The derivation Cramer-Rao lower bound for parameters of signals multiplied on window functions

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Abstract. The article focuses on the refinement of the Cramer-Rao lower boundary for the single-tone signal amplitude for estimation methods using window functions. The Cramer-Rao lower boundary allows finding the minimum variance of the signal parameter estimate. There are formulas for variances estimations of the amplitude, frequency, and phase of harmonics of harmonic signals, but the known methods for their finding, including algorithms based on the maximum likelihood method, show results above this boundary. The increased variance of the estimate occurs due to the application of window functions on the original signal. The estimation of the accuracy of the parameters harmonics in case using the window functions usually done with a numerical simulation. In the article, the authors derive a formula for the minimum variance for amplitudes of harmonics in the case of using the window functions. This derivation allows us to understand the mathematical meaning of the broadening the Cramer-Rao lower boundary when using window functions, and the resulting formula brings a faster and more accurate estimation of an amplitude accuracy, in comparison with numerical simulation. According to the results of the experiment the calculations by the proposed formula and the numerical experiment data are the same.

1. Introduction

One of the areas of application of digital signal processing methods is spectral analysis. Discrete Fourier Transform (DFT) is a tool for analyzing discrete signals. To reduce the calculation time, the Fast Fourier Transform (FFT) is usually used, which allows performing the DFT with a relatively small number of operations [1-3]. DFT allows you to find parameters of harmonics with frequencies that are multiples of the fundamental frequency. To find the spectral components, the frequencies of which are between these harmonics, a large number of methods have been proposed, which can be conditionally divided into two groups: interpolation and correlation. Correlation methods provide greater accuracy, while interpolation methods allow obtaining a result with good accuracy using a relatively small number of operations. The maximum possible accuracy of estimating the parameters of spectral components can be determined using the Cramer-Rao inequality (Cramer-Rao lower boundary). An unbiased estimate, that reaches the lower Cramer-Rao bound, is called effective. It provides the smallest mean square error among the unbiased estimates and is called the minimum variance unbiased (MVU) estimate. Algorithms for obtaining an MVU estimate should estimate the parameters based on the maximum likelihood estimation (MLE) function [4]. It is known [5] that the maximum likelihood estimates are asymptotically consistent and efficient, that is, their variance are the same with the Cramer-Rao lower bound. The source [6] classifies existing estimation algorithms.

2. Statement of the problem
As a result of experiments presented in several publications [7, 8] and performed by the authors, the accuracy of estimating the amplitude of signal harmonics when using window functions is lower than the theoretical one, defined as the Cramer-Rao boundary. This effect was even observed in those cases when the methods of correlation analysis were used, which, as is known from radio engineering, implement reception according to the principle of maximum likelihood.

The paper investigates the causes of this phenomenon and implements a mathematical model for methods for evaluating harmonics, which makes it possible to determine the real accuracy of the amplitude found using these methods.

3. Mathematical model for estimating the accuracy of finding the amplitude of harmonics

In general, the Cramer-Rao boundary is defined as follows [5, 6]:

$$\text{var}(\theta) \geq \frac{1}{\text{I}(\theta)}$$

(1)

where $\theta$ is estimated parameter, $\text{var}(\theta)$ is the variance of an unbiased parameter estimate, $E$ is average value, $p(x; \theta)$ is likelihood function.

Formula (1) under the regularity condition of the likelihood function can be divided into parts and rewritten in the following form:

$$\text{var}(\theta) \geq \frac{1}{I(\theta)}$$

(2)

where $I(\theta) = -E\left[\frac{\partial^2 \ln p(x; \theta)}{\partial \theta^2}\right]$ is the amount of information on Fischer was obtained as a result of observation.

Formula (2) reveals the physical meaning of the Cramer-Rao lower boundary: the variance of the parameter estimate is inverse proportional to the amount of information obtained as a result of observation.

When observing a discrete signal overlaid with additive white Gaussian noise, formula (1) can be rewritten as:

$$\text{var}(\theta) \geq \frac{\sigma^2}{\sum_{n=0}^{N} \left(\frac{\partial x(n; \theta)}{\partial \theta}\right)^2}$$

(3)

where $\sigma$ is the standard deviation of the noise, $N$ is number of samples in a discrete signal, $x(n; \theta)$ is observed signal.

In our case, the observed signal is described by the following formula:

$$x_n = A \cos(2 \pi f n + \phi) + w_{\text{GN}}, n \in 0,\ldots,n-1$$

(4)

where $A$ is signal amplitude, $f$ is frequency, $\phi$ is phase, $w_{\text{GN}}$ is white Gaussian noise.

