1. Introduction

Let $\bar{\rho}: G_\mathbb{Q} \to \text{GL}_2(k)$ be an absolutely irreducible modular Galois representation over a finite field $k$ of characteristic $p$. Assume further that $\bar{\rho}$ is $p$-ordinary and $p$-distinguished in the sense that the restriction of $\bar{\rho}$ to a decomposition group at $p$ is reducible and non-scalar. The Hida family $\mathcal{H}(\bar{\rho})$ of $\bar{\rho}$ is the set of all $p$-ordinary $p$-stabilized newforms $f$ with mod $p$ Galois representation isomorphic to $\bar{\rho}$. (If $\bar{\rho}$ is unramified at $p$, then one must also fix an unramified line in $\bar{\rho}$ and require that the ordinary line of $f$ reduces to this fixed line.) These newforms are a dense set of points in a certain $p$-adic analytic space of overconvergent eigenforms, consisting of an intersecting system of branches (i.e. irreducible components) $T(\mathfrak{a})$ indexed by the minimal primes $\mathfrak{a}$ of a certain Hecke algebra.

To each modular form $f \in \mathcal{H}(\bar{\rho})$ one may associate the Iwasawa invariants $\mu^\text{an}(f)$, $\lambda^\text{an}(f)$, $\mu^\text{alg}(f)$, and $\lambda^\text{alg}(f)$. The analytic (resp. algebraic) $\lambda$-invariants are the number of zeroes of the $p$-adic $L$-function (resp. of the characteristic power series of the dual of the Selmer group) of $f$, while the $\mu$-invariants are the exponents of the powers of $p$ dividing the same objects. In this paper we prove the following results on the behavior of these Iwasawa invariants as $f$ varies over $\mathcal{H}(\bar{\rho})$.

**Theorem 1.** Fix $\ast \in \{\text{alg}, \text{an}\}$. If $\mu^\ast(f_0) = 0$ for some $f_0 \in \mathcal{H}(\bar{\rho})$, then $\mu^\ast(f) = 0$ for all $f \in \mathcal{H}(\bar{\rho})$. (We then write simply $\mu^\ast(\bar{\rho}) = 0$.)

It is conjectured by Greenberg that if a $p$-ordinary modular form $f$ of weight two has a residually irreducible Galois representation, then $\mu^\text{an}(f) = \mu^\text{alg}(f) = 0$; Theorem 1 thus shows that this conjecture is equivalent to the corresponding conjecture for modular forms of arbitrary weight.

**Theorem 2.** Fix $\ast \in \{\text{alg}, \text{an}\}$ and assume that $\mu^\ast(\bar{\rho}) = 0$. Let $f_1, f_2 \in \mathcal{H}(\bar{\rho})$ lie on the branches $T(\mathfrak{a}_1), T(\mathfrak{a}_2)$ respectively. Then

\[
\lambda^\ast(f_1) - \lambda^\ast(f_2) = \sum_{\ell | N_1N_2} e_\ell(\mathfrak{a}_2) - e_\ell(\mathfrak{a}_1);
\]

here the sum is over all primes dividing the tame level of $f_1$ or $f_2$ and $e_\ell(\mathfrak{a}_j)$ is a certain explicit non-negative invariant of the branch $T(\mathfrak{a}_j)$ and the prime $\ell$.

The first (resp. second) theorem corresponds to Theorems 3.3.2 and 4.4.5 (resp. Theorems 3.3.3 and 4.4.7) with trivial twist by the mod $p$ cyclotomic character.

Note that the right-hand side of (1.1) is identical both algebraically and analytically. In particular, using work of Kato on the main conjecture of Iwasawa theory for modular forms we obtain the following result.

**Corollary 1.** Assume that $\mu^\text{alg}(\bar{\rho}) = \mu^\text{an}(\bar{\rho}) = 0$. If the main conjecture holds for some $f_0 \in \mathcal{H}(\bar{\rho})$, then it holds for all $f \in \mathcal{H}(\bar{\rho})$. 

The corollary in particular reduces the main conjecture to the case of modular forms of weight two, together with the conjecture on the vanishing of the \( \mu \)-invariant. We also obtain the following result on the variation of \( \lambda \)-invariants in a Hida family.

**Corollary 2.** Fix \( * \in \{ \text{alg}, \text{an} \} \) and assume that \( \mu^*(\bar{\rho}) = 0 \).

1. \( \lambda^*(\cdot) \) is constant on branches of \( \mathcal{H}(\bar{\rho}) \).
2. \( \lambda^*(\cdot) \) is minimized on the branches of \( \mathcal{H}(\bar{\rho}) \) of minimal tame level.

This work was motivated by the paper [15] of Greenberg and Vatsal where Theorems 1 and 2 were obtained for \( p \)-ordinary modular forms of weight two. Our results here help to illuminate the results of [15]; indeed, if two congruent elliptic curves have different \( \lambda \)-invariants, it follows from these results that they must lie on two branches of the associated Hida family with different ramification behavior. One may thus think of the change of \( \lambda \)-invariants in terms of “jumps” as one moves from one branch to another at crossing points (which necessarily occur at non-classical\(^1\) eigenforms).

1.1. **Iwasawa theory.** The papers [11] of Greenberg and [27] of Mazur introduce the following point of view on Iwasawa theory: If \( X \) is a \( p \)-adic analytic family of \( p \)-adic representations of \( G_{\mathbb{Q}} \) interpolating a collection of motivic representations, then there should exist an analytic function (or perhaps a section of an invertible sheaf) \( L \), which we call a \( p \)-adic \( L \)-function, defined on \( X \), and a coherent sheaf \( H \) (the sheaf of Selmer groups) on \( X \), such that the zero locus of \( L \) coincides with the codimension one part of the support of \( H \) (thought of as a Cartier divisor on \( X \)).

The function \( L \) should \( p \)-adically interpolate the (suitably modified) values at \( s = 1 \) of the classical \( L \)-functions attached to the Galois representations at motivic points of \( X \), while at such a motivic point \( x \in X \), the Selmer group \( H_x \) (the fibre of \( H \) at \( x \)) should coincide with the Bloch–Kato Selmer group (computed with respect to the deRham condition at \( p \)).

The \( L \)-function considered above is what is usually known as the analytic \( p \)-adic \( L \)-function in classical Iwasawa theory; the equation of the codimension one part of the support of \( H \) is usually known as the algebraic \( p \)-adic \( L \)-function. The equality of the zero locus of \( L \) with the divisorial part of the support of \( H \) is thus a statement of the main conjecture of Iwasawa theory for the family of Galois representations \( X \).

In this paper, we restrict to the case of two dimensional nearly ordinary modular representations, in which case the space \( X \) can be described via Hida theory. Consider a nearly ordinary residual representation; such a representation may be written in the form \( \bar{\rho} \otimes \omega^i : G_{\mathbb{Q}} \to \text{GL}_2(k) \), where \( \bar{\rho} \) is as above, \( \omega \) denotes the mod \( p \) cyclotomic character, and \( 0 \leq i \leq p - 1 \). As \( \bar{\rho} \) is ordinary, we may (and do) fix a \( p \)-stabilisation of \( \bar{\rho} \otimes \omega^i \): a choice of one dimensional \( G_{\mathbb{Q}_p} \)-invariant quotient of \( \bar{\rho} \otimes \omega^i \).

Consider the universal deformation space \( Y^\text{ord} \) parameterising all nearly ordinary deformations of the \( p \)-stabilized representation \( \bar{\rho} \otimes \omega^i \) which are unramified at primes in \( \Sigma \cup \{ p \} \) for some fixed finite set of primes \( \Sigma \). Any such deformation is of the form \( \rho \otimes \omega^i \otimes \chi \), where \( \rho \) is an ordinary deformation of \( \bar{\rho} \), and \( \chi \) is a character of \( \Gamma := 1 + p\mathbb{Z}_p \). (We regard \( \chi \) as a Galois character by composing it with

\(^1\)To avoid circumlocutions, we adopt the convention that weight one is a non-classical weight.
the projection onto $\Gamma$ of the $p$-adic cyclotomic character.) If we let $Y_{\Sigma}^{\text{ord}}$ denote the universal deformation space parameterising all ordinary deformations of the $p$-stabilized representation $\bar{\rho}$ which are unramified at primes in $\Sigma \cup \{p\}$, then $Y_{\Sigma}^{\text{ord}}$ is equal to the product (as formal schemes) of $Y_{\Sigma}^{\text{ord}}$ and $\text{Spf} \left( \mathbb{Z}_p[[\Gamma]] \right)$. By results of Wiles and Taylor-Wiles [34, 33], as strengthened by Diamond [6], the deformation spaces $Y_{\Sigma}^{\text{n,ord}}$ can be identified with various local pieces of the universal ordinary Hecke algebra constructed by Hida (at least under mild hypotheses on $\bar{\rho}$).

If we allow $\Sigma$ to vary, the spaces $Y_{\Sigma}^{\text{n,ord}}$ form a formal Ind-scheme over $W(k)$ parameterizing nearly ordinary deformations of $\bar{\rho}$ of arbitrary conductor. Let $Y_{\Sigma}^{\text{n,ord}}$ be an irreducible component of this formal Ind-scheme: we may write $Y_{\Sigma}^{\text{n,ord}}$ as the product of $Y_{\Sigma}^{\text{ord}}$ (an irreducible component of the formal Ind-scheme parameterizing ordinary deformations of $\bar{\rho}$) and $\text{Spf} \left( \mathbb{Z}_p[[\Gamma]] \right)$. The motivic points are dense on $Y_{\Sigma}^{\text{n,ord}}$ and all have the same tame conductor, which we will denote by $N$. If $T_N^{\text{new}}$ denotes the new-quotient of the Hida Hecke algebra of level $N$, then $Y_{\Sigma}^{\text{ord}}$ corresponds to a minimal prime $a$ of $T_N^{\text{new}}$, and we set $X_{\Sigma}^{\text{ord}} = \text{Spf} \left( T_N^{\text{new}} / a \right)$ and $X_{\Sigma}^{\text{n,ord}} = \text{Spf} \left( T_N^{\text{new}} / a [[\Gamma]] \right)$. (We point out that $X_{\text{ord}}$ is nearly the same as $X_{\text{ord}}$ – and hence $X_{\text{n,ord}}$ is nearly the same as $X_{\text{n,ord}}$. Indeed, $X_{\text{ord}}$ is a finite cover of $Y_{\text{ord}}$ and maps isomorphically to it after inverting $p$; see section 2.4 for details.)

The generalities of Iwasawa theory discussed above apply to this space $X_{\text{n,ord}}$. In section 4 we give a construction of the analytic $p$-adic $L$-function on $X_{\text{n,ord}}$; as is usual, we regard it as a “function of two-variables” – one variable moving along $X_{\text{ord}}^*$, and the other along $\text{Spf} \left( \mathbb{Z}_p[[\Gamma]] \right)$. In our discussion of Selmer groups we do not go so far as to construct a coherent sheaf of Selmer groups over the space $X_{\text{n,ord}}$; for this, the reader should refer to recent and forthcoming work of Ochiai [29].

An important feature of the situation we consider is that the nearly ordinary deformation space contains many different irreducible components. The main goal of this paper is to describe how to pass Iwasawa theoretic information (such as knowledge of the main conjecture) between components.

1.2. Two-variable $p$-adic $L$-functions. To establish the analytic parts of Theorems 1 and 2 we make use of a two-variable $p$-adic $L$-function (playing the role of $L$ above). As already remarked upon in [14] one variable is a parameter of wild character space (i.e. the standard cyclotomic variable) and the second variable runs through the Hida family (which is a finite cover of weight space) of some fixed residual representation $\bar{\rho}$. Of the many constructions of two-variable $p$-adic $L$-functions (see [14], [32], [30]), our construction follows most closely that of Mazur [28] and Kitigawa [22], with two main differences.

The first difference is that by using the fact that the Hecke algebras are known to be Gorenstein (as in [24]) we may construct these $L$-function with fewer assumptions on $\bar{\rho}$. The second difference is that we do not limit ourselves to working solely on a part of the Hida family parameterising newforms of some fixed tame level.

For each branch $Y$ of the Hida family we construct a two-variable $L$-function along $Y$, which at each classical point specialises to the $p$-adic $L$-function of the corresponding newform computed with respect to its canonical period. (Actually, the two-variable $L$-function is defined not on $Y$, but on the partial normalisation $\overline{X}$ of $Y$ discussed above in [14].) However, for any finite set of primes $\Sigma$ not containing $p$, we also construct a two-variable $L$-function along $Y_{\Sigma}$ (the Hida family of $\bar{\rho}$ with ramification only at the primes of $\Sigma$). If we specialize this two-variable $p$-adic $L$-function at a classical point, we obtain a non-primitive $p$-adic $L$-function attached.
to the corresponding newform (i.e. the usual $p$-adic $L$-function stripped of its Euler factors at primes of $\Sigma$). In fact, we prove that this non-primitivy occurs in families: the $p$-adic $L$-function on the Hida family $Y_\Sigma$, restricted to some branch $Y$, is equal to the $p$-adic $L$-function of that branch stripped of its two variable Euler factors at the primes of $\Sigma$.

With this construction in hand, the analytic part of Theorem 1 follows immediately. Indeed, the vanishing of the analytic $\mu$-invariant for any particular form $f$ in the Hida family is equivalent to the vanishing of the $\mu$-invariant of the two-variable $p$-adic $L$-function along the branch $Y$ containing $f$, which in turn is equivalent to the vanishing of the $\mu$-invariant of the non-primitive two-variable $L$-function on $Y_\Sigma$ for any sufficiently large choice of $\Sigma$ (i.e. such that $Y_\Sigma$ contains $Y$). This last condition is independent of $f$, and so is equivalent to the vanishing of the $\mu$-invariant for every modular form in the family.

The analytic part of Theorem 2 also follows from this construction. The analytic $\lambda$-invariant of a modular form $f$ is equal to the $\lambda$-invariant of the two-variable $L$-function attached to the branch containing $f$. To compare the $\lambda$-invariants of two branches, we choose $\Sigma$ large enough so that $Y_\Sigma$ contains both of these branches. We then relate each $\lambda$-invariant to the $\lambda$-invariant of the two-variable $p$-adic $L$-function attached to $Y_\Sigma$. As explained above, the difference of the two $\lambda$-invariants is then realized in terms of the $\lambda$-invariants of certain Euler factors. The quantity $e_\ell(a_i)$ appearing in the formula of Theorem 2 is precisely the $\lambda$-invariant of the Euler factor at $\ell$ along the branch corresponding to $a_i$.

We point out that we found it necessary to use two-variable $L$-functions and Hida theory in order to compare the $\lambda$- and $\mu$-invariants of two congruent modular forms of different weights. It would be interesting to know if these results could be obtained with different methods that do not depend upon using families of modular forms.

1.3. Residual Selmer groups. Let $\bar{\rho} : G_\mathbb{Q} \to \text{GL}_2(k)$ be as above. For any $f \in \mathcal{H}(\bar{\rho})$, say with Fourier coefficients in the finite extension $K$ of $\mathbb{Q}_p$, Greenberg has defined the Selmer group $\text{Sel}(\mathbb{Q}_\infty, \rho_f)$ as the subgroup of $H^1(\mathbb{Q}_\infty, A_f)$ cut out by local conditions, all of which are as strong as possible except for that at $p$; here $\mathbb{Q}_\infty$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and $A_f$ is $(K/O_K)^2$ with Galois action via $\rho_f$. If one defines $\text{Sel}(\mathbb{Q}_\infty, \bar{\rho})$ in the analogous way, one is confronted with the fundamental problem that the $\pi$-torsion on $\text{Sel}(\mathbb{Q}_\infty, \rho_f)$ may be larger than $\text{Sel}(\mathbb{Q}_\infty, \bar{\rho})$; this is, of course, precisely the reason why congruent modular forms need not have isomorphic Selmer groups.

In [15] this issue is overcome by introducing non-primitive Selmer groups of $\bar{\rho}$; essentially, if $f_1, f_2 \in \mathcal{H}(\bar{\rho})$ have tame levels $N_1$ and $N_2$ respectively, then the $\pi$-torsion on $\text{Sel}(\mathbb{Q}_\infty, \rho_{f_1})$ and $\text{Sel}(\mathbb{Q}_\infty, \rho_{f_2})$ can both be compared to the non-primitive Selmer group of $\bar{\rho}$ obtained by ignoring the local conditions at primes dividing $N_1N_2$.

Although this approach can also be made to work in the higher weight case, here we proceed somewhat differently. Following Mazur [20], we allow non-strict, but not necessarily vacuous, local conditions $S$ on the cohomology of $\bar{\rho}$, resulting in a family of residual Selmer groups $\text{Sel}_S(\mathbb{Q}_\infty, \bar{\rho})$. We show that for any $f \in \mathcal{H}(\bar{\rho})$ there is a local condition $S(f)$ such that

$$\text{Sel}_S(f)(\mathbb{Q}_\infty, \bar{\rho}) \cong \text{Sel}(\mathbb{Q}_\infty, \rho_f)[\pi].$$
It follows that the Iwasawa invariants of \( f \) can be recovered from the residual Selmer group \( \text{Sel}_{S(f)}(\mathbb{Q}_\infty, \bar{\rho}) \).

In fact, if \( f \) has tame level equal to the conductor of \( \bar{\rho} \), we show that \( \text{Sel}_{S(f)}(\mathbb{Q}_\infty, \bar{\rho}) \) agrees with the naive residual Selmer group \( \text{Sel}(\mathbb{Q}_\infty, \bar{\rho}) \). Since such \( f \) always exist by level lowering, we are then able to use duality results to show that the difference

\[
\dim_k \text{Sel}_{S(f)}(\mathbb{Q}_\infty, \bar{\rho}) - \dim_k \text{Sel}(\mathbb{Q}_\infty, \bar{\rho})
\]

is precisely given by the dimensions of the local conditions \( S(f) \). The main algebraic results follow from this.

We note that Hida theory plays little overt role in these results. In particular, these methods may well apply more generally for algebraic groups other than \( \text{GL}_2 \). The key inputs are automorphic descriptions of the Galois representation at ramified primes, level lowering results, and the fact that the Selmer groups of interest are \( \Lambda \)-cotorsion; the remainder of the argument is essentially formal.

1.4. \( L \)-functions modulo \( p \). In the paper [25], Mazur raises the question of whether one can define a mod \( p \) \( L \)-function attached to the residual representation \( \bar{\rho} \) and a choice of tame conductor \( N \) (divisible by the tame conductor of \( \bar{\rho} \)). The construction discussed in [22] gives a positive answer to this question, with the caveat that the appropriate extra data is not simply a choice of tame conductor \( N \), but rather the more precise data of a component \( Y \) of the universal deformation space of \( \bar{\rho} \). We may then specialise the \( p \)-adic \( L \)-function at the closed point of the partial normalisation \( X \) of \( Y \) on which it is defined, so as to obtain a mod \( p \) \( L \)-function attached to \( \bar{\rho} \) and \( Y \).

We also show that the local condition \( S(f) \) discussed in the preceding section depends only on the component \( Y \) of the Hida family that contains \( f \); we may thus write \( S(Y) \) for \( S(f) \). Assuming that both the analytic and algebraic \( \mu \)-invariant of \( \bar{\rho} \) vanish, Theorem [2] then shows that the main conjecture for any member of \( \mathcal{H}(\bar{\rho}) \) is equivalent to the following mod \( p \) main conjecture (for one, or equivalently every, choice of \( Y \)):

\[
\text{Mod } p \text{ Main Conjecture. Let } Y \text{ be a branch of } \mathcal{H}(\bar{\rho}). \text{ Then the mod } p \text{ } L \text{-function } L_p(Y) \text{ of } Y \text{ is non-zero, the Selmer group } \text{Sel}_{S(Y)}(\mathbb{Q}_\infty) \text{ is finite, and }
\lambda(L_p(Y)) = \dim_k \text{Sel}_{S(Y)}(\mathbb{Q}_\infty, \bar{\rho}).
\]

1.5. Overview of the paper. In the following section, we recall the Hida theory used in this paper. In particular, we discuss the Hida family attached to a residual representation \( \bar{\rho} \), its decomposition into irreducible components and the various Galois representations attached to this family, with an emphasis on the integral behavior.

In the third section, we prove the algebraic parts of Theorems [1] and [2] via the theory of residual Selmer groups and the use of level lowering to reduce to the minimal case.

In the fourth section, we construct two-variable \( p \)-adic \( L \)-functions on the Hida family of \( \bar{\rho} \) and on each of its irreducible components. By relating these two constructions to each other and to the construction of classical \( p \)-adic \( L \)-functions, we obtain proofs of the analytic parts of Theorems [1] and [2].

In the final section we give applications to the main conjecture. We discuss some explicit examples to illustrate the general theory, including the congruence modulo 11 between the elliptic curve \( X_0(11) \) and the modular form \( \Delta \).
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Notation. We write $G_\mathbb{Q}$ for the absolute Galois group of $\mathbb{Q}$; if $\Sigma$ is a finite set of
places of $\mathbb{Q}$, we write $\mathbb{Q}_\Sigma$ for the maximal extension of $\mathbb{Q}$ unramified away from
$\Sigma$. We fix an odd prime $p$ and let $\mathbb{Q}_\infty$ denote the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.
For each prime $\ell$ of $\mathbb{Q}$ (resp. place $v$ of $\mathbb{Q}_\infty$) fix a decomposition group $G_\ell \hookrightarrow G_\mathbb{Q}$
(resp. $G_v \hookrightarrow G_{\mathbb{Q}_\infty}$) with inertia group $I_\ell$ (resp. $I_v$) such that $G_v \subseteq G_\ell$ whenever $v$
divides $\ell$; note that $I_\ell = I_v$ for a place $v$ dividing a prime $\ell \neq p$. Let $v_p$ denote the
unique place of $\mathbb{Q}_\infty$ above $p$.

We write $\mathbb{C}_\ell : G_\ell \rightarrow Z_p^\times$ for the $p$-adic cyclotomic character. We let $\Gamma$ denote the
group of 1-units in $Z_p^\times$; the cyclotomic character induces a canonical isomorphism
$\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong \Gamma$. If $\mathcal{O}$ is a $\mathbb{Z}_p$-algebra, we let $\Lambda_{\mathcal{O}}$ denote the completed group
ring $\mathcal{O}[\Gamma]$; we simply write $\Lambda$ for $\Lambda_{\mathbb{Z}_p}$. We denote the natural map $\Gamma \rightarrow \Lambda^\times$ by
$\gamma \mapsto \langle \gamma \rangle$. Recall that $\Lambda$ is a complete local ring of dimension two, non-
canonically isomorphic to the power series ring $Z_p[[T]]$. (Such an isomorphism is obtained
by choosing a topological generator $\gamma$ of $\Gamma$ and mapping $\langle \gamma \rangle$ to $T + 1$.)

We let $\ell$ denote the field of fractions of $\Lambda$.

We let $\Delta$ denote the group of cyclotomic units of $Z_p^\times$; there is then an isomorphism
$\Gamma \times \Delta \cong Z_p^\times$. We let $\omega$ denote the inclusion of $\Delta$ in $Z_p^\times$, so that
$\text{Hom}(\Delta, Z_p^\times) = \{\omega^i \mid 0 \leq i \leq p - 2\}$.

If $Z_p,(i)$ denotes $Z_p$ regarded as a $\Delta$-module via the character $\omega^i$, then there is a
natural isomorphism of $\mathbb{Z}_p[\Delta]$-algebras $Z_p[\Delta] \cong \prod_{i=0}^{p-2} Z_p,(i)$. Combining this with the
natural isomorphism $Z_p[[Z_p^\times]] \cong Z_p[\Delta] \otimes_{\mathbb{Z}_p} \Lambda$, we obtain an isomorphism of $Z_p[[Z_p^\times]]$-algebras

$$Z_p[[Z_p^\times]] \cong \prod_{i=0}^{p-2} \Lambda_{(i)},$$

where $\Lambda_{(i)}$ denotes a copy of $\Lambda$, enhanced to a $Z_p[[Z_p^\times]]$-algebra by having $\Delta$ act
through $\omega^i$.

If $\wp$ is a height one prime of $\Lambda$, then we write $\mathcal{O}(\wp) := \Lambda/\wp$. Note that $\mathcal{O}(\wp)$ is
a complete local domain of dimension one, either isomorphic to $F_p[[\Gamma]]$ (if $\wp = p\Lambda$)
or else free of finite rank over $Z_p$ (in which case we will say that $\wp$ is of residue
characteristic zero). The natural embedding $\Gamma \rightarrow \Lambda^\times$ induces a character $\kappa_{\wp} : \Gamma \rightarrow
\mathcal{O}(\wp)^\times$. We say that $\wp$ is classical if it is of residue characteristic zero, and if there
is a finite index subgroup $\Gamma'$ of $\Gamma$ such that $\kappa_{\wp}$, when restricted to $\Gamma'$, coincides with
the character $\gamma \mapsto \gamma^k \in Z_p^\times \subset \mathcal{O}(\wp)^\times$, for some integer $k \geq 2$. If we wish to specify $k$,
then we will say that $\wp$ is classical of weight $k$.

More generally, if $\wp$ is a height one prime ideal of a finite $\Lambda$-algebra $\mathcal{T}$ then we
will again write $\mathcal{O}(\wp) := \mathcal{T}/\wp$, and if $\wp' := \wp \cap \Lambda$, we will write $\kappa_{\wp'} := \kappa_{\wp}' : \Gamma \rightarrow
\mathcal{O}(\wp')^\times \subset \mathcal{O}(\wp)^\times$. We say that $\wp$ is classical (of weight $k$) if $\wp'$ is classical (of weight $k$).

If $N$ is a natural number prime to $p$, then we will write $Z_p,N := (Z/N)^\times \times Z_p^\times$. We
will have occasion to consider the completed group ring $Z_p[[Z_p,N]]$. In this situation,
we will extend the diamond bracket notation introduced above, and write $x \mapsto \langle x \rangle$
to denote the natural map $Z_p,N \rightarrow Z_p[[Z_p,N]]^\times$. 

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2. Hida theory

2.1. The universal ordinary Hecke algebra. We begin by briefly recalling Hida’s construction of the universal ordinary Hecke algebra of tame level $N$.

