Dissipative Kerr solitons generated in high-Q optical microresonators provide unique opportunities for different up-to-date applications. Increasing the generation efficiency of such signals is a problem of paramount importance. We provided comprehensive analysis and found optimal conditions providing maximal theoretical pump-to-comb conversion efficiency (up to 100%) for the cases of free-running and self-injection-locked pump lasers. The discrepancy between numerical and analytical solutions is revealed. The dependence of the optimal coupling rate on the pump power was studied, in addition to the trade-off relations balancing the efficiency versus the number of comb lines. The methods to increase the total comb power were also discussed.

Microresonator-based frequency combs or microcombs [1–3] are perspective tools for modern photonics [4–8]. Coherent frequency combs or dissipative solitons in temporal representation are of particular interest. Unfortunately, the most accessible coherent frequency combs in the form of bright dissipative Kerr solitons [8] is realizable in the anomalous Group Velocity Dispersion (GVD) and suffer from a low pump-to-comb conversion efficiency, which had long been commonly known to be no more than 10% [8, 9]. To overcome this limitation, special feedback schemes [10, 11], normal dispersion usage [12–14] and pulsed pumping [15] have been suggested. However, recent studies demonstrated that it is possible to reach the values of 20–40% [16, 17] even without any special schemes, showing that this problem has not been fully investigated, and no fundamental limit of the conversion efficiency has been clearly defined. Although previous studies concerned the dependence of the pump conversion efficiency on the microresonator’s Free Spectral Range (FSR) [17], the input power and the coupling rate [18, 19], those dependencies were usually treated independently, and only general trends were shown. In reality, both FSR and coupling influence the threshold power – the former through the mode volume, the latter through the loaded quality factor. Furthermore, restrictions should be applied on the coupling and the pump power in order to keep the system above the generation threshold. This suggests a nontrivial optimization problem that has never been addressed. In this paper, we report for the first time, up to our knowledge, how the pump power, coupling rate, and dispersion interplay to allow up to a 100% theoretical high-conversion without any additional schemes, but at the cost of the comb line number. The analysis is performed for the cases of the free-running [20] and self-injection-locked [21–23] pump lasers.

Two main definitions of comb generation efficiency can be found in literature: the pump to total comb (pump to soliton) $\eta_{p2s}$ [17, 18, 24] and pump to comb (sidebands) $\eta_{p2c}$ [11, 12]. The first variant is more straightforward from a theoretical point of view, as the field can be naturally decomposed into a soliton part and a background part. The average output power of solitons can be easily calculated by summing the components of the hyperbolic-secant-squared shaped comb power [16, 20], and recalculating to the output [Supplementary 1]:

$$p_{\text{out, soliton}} = \frac{8\eta^2}{\pi} \sqrt{D_2 \zeta_{\text{eff}}} P_{th},$$

(1)

where $D_2 = 2D_2/\kappa$ is the normalized GVD coefficient (assuming the microresonator eigenfrequencies $\omega_{\mu} = \omega_0 + D_1 \mu + D_2 \mu^2 + \ldots$, the eigenfrequency closest to the pump $\omega_0$, the microresonator FSR $D_1$ and the relative mode number from the pumped mode $\mu$), $\kappa$ is the pumped mode linewidth, $\eta = \kappa_c/\kappa$ is the normalized coupling rate or coupling efficiency ($\eta = 1/2$ for critical coupling), $\kappa_c$ is the coupling rate, $\zeta_{\text{eff}} = -2(\omega_{\text{gen}} - \omega_0)/\kappa$ is the normalized pump frequency detuning (with $\omega_{\text{gen}}$ the laser generation frequency).

