Topological superconductor to Anderson localization transition in one-dimensional incommensurate lattices

Xiaoming Cai, Li-Jun Lang, Shu Chen, and Yupeng Wang

1Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
2State Key Laboratory of Magnetic Resonance and Atomic and Molecular Physics, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, Wuhan 430071, China

(Dated: February 21, 2022)

PACS numbers: 03.65.Vf, 71.10.Pm, 72.15.Rn

Introduction.- Topological superconductors (TSCs) have attracted intense recent studies, as they are promising candidates for the practical realization of Majorana fermions [1-7]. Among various proposals, the one-dimensional (1D) TSC in nanowires with strong spin-orbit interactions and proximity-induced superconductivity [6, 7] provides experimental feasibility on the detection of Majorana fermions in hybrid superconductor-semiconductor wires [8-10], which has stimulated great enthusiasm in exploring physical properties of topological superconductors. A key feature of a 1D TSC is the emergence of edge Majorana fermions (MFs) at ends of the superconducting (SC) wire as a result of bulk-boundary correspondence. A prototype model unveiling topological features of the 1D TSC is given by the effective spinless p-wave SC model studied originally by Kitaev [2].

As the TSC is protected by the particle-hole symmetry, the topological phase is expected to be immune to perturbations of weak disorder [11]. Nevertheless, a strong disorder may destroy the SC phase and induce a transition to the Anderson insulator. Localization in 1D SC system in the presence of disorder has been an active research field in the past decades [12-15]. The theoretical studies have unveiled that the particle-hole symmetry in the SC system plays an important role in the problem of the Anderson localization [12]. Due to the existence of a finite SC gap, the interplay of disorder and superconductivity leads to a topological phase transition from topological SC phase to a topologically trivial localized phase when the strength of disorder increases over a critical value.

So far, most theoretical work for the Anderson localization in 1D TSCs focuses on the random disorder [13-17], disorder produced by incommensurate potentials is concerned only very recently [18, 19]. While Ref. [18] explores the TS phase by tuning the chemical potential in 1D quantum wire with spin-orbit interaction in proximity to a superconductor under incommensurate modulation, we focus our study on the transition from TS phase to Anderson localization purely induced by the incommensurate potential for a 1D p-wave superconductor system. In the absence of superconductivity, the localization transition driven by the incommensurate potential occurs at a finite disorder strength which can be exactly determined by a self-duality mapping [20], whereas an arbitrary weak random disorder induces the Anderson localization in one dimension. The incommensurate potential can now be engineered with ultracold atoms loaded in 1D bichromatic optical lattices [21], opening the experimental way to study the localization properties of quasi-periodic systems. In this work, we shall study the interplay of the incommensurate potential and topologically protected superconductivity in the 1D p-wave SC model and determine the phase boundary of TSC to localization transition exactly. The tunability of the incommensurate potential [21] provides a potential way to experimentally study the controllable disorder effect in TSCs realizable in cold atom systems [22].

Model of p-wave superconductor with incommensurate potential.- The 1D p-wave superconductor in the incommensurate lattices is described by the following Hamiltonian:

\[ H = \sum_i \left[ \left( -t\hat{c}_i^\dagger \hat{c}_{i+1} + \Delta \hat{c}_i \hat{c}_{i+1} + H.c. \right) + V_i \hat{n}_i \right], \]

where \( \hat{n}_i = \hat{c}_i^\dagger \hat{c}_i \) is the particle number operator and \( \hat{c}_i^\dagger \) (\( \hat{c}_i \)) the creation (annihilation) operator of fermions. Here the nearest-neighbor hopping amplitude \( t \) and the p-wave pairing amplitude \( \Delta \) are taken as real constants, whereas the incommensurate potential

\[ V_i = V \cos(2\pi \alpha) \]
Bogoliubove-de Gennes (BDG) transformation [23, 24]:

determine the phase diagram of the system.

and then edge MFs [2]. In this work, we shall study the interplay

behavior: the gap reaches a minimum, which approaches

the system with \( |V| < 2t \) characterized by the presence of
elements of spectra. To see it more clearly, we show

\[ \Delta = 0 \] [20], while the Hamiltonian describes the Kitaev’s

p-wave SC model for \( \alpha = 0 \) [2]. For \( \Delta = 0 \), the system

undergoes a delocalization to localization transition at \( V = 2t \). On the other hand, the uniform p-wave SC

system with \( V_i = V \) undergoes a topological phase transition at

\[ V > V_c \] in the regime close to the

transition point. For cases with different irrational \( \alpha \), we find similar phenomena and the transition point does not depend on the specific choice of \( \alpha \) [25].

