), for every \(\varphi \in C^1([\epsilon, T] \times \Omega)\), we have
\[
\lim_{n \to \infty} \int_\Omega \int_\epsilon^T \frac{du_n}{dt} \varphi(t, P) dt dQ \quad (83)
\]
Integrating by parts the left-hand side, we get
\[
\int_\Omega \int_\epsilon^T \frac{du_n}{dt} \varphi(t, P) dt dQ \\
= \int_\Omega \left[ u_n(T, P) \varphi(T, P) - u_n(0, P) \varphi(0, P) \right] dQ \\
- \int_\Omega \int_\epsilon^T u_n(t, P) \frac{d\varphi}{dt} (t, P) dt dQ. 
\]
From (47) or (48), we have
\[
\lim_{n \to \infty} \int_\Omega \int_\epsilon^T \frac{du_n}{dt} \varphi(t, P) dt dQ \\
= \int_\Omega \left[ u(T, P) \varphi(T, P) - u(0, P) \varphi(0, P) \right] dQ \\
- \int_\Omega \int_\epsilon^T u(t, P) \frac{d\varphi}{dt} (t, P) dt dQ. 
\]
From the uniqueness of weak limit, we get \(v = du/dt\) a.e.

From the convergence of the sequence \(\{u_{h_n}\}\) to \(u\) in \(L^2([\epsilon, T] \times \Omega)\) and the weak convergence of the subsequence \(\{du_{h_n}/dt\}\) to \(du/dt\) in \(L^2([\epsilon, T] \times \Omega)\), we deduce that the whole sequence \(\{du_h/dt\}\) weakly converges to \(du/dt\) in \(L^2([\epsilon, T] \times \Omega)\).

We now prove condition (iii). It is an easy consequence of (i) and (ii). In fact \(A_h u_h = (du_h/dt) - J(u_h)\); taking the weak limit in \(L^2([\epsilon, T] \times \Omega)\), we get the thesis.

We now prove condition (iv). From (i), (iii), and the property of the scalar product in \(L^2([\epsilon, T] \times \Omega)\), we get that
\[
\lim_{h \to \infty} (A_h u_h, u_h)_{L^2([\epsilon, T] \times \Omega)} = (A u, u)_{L^2([\epsilon, T] \times \Omega)}. 
\]
That is,
\[
\lim_{h \to \infty} \int_\epsilon^T \int_\Omega A_h u_h(t, P) u_h(t, P) dQ \\
\rightarrow \int_\epsilon^T \int_\Omega A u(t, P) u(t, P) dQ. 
\]
From the relation between a Dirichlet form and the associated generator, it follows that
\[
\lim_{h \to \infty} \int_\epsilon^T E^{(h)}(u_h) dt = \int_\epsilon^T E(u) dt. 
\]
There exists a constant \(c\) such that
\[
\int_\epsilon^T \int_\Omega |Du_h(t, P)|^2 dQ dt + \int_\epsilon^T E_{S_h}(u_h) dt \leq c, 
\]
for every \(h \in \mathbb{N}\).