Note that we consider anomalous GVD ($D_2 > 0$) and use the negative detuning definition, where the solitons exist at positive $\zeta_{\text{eff}}$. The parametric instability threshold power $P_{th}$ is defined as follow:

$$P_{th} = \frac{\omega_0 n_2^2 V_{\text{eff}}}{8 c n_2 Q^2 \eta} = \frac{4 P_0 D_1}{27} \frac{1}{\eta(1-\eta)^2},$$

(2)

where $V_{\text{eff}} = \frac{\omega_0^2 n_2^2}{c} = \frac{\kappa_c}{\kappa}$ is the effective volume, and $\eta$ is the coupling efficiency.
where \( Q = \omega_0 / \kappa \) is the loaded quality factor, \( n_2 \) is the Kerr nonlinear refractive index, \( n_c \) is the microresonator mode group index, \( V_{\text{eff}} \) is the microresonator effective mode volume. In this context, it is convenient to have a dimensional constant independent of the coupling efficiency. Therefore we introduce the minimal threshold power \( P_0 = \frac{\pi^2 n_c^2 V_{\text{eff}}}{2 \kappa_0^2 c \epsilon_0 n_2} \), where \( \kappa_0 = \kappa - \kappa_c \) is the microresonator intrinsic decay rate. Minimal threshold power is achieved at \( \eta = 1/3 \), which corresponds to the undercoupling regime, while the threshold power at critical coupling is 1.2 times higher. Remarkably, fulfilling the criterion \( \eta = 1/3 \) coincides with the optimal coupling efficiency for the self-injection locking at weak backscattering [25]. Then, dividing Eq. (1) by the input power at the coupler \( P \), we immediately get \( \eta_{p2c} \).

Experimentally, the soliton parameters are difficult to measure in the time domain because of the ultra-broad bandwidth required for their detection, while the optical power spectrum is usually easier to measure and enough to determine the average power of the soliton thanks to Parseval’s theorem. The spectral representation of the solitons results in a frequency comb with a uniform phase under a hyperbolic-secant envelope, while the representation of the solitons results in a frequency comb with a constant background field results in addition to the central line saturates near the maximum soliton number \( N_{\text{max}} \) set by the minimum distance at which solitons do not interact with each other [9]. Equation Eq. (3) shows that both efficiencies go down as the square of the pump amplitude, suggesting it to be as low as possible.

Generally, the detuning \( \zeta_{\text{eff}} \) is a free parameter. Obviously, it should be as large as possible for higher efficiency. Additionally, a stabilization of \( \zeta_{\text{eff}} \) is required to ensure the locking of the system to the desired regime despite the influence of various fluctuations [20]. The easiest way to achieve this stabilization is the self-injection locking (SIL) mechanism [26], which locks pump frequency to the microresonator eigenfrequency. As a result, we have two key ways to fix the range of the optimal detuning: (i) the maximum possible detuning enabling soliton generation (cut-off) \( \zeta_{\text{eff}}^{\max} = \pi^2 f^2 / 8 \) and (ii) the maximum detuning allowed by nonlinear SIL regime \( \zeta_{\text{eff}} \approx 3 (f^2 / 2)^{1/3} = (f^2 / 2)^{-1/3} \) [16]. We can see that \( \zeta_{\text{eff}} \approx \zeta_{\text{eff}}^{\max} \). For small pump powers, which are optimal for high generation efficiency, those both boundary limits tend to converge to \( \zeta \approx 1.1 \) with \( \zeta_{\text{eff}}^{\max} \) hard (or impossible) to reach.

It is important to note that both the SIL and maximal detunings do not depend on the GVD coefficient. Thus, the pump-to-comb efficiency, Eq. (3), has a global maximum over the dispersion coefficient

\[
\eta_{p2c} = \frac{8 \eta^2 \zeta_{\text{eff}}}{f^2 \pi^2} \text{ at } \frac{d_2}{f^2} = \frac{4 \zeta_{\text{eff}}}{\pi^2},
\]

while the pump-to-soliton efficiency \( \eta_{p2s} \) is monotonic over \( d_2 \). We can see that the global maximum of pump-to-comb efficiency \( \eta_{p2c} \) is 100 % at \( \eta = 1 \) regardless of the pump power if we lock to the maximal detuning [see red dashed line in Fig. 1] and 90.9 % for the SIL state at near-threshold pump [see green dashed line in Fig. 1]. Note that the comb width above 3 dB level in this regime is \( N_{\text{lines}} = \frac{1}{2} \sqrt{\frac{f^2}{d_2}} \times 2 \arccosh \sqrt{2} \approx 1.76 \) in units of the intermode distance. This number effectively shows a number of comb line pairs around the pump above the 3 dB level. Here we come to the trade-off problem between the efficiency and the number of comb lines.

