Partial difference equations over compact Abelian groups, II: step-polynomial solutions

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Abstract

This paper continues an earlier work on the structure of solutions to two classes of functional equation. Let $Z$ be a compact Abelian group and $U_1, \ldots, U_k \leq Z$ be closed subgroups. Given $f : Z \rightarrow \mathbb{T}$ and $w \in Z$, one defines the differenced function

$$d_w f(z) := f(z + w) - f(z).$$

In this notation, we shall study solutions to the system of difference equations

$$d_{u_1} \cdots d_{u_k} f \equiv 0 \quad \forall (u_1, \ldots, u_k) \in \prod_{i \leq k} U_i,$$

and to the zero-sum problem

$$f_1 + \cdots + f_k = 0$$

for functions $f_i : Z \rightarrow \mathbb{T}$ that are $U_i$-invariant for each $i$.

Our main result is that solutions to either problem can always be decomposed into pieces which either solve a simpler system of equations, or have some special ‘step polynomial’ structure.

It has previously been shown that the $Z$-modules of solutions to these problems can be described using a general theory of ‘almost modest $\Delta$-modules’. Various features of the global structure of these solution $Z$-modules could then be extracted from results about the closure of this general class under certain natural operations, such as forming cohomology groups.

This work follows a similar pattern. We will augment the definition of ‘almost modest $\Delta$-modules’ further, to isolate a subclass in which elements

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*Research supported by a fellowship from the Clay Mathematics Institute
can be represented by the desired ‘step polynomials’. We shall then find that this subclass is closed under the same operations as in that previous work, which will again lead to the desired conclusions about the solutions to PDCEs and zero-sum problems.
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1 Introduction

This paper continues the work of [1] (‘Part I’). As in that paper, let $Z$ be a compact metrizable Abelian group, $U = (U_1, \ldots, U_k)$ a tuple of closed subgroups of $Z$, and $A$ an Abelian Lie group (often $A = T := \mathbb{R}/\mathbb{Z}$). Given a measurable function $f : Z \rightarrow A$ and an element $w \in Z$, we define the associated differenced function to be

$$d_w f(z) := f(z - w) - f(z).$$

Part I introduced two classes of functional equation for such functions $f : Z \rightarrow A$.

- The partial difference equation, or PD$^\infty$E, associated to $Z$ and $U$ is the system

$$d_{u_1} \cdots d_{u_k} f = 0 \quad \forall u_1 \in U_1, w_2 \in U_2, \ldots, u_k \in U_k$$

(since we quotient by functions that vanish a.e., this means formally that for strictly every $u_1, \ldots, u_k$, the left-hand side is a function $Z \rightarrow A$ that vanishes at almost every $z$).

- The zero-sum problem associated to $Z$ and $U$ is the problem of solving the equation

$$f_1(z) + \cdots + f_k(z) = 0 \quad \text{for } m_Z\text{-a.e. } z$$

among $k$-tuples of measurable functions $f_i : Z \rightarrow A$ such that each $f_i$ is $U_i$-invariant. A tuple $(f_i)^k_{i=1}$ satisfying (3) is a zero-sum tuple of functions.

For each of these problems, the set of solutions (that is, functions $f$ satisfying the PD$^\infty$E in the first case, and zero-sum tuples $(f_1, \ldots, f_k)$ in the second) form a Polish $Z$-module. Part I was concerned with the global structure of these Polish $Z$-modules, and established a basic description of them as two families of examples of ‘almost modest $\Delta$-modules’.
In this paper, our interest will be in the form of the individual functions that solve these equations. Our main result is already suggested by the several examples analyzed in the Introduction and Section 12 of Part I. Here it will suffice to recall just one of the most complex among those examples. Recall that $\lfloor \cdot \rfloor$ denotes the integer-part function on $\mathbb{R}$, and that $\{ \cdot \}$ denotes the canonical selector $\mathbb{T} \to [0, 1) \subset \mathbb{R}$. Then Example 12.7 in Part I showed that the functions $\sigma, c : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}$ defined by

$$
\sigma(s, x) := \{s_1\}\{x_2\} - \{s_2\}\{x_1 + s_1\} \mod 1
$$

and

$$
c(s, t) = \{s_1\}\{t_2\} - \{t_1\}\{s_2\} \mod 1
$$

satisfy the equation

$$
\sigma(t, x) + \sigma(s, x + t) = \sigma(s, x) + \sigma(t, x + s) + c(s, t),
$$

which, as explained there, may be read as giving a zero-sum quintuple of functions on $\mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$, with each function invariant under a different subgroup of this group.

The main result of this paper will be that, in a sense to be made precise below, the basic ‘building block’ solutions to PD$^\infty$Es or zero-sum problems may always be (chosen to be) assembled out of $\lfloor \cdot \rfloor$ and $\{ \cdot \}$ along these lines. To formulate these theorems properly, we will introduce in Section 3 a precise notion of ‘step polynomial’ functions on a compact Abelian group. For now let us simply remark that a ‘step polynomial’ on $\mathbb{Z}$ is a function for which there is a partition of $\mathbb{Z}$ into ‘geometrically-simple’ pieces such that, on each of those pieces, the function is given by a polynomial in some fractional parts of characters of $\mathbb{Z}$.

In the first place, step polynomials will appear as representative cocycles of the cohomology groups $H^p_m(\mathbb{Z}, \mathbb{T})$, which played a key rôle in the analysis in Part I. A precursor to this fact can be found as [2, Proposition 9.4], and we give a complete proof in Subsection 5.2 below. Our main results, Theorems A and B below, assert that step polynomials similarly appear as a complete list of solutions to both PD$^\infty$Es and zero-sum problems, modulo degenerate solutions. Those theorems also give the related fact that, if a degenerate PD$^\infty$E-solution (resp. zero-sum tuple) happens to be a step polynomial, then it may be decomposed into PD$^\infty$E-solutions (resp. zero-sum tuples) corresponding to simpler equations so that the summands are also step polynomials. Conclusions of this second kind will be an essential inclusion in some of the inductive proofs that will lead to Theorems A and B, as well as having some interest in their own right.

Theorems A and B below are best stated using the language of $\Delta$-modules from Part I, and in particular the PD$^\infty$E-solution and zero-sum $\Delta$-modules associated to
a given base group $Z$ and subgroup-tuple $U = (U_1, \ldots, U_k)$. Fix $Z$ and $U$, and assume that $U_1 + \cdots + U_k = Z$. (If this is not so, then the PD$^{\text{ce}}$E and zero-sum problems may simply be solved on each coset of $U_1 + \cdots + U_k$ independently, as discussed in Part I.) Let $\mathcal{M} = (M_e)_{e \subseteq [k]}$ and $\mathcal{N} = (N_e)_{e \subseteq [k]}$ be the $\Delta$-modules of PD$^{\text{ce}}$E solutions and zero-sum tuples associated to this $Z$ and $U$ (Subsection I.5.4). Recall the main results of Part I: in this setup, the submodules $\partial_{M} (M_{k-1})$ of degenerate PD$^{\text{ce}}$E solutions, and $\partial_{N} (N_{k-1})$ of degenerate zero-sum tuples, are relatively open and co-countable in $M_{[k]}$ and $N_{[k]}$ respectively.

Also, before proceeding, we should recall from Corollaries A$^\prime$ and B$^\prime$ in Part I that in case $A$ is a Euclidean space, these $\Delta$-modules have vanishing structural homology except in the lowest nonvanishing position of each structure complex. As a result, PD$^{\text{ce}}$E-solutions for $k \geq 2$ and zero-sum tuples for $k \geq 3$ could all be expressed quite easily in terms of degenerate solutions for Euclidean $A$. On the other hand, some of the auxiliary theory to be developed later fails for some Euclidean modules, so we will simply exclude these now.

**Theorem A** Suppose that $U_1 + \cdots + U_k = Z$ and that $A$ is a compact-by-discrete $Z$-module (this includes all compact and discrete modules). The cosets of $\partial_{M} (M_{k-1})$ in $M_{[k]}$ all contain representatives that are step polynomials. Also, if an element of $\partial_{M} (M_{k-1})$ is a step polynomial, then it is the image of a $k$-tuple in $M_{[k-1]}$ consisting of step polynomials.

If $f : Z \to \mathbb{T}$ solves the PD$^{\text{ce}}$E-system associated to $U$, then by an iterative appeal to Theorem A it can be decomposed as

$$f = \sum_{e \subseteq [k]} f_e,$$

where:

- $f_e$ solves the PD$^{\text{ce}}$E-system associated to the sub-tuple $(U_i)_{i \in e}$;
- each $f_e$ is a step polynomial on every coset of $\sum_{i \in e} U_i$.

**Theorem B** Suppose that $U_1 + \cdots + U_k = Z$ and that $A$ is a compact-by-discrete $Z$-module. The cosets of $\partial_{N} (N_{k-1})$ in $N_{[k]}$ all contain representatives that are step polynomials. Also, if an element of $\partial_{N} (N_{k-1})$ is a step polynomial, then it is the image of a $k$-tuple in $N_{[k-1]}$ consisting of step polynomials.

Similarly to the above, this implies that any zero-sum tuple $(f_i^k)_{i=1}^k$ as in (3) can be decomposed as

$$(f_i^k)_{i=1}^k = \sum_{e \subseteq [k], |e| \geq 2} (g_{e,i}^k)_{i=1}^k$$

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where:

- \((g_{e,i})_{i=1}^{k}\) is a zero-sum tuple for every \(e\);
- \(g_{e,i} = 0\) if \(i \notin e\);
- each \(g_{e,i}\) is a step polynomial on every coset of \(\sum_{i \in e} U_i\).

As they stand, Theorems A and B are vacuous in case \(Z\) is a finite group, since every function on a finite Abelian group may be written as a step polynomial. However, knowing these results for arbitrary compact Abelian groups, we will be able to extract some (highly ineffective) quantitative dependence via a compactness argument. This will promise some control on the ‘complexity’ of the functions involved, nontrivial even for finite groups.

To formulate this, we next invoke a notion of complexity of step polynomial functions. That notion will not be defined until Section 10 where the theorem will be proved, but intuitively it bounds the number of cells, the degrees and the coefficients involved in specifying the step polynomial.

**Theorem C**  For every \(k \geq 1\) there is an \(\varepsilon > 0\) such that for every \(d \in \mathbb{N}\) there is a \(D \in \mathbb{N}\) for which the following holds. Let \(Z\) be a compact metrizable Abelian group, let \(U = (U_i)_{i=1}^{k}\) be a tuple of subgroups of \(Z\), and let \(\mathcal{M} = (M_e)_e\) be the associated \(\mathbb{T}\)-valued PD\(^{\infty}\)E-solution \(\Delta\)-module. If \(f \in M_{[k]}\) is such that \(d_0(f, g) < \varepsilon\) for some step polynomial \(g \in \mathcal{F}(Z)\) of complexity at most \(d\), then \(f \in f' + \partial_k(M_{[k-1]})\) for some step polynomial \(f' \in M_{[k]}\) of complexity at most \(D\). That is, in the decomposition (3) one may choose \(f_{[k]}\) to have complexity at most \(D\).

Thus, if a function solves a PD\(^{\infty}\)E, and can be well-approximated by a step polynomial of a certain complexity, then it agrees with a PD\(^{\infty}\)E-solution which itself has a bound on its complexity, up to a degenerate solution. Since we will see that any element of \(\mathcal{F}(Z)\) may be approximated in \(d_0\) by step polynomials (this will follow easily from Lemma 3.5), in principle this can be applied to a set of representatives for each class in \(M_{[k]}/\partial_k(M_{[k-1]})\), and so gives a quantitative enhancement of the first conclusion of Theorem A.

It will be clear that the analog of Theorem C holds also for zero-sum tuples, and can be proved in the same way, but we will not spell out those details separately. The same argument should also work for other Lie target modules \(A\), but we will also set this aside for the sake of brevity.

Interestingly, there do not seem to be analogous quantitative versions of the second assertions in Theorems A and B. In the setting of Theorem A, if \(Z\) and
each $U_i$ are fixed compact Abelian Lie groups, then given a step polynomial $f \in \partial_k(M^{(k-1)})$, one can bound the minimal complexity of pre-images $(f_1, \ldots, f_k) \in \partial_k^{-1}\{f\}$ in terms of only the complexity of $f$. However, it can happen that this bound must deteriorate as one considers increasingly ‘complicated’ Lie groups $Z$ and $U$, and such a simple bound is actually impossible for some infinite-dimensional $Z$ and $U$. This will be discussed further, and witnessed by examples, in Subsection 10.3.

Since our proof of Theorem C is by compactness-and-contradiction, it does not give explicit bounds. One could presumably extract such bounds by making all steps of the proof suitably quantitative, but even then one would expect them to be extremely poor. On the other hand, even if one ignores the topological aspects of the proof, these results have a similar flavour to more classical structure theorems in real algebraic (or, more appropriately, semi-algebraic) geometry, where quantitative bounds are well-known to grow extremely rapidly: see, for instance, the classical monograph [4], or [3].

2 Background and basic definitions

We shall refer to [1] as ‘Part I’, and will freely use the definitions and results of that paper. They will be cited by prepending ‘I’ to their numbering in that paper: for instance, ‘Theorem X.Y in Part I’ will be ‘Theorem I.X.Y’.

2.1 Compact and Polish Abelian groups

Like Part I, this work will focus on measurable functions defined on a compact metrizable Abelian group. Assuming metrizability incurs no loss of generality, as explained in Section I.1.2.

A function from such a group to a Polish space is ‘measurable’ if it is measurable with respect to the Haar-measure completion of the Borel $\sigma$-algebra; it is ‘Borel’ if it is Borel measurable. We will freely use the Measurable Selector Theorem for the former notion of measurability.

Many of our functions will take values in an Abelian Lie group. By this we understand a locally compact, second-countable Abelian group whose topology is locally Euclidean. It need not be connected or compactly generated.

If $Z$ is a compact metrizable group and $M$ is any Polish Abelian group, then, as in Part I, the space of Haar-a.e. equivalence classes of measurable functions $Z \rightarrow M$ is denoted $\mathcal{F}(Z, M)$, and is regarded as another Polish Abelian group with the topology of convergence in probability. We also abbreviate $\mathcal{F}(Z, \mathbb{T}) =: \mathcal{F}(Z)$. We shall largely assume basic harmonic analysis for locally compact Abelian
groups: a suitable reference is [10]. In particular, we shall sometimes make use of the following, both of which follow from Theorems (9.8) and (24.7) in [10].

**Theorem 2.1** (Structure of Abelian Lie groups). If $A$ is an Abelian Lie group, then

$$A \cong \mathbb{R}^n \oplus T^d \oplus D$$

for some countable discrete Abelian group $D$. If, in addition, $A$ is compactly-generated, then we may take $D = \mathbb{Z}^s \oplus F$ for some $d \geq 0$ and finite Abelian group $F$.

**Theorem 2.2** (Structure of compact Abelian groups). If $Z$ is a compact Abelian group, then it has a sequence of closed subgroups $U_1 \supseteq U_2 \supseteq \cdots$ such that $\bigcap_{i \geq 1} U_i = \{0\}$ and each quotient $Z/U_i$ is a Lie group. Given any such sequence, every character on $Z$ is lifted from a character on $Z/U_i$ for some $i$: equivalently,

$$\hat{Z} = \bigcup_{i \geq 1} \hat{Z} \cap U_i^\perp.$$  

In addition to standard notation from harmonic analysis, the following additional terms will be convenient later.

If $Z$ is a compact Abelian group, then an **enlargement** of it is another compact Abelian group $Z'$ containing $Z$ as a subgroup.

An **affine function** on $Z$ is a function $f : Z \rightarrow T$ of the form $\theta + \chi$ for some $\theta \in T$ and $\chi \in \hat{Z}$. The affine functions form a closed, translation-invariant subgroup $A(Z) \leq F(Z)$. More generally, if $Y$ is another compact Abelian group then an **affine map** $f : Z \rightarrow Y$ is of the form $f(z) = y_0 + \chi(z)$ for some $y_0 \in Y$ and continuous homomorphism $\chi : Z \rightarrow Y$. It is an **affine isomorphism** if $\chi$ is an isomorphism.

A **torus** is a compact group isomorphic to $T^d = \mathbb{R}^d/\mathbb{Z}^d$ for some $d$. For $T^d$ itself there is a canonical choice of fundamental domain, $[0, 1)^d \subset \mathbb{R}^d$. For any $t \in T^d$, we let $\{t\}$ denote its unique representative in $[0, 1)^d$. If $Z$ is a torus and $\chi : Z \rightarrow T^d$ is a choice of affine isomorphism, then $\{\chi\} : Z \rightarrow [0, 1)^d$ denotes the composition of $\chi$ with $\{\cdot\}$.

### 2.2 Functions, sets and partitions

Let $X$, $Y$ and $Z$ be sets, and let $\chi : X \rightarrow Y$ and $\gamma : X \rightarrow Z$ be functions. Then $\chi$ **factorizes through** $\gamma$ if $\chi = f \circ \gamma$ for some $f : Z \rightarrow Y$; equivalently, if the level-set partition of $\gamma$ refines that of $\chi$.

If $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of any set, then $\mathcal{P} \vee \mathcal{Q}$ denotes their common refinement, and $\mathcal{P} \preceq \mathcal{Q}$ denotes that $\mathcal{Q}$ is already a refinement of $\mathcal{P}$. If $\mathcal{P}$ is a
partition of a set $S$ and $T \subseteq S$, then
\[ \mathcal{P} \cap T := \{ C \cap T \mid C \in \mathcal{P} \}, \]
a partition of $T$. If $\mathcal{P}$ is a partition of an Abelian group $Z$, and $z \in Z$, then
\[ \mathcal{P} - z := \{ C - z \mid Z \in \mathcal{P} \}. \]
Given also a subgroup $W \leq Z$, the partition $\mathcal{P}$ is $W$-invariant if $\mathcal{P} - w = \mathcal{P}$ for all $w \in W$; of course, this does not require that the individual cells of $\mathcal{P}$ be $W$-invariant.

If $\mathcal{P}$ and $\mathcal{Q}$ are Borel partitions of a compact Abelian group $Z$, then $\mathcal{P}$ almost refines $\mathcal{Q}$ if there is a Borel subset $Y \subseteq Z$ with $m_Z(Y) = 1$ and $\mathcal{P} \cap Y \supseteq \mathcal{Q} \cap Y$; they are almost equal if $\mathcal{P} \cap Y = \mathcal{Q} \cap Y$ for such a $Y$.

Relatedly, if $Z$ is a set and $U$ and $V$ are any covers of it (not necessarily partitions), then $V$ is subordinate to $U$ if for every $U \in U$ there is a $V \in V$ such that $U \subseteq V$. This will also be denoted by $U \succeq V$, as it is obviously equivalent to refinement in case $U$ and $V$ are partitions.

Lastly, if $\mathcal{P}$ is a partition of $S$ then $\sim_\mathcal{P}$ denotes the corresponding equivalence relation on $S$:
\[ s \sim_\mathcal{P} t \iff s, t \text{ lie in same cell of } \mathcal{P}. \]

### 3 Step functions and step polynomials

The section introduces the ‘step polynomials’ that appear in the formulations of Theorems A and B, and builds up some basic theory for them.

#### 3.1 Quasi-polytopal partitions and step-affine maps

Our convention below is that a convex polytope in a Euclidean space may contain some of its facets and not others, and may lie in a proper affine subspace.

**Definition 3.1** (Quasi-polytopal partitions and step functions). *If $Z$ is a compact Abelian group and $\mathcal{P}$ is a partition of $Z$, then $\mathcal{P}$ is quasi-polytopal (‘q.-p.’) if there are*

- an affine map $\chi : Z \rightarrow \mathbb{T}^d$,
- a partition $\Omega$ of $[0, 1)^d$ into convex polytopes

such that $\mathcal{P} \leq \{ \chi \}^{-1}(\Omega)$.

A subset $C \subseteq Z$ is *q.-p. if the partition $\{ C, Z \setminus C \}$ is q.-p.*

For any set $S$, a function $f : Z \rightarrow S$ is step if its level-set partition $\{ f^{-1}(s) \mid s \in S \}$ is q.-p.
The following properties are immediate.

**Lemma 3.2.** Quasi-polytopal partitions have the following properties:

- If \( \theta : Z \rightarrow Y \) is affine and \( \mathcal{P} \) is a q.-p. partition of \( Y \), then \( \theta^{-1}(\mathcal{P}) \) is a q.-p. partition of \( Z \).
- If \( \mathcal{P} \) and \( \mathcal{Q} \) are partitions of \( Z \) with \( \mathcal{Q} \) q.-p., and \( \mathcal{P} \preceq \mathcal{Q} \), then \( \mathcal{P} \) is q.-p. \( \blacklozenge \)

**Lemma 3.3.** If \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are two q.-p. partitions of \( Z \), then \( \mathcal{P}_1 \lor \mathcal{P}_2 \) is also q.-p.

*Proof.* For \( i = 1, 2 \), let \( \chi_i : Z \rightarrow \mathbb{T}^{d_i} \) be affine maps and \( \Omega_i \) be partitions of \( [0,1)^{d_i} \) into convex polytopes such that \( \mathcal{P}_i \preceq \{ \chi_i \}^{-1}(\Omega_i) \). Then \( (\chi_1, \chi_2) : Z \rightarrow \mathbb{T}^{d_1+d_2} \) is also affine; the product partition \( \Omega_1 \otimes \Omega_2 \) (whose cells are products of cells from \( \Omega_1 \) and \( \Omega_2 \)) is a partition of \( [0,1)^{d_1+d_2} \) into convex polytopes; and \( \mathcal{P}_1 \lor \mathcal{P}_2 \preceq \{ (\chi_1, \chi_2) \}^{-1}(\Omega_1 \otimes \Omega_2) \). \( \square \)

**Lemma 3.4.** If \( \mathcal{P} \) is a partition of a compact Abelian Lie group \( Z \), and \( Z_0 \leq Z \) is the identity component, then \( \mathcal{P} \) is q.-p. if and only if \( (\mathcal{P} - z) \cap Z_0 \) is a q.-p. partition of \( Z_0 \) for every \( z \in Z \).

*Proof.* If \( \mathcal{P} \) is q.-p. and \( z \in Z \), then the map \( \theta : Z_0 \rightarrow Z : z_0 \mapsto z_0 + z \) is affine, and \( (\mathcal{P} - z) \cap Z_0 = \theta^{-1}(\mathcal{P}) \), so this direction follows from Lemma 3.2.

On the other hand, let \( \mathcal{P}' \) be the partition of \( Z \) into cosets of \( Z_0 \), and let \( \chi : Z \rightarrow \mathbb{T}^{d} \) be a homomorphism whose kernel equals \( Z_0 \) (such exists, because characters separate points). Then \( \mathcal{P}' \) is a partition of \( [0,1)^d \) into sufficiently small boxes, then \( \mathcal{P}' = \{ \chi \}^{-1}(\mathcal{P}') \).

Let \( z_1, \ldots, z_m \) be a cross-section of \( Z_0 \) in \( Z \). If \( (\mathcal{P} - z_i) \cap Z_0 \) is q.-p. for each \( i \), then there are affine maps \( \theta_i : Z_0 \rightarrow \mathbb{T}^{D_i} \) and convex polytopal partitions \( \Omega_i'' \) of \( [0,1)^{D_i} \) such that \( (\mathcal{P} - z_i) \cap Z_0 \preceq \{ \theta_i \}^{-1}(\Omega_i'') \). Let \( \theta_i' : Z \rightarrow \mathbb{T}^{D_i} \) be an extension of \( \theta_i \) to all of \( Z \), and let \( \theta_i'' \) be the composition of \( \theta_i' \) with rotation by \( z_i \). Then

\[
\mathcal{P} \preceq \mathcal{P}' \lor \bigvee_{i \leq m} \{ \theta_i'' \}^{-1}(\Omega_i''),
\]

and this latter is q.-p. by Lemma 3.3 so \( \mathcal{P} \) is also q.-p. \( \square \)

**Lemma 3.5.** If \( \mathcal{U} \) is an open cover of a compact metrizable Abelian group \( Z \), then it is subordinate to some q.-p. partition.

*Proof.* Choosing a suitable metric on \( Z \), Theorem 2.2 and Lebesgue’s Number Lemma imply that there are a Lie group quotient \( q : Z \rightarrow Z_1 \) and an open cover \( \mathcal{V} \) of \( Z_1 \) such that \( \mathcal{U} \) is subordinate to \( q^{-1}(\mathcal{V}) \). This justifies assuming that \( Z \) is a Lie group, but in that case a partition of \( Z \) into connected components, and then of
each connected component into sufficiently small boxes, has the desired property, by another appeal to Lebesgue’s Number Lemma.

We will next introduce a class of maps that respect \( q \)-p. partitions. We will first define the corresponding maps on Euclidean spaces, before returning to compact groups.

**Definition 3.6.** If \( Q \subseteq \mathbb{R}^d \) is a convex polytope in a Euclidean space, then a map \( f : Q \to \mathbb{R}^r \) is **step-affine** if there are a partition \( \mathcal{Q} \) of \( Q \) into further convex polytopes, and, for every \( C \in \mathcal{Q}, \) an affine map \( \ell_C : \mathbb{R}^d \to \mathbb{R}^r, \) such that \( f|_C = \ell_C|_C. \)

**Lemma 3.7.** If \( f : Q \to \mathbb{R}^r \) is step-affine and \( A : \mathbb{R}^r \to \mathbb{R}^s \) is affine, then \( A \circ f \) is step-affine.

**Lemma 3.8.** If \( f : Q \to \mathbb{R}^r \) and \( g : Q \to \mathbb{R}^s \) are step-affine, then so are \( (f, g) : Q \to \mathbb{R}^{r+s} \) and, in case \( r = s, \) \( f + g : Q \to \mathbb{R}^r. \)

**Proof.** If two partitions of \( Q \) consist of convex polytopes, then so does their common refinement.

**Lemma 3.9.** If \( f : Q \to \mathbb{R}^s \) is step-affine and \( \mathcal{R} \) is a pairwise-disjoint collection of convex polytopes in \( \mathbb{R}^s \) which covers \( f(Q), \) then \( f^{-1}(\mathcal{R}) \) is a partition of \( Q \) which has a refinement into convex polytopes.

**Proof.** Letting \( \mathcal{Q} \) be as in Definition 3.6, it is obvious that \( C \cap f^{-1}(\mathcal{R}) \) consists of convex polytopes for each \( C \in \mathcal{Q}. \)

**Corollary 3.10.** If \( f : Q \to \mathbb{R}^s \) and \( g : R \to \mathbb{R}^p \) are step-affine and \( f(Q) \subseteq R \subseteq \mathbb{R}^s, \) then \( g \circ f \) is step-affine.

**Lemma 3.11.** If \( f : Q \to \mathbb{R}^s \) is step-affine and \( D \subseteq Q \) is a finite union of convex sub-polytopes of \( Q, \) then \( f(D) \) is also finite union of convex polytopes.

**Proof.** Clearly we may assume that \( D \) is a single convex polytope. Let \( \mathcal{Q} \) be the partition appearing in Definition 3.6. Then \( f(C \cap D) \) is an affine image of a convex polytope for each \( C \in \mathcal{Q}, \) so their union has the required form.

In the first place, the relevance of step-affine maps to the setting of compact Abelian Lie groups stems from the following.

**Lemma 3.12.** If \( \alpha : \mathbb{T}^d \to \mathbb{T}^s \) is an affine map, then there is a step-affine map \( f : [0, 1)^d \to [0, 1)^s \) which makes the following diagram commute:
Proof. By Lemma 3.8, we may argue coordinate-wise, and hence assume $s = 1$. In this case there are $\theta \in \mathbb{T}$ and $n_1, \ldots, n_d \in \mathbb{Z}$ such that

$$\alpha(t_1, \ldots, t_d) = \theta + n_1 t_1 + \ldots + n_d t_d,$$

using the obvious coordinates in $\mathbb{T}^d$, and hence

$$\{\alpha\}(t_1, \ldots, t_d) = \{\theta + n_1 t_1 + \ldots + n_d t_d\}.$$

Let $\mathcal{Q}$ be the partition of $[0, 1)^d$ into the sets

$$C_m := \{(x_1, \ldots, x_d) \in [0, 1)^d \mid \{\theta\} + n_1 x_1 + \ldots + n_d x_d \in [m, m + 1]\}$$

for integers $-|n_1| - \cdots - |n_d| - 1 \leq m \leq |n_1| + \cdots + |n_d| + 1$. This is a partition into convex polytopes, and on the set $C_m$ the desired function $f$ agrees with the affine function

$$\ell_{C_m}(x_1, \ldots, x_d) = \{\theta\} + n_1 x_1 + \ldots + n_d x_d - m.$$

This has the some useful corollaries.

**Corollary 3.13.** If $\Psi$ is a $q$.-p. partition of $\mathbb{T}^d$, then there is a partition $\mathcal{Q}$ of $[0, 1)^d$ into convex polytopes such that $\Psi \preceq \{\cdot\}^{-1}(\mathcal{Q})$.

**Proof.** Definition 3.1 gives an affine map $\alpha : \mathbb{T}^d \rightarrow \mathbb{T}^s$ and a partition $\mathcal{R}$ of $[0, 1)^s$ into convex polytopes such that $\Psi \preceq \{\alpha\}^{-1}(\mathcal{R})$. Now Lemma 3.12 gives a step-affine map $f : [0, 1)^d \rightarrow [0, 1)^s$ which makes the diagram in that lemma commute. Having done so, Lemma 3.9 gives a convex-polytopal partition $\mathcal{Q} \succeq f^{-1}(\mathcal{R})$, and so the commutativity of that diagram gives $\Psi \succeq \{\cdot\}^{-1}(f^{-1}(\mathcal{R})) \preceq \{\cdot\}^{-1}(\mathcal{Q})$.

**Corollary 3.14.** If $\Psi$ is a $q$.-p. partition of a compact Abelian Lie group $Z$, and $\Psi_1$ is the refinement consisting of all connected components of cells of $\Psi$, then $\Psi_1$ is also $q$.-p.
Proof. First, if $Z_0 \leq Z$ is the identity component, and $\mathcal{R}$ is the partition of $Z$ into $Z_0$-cosets, then clearly $\mathcal{P}_1 \succeq \mathcal{P} \lor \mathcal{R}$, and $\mathcal{P} \lor \mathcal{R}$ is still q.-p. We may therefore consider $Z_0$-cosets separately, or, equivalently, assume that $Z = Z_0$. In this case, assuming $Z = \mathbb{T}^d$, if $\mathcal{Q}$ is the partition provided by the previous corollary, then every cell of $\mathcal{Q}$ is connected, and the map $\{\cdot\}^{-1} : [0, 1)^d \to \mathbb{T}^d$ is just the restriction of the covering map $\mathbb{R}^d \to \mathbb{T}^d$, hence continuous, so $\{\cdot\}^{-1}(\mathcal{Q})$ also refines $\mathcal{P}_1$.

Remark. The previous lemma and corollary are false for general compact Abelian groups. For example, for the totally disconnected group $Z := (\mathbb{Z}/2\mathbb{Z})^\mathbb{N}$, the partition $\{Z\}$ is q.-p., but the partition into singletons is certainly not.

Corollary 3.15. If $\alpha : Z \to Y$ is an affine map of compact Abelian groups with $Y$ a Lie group, and $C \subseteq Z$ is a q.-p. set, then $q(C) \subseteq Y$ is a q.-p. set.

Proof. Since $C$ is q.-p., it is lifted from some Lie group quotient of $Z$. Also, $\alpha$ must factorize through such a quotient of $Z$, since $Y$ is Lie. Combining these two quotients, we may assume $Z$ itself is Lie.

Having done so, the result will be true for $C$ if it is true for $C \cap (z + Z_0)$ for every identity-component coset $z + Z_0 \leq Z$, so we may assume $C \subseteq Z_0$, and hence that $Z$ is a torus. This implies that the image $\alpha(Z)$ lies in an identity-component coset of $Y$, so we may assume that $Y$ is also a torus.

Finally, letting $Z = \mathbb{T}^d$ and $Y = \mathbb{T}^s$, the result follows by Lemma 3.11 and the commutative diagram of Lemma 3.12.

We can now proceed from Definition 3.6 to the following.

Definition 3.16 (Step-affine map). Let $Z$ be a compact Abelian group.

- A map $f : Z \to \mathbb{R}^s$ is step-affine if it equals $f_0 \circ \{\chi\}$ for some affine map $\chi : Z \to \mathbb{T}^d$ and step-affine map $f_0 : [0, 1)^d \to \mathbb{R}^s$.

- A map $f : Z \to \mathbb{T}$ is step-affine if it equals $\psi \circ f_1$ for some homomorphism $\psi : \mathbb{R} \to \mathbb{T}$ and some step-affine $f_1 : Z \to \mathbb{R}$.

- Finally, if $Y$ is another compact Abelian group, then a map $f : Z \to Y$ is step-affine if $\chi \circ f$ is step-affine for every $\chi \in \mathcal{A}(Y)$.

Let $\mathcal{F}_{sa}(Z, Y)$ denote the subset of step-affine members of $\mathcal{F}(Z, Y)$, for $Y$ equal to either $\mathbb{R}^s$ or another compact Abelian group.

In case $f : Z \to \mathbb{R}$ is step-affine, the q.-p. partition implicit in its definition will be said to control $f$.

The following lemma is obvious from the properties of Euclidean step-affine maps established above.
Lemma 3.17. Step-affine functions have the following properties.

- Sums and scalar multiples of step-affine functions \( Z \to \mathbb{R}^s \) are still step-affine.

- If \( Y \) is compact Abelian, then sums of step-affine functions \( Z \to Y \) are still step-affine, and so \( F_{sa}(Z,Y) \) is a subgroup of \( F(Z,Y) \).

- If \( Y \) is compact Abelian and \( S \subseteq \hat{Y} \) is a generating set for \( \hat{Y} \), then \( \psi : Z \to Y \) is step-affine if and only if \( \chi \circ \psi \) is step-affine for every \( \chi \in S \).

- If \( Y \) is a compact Abelian Lie group, then any step-affine map \( Z \to Y \) factorizes through a Lie-group quotient of \( Z \).

- If \( f : Z \to \mathbb{R}^s \) is step-affine, then so is \( \{f\} \).

- If \( f : Z \to T \), then \( f \) is step-affine if and only if \( \{f\} : Z \to [0,1) \) is step-affine.

- If \( f : Z \to \mathbb{R} \) is step-affine, and \( \mathcal{R} \) is a locally finite partition of \( \mathbb{R} \) into intervals, then \( f^{-1}(\mathcal{R}) \) is a q.-p. partition of \( Z \).

Lemma 3.18. If \( Y \) and \( Z \) are compact Abelian groups, then affine maps \( Z \to Y \) are step-affine.

Proof. Composing with a character, it suffices to prove this in case \( Y = T \). However, if \( \chi \in \mathcal{A}(Z) \), then

\[
\chi(z) = \{\chi(z)\} + Z,
\]

and \( \{\chi\} : Z \to \mathbb{R} \) is clearly step-affine. \( \square \)

The next result is a natural extension of Lemma 3.12

Lemma 3.19. If \( Z \) is a compact Abelian Lie group and \( f : Z \to T^d \) is step-affine, then there are an affine map \( \chi : Z \to T^D \) and a step-affine map \( f_0 : [0,1)^D \to [0,1]^d \) such that the following diagram commutes:

\[
\begin{array}{ccc}
Z & \xrightarrow{f} & T^d \\
\{\chi\} \downarrow & & \downarrow \{\} \\
[0,1)^D & \xrightarrow{f_0} & [0,1)^d.
\end{array}
\]

Proof. Arguing coordinate-wise, it suffices to prove this when \( d = 1 \), in which case \( \{f\} : Z \to \mathbb{R} \) is step-affine. The desired conclusion is now just the definition of step-affine maps to \( \mathbb{R} \). \( \square \)
Lemma 3.20. If $X$, $Y$ and $Z$ are compact Abelian groups, and $\varphi : X \to Y$ and $\psi : Y \to Z$ are step-affine, then so is $\psi \circ \varphi : X \to Z$.

Proof. We treat this in two steps.

Step 1. If $\psi$ is actually affine, and $\chi \in A(Z)$, then $\chi \circ \psi \in A(Y)$, so in this case the result follows at once from the definition.

Step 2. For the general case, after composing with a character we may assume $Z = \mathbb{T}$. Having done so, by definition there are an affine map $\chi : Y \to \mathbb{T}^d$ and a step-affine map $\psi_0 : \mathbb{T}^d \to \mathbb{T}$ such that $\psi = \psi_0 \circ \chi$. Replacing $\varphi$ with $\chi \circ \varphi$ (justified by Step 1) and $\psi$ with $\psi_0$, we may therefore assume that $Y = \mathbb{T}^d$. Having done so, Definition 3.16, Lemma 3.12 and Lemma 3.19 give a commutative diagram

$$
\begin{array}{cccccc}
X & \xrightarrow{\varphi} & \mathbb{T}^d & \xrightarrow{\psi} & \mathbb{T} \\
\downarrow{\chi} & & \downarrow{\theta} & & \downarrow{\text{mod } 1} \\
[0,1)^D & \xrightarrow{f_0} & [0,1)^d & \xrightarrow{f_1} & [0,1)^s \\
\end{array}
$$

in which $\chi : X \to \mathbb{T}^D$ and $\theta : \mathbb{T}^d \to \mathbb{T}^s$ are affine and $f_0 : [0,1)^D \to [0,1)^d$, $f_1 : [0,1)^d \to [0,1)^s$ and $f_2 : [0,1)^s \to \mathbb{R}$ are step affine.

Reading around the bottom row of this diagram, the proof is completed by an appeal to Corollary 3.10.

Lemma 3.21. If $\mathcal{P}$ is a q.-p. partition of $Y$ and $\psi : Z \to Y$ is step-affine, then $\psi^{-1}(\mathcal{P})$ is a q.-p. partition of $Z$.

Proof. By definition we may assume that $\mathcal{P} = \{\chi\}^{-1}(\Omega)$ for some affine map $\chi : Y \to \mathbb{T}^d$ and some partition $\Omega$ of $[0,1)^d$ into convex polytopes. Replacing $\psi$ with $\chi \circ \psi$, we may therefore assume $Y = \mathbb{T}^d$. Now the definitions and Lemma 3.19 give a commutative diagram

$$
\begin{array}{cccc}
Z & \xrightarrow{\psi} & \mathbb{T}^d \\
\downarrow{\theta} & & \downarrow{\text{mod } 1} \\
[0,1)^D & \xrightarrow{f_1} & [0,1)^d \\
\end{array}
$$

for some affine $\theta$ and step-affine $f_1$. Reading counterclockwise around this diagram and applying Lemma 3.9 completes the proof. \qed
3.2 Step-affine cross-sections

Step-affine maps also appear as cross-sections of quotient maps. This important result will require a little more work.

**Proposition 3.22.** Suppose that $W \leq Z$ is an inclusion of tori, and let $q : Z \rightarrow Z/W$ be the quotient. Then $q$ has a step-affine cross-section $\sigma = (\sigma_1, \ldots, \sigma_d) : Z/W \rightarrow Z$, and there is a step-affine $W$-equivariant map $Z \rightarrow W$.

**Proof.** We may identify $Z = \mathbb{T}^d$, and then the quotient $Z/W$ may be identified with another torus, say $\mathbb{T}^r$.

Let $\pi : \mathbb{R}^d \rightarrow \mathbb{T}^d$ be the universal cover, and let $Q : \mathbb{R}^d \rightarrow \mathbb{R}^r$ be the lift of $q$ to the universal covers. Then $Q$ is a surjective linear map, so it has a linear cross-section $A : \mathbb{R}^r \rightarrow \mathbb{R}^d$. Let $\mathfrak{P}$ be the partition of $\mathbb{T}^r$ that is the pullback under $A \circ \{\}$ of the partition of $\mathbb{R}^d$ into half-open unit cubes. Then $\mathfrak{P}$ is q.-p., and for each $D \in \mathfrak{P}$ the map $A[D]$ has image contained in a single fundamental domain of $\pi$. Composing with $\pi$ identifies each of these restrictions of $A$ with a map $D \rightarrow \mathbb{T}^d$, which gives a suitable step-affine cross-section.

Finally, we obtain an equivariant selector $\xi : Z \rightarrow W = \ker q$ by setting $\xi(t) = z - \sigma(q(z))$.

If $\sigma$ is step-affine, then so is this, by Lemmas 3.20 and 3.17.

