LOWEST ENERGY BAND FUNCTION FOR MAGNETIC STEPS

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ABSTRACT. We study the Schrödinger operator in the plane with a step magnetic field function. The bottom of its spectrum is described by the infimum of the lowest eigenvalue band function, for which we establish the existence and uniqueness of the non-degenerate minimum. We discuss the curvature effects on the localization properties of magnetic ground states, among other applications.

1. INTRODUCTION

1.1. The planar magnetic step operator. Let \( a \in [-1, 1) \setminus \{0\} \). We define the self-adjoint magnetic Schrödinger operator on the plane

\[
L_a = \frac{\partial^2}{\partial x_2^2} + \left( \frac{\partial}{\partial x_1} + i a \sigma(x_2) \right)^2, \quad (x_1, x_2) \in \mathbb{R}^2,
\]

where \( \sigma \) is a step function defined as follows

\[
\sigma(x_1, x_2) = 1_{\mathbb{R}^+}(x_2) + a 1_{\mathbb{R}^-}(x_2).
\]

The operator \( L_a \) is invariant w.r.t. translations in the \( x_1 \)-direction, then it can be fibered and reduced to a family of 1D Schrödinger operators on \( L^2(\mathbb{R}) \), \( h_a[\xi] \), after a Fourier transform along the \( x_1 \)-axis (see \[14, 20\]). The fiber operators \( h_a[\xi] \), parametrized by \( \xi \in \mathbb{R} \), are defined in Section 1.2.

We have the following link between the spectra of the operators \( L_a \) and \( h_a[\xi] \) (see \[14\] and \[10, Section 4.3\]):

\[
\mathrm{sp}(L_a) = \bigcup_{\xi \in \mathbb{R}} \mathrm{sp}(h_a[\xi]).
\]

Consequently, the bottom of the spectrum of \( L_a \), denoted by \( \beta_a \), can be computed by minimizing the ground state energies of the fibered operators \( h_a[\xi] \) (see (1.10) below).

1.2. The lowest energy band function. Let \( a \in [-1, 1) \setminus \{0\} \). For all \( \xi \in \mathbb{R} \), we introduce the operator

\[
h_a[\xi] = -\frac{d^2}{d\tau^2} + V_a(\xi, \tau),
\]

with the potential \( V_a(\xi, \tau) = (\xi + \sigma(\tau))^2 \), where

\[
\sigma(\tau) = 1_{\mathbb{R}^+}(\tau) + a 1_{\mathbb{R}^-}(\tau).
\]

The domain of \( h_a[\xi] \) is given by :

\[
\mathrm{Dom}(h_a[\xi]) = \left\{ u \in B^1(\mathbb{R}) : \left( -\frac{d^2}{d\tau^2} + V_a(\xi, \tau) \right) u \in L^2(\mathbb{R}) \right\},
\]

where the space \( B^n(I) \) is defined for a positive integer \( n \) and an open interval \( I \subset \mathbb{R} \) as follows

\[
B^n(I) = \{ u \in L^2(I) : \tau^i \left( \frac{d^j u}{d\tau^j} \right) \in L^2(I), \forall i, j \in \mathbb{N} \text{ s.t. } i + j \leq n \}.
\]
The quadratic form associated to $h_a[\xi]$ is
$$q_a[\xi](u) = \int_{\mathbb{R}} \left( |u'(\tau)|^2 + V_a(\xi, \tau)|u(t)|^2 \right) d\tau$$
(1.6)
defined on $B^1(\mathbb{R})$. The operator $h_a[\xi]$ is with compact resolvent. We introduce the lowest eigenvalue of this operator (lowest band function)
$$\mu_a(\xi) = \inf_{u \in B^1(\mathbb{R}), u \neq 0} \frac{q_a[\xi](u)}{\|u\|^2_{L^2(\mathbb{R})}}.$$  
(1.7)
This is a simple eigenvalue, to which corresponds a unique positive $L^2$-normalized eigenfunction, $\varphi_{a,\xi}$, i.e. satisfying (see [2, Proposition A.2]).
$$\varphi_{a,\xi} > 0, \ (h_a[\xi] - \mu_a[\xi])\varphi_{a,\xi} = 0 \ \& \ \int_{\mathbb{R}} |\varphi_{a,\xi}(\tau)|^2 d\tau = 1.$$  \ (1.8)
Moreover, the above eigenvalue and eigenfunction depend smoothly on $\xi$ (see [7, 14]),
$$\xi \mapsto \mu_a(\xi) \ \& \ \xi \mapsto \varphi_{a,\xi}$$
are in $C^\infty$.
(1.9)
We introduce the step constant (at $a$) as follows
$$\beta_a := \inf_{\xi \in \mathbb{R}} \mu_a(\xi),$$
(1.10)
along with the celebrated de Gennes constant
$$\Theta_0 := \beta_{-1}. $$
(1.11)
Our main result is the following.

**Theorem 1.1.** Given $a \in (-1, 0)$, there exists a unique $\zeta_a \in \mathbb{R}$ such that
$$\beta_a = \mu_a(\zeta_a).$$
Furthermore, the following holds.

1. $\zeta_a < 0$ and satisfies $\mu''_a(\zeta_a) > 0$.
2. $|a|\Theta_0 < \beta_a < \Theta_0$.
3. The ground state $\phi_a := \varphi_{a,\zeta_a}$ satisfies $\phi'_a(0) < 0$.

**Remark 1.2.**

1. The existence of the minimum $\zeta_a$ was known earlier [2, 14]. Our contribution establishes the uniqueness of $\zeta_a$ and that it is a non-degenerate minimum. These new properties were only conjectured in [14] based on numerical computations.
2. The case $a = -1$ is perfectly understood and can be reduced to the study of the de Gennes model (family of harmonic oscillators on the half-axis with Neumann condition at the origin). In this case, we know the existence of the unique and non-degenerate minimum $\zeta_{-1} = -\sqrt{\Theta_0}$, and that the ground state $\phi_{-1}$ is an even function with a vanishing derivative at the origin ($\phi'_{-1}(0) = 0$).
3. Our comparison result $\beta_a < \Theta_0$ is also new. It was conjectured in [2] based on numerical computations. This comparison has an interesting application to the existence of superconducting magnetic edge states (see Section 4.4).
4. The sign of $\phi'_a(0)$ has an important application too, namely in precisng the localization properties of ground states for the Schrödinger operator with magnetic steps and in the large field asymptotics. That will be discussed in Section 4.3.
5. In the case $a \in (0, 1)$, we have $\beta_a = a$ and $\mu_a(\cdot)$ does not achieve a minimum.

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1. Many thanks to V. Bonnaillie-Noël for the numerical computations and Fig. 5 in [2].
2. The Robin model on the half line

We discuss in this section a model operator introduced in [18, 16]. Let $\xi$ and $\gamma$ be two real parameters. We introduce the family of harmonic oscillators on $\mathbb{R}_+$,

$$H[\gamma, \xi] = -\frac{d^2}{d\tau^2} + (\tau + \xi)^2,$$

(2.1)

with the following operator domain (accommodating functions satisfying the Robin condition at the origin)

$$\text{Dom}(H[\gamma, \xi]) = \{ u \in B^2(\mathbb{R}_+) : u'(0) = \gamma u(0) \}.$$ 

(2.2)

The quadratic form associated to $H[\gamma, \xi]$ is

$$B^1(\mathbb{R}_+) \ni u \mapsto q[\gamma, \xi](u) = \int_{\mathbb{R}_+} \left( |u'(\tau)|^2 + (\tau + \xi)^2 |u(\tau)|^2 \right) d\tau + \gamma |u(0)|^2.$$ 

The operator $H[\gamma, \xi]$ is with compact resolvent, hence its spectrum is an increasing sequence of eigenvalues $\lambda_j(\gamma, \xi)$, $j \in \mathbb{N}^*$. Furthermore, these eigenvalues are simple (see [10, Section 3.2.1] for the argument). Consequently, we introduce the corresponding orthonormal family of eigenfunctions $u^j_{\gamma, \xi}$ satisfying

$$u^j_{\gamma, \xi}(0) > 0.$$ 

(2.3)

The perturbation theory ensures that the functions

$$\xi \mapsto \lambda^j(\gamma, \xi), \ \xi \mapsto u^j_{\gamma, \xi}, \ \gamma \mapsto \lambda^j(\gamma, \xi), \ \text{and} \ \gamma \mapsto u^j_{\gamma, \xi} \text{ are } C^\infty.$$ 

(2.4)

