On the asymptotic geometry of the hyperbolic plane

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Abstract

Asymptotic subcone of an unbounded metric space is another metric space, capturing the structure of the original space at infinity. In this paper we define a functional metric space \( S \) which is an asymptotic subcone of the hyperbolic plane. This space is a real tree branching at every its point. Moreover, it is a homogeneous metric space such that any real tree with countably many vertices can be isometrically embedded into it. This implies that every such tree is also an asymptotic subcone of the hyperbolic plane.

1 Introduction and main results

Let \( (X, d_X) \) be a metric space \( X \) with a distance function \( d_X \). Suppose that \( X \) has infinite diameter, i.e. the function \( d_X \) is unbounded. Then one may ask what is the structure of the space \( X \) "at infinity". Intuitively, structure at infinity is what is seen if one looks at the space \( X \) from an infinitely far point (see [Gr2]). M. Gromov suggested several

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ways to treat this notion rigorously. In this paper we follow one of them.

**Definition 1.1.** Let \((X, d_X)\) be a metric space with an infinite diameter. A metric space \((T, d_T)\) can be *isometrically embedded at infinity* into the space \(X\) if for every point \(t \in T\) there exists an infinite sequence \(\{x_i^t\}, i = 1, 2, \ldots\) of points in \(X\), such that for some fixed sequence of positive \(\varepsilon_i \to 0\)

\[
\lim_{i \to \infty} \varepsilon_i \cdot d_X(x_i^{t_1}, x_i^{t_2}) = d_T(t_1, t_2)
\]

(1.2)

for every \(t_1, t_2 \in T\).

In other words, to every point of the space \(T\) we may put in correspondence a sequence of points in \(X\) going to infinity, such that the “normalized” pairwise distances between the sequences in \(X\) tend to the distances between the corresponding points in \(T\).

The Definition 1.1. somehow clarifies what the “structure at infinity” means but it is still too difficult to work with it.

In order to proceed we need to define a certain class of “geometrically simpler” metric spaces — *geodesic* metric spaces (see [Gr1], [GhH]):

**Definition 1.3.** Let \(x_0, x_1\) be two points of the metric space \(X\) and let \(a = d_X(x_0, x_1)\) be the distance between them. A *geodesic segment* in \(X\) connecting \(x_0\) and \(x_1\) is an isometric inclusion \(g : [0, a] \to X\) such that \(g(0) = x_0, g(a) = x_1\). The image of this inclusion sometimes is also called a geodesic segment. A metric space \(X\) is *geodesic* if for every two points \(x_1, x_2 \in X\) there exists a (not necessarily unique) geodesic segment connecting these two points.

For example, every complete Riemannian manifold is a geodesic space due to Hopf-Rinow theorem. All metric spaces which appear in this paper are geodesic.

**Definition 1.4.** A metric space \(X_0\) is an *asymptotic subcone* of the space \(X\) if \(X_0\) is geodesic and its every finite subset of points can be isometrically embedded at infinity into the space \(X\).

Asymptotic subcones were introduced in [Gr1] in the context of *hyperbolic* metric spaces (see [Gr1], [GhH]). There are several equivalent formal definitions of a hyperbolic metric space but all of them demand additional non–trivial geometric explanations. We are not
presenting any of them here since throughout this paper we deal with the familiar Lobachveskian (hyperbolic) plane which is the most well–known example of a hyperbolic metric space.

In fact, there is another very well–known class of hyperbolic spaces — these are 0-hyperbolic spaces (see [Gr1], [GhH] for the definition of δ-hyperbolicity), or real trees:

**Definition 1.5.** A metric space \( X \) is a real tree if it satisfies the following conditions: 1) For every two distinct points of the space there exists a unique geodesic segment joining them. 2) If two geodesic segments \([a, b],[b, c]\) have exactly one endpoint \( b \) in common, their union is also a geodesic segment.

Real trees play the key role in the asymptotic geometry of the hyperbolic metric spaces. There is a general theorem that every asymptotic subcone of a hyperbolic space is a real tree (see [GhH]). Moreover, as it was stated by Gromov (see [Gr1], [GhH]), if every asymptotic subcone of a metric space is a real tree than this space is hyperbolic. Therefore, one may define hyperbolic spaces as metric spaces whose all asymptotic subcones are real trees. For the hyperbolic groups Gromov considered this definition as the most intuitive ([Gr1]).

