A Conditional Explicit Result for the Prime Number Theorem in Short Intervals

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Abstract

This paper gives an explicit bound for the prime number theorem in short intervals under the assumption of the Riemann hypothesis.

1 Introduction

The von Mangoldt function is defined as

$$\Lambda(n) = \begin{cases} \log p & n = p^m, \ p \ \text{is prime}, \ m \in \mathbb{N} \\ 0 & \text{otherwise}, \end{cases}$$

and we will consider the sum $\psi(x) = \sum_{n \leq x} \Lambda(n)$. The prime number theorem (PNT) is the statement $\psi(x) \sim x$ as $x \to \infty$. For the PNT in short intervals, it is known that

$$\psi(x + h) - \psi(h) \sim h$$

provided that $h$ grows suitably with respect to $x$. Heath-Brown [9] has shown that one can take $h = x^{\frac{7}{12} - \epsilon}$ provided that $\epsilon \to 0$ as $x \to \infty$. Assuming
the Riemann hypothesis (RH), Selberg [14] showed that (1) is true for any 
h = h(x) such that \( h/(x^{1/2} \log x) \to \infty \) as \( x \to \infty \). On the other hand, Maier [11] has shown that the statement is false for \( h = (\log x)^{\lambda} \) for any \( \lambda > 1 \).

In this paper we prove the following explicit version of Selberg’s result.

**Theorem 1.** Assuming RH, for any \( h \) satisfying \( \sqrt{x} \log x \leq h \leq x^{3/4} \) and all \( x \geq e^{10} \) we have

\[
|\psi(x+h) - \psi(x) - h| < \frac{1}{\pi} \sqrt{x} \log x \log \left( \frac{h}{\sqrt{x} \log x} \right) + 2\sqrt{x} \log x. \tag{2}
\]

Selberg’s result follows from Theorem 1 for any \( h = f(x) \sqrt{x} \log x \) with unbounded \( f(x) = o(x) \), in that we would have

\[
|\psi(x+h) - \psi(x) - h| \ll \sqrt{x} \log x \log f(x) = o(h).
\]

For \( h = c \sqrt{x} \log x \), Theorem 1 implies Cramér’s [6] result on primes in the interval \((x, x+h)\) for all sufficiently large \( x \) and \( c \). In an earlier paper [7], the author showed that \( c = 1 + \epsilon \) is suitable for any \( \epsilon > 0 \) and for all sufficiently large \( x \). Carneiro, Milinovich and Soundararajan [4] have since shown that we can take \( c = 22/55 \) for all \( x \geq 4 \). The same methods used in [7] are applied to reach Theorem 1. As such, it could be possible to sharpen Theorem 1 using the techniques in [4].

The closest result to Theorem 1 is the following from Schoenfeld [13].

**Theorem 2.** Assuming RH, for \( x \geq 73.2 \) we have

\[
|\psi(x) - x| < \frac{1}{8\pi} \sqrt{x} \log^2 x. \tag{3}
\]

Schoenfeld’s result confirms Selberg’s theorem for the slightly stronger condition of \( h/(\sqrt{x} \log^2 x) \to \infty \). One also has from the above

\[
|\psi(x+h) - \psi(x) - h| < \frac{1}{4\pi} \sqrt{x+h} \log^2 (x+h).
\]

When \( x \) is sufficiently large, Theorem 1 improves the leading constant in this bound for any choice of \( h \leq x^{0.735} \).
2 Proof of Theorem 1

2.1 A smooth explicit formula

The Riemann–von Mangoldt explicit formula relates $\psi(x)$ to the zeros of the Riemann zeta-function $\zeta(s)$ (e.g. see Ingham [10]). For all non-integer $x > 0$,

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2} \log(1 - x^{-2}), \quad (4)$$

where the sum is over all non-trivial zeroes $\rho = \beta + i\gamma$ of $\zeta(s)$. We define the weighted sum

$$\psi_1(x) = \sum_{n \leq x} (x - n)\Lambda(n) = \int_2^x \psi(t)dt \quad (5)$$

and use the following explicit formula, proved in [7] (see also Thm. 28 of [10]).

