CONVERGENCE TO A GRIM REAPER FOR A CURVATURE FLOW WITH VARIABLE BOUNDARY ANGLES

BENDONG LOU†, XIAOLIU WANG‡ AND LIXIA YUAN†,§

Abstract. We consider a curvature flow $V = H$ in a band domain $\Omega := [-1,1] \times \mathbb{R}$, where, for a graphic curve $\Gamma_t$, $V$ denotes its normal velocity and $H$ denotes its curvature. If $\Gamma_t$ contacts the two boundaries $\partial_{\pm}\Omega$ of $\Omega$ with constant angles, Altschuler and Wu [2] proved that $\Gamma_t$ converges to a grim reaper contacting $\partial_{\pm}\Omega$ with the same prescribed angles. In this paper we consider the case where $\Gamma_t$ contacts $\partial_{\pm}\Omega$ with a slope equaling to $\pm 1$ times of its height, respectively. We first obtain uniform interior gradient estimates for the solution by using the so-called zero number argument, and then prove that, as $t \to \infty$, $\Gamma_t$ converges in $C^{2,1}_{loc}((-1,1) \times \mathbb{R})$ topology to the grim reaper with span $(-1,1)$.

1. Introduction

Consider the following curvature flow

\begin{equation}
V = H \quad \text{on} \quad \Gamma_t \subset \Omega,
\end{equation}

in the band domain $\Omega := \{(x,y) : -1 \leq x \leq 1, y \in \mathbb{R}\}$ in $\mathbb{R}^2$, where, $\Gamma_t$ is a family of simple curves in $\Omega$ which contact the boundaries $\partial_{\pm}\Omega := \{\pm 1\} \times \mathbb{R}$ of $\Omega$ with prescribed angles (see details below), $V$ and $H$ denote the normal velocity and the curvature of $\Gamma_t$, respectively.

The equation (1.1) is an important model in phase transition problems. It is also used to describe the motion of front arising from the singular limit of the Allen-Cahn equations. When the boundary problem is considered, there will be a contact angle condition for the front at the intersection with the domain boundaries (see, for example, [18]). In case $\Gamma_0$ is a $C^1$ graph on $[-1,1]$, it is easily seen that $\Gamma_t$ is the graph of a function $y = u(x,t)$ for each $t$ and

\begin{align*}
V &= \frac{u_t}{\sqrt{1 + u_x^2}}, \\
H &= \frac{u_{xx}}{(1 + u_x^2)^{3/2}}.
\end{align*}

Hence, our problem can be expressed as

\begin{equation}
\begin{cases}
  u_t = \frac{u_{xx}}{1 + u_x^2}, & -1 < x < 1, \quad t > 0, \\
  u_x(-1,t) = g_-, & u_x(1,t) = g_+, \quad t > 0,
\end{cases}
\end{equation}

where $g_-, g_+$ denote the boundary contact conditions and will be made clear later.

In 1989, Huisken [13] considered (1.2) with $g_+ = 0$ (actually, its higher dimensional version) and proved that the solution converges to a constant as $t \to \infty$. In 1993, Altschuler and Wu [2] studied (1.2) with $g_- < 0 < g_+$ being constants, and proved that any solution converges to a traveling wave solution (which is also called a translating solution, or a grim reaper in
one dimensional case). In 1994 they [3] extended their result to two dimension. In 2012, Cai and Lou [4] considered (1.2) with $g_{\pm}$ being almost periodic functions of $u$, and proved that any solution converges to an almost periodic traveling wave. Recently, Yuan and Lou [19] considered a more general case, that is, $g_{\pm} = g_{\pm}(u)$ are asymptotic periodic functions as $u \to \pm \infty$. They constructed some entire solutions connecting two periodic traveling waves. In 2012, Chou and Wang [7] considered (1.2) with Robin boundary conditions, and present various asymptotic behavior for the solutions.

Besides the above mentioned papers, other works related to the mean curvature flow (1.1), as well as its anisotropic analogues, in domains with boundaries include Matano et al. [16, 17] for problems in band domain with undulating boundaries; [5, 10, 11, 13] for self-similar solutions in sectors on the plane; [6, 12] for problems on the half space, etc.

Inspired by [2, 3, 4, 7, 19] etc., in this paper we consider the problem (1.2) with $g_{\pm} = \pm u$, that is, consider the following problem

$$
(1.3) \quad \begin{cases}
    u_t = \frac{u_{xx}}{1 + u_x^2}, & -1 < x < 1, \quad t > 0, \\
    u_x(\pm 1, t) = \pm u(\pm 1, t), & t > 0, \\
    u(x, 0) = u_0(x).
\end{cases}
$$

