Compton light pressure and spectral imprint of relic radiation on cosmic electrons

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A fully QED/relativistic theory of light pressure of CMB radiation and Fokker-Planck equation for electron distribution combined with cosmologic relation for CMB temperature, T, yields analytic results for the evolution of the distribution over large span of time and energies. A strong imprint of CMB on electrons transpires via formation of “frozen non-equilibrium” state of electrons in current epoch, and possible existence of cutoff and narrow spectral lines as remnants of high-T sources.

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Beginning with its discovery [1], light pressure, in particular that by an isotropic Cosmic Microwave Background (CMB) [2] on charged particles, resulting in the loss of their momentum/energy [3] via inverse Compton scattering [4], played a substantial role in astrophysics and cosmology. It strongly affects high-energy baryons’ fast decay facilitated by the pion production and high-energy photons via secondary production of virtual pairs, imposing an upper limit [5] on cosmic rays energy. One would expect even stronger CMB interaction with electrons, since due to their fundamental nature, their momentum loss can be treated more thoroughly using the QED theory [6] of photon-electron scattering.

In this Letter, we show that very interesting effects in such an interaction can be elicited at both low- and high-energy domains. At low-energy, we predict the formation of a “frozen non-equilibrium” state of electrons as the universe expands (including present epoch): due to its cooling, CMB fails to enforce thermal equilibrium on electrons, and let them keep constant temperature for- ever (10 −20K). At high-energies (>EC ∼1014eV) we predict transformation of initial thermal electron spectra into narrow lines followed by a cutoff near E C. We limit our consideration here only to the momentum decay due to CMB and do not consider other evolution channels (such as e. g. synchrotron radiation due to galactic magnetic fields, secondary effects due to decay of protons, anisotropy fluctuations due to Sachs-Wolfe effect, etc).

Toolkit. A major tool here is a light pressure F by a black-body isotropic radiation (in particular, CMB) on an electron. In [7] we derived a general QED/relativistic formula for F based on Lorentz transformation of an arbitrary spectrum ρ(ε) of (dimensionless) photon energies ε = ℏω/m0c2 (here m0 is the rest mass of an electron), in a frame, L, where the radiation is isotropic, upon transition from that frames to the frame, R, where the particle is at rest. The theory is valid also for any energy dependence of a cross-section σ(ε) of scattering of an ω-photon at a particle. For a particular case of

(1) a black-body (Planck) radiation of an arbitrary temperature T, with its spectral, ρBB, and total energy, WBB = W C∫0∞ερBB(ε)dε in the L-frame being

\[ \rho_{BB}(\epsilon)d\epsilon = \frac{8\pi\epsilon^2d\epsilon}{e^{\epsilon/T} - 1}, \quad W_{BB} = \frac{8\pi^5}{15}W_C\theta^4 \] (1)

where \( \theta = k_B T/m_0c^2 \) is a dimensionless temperature (with \( T = T_0 \approx 2.725K \), and \( \theta = \theta_0 \approx 0.534 \times 10^{-9} \) for present CMB), \( k_B \) is the Boltzmann constant, \( W_C = m_0c^2/\lambda_C^3 \) is a “Compton energy density”, and \( \lambda_C = 2\pi\hbar/m_0c \) is the Compton wavelength.

(2) an electron as a scattering particle, with its cross-section, \( \sigma(\epsilon) \), described by Klein-Nishina theory [6] accounting for virtual electron-positron pair creation/annihilation in the 1-st order of \( \alpha = e^2/m_0c^2 \approx 1/137 \), for any \( \epsilon \), so \( \sigma \approx \sigma_0 = (8\pi/3)\sqrt{2} \) at \( \epsilon \ll 1 \) is the Thompson cross-section of electron, where \( \sigma_0 = e^2/m_0c^2 \) is the classical electron EM-radius, and \( \sigma = \sigma_0(3/8\epsilon)\ln(2\epsilon) + 1/2 \) at \( \epsilon \gg 1 \) in a Compton domain, we found a simple and precise analytic approximation [7]

\[ f(\mu, \theta) = \frac{d\mu}{dt} \approx \frac{\mu\theta^3}{q} \ln(1 + K_C); \quad K_C = \gamma \theta q \] (2)

where \( K_C \) is a “Compton factor”, \( q = 10.0 \) is a numerical fitting parameter, and \( t_C \) is a “Compton time scale”:

\[ t_C = 135\lambda_C /64\pi^4\alpha^2c \approx 3.2515 \times 10^{-18}s; \quad t_C \propto h^3 \] (3)