**Lemma 3.** For non-integer $x > 0$ we have

$$\psi_1(x) = \frac{x^2}{2} - \frac{x^{\rho+1}}{\rho(\rho+1)} - x \log(2\pi) + \epsilon(x) \quad (6)$$

where

$$1.545 < \epsilon(x) < 2.069.$$ 

The bound on $\epsilon(x)$ has been reduced from [7], as we can write

$$\epsilon(x) = 2 \log 2\pi - 2 + \sum_{\rho} \frac{2^{\rho+1}}{\rho(\rho+1)} - \frac{1}{2} \int_2^x \log(1 - t^{-2})dt$$

$$< 2 \log 2\pi - 2 + 2^2(\gamma + 2 - \log 4\pi) + \log \frac{3\sqrt{3}}{4} < 2.069$$

and

$$\epsilon(x) > 2 \log 2\pi - 2 - 2^2(\gamma + 2 - \log 4\pi) > 1.545.$$ 

Using a linear combination of equation (5), we can examine the distribution of prime powers in the interval $(x, x+h)$. For $2 \leq \Delta < \sqrt{x} \log x \leq h \leq x$, let

$$w(n) = \begin{cases} 
(n - x + \Delta)/\Delta & : x - \Delta \leq n \leq x \\
1 & : x \leq n \leq x + h \\
(x + h + \Delta - n)/\Delta & : x + h \leq n \leq x + h + \Delta \\
0 & : \text{otherwise.}
\end{cases} \quad (7)$$

This leads to the identity
\[
\sum_n \Lambda(n)w(n) = \frac{1}{\Delta}(\psi_1(x + h + \Delta) - \psi_1(x + h) - \psi_1(x) + \psi_1(x - \Delta)),
\]

which can be verified by expanding both sides. Notice that over \(x \leq n \leq x + h\), the sum on the LHS is equal to \(\psi(x + h) - \psi(x)\). We thus aim to estimate this expression by bounding the RHS of (7). Using Lemma 3 in the above equation gives the following.

**Lemma 4.** Let \(2 \leq \Delta < h \leq x\) with \(x \notin \mathbb{Z}\). Then

\[
\sum_n \Lambda(n)w(n) = h + \Delta - \frac{1}{\Delta} \sum \rho S(\rho) + \epsilon(\Delta)
\]

where

\[
S(\rho) = \frac{(x + h + \Delta)^{\rho + 1} - (x + h)^{\rho + 1} - x^{\rho + 1} + (x - \Delta)^{\rho + 1}}{\rho(\rho + 1)}
\]

and

\[
|\epsilon(\Delta)| < \frac{21}{20\Delta}
\]

It remains to estimate the sum over zeros. We will split it into three sums,

\[
\sum \rho S(\rho) = \left( \sum_{|\gamma| \leq \alpha x / h} + \sum_{\alpha x / h < |\gamma| < \beta x / \Delta} + \sum_{|\gamma| \geq \beta x / \Delta} \right) S(\rho)
\]

where \(\alpha > 0\) and \(\beta > 0\) are parameters we can later optimise over.

**Lemma 5.** Let \(2 \leq \Delta < h \leq x\) and assume RH. We have

\[
\left| \sum_{|\gamma| \geq \beta x / \Delta} S(\rho) \right| < \frac{4\Delta(x + h + \Delta)^{3/2}}{\pi \beta x} \log(\beta x / \Delta)
\]

provided that \(\beta x / \Delta \geq \gamma_1 = 14.13\ldots\), the ordinate of the first zero of \(\zeta(s)\).

**Proof.** On RH, one has

\[
|S(\rho)| \leq \frac{4(x + h + \Delta)^{3/2}}{\gamma^2}.
\]
The result follows from Lemma 1(ii) of Skewes [15], that for all $T \geq \gamma_1$,
\[
\sum_{\gamma \geq T} \frac{1}{\gamma^2} < \frac{1}{2\pi} \log \frac{T}{T}.
\]

The following lemmas require estimates on the zero-counting function $N(T)$, which counts the number of zeros of $\zeta(s)$ in the critical strip $0 < \beta < 1$ with $0 < \gamma \leq T$. Backlund [1] showed that $N(T) = P(T) + Q(T)$, where
\[
P(T) := \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}
\]
and $Q(T) = O(\log T)$. Hasanalizade, Shen, and Wong [8, Cor. 1.2] have given the most recent explicit version of this, of
\[
|Q(T)| \leq R(T) = a_1 \log T + a_2 \log \log T + a_3
\]
with $a_1 = 0.1038$, $a_2 = 0.2573$, and $a_3 = 9.3675$, for all $T \geq e$.