The optimum described by Eq. (4) gives \( d_2 \geq 0.45 \), while in most reported experiments, \( d_2 < 0.1 \). Furthermore, \( d_2 \geq 1 \) provides \( N_{\text{sol}}^{\max} < 1 \), coinciding with the state when the width of the
hypercritical-sect-squared soliton becomes comparable with the microresonator circumference. Although the soliton still exists for greater dispersion values, lower pump, and detunings (for SIL optimum), hyperbolic secant ceases to be a proper stationary solution, and the formula Eq. (1) becomes inaccurate. So, we performed the numerical analysis of dissipative Kerr soliton properties using Lugiato-Lefever equation formalism [27, 28]. This study showed that the expression Eq. (3) is accurate only close to (but not exactly at) the cut-off detunings and \( f > 2 \) [see Fig. 1, solid lines]. For SIL detuning, there is also a possibility to reach 100% efficiency [see Fig. 1, green lines], found numerically. More details are presented in Supplementary material 2.

To get more deep insight into the problem, we continue studying the Eq. (3) for the near-cut-off detuning. First, the normalized pump amplitude should be greater than 1 for the soliton generation to be possible and then \( f_x \approx 2 \) for the analytics to be valid. Secondly, the soliton should fit the ring \( d_2 < 1 \). This restriction also makes the number of lines \( N_{\text{lines}} > 1 \) (for \( \zeta_{\text{eff}} \approx 1 \)), which is quite close to the natural demand for having more than one line pair in the comb around the pump. Last but not least, the coupling efficiency \( \eta \in [0;1] \) by definition. Outstandingly, these restrictions are independent on the pump frequency detuning and conversion efficiency definition choice.

Figure 2 shows the dependence of \( \eta_{\text{p2c}} \) on the coupling rate and internal losses for the cut-off detuning and fixed other parameters (silicon nitride [16]). We can see that both \( \kappa_c \) and \( \kappa_0 \) cannot be chosen arbitrarily. We can introduce \( \kappa_{\max} = \sqrt{\frac{P_0}{f_x f_{\text{c}}} \kappa_0} \), which bounds \( \kappa_c \) [see horizontal part of the red line in Fig. 2] and \( \sqrt{4/27} \kappa_{\max} \) bounds \( \kappa_0 \) [see intersection of the red line and red dash-dotted line in Fig. 2]. Note that the \( \kappa_{\max} \) actually does not depend on \( \kappa_0 \) as \( P_0 \propto \kappa_0^2 \) and thus consists only of other trivial parameters and pump power. Another important observation is that the maximum efficiency always lies on the boundary \( f = 1 \) curve, confirming our previous speculations. The numerical study [see Supplementary 2] showed that the analytical solution is not accurate for low \( f < f_x \), after which the efficiency drops. So, for both definitions of the efficiency and both maximum detuning values, we can use \( f = f_x \) (e.g. the power should be minimized closer to the threshold) and strong overcoupling \( \eta = 1 \) to maximize the conversion. The global maximum for the given system parameters can be estimated with Eq. (3) with \( f = f_x, \eta = 1 \) and \( d_2 = 2D_2/\kappa_{\max} \) [See also Supplementary 3].

