How to efficiently select an arbitrary Clifford group element

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We give an algorithm which produces a unique element of the Clifford group on \( n \) qubits \( (C_n) \) from an integer \( 0 \leq i < |C_n| \) (the number of elements in the group). The algorithm involves \( O(n^3) \) operations. It is a variant of the subgroup algorithm by Diaconis and Shahshahani \( [5] \) which is commonly applied to compact Lie groups. We provide an adaption for the symplectic group \( Sp(2n,F_2) \) which provides, in addition to a canonical mapping from the integers to group elements \( g \), a factorization of \( g \) into a sequence of at most \( 4n \) symplectic transvections. The algorithm can be used to efficiently select random elements of \( C_n \) which is often useful in quantum information theory and quantum computation. We also give an algorithm for the inverse map, indexing a group element in time \( O(n^3) \).

I. INTRODUCTION

The Clifford group (which we will define carefully below) is of great interest in the field of quantum information and computation. Though the group is not universal for quantum computation \( [9] \), it is central in the field of quantum error-correction codes \( [8] \), and the use of random elements of the Clifford group has numerous applications, from establishing bounds on quantum capacities \( [2] \) to randomized benchmarking \( [7, 10, 11] \) to data hiding \( [6] \). Most of these applications depend on the useful fact that the uniform distribution over Clifford group elements constitutes a 2-design for the unitary group, that is, reproduces the second moments of a Haar-random unitary (see \( [2, 4, 6] \)).

There are many ways of choosing a random Clifford element. The most straightforward is to simply write down all the elements of the group, and then pick randomly from the list. This quickly becomes impractical because the cardinality of the group

\[
|C_n| = 2^{n^2 + 2n} \prod_{j=1}^{n} (4^j - 1) \tag{1}
\]

grows quickly with the number of qubits \( n \) \( [15] \). Other (approximate) methods have been proposed: In \( [6] \) a method is given requiring time \( O(n^4) \) and producing an approximately random Clifford, and \( [4] \) gives a method that produces an \( \epsilon \)-approximate unitary 2-design based on Cliffords (consisting of only \( n \log 1/\epsilon \) gates).

Our method gives a canonical mapping of consecutive integers to a Clifford group element. Picking a random element is equivalent then to picking a random integer of the size of the group. We give both \( O(n^4) \) and \( O(n^3) \) algorithms for computing the group element from the associated integer. We also give a \( O(n^3) \) algorithm realizing the inverse map, i.e., taking group elements to integers.

A. The Pauli, Clifford, and Symplectic groups

The Pauli group \( \mathcal{P}_n \) on \( n \) qubits is generated by single-qubit Pauli operators \( X_j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Y_j = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, Z_j = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) acting on the \( j \)th qubit, for \( j = 1, \ldots, n \). Consider the normalizer \( \mathcal{N}(\mathcal{P}_n) = \{ U \in U(2^n) \mid U \mathcal{P}_n U^\dagger = \mathcal{P}_n \} \) of \( \mathcal{P}_n \) in the group of unitaries \( U(2^n) \). The Clifford group \( C_n \) is this normalizer, neglecting the global phase: \( C_n = \mathcal{N}(\mathcal{P}_n)/U(1) \). Any element \( U \in C_n \) is uniquely determined up to a global phase by its action on conjugation on the generators of \( \mathcal{P}_n \), i.e. the list of parameters \( (\alpha, \beta, \gamma, \delta, r, s) \) where \( \alpha, \beta, \gamma, \delta \) are \( n \times n \) matrices of bits, and \( r, s \) are \( n \)-bit vectors defined by

\[
UX_jU^\dagger = (-1)^{\alpha_j} \prod_{i=1}^{n} X_i^{\alpha_i j} Z_i^{\beta_i j} \quad \text{and} \quad UZ_jU^\dagger = (-1)^{\delta_j} \prod_{i=1}^{n} X_i^{\gamma_i j} Z_i^{\delta_i j}. \tag{2}
\]
Note that because unitaries preserve commutation relations among the generators not all values for the matrices \(\alpha, \beta, \gamma, \delta\) are allowed. This is what makes picking a random element of the group nontrivial. By \(\mathbb{P}\), the task of drawing a random Clifford element can be rephrased as that of drawing from the corresponding distribution of parameters \((\alpha, \beta, \gamma, \delta, r, s)\) describing such an element.