**Lemma 6.** Let $2 \leq \Delta < h \leq x$ and assume RH. We have
\[
\left| \sum_{|\gamma| \leq \alpha x/h} S(\rho) \right| < \frac{\alpha x(h + \Delta)\Delta}{\pi h \sqrt{x - \Delta}} \log(\alpha x/h).
\]

**Proof.** We can write
\[
S(\rho) = \int_{x+h}^{x+h+\Delta} \int_{u-h-\Delta}^{u} t^{\rho-1}dtdu,
\]
so, under RH, one has
\[
|S(\rho)| < \frac{(h + \Delta)\Delta}{\sqrt{x - \Delta}}.
\]
With (8), we can use
\[
N(T) < \frac{T \log T}{2\pi},
\]
from which the result immediately follows. \qed

For the middle sum of (7), we will use the following lemma. It follows directly from Lemma 3 of [2], in whose notation we use $\phi(\gamma) = \gamma^{-1}$, and takes constants $A_0$ and $A_1$ from Trudgian [16, Thm. 2.2] and $A_2$ from [2, Lem. 2].
Lemma 7. For $2\pi \leq T_1 \leq T_2$ we have

$$\sum_{T_1 < \gamma < T_2} \frac{1}{\gamma} = \frac{1}{4\pi} \log \frac{T_2}{T_1} \log \frac{T_2 T_1}{4\pi^2} + \frac{Q(T_2)}{T_2} - \frac{Q(T_1)}{T_1} + E(T_1), \quad (9)$$

where $|Q(T)| \leq R(T)$, defined in (8), and

$$|E(T)| \leq \frac{2A_1 \log T + 2A_0 + A_1 + A_2}{T^2}$$

with $A_0 = 2.067$, $A_1 = 0.059$, $A_2 = 1/150$.

Lemma 8. Let $2 \leq \Delta < h \leq x$ and assume RH. For $\alpha x/h \geq 15$ we have

$$\left| \sum_{\alpha x/h < \gamma < \beta x/\Delta} S(\rho) \right| < \Delta(x + h + \Delta)^{1/2} \left( \frac{1}{\pi} \log \left( \frac{\beta h}{\alpha \Delta} \right) \log \left( \frac{\alpha \beta x^2}{4\pi^2 h \Delta} \right) + 5.4 \right).$$

Proof. We can write

$$S(\rho) = \frac{1}{\rho} \left( \int_{x+h}^{x+h+\Delta} t^\rho dt - \int_{x-\Delta}^{x} t^\rho dt \right),$$

and so bounding trivially gives

$$|S(\rho)| \leq \frac{2(x + h + \Delta)^{1/2}}{|\gamma|}.$$

It follows that

$$\left| \sum_{\alpha x/h < \gamma < \beta x/\Delta} S(\rho) \right| \leq 4(x + h + \Delta)^{1/2} \Delta \sum_{\alpha x/h < \gamma < \beta x/\Delta} \frac{1}{\gamma},$$

on which we apply Lemma 7 and bound the smaller order terms with the assumption of $T_1 \geq 15$ to obtain the result. Note that the bound on $T_1$ is to reduce the constant 5.4, but not restrict $\alpha$ too much. \qed

2.2 Bounding the PNT in intervals

From Lemma 4 we can write

$$\left| \psi(x + h) - \psi(x) - h \right| < \frac{1}{\Delta} \sum_{\rho} S(\rho) + \Delta + \frac{21}{20\Delta} + \sum_{x-\Delta < n \leq x} w(n)\Lambda(n)$$

$$\quad + \sum_{x+h<n\leq x+h+\Delta} w(n)\Lambda(n).$$
As the smooth weight has \(|w(n)| \leq 1\), the above bound is no greater than

\[
\frac{1}{\Delta} \left| \sum_{\rho} S(\rho) \right| + \Delta + \frac{21}{20\Delta} + 2 \sum_{x+h<p^k \leq x+h+\Delta, k \geq 1} \log p. \tag{10}
\]

