Universal moduli spaces of vector bundles and the log-minimal model program on the moduli of curves

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Abstract

Recent work on the log minimal model program for the moduli space of curves, as well as past results of Caporaso, Pandharipande, and Simpson motivate an investigation of compactifications of the universal moduli space of slope semi-stable vector bundles over moduli spaces of curves arising in the Hassett-Keel program. Our main result is the construction of a universal moduli space of slope semi-stable sheaves which compactifies the moduli space of vector bundles over the moduli space of pseudo-stable curves.

1 Introduction

Moduli spaces of vector bundles over smooth curves have long been a subject of interest in algebraic geometry. Recall that due to the work of Mumford, Newstead, and Seshadri there is a projective moduli space \( U_{e,r}(C) \) of slope semi-stable vector bundles of degree \( e \) and rank \( r \) over \( C \). The result has since been generalized to other settings by many others. In particular, in \([\text{Sim94}]\), Simpson constructed a moduli space of slope semi-stable sheaves on families of polarized projective schemes.

As a consequence of Simpson’s result, there is a moduli space of slope semi-stable vector bundles for the universal curve \( C_g \to M_g \) over the moduli space of smooth, automorphism-free curves, polarized by the relative canonical bundle (for \( g \geq 2 \)). Though the scheme \( \overline{M}_g \) does not admit a universal curve, it is natural to ask whether there is a universal moduli space of slope semi-stable bundles over \( \overline{M}_g \) which compactifies the moduli space over the space of automorphism-free curves. Caporaso and Pandharipande affirmatively answered this question in the case \( r = 1 \) and for general rank, respectively in \([\text{Pan96}, \text{Cap94}]\).

Specifically, in \([\text{Pan96}]\), Pandharipande constructed a compactification for \( g \geq 2 \)

\[
U_{e,r,g} \to \overline{M}_g
\]
parametrizing slope semistable, torsion-free sheaves of uniform rank $r$ and degree $d$, with a dense open subset identified with a subset of Simpson’s moduli space.

More recently, with the aim of providing a modular interpretation for the canonical model of the moduli space of curves, there has been interest in understanding alternate modular compactifications of the moduli space of curves. The Hassett-Keel program outlines a principle for applying the log-minimal model program to the moduli space of genus $g$ curves to obtain a modular interpretation of the canonical model by studying spaces of the form

$$\overline{M}_g(\alpha) := \text{Proj} \left( \bigoplus_n H^0(\overline{M}_g, n(K_{\overline{M}_g} + \alpha \Delta)) \right),$$

where $\Delta$ is the boundary divisor in $\overline{M}_g$ and $\alpha \in [0, 1] \cap \mathbb{Q}$. One has $\overline{M}_g(1) = \overline{M}_g$ and $\overline{M}_g(0)$ equal to the canonical model of $M_g$ for $g \gg 0$ (HM82, EH87, Far09).

The first steps in the program have been worked out in [HH09, HH13, AFSvdW13]. We direct the reader as well to [FS10, Hye10] for more details.

For $\alpha = 9/11$, Hassett and Hyeon showed (HH09) that the first birational modification is a divisorial contraction,$$
\overline{M}_g(9/11) \sim \overline{M}^{ps}_g,$$
where $\overline{M}^{ps}_g$ is Schubert’s moduli space of pseudo-stable curves (see [Sch91]). The contraction essentially replaces elliptic tails with cusps.

It is natural to ask for a given $\alpha$ if there exists a universal moduli space of slope semi-stable sheaves

$$\overline{U}_{e,r,g}(\alpha) \to \overline{M}_g(\alpha).$$

The answer for $7/10 < \alpha \leq 9/11$ is our main theorem.

**Theorem.** For genus $g \geq 2$, there exists an irreducible moduli space

$$\overline{U}^{ps}_{e,r,g} \to \overline{M}^{ps}_g$$

parametrizing slope semi-stable, torsion-free sheaves of degree $e$ and uniform rank $r$ on pseudo-stable curves.

The case $r = 1$ was established in [BFMV11] using a different approach more in line with that of [Cap94]. We now outline our strategy for proving the result.

