Lectures on Strings and Dualities

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In this set of lectures I review recent developments in string theory emphasizing their non-perturbative aspects and their recently discovered duality symmetries. The goal of the lectures is to make the recent exciting developments in string theory accessible to those with no previous background in string theory who wish to join the research effort in this area. Topics covered include a brief review of string theory, its compactifications, solitons and D-branes, black hole entropy and web of string dualities. (Lectures presented at ICTP summer school, June 1996)
In this series of lectures I attempt to give an overview of a subject roughly characterized as string duality. String theory is by now a vast subject with almost three decades of active research contributing to its development. The past couple of years have seen a tremendous growth in our understanding of string theory, due to a better understanding of its non-perturbative properties. These developments have been so rapid that for a beginner it may appear an impossible dream to join active research in this exciting field of physics. Here I hope to make a modest effort towards bringing these highly advanced aspects of string theory to a wider audience of physicists which have no previous background in string theory. Perhaps this will make the beginners feel less shy about joining this very exciting area of fundamental physics.

It is rather difficult (and probably impossible) to give an overview of string theory and all the recent developments in the relatively short space (and time!) available here. My aim has been to present here some of the major highlights with as much background as is needed to appreciate their importance. I hope that the interested reader will continue by reading more in depth discussions available in the literature. Luckily there have been many excellent recent review articles in this field [1] to which I refer the interested reader to for different emphasis on different aspects of string theory dualities.

The organization of this paper is as follows: In section one I make a lightning review of string theories. In section two I discuss compactification of string theories (and in particular on tori, K3 manifold and Calabi-Yau threefolds). In section three I discuss aspects of solitonic states in string theory. In section four I discuss certain implications of the existence of solitons for non-perturbative aspects of string theory and entropy of extremal black holes. In section five I discuss the web of string dualities. In section six I end with some speculations and open questions.

I have decided to put no references in the main text (in the spirit of giving an ‘informal’ talk). Apart from the above mentioned review articles and references therein, let me list some sample references for each section covered which by no means is complete nor necessarily the original works. For sections one and two a related reference is [2][3] and for K3 compactifications [4]. For section three see, for example, [5][6][7]. For section four see, for example, [8][9][10][11] and for stringy aspects of extremal black holes see [12][13]. For section five see [14][15][16][17][18][19][20].
1. Lightning Review of String Theory

In this section we will give a brief overview of string theory. The interested reader is strongly advised to study this subject more thoroughly than presented here. String theory is a description of dynamics of objects with one spatial direction, which we parameterize by $\sigma$, propagating in a space parameterized by $X^\mu$. The worldsheet of the string is parameterized by coordinates $(\sigma, \tau)$ where each $\tau = \text{constant}$ denotes the string at a given time. The amplitude for propagation of a string from an initial configuration to a final one is given by sum over worldsheets which interpolate between the two string configurations weighed by $\exp(iS)$ where

$$S = \int d\sigma d\tau \partial_i X^\mu \partial_i X^\nu G_{\mu\nu}(X)$$

where $G_{\mu\nu}$ is the metric on spacetime and $i$ runs over the $\sigma$ and $\tau$ directions. Note that by slicing the worldsheet we will get configurations where a single string splits to a pair or vice versa, and combinations thereof (consider for example the worldsheet configuration which looks like a “pant”).

If we consider propagation in flat spacetime where $G_{\mu\nu} = \eta_{\mu\nu}$ the fields $X^\mu$ on the worldsheet, which describe the position in spacetime of each bit of string, are free fields and satisfy the 2d equation

$$\partial_i \partial^i X^\mu = \left(\partial_\tau^2 - \partial_\sigma^2\right)X^\mu = 0$$

The solution of which is given by

$$X^\mu(\sigma, \tau) = X^\mu_L(\tau + \sigma) + X^\mu_R(\tau - \sigma)$$

In particular notice that the left- and right-moving degrees of freedom are essentially independent. There are two basic types of strings: Closed strings and Open strings depending on whether the string is a closed circle or an open interval respectively. If we are dealing with closed strings the left- and right-moving degrees of freedom remain essentially independent but if are dealing with open strings the left-moving modes reflecting off the left boundary become the right-moving modes--thus the left- and right-moving modes are essentially identical in this case. In this sense an open string has ‘half’ the degrees of freedom of a closed string and can be viewed as a ‘folding’ of a closed string so that it looks like an interval.

There are two basic types of string theories, bosonic and fermionic. What distinguishes bosonic and fermionic strings is the existence of supersymmetry on the worldsheet. This
means that in addition to the coordinates $X^\mu$ we also have anti-commuting fermionic coordinates $\psi^\mu_{L,R}$ which are spacetime vectors but fermionic spinors on the worldsheet whose chirality is denoted by subscript $L, R$. The action for superstrings takes the form

$$S = \int \partial_L X^\mu \partial_R X^\mu + \psi_R^\mu \partial_L \psi_R^\mu + \psi_L^\mu \partial_R \psi_R^\mu.$$  

There are two consistent boundary conditions on each of the fermions, periodic (Ramond sector) or anti-periodic (Neveu-Schwarz sector) (note that the coordinate $\sigma$ is periodic).

A natural question arises as to what metric we should put on the worldsheet. In the above we have taken it to be flat. However in principle there is one degree of freedom that a metric can have in two dimensions. This is because it is a $2 \times 2$ symmetric matrix (3 degrees of freedom) which is defined up to arbitrary reparametrization of 2d spacetime (2 degrees of freedom) leaving us with one function. Locally we can take the 2d metric $g$ to be conformally flat

$$g_{ij} = \exp(\phi) \eta_{ij}.$$  

Classically the action $S$ does not depend on $\phi$. This is easily seen by noting that the properly coordinate invariant action density goes as $\sqrt{gg^{ij} \partial_i X \partial_j X}$ and is independent of $\phi$ only in $d = 2$. This is rather nice and means that we can ignore all the local dynamics associated with gravity on the worldsheet. This case is what is known as the critical string case which is the case of most interest. It turns out that this independence from the local dynamics of the worldsheet metric survives quantum corrections only when the dimension of space is 26 in the case of bosonic strings and 10 for fermionic or superstrings. Each string can be in a specific vibrational mode which gives rise to a particle. To describe the totality of such particles it is convenient to go to ‘light-cone’ gauge. Roughly speaking this means that we take into account that string vibration along their worldsheet is not physical. In particular for bosonic string the vibrational modes exist only in 24 transverse directions and for superstrings they exist in 8 transverse directions.

Solving the free field equations for $X, \psi$ we have

$$\partial_L X^\mu = \sum_n \alpha^\mu_{-n} e^{-in(\tau+\sigma)}$$

$$\psi^\mu_L = \sum_n \psi^\mu_{-n} e^{-in(\tau+\sigma)}$$

and similarly for right-moving oscillator modes $\tilde{\alpha}^\mu_{-n}$ and $\tilde{\psi}^\mu_{-n}$. The sum over $n$ in the above runs over integers for the $\alpha_{-n}$. For fermions depending on whether we are in the R
sector or NS sector it runs over integers or integers shifted by $\frac{1}{2}$ respectively. Many things decouple between the left- and right-movers in the construction of a single string Hilbert space and we sometimes talk only about one of them. For the open string Fock space the left- and right-movers mix as mentioned before, and we simply get one copy of the above oscillators.

A special role is played by the zero modes of the oscillators. For the $X$-fields they correspond to the center of mass motion and thus $\alpha_0$ gets identified with the left-moving momentum of the center of mass. In particular we have for the center of mass

$$X = \alpha_0(\tau + \sigma) + \tilde{\alpha}_0(\tau - \sigma)$$

where we identify

$$(\alpha_0, \tilde{\alpha}_0) = (P_L, P_R)$$

Note that for closed string, periodicity of $X$ in $\sigma$ requires that $P_L = P_R = P$ which we identify with the center of mass momentum of the string.

In quantizing the fields on the strings we use the usual commutation relations

$$[\alpha^\mu_n, \alpha^\nu_m] = n\delta_{m+n,0}\eta^{\mu\nu}$$

$$[\psi^\mu_n, \psi^\nu_m] = \eta^{\mu\nu}\delta_{m+n,0}$$

We choose the negative moded oscillators as creation operators. In constructing the Fock space we have to pay special attention to the zero modes. The zero modes of $\alpha$ should be diagonal in the Fock space and we identify their eigenvalue with momentum. For $\psi$ in the NS sector there is no zero mode so there is no subtlety in construction of the Hilbert space. For the $R$ sector, we have zero modes. In this case the zero modes form a Clifford algebra

$$[\psi^\mu_0, \psi^\nu_0] = \eta^{\mu\nu}.$$ 

This implies that in these cases the ground state is a spinor representation of the Lorentz group. Thus a typical element in the fock space looks like

$$\alpha^L_{-m_1}...\psi^L_{-n_k}|P_L, a\rangle \otimes \alpha^R_{-m_1}...\psi^R_{-m_r}|P_R, b\rangle$$

where $a, b$ label spinor states for R sectors and are absent in the NS case; moreover for the bosonic string we only have the left and right bosonic oscillators.
It is convenient to define the total oscillator number as sum of the negative oscillator numbers, for left- and right- movers separately. \( N_L = n_1 + \ldots + n_k + \ldots, N_R = m_1 + \ldots + m_r + \ldots \). The condition that the two dimensional gravity decouple implies that the energy momentum tensor annihilate the physical states. The trace of the energy momentum tensor is zero here (and in all compactifications of string theory) and so we have two independent components which can be identified with the left- and right- moving hamiltonians \( H_{L,R} \) and the physical states condition requires that

\[
H_L = N_L + \frac{1}{2}P_L^2 - \delta_L = 0 = H_R = N_R + \frac{1}{2}P_R^2 - \delta_R
\]

(1.2)

where \( \delta_{L,R} \) are normal ordering constants which depend on which string theory and which sector we are dealing with. For bosonic string \( \delta = 1 \), for superstrings we have two cases: For \( NS \) sector \( \delta = \frac{1}{2} \) and for the \( R \) sector \( \delta = 0 \). The equations (1.2) give the spectrum of particles in the string perturbation theory. Note that \( P_L^2 = P_R^2 = -m^2 \) and so we see that \( m^2 \) grows linearly with the oscillator number \( N \), up to a shift:

\[
\frac{1}{2}m^2 = N_L - \delta_L = N_R - \delta_R
\]

(1.3)

The number of states at oscillator number \( N \) grows as \( \exp(\sqrt{cN}) \) for some \( c \), due to a large number of ways we can get states at level \( N \) by using oscillators. Since \( m \propto \sqrt{N} \) we learn that the number of states for large mass \( m \) grows as \( \exp(\beta_H m) \) (for some constant \( \beta_H \)). We thus have a Hagedorn behaviour and a phase transition is expected at temperature \( T = 1/\beta_H \).

Let us analyze the low lying states for bosonic and superstrings.

1.1. Low Lying States of Bosonic Strings

Let us consider the left-mover excitations. Since \( \delta = 1 \) for bosonic string, (1.3) implies that if we do not use any string oscillations, the ground state is tachyonic \( \frac{1}{2}m^2 = -1 \). This clearly implies that bosonic string by itself is not a good starting point for perturbation theory. Nevertheless in anticipation of a modified appearance of bosonic strings in the context of heterotic strings, let us continue to the next state.

If we consider oscillator number \( N_L = 1 \), from (1.3) we learn that excitation is massless. Putting the right-movers together with it, we find that it is given by

\[
\alpha_{-1}^{\mu} \tilde{\alpha}_{-1}^{\nu} |P\rangle
\]
What is the physical interpretation of these massless states? The most reliable method is to find how they transform under the little group for massless states which in this case is $SO(24)$. If we go to the light cone gauge, and count the physical states, which roughly speaking means taking the indices $\mu$ to go over spatial directions transverse to a null vector, we can easily deduce the content of states. By decomposing the above massless state under the little group of $SO(24)$, we find that we have symmetric traceless tensor, anti-symmetric 2-tensor, and the trace, which we identify as arising from 26 dimensional fields

$$g_{\mu\nu}, B_{\mu\nu}, \phi$$

the metric, the anti-symmetric field $B$ and the dilaton. This triple of fields should be viewed as the stringy multiplet for gravity. The quantity $\lambda = \exp[-\phi]$ is identified with the the string coupling constant. What this means is that a worldsheet configuration of a string which sweeps a genus $g$ curve, which should be viewed as $g$-th loop correction for string theory, will be weighed by $\exp(-2(g-1)\phi) = \lambda^{2g-2}$. The existence of the field $B$ can also be understood (and in some sense predicted) rather easily. If we have a point particle it is natural to have it charged under a gauge field, which introduces a term $\exp(i \int A)$ along the worldline. For strings the natural generalization of this requires an anti-symmetric 2-form to integrate over the worldsheet, and so we say that the strings are charged under $B_{\mu\nu}$ and that the amplitude for a worldseet configuration will have an extra factor of $\exp(i \int B)$.

Since bosonic string has tachyons we do not know how to make sense of that theory by itself.

1.2. Low Lying States of Type II Superstrings

Let us now consider the light particle states for superstrings. We recall from the above discussion that there are two sectors to consider, NS and R, separately for the left- and the right-movers. As usual we will first treat the left- and right-moving sectors separately and then combine them at the end. Let us consider the NS sector for left-movers. Then the formula for masses (1.2) implies that the ground state is tachyonic with $\frac{1}{2}m^2 = \frac{-1}{2}$. The first excited states from the left-mover are massless and corresponds to $\psi^{\mu}_{\frac{1}{2}1/2}|0\rangle$, and so is a vector in spacetime. How do we deal with the tachyons? It turns out that summing over the boundary conditions of fermions on the worldsheet amounts to keeping the states with a fixed fermion number $(-1)^F$ on the worldsheet. Since in the NS sector the number of
fermionic oscillator correlates with the integrality/half-integrality of \( N \), it turns out that the consistent choice involves keeping only the \( N = \text{half-integral} \) states. This is known as the GSO projection. Thus the tachyon is projected out and the lightest left-moving state is a massless vector.

For the R-sector using (1.2) we see that the ground states are massless. As discussed above, quantizing the zero modes of fermions implies that they are spinors. Moreover GSO projection, which is projection on a definite \((-1)^F\) state, amounts to projecting to spinors of a given chirality. So after GSO projection we get a massless spinor of a definite chirality. Let us denote the spinor of one chirality by \( s \) and the other one by \( s' \).