Observing that the p-wave fermion model corresponds to the transverse XY model with a randomly (irrationally) modulated transverse field [26, 27]:

\[ H = \sum_{n} \left[ J_x \sigma_i^x \sigma_{i+1}^x + J_y \sigma_i^y \sigma_{i+1}^y \right] + \sum_{i} h_i \sigma_i^z \],

with the identification of \( J_x = (t+\Delta)/2 \), \( J_y = (t-\Delta)/2 \) and \( h_i = -V_i/2 \), we can identify the phase transition by calculating the correlation function

\[ C_{ij} = \langle \sigma_i^x \sigma_j^x \rangle \].

In the language of quantum spin model, the ferromagnetic phase is characterized by the long-range order of the correlation function

\[ \langle \sigma_i^x \sigma_j^x \rangle_{|i-j| \to \infty} = A \] with \( A \) being a nonzero positive number. In the original fermion representation, \( \sigma_i^x = \]
To characterize the localization transition, we define the quantity of the inverse participation ratio (IPR) as

\[ IPR = \sum_{n} \left( \frac{1}{n} \right)^{\frac{1}{5}} \left| \langle \psi | \hat{c}_n \hat{c}_{n+i} \hat{c}_{n+i} \hat{c}_{n} \rangle \right|^2 \]

as the mean inverse participation ratio (MIPR) as

\[ MIPR = \sum_{n=1}^{L} P_n / L \]

to characterize the localization of the ground state. As shown in Fig. 2c, the MIPR increases monotonically with the increase of \( V \). At \( V = 2 + 2\Delta \), the MIPR has a sudden increase which characterizes a localization transition. As a comparison, we note that the localization transition does not occur for the commensurate potential system with a rational \( \alpha \) [30], for which the wave functions of a periodic system take the Bloch's form and are extended for arbitrary \( V \).

We then make finite size analysis by calculating the transition points for systems with different sizes. As shown in Fig. 3, the value of transition points \( V_c(L) \) for systems with \( \Delta = 0.5 \) oscillates around 3.0. Defining

\[ V_{ave} = \sum_{L=L_{min}}^{L_{max}} V_c(L) / (L_{max} - L_{min}) \]

we calculate the average of \( V_c(L) \) for different \( L_{max} \) and \( L_{min} \) and find that \( V_{ave} \) is about 3.0040 with \( 0.0005 \) being very close to 3. The change of the gap size at \( V = 2.5 \) and \( V = 3.5 \) is shown in the inset of Fig. 3, which indicates that the gap is finite in the regime of \( V < V_c \), whereas the narrow gap in the regime of \( V > V_c \) approaches zero in the large \( L \) limit.

We also check systems with different \( \Delta \) and find similar behaviors, i.e., \( V_c(L) - 2\Delta \) oscillates with \( L \) and approaches to 2.0 in the large \( L \) limit.

Next we make analytical derivation of the critical value \( V_c \) in the large \( L \) limit [23]. Rewriting the Hamiltonian as the form of

\[ H = \sum_{ij} \left[ c_i^\dagger c_j + \frac{1}{2} \left( c_i^\dagger B_{ij} c_j + \text{h.c.} \right) \right] \]

where \( A \) is a Hermitian matrix and \( B \) is an antisymmetric matrix, we can obtain the excitation spectrum \( \Lambda_n \) by solving the secular equation \( \det[(A + B)(A - B) - \Lambda_n^2] = 0 \) [24, 27]. Since the excitation gap approaches zero at the phase transition point, \( V_c \) can be determined by the condition of \( \det[(A - B)(A + B)] = 0 \). Using the relation \( \det(A - B) = \det(A - B)^T = \det(A + B) \), we can determine \( V_c \) by \( \det(A - B) = 0 \), which leads to the constraint condition

\[ \prod_{i=1}^{L} \cos(2\pi \alpha i) = \left( \frac{\Delta + t}{V} \right)^L \]

in the limit of \( L \to \infty \). Taking logarithm of the above equation and replacing the summation by integral, we can get \( V_c = 2(\Delta + t)e^{2\pi n / L} \) with \( n \) being the integer. For the real solution of \( V_c \), we have \( |V_c| = 2(\Delta + t) \), which is consistent with our numerical result.

