Decentralized Optimal Control in Multi-lane Merging for Connected and Automated Vehicles

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Abstract—We address the problem of optimally controlling Connected and Automated Vehicles (CAVs) arriving from two multi-lane roads and merging at multiple points where the objective is to jointly minimize the travel time and energy consumption of each CAV subject to speed-dependent safety constraints, as well as speed and acceleration constraints. This problem was solved in prior work for two single-lane roads. A direct extension to multi-lane roads is limited by the computational complexity required to obtain an explicit optimal control solution. Instead, we propose a general framework that converts a multi-lane merging problem into a decentralized optimal control problem for each CAV in a less-conservative way. To accomplish this, we employ a joint optimal control and barrier function method to efficiently get an optimal control for each CAV with guaranteed satisfaction of all constraints. Simulation examples are included to compare the performance of the proposed framework to a baseline provided by human-driven vehicles with results showing significant improvements in both time and energy metrics.

I. INTRODUCTION

Traffic management at merging points (usually, highway on-ramps) is one of the most challenging problems within a transportation system in terms of safety, congestion, and energy consumption, in addition to being a source of stress for many drivers [1], [2], [3]. Advances in next-generation transportation technologies and the emergence of Connected and Automated Vehicles (CAVs) have the potential to drastically improve a transportation network’s performance by better assisting drivers in making decisions, ultimately reducing energy consumption, air pollution, congestion and accidents.

Most research work just focuses on the single lane merging problem [4], [5], [6], with limited work done in the multi-lane merging problem. In our recent work [7], we addressed the merging problem through a decentralized optimal control (OC) formulation and derived explicit analytical solutions for each CAV when no constraints are active. We have extended the solution to include constraints, in which case the computational cost depends on the number of constraints becoming active; we have found this to become potentially prohibitive for a CAV to determine through on-board resources. In addition, our analysis has thus far assumed no noise in the vehicle dynamics and sensing measurements, and the dynamics have precluded nonlinearities.

To address the limitations above, one can adopt Model Predictive Control (MPC) (e.g., [8], [4], [9]) or the Control Barrier Function (CBF) method [10], [11]. MPC is very effective for problems with simple (usually linear or linearized) dynamics. Unlike MPC, the CBF method does not use states as decision variables in its optimization process; instead, any continuously differentiable state constraint is mapped onto a new constraint on the control input. A control input that satisfies this new constraint is guaranteed to also satisfy the original constraint. We have adopted this approach to the single-lane merging problem in recent work [12] and shown that it provides good approximations of the analytically obtained OC solutions. To account for both optimality and computational complexity, we developed a joint optimal control and barrier function (OCBF) controller in [13] for a two-lane merging problem. The implementation of this approach is hard for multi-lane merging, especially in determining the safety constraints that a CAV has to satisfy. The common approach to avoid such complex safety constraint determination is to treat an entire conflict area as a point, which is conservative (e.g., for an intersection, see [14]). Alternatively, the conflict area can be partitioned according to lane intersections and a tree search approach may be used to find a feasible path for each CAV [15]; this approach is limited by the computational complexity due to the high-dimensional search space involved.

The contribution of this paper is to show how we can transform a multi-lane merging problem into a multi-point merging problem in a simpler and less-conservative way. Specifically, we first determine the merging points that a CAV must pass through and construct queueing tables maintained by a coordinator associated with the merging area. Using a simple search through these tables, we determine the safe merging and rear-end safety constraints that a CAV has to satisfy, hence transforming the multi-lane merging problem into a decentralized optimal control problem for each CAV. Finally, we use the aforementioned OCBF method to solve these optimal control problems. The main advantages of the proposed framework lie in the optimality it provides, its computational efficiency, safety guarantees, and good generalization properties for more complex traffic scenarios. Simulation results of the proposed framework have shown significantly better performance compared to human drivers.