Note also that given the list \((\alpha, \beta, \gamma, \delta, r, s)\), there is a classical algorithm for compiling a circuit implementing \(U\) which is composed of \(O(n^2/\log n)\) gates from the gate set \(\{H, \text{CNOT}, P\}\), see [1]. A simpler and more (time-) efficient algorithm was proposed earlier in [8]; it essentially performs a form of Gaussian elimination, has runtime \(O(n^3)\) and produces a circuit with \(O(n^4)\) gates.

The group \(\mathcal{C}_n/\mathcal{P}_n\) has a particularly simple form: we have
\[
\mathcal{C}_n/\mathcal{P}_n \cong \text{Sp}(2n, \mathbb{F}_2) \equiv \text{Sp}(2n)
\]
where the latter is the symplectic group on \(\mathbb{F}_2^{2n}\), i.e., the group of \(2n \times 2n\) matrices \(S\) with entries in the two-element field \(\mathbb{F}_2\) such that
\[
SA(n)S^T = \Lambda(n) \equiv \bigoplus_{i=1}^{n} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

In this expression, the block-diagonal matrix \(\Lambda(n)\) defines the symplectic inner product \(\langle v, w \rangle = v^T \cdot \Lambda(n) w\) on \(\mathbb{F}_2^{2n}\). Preservation of the symplectic inner product \(\mathbb{I}\) is equivalent to the preservation of commutation relations between the generators of \(\mathcal{P}_n\) when acted on by conjugation with the corresponding unitary. Explicitly, if a representative \(U \in \mathcal{C}_n/\mathcal{P}_n\) acts as \(\mathbb{P}\), then the corresponding symplectic matrix \(S\) has entries
\[
\begin{align*}
(\alpha_{j1}, \beta_{j1}, \ldots, \alpha_{jn}, \beta_{jn}) & \text{ in column } 2j - 1 \\
(\gamma_{j1}, \delta_{j1}, \ldots, \gamma_{jn}, \delta_{jn}) & \text{ in column } 2j, \text{ for } j = 1, \ldots, n.
\end{align*}
\]

Eq. (3) gives an important simplification to our algorithm, directly implying the following lemma:

**Lemma 1.** Specifying an arbitrary element of the Clifford group is equivalent to specifying an element of the Pauli group and also an element from the symplectic group.

Specifying an element of the Pauli group (up to an overall phase) simply requires picking the bitstrings \(r, s\), which is trivial. We therefore concentrate on how to specify elements from the symplectic group henceforth.

### B. Symplectic Gram-Schmidt procedure

We will make use of a simple generalization of the Gram-Schmidt orthogonalization procedure over the symplectic inner product. The basic step in this procedure takes as input a set of vectors \(\Omega \subset \mathbb{F}_2^{2n}\) and a vector \(v \in \Omega\). If \(\langle v, f' \rangle = 0\) for all \(f' \in \Omega \setminus \{v\}\), the output is the set \(\Omega' = \Omega \setminus \{v\}\). Otherwise, the output is a vector \(w \in \Omega \setminus \{v\}\) such that the pair \((v, w) \in \mathcal{S}_n\) is symplectic (that is, satisfies \(\langle v, w \rangle = 1\)) and a set \(\Omega'\) such that

(i) \(\Omega\) and \(\Omega' \cup \{v, w\}\) span the same space, \(|\Omega'| \leq |\Omega| - 2\), and
(ii) \(\langle v, f' \rangle = \langle w, f' \rangle = 0\) for all \(f' \in \Omega'\).

The vector \(w\) and \(\Omega'\) are obtained by first choosing \(w \in \Omega \setminus \{v\}\) such that \(\langle v, w \rangle = 1\) and subsequently inserting the vector \(f + \langle v, f \rangle w + \langle w, f \rangle v\) into \(\Omega'\) for each \(f \not\in \Omega \setminus \{v, w\}\).

Repeatedly picking a vector \(v\) (arbitrarily) in the resulting set \(\Omega'\) and reapplying this basic step yields a symplectic basis of the space spanned by the original set of vectors \(\Omega\). In particular, for any non-zero vector \(v \in \mathbb{F}_2^{2n}\), a symplectic basis \((v_1, w_1, v_2, w_2, \ldots, v_n, w_n)\) of \(\mathbb{F}_2^{2n}\), i.e., a basis satisfying
\[
\langle v_j, w_k \rangle = \delta_{j,k} \quad \text{and} \quad \langle v_j, v_k \rangle = \langle w_j, w_k \rangle = 0
\]
with \(v_1 = v\) can be obtained starting from \(\Omega = \{v\} \cup \{e_1, \ldots, e_{2n}\}\), where \(e_1, \ldots, e_{2n} \in \mathbb{F}_2^{2n}\) are the standard basis vectors of \(\mathbb{F}_2^{2n}\). The complexity of this procedure is easily seen to be \(O(n^3)\).
Our algorithm is an adaptation of a method for generating random matrices from the classical compact Lie groups by Diaconis and Shahshahani [5] (also see [12] for a nice description). In [5], a method for the Lie group Sp(2n, C) is given which partly relies on the fact that its group elements can be represented as n × n matrices with entries in the quaternions. In our case, we do not have this tool at our disposal since we are working over a finite field. Getting an efficient algorithm therefore requires some additional effort.