The property of being a real tree does not give a complete description of a metric space. Different real trees can be very much unlike each other. Therefore in order to describe asymptotic subcones of a particular hyperbolic metric space (a Lobachevskian plane in our case) it is not satisfactory to say they are just real trees, a far more “explicit” construction is desirable.

We found only one example of an asymptotic subcone of a hyperbolic plane in the literature — a star–shaped tree formed by \( k \) segments with a common vertex (see [GhH]). Clearly, the asymptotic geometry of a hyperbolic plane is much richer. Hyperbolic plane is a homogeneous metric space. It is quite natural to look for asymptotic subcones sharing this property. Such a subcone should be real tree branching at every its point — already an object which is quite difficult to imagine. We also want our subcone to contain all “simple” examples of asymptotic subcones (it follows from the Definition 1.4. that every geodesic subset of an asymptotic subcone of the space \( X \) is itself an asymptotic subcone of \( X \)). As a criterion of “simplicity” we choose countability of the number of vertices of the real tree:

**Definition 1.6.** A real tree is thick if it allows an isometric inclusion
of any real tree with countably many vertices.

Thus, we want to “materialize” an asymptotic subcone of a hyperbolic plane which is a homogeneous and thick real tree. The surprising fact is that such a substantial part of the “structure at infinity” of the Lobachevskian plane can be described as a certain simple functional space.

**Definition 1.7.** Let $S$ be the set of all continuous real functions $f(t)$ defined on a finite interval $[0, \rho]$, $0 \leq \rho < \infty$ (each function has its own $\rho$), such that $f(0) = 0$ for all $f \in S$. Define the following metric on $S$:

$$d_S(f_1, f_2) = (\rho_1 - s) + (\rho_2 - s),$$

(1.8)

where $[0, \rho_i]$ is the domain of the function $f_i$, $i = 1, 2$ and

$$s = \sup\{t | f_1(t') = f_2(t') \ \forall t' < t\}.$$

This defines the metric space $S$. The number $s$ is called the moment of segregation of the functions $f_1(t), f_2(t)$.

Let us formulate the main result of our paper:

**Theorem 1.9.** The space $S$ is a thick real tree and a homogeneous metric space. It is an asymptotic subcone of the hyperbolic plane.

We prove this theorem in the next two sections. In the last section we show that the completion of an asymptotic subcone of a metric space is itself an asymptotic subcone of a metric space (motivated by the fact that the space $S$ is non–complete). The paper is completed by two appendices.

**Remark 1.10.** Most of the results of the present paper were announced in [PSh]. Some constructions introduced here were used in [Sh] to describe the asymptotic cone, or the asymptotic space (see [Gr0], [Gr2]) of the Lobachevskian plane by means of non–standard analysis (see [D]).

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2 Properties of the metric space $S$

In this section we study some properties of the metric space $S$.

**Lemma 2.1.** The function $d_S$ defined by (1.8) is a metric.

**Proof.** We need to check that the metric (1.8) satisfies the triangle inequality. Let $f_1, f_2, f_3$ be three functions in $S$, $\rho_1, \rho_2, \rho_3$ be the lengths of their domains and $s_{12}, s_{13}, s_{23}$ be their segregation moments. We may always assume that $s_{12} \leq s_{13} \leq s_{23}$ and $s_{12} = s_{13}$. Therefore,

$$d_S(f_1, f_2) + d_S(f_1, f_3) = 2\rho_1 + \rho_2 + \rho_3 - 4s_{12} \geq \rho_2 + \rho_3 - 2s_{23} = d_S(f_2, f_3).$$

The two other inequalities are proved similarly.

**Lemma 2.2.** The space $S$ is a real tree.

**Proof.** Let $f_1, f_2 \in S$ be two arbitrary functions, $\rho_1, \rho_2$ be their domains and $s$ be their moment of segregation. By (1.8) $d_S(f_1, f_2) = \rho_1 + \rho_2 - 2s$. Consider the following inclusion $g : [0, \rho_1 + \rho_2 - 2s] \to X$:

$$g(x) = \begin{cases} 
\{f_1(t), 0 \leq t \leq \rho_1 - x\}, 0 \leq x \leq \rho_1 - s; \\
\{f_2(t), 0 \leq t \leq x + 2s - \rho_1\}, \rho_1 - s \leq x \leq \rho_1 + \rho_2 - 2s.
\end{cases}$$