Eq. (2) remains true in the entire span of momenta, \( \mu \in (0, \mu_{pl}) \), where \( \mu_{pl} = k_B T_{pl}/m_0c^2 \approx 2.4 \times 10^{22} \) is the highest momentum in the universe related to the Planck temperature, \( T_{pl} \approx 1.417 \times 10^{32}K \). The Thompson domain corresponds to \( K_C \ll 1 \) (hence \( \theta \ll 1 \)), with

\[ f \approx -\mu\gamma \theta^4, \quad \gamma = \sqrt{1 + \mu^2} \] (4)

consistent with a well known result (see e. g. [8]); note that it is still good for relativistic case, \( |\mu| \sim \gamma \gg 1 \), as long as \( |\mu| \ll \theta^{-1} \). The Compton (QED) domain is defined by \( K_C \gg 1 \), and its threshold, \( K_C = 1 \), for present CMB corresponds to the energy \( E_C \sim 10^{14}eV \).

While Eq. (2) can be directly used to calculate the decay of momentum \( \mu(t) \) for a given initial condition (see below), the temporal evolution of electron distribution should be found from a Fokker-Planck equation for the
diffusion in the momentum space [9]. We define a distribution function, \( g^{(e)}(\mu, t) \) of electrons as the number of electrons per elements of solid angle \( d\Omega \), momentum, \( d\mu \), within a unity of coordinate space, and a density number, \( n^{(e)}(\mu, t) = 4\pi\mu^2 g^{(e)}(\mu, t) \), and note that in the expanding space/universe, we need to use an also expanding unity of coordinate space. Assuming then that (a) the electron distribution is isotropic, same as CMB, (b) the total number of electrons is invariant, \( \int_{-\infty}^{\infty} n^{(e)} d\mu = \text{inv} \), and (c) the thermal equilibrium of a relativistic gas at any \( \theta = \text{const} \) is due to the Maxwell-Jüttner (MJ) distribution [10],

\[
\rho^{(e)}_{\text{MJ}} = e^{-\gamma/\theta |\theta K_2(1/\theta)|^{-1}} \tag{5}
\]

where \( K_2 \) is the modified Bessel function of the second order, with MJ being a relativistic generalization of the Maxwell-Boltzmann (MB) distribution,

\[
\rho^{(e)}_{\text{MB}} = e^{-\mu^2/2\theta} \tag{6}
\]

we found [7] a Fokker-Planck equation for \( g^{(e)}(\mu, t) \), as

\[
\frac{\mu^2 \partial [g^{(e)}]}{\partial (t/T_C)} + \frac{\partial}{\partial \mu} \left\{ \mu^2 f(\mu, t) \left[ g^{(e)} + \theta(t) \frac{\gamma g^{(e)}}{\mu} \right] \right\} = 0 \tag{7}
\]

In non-relativistic case \( |\gamma| \approx 1 \) in Eq. (4)], it comes to

\[
\frac{\mu^2 \partial [g^{(e)}]}{\partial (t/T_C)} = \theta^4(t) \frac{\partial}{\partial \mu} \left\{ \mu^3 \left[ g^{(e)} + \theta(t) \frac{\mu g^{(e)}}{\mu} \right] \right\} \tag{8}
\]