The core of these algorithms is called the subgroup algorithm, which is most easily explained for a finite group G with a nested chain of subgroups

\[ G_1 \subset G_2 \subset \cdots \subset G_{n-1} \subset G_n = G. \tag{7} \]

In this situation, the map

\[ G_n/G_{n-1} \times G_{n-1}/G_{n-2} \times \cdots \times G_2/G_1 \times G_1 \rightarrow G \]

\[ ([g_n], [g_{n-1}], \ldots, [g_2], g_1) \rightarrow g_n g_{n-1} \cdots g_1 \]

is an isomorphism. In particular, each g ∈ G has a unique representation as \( g_n g_{n-1} \cdots g_1 \) with \( [g_j] \in G_j/G_{j-1} \) for \( j = 2, \ldots, n \) and \( g_1 \in G_1 \). This implies that given an element \( g_j \in G_j \) representing a uniformly random coset \([g_j] \in G_j/G_{j-1}\) for every \( j = 2, \ldots, n \), and a uniformly chosen random element \( g_1 \in G_1 \), we can obtain a uniformly distributed element of G by taking the product.

In our case we take \( G_j = \text{Sp}(2j) \) where the embedding \( \text{Sp}(2(j-1)) \rightarrow \text{Sp}(2j) \) is given by \( S \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus S \).

Furthermore, it is easy to see that there is a one-to-one correspondence between the set

\[ S_n := \{(v, w) \in \mathbb{F}_2^{2n} \times \mathbb{F}_2^{2n} \mid \langle v, w \rangle = 1\} \]

of symplectic pairs of vectors and the cosets \( \text{Sp}(2n)/\text{Sp}(2(n-1)) \). More precisely, let \( S_{v,w} \in \text{Sp}(2n) \) be a symplectic matrix with \( v \) in the first and \( w \) in the second column for any symplectic pair \( (v, w) \in S_n \) (we show below how to find such a matrix). Then

\[ S_n \rightarrow \text{Sp}(2n)/\text{Sp}(2(n-1)) \]

\[ (v, w) \mapsto [S_{v,w}] \] \tag{8}

establishes the claimed one-to-one correspondence [16] between \( S_n \) and \( \text{Sp}(2n)/\text{Sp}(2(n-1)) \), where we write \( [S] = S : \text{Sp}(2(n-1)) \) for the coset represented by \( S \).

**Remark 1.** Another way to think of the subgroup algorithm for the symplectic group is the following: The coset \( G_n/G_{n-1} \) will simply be represented by a symplectic pair \( (v, w) \in S_n \) along with an arbitrary basis for the space orthogonal to \( v \) and \( w \). Both our algorithms will proceed by picking out such a symplectic pair, then repeating in the orthogonal space. It is apparent that this will give the canonical mapping we require. What remains is to find an efficient algorithm for computing \( v, w \) and the orthogonal space.

II. ALGORITHMS

We will give two solutions to giving a canonical mapping of integers to Sp(2n). The first is based on symplectic Gaussian elimination, but has complexity \( O(n^4) \). It is mainly of didactical interest. The second algorithm uses symplectic transvections and achieves a complexity \( O(n^3) \). Note that these algorithms do not give the same canonical mapping. We also provide an algorithm for the inverse problem, finding the integer associated with a member of SP(2n).

A. An algorithm with runtime \( O(n^4) \) based on Gaussian elimination

We present an algorithm SYMPELCTIC\((n, i)\) which produces the \( i \)th symplectic matrix \( S_i \in \text{Sp}(2n) \). The algorithm is described in Fig. [II].

