Index-free Heat Kernel Coefficients

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Abstract

Using index-free notation, we present the diagonal values $a_j(x, x)$ of the first five heat kernel coefficients $a_j(x, x')$ associated with a general Laplace-type operator on a compact Riemannian space without boundary. The fifth coefficient $a_5(x, x)$ appears here for the first time. For the special case of a flat space, but with a gauge connection, the sixth coefficient is given too. Also provided are the leading terms for any coefficient, both in ascending and descending powers of the Yang-Mills and Riemann curvatures, to the same order as required for the fourth coefficient. These results are obtained by directly solving the relevant recursion relations, working in Fock-Schwinger gauge and Riemann normal coordinates. Our procedure is thus noncovariant, but we show that for any coefficient the ‘gauged’ respectively ‘curved’ version is found from the corresponding ‘non-gauged’ respectively ‘flat’ coefficient by making some simple covariant substitutions. These substitutions being understood, the coefficients retain their ‘flat’ form and size. In this sense the fifth and sixth coefficient have only 26 and 75 terms respectively, allowing us to write them down. Using index-free notation also clarifies the general structure of the heat kernel coefficients. In particular, in flat space we find that from the fifth coefficient onward, certain scalars are absent. This may be relevant for the anomalies of quantum field theories in ten or more dimensions.
1 Introduction

The heat kernel method has become a ubiquitous tool in both mathematics and physics (see [1] for a recent overview). In mathematics it appears e.g. in the study of the spectral geometry of a Laplace-type differential operator on a Riemannian space and in the proof of index theorems [2, 3]. In physics, the euclidean one-loop effective action for a given quantum field theory can be expressed in terms of the determinant of such a differential operator [4], which in turn can be written in terms of the associated heat kernel. The heat kernel therefore appears in many places, from quantum gravity [5] to chiral perturbation theory [6]. Anomalies can also be studied with the heat kernel method (see [6, 7]). Physicists frequently refer to it as the Schwinger-DeWitt [4, 5] or proper-time method. Exact expressions for the heat kernel exist only for special spaces. In the general case one may use its asymptotic expansion in the proper-time. The coefficients in this expansion are the so-called heat kernel coefficients (see sect 2 for a precise definition).

Several methods have been developed to find the heat kernel coefficients (see e.g. [3] for a review). DeWitt [5] determined the first two coefficients with a covariant recursive method. Sakai [6] relied on Riemannian coordinates to find the third coefficient in the scalar case (i.e. a single scalar field on a curved space). For the general case, this coefficient was found by Gilkey [7] using a noncovariant pseudo-differential-operator technique. The integrated and traced fourth and fifth coefficients for an arbitrary field theory in flat space were found in [9] through the evaluation of a noncovariant Feynman graph. Avramidi [9] presented a new covariant nonrecursive procedure and found the fourth coefficient for the general case (for the scalar case see also [10]). More recently, string-inspired world line path integral methods have been used [11] to determine the functional trace of the first eight heat kernel coefficients for the case of a matrix potential in flat space without gauge connection (see also [12]).

In this paper the explicit diagonal value of the fifth heat kernel coefficient in the general case is presented for the first time. In physics this coefficient is of importance e.g. in analysing the short distance behavior and anomalies of ten-dimensional quantum field theories (see [13]). However, the number of terms in the higher heat kernel coefficients grows rapidly, leading one to expect more than a thousand terms for the fifth coefficient. This would seem to preclude writing down this coefficient in an intelligable form. Indeed, a computer would appear to be an essential piece of equipment in determining and storing the fifth coefficient. Contrary to these expectations, we will show here that with a suitable index-free notation one obtains a compact expression for this coefficient, containing only 26 terms.

Using standard matrix notation for the field indices, the heat kernel coefficients are

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1 An alternative is the so-called covariant perturbation theory [4]. It provides a (partial) summation of the Schwinger-DeWitt series and can account for nonlocal effects.

2 We frequently abbreviate ‘diagonal value of a heat kernel coefficient’ to ‘coefficient’.

3 Even restricting to the a single scalar field on a curved space, the $j$-th coefficient already contains 1, 4, 17, 92, 668 $R^j$-terms for $j = 1, 2, 3, 4, 5$ (see appendix A of [15]).
scalars and this suggests that it may be possible to write them in a form which is free of spacetime indices as well. Since the heat kernel coefficients are covariant, we may determine them in a special gauge and adapted coordinates. Following earlier authors, we select the Fock-Schwinger gauge and Riemann normal coordinates. Examination of the relevant recursion relations then shows that only certain maximally symmetrized multiple (covariant) derivatives of the matrix potential and Yang-Mills or Riemann curvature tensors appear. The heat kernel coefficients being formal scalars, it turns out that we need to keep track of the rank of these tensors only (note the similarity with totally antisymmetric tensors, i.e. differential forms). This provides the basis for our index-free notation. Using instead a fully covariant method, Avramidi has arrived at similar conclusions by expanding in so-called covariant Taylor series, obtaining in this way the first four coefficients. Our non-covariant procedure seems to be no less efficient and, by maintaining index-free notation and manifest hermiticity, yields even compactier answers for the heat kernel coefficients. Thus, in the special case of a flat space with a gauge connection, we can also present the answer for the sixth coefficient. It contains only 75 terms.

Using index-free notation also permits us to investigate the general structure of the heat kernel coefficients. Without actually solving the recursion relations, we can show that certain Lorentz scalars are absent from all coefficients. In flat space, this happens for the first time in the fifth coefficient. This may therefore be relevant for the anomalies of quantum field theories in ten or more dimensions.

As we already mentioned above, a brute force approach with a computer algebra program would have produced an unwieldy result. Our results were obtained without the aid of a computer. However, it is relatively easy to program our index-free method and we used FORM and Mathematica to run some checks.

An outline of this paper is as follows. In sect 2 we recall the main features of the heat kernel method alias the Schwinger-DeWitt formalism. In sect 3 we introduce our index-free notation and use it to determine first the heat kernel coefficients in flat space without gauge connection, expanding in powers of either derivatives (sect 3.1) or of the matrix potential (sect 3.2). In sect 3.1 we also use our index-free notation to prove that certain scalars are absent from all heat kernel coefficients. In sect 4 we show that the corresponding heat kernel coefficients with a gauge connection can be obtained from a simple covariantization process. This involves not only replacing partial derivatives by covariant ones but also adding new field strength dependent terms. The latter kind of terms are shown to arise only as shifts in the potential and in its covariant derivatives. These shifts being understood, the heat kernel coefficients do not change their form upon ‘turning on the gauge field’. In sect 5 we employ Riemann normal coordinates to generalize to a curved space. In particular, we present in subsect 5.1 an explicit expansion of the vielbein to all orders in these coordinates (such an expansion, usually given for the metric, but only to some finite order in the normal coordinates can be found in many places, e.g. [3, 21]). We give a similar result in subsect 5.2 for the gauge connection to all orders in normal coordinates. Based on this, we find in subsect 5.3 the explicit form for any heat kernel coefficient up to and including terms of fourth order in the Yang-Mills and Riemann curvatures. The
curved coefficients can be obtained from the corresponding gauged but flat coefficients via further simple covariant substitutions. To complete the fifth coefficient, we need to find the few terms of fifth order in the curvatures. This we do in subsect 5.4 by specializing to a locally symmetric space. In sect 6 we present the explicit answers for the first five coefficients. We indicate how to return to more conventional notation and compare with earlier results. Our conclusions are given in sect 7. Several appendices follow (in particular, the sixth coefficient in flat space is given in appendix C).

2 Schwinger-DeWitt formalism

Consider a set of fields $\phi_i(x)$, $i = 1 \ldots n$, defined over a compact $d$-dimensional Riemannian manifold with coordinates $x^\mu$, $\mu = 1 \ldots d$ and metric $g_{\mu\nu}$ (see appendix A for our notation and conventions). The fields are acted upon by a Laplace-type wave operator $\Delta$

$$\Delta = -\nabla^2 - X , \quad \nabla^2 = g^{\mu\nu}\nabla_\mu \nabla_\nu \quad (2.1)$$

Here the covariant derivative $\nabla$ includes connection terms as needed for the fields $\phi_i$. $X$ is a hermitian $n \times n$ matrix potential (we suppress the field or ‘bundle’ indices). The wave operator $\Delta$ is hermitian with respect to the inner product

$$(\phi, \psi) = \int d^d x \sqrt{g} \phi^* \psi \quad (2.2)$$

For most bosonic gauge field theories of interest one can achieve a wave operator of Laplace-type as in (2.1) by a suitable gauge choice. For fermionic (gauge) fields one squares the wave operator to obtain again (2.1) (see [19] for wave operators not of this form).

Following Schwinger and DeWitt, we introduce the proper-time parameter $\tau$ and define the heat kernel $K$ associated with $\Delta$ by

$$\left( \frac{\partial}{\partial \tau} + \Delta \right) K(x, x'; \tau) = 0 , \quad K(x, x'; 0) = I\delta(x, x') \quad (2.3)$$

where $I$ is the $n \times n$ unit matrix and the bi-scalar $\delta$ function is defined by

$$\int d^d x \sqrt{g} \delta(x, x') \phi(x) = \phi(x') \quad (2.4)$$

for any scalar field $\phi$. As we mentioned in the introduction, an exact solution for the kernel $K$ exists only for special spaces. We instead make DeWitt’s ansatz

$$K(x, x'; \tau) = (4\pi\tau)^{-d/2} D(x, x')^{1/2} e^{-\sigma(x, x')/2\tau} \sum_{j=0}^\infty a_j(x, x') \frac{\tau^j}{j!} \quad (2.5)$$

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Footnote: The fields can be considered to be sections of a smooth vector bundle. The heat kernel coefficients do not explicitly depend on the dimension or signature of space(time). We assume space to have no boundary (for the case with boundary, see e.g. [20]).
which is known to be an asymptotic expansion in $\tau$. Note our unconventional normalization for the heat kernel coefficients $a_j$, which however agrees with [9]. They transform as scalar densities of weight $-1/2$ at both $x$ and $x'$. The bi-scalar $\sigma$ is the geodetic interval (one half of the distance squared between $x$ and $x'$) and satisfies
\[ \sigma;^\mu \sigma;_\mu = 2\sigma, \quad [\sigma] \equiv \sigma(x,x) = 0 \]  
(2.6)

where we use Synge’s bracket notation to indicate evaluation on the diagonal. The bi-scalar $D$ is the Van Vleck-Morette determinant defined by
\[ D(x,x') = g^{-1/2} \det(-\sigma_{\mu\nu}) g'^{-1/2} \]  
(2.7)

where a prime refers to the point $x'$. It satisfies
\[ (2\sigma;^\mu \nabla_\mu + \sigma;^\mu - d) D^{1/2} = 0, \quad [D] = 1 \]  
(2.8)

Inserting (2.3) into (2.3) and using (2.6) and (2.8), one finds that the heat kernel coefficients must satisfy the following recursion relations for $j \geq 0$
\[ (\sigma;^\mu \nabla_\mu + j)a_j = -jD^{-1/2} \Delta D^{1/2} a_{j-1} , \quad [a_0] = I \]  
(2.9)

where it is to be understood that $a_{-1}$ vanishes. Note that whereas $\sigma$ and $D$ depend only on the metric, the $a_j$ are matrix valued and depend in addition on the detailed form of the wave operator. The hermiticity of the wave operator implies
\[ a_j(x,x')^\dagger = a_j(x',x) \Rightarrow [a_j]^\dagger = [a_j] \]  
(2.10)

To keep this property manifest we introduce the following notation: for any matrix valued function $F$ we define
\[ \{F\} \equiv F + F^\dagger \quad \text{if} \quad F \neq F^\dagger \quad \text{else} \quad F \]  
(2.11)

Thus $\{F\} = F$ when $F$ is selfadjoint. Frequently one is interested in the functional trace of the heat kernel coefficients
\[ b_j \equiv \text{Tr} a_j \equiv \text{tr} \int d^d x \sqrt{g} a_j(x,x) \]  
(2.12)

where $\text{tr}$ denotes the matrix trace over field indices only. To determine e.g. chiral anomalies one would need the non-traced version. In this paper we will determine the non-traced diagonal heat kernel coefficients.

3 Flat space without gauge connection

In a flat space without gauge connection the recursion relations (2.9) for the heat kernel coefficients become
\[ (x^\mu \partial_\mu + j)a_j = j(\partial^2 + X)a_{j-1} , \quad [a_0] = I \]  
(3.1)
Here we have set $x'$ to zero (we will not differentiate at $x'$). We are mostly interested in the diagonal values of the heat kernel coefficients. However, it is easy to see that this in turn requires knowledge of some derivatives of preceding heat kernel coefficients on the diagonal. Indeed, taking the diagonal value of (3.1) yields

$$[a_j] = [\partial^2 a_{j-1}] + X [a_{j-1}]$$

(3.2)

Thus in particular

$$[a_1] = X$$

(3.3)

but for $j > 1$ we must first find $a_{j-1}$ and $\partial^2 a_{j-1}$ on the diagonal. Applying $\partial^2$ to (3.1) and then going on the diagonal gives

$$\left[\partial^2 a_{j-1}\right] = \frac{j - 1}{j + 1} \left( [\partial^2 \partial^2 a_{j-2}] + X [\partial^2 a_{j-2}] + 2X,^\mu [\partial,^\mu a_{j-2}] + X,^\mu [a_{j-2}] \right)$$

(3.4)

Setting $j = 2$ and substituting the result in (3.2), we find

$$[a_2] = \frac{1}{3} \partial^2 X + X^2$$

(3.5)

but for $j > 2$ we require some new diagonal values of derivatives of $a_{j-2}$. This recursive procedure ends after $j$ steps since the diagonal value of any derivative of $a_0$ vanishes. Differentiating (3.1) $n$ times and taking the diagonal value yields

$$[\partial_{\mu_1} \ldots \partial_{\mu_n} a_j] = -\frac{j}{j + n} [\partial_{\mu_1} \ldots \partial_{\mu_n} \Delta a_{j-1}]$$

(3.6)

To solve these recursion relations in an effective way, we introduce a short hand notation for them. Using comma notation for partial derivatives, but writing only the number $n$ of uncontracted derivatives taken, we can abbreviate (3.6) as

$$a_{j,n} = \frac{j}{j + n} \left( a_{j-1,(2),n} + (X a_{j-1})_n \right)$$

(3.7)

Here, the index $n$ stands for all partial derivatives taken on the left hand side of (3.6). The sans serif symbols serve to emphasize that we are using this short hand notation and at the same time imply evaluation on the diagonal. The $\partial^2$ on the right hand side has been abbreviated to a 2 in parenthesis. Of course one must first distribute the $n$ derivatives over the factors of the second term in (3.7) before taken the diagonal value. Doing so yields

$$a_{j,n} = \frac{j}{j + n} \left( a_{j-1,(2),n} + \sum_{p=0}^{n} \binom{n}{p} X_{(p) a_{j-1,n-p}} \right)$$

(3.8)

where the parenthesis around the $p$ plus $n - p$ indices imply total symmetrization.