The condition in (2.3) determines the normalized eigenfunction uniquely, because $u^j_{\gamma, \xi}(0) \neq 0$, otherwise it will vanish everywhere by Cauchy’s uniqueness theorem, since $(u^j_{\gamma, \xi})'(0) = \gamma u^j_{\gamma, \xi}(0)$ and

$$-\frac{d^2}{d\tau^2} u^j_{\gamma, \xi} + (\tau + \xi)^2 u^j_{\gamma, \xi} = \lambda^j(\gamma, \xi) u^j_{\gamma, \xi} \text{ on } \mathbb{R}_+.$$ 

The first partial derivatives of the eigenvalues with respect to $\xi$ and $\gamma$ are as follows (see [18, 16])

$$\partial_\xi \lambda^j(\gamma, \xi) = (\lambda^j(\gamma, \xi) - \xi^2 + \gamma^2)|u^j_{\gamma, \xi}(0)|^2,$$

(2.5)

$$\partial_\gamma \lambda^j(\gamma, \xi) = |u^j_{\gamma, \xi}(0)|^2.$$ 

(2.6)

Using the min-max principle, the lowest eigenvalue is defined as follows:

$$\lambda(\gamma, \xi) := \lambda^1(\gamma, \xi) = \inf_{u \in B^1(\mathbb{R}), u \neq 0} \frac{q[\gamma, \xi](u)}{\|u\|_{L^2(\mathbb{R})}^2}.$$ 

(2.7)

Note that the normalized ground state, $u_{\gamma, \xi}$, does not change sign on $\mathbb{R}_+$, and hence it is positive by our choice in (2.3).

For $\gamma \in \mathbb{R}$, we introduce the de Gennes function,

$$\Theta(\gamma) := \inf_{\xi \in \mathbb{R}} \lambda(\gamma, \xi).$$ 

(2.8)

**Theorem 2.1.** ([7, 18])

The following statements hold

1. For all $\xi \in \mathbb{R}$, $\gamma \mapsto \lambda(\gamma, \xi)$ is increasing.
2. For all $\gamma \in \mathbb{R}$, $\lim_{\xi \to -\infty} \lambda(\gamma, \xi) = 1$ and $\lim_{\xi \to +\infty} \lambda(\gamma, \xi) = +\infty$. 


For all $\gamma \in \mathbb{R}$, the function $\xi \mapsto \lambda(\gamma, \xi)$ admits a unique minimum attained at
\[
\xi(\gamma) := -\sqrt{\Theta(\gamma) + \gamma^2}.
\] (2.9)

Furthermore, this minimum is non-degenerate, $\partial_\xi^2 \lambda(\gamma, \xi(\gamma)) > 0$.

For all $\gamma \in \mathbb{R}$, $-\gamma^2 \leq \Theta(\gamma) < 1$.

The Neumann realization. The particular case where $\gamma = 0$ corresponds to the Neumann realization of the operator $H[0, \xi]$, denoted by $H^N[\xi]$, with the associated quadratic form $q^N[\xi] = q[0, \xi]$. The first eigenvalue of $H^N[\xi]$ is denoted by
\[
\lambda^N(\xi) = \inf \text{sp}(H^N[\xi]) = \lambda(0, \xi),
\] (2.10)
with the corresponding positive $L^2$-normalized eigenfunction $u^N_\xi := u_{0, \xi}$.

By a symmetry argument [2, 14], we get that the step constant $\beta_{-1}$ (in (1.10)) satisfies
\[
\Theta_0 := \beta_{-1} = \Theta(0).
\] (2.11)

This universal value $\Theta_0$ is often named the de Gennes constant in the literature [9, 10] and satisfies $\Theta_0 \in \left(\frac{1}{2}, 1\right)$. Numerically (see [5]), one finds $\Theta_0 \sim 0.59$. Note that the non-degenerate minimum $\xi_0 := \xi(0)$ of $\mu^N(\cdot)$ satisfies $\xi_0 = -\sqrt{\Theta_0}$.

3. The step model on the line

We analyze the band function $\mu_a(\cdot)$ introduced in (1.7) along with the positive normalized ground state $\varphi_{a, \xi}$.

Note that we are focusing on the interesting situation where $a \in (-1, 0)$. As mentioned earlier, for $a \in (0, 1)$, the minimum of $\mu_a(\cdot)$ is not achieved and the step constant $\beta_a = a$ [2, 14]; while for $a = -1$, the case reduces to the de Gennes model and $\beta_{-1} = \Theta_0$.

3.1. Preliminaries. Left with the situation $a \in (-1, 0)$, it is known that a minimum $\zeta_a$ exists and must be negative, $\zeta_a < 0$ [2, Prop. A.7]; our Theorem 1.1 sharpens this by establishing that the minimum is unique and non-degenerate. To prove this, new comparison estimates of the step constant $\beta_a$ are needed which improve the existing estimates in the literature [2, 14].

The existence of a minimum is due to the behavior at infinity of the band function $\mu_a(\cdot)$, namely,
\[
\lim_{\xi \to -\infty} \mu_a(\xi) = |a| \quad \text{and} \quad \lim_{\xi \to +\infty} \mu_a(\xi) = +\infty,
\]
and the following estimates on the step constant,
\[
|a| \Theta_0 < \beta_a < |a|.
\] (3.1)

Note that the lower bound (3.1) results from a simple comparison arguments using the min-max principle (see [2, Prop. A.6]); the upper bound is more tricky and relies on the construction of a trial state related to the Robin model introduced in Section 2 (see e.g. [2, Thm. 2.6]). Finally, we recall the expression for the derivative of $\mu_a(\cdot)$ established in [15] (see also [2, Prop. A.4]).
\[
\mu_a'(\xi) = \left(1 - \frac{1}{a}\right) \left(\varphi_{a, \xi}(0)^2 + (\mu_a(\xi) - \xi^2) \varphi_{a, \xi}(0)^2\right).
\] (3.2)
3.2. Comparison with the de Gennes constant.

Proposition 3.1. Let \( a \in (-1, 0) \). For \( \beta_a \) and \( \Theta_0 \) as in (1.10) and (2.11) respectively, we have

\[ \beta_a < \Theta_0. \]

Proof. If \( a \in [-\Theta_0, 0) \), then (3.1) yields that \( \beta_a < \Theta_0 \) and the conclusion of Proposition 3.1 follows in this particular case.

In the sequel, we fix \( a \in (-1, \Theta_0) \). For all \( \xi \in \mathbb{R} \), we denote by \( u(\cdot; \xi) = u^N_\xi(\cdot) \) the positive ground state of the de Gennes model (corresponding to the eigenvalue \( \lambda^N(\xi) \) in (2.10)). We introduce the function \( g_\xi \) on \( \mathbb{R} \) as follows:

\[ g_\xi(t) = \begin{cases} u(t; \xi), & \text{if } t \geq 0, \\ cu(t; \xi/\sqrt{|a|}), & \text{if } t < 0, \end{cases} \]

with \( c = c_\xi := u(0; \xi)/u(0; \xi/\sqrt{|a|}) > 0 \) so that \( g_\xi(0^-) = g_\xi(0^+) \). We observe that \( g_\xi \) is in the form domain of the operator \( h_a[\xi] \). Performing an elementary scaling argument, we get

\[
g_a[\xi](g_\xi) = \lambda^N(\xi) \int_{\mathbb{R}_+} |g_\xi(t)|^2 \, dt + |a| \lambda^N\left(\frac{\xi}{\sqrt{|a|}}\right) \int_{\mathbb{R}_-} |g_\xi(t)|^2 \, dt
\]

\[ = \lambda^N(\xi) \int_{\mathbb{R}} |g_\xi(t)|^2 \, dt + \left(|a| \lambda^N\left(\frac{\xi}{\sqrt{|a|}}\right) - \lambda^N(\xi)\right) \int_{\mathbb{R}_-} |g_\xi(t)|^2 \, dt.\]

We choose now \( \xi = \xi_0 := -\sqrt{\Theta_0} \) corresponding to \( \Theta_0 \) in (2.11). That way, we get \( \lambda^N(\xi_0) = \Theta_0 \) and

\[ g_a[\xi_0](g_{\xi_0}) = \Theta_0 \int_{\mathbb{R}} |g_{\xi_0}(\tau)|^2 \, d\tau + f(|a|) \int_{\mathbb{R}_-} |g_{\xi_0}(\tau)|^2 \, d\tau,
\]

where \( f(x) := x \lambda^N\left(\frac{\xi_0}{\sqrt{x}}\right) - \Theta_0 \), for \( x \in (\Theta_0, 1) \). By the min-max principle

\[ \beta_a \leq \frac{g_a[\xi_0](g_{\xi_0})}{\|g_{\xi_0}\|_{L^2(\mathbb{R})}^2} \leq \Theta_0 + f(|a|) \int_{\mathbb{R}_-} |g_{\xi_0}(\tau)|^2 \, d\tau \int_{\mathbb{R}} |g_{\xi_0}(\tau)|^2 \, d\tau. \]

To get that \( \beta_a < \Theta_0 \), it suffices to prove that \( f(x) < 0 \), for \( x \in (\Theta_0, 1) \).