(2.3)

This inclusion is isometric and clearly unique with such property, therefore the condition 1) of the Definition 1.5. is verified. In order to check the condition 2) we note that any two geodesic segments $[f, g], [g, h]$ may have exactly one point $g$ in common if and only if the function $h$ is the extension of the function $g$ and segregates from it not earlier than from the function $f$, or, symmetrically, if the function $f$ is the extension of the function $g$ and segregates from it not earlier than from the function $h$. In both cases the formula (2.3) implies that $[f, h]$ is also a geodesic segment. Therefore $S$ is a real tree which completes the proof.

**Lemma 2.4.** The space $S$ is a thick tree.

**Proof.** Let $T$ be an arbitrary real tree with countably many vertices. We "brush" this tree in the following way. Fix some isomorphism between natural numbers and the set of all vertices. Let $a_{ij}$ be the
distance between vertices corresponding to the numbers $i$ and $j$, and
\{k_n\} be an infinite strictly increasing sequence of natural numbers.
Now we build a mapping from $T$ into $S$. Let the vertex 1 go to zero
(by zero we denote the function defined and equal to 0 at the single
point 0). The vertex 2 goes to a linear function $f(t) = k_1 t$ defined
on the interval $[0, a_{12}]$. In order to find the image of the vertex 3
we find from $a_{12}, a_{13}$ and $a_{23}$ where it branches from 1 and 2; let $s_3$
be the abscissa of this point. Therefore on the interval $[0, s_3]$ it is
already defined and on the interval $[s_3, a_{13}]$ we set it to be linear
with the angular coefficient $k_2$ (the free term is found from continuity).
Repeating the same inductive algorithm for all $n$ (if $n − 1$ vertices are
already built we find the abscissa $s_n$ of its point of segregation from
the already built tree and continue the function by setting it linear
with the coefficient $k_{n−1}$ on the interval $[s_n, a_{1n}]$) we get an inclusion
of $T$ into $S$. It is isometric by construction since the sequence \{k_n\}
is strictly increasing and hence segregation is defined correctly.

Now let us prove that the space $S$ is homogeneous, i.e. for every
two its points there exists a one-to-one isometry moving one point to
another.

**Lemma 2.5.** The metric space $S$ is homogeneous.

**Proof.** Clearly it is sufficient to construct a one-to-one isometry $F$
which moves any function to zero. Denote the preimage of zero by
$f_0(t)$, let $[0, \rho]$ be its domain. Let $f(t)$ be any other function with
the domain $[0, a + b]$, where $a$ is the moment of segregation of the
functions $f_0(t), f(t)$. If $a < \rho$ then the image of $f$ is given by the
function $F(f(t))$, such that $F(f(t)) = 0$ on $[0, \rho − a]$ and $F(f(t)) =
f(t − \rho + 2a) − f_0(a)$ on $[\rho − a, \rho − a + b]$. If $a = \rho$, i.e. the function $f$
is a “continuation” of $f_0$, the construction is more complicated. Let
us choose some infinite sequence of continuous functions \{g_n(t)\} such
that $g_1(t)$ is identically zero and for any two elements of this sequence
their moment of segregation is zero. For example we can take the sequence
\[
g_n(t) = \frac{(2^n − 1)t}{2^n}, n = 0, 1, 2, ...
\]
Consider the function $F^*(f(t)) = f(t + \rho) − f_0(\rho)$ defined on $[0, b]$. If
there exists $0 < c < b$ such that $F^*(f(t)) = g_n(t)$ identically on $[0, c]$
for some $n \geq 0$ then $F(f(t)) = F^*(f(t)) + g_{n+1}(t)$ on $[0, b]$, otherwise
simply $F(f(t)) = F^*(f(t))$. One can easily check that $F$ is indeed an isometry.

As it was mentioned in the introduction, homogeneity is a very important feature of the space $S$ since we are interested in it as in the model of the asymptotic space of the hyperbolic plane, and such a model can not be considered “good” if it does not preserve such a fundamental property of the initial space. At the same time the question about the relations between the isometries of $S$ and of the initial hyperbolic plane remains open.