II. PROBLEM FORMULATION

The multi-lane merging problem arises when traffic must be joined from two different roads, usually associated with a main and a merging road as shown in Fig.1. Each road
has two lanes (as we will see, the same modeling method can be applied to more than two lanes). We label the lanes $l_1, l_2$ and $l_3, l_4$ for the main and merging roads, respectively, with corresponding origins $O_1, O_2, O_3, O_4$. The CAVs in lane $l_2$ can change lanes to $l_1$. In addition, the CAVs in lane $l_3$ have the option to merge into either lane $l_1$ or $l_2$ (the main benefit being that the CAV in $l_3$ can surpass a group of CAVs in $l_4$ when $l_4$ is congested). Finally, the CAVs in lane $l_4$ can only merge to $l_2$.

In our original single-lane merging problem [7] only lanes $l_2, l_4$ are involved and the only merging point is $M_3$ in Fig.1. Here, CAVs from lanes $l_1, l_2, l_3, l_4$ may merge at the three fixed merging points $M_2, M_3, M_4$. In addition, a CAV from lane $l_2$ may merge into $l_1$ at an arbitrary merging point $M_{11}$, as long as this point is located prior to $M_2$. We consider the case where all traffic consists of CAVs randomly arriving at the four lanes joined at the Merging Points (MPs) $M_{11}, M_2, M_3, M_4$ where a collision may occur. The road segment from $O_3$ or $O_4$ to the merging point $M_3$ has a length $L_3$ and is called the Control Zone (CZ). The segment from $O_1$ to $M_{11}$ for CAV $i$ has a length $L_{i1}$ which is variable and depends on $i$). The segment from $O_2$ or $O_3$ to $M_2$ has a length $L_2$.

We assume that CAVs do not overtake each other in the CZ (unless so dictated by the CAV’s controller to be developed in the sequel), that $L_{i1} < L_2$, and that the merging point $M_4$ is within the CZ. Moreover, note that if the controller determines that a CAV needs to change lanes from $l_2$ to $l_1$, then it has to travel an additional distance; we assume that this extra distance is a constant $l > 0$. The same constant applies to CAVs in lane $l_4$ which choose to merge into $l_1$ at $M_4$ (as opposed to merging into $l_2$).

A coordinator (typically a Road Side Unit (RSU)) is associated with the MP $M_3$ whose function is to maintain First-In-First-Out (FIFO) queues of all CAVs regardless of lanes based on their arrival time at the CZ and to ensure real-time communication with the CAVs that are in the CZ as well as the last one leaving the CZ (in particular, the coordinator does not make control decisions; this is done in decentralized fashion on-board each CAV). The FIFO assumption (so that CAVs cross the MP in their order of arrival) is made for simplicity and often to ensure fairness; however, it can be relaxed through dynamic resequencing schemes as described, for example, in [14], [16]. Since we have two lanes in the main road, we need two queues to manage each CAV sequence leaving the CZ via $l_1$ and $l_2$ respectively, as shown in Fig. 1. Note that the number of queues equals the number of lanes in the main road, thus this framework can be easily extended to other multi-lane road traffic configurations, such as intersections.

Let $s_1(t), s_2(t)$ be the sets of the FIFO-ordered CAV indices associated with the two possible CZ exit lanes $l_1$ and $l_2$. To maintain a single unique index for each CAV, let $n > 0$ be a large enough integer representing the road capacity over $L_3$ in terms of the number of CAVs that can be accommodated. Then, let the set of possible CAV indices in $s_2(t)$ be $\{0, 1, \ldots, n - 1\}$ and that in $s_1(t)$ be $\{n, n + 1, \ldots, 2n - 1\}$. Thus, CAV $n + j$ ($j \in \mathbb{N}$) belongs to $s_1(t)$. The CAVs indexed by $n$ or $0$ are the ones that have just left the CZ from $l_1, l_2$ respectively. Let $N_1(t), N_2(t)$ be the cardinalities of $s_1(t), s_2(t)$, respectively. Observe that the CAVs in any one queue may have a physical conflict (i.e., collisions may happen) with the CAVs in the other queue only in lanes $l_2, l_3$, but not in lanes $l_1, l_4$. Thus, we assign a newly arriving CAV according to the following cases:

(i) If a CAV arrives at time $t$ at lane $l_1$, it is assigned to $s_1(t)$ with an index $n + N_1(t)$.

