Four Dimensional Plane Wave String Solutions with Coset CFT Description

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Abstract

We present a number of $D = 4$ bosonic and heterotic string solutions with a covariantly constant null Killing vector which, like the solution of Nappi and Witten (NW), correspond to (gauged) WZW models and thus have a direct conformal field theory interpretation. A class of exact plane wave solutions (which includes the NW solution) is constructed by ‘boosting’ the twisted products of two $D = 2$ ‘cosmological’ or ‘black-hole’ cosets related to $(G \otimes G')/(H \otimes H')$ ($G, G’ = SL(2, \mathbb{R})$ or $SU(2)$; $H, H’ = SO(1, 1)$ or $SO(2)$) gauged WZW models. We describe a general limiting procedure by which one can construct new solutions with a covariantly constant null Killing vector starting with known string backgrounds. By applying a non-abelian duality transformation to the NW model we find a $D = 4$ solution which admits a covariantly constant null Killing vector but is not a plane wave. Higher dimensional bosonic backgrounds with isometries can be interpreted as lower dimensional backgrounds with extra gauge fields. Some of them are at the same time solutions of the heterotic string theory. In particular, the NW model represents also a $D = 3$ gravi-electromagnetic heterotic string plane wave. In addition to the (1,1) supersymmetric embeddings of bosonic solutions we construct a number of non-trivial (1,0) supersymmetric exact $D = 4$ heterotic string plane wave solutions some of which are related (by a boost and analytic continuation) to limiting cases of $D = 4$ heterotic black hole solutions.

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1. Introduction

In order to be able to study various issues (e.g. singularities) of gravitational physics in string theory at least to first order in the string coupling expansion one needs exact (in $\alpha'$) classical solutions of string equations which admit an explicit conformal field theory (CFT) interpretation. Knowing the corresponding CFT not only makes possible in principle to determine the operator content and scattering amplitudes but is also crucial in order to give an adequate ‘stringy’ space-time interpretation to a string solution which should go beyond the naive picture of a background parametrised just by a set of ‘massless’ fields $(G_{\mu\nu}, B_{\mu\nu}, \Phi)$.

The only two classes of exact classical solutions which are known at present in bosonic string theory are the plane-wave type backgrounds (more generally, backgrounds with a covariantly constant null Killing vector and flat or ‘conformal’ transverse part, see, e.g. [1][2][3][4][5][6][7]) and the backgrounds obtained from gauged WZW models (see e.g. [8][9][10][11][12][13][14][15][16][17]). Only the latter class of solutions has a direct CFT interpretation (in terms of coset $G/H$ models [18][19]). It was recently understood [20] that the two classes actually intersect. As a result, some representatives of the first class can be given a CFT interpretation by identifying them with members of the second class related to non-semi-simple groups [20][21][22][23][24][25][1].

A subclass of backgrounds with a covariantly constant null Killing vector and flat transverse part (plane wave backgrounds) can be described by the following $\sigma$-model action ($i, j = 1, 2, \ldots, N$)

\[
S(u, v, x) = \frac{1}{\pi \alpha'} \int d^2z \left[ 2\partial u \bar{\partial} v + (g_{ij} + b_{ij})(u) \partial x^i \bar{\partial} x^j \right] - \frac{1}{8\pi} \int d^2z \sqrt{\gamma} R^{(2)}(u) \Phi(u),
\]

(1.1)

1 The basic solution of [20] is the WZW action for the non-semi-simple group $E_2^c$ – a central extension of the Euclidean group in two dimensions [26][27]. Further studies of this model can be found in [21][22][28]. Larger classes of WZW models based on non-semi-simple groups were constructed in [23] using a contraction method and directly in [24]. The explicit form of the action of these models can be found in [25] (see also [24]). A class of coset models obtained by a particular gauging of the general model of [23] and equivalent to non-abelian duals [29] of WZW actions was found in [28].
where the functions $g_{ij}, b_{ij}, \Phi$ are subject to the equation

$$g_{ij} \ddot{g}_{ij} - \frac{1}{2} g^{ij} g^{mn} \ddot{g}_{im} \ddot{g}_{jn} + \frac{1}{2} g^{ij} g^{mn} \dot{b}_{im} \dot{b}_{jn} + 2 \ddot{\Phi} = 0 ,$$

which expresses the condition of conformal invariance of (1.1) to all orders in $\alpha'$.2

Since one is ultimately interested in four dimensional solutions it is important to find which of the exact $D = N + 2 = 4$ backgrounds (1.1),(1.2) (in addition to the already known $D = 4$ model of [20]) admit direct CFT interpretation in terms of $G/H$ cosets.3

This is one of the aims of the present paper.

The idea we shall use is the following (see also [22]). We shall start with the known $D = 4$ solutions corresponding to gauged WZW models and having the structure

$$S(t, r, y) = \frac{1}{\pi \alpha'} \int d^2 z [-\partial t \bar{\partial} t + \partial r \bar{\partial} r + (G_{ij} + B_{ij})(t, r) \partial y^i \partial y^j]$$

$$- \frac{1}{8\pi} \int d^2 z \sqrt{\gamma} R^{(2)} \Phi'(t, r),$$

and identify the models in the class (1.1) which are related to (1.3) by performing a singular coordinate transformation and a rescaling of $\alpha'$

$$r = \epsilon v + u , \quad t = u , \quad y^i = \sqrt{\epsilon} x^i , \quad \alpha' \to \epsilon \alpha' , \quad \epsilon \to 0 ,$$

$$G_{ij} + B_{ij})(u, u) = (g_{ij} + b_{ij})(u) , \quad \Phi'(u, u) = \Phi(u) .$$

Since the $u, v$-term in the resulting action (1.1) is boost-invariant, we can also introduce a free parameter by rescaling $u$ in $g_{ij}, b_{ij}, \Phi$. We can also make a translation of $u$ by a

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2 The exactness of this condition is easy to understand by noting that higher-order corrections should depend on $\alpha'$ while $\alpha'$ in (1.1) can be given an arbitrary value by rescaling $v$ and $x^i$, i.e. by a coordinate transformation. That implies, in particular, that the central charge of the model should be equal to its free-theory value $c = D = N + 2$.

3 One can prove either by a direct computation [30] or by using a theorem in [31] that no WZW model based on a non-simple group [20][32], other than the one based on $E_6^\times [20]$, or its analytic continuations, can be constructed in four dimensions.

4 We are assuming that there are two commuting isometries (shifts of $y^i$). A number of such models are known (e.g. cosets of products of $SL(2, \mathbb{R})$ and $SU(2)$, see e.g. [11][14][15][16]) and we will mention them in the appropriate parts of the paper.
constant in one of two arguments of \( G, B, \Phi' \). As discussed in Appendix A similar limiting procedure can be applied to a more general class of \( \sigma \)-models (an example of such model will be considered in Section 5).

If the background fields in (1.3) satisfy the string equations (which are covariant relations) the background fields in (1.1) obtained by the (singular) coordinate transformation (1.4), (1.5) are guaranteed to satisfy the same equations, i.e. the conformal invariance condition (1.2). Note that to get the exact expressions for \( g, b, \Phi \) it is enough to start with just the leading-order (one-loop) expressions for \( G, B, \Phi' \): \( \alpha' \)-dependent terms in \( G, B, \Phi' \) present in the ‘conformal’ or ‘CFT’ scheme (the one in which the tachyon equation has the standard Klein-Gordon form with no \( \alpha' \)-corrections, see e.g. [33][34][35]) in any case would drop out in the limit \( \epsilon \to 0 \) in (1.4). Let us mention also that the \( O(2, 2) \) duality rotations of the models (1.3) (in the directions of the isometries \( y^i \)) directly correspond to the duality rotations of the models (1.1) [28].

Having identified a model (1.1) which originates from a model (1.3) we can now reconstruct a (non-semi-simple) coset CFT which is behind (1.1) by taking the corresponding limit in the (semi-simple) coset CFT corresponding to (1.3) [3].

The examples of \( D = 4 \) plane-wave backgrounds which admit an explicit coset CFT interpretation will be given in Sections 2 and 3. All of them can be constructed either directly (by various gaugings based on the \( E_2 \) WZW model of [20]) or by using the above limiting prescription. For simplicity, in some cases we shall start directly from an non-semi-simple coset CFT and only mention the original model which gives it in the singular limit, whereas in other cases we shall obtain the result via the limiting method and only mention the underlying CFT.

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\[ ^5 \text{The limit in (1.4) with } r, t, y \text{ identified with coset coordinates implies the corresponding limit in the coset current algebra construction (i.e. in the OPE's or in the Lie algebra relations) and vice versa. For the WZW models of } \frac{\mathbb{E}_8}{\mathbb{F}_4} \text{ and the coset models of } \frac{\mathbb{E}_8}{\mathbb{F}_4} \text{ such a relation has been found in [25].} \]
In Section 4 we shall study a number of $D \leq 6$ plane wave solutions in bosonic string theory which have explicit coset CFT counterparts. Since ‘extra’ dimensions will be compact and isometric the corresponding backgrounds can be given a Kaluza-Klein interpretation as $D \leq 4$ ‘plane wave with gauge field’ solutions. Some of these bosonic backgrounds can be also identified with $D \leq 4$ plane wave solutions in the heterotic string theory. We shall discuss a number of examples of such heterotic string solutions which are closely related to $D = 4$ black hole solutions (for a previous discussion of plane wave solutions in the heterotic string theory see [1][7]). In many cases the prescription (1.4) can be applied to obtain a plane wave solution from a ‘cosmological’ solution which is related by analytic continuation to the part of a black hole solution inside the horizon. As an example, we shall also consider a $D = 4$ bosonic plane wave related to a bosonic $D = 4$ ‘charged black hole’ background (with non-trivial axionic and electric charges) based on the coset $[SU(2) \otimes SL(2,IR) \otimes U(1)]/[U(1) \otimes U(1)]$.

In Section 5 we shall present another $D = 4$ solution which also admits an exact CFT description. It belongs to the general class of backgrounds with a covariantly constant null Killing vector but is not a plane wave. It can be obtained by a non-abelian duality transformation [29] (see also [37][38][39][25][40]) applied to the $SU(2)$ WZW model and a singular limit, or directly by the non-abelian duality transformation of the $E_2^2$ WZW model of [27].

