Abstract

We study the dynamics of iterates at the transition to chaos in the logistic map and find that it is constituted by an infinite family of Mori’s $q$-phase transitions. Starting from Feigenbaum’s $\sigma$ function for the diameters ratio, we determine the atypical weak sensitivity to initial conditions $\xi_t$ associated to each $q$-phase transition and find that it obeys the form suggested by the Tsallis statistics. The specific values of the variable $q$ at which the $q$-phase transitions take place are identified with the specific values for the Tsallis entropic index $q$ in the corresponding $\xi_t$. We describe too the bifurcation gap induced by external noise and show that its properties exhibit the characteristic elements of glassy dynamics close to vitrification in supercooled liquids, e.g. two-step relaxation, aging and a relationship between relaxation time and entropy.

1 Introduction

The logistic equation was introduced in 1845 by the Belgian mathematician and sociologist Pierre-François Verhulst to model the growth of populations limited by finite resources [1]. The discrete time variable version of Verhulst’s
growth law, the logistic map, has become a foundation stone for the theory of nonlinear dynamics. The logistic map is the archetypal example of how the use as starting point of simple non-linear discrete maps have often led to significant developments in the theory of non-linear dynamical systems [2]. The discovery of the universal properties associated to the renowned period-doubling and intermittency routes to chaos displayed by the logistic map, similar to those of conventional critical phenomena in statistical physics, triggered, about three decades ago, an upsurge of activity in the field and now both routes, as well as many other remarkable features displayed by the logistic map, are well understood.

Here we might argue that at the present time the logistic map is again becoming a prototypical model system. This time for the assessment of the validity and understanding of the reasons for applicability of the nonextensive generalization of the Boltzmann Gibbs (BG) statistical mechanics [3], [4]. This is a formalism assumed to be appropriate for circumstances where the system is out of the range of validity of the canonical BG theory. And these circumstances are believed in some cases to be a breakdown in the chain of increasing randomness from non-ergodicity to completely developed phase-space mixing.

The logistic map contains infinite families of critical attractors at which the ergodic and mixing properties breakdown. These are the tangent bifurcations and the accumulation point(s) of the pitchfork bifurcations, the so-called onset of chaos [2]. At each of the map critical attractors the Lyapunov coefficient $\lambda_1$ vanishes, and the sensitivity to initial conditions $\xi_t$ for large iteration time $t$ ceases to obey exponential behavior, exhibiting instead power-law or faster than exponential behavior. The pitchfork bifurcations are also critical attractors at which the negative Lyapunov coefficient of periodic orbits goes to zero. There are other attractors at which the Lyapunov coefficient diverges to minus infinity, where there is faster than exponential convergence of orbits. These are the superstable attractors located between successive pitchfork bifurcations and their accumulation point is also the onset of chaos.

Here we review briefly specific and rigorous results on the dynamics associated to the critical attractors of the logistic map, or of its generalization to non-linearity of order $z > 1$, $f_{\mu}(x) = 1 - \mu |x|^z$, $-1 \leq x \leq 1$, $0 \leq \mu \leq 2$. (The phase space variable is $x$, the control parameter is $\mu$ and the conventional logistic map corresponds to $z = 2$). Our results relate to the anomalous sen-
sitivity to initial conditions at the onset of chaos, the associated spectrum of Tsallis $q$-Lyapunov coefficients, and the relationship of these with the Mori $q$-phase transitions [5], one of which was originally observed numerically for the Feigenbaum attractor [5] [6]. In particular, we identify the Mori singularities in the Lyapunov spectra with the appearance of special values for the Tsallis entropic index $q$. As the properties of the logistic map are very familiar and well understood it is of interest to see how previous knowledge fits in with the new perspective.

Tsallis suggested [7] that for critical attractors $\xi_t$ (defined as $\xi_t(x_0) \equiv \lim_{\Delta x_0 \to 0} (\Delta x_t/\Delta x_0)$ where $\Delta x_0$ is the initial separation of two orbits and $\Delta x_t$ that at time $t$), has the form

$$\xi_t(x_0) = \exp_q[\lambda_q(x_0) t] \equiv [1 - (q - 1)\lambda_q(x_0) t]^{-1/(q-1)},$$

that yields the customary exponential $\xi_t$ with $\lambda_1$ when $q \to 1$. In Eq. (1) $q$ is the entropic index and $\lambda_q$ is the $q$-generalized Lyapunov coefficient; $\exp_q(x) \equiv [1 - (q - 1)x]^{-1/(q-1)}$ is the $q$-exponential function. Tsallis also suggested [7] that the Pesin identity $K_1 = \lambda_1$ (where the rate of entropy production $K_1$ is given by $K_1 t = S_1(t) - S_1(0)$, $t$ large, and $S_1 = -\sum_i p_i \ln p_i$) would be generalized to $K_q = \lambda_q$, where the $q$-generalized rate of entropy production $K_q$ is defined via $K_q t = S_q(t) - S_q(0)$, $t$ large, and where

$$S_q \equiv \sum_i p_i \ln_q \left( \frac{1}{p_i} \right) = \frac{1 - \sum_i W_i^q}{q - 1}$$

is the Tsallis entropy; $\ln_q y \equiv (y^{1-q} - 1)/(1 - q)$ is the inverse of $\exp_q(y)$.