First, we establish notation and several local results concerning sheaves on singular curves. The next section involves the fiberwise GIT problem; i.e., the moduli problem for a fixed projective, Gorenstein curve $C$. In [Sim94], Simpson constructed a compactification of the moduli space of slope semi-stable vector bundles over $C$ in greater generality. His construction utilizes an asymptotic description of the GIT stability of sheaves. Specifically, he studies the component $Q$ of the Quot scheme containing slope semi-stable, rank $r$, torsion-free sheaves of degree $d$ on $C$. Studying a very ample linearization $O_Q(1)$, he proves
that for $N \gg 0$, GIT (semi-)stability with respect to $O_Q(N)$ is equivalent to slope (semi-)stability. Our attention is restricted to the component $Q^r$ of $Q$ containing sheaves of uniform rank $r$. Our result strengthens the application of Simpson’s result for the case of Gorenstein curves to $Q^r$. We provide specific bounds for $N$, depending only on $d$, $g$, $r$, and the singularity type of $C$. We wish to study GIT stability for the analogous problem over families of curves. Our approach employs a variation of GIT argument to reduce the problem to GIT stability for an individual curve. The uniformity of our bounds is the key point which allows this argument to work. The result for nodal curves was established by Pandharipande [Pan96]. Our argument closely follows his, but differs in two ways:

1. We construct the moduli of vector bundles over a fixed Gorenstein curve $C$, using an arbitrary polarization of the curve.

2. In the construction of the universal moduli space, the curves we consider are pseudo-stable, not stable. This complicates certain bounds and is dealt with mostly in Section 2.2.

At this point, we construct our moduli space. Our construction is a GIT quotient of a certain Quot scheme. The main difficulty lies in characterizing the semi-stable and stable loci. By shifting the weight of the group action almost entirely onto the curve, the stability of a pair can be ascertained by studying only the stability of the sheaf. A key condition that allows our argument to work is the fact that there are no strictly semi-stable points in $\mathcal{M}^{ss}$.

Lastly, we provide a functorial description of our moduli problem, and prove that our GIT quotient is the categorical moduli space. Here we also demonstrate the irreducibility of the quotient.

Moving forward, we would like to construct analogous moduli spaces over each of the Hassett-Keel moduli spaces. As mentioned above, one of the first obstructions to applying the same techniques is the existence of strictly semi-stable curves. Additionally, most of the techniques in this paper require the assumption that the curves are all reduced. Recent work by Chen and Kass ([CK11]) has hinted at a way forward which we are actively pursuing.

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2 Preliminaries

In this section, we make our definitions and state our conventions. We also include here various technical results from other sources for completeness.
2.1 Notations and Conventions

**Notation 2.1.1** (Curve). A *curve* is a proper one-dimensional scheme over the complex numbers. The *genus* of a curve $C$ will refer to $h^1(C, \mathcal{O}_C)$.

**Definition 2.1.2.** For a reduced curve $C$, the singularity type of $C$ is defined to be

$$T = T(C) := \{[\hat{O}_{C,x}] : x \in C\},$$

where $[\hat{O}_{C,x}]$ denotes the isomorphism class of the ring $\hat{O}_{C,x}$.

**Remark 2.1.3.** As there are only a finite number of singular points in a given curve, and the complete local ring on a smooth point is the completion of a polynomial ring, there are only a finite number of isomorphism classes of rings in $T(C)$. Given a set $\mathcal{T}$ of isomorphism classes of complete local rings, we say that a curve has at worst singularities of type $\mathcal{T}$ if $T(C) \subset \mathcal{T}$.

**Definition 2.1.4** (Rank and Degree). Let $(C, L)$ be a polarized curve of genus $g$ with $\deg L = d$ and let $F$ be a coherent sheaf on $C$. If $\Phi(t) = \chi(F \otimes L^t)$, the *rank* and *degree* of $F$ with respect to $L$ are the unique pair $(r, e)$ such that

$$\Phi(t) = e + r(1 - g) + rdt.$$ 

We will refer to these numbers as $\text{rank}_L F$, $\text{deg}_L F$.

The pair of degree and rank with respect to the polarization are integers. In fact, we have

$$\chi(C, F \otimes L^t) = \frac{r}{n!} t^n + \ldots,$$

where $n = \dim \text{Supp} F$. We will make extensive use of the fact that, for a coherent sheaf $F$ and a line bundle $M$, we have

$$\text{rank}_L (F \otimes M) = \text{rank}_L F, \quad \text{deg}_L (F \otimes M) = \text{deg}_L (F) + \text{rank}_L (F) \text{deg}(M).$$

**Definition 2.1.5.** A sheaf $F$ on $C$ is said to be of uniform rank if there exists an integer $r$ such that for every component $C_i$ of $C$, $\text{rank} F|_{C_i} = r$.