In this case, the prescribed boundary slopes are $\pm u$. The global well-posedness of the problem (1.3) is studied in a standard way. For any time-global solution moving upward to infinity, $u_x$ is unbounded since the boundary angle becomes larger and larger. This will be the main difficulty in our approach. Indeed, the problem (1.3) has been studied as a special case in [7], where the curvature flow has a general Robin boundary condition. In [7], the authors did not obtain the convergence of solution to (1.3) and left it as an open problem. Since the boundary gradients are unbounded, it is natural to consider the convergence of the solution in $L^\infty_{loc}((-1, 1))$ topology. This, however, also needs some uniform (in $t$) interior gradient estimates. The well known results in this field as in [9] are not applied here since they depend on the boundedness of $u$. Instead, we will use the so-called zero number argument (i.e., zero number diminishing properties, cf. [1, 15]) for one dimensional parabolic equations to derive the uniform bounds for the gradient of the solution in any interior domain (see details in sections 4 and 5). Furthermore, as can be expected, the profile of the solution might converge to a traveling wave with infinite slope near the boundaries, which should be the grim reaper with span in $(-1, 1)$, that is,

$$
(1.4) \quad \varphi_0(x) + \frac{\pi}{2} t \quad \text{with} \quad \varphi_0(x) := -\frac{2}{\pi}\ln \left[\cos \left(\frac{\pi}{2} x\right)\right] \quad (x \in (-1, 1)).
$$

Actually we will show that this is true.

**Theorem 1.1.** Assume $u_0(x) \in C^1([-1, 1])$ with

$$
(1.5) \quad u_0(x) \geq 1 \quad (x \in [-1, 1]), \quad u_0'(\pm 1) = \pm u_0(\pm 1).
$$

Then the problem (1.3) has a time-global classical solution $u(x, t)$. It moves upward to infinity and, for some $K_0 \in \mathbb{R}$,

$$
(1.6) \quad u(x, t) - \varphi_0(x) - \frac{\pi}{2} t - K_0 \to 0 \quad \text{as} \quad t \to \infty,
$$

in the topology of $C^2_{loc}((-1, 1) \times \mathbb{R})$.

This paper is arranged as follows. In section 2, as preliminaries, we present traveling waves (grim reapers) contacting the boundaries of $\Omega$ with various different constant angles. In section 3, we give some rough a priori estimates and then show the time-global existence for the solution of (1.3). In section 4, we consider symmetric solutions of (1.3). First we present precise estimates for $u_x$ by using the zero-number argument, and then show its convergence to the grim reaper.
Finally, in section 5 we consider general solutions of (1.3) which are not necessarily symmetric. By some further uniform interior estimates we show its convergence to the grim reaper.

2. Traveling Waves

First we give a definition.

**Definition 2.1.** A function $u(x,t)$ satisfying

$$
\begin{align*}
    & u_t \leq \frac{u_{xx}}{1 + u^2_x}, \quad -1 < x < 1, \quad t > 0, \\
    & u_x(1,t) \leq u(1,t), \quad u_x(-1,t) \geq -u(-1,t), \quad t > 0,
\end{align*}
$$

is called a lower solution of (1.3). A function $u(x,t)$ satisfying the reversed inequalities is called an upper solution of (1.3).

As preliminaries, we study the following problem

$$
\begin{align*}
    & u_t = \frac{u_{xx}}{1 + u^2_x}, \quad -1 < x < 1, \quad t \in \mathbb{R}, \\
    & u_x(\pm 1,t) = \pm h, \quad t \in \mathbb{R}.
\end{align*}
$$

For each $h > 0$, a traveling wave of (2.1) (also called a translating solution in [2]) is a special solution of the form

$$
    u(x,t) = \varphi(x;h) + c(h)t.
$$

Substituting this formula into (2.1) we easily obtain

$$
\begin{align*}
    & \varphi(x;h) := -\frac{1}{c(h)} \ln \left[ \cos \left( c(h)x \right) \right], \quad c(h) := \arctan h, \\
    & \varphi(x;1) + c(1)t + 1 - \varphi(1;1).
\end{align*}
$$

$\varphi$ is called a grim reaper in [2]. Note that

$$
\begin{align*}
    & \varphi(\pm 1; h) = \frac{\ln(1 + h^2)}{2 \arctan h}, \quad \varphi_x(\pm 1; h) = \pm h.
\end{align*}
$$

Hence, for any $M \in \mathbb{R}$, $\varphi(x;h) + c(h)t + M$ is a lower solution of (1.3) when

$$
    h \leq \varphi(1;h) + c(h)t + M.
$$

It is an upper solution if the reversed inequality holds.

Besides the traveling waves of (2.1), we have another grim reaper $\varphi_0(x) + \frac{\pi}{2} t$ with $\varphi_0$ defined by (1.4). Note that the definition domain of $\varphi_0(x)$ is $(-1,1)$, that is, this grim reaper lies completely in $\Omega$. In what follows we will use the above grim reapers to give a priori estimates for the solution of (1.3).

3. Global Well-posedness of (1.3)

Assume $u(x,t)$ is a classical solution of (1.3) in the time-interval $[0,T]$ for some $T > 0$. We first give its $L^\infty$ estimate.

**Lemma 3.1.** There exist $C_1, C_2 > 0$ with $C_2$ depending on $T$ such that

$$
\begin{align*}
    & c(1)t - C_1 \leq u(x,t) \leq C_2(T), \quad x \in [-1,1], \quad t \in [0,T].
\end{align*}
$$

**Proof.** Assume, for some $M_0 > 0$,

$$
\begin{align*}
    & 1 \leq u_0(x) \leq M_0, \quad x \in [-1,1].
\end{align*}
$$

Then

$$
\begin{align*}
    & u(x,t) := \varphi(x;1) + c(1)t + 1 - \varphi(1;1)
\end{align*}
$$
is a lower solution of (1.3) and so, by the comparison principle we have

\[(3.3) \quad u(x, t) = \varphi(x; 1) + c(1)t + 1 - \varphi(1; 1) \leq u(x, t), \quad x \in [-1, 1], \quad t > 0,\]

which leads to the first inequality in (3.1).