3 Asymptotic subcones of the hyperbolic plane

Let $X$ be the hyperbolic plane. For the convenience of computations we use its Poincare unit disc model

$$X = \{x \in C : |x| < 1 \}.$$ 

For every point $x \in X$, denote by $\rho$ the non-Euclidean distance between $x$ and 0, the centre of $X$, and by $\varphi$ the polar angle; thus, $(\rho, \varphi)$ are the non-Euclidean polar coordinates of the point $x$.

The Euclidean distance between 0 and $x$ is denoted by $r$.

There is the following relation between $\rho$ and $r$ (see [Be]):

$$r = \frac{e^\rho - 1}{e^\rho + 1}. \quad (3.1)$$

The distance between the two points $x_1, x_2 \in X$ is given by the formula ([Be]):

$$d_X(x_1, x_2) = \ln \frac{1 - |x_1 - x_2|}{1 + |x_1 - x_2|} \quad (3.2)$$

We rewrite this formula in Euclidean polar coordinates. If $x_1 = (r_1, \varphi_1)$, $x_2 = (r_2, \varphi_2)$ then the formula (3.2) transforms into the following:

$$d_X(x_1, x_2) = \ln \sqrt{\frac{1+(r_1 r_2)^2-2 r_1 r_2 \cos(\varphi_1-\varphi_2)}{r_1^2 + r_2^2 - 2 r_1 r_2 \cos(\varphi_1-\varphi_2)}} + 1 \quad (3.3)$$

We omit the proof of this statement.
After a number of elementary transformations of (3.3) we get:

\[ d_X(x_1, x_2) = \ln \frac{1 + A}{1 - A}, \quad (3.4) \]

where

\[ A^2 = 1 - \frac{8}{(2 - \beta^2)(t + 1/t)^2 + \beta^2(s + 1/s)^2}, \]
\[ \beta^2 = 1 - \cos(\varphi_1 - \varphi_2), \quad s^2 = e^{\rho_1 + \rho_2}, \quad t^2 = e^{\rho_1 - \rho_2}. \]

In order to prove that the space \( S \) is an asymptotic subcone of the hyperbolic plane we introduce an auxiliary metric space \( D \):

**Definition 3.5.** Consider the set of all real functions \( f(t) \) defined on a finite semi-interval \([0, \rho)\), \( 0 \leq \rho < \infty \) (\( \rho \) also depends on the function), such that \( f(0) = 0 \) and \( f(t) = 0 \) everywhere but at a finite number of points. Denote the metric space \( D \) by endowing this set with the metric \( d_D \), whose expression is given by the formula (1.8).

The space \( D \) is also a real tree (the proof is exactly the same as of the Lemma 2.2.). In fact, it can be isometrically included into the space \( S \) but is not isometric it (and hence “smaller” than the space \( S \)). This statement is proved in the Appendix A.

**Lemma 3.6.** The space \( D \) is an asymptotic subcone of the hyperbolic plane.

**Proof.** Let \( f_1(t), f_2(t) \) be two arbitrary functions from \( D \), let \( \rho_1 \geq \rho_2 \) be the lengths of their domains, \( \{s_{i,1}, \ldots, s_{i,N_i}\} \) be their supports and \( f_i(s_{i,k}) = a_{i,k}, \ k = 1, \ldots, N_i, \ i = 1, 2 \). Consider the following two sequences of points of the hyperbolic plane: \( x_i(n) = (\rho_i(n), \varphi_i(n)) \), where \( \rho_i(n) = \rho_i/\varepsilon_n, \ \varphi_i(n) = \sum_{k=0}^{N_i} a_{i,k} e^{-s_{i,k}/\varepsilon_n}, \ i = 1, 2 \), where \( \{\varepsilon_n\} \) is an arbitrary sequence of positive numbers tending to zero. Then after a simple asymptotic analysis of the formula (3.4) we get

\[
\lim_{n \to \infty} \varepsilon_n d_X(x_1(n), x_2(n)) = \lim_{n \to \infty} \varepsilon_n \ln(e^{(\rho_1 - \rho_2)/\varepsilon_n} + e^{-2s/\varepsilon_n} e^{(\rho_1 + \rho_2)\varepsilon_n}) = \\
= \rho_1 + \rho_2 - 2s,
\]

where \( s \) is exactly the moment of segregation of the functions \( f_1(t), f_2(t) \). Therefore,

\[
\lim_{n \to \infty} \varepsilon_n d_X(x_1(n), x_2(n)) = d_D(f_1, f_2).
\]
Comparing this with (1.2) and recalling that $D$ is a geodesic space being a real tree completes the proof of our lemma.