**Lemma 2.1.1** ([Ser82](#2222), Corollary 8, p. 152). Let $(C, L)$ be a polarized curve with $\deg L = d$. For the irreducible components of $C$, $\{C_i\}$, denote by $L_i$ the restriction $L|_{C_i}$. Let $d_i = \deg L_i$. Then for any coherent sheaf $F$, we have

$$\chi(F \otimes L^t) = \chi(F) + t \sum_i r_i d_i,$$

where $r_i := \dim_{k(\eta_i)} F|_{C_i} \otimes k(\eta_i)$ and $\eta_i$ is the generic point of $C_i$.

**Remark 2.1.6.** An immediate consequence is that $\sum_i r_i d_i = rd$. In particular, because $d_i > 0$ for every $i$, $r_i \leq rd$. Moreover, when $C$ is irreducible, the rank of $F$ is equal to the dimension of the stalk at the generic point of $C$. 


A coherent sheaf \( F \) on \( C \) is said to be pure if for every non-zero subsheaf \( F' \subset F \), the dimension of the support of \( F' \) is equal to the dimension of the support of \( F \). A coherent sheaf \( F \) on \( C \) is said to be torsion-free if it is pure and the support of \( F \) is equal to \( C \).

**Definition 2.1.7.** Let \((C, L)\) be a polarized curve and \( F \) a torsion-free sheaf on \( C \). \( F \) is said to be slope stable (slope semi-stable) with respect to \( L \) if for every nonzero, proper subsheaf \( 0 \to E \to F \),

\[
\chi(E) \sum s_i d_i < (\leq) \chi(F) \sum r_i d_i,
\]

where \( s_i \) and \( r_i \) denote the ranks of \( E \) and \( F \) on each irreducible component of \( C \), and \( d_i \) is the degree of \( L \) restricted to each irreducible component.

**Remark 2.1.8.** For torsion-free sheaves of uniform rank, rank and degree are independent of the polarization. Indeed, we see from Remark 2.1.6 that the generic rank of such a sheaf must agree with the rank with respect to a polarization, and then the degrees must also agree.

**Remark 2.1.9.** If \((C, L)\) is a polarized nonsingular curve and \( E \) is a vector bundle on \( C \), then \( E \) is slope-stable (slope-semistable) with respect to \( L \) if and only if for each nonzero subsheaf \( 0 \to E \to F \),

\[
\frac{\text{degree}(E)}{\text{rank}(E)} < (\leq) \frac{\text{degree}(F)}{\text{rank}(F)}.
\]

Note that slope-stability for a torsion-free sheaf of uniform rank is independent of the polarization.

### 2.2 Sheaves on Singular Irreducible Curves

Let \( A \) be a ring, \( \tilde{A} \) its integral closure. Recall that the conductor ideal of \( A \) is the largest ideal \( \mathfrak{c} \subset A \) such that \( \mathfrak{c} \tilde{A} \subset A \).

**Note 2.2.1.** The support of \( A/\mathfrak{c} \) is the set of primes where \( A \to \tilde{A} \) fails to be an isomorphism. When \( A \) is a local Noetherian ring with maximal ideal \( \mathfrak{m} \) it is well-known that we have \( \sqrt{\mathfrak{c}} = \mathfrak{m} \) and \( \mathfrak{m}^n \subset \mathfrak{c} \subset \mathfrak{m} \) for some \( n \). If \( A \) is a Noetherian local ring essentially of finite type over a field with residue field \( k \), the ring \( A/\mathfrak{c} \) is a finite-dimensional \( k \)-vector space. The dimension of \( A/\mathfrak{c} \) is a local analytic invariant in the sense that completing along the maximal ideal does not change the dimension:

\[
\dim_k A/\mathfrak{c} = \dim_k \tilde{A}/\mathfrak{c}.
\]

The result follows from the fact that with the hypotheses on \( A \) the conductor of the completion of \( A \) is the completion of the conductor of \( A \). We will make use of the conductor ideal in the sequel to construct a bound on certain quotients of coherent sheaves:

\[
\dim_k \mathcal{F}/m_x \mathcal{F},
\]

where \( \mathcal{F} \) is a sheaf, and \( m_x \) the ideal of a point of a curve.
We will make use of several results from [Ses82], which we include here for convenience. For the following results, let $r > 0$ be an integer, and $Y$ an irreducible projective curve over $\mathbb{C}$. Let $x \in Y$ be a point and $\nu : Y' \to Y$ the normalization of $Y$. Let $x_1, \ldots, x_p$ be the points of $Y'$ over $x$. There is a canonical isomorphism

$$\overline{\mathcal{O}}_{Y,x} = \mathcal{O}_{Y,\nu,x_1} \cap \cdots \cap \mathcal{O}_{Y,\nu,x_p},$$

where $\overline{\mathcal{O}}_{Y,x}$ is the integral closure of $\mathcal{O}_{Y,x}$ (see [HS06, Corollary 2.1.13]) and the intersection takes place in the field of fractions of $Y'$. The maximal ideals of $\overline{\mathcal{O}}_{Y,x}$ are

$$m'_{x_i} := m_{x_i} \cap \left( \bigcap_{j=1}^p \mathcal{O}_{Y,\nu,x_j} \right).$$