Let \( x \in (\Theta_0, 1) \) and \( \alpha = \frac{\xi_0}{\sqrt{x}} \in (-1, \xi_0) \). By (2.5) (applied for \( j = 1 \) and \( \gamma = 0 \)), we can write

\[ f(x) = x(\lambda^N(\alpha) - \alpha^2) = x \frac{\lambda^N(\alpha)}{|u^N_\alpha(0)|^2}. \]

Since \( \alpha \in (-1, \xi_0) \) and \( \lambda^N(\cdot) \) is monotone decreasing on the interval \((-1, \xi_0)\), we deduce that \( (\lambda^N)'(\alpha) < 0 \) and eventually \( f(x) < 0 \) as required. \( \square \)

3.3. Variation of the ground state near zero. We pick any \( \zeta_a \in \mu^{-1}_a(\beta_a) \) so that \( \beta_a = \mu_a(\zeta_a) \), and denote by \( \phi_a = \varphi_{a, \zeta_a} \) the positive normalized ground state for \( \beta_a \) (so we are suppressing the dependence of the ground state on \( \zeta_a \)). We determine the sign of the derivative of \( \phi_a \) at the origin, thereby yielding that the ground state is a decreasing function in a neighborhood of 0. This result will be crucial in deriving the sign of some moments in Section 4.1 later.

Proposition 3.2. For all \( a \in (-1, 0) \) and \( \zeta_a \in \mu^{-1}_a(\beta_a) \), the positive normalized ground state \( \phi_a = \varphi_{a, \zeta_a} \) satisfies \( \phi'_a(0) < 0 \).
Proof. The proof relies on a comparison procedure with the Robin model. Let $\gamma_a = \phi_a''(0)/\phi_a(0)$. Since the ground state $\phi_a$ is positive, it suffices to prove that $\gamma_a < 0$. The eigenvalue equation $\mathfrak{h}_a[\zeta_a] \phi_a = a \phi_a$ written on $\mathbb{R}_+$ is

$$
\begin{cases}
-\phi_a''(\tau) + (\tau + \zeta_a)^2 \phi_a(\tau) = \beta_a \phi_a(\tau), & \tau > 0, \\
\phi_a'(0) = \gamma_a \phi_a(0),
\end{cases}
$$

(3.4)

Consequently, $\phi_a$ is an eigenfunction of the Robin operator $H[\gamma_a, \zeta_a]$, defined in (2.2), with a corresponding eigenvalue $\beta_a$. Using the min-max principle, we have

$$\beta_a \geq \lambda(\gamma_a, \zeta_a)$$

(3.5)

where $\lambda(\gamma_a, \zeta_a)$ is defined in (2.7).

If $\gamma_a \geq 0$, then by Theorem 2.1, Proposition 3.1 and (2.11), we get

$$\lambda(\gamma_a, \zeta_a) \geq \gamma(0, \zeta_a) = \lambda^N(\zeta_a) \geq \Theta_0 > \beta_a,$$

thereby contradicting (3.5). This proves that $\gamma_a < 0$. \hfill \Box

3.4. Uniqueness and non-degeneracy of the minimum. Now, we establish that the minimum of $\mu_a(\cdot)$ is unique and non-degenerate. The key in our proof is a tricky connection with the Robin model.

Proposition 3.3. For all $a \in (-1, 0)$,

$$\exists \zeta_a < 0, \mu_a^{-1}(\beta_a) = \{\zeta_a\} \& \mu_a''(\zeta_a) > 0,$$

where $\mu_a(\cdot)$ and $\beta_a$ are the eigenvalues introduced in (1.7) and (1.10) respectively.

Proof. First, note that $\mu_a^{-1}(\beta_a) \subset \mathbb{R}$ and is non-empty, by [2, Proposition A.7]. Hence, it suffices to prove that any negative critical point is a non-degenerate local minimum.

Let $\eta < 0$ be a critical point of $\mu_a(\cdot)$ (i.e., $\mu_a'(<\eta) = 0$). For all $\xi \in \mathbb{R}$, we introduce

$$\gamma(\xi) = \gamma_a(\xi) := \varphi'_{\xi,a}(0)/\varphi_{\xi,a}(0),$$

(3.6)

where $\varphi_{\xi,a}$ is the positive normalized ground state of the operator $\mathfrak{h}_a[\xi]$, which is now an eigenfunction for the Robin problem

$$
\begin{cases}
-\varphi''_{\xi,a}(\tau) + (\tau + \xi)^2 \varphi_{\xi,a}(\tau) = \mu_a(\xi) \varphi_{\xi,a}(\tau), & \tau > 0, \\
\varphi'_{\xi,a}(0) = \gamma(\xi) \varphi_{\xi,a}(0).
\end{cases}
$$

(3.7)

Using this for $\xi = \eta$, we can pick $j = j(\eta) \in \mathbb{N}$ such that $\mu_a(\eta) = \lambda^j(\gamma(\eta), \eta)$, the $j$th min-max eigenvalue of $H[\gamma(\xi), \xi]$. By the continuity of the involved functions and the simplicity of the eigenvalue $\lambda^j(\gamma(\eta), \eta)$, we can pick $\epsilon = \epsilon(\eta) > 0$ such that

$$\text{for all } \xi \in (\eta - \epsilon, \eta + \epsilon), \mu_a(\xi) = \lambda^j(\gamma(\xi), \xi).$$

(3.8)

Hence, by (2.5), (2.6) and differentiation in (3.8) w.r.t. $\xi$ we get

$$\mu'_a(\xi) = \partial_\xi \lambda^j(\gamma(\xi), \xi)$$

$$= (\lambda^j(\gamma(\xi), \xi) - \xi^2 + \gamma^2(\xi) |u^j_{\gamma(\xi), \xi}(0)|^2 + \gamma'(\xi)|u^j_{\gamma(\xi), \xi}(0)|^2).$$

(3.9)

Since $\mu'_a(\eta) = 0$, we infer from (3.2) and (3.8) that

$$\lambda^j(\gamma(\eta), \eta) - \eta^2 + \gamma(\eta)^2 = \mu_a(\eta) - \eta^2 + \gamma(\eta)^2 = \frac{\mu'_a(\eta)}{\varphi_{0,a}(0)} = 0.$$

(3.10)

Inserting this into (3.9) after setting $\xi = \eta$, we get (thanks to (2.3))

$$\gamma'(\eta) = 0.$$
This result will be used in the computation of $\mu''_a(\eta)$ below. In fact, differentiation in (3.2) w.r.t. $\xi$ yields
$$\mu''_a(\xi) = \left(1 - \frac{1}{a}\right)(\mu_a(\xi) - \xi^2 + \gamma(\xi)^2)\partial_{\xi} \varphi^2_{\xi,a}(0) + (\mu''_a(\xi) - 2\xi + 2\gamma(\xi)\partial_{\xi} \gamma(\xi)) \varphi^2_{\xi,a}(0).$$

Considering again $\xi = \eta$, we get
$$\mu''_a(\eta) = 2\left(\frac{1}{a} - 1\right) \eta \varphi^2_{\eta,a}(0).$$

In the above equation, we used (3.2), (3.10) and (3.11). Recall that we take $\eta < 0$ and $a \in (-1, 0)$, hence
$$\mu''_a(\eta) > 0,$$
and this holds for any negative critical point, $\eta$, of $\mu_a(\cdot)$. This finishes the proof. \hfill \Box

3.5. **Proof of the main result.** Theorem 1.1 now follows by collecting Propositions 3.3, 3.2 and 3.1.