To check on the above, we have analyzed recently [8]-[11] both the pitchfork and tangent bifurcations and the onset of chaos of the logistic map and found that indeed the Tsallis suggestions hold for these critical attractors, though with some qualifications. For the case of the tangent bifurcation it is important to neglect the feedback mechanism into the neighborhood of the tangency to avoid the crossover out of the $q$-exponential form for $\xi_t$. As we explain below, for the onset of chaos there is a multiplicity of $q$-indexes, appearing in pairs $q_j$ and $Q_j = 2 - q_j$, $j = 0, 1, \ldots$, and a spectrum of $q$-Lyapunov coefficients $\lambda_{q_j}^{(k,l)}$ and $\lambda_{Q_j}^{(k,l)}$ for each $q_j$ and $Q_j$, respectively. (The superindex $(k,l)$ refers to starting and final trajectory positions). The dynamics of the attractor confers the $q$-indexes a decreasing order of importance. Retaining only the dominant indexes $q_0$ and $Q_0$ yields a quite
reasonable description of the dynamics and considering the next few leading indexes provides a satisfactorily accurate account.

Although in brief, we also describe our finding [12] that the dynamics at the noise-perturbed edge of chaos in logistic maps is analogous to that observed in supercooled liquids close to vitrification. The three major features of glassy dynamics in structural glass formers, two-step relaxation, aging, and a relationship between relaxation time and configurational entropy, are displayed by orbits with vanishing Lyapunov coefficient. The known properties in control-parameter space of the noise-induced bifurcation gap play a central role in determining the characteristics of dynamical relaxation at the chaos threshold.

2 Critical attractors in the logistic map

For our purposes it is convenient to recall some essentials of logistic map properties. The accumulation point of the period doublings and also of the chaotic band splittings is the Feigenbaum attractor that marks the threshold between periodic and chaotic orbits, at \( \mu_\infty(z) \), with \( \mu_\infty = 1.40115... \) when \( z = 2 \). The locations of period doublings, at \( \mu = \mu_n < \mu_\infty \), and band splittings, at \( \mu = \hat{\mu}_n > \mu_\infty \), obey for large \( n \) power laws of the form \( \mu_n - \mu_\infty \sim \delta(z)^{-n} \) and \( \mu_\infty - \hat{\mu}_n \sim \delta(z)^{-n} \), with \( \delta = 0.46692... \) when \( z = 2 \), which is one of the two Feigenbaum’s universal constants. For our use below we recall also the sequence of parameter values \( \bar{\mu}_n \) employed to define the diameters \( d_n \) of the bifurcation forks that form the period-doubling cascade sequence. At \( \mu = \bar{\mu}_n \) the map displays a ‘superstable’ periodic orbit of length \( 2^n \) that contains the point \( x = 0 \). For large \( n \) the distances to \( x = 0 \), of the iterate positions in such \( 2^n \)-cycle that are closest to \( x = 0 \), \( d_n \equiv f_{\bar{\mu}_n}^{(2^n-1)}(0) \), have constant ratios \( d_n / d_{n+1} = -\alpha(z) \); \( \alpha = 2.50290... \) when \( z = 2 \), which is the second of the Feigenbaum’s constants. A set of diameters with scaling properties similar to those of \( d_n \) can also be defined for the band splitting sequence [2]. Other diameters in the \( 2^n \)-supercycles are defined as the distance of the \( m \)th element \( x_m \) to its nearest neighbor \( f_{\bar{\mu}_n}^{(2^n-1)}(x_m) \). That is

\[
d_{n,m} \equiv f_{\bar{\mu}_n}^{(m+2^n-1)}(0) - f_{\bar{\mu}_n}^{(m)}(0), \quad m = 0, 1, 2, \ldots, \quad (3)
\]
with \( d_{n,0} = d_n \). Feigenbaum [13] constructed the auxiliary function

\[
\sigma_n(m) = \frac{d_{n+1,m}}{d_{n,m}}
\]  

(4)

to quantify the rate of change of the diameters and showed that it has finite (jump) discontinuities at all rationals, as can be seen by considering the variable \( y = m/2^{n+1} \) with \( n \) large (and therefore omitting the subindex \( n \)). One obtains [13] [2] \( \sigma(0) = -1/\alpha \), but \( \sigma(0^+) = 1/\alpha^z \), and through the antiperiodic property \( \sigma(y + 1/2) = -\sigma(y) \), also \( \sigma(1/2) = 1/\alpha \), but \( \sigma(1/2 + 0^+) = -1/\alpha^z \). Other discontinuities in \( \sigma(y) \) appear at \( y = 1/4, 1/8, \) etc. As these decrease rapidly in most cases it is only necessary to consider the first few.