Next we consider the upper bound. Since

\[h - \varphi(\pm 1; h) = h - \frac{\ln(1 + h^2)}{2 \arctan h} \to \infty \quad \text{as} \quad h \to \infty,\]

there exists \(h = h_T\) large such that

\[(3.4) \quad h_T > \varphi(\pm 1; h_T) + \frac{\pi}{2} T + M_0.\]

Set

\[\overline{u}(x, t) := \varphi(x; h_T) + c(h_T)t + M_0, \quad x \in [-1, 1], \quad t > 0.\]

We verify that \(\overline{u}\) is an upper solution of (1.3) in the time interval \([0, T]\). In fact, \(\overline{u}\) satisfies the equation in (1.3). Moreover, for \(t \in [0, T]\),

\[\overline{u}(1, t) < \varphi(1; h_T) + \frac{\pi}{2} t + M_0 < h_T = \overline{u}(x, 1, t),\]

by (3.4), and

\[\overline{u}(x, 0) = \varphi(x; h_T) + M_0 \geq M_0 \geq u_0(x), \quad x \in [-1, 1].\]

Hence, \(\overline{u}\) is an upper solution of (1.3). By comparison principle we have

\[u(x, t) \leq \overline{u}(x, t) \leq \varphi(1; h_T) + M_0 + \frac{\pi}{2} T, \quad x \in [-1, 1], \quad t \in [0, T].\]

This proves the second inequality of (3.1). \(\square\)

Next, we give the gradient estimate.

**Lemma 3.2.** Let \(u(x, t)\) be a solution of (1.3) in the time interval \([0, T]\). Then there exist \(C_3(T)\) such that

\[|u_x(x, t)| \leq C_3(T), \quad x \in [-1, 1], \quad t \in [0, T].\]

**Proof.** From the above lemma, we see that

\[|u_x(\pm 1, t)| = |u(\pm 1, t)| \leq C_1 + C_2(T), \quad t \in [0, T].\]

Using the maximum principle for \(u_x\) we see that

\[|u_x(x, t)| \leq C_3(T) := \max\{\|u'_0\|_C, C_1 + C_2(T)\}, \quad x \in [-1, 1], \quad t \in [0, T].\]

This proves the lemma. \(\square\)

With the above a priori estimates in hand, by using the standard parabolic theory we obtain the time-global existence of the classical solution \(u(x, t)\). Its uniqueness is proved in the standard way by using the maximum principle.

### 4. Symmetric Solutions

In this section we consider symmetric solutions. More precisely, we consider the case where \(u_0(x) \in C^1([-1, 1])\) satisfying the following conditions:

\[(4.1) \quad u_0(x) = u_0(-x), \quad 1 \leq u_0(x) \leq M_0, \quad u'_0(x) < \varphi'_0(x) \quad \text{for} \quad x \in (0, 1), \quad u'_0(1) = u_0(1).\]

In this case, \(u(x, t)\) is even in \(x\). To study the convergence of \(u\) (actually, the convergence of \(u(x, t) - u(0, t)\)), we need further estimate for \(u_x\). We will do this by the so-called zero number argument.
4.1. Precise lower gradient estimate. In this part, we show that “the gradient is not too small”.

Fix an $h_0 > 0$. For any $h \geq M_0 + h_0$ we see that
\[ \bar{u}(x, t; h) := \varphi(x; h_0) + c(h_0)t + h \]
is a lower solution of (1.3). Denote the union of the graphs of $\bar{u}(x, t; h)$ by
\[ D(t) := \{(x, \bar{u}(x, t; h)) \mid x \in [-1, 1], \ h \geq M_0 + h_0\} = \{(x, y) \mid x \in [-1, 1], \ y \geq \bar{u}(x, t; M_0 + h_0)\}. \]
Then $D(t)$ is the upper half of $\Omega$ with bottom $\{(x, \bar{u}(x, t; h)) \mid x \in [-1, 1]\}$, which moves upward with speed $c(h_0)$.

We now construct another lower solution below the real solution $u$ such that it moves faster than $D(t)$, and so pushes $u$ entering the domain $D(t)$ for large $t$. Then, by considering the numbers of the contact points between $u$ and $u(x, t; h)$, we can obtain the desired gradient estimate, which implies that $u_x$ is not too small for $x \in (0, 1)$.