(ii) If a CAV arrives at time $t$ at lane $l_2$, a decision is made (as described later) on whether it exits the CZ through $l_2$ or switches to $l_1$ at $L_{i1}$. This CAV is assigned to both $s_1(t)$ and $s_2(t)$ with the index $N_2(t)$ if it chooses to stay in $l_2$ (e.g., CAV 2 in Fig. 1) or the index $n + N_1(t)$ if it switches to $l_1$ (e.g., CAV $n + 3$ in Fig. 1).

(iii) If a CAV arrives at time $t$ at lane $l_3$, it is assigned to both $s_1(t)$ and $s_2(t)$ with the index $n + N_1(t)$ if the control decision is to merge to lane $l_1$ or the index $N_2(t)$ if it merges to lane $l_2$.

(iv) If a CAV arrives at time $t$ at lane $l_4$, it is assigned to $s_2(t)$ with the index $N_2(t)$.

Note that in the above case (ii), the index of the CAV arriving at $l_2$ is dropped from $s_2(t)$ (or $s_1(t)$) after it changes its lane to $l_1$ at $L_{i1}$ (or passes $M_2$). In the above case (iii), the index of the CAV arriving at lane $l_3$ is dropped from $s_1(t)$ (or $s_2(t)$) after it passes $M_2$ if it chooses to merge into $l_2$ (or $l_1$). In summary, the index of any CAV arriving at $O_2$ or $O_3$ will be dropped from queue $s_1(t)$ or $s_2(t)$ after it passes its first MP. All CAV indices in $s_2(t)$ decrease by one when a CAV passes MP $M_3$ and the CAV whose index becomes $-1$ is dropped (similarly for $s_1(t)$, the CAV leaving the CZ through $M_4$ whose index becomes $n - 1$ is dropped). Observe that this scheme allows any CAV $i \in s_1(t)$ to look up only queue table $s_1(t)$ (similarly for $s_2(t)$ if $i \in s_2(t)$) in order to identify all possible collisions with other CAVs, without any need to consider the other queue.

The vehicle dynamics for each CAV $i \in s_1(t) \cup s_2(t)$ along the lane to which it belongs takes the form

$$\begin{align*}
\dot{x}_i(t) &= v_i(t) + w_{i,1}(t) \\
\dot{v}_i(t) &= v_i(t) + w_{i,2}(t),
\end{align*}$$

(1)

where $x_i(t)$ denotes the distance to the origin $O_1$ or $O_2, O_3, O_4$ along the lane that $i$ is located in when it enters the CZ, $v_i(t)$ denotes the velocity, and $u_i(t)$ denotes the control input (acceleration). Moreover, $w_{i,1}(t), w_{i,2}(t)$ denote two random processes defined in an appropriate probability space to capture possible noise. We consider two objectives for each CAV subject to three constraints, as detailed next.

**Objective 1** (Minimize travel time): Let $t^0$ and $t^0_i$ denote the time that CAV $i \in s_1(t) \cup s_2(t)$ arrives at the origin $O_1$ or $O_2, O_3, O_4$ and the time that CAV $i$ leaves the CZ (through either $M_3$ or $M_4$), respectively. We wish to minimize the travel time $t^0 = t^0_i - t^0$ for CAV $i$.

**Objective 2** (Minimize energy consumption): We also wish to minimize the energy consumption for each CAV
\( i \in S_1(t) \cup S_2(t) \) expressed as

\[
\min_{u(t)} \int_{t_0}^{t_m} C(u_i(t))dt, \tag{2}
\]

where \( C(\cdot) \) is a strictly increasing function of its argument.

**Constraint 1** (Safety constraint): Let \( i_p \) denote the index of the CAV which physically immediately precedes \( i \in S_1(t) \cup S_2(t) \) in the CZ (if one is present). We require that the distance \( z_{i,i_p}(t) \equiv x_{i_p}(t) - x_i(t) \) be constrained by:

\[
z_{i,i_p}(t) \geq \varphi v_i(t) + \delta, \quad \forall t \in [t_i^0, t_i^{m}], \tag{3}
\]

where \( \varphi \) denotes the reaction time (as a rule, \( \varphi = 1.8 \) is used, e.g., [17]). If we define \( z_{i,i_p} \) to be the distance from the center of CAV \( i \) to the center of CAV \( i_p \), then \( \delta \) is a constant determined by the length of these two CAVs (generally dependent on \( i \) and \( i_p \), but taken to be a constant over all CAVs for simplicity).