An important factor of our work is that the sensitivity to initial conditions \( \xi_t(x_0) \) can be evaluated for trajectories within the Feigenbaum attractor via consideration of the discontinuities of \( \sigma_n(m) \). If the initial separation is chosen to be a diameter \( \Delta x_0 = d_{n,m} \) and the final time \( t \) is chosen to have the form \( t = 2^n - 1 \), then \( \Delta x_t = d_{n,m+2^n-1} \), and \( \xi_t(x_0) \) can be written [14] as

\[
\xi_t(m) \simeq \left| \frac{\sigma_n(m-1)}{\sigma_n(m)} \right|^n, \ t = 2^n - 1, \ n \ \text{large}.
\]  

(5)

For clarity of presentation, we shall only use absolute values of positions so that the dynamics of iterates do not carry information on the self-similar properties of “left” and “right” symbolic dynamic sequences [2]. This choice does not affect results on the sensitivity to initial conditions.

3 Mori’s \( q \)-phase transitions in the logistic map

During the late 1980’s Mori and coworkers developed a comprehensive thermodynamic formalism to characterize drastic changes at bifurcations and at other singular phenomena in low dimensional maps [5]. The formalism was also adapted to the study of critical chaotic attractors and was illustrated by considering the specific case of the onset of chaos in the logistic map [5] [6] [15]. For critical attractors the scheme involves the evaluation of fluctuations
of the generalized finite-time Lyapunov coefficient

\[ \lambda(t, x_0) = \frac{1}{\ln t} \sum_{i=0}^{t-1} \ln \left| \frac{df_{\mu_\infty}(x_i)}{dx_i} \right|, \quad t \gg 1. \] \tag{6}

Notice the replacement of the customary \( t \) by \( \ln t \) above, as the ordinary Lyapunov coefficient

\[ \lambda_1(x_0) = \lim_{t \to \infty} \frac{1}{t} \sum_{i=0}^{t-1} \ln \left| \frac{df_{\mu_\infty}(x_i)}{dx_i} \right| \] \tag{7}

vanishes at \( \mu_\infty, t \to \infty \).

The probability density distribution for the values of \( \lambda \), \( P(\lambda, t) \), is written in the form

\[ P(\lambda, t) = t^{-\psi(\lambda)} P(0, t), \quad t \gg 1, \] \tag{8}

where \( \psi(\lambda) \) is a concave spectrum of the fluctuations of \( \lambda \) with minimum \( \psi(0) = 0 \) and is obtained as the Legendre transform of the 'free energy' function \( \phi(q) \), defined as

\[ \phi(q) \equiv -\lim_{t \to \infty} \frac{1}{\ln t} \ln Z(t, q), \] \tag{9}

where \( Z(t, q) \) is the dynamic partition function

\[ Z(t, q) \equiv \int d\lambda \ P(\lambda, t) \ t^{-(q-1)\lambda}. \] \tag{10}

The 'coarse-grained' function of generalized Lyapunov coefficients \( \lambda(q) \) is given by \( \lambda(q) \equiv d\phi(q)/dq \) and the variance \( v(q) \) of \( P(\lambda, t) \) by \( v(q) \equiv d\lambda(q)/dq \) \cite{5} \cite{6}. Notice the special weight \( t^{-(q-1)\lambda} \) in the partition function \( Z(t, q) \) as this shapes the quantities derived from it. These functions are the dynamic counterparts of the Renyi dimensions \( D_q \) and the spectrum \( f(\tilde{\alpha}) \) that characterize the geometric structure of the attractor.

A "\( q \)-phase" transition is indicated by a section of linear slope \( m_c = 1 - q_c \) in the spectrum (free energy) \( \psi(\lambda) \), a discontinuity at \( q_c \) in the Lyapunov function (order parameter) \( \lambda(q) \), and a divergence at \( q_c \) in the variance (susceptibility) \( v(q) \). For the onset of chaos at \( \mu_\infty(z = 2) \) a single \( q \)-phase transition was numerically determined \cite{5} \cite{6} \cite{15} and found to exhibit a value close to \( m_c = -(1 - q_c) \simeq -0.7 \); arguments were provided for this value to
be \( m_c = -(1 - q_c) = -\ln 2 / \ln \alpha = -0.7555 \ldots \) Our analysis described below shows that the older results give a broad picture of the dynamics at the Feigenbaum attractor and that actually an infinite family of \( q \)-phase transitions of decreasing weights take place at \( \mu_\infty \).

4 Tsallis dynamics at the edge of chaos

By taking as initial condition \( x_0 = 0 \) we found that the resulting orbit consists of trajectories made of intertwined power laws that asymptotically reproduce the entire period-doubling cascade that occurs for \( \mu < \mu_\infty \) \([9]\) \([11]\). This orbit captures the properties of the so-called 'superstable' orbits at \( \mu_\infty < \mu_\infty \), \( n = 1, 2, \ldots \) \([2]\) (see Fig. 1), and can be used as reference to read all other orbits within the attractor. At \( \mu_\infty \) the Lyapunov coefficient \( \lambda_1 \) vanishes and in its place there appears a spectrum of \( q \)-Lyapunov coefficients \( \lambda_q^{(k,l)} \). This spectrum was originally studied in Refs. \([15]\) and \([6]\) and our recent interest has been to study its properties in more detail to examine their relationship with the Tsallis statistics. Recent analytical results about the \( q \)-Lyapunov coefficients and the \( q \)-generalized Pesin identity are given in Refs. \([9]\) and \([11]\).