To construct another lower solution below $u$, we notice by [3.1] that $u(x, t)$ moves up to infinity. Hence for any given $h^0 > h_0$, there exists $t^0$ large such that
\[ u(x, t^0) > \varphi(x; h^0) + h^0 \geq h^0, \quad x \in [-1, 1]. \]
Hence,
\[ \bar{u}^*(x, t) := \varphi(x; h^0) + h^0 + c(h^0)t \]
is also a lower solution of (1.3), and by the comparison principle we have
\[ \bar{u}^*(x, t) \leq u(x, t + t^0), \quad x \in [-1, 1], \ t \geq 0. \] (4.2)

Since $c(h^0) > c(h_0)$ we see that for all large $t$ (to say, $t \geq T^0$), $\bar{u}^*(x, t)$ rushes into the domain $D(t)$. So does $u(x, t + t^0)$ due to (1.2). In other words, when $t \geq t^0 + T^0$, $u(x, t)$ contacts $\bar{u}(x, t; h)$ for some $h \geq M_0 + h_0$ at some points. We now study the number of the contact points between them. Set, for each $h \geq M_0 + h_0$,
\[ \eta(x, t; h) := \bar{u}(x, t; h) - u(x, t) = \varphi(x; h_0) + c(h_0)t + h - u(x, t). \]
By $h \geq M_0 + h_0 > u_0(x)$, we see that the number of the zeros of the function $\eta(\cdot, t; h)$, denoted by $Z[\eta(\cdot, t; h)]$, is zero for small $t > 0$. If, for some $t_\ast > 0$, $\eta(\pm 1, t; h) > 0$ for all $t \in [0, t_\ast)$ and $\eta(\pm 1, t_\ast; h) = 0$, then
\[ u(1, t_\ast) = \varphi(1; h_0) + c(h_0)t_\ast + h > h_0, \]
and so
\[ \eta_x(1, t_\ast; h) = \varphi_x(1; h_0) - u_x(1, t_\ast) = h_0 - u(1, t_\ast) < 0, \quad \eta_x(-1, t_\ast; h) = -h_0 + u(-1, t_\ast) > 0. \]
This implies that $\eta(\cdot, t; h)$ has no zero for $t \in [0, t_\ast)$ but has two non-degenerate zeros at $t = t_\ast$.

Using the so-called zero number properties (cf. [1] and [13]) we see that
\[ Z[\eta(\cdot, t; h)] = 0, \quad 1 \text{ or } 2. \]
It is 0 when $t$ is very large or very small. Hence there exist $t_2 > t_1 > 0$ such that
\[ \eta(\cdot, t; h) \begin{cases} \text{no zero,} & \text{for } t > t_2; \\ \text{one degenerate zero,} & \text{for } t = t_2; \\ \text{two non-degenerate zeros,} & \text{for } t \in [t_1, t_2). \end{cases} \]
In particular, in the time interval $[t_1, t_2)$, the two non-degenerate zeroes (denote by $\pm \tilde{x}(t, h)$) satisfy
\[ -1 \leq -\tilde{x}(t, h) < \tilde{x}(t, h) \leq 1, \quad u_x(-\tilde{x}, t) < \varphi_x(-\tilde{x}; h_0) < 0, \quad 0 < \varphi_x(\tilde{x}; h_0) < u_x(\tilde{x}, t). \]
Note that the graphs of \( u(x, t; h) \) for all \( h \geq M_0 + h_0 \) form a solid \( D(t) \) and graph of \( u \) is immersed in \( D(t) \) for all large \( t \). Hence, for any large \( t_1 > 0 \) and any \( x_1 \in [0, 1] \), there exists a unique \( h = \tilde{h}(t_1, x_1) \) such that

\[
u(x, t_1; \tilde{h}) := \varphi(x; h_0) + c(h_0)t_1 + \tilde{h}
\]

contacts \( u(x, t_1) \) exactly at \( \pm x_1 \), and so \( u_x(x_1, t_1) = \varphi_x(x_1; h_0) = 0 \) when \( x_1 = 0 \) and

\[
\varphi_x(x_1; h_0) < u_x(x_1, t_1),
\]
when \( x_1 \in (0, 1] \). In this sense, we say that “the gradient of \( u \) is not too small”.

4.2. Precise upper gradient estimate. We now show that “the gradient of \( u \) is not too large”.

By our assumption \( u_0'(x) < \varphi_0'(x) \) for \( x \in (0, 1) \), we see that \( u_0(x) \leq \varphi_0(x) + u_0(0) \) in \((-1, 1)\), and so

\[
u(x, t) < \varphi_0(x) + u_0(0) + \frac{\pi}{2} t, \quad x \in (-1, 1), \ t > 0
\]

by the comparison principle. On the other hand, for any \( r < u_0(0) \), \( \varphi_0(\cdot) + r - u(x, 0) \) has exactly two non-degenerate zeros. Since \( \zeta(x, t) := \varphi_0(x) + \frac{\pi}{2} t + r - u(x, t) \) satisfies a linear parabolic equation whose coefficients are bounded in any compact interval of \((-1, 1) \times (0, \infty)\), we can use the zero number properties to conclude that, for any \( t > 0 \), either (1) \( \zeta(\cdot, t) \) has two non-degenerate zero \( \pm \rho(t) \) with \( \rho(t) \in (0, 1) \); or (2) \( \zeta(\cdot, t) \) has a unique degenerate zero 0; or (3) \( \zeta(\cdot, t) \) is positive, and has no zeros.