**Constraint 2** (Safe merging): Let \( t_i^{m_p}, p \in \{1, 2, 3, 4\} \) denote the arrival time of CAV \( i \in S_1(t) \cup S_2(t) \) (note that CAV \( i \) will only pass at most two of these MPs at the merging points \( M_{1,2}, M_{2,3}, M_{3,4}, M_4 \), respectively. There should be enough safe space at these MPs for a merging CAV \( i \) to cut in, i.e.,

\[
z_{i,j}(t_i^{m_p}) \geq \varphi v_i(t_i^{m_p}) + \delta, \quad p \in \{1, 2, 3, 4\}, \tag{4}
\]

where \( j \in S_1(t) \cup S_2(t) \) is the CAV that may collide with \( i \) (\( j \) may not exist) at the merging points \( M_{1,2}, M_{2,3}, M_{3,4}, M_4 \). Observe that since a CAV crosses at most two of the four MPs, CAV \( i \) only needs to satisfy the safe merging constraints above corresponding to the MPs that it will actually cross (e.g., CAV 1 in Fig. 1 only needs to satisfy the third constraint in (4)). The index \( j \) corresponding to each \( i \) is generally hard to determine; we resolve this issue in the next section through a conflict-point-based method.

**Constraint 3** (Vehicle limitations): Finally, there are constraints on the speed and control for each \( i \in S_1(t) \cup S_2(t) \):

\[
v_{\min} \leq v_i(t) \leq v_{\max}, \quad \forall t \in [t_i^0, t_i^{m}],
\]

\[
u_{i,\min} \leq u_i(t) \leq u_{i,\max}, \quad \forall t \in [t_i^0, t_i^{m}], \tag{5}
\]

where \( v_{\max} > 0 \) and \( v_{\min} \geq 0 \) denote the maximum and minimum speed allowed in the CZ, while \( u_{i,\min} < 0 \) and \( u_{i,\max} > 0 \) denote the minimum and maximum control for each CAV \( i \), respectively.

A common way to minimize energy consumption is by minimizing the control input effort \( u^2_i(t) \). By normalizing travel time and \( u^2_i(t) \), and using \( \alpha \in [0, 1] \), we construct a convex combination as follows:

\[
J_i(u_i(t)) = \beta(t_i^{m} - t_i^{0}) + \int_{t_i^{0}}^{t_i^{m}} \frac{u_i^2(t)}{2} dt, \tag{6}
\]

where \( \beta = \frac{\alpha \max u_{\max}^2 u_{\min}^2}{2(1-\alpha)} \) is a weight factor that can be adjusted through \( \alpha \in [0, 1] \) to penalize travel time relative to the energy cost. Then, we have the following problem formulation:

**Problem 1:** For each CAV \( i \in S_1(t) \cup S_2(t) \) governed by dynamics (1), determine a control law such that (6) is minimized subject to (1), (3), (4), (5), given \( t_i^{0} \) and the initial and final conditions \( x_i(t_i^{0}) = 0, v_i(t_i^{0}), x_i(t_i^{m}) \).

**III. MULTI-LANE MERGING PROBLEM SOLUTION**

We now show how to decompose Problem 1 into a multi-point merging problem for each CAV and use the CBF method to account for constraints while tracking a CAV trajectory obtained through OC. We also take advantage of the robustness to noise that the CBF approach offers.

However, determining the exact merging constraints in (4) that a CAV \( i \in S_1(t) \cup S_2(t) \) has to satisfy is challenging since there are four lanes and the traffic is asymmetric. This is even harder for more lanes and other scenarios, such as intersections. Using the approach introduced in [7] and
considering the multi-lane merging problem in Fig. 1, there are 15 cases, making this hard to implement. Moreover, this approach does not scale well for more complicated cases. Therefore, we propose a conflict-point based approach to simplify this process, as described next.

A. Lane Merging Determination Strategy

When a new CAV $i \in S_1(t) \cup S_2(t)$ arrives at $O_2$ or $O_3$, it has the option of exiting the CZ through lane $l_1$ or $l_2$. In addition, if it arrives at $O_2$ and decides to merge to $l_1$, it must also determine the location of the variable MP $M_{i,1}$.