Now, consider a first approximation to the function \( \sigma_n(m) \) for \( n \) large, applicable to general non-linearity \( z > 1 \). This is to assume that half of the diameters scale as \( \alpha \) (as in the most sparse region of the attractor) while the other half scale as \( \alpha_0 = \alpha^z \) (as in the most crowded region of the attractor). This approximation captures the effect on \( \xi_t \) of the most dominant trajectories within the attractor. With these two scaling factors \( \sigma_n(m) \) becomes the periodic step function

\[
\frac{1}{\sigma_n(m)} = \begin{cases} 
\alpha_0 = \alpha^z, & 0 < m \leq 2^{n-1}, \\
\alpha, & 2^{n-1} < m \leq 2 \cdot 2^{n-1}, \\
-\alpha_0 = -\alpha^z, & 2 \cdot 2^{n-1} < m \leq 3 \cdot 2^{n-1}, \\
-\alpha, & 3 \cdot 2^{n-1} < m \leq 4 \cdot 2^{n-1}, \ldots 
\end{cases} \tag{11}
\]

Use of Eqn. (11) into Eqn. (5) for the sensitivity \( \xi_t(m) \) yields the result

\[
\xi_t(m) = \begin{cases} 
\alpha^{-(z-1)n}, & m = (2k + 1)2^{n-1}, \\
\alpha^{(z-1)n}, & m = (2k + 2)2^{n-1}.
\end{cases} \tag{12}
\]
Figure 1: Absolute values of positions in logarithmic scales of iterations $\tau$ for a trajectory at $\mu_\infty$ with initial condition $x_0 = 0$. The numbers correspond to iteration times. The power-law decay of the time subsequences can be clearly appreciated.

where $k = 0, 1, \ldots$ and where the final observation time is of the form $t = l2^n - 1, l = 1, 2, \ldots$. With the introduction of the total time variable $\tau \equiv m + 1 + t$ Eq. (12) can be rewritten in terms of the $q$-exponential functions

$$
\xi_\tau(m) = \begin{cases} 
[1 + (1 - q_0)\lambda_{\phi_0}^{(k,l)}]^{1/(1-q_0)}, & m = (2k + 2)2^{n-1}, \\
[1 + (1 - Q_0)\lambda_{\phi_0}^{(k,l)}]^{1/(1-Q_0)}, & m = (2k + 1)2^{n-1},
\end{cases}
$$

(13)

where

$$
q_0 = 1 - \frac{\ln 2}{\ln \alpha_0 / \alpha} = 1 - \frac{\ln 2}{(z - 1) \ln \alpha},
$$

(14)

$$
\lambda_{\phi_0}^{(k,l)} = \frac{(z - 1) \ln \alpha}{(k + 2l + 1) \ln 2},
$$

(15)

$$
Q_0 = 1 + \frac{\ln 2}{\ln \alpha_0 / \alpha} = 1 + \frac{\ln 2}{(z - 1) \ln \alpha},
$$

(16)
\[ \lambda_{Q_0}^{(k,l)} = -\frac{2(z-1)\ln \alpha}{(2k + 4l + 1)\ln 2}. \] (17)

Notice that \( Q_0 = 2 - q_0 \). For \( z = 2 \) one obtains \( Q_0 \simeq 1.7555 \) and \( q_0 \simeq 0.2445 \), this latter value agrees with that obtained in several earlier studies [7] [16] [9] [11]. The two scales considered describe correctly only trajectories that start at the most sparse region of the multifractal \((x \simeq 0)\) and terminate at its most crowded region \((x \simeq 1)\), or the inverse. (This is why we obtain the two conjugate values \( q \) and \( Q = 2 - q \), as the inverse of the \( q \)-exponential function satisfies \( \exp_q(y) = 1/\exp_{2-q}(-y) \). The vertical lines in Fig. 2a represent the ranges of values obtained when \( z = 2 \) for \( \lambda_{q_0}^{(k,l)} \) and \( \lambda_{Q_0}^{(k,l)} \). See Ref. [14] for more details.

![Figure 2: \( q \)-phase transitions with index values \( q_0 \) and \( Q_0 = 2 - q_0 \) obtained from the main discontinuity in \( \sigma_n(m) \). See text for details.](image)

We consider the next discontinuities of importance in \( \sigma_n(m) \). Independently of the number of discontinuities taken into account one obtains \( q \)-exponential forms for \( \xi_t \). The value of \( \sigma(y = m/2^{n+1}) \) at \( y = 1/4 \) measures \( 1/\alpha_1 \), with \( \alpha_1 \simeq 5.4588 \) for \( z = 2 \), and it is associated to one 'midway' region between the most crowded and most sparse regions of the attractor (the other 'midway' region being associated to \( \sigma(3/4) \)). With three scaling factors, \( \alpha, \alpha_0 \) and \( \alpha_1 \), we have three values for the \( q \) index, \( q_0, q_1 \) and \( q_2 \) (together with the conjugate values \( Q_0 = 2 - q_0, Q_1 = 2 - q_1 \) and \( Q_2 = 2 - q_2 \).
for the inverse trajectories). To each value of \( q \) there is a set of \( q \)-Lyapunov coefficients running from a maximum \( \lambda_{q, \text{max}} \) to zero (or a minimum \( \lambda_{Q, \text{min}} \) to zero). The results when \( z = 2 \) for the ranges of values obtained for the \( q \)-Lyapunov coefficients are shown as the vertical lines in Figs. 2a, 3a and 4a. Similar results are obtained for the case of four discontinuities in \( \sigma_n(m) \), etc.