Now let us combine the left- and right-moving sectors together. Here we run into two distinct possibilities: A) The GSO projections on the left- and right-movers are different and lead in the R sector to ground states with different chirality. B) The GSO projections on the left- and right-movers are the same and lead in the R sector to ground states with the same chirality. The first case is known as type IIA superstring and the second one as type IIB. Let us see what kind of massless modes we get for either of them. From \( \text{NS} \otimes \text{NS} \) we find for both type IIA,B

\[
\text{NS} \otimes \text{NS} \rightarrow v \otimes v \rightarrow (g_{\mu \nu}, B_{\mu, \nu}, \phi)
\]

From the \( \text{NS} \otimes R \) and \( R \otimes \text{NS} \) we get the fermions of the theory (including the gravitinos). However the IIA and IIB differ in that the gravitinos of IIB are of the same chirality, whereas for IIA they are of the opposite chirality. This implies that IIB is a chiral theory whereas IIA is non-chiral. Let us move to the \( R \otimes R \) sector. We find

\[
\text{IIA} : R \otimes R = s \otimes s' \rightarrow (A_\mu, C_{\mu \nu \rho})
\]

\[
\text{IIB} : R \otimes R = s \otimes s \rightarrow (\chi, B'_{\mu \nu}, D_{\mu \nu \rho \lambda}) \quad (1.5)
\]

where all the tensors appearing above are fully antisymmetric. Moreover \( D_{\mu \nu \rho \lambda} \) has a self-dual field strength \( F = dD = *F \). It turns out that to write the equations of motion in a unified way it is convenient to consider a generalized gauge fields \( A \) and \( B \) in the IIA and IIB case respectively by adding all the fields in the RR sector together with the following properties: i) \( A(B) \) involve all the odd (even) dimensional antisymmetric fields. ii) the equation of motion is \( dA = *dA \). In the case of all fields (except \( D_{\mu \nu \lambda \rho} \)) this equation allows us to solve for the forms with degrees bigger than 4 in terms of the lower ones and moreover it implies the field equation \( d * dA = 0 \) which is the familiar field equation for the gauge fields. In the case of the \( D \)-field it simply gives that its field strength is self-dual.
1.3. Open Superstring: Type I string

In the case of type IIB theory in 10 dimensions, we note that the left- and right-moving degrees of freedom on the worldsheet are the same. In this case we can ‘mod out’ by a reflection symmetry on the string; this means keeping only the states in the full Hilbert space which are invariant under the left-/right-moving exchange of quantum numbers. This is simply projecting the hilbert space onto the invariant subspace of the projection operator \( P = \frac{1}{2}(1 + \Omega) \) where \( \Omega \) exchanges left- and right-movers. \( \Omega \) is known as the orientifold operation as it reverses the orientation on the worldsheet. Note that this symmetry only exists for IIB and not for IIA theory (unless we accompany it with a parity reflection in spacetime). Let us see which bosonic states we will be left with after this projection. From the NS-NS sector \( B_{\mu\nu} \) is odd and projected out and thus we are left with the symmetric parts of the tensor product

\[
NS - NS \rightarrow (v \otimes v)_{symm.} = (g_{\mu\nu}, \phi)
\]

From the R-R sector since the degrees of freedom are fermionic from each sector we get, when exchanging left- and right-movers an extra minus sign which thus means we have to keep anti-symmetric parts of the tensor product

\[
R - R \rightarrow (s \otimes s)_{anti-symm.} = \tilde{B}_{\mu\nu}
\]

This is not the end of the story, however. In order to make the theory consistent we need to introduce a new sector in this theory involving open strings. This comes about from the fact that in the R-R sector there actually is a 10 form gauge potential which has no propagating degree of freedom, but acquires a tadpole. Introduction of a suitable open string sector cancels this tadpole.

As noted before the construction of open string sector Hilbert space proceeds as in the closed string case, but now, the left-moving and right-moving modes become indistinguishable due to reflection off the boundaries of open string. We thus get only one copy of the oscillators. Moreover we can associate ‘Chan-Paton’ factors to the boundaries of open string (very much as in the picture for mesons made of quark-antiquark pairs connected by flux lines). To cancel the tadpole it turns out that we need 32 Chan-Paton labels on each end. We still have two sectors corresponding to the NS and R sectors. The NS sector gives a vector field \( A_\mu \) and the R sector gives the gaugino. The gauge field \( A_\mu \) has two additional labels coming from the end points of the open string and it turns out that the
left-right exchange projection of the type IIB theory translates to keeping the antisymmetric component of $A_\mu = -A^T_\mu$, which means we have an adjoint of $SO(32)$. Thus all put together, the bosonic degrees of freedom are

$$(g_{\mu\nu}, \tilde{B}_{\mu\nu}, \phi) + (A_\mu)_{SO(32)}$$

We should keep in mind here that $\tilde{B}$ came not from the NS-NS sector, but from the R-R sector.

1.4. Heterotic Strings

Heterotic string is a combination of bosonic string and superstring, where roughly speaking the left-moving degrees of freedom are as in the bosonic string and the right-moving degrees of freedom are as in the superstring. It is clear that this makes sense for the construction of the states because the left- and right-moving sectors hardly talk with each other. This is almost true, however they are linked together by the zero modes of the bosonic oscillators which give rise to momenta $(P_L, P_R)$. Previously we had $P_L = P_R$ but now this cannot be the case because $P_L$ is 26 dimensional but $P_R$ is 10 dimensional. It is natural to decompose $P_L$ to a $10+16$ dimensional vectors, where we identify the 10 dimensional part of it with $P_R$. It turns out that for the consistency of the theory the extra 16 dimensional component should belong to the root lattice of $E_8 \times E_8$ or a $Z_2$ sublattice of $SO(32)$ weight lattice. In either of these two cases the vectors in the lattice with $(\text{length})^2 = 2$ are in one to one correspondence with non-zero weights in the adjoint of $E_8 \times E_8$ and $SO(32)$ respectively. These can also be conveniently represented (through bosonization) by 32 fermions: In the case of $E_8 \times E_8$ we group them to two groups of 16 and consider independent NS, R sectors for each group. In the case of $SO(32)$ we only have one group of 32 fermions with either NS or R boundary conditions.

Let us tabulate the massless modes using (1.3). The right-movers can be either NS or R. The left-moving degrees of freedom start out with a tachyonic mode. But (1.3) implies that this is not satisfying the level-matching condition because the right-moving ground state is at zero energy. Thus we should search on the left-moving side for states with $L_0 = 0$ which means from (1.3) that we have either $N_L = 1$ or $\frac{1}{2}P^2_L = 1$, where $P_L$ is an internal 16 dimensional vector in one of the two lattices noted above. The states with $N_L = 1$ are

$$16 \oplus v$$
where 16 corresponds to the oscillation direction in the extra 16 dimensions and \( v \) corresponds to vector in 10 dimensional spacetime. States with \( \frac{1}{2} P_L^2 = 1 \) correspond to the non-zero weights of the adjoint of \( E_8 \times E_8 \) or \( SO(32) \) which altogether correspond to 480 states in both cases. The extra 16 \( N_L = 1 \) modes combine with these 480 states to form the adjoints of \( E_8 \times E_8 \) or \( SO(32) \) respectively. The right-movers give, as before, a \( v \oplus s \) from the NS and R sectors respectively. So putting the left- and right-movers together we finally get for the massless modes

\[
(v \oplus \text{Adj}) \otimes (v \oplus s)
\]

Thus the bosonic states are \((v \oplus \text{Adj}) \otimes v\) which gives

\[
(g_{\mu\nu}, B_{\mu\nu}, \phi; A_\mu)
\]

where the \( A_\mu \) is in the adjoint of \( E_8 \times E_8 \) or \( SO(32) \). Note that in the \( SO(32) \) case this is an identical spectrum to that of type I strings.

1.5. Summary

Just to summarize, we have found 5 consistent strings in 10 dimensions: Type IIA with \( N = 2 \) non-chiral supersymmetry, type IIB with \( N = 2 \) chiral supersymmetry, type I with \( N=1 \) supersymmetry and gauge symmetry \( SO(32) \) and heterotic strings with \( N=1 \) supersymmetry with \( SO(32) \) or \( E_8 \times E_8 \) gauge symmetry. Note that as far as the massless modes are concerned we only have four inequivalent theories, because heterotic \( SO(32) \) theory and Type I theory have the same light degrees of freedom. We shall return to this point later, when we discuss dualities. In discussing compactifications it is sometimes natural to divide the discussion between two cases depending on how many supersymmetries we start with. In this context we will refer to the type IIA and B as \( N = 2 \) theories and Type I and heterotic strings as \( N = 1 \) theories.

2. String Compactifications

So far we have only talked about superstrings propagating in 10 dimensional Minkowski spacetime. If we wish to connect string theory to the observed four dimensional spacetime, somehow we have to get rid of the extra 6 directions. One way to do this is by assuming that the extra 6 dimensions are tiny and thus unobservable in the present
day experiments. In such scenarios we have to understand strings propagating not on ten
dimensional Minkowski spacetime but on four dimensional Minkowski spacetime times a
compact 6 dimensional manifold $K$. In order to gain more insight it is convenient to con-
sider compactifications not just to 4 dimensions but to arbitrary dimensional spacetimes,
in which case the dimension of $K$ is variable.

The choice of $K$ and the string theory we choose to start in 10 dimensions will lead
to a large number of theories in diverse dimensions, which have different number of super-
symmetries and different low energy effective degrees of freedom. In order to get a handle
on such compactifications it is useful to first classify them according to how much super-
symmetry they preserve. This is useful because the higher the number of supersymmetry
the less the quantum corrections there are.

If we consider a general manifold $K$ we find that the supersymmetry is completely
broken. This is the case we would really like to understand, but it turns out that string
theory perturbation theory always breaks down in such a situation; this is intimately
connected with the fact that typically cosmological constant is generated by perturbation
theory and this destabilizes the Minkowski solution. For this reason we do not even have a
single example of such a class whose dynamics we understand. Instead if we choose $K$ to
be of a special type we can preserve a number of supersymmetries.

For this to be the case, we need $K$ to admit some number of covariantly constant
spinors. This is the case because the number of supercharges which are ‘unbroken’ by
compactification is related to how many covariantly constant spinors we have. To see this
note that if we wish to define a \textit{constant} supersymmetry transformation, since a spacetime
spinor, is also a spinor of internal space, we need in addition a constant spinor in the
internal compact directions. The basic choices are manifolds with trivial holonomy (flat
tori are the only example), $SU(n)$ holonomy (Calabi-Yau n-folds), $Sp(n)$ holonomy (4n di-

dimensional manifolds), 7-manifolds of $G_2$ holonomy and 8-manifolds of $Spin(7)$ holonomy.
The case mostly studied in physics involves toroidal compactification, $SU(2) = Sp(1)$
holonomy manifold (the 4-dimensional $K^3$), $SU(3)$ holonomy (Calabi-Yau 3-folds)\footnote{Calabi-Yau 4-folds have also recently appeared in connection with F-theory compactification to 4 dimensions as we will briefly discuss later.}. The cases of $Sp(2)$ holonomy manifolds (8 dimensional) and $G_2$ and $Spin(7)$ end up giving us compactifications below 4 dimensions. Here we will review the cases of toroidal compacti-

fication, $K^3$ compactification and Calabi-Yau 3-fold compactification.
2.1. Toroidal Compactifications

The space with maximal number of covariantly constant spinors is the flat torus $T^d$. This is also the easiest to describe the string propagation in. The main modification to the construction of the Hilbert space from flat non-compact space in this case involves relaxing the condition $P_L = P_R$ because the string can wrap around the internal space and so $X$ does not need to come back to itself as we go around $\sigma$. In particular if we consider compactification on a circle of radius $R$ we can have

$$(P_L, P_R) = \left( \frac{n}{2R} + mR, \frac{n}{2R} - mR \right)$$

Here $n$ labels the center of mass momentum of the string along the circle and $m$ labels how many times the string is winding around the circle. Note that the spectrum of allowed $(P_L, P_R)$ is invariant under $R \rightarrow 1/2R$. All that we have to do is to exchange the momentum and winding modes $(n \leftrightarrow m)$. This symmetry is a consequence of what is known as $T$-duality and as you can see it is a relatively simple fact to understand. If we compactify on a d-dimensional torus $T^d$ it can be shown that $(P_L, P_R)$ belong to a $2d$ dimensional lattice with signature $(d, d)$. Moreover this lattice is integral, self-dual and even. Evenness means, $P_L^2 - P_R^2$ is even for each lattice vector. Self-duality means that any vector which has integral product with all the vectors in the lattice sits in the lattice as well. It is an easy exercise to check these condition in the one dimensional circle example given above. Note that we can change the radii of the torus and this will clearly affect the $(P_L, P_R)$. Given any choice of a d-dimensional torus compactifications, all the other ones can be obtained by doing an $SO(d,d)$ Lorentz boost on $(P_L, P_R)$ vectors. Of course rotating $(P_L, P_R)$ by an $O(d) \times O(d)$ transformation does not change the spectrum of the string states, so the totality of such vectors is given by

$$\frac{SO(d,d)}{SO(d) \times SO(d)}$$

Some Lorentz boosts will not change the lattice and amount to relabeling the states. These are the boosts that sit in $O(d,d; Z)$ (i.e. boosts with integer coefficients), because they can be undone by choosing a new basis for the lattice by taking an integral linear combination of lattice vectors. So the space of inequivalent choices are actually given by

$$\frac{SO(d,d)}{SO(d) \times SO(d) \times O(d,d; Z)}$$
The $O(d, d; \mathbb{Z})$ generalizes the T-duality considered in the 1-dimensional case. So far our discussion is general in that we have described only the bosonic degrees of freedom of the string. In discussing further aspects it is useful to divide the discussion to the toroidal compactification of $N = 2$ supersymmetric theories (IIA and IIB) and $N = 1$ case (Type I and heterotic).

2.2. $N = 2$ theories on $T^d$

Once we compactify type IIA and IIB on a circle both theories have the same low energy degrees of freedom. Actually they are isomorphic to each other; the T-duality discussed before, $R \rightarrow \frac{1}{R}$, exchanges the two theories. This is because the operation of $R \rightarrow \frac{1}{R}$ is accompanied on the fermions along the circle direction by

$$\psi_L \rightarrow -\psi_L$$

$$\psi_R \rightarrow \psi_R$$

This in particular means that the product of all the left moving fermions, which defines the left-moving chirality, is switched by an overall minus sign, and thus under the GSO projection we keep the opposite chirality to what we had. This therefore exchanges $IIA$ and $IIB$. For more general compactification on $T^d$ the part of the T-duality group which does not exchange the two theories is $SO(d, d; \mathbb{Z})$; the elements of T-duality which are in $O(d, d; \mathbb{Z})$ but not in $SO(d, d; \mathbb{Z})$ will exchange IIA and IIB and thus are not symmetries of either one.

In compactifying type II strings on tori, the scalars parameterized by the coset $SO(d, d)/SO(d) \times SO(d)$ correspond to choices of the metric of the torus ($d(d+1)/2$ degrees of freedom) and the anti-symmetric field $B_{ij}$ on the torus ($d(d-1)/2$ degrees of freedom). As we see these do modify the string Hilbert space spectrum. However there are other choices we have to make by turning on expectation values of the RR anti-symmetric tensor fields with zero field strength. Since none of the string modes couple to the anti-symmetric potentials and only couple to the field strength, this will not affect the perturbative string spectrum. It will however affect the solitonic spectrum of string states.