Finally, when tackling the dynamics of CMB temperature, \( T \), due to universe expansion, we recall that it is related to the redshift \( z \) as \( T(t)/T_0 = 1 + z(t) \) (\( T_0 \) is a present value), and thus is governed by a standard cosmological relation [11,12]:

\[
dT/dt = -TH_0[\Omega_\Lambda + \Omega_M(T/T_0)^3 + \Omega_R(T/T_0)^4]^{1/2} \tag{9}
\]

where \( H_0 \) is a present Hubble constant (with \( H_0 \approx 4.414 \times 10^{-17} \) s being an approximate age of the universe), \( \Omega \)'s are the fractions of respective forms of energy in critical energy density (it is a common convention that our universe is flat, hence \( \Omega_\Lambda + \Omega_M + \Omega_R = 1 \)) with commonly accepted values \( \Omega_\Lambda \approx 0.7 \) (a vacuum energy density fraction, or cosmological (or dark energy) constant, a major contributor to the current rate of the universe expansion), and \( \Omega_R \approx 0.85 \times 10^{-4} \) – radiation, or relativistic fraction, dominant at the earliest stage of the universe, and non-relativistic, or “matter” fraction \( \Omega_M \approx 0.3 \).

**Frozen non-equilibrium.** How promptly an electron distribution equilibrates with changing CMB temperature, \( \theta(t) \)? To find this out, we compare time scales of both of them. That of CMB is roughly the age of universe, \( t_U \), at a given CMB temperature, \( \theta \), i.e. \( t_U(\theta) = \int_0^{\theta} 2d\theta/(d\theta/dt) \). Using Eq. (2), we evaluate the time scale as inverse momentum decay rate, \( t_\mu = \mu/[d\mu/dt] \), at the peak of distribution \( \rho^{(e)}(\mu, t) \) for a given equilibrium, and then solve the equation \( t_U(\theta) = t_\mu(\theta) \) for a split-point \( \theta = \theta_{\text{spl}} \) numerically. With \( \Omega_M >> \Omega_R \), we found then that \( T_{\text{spl}}/T_0 \ll 10 \), which is consistent with detailed calculations, Fig. 2, and the split occurred at \( t \ll 0.05 H_0^{-1} \). The main point here is that it falls far within Thompson domain, \( \theta_{\text{spl}} \lesssim 0.5 \times 10^{-8} \ll 1 \), so that the electron kinetics could be described by classical Eq. (8).

In the earlier epoch, the thermalization of electrons happened almost instantaneously, so that their distribution is described by Eq. (5) and (6) with the temperature, \( \theta(t) \), of this distribution following almost exactly the CMB temperature, \( \theta(t) \). To investigate what happened after they start diverging near \( \theta = \theta_{\text{spl}} \), we need to solve Eq. (8) with an initial condition given by MB-distribution (6) at any point \( 1 \gg \theta \gg \theta_{\text{spl}} \). Most luckily, that partial derivative equation happens to have an exponential MB-distribution Eq. (6) as a self-similar solution [13], where the temperature \( \theta \) has to be replaced by an electron temperature, \( \theta_e(t) \), as yet unknown function of time, and thus Eq. (8) can be reduced to an ordinary differential equation for \( \theta_e(t) \), where the CMB temperature, \( \theta(t) \), could still be an arbitrary function of time:

\[
d\theta_e/dt = -2\theta^4(t)[\theta_e(t) - \theta(t)]/t_C \tag{10}
\]

Eqs. (9) and (10) can now be used to solve the dynamics of both \( \theta = k_B T/m_0 c^2 \) and \( \theta_e = k_B T_e/m_0 c^2 \). It suffices, however, to find \( T_e \) as function of \( T \); eliminating the time \( t \) by dividing Eq. (10) by (9), we get then a single equation in the phase space of \( \Theta = T/T_0 \), \( \theta_e/T_0 \) as:

\[
d\theta_e/d\Theta = \chi \Theta^3(\Theta - \Theta) / (\Omega_\Lambda + \Omega_M \Theta^3)^{1/2} \tag{11}
\]

where we dropped the term \( \Omega_\Lambda \Theta^4 \), which is negligible at \( \Theta \lesssim 10^2 \), and introduced a “QM+cosmic” parameter

\[
\chi = 2(k_B T_0/m_0 c^2)^3/t_C H_0 \approx 1.21 \times 10^{-2} \approx 5\alpha/3 \tag{12}
\]

Note that with known precision of \( H_0 \), \( \chi \) is well approximated by \( 5\alpha/3 \); it would be surprising and revealing if that is not a chance coincidence. The boundary condition for the solution of Eq. (11) is \( (\Theta_e - \Theta) \to 0 \) at \( \Theta \to \infty \).