Given an integer $k$, we let $S_k(Np^\infty, \mathbb{Z}_p)$ denote the space of weight $k$ cusp forms that are on $\Gamma_1(Np^r)$ for some $r \geq 0$, and whose $q$-expansion coefficients (at the cusp $\infty$) lie in $\mathbb{Z}_p$. We define an action of $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ on $S_k(Np^\infty, \mathbb{Z}_p)$ by taking the product of the nebentypus action with the character $\gamma \mapsto \gamma^k$; in this way $S_k(Np^\infty, \mathbb{Z}_p)$ becomes a $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$-module. Also, for each prime $\ell \neq p$, the Hecke operator $T_\ell$ acts on $S_k(Np^\infty, \mathbb{Z}_p)$, as does the Hecke operator $U_p$. Finally, the group $(\mathbb{Z}/N)^\times$ acts on $S_k(Np^\infty, \mathbb{Z}_p)$, via the nebentypus action. Thus if we write

$$H := \mathbb{Z}_p[[\mathbb{Z}_p^\times]][\{T_\ell\}_{\ell \neq p}, U_p],$$

then $S_k(Np^\infty, \mathbb{Z}_p)$ is naturally an $H$-module. More generally, for any $\mathbb{Z}_p$-algebra $R$, we write $S_k(Np^\infty, R) := S_k(Np^\infty, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} R$. The $H$-action on $S_k(Np^\infty, \mathbb{Z}_p)$ extends uniquely to an $R$-linear action on $S_k(Np^\infty, R)$.

Taking $q$-expansions yields an injection of $\mathbb{Q}_p$-algebras

$$\bigoplus_{k \geq 0} S_k(Np^\infty, \mathbb{Q}_p) \rightarrow \mathbb{Q}_p[[q]];$$

we let $D$ denote the $\mathbb{Z}_p$-subalgebra of the source of (2.1) obtained as the preimage of the subalgebra $\mathbb{Z}_p[[q]]$ of its target. (This is a variant of Katz’s ring of divided congruences [21].) If $D$ denotes the $p$-adic completion of $D$, then $\hat{D}$ is naturally isomorphic to Katz’s ring of generalised $p$-adic modular functions [21]. One shows that the action of $H$ on the source of (2.1) (induced by the action on each of the direct summands) restricts to an action of $H$ on $D$. This in turn induces an action of $H$ on $\hat{D}$.

Hida has defined the ordinary projector $e^{\text{ord}}$ acting on $\hat{D}$; it cuts out the submodule $\hat{D}^{\text{ord}}$ of $\hat{D}$ on which $U_p$ acts as an isomorphism. The projector $e^{\text{ord}}$ commutes with the operators in $H$, and so $\hat{D}^{\text{ord}}$ is closed under the action of this algebra. The action of $H$ on $\hat{D}$ induces a map

$$H \rightarrow \text{End}_{\mathbb{Z}_p}(\hat{D}^{\text{ord}}).$$

Definition 2.1.1. We let $T_N$ denote the image of the map (2.2), and refer to $T_N$ as the universal ordinary Hecke algebra of tame level $N$.

Before stating Hida’s results concerning $T_N$, we introduce some additional notation. For any $\mathbb{Z}_p$-algebra $R$, and any integer $k$, we let $S_k(Np^\infty, R)^{\text{ord}}$ denote the $R$-submodule of $S_k(Np^\infty, R)$ obtained as the image of the idempotent $e^{\text{ord}}$. This is an $R$-submodule of $S_k(Np^\infty, R)$, closed under the action of $H$, and by construction, the action of $H$ on $S_k(Np^\infty, R)^{\text{ord}}$ factors through its quotient $T_N$.

If $\kappa$ is any element of $\text{Hom}(\Gamma, R^\times)$, we let $S_k(Np^\infty, R)^{\text{ord}}[\kappa]$ denote the $R$-submodule of $S_k(Np^\infty, R)^{\text{ord}}$ on which $\Gamma$ acts via the character $\kappa$. Note that this is in fact a finitely generated $R$-module, because the $p$-part of the conductor of a $p$-ordinary eigenform is bounded by the $p$-part of the conductor of its nebentypus (unless the eigenform has trivial $p$-nabentypus, in which case the $p$-part of its conductor is bounded by $p$).

Theorem 2.1.2.
The algebra \( T_N \) is free of finite rank over \( \Lambda \). (It is regarded as a \( \Lambda \)-algebra via the inclusion \( \Lambda \subset \mathbb{Z}_p[[Z_{p,N}^\infty]] \).)

(2) If \( \psi \) is a classical height one prime in \( \Lambda \) of weight \( k \), then the surjection

\[
\mathcal{H} \to T_N \to T_N/\psi T_N
\]

identifies \( T_N/\psi T_N \) with the quotient of the Hecke algebra \( \mathcal{H} \) that acts faithfully on the space \( S_k(Np^\infty, \mathcal{O}(\psi))^{\text{ord}}[\kappa_\psi] \).

**Proof.** See [18, Thm. 3.1, Cor. 3.2], and [17, Thm. 1.1, Thm. 1.2]. \( \square \)

Part (1) of this theorem implies that \( T_N \) is a complete Cohen-Macaulay semi-local ring of dimension two.

It follows from part (2) of the theorem, together with the usual duality between spaces of modular forms and Hecke algebras, that a classical height one prime ideal \( \mathfrak{p} \) in \( T_N \) gives rise to a normalised Hecke eigenform in \( S_k(Np^\infty, \mathcal{O}(\psi))^{\text{ord}}[\kappa_\psi] \). We denote this eigenform by \( f_\mathfrak{p} \).

Suppose conversely that \( f \) is a normalised Hecke eigenform in \( S_k(Np^\infty, K)^{\text{ord}}[\kappa] \), for some finite extension \( K \) of \( \mathbb{Q}_p \), and some character \( \kappa : \Gamma \to K^\times \). If \( \mathcal{O}(f) \) denotes the finite extension of \( \mathbb{Z}_p \) generated by the \( q \)-expansion coefficients of \( f \), then \( \kappa \) in fact takes values in \( \mathcal{O}(f)^\times \), and \( f \) lies in \( S_k(Np^\infty, \mathcal{O}(f))^{\text{ord}}[\kappa] \). The Hecke action on \( f \) induces a homomorphism \( \mathcal{H} \to \mathcal{O}(f) \), which factors as

\[
\mathcal{H} \to T_N \to T_N/\mathfrak{p} \mathcal{O} \cong \mathcal{O}(f)
\]

for some weight \( k \) classical height one prime ideal \( \mathfrak{p} \) of \( T_N \), and some isomorphism \( \iota : \mathcal{O}(\mathfrak{p} \mathcal{O}) \cong \mathcal{O}(f) \). By construction, the isomorphism \( \iota \) identifies the normalised eigenform \( f_{\mathfrak{p} \mathcal{O}} \) corresponding to \( \mathfrak{p} \mathcal{O} \) with \( f \), and the character \( \kappa_{\mathfrak{p} \mathcal{O}} \) with the character \( \kappa \).

To summarise, the classical height one primes in \( T_N \) correspond to Galois conjugacy classes of \( p \)-ordinary normalised Hecke eigenforms of tame level \( N \) and weight \( k \geq 2 \) defined over an algebraic closure of \( \mathbb{Q}_p \).

We close this section by remarking that it follows from Theorem 2.2 that the algebra \( T_N \) acts faithfully on the space \( S_k(Np^\infty)^{\text{ord}} \), for any (fixed) weight \( k \geq 2 \). (Part (2) of the theorem shows that any element that annihilates this space lies in the intersection of the ideals \( \mathfrak{p} T_N \), where \( \mathfrak{p} \) ranges over all classical height one prime ideals of weight \( k \). Since \( T_N \) is free over \( \Lambda \), and since the intersection of all such ideals \( \mathfrak{p} \) is the zero ideal of \( \Lambda \), we see that any such element must vanish.) Thus we could have defined \( T_N \) by considering any fixed weight \( k \geq 2 \), rather than taking the direct sum over all weights. (This is the approach to defining \( T_N \) taken in [18].) Similarly, we could have defined \( T_N \) by forming the direct sum over all weights \( k \) of the spaces of cusp forms with trivial \( p \)-power nebentypus and \( \mathbb{Q}_p \) coefficients, and then forming the analogue of \( D \) and \( \hat{D} \) above. (This is the approach to defining \( T_N \) taken in [17].) We chose to incorporate all weights and \( p \)-power levels in our definition of \( T_N \) so as to have specialisations to all weights and nebentypus characters be made apparent from the outset.

### 2.2. The new quotient

Given an integer \( k \), we denote by \( S_k(Np^\infty, \mathbb{Z}_p)^{\text{ord}} \) the subspace of \( S_k(Np^\infty, \mathbb{Z}_p)^{\text{ord}} \) that consists of forms that are spanned (over \( \mathbb{Q}_p \)) by forms that are old at level \( N \) (i.e. that arise from \( p \)-ordinary forms of level \( p^\infty M \), for some proper divisor \( M \) of \( N \), by applying one of the various degeneracy maps \( q \mapsto q^d \), corresponding to the divisors \( d \) of \( N/M \)). Note that \( S_k(Np^\infty, \mathbb{Z}_p)^{\text{ord}} \) is
closed under the action of $T_N$ on $S_k(Np^\infty, Z_p)^{\text{ord}}$. We let $S_k(Np^\infty, Z_p)^{\text{new}}$ denote the $T_N$-module obtained as the quotient of $S_k(Np^\infty, Z_p)^{\text{ord}}$ by its submodule $S_k(Np^\infty, Z_p)^{\text{old}}$. Note that by construction $S_k(Np^\infty, Z_p)^{\text{ord}}$ is $p$-torsion free.

If $R$ is any $Z_p$-algebra, then we may apply an extension of scalars from $Z_p$ to $R$ to each of the modules of oldforms and newforms constructed in the preceding paragraph. We denote the resulting modules in the obvious way; they sit in the short exact sequence

$$0 \to S_k(Np^\infty, R)^{\text{ord}} \to S_k(Np^\infty, R)^{\text{ord}} \to S_k(Np^\infty, R)^{\text{new}} \to 0.$$ 

If $\kappa \in \text{Hom}(\Gamma, R^\times)$, let $S_k(Np^\infty, R)^{\text{new}}[\kappa]$ denote the submodule of $S_k(Np^\infty, R)^{\text{new}}$ on which $\Gamma \subset \Lambda^\times \subset T_N^\times$ acts through $\kappa$.

**Definition 2.2.1.** We let $T_N^{\text{new}}$ denote the maximal quotient of $T_N$ that acts faithfully on any one (or equivalently all) of the modules $S_k(Np^\infty, Z_p)^{\text{ord}}$. (The claimed equivalence follows from the discussion at the end of the preceding section.)

The following result is the analogue for $T_N^{\text{new}}$ of Theorem 2.1.2. We include a proof, since it is phrased slightly differently to the corresponding results in [18] and [17].

**Theorem 2.2.2.**

1. $T_N^{\text{new}}$ is a finite and reduced torsion free $\Lambda$-algebra.
2. The classical height one primes of $T_N^{\text{new}}$ correspond (under pull-back) to those classical height one primes $\wp$ of $T_N$ for which the corresponding normalised eigenform $f_\wp$ (as described in the discussion at the end of section [2.1]) is of tame conductor $N$.
3. If $\wp$ is a classical height one prime ideal of $\Lambda$, then $T_N^{\text{new}} \otimes \Lambda_\wp$ is a finite étale extension of the discrete valuation ring $\Lambda_\wp$.

**Proof.** Let us define $T_N^{\text{old}}$ in analogy to $T_N^{\text{new}}$, and let $I^{\text{old}}$ denote the kernel of the surjection $T_N \to T_N^{\text{old}}$. Recalling that $\mathcal{L}$ denotes the field of fractions of $\Lambda$, consider the short exact sequence

$$0 \to I^{\text{old}} \otimes \Lambda \mathcal{L} \to T_N \otimes \Lambda \mathcal{L} \to T_N^{\text{old}} \otimes \Lambda \mathcal{L} \to 0.$$ 

The tensor product $T_N \otimes \Lambda \mathcal{L}$ is a product of Artin local $\mathcal{L}$-algebras; more precisely, we have an isomorphism

$$T_N \otimes \Lambda \mathcal{L} \cong \prod_{j \in J} (T_N)_{a_j},$$

where $\{a_j\}_{j \in J}$ is the set of minimal primes of $T_N$. From [18] Cor. 3.3] we see that $I^{\text{old}}$ has trivial intersection with the nilradical of $T_N \otimes \Lambda \mathcal{L}$, and thus that $I^{\text{old}} \otimes \Lambda \mathcal{L}$ is isomorphic to a product $\prod(T_N)_{a_j}$, where $a_j$ ranges over a subset of the set of minimal primes, indexed by some subset $J'$ of $J$. The components $(T_N)_{a_j}$ for $j \in J'$ are precisely the primitive components of $T_N \otimes \Lambda \mathcal{L}$, in the terminology of [18]. These components are finite field extensions of $\mathcal{L}$ (since they have trivial nilradical).

Write $T_N \otimes \Lambda \mathcal{L}$ as a product of two factors, as follows:

$$T_N \otimes \Lambda \mathcal{L} \cong \left( \prod_{j \in J'} (T_N)_{a_j} \right) \times \left( \prod_{j \in J \setminus J'} (T_N)_{a_j} \right).$$
We let $\mathbf{T}$ denote the free closure (in the terminology of [18]) of the image of $T_N$ under the projection onto the first factor. If $\varphi$ is a classical height one prime of $\Lambda$, then part (2) of Theorem 2.1.2 identifies $(T_N/\varphi T_N)[1/p]$ with the $\mathbb{Q}_p$-Hecke algebra acting on the space $S_k(Np\infty, \mathcal{O}(\varphi)[1/p]^{\text{ord}}[\kappa_\varphi])$. We see from [18] Cor. 3.7 that under this isomorphism, the surjective map $(T_N/\varphi T_N)[1/p] \to (T/\varphi T)[1/p]$ is identified with the surjection from the $\mathbb{Q}_p$-Hecke algebra acting on $S_k(Np\infty, \mathcal{O}(\varphi)[1/p]^{\text{ord}}[\kappa_\varphi])$ onto the $\mathbb{Q}_p$-Hecke algebra acting on $S_k(Np\infty, \mathcal{O}(\varphi)[1/p]^{\text{ord}}[\kappa_\varphi])$.

Thus we see that the image of $T_N$ under the map $T_N \to T$, that is, the image of $T_N$ in $\prod_{\ell \in \mathcal{P}} (T_N)_{\ell}$, is precisely equal to $T_{\text{new}}$. Since this product is a product of fields, we see that $T_{\text{new}}$ is $\Lambda$-torsion free and reduced. Being a quotient of the finite $\Lambda$-module $T_N$, it is also finite over $\Lambda$, and so (1) is proved. We also see that $T_{\text{new}}/(T_{\text{new}} \cap \psi T)$ maps isomorphically onto the Hecke algebra acting on $S_k(Np\infty, \mathcal{O}(\varphi))_{\text{new}}^{\text{ord}}[\kappa_\varphi]$, proving (2). (Note that since $T$ contains $T_{\text{new}}$ with finite index, the height one primes of $T$ correspond bijectively to those of $T_{\text{new}}$ under restriction.)

Since $T$ contains $T_{\text{new}}$ with finite index, the natural map $T_{\text{new}} \otimes \Lambda \varphi \to T \otimes \Lambda \varphi$ is an isomorphism. Hecke algebras acting on spaces of newforms are reduced, and so we see that the quotient $(T_{\text{new}} \otimes \Lambda \varphi)/\psi(T_{\text{new}} \otimes \Lambda \varphi) \cong (T \otimes \Lambda \varphi)/\psi(T \otimes \Lambda \varphi)$ is reduced. This proves part (3) (which is simply a rewording of [17] Cor. 1.4)). □

2.3. Galois representations. In this section we recall the basic facts regarding Galois representations attached to Hida families. As above, we fix a tame level $N$ prime to $p$. Recall that $\mathcal{L}$ denotes the field of fractions of $\Lambda$.

Theorem 2.3.1. There is a continuous Galois representation

$$\rho : G_\mathbb{Q} \to \text{GL}_2(T_{\text{new}} \otimes \mathcal{L}),$$

caracterised by the following properties:

(1) $\rho$ is unramified away from $Np$.

(2) If $\ell$ is a prime not dividing $Np$ then $\rho(\text{Frob}_\ell)$ has characteristic polynomial equal to $X^2 - T_{\ell}X + (\ell)^{-1} \in T_{\text{new}}[X]$. (Recall that $(\ell)$ denotes the unit in $\mathbb{Z}_p[[\mathbb{Z}_p^{\times}]]$ corresponding to the element $\ell \in \mathbb{Z}_p^{\times}$.)

The representation $\rho$ satisfies the following additional properties:

(3) $\rho$ is absolutely irreducible.

(4) The determinant of $\rho$ is equal to the following character:

$$G_\mathbb{Q} \cong \hat{\mathbb{Z}}^{\times} \to \mathbb{Z}^{\times}_{p,N} \xrightarrow{x \mapsto x^{-1}} \mathbb{Z}_p[[\mathbb{Z}_p^{\times}]]^{\times} \subset (T_{\text{new}}^{\times})^{\times}.$$

(Here the first arrow is the full cyclotomic character.)

(5) The space of $I_p$-coinvariants of $\rho$ is free of rank one over $T_{\text{new}} \otimes \mathcal{L}$, and $\text{Frob}_p$ acts on this space through the eigenvalue $U_p \in T_{\text{new}}^{\times}$.

Proof. See [17] Thm. 2.1 and [14] Thm. 2.6]. □

From the Galois representation $\rho$ we may construct various related Galois representations. We recall some of these constructions here, and introduce the corresponding notation.

First, we may localise $\rho$ in various ways. If $a$ is a minimal prime ideal of $T_{\text{new}}$, then we let $\rho_a$ denote the representation

$$\rho_a : G_\mathbb{Q} \to \text{GL}_2((T_{\text{new}}^a))_a$$
obtained by composing $\rho$ with the projection $T_{\text{new}}^{\text{new}} \otimes A \rightarrow (T_{\text{new}}^{\text{new}})_\mathfrak{a}$.

If $\mathfrak{m}$ is a maximal ideal of $T_{\text{new}}^{\text{new}}$, then the localisation $(T_{\text{new}}^{\text{new}})_{\mathfrak{m}}$ is a direct factor of $T_{\text{new}}^{\text{new}}$, and so $(T_{\text{new}}^{\text{new}})_{\mathfrak{m}} \otimes A \mathcal{L}$ is a direct factor of $T_{\text{new}}^{\text{new}} \otimes A \mathcal{L}$. We let $\rho_{\mathfrak{m}}$ denote the representation

$$\rho_{\mathfrak{m}} : G_Q \rightarrow \text{GL}_2((T_{\text{new}}^{\text{new}})_{\mathfrak{m}} \otimes A \mathcal{L})$$

obtained by composing $\rho$ with the projection $T_{\text{new}}^{\text{new}} \otimes A \mathcal{L} \rightarrow (T_{\text{new}}^{\text{new}})_{\mathfrak{m}} \otimes A \mathcal{L}$.

If $T_{\text{new}}^{\text{new}}$ denotes the normalisation of $T_{\text{new}}^{\text{new}}$, then we may descend $\rho$ to a two dimensional representation defined over $T_{\text{new}}^{\text{new}}$. If $\varphi$ is a classical height one prime ideal in $T_{\text{new}}^{\text{new}}$, then we have an isomorphism $(T_{\text{new}}^{\text{new}})_{\varphi} \cong (T_{\text{new}}^{\text{new}})_{\varphi}$, by Theorem 2.2.2 part (3), and consequently may also descend $\rho$ to a two dimensional representation over $(T_{\text{new}}^{\text{new}})_{\varphi}$. Reducing this representation modulo the maximal ideal of this local ring, we obtain a representation $\bar{\rho}_\varphi : G_Q \rightarrow \text{GL}_2((O(\varphi)[1/p])$. Part (2) of Theorem 2.3.1 shows that this is the usual absolutely irreducible two dimensional Galois representation attached to the newform $f_\varphi$. We may also use an integral model of $\rho$ to construct a residual representation $\bar{\rho}_{\mathfrak{m}}$ attached to any maximal ideal of $T_{\text{new}}^{\text{new}}$. We recall this construction in more detail in Theorem 2.3.4 below.

If $\mathfrak{a}$ is a minimal prime of $T_{\text{new}}^{\text{new}}$, then the representation $\rho_{\mathfrak{a}}$ takes values in a field, and so we may define its tame conductor by the usual formula. That is, if $\rho_{\mathfrak{a}}$ acts on the two dimensional $(T_{\text{new}}^{\text{new}})_{\mathfrak{a}}$-vector space $V$, then exponent of a prime $\ell \neq p$ in the tame conductor is equal to

$$\dim V - \dim V_{I_\ell} + \int_0^\infty \left( \dim V - \dim V_{I_\ell^u} \right) \, du,$$

where $\{I_\ell^u\}_{u \geq 0}$ denotes the usual filtration of $I_\ell$ by higher ramification groups, indexed by the upper numbering. (Note that this formula is often expressed in terms of invariants under the ramification groups, rather than coinvariants. One obtains the same value with either formulation, since $\dim V_{I_\ell^u} = \dim V_{I_\ell^p}$ for any value of $u$.)

**Proposition 2.3.2.** If $\mathfrak{a}$ is any minimal prime of $T_{\text{new}}^{\text{new}}$, then the tame conductor of $\rho_{\mathfrak{a}}$ is equal to $N$.

**Proof.** Write $\hat{T}$ to denote the normalisation of $T_{\text{new}}^{\text{new}}/\mathfrak{a}$; this is the component of the normalisation of $T_{\text{new}}^{\text{new}}$ cut out by the minimal prime $\mathfrak{a}$. Since it is Cohen-Macaulay (being normal and of dimension two) and finite over $\Lambda$, it is in fact finite flat over $\Lambda$ by [21 Thm. 23.1].

Let $V$ denote a two dimensional vector space over the fraction field of $T_{\text{new}}^{\text{new}}/\mathfrak{a}$ on which $\rho_{\mathfrak{a}}$ acts, and let $M$ be a free rank two $T$-lattice in $V$ invariant under $G_Q$. If $\varphi$ is a classical height one prime of $T_{\text{new}}^{\text{new}}$ containing $\mathfrak{a}$, then as was observed above, the injection

$$(T_{\text{new}}^{\text{new}}/\mathfrak{a})_{\varphi} \rightarrow \hat{T}_\varphi$$

is an isomorphism. Moreover, $G_Q$ acts on the $\mathcal{O}(\varphi)[1/p]$-vector space $V(\varphi) := (M/\varphi M)[1/p]$ via the usual Galois representations $\bar{\rho}_{\varphi}$ attached to newform $f_\varphi$ of tame conductor $N$.

The ratio of the tame conductor of $\rho_{\mathfrak{a}}$ and the tame conductor of $\bar{\rho}_{\varphi}$ (that is, $N$) is equal to

$$\prod_{\ell \neq p} \ell^{\dim V(\varphi)_{I_\ell} - \dim V_{I_\ell}},$$

(2.3)
furthermore, each of the exponents appearing in this expression is non-negative (i.e. \( \dim V(\varphi)_{I_\ell} \) bounds \( \dim V_{I_\ell} \) from above). (See \[23\] §1). Thus to prove the proposition, we must show that \( \dim V(\varphi)_{I_\ell} = \dim V_{I_\ell} \) for each prime \( \ell \neq p \).

If \( \dim V(\varphi)_{I_\ell} = 2 \), then \( \ell \) does not divide \( N \). In this case, \( \rho_a \) is unramified at \( \ell \), and so \( \dim V_{I_\ell} = 2 \) as well. Conversely, if \( \dim V_{I_\ell} = 2 \), then the same is true of \( \dim V(\varphi)_{I_\ell} \) (since the latter dimension bounds the former dimension from above). If \( \dim V(\varphi)_{I_\ell} = 0 \), then also \( \dim V_{I_\ell} = 0 \) (again, since the latter dimension is bounded above by the former). Thus it remains to show that if \( \dim V(\varphi)_{I_\ell} > 0 \), then \( \dim V_{I_\ell} > 0 \). This we now do.

Note that the formula \[23\] (and the fact that the tame conductor of \( V(\varphi) \) is equal to \( N \), and thus is independent of the particular choice of \( \varphi \) containing \( a \)) shows that \( \dim V(\varphi)_{I_\ell} \) is independent of the choice of the classical height one prime \( \varphi \) containing \( a \). Thus we may assume that \( \dim V(\varphi)_{I_\ell} \) is positive for every classical height one prime \( \varphi \) containing \( a \). There is a natural map \( I_\ell \to \mathbb{Z}_p(1) \), given by projection onto the \( p \)-Sylow subgroup of the tame quotient of \( I_\ell \). Let \( J_\ell \) denote the kernel of this projection. Since \( J_\ell \) has order prime to \( p \), it acts on \( M \) through a finite quotient of order prime to \( p \), and so certainly \( \dim V_{I_\ell} = \dim V(\varphi)_{I_\ell} \) for any classical height one prime \( \varphi \). Let \( \sigma \) denote a topological generator of \( \mathbb{Z}_p(1) \), and consider the matrix \( \rho_a(\sigma) - I \) acting on \( V_{I_\ell} \). (Here \( I \) denotes the identity matrix.) By assumption, the determinant of this matrix (which lies in \( \mathbb{T} \)) lies in \( \varphi \mathbb{T} \) for every classical height one prime \( \varphi \) of \( \mathbb{T} \). Thus it lies in \( \varphi' \mathbb{T} \) for every classical height one prime \( \varphi' \) of \( \Lambda \). (Recall that these primes are unramified in \( \mathbb{T} \), by part (3) of Theorem \[22\]) Since \( \mathbb{T} \) is finite flat over \( \Lambda \), we see that this determinant vanishes, and thus that \( \rho_a(\sigma) \) admits a non-zero coinvariant quotient of \( V_{I_\ell} \). This proves that \( \dim V_{I_\ell} > 0 \), as required.

As a byproduct of the proof of the preceding proposition, we also obtain the following useful result.

**Proposition 2.3.3.** Let \( \varphi \) be a classical height one prime ideal in \( \mathbb{T}_{N_{\text{new}}}^{\text{new}} \), and let \( L \) denote a choice of two dimensional \( (\mathbb{T}_{N_{\text{new}}}^{\text{new}})_{\varphi} \)-lattice on which \( \rho \) acts, so that \( L/\varphi L \) is the two dimensional \( \mathcal{O}(\varphi)[1/p] \)-vector space on which \( \bar{\rho}_{\varphi} \) acts. If \( \ell \) is any prime distinct from \( p \), then \( L_{I_\ell} \) is a free \( (\mathbb{T}_{N_{\text{new}}}^{\text{new}})_{\varphi} \)-module, and there are natural isomorphisms \( L_{I_\ell} \otimes_{\Lambda} \mathcal{L} \to (L \otimes_{\Lambda} \mathcal{L})_{I_\ell} \) and \( L_{I_\ell}/\varphi L_{I_\ell} \to (L/\varphi L)_{I_\ell} \).