**Corollary 3.23.** Suppose that $W \leq Z$ is an inclusion of compact Abelian groups. Then there is a step-affine $W$-equivariant map $Z \rightarrow W$.

**Proof.** We first treat a special case.

**Step 1.** Suppose that $Z/W$ is finite-dimensional (that is, isomorphic to a subgroup of a torus). By the argument in the previous proof, it suffices instead to find a step-affine cross-section $\sigma : Z/W \rightarrow Z$. Theorem 2.2 gives a decreasing sequence of closed subgroups $U_1 \supseteq U_2 \supseteq \cdots$ in $Z$ whose intersection is $\{0\}$, and such that each $Z/U_i$ is finite-dimensional. Intersecting each of them with $W$ if necessary, we may assume that $W := U_0 \supseteq U_1$. Let $q_i : Z \rightarrow Z/U_i$ be the quotient homomorphism, and similarly $q_{ij} : Z/U_i \rightarrow Z/U_j$ whenever $i \geq j$. By Proposition 3.22 there are step-affine cross-sections

$$Z/W \overset{\sigma_1}{\rightarrow} Z/U_1 \overset{\sigma_2}{\rightarrow} Z/U_2 \overset{\sigma_3}{\rightarrow} \cdots$$

of $q_{21}, q_{32}, \ldots$. It is now routine to check that the partial compositions $\sigma_{1,j} := \sigma_j \circ \sigma_{j-1} \circ \cdots \circ \sigma_1 : Z/W \rightarrow Z/U_j$ converge uniformly. Their limit $\sigma : Z/W \rightarrow$
is a step-affine cross-section, because any $\chi \in \mathcal{A}(Z)$ factorizes through some $\chi' \in \mathcal{A}(Z/U_i)$, hence
\[ \chi \circ \sigma = \chi' \circ \sigma_{1,i}, \]
and $\sigma_{1,i}$ is step affine by construction.

Step 2. We now prove an auxiliary selection result. Suppose that $V, W \leq Z$ are closed subgroups that both have Lie-group quotients, and that $W + V = Z$, so the quotient map $q : Z \twoheadrightarrow Z/V$ satisfies $q(W) = q(Z)$. Then there is a $W$-equivariant map $\sigma : Z \twoheadrightarrow W$ with the additional property that $q \circ \sigma = q$, so the quotient $q$ `cannot see' the operation of $\sigma$.

To see this, let $\sigma_1 : Z \twoheadrightarrow W$ be a step-affine $W$-equivariant map as given in Step 1, and let $\tau : Z/V \cong W/(W \cap V) \twoheadrightarrow W$ be a step-affine selector. Then the map $\sigma := \sigma_1 - \tau \circ q \circ (\sigma_1 - \text{id}_Z)$ is still step-affine and has all the required properties. Let us refer to this as a $V$-stable $W$-equivariant map $Z \twoheadrightarrow W$.

Step 3. Now assume general $W \leq Z$. We will construct a step-affine $W$-equivariant map $Z \twoheadrightarrow W$. Let $U_1 \supseteq U_2 \supseteq \cdots$ be as in Step 1, and now let $W_i := W + U_i$ and $W_0 := Z$. This is decreasing sequence of closed subgroups of $Z$ whose intersection is $W$, and such that $Z/W_i$ is finite-dimensional for each $i$. By Step 2, there are $U_i$-stable $W_i$-equivariant (hence also $W$-equivariant) step-affine maps $\tau_i : W_{i-1} \twoheadrightarrow W_i$ for all $i \geq 1$. Now let $\tau_{1,i} := \tau_i \circ \ldots \circ \tau_1 : Z \twoheadrightarrow W_i$. The $U_i$-stability implies that these compositions converge uniformly (because their compositions with quotienting by any given $U_j$ stabilize once $i \geq j$). Their limit is the desired $W$-equivariant step-affine map $\tau : Z \twoheadrightarrow W$. $\square$

Example 3.24. For the two-fold covering homomorphism
\[ T \xrightarrow{x^2} T, \]
the obvious selector
\[ \theta \mapsto e(\{\theta\}/2) : T \longrightarrow e([0,1/2)) \subset T, \]
where $e : \mathbb{R} \longrightarrow T$ is the usual quotient homomorphism, is step-affine. $\square$

Example 3.25. The restriction map arising from the inclusion $Z \subset \hat{Q}$ defines a homomorphism
\[ \hat{Q} \longrightarrow \hat{Z} \cong T, \]
where $\hat{Q}$ is given its discrete topology so that $\hat{Q}$ is compact ($\hat{Q}$ is sometimes called the solenoid). In this case one has also a `natural' selector, which to $\theta \in T$ assigns the element of the solenoid defined by
\[ \varphi_\theta(p/q) = e(p\{\theta\}/q) \]
with \( e \) the same quotient homomorphism as in the previous example. Now any \( \chi \in \hat{Q} \) takes the form of evaluation at some \( p/q \in Q \), so the above formula shows that the composition \( \theta \mapsto \varphi_0 \mapsto \chi(\varphi_0) \) is step-affine.

Topologically, \( \hat{Q} \) is a bundle of copies of the Cantor set over \( S^1 \), and the above selector embeds \([0, 1)\) into this as a cross-section of the base map from the total space onto \( S^1 \).

### 3.3 Step polynomials

Again let \( Z \) be a compact metrizable Abelian group and \( A \) an Abelian Lie group. We next introduce step polynomials, which form a natural generalization of step affine maps.

**Definition 3.26.** If \( Q \subseteq \mathbb{R}^d \) is a convex polytope, then a function \( f : Q \rightarrow \mathbb{R}^r \) is a **step polynomial** if there is a partition \( \mathcal{P} \) of \( Q \) into convex sub-polytopes and, for each \( C \in \mathcal{P} \), a polynomial \( p_C : \mathbb{R}^d \rightarrow \mathbb{R}^r \) such that \( f|_C = p_C|_C \).

A step polynomial \( f : Q \rightarrow \mathbb{R}^r \) is **basic** if \( f = g \cdot 1_{R^r} \) for some polynomial \( g : \mathbb{R}^d \rightarrow \mathbb{R}^r \) and convex sub-polytope \( R \subseteq Q \).

**Definition 3.27 (Step polynomial).** If \( Z \) is a compact Abelian group and \( A \) is an Abelian Lie group, then a map \( f : Z \rightarrow A \) is a **step polynomial** if it is a composition

\[
Z \xrightarrow{\chi} [0, 1)^d \xrightarrow{f_0} \hat{A} \xrightarrow{\psi} A,
\]

where

- \( \chi : Z \rightarrow \mathbb{T}^d \) is affine,
- \( \hat{A} \) is a closed subgroup of \( \mathbb{R}^r \) for some \( r \),
- \( f_0 : [0, 1)^d \rightarrow \mathbb{R}^r \) is a step polynomial with image contained in \( \hat{A} \),
- and \( \psi : \hat{A} \rightarrow A \) is a continuous homomorphism.

The step polynomial \( f \) is **basic** if \( f_0 \) may be taken to be basic in the above definition.

The set of (Haar-a.e. equivalence classes of) step polynomials in \( \mathcal{F}(Z, A) \) is denoted \( \mathcal{F}_{sp}(Z, A) \).

Clearly any step polynomial decomposes as a finite sum of basic step polynomials. Also, if \( Z \) is an Abelian Lie group with identity component \( Z_0 \leq Z \) and \( f : Z \rightarrow \mathbb{R}^d \) is a step polynomial, then an easy exercise shows that \( f \cdot 1_{z+Z_0} \) is also a step polynomial for every coset \( z+Z_0 \). Combining these facts, if \( Z \) is Lie then we may always decompose \( f \) into basic step polynomials supported on single identity-component cosets.
Remark. Having reached this definition, it is high time we drew attention to the overlap between this section and Bergelson and Leibman’s work in [5]. Their interest is in the study of bounded ‘generalized polynomials’ from $\mathbb{Z}^d$ to $\mathbb{R}$. Generalized polynomials $\mathbb{Z}^d \rightarrow \mathbb{R}$ form the smallest class which contains the linear functions and is closed under addition, multiplication, and also taking integer parts. Generalized polynomials arise naturally in various problems from equidistribution and additive combinatorics (see, for instance, [8], and the many further references in [5]). The paper [5] seeks to develop a general structure theory for them. This is achieved by proving that every bounded generalized polynomial can be obtained by sampling along an orbit of a $\mathbb{Z}^d$-action by rotations on a compact nilmanifold, where the function sampled is essentially what our terminology would call a step polynomial on that nilmanifold.

Insofar as any compact connected nilmanifold can be described as a tower of topological circle extensions, and has a natural ‘coordinate system’ which identifies it with some cube $[0, 1)^d$, the study of such functions on nilmanifolds forms a natural generalization of our work on step polynomials on compact Abelian Lie groups. The concerns of [5] are fairly disjoint from ours, and so are the results, but that paper does strongly suggest that most of the work of this section could be generalized to their setting. It might even be worth looking for an abstract category of ‘step-polynomial spaces’, whose objects are spaces that admit a coordinate system based on finitely many Euclidean cubes, and whose morphisms are an abstract characterization of ‘step polynomial mappings’ between such spaces. This would also bear comparison with the abstract study of ‘nilspaces’ in [7]. Such a theory would put us in the realm of quite general semi-algebraic geometry (see [4], Chapter 2) for a good introduction), but I do not know whether ideas from that theory could shed additional light on the kind of work that we will do below. Our interest in PD$^\infty$Es and zero-sum problems could also be generalized, by replacing the rotation-actions of subgroups of $\mathbb{Z}$ with the actions of commuting nilpotent subgroups of a nilpotent Lie group $G$ on a compact nilmanifold $G/\Gamma$ (a similar proposal was already discussed in Subsection I.14.4), and so there might generalizations of Theorems A and B to that setting.

Lemma 3.28. A sum of two step polynomials is a step polynomial, so $\mathcal{F}_{sp}(Z, A)$ is a subgroup of $\mathcal{F}(Z, A)$.

Proof. Let

$$Z \xrightarrow{(x_i)} (0, 1)^{d_i} \xrightarrow{f_i} \widehat{A}_i \xrightarrow{\psi_i} A$$

for $i=1, 2$ be factorizations of our two step polynomials as given by Definition 3.27. Then their sum factorizes as

$$Z \xrightarrow{(x_1, x_2)} (0, 1)^{d_1+d_2} \xrightarrow{(f_1 \pi_1, f_2 \pi_2)} \widehat{A}_1 \oplus \widehat{A}_2 \xrightarrow{\psi_1 \phi_1 + \psi_2 \phi_2} A,$$
where \( \pi_i : [0, 1)^{d_1 + d_2} \to [0, 1)^{d_i} \) is the projection onto the first (resp. last) \( d_i \) coordinates for \( i = 1 \) (resp. \( i = 2 \)), and also \( q_i : \widehat{A}_1 \oplus \widehat{A}_2 \to \widehat{A}_i \) are the coordinate projections. This is clearly a factorization into ingredients of the required kind.

**Lemma 3.29.** If \( f : Z \to A \) is a step polynomial, then there is a q.-p. partition of \( Z \) such that \( f|C \) extends to a uniformly continuous function on \( \overline{C} \) for every \( C \in \mathcal{P} \).

**Proof.** Let \( f = \psi \circ f_0 \circ \{ \chi \} \), and let \( \Omega \) be the convex polytopal partition of \([0, 1)^d\) associated to \( f_0 \) as in Definition [3.26]. Then \( \mathcal{P} := \{ \chi \}^{-1}(\Omega) \) is the desired q.-p. partition of \( Z \).

**Corollary 3.30.** If \( A \) is discrete, then a step polynomial \( f : Z \to A \) is a step function.

**Proof.** This follows by combining the preceding lemma and Corollary [3.14].

**Lemma 3.31.** If \( \xi \in \mathcal{F}_{\text{sa}}(Z, Y) \) and \( f : Y \to A \) is a step polynomial, then \( f \circ \xi \) is a step polynomial.

**Proof.** Clearly every step polynomial factorizes through an affine map to a torus, so we may assume \( Y = \mathbb{T}^d \).

Let \( f = \psi \circ f_0 \circ \chi \) as in Definition [3.27]. Lemma [3.19] provides a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\xi} & \mathbb{T}^d \\
\{ \theta \} \downarrow & & \downarrow \{ \chi \} \\
[0, 1)^D & \overset{f_0}{\underset{\psi \circ f_0}{\to}} & A,
\end{array}
\]

where \( \theta \) is affine and \( \xi_0 \) is step-affine. The proof is completed by observing that \( f_0 \circ \xi_0 : [0, 1)^D \to \widehat{A} \) is step polynomial, which is clear from the definitions.

A fact about step polynomials that will be used repeatedly is the ability to lift them through target-module homomorphisms.

**Lemma 3.32.** Let \( q : A \to B \) be a continuous epimorphism of Abelian Lie groups, and \( Z \) a compact Abelian group. For any step polynomial \( f : Z \to B \) there is a step polynomial \( F : Z \to A \) such that \( f = q \circ F \).

**Proof.** By Theorem [2.1] we may assume that

\[ B = \mathbb{R}^r \oplus \mathbb{T}^d \oplus D, \]
with $D$ discrete. Correspondingly, one may decompose $f$ as $f_1 + f_2 + f_3$, where each summand takes values within one of the direct summands on the right above. Each $f_i$ is the composition of $f$ with a coordinate-projection, so is still a step polynomial. It therefore suffices to lift each $f_i$ separately.

Firstly, $f_3$ takes values in a discrete group. It is therefore constant on each cell of some $q$-p. partition, so one may simply choose of lift of that constant value on each cell separately.

Second, $f_1$ takes values in $R$. Consider the analogous decomposition $A = R' \oplus T' \oplus D'$. Let $q_1 : A \to R'$ be the composition of $q$ with the projection from $B$ onto its summand $R'$. The image $q_1(T')$ must be a compact subgroup of $R'$, hence it must equal $0$. Also, the image $q_1(D')$ is countable, so since $q_1$ is onto, the image $q(R')$ is a co-countable vector subspace of $R'$. It is therefore equal to $R'$. By linear algebra, this implies that there is a linear function $M : R' \to R' \leq A$ which is a cross-section of $q_1$, and now $M \circ f_1$ is the desired lift of $f_1$.

Finally, the image $q(T')$ is a closed, connected subgroup of $T$. Is is therefore a subtorus, and so we may split $T$ further as $q(T') \oplus T$ for some complementary subtorus $T \leq T$. Correspondingly we may decompose $f_2 = f_{21} + f_{22}$ and lift each summand separately. For $f_{21}$, Corollary 3.23 gives a step-affine cross-section $\sigma : q(T') \to T'$, so the composition $\sigma \circ f_{21}$ is a suitable lift of $f_{21}$. For $f_{22}$, observe that the composition

$$R' \leq A \xrightarrow{q} B \to T' \to T' / q(T') \cong T$$

is surjective: its image is a $\sigma$-compact (hence Borel) and co-countable subgroup of $T$, so must equal $T$. This gives a continuous homomorphism $R' \to T$, which therefore lifts to a linear epimorphism of the universal covers $Q : R' \to \hat{T}$. Composition $f_{22}$ with a step-affine fundamental domain $T \to \hat{T}$ and then with a linear $Q$-cross-section $\hat{T} \to R'$ completes the proof.

Remark. This result has quietly made an appeal to our assumption of second-countability for Abelian Lie groups. Without it, one could consider the uncountable group $Z^\oplus T$ with its discrete topology and the map

$$q : Z^\oplus T \to T : (z_t)_{t \in T} \mapsto \sum_{t \in T} z_t \cdot t.$$

This $q$ is a continuous epimorphism, but the identity map $T \to T$ has no lift through $q$ to a step polynomial $T \to Z^\oplus T$. ▷

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3.4 Slicing

Suppose that $X$ and $Y$ are compact Abelian groups and that $A$ is an Abelian Lie group. In general, a step polynomial $f : X \times Y \to A$ cannot be viewed in any simple way as a step function $X \to \mathcal{F}(Y, A)$, even if $A$ is discrete.

Example 3.33. If $f : T^2 \to \{0, 1\}$ is the step function $(s, t) \mapsto \lfloor \{s\} + \{t\} \rfloor$, then regarded as a function $T \to \mathcal{F}(T, Z)$ it is continuous and injective, so its level-set partition is not even finite, let alone quasi-polytopal.

However, it will be important later that this ‘slicing’ operation gives rise to a function that is continuous on the cells of a q.-p. partition. This will be the main result of the present subsection. The following notion will be useful both here and later in the paper.

**Definition 3.34.** If $W \leq Z$ is an inclusion of Abelian groups, $S$ is any set, $f : Z \to S$ and $z \in Z$, then the **restriction of $f$ to $W$ at $z$** is the function

$$
\text{res}_{z,W} f := (R_z f)|_W : W \to S : w \mapsto f(w - z).
$$

Similarly, if $\mathcal{P}$ is a partition of $Z$ then

$$
\text{res}_{z,W} \mathcal{P} := (\mathcal{P} + z) \cap W.
$$

**Lemma 3.35.** Let $Q \subseteq \mathbb{R}^{d+r}$ be a bounded convex polytope, and suppose that $f = g \cdot 1_Q : \mathbb{R}^{d+r} \to \mathbb{R}$ is a basic step polynomial supported on $Q$. Let $\Pi : \mathbb{R}^{d+r} \to \mathbb{R}^r$ be the projection onto the last $r$ coordinates. Then the sliced function

$$
\mathbb{R}^r \to \mathcal{F}(\mathbb{R}^d, \mathbb{R}) : v \mapsto f(\cdot, v)
$$

is identically zero on $\mathbb{R}^r \setminus \Pi(Q)$ and is uniformly continuous on $\text{int} \Pi(Q)$.

**Proof.** The first assertion is obvious, so we focus on the second.

Since $g$ is uniformly continuous and uniformly bounded on the bounded set $\overline{Q}$, it suffices to prove this for $1_Q$ alone.

Let $B \supseteq Q$ be an open ball containing $\overline{Q}$. By an easy calculation, the sliced function

$$
v \mapsto 1_B(\cdot, v)
$$

is uniformly continuous on the whole of $\mathbb{R}^r$.

Now, $Q$ is defined by a finite intersection of half-spaces in $\mathbb{R}^{d+r}$, say $Q = H_1 \cap \cdots \cap H_m$. This implies that $1_Q = (1_{B \cap H_1}) \cdots (1_{B \cap H_m})$, and each function
here has bounded support and is uniformly bounded by 1, so the result will follow
if we know that each of the sliced functions
\[ v \mapsto 1_{B \cap H_i}(\cdot, v), \quad i = 1, 2, \ldots, m, \]
is uniformly continuous on \( \text{int} \, \Pi(B) \).

Now, on the one hand, if the bounding hyperplane \( \partial H_i \) is not of the form \( \mathbb{R}^d \times V \) for any hyperplane \( V \leq \mathbb{R}^r \), then this sliced function is actually continuous on the entire \( \mathbb{R}^r \).

On the other, if \( \partial H_i \) equals \( \mathbb{R}^d \times V \) for such a hyperplane \( V \leq \mathbb{R}^r \), then \( H_i \) itself equals \( \mathbb{R}^d \times K \) for some half-space \( K \subseteq \mathbb{R}^r \), and in this case the sliced function of \( 1_{B \cap H_i} \) is uniformly continuous on \( \text{int} \, K \), since it agrees with the sliced function of \( 1_B \) there.

Let \( I \subseteq \{1, 2, \ldots, m\} \) be the set of indices \( i \) for which the second of the possibilities above obtains. Then we have shown that the function in question is uniformly continuous on \( \Pi(B) \cap \bigcap_{i \in I} \text{int} \, K_i \).

This clearly contains (and is often equal to) \( \text{int} \, \Pi(Q) \).

**Corollary 3.36.** If \( A \subseteq \mathbb{R}^d \) and \( B \subseteq \mathbb{R}^r \) are bounded convex polytopes of positive measure (equivalently, nonempty interior), and \( f : A \times B \rightarrow \mathbb{R} \) is a step polynomial, then there is a partition \( \mathcal{P} \) of \( B \) into convex sub-polytopes such for each \( C \in \mathcal{P} \), the sliced function

\[ C \mapsto \mathcal{F}(A, \mathbb{R}) : v \mapsto f(\cdot, u) \]

extends to a uniformly continuous function on \( \overline{C} \).

**Proof.** If \( f \) is a basic step polynomial supported on a convex polytope \( Q \subseteq A \times B \), then the previous lemma gives that the sliced function is uniformly continuous on the relative interiors in \( B \) of both \( \Pi(Q) \) and \( B \setminus \Pi(Q) \).

A general \( f \) may be written as a finite sum of basic step polynomials, say \( f = f_1 + \cdots + f_m \). Let \( Q_1, \ldots, Q_m \) be convex polytopal supports for \( f_1, \ldots, f_m \), and let \( \mathcal{P}_1 \) be a convex-polytopal refinement of the partition generated by the polytopes \( \Pi(Q_1), \ldots, \Pi(Q_m) \) together with all their facets, regarded as separate polytopes also.

It follows that the sliced function is uniformly continuous on the relative interior of every cell of \( \mathcal{P}_1 \). This implies the desired conclusion for the cells of \( \mathcal{P}_1 \) that have nonempty relative interiors, but does not handle the facets that lie in \( \partial \Pi(Q_1) \cup \cdots \cup \partial \Pi(Q_m) \). However, if a cell of \( \mathcal{P}_1 \) has no relative interior,
then it lies in some choice of co-dimension-1 affine subspace of $\mathbb{R}^r$. For each of these lower-dimensional subspaces $V$, we may now simply repeat the previous construction for the restriction $f|(A \times (B \cap V))$. An induction on $r$ completes the proof.

**Proposition 3.37.** Suppose that $A$ is an Abelian Lie group, $W \leq Z$ is an inclusion of compact Abelian groups, and $f : Z \rightarrow A$ is a step polynomial. Then there is a $W$-invariant q.-p. partition $\Omega$ of $Z$ such that the map

$$z \mapsto \text{res}_{z,W}f$$

is continuous from $C$ to $F(W, A)$ for every $C \in \Omega$.

**Proof.** This is treated in two steps.

**Step 1.** Suppose first that $Z = T^d + r$ and that $W$ is a connected subgroup. In this case we may choose a coordinate system so that $W = T^d \times \{0^r\}$ for some $d$ and $r$. For this subgroup we will actually obtain a q.-p. partition lifted from $Z/W$.

On the one hand, for each fixed $z$ the function

$$W \rightarrow F(W, A) : w \mapsto \text{res}_{z+w,W}f$$

is uniformly continuous, with modulus of continuity not depending on $z$. This is because one simply has $\text{res}_{z+w,W}f = R_w(\text{res}_{z,W}f)$ for $w \in W$, and each $\text{res}_{z,W}f$ is a step polynomial defined by a bounded number of convex sets and polynomials of bounded degrees and coefficients, independently of $z$.

Therefore, letting $V := \{0^d\} \times T^r$, the obvious complement of $W$, it suffices to show that the function

$$V \rightarrow F(W, A) : v \mapsto \text{res}_{v,W}f$$

has uniformly continuous restriction to each cell of some q.-p. partition of $V$. However, applying the fundamental-domain map $\{\cdot\} : T^{d+r} \rightarrow [0,1)^{d+r}$, this just becomes the assertion of the preceding corollary.

**Step 2.** For the case of general $W \leq Z$, Definition 3.27 gives an affine map $\alpha : Z \rightarrow T^D$ and a step polynomial $f_1 : T^D \rightarrow A$ such that $f = f_1 \circ \alpha$. It easily follows that

$$\text{res}_{z,W}f = (\text{res}_{\alpha(z),\alpha(W)}f_1) \circ \alpha \text{ on } W.$$  

On the other hand, if $\Omega$ is an $\alpha(W)$-invariant q.-p. partition of $T^d$, then $\alpha^{-1}(\Omega)$ is a $W$-invariant q.-p. partition of $Z$. It therefore suffices to prove the result instead for the inclusion $\alpha(W) \leq T^d$; or, equivalently, to assume that $Z = T^d$.  

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Finally, letting $W_0$ be the identity component of $W$, we see for any $z$ that $\text{res}_{z,W} f$ is a sum of $[W : W_0]$-many function supported on cosets of $W_0$, each of which may be identified with $\text{res}_{z+w',W_0} f$ for some fixed $w' \in W$. If a partition of $Z$ is $W_0$-invariant, then it has finitely many $W$-orbits, so taking their common refinement produces a $W$-invariant partition. The result for $\text{res}_{z,W} f$ therefore follows from the result for $\text{res}_{z+w',W_0} f$ for each $w'$ in some finite cross-section of $W_0$ in $W$, which brings us back to the case treated in Step 1. \qed

3.5 Integrating slices of step polynomials

The following result will be of great importance later. It strikes me as something that is probably known, but I have not been able to find a suitable reference.

**Proposition 3.38.** Suppose that $C \subseteq \mathbb{R}^n \times \mathbb{R}^m$ is a bounded convex polytope and that $p : C \rightarrow \mathbb{R}$ is a step polynomial, which we extend by 0 outside $C$. Then the integrated function

$$q(u) := \int_{\mathbb{R}^m} p(u, v) \, dv$$

is a step polynomial of bounded support.

**Proof.** It suffices to prove this result for $m = 1$, since the general case may then be recovered by integrating out the last $m$ coordinates in turn.

Next, by decomposing $f$ into basic step polynomials, it suffices to assume that $p$ is the restriction to $C$ of a genuine polynomial. Because $C$ is bounded, it is obvious that $q$ has bounded support; it remains to prove that it is a step polynomial.

Clearly the result is unaffected if we replace $C$ with $\overline{C}$, so we may assume $C$ is closed. It is then defined by some intersection of closed linear inequalities, so we may write

$$C = \bigcap_{i=1}^r H_i \quad \text{with} \quad H_i = \{(u, v) \mid a_i \cdot u + b_i v_i \leq \alpha_i\}$$

for some $(a_i, b_i) \in (\mathbb{R}^n \times \mathbb{R}) \setminus \{(0, 0)\}$ and $\alpha_i \in \mathbb{R}$, $i = 1, 2, \ldots, r$. For each $j \leq r$, let also

$$C_j := \bigcap_{i \in [r] \setminus \{j\}} H_i.$$

We may assume that the above representation of $C$ is irredundant, meaning that for each $i$, the simpler intersection $\overline{C}_i$ is strictly larger than $C$. This is equivalent to the assertion that each of the hyperplanes $\partial H_i$ intersects the interior of the corresponding $C_i$. Knowing this, it follows that each of these intersections $\partial H_i \cap C_i$ is a bounded convex polytope with non-empty interior relative to $\partial H_i$. \hfill \qed
By convexity and boundedness, the slice 

\[ I_u := \{ v \mid (u, v) \in C \} \]

is a closed bounded interval for every \( u \), and the set 

\[ D := \{ u \mid I_u \neq \emptyset \} \]

is also a bounded convex polytope. Each \( u \in D \) may be labeled by a pair \((i_1(u), i_2(u))\) of elements of \([r]\) with the property that \( \partial H_{i_1(u)} \) contains the lower end-point of \( I_u \), and \( \partial H_{i_2(u)} \) contains the upper end-point.

It is now easy to see that this choice of \((i_1(u), i_2(u))\) is unique for a.e. \( u \in D \), and that it defines a partition of \( D \), say \( \mathfrak{P} \), into convex sub-polytopes. The proof will be finished by showing that \( g \) agrees with a polynomial on each of these sub-polytopes.

Indeed, suppose that \( D' = \{ u \mid (i_1(u), i_2(u)) = (i_1, i_2) \} \in \mathfrak{P} \). Then we may write

\[ I_u = [\psi_1(u), \psi_2(u)] \]

for some affine functions \( \psi_1, \psi_2 : D' \to \mathbb{R} \), where \( \psi_s \) is the function whose graph is \( \partial H_{i_s} \cap (D' \times \mathbb{R}) \) for \( s = 1, 2 \). For our integral, this now gives

\[ q(u) = \int_{\psi_1(u)}^{\psi_2(u)} p(u, v) \, dv \quad \forall u \in D'. \]

We will complete the proof by induction on \( \deg p \). If \( p \) is a constant, then we simply obtain

\[ q(u) = \psi_2(u) - \psi_1(u) \quad \text{on } D', \]

which is affine. So now suppose that \( \deg p \geq 1 \). We will use the inductive hypothesis to prove that the gradient \( \nabla q \) is a polynomial function on \( D' \). Indeed, the rules for differentiation under an integral give

\[ \nabla q(u) = \int_{\psi_1(u)}^{\psi_2(u)} \nabla_u p(u, v) \, dv + p(u, \psi_2(u)) \cdot \nabla \psi_2 - \psi_1 \cdot \nabla \psi_1, \]

so we can apply the inductive hypothesis to the first term here, and observe directly that the second and third are polynomials on \( D' \).

**Corollary 3.39.** If \( Y \) and \( Z \) are compact Abelian groups and \( f : Y \times Z \to \mathbb{R} \) is a step polynomial, then so is the integrated function

\[ g(y) := \int_Z f(y, z) \, dz. \]
Proof. Since $f$ factorizes through a Lie-group quotient, and then $g$ does the same, we may assume that $Y$ and $Z$ are both Lie groups. Since a function on an Abelian Lie group is a step polynomial if this is so of its restriction to every identity-component coset, we may assume further that $Y$ is a torus. On the other hand, if $Z_0$ is the identity-component of $Z$, then $g$ is the sum of $[Z : Z_0]$-many integrals over cosets of $Z_0$, so it suffices to prove the result for each of these, and hence assume that $Z$ is also a torus.

Finally, if $Z = T^m$ and $Y = T^n$, then applying the map $\{ \cdot \} : Y \times Z \rightarrow [0, 1)^{m+n}$ turns the definition of $g$ into

$$g(y) = \int_{R^m} p(\{ y \}, v) \, dv$$

for some step polynomial $p : R^{m+n} \rightarrow R$ supported on $[0, 1)^{m+n}$, which is treated by the preceding proposition.

3.6 A special class of Lie modules

In the sequel we will often consider step polynomials $Z \rightarrow A$ for an Abelian Lie group $A$ that is already equipped with an action of $Z$. In this setting the following special class of Lie modules will become important.

Definition 3.40. Let $A$ be an Abelian Lie group, $Z$ a compact Abelian group and $T : Z \rightarrow \text{Aut} \, A$ an action. Then this action is step-polynomial, or ‘s.-p.’, and $A$ is a s.-p. $Z$-module, if, for every compact Abelian group $Y$ and s.-p. function $f : Y \rightarrow A$, the function

$$F : Z \times Y \rightarrow A : (z, y) \mapsto T^z f(y)$$

is also s.-p.

This includes the requirement that the orbit maps $z \mapsto T^z a$ be s.-p. for all $a \in A$. Later this will turn out to be equivalent, but we will use the condition in the stronger form above.

Example 3.41. The trivial action of $Z$ on $A$ is always s.-p. More generally, an action that factorizes through a quotient $q : Z \rightarrow \hat{Z}_1$ to a finite group is always s.-p., because the partition of $Z$ into the cosets of $\ker q$ is q.-p.

Example 3.42. The rotation action $\text{rot} : T \rightarrow \text{O}(2) \leq \text{Aut} \, \mathbb{R}^2$ given by

$$\text{rot}(t) = \begin{pmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{pmatrix}$$

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is not s.-p., because \( \sin \) and \( \cos \) are not s.-p. functions \( T \to \mathbb{R} \).

More generally, a **full rotation action** of some \( Z \) is an action isomorphic to \( \text{rot} \circ \chi : Z \curvearrowright \mathbb{R}^2 \) for some \( \chi \in \hat{Z} \) for which \( \chi(Z) = \mathbb{T} \) (equivalently, \( \chi \) is non-torsion in \( \hat{Z} \)). It is easy to see that any full rotation action is not s.-p. \( \Box \)

In fact, it will turn out that full rotation actions are the unique obstructions to a Lie module being s.-p.

**Proposition 3.43.** A Lie \( Z \)-module \( T : Z \curvearrowright A \) is s.-p. unless it has a closed submodule which is a full rotation action.

The proof of this will be given in Subsection 5.2 once we have collected some necessary results about group cohomology. However, we can immediately deduce the following (which will, of course, not be used in the proofs of Subsection 5.2).

**Corollary 3.44.** If \( A \leq B \xrightarrow{q} C \) is a short exact sequence in \( \text{PMod}(Z) \), then \( B \) is s.-p. if and only if both \( A \) and \( C \) are s.-p.

**Proof.** It is classical that this is true of Lie modules, so we may focus on those. Let \( T \) denote any of these three actions.

First suppose that \( B \) is s.-p. Any step polynomial \( Y \to A \) is also a step polynomial \( Y \to B \), so \( A \) is also s.-p. On the other hand, if \( f : Y \to C \) is a step polynomial, then Lemma 3.32 gives a s.-p. lift of it \( F : Y \to B \). It follows that

\[
T^z f(y) = q(T^z F(y))
\]

is a homomorphic image of a step polynomial, hence is a step polynomial.

On the other hand, suppose that \( B \) is not s.-p. Then Proposition 3.43 gives a full rotation \( Z \)-submodule \( B_1 \leq B \). Since full rotation modules are irreducible (indeed, the orbit of any non-zero element of \( B_1 \) has \( \mathbb{Z} \)-span which is dense in \( B_1 \)), it follows that either \( B_1 \leq A \), in which case \( A \) is not s.-p.; or \( q|B_1 : B_1 \to q(B_1) \leq C \) is an isomorphism, in which case \( C \) is not s.-p. \( \square \)

## 4 Semi-functional modules and step polynomials

### 4.1 Functional and semi-functional modules

**Definition 4.1** (Functional modules). Suppose that \( Z \) is a compact Abelian group, that \( X \) is another compact Abelian group with a distinguished homomorphism \( \alpha : Z \to X \), and that \( A \) is a Lie \( Z \)-module, say with action \( T_A \). Then a **functional \( Z \)-module with fibre \( A \) and base \( X \)** is a closed \( Z \)-submodule of \( \mathcal{F}(X,A) \), equipped with the diagonal \( Z \)-action:

\[
z \cdot f(x) := T_A^z(f(x - \alpha(z))).
\]
Often it will be important that $X = X_0 \times Z$ for some $X_0$, with the obvious homomorphism $Z \hookrightarrow X$. In this case we shall refer instead to a functional $Z$-module with fibre $A$ and dummy $X_0$.

Much of our later work will concern quotients of pairs of functional modules.

**Definition 4.2** (Semi-functional modules). A semi-functional $Z$-module is a short exact sequence

$$0 \rightarrow P \xrightarrow{i} Q \rightarrow M \rightarrow 0$$

in which $P \leq Q \leq \mathcal{F}(X, A)$ are functional $Z$-modules for some $X$ and $A$, and $i$ is the inclusion. This sequence will usually be abbreviated to

$$P \leq Q \rightarrow M.$$ 

Often, our emphasis will be on properties of the quotient module $M$ appearing here. For this reason we add the following definition.

**Definition 4.3.** If $M$ is a given Polish $Z$-module, then a semi-functional presentation of $M$ is a semi-functional $\Delta$-module $P \leq Q \rightarrow M$.

**Definition 4.4** (Semi-functional morphism). Given semi-functional $Z$-modules $P \leq Q \rightarrow M$ and $P' \leq Q' \rightarrow M'$, a semi-functional morphism between them is a commutative diagram

$$\begin{array}{ccc}
P & \xrightarrow{\psi_1} & P' \\
\downarrow{\text{incl}} & & \downarrow{\text{incl}} \\
Q & \xrightarrow{\psi_2} & Q' \\
\downarrow{\psi_3} & & \downarrow{\psi_3} \\
M & \xrightarrow{\psi_3} & M',
\end{array}$$

where ‘incl’ abbreviates ‘inclusion’.

Given a morphism $M \xrightarrow{\psi_3} M'$ in the category $\text{PMod}(Z)$, a semi-functional presentation of $\psi_3$ is a commutative diagram as above, in which the vertical columns are therefore semi-functional presentations of $M$ and $M'$.

Note that in the above diagram one must have $\psi_1 = \psi_2|P$, so properties of $\psi_1$ often follow from those of $\psi_2$.

**Example 4.5.** Whenever $q : Z \rightarrow Y$ is a surjective homomorphism and $A$ is a Lie $Y$-module, the $Y$-module $\mathcal{F}(Y, A)$ pulls back to a functional $Z$-module

$$\mathcal{F}(Y, A) \circ q := \{ f \circ q | f \in \mathcal{F}(Y, A) \} \leq \mathcal{F}(Z, \mathfrak{a})$$,
where $A^q$ denotes $A$ endowed with the action of $Z$ obtained by composition with $q$. The same construction may be applied to any other functional $Y$-module.

Example 4.6. Most other examples will arise by repeatedly forming images or kernels of suitable homomorphisms.

For instance, given $A$, $Z$ and a subgroup-tuple $U = (U_1, \ldots, U_k)$ in $Z$, the module of associated PD$^\infty E$-solutions,

$$M := \{ f \in F(Z, A) \mid d^{U_1} \cdots d^{U_k} f = 0 \},$$

is functional. It is the kernel of the $Z$-module homomorphism

$$F(Z, A) \longrightarrow F(U_1 \times \cdots \times U_k \times Z, A) : f \mapsto d^{U_1} \cdots d^{U_k} f,$$

where the target module has dummy $U_1 \times \cdots \times U_k$.

Within $M$, one also finds the submodules $M_i$, $i = 1, 2, \ldots, k$, of functions that satisfy one of the obvious simplified equations:

$$M_i := \{ f \in F(Z, A) \mid d^{U_1} \cdots \hat{d}^{U_i} \cdots d^{U_k} f = 0 \}.$$

From these one forms the submodule $M_0 := M_1 + \cdots + M_k$ of `degenerate' solutions to the original PD$^\infty E$. Each $M_i$ is also obtained as the kernel of a homomorphism defined by repeated differencing; and then $M_0$ is the image of $M_1 + \cdots + M_k$ under the sum-homomorphism. The results of Part I include that $M_0$ is closed, so it is another example of a functional $Z$-module. As a result, the inclusion $M_0 \subseteq M$ becomes an example of a semi-functional $\Delta$-module.

This construction was studied in several of the specific worked examples in Part I, where $M/M_0$ could be computed explicitly in terms of some cohomology groups.

4.2 Semi-functional modules with step-polynomial representatives

Let $Z$ be a compact Abelian group and $A$ an Abelian Lie group.

Definition 4.7. If $P \subseteq F(Z, A)$ is any subgroup, then its s.-p. subgroup relative to $(Z, A)$ is the subgroup $P \cap F_{sp}(Z, A)$.

Usually, $Z$ and $A$ will be left to the reader's understanding, and the s.-p. subgroup will be denoted by $P_{sp}$. To be precise, however, this subgroup does depend on a particular choice of $Z$ and $A$.

Definition 4.8 (Complexity-bounded homomorphisms). Suppose that $A$ and $A'$ are Abelian Lie groups, that $Z$ and $Z'$ are compact Abelian groups, and that
\( P \leq \mathcal{F}(Z, A) \) and \( Q \leq \mathcal{F}(Z', A') \) are closed subgroups. Then a continuous homomorphism \( \varphi : P \rightarrow Q \) is **complexity-bounded relative to \((Z, A)\) and \((Z', A')\)** if

\[
\varphi(P_{sp}) \subseteq Q_{sp}.
\]

Once again, the relevant choice of \((Z, A)\) and \((Z', A')\) will usually be obvious, and will be suppressed.

Similarly, a semi-functional morphism as in Definition 4.4 is **complexity-bounded** if the first two horizontal arrows of its commutative diagram are complexity-bounded.

Note that this property can be weaker or stronger, depending on whether \( P \) contains many step polynomials. In the examples that will concern us, our modules will be replete with step polynomials (for instance, they will be dense in probability), and so complexity-bounded homomorphisms will be rather special. However, we should note in passing that I do now know the answer to the following basic question.

**Question 4.9.** Are there an Abelian Lie group \( A \) with trivial \( \mathbb{T} \)-action and a functional \( \mathbb{T} \)-submodule \( Q \leq C(\mathbb{T}, A) \) such that \( Q_{sp} \) is not dense in \( Q \) for convergence in probability?