Moreover, $\overline{\mathcal{O}}_{Y,x}$ is principal and each ideal of $\overline{\mathcal{O}}_{Y,x}$ can be expressed uniquely as a product of powers of these maximal ideals. This follows because $\overline{\mathcal{O}}_{Y,x}$ is a Dedekind domain. Let $\mathcal{O}_Y(1)$ denote a fixed ample line bundle on $Y$.

**Lemma 2.2.1** ([Ses82, Lemma 5, p. 149]). Let $F$ be a coherent torsion-free sheaf of rank $r$ over and irreducible curve $Y$. There exists an integer $m$ such that there is an injection of sheaves

$$F(-m) \to \mathcal{O}_Y^r.$$

**Lemma 2.2.2** ([Ses82, Lemma 6, p. 149]). Let $M$ be a torsion-free $\mathcal{O}_{Y,x}$-module of rank $r$. By Lemma 2.2.1, we may assume $M \subset \mathcal{O}_Y^r$. There exist elements $m_1, \ldots, m_r$ of $M$ such that

$$\nu^{-1}M \cong \bigoplus_{i=1}^r \overline{\mathcal{O}}_{Y,x}m_i,$$

where $\nu^{-1}M$ denotes submodule generated by the image of $M$ in $\overline{\mathcal{O}}_{Y,x}$.

We arrive at the key result of this subsection. We are interested in the interaction of singularities on a curve with sheaves on the curve. The following lemma allows us to bound the dimension of certain quotients in terms of analytic invariants of the curve in question. The lemma is almost entirely the same as that [Ses82, Lemma 7, p. 150], but the proof there only concerns curves with at worst nodal singularities. The introduction of the conductor ideal allows us the extend the result to more general singularities.

**Lemma 2.2.3.** Let $M$ be an $\mathcal{O}_{Y,x}$-module, torsion-free of rank $r$ and $\mathfrak{c}$ the conductor ideal of $\mathcal{O}_{Y,x}$. Then

$$\dim_k (M/m_xM) \leq \left( \dim_k \mathcal{O}_{Y,x}/\mathfrak{c} + \dim_k \overline{\mathcal{O}}_{Y,x}/\mathcal{O}_{Y,x} \right) \cdot r.$$
Proof. By Lemma 2.2.2 we may write

\[ \nu^{-1}M = \bigoplus_{i=1}^{r} \mathcal{O}_{Y,x}m_i \]

for some \( m_1, \ldots, m_r \in M \). Thus, because \( \nu^{-1}M = \nu^{-1}\mathfrak{c}M \),

\[
\dim_k(\nu^{-1}M/\nu^{-1}\mathfrak{c}M) \leq r \dim_k \mathcal{O}_{Y,x}/\mathfrak{c},
\]

\[
\quad \leq (\dim_k \mathcal{O}_{Y,x}/\mathcal{O}_{Y,x} + \dim_k(\mathcal{O}_{Y,x}/\mathfrak{c})) r.
\]

Now we need to connect the result to \( M/\mathfrak{m}_xM \). Note that

\[
\dim_k M/\mathfrak{m}_xM + \dim_k \mathfrak{m}_xM/\mathfrak{c}M = \dim_k M/\mathfrak{c}M.
\]

In particular, \( \dim_k M/\mathfrak{m}_xM \leq \dim_k M/\mathfrak{c}M \). We have the following diagram of \( \mathcal{O}_{Y,x} \)-modules:

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathfrak{c}M & \rightarrow & M & \rightarrow & M/\mathfrak{c}M & \rightarrow & 0 \\
0 & \rightarrow & \nu^{-1}\mathfrak{c}M & \rightarrow & \nu^{-1}M & \rightarrow & \nu^{-1}M/\nu^{-1}\mathfrak{c}M & \rightarrow & 0
\end{array}
\]