**Proof.** We let \( a \) denote the minimal ideal contained in \( \varphi \) (unique, since \( (\mathbb{T}_{N_{\text{new}}}^{\text{new}})_{\varphi} \) is a discrete valuation ring), and employ the notation introduced in the proof of the preceding proposition. The lattice \( L \) is a free rank two \( (\mathbb{T}_{N_{\text{new}}}^{\text{new}})_{\varphi} \)-module, which we may regard as being embedded in a \( G_{\mathbb{Q}} \)-equivariant fashion in \( \mathcal{V} \).

For general reasons, the composite surjection \( L \to L/\varphi L \to (L/\varphi L)_{I_\ell} \) induces an isomorphism \( L_{I_\ell}/\varphi L_{I_\ell} \cong (L/\varphi L)_{I_\ell} \). Thus a minimal set of generators for \( L_{I_\ell} \) as a \( (\mathbb{T}_{N_{\text{new}}}^{\text{new}})_{\varphi} \)-module contains at most \( \dim(L/\varphi L)_{I_\ell} \) elements. On the other hand the surjection \( \mathcal{V} \to \mathcal{V}_{I_\ell} \) induces a surjection \( L_{I_\ell} \otimes_{\Lambda} \mathcal{L} \to \mathcal{V}_{I_\ell} \), and hence \( L_{I_\ell} \otimes_{\Lambda} \mathcal{L} \) is of dimension at least \( \dim \mathcal{V}_{I_\ell} \). The proof of Proposition \[23\] shows that \( \dim(L/\varphi L)_{I_\ell} = \dim \mathcal{V}_{I_\ell} \), and hence (taking into account the fact that \( (\mathbb{T}_{N_{\text{new}}}^{\text{new}})_{\varphi} \) is a discrete valuation ring) we may conclude that \( L_{I_\ell} \) is free over \( (\mathbb{T}_{N_{\text{new}}}^{\text{new}})_{\varphi} \), of rank equal to \( \dim \mathcal{V}_{I_\ell} \). Thus the surjection \( L_{I_\ell} \otimes_{\Lambda} \mathcal{L} \to \mathcal{V}_{I_\ell} \) is an isomorphism, as claimed.

The next result describes the residual representation \( \bar{\rho}_m \) attached to a maximal ideal of \( \mathbb{T}_{N_{\text{new}}}^{\text{new}} \).
Theorem 2.3.4. If \( \mathfrak{m} \) is a maximal ideal of \( T_N^{\text{new}} \), then attached to \( \mathfrak{m} \) is a semi-simple representation \( \tilde{\rho}_\mathfrak{m} : G_{\mathbb{Q}} \to \text{GL}_2(T_N^{\text{new}}/\mathfrak{m}) \), uniquely determined by the properties:

1. \( \tilde{\rho}_\mathfrak{m} \) is unramified away from \( Np \).
2. If \( \ell \) is a prime not dividing \( Np \) then \( \tilde{\rho}_\mathfrak{m}(\text{Frob}_\ell) \) has characteristic polynomial equal to \( X^2 - T_\ell X + (\ell \ell^{-1} \mod \mathfrak{m}) \in (T_N^{\text{new}}/\mathfrak{m})[X] \).

Furthermore, \( \tilde{\rho}_\mathfrak{m} \) satisfies the following condition:

3. The restriction of \( \tilde{\rho}_\mathfrak{m} \) to \( D_p \) has the following shape (with respect to a suitable choice of basis):

\[
\begin{pmatrix}
\chi & * \\
0 & \psi
\end{pmatrix},
\]

where \( \chi \) and \( \psi \) are \((T_N^{\text{new}}/\mathfrak{m})^\times\)-valued characters of \( D_p \), such that \( \psi \) is unramified and \( \psi(\text{Frob}_p) = U_p \mod \mathfrak{m} \).

Proof. The representation \( \tilde{\rho}_\mathfrak{m} \) is constructed in the usual way, by choosing an integral model for \( \rho \) over \( T_N^{\text{new}} \), reducing this model modulo a maximal ideal \( \hat{\mathfrak{m}} \) lifting \( \mathfrak{m} \), semi-simplifying, and then descending (if necessary) from \( T_N^{\text{new}}/\mathfrak{m} \) to \( T_N^{\text{new}}/\mathfrak{m} \). The stated properties follow from the corresponding properties of \( \rho \).

Proposition 2.3.5. If \( \mathfrak{m} \) is a maximal ideal of \( T_N^{\text{new}} \) for which the associated residual representation \( \tilde{\rho}_\mathfrak{m} \) is irreducible, then \( \rho_\mathfrak{m} \) admits a model over \( (T_N^{\text{new}})_\mathfrak{m} \) (which we denote by the same symbol)

\[
\rho_\mathfrak{m} : G_{\mathbb{Q}} \to \text{GL}_2((T_N^{\text{new}})_\mathfrak{m}),
\]

unique up to isomorphism. Furthermore, in this model, the space of \( I_p \)-coinvariants is free of rank one over \( (T_N^{\text{new}})_\mathfrak{m} \).

Proof. This follows from the irreducibility of \( \tilde{\rho}_\mathfrak{m} \), and the fact that the traces of \( \rho_\mathfrak{m} \) lie in \((T_N^{\text{new}})_\mathfrak{m} \). (See [2].)

Suppose that \( \mathfrak{m} \) is a maximal ideal of \( T_N^{\text{new}} \) satisfying the hypothesis of the preceding proposition, and let \( M \) denote a choice of free rank two \((T_N^{\text{new}})_\mathfrak{m}\)-module on which the representation \( \rho_\mathfrak{m} \) acts. The group \( G_{\mathbb{Q}} \) then acts on the quotient \( M/\mathfrak{m}M \) via the residual representation \( \tilde{\rho}_\mathfrak{m} \). Since the space \( M_{I_p} \) of \( I_p \)-coinvariants of \( M \) is a free rank one quotient of \( M \), we see that \((M_{I_p})/\mathfrak{m}(M_{I_p})\) is a one dimensional quotient of the \( I_p \)-coinvariants \((M/\mathfrak{m}M)_{I_p}\) of \( M/\mathfrak{m}M \).

Definition 2.3.6. If \( k \) is a finite field of characteristic \( p \), and \( \tilde{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(k) \) is a continuous two dimensional Galois representation defined over \( k \), acting on the two dimensional \( k \)-vector space \( V \), say, then a \( p \)-stabilisation of \( \tilde{\rho} \) consists of the choice of a one dimensional quotient of the space \( V_{I_p} \) of \( I_p \)-coinvariants of \( V \).

Definition 2.3.7. The discussion preceding Definition 2.3.6 shows that if \( \mathfrak{m} \) is a maximal ideal of \( T_N^{\text{new}} \) for which \( \tilde{\rho}_\mathfrak{m} \) is irreducible, then \( \rho_\mathfrak{m} \) comes equipped with a natural \( p \)-stabilisation. We refer to this as the canonical \( p \)-stabilisation of \( \tilde{\rho}_\mathfrak{m} \) attached to the maximal ideal \( \mathfrak{m} \).

2.4. The reduced Hida algebra.

Definition 2.4.1. For any level \( N \), we let \( T_N' \) denote the \( \Lambda \)-subalgebra of \( T_N \) generated by the Hecke operators \( T_\ell \) for \( \ell \) prime to \( Np \), together with the operator \( U_p \).
Since $T'_N$ is a subalgebra of the finite flat $\Lambda$-algebra $T_N$, it is certainly finite and torsion free over $\Lambda$. It turns out that $T'_N$ is also reduced. (Being reduced is a standard property of Hecke algebras in which we omit the operators indexed by the primes dividing the level.) In fact, we can be somewhat more precise.

If $M$ is a divisor of $N$, then restricting the action of the prime-to-$N$ Hecke operators to ordinary forms of level dividing $M$ yields a surjective map of $\Lambda$-algebras $T'_N \to T'_M$. Composing this with the composite $T'_M \subset T_M \to T_{\text{new}}^M$ yields a map

$$T'_N \to T_{\text{new}}^M. \tag{2.4}$$

Taking the product of these maps over all divisors $M$ of $N$, we obtain a map

$$T'_N \to \prod_{M \mid N} T_{\text{new}}^M. \tag{2.5}$$

Proposition 2.4.2.

1. The map (2.5) is injective, and induces an isomorphism after localising at any classical height one prime ideal of $\Lambda$ (and hence also after tensoring over $\Lambda$ with its fraction field $\mathcal{L}$).

2. If $\wp$ is a classical height one prime ideal of $\Lambda$, then $T'_N \otimes_\Lambda \Lambda_\wp$ is a finite étale extension of the discrete valuation ring $\Lambda_\wp$.

Proof. It is an easy consequence of the theory of newforms for ordinary families developed in [18] that the map (2.5) is injective, and that it induces an isomorphism after tensoring with $\mathcal{L}$ over $\Lambda$. We wish to check that the same is true if we tensor with $\Lambda_\wp$ over $\Lambda$ for any classical height one prime $\wp$ of $\Lambda$. For this, it suffices to replace $T'_N$ and each of the $\Lambda$-algebras $T_{\text{new}}^M$ by their free closures (in the terminology of [18]). We denote these free closures by $\tilde{T}'_N$ and $\tilde{T}_{\text{new}}^M$ respectively.

It is proved in [18], and recalled in the proof of Theorem 2.2.2, that the isomorphism of part (2) of Theorem 2.1.2 induces an isomorphism of $(\tilde{T}_{\text{new}}^M/\wp \tilde{T}_{\text{new}}^M)[1/p]$ with the $\mathbb{Q}_p$-Hecke algebra acting on the space of newforms

$$S_k(M p^\infty, \mathcal{O}(\wp)[1/p])_{\text{ord}}^\ord [\kappa_\wp].$$

Strong multiplicity one for newforms, together with the standard duality between spaces of modular forms and Hecke algebras, then shows that the map

$$(T'_N/\wp \tilde{T}'_N)[1/p] \to \prod_{M \mid N} (T_{\text{new}}^M/\wp \tilde{T}_{\text{new}}^M)[1/p]$$

is surjective. Thus the map

$$T'_N \otimes_\Lambda \Lambda_\wp \to \prod_{M \mid N} \tilde{T}_{\text{new}}^M \otimes_\Lambda \Lambda_\wp$$

is a surjection of finite free $\Lambda_\wp$-modules. Since it induces an isomorphism after extending scalars to $\mathcal{L}$, it must be an isomorphism. This completes the proof of part (1). Part (2) is an immediate consequence of part (1), together with part (3) of Theorem 2.2.2. $\square$

Taking the product of the Galois representations

$$G_{\mathbb{Q}} \to \text{GL}_2(T_{\text{new}}^M \otimes_\Lambda \mathcal{L})$$
given by Theorem 2.3.1 as $M$ ranges over all divisors of $N$, and taking into account part (1) of the preceding proposition, we obtain a Galois representation

$$\rho : G_Q \to \text{GL}_2(T'_N \otimes_A \mathcal{L})$$

satisfying the analogue of Theorem 2.3.1.

Just as in the preceding section, we may reduce $\rho$ modulo a height one prime ideal $\mathfrak{p}$ or a maximal ideal $\mathfrak{m}$ of $T'_N$. We denote the corresponding residual representations by $\bar{\rho}_\mathfrak{p}$ and $\bar{\rho}_\mathfrak{m}$ respectively. Similarly, we may localise $T'_N$ at any maximal ideal $\mathfrak{m}$, and obtain a corresponding representation

$$\rho_\mathfrak{m} : G_Q \to \text{GL}_2((T'_N)_\mathfrak{m} \otimes_A \mathcal{L}).$$

If $\bar{\rho}_\mathfrak{m}$ is irreducible, then the analogue of Proposition 2.3.5 holds (by the same appeal to the results of [2]), and so we obtain a uniquely determined representation

$$\rho_\mathfrak{m} : G_Q \to \text{GL}_2((T'_N)_\mathfrak{m}).$$

2.5. Hecke eigenvalues. Fix a tame level $N$, a maximal ideal $\mathfrak{m}$ of $T'_N$, a divisor $M$ of $N$, and a maximal ideal $\mathfrak{n}$ of $T'^{\text{new}}_M$ that pulls back to $\mathfrak{m}$ under the map $\mathfrak{m} \mapsto \mathfrak{n}$, and hence induces a local map $(T'_N)_\mathfrak{m} \to (T'^{\text{new}}_M)_\mathfrak{n}$. We will let $T$ denote the image of this map; by construction $(T'^{\text{new}}_M)_\mathfrak{n}$ is obtained from $T$ by adjoining the Hecke operators $T_\ell$ for those primes $\ell$ dividing $N$, together with the diamond operators corresponding to elements of $(\mathbb{Z}/M)^\times$.

The residual representations $\bar{\rho}_\mathfrak{m}$ and $\bar{\rho}_\mathfrak{n}$ are isomorphic, up to a possible extension of scalars; we assume that they are irreducible (equivalently, absolutely irreducible, since the residue characteristic $p$ is odd). It follows from the results of [2] that the Galois representation $\rho_{\mathfrak{n}} : G_Q \to \text{GL}_2((T'^{\text{new}}_M)_\mathfrak{n})$ is in fact defined over $T$. If $\ell$ does not divide $M$, then $\rho_{\mathfrak{n}}$ is unramified at $\ell$, and $T_\ell$ is equal to the trace of $\rho_{\mathfrak{n}}(\text{Frob}_\ell)$. In particular, for such primes $\ell$, the Hecke operator $T_\ell$ lies in $T$. The map $(T'_N)_\mathfrak{m} \to (T'^{\text{new}}_M)_\mathfrak{n}$ thus factors through a surjection $(T'_N)_\mathfrak{m} \to (T'_M)_\mathfrak{n}$ (where we denote the maximal ideal in $T'_M$ induced by $\mathfrak{m}$ by the same letter). Thus we may suppose that $M = N$, and we do so from now on. The diamond operators given by $(\mathbb{Z}/N)^\times$ also lie in $T$, since these can be recovered by evaluating the determinant of $\rho_{\mathfrak{n}}$ on Frobenius elements. Thus we see that $(T'^{\text{new}}_N)_\mathfrak{n}$ is generated by adjoining to $T$ the Hecke operators $T_\ell$, for $\ell$ dividing $N$. In the remainder of this section we explain the representation-theoretic interpretation of these Hecke operators, and also consider the related question of whether or not the inclusion $T \subset (T'^{\text{new}}_N)_\mathfrak{n}$ is an equality.

Before proceeding further, it will be helpful to recall the Galois representation-theoretic description of the Hecke eigenvalues of a classical newform. Thus we let $f$ denote a classical newform over $\mathbb{Q}$ of some weight, and of tame level $N$, let $\rho_f$ denote the $p$-adic Galois representation attached to $f$, and let $a_\ell(f)$ denote its Hecke eigenvalues. Let $V$ be a two dimensional vector space on which the $p$-adic Galois representation $\rho_f$ acts. If $\ell$ does not divide $N$, then $\rho_f$ is unramified at $\ell$, and $a_\ell(f)$ is equal to the trace of $\rho_f(\text{Frob}_\ell)$. For each prime $\ell$ dividing $N$, the space $V_\ell$ of $I_\ell$-coinvariants in $V$ is either trivial or one dimensional. In the first case, one has that $a_\ell(f) = 0$. In the second case, one has that $a_\ell(f)$ is equal to the eigenvalue of $\text{Frob}_\ell$ acting on $V_\ell$. If $\mathcal{O}(f)$ denotes the finite extension of $\mathbb{Z}_p$ generated by the Fourier coefficients of $f$, then considerations of an integral model for $\rho_f$ show that this eigenvalue is a unit in $\mathcal{O}(f)$. Thus we see that, for the primes $\ell$ dividing $N$, if $a_\ell(f)$ lies in the maximal ideal of $\mathcal{O}(f)$, then it actually vanishes.
We are now in a position to prove the following lemma.

**Lemma 2.5.1.** If \( \ell \) is a prime dividing \( N \), then the following are equivalent:

1. \( T_\ell \) lies in the maximal ideal \( n \).
2. There is a classical height one prime ideal \( \wp \) of \( (T^\text{new}_N)_n \) such that \( T_\ell \equiv 0 \mod \wp \).
3. For every classical height one prime ideal \( \wp \) of \( (T^\text{new}_N)_n \), we have \( T_\ell \equiv 0 \mod \wp \).
4. We have \( T_\ell = 0 \) in \( (T^\text{new}_N)_n \).
5. Letting \( V \) denote a free rank two \( (T^\text{new}_N)_n \otimes_\Lambda L \)-module on which \( \rho_n \) acts, we have \( V_{\lambda, \ell} = 0 \).

If these equivalent conditions do not hold, then we have that \( V_{\lambda, \ell} \) is free of rank one over \( (T^\text{new}_N)_n \otimes_\Lambda L \), and \( T_\ell \) is equal to the eigenvalue of \( \text{Frob}_\ell \) acting on this free rank one module.

**Proof.** It is clear that (4) \( \implies \) (3) \( \implies \) (2) \( \implies \) (1). Condition (1) implies that \( a_\ell (f_\wp) \) is contained in the maximal ideal of \( \mathcal{O}(f_\wp) = \mathcal{O}(\wp) \) for each classical height one prime \( \wp \) of \( (T^\text{new}_N)_n \), and thus (by the discussion preceding the statement of the lemma) that \( a_\ell (f_\wp) = 0 \) for each such prime. Thus (1) implies (3). Proposition 2.4.3, together with the discussion preceding the statement of the lemma, shows that (3) and (5) are equivalent. Let \( (T^\text{new}_N)_n \) denote the free closure of \( (T^\text{new}_N)_n \) as a \( \Lambda \)-module. If condition (3) holds, then (taking into account part (3) of Theorem 2.2.2) we find that

\[
T_\ell \in \bigcap_{\wp \text{ classical ht. } 1 \text{ in } \Lambda} \wp'(T^\text{new}_N)_n,
\]

and thus that \( T_\ell = 0 \). Thus (3) implies (4), and this completes the proof of the claimed equivalences.

Suppose now that these equivalent conditions do not hold. The remarks preceding the statement of the lemma, together with Proposition 2.4.3, show that in this case the space \( V_{\lambda, \ell} \) is free of rank one over \( (T^\text{new}_N)_n \otimes_\Lambda L \), as claimed. Let \( A_\ell \in (T^\text{new}_N)_n \otimes_\Lambda L \) denote the eigenvalue of \( \text{Frob}_\ell \) acting on this module. Again, Proposition 2.4.3 shows that this module admits a model over \( (T^\text{new}_N)_{\wp'} \) for each classical height one prime ideal of \( (T^\text{new}_N)_n \), and thus that in fact \( A_\ell \in (T^\text{new}_N)_{\wp'} \) for each such \( \wp' \). The discussion preceding the statement of the lemma, together with part (3) of Theorem 2.2.2, then shows that

\[
T_\ell - A_\ell \in \bigcap_{\wp' \text{ classical ht. } 1 \text{ in } \Lambda} \wp'(T^\text{new}_N)_n \otimes_\Lambda \Lambda_{\wp'}.
\]

This implies that \( T_\ell - A_\ell = 0 \), and completes the proof of the remaining claim of the lemma.

Recall that \( T \) denotes the image of \( (T^\text{new}_N)_m \) in \( (T^\text{new}_N)_n \). We now address the issue of whether or not \( T \) equals all of \( (T^\text{new}_N)_n \). We note that the remainder of this section is tangential to the main arguments of the paper.

**Proposition 2.5.2.** Assume that \( \bar{\rho}_m \) is irreducible and that the characteristic polynomial of any element in the image of \( \bar{\rho}_m \) splits over \( T^\text{new}_N/_m \) if and only if it splits over \( T^\text{new}_N/_n \). Then for any prime \( \ell \) prime to \( p \) we have that \( T_\ell \) lies in \( T \), unless \( \ell \) satisfies both of the following conditions:
Furthermore, if these conditions hold, then $\ell \equiv 1 \mod p$.

**Remark 2.5.3.** Note that the condition on characteristic polynomials above holds if the map $T'_N/m \to T'_{N'}/n$ is an isomorphism, or if the characteristic polynomial of every element in the image of $\overline{\rho}_m$ already splits over $T'_N/m$.

**Proof.** Let $V'$ denote a two dimensional vector space on which $\rho_m$ acts. If $N(\overline{\rho}_m)$ denotes the tame conductor of $\overline{\rho}_m$, then [28] §1 implies that

$$N/N(\overline{\rho}_m) = \prod_{\ell \neq p} \ell^{\dim V_{\ell} - \dim \nabla_{\ell}}.$$ 

If $\dim V_{\ell} = 0$, then Lemma 2.5.1 shows that $T_{\ell} = 0$, and so there is nothing to prove. If $\dim V_{\ell} = \dim V_{\ell}$, then arguments similar to those used in the proofs of Remarks 2.9 and 2.11 of [34] prove that $T_{\ell} \in T$.

It remains to consider primes $\ell$ for which $V_{\ell}$ is one dimensional, while $\overline{\rho}_m$ is unramified at $\ell$. In such a case, we see that $T_{\ell}$ reduces modulo $n$ to one of the two roots of the characteristic polynomial of $\overline{\rho}_m(\text{Frob}_{\ell})$. By hypothesis this characteristic polynomial splits over the residue field $T'_N/m$. If it has distinct roots, then Hensel’s lemma shows that $T_{\ell} \in T$. Thus if $T_{\ell} \not\in T$, then the characteristic polynomial of $\overline{\rho}_m(\text{Frob}_{\ell})$ must have repeated roots.

Let us finally explain how to conclude that $\ell \equiv 1 \mod p$. If $\varphi$ is any classical height one prime of $T$, then recall that $\overline{\rho}_\varphi$ is the $p$-adic Galois representation attached to the classical newform $f_\varphi$ of level $N$. Proposition 2.3.3 shows that $\overline{\rho}_\varphi$ has one dimensional $I_{\ell}$-coinvariants. It is then well-known that there is either an isomorphism

$$(\overline{\rho}_\varphi|_{D_{\ell}})^{ss} \cong \psi \epsilon \oplus \chi,$$

where $\psi$ denotes an unramified character and $\epsilon$ the $p$-adic cyclotomic character (the special case), or

$$(\overline{\rho}_\varphi|_{D_{\ell}})^{ss} \cong \chi \oplus \psi$$

where $\chi$ is tamely ramified and $\psi$ is unramified (the principal series case). In the first case, since $\overline{\rho}_m(\text{Frob}_{\ell})$ acts on $(\overline{\rho}_\varphi|_{D_{\ell}})^{ss}$ as a scalar, we see that $\epsilon(\text{Frob}_{\ell}) = \ell \equiv 1 \mod p$. In the second case, since $\overline{\rho}_m$ is unramified at $\ell$, we see that $\chi|_{I_{\ell}} \equiv 1 \mod p$; the group $D_{\ell}$ admits a ramified character with this property only if $\ell \equiv 1 \mod p$. □

If we are willing to invert $p$, then we obtain the following more definitive result.

**Proposition 2.5.4.** The inclusion $T[1/p] \subset (T'_{N'})^n[1/p]$ is an equality.

**Proof.** As in the proof of the preceding proposition, it suffices to show that if $\ell$ is a prime at which $\overline{\rho}_m$ is unramified and for which $V_{\ell}$ is one dimensional, then $T_{\ell}$ lies in $T[1/p]$. We may factor $T[1/p]$ as a product $T_{sp} \times T_{ps}$, where $T_{sp}$ is the factor on which the diamond operators at $\ell$ act trivially, and $T_{ps}$ is a the factor on which these diamond operators act non-trivially. (The subscripts are for “special” and “principal series”, respectively.)

By specialising at classical height one primes, and arguing as in the proof of Lemma 2.5.1, we find that over $T_{ps}$, the associated two dimensional Galois representation has a free rank one space of $I_{\ell}$-invariants, and that $T_{\ell}$ is equal to the eigenvalue of $\text{Frob}_{\ell}$ acting on this one dimensional space of invariants. In
particular, we find that $T_\ell \in T_{ps}$. A similar specialisation argument shows that $T_\ell^2 = \ell^{-1} \det \rho(\text{Frob}_\ell)$ in $T_{sp}$. An application of Hensel’s lemma then shows that $T_\ell$ lies in the image of $T$ in $T_{sp}$, and so in particular in $T_{sp}$. □

2.6. The reduced Hida algebras attached to $\bar{\rho}$. If $k$ is a finite field of characteristic $p$, and $\bar{\rho} : G_\mathbb{Q} \to GL_2(k)$ is a continuous two dimensional Galois representation defined over $k$, then we say that $\bar{\rho}$ is ordinary if it satisfies condition (3) of Theorem 2.3.4. We say that $\bar{\rho}$ is $p$-distinguished if furthermore the characters $\chi$ and $\psi$ appearing in the statement of condition (3) are distinct. Clearly $\bar{\rho}$ admits a $p$-stabilisation, in the sense of Definition 2.3.6, if and only if $\bar{\rho}$ is ordinary. If $\bar{\rho}$ is furthermore $p$-distinguished, then $\bar{\rho}$ admits at most two choices of $p$-stabilisation (and does admit two such choices precisely when the determinant of $\bar{\rho}$ is unramified at $p$).

Let us now fix such a representation $\bar{\rho} : G_\mathbb{Q} \to GL_2(k)$, and let $\mathbf{V}$ be a two dimensional $k$-vector space on which $\bar{\rho}$ acts. We assume that $\bar{\rho}$ is irreducible, odd, ordinary and $p$-distinguished, and we fix a choice of $p$-stabilisation of $\bar{\rho}$. We assume that $k$ is equal to the field generated by the traces of $\bar{\rho}$. (If it were not, we could replace $k$ by this field of traces, and descend $\bar{\rho}$ to the smaller field.) Finally, we suppose that $\bar{\rho}$ is modular (i.e. that it arises as the residual representation attached to a modular form of some weight and level defined over $\bar{\mathbb{Q}}_p$). Our goal in this section is to define the reduced ordinary Hecke algebras $T_{\Sigma}(\bar{\rho})$ attached to an ordinary residual representation, and to describe its basic properties.