**Example 4.10.** Suppose there are

- a continuous homomorphism \( \kappa : A \rightarrow A' \), and
- a family of continuous epimorphisms \( \zeta_1, \ldots, \zeta_k : Z' \rightarrow Z \)

such that

\[
\varphi(f)(z') = \sum_{i=1}^{k} \kappa(f(\zeta_i(z'))), \quad \forall f \in P. \tag{6}
\]

Homomorphisms of this form are easily checked to be complexity-bounded. \( \triangleright \)

In Example 4.6, several functional \( \Delta \)-modules arising in the study of \( \text{PD}\)-Es were obtained using images and kernels of homomorphisms, and all of those homomorphisms took the form (6). However, it would be interesting to know whether kernels of complexity-bounded homomorphisms suffice.

**Question 4.11.** In the notation of Example 4.6, is there a complexity-bounded \( Z \)-module homomorphism \( \varphi : \mathcal{F}(Z, A) \rightarrow \mathcal{F}(X, A') \) for some base \( X \) and fibre \( A' \) such that \( M_0 = \ker \varphi \)? Can \( \varphi \) be of the form (6)? \( \triangleright \)

We will also need an approximate reversal of Definition 4.8.
Definition 4.12 (Step-polynomial pre-images). Let $P \leq \mathcal{F}(Z, A)$ and $Q \leq \mathcal{F}(Z', A')$ be as in Definition 4.8, and let $\varphi : P \rightarrow Q$ be a continuous homomorphism. Then $\varphi$ admits s.-p. pre-images relative to $(Z, A)$ and $(Z', A')$ if

$$(\varphi(P))_{sp} \subseteq \varphi(P_{sp}) :$$

that is, whenever $f \in \varphi(P)$ is a step polynomial, there is some $g \in P_{sp}$ with $\varphi(g) = f$.

Clearly, if $\varphi$ is invertible, then $\varphi$ has s.-p. pre-images if and only if $\varphi^{-1}$ is complexity-bounded.

The following elementary lemma will generally be used without comment.

Lemma 4.13. Let $P$, $Q$ and $R$ be functional $Z$-modules (each with their own fibres and bases), and let $\varphi : P \rightarrow R$ and $\psi : R \rightarrow Q$ be complexity-bounded homomorphisms. If both $\varphi$ and $\psi$ admit s.-p. pre-images then so does $\varphi \circ \psi$. If $\varphi \circ \psi$ admits s.-p. pre-images, then so does $\psi$, and if $\psi$ is injective then so does $\varphi$.

We next consider the behaviour of the above properties under co-induction. The notion of step polynomial is stable under co-induction only if one restricts attention to Lie modules $A$ that are s.-p.: this is why s.-p. modules were introduced.

To be precise, suppose that $P \leq \mathcal{F}(X, A)$ is a functional $Z$-module and that $Z' \geq Z$ is an enlargement. Then we have

$$\text{Coind}_{Z'}^Z P := \mathcal{F}(Z', P)^Z \leq \mathcal{F}(Z', \mathcal{F}(X, A))^Z.$$ 

Henceforth we will always identify the right-hand module here with $\mathcal{F}(X \times Z', A)^Z$, where $Z$ acts diagonally on $X$, $Z'$ and $A$. This identifies $\text{Coind}_{Z'}^Z P$ as a functional $Z'$-module in $\mathcal{F}(X \times Z', A)$, with fibre $A$ and dummy $X$, and its step polynomial elements will always be understood relative to $(X \times Z', A)$.

Definition 4.14 (Strong complexity boundedness and s.-p. pre-images). Suppose that $P \leq \mathcal{F}(X, A)$ and $Q \leq \mathcal{F}(X', A')$ are functional $Z$-modules with bases $X$ and $X'$ and fibres $A$ and $A'$, and that $\varphi : P \rightarrow Q$ is a $Z$-module homomorphism. Then $\varphi$ is strong complexity-bounded (resp. admits strong s.-p. pre-images) if, for every enlargement $Z' \geq Z$, the homomorphism

$$\text{Coind}_{Z'}^Z \varphi : \text{Coind}_{Z'}^Z P \rightarrow \text{Coind}_{Z'}^Z Q$$

is complexity-bounded (resp. admits s.-p. pre-images) relative to $(X \times Z', A)$ and $(X' \times Z', A')$. 

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Suppose again that $\alpha : Z \rightarrow X$ is a base, $A$ is a Lie $Z$-module, and $Z' \geq Z$ is an enlargement. In this case $\text{Coind}^{Z'}_Z F(X, A) = F(Z' \times X, A)^Z$ has an obvious isomorphism to $F(Z' \times_X X, A)$, where 

$$Z' \times_X X := (Z' \times X)/\{(z, \alpha(z)) \mid z \in Z\},$$

equipped with the homomorphism $\alpha' : Z' \rightarrow Z' \times_X X$ that sends $z'$ to the image of $(z', 0)$ in $Z' \times_X X$. The importance of s.-p. modules is that for these, this module isomorphism can be complexity-bounded.

**Lemma 4.15.** Let $A$ be a Lie $Z$-module, thought of as a functional module with base 0 (the trivial group) and fibre $A$. Then $A$ is s.-p. if the obvious isomorphism $\text{Coind}^{Z'}_Z A \cong A$ is complexity-bounded. In this case there are complexity-bounded isomorphisms with s.-p. pre-images

$$\text{Coind}^{Z'}_Z F(X, A) \cong F(Z' \times_X X, A)$$

for every base $X$ and every enlargement $Z' \geq Z$.

**Proof.** Let $T : Z \curvearrowright A$ be the action. If $A \rightarrow \text{Coind}^{Z'}_Z A$ is complexity-bounded, then for every $a \in A$ the function $z \mapsto T^z a$ must be s.-p. This precludes a full-rotation submodule, so $A$ is s.-p., by Proposition 3.43.

Now consider $\alpha : Z \rightarrow X$ and $Z'$, and let $\overline{\alpha} : Z \rightarrow Z' \times_X X$ be the diagonal embedding and $q : Z' \times X \rightarrow Z' \times_Z X$ be the quotient homomorphism by $\overline{\alpha}(Z)$. Corollary 3.23 gives a step-affine cross-section $\sigma$ of $q$, and using this one may define a step-affine $Z$-equivariant map $\lambda : Z' \times X \rightarrow Z$ by the equation

$$(z', x) = \sigma(q(z', x)) + \overline{\alpha}(\lambda(z', x)).$$

Now a suitable $Z'$-module isomorphism

$$\Phi : F(Z' \times_X X, A) \rightarrow F(Z' \times X, A)^Z$$

is given by

$$\Phi(f)(z', x) := T^{\lambda(z', x)} f(q(z', x)).$$

This $\Phi$ is complexity-bounded because $q$ and $\lambda$ are step-affine and $T$ is s.-p. On the other hand, $\Phi$ also has s.-p. pre-images, because if $F : Z' \times X \rightarrow A$ is a $Z$-equivariant step polynomial, then for a.e. $z_0 \in Z$ one knows that

$$F(z', x) = T^{\lambda(z', x) - z_0} F(\sigma(q(z', x)) + \overline{\alpha}(z_0))$$

$$= T^{\lambda(z', x)} (T^{-z_0} F(\sigma(q(z', x)) + \overline{\alpha}(z_0)))$$

for a.e. $(z', x)$.

For any choice of $z_0$ with this property, $F$ agrees a.e. with $\Phi(f)$, where $f(x, z') := T^{-z_0} F(\sigma(q(z', x)) + \overline{\alpha}(z_0))$ is still a step polynomial. □
**Corollary 4.16.** If \( \varphi : A \to B \) is a closed homomorphism of s.-p. \( Z \)-modules, then it has strong s.-p. pre-images.

**Proof.** Clearly we may assume \( B = \varphi(A) \). Letting \( Z' \geq Z \) be any enlargement, this is the assertion that any \( f \in \mathcal{F}_{sp}(Z', B)^Z \) may be lifted to some \( \mathcal{F}_{sp}(Z', A)^Z \). By the previous lemma, this is equivalent to the existence of a lift in \( \mathcal{F}_{sp}(Z'/Z, A) \) for every element of \( \mathcal{F}_{sp}(Z'/Z, B) \), and this is given by Lemma 3.32.

**Proposition 4.17.** Suppose that

- \( W \leq Z \) is an inclusion of compact Abelian groups,
- \( A \) is a s.-p. \( W \)-module,
- and
  \[
P_0 \leq Q_0 \leq \mathcal{F}(X, A)
  \]
  is a semi-functional \( W \)-module with base \( W \to X \) and fibre \( A \) such that \( Q_0/P_0 \) is discrete.

Let \( P := \text{Coind}_W^Q P_0, Q := \text{Coind}_W^Z Q_0 \). Finally, suppose that \( f \in Q \) is a step polynomial when regarded as an element of

\[
\text{Coind}_W^Z \mathcal{F}(X, A) \cong \mathcal{F}(X \times_W Z, A)
\]

(where this last isomorphism can be chosen complexity-bounded in both directions, by Lemma 4.15).

Then there is a \( W \)-invariant q.-p. partition \( \mathcal{P} \) of \( Z \) such that the function

\[
Z \to Q_0/P_0 : z \mapsto \text{res}_{(0,z), X} f + P_0
\]

is constant on each cell of \( \mathcal{P} \). Hence, the quotient homomorphism

\[
\text{Coind}_W^Z Q_0 \to \text{Coind}_W^Z (Q_0/P_0)
\]

is complexity-bounded.

**Proof.** Clearly \( X \) may be identified with a subgroup of \( X \times_W Z \). Having done so, Proposition 3.37 gives an \( X \)-invariant q.-p. partition \( \mathcal{P}_1 \) of \( X \times_W Z \) such that for each \( C \in \mathcal{P}_1 \), the map

\[
\Phi_C : C \to \mathcal{F}(X, A) : (x, z) \mapsto \text{res}_{(x,z), X} f
\]

is continuous.
Since $f$ was a step polynomial, there are Lie-group quotients $q : Z \rightarrow \mathbb{Z}$ and $q_1 : X \rightarrow X$ such that $f$ factorizes as $\overline{f} \circ (q_1 \times q)$ for some step polynomial

$$\overline{f} : X \times_{\mathbb{Z}} Z \rightarrow A,$$

and such that $\mathcal{P}_1$ is lifted from an $X$-invariant q.-p. partition $\mathcal{P}_1$ of $X \times_{\mathbb{Z}} Z$. It follows that the above maps $\Phi_{D}$ are pulled back from the corresponding maps

$$\overline{\Phi}_D : D \rightarrow F(X, A) : (\bar{x}, \bar{z}) \mapsto \text{res}_{\bar{x}, \bar{z}} \overline{f}, \quad D \in \mathcal{P}_1.$$

Now Lemma 3.14 provides that the refinement $\mathcal{P}_2$ of $\mathcal{P}_1$ into its connected components is still q.-p., and of course it is still $X$-invariant. By restricting the above maps $\overline{\Phi}_D$ to these connected components, we may therefore assume that $\mathcal{P}_2 = \mathcal{P}_1$. Having done so, each $D \in \mathcal{P}_1$ is connected, so has connected image under the continuous map $\overline{\Phi}_D$. Since $P_0$ is a clopen subgroup of $Q_0$, this implies that upon lifting back to $Q_0$, each of these connected $\overline{\Phi}_D$-images lies in a single coset of $P_0$.

Letting $\mathcal{P} := \mathcal{P}_1 \cap (\{0\} \times_{W} Z)$ completes the proof. \qed

Remark. In the examples that arise later where $A$ is a discrete group, every semi-functional $Z$-module $P \leq Q \leq F(Z, A)$ will have $P$ relatively open inside $Q$. I do not know whether there are examples with discrete $A$ for which this is not the case.

Suppose now that $P \leq Q \leq F(X, A)$ is an inclusion of functional $Z$-modules with base $X$ and fibre $A$.

**Definition 4.18 (Step-polynomial representatives).** The inclusion $P \leq Q$ (or the short exact sequence $P \leq Q \rightarrow Q/P$) has (resp. strong) s.-p. representatives if the quotient $Q/P$ is an s.-p. $Z$-module and the quotient homomorphism $Q \rightarrow Q/P$ admits (resp. strong) s.-p. pre-images, where we interpret $(Q/P)_{sp} = Q/P$.

If $A$ is discrete, we may refer instead to step-function representatives.

Given our convention that in an s.-p. module such as $Q/P$, all elements are step polynomial, we see that if $P \leq Q$ has s.-p. quotient, then it has s.-p. representatives if and only if $P + Q_{sp} = Q$.

Remark. Although we will not need it in the sequel, the following fact about s.-p. representatives may help build the reader’s understanding.

**Lemma 4.19.** If $P \leq Q \leq F(Y, A)$ are closed subgroups and $P + Q_{sp} = Q$, then $Q/P$ is locally compact.
Sketch proof. Let $\chi_1, \chi_2, \ldots$ be an enumeration of all homomorphisms from $Y$ to finite-dimensional tori, and now for $f \in \mathcal{F}_{sp}(Y, A)$, let $\text{Cplx}(f)$ denote the sum of the least $i$ such that $f$ factorizes through $\chi_i$, and of the number of linear inequalities and of all the degrees and coefficients needed to specify the convex polytopes and polynomials that go into the definition of $f$, if one uses that $\chi_i$ in the definition (this is similar to the notion of complexity that will be set up in Definition 10.1, except that $\text{Cplx}$ also keeps track of which quotient $\chi_i$ one uses to factorize $f$).

Let $q : Q \rightarrow Q/P$ be the quotient homomorphism, and for each $n$ let

$$Q_{\leq n} := \{ f \in Q_{sp} | \text{Cplx}(f) \leq n \}.$$ 

An easy argument shows that each $Q_{\leq n}$ compact (or see Corollary 10.5), and hence $Q/P = q(Q_{sp}) = \bigcup_{n \geq 1} q(Q_{\leq n})$ is $\sigma$-compact. By the Baire Category Theorem, some $q(Q_{\leq n})$ must be co-meager inside some nonempty open set $U$ in $Q/P$. This now implies that $q(Q_{\leq n}) - q(Q_{\leq n})$ is a compact neighbourhood of the identity in $Q/P$, so $Q/P$ is locally compact.

In view of this, the assumption that $Q/P$ is s.-p. in Definition 4.18 is only slightly more restrictive than simply asserting directly that $P + Q_{sp} = Q$. ◊

Lemma 4.20. Suppose that $P \leq Q \leq \mathcal{F}(X, A)$ is a semi-functional $\mathbb{Z}$-module with $Q/P$ discrete and with step-function representatives, and that $D \leq \mathcal{F}(X, A)$ is another submodule such that $P + D$ is closed. Then $P + D \leq Q + D$ has step-function representatives.

Proof. If $q + d \in Q + D$, then it equals $q' + p + d$ for some step function $q' \in Q$ and some $p \in P$, so it agrees with $q'$ modulo $P + D$. ◊

The following result is more subtle. It will be used repeatedly later.

Lemma 4.21. Suppose that $P \leq Q$ is a semi-functional $\mathbb{Z}$-module with base $\alpha : \mathbb{Z} \rightarrow X$ and s.-p. fibre $A$ for which $Q/P$ is discrete. If $P \leq Q$ has s.-p. representatives, then it has strong s.-p. representatives.

Proof. Given $g \in Q$, let $\overline{g}$ be its class in $Q/P$, and let $T$ be the action $Z \lhd A$.

Let $Z' \geq Z$ and let $f$ be a step-polynomial in $\text{Coind}^{Z'}_{\mathbb{Z}}(Q/P) \subseteq \mathcal{F}(Z', Q/P)$. Since $Q/P$ is discrete, $f$ is a step function (Corollary 3.30), and its level-set partition $\mathcal{P}$ is a q.-p. partition of $Z'$. Since $f$ is $Z$-equivariant, $\mathcal{P}$ is $Z$-invariant.

For each $C \in \mathcal{P}$, $f|\mathcal{C}$ takes a single value in $Q/P$, say equal to $\overline{g_C} \in Q/P$ for some choice of $g_C \in Q$. By the equivariance of $f$,

$$g_{C + \mathbb{Z}}(\cdot) = T^z g_C((\cdot - \alpha(z)) \quad \forall z \in Z. \quad (7)$$

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Using this, the value of \( f \) on every cell of \( \mathcal{P} \) is determined if one knows it only on a representative of each \( Z \)-orbit on \( \mathcal{P} \).

Let \( C(z') \) denote the cell of \( \mathcal{P} \) that contains \( z' \in Z' \); this is manifestly a step function of \( z' \). Also, let \( \tau : Z' \to Z \) be a step-affine \( Z \)-equivariant function, as provided by Corollary 3.23.

Now define \( F : Z' \times X \to A \) by
\[
F(z', x) := T^{\tau(z')}\left( g_C(z' - \tau(z'))(x - \alpha(\tau(z'))) \right) = T^{\tau(z')}\left( g_C(z' - \tau(z'))(x - \alpha(\tau(z'))) \right).
\]
This is clearly a step polynomial. A simple calculation gives
\[
F(z' + z_1, x + \alpha(z_1)) = T^{\tau(z' + z_1)}\left( g_C(z' + z_1 - \tau(z' + z_1))((x + \alpha(z_1)) - \alpha(\tau(z' + z_1))) \right)
\]
\[
= T^{z_1} T^{\tau(z')}\left( g_C(z' - \tau(z'))(x - \alpha(\tau(z'))) \right) = T^{z_1} F(z', x) \quad \forall z_1 \in Z,
\]
and also (7) gives
\[
F(z', \cdot) = T^{\tau(z')} g_C(z' - \tau(z'))(\cdot - \alpha(\tau(z'))) = g_{C(z')} = f(z').
\]
So \( F \in \text{Coind}_{Z'}^Z Q \) is a step polynomial whose image in \( \text{Coind}_{Z'}^Z Q/P \) equals \( f \).

In a few places, the preceding lemma will be applied via the following rather fiddly corollary.

**Corollary 4.22.** Let \( P \leq R \) be a semi-functional \( Z \)-module for which \( R/P \) is discrete, let \( Q \) be another functional \( Z \)-module, and let \( \alpha : Q \to R \) be a complexity-bounded homomorphism. If the combined homomorphism
\[
\varphi : P \oplus Q \to R : (p, q) \mapsto p + \alpha(q)
\]
has s.-p. pre-images, then it has strong s.-p. pre-images.

**Proof.** Let \( \psi : R \to R/P \) be the quotient homomorphism, and let \( A := \psi(R_{sp}) \leq R/P \). This is discrete, since \( R/P \) is discrete. Let \( R_1 := \psi^{-1}(A) = R_{sp} + P \), a closed submodule of \( R \); and now let \( Q_1 := \alpha^{-1}(R_1) \), a closed submodule of \( Q \). Since \( P \leq R_1 \), one has \( P \oplus Q_1 = \varphi^{-1}(R_1) \).

Let \( \psi_1 := \psi|_{R_1} : R_1 \to A \) and \( \varphi_1 := \varphi|(P \oplus Q_1) \). By construction, \( \psi_1 \) has s.-p. representatives, and \( \varphi_1 \) has them by assumption, so Lemma 4.13 gives that \( \psi_1 \circ \varphi_1 : P \oplus Q_1 \to A \) has s.-p. representatives. Since \( A \) is discrete, Lemma 4.21 implies that \( \psi_1 \circ \varphi_1 \) has strong s.-p. representatives.

Now let \( Z' \geq Z \) be an enlargement, and suppose that \((p, q) \in \text{Coind}_{Z'}^Z P \oplus \text{Coind}_{Z'}^Z Q\) are such that \( p + \alpha(q) \in (\text{Coind}_{Z'}^Z R)_{sp} \) (where we commit the slight abuse of writing \( \alpha \) in place of \( \text{Coind}_{Z'}^Z \alpha \)).
Lemma 3.31 gives that the restriction of $p + \alpha(q)$ to every $Z$-coset in $Z'$ is a step polynomial, so $p + \alpha(q)$ must actually lie in $\text{Coind}^Z_{Z/R} R_1$.

Since $p + \alpha(q)$ is a step polynomial, Proposition 4.17 gives that its image $\psi_1(p + \alpha(q)) = \psi_1(\varphi_1(p,q)) \in \text{Coind}^Z_{Z/R} A$ is a step function. Since $\psi_1 \circ \varphi_1$ has strong s.-p. representatives, as argued above, this gives us some $(p',q') \in P_{sp} \oplus (Q_1)_{sp}$ such that $p + \alpha(q) = p' + \alpha(q') + p''$ for some $p'' \in P$. However, this now forces $p'' = (p + \alpha(q)) - (p' + \alpha(q'))$ to be a step polynomial too, so $p + \alpha(q) = (p' + p'') + \alpha(q')$ with $p' + p'' \in P_{sp}$ and $q' \in Q_{sp}$, as required.

5 Cohomology with step polynomials

5.1 Cohomology with relative cocycles

As in Part I, a central rôle in this paper will be played by measurable cohomology for compact Abelian groups. A more complete overview of this was given in Section I.3. Here we focus on some properties that specifically concern step polynomials and semi-functional modules. We will use the same notation of cocycles, coboundaries and boundary operators as in Part I.

When a Polish $W$-module $M$ is given as a quotient of two other Polish $Z$-modules, one can present the group $H^p_m(W,M)$ in terms of those other modules. The following definition is in much the same the spirit as relative homology and cohomology in algebraic topology [9, Sections 2.1 and 3.1].

**Definition 5.1** (Relative cocycles). Given a short exact sequence

$$P \hookrightarrow Q \twoheadrightarrow M$$

in $\text{PMod}(W)$, a **relative cocycle from** $W$ **to** $(P,Q)$ **in degree** $p$ is an element of the module

$$Z^p(W,P,Q) := \{ f \in C^p(W,Q) \mid df \text{ takes values in } P \},$$

and a **relative coboundary from** $W$ **to** $(P,Q)$ **in degree** $p$ is an element of the module

$$B^p(W,P,Q) := C^p(W,P) + B^p(W,Q).$$

Just as for the usual groups of cocycles and coboundaries, $Z^p(W,P,Q)$ is always a closed subgroup of $C^p(W,Q)$, and the subgroup $B^p(W,P,Q)$ is contained in $Z^p(W,P,Q)$ but may not be closed.

These modules can be used to give an alternative presentation of the cohomology groups $H^p_m(W,Q/P)$. This is based on the following auxiliary lemma.
Lemma 5.2. Suppose that $\psi_i : P_i \hookrightarrow Q_i$ are continuous injective homomorphisms of Polish groups for $i = 1, 2$ with quotient groups $Q_i/\psi_i(P_i) = M_i$ (which are not assumed Hausdorff). Suppose also that $\varphi^P : P_1 \longrightarrow P_2$ and $\varphi^Q : Q_1 \longrightarrow Q_2$ are continuous homomorphisms which give rise to a commutative diagram

$$
\begin{array}{ccc}
P_1 & \xrightarrow{\varphi^P} & P_2 \\
\downarrow{\psi_1} & & \downarrow{\psi_2} \\
Q_1 & \xrightarrow{\varphi^Q} & Q_2 \\
\downarrow & & \downarrow \\
M_1 & \longrightarrow & M_2.
\end{array}
$$

Then the algebraic isomorphism

$$\text{coker}(M_1 \longrightarrow M_2) \cong \frac{Q_2}{\varphi^Q(Q_1) + \psi_2(P_2)}$$

is also a topological isomorphism.

Proof. The construction of this algebraic isomorphism is a standard diagram chase. It remains to prove that it is continuous in each direction.

The continuity from right to left is obvious from the definition of the quotient topology, because the composition of quotient maps

$$Q_2 \longrightarrow M_2 \longrightarrow \text{coker}(M_1 \longrightarrow M_2)$$

is continuous by definition and its kernel is $\varphi^Q(Q_1) + \psi_2(P_2)$. Similarly, from left to right we see that $\text{im}(M_1 \longrightarrow M_2)$ is the kernel of the homomorphism

$$M_2 = Q_2/\psi_2(P_2) \longrightarrow Q_2/(\varphi^Q(Q_1) + \psi_2(P_2)),$$

so the map is continuous in this direction also, by the definition of the topology on the cokernel. \qed

Lemma 5.3. In the setting of the above definition, the inclusion

$$B^p(W, P, Q) \leq Z^p(W, P, Q)$$

has quotient topologically isomorphic to $\Pi^0_n(W, Q/P)$.

Proof. Quotienting both $B^p(W, P, Q)$ and $Z^p(W, P, Q)$ by the common subgroup $C^p(W, P)$ gives a commutative diagram
The first of these vertical maps is obviously surjective, and a simple application of the Measurable Selector Theorem shows that the second is also surjective. Since $B^p(W, P, Q)$ contains $\ker(Z^p(W, P, Q) \rightarrow C^p(W, Q/P)) = C^p(W, P)$, it follows that the sequence

$$0 \rightarrow B^p(W, P, Q) \rightarrow Z^p(W, P, Q) \rightarrow H^p_m(W, Q/P) \rightarrow 0$$

is algebraically exact. Topological isomorphism follows from Lemma 5.2.}

### 5.2 Cohomology with step polynomial cocycles

Much of our interest will be in cocycles and coboundaries that are step polynomials.

**Definition 5.4.** If $A$ is a Lie $Z$-module and $p \geq 0$, then $C^p_{sp}(Z, A)$ is the set of cochains in $C^p(Z, A)$ that are step polynomials $Z^p \rightarrow A$. The **s.-p. cocycles** are the elements of

$$Z^p_{sp}(Z, A) := Z^p(Z, A) \cap C^p_{sp}(Z, A).$$

**Lemma 5.5.** If $A$ is a s.-p. $Z$-module, then $d : C^p(Z, A) \rightarrow C^{p+1}(Z, A)$ is complexity-bounded for every $p \geq 0$.

**Proof.** Recall the defining formula from Section I.3.1:

$$df(z_1, \ldots, z_{p+1}) := z_1 \cdot f(z_2, \ldots, z_{p+1}) + \sum_{i=1}^{p} (-1)^p f(z_1, \ldots, z_i + z_{i+1}, \ldots, z_{p+1}) + (-1)^{p+1} f(z_1, \ldots, z_p). \quad (8)$$

If $f$ is a step polynomial, then every term on the right is obviously a step polynomial, except for the first, which is a step polynomial because $A$ was assumed to be s.-p.

Given the above lemma, we can now make the following definition.
**Definition 5.6** (Step-polynomial cohomology). If $A$ is a s.-p. $\mathbb{Z}$-module, then its step-polynomial cohomology groups are defined by

$$H^p_{sp}(\mathbb{Z}, A) := \frac{\ker(d|C^p_{sp}(\mathbb{Z}, A))}{d(C^{p-1}_{sp}(\mathbb{Z}, A))} \text{ for } p \geq 0.$$  

Since we always have $C^p_{sp}(\mathbb{Z}, A) \subseteq C^p(\mathbb{Z}, A)$, there are natural comparison homomorphisms

$$H^*_{sp}(\mathbb{Z}, A) \rightarrow H^*_{m}(\mathbb{Z}, A).$$

The important result here is the following.

**Proposition 5.7.** The comparison homomorphisms are isomorphisms for any s.-p. $\mathbb{Z}$-module $A$.

More concretely, this asserts that the following hold for all $p \geq 0$:

1. every cocycle $f \in Z^p(\mathbb{Z}, A)$ is cohomologous in $H^*_m$ to a cocycle which is a step polynomial $\mathbb{Z}^p \rightarrow A$;

2. if $f \in C^p(\mathbb{Z}, A)$ is such that $df \in B^{p+1}(\mathbb{Z}, A)$ is a step polynomial $\mathbb{Z}^{p+1} \rightarrow A$, then there is a step polynomial $f' : \mathbb{Z}^p \rightarrow A$ such that $df = df'$.

Proposition 5.7 will first be proved for discrete modules, and then the general case will be obtained from that one.

The key to the proposition is to define another new cohomology theory which obviously agrees with $H^*_m(Z, A)$ in case $A$ is discrete, but admits a more ready comparison with $H^*_m(\mathbb{Z}, M)$ for general Polish (not just s.-p.) $\mathbb{Z}$-modules $M$. Similar techniques were used to prove various cocycle-regularity results for the theory $H^*_m$ in [2].

**Definition 5.8.** For any Polish $\mathbb{Z}$-module $M$, an element of $C^p(\mathbb{Z}, M)$ is almost-step if it is a uniform limit of step functions $\mathbb{Z}^p \rightarrow M$. The set of these is denoted by $C^p_{as}(\mathbb{Z}, M)$, and we set $Z^p_{as}(\mathbb{Z}, M) := Z^p(\mathbb{Z}, M) \cap C^p_{as}(\mathbb{Z}, A)$.

**Lemma 5.9.** Suppose that $M \in PMod(\mathbb{Z})$ and $f : \mathbb{Z} \rightarrow M$ is a function for which there is a q.-p. partition $\mathfrak{P}$ of $Z$ such that $f|C$ has a uniformly continuous extension to $C$ for every $C \in \mathfrak{P}$. Then $f$ is almost step.

**Proof.** Fix $\varepsilon > 0$, and let $d$ be a suitable metric on $M$. Since $\mathfrak{P}$ has only finitely many cells, our assumption gives a finite open cover $\mathcal{U}$ of $Z$, say by balls of sufficiently small radius for some choice of metric on $Z$, such that

$$y \sim_\mathfrak{P} z \text{ and } z, y \in U \in \mathcal{U} \implies d(f(y), f(z)) < \varepsilon.$$
Now let $Q$ be a q.-p. partition to which $U$ is subordinate (Lemma 3.5), and let $R := \mathcal{P} \cap Q$. Then the above property allows us to choose a function $g$ which is constant on every cell of $R$ and takes values uniformly within $\varepsilon$ of those of $f$ everywhere.

**Corollary 5.10.** If $M \in \text{PMod}(Z)$ and $f : Z^p \to M$ is almost step, then $df : Z^{p+1} \to M$ is almost step.

**Proof.** Looking again at formula (8), it is clear that if $g_n \to f$ uniformly, then $dg_n \to df$ uniformly, so it suffices to prove this in case $f$ is strictly a step function.

In this case, every summand in (8) is obviously also step except for the first:

$$z_1 \cdot f(z_2, \ldots, z_{p+1}).$$

If $\mathcal{P}$ is a q.-p. partition of $Z^p$ that directs $f$, and $\Omega$ is its lift through the coordinate projection $Z^{p+1} \to Z^p$ which omits the first coordinate, then this function is uniformly continuous on every cell of $\Omega$, so it is almost step by the preceding lemma.

Finally, it is obvious that a sum of almost step functions is almost step. \qed

This corollary now enables the following definition.

**Definition 5.11** (Almost-step cohomology). For any $M \in \text{PMod}(Z)$, its **almost-step cohomology groups** are defined by

$$H^p_{\text{as}}(Z, M) := \frac{\ker(d|C^p_{\text{as}}(Z, M))}{d(C_{\text{sp}}^{p-1}(Z, M))}.$$

In case $A$ is discrete, a uniformly convergent sequence of functions $Z^p \to A$ must actually stabilize. Therefore we obviously have

$$C^p_{\text{as}}(Z, A) = C^p_{\text{sp}}(Z, A) \quad \text{and} \quad H^p_{\text{as}}(Z, A) = H^p_{\text{sp}}(Z, A)$$

for discrete $A$. However, if $A$ is a non-discrete s.-p. module, then the class of almost-step functions $Z^p \to A$ may be strictly larger than the class of step polynomials $Z^p \to A$, and so we do not know at this stage that $H^p_{\text{as}}(Z, A) = H^p_{\text{sp}}(Z, A)$ for general s.-p. modules $A$. On the other hand, an almost-step function is certainly Borel, so once again there are obvious comparison homomorphisms

$$H^*_\text{as}(Z, M) \to H^*_\text{as}(Z, M)$$

for any $M \in \text{PMod}(Z)$.

In view of these remarks, the discrete-modules case of Proposition 5.7 is a consequence of the following more general result:
Proposition 5.12. The comparison maps \( H^p(Z, M) \rightarrow H^m(Z, M) \) are isomorphisms for every \( M \in P\text{Mod}(Z) \).

The key difference between Propositions 5.7 and 5.12 is that the latter holds on the whole of \( P\text{Mod}(Z) \). This means that the required isomorphisms may be checked by using the abstract characterization of \( H^m(Z, -) \) on that category in terms of Buchsbaum’s criteria [6]. This kind of argument was discussed in more detail in Subsection I.3.1. One cannot argue directly about \( H^*_{as} \) in this way, because this theory does not make sense on the whole of \( P\text{Mod}(Z) \) and so trivially cannot satisfy Buchsbaum’s axioms.

The axioms we need to verify are those listed in Subsection I.3.1: correct interpretation in degree zero; effaceability; and the long exact sequence. We shall prove that \( H^*_{as}(Z, -) \) has these properties after some preliminary lemmas.

Lemma 5.13. If \( A \rightarrow B \) is a short exact sequence of Polish Abelian groups, \( Z \) is a compact metrizable Abelian group and \( g : Z \rightarrow B \) is a uniform limit of step functions, then it has a lift \( Z \rightarrow A \) which is also a uniform limit of step functions.

Proof. Let \( d \) be a bounded complete group metric generating the topology of \( A \), let \( d/\infty \) be the resulting quotient metric on \( B \), and let \( d/\infty \) be the uniform metric on functions \( Z \rightarrow B \) that arises from \( d \). Let \( g_n : Z \rightarrow B \) be a sequence of step functions such that \( d/\infty(g_n, g_{n+1}) < 2^{-n-1} \) for all \( n \), and let \( \mathcal{P}_n \) be a q.p. partition that refines the level-set partition of \( g_n \) for each \( n \). By replacing each \( \mathcal{P}_n \) with \( \bigvee_{m \leq n} \mathcal{P}_m \) if necessary, we may assume that each \( \mathcal{P}_n \) refines its predecessors.

We now construct lifts \( \hat{g}_n : Z \rightarrow A \) of the functions \( g_n \) recursively as follows.

To begin, let \( \hat{g}_1 \) be any lift of \( g_1 \) which is constant on the cells of \( \mathcal{P}_1 \). Now assume that \( g_m \) has already been defined for each \( m \leq n \). Since \( d/\infty(g_n, g_{n+1}) < 2^{-n+1} \), and since \( g_n \) and \( g_{n+1} \) are both constant on every cell of \( \mathcal{P}_{n+1} \), by the definition of \( d \) we may choose a lift of the value of \( g_{n+1} \) on each of these cells which lies within \( 2^{-n+2} \) of the previously-chosen value of \( \hat{g}_n \) on this cell.

These selections define the lifts \( \hat{g}_n \). The construction gives \( d/\infty(\hat{g}_n, \hat{g}_m) < 2^{-n-m+4} \) for all \( n \) and \( m \), so these lifts form a Cauchy sequence in the uniform topology. Letting \( \hat{g} \) be their uniform limit, this completes the proof.

Corollary 5.14. If \( M \hookrightarrow N \rightarrow P \) is a short exact sequence in \( P\text{Mod}(Z) \), then one has a long exact sequence

\[ 0 \rightarrow M^Z \rightarrow N^Z \rightarrow P^Z \rightarrow H^1_{as}(Z, M) \rightarrow H^1_{as}(Z, N) \rightarrow H^1_{as}(Z, P) \rightarrow \cdots \]

Proof. The usual construction works here (c.f. [12]), provided all of the homomorphisms can be defined within the class of almost-step functions. This is obvious
for the compositions with the homomorphisms $A \to B$ and $B \to C$, and for the switchback homomorphisms it follows from the preceding Lemma 5.13 and Corollary 5.10.

**Lemma 5.15.** If $Y$ and $Z$ are compact metric Abelian groups, $M$ is a Polish Abelian group with translation-invariant metric $d$, and $f : Y \times Z \to M$ is an almost-step function, then the function $F : Y \to \mathcal{F}(Z, M)$ defined by

$$F(y)(z) := f(y, z)$$

is also an almost-step function.

**Proof.** Let $g_n \to f$ be a uniformly convergent sequence of step functions, and define $G_n : Y \to \mathcal{F}(Z, M)$ by

$$G_n(y)(z) := g_n(y, z).$$

It is easy to see that we now have uniform convergence $G_n \to F$ for the convergence-in-probability metric $d_0$ on $\mathcal{F}(Z, M)$. The proof is finished by showing that each $G_n$ is an almost-step function (although possibly not a step function: consider again Example 3.33). This holds because Proposition 3.37 gives a q.p. partition $\mathfrak{P}$ of $Y$ such that $G_n$ restricts to a uniformly continuous function on each cell of $\mathfrak{P}$, so we may apply Lemma 5.9 again.

**Corollary 5.16.** The inclusion $M \hookrightarrow \mathcal{F}(Z, M)$ is effacing for $\Pi^p_{\text{as}}(Z, -)$ for any $p \geq 1$ and $M \in \text{PMod}(Z)$.

**Proof.** This is similar to the proof of Lemma I.3.1. If $f : Z^p \to M$ is almost-step an satisfies $df = 0$, then the function $F : Z^{p-1} \to \mathcal{F}(Z, M)$ defined by

$$F(z_1, \ldots, z_{p-1})(z) := (-1)^p f(z_1, z_2, \ldots, z_{p-1}, z - z_1 - \cdots - z_{p-1})$$

is almost-step by the preceding lemma, and a direct calculation gives that $dF = f$ among $\mathcal{F}(Z, M)$-valued cochains.

**Proof of Proposition 5.12.** The theory $\Pi^p_{\text{as}}(Z, -)$ satisfies the long exact sequence axiom and the effaceability axiom by Corollaries 5.14 and 5.16 respectively. It remains to check its interpretation in degree 0. However, one of course has $C^0_{\text{as}}(Z, M) = C^0(Z, M) = M$, and so $\mathcal{Z}^0_{\text{as}}(Z, M) = M^Z$, as required.

As explained above, this already proves the discrete-groups case of Proposition 5.7. We next consider the case of Euclidean modules in that proposition.
Proof of Proposition 5.7 for Euclidean modules. Let $A$ be a Euclidean space with an s.-p. $Z$-action. We certainly have $H_0^0(Z, A) = A^Z = H_0^m(Z, A)$. On the other hand, by [2, Theorem A], we have $H_p^m(Z, A) = 0$ for all $p \geq 1$, so the comparison homomorphism is trivially surjective. It remains to prove that it is injective.

Suppose that $f \in Z_p^{sp}(Z, A)$. Since step polynomials are bounded, we may efface this cocycle as in the classical cohomology of finite groups: one has $f = dg$, where

$$g(z_1, \ldots, z_{p-1}) := (-1)^p \int_Z f(z_1, \ldots, z_{p-1}, z) \, dz.$$ 

However, this $g$ is also a step polynomial, by Corollary 3.39, so the proof is complete.

Theorem 2.1 asserts that any Abelian Lie group may be built from discrete, Euclidean and toral ingredients. We have proved Proposition 5.7 for discrete and Euclidean modules, and will now see how these cases can be combined to obtain the general theorem.

Lemma 5.17. Suppose that $A \leq B \xrightarrow{q} C$ is a short exact sequence consisting of s.-p. $Z$-modules. If the comparison maps of Proposition 5.7 are isomorphisms for $A$ and $B$, then the same holds also for $C$.

Proof. This is effectively a long-exact-sequence calculation in $H_{sp}^*$, although an elementary treatment seems quickest.

Step 1. Suppose $\sigma \in Z^p(Z, C)$. By a measurable selection, it equals $q\sigma$ for some $\sigma \in C^p(Z, B)$, and $d\sigma$ must take values in $A$. Therefore Proposition 5.7 for $A$ gives $d\sigma = \tau + d\alpha$ for some $\tau \in Z^{p+1}_{sp}(Z, A)$ and $\alpha \in C^p(Z, A)$. This implies that $d(\sigma - \alpha)$ is a step polynomial in $B^{p+1}(Z, B)$, so by Proposition 5.7 for $B$, it equals $d\sigma'$ for some $\sigma' \in C^p_{sp}(Z, B)$.

Therefore $\sigma - \alpha - \sigma' \in Z^p(Z, B)$, and since $d\tau$ takes values in $A$, so $d\tau = d(\tau - d\alpha)$ is both a step polynomial and an $A$-valued coboundary. Therefore, by Proposition 5.7 for $A$, it equals $d\tau'$ for some $\tau' \in C^p_{sp}(Z, A)$. This now implies that $\sigma + d\beta$ for some $\beta \in C^{p-1}(Z, B)$. This gives

$$\sigma = q(\sigma' + \sigma'') + d(q\alpha)$$

(because $q\alpha = 0$), where $q(\sigma' + \sigma'')$ is a step polynomial.