Of particular relevance is the surjection \( \mathfrak{c}M \rightarrow \nu^{-1}\mathfrak{c}M \); this is due to \( \mathfrak{c} \) being the conductor: \( \nu^{-1}\mathfrak{c}M \) is generated by elements of the form \( acm \), where \( a \in \mathcal{O}_{Y,x} \), \( c \in \mathfrak{c} \), and \( m \in M \). But \( \mathfrak{c} \) is the conductor, and so \( ac \in \mathfrak{c} \). Thus, \( \nu^{-1}\mathfrak{c}M \) is generated by elements of the form \( cm \). The morphism \( M \rightarrow \nu^{-1}M \) is an inclusion, and so, by the snake lemma, we deduce that

\[ 0 \rightarrow M/\mathfrak{c}M \rightarrow \nu^{-1}M/\nu^{-1}\mathfrak{c}M. \]

Thus, we have

\[ \dim_k M/\mathfrak{m}_xM \leq \dim_k M/\mathfrak{c}M \leq (\dim_k \mathcal{O}_{Y,x}/\mathcal{O}_{Y,x} + \dim_k(\mathcal{O}_{Y,x}/\mathfrak{c})) r, \]

as claimed. \( \square \)

The statement and proof of Lemma 2.2.3 assume \( Y \) is irreducible. We have the following bound for reducible \( Y \).

Corollary 2.2.4. Suppose \( Y \) is reducible and polarized by \( L \rightarrow Y \) with \( \deg L = d \). Let \( \mathcal{F} \) be a coherent \( \mathcal{O}_Y \)-module of multirank \((r_1, \ldots, r_p)\). Then for any \( y \in Y \),

\[ \dim_k (\mathcal{F}_y/m_y\mathcal{F}_y) \leq d\delta \max_i(r_i), \]

where

\[ \delta = \max_{A \in T(Y)} (\dim_k A/\mathfrak{c} + \dim_k A/A). \]

Proof. This follows by bounding the restriction of \( \mathcal{F} \) to each component of \( Y \) \((\text{Ses82}, \text{p. 152})\) and observing that \( d \) is greater than the number of irreducible components of \( Y \). \( \square \)
3 Fiberwise Stability

Our main result for this section is the calculation of the fiberwise GIT quotient. We compare GIT stability for sheaves on a fixed curve with slope stability:

**Theorem 3.0.5.** Let \( g \geq 2 \), \( r \), and \( d \) be integers. Define \( \Phi(t) = e + r(1-g) + rd/t \), and let \( n = \Phi(0) \). Let \((C, L)\) be a polarized reduced Gorenstein curve such that \( \deg L = d \) and let \( t(d, g, r, e) \) have the property that for all \( t > t(d, g, r, e) \), the morphism

\[
i_t : \text{Quot}^{\Phi, L}_{C^n \otimes \mathcal{O}_C/C/\text{Spec} \ C} \to G(\Phi(t), (C^n \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t))^*))
\]

is a closed embedding.

Let \( T \) be the singularity type of \( C \). There exists integers \( E(d, g, r, T) \) and \( t(d, g, r, e) \) (see Definition 3.5.1 for an explicit description) such that for all \( e > E(d, g, r, T) \) and \( t > t(d, g, r, e) \), we have the following:

if \( \xi \in \text{Quot}^{\Phi, L}_{C^n \otimes \mathcal{O}_C/C/\text{Spec} \ C} \) corresponds to a quotient

\[
\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,
\]

where \( E \) is of rank \( r \), then \( \xi \) is GIT stable (semi-stable) with respect to the \( SL_n \)-linearization determined by \( i_t \) if and only if \( E \) is a slope stable (semi-stable) torsion-free sheaf on \( C \) and the induced function

\[
\mathbb{C}^{\chi(E)} \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)
\]

is an isomorphism.

A few remarks are in order.

**Remark 3.0.2.** Theorem 3.0.5 is a direct generalization of [Pan96, Thm. 2.1.1]. Here, we remove the hypotheses that the curve be Deligne-Mumford stable, and that the polarization be given by the dualizing bundle (in later applications to pseudo-stable curves, the polarization is not seen by the moduli problem).

We will show later that the space \( \overline{U}_{e,r}(C) \) is a subspace of the moduli space of sheaves described in [Sim94, Thm. 1.21]. While Simpson’s result provides a moduli space of sheaves in a more general setting, our approach provides a uniform bound \( \hat{t}(d, g, r, e) \), which depends only on these discrete invariants. Simpson established that, for a given curve and high degree, slope stability is equivalent to GIT stability. Our uniform bound allows us to perform this classification for all curves in consideration at once, and therefore apply the result to families of sheaves over families of curves. In addition, we restrict our attention to sheaves of uniform rank.