We write the result of replacing $n$ by $2n$ and contracting all derivatives as

$$a_{j,(2n)} = \frac{j}{j + 2n} \left( a_{j-1,(2n+2)} + \sum_{p=0}^{2n} \binom{2n}{p} X_{(p) a_{j-1,2n-p}} \right)$$

(3.9)
where the parenthesis now imply not only total symmetrization, but also full contraction. Here we introduced the following notation: for any functions $F(x)$, $G(x)$ and $H(x)$ we define

$$
F_{(2n)} = F_{(\mu_1\mu_2...\mu_{2n})}(0)
$$

$$
F_{(kG_{2n-k})} = F_{(\mu_1...\mu_k)}(0) G_{\mu_{k+1}...\mu_{2n}}(0) \delta^{\mu_1\mu_2}...\delta^{\mu_{2n-1}\mu_{2n}}
$$

$$
F_{(kG_{2n-k})} = F_{(\mu_1...\mu_k)}(0) G_{\mu_{k+1}...\mu_{2n}}(0) H_{\mu_{k+1}\ell+1...\mu_{2n}}(0) \delta^{\mu_1\mu_2}...\delta^{\mu_{2n-1}\mu_{2n}}
$$

with similar expressions in case the $2^n$ derivatives are distributed over yet more factors. Unless otherwise noted, the use of parenthesis as on the left hand side of (3.10) indicates that the $2^n$ derivatives are to be totally symmetrized and fully contracted. Note that

$$
F_{(2n)} = (\partial^2)^n F(0)
$$

so in this case the symmetrization in (3.10) is superfluous (it is relevant when a gauge connection is present: see (4.14)). The notation of (3.11) is due to Avramidi [9] (evaluation at the origin is not implied there). We generalized it here by allowing distribution of the contracted and symmetrized derivatives over any number of factors. Note that e.g.

$$
F_{(1G_1)} = F_{,\mu}(0) G_{,\mu}(0)
$$

$$
F_{(3G_3)} = F_{,\mu}(0) G_{,\mu\nu\nu}(0) = F_{(1G_1)(2)}
$$

$$
F_{(2G_2)} = \frac{1}{3} F_{,\mu\mu}(0) G_{,\mu\nu}(0) + \frac{2}{3} F_{,\mu\nu}(0) G_{,\mu\nu}(0)
$$

The general rule for such a reduction is given in (A.9).

3.1 Expansion in derivatives

We first discuss some generic features of the derivative expansion, which can be understood without solving the recursion relations. Using only dimensional analysis, it follows that through second order in derivatives the diagonal values of the heat kernel coefficients must have the structure

$$
[a_j] = \alpha_j X^j + \sum_{k=1}^{j-1} \alpha_{j\ell} X^{k-1} X_{,\mu\mu} X^{j-k-1} + \sum_{k=1}^{j-2} \sum_{\ell=1}^{j-k-1} \alpha_{j\ell k} X^{k-1} X_{,\mu\mu} X^{j-k-\ell-1} X_{,\mu\mu} X^{\ell-1} + ...
$$

(3.13)

The $\alpha$’s are numerical coefficients which, in order for (3.13) to be hermitian, must satisfy

$$
\alpha_{j\;j-k} = \alpha_{j\;k} , \quad \alpha_{j\;\ell k} = \alpha_{j\;k\ell}
$$

(3.14)

These expectations are borne out by our results. In particular, all terms in (3.13) will turn out to have nonvanishing coefficients. At fourth order in derivatives, dimensional
analysis allows 11 ways of distributing the derivatives, namely

\[
\begin{align*}
X_{\mu\nu\rho\sigma} & , & X_{\mu\rho\sigma\nu} & , & X_{\nu\rho\mu\sigma} \\
X_{\mu\nu\rho\sigma} & , & X_{\mu\nu\rho\sigma} & , & X_{\mu\nu\rho\sigma} \\
X_{\mu\nu\rho\sigma} & , & X_{\mu\nu\rho\sigma} & , & X_{\mu\nu\rho\sigma} \\
X_{\mu\nu\rho\sigma} & , & X_{\mu\nu\rho\sigma} & , & X_{\mu\nu\rho\sigma} \\
X_{\mu\nu\rho\sigma} & , & X_{\mu\nu\rho\sigma} & , & X_{\mu\nu\rho\sigma} \\
X_{\mu\nu\rho\sigma} & , & X_{\mu\nu\rho\sigma} & , & X_{\mu\nu\rho\sigma}
\end{align*}
\] (3.15)

Here and through (3.19) below, we list only equivalence classes of Lorentz scalars, where two scalars are considered equivalent if they become equal upon omission of all undifferentiated factors \(X\) and/or reversing the order of the factors. We therefore do not write the curly brackets as in (2.11).

We now claim that two of the Lorentz scalars in (3.15) cannot appear in the heat kernel coefficients. To prove our assertion, we need not solve the recursion relations explicitly. It suffices to keep track of the way the derivatives get distributed over the various factors as the recursion proceeds. Hence, we omit numerical factors as well as any undifferentiated matrix \(X\). We may even drop the ordering labels \(j\) respectively \(j-1\) in (3.8) and thus abbreviate \(a_{j,n}\) to \(a_n\), where \(n\) is the number of derivatives. Writing only Lorentz scalars, we thus have the following chain of substitutions

\[
a \rightarrow a^{(2)}
\]

\[
a_{(2n)} \rightarrow a_{(2n+2)} + \sum_{p=1}^{2n} X_{(p)a_{2n-p}} , \quad n \geq 1
\] (3.16)

\[
X_{(p)a_{2n-p}} \rightarrow X_{(p)a_{2n-p}(2)} + \sum_{q=1}^{2n-p} X_{(p)X_qa_{2n-p-q}} , \quad p = 1 \ldots 2n
\]

etc. We truncate this hierarchy at level \(2N\) in derivatives, i.e. we drop any term with more than \(2N\) derivatives (note that for the \(j\)-th coefficient, \(N \leq j - 1\)). To be definite, consider the case \(N = 2\) where the first few steps are

\[
a \rightarrow a^{(2)}
\]

\[
\rightarrow a^{(4)} + X_{(1)a_1} + X_{(2)a}
\]

\[
\rightarrow X_{(4)} + X_{(3)a_1} + X_{(2)a_2} + X_{(1)a_3} + X_{(1)X_1a} + X_{(2)a}
\] (3.17)

We used\(^6\) that \(X_{(1)a_1} = X_{(1)a_3}\), see (3.12) Iterating once more, we see that at fourth order in derivatives a generic heat kernel coefficient can contain only the following 9 equivalence classes of Lorentz scalars

\[
X_{(4)}
\]

\(^5\)By working with real fields only, we can always arrange \(X\) to be a real symmetric matrix. The same is true for its derivatives and \(\{F\}\) as in (2.11) then means that, unless \(F\) is a palindrome, we must add to \(F\) its transpose.

\(^6\)Note that in general we have \[X_{(p)a_{2n-p}(2)} \neq X_{(p)a_{2n-p+2}}.\] Thus, before being able to iterate again, we must ‘remove the box’. See (3.30) for an example and app. B for the general solution. Since we shall not go beyond \(N = 2\) this complication is irrelevant here.
\[
\begin{align*}
X_{(3)X_1} & , \ X_{(2)X_2} , \ X_{(2)X_2} \\
X_{(1)X_2X_1} & , \ X_{(2)X_2X_1} , \ X_{(2)X_1X_1} \\
X_{(1)X_1X_1X_1} & , \ X_{(1)X_1X_1X_1}
\end{align*}
\] (3.18)

Comparison with (3.15) shows that expressions with overlapping sets of symmetrized and contracted derivatives, namely
\[
\begin{align*}
X_{(1)X^{(2)}X_1} & \equiv X_{\mu\nu}X_{\mu\mu}X_{\nu} \\
X_{(1)X^{(1)}X_1} & \equiv X_{\mu\nu}X_{\mu\rho}X_{\nu}X_{\rho} \quad \text{or} \quad X_{(1)X^{(1)}X_1} \equiv X_{\mu\nu}X_{\mu\rho}X_{\rho}X_{\nu}
\end{align*}
\] (3.19)

are absent. This holds for all heat kernel coefficients (recall that we dropped any undifferentiated matrix \(X\)). The absence of the first (second and third) entry in (3.19) can be first observed in \(a_5\) (respectively \(a_6\)). This may hence be relevant to the short distance behavior and anomalies of quantum field theories in ten or more dimensions. We conclude that at fourth order in derivatives only 9 of the 11 \textit{a priori} allowed scalars appear. This is only a small reduction in the number of terms, but at higher orders the relative number of such absent Lorentz scalars increases rapidly.

At sixth order in derivatives we find that dimensional analysis and hermiticity would permit 85 different Lorentz scalars similar to those in (3.15). However, continuing (3.17), we discover that only 53 of these scalars can actually appear in the heat kernel coefficients.

We now give our explicit solution for \(a_j\) through fourth order in derivatives. To have manifest hermiticity, we use here the convention that in an \(N\)-fold sum with \(k_1, \ldots, k_N\) as summation variables (below \(k_1, k_2, \ldots = k, \ell, \ldots\) etc, but skip \(o\)), the label \(k_{N+1}\) has by definition the value \((k_N)^\text{max} - k_N + 1\) (so e.g. in the second line below, \(\ell \equiv j - k\)).

\[
\begin{align*}
a_j & = X_j^j \\
& + \sum_{k=1}^{j-1} \sum_{\ell=1}^{k-1} \frac{k\ell}{j + 1} X^{k-1}X_{(2)}X^{\ell-1} \\
& + \sum_{k=1}^{j-2} \sum_{\ell=1}^{k-2} \frac{2k\ell}{j + 1} X^{k-1}X_{(1)} X_{m-1}X_{(1)}X^{\ell-1} \\
& + \sum_{k=1}^{j-2} \frac{k(k+1)\ell(\ell+1)}{2(j + 1)(j + 2)} X^{k-1}X_{(4)}X^{\ell-1} \\
& + \sum_{k=1}^{j-3} \sum_{\ell=1}^{k-3} \frac{4k\ell m}{j + 1} X^{k-1}X_{(2)}X_{m-1}X_{(2)}X^{\ell-1} \\
& \quad + \sum_{m=1}^{j-4} \sum_{\ell=1}^{k-3} \frac{2k\ell m}{j + 1} \{X^{k-1}X_{(2)}X_{m-1}X_{(1)}X^{\ell-1}\} \\
& + \sum_{k=1}^{j-4} \sum_{\ell=1}^{k-3} \sum_{m=1}^{j-5} \frac{2k\ell m}{j + 1} \{X^{k-1}X_{(2)}X_{m-1}X_{(1)}X^{\ell-1}\}
\end{align*}
\]
yield an expression of the form
\[ a_{j,n} = \frac{j}{j+n} \left( a_{j-1,n(2)} + (X a_{j-1})_n \right) \]  \hspace{1cm} (3.23)

With the restriction of at most four derivatives, this result constitutes an explicit solution for all heat kernel coefficients in flat space without a gauge connection. It is complete for \( j \leq 3 \) and yields all terms but one in \( a_4 \). To find the coefficient of the ‘missing’ \( X_{(6)} \) term in \( a_4 \) it is best to expand in powers of the matrix potential. This will be presented in the next subsection. Also note that the last two terms in (3.20) do not appear until \( a_6 \). Taken together, (3.20) and (3.31) yield the complete answer for the first six heat kernel coefficients. We refer to sect 6 for the explicit answers for the first five coefficients (to obtain the flat space results, replace each \( \alpha \) by an \( X \) and omit all hats and daggers). The result for \( a_6 \) is presented in appendix C.

### 3.2 Expansion in the potential

We return to (3.7), now paying attention to the order in the matrix potential rather than in derivatives. Thus we are looking for an expansion that starts as

\[ [a_j] = \alpha_j' (\partial^2)^{j-1} X + O(X^2) \]  \hspace{1cm} (3.21)

\( \alpha_j' \) being some \( j \)-dependent numerical factor. We will show that the recursion relations yield an expression of the form

\[ a_j = \alpha_j' X_{(2j-2)} + \sum_k \sum_{\ell} \alpha_{j,k}(2k-\ell)X_{(2j-2k-4)}^{(2k-\ell)} \]  \hspace{1cm} (3.22)

where we used our shorthand notation (the exponents \( 2\ell - p \) and \( p \) on the second and third factor of the last term also count derivatives). This is not manifestly hermitian and verifying hermiticity thus provides a strong check on our results for the \( \alpha' \) coefficients. Furthermore, at third order in \( X \), (3.22) shows overlapping sets of derivatives. The results of the previous subsection imply that it must be possible to remove these overlapping terms. Below we show how this can be done.