Step 2. Now suppose $\sigma \in C^{p-1}(Z, C)$ is such that $d\sigma$ is a step polynomial, and let $\alpha \in C^{p-1}(Z, B)$ be a measurable lift of $\sigma$. Lemma 3.32 gives a lift $\tau \in C^p_{sp}(Z, B)$ of $d\alpha$, and it follows that $d\tau$ must take values in $A$. On the other hand, $\tau - d\alpha$ takes values in $A$, so $d\tau = d(\tau - d\alpha)$ is both a step polynomial and an $A$-valued coboundary. Therefore, by Proposition 5.7 for $A$, it equals $d\tau'$ for some $\tau' \in C^p_{sp}(Z, A)$. This now implies that $\sigma + d\beta$ for some $\beta \in C^{p-1}(Z, B)$. This gives

$$\sigma = q(\sigma' + \sigma'') + d(q\alpha)$$

(because $q\alpha = 0$), where $q(\sigma' + \sigma'')$ is a step polynomial.
of Proposition 5.7 for $A$, this must be equal to $\sigma + d\beta$ for some $\sigma \in Z_{sp}^p(Z, A)$ and $\beta \in C^{p-1}(Z, A)$, so $d\alpha = (\tau - \tau' - \sigma) - d\beta$, where $\tau - \tau' - \sigma$ is an $B$-valued step polynomial. Therefore, by Proposition 5.7 for $B$, we have $d(\alpha + \beta) = d\gamma$ for some $\gamma \in C_{sp}^{p-1}(Z, B)$. Since $\beta$ takes values in $A$, this implies that

$$d\alpha = dq\gamma$$

with $q\gamma \in C_{sp}^{p-1}(Z, C)$, as required. $\square$

Lemma 5.18. Suppose that $A \leq B \xrightarrow{q} C$ is a short exact sequence consisting of s.-p. $Z$-modules. If the comparison maps of Proposition 5.7 are isomorphisms for $A$ and $C$, then the same holds also for $B$.

Proof. This follows very similar lines to the previous proof.

Step 1. First suppose that $\sigma \in Z^p(Z, B)$. Then $q\sigma \in Z^p(Z, C)$, so by assumption $q\sigma = \tau + d\beta$ for some $\tau \in Z_{sp}^p(Z, C)$ and $\beta \in C^{p-1}(Z, C)$.

Applying Lemma 3.32 to $\tau$, it equals $q\tau$ for some $\tau \in C_{sp}^p(Z, B)$. Also, a measurable selection gives $\beta \in C^{p-1}(Z, B)$ such that $\beta = q\beta$. Hence $\sigma_1 := \sigma - \tau - d\beta$ takes values in $A$, and also $d\sigma_1 = -d\tau \in B_{sp}^{p+1}(Z, A)$. Because Proposition 5.7 holds for $A$, this implies that $d\sigma_1 = d\kappa$ for some $\kappa \in C_{sp}^{p+1}(Z, A)$, and so $\sigma_2 := \sigma - \tau - \kappa - d\beta$ is a $p$-cocycle taking values in $A$. By another use of Proposition 5.7 for $A$, it equals $\kappa' + d\beta'$ for some $\kappa' \in Z_{sp}^p(Z, A)$, and $\beta' \in C^{p-1}(Z, A)$. Putting these facts together gives

$$\sigma = (\tau + \kappa + \kappa') + d(\beta + \beta'),$$

where $\tau + \kappa + \kappa'$ is a step polynomial.

Step 2. Secondly, suppose $\alpha \in C^{p-1}(Z, B)$ is such that $d\alpha$ is a step polynomial. Then $d(q\alpha)$ is a step-polynomial, so by Proposition 5.7 for $C$ we have $d(q\alpha) = d\beta$ for some $\beta \in C_{sp}^{p-1}(Z, C)$. Now Lemma 3.32 gives a lift $\beta \in C_{sp}^{p-1}(Z, B)$ of $\beta$, for which we must have $d(q\alpha) = d(q\beta)$, so $q(\alpha - \beta) \in Z^{p-1}(Z, C)$. By another appeal to Proposition 5.7 and Lemma 3.32, this implies $\alpha - \beta = \tau + d\gamma + \kappa$, with $\tau \in C_{sp}^{p-1}(Z, B)$, $\gamma \in C^{p-2}(Z, B)$ and $\kappa \in C^{p-1}(Z, A)$.

This now requires that $d\kappa \in B^p(Z, A)$ be a step polynomial, and so by Proposition 5.7 for $A$ we have $d\kappa = d\kappa'$ for some $\kappa' \in C_{sp}^{p-1}(Z, A)$. Combing this with the above, we obtain $d\alpha = d(\beta + \tau + \kappa')$, where $\beta + \tau + \kappa'$ is a step polynomial. $\square$

Proof of Proposition 5.7 for toral modules. Let $A = T^d$, so one has a presentation

$$Z^d \hookrightarrow \mathbb{R}^d \rightarrow A.$$
This presentation is preserved by any element of $\text{Aut} \mathbb{T}^d \cong \text{SL}_d(\mathbb{Z})$, hence, in particular, by the $\mathbb{Z}$-action. Since Proposition 5.7 has already been proved for $\mathbb{Z}^d$ and $\mathbb{R}^d$, Lemma 5.17 gives it also for $A$.

**Completed proof of Proposition 5.7.** By Theorem 2.1 an s.-p. $\mathbb{Z}$-module $A$ is always isomorphic as a Lie group to

$$\mathbb{R}^s \oplus \mathbb{T}^d \oplus D$$

for some $s, d \geq 0$ and discrete group $D$. Moreover, the sequence of inclusions

$$\mathbb{T}^d \leq \mathbb{R}^s \oplus \mathbb{T}^d \leq \mathbb{R}^s \oplus \mathbb{T}^d \oplus D$$

must be preserved by any topological automorphism of $\mathbb{R}^s \oplus \mathbb{T}^d \oplus D$, because $\mathbb{R}^s \oplus \mathbb{T}^d$ is the connected component of the identity, and within that $\mathbb{T}^d$ is the unique maximal compact subgroup.

Therefore, since Proposition 5.7 has already been proved for discrete, toral and Euclidean s.-p. modules, we obtain it for the $\mathbb{Z}$-action on $\mathbb{R}^s \oplus \mathbb{T}^d$, and thence on the whole of $A$, by two applications of Lemma 5.18.

Before leaving this section, we can now complete the proof of Proposition 3.43. The rôle of cohomology here results from the following description of actions on products of Abelian Lie groups. It is a special case of the measurable analog of the classical interpretation of degree-1 cohomology, and can be proved in just the same way. The details of that general result can be found in [12], or, a little more gently, in [11, Section 4.7].

**Proposition 5.19.** Suppose that $A$ and $C$ are Abelian Lie groups, that $B = A \times C$, and that $T_A$, $T_B$ and $T_C$ are $\mathbb{Z}$-actions on these groups fitting into a short exact sequence

$$T_A \leq T_B \rightarrow T_C$$

in $\text{PMod}(\mathbb{Z})$, which may not be split. Consider $\text{Hom}(C, A)$ as a $\mathbb{Z}$-module with the action

$$\tilde{T}^z \varphi := T_A^z \circ \varphi \circ T_C^{-z} \quad \text{for } \varphi \in \text{Hom}(C, A).$$

Then there is a 1-cocycle $\sigma : \mathbb{Z} \rightarrow \text{Hom}(C, A)$ such that $T_B$ is isomorphic to the action $T : \mathbb{Z} \curvearrowright A \times C$ defined by

$$T^z(a, c) = (T_A^z a + \sigma(z)(c), T_C^z c).$$

Two such actions, corresponding to $\sigma$ and $\sigma'$, are isomorphic if and only if $\sigma - \sigma' \in \mathcal{B}^1(\mathbb{Z}, \text{Hom}(C, A))$, and so $\mathbb{Z}$-module extensions of $T_A$ by $T_C$ are classified by $H^1_{\text{in}}(\mathbb{Z}, \text{Hom}(C, A))$.  \hfill \Box
Lemma 5.20. If $A$ is a discrete Abelian group or a compact Abelian Lie group, then any $Z$-action on $A$ is s.-p. 

Proof. First suppose that $A$ is discrete and $T : Z \curvearrowright A$ is a continuous action. Then every $T$-orbit must be finite. It follows that every finite-subset of $A$ is contained in a finitely-generated, $Z$-invariant submodule of $A$. Since a step polynomial from a compact group always has pre-compact image, this means we may consider only finitely-generated discrete modules. In this case, $T$ factorizes through a finite quotient of $Z$, so it is certainly s.-p.

On the other hand, if $A$ is a compact Lie $Z$-module, then $A = \hat{A}$ is the Pontryagin dual of a finitely-generated discrete $Z$-module, so again the action factorizes through a finite quotient of $Z$.

Lemma 5.21. Every compact-by-discrete Lie $Z$-module is s.-p.

Proof. By Theorem 2.1, such a module always takes the form $T^d \times C$ for some torus $d \geq 1$ and discrete module $C$, and now the action must be of the form described in Proposition 5.19:

$$T^z(a, c) = (S^z a + \sigma(z)(c), T^z c).$$

As for the previous proof, for any given s.-p. map $Y \longrightarrow T^d \times C$ we may restrict our attention to a finitely-generated $Z$-invariant subgroup of $C$, and so assume that $C$ is itself finitely-generated. In this case, $\text{Hom}(C, T^d)$ is also a compact Lie group, and so it is an s.-p. module, by the preceding lemma. Therefore, by Proposition 5.19 we may take $\sigma : Z \longrightarrow \text{Hom}(C, T^d)$ to be a step polynomial. If $f_1 : Y \longrightarrow T^d$ and $f_2 : Y \longrightarrow C$ are step polynomials and $f := (f_1, f_2)$, then this gives that

$$F(z, y) := T^z f(y) = (S^z f_1(y) + \sigma(z)(f_2(y)), T^z f_2(y))$$

is also s.-p.. This follows by the previous lemma, together with the fact that $f_2$ is constant on each cell of some q.-p. partition of $Z$.

Proof of Proposition 3.43 By another appeal to Theorem 2.1, any Abelian Lie group is of the form $\mathbb{R}^d \times A$ for some compact-by-discrete group $A$. If $T$ is a $Z$-action on this product, it must be of the form described in Proposition 5.19. However, in this case $\text{Hom}(A, \mathbb{R}^d)$ is itself a Euclidean space, and so [2] Theorem A] gives $H^1_m(Z, \text{Hom}(A, \mathbb{R}^d)) = 0$. Hence $T$ must split as $U \oplus T_A$ for some linear representation $U : Z \longrightarrow \text{Aut} \mathbb{R}^d$.

Arguing coordinate-wise, it is clear that $T$ is s.-p. if and only if both $U$ and $T_A$ are s.-p. The latter is always the case by the preceding lemma, so we need only analyze when $U$ is s.-p.
As is standard, such a linear representation is always isomorphic to an orthogonal representation, and this then decomposes into a finite direct sum \( \rho_0 \oplus \bigoplus_i (\text{rot} \circ \chi_i) \), where \( \rho_0 \) factorizes through a finite quotient of \( Z \) and each \( \text{rot} \circ \chi_i \) is a full rotation factor. It is now clear that this is s.-p. if and only if there are no summands of the second kind. This completes the proof.

5.3 Semi-functional inclusions and cohomology

Our next step is to generalize some of the ideas of the preceding section to semi-functional modules. The following definition fits naturally into the framework of Subsection 4.2.

Let \( P \leq Q \leq \mathcal{F}(X, A) \) be a semi-functional \( Z \)-module, where \( X \) is a base and \( A \) is a s.-p. \( Z \)-module. Observe that for each \( p \geq 0 \), the groups \( \mathcal{Z}_p^p(W, Q, P) \) and \( \mathcal{B}_p^p(W, Q, P) \) have natural interpretations as subgroups of \( \mathcal{F}(W^p \times X, A) \).

**Definition 5.22.** For each \( p \geq 0 \), the s.-p. relative \( p \)-cocycles associated to \( P \leq Q \rightarrow Q/P \) are the elements of the group

\[
\mathcal{Z}_{sp}^p(W, Q, P) := \mathcal{Z}_p^p(W, Q, P) \cap \mathcal{F}_{sp}(W^p \times X, A).
\]

Similarly, the s.-p. relative \( p \)-coboundaries are the elements of the sum

\[
\mathcal{B}_{sp}^p(W, Q, P) := d(\mathcal{C}_{sp}^{p-1}(W, Q)) + \mathcal{C}_p^p(W, P).
\]

Clearly \( \mathcal{B}_{sp}^p \subseteq \mathcal{Z}_{sp}^p \).

**Definition 5.23.** Let \( P \leq Q \leq \mathcal{F}(X, A) \) be a semi-functional \( Z \)-module as above. It admits (resp. strong) s.-p. coboundary solutions if

\[
\mathcal{C}_{sp}^{p-1}(W, Q) \oplus \mathcal{C}_p^p(W, P) \rightarrow \mathcal{B}_p^p(W, Q, P) : (f, g) \mapsto df + g,
\]

regarded as a homomorphism of functional \( Z \)-modules, admits (resp. strong) s.-p. pre-images for every \( W \leq Z \) and every \( p \geq 0 \).

Later, we shall also handle examples of semi-functional \( \Delta \)-modules \( P \leq Q \) for which the cohomology groups \( \mathcal{H}^p_{\text{in}}(W, Q/P) \) may be interpreted as functional, and so that the resulting quotient homomorphisms

\[
\mathcal{Z}_p^p(W, Q, P) \rightarrow \mathcal{H}^p_{\text{in}}(W, Q/P)
\]

are complexity-bounded and have strong s.-p. pre-images. However, in these cases there will be some delicacy in the choice of base for these functional modules (it can depend on \( W \)), and so it seems easier to avoid making a general definition to describe this situation.
We will shortly derive our first examples of these phenomena from Proposition 5.7. First, we must reconsider the Shapiro Lemma in terms of relative cocycles and coboundaries. The Shapiro Lemma is a classical tool from group cohomology which described how that theory transforms under co-induction. Its version for measurable group cohomology of compact Abelian groups was recalled in Subsection I.3.3. If $W \leq Z$ and $M \in \text{PMod}(W)$, the Shapiro Isomorphisms are a sequence
\[
\psi_p : H^p_m(W, M) \xrightarrow{\cong} H^p_m(Z, \text{Coind}^Z_W M), \quad p \geq 0.
\]
The proof gives a canonical sequence of isomorphisms between these cohomology groups in each degree, although they are not implemented by canonical maps at the level of cochains. Nevertheless, we will prove that this isomorphism does respect complexity in functional modules, in the following sense.

**Proposition 5.24.** Suppose that $P \leq Q \rightarrow M$ is a semi-functional $W$-module with base $X$ and fibre $A$. Then the isomorphism from right to left above has a semi-functional presentation
\[
\begin{array}{ccc}
B^p(W, Q, P) & \xrightarrow{\text{incl}^p} & B^p(Z, \text{Coind}^Z_W Q, \text{Coind}^Z_W P) \\
\downarrow \Psi_p & & \downarrow \text{incl}^p \\
Z^p(W, Q, P) & \xrightarrow{\Psi_p} & Z^p(Z, \text{Coind}^Z_W Q, \text{Coind}^Z_W P) \\
\downarrow & & \downarrow \\
H^p_m(W, M) & \xrightarrow{\psi_p} & H^p_m(Z, \text{Coind}^Z_W M)
\end{array}
\]
which is complexity-bounded. On the other hand, any step polynomial in
\[
Z^p(W, \text{Coind}^Z_W Q, \text{Coind}^Z_W P)
\]
lies in $\Psi_p(Z_{\text{sp}}^p(W, Q, P))$ up to a relative coboundary. Therefore, under $\psi_p$, classes in $H^p_m(W, M)$ that have s.-p. representatives in $Z^p(W, Q, P)$ correspond to classes in $H^p_m(Z, \text{Coind}^Z_W M)$ that have s.-p. representatives in
\[
Z^p(W, \text{Coind}^Z_W Q, \text{Coind}^Z_W P).
\]

This will be proved after the following auxiliary result.

**Lemma 5.25.** Let $P \leq Q \leq \mathcal{F}(X, A)$ be a semi-functional $Z$-module with base $X$ and fibre $A$, and let $f \in Z^p_{\text{sp}}(Z, Q, P)$. Then there is another element $g \in Z^p_{\text{sp}}(Z, Q, P)$ such that
\[
m_{Z^p \times X} \{ g = f \} = 1
\]

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and
\[ dg(z_1, \ldots, z_{p+1})(\cdot) \in P \quad \text{for strictly every } (z_1, \ldots, z_{p+1}) \in Z^{p+1}. \]

This lemma will be proved by constructing a \( g \) with the following property:

**Definition 5.26.** If \( Z \) is a compact Abelian group, \( z \in Z \), \( M \) is a Polish space and \( g : Z \to M \) is measurable, then \( g \) takes an essential value at \( z \) if for all neighbourhoods \( U \) of \( z \) and \( V \) of \( g(z) \) one has
\[ m_Z(U \cap g^{-1}(V)) > 0. \]

The function \( g \) takes only essential values if this is true at strictly every \( z \).

**Proof of Lemma 5.25**. Step 1. Since \( f \) is a step polynomial, it is lifted from some Lie-group quotients of \( Z \) and \( X \), so we may assume these are themselves Lie groups. Let \( Z_0 \) and \( X_0 \) be their respective identity components, let \( z \) and \( x \) be their respective Lie algebras, and let \( \exp : z \to Z_0 \) denote the exponential map, and similarly for other groups.

Step 2. Proposition 3.37 gives a \( q \)-p. partition \( P_1 \) of \( Z \times X \) which is \( X \)-invariant and such that the function
\[ (z_1, \ldots, z_p, x) \mapsto \text{res}(-z_1, \ldots, -z_p, x), Xf = f(z_1, \ldots, z_p)(\cdot - x) \]
restricts to a uniformly continuous function on each cell of \( P_1 \). Letting \( \mathfrak{P} \) be the pull-back of \( P_1 \) under the coordinate embedding \( Z^p \to Z^p \times X \), it follows that as a function \( Z^p \to F(X, A) \), \( g \) itself has uniformly continuous restriction to each cell of \( \mathfrak{P} \).

Step 3. After refining \( \mathfrak{P} \) if necessary, we may also assume the following:

- for each coset \( z + Z_0^p \), where \( z \in Z^p \), the partition \( (z + Z_0^p) \cap \mathfrak{P} \) consists of finitely many subsets of \( Z_0^p \) that are all images under \( \exp \) of convex polytopes in \( z^{\oplus p} \);

- for each of these convex-polytopal images \( \exp(Q) \in (z + Z_0^p) \cap \mathfrak{P} \), the \( \exp \)-image of each facet of \( Q \) is also an element of \( (z + Z_0^p) \cap \mathfrak{P} \).

It follows that for \( v \in z^{\oplus p} \), if \( \mathbb{R} \cdot v \) does not lie in a translate of any of the affine subspaces generated by one of the proper facets of those polytopes, then it has the following property:
(P): For strictly every \( z \in \mathbb{Z}^p \), there are a neighbourhood \( U \) of \( v \) in \( \mathbb{Z}^p \) and an \( \varepsilon > 0 \) such that the set
\[
\{ z - \exp(tu) \mid u \in U \text{ and } 0 < t < \varepsilon \}
\]
is contained in a single cell of \( \mathcal{P} \).

Geometrically, this is asserting that if one considers a cone with apex at \( z \) and axis in direction \( v \), and if the cone is small enough and narrow enough, then it lies entirely in a single cell of \( \mathcal{P} \), possibly except for the apex itself.

In particular, this holds for Lebesgue-almost every \( v \in z \oplus \mathbb{Z}^p \). Moreover, if we write \( C_v(z) \) for the cell of \( \mathcal{P} \) that contains this little cone for a given \( z \), and we write \( C(z) \) for the cell of \( \mathcal{P} \) that contains \( z \) itself, then \( C_v(z) \) depends only on \( C(z) \), by the second bullet-point above. This therefore defines a map \( \mathcal{P} \rightarrow \mathcal{P} : C \mapsto C_v \), under which we have \( C_v = C \) if and only if \( C \) has maximal dimension, and hence nonempty interior. If \( \dim C < \dim \mathbb{Z}^p \), then \( C_v \) will be one of the cells of maximal dimension which is adjacent to \( C \).

Step 4. Now choose a vector \( v \in \mathbb{Z}^p \) with the above property, and let
\[
g(z)(x) := \lim_{t \downarrow 0} f(z - \exp(tv))(x) \quad \text{for } (z, x) \in \mathbb{Z}^p \times X.
\]
Because \( g \) is uniformly continuous on each cell of \( \mathcal{P} \), for strictly every \( z \) this limit is well-defined for a.e. \( x \). It is still a step polynomial, because if \( \Omega \) is a q.-p. partition of \( \mathbb{Z}^p \times X \) that directs \( f \), then on each cell \( D \in \Omega \) the new function \( g \) is simply obtained from the Euclidean-space polynomial that defined \( f \) on either \( D \) or some cell adjacent to \( D \).

Moreover, this \( g \) is equal to \( f \) a.e., because a.e. point \( (z, x) \in \mathbb{Z}^p \times X \) lies in the interior of some \( D \in \Omega \) of maximal dimension, in which case the shifted point \( (z - \exp(tv), x) \) also lies in \( \text{int } D \) for all sufficiently small \( t \).

Step 5. The proof is completed by showing that \( dg(z_1, \ldots, z_p)(\cdot) \in P \) for strictly every \( (z_1, \ldots, z_p) \). The key to this is that the function
\[
\mathbb{Z}^p \rightarrow \mathcal{F}(X, A) : z \mapsto g(z)(\cdot)
\]
takes only essential values. This is because, for every \( z \in \mathbb{Z}^p \), if one chooses a small enough open set \( U_z \) according to property (P), then
\[
g(z) = \lim_{t \downarrow 0} f(z - \exp(tu)) = \lim_{t \downarrow 0} g(z - \exp(tu)) \quad \forall u \in U_z.
\]
This implies that the set \( \{ g(z')(\cdot) \mid z' \in A_{z, \varepsilon} \} \subseteq \mathcal{F}(X, A) \) converges to the singleton \( \{ g(z)(\cdot) \} \) as \( \varepsilon \downarrow 0 \), where
\[
A_{z, \varepsilon} := \{ z - \exp(tu) \mid u \in U_z, \ t \in (0, \varepsilon) \}.
\]
However, for any neighbourhood $V$ of $z$, we can now choose $\varepsilon > 0$ so that $A_{z,\varepsilon} \subseteq V$ and $m_{Z^p}(A_{z,\varepsilon}) > 0$, giving that $g(z)(\cdot)$ is an essential value.

Finally, formula (8) easily implies that $dg$ must also take essential values strictly everywhere. Therefore, since $dg$ takes values in $P$ almost everywhere, it must do so strictly everywhere. \qed

Proof of Proposition 5.24. This is most easily seen by recalling an explicit formula for a suitable homomorphism $\Psi$. This was given in Part I following Definition I.3.6:

$$\Psi_p(f)(z_1, \ldots, z_p)(z) = \sigma(-z) \cdot (f(\sigma(z_1 - z) - \sigma(-z), \sigma(z_2 + z_1 - z) - \sigma(z_1 - z), \ldots, \sigma(z_1 + \ldots + z_p - z) - \sigma(z_1 + \ldots + z_{p-1} - z)))$$

where $\sigma$ is a $W$-equivariant map $Z \to W$. In Part I, this formula was applied to elements of $Z^p(W, M)$, but it clearly also has the necessary properties we need on $Z_p(W, Q, P)$, in which case both sides of the above are elements of $Q$ for each $(z_1, \ldots, z_p, z)$. If we choose $\sigma$ to be a step-affine map, as provided by Corollary 3.23, then $\Psi_p$ is manifestly complexity-bounded.

On the other hand, if $f \in Z^p(Z, \text{Coind}^Z_W Q, \text{Coind}^Z_W P)$ is s.-p. as an element of $\mathcal{F}(Z^{p+1}, Q) \subseteq \mathcal{F}(Z^{p+1} \times X, A)$, then by the preceding lemma it is a.e. equal to a step polynomial that satisfies the relative cocycle equation strictly everywhere. As recalled after Definition I.3.6, in this case it has the same cohomology class in $H_m(Z, \text{Coind}^Z_W M)$ as $\Psi_p(f')$, where $f' = f|_{W^p \times \{0\}}$, which is an element of $Z_{sp}^p(W, Q, P)$. \qed

Corollary 5.27. Let $Y, W \leq Z$ be compact Abelian groups such that $W + Y = Z$, and let $A$ be an s.-p. $Y$-module. Then the functional $Z$-module $Q := \text{Coind}^Z_W A$ admits strong s.-p. coboundary solutions. Also, $H_m(W, Q)$ is an s.-p. $Z$-module, and the quotient homomorphism

$$Z^p(W, Q) \to H_m^p(W, Q)$$

is complexity-bounded and has strong s.-p. pre-images.

Proof. Let $A'$ denote $A$ with action restricted to the subgroup $W \cap Y$. Using that $W + Y = Z$, one has the following isomorphism of Polish Abelian groups:

$$\text{Coind}^Z_W A = \mathcal{F}(Z, A)^Y \cong \mathcal{F}(W, A')^{W \cap Y} = \text{Coind}^W_{W \cap Y} A'.$$

This isomorphism is complexity-bounded in both directions, by Lemma 4.15, and preserves the action of the subgroup $W$, although possibly not of the whole of $Z$. 54
Replacing $Y$ with $W \cap Y$ and $Z$ with $W$, we have therefore reduced our task to the case $W = Z$.

In this case, the Shapiro Isomorphism and Proposition 5.24 give a commutative diagram

\[
\begin{array}{ccc}
\mathcal{B}^p(Y, A) & \rightarrow & \mathcal{B}^p(Z, \text{Coind}^Z_Y A) \\
\text{incl}_W & \downarrow & \text{incl}_W \\
\mathcal{Z}^p(Y, A) & \rightarrow & \mathcal{Z}^p(Z, \text{Coind}^Z_Y A) \\
\downarrow & & \downarrow \\
H^m_m(Y, A) & \cong & H^m_m(Z, \text{Coind}^Z_Y A),
\end{array}
\]

where the middle horizontal arrow is complexity-bounded and has strong s.-p. pre-images.

If $p = 0$, then

\[H^0_m(Y, A) \cong A^Y \cong H^0_m(Z, \text{Coind}^Z_Y A),\]

endowed with the trivial $Z$-action. Since this is a Lie module with trivial action, it is s.-p. On the other hand, if $p \geq 1$, then Proposition I.3.3 gives that $H^p_m(Y, A) \cong H^p_m(Z, \text{Coind}^Z_Y A)$ is discrete, so it is s.-p. by Lemma 5.20.

Since all elements of an s.-p. module are s.-p., the quotient homomorphism $\mathcal{Z}^p(Z, Q) \rightarrow H^p_m(Z, Q)$ is trivially complexity-bounded, so it remains to prove strong s.-p. pre-images.

If $p = 0$, then this homomorphism is just the identity $\mathcal{Z}^0(Z, A) \rightarrow A^Y$, so the result is obvious.

On the other hand, if $p \geq 1$, then $H^p_m(Z, Q)$ is discrete, as recalled above. It therefore suffices to prove s.-p. pre-images, from which strong s.-p. pre-images follow by Lemma 4.21.

To finish, simply observe that Proposition 5.7 has given step-polynomial representatives for every cohomology class in $H^m_m(Y, A)$, and now the middle row of the commutative diagram above converts these into step-polynomial representatives for every cohomology class in $H^p_m(Z, Q)$, as required.

**Corollary 5.28.** Suppose that $Y \leq Z$ and that $P_0 \leq Q_0$ is a functional $Y$-module whose quotient $A_0 := Q_0/P_0$ is s.-p. and which has strong s.-p. representatives. Let $P := \text{Coind}^Z_Y P_0$, $Q := \text{Coind}^Z_Y Q_0$ and $A := \text{Coind}^Z_Y A_0$. Then the semi-functional $Z$-module $P \leq Q$ admits strong s.-p. coboundary solutions.

**Proof:** Let $W \leq Z$. Because $Z$ was already an arbitrary enlargement of $Y$, we need not consider a further enlargement of $Z$ in the proof.
Let $\overline{\sigma}$ denote the composition of $\sigma$ with the quotient map $Q \to A$, and similarly. Then $\overline{\sigma} = d^W\overline{f}$ is an element of $(B^p(W,A))_{sp}^-$, and the module $A$ admits strong s.-p. coboundary solutions, by the preceding corollary. Therefore $\overline{\sigma} = d^W\overline{f}$ for some $\overline{g} \in C^p_{sp}((W,A))_{sp}^-$, and the module $A_{sp}$ admits strong s.-p. coboundary solutions, by the preceding corollary. Therefore $\sigma = d^Wg$ for some $g \in C^{p-1}_{sp}(W,Q)$. Since $Q_0 \to A_0$ has strong s.-p. representatives, there is some $g \in C^{p-1}_{sp}(W,Q)$ whose image is $\overline{g}$, and now $\sigma - d^Wg$ is $P$-valued, as required.

Before leaving this subsection, it seems worth recording the following basic question, which I do not know how to approach.

**Question 5.29.** Are there a compact Abelian group $Z$, a closed submodule $Q \leq C(Z,\mathbb{T})$ and an integer $p \geq 1$ for which $B^p(Z,Q)$ is not closed in $\mathcal{Z}^p(Z,Q)$?

### 5.4 Some specialized results about co-discrete functional inclusions

This subsection will analyze the behaviour of cohomology groups in a rather special situation, which will be encountered frequently during our analysis of cohomology $\Delta$-modules later.

Suppose that $Y, W \leq Z$ are two closed subgroups of a compact Abelian group, and that

$$P_0 \leq Q_0 \leq R_0 \leq C(Y,A)$$

are inclusions of closed functional $Y$-modules, where $A$ is a s.-p. $Y$-module. Let

$$P_1 := \text{Coind}^{Y+W}_Y P_0, \quad Q_1 := \text{Coind}^{Y+W}_Y Q_0, \quad \text{and} \quad R_1 := \text{Coind}^{Y+W}_Y R_0$$

and

$$P := \text{Coind}^{Z}_Y P_0, \quad Q := \text{Coind}^{Z}_Y Q_0, \quad \text{and} \quad R := \text{Coind}^{Z}_Y R_0.$$

These data give rise to a short exact sequence $Q/P \hookrightarrow R/P \twoheadrightarrow R/Q$, and hence a cohomology long exact sequence

$$\cdots \to H^p_{in}(W,Q/P) \to H^p_{in}(W,R/P) \to H^p_{in}(W,R/Q) \to H^{p+1}_{in}(W,Q/P) \to \cdots.$$ 

Each of these cohomology groups may be presented using relative cocycles and coboundaries. This subsection will examine how step polynomial relative cocycles fare under the various homomorphisms of the long exact sequence, given suitable assumptions on the original modules.

We make the standing assumption in this subsection that all of these cohomology groups are Polish; when we apply these results later, that fact will be justified by results from Part I.
Lemma 5.30. Assume that $Q_0 \leq R_0$ has strong s.-p. representatives and that $P_0 \leq Q_0$ admits strong s.-p. relative coboundary solutions. Also, assume that $W + Y = Z$. Let $p \geq 0$, and consider the semi-functional $Z$-module

$$\mathcal{Z}(W, Q, P) + \mathcal{B}(W, R, P) \leq \mathcal{Z}(W, R, P) \to \coker(\mathcal{H}_{m}^p(W, Q/P) \to \mathcal{H}_{m}^p(W, R/P)).$$

Then the cokernel appearing here is s.-p., and discrete in case $p \geq 1$ or $R_0/Q_0$ is discrete, and the quotient homomorphism admits strong s.-p. representatives.

In view of the long exact sequence axiom, we could equivalently have written the presentation

$$\mathcal{Z}(W, Q, P) + \mathcal{B}(W, R, P) \leq \mathcal{Z}(W, R, P) \to \text{im} \left( \mathcal{H}_{m}^p(W, R/P) \to \mathcal{H}_{m}^p(W, R/Q) \right),$$

and we shall not distinguish between these formulations when we apply this lemma later.

Proof. First suppose $p = 0$, so that the cokernel in question is

$$(R/P)^W/(Q/P)^W.$$  

This is clearly Polish, and is identified with a closed submodule of $(R/Q)^W = (\text{Coind}^2_{Y} Q_0/Q_0)^W$, which is isomorphic to $(R_0/Q_0)^Y \cap W$, since $Y + W = Z$. It is therefore also s.-p. It has strong s.-p. representatives, simply because the homomorphism $R_0 \to R_0/Q_0$ is assumed to have strong s.-p. representatives, and we are considering the restriction of this homomorphism to the pre-image of $(R_0/Q_0)^Y \cap W$.

So now let $p \geq 1$. The conclusion of discreteness follows because the long exact sequence corresponding to $Q/P \leq R/Q \to R/Q$ gives a continuous injection

$$\coker(\mathcal{H}_{m}^p(W, Q/P) \to \mathcal{H}_{m}^p(W, R/P)) \to \mathcal{H}_{m}^p(W, R/Q),$$

and this target is topologically isomorphic to $\mathcal{H}_{m}^p(W \cap Y, R_0/Q_0)$ by the Shapiro Isomorphism, so it is discrete (this was essentially the proof of Lemma I.3.7). It is therefore s.-p. by Lemma [5.20].

Also, owing to this discreteness, by Lemma [4.21] it suffices to find s.-p. representatives without any enlargement of $Z$. This will be obtained along similar lines to the proof of Lemma I.3.7.

The Shapiro Isomorphism gives
\[ H^p_m(W \cap Y, Q_0/P_0) \cong H^p_m(W \cap Y, R_0/P_0) \]
\[ H^p_m(W, Q/P) \cong H^p_m(W, R/P), \]
where \( \Lambda \) is the name of this commutative square. By Proposition 5.24, the second downward isomorphism here can be extended to a complexity-bounded semifunctional morphism
\[ B^p(W \cap Y, R_0,P_0) \cong B^p(W,R,P) \]
\[ Z^p(W \cap Y, R_0,P_0) \cong Z^p(W,R,P) \]
\[ H^p_m(W \cap Y, R_0/P_0) \cong H^p_m(W, R/P). \]

It therefore suffices to show the the cokernel of the top row of the commutative square \( \Lambda \) has s.-p. representatives in \( Z^p(W \cap Y, R_0, P_0) \), since the middle row of this second diagram will convert those into s.-p. representatives for the desired cokernel. Equivalently, this reduces our task to the case \( Y = Z \), and hence \( P = P_0 \) and so on. We now adopt this simplification.

Having done so, since \( R/Q \) is s.-p., Corollary 5.27 gives that the quotient
\[ B^p(W, R/Q) \leq Z^p(W, R/Q) \rightarrow H^p_m(W, R/Q) \]
has strong s.-p. representatives. Therefore, by Lemma 4.13 again, the same is true of the quotient
\[ B^p(W, R/Q) \leq \varphi(Z^p(W, R/P)) \rightarrow \text{img}(H^p_m(W, R/P) \rightarrow H^p(W, R/Q)) \]
\[ = \ker(H^p_m(W, R/Q) \rightarrow H^{p+1}_m(W, Q/P)), \]
where \( \varphi : R/P \rightarrow R/Q \) is the quotient homomorphism, and the image \( \varphi(Z^p(W, R/P)) \) is closed because we assumed that \( H^p_m(W, R/Q) \) are \( H^{p+1}_m(W, Q/P) \) are Polish.

Finally, suppose that \( \sigma \in \varphi(Z^p(W, R/P)) \cap Z^p_{sp}(W, R/Q) \). Because \( Q \leq R \) has strong s.-p. representatives, there is \( \sigma_1 \in C^p_{sp}(W, R) \) whose quotient \( \overline{\sigma_1} \in Z^p_{sp}(W, R/Q) \) is equal to \( \sigma \), which implies that \( \sigma_1 \in Z^p_{sp}(W, R, Q) \). By assumption, there is also some \( \sigma_2 \in Z^p(W, R, P) \) for which \( \overline{\sigma_2} = \sigma \).

Therefore \( \beta := \sigma_1 - \sigma_2 \in C^p(W, Q) \), and \( d\sigma_1 = d\sigma_2 + d\beta \). Since \( \sigma_1 \) is a step polynomial, so is \( d\sigma_1 \). On the other hand, we know that \( d\sigma_2 \) takes values in \( P \), so this equation implies that \( d\sigma_1 \in (B^p(W, Q, P))_{sp} \). By the assumption that
\( P \leq Q \) admits s.-p. coboundary solutions, it follows that \( d\sigma_1 = d\tau \mod P \) for some \( \tau \in C^p_{\text{sp}}(W, Q) \). Now letting \( \sigma_3 := \sigma_1 - \tau \), it follows that \( \sigma_3 \) is also a lift of \( \sigma \), that it is a step polynomial, and that it lies in \( Z^p_{\text{sp}}(W, R, P) \). This is the desired step-polynomial representative.

**Proposition 5.31.** Let \( p \geq 0 \), assume again that \( Q_0 \leq R_0 \) has strong s.-p. representatives, and now assume in addition that \( \ker(H^p_m(W, Q_1/P_1) \rightarrow H^p_m(W, R_1/P_1)) \) is discrete. Then the homomorphism

\[
\Phi : B^p(W, Q, P) \oplus C^{p-1}(W, R) \rightarrow B^p(W, R, P) : (\tau, f) \mapsto \tau + d^W f
\]

has strong s.-p. pre-images.

**Remark.** At this point, the assumption of discrete kernel for \( H^p_m(W, Q_1/P_1) \rightarrow H^p_m(W, R_1/P_1) \) has no clear relation to the theory built so far. It will arise naturally during the course of other proofs later. \( \triangleright \)

**Lemma 5.32.** In the setting of Proposition 5.31 the composition of \( \Phi \) with the quotient

\[
\Psi : B^p(W, R, P) \rightarrow B^p(W, R/Q) = \text{Coind}^Z_{W} B^p(W, R_1/Q_1)
\]

has strong s.-p. representatives, where one identifies the target as a submodule of \( \mathcal{F}(W^p \times Z, R_1/Q_1) \).

**Proof.** Let \( \psi : R \rightarrow R/Q \) be the quotient homomorphism, and let \( (\tau, f) \in B^p(W, Q, P) \oplus C^{p-1}(W, R) \) and \( \sigma := \tau + d^W f \) be such that

\[
\overline{\sigma} := \psi\sigma \in B^p_{\text{sp}}(W, R/Q).
\]

Then, by Corollary 5.27 there is some \( \overline{f_1} \in C^{p-1}_{\text{sp}}(W, R/Q) \) such that \( d^W \overline{f_1} = \overline{\sigma} \). Since \( Q_0 \leq R_0 \) was assumed to have strong s.-p. representatives, there is \( f_1 \in C^{p-1}_{\text{sp}}(W, R) \) such that \( \psi f_1 = \overline{f_1} \), and hence \( d^W f - d^W f_1 = \sigma - d^W f_1 \) lies in \( \ker(\Psi \circ \Phi) \). \( \square \)

**Proof of Proposition 5.31.** Let \( \sigma := \tau + d^W f \), and suppose it is a step polynomial. After adjusting \( \sigma \) by the coboundary \( d^W f_1 \) provided by the previous lemma, we may assume in addition that \( \sigma \) is \( Q \)-valued, so lies in \( B^p(W, R, P) \cap Z^p(W, Q, P) \).
This is equivalent to asserting that \( f \in Z_p^{-1}(W, R, Q) \), so the proof will be completed by showing that the homomorphism

\[
\Phi_1 : B^p(W, Q, P) \oplus Z_p^{-1}(W, R, Q) \rightarrow B^p(W, R, P) \cap Z_p(W, Q, P) : \\
(\tau, f) \mapsto \tau + d^W f
\]

has strong s.-p. pre-images.

However, it now suffices to prove this in the special case \( Z = Y + W \), for doing so will verify all the hypotheses of Corollary 4.22. This is because, in case \( Z = Y + W \), we have \( P_1 = P \) etc., and

\[
\frac{B^p(W, R, P) \cap Z_p^p(W, Q, P)}{B^p(W, Q, P)} \cong \ker \left( H^p_m(W, Q_1) \rightarrow H^p_m(W, R_1/P_1) \right)
\]

is discrete by assumption; and because the boundary homomorphism \( d \) is complexity-bounded. So now suppose \( Z = Y + W \).