4. Applications

4.1. Moments.

Fix $a \in [-1, 0]$ and consider $\beta_a$ as in (1.10), the ground state $\phi_a$, and $\zeta_a$ the unique minimum of $\mu_a(\cdot)$ (see Theorem 1.1 and Remark 1.2). We can invert the operator $\mathfrak{h}_a[\zeta_a] - \beta_a$ on the functions orthogonal to the ground state $\phi_a$, thereby leading to the introduction of the regularized resolvent (see e.g. [10, Lemma 3.2.9]):
$$\mathfrak{R}_a(u) = \begin{cases} 0 & \text{if } u \parallel \phi_a \\ (\mathfrak{h}_a[\zeta_a] - \beta_a)^{-1}u & \text{if } u \perp \phi_a. \end{cases} \quad (4.1)$$

The construction of certain trial states in Sec. 4.2 below requires inverting $\mathfrak{h}_a[\zeta_a] - \beta_a$ on functions involving $(\zeta_a + \sigma(\tau))\phi_a(\tau)$, for positive integers $n$, with $\sigma(\cdot)$ introduced in (1.4). We are then lead to investigate the following moments

$$M_n(a) = \int_{-\infty}^{+\infty} \frac{1}{\sigma(\tau)} (\zeta_a + \sigma(\tau))^{n}\phi_a(\tau)^2 d\tau,$$

**Proposition 4.1.** For $a \in [-1, 0)$, we have

$$M_1(a) = 0, \quad (4.2)$$
$$M_2(a) = -\frac{1}{2} \beta_a \int_{-\infty}^{+\infty} \frac{1}{\sigma(t)} |\phi_a(t)|^2 dt - \frac{1}{4} (\frac{1}{a} - 1) \zeta_a \phi_a(0) \phi'_a(0), \quad (4.3)$$
$$M_3(a) = \frac{1}{3} \left(\frac{1}{a} - 1\right) \zeta_a \phi_a(0) \phi'_a(0). \quad (4.4)$$

**Remark 4.2.**

1. (Feynman-Hellmann) We have (see e.g. [2, Eq. (A.9)])

$$\zeta_a + \sigma(\tau) \phi_a(\tau) \perp \phi_a(\tau) \text{ in } L^2(\mathbb{R}). \quad (4.5)$$

Furthermore, since $M_1(a) = 0$, we get further that $\frac{1}{\sigma(\tau)} (\zeta_a + \sigma(\tau)) \phi_a \perp \phi_a$ too. Combined together, we see that

$$(\zeta_a + a\tau) \phi_a \perp \phi_a \text{ in } L^2(\mathbb{R}_-) \& (\zeta_a + \tau) \phi_a \perp \phi_a \text{ in } L^2(\mathbb{R}_+)$$

which is consistent with (3.4), since by (3.2) and (2.5), $\zeta_a$ is a critical point of the corresponding Robin band function $\lambda^j(\gamma_a, \cdot)$.

2. As a consequence of Theorem 1.1, $M_3(a) = 0$ for $a = -1$, and it is negative for $-1 < a < 0$, which is consistent with [4].
Proof. In an analogous manner to [4], we define the operator
\[ L = h_a[\zeta] - \beta_a = -\frac{d^2}{d\tau^2} + (\zeta_a + \sigma(\tau))^2 - \beta_a. \]
Pick an arbitrary smooth function on \( \mathbb{R} \setminus \{0\} \) and set \( v = 2p\phi_a' - p'\phi_a \). We check that
\[ Lv = \left( p^{(3)} - 4((\zeta_a + \sigma)^2 - \beta_a)p' - 4\sigma(\zeta_a + \sigma)p \right) \phi_a. \]  
(4.6)
Noting that \( L\phi_a = 0 \), we obtain by an integration by parts,
\[ \int_{-\infty}^{+\infty} \phi_aLv\,d\tau = \int_{-\infty}^{+\infty} vL\phi_a\,d\tau - \phi_a(0)v'(0^-) + \phi_a(0)v'(0^+) + \phi'_a(0)v(0^-) - \phi'_a(0)v(0^+). \]
\[ = -\phi_a(0)v'(0^-) + \phi_a(0)v'(0^+) + \phi'_a(0)v(0^-) - \phi'_a(0)v(0^+). \]  
(4.7)
Take \( p = 1/\sigma^2 \), then a simple computation, using (4.6) and (4.7), yields
\[ M_1(a) = \frac{1}{2} \left( 1 - \frac{1}{a^2} \right) (\beta_a - \zeta_a^2) \phi_a(0)^2 + \phi'_a(0)^2). \]
The definition of \( \zeta_a \) ensures that \( \mu'(\zeta_a) = 0 \). Hence, by (3.9)
\[ (\beta_a - \zeta_a^2) \phi_a(0)^2 + \phi'_a(0)^2 = 0. \]  
(4.8)
Consequently, \( M_1(a) = 0 \).

Now, inserting \( p = \frac{1}{\sigma^2}(\zeta_a + \sigma t)^2 \) into (4.6)–(4.8), we establish (4.3).

A similar computation as above, with the choice \( p = \frac{1}{\sigma^2}(\zeta_a + \sigma t)^3 \), gives
\[ M_2(a) = \frac{2}{3} \beta_a M_1(a) + \frac{1}{3} \left( \frac{1}{a} - 1 \right) \zeta_a \phi_a(0)\phi'_a(0). \]
Having \( M_1(a) = 0 \), we get (4.4). \( \square \)

4.2. A model operator in a weighted space.

The operator \( h_a[\zeta] \) is not sufficient for the understanding of the geometry’s influence on the spectrum, as we shall do in Section 4.3 below. For that reason, we introduce a somehow more complicated operator accounting for the curvature term. This is very similar to the setting of the magnetic Neumann Laplacian [12].

We fix \( a \in (-1, 0) \), \( \delta \in (0, \frac{1}{12}) \), \( M > 0 \) and \( h_0 > 0 \) such that, for all \( h \in (0, h_0) \),
\[ Mh^{\frac{1}{2} - \delta} < \frac{1}{4}. \]
That way, for \( t \in [-M, M] \), we can introduce the positive function \( a_h = (1 - \frac{h}{\sqrt{\tau}}) \) and the Hilbert space \( L^2((-h^\delta, h^\delta); a_h\,d\tau) \) with the weighted inner product
\[ \langle u, v \rangle = \int_{-h^\delta}^{h^\delta} u(\tau)v(\tau) (1 - \frac{h}{\sqrt{\tau}})\,d\tau. \]
For \( \xi \in \mathbb{R} \), we introduce the self-adjoint operator
\[ \mathcal{H}_{a,\xi, t, h} = -\frac{d^2}{d\tau^2} + (\sigma + \xi)^2 + \frac{1}{h} (1 - \frac{h}{\sqrt{\tau}})^{-1} \partial_\tau + 2\mathcal{H} \left( \sigma + \xi - \frac{h^\frac{1}{2} \sigma^2}{2} \right)^2 \]
\[ - \frac{h^\frac{1}{2} \sigma^2}{4} + \frac{h^\frac{1}{2} \sigma^2}{2} \right) \left( \sigma + \xi - \frac{h^\frac{1}{2} \sigma^2}{2} \right)^2 \]
where \( \sigma(\cdot) \) is the function in (1.4). The domain of definition of this operator is
\[ \text{Dom}(\mathcal{H}_{a,\xi, t, h}) = \{ u \in H^2((-h^\delta, h^\delta); a_h\,d\tau) : u(\pm h^\delta) = 0 \}. \]  
(4.10)
The operator \( \mathcal{H}_{a,\xi, t, h} \) is the Friedrichs extension in \( L^2((-h^\delta, h^\delta); a_h\,d\tau) \) associated to the quadratic form \( q_{a,\xi, t, h} \) defined by
\[ q_{a,\xi, t, h}(u) = \int_{-h^\delta}^{h^\delta} \left( |u'(\tau)|^2 + (1 + 2h^{\frac{1}{2} \tau}) (\sigma + \xi - \frac{h^\frac{1}{2} \sigma^2}{2})^2 \right) u^2(\tau) d\tau. \]
The operator $\mathcal{H}_{a,ξ,t,h}$ is with compact resolvent. We denote by $\left(\lambda_n(\mathcal{H}_{a,ξ,t,h})\right)_{n\geq 1}$ its sequence of min-max eigenvalues.

By Theorem 1.1, $\mu_a(\cdot)$ has a unique minimum $\beta_a$ (attained at $ξ_a$) which is non-degenerate, and the moment $M_3(a)$ in (4.4) is negative, thereby allowing us to derive the following result on the ground state energy of $\mathcal{H}_{a,ξ,t,h}$.

**Proposition 4.3.** Let $β_{a,t,h} := \inf_{ξ \in \mathbb{R}} \lambda_1(\mathcal{H}_{a,ξ,t,h})$. Then, as $h \to 0+$,

$$β_{a,t,h} = β_a + ξ M_3(a) h^{\frac{1}{2}} + \mathcal{O}(h^{\frac{3}{2}})$$

uniformly with respect to $ξ \in [-M,M]$.

**Proof.** We will present the outline of the proof to show the role of Theorem 1.1. A similar approach was detailed in [12, Theorem 11.1] (see also [17, Section 4.2]). By the min-max principle, there exists $C > 0$ such that for all $n \geq 1$, $ξ \in \mathbb{R}$ and $h \in (0,h_0)$,

$$|λ_n(\mathcal{H}_{a,ξ,t,h}) - λ_n(\mathcal{H}_{a,ξ})| ≤ Ch^{\frac{1}{2} - 2δ}(1 + λ_n(\mathcal{H}_{a,ξ})), \quad (4.11)$$

where $\mathcal{H}_{a}[ξ]$ is the fiber operator in (1.1). Consequently, we may find a constant $z(a) > 0$ such that

for $|ξ - ξ_a| ≥ z(a)h^{\frac{1}{4} - δ}$, $λ_1(\mathcal{H}_{a,ξ,t,h}) ≥ β_a + h^{\frac{1}{2} - 2δ}$. \quad (4.12)

Note that (4.12) is a consequence of the fact that $ξ_a$ is a non-degenerate minimum of $μ_a(\cdot)$.