Some concluding remarks will be made in Section 6. A generalisation of the limiting procedure (1.4),(1.5) leading to backgrounds with a covariantly constant null Killing vector will be given in Appendix A, where, as an example, we present a five dimensional plane wave solution related to the Schwarzchild black hole. In Appendix B we shall discuss a CFT description of a background which is dual to the flat $D = 2$ space and which represents the ‘transverse’ part of the solution of Section 5. In Appendix C we shall show that the charged black string model [41] can be interpreted as a $D = 3$ bosonic model corresponding to the $D = 2$ charged black hole solution [42] of the heterotic theory. This result will be used in Section 4.
2. Simplest examples: $D = 3$ gauged WZW and $D = 4$ WZW models

2.1. $D = 3$ coset models

To illustrate the above limiting prescription (1.4) let us first consider the $D = 3$ backgrounds (which form an obvious subset of $D = 4$ models with one ‘spectator’ dimension). Let us start with the flat space model (1.3) in cylindrical coordinates

$$S(t, r, y) = \frac{1}{\pi \alpha'} \int d^2z \left[ -\partial t \bar{\partial}t + \partial r \bar{\partial}r + r^2 \partial y \bar{\partial}y \right] - \frac{1}{8\pi} \int d^2z \, \sqrt{\gamma} R^{(2)} \Phi_0 .$$

(2.1)

The singular boost (1.4) gives the following flat background (called ‘null orbifold’ in [43])

$$S(u, v, x) = \frac{1}{\pi \alpha'} \int d^2z \left[ 2 \partial u \bar{\partial}v + u^2 \partial x \bar{\partial}x \right] - \frac{1}{8\pi} \int d^2z \, \sqrt{\gamma} R^{(2)} \Phi_0 .$$

(2.2)

If we start with the leading-order dual to (2.1)

$$S(t, r, y) = \frac{1}{\pi \alpha'} \int d^2z \left[ -\partial t \bar{\partial}t + \partial r \bar{\partial}r + r^{-2} \partial y \bar{\partial}y \right] - \frac{1}{8\pi} \int d^2z \, \sqrt{\gamma} R^{(2)} (\Phi_0 + 2 \ln r)$$

(2.3)

we get another (now curved) background in the class (1.1) which is the exact dual image [28] of (2.2)

$$S(u, v, x) = \frac{1}{\pi \alpha'} \int d^2z \left[ 2 \partial u \bar{\partial}v + u^{-2} \partial x \bar{\partial}x \right] - \frac{1}{8\pi} \int d^2z \, \sqrt{\gamma} R^{(2)} (\Phi_0 + 2 \ln u) .$$

(2.4)

These models can be considered as limiting cases of the ‘neutral black string’ one. If we start with the $SL(2, \mathbb{R})/U(1) \otimes \mathbb{R}$ model in a particular coordinate region

$$S(t, r, y) = \frac{1}{\pi \alpha'} \int d^2z [ -\partial t \bar{\partial}t + \partial r \bar{\partial}r + a^{-2} \tanh^2 ar \partial y \bar{\partial}y ]$$

$$- \frac{1}{8\pi} \int d^2z \sqrt{\gamma} R^{(2)} (\Phi_0 + 2 \ln \cosh ar)$$

(2.5)

which in the formal limit $a \to 0$ reduces to (2.1) (note that the zero central charge condition restricts the value of $a$: $\alpha' a^2 = (26 - D)/6$) we find

$$S(u, v, x) = \frac{1}{\pi \alpha'} \int d^2z \left[ 2 \partial u \bar{\partial}v + a^{-2} \tanh^2 au \partial x \bar{\partial}x \right]$$

$$- \frac{1}{8\pi} \int d^2z \, \sqrt{\gamma} R^{(2)} (\Phi_0 + 2 \ln \cosh au) .$$

(2.6)
Here $a$ is a true free parameter (it can be absorbed into a rescaling of $u, v$ and $x$). The $a \to 0$ limit of this model is the ‘null orbifold’ (2.2).

Starting with the $SL(2, \mathbb{R})/U(1) \otimes \mathbb{R}$ model in the dual to (2.7) region gives the model dual to (2.6) with $a^{-1} \tanh \to a \coth, \cosh \to \sinh$ (i.e. with $au \to au + i\pi/2$) which includes (2.4) as the $a \to 0$ limit. If instead of $SL(2, \mathbb{R})$ cosets we start with $SU(2)$ ones we find (2.6) with hyperbolic functions replaced by trigonometric ones, i.e. with $a \to ia$. The model (2.6) was obtained in [21] by direct gauging of the four-dimensional $E_5^2$ WZW model of [21].

We conclude that the $D = 3$ plane wave type models which have coset CFT interpretation are represented by

$$S(u, v, x) = \frac{1}{\pi \alpha'} \int d^2 z \left[ 2 \partial u \bar{\partial} v + g(u) \partial x \bar{\partial} x \right] - \frac{1}{8\pi} \int d^2 z \sqrt{\gamma R^{(2)}} [\Phi_0 + 2 \ln f(u)], \quad (2.7)$$

where the functions $g(u)$ and $f(u)$ can take the following pairs of values

$$g(u) = 1, \quad u^2, \quad \tanh^2 u, \quad \tan^2 u, \quad u^{-2}, \quad \coth^2 u, \quad \cot^2 u,$$

$$f(u) = 1, \quad 1, \quad \cosh u, \quad \cos u, \quad u, \quad \sinh u, \quad \sin u. \quad (2.8)$$

The first two cases here correspond to flat spaces. The free parameter $a$ can be reintroduced by shifting $u \to au$ in (2.8). Applying the limiting procedure similar to (1.4) directly to $SL(2, \mathbb{R})$ WZW model does not give a new $D = 3$ model: one finds a background with a degenerate metric (i.e. effectively a two-dimensional space). This is consistent with the absence of non-semi-simple $D = 3$ Lie algebras with a non-degenerate invariant bilinear form [30] [31].

2.2. $D = 4$ WZW models

Let us now turn to $D = 4$ models. It was shown in [22] that the first non-trivial example [20] can be obtained by taking the singular limit (1.4) of the WZW action for $SU(2)_k \otimes \mathbb{R}$. Indeed, if we parametrize the $SU(2)$ group element as

$$g = e^{i\alpha_L \phi} e^{i\alpha_R \phi} e^{i\alpha_L \theta_L} e^{i\alpha_R \theta_R} \quad (2.9)$$
and the translational factor $\mathbb{R}$ in terms of the time-like coordinate $t$, then the WZW action is given by (we omit the constant dilaton term)

$$S = \frac{k}{4\pi} \int d^2 z \left( -\partial t \partial \bar{t} + \partial \phi \partial \bar{\phi} + \partial \theta_L \partial \bar{\theta}_L + \partial \theta_R \partial \bar{\theta}_R + 2 \cos \phi \partial \theta_L \partial \bar{\theta}_R \right).$$

(2.10)

This action has the same form as (1.3) with $r = \phi$, $y^i = (\theta_L, \theta_R)$, $\alpha' = 4/k$. After making the redefinition (1.4),(1.5) and taking the limit $\epsilon \to 0$ the action (2.10) becomes identical to the action for the $E_c^2$ WZW model \cite{20} ($\alpha' = 2$)

$$I_0(g) = \frac{1}{2\pi} \int d^2 z \left[ 2 \partial v \partial \bar{u} + \partial x_1 \partial \bar{x}_1 + \partial x_2 \partial \bar{x}_2 + 2 \cos u \partial x_1 \partial \bar{x}_2 \right],$$

(2.11)

which is a particular member of the class (1.1).

Similar model (related by analytic continuation $v \to -iv, u \to iu$) is obtained by starting with the WZW action for $SL(2, \mathbb{R})_k \otimes U(1)$ where the translational factor now is space-like (coordinate $r$) and the role of time is played by the non-compact coordinate ($i\phi$) in the analog of (2.3) for $SL(2, \mathbb{R})$. We finish with (2.11) with cos $u$ replaced by cosh $u$, and the corresponding symmetry group is $Pc^2$ (i.e. the extension of the Poincaré group in two dimensions). Now, however, there is an extra time-like direction in the $(x_1, x_2)$ plane so that this model is not interesting by itself (but it is useful as a starting point for constructing cosets, see subsec. 3.1). Let us also note that (2.11) can be obtained by starting with the $SL(2, \mathbb{R})_{-k} \otimes U(1)$ action in a particular patch (i.e. with (2.10) with $\theta_L \to i\theta_L, \theta_R \to i\theta_R$ and $k \to -k$).

Other $D = 4$ models of the type (1.1) can be found by applying abelian $O(2, 2)$ duality rotations to the $SU(2)_k \otimes \mathbb{R}$ (or $SL(2, \mathbb{R})_{-k} \otimes U(1)$) WZW model and taking the singular limit (1.4). The resulting models will be duality rotations of (2.11). Since the abelian duality transformations of a WZW model for a group $G$ can be represented \cite{14} as $(G \otimes H)/H$ gauged WZW models we are guaranteed to have a CFT description for the resulting backgrounds. Other non-trivial examples \cite{28} are obtained by starting from the $O(2, 2)$ duals of $SU(2)$ or $SL(2, \mathbb{R})$ WZW models equivalent to $[SU(2) \otimes U(1)]/U(1)$ or
[\text{SL}(2, \mathbb{R}) \otimes \mathbb{R}] / \mathbb{R}$ gauged WZW models (the latter is the charged black string model of [11]). We shall discuss the CFT interpretation of the corresponding plane-wave solutions explicitly in the next section.

We can also start with a product of two $D = 2$ cosets $G/H \otimes G'/H'$ (where $G, G' = \text{SL}(2, \mathbb{R})$ or $\text{SU}(2)$ and $H, H' = \mathbb{R}$ or $U(1)$) with signature $(-+++)$, i.e. with the model of the type ([1.3]). More general models correspond to its $O(2, 2)$ dual versions (see, e.g., [11] [16] [15] [47] [48]), i.e. the ‘twisted’ gauged WZW models for $(G \otimes G')/H \otimes H'$. The resulting models ([1.4]) will also be discussed in Section 3.

Finally, there is a possibility to first apply the non-abelian duality transformation [29] [38] to $\text{SU}(2)$ or $\text{SL}(2, \mathbb{R})$ WZW models and then consider the limit analogous to (1.4) (see Appendix A). The resulting model will have a covariantly constant null Killing vector but will not belong to the class ([1.4]) and will be discussed in detail in Section 5.

3. $D = 4$ plane wave solutions corresponding to gauged WZW models

In this section we shall give explicit examples of four dimensional plane-wave type solutions with a particular emphasis on their description in terms of CFT’s. All of them will correspond to gauged WZW models based on non-semi-simple groups.

3.1. $D = 4$ plane-wave solution related to ‘black string’ background

Let start with the current algebra for the direct product theory $\text{SU}(2)_k \otimes \text{U}(1) \otimes U'(1)$. We will perform a contraction [23] with respect to the generator $J_3 \in \text{su}(2)$ and $J'_0 \in u'(1)$ and gauge the abelian symmetry generated by the linear combination $J_1 \in \text{su}(2)$ and $J_0 \in u(1)$. The result of the contraction is the direct product of the current algebra for $E^\mathbb{C}_2$ [20] and a $U(1)$ factor

$$
JP_i \sim \frac{\epsilon_{ij} P_j}{z - w}, \quad P_i P_j \sim \frac{\epsilon_{ij} F}{z - w} + \frac{\delta_{ij}}{(z - w)^2} \quad (i = 1, 2)
$$

$$
JF \sim \frac{1}{(z - w)^2}, \quad P_0 P_0 \sim \frac{1}{(z - w)^2}.
$$

(3.1)
Let us consider now the coset theory \([E_2^c \otimes U(1)]/U(1)\) with the stress tensor given by

\[
T = \frac{1}{2} : (P_i^2 + 2JF - F^2) : + \frac{1}{2} : P_0^2 : - \frac{(P_1 + q_0 P_0)^2}{2(1 + q_0^2)},
\]

(3.2)

where the constant \(q_0\) parametrizes the embedding of \(H = U(1)\) into \(G = E_2^c \otimes U(1)\).