We analyze the algorithm step by step. Step 1 sets \( s \) to be the number of different choices of nonzero bitstrings of length \( n \) and \( k \) to be a choice of one of them based on the input \( i \). Step 2 creates the vector \( v_1 \) corresponding
Lemma 2. Let \( x, y \in \mathbb{F}_2^{2n} \setminus \{0\} \) be two non-zero vectors. Then
\[
y = Z_h x \quad \text{for some } h \in \mathbb{F}_2^{2n}
\]
\[
y = Z_{h_1} Z_{h_2} x \quad \text{for some } h_1, h_2 \in \mathbb{F}_2^{2n}
\]
In other words, \( x \) can be mapped to \( y \) by at most two transvections. Furthermore, there is an algorithm that outputs either \( h \) satisfying (10) or \((h_1, h_2)\) satisfying (11) in time \( O(n) \).
Proof. If \( x = y \), the algorithm outputs \( h = 0 \). Otherwise, it computes \( \langle x, y \rangle \) and proceeds as follows:

(i) if \( \langle x, y \rangle = 1 \), the algorithm outputs \( h = x + y \). It is easy to check that this has the required property (10).

(ii) if \( \langle x, y \rangle = 0 \), the algorithm computes some \( z \in \mathbb{F}_2^n \) such that \( \langle x, z \rangle = \langle z, y \rangle = 1 \). Concretely, this is achieved e.g., by trying to locate an index \( j \in \{1, \ldots, 2n\} \) such that \( \langle x_{2j-1}, x_{2j} \rangle \neq (0,0) \) and \( \langle y_{2j-1}, y_{2j} \rangle \neq (0,0) \). If such an index \( j \) is found, then there is a pair \( (v, w) \in \mathbb{F}_2^n \) such that \( x_{2j-1}v + x_{2j} = y_{2j-1}w + y_{2j} = 1 \) and we set \( z = x + ve_{2j-1} + we_{2j} \). Otherwise, there must be two distinct indices \( j, k \in \{1, \ldots, 2n\} \) such that \( \langle x_{2j-1}, x_{2j} \rangle \neq (0,0) \), \( \langle y_{2j-1}, y_{2j} \rangle = (0,0) \) and \( \langle x_{2k-1}, x_{2k} \rangle = (0,0) \), \( \langle y_{2k-1}, y_{2k} \rangle \neq (0,0) \) since \( x \) and \( y \) are non-zero. Then there are pairs \( (v, w), (v', w') \in \mathbb{F}_2^n \) such that \( x_{2j-1}v + x_{2j} = y_{2k-1}w' + y_{2k}v' = 1 \) and we set \( z = x + ve_{2j-1} + we_{2j} + v'e_{2k-1} + w'e_{2k} \).

This reduces the problem to (i) (mapping \( x \) to \( z \) and \( z \) to \( y \)); the algorithm outputs \( h_1 = x + z \) and \( h_2 = z + y \) and (11) follows.

Our improved algorithm based on transvections is shown in Fig. 2. Python code that implements it can be found in the appendix. We now analyze it step by step. Step 1 sets \( s \) to be the number of different choices of nonzero bitstrings of length \( n \) and \( k \) to be a choice of one of them based on the input \( i \). Step 2 creates the vector \( f_1 \) corresponding to \( k \). So far this is just as in the original \textsc{Symplectic}, save that \( v_1 \) is now named \( f_1 \). Step 3 computes the transvection(s) that transform the first standard basis vector \( e_1 \) to \( f_1 \). This can be done efficiently using the algorithm of Lemma 2. Step 4 again picks out the bits that will specify a vector \( T'Te_2, \) computed subsequently) which forms a symplectic pair with \( f_1 \). Step 5 and 6 find the transvection or pair of transvections \( T' \) with the property that \( T'Te_1 = f_1 \) and \( T'Te_2 \) is an arbitrary vector forming a symplectic pair with \( f_1 \). Thus, by (2), \( g_0 \equiv T'T \) represents a unique coset \([g_n]\) as required for the subgroup algorithm.

To see this it is convenient to define the vectors \( f_\ell = Te_\ell \) for \( \ell = \{1, \ldots, 2n\} \) corresponding to the images of the standard basis vectors. Observe that \( (f_1, f_2, \ldots, f_{2n-1}, f_{2n}) \) is a symplectic basis. We will show that

\[
T'Te_1 = f_1 \quad \text{ and } \quad T'Te_2 = bf_1 + f_2 + \sum_{\ell=3}^{2n} b_\ell f_\ell . \tag{12}
\]

By linearity, the vector \( h_0 \) computed in step 5 of the algorithm has the form \( h_0 = f_1 + \sum_{k=3}^{2n} b_k f_k \). In particular, we get \( \langle f_1, h_0 \rangle = 0 \) and \( \langle f_2, h_0 \rangle = 1 \), which implies

\[
Z_{h_0}e_1 = Z_{h_0}f_1 = f_1 \quad \text{ and } \quad Z_{h_0}e_2 = Z_{h_0}f_2 = f_1 + f_2 + \sum_{k=3}^{2n} b_k f_k \tag{13}
\]
returns the index $i, 0 \leq i < 2^{n^2} \prod_{j=1}^{n}(4^j - 1)$ of a group element $g_n \in \text{Sp}(2n)$.