This signal contains three unknown values (amplitude, frequency, and phase). However, as shown in [5], provided that the signal frequency is not near frequencies equal to zero or the sampling frequency, the impact of the signal frequency and phase inaccuracies on the amplitude inaccuracy is negligible.

Neglecting the frequency and phase inaccuracies, we obtain the following formula for the variance of the estimate of the harmonic amplitude in additive white Gaussian noise:

$$I(A) = \sum_{n=0}^{N-1} \left(\frac{\partial x}{\partial A}\right)^2 = \frac{1}{\sigma^2} \sum_{n=0}^{N-1} \cos^2(2 \pi f n + \phi) \approx \frac{N}{2 \sigma^2}$$

(5)

After applying the window function, the signal will be rewritten as:
\[
x_n' = w_n \ast A \cos(2 \pi f_n + \phi) + w_n \ast \text{wgn}, n \in 0, ..., n-1
\]  
where \( w \) is window function.

Let us consider the terms in formula (6) separately. The term \( w_n \ast \text{wgn} \) will be a normally distributed random variable with the mean value of zero. Let's determine the variance of this random variable.

In all \( n \in 0, ..., n-1 \) the deviation from zero will be normally distributed with an average deviation equal to \( w_n \ast \sigma \). The mean square deviation at each point will be equal to \( w_n^2 \ast \sigma^2 \). Averaging over all \( n \) we obtain that the variance of the random variable \( w_n \ast \text{wgn} \) will be:

\[
\sigma^2 = \frac{\sum_{n=0}^{n-1} w_n^2}{N} \ast \sigma^2
\]

Now, to find the variance of the amplitude estimation, we can use formula (3) and substitute the new noise variance \( \sigma^2 \) into it.

Consider the denominator of the right side of the formula. To simplify the calculations, we will use the Cramer-Rao lower boundary for transformed parameters [5, 6]. If the value \( \alpha \), to be estimated is related to the value of \( \theta \), for which the Cramer-Rao lower boundary is known, by the relation \( \alpha = \alpha(\theta) \), then the Cramer-Rao boundary for the value of \( \alpha \) can be found by the formula:

\[
\var(\theta) \geq \frac{\left\langle \left( \frac{\partial \alpha}{\partial \theta} \right)^2 \right\rangle}{-E\left[ \frac{\partial^2 \ln L(\alpha, \theta)}{\partial \alpha \partial \theta} \right]}
\]

The multiplication of the signal by the window function results in the convolution of each harmonic of the signal with the spectrum of the window function. In the case of a single harmonic, its amplitude is multiplied by the zero harmonics of the window function. The zero harmonics of the signal is a constant component; its value for the window function can be found by the formula:

\[
\var(A) \geq \frac{2 \sigma^2}{N} \sum_{n=0}^{n-1} w_n^2
\]

From a practical point of view, it makes sense to isolate the second part from this formula. Let's introduce the term «window coefficient»:

\[
F_w = \frac{\sum_{n=0}^{n-1} w_n^2}{\sum_{n=0}^{n-1} w_n}
\]

This coefficient shows how much the estimate of the harmonic amplitude will deteriorate when using the window \( w \).

4. Experimental results

To check the correctness of the considered formulas, we simulated the measurement of the amplitude of harmonics in a single-tone signal. Formulas (7) (estimate of the variance of white noise after imposing a window on it) and (9) (estimate of the variance of the harmonic amplitude) were experimentally verified. For formula (9), two graphs were built – the variance versus the window and the noise level.

The dependence of the variance of a signal with noise on the variance of a signal with a superimposed window (Kaiser window with the kaiser_beta parameter) is shown in Figure 2.
The main part of the program for evaluating the variance of white noise after overlaying a window is shown below:

```python
window_tmp = np.kaiser(n_point + 1, kaiser_beta)
window = window_tmp[0:n_point]
window_mean_2 = np.sum([window[i]**2 for i in range(n_point)])/n_point

for i_var in range(n_var):
    for i_test in range(n_test):
        noise = np.random.normal(0, np.sqrt(variances[i_var]), n_point)
        real_variances[i_var] += np.var(noise * window)
real_variances /= n_test
```

**Figure 1.** Estimation of the variance of white noise after imposing a window on it.

Here `np` is the imported numpy library, `variances` – an source array white noise variances.