The numerical solutions of Eq. (11) are depicted at Fig. 1. They clearly show that below \( \Theta \approx 20 \), CMB has “dropped the ball” and cannot enforce thermal equilibrium on cosmic electrons, whose temperature got eventually frozen at some non-equilibrium level (\( T_{\text{inf}} = 16.3 K \) for \( \Omega_\Lambda = 0.7 \)) till “the end of time”. This brings up a new facet to the issue of “heat death” of the universe. [Note, however, that by our definition of the density number \( \rho^{(e)} \) the spacing between electrons increases as \( \Theta^{-1}(t) \).]

This frozen state is fully developed by the present day, regardless of specific values of \( \Omega \)'s in Eqs. (9) or (11). A good analytical approximation for \( T_{\text{inf}} \) and the solution of Eq. (11) for various \( \Omega \)'s is found as [14]

\[
\Theta_e = \left( \Theta_0^{5/2} + \Theta_{\text{inf}}^{5/2} \right)^{2/5} \tag{13}
\]

\[
\Theta_{\text{inf}} = \frac{T_{\text{inf}}}{T_0} = \left( \frac{15(\Omega_\Lambda + 3\chi)}{10} \right)^{2/5} \tag{14}
\]
Thus conceivable measurements of $T_\infty$ in deep space may offer an alternative way to evaluate $\Omega_\Lambda \approx 1 - \Omega_M$. 

**Narrow lines and cutoff in cosmic electron spectra?** At the opposite, high-energy end of electron spectrum, it could be expected that, similarly to baryons, CMB might strongly affect it, albeit due to different mechanism, and do it on a much faster time-scale, so we can even assume $\theta = \text{const} = \theta_0 \ll 1$. To illustrate that, we consider the dynamics of the momentum $\mu(t)$ whose implicit solution for a given $\theta$, is provided by $t(\mu) = t_C \int d\mu/f(\mu, \theta)$, Eq. (2). The integration here can be done numerically, yet to gain the insights provided by analytical results, it would be nice to have a “good” model function $f_M$ that is very close to the one in Eq. (2) in the domain of interest, and at that has (a) an analytical integrability of $\int d\mu/f$, and (b) explicit “reversibility” of resulting functions $t(\mu) \leftrightarrow \mu(t)$. For $K_C \ll 1$, Eq. (4) satisfies these conditions and is fully integrable [7]. But to cover both the upper (and largest) part of Thompson domain, $\mu \gg 1$, and at the same time – the entire immensely larger, Compton domain, $K_C > 1$, another greatly useful interpolation model is found as

$$f_M(\mu, \theta) = -\left(\frac{\theta^3}{y}\right) y (1 + \ln y) \left\{1 + \ln (1 + \ln y) \right\}^{-2}$$

\begin{equation}
\text{(15)}
\end{equation}

where $y = 1 + \mu/\mu_C$ with $\mu_C = 1/g\theta \gg 1, \mu > 1$. For $\theta = \theta_0$, we have $|f - f_M/f| < 0.01$, for any $\mu > 7$. At $1 \ll \mu \ll \mu_C$, Eq. (15) yields $f_M \approx -\theta^4\mu^2$, which is consistent with Eq. (3) at $\mu \approx \gamma \gg 1, \mu \gg 1$ for relativistic case. Yet this is more than enough if $\theta \ll 1$ by insuring that momentum decay can be continually traced from far Compton to low Thompson domains. Thus Eqs. (4) and (15) smoothly cover the entire span $\mu \in (0, \mu_{in})$, as their areas of validity overlap by orders of magnitude in $\mu$ if $\theta \ll 1$. The momentum decay from initial $\mu = \mu_{in}$ at $\tau = 0$, via integration $d\mu/d\tau = f_M/t_C$, $\theta = \text{const}$ is