We let $N(\bar{\rho})$ denote the tame conductor of $\bar{\rho}$. If $\ell \neq p$ is prime, then define

$$m_\ell = \dim_k \mathbf{V}_\ell,$$

and for any finite set of primes $\Sigma$ that does not contain $p$, write

$$N(\Sigma) = N(\bar{\rho}) \prod_{\ell \in \Sigma} \ell^{m_\ell}.$$

Theorem 2.6.1. There is a unique maximal ideal $\mathfrak{m}$ of $T_{N(\Sigma)}$ such that $\bar{\rho}_\mathfrak{m}$, with its canonical $p$-stabilisation, is isomorphic to $\bar{\rho}$, with its given $p$-stabilisation.

Proof. The uniqueness is clear. The existence follows from the assumption that $\bar{\rho}$ is modular, and the results of [5]. □

Proposition 2.6.2. If $\mathfrak{m}$ denotes the maximal ideal of $T_{N(\Sigma)}$ of the preceding theorem, then there is a unique maximal ideal $\mathfrak{n}$ of $T_{N(\Sigma)}$ satisfying the following conditions:

1. $\mathfrak{n}$ lifts $\mathfrak{m}$.
2. $T_\ell \in \mathfrak{n}$ for each $\ell \in \Sigma$.
3. The natural map of localisations $(T_{N(\Sigma)})_\mathfrak{m} \to (T_{N(\Sigma)})_\mathfrak{n}$ is an isomorphism of $\Lambda$-algebras.

In particular, $(T_{N(\Sigma)})_\mathfrak{m}$ is a finite flat $\Lambda$-algebra. Also, the image of $T_\ell$ in the localisation $(T_{N(\Sigma)})_\mathfrak{n}$ in fact vanishes for each $\ell \in \Sigma$.

Proof. This is a variant of [34, Prop. 2.1.5], and is proved in an analogous manner. The second to last claim follows from the rest of the proposition, together with part (1) of Theorem 2.1.2. □
Definition 2.6.3. We let $T_{\Sigma}(\bar{\rho})$ (or simply $T_{\Sigma}$, if $\bar{\rho}$ is understood) denote the localisation of $T_{N(\Sigma)}$ at the maximal ideal whose existence is guaranteed by Proposition 2.6.2. We let $\rho_{\Sigma}: G_{\mathbb{Q}} \to \text{GL}_2(T_{\Sigma})$ denote the Galois representation attached to this local factor of $T_{N(\Sigma)}$, as discussed in section 2.6.3. Recall that $\rho_{\Sigma}$ is characterised by the following property: If $\ell$ is a prime not dividing $N(\Sigma)p$, then $\rho_{\Sigma}(\text{Frob}_{\ell})$ has trace equal to $T_{\ell} \in T_{\Sigma}$.

Taking into account Proposition 2.6.2, we see that $T_{\Sigma}$ is a reduced and finite flat $\Lambda$-algebra. Note that if $\Sigma \subset \Sigma'$ then $N(\Sigma) | N(\Sigma')$, and the natural surjection $T_{N(\Sigma')} \to T_{N(\Sigma)}$ induces a surjection $T_{\Sigma'} \to T_{\Sigma}$. The Galois representations $\rho_{\Sigma'}$ and $\rho_{\Sigma}$ are evidently compatible with this surjection.

We refer to Spec $T_{\Sigma}$ as the universal ordinary family of newforms, or sometimes simply “the Hida family”, minimally ramified outside $\Sigma$, attached to $\bar{\rho}$ and our chosen $p$-stabilisation. Localising $\rho_{\Sigma}$ over Spec $T_{\Sigma}$, we obtain a two dimensional vector bundle on which $G_{\mathbb{Q}}$ acts, which we refer to as the universal family of Galois representations over Spec $T_{\Sigma}$. If $\Sigma \subset \Sigma'$, then the surjection $T_{\Sigma} \to T_{\Sigma'}$ induces a closed embedding Spec $T_{\Sigma'} \to$ Spec $T_{\Sigma}$, and the universal family of Galois representations on the target pulls back to the universal family of Galois representations on the source. If we consider all $\Sigma$ simultaneously, then we obtain an Ind-scheme

$$
\text{“lim" Spec } T_{\Sigma},$$

and a family of two dimensional Galois representations lying over it. We refer to this Ind-scheme as the universal ordinary family of newforms, or simply “the Hida family”, attached to $\bar{\rho}$ and its chosen $p$-stabilisation.

The Hida family corresponding to $\Sigma = \emptyset$ will play a special role; we refer to it as the minimal Hida family attached to $\bar{\rho}$ and our chosen $p$-stabilisation.

When $\bar{\rho}$ is irreducible after restriction to the quadratic field of discriminant $\pm p$, the results of Wiles and Taylor–Wiles [34, 33], as strengthened by Diamond [6], in fact allow us to identify the rings $T_{\Sigma}$, and their accompanying Galois representations $\rho_{\Sigma}$, with certain universal deformation rings, and their accompanying universal Galois representations, attached to the residual representation $\bar{\rho}$.

Specifically, let $R_{\Sigma}$ denote the universal deformation ring parameterizing lifts of $\bar{\rho}$ which are ordinary at $p$, and whose tame conductor coincides with that of $\bar{\rho}$ at primes not in $\Sigma \cup \{p\}$. The representation $\rho_{\Sigma}$ induces a map $R_{\Sigma} \to T_{\Sigma}$ which by [6] is an isomorphism after tensoring with the quotient of $\Lambda$ by any classical height one prime; it follows that in fact $R_{\Sigma} \to T_{\Sigma}$ is an isomorphism, as claimed.

2.7. Branches. We now prove some results concerning the irreducible components of the Hida family attached to $\bar{\rho}$.

Definition 2.7.1. If $a$ is a minimal prime ideal in $T_{\Sigma}$, for any $\Sigma$ as above, then we will write $T(a) := T_{\Sigma}/a$. Note that $T(a)$ is a local domain, finite over $\Lambda$. We will write $K(a)$ to denote the fraction field of $T(a)$. (Thus there is an isomorphism $K(a) \cong T(a) \otimes_{\Lambda} \mathcal{L}$.) We let $\rho(a)$ denote the Galois representation

$$
\rho(a): G_{\mathbb{Q}} \to \text{GL}_2(T(a))
$$

induced by $\rho_{\Sigma}$.

Proposition 2.7.2. If $a$ is a minimal prime of $T_{\Sigma}$, for any $\Sigma$ as above, there is a unique divisor $M$ of $N(\Sigma)$ and a unique minimal prime $a' \subseteq T_{\Sigma}^{\text{new}}$ sitting over $a$.
such that

\[
\begin{array}{ccc}
T_\Sigma & \longrightarrow & T'_{N(\Sigma)} \\
\downarrow & & \downarrow \\
T_\Sigma/a & = & T(a) \longrightarrow T'_{M}/a'
\end{array}
\]

commutes.

Proof. Since \(T_\Sigma\) is finite over \(\Lambda\), the minimal primes of \(T_\Sigma\) are in bijection with the local components of \(T_\Sigma \otimes_\Lambda L\). Since \(T_\Sigma\) is a local factor of \(T'_{N(\Sigma)}\), these local components are included in the local components of \(T'_{N(\Sigma)} \otimes_\Lambda L\). By part (1) of Proposition 2.4.2, we have that

\[
T'_{N(\Sigma)} \otimes_\Lambda L \cong \prod_{M|N(\Sigma)} T'_{M} \otimes_\Lambda L.
\]

The local components of \(\prod_{M|N(\Sigma)} T'_{M} \otimes_\Lambda L\) are in one-to-one correspondence with its minimal primes. Thus, our given minimal prime \(a\) gives rise to a minimal prime of this ring. However, any such minimal prime corresponds to a minimal prime \(a'\) in \(T'_{M}\) for some \(M \mid N(\Sigma)\), which establishes the proposition.

\[\square\]

**Definition 2.7.3.** In the context of the preceding proposition, we refer to \(M\) as the tame conductor attached to \(a\) (or to the irreducible component of \(\text{Spec } T_\Sigma\) corresponding to \(a\)), and write \(N(a)\) for \(M\).

We write \(T(a)^\circ := T'_{N(\Sigma)}/a'\); Proposition 2.7.2 then gives rise to an embedding of local domains \(T(a) \rightarrow T(a)^\circ\). (To see that \(T(a)^\circ\) is local, note that it is a complete finite \(\Lambda\)-algebra, and hence a product of local rings. Being a domain, it must be local.)

If \(a\) is a minimal prime of \(T_\Sigma\) for which the tame conductor \(N(a)\) is minimal (i.e. equal to \(N(\bar{p})\)), then the embedding \(T(a) \rightarrow T(a)^\circ\) is an isomorphism. (This follows from the arguments used in the proofs of Remarks 2.9 and 2.11 of [34].) More generally, Proposition 2.4.2 shows that the induced embedding \(T(a)[1/p] \rightarrow T(a)^\circ[1/p]\) is an isomorphism. In particular, the embedding \(T(a) \rightarrow T(a)^\circ\) induces an isomorphism on fraction fields, and we will use this isomorphism to identify the fraction field of \(T(a)^\circ\) with \(K(a)\). One can thus think of \(T(a)^\circ\) as a partial normalisation of \(T(a)\) inside \(K(a)\), which coincides with \(T(a)\) after inverting \(p\).

We next observe that at the generic point of the \(a\)-component of \(\text{Spec } T_\Sigma\), the universal Galois representation has conductor equal to the conductor of \(a\).

**Corollary 2.7.4.** The Galois representation \(\rho(a)\) has tame conductor equal to \(N(a)\). (Here we define the tame conductor by regarding \(\rho(a)\) as a Galois representation defined over the field of fractions \(K(a)\) of \(T(a)\).)

Proof. This follows from Propositions 2.3.2 and 2.4.2.

The next result deals with the classical height one primes in \(T_\Sigma\).

**Proposition 2.7.5.** Let \(\wp\) be any classical height one prime ideal of \(T_\Sigma\).

1. The ring \(T_\Sigma\) is étale over \(\Lambda\) (and so regular) in a neighbourhood of \(\wp\); consequently \(\wp\) contains a unique minimal prime \(a\) of \(T_\Sigma\), and the natural map of localisations \((T_\Sigma)_{\wp} \rightarrow T(a)_{\wp}\) is an isomorphism.
(2) Thinking of \( \varphi \) as a height one prime of \( T(\mathfrak{a}) \), the map \( T(\mathfrak{a}) \to T(\mathfrak{a})^\circ \) is an isomorphism in a neighbourhood of \( \varphi \). Consequently, there is a unique height one prime \( \varphi' \) of \( T(\mathfrak{a})^\circ \) that pulls back to \( \varphi \) under the this map, and the map of localisations \( T(\mathfrak{a})_{\!\varphi} \to T(\mathfrak{a})_{\!\varphi'}^\circ \) is an isomorphism.

Proof. Both claims follow directly from Proposition 2.22 (Part (2) also follows from the fact that \( p \notin \varphi \), and the equality \( T(\mathfrak{a})[1/p] = T(\mathfrak{a})^\circ[1/p] \).

In the situation of the preceding proposition, we write \( \mathcal{O}(\varphi)^\circ := \mathcal{O}(\varphi') \); this is a finite extension of \( \mathcal{O}(\varphi) \). Recall that we have defined the classical newform \( f_\varphi' \) attached to the height one prime ideal \( \varphi' \) of \( T(\mathfrak{a})^\circ \) (thought of as a classical height one prime of \( T_{\!\text{new}}(\mathfrak{a}) \)). We write \( f_\varphi := f_\varphi' \); part (2) of Theorem 2.22 implies that \( f_\varphi \) lies in \( S_\kappa(N(\mathfrak{a})p^\infty, \mathcal{O}(\varphi)^\circ) \).

2.8. \( \Lambda \)-adic modular forms and Euler factors. For each minimal prime \( \mathfrak{a} \) of \( T_\Sigma \), we can define a formal \( q \)-expansion along the partial normalisation \( \text{Spec} \, T(\mathfrak{a})^\circ \) of the component \( \text{Spec} \, T(\mathfrak{a}) \) of \( \text{Spec} \, T_\Sigma \) that interpolates the \( q \)-expansions of the newforms \( f_\varphi \) obtained from the classical primes \( \varphi \) in \( \text{Spec} \, T(\mathfrak{a}) \). Namely, if we write \( T(\mathfrak{a})^\circ = T_{\!\text{new}}(\mathfrak{a})/\mathfrak{a}' \), as in Definition 2.23, then we define \( f(\mathfrak{a}, q) \in T(\mathfrak{a})^\circ[[q]] \) via

\[
 f(\mathfrak{a}, q) = \sum_{n \geq 1} (T_n \mod \mathfrak{a}') q^n.
\]

(Here we have written \( T_n \) rather than \( U'_p T_{n'} \), when \( n \) is of the form \( n = n'p^r \) with \( (n', p) = 1 \), for the sake of uniformity of notation.)

An alternative way of describing this formal \( q \)-expansion along \( \text{Spec} \, T(\mathfrak{a})^\circ \) is to describe the corresponding Euler factors.

Definition 2.8.1. Let \( \mathfrak{a} \) be a minimal prime of \( T_\Sigma \). As above, write \( T(\mathfrak{a})^\circ \cong T_{\!\text{new}}(\mathfrak{a})/\mathfrak{a}' \). For each prime \( \ell \neq p \), define the reciprocal Euler factor \( E_\ell(\mathfrak{a}, X) \in T(\mathfrak{a})^\circ[X] \) via the usual formula:

\[
 E_\ell(\mathfrak{a}, X) := \begin{cases} 
 (1 - (T_\ell \mod \mathfrak{a}') X + (\ell)\ell^{-1} X^2) & \text{if } \ell \text{ is prime to } N(\mathfrak{a}) \\
 (1 - (\overline{T_\ell} \mod \mathfrak{a}') X) & \text{otherwise.}
\end{cases}
\]

For the sake of completeness, define \( E_p(\mathfrak{a}, X) = (1 - (U_p \mod \mathfrak{a}') X) \).

In terms of these reciprocal Euler factors, the formal \( q \)-expansion \( f(\mathfrak{a}, q) \) is characterised by the fact that its “formal Mellin transform” is equal to the formal Dirichlet series

\[
 \prod_\ell E_\ell(\mathfrak{a}, \ell^{-s})^{-1}.
\]

We will now give a Galois theoretic description of these reciprocal Euler factors.

Proposition 2.8.2. Let \( \mathfrak{a} \) be a minimal prime of \( T_\Sigma \), and let \( \mathcal{V} \) denote a two dimensional \( K(\mathfrak{a}) \)-vector space on which the Galois representation \( \rho(\mathfrak{a}) \) acts. If \( \ell \neq p \) is prime, then the Euler factor \( E_\ell(\mathfrak{a}, X) \in K(\mathfrak{a})[X] \) is equal to the determinant \( \det(\text{Id} - \text{Frob}_\ell X | V_\ell) \). (Here \( V_\ell \) denotes the space of \( I_\ell \)-coinvariants in \( \mathcal{V} \).)

Proof. This follows from Lemma 2.21 \( \square \)

Since \( T(\mathfrak{a})^\circ \subset T(\mathfrak{a})[1/p] \), we may in particular regard the reciprocal Euler factors \( E_\ell(\mathfrak{a}, X) \) and the formal \( q \)-expansion \( f(\mathfrak{a}, q) \) as varying over \( \text{Spec} \, T(\mathfrak{a})[1/p] \). Let us close this section by signaling a phenomena which will be fundamental to
all that follows: the reciprocal Euler factors $E_\ell(a, X)$ (or equivalently the formal $q$-expansions $f(a, q)$) do not extend in a well-defined fashion over $\text{Spec} T_\Sigma[1/p]$. In general, if $\varphi$ is a (necessarily non-classical) height one prime lying in the intersection of two different components of $\text{Spec} T_\Sigma[1/p]$, say $\text{Spec} T(a_1)[1/p]$ and $\text{Spec} T(a_2)[1/p]$, then $E_\ell(a_1, X)$ and $E_\ell(a_2, X)$ (and hence $f(a_1, q)$ and $f(a_2, q)$) may have different specialisations in $O(\varphi)[1/p][X]$ (respectively $O(\varphi)[1/p][[q]]$) for certain values of $\ell$ dividing $N(a_1)$ or $N(a_2)$.

3. Algebraic Iwasawa invariants

3.1. Selmer groups of modular forms. Let $f = \sum a_n q^n$ be a $p$-ordinary and $p$-stabilized newform of weight $k \geq 2$, tame level $N$, and character $\chi$. Let $K$ denote the finite extension of $\mathbb{Q}_p$ generated by the Fourier coefficients of $f$ and let $O$ denote the ring of integers of $K$; we write $k$ for the residue field and fix also a uniformizer $\pi$ of $O$. Let

$$\rho_f : G_\mathbb{Q} \to \text{GL}_2(K)$$

be the corresponding Galois representation, characterized by the fact that the characteristic polynomial under $\rho_f$ of an arithmetic Frobenius at a prime $\ell \nmid N p$ is

$$X^2 - a_\ell X + \chi(\ell)\ell^{k-1}.$$  

By [14, Thm. 2.6] the restriction of $\rho_f$ to $G_p$ is of the form

$$(3.1) \quad \rho|_{G_p} \cong \begin{pmatrix} \varepsilon^{k-1}\chi\varphi^{-1} & * \\ 0 & \varphi \end{pmatrix}$$

with $\varphi : G_p \to O^\times$ the unramified character sending an arithmetic Frobenius to $a_p$.

We assume that the semisimple residual representation

$$\bar{\rho}_f : G_\mathbb{Q} \to \text{GL}_2(k)$$

is absolutely irreducible. It follows that, up to conjugation by $\text{GL}_2(O)$, there is a unique integral model

$$\rho_f : G_\mathbb{Q} \to \text{GL}_2(O)$$

of $\rho_f$, which we now fix. For $0 \leq i \leq p - 2$ let $A_{f,i}$ denote a cofree $O$-module of corank 2 with $G_\mathbb{Q}$-action via $\rho_f \otimes \omega^i$. We obtain from [14] and [16, Prop. 12.1] an $O[G_p]$-equivariant exact sequence

$$(3.2) \quad 0 \to (K/O)(\varepsilon^{k-1}\chi\omega^i\varphi^{-1}) \to A_{f,i} \to (K/O)(\omega^i\varphi) \to 0.$$  

We write $A^i_{f,i}$ (resp. $A^q_{f,i}$) for the submodule (resp. quotient module) of $A_{f,i}$ in the above sequence.

For a place $v$ of $\mathbb{Q}_\infty$ define

$$H^1_s(\mathbb{Q}_\infty, v, A_{f,i}) = \begin{cases} H^1(\mathbb{Q}_\infty, v, A_{f,i}) & v \neq v_p; \\
\text{im}(H^1(\mathbb{Q}_\infty, v_p, A_{f,i}) \to H^1(I_{v_p}, A^q_{f,i})) & v = v_p.
\end{cases}$$

Following [10], the Selmer group of $A_{f,i}$ is defined by

$$\text{Sel}(\mathbb{Q}_\infty, A_{f,i}) = \ker(H^1(\mathbb{Q}_\infty, A_{f,i}) \to \prod_v H^1_s(\mathbb{Q}_\infty, v, A_{f,i}))$$

$$= \ker(H^1(\mathbb{Q}_\Sigma/\mathbb{Q}_\infty, A_{f,i}) \to \prod_{v \in \Sigma} H^1_s(\mathbb{Q}_\infty, v, A_{f,i}))$$

for any finite set of places $\Sigma$ containing $v_p$, all archimedean places, and all places dividing $N$. We regard $\text{Sel}(\mathbb{Q}_\infty, A_{f,i})$ as a $\Lambda_\mathbb{O}$-module via the natural action of $\Gamma$. 
Recall that \( \mu_{\text{alg}}(f, \omega^i) \) (resp. \( \lambda_{\text{alg}}(f, \omega^i) \)) is defined to be the largest power of \( \pi \) dividing (resp. the number of zeroes of) the characteristic power series of the \( \Lambda_\Omega \)-dual of \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i}) \) (assuming that this Selmer group is \( \Lambda_\Omega \)-cotorsion).

**Theorem 3.1.1.** Let \( f \) be a \( p \)-ordinary and \( p \)-stabilized newform with \( \tilde{\rho}_f \) absolutely irreducible. Then \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i}) \) is co-finitely generated, \( \Lambda_\Omega \)-cotorsion, and has no proper \( \Lambda_\Omega \)-submodules of finite index. Furthermore, \( \mu_{\text{alg}}(f, \omega^i) \) vanishes if and only if \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i})[\pi] \) is finite. If this is the case, then \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i}) \) is \( \mathcal{O} \)-divisible and

\[
\lambda_{\text{alg}}(f, \omega^i) = \dim_k \text{Sel}(\mathbb{Q}_\infty, A_{f,i})[\pi].
\]

**Proof.** It is shown in [10, Prop. 6] that Selmer groups are always co-finitely generated. The fact that they are also \( \Lambda_\Omega \)-cotorsion for modular forms is proven in [20]. The equivalence of the vanishing of \( \mu_{\text{alg}}(f, \omega^i) \) and the finiteness of \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i})[\pi] \) is now an immediate consequence of the structure theory of \( \Lambda_\Omega \)-modules.

The proof of [12, Prop. 4.14] easily adapts to show that \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i}) \) has no proper \( \Lambda_\Omega \)-submodules of finite index. (As \( A_{f,i} \) need not be self-dual, this also requires the fact that \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i}) \) is \( \mathcal{O} \)-cotorsion, which follows since \( A_{f,i}^* \) is also modular.) When \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i})[\pi] \) is finite, the maximal \( \mathcal{O} \)-divisible \( \Lambda_\Omega \)-submodule of \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i}) \) has finite index, so that it must coincide with \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i}) \); that is, \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i}) \) is divisible. It now follows again from the structure theory of \( \Lambda_\Omega \)-modules that as \( \mathcal{O} \)-modules

\[
\text{Sel}(\mathbb{Q}_\infty, A_{f,i}) \cong (\mathcal{K}/\mathcal{O})^{\lambda_{\text{alg}}(f, \omega^i)},
\]

which proves the last statement. \( \square \)

We close this section with a useful result on the local invariants \( H^0(G_v, A_{f,i}) \) for places \( v \) dividing primes \( \ell \neq p \). Let \( \text{cond}_v(\tilde{\rho}_f) \) denote the exponent of the highest power of \( \ell \) that divides the conductor of \( \tilde{\rho}_f \).

**Lemma 3.1.2.** Let \( v \) be a place of \( \mathbb{Q}_\infty \) dividing a prime \( \ell \neq p \). If \( \text{cond}_v(\tilde{\rho}_f) = \text{ord}_v(N) \), then \( H^0(G_v, A_{f,i}) \) is \( \mathcal{O} \)-divisible for all \( i \).

**Proof.** By the invariance of the Swan conductor under reduction (see [23, §1]), we have

\[
\text{cond}_v(\rho_f) - \text{cond}_v(\tilde{\rho}_f) = \dim_k A_f[\pi]^I_{\ell} - \dim_K V_f^{I_{\ell}}
\]

where \( V_f \) is a two dimensional \( K \)-vector space with \( G_{\mathbb{Q}} \)-action via \( \rho_f \). Since \( \text{cond}_v(\rho_f) \) equals \( \text{ord}_v(N) \) by [11], we see that the hypothesis of the lemma is equivalent to the equality

\[
\dim_k A_f[\pi]^I_{\ell} = \dim_K V_f^{I_{\ell}}.
\]

It follows easily from this that \( A_f^I = A_f^{I_{\ell}} \) is \( \mathcal{O} \)-divisible. As the \( G_v/I_v \)-invariants of an \( \mathcal{O} \)-divisible \( G_v/I_v \)-module are again \( \mathcal{O} \)-divisible, we conclude that \( H^0(G_v, A_f) \) is divisible, as claimed. Since \( \omega \) is unramified at \( \ell \), the above argument works for \( A_{f,i} \) as well. \( \square \)

3.2. **Residual Selmer groups.** Let \( \tilde{A} \) denote a two-dimensional \( k \)-vector space equipped with a continuous \( k \)-linear action of \( G_{\mathbb{Q}} \) and a \( k[G_p] \)-equivariant exact sequence

\[
0 \to \tilde{A}' \to \tilde{A} \to \tilde{A}'' \to 0
\]

with \( \tilde{A}' \) and \( \tilde{A}'' \) one-dimensional. We make the following assumptions on this data.
(1) $\bar{A}$ is absolutely irreducible as a $k[G]$-module;
(2) The $k$-vector space $A''$ is unramified as a $G_p$-module;
(3) $\bar{A}$ is $p$-distinguished in the sense that the representation of $G_p$ on $\bar{A}$ is non-scalar;
(4) $\bar{A}$ is modular: there exists a totally ramified extension $K'$ of $K$, a $p$-stabilized newform $f \in \mathcal{O}'[[q]]$ of weight $k \geq 2$, and a $k[G]$-isomorphism $\bar{A} \cong A_f[\pi']$ identifying $A'$ with $A'_f[\pi']$ in the notation of the previous section; here $\pi'$ is a uniformizer of the ring of integers $\mathcal{O}'$ of $K'$.

We will study Selmer groups of $\bar{A}$ and its cyclotomic twists $\bar{A} \otimes \omega^i$ with respect to various local conditions.

**Definition 3.2.1.** A finite/singular structure $\mathcal{S}$ on $\bar{A} \otimes \omega^i$ is a choice of $k$-subspaces

$$(3.4) \quad H^1_{f,S}(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i) \rightarrow H^1(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i)$$

for each place $v$ of $\mathbb{Q}_\infty$, subject to the restrictions:

1. $H^1_{f,S}(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i) = 0$ for almost all $v$;
2. $H^1_{f,S}(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i) = \ker(H^1(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i) \rightarrow H^1(I_{\omega^i}, A'' \otimes \omega^i))$.

(Note that we are not allowing any variation in the choice of condition at $v_p$.) We define $H^1_{f,S}(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i)$ as the cokernel of $(3.4)$. The $\mathcal{S}$-Selmer group of $\bar{A} \otimes \omega^i$ is

$$\text{Sel}_{\mathcal{S}}(\mathbb{Q}_\infty, \bar{A} \otimes \omega^i) = \ker(H^1(\mathbb{Q}_\infty, \bar{A} \otimes \omega^i) \rightarrow \prod_v H^1_{f,S}(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i))$$

for any finite set of places $\Sigma$ containing $p$, all archimedean places, all places at which $\bar{A}$ is ramified, and all places for which $H^1_{f,S}(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i)$ does not vanish.

We will be especially interested in two kinds of finite/singular structures. First, the minimal structure $\mathcal{S}_{\text{min}}$ on $\bar{A} \otimes \omega^i$ is given by

$$H^1_{f,\mathcal{S}_{\text{min}}}(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i) = 0$$

for $v \neq v_p$.