For convenience, we repeat our starting point, eq (3.7)
The first term on the right hand side, which is of zeroth order in $X$, can be eliminated by substituting for it from the left hand side, i.e. replace $j \to j - 1$, $n \to n + 2$ and contract one pair of derivatives

$$a_{j-1,n,(2)} = \frac{j - 1}{j + n + 1} \left( a_{j-2,n,(4)} + (X a_{j-2},n,(2)) \right)$$

(3.24)

Iterating this yields

$$a_{j,n} = \sum_{k=0}^{j-1} (X a_{j-k-1},n,(2k)) \prod_{q=0}^{k} \frac{j - q}{j + n + q} = \sum_{k=0}^{j-1} \left( \frac{j}{j + k + 1} \right) (X a_{j-k-1},n,(2k))$$

(3.25)

Clearly, each term is now at least of first order in $X$. We prefer to reverse the order of summation and write this as

$$a_{j,n} = \sum_{k=0}^{j-1} C_{j,k}^{n} (X a_{k},n,(2j-2k-2))$$

(3.26)

where we defined the combinatorical coefficients $C$ by

$$C_{j,k}^{n} = \frac{j}{(2j-k+n-1)}$$

(3.27)

Now separate off the $k = 0$ term and distribute the derivatives over $X a_{k}$ to obtain

$$a_{j,n} = C_{j,0}^{n} X_{n,(2j-2)} + \sum_{k=1}^{j-1} \sum_{p=0}^{2j'+n} \left( \frac{2j'+n}{p} \right) C_{j,k}^{n} X_{(2j'+\hat{n}-p)a_{k,p}}$$

(3.28)

with $j' \equiv j - k - 1$. The set of indices $\mu_{1}\ldots \mu_{n}$ is labeled $\hat{n}$ here to indicate that the elements of this set are to be included in the indicated symmetrization, but they remain uncontracted. By substituting for $a_{k,p}$ from the left hand side we find

$$a_{j,n} = C_{j,0}^{n} X_{n,(2j-2)} + \sum_{k=1}^{j-1} \sum_{p=0}^{2j'+n} \left( \frac{2j'+n}{p} \right) C_{j,k}^{n} C_{k,0}^{p} X_{(2j'+\hat{n}-p)a_{k,p}}(2k-2)$$

$$+ \sum_{k=2}^{j-1} \sum_{\ell=1}^{k-1} \sum_{p=0}^{2j'+n} \left( \frac{2j'+n}{p} \right) C_{j,k}^{n} C_{k,\ell}^{p} X_{(2j'+\hat{n}-p)(X a_{\ell})p}(2k')$$

(3.29)

with $k' \equiv k - \ell - 1$. This shows explicitly the terms of second order in $X$. To proceed to third order, we take $n = 0$ and distribute the derivatives over $X a_{\ell}$. In general requires two binomial sums, one for each of the sets of derivatives marked $p$ and $2k'$, and leads to overlapping sets of derivatives as in $3.22$. To avoid this and to allow us to lump the two sets of derivatives together, we use that for any functions $F(x)$ and $G(x)$

$$F_{1}(G_{1}(2)) = F_{1}(G_{3})$$

$$F_{2}(G_{2}(2)) = \frac{5}{6} F_{2}(G_{4}) + \frac{1}{6} F_{2}(G_{4})$$

(3.30)
etc. Choosing $F = X$, $G = Xa_1$ and taking $x = 0$, the first identity shows that we can trivially lump the derivatives together in computing $a_5$ and the second identity shows how to achieve the same for $a_6$. In appendix B, we show how to do this for arbitrary values of $j$. Restricting for simplicity here to $j \leq 5$, we obtain as our final result

$$a_j = \frac{1}{\binom{2j-1}{j}} X(2j-2) + \sum_{k=1}^{[j/2]} \sum_{p=0}^{j-2k} \binom{2j'}{p} C^0_{j,k} C^p_{k,0} \{X(2j'-p)X_p(2k-2)\}$$

$$+ \sum_{k=2}^{j-1} \sum_{\ell=1}^{2j'} \sum_{p=0}^{2k'+p} \sum_{q=0}^{2k'} \binom{2j'}{p} \binom{2k'}{q} C^0_{j,k} C^p_{k,\ell} C^q_{\ell,0} X(2j'-p)X_{2k'+p-q}X_q(2\ell-2)$$

$$+ O(X^4)$$

(3.31)

with $j' \equiv j - k - 1$, $k' \equiv k - \ell - 1$ and the $C$-symbols were defined in (3.27). Note that we rewrote the second order terms in manifestly hermitian form, the range of the double sum having been correspondingly restricted (we use $\lfloor n \rfloor$ to denote the integer part of $n$). Expression (3.31) is one of the main results of this paper.

4 Flat space with gauge connection

We will show that the heat kernel coefficients in the presence of a gauge connection can be obtained from their ‘trivial’ counterparts without such a connection by making simple covariant substitutions of the kind

$$\partial_{\mu_1} \ldots \partial_{\mu_j} X \rightarrow \nabla_{(\mu_1} \ldots \nabla_{\mu_j)} X + F\text{-dependent terms}$$

(4.1)

Here partial derivatives are turned into totally symmetrized covariant derivatives and in general there are additional field strength dependent terms.

Denote the nonabelian vector connection by $A$. The associated covariant derivative and field strength are defined by

$$\nabla_\mu = \partial_\mu + A_\mu \quad , \quad F_{\mu\nu} = [\nabla_\mu, \nabla_\nu]$$

(4.2)

We take $\nabla$ and thus $F$ to be antihermitian. The heat kernel coefficients satisfy

$$(x^\mu \nabla_\mu + j) a_j = j (\nabla^2 + X) a_{j-1} \quad , \quad [a_0] = I$$

(4.3)

To solve these recursion relations, we find it convenient to work in Fock-Schwinger gauge

$$x^\mu A_\mu(x) = 0$$

(4.4)

This is equivalent to the requirement that all partial derivatives of the gauge connection vanish upon total symmetrization at the origin, i.e. for $j \geq 1$

$$A_{(\mu_1, \ldots \mu_j)}(0) = 0$$

(4.5)
In particular, the gauge field vanishes at the origin and for any \( n \geq 0 \)
\[
(\partial^2)^n \partial \cdot A(0) = 0 \quad (4.6)
\]
In Fock-Schwinger gauge the recursion relations simplify to (cf eq (3.1))
\[
(x^\mu \partial_\mu + j) a_j = j (\partial^2 + X) a_{j-1} \quad , \quad [a_0] = I \quad (4.7)
\]
where we defined the differential operator
\[
\hat{X} = X + A^\mu \partial_\mu + 2A^\mu A_\mu \quad (4.8)
\]
Let \( Z \) be its non-differential-operator part, i.e.
\[
Z = X + A^\mu \partial_\mu + A^\mu A_\mu \quad (4.9)
\]
We will soon need a covariant expression for the partial derivatives of \( Z \) at the origin. It is well known and easily verified that in Fock-Schwinger gauge the partial derivatives of the gauge field at this point have covariant values given for \( j \geq 1 \) by
\[
A^\nu_{\mu_1...\mu_j}(0) = \frac{j}{j+1} F_{(\mu_1...\mu_j)}(0) \quad (4.10)
\]
Here we use semicolon notation for Yang-Mills covariant derivatives. Eq (4.10) implies
\[
A^\nu_{\mu_1...\mu_j}(0) = \frac{j+1}{j+2} F_{(\nu;\mu_1...\mu_j)}(0) \quad , \quad j \geq 1 \quad (4.11)
\]
\[
A^2_{\mu_1...\mu_j}(0) = \frac{1}{(j+1)(j+2)(j+3)} k(j-k) F_{(\mu_1...\mu_k;\mu_{k+1}...\mu_j)}(0) F_{(\mu_{k+1}...\mu_j)}(0) \quad , \quad j \geq 2 \quad (4.13)
\]
If we now define new covariant sans serif symbols \( Y_j \)
\[
Y_j^\nu \equiv \frac{j}{j+1} F_{(\mu_1...\mu_j)}(0) \quad (4.12)
\]
which are to be treated formally as vectors, then we can abbreviate (4.10), (4.11) as
\[
A^\nu_{\mu_1...\mu_j}(0) = Y_j^\nu \quad , \quad A^\nu_{\nu\mu_2...\mu_j}(0) = Y_j^\nu \quad , \quad j \geq 1 \quad (4.14)
\]
\[
A^2_{\mu_1...\mu_j}(0) = \frac{1}{(j+1)(j+2)(j+3)} k(j-k) Y_{(kY_{j-k})} \quad , \quad j \geq 2 \quad (4.13)
\]
We further note that at the origin, due to (4.5), we can immediately covariantize the partial derivatives of the matrix potential as follows
\[
X_j \equiv X_{\mu_1...\mu_j}(0) = X_{(\mu_1...\mu_j)}(0) \quad (4.14)
\]
Here we used the same symbol \( X_j \) as in sect 3 to mean now a totally symmetrized \( j \)-fold covariant derivative of the matrix potential at the origin. Thus, the desired covariant expression for \( any \) partial derivative of \( Z \) at the origin is
\[
Z_j \equiv Z_{\mu_1...\mu_j}(0) = X_j + Y_j^\nu + \frac{1}{(j+1)(j+2)(j+3)} k(j-k) Y_{(kY_{j-k})} \quad (4.15)
\]

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No implicit contractions occur here except for the scalar product between $Y_k$ and $Y_{j-k}$. Replacing $j$ by $2j$ and contracting all indices yields

$$Z_{(2j)} = X_{(2j)} + \sum_{k=1}^{2j-1} \binom{2j}{k} Y_k Y_{\ell} = X_{(2j)} + \sum_{k=1}^{j} \binom{2j}{k} \{Y_k Y_{\ell}\} \quad (4.16)$$

where $\ell = 2j - k$ is to be understood. In the second expression we used the notation of (2.11) to obtain a manifestly hermitian result. In sect. 5 we determine the generalization of (4.13, 4.16) to a curved space. In that case we do not find a closed expression which holds for all values of $j$.

### 4.1 Expansion in derivatives

Returning to our short hand notation, we have

$$a_{j,n} = \frac{j}{j+n} \left( a_{j-1,(2),n} + (\hat{X} a_{j-1}, n) \right) \quad (4.17)$$

Similar to (3.16), we now have the following chain of substitutions (we again omit numerical factors and ordering labels for the heat kernel coefficients, but this time we keep undifferentiated matrices $X$)

$$a \rightarrow a + X a$$

$$a_{(2n)} \rightarrow a_{(2n+2)} + (\hat{X} a)_{(2n)} \quad , \quad n \geq 1 \quad (4.18)$$

etc. In the first line we used that $\hat{X}$ at the origin equals $X$. To remove $\hat{X}$ from the second line of (4.18) as well, we use that at the origin

$$(\hat{X} a)_{(2n)} = (Z^\dagger a)_{(2n)} \quad , \quad n \geq 0 \quad (4.19)$$

Note that the right hand side no longer contains a differential operator. To prove this, we use $\hat{X} = Z + 2A^\mu \partial_\mu$ and $Z^\dagger = Z - 2A^\mu \partial_\mu$ and note that for any function $F$ one has (see appendix D for the proof)

$$(\partial^2)^n \partial_\mu (A^\mu F)(0) = 0 \quad , \quad n \geq 0 \quad (4.20)$$

Taking for $F$ a heat kernel coefficient, (4.19) follows. Adding one more step to the hierarchy (4.18), we obtain

$$a \rightarrow a + X a$$

$$a_{(2n)} \rightarrow a_{(2n+2)} + \sum_{p=0}^{2n} Z_{(p)}^\dagger a_{(2n-p)} \quad , \quad n \geq 1 \quad (4.21)$$

$$Z_{(p)}^\dagger a_{(2n-p)} \rightarrow Z_{(p)}^\dagger a_{(2n-p)}(2) + Z_{(p)}^\dagger (\hat{X} a)_{(2n-p)} \quad , \quad p = 0 \ldots 2n$$
Except for \( p = 0 \) and \( p = 2n \), \([4.19]\) can not be used to eliminate \( \hat{X} \) from the last term. Instead we proceed as follows (write \( r \) for \( 2n - p \) and keep binomial coefficients here)

\[
Z_{(p)}(\hat{X} a) = \sum_{q=0}^{r} \binom{r}{q} Z_{(p)}(Z_q a_r - q + 2A_{q}^\nu a_r - q) \nu
\]

\[
= \sum_{q=0}^{r} \binom{r}{q} Z_{(p)} Z_q a_r - q + \sum_{q'=0}^{r-1} \binom{r}{q'+1} Z_{(p)} 2A_{q'+1}^\nu a_r - q - 1) \nu
\]

\[
= \sum_{q=0}^{r} \binom{r}{q} Z_{(p)}(Z_q a_r - q + \frac{r-q}{q+1} 2A_{q+1}^\nu a_r - q - 1) \nu
\]

\[
\equiv \sum_{q=0}^{r} \binom{r}{q} Z_{(p)} Z_q a_r - q \quad (4.22)
\]

In the second line we shifted the summation index \( q \) so as to collect terms with the same number of derivatives on the heat kernel coefficient. In the third line we can use \([4.10]\) to replace \( A_{q+1} \) by \( Y_{q+1} \) and thus obtain a covariant answer. The last line defines \( Z_q \) in terms of the previous line. The hatted \( Z \) is designed to absorb the new field strength dependent term (this term exists even for \( q = 0 \) and we will write \( \hat{Z} \) for \( \hat{Z}_0 \).

In case \( r = 0 \) too, \( \hat{Z}_0 = X \). With this definition of \( \hat{Z} \) we can replace the last line of the hierarchy \([4.21]\) by

\[
Z_{(p)}(a_{2n-p}) \rightarrow Z_{(p)}(a_{2n-p}(2)) + \sum_{q=0}^{2n-p} Z_{(p)} \hat{Z}_q a_{2n-p-q} \quad , \quad p = 0 \ldots 2n \quad (4.23)
\]

Iterating once more does not bring new features\footnote{The \( q' = -1 \) term is absent because the gauge connection vanishes at the origin, see \([4.5]\). \footnote{Other than those remarked upon in footnote 6.}} and we arrive at the following conclusion.

The diagonal values of the heat kernel coefficients in the presence of a gauge connection are obtained from those without this gauge connection by making the following covariant substitutions in \([3.20]\)

\[
X_{(2j)} \rightarrow Z_{(2j)}
\]

\[
X_{(j)X_k} \rightarrow Z_{(j)Z_k}
\]

\[
X_{(j)X_nX_k} \rightarrow Z_{(j)Z_nZ_k}
\]

\[
X_{(j)X_nX_k} \rightarrow Z_{(j)Z_nZ_k}
\]

etc, where \( j, k \geq 1,\ n, p, \ldots \geq 0,\ Z_j \) is defined in \([4.14]\) and the \( \hat{Z} \) act to their right as follows

\[
\hat{Z}_n Z_k = Z_n Z_k + \frac{2}{n+1} k Y_{n+1}^\nu (Z_{k-1}) \nu
\]

\[
\hat{Z}_p Z_n Z_k = Z_p \hat{Z}_n Z_k + \frac{2}{p+1} Y_{p+1}^\nu (n \hat{Z}_{n-1}^\nu Z_k + k \hat{Z}_n Z_{k-1}) \nu \quad (4.25)
\]
Since they are unaffected, we omitted $Z_{ij} \ldots$ from the left of each term in (4.25), the dots representing possible further $\hat{Z}$. Note that $\hat{Z}_n$ acts on each one of the $k$ indices on $Z_k$ in the same way, hence the factor $k$. Similarly $\hat{Z}_p$ acts on $n$ plus $k$ indices, etc. Thus, upon ‘turning on the gauge field’, (3.20) becomes

$$a_j = \hat{X}^j + \sum_{k=1}^{j-1} \frac{k\ell}{j+1} X^{k-1} Z_{(2)}^j X^\ell + \sum_{k=1}^{j-2} \sum_{\ell=1}^{j-k-1} \frac{2k\ell}{j+1} X^{k-1} Z_{(1)}^j \hat{Z}^m \hat{Z}_{(1)}^\ell X^\ell + \ldots$$  (4.26)

Note that only an $X$ which is ‘sandwiched’ between $X_j$ and $X_k$ gets replaced by a $\hat{Z}$. Formally, the substitutions in (4.24) do not change the number of terms or their numerical coefficients. In this sense the heat kernel coefficients retain their original appearance in the gauging process. If we use the curly bracket notation of (2.11), we should take note that for $j \neq k$

$$\{Z_{ij}^j \hat{Z}_n Z_k\} = Z_{ij}^j \hat{Z}_n Z_k + Z_{ik}^i \hat{Z}_n Z_j$$  (4.27)

with no dagger on $\hat{Z}_n$ in the second term (this can be shown to be hermitian). Finally, repeating the steps of (3.17), we find that the covariant analogues of (3.19) are absent.