Let us begin with the first issue. Determining the lane from which a CAV should exit the CZ may be addressed using the optimal dynamic resequencing method from [16]. This approach becomes computationally intensive; for example in the single-lane merging problem we have found this to require 3 to 30 sec in MATLAB [16], and this will generally increase in the multi-lane merging problem at hand. Although this remains an option, in this paper we focus on computational efficiency by adopting the following lane-merging decision strategy: we seek to balance the expected number of CAVs in the two lanes in order to improve the cost on average. In a queueing-theoretic context, this implies adopting a shortest-queue-first policy which is known to be (6) on average. In a queueing-theoretic context, this implies determining the location of the variable MP $M_{i,1}$.

Next, we address the issue of selecting the location of the MP $M_{i,1}$ for a CAV $i$ arriving at $O_2$, if its decision is $i \in S_1(t)$ above. There are three important observations to make: (i) The unconstrained optimal control for such $i$ is independent of the location of $M_{i,1}$ since we have assumed that lane-changing will only induce a fixed extra length $l$. (ii) The OC solution under the first safe-merging constraint in (4) is better (i.e., lower cost in (6)) than one which includes an active rear-end safety constrained arc in its optimal trajectory. This is because the former applies only to a single time instant $t_i^0$, whereas the latter requires the constraint (3) to be satisfied over all $t \in [t_i^0, t_i^m]$. It follows that the merging point $M_{i,1}$ should be as close as possible to $M_2$ (i.e., $L_{i,1}$ should be as large as possible), since the merging constraint between $i$ and $i - 1$ would become a rear-end safety constraint after $M_{i,1}$. (iii) In addition, CAV $i$ arriving at $O_2$ may also be constrained by its physically preceding CAV $i_p$ (if one exists) in lane $l_2$. In this case, CAV $i$ needs to consider both the rear-end safety constraint with $i_p$ and the safe-merging constraint with $i - 1$. Thus, the solution is more constrained (hence, more sub-optimal) if $i$ stays in lane $l_2$ after the rear-end safety constraint due to $i_p$ becomes active. We conclude that in this case CAV $i$ should merge to lane $l_1$ when the rear-end safety constraint with $i_p$ in lane $l_2$ first becomes active, i.e., $L_{i,1}$ is determined by

$$L_{i,1} = x_i^*(t_i^0) \quad (8)$$

where $x_i^*(t)$ denotes the unconstrained optimal trajectory of CAV $i$ (as determined in Sec. III-C), and $t_i^m \geq t_i^0$ is the time instant when the rear-end safety constraint first becomes active between $i$ and $i_p$ in lane $l_2$; if this constraint never becomes active, then $t_i^m = t_i^m$. The value of $t_i^m$ is determined from (3) by

$$x_i^*(t_i^0) - x_i^*(t_i^m) = \varphi_i^*(t_i^m_i) + \delta, \quad (9)$$

where $x_i^*(t)$, $v_i^*(t)$ are the unconstrained optimal trajectory and optimal speed respectively of CAV $i_p$. If, however, CAV $i_p$’s optimal trajectory includes a constrained arc, then (9) is only an approximation (in fact, an upper bound) of $t_i^m$. In summary, if CAV $i$ never encounters a point on $l_2$ where its rear-end safety constraint becomes active, we set $L_{i,1} = L_2$, otherwise $L_{i,1}$ is determined through (8)-(9).

B. Merging Constraint Determination Strategy

The CAVs arriving at lanes $l_2, l_3$ will pass two MPs. On the other hand, CAVs arriving at lane $l_1$ will pass either one or two MPs (depending on whether $i$ and $i - 1$ are in the same lane or not), whereas all CAVs arriving at $l_1$ will pass only MP $M_3$. Moreover, CAVs arriving at lanes $l_2, l_3$ may pass through different MPs, depending on which lane they choose to merge into following the strategy presented in the last subsection. Since all MPs that a CAV has to pass are now determined, we augment the FIFO queues in Fig. 1 with the original lane and the MP information for each CAV as shown in Fig. 2. The current and original lanes are shown in the third and fourth column, respectively. The last two columns indicate the first and second MPs for each CAV (note that all CAVs arriving at lane $l_4$ and some CAVs arriving at lane $l_1$ have only one MP, in which case the first MP is left blank).