Note that, for any \( t > 0 \), the graph of \( u(\cdot, t) \) is immersed in

\[
\mathcal{E}(t) := \left\{(x, \varphi_0(x) + r + \frac{\pi}{2} t) \bigg| x \in (-1, 1), \ r \leq u_0(0)\right\}.
\]

Hence, for any \( t_1 > 0 \) and any \( x_1 \in (0, 1) \), there exists a unique \( r = R(t_1, x_1) < u_0(0) \) such that \( \zeta(\cdot, t_1) \) with \( r = R(t_1, x_1) \) has zeros exactly at \( x = \pm x_1 \). Consequently,

\[
u_x(x_1, t_1) < \varphi_0'(x_1).
\]

Since \( t_1 > 0 \) and \( x_1 \in (0, 1) \) are arbitrarily given, we actually obtain the following result.

Lemma 4.1. Assume \( u_0 \) and that \( u(x, t) \) is the solution of \( (1.3) \). Then

\[
u_x(x, t) < \varphi_0'(x), \quad x \in (0, 1), \ t > 0.
\]

In this sense, we say that “the gradient of \( u \) is not too large”.

4.3. Convergence of the solution. Assume \( u(x, t) \) is a symmetric solution starting from an initial data satisfying \( (1.1) \). Let \( \{t_n\} \) be a time sequence with \( t_n \to \infty (n \to \infty) \). Set

\[
u_n(x, t) := u(x, t + t_n) - u(0, t_n), \quad x \in [-1, 1], \ -t_n < t < \infty.
\]

For any given small \( \varepsilon > 0 \), by the above results in this section we have

\[
u_x(x; h_0) < u_{nx}(x, t) < \varphi_0'(x), \quad x \in (0, 1 - \varepsilon], \ n \gg 1,
\]

for any \( h_0 > 0 \). Combining with Lemma 3.1 we have

\[
\|u_n(x, t)\|_{C^{1, \alpha}([\varepsilon - 1, 1 - \varepsilon] \times [-T, T])} \leq C_1(\varepsilon, T),
\]

for any \( T > 0 \). By the \( L^p \) estimates, Sobolev embedding theorem and the Schauder estimate we have, for any \( \alpha \in (0, 1) \),

\[
\|u_n(x, t)\|_{C^{2+\alpha, 1}}(\varepsilon - 1, 1 - \varepsilon] \times [-T, T]) \leq C_2(\varepsilon, T).
\]
Finally, since $\psi$ and so we conclude that

$$
\psi((x - 1, 1 - \varepsilon) \times [-T, T]) \rightarrow 0 \quad (i \to \infty).
$$

Using the Cantor’s diagonal argument, we see that there exist a function $U_\varepsilon \in C^{2+\alpha,1+\frac{\beta}{2}}([-1,1] \times [-T, T])$ such that

$$
\|U_n - U_{T, \varepsilon}\|_{C^{2+\alpha,1+\frac{\beta}{2}}([-1,1-\varepsilon] \times [-T, T])} \to 0 \quad (i \to \infty).
$$

Moreover,

$$
\psi((x - 1, 1 - \varepsilon) \times [-T, T]) \rightarrow 0 \quad (i \to \infty),
$$

and a subsequence of $\{U_n\}$ (denoted it again by $\{u_n\}$) such that

$$
u_n \to U \quad (i \to \infty), \quad \text{in } C^{2+\alpha,1+\frac{\beta}{2}}((-1,1) \times \mathbb{R}) \text{ topology.}
$$

Moreover, $U(x,t)$ is an entire solution of the equation in (1.3) with $U(0,0) = 0$ and, by (4.6),

$$
\varphi_x(x;h_0) \leq U_x(x,t) \leq \varphi'(0)(x), \quad x \in [0,1], \ t \in \mathbb{R}.
$$

Since $h_0 > 0$ can be as large as possible and since

$$
\varphi_x(x;h_0) \to \varphi'(0)(x) \quad \text{as} \quad h_0 \to \infty,
$$

we conclude that

$$
U_x(x,t) = \varphi'(0)(x), \quad x \in (-1,1),
$$

and so

$$
U(x,t) = \varphi(0)(x) + C(t), \quad x \in (-1,1), \ t \in \mathbb{R}.
$$

Finally, since $U$ is an entire solution of the equation in (1.3), we see that $C'(t) = \frac{\pi}{2}$ and thus

$$
U(x,t) = \varphi(0)(x) + \frac{\pi}{2} t, \quad x \in (-1,1), \ t \in \mathbb{R}.
$$

The above result implies that $\{u_n\}$ converges to the special grim reaper $\varphi(0)(x) + \frac{\pi}{2} t$. Since this grim reaper is unique and the time sequence $\{t_n\}$ is arbitrarily given, we actually obtain the following result.

**Theorem 4.2.** Assume $u(x,t)$ is the time-global solution of (1.3) with initial data $u_0(x)$ satisfying (4.1). Then, for any $\alpha \in (0,1)$,

$$
u(x,t+s) - u(0,t) \to \varphi(0)(x) + \frac{\pi}{2} t, \quad \text{as} \quad s \to \infty,
$$

in the $C^{2+\alpha,1+\frac{\beta}{2}}((-1,1) \times \mathbb{R})$ topology.

5. General Solutions

The conclusion in the previous section holds only for symmetric solutions. We consider general solutions in this section, that is, we assume $u_0$ satisfies (1.5) in this section.