In order to find the \(\sigma\)-model corresponding to the above coset CFT we parametrize the group element \(g \in G\) as

\[
g = \text{diag}(g_1, g_2), \quad g_1 = e^{ix_1 P_1} e^{iu J} e^{ix_2 P_1} e^{iv F} \in E_2^c, \quad g_2 = e^{i \phi P_0} \in U(1).
\]

(3.3)

The action of the \([E_2^c \otimes U(1)]/U(1)\) gauged WZW model reads

\[
S = I_0(g_1) + \frac{1}{\pi} \int d^2z \left[ \frac{1}{2} \partial \phi \bar{\partial} \phi + iA (\sqrt{q_0} \bar{\partial} \phi + \bar{\partial} x_1 + \cos u \bar{\partial} x_2) \\
- i \bar{A} (\sqrt{q_0} \partial \phi + \partial x_2 + \cos u \partial x_1) + A \bar{A} (1 + \cos u + 2q_0) \right]
\]

(3.4)

where the action \(I_0(g_1)\) for the \(E_2^c\) WZW model is given in (2.11). The gauge transformations that leave (3.4) invariant are

\[
\delta x_1 = \delta x_2 = \nu, \quad \delta u = \delta v = 0, \quad \delta \phi = 2\sqrt{q_0} \nu, \quad \delta A = i \partial \nu, \quad \delta \bar{A} = -i \bar{\partial} \nu.
\]

(3.5)

After fixing the gauge \(\phi = 0\), integrating over the gauge fields and changing the variables

\[
x_1 \to \frac{1}{2} \left( \frac{x_1}{\sqrt{q_0}} + \frac{x_2}{\sqrt{1 + q_0}} \right), \quad x_2 \to \frac{1}{2} \left( \frac{x_1}{\sqrt{q_0}} - \frac{x_2}{\sqrt{1 + q_0}} \right)
\]

(3.6)

we obtain the \(\sigma\)-model action (1.1) with the background fields given by (we rescale \(u \to 2u, \ v \to v/2\))

\[
ds^2 = 2dvdu + \frac{\cos^2 u}{q_0 + \cos^2 u} \, dx_1^2 + \frac{\sin^2 u}{q_0 + \cos^2 u} \, dx_2^2
\]

\[
b_{12} = \frac{\sqrt{q_0(q_0 + 1)}}{q_0 + \cos^2 u}
\]

\[
\Phi = \ln(q_0 + \cos^2 u) + \text{const.}
\]

(3.7)

For \(q_0 = 0\) the above solution reduces to the direct product of the coset model \(E_2^c/U(1)\) of [21] and a free boson, or, equivalently, to a particular case of (2.7),(2.8) with an extra
free dimension. The background (3.7) with \(q_0 \neq 0\) can be found from the latter one by a duality transformation.

The solution (3.7) can be also obtained by applying the limiting procedure of Section 1 to the direct product theory of the ‘Euclidean’ black string model [41] and a translation, i.e. \([SU(2) \otimes U(1)]/U(1) \otimes \mathbb{R}\).

The analytic continuation \(u \rightarrow iu, v \rightarrow -iv, x_2 \rightarrow ix_2\) and \(q_0 \rightarrow -q_0\) is changes \(E_2^c\) into \(P_2^c\) and the compact \(U(1)\) factors into the non-compact \(SO(1,1)\) ones. The trigonometric functions are then replaced by their hyperbolic counterparts and the background becomes asymptotically (for large \(u\)) Minkowski with zero antisymmetric tensor and linear dilaton (\(\Phi\) depends only on \(u\) and thus does not contribute to the central charge).

A number of related solutions can be found by taking various limits of the parameters in (3.7) and making analytic continuations. For example, the special limit of (3.7) : \(q_0 = -1 - q^2 \epsilon, u \rightarrow \sqrt{\epsilon} u, x_1 \rightarrow i \sqrt{\epsilon} x_1, x_2 \rightarrow ix_2\) gives the following interesting solution

\[
d s^2 = 2 d v d u + \frac{1}{u^2 + q^2} d x_1^2 + \frac{u^2}{u^2 + q^2} d x_2^2
\]
\[
b_{12} = \frac{q}{u^2 + q^2}
\]
\[
\Phi = \ln(u^2 + q^2) + \text{const.}
\]

It can be obtained by gauging other combinations of currents in the direct product \(E_2^c \otimes U(1)\), or by applying a limiting procedure similar to (1.4) to the case of the direct product of \(E_2^c/U(1)\) model (see eq.(2.8) of [22]), and a translation \(\mathbb{R}\), or by applying a duality (combined with a coordinate) transformation to Minkowski space [22]. The direct product models based on (2.7),(2.8) with \(g(u) = u^2\) or \(u^{-2}\) are special limits of this solution.

3.2. \(D = 4\) plane-wave solutions related to twisted products of two \(D = 2\) cosets

Let us start with the current algebra for the direct product \(SU(2)_k \otimes SL(2, \mathbb{R})_{-k'}\). The OPE’s for the currents \(\{I_3, I_i\} \in su(2)_k\) \((i = 1, 2)\) are

\[
I_i I_j \sim \frac{i \delta_{ij} I_3}{z - w} + \frac{k \delta_{ij}}{2(z - w)^2}, \quad I_3 I_i \sim \frac{i \delta_{ij} I_j}{z - w}, \quad I_3 I_3 \sim \frac{k}{2(z - w)^2},
\]
and for \( \{J_3, J_i\} \in \mathfrak{sl}(2, \mathbb{R})_{-k'} \ (i = 1, 2) \) are

\[
J_i J_j \sim \frac{i \epsilon_{ij} J_3}{z - w} + \frac{k' \delta_{ij}}{2(z - w)^2}, \quad J_3 J_i \sim \frac{-i \epsilon_{ij} J_3}{z - w}, \quad J_3 J_3 \sim \frac{-k'}{2(z - w)^2}.
\] (3.10)

The stress energy tensor of the direct product theory and the corresponding central charge are

\[
T = \frac{I_i^2 + I_3^2}{k + 2} + \frac{J_i^2 - J_3^2}{k' - 2}, \quad c = \frac{3k}{k + 2} + \frac{3k'}{k' - 2}.
\] (3.11)

We shall do a contraction with respect to the generators \( I_3 \in su(2) \) and \( J'_3 \in \mathfrak{sl}(2) \). We define a new basis \( \{I_3, I_i, J_3, J_i\} \rightarrow \{J, F, P^\alpha_i\} \), \( (i, \alpha = 1, 2) \) as

\[
J = I_3 + a J_3, \quad F = \epsilon(I_3 - a J_3), \quad P^1_i = \sqrt{2} \epsilon I_i, \quad P^2_i = \sqrt{2} \epsilon a J_i
\]

\[
k = \frac{1}{\epsilon}, \quad k' = \frac{1}{a^2 \epsilon}, \quad a^2 = \frac{k}{k'},
\] (3.12)

where \( a \) is a free parameter. Then in the singular limit the OPE’s of the resulting six dimensional current algebra (which will be denoted by \( su(2)_{\frac{3}{2}} \)) are

\[
J P^\alpha_i \sim \frac{i \epsilon_{ij} \lambda_\alpha P^\alpha_j}{z - w}, \quad P^\alpha_i P^\beta_j \sim \frac{i \delta_{\alpha \beta} \epsilon_{ij} \lambda_\alpha F}{z - w} + \frac{\delta_{\alpha \beta} \delta_{ij}}{(z - w)^2}, \quad JF \sim \frac{1}{(z - w)^2}.
\] (3.13)

where \( \lambda_\alpha = (1, a) \). The corresponding stress tensor and central charge are

\[
T = \frac{1}{2} : [P^\alpha_i P^\alpha_i + 2JF - (1 + a^2)F^2] : , \quad c = 6.
\] (3.14)

Let us note that, our contraction, though similar in spirit, differs from that of \([23]\). In particular, notice the presence of the ‘contraction’ parameter \( a \). To find the corresponding \( \sigma \)-model we parametrize the group element as

\[
g = e^{i P^\alpha_i x^\alpha_i} e^{iuJ} e^{ix^\alpha_1 P^\alpha_1} e^{ivF}.
\] (3.15)

In the case of \( a = 1 \) (‘diagonal’ contraction) the algebra (3.13) can be also obtained by contracting the \( so(4) \) algebra as in \([23]\) with respect to an \( so(2) \) subalgebra and neglecting one free field (see \([24]\), App.C). In the notation of \([25]\) \( su(2)_{\frac{3}{2}} \simeq so^*(4)_{so(2)} \).
Then the WZW action reads\footnote{Throughout this paper we shall use the notation $u' = a + d$, where $a$ and $d$ are constant free parameters. This action is conformal for arbitrary values of $a$ and $d$, being a solution of (1.2). While the origin of $a$ is clear from the current algebra (3.12) this is not so for the parameter $d$. In fact, when $d \neq 2\pi n$, $n \in \mathbb{Z}$ the corresponding CFT is not the current algebra (3.13) for $SU_2^c$. To understand the origin of the free parameter $d$ one should start with the eight dimensional WZW action for the direct product $E_8^c \otimes E_8^c$ and gauge the subgroup generated by $F_1 + F_2$. Because this subgroup is nilpotent there will be a constraint condition to be solved (besides the usual gauge fixing). Its most general solution will contain a parameter related to $d$.}

$$I_0(g) = \frac{1}{2\pi} \int d^2z \left[ 2\partial v \bar{\partial} u + \partial x_1^\alpha \bar{\partial} x_1^\alpha + \partial x_2^\alpha \bar{\partial} x_2^\alpha + 2\cos u \partial x_1^1 \bar{\partial} x_2^1 + 2\cos u' \partial x_1^2 \bar{\partial} x_2^2 \right]. \quad (3.16)$$

To obtain a four-dimensional model we should gauge a two-dimensional subgroup. We choose to gauge a symmetry that acts in a left-right asymmetric fashion \footnote{While the origin of $a$ is clear from the current algebra (3.12) this is not so for the parameter $d$. In fact, when $d \neq 2\pi n$, $n \in \mathbb{Z}$ the corresponding CFT is not the current algebra (3.13) for $SU_2^c$. To understand the origin of the free parameter $d$ one should start with the eight dimensional WZW action for the direct product $E_8^c \otimes E_8^c$ and gauge the subgroup generated by $F_1 + F_2$. Because this subgroup is nilpotent there will be a constraint condition to be solved (besides the usual gauge fixing). Its most general solution will contain a parameter related to $d$.} on the group element, i.e. $\delta g = i\nu_\alpha (t_1^\alpha g - g t_1^\alpha), \nu_\alpha = (\nu_1, \nu_2)$, where

$$t_1^1 = P_1^1 + qP_1^2, \quad t_2^1 = -qP_1^1 + P_1^2, \quad t_1^2 = P_1^1 - qP_1^2, \quad t_2^2 = qP_1^1 + P_1^2, \quad (3.17)$$

with $q$ being a ‘twisting’ parameter. Then the gauged WZW action becomes (we suppress the indices $(\alpha, \beta)$)

$$S = I_0(g) + \frac{1}{\pi} \int d^2z [iA(m_1(q)\bar{\partial} x_1 + m_2(q)\bar{\partial} x_2) - i\bar{A}(m_2(-q)\partial x_1 + m_1(-q)\partial x_2) + AM\bar{A}] \quad (3.18)$$

$$m_1(q) = \begin{pmatrix} 1 & q \\ -q & 1 \end{pmatrix}, \quad m_2(q) = \begin{pmatrix} \cos u & q \cos u' \\ -q \cos u & \cos u' \end{pmatrix}, \quad M = \begin{pmatrix} (\cos u - 1) - q^2(\cos u' + 1) & q(\cos u + \cos u') \\ -q(\cos u + \cos u') & (\cos u' - 1) - q^2(\cos u + 1) \end{pmatrix}. \quad (3.19)$$