### 4.2 Expansion in the potential

Upon replacing $X$ in sect. 3.2 by the differential operator $\hat{X}$ and retracing our steps, we find that the heat kernel coefficients for the case with a gauge connection are obtained from those in (3.31) through the same substitutions as in (4.24). Thus

$$a_j = \frac{1}{(2j-1)} Z_{(2j-2)} + \sum_{k=1}^{j/2} \sum_{p=0}^{j-2k} \binom{2j'}{p} C_0^{jk} C_0^{kp} \{Z_{(2j'-p)} Z_{(2k-2)}\}$$

$$+ \sum_{k=2}^{j-1} \sum_{\ell=1}^{j-k-1} \sum_{p=0}^{2j'} \binom{2j'}{p} \binom{2k' + p}{q} C_0^{jk} C_0^{k\ell} C_0^{pq} Z_{(2j'-p)} \hat{Z}_{(2k'+p-q)} Z_{(2\ell-2)} + O(Z^4)$$  (4.28)

with $j' \equiv j - k - 1$, $k' \equiv k - \ell - 1$ and the $C$-symbols were defined in (3.27). The action of $\hat{Z}_n$ was defined in (4.25) (it does not act on the set of indices labeled $(2\ell - 2)$).

### 5 Curved space

In this section we shall generalize our results to a curved space with metric $g_{\mu\nu}$. Riemann normal coordinates $x^\mu$ can be defined by\textsuperscript{9}

$$g_{\mu\nu}(0) = \delta_{\mu\nu}, \quad x^\mu g_{\mu\nu}(x) = x^\mu g_{\mu\nu}(0)$$  (5.1)

\textsuperscript{9} We use the same symbol for normal coordinates as for general coordinates. This should not cause confusion. In these coordinates and at the origin there is no need to
which is equivalent to
\[ g_{\mu\nu}(0) = \delta_{\mu\nu}, \quad g_{\mu(\mu_1,\mu_2,\ldots,\mu_j)}(0) = 0, \quad j \geq 2 \] (5.2)
These properties also hold for the inverse metric \( g^{\mu\nu} \). Taking the origin of our normal coordinate system to coincide with the point \( x' \), we have
\[ \sigma(x,0) = \frac{1}{2} x^2 = \frac{1}{2} x^\mu x^\nu \delta_{\mu\nu}, \quad \mathcal{D}(x,0) = g(x)^{-1/2} \] (5.3)
and DeWitt’s ansatz (2.3) for the heat kernel becomes
\[ K(x,0;\tau) = (4\pi\tau)^{-d/2} g(x)^{-1/4} e^{-x^2/4\tau} \sum_{j=0}^{\infty} \frac{a_j(x,0)}{j!} \tau^j \] (5.4)
Using (5.3), the recursion relations (2.9) become
\[ (x^\mu \partial_\mu + j) a_j = j (\partial_\mu g^{\mu\nu} \partial_\nu + 2A^\mu \partial_\mu + Z) a_{j-1}, \quad [a_0] = I \] (5.5)
Here
\[ Z \equiv Z^M + Z^S, \quad Z^M \equiv X + A^\mu A_\mu, \quad Z^S \equiv \frac{1}{2} B^\mu_{\mu} - \frac{1}{4} B^\mu_{\nu} B_{\nu} \] (5.6)
where we defined
\[ B \equiv \ln \mathcal{D} = - \ln \sqrt{g} \] (5.7)
Note the position of the explicit inverse metric in (5.3). The quantities \( Z^M \) and \( Z^S \) are the matrix and scalar parts of \( Z \) respectively. Also note the similarity between \( Z^M - X \) and \( Z^S \). In the flat space limit \( Z^S \) vanishes and \( Z^M \) reduces to the quantity earlier defined as \( Z \) in (4.9). We now reserve the symbol \( Z \) to mean the sum of \( Z^M \) and \( Z^S \).

### 5.1 Scalar part

We write \( Z^S \) with explicit inverse metric as
\[ Z^S = \frac{1}{2} (g^{\mu\nu} B_\nu)_{,\mu} - \frac{1}{4} B_{\mu,\nu} g^{\mu\nu} B_{\nu} \] (5.8)
Our task is to find a covariant expression for the partial derivatives of \( Z^S \) at the origin of the normal coordinate system. This in turn requires the expansion for the (inverse) metric in normal coordinates, which is a well-known problem. We find it easier to work with the vielbein, the metric being defined as usual by
\[ g_{\mu\nu} = e_\mu^a e_\nu^b \delta_{ab} \] (5.9)

Distinguish co- from contravariant indices. We exploit this to position indices in such a way as to cause minimal clutter. Normal coordinates \( x^\mu \) may alternatively be defined in terms of the affine connection by \( x^\mu x^\nu \Gamma^\lambda_{\mu\nu}(x) = 0 \) (which is equivalent to \( \Gamma^\lambda_{(\mu_1,\mu_2,\ldots,\mu_j)}(0) = 0 \) for \( j \geq 2 \)). Note the similarity with the Fock-Schwinger gauge (4.4).
where \( a \) and \( b \) are tangent-space indices. A recursive formula for the vielbein in normal coordinates was given in \([10]\). Here we shall give its solution to all orders in the curvature. Using matrix notation for the vielbein, i.e.

\[
(E_j)^a_\mu \equiv e^a_\mu \ldots \mu_j(0)
\]  

and defining the following sans serif curvature symbols (compare with \((4.12)\))

\[
K^\mu_\nu = \frac{j-1}{j+1} R(\mu_1 \mu_2 \ldots \mu_j)(0) , \ j \geq 2
\]  

we can write the recursive formula of \([10]\) for the vielbein as follows

\[
E_0 = I , \ E_1 = 0 , \ E_j = -K_j - \sum_{k=2}^{j-2} \binom{j}{k} \frac{k(k+1)}{j(j+1)} K_k E_k , \ j \geq 2
\]  

where \( I \) is the \( d \times d \) unit matrix and total symmetrization on the \( k + \ell \equiv j \) indices is implied. We treat \( K_j \) as a symmetric matrix in the index pair \( \mu \nu \). Taking the trace of such a matrix yields

\[
\text{tr}[K_j] = \frac{j-1}{j+1} R(\mu_1 \mu_2 \ldots \mu_j)(0) , \ j \geq 2
\]  

where the Ricci tensor and its covariant derivatives appear. Note that we traced over the world indices. This is not to be confused with the trace over field or ‘bundle’ indices which does not occur in this paper. In taking the trace over a product of \( K \)-matrices, say \( \text{tr}[K_j K_k K_\ell] \), total symmetrization on the \( j + k + \ell \) indices is implied. Such traces appear in our final expression for the heat kernel coefficients in curved space, e.g.

\[
\text{tr}[K(2K_2)] = \frac{1}{9} R(\kappa^\mu_\nu(0) R_\lambda^\nu_\lambda)(0) = \frac{1}{27} R^{\kappa\lambda} R_{\kappa\lambda} + \frac{1}{18} R^{\kappa\lambda\mu\nu} R_{\kappa\lambda\mu\nu}
\]  

where the parenthesis on the left hand side imply symmetrization and contraction (see \((A.9)\) for the general case). In this way we maintain an index-free notation. Returning now to the recursion formula \((5.12)\), the first few cases are

\[
E_2 = -K_2 , \ E_3 = -K_3 , \ E_4 = -K_4 + \frac{9}{5} K_2 K_2 , \ E_5 = -K_5 + 2K_2 K_3 + 4K_3 K_2
\]  

where the parenthesis mean symmetrization only. Note that for \( j \geq 5 \) the \( E_j \) are not symmetric matrices. By iteration of \((5.12)\) we obtain as solution to all orders

\[
E_j = -K_j - \sum_{n=1}^{\lfloor j/2 \rfloor - 1} (-1)^n \sum_{k_1=2}^{j-2n} \sum_{k_2=2}^{j-2n+2} \ldots \sum_{k_n=2}^{j-n} \binom{j}{k_1,\ldots,k_n} \left( \prod_{i=1}^n \frac{k_i(k_i+1)}{j_i(j_i+1)} K_{k_i} \right) K_{j+n+1}
\]

\[
j_1 = j , \ j_i = j - \sum_{\ell=1}^{i-1} k_\ell , \ i \geq 2
\]  

\[\text{Except for the normalization factor, this agrees with the definition in } [9]. \text{ See } (A.2)\]  

for our curvature conventions.
where total symmetrization of the \( j \) indices is to be understood. To the best of our knowledge, such an explicit expression for the vielbein in normal coordinates was not available in the literature up to now\(^{11}\). For the inverse metric it follows that

\[
g^{\mu\nu} = 2 K^{\mu\nu} + 2 \sum_{k=2}^{[j/2]} \binom{j}{k} \alpha_{k\ell} \left\{ K_k K_\ell \right\}^{\mu\nu} + O(K^3) \quad \alpha_{k\ell} \equiv 1 + \frac{k\ell}{j(j+1)} \tag{5.17}
\]

where \( \ell \equiv j - k \) (see \([11, 12]\) for our summation conventions). Finding the heat kernel coefficients through fourth order in \( K \) turns out to require knowledge of the inverse metric through second order only. See appendix E. Explicit expansions for the metric and its inverse to some finite order in normal coordinates are well known, see e.g. \([8, 21]\) and our eq \((E.4)\). We next use

\[
B(x) = - \text{tr} \ln E = - \sum_{k=1}^{\infty} \frac{1}{k} \text{tr}[(I - E)^k]
\]

Thus \( B \) and its first derivative vanish at the origin and for \( j \geq 2 \), after collecting terms of the same degree in \( K \), we find (cf \((E.5)\))

\[
B_j = \text{tr}[K_j] + \sum_{k=2}^{[j/2]} \binom{j}{k} \beta_{k\ell} \tilde{\text{tr}}[K_k K_\ell] + \sum_{k=2}^{[j/2]} \sum_{\ell=k}^{[j-k-2]} \binom{j}{\ell} \beta_{k\ell m} \tilde{\text{tr}}[K_k K_\ell K_m] + O(K^5) \tag{5.19}
\]

where \( \tilde{\text{tr}} \) is a weighted trace defined below and the lower limit \( k'' \) means: when \( \ell = k \), reduce the upper limit for the sum over \( m \) to \([j/2] - k \). The coefficients are given by

\[
\beta_{k\ell} = 1 - P_2 \frac{k(k+1)}{j(j+1)} = \frac{2k\ell}{j(j+1)}
\]

\[
\beta_{k\ell m} = 2 - P_3 \frac{k(k+1)(j+\ell+1)}{j(j+1)(j-\ell+1)} = \frac{4k\ell m(j^2 + 4j + 3k + \ell + \ell m + mk)}{j(j+1)D_{k\ell}D_{\ell m}D_{m k}}
\]

\[
\beta_{k\ell m n} = 2 - P_4 \left[ \frac{k(k+1)}{(k+\ell)D_{k\ell}} \left( 1 - \frac{n(n+1)(j+m+1)}{j(j+1)D_{k\ell m}} - \frac{m(m+1)}{2(m+n)D_{mn}} \right) \right]
\]

\[
D_{k\ell} \equiv k + \ell + 1 \quad , \quad D_{k\ell m} \equiv k + \ell + m + 1 \tag{5.20}
\]

Here \( P_N \) denotes the group consisting of all cyclic and anticyclic permutations on \( N \) objects (so \( |P_2| = 2 \), \( |P_N| = 2N \) for \( N > 2 \)) and its action is defined by

\[
P_N[f(k_1, \ldots, k_N)] = \sum_{\pi} f(k_{\pi(1)}, \ldots, k_{\pi(N)}) \quad , \quad j \equiv \sum_{i=1}^{N} k_i \tag{5.21}
\]

where the sum runs over all cyclic and anticyclic permutations. We used here that the trace of a product of symmetric matrices is invariant under such permutations

\(^{11}\)We have been informed by C. Schubert and U. Müller that they have obtained results equivalent to \((5.16)\). Details will be published elsewhere \([29]\).
in order to simplify the sums. Thus, in \( \text{tr}[K_k K_{\ell} K_m] \) the order of the matrices is irrelevant and we can assume without loss of generality that \( k \leq \ell \leq m \). Similarly, in \( \text{tr}[K_k K_{\ell} K_m K_n] \) we can assume \( k \) to be the smallest label and \( \ell \leq n \) (when \( \ell = k \) we arrange \( m \leq n \)). Ordering subtleties do not occur before fourth order, where we must distinguish between \( \text{tr}[K_2 K_2 K_3 K_3] \) and \( \text{tr}[K_2 K_3 K_2 K_3] \). The weighted trace \( \tilde{\text{tr}} \) is defined by

\[
\tilde{\text{tr}}[K_{k_1} K_{k_2} \ldots K_{k_N}] = \frac{S}{|P_N|} \text{tr}[K_{k_1} K_{k_2} \ldots K_{k_N}]
\]

with \( S \) the total number of distinguishable cyclic and anticyclic permutations (including the identity) of the product \( K_{k_1} K_{k_2} \ldots K_{k_N} \). We refer to appendix F for a table containing all symmetry factors for \( N \leq 4 \). Substituting (5.19) and (5.17) into (5.8) we find for any \( j \geq 0 \) but with a cutoff at third order in \( K \) (see appendix E for the details and (6.4) for examples)

\[
Z_{j}^{S} = \frac{1}{2} \text{tr}[K_{j \mu \mu}] + \sum_{k=2}^{j} \binom{j+1}{k} K_{k \mu \nu} \text{tr}[K_{j-k} \mu \nu]
\]

\[
+ \sum_{k=2}^{j} \sum_{\ell=1}^{k-1} \binom{k-2}{\ell} \frac{j+1}{j+3} \text{tr}[K_{\mu \kappa} K_{\mu \ell}] - \frac{1}{2} \text{tr}[K_{\mu \ell}] \text{tr}[K_\ell]
\]

\[
+ \sum_{k=2}^{j} \sum_{\ell=1}^{k-1} \sum_{\mu=1}^{k} \binom{k}{\ell} \binom{k+1}{k-\mu} \beta_{k \ell m} \text{tr}[K_{\mu-1} K_m K_{m \ell-1}]
\]

\[
- \sum_{k=1}^{j} \sum_{\ell=2}^{k-1} \binom{j}{k} \binom{k}{\ell} \frac{\ell}{j-k+2} \text{tr}[K_{\mu \ell}] \text{tr}[K_{\ell} K_{\ell}] + \sum_{k=1}^{j} \sum_{\ell=2}^{k} \binom{j}{k} \binom{k-1}{\ell} \text{tr}[K_{\mu \ell}] \text{tr}[K_{\ell}]
\]

\[
+ \sum_{k=2}^{m} \binom{j+1}{k} \sum_{\ell=2}^{m} \binom{m}{\ell} \frac{\ell}{k(k+1)} \text{tr}[K_{\ell} K_{\ell}] \text{tr}[K_{j-k} \mu \nu]
\]

\[
+ \sum_{k=2}^{j} \sum_{\ell=2}^{k} \binom{j+1}{k} \binom{k-1}{\ell} \frac{2\ell}{j-k+3} \text{tr}[K_{\ell} K_{j-k-\mu \nu}]
\]

\[
+ O(K^4)
\]