The largest term in this bound comes from the sum over \(\rho\), in particular, the section estimated in Lemma 8. Larger \(\Delta\) results in a smaller main-term constant, so we will set \(\Delta = C \sqrt{x} \log x\) and later choose an optimal value of \(C \in (0,1)\). The reason for not taking larger \(\Delta\) is two-fold: to keep \(\Delta < h\) and ensure the smaller terms in (10) are \(O(\sqrt{x} \log x)\).

To bound the sum over prime powers we can use Montgomery and Vaughan’s version of the Brun–Titchmarsh theorem for primes in intervals \[12, Eq. 1.12\]. Defining \(\theta(x) = \sum_{p \leq x} \log p\), equation (1.12) of \[12\] implies

\[
\theta(x+h) - \theta(x) = \sum_{x<p \leq x+h} \log p \leq \frac{2h \log(x+h)}{\log h}.
\]

The contribution from higher prime powers is relatively small, and can be bounded with explicit estimates on the difference between the Chebyshev functions \(\psi(x)\) and \(\theta(x)\). Costa Pereira \[5, Thm. 2,4,5\] gives lower bounds for different ranges of \(x\). These can be combined into

\[
\psi(x) - \theta(x) > 0.999x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{5}}, \tag{11}
\]

for all \(x \geq 2187\). Broadbent et al. \[3, Cor. 5.1\] give

\[
\psi(x) - \theta(x) < \alpha_1 x^{\frac{1}{2}} + \alpha_2 x^{\frac{3}{5}} \tag{12}
\]

with \(\alpha_1 = 1 + 1.93378 \cdot 10^{-8}\) and \(\alpha_2 = 2.69\) for all \(x \geq e^{10}\). Thus, we have

\[
\psi(x+h+\Delta) - \psi(x+h) \leq \theta(x+h+\Delta) - \theta(x+h) + E_1(x) \leq \frac{2\Delta \log(x+h+\Delta)}{\log \Delta} + E_1(x)
\]

where \(E_1(x) = \alpha_1(x+h+\Delta)^{\frac{1}{2}} + \alpha_2(x+h+\Delta)^{\frac{3}{5}} - 0.999(x+h)^{\frac{1}{2}} - \frac{2}{3}(x+h)^{\frac{3}{5}}\), and is bounded by \(E_1(x) \leq \beta_1 x^{\frac{1}{2}} + \beta_2 x^{\frac{3}{5}}\) with

\[
\beta_1 = \sqrt{3}\alpha_1 - 0.999 \quad \text{and} \quad \beta_2 = 3\alpha_2 - \frac{2}{3}.
\]
Here and hereafter, let \( x_0 = e^{10} \). For \( x \geq x_0 \) we can bound the smaller order terms in (10),

\[
\Delta + \frac{21}{20 \Delta} + 2 \sum_{x+h < p^k \leq x+h+\Delta} \log p < K_1 \sqrt{x} \log x
\]

where, for \( h \leq x^t \) with \( t < 1 \),

\[
K_1 = C + \frac{4C \log(x_0 + 2x^t)}{\log(C\sqrt{x_0 \log x_0})} + \frac{2\beta_1}{x_0 \log x_0} + \frac{2\beta_2}{x_0^2 \log x_0} + \frac{21}{20C x_0 \log^2 x_0}.
\]

This, along with Lemmas 5 and 6 allow us to bound

\[
\left| \psi(x + h) - \psi(x) - h \right| < \frac{1}{\Delta} \left| \sum_{\alpha x/h < |\gamma| < \beta x/\Delta} S(\rho) \right| + E(x,h,\Delta) \quad (13)
\]

where

\[
E(x,h,\Delta) = K_1 \sqrt{x} + \frac{\alpha x(h + \Delta)}{\pi h \sqrt{x - \Delta}} \log \left( \frac{\alpha x}{h} \right) + \frac{4(x + h + \Delta)^{3/2}}{\pi \beta x} \log \left( \frac{\beta x}{\Delta} \right).
\]