5.1. **Interior estimates.** We choose a smooth, even function $\psi(x)$ with

$$
0 < \psi'(x) < \varphi'(0)(x), \quad 0 < \psi(x) < u_0(x), \quad \psi''(x) > 0 \quad \text{for} \ x \in (0,1), \quad \psi'(1) = \psi(1).
$$

Then $\psi$ satisfies (4.1) except for $\psi(x) \geq 1$. One example is $\psi(x) = \delta \left[ \sqrt{2} - \frac{4}{\pi+4} \cos \frac{\pi x}{4} \right]$ for small $\delta > 0$. Denote the solution of (1.3) with $u(x,0) = \psi(x)$ by $u(x,t;\psi)$. Then it is symmetric, $u_{xx}(x,t;\psi) > 0$ due to $u_t(x,t;\psi) > 0$, and it satisfies all the conclusions in the previous section. Furthermore, it moves upward monotonically to infinity, so we have

$$
u_0(x) < u(x,T;\psi), \quad x \in [-1,1],
$$

for some positive $T$. Thus, by the comparison principle we have

$$
u(x,t;\psi) < u(x,t;u_0) < u(x,t+T;\psi), \quad x \in [-1,1], \ t > 0.
$$

This formula gives the $L^\infty$ estimate for $u(x,t;u_0)$. 
In what follows, we want to present a uniform interior gradient estimate. First we prove a lemma.

**Lemma 5.1.** For any small \( \varepsilon \in (0, \frac{1}{2}) \) and any \( t > 0 \), there hold
\[
(5.2) \quad \min_{1 - \varepsilon \leq x \leq 1 - \varepsilon} |u_x(x, t; u_0)| < M_1 := \varepsilon^{-1} \left[ \varphi_0(1 - \varepsilon) + \frac{\pi}{2} T \right],
\]
\[
(5.3) \quad \min_{\varepsilon - 1 \leq x \leq 2\varepsilon - 1} |u_x(x, t; u_0)| < M_1.
\]

**Proof.** We only prove the first inequality since the second one is proved similarly. Assume by contradiction that, for some \( t = t_0 > 0 \),
\[
|u_x(x, t_0; u_0)| \geq M_1, \quad x \in [1 - 2\varepsilon, 1 - \varepsilon].
\]
Integrating it over \([1 - 2\varepsilon, 1 - \varepsilon]\) we obtain
\[
(5.3) \quad u(1 - \varepsilon, t_0; u_0) - u(1 - 2\varepsilon, t_0; u_0) \geq \varphi_0(1 - \varepsilon) + \frac{\pi}{2} T.
\]

On the other hand, by \((5.1)\) we have
\[
(5.4) \quad u(1 - \varepsilon, t_0; u_0) - u(1 - 2\varepsilon, t_0; u_0) < u(1 - \varepsilon, t_0 + T; \psi) - u(1 - 2\varepsilon, t_0; \psi) \leq u(1 - \varepsilon, t_0 + T; \psi) - u(0, t_0; \psi)
\]
since \(u(\cdot, t; \psi)\) is convex and symmetric. By Lemma 4.1 \( u(x, t_0; \psi) < \varphi_0(x) + u(0, t_0; \psi) \), and so by comparison we have
\[
u(x, t_0 + T; \psi) < \varphi_0(x) + \frac{\pi}{2} T + u(0, t_0; \psi).
\]
In particular, at \( x = 1 - \varepsilon \) we have
\[
u(1 - \varepsilon, t_0 + T; \psi) - u(0, t_0; \psi) < \varphi_0(1 - \varepsilon) + \frac{\pi}{2} T.
\]
This contradicts \((5.3)\) and \((5.4)\). This proves the lemma. \(\square\)

Using this lemma we can prove the following interior gradient estimates.

**Lemma 5.2.** For any small \( \varepsilon > 0 \), there exists \( T_\varepsilon > 0 \) such that
\[
(5.5) \quad |u_x(x, t; u_0)| \leq M_2, \quad -1 + 2\varepsilon < x < 1 - 2\varepsilon, \quad t > T_\varepsilon,
\]
\[
where \( M_2 := \max\{ M_1, \|u_x(\cdot; T_\varepsilon; u_0)\|_{L^\infty}\} \) and \( M_1 \) is that in \((5.2)\).
\]