Now, since \( \sigma = \tau + d^W f \) with \( \tau \) itself an \( (R, P) \)-relative coboundary, we have

\[
\sigma = d^W g \mod C^p(W, P)
\]

for some \( g \in C^{p-1}(W, R) \). Letting \( \psi : R \rightarrow R/Q \) be the quotient homomorphism as above and letting \( \overline{g} := \psi g \), it follows that \( \overline{\sigma} \in Z_{p}^{-1}(W, R/Q) \). Therefore Corollary 5.27 gives that

\[
\overline{g} = \overline{\alpha} + d^W \overline{\beta}
\]

for some \( \overline{\beta} \in C^{p-2}(W, R/Q) \) and some \( \overline{\alpha} \in Z_{sp}^{-1}(W, R/Q) \).

Using again the assumption that \( Q_0 \leq R_0 \) has strong s.-p. representatives, we obtain a lift \( \alpha \in C^{p-1}_{sp}(W, R) \) of \( \overline{\alpha} \). Letting also \( \beta \in C^{p-2}(W, R) \) be any measurable lift of \( \overline{\beta} \), this implies that

\[
g = \alpha + d^W \beta + h
\]

for some \( h \in C^{p-1}(W, Q) \). Therefore

\[
\sigma = d^W \alpha + d^W h \mod C^p(W, P) = d^W \alpha \mod B^p(W, Q, P).
\]

Since \( \alpha \in C^{p-1}_{sp}(W, R) \), it follows that

\[
\tau' := \sigma - d^W \alpha \in B^p(W, Q, P)
\]

must also be a step polynomial. Now the decomposition \( \sigma = \tau' + d^W \alpha \) shows that \( \Phi \) has s.-p. pre-images, as required. \( \square \)
6 Semi-functional complexes

Section I.4 studied complexes of Polish modules. The main result was a topological addendum to the Snake Lemma relating the topological structure of the homologies in three complexes that form a short exact sequence.

Here we will consider similar issues for complexes of semi-functional modules. For simplicity we focus exclusively on bounded complexes (that is, having finite length and starting and ending with 0), since only these will be needed later.

**Definition 6.1** (Semi-functional complex). A semi-functional complex is a diagram of the form

\[
\begin{array}{cccccccccccc}
0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & \cdots & \longrightarrow & P_k & \longrightarrow & 0 \\
& & \downarrow\text{incl} & & \downarrow\text{incl} & & \downarrow\text{incl} & & \downarrow\text{incl} & & \\
0 & \longrightarrow & Q_1 & \longrightarrow & Q_2 & \longrightarrow & \cdots & \longrightarrow & Q_k & \longrightarrow & 0 \\
& & \downarrow\alpha_1 & & \downarrow\alpha_2 & & \downarrow\alpha_3 & & \cdots & & \downarrow\alpha_k & & \downarrow\alpha_{k+1} & & \downarrow0 \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & \cdots & \longrightarrow & M_k & \longrightarrow & 0,
\end{array}
\]

where every column \(P_i \leq Q_i \rightarrow M_i\) is a semi-functional module with the same base \(X\), but possibly different fibres \(A_i\), and where the fibres \(A_i\) are all s.-p. modules.

Such a diagram will often be abbreviated to \(P_{\bullet} \leq Q_{\bullet} \rightarrow M_{\bullet}\), when no confusion can arise.

**Remark.** Both the insistence on a common base \(X\) and the freedom to use different fibres \(A_i\) will be important later.

Of course, the morphisms \(\alpha_i\) in the middle row of this diagram determine all the other horizontal morphisms.

When considering such a complex, our focus will be on the structure of the quotient sequence \(M_{\bullet}\). This is the reason for the following.

**Definition 6.2.** In a semi-functional complex as above, its homology is the homology of the bottom row: in position \(i\) this is

\[
\frac{\ker(M_i \rightarrow M_{i+1})}{\img(M_i \rightarrow M_{i+1})} \cong \frac{\alpha_{i+1}^{-1}(P_i)}{\alpha_i(Q_{i+1}) + P_i}
\]

(where this is easily seen to be a topological isomorphism, even if these quotients are not Hausdorff, as in Lemma 5.2).

For instance, it has discrete homology if this quotient is topologically discrete for each \(i\).
6.1 Step polynomials and meekness

Let $P_\bullet \leq Q_\bullet \to M_\bullet$ be a semi-functional complex as in Definition 6.1. We will need to refer to several properties concerning step-polynomial elements of the modules appearing there.

**Definition 6.3** (Complexity-bounded complex). The semi-functional complex is **complexity-bounded** if every morphism $\alpha_\ell$ in the middle row is complexity-bounded.

**Definition 6.4** (Strong s.-p. representatives). Let $1 \leq \ell \leq k$. The semi-functional complex has **strong s.-p. representatives at position $\ell$** if $\alpha_\ell(Q_{\ell-1}) + P_\ell$ is closed in $Q_\ell$ (equivalently, if the bottom-row morphism from position $\ell - 1$ to $\ell$ is closed), if the homology group 
$$\frac{\ker(M_\ell \to M_{\ell+1})}{\text{img}(M_{\ell-1} \to M_\ell)}$$
is an s.-p. module, and if the semi-functional module
$$\alpha_\ell(Q_{\ell-1}) + P_\ell \leq \alpha_{\ell+1}^{-1}(P_{\ell+1}) \to \frac{\ker(M_\ell \to M_{\ell+1})}{\text{img}(M_{\ell-1} \to M_\ell)}$$has complexity-bounded quotient map and strong s.-p. representatives.

The complex has **strong s.-p. representatives** if this holds at all positions.

**Definition 6.5** (Finite-complexity decompositions). Let $1 \leq \ell \leq k$. The semi-functional complex has **finite-complexity decompositions at position $\ell$** if the homomorphism
$$Q_{\ell-1} \oplus P_\ell \to Q_\ell : (q,p) \mapsto \alpha_\ell(q) + p$$admits strong s.-p. pre-images. The complex has **finite-complexity decompositions** if this holds at all positions.

Written out in full, this last definition asserts that for any enlargement $Z' \geq Z$, one has
$$(\alpha_\ell(\text{Coind}_Z^{Z'} Q_{\ell-1}) + \text{Coind}_Z^{Z'} P_\ell)_{\text{sp}} \leq \alpha_\ell((\text{Coind}_Z^{Z'} Q_{\ell-1})_{\text{sp}}) + (\text{Coind}_Z^{Z'} P_\ell)_{\text{sp}}$$
(where we abusively abbreviate $\text{Coind}_Z^{Z'} \alpha_\ell$ to $\alpha_{\ell}$).

Putting Definitions 6.4 and 6.5 together, the semi-functional complex $P_\bullet \leq Q_\bullet \to M_\bullet$ has both strong s.-p. representatives and finite-complexity decompositions at position $\ell$ if and only if its homology group $H_\ell^M$ at position $\ell$ is s.-p., and all homomorphisms in the exact sequence
$$Q_{\ell-1} \oplus P_\ell \to \alpha_{\ell+1}^{-1}(P_{\ell+1}) \to H_\ell^M \to 0$$
admit strong s.-p. pre-images.

The possibility of an enlarged group \( Z' \) is most important in the definitions above. Later on we will be working with complexes of modules for some \( Z \) that arise naturally by co-induction from a complex over some subgroup of \( Z \), and we will need to know that the relevant properties of those complexes are stable under that co-induction.

We now come to the class of complexes that will play the most central rôle in the sequel. It builds on the notion of almost-discrete homology from Definition I.4.2.

**Definition 6.6 (Meek and almost meek complexes).** Let \( P_\bullet \leq Q_\bullet \to M_\bullet \) be a semi-functional complex as above, and let \( 0 \leq \ell_0 \leq k \). Then the complex is \( \ell_0 \)-almost meek (resp. \( \ell_0 \)-meek) if

- it has \( \ell_0 \)-almost (resp. \( \ell_0 \)-strictly) discrete homology, and its homology groups are all s.-p. \( Z \)-modules,
- it has strong s.-p. representatives, and
- it has finite-complexity decompositions.

As with modesty, we will sometimes refer to \( \ell_0 \)-meekness as strict \( \ell_0 \)-meekness, to contrast it with almost meekness.

**Remark.** Considering Definition I.4.2 alongside the above, one sees that the class of strictly \( \ell_0 \)-meek complexes is precisely the intersection of the \( \ell_0 \)-almost meek complexes with those that have strictly discrete homology.

### 6.2 Concatenations

The main results of this subsection are relatives of Lemma I.4.3—I.4.5. They prove in various cases that if one has a short exact sequence of semi-functional complexes, and two of them are almost meek, then so is the third. In the cases of interest, the short exact sequence arises from the following structure.

**Definition 6.7 (Concatenation).** A **concatenation** of semi-functional complexes is a sequence of functional-module inclusions

\[
P_i \leq R_i \leq Q_i \leq \mathcal{F}(X, A_i), \quad i = 0, 1, \ldots, k + 1,
\]

where \( Q_0 = Q_{k+1} = 0 \), together with morphisms \( \alpha_i : Q_{i-1} \to Q_i \) such that \( \alpha_i(R_{i-1}) \leq R_i \) and \( \alpha_i(P_{i-1}) \leq P_i \) for each \( i \).
Given a concatenation as above, one may form three different semi-functional complexes:

\[ P_{\bullet} \leq R_{\bullet} \rightarrow M_{\bullet}, \]

\[ P_{\bullet} \leq Q_{\bullet} \rightarrow N_{\bullet} \]

and

\[ R_{\bullet} \leq Q_{\bullet} \rightarrow L_{\bullet} \cong N_{\bullet}/M_{\bullet}. \]

If these quotients are all Hausdorff, hence Polish, then they fit into a short exact sequence of Polish complexes, as examined in Subsection I.4.2.

Let \( H^M_{\bullet}, H^N_{\bullet} \) and \( H^L_{\bullet} \) denote the sequences of homology groups of these three semi-functional complexes. If one co-induces to some enlargement \( Z' \geq Z \), then

\[ H^\text{Coind} Z'_Z (\alpha_{\ell+1}^{-1}(P_{\ell+1})) \]

\[ \cong \text{Coind} Z'_Z H^M_{\ell}, \]

and similarly for \( N_{\bullet} \) and \( L_{\bullet} \). A routine check shows that these isomorphisms are always topological. If one knows that \( R_{\ell}/P_{\ell} \) and \( Q_{\ell}/R_{\ell} \) are s.-p. for every \( \ell \), Corollary 4.16 promises that if the homomorphisms in the long exact sequence of s.-p. modules given by the Snake Lemma,

\[ \cdots \rightarrow H^M_{\ell} \rightarrow H^N_{\ell} \rightarrow H^L_{\ell} \rightarrow H^M_{\ell+1} \rightarrow H^N_{\ell+1} \rightarrow H^L_{\ell+1} \rightarrow \cdots \]  

(9)

are all closed, then they all admits strong s.-p. pre-images.

The results we need about the semi-functional modules arising from a concatenation are contained in the following three lemmas.

**Lemma 6.8.** Consider a concatenation of semi-functional complexes as above, and the resulting short exact sequence. Assume that all three of these semi-functional complexes have all morphisms closed and have Hausdorff homology groups, and that the first and second of them are \( \ell_0 \)-almost (resp. strictly) meek. Then the third is also \( \ell_0 \)-almost (resp. strictly) meek.

**Lemma 6.9.** In the above short exact sequence of closed complexes, assume that all three of these semi-functional complexes have all morphisms closed and have Hausdorff homology groups, and that the first and third of them are \( \ell_0 \)-almost (resp. strictly) meek. Then the second is also \( \ell_0 \)-almost (resp. strictly) meek.

**Lemma 6.10.** In the above short exact sequence of closed complexes, assume that all three of these semi-functional complexes have all morphisms closed and have Hausdorff homology groups, and that the second and third of them are \( \ell_0 \)-almost (resp. strictly) meek, and that \( M_i = 0 \) for all \( i < \ell_0 + 1 \). Then the first complex is \( (\ell_0 + 1) \)-almost (resp. strictly) meek.
Similarly to Lemmas I.4.3–I.4.5, the proofs amount to tracing through the steps in the classical Snake Lemma, carefully but quite mechanically, keeping track of where one can find step polynomial representatives. There are really no surprises: the reader familiar with the Snake Lemma might prefer to skip (at least) two of the proofs below, on the understanding that they all involve just the same ideas. They are also rather similar to the proofs of Lemmas 5.17 and 5.18.

Proof of Lemma 6.8. The assertion of (almost) modesty comes from Lemma I.4.3. Also, given that all the homology groups $H^M_\bullet$ and $H^N_\bullet$ are s.-p. and the long exact sequence $\mathcal{E}$ consists of closed homomorphisms, Corollary 3.44 gives that all of the homology groups $H^L_\bullet$ are s.-p.

We focus on the second and third conditions in Definition 6.1.

Let $Z' \geq Z$ be an enlargement. The proof will mostly involve co-induced modules such as $\text{Coind}^Z_{Z'}Q_\ell$, but, to lighten notation, for morphisms (not modules) we will abbreviate $\text{Coind}^Z_{Z'}Z_\alpha_\ell$ to $\alpha_\ell$, and similarly, since no confusion should arise.

Strong s.-p. representatives. Given $q \in \text{Coind}^Z_{Z'}Q_\ell$, let $\overline{q}$ be its image in $\text{Coind}^Z_{Z'}H^L_\ell$, and let $\hat{q}$ be its image in $\text{Coind}^Z_{Z'}H^N_\ell$. Similarly, given $r \in \text{Coind}^Z_{Z'}R_\ell$, let $\overline{r}$ be its image in $\text{Coind}^Z_{Z'}H^M_\ell$.

Now let $q \in \text{Coind}^Z_{Z'}Q_\ell$ be an element such that $r := \alpha_{\ell+1} q \in \text{Coind}^Z_{Z'}R_{\ell+1}$, and such that $\overline{q} \in \text{Coind}^Z_{Z'}H^L_\ell$ is a step polynomial. Then $\alpha_{\ell+2} \overline{r} = 0$, so in particular $r \in \alpha_{\ell+2}^{-1}(\text{Coind}^Z_{Z'}P_\ell)$.

Also, $\overline{r}$ is s.-p., because it is the image of $\overline{q}$ under a homomorphism co-induced to $Z'$ from the switchback homomorphism $H^L_\ell \rightarrow H^M_{\ell+1}$ from $\mathcal{E}$, which is a closed homomorphism of Lie groups and therefore complexity-bounded. Therefore, since $M_\bullet$ has strong s.-p. representatives, we may write

$$r = r_1 + \alpha_{\ell+1}(r_2) + p_1$$

for some $r_1 \in (\text{Coind}^Z_{Z'}R_{\ell+1})_{\text{sp}}$, some $r_2 \in \text{Coind}^Z_{Z'}R_\ell$ and some $p_1 \in \text{Coind}^Z_{Z'}P_{\ell+1}$.

Now, since $r_2 \in \text{Coind}^Z_{Z'}R_\ell$, we find that $q - r_2$ has both of the same properties as $q$, and we may replace $q$ with $q - r_2$ without disrupting our desired conclusion. We may therefore assume that $r_2 = 0$.

However, now

$$\alpha_{\ell+1}(q) = r_1 + p_1,$$

so re-arranging this gives

$$\alpha_{\ell+1}(q) - p_1 \in (\text{Coind}^Z_{Z'}Q_{\ell+1})_{\text{sp}}.$$

Because $N_\bullet$, and hence also $\text{Coind}^Z_{Z'}N_\bullet$, admits finite-complexity decompositions, there are some $q_1 \in (\text{Coind}^Z_{Z'}Q_\ell)_{\text{sp}}$ and $p_2 \in (\text{Coind}^Z_{Z'}P_{\ell+1})_{\text{sp}}$ for which

$$\alpha_{\ell+1}(q) - p_1 = \alpha_{\ell+1}(q_1) - p_2 = r_1.$$
Let \( q_2 := q - q_1 \). It follows that \( \overline{q_2} \) is still a step polynomial. Since the homomorphism \( H^N_\ell \to H^L_\ell \) has image equal to the kernel of a continuous homomorphism \( H^L_\ell \to H^M_{\ell+1} \), the former homomorphism is closed, and so Corollary 3.32 and Lemma 3.32 give a step-polynomial \( \hat{q}_3 \in \text{Coind}_{\ell}^Z H^N_\ell \) which lifts \( \overline{q_2} \).

Lastly, because \( N_\ast \) has strong s.-p. representatives, we may find some step polynomial \( q_3 \in (\text{Coind}_{\ell}^Z Q_\ell)_{sp} \) whose image in \( \text{Coind}_{\ell}^Z H^N_\ell \) equals \( \hat{q}_3 \). Therefore \( q_1 + q_3 \) is a step polynomial in \( Q_\ell \), and \( \overline{q_1 + q_3} = \overline{q} \), as required.

**Finite-complexity decompositions.** Suppose \( q \in \text{Coind}_{\ell}^Z Q_\ell \) and \( r \in \text{Coind}_{\ell}^Z R_{\ell+1} \) are such that

\[
q' := \alpha_{\ell+1}(q) + r \in (\text{Coind}_{\ell}^Z Q_{\ell+1})_{sp}.
\]

Then

\[
\alpha_{\ell+2}(r) = \alpha_{\ell+2}(q')
\]

is also a step polynomial. Because \( M_\ast \) admits finite-complexity decompositions, this gives \( r - r_1 \in \alpha_{\ell+2}^{-1}(\text{Coind}_{\ell}^Z P_{\ell+2}) \) for some \( r_1 \in (\text{Coind}_{\ell}^Z R_{\ell+2})_{sp} \).

Replacing \( q' \) with \( q' - r_1 \) and \( r \) with \( r - r_1 \), this means we may assume in addition that \( r \in \alpha_{\ell+2}^{-1}(\text{Coind}_{\ell}^Z P_{\ell+2}) \), and hence the same for \( q' \). Therefore \( q' \) has a class \( \hat{q}' \) in \( \text{Coind}_{\ell}^Z H^N_{\ell+1} \), which is step polynomial because this quotient homomorphism is complexity-bounded, and this class agrees with the image of \( \overline{r} \) under the homomorphism \( \text{Coind}_{\ell}^Z H^M_{\ell+1} \to \text{Coind}_{\ell}^Z H^N_{\ell+1} \).

That last homomorphism admits strong s.-p. pre-images, by Corollary 4.16 and Lemma 3.32 so this gives some \( \hat{r}_2 \in (\text{Coind}_{\ell}^Z H^M_{\ell+1})_{sp} \) such that \( \hat{r}_2 = \hat{r} \). Hence, since \( M_\ast \) has strong s.-p. representatives, \( r = r_2 + \alpha_{\ell+1}(q_2) + p_2 \) for some \( r_2 \in (\text{Coind}_{\ell}^Z R_{\ell+2})_{sp} \), \( q_2 \in \text{Coind}_{\ell}^Z Q_\ell \) and \( p_2 \in \text{Coind}_{\ell}^Z P_{\ell+1} \).

Therefore, making another replacement of \( q' \) with \( q' - r_2 \) and of \( r \) with \( r - r_2 \), equation (10) has been simplified to

\[
q' := \alpha_{\ell+1}(q + q_2) + p_2.
\]

Finally, the fact that \( N_\ast \) admits finite-complexity decompositions shows that this must equal \( \alpha_{\ell+1}(q_3) + p_3 \) for some \( q_3 \in (\text{Coind}_{\ell}^Z Q_\ell)_{sp} \) and \( p_3 \in (\text{Coind}_{\ell}^Z P_{\ell+1})_{sp} \), so this completes the proof. \( \square \)

**Proof of Lemma 6.9** Let \( Z' \geq Z \). Once again we will suppress the functor \( \text{Coind}_{\ell}^Z (-) \) in our notation for morphisms.

**Strong s.-p. representatives** Let \( q \in \text{Coind}_{\ell}^Z Q_\ell \) be an element such that \( p := \alpha_{\ell+1}q \in \text{Coind}_{\ell}^Z P_{\ell+1} \), and such that \( \hat{q} \in \text{Coind}_{\ell}^Z H^N_\ell \) is a step polynomial.

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Then certainly $\eta \in \operatorname{Coind}Z H^M_\ell$ is a step polynomial, so since $L_\bullet$ has strong s.-p. representatives there is a decomposition

$$q = q_1 + \alpha_\ell(q_2) + r$$

for some $q_1 \in (\operatorname{Coind}Z Q_\ell)_{sp}$, $q_2 \in \operatorname{Coind}Z Q_{\ell-1}$ and $r \in \operatorname{Coind}Z R_\ell$.

Applying $\alpha_{\ell+1}$ to this decomposition and re-arranging, it gives that

$$\alpha_{\ell+1}(r) - p = \alpha_{\ell+1}(q_1),$$

which is a step polynomial. Therefore, because $M_\bullet$ has finite-complexity decompositions, there are $r_1 \in (\operatorname{Coind}Z R_\ell)_{sp}$ and $p_1 \in (\operatorname{Coind}Z P_{\ell+1})_{sp}$ such that

$$\alpha_{\ell+1}(r) - p = \alpha_{\ell+1}(r_1) - p_1.$$

Replacing $q_1$ with $q_1 + r_1$ and $r$ with $r - r_1$, we may assume in our decomposition of $q$ that $r \in \alpha_{\ell+1}^{-1}(\operatorname{Coind}Z P_{\ell+1})$. However, having done so, it follows that $\alpha_{\ell+1}(q_1)$ also lies in $\operatorname{Coind}Z P_{\ell+1}$, so now replacing $q$ with $q - q_1 - \alpha_\ell(q_2)$, we may assume that $q_1 + \alpha_\ell(q_2) = 0$. Our decomposition has therefore reduced to the assertion that $q = r \in \operatorname{Coind}Z R_\ell$.

Now, we assumed that $\tilde{q} \in \operatorname{Coind}Z H^N_\ell$ is s.-p., and if $q \in \operatorname{Coind}Z R_\ell$ then it is in the image of the homomorphism $\operatorname{Coind}Z H^M_\ell \rightarrow \operatorname{Coind}Z H^N_\ell$. Therefore Lemma [3.32] gives a step polynomial $\tilde{r}_2 \in \operatorname{Coind}Z H^M_\ell$ whose image is $\tilde{q}$. Because $M_\bullet$ has strong s.-p. representatives, this $\tilde{r}_2$ is now the image of some $r_2 \in (\operatorname{Coind}Z R_\ell)_{sp}$. Overall, this gives that

$$q \in r_2 + \alpha_\ell(\operatorname{Coind}Z Q_{\ell-1}) + \operatorname{Coind}Z P_\ell,$$

as required.

Finite-complexity decompositions. Suppose $(q, p) \in \operatorname{Coind}Z Q_\ell \oplus \operatorname{Coind}Z P_{\ell+1}$ are such that $q' := \alpha_{\ell+1}(q) + p \in (\operatorname{Coind}Z Q_{\ell+1})_{sp}$.

Because $L_\bullet$ has finite-complexity decompositions, there is

$$(q_1, r_1) \in (\operatorname{Coind}Z Q_\ell \oplus \operatorname{Coind}Z R_{\ell+1})_{sp}$$

such that

$$\alpha_{\ell+1}(q) + p = \alpha_{\ell+1}(q_1) + r_1.$$

We may therefore replace $q - q_1$, and assume that

$$\alpha_{\ell+1}(q) + p = r_1 \in (\operatorname{Coind}Z R_{\ell+1})_{sp}.$$
This implies that \( r_1 \in \alpha_{\ell+2}^{-1}(\operatorname{Coind}_Z^{Z'} P_{\ell+2}) \), and that its image \( \tilde{r}_1 \in \operatorname{Coind}_Z^{Z'} H_{\ell+1}^M \) is s.-p. Also, this \( \tilde{r}_1 \) lies in the image of the homomorphism \( \operatorname{Coind}_Z^{Z'} H_{\ell}^L \rightarrow \operatorname{Coind}_Z^{Z'} H_{\ell+1}^M \), so by Lemma \( \ref{Lemma3.32} \), and because \( L_\bullet \) has strong s.-p. representatives, there are some \( q_2 \in (\operatorname{Coind}_Z^{Z'} Q_\ell)^{\text{sp}} \), \( r_2 \in \operatorname{Coind}_Z^{Z'} R_\ell \) and \( p_2 \in \operatorname{Coind}_Z^{Z'} P_{\ell+1} \) such that

\[
 r_1 = \alpha_{\ell+1}(q_2 + r_2) + p_2. 
\]

Inserting this above now gives

\[
 \alpha_{\ell+1}(q) + p = \alpha_{\ell+1}(q_2) + \alpha_{\ell+1}(r_2) + p_2, 
\]

so replacing \( q \) with \( q - q_2 \), we may assume that \( r_1 = \alpha_{\ell+1}(r_2) + p_2 \). A final appeal to the property of finite-complexity decompositions for \( M_\bullet \) completes the proof.

**Proof of Lemma 6.10** Let \( Z' \geq Z \). Once again we will suppress the functor \( \operatorname{Coind}_Z^{Z'} (-) \) in our notation for morphisms.

**Strong s.-p. representatives** Suppose \( r \in \operatorname{Coind}_Z^{Z'} R_\ell \cap \alpha_{\ell+1}^{-1}(\operatorname{Coind}_Z^{Z'} P_{\ell+1}) \) is such that \( \tilde{r} \in (\operatorname{Coind}_Z^{Z'} H_{\ell}^M)_{\text{sp}} \). Then its image \( \tilde{r} \) lies in \( (\operatorname{Coind}_Z^{Z'} H_{\ell}^N)_{\text{sp}} \), so since \( N_\bullet \) has s.-p. representatives there are \( q \in (\operatorname{Coind}_Z^{Z'} Q_\ell)_{\text{sp}} \), \( q_1 \in \operatorname{Coind}_Z^{Z'} Q_{\ell-1} \) and \( p \in \operatorname{Coind}_Z^{Z'} P_\ell \) such that

\[
 r = q + \alpha_\ell(q_1) + p. 
\]

Re-arranging, this implies that

\[
 q = \alpha_\ell(-q_1) + (r - p) \in (\alpha_\ell(\operatorname{Coind}_Z^{Z'} Q_{\ell-1}) + \operatorname{Coind}_Z^{Z'} R_\ell)_{\text{sp}}, 
\]

so since \( L_\bullet \) has finite-complexity decompositions, we have \( q = \alpha_\ell(q_2) + r_1 \) for some \( (q_2, r_1) \in (\operatorname{Coind}_Z^{Z'} Q_{\ell-1} \oplus \operatorname{Coind}_Z^{Z'} R_\ell)_{\text{sp}} \). Therefore

\[
 r = \alpha_\ell(q_1 + q_2) + p + r_1. 
\]

This implies that we also have \( \alpha_{\ell+1}(r_1) \in \operatorname{Coind}_Z^{Z'} P_{\ell+1} \). We may therefore replace \( r \) with \( r - r_1 \), and so assume that \( r_1 = 0 \). We also set \( q_3 := q_1 + q_2 \).

It now follows that \( q_3 \in \alpha_{\ell-1}^{-1}(\operatorname{Coind}_Z^{Z'} R_\ell) \), and that the image of \( \tilde{q}_3 \in \operatorname{Coind}_Z^{Z'} H_{\ell-1}^L \) under the switchback homomorphism \( \operatorname{Coind}_Z^{Z'} H_{\ell-1}^L \rightarrow \operatorname{Coind}_Z^{Z'} H_{\ell}^M \) is \( \tilde{r} \), which is s.-p. Therefore there is some \( \tilde{q}_4 \in (\operatorname{Coind}_Z^{Z'} H_{\ell-1}^L)_{\text{sp}} \) with the same switchback-image in \( \operatorname{Coind}_Z^{Z'} H_{\ell}^M \), and since \( L_\bullet \) has strong s.-p. representatives we may in fact assume \( q_4 \in (\operatorname{Coind}_Z^{Z'} Q_{\ell-1})_{\text{sp}} \).
This equality of images implies that
\[ q_3 - q_4 = r_2 + \alpha_{\ell-1}(q_5) \in \text{Coind}_Z^\ell R_{\ell-1} + \alpha_{\ell-1}(\text{Coind}_Z^\ell Q_{\ell-2}), \]
and hence
\[ r = (\alpha_\ell(q_4) + r_1) + \alpha_\ell(r_2) + p. \]
This lies in \((\text{Coind}_Z^\ell R_\ell)_{sp} + \alpha_\ell(\text{Coind}_Z^\ell R_{\ell-1}) + \text{Coind}_Z^\ell P_\ell, \) as required.

**Finite-complexity decompositions.** Suppose \((r,p) \in \text{Coind}_Z^\ell R_\ell \oplus \text{Coind}_Z^\ell P_{\ell+1}\)
is such that
\[ r' := \alpha_{\ell+1}(r) + p \in (\text{Coind}_Z^\ell R_{\ell+1})_{sp}. \]
Since \(N_*\) has finite-complexity decompositions, this gives
\[ r' = \alpha_{\ell+1}(q) + p_1 \]
for some \((q,p_1) \in (\text{Coind}_Z^\ell Q_\ell \oplus \text{Coind}_Z^\ell P_{\ell+1})_{sp}. \)
Re-arranging, one obtains
\[ \alpha_{\ell+1}(r - q) = p_1 - p \in \text{Coind}_Z^\ell P_{\ell+1}. \]
Since \(N_*\) has s.-p. representatives, this gives
\[ r - q = q_1 + \alpha_{\ell-1}(q_2) + p_2 \]
for some \(q_1 \in (\text{Coind}_Z^\ell Q_\ell)_{sp},\) \(q_2 \in \text{Coind}_Z^\ell Q_{\ell-1}\) and \(p_2 \in \text{Coind}_Z^\ell P_\ell. \) Re-arranging and replacing \(q_1\) with \(q_1 + q,\) this becomes
\[ r = q_1 + \alpha_{\ell-1}(q_2) + p_2. \]
This implies that \(\hat{r} \in \text{Coind}_Z^\ell H_\ell^N\) is s.-p., and of course it lies in the image of \(\text{Coind}_Z^\ell H_\ell^M \to \text{Coind}_Z^\ell H_\ell^N.\)
Therefore, by Lemma 3.32 there is some \(\hat{r}_1' \in (\text{Coind}_Z^\ell H_\ell^M)_{sp}\) whose image in \(\text{Coind}_Z^\ell H_\ell^N\) equals \(\hat{r};\) and now, because \(M_*\) has s.-p. representatives, we may assume that \(r_1 \in \text{Coind}_Z^\ell R_\ell\) itself is a step polynomial. Since \(\hat{r} = \hat{r}_1',\) we have obtained
\[ r = r_1 + \alpha_{\ell-1}(q_3) + p_3 \]
for some \((q_3,p_3) \in \text{Coind}_Z^\ell Q_{\ell-1} \oplus \text{Coind}_Z^\ell P_\ell.\) Applying \(\alpha_{\ell+1},\) this gives
\[ r' = \alpha_{\ell+1}(r_1) + (\alpha_{\ell+1}(p_3) + p). \]
This implies that \(\alpha_{\ell+1}(p_3) + p \in (\text{Coind}_Z^\ell P_{\ell+1})_{sp},\) so gives the desired decomposition. \qed
7 Semi-functional $\Delta$-modules

7.1 Definitions

**Definition 7.1 (Functional $\Delta$-modules).** Suppose that $Z$, $Y$ and $U$ are as in the Introduction, $A$ is an Abelian Lie group, and $X$ is another compact Abelian group. A $(Z, Y, U)$-$\Delta$-module $(P_e)_e$ is **functional with fibre $A$ and dummy $X$** if

i) each $P_e$ is of the form $\operatorname{Coind}_{Y + U_e}^Z P^\circ_e$, where $P^\circ_e$ is a functional $(Y + U_e)$-module with fibre $A$ (with the trivial action) and dummy $X$ (this works out to give $P_e \leq F(X \times Z, A)$);

ii) each $\varphi^P_{ae} : P_a \rightarrow P_e$ is simply an inclusion between two submodules of $F(X \times Z, A)$.

iii) each derivation-lift $\bar{\nabla}^{e, e\setminus i} : P_e \rightarrow Z^1(U_i, P_{e\setminus i})$ is a complexity-bounded homomorphism.

**Example 7.2.** The principle examples from Part I, $\Delta$-modules of PD$^{ce}$E-solutions and zero-sum tuples, are clearly of this kind. ⊳

In Subsection 4.1, semi-functional modules were defined as short exact sequences constructed from functional modules and their submodules. The following is the analog in the category of $\Delta$-modules.

**Definition 7.3 (Semi-functional $\Delta$-modules).** A **semi-functional** $(Z, Y, U)$-$\Delta$-module with fibre $A$ and dummy $X$ is a short exact sequence

$$0 \rightarrow P^{(te)}_e \rightarrow Q_e \rightarrow M \rightarrow 0$$

of $(Z, Y, U)$-$\Delta$-modules, where

- $\mathcal{Q} = (Q_e)_e$ is a functional $(Z, Y, U)$-$\Delta$-module with fibre $A$ and dummy $X$, and $\mathcal{P} = (P_e)_e$ is a $(Z, Y, U)$-$\Delta$-submodule of $\mathcal{Q}$ (with the restrictions of the structure morphisms and derivation-lifts of $\mathcal{Q}$);

- $i_e : P_e \rightarrow Q_e$ is the inclusion map for every $e \subseteq [k]$.

This situation will often be denoted by

$$\mathcal{P} \leq \mathcal{Q} \rightarrow M.$$

The individual modules $P_e$ and $Q_e$ are the **constituents** of this $\Delta$-module.
One could alternatively define a semi-functional $\Delta$-module more directly in terms of Definition 4.2, as a ‘$\Delta$-module in the category of semi-functional modules’. The above choice leads to slightly lighter notation later on, because our emphasis will be on properties of the quotient $\mathcal{M}$, with $\mathcal{P}$ and $\mathcal{Q}$ playing an auxiliary rôle in their description.

**Definition 7.4.** If $\mathcal{M}$ is a given $\Delta$-module, then a **semi-functional presentation** of $\mathcal{M}$ is a semi-functional $\Delta$-module $\mathcal{P} \leq \mathcal{Q} \rightarrow \mathcal{M}$.

Also, given a property $P$ of $\Delta$-modules, such as modesty or almost modesty, we write that a semi-functional $\Delta$-module $\mathcal{P} \leq \mathcal{Q} \rightarrow \mathcal{M}$ has $P$ if this is true of the quotient $\mathcal{M}$ (but not necessarily $\mathcal{P}$ or $\mathcal{Q}$).

All the $\Delta$-modules that will appear later have semi-functional presentations. This will allow us to set up a formalism for tracking step-polynomial representatives. It seems likely that one could construct some abstract $\Delta$-modules that do not have semi-functional presentations, but I have not pursued this question.

Definition 7.4 naturally leads to the following.

**Definition 7.5.** Suppose that $\mathcal{M}$ and $\mathcal{M}'$ are $(Z,Y,U)$-$\Delta$-modules and that $\Phi = (\Phi_e)_e : \mathcal{M} \rightarrow \mathcal{M}'$ is a $\Delta$-module isomorphism. Then a **semi-functional presentation** of $\Phi$ is a commutative diagram of $\Delta$-morphisms

$$
\begin{array}{ccc}
\mathcal{P} & \longrightarrow & \mathcal{P}' \\
\downarrow^{\text{incl}} & & \downarrow^{\text{incl}} \\
\mathcal{Q} & \longrightarrow & \mathcal{Q}'
\end{array}
$$

in which the columns are semi-functional presentations of $\mathcal{M}$ and $\mathcal{M}'$.

This presentation is **complexity-bounded** (resp. has s.-p. **pre-images** or **strong s.-p. pre-images**) if this is true of each $\Psi_e$.

We can now introduce the class of $\Delta$-modules within which most of our subsequent arguments will take place. To make this definition, observe that if

$$
\mathcal{P} \leq \mathcal{Q} \rightarrow \mathcal{M}
$$

is a $(Z,Y,U)$-$\Delta$-module, and if $e \subseteq [k]$, then the structure complexes of $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{M}$ at $e$ fit together into a short exact sequence of complexes. It is a semi-functional complex in the sense of Definition 6.1, all of which is co-induced over $Y + U_e$. 

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**Definition 7.6.** We will refer to this semi-functional complex as the **structure complex** of \( \mathcal{P} \leq \mathcal{D} \to \mathcal{M} \) at \( e \).

We will carry over many of our notational practices from Part I when working with structure complexes. For example, we shall generally write \( \partial_k \) or \( \partial_{e,\ell} \) for the boundary homomorphisms of the structure complex (at \( [k] \) or \( e \), respectively), without notating the \( \Delta \)-module they belong to, since this will be clear from the context.

**Definition 7.7** (Meek and almost meek \( \Delta \)-modules). Suppose that \( 0 \leq \ell_0 \leq k \). If

\[
\mathcal{P} \leq \mathcal{D} \to \mathcal{M}
\]

is a semi-functional \( \Delta \)-module, then it is \( \ell_0 \)-meek (resp. \( \ell_0 \)-almost meek) if it is Polish and for each \( e \subseteq [k] \), its structure complex at \( e \) is co-induced over \( Y + U_e \) from an \( \ell_0 \)-meek (resp. \( \ell_0 \)-almost meek) semi-functional complex of \((Y + U_e)\)-modules.

Almost meek \( \Delta \)-modules should be thought of as \( \Delta \)-modules which are ‘explicit’ (in that they are semi-functional), and for which the structure complexes are ‘nearly exact’ (in the sense of almost modesty) and have homology represented by step polynomials. One of our main results, Theorem 8.1 below, will assert that this class is closed under forming cohomology, and in proving that result we will find that the different parts of the definition of meekness frequently interact: each part will be needed in proving some of the others. This is why we have packaged them together under a single term.

### 7.2 Recap of some basic constructions

The direct sum of two \( \Delta \)-modules appeared in Definition I.5.9. The following requires only a trivial check, which we omit.

**Lemma 7.8.** If \( \mathcal{P}_i \leq \mathcal{D}_i \to \mathcal{M}_i \) are semi-functional (resp. \( \ell_0 \)-meek, \( \ell_0 \)-almost meek) for \( i = 1, 2 \), then so is

\[
\mathcal{P}_1 \oplus \mathcal{P}_2 \leq \mathcal{D}_1 \oplus \mathcal{D}_2 \to \mathcal{M}_1 \oplus \mathcal{M}_2. \quad \square
\]

Two more subtle constructions with \( \Delta \)-modules were introduced in Section I.6. The first of these was aggrandizement: Definition I.6.1.

Suppose that \( c \subseteq e \) is an inclusion of finite sets, that \( Y \leq Z \) is an inclusion of compact Abelian groups, and that \( U = (U_i)_{i \in e} \) is a family of closed subgroups of \( Z \); let \( U \downharpoonright c \) be the subfamily \((U_i)_{i \in c}\), and let \( \mathcal{P} = (P_e)_{e} \) be a functional
(Z, Y, U ↾c)-Δ-module with fibre \( A \) and dummy \( X \). Then Definition I.6.1 gives immediately that \( \text{Ag}_c^e \mathcal{P} \) is a functional (Z, Y, U)-Δ-module with the same fibre and dummy.

Now let \( \mathcal{P} \leq \mathcal{D} \to \mathcal{M} \) be a semi-functional (Z, Y, U ↾c)-Δ-module. Then each of \( \mathcal{P}, \mathcal{D} \) and \( \mathcal{M} \) has an aggrandizement to \( U \), the first two of which are still functional. These fit into a new semi-functional Δ-module

\[
\text{Ag}_c^e \mathcal{P} \leq \text{Ag}_c^e \mathcal{D} \to \text{Ag}_c^e \mathcal{M}.
\]

This gives a semi-functional presentation of \( \text{Ag}_c^e \mathcal{M} \). Explicitly, the items appearing here are as follows:

- (modules:) \( P_{a \cap c} \leq Q_{a \cap c} \to M_{a \cap c} \) for \( a \subseteq e \);
- (structure morphisms:) in the middle row, \( \varphi_{a \cap c, b \cap c}^e \) for \( a \subseteq b \subseteq e \), and similarly in the other rows;
- (derivation-lifts:) in the middle row, \( \tilde{\nabla}_{\mathcal{D}, a \cap c, (a \cap c) \setminus i} \) for \( a \subseteq e \) and \( i \in e \), and similarly in the other rows.

It is clear that this construction satisfies all the properties of a semi-functional Δ-module.