1. Take the first two columns of $g_n$ and call them $v$ and $w$.

2. Using Lemma 2, compute a vector $h_i$ corresponding to a transvection $T = Z_{h_i}$ or a pair of vectors $(h_1, h_2)$ corresponding to a product $T = Z_{h_1}Z_{h_2}$ of two transvections such that $Tv = e_1$ is the first standard basis vector.

3. Compute $Tw = be_1 + c_2 + \sum_{i = 3}^{2n} b_i c_i$, i.e., $b$ and $\{b_i\}_{i = 3}^{2n}$. Set $h_0 = e_1 + \sum_{i = 3}^{2n} b_i c_i$.

4. Compute $z_v = \text{int}(v) - 1$, where $\text{int}(v)$ is the integer whose binary expansion is $v$. Also compute $z_w = \text{int}(i(a, b_3, b_4, \ldots, b_{2n}))$ and $c_{v,w} = z_w \cdot (2^{2n} - 1) + z_v$.

5. If $n = 1$, return $c_{v,w}$ as the result.

6. If $b = 0$, compute the matrix $g' = Z_{h_0} Z_{h_0} T g_n$.

7. Define $g_{n-1}$ as the $2(n-1) \times 2(n-1)$ matrix obtained by removing the first two columns and rows from $g'$. Return SYMplecticInverse$(n - 1, g_{n-1}) \cdot N(n) + c_{v,w}$, where $N(n) = 2^{n^2 - 1} (2^{2n} - 1)$.

by the definition of transvections. Consider the case when $b = 1$. Then $T^t T = Z_{h_0} T$ and \([13]\) reduces to \([12]\), as claimed. On the other hand, if $b = 0$, then $T^t T = Z_{f_1} Z_{h_0} T$ and we can use \([13]\) to compute

$$T^t T e_1 = Z_{f_1} Z_{h_0} T e_1 = Z_{f_1} f_1 = f_1$$

and

$$T^t T e_2 = Z_{f_1} Z_{h_0} T e_2 = Z_{f_1} (f_1 + f_2 + \sum_{k=3}^{2n} b_k f_k) = f_2 + \sum_{k=3}^{2n} b_k f_k ,$$

confirming \([12]\).

Finally, step 7 multiplies $T^t T$ by a symplectic of the next smaller size in the chain, if necessary, and returns the answer. This multiplication takes $O(n^3)$ time because $T^t T$ is a product of transvections associated with known vectors. Since there are $n$ recursions, the total complexity is $O(n^3)$.

C. An algorithm for the inverse problem with runtime $O(n^3)$

Consider the inverse problem: given a group element $g_n \in \text{Sp}(2n)$, we would like to associate to it a unique index $i = i(g_n)$ where $0 \leq i < |\text{Sp}(2n)| = 2^{n^2} \prod_{j=1}^{n}(4^j - 1)$. With similar reasoning as before, we can construct an efficient algorithm achieving this. It is shown in Fig. 3 and will be referred to as SYMplecticInverse. It implements the exact inverse map of the map $i \mapsto \text{SYMplecticImproved}(n, i)$ defined by the algorithm in Fig. 2 and runs in time $O(n^3)$.

Given a matrix $g_n \in \text{Sp}(2n)$, the algorithm SYMplecticInverse proceeds recursively by factoring the given group element into representatives of cosets. Clearly, by definition of $\mathcal{S}_n$, the coset in $\text{Sp}(2n)/\text{Sp}(2(n-1))$ can be read off from the first two columns $(v, w)$ of $g_n$ (Step 1). To uniquely index different cosets, the algorithm relies on the transvection $T$ computed in step 2. After step 3, the non-zero vector $v$, together with the (arbitrary) bits $b, \{b_i\}_{i = 3}^{2n}$, uniquely specify the symplectic pair $(v, w)$ (and hence a coset). In step 4, this is used to compute an associated (unique) number $c_{v,w}$, where $0 \leq c_{v,w} < N(n)$ and where $N(n)$ is the number of different cosets in $\text{Sp}(2n)/\text{Sp}(2(n-1))$. If $n = 1$, the number $c_{v,w}$ already indexes a unique group element in $\text{Sp}(2)$, and no recursion is necessary (step 5).