We call $D$ the *discrete subcone* of the hyperbolic plane.

Now we are able to complete the proof of our main result.

**Theorem 3.7.** The space $S$ is an asymptotic subcone of the hyperbolic plane.

**Proof.** Let $f_k(t), k = 1, \ldots, n$ be an arbitrary finite subset of the space $S$. Denote by $[0, \rho_k]$ the domains of the functions $f_k(t), k = 1, \ldots, n$ and by $s_{k_1k_2}$ the moments of segregation of the functions $f_{k_1}$ and $f_{k_2}, k_1, k_2 = 1, \ldots, n$. We choose $\frac{n(n-1)}{2}$ pairwise distinct points $t_{k_1k_2}^1$ on the positive half-line such that

$$f_{k_1}(t_{k_1k_2}) \neq f_{k_2}(t_{k_1k_2}) \quad \text{and} \quad t_{k_1k_2}^1 - s_{k_1k_2} < 1/4 \quad (3.8)$$

for every two different $k_1, k_2 = 1, \ldots, n$ (the first condition should be checked only if $t_{k_1k_2}^1 \leq \min(\rho_{k_1}, \rho_{k_2})$ since otherwise it does not make sense). Let us put down these points in the growing order and denote the resulting ordered set by $\{t_j^1\}$: $t_{j_1}^1 > t_{j_2}^1$ for all $j_1 > j_2, j_1, j_2 = 1, \ldots, \frac{n(n-1)}{2}$. Take an arbitrary sequence of positive numbers $\{\varepsilon_i\}, i = 1, 2, \ldots$ tending to zero. Consider the following sequences of points of the hyperbolic plane:

$$x_{i,k}^1 = (\rho_k/\varepsilon_i, \varphi_{i,k}^1(\varepsilon_i)),$$

where

$$\varphi_{i,k}^1(\varepsilon_i) = \sum_{j=1}^{J_k^1} e^{-t_j^1/\varepsilon_i} f_k(t_j^1), \quad k = 1, \ldots, n,$$

where $J_k^1$ is the number of points $t_j^1$ which lie in the domain $[0, \rho_k]$ of the function $f_k$; if there are no such points set $\varphi_{i,k}^1(\varepsilon_i) \equiv 0$.

Denote by $g_k^1(t), k = 1, \ldots, n$, the functions belonging to the discrete subcone $D$ such that each $g_k^1(t)$ has the same domain $[0, \rho_k]$ as $f_k(t)$ and

$$g_k^1(t) = \begin{cases} f_k(t), & t = t_j^1, j = 1, \ldots, J_k^1, \\ 0, & \text{otherwise}. \end{cases}$$

Then, by lemma 3.6 there exists a number $I_1$, such that for all $k_1, k_2 = 1, \ldots, n$:

$$|\varepsilon_{I_1} d_X(x_{I_1,k_1}^1, x_{I_1,k_2}^1) - d_D(g_{k_1}^1, g_{k_2}^1)| \leq 1/2. \quad (3.9)$$
Therefore, by the choice of the points $t_{k_1k_2}^1$ we have:

$$|\varepsilon_{I_1}dX(x_{I_1,k_1}^1, x_{I_1,k_2}^1) - d_S(f_{k_1}, f_{k_2})| \leq$$

$$\leq |\varepsilon_{I_1}dX(x_{I_1,k_1}^1, x_{I_1,k_2}^1) - d_D(g_{k_1}^1, g_{k_2}^1)| + |d_D(g_{k_1}^1, g_{k_2}^1) - d_S(f_{k_1}, f_{k_2})| \leq$$

$$\leq 1/2 + 2|t_{k_1k_2}^1 - s_{k_1k_2}| < 1/2 + 2 \cdot 1/4 = 1.$$  

Now we take new points $t_{k_1k_2}^2$ which satisfy the condition (3.8) with 1/8 instead of 1/4 in the right-hand side and get the new set of points $\{t_{k_1k_2}^2\}$, the new sequences $x_{i,k}^2$ and the new functions $g_{k}^2(t) \in D$, $k = 1, \ldots, n$.