5  A family of \( q \)-phase transitions at the edge of chaos

As a function of the variable \(-\infty < q < \infty\) the \( q \)-Lyapunov coefficients obtained in the previous section are functions with two steps with the jumps located at \( q = q_i = 1 - \ln 2 / \ln \alpha_i(z) / \alpha(z) \) and \( q = Q_i = 2 - q_i \). Immediate contact can be established with the formalism developed by Mori and coworkers and the \( q \) phase transition obtained in Ref. [6]. Each step function for \( \lambda(q) \) can be integrated to obtain the spectrum \( \phi(q) \) (\( \lambda(q) \equiv d\phi/dq \)) and from this its Legendre transform \( \psi(\lambda) \) (\( \equiv \phi - (1 - q)\lambda \)). We illustrate this first with \( \sigma_n(m) \) obtained with two scale factors, as in Eq. (11). We show numerical values for the case \( z = 2 \). From Eqs. (14) to (17) we obtain

\[
\lambda(q) = \begin{cases} 
\lambda_{q_0, \text{max}}, & q \leq q_0 = 1 - \ln 2 / (z - 1) \ln \alpha \simeq 0.2445, \\
0, & q_0 < q < Q_0, \\
\lambda_{Q_0, \text{min}}, & q \geq Q_0 = 2 - q_0 \simeq 1.7555,
\end{cases} \tag{18}
\]

where \( \lambda_{q_0, \text{max}} = \ln \alpha / \ln 2 \simeq 1.323 \) and \( \lambda_{Q_0, \text{min}} = -2 \ln \alpha / \ln 2 \simeq -2.646 \).

The free energy functions \( \phi(q) \) and \( \psi(\lambda) \) that correspond to Eq. (18) are given by

\[
\phi(q) = \begin{cases} 
\lambda_{q_0, \text{max}}(q - q_0), & q \leq q_0, \\
0, & q_0 < q < Q_0, \\
\lambda_{Q_0, \text{min}}(q - Q_0), & q \geq Q_0,
\end{cases}
\]

and

\[
\psi(\lambda) = \begin{cases} 
(1 - Q_0)\lambda, & \lambda_{Q_0, \text{min}} < \lambda < 0, \\
(1 - q_0)\lambda, & 0 < \lambda < \lambda_{q_0, \text{max}}.
\end{cases}
\]

We show these functions in Fig. 2. The constant slopes of \( \psi(\lambda) \) represent the \( q \)-phase transitions associated to trajectories linking two regions of the attractor, that in this case are its most crowded and most sparse, and their
values $1 - q_0$ and $q_0 - 1$ correspond to those obtained for the $q$-exponential $\xi_t$ Eq. (13). The slope $q_0 - 1 \simeq -0.7555$ coincides with that originally detected by Mori and colleagues [5].

When we consider also the next discontinuity of importance in $\sigma_n(m)$, at $\sigma(1/4) = 1/\alpha_1$, we obtain a set of two $q$-phase transitions for each of the three values of the $q$ index, $q_0$, $q_1$ and $q_2$. We show in Figs. 2, 3 and 4 the functions $\lambda(q)$, $\phi(q)$, and $\psi(\lambda)$ obtained for these three cases. The parameter values for the $q$-phase transitions at $1 - q_0$ and $q_0 - 1$ appear again, but now we have also two other sets at $1 - q_1$ and $q_1 - 1$, and at $1 - q_2$ and $q_2 - 1$, that correspond, respectively, to orbits that link a ‘midway’ region of the attractor with the most sparse region, and with the most crowded region of the attractor.

![Diagram](image)

Figure 3: $q$-phase transitions with index values $q_1$ and $Q_1 = 2 - q_1$ obtained from the main two discontinuities in $\sigma_n(m)$. See text for details.

### 6 Noisy dynamics at the edge of chaos

Consider now the logistic map $z = 2$ in the presence of additive noise

$$x_{t+1} = f_\mu(x_t) = 1 - \mu x_t^2 + \chi_t \varepsilon, \ -1 \leq x_t \leq 1, \ 0 \leq \mu \leq 2, \quad (19)$$
Figure 4: $q$-phase transitions with index values $q_2$ and $Q_2 = 2 - q_2$ obtained from the main two discontinuities in $\sigma_n(m)$. See text for details.