Now, we estimate $λ_1(\mathcal{H}_{a,ξ,t,h})$ for $|ξ - ξ_a| ≤ z(a)h^{\frac{1}{4} - δ} \ll 1$. By (4.11), the simplicity of the eigenvalues $λ_n(\mathcal{H}_{a,ξ})$ and the continuity of the function $ξ \mapsto λ_n(\mathcal{H}_{a,ξ})$, we know that as $h \to 0+$,

$$λ_1(\mathcal{H}_{a,ξ,t,h}) = β_a + o(1) \& λ_2(\mathcal{H}_{a,ξ,t,h}) = λ_2(\mathcal{H}_{a,ξ}) + o(1),$$

with

$$λ_2(\mathcal{H}_{a,ξ}) > λ_1(\mathcal{H}_{a,ξ}) \equiv β_a. \quad (4.13)$$

One may construct a formal eigen-pair $(λ_{a,ξ,t,h}^{app}, f_{a,ξ,t,h}^{app})$ of the operator $\mathcal{H}_{a,ξ,t,h}$, with

$$λ_{a,ξ,t,h}^{app} = c_0 + c_1(ξ - ξ_a) + c_2(ξ - ξ_a)^2 + c_3 h^{1/2}$$

and

$$f_{a,ξ,t,h}^{app} = u_0 + (ξ - ξ_a)u_1 + (ξ - ξ_a)^2 u_2 + h^{1/2}u_3. \quad (4.14)$$

Expanding $R_h := (\mathcal{H}_{a,ξ,t,h} - λ_{a,ξ,t,h}^{app})^{-1}_{a,ξ,t,h}$ in powers of $(ξ - ξ_a)$ and $h^{1/2}$, one can choose $(c_i, u_i)_{0 ≤ i ≤ 3}$ so as the coefficients of the $h^{1/2}$ and $(ξ - ξ_a)^j$ terms, $j = 0, 1, 2$, vanish. We choose

$$c_0 = β_a, \quad u_0 = φ_a$$

$$c_1 = 0, \quad u_1 = -2R_a v_1, \quad v_1 := (στ + ξ_a)φ_a \perp φ_a$$

$$c_2 = 1 - 4 \int_{-∞}^{+∞} (στ + ξ_a)φ_a R_a[(στ + ξ_a)φ_a] dt, \quad u_2 = R_a v_2,$$

$$v_2 := 4(στ + ξ_a)R_a[(στ + ξ_a)φ_a] + (c_2 - 1)φ_a \perp φ_a$$

$$c_3 = ξ M_3(a), \quad u_3 = R_a v_3,$$

$$v_3 := -ξ \left( φ_τ + \frac{1}{σ} (στ + ξ_a)^2 - \frac{c_2}{σ} (στ + ξ_a) \right) φ_a + c_3 φ_a \perp φ_a,$$

where $R_a \in \mathcal{L}(L^2(\mathbb{R}))$ is the regularized resolvent introduced in (4.1). That the functions $v_1, v_2, v_3$ are orthogonal to $φ_a$ is ensured by our choice of $c_1, c_2, c_3$, the expressions of the moments in Proposition 4.1, and the first item in Remark 4.2.
Eventually, using \( \chi(h^3 \tau)f_{app}^{a,\xi,t,h} \) as a quasi-mode, with \( \chi \) a cut-off function introduced to insure the Dirichlet condition at \( \tau = \pm h^{-\delta} \), we get by the spectral theorem and (4.13),

\[
\lambda_1(H_{a,\xi,t,h}) = c_0 + c_2(\xi - \zeta_a)^2 + c_3 h^{1/2} + O\left(\max(h^{1/2}|\xi - \zeta_a|, |\xi - \zeta_a|^3, h)\right). \quad (4.15)
\]

Note that, for \( |\xi - \zeta_a| \leq z(a) h^{1/2} \), we have

\[
O\left(\max(h^{1/2}|\xi - \zeta_a|, |\xi - \zeta_a|^3, h)\right) = O(h^{3(1/2 - \delta)}).
\]

In order to minimize over \( \xi \), we observe that the constant \( c_2 \) can be expressed in the pleasant form \(^2\)

\[
c_2 = \frac{1}{2} \partial^2 \mu_a(\zeta_a),
\]

hence \( c_2 > 0 \) by Theorem 1.1. So, we get from (4.12) and (4.13),

\[
\inf_{\xi \in \bar{\mathbb{R}}} \lambda_1(H_{a,\xi,t,h}) = c_0 + c_3 h^{1/2} + O(h^{3(1/2 - \delta)}). \quad (4.16)
\]

To improve the error in (4.16), notice that, by (4.15), it is enough to minimize over \( \{ |\xi - \zeta_a| \leq h^{1/2} \} \), thereby finishing the proof of Theorem 4.3. \( \square \)

**Remark 4.4.** The approximate eigen-pair \((\lambda_{app}^{H_{a,\xi,t,h}}, f_{app}^{H_{a,\xi,t,h}})\) in (4.14) does not depend on the parameter \( \delta \) introduced in (4.10). Moreover, we have, for \( |\xi - \zeta_a| < 1 \),

\[
\left\| (H_{a,\xi,t,h} - \lambda_{app}^{H_{a,\xi,t,h}}) f_{app}^{H_{a,\xi,t,h}} \right\|_{L^2(\mathbb{R})} = O\left(\max(h^{1/2}|\xi - \zeta_a|, |\xi - \zeta_a|^3, h)\right).
\]

### 4.3. Magnetic edge & semi-classical ground state energy.

With the precise estimate for the ground state energy of weighted operator of Section 4.2 in hand, we can inspect edge states for the Dirichlet Laplace operator with a magnetic step field.

#### 4.3.1. Magnetic edge, the domain and the operator.

Consider a smooth planar curve \( \Gamma \subset \mathbb{R}^2 \) that splits \( \mathbb{R}^2 \) into two disjoint unbounded open sets, \( P_{\Gamma,1} \) and \( P_{\Gamma,2} \). We will refer to \( \Gamma \) as the magnetic edge, since we are going to consider magnetic fields having a jump along \( \Gamma \) (see Fig. 1).

Now consider an open bounded simply connected subset \( \Omega \) of \( \mathbb{R}^2 \), with smooth boundary \( \partial \Omega \) of class \( C^1 \), and assume that

1. \( \Gamma \) intersects \( \partial \Omega \) at two distinct points \( p \) and \( q \), and the intersection is transversal, i.e. \( T_{\partial \Omega} \times T_{\Gamma} \neq 0 \) on \( \{ p, q \} \), where \( T_{\partial \Omega} \) and \( T_{\Gamma} \) are respectively unit tangent vectors of \( \partial \Omega \) and \( \Gamma \).  
2. \( \Omega_1 := \Omega \cap P_{\Gamma,1} \neq \emptyset \) and \( \Omega_2 := \Omega \cap P_{\Gamma,2} \neq \emptyset \).

Fix \( a \in (-1, 0) \). Let \( F_a \in H^1(\Omega, \mathbb{R}^2) \) be a magnetic potential with the corresponding scalar magnetic field:

\[
\text{curl} F_a = B_a := 1_{\Omega_1} + a 1_{\Omega_2}. \quad (4.17)
\]

We consider the Dirichlet realization of the self-adjoint operator in the domain \( \Omega \)

\[
P_{h,a} = -(h \nabla - i F_a)^2 = -h^2 \Delta + ih(\text{div} F_a + F_a \cdot \nabla) + |F_a|^2,
\]

with domain

\[
\text{Dom}(P_{h,a}) = \{ u \in L^2(\Omega) : (h \nabla - i F_a)^j u \in L^2(\Omega), j \in \{ 1, 2 \}, u|_{\partial \Omega} = 0 \},
\]

and quadratic form

\[
q_{h,a}(u) = \int_{\Omega} (|(h \nabla - i F_a) u|^2 dx \quad (u \in H^1_0(\Omega)). \quad (4.18)
\]

\(^2\)Using the Feynman-Hellmann formula \( \mu'_a(\xi) = \langle(\zeta_a + \sigma(\tau)\tau)\varphi_{a,\xi}, \varphi_{a,\xi} \rangle \) [2, Eq. (A.9)].
The bottom of the spectrum of this operator is introduced as follows

$$\lambda_1(\mathcal{P}_{h,a}) = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} \frac{q_{h,a}(u)}{\|u\|^2_{L^2(\Omega)}}.$$  (4.19)

4.3.2. Frenet coordinates near the magnetic edge.

We introduce the Frenet coordinates near $\Gamma$. We refer the reader to [10, Appendix F] and [2] for a similar setup.