This expression is complete for \( j \leq 5 \). The coefficient \( \beta_{k \ell m} \) which appears here is obtained by replacing \( j \) by \( j+2 \) in (5.20) and \{\( K_k K_{\ell} \)\} is defined as in (2.11). The prime on \([j/2]\) in the second sum implies division by 2 when \( j \) is even and this limit is reached. Similarly, the lower limit \( \ell = k' \) means division by 2 when \( \ell = k \). We recall that total symmetrization on all indices within a given trace (except for those being traced over) is to be understood and thus one has e.g.

\[
\text{tr}[K_{\mu \ell} K_{\mu \ell}] = \text{tr}[K_{\mu \ell} K_{\mu \ell}]
\]

\[\text{(5.24)}\]

\[\text{Thus in obtaining the logarithm of the Van Vleck-Morette determinant through ninth order in the normal coordinates, see (E.3), we may treat the } K_j \text{ as commuting objects!}\]
Replacing \( j \) by \( 2j \) in (5.23) and contracting fully, the second and last two terms vanish. Keeping also terms of fourth order in \( K \) we find (cf (5.5))

\[
Z^S_{(2j)} = \frac{1}{2} \text{tr}[K_{(2j+2)}] \\
+ \sum_{k=1}^{j'} \frac{2j}{k} \left( \frac{2j+1}{2j+3} \text{tr}[K_{(k+1)}K_{(\ell+1)}] - \frac{1}{2} \text{tr}[K_{\mu}K_{\ell}K_{\mu}] \right) \\
+ \sum_{k=2}^{[2j+2]} \sum_{\ell=k}^{[2j-k+2]} \left( \frac{2j+2}{k,\ell} \right) \frac{1}{2} \beta_{k\ell m} \tilde{\text{tr}}[K_{(k)K_{m}K_{m}}] \\
- \sum_{k=1}^{[2j-3]} \sum_{\ell=1}^{[2j-k-1]} \gamma_{\ell m} \text{tr}[K_{\mu}K_{m}K_{\ell}] \\
+ \sum_{k=1}^{[j+1]} \sum_{\ell=k+1}^{2j-k-2+\ell} \left( \frac{2j+2}{k,\ell, m} \right) \frac{1}{2} \beta_{k\ell m} \text{tr}[K_{(k)K_{m}K_{m}}] \\
- \sum_{k=1}^{[j-k]} \sum_{\ell=k+1}^{[j-k-1]} \left( \frac{2j+2}{k,\ell, m} \right) \gamma_{\ell m} \text{tr}[K_{\mu}K_{m}K_{\ell}] \\
- \sum_{k=1}^{[j-k]} \sum_{\ell=k+1}^{[j-k-1]} \left( \frac{2j}{k,\ell, m} \right) \gamma_{\ell m} \text{tr}[K_{\mu}K_{m}K_{\ell}] \\
- \sum_{k=1}^{[j-k]} \sum_{\ell=k+1}^{[j-k-1]} \left( \frac{2j}{k,\ell, m} \right) \gamma_{\ell m} \text{tr}[K_{\mu}K_{m}K_{\ell}] \\
- \sum_{k=1}^{[j-k]} \sum_{\ell=k+1}^{[j-k-1]} \left( \frac{2j}{k,\ell, m} \right) \gamma_{\ell m} \text{tr}[K_{\mu}K_{m}K_{\ell}] \\
+ O(K^5)
\]

This expression is complete for \( j \leq 3 \). The \( \alpha_{mn} \) are defined in (5.17) (with \( j \to m + n \equiv 2j - k - \ell \)) and the \( \beta \)-coefficients are those of (5.20) with \( j \to 2j + 2 \). The \( \gamma \)-coefficients are defined by

\[
\gamma_{k\ell} \equiv \frac{\ell}{k+\ell+2}, \quad \gamma_{k\ell m} \equiv \frac{k+\ell+m+1}{2(k+1)} \beta_{k+1\ell m}
\]

The upper limit \( j' \) means: divide by 2 when \( k = j \). A lower limit \( k' \) means: divide by 2 when \( \ell = k \). The upper limit \( (j-k)' \) means: divide by 2 when \( \ell = j-k \) and \( m = k \). Finally, the lower limit \( k'' \) means: when \( \ell = k \), reduce the upper limit for the sum over \( m \) to \( j-k+1 \).
5.2 Matrix part

Abbreviate the $j$-th partial derivative of the gauge connection at the origin by

$$A_j = A_{\mu_1\ldots\mu_j}(0) \quad (5.27)$$

Then the recursion for the gauge connection is given by (until further notice, parenthesis imply symmetrization only)

$$A_0 = 0 \ , \ A_j = Y_j + \sum_{k=2}^{j-1} \binom{j}{k} \frac{\ell + 1}{j + 1} E_{(kY_\ell)} \ , \ j \geq 1 \quad (5.28)$$

We iterate this and use \( (5.16) \) to obtain as solution to all orders

$$A_j = Y_j + \sum_{n=1}^{\lfloor i/2 \rfloor} (-1)^n \sum_{k=1}^{j-2n} \sum_{i=2}^{j-n-2} \binom{j}{k} j^{k+1} \sum_{i=1}^{n-1} \frac{k_i(k_i+1)}{j_i(j_i+1)} K_{k_i} K_{j_n}$$

$$j_i \equiv j - \sum_{\ell=0}^{i-1} k_\ell \ , \ k_0 \equiv k \quad (5.29)$$

where symmetrization on the $j$ indices is to be understood. We used that, since the $K_j$ are symmetric matrices, we may write them to the right of the vector $Y_{k_0}$ in the reverse order. Through second order in $K$, \( (5.29) \) reads \( cf \ (5.3) \)

$$A_j = Y_j - \sum_{k=1}^{j-2} \binom{j}{k} \frac{k + 1}{j + 1} Y_{(kK_\ell)}$$

$$+ \sum_{k=1}^{j-4} \sum_{\ell=2}^{j-k-2} \binom{j}{k, \ell} \frac{(k+1)\ell(\ell+1)}{(j+1)(j-k)(j-k+1)} Y_{(kK_\ell K_m)} + O(K^3) \quad (5.30)$$

We thus find (this is exact for $j \leq 5$; see \( (5.2) \) for examples)

$$Z^{(j)} = X_j + Y_{(j)} + \sum_{k=1}^{j-1} Y_{(kY_\ell)} + \sum_{k=2}^{j} \binom{j + 1}{k} \frac{j + k + 2}{j + 2} K_{(\mu \nu \omega)} Y_{(k_\mu \nu j-k)}$$

$$+ \sum_{k=1}^{\lfloor j/2 \rfloor} \sum_{\ell=k}^{j-k-2} \binom{j}{k, \ell} \left( \frac{m}{k + m + 1} + \frac{m}{\ell + m + 1} \right) \{ Y_{(kK_\ell Y_\ell)} \}$$

$$+ \sum_{k=2}^{j-2} \sum_{\ell=2}^{j-k} \binom{j + 1}{k, \ell} \left( 3\ell + m + 1 - \frac{\ell(\ell+1)}{(j+2)(k+\ell+1)} - \frac{k(k+1)}{(k+\ell)(k+\ell+1)} \right)$$

$$\times (K_{(kK_\ell)} Y_{(\mu \nu \omega)^{j-k-\ell}}) + O(K^3 Y_{2Y_2}) \quad (5.31)$$

If we now replace $j$ by $2j$ in \( (5.31) \) and contract all indices, the second, fourth and last term vanish. Including terms of fourth order in curvatures, we find \( cf \ (6.3) \)

$$Z^{(2j)} = X_{(2j)} + \sum_{k=1}^{j} \binom{2j}{k} Y_{(kY_\ell)}$$

$$+ \sum_{k=1}^{j-1} \sum_{\ell=k}^{2j-k-2} \binom{2j}{k, \ell} \left( \frac{m}{k + m + 1} + \frac{m}{\ell + m + 1} \right) \{ Y_{(kK_\ell Y_\ell)} \}$$

$$+ \sum_{k=1}^{j-2} \sum_{\ell=k}^{2j-k-\ell-2} \binom{2j}{k, \ell, m} \sum_{m=2}^{mn} \frac{m n N_{k_l m}}{D_{k m} D_{k m} D_{m n} D_{m n} D_{l m n}} \{ Y_{(kK_\ell K_m Y_\ell)} \} + O(K^3) \quad (5.32)$$
where the \(D\)-symbols were defined in (5.20) and

\[
N_{k\ell m} = 5 + 5k\ell + 12j + 16jmn + \{6m + (k + m)(9m + 10n) + k^2(\ell + m + 2n + 2) + (3k + 4\ell + 2m)m^2 + 4k\ell m\}_5 + \{6m + (k + m)(9m + 10n) + k^2(\ell + m + 2n + 2) + (3k + 4\ell + 2m)m^2 + 4k\ell m\}_S
\]

\[
\{ f(k, \ell, m, n) \}_S \equiv f(k, \ell, m, n) + f(\ell, k, m, n) \quad (5.33)
\]

We recognize the first line of (5.32) as the flat spacetime result (4.16). As we already mentioned there, we are not able to give a closed expression to all orders in \(K\) here. However, the terms of \(N\)-th order in \(K\) in \(Z^M_{(2j)}\) will be of the form

\[
\{ Y(k K m_1 \ldots K m_N Y_0) \} \quad , \quad 2j = k + \ell + \sum_{i=1}^{N} m_i \quad , \quad k \leq \ell \quad (5.34)
\]

and this shows that \(Z^M\) has a much simpler structure than \(Z^S\).

### 5.3 Expansion in derivatives

We retrace the steps of sect 4.1, starting with (compare with (4.17))

\[
a_{j,n} = \frac{j}{j+n} \left( a_{j-1,\mu,n} + (\hat{X} a_{j-1}),_n \right) \quad (5.35)
\]

where now

\[
\hat{X} = Z + 2A^\mu \partial_\mu = Z^M + Z^S + 2A^\mu \partial_\mu \quad (5.36)
\]

Except for \(n = 0\) or 1, we can not replace the explicit pair of indices in (5.35) by our shorthand notation as in (4.17). However, for even values of \(n\) and after full contraction we can use the following lemma: for any scalar \(F(x)\)

\[
F_{\mu (2n)} = F_{(2n+2)} \quad , \quad n \geq 0 \quad (5.37)
\]

This follows immediately from the defining properties of the inverse metric in normal coordinates, see (5.2). Using this lemma, we obtain the following chain of substitutions (omit numerical factors and ordering labels, but keep undifferentiated \(Z\)'s)

\[
a \rightarrow a_{(2)} + Z a
\]

\[
a_{(2n)} \rightarrow a_{(2n+2)} + (\hat{X} a_{(2n)}) \quad , \quad n \geq 1 \quad (5.38)
\]

\[
Z_{(p a_{2n-p})} \rightarrow Z_{(p a_{2n-p})\mu} + Z_{(p (\hat{X} a_{2n-p})\mu)} \quad , \quad p = 0 \ldots 2n
\]

etc, where

\[
Z = Z^M + Z^S = X + \frac{1}{2} \text{tr}[K_{(2)}] = X(0) + \frac{1}{6} R(0) \quad (5.39)
\]

In the second step we used that (4.20), with the understanding that \(\partial^2 = \partial_\mu \partial_\mu\), remains true in normal coordinates. We can remove \(\hat{X}\) from the last line exactly as
in the first three lines of (4.22). However, the first term on the right hand side in the last line of (5.38) yields a new term as compared to flat space \((r \equiv 2n - p)\)

\[
Z_{(p}^\dagger a_{r)}^\mu \equiv Z_{(p}^\dagger (h^{\mu\nu}a_{|\nu|})_{r}\mu \\
= -\frac{p}{r+1} Z_{\mu(p-1}^\dagger (h^{\mu\nu}a_{|\nu|})_{r+1)} \\
= -\frac{p}{r+1} \sum_{q=0}^{r+1} \binom{r+1}{q} Z_{\mu(p-1}^\dagger h^{\mu\nu}q a_{r-q+1}\nu \\
= -\sum_{q=0}^{r} \binom{r}{q} \frac{p(r-q)}{(q+1)(q+2)} Z_{\mu(p-1}^\dagger g^{\mu\nu}q+2 a_{r-q+1}\nu}
\]  

(5.40)

The first step merely defines \(h^{\mu\nu} \equiv g^{\mu\nu} - \delta^{\mu\nu}\). The vertical bars indicate that the index \(\nu\) is not to be included in the indicated symmetrization. In the second step we use that, due to (5.2)

\[
Z_{(p}^\dagger (h^{\mu\nu}a_{|\nu|})_{r}\mu) = 0
\]  

(5.41)

Writing this out with respect to the position of the index \(\mu\) shows the equality of the first two lines of (5.40). Next we use the binomial theorem and in the last step we shift \(q\) and use that \(h\) and its first derivative vanish at the origin. We absorb this new term through a redefinition of \(\hat{Z}_q\) in (4.22) and arrive at the following conclusion.