Let us count how many total parameters we have in specifying the choice of a vacuum. These would correspond to massless scalars in the non-compact theory. Just as an example let us consider the compactification of type IIA strings on $T^4$. Then we have $4 \times 4 = 16$ parameters specifying the metric of $T^4$ as well as the anti-symmetric field on it. In addition
we have 4 parameters specifying the choice for the Wilson line of \( A_\mu \) and 4 parameters specifying the choice for the constant modes of \( C_{\mu\nu\rho} \). So altogether we have \( 16 + 4 + 4 = 24 \) massless scalars coming from the choice of the compactification. Noting that we also have one scalar corresponding to the dilaton in 10 dimensions we have altogether 25 massless scalars. The kinetic term for 16 of these scalars corresponding to the internal metric and antisymmetric \( B \)-field is given by \( SO(4,4)/SO(4) \times SO(4) \) coset metric. One could ask about what the metric looks like for the other parameters? It turns out that supersymmetry alone predicts that the metric of the total 25 dimensional space is that of the coset

\[
SO(5,5)/SO(5) \times SO(5)
\]

This appears somewhat mysterious from the viewpoint of string theory, because the origin of various scalars are so different. The above coset unifies dilaton, RR tensor fields and the metric degrees of freedom of the compactification into one single object! The simplest way to understand this result of supergravity is as follows: The low energy degrees of freedom, which is all that is needed in the description of the coset, can also be obtained from the compactification of \( N = 1 \) supergravity in eleven dimensions on a 5-dimensional torus. The choice of the metric on \( T^5 \) up to an overall scale involves the coset space

\[
SL(5)/SO(5)
\]

The group \( SL(5) \) and the group visible from string perturbation theory, namely \( SO(4,4) \) intersect on \( SL(4) \). The smallest group containing both is \( SO(5,5) \). This gives the coset description given above, a fact which also follows from supersymmetry arguments alone. Note that this same reasoning would lead us to postulate the global identifications on this coset space to be \( SO(5,5;\mathbb{Z}) \) because the identification from string theory side is \( SO(4,4;\mathbb{Z}) \) and from the compactification from \( N = 1 \) theory in eleven dimensions \( SL(5;\mathbb{Z}) \) which together generate \( SO(5,5;\mathbb{Z}) \). Thus if we believe that the compactification of the 11 dimensional theory is somehow equivalent to type II strings, as we will discuss later, we would be led to the coset space

\[
SO(5,5)/SO(5) \times SO(5) \times SO(5,5;\mathbb{Z}).
\]

Note that the elements in \( SO(5,5;\mathbb{Z}) \) not contained in \( SO(4,4;\mathbb{Z}) \) act non-trivially on the dilaton of type IIA strings. This symmetry cannot be directly checked in string perturbation theory because it typically takes small values of the string coupling to large values,
inaccessible by string perturbation techniques. We will return to evidence for this kind of non-perturbative symmetry, known as U-dualities later on in this paper.

The same construction generalizes to compactification on $T^d$. In this case the T-duality group of string theory is $SO(d, d)$ and that of the 11 dimensional theory is $SL(d+1)$, which have the common subgroup $SL(d)$. Together they form the group $E_{d+1}$ which is the maximally non-compact form of the exceptional group. Note that upon compactification of all spatial directions we end up with $E_{10}$ which is the hyperbolic Kac-Moody algebra. This suggests that $E_{10}$ should play a prominent role in an “invariant” way of thinking about type II string theories.

2.3. Compactifications of $N = 1$ theories on $T^d$

If we compactify the heterotic strings ($E_8 \times E_8$ or $SO(32)$) or Type I theory on $T^d$, in addition to the choice of the metric $g_{ij}$ of $T^d$ and the antisymmetric field $B_{ij}$ on it, whose moduli form the same coset as for the type II case discussed above $SO(d, d)/SO(d) \times SO(d)$, we also have to choose Wilson lines which lie in the Cartan of these gauge groups. Given that in all these cases the rank of the group is 16 the local structure turns out to be the coset space

$$\frac{SO(d+16, d)}{SO(d+16) \times SO(d)}.$$

Note that the dimension of this space is $16d$ bigger than that for the type II case, which signifies the choice of 16 $U(1)$ Wilson lines along $d$ different directions. For the global aspects of identification, the heterotic case is very simple and leads to $SO(d + 16, d; \mathbb{Z})$ symmetry in the denominator above. In fact the construction of Hilbert space is particularly simple and the only difference from the 10 dimensional case is that now the left and right momenta can lie on a self-dual even lorentzian lattice of signature $(16+d, d)$, denoted by $\Gamma^{16+d,d}$. In other words

$$(P_L, P_R) \in \Gamma^{16+d,d}$$

It is known that this lattice is unique. The only choice comes from the ‘choice of polarization’, which corresponds to how we project to left- and right- momenta. The moduli space for this choice is precisely

$$\frac{SO(16 + d, d; \mathbb{R})}{SO(d+16; \mathbb{R}) \times SO(d; \mathbb{R}) \times SO(16+d,d;\mathbb{Z})}$$
This uniqueness implies in particular that upon toroidal compactification (in fact even upon compactification on a circle) the $E_8 \times E_8$ heterotic string and $SO(32)$ heterotic string become isomorphic, i.e. T-dual to one another.

For the type I string the local part of the moduli space is easily seen to be the same and it is conjectured to also have the same global T-duality group $SO(d + 16, d : \mathbb{Z})$.

Note that apart from the above moduli, the expectation value of the dilaton, $\phi$ which plays the role of the string coupling constant, is part of the moduli of vacua of this theory. For $d \leq 4$ there are extra scalars. Let us consider the case where $d = 4$. In this case we have an $N = 4$ supersymmetric theory. In addition to the above moduli the anti-symmetric field in 4 dimension is dual to a scalar and so combining this with the dilaton we have a complex field. Moreover the $N = 4$ supergravity fixes the metric on that space to be exactly the same metric as the upper-half plane. The moduli space of this theory in $d = 4$ is locally given by

$$\frac{SL(2; \mathbb{R})}{SO(2; \mathbb{R})} \times \frac{SO(22, 6; \mathbb{R})}{SO(22; \mathbb{R}) \times SO(6; \mathbb{R})}$$

It is natural to speculate that the global symmetry of the first factor is $SL(2, \mathbb{Z})$. The non-trivial element of this duality acts to invert the four dimensional string coupling constant and conjecturing such a symmetry is consistent with the Olive-Montonen strong/weak duality conjecture for $N = 4$ field theories. We shall return to this point later.

2.4. Compactifications on $K3$

The four dimensional manifold $K3$ is the only manifold in four dimensions, besides $T^4$, which admits covariantly constant spinors. In fact it has exactly half the number of covariantly constant spinors as on $T^4$ and thus preserves half of the supersymmetry that would have been preserved upon toroidal compactification. More precisely the holonomy of a generic four manifold is $SO(4)$. If the holonomy resides in an $SU(2)$ subgroup of $SO(4)$ which leaves an $SU(2)$ part of $SO(4)$ untouched, we end up with one chirality of $SO(4)$ spinor being unaffected by the curvature of $K3$, which allows us to define supersymmetry transformations as if $K3$ were flat (note a spinor of $SO(4)$ decomposes as $(2, 1) \oplus (1, 2)$ of $SU(2) \times SU(2)$). Before considering compactifications of various superstrings on $K3$ it is convenient to talk about various geometric properties of $K3$.  

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2.5. Aspects of K3 manifold

There are a number of realizations of K3, which are useful depending on which question one is interested in. We shall describe all these different viewpoints in turn. Perhaps the simplest description of it is in terms of orbifolds. This description of K3 is very close to toroidal compactification and differs from it by certain discrete isometries of the $T^4$ which are used to (generically) identify points which are in the same orbit of the discrete group. The other description of K3 describes a 19 complex parameter family of K3 defined by an algebraic equation. Finally an 18 complex parameter subspace of K3 surfaces admit elliptic fibrations (with a section) which is an intermediate between the toroidal orbifold limit of K3 and its algebraic description.

2.6. K3 as an orbifold of $T^4$

Consider a $T^4$ which for simplicity we take to be parametrized by four real coordinates $x_i$ with $i = 1, \ldots, 4$, subject to the identifications $x_i \sim x_i + 1$. It is sometimes convenient to think of this as two complex coordinates $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$ with the obvious identifications. Now we identify the points on the torus which are mapped to each other under the $Z_2$ action (involution) given by reflection in the coordinates $x_i \rightarrow -x_i$, which is equivalent to

$$z_i \rightarrow -z_i$$

Note that this action has $2^4 = 16$ fixed points given by the choice of midpoints or the origin in any of the four $x_i$. The resulting space is singular at any of these 16 fixed points because the angular degree of freedom around each of these points is cut by half. Put differently, if we consider any primitive loop going ‘around’ any of these 16 fixed point, it corresponds to an open curve on $T^4$ which connects pairs of points related by the $Z_2$ involution. Moreover the parallel transport of vectors along this path, after using the $Z_2$ identification, results in a flip of the sign of the vector. This is true no matter how small the curve is. This shows that we cannot have a smooth manifold at the fixed points.

One of the nice things about the orbifold limit of K3 is that the construction of the string Hilbert space is relatively simple (known as the orbifold construction): we start with the Hilbert space of $T^4$ compactification. The $Z_2$ action on the torus will induce a $Z_2$ action on the Hilbert space. We project unto the $Z_2$ invariant subspace. This is not the end of story for closed strings on K3: We will have to consider open strings on $T^4$ which begin and end at points identified under the $Z_2$ map. This is called the twisted sector. In
this case we have 16 choices of the center of mass of the twisted string state. Moreover the bosonic oscillators are shifted by $1/2$ (because we have anti-periodic boundary conditions induced by $x_i(0) = -x_i(2\pi)$). Clearly these modifications are rather minor and it is straightforward to consider the twisted sector. The only point is that now we have no zero modes corresponding to left or right momenta. In fact the center of mass of the twisted string is frozen at any of the 16 fixed points of the $Z_2$ action. To complete the orbifold construction we still have to project unto the $Z_2$ invariant subsector even in the twisted sector.

As noted above this is a singular limit of $K3$, due to the singularities of the metric at the 16 fixed points. It is natural to ask how these singularities can be remedied.

To answer the first question it is convenient to use complex coordinates. Near a fixed point $(z_1, z_2) = (0, 0)$, where the identification is $(z_1, z_2) \rightarrow (-z_1, -z_2)$, it is natural to choose coordinates that are invariant under this action. To do this we use the complex coordinates

$$u = z_1 z_2$$
$$v = z_1^2$$
$$w = z_2^2$$

Clearly these are the only basic $Z_2$ invariant combinations of these coordinates. However there is a relation between them:

$$u^2 = vw$$

Since manifolds can be identified by the functions on them, the above equation can also be viewed as being equivalent to the $K3$ manifold near the singular locus. By a simple change of complex coordinates this can also be written as

$$f = u^2 + \tilde{v}^2 + \tilde{w}^2 = 0$$

(2.1)

It is easy to see that at $u = \tilde{v} = \tilde{w} = 0$ the manifold given by the above equation is singular (by considering the rank of $df$ at that point). This of course was expected as it is equivalent to the fixed point of the $Z_2$ action discussed above. This way of writing the singularity also makes clear that there is a 2-sphere hidden in the singularity. Namely, if we consider the sublocus of the manifold where $u, \tilde{v}, \tilde{w}$ are real, then $f = 0$ is the equation for a sphere of zero size. In other words we have a vanishing sphere at this singularity.
Now we come to how this singularity can be resolved. In general this can be done in two ways. Given the discussion of the existence of the singularity at the origin it is natural to eliminate that point from the manifold by slightly changing the defining equation of the manifold. In other words we can deform the complex structure of the manifold by considering

\[ f = u^2 + \tilde{v}^2 + \tilde{w}^2 = \epsilon \]

In this way the point \( u = \tilde{v} = \tilde{w} = 0 \) is eliminated from the manifold and in fact the manifold is now smooth. Another way this singularity can be remedied is by giving the sphere defined by \( f = 0 \) a finite volume. This is known as blowing up the singularity. Changing the volumes of the complex manifold are known as Kahler deformations. So in this case we say that the singularity of the manifold has been repaired by deformation of the Kahler structure. It turns out that in the special case of K3 these two operations are not unrelated. In fact if we take \( \epsilon \) to be real and view it as radius squared of the sphere, the above equation can also be viewed as giving the vanishing sphere a finite volume. Only in the case of K3 the deformation by Kahler and complex structure can be related (this has to do with the fact that K3 is hyperkahler). In more general cases, such as Calabi-Yau threefolds, the deformations of Kahler and complex structures are distinguishable and repairing singularities by using them in general lead to topologically distinct manifolds.

The above singularity is known as an A1 singularity. There is a simple generalization of it which occurs when we locally identify:

\[ z_1 \rightarrow \omega z_1 \]
\[ z_2 \rightarrow \omega^{-1} z_2 \]

where \( \omega \) is an \( n \)-th root of unity. This space is known as an ALE (asymptotically locally Euclidean) space. In this case we define the invariant coordinates as

\[ u = z_1 z_2 \]
\[ v = z_1^n \]
\[ w = z_2^n \]

with the identification

\[ u^n = vw \]
which defines the local description of the singularity. To resolve the singularity we introduce terms of lower order in $u$ and consider

$$\prod_{i=1}^{n}(u - a_i) = vw = -\tilde{v}^2 - \tilde{w}^2$$

If any of the two $a_i$ are equal then we have a singularity of the type $A_1$ discussed before and we obtain a vanishing 2-sphere. When all $a_i$ are distinct the singularity is completely resolved. In general to any pair of $a_i$ we can associate a vanishing 2-sphere. If we wish to express the homology class they represent all of them can be represented as a linear combination of $n - 1$ vanishing 2-spheres. For example, we can take all $a_i$ to be real and order the index of $a_i$ to be in an increasing order. We then take the $j$-th generator $S_j$ to be the vanishing sphere associated with $a_j$ approaching $a_{j+1}$. If we consider the image of $S_j$ on the $u$-plane it corresponds to an interval running from $a_j$ to $a_{j+1}$. In this way it is also easy to see that the $S_j$ and $S_{j+1}$ intersect one another at one point, namely $u = a_{j+1}, \tilde{v} = \tilde{w} = 0$. If for every generator $S_j$ we consider a node and for every intersection between adjacent vanishing spheres draw a line connecting them we obtain the $A_{n-1}$ Dynkin diagram. Indeed this singularity is known as $A_{n-1}$ singularity. Note that again as in the $A_1$ case we can consider blowing up the singularity instead of deforming its complex structure. However again in the case of $K3$ this turns out to be equivalent to deformation.

There are extensions of this singularity associated with modding out the $(z_1, z_2)$ complex space by Dihedral and exceptional subgroups of $SU(2)$. In these cases the story is very similar to what we discussed above. When we consider the generators for the vanishing spheres and their intersections we get the Dynkin diagram for the D and E series (corresponding respectively to the Dihedral and exceptional subgroups of $SU(2)$).