Let $s \mapsto M(s) \in \Gamma$ be the arc length parametrization of $\Gamma$ such that

- $\nu(s)$ is the unit normal of $\Gamma$ at the point $M(s)$ pointing to $P_{\Gamma,1}$;
- $T(s)$ is the unit tangent vector to $\Gamma$ at the point $M(s)$, such that $(T(s), \nu(s))$ is a direct frame, i.e. $\det(T(s), \nu(s)) = 1$.

Now, we define the curvature $k$ of $\Gamma$ as follows $T'(s) = k(s)\nu(s)$. For $\epsilon > 0$, we define the transformation

$$\Phi : \mathbb{R} \times (-\epsilon, \epsilon) \ni (s, t) \mapsto M(s) + t\nu(s) \in \Gamma_\epsilon := \{x \in \mathbb{R}^2 : \text{dist}(x, \Gamma) < \epsilon\}.$$  (4.20)

We pick $\epsilon$ sufficiently small so that $\Phi$ is a diffeomorphism, whose Jacobian is

$$a(s, t) := J_\Phi(s, t) = 1 - tk(s).$$  (4.21)

There is a natural correspondence between functions in $H^1(\Gamma_\epsilon)$ and those in $H^1(\mathbb{R} \times (-\epsilon, \epsilon))$. In fact, to every $u \in H^1(\Gamma_\epsilon)$ we assign $\tilde{u} \in H^1(\mathbb{R} \times (-\epsilon, \epsilon))$

$$\tilde{u}(s, t) = u(\Phi(s, t)),$$  (4.22)

and vice versa.

The vector field $\mathbf{F}_a$ can be extended in a natural manner to a vector field in $H^1(\mathbb{R}^2)$. Seen as a vector field on $\Gamma_\epsilon$, it gives rise to a vector field on $\mathbb{R} \times (-\epsilon, \epsilon)$ as follows

$$\mathbf{F}_a(x) = (F_{a,1}(x), F_{a,2}(x)) \mapsto \tilde{\mathbf{F}}_a(s, t) = (\tilde{F}_{a,1}(s, t), \tilde{F}_{a,2}(s, t)),$$

where

$$\tilde{F}_{a,1}(s, t) = a(s, t)\mathbf{F}_a(\Phi(s, t)) \cdot T(s) \quad \text{and} \quad \tilde{F}_{a,2}(s, t) = \mathbf{F}_a(\Phi(s, t)) \cdot \nu(s).$$  (4.23)
Finally, we note the change of variable formula (for functions compactly supported in $\Gamma$):
\[
\int_{\Gamma} |u|^2 \, dx = \int_{\mathbb{R}} \int_{-\varepsilon}^{\varepsilon} |\tilde{u}|^2 \, a \, dt \, ds \& \\
\int_{\Gamma} |(h\nabla - i\mathbf{F}_a)u|^2 \, dx = \int_{\mathbb{R}} \int_{-\varepsilon}^{\varepsilon} \left( a^{-2}|(h\partial_x - i\tilde{F}_{a,1})\tilde{u}|^2 + |(h\partial_t - i\tilde{F}_{a,2})\tilde{u}|^2 \right) a \, dt \, ds.
\] (4.24)

4.3.3. Ground state energy and curvature of the magnetic edge.

We introduce the maximal curvature of $\Gamma$ in $\Omega$ as follows
\[
k_{\Omega}^{\max} = \max_{x \in \Gamma \cap \Omega} \left( k(\Phi^{-1}(x)) \right).
\] (4.25)

**Theorem 4.5.** There exist positive constants $c_a, C_a, h_a$ such that the ground state energy in (4.19) satisfies, for all $h \in (0, h_a),
\[-c_a h^\frac{3}{2} \leq \lambda_1(P_{h,a}) - (\beta_a h + M_3(a)k_{\Omega}^{\max} h^\frac{3}{2}) \leq C_a h^\frac{7}{4}.
\]

4.3.4. Upper bound on the ground state energy.

This will be done by the construction of a trial state involving an appropriate gauge transformation in the Frenet coordinates that we recall below.

**Lemma 4.6.** For $x_0 = \Phi(s_0, 0) \in \Gamma$ and $0 < \ell < \varepsilon$, we introduce the neighborhood $\mathcal{N}(x_0, \ell) = \{\Phi(s, t) : |s - s_0| < \ell \& \, |t| < \ell\}$. There exists a function $\omega_{\ell} \in \mathcal{N}(x_0, \ell)$ such that the vector potential $\tilde{F}_{a,\ell} : \mathbb{R} \rightarrow C(\mathbb{R})$, defined on $\mathcal{N}(x_0, \ell)$, satisfies
\[
\tilde{F}_{a,\ell}^{\new}(s, t) = \begin{cases} 
- (t - \frac{\ell^2}{2} k(s)) & \text{if } t > 0 \\
- a(t - \frac{\ell^2}{2} k(s)) & \text{if } t < 0
\end{cases} \quad \& \quad \tilde{F}_{a,\ell}^{\new}(s, t) = 0.
\] (4.26)

Now pick $x_0 = \Phi(s_0, 0) \in \Gamma \cap \Omega$ such that $k(s_0) = k_{\Omega}^{\max}$. Select $x_h = \Phi(s_h, 0) \in \Gamma \cap \Omega$ so that $|s - s_h| = h^{1/2}$. We introduce the trial state $u$ defined in the Frenet coordinates as follows
\[
u(\Phi(s, t)) = \tilde{u}(s, t) = c_h \chi \left( \frac{s - s_h}{h^{1/2}} \right) \chi \left( \frac{t}{h^{1/2}} \right) f_{a,c_h,k(s_0),h}^{\app}(h^{-1/2} t) \exp \left( \frac{t(\zeta_a s - \omega(s, t))}{h^{1/2}} \right),
\] (4.27)

where $\omega = \omega_{\ell}$ is the gauge function introduced in Lemma 4.6 for $\ell = 2h^{1/4}$, $f_{a,c_h,k(s_0),h}^{\app}$ is the approximate 1D eigenfunction introduced in (4.14) with $\xi = \zeta_h$, $\chi$ is a cut-off function and $c_h > 0$ is a constant selected so that the $L^2$-norm of $u$ in $\Omega$ is equal to 1. We choose the cut-off function as follows:
\[
\chi \in C^\infty_c(\mathbb{R}), \, \text{supp} \, \chi \subset [-1, 1], \, \chi = 1 \text{ on } [-1/2, 1/2].
\]

Then, we can compute $q_{h,a}(u)$ and get
\[
\lambda_1(P_{h,a}) \leq \frac{q_{h,a}(u)}{\|u\|_{L^2(\Omega)}} \leq \beta_a h + k(s_0)M_3(a)h^{3/2} + \mathcal{O}(h^{7/4}).
\]

4.3.5. Concentration near the magnetic edge. Fix $R_0 > 1$ and consider a partition of unity
\[
\sum_{j=1}^{N_h} \chi_{h,j}^2 = 1 \text{ in } \Gamma_{R_0h^{1/2}}
\]
such that
\[ \text{supp } \chi_{h,j} \subset \mathcal{N}(x_j, R_0 h^{1/2}) \text{ and } \sum_{j=1}^{N_h} |\nabla \chi_{j,h}|^2 = \mathcal{O}(R_0^{-2} h^{-1}). \]

Also, we assume that \( x_1 = p, x_{N_h} = q \), where \( \{p, q\} = \Gamma \cap \partial \Omega \).

We introduce another partition of unity \( \sum_{i=1}^{2} \varphi_{i,h}^2 = 1 \) in \( \mathbb{R}^2 \) such that \( \text{supp} \varphi_{1,h} \subset \mathbb{R}^2 \setminus \Gamma R_0 h^{1/2} \) and \( \sum_{i=1}^{2} |\nabla \varphi_{i,h}|^2 = \mathcal{O}(R_0^{-2} h^{-1}) \).