Hence
\[
\|Du_h\|_{L^2([\epsilon, T] \times \Omega)} \leq c. 
\]
There exists a subsequence \(Du_{h_n}\) weakly converging to \(w\) in \(L^2([\epsilon, T] \times \Omega)^2\). We now prove that
\[
\int_\epsilon^T \int_\Omega |Du(t, P)|^2 dQ dt \\
\rightarrow \int_\epsilon^T \int_\Omega |Du(t, P)|^2 dQ dt. 
\]
From Theorem 19, it follows in particular that \(u_{h_n}\) converges to \(u\) in \(L^2([\epsilon, T] \times \Omega)\); hence \(w = Du\) and \(u_{h_n} \to u\) in \(L^2([\epsilon, T]; H^1_0(\Omega))\); in particular (91) holds. We now prove assertion (iv) as
\[
0 \leq \int_\epsilon^T \|u_h(t) - u(t)\|^2_{H^1_0(\Omega)} dt \\
= \int_\epsilon^T dt \int_\Omega |Du_h(t, P) - Du(t, P)|^2 dQ \\
= \int_\epsilon^T dt \left( |Du_h(t, P)|^2 + |Du(t, P)|^2 \\
- 2 Du_h(t, P) Du(t, P) \right) dQ \\
= \int_\epsilon^T \left( E^{(h)}(u_h) - E_{S_h}(u_h) \right) dt \\
+ \int_\epsilon^T dt \left( |Du(t, P)|^2 - 2 Du_h(t, P) Du(t, P) \right) dQ. 
\]
Taking the upper limit as \( h \to \infty \), we have
\[
0 \leq \lim \sup \left\| u_h(t) - u(t) \right\|_{L^2([\epsilon, T]; H^\gamma(Q))}^2 \\
\leq \int_\epsilon^T E[u] \, dt \\
- \lim \inf_{h \to \infty} \int_\epsilon^T E_{g_h}[u_h] \, dt \\
+ \int_\epsilon^T \int_Q |Du(t, P)|^2 \, dQ \\
- 2 \int_\epsilon^T \int_Q |Du(t, P)|^2 \, dQ \\
\leq c_0 \int_\epsilon^T E_S[u] \, dt - \lim \inf_{h \to \infty} \int_\epsilon^T E_{g_h}[u_h] \, dt
\]
(93)
where the last inequality follows from (4.9) in [21]. Hence the sequence \( \{u_h\} \) converges to \( u \) in \( L^2([\epsilon, T]; H^\gamma_0(Q)) \) and therefore \( \{Du_h\} \) converges to \( Du \) in \( L^2([\epsilon, T]; (L^2(Q))^3) \). □

**Proposition 24.** Let \( u \) and \( u_h \) be the solutions of problems (P) and \((P_h)\), respectively. Then \( u \) and \( u_h \in H^1([\epsilon, T] \times Q) \).

**Proof.** We prove the thesis for \( u \). From Theorem 18, it follows that \( u \in C([\epsilon, T]; \mathcal{D}(A)) \) and \( (du/dt) \in C([\epsilon, T]; L^2(Q)) \). Since \( \mathcal{D}(A) \subset V(Q, S) \subset H^\gamma_0(Q) \), we obtain \( u \in C([\epsilon, T]; H^\gamma_0(Q)) \); hence \( Du \in C([\epsilon, T]; (L^2(Q))^3) \). The thesis follows as \( C([\epsilon, T]; L^2(Q)) \subset L^2([\epsilon, T] \times Q) \). The result for \( u_h \) can be proved analogously. □

**Appendices**

Here we recall some definitions of functional spaces and trace results.

**A. Sobolev Spaces**

Let \( Q \) be a polyhedral domain; just to fix the ideas, the parallelepiped is as in Section 2. For every integer \( h \geq 1 \), let \( S_h \) be the prefractal surface approximating the Koch-type surface \( S \) and let us define every affine “face” of \( S_h \) by \( S_h^0 \); \( S_h \) divides \( Q \) into two subsets \( Q_h^1 \) and \( Q_h^2 \).

By \( L^p(Q), p > 1 \) we denote the Lebesgue space with respect to the Lebesgue measure on subsets of \( \mathbb{R}^3 \), which will be left to the context whenever that does not create ambiguity. Let \( \mathcal{F} \) be a closed set of \( \mathbb{R}^3 \); by \( C(\mathcal{F}) \) we denote the space of continuous functions on \( \mathcal{F} \); by \( C_0(\mathcal{F}) \) we denote the space of continuous functions vanishing on \( \partial \mathcal{F} \). Let \( \mathcal{G} \) be an open set of \( \mathbb{R}^3 \); by \( H^1(\mathcal{G}) \) we denote the usual Sobolev spaces (see Necas [38]); \( H^1_0(\mathcal{G}) \) is the closure of \( \mathcal{D}(\mathcal{G}) \) (the smooth functions with compact support on \( \mathcal{G} \)), with respect to the \( \| \cdot \|_{H^1} \)-norm. In the following, we will make use of trace spaces on boundaries of polyhedral domains of \( \mathbb{R}^3 \).