For \( \sqrt{x} \log x \leq h \leq x^t \) we have

\[
E(x,h,\Delta) \leq K_1 \sqrt{x} + \frac{2\alpha x}{\pi \sqrt{x - C\sqrt{x \log x}}} \log \left( \frac{\alpha \sqrt{x}}{\log x} \right) + \frac{4(x + x^t + C\sqrt{x \log x})^{3/2}}{\pi \beta x} \log \left( \frac{\beta \sqrt{x}}{C \log x} \right) \leq K_2 \sqrt{x} \log x,
\]

where, for \( x \geq x_0 \geq e^{3/C} \) and \( 0 < \alpha \leq 5 \), we can take

\[
K_2 = \frac{K_1}{\log x_0} + \frac{\alpha}{\pi} + \frac{2(x_0 + x_0^t + C\sqrt{x_0 \log x_0})^{3/2}}{\pi \beta x_0^{3/2}}.
\]

The first term in (13) can be estimated with Lemma 8 so that

\[
\frac{1}{\Delta} \left| \sum_{\alpha x/h < |\gamma| < \beta x/\Delta} S(\rho) \right| < (x + h + \Delta)^{1/2} \left( \frac{1}{\pi} \log \left( \frac{\beta h}{\alpha \Delta} \right) \log \left( \frac{\alpha \beta x^2}{4\pi^2 h \Delta} \right) + 5.4 \right)
\]

\[
< \frac{\sqrt{x}}{\pi} \log x \log \left( \frac{h}{\sqrt{x \log x}} \right) + K_3 \sqrt{x} \log x,
\]
in which, assuming $100e^{-10} \leq \frac{\alpha \beta}{4\pi^2C} \leq 100$, we can take

$$K_3 = \frac{1}{\pi} \log \left( \frac{\beta}{\alpha C} \right) \log \left( \frac{\alpha \beta x_0}{4\pi^2C\log^2 x_0} \right) \frac{1}{\log x_0}$$

$$+ \frac{x_0^{t/2-1/2}}{\pi \log x_0} \log \left( \frac{\beta x_0^{t-1/2}}{\alpha C \log x_0} \right) \log \left( \frac{\alpha \beta x_0}{4\pi^2C \log^2 x_0} \right)$$

$$+ \frac{\sqrt{C}}{\pi x_0^{1/4} \log x_0} \log \left( \frac{\beta x_0^{t-1/2}}{\alpha C \log x_0} \right) \log \left( \frac{\alpha \beta x_0}{4\pi^2C \log^2 x_0} \right)$$

$$+ \frac{5.4}{\log x_0} \left( 1 + x_0^{t-1} + \frac{C \log x_0}{\sqrt{x_0}} \right)^{1/2}.$$  

Note that the assumption for $\alpha$ and $\beta$ is to ensure certain terms are bounded for all $x \geq x_0$. Combining estimates, we have

$$\left| \psi(x + h) - \psi(x) - h \right| < \frac{\sqrt{x}}{\pi} \log x \log \left( \frac{h}{\sqrt{x} \log x} \right) + K_4 \sqrt{x} \log x, \quad (14)$$

where $K_4 = K_3 + K_2$. It remains to optimise over the parameters. Before deciding these values, recall that we have made the assumptions $\beta \leq 10C$,

$$\frac{15h}{x} \leq \alpha \leq 5, \quad \beta \geq \gamma \frac{C \log x}{\sqrt{x}}, \quad C \alpha < \beta \leq \alpha, \quad \text{and} \quad \frac{100}{e^{10}} \leq \frac{\alpha \beta}{4\pi^2C} \leq 100.$$  

The restriction on $\alpha$ will be satisfied for all $\sqrt{x} \log x \leq h \leq x^{3/4}$ if we take $\alpha \geq 15x_0^{-3/4}$. Optimising over $C$, $\alpha$, and $\beta$ to minimise $K_4$, we find that choosing $C = 0.25$ and $\alpha = \beta = 1.35$ allows us to take $K_4 = 2$ for all $x \geq x_0$.

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