One of the key results about aggrandizements was Corollary I.6.3, which related the structure complexes of \( \text{Ag}_c^e \mathcal{M} \) to those of \( \mathcal{M} \) itself: if \( e \subseteq c \), then the structure complexes at \( e \) are the same; and if \( e \nsubseteq c \), then the structure complex of \( \text{Ag}_c^e \mathcal{M} \) at \( e \) is split.

We will need to re-use this splitting in the present paper, and in doing so we will need the following additional information.

**Lemma 7.9.** Let \( c \) and \( \mathcal{P} \leq \mathcal{D} \to \mathcal{M} \) be as above, and let \( e \subseteq [k] \) with \( e \nsubseteq c \), The structure complexes of \( \text{Ag}_c^e \mathcal{P}, \text{Ag}_c^e \mathcal{D} \) and \( \text{Ag}_c^e \mathcal{M} \) at \( e \) have sequences of splitting homomorphisms that are consistent (in the sense that they commute with the inclusions and quotients), and complexity-bounded for \( \mathcal{P} \) and \( \mathcal{D} \).

**Proof.** It suffices to treat the case \( e = [k] \). An explicit sequence of splitting homomorphisms was constructed in the proof of Lemma I.6.2 (the ‘Homotopical Lemma’): pick \( s \in [k] \setminus c \), and now for \( \ell = 0, 1, \ldots, k \) define the homomorphism

\[
\xi_\ell^e : \bigoplus_{|b| = \ell + 1} Q_{e \cap c} \to \bigoplus_{|a| = \ell} Q_{a \cap c}
\]

by setting

\[
(\xi_\ell^e ((q_b)_{|b| = \ell + 1}))_a := \begin{cases} 0 & \text{if } s \in a \\ \text{sgn}(a \cup s : a)q_{a \cup s} & \text{if } s \nsubseteq a, \end{cases}
\]

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and analogously for $\mathcal{P}$ and $\mathcal{M}$. It is clear that $\xi^\mathcal{P}_\ell$ restricts to $\xi^\mathcal{P}_\ell$ on $\bigoplus_{|b|=\ell+1} P_{b \cap c}$, and that it agrees with $\xi^\mathcal{M}_\ell$ upon quotienting its domain by $\bigoplus_{|b|=\ell+1} P_{b \cap c}$ and its target by $\bigoplus_{|a|=\ell} P_{a \cap c}$. Since $\xi^\mathcal{P}_\ell$ is manifestly complexity-bounded, these sequences of splitting homomorphisms have the desired properties.

Subsection I.6.2 introduced the operations of restriction and reduction. Now let $\mathcal{P} \leq \mathcal{D} \rightarrow \mathcal{M}$ be a semi-functional $(Z,Y,U)$-$\Delta$-module, and let $c \subseteq [k]$. Following Definition I.6.4, the restrictions $\mathcal{P} \mid_c$ and $\mathcal{D} \mid_c$ are functional $(Z,Y,U \mid_c)$-$\Delta$-modules, and they still form a short exact sequence (hence, a semi-functional $\Delta$-module)

$$\mathcal{P} \mid_c \leq \mathcal{D} \mid_c \rightarrow \mathcal{M} \mid_c. $$

The ingredients here as follows:

- (modules:) $P_a \leq Q_a \rightarrow M_e$ for $a \subseteq c$.
- (structure morphisms:) $\varphi_{a,b}$ for $a \subseteq b \subseteq c$.
- (derivation-lifts:) $\tilde{\nabla}^{a,a \backslash i}$ for $a \subseteq c$ and $i \in c$.

This construction was then combined with aggrandizement to define reduction: Definition I.6.5. The reduction of $\mathcal{M}$ at $c$ is the $\Delta$-module

$$\mathcal{M} \mid_c := \text{Ag}^{[k]}(\mathcal{M} \mid_c),$$

and putting together the remark above shows that one obtains a semi-functional $\Delta$-module

$$\mathcal{P} \mid_c \leq \mathcal{D} \mid_c \rightarrow \mathcal{M} \mid_c.$$

Lemma I.6.6 identified the structure complexes of $\mathcal{M} \mid_c$ in terms of those of $\mathcal{M}$. In Corollary I.6.7 this implied that the properties of almost or strict modesty are inherited by this operation, and similarly one deduces the same for meekness.

**Lemma 7.10.** If $\mathcal{P} \leq \mathcal{D} \rightarrow \mathcal{M}$ is a semi-functional $(Z,Y,U)$-$\Delta$-module and $c \subseteq [k]$, then $\mathcal{M} \mid_c$ is $\ell_0$-almost (resp. strictly) meek if and only if $\mathcal{M} \mid_c$ is $\ell_0$-almost (resp. strictly) meek, and both are implied if $\mathcal{M}$ itself is $\ell_0$-almost (resp. strictly) meek.

**Proof:** Mostly this is clear from the identification of the structure complexes of the restricted and reduced $\Delta$-modules (see Lemma I.6.6). The only remaining observation one needs is that if $e \not\subseteq c$, then the structure complexes of $\mathcal{P} \mid_c$, $\mathcal{D} \mid_c$ and $\mathcal{M} \mid_c$ at $e$ form a semi-functional complex which is split, and has splitting homomorphisms that are complexity-bounded in the first and second rows by Lemma 7.9.
The fact that these complexes are split implies that the property of step-polynomial representatives is vacuously satisfied for the homology in these structure complexes. The fact that the splitting homomorphisms are complexity-bounded implies that one can obtain bounded-complexity representatives in the structure complexes of $\mathcal{P}_{\mathcal{C}}$ and $\mathcal{Q}_{\mathcal{C}}$ by applying those splitting homomorphisms. □

### 7.3 Short exact sequences and concatenations

Section I.8 introduced short exact sequences of $(Z, Y, U) - \Delta$-modules, such as

$$0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{L} \to 0. \quad (11)$$

Its main result, Proposition I.8.2, related the almost (resp. strict) modesty of two of these $\Delta$-modules to the almost (resp. strict) modesty of the third.

We now define concatenations of semi-functional $\Delta$-modules, in analogy with Definition 6.7, and use the results of Section 6 to relate the properties of three semi-functional $\Delta$-modules arising from a concatenation.

**Definition 7.11 (Concatenation).** A concatenation of functional $(Z, Y, U) - \Delta$-modules is a pair of inclusions of closed functional $(Z, Y, U) - \Delta$-modules

$$(P_e)_{\mathcal{C}} \leq (R_e)_{\mathcal{C}} \leq (Q_e)_{\mathcal{C}},$$

all having the same auxiliary group $X$ and fibre $A$, and where we assume that the structure morphisms and derivation-lifts are all compatible under these inclusions.

From a concatenation as above, similarly to the situation with semi-functional complexes, one may form the semi-functional $\Delta$-modules

$$(P_e)_{\mathcal{C}} \hookrightarrow (R_e)_{\mathcal{C}} \to (R_e/P_e)_{\mathcal{C}} = \mathcal{M},$$

$$(P_e)_{\mathcal{C}} \hookrightarrow (Q_e)_{\mathcal{C}} \to (Q_e/P_e)_{\mathcal{C}} = \mathcal{N}$$

and

$$(R_e)_{\mathcal{C}} \hookrightarrow (Q_e)_{\mathcal{C}} \to (Q_e/R_e)_{\mathcal{C}} = \mathcal{L},$$

and these now fit into the commutative diagram

$$
\begin{array}{ccc}
(P_e)_{\mathcal{C}} & \to & (P_e)_{\mathcal{C}} \\
\downarrow & & \downarrow \\
(R_e)_{\mathcal{C}} & \to & (Q_e)_{\mathcal{C}} \\
\downarrow & & \downarrow \\
0 & \to & (R_e/P_e)_{\mathcal{C}} \\
\end{array}
\begin{array}{ccc}
& & \to \\
& & \downarrow \\
& & (Q_e/P_e)_{\mathcal{C}} \\
& & \downarrow \\
& & (Q_e/R_e)_{\mathcal{C}} \\
& & \to \\
\end{array}
\begin{array}{c}
0
\end{array}
$$
whose bottom row is a short exact sequence as in (11).

**Proposition 7.12.** For a concatenation of semi-functional $\Delta$-modules as above, the following implications hold.

1. If $\mathcal{M}$ and $\mathcal{N}$ are $\ell_0$-almost (resp. strictly) meek, then so is $\mathcal{L}$.
2. If $\mathcal{M}$ and $\mathcal{L}$ are $\ell_0$-almost (resp. strictly) meek, then so is $\mathcal{N}$.
3. If $\mathcal{N}$ and $\mathcal{L}$ are $\ell_0$-almost (resp. strictly) meek, and $M_e = 0$ whenever $|e| = \ell_0$, then $\mathcal{M}$ is $(\ell_0 + 1)$-almost (resp. strictly) meek.

**Proof.** For each $e \subseteq [k]$, the structure complexes of $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{Q}$ at $e$ fit into a concatenation of structure complexes. The conclusions now follow from Lemmas 6.8–6.10. 

Part (1) of Proposition 7.12 asserts that the properties $\ell_0$-almost and -strict meekness are preserved under the following operation.

**Definition 7.13.** Given a concatenation as above, the semi-functional $\Delta$-module $\mathcal{L}$ will be referred to as the quotient of $\mathcal{N}$ by $\mathcal{M}$, and denoted $\mathcal{N}/\mathcal{M}$.

This quotient may also sometimes be written as

$$\mathcal{N}/\mathcal{M} : (R_e)_e \hookrightarrow (Q_e)_e \twoheadrightarrow (N_e/M_e)_e.$$  \hfill (12)

### 7.4 Complexity and pure semi-functional $\Delta$-modules

Some of the more substantial proofs in Part I were by induction on the subgroup-data $(Z,Y,U)$ directing a $\Delta$-module of interest. This induction was organized using a partial order on those data. We will meet more proofs like this below, so we quickly recall that partial order now.

Given two tuples of subgroup-data $(Z,Y,U) = (U_i)_{i=1}^k$ and $(Z',Y',U') = (U'_i)_{i=1}^{k'}$, we have $(Z,Y,U) \preceq (Z',Y',U')$ if

- either $k < k'$,
- or $k = k'$, but $|\{i \leq k \mid Y \geq U_i\}| > |\{i \leq k' \mid Y' \geq U'_i\}|$.

We have $(Z,Y,U) \preceq (Z',Y',U')$ if either $(Z,Y,U) \preceq (Z',Y',U')$ or they are equal. We usually refer to this partial order using ‘$\preceq$’. It is easily seen to be a well-ordering, although not a total ordering.

The minimal tuples for this ordering are those of the form $(Z,Y,\ast)$, where $\ast$ denotes the empty tuple. However, in most of our proofs by $\preceq$-induction, the
downwards movement in the order \( \preceq \) stops before one reaches such a minimal tuple. Rather, one must argue directly, without the \( \preceq \)-inductive hypothesis, for any tuple for which \( Y \geq U_i \) for all \( i \). Such a tuple is called ‘pure’, and a \((Z, Y, U)\)-\( \Delta \)-module is ‘pure’ if \((Z, Y, U)\) is pure.

In fact, Subsection I.7.2 showed that pure almost-modest \( \Delta \)-modules have quite a simple structure, which made those parts of the inductive proofs very simple. The same happens for almost meek semi-functional \( \Delta \)-modules, as we shall show now.

**Proposition 7.14.** If \((Y, Y, U)\) is pure (thus, \( Z = Y \)), and \( \mathcal{P} \leq \mathcal{D} \rightarrow \mathcal{M} \) is an \( \ell_0 \)-almost (resp. strictly) meek semi-functional \((Y, Y, U)\)-\( \Delta \)-module, then the semi-functional \( Y \)-module

\[
P_e \leq Q_e \rightarrow M_e
\]

has s.-p. (resp. discrete) quotient and strong s.-p. representatives for every \( e \subseteq [k] \).

If \((Z, Y, U)\) is pure, then an \( \ell_0 \)-almost (resp. strictly) meek semi-functional \((Z, Y, U)\)-\( \Delta \)-module is co-induced from an \( \ell_0 \)-almost (resp. strictly) meek semi-functional \((Y, Y, U)\)-\( \Delta \)-module.

**Proof.** Consider first a \((Y, Y, U)\)-\( \Delta \)-module \( \mathcal{P} \leq \mathcal{D} \rightarrow \mathcal{M} \).

If \(|e| = \ell_0\), then all the asserted structure for \( P_e \leq Q_e \rightarrow M_e \) is directly contained in the definition of almost (resp. strict) meekness. If \(|e| \geq \ell_0 + 1\), then the top end of its structure complex at \( e \) looks like

\[
\cdots \rightarrow \bigoplus_{a \in \langle |e|-1 \rangle} P_a \rightarrow P_e
\]

\[
\cdots \rightarrow \bigoplus_{a \in \langle |e|-1 \rangle} Q_a \rightarrow Q_e
\]

\[
\cdots \rightarrow \bigoplus_{a \in \langle |e|-1 \rangle} M_a \rightarrow M_e.
\]

The last homomorphism of the bottom row has closed and co-discrete image, by the assumption of almost modesty. From this and Corollary 3.44, it follows by induction on \(|e|\) that every \( M_e \) is an s.-p. \( Y \)-module, and discrete in case \( \mathcal{M} \) is strictly modest.

Induction on \(|e|\) is also used to see that the semi-functional modules \( P_e \leq Q_e \rightarrow M_e \) have strong s.-p. representatives for any \( e \subseteq [k] \). The base case \(|e| = \ell_0\) is already handled above, so suppose \(|e| \geq \ell_0 + 1\), suppose that \( m = q + \text{Coind}_Y^Z P_e \in (\text{Coind}_Y^Z M_e)_{\text{sp}} \) for some enlargement \( Z \geq Y \). Then Lemma 4.21 and the almost meekness of \( \mathcal{M} \) give that \( q = q_1 + \alpha_{|e|}(q_2) + p \) for some \( q_1 \in \text{sp} \).
(Coind\^Z Q_\epsilon)_{sp}, q_2 \in \bigoplus_{a \in \{1, \ldots, e\}} \text{Coind}\^Z Q_a$ and $p \in \text{Coind}\^Z P_\epsilon$. Replacing $q$ with $q_2 - q_1$, we may therefore assume that $q = \alpha_{|\epsilon|}(q_2)$ maps to a step polynomial in $\text{Coind}\^Z M_\epsilon$. However, this implies that the class $q_2 + \bigoplus_{a \in \{1, \ldots, e\}} \text{Coind}\^Z P_a$ in $\bigoplus_{a \in \{1, \ldots, e\}} \text{Coind}\^Z M_a$ maps to a step polynomial in $\text{Coind}\^Z M_\epsilon$, and so by Lemma[4.16] we may choose that class in $\bigoplus_{a \in \{1, \ldots, e\}} \text{Coind}\^Z M_a$ to be a step polynomial. Finally, by the inductive hypothesis on $|\epsilon|$, we may now choose $q_2$ itself, and hence $\alpha_{|\epsilon|}(q_2)$, to be a step polynomial.

For general $Z \geq Y$, the assertions for a pure semi-functional $(Z, Y, U)$-$\Delta$-module follow directly from the definition and the results above. \hfill \Box

8 Cohomology $\Delta$-modules

This section will consider another construction with $\Delta$-modules from Part I: cohomology $\Delta$-modules. Similarly to Part I, the analysis of these is the most technically complicated part of this paper.

If $\mathcal{M}$ is a Polish $(Z, Y, U)$-$\Delta$-module, $W \leq Z$ and $\beta \geq 0$, then one may form a new $(Z, Y + W, U)$-$\Delta$-module by applying the cohomology functor $H^p_m(W, -)$ to all the modules and morphisms of $\mathcal{M}$. This construction was the subject of Section I.9, where it was shown to preserve the property of (almost) modesty (Theorem I.9.5).

Now suppose that $\mathcal{P} \leq \mathcal{L} \rightarrow \mathcal{M}$ is a semi-functional presentation of $\mathcal{M}$, with dummy $X$ and fibre $A$. This can be used to construct a semi-functional presentation of $H^p_m(W, \mathcal{M})$. This does not arise by applying $H^p_m(W, -)$ to each $\Delta$-module in the presentation separately. Rather, the natural presentation is suggested and justified by Lemma[5.3]

$$B^p(W, Q_\epsilon, P_\epsilon) \leq Z^p(W, Q_\epsilon, P_\epsilon) \rightarrow H^p_m(W, M_\epsilon) \quad \text{for each } \epsilon \subseteq [k].$$

This is a short exact sequence of topological modules by Lemma[5.3] In all the cases of interest to us, the results of Part I (in particular, Theorem I.9.5) will imply that each $H^p_m(W, M_\epsilon)$ is Hausdorff, and hence that $B^p(W, Q_\epsilon, P_\epsilon)$ is closed.

Knowing this, all the required properties of this presentation follow directly from Definition I.9.2. It is easy to check that $(B^p(W, Q_\epsilon, P_\epsilon)_e$ and $(Z^p(W, Q_\epsilon, P_\epsilon)_e$ are both functional $(Z, W + Y, U)$-$\Delta$-submodules of the constant $(Z, W + Y, U)$-$\Delta$-module $\mathcal{F}(W^p \times X \times Z, A)$. The structure morphisms are simply the inclusions $B^p(W, Q_a, P_a) \leq B^p(W, Q_\epsilon, P_\epsilon)$ and $Z^p(W, Q_a, P_a) \leq Z^p(W, Q_\epsilon, P_\epsilon)$ for $a \subseteq \epsilon$.

(which are inclusions because $P_a \leq P_\epsilon$, $P_\epsilon \leq Q_\epsilon$ and $Q_a \leq Q_\epsilon$). The derivation-lifts are inherited from $\mathcal{L}$ as was explained before Definition I.9.2.

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With these facts established, we will usually abbreviate the above presentation of $H^p(W, \mathcal{M})$ to
\[ B^p(W, D, \mathcal{P}) \leq \mathcal{Z}^p(W, D, \mathcal{P}) \rightarrow H^p(W, \mathcal{M}). \quad (13) \]

**Theorem 8.1.** If $\mathcal{P} \leq D \rightarrow \mathcal{M}$ is $\ell_0$-almost meek, then the cohomology semi-functional $\Delta$-module in (13) is $\ell_0$-almost meek. It is strictly meek in case either $\mathcal{P} \leq D \rightarrow \mathcal{M}$ is strictly meek or $p \geq 1$.

The proof will require that we also establish the following, which is of interest in its own right.

**Theorem 8.2 (Step-polynomial coboundary solutions).** Let $\mathcal{P} \leq D \rightarrow \mathcal{M}$ be an $\ell_0$-almost meek $(Z, Y, U)$-$\Delta$-module, and let $W \leq Z$ be a closed subgroup. Then the semi-functional $W$-module $P_e \leq Q_e \rightarrow M_e$ admits strong s.-p. coboundary solutions for every $e \subseteq [k]$.

This provides an extension of the first part of Corollary [5.27] to the setting of modest $\Delta$-modules. More explicitly, it asserts that for all $W \leq Z$, $e \subseteq [k]$ and $p \geq 0$, if $\sigma \in (B^p(W, Q_e, P_e))_{\text{sp}}$, then there are
\[ f \in C^p_{\text{sp}}(W, Q_e) \quad \text{and} \quad g \in C^p_{\text{sp}}(W, P_e) \]
such that
\[ d^W f = \sigma + g, \quad (14) \]
and similarly after co-induction to an arbitrary enlargement of $Z$.

**Notation**

We will adopt here essentially the same notation as in Section I.9. In addition to those conventions, let
\[ P^{(\ell)} := \bigoplus_{|e| = \ell} P_e \quad \text{and} \quad Q^{(\ell)} := \bigoplus_{|e| = \ell} Q_e, \]
giving presentations
\[ P^{(\ell)} \hookrightarrow Q^{(\ell)} \rightarrow M^{(\ell)}. \]

If $M = I_{\ell-1}$ or $K_{\ell}$ (the position-$\ell$ image or kernel, respectively, in the top structure complex of $\mathcal{M}$), both of which are submodules of $M^{(\ell)}$, let
\[ P^M := P^{(\ell)}. \]
and let $Q^M$ be the submodule of $Q^{(l)}$ that maps onto $M$ under the quotient by $P^M$. This gives further presentations

$$P^M \hookrightarrow Q^M \twoheadrightarrow M$$

for each such $M$.

The above definition is equivalent to

$$Q^{I_{\ell-1}} = \partial_\ell(Q^{(l-1)}) + P^{(l)}$$

and

$$Q^{K_\ell} = \{ q \in Q^{(l)} | \partial_{\ell+1}(q) \in P^{(\ell+1)} \}.$$

It is obvious from this definition that $Q^{K_\ell}$ is closed in $Q^{(l)}$, since $\partial_{\ell+1}$ is continuous. We also know that $Q^{I_{\ell-1}}$ is closed in $Q^{(l)}$ from the structural closure assumption in the definition of modesty.

In terms of relative coboundaries and relative cocycles, the top structure complex for $H_p^m(W, M)$ has the following presentation:

$$
\begin{array}{cccccccccccc}
0 & \longrightarrow & B^p(W, Q, P_0) & \overset{\partial_1^u}{\longrightarrow} & B^p(W, Q^{(1)}, P^{(1)}) & \overset{\partial_2^u}{\longrightarrow} & \cdots & \overset{\partial_k^u}{\longrightarrow} & B^p(W, Q^{[k]}, P^{[k]}) & \longrightarrow & 0 \\
0 & \longrightarrow & Z^p(W, Q, P_0) & \overset{\partial_1^q}{\longrightarrow} & Z^p(W, Q^{(1)}, P^{(1)}) & \overset{\partial_2^q}{\longrightarrow} & \cdots & \overset{\partial_k^q}{\longrightarrow} & Z^p(W, Q^{[k]}, P^{[k]}) & \longrightarrow & 0 \\
0 & \longrightarrow & H^p_m(W, M_0) & \overset{\partial_1^*}{\longrightarrow} & H^p_m(W, M^{(1)}) & \overset{\partial_2^*}{\longrightarrow} & \cdots & \overset{\partial_k^*}{\longrightarrow} & H^p_m(W, M^{[k]}) & \longrightarrow & 0,
\end{array}
$$

where we use the notation $\partial_1^P$, $\partial_2^Q$ and $\partial_1^*$ := $H^p_m(W, \partial_1)$ for the obvious structure morphisms.

### 8.1 Setting up the induction

Theorems 8.1 and 8.2 will be proved by an outer induction on $(Z, Y, U)$ (in the partial order $\preceq$), and an inner induction along the positions in the top structure complex.

To carry out this induction, we will need to formulate a much more precise inductive hypothesis, which also gives information about the cohomology groups $H^p_m(W, I_\ell)$ and $H^p_m(W, K_\ell)$. The formulation will take the form of Proposition 8.4 below.

To lighten notation, we will now adopt the abbreviation $H^p(-) := H^p_m(W, -)$ (fixing $W$ for the time being), as in Section I.9.

First, it will be convenient to introduce another relative of Definition 7.7.
**Definition 8.3.** Suppose that $0 \leq \ell_0 \leq \ell \leq k$. If

$$\mathcal{P} \leq \mathcal{D} \to \mathcal{M}$$

is a semi-functional $\Delta$-module, then it is $(\ell_0, \ell)$-stunted (resp. $(\ell_0, \ell)$-stunted almost meek) if

- it is $\ell_0$- (resp. $\ell_0$-almost) modest,
- its every nontrivial restriction is $\ell_0$-almost meek,
- all derivation-lifts of $\mathcal{D}$ are complexity-bounded,
- and its top structure complex is co-induced over $Y + U[k]$ from a semi-functional complex of $(Y + U[k])$-modules which has
  - strong s.-p. representatives at position $j$ for every $j \leq \ell - 1$,
  - and finite-complexity decompositions at position $j$ for every $j \leq \ell$.

**Remark.** The complexity-boundedness assumption for the derivation-lifts is contained in the meekness of the nontrivial restrictions, except for the derivation-lifts $\nabla^{(k),[k]} : M[k] \to M_{[k]}/i$. This is different from Definition 7.7 only in the top structure complex, where the new definition ‘stops at position $\ell$’. Notice the wrinkle that in Definition 8.3 finite-complexity decompositions run up to position $\ell$, but s.-p. representatives are assumed only up to position $\ell - 1$. This is a somewhat arbitrary choice, but it seems to fit best into the coming induction along the top structure complex. An $(\ell_0, k)$-stunted almost meek semi-functional $\Delta$-module is very nearly $\ell_0$-almost meek as in Definition 7.7: in that case, the only missing condition is strong s.-p. representatives at the very end of the top structure complex.

**Proposition 8.4.** Fix $(Z,Y,U)$, $0 \leq \ell_0 \leq \ell \leq k$, and an $(\ell_0, \ell)$-stunted almost meek semi-functional $(Z,Y,U)$-$\Delta$-module $\mathcal{P} \leq \mathcal{D} \to \mathcal{M}$.

The following hold for every closed $W \leq Z$ and all $j \in \{\ell_0 - 1, \ell_0, \ldots, \ell - 1\}$.

(i) The top structure complex of

$$\mathcal{B}^p(W, \mathcal{D}, \mathcal{P}) \leq \mathcal{Z}^p(W, \mathcal{D}, \mathcal{P}) \to \mathcal{H}^p(\mathcal{M})$$

has strong s.-p. representatives at position $j$ for every $p \geq 1$ (note that we do not allow $p = 0$ here; that will need to be treated separately).
(ii) The top structure complex of
\[ \mathcal{B}^p(W, \mathcal{Q}, \mathcal{P}) \leq \mathcal{Z}^p(W, \mathcal{Q}, \mathcal{P}) \to H^p(\mathcal{M}) \]
 admits finite-complexity decompositions at position \( j + 1 \) for all \( p \geq 1 \)
(again, not \( p = 0 \)).

(iii) If \( \sigma \in Z^p(W, Q^{K_j}, P^{(j)}) \) has the property that
\[ (\sigma + B^p(W, Q^{(j)}, P^{(j)})) \cap Z^{sp}_p(W, Q^{(j)}, P^{(j)}) \neq \emptyset, \]
then in fact
\[ (\sigma + B^p(W, Q^{(j)}, P^{(j)})) \]
\[ \cap (Z^{sp}_p(W, Q^{K_j}, P^{(j)}) + \partial_* (Z^p(W, Q^{(j-1)}, P^{(j-1)}))) \neq \emptyset. \]

(iv) The semi-functional module \( P^{(j+1)} \leq Q^{I_j} \to I_j \) admits strong s.-p. solutions to the relative coboundary equation over \( W \) in every degree \( p \geq 0 \).

(v) The semi-functional module \( P^{(j+1)} \leq Q^{K_{j+1}} \to K_{j+1} \) admits strong s.-p. solutions to the relative coboundary equation over \( W \) in every degree \( p \geq 0 \).

(vi) The quotient homomorphism
\[ Z^p(W, Q^{I_j}, P^{(j)}) \to \text{coker}(H^p(M^{(j)}) \to H^p(I_j)) \]
has target that is co-induced from a s.-p. \((Y + W + U_{[k]})\)-module, and has strong s.-p. representatives for all \( p \geq 0 \).

(vii) The quotient homomorphism
\[ Z^p(W, Q^{K_{j+1}}, P^{(j+1)}) \to \text{coker}(H^p(M^{(j)}) \to H^p(K_{j+1})) \]
has target that is co-induced from a s.-p. \((Y + W + U_{[k]})\)-module, and has strong s.-p. representatives for all \( p \geq 0 \).

(viii) The quotient homomorphism
\[ Z^p(W, Q^{K_{j+1}}, P^{(j+1)}) \cap B^p(W, Q^{(j+1)}, P^{(j+1)}) \]
\[ \to \ker (H^p(K_{j+1}) \to H^p(M^{(j+1)})) \]
has target that is co-induced from a s.-p. \((Y + W + U_{[k]})\)-module, and has strong s.-p. representatives for all \( p \geq 0 \).
The inductive proof of Proposition 8.4 will be broken into several stages, occupying the next three subsections.

- The next subsection will assume that Proposition 8.4 is known up to a given \((Z, Y, U)\) and \(\ell\), and will deduce from that hypothesis some other auxiliary properties of \((\ell_0, \ell)\)-stunted almost meek semi-functional \(\Delta\)-modules.

- Then, Subsection 8.3 will explain the inductive proof of Proposition 8.4 itself. For each of the properties (i)—(viii) in turn, we will show how it follows from the conjunction of those same properties coming from earlier steps in the induction. These deductions will use the auxiliary results of Subsection 8.2, hence the need to interpose those other subsections. Subsection 8.3 will also give the completion of the proofs of Theorems 8.1 and 8.2.

In order to keep track of our position in the induction, it will be handy to give a name to the following assumption for a given tuple \((Z, Y, U)\) and a given \(\ell \leq k\):

\[(\text{Ind})^{Z,Y,U,\ell}.\]

Theorems 8.1 and 8.2 are already known for any almost meek semi-functional \((Z_1, Y_1, U_1)\)-\(\Delta\)-module for which \((Z_1, Y_1, U_1) \preceq (Z, Y, U)\), and Proposition 8.4 is known for any \((\ell'_0, \ell_1)\)-stunted almost meek semi-functional \((Z, Y, U)\)-\(\Delta\)-module for which \(\ell'_0 \leq \ell_1 \leq \ell\).

### 8.2 First part of the induction

This subsection is roughly a counterpart to Subsection I.9.2.

The main result of this subsection is the following deduction from \((\text{Ind})^{Z,Y,U,\ell}\).

It says that, in some cases, the property of being stunted meek is self-strengthening.

**Proposition 8.5.** [Self-strengthening of stunted almost meekness] Assume \((\text{Ind})^{Z,Y,U,\ell}\).

Suppose that \(0 \leq \ell_0 \leq \ell \leq k - 1\), and let \(\mathcal{P} \subseteq \mathcal{M} \to \mathcal{N}\) be an \((\ell_0, \ell)\)-stunted meek semi-functional \((Z, Y, U)\)-\(\Delta\)-module. Then the following also hold:

(a) If \(Y + U_{[k]} = Z\), then the top structure complex has s.-p. representatives at position \(\ell\).

(b) The top structure complex has finite-complexity decompositions at position \(\ell + 1\).

Thus, the semi-functional \(\Delta\)-module is actually \((\ell_0, \ell + 1)\)-stunted meek.

Remark. The proof will make essential use of the discreteness of the structural homology here, and so we cannot weaken ‘modest’ to ‘almost modest’ in the assumptions. This discreteness will provide one of the hypotheses for some appeals to Proposition 4.17.
Proposition 8.5 will be used during the inductive proof of Proposition 8.4, specifically for the inductive proof of assertions (i) and (ii) of that proposition.

In the notation of the statement of Proposition 8.4, that inductive proof will involve applying Proposition 8.5 to the semi-functional $\Delta$-modules
\[
\left( \tilde{\mathcal{P}} \leq \tilde{\mathcal{Q}} \to \tilde{\mathcal{M}} \right) = \left( B^p(W, \mathcal{Q}) \leq Z^p(W, \mathcal{P}) \to \Pi^p_{\mathcal{M}}(W, \mathcal{M}) \right)
\] (15)
for some $p \geq 1$, not to $(\mathcal{P} \leq \mathcal{Q} \to \mathcal{M})$ itself. We have therefore added tildes to the notation of Proposition 8.5 to distinguish $\tilde{\mathcal{M}}$ from the $\mathcal{M}$ of Proposition 8.4.

We will see in the present subsection that the proof of Proposition 8.5 involves forming the further cohomology $\Delta$-modules $H^q_m(U_i, \tilde{\mathcal{M}})$ for some $i \in [k]$, which in the case (15) would be
\[
\Pi^p_{\mathcal{M}}(U_i, H^p_m(W, \mathcal{M})).
\]
In fact, if one were to write out several consecutive steps of the induction for Proposition 8.4 explicitly, one would see that they involve $\Delta$-modules obtained from longer and longer compositions of cohomology functors. This is one reason why the overall induction is rather lengthy. We have separated Proposition 8.5 into its own subsection, and given it its own notation, so that we do not have to keep track of more than one cohomology functor at any one time.

Given $(Z, Y, U)$, the whole structure in Proposition 8.5 is already co-induced from $Y + U_{[k]}$ to $Z$. Therefore, although (b) is an assertion about strong s.-p. pre-images, we will lose no generality if we simply work with functions on $Z$, rather than introducing a further enlargement. This will not be remarked again during the proof.

Also, if $\ell_0 = k$, then $\tilde{Q}^{(\ell)} = \tilde{P}^{(\ell)}$, so Proposition 8.5 is vacuously true. Hence we may assume $\ell_0 \leq k - 1$.

Having fixed $(Z, Y, U)$, let (a) and (b) denote the conjunction of all the assertions (a) and (b) for subgroup tuples $(Z_1, Y_1, U_1) \not\subseteq (Z, Y, U)$ and all admissible $\ell$.

It seems clearest to present the proof of Proposition 8.5 in several separate steps.

**Proof of Proposition 8.5 in pure case.** This is our main application of Proposition 7.14. Firstly, applying that proposition to each nontrivial restriction of $\mathcal{P} \leq \mathcal{Q} \to \mathcal{M}$, we see that it is co-induced from an $(\ell_0, \ell)$-stunted meek semi-functional $(Y, Y, U)$-$\Delta$-module, say, $\tilde{\mathcal{P}} \leq \tilde{\mathcal{Q}} \to \tilde{\mathcal{M}}^\circ$.

Suppose that $f \in \tilde{Q}^{(\ell)}$ and $p \in \tilde{P}^{(\ell+1)}$ are such that $q := \partial_{\ell+1} f + p$ is a step polynomial. Let $\tilde{f}$ be the image of $f$ in $\tilde{M}^{(\ell)}$. By Proposition 7.14 and 8.17, the fact that $q$ is a step polynomial implies that its image
\[
q + \tilde{P}^{(\ell+1)} = \partial_{\ell+1} \tilde{f} \in \tilde{M}^{(\ell+1)} = \text{Coind}_{Y} \tilde{M}^{\circ(\ell+1)}
\]
is a step function. The homomorphism $\partial_{\ell+1} \colon \widetilde{M}^{(\ell)} \to \widetilde{M}^{(\ell+1)}$ is co-induced from a closed homomorphism of s.-p. $Y$-modules, so Corollary 4.16 gives some $\overline{g} \in \widetilde{M}_{\text{sp}}^{(\ell)}$ such that $\partial_{\ell+1} \overline{g} = \partial_{\ell+1} \overline{f}$.

Now, $\widetilde{Q}^{(\ell)} \to \widetilde{M}^{(\ell)}$ is a direct sum of quotient maps that have strong s.-p. pre-images, by Proposition 7.14. We may therefore lift $\overline{g}$ to some $g \in \widetilde{Q}_{\text{sp}}^{(\ell)}$. This satisfies $\partial_{\ell+1}(f - g) \in \widetilde{P}^{(\ell+1)}$, hence $q = \partial_{\ell+1}g \mod \widetilde{P}^{(\ell+1)}$, as required for conclusion (b).

Finally, if $Y + U[k] = Y = Z$, then $\widetilde{P}^{(\ell)} \leq \widetilde{Q}^{(\ell)}$ itself is co-discrete and has s.-p. representatives, and of course $\widetilde{P}^{(\ell)} \leq \partial_{\ell+1}(\widetilde{P}^{(\ell+1)}) \leq \widetilde{Q}^{(\ell)}$, so conclusion (a) follows by applying Lemma 4.13 to these inclusions.

The remaining cases of Proposition 8.5 will be proved by $\preceq$-induction. The inductive steps will make use of a few auxiliary lemmas.

**Lemma 8.6.** Under (Ind)$^Z,Y,U,\ell$, if $\widetilde{\mathcal{P}} \leq \widetilde{\mathcal{Q}} \to \widetilde{\mathcal{M}}$ is a semi-functional $(Z,Y,U)$-$\Delta$-module whose every nontrivial restriction is $\ell_0$-almost meek, then for every $j \leq k - 1$, the $(\widetilde{Q}^{(j)}, \widetilde{P}^{(j)})$-relative coboundary equation admits strong s.-p. solutions, as in Theorem 8.2.

**Proof.** Since $\widetilde{Q}^{(j)} = \bigoplus_{[a] = j} \tilde{Q}_a$ and similarly for $\widetilde{P}^{(j)}$, and since $j \leq k - 1$, this is just a direct sum of relative coboundary equations from nontrivial restrictions of $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{Q}}$. Each nontrivial restriction is directed by $(Z,Y,U|_c)$ for some $c \subseteq [k]$, and this strictly precedes $(Z,Y,U)$ in the complexity order, so we may apply a case of Theorem 8.2 contained in (Ind)$^Z,Y,U,\ell$.

**Lemma 8.7.** Under (Ind)$^Z,Y,U,\ell$, if $0 \leq \ell_0 \leq \ell \leq k - 1$ and $\widetilde{\mathcal{P}} \leq \widetilde{\mathcal{Q}} \to \widetilde{\mathcal{M}}$ is an $(\ell_0, \ell)$-stunted almost meek semi-functional $(Z,Y,U)$-$\Delta$-module, and if $W \leq Z$, then the following hold:

(A) If $Y + W + U[k] = Z$, then

$$
\partial_{j+1}^{-1}(B^1(W, \widetilde{Q}^{(j+1)}, \widetilde{P}^{(j+1)}))
\subseteq Z_{\text{sp}}^1(W, \widetilde{Q}^{(j)}, \widetilde{P}^{(j)}) + \partial_j(Z^1(W, \widetilde{Q}^{(j-1)}, \widetilde{P}^{(j-1)})) + B^1(W, \widetilde{Q}^{(j)}, \widetilde{P}^{(j)})
$$

for all $j \leq \ell - 1$.

(B) In general, the homomorphism

$$
Z^1(W, \widetilde{Q}^{(j-1)}, \widetilde{P}^{(j-1)}) \oplus \widetilde{Q}^{(j)} \oplus C^1(W, \widetilde{P}^{(j)}) : (\sigma, q, f) \mapsto \partial_j \sigma + d^W q + f
$$

has strong s.-p. pre-images for all $j \leq \ell$. 85
Proof. As usual, for part (B), it suffices to handle the case of general $Y + W + U_{[k]} \leq Z$, without introducing another enlargement of $Z$.

The semi-functional $(Z, Y + W, U)\Delta$-module

$$B^1(W, \mathcal{D}, \mathcal{P}) \leq Z^1(W, \mathcal{D}, \mathcal{P}) \rightarrow H^1_{\text{m}}(W, \mathcal{M})$$

satisfies the same assumptions as $\mathcal{M}$ itself in the statement of Proposition 8.5. It is $\ell_0$-strictly modest, by Theorem I.9.5. It is still $(\ell_0, \ell)$-stunted almost meek, by a case of Proposition 8.4 contained in $(\text{Ind})_{Z, Y, U, \ell}$, and in fact it is $(\ell_0, \ell)$-stunted meek owing to its strict modesty. Also, the derivation-lifts of $Z^1(W, \tilde{Q}, \tilde{P})$ are obtained explicitly from those of $\mathcal{D}$ as described at the start of this section, so they are all complexity-bounded.

However, because it is $(\ell_0, \ell)$-stunted meek, it has strong s.-p. representatives in its top structure complex in all positions $j \leq \ell - 1$. This is precisely assertion (A).

Similarly, our semi-functional presentation of $H^1_{\text{m}}(W, \mathcal{D}, \mathcal{P})$ has finite-complexity decompositions in its top structure complex in all positions $j \leq \ell$. This implies that if $\partial \sigma + dW q + f$ is a step polynomial, then we may assume that $\sigma$ and $dW q + f$ are step polynomials. Since $j \leq k - 1$, now Lemma 8.6 implies that $q$ and $f$ may also be assumed to be step polynomials, so this completes the proof of (B).

Lemma 8.8. Assume $(\text{Ind})^{Z, Y, U, \ell}, (a)_{\geq}$ and $(b)_{\geq}$, and let $\tilde{\mathcal{P}} \leq \tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{M}}$ satisfy the assumptions of Proposition 8.5. Suppose also that $W \leq Z$ is such that $(Z, Y + W, U) \not\leq (Z, Y, U)$. Then the following hold:

(C) If $Y + W + U_{[k]} = Z$, then

$$Z^0(W, \tilde{Q}^\ell, \tilde{P}^\ell) \cap \partial_{\ell+1}^{-1}(\tilde{P}^{\ell+1}) \subseteq Z^0_{\text{sp}}(W, \tilde{Q}^\ell, \tilde{P}^\ell) + \partial_{\ell}(Z^0(W, \tilde{Q}^{\ell-1}, \tilde{P}^{\ell-1})) + \tilde{P}^\ell.$$

(D) In general, the homomorphism

$$Z^0(W, \tilde{Q}^\ell, \tilde{P}^\ell) \oplus \tilde{P}^{\ell+1} \rightarrow Z^0(W, \tilde{Q}^{\ell+1}, \tilde{P}^{\ell+1}) : (\tau, p) \mapsto \partial_{\ell+1} \tau + p$$

has strong s.-p. pre-images.