By \( H^1_0(S_h) \) we denote the closure in \( H^1(S_h) \) of the set
\[
\{ v_{|\partial Q_h^k} : v \in C_c(\partial^\infty Q_h^2) ; \quad v \text{ vanishes in a neighborhood of } S_h \}.
\]
(4.1)

By \( H^r(S_h), 0 < r < 1 \) we denote the Sobolev space on \( S_h \), defined by local Lipschitz charts as in Necas [38].

It is to be pointed out that the Sobolev space \( H^r(S_h) \) (defined in [38]) coincides, with equivalent norms, with the trace space defined in Buffa and Ciarlet in [37] (see also [39] for the case of polygonal boundaries).

When \( r > 1 \), the trace spaces on nonsmooth boundaries can be defined in different ways; we now recall two trace theorems, specialized to our case, referring to [40] and [41] for a more general discussion.

For \( f \) in \( H^2(\mathcal{G}) \), we put
\[
y_0 f (P) = \lim_{r \to 0} \frac{1}{|B(P, r) \cap \mathcal{G}|} \int_{B(P, r) \cap \mathcal{G}} f (Q) \, dQ
\]
(4.2)
at every point \( P \in \mathcal{F} \), where the limit exists. It is known that the limit \( (A.2) \) exists at quasi every \( P \in \mathcal{F} \) with respect to the \((1,2)-\text{capacity} \) [42].

We now recall the results of Theorem 3.1 in [36] specialized to our case, referring to [41] for a more general discussion.

**Proposition A.1.** Let \( \mathcal{G} \) denote, respectively, \( Q, Q^1_h, \) and \( Q^2_h \) and let \( \Gamma \) denote \( S_h, \partial Q^1_h, \partial Q^2_h, \) and \( \partial Q \). Then \( H^{1/2}(\Gamma) \) is the trace space to \( \Gamma \) of \( H^1(\mathcal{G}) \) in the following sense:

(i) \( y_0 \) is a continuous and linear operator from \( H^1(\mathcal{G}) \) to \( H^{1/2}(\Gamma) \);

(ii) there is a continuous linear operator \( \text{Ext} \) from \( H^{1/2}(\Gamma) \) to \( H^1(\mathcal{G}) \), such that \( y_0 \circ \text{Ext} \) is the identity operator in \( H^{1/2}(\Gamma) \).

**B. Besov Spaces**

**Definition B.1.** Let \( \mathcal{F} \subset \mathbb{R}^D \) be a closed nonempty subset. It is a \( d \)-set \((0 < d < D)\) if there exists a Borel measure \( \mu \) with \( \mu(S) < \infty \) on \( \mathcal{F} \) such that, for some constants \( c_1 = c_1(\mathcal{F}) > 0 \) and \( c_2 = c_2(\mathcal{F}) > 0 \),
\[
c_1 r^d \leq \mu (B(P, r)) \leq c_2 r^d \quad (P \in \mathcal{F}, 0 < r \leq 1) .
\]
(5.1)
Such a \( \mu \) is called a \( d \)-measure on \( \mathcal{F} \).

**Proposition B.2.** The set \( F \) is a \( d \)-set with \( d = d_f \). The measure \( \mu_F \) is a \( d \)-measure. The layer \( S \) is a \( d \)-set with \( d = d_j + 1 \). The measure \( m \) is a \( d \)-measure.

See [23, 26].

We now come to the definition of the Besov spaces. Actually there are many equivalent definitions of these spaces;
see, for instance, [43, 44]. We recall here the one which best fits our aims and we will restrict ourselves to the case \( \alpha \) positive and noninteger, \( p = q = 2 \); the general setting is being much more involved; see [44].

Let \( \mathcal{T} \) be a \( d \)-set in \( \mathbb{R}^D \).

Let \( \alpha > 0 \) be noninteger, \( k = \lceil \alpha \rceil \) the integer part of \( \alpha \), and \( j \) a \( D \)-dimensional multi-index of length \( |j| \leq k \).