Now we build the pump-to-comb efficiency map. After we fix the pump amplitude \( f = f_x \) the detuning values \( \zeta_{\text{eff}} \) and \( \eta_{\text{eff}} \) become fixed, and the expression Eq. (3) has two free parameters: dispersion coefficient \( d_2 \) and coupling coefficient \( \eta \). The last one is the easiest to be tuned. However, the first parameter also depends on the coupling, so it is better to extract it explicitly for further optimization. Considering Eq. (2), the optimal pumping regime \( f = f_x \) imposes restrictions to the coupling

\[
\kappa_c = \frac{\kappa_c}{\kappa_0} = \frac{3}{\sqrt{2}} \frac{f_x}{f_{\text{c}}} \sin \left( \frac{1}{3} \arccos \left( \frac{f_x}{f_{\text{c}}} + \frac{\pi}{6} \right) \right) - 1,
\]

where we introduce normalized pump amplitude at minimal threshold \( f_0 = \sqrt{P_0/P_{\text{th}}} \) for simplicity. Formula Eq. (5) directly connects the required pump power with the optimal coupling level without using other parameters. This dependence is shown in Fig. 3. For large pump \( f_0/f_x \gg 3 \) it tends to \( \kappa_c = 3 \frac{\sqrt{3}}{2} \frac{f_x}{f_{\text{c}}} - \frac{\pi}{2} \). Note, the minimal possible pump power \( P = P_0 f_x^2 \) to maintain \( f = f_x \) is obtained at subcritical coupling \( \kappa_c = \kappa_0/2 \) which is in agreement with Eq. (2).

As we fix the detuning and combine Eq. (5) and Eq. (3), this dependence can be plotted as a colormap, having only two free parameters: \( d_2 = D_2/\kappa_0 \) and \( \kappa_c = \kappa_c/\kappa_0 \) [see Fig. 4]. We can see that for fixed dispersion value there is an optimum over the coupling value [green line in Fig. 4]. It can be shown that for small GVD, it tends to \( \kappa_c/\kappa_0 \approx 4 \). In Fig. 4 we fix \( f_x = 1 \) as the most illustrative case though it is not quite correct. For greater \( f_x \) the global maximum will shift up and eventually above \( d_2 = 1 \) line [see Supplementary Fig S4]. Another essential way to represent the efficiency map is to recalculate the \( d_2 \) into the number of comb lines above 3 dB level for given \( f_0 \). So, we present the map of pump-to-comb efficiency in coordinates \( f_0-N_{\text{lines}} \) in Fig. 5. One can see that the pump-to-comb efficiency is reduced with the line number increase, and its growth saturates with the power increase.
The increased pump power (for fixed threshold) provides greater detuning values; however, this effect is weaker. The threshold pumping regime (so that the coupling is changed with the pump power, according to Eq. (5)) is always superior to the case of fixed coupling [See Supplementary Fig. S5]. If the $P_0$ is increased through the microresonator intrinsic loss $\kappa_0$ the optimal curve goes up. This means that by increasing both the pump and the threshold power for higher output comb power, the microresonator intrinsic losses should never be increased.

It is also useful to add the line corresponding to a variation of the intrinsic losses to the previous efficiency maps in Figs. 4 and 5. We should note that both $N_{lines}$ and $P/P_0$ depend on the microresonator intrinsic loss rate $\kappa_0$, and if we tune it having other parameters fixed, the working point will follow the curve $d_2 = f_0D_2/k_{max}$ where $f_0 \propto 1/\kappa_0$. This path is demonstrated in Figs. 5 and 4 for $D_2/2\pi = 14.3 \text{ MHz, } k_{max}/2\pi = 1 \text{ GHz}$ with pink curve and an arrow shows the way of $\kappa_0$ increase. This line also represents the same configurations as the upper branch of the red line of Fig. 2. Such analysis also shows that the high intrinsic losses are not optimal for conversion efficiency.