where $\chi_t$ is the random variable with average $\langle \chi_t\chi_{t'} \rangle = \delta_{t,t'}$, and $\varepsilon$ measures the noise intensity [2] [17]. Except for a set of zero measure, all the trajectories with $\mu_\infty(\varepsilon = 0)$ and initial condition $-1 \leq x_0 \leq 1$ fall into the attractor with fractal dimension $d_f = 0.5338...$. These trajectories represent nonergodic states, since as $t \to \infty$ only a Cantor set of positions is accessible out of the total phase space. For $\varepsilon > 0$ the noise fluctuations erase the fine features of the periodic attractors as these widen into bands similar to those in the chaotic attractors, yet there remains a definite transition to chaos at $\mu_\infty(\varepsilon)$ where the Lyapunov exponent $\lambda_1$ changes sign. The period doubling of bands ends at a finite value $2^{N(\varepsilon)}$ as the edge of chaos transition is approached and then decreases at the other side of the transition. This effect displays scaling features and is referred to as the bifurcation gap [2] [17]. When $\varepsilon > 0$ the trajectories visit sequentially a set of $2^n$ disjoint bands or segments leading to a cycle, but the behavior inside each band is completely chaotic. These trajectories represent ergodic states as the accessible positions have a fractal dimension equal to the dimension of phase space. Thus the removal of the noise $\varepsilon \to 0$ at $\mu_\infty$ leads to an ergodic to nonergodic transition in the map.

In the absence of noise ($\varepsilon = 0$) the diameter positions $x_{2^n} = d_n = \alpha^{-n}$ visited at times $\tau = 2^n$ by the trajectory starting at $x_0 = 1$ is given by the
Feigenbaum fixed-point map solution \( g(x) \),

\[
x_\tau = \left| g^{(r)}(x_0) \right| = \tau^{-1/1-q_0} \left| g(\tau^{1/1-q_0} x_0) \right|,
\]

that in turn is obtained from the \( n \to \infty \) convergence of the \( 2^n \)th map composition to \((-\alpha)^{-n} g(\alpha^n x)\) with \( \alpha = 2^{1/(1-q_0)} \). When \( x_0 = 0 \) one obtains in general [9]

\[
x_\tau = \left| g^{(2l+1)}(0) g^{(2^n-1)}(0) \right| = \left| g^{(2l+1)}(0) \right| \alpha^{-n}, \tau = (2l + 1)2^n, \ l, n = 0, 1, ...
\]

When the noise is turned on (\( \varepsilon \) always small) the \( 2^n \)th map composition converges instead to

\[
(-\alpha)^{-n} \left[ g(\alpha^n x) + \chi \varepsilon \kappa^n G_\Lambda(\alpha^n x) \right],
\]

where \( \kappa \) a constant whose numerically determined [18], [19] value \( \kappa \simeq 6.619 \) is well approximated by \( \nu = 2\sqrt{2} \alpha (1 + 1/\alpha^2)^{-1/2} \), the ratio of the intensity of successive subharmonics in the map power spectrum [19], [2]. The connection between \( \kappa \) and the \( \varepsilon \)-independent \( \nu \) stems from the necessary coincidence of two ratios, that of noise levels causing band-merging transitions for successive \( 2^n \) and \( 2^{n+1} \) periods and that of spectral peaks at the corresponding parameter values \( \mu_n \) and \( \mu_{n+1} \) [19], [2]. Following the same procedure as above we see that the orbits \( x_\tau \) at \( \mu_\infty(\varepsilon) \) satisfy, in place of Eq. (20), the relation

\[
x_\tau = \tau^{-1/1-q_0} \left| g(\tau^{1/1-q_0} x) + \chi \varepsilon \tau^{1/1-r} G_\Lambda(\tau^{1/1-q_0} x) \right|,
\]

where \( G_\Lambda(x) \) is the first-order perturbation eigenfunction, and where \( r = 1 - \ln 2 / \ln \kappa \simeq 0.6332 \). So that use of \( x_0 = 0 \) yields

\[
x_\tau = \tau^{-1/1-q_0} \left| 1 + \chi \varepsilon \tau^{1/1-r} \right|
\]

or

\[
x_t = \exp_{2-q_0} (-\lambda_{q_0} t) \left[ 1 + \chi \varepsilon \exp_r(\lambda_r t) \right]
\]

where \( t = \tau - 1, \ \lambda_{q_0} = \ln \alpha / \ln 2 \) (\( \lambda_{q_0,\max} \) of the previous section) and \( \lambda_r = \ln \kappa / \ln 2 \).

At each noise level \( \varepsilon \) there is a 'crossover' or 'relaxation' time \( t_x = \tau_x - 1 \) when the fluctuations start destroying the detailed structure imprinted by
the attractor on the orbits with \( x_0 = 0 \). This time is given by \( \tau_x = \varepsilon^{r-1} \), the time when the fluctuation term in the perturbation expression for \( x_\tau \) becomes \( \varepsilon \)-independent, i.e.

\[ x_{\tau_x} = \tau_x^{-1/1-q_0} |1 + \chi| . \] (26)

Thus, there are two regimes for time evolution at \( \mu_\infty(\varepsilon) \). When \( \tau < \tau_x \) the fluctuations are smaller than the distances between adjacent subsequence positions of the noiseless orbit at \( \mu_\infty(0) \), and the iterate positions in the presence of noise fall within small non overlapping bands each around the \( \varepsilon = 0 \) position for that \( \tau \). In this regime the dynamics follows in effect the same subsequence pattern as in the noiseless case. When \( \tau \sim \tau_x \) the width of the fluctuation-generated band visited at time \( \tau_x = 2^N \) matches the distance between two consecutive diameters, \( d_N - d_{N+1} \) where \( N \sim -\ln \varepsilon / \ln \kappa \), and this signals a cutoff in the advance through the position subsequences. At longer times \( \tau > \tau_x \) the orbits are unable to stick to the fine period-doubling structure of the attractor. In this 2nd regime the iterate follows an increasingly chaotic trajectory as bands merge progressively. This is the dynamical image - observed along the time evolution for the orbits of a single state \( \mu_\infty(\varepsilon) \) - of the static bifurcation gap first described in the map space of position \( x \) and control parameter \( \mu \) [17], [18].