The gauge transformations that leave (3.19) invariant are

$$\delta x_1 = -m_1(-q)\nu, \quad \delta x_2 = m_1(q)\nu, \quad \delta u = \delta v = 0, \quad \delta A = -i\partial \nu, \quad \delta \bar{A} = -i\bar{\partial} \nu.$$
Fixing the gauge as $x_2^* = 0$ and integrating out the gauge fields gives (we change the coordinates $x_1^\alpha \to m_1(q)x_1^\alpha$, rename $x_1^\alpha \to x_\alpha$ and rescale $u \to 2u$, $v \to v/2$)

$$ds^2 = 2dvdu + \Delta^{-1} (\cos^2 u \sin^2 u' \, dx_1^2 + \cos^2 u' \sin^2 u \, dx_2^2)$$

$$b_{12} = q \Delta^{-1} \cos^2 u \cos^2 u' , \quad \Phi = \ln \Delta + \text{const.} \quad (3.20)$$

$$\Delta \equiv \sin^2 u \sin^2 u' + q^2 \cos^2 u \cos^2 u' , \quad u' \equiv au + d .$$

Notice that for zero twisting $q = 0$ (3.20) corresponds to a plane wave which can be obtained from the direct product $SU(2)/U(1) \otimes SL(2, \mathbb{R})/\mathbb{R}$ by performing the singular coordinate transformation (1.4),(1.5). The solution (3.20) can be obtained from that direct product model by performing a one-parameter $O(2,2)$ duality transformation, or by ‘boosting’ according to (1.4) the backgrounds of [14] [15] [47]. Note that the solution (3.7) can be obtained from (3.20) after we rescale $x_i \to \sqrt{2(q^2 - 1)} x_i$, set $d = \pi/4$, $q_0 = 1/(q^2 - 1)$ and take the limit $a \to 0$.

The background (3.20) and its various analytic continuations and limits can be represented also in the following general form

$$ds^2 = 2dvdu + \frac{g_1(u')}{g_1(u')g_2(u) + q^2} \, dx_1^2 + \frac{g_2(u)}{g_1(u)g_2(u') + q^2} \, dx_2^2$$

$$b_{12} = \frac{q}{g_1(u')g_2(u) + q^2}$$

$$\Phi = \ln \left(g_1^2(u')f_1^2(\bar{u}')f_2^2(u)[g_1(u')g_2(u) + q_0^2] \right) + \text{const.} \quad (3.21)$$

where the functions $g_i, f_i$ can take any pairs of values in (2.8) (3.21) is related by (1.4) to a similar background in [17]). In particular, the solution (3.7) is a special case of (3.21) with $g_1 = 1$, $g_2 = \tan^2 u$. Also, (3.8) is a special case of (3.21) with $g_1 = 1$, $g_2 = u^2$, i.e. the background (3.8) is actually dual to the flat space.

4. Bosonic and heterotic plane wave solutions with abelian gauge fields

In this section we shall consider a number of $D \leq 6$ plane wave solutions in bosonic string theory which have explicit coset CFT counterparts. If ‘extra’ dimensions are compact and represent isometric directions the corresponding backgrounds can be given a
Kaluza-Klein interpretation as $D \leq 4$ plane waves with extra gauge fields. Some of these solutions can be also identified with $D \leq 4$ plane wave solutions in the heterotic string theory. Since the special heterotic plane wave backgrounds we shall find below will be related to the exact bosonic plane wave solutions, they will represent the exact (all order in $\alpha'$) solutions of the heterotic string theory. In addition to the obvious $(1,1)$ supersymmetric versions of the solutions of the previous sections we shall find non-trivial ‘asymmetric’ solutions with $(1,0)$ world sheet supersymmetry.

4.1. General remarks

As in Sections 2,3 one can construct bosonic plane wave solutions by starting with the known $G/H$ gauged WZW models and applying the limiting procedure (1.4),(1.5). If one is interested in the dimensions of the resulting $\sigma$-model configuration space being $D = 5, 6$ (with one time-like direction) a few simplest choices for $G/H$ are again products of cosets of $SL(2, \mathbb{R}),$ $SU(2)$ and $U(1)$’s and various ‘twisted’ gaugings (duality rotations), e.g., for $D = 5 :$ 

$\quad SL(2, \mathbb{R})/U(1) \otimes SU(2),$ 

$\quad [SL(2, \mathbb{R}) \otimes SU(2) \otimes U(1)]/[U(1) \otimes U(1)],$ 

and for $D = 6 :$ 

$\quad SL(2, \mathbb{R}) \otimes SU(2),$ 

$\quad [SL(2, \mathbb{R}) \otimes SU(2) \otimes U(1)]/U(1),$ 

$\quad [SL(2, \mathbb{R}) \otimes SU(2) \otimes U(1)])/[U(1) \otimes U(1)],$ 

... . The conformal $\sigma$-models obtained from such gauged WZW models can be represented as

$$S = \frac{1}{\pi\alpha'} \int d^2z[(G_{\mu\nu} + B_{\mu\nu})(x)\partial x^\mu \partial x^\nu + A_{n\mu}(x)\partial x^\mu \partial Z^n + \bar{A}_{n\mu}(x)\partial x^\mu \partial \bar{Z}^n + G_{mn}(x)\partial Z^m \partial Z^n ] - \frac{1}{8\pi} \int d^2z \sqrt{\gamma} R^{(2)} \Phi(x),$$

(4.1)
i.e. in the form of an action of a bosonic string propagating in $D = 4+r$ space-time $(x^\mu, Z^m)$ $(\mu, \nu = 0, 1, 2, 3, \ m, n = 1, ..., r)$ with $G_{\mu\nu} = \frac{1}{2}(A_{n\mu} + \bar{A}_{n\mu})$ and $B_{\mu\nu} = \frac{1}{2}(A_{n\mu} - \bar{A}_{n\mu}).$ Since the background fields do not depend on the coordinates $Z^n$ one may interpret such a model in a Kaluza-Klein manner as a $D = 4$ background with non-trivial gauge (and scalar) fields. One can represent (4.1) in the form which is invariant under the the space-time gauge transformations of $A^{n(+)}_{\mu}, A^{n(-)}_{\mu}$ (see below) if one also shifts $Z^n$ and makes the
‘anomalous’ transformations of $B_{\mu\nu}$

\[
S = \frac{1}{\pi \alpha'} \int d^2 z \left[ (G_{\mu\nu}^{(4)} + B_{\mu\nu})(x) \partial x^\mu \partial x^\nu + Z^n \mathcal{F}_{n\mu\nu}(x) \partial x^\mu \partial x^\nu \\
\qquad + G_{nm}(x) (\partial Z^n + A_{\mu}^{n(+)}(x) \partial x^\mu)(\partial Z^m + A_{\mu}^{m(+)}(x) \partial x^\mu) \right] \tag{4.2}
\]

\[- \frac{1}{8\pi} \int d^2 z \sqrt{\gamma} R^{(2)} \Phi(x),\]

where

\[
A_{\nu\mu}^{(\pm)} \equiv \frac{1}{2} (A_{\nu\mu} \pm \bar{A}_{\nu\mu}), \quad A_{\mu}^{m(+)} \equiv G^{mn} A_{\nu\mu}^{(+)}, \quad \mathcal{F}_{n\mu\nu} \equiv \partial_\nu A_{n\mu}^{(-)} - \partial_\mu A_{n\nu}^{(-)}
\]

and

\[
G_{\mu\nu}^{(4)} = G_{\mu\nu} - G_{mn} A_{\mu}^{m(+)} A_{\nu\mu}^{(+)}
\]

is the gauge-invariant $D = 4$ ‘Kaluza-Klein’ metric.

One can also try to interpret (4.1) (e.g., with $\bar{A}_\mu = 0$) as a bosonic part of the heterotic string action coupled to $A_{\nu\mu}$ with $Z^n$ being now chiral scalars representing the ‘right’ internal sector of the heterotic string (see, e.g., [49]. In this case, however, it is not a priori clear why the full action (with the additional ‘left’ fermionic terms implied by (1,0) supersymmetry) should also be conformally invariant, i.e. should represent a solution of the heterotic string theory.

In general, there are several ways to construct heterotic string solutions related to gauged WZW models. One can start with a bosonic coset, consider it formally as a solution of the superstring theory (conformal (1,1) supersymmetric model) and embed this superstring solution into the (1,1) supersymmetric subset of solutions of the heterotic string theory by adding the gauge field background equal to the (generalised) Lorentz connection (see [50] and refs. there). In this way one cancels the $2d$ gauge anomaly which otherwise makes direct (1,0) supersymmetric version of $G/H$ gauged WZW model inconsistent [50] (for a discussion of a potential unitarity problem and some earlier refs. see also [51]).
solution \([9]\) of the bosonic string theory one needs to add a \(U(1)\) gauge field background

\[
ds^2 = dr^2 + a^{-2} \tanh^2 ar \, dy^2 , \quad \Phi = \Phi_0 + \ln \cosh^2 ar , \quad A_y = - \frac{1}{\cosh^2 ar} . \quad (4.4)
\]

This solution can be related to the ‘charged \(D = 2\) black hole’ (or ‘neutral black string’) solution of \([52]\) (the bosonised form of the heterotic string action can be identified with the model \([4.1]\) where \(D = 3\), \(r = 1\), \(G_{mn} = 1\) and \(\tilde{A}_\mu = -A_\mu\) so that the Kaluza-Klein redefinition of the metric \((4.3)\) is trivial, cf. \((C.1)\) with \(q = 0\)).

For example, the bosonic plane wave solutions of Sects. 2,3 are obviously also the solutions of the superstring theory and as such can be embedded into the set of solutions of the heterotic string theory by supplementing them with gauge field backgrounds equal to the values of the Lorentz connection.

Another possibility is to consider directly the \((1,0)\) supersymmetric extension of the group \(G\) WZW theory or, more generally, of the chiral gauged \(G/H\) WZW theory \([53]\) (with the dimension of the configuration space being still equal to \(\dim G\) and thus the \(2d\) anomaly being harmless). One can also construct ‘heterotic cosets’ by starting with the \((1,0)\) supersymmetric version of anomalously gauged \((1,0)\) supersymmetric WZW model and adding internal ‘right’ fermions in order to cancel the total \(2d\) gauge anomaly \([54]\). Once bosonised, the total heterotic string action should be equivalent to a particular bosonic anomaly-free gauged WZW model \([55][54][54]\).

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8 The \(G/H\) chiral gauged WZW model with an abelian subgroup \(H\) is equivalent to a particular \((G \otimes H)/H\) gauged WZW model, or to a particular duality rotation of \(G\) WZW model \([53]\). Analogous relation is probably true also in the case of a non-abelian \(H\).