**Proof.** Since \( u(x, t; u_0) \to \infty \) as \( t \to \infty \), there exists \( T' > 0 \) large such that \( u(\pm 1, t; u_0) > M_1 \) for all \( t > T' \). Denote \( \zeta(x, t) := u_x(x, t; u_0) - M_1 \), then \( \zeta \) solves
\[
\zeta_t = \frac{\zeta_{xx}}{1 + u_x^2} - \frac{2u_x}{(1 + u_x^2)^2} \zeta^2, \quad -1 < x < 1, \quad t > 0,
\]
and \( \zeta(1, t) > 0 > \zeta(-1, t) \) for \( t > T' \). Using the zero number properties (cf. \(1\)) we conclude that, for some \( T_\varepsilon > T' \), the function \( \zeta(\cdot, t) \) has only non-degenerate zeros for \( t \geq T_\varepsilon \). Denote the largest zero of \( \zeta(\cdot, t) \) in \((-1, 1)\) by \( \rho_+(t) \). Due to the non-degeneracy of \( \rho_+(t) \) we see that \( x = \rho_+(t) \) is a continuous curve. Moreover, \((5.2)\) in the previous lemma indicates that \( \rho_+(t) > 1 - 2\varepsilon \). In a similar way we can find another continuous curve \( x = \rho_-(t) \) for \( t > T_\varepsilon \) \((T_\varepsilon \) can be chosen larger if necessary), with \( \rho_-(t) \in (-1, -1 + 2\varepsilon) \), such that \( u_x(\rho_-(t), t) = -M_1 \) for \( t > T_\varepsilon \). Then, using the maximum principle for \( u_x \) in the domain \( D(T_\varepsilon) := \{(x, t) \mid \rho_-(t) < x < \rho_+(t), \ t > T_\varepsilon\} \) we conclude that \( |u_x(x, t; u_0)| \leq M_2 \) in \( D(T_\varepsilon) \). The estimate \((5.5)\) then follows from the fact \( \rho_-(t) < -1 + 2\varepsilon < 1 - 2\varepsilon < \rho_+(t) \) for \( t > T_\varepsilon \). \(\square\)
5.2. Convergence of general solutions. Let \( \{t_n\} \) be any time sequence with \( t_n \to \infty \), we consider the solution sequence \( \{u(x,t + t_n; u_0) - u(0,t_n; \psi)\} \).

For any given small \( \varepsilon > 0 \) and any \( \tau > 0 \), let \( T_\varepsilon \) be as that in Lemma 5.2 then (5.1) and (5.5) imply that, for all large \( n \), \( u(x,t + t_n; u_0) - u(0,t_n; \psi) \) is bounded in \( C^{1,0}([2\varepsilon - 1, 1 - 2\varepsilon] \times [-\tau, \tau]) \) norm, and the bounds are independent of \( n \). Using the standard parabolic theory we can even have the \( C^{2+\alpha,1+\frac{\alpha}{2}}([2\varepsilon - 1, 1 - 2\varepsilon] \times [-\tau, \tau]) \) (for any \( \alpha \in (0,1) \)) bounds for the solution sequence, also uniform in \( n \), and so we can find a convergent subsequence. Taking \( \varepsilon \to 0 \) and \( \tau \to \infty \), and using the Cantor’s diagonal argument we conclude that, there is a subsequence of \( \{t_n\} \), denoted is again by \( \{t_n\} \), such that

\[
\tag{5.6}
 u(x,t + t_n; u_0) - u(0,t_n; \psi) \to W(x,t) \quad \text{in } C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}((-1,1) \times \mathbb{R}) \quad \text{topology,}
\]

for some entire solution \( W \) of the equation in (1.3). On the other hand, as a consequence of Theorem 4.2 we have, as \( n \to \infty \),

\[
 u(x,t + t_n; \psi) - u(0,t_n; \psi) \to \varphi_0(x) + \frac{\pi}{2} t, \quad u(x,t + T + t_n; \psi) - u(0,t_n; \psi) \to \varphi_0(x) + \frac{\pi}{2} (t + T).
\]

Hence, we conclude from (5.1) that

\[
\tag{5.7}
 \varphi_0(x) + \frac{\pi}{2} \leq W(x,t) \leq \varphi_0(x) + \frac{\pi}{2} t + \frac{\pi}{2} T, \quad x \in (-1,1), \ t \in \mathbb{R}.
\]

Denote

\[
\theta(x,t) := \arctan u_x(x,t), \quad x \in [-1,1], \ t > 0.
\]

Then \( \theta \) satisfies

\[
\begin{align*}
\theta_t &= \cos^2 \theta \cdot \theta_{xx}, \\
\theta(\pm 1,t) &= \pm \arctan u(\pm 1,t), \quad t > 0.
\end{align*}
\]

The global existence of \( u \) implies that \( \theta \) exists for all \( t > 0 \). For any \( T > 0 \), by Lemma 3.1 we have

\[
\tag{5.8}
|\theta(x,t)| \leq \arctan[C(T)], \quad x \in [-1,1], \ t \in [0,T],
\]

and \( \theta(\pm 1,t) = \pm \arctan u(\pm 1,t) \to \pm \frac{\pi}{2} \) as \( t \to \infty \). Moreover, by (5.6) we have

\[
\theta(x,t_n + t) \to \Theta(x,t) := \arctan[W_x(x,t)] \quad \text{in } C^{2+\alpha,1+\frac{\alpha}{2}}_{loc}((-1,1) \times \mathbb{R}) \quad \text{topology}.
\]

1. We claim that \( \Theta(x,t) \) is a stationary solution of \( \theta_t = \cos^2 \theta \cdot \theta_{xx} \). Without loss of generality we only need to prove that \( \Theta(x,0) \) is a stationary one. Denote \( a := \Theta_x(0,0), \ b := \Theta(0,0) \) and \( v(x) := ax + b \), then \( v(x) \) is a stationary solution of \( \theta_t = \cos^2 \theta \cdot \theta_{xx} \), and the function \( \xi(x,t) := \theta(x,t) - v(x) \) satisfies one of the following boundary conditions:

- in case \( v(-1) > -\frac{\pi}{2} \), \( v(1) \geq \frac{\pi}{2} \): \( \xi(-1,t) < 0, \xi(1,t) < 0 \) for all large \( t \);
- in case \( v(-1) > -\frac{\pi}{2} \), \( v(1) < \frac{\pi}{2} \): \( \xi(-1,t) > 0, \xi(1,t) > 0 \) for all large \( t \);
- in case \( v(-1) \leq -\frac{\pi}{2} \), \( v(1) \geq \frac{\pi}{2} \): \( \xi(-1,t) > 0, \xi(1,t) < 0 \) for all large \( t \);
- in case \( v(-1) \leq -\frac{\pi}{2} \), \( v(1) < \frac{\pi}{2} \): \( \xi(-1,t) < 0, \xi(1,t) > 0 \) for all large \( t \).

Moreover, \( \xi \) satisfies the linear equation \( \xi_t = \cos^2 \theta \cdot \xi_{xx} \). Hence the zero number argument is applied to conclude that \( \xi(x,t) \) has no degenerate zeros for all large \( t \). Then using a similar argument as in the proof of [S] Lemma 2.6 we conclude that the \( \omega \)-limit \( \Theta(x,t) - v(x) \) of \( \theta(x,t) - v(x) \) either satisfies (1) \( \Theta(x,t) \equiv v(x) \), or satisfies (2) \( \Theta(x,t) \not\equiv v(x) \) and \( \Theta(x,t) - v(x) \) has no degenerate zeros for each \( t \in \mathbb{R} \). Case (2), however, contradicts the definition of \( v(x) \). Therefore, only case (1) holds. This prove our claim.

2. Next we show that \( \Theta(x,t) \equiv \frac{\pi}{2} x \). In the previous step, we have shown that \( \Theta(x,t) \equiv v(x) \equiv ax + b \) for some \( a, b \in \mathbb{R} \). If \( v(\pm 1) = \pm \frac{\pi}{2} \), then the conclusion is proved. On the contrary, we assume, without loss of generality that, for some \( x_0 \in (0,1) \) and some small \( \delta > 0 \), there holds
\[\Theta(x, t) \equiv v(x) < \frac{\pi}{2}x - \pi\delta \text{ in } x \in [x_0, 1).\] For any given small \(\epsilon > 0\), since \(|\theta(x, t_n) - \Theta(x, 0)| \leq \frac{1}{2}\pi\delta\) in \([-1 + \epsilon, 1 - \epsilon]\) when \(n\) is sufficiently large, we have

\[\arctan u(x, t_n) = \theta(x, t_n) \leq \Theta(x, 0) + \frac{1}{2}\pi\delta = v(x) + \frac{1}{2}\pi\delta < \frac{\pi}{2}(x - \delta), \quad x \in [-1 + \epsilon, 1 - \epsilon].\]

Integrating the above inequalities over \([x_0, 1 - \epsilon]\) we have

\[u(1 - \epsilon, t_n) \leq u(x_0, t_n) + \int_{x_0}^{1-\epsilon} \tan \left[\frac{\pi}{2} (x - \delta)\right] dx = u(x_0, t_n) + \varphi_0(1 - \epsilon - \delta) - \varphi_0(x_0 - \delta).\]

Taking limit as \(n \to \infty\) we have

\[W(1 - \epsilon, 0) \leq W(x_0, 0) + \varphi_0(1 - \epsilon - \delta) - \varphi_0(x_0 - \delta).\]

Combining with (5.7) we have

\[\varphi_0(1 - \epsilon) \leq W(x_0, 0) + \varphi_0(1 - \epsilon - \delta) - \varphi_0(x_0 - \delta).\]

This is a contradiction as \(\epsilon \to 0\). This proves our claim, and so \(\Theta(x, t) \equiv \frac{\pi}{2}x\).

3. As an \(\omega\)-limit of \(\theta\), the function \(\Theta(x, t) \equiv \frac{\pi}{2}x\) is unique. Hence, \(\theta(x, t) \to \frac{\pi}{2}x\) as \(t \to \infty\). Equivalently, \(W(x, t) \equiv \tan[\frac{\pi}{2}x]\), and so

\[W(x, t) = \varphi_0(x) + K(t).\]

Since \(W(x, t)\) is an entire solution, by its equation we have \(W_t(x, t) \equiv \frac{\pi}{2}\), and so \(K(t) = \frac{\pi}{2}t + K_0\) for some \(K_0 \in \mathbb{R}\). By (5.7) we actually have \(K_0 \in [0, \frac{\pi}{2}T]\). Finally, we obtain the following proposition, which completes the proof of our main theorem.

**Proposition 5.3.** There exists \(K_0 \in [0, \frac{\pi}{2}T]\) such that

\[W(x, t) = \varphi_0(x) + \frac{\pi}{2}t + K_0, \quad x \in (-1, 1), \ t \in \mathbb{R}.\]

**Remark 5.4.** We see from sections 4 and 5 that the zero number argument plays a key role in the uniform interior gradient estimates, which does not rely on the boundedness of \(u\) and can be extended to other problems with unbounded solutions (but only for one dimensional problems).

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