Before moving to other descriptions of $K3$ let us note that the cohomology classes of $K3$ are very visible in the orbifold construction. We decompose the cohomology classes according to the degree of holomorphic and anti-holomorphic forms $(p, q)$ and denote the number in each class by $h^{p,q}$. We get from the untwisted sector 8 elements corresponding to the $Z_2$ invariant forms on $T^4$, namely,

$$1, dz_1 \wedge dz_2, d\bar{z}_1 \wedge d\bar{z}_2, dz_1 \wedge d\bar{z}_2, dz_2 \wedge d\bar{z}_1, dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_j$$

$$dz_i \wedge d\bar{z}_j$$

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From the twisted sectors, after resolving we obtain from each a contribution to $h^{1,1}$. This is the cohomology vanishing 2-sphere hidden in the fixed points, discussed above. So we obtain altogether 16 elements in $h^{1,1}$ coming from the twisted sector. Note that the number of Kahler deformations is the dimension of $h^{1,1}$ which altogether is 20. Also the number of complex deformations for $K3$ is also given by $h^{1,1}$ which is 20 (this will be made more precise below); 16 of them correspond to resolving the singularity near the fixed points and 4 of them correspond to changing the complex structure of the underlying $T^4$. Note that from the cohomology computation above we learn in particular that the Euler characteristic of $K3$ is $\chi(K3) = 24$.

2.7. $K3$ as a complex surface in $\mathbf{P}^3$

When we move away from the orbifold points of $K3$ the description of the geometry of $K3$ in terms of the properties of the $T^4$ and the $Z_2$ twist become less relevant, and it is natural to ask about other ways to think about $K3$. In general a simple way to define complex manifolds is by imposing complex equations in a compact space known as the projective $n$-space $\mathbf{P}^{n}$. This is the space of complex variables $(z_1, ..., z_{n+1})$ excluding the origin and subject to the identification

$$(z_1, ..., z_{n+1}) \sim \lambda(z_1, ..., z_{n+1}) \quad \lambda \neq 0$$

One then considers the vanishing locus of a homogeneous polynomial of degree $d$, $W_d(z_i) = 0$ to obtain an $n - 1$ dimensional subspace of $\mathbf{P}^{n}$. An interesting special case is when the degree is $d = n + 1$. In this case one obtains an $n - 1$ complex dimensional manifold which admits a Ricci-flat metric. This is the case known as Calabi-Yau. For example, if we take the case $n = 2$, by considering cubics in it

$$z_1^3 + z_2^3 + z_3^3 + az_1z_2z_3 = 0$$

we obtain an elliptic curve, i.e. a torus of complex dimension 1 or real dimension 2. The next case would be $n = 3$ in which case, if we consider a quartic polynomial in $\mathbf{P}^3$ we obtain the 2 complex dimensional $K3$ manifold:

$$W = z_1^4 + z_2^4 + z_3^4 + z_4^4 + \text{deformations} = 0$$

There are 19 inequivalent quartic terms we can add. This gives us a 19 dimensional complex subspace of 20 dimensional complex moduli of the $K3$ manifold. Clearly this way
of representing $K3$ makes the complex structure description of it very manifest, and makes the Kahler structure description implicit.

Note that for a generic quartic polynomial the $K3$ we obtain is non-singular. This is in sharp contrast with the orbifold construction which led us to 16 singular points. It is possible to choose parameters of deformation which lead to singular points for $K3$. For example if we consider

$$z_1^4 + z_2^4 + z_3^4 + z_4^4 + 4z_1z_2z_3z_4 = 0$$

it is easy to see that the resulting $K3$ will have an $A_1$ singularity (one simply looks for non-trivial solutions to $dW = 0$).

There are other ways to construct Calabi-Yau manifolds and in particular $K3$’s. One natural generalization to the above construction is to consider weighted projective spaces where the $z_i$ are identified under different rescalings. In this case one considers quasi-homogeneous polynomials to construct submanifolds.

2.8. $K3$ as an elliptic manifold

A certain subset of $K3$ manifolds admit elliptic fibration. What this means is that locally they look like a two torus times a complex plane which we denote by $z$. The complex structure of the torus varies holomorphically as a function of $z$, and over 24 points on the $z$-sphere the elliptic fiber degenerates. Let us describe this in more detail.

A 2 real dimensional torus (also known as elliptic curve) can be viewed as a double cover of the sphere branched over 4 points. Let us denote the complex coordinate parametrizing the sphere by $x$. Then an equation of the form

$$y^2 = P_4(x)$$

defines a torus where $P_4(x)$ is a polynomial of degree 4 in $x$. The fact that this is a double cover of the plane is because for each $x$ there are two values of $y$, related by a change in sign, satisfying the above equation. The four branch points are where $P_4(x) = 0$, in which case we get only one value of $y$ over it. It is convenient to take one of the branch points to infinity and define the torus through

$$y^2 = x^3 + fx + g$$

where with no loss of generality we have taken the coefficient of $x$ and $y$ to be one, and we have eliminated the terms proportional to $x^2$ by a shift in $x$. The complex coefficients
If $f, g$ vary the complex structure of the torus. However the complex structure of the two torus is one dimensional. This can be seen by the parallelogram construction of the torus which up to rescaling we can choose one length of the parallelogram to correspond to the point 1 on the complex plane and the other to a complex parameter $\tau$. If we rescale $(f, g) \rightarrow (\lambda^4 f, \lambda^6 g)$, the above equation is invariant if we redefine $(x, y) \rightarrow (\lambda^2 x, \lambda^3 y)$.

Note that when the cubic $x^3 + fx + g$ has double roots the torus is degenerate. This is because near such points we can write the torus as $y^2 = (x - x_0)^2$, which is a one-dimension lower singularity of the $A_1$ type discussed above. This corresponds to a vanishing circle (instead of the 2-sphere). The discriminant of the cubic, where the elliptic curve is singular is given by

$$\Delta = 4f^3 + 27g^2 = 0$$

Now we come to the elliptic description of $K3$. In view of the generality of the ‘fibration’ picture, let us be a bit more general. Suppose we have a string compactification $X$ whose data is parameterized by a set of parameters $\mathcal{M}$ (i.e., its moduli space). By fibering this over another space, we mean that these parameters vary “slowly” over another space, known as the base of the fibration. In other words we let the parameters of compactification on $X$, such as complex structure and radii, become functions over the base. Having said that, if we wish to have an elliptic fibration over the sphere, all we have to do is to make the $f$ and $g$ given in defining an elliptic curve be functions of the sphere, which we denote by the $z$ parameter. In particular if we take $f(z)$ to be a polynomial of degree 8 and $g(z)$ to be a polynomial of degree 12 we obtain a $K3$ surface:

$$y^2 = x^3 + f(z)x + g(z)$$

The condition that the degrees of $f$ and $g$ should be in the ratio of 2 to 3 follows from the fact that as $z \rightarrow \infty$ we generically wish to have a non-singular elliptic curve (in which case for large $z$ we can redefine $x$ and $y$ to see that we obtain a fixed elliptic fibration at infinity). Note the the discriminant $\Delta$ is in this case a polynomial of degree 24. This means that over 24 points on the sphere the elliptic fiber develops a degeneration of the form

$$xy = (z - z_0)$$

where the elliptic fiber is degenerate at $z = z_0$, where a 1-cycle (i.e. a circle) has shrunk to a point. Note that the total space including $(x, y, z)$ subject to the above equation is not singular. In fact we can use $x$ and $y$ to coordinatize this region and read off what $z$
is from the above equation. It is only the fibration that becomes singular. The situation would be different, however if more than one fiber becomes singular at the same point. In particular if we have \( n \) singular fibers at the same point, i.e., if we locally have

\[ xy = (z - z_0)^n \]

we have an \( A_{n-1} \) singularity of \( K3 \).

Note that the complex dimension of moduli of elliptic \( K3 \)'s is given by the number of parameters which go into defining \( f \) (9) plus the number of parameters for \( g \) (13) minus the \( SL(2, C) \) action on the \( z \) sphere (3) minus the overall rescaling of the equation defining elliptic curve (1), which gives us altogether 18 complex parameters. The dimension of Kahler deformation of \( K3 \) consistent with preservation of elliptic fibration is 2, one from the Kahler class (size) of the \( z \)-sphere, and one from the Kahler class of the elliptic fiber.

2.9. Moduli space of \( K3 \)

As discussed before \( K3 \) has 20 complex deformations and 20 Kahler deformations. To be more precise, if we consider the moduli of metrics on \( K3 \) with \( SU(2) \) holonomy this space is 58 real dimensional space. The way this arises is that we have 40 real parameters from the choice of complex structure and 20 real parameters from the choice of Kahler structures. However there is a sphere worth of redundancy in that for every Ricci-flat metric on \( K3 \) we can choose a sphere worth of choice of complex structures consistent with it. Thus the space is 58 dimensional. It turns out that this moduli space of geometric \( K3 \)'s is isomorphic to

\[ \mathcal{M}^{\text{geometric}} = \frac{SO(19, 3)}{SO(19) \times SO(3) \times SO(19, 3; \mathbb{Z})} \times \mathbb{R}^+ \]

The easiest way to understand these results is that if we consider the elements of \( H_2(K3) \), as discussed above, this space has dimension 22, corresponding to the fact that \( h^{2,0} = h^{0,2} = 1 \) and \( h^{1,1} = 20 \). Moreover since the dimensions of these cycles is half the dimension of the total space we can consider an intersection form on this 22 dimensional space corresponding to the intersection between these cycles. It turns out this space is the same as the Narain lattice \( \Gamma^{19,3} \). For a fixed metric on \( K3 \) we can use the duality by the \( * \)-operation. This gives a polarization on this lattice, exactly as the choice of left- and right-moving momenta arise for toroidal compactifications. It turns out that except for the overall rescaling of the \( K3 \) metric which does not affect this polarization, the rest of the choice of metric on \( K3 \) is
faithfully represented by the change in the polarization of the \( H_2 \) lattice and this gives rise
to the above moduli space. The \( R^+ \) above corresponds to overall rescaling of the metric.

This is not the end of the story if we are considering string propagation on \( K3 \). In this case for each element of \( H_2(K3) \) we can turn on the fields \( B_{\mu\nu} \). This space for \( K3 \) is 22 dimensional. Thus we have 22 additional real parameters and that makes the total
dimension of moduli space \( 58 + 22 = 80 \). It turns out that this moduli space is isomorphic
to
\[
\mathcal{M}^{\text{stringy}} = \frac{SO(20, 4)}{SO(20) \times SO(4) \times SO(20, 4; \mathbb{Z})}
\]
Note that the full homology lattice of \( K3 \) with the natural intersection pairing is isomorphic
to \( \Gamma^{20,4} \) lattice. This result on stringy moduli space of \( K3 \) can be interpreted by saying
that for strings the polarization on the full homology (not just the middle homologies) are
relevant.

For elliptic \( K3 \)'s as noted above the moduli space is 18 complex dimensional plus 2
real parameters corresponding to the Kahler classes of the base and fiber. This space is
isomorphic to
\[
\mathcal{M}^{\text{elliptic}} = \frac{SO(18, 2)}{SO(18) \times SO(2) \times SO(18, 2; \mathbb{Z})} \times R^+ \times R^+
\]
This result can be interpreted by noting that the choice of the elliptic fiber amounts to
choosing a null vector in the \( H_2 \) lattice, which has to be preserved by the elliptic metrics
on \( K3 \) and this reduces the group acting on moduli space from \( SO(19, 3) \) to \( SO(18, 2) \)
via a null reduction (note that the self-intersection of the elliptic fiber is zero and thus
corresponds to a null vector).

2.10. \( N = 2 \) theories on \( K3 \)

Let us now consider \( N = 2 \) theories compactified on \( K3 \). The two choices will lead
to inequivalent theories in 6 dimensions, this is because the chirality of the covariantly
constant spinor is correlated with the 6 dimensional chirality. Thus if we start with the
chiral \( N = 2 \) theory as is the case for type IIB strings, we end up with a chiral theory in
6 dimensions, whereas if we start with the non-chiral type IIA theory we will end up with
the non-chiral theory in 6 dimensions, which has exactly the same supersymmetry as the
\( N = 1 \) theories on \( T^4 \).

Let us first consider the moduli space of type IIA on \( K3 \). Recall that the bosonic fields
are \( g_{\mu\nu}, B_{\mu\nu}, \phi, A_\mu, C_{\mu\nu\rho} \). The choices of \( g_{\mu\nu} \) and \( B_{\mu\nu} \) on \( K3 \) lead to the stringy moduli of
$K3$ discussed above. Since $H_1(K3)$ and $H_3(K3)$ are trivial we cannot make any choice for the vacuum solutions for $A_\mu$ or $C_{\mu\nu\rho}$ fields (which preserve supersymmetry with zero field strength). We still have the choice of string coupling which is a positive real parameter. Thus the moduli space is

$$\mathcal{M}^{IIA} = \frac{SO(20,4)}{SO(20) \times SO(4) \times SO(20,4;\mathbb{Z})} \times R^+$$

Note that this is exactly the same as the moduli space of $N = 1$ theories (type I or heterotic theory) on $T^4$. As we will discuss later it turns out that this is not accidental!

As for type IIB on $K3$ in addition to the stringy moduli space of $K3$ and the dilaton we note that we have the choice for the $\chi, \tilde{B}_{\mu\nu}$ and $D_{\mu\nu\rho\lambda}$ coming from the RR sector. The choices for the vevs of these fields with zero field strength correspond to even cohomology elements of $K3$ and so there are 24 parameters. Thus the total moduli space has dimension $80 + 1 + 24 = 105$. It turns out that by supersymmetry arguments this full moduli space is simply given by

$$\mathcal{M}^{\text{typeIIB}} = \frac{SO(21,5)}{SO(21) \times SO(5) \times SO(21,5;\mathbb{Z})}$$

(where the integral modding out in the denominator is conjectural).

2.11. $N=1$ theories on $K3$

We now consider compactification of $N = 1$ theories on $K3$. Recall that the bosonic fields for them is given by $g_{\mu\nu}, B_{\mu\nu}, \phi, A_\mu$ where the gauge field belongs to $SO(32)$ group or $E_8 \times E_8$. As far as the moduli for $g_{\mu\nu}$ and $B_{\mu\nu}$ are concerned we get the same story as in the type II compactifications. Now, however we can choose non-vanishing configuration for the gauge fields without breaking supersymmetry. The reason is that if we consider gauge field configurations on $K3$ satisfying self-duality configuration:

$$F = *F$$

then they preserve the same supersymmetry as the metric configuration on $K3$. These configurations of gauge fields are of course the standard instantons of gauge theory. In fact not only we can turn on these gauge fields we have to turn them on if we wish to have a consistent solution. This is because the $N = 1$ theories have an equation of motion of the form

$$dH = Tr F \wedge F - Tr R \wedge R$$

(2.2)
where $H = dB$ is the field strength for the anti-symmetric field. This equation implies that, in the absence of singularities for the $H$ field, since the integral over $K3$ of the left hand side vanishes, the instanton number of the gauge field should be related to a topological computation involving the curvature, which turns out to be $-\frac{1}{2}p_1(R)$ where $p_1$ is the first Pontrjagin class. For $K3$ this is 24. We thus learn that we have to consider instanton number 24 configuration for the gauge field. For $E_8 \times E_8$ this total instanton number can be distributed among the two $E_8$’s, with instanton numbers $(12 + n, 12 - n)$.

In principle we can also consider configurations where the integral of $dH$ over $K3$ is not zero, i.e.,

$$\int_{K3} dH = m$$

The meaning of this singularity will be discussed in the next section where we will discuss p-branes. Let us just state that what this configuration means is that we have $m$ 5-branes. If we choose this, then we have $24 - m$ instantons to put on the gauge groups.