The diagonal values of the heat kernel coefficients in curved space are obtained from those in flat space by formally making the same substitutions as in (4.24), where \(Z_j\) now stands for \(Z^M_j + Z^S_j\) (see (5.23, 5.31) and the action of \(\hat{Z}\) is defined by

\[
Z_{(j}^\dagger Z_{n}^\mu Z_{k)} = Z_{(j}^\dagger Z_{n}^\mu Z_{k)} + \frac{2k}{n+1} Z_{(j}^\dagger A_{n+1}^\mu Z_{k-1)\nu} - \frac{j\kappa}{(n+1)(n+2)} Z_{\mu(j-1}^\dagger g^{\mu\nu}n+2 Z_{k-1)\nu}
\]  

(5.42)

etc, where the (contravariant) gauge connection and inverse metric follow from (5.17) and (5.30).

These redefinitions being understood, the heat kernel coefficients formally remain unchanged upon going to curved spacetime.

### 5.4 Locally symmetric space

The results of the previous sections nearly suffice to find \(a_5\). The few missing terms are of fifth order in the Yang-Mills and Riemann curvature. They are most easily found by considering a locally symmetric space, i.e. a space with covariantly constant curvatures. In that case one can find closed expressions which hold through all orders in these curvatures. This situation has been considered in detail by Avramidi \[24\]. Here, we merely want to find those terms in a given heat kernel coefficient for a general curved space which do not contain explicit covariant derivatives. We therefore do not take account of the consequences of the requirement that the Riemann curvature is covariantly constant (i.e. \(\nabla R = 0\) would imply \([\nabla, \nabla]R = 0\)).
Thus, consider the case where only $X_0$, $Y_1$, and $K_2$ are nonvanishing. Define

$$Y_\nu(x) = \frac{1}{2} F_{\mu\nu}(0) x^\mu, \quad K_{\mu\nu}(x) = \frac{1}{3} R_{\mu\nu\rho\sigma}(0) x^\rho x^\sigma$$  \hspace{1cm} (5.43)$$

The vielbein is then given to all orders in $K$ by (see (5.16)

$$E[K] = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} (-3K)^k = \frac{\sinh S}{S}, \quad S \equiv \sqrt{-3K}$$  \hspace{1cm} (5.44)$$

Note that the vielbein is an even function of $S$ and hence it depends only on $K$. Since $K$ is a symmetric matrix, so is $E$. It follows that the inverse metric simply equals $E^{-2}$. Differentiating $B = -\text{tr} \ln E$ we find

$$B_{,\mu} = \frac{1}{2} \text{tr}[K_{,\mu} L], \quad L[K] \equiv \frac{3}{S} (\coth S - \frac{1}{S})$$  \hspace{1cm} (5.45)$$

where $L$ is an even function$^{13}$ of $S$. In general, the commutator of $K$ and $K_{,\mu}$ does not vanish, but the trace insures that (5.45) holds. Differentiating $L$ we find

$$L_{,\mu} = \langle K_{,\mu} L \rangle, \quad \langle K_{,\mu} K^n \rangle \equiv \frac{1}{n} \sum_{k=0}^{n} K^k K_{,\mu} K^{n-k}$$  \hspace{1cm} (5.46)$$

With this notation, the result for $Z^S$ in a locally symmetric space reads

$$Z^S[K] = \frac{1}{4} (E^{-2})^{\mu\nu} \left( \text{tr}[K_{,\mu\nu} L + K_{,\mu} L_{,\nu}] - \frac{1}{4} \text{tr}[K_{,\mu} L] \text{tr}[K_{,\nu} L] \right) + \frac{1}{4} (E^{-2} K_{,\mu} L)^{\mu\nu} \text{tr}[K_{,\nu} L]$$  \hspace{1cm} (5.47)$$

To obtain a similar result for $Z^M$ we note that the covariant gauge connection is given by

$$A[Y, K] = \left( \frac{\sinh S/2}{S/2} \right)^2 Y$$  \hspace{1cm} (5.48)$$

The contravariant gauge connection is therefore

$$A[Y, K] = (\text{sech} S/2)^2 Y$$  \hspace{1cm} (5.49)$$

and we obtain the following result for $Z^M$ in a locally symmetric space

$$Z^M[X, Y, K] = X - \partial \cdot (\tanh S/2)^2 Y + Y \left( \frac{\tanh S/2}{S/2} \right)^2 Y$$

$$= X + \partial \cdot \left( \frac{3}{4} K + \frac{3}{8} K^2 + \ldots \right) Y + Y (I + \frac{1}{2} K + \frac{17}{80} K^2 + \frac{93}{1120} K^3 + \ldots) Y$$  \hspace{1cm} (5.50)$$

$^{13}$Except for the extra factor $3/S$, this is the well known Langevin function.
6 Explicit results for $a_1$ through $a_5$

As an application, we give here the explicit results for the diagonal values of the first five heat kernel coefficients, obtained from (4.24), (4.28) and (5.42). The number of terms equals 1, 2, 4, 10 and 26 respectively. The sixth coefficient in flat space can be found in appendix C.

\[
\begin{align*}
a_1 &= Z \\
a_2 &= Z^2 + \frac{1}{3}Z(2) \\
a_3 &= Z^3 + \frac{1}{2}(Z_1 Z(2)) + \frac{1}{2}Z(1)Z_1 + \frac{1}{10}Z(4) \\
a_4 &= Z^4 + \frac{3}{5}(Z^2 Z(2)) + \frac{4}{5}Z Z(3) + \frac{2}{5}Z_1 Z_2 + \frac{4}{5}(ZZ_1)^2 + \frac{4}{5}(Z_1 Z_2) \\
a_5 &= Z^5 + \frac{2}{3}(Z^2 Z(2)) + \{Z^2 Z(2)\} + \frac{1}{3}Z_1 Z_1 + \frac{2}{3}(ZZ_1) Z_2 + \frac{2}{3}(Z_1 Z_2) \\
&\quad+ \frac{4}{3}Z(Z_1 Z_2) + \frac{2}{7}(Z^2 Z(4)) + \frac{3}{7}Z Z(4) + \frac{6}{7}(ZZ_2 Z(2)) + \frac{3}{7}Z_2 Z(2) + \frac{1}{7}Z_1 Z_1 \\
&\quad+ \frac{16}{21}(ZZ_1 Z_3) + \frac{16}{7}Z Z_2 Z_2 + \frac{6}{7}(Z Z_2 Z_2) + \frac{1}{3}(Z_1 Z_2) Z_2 + \frac{1}{3}(Z_1 Z_2) + \frac{1}{14}(Z_1 Z_2) \\
&\quad+ \frac{5}{14}(Z_1 Z_2) + \frac{5}{14}Z_1 Z_3 + \frac{2}{21}Z_2 Z_4 + \frac{4}{21}Z_3 Z_3 + \frac{1}{126}Z_4
\end{align*}
\]  

The $Z$'s appearing here were defined in the previous sections, but to be quite explicit and to avoid misunderstanding of our conventions, we give them below. We have $Z_j = Z^{M_j} + Z^{S_j}$ with the matrix quantities, see (5.31), given by

\[
\begin{align*}
Z^{M_1} &= X_1 + Y_{(\mu \nu ;1)} \\
Z^{M_2} &= X_2 + Y_{(\mu \nu ;2)} + 2Y_{(1 \nu ;1)} + \frac{9}{12}K_{(2 \nu \nu ;1)} \\
Z^{M_3} &= X_3 + Y_{(\mu \nu ;3)} + 3\{Y_{(1 \nu ;2)}\} + \frac{2}{3}(21K_{(2 \nu \nu ;1)} + 16K_{(3 \nu \nu ;1)}) \\
Z^{M_4} &= X_4 + Y_{(\mu \nu ;4)} + 4\{Y_{(2 \nu ;2)}\} + 6Y_{(2 \nu ;2)} + 12Y_{(1 \nu ;1)} \\
&\quad+ \frac{5}{3}(8K_{(2 \nu \nu ;1)} + 9K_{(3 \nu \nu ;1)} + (5K_{(4 \nu ;1)} + 9K_{(2 \nu \nu ;1)}) \\
Z^{M_5} &= X_5 + Y_{(\mu \nu ;5)} + 5\{Y_{(1 \nu ;3)}\} + 10\{Y_{(1 \nu ;3)}\} + 24Y_{(1 \nu ;1)} + 9\{Y_{(1 \nu ;1)}\} \\
&\quad+ \frac{3}{3}(45K_{(2 \nu \nu ;3)} + 200K_{(3 \nu \nu ;2)} + 9(5K_{(4 \nu ;1)} + 31K_{(2 \nu \nu ;1)} + 1) \\
&\quad+ 4(6K_{(5 \nu ;1)} + 51K_{(3 \nu ;2)} + 60K_{(2 \nu ;3)} + 1)
\end{align*}
\]  

In (6.2) the parenthesis around the indices imply symmetrization only (but note that $Z^{M_4}$ and $Z^{M_5}$ appear in (6.1) with at least one respectively two contracted pair(s) of
indices). All terms in (6.2) with explicit indices vanish upon full contraction. From (5.32) we find

\[
Z^M_{(2)} = X_{(2)} + 2Y_{(1\ Y_1)} \\
Z^M_{(4)} = X_{(4)} + 4\{Y_{(1\ Y_3)}\} + 6Y_{(2\ Y_2)} + 12Y_{(1\ K_2\ Y_1)} \\
Z^M_{(6)} = X_{(6)} + 6\{Y_{(1\ Y_6)}\} + 15\{Y_{(2\ Y_4)}\} + 24Y_{(3\ Y_3)} + 40Y_{(1\ K_4\ Y_1)} + 66\{Y_{(1\ K_3\ Y_2)}\} + 50\{Y_{(1\ K_2\ Y_3)}\} + 72Y_{(2\ K_2\ Y_2)} + 153Y_{(1\ K_2\ K_2\ Y_1)} \\
Z^M_{(8)} = X_{(8)} + 8\{Y_{(1\ Y_7)}\} + 28\{Y_{(2\ Y_6)}\} + 56\{Y_{(3\ Y_5)}\} + 70Y_{(4\ Y_4)} + 84Y_{(1\ K_6\ Y_1)} + 225\{Y_{(1\ K_5\ Y_2)}\} + \frac{980}{3}\{Y_{(1\ K_4\ Y_3)}\} + 273\{Y_{(1\ K_3\ Y_4)}\} + 126\{Y_{(1\ K_2\ Y_5)}\} + 480Y_{(2\ K_4\ Y_2)} + 520\{Y_{(2\ K_3\ Y_3)}\} + 288\{Y_{(2\ K_2\ Y_4)}\} + \frac{1120}{3}Y_{(3\ K_2\ Y_3)} + \frac{6336}{5}Y_{(1\ K_3\ K_2\ Y_1)} + 870\{Y_{(1\ K_2\ K_4\ Y_1)}\} + 1440\{Y_{(1\ K_2\ K_3\ Y_2)}\} + \frac{6966}{5}\{Y_{(1\ K_3\ K_2\ Y_2)}\} + 1092\{Y_{(1\ K_2\ K_2\ Y_3)}\} + \frac{7776}{5}Y_{(2\ K_2\ K_2\ Y_2)} + 3348Y_{(1\ K_2\ K_2\ K_2\ Y_1)}
\]

The last term in \(Z^M_{(8)}\) was obtained from (5.50). From (5.23) we find that the scalar quantities are given explicitly by

\[
Z^S_1 = \frac{1}{2} \text{tr}[K_{1\mu\nu}] \\
Z^S_2 = \frac{1}{2} \text{tr}[K_{2\mu\nu} + \frac{6}{5}K_{2\mu\nu}] - \frac{1}{2} \text{tr}[K_{\mu(1)} \text{tr}[K_{1\mu\nu}] + 3K_{2\mu\nu} \text{tr}[K_{1\mu\nu}]] \\
Z^S_3 = \frac{1}{2} \text{tr}[K_{3\mu\nu} + 4K_{2\mu\nu}] - \frac{3}{2} \text{tr}[K_{\mu(1)} \text{tr}[K_{2\mu\nu}] + 4K_{3\mu\nu} \text{tr}[K_{1\mu\nu}] + 6K_{2\mu\nu} \text{tr}[K_{1\mu\nu}]] \\
Z^S_4 = \frac{1}{14} \text{tr}[7K_{4\mu\nu} + 40K_{2\mu\nu} + 30K_{3\mu\nu} + 48K_{2\mu\nu}] - \frac{3}{2} \text{tr}[K_{\mu(2)} \text{tr}[K_{2\mu\nu}] + \frac{5}{2} \text{tr}[K_{\mu(1)} \text{tr}[5K_{3\mu} + 6K_{2\mu}]] - 6 \text{tr}[K_{\mu(1)} \text{tr}[K_{2\mu\nu} \text{tr}[K_{1\mu\nu}]] + 10K_{3\mu\nu} \text{tr}[K_{1\mu\nu}]] + (5K_{4\mu\nu} + 36K_{2\mu\nu} \text{tr}[K_{\mu\nu}]) + 2K_{2\mu\nu} \text{tr}[5K_{2\mu\nu} + 6K_{2\mu\nu}]] \\
Z^S_5 = \frac{1}{2} \text{tr}[K_{5\mu\nu} + \frac{15}{2}K_{2\mu\nu} + 15K_{2\mu\nu} + 48K_{2\mu\nu} + 15K_{3\mu\nu}] \\
- \frac{5}{2} \text{tr}[K_{\mu(1)} \text{tr}[K_{4\mu\nu} + 4K_{2\mu\nu}]] - \text{tr}[K_{\mu(2)} \text{tr}[5K_{3\mu} + 6K_{2\mu}]] - 30 \text{tr}[K_{\mu(1)} \text{tr}[K_{2\mu\nu}]] - 10 \text{tr}[K_{\mu(1)} \text{tr}[K_{3\mu\nu} \text{tr}[K_{1\mu\nu}]] + 15K_{2\mu\nu} \text{tr}[K_{3\mu\nu} + 4K_{2\mu\nu}]] + 4K_{3\mu\nu} \text{tr}[5K_{2\mu\nu} + 6K_{2\mu\nu}]] + 3(5K_{4\mu\nu} + 36K_{2\mu\nu} \text{tr}[K_{\mu\nu}]) + 6(5K_{5\mu\nu} + 12K_{2\mu\nu} + 12K_{3\mu\nu}) \text{tr}[K_{\mu\nu}]]
\]

where the parenthesis mean symmetrization only. Upon full contraction we find from (5.23)

\[
Z^S_{(2)} = \frac{1}{2} \text{tr}[K_{(4)} + \frac{6}{5}K_{(2\ K_2)}] - \frac{1}{2} \text{tr}[K_{\mu(1)} \text{tr}[K_{1\mu\nu}]]
\]
where the seven terms of fifth order were found from (7.47) and also from expanding $B$ through tenth order in the normal coordinates (see appendix E).