We now outline the strategy of the proof. We begin with GIT destabilization arguments. Specifically, we demonstrate that any quotient which is not slope-semistable, torsion-free, with an induced isomorphism on global sections is GIT unstable. Then, we study GIT stabilization. We establish that the remaining
sheaves are GIT semi-stable, and show that GIT stability is equivalent to slope-stability. The bulk of the argument is a straightforward extension of the argument presented in [Pan96]. The inclusion of pseudo-stable curves primarily affects the statement and proof of Proposition 3.4.1, where a more subtle bound of the numerical properties of singularities is required.

3.1 Numerical Criterion for Grassmannians

The following lemma, adapted from [Pan96], provides a useful reformulation of the Hilbert-Mumford numerical criterion for Grassmannians. The lemma constitutes the principal technical tool we use for destabilizing sheaves on curves.

Lemma 3.1.1 ([Pan96, Lemma 2.3.1]). Let \( g \geq 2, r, \) and \( d \) be integers. Define \( \Phi(t) = e + r(1 - g) + rdt \), and let \( n = \Phi(0) \). Let \((C, L)\) be a polarized reduced Gorenstein curve such that \( \deg L = d \) and let \( \hat{t}(d, g, r, e) \) have the property that for all \( t > \hat{t}(d, g, r, e) \), the morphism

\[
i_t : \text{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C/\text{Spec} \mathbb{C}} \to \mathcal{G}(\Phi(t), (\mathbb{C}^n \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)))^*)
\]

is a closed embedding.

Let \( \xi \in \text{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C/\text{Spec} \mathbb{C}} \) correspond to a quotient

\[
\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0.
\]

Let \( U \subset \mathbb{C}^n \) be a subspace, and define \( W := \text{im} (U \otimes H^0(C, \mathcal{O}_C)) \subset H^0(C, E) \). Let \( G \) be the subsheaf of \( E \) generated by \( W \). Then if

\[
\frac{\dim(U)}{n} > \frac{h^0(C, G \otimes L^t)}{\Phi(t)},
\]

\( \xi \) is GIT unstable.

3.2 Destabilization

We recall our setup. Let \( g \geq 2, r, \) and \( d \) be integers. Define \( \Phi(t) = e + r(1 - g) + rdt \), and let \( n = \Phi(0) \). Let \((C, L)\) be a polarized reduced Gorenstein curve such that \( \deg L = d \) and let \( \hat{t}(d, g, r, e) \) have the property that for all \( t > \hat{t}(d, g, r, e) \), the morphism

\[
i_t : \text{Quot}^{\Phi,L}_{\mathbb{C}^n \otimes \mathcal{O}_C/C/\text{Spec} \mathbb{C}} \to \mathcal{G}(\Phi(t), (\mathbb{C}^n \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)))^*)
\]

is a closed embedding.

Our strategy for understanding the points of the GIT quotient is to begin with the destabilization of certain points. First, we demonstrate that semi-stability requires the presentation of a sheaf to induce an injection on global sections. This allows us to then place bounds on the size of torsion. Next, we destabilize slope-unstable sheaves, and finally demonstrate that the presentation must be surjective on global sections. Combined with the bound on torsion, we
will see that GIT semi-stable sheaves must be torsion-free. In summary, by the end of the section we will have established that for large $e$ and $t$ (with explicit bounds), GIT semi-stable sheaves must be torsion-free, slope semi-stable, and have a presentation which induces an isomorphism on global sections.

### 3.2.1 Injectivity

We immediately destabilize a large class of points.

**Proposition 3.2.1** ([Pan96, Prop. 2.3.1]). Let $\xi \in \text{Quot}_{\mathbb{C}^n \otimes O_C / \text{Spec } \mathbb{C}}^{\Phi, L}$ correspond to a quotient

$$\mathbb{C}^n \otimes O_C \to E \to 0.$$  

Then if $\mathbb{C}^n \otimes H^0(C, O_C) \to H^0(C, E)$ is not injective, $\xi$ is GIT unstable.

**Proof.** If the morphism on global sections is not injective, let $U \otimes H^0(C, O_C)$ be its kernel. Because its image generates the zero sheaf, we have

$$\frac{\dim(U)}{n} > 0,$$

and we may conclude that $\xi$ is unstable by Lemma 3.1.1. □

### 3.2.2 Torsion

The main result of this section places restrictions on the size of torsion in GIT semi-stable sheaves. Once we establish that semi-stability of a presentation of a sheaf requires an isomorphism on global sections, we will conclude that sheaves with torsion are GIT unstable.