\begin{equation}
\frac{\mu(\tau)}{\mu_C} = \exp\left\{\exp\left\{\frac{(\tau_0 + \tau)^2}{4} + 1 - \frac{\tau_0 + \tau}{2}\right\} - 1\right\} - 1
\end{equation}

\begin{equation}
\text{(16)}
\end{equation}

where $\tau = (\theta^3/q)t/t_C$ and

$$\tau_0 = s^{-1} - s \quad \text{with} \quad s = \ln[1 + \ln(1 + \mu_{in}/\mu_C)]$$

\begin{equation}
\text{(17)}
\end{equation}

For $K_C \approx \mu_{in}/\mu_C = 2 \times 10^5$ (or initial energy $E_{in} \sim 2 \times 10^{19} \text{eV}$ slightly below the highest particle energy $\sim 5 \times 10^{19} \text{eV}$, observed in cosmic rays [15]), $\mu(\tau)$ is depicted in Fig. 2, curve 1. The time $\int f_{\mu_C}^\mu d\mu/f_M$ or an electron to lose about $2 \times 10^5$ of its momentum during the “Compton phase” $\mu_{in} \rightarrow \mu_C$ is $\Delta t_C \sim 5.3 \times 10^1 s$, which is by 6 orders of magnitude shorter than the age of universe (and thus justifies our assumption of $\theta \approx \text{const}$), whereas immediately after that, within the same period, $\mu$ loses much less than a factor of magnitude. (For $E_{in} < 5 \times 10^{19} \text{eV}$, this time is even shorter.) As $\mu$ keeps decaying from $\mu_{in}$ down to a relativistic threshold, $\mu = 1$, its dynamics slows down tremendously, down to a frozen non-equilibrium at lower $\mu$.

These results call for the study of the evolution of electron spectra at the energies far exceeding that of equilibrium. At that, the last term in a Fokker-Planck Eq. (7) can be omitted since $\theta_0 \ll \theta_{in}$, so that in terms of number density $\rho^{(e)} \propto \mu^2 g^{(e)}$, it can be reduced to

\begin{equation}
t_C \partial \rho^{(e)}/\partial t + \partial (f \rho^{(e)})/\partial \mu = 0
\end{equation}

\begin{equation}
\text{(18)}
\end{equation}

which is essentially a continuity-like equation. Again, it is fully integrable, and its general solution is

$$\rho^{(e)} = \Phi(\xi - t)/f(\mu), \quad \text{with} \quad \xi = t_C \int d\mu/f$$

\begin{equation}
\text{(19)}
\end{equation}

where $\Phi(x)$ is an arbitrary function of $x$ defined here by initial conditions, e. g. a MJ-distribution with $\theta_{in} \gg 1$. A resulting analytic solution for $\rho^{(e)}(\mu, \tau)$ with $f = f_M$, Eq. (15), for $\rho^{(e)}$ vs $\mu$ for various $\tau = (\theta^3/q)t/t_C$ is plotted in Fig. 3 for initial temperature, $\theta_{in} = 10^5 \mu_C$. 

![FIG. 1: Normalized temperature of electrons $T_e/T_0$, vs that of CMB, $T/T_0$, for various values of cosmological constant, $\Omega_\Lambda$, and asymptotic electron temperatures, $T_e$, at $t \rightarrow \infty$. A star marks the data for a commonly accepted model [11,12].](image1)

![FIG. 2: Normalized momentum, $\mu/\mu_C$, and position of the peak of density distribution (curve 1), and the peak intensity of the distribution, $\rho_{pk}/(\rho_{pk})_{in}$, vs normalized time, $\tau$ (curve 2). Dashed lines – respective asymptotics for $\tau \gg 1$.](image2)
FIG. 3: Evolution of normalized density distribution of electrons, \( \rho^{(\infty)}(\mu)/\rho^{(0)}(\mu) \) in normalized time, \( \tau \), beginning with the initial, Maxwell-Jüttner (MJ) distribution at \( k_B T_{in} = 10^{19} \text{eV} \), from \( \tau = 0 \) to \( \tau = 9 \) (\( \Delta \tau = 1 \rightarrow \Delta t \approx 2 \times 10^{13} \text{s} \)).