If $n > 1$, the algorithm constructs a symplectic matrix $V$ such that

$$g' := V g_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus g_{n-1}$$

and returns the value SYMplecticInverse$(n - 1, g_{n-1}) \cdot N(n) + c_{v,w}$ (Step 7). This number encodes both $c_{v,w}$, i.e., the coset in $\text{Sp}(2n)/\text{Sp}(2(n-1))$, as well as all the cosets in the chain of subgroups.
It is clear that this algorithm has runtime $O(n^3)$ if the matrix product in step 6 is computed using the vectors specifying the transvections. It remains to show that the matrix $g'$ constructed in step 6 has property (14).

By definition, we have $g_n e_1 = v$ and $g_n e_2 = w$. In particular, the definition of $T$, the coefficients $b$, $(b_{\ell})_{\ell=3}^{2n}$ and $h_0$ give

$$(Tg_n)e_1 = e_1$$

$$(Tg_n)e_2 = be_1 + e_2 + \sum_{\ell=3}^{2n} b_{\ell}e_\ell = h_0 + (b-1)e_1 + e_2.$$  

Since $\langle e_1, h_0 \rangle = 0$, $\langle h_0, h_0 \rangle = 0$ and $\langle e_2, h_0 \rangle = 1$, this implies

$$(Z_{h_0}Tg_n)e_1 = e_1$$

$$(Z_{h_0}Tg_n)e_2 = (b-1)e_1 + e_2.$$  

This shows that if $b = 1$, then $g' = Z_{h_0}Tg_n$ has the required property. If $b = 0$, we use the fact that $Z_{e_i}e_1 = e_1$ and $Z_{e_i}(e_1 + e_2) = e_2$ to conclude that $g' = Z_{e_i}Z_{h_0}Tg_n$ has the desired form.

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[1] S. Aaronson and D. Gottesman. Improved simulation of stabilizer circuits. Phys. Rev. A, 70:052328, Nov 2004.
[2] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters. Mixed-state entanglement and quantum error correction. Phys. Rev. A, 54:3824–3851, Nov 1996.
[3] A. R. Calderbank, E. M. Rains, P. M. Shor, and N. J. A. Sloane. Quantum error correction via codes over GF(4). IEEE Trans. Inf. Th., 44(4):1369 –1378, July 1998.
[4] C. Dankert, R. Cleve, J. Emerson, and E. Livine. Exact and approximate unitary 2-designs and their application to fidelity estimation. Phys. Rev. A, 80:012304, Jul 2009.
[5] P. Diaconis and M. Shahshahani. The subgroup algorithm for generating uniform random variables. Probability in the Engineering and Informational Sciences, 1:15–32, 1987.
[6] D. DiVincenzo, D.W. Leung, and B.M. Terhal. Quantum data hiding. IEEE Trans. Inf. Th., 48(3):580–599, 2002.
[7] J. Emerson, R. Alicki, and K. yczkowski1. Scalable noise estimation with random unitary operators. Journal of Optics B: Quantum and Semiclassical Optics, 7:S357, 2005.
[8] D. Gottesman. Stabilizer Codes and Quantum Error Correction. PhD thesis, Caltech, 1997.
[9] D. Gottesman. The heisenberg representation of quantum computers. arXiv:quant-ph/9807006, Group22: Proceedings of the XXII International Colloquium on Group Theoretical Methods in Physics, eds. S. P. Corney, R. Delbourgo, and P. D. Jarvis, pp. 32-43 (Cambridge, MA, International Press, 1999), 1998.
[10] E. Knill, D. Leibfried, R. Reichle, J. Britton, R. B. Blakestad, J. D. Jost, C. Langer, R. Ozeri, S. Seidelin, and D. J. Wineland. Randomized benchmarking of quantum gates. Phys. Rev. A, 77:012307, Jan 2008.
[11] Easwar Magesan, J. M. Gambetta, and Joseph Emerson. Scalable and robust randomized benchmarking of quantum processes. Phys. Rev. Lett., 106:180504, May 2011.
[12] F. Mezzadri. How to generate random matrices from the classical compact groups. Notices of the AMS, 54:592–604, 2007.
[13] M. Ozols. Clifford group, 2008. http://home.lu.lv/~sd2008/papers/essays/Clifford group [paper]53.pdf.
[14] A. Salam, E. Al-Aidarous, and A. El Farouk. Optimal symplectic Householder transformations for SR decomposition. Linear Algebra and its Applications, 429(56):1334 – 1353, 2008.
[15] As pointed out in [13], this does not agree with e.g., R, since R assumes that $C_\alpha$ is generated by $H, P$ and $CNOT$, and these generate additional phases because $(PH)^3$ $= e^{i\pi/4}$ resulting in an additional factor of 8. This extra phase, irrelevant to quantum mechanics, is needed in order to write down a unitary representation of the group.
[16] To show that this is well-defined, suppose that $[S_{v, w}] = [S_{v', w'}]$, then $S_{v, w}^{-1}S_{v', w'} \in \{0, 1\} \oplus Sp(2(n-1))$, and it follows immediately that the first two standard basis vectors $e_1, e_2$ are mapped identically under $S_{v, w}$ and $S_{v', w'}$, i.e., $v = S_{v, w}e_1 = S_{v', w'}e_1 = v'$ and $w = S_{v, w}e_2 = S_{v', w'}e_2 = w'$. To show that this parameterization is injective, suppose $[S_{v, w}] \neq [S_{v', w'}]$. Then we must have $(v, w) \neq (v', w')$ since otherwise $S_{v, w}^{-1}S_{v', w'} \in \{0, 1\} \oplus Sp(2(n-1))$, a contradiction.
Appendix: Python code implementing SYMLECTICimproved and SYMLECTICinverse