9 The bosonic version of the action of ‘\(S^2\) plus monopole’ solution of \([55]\) can thus be essentially identified \([55][54]\) with the action of \(SU(2)\) WZW model (the original suggestion to reinterpret a group \(G\) WZW action as a sigma model on \(G/H\) with an \(H\)- gauge field background was made in \([50]\), see also \([57]\)). For other examples see \([54][58]\). In particular, the bosonic action of the \([SL(2, \mathbb{R}) \otimes SU(2)]/[U(1) \otimes U(1)]\) heterotic model of \([54]\) can be also considered as the action of a \([SL(2, \mathbb{R}) \otimes SU(2) \otimes U(1) \otimes U(1)]/[U(1) \otimes U(1)]\) bosonic gauged WZW model.
Thus in order to interpret a particular bosonic $G/H$ coset as representing a heterotic string solution one should be able to identify the corresponding sigma model \((4.1)\) with the full bosonised action of the heterotic string in a given background \([53,54]\) (for some related earlier refs. see \([59]\)). Namely, \((4.2)\) should be equivalent to a bosonised version of a special case of the heterotic string action \([60]\)

\[
S = \frac{1}{\pi \alpha'} \int d^2z \left[ (G^{(4)}_{\mu\nu} + B_{\mu\nu})(x) \partial x^\mu \bar{\partial} x^\nu + i\bar{\psi}_a \left[ \delta^a_b \bar{\partial} + \omega^a_{b\mu}(x) \bar{\partial} x^\mu \right] \psi^b + i\bar{\Psi}_I \left[ \delta^I_J \partial + A^I_{J\mu}(x) \partial x^\mu \right] \Psi^J + F^I_{Jab}(x) \bar{\Psi}_I \psi^a \bar{\psi}^b \right].
\]

The $D = 4$ heterotic string background is then given by $G^{(4)}_{\mu\nu}, B_{\mu\nu}, A_{n\mu}$ and $\Phi$. Note that the scalar background $G_{mn}(x)$ (which is present in the corresponding bosonic Kaluza-Klein solution) is absent in the heterotic action; it appears only after \((4.5)\) is bosonised and put in the form of \((4.1)\) (see also \([54]\)).

4.2. Heterotic string plane wave solutions

Let us start with $D = 3$ examples, first reviewing how one can represent the $SU(2)$ WZW action in the form \((1.1)\), i.e. as a $\sigma$-model on $S^2$ with a $U(1)$ monopole coupling (see \([56,55,54]\)). Using the parametrisation \((2.9)\) one finds (cf. \((1.1),(1.3)\))

\[
S = \frac{k}{4\pi} \int d^2z \left( \partial \phi \bar{\partial} \phi + \partial \theta_L \bar{\partial} \theta_L + \partial \theta_R \bar{\partial} \theta_R + 2 \cos \phi \partial \theta_L \bar{\partial} \theta_R \right)
\]

\[
= \frac{k}{\pi} \int d^2z \left( \partial \phi' \bar{\partial} \phi' + \sin^2 \phi' \partial \varphi \bar{\partial} \varphi + \partial Z' \bar{\partial} Z' - 2 \sin^2 \phi' \partial \varphi \bar{\partial} Z' \right)
\]

\[
= \frac{k}{4\pi} \int d^2z \left[ \partial \phi \bar{\partial} \phi + (\sin^2 \phi + \frac{1}{4}A_\varphi \bar{A}_\varphi) \partial \varphi \bar{\partial} \varphi + \partial Z \bar{\partial} Z + A_\varphi \partial \varphi \bar{\partial} Z \right]
\]

where $\phi = 2\phi', \varphi = \theta_L, Z = 2Z' = \theta_L + \theta_R$ and

\[
ds^2 = d\phi^2 + \sin^2 \phi \, dx^2, \quad \Phi = \Phi_0, \quad A_x = 2(\cos \phi - 1).
\]

We are now able to interpret the $E_2^2$ WZW model \((2.11)\) of \([20]\) as a $D = 3$ heterotic string plane wave solution with a vector field.\(^{10}\) In fact, adding an extra time direction $t$

\(^{10}\) Since the internal coordinate should be periodic one should also factorize over a discrete subgroup, as in \([55]\). The same remark holds for all models in this section.
as in (2.10) and combining it with $\phi$ according to (1.4) we get another equivalent representation for (2.11) (cf. (4.1),(4.3))

$$S = \frac{k}{4\pi} \int d^2 z \left[ 2\partial \bar{v} u + \left( \sin^2 u + \frac{1}{4} A_x A_x \right) \partial x \partial \bar{x} + \partial Y \partial \bar{Y} + A_x(u) \partial x \partial \bar{Y} \right], \quad (4.8)$$
i.e. the following $D = 3$ background

$$ds^2 = 2dvdu + \sin^2 u \, dx^2, \quad \Phi = \Phi_0, \quad A_x = 2(\cos u - 1). \quad (4.9)$$

Analytic continuation gives related solution with $\sin \to -\sinh, \cos \to \cosh$ (which is found directly if one replaces the monopole model (4.6) by the $D = 2$ anti de Sitter one obtained from $SL(2, \mathbb{R})$ WZW theory [61]). We also get a $D = 3$ heterotic plane wave solution by ‘boosting’ (cf. (2.3),(2.4)) the product of the $D = 2$ black hole solution (4.4) with a time line.

Another $D = 3$ plane wave solution is obtained by starting with the $D = 2$ ‘charged black hole’ heterotic string solution (the solution (4.4) is related to a special case of this background) [42]

$$ds^2 = f^{-1} dr^2 - f dy^2, \quad f = 1 - 2me^{-Qr} + p^2 e^{-2Qr}, \quad (4.10)$$

$$\Phi = \Phi_0 + Qr, \quad A_y = -\sqrt{2} p e^{-Qr}.$$  

It can be given a world sheet interpretation in terms of the $D = 3$ bosonic conformal $\sigma$-model (4.1) by starting with the action of the $D = 3$ charged black string ($[SL(2, \mathbb{R}) \otimes U(1)]/U(1)$ coset) model [41] or by constructing $SL(2, \mathbb{R})$ ‘heterotic’ cosets [54] (related background with a non-constant scalar field can be interpreted a $D = 2$ bosonic Kaluza-Klein background [61]). Considering the region between the horizons and singularity one can represent (4.10) as a cosmological solution [62]

$$ds^2 = -dt^2 + g(t) \, dx^2, \quad g(t) = \frac{\cosh^2 bt \sinh^2 bt}{(\cosh^2 bt + \gamma \sinh^2 bt)^2}, \quad (4.11)$$

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\[\Phi(t) = \Phi_0' + \ln(\cosh^2bt + \gamma \sinh^2bt) , \quad A_x(t) = -\frac{\sqrt{-2\gamma}}{\left(\gamma + 1\right) (\cosh^2bt + \gamma \sinh^2bt)} ,\]

where \(x = i(1 + \gamma)y, \quad Q = -2ib, \quad p^2/m^2 = -4\gamma/(1 - \gamma)^2.\) Taking special limits and making analytic continuations one can relate (4.11) to the two other \(D = 2\) heterotic solutions (4.4) and (4.7) discussed above. The background (4.4) is reproduced when \(p = 0, \quad Q = \sqrt{2/\gamma}, \quad b = ia\) and \(\gamma \to 0\) (comparing (4.11) with (4.4) one needs to take into account a rescaling of the gauge potential due to a rescaling of coordinates). Another limit \(\gamma \to -1\) leads to the solution related to (4.7) by \(t \to i\phi\) (assuming one absorbs an infinite constant in the gauge potential into a gauge transformation and rescales \(k\) in (4.6)). This relation is explained in Appendix C where we show that the \(D = 3\) model representing the bosonised form of the heterotic solution (1.11) is equivalent to the \(D = 3\) charged black string [11] or \([SL(2, \mathbb{R}) \otimes U(1)]/U(1)\) gauged WZW model.

Adding a spatial direction to (4.11) and applying (1.4) we finish with the \(D = 3\) heterotic plane wave

\[ds^2 = 2dvdu + g(u) \, dx^2 , \quad \Phi = \Phi(u) , \quad A_x = A_x(u) , \quad (4.12)\]

which includes (4.9) as a particular case. Given the relation of Appendix C between (4.11) and the charged black string background one can also relate the corresponding plane wave solutions (1.12) and (3.7) in a similar way (namely, (3.7) is the \(D = 4\) bosonic background which is the bosonized version of the \(D = 3\) heterotic solution (4.12)).

We can now construct \(D = 4\) heterotic plane waves by starting with the direct products of \(D = 2\) heterotic solutions (4.4),(4.7),(4.11). Since, as was noted above, (1.11) is the most general solution, we get the most general ‘direct product’ \(D = 4\) model by tensoring the two models (1.11). In view of the result of the Appendix C the corresponding \(D = 6\) bosonic model is equivalent (up to analytic continuations) to the product of \([SL(2, \mathbb{R}) \otimes U(1)]/U(1)\) and \([SU(2) \otimes U(1)]/U(1)\) models with different values of ‘mixing’ parameters \(\gamma_1, \gamma_2\) and levels \(k', k\) (or parameters \(b', b\)). The plane wave solution obtained after applying (1.4) to such product will be (cf. (3.21))

\[ds^2 = 2dvdu + g_1(u') \, dx_1^2 + g_2(u) \, dx_2^2 , \quad \Phi = \Phi_1(u') + \Phi_2(u) , \quad (4.13)\]
\[ A_1 = A_x(u') , \quad A_1 = A_x(u) , \quad u' = au + d , \quad a^2 = k/k' , \]

where the functions \( g, \Phi, A_x \) are the same as in (4.11) with \( b t \to u, \gamma \to \gamma_i \).

Let us consider explicitly some particular cases of (4.13). For example, a ‘throat’ limit of the \( D = 4 \) black hole solution of [63] is described by the product of the \( D = 2 \) black hole (4.4) and the monopole theory (4.6) [55] (note that the gauge field background in (4.4) can be ignored in the leading order approximation since it can be considered as being higher order in \( \alpha' \) but is essential in order to have an exact heterotic string solution [50]). The resulting background (equivalent to the \( D = 6 \) bosonic plane wave solution obtained from the coset \([ SL(2, \mathbb{R})_{-k'} \otimes U(1)]/U(1) \otimes SU(2)_k \), i.e. from the direct product of the model of [52] and \( SU(2) \) WZW model) can be represented as a \( D = 4 \) heterotic plane wave (cf. (3.20),(3.21),(4.13))

\[
\begin{align*}
\text{ds}^2 &= 2dvdu + \tanh^2 u' \, dx_1^2 + \sin^2 u \, dx_2^2 , \quad \Phi = \Phi_0 + \ln \cosh^2 u' \\
A_1 &= -\frac{1}{\cosh^2 u'} , \quad A_2 = 2(\cos u - 1).
\end{align*}
\]

The \( D = 6 \) bosonic plane wave (with the action \( S' \) is equivalent to (3.16)) which one obtains from the \( SL(2, \mathbb{R})_{-k'} \otimes SU(2)_k \) WZW model

\[
S = \frac{k'}{4\pi} \int d^2z \left[ -\partial \phi \bar{\partial} \phi + \partial \theta_L \bar{\partial} \theta_L + \partial \theta_R \bar{\partial} \theta_R + 2 \cos \phi \, \partial \theta_L \bar{\partial} \theta_R \right] \\
+ a^2 \left[ \partial \phi' \bar{\partial} \phi' + \partial \theta'_L \bar{\partial} \theta'_L + \partial \theta'_R \bar{\partial} \theta'_R + 2 \cos \phi' \, \partial \theta'_L \bar{\partial} \theta'_R \right],
\]

\[
S' = \frac{1}{\pi \alpha'} \int d^2z \left[ 2 \partial \bar{\partial} u + \partial x_1 \bar{\partial} x_1 + \partial x_2 \bar{\partial} x_2 + \partial x_3 \bar{\partial} x_3 + \partial x_4 \bar{\partial} x_4 \\
+ 2 \cos u' \, \partial x_1 \bar{\partial} x_2 + 2 \cos u \, \partial x_3 \bar{\partial} x_4 \right],
\]

corresponds to the following \( D = 4 \) heterotic string solution

\[
\begin{align*}
\text{ds}^2 &= 2dvdu + \sin^2 u' \, dx_1^2 + \sin^2 u \, dx_2^2 , \quad \Phi = \Phi_0 \\
A_1 &= 2(\cos u' - 1) , \quad A_2 = 2(\cos u - 1).
\end{align*}
\]

This background can be also related (by analytic continuation and boost) to the Bertotti-Robinson solution of [61], i.e. to the direct product of the ‘anti de Sitter’ and ‘monopole’ \( D = 2 \) solutions.
Another special case of (4.13) is obtained by starting with the direct product of (4.10) and (4.6) (which is related to the bosonic solution describing a ‘throat’ limit of the electrically charged $D = 4$ black hole of [61]). The resulting background is an obvious superposition of (4.9) and (4.12).