In conclusion, an analytical guideline has been proposed to optimize the comb generation efficiency in a microresonator. The theory shows the possibility of reaching 100%. The analytics can predict the optimal parameters for pumps above $f = 2$. For a lower pump, optimal detunings will be slightly lower and dispersion slightly higher. Generally, the system should be pumped not far from the generation threshold ($f \approx 1.6$). This regime has a smaller optimal dispersion value and maximal efficiency for most detuning values and ensures no wasted pump power – more pump power does not change the soliton amplitude and average power. For this purpose, the coupling should be increased or the pump power should be reduced. In practical terms, a trade-off between the efficiency and comb line number should be chosen at the design stage, forcing the appropriate dispersion profile to be tailored (from geometry and material engineering) for an achievable coupling. However, it should be noted that dispersion engineering can also affect the threshold power and coupling due to the changes in the effective mode volume and effective index. Additionally, the regime of the self-injection locking of the pump laser can help to maintain the necessary pump frequency detuning since, at threshold pumping, the soliton existence range in the spectral domain is quite narrow. Though the analytical theory was found to fail in the low pump – low detuning region, numerical simulations show the optimum point near $f \approx 1.6, d_2 \approx 0.95$. Finally, to increase the absolute comb power, the same strategy should be applied, with the only difference being that the generation threshold power should be maximized. To this aim, the safest approach is to increase the coupling efficiency significantly while keeping the intrinsic losses low.

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**Supplemental document.** See Supplement for supporting content.

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**Fig. 5.** Pump-to comb-efficiency vs. $f_0$ and comb lines number calculated via Eq. (3) and Eq. (5). Additional lines and parameters the same as in Fig. 4.

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Detailed analysis of ultimate soliton microcomb generation efficiency: supplemental document

1. NORMALIZATION

Following [1], we proceed from Maxwell equations with electric field \( E = [V/m] \) to the normalized equations for the modes

\[
E(t) \propto \sum_{\mu} \sqrt{\frac{4\pi^2 \kappa}{3\chi^2 \omega \mu}} a_{\mu}(t) e^{-\mu t - i\omega D_{\mu t}} \propto [V/m].
\]  

(S1)

Here \( \kappa \) is group refraction index, \( \chi \propto [2\pi Hz] \) is the loss rate (resonance FWHM), \( \chi_3 \propto [m^2/V^2] \) is third-order nonlinear optical susceptibility. It is also usually changed to the Kerr nonlinearity coefficient \( n_2 = \frac{3\chi_3}{4\epsilon_0 c n_0^2} \propto [m^2/W] \). Thus for the new variable dimension, we get unity and normalization coefficient is

\[
\sqrt{\frac{\kappa}{\chi_22\text{mod}}} = \sqrt{\frac{8\eta P_{th}}{2\chi n_0^2 n_{\text{eff}} V_{\text{mod}}}},
\]

where we also used the effective index \( n_{\text{eff}} \), mode volume \( V_{\text{eff}} \) and threshold power [2]

\[
P_{\text{th}} = \frac{\kappa^2 n_2 n_{\text{eff}} V_{\text{eff}}}{8\epsilon_0 n_2^2} \propto [W].
\]

(S2)

We also note that usually \( n_{\text{eff}} = n_g \) is assumed.

The equation for \( a_{\mu} \) can be transformed to a Lugiato–Lefever equation (LLE) [1, 2] thus its soliton part of solution can be written in spectral representation as

\[
a_{\mu} = \sqrt{\frac{d_2}{2}} \text{sech} \left( \frac{\pi \mu}{2} \sqrt{\frac{d_2}{\zeta_{\text{eff}}}} \right) e^{i\varphi_{\text{sol}}}
\]

(S3)

\[
\varphi_{\text{sol}} = \arctan \left( \frac{\sqrt{2\pi^2 f^2 \zeta_{\text{eff}}^2 - 16\zeta_{\text{eff}}^2}}{4\zeta_{\text{eff}}}, 0 \right).
\]

(S4)

where \( d_2 \) is normalized dispersion and \( \zeta_{\text{eff}} \) is detuning. All quantities here are dimensionless. We should note that Eq. (S3) is exact only for an Nonlinear Schrödinger Equation that is undumped undriven LLE, being a limit case at \( \zeta_{\text{eff}} \gg 1 \). This limitation is also in line with a natural demand for a soliton width to be smaller than the microresonator circumference \( \sqrt{d_2 / \zeta_{\text{eff}}} \ll 2\pi \).