**Topological features of the topological SC phase.** To characterize the topological properties of the SC phase, we seek the zero-mode solution of the system under open boundary conditions (OBC). As shown in Fig. 4a, we plot the quasi-particle spectra of BDG equations under OBC. In comparison with the spectra under PBC, an obvious feature is the the presence of the zero mode solution in the gap regime. The enlarged \( \Lambda_1 \) is shown in the inset of Fig. 4a, which indicates a sudden increase in \( \Lambda_1 \) for \( V > 3 \). Here the zero mode solution corresponds to the Majorana edge state with MFs localized at ends of 1D wires. To see it clearly, we introduce the Majorana operators \( \gamma^A_i = \hat{c}_i + \hat{c}_i^\dagger \) and \( \gamma^B_i = (\hat{c}_i - \hat{c}_i^\dagger) / \sqrt{2} \), which fulfill the relations \( \gamma^A_i \gamma^A_j = \gamma^A_j \gamma^A_i \) and anticommutation relations \( \{ \gamma^A_i, \gamma^B_j \} = 2\delta_{ij} \delta_{\alpha\beta} \) with \( \alpha \) and \( \beta \) taking \( A \) or \( B \), and rewrite the
quasi-particle operators as

\[ \eta_n^\dagger = \frac{1}{2} \sum_{i=1}^L [\phi_{n,i} \gamma_i^A - i\psi_{n,i} \gamma_i^B], \]

where \( \phi_{n,i} = (u_{n,i} + v_{n,i}) \) and \( \psi_{n,i} = (u_{n,i} - v_{n,i}) \). Typical distributions of \( \phi_i \) and \( \psi_i \) for the lowest excitation solution of \( A_1 \) are shown in Fig.4b and Fig.4c. When \( V < V_c \), \( \phi_i \) (\( \psi_i \)) is located at the left (right) end and decays very quickly away from the left (right) edge. As \( V \) deviates farther from the transition point \( V_c \), the edge mode decays more quickly. Since there is no overlap in the Hamiltonian (1) can be represented as \[ H = \frac{1}{2} \sum_{j,m=1}^{2L} A_{jm} \gamma_j \gamma_m \] with \( A_{jm} = A_{mj}^* \), where \( L \) is the number of lattice sites, \( A \) is a skew-symmetric matrix, \( \gamma_j \) is defined as \( \gamma_j = \gamma_j^A, \gamma_{2j} = \gamma_j^B \) and \( \{ \gamma_j, \gamma_m \} = 2\delta_{jm} \). The nonzero matrix elements are given by \( A_{2j-1,2j} = -A_{2j,2j-1} \) and \( A_{2j,2j} = \cos(2\pi j \alpha) \). As shown in Fig.5, the \( Z_2 \) topologically non-trivial phase is characterized by \( M = -1 \), whereas the \( Z_2 \) topologically trivial phase corresponds to \( M = 1 \). For the system with \( V < 2 + 2\Delta \), the \( Z_2 \) number \( M = -1 \) and the system is in the topologically non-trivial phase, while for the system with \( V > 2 + 2\Delta \), the \( Z_2 \) number \( M = 1 \) and the system is in the topologically trivial phase. As the strength of \( V \) increases, a topological phase transition happens.

Summary.- In summary, we study the effect of disorder produced by the incommensurate potential in 1D p-wave superconductors which support a topological SC phase with Majorana edge states. Increasing the strength of disorder destroys the topological SC phase and drives the system into a Anderson localized state. The phase transition driven by the disorder is identified by analyzing the change of gap, the long-range order of the correlation function of nonlocal operators and the IPR which characterizes the spacial localization of wavefunctions. The transition point is exactly determined both numerically and analytically. A \( Z_2 \) topological invariant is also used to identify the transition from the topological SC phase, which has emergent Majorana edge states for the system with OBC, to the topologically trivial localized state.

This work has been supported by National Program for Basic Research of MOST, NSF of China under Grants No.11174360 and No.11121063, and 973 grant.

Note added. During the preparation of this manuscript...
we became aware of a preprint on similar topics [31].