7 Analogy with glassy dynamics

We recall the main dynamical properties displayed by supercooled liquids on approach to glass formation. One is the growth of a plateau and for that reason a two-step process of relaxation, as presented by the time evolution of correlations e.g. the intermediate scattering function \( F_k \) [20]. This consists of a primary power-law decay in time \( t \) (so-called '\( \beta \)' relaxation) that leads into the plateau, the duration \( t_x \) of which diverges also as a power law of the difference \( T - T_g \) as the temperature \( T \) decreases to a critical value \( T_g \). After \( t_x \) there is a secondary power law decay (so-called '\( \alpha \)' relaxation) that leads to a conventional equilibrium state [20]. A second characteristic dynamical property of glasses is the loss of time translation invariance, a characteristic known as aging [21]. The time decay of relaxation functions and correlations display a scaling dependence on the ratio \( t/t_w \) where \( t_w \) is a waiting time. A third distinctive property is that the experimentally
observed relaxation behavior of supercooled liquids is described, via regular heat capacity assumptions [20], by the so-called Adam-Gibbs equation,

\[ t_x = A \exp(B/TS_c), \]  

(27)

where the relaxation time \( t_x \) can be identified with the viscosity, and the configurational entropy \( S_c \) is related to the number of minima of the fluid’s potential energy surface (and \( A \) and \( B \) are constants).

Returning to the map, phase space is sampled at noise level \( \varepsilon \) by orbits that visit points within the set of \( 2^N \) bands of widths \( \sim \varepsilon \), and this takes place in time in the same way that period doubling and band merging proceeds in the presence of a bifurcation gap when the control parameter is run through the interval \( 0 \leq \mu \leq 2 \). That is, the trajectories starting at \( x_0 = 0 \) duplicate the number of visited bands at times \( \tau = 2^n, n = 1, ..., N \), the bifurcation gap is reached at \( \tau_x = 2^N \), after which the orbits fall within bands that merge by pairs at times \( \tau = 2^{N+n}, n = 1, ..., N \). The sensitivity to initial conditions grows as \( \xi_t = \exp(q_0(\lambda q_0 t)) (q_0 = 1 - \ln 2/\ln \alpha < 1) \) for \( t < t_x \), but for \( t > t_x \) the fluctuations dominate and \( \xi_t \) grows exponentially as the trajectory has become chaotic and so one anticipates an exponential \( \xi_t \) (or \( q = 1 \)). We interpret this behavior to be the dynamical system analog of the '\( \alpha \)' relaxation in supercooled fluids. The plateau duration \( t_x \to \infty \) as \( \varepsilon \to 0 \). Additionally, trajectories with initial conditions \( x_0 \) not belonging to the attractor exhibit an initial relaxation stretch towards the plateau as the orbit falls into the attractor. This appears as the analog of the '\( \beta \)' relaxation in supercooled liquids. See [12].

Next, we determine the entropy of the orbits starting at \( x_0 = 0 \) as they enter the bifurcation gap at \( t_x(\varepsilon) \) when the maximum number \( 2^N \) of bands allowed by the fluctuations is reached. The entropy \( S_c(\mu_\infty, 2^N) \) associated to the state at \( \mu_\infty(\varepsilon) \) has the form

\[ S_c(\mu_\infty, 2^N) = 2^N \varepsilon s, \]  

(28)

since each of the \( 2^N \) bands contributes with an entropy \( \varepsilon s \) where

\[ s = - \int_{-1}^{1} p(\chi) \ln p(\chi) d\chi, \]  

(29)

and where \( p(\chi) \) is the distribution for the noise random variable. In terms of \( t_x \), given that \( 2^N = 1 + t_x \) and \( \varepsilon = (1 + t_x)^{-1/1-r} \), one has

\[ S_c(\mu_\infty, t_x)/s = (1 + t_x)^{-r/1-r} \]  

(30)
or, conversely,
\[
t_x = (s/S_c)^{(1-\nu)/\nu}.
\] (31)
Since \( t_x \approx \varepsilon^{-1}, \) \( r - 1 \approx -0.3668 \) and \( (1 - r)/r \approx 0.5792 \) then \( t_x \to \infty \) and \( S_c \to 0 \) as \( \varepsilon \to 0, \) i.e. the relaxation time diverges as the 'landscape' entropy vanishes. We interpret this relationship between \( t_x \) and the entropy \( S_c \) to be the dynamical system analog of the Adam-Gibbs formula for a supercooled liquid. See [12]. Notice that Eq.(31) is a power law in \( S_c^{-1} \) while for structural glasses it is an exponential in \( S_c^{-1} \) [20].