For the travelling wave in a waveguide, we have an optical power \( P = \epsilon c n_0 |E|^2 S / 2 \propto [W] \), where \( S \) is the guide cross-section area. Reversing the normalization, we get the modal fields

\[
P_{\mu} = \epsilon c n_0 |S| / 2 \epsilon_0 \kappa n_0 n_{\text{eff}} V \frac{d_2}{2} \text{sech}^2 \left( \frac{\pi \mu}{2} \sqrt{\frac{d_2}{\zeta_{\text{eff}}}} \right) =
\]

\[
= c \frac{4\eta P_{th} d_2}{\kappa n_{\text{eff}} L} \text{sech}^2 \left( \frac{\pi \mu}{2} \sqrt{\frac{d_2}{\zeta_{\text{eff}}}} \right) \propto [W]
\]

(S5)

Using the transmittance coefficient from the ring resonator to the straight waveguide \( T^2 = \eta t_{\text{rt}} \), where \( t_{\text{rt}} = n_{\text{eff}} L / c \) is the round-trip time [3], we obtain

\[
P_{\mu}^{\text{out}} = 4\eta^2 P_{th} \frac{d_2}{2} \text{sech}^2 \left( \frac{\pi \mu}{2} \sqrt{\frac{d_2}{\zeta_{\text{eff}}}} \right) \propto [W]
\]

(S6)

So the total output power is

\[
P_{\text{out}} = \sum_{\mu} P_{\mu}^{\text{out}} \approx \int P_{\mu}^{\text{out}}(\mu) d\mu = \frac{8\eta^2}{\pi} \sqrt{d_2 \zeta_{\text{eff}}} P_{th} \propto [W]
\]

(S7)
as $\mu$ is dimensionless number. To get the energy of soliton, we need to integrate the total output power over round-trip time

$$E = \int_0^{t_{rt}} P_{out} dt = P_{out} t_{rt} \propto |f|$$

(S8)

The total sideband power we can simply deduce the central line power with $\mu = 0$ in Eq. (S6) from the total Eq. (S7). Finally, dividing by the input power, we obtain (3) of the main text. The above formulas becomes inaccurate when $d_2 \approx 1$, which indicates the moment when the width of the hyperbolic secant-squared soliton becomes comparable with the microresonator circumference and the summation and the integration of hyperbolic sech-squared spectrum become different. For low pump and low detuning, this formula is also inaccurate following Eq. (S3) failure. Nevertheless, for a half of the region of interest (high-pump near-cut-off detuning), the above formalism is applicable. In next section, a numerical study is performed to give an insight on the above analytics validity.

2. THEORY AND NUMERICAL MODELING

Here we compare the results based on the above theory [2, 4] with the numerical modelling using the pulse propagation in Lugiato-Lefever equation formalism [5, 6]. In this study we find that the solution Eq. (S3) actually diverges with the numerically obtained waveforms near cut-off
detuning at low pump power. Figure S1 shows the simulated and analytical solitons for the cut-off and SIL detunings with corresponding optimal dispersion. It is interesting to note, that though the theoretical soliton profile changes significantly for the case of cut-off detuning, its efficiency remains the same (see Fig. 1). This suggests that the analytical soliton has wrong phase dependence for low $f$.

In Fig. S2, the simulated efficiency values (left) and their difference with the analytical values (right) are shown. We can see that as the pump amplitude goes to 1 the analytic solution starts to overestimate the numerical one [see also left panel of Fig. S3]. At the same time, the values at small detunings (e.g. in self-injection locking regime) are underestimated [see also right panel of Fig. S3]. For $f < 1.5$ the efficiency is always less than 90%. The maximal value of 100% appears at $f = 1.6$ at the SIL detuning and then travels to the cut-off, reaching it near $f = 2$, where the analytical hyperbolic secant squared solution becomes precise. Generally, in the region of bad analytical approximation $f < 2$ the optimal detunings will be slightly lower than analytically predicted and dispersion – slightly higher. The highest concentration of the high-efficiency regime points is near $f \approx 1.6$.