![Figure 1](image1.png)

**Figure 2.** Dependence of the variance of noise after overlaying the window on the window parameter.

The parameters `kaiser_beta=5` and `n_point = 1000` and `n_test = 2` were used. The solid line shows the calculation results, the dashed line shows the results of the simulation by formula (7).

![Figure 2](image2.png)

**Figure 3.** Dependence of the variance of the amplitude estimate on
Simulation results for other parameters are similar. The simulation of the amplitude estimation using different windows was modeled using the Kaiser window with different beta parameters. Window properties with different beta levels are comparable to properties of all other windows, so the results obtained are valid for all windows. The simulation results are shown in Figure 3.

```python
for i_betas in range(n_beta):
    kaiser_beta = betas[i_betas]
    window_tmp = np.kaiser(n_point + 1, kaiser_beta)
    window = window_tmp[0:n_point]
    window_spec = np.fft.fft(window) / n_point
    window_factors[i_betas] = np.sum(window) / n_point  # abs(window_spec[0])
    window_mean_2[i_betas] = np.sum([window[i] ** 2 for i in range(n_point)]) / n_point

for i_test in range(n_test):
    noise = np.random.normal(0, np.sqrt(variance), n_point)
    signal = amp * np.sin([2 * np.pi * freq * i / n_point + np.pi/4 for i in range(n_point)])
    signal_noise_win = (signal + noise) * window
    spectrum = 2 * np.fft.fft(signal_noise_win) / n_point
    amp_est = abs(spectrum[freq]) / window_factors[i_betas]
    errors_var[i_betas] += (abs(amp_est) - amp) ** 2
```

The solid line shows the calculations by formula (9), the dashed line shows the simulation results. As the number of tests increases, the lines become more and more the same.

The simulation results are shown in Figure 5.

```python
window_tmp = np.kaiser(n_point + 1, kaiser_beta)  # create symmetric window
window = window_tmp[0:n_point]
window_spec = np.fft.fft(window) / n_point
window_factors = np.sum(window)/ n_point  # abs(window_spec[0])
window_mean_2 = np.sum([window[i] ** 2 for i in range(n_point)]) / n_point

for i_var in range(n_var):
    for i_test in range(n_test):
        noise = np.random.normal(0, np.sqrt(variances[i_var]), n_point)
```

**Figure 4.** Amplitude estimation using different windows.

**Figure 5.** A variance of the amplitude estimate versus noise variance.

A program for simulating amplitude estimation at various noise levels is shown below:
```python
signal = amp * np.sin([2 * np.pi * freq * i / n_point for i in range(n_point)])
signal_noise_win = (signal + noise) * window
spectrum = 2 * np.fft.fft(signal_noise_win) / n_point
amp_est = abs(spectrum[freq]) / window_factors
errors_var[i_var] += (abs(amp_est) - amp) ** 2
errors_var /= n_test
```

**Figure 6.** Amplitude Estimates at Various Noise Levels.

The solid line shows the results of calculations by formula (9), the dashed line shows the simulation results, and the dotted line shows the Cramer-Rao boundary.

5. Experimental results

Estimating the accuracy of the results for data that are in themselves an estimate of the accuracy is rather difficult to perform. To check the accuracy and reliability of the results, all experiments were repeated with a different number of tests (the \( n_{\text{test}} \) value in the programs). In all three experiments, with an increase in the number of tests, the experimental curves smoothed out and became visually indistinguishable from the calculated lines.

The experiments were carried out with different input parameters (number of points, amplitude, frequency and phase of signals, noise variance, or window beta). In no case was the deviation of the experimental results from the results obtained by the proposed formula recorded.

Thus, the results of modeling the algorithm for estimating the amplitude of the harmonic in noise conditions with multiplied window function confirm the obtained formulas for estimating the variance of the estimate of the amplitude.

6. Conclusion

In this work, a formula is obtained for estimating the variance of the amplitude when it is estimated using the optimal unbiased method using a window function.

This formula gives an estimate of the variance using the method that introduces the best-unbiased estimate. These methods include FFT-based correlation analysis, which is most often used in solving problems of spectral analysis.

The proposed formula can be used when choosing a window function for solving problems of spectral analysis. The accuracy of the resulting amplitude estimate is not the only criterion for this choice, but this criterion is often the most significant.

It can also be used at the stage of assessing the accuracy of the developed signal spectrum analysis algorithm, for estimating the noise level and in other related tasks.

The approach used in this work for finding the variance of the result of a signal processing algorithm can be applied to other signal processing tasks in which window functions or other preliminary signal transformations are used.

7. References

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