To return to conventional notation we use

$$Z = X + \frac{1}{6} R$$

$$Z_\mu = X_\mu + \frac{1}{3} F_\mu^{\nu \rho} \varepsilon_{\rho \nu} + \frac{1}{6} R_{\mu \nu}$$

$$Z_{(2)} = X_{\mu \nu} + \frac{1}{2} F^{\mu \nu} F_{\mu \nu} + \frac{1}{5} R_{\mu \nu} + \frac{1}{12} R^2 - \frac{1}{30} R_{\mu \nu} R_{\mu \nu} + \frac{1}{30} R^{\kappa \lambda \mu \nu} R_{\kappa \lambda \mu \nu}$$
etc (these suffice for $a_1$ and $a_2$). In $a_4$ respectively $a_5$ we have e.g.

$$Z_{(1)}^{*} \hat{Z} Z_{1} = Z_{\mu}^{*}(Z g_{\mu\nu} + F_{\mu\nu} - \frac{1}{3} R_{\mu\nu}) Z_{\nu}$$

$$Z_{(2)}^{*} \hat{Z} Z_{2} = \frac{1}{3} Z_{(2)} Z Z_{(2)} + \frac{2}{3} Z_{\kappa \mu}^{*}(Z g_{\mu\nu} + 2 F_{\mu\nu} - \frac{2}{3} R_{\mu\nu}) Z_{\kappa \nu} + \frac{4}{9} Z_{\kappa \mu}^{*} R_{\kappa \lambda \mu \nu} Z_{\lambda \nu}$$

Using such translations we have verified that our results for the first four coefficients agree with earlier authors, in particular [5], [7] and [9], after accounting for differences in conventions and the occasional typographical mistake.

Finally, to illustrate the compactness of our notation, consider the special case of a scalar field in a Ricci flat space. Then only the Weyl tensor and its covariant derivatives can appear in the heat kernel coefficients. From appendix D of [15] we can read off the total number of general coordinate scalars in that case. Thus, for $j = 2, 3$ and $4$ we expect $1$, $3$ and $12$ terms in $a_j$, respectively. In our notation, see (5.13), Ricci-flatness implies that $\text{tr}[K_j]$ vanishes. Only a few terms remain then in (5.23, 5.25). In particular, $Z$ and $Z_1$ vanish and $a_2$ through $a_4$ contain only

$$a_2 : \text{tr}[K_{(2)K_2}]$$
$$a_3 : \text{tr}[K_{(2)K_3}] , \text{tr}[K_{(3)K_3}] , \text{tr}[K_{(2)K_2K_2}]$$
$$a_4 : \text{tr}[K_{(2)K_6}] , \text{tr}[K_{(3)K_3}] , \text{tr}[K_{(4)K_4}] , \text{tr}[K_{(2)K_2K_4}] , \text{tr}[K_{(2)K_3K_3}] , \text{tr}[K_{(2)K_2K_2}] , \text{tr}[K_{(4)K_4}], \text{tr}[K_{(2)K_2K_4}], \text{tr}[K_{(2)K_3K_3}], \text{tr}[K_{(2)K_2K_2}], (\text{tr}[K_{(2)K_2}])^2$$

Here there are 1, 3 and 9 terms respectively, so that, starting with $a_4$, our notation is not only index-free, but also generates less terms. This becomes even more pronounced for $a_5$ where [15] informs us that in the Ricci-flat case there are 67 terms, however in our notation there are only 17 terms.

### 7 Conclusions

We have presented the explicit diagonal values of the first five (six) heat kernel coefficients for a general Laplace-type operator on a Riemannian (respectively flat) space. To solve the pertinent recursion relations, we relied not only on well known techniques (matrix notation for field indices, Fock-Schwinger gauge and Riemann normal coordinates), but also used a new notation free of spacetime indices. It is this latter compact notation which allows us to write down the fifth and sixth coefficient in the first place. Insisting also on manifest hermiticity of the results, the fifth (sixth) coefficient has 26 (respectively 75) terms. They were presented here for the first time. Beyond these coefficients, the leading terms – to the same order in derivatives or curvatures as needed for the fourth coefficient – for any heat kernel coefficient were given in the general case. To determine the heat kernel coefficients, we have found it useful to proceed in a few steps, starting from the simplest case of a flat spacetime without a gauge field. We could show that ‘turning on a gauge field’ and next ‘curving
spacetime’ is taken care of by specific covariant substitutions in the flat space coefficients. With this ‘dressing up’ understood, the number of terms and their numerical prefactors do not change in the process.

In our notation, a typical term in a heat kernel coefficient consists of a maximally symmetrized (product of) covariant derivatives of the basic curvatures, made into a scalar by contracting all indices. Consider e.g. the following term

\[ \text{tr}[K_{i3}K_3K_{i4}] = \frac{3}{20} R^\alpha_{\beta \kappa \lambda} R^{\beta \gamma \mu \nu} R^{\gamma \alpha \nu \rho \mu} \]  

which appears in the fifth coefficient. Since we do not integrate over spacetime, partial integrations are not permitted. Furthermore, as long as we do not write out the symmetrization, the Bianchi identity cannot be used here. Thus our notation gives a certain degree of uniqueness to the appearance of the heat kernel coefficients, missing in a more conventional notation with explicit spacetime indices.

Of course, depending on the application one has in mind, our notation may or may not be useful. We plan to calculate the chiral anomaly based on our results, i.e. essentially evaluate the spinor-trace \( \text{tr}[\gamma_{2j+1}a_j] \) in \( d = 2j \) dimensions. In this case elegant and complete results are known \[25\], using the language of differential forms. It should be interesting to see how this can be related to our use of symmetric tensors. Also the gravitational anomalies \[26\] in \( d = 4j + 2 \) are computable in our framework. Possibly, the absence of certain terms from the heat kernel coefficients found here, see (3.19), plays a role in this connection.

Another task, now in progress \[30\], is to work out the functional trace of the diagonal heat kernel coefficients given here. In that case we can maintain our index-free notation (e.g. the integral of (7.1) is easily seen to vanish without writing out the indices). Although it would probably hold few surprises, cf \[18\], the result for the integrated and traced fifth coefficient could be used to study for the first time the one-loop short distance divergences of ten-dimensional supergravity.

As this paper neared completion, we were informed by the authors of \[11\] that they had extended their results so as to include gauge fields. In \[31\] they present the functional trace of the first six heat kernel coefficients for this case in flat spacetime, reduced to a so-called minimal basis. It should be possible to compare their result to the sixth heat kernel coefficient as presented here, after taking its trace and integrating it.

Finally, to avoid misunderstanding, we should mention that a recent preprint \[32\] with the title “The \( a_5 \) heat kernel coefficient on a manifold with boundary” is not concerned with the fifth heat kernel coefficient as presented here. Rather, on a manifold with boundary the expansion \[23\] is in powers of \( \sqrt{\tau} \) rather than \( \tau \) and therefore our \( a_j \) corresponds to \( a_{2j} \) of \[32\]. Thus, in the terminology of \[32\] we have determined here ‘the volume part of \( a_{10} \).

I would like to thank J.-P. Börnusen for his assistance in using Mathematica to verify some of the results presented here. I am also grateful to I. Avramidi for useful correspondence.
A Notation

We use Greek letters for spacetime indices. Indices $j, k, \ell, \ldots$ simply enumerate various objects. We take differential operators to act on everything to their right. If this is not intended, we use comma (semicolon) notation for partial (covariant) derivatives

$$F_{,\mu} \equiv \left[ \partial_{\mu}, F \right], \quad F;_{\mu} \equiv \left[ \nabla_{\mu}, F \right] \quad (A.1)$$

A semicolon denotes simultaneous gauge and gravitational covariant differentiation.

We use $a \equiv b$ to define $a$ in terms of $b$. Our curvature conventions are fixed by ($V_{\lambda}$ is a gauge singlet)

$$[\nabla_{\nu}, \nabla_{\mu}] V_{\lambda} = 2 V_{\lambda;[\mu\nu]}, \quad R_{\lambda\nu} \equiv R^{\kappa}_{\lambda\kappa\nu}, \quad R \equiv R^{\lambda}_{\lambda} \quad (A.2)$$

In Fock-Schwinger gauge and Riemannian coordinates, see (4.4) and (5.1), we have

$$F_{,\mu_1 \ldots \mu_j}(0) = F_{(\mu_1 \ldots \mu_j)}(0) \quad (A.3)$$

for any, possibly matrix valued, general coordinate scalar $F(x)$. This property allows one to immediately covariantize the partial derivatives. Due to the total symmetry here it is convenient and sufficient to keep track of only the number of indices, leading us to define the sans serif symbols

$$F_{j} = F_{(\mu_1 \ldots \mu_j)}(0) \quad (A.4)$$

We thus use an index-free notation for totally symmetrized tensors. At the same time, such a symbol implies evaluation at the origin. Following Avramidi, we use parenthesis around an even enumerative label to indicate not only total symmetrization, but also full contraction as in

$$F_{(2j)} = F_{(\mu_1 \mu_2 \ldots \mu_j \mu_{j+1} \ldots \mu_{2j})}(0) \quad (A.5)$$

Since the metric at the origin is flat there is no need to write the indices in their covariant respectively contravariant positions. *Par abus de language* an odd label enclosed in parenthesis denotes symmetrization and contraction of all but one of the involved indices. This will occur only if there is exactly one other factor with an odd label, as e.g. in

$$F_{(3)} G_{(3)} = F_{(\kappa\mu\rho)}(0) G_{(\kappa\nu\sigma)}(0) \quad (A.6)$$

We generalize Avramidi’s notation by allowing such a simultaneous symmetrization and contraction to extend over several factors, e.g.

$$F_{(kG_{2j-k})} = F_{(\mu_1 \ldots \mu_k)}(0) G_{(\mu_{k+1} \ldots \mu_{2j})}(0) \delta^{\mu_1 \mu_2} \ldots \delta^{\mu_{2j-1} \mu_{2j}} \quad (A.7)$$

If desired, such an expression can be written out in such a way that the total symmetrizations extend only over the individual factors, as e.g. in

$$F_{(2G_2)} = \frac{1}{3} F_{(\mu\nu)}(0) G_{(\mu\nu)}(0) + \frac{2}{3} F_{(\mu\nu)}(0) G_{(\mu\nu)}(0) \quad (A.8)$$

\[\text{Parenthesis around } j \text{ explicit indices denotes total symmetrization (with division by } j!).\]
In general, expanding (A.7) in this way yields at first \((2j - 1)!!\) terms. However, due to the total symmetry of the covariant derivatives acting on each factor and assuming that \(k \leq j\), there remain only \([k/2] + 1\) terms, \([k/2]\) being the maximal number of self-contractions possible for \(F_k\). Finding the coefficient of each term is a combinatorial problem with the following solution

\[
F_{(2j)G_{2k}} = \frac{1}{(2j + 2k - 1)!!} \sum_{\ell=0}^{j} P_{j-\ell}^{2j} P_{2k-\ell}^{2k} (2\ell)! F^{(2j-2\ell)}_{\mu_1...\mu_{2\ell}} G^{(2k-2\ell)}_{\mu_1...\mu_{2\ell}} \\
F_{(2j+1)G_{2k+1}} = \frac{1}{(2j + 2k + 1)!!} \sum_{\ell=0}^{j} P_{j-\ell}^{2j+1} P_{2k+1-\ell}^{2k+1} (2\ell + 1)! F^{(2j-2\ell)}_{\mu_1...\mu_{2\ell+1}} G^{(2k-2\ell)}_{\mu_1...\mu_{2\ell+1}} \\
P_k^j \equiv \binom{j}{2k}(2k-1)!! \quad , \quad k \leq \lfloor j/2 \rfloor \quad , \quad (-1)!! \equiv 1 \quad \text{(A.9)}
\]

Here \(P_k^j\) is the number of ways in which one can choose \(k\) pairs out of \(j\) objects. The covariant derivatives on \(F\) respectively \(G\) on the right hand side are understood to have been totally symmetrized. In general there will be more than two factors, but in practice these are easily taken care of, as e.g. in

\[
F_{(2G_2H_2)} = \frac{1}{15} F_{(2)G_{(2)}} H_{(2)} + \frac{2}{15} (F_{(2)G_{(\mu\nu)}} H_{(\mu\nu)} + 2\, \text{more}) + \frac{8}{15} F_{(\mu\nu)G_{(\nu\rho)}} H_{(\rho\mu)} \quad \text{(A.10)}
\]

Multinomial coefficients are defined as usual by

\[
\binom{j}{k_1, \ldots, k_N} = \frac{j!}{k_1! \ldots k_{N+1}!} \quad , \quad k_{N+1} \equiv j - \sum_{n=1}^{N} k_n \quad \text{(A.11)}
\]

If the index \(k_{N+1}\) appears in the summand of a sum involving this multinomial coefficient, then it is understood to have the indicated value. This convention applies also to alphabetic ordering as e.g. in

\[
\sum_{k=0}^{j} \binom{j}{k} F_k G_{\ell} \equiv \sum_{k=0}^{j} \binom{j}{k} F_k G_{j-k} \quad \text{(A.12)}
\]

## B Moving boxes

In the main text we gave an expression, eq (B.31), for the diagonal values of the heat kernel coefficients \(a_j\) through third order in the matrix potential \(X\) which holds only for \(j \leq 5\). We indicated in (B.30) how to obtain \(a_6\) as well without generating terms with overlapping derivatives. In the general case we rely on the following lemma to move the boxes from \(G\) to \(F\)

\[
F_{(2n)G_{2p}(2q-2p)} = \sum_{j=0}^{n} \beta_j(n, p, q) F_{(2n-2j)(2j)G_{2q}} \quad , \quad n \leq p \leq q \quad \text{(B.1)}
\]

\[
\beta_j(n, p, q) = \binom{q - p}{n - j} \frac{f(j, p, q)}{f(n, q, p)} \quad , \quad f(j, p, q) = (2j + 2q - 1)!! P_{p-j}^{2p} 
\]

\[\text{Multinomial coefficients are defined as usual by}\]

\[
\binom{j}{k_1, \ldots, k_N} = \frac{j!}{k_1! \ldots k_{N+1}!} \quad , \quad k_{N+1} \equiv j - \sum_{n=1}^{N} k_n \quad \text{(A.11)}
\]