It is also possible to go beyond the direct product models (4.13). One can, in fact, construct various conformal $D = 6$ bosonic cosets by applying the $O(4,4)$ duality rotations to (4.15), (4.16) (cf. (3.21)). It is not clear a priori which of them can be re-interpreted as $D = 4$ heterotic string solutions (duality may not ‘commute’ with bosonisation/fermionisation). An example of a non-direct-product model where this is possible was given in [54]. It is straightforward to write down a plane wave background which is related by (1.4) to the $D = 4$ ‘dyonic black hole’ heterotic string solution of [54]. Equivalent $D = 6$ bosonic plane wave is obtained from a particular \[ SL(2, \mathbb{R}) \otimes SU(2) \otimes U(1) \otimes U(1) / [U(1) \otimes U(1)] \] bosonic gauged WZW model.

4.3. A $D = 4$ bosonic plane wave background with $U(1)$ gauge field

As an example of a less trivial bosonic model which is not just a direct product of lower dimensional ones let us consider the plane wave analog of the bosonic ‘charged black hole’ solution of [36] based on the bosonic \[ SL(2, \mathbb{R})_{-k} \otimes SU(2)_{k'} \otimes U(1) \] gauged WZW model. The corresponding $\sigma$-model is given by (4.1) with $D = 5$ (i.e. $r = 1$), $G_{mn} = 1$, $A_\mu \neq 0$, and $\bar{A}_\mu = 0$. It can be interpreted in a Kaluza-Klein manner as a $D = 4$ bosonic background with an extra gauge field. Note that the Kaluza-Klein redefinition of the metric (4.3) here is non-trivial (this redefinition was ignored in [36] so the analysis of properties of the resulting $D = 4$ background is to be reconsidered). In contrast to the bosonic backgrounds of the previous subsection this background does not at the same time represent a $D = 4$ solution of the heterotic string theory.
The non-vanishing $D = 4$ components of the (unredefined) metric $G_{\mu\nu}$, antisymmetric tensor, dilaton and vector potential in (4.1) are

\[ ds^2 = -\frac{\sinh^2 \rho (1 + Q^2 \sin^2 \theta)}{\cosh^2 \rho + Q^2 \sin^2 \theta} dt^2 + a^2 d\rho^2 + d\theta^2 + \frac{\cosh^2 \rho \sin^2 \theta}{\cosh^2 \rho + Q^2 \sin^2 \theta} d\phi^2 \]

\[ = -\frac{(r-1)(1 + Q^2 \sin^2 \theta)}{r + Q^2 \sin^2 \theta} dt^2 + \frac{a^2 dr^2}{4r(r-1)} + d\theta^2 + \frac{r \sin^2 \theta}{r + Q^2 \sin^2 \theta} d\phi^2 \]

\[ B_{t\phi} = \frac{Q(r-1)\sin^2 \theta}{r + Q^2 \sin^2 \theta}, \quad \Phi = \ln(r + Q^2 \sin^2 \theta) + \text{const.} \]

\[ A_t = 2Q\sqrt{Q^2 + 1} \frac{(r-1)\sin^2 \theta}{r + Q^2 \sin^2 \theta}, \quad A_\phi = -2\sqrt{Q^2 + 1} \frac{r \sin^2 \theta}{r + Q^2 \sin^2 \theta}, \]

where $a^2 = k/k'$ and $r \equiv \cosh^2 \rho > 1$. The proper gauge-invariant $D = 4$ metric $G^{(4)}_{\mu\nu}$ is given by (4.3). The $Q = 0$ limit of the model is equivalent to the direct product of the $D = 2$ black hole and ‘$S^2$ plus monopole’ model (4.4) and thus is the same as the limiting case of the $D = 4$ black solution of [63]. The absence of an extra gauge field component (cf. (4.4)) implies that (4.18) is not an exact solution of the heterotic string theory.

If we formally set (cf. (1.4), (1.5))

\[ \rho = iu, \quad \theta = e \alpha^{-1} + au + d, \quad t = -\sqrt{\epsilon} x_1, \quad \phi = \sqrt{\epsilon} x_2, \quad Z \to \sqrt{\epsilon} Z, \quad \alpha' \to \epsilon \alpha', \]

and take the limit $\epsilon \to 0$ we obtain the following plane wave background with $Q, a$ and $d$ as free parameters ($u' \equiv au + d$)

\[ ds^2 = 2dudv + \frac{\sin^2 u (1 + Q^2 \sin^2 u')}{\cos^2 u + Q^2 \sin^2 u'} dx_1^2 + \frac{\cos^2 u \sin^2 u'}{\cos^2 u + Q^2 \sin^2 u'} dx_2^2 \]

\[ b_{12} = Q \frac{\sin^2 u \sin^2 u'}{\cos^2 u + Q^2 \sin^2 u'}, \quad \Phi = \ln(\cos^2 u + Q^2 \sin^2 u') + \text{const.} \]

\[ A_1 = 2Q\sqrt{Q^2 + 1} \frac{\sin^2 u \sin^2 u'}{\cos^2 u + Q^2 \sin^2 u'}, \quad A_2 = -2\sqrt{Q^2 + 1} \frac{\cos^2 u \sin^2 u'}{\cos^2 u + Q^2 \sin^2 u'}. \]

---

11 We make a particular choice of possible free parameters with $Q$ being a real number (related to the electric and axionic charges [36]). We correct two erroneous coefficients (in front of $B_{t\phi}$ and $A_\phi$) in the final expressions of [36].

12 As in the case of all other black-hole type backgrounds to obtain a plane wave we actually consider the region below the horizon, i.e. we first make the analytic continuation $\rho \to ip$. In this case (4.18) takes the form of a cosmological metric: $t$ becomes space-like and $\rho$ plays the role of a time coordinate.
Similar backgrounds are obtained by analytic continuations in $Q, a, d$ and the coordinates. It is straightforward to write down explicitly the resulting world-sheet action (4.1) as a sigma model (1.1) corresponding to a $D = 5$ plane-wave solution and to check that the all-order conformal invariance condition (1.2) is satisfied. The CFT description of this solution is based on the coset $[SU(2)_c \otimes U(1)]/[U(1) \otimes U(1)]$ which has $c = 5$, cf. Sect.3.2.

Computing the $D = 4$ metric $G^{(4)}_{\mu\nu}$ (4.3) of this bosonic ‘plane wave with a gauge field’ background we get

\[
(ds^2)^{(4)} = 2dvdu + \sin^2 u [\cos^2 u (1 + Q^2 \sin^2 u') + Q^2 \sin^2 u' (1 + \cos^2 u - \sin^2 u \sin^2 u')] \frac{dx_1^2}{(\cos^2 u + Q^2 \sin^2 u')^2}
\]

\[
+ \cos^2 u \sin^2 u' (\cos^2 u \cos^2 u' + Q^2 \sin^2 u \sin^2 u') \frac{dx_2^2}{(\cos^2 u + Q^2 \sin^2 u')^2}
\]

\[
+ 2Q(Q^2 + 1) \frac{\sin^2 u \cos^2 u \sin^4 u'}{(\cos^2 u + Q^2 \sin^2 u')^2} dx_1 dx_2 .
\]

5. Solution with a covariantly constant null Killing vector: non-abelian dual of $E^c_2$ WZW model

Let us consider the non-abelian duality transformation on the WZW action for $E^c_2$ (2.11) with respect to the group $E^c_2$ itself. We start with the action \[ S_{\text{dual}} = I_0(g) + \frac{1}{\pi} \int d^2 z \text{Tr} [A\bar{\partial}gg^{-1} - \bar{A}g^{-1}\partial g + Ag\bar{A}g^{-1} - A\bar{A}] \]

\[ - 2i\lambda(\partial A - \bar{\partial}A - [A, \bar{A}]), \] (5.1)

where the group element $g \in E^c_2$ is parametrized as in (3.3). In the $E^c_2$- Lie algebra basis \{$P_i, J, F$\} the Lagrange multipliers and the gauge fields are $\lambda = \{\lambda_i, \lambda_3, \lambda_4\}$, $A = \{A_1, A_3, A_4\}$. From the infinitesimal gauge transformations $\delta g = [g, i\nu]$, $\delta \lambda = [\lambda, i\nu]$, where $\nu = \{\nu_i, \nu_3, \nu_4\}$ one finds

\[
\delta x_1 + \delta x_2 = \tan u \frac{u}{2} [2\nu_2 + (x_2 - x_1) \nu_3], \quad \delta x_1 - \delta x_2 = -2\nu_1 + (x_1 + x_2) \cot \frac{u}{2} \nu_3
\]

\[
\delta v = -(x_1 + x_2) \nu_2 + x_1(x_1 + x_2 \cos u) \nu_3, \quad \delta u = 0
\]

\[
\delta \lambda_i = \epsilon_{ij} (\lambda_3 \nu_j - \nu_3 \lambda_j), \quad \delta \lambda_3 = 0, \quad \delta \lambda_4 = -\epsilon_{ij} \lambda_i \nu_j.
\]
The corresponding transformations for the gauge fields are

\[
\delta A_i = -i \partial \nu_i + \epsilon_{ij} (A_3 \nu_j - \nu_3 A_j) , \quad \delta A_3 = -i \partial \nu_3 , \quad \delta A_4 = -i \partial \nu_4 - \epsilon_{ij} A_i \nu_j , \quad (5.3)
\]

and similarly for \( \bar{A} \). A convenient gauge choice is \( x_1 = x_2 = \lambda_2 = 0 \). In this gauge the action (5.1) takes the following explicit form (we shift \( \lambda_4 \to \lambda_4 + v \))

\[
S = \frac{1}{\pi} \int d^2 z \left[ \partial \nu \bar{\nu} - i A_1 \partial \lambda_1 - i A_3 \partial \lambda_4 + i A_4 (\partial \nu - \partial \nu_3) + i \bar{A}_1 \partial \lambda_1 + i \bar{A}_3 \partial \lambda_4 - i \bar{A}_4 (\partial \nu - \partial \nu_3) + \left( -2 \sin^2 \frac{u}{2} \delta_{ij} + (\sin \nu - \nu_3) \epsilon_{ij} \right) A_i \bar{A}_j - \lambda_1 (A_2 A_3 - A_3 A_2) \right] . \quad (5.4)
\]

After integrating over the gauge fields\(^\text{14}\) we obtain a \( \sigma \)-model action with the following couplings (we introduce new coordinates \( x_1, x_2 \) with \( x_1 = \lambda_1 \) and \( x_2 = -\lambda_4 \))

\[
ds^2 = 2d\nu d\nu + \frac{1}{x_1^2 \sin^2 \frac{u}{2}} \left[ 4 \sin^4 \frac{u}{2} dx_2^2 + (x_1 dx_1 + (\sin \nu - \nu_3) dx_2)^2 \right] \quad (5.5)
\]

\[
\Phi = \ln(x_1^2 \sin^2 \frac{u}{2}) + \text{const} . , \quad B_{\mu\nu} = 0 .
\]

This solution, though not a plane wave, belongs to a general class of backgrounds with a covariantly constant null Killing vector [65][3]. While the abelian duality transformations do not lead us out of the class (1.1) of plane wave backgrounds [26] this is no longer true for the non-abelian duality. The solution (5.5) can be also obtained by applying the limiting procedure of Appendix A to the direct product of the time line and the background of the non-abelian dual to the \( SU(2) \) WZW action (see eqs. (6.10),(6.11) of [36]).