Pick an arbitrary \( u \in H_0^1(\Omega) \). We extend \( u \) by 0 on \( \mathbb{R}^2 \setminus \Omega \). Notice that
\[
q_{h,a}(u) = \sum_{i=1}^{2} q_{h,a}(\varphi_{i,h} u) - h^2 \sum_{i=1}^{2} \|\nabla \varphi_{i,h} u\|^2 \\
= q_{h,a}(\varphi_{1,h} u) + \sum_{j=1}^{N_h} \left( q_{h,a}(\varphi_{2,h} \chi_{j,h} u) - h^2 \|\nabla \chi_{j,h} \varphi_{2,h} u\|^2 \right) - h^2 \sum_{i=1}^{2} \|\nabla \varphi_{i,h} u\|^2 \\
= q_{h,a}(\varphi_{1,h} u) + \sum_{j=1}^{N_h} q_{h,a}(\varphi_{2,h} \chi_{j,h} u) - \mathcal{O}(R_0^{-2} h).
\]

We bound from below each \( q_{h,a}(\varphi_{2,h} \chi_{j,h} u) \) as follows (see [1])
\[ q_{h,a}(\varphi_{2,h} \chi_{j,h} u) \geq (\beta_a h - \mathcal{O}(h^{3/2})) \|\varphi_{2,h} \chi_{j,h} u\|^2. \]

Since \( \text{curl } F_a \) is constant away from \( \Gamma \), we bound \( q_{h,a}(\varphi_{1,h} u) \) from below as follows
\[ q_{h,a}(\varphi_{1,h} u) \geq \int_{\Omega} |\text{curl } F_a| |\varphi_{1,h} u|^2 \, dx \geq |a|h \|\varphi_{1,h} u\|^2. \]

Summing up, we deduce the following lower bound on the quadratic form
\[ q_{h,a}(u) \geq \int_{\Omega} \left( U_{h,a}(x) - \mathcal{O}(R_0^{-2} h) \right) |u(x)|^2 \, dx \quad (u \in H_0^1(\Omega)), \]
where
\[ U_{h,a}(x) = \begin{cases} |a|h & \text{if } \text{dist}(x, \Gamma) > R_0 h^{1/2} \\ \beta_a h & \text{if } \text{dist}(x, \Gamma) < R_0 h^{1/2}. \end{cases} \]

This allows us to do Agmon estimates and arrive at the following decay property of eigenfunctions \( u_h \) with eigenvalues \( z_h \leq \beta_a h + o(h) \):
\[
\int_{\Omega} \left( |u_h|^2 + h^{-1} |(h \nabla - i F_a) u_h|^2 \right) \exp \left( \frac{\alpha \text{dist}(x, \Gamma)}{h^{1/2}} \right) \, dx \leq C \|u_h\|_{L^2(\Omega)}, \quad (4.28)
\]
for some positive constants \( \alpha \) and \( C \).

As a consequence of (4.28) (and the inequality \( e^z \geq \frac{n!}{z!} \) for \( z \geq 0 \)), we get for any positive integer \( n \),
\[
\int_{\Omega} \left( \text{dist}(x, \Gamma) \right)^n \left( |u_h|^2 + h^{-1} |(h \nabla - i F_a) u_h|^2 \right) \, dx \leq C_n h^{n/2} \|u_h\|_{L^2(\Omega)}^2, \quad (4.29)
\]
for a positive constant \( C_n \).
4.3.6. Lower bound on the ground state energy. Pick a ground state $u_h$ of $\lambda_1(\mathcal{P}_{h,a})$ and extend it by 0 on $\mathbb{R}^2 \setminus \Omega$. We will bound the quadratic form from below as follows

$$q_{h,a}(u_h) \geq \left( \beta_a h + M_3(a) k_{\max}^\Omega h^\frac{2}{3} - O(h^\frac{5}{2}) \right) \| u_h \|_{L^2(\Omega)}^2.$$  \hspace{1cm} (4.30)

Set $\epsilon_h = h^{\frac{1}{2} - \delta}$, with $\delta \in (0, \frac{1}{12})$. Consider two partitions of unity

$$\sum_{i=1}^2 \varphi_{j,i}^2 = 1 \text{ in } \mathbb{R}^2, \quad \text{supp} \varphi_{1,h} \subset \mathbb{R}^2 \setminus \Gamma_{h,a}, \quad \sum_{i=1}^2 \| \nabla \varphi_{i,h} \|^2 = O(h^{2\delta - 1}),$$

and, for a fixed $\rho \in (0, \frac{1}{2})$,

$$\sum_{j=1}^{N_h} \chi_{h,j}^2 = 1 \text{ in } \Gamma_{h}, \quad \text{supp} \chi_{h,j} \subset \mathcal{N}(x_j, h^\rho), \quad \sum_{j=1}^{N_h} \| \nabla \chi_{j,h} \|^2 = O(h^{-2\rho}),$$

with $x_j = \Phi(s_j, 0) \in \Gamma \cap \overline{\Omega}$, $x_1 = p$, $x_2 = q$ and $\{p, q\} = \Gamma \cap \partial \Omega$. Set $w_h = \varphi_{2,h} u_h$.

By (4.28)

$$\| w_h \|_{L^2(\Omega)} = \| u_h \|_{L^2(\Omega)} + O(h^{\infty}) \& q_{h,a}(w_h) = q_{h,a}(u_h) + O(h^{\infty}). \hspace{1cm} (4.31)$$

Now, we decompose $q_{h,a}(w_h)$ via the partition of unity along $\Gamma$ as follows

$$q_{h,a}(w_h) = \sum_{j=1}^{N_h} q_{h,a}(w_{h,j}) + O(h^{-2\rho}) \| w_h \|_{L^2(\Omega)}^2 \text{ with } w_{h,j} = \chi_{h,j} \varphi_{2,h} u_h. \hspace{1cm} (4.32)$$

Performing a local gauge transformation in $\mathcal{N}(x_j, h^\rho)$ as in Lemma 4.6, we get a new function $\tilde{w}_{h,j}$ such that

$$q_{h,a}(w_{h,j}) = \int_{\mathbb{R}} \int_{-\epsilon_h}^{\epsilon_h} \left( a^{-2} \left( \left| \partial_s + i\sigma t - \frac{\sigma t^2}{2} k(s) \right| \tilde{w}_{h,j} \right)^2 + h^2 \left| \partial_t \tilde{w}_{h,j} \right|^2 \right) \ dt \ ds$$

In every $\mathcal{N}(x_j, h^\rho)$, we expand

$$\kappa(s) = \kappa_j + O(h^\rho), \quad a = 1 - t\kappa_j + O(h^\rho t), \quad a^{-2} = 1 + 2\kappa_j t + O(h^\rho t),$$

where,

$$\kappa_j := \kappa(\bar{s}_j) = \min_{|s-s_j| \leq h^\rho} \kappa(s), \quad x_j = \Phi(s_j, 0 \& \bar{s}_j \in \{s-s_j \leq h^\rho\}. \hspace{1cm} (4.33)$$

For every integer $n \geq 0$, we write by (4.29),

$$\sum_{j=1}^{N_h} \int_{\mathbb{R}} \int_{-\epsilon_h}^{\epsilon_h} |t|^n \left| \partial_s \tilde{w}_{h,j} \right|^2 \ dt \ ds \leq \tilde{C}_n h^{\frac{n}{2}} \| u_h \|_{L^2(\Omega)}^2$$

$$\sum_{j=1}^{N_h} \int_{\mathbb{R}} \int_{-\epsilon_h}^{\epsilon_h} h^2 |t|^n \left| \partial_t \tilde{w}_{h,j} \right|^2 \ dt \ ds \leq \tilde{C}_n h^{1+\frac{n}{2}} \| u_h \|_{L^2(\Omega)}^2$$

$$\sum_{j=1}^{N_h} \int_{\mathbb{R}} \int_{-\epsilon_h}^{\epsilon_h} |t|^n \left( \left( \partial_s + i\sigma t - \frac{\sigma t^2}{2} k(s) \right) \tilde{w}_{h,j} \right)^2 \ dt \ ds \leq \tilde{C}_n h^{1+\frac{n}{2}} \| u_h \|_{L^2(\Omega)}^2.$$