The moduli space for compactification of $N = 1$ theories will in addition to the moduli describing the geometry of $K3$ includes moduli describing the choices of gauge bundle. This makes the description more complicated and we will not discuss it here.

2.12. Calabi-Yau threefolds

We have already noted that the next case of interest involves compactification on manifolds of $SU(3)$ holonomy. This preserves 1/4 of the supersymmetry. In particular if we compactify $N = 2$ theories on Calabi-Yau threefolds we obtain $N = 2$ theories in $d = 4$, whereas if we consider $N = 1$ theories we obtain $N = 1$ theories in $d = 4$. In the latter case, as discussed in the context of $K3$ compactifications we have to choose the instanton class of the bundle to be in the same class as the tangent bundle on the manifold. Let us first consider some simple classes of Calabi-Yau threefolds and then we discuss some aspects of them in the cases of $N = 2$ and $N = 1$ theories.

Calabi-Yau threefolds are manifolds with $SU(3)$ holonomy. In particular if we wish to construct them as toroidal orbifolds we need to consider six dimensional tori, three complex dimensional, which have discrete isometries residing in $SU(3)$ subgroup of the $O(6) = SU(4)$ holonomy group. A simple example is if we consider the product of three copies of $T^2$ corresponding to the Hexagonal lattice and mod out by a simultaneous $Z_3$ rotation on each torus (this is known as the ‘$Z$-orbifold’). This $Z_3$ transformation has 27 fixed points which can be blown up to give rise to a smooth Calabi-Yau. We can also
consider description of Calabi-Yau threefolds in algebraic geometry terms for which the complex deformations of the manifold can be typically realized as changes of coefficients of defining equations, as in the $K3$ case. For instance we can consider the projective 4-space $\mathbf{P}^4$ defined by 5 complex not all vanishing coordinates $z_i$ up to overall rescaling, and consider the vanishing locus of a homogeneous degree 5 polynomial

$$P_5(z_1, \ldots, z_5) = 0$$

This defines a Calabi-Yau threefold, known as the quintic three-fold. This can be generalized to the case of product of several projective spaces with more equations. Or it can be generalized by taking the coordinates to have different homogeneity weights. This will give a huge number of Calabi-Yau manifolds.

We can also consider Calabi-Yau manifolds which admit elliptic fibrations, as was the case for elliptic $K3$’s. For example we can consider a space which as the base contains $\mathbf{P}^1 \times \mathbf{P}^1$, with an elliptic fiber over it. Let us denote the coordinates of $\mathbf{P}^1 \times \mathbf{P}^1$ by $(z_1, z_2)$. Then we can write the equation denoting the complex structure of the elliptic fiber by

$$y^2 = x^3 + f_{(8,8)}(z_1, z_2)x + g_{(12,12)}(z_1, z_2).$$

where $f_{(8,8)}$ and $g_{(12,12)}$ are polynomials of bidegree 8 and 12 in $(z_1, z_2)$ respectively.

In various applications it is important to study the moduli space of Calabi-Yau manifolds. This generically splits to complex deformations and Kahler deformations. The number of Kahler deformations is given by $h^{1,1}$ of the manifold corresponding to the choice of the Kahler form. Together with the choice of the $B_{\mu\nu}$ field the dimension of the Kahler deformation is given by $h^{1,1}$ complex parameters. The number of complex deformations is given by $h^{2,1}$ of the Calabi-Yau and is typically realized by complex coefficients in the defining equations. Let us enumerate these for the examples of Calabi-Yau manifolds presented above.

For the $\mathbb{Z}$-orbifold, from the untwisted sector, we obtain $h^{1,1}_u = 9$ corresponding to the choice $dz_i \wedge d\bar{z}_j$ for $i, j = 1, 2, 3$, and from the twisted sector we obtain $h^{1,1}_t = 27$ additional contributions (corresponding to blowing up the singularity at each of the 27 fixed points, which locally replaces the neighborhood of the fixed point with a line bundle over $\mathbf{P}^2$ which has $h^{1,1} = 1$), we thus obtain altogether $h^{1,1} = 36$ parameters. For $\mathbb{Z}$-orbifold $h^{2,1} = 0$ and thus there are no complex deformations. This is easy to see because the $\mathbb{Z}_3$ symmetry
exists only for a unique complex structure on $T^2 \times T^2 \times T^2$ (and there are no additional complex structure parameters arising from the twisted sector).

For the example of quintic threefold it is easy to count the complex deformations realized as coefficients of a degree 5 homogeneous polynomial. This gives 101 complex parameters, which is in agreement with $h^{2,1} = 101$. The dimension of $h^{1,1} = 1$ which means that there is only one Kahler class, and that corresponds to the overall size of the quintic. Finally for the elliptically fibered Calabi-Yau, over $\mathbb{P}^1 \times \mathbb{P}^1$, we have $h^{1,1} = 3$ coming from the three Kahler classes of the size of each $\mathbb{P}^1$ plus the size of the elliptic fiber. To count $h^{2,1}$ note that there are $9 \cdot 9$ coefficients in $f_{(8,8)}$ and $13 \cdot 13$ coefficients in defining $g_{(12,12)}$, and subtracting the $SL(2, \mathbb{C})$ symmetry of each $\mathbb{P}^1$ and the overall rescaling of the equation, we find $9 \cdot 9 + 13 \cdot 13 - 3 \cdot 2 - 1 = 243$. This in particular means that $h^{2,1} = 243$.

2.13. $N=2$ theories on Calabi-Yau 3-folds

As mentioned above if we consider $N=2$ theories on Calabi-Yau threefolds we obtain $N = 2$ supersymmetric theories in four dimensions. An $N = 2$ quantum field theory contains hypermultiplets and vector multiplets in addition to the $N = 2$ supergravity multiplet. If we consider type IIA strings on Calabi-Yau threefolds the number of vector multiplets is $h^{1,1}$ and the number of hypermultiplets is $h^{2,1}$. To see this note that expectation values for the scalar in the vector multiplet contain a complex field which can be identified with the choice of the Kahler class and the choice of the NS-NS $B_{\mu \nu}$ in the internal space. Similarly the four component expectation values for a hypermultiplet can be identified with the complex structure of the manifold ($h^{2,1}$ complex variables) together with the expectation values of the anti-symmetric three form ($h^{2,1}$). Two more choices for the antisymmetric three form ($h^{3,0} + h^{0,3}$) as well as the choice of the string coupling and the dual to the spacetime components of $B_{m \nu}$ (the axion) forms an extra hypermultiplet, we thus end up with $h^{1,1}$ vector multiplets and $h^{2,1} + 1$ hypermultiplets. If we consider instead type IIB strings the role of complex and Kahler structures of Calabi-Yau are exchanged and we end up with $h^{2,1}$ vector multiplets and $h^{1,1} + 1$ hypermultiplets.

An important aspect of type IIA and type IIB strings on Calabi-Yau threefolds is the notion of mirror symmetry. The simplest way to view this is to note that many Calabi-Yau threefolds can be viewed (roughly speaking) as a fibered space, with a $T^3$ fiber and an $S^3$ base. The inversion of the volume of the $T^3$ fiber leads to another Calabi-Yau manifold which we identify with the mirror Calabi-Yau. In this way the complex and
Kahler structures get exchanged (note that this is the natural generalization of the 1-fold case where we consider an $T^1 = S^1$ fiber over $S^1$ and do scale inversion on the fiber $T^1$). If we exchange a Calabi-Yau manifold with its mirror we have the exchange of

$$h^{1,1} \leftrightarrow h^{2,1}$$

In particular type IIA on one Calabi-Yau is equivalent to type IIB on its mirror.

2.14. $N=1$ strings on Calabi-Yau threefolds

If we consider $N = 1$ strings (i.e. heterotic strings or Type I strings) on Calabi-Yau threefolds we obtain an $N = 1$ supersymmetric theory in four dimensions. As discussed in the context of $K3$ compactification we have to choose a gauge field configuration whose instanton class $\text{Tr} F \wedge F$ is in the same class as (minus one half) the Pontrjagin class of the Calabi-Yau threefold which is proportional to $\text{Tr} R \wedge R$. There are many ways to fulfill this. One particularly simple way is to consider an $SU(3)$ subgroup of the gauge group and choose the gauge connection to be the same as the spin connection on the threefold. For example, in the context of $E_8 \times E_8$ heterotic strings, if we choose an $SU(3)$ subgroup of one of the $E_8$’s we will end up breaking it to $E_6$. Moreover the charged matter are in the $27$ and $\overline{27}$ representations of $E_6$. The number of $27$’s is given by $h^{1,1}$ and those of $\overline{27}$ by $h^{1,2}$. There will also be many neutral fields. In particular $h^{1,1} + h^{2,1}$ of them will correspond to deforming the (complexified) Kahler structure and the complex structure of the threefold. There will also be some neutral scalar fields which correspond to deforming the gauge connection away from its identity with spin connection (note that any continuous deformation of the gauge connection will still be consistent with the topological condition that instanton class of the bundle agree with half the Pontrjagin class of the manifold). In the $N = 1$ case, as is well known there could be non-perturbative superpotentials generated and the above spectrum is the spectrum of massless modes before taking into account such corrections, which could in principle give mass to some of them.

3. Solitons and String Theory

Solitons arise in field theories when the vacuum configuration of the field has a non-trivial topology which allows non-trivial wrapping of the field configuration at spatial infinity around the vacuum manifold. These will carry certain topological charge related to the ‘winding’ of the field configuration around the vacuum configuration. Examples of solitons
include magnetic monopoles in four dimensional non-abelian gauge theories with unbroken $U(1)$, cosmic strings and domain walls. The solitons naively play a less fundamental role than the fundamental fields which describe the quantum field theory. In some sense we can think of the solitons as ‘composites’ of more fundamental elementary excitations. However as is well known, at least in certain cases, this is just an illusion. In certain cases it turns out that we can reverse the role of what is fundamental and what is composite by considering a different regime of parameter. In such regimes the soliton may be viewed as the elementary excitation and the previously viewed elementary excitation can be viewed as a soliton. A well known example of this phenomenon happens in 2 dimensional field theories. Most notably the boson/fermion equivalence in the two dimensional sine-Gordon model, where the fermions may be viewed as solitons of the sine-Gordon model and the boson can be viewed as a composite of fermion-anti-fermion excitation. Another example is the T-duality we have already discussed in the context of 2d worldsheet of strings which exchanges the radius of the target space with its inverse. In this case the winding modes may be viewed as the solitons of the more elementary excitations corresponding to the momentum modes. As discussed before $R \rightarrow 1/R$ exchanges momentum and winding modes. In anticipation of generalization of such dualities to string theory, it is thus important to study various types of solitons that may appear in string theory.

As already mentioned solitons typically carry some conserved topological charge. However in string theory every conserved charge is a gauge symmetry. In fact this is to be expected from a theory which includes quantum gravity. This is because the global charges of a black hole will have no influence on the outside and by the time the black hole disappears due to Hawking radiation, so does the global charges it may carry. So the process of formation and evaporation of black hole leads to a non-conservation of global charges. Thus for any soliton, its conserved topological charge must be a gauge charge. This may appear to be somewhat puzzling in view of the fact that solitons may be point-like as well as string-like, sheet-like etc. We can understand how to put a charge on a point-like object and gauge it. But how about the higher dimensional extended solitonic states? (note that if we view the higher dimensional solitons as made of point-like structures the soliton has no stability criterion as the charge can disintegrate into little bits)

Let us review how it works for point particles (or point solitons): We have a 1-form gauge potential $A_\mu$ and the coupling of the particle to the gauge potential involves weighing the worldline propagating in the spacetime with background $A_\mu$ by

$$ Z \rightarrow Z \exp(i \int A) $$
where $\gamma$ is the world line of the particle. The gauge principle follows from defining an action in terms of $F = dA$:

$$S = \int F \wedge *F$$

which $*F$ is the dual of the $F$, where we note that shifting $A \rightarrow d\epsilon$ for arbitrary function $\epsilon$ will not modify the action. Suppose we now consider instead of a point particle a $p$-dimensional extended object. In this convention $p = 0$ corresponds to the case of point particles and $p = 1$ corresponds to strings and $p = 2$ corresponds to membranes, etc. We shall refer to $p$-dimensional extended objects as $p$-branes (generalizing ‘membrane’). Note that the worldvolume of a $p$-brane is a $p + 1$ dimensional subspace $\gamma_{p+1}$ of spacetime. To generalize what we did for the case of point particles we introduce a gauge potential which is a $p + 1$ form $A_{p+1}$ and couple it to the charged $p + 1$ dimensional state by

$$Z \rightarrow Z \exp(i \int_{\gamma_{p+1}} A_{p+1})$$

Just as for the case of the point particles we introduce the field strength $F = dA$ which is now a totally antisymmetric $p + 2$ tensor. Moreover we define the action as in (3.1), which possesses the gauge symmetry $A \rightarrow d\epsilon$ where $\epsilon$ is a totally antisymmetric tensor of rank $p$.

3.1. Magnetically Charged States

The above charge defines the generalization of electrical charges for extended objects. Can we generalize the notion of magnetic charge? Suppose we have an electrically charged particle in a theory with spacetime dimension $d$. Then we measure the electrical charge by surrounding the point by an $S^{d-2}$ sphere and integrating $*F$ (which is a $d - 2$ form) on it, i.e.

$$Q_E = \int_{S^{d-2}} *F$$

Similarly it is natural to define the magnetic charge. In the case of $d = 4$, i.e. four dimensional spacetime, the magnetically charged point particle can be surrounded also by a sphere and the magnetic charge is simply given by

$$Q_M = \int_{S^2} F.$$

Now let us generalize the notion of magnetic charged states for arbitrary dimensions $d$ of spacetime and arbitrary electrically charged $p$-branes. From the above description it is clear that the role that $*F$ plays in measuring the electric charge is played by $F$ in
measuring the magnetic charge. Note that for a $p$-brane $F$ is $p + 2$ dimensional, and $*F$ is $d - p - 2$ dimensional. Moreover note that a sphere surrounding a $p$-brane is a sphere of dimension $d - p - 2$. Note also that for $p = 0$ this is the usual situation. For higher $p$, a $p$-dimensional subspace of the spacetime is occupied by the extended object and so the position of the object is denoted by a point in the transverse $(d - 1) - p$ dimensional space which is surrounded by an $S^{d-p-2}$ dimensional sphere.

Now for the magnetic states the role of $F$ and $*F$ are exchanged:

$$F \leftrightarrow *F$$

To be perfectly democratic we can also define a magnetic gauge potential $\tilde{A}$ with the property that

$$d\tilde{A} = *F = *dA$$

In particular noting that $F$ is a $p + 2$ form we learn that $*F$ is an $d - p - 2$ form and thus $\tilde{A}$ is an $d - p - 3$ form. We thus deduce that the magnetic state will be an $d - p - 4$-brane (i.e. one dimension lower than the degree of the magnetic gauge potential $\tilde{A}$). Note that this means that if we have an electrically charged $p$-brane, with a magnetically charged dual $q$-brane then we have

$$p + q = d - 4 \quad (3.2)$$

This is an easy sum rule to remember. Note in particular that for a 4-dimensional spacetime an electric point charge ($p = 0$) will have a dual magnetic point charge ($q = 0$). Moreover this is the only spacetime dimension where both the electric and magnetic dual can be point-like.