![Fig. 2. The extended coordinator queue tables.](image-url)

When a new CAV $i$ arrives at $O_1$ (or $O_2, O_3, O_4$) and has determined whether it will merge into another lane or not (based on the last subsection), it looks up the extended queue tables in Fig. 2 which already contain all prior CAV state and MP information. If $i \in S_1(t)$, it looks up the extended FIFO queue $S_1(t)$, otherwise, it looks up $S_2(t)$. From the current lane column in Fig. 2, CAV $i$ can determine its current physically immediately preceding CAV $i_p$ if one exists. Moreover, CAV $i$ can determine the safe-merging constraints that it should satisfy (i.e., with respect to which CAV $j$ in (4) in the queue) upon its arrival at any origin.

The precise process through which each arriving CAV $i$ looks up each queue $S_1(t)$ and $S_2(t)$ in Fig. 2 is a follows.
CAV $i$ compares its original lane and MP information to that of every CAV in each queue starting with the last row and moving up. Depending on which column (among the last three columns) matches first, there are four possible cases (a much smaller number than 15 if the approach in [7], [12], [13] were followed). This process terminates the first time that any one of these four cases is satisfied at some row. If that does not happen, this implies that CAV $i$ does not have to satisfy any safe-merging constraint. Let $\text{type}(i) \in \{1,2\}$ be such that $\text{type}(i) = 1$ if $i \geq n$ and $\text{type}(i) = 2$ otherwise.

Then, the four cases are:

1. All last three columns match first.
2. $[1^{st} \text{ MP column matches with } j \in S_1(t) \text{ (or } S_2(t)\text{) first}] \& [\text{type}(i) = \text{type}(j)]$.
3. $[1^{st} \text{ MP column matches with } j \in S_1(t) \text{ (or } S_2(t)\text{) first}] \& [\text{type}(i) \neq \text{type}(j)]$.
4. The $2^{nd}$ MP column matches first.

If none of the four cases above is satisfied, then CAV $i$ does not have to satisfy any safe-merging constraint. In summary, a newly arriving CAV may have to satisfy at most three safety (or safe-merging) constraints in Fig. 1. If the corresponding $k$ or $i_p$ is not found in the above cases, then the related safe-merging or safety constraint is skipped.

**Updating $S_1(t)$ and $S_2(t)$**. Observe that while the MP information in the last two columns of each queue in Fig. 2 remains unchanged, the same is not true for the current lane information. More precisely, the two queues need to be updated whenever one of the following four events takes place: (i) A new CAV arrives at the CZ and is added to one or both queues. (ii) A CAV $i \in S_2(t)$ (or $S_1(t)$) leaves the CZ causing the index of any CAV $j \in S_1(t) \cup S_2(t)$ with $\text{type}(j) = 2$ (or $\text{type}(j) = 1$) to decrease by 1 and the CAV whose index is $-1$ (or $n-1$ in $S_1(t)$) is removed from $S_2(t)$ (or $S_1(t)$). Note that CAV $-1$ only appears in $S_2(t)$ (CAV $n-1$ only appears in $S_1(t)$), as discussed in Sec. II. (iii) A CAV changes lanes, causing an update in the current lane column in Fig. 2. This event is important because the value of $i_p$ for any CAV $i$ already in a queue may change, since its original $i_p$ may merge into another lane. (iv) A CAV overtake event when a CAV passes $M_3$ or $M_4$. This may occur when a CAV $i \in S_2(t)$ (or $S_1(t)$) overtakes $i-1 \in S_1(t) \cup S_2(t)$ when the two CAVs pass different MPs without conflict. Thus, if $i$ passes $M_3$ or $M_4$ and $i-1$ is still in one of the queues, we need to re-order $S_2(t)$ (or $S_1(t)$) according to the incremental position order, so that CAV $i+1$ can properly identify its $(i+1)_p$. For example, consider $i = 4$, $i-1 = n+3$, $i+1 = 5$ in queue $S_2(t)$ of Fig. 1. CAV 4 can overtake $n+3$, and its current lane will become $l_2$ when it passes $M_3$. When this happens, CAV 5 may mistake CAV 4 as its $i_p$ by looking at the new current lane entry for it, which is now in $l_2$. In reality, $i_p = n+3$ as long as CAV $n+3$ is still in lane $l_2$. This is avoided by re-ordering queue $S_2(t)$ according to the position information when this event occurs (i.e., swapping rows for CAVs 4 and $n+3$).