**Proposition 3.2.2** ([Pan96, Prop. 3.2.1]). There exists a bound $t_0(d, g, r, e) > i(d, g, r, e)$ such that for each $t > t_0(d, g, r, e)$ and $\xi \in \text{Quot}_{\mathbb{C}^n \otimes O_C / \text{Spec } \mathbb{C}}^{\Phi, L}$ corresponding to

$$\mathbb{C}^n \otimes O_C \to E \to 0,$$

if \( \text{im} \left( \mathbb{C}^n \otimes H^0(C, O_C) \right) \cap H^0(C, \tau_E) \neq 0 \), where $\tau_E$ is the torsion subsheaf of $E$, $\xi$ is GIT-unstable.

The proof of the proposition relies on the following lemma.

**Lemma 3.2.3** ([Pan96, Lemma 3.1.1]). Let $R > 0$ be an integer. There exists an integer $b(d, g, R)$ with the following property:

- If $E$ is a coherent sheaf on $C$ such that
  - (1) $E$ is generated by global sections.
  - (2) $E$ has generic rank less than $R$ on every irreducible component of $C$.

Then $h^1(C, E) < b(d, g, R)$. In particular, we may take $b(d, g, R) = dgR + 1$.

Now we will prove the proposition. The plan is to find a subspace of $\mathbb{C}^n \otimes H^0(C, O_C)$ which maps to torsion, and then apply Lemma 3.1.1.
Proof of Proposition 3.2.2. We now sketch a proof. Let $U \subset \mathbb{C}^n$ be a one-dimensional subspace such that $W := \text{im } (U \otimes H^0(C, \mathcal{O}_C)) \subset H^0(C, \tau_E)$. Let $G$ be the subsheaf of $E$ generated by $W$. For all $t$ we have

$$h^0(C, G \otimes L^t) \leq h^0(C, \tau_E \otimes L^t) = h^0(C, \tau_E).$$

Now, to apply Lemma 3.2.3 to $E$, we must demonstrate that

1. $E$ is generated by global sections, and
2. there is an $R > 0$ which bounds the multirank of $E$.

The first is true by assumption, and the second is given by $rd + 1$ (see Remark 2.1.6). Thus, there is a $b(d, g, R)$ such that $h^1(C, E) < b(d, g, R)$. We then have

$$h^0(C, \tau_E) \leq h^0(C, E) = \chi(E) + h^1(C, E) < \chi(E) + b(d, g, R) = \Phi(0) + b(d, g, R).$$

The result now follows for $t$ satisfying

$$\Phi(t) > n \cdot (\Phi(0) + b(d, g, rd + 1)),$$

by Lemma 3.1.1.

3.2.3 Slope Stability

In this subsection, we destabilize certain slope-unstable sheaves. The primary result is

**Proposition 3.2.4 ([Pan96, Prop. 4.1.1]).** There exist bounds $e_1(d, g, r) > r(g-1)$ and $t_1(d, g, r, e) > t(d, g, r, e)$ such that for any pair $e > e_1(d, g, r)$ and $t > t_1(d, g, r, e)$, the following holds: If $\xi \in \text{Quot}_{\mathbb{C}^n \otimes \mathcal{O}_C/C/\text{Spec } \mathbb{C}}(E)$ corresponds to $\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0$, where the induced morphism on global sections is an isomorphism and $E$ is a slope-unstable torsion-free sheaf, then $\xi$ is GIT unstable with respect to the linearization induced by $t_1$.

As usual, our goal is to apply Lemma 3.1.1. We will require two lemmas whose proof we omit. They include references which will allow the interested reader to fill in the details. The first allows us to destabilize a slope-unstable sheaf by a subsheaf which is nearly generated by global sections:

**Lemma 3.2.5 ([Pan96, Lemma 4.2.1]).** Assume the hypotheses of Proposition 3.2.4. There exists an integer $e_1(d, g, r) > r(g-1)$ such that for each $e > e_1(d, g, r)$, the following holds:

if $\xi \in \text{Quot}_{\mathbb{C}^n \otimes \mathcal{O}_C/C/\text{Spec } \mathbb{C}}(E)$ corresponds to $\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0$,
where $E$ is slope-unstable and torsion-free, then there exists a nonzero, proper, destabilizing subsheaf $0 \to F \to E$ and an exact sequence

$$0 \to F \to F \to \tau \to 0,$$

where $\tau$ is generated by global sections and $\tau$ is torsion.