By construction, (5.5) must satisfy the one-loop conformal invariance condition but is not an exact solution to all orders in the \( \alpha' \)- expansion in the standard ‘conformal’ scheme.

\(^{13}\) This choice introduces a non-trivial Faddeev-Popov factor \( \sim \lambda_1 \tan \frac{u}{2} \) in the path integral measure. It can be verified using the appropriate expressions below that when this factor is combined with the measure of the original WZW model (in our case this is the Haar measure \( \sim \sin \nu \)) it gives \( e^\Phi \sqrt{-G} \), as expected [34][10].

\(^{14}\) Note that the fields \( A_4, \bar{A}_4 \) appear in (5.4) only linearly, imposing the \( \delta \)-function condition \( \lambda_3 = u + d \), where \( d = \text{const} . \). This should have been expected since we could have gauge fixed only three parameters even though the gauge group \( E_2^c \) is a four dimensional one (notice that \( \lambda_3 \) is inert under the gauge transformation (5.2)).
The condition of conformal invariance of a background with a covariantly constant null Killing vector and $\nu$-independent dilaton is that the ‘transverse’ parts of its fields (obtained by setting $u = \text{const.}$) should also represent a conformal theory, i.e. should satisfy the conformal invariance equations to all orders \[5\]. In the present case \((5.5)\) \((a, b = \text{const.})\)

\[
\begin{align*}
\frac{ds^2_\perp}{x_1^2} &= \frac{1}{x_1^2} [4a^2 dx_2^2 + 1/4 (x_1 dx_1 + b dx_2)^2] , \quad \Phi_\perp = \ln x_1^2 + \text{const.} . \\
\end{align*}
\] (5.6)

By a shift $x_2 \rightarrow c_1 x_2 + c_2 x_1^2$ and rescaling of $x_1$ \((5.6)\) can be put into the form

\[
\begin{align*}
\frac{ds^2_\perp}{x_1^2} &= dx_1^2 + \frac{1}{x_1^2} dx_2^2 , \quad \Phi_\perp = \ln x_1^2 + \text{const.} , \\
\end{align*}
\] (5.7)

which is one-loop conformally invariant, being dual to the flat $D = 2$ Euclidean space. However, it receives $\alpha'$-corrections in the standard ‘conformal’ scheme (in which there are $\alpha'$-corrections the $SL(2, \mathbb{R})/U(1)$ Euclidean $D = 2$ black hole background. As for the $D = 2$ black hole (and for the $D = 3$ black string and, most probably, for all gauged WZW models) \([66][35]\) there exists a special scheme in which the leading-order solution \((5.7)\) (and thus also \((5.3)\)) is actually the exact solution to all orders in $\alpha'$.

The CFT description of \((7.5)\) is based on the coset $G^c/G$ with $G = E_2^c$ \([13]\). The presence of the $\alpha'$-corrections to the non-abelian dual \((5.5)\) in the standard ‘conformal’ scheme is consistent with the coset description \([25]\) and is not in conflict with the result of \([28]\) (which only states that the abelian $O(d,d)$ duality transformations performed on exact plane wave solutions \((1.1),(1.2)\) give exact solutions belonging to the same class \((1.1)\)).

15 The general solution of the one-loop conformal invariance equations in two dimensions in the conformal gauge $ds^2 = \Omega^{-1} dz d\bar{z}$, $\Phi = \ln \Omega$ has $\Omega = A z \bar{z} + B z + C \bar{z} + D$. The equation for the central charge gives $c = 2 + \delta c$, with $\delta c \sim A$. If $A \neq 0$ then shifting the coordinates $z$, $\bar{z}$ we can set the constants $B$ and $C$ equal to zero. This solution is described by the $SU(2)/U(1)$ CFT. If we demand that $\delta c = 0$ then by a change of coordinates we obtain \((5.7)\). The CFT description of \((5.7)\) is given by the coset $E_2^c/[\mathbb{R} \otimes U(1)]$ \([22]\). The all-order solution that corresponds to a correlated limit taken in the exact background for the coset $SU(2)/U(1)$ \([33]\) can be found in \([22]\) (see also Appendix B).

16 More generally, according to \([25]\) the non-abelian duality transformations of the WZW action for a group $G \otimes U(1)^{\text{dim}(G/H)}$ are equivalent to gauged WZW models $G^c/H$ where by $G^c/H$ we denoted the non-semi-simple group obtained by contracting the direct product group $G \otimes H$ as in \([23]\), and where we gauge the diagonal subgroup.
6. Concluding remarks

In this paper we have presented a number of $D = 4$ plane-wave-type bosonic and heterotic string solutions which admit explicit coset CFT description, thus supplementing the previously discussed $D = 4$ example \cite{20}. The knowledge of underlying CFT is very important in order to study the properties of these backgrounds (see \cite{21}). For example, the conformal algebra fixes the form of the equations for the string modes. We find, for example, that the tachyonic equation does not receive $\alpha'$ corrections in the standard ‘CFT’ scheme.\footnote{In general, the tachyon equation on a plane-wave background may contain $\alpha'$ corrections since the argument that a rescaling of $\alpha'$ in (1.1) is equivalent to a rescaling of $x^i$ does not apply once an $x$-dependent tachyon coupling $T(x)$ is added to the action.} For the plane wave backgrounds \cite{11} the Klein-Gordon-type equation

$$-\frac{1}{e^\Phi\sqrt{-G}} \partial_\mu (e^\Phi\sqrt{-G} G^{\mu\nu} \partial_\nu) \Psi(x) = E \Psi(x), \quad (6.1)$$

can be solved explicitly (see also \cite{57} [2] \cite{58})

\[ \Psi_{p,k_i}(u,v,x^i) = C (e^\Phi\sqrt{g})^{-1/2} e^{g^{ij}y^i} e^{ik_i x_i} e^\frac{1}{2} [Eu - \int d^2u g^{ij}(u)k_i k_j]. \quad (6.2) \]

Let us mention also that it is possible to find explicitly the bosonisation rules and free field description for all of our examples since they all are related in one way or another to cosets of $SU(2)_k$. The bosonisation rules for $E^c_2$ (see \cite{21}) can be very easily derived from the known ones for $SU(2)_k \otimes \mathbb{R}$ (however, a relation of the corresponding representation theories is unclear).

Note added

While this paper was in preparation there appeared an interesting preprint \cite{69} where some $D = 4$ bosonic plane-wave type backgrounds with a covariantly constant null Killing vector were given a CFT interpretation in terms of the coset model $(E^c_2 \otimes E^c_2)/E^c_2$. From the point of view of the present paper such solutions should correspond to a limit (similar...
to (1.4)) of the backgrounds related to the coset $[SU(2) \otimes SU(2) \otimes \mathbb{R}^2]/[SU(2) \otimes \mathbb{R}]$. The bosonic solutions of our Section 3 are different from the solutions of [69]. At the same time, our solution (5.5) can be obtained from eq.(5.10) of [69] by taking the limit, $\kappa = 1 + \epsilon$, $\epsilon \to 0$ and rescaling the coordinates. This could be expected in view of the general result of [25] about a relation between non-abelian duals and singular limits of some special cosets (see footnote 16).

**Appendix A. Limiting procedure for a general class of $\sigma$-models**

The discussion of a limiting procedure (1.4) in Section 1 can be repeated for a more general class of $\sigma$-models with the aim to obtain other backgrounds with a covariantly constant null Killing vector. Consider the $\sigma$-model action

$$S = \frac{1}{\pi\alpha'} \int d^2z \left[ (G_{ij} + B_{ij})(\phi, y) \, \partial y^i \partial y^j + (G_{ab} + B_{ab})(\phi, y) \, \partial \phi^a \partial \phi^b ight.$$  
$$+ A_{ia}(\phi, y) \, \partial y^i \partial \phi^a + B_{ai}(\phi, y) \, \partial \phi^a \partial y^i \big] - \frac{1}{8\pi} \int d^2z \sqrt{\gamma} R^{(2)} \Phi(\phi, y), \quad (A.1)$$

where we split $D$ coordinates into two sets $\phi^a$ $(a = 1, 2)$ and $y^i$ $(i = 1, ..., N)$ and make the coordinate transformation \{\(\phi^a, y^i\) \rightarrow \{u, v, x^i\}\}

$$\phi^1 = \epsilon \, v + u, \quad \phi^2 = u, \quad y^i = f^i(\epsilon; x), \quad \alpha' \rightarrow \epsilon \alpha'. \quad (A.2)$$

The set of functions \{\(f^i(\epsilon; x)\}\} must be chosen in such a way that the limit $\epsilon \to 0$ is well defined and the new action corresponds to a string background with a covariantly constant null Killing vector

$$S = \frac{1}{\pi\alpha'} \int d^2z \left[ 2\partial v \partial u + (g_{ij} + b_{ij})(u, x) \, \partial x^i \partial x^j + a_i(u, x) \, \partial x^i \partial u ight.$$  
$$+ b_i(u, x) \, \partial u \partial x^i \big] - \frac{1}{8\pi} \int d^2z \sqrt{\gamma} R^{(2)} \Phi(u, x). \quad (A.3)$$

If the original action (A.1) is exactly conformal to all orders in $\alpha'$ then the resulting one (A.3) will be exactly conformal as well. However, if the original action represents a leading
order solution, then the fact that in the limit $\epsilon \to 0$ we have $\alpha' \to 0$ does not necessarily imply that the resulting action will correspond to an exact solution to all orders in $\alpha'$ (note, in particular, that the inverse transformation to (A.2) is not defined in the limit $\epsilon \to 0$). Still, this is the case when the resulting background fields depend only on the light cone variable $u$ and not on $x^i$. Since the central charge of the original model (A.1) is given by a power series in $\alpha'$ the central charge of the new model (A.3) will be $c = D = N + 2$, i.e. will be equal to the number of degrees of freedom. Such cases were considered in Sections 2,3,4. The background from Section 5 which explicitly depends on $x^i$ is an example of a leading-order solution.