* Corresponding author, schen@aphy.iphy.ac.cn

[1] C. W. J. Beenakker, Annu. Rev. Con. Mat. Phys. 4, 113 (2013).
[2] A. Y. Kitaev, Phys. Usp. 44, 131 (2001).
[3] D. A. Ivanov, Phys. Rev. Lett. 86, 268 (2001).
[4] M. Stone and S.-B. Chung, Phys. Rev. B 73, 014505 (2006).
[5] L. Fu and C. L. Kane, Phys. Rev. Lett. 100, 096407 (2008).
[6] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. 105, 077001 (2010).
[7] Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. 105, 177002 (2010).
[8] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, Science 336, 1003 (2012).
[9] M. T. Deng, C. L. Yu, G. Y. Huang, M. Larsson, P. Caroff, and H. Q. Xu, Nano Lett. 12, 6414 (2012).
[10] A. Das, Y. Ronen, Y. Most, Y. Oreg, M. Heiblum, and H. Shtrikman, Nat. Phys. 8, 887 (2012).
[11] A. C. Potter and P. A. Lee, Phys. Rev. Lett. 105, 227003 (2010).
[12] A. Altland and M. R. Zimbauer, Phys. Rev. B 55, 1142 (1997).
[13] O. Motrunich, K. Damle, and D. A. Huse, Phys. Rev. B 63, 224204 (2001).
[14] P. W. Brouwer, A. Furusaki, I. A. Gruzberg, and C. Mudry, Phys. Rev. Lett. 85, 1064 (2000); P. W. Brouwer, A. Furusaki, and C. Mudry, Phys. Rev. B 67, 014530 (2003).
[15] I. A. Gruzberg, N. Read, and S. Vishveshwara, Phys. Rev. B 71, 245124 (2005).
[16] A. Lobos, R. Lutchyn, and S. Das Sarma, Phys. Rev. Lett. 109, 146403 (2012).
[17] P. W. Brouwer, M. Dukheim, A. Romita, and F. von Oppen, Phys. Rev. Lett. 107, 196804 (2011); Phys. Rev. B 84, 144526 (2011).
[18] M. Tezuka and N. Kawakami, Phys. Rev. B 85, 140508(R) (2012).
[19] M. Tezuka and A. M. Garcia-Garcia, Phys. Rev. A 82, 043613 (2010).
[20] S. Aubry and G. André, Ann. Isr. Phys. Soc. 3, 133 (1980).
[21] G. Roati, C. D’Errico, L. Fallani, M. Fattori, C. Fort, M. Zaccanti, G. Modugno, M. Modugno, and M. Inguscio, Nature (London) 453, 895 (2008).
[22] L. Jiang, et. al., Phys. Rev. Lett. 106, 220402 (2011).
[23] P. G. de Gennes, Superconductivity of Metals and Alloys (Benjamin, New York, 1966).
[24] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (N.Y.) 16, 407 (1961).
[25] See Supplemental Material for more details.
[26] D. S. Fisher, Phys. Rev. B 51, 6411 (1995).
[27] A. P. Young, and H. Rieger, Phys. Rev. B. 53, 8486 (1996).
[28] G.-L. Ingolda, A. Wobst, Ch. Aulbach, and P. Hänggi, Eur. Phys. J. B 30, 175 (2002).
[29] D. J. Thouless, Phys. Rep. 13, 93 (1974); M. Schreiber, J. Phys. C 18, 2493 (1985); Y. Hashimoto, K. Niizeki, Y. Okabe, J. Phys. A 25, 5211 (1992).
[30] L.-J. Lang and S. Chen, Phys. Rev. B. 86, 205135 (2012).
[31] W. DeGottardi, D. Sen, and S. Vishveshwara, arXiv:1208.0015 (2012).
[32] I. S. Gradshteyn, and I. M. Ryzhik, ‘Table of integrals, series, and products’, Academic Press (2007), 7th edition, p891.
I. Examples for different irrational $\alpha$

In the main text, we discuss the topological superconductor to Anderson localization transition in one-dimensional incommensurate lattices by considering the case with $\alpha = (\sqrt{5} - 1)/2$ (inverse golden ratio). The conclusions obtained in the main text do not depend on the choice of the inverse golden ratio. In the supplemental material, we give two more examples for the incommensurate potential with some other irrational values. To give concrete examples, we show the variation of $\Lambda_1 = \Delta g / 2$ versus $V$ in the regime close to the transition point for $\alpha = \sqrt{3}/2$ and $\alpha = \sqrt{2}/2$ in Fig.6. As shown in the figure, for both cases with $\alpha = \sqrt{3}/2$ and $\alpha = \sqrt{2}/2$, the gap vanishes at about $V_c = 2 + 2\Delta$ for different $\Delta$. In comparison with Fig.2(a) in the main text, we can see that the transition point does not depend on the choice of $\alpha = (\sqrt{5} - 1)/2$ as long as $\alpha$ being the irrational number. For the case of $\Delta = 0$, it is known that the transition point from extended states to localized states at $V_c = 2$ does not depend on the special choice of inverse golden ratio.