or \( k_B T_{in} = 10^{19} \text{eV} \). A curve at \( \tau = 0 \) depicts an initial MJ-distribution, \( \rho^{(\infty)}_{in}(\mu) \propto \mu^2 e^{-\mu/\theta_{in}} \), which peaks at \( \mu = 2\theta_{in} \), i.e., \( E_{in} = 2 \times 10^{19} \text{eV} \), same as for a single electron example. A transient peak at \( \mu = \mu_{pk} \) moves fast in the beginning, but slows down tremendously as it reaches \( \mu_c(\theta) \). Its motion coincides with the timeline of a single electron with \( \mu_{in} = 2\theta_{in} \), see curve 1 in Fig. 2, whereas its intensity \( \rho_{pk}(\tau) \), curve 2 in Fig. 2, goes up orders of magnitude higher than that of the initial MJ-distribution; at \( \mu < \mu_c \), \( \rho_{pk} \propto \tau^2 \). Its width narrows down respectively, \( \Delta \mu / \rho_{pk} \propto \tau^{-2} \) so that for e.g. \( \mu = 20 \) (\( E = 10 \text{MeV} \)), it reaches \( \Delta E \approx 2K \text{eV} \) i.e. \( \Delta \mu / \rho_{pk} \sim 2 \times 10^{-4} \), compared to the initial relative width \( \Delta \mu_{in} / \mu_{in} \sim 1.4 \). Notice that even before strong line-narrowing, there is a sharp cutoff at the upper part of the spectrum. This collapse and cutoff are due to a "pile-up" effect, whereby a leading downward front moves slower than a trailing one, resulting in the line squeezing; it is reminiscent of a shock precursor formation in astrophysics [16] and Coulomb explosion [17].

These lines would indicate signals from far and hot sources; most likely they will be very weak. Their detection may necessitate the development of high-resolution spectral techniques. More detailed study may need expanding Eqs. (7) and (18) into anisotropic F-P equations for data analysis. The averaging over many sources is expected however to be isotropic, although the observed line might be broaden up similarly to the inhomogeneous line broadening in laser physics [18]. Another major common feature to search for in these spectra is a sharp cutoff near the Compton threshold, \( E_C \sim 10^{19} \text{eV} \).

In conclusion, we showed that a diminished light pressure on electrons by CMB and ensuing low rate of their energy decay should result in the formation of their frozen non-equilibrium state of \( T \sim 10 - 20K \) as the universe expands long before the current epoch. We also predicted the implosion of high-\( T \) sources electron spectra into narrow lines and cutoff formation due to pile-up effect.

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[13] In fact, Eq. (8) can be solved the same way for an arbitrary initial function \( \rho^{(n)}_0(\mu) \), by decomposing it into exponential components via Laplace transformation, and looking for the solutions of Eq. (10) with initial magnitude \( \theta_{in} \) different for each of the components.
[14] If one of \( \Omega \)'s in Eq. (11) is unity (hence the other is zero), Eq. (11) can be integrated in close form in terms of \( \Gamma \)-function. In particular, for \( \Omega = 1 \) (and thus the lowest possible temperature \( T_{\infty} \)), its exact solution is \( \Theta = \Theta + (1/4)(1/\chi)^{1/4}\exp(\chi \theta/4) \Gamma(1/4, \chi \theta^4/4) \), where \( \Gamma(a, x) \) is an upper incomplete \( \Gamma \)-function, and \( T_{\infty} / T_0 = (64x)^{1/4} \Gamma(1/4) \approx 3.86 \), where \( \Gamma(a) = \Gamma(a, 0) \) is a complete \( \Gamma \)-function. Thus \( T_{\infty} \approx 10.52K \), which is consistent with the numeric calculations, Fig. 1, and Eq. (14).
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