We can apply this procedure to other leading-order solutions with an idea to get exact plane-wave solutions or leading-order solutions with covariantly constant null Killing vector. Consider, for example, the following class of metrics

$$\alpha' ds^2 = -f(r)dt^2 + h(r)dr^2 + G_{ij}(r,t,y)dy^idy^j$$

(A.4)

and change the coordinate $r \to z$ to put (A.4) in the form

$$\alpha' ds^2 = F(z)(-dt^2 + dz^2) + G'_{ij}(z,t,y)dy^idy^j.$$  

(A.5)

If we now set $z = \epsilon v + u$, $t = u$, rescale $\alpha'$ by $\epsilon$, assume that there exists an appropriate redefinition of $y^i$ (see (A.2)) and take the limit $\epsilon \to 0$ we finish with ($\alpha' = 1$)

$$ds^2 = 2F(u)dudv + g_{ij}(u,x)dx^idx^j,$$

(A.6)

or a background with a null Killing vector (further redefinition of $u$ puts it in the standard form).

Similar procedure can be applied to various black-hole type solutions (cf. (4.10), (4.11), (4.12)). In the case of the $D = 4$ Schwarzchild black hole it gives the rather trivial (flat) background

$$ds^2 = 2dudv + u^2(dx_1^2 + dx_2^2)$$

A different singular coordinate transformation procedure for generating new string solutions from known ones was discussed in [70].
which is an obvious solution of (1.2) and a special case of (3.21) (with \( q = 0 \) and \( g_0, f_i \) corresponding to the second flat case in (2.8)). A less trivial example is found if we start with the five-dimensional solution that is the direct product of the \( D = 4 \) Schwarzchild black hole with an additional space-like direction \( y \)

\[
\alpha' ds^2 = -(1 - \frac{m}{r}) dt^2 + (1 - \frac{m}{r})^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta \ d\phi^2) + dy^2 .
\]

Replacing \( r \) by a new coordinate \( z \) defined by (we consider the region inside the horizon)

\[ z = -m(\psi + \frac{1}{2} \sin 2\psi) , \quad r \equiv m \cos^2 \psi , \]

we then set

\[
\alpha' \rightarrow \epsilon \alpha' , \quad z = -\epsilon v + u , \quad y = u ,
\]

\[
t = \sqrt{\epsilon} x_1 , \quad \theta = \sqrt{\epsilon} \rho , \quad x_2 = m \rho \cos \phi , \quad x_3 = m \rho \sin \phi ,
\]

take the limit \( \epsilon \rightarrow 0 \) (cf. (A.2)), and define \( \psi(u) \) as a solution of \( u = -m(\psi + \frac{1}{2} \sin 2\psi) \). The result is the following \( D = 5 \) plane-wave background (which of course is a solution of (1.2))

\[
ds^2 = 2dvdu + \tan^2 \psi(u) \ dx_1^2 + \cos^4 \psi(u) \ (dx_2^2 + dx_3^2) .
\]

Appendix B. Coset CFT description of background dual to \( D = 2 \) flat spacetime

Let us consider the \( SL(2,\mathbb{R})_k/\text{SO}(1,1) \) current algebra theory and perform the Inönü-Wigner-type contraction: \( J_\pm \rightarrow P_\pm/\sqrt{\epsilon} , \quad J_0 \rightarrow F/\epsilon \) and \( k \rightarrow k/\epsilon \). The resulting OPE’s, stress tensor and central charge are

\[
P_+ P_- \sim \frac{F}{z - w} + \frac{k}{(z - w)^2} , \quad FP_\pm \sim 0 , \quad FF \sim 0
\]

\[
T = \frac{1}{2k} : (P_+ P_- + P_- P_+ - \frac{1}{k} F^2) : , \quad c = 2 .
\]

The constant \( k \) can be rescaled by redefining the current algebra generators but it is useful to keep it (note that we cannot drop the \( F^2 \) term in the stress tensor since it is needed
for the closure of the Virasoro algebra). The only current that has a regular OPE with the stress tensor is $F$ (in the original coset this was the property of $J_0$). We can find the $\sigma$-model corresponding to the stress tensor in (B.1) using the algebraic operator method [33][34]. Representing the zero modes of the currents as first order differential operators

$$
P_+ = -\frac{1}{a}(x_1 + x_2) \frac{\partial}{\partial x_1} - \frac{\partial}{\partial a}, \quad P_- = a \frac{\partial}{\partial x_2}, \quad F = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \quad \text{(B.2)}$$

and restricting [34] the $L_0$ (‘Klein-Gordon’) operator to states satisfying $\frac{\partial}{\partial a} \Psi = 0$ we obtain the following $D = 2$ metric and dilaton

$$
 ds^2 = \frac{1}{(x_1 + x_2)(x_1 + x_2 - 2/k)} \left[ -\frac{1}{k} (dx_1^2 + dx_2^2) + 2(x_1 + x_2 - 1/k) \, dx_1 dx_2 \right] + \Phi = \frac{1}{2} \ln \left[ (x_1 + x_2)(x_1 + x_2 - 2/k) \right] + \text{const.} \quad \text{(B.3)}
$$

One can diagonalize the metric in the region $x_1 + x_2 < 0$ (changing the coordinates $x_1 = -\frac{1}{4}t^2 + \frac{1}{2}x$, $x_2 = -\frac{1}{4}t^2 - \frac{1}{2}x$)

$$
 ds^2 = -dt^2 + \frac{1}{t^2 + 4/k} \, dx^2, \quad \Phi = \frac{1}{2} \ln[t^2(t^2 + 4/k)] + \text{const.} \quad \text{(B.4)}
$$

This metric has a cosmological interpretation of a Universe that starts from a collapsed state at $t = -\infty$, expands and reaches its maximum size at $t = 0$ and then recollapses at $t = \infty$ (see also [71][72][73]). As long as $k$ is finite, there is no curvature singularity at $t = 0$. This is reminiscent of the analogous situation for the 2D black hole [34][74]. In the region where $x_1 + x_2 > 0$ we analytically continue $t \rightarrow i\rho$ and rename $x \rightarrow t$. Then a curvature singularity appears at $\rho = \sqrt{4/k}$. This corresponds to a naked singularity region of the $D = 2$ black hole. For $k \rightarrow \infty$ the analytically continued version of (B.4) coincides with (5.7). The background (B.4) is thus dual to the $D = 2$ flat space-time with a constant dilaton.
Appendix C. Relation between $D = 3$ bosonic charged black string background and $D = 2$ heterotic charged black hole solution

We would like to identify the $D = 3$ $\sigma$-model corresponding to the $[SL(2, \mathbb{R}) \otimes \mathbb{R}] / \mathbb{R}$ gauged WZW model (charged black string model [41]) with the bosonised form (see Section 4.1 and [53, 54]) of the heterotic string action in the charged black hole background [42]. Namely, the aim will be to represent the former model in the ‘Kaluza-Klein’ form (4.1), (4.2) and to identify the metric, gauge potential and dilaton of (4.1) (4.1) with the corresponding couplings in (4.2), (1.3).

The action of the $[SL(2, \mathbb{R}) \otimes \mathbb{R}] / \mathbb{R}$ (axially) gauged WZW model [41] can be represented in the form (see, e.g., [35]; we use the Euler angle coordinates on $SL(2, \mathbb{R})$, $\theta = \frac{1}{2}(\theta_L - \theta_R)$, $\tilde{\theta} = \frac{1}{2}(\theta_L + \theta_R)$ and fix the gauge where the extra $\mathbb{R}$-field is zero)

$$S(r, \theta, \tilde{\theta}) = \frac{k}{\pi} \int d^2 z [G_{\mu\nu} \partial x^\mu \partial x^\nu + B_{\theta \tilde{\theta}} (\partial \theta \partial \tilde{\theta} - \partial \tilde{\theta} \partial \theta)] ,$$

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{1}{4} dr^2 + (1 + q) \frac{C - 1}{C + 1 + 2q} d\theta^2 - q \frac{C + 1}{C + 1 + 2q} d\tilde{\theta}^2 ,$$

$$B_{\theta \tilde{\theta}} = \frac{q(C - 1)}{C + 1 + 2q} , \quad \Phi = \Phi_0 + \ln(C + 1 + 2q) , \quad C \equiv \cosh r ,$$

where $q$ is a free parameter. Changing the coordinates to $r, \theta, Z$

$$\theta \rightarrow \theta + \frac{q}{q + 1} Z , \quad \tilde{\theta} \rightarrow Z ,$$

we can re-write the action (C.1) as (4.1) where $x^\mu = (r, \theta)$ and

$$G_{\theta \tilde{\theta}} = (1 + q) \frac{C - 1}{C + 1 + 2q} , \quad G_{ZZ} = - \frac{q}{q + 1} ,$$

$$A_\theta = \frac{q}{q + 1} G_{\theta \theta} + B_{\theta \tilde{\theta}} = \frac{2q(C - 1)}{C + 1 + 2q} , \quad \bar{A}_\theta = \frac{q}{q + 1} G_{\theta \theta} - B_{\theta \tilde{\theta}} = 0 .$$

Shifting $B_{\theta \tilde{\theta}}$ by a constant leads to a constant shift of the gauge potentials. Computing the gauge-invariant $D = 2$ ‘Kaluza-Klein’ metric (4.3) we find

$$G_{\theta \theta}^{(2)} = G_{\theta \theta} - \frac{1}{4} G_{ZZ}^{-1} A_\theta^2 = \frac{(q + 1)^2 (C^2 - 1)}{(C + 1 + 2q)^2} ,$$

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i.e. the following $D = 2$ background

$$
\begin{align*}
    ds^2 &= \frac{1}{4} dr^2 + \frac{(q + 1)^2 (\cosh^2 r - 1)}{(\cosh r + 1 + 2q)^2} d\theta^2, \\
    \Phi &= \Phi_0 + \ln(\cosh r + 1 + 2q),
\end{align*}
$$

$$
A_\theta = \frac{2q(C - 1)}{C + 1 + 2q} = 2q - \frac{4q(q + 1)}{C' + 1 + 2q}, \tag{C.7}
$$

which becomes equivalent to the analytic continuation of (4.10) or (4.11) with $\gamma = -q/(q + 1)$ after we drop a constant in the gauge potential and rescale the coordinates (a similar to (C.7) form of (4.11) appeared in [54]). The background (C.7) was also obtained in [54] by a ‘heterotic’ gauging of $SL(2, \mathbb{R})$. The above discussion implies that the model of [54] when viewed as a specific conformal bosonic model is actually equivalent to the $[SL(2, \mathbb{R}) \otimes \mathbb{R}] / \mathbb{R}$ gauged WZW model.

The model (C.1)–(C.3) has two obvious limits: $q = 0$ (neutral black string or $SL(2, \mathbb{R}) / \mathbb{R} \otimes \mathbb{R}$ gauged WZW model) and $q = \infty$ ($SL(2, \mathbb{R})$ WZW model). This explains the observation in Sect.4.2 that the corresponding limits $\gamma = 0$ and $\gamma = -1$ of the $D = 2$ background (4.11) are equivalent to the exact $(1, 1)$ supersymmetric heterotic black hole solution (4.4) (which, from the bosonised $D = 3$ point of view is related to the model of [52] or, after change of coordinates, to the neutral black string) and the monopole model (4.7) (related to $SU(2)$ WZW theory).
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