Note that a $p$-brane wrapped around an $r$-dimensional compact object will appear as a $p - r$-brane for the non-compact spacetime. This is in accord with the fact that if we decompose the $p+1$ gauge potential into an $(p+1-r)+r$ form consisting of an $r$-form in the compact direction we will end up with an $p+1-r$ form in the non-compact directions. Thus the resulting state is charged under the left-over part of the gauge potential. A particular case of this is when $r = p$ in which case we are wrapping a $p$-dimensional extended object about a $p$-dimensional closed cycle in the compact directions. This will leave us with point particles in the non-compact directions carrying ordinary electric charge under the reduced gauge potential which now is a 1-form.
3.2. String Solitons

From the above discussion it follows that the charged states will in principle exist if there are suitable gauge potentials given by \( p + 1 \)-forms. Let us first consider type II strings. Recall that from the NS-NS sector we obtained an anti-symmetric 2-form \( B_{\mu \nu} \). This suggests that there is a 1-dimensional extended object which couples to it by

\[
\exp(i \int B)
\]

But that is precisely how \( B \) couples to the worldsheet of the fundamental string. We thus conclude that the fundamental string carries electric charge under the antisymmetric field \( B \). What about the the magnetic dual to the fundamental string? According to (3.2) and setting \( d = 10 \) and \( p = 1 \) we learn that the dual magnetic state will be a 5-brane. Note that as in the field theories we expect that in the perturbative regime for the fundamental fields, the solitons be very massive. This is indeed the case and the 5-brane magnetic dual can be constructed as a solitonic state of type II strings with a mass per unit 5-volume going as \( 1/g^2 \) where \( g \) is the string coupling. We shall discuss below some relationship between these 5-branes and the type II theories on ALE space with \( A_n \) singularities.

Let us also recall from our discussion in section 1 that type II strings also have anti-symmetric fields coming from the R-R sector. In particular for type IIA strings we have 1-form \( A_\mu \) and 3-form \( C_{\mu \nu \rho} \) gauge potentials. Note that the corresponding magnetic dual gauge fields will be 7-forms and 5-forms respectively (which are not independent degrees of freedom). We can also include a 9-form potential which will have trivial dynamics in 10 dimensions. Thus it is natural to define a generalized gauge field \( A \) by taking the sum over all odd forms and consider the equation \( \mathcal{F} = * \mathcal{F} \) where \( \mathcal{F} = dA \). A similar statement applies to the type IIB strings where from the R-R sector we obtain all the even-degree gauge potentials (the case with degree zero can couple to a “\(-1\)-brane” which can be identified with an instanton, i.e. a point in spacetime). We are thus led to look for \( p \)-branes with even \( p \) for type IIA and odd \( p \) for type IIB which carry charge under the corresponding RR gauge field. It turns out that surprisingly enough the states in the elementary excitations of string all are neutral under the RR fields. We are thus led to look for solitonic states which carry RR charge. Indeed there are such \( p \)-branes and they are known as \( D \)-branes, as we will now review.
3.3. D-branes: The carriers of RR charge

In the context of field theories constructing solitons is equivalent to solving classical field equations with appropriate boundary conditions. For string theory the condition that we have a classical solution is equivalent to the statement that propagation of strings in the corresponding background would still lead to a conformal theory on the worldsheet of strings, as is the case for free theories.

In search of such stringy p-branes, we are thus led to consider how could a p-brane modify the string propagation. Consider an \( p + 1 \) dimensional plane, to be identified with the worldvolume of the p-brane. Consider string propagating in this background. How could we modify the rules of closed string propagation given this \( p + 1 \) dimensional sheet? The simplest way turns out to allow closed strings to open up and end on the \( p + 1 \) dimensional worldvolume. In other words we allow to have a new sector in the theory corresponding to open string with ends lying on this \( p + 1 \) dimensional subspace. This will put Dirichlet boundary conditions on \( 10 - p - 1 \) coordinates of string endpoints. Such \( p \)-branes are called \( D \)-branes, with \( D \) reminding us of Dirichlet boundary conditions. In the context of type IIA,B we also have to specify what boundary conditions are satisfied by fermions. This turns out to lead to consistent boundary conditions only for \( p \) even for type IIA string and \( p \) odd for type IIB.\(^2\) Moreover it turns out that they do carry the corresponding RR charge.

Quantizing the new sector of type II strings in the presence of D-branes is rather straightforward. We simply consider the set of oscillators as before, but now remember that due to the Dirichlet boundary conditions on some of the components of string coordinates, the momentum of the open string lies on the \( p + 1 \) dimensional worldvolume of the D-brane. It is thus straightforward to deduce that the massless excitations propagating on the D-brane will lead to the dimensional reduction of \( N = 1 \) \( U(1) \) Yang-Mills from \( d = 10 \) to \( p + 1 \) dimensions. In particular the \( 10 - (p + 1) \) scalar fields living on the D-brane, signify the D-brane excitations in the \( 10 - (p + 1) \) transverse dimensions. This tells us that the significance of the new open string subsector is to quantize the D-brane excitations.

An important property of D-branes is that when \( N \) of them coincide we get a \( U(N) \) gauge theory on their worldvolume. This follows because we have \( N^2 \) open string subsectors going from one D-brane to another and in the limit they are on top of each other all will

\(^2\) Note that this is a consequence of the fact that for type IIA(B), left-right exchange is a symmetry only when accompanied by a \( Z_2 \) spatial reflection with determinant \(-1(+1)\).
have massless modes and we thus obtain the reduction of $N = 1$ $U(N)$ Yang-Mills from $d = 10$ to $d = p + 1$.

If we consider the tension of D-branes they are proportional to $1/g$ where $g$ is the string coupling constant. Note that as expected at weak coupling they have a huge tension. An important property of D-branes is that they are BPS states. A BPS state is a state which preserves a certain number of supersymmetries and as a consequence of which one can show that their mass (per unit volume) and charge are equal. This in particular guarantees their absolute stability against decay.

3.4. Fivebranes and ALE spaces

In the context of $K3$ compactification we discussed $A_{n-1}$ singularities as corresponding to $n$ cosmic strings coming together on a point $z_0$ on the base $z$-sphere. Let us consider a three cycle $C$ consisting of a circle $\gamma$ around $z_0$ and the $T^2$ fiber above it. Note that as we go around the cycle the complex modulus of $T^2$ shifts by $\tau \to \tau + n$. Now let us do T-duality on the cycle of $T^2$ which vanishes as we approach $z_0$. In this way we exchange the role of complex moduli of torus with its (complexified) Kahler modulus. In particular as we go around $\gamma$ now we have $B \to B + n$ where $B$ corresponds to the antisymmetric field with components along $T^2$. This implies that

$$\int_C dB = n$$

which shows that we now have a source of $n$-units of magnetic charge for $B$, i.e. $n$ units of 5-brane charge. Note however in showing this relation we used $R \to 1/R$ once, and so this exchanges type IIA with type IIB. We thus learn that $n$ 5-branes of type IIA(B) is equivalent to type IIB(A) in the presence of an $A_{n-1}$ ALE singularity.

From these facts we can also deduce some aspects of the theory living on the fivebrane for type IIA and type IIB strings. In particular we learn that for type IIB 5-brane, the bosonic fields include a $U(1)$ vector field together with 4 scalars, and for type IIA 5-brane, the bosonic fields include an antisymmetric two form with self-dual field strength, and five scalars.
3.5. \( N = 1 \) Theories and p-branes

For \( N = 1 \) theories, the only antisymmetric gauge potential is the two-form \( B_{\mu\nu} \). We thus expect a 1-brane and a dual 5-brane. For heterotic string theory the 1-brane is identified with the closed string itself. In the type I case the 1-brane comes from the RR sector and is not identified with the fundamental string (note that the closed string for type I can decay to open string and thus does not couple to a conserved charge). The carrier of the 1-brane charge is a D1-brane. Moreover the dual magnetic state is a D5-brane. From the viewpoint of D-branes, the existence of the open string subsector for type I strings with 32 Chan-Paton indices can be viewed as follows: We consider the orientifold of type IIB. This induces a tadpole for a 10-form gauge potential (which is not dynamical) from the RR-sector which needs to be canceled by placing 32 9-branes. The strings ending on the 32 D9-branes is the open string sector of type I strings.

3.6. D-branes and T-duality

Consider type II strings with a D-brane wrapped around a circle and consider doing T-duality along the circle. We are interested in knowing what happens to D-brane after doing T-duality. Note that this exchanges type IIA and B, and thus the dimension of D-brane should change by an odd number. In fact what happens is that it goes down by one. This is because under T-duality the Dirichlet and Neumann boundary conditions get exchanged. Thus if a D-brane is wrapped around the circle the open strings ending on it have Neumann boundary conditions along the circle and after T-duality they become Dirichlet boundary condition, thus decreasing the worldvolume dimension of the D-brane by one unit. Similarly if the D-brane was not wrapped around the circle, and its position was represented by a point on the circle, after T-duality the Dirichlet boundary conditions turn to Neumann and we end up with one higher dimensional D-brane.

For type I string T-duality works in a slightly more delicate way: For example consider compactification of type I string on a circle. We take the radius of the circle to very small size, and by T-duality we obtain a theory which now has 32 D8-branes, instead of 32 D9-branes (as discussed above). Their position is labeled by 32 points which are symmetric under the \( Z_2 \) reflection on a circle and moving the positions correspond to changing the Wilson lines of the \( SO(32) \) theory along the circle. But there are also two additional special points which are the points along the circle fixed by the \( Z_2 \) involution. The best way to understand this theory is to consider type IIA theory compactified on a circle and
mod out by the orientifold symmetry which at the same time acts by a reflection on the circle. The two fixed points of this action are known as the orientifold plane. When the 32 D8 branes are on one or the other orientifold plane we have the $SO(32)$ gauge symmetry restored. In the generic position of the D8-branes that will be broken to $U(1)^{16}$.

4. Applications of D-branes

We have learned that type IIA,B strings admit a collection of D-branes. It is natural to ask what kinds of properties they lead to.

We have discussed that in compactification of string theory we often end up with singular limits of manifolds when some cycles shrink to zero size. A typical example is when an $n$-dimensional sphere shrinks to zero size. What is the physical interpretation of this singularity?

Suppose we consider for concreteness an $n$-dimensional sphere $S^n$ with volume $\epsilon \to 0$. Then the string perturbation theory breaks down when $\epsilon << \lambda$ where $\lambda$ is the string coupling constant. If we have $n$-brane solitonic states such as D-branes then we can consider a particle solitonic state corresponding to wrapping the $n$-brane on the vanishing $S^n$. The mass of this state is proportional to $\epsilon$, which implies that in the limit $\epsilon \to 0$ we obtain a massless soliton. An example of this is when we consider type IIA compactification on $K3$ where we develop an $ADE$ singularity. Then by wrapping D2-branes around vanishing $S^2$'s of the $ADE$ singularity we obtain massless states, which are vectors and charged under the Cartan $U(1)^r$ of the $ADE$. This in fact implies that in this limit we obtain enhanced $ADE$ gauge symmetry. Had we been considering type IIB on $K3$ near an $ADE$ singularity, the lightest mode would be obtained by wrapping a D3-brane around vanishing $S^2$'s, which leaves us with a string state with tension of the order of $\epsilon$. The dynamics in this regime is not well understood, as the light degrees of freedom are string-like rather than particle-like. This kind of regime which exists in other examples of compactifications as well is called the phase with tensionless strings.$^3$

Another interesting example along the lines above occurs when we consider Calabi-Yau threefold compactification of type IIB strings, in the presence of vanishing $S^3$'s. If an $S^3$ vanishes we obtain a massless particle by wrapping a D3 brane around $S^3$, which is a

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$^3$ We could have considered higher dimensional D-branes wrapping around the vanishing cycles, but in such cases by dimensional analysis one can see that the relevant mass scale would be smallest for the smallest dimension D-brane.
charged hypermultiplet of an $N = 2$ supersymmetric $U(1)$ theory. Suppose we have for instance 2 vanishing $S^3$'s as we change the scalar vev for the $U(1)$ vector multiplet (which controls the size of the $S^3$'s). Then we obtain a $U(1)$ theory with two charged massless hypermultiplets. We can then consider Higgsing the $U(1)$ and we are left with one massless hypermultiplet. This simple physical process has an interesting mathematical application: It corresponds to replacing the two shrunk $S^3$'s by two growing $S^2$'s! In other words the topology of the Calabi-Yau has changed and we obtain a manifold with one less $h^{2,1}$ and one more $h^{1,1}$. In this way a singular transition from one manifold to the other can be given a smooth and simple physical description.

In the above applications the D-brane is mainly wrapped about the internal direction and as far as the non-compact spacetime is concerned it behaves as a point-like or sometimes string-like object. However there is another useful application of p-branes and that is in mimicking the singularities of the compactified spacetime. In this case the p-brane will be filling the spacetime. One particularly simple application of this is as follows: By the discussion about the relation between 5-branes and $A_{n-1}$ singularities, since type IIA in the presence of $A_{n-1}$ singularity acquires $SU(n)$ gauge symmetry we thus conclude, by T-duality, that type IIB in the presence of $n$ coinciding 5-branes acquires also $SU(n)$ gauge symmetry in the 5+1 dimensional worldvolume of the coinciding 5-branes. This sounds very much like what we would have expected if instead of the 5-brane associated to the $NS-NS$ $B$ field, we were considering Dirichlet 5-brane. This is the first hint that there is a symmetry exchanging the two.

4.1. Counting Degeneracy of D-branes: Applications to Black-hole physics

Let us consider type IIA theory compactified on $K3$. Let us consider how many D2-branes (Dirichlet 2-branes) are there with some specific charges. In other words we are interested in counting the number of BPS states with specific charges given by the choice of an element in the second homology of $K3$ which represents the homology class of a D2-brane. Instead of being general let us consider an elliptic $K3$ and consider D2-branes having specific charges, corresponding to the wrapping of the D2-brane $n$-times over the elliptic fiber and once over the base.

In fact it is quite simple to find the appropriate configuration of such D2-brane: It consists of a degenerate Riemann surface consisting of a sphere (the base of the elliptic fibration) together with $n$ tori connected to it over $n$ points on the sphere. However there is a moduli space of such configurations: In fact we can move the $n$ points as we wish
over the base sphere. Moreover on a D2-brane lives a $U(1)$ gauge field and so as far as ground state configurations are concerned we can consider wilson lines of the $U(1)$ over this Riemann surface, which corresponds to choosing wilson lines on each of the $n$-tori. The choice of a $U(1)$ Wilson line on a torus is equivalent to the choice of a point on the dual torus (which can be easily understood by applying T-duality to the torus turning the D2-brane to D0-brane). Thus as we move the base points together with the wilson line on the $T^2$ above it, we span the (dual) $K3$. Thus we find that the moduli of configuration for the above D2-brane corresponds to the choice of $n$ points on $K3$. However the order of the points is irrelevent and so we can write

$$\mathcal{M}_{1,n} = \text{Sym}^n K3$$

where $\mathcal{M}_{1,n}$ denotes the moduli space of the configurations of D2-brane with charges $(1,n)$ corresponding to wrapping about the base and fiber, and $\text{Sym}^n K3$ denotes the $n$-fold symmetric product of $K3$, i.e. $K3^\otimes n / S_n$ where $S_n$ is the permutation group on $n$-objects.