Next let $f$ be a newform as in (4) above. We define the induced structure $\mathcal{S}(f,i)$ on $\bar{A} \otimes \omega^i$ by setting

$$H^1_{f,\mathcal{S}(f,i)}(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i) = \ker(H^1(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i) \rightarrow H^1(\mathbb{Q}_\infty,v, A_{f,i}))$$

for all places $v$ of $\mathbb{Q}_\infty$. Note that by the definition of $H^1_{f}(\mathbb{Q}_\infty,v, A_{f,i})$ we have

$$H^1_{f,\mathcal{S}(f,i)}(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i) = \ker H^1(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i) \rightarrow H^1(I_{\omega^i}, A'' \otimes \omega^i)$$

in fact, since $H^0(I_{\omega^i}, A''_{f,i})$ is either $A''_f$ (for $i = 0$) or else zero (for $i \neq 0$), the latter map is injective, so that

$$H^1_{f,\mathcal{S}(f,i)}(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i) = \ker H^1(\mathbb{Q}_\infty,v, \bar{A} \otimes \omega^i) \rightarrow H^1(I_{\omega^i}, A'' \otimes \omega^i)$$

as required.
Lemma 3.2.2. For \( v \nmid p \) we have
\[
H^1_{f,S(f, i)}(\mathbb{Q}_\infty, \bar{A} \otimes \omega^i) = \text{im}(A^{G_\pi}_f / \pi \hookrightarrow H^1(\mathbb{Q}_\infty, \bar{A} \otimes \omega^i))
\]
In particular, \( H^1_{f,S(f, i)}(\mathbb{Q}_\infty, \bar{A} \otimes \omega^i) = 0 \) if \( \bar{A} \otimes \omega^i \) is unramified at \( v \neq v_p \) so that \( S(f, i) \) is a finite/singular structure. The natural map
\[
\text{Sel}_{S(f, i)}(\mathbb{Q}_\infty, \bar{A} \otimes \omega^i) \to \text{Sel}(\mathbb{Q}_\infty, A_{f, i})[\pi]
\]
is an isomorphism.

Proof. The lemma follows from various exact sequences in cohomology coming from the exact sequence
\[
0 \to \bar{A} \otimes \omega^i \to A_{f, i} \xrightarrow{\pi} A_{f, i} \to 0
\]
together with Lemma 3.1.2. We leave the details to the reader. \( \square \)

Proposition 3.2.3. Let \( \bar{A} \) be as above. Then there exists a newform \( f \) as in (4) above such that the finite/singular structures \( S_{\text{min}} \) and \( S(A_{f, i}) \) coincide under the isomorphism \( \bar{A} \otimes \omega^i \cong A_{f, i}[\pi^j] \) for any \( i \).

Proof. Let \( N \) denote the tame conductor of the Galois representation \( \bar{A} \). As \( f \) is \( p \)-distinguished, by [3 Thm. 6.4] there exists a finite totally ramified extension \( K' \) of \( K \) and a (not necessarily unique) \( p \)-stabilized newform \( f \in K'[[q]] \) of tame level \( N \) and weight 2 satisfying the condition of (4). (The final condition in (4) is in fact already automatic from the more standard level lowering result [5 Thm. 1.1] unless \( \bar{A} \) is an unramified \( G_p \)-module, in which case the Selmer case of [5 Thm. 6.4] ensures the existence of such an \( f \).)

Since the tame conductor of \( f \) equals the conductor of \( \bar{A} \), by Lemma 3.1.2, \( A^{G_\pi}_f \) is divisible for any place \( v \neq v_p \). It then follows from Lemma 3.2.2 that the structure \( S(A_{f, i}) \) on \( A_{f, i}[\pi^j] \cong \bar{A} \) agrees with \( S_{\text{min}} \), as desired. \( \square \)

The next proposition, which follows from a result of Greenberg, is crucial to our method.

Proposition 3.2.4. For \( f \) be as above there is an exact sequence
\[
0 \to \text{Sel}_{S(f, i)}(\mathbb{Q}_\infty, \bar{A} \otimes \omega^i) \to H^1(\mathbb{Q}_\Sigma / \mathbb{Q}_\infty, \bar{A} \otimes \omega^i) \to \prod_{v \in \Sigma} H^1_{s,S(f, i)}(\mathbb{Q}_{\infty, v}, \bar{A} \otimes \omega^i) \to 0
\]
for any finite set of places \( \Sigma \) containing \( v_p \), all archimedean places and all places dividing the tame level of \( f \).

Proof. Since \( A_{f, i} \) is odd and \( \text{Sel}(\mathbb{Q}_\infty, A_{f, i}) \) is \( \Lambda_{\mathcal{O}} \)-cotorsion, by [15 Prop. 2.1] there is an exact sequence
\[
0 \to \text{Sel}(\mathbb{Q}_\infty, A_{f, i}) \to H^1(\mathbb{Q}_\Sigma / \mathbb{Q}_\infty, A_{f, i}) \to \prod_{v \in \Sigma} H^1_{s}(\mathbb{Q}_{\infty, v}, A_{f, i}) \to 0
\]
with \( \Sigma \) as above. As \( \text{Sel}(\mathbb{Q}_\infty, A_{f, i}) \) is \( \mathcal{O} \)-divisible the \( \pi \)-torsion of this sequence is an exact sequence
\[
0 \to \text{Sel}(\mathbb{Q}_\infty, A_{f, i})[\pi] \to H^1(\mathbb{Q}_\Sigma / \mathbb{Q}_\infty, A_{f, i})[\pi] \to \prod_{v \in \Sigma} H^1_{s}(\mathbb{Q}_{\infty, v}, A_{f, i})[\pi] \to 0.
\]
By Lemma 3.2.2 and the definition of the induced structure this sequence identifies with that of the proposition. \( \square \)
Our main algebraic theorems are consequences of the next result.

**Corollary 3.2.5.** Let $\bar{A}$ be as above and let $f$ be a $p$-stabilized newform of tame level $N_f$ such that $A_f[\pi] \cong \bar{A}$ as in (4). Then the sequence

$$0 \to \text{Sel}_{\text{min}}(Q_\infty, \bar{A} \otimes \omega^i) \to \text{Sel}(Q_\infty, A_{f,i})[\pi] \to \prod_{v|N_f} A_{f,i}^{G_v}/\pi \to 0$$

is exact for any $i$.

**Proof.** The exactness of (3.5) follows from the definitions and Lemma 3.2.2 except for the surjectivity of the last map. For this, by Proposition 3.2.3 there exists a totally ramified extension $K'/K$ and a $p$-stabilized newform $f_0$ over $K'$ such that $(\bar{A} \otimes \omega^i, S_{\text{min}}) \cong (A_{f_0,i}[\pi^2], S(A_{f_0,i}))$. The surjectivity is then a formal consequence of the exact sequence of Proposition 3.2.4 applied to both $A_{f_0,i}$ and $A_{f,i}$ with $\Sigma = \{v \mid N_f p\}$. \(\square\)

3.3. **Algebraic Iwasawa invariants.** Let $\bar{A}$ be as in the previous section. We write $\bar{\rho} : G_Q \to \text{GL}_2(k)$ for the corresponding Galois representation. We now use Corollary 3.2.5 to study the relations between Selmer groups of newforms in the Hida family of $\bar{\rho}$. We first consider the behavior of the corresponding finite/singular structures on $\bar{A}$.

**Lemma 3.3.1.** Let $f$ be a newform on the branch $T(a)$ of the Hida family of $\bar{\rho}$. Then the finite singular structure $S(f,i)$ depends only on the branch $T(a)$ and $i$.

**Proof.** Fix a place $v \neq v_p$ dividing the rational prime $\ell$; we must show that $H^1_{f, S(f,i)}(Q_\infty, v, \bar{A} \otimes \omega^i)$ is independent of $f$ on the branch $T(a)$. Consider the exact sequence

$$0 \to A_{f,i}^{G_v}/\pi \to H^1(Q_\infty, v, \bar{A} \otimes \omega^i) \to H^1(Q_\infty, v, A_{f,i})[\pi] \to 0$$

defining the local condition. By [15] Prop. 2.4, $H^1(Q_\infty, v, A_{f,i})[\pi]$ has $k$-dimension equal to the multiplicity of $\omega^i$ in the residual representation of the unramified $G_\ell$-representation $(V_f)_{I_\ell}$. By the proof of Proposition 3.2.2 the dimension of $(V_f)_{I_\ell}$ depends only on the branch $T(a)$. In particular, if $(V_f)_{I_\ell} = 0$, then

$$H^1_{f, S(f,i)}(Q_\infty, v, \bar{A} \otimes \omega^i) = H^1(Q_\infty, v, \bar{A} \otimes \omega^i)$$

is certainly independent of $f$, while if $(V_f)_{I_\ell} = V_f$, then Lemma 3.3.2 shows that

$$H^1_{f, S(f,i)}(Q_\infty, v, \bar{A} \otimes \omega^i) \cong (A_{f,i})^{G_v}/\pi = 0$$

is again independent of $f$.

To prove the lemma it thus suffices to consider the case that $(V_f)_{I_\ell}$ has dimension one. In this case $f$ at $\ell$ is either:

1. principal series associated to a ramified character $\chi_1$ and an unramified character $\chi_2$;
2. special associated to an unramified character $\chi$.

We claim first that which of (1) or (2) holds is constant on the branch $T(a)$. Indeed, the semi-simplification $\rho(a)|_{G_\ell}$ is a sum of two characters. As the restriction of these characters to $I_\ell$ has finite image, the conductor of $\rho(a)|_{G_\ell}$ does not change on reduction modulo any height one prime of $T(a)$. It follows that (1) holds for all forms on the branch $T(a)$ if and only if $\rho(a)|_{G_\ell}$ is ramified; that is, which of (1) and (2) holds is constant on the branch $T(a)$. 


Let $A(a)$ denote $\text{Hom}_{\mathbb{Z}_p}(T(a)^2, \mathbb{Q}_p/\mathbb{Z}_p)$ endowed with a $G_{\mathbb{Q}}$-action via $\rho(a)$. We claim that the map

$$H^1(G_v, A(a)_{(i)}[\varphi_f]) \to H^1(G_v, A(a)_{(i)})[\varphi_f]$$

is an isomorphism; here $\varphi_f$ is the height one prime of $T(a)$ corresponding to $f$ and $A(a)_{(i)}$ denotes $A(a) \otimes \omega^i$. Indeed, since which of (1) and (2) holds is constant on $T(a)$, the multiplicity $\nu$ of $\omega^{1-i}$ in the $G_v/I_n$-representation $A(a)[\varphi_f]_{I_n}$ is equal to that in $A(a)_{I_n}$. Computing as in \cite[p. 38]{15}, it follows that the map above is dual to the natural map

$$T(a)^{\nu}/\varphi_f \to (T(a)/\varphi_f)^{\nu}$$

which is visibly an isomorphism.

It follows that

$$H^1_{T,S(f,i)}(G_v, \mathbb{A} \otimes \omega^i) := \ker \left( H^1(G_v, \mathbb{A} \otimes \omega^i) \to H^1(G_v, A(a)_{(i)}[\varphi_f]) \right)$$

$$= \ker \left( H^1(G_v, \mathbb{A} \otimes \omega^i) \to H^1(G_v, A(a)_{(i)})[\varphi_f] \right)$$

$$= \ker \left( H^1(G_v, \mathbb{A} \otimes \omega^i) \to H^1(G_v, A(a)_{(i)}) \right)$$

which is certainly independent of $f$. □

For a branch $T(a)$ of the Hida family of $\rho$ and a place $v \neq v_p$, we may now define

$$\delta_v(a, \omega^i) := \dim_k H^1_{T,S(f,i)}(\mathbb{Q}_\infty, \mathbb{A} \otimes \omega^i) = \dim_k A^G_{f,i}/\pi$$

for any form $f$ lying on the branch $T(a)$.

We say that

$$\mu^\text{alg}(\bar{\rho}, \omega^i) = 0$$

if $\text{Sel}_{\min}(\mathbb{Q}_\infty, \mathbb{A} \otimes \omega^i)$ is finite dimensional over $k$; if this is the case we define

$$\lambda^\text{alg}(\bar{\rho}, \omega^i) = \dim_k \text{Sel}_{\min}(\mathbb{Q}_\infty, \mathbb{A} \otimes \omega^i)$$.  

**Theorem 3.3.2.** Let $\bar{\rho}$ be as above. Let $f$ be any newform in the Hida family of $\bar{\rho}$. Then $\mu^\text{alg}(\bar{\rho}, \omega^i) = 0$ if and only if $\mu^\text{alg}(f, \omega^i) = 0$.

**Proof.** By Theorem 3.1.1, $\mu^\text{alg}(f, \omega^i)$ vanishes if and only if $\text{Sel}(\mathbb{Q}_\infty, A_{f,i})[\pi]$ is finite dimensional. By Corollary 3.2.3, this is equivalent to the finite dimensionality of $\text{Sel}_{\min}(\mathbb{Q}_\infty, \mathbb{A} \otimes \omega^i)$, as claimed. □

**Theorem 3.3.3.** Let $\bar{\rho}$ be as above. Assume that $\mu^\text{alg}(\bar{\rho}, \omega^i) = 0$.

1. Let $f$ be a newform of level $N_f$ lying on the branch $T(a)$ of the Hida family of $\bar{\rho}$. Then

$$\lambda^\text{alg}(f, \omega^i) = \lambda^\text{alg}(\bar{\rho}, \omega^i) + \sum_{v \mid N_f} \delta_v(a, \omega^i).$$

In particular, $\lambda^\text{alg}(f, \omega^i)$ depends only on the branch $T(a)$ of $f$; we write $\lambda^\text{alg}(a, \omega^i)$ for this value.

2. Let $T(a_1)$ and $T(a_2)$ be two branches of the Hida family of $\bar{\rho}$. Then

$$\lambda^\text{alg}(a_1, \omega^i) - \lambda^\text{alg}(a_2, \omega^i) = \sum_{v \neq v_p} \delta_v(a_1, \omega^i) - \delta_v(a_2, \omega^i).$$

**Proof.** The first statement is immediate from Corollary 3.2.3 and the definition of $\delta_v(a, \omega^i)$. The second statement follows from the first. □
4. Two-variable p-adic L-functions

4.1. Constructions of p-adic L-functions. If $N$ is an integer prime to $p$, then we let $T_N^*$ denote the analogue of $T_N^*$ constructed with respect to all modular forms, rather than just cusp forms (so that $T_N$ is the quotient of $T_N^*$ that acts on cusp forms). Suitable analogues of the results stated for $T_N$ in section 2.1 hold for $T_N^*$. Furthermore, if $m$ is a maximal ideal of $T_N^*$ for which the associated residual Galois representation is irreducible, then the natural map of localisations $(T_N^*)_m \to (T_N)_m$ will be an isomorphism. For this reason, we will ultimately have no need of the ring $T_N$. However, it is convenient to introduce it, since it is this ring, rather than $T_N$, that naturally acts on some of the homology groups that we will now introduce.

For any level $M$, we let $X_1(M)$ denote the closed modular curve of level $\Gamma_1(M)$, and let $C_1(M)$ denote its set of cusps. We have a short exact sequence of Hecke modules

\[(4.1) \quad 0 \to H_1(X_1(M); \mathbb{Z}_p) \to H_1(X_1(M), C_1(M); \mathbb{Z}_p) \to \tilde{H}_0(C_1(M); \mathbb{Z}_p) \to 0.\]

(This is a part of the relative homology sequence of the pair $(X_1(M), C_1(M))$; the tilde over the $\tilde{H}_0$ denotes reduced homology.) If we localise this sequence at a maximal ideal in the Hecke algebra corresponding to an irreducible residual $\mathbb{Q}$-representation, then the localisation of $\tilde{H}_0(C_1(M); \mathbb{Z}_p)$ will vanish, and the two $H_1$ terms will become isomorphic.

To ease notation we will write

\[H_1(M; \mathbb{Z}_p) := H_1(X_1(M); \mathbb{Z}_p)\]

and

\[H_1(M, \{\text{cusps}\}; \mathbb{Z}_p) := H_1(X_1(M); C_1(M); \mathbb{Z}_p).\]

We now consider \((4.1)\) with $M = Np^r$ ($r \geq 0$). Passing to ordinary parts (an exact functor) we obtain a short exact sequence of $T_N^*$-modules. If we fix a maximal ideal $m$ of $T_N^*$ for which the Galois representation $\tilde{\rho}_m$ is irreducible, then the above comments show that we obtain an isomorphism

\[(H_1(Np^r; \mathbb{Z}_p)_{\text{ord}})_m \cong (H_1(Np^r, \{\text{cusps}\}; \mathbb{Z}_p)_{\text{ord}})_m.\]

Passing to the projective limit in $r$, we obtain a corresponding isomorphism of $(T_N^*)_m \cong (T_N)_m$-modules

\[\lim_{r} (H_1(Np^r; \mathbb{Z}_p)_{\text{ord}})_m \cong \lim_{r} ((H_1(Np^r, \{\text{cusps}\}; \mathbb{Z}_p)_{\text{ord}})_m.\]

We denote these isomorphic modules by $M_m$. There is an additional piece of structure on $M_m$ that we should mention: the action of complex conjugation on the modular curves $X_1(Np^r)$ induces an action of complex conjugation on homology which commutes with the Hecke action. Thus we obtain an action of complex conjugation on $M_m$. We let $M_m^\pm$ denote the $\pm$-eigenspace for this action. Since $p$ is odd, we have $M_m = M_m^+ \oplus M_m^-$. 

Proposition 4.1.1. If $\tilde{\rho}_m$ is irreducible and $p$-distinguished, then each of the $(T_N^*)_m$-modules $M_m^\pm$ is free of rank one.

Proof. Let $J_1(Np)$ denote the Jacobian of $X_1(Np)$. By [24] Thm. 2.1 and Theorem 2.1.2, the ordinary $m$-torsion $J_1(Np)(\mathbb{Q})[m]_{\text{ord}}$ is free of rank 2 over $T_N/\mathfrak{m}$. On the
we conclude that \( \text{Hom} (M, Q_p/Z_p)[m] \cong H^1 (N p; Q_p/Z_p)_{\text{ord}}[m] \)
so that
\[
\text{Hom} (M, Q_p/Z_p)[m] \cong H^1 (N p; Q_p/Z_p)_{\text{ord}}[m] \cong H^1 (N p; F_p)_{\text{ord}}[m].
\]
Since
\[
J_1 (N p)[Q][p] \cong H^1 (N p; F_p),
\]
we conclude that \( \text{Hom} (M, Q_p/Z_p)[m] \), and thus also its dual \( M_m/mM_m \), is free of rank 2 over \( T_m \). As \( M_m \otimes Q_p \) is free of rank 2 over \( (T_m)_m \otimes Q_p \), it follows that \( M_m \) is free of rank 2 over \( (T_m)_m \). Passing to \( \pm \)-subspaces establishes the proposition. 

The advantage of the second description of \( M_m \), as a limit of relative homology groups, is that the corresponding relative homology classes admit a description via modular symbols. More precisely, for fixed \( r \), there is a map
\[
P^1 (Q) \to H_1 (N p^r; \{ \text{cusps} \}; Z_p)
\]
defined by sending the element \( a \in P^1 (Q) \) to the homology class corresponding to the image of the path \([\infty, a]\) on the modular curve \( X_1 (N p^r) \). This map is compatible with varying \( r \), and so projecting to ordinary parts, localising at \( m \), and then passing to the limit, we obtain a map
\[
P^1 (Q) \to M_m,
\]
which we denote by \( a \mapsto \{ \infty, a \} \). This map allows us to define an \( M_m \)-valued measure on \( Z_p^\times \) in the usual way.

**Definition 4.1.2.** For any open subset \( a + p^r Z_p \) of \( Z_p^\times \), we define
\[
\mu (a + p^r Z_p) = U_p^{-r} \{ \infty, a/p^r \} \in M_m.
\]

**Proposition 4.1.3.** The function \( \mu \) is a measure (i.e. it is additive).

**Proof.** This is standard. \qed

Recall that the completed group ring \( Z_p[[Z_p^\times]] \) may naturally be regarded as the space of \( Z_p \)-valued measures on \( Z_p^\times \). We may thus regard \( \mu \) as defining an element \( L(m, N) \in M_m \hat{\otimes} Z_p[[Z_p^\times]] \). (Here the tensor product is completed with respect to the usual (i.e. profinite topology) on \( Z_p[[Z_p^\times]] \) and the \( p \)-adic topology on \( M_m \).

We may decompose \( L(m, N) \) under the action of complex conjugation to get a pair of elements
\[
L^\pm (m, N) \in M_m^\pm \hat{\otimes} Z_p[[Z_p^\times]].
\]

If \( \wp \) is a height one prime ideal of \( (T_N)_m \), then we may reduce \( L(m, N)^\pm \) modulo \( \wp \) to obtain elements \( L^\pm (m, N)(\wp) \in M_m^\pm /\wp M_m^\pm \hat{\otimes} Z_p[[Z_p^\times]] \). We may also reduce \( L(m, N)^\pm \) modulo \( m \), and so obtain elements \( L^\pm (m, N) \in M_m^\pm /mM_m^\pm \hat{\otimes} Z_p[[Z_p^\times]] \).

We now assume that the hypotheses of Proposition 4.1.1 are satisfied. That result shows that we may choose an isomorphism
\[
(T_N)_m \cong M_m^\pm,
\]
and so regard \( L^\pm (m, N) \) as an element of \( (T_N)_m \hat{\otimes} Z_p[[Z_p^\times]] \cong (T_N)_m[[Z_p^\times]] \). For any height one prime \( \wp \) in \( (T_N)_m \) we may regard \( L^\pm (m, N)(\wp) \) as an element of
\[ \mathcal{O}(\varphi) \otimes \mathbb{Z}_p, \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \cong \mathcal{O}(\varphi)[[\mathbb{Z}_p^\times]]. \] Finally, \((T_N)_m/m \cong k\), and so we may regard \( \tilde{L}^\pm(m, N) \) as an element of \( k \otimes \mathbb{Z}_p, \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \cong k[[\mathbb{Z}_p^\times]] \).

**Proposition 4.1.4.** If \( \varphi \) is a classical height one prime ideal in \((T_N)_m\), then

\[ L^\pm(m, N)(\varphi) \in \mathcal{O}(\varphi)[[\mathbb{Z}_p^\times]] \]

is the usual analytic \( p \)-adic \( L \)-function attached to the corresponding normalised eigenform

\[ f_\varphi \in S_k(Np^{\infty}, \mathcal{O}(\varphi))^{\text{ord}}[\kappa_\varphi] \]

(\text{computed with respect to a canonical period}).

**Proof.** This is the standard comparison between the specialisation of a two-variable \( L \)-function arising from a Hida family and a one-variable \( p \)-adic \( L \)-function, and will be clear once we give a one-variable construction analogous to our two-variable construction. Let \( T_{r,k} \) denote the ordinary part of the Hecke algebra acting on \( S_k(Np^r) \) and let

\[ M_{r,k} = H_1(Np^r, \{\text{cusps}, L_{k-2}(\mathbb{Z}_p)\}). \]

Choose \( r \) and \( k \) so that the modular form \( f_\varphi \) corresponds to some prime ideal \( \varphi_{r,k} \) of \( T_{r,k} \) and let \( m_{r,k} \) be the maximal ideal of \( T_{r,k} \) containing \( \varphi_{r,k} \). There is an \((M_{r,k})_{m_{r,k}}\)-valued measure defined by

\[ \mu_{r,k}(a + p^n \mathbb{Z}_p) = U_p^{-n} \{\infty, a/p^n\}; \]

we write

\[ L_p^\pm(m_{r,k}, N) \in (M_{r,k})_{m_{r,k}}[[\mathbb{Z}_p^\times]] \]

for its associated power series.

We claim that \((M_{r,k})_{m_{r,k}}^\pm\) is a free \((T_{r,k})_{m_{r,k}}\)-module of rank one. First note that by Theorem 2.1.2,

\[ T/\omega_{r,k}T \cong T_{r,k} \]

where \( T = T_N \) and \( \omega_{r,k} \) is the product of all classical primes \( \varphi \subset \Lambda \) of weight \( k \) and such that \( \kappa_\varphi \) restricted to \( 1 + p^\infty \mathbb{Z}_p \) is trivial. Furthermore, by [19, Theorem 1.9],

\[ M_{m}/\omega_{r,k}M_{m} \cong (M_{r,k})_{m_{r,k}}. \]

Thus, since \((M_{r,k})_{m_{r,k}}^\pm\) is a free \( T_m \)-module of rank one (Proposition 4.1.1), it follows that \((M_{r,k})_{m_{r,k}}^\pm\) is a free \((T_{r,k})_{m_{r,k}}\)-module of rank one as well.

Our fixed identification \((M_{m})_{m_{r,k}}^\pm \cong T_m\) now gives an identification \((M_{r,k})_{m_{r,k}}^\pm \cong (T_{r,k})_{m_{r,k}}\) and we can view \( L_p^\pm(m_{r,k}, N) \) as an element of \((T_{r,k})_{m_{r,k}}[[\mathbb{Z}_p^\times]]\). The reduction of \( L_p^\pm(m_{r,k}, N) \) mod \( \varphi_{r,k} \), viewed in \( T_{r,k}/\varphi_{r,k}[[\mathbb{Z}_p^\times]] \cong \mathcal{O}(\varphi)[[\mathbb{Z}_p^\times]] \), is simply the \( p \)-adic \( L \)-function of \( f_\varphi \) computed with respect to a canonical period. (This last claim follows from the definition of canonical period and the fact that this power series has the correct interpolation property.)

By construction

\[ L^\pm(m, N) \equiv L^\pm(m_{r,k}, N) \mod \omega_{r,k} \]

and thus

\[ L^\pm(m, N)(\varphi) \equiv L^\pm(m_{r,k}, N) \mod \varphi_{r,k}. \]

The right hand side of the above equation is the \( p \)-adic \( L \)-function of \( f_\varphi \); the proposition follows. (Note that the usual ambiguity of the canonical period coming from the choice of an isomorphism \((M_{r,k})_{m_{r,k}}^\pm \cong (T_{r,k})_{m_{r,k}} \) is controlled along the Hida family by the single identification \( M_{m}^\pm \cong T_m \).)

\(\square\)
4.2. Two variable $L$-functions on branches of the Hida family. Fix an irreducible, $p$-ordinary, $p$-distinguished, modular Galois representation $\bar{\rho}: G_{\mathbf{Q}} \to \text{GL}_2(k)$, equipped with a chosen $p$-stabilisation (which we will suppress in our discussion and notation), as in section 2.6. Our goal in this section is to define a $p$-adic $L$-function varying over each component of the Hida family of $\bar{\rho}$, which at any classical height one prime $\wp$ specialises to the $p$-adic $L$-function of the newform $f_{\wp}$.