That way we get

$$\sum_{j=1}^{N_h} q_{h,a}(w_{h,j}) \geq \sum_{j=1}^{N_h} \int_{\mathbb{R}} \int_{-\epsilon_h}^{\epsilon_h} \left( 1 + 2\kappa_j t \right) \left( \left( \partial_s + i\sigma t - \frac{\sigma t^2}{2} k_j \right) \tilde{w}_{h,j} \right)^2$$

$$+ h^2 \left| \partial_t \tilde{w}_{h,j} \right|^2 \left( 1 - t\kappa_j \right) \ dt \ ds - O(h^{\frac{3}{2} + \rho}) \hspace{1cm} (4.34)$$
In each $\{ |s - s_j| < h^\rho \} \cap \{ |t| < h^{\frac{1}{2} - \delta} \},$ we perform a partial Fourier transform w.r.t. $s$ and the scaling $t \mapsto \tau = h^{-\frac{1}{2}} t$. We then reduce to the setting of Proposition 4.3 and get, after summing over $j,$
\begin{align*}
\sum_{j=1}^{N_h} q_{h,a}(w_{h,j}) & \geq h \sum_{j=1}^{N_h} \left( \beta_a + M_3(a) \kappa_j h^{\frac{3}{2}} + \mathcal{O}(h^{\frac{3}{2}}) - \mathcal{O}(h^{\frac{1}{2} + \rho}) \right) \|w_{h,j}\|_{L^2(\Omega)}^2. \quad (4.35)
\end{align*}
Noticing that $\sum_{j=1}^{N_h} \|w_{h,j}\|_{L^2(\Omega)}^2 = \|w_h\|_{L^2(\Omega)}^2,$ the following holds
\begin{align*}
\sum_{j=1}^{N_h} q_{h,a}(w_{h,j}) & \geq h \int_{\mathbb{R}} \int_{-\epsilon_h}^{\epsilon_h} \left( \beta_a + M_3(a) \kappa(s) h^{\frac{3}{2}} - \mathcal{O}(h^{\frac{3}{2}}) - \mathcal{O}(h^{\frac{1}{2} + \rho}) \right) |\tilde{w}_h|^2 (1 - t \kappa(s)) dt ds
\end{align*}
since $M_3(a) < 0,$ by Proposition 4.1, and $\kappa_j \leq \kappa(s)$ in the support of $w_{h,j},$ by (4.33). Inserting this into (4.32), we get
\begin{align*}
q_{h,a}(w_h) & \geq \int_{\mathbb{R}} \int_{-\epsilon_h}^{\epsilon_h} \left( \beta_a h + M_3(a) \kappa(s) h^{\frac{3}{2}} - \mathcal{O}(\max(h^{\frac{7}{2}}, h^{\frac{3}{2} + \rho}, h^{2-2\rho})) \right) |\tilde{w}_h|^2 dt ds.
\end{align*}
Now, by (4.31), we get
\begin{align*}
\lambda_1(P_{h,a}) & \geq \beta_a h + M_3(a) \kappa_{\text{max}} h^{\frac{3}{2}} - \mathcal{O}(\max(h^{\frac{7}{2}}, h^{\frac{3}{2} + \rho}, h^{2-2\rho})).
\end{align*}
Optimizing, we choose $\rho = \frac{1}{6}$ and get that the remainder is $\mathcal{O}(h^{\frac{5}{2}}).$

**Remark 4.7.** Let us introduce the potential
\[ U^\Gamma_{h,a}(x) = \begin{cases} 
|a|h & \text{if dist}(x, \Gamma) > 2h^{\frac{1}{2}} \\
\beta_a h + M_3(a) \kappa(s) h^{\frac{3}{2}} & \text{if dist}(x, \Gamma) < 2h^{\frac{1}{2}} & & \text{& } x = \Phi(s, t). 
\end{cases} \]
Then, repeating the foregoing proof (with $\rho = \frac{1}{6}$) on the Schrödinger operator
\[ P_{h,a} - U^\Gamma_{h,a}, \]
we get that its ground state energy satisfies
\[ \lambda(h, a, \Gamma) \geq -c h^{\frac{5}{2}} \]
for some positive constant $c.$ Therefore, we deduce that, for any $u \in H_0^1(\Omega),$ the following inequality holds
\begin{align*}
q_{h,a}(u) & \geq \int_{\Omega} \left( U^\Gamma_{h,a}(x) - c h^{\frac{5}{2}} \right) |u|^2 dx.
\end{align*}
(4.36)
The inequality in (4.36) yields that the ground states of $P_{h,a}$ are localized near the set of maximal magnetic edge curvature, $\Pi_\Gamma = \{ \kappa(s) = \kappa_{\text{max}}(s) \}.$ We omit the details and refer the reader to [10, Thm. 8.3.4].

### 4.4. Superconductivity along the magnetic edge.

The new estimate $\beta_a < \Theta_0$ in Theorem 1.1 gives an integrated description of the nucleation of superconductivity in type-II superconductors subject to magnetic steps fields with certain intensity, considered for instance in [2].

In the context of superconductivity, the set $\Omega$ introduced in Section 4.3 models the horizontal cross section of a cylindrical superconductor-sample, with a large characteristic parameter $\kappa$ and submitted to the magnetic field $HB_a,$ where $B_a$ is as in (4.17), $a \in (-1, 0),$ and the parameter $H > 0$ measures the intensity of the
magnetic field. The superconducting properties of the sample are described by the minimizing configurations of the following Ginzburg–Landau (GL) energy functional:

\[ \mathcal{E}_{\kappa,H}(\psi, \mathbf{A}) = \int_{\Omega} \left( |(\nabla - i\kappa \mathbf{A})\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right) dx + \kappa^2 H^2 \int_{\Omega} |	ext{curl} \mathbf{A} - B_n|^2 dx, \]

where \( \psi \in H^1(\Omega; \mathbb{C}) \) is the order parameter, and \( \mathbf{A} \in H^1(\Omega; \mathbb{R}^2) \) is the induced magnetic field. For a fixed \((\kappa, H)\), the infimum of the energy–the ground state energy–is attained by a minimizer \((\psi^{\text{GL}}, \mathbf{A}^{\text{GL}})_{\kappa,H}\).

In [2], the limit profile of \(|\psi^{\text{GL}}|^4\) is determined in the sense of distributions in the regime where \( H = b\kappa \) and \( \kappa \to +\infty \), with \( b > \frac{1}{|a|} \) a fixed constant. More precisely, the following convergence holds

\[ \kappa \mathcal{T}_{\kappa}^b \to \mathcal{T}^b \text{ in } \mathcal{D}'(\mathbb{R}^2), \text{ as } \kappa \to +\infty, \]

where

\[ C^\infty_c(\mathbb{R}^2) \ni \varphi \mapsto \mathcal{T}_{\kappa}^b(\varphi) = \int_{\Omega} |\psi^{\text{GL}}|^4 \varphi \, dx \]

and the limit distribution \( \mathcal{T}^b \) is defined via three distributions related to the edges \( \Gamma, \Gamma_1 = (\partial\Omega_1) \cap (\partial\Omega) \) and \( \Gamma_2 = (\partial\Omega_2) \cap (\partial\Omega) \) as follows

\[ C^\infty_c(\mathbb{R}^2) \ni \varphi \mapsto \mathcal{T}^b(\varphi) = -2b^{-\frac{1}{2}} \left( \mathcal{T}^{b}_{\Gamma}(\varphi) + \mathcal{T}^{b}_{\Gamma_1}(\varphi) + \mathcal{T}^{b}_{\Gamma_2}(\varphi) \right), \]

with

\[ \mathcal{T}^{b}_{\Gamma}(\varphi) := \epsilon_a(b) \int_{\Gamma} \varphi \, ds_{\Gamma}, \quad \mathcal{T}^{b}_{\Gamma_1}(\varphi) = E_{\text{surf}}(b) \int_{\Gamma_1} \varphi \, ds \]

and

\[ \mathcal{T}^{b}_{\Gamma_2}(\varphi) = |a|^{-\frac{1}{2}} E_{\text{surf}}(b|a|) \int_{\Gamma_2} \varphi \, ds. \]

The effective energies \( \epsilon_a \) and \( E_{\text{surf}} \) correspond respectively to the contribution of the magnetic edge \( \Gamma \) and the boundary \( \partial\Omega \) (see [2, 6] for the precise definitions). They have the following properties:

- \( \epsilon_a(b) = 0 \) if and only if \( b \geq 1/\beta_a \).
- \( E_{\text{surf}}(b) = 0 \) if and only if \( b \geq 1/\Theta_0 \).

Based on the results above, a detailed discussion on the distribution of superconductivity near \( \Gamma \cup \partial\Omega \) has been done in [2, Section 1.5]. This discussion mainly relies on the order of the values \(|a|\Theta_0, \beta_a, \Theta_0 \). With the existing estimates in this paper (and [2]), we have

\[ |a|\Theta_0 < \beta_a < \min(\Theta_0, |a|) \text{ for } a \in (-1, 0). \]

Consequently, we observe that (see Fig 2 for illustration)

- \( \mathcal{T}^b = 0 \) for \( b \geq b_{\epsilon,3} := \frac{1}{|a|\Theta_0} \).
\[ T_{\Gamma_1}^b = T_{\Gamma}^b = 0 \quad \& \quad T_{\Gamma_2}^b \neq 0 \quad \text{for} \quad b_{c,2} := \frac{1}{\|a\|} \leq b < b_{c,3} ; \]

\[ T_{\Gamma_1}^b = 0, \quad T_{\Gamma_1}^b \neq 0 \quad \& \quad T_{\Gamma_2}^b \neq 0 \quad \text{for} \quad b_{c,1} := \max\left(\frac{1}{|\Theta_0|}, \frac{1}{\|a\|}\right) \leq b < b_{c,2} . \]

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