# canonical ordering of symplectic group elements
# from "How to efficiently select an arbitrary clifford group element"
# by Robert Koenig and John A. Smolin
#

from numpy import *
from time import clock
def directsum(m1, m2):
    n1 = len(m1[0])
    n2 = len(m2[0])
    output = zeros((n1+n2, n1+n2), dtype=int8)
    for i in range(0, n1):
        for j in range(0, n1):
            output[i, j] = m1[i, j]
    for i in range(0, n2):
        for j in range(0, n2):
            output[i+n1, j+n1] = m2[i, j]
    return output

# # # # # # # # # end directsum

def inner(v, w):
    # symplectic inner product
    t = 0
    for i in range(0, size(v)>>1):
        t += v[2*i]*w[2*i+1]
        t += w[2*i]*v[2*i+1]
    return t%2

def transvection(k, v):
    # applies transvection Zk to v
    return (v+inner(k, v)*k)%2

def int2bits(i, n):
    # converts integer i to an length n array of bits
    output = zeros(n, dtype=int8)
    for j in range(0, n):
        output[j] = i&1
        i >>= 1
    return output

def findtransvection(x, y):
    # finds h1, h2 such that y = Z_h1 Z_h2 x
    # Lemma 2 in the text
    # Note that if only one transvection is required output[1] will be
    # zero and applying the all-zero transvection does nothing.
    output = zeros((2, size(x)), dtype=int8)
    if array_equal(x, y):
        return output
    if inner(x, y) == 1:
        output[0] = (x+y)%2
        return output
    #
    # find a pair where they are both not 00
    z = zeros(size(x))
    for i in range(0, size(x)>>1):
        ii = 2*i
        if (((x[ii]+x[ii+1]) != 0) and ((y[ii]+y[ii+1]) != 0)):  # found the pair
\[ z[ii] = (x[ii] + y[ii]) \mod 2 \]
\[ z[ii+1] = (x[ii+1] + y[ii+1]) \mod 2 \]

if \((z[ii] + z[ii+1]) == 0\): # they were the same so they added to 0
    \[ z[ii+1] = 1 \]
else:
    \[ z[ii] = 1 \]

output[0] = (x + z) \mod 2
output[1] = (y + z) \mod 2

return output

# didn't find a pair
# so look for two places where x has 00 and y doesn't, and vice versa
#
# first y==00 and x doesn't
for i in range(0, size(x) \gg 1):
    ii = 2*i
    if ((x[ii] + x[ii+1]) != 0) and ((y[ii] + y[ii+1]) == 0):  # found the pair
        if x[ii] == x[ii+1]:
            \[ z[ii+1] = 1 \]
        else:
            \[ z[ii+1] = x[ii] \]
        \[ z[ii] = x[ii+1] \]
        break