Next we show finite size analysis of the lowest excitation energy, $\Lambda_1$, in the regime of $V > V_c$ for cases with $\alpha = \sqrt{3}/2$ and $\alpha = \sqrt{2}/2$. For systems with $\Delta = 0.5$, the change of $\Lambda_1$ at $V = 3.5$ is shown in Fig.7 which indicates that the narrow gap in the regime of $V > V_c$ approaches zero in the large $L$ limit in a similar way as the case of $\alpha = (\sqrt{5} - 1)/2$.

II. Details for the derivation of phase transition point

Under periodic boundary conditions, $\hat{c}_{L+1}^\dagger = \hat{c}_1^\dagger$, the Hamiltonian (1) in the main text can be represented as

$$H = \sum_{i=1}^{L} [-t\hat{c}_i^\dagger \hat{c}_{i+1} + \Delta \hat{c}_i \hat{c}_{i+1} + \text{h.c.} + V_i \hat{n}_i]$$

$$= \sum_{ij} [\hat{c}_i^\dagger A_{ij} \hat{c}_j + \frac{1}{2} (\hat{c}_i^\dagger B_{ij} \hat{c}_j^\dagger + \text{h.c.})], \quad (7)$$

![FIG. 6: The lowest excitation energy, $\Lambda_1$, versus $V - 2\Delta$ for the system with $L = 500$, $\alpha = \sqrt{3}/2$ and $\alpha = \sqrt{2}/2$.](image)
FIG. 7: The finite size analysis of $\Lambda_1$ for the system with $V = 3.5$, $\Delta = 0.5$ and different $\alpha$.

where $A, B$ are $L \times L$ ($L$ is the number of lattice sites) matrices as

$$A = \begin{pmatrix} V_1 & -t & \cdots & -t \\ -t & V_2 & -t \\ & \ddots & \ddots & -t \\ -t & -t & \cdots & V_L \end{pmatrix}, B = \begin{pmatrix} 0 & -\Delta & \cdots & \Delta \\ \Delta & 0 & -\Delta \\ & \ddots & \ddots & -\Delta \\ -\Delta & \Delta & 0 \end{pmatrix}$$

with $V_i = V \cos(2\pi \alpha i)$.

The excitation spectrum $\Lambda_n$ is a solution of the secular equation, \[24\]

$$\det[(A + B)(A - B) - \Lambda_n^2] = 0.$$  

For the system undergoing a topological quantum phase transition, the excitation gap has to be closed at the phase transition point. So the above equation must have a solution $\Lambda_n = 0$ at the critical point, i.e., $\det[(A - B)(A + B)] = 0$. Notice that

$$\det(A - B) = \det(A - B)^T = \det(A + B),$$

so the phase transition point can be determined by the following requirement

$$\det(A - B) = 0.$$  \(8\)

Now we need calculate the determinant of matrix $A - B$, that is

$$\det(A - B) = -(t - \Delta)^L - (t + \Delta)^L + V^L \prod_{i=1}^L \cos(2\pi \alpha i)$$

$$+ V^{L-2}(\Delta^2 - t^2) \sum_{j=1}^L \prod_{i=1 \atop i \neq j, j+1}^L \cos(2\pi \alpha i) + V^{L-4}(\Delta^2 - t^2)^2 \sum_{j_1=1}^L \sum_{j_2=j_1+2}^L \prod_{i=1 \atop i \neq j_1, j_1+1, j_2, j_2+1}^L \cos(2\pi \alpha i)$$

$$+ \cdots + V^{L-2n}(\Delta^2 - t^2)^n \sum_{j_s=1}^L \prod_{i=1 \atop i \neq j_s, j_s+1}^L \cos(2\pi \alpha i) + \cdots.$$  \(9\)
We note that the inverse golden ratio \((\sqrt{5} - 1)/2\) can be approached by a set of rational numbers \(F_{n-1}/F_n\) when \(n \to \infty\), where \(F_n\) is the Fibonacci sequence given by \(F_n = F_{n-1} + F_{n-2}\) with \(F_1 = F_0 = 1\). As an irrational number can always be approximated by some rational number, without loss of generality, we take \(\alpha = p/q\) (\(p, q\) are co-prime integers) with \(q \to \infty\) to approach a given irrational number. For a finite size system, it is physically impossible to distinguish between an irrational \(\alpha\) and its rational approximation \(p/q\) as long as \(q \geq L\). To make progress, we shall evaluate the determinant of the \(L \times L\) matrix \(A - B\), by taking \(\alpha = p/q\) with \(L = q\) (for the inverse golden ratio \(L = F_n\)).