If the index \(k_{N+1}\) appears in the summand of a sum involving this multinomial coefficient, then it is understood to have the indicated value. This convention applies also to alphabetic ordering as e.g. in

\[
\sum_{k=0}^{j} \binom{j}{k} F_k G_{\ell} \equiv \sum_{k=0}^{j} \binom{j}{k} F_k G_{j-k} \quad \text{(A.12)}
\]
with a similar expression for the case $F_{(2n+1)}G_{2p+1}(2q-2p-2)$. Note that this can only be done when all indices are contracted. To prove the lemma, expand both sides using (1.3) and compare. We also note the following exceptional case

$$F_{(k+2n)}G_{k}(2n) = F_{(2n)}(kG_{k+2n}) \quad (B.2)$$

### C a₆ in flat space

Expressions (3.20), (3.31), (4.24) and (4.28) suffice to write down the sixth heat kernel coefficient in a flat space, but with a gauge connection. We obtain

$$a₆ = Z^6 + \frac{5}{7}Z\{Z^3Z(2)\} + \frac{3}{7}Z\{Z^3Z(2)Z\} + \frac{9}{7}Z^2Z(2)Z^2 + \frac{2}{7}Z\{Z^3Z(1)iZ_1\} + \frac{4}{7}Z\{ZZ(1)iZ^2Z_1\}$$

$$+ \frac{6}{7}Z^2Z(1)iZ_1 + \frac{8}{7}Z\{ZZ(1)iZ_1\} + \frac{8}{7}Z\{Z^2Z(1)iZ_1\} + \frac{12}{7}Z^2Z(1)iZ_1 + \frac{5}{14}Z^2Z(4)$$

$$+ \frac{9}{14}Z\{Z^2Z(4)Z\} + \frac{12}{7}Z\{ZZ(1)iZ_3\} + \frac{15}{14}Z\{Z^2Z(1)iZ_3\} + \frac{9}{7}Z\{Z^2Z(3)Z\} + \frac{5}{7}Z\{ZZ(1)iZ_3\}$$

$$+ \frac{6}{7}Z\{ZZ(3)iZ_1\} + \frac{5}{14}Z\{Z(1)iZ^2Z_3\} + \frac{9}{7}Z\{Z^2Z(2)Z_2\} + \frac{3}{7}Z^2Z(2)Z_2 + \frac{3}{7}Z^2Z(2)^2 + \frac{3}{7}Z^2Z(2)$$

$$+ \frac{3}{7}Z(2)Z^2Z(2) + \frac{9}{14}Z\{ZZ(2)iZ_2\} + \frac{4}{7}Z\{ZZ(2)Z_2\} + \frac{27}{14}Z\{ZZ(1)iZ_2\} + \frac{4}{7}ZZ(2)Z_2$$

$$+ \frac{9}{14}Z\{Z(1)iZ_1Z_2\} + \frac{4}{7}Z\{Z(1)iZ_1Z_2\} + \frac{12}{7}Z\{ZZ(1)iZ_2\} + \frac{8}{7}Z\{ZZ(1)iZ_3\}(2)$$

$$+ \frac{27}{14}Z\{ZZ(1)iZ_1\} + \frac{4}{7}Z\{ZZ(2)Z_2\} \{Z(1)iZ_1\} + \frac{6}{7}Z\{Z(1)iZ_2\} + \frac{5}{7}Z\{Z(1)iZ_3\}$$

$$+ \frac{18}{7}Z\{ZZ(1)iZ_2\} + \frac{9}{7}Z\{Z(1)iZ_2\} + \frac{27}{14}Z(1)iZ_1Z_1 + \frac{4}{7}Z(1)iZ_1Z(1)$$

$$+ \frac{9}{42}Z^2Z(6) + \frac{4}{21}Z\{ZZ(1)iZ_3\} + \frac{10}{21}Z\{ZZ(1)iZ_3\} + \frac{5}{21}Z\{Z(1)iZ_3\}$$

$$+ \frac{10}{21}Z\{Z(1)iZ_3\} + \frac{25}{28}Z\{Z(1)iZ_4\} + \frac{20}{21}Z\{Z(2)iZ_2\} + \frac{20}{21}Z\{Z(3)iZ_3\} + \frac{5}{21}Z\{Z(3)iZ_3\}$$

$$+ \frac{25}{28}Z\{Z(1)iZ_4\} + \frac{10}{7}Z\{Z(1)iZ_3\} + \frac{10}{7}Z\{Z(1)iZ_3\} + \frac{10}{7}Z\{Z(1)iZ_3\} + \frac{10}{7}Z\{Z(1)iZ_3\}$$

$$+ \frac{40}{21}Z\{Z(2)iZ_2\} + \frac{3}{14}Z\{ZZ(1)iZ_3\} + \frac{9}{28}Z\{Z(2)iZ_4\} + \frac{5}{28}Z\{ZZ(2)iZ_4\} + \frac{3}{14}Z\{Z(3)iZ(3)\}$$

$$+ \frac{3}{7}Z\{Z(3)iZ_3\} + \frac{5}{28}Z\{Z(1)iZ_3\} + \frac{9}{14}Z\{Z(1)iZ_3\} + \frac{9}{14}Z\{Z(1)iZ_3\} + \frac{9}{14}Z\{Z(1)iZ_3\}$$

$$+ \frac{3}{7}Z(3)iZ_3 + \frac{2}{7}Z(3)iZ_3 + \frac{2}{7}Z(3)iZ_3 + \frac{2}{7}Z(3)iZ_3 + \frac{2}{7}Z(3)iZ_3 + \frac{2}{7}Z(3)iZ_3$$

$$+ \frac{3}{7}Z(3)iZ_3 + \frac{3}{7}Z(3)iZ_3 + \frac{3}{7}Z(3)iZ_3 + \frac{3}{7}Z(3)iZ_3 + \frac{3}{7}Z(3)iZ_3 + \frac{3}{7}Z(3)iZ_3$$

$$+ \frac{1}{3}Z(3)iZ_3 + \frac{1}{3}Z(3)iZ_3 + \frac{5}{6}Z(2)iZ_3 + \frac{5}{42}Z(3)iZ_3 + \frac{3}{14}Z(3)iZ_3 + \frac{1}{462}Z(10)$$

The Z’s appearing here were defined in (1.13), (1.16) and (1.23). Note in particular that $Z = X$. There is a total of 75 terms. In curved space, $a₆$ will look exactly the same, but in that case we do not know the values for some of the Z’s.
D Consequences of Fock-Schwinger gauge

To prove (4.20), we start from

\[(A_{(\mu_1 F),\mu_2...\mu_j})(0) = 0\]  \hfill (D.1)

which holds for any function \(F\) in Fock-Schwinger gauge, see (4.5). Now take \(j = 2n + 2\), contract all indices and write out with respect to the index of the gauge connection. It is essential for the proof that, due to the full contraction, all indices here are dummies.

A further consequence of (4.5) is the following. If we define \(\check{Z}\) to act to its left as follows

\[Z^j(\jmath^{\dagger} \check{Z}) \equiv Z^j(\jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z}) = Z^j(\jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z}) - \frac{2j}{n+1} Z^{(j-1)^\dagger} Y_{n+1}^\nu\]  \hfill (D.2)

where we left out inessential factors to the right, then we have the following alternative notations

\[Z^j(\jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z}) \quad Z^j(\jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z}) = Z^j(\jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z}) \]  \(\jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z}) = Z^j(\jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z})\]  \hfill (D.3)

In the last case the ‘check’ notation has the advantage that it generates less terms than the ‘hat’ notation, namely

\[Z^j(\jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z}) = Z^j(\jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z} \jmath^{\dagger} \check{Z}) - \frac{4j^k}{(p+1)(n+1)} Z^{(j-1)^\dagger} Y_{p+1}^\mu Y_{n+1}^\nu Z_{k-1}^\nu\]  \hfill (D.4)

E How to get your Z’s

Here we shall give some details on how to obtain (5.23). The steps for (5.31) are very similar so we omit them. Our starting point is (5.8). We take \(j\) partial derivatives of this expression and evaluate at the origin \(m \equiv j - k - \ell\) to obtain

\[Z^S = \frac{1}{2} \sum_{k=0}^{j} \binom{j+1}{k} \epsilon^{\mu\nu}(k B_{j-k})_{\mu
u} - \frac{1}{4} \sum_{k=1}^{j-1} \sum_{\ell=1}^{j-k} \binom{j}{k,\ell} B_{(k B_{j-k})_{\mu
u}} \]  \hfill (E.1)

where we used that \(B\) and its first derivative vanish there. Separate off the terms with undifferentiated inverse metric

\[Z^S = \frac{1}{2} B_{\mu j} + \frac{1}{2} \sum_{k=2}^{j} \binom{j+1}{k} \epsilon^{\mu\nu}(k B_{j-k})_{\mu
u} \]  \hfill (E.2)

- \frac{1}{4} \sum_{k=1}^{j-1} \binom{j}{k} B_{(k B_{j-k})_{\mu
u}} - \frac{1}{4} \sum_{k=1}^{j-3} \sum_{\ell=1}^{j-k-2} \binom{j}{k,\ell} B_{(k B_{j-k})_{\mu
u}} \]  \hfill (E.2)
Inserting (5.17) and (5.19) and collecting terms of the same order in the curvatures \( K \) and \( Y \), we obtain (5.23). Up to here all contractions were explicitly indicated and the parenthesis denoted symmetrization only. If we now replace \( j \) by \( 2j \) and fully contract, the second term in (E.2) vanishes due to (5.2) and we find
\[
Z^S_{(2j)} = \frac{1}{2} B_{(2j+2)} - \frac{1}{2} \sum_{k=1}^{2j-1} \binom{2j}{k} B_{\mu(k B_{\ell})} = \frac{1}{4} \sum_{k=1}^{2j-3} \sum_{\ell=1}^{2j-k-2} \binom{2j}{k, \ell} \mu(k g^{\mu\nu} B_{\ell}) (E.3)
\]
This yields (5.25). Thus we see that, since \( B \) is at least of first order in curvature, finding the heat kernel coefficients through order \( n \) in the curvature requires the inverse metric only through order \( n - 2 \).

Writing \( G \) for the inverse metric, we find through sixth order in the normal coordinates
\[
G(x) = 1 + 2K_2 \frac{x^2}{2!} + 2K_3 \frac{x^3}{3!} + 2(K_4 + \frac{36}{5}K_2^2) \frac{x^4}{4!} + 2(K_5 + 12K_2 K_3) \frac{x^5}{5!} + O(x^7)
\]
For the logarithm of the Van Vleck-Morette determinant we find through tenth order in the normal coordinates
\[
B(x) = \text{tr} [K_2 \frac{x^2}{2!} + K_3 \frac{x^3}{3!} + (K_4 + \frac{6}{5}K_2^2) \frac{x^4}{4!} + (K_5 + 4K_2 K_3) \frac{x^5}{5!} + (K_6 + \frac{40}{7}K_2 K_4 + \frac{28}{3}K_2 K_5 + \frac{70}{9}K_3 K_5 + \frac{140}{3}K_3 K_4 + \frac{272}{3}K_2^2 K_4 + \frac{400}{3}K_2 K_3^2 + \frac{432}{5}K_4) \frac{x^6}{6!} + (K_7 + \frac{15}{2}K_2 K_5 + 15K_3 K_4 + 48K_2^2 K_3) \frac{x^7}{7!} + (K_8 + \frac{28}{3}K_2 K_6 + \frac{70}{9}K_3 K_6 + \frac{140}{9}K_4 K_4 + \frac{272}{3}K_2^2 K_4 + \frac{400}{3}K_2 K_3^2 + \frac{432}{5}K_4) \frac{x^8}{8!} + (K_9 + \frac{56}{5}K_2 K_7 + \frac{168}{5}K_3 K_7 + 56K_4 K_5 + \frac{756}{5}K_2^2 K_5 + 584K_2 K_3 K_4 + 144K_3^3 + \frac{5184}{5}K_2^3 K_3) \frac{x^9}{9!} + (K_{10} + \frac{144}{11}K_2 K_8 + \frac{504}{11}K_3 K_7 + \frac{1008}{11}K_4 K_6 + \frac{630}{11}K_5 K_5 + \frac{12768}{55}K_2^2 K_6 + \frac{12180}{11}K_2 K_3 K_5 + \frac{8000}{11}K_2^2 K_4 + 1080K_3^2 K_4 + \frac{26880}{11}K_2^3 K_4 + 3576K_2^2 K_3^2 + \frac{19320}{11}K_2 K_3 K_2 K_3 + \frac{20736}{11}K_2^5) \frac{x^{10}}{10!}] + O(x^{11})
\]
For the gauge connection (with covariant index) it suffices to know that
\[
A(x) = Y_1 x^1 + Y_2 \frac{x^2}{2!} + (Y_3 + \frac{3}{2}K_2 Y_1) \frac{x^3}{3!} + (Y_4 - \frac{18}{5}K_2 Y_2 - \frac{8}{5}K_3 Y_1) \frac{x^4}{4!} + (Y_5 - \frac{20}{3}K_2 Y_3 - 5K_3 Y_2 - \frac{1}{3}(5K_4 - 9K_2^2) Y_1) \frac{x^5}{5!} + (Y_6 - \frac{75}{7}K_2 Y_4 - \frac{80}{7}K_3 Y_3 - \frac{9}{7}(5K_4 - 9K_2^2) Y_2 - \frac{12}{7}(K_5 - 2K_3 K_2 - 4K_2 K_3) Y_1) \frac{x^6}{6!} + O(x^7)
\]
To convert the above expansions to conventional notation, use (4.12) and (5.11).
F Symmetry factors

Below we give a table containing all symmetry factors $S/P_N$, defined in (5.22), for $N \leq 4$. The list $k_1 \ldots k_N$ in the second column is an abbreviation for $\text{tr}[K_{k_1} \ldots K_{k_N}]$. It is to be understood that $k, \ell, m$ and $n$ are different integers, each having at least the value 2.

| $N$ | $k_1 \ldots k_N$ | $S$ | $S/P_N$ |
|-----|------------------|-----|---------|
| 2   | $k k$            | 1   | 1/2     |
|     | $k \ell$         | 2   | 1       |
| 3   | $k k k$          | 1   | 1/6     |
|     | $k k \ell$       | 3   | 1/2     |
|     | $k \ell m$       | 6   | 1       |
| 4   | $k k k k$        | 1   | 1/8     |
|     | $k \ell k \ell$  | 2   | 1/4     |
|     | $k k k \ell$     | 4   | 1/2     |
|     | $k k \ell \ell$  | 4   | 1/2     |
|     | $k \ell k m$     | 4   | 1/2     |
|     | $k k \ell m$     | 8   | 1       |
|     | $k \ell m n$     | 8   | 1       |
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