C. Joint Optimal and Barrier Function Controller

Once a newly arriving CAV $i \in S_1(t) \cup S_2(t)$ has determined all the safe merging constraints it has to satisfy as described in the last subsection, it can solve problem (6) subject to these constraints along with the rear-end safety constraint (3) and the state limitations (5). Obtaining a solution to this constrained optimal control problem is computationally intensive in the single-lane merging problem [7], and is obviously more computationally intensive in the multi-lane merging problem, since a CAV may have to satisfy two safe-merging constraints. Therefore, we will employ the joint optimal control and barrier function (OCBF) controller developed in [13] to account for all constraints.

We begin by noting that the distances from $O_2, O_3, O_4$ to $M_2$ or $M_3$ are all the same, while the distances from $O_1, O_2$ to $M_{1,2}$ or $M_4$ (or from $O_1, O_3, O_4$ to $M_2$) are different since the lane change behavior will induce an extra $l$ distance (a CAV moving from $M_2$ to $M_4$ is equivalent to a lane change). Therefore, we need to perform a coordinate transformation for those CAVs that are in different lanes (e.g., $l_2$ and $l_1$) and will merge into the same lane (e.g., $l_1$). In other words, when $i \in S_1(t)$ obtains information for $j \in S_1(t)$ from queue 1, the position information $x_j(t)$ is transformed by (using the original lane information in Fig. 2):

$$x_j(t) := \begin{cases} x_j(t)+l, & \text{if } [i \in l_2 \text{ or } l_3] \& [i-1 \in l_1], \\ x_j(t)+l, & \text{if } [i \in l_1] \& [i-1 \in l_2 \text{ or } l_3], \\ x_j(t), & \text{Otherwise}. \end{cases}$$

(10)

Next, we briefly review the OCBF approach in [13] as it applies to our problem. Problem (6) was solved in [7] for the single-lane merging problem and no noise in (1) and the unconstrained solution gives the following optimal control, speed, and position trajectories:

$$u^*_t(t) = a_i t + b_i$$

$$v^*_t(t) = \frac{1}{2} a_i t^2 + b_i t + c_i$$

$$x^*_t(t) = \frac{1}{6} a_i t^3 + \frac{1}{2} b_i t^2 + c_i t + d_i$$

(11) (12) (13)

where $a_i, b_i, c_i$ and $d_i$ are constants that can be solved along with $t^*_m$ by the initial and terminal constraints [7].

The OCBF controller aims to track the OC solution (11)-(13) while satisfying all constraints (3), (5) and (4). To accomplish this, first let $x_t(t) \equiv (x_t(t), v_t(t))$. Referring to the vehicle dynamics (1), let $f(x_t(t)) = [x_t(t), 0]^T$ and $g(x_t(t)) = [0, 1]^T$. Each of the seven constraints in (3), (5) and (4) can be expressed as $b_k(x_t(t)) \geq 0, k \in \{1,\ldots,7\}$ where each $b_k(x_t(t))$ is a CBF. For example, we have $b_1(x_t(t)) = z_{s_i}(t) - \varphi v_t(t) - \delta$ for the rear-end safety constraint (3). As an alternative, a Control Lyapunov Function (CLF) [10] $V(x_t(t))$ can also be used to track (stabilize) the optimal speed trajectory (12) through a CLF constraint. Therefore, the OCBF controller solves the following problem:

$$\min_{u_t(t)} \int_{t_0}^{t^*_m} \left( \frac{1}{2} (u_t(t) - u_{ref}(t))^2 \right) dt,$$