Similarly, there exists a number $I_2$, greater than $I_1$, such that

$$|\varepsilon_{I_2}dX(x_{I_2,k_1}^2, x_{I_2,k_2}^2) - d_D(g_{k_1}^2, g_{k_2}^2)| \leq 1/4$$

for all $k_1, k_2 = 1, \ldots, n$, and therefore

$$|\varepsilon_{I_2}dX(x_{I_2,k_1}^2, x_{I_2,k_2}^2) - d_S(f_{k_1}, f_{k_2})| \leq 1/4 + 2|t_{k_1k_2}^2 - s_{k_1k_2}| < 1/2$$

for all $k_1, k_2 = 1, \ldots, n$.

Continuing this procedure (which is possible due to the Lemma 3.6. and the assumption that $f_k(t)$ are continuous functions) we get the sequence $\{I_N\}$, $I_N \to \infty$ as $N \to \infty$, the sequences $\{\varepsilon_{I_N}\} \to 0$ (or just $\{\varepsilon_N\}$), $\{x_{I_N,k}^N\}$ (or just $\{x_{N,k}\}$), and the functions $g_{k}^N(t) \in D$, $k = 1, \ldots, n$ such that

$$|\varepsilon_N dX(x_{N,k_1}, x_{N,k_2}) - d_S(f_{k_1}, f_{k_2})| \leq$$

$$\leq |\varepsilon_N dX(x_{N,k_1}, x_{N,k_2}) - d_D(g_{k_1}^N, g_{k_2}^N)| + 2|t_{k_1k_2}^N - s_{k_1k_2}| <$$

$$< 1/2^N + 2 \cdot 1/2^{N+1} = 1/2^{N-1}$$

Thus we have proved that there exists a limit

$$\lim_{N \to \infty} \varepsilon_N dX(x_{N,k_1}, x_{N,k_2}) = d_S(f_{k_1}, f_{k_2})$$

for all $k_1, k_2 = 1, \ldots, n$. Therefore $S$ is indeed an asymptotic subcone of the hyperbolic plane.

The Theorem 3.7. together with the Lemmas 2.2., 2.4. and 2.5. completes the proof of the Theorem 1.9.

We call $S$ the continuous subcone of the hyperbolic plane. In fact it is an asymptotic subcone of an infinitely narrow neighborhood of a
single half-line on the hyperbolic plane, as follows from the construction of the discrete subcone \(D\).

Some modifications of the construction of the continuous subcone \(S\) are considered in the Appendix B.

We would like to conclude this section with a simple corollary of the main result which however reflects the “wealth” of the space \(S\):

**Corollary 3.10.** Any real tree with countably many vertices is an asymptotic subcone of the hyperbolic plane.

**Proof.** Indeed, every geodesic subspace of an asymptotic subcone is itself an asymptotic subcone. Real trees are geodesic spaces and by Lemma 2.4 any real tree with countably many vertices can be isometrically included into the continuous subcone \(S\). Therefore every such tree is an asymptotic subcone of the hyperbolic plane.

## 4 Completions of the asymptotic subcones

The metric spaces \(S\) and \(D\) are non-complete metric spaces. For \(D\) this is obvious; for \(S\) it follows from the following example. Consider a sequence \(\{f_k(t)\}\) of functions, defined on the segment \([0, 1 - 2^{-k}]\), and equal to \(f_k(t) = \sin(1/(1-t))\) on its domain of definition. Clearly, this sequence has no limit in the space \(S\) as \(k \to \infty\).

Therefore, it is reasonable to consider the completion \(\bar{S}\) of the continuous subcone \(S\). Though it does not have such a simple functional description as the original one, it is also an asymptotic subcone of the hyperbolic plane due to the following simple general theorem (since we could not find it in the literature we found it appropriate to state it here):

**Theorem 4.1.** A completion of an asymptotic subcone of a metric space is also an asymptotic subcone of this metric space.

**Proof.** Let \(X\) be our metric space, \(X_0\) — its asymptotic subcone and \(\bar{X}_0\) — the completion of \(X_0\). Let \(f_1, f_2, ..., f_k\) be a finite number of points in \(\bar{X}_0\). By definition,

\[
f_i = \lim_{n \to \infty} \varphi_i^n,
\]
where $\varphi^n_i \in X_0$, $n \in \mathbb{N}$, $i = 1, ..., k$.