If \( f \) and \( \{ f^{(j)} \} \) are functions defined \( \mu \text{-a.e. on } \mathcal{T} \), we set

\[
R_j(P, P') = f^{(j)}(P) - \sum_{|j|=k} f^{(j+1)}(P') \bigg( P - P' \bigg)^j, \tag{B.2}
\]

where \( f^{(0)} = f \) and \( l \) denotes a \( D \)-dimensional multi-index. We now define the Besov space as \( B^{2,2}_\alpha(\mathcal{T}) = B^{2,2}(\mathcal{T}, \mu) \).

**Definition B.3.** One says that \( f \in B^{2,2}_\alpha(\mathcal{T}) \) if there exists a family \( \{ f^{(j)} \} \) with \( |j| \leq k \), as above, such that \( f^{(j)} \in L^2(\mathcal{T}, \mu) \) and \( \|a_n\|_1 < \infty \), where \( a_n \) is the smallest number such that

\[
\left( 3^n d \int_{|p - p'| < 3^{-n}} |R_j(P, P')|^2 \, d\mu(P) \, d\mu(P') \right)^{1/2} \leq 3^{-n(\alpha - |j|)} a_n. \tag{B.3}
\]

The norm of \( f \) in \( B^{2,2}_\alpha(\mathcal{T}) \) is

\[
\|f\|_{B^{2,2}_\alpha(\mathcal{T})} = \|\|a_n\|_1 \|_1. \tag{B.4}
\]

The family \( \{ f^{(j)} \} \) in the previous definition is uniquely determined by \( f \), as shown in [44], for \( d \)-sets with \( d > D - 1 \).

Let us note that for \( 0 < \alpha < 1 \) the norm \( \|f\|_{B^{2,2}_\alpha(\mathcal{T})} \) can be written as

\[
\|f\|_{2,\mu} + \left( \sum_{n=0}^{\infty} 3^n d^{2x_2} \times \int \int_{|p - p'| < 3^{-n}} |f(P) - f(P')|^2 \, d\mu(P) \, d\mu(P') \right)^{1/2}. \tag{B.5}
\]

**Proposition B.4.** Let \( \mathcal{T} \) be a \( d \)-set, \( \mathcal{T} \subset \tilde{Q} \). Let \( s > (3 - d)/2 \), \( s - (3 - d)/2 \neq \mathbb{N} \); then \( B^{2,2}_{s-(3-d)/2}(\mathcal{T}) \) is the trace space of \( \mathcal{T} \) of \( H^s(Q) \) in the following sense:

(i) \( \gamma_0 \) is a continuous linear operator from \( H^s(Q) \) to \( B^{2,2}_{s-(3-d)/2}(\mathcal{T}) \);

(ii) there is a continuous linear operator \( \text{Ext} \) from \( B^{2,2}_{s-(3-d)/2}(\mathcal{T}) \) to \( H^s(Q) \) such that \( \gamma_0 \circ \text{Ext} \) is the identity operator in \( B^{2,2}_{s-(3-d)/2}(\mathcal{T}) \).

For the proof, we refer to Theorem 1 of Chapter VII in [44]; see also [43].

From Proposition B.4, it follows that when \( \mathcal{T} = S \) and \( s = 1 \) the trace space of \( H^1(Q) \) is \( B^{2,2}_{1/2}(S) \).

Let \( \beta = d_f/2 \). The space \( B^{2,2}_{\beta,0}(S) \) is a subspace of \( B^{2,2}_\beta(S) \); more precisely

\[
B^{2,2}_{\beta,0}(S) = \{ u \in L^2(S, m) \mid \text{there exists } v \in H^1_0(Q) \text{ such that } \gamma_0 v = u \text{ on } S \} \tag{B.6}
\]

equipped with the norm

\[
\|u\|_{B^{2,2}_{\beta,0}(S)} = \inf \|v\|_{H^1(Q)} : v \in H^1_0(Q), \quad \gamma_0 v = u, \quad \text{on } S. \tag{B.7}
\]

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgment**

This research was partially supported under the Grant no. 1109356 by Fractal Fibers and Singular Homogenization National Science Foundation.

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