To find the number of D2 branes we have to quantize the above moduli space. The analogy to keep in mind is that the moduli space of a point particle in a box is given by the choice of a point in the box, and quantizing that means choosing momentum modes, the ground states would correspond to zero momentum states. In the supersymmetric case that we are considering the number of ground states of this quantum system is given by the cohomology of $\mathcal{M}_{1,n}$. It is relatively simple to count the dimension of cohomology for $\mathcal{M}_{1,n}$ using orbifold realization of symmetric products (which tells us how to deal with the singularities of the space). For $n = 1$ we simply get the cohomology of $K3$ which is 24 dimensional. Let us symbolically denote these states by

$$|i\rangle = \alpha^{i}_{-1}|0\rangle$$

where $i = 1, \ldots, 24$. For $n = 2$, we should consider the ground states of $K3 \times K3 / Z_2$. From the untwisted sector we obtain the symmetrization of two copies of the above state, i.e.

$$|(i,j)\rangle = \alpha^{i}_{-1} \alpha^{j}_{-1}|0\rangle$$

where writing it in terms of “creation operators” has the advantage of making it manifest that we symmetrize over them. However now we also have the twisted sector where the two $K3$’s coincide. The ground states of the twisted sector consists thus of a single copy
of $K3$, and so its cohomology is again in correspondence with that of $K3$. These states we can represent by

$$\alpha^i_{-2}|0\rangle$$

where we have introduced new oscillators to keep track of the fact that this comes from two copies of $K3$ on top of each other. We thus see that for the $n = 2$ case we get exactly the same number of states as the physical states of bosonic oscillators! Continuing in this way we find that for $\mathcal{M}_{1,n}$ we find as many cohomology elements as the $n$-th level degeneracy of bosonic oscillator fock space! This sounds very remarkable! We are seeing a hint that somehow bosonic oscillators should be related to type IIA strings on $K3$. We will have more to say about this when we come to a discussion of string dualities in the next section.

Now let us consider further compactification on a circle down to 5 dimensions. In this case we can dualize on the radius of circle and we obtain an equivalent type IIB compactification on $K3 \times S^1$. The configuration of D2-branes we have been considering now get mapped to D3-branes wrapped around cycles of $K3$ and the circle. If we consider the limit where the size of $K3$ is small, the effective theory on the $D3$ brane is equivalent to a $1 + 1$ dimensional sigma model where the target space is $\mathcal{M}_{1,n}$, where the spatial direction of the $1 + 1$ dimensional theory is the $S^1$.

In the effective theory of type IIB compactified on $K3 \times S^1$, there are various kinds of gauge fields and charges. The RR gauge field is one source which couples to D-brane charges. We can also consider the Kaluza-Klein (KK) $U(1)$ field corresponding to the gauge field $A_\mu = g_{\mu \theta}$ where $\theta$ is the circle direction. We can consider BPS states with definite D-brane charge and $U(1)$ charge under the KK $U(1)$. We can also consider extremal black holes which carry KK charge as well as D-brane charge. It is natural to ask if the Bekenstein-Hawking entropy of the black hole which is $1/4$ of the area (in this case volume) of the horizon, is valid. A direct computation of the black hole metric shows that the expected entropy is

$$S_{\text{macroscopic}} = 2\pi \sqrt{Q_H n}$$

where $Q_H$ denotes the KK charge and $n$ is related to the same D-brane charge we have been considering. To find the analog of these states in the D-brane worldvolume we simply have to consider states with a definite momentum along the circle direction (Note that the $U(1)$ gauge symmetry is the same as translation along the circle). This implies that we should consider states in the effective $1 + 1$ dimensional theory where $L_0 - T_0 = Q_H$. To preserve BPS condition we need to consider states excited only on left-movers, i.e. $L_0 \neq 0, T_0 = 0$. 41
For large enough $Q_H$ the number of such states is given by the asymptotic growth of string degeneracy for a supersymmetric sigma model on a manifold of real dimension $4n$ (which is the dimension of $\mathcal{M}_{1,n}$). This gives the central charge $6n$ (taking account of fermions) and the asymptotic growth is thus

$$S_{\text{microscopic}} = 2\pi \sqrt{\frac{cN}{6}} = 2\pi \sqrt{Q_H n}$$

in agreement with the macroscopic entropy (4.1). This kind of match between black hole entropy and string state entropy can be generalized to many similar cases. A particularly easy case is type II compactification on tori where the constructions are particularly simple.

5. Web of String Dualities

We have already mentioned that T-duality connects certain string theories together: Type IIA and B get exchanged under T-duality. So does heterotic string $E_8 \times E_8$ and $SO(32)$ heterotic string, which get exchanged under T-duality. Thus up to compactification and the use of T-duality, which can be understood using string perturbation theory, we only have 3 inequivalent string theories: Type II, Type I and heterotic strings. It is natural to ask if these three are also connected in some way with each other. If there are any connections they should be realized in a non-perturbative way, because perturbative symmetries have already been accounted for by T-duality.

5.1. Type I/heterotic duality

Let us first consider a possible relation between type I string and heterotic strings. In 10 dimensions, the $N = 1$ theory with $SO(32)$ gauge symmetry has two different realizations as a string theory: Type I strings, as well as $SO(32)$ heterotic strings. As already discussed these theories both have the same matter content. However they cannot be trivially related, because they have completely different perturbative properties. There is one scalar in the theory whose expectation value plays the role of the coupling constant. It is thus natural to speculate that the strong coupling limit of one theory is equivalent to the weak coupling limit of the other. There is no way that we know at the present to prove such a conjecture, because it necessarily means going beyond perturbative definition of the theory and at the present we do not have a good understanding of how to do this. However there are some consequences of this duality that can in particular be checked.
One particularly nice check is to see how we can identify the heterotic string as a soliton in the type I string. Type I has an antisymmetric 2-form gauge field coming from the RR sector. A D1 brane is a source for this field. It is natural to ask if the fields propagating on this string are in any way similar to that of heterotic strings. We consider an infinitely long D1 brane. This introduces 2 new sectors in the type I theory: D1-D1 open strings and D1-D9 open strings (recall that the 32 Chan-Paton factors can be viewed as having 32 D9 branes). From the D1-D1 sector we obtain (after orientifold projection) 8 left and right-moving bosonic degrees of freedom as well as 8 right-moving fermionic degrees of freedom. From the D1-D9 sector we obtain 32 left-moving fermions. These are precisely the worldsheet degrees of freedom for the SO(32) heterotic strings in its fermionic formulation.

5.2. Type II/Type II U-duality and M-theory

As already remarked in connection with toroidal compactification of type II strings given that the space of expectation values for the scalars is of the form of a coset space $G/H$, and T-duality is a discrete subgroup of $G$ giving further identification of this space. This suggests that there may be further identifications of this space which have non-perturbative origin. It is in fact natural to expect that the identification space is maximal, i.e. $G(Z)$, an integral version of the group $G$, leading to the moduli space $G/H \times G(Z)$. The group $G(Z)$ is called the U-duality group.

There are two distinct ways this can be motivated further: One is to assume an eleven dimensional $N = 1$ theory exists. If so, as mentioned before, toroidal compactifications on $T^{d+1}$ which can also be obtained by type II string compactification on $T^d$ will give both an $SL(d+1, Z)$ symmetry from the $T^{d+1}$ torus and an $SO(d, d)$ symmetry from the type II string T-duality. These groups do not commute and lead to the group $E_d(Z)$ as discussed before. This identification in particular suggest that type IIA string which is a non-chiral $N = 2$ theory in $d = 10$ should be identified with a circle compactification of an 11-dimensional $N = 1$ theory, which is called the M-theory.

In 11 dimensional $N = 1$ theory the bosonic fields are very simple: In addition to the metric there is a three index antisymmetric field $C_{\mu\nu\rho}$ which couples to a 2 dimensional membrane. Upon compactification on a circle, we obtain in addition to the metric and the

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4 The inverse question is more difficult because type I string does not couple to a conserved charge as it has both closed and open string sectors.
three index field in 10 dimension, one scalar corresponding to the size of the circle, one vector corresponding to the off-diagonal components of the metric and an anti-symmetric two form by considering one index of $C$ to be along the circle. These are, as expected, the bosonic fields of type IIA strings. Note in particular that the coupling constant $\lambda$ of type IIA gets identified with the radius $R$ of the circle. In particular the precise relation (which can be easily obtained by relating the eleven dimensional fields to ten dimensional fields appearing in type IIA) in terms of the string coupling constant turns out to be

$$R^3 = \lambda^2$$

and thus the eleven dimensional theory $R \to \infty$ should emerge in the strong coupling regime $\lambda \to \infty$. As a further evidence for the existence of M-theory and its equivalence to type IIA, one can ask what would the Kaluza-Klein modes along the circle be related to in the 10 dimensional theory. Given the relation between the fields in M-theory and type IIA strings these would be the states which are charged under the vector field 1-form coming from the RR sector. They couple to D0-branes. We are thus naturally led to view the D0-branes (and their bound states) as the KK states of M-theory.

There is another way to motivate the U-duality group. Consider type IIB in 10 dimensions. For this theory $G = SL(2)$. In this case the coupling constant $\lambda$ and the scalar in the RR sector $\chi$ combine to form a complex field $\tau = \frac{1}{\lambda} + \chi$, which transforms according to the Mobius transformations under $SL(2,\mathbb{Z})$. The symmetry $\tau \to \tau + 1$ is the statement that the RR scalar $\chi$ is a periodic field–this follows from the existence of the magnetic dual D7-branes. The other conjectured symmetry $\tau \to -1/\tau$ is much more interesting and involves a strong/weak coupling duality. This transformation exchanges the two antisymmetric 2-forms of the NS-NS sector and the RR sector. Given that one couples to the fundamental string and the other to the D1-string, it implies that at strong couplings the D1-strings should behave the same way as the fundamental string does at weak coupling. Similarly the D5-brane gets exchanged with the NS 5-brane. As a check on this note that we already had seen that when $k$ NS 5-branes of type IIB coincide we obtain a $U(k)$ symmetry by its T-duality with type IIA on $A_{k-1}$ ALE space. It turns out that in this strong/weak duality exchange the D3-brane gets mapped to itself. If we consider $N$ parallel D3-branes we get $U(N)$ gauge symmetry and the invariance of D3-brane under strong/weak duality gets mapped to the conjectured Olive/Montonen duality on the worldvolume of D3-brane, which is an $N = 4$ theory in $d = 4$. As another check we
can ask whether the degrees of freedom on the D1-string look like that of the fundamental type IIB strings. If we consider an infinitely extended D1 string we have one extra open string D1-D1 sector. This gives rise to 8 left- and 8 right-moving bosons and fermions on the worldsheet, the same degrees of freedom as expected for type IIB strings. This is a strong evidence for the existence of $SL(2, \mathbb{Z})$ symmetry for 10 dimensional type IIB.

If we compactify type IIB on a circle the T-duality relates it to type IIA on an inverse sized circle. The $SL(2, \mathbb{Z})$ symmetry of type IIB will be the U-duality group for type IIA on a circle. This is the same as the symmetry expected from M-theory compactification on $T^2$, for which the $SL(2, \mathbb{Z})$ symmetry is just the global reparametrization of the torus (choice of two basis vectors for the lattice defining the torus). Upon further compactification the U-duality works as explained above. Thus the U-duality symmetry is a consequence of type IIB strong/weak duality and T-duality of type II strings.

5.3. $E_8 \times E_8$ and M-theory orbifold

If we consider the 5 string theories in 10 dimensions, we can ask what is the limit of strong coupling in each case. We have answered this question for four cases: Type IIB is self-dual, type ISO(32) and heterotic SO(32) are dual to one another, type IIA grows an extra dimension and becomes the 11-dimensional M-theory. What about $E_8 \times E_8$? It is easy to argue that it cannot be self-dual (by a simple field redefinition involving strong/weak exchange the theory does not come back to itself). So what could its strong coupling limit look like? There is another question which we could ask: If we believe in the existence of M-theory in 11 dimensions, it is natural to consider its compactification on a circle modded out by a $Z_2$ reflection. It is easy to see that this preserves half the supersymmetry and we end up in 10 dimensions with a theory with $N = 1$ supersymmetry. It is tempting to identify this as the strong coupling limit of $E_8 \times E_8$ heterotic string. Note that we also have two special fixed points on the circle. This conjecture is consistent with the presence of these fixed points (which fill the 10 dimensional spacetime) each of which is identified with a 9-brane carrying the $E_8$ gauge symmetry. So the fact that we have two factors in this heterotic theory would be related to the fact that we have two fixed points on a circle modded out by reflection! There are various hints that this is the right picture.
5.4. Evidence for F-theory

As we have seen Type IIA and heterotic \( E_8 \times E_8 \) in ten dimensions can be viewed as arising from a more fundamental theory in 11 dimensions, the M-theory. How about type IIB and the \( N = 1 \) theory with gauge group \( SO(32) \) (which has realizations as type I or heterotic \( SO(32) \) theory)? In addition the fact that type IIB has an \( SL(2, \mathbb{Z}) \) symmetry in ten dimensions, begs for a geometric explanation. From the M-theory viewpoint if we go to 9 dimensions as explained above we can see the \( SL(2, \mathbb{Z}) \) symmetry as a geometric symmetry of a torus. But in order to go to the 10 dimensional limit we have to take the limit of zero size torus, which gets us away from the domain of validity of geometric description of M-theory. Similar statement applies to the \( N = 1 \) theory with \( SO(32) \) gauge group and how it will have an M-theory realization. Given the \( SL(2, \mathbb{Z}) \) symmetry of type IIB in ten dimensions, it is natural to postulate the existence of a 12 dimensional theory, F-theory, which upon compactification on \( T^2 \), with modulus \( \tau \), would give rise to type IIB in 10 dimensions, or upon compactification on a cylinder (\( Z_2 \) orbifold of \( T^2 \)) would give \( SO(32) \) theory. Note that then not only the strong weak duality of type IIB would be manifest, but also the fact that a finite cylinder has two inequivalent limits gives rise to two theories that are connected, thus explaining type I/heterotic duality.

This picture is rather suggestive, but various obstacles would have to be overcome. For one thing, there is no known covariant 12 dimensional supergravity theory (and there are various no-go theorems). There are further hints in connection with integrable structures with signature \((2, 2)\) that a \((10, 2)\) signature for F-theory is rather natural to expect. Perhaps the most reasonable thing to expect for F-theory is a partially twisted topological theory in 12 dimensions whose BRST invariant states are the 10 dimensional degrees of freedom of type IIB strings.

Note that if we compactify type IIB on \( S^1 \), it should be equivalent to M-theory on \( T^2 \). Moreover type IIB itself should be identified with F-theory on \( T^2 \). Thus we learn that F-theory on \( T^2 \times S^1 \) is the same as M-theory on \( T^2 \).