(These two parts are labeled to avoid conflict with Lemma 8.7.)

Proof. The semi-functional $(Z, Y + W, U)\Delta$-module

$$\tilde{\mathcal{P}} \leq Z^0(W, \mathcal{D}, \mathcal{P}) \rightarrow \mathcal{M}^W$$
satisfies the same assumptions as \( \widetilde{\mathcal{M}} \) itself in the statement of Proposition 8.5. It is still \( \ell_0 \)-modest, by Theorem I.9.5. It is still \((\ell_0, \ell)\)-stunted meek, by a case of Proposition 8.4 contained in (Ind)\(Z,Y,U,\ell\). Also, its derivation-lifts are simply restrictions of those of \( \widetilde{\mathcal{M}} \), so are still complexity-bounded.

However, we have assumed that \((Z, Y + W, U) \preceq (Z, Y, U)\), so (a) \(\preceq\) and (b) \(\preceq\) applied to \( \widetilde{\mathcal{M}}^W \) give (C) and (D), respectively. □

**Proof of (a).** We may restrict our attention to the non-pure case here. We now assume that \((Z, Y + U)[k] = Z\). Suppose \(i \in [k]\) is such that \(Y \npreceq U_i\). Let \(f \in \partial_{\ell+1}^{-1}(\widetilde{P}^{(\ell+1)})\).

Step 1. Let \(\nabla^U_i\) denote the derivation-lift \(\nabla^e,e,e\setminus i\) for any \(e\), or any direct sum over several \(e\)s. Applying \(\nabla^U_i\) to the above equation gives

\[
\sigma_0 := \nabla^U_i f \in Z^1(U_i, \partial_{\ell+1}^{-1}(\widetilde{P}^{(\ell+1)})).
\]

Applying the splitting homomorphism \(\widetilde{Q}^{(\ell+1)} \to \widetilde{Q}^{(\ell)}\) from Lemma 7.9 to this, one obtains

\[
\sigma_0 = \tau_0 + \partial_{\ell} \sigma'_0,
\]

where \(\tau_0 \in C^1(U_i, \widetilde{P}^{(\ell)}\) (coming from \(\partial_{\ell+1} \nabla^U_i f\)) and \(\sigma'_0 \in Z^1(U_i, \widetilde{Q}^{(\ell-1)}\) (the remainder in the complement of the splitting).

Step 2. Now, let \(\varphi^e := (\varphi^e_e)_{\ell} \) be the collection of structure morphisms \(\varphi^e_e := \varphi^e_e \to \widetilde{Q}_e\) (which are just inclusions for a functional \(\Delta\)-module, hence complexity-bounded). Let \(\sigma := \varphi^\ell \sigma_0, \tau := \varphi^\ell \tau_0\) and \(\sigma' := \varphi^\ell \sigma'_0\).

Applying \(\varphi^e\) to (16) gives \(d^U_i f = \partial_{\ell} \sigma' + \tau\). After re-arranging, this gives that

\[
\sigma' \in \partial_{\ell}^{-1}(B^1(U_i, \widetilde{Q}^{(\ell)}, \widetilde{P}^{(\ell)})).
\]

Since \(Y + U[k] = Z\) and \((Z, Y + U, U) \preceq (Z, Y, U)\), the hypotheses of our \(\preceq\)-induction now allow us to apply part (A) of Lemma 8.7. This gives

\[
\sigma' = \alpha + \partial_{\ell-1} \xi + d^U_i h + \kappa
\]

for some

\[
\alpha \in Z^1_{sp}(U_i, \widetilde{Q}^{(\ell-1)}, \widetilde{P}^{(\ell-1)}), \quad \xi \in Z^1(U_i, \widetilde{Q}^{(\ell-2)}, \widetilde{P}^{(\ell-2)}),
\]

\[
h \in \widetilde{Q}^{(\ell-1)} \quad \text{and} \quad \kappa \in C^1(U_i, \widetilde{P}^{(\ell-1)}).
\]
Applying $\partial_k$ to this decomposition of $\sigma'$ and substituting into our equation for $dU_i f$ gives
\[ dU_i f = \partial_k \alpha + (\tau + \partial_k \kappa) + dU_i \partial_k h. \]
Let us replace $\tau$ with $\tau + \partial_k \kappa$, which does not alter the fact that $\tau \in C^1(U_i, \bar{P}(\ell))$, and so assume that $\kappa = 0$.

**Step 3.** Since $\partial_k h \in \partial_k(\bar{Q}(\ell-1))$, we may also replace $f$ by $f - \partial_k h$ without altering our desired conclusion about $f$, and hence also assume that $h = 0$ above. This reduces us to the case in which
\[ dU_i f = \alpha + \tau \]
for some $\alpha \in Z_{sp}^1(U_i, \bar{Q}(\ell), \bar{P}(\ell))$ and $\tau' \in C^1(U_i, \bar{P}(\ell))$.

Since this is a relative coboundary equation for the direct sum $\bar{M}(\ell)$, and $\ell \leq k-1$, Lemma 8.6 gives some $f' \in \bar{Q}_{sp}^{(\ell)}$ for which $dU_i f = dU_i f'$ modulo $C^1(U_i, \bar{P}(\ell))$. Letting $f_1 := f - f'$, it suffices to prove that
\[ f_1 := \bar{Q}_{sp}^{(\ell)} + \partial_k(\bar{Q}(\ell-1)) + \bar{P}(\ell). \]
Our initial hypothesis that $f \in \partial_{\ell+1}^{-1}(\bar{P}(\ell+1))$ implies that
\[ \partial_{\ell+1} f_1 \in \partial_{\ell+1}(-f') + \bar{P}(\ell+1) \subseteq \bar{Q}_{sp}^{(\ell+1)} + \bar{P}(\ell+1), \]
and the choice of $f'$ gives $dU_i f_1 \in C^1(U_i, \bar{P}(\ell))$: that is, $f_1 \in Z^0(U_i, \bar{Q}(\ell), \bar{P}(\ell))$.

Since $(Z, Y + U_i, U) \not\subseteq (Z, Y, U)$, conclusion (D) of Lemma 8.8 gives that the homomorphism
\[ \partial_{\ell+1} + \text{id}_{\bar{P}(\ell+1)} : Z^0(U_i, \bar{Q}(\ell), \bar{P}(\ell)) \oplus \bar{P}(\ell+1) \rightarrow Z^0(U_i, \bar{Q}(\ell+1), \bar{P}(\ell+1)) \]
has strong s.-p. pre-images. Therefore we have
\[ \partial_{\ell+1} f_1 = \partial_{\ell+1} f_2 + p \]
for some
\[ f_2 \in Z_{sp}^0(U_i, \bar{Q}(\ell), \bar{P}(\ell)) \quad \text{and} \quad p \in \bar{P}(\ell+1). \]
It follows that $f_1 - f_2 \in \partial^{-1}_{\ell+1}(\bar{P}(\ell+1)) \cap Z^0(U_i, \bar{Q}(\ell), \bar{P}(\ell))$, and so part (C) of Lemma 8.8 gives
\[ f_1 - f_2 \in Z_{sp}^0(U_i, \bar{Q}(\ell), \bar{P}(\ell)) + \partial_k(Z^0(U_i, \bar{Q}(\ell-1), \bar{P}(\ell-1))) + \bar{P}(\ell). \]
Hence
\[ f = f' + (f_1 - f_2) + f_2 \in \bar{Q}_{sp}^{(\ell)} + \partial_k(\bar{Q}(\ell-1)) + \bar{P}(\ell), \]
as required. \qed
The proof of (b) will begin with another auxiliary lemma.

**Lemma 8.9.** Assume \((\text{Ind})^{Z,Y,U}\), \((a)\) and \((b)\), suppose that \((Z, Y, U)\) is not pure and \(Y \not\geq U_i\), and suppose that \(f \in \tilde{Q}^{(\ell)}\) and \(p \in \tilde{P}^{(\ell+1)}\) are such that

\[
g := \partial_{\ell+1}f + p
\]

is a step polynomial. Then there is an \(f' \in \tilde{G}_{s_p}^{(\ell)}\) such that

\[
g - \partial_{\ell+1}f' \in Z^0(U_i, \tilde{Q}^{(\ell+1)}, \tilde{P}^{(\ell+1)}).\]

**Proof.** Step 1. As above, we start by applying \(\tilde{\nabla}_{U_i}\) to this equation. This gives

\[
\tilde{\nabla}_{U_i}g = \partial_{\ell+1}\sigma + \tilde{\nabla}_{U_i}p \in Z^1_s(U_i, \tilde{Q}^{(\ell+1)}),
\]

where

\[
\sigma := \tilde{\nabla}_{U_i}f \in Z^1(U_i, \tilde{Q}^{(\ell)}).
\]

Re-arranging (17) gives

\[
\partial_{\ell+1}\sigma = \tilde{\nabla}_{U_i}g - \tilde{\nabla}_{U_i}p.
\]

Applying the splitting homomorphism \(\tilde{\nabla}^{(\ell+1)} \rightarrow \tilde{Q}^{(\ell)}\) from Lemma 7.9 to this, one obtains

\[
\sigma = \sigma' + \tau'_0 + \partial_{\ell}\sigma''_0, \quad \text{and hence} \quad \partial_{\ell+1}\sigma = \partial_{\ell+1}\sigma'_0 + \partial_{\ell+1}\tau'_0, \quad (19)
\]

where \(\sigma'_0 \in Z^1_s(U_i, \tilde{Q}^{(\ell)}\) (coming from \(\tilde{\nabla}_{U_i}g\)), \(\tau'_0 \in C^1(U_i, \tilde{P}^{(\ell)}\) (coming from \(\tilde{\nabla}_{U_i}p\)) and \(\sigma''_0 \in Z^1(U_i, \tilde{Q}^{(\ell-1)}\) (the remainder in the complement of the splitting).

Step 2. Let \(\varphi^+ = (\varphi^+)_e\) be as in the previous proof, and let \(\sigma' := \varphi^\ast\sigma'_0\), \(\tau' := \varphi^\ast\tau'_0\) and \(\sigma'' := \varphi^\ast\sigma''_0\). Applying \(\varphi^+\) to (17) and (19) and combining them, we obtain

\[
d^{U_i}g = \partial_{\ell+1}\sigma' + \partial_{\ell+1}\tau' + d^{U_i}p, \quad (20)
\]

where we still have \(\sigma' \in Z^1_s(U_i, \tilde{Q}^{(\ell)}\). On the other hand, applying \(\varphi^+\) to (18) and (19) gives

\[
\sigma' = \varphi^\ast\sigma'_0 = \partial_{\ell}(-\sigma'') + (d^{U_i}f - \tau') \in \partial_{\ell}(Z^1(U_i, \tilde{Q}^{(\ell-1)})) + B^1(U_i, \tilde{Q}^{(\ell)}\), \tilde{P}^{(\ell)}).
\]
Step 3. Since \( \sigma' \) is a step polynomial, we can combine the last equation above with part (B) of Lemma 8.7 to obtain that
\[
\sigma' = \partial_\ell \sigma''' + d^U_i f' + \tau''
\]
for some
\[
\sigma''' \in Z^1_{sp}(U_i, \tilde{Q}_s^{(\ell-1)}, \tilde{P}^{(\ell-1)}), \quad f' \in \tilde{Q}_s^{(\ell)} \quad \text{and} \quad \tau'' \in C^1_{sp}(U_i, \tilde{P}^{(\ell)}).
\]
Substituting this new decomposition of \( \sigma' \) into (20) gives
\[
d^U_i g = d^U_i \partial_\ell f' + (\partial_\ell (\tau' + \tau'') + d^U_i p).
\]
This now implies that \( d^U_i (g - \partial_\ell f') \) is \( \tilde{P}^{(\ell+1)} \)-valued, as required.

Proof of (b). Again, we restrict our attention to the non-pure case, and let \( i \in [k] \) be such that \( Y \not\subseteq U_i \).

Step 1. Suppose now that \( g \in \tilde{Q}_s^{(\ell+1)} \) and
\[
g = \partial_\ell f + p \quad \text{for some} \quad f \in \tilde{Q}_s^{(\ell)}, \quad p \in \tilde{P}^{(\ell+1)}. \tag{21}
\]
Letting \( f' \) be as given by Lemma 8.9 we see that \( g' := g - \partial_\ell f' \) is still a step polynomial, satisfies the same assumptions as \( g \), and lies in \( Z^0(U_i, \tilde{Q}_s^{(\ell+1)}, \tilde{P}^{(\ell+1)}) \).

We may therefore assume that \( g \) actually lies in this smaller module.

To finish the proof, we will reduce this further to the case when in addition \( f \in Z^0(U_i, \tilde{Q}_s^{(\ell)}, \tilde{P}^{(\ell)}) \). Having done so, our task is completed by conclusion (D) of Lemma 8.8.

Step 2. Let
\[
R := \partial_{\ell+1}^{-1}(Z^0(U_i, \tilde{Q}_s^{(\ell+1)}, \tilde{P}^{(\ell+1)}))
\]
and
\[
P' := \partial_{\ell+1}(Z^0(U_i, \tilde{Q}_s^{(\ell)}, \tilde{P}^{(\ell)})) + \tilde{P}^{(\ell+1)}.
\]
Then the required reduction will follow if we show that the homomorphism
\[
\Phi : R \oplus P' \longrightarrow Z^0(U_i, \tilde{Q}_s^{(\ell+1)}, \tilde{P}^{(\ell+1)}) \cap \partial_{\ell+2}^{-1}(\tilde{P}^{(\ell+2)}) : (r, p') \mapsto \partial_{\ell+1} r + p'
\]
has strong s.-p. pre-images. Indeed, if \( g = \partial_\ell f + p \) as in (21) is a step polynomial and lies in \( Z^0(U_i, \tilde{Q}_s^{(\ell+1)}, \tilde{P}^{(\ell+1)}) \), then this conclusion about \( \Phi \) will enable us to remove \( \partial_\ell r \) for some \( r \in R_{sp} \), and so assume that \( g = \partial_\ell f' + p' \) for some \( f' \in Z^0(U_i, \tilde{Q}_s^{(\ell)}, \tilde{P}^{(\ell)}) \), as required for the reduction.

Step 3. However, with the reformulation in Step 2, it suffices to make the required reduction under the extra assumption that \( Y + U_{[k]} = Z \). This is because doing so will verify the hypotheses of Corollary 4.22 along with the following observations:
• if \( Y + U_{[k]} = Z \) then the inclusion
\[
P' \leq Z^0(U_i, \tilde{Q}^{(\ell+1)}, \tilde{P}^{(\ell+1)}) \cap \partial_{\ell+2}(\tilde{P}^{(\ell+2)})
\]
has discrete quotient, by the modesty of \( \tilde{M}^{U_i} \), and
• the boundary homomorphism \( \partial_{\ell} \) is complexity-bounded.

**Step 4.** So now assume \( Y + U_{[k]} = Z \). Since \( d^{U_i}g \) takes values in \( \tilde{P}^{(\ell+1)} \), we have
\[
\sigma := d^{U_i}f \in Z^1(U_i, \partial_{\ell+1}^{-1}(\tilde{P}^{(\ell+1)})).
\]
Now, the modesty of \( \tilde{M} \) and conclusion (a) (which applies in case \( Y + U_{[k]} = Z \))
give that the semi-functional module
\[
\tilde{Q}^{I_{\ell-1}} = \partial_{\ell}(\tilde{Q}^{(\ell-1)}) + \tilde{P}^{(\ell)} \leq \partial_{\ell+1}^{-1}(\tilde{P}^{(\ell+1)})
\]
has discrete quotient, and has strong s.-p. representatives. Also, conclusion (iv)\(_{\ell-1}\) of Proposition 8.4 (which is included in \( \text{Ind}^{Z,Y,U,\ell} \))
gives that the inclusion
\[
\tilde{P}^{(\ell)} \leq \tilde{Q}^{I_{\ell-1}}
\]
admits strong s.-p. coboundary solutions. We may therefore apply Lemma 5.30 to conclude that
\[
d^{U_i}f = \sigma_1 + \partial_{\ell}\tau_1 + d^{U_i}f_1 + p_1,
\]
where
\[
\sigma_1 \in Z^1_{sp}(U_i, \partial_{\ell+1}^{-1}(\tilde{P}^{(\ell+1)}), \tilde{P}^{(\ell)}), \quad \tau \in Z^1(U_i, \tilde{Q}^{I_{\ell-1}}, \tilde{P}^{(\ell)})
\]
and \( f_1 \in \partial_{\ell+1}^{-1}(\tilde{P}^{(\ell+1)}) \).

Also, conclusion (vi)\(_{\ell-1}\) of Proposition 8.4 allows us to decompose
\[
\tau = \partial_{\ell}\tau_1 + \tau' + d^{U_i}f' + p_1
\]
for some
\[
\tau_1 \in Z^1(U_i, \tilde{Q}^{(\ell-1)}, \tilde{P}^{(\ell-1)}), \quad \tau' \in Z^1_{sp}(U_i, \tilde{Q}^{I_{\ell-1}}, \tilde{P}^{(\ell)}),
\]
\[
f' \in \tilde{Q}^{I_{\ell-1}} \quad \text{and} \quad p_1 \in C^1(U_i, \tilde{P}^{(\ell)}).
\]

**Step 5.** Comparing with our decomposition of \( d^{U_i}f \), we may now combine \( \tau' \) with \( \sigma_1 \) and \( f' \) with \( f_1 \) to obtain
\[
d^{U_i}f = \sigma_1 + \partial_{\ell}\tau_1 + d^{U_i}f_1 + p_1,
\]
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where $\sigma_1$, $\tau_1$, $f_1$ and $p_1$ have the same properties as above.

Replacing $f$ with $f - f_1$, which does not change its $\partial_{\ell+1}$-image modulo $\tilde{P}(\ell+1)$, we may also assume that $f_1 = 0$.

Re-arranging again, this now asserts that

$$-\sigma_1 = \partial_{\ell} \tau_1 + d^{U_i}(-f) + p_1$$

is a step polynomial, so by part (B) of Lemma [8.7] we obtain step-polynomial counterparts $\tau'_1$, $f'$ and $p'_1$ such that

$$\partial_{\ell} \tau_1 + d^{U_i}(-f) + p_1 = \partial_{\ell} \tau'_1 + d^{U_i}(-f') + p'_1.$$

In the first place, this implies that $d^{U_i} \partial_{\ell+1} f' = d^{U_i} \partial_{\ell+1} f = d^{U_i} g$ modulo $\tilde{P}(\ell+1)$, so $\partial_{\ell+1} f' \in Z^0(U_i, \tilde{Q}(\ell+1), \tilde{P}(\ell+1))$, as well as being a step polynomial. We may therefore replace $g$ with $g - \partial_{\ell+1} f'$ and $f$ with $f - f'$, and so assume that $f' = 0$.

**Step 6.** Having done so, we now obtain $d^{U_i} f = \partial_{\ell} \tau_3 + p_3$ for some $\tau_3 \in Z^1(U_i, \tilde{Q}(\ell), \tilde{P}(\ell))$ and $p_3 \in C^1(U_i, \tilde{P}(\ell))$. Therefore $\tau_3$ defines a cohomology class in the kernel of

$$H^1_m(U_i, \tilde{M}^{(\ell-1)}) \to H^1_m(U_i, \tilde{M}^{(\ell)}).$$

By part (A) of Lemma [8.7] this gives that

$$\partial_{\ell} \tau_3 = \partial_{\ell} (\tau_4 + d^{U_i} h) \mod C^1(U_i, \tilde{P}(\ell))$$

for some $\tau_4 \in Z^1_{sp}(U_i, \tilde{Q}(\ell-1), \tilde{P}(\ell-1))$ and $h \in \tilde{Q}(\ell-1)$. We may therefore adjust $p_3$ above, and so write $\tau_4 + d^{U_i} h$ in place of $\tau_3$.

**Step 7.** Finally, this gives that the relative coboundary

$$d^{U_i} (f - \partial_{\ell} h) - p_4$$

is a step polynomial. This falls into a case of Theorem [8.2] contained in Assumption (Ind)$_{Z,Y,U}$, which implies that there is another step polynomial $f'' \in \tilde{Q}(\ell)$ such that $d^{U_i} (f - \partial_{\ell} h) = d^{U_i} f''$ modulo $\tilde{P}(\ell)$. Once again, this means we may replace $g$ with $g - \partial_{\ell+1} f''$ and $f$ with $f - f'' - \partial_{\ell} h$, and so finally make the reduction to the case $f \in Z^0(U_i, \tilde{Q}(\ell), \tilde{P}(\ell))$.

8.3 Completing the inductive proof

This subsection will complete the inductive proof of Proposition [8.4] and Theorems [8.1] and [8.2].

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As explained in Subsection 8.1, we will proceed by an outer induction on \((Z, Y, U)\) (using the order \(\preceq\)) and an inner induction on \(\ell\). This will follow a similar pattern to the proof of modesty in Subsection 1.9.3.

The steps of this induction, including the base clause, are of two kinds:

- **(Outer induction steps)** In some cases, given \((Z, Y, U)\), our inductive hypothesis is that Proposition 8.4 holds for any stunted almost meek semi-functional \((Z_1, Y_1, U_1)\)-\(\Delta\)-module for which \((Z_1, Y_1, U_1) \preceq (Z, Y, U)\), and we must show that it holds for any \((\ell_0, \ell_0)\)-stunted almost meek semi-functional \((Z, Y, U)\)-\(\Delta\)-module. We can interpret the base clause of the overall induction as falling into this case, by letting the inductive hypothesis be empty.

- **(Inner inductive steps)** In the remaining cases, we shall again adopt assumption \((\text{Ind})^{Z, Y, U, \ell}\), where \(U\) is a \(k\)-tuple and \(\ell \leq k - 1\). From this we will deduce the next instance of Proposition 8.4, which applies to \((\ell_0, \ell_0 + 1)\)-stunted almost meek semi-functional \((Z, Y, U)\)-\(\Delta\)-modules. Proposition 8.5 will be used in the course of this deduction.

These two kinds of step will be explained separately below, and then it will be shown how they combine to give the full inductive proof of Proposition 8.4 and Theorems 8.1 and 8.2.

All the structures appearing in these arguments are already co-induced from \(Y + W + U_{[k]}\) to \(Z\), so we will not trouble to introduce a further enlargement of \(Z\), even though our assertions are mostly about strong s.-p. pre-images.

**Outer induction steps**

Given \((Z, Y, U)\), suppose that Proposition 8.4 is known for any directing tuple of groups which strictly precedes \((Z, Y, U)\) in the order \(\preceq\), and suppose that \(\mathcal{P} \leq \mathcal{M}\) is an \((\ell_0, \ell_0)\)-stunted almost meek semi-functional \(\Delta\)-module for some \(\ell_0 \leq k\).

In this case, assertions (i)\(\ell_0 - 1\), (ii)\(\ell_0 - 1\), (iii)\(\ell_0 - 1\), (iv)\(\ell_0 - 1\) and (vi)\(\ell_0 - 1\) hold vacuously, because \(M(\ell_0 - 1) = I_{\ell_0 - 1} = (0)\). It remains to prove the others.

**Proof of (v)\(\ell_0 - 1\), (vii)\(\ell_0 - 1\) and (viii)\(\ell_0 - 1\).** If \(\ell = \ell_0 - 1\), then \(M(\ell) = 0\). The assumption of \((\ell_0, \ell_0)\)-stunted almost meekness for \(\mathcal{P} \leq \mathcal{M}\) itself implies that

\[ P(\ell_0) \leq Q^{K_{\ell_0}} \rightarrow K_{\ell_0} \]

is co-induced from a semi-functional \((Y + U_{[k]})\)-module which has s.-p. quotient and strong s.-p. representatives. Therefore, \(\text{Proof of (v)\(\ell_0 - 1\)}\) is an instance of Corollary 5.28.
and (vii)\(_{\ell_0-1}\) follows by the special case of Lemma 5.30 in which the smaller and middle modules are equal.

On the other hand, conclusion (viii)\(_{\ell_0-1}\) asserts that

\[ Z_p(W, \mathcal{Q}_K^{\ell_0}, P^{(\ell_0)}) \cap B_p(W, \mathcal{Q}^{(\ell_0)}, P^{(\ell_0)}) \rightarrow \ker \left( \text{H}^p(K_{\ell_0}) \rightarrow \text{H}^p(M^{(\ell_0)}) \right) \]

has s.-p. target and strong s.-p. representatives. This is simply the restriction of the quotient homomorphism

\[ \Phi : Z_p(W, \mathcal{Q}_K^{\ell_0}, P^{(\ell_0)}) \rightarrow \text{H}^p(K_{\ell_0}) \]

to the closed submodule

\[ \Phi^{-1} \left( \ker \left( \text{H}^p(K_{\ell_0}) \rightarrow \text{H}^p(M^{(\ell_0)}) \right) \right) \leq Z_p(W, \mathcal{Q}_K^{\ell_0}, P^{(\ell_0)}), \]

so both of the desired conclusions follow from (vii)\(_{\ell_0-1}\). □

**Inner induction steps**

Now suppose that \( \ell_0 \leq \ell \leq k - 1 \), adopt assumption (Ind)\(_{Z,Y,U,\ell}\), and let \( \mathcal{P} \leq \mathcal{Q} \rightarrow \mathcal{M} \) be \((\ell_0, \ell+1)\)-stunted almost meek. Of course, this implies that it is also \((\ell_0, \ell)\)-stunted almost meek.

Underlying the proofs in this case is a careful analysis of the homomorphism

\[ \text{H}^p_{\text{in}}(W, I_\ell) \rightarrow \text{H}^p_{\text{in}}(W, K_{\ell+1}) \]

at the level of individual relative cocycles, based on the results about semi-functional inclusions in Subsection 5.3.

**Proof of (i)\(_\ell\) and (ii)\(_\ell\).** Assumption (Ind)\(_{Z,Y,U,\ell}\) gives that

\[ B^p(W, \mathcal{Q}, \mathcal{P}) \leq Z^p(W, \mathcal{Q}, \mathcal{P}) \rightarrow \text{H}^p_{\text{in}}(W, \mathcal{M}) \]

is \((\ell_0, \ell)\)-stunted almost meek, and actually \((\ell_0, \ell)\)-stunted meek for \( p \geq 1 \), because it is strictly modest by Theorem I.9.5.

This now gives the hypotheses needed to apply Proposition 8.5. The conclusions (a) and (b) for the above presentation of \( \text{H}^p_{\text{in}}(W, \mathcal{M}) \) are precisely (i)\(_\ell\) and (ii)\(_\ell\), respectively. □

**Proof of (iii)\(_\ell\).** This conclusion is trivial if \( p = 0 \), so suppose \( p \geq 1 \). Consider the composition

\[ \text{H}^p(M^{(\ell-1)}) \rightarrow \text{H}^p(K_{\ell}) \rightarrow \text{H}^p(M^{(\ell)}), \]

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and let

$$A := \ker \left( H^{p}(M^{(\ell - 1)}) \rightarrow H^{p}(K_{\ell}) \right)$$

and

$$B := \ker \left( H^{p}(M^{(\ell - 1)}) \rightarrow H^{p}(M^{(\ell)}) \right).$$

Since $p \geq 1$, we obtain a map $\psi : A \rightarrow B$ which is co-induced over $Y + W + U_{[k]}$ from a homomorphism of discrete $(Y + W + U_{[k]})$-modules.

Let $[\sigma]$ be the image in $A$ of the class $[\sigma] \in H^{p}(K_{\ell})$, and let $[\tilde{\sigma}]$ be its further image in $B$. That further image is equal $\psi([\sigma])$, but is also equal to the image of any other element of $Z^{p}(W, Q^{(\ell)}, P^{(\ell)})$ which is cohomologous to $\sigma$.

By assumption, there is such another element of $Z^{p}(W, Q^{(\ell)}, P^{(\ell)})$ which is a step polynomial. In addition, the quotient map

$$Z^{p}(W, Q^{(\ell)}, P^{(\ell)}) \rightarrow B$$

is co-induced from a quotient map of a semi-functional $(Y + W + U_{[k]})$-module with discrete quotient, by the modesty of $H^{p}(.\mathcal{M})$ (Theorem I.9.5). Therefore Proposition 4.17 implies that $[\tilde{\sigma}]$ is a step-polynomial in $\psi(A) \leq B$.

Therefore, by Corollary 4.16, there is another class $[\tau] \in A$ which is a step polynomial and such that $\psi([\tau]) = \psi([\sigma])$.

However, now conclusion (vii) implies that for this last conclusion, we may actually choose $\tau$ itself to lie in $Z^{p}_{sp}(W, Q^{K_{\ell}}, P^{(\ell)})$. The conclusion of equal $\psi$-images gives that $\sigma$ is cohomologous in $Z^{p}(W, Q^{(\ell)}, P^{(\ell)})$ to $\tau + \partial_{\ell} \kappa$ for some $\kappa \in Z^{p}(W, Q^{(\ell - 1)}, P^{(\ell - 1)})$, as required.

\[\Box\]

**Proof of (iv), using (iii).** Once again, this conclusion is trivial if $p = 0$, so suppose $p \geq 1$.

Suppose $f \in C^{p-1}(W, Q^{I_{\ell}})$ and $p \in C^{p}(W, P^{(\ell + 1)})$ are such that $\sigma := d^{W} f + p$ is a step polynomial. Since $Q^{I_{\ell}} = \partial_{\ell + 1}(Q^{(\ell)}) + P^{(\ell + 1)}$, and this sum of modules is Polish, we may measurably decompose $f$ into two corresponding summands. By absorbing the second of these summands into $p$, we may therefore assume that $f = \partial_{\ell + 1} \tilde{f}$ for some measurable $\tilde{f} : W^{p-1} \rightarrow Q^{(\ell)}$.

Because $\mathcal{P} \leq \mathcal{Q} \rightarrow \mathcal{M}$ admits strong s.-p. representatives (by the assumption of its meekness), we can write $\sigma = \partial_{\ell + 1} (\tilde{\sigma}) + \tau$ for some $\tilde{\sigma} : W^{p} \rightarrow Q^{(\ell)}$ and $\tau : W^{p} \rightarrow P^{(\ell)}$ that are both step polynomials. (We are not yet able to guarantee that the lift $\tilde{\sigma}$ is still a relative cocycle.)

**Step 1.** We may now replace $p$ with $p + \tau$, and so write

$$d^{W} \partial_{\ell + 1} \tilde{f} = \partial_{\ell + 1} (\tilde{\sigma}) + p \implies \partial_{\ell + 1} (d^{W} \tilde{f} - \tilde{\sigma}) = p,$$

(22)
so \( \sigma' := d^W \tilde{f} - \bar{\sigma} \) must take values in \( Q^{K_\ell} \). Its coboundary is \( d^W \sigma' = -d^W \bar{\sigma} \), which is a step polynomial.

Therefore, by \((v)_{\ell-1}\), there is some step polynomial \( \sigma'' : W^p \to Q^{K_{\ell+1}} \) with \( d^W \sigma'' = d^W \sigma' \mod P(\ell) \).

Since \( \sigma'' \) takes values in \( Q^{K_{\ell+1}} \), we have \( \partial_{\ell+1}(\bar{\sigma} + \sigma'') = \partial_{\ell+1}(\bar{\sigma}) \). We may therefore replace \( \bar{\sigma} \) with \( \bar{\sigma} + \sigma'' \), and hence assume that \( d^W \bar{\sigma} \in C^{p+1}(W, P(\ell)) \), i.e., that \( \bar{\sigma} \in \mathbb{Z}^p_{sp}(W, Q(\ell), P(\ell)) \).

**Step 2.** Consider again \( \sigma' = d^W \tilde{f} - \bar{\sigma} \). The conclusion of Step 1 asserts that under the homomorphism

\[
H^p(K_\ell) \to H^p(M(\ell)),
\]

this \( \sigma' \) has image which is equivalent to the step-polynomial relative cocycle \( \bar{\sigma} \). Therefore \((iii)_\ell\) gives \( \sigma'' \in \mathbb{Z}^p_{sp}(W, Q^{K_\ell}, P(\ell)) \) and \( \kappa \in \mathbb{Z}^p(W, Q^{(\ell-1)}, P(\ell-1)) \) for which

\[
\bar{\sigma} - \sigma'' - \partial_\ell \kappa = (-\sigma' - \sigma'' - \partial_\ell \kappa) + d^W f \in B^p(W, Q(\ell), P(\ell)).
\]

**Step 3.** The end of Step 2 implies that \( \partial_\ell \kappa \) agrees with a step polynomial relative cocycle modulo \( B^p(W, Q(\ell), P(\ell)) \). Now property \((ii)_{\ell-1}\) gives some \( \kappa' \in \mathbb{Z}^p_{sp}(W, Q^{(\ell-1)}, P(\ell-1)) \) such that \( \partial_\ell \kappa = \partial_\ell \kappa' \mod B^p(W, Q(\ell), P(\ell)) \).

**Step 4.** Finally, this implies that \( \bar{\sigma} - \sigma'' - \partial_\ell \kappa' \) is both an element of \( B^p(W, Q(\ell), P(\ell)) \), and a step polynomial. It therefore falls within the domain of Theorem [5,2] applied to each direct summand in the semi-functional module \( P(\ell) \leq Q(\ell) \to M(\ell) \). Since \( \ell \leq k - 1 \), all of those direct summands arise from nontrivial restrictions of \( \mathcal{P} \leq \mathcal{M} \), and so the hypothesis of our overall \( \leq \)-induction include that case of Theorem [5,2]. It gives that \( \bar{\sigma} - \sigma'' - \partial_\ell \kappa' = d^W f'' + g \) for some \( f'' \in C^{p-1}_{sp}(W, Q(\ell)) \) and \( g \in C^p_{sp}(W, P(\ell)) \). Applying \( \partial_\ell + 1 \), and recalling that \( \sigma'' \) is \( Q^{K_\ell}\)-valued, this now gives

\[
\sigma = \partial_{\ell+1}(\bar{\sigma}) + \tau = d^W \partial_{\ell+1}(f'') + (\partial_{\ell+1}(g) + \tau),
\]

which is of the form desired. \( \square \)

**Proof of \((v)_\ell\), using \((iv)_{\ell}\).** This conclusion is trivial if \( p = 0 \), so suppose \( p \geq 1 \). The semi-functional inclusion \( Q^{I_{\ell}} \hookrightarrow Q^{K_{\ell+1}} \) is co-induced from a semi-functional \((Y + U_{[k]})\)-module with discrete quotient, since \( \ell_0 \leq \ell \). Therefore we may apply Proposition [5,3] to it. Given \( \sigma \in B^p_{sp}(W, Q^{K_{\ell+1}}, P(\ell+1)) \), let \( f \in C^{p-1}_{sp}(W, Q^{K_{\ell+1}}) \) be the function provided by that proposition. This gives that

\[
\sigma - d^W f \in B^p_{sp}(W, Q^{I_{\ell}}, P(\ell+1)).
\]
Since it suffices to prove the desired conclusion for \( \sigma - d^W f \) in place of \( \sigma \), our assumption of (iv) completes the proof.

Proof of (vi). All modules appearing here are co-induced over \( Y + W + U_{[k]} \). In particular, by conclusion (i) of Proposition I.9.12, the target cokernel here is co-induced from a discrete (and hence s.-p.) \( (Y + W + U_{[k]}) \)-module. Therefore, by Lemma 4.21 strong s.-p. representatives will follow if one only obtains s.-p. representatives in the case \( Z = Y + W + U_{[k]} \), so we now make that extra assumption.

Let \( \sigma \in Z^p(W, Q^{I_\ell}, P^{(\ell+1)}) \), and choose a lift \( \alpha \in C^p(W, Q^{I_\ell}, P^{(\ell+1)}) \) so that \( \sigma = \partial_{\ell+1}(\alpha) \). Since \( d^W \alpha \) is a cocycle taking values in \( K_\ell \), by (viii) \( \ell - 1 \) it equals \( \tau + d^W \beta \) modulo \( C^{p+1}(W, P^{(\ell)}) \) for some \( \tau \in Z_{sp}^p(W, Q^{I_\ell}, P^{(\ell+1)}) \) and some \( \beta \in C^p(W, Q^{K_\ell}) \).

Now an appeal to Lemma 8.6 gives \( h \in C_{sp}^p(W, Q^{I_\ell}) \) such that \( \tau = d^W h \) modulo \( C^{p+1}(W, P^{(\ell)}) \) (recall that \( \ell \leq k - 1 \) in Proposition 8.4).

It now suffices to show that \( \partial_{\ell+1}(\alpha - h) \) is a \( (Q^{I_\ell}, P^{(\ell+1)}) \)-valued relative cocycle whose cohomology class lies in the image of the homomorphism

\[
H^p_m(W, M^{(\ell)}) \to H^p_m(W, I_\ell),
\]

for this will imply that the step polynomial \( \partial_{\ell+1} h \) has the same image as \( \sigma \) in the target cokernel of (vi). However, \( \partial_{\ell+1}(\alpha - h) = \partial_{\ell+1}(\alpha - h - \beta) \), because \( \partial_{\ell+1}(\beta) = 0 \), and for this the above relations give

\[
d^W (\alpha - h - \beta) \in C^{p+1}(W, P^{(\ell)}), \quad \text{i.e.} \quad \alpha - h - \beta \in Z^p(W, Q^{I_\ell}, P^{(\ell+1)}).
\]

Proof of (vii), using (iv) \( \ell \lor (vi) \). Since \( \ell \geq \ell_0 \), part (iii) of Proposition I.9.12 has shown that the target cokernel here is co-induced from a discrete (and hence s.-p.) \( (Y + W + U_{[k]}) \)-module. It therefore suffices to prove s.-p. representatives in case \( Z = Y + W + U_{[k]} \).

Let \( D := B^p(W, Q^{K_{\ell+1}}, P^{(\ell+1)}) \). By Lemma 4.13 one need only obtain s.-p. representatives for the quotients corresponding to both of the inclusions

\[
\partial_{\ell+1}(Z^p(W, Q^{I_\ell}, P^{(\ell)})) + D \leq Z^p(W, Q^{I_\ell}, P^{(\ell+1)}) + D \leq Z^p(W, Q^{K_{\ell+1}}, P^{(\ell+1)}),
\]

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where all these modules are known to be closed because they are continuous pre-images of closed subgroups of the Hausdorff group $H^p(K_{\ell+1})$, by another appeal to Proposition I.9.12.

S.-p. representatives for the first of these quotients follow by Lemma 4.20 and property (vi). For the second quotient, they are given by Lemma 5.30. The conditions of that lemma are satisfied because $Q^K_{\ell+1} \to K_{\ell+1}/I_{\ell}$ has s.-p. target and strong s.-p. representatives, by the almost meekness of $\mathcal{M}$, and because (iv)$\ell$ gives that $P^{(\ell+1)} \leq Q^{I_{\ell}}$ admits s.-p. coboundary solutions.

\[\square\]

Proof of (viii)$\ell$, using (i)$\ell \lor (ii)$ and (vii)$\ell$. The kernel in question is trivial in case $\ell = 0$, so assume $\ell \geq 1$. In that case it is co-induced from a discrete (hence s.-p.) $(Y + W + U_{[k]})$-module (part (v)$\ell$ of Proposition I.9.12), so, as previously, we may assume $Z = Y + W + U_{[k]}$ in looking for s.-p. representatives.

Suppose that $\sigma \in Z^p(W, Q^{K_{\ell+1}}, P^{(\ell+1)}) \cap B^p(W, Q^{(\ell+1)}, P^{(\ell+1)})$.

Step 1. By property (vii)$\ell$, we may decompose

$\sigma = \sigma_1 + \partial_{\ell+1}(\tau) + d^W \beta$

with

$\sigma_1 \in Z^p_{sp}(W, Q^{K_{\ell+1}}, P^{(\ell+1)})$, \quad $\tau \in Z^p(W, Q^{(\ell)}, P^{(\ell)})$

and

$\beta \in C^{p-1}(W, Q^{K_{\ell+1}}, P^{(\ell+1)})$.