Under the condition of \(\alpha \in \mathbb{R}\), we can infer that Eq. (10) is still valid for irrational \(\alpha\)’s with \(L \to \infty\). In the limit of \(L \to \infty\), the condition to find the critical point, Eq. (8), can be rewritten as

\[
\sum_{i=1}^{L} \cos(2\pi i) \cos(2\pi (i+1))/(\pi p) = 0 \quad \text{and} \quad \sum_{i=1}^{L} \cos(2\pi i) \cos(2\pi (i+2))/(\pi p) = 0.
\]

For larger \(L\), one can also find that all the terms except \(n = 0\) and \(L/2\), only leaving \(V L \prod_{i=1}^{L} \cos(2\pi i)\) and \((\Delta^2 - t^2)^{L/2}\), i.e.

\[
\det(A - B) = \begin{cases} 
0 & \text{for odd } L \\
-\prod_{i=1}^{L} V \cos(2\pi i) - (\Delta - t)^{L} - (\Delta + t)^{L}, & \text{for odd } L \\
0 & \text{for odd } L \\
-\prod_{i=1}^{L} V \cos(2\pi i) - (\Delta - t)^{L} + (\Delta + t)^{L}, & \text{for even } L.
\end{cases}
\]

From the above discussion, we can infer that Eq. (10) is still valid for irrational \(\alpha\)’s with \(L = q \to \infty\). In the limit of \(L \to \infty\), the condition to find the critical point, Eq. (8), can be rewritten as

\[
\begin{cases} 
\prod_{i=1}^{L} \frac{V}{\Delta + t} \cos(2\pi i) - (\Delta + t)^{L} - 1 = 0, & \text{for odd } L \\
\prod_{i=1}^{L} \frac{V}{\Delta - t} \cos(2\pi i) - (\Delta - t)^{L} + (\Delta + t)^{L/2} - 1 = 0, & \text{for even } L.
\end{cases}
\]

Without loss of generality, we suppose that \(\Delta, t > 0\), and the above equations reduce to

\[
\prod_{i=1}^{L} \cos(2\pi i) = \left(\frac{\Delta + t}{V}\right)^{L}.
\]

(11)

in the limit of \(L \to \infty\). The above equation is equivalent to

\[
\sum_{i=1}^{L} \ln \cos(2\pi i) = \ln \left(\frac{\Delta + t}{V}\right)^{L} + i2\pi n, \quad n \in \mathbb{Z}
\]

or

\[
\frac{1}{L} \sum_{i=1}^{L} \ln \cos(2\pi i) = \ln \left(\frac{\Delta + t}{V}\right) + i2\pi n/L, \quad n \in \mathbb{Z}.
\]

(12)

For the irrational \(\alpha\), we can replace the summation on the left hand side of the above equation by integral in the limit \(L \to \infty\), that is

\[
\frac{1}{L} \sum_{i=1}^{L} \ln \cos(2\pi i) \to \int_{0}^{1} \ln \cos(2\pi \alpha L x) dx
\]

\[
= \frac{1}{2\pi \alpha L} \int_{0}^{2\pi \alpha L} \ln \cos(x) dx = -\frac{1}{2\pi \alpha L} \mathcal{L}(2\pi \alpha L),
\]

where \(\mathcal{L}(x)\) is the Lobachevskiy’s function \([32]\) defined as

\[
\mathcal{L}(x) = -\int_{0}^{x} \ln \cos(t) dt = x \ln 2 - \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\sin(2kx)}{k^2}.
\]
Therefore, we get
\[
\lim_{L \to \infty} \frac{1}{L} \sum_{i=1}^{L} \ln \cos(2\pi ai) = -\ln 2.
\]
So the critical potential strength $V_c$ is determined by
\[
-\ln 2 = \ln \left( \frac{\Delta + t}{V_c} \right) + \frac{i2\pi n}{L}, \quad n \in \text{Integer}.
\]
For the real solution of $V_c$, we have
\[
|V_c| = 2(\Delta + t), \quad (13)
\]
which gives the topological quantum phase transition point from topological superconductor to Anderson localization. The above derivation is not limited to the case of $\Delta, t > 0$. For general cases, similar derivation can be directly followed and $V_c$ is given by
\[
|V_c| = 2(|\Delta| + |t|), \quad (14)
\]