5.5. String/String Duality

So far we have seen how the \( N = 2 \) theories, i.e. type II strings are related and behave for strong couplings. We have also seen how \( N = 1 \) theories are related and behave at strong couplings. In particular by the time we come to 9 dimensions there is only one \( N = 2 \) and one \( N = 1 \) theory (up to variation of moduli). It is natural to relate also the
$N = 2$ theories to $N = 1$ theories. In order to do this we will have to cut down the number of supersymmetries. The most natural way to do this in string theory is to go down to 6 dimensions and consider type IIA strings compactified on $K3$ and compare that with $N = 1$ strings (heterotic or type I) on $T^4$. In this case both theories in 6 dimensions will have $N = 2$ (non-chiral) supersymmetry. Moreover the moduli of scalars for both theories is

$$\frac{SO(20, 4)}{SO(20) \times SO(4) \times SO(20, 4; \mathbb{Z})} \times \mathbb{R}^+$$

where $R^+$ denotes the coupling constant. It is natural to conjecture that the strong coupling limit of one is equivalent to the weak coupling limit of the other. We have already seen a number of evidence for this statement. In particular we have seen that when $K3$ develops ADE singularity we obtain a non-perturbative enhanced gauge symmetry on the type IIA side due to wrapped D2 branes. This is also matched with a perturbative enhancement of gauge symmetry on the heterotic side. We identify $(P_L, P_R)$ Narain momenta of the heterotic string with a lattice element of the homology of $K3$. Having a vanishing 2-cycle for $K3$ is the same statement as $P_R = 0, P_L^2 = 2$ vector on the homology lattice of $K3$. We can also now explain why in computing the BPS states of type IIA strings on $K3$ we encountered the partition function of bosonic string: The same BPS states on the heterotic side would map to perturbative BPS states which are elementary string excitations which on the right-movers are on the ground state (i.e. no oscillators excited) and on the left-side are the excited oscillator states of the bosonic string, thus explaining the mysterious appearance of the partition function of bosonic strings in the degeneracy of BPS states for type IIA on K3 that we encountered. This is the strongest evidence for this duality (which is sometimes known as the string/string duality).

We can push this duality up in dimension and connect it with the other dualities already discussed: First note that type IIA on $K3$ is by definition M-theory on $K3 \times S^1$. This being equivalent to heterotic on $T^4$ implies, upon deletion of an $S^1$ from both sides, that M-theory on $K3$ is equivalent to heterotic on $T^3$. One evidence in favour of this is that they both have the same duality symmetry $SO(19, 3)$, which in the case of M-theory is the symmetry of moduli of Ricci-flat metrics on $K3$. This gets identified with the heterotic

\footnote{Note that the notion of strong coupling limit is a dimension dependent statement because the volume of the internal compactification rescales the value of the effective coupling in the lower dimension by $\frac{1}{\lambda^2} \rightarrow \frac{V}{\lambda^2}$.}
T-duality upon compactification on $T^3$. Moreover the overall size of $K3$ gets identified with the inverse of coupling of the heterotic theory.

This can be pushed up one more step as follows: Consider M-theory on elliptic $K3$’s, which we have already discussed in detail. Now consider the limit in which the size of the $T^2$ fiber goes to zero. Note that M-theory on $T^2$ in the limit of zero size is identified with type IIB in 10 dimensions. Thus we should obtain in this limit a compactification of type IIB on a sphere times a large radius $S^1$. The sphere is the base of the elliptic fibration, where the string coupling constant $\tau$ undergoes $SL(2,\mathbb{Z})$ monodromies. Or put differently we can view it as F-theory on elliptic $K3$. The T-duality group for elliptic $K3$ is $SO(18,2)$ and this again matches the T-duality group of heterotic string on $T^2$ (which is what one is led to again by deletion of the extra circle). At this stage we can also connect this duality to the dualities already discussed in 10 dimensions: Consider the limit of $K3 = T^2 \times T^2/\mathbb{Z}_2$, which is elliptic, in the context of F-theory compactification. Note that now we can take the fiber $T^2$ to correspond to a fixed complex structure with a large value of $\tau$. Given the interpretation of $\tau$ as the coupling constant of type IIB this point should be describable in terms of a perturbative type IIB compactification on $\frac{T^2}{\mathbb{Z}_2}$. For this to preserve half the supersymmetry, the $\mathbb{Z}_2$ must also exchange the left- and right- movers of strings. In other words in this limit we are led to the duality between an orientifold of type IIB on $T^2$ and the heterotic strings on $T^2$. If we take the size of the $T^2$ on type IIB side to be small we would be led back to type I theory in 10 dimensions compactified on $T^2$, being equivalent to the heterotic strings on $T^2$. By elimination of the extra $T^2$ we are thus led to heterotic/type I duality in 10 dimensions. Thus heterotic ($T^4$)/type IIA ($K3$) duality is a consequence of type I/heterotic duality in 10 dimensions. Of course we could logically reverse the arrows and say that type I/heterotic duality is a consequence of type IIA/heterotic duality in 6-dimensions.

5.6. The Adiabatic Principle and the web of String Dualities

In the preceding subsection we have seen the power of using fiberwise dualities to derive new dualities. This means that we start with two theories which are dual to one another. If we now consider varying the parameters of these two theories over a base space then, at least if we vary the parameters slowly, we expect the duality to continue to hold on the lower dimensional space. It turns out that this is a powerful principle to connect various string dualities in different dimensions to one another. It seems to be applicable even if the adiabaticity assumption is not strictly valid and the parameters vary
rapidly in some regions over the base. It would be very important to understand why this principle works! Note that whenever we get some new physics in lower dimension using this principle which does not follow in a trivial way from the dimensional reduction of the higher dualities, the new physics is hidden at the points where the adiabaticity assumption fails. At any rate, taking the adiabaticity principle for granted we can now generate new dualities and new physics in lower dimensions.

Let us just give one example of the application of this principle: Consider F-theory on $K3$ duality with heterotic on $T^2$. Now we can fiber both of these spaces over a $\mathbb{P}^1$, in such a way that on the heterotic side we get a $K3$ compactification of heterotic strings. On the F-theory side we get a Calabi-Yau three-fold which is an elliptic fibration over a base space which is itself a $\mathbb{P}^1$ fiber space over $\mathbb{P}^1$. Given a base space the elliptic Calabi-Yau is fixed (up to variation of moduli). Thus the number of inequivalent such choices is related to the number of inequivalent $\mathbb{P}^1$ fiber spaces over $\mathbb{P}^1$. These are parameterized by an integer $n$, and are known as the Hirzebruch surface $F_n$. The case of the elliptic CY 3-fold over $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is the one we have already discussed in detail in section 2. For $n \leq 12$ there does exist a smooth Calabi-Yau. On the heterotic side it turns out that this $n$ is mapped to the choice of the splitting of the instanton number between the two $E_8$’s. Recall that we need the total number of instantons to be 24. The two instanton numbers turn out to be $12 + n, 12 - n$. If we further compactify on $S^1$ we find duality between M-theory on the corresponding Calabi-Yau threefold with heterotic strings on $K3 \times S^1$. Continuing one more step we find duality between type IIA on Calabi-Yau threefold with heterotic strings on $K3 \times T^2$. These will give $N = 2$ theories in four dimensions. For example we learn that type IIA compactification on elliptic Calabi-Yau over $F_0$ is equivalent to $E_8 \times E_8$ heterotic strings on $K3 \times T^2$ with instanton numbers 12 for each $E_8$. It turns out that the moduli space of the vector multiplet on the type IIA side does not receive quantum string corrections (since the string coupling is in a hypermultiplet) and thus exact non-trivial results for the moduli of $N = 2$ field theories can be obtained from classical string considerations on type IIA side. Moreover the moduli of hypermultiplets on the heterotic side does not receive quantum corrections because the heterotic string coupling constant is in a vector multiplet. Thus the classical moduli of hypermultiplets on the heterotic side gives the exact quantum corrected result for the hypermultiplet moduli of type IIA. This phenomenon is also known as the ‘second quantized mirror symmetry’.

We can also consider the fibration of the 8 dimensional heterotic/F-theory on a two complex dimensional base, in which case we obtain duality between F-theory on Calabi-Yau fourfold with heterotic strings on Calabi-Yau threefold. This is a duality between two theories with $N = 1$ in $d = 4$. 

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6. Speculations and Open Questions

In this paper we have reviewed some aspects of string theories with emphasis on their solitonic/non-perturbative aspects. We have also seen that duality of one string theory with another can occur in a rather non-trivial way. Due to accumulation of massive evidence in their favor these dualities have the status of conjectural facts! These dualities exchange what looks like classical results in one theory to highly non-trivial quantum results on the other side. It clearly demonstrates the fact that what constitutes quantum corrections is partly a matter of convention. This suggests a rather revolutionary reformulation of quantum mechanics. This is a natural extension of various relativity principles we have learned in physics in this century: In general theory of relativity we have learned that many physically reasonable statements make sense only when we specify relative to which observers those statements are made. In quantum mechanics we learn that the very notion of reality is relative to an experimental setup. Here we are learning that the notion of quantum versus classical is also relative to which theory we measure it from.

One basic fact we have seen many times in the context of string duality is that in each regime of parameter space there is at most one simple description of the physics. This is somewhat similar to the complementarity principle of Bohr in the context of quantum mechanics: His view was that depending on the experiment one should either use the wave picture or the particle picture but not both at the same time.\footnote{We know that quantum mechanics ultimately developed in a direction somewhat different from this view of Bohr, in that the wave description is always the correct description. In the case of string dualities we have strong evidence that none of the string theories (nor M or F theory for that matter) are supreme to any other and each one is as fundamental as any other. This suggests that perhaps Bohr’s original view of complementarity may have more merits than the present view of quantum mechanics leads us to believe.}

Despite the tremendous progress made recently, we should not lose sight of the fact that the most interesting physical questions are still unanswered by these developments. In particular a pessimist might say that the recent discovery of the non-perturbative consistency of string theories in various dimensions with varying number of supersymmetries inevitably leads to the conclusion that the universe we live in is not apparently selected by consistency conditions alone! This may also make the precise realization of our universe in the context of string theory more difficult. In fact since we do not have a single example of a non-supersymmetric string vacuum which we can claim to understand deeply enough, we
are still rather far from the ultimate goal of string theory. This is also connected with the fact that non-supersymmetric theories will have no known criterion to rule out a non-zero cosmological constant, in apparent contradiction with the observed universe. Clearly we have still a lot to learn.

Another aspect of non-supersymmetric situations which we do not know much about involve aspects of black hole far away from being extremal (i.e. far from being BPS). In these situations we run into the usual puzzles of black holes (the information puzzle in particular). It seems natural to speculate that the two puzzles of cosmological constant and black hole information puzzle are related to one another, and what resolves one will also resolve the other issue. At any rate we have not made any real progress on either of these fronts.

We have learned a great deal about non-perturbative aspects of string theory. One of the main reasons we believe in duality is that string theory is a concrete theory with concrete properties and any statement about duality will have many consequences which can be readily checked using tools available in string theory. The same cannot be said yet, unfortunately, about M-theory nor F-theory. I thus believe that until we have a deeper understanding of these non-perturbative aspects of string theory and what is the right way to think about them, it would be prudent to continue calling the general subject we are studying by its own name: ‘string theory’!

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References

[1] J. Schwarz, “Lectures on Superstring and M Theory Dualities,” hep-th/9607201; J. Polchinski, “String Duality—A Colloquium,” hep-th/9607050; J. Polchinski, “TASI Lectures on D-Branes,” hep-th/9611050; A. Sen, “Unification of String Dualities,” hep-th/9609176; P. Townsend, “Four Lectures on M-theory,” hep-th/9612121.
H. Ooguri and Z. Yin, “TASI Lectures on Perturbative String Theories,” hep-th/9612254; M. Duff, “Supermembranes,” hep-th/9611203; M. Douglas, “Superstring Dualities, Dirichlet Branes and the Small Scale Structure of Space,” hep-th/9610041.
W. Lerche, “Introduction to Seiberg-Witten Theory and its Stringy Origin,” hep-th/9611190.
P. Aspinwall, “K3 Surfaces and String Duality,” hep-th/9611137; G. Horowitz, “The Origin of Black Hole Entropy in String Theory,” gr-qc/9604051; B. Greene, “String Theory on Calabi-Yau Manifolds,” hep-th/9702155.
[2] M. Green, J.H. Schwarz and E. Witten, “Superstring Theory,” (2 volumes), Cambridge University Press (1986).
[3] H. Ooguri and Z. Yin, “TASI Lectures on Perturbative String Theories,” hep-th/9612254.
[4] P. Aspinwall, “K3 Surfaces and String Duality,” hep-th/9611137.
[5] J. Polchinski, “TASI Lectures on D-Branes,” hep-th/9611050.
[6] M. Douglas, “Superstring Dualities, Dirichlet Branes and the Small Scale Structure of Space,” hep-th/9610041.
[7] A. Strominger, “Massless Black Holes and Conifolds in String Theory,” hep-th/9510135.
B. Greene, D. Morrison and A. Strominger, “Black Hole Condensation and the Unification of String Vacua,” hep-th/9504145.
[8] E. Witten, “Bound States Of Strings And p-Branes,” Nucl. Phys. B460 (1996) 335, hep-th/9510135.
[9] M. Bershadsky, V. Sadov and C. Vafa, “D-Strings on D-manifolds,” Nucl. Phys B463 (1996) 398, hep-th/.
M. Bershadsky, V. Sadov and C. Vafa, “D-branes and topological field theories,” Nucl. Phys. B463 (1996) 420, hep-th/.
[10] M. Douglas, D. Kabat, P. Pouliot and S. Shenker, “D-branes and Short Distances in String Theory,” hep-th/9608024.
[11] R. Dijkgraaf, E. Verlinde and H. Verlinde, “BPS Quantization of the Five-Brane,” hep-th/9604053.

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[12] A. Strominger and C. Vafa, “Microscopic origin of the Bekenstein-Hawking entropy,” Phys. Lett. B383 (1996) 44, hep-th 9601029

[13] G. Horowitz, “The Origin of Black Hole Entropy in String Theory,” gr-qc/9604051

[14] C. Hull and P. Townsend, “Unity of Superstring Dualities,” Nucl. Phys. B438 (1995) 109, hep-th/9410167

[15] E. Witten, “String Theory Dynamics in Various Dimensions,” Nucl. Phys. B443 (1995) 85, hep-th/9503124.

[16] S. Kachru and C. Vafa, “Exact Results for N=2 Compactifications of Heterotic Strings,” Nucl. Phys. B450 (1995) 69, hep-th/9505105.

[17] P. Horava and E. Witten, “Heterotic and Type I String Dynamics from Eleven Dimensions,” hep-th/9510209.

[18] J. Polchinski and E. Witten, “Evidence for Heterotic - Type I String Duality,” Nucl. Phys. B460 (1996) 525, hep-th/9510168.

[19] C. Vafa, “Evidence for F-theory,” Nucl. Phys B469 (1996) 403, hep-th/9602022.

[20] A. Sen, “Unification of String Dualities,” hep-th/9609176.