Choose a number $N_1 = N_1(1/4)$ such that

$$d_{\bar{X}_0}(\varphi^n_i, f_i) < 1/4$$

for all $i = 1, .., k$, $n \geq N_1$. Then for all $l, m = 1, .., k$ and $n \geq N_1$ we get

$$|d_{\bar{X}_0}(f_l, f_m) - d_{X_0}(\varphi^n_l, \varphi^n_m)| \leq d_{\bar{X}_0}(\varphi^n_l, f_l) + d_{\bar{X}_0}(\varphi^n_m, f_m) < \\
< \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

(4.2)

The space $X_0$ is an asymptotic subcone of $X$. Therefore for its finite subset $\varphi^{N_1}_i$, $i = 1, .., k$ there exists a sequence $\{\varepsilon_j\}$ of positive numbers tending to zero, and $k$ sequences of points of the space $X$, $\{x^{i,N_1}_j\}$, $i = 1, .., k$, such that for some $J_1 = J_1(N_1, 1/2)$

$$|\varepsilon_j d_X(x^{l,N_1}_j, x^{m,N_1}_j) - d_{X_0}(\varphi^{N_1}_l, \varphi^{N_1}_m)| < \frac{1}{2} \quad \forall j \geq J_1.$$  

(4.3)

Therefore, by (4.2) and (4.3) we get:

$$|\varepsilon_{J_1} d_X(x^{l,N_1}_{J_1}, x^{m,N_1}_{J_1}) - d_{\bar{X}_0}(f_l, f_m)| < \frac{1}{2} + \frac{1}{2} = 1$$

for all $l, m = 1, .., k$.

Next, we choose $N_2 = N_2(1/8)$ and $J_2 = J_2(N_2, 1/4)$ and similarly obtain

$$|\varepsilon_{J_2} d_X(x^{l,N_2}_{J_2}, x^{m,N_2}_{J_2}) - d_{\bar{X}_0}(f_l, f_m)| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

for all $l, m = 1, .., k$. Let us note that we can always choose $N_2$ and $J_2$ in such a way that $N_2 > N_1$ and $\varepsilon_{J_2} < \varepsilon_{J_1}/2$.

Continuing this procedure analogously we get the sequence $\{\varepsilon_{J_r}\}$ of positive numbers tending to zero and $k$ sequences of points in $X$, $\{x^{i,N_r}_{J_r}\}$, $i = 1, .., k$, such that

$$|\varepsilon_{J_r} d_X(x^{l,N_r}_{J_r}, x^{m,N_r}_{J_r}) - d_{\bar{X}_0}(f_l, f_m)| < \frac{1}{2^r} + \frac{1}{2^r} = \frac{1}{2^{r-1}}$$

for all $l, m = 1, .., k$, which implies
\[
\lim_{r \to \infty} \varepsilon_J d_X(x_{Jr}^{m,r}, x_{Jr}^{m,r}) = d_{X_0}(f_l, f_m), \quad l, m = 1, \ldots, k. \quad (4.4)
\]

The relation (4.4) exactly means that \( X_0 \) is an asymptotic subcone of the metric space \( X \), which completes the proof of the theorem.

**Appendix A**

We have shown that the spaces \( S \) and \( D \) are real trees. These trees are branching at every point and the cardinal number of the set of their vertices is continuum (for \( D \) this is clear and for \( S \) it follows from the fact that every continuous function is defined by its values at the rational points). As a metric space \( S \) is “larger” than \( D \), as it is shown by the following

**Lemma A.1.** The space \( D \) can be isometrically included into the space \( S \) but is not isometric to it.

**Proof.** Let us show that \( D \) can be isometrically included into \( S \). Let \( \gamma(t) \) be any element of \( D \), \( Z_{\gamma} = \{a_1 < a_2 < \ldots < a_N\} \) be the set of points where \( \gamma(t) \) is non-zero, and \([0, \rho_0) \) be the domain of \( \gamma(t) \). Set \( a_0 = 0, a_{N+1} = \rho_0 \). Consider the following map \( F : D \to S \):

\[
F[\gamma(t)](t) = \sum_{i=0}^{k} \gamma(a_i)(t - a_i), \quad a_k \leq t \leq a_{k+1}, \quad k = 0, \ldots, N. \quad (A.2)
\]

It is easy to see that \( F \) is an isometric inclusion of \( D \) into \( S \).