(14)

subject to the vehicle dynamics (1), the CBF constraints for (3), (5) and (4) and the CLF constraint for the speed tracking.
TABLE I
COMPARISON OF OC, CBF AND OCBF (WITH NOISE)

| Method | α | Noise | Ave. time(s) | Ave. $\nu_i^2(t)$ | Ave. obj. |
|--------|----|-------|--------------|-----------------|-----------|
| CBF    | N/A| N/A   | 14.7539      | 19.7241         | N/A       |
| Vissim | 0.01| N/A   | 31.5351      | 17.0415         | 19.2993   |
| OCBF   | yes| no    | 22.6763      | 6.7674          | 8.4558    |
|        | yes| yes   | 22.7636      | 8.8133          | 10.4780   |
| Vissim | 0.25| N/A   | 16.1811      | 10.4780         | N/A       |
| OCBF   | yes| no    | 16.4412      | 38.3094         | 38.3094   |
|        | yes| yes   | 16.331 | 11.2944    | 19.0546   |
| Vissim | 0.40| N/A   | 38.8611      | 107.3404        | N/A       |
| OCBF   | yes| no    | 40.8611      | N/A             | 11.2944   |
|        | yes| yes   | 44.6412      | 54.5177         | 54.5177   |

The obvious selection for speed and acceleration reference signals is $v_{ref}(t) = v_i^*(t)$, $u_{ref}(t) = u_i^*(t)$, but we select $v_{ref}(t) = \nu_i^*(t)$, $u_{ref}(t) = \nu_i^*(t)$ so as to provide position feedback to automatically reduce (or eliminate) the tracking position error, since the optimal solutions in (11)-(13) depend on the position (alternative forms of $v_{ref}(t)$, $u_{ref}(t)$ are possible as shown in [13]). The OCBF control can also deal with constraint violation due to noise in the dynamics included in (1) [13].

Remark (Framework Generalization). We can generalize the framework of any traffic scenario that involves multiple lanes leading to conflict zones beyond the merging configuration of Fig. 1. For example, in a 4-way intersection with one lane in each road, the number of possible cases is 56, but a CAV can easily find all the safe merging constraints (at most 5) that it needs to satisfy by looking up the extended queue similar to Table 2.

IV. SIMULATION RESULTS

All controllers have been implemented using MATLAB and we have used the Vissim microscopic multi-modal traffic flow simulation tool as a baseline for the purpose of making comparisons between our controllers and human-driven vehicles adopting standard car-following models used in Vissim. We used QUADPROG for solving QPs of the form (14) and ODE45 to integrate the vehicle dynamics.

Referring to Fig. 1, CAVs arrive according to Poisson processes with rates 2000 CAVs per hour and 1200 CAVs per hour for the main and merging roads, respectively. The initial speed $v_i(t)$ is also randomly generated with a uniform distribution over $[15 \text{ m/s}, 20 \text{ m/s}]$ at the origins $O$ and $O'$, respectively. The parameters for (14) and (1) are: $L_2 = 400 \text{ m}$, $L_3 = 407 \text{ m}$, $L_4 = 406.0622 \text{ m}$, $l = 0.9378 \text{ m}$, $\varphi = 1.8$, $\delta = 0$, $\max = 3.924 \text{ m/s}$, $\min = -5.886 \text{ m/s}$, $\max = 30 \text{ m/s}$, $\min = 0 \text{ m/s}$. We consider uniformly distributed noise processes (in [-2, 2] m/s for $w_{1,1}(t)$ and in [-0.05, 0.05] $m/s^2$ for $w_{1,2}(t)$).

We compare the simulation results between Vissim (human driver), the CBF method [12] (by setting $u_{ref}(t) = 0$ and $v_{ref}(t) = v_{max}$ in (14)) and the OCBF method, as shown in Table I. The CBF method is aggressive in travel time, and thus has larger energy consumption than both the OCBF method and human drivers. The OCBF method does better in both metrics than human drivers in Vissim, and achieves about 50% improvement in the objective function (6) under all three different trade-off parameters $\alpha$ (recall that $\alpha$ trades off travel time and energy in (6)).

V. CONCLUSIONS

We have shown how to transform a multi-lane merging problem into a decentralized optimal control problem, and combine OC with the CBF method to solve the merging problem for CAVs in order to deal with cases where the OC solution becomes difficult to obtain, as well as to handle the presence of noise in the vehicle dynamics by exploiting the ability of CBFs to add robustness to an OC controller. Remaining challenges include research on resequencing and extensions to large traffic networks.

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