Last, we examine the aging scaling property of the trajectories \( x_t \) at \( \mu_\infty(\varepsilon). \) The case \( \varepsilon = 0 \) is more readily appraised because this property is actually built into the position subsequences \( x_\tau = |g^{(\tau)}(0)|, \tau = (2l + 1)2^n, l, n = 0, 1, \ldots \) These subsequences are relevant for the description of trajectories that are 'detained' at a given attractor position for a waiting period of time \( t_w \) and then 'released' to the normal iterative procedure. We chose the holding positions to be any of those along the top band shown in Fig. 1 for a waiting time \( t_w = 2l + 1, l = 0, 1, \ldots \) Notice that, as shown in Fig. 1, for the \( x_0 = 0 \) orbit these positions are visited at odd iteration times. The lower-bound positions for these trajectories are given by those of the subsequences at times \( (2l + 1)2^n \) (see Fig. 1). Writing \( \tau \) as \( \tau = t_w + t \) we have that \( t/t_w = 2^n - 1 \) and
\[
x_{t+t_w} = g^{(t_w)}(0)g^{(t/t_w)}(0)
\] (32)
or
\[
x_{t+t_w} = g^{(t_w)}(0) \exp_q(-\lambda_q t/t_w).
\] (33)
This property is gradually modified when noise is turned on. The presence of a bifurcation gap limits its range of validity to total times \( t_w + t < t_x(\varepsilon) \) and so progressively disappears as \( \varepsilon \) is increased. See Ref. [12].

8 Concluding remarks

We have re-examined the dynamical properties at the onset of chaos in the logistic map and obtained further understanding about their nature. We exhibited links between original developments, such as Feigenbaum’s \( \sigma \) function, Mori’s \( q \)-phase transitions and the noise-induced bifurcation gap, with more recent advances, such as \( q \)-exponential sensitivity to initial conditions
The dynamics at the edge of chaos is anomalous because it is an incipient chaotic attractor with vanishing ordinary Lyapunov coefficient $\lambda_1$. Chaotic orbits possess a time irreversible property that stems from mixing in phase space and loss of memory, but orbits within critical attractors are non-mixing and have no loss of memory. As a classic illustration of the latter case the attractor at the onset of chaos presents dynamical properties with self-similar structure that result in a set of power laws for the sensitivity to initial conditions. We determined exact analytical expressions for $\xi_t$.

Our most striking finding is that the dynamics at the onset of chaos is constituted by an infinite family of Mori’s $q$-phase transitions, each associated to orbits that have common starting and finishing positions located at specific regions of the attractor. Each of these transitions is related to a discontinuity in the $\sigma$ function of ‘diameter ratios’, and this in turn implies a $q$-exponential $\xi_t$ and a spectrum of $q$-Lyapunov coefficients for each set of orbits. The transitions come in pairs with specific conjugate indexes $q$ and $Q = 2 - q$, as these correspond to switching starting and finishing orbital positions. Since the amplitude of the discontinuities in $\sigma$ diminishes rapidly, in practical terms there is only need of evaluation for the first few of them. The dominant discontinuity is associated to the most crowded and sparse regions of the attractor and this alone provides a very reasonable description, as found in earlier studies [7] [16] [9] [11]. Thus, the special values for the Tsallis entropic index $q$ in $\xi_t$ are equal to the special values of the variable $q$ in the formalism of Mori and colleagues at which the $q$-phase transitions take place.

As described, the dynamics of noise-perturbed logistic maps at the chaos threshold presents the characteristic features of glassy dynamics observed in supercooled liquids. In particular our results are [12]: i) The two-step relaxation that takes place when $\varepsilon \to 0$ is obtained in terms of the bifurcation gap properties, specifically, the plateau duration $t_x$ is given by a power law in the noise amplitude $\varepsilon$. ii) The map analog of the Adam-Gibbs law is given also as a power-law relation between $t_x(\varepsilon)$ and the entropy $S_c(\varepsilon)$ associated to the noise widening of chaotic bands. iii) The trajectories at $\mu_\infty(\varepsilon \to 0)$ are shown to obey a scaling property, characteristic of aging in glassy dynamics, of the form $x_{t+t_w} = h(t_w)h( t/t_w)$ where $t_w$ is a waiting time.

The limit of vanishing noise amplitude $\varepsilon \to 0$ (the counterpart of the limit $T - T_g \to 0$ in the supercooled liquid) brings about loss of ergodicity. This nonergodic state with $\lambda_1 = 0$ corresponds to the limiting state, $\varepsilon \to$
0, \( t_x \to \infty \), for a family of small \( \varepsilon \) states with glassy properties, which are expressed for \( t < t_x \) via the \( q \)-exponentials of the Tsallis formalism. It has been suggested on several occasions [22] [23] that the setting in which nonextensive statistics appears to come out is linked to the prevalence of nonuniform convergence, such as that involving the thermodynamic \( N \to \infty \) and the infinitely large time \( t \to \infty \) limits. Here a similar situation happens, that is, if \( \varepsilon \to 0 \) is taken before \( t \to \infty \) a nonergodic orbit restrained to the Feigenbaum attractor and with fully-developed glassy properties is obtained, whereas if \( t \to \infty \) is taken before \( \varepsilon \to 0 \) a chaotic orbit with \( q = 1 \) would be observed.

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