Step 2. Since $\sigma$ itself represents an $M^{(\ell+1)}$-valued coboundary, we must have

$\partial_{\ell+1}(\tau) \in -\sigma_1 + B^p(W, Q^{(\ell+1)}, P^{(\ell+1)})$.

Therefore property (ii)$\ell$ (applicable because $\ell \geq 1$) gives some $\tau_1 \in Z^p_{sp}(W, Q^{(\ell)}, P^{(\ell)})$ such that

$\partial_{\ell+1}(\tau) = \partial_{\ell+1}(\tau_1) \mod B^p(W, Q^{(\ell+1)}, P^{(\ell+1)})$,

so that $\tau - \tau_1$ represents a class in

$\ker \left( H^p_m(W, M^{(\ell)}) \to H^p_m(W, M^{(\ell+1)}) \right)$.

By property (i)$\ell$ (applicable because $\ell \geq 1$), this kernel has s.-p. representatives over

$\text{img}(H^p_m(W, M^{(\ell-1)}) \to H^p_m(W, M^{(\ell)}))$.

Therefore we may write

$\tau = \tau_1 + \tau_2 + \partial_\ell(\gamma) + d^W \alpha$.

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for some $\tau_2 \in Z_{sp}(W, Q^{(\ell)}, P^{(\ell)})$, $\gamma \in Z^p(W, Q^{(\ell-1)}, P^{(\ell-1)})$ and $\alpha \in C^{p-1}(W, Q^{(\ell-1)}, P^{(\ell-1)})$.

**Step 3.** Combining these expressions for $\sigma$ and $\tau$ and applying $\partial_{\ell+1}$, one obtains

$$\sigma = \sigma_1 + \partial_{\ell+1}(\tau) + dW\beta = \sigma_1 + \partial_{\ell+1}(\tau_1 + \tau_2) + dW(\beta + \partial_{\ell+1}(\alpha)).$$

So $\sigma$ is cohomologous in $H^p_m(W, K_{\ell+1})$ to

$$\sigma_1 + \partial_{\ell+1}(\tau_1 + \tau_2) \in Z^p(W, Q^{K_{\ell+1}}, P^{(\ell+1)}),$$

which is also a step polynomial.

**Completion**

**Completed proof of Proposition 8.4 and Theorems 8.2 and 8.1** These are proved together by an outer induction on $(Z, Y, U)$.

First, given $(Z, Y, U)$, if all three results are known for any stunted, almost or strictly meek semi-functional $(Z_1, Y_1, U_1)$-$\Delta$-module for which $(Z_1, Y_1, U_1) \not\subset (Z, Y, U)$, then the outer induction step above proves Proposition 8.4 for any $(\ell_0, \ell_0)$-stunted almost meek semi-functional $(Z, Y, U)$-$\Delta$-module. This includes the base clause of our outer induction, for which the assumption here is vacuous.

Next, having shown this, we can run an inner induction on $\ell = \ell_0, \ldots, k$ using all the implications above. This proves that Proposition 8.4 actually holds for any $(\ell_0, \ell)$-stunted almost meek semi-functional $(Z, Y, U)$-$\Delta$-module for which $\ell \leq k$.

To complete the next step of the outer induction, and hence the proof, let us see why Proposition 8.4 for this $(Z, Y, U)$ and $\ell = k$ implies Theorems 8.1 and 8.2 for this $(Z, Y, U)$.

First, Theorem 8.2 is given by property (v) in Proposition 8.4 since $K_k = M_{[k]}$.

In case $p \geq 1$, strong s.-p. representatives are given in the top structure homology of $H^p_m(W, \mathcal{M})$ by properties (i)$_\ell$, $\ell_0 - 1 \leq \ell \leq k - 1$. In case $p = 0$, this assertion for $\ell \leq k - 1$ is that the inclusion

$$\text{img}((M^{(\ell-1)})^W \rightarrow (M^{(\ell)})^W) \leq \ker((M^{(\ell)})^W \rightarrow (M^{(\ell+1)})^W) = K^W_\ell$$

has strong s.-p. representatives, and this is given by property (vii)$_{\ell-1}$ of Proposition 8.4.

Next, strong s.-p. representatives are given at the last position of the top structure complex by property (vii)$_{k-1}$, since $K_k = M_{[k]}$. 99
Lastly, we must prove finite-complexity decompositions. If \( p \geq 1 \) then this is given by (ii) of Proposition 8.4 for \( \ell_0 \leq \ell \leq k \), so suppose \( p = 0 \), and suppose \((f, p) \in Z^0(W, Q^{(\ell)}, P^{(\ell)}) \oplus P^{(\ell+1)}\) are such that \( \partial_{\ell+1} f + p \) is a step polynomial for some \( \ell \leq k - 1 \).

By finite-complexity decompositions in \( \mathcal{P} \leq \mathcal{Z} \to \mathcal{M} \) itself, there are some \( g \in Q^{(\ell)} \) and \( p' \in P^{(\ell+1)} \) such that \( \partial_{\ell+1} f + p = \partial_{\ell+1} g + p' \). Hence \( k := f - g \) lies in \( Q^{K_\ell} \), and so \( d^W k = -d^W g \) is, on the one hand, a \( Q^{K_\ell} \)-valued relative coboundary, and on the other a step polynomial. Therefore property (v) \( \ell - 1 \) of Proposition 8.4 gives \( d^W k = d^W k' + p'' \) for some \( k' \in Q^{K_\ell} \) and \( p'' \in P^{(\ell)} \). Therefore \( d^W (g + k') = 0 \), and hence \( g + k' \in Z^0(W, Q^{(\ell)}, P^{(\ell)}) \) is the desired s.-p. pre-image. \( \square \)

9 Partial difference equations and zero-sum tuples

In Part I, PD\(^{\text{ce}}\)E-solution \( \Delta \)-modules and zero-sum \( \Delta \)-modules were shown to be 1-almost and 2-almost modest, respectively, and this implied the main structural results of that paper. This was proved by showing how a general PD\(^{\text{ce}}\)E-solution \( \Delta \)-module, resp. zero-sum \( \Delta \)-module, can be constructed out of simpler \( \Delta \)-modules using cohomology and short exact sequences: since all of those preserve the desired almost modest structure, the result followed.

We will now prove Theorems A and B along similar lines, using the general results of the preceding sections about the preservation of almost meekness.

**Proof of Theorem A.** Proposition 3.43 implies that if \( A \) is compact-by-discrete, then it is s.-p.; this is the key assumption for the proof.

Fix \( Z \) and \( U = (U_1, \ldots, U_k) \), and let us recall the description of the solution \( \Delta \)-module for the resulting PD\(^{\text{ce}}\)E given in Subsection I.1.1. For each \( j \in \{0, 1, \ldots, k\} \), let \( U^{(j)} = (U^{(j)}_\ell)_{\ell=1}^k \) be the subgroup tuple defined by

\[
U^{(j)}_\ell := \begin{cases} 
U_\ell & \text{if } \ell \leq j \\
\{0\} & \text{if } \ell > j,
\end{cases}
\]

and let \( \mathcal{M}^j \) be the solution \( \Delta \)-module of the PD\(^{\text{ce}}\)E directed by \( U^{(j)} \). In particular, \( \mathcal{M}^k \) is the \( \Delta \)-module we wish to analyze.

It follows from this definition that

\[
M^0_0 = 0 \quad \text{and} \quad M^0_e = F(Z, A) \quad \text{whenever } e \neq \emptyset.
\]

This is easily seen to be both 1-almost modest and 1-almost meek as a \((Z, 0, U^{(0)})\)-\( \Delta \)-module: indeed, its homology is equal to \( F(Z, A) \) in position \((e, 1)\) whenever
Also, let \( M^j := M^j_{s,p} \) for each \( j \). In Subsection I.10.1, it was next shown that all these solution \( \Delta \)-modules \( M^j \) and their reductions \( M^j_{s,p} \) are related as follows. For each \( j \), if \( M^j \) is 1-almost modest, then the \((Z, 0, U^{(j)})\)-\( \Delta \)-module \( M^j \) and the \((Z, U^{(j+1)}, \Delta U^{(j)})\)-\( \Delta \)-module \((M^j, M^{j+1})U^{j+1}\) are both also \((Z, 0, U^{(j+1)})\)-\( \Delta \)-modules, and are still 1-almost modest after this re-interpretation. The same assertion holds for 1-almost meekness, since this re-interpretation does not affect whether any of the relevant functions are step polynomials. Then, for each \( j \in \{0, 1, \ldots, k \} \), one obtains a short exact sequence of \((Z, 0, U^{(j+1)})\)-\( \Delta \)-modules:

\[
M^j \hookrightarrow M^{j+1} \rightarrow (M^j / M^{j+1})U^{j+1}.
\]

Since we have seen that \( M^0 \) is a 1-almost meek \((Z, 0, U^{(0)})\)-\( \Delta \)-module, it now follows by induction on \( j \), using Corollary 7.10, Proposition 7.12, and Theorem 8.1, that \( M^j \), \( M^j_{s,p} \) and then \((M^j / M^{j+1})U^{j+1}\) are 1-almost meek for every \( j \). Once we reach \( j = k \), this includes the conclusion we wanted.

\[ \text{Proof of Theorem B.} \]

Once again, it suffices to assume \( A \) is a s.-p. module. Let \( U^{(j)} \) for \( j \in \{0, 1, \ldots, k \} \) be as before, and now for each \( j \) let \( N^j \) be the zero-sum \( \Delta \)-module associated to \((Z, U^{(j)})\). In this setting, Section I.10.3 showed that \( N^0 \) is 2-almost modest, and that these \( N^j \) are related to one another in just the same way as the PD\( \text{ce} \)Es-solution modules \( M^j \) in the proof of Theorem A. Since \( N^0 \) is also easily seen to be 2-almost meek, the rest of the proof is now just like the proof of Theorem A.

\[ \text{Remark.} \]

Subsection I.10.2 extended the study of PD\( \text{ce} \)Es-solution \( \Delta \)-modules to certain systems of multiple PD\( \text{ce} \)Es, and showed that the solution \( \Delta \)-modules are also almost modest in that case. One can show that they are almost meek in just the same way, so once again the ‘basic’ solutions to those problems are step-polynomial. This argument requires only the juxtaposing of the ideas from Subsection I.10.2 and the above proof, so we leave the details to the interested reader.

With Theorem A in hand, the following is an easy corollary. The analogous corollary for Theorem B can be proved along exactly the same lines.

\[ \text{Corollary 9.1.} \]

If \( \mathcal{M} \) is the PD\( \text{ce} \)E-solution \( \Delta \)-module corresponding to \( Z \) and \( U \), so that \( M[k] \) is the module of solutions, then \((M[k])_{sp} \) is dense in \( M[k] \).

\[ \text{Proof.} \]

We prove this by induction on \( k \). When \( k = 1 \), the solutions are just measurable functions lifted from \( Z/U_1 \), which may of course be approximated by lifts of step polynomials on \( Z/U_1 \). So now suppose the result is known for all PD\( \text{ce} \)Es
of degree less than some $k \geq 2$, and that $f \in M[k]$. By Theorem A, there is a decomposition

$$f = f_0 + \sum_{i=1}^{k} f_i,$$

where $f_i \in M[k] \setminus i$ for $i = 1, 2, \ldots, k$, and where the restriction $f_0|(z + U[k])$ is a step polynomial for each coset $z + U[k]$.

Each $f_i$ may be approximated by a step polynomial solution to its respective simpler PD$^cE$, by the hypothesis of our induction on $k$, so it suffices to find a step-polynomial approximation to $f_0$. Let $\xi : Z/U[k] \to Z$ be a step-affine cross-section (Corollary 3.23), and let $\sigma(z) := z - \xi(z + U[k])$ be the corresponding equivariant map $Z \to U[k]$. Let $\mathcal{P}$ be a measurable partition $Z/U[k]$ into positive-measure sets, and let $(\mathfrak{s}_C)_{C \in \mathcal{P}}$ be a selection of a point $\mathfrak{s}_C \in C$ for each $C \in \mathcal{P}$, uniformly at random using Haar measure, independently for different cells $C$. Also, given $\mathfrak{s} \in Z/U[k]$, let $C(\mathfrak{s})$ be the cell of $\mathcal{P}$ that contains it. Then simple measure theory shows that if $\mathcal{P}$ is sufficiently fine (that is, having all cells of small enough diameter), then

$$f_0(z) \approx f_0(\mathfrak{s}_C(z + U[k]) + \xi(z))$$

with high probability in $z \in Z$ and in the choice of $(\mathfrak{s}_C)_{C \in \mathcal{P}}$. In particular, one can choose $(\mathfrak{s}_C)_{C}$ for which the above holds with high probability in $z$. This still holds if, in addition, we choose $\mathcal{P}$ to be a q.-p. partition. For fixed $(\mathfrak{s}_C)_{C}$, this now gives a function on the right-hand side above which is a step-polynomial approximation to $f_0$. This new function, say $f'_0$, is also still a solution the desired PD$^cE$, because for every coset $z + U[k]$ there is another coset $z' + U[k]$ such that $f'_0|(z + U[k])$ is just a copy of $f_0|(z' + U[k])$. 

10 Steps towards quantitative bounds

Our next goal is Theorem C, the quantitative extension of Theorem A. Before proving it we must introduce the notion of ‘complexity’ for a step polynomial, and then prove some of its basic properties. This notion is not canonical, and other possibilities will be discussed following the proofs, but it is essentially the only notion for which I can prove a version of Theorem C.

10.1 Complexity and its basic consequences

Recall from Subsection 3.3 that a function $f : Z \to \mathbb{T}$ is a step polynomial if and only if $\{f\} : Z \to \mathbb{R}$ is a step polynomial, and that any step polynomial
$[0,1]^d \rightarrow \mathbb{R}$ is a sum of basic step polynomials, one for each cell of some convex polytopal partition of $[0,1]^d$ that controls it.

**Definition 10.1** (Complexity). Fix $D \in \mathbb{N}$. If $C \subseteq [0,1]^d$ is a convex polytope, then it has complexity at most $D$ relative to $[0,1]^d$ if $C = [0,1]^d \cap H_1 \cap \cdots \cap H_D$ for some open or closed half-spaces $H_i \subseteq \mathbb{R}^d$, $i = 1,2,\ldots,D$.

A basic step polynomial $f : [0,1]^d \rightarrow \mathbb{R}$ has basic complexity at most $D$ if

- $d \leq D$, and
- $f$ may be represented as $p \cdot 1_C$, where $p : \mathbb{R}^d \rightarrow \mathbb{R}$ is a polynomial of degree at most $D$ and with absolute values of all coefficients bounded by $D$, and $C \subseteq [0,1]^d$ is a convex polytope of complexity at most $D$ relative to $[0,1]^d$.

A step polynomial $f : [0,1]^d \rightarrow \mathbb{R}$ has complexity at most $D$ if it is a sum of at most $D$ basic step polynomials, all of basic complexity at most $D$. The least $D$ for which this holds is denoted $\text{cplx}(f)$.

Finally, if $Z$ is a compact Abelian group, then a function $f : Z \rightarrow \mathbb{T}$ has complexity at most $D$ if $\{ f \} = f_0 \circ \{ \chi \}$ for some affine map $\chi : Z \rightarrow \mathbb{T}^d$ and some $f_0 : [0,1]^d \rightarrow \mathbb{R}$ of complexity at most $D$. The least $D$ for which there is such a factorization is denoted $\text{cplx}(f)$.

In either setting, if $f$ is not a step polynomial then we set $\text{cplx}(f) := \infty$.

Our first estimate for this notion is obvious from the definition.

**Lemma 10.2.** If $g_0, h_0 : [0,1]^d \rightarrow \mathbb{R}$ are step polynomials, then $\text{cplx}(g_0 + h_0) \leq \text{cplx}(g_0) + \text{cplx}(h_0)$. \hfill $\blacksquare$

The next estimate requires a little more work.

**Lemma 10.3.** For each $D \in \mathbb{N}$ there is a $D' \in \mathbb{N}$ with the following property. Let $d \leq D$, let $Z$ be a compact Abelian group and let $\chi : Z \rightarrow \mathbb{T}^d$ be an affine map. If $g_0 : [0,1]^d \rightarrow \mathbb{R}$ has complexity at most $D$, $g := g_0 \circ \{ \chi \}$ and $z \in Z$, then there is some $h_0 : [0,1]^d \rightarrow \mathbb{R}$ of complexity at most $D'$ such that $R_z g = h_0 \circ \{ \chi \}$.

**Proof.** First suppose that $Z = \mathbb{T}^d$ and $\chi$ is the identity. By summing at most $D$ terms, it suffices to prove this when $g = p \cdot 1_C$ is a basic step polynomial of basic complexity at most $D$. However, now the coordinates of $\{ z \} = \{ z_1,\ldots,z_d \}$ dissect $[0,1]^d$ into at most $2^d \leq 2^D$ smaller boxes, say $Q_\omega$ for $\omega \in \{0,1\}^d$, and for each $\omega$ one has a vector $v_\omega \in \mathbb{R}^d$ such that the following diagram commutes:

$$\begin{array}{ccc}
\{ \cdot \}^{-1}(Q_\omega) & \xrightarrow{w \mapsto u + z} & \{ \cdot \}^{-1}(Q_\omega) - z \\
\downarrow & & \downarrow \\
Q_\omega & \xrightarrow{w \mapsto u + v_\omega} & Q_\omega + v_\omega.
\end{array}$$
It follows that for each $\omega$, the function $R_z((p \cdot 1_{C \cap Q_\omega}) \circ \{\cdot\})$ is still a basic polynomial of complexity at most $D + d \leq 2D$ (since $C \cap Q_\omega$ could require up to $D + d$ linear inequalities to define it). Summing over $\omega$ completes the proof when $\chi = \text{id}_{T^d}$.

Finally, for general $\chi$, simply observe that we may factorize $g = g_1 \circ \chi$, where $g_1 := g_0 \circ \{\cdot\}$ satisfies the assumption of the special case above. Since $R_z g = R_{\chi(z)} g_1 \circ \chi$, the result now follows from that special case.

**Lemma 10.4 (Compactness).** If $d \leq D$ then the set

$$\{ f \circ \{\cdot\} \mid f : [0, 1]^d \to \mathbb{R}, \text{ cplx}(f) \leq D \} \subseteq F(T^d)$$

is compact for the topology of convergence in probability for the measure $m_{T^d}$.

**Proof.** It suffices to prove this for the set of all compositions $f_0 \circ \{\cdot\}$ with $f_0$ a basic polynomial of basic complexity at most $D$, and hence for the set of all functions $(p \cdot 1_C) \circ \{\cdot\}$ where $p$ and $C$ satisfy the bounds in the definition of basic complexity. Also, since $\{\cdot\} : T^d \to [0, 1]^d$ defines an isomorphism of measure spaces, it suffices to prove this compactness on $[0, 1]^d$ itself. This conclusion now follows because these data are specified by $O_D(1)$ coefficients for $p$, all lying in $[-D, D]$, and by at most $D$ open or closed half-spaces that intersect $[0, 1]^d$, and the set of all such half-spaces is compact.

**Corollary 10.5.** Let $d \leq D$, let $Z$ be a compact Abelian Lie group and let $\chi : Z \to T^d$ be affine. Then the set

$$\{ f \circ \chi \mid f : [0, 1]^d \to \mathbb{R}, \text{ cplx}(f) \leq D \} \subseteq F(Z)$$

is pre-compact for the topology of convergence in $m_Z$-probability, and its closure is contained in the set

$$\{ g \in F(Z) \mid \text{cplx}(g) \leq D' \}$$

for some $D' \in \mathbb{N}$.

**Proof.** It clearly suffices to prove this for each coset of the identity component $Z_0 \leq Z$ separately, so we may assume that $Z = T^r$ for some $r$. Now Lemma 3.12 gives a commutative diagram

$$
\begin{array}{ccc}
T^r & \xrightarrow{\chi} & T^d \\
\{\} \downarrow & & \{\} \\
[0, 1]^r & \xrightarrow{\psi} & [0, 1]^d
\end{array}
$$

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for some step-affine map \( \psi : [0, 1)^r \rightarrow [0, 1)^d \). It is also clear that if \( f : [0, 1)^d \rightarrow \mathbb{R} \) has \( \text{cplx}(f) \leq D \), then \( f \circ \psi \) has complexity bounded by some \( D' \) that depends only on \( \psi \) and \( D \). Therefore the set of functions in question is contained in

\[
\{ \{ f' \} \circ \{ \cdot \} \mid f' : [0, 1)^r \rightarrow \mathbb{R}, \; \text{cplx}(f') \leq D' \},
\]

so the result follows from Lemma 10.4.

\[\blacksquare\]

**Lemma 10.6 (Bounded-complexity and equidistribution).** Let \( Z \) be a compact Abelian Lie group with a translation-invariant metric \( \rho \), let \( \chi : Z \rightarrow \mathbb{T}^d \) be affine, and let \( D \in \mathbb{N} \) be fixed. Then for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that for any closed subgroup \( Y \leq Z \), if \( Y \) is \( \delta \)-dense in \( Z \) for the metric \( \rho \), then

\[
|d^Y_Y(0, g|_Y) - d^Z_Z(0, g)| < \varepsilon
\]

whenever \( g = g_0 \circ \{ \chi \} \) and \( g_0 : [0, 1)^d \rightarrow \mathbb{R} \) has complexity at most \( D \), where \( d^Y_Y \) and \( d^Z_Z \) are the metrics of convergence in \( m_Y \)-probability and \( m_Z \)-probability, respectively.

**Proof:** For \( \delta \) sufficiently small, it suffices to prove this on each connected component of \( Z \), so we may assume that \( Z = \mathbb{T}^r \). Having done so, we may argue from Lemma 3.12 as in the proof of Corollary 10.5 to pull everything back to \( \mathbb{T}^r \) itself, and assume that \( \chi \) is the identity.

Lastly, it suffices to prove the result for \( g = g_0 \circ \{ \cdot \} \) with \( g_0 \) a basic step polynomial of basic complexity at most \( D \), say \( g_0 = p \cdot 1_C \) as in Definition 10.1.

At this point the result is clear, because

- on the one hand, the resulting function \((p \cdot 1_C) \circ \{ \cdot \} : \mathbb{T}^d \rightarrow \mathbb{R}\) is locally Lipschitz, with constant bounded in terms of \( D \), on both the interior and exterior of \( \{ \cdot \}^{-1}(C) \),

- and on the other, \( C \subseteq [0, 1)^d \) is a convex set with boundary contained in a union of at most \( D \) portions of hyperplanes in \( \mathbb{R}^d \), implying that

\[
m_{[0, 1)^d}(\{ v \mid \text{dist}(v, \partial C) < \delta \}) \rightarrow 0 \quad \text{as } \delta \downarrow 0
\]

at a rate that can be bounded in terms of \( D \) alone.

\[\blacksquare\]
10.2 Proof by compactness

Theorem C will also be proved by compactness and contradiction. It will involve the Vietoris topology on the set of closed subgroups of a torus $\mathbb{T}^d$, which is a compact and metrizable topology since $\mathbb{T}^d$ is compact and metrizable. Letting $\rho$ be any standard choice of metric on $\mathbb{T}^d$ (such as the quotient of the Euclidean metric on $\mathbb{R}^d$), if $Z_n \rightarrow Z$ as subsets of $\mathbb{T}^d$ with this topology, then, by definition, for every $\epsilon > 0$ there is an $n_0$ such that $Z_n$ is $\epsilon$-dense in $Z$ for $\rho$ for all $n \geq n_0$. It is also standard that this implies $m_{Z_n} \rightarrow m_Z$ in the vague topology.

In this setting we will also need the following elementary lemma.

**Lemma 10.7.** If $Z_n$ is a sequence of closed subgroups of $\mathbb{T}^d$ tending to another subgroup $Z$ in the Vietoris topology, then $Z_n \leq Z$ for all sufficiently large $n$.

**Proof.** This is most easily seen in the Pontryagin dual. The Vietoris convergence $Z_n \rightarrow Z$ implies the vague convergence $m_{Z_n} \rightarrow m_Z$, and hence the convergence $Z_n^\perp \rightarrow Z^\perp$ as subsets of $\hat{\mathbb{T}}^d = \mathbb{Z}^d$, in the sense of eventual agreement at any given point of $\mathbb{Z}^d$. Since all subgroups of $\mathbb{Z}^d$ are generated by at most $d$ elements, this means that $Z_n^\perp$ must eventually contain a set of generators for $Z^\perp$, at which point one has $Z_n^\perp \geq Z^\perp$ and hence $Z_n \leq Z$. \hfill $\Box$

**Proof of Theorem C.** We will first select an $\epsilon$ depending only on $k$. The keys to this are the quantitative results from Part I.

Firstly, Theorem I.C implies that there is non-decreasing function $\kappa : (0, \infty) \rightarrow (0, \infty)$, depending only on $k$ and tending to 0 at 0, such that if $\mathcal{M}$ is the solution $\Delta$-module for the PD$^{ce}$E associated to some $Z$ and subgroup-tuple $U = (U_1, \ldots, U_k)$, and if $f \in \mathcal{F}(Z)$ is such that

$$d_0(0, d^{U_1} \cdots d^{U_k} f) < \epsilon$$

in $\mathcal{F}(U_1 \times \cdots \times U_k \times \mathbb{Z})$, then $d_0(f, M_{[k]}) < \kappa(\epsilon)$.

Second, Theorem I.A' promises some $\eta > 0$, depending only on $k$, such that if $f' \in M_{[k]}$ and $d_0(0, f') < \eta$, then in fact $f' \in \partial_k(M^{(k-1)})$.

Putting these facts together, we may now choose $\epsilon > 0$ so small that

$$\epsilon + \kappa(2^{k+1}\epsilon) < \eta.$$

This still depends only on $k$. We will prove that this $\epsilon$ has the property asserted in Theorem C.

This will be proved by assuming otherwise, and deriving a contradiction from Theorem A and a compactness argument. Thus, suppose that $Z_n$ is a sequence of compact Abelian groups and $U_n = (U_{n,i})_{i=1}^k$ a sequence of tuples of subgroups.
Let $\mathcal{M}_n = (M_{n,e})_e$ be the PD$^\infty$E solution $\Delta$-module associated to $U_n$, and let $f_n \in M_{n,[k]}$ and $g_n \in \mathcal{F}_{sp}(Z_n, T)$ be sequences such that
\[
\text{cplx}(g_n) \leq d \quad \text{and} \quad d_0(f_n, g_n) < \varepsilon
\]
for all $n$, but on the other hand such that
\[
\min\{\text{cplx}(f') \mid f' \in f_n + \partial_k(M_n^{(k-1)})\} \to \infty.
\]

For each $n$, let $\chi_n : Z_n \to \mathbb{T}^{r_n}$ be an affine map and $g''_n : [0, 1)^{r_n} \to \mathbb{R}$ be a step polynomial such that $g_n = g''_n \circ \{\chi_n\} \mod \mathbb{Z}$ and $\text{cplx}(g''_n) = \text{cplx}(g_n)$. Also, let
\[
g'_n := g''_n \circ \{\cdot\} \mod \mathbb{Z} : \mathbb{T}^{r_n} \to \mathbb{T},
\]
so $g_n = g'_n \circ \chi_n$.

**Step 1.** Since $r_n \leq d$ for every $n$, after passing to a subsequence we may suppose that

- $r_n = r$ for every $n$,
- $Z'_n := \chi_n(Z_n) \to Z'$ for some $Z' \leq \mathbb{T}^r$ in the Vietoris topology,
- similarly, $U'_{n,i} := \chi_n(U_{n,i}) \to U'_i \leq Z'$ for each $i \leq k$ in the Vietoris topology.

Lemma 10.7 implies that $Z'_n \leq Z'$ and $U'_{n,i} \leq U'_i$ for all sufficiently large $n$; by omitting finitely many terms of our sequences, we may assume this holds for all $n$.

Using Corollary 10.5, another passage to a subsequence now allows us to assume in addition that $d_0(g'_n|_{Z'}, g') \to 0$ in $\mathcal{F}(Z')$, where $g' : Z' \to \mathbb{T}$ is another step polynomial.

Let $\mathcal{M} = (M'_e)_e$ be the solution $\Delta$-module for the PD$^\infty$E associated to $Z'$ and $U' = (U'_1, \ldots, U'_k)$.

**Step 2.** Since $d_0(f_n, g_n) < \varepsilon$ and $f_n \in M_{n,[k]}$, we have
\[
d_0(0, d_{u_1} \cdots d_{u_k} g_n) < 2^k \varepsilon \quad \text{in } \mathcal{F}(Z_n)
\]
for all $(u_1, \ldots, u_k) \in \prod_i U_{n,i}$, for all $n \geq 1$. Equivalently,
\[
d_0(0, (d_{u'_1} \cdots d_{u'_k} g'_n)|_{Z'_n}) < 2^k \varepsilon \quad \text{in } \mathcal{F}(Z'_n)
\]
(23) for all $(u'_1, \ldots, u'_k) \in \prod_i U'_{n,i}$, for all $n \geq 1$. 

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Moreover, by the density result of Corollary 9.1, we may assume in addition that each function $h_n$ is defined, each $h_n$ is still a step polynomial on $Z_n^i$ of complexity at most $d$, applying each of Lemmas 10.2 and 10.3 $k$ times gives that also

$$d_{u_1'} \cdots d_{u_k'} g_n = g''_{n,u_1',\ldots,u_k'} \circ \{ \cdot \} \mod \mathbb{Z}$$

for some functions $g''_{n,u_1',\ldots,u_k'} : [0,1]^d \rightarrow \mathbb{R}$ having complexity at most $d$, applying each of Lemmas 10.2 and 10.3 $k$ times gives that also

$$d_{u_1'} \cdots d_{u_k'} g_n = g''_{n,u_1',\ldots,u_k'} \circ \{ \cdot \} \mod \mathbb{Z}$$

and the fact that $g''_{n,u_1',\ldots,u_k'}$ is dense in $\mathbb{R}$ whose complexity is bounded in terms of $D$ and $k$, uniformly in $(u_1',\ldots,u_k')$. On the other hand, $Z_n^i$ equidistributes in $Z'$. Given this, Lemma 10.6 and (23) imply that

$$d_0(0, d_{u_1'} \cdots d_{u_k'} g_n) < 2^k \varepsilon \quad \text{in } \mathcal{F}(Z')$$

for all $(u_1',\ldots,u_k') \in \prod_i U_n^i$, for all sufficiently large $n$.

Next, since $d_0(g_n', g') \rightarrow 0$ in $\mathcal{F}(Z')$, the estimate (24) implies that

$$d_0(0, d_{u_1'} \cdots d_{u_k'} g') < 2^{k+1/2} \varepsilon \quad \text{in } \mathcal{F}(Z')$$

for all $(u_1',\ldots,u_k') \in \prod_i U_n^i$, for all sufficiently large $n$. Finally, since $\bigcup_{n \geq 1} U_n^i$ is dense in $U_i'$ for each $i$, this turns into

$$d_0(0, d_{u_1'} \cdots d_{u_k'} g') < 2^{k+1} \varepsilon \quad \text{in } \mathcal{F}(Z') \quad \forall (u_1',\ldots,u_k') \in \prod_i U_i'$$

$$\implies d_0(0, d_{u_1'} \cdots d_{u_k'} g') < 2^k \varepsilon \quad \text{in } \mathcal{F}(U_1' \times \cdots \times U_k' \times Z').$$

**Step 3.** Having reached this last estimate, we may apply Theorem I.C to conclude that there is some $h' \in M^i_{[k]}$ such that

$$d_0(g', h') < \kappa(2^{k+1} \varepsilon).$$

Moreover, by the density result of Corollary 9.1 we may assume in addition that this $h'$ is a step polynomial on $Z'$. 

**Step 4.** Finally, since $Z_n^i \leq Z'$ and $U_n^i \leq U_i'$ for all $i$ and $n$, the pulled-back functions

$$h_n := h' \circ \chi_n$$

are all well-defined, each $h_n$ solves the PDCE associated to $Z_n$ and $U_n$, and each $h_n$ is still a step polynomial on $Z_n$ of complexity at most cplx$(h')$ (since pre-composing with an affine function cannot increase complexity according to Definition 10.1). Now another appeal to Lemma 10.6, the equidistribution of $Z_n^i$ in $Z'$ and the fact that $g_n' \rightarrow g' \circ h'$ in $\mathcal{F}(Z')$ give that

$$d_0(g_n, h_n) \leq d_0(g_n' \circ \chi_n, g' \circ h_n) + d_0(g' \circ \chi_n, h' \circ \chi_n) < \kappa(2^{k+1} \varepsilon)$$

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for all \( n \) sufficiently large. Therefore

\[ d_0(f_n, h_n) < \varepsilon + \kappa(2^{k+1}\varepsilon) \text{ in } \mathcal{F}(Z_n). \]

However, we chose this right-hand side to be less than \( \eta \), so this implies \( h_n \in f_n + \partial h(M^{(k-1)}_n) \), yet \( \text{cplx}(h_n) = \text{cplx}(h') \) remains bounded as \( n \to \infty \). This contradicts our initial assumptions, and so completes the proof. \( \square \)

### 10.3 Discussion

Instead of proving Theorem C by compactness and contradiction, it is tempting to try to keep track of the complexities of the step polynomials that appear as we build up the machinery of semi-functional \( \Delta \)-modules, and so obtain more effective versions of our results about concatenations, cohomology \( \Delta \)-modules, and so on. (In principle, our proof by compactness could be forced to yield explicit bounds by quantifier-elimination, but they would be atrocious.)

Unfortunately, the most naïve version of this idea cannot be carried out, because some of the necessary results are not true.

**Example 10.8.** Let \( Z = (\mathbb{Z}/p\mathbb{Z})^2 \times \mathbb{T} \) for some large prime \( p \), and let \( \chi : \mathbb{T} \to \mathbb{T} \) be the identity character. Let

\[
U_1 := ((1, 0) \cdot (\mathbb{Z}/p\mathbb{Z})) \times \mathbb{T}, \quad U_2 := ((1, -1) \cdot (\mathbb{Z}/p\mathbb{Z})) \times \mathbb{T}
\]

and

\[
U_3 := ((0, 1) \cdot (\mathbb{Z}/p\mathbb{Z})) \times \mathbb{T},
\]

and define \( f : Z \to \mathbb{T} \) by

\[
f(s_1, s_2, t) := \left\lfloor \frac{s_1}{p} + \frac{s_2}{p} \right\rfloor \cdot p \cdot \chi(t),
\]

where \( \{s_1\} \) now denotes the element of \( \{0, 1, \ldots, p-1\} \) that represents \( s_1 \mod p \).

An easy check shows that

\[
f(s_1, s_2, t) = \{s_1\} \chi(t) + \{s_2\} \chi(t) - \{s_1 + s_2\} \chi(t),
\]

(25)

so \( f \) is a degenerate solution of the PD\textsuperscript{co}E associated to \( (U_1, U_2, U_3) \).

This \( f \) has complexity bounded independently of \( p \). Indeed, one may simply observe that \( f = f_1 \circ \pi_p \), where

\[
\pi_p : (\mathbb{Z}/p\mathbb{Z})^2 \times \mathbb{T} \to \mathbb{T}^3 : (s_1, s_2, t) \mapsto (\{s_1\}/p \text{ mod } 1, \{s_2\}/p \text{ mod } 1, pt)
\]

is a homomorphism, and

\[
f_1(s_1', s_2', t') = \left\lfloor s_1' + s_2' \right\rfloor \cdot \chi(t').
\]

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Therefore $\text{cplx}(f) \leq \text{cplx}(f_1)$, and this does not depend on $p$. However, the summands in (25) have complexity that grows as $p$ increases, because each of $\{s_1\}, \{s_2\}$ and $\{s_1 + s_2\}$ can take real values as large as $p - 1$. (This is not a complete proof that the summands have growing complexity, but can be turned into one with a little extra work. Alternatively, Corollary [10.5] shows that if the complexities of these summands were bounded uniformly in $p$, then one could factorize them through affine maps to a fixed torus so that they are all pulled back from some pre-compact family of step polynomials, and one can also check by hand that this is not possible.) Similar reasoning also gives that in this case there is no other representation as in (25) in which all summands have complexity bounded independently of $p$.

Before leaving this example, let us note that the function $f$ above is built from the function

$$(\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow \mathbb{Z} : (s_1, s_2) \longrightarrow p\{s_1\}/p + \{s_2\}/p,$$

which is an element of $B^2(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z})$. This latter can also be cast as an example in which a low-complexity 2-coboundary is not the coboundary of a low-complexity 1-cochain, and from that perspective it can also be used to rule out certain complexity-dependences in Theorem 8.2.

The upshot of this example is that, although we have proved that a PD$^{\omega}$E-solution $\Delta$-module $M$ admits finite-complexity decompositions in its top structure complex, if one has a step polynomial $f \in \partial_k(M^{(k-1)})$ then it may not be possible to control the minimal complexity of $g \in \partial_{k-1}^{-1}\{f\}$ only in terms of $\text{cplx}(f)$. At the very least, this control must also depend on the choice of $Z$, and if $Z$ is infinite-dimensional then there may be no such control at all (for instance, by taking an infinite product of increasingly bad examples).

The best one can obtain is a control on the minimal complexity of $g \in \partial_{k-1}^{-1}\{f\}$ that depends on both $\text{cplx}(f)$, and on an a priori bound on how easily some $g' \in \partial_{k-1}^{-1}\{f\}$ may be approximated by a controlled-complexity step function. This would be much like the statement of Theorem C. However, this is strictly stronger than a control on the ability to approximate $f$ itself (the above example witnesses this, too), and it seems to me a very complicated challenge to develop effective versions of the main results for $\Delta$-modules that take this technical necessity account.

**Question 10.9.** In the setting above, if one fixes a Lie group $Z$ and subgroup-tuple $U$, can one bound $\min\{\text{cplx}(g) \mid g \in \partial_{k}^{-1}\{f\}\}$ only in terms of $\text{cplx}(f)$?

Looking again at Example [10.8], the summands in (25) had large complexity (in the sense of Definition [10.1]) only by virtue of requiring polynomials with large coefficients in their representation as step polynomials. Those polynomials may
still be taken to be quadratic, and the directing q.-p. partitions involve only a small
number of cells.

**Question 10.10.** *Is there a version of Theorem C in which one controls only the
complexities of directing q.-p. partitions and the degrees of polynomials, but not
the size of their coefficients?*

I do not know how to approach this question. Subsections [10.1] and [10.2] both
made essential use of some consequences of low complexity for the ‘regularity’
of step polynomials, and this will not hold if one allows very large coefficients.
If one attempts to avoid the proof by contradiction-and-compactness, and instead
keep track of this alternative kind of complexity through the work of all the pre-
vious sections, one still runs into this problem, because the regularity properties
of step polynomials were also implicitly involved in the cohomological results of
Subsection 5.2.

Finally, recall from Theorems I.A and I.B that if \( Z \) is a Lie group, then the
structural homology of the \( \Delta \)-modules \( \mathcal{M} \) and \( \mathcal{N} \) is finitely generated in all posi-
tions above 1 (resp. 2). Coupled with Theorems A and B of the present paper, this
immediately implies the following.

**Corollary 10.11.** Let \( Z \) be a Lie group, \( U \) a tuple of at least three subgroups of
\( Z \), \( \mathcal{M} \) the associated PD\(^{\infty} \)-solution \( \Delta \)-module, and \( \mathcal{N} \) the associated zero-sum
\( \Delta \)-module. Then there is some \( D \in \mathbb{N} \) such that every \( f \in \mathcal{M}_{[k]} \) (resp. \( f \in \mathcal{N}_{[k]} \)) is
of the form

\[
    f = n_1 f_1 + \cdots + n_{\ell} f_{\ell} + g,
\]

where \( n_i \in \mathbb{Z} \), \( f_i \in \mathcal{M}_{[k]} \) (resp. \( f_i \in \mathcal{N}_{[k]} \)) has complexity at most \( D \) for all \( i \leq \ell \),
and \( g \in \partial_k(M^{(k-1)}) \) (resp. \( g \in \partial_k(N^{(k-1)}) \)).

Of course, we cannot control the size of the coefficients \( n_i \) in this corollary, and
so we do not control the complexity of the whole sum \( n_1 f_1 + \cdots + n_{\ell} f_{\ell} \). This begs
the following question, which I suspect lies far beyond the methods of the present
paper.

**Question 10.12.** *Does the preceding corollary hold for any compact Abelian \( Z \),
with a choice of \( D \) that does not depend on \( Z \) or \( U \), but now with no control over
either the coefficients \( n_i \) or the number of summands \( \ell \)?*

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