We begin by fixing a finite set of primes $\Sigma$, and considering the Hecke algebra $T_\Sigma$ associated to $\bar{\rho}$ as in section 2.6. Proposition 2.6.2 yields an isomorphism $T_\Sigma \cong (T_{N(\Sigma)})_n$ for a certain maximal ideal $n$ of $T_{N(\Sigma)}$. The construction of the preceding section defines elements $L(n, N(\Sigma), \omega^i) \in (T_{N(\Sigma)})_n \hat{\otimes} \mathbf{Z}_p \Lambda(i)$ (well-defined up to multiplication by an element of $(T_{N(\Sigma)})_n^\times$).

**Definition 4.2.1.**

1. Set $L_\Sigma(\bar{\rho}, \omega^i)$ equal to the element of $T_\Sigma \hat{\otimes} \mathbf{Z}_p \Lambda(i)$ arising from $L(n, N(\Sigma), \omega^i)$ via the isomorphism $T_\Sigma \cong (T_{N(\Sigma)})_n$. (This is well-defined up to a unit of $T_\Sigma$.)
2. Set $L_\Sigma(\bar{\rho}, \omega^i)(\wp)$ equal to the element of $\mathcal{O}(\wp) \hat{\otimes} \mathbf{Z}_p \Lambda(i)$ obtained as the reduction of $L_\Sigma(\bar{\rho}, \omega^i)$ modulo $\wp$ for any height one prime ideal $\wp$ of $T_\Sigma$. (This is well-defined up to a unit in $\mathcal{O}(\wp)$.)
3. Set $\overline{L}_\Sigma(\bar{\rho}, \omega^i)$ equal to the element of $k \hat{\otimes} \mathbf{Z}_p \Lambda(i)$ obtained as the reduction of $L_\Sigma(\bar{\rho}, \omega^i)$ modulo the maximal ideal of $T_\Sigma$. (This is well-defined up to a unit of $k$.)

If $\wp$ is a classical height one prime ideal of $T_\Sigma$, then $\wp$ corresponds to a classical height one prime ideal $\wp'$ of $T_{N(\Sigma)}$ (via the isomorphism of Proposition 2.6.2), and we write $g_{\wp'} := f_{\wp'}$ to denote the normalised eigenform in $S_k(N(\Sigma)p^{\infty}, \mathcal{O}(\wp')^{\text{ord}})[\kappa_{\wp'}]$ corresponding to $\wp'$ via part (2) of Theorem 2.1.2. If $a$ denotes the (unique) minimal prime of $T_\Sigma$ contained in $\wp$, then we may also form the normalised eigenform $f_{\wp} \in S_k(N(a)p^{\infty}, \mathcal{O}(\wp)^{\text{ord}})[\kappa_{\wp}]$. Note that in general $f_{\wp}$ and $g_{\wp}$ are not equal. Indeed, they are eigenforms for different Hecke algebras: $f_{\wp}$ is an eigenform for $T_{N(a)}$, while $g_{\wp}$ is an eigenform for $T_{N(\Sigma)}$. Proposition 2.6.2 makes it clear how they are related: $g_{\wp}$ is the normalised oldform obtained from $f_{\wp}$ by “removing the Euler factor” at each of the primes $\ell \in \Sigma$. Proposition 4.1.3 shows that $L_\Sigma(\bar{\rho}, \omega^i)(\wp)$ is the $p$-adic $L$-function attached to $g_{\wp}$.

We now turn to constructing a $p$-adic $L$-function $L(\bar{\rho}, a, \omega^i) \in (T_{N(a)})^{\hat{\otimes}} \mathbf{Z}_p \Lambda(i)$ for each minimal prime $a$ of $T_\Sigma$, whose specialisation at each classical height one prime $\wp$ of $T(a)$ will be equal to the $p$-adic $L$-function of $f_{\wp}$. For this, we recall the natural isomorphism $T(a)_{\wp} \cong T_{N(a)}^{\text{new}}/a^{\wp}$ of Definition 2.7.3 which gives rise to the
composite surjection
\[(4.4) \quad T_{N(a)} \to T_{N(a)}^{'new} \to T_{N(a)/a^{'}} \cong T(a)^{\circ} .\]

If we let \(m\) denote the maximal ideal of \(T_{N(a)}\) obtained as the preimage of the maximal ideal of \(T(a)^{\circ}\) under \((4.3)\), then the construction of the preceding section yields an \(L\)-function \(L(m, N(a), \omega^i) \in T_{N(a)} \otimes \mathbb{Z}_p \Lambda(i)\). The surjection \((4.4)\) induces a corresponding surjection
\[T_{N(a)} \otimes \mathbb{Z}_p \Lambda(i) \to T(a)^{\circ} \otimes \mathbb{Z}_p \Lambda(i) .\]

**Definition 4.2.2.** We let \(L(\hat{\rho}, a, \omega^i)\) denote the image of \(L(m, N(a), \omega^i)\) under the preceding surjection.

Proposition \(4.1.4\) shows that the specialisation of \(L(\hat{\rho}, a, \omega^i)\) at any classical height one prime ideal \(\varphi\) of \(T(a)\) is equal to the \(p\)-adic \(L\)-function \(L_p(f_{\varphi})\) of the associated newform \(f_{\varphi}\).

### 4.3. Comparisons

Our next task is to compare \(L(\hat{\rho}, a, \omega^i)\) and \(L_{\Sigma}(\hat{\rho}, \omega^i)\) mod \(a\). Each of these two \(p\)-adic \(L\)-functions lies in \(T(a)^{\circ} \otimes \mathbb{Z}_p \Lambda(i)\). The interpolation property satisfied by classical \(p\)-adic \(L\)-functions shows that for any height one prime \(\varphi\) of \(T(a)\), the \(L\)-functions \(L_p(f_{\varphi})\) and \(L_p(g_{\varphi})\) agree up to multiplication by a certain product of Euler factors (reflecting the Euler factors removed from \(f_{\varphi}\) to obtain \(g_{\varphi}\) and an element of \(\mathcal{O}(\varphi)/[1/p]^{\times}\) (reflecting the ratio of the canonical period of \(f_{\varphi}\) and the canonical period of \(g_{\varphi}\)). We will show that in fact the ratio of these canonical periods lies in \((\mathcal{O}(\varphi)^{\times})^{\times}\), and hence that \(L_p(f_{\varphi})\) and \(L_p(g_{\varphi})\) agree up to multiplication by a product of Euler factors, and the inevitable ambiguity of a unit of \(\mathcal{O}(\varphi)^{\circ}\). Furthermore, we will show that this occurs uniformly in Hida families; in other words, that \(L_{\Sigma}(\hat{\rho}, \omega^i)\) mod \(a\) and \(L(\hat{\rho}, a, \omega^i)\) agree up to multiplication by a product of Euler factors, and a unit in \(T(a)^{\circ}\).

We begin by defining the relevant Euler factors. Recall the Euler factors \(E_{\ell}(a, X)\) in \(T(a)^{\circ}[X]\) defined in section \(2.8\). Since \(\ell\) is prime to \(p\), we may regard \(\ell\) as an element of \(\mathbb{Z}_p^\times\), and so obtain a corresponding unit element \(\langle \ell \rangle \in \mathbb{Z}_p[[\mathbb{Z}_p^\times]]\). We may substitute \(\langle \ell \rangle^{-1}\) in place of \(X\) in the Euler factor \(E_{\ell}(a, X)\), and so obtain an element
\[E_{\ell}(a, \langle \ell \rangle^{-1}) \in T(a)^{\circ}[\mathbb{Z}_p^\times] \cong T(a)^{\circ} \otimes \mathbb{Z}_p[\mathbb{Z}_p^\times].\]

Recall that for \(0 \leq i \leq p - 2\), we write \(\langle \ell \rangle_i\) to denote the projection of \(\langle \ell \rangle \in \mathbb{Z}_p[[\mathbb{Z}_p^\times]]\) under the surjection \(\mathbb{Z}_p[[\mathbb{Z}_p^\times]] \to \Lambda(i)\). We write \(E_{\ell}(a, \langle \ell \rangle_i^{-1})\) to denote the corresponding element of \(T(a)^{\circ} \otimes \mathbb{Z}_p \Lambda(i)\).

**Definition 4.3.1.** If \(a\) is a minimal prime of \(T_{\Sigma}\), then we write
\[E_{\Sigma}(a) := \prod_{\ell \in \Sigma} E_{\ell}(a, \langle \ell \rangle_i^{-1}) \in T(a)^{\circ} \otimes \mathbb{Z}_p[\mathbb{Z}_p^\times],\]
and
\[E_{\Sigma}(a, \omega^i) := \prod_{\ell \in \Sigma} E_{\ell}(a, \langle \ell \rangle_i^{-1}) \in T(a)^{\circ} \otimes \mathbb{Z}_p \Lambda(i) .\]

Since \(L_{\Sigma}(\hat{\rho}, \omega^i)\) and \(L(\hat{\rho}, a, \omega^i)\) are constructed using modular symbols of levels \(N(\Sigma)\) and \(N(a)\) respectively, in order to compare them, we will need to be able to compare the corresponding Hecke algebras. Recall that by construction \(T_{\Sigma} = (T_{N(\Sigma)})^m\) for a certain maximal ideal \(m\) of \(T_{N(\Sigma)}\), and that \(n\) is a certain maximal ideal of \(T_{N(\Sigma)}\) lying over \(m\) with the property that the map \((T_{N(\Sigma)})^m \to \)
Lemma 4.4.2. Let \((T_{N(\Sigma)})_n\) be an isomorphism. We let \(m\) denote the preimage in \(T_{N(a)}\), under the surjection \((4.2)\), of the maximal ideal of \(T(a)\). Altogether, we have the following diagram of maps between the various Hecke algebras:

\[
\begin{array}{ccc}
(T'_{N(\Sigma)})_{m'} & \xrightarrow{\sim} & (T_{N(\Sigma)})_n \\
\downarrow & & \downarrow \\
(T_{N(a)})_m & \longrightarrow & T(a).
\end{array}
\]

By inverting the upper horizontal isomorphism, we obtain a map \((T_{N(\Sigma)})_n \rightarrow T(a)\). This is a homomorphism of \(T'_{N(\Sigma)}\)-algebras, when we equip the source and target with the \(T'_{N(\Sigma)}\)-algebra structure provided by the diagram \((4.2)\).

We write \(M(N(\Sigma))_n\) and \(M(N(a))_m\) to denote the modules of \(p\)-adic modular symbols constructed in the preceding section, for the indicated choice of tame level, and localised at the indicated maximal ideal. The following result provides the key to comparing the \(L\)-functions \(L_{\Sigma}(\bar{\rho}, \omega')\) mod \(a\) and \(L(\bar{\rho}, a, \omega')\). We postpone its proof to the section 4.5.

**Theorem 4.3.2.** There is an isomorphism of \(T(a)\)-modules

\[
T(a)^{\circ} \otimes_{(T_{N(\Sigma)})_n} M(N(\Sigma))_n \cong T(a)^{\circ} \otimes_{(T_{N(a)})_m} M(N(a))_m,
\]

compatible with the action of complex conjugation, and having the property that the induced isomorphism

\[
T(a)^{\circ} \otimes_{(T_{N(\Sigma)})_n} M(N(\Sigma))_n \otimes \mathbb{Z}_p \mathbb{Z}_p[[Z_p^\infty]] \cong T(a)^{\circ} \otimes_{(T_{N(a)})_m} M(N(a))_m \otimes \mathbb{Z}_p \mathbb{Z}_p[[Z_p^\infty]]
\]

maps the element \(1 \otimes \mathcal{L}(n, N(\Sigma))\) of the source to the product \((1 \otimes \mathcal{L}(m, N(a)) E_{\Sigma}(a))\) in the target.

**Corollary 4.3.3.** There is a unit \(u \in T(a)^{\circ}\) such that

\[
L_{\Sigma}(\bar{\rho}, \omega') \equiv u \cdot L(\bar{\rho}, a, \omega') E_{\Sigma}(a, \omega') \pmod{a}.
\]

**Proof.** This follows immediately from the preceding theorem. \(\square\)

4.4. Iwasawa invariants. If we choose an isomorphism

\[
\Lambda \cong \mathbb{Z}_p[[T]],
\]

and hence an isomorphism \((T_N)_m \otimes \mathbb{Z}_p \Lambda_{\Sigma} \cong (T_N)_m[[T]]\), then we may regard \(L(m, N, \omega')\) as an element of \(T_m[[T]]\). We are of course interested in the Iwasawa invariants of such power series. We begin this section by reviewing the definitions and basic properties of such invariants.

**Definition 4.4.1.** If \(R\) is a ring and \(f(T) \in R[[T]]\) is a one-variable power series with coefficients in \(R\), then we define the content of \(f(T)\) to be the ideal \(I(f(T)) \subset R\) generated by the coefficients of \(f(T)\).

The proof of the next lemma is straightforward.

**Lemma 4.4.2.** Let \(f(T) \in R[[T]]\).

1. If \(\alpha : R[[T]] \rightarrow R[[T]]\) is any automorphism of \(R[[T]]\), then \(I(f(T)) = I(\alpha(f(T)))\).
2. If \(u(T) \in R[[T]]^\times\), then \(I(u(T)f(T)) = I(f(T))\).
(3) If \( f(T) = g(T)h(T) \) is an equation in \( A[[T]] \), then if any two of the three power series \( f(T) \), \( g(T) \), and \( h(T) \) have unit content, so does the third.

(4) If \( \phi : R \to S \) is a morphism of rings, and if \( f(T) \in S[[T]] \) denotes the image of \( f(T) \) under the induced map \( R[[T]] \to S[[T]] \), then \( I(\phi(f(T))) = \phi(I(f(T))) \).

In particular, regarding \( L(m,N,\omega^i) \) as a power series, we may consider its content \( I(L(m,N,\omega^i)) \), an ideal of \((T_N)_m \). Parts (1) and (2) of the preceding lemma show that this ideal is in fact independent of the choice of the isomorphisms \((4.3)\) and \((4.6)\).

For each height one prime \( \wp \), we may likewise form an element \( L(m,N,\omega^i)\wp \in O(\wp)[[T]] \) and the corresponding ideal \( I(L(m,N,\omega^i)(\wp)) \) in \( O(\wp) \). Then part (4) of Lemma \( \ref{lem:height-one-props} \) shows that this ideal can equally well be regarded as the image of \( I(L(m,N,\omega^i)) \) under the surjection \((T_N)_m \to O(\wp)\).

Finally, we may construct an element \( L(m,N,\omega^i) \in k[[T]] \), and consider the corresponding ideal \( I(L(m,N,\omega^i)) \) of \( k \) generated by its coefficients. Again, we may also regard \( I(L(m,N,\omega^i)) \) as the image of \( I(L(m,N,\omega^i)) \) under the surjection \((T_N)_m \to (T_N)_m/m \cong k \). Note that since \( k \) is field, the ideal \( I(L(m,N,\omega^i)) \) is either zero, or all of \( k \).

The next result is an immediate consequence of the fact that \((T_N)_m \) is a local ring.

**Proposition 4.4.3.** Fix \( i, 0 \leq i \leq p-2 \). The following are equivalent:

1. \( I(L(m,N,\omega^i)) = (T_N)_m \).
2. \( I(L(m,N,\omega^i)(\wp)) = O(\wp) \) for some height one prime \( \wp \) of \((T_N)_m \).
3. \( I((m,N,\omega^i)(\wp)) = O(\wp) \) for every height one prime \( \wp \) of \((T_N)_m \).
4. \( I(L(m,N,\omega^i)) \) is non-zero (and hence equal to \( k \)).

We remark that if \( \wp \) is a classical height one prime ideal in \((T_N)_m \), then the length of the quotient \( O(\wp)/I(L(m,N,\omega^i)(\wp)) \) is related to the usual \( \mu \)-invariant of the \( p \)-adic \( L \)-function \( L(m,N,\omega^i)(\wp) \in O(\wp)[[T]] \). In particular, this \( \mu \)-invariant vanishes if and only if \( I(L(m,N,\omega^i)(\wp)) \) is the unit ideal of \( O(\wp) \).

Analogously, we define the ideals

\[
I(L_\Sigma(\bar{\rho},\omega^i)), I(L_\Sigma(\bar{\rho},\omega^i)(\wp)), \text{ and } I(T_\Sigma(\bar{\rho},\omega^i))
\]

of \( T_\Sigma \), \( O(\wp) \), and \( k \) respectively as the content of the elements

\[
L_\Sigma(\bar{\rho},\omega^i), L_\Sigma(\bar{\rho},\omega^i)(\wp), \text{ and } T_\Sigma(\bar{\rho},\omega^i).
\]

All of the above results apply equally well to these ideals since they are constructed out of \( I(L(m,N,\omega^i)) \) for some choice of \( m \) and \( N \).

We now compute the content of the reciprocal Euler factor \( E_\Sigma(a,\omega^i) \).

**Lemma 4.4.4.** The element \( E_\Sigma(a,\omega^i) \) of \( T(a)^\circ \otimes_{\mathbb{Z}_p} \Lambda_{(i)} \) has unit content.

**Proof.** Part (3) of Lemma \( \ref{lem:height-one-props} \) shows that it suffices to prove that each of the reciprocal Euler factors \( E_\Sigma(a,\ell^{-1}) \) has unit content. If \( \gamma \) is a topological generator of \( \Gamma \), then we may write \( \ell^{-1}\omega(\ell) = \gamma^{up^n} \), for some \( u \in \mathbb{Z}_p^\times \) and some integer \( n \geq 0 \).

If we choose our isomorphism \((4.4)\) so that \( \gamma \mapsto 1 + T \), then we see that

\[
E_\ell(a,\langle \ell \rangle_i)^{-1} = \begin{cases} 
(1 - \ell \ell^u)(1 + T)^{up^n} + \langle \ell \rangle \omega^{-2i}(1 + T)^{2up^n} & \text{if } \ell \text{ is prime to } N(a) \\
(1 - \ell \ell^u)(1 + T)^{up^n} & \text{otherwise.}
\end{cases}
\]
In the second case either the constant term is a unit (if $T_{c}$ is not a unit) or else the coefficient of $T^{p^a}$ is a unit (if $T_{c}$ is a unit). In the first case, we may compute the content after making the substitution $(1 + T)^{u} \mapsto 1 + T$, in which case one sees immediately that the coefficient of $T^{2p^n}$ is a unit. □

The following theorem establishes that if the $\mu$-invariant vanishes for one form in a Hida family then it vanishes for every form in that family.

**Theorem 4.4.5.** The following are equivalent:

1. There is one ordinary newform $f$ in the Hida family of $\bar{\rho}$ for which the $p$-adic $L$-function $L_p(f, \omega_i)$ has vanishing $\mu$-invariant.
2. For every ordinary newform $f$ in the Hida family of $\bar{\rho}$, the $p$-adic $L$-function $L_p(f, \omega_i)$ has vanishing $\mu$-invariant.
3. For one irreducible component $T(a)$ of the Hida family of $\bar{\rho}$, the $p$-adic $L$-function $L(\bar{\rho}, a, \omega_i)$ has unit content.
4. For every irreducible component $T(a)$ of the Hida family of $\bar{\rho}$, the $p$-adic $L$-function $L(\bar{\rho}, a, \omega_i)$ has unit content.

**Proof.** Let $T(a)$ be an irreducible component of the Hida family of $\bar{\rho}$. Since $T(a)^{\circ}$ is a local ring, we deduce from Corollary 4.3.3 that the following are equivalent: $L(\bar{\rho}, a, \omega_i)$ has unit content; the $p$-adic $L$-function $L_p(f_{\varphi}, \omega_i)$ has unit content (i.e. trivial $\mu$-invariant) for one classical height one prime $\varphi$ of $T(a)$; the $p$-adic $L$-function $L_p(f_{\varphi}, \omega_i)$ has unit content for every classical height one prime $\varphi$ of $T(a)$. (Compare Proposition 4.4.3.) To complete the proof of the proposition, it suffices to show that if $T(a_1)$ and $T(a_2)$ are two irreducible components of the Hida family of $\bar{\rho}$, then $L(\bar{\rho}, a_1, \omega_i)$ has unit content if and only if the same holds true for $L(\bar{\rho}, a_2, \omega_i)$. We may choose $\Sigma$ such that each of $T(a_1)$ and $T(a_2)$ is an irreducible component of $T_\Sigma$. Lemmas 4.4.2 (3) and 4.4.4 then show that each of $L(\bar{\rho}, a_1, \omega_i)$ and $L(\bar{\rho}, a_2, \omega_i)$ have unit content if and only if the same is true of $L_{\Sigma_i}(\bar{\rho}, \omega_i)$. □

If the equivalent conditions of the preceding theorem hold, then we write
\[
\mu_{\text{an}}(\bar{\rho}, \omega_i) = 0.
\]
(As usual, our notation suppresses the choice of $p$-stabilisation of $\bar{\rho}$.) In the case when the $\mu$-invariant vanishes we can further study the $\lambda$-invariant of these power series. (Note that for a local ring which is not a discrete valuation ring one can only define the $\lambda$-invariant for power series of unit content.)

**Definition 4.4.6.** If $A$ is a local ring, and $f(T) \in A[[T]]$ is a power series having unit content, then we define the $\lambda$-invariant $\lambda(f(T))$ to be the smallest degree in which $f(T)$ has a unit coefficient.

We remark that if $\phi : A \to B$ is a local morphism of complete local rings, and if $f(T)$ is an element of $A[[T]]$ having unit content, then $\lambda(f(T)) = \lambda(\phi(f(T)))$.

For $T(a)$ an irreducible component of the Hida family of $\bar{\rho}$ with $\mu_{\text{an}}(\bar{\rho}, \omega_i) = 0$, by Theorem 4.4.3 we have that $L(\bar{\rho}, a, \omega_i)$ has unit content. We can therefore define the analytic $\lambda$-invariant of a branch by
\[
\lambda_{\text{an}}(\bar{\rho}, a, \omega_i) = \lambda(L(\bar{\rho}, a, \omega_i)).
\]
Our main results on analytic $\lambda$-invariants in Hida families are as follows.

**Theorem 4.4.7.** Assume that $\mu_{\text{an}}(\bar{\rho}, \omega_i) = 0$. 
(1) For any given irreducible component $T(a)$ of the Hida family of $\bar{\rho}$, the $\lambda$-invariant of $L_p(f_\psi, \omega_i)$ takes on the constant value of $\lambda(a, \omega_i)$ as $\psi$ varies over all classical height one primes of $T(a)$.

(2) For any two irreducible components $T(a_1), T(a_2)$ of the Hida family of $\bar{\rho}$, we have that
\[ \lambda^{an}(a_1, \omega_i) - \lambda^{an}(a_2, \omega_i) = \sum_{\ell \neq p} (e_\ell(a_2, \omega_i) - e_\ell(a_1, \omega_i)) \]
where $e_\ell(a, \omega_i) = \lambda(E_\ell(a, \{\ell\}, i))$.

Proof. The first part follows from the remarks following Definition 4.2.2 and Definition 4.4.6. For the second part, choose $\Sigma$ large enough so that both $T(a_1)$ and $T(a_2)$ are irreducible components of $T_\Sigma$. Then, by Corollary 4.3.3 we have that
\[ \lambda(L(\Sigma, \bar{\rho}, \omega_j)) = \lambda^{an}(a_j, \omega_j) + \sum_{\ell \in \Sigma} e_\ell(a_j, \omega_j) \]
for $j = 1, 2$. Our formula then follows since $e_\ell(a_1, \omega_i) = e_\ell(a_2, \omega_i)$ for $\ell \notin \Sigma$. \(\square\)

We remark that in section 5 we will see that the formulas of the preceding theorem compare well with the formulas of Theorem 3.3.3.

4.5. Proof of Theorem 4.3.2. In this section, we present the proof of Theorem 4.3.2. We will utilise the notation introduced prior to the statement of the theorem, and begin the proof by introducing some additional notation.

If $M \geq 1$ is an integer, $d$ is a divisor of $M$, and $d'$ is a divisor of $d$, then we let $B_{d,d'} : X_1(M) \rightarrow X_1(M/d)$ denote the map induced by the map $\tau \mapsto d'\tau$ on the upper half-plane. (It will be convenient in the following discussion to be able to omit the level $M$ from the notation, and hence we do so.) This map induces a corresponding map
\[ (B_{d,d'})_* : H_1(M, \{\text{cusps}\}; \mathbb{Z}_p) \rightarrow H_1(M/d, \{\text{cusps}\}; \mathbb{Z}_p) \]
on relative homology groups.

If $\ell$ is any prime distinct from $p$, then we let $e_\ell$ denote the largest power of $\ell$ dividing $N(\Sigma)/N(a)$. We have the inequality $0 \leq e_\ell \leq 2$. Also, $e_\ell = 0$ unless $\ell \in \Sigma$, while $e_\ell = 2$ only if $\ell \in \Sigma$ and $\ell$ is prime to $N(a)$. For each such prime $\ell$, we write
\[ e(\ell) := \begin{cases} 1 & \text{if } e_\ell = 0 \\ ((B_{\ell,1})_* - \ell^{-1}T_\ell(B_{\ell,\ell})_*) & \text{if } e_\ell = 1 \\ ((B_{\ell',1})_* - \ell^{-1}T_\ell(B_{\ell',\ell})_*) + \ell^{-3}(\ell)(B_{\ell',\ell^2})_* & \text{if } e_\ell = 2. \end{cases} \]
Now write $\Sigma = \{\ell_1, \ldots, \ell_n\}$, and for any $r \geq 1$ define
\[ e_r : H_1(N(\Sigma)p^r, \{\text{cusps}\}; \mathbb{Z}_p)_{\text{ord}} \rightarrow H_1(N(a)p^r, \{\text{cusps}\}; \mathbb{Z}_p)_{\text{ord}} \]
by $e_r = e(\ell_n) \circ \cdots \circ e(\ell_1)$.

Let us explain this formula. For any $i = 1, \ldots, n$, write $N_i = N(\Sigma)/\ell_1^{e_1} \cdots \ell_i^{e_i}$. In the formula for $e_r$, the map $e(\ell_i)$ is taken to be the map
\[ H_1(N_{i-1}p^r, \{\text{cusps}\}; \mathbb{Z}_p)_{\text{ord}} \rightarrow H_1(N_ip^r, \{\text{cusps}\}; \mathbb{Z}_p)_{\text{ord}} \]
given by the stated formula for $e(\ell_i)$. (The symbol $T_{\ell_i}$ in the formula for $e(\ell_i)$ is understood to stand for the corresponding Hecke operator acting in level $N_ip^r$.) It is easily verified that the map $e_r$ is in fact independent of the choice of ordering of the elements of $\Sigma$.  