Now, assume that \( \Phi \) is an an isometric inclusion of \( S \) into \( D \) and let \( \Phi[f(t)](t) = \gamma(t) \) for some \( f(t) \in S \) where \( \gamma(t) \in D \) is defined as above.

Take some \( 0 < \varepsilon_0 < \rho_0 - m_0 \), where \( m_0 = a_N \), and consider two functions \( g_1(t), h_1(t) \) in the \( \varepsilon_0 \)-neighborhood of \( f(t) \) in \( S \), such that none of the functions \( f(t) \), \( g_1(t) \), \( h_1(t) \) is the extension of any of the other two. Denote \( \alpha_1(t) = \Phi[g_1(t)](t), \beta_1(t) = \Phi[h_1(t)](t) \).

Let us show that \( Z_\gamma \subset Z_{\alpha_1}, Z_\gamma \subset Z_{\beta_1} \). If, for instance, \( Z_\gamma \) is not a subset of \( Z_{\alpha_1} \), then \( d_D(\gamma, \alpha_1) \geq \rho_0 - m_0 > \varepsilon_0 \), which contradicts with the choice of \( \alpha_1(t) \); the similar argument is valid for \( \beta_1(t) \). At least one of these inclusions is proper; indeed, if \( Z_{\alpha_1} = Z_{\beta_1} = Z_\gamma \) then one
of the functions $\alpha_1(t)$, $\beta_1(t)$, $\gamma(t)$ is the extension of the two others which is impossible since $\Phi$ is an isometry and none of the functions $f(t)$, $g_1(t)$, $h_1(t)$ is the extension of any of the other two. Let this proper inclusion be $Z_\gamma \subseteq Z_{\alpha_1}$. Then the set $Z_{\alpha_1}$ consists of at least $N + 1$ points.

Repeat this construction taking $g_1$ instead of $f$, and instead of $\varepsilon_0$ taking $0 < \varepsilon_1 < \min(\varepsilon_0/2, \rho_1 - m_1)$, where $[0, \rho_1)$ is the domain of $\alpha_1(t)$ and $m_1$ is the maximal element in $Z_{\alpha_1}$. Similarly, we shall get the new function $\alpha_2(t) \in D$ such that $Z_{\alpha_2}$ consists of at least $N + 2$ points.

Continuing this procedure we obtain a sequence $\{g_i(t)\}$ in $S$ and the corresponding sequence $\{\alpha_i(t)\}$ in $D$ such that $Z_{\alpha_i}$ consists of at least $N + i$ points. When $i \to \infty$ the functions $g_i(t) \to g(t)$, where $g(t) \in S$ since $\varepsilon_i < \varepsilon_0/2^i$ for all $i = 1, 2, \ldots$ and therefore the length of the domain of $g(t)$ is finite — it is not greater than $r+2\varepsilon_0$, where $r$ is the length of the domain of $f(t)$. Consider the image of $g(t)$ under the isometry $\Phi$; it is equal to $\alpha(t) = \lim_{i \to \infty} \alpha_i(t)$. By our construction $Z_{\alpha} \supseteq Z_{\alpha_i}$ for all $i = 1, 2, \ldots$, therefore $Z_{\alpha}$ consists of an infinite number of points. But this contradicts with the fact that $\alpha(t) \in D$, and hence $S$ and $D$ are non-isometric. This completes the proof of our lemma.

**Appendix B**

Instead of continuous functions with bounded domain one could take generalized functions of bounded domain with the distance defined by (1.8). Such generalized functions are of finite order (see [GeS]), i.e. they can be represented as finite sums of generalized derivatives of continuous functions. One may check that integration preserving the condition $f(0) = 0$ is an isometry with respect to our metric (compare this with the formula (A.2.) — in fact we have represented each element of the discrete subcone $D$ as a finite sum of $\delta$-functions and integrated twice). Hence every space of all generalized functions of order less than some finite $m > 0$ is isometric to $S$: its isometric inclusion into $S$ is obtained by integrating $m$ times every its element as described above, and isometric inclusion of $S$ into such space can be given by an identical map. Similarly, if we take $C^m$-smooth functions, we also get an isometric space. Therefore, all such functional spaces are isometric asymptotic subcones of the hyperbolic plane.
The construction of $S$ can be also generalized for the Lobachevskian space of arbitrary dimension $n$. In this case instead of scalar functions one has to take $n$-component vector functions.

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