3. ANALYTICAL MODEL

First we obtain the expression of the pump to comb efficiency for a fixed pump, while varying the microresonator losses and coupling [Fig. 2 of the main text]. To increase the level of generalization, we normalize all rates and dispersion to the maximal coupling rate $\kappa_{\text{max}} = \sqrt{\frac{2P}{4\hbar \nu_x}} = \sqrt{\frac{8\nu_x}{\hbar}} \frac{1}{f_x}$. Then for the pump to comb efficiency (3), we get

$$\eta_{p2c}^{f=f_x} = \frac{4\kappa_c}{f_x (\kappa_0 + \kappa_c)} \left( \sqrt{d_2 \kappa_c - \frac{d_2}{f_x} (\kappa_0 + \kappa_c)} \right).$$

(S9)

Here $\kappa_c = \kappa / \kappa_{\text{max}}$, and $d_2 = D_2 / \kappa_{\text{max}}$. We also used $\zeta_{\text{eff}} = \frac{\kappa / f_x}{\kappa_{\text{max}}} \kappa_0$ as we showed in previous section that the theory works better near cut-off. Maximum value for the Eq. (S9), found in $(\kappa_c, \kappa_0) \rightarrow (\kappa_{\text{max}}, 0)$, can be written as $\eta_{p2c}^{f=f_x} \approx 4(\sqrt{d_2} - \bar{d}_2 / f_x) / f_x$. For typical 1 THz Si$_3$N$_4$ microresonator [4], the $P_0 \approx 9$ mW and $\kappa_0 = 188$ MHz. Then for the pump power $P \approx 40$ mW, the maximal coupling rate is $\kappa_{\text{max}} \approx 1$ GHz. Having $D_2 \approx 14.3$ MHz and $f_x = 1.5$ we get $\eta_{p2c}^{f=f_x} = 30\%$ for single-soliton comb.

Combining the expressions (3) and (5), we obtain the formula for the pump to comb efficiency dependence on pump amplitude normalized to minimal threshold $f_0$, normalized coupling rate...
\[ \eta_{\text{p,2c}}^{f=f_x} = \frac{8k_{c}f_{x}^{2}}{27f_{0}^{2}} \left( f_{x} \sqrt{6d_{2} \left( \frac{f_{0}^{2}k_{c}}{4f_{x}^{2}} \right)^{1/3} - d_{2}} \right) . \] 

Here \( k_{c} = \kappa_{c}/\kappa_{0} \), \( d_{2} = D_{2}/\kappa_{0} \) and we also used \( \xi_{\text{eff}} = \frac{k_{c}}{f_{x}^{2}} \) as well. Figure S4 shows the efficiency map for \( f_{x} = 2 \). Comparing it with the Fig. 4, 5 we can see that the maximum goes behind the \( d_{2} = 1 \) with growing \( f_{x} \). The other important observation is that the number of lines grows with \( f_{x} \).

4. POWER CONSIDERATION

For some important applications, the efficiency can be a less concern than the absolute power of the generated comb. The formula (1 in main text) suggests that it does not depend on the pump power.
power directly and increases mostly with the threshold power. The increased pump power (for fixed threshold) provides greater possible detuning values; however, this effect is weaker. In Fig. S5, the total comb power without pumped mode normalized to $P_0 (P_{\text{out}} = \eta p^2 c P)$ is shown depending on the normalized pump power $f_0^2$ for different coupling rates. The thick blue line corresponds to the threshold pumping case $f = 1$, so that the coupling is changed with the pump power, according to expression (4) of main text (solid line for maximum detuning and dashed – for locked detuning). We can see that this regime is always superior to the case of fixed coupling [see green, purple and orange lines]. The lines in Fig. S5 do not change if the minimal threshold power $P_0$ is changed by means of the material parameters. If the $P_0$ is increased by means of the microresonator intrinsic loss $\kappa_0$, then the optimal curve goes up. This means that though we should increase both pump and the threshold power for higher output comb power, the microresonator intrinsic losses should be never increased.

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