For any tame level $M$ we let $(T_M^*)'$ denote the $\Lambda$-subalgebra of the ordinary Hecke algebra $T_M^*$ generated by the Hecke operators prime to $M$. If we regard the source and target of $\epsilon_r$ as $(T_{N(\Sigma)}^*)'$-modules via the inclusion $(T_{N(\Sigma)}^*)' \subset T_{N(\Sigma)}^*$ and the natural map $(T_{N(\Sigma)}^*)' \rightarrow (T_{N(a)}^*)' \subset T_{N(a)}^*$, then $\epsilon_r$ is immediately seen to be $(T_{N(\Sigma)}^*)'$-linear.

As $r$ varies, the sources and targets of the maps $\epsilon_r$ each form a projective system, and the maps $\epsilon_r$ are evidently compatible with the projection maps on source and target. Thus, passing to the limit in $r$, we obtain a $(T_{N(\Sigma)}^*)'$-linear map

$$
\epsilon_\infty : \lim_{r} H_1(N(\Sigma)p^r, \{\text{cusps}\}; \mathbb{Z}_p)^{\text{ord}} \rightarrow \lim_{r} H_1(N(a)p^r, \{\text{cusps}\}; \mathbb{Z}_p)^{\text{ord}}.
$$

Recall that the source and target of this map are denoted by $M(N(\Sigma))$ and $M(N(a))$ respectively.

We may regard each of the maximal ideals $m'$, $n$, and $m$ equally well as maximal ideals of $(T_{N(\Sigma)})'$, $T_{N(\Sigma)}$, and $T_{N(a)}$. If we localise $\epsilon_\infty$ with respect to $m'$, we obtain a map

$$(4.7) \quad M(N(\Sigma))_{m'} \rightarrow M(N(a))_{m'}.$$

Now $n$ and $m$ each pull back to $m'$ under the natural maps $T_{N(\Sigma)}' \rightarrow T_{N(\Sigma)}$, and $T_{N(\Sigma)}' \rightarrow T_{N(a)}$, and so the localisations $(T_{N(\Sigma)})_n$ and $(T_{N(a)})_m$ are local factors of the complete semi-local rings $(T_{N(\Sigma)})_{m'}^\otimes (T_{N(\Sigma)})_n$, and $(T_{N(a)})_{m'}^\otimes (T_{N(a)})_m$.

Thus the localisations $M(N(\Sigma))_n$ and $M(N(a))_m$ appear naturally as direct factors of $M(N(\Sigma))_{m'}$ and $M(N(a))_{m'}$ respectively, and so the map $(4.7)$ induces a map

$$
M(N(\Sigma))_n \rightarrow M(N(a))_n.
$$

Tensoring the source of this map with $T(a)^\circ$ over $(T_{N(\Sigma)})_n$, and the target with $T(a)^\circ$ over $(T_{N(a)})_m$, we obtain a $T(a)^\circ$-linear map

$$(4.8) \quad T(a)^\circ \otimes (T_{N(\Sigma)})_n M(N(\Sigma))_n \rightarrow T(a)^\circ \otimes (T_{N(a)})_m M(N(a))_m.$$

We claim that this map satisfies the requirements of Theorem 4.3.2.

We first observe that this map satisfies the claimed property with regard to the $L$-functions. This follows from the an explicit calculation of the effect of the maps $\epsilon_\ell$ on modular symbols. In fact, one easily shows that for any character $\chi$ of conductor $p^n$ and $\alpha = \sum_{a \in (\mathbb{Z}/p^n)^\times} \chi(a) \{a/p^n, \infty\}$, we have that

$$
\epsilon_\ell(\alpha) = \prod_{i \text{ s.t. } \ell^{e_i} = 1} (1 - \chi^{-1}(\ell)\ell^{-1}T_\ell) \prod_{i \text{ s.t. } \ell^{e_i} = 2} (1 - \chi^{-1}(\ell)\ell^{-1}T_\ell + \chi^{-2}(\ell)\ell^{-3}(\ell)) \cdot \alpha
$$

(In the left hand side of this equation, the modular symbols $\{a/p^n, \infty\}$ are regarded as lying in $H_1(N(\Sigma)p^r, \{\text{cusps}\}; \mathbb{Z}_p)$, and the right hand side, they are regarded as lying in $H_1(N(a)p^r, \{\text{cusps}\}; \mathbb{Z}_p)$.) Passing to the limit in $r$, and taking into account the fact that $\chi$ is an arbitrary Dirichlet character of $p$-power conductor, we conclude that the isomorphism $(4.3)$ has the required effect on $L$-functions.

We now turn to showing that $(4.3)$ is an isomorphism. Note that Proposition 4.1.1 shows that both source and target are free of rank two over $T(a)^\circ$. Thus to see that this map is an isomorphism, it suffices to check that it induces a surjection after being reduced modulo the maximal ideal of $T(a)^\circ$. To do this, we first mod out by a classical prime of weight two and then by the full maximal ideal.
Let $\wp$ denote the classical height one prime in $\Lambda$ of weight two for which $\kappa_\wp$ is trivial. If we tensor each side of (4.8) by $\Lambda/\wp$ over $\Lambda$, we obtain the map

$$(4.9) \quad T(a) \otimes (T_{N(\Sigma)})_n H_1(N(\Sigma)p; \mathbb{Z}_p)^{\text{ord}} \to T(a) \otimes (T_{N(\Sigma)})_m H_1(N(a)p; \mathbb{Z}_p)^{\text{ord}}$$

induced by localising the source and target of $\epsilon_1$ at $n$ and $m$ respectively, and then extending scalars to $T(a)^\circ$. (Here, as in Proposition 4.3.1, we are using $[17]$ Thm. 3.1.)

Thus, the reduction modulo the maximal ideal of $T(a)^\circ$ of (4.8) coincides with the map

$$(4.10) \quad (T_{N(a)/m}) \otimes_{(T_{N(\Sigma)})/n} (H_1(N(\Sigma)p; \mathbb{Z}_p)^{\text{ord}}/nH_1(N(\Sigma)p; \mathbb{Z}_p)^{\text{ord}}) \to H_1(N(a)p; \mathbb{Z}_p)^{\text{ord}}/mH_1(N(a)p; \mathbb{Z}_p)^{\text{ord}}$$

of $T_{N(a)/m}$-vector spaces induced by (4.9).

Rather than showing directly that (4.10) is surjective, we will show that the corresponding dual map

$$(4.11) \quad H^1(N(a)p; F_p)^{\text{ord}}[m] \to (T_{N(a)/m}) \otimes_{(T_{N(\Sigma)})/n} (H^1(N(\Sigma)p; F_p)^{\text{ord}}[n])$$

is injective. (In writing the dual of (4.11) in this form, we have implicitly fixed an isomorphism of one dimensional $T_{N(a)/m}$-vector spaces between $T_{N(a)/m}$ and its $(T_{N(\Sigma)})^{m^*}$-linear dual. We will suppress this choice of isomorphism here and below.) This map may be written as a composite

$$H^1(N(a)p; F_p)^{\text{ord}}[m] \to (T_{N(a)/m}) \otimes_{(T_{N(\Sigma)})/n} (H^1(N(\Sigma)p; F_p)^{\text{ord}}[m']) \to (T_{N(a)/m}) \otimes_{(T_{N(\Sigma)})/n} (H^1(N(\Sigma)p; F_p)^{\text{ord}}[n]).$$

To explain this, we first recall that since the localisation $(T_{N(\Sigma)})_n$ is a local factor of the tensor product $(T_{N(\Sigma)})_n^m \otimes (T_{N(\Sigma)})_n$, for which the natural map $(T_{N(\Sigma)})_m \to (T_{N(\Sigma)})_n$ is an isomorphism, the residue field $T_{N(\Sigma)/n}$ is a local factor of the Artin local ring $T_{N(\Sigma)/m}T_{N(\Sigma)}$; write

$$T_{N(\Sigma)/m}T_{N(\Sigma)} \cong T_{N(\Sigma)/n} \times A,$$ 

where $A$ denotes the product of the remaining local factors. This decomposition induces a corresponding decomposition

$$B \cong B[n] \times A \otimes T_{N(\Sigma)/m}T_{N(\Sigma)} B$$

for any $T_{N(\Sigma)/m}T_{N(\Sigma)}$-module $B$. The third arrow of (4.12) is precisely projection onto the first two of the factors in (4.13), with

$$B = (T_{N(a)/m}) \otimes_{T_{N(\Sigma)/n}} (H^1(N(\Sigma)p; F_p)^{\text{ord}}[m']).$$

The second arrow is induced by dualising the reduction modulo $m'$ of the map (4.12), and the first arrow is induced from the obvious inclusion, or if the reader prefers, is obtained by dualising the surjection

$$(4.14) \quad (T_{N(a)/m}) \otimes_{T_{N(\Sigma)/n}} (H^1(N(a)p; \mathbb{Z}_p)^{\text{ord}}/m'H_1(N(a)p; \mathbb{Z}_p)^{\text{ord}}) \to H_1(N(a)p; \mathbb{Z}_p)^{\text{ord}}/mH_1(N(a)p; \mathbb{Z}_p)^{\text{ord}}.$$

**Lemma 4.5.1.** The map (4.11) is injective if and only if the composite of the first two arrows of (4.12) is injective.
Proof. The only if statement is clear. In order to prove the other statement, we first observe that the second arrow of (4.12) is given by the cohomological version of the map $\epsilon_1$. More precisely, if for each $\ell \in \Sigma$ we define

$$
\epsilon_\ell^* := \begin{cases} 
1 & \text{if } \epsilon_\ell = 0 \\
B_{\ell,1}^* - B_{\ell,\ell}^* \ell^{-1}T_\ell & \text{if } \epsilon_\ell = 1 \\
B_{\ell,2}^* - B_{\ell,\ell}^* \ell^{-1}T_\ell + B_{\ell,2,\ell}^* \ell^{-3}(\ell) & \text{if } \epsilon_\ell = 2 
\end{cases}
$$

(where $B_{d,d'}^*$ denotes the map on cohomology induced by the degeneracy map $B_{d,d'}$), and define

$$
\epsilon^* = \epsilon_{\ell_1}^* \circ \cdots \circ \epsilon_{\ell_n}^* : H^1(N(a)p; F_p)^{\text{ord}} \to H^1(N(\Sigma)p; F_p)^{\text{ord}}.
$$

Then the second arrow of (4.12) is obtained from the map $\epsilon^*$ by passing to the kernel of $m'$ in the source and target, and then extending scalars from $T_{N(\Sigma)}/n$ to $T_{N(\Sigma)}/m$.

One then checks that any element in the image of $\epsilon^*$ is annihilated by the Hecke operators $T_\ell$, for those $\ell \in \Sigma$. On the other hand, Proposition 2.6.2 shows that the maximal ideal $n$ is uniquely characterised by the property of containing these operators. Thus the image of the second arrow of (4.12) lies in the local factor

$$(T_{N(\Sigma)}/m) \otimes_{T_{N(\Sigma)}/n} (H_1(N(\Sigma)p; F_p)^{\text{ord}}[n])$$

of

$$(T_{N(\Sigma)}/m) \otimes_{T_{N(\Sigma)}/n} (H_1(N(\Sigma)p; F_p)^{\text{ord}}[m']) .$$

This proves the lemma. \hfill \Box

By the preceding lemma, we are reduced to proving that the composite of the first two arrows of (4.12) is injective. It will be notationally easier to deal with each of the maps $\epsilon_\ell$, separately, and so we put ourselves in the following more general situation. We consider a natural number $M$ prime to $p$ and a prime $\ell$ distinct from $p$. We define $n_\ell = 1$ or $2$ according to whether or not $\ell$ divides $M$, and write $N := \ell^{n_\ell}M$. We let $m$ denote a maximal ideal of $T_M$ for which $\bar{\rho}_m$ is irreducible and $p$-distinguished, and let $m'$ denote the pull back of $m$ under the natural map $T_N \to T_M$. The map $\epsilon_\ell^*$ defined in the proof of Lemma 4.5.1 induces a $T_M/m$-linear map

$$H^1(Mp; F_p)^{\text{ord}}[m] \to (T_M/m) \otimes_{T_{m'}/m'} (H^1(Np; F_p)^{\text{ord}}[m']).$$

Lemma 4.5.2. The map (4.15) is injective.

The proof of Lemma 4.5.1 shows that elements in the image of (4.15) are annihilated by $T_\ell$, and hence are in fact annihilated by a maximal ideal of the full Hecke algebra $T_N$. Thus we may apply Lemma 4.5.2 inductively to establish the injectivity of the composite of the first two arrows of (4.12), and thereby complete the proof of the theorem.

Proof of Lemma 4.5.2. It will be convenient to bring the residue field $T_M/m$ inside the coefficients of cohomology. To do this, we choose a finite field $k$ containing $T_M/m$, and tensor the source and target of (4.15) with $k$ over $F_p$, to obtain a map of $k \otimes_{F_p} T_M/m$-modules. The chosen inclusion of $T_M/m$ determines a projection

$$k \otimes_{F_p} T_M/m \to k,$$
which realises $k$ as a local factor of $k \otimes F_p T_M / \mathfrak{m}$. Projecting onto this local factor, we recover our original map (4.15), but rewritten as

\begin{equation}
H^1(M; p; k)_{\ord} \rightarrow H^1(N; p; k)_{\ord}[\mathfrak{m}_k].
\end{equation}

Here we regard the source as a $W(k) \otimes Z_p T_M$-module, and the target as a $W(k) \otimes Z_p T_N$-module. Also, we have written $\mathfrak{m}_k$ to denote the maximal ideal of $W(k) \otimes Z_p T_M$ that is the kernel of the composite

\[ W(k) \otimes Z_p T_M \rightarrow k \otimes F_p T_M / \mathfrak{m} \rightarrow k \]

(where the second arrow is given by the projection (4.16)), and $\mathfrak{m}_k$ to denote the maximal ideal of $W(k) \otimes Z_p T_N$ that is the kernel of the composite

\[ W(k) \otimes Z_p T_N \rightarrow k \otimes F_p T_N / \mathfrak{m'} \rightarrow k \otimes F_p T_M / \mathfrak{m} \rightarrow k \]

(where the second arrow is obtained by tensoring the injection $T_N / \mathfrak{m'} \rightarrow T_M / \mathfrak{m}$ by $k$ over $F_p$, and the third arrow is given by the projection (4.16)). We must show that the map (4.17) is injective.

Our argument will rely on the results of [34, §2.2], which extend a well-known result of Ihara. Recall that the standard practice for analysing the map (4.17) is to factor the map $\epsilon_t$ and so to write (4.17) as a composite

\[ H^1(M; p; k)_{\ord}[\mathfrak{m}_k] \xrightarrow{\alpha_t} (H^1(M; p; k)_{\ord}[\mathfrak{m}_k])^{n_\ell+1} \xrightarrow{\beta_t} H^1(N; p; k)_{\ord}[\mathfrak{m}_k], \]

where

\[ \alpha_t = \begin{cases} (1, -T_\ell^{-1}T_\ell) & \text{if } n_\ell = 1 \\ (1, -T_\ell^{-1}T_\ell, T_\ell^{-3}(\ell)) & \text{if } n_\ell = 2 \end{cases} \]

and

\[ \beta_t = \begin{cases} B_{i_1}^* \pi_1 + B_{i_2}^* \pi_2 & \text{if } n_\ell = 1 \\ B_{i_2}^{* \ell} \pi_1 + B_{i_2}^{* \ell^2} \pi_2 + B_{i_2}^{* \ell} \pi_3 & \text{if } n_\ell = 2, \end{cases} \]

with $\pi_i$ denoting projection onto the $i$th factor of the product

\[ (H^1(N; p; k)_{\ord}[\mathfrak{m}_k])^{n_\ell+1}. \]

The map $\alpha_t$ is manifestly injective, and in the case $n_\ell = 2$, Wiles has shown that the map $\beta_t$ is also injective. (See in particular the discussion at the top of [34, p. 497], or the discussion of Wiles’ results provided by [7, §§4.3] and [3, §§4.4, 4.5].) Thus when $n_\ell = 2$, the lemma is proved.

However, if $n_\ell = 1$, then the situation is more complicated. Lemma 2.5 of [34] provides an exact sequence

\[ H^1((M/\ell); p; k)_{\ord}[\mathfrak{m}_k] \xrightarrow{\gamma_t} (H^1(M; p; k)_{\ord}[\mathfrak{m}_k])^{2} \xrightarrow{\beta_t} H^1(N; p; k)_{\ord}[\mathfrak{m}_k], \]

where $\gamma_t = (B_{i_1}^* - B_{i_1}^*)$. (Here the subscript $\mathfrak{m}_k'$ denotes that we have localised at this maximal ideal.)

Let $x \in H^1((M/\ell); p; k)[\mathfrak{m}]$ be an element in the kernel of (4.17). Then (3.3) shows that we may find $y \in H^1((M/\ell); p; k)$ such that $(x, -T_\ell x) = (B_{i_1}^* y, -B_{i_1}^* y)$. In particular, $B_{i_1}^* y = x$ is annihilated by $\mathfrak{m}$, and so is an eigenvector for the full Hecke algebra $T_M$. The following result shows that $B_{i_1}^* y = 0$, and hence that $x = 0$. This completes the proof of the lemma.

\[ \square \]
Lemma 4.5.3. Let \( D \) be a natural number prime to \( p \), let \( \ell \) be a prime distinct from \( p \), and consider the map \( B_{\ell}^*: H^1(D; \bar{\mathbf{F}}_p)^{\text{ord}} \to H^1(D\ell; \bar{\mathbf{F}}_p)^{\text{ord}} \). If \( y \) is a class in the domain, with the property that \( B_{\ell}^*y \) is an eigenvector for the full Hecke ring \( T_{D\ell} \), corresponding to a maximal ideal for which the attached Galois representation into \( \text{GL}_2(\mathbf{F}_p) \) is irreducible and \( p \)-distinguished, then \( B_{\ell}^*y = 0 \).

Proof. We will prove the lemma by comparing the map \( B_{\ell}^* \) on cohomology classes with the corresponding map on modular forms \( \text{mod} \ p \). For such modular forms, the analogue of the lemma follows immediately from a consideration of \( q \)-expansions. For the comparison with modular forms \( \text{mod} \ p \), we follow the discussion in the proof of [34] Thm. 2.1.

Let \( m \) denote the maximal ideal in \( \bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D\ell} \) describing the action of the Hecke operators on the eigenvector \( B_{\ell}^*y \). Let \( m' \) denote the intersection of \( m \) with the subring \( \bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D\ell} \) of \( \bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D\ell} \). We may regard each of \( H^1(D; \bar{\mathbf{F}}_p)^{\text{ord}} \) and \( H^1(D\ell; \bar{\mathbf{F}}_p)^{\text{ord}} \) as \( \bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D\ell} \)-modules (the former via the natural map \( \bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D\ell} \to \bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D\ell}' \)), and the map \( B_{\ell}^* \) is \( \bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D\ell} \)-linear. If \( y_{m'} \) denotes the projection of \( y \) onto the localisation \( H^1(D; \bar{\mathbf{F}}_p)^{\text{ord}} \) at \( m' \), then \( B_{\ell}^*y_{m'} = B_{\ell}^*y \) (since \( B_{\ell}^*y \) is assumed to be annihilated by \( m \), and so in particular by \( m' \)). Also, there is a natural isomorphism

\[
H^1(D; \bar{\mathbf{F}}_p)^{\text{ord}} \cong \prod_{n \subset m'(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D})} H^1(D; \bar{\mathbf{F}}_p)^{\text{ord}},
\]

where \( n \) ranges over all maximal ideals of \( \bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D} \) containing \( m'(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D}) \). Thus to show that \( B_{\ell}^*y = 0 \), it suffices to show that \( B_{\ell}^*y_n = 0 \) for each such maximal ideal \( n \), where \( y_n \) denotes the projection of \( y \) onto \( H^1(D; \bar{\mathbf{F}}_p)^{\text{ord}}_n \). Thus for the duration of the proof we assume that \( y \in H^1(D; \bar{\mathbf{F}}_p)^{\text{ord}}_n \), for some maximal ideal \( n \) of \( \bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D} \) lying over \( m' \).

As in the proof of [34] Thm. 2.1, we consider two cases: that in which the mod \( p \) diamond operators have non-trivial image in \((\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} T_{D\ell})/m, \) and that in which they have trivial image.

In order to treat the first case, we begin by noting that the commutative diagram

\[
\begin{array}{ccc}
H^1(D; \bar{\mathbf{F}}_p)^{\text{ord}}_n & \xrightarrow{B_{\ell}^*} & H^1(D\ell; \bar{\mathbf{F}}_p)^{\text{ord}}_m \\
\cong & & \\
(\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(D\ell)[p]^{\text{ord}})_n & \xrightarrow{B_{\ell}^*} & (\bar{\mathbf{F}}_p \otimes_{\mathbf{F}_p} J_1(D\ell)[p]^{\text{ord}})_m,
\end{array}
\]

in which the lower horizontal arrow is induced by the map of Jacobians arising by Picard functoriality applied to the degeneracy map \( B_{\ell} : \chi_1(D\ell) \to \chi_1(D) \), allows us to replace curves with coefficients in \( \bar{\mathbf{F}}_p \) with a consideration of \( p \)-torsion in the corresponding Jacobians ( tensored by \( \bar{\mathbf{F}}_p \) over \( \mathbf{F}_p \)). (The superscript ord on \( J_1(D\ell)[p] \) and \( J_1(D\ell)[p][\ell] \) denotes the localisation of these \( p \)-torsion modules at the \( p \)-ordinary part of the Hecke algebra, or equivalently, the image of the \( p \)-torsion modules under Hida’s idempotent \( e^{\text{ord}} \).)

Let \( \bar{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(\bar{\mathbf{F}}_p) \) denote the residual Galois representation attached to each of \( m \) and \( n \). From [34] Thm. 2.1 and its proof we conclude that there are
isomorphisms of Galois modules
\[(\bar{F}_p \otimes_{F_p} T_D)_n/p(\bar{F}_p \otimes_{F_p} T_D)_n \otimes_{F_p} \bar{\rho} \cong (\bar{F}_p \otimes_{F_p} J_1(Dp)[p^{ord}])_n\]
and
\[(\bar{F}_p \otimes_{F_p} T_{Dt})_m/p(\bar{F}_p \otimes_{F_p} T_{Dt})_m \otimes_{F_p} \bar{\rho} \cong (\bar{F}_p \otimes_{F_p} J_1(Dtp)[p^{ord}])_m.\]

Also, passing to $p$-torsion in the short exact sequence \[\text{[34, (2.2)]}\] for each of the maximal ideals $m$ and $n$ yields short exact sequences
\[0 \to (\bar{F}_p \otimes_{F_p} J_1(Dp)[p^{ord}])_n^0 \to (\bar{F}_p \otimes_{F_p} J_1(Dp)[p^{ord}])_n \to (\bar{F}_p \otimes_{F_p} J_1(Dp)[p^{ord}])_n^E \to 0\]
and
\[0 \to (\bar{F}_p \otimes_{F_p} J_1(Dtp)[p^{ord}])_m^0 \to (\bar{F}_p \otimes_{F_p} J_1(Dtp)[p^{ord}])_m \to (\bar{F}_p \otimes_{F_p} J_1(Dtp)[p^{ord}])_m^E \to 0.\]

The lower horizontal arrow $B_{\ell,\ell}^*$ of \[\text{[41, Thm. 2.1]}\] is $G_Q$-equivariant, and also induces a morphism between these short-exact sequences.

If $B_{\ell,\ell}$ is non-zero, then, since $\bar{\rho}$ is an irreducible $G_Q$-module, we may find $\sigma \in G_Q$ such that $\sigma(B_{\ell,\ell})$ has non-zero image in $(\bar{F}_p \otimes_{F_p} J_1(Dtp)[p^{ord}])_m^E$. Replacing $y$ by the image of $\sigma(y)$ in $(\bar{F}_p \otimes_{F_p} J_1(Dp)[p^{ord}])_n^E$, we are reduced to proving the following claim.

**Claim:** If $y \in (\bar{F}_p \otimes_{F_p} J_1(Dp)[p^{ord}])_n^E$ is such that $B_{\ell,\ell}^* y \in (\bar{F}_p \otimes_{F_p} J_1(Dp)[p^{ord}])_n^E$ is annihilated by $m$, then $B_{\ell,\ell}^* y$ vanishes.

Passing to the special fibre of the Néron model of each of $J_1(Dp)$ and $J_1(Dtp)$ over $\mathbb{Z}_p[\zeta_p]$ (where $\zeta_p$ denotes a primitive $p$th root of unity), the discussion in the proof of \[\text{[34, Thm. 2.1]}\] yields a commutative diagram in which the horizontal arrows are isomorphisms:
\[
\begin{array}{ccc}
(\bar{F}_p \otimes_{F_p} J_1(Dp)[p^{ord}])_n^E & \cong & (H^0(\Sigma_1(D)\mu, \Omega^1) \oplus H^0(\Sigma_1(D)^{et}, \Omega^1))_n^{ord} \\
\downarrow B_{\ell,\ell}^* & & \downarrow B_{\ell,\ell}^* \\
(\bar{F}_p \otimes_{F_p} J_1(Dtp)[p^{ord}])_m^E & \cong & (H^0(\Sigma_1(D\ell)\mu, \Omega^1) \oplus H^0(\Sigma_1(D\ell)^{et}, \Omega^1))_m^{ord}.
\end{array}
\]

(For any $M$ prime to $p$, we denote by $\Sigma_1(M)\mu$ and $\Sigma_1(M)^{et}$ the base-change from $\bar{F}_p$ to $F_p$ of the two components of the special fibre of the canonical model of $X_1(Mp)$ over $\mathbb{Z}_p[\zeta_p]$.) Thus to prove the claim, it suffices to show that if $z$ is a non-zero element of $(H^0(\Sigma_1(D)^{et}, \Omega^1))_n^{ord}$ and $B_{\ell,\ell}^*$ cannot lie in the $m$-eigenspace of $(H^0(\Sigma_1(D\ell)^{et}, \Omega^1))_m^{ord}$. This follows from the $q$-expansion principal; more precisely, the $q$-expansion of $B_{\ell,\ell}^*$ at the cusp $\infty$ involves only powers of $q'$, and so cannot be an eigenform for the full Hecke algebra $T_{Dt}$.