# finally x==00 and y doesn't
for i in range(0, size(x) \gg 1):
    ii = 2*i
    if ((x[ii] + x[ii+1]) == 0) and ((y[ii] + y[ii+1]) != 0):  # found the pair
        if y[ii] == y[ii+1]:
            \[ z[ii+1] = 1 \]
        else:
            \[ z[ii+1] = y[ii] \]
        \[ z[ii] = y[ii+1] \]
        break

output[0] = (x + z) \mod 2
output[1] = (y + z) \mod 2
return output

########################################################################
end findtransvection

def symplectic(i,n): # output symplectic canonical matrix i of size 2nX2n
    # Note, compared to the text the transpose of the symplectic matrix
    # is returned. This is not particularly important since
    # Transpose(g in Sp(2n)) is in Sp(2n)
    # but it means the program doesn’t quite agree with the algorithm in the
    # text. In python, row ordering of matrices is convenient, so it is used
    # internally, but for column ordering is used in the text so that matrix
    # multiplication of symplectics will correspond to conjugation by
    # unitaries as conventionally defined Eq. (2). We can’t just return the
    # transpose every time as this would alternate doing the incorrect thing
    # as the algorithm recurses.
    nn=2*n  # this is convenient to have
    # step 1
    s=((-1)**(i))
    k=(i/s)+1
    i/=s

    #
# step 2
f1=int2bits(k,nn)
#
# step 3
e1=zeros(nn,dtype=int8) # define first basis vectors
e1[0]=1
T=findtransvection(e1,f1) # use Lemma 2 to compute T
#
# step 4
# b[0]=b in the text, b[1]...b[2n-2] are b_3...b_{2n} in the text
bits=int2bits(i%(1<<(nn-1)),nn-1)
#
# step 5
eprime=copy(e1)
for j in range(2,nn):
eprime[j]=bits[j-1]
h0=transvection(T[0],eprime)
h0=transvection(T[1],h0)
#
# step 6
if bits[0]==1:
f1*0=0
# T’ from the text will be Z_f1 Z*h0. If f1 has been set to zero
# it doesn’t do anything
# We could now compute f2 as said in the text but step 7 is slightly
# changed and will recompute f1,f2 for us anyway
#
# step 7
# define the 2x2 identity matrix
id2=zeros((2,2),dtype=int8)
id2[0,0]=1
id2[1,1]=1
#
if n!=1:
g=directsum(id2,symplectic(i>>(nn-1),n-1))
else:
g=id2
#
for j in range(0,nn):
g[j]=transvection(T[0],g[j])
g[j]=transvection(T[1],g[j])
g[j]=transvection(h0,g[j])
g[j]=transvection(f1,g[j])
#
return g

def bit2int(b,nn): # converts an nn-bit string b to an integer between 0 and 2^n-1
output=0
tmp=1
for j in range(0,nn):
if b[j]==1:
    output=output+tmp
tmp=tmp*2
return output

def numberofcosets(n): # returns the number of different cosets
x=power(2,2*n-1)*(power(2,2*n)-1)
return x;

def numberofsymplectic(n): # returns the number of symplectic group elements
x=1;
for j in range(1,n+1):
    x=x*numberofcosets(j);
return x;

def symplecticinverse(n,gn): # produce an index associated with group element gn
    nn=2*n  # this is convenient to have
    v=gn[0];
    w=gn[1];
    e1=zeros(nn,dtype=int8) # define first basis vectors
    e1[0]=1
    T=findtransvection(v,e1); # use Lemma 2 to compute T
    tw=copy(w)
    tw=transvection(T[0],tw)
    tw=transvection(T[1],tw)
    b=tw[0]:
    h0=zeros(nn,dtype=int8)
    h0[0]=1
    h0[1]=0
    for j in range(2,nn):
        h0[j]=tw[j]
    bb=zeros(nn−1,dtype=int8)
    bb[0]=b;
    for j in range(2,nn):
        bb[j−1]=tw[j];
    zv=bits2int(v,nn)−1; # number between 0...2^(2n)−2
    # indexing non-zero bitstring v of length 2n
    zw=bits2int(bb,nn−1); # number between 0..2^(2n−1)−1
    # indexing w (such that v,w is symplectic pair)
    cvw=zv*(power(2, 2*n)−1)+zv;
    # cvw is a number indexing the unique coset specified by (v,w)
    # it is between 0...2^(2n−1)*(2^(2n)−1)=numberofcosets(n)−1
    if n==1:
        return cvw
    gprime=copy(gn);
    if b==0:
for j in range(0, nn):
gprime[j] = transvection(T[1], transvection(T[0], gn[j]))
gprime[j] = transvection(h0, gprime[j])
gprime[j] = transvection(e1, gprime[j])
else:
    for j in range(0, nn):
        gprime[j] = transvection(T[1], transvection(T[0], gn[j]))
        gprime[j] = transvection(h0, gprime[j])

# step 7

gnew=gprime[2:nn,2:nn]; # take submatrix
gnidx=symplecticinverse(n-1,gnew)*numberofcosets(n)+cvw;
return gnidx

######### end symplecticinverse