The second case (when the mod $p$ diamond operators are trivial modulo $m$) is treated similarly, using the corresponding results from the proof of \[\text{[34, Thm. 2.1]}\].

\[\square\]
5. APPLICATIONS TO THE MAIN CONJECTURE AND EXAMPLES

5.1. The main conjecture. Let \( O \) be the ring of integers of a finite extension \( K \) of \( \mathbb{Q}_p \) and let \( f \in O[[q]] \) be a \( p \)-ordinary eigenform such that the residual representation \( \bar{\rho}_f \) is irreducible. Let \( L_p^{\text{alg}}(f, \omega^i) \in \Lambda_O \) denote a generator of the characteristic power series of the \( \Lambda_O \)-dual of the Selmer group \( \text{Sel}(\mathbb{Q}_\infty, A_{f,i}) \) of Section 3.1. Let \( L_p^{\text{an}}(f, \omega^i) \in \Lambda_O \) denote the usual \( p \)-adic \( L \)-function of \( f \otimes \omega^i \) (computed with respect to some canonical period). The main conjecture of Iwasawa theory in this context is the following; it is independent of the particular choice of coefficient field \( K \) over which \( f \) is defined.

Conjecture 5.1.1. There is a unit \( u \in \Lambda^\times_O \) such that

\[
L_p^{\text{alg}}(f, \omega^i) \cdot u = L_p^{\text{an}}(f, \omega^i).
\]

We remark that the Selmer group in the above conjecture is not the Bloch–Kato Selmer group in which the local condition at \( p \) is defined via crystalline periods, but instead is the Greenberg Selmer group in which the local condition at \( p \) is defined via the ordinary filtration as in Section 3.1. The former group is always contained in the latter group, and the quotient is trivial unless the analytic \( p \)-adic \( L \)-function has a trivial zero, in which case it has corank one (see [10, pp. 108-109]). It is for this reason that in our statement of the main conjecture we do not need to consider separately the case of trivial zeroes.

The following deep theorem of Kato [20] establishes one divisibility of the main conjecture.

Theorem 5.1.2 (Kato). There is a \( u \in \Lambda \otimes \mathbb{Q}_p \) such that

\[
L_p^{\text{alg}}(f, \omega^i) \cdot u = L_p^{\text{an}}(f, \omega^i).
\]

In particular, to verify the main conjecture for \( f \otimes \omega^i \) it suffices to check that

\[
\mu^{\text{alg}}(f, \omega^i) = \mu^{\text{an}}(f, \omega^i) \quad \text{and} \quad \lambda^{\text{alg}}(f, \omega^i) = \lambda^{\text{an}}(f, \omega^i).
\]

Our results in Sections 3 and 4 yield the following result.

Theorem 5.1.3. Let \( k \) be a finite field of characteristic \( p \) and let \( \bar{\rho} : G_{\mathbb{Q}} \to \text{GL}_2(k) \) be an irreducible, modular, \( p \)-ordinary and \( p \)-distinguished representation; as always we fix a choice of \( p \)-stabilization. Suppose that

\[
\mu^{\text{alg}}(f_0, \omega^i) = \mu^{\text{an}}(f_0, \omega^i) = 0 \quad \text{and} \quad \lambda^{\text{alg}}(f_0, \omega^i) = \lambda^{\text{an}}(f_0, \omega^i)
\]

for some \( f_0 \) in the Hida family attached to \( \bar{\rho} \) and some \( i \). Then

\[
\mu^{\text{alg}}(f, \omega^i) = \mu^{\text{an}}(f, \omega^i) = 0 \quad \text{and} \quad \lambda^{\text{alg}}(f, \omega^i) = \lambda^{\text{an}}(f, \omega^i)
\]

for every \( f \) in the Hida family attached to \( \bar{\rho} \).

Before giving a proof, we first state an immediate corollary of Theorem 5.1.3 and Kato’s result.

Corollary 5.1.4. Let \( \bar{\rho} \) be as above and suppose that \( \mu^{\text{alg}}(\bar{\rho}, \omega^i) = \mu^{\text{an}}(\bar{\rho}, \omega^i) = 0 \) for some \( i \). If the main conjecture holds for \( f_0 \otimes \omega^i \) for one form \( f_0 \) in the Hida family of \( \bar{\rho} \), then the main conjecture holds for \( f \otimes \omega^i \) for every form \( f \) in the Hida family of \( \bar{\rho} \).
Since every Hida family contains a form of weight two, this corollary in particular reduces the main conjecture to the case of weight two and to the conjecture on the vanishing of the \(\mu\)-invariants.

The proof of Theorem \[5.1.3\] is based on the following lemma which relates the invariants \(\delta_v(a, \omega^i)\) and \(e_\ell(a, \omega^i)\) of section \[3.3\] and Theorem \[4.4.7\].

**Lemma 5.1.5.** Let \(a_1\) and \(a_2\) be minimal primes of \(T_\Sigma\). For any prime \(\ell \neq p\)
\[
\sum_{v|\ell} \delta_v(a_1, \omega^i) - \delta_v(a_2, \omega^i) = e_\ell(a_2, \omega^i) - e_\ell(a_1, \omega^i)
\]
where the sum is taken over all primes \(v\) of \(\mathbb{Q}_\infty\) over \(\ell\).

**Proof.** To prove the lemma it suffices to see that for any minimal prime \(a\) of \(T_\Sigma\), the sum
\[
e_\ell(a, \omega^i) + \sum_{v|\ell} \delta_v(a, \omega^i)
\]
is independent of \(a\). For this, fix a classical newform \(f\) on the branch \(T(a)\). Consider first the group \(H^1(\mathbb{Q}_\infty, v, A_{f, i})\). Since \(H^1(\mathbb{Q}_\infty, v, A_{f, i})\) is divisible (as \(G_v\) has \(p\)-cohomological dimension one) we have
\[
\dim_k H^1(\mathbb{Q}_\infty, v, A_{f, i})[\pi] = \lambda(\text{char}_{\Lambda_\infty}(H^1(\mathbb{Q}_\infty, v, A_{f, i})^\vee))
\]
where \(M^\vee = \text{Hom}(M, \mathbb{Q}_p/\mathbb{Z}_p)\).

By \[15, Proposition 2.4\] we have
\[
\text{char}_{\Lambda_\infty}(\oplus_{v|\ell} H^1(\mathbb{Q}_\infty, v, A_{f, i})^\vee) = E_\ell(f, (\ell)_i^{-1}) \cdot \Lambda_\infty.
\]
Since \(E_\ell(f, (\ell)_i^{-1})\) is simply \(E_\ell(a, (\ell)_i^{-1})\) mod \(\varphi_f\), we conclude that
\[
\sum_{v|\ell} \dim_k H^1(\mathbb{Q}_\infty, v, A_{f, i})[\pi] = e_\ell(a, \omega^i).
\]

Consider now the exact sequence
\[
0 \to A_{f, i}^\vee/\pi \to H^1(\mathbb{Q}_\infty, v, \tilde{A} \otimes \omega^i) \to H^1(\mathbb{Q}_\infty, v, A_{f, i})[\pi] \to 0.
\]
Since the first term has \(k\)-dimension \(\delta_v(a, \omega^i)\) and the second term is certainly independent of \(a\), the lemma now follows from \[5.1\].

**Proof of Theorem 5.1.3.** Let \(f\) be a form in the Hida family of \(\tilde{\rho}\). The vanishing of the \(\mu\)-invariants of \(f\) is immediate from that for \(f_0\) and Theorems \[3.3.2\] and \[4.4.5\].

By Lemma \[5.1.5\] and Theorems \[3.3.3\] and \[4.4.7\] we see also that
\[
\lambda^{\text{alg}}(f, \omega^i) - \lambda^{\text{alg}}(f_0, \omega^i) = \lambda^{\text{an}}(f, \omega^i) - \lambda^{\text{an}}(f_0, \omega^i).
\]

Since we are assuming the main conjecture for \(f_0 \otimes \omega^i\), we have that
\[
\lambda^{\text{alg}}(f_0, \omega^i) = \lambda^{\text{an}}(f_0, \omega^i),
\]
so that it follows that
\[
\lambda^{\text{alg}}(f, \omega^i) = \lambda^{\text{an}}(f, \omega^i)
\]
as desired.
5.2. Raising the level. We conclude our general discussion with some results on branches of Hida families. It is well-known (see [3, 4]) which levels can occur among forms in the Hida family of \( \bar{\rho} \). In the next proposition we list the cases in which the invariant \( \delta_v(\cdot, \omega^i) \) increases.

**Proposition 5.2.1.** Let \( \bar{\rho} : G_\mathbb{Q} \to \text{GL}_2(k) \) be an irreducible, \( p \)-ordinary and \( p \)-distinguished modular Galois representation; assume further that \( \bar{\rho} \) is ramified at \( p \). Let \( \Sigma \) be some finite set of primes not containing \( p \) and let \( \ell \neq p \) be a prime at which \( \bar{\rho} \) is unramified. Set \( \Sigma' = \Sigma \cup \{ \ell \} \) and let \( T_{\Sigma}, T_{\Sigma'} \) be the Hida algebras associated to \( \bar{\rho} \). Let \( a_\ell = \text{Trace} \bar{\rho}(Frob_\ell) \) and \( c_\ell = \text{det} \bar{\rho}(Frob_\ell) \), both viewed as elements of \( k \).

Then for every branch \( T(a) \) of \( T_{\Sigma} \):

1. If there is an \( i \) such that \( a_\ell = \ell^{-i} + \ell^{1-i} \) and \( c_\ell = \ell^{1-2i} \), then there is a branch \( T(b) \) of \( T_{\Sigma'} \) such that
   \[
   N(b) = N(a)\ell \text{ and } \delta_v(b, \omega^i) = 1
   \]
   for all places \( v \) dividing \( \ell \).

2. If \( \ell \equiv 1 \pmod{p} \) and \( a_\ell = c_\ell + 1 \), then there is a branch \( T(b) \) of \( T_{\Sigma'} \) (distinct from the branch provided by (1)) such that
   \[
   N(b) = N(a)\ell \text{ and } \delta_v(b, \omega^i) = 1
   \]
   for all places \( v \) dividing \( \ell \) and all \( i \).

3. If \( \ell \equiv 1 \pmod{p} \), \( a_\ell = 2 \), and \( c_\ell = 1 \), then there are distinct branches \( T(b_1) \) and \( T(b_2) \) of \( T_{\Sigma'} \) such that
   \[
   N(b_1) = N(b_2) = N(a)\ell^2 \text{ and } \delta_v(b_1, \omega^i) = \delta_v(b_2, \omega^i) = 2
   \]
   for all places \( v \) dividing \( \ell \) and all \( i \).

4. If \( \ell \equiv -1 \pmod{p} \), \( a_\ell = 0 \), and \( c_\ell = -1 \), then there is a branch \( T(b) \) of \( T_{\Sigma} \) such that
   \[
   N(b) = N(a)\ell^2 \text{ and } \delta_v(b, \omega^i) = 1
   \]
   for all places \( v \) dividing \( \ell \) and all \( i \).

Note in particular that if (3) holds, then (1) and (2) also hold, so that the proposition exhibits four distinct non-minimal branches of \( T_{\Sigma'} \) for each branch of \( T_{\Sigma} \).

**Proof.** If (1) holds, then one sees easily that the eigenvalues of \( \text{Frob}_\ell \) on \( \bar{\rho} \) equal \( \ell^{1-i} \) and \( \ell^{-i} \). It follows that there exists a ramified representation

\[
\tau_\ell : G_\ell \to \text{GL}_2(\mathbb{Z}_p)
\]

such that \( \tau_\ell k \cong \bar{\rho}|_{G_\ell} \) and

\[
\tau_\ell = \begin{pmatrix}
\omega^{1-i} & * \\
0 & \omega^{-i}
\end{pmatrix}.
\]

By [4, Theorem 1] (which applies since \( \bar{\rho} \) is modular of weight 2 by [4, Theorem 1.1]) there exists a newform \( g \) of tame level \( N\ell \) in the Hida family of \( \bar{\rho} \) such that \( \rho_g|_{I_\ell} \cong \tau_\ell|_{I_\ell} \). Since \( \bar{\rho} \otimes \omega^i \) is unramified at \( \ell \) with Frobenius eigenvalues \( \ell \) and 1, one computes easily that

\[
\delta_v(A_g, \omega^i) = 1
\]

for any place \( v \) dividing \( \ell \). In particular, the branch \( a_1 \) on which \( g \) lies satisfies the requirements of the proposition.
The proofs for (2) (principal series), (3) (either special or principal series) and (4) (supercuspidal) are entirely similar; the conditions on \( \ell, a_\ell, \) and \( c_\ell \) are simply those obtained by combining the conditions of [3, p. 435] for ramified lifts of the appropriate type to exist with the requirement that \( \overline{\rho}  \) (Frob) has a trivial eigenvalue. Note also that \( \omega|_{G_\ell} \) is trivial (resp. quadratic) in (2) and (3) (resp. (4)), so that the choice of \( i \) is irrelevant. Finally, the branches in (1) and (2) (resp. in (3)) are distinct since one is special at \( \ell \) and the other is principal series at \( \ell \).

It would be interesting to determine when the branches of the preceding proposition intersect. The following proposition uses \( \Lambda \)-adic level raising to give some insight into this question.

**Proposition 5.2.2.** We maintain the hypotheses of the previous proposition. Let \( \mathfrak{a} \) be a minimal prime of \( \mathcal{T}_\Sigma \) such that

\[
(T_\ell \mod \mathfrak{a}')^2 - \ell^{-2} (\ell)(\ell + 1)^2
\]

is not a unit in \( T(\mathfrak{a})^\circ \). (Recall that \( \mathfrak{a}' \) is a minimal prime of \( T_{\mathfrak{a}}^\text{new} \), sitting over \( \mathfrak{a} \) and that \( T(\mathfrak{a})^\circ \cong T_{\mathfrak{a}}^\text{new} / \mathfrak{a}' \).) Then there exists some minimal prime \( \mathfrak{b} \) of \( T_{\Sigma'} \) with \( N(\mathfrak{b}) = N(\mathfrak{a}) \ell \) and a height one prime \( \varphi \) of \( T_{\Sigma'} \) such that \( T(\mathfrak{a}) \) and \( T(\mathfrak{b}) \) cross at \( \varphi \). (That is, both \( \mathfrak{b} \) and the pre-image of \( \mathfrak{a} \) under the natural map \( T_{\Sigma'} \rightarrow T_{\Sigma} \) contain \( \varphi \).)

**Proof.** Attached to the minimal prime \( \mathfrak{a} \) we have the \( \Lambda \)-adic modular form

\[
f(\mathfrak{a}, q) = \sum_{n \geq 1} (T_n \mod \mathfrak{a}') q^n.
\]

By [4, Theorem 6C], there exists a finite extension \( R \) of \( T(\mathfrak{a})^\circ \), a \( \Lambda \)-adic modular form \( g(q) = \sum_{n \geq 1} b_n q^n \in R[[q]] \) new of level \( N(\mathfrak{a}) \ell \), and a height one prime \( \varphi' \) of \( R \) such that

\[
(T_n \mod \mathfrak{a}') \equiv b_n \pmod{\varphi'}.
\]

for each \( n \) relatively prime to \( N(\mathfrak{a}) \ell \).

The \( \Lambda \)-adic form \( g(q) \) corresponds to some minimal prime ideal \( \mathfrak{b} \) of \( T_{\Sigma} \). Indeed, the map \( \mathcal{H} \rightarrow R \) that sends \( T_n \) to \( b_n \) factors through \( T_{\mathfrak{a}}^\text{new} / \mathfrak{a}' \). Let \( \mathfrak{b}' \subseteq T_{\mathfrak{a}}^\text{new} / \mathfrak{a}' \) denote the kernel of this map; it is necessarily a minimal prime since both the source and the target are finite extensions of \( \Lambda \). The preimage of \( \mathfrak{b}' \) in \( \prod_{M|N(\Sigma')} T_M^\text{new} \) is a minimal prime and thus, by Proposition 5.2.2, corresponds to some minimal prime of \( T_{\mathfrak{a}}^\text{new} / \mathfrak{a}' \). Since the residual representation attached to \( g(q) \) equals \( \overline{\rho} \), this minimal prime yields a minimal prime of \( T_{\Sigma'} \) which we denote by \( \mathfrak{b} \). Note that by construction \( N(\mathfrak{b}) = N(\mathfrak{a}) \ell \).

We thus have a map \( T_{\Sigma'} \rightarrow R \) (sending \( T_n \) to \( b_n \)) with kernel \( \mathfrak{b} \). If let \( \tilde{\mathfrak{a}} \) denotes the pre-image of \( \mathfrak{a} \) in \( T_{\Sigma'} \), then we also have a map \( T_{\Sigma'} \rightarrow R \) with kernel \( \tilde{\mathfrak{a}} \) (given by reducing mod \( \tilde{\mathfrak{a}} \) and then embedding the image \( T(\mathfrak{a}) \) into \( R \)). By (5.2), the reduction of these two maps modulo \( \varphi' \) are the same. If we let \( \varphi \) denote the kernel of either of these maps modulo \( \varphi' \), then \( \varphi \) is a height one prime contained in both of \( \mathfrak{b} \) and \( \tilde{\mathfrak{a}} \), as desired.

The ideal \( \varphi \) of the previous proposition could potentially be a prime ideal lying over the principal ideal \( (p) \). It would be interesting to determine whether or not these branches actually meet at a prime ideal of residue characteristic zero.
5.3. Examples.

Example 5.3.1. Set $p = 11$ and let $f$ denote the weight 2 newform associated to the elliptic curve $X_0(11)$. It follows from the fact that $f$ is the only newform of weight 2 and level dividing 11 that $T := T_{\Sigma}(\bar{\rho}_f) \cong \Lambda$ for $\Sigma = \emptyset$. Thus for $k \geq 2$, there is a unique newform $f_k$ of weight $k$ and level dividing 11 that is congruent to $f$ modulo 11. For example, $f_{12}$ is the 11-ordinary, 11-stabilized oldform of level 11 attached to the Ramanujan $\Delta$-function.

The $p$-adic $L$-function of this family was studied in detail in [14]. Also, in [13], the congruence between $X_0(11)$ and $\Delta$ was exploited to gain information about the Selmer groups of both forms. We review below what is known in these examples and then go on to study the other branches of this Hida family with tame conductor greater than one.

We first verify the main conjecture for $f$. Since $X_0(11)$ has split multiplicative reduction at $p = 11$, the $p$-adic $L$-function $L_p^\an(f, T)$ has a trivial zero at $T = 0$. By the Mazur, Tate, and Teitelbaum conjecture (proved by Greenberg and Stevens [12, Proposition 3.7]) we have an explicit formula for the derivative of $L_p^\an(f, T)$ at 0; that is,

$$\frac{d}{dT} L_p^\an(f, T)\bigg|_{T=0} = \frac{L_p(f)}{\log_p(\gamma)} \cdot \frac{L(f, 1)}{\Omega_f}. $$

(Here, $\gamma$ is a topological generator of $1 + p\mathbb{Z}_p$ that is implicitly chosen by writing the $p$-adic $L$-function in the $T$-variable; the appearance of the extra factor of $\log_p(\gamma)$ above is accounted for by the fact that we have written this formula in the $T$-variable rather than the more standard $s$-variable.) One checks that $L_p(f)$ and $\log_p(\gamma)$ are exactly divisible by 11 while $L(f, 1)/\Omega_f$ is an 11-adic unit. Thus $L_p^\an(f, T)$ is a unit multiple of $T$ so that $\lambda^\an(f) = 1$ and $\mu^\an(f) = 0$.

Since $L(f, 1)/\Omega_f$ is an 11-adic unit, by an Euler system argument of Kolyvagin, the classical 11-adic Selmer group $\text{Sel}_p(X_0(11)/\mathbb{Q})$ of the elliptic curve $X_0(11)$ vanishes. It thus follows from a refined control theorem of Greenberg ([12 Proposition 3.7]) that $\text{Sel}_p(X_0(11)/\mathbb{Q}_\infty)$ vanishes as well. The latter group agrees with the Bloch–Kato Selmer group of $f$; as $f$ has a trivial zero, it follows that the Greenberg Selmer group $\text{Sel}_p(\mathbb{Q}_\infty, A_f)$ is simply $\mathbb{Q}_p/\mathbb{Z}_p$ with trivial Galois action. (This example is worked out in detail in [13, Example 3].) Thus $L_p^\rig(f, T)$ is a unit multiple of $T$, so that $\lambda^\rig(f) = 1$ and $\mu^\rig(f) = 0$. In particular, this verifies the main conjecture for $X_0(11)$ at $p = 11$.

Corollary 5.1.4 now shows that the main conjecture holds at $p = 11$ for each form $f_k$ with $k \geq 2$, and Theorem 5.1.5 shows that

$$\lambda^\an(f_k) = \lambda^\rig(f_k) = 1 \text{ for } k \geq 2. $$

Note that the $p$-adic $L$-function corresponding to $X_0(11)$ is the only $p$-adic $L$-function in the Hida family with a trivial zero. Nonetheless, the existence of this trivial zero forces every $p$-adic $L$-function in the family to have at least one zero.

For $k \equiv 2 \pmod{10}$, the unique zero of $L_p^\an(f_k)$ can be explained by the functional equation of the $p$-adic $L$-function. Namely, the sign of this functional equation is $-1$ for such $k$ and thus $L_p^\an(f_k)$ vanishes at the character that sends $x$ to $x^{k-2}$. Using this observation, we can determine the two-variable $p$-adic $L$-function attached to $\bar{\rho}_f$ (which we denote by $L(\bar{\rho}_f)$). By Theorem 4.4.7 we have that $\lambda(L(\bar{\rho}_f)) = 1$ and hence, under the identification of [14],

$$L(\bar{\rho}_f) = (T - a_0) \cdot u(T). $$
with \(a_0 \in T\) and \(u(T) \in T[[T]]\) a unit. Since \(L(\rho_f)\) specializes at weight \(k\) to \(L_p(f_k)\) (Proposition 4.3.3), we know that \(a_0\) specializes at weight \(k\) to the unique zero of \(L_p(f_k)\). In particular, for \(k \equiv 2 \pmod{10}\), we have that \(a_0\) specializes to \(\gamma^{\frac{2}{10}}\).

Thus, \(a_0 = \gamma^{-1}(\gamma)^{\frac{2}{10}}\) since this element has the correct specialization for infinitely many \(k\).

So far we have only applied our results to minimal lifts of \(\bar{\rho}_f\); however, Corollary 5.1.2 applies to every modular form lifting this representation. For instance, we can use Proposition 5.2.1 to explicitly find primes \(\ell\) to add to the level that will produce modular forms that have higher \(\lambda\)-invariants and for which we still know the main conjecture.

The first prime that satisfies condition (3) (and thus (1) and (2)) of Proposition 5.2.1 is \(\ell = 1321\). If we set \(\Sigma = \{1321\}\) then, \(T_\Sigma\) will have four distinct branches \(T(a_1), \ldots, T(a_4)\) with \(\lambda(a_1) = \lambda(a_2) = 2\) and \(\lambda(a_3) = \lambda(a_4) = 3\). (Here and in what follows, we do not distinguish between the analytic and algebraic \(\lambda\)-invariants in cases where the main conjecture is known to be true.) Note that 1321 is inert in the cyclotomic \(\mathbb{Z}_{11}\)-extension of \(\mathbb{Q}\) and so there is a unique prime \(v\) of \(\mathbb{Q}_\infty\) sitting over 1321.

The prime \(\ell' = 2113\) is also inert in \(\mathbb{Q}_\infty / \mathbb{Q}\) and satisfies condition (3) of Proposition 5.2.1. If \(\Sigma' = \{1321, 2113\}\), we then have that \(T_{\Sigma'}\) possesses 20 additional branches with \(\lambda\)-invariants ranging between 3 and 5.

**Example 5.3.2.** We conclude by considering an example of Greenberg and Vatsal in [15]. Consider the elliptic curves

\[
E_1 : y^2 = x^3 + x - 10 \\
E_2 : y^2 = x^3 - 584x + 5444,
\]

of conductors 52 and 364 respectively, which are ordinary and congruent mod 5. One computes that \(\lambda_m(E_1) = \mu_m(E_1) = 0\) so that Kato’s divisibility (Theorem 3.3.3) yields the main conjecture for \(E_1\) at \(p = 5\). The main conjecture for \(E_2\) at \(p = 5\) follows and in this case we have \(\lambda(E_2) = 5\). (This example is discussed in detail in [15] pp. 22,44.)

We now examine this congruence from the point of view of Hida theory. Let \(f_i\) denote the newform of weight 2 associated to \(E_i\). One checks that \(f_1\) is not congruent modulo 5 to any other modular forms of weight two and level dividing \(2^2 \cdot 5 \cdot 13^2\) (using [31], for example). It follows that we have \(T_\Sigma \cong \Lambda\) for \(\Sigma = \{2, 13\}\). For consistency of notation, let us denote the unique irreducible component of this space as \(T(a_1)\). Moreover, since \(\mu(f_1) = \lambda(f_1) = 0\), the two-variable \(p\)-adic \(L\)-function attached to \(T_{\Sigma}\) is a unit and \(\lambda(a_1) = 0\).

The prime \(\ell = 7\) satisfies condition (1) of Proposition 5.2.1 (with \(i = 0\)). Thus, if \(\Sigma' = \{2, 7, 13\}\), then \(T_{\Sigma'}\) contains an irreducible component \(T(a_2)\) not contained in \(T_{\Sigma}\). Moreover, since \(f_2\) is the unique normalized newform congruent modulo 5 to \(f_1\) with level dividing \(2^2 \cdot 5 \cdot 7^2 \cdot 13^2\), it follows that \(T_{\Sigma'}\) has rank 2 over \(\Lambda\). Hence, \(T(a_2)\) is the only branch of \(T_{\Sigma'}\) not coming from \(T_{\Sigma}\). By Theorems 5.3.3 and 4.4.7, we have that \(\lambda(a_2) = 5\). Since \(f_2\) sits on the branch \(T(a_2)\), it follows that \(\lambda(f_2) = 5\).

We close with some questions. Proposition 5.2.2 establishes that the branches \(T(a_1)\) and \(T(a_2)\) must cross at some (non-classical) height one prime, but does not exclude the possibility that they cross at the prime \((p)\). Do they in fact cross at a prime of residue characteristic zero, and if so could one compute \(p\)-adic approximations of this prime? How many such crossing points do these two branches share?
It appears at present that little is known about the shape of these Hida families when multiple branches appear (even in any particular case).

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