We use Lemma 3.2.5 to prove the following lemma. This lemma produces a sheaf which destabilizes according to the Numerical Criterion for Grassmannians:

**Lemma 3.2.6** ([Pan96, Lemma 4.2.2]). Assume the hypotheses of Proposition 3.2.4. Let $e > e_1(d, g, r)$ as described in Lemma 3.2.5. There exists an integer $t_1(d, g, r, e) > t_1(d, g, r, e)$ such that for all $t > t_1(d, g, r, e)$, the following holds:

If $\xi \in \text{Quot}_{\mathbb{C}^n \otimes \mathcal{O}_C / \text{Spec } \mathbb{C}} \mathcal{L}$ corresponds to

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where $E$ is a slope-unstable, torsion-free sheaf on $C$, then there exists a nonzero proper subsheaf $0 \to F \to E$ such that

$$\frac{h^0(C, F)}{n} \geq \frac{h^0(C, F \otimes L^t)}{\Phi(t)}.$$

Now we use Lemma 3.2.6 to prove Proposition 3.2.4.

**Proof of Proposition 3.2.4.** First, let $e > e_1(d, g, r)$ be as defined in Lemma 3.2.5. Now, taking $e > e_1(d, g, r)$, let $t_1(d, g, r, e)$ and $F$ be defined as in Lemma 3.2.6. Let $t > t_1(d, g, r, e)$.

Let $U \otimes H^0(C, \mathcal{O}_C) \subset \mathbb{C}^n \otimes H^0(C, \mathcal{O}_C)$ be the preimage of $H^0(C, F)$. We have $\mathbb{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$ by the hypothesis of the proposition, so $\dim U = h^0(C, F)$. Let $G$ be the subsheaf of $F$ generated by global sections. Certainly $h^0(C, G \otimes L^t) \leq h^0(C, F \otimes L^t)$, and so we have

$$\frac{\dim(U)}{n} = \frac{h^0(C, F)}{n} > \frac{h^0(C, F \otimes L^t)}{\Phi(t)} \geq \frac{h^0(C, G \otimes L^t)}{\Phi(t)},$$

which proves the proposition by Lemma 3.1.1. □

**3.2.4 Surjectivity**

In this subsection, we establish the surjectivity of the global sections morphism on GIT semistable sheaves. If $\xi \in \text{Quot}_{\mathbb{C}^n \otimes \mathcal{O}_C / \text{Spec } \mathbb{C}} \mathcal{L}$ corresponds to

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

because $n = \chi(E)$, we always have $n \leq h^0(C, E)$. In particular, if $H^0(C, \mathbb{C}^n \otimes \mathcal{O}_C) \to H^0(C, E)$ is a surjection, we have $n \geq h^0(C, E)$, which implies that
$h^1(C, E) = 0$. On the other hand, if $h^1(C, E) = 0$, then $n = \chi(E)$. Excluding GIT unstable sheaves, we may assume by Proposition 3.2.1 that the global sections morphism is an injective morphism of finite-dimensional vector spaces of the same dimension, and therefore an isomorphism. Our main result for this subsection thus concerns $h^1(C, E)$:

**Proposition 3.2.7** ([Pan96, Prop. 5.2.1]). There exist bounds $e_2(d, g, r) > r(g - 1)$ and $t_2(d, g, r, e) > \hat{t}(d, g, r, e)$ such that if $\xi \in \text{Quot}^{P,L}_{\mathbb{C}^n \otimes \mathcal{O}_C / \text{Spec} \mathbb{C}}$ corresponds to

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where $h^1(C, E) \neq 0$, then $\xi$ is GIT unstable.

**Proof.** We sketch the proof. Assume the conditions of Lemma 3.2.8 and Lemma 3.2.9 below are satisfied. By Proposition 3.2.1, we may assume that the global sections map $\psi$ is injective. By Proposition 3.2.2, we may assume that the codimension of the image of $\psi$ is bounded by the dimension of the space of torsion sections. Now Lemma 3.2.9 produces a subspace which destabilizes $\xi$ by Lemma 3.1.1. □

**Lemma 3.2.8** ([Pan96, Lemma 5.1.1]). There exists an integer $e_2(d, g, r) > r(g - 1)$ such that for each $e > e_2(d, g, r)$ the following holds:

Suppose that $E$ is a coherent sheaf on $C$ having Hilbert polynomial $\Phi(t)$ with respect to $L$, and $\tau$ is the maximal torsion sub-sheaf of $E$. If

1. $h^1(C, E) \neq 0$,
2. $\chi(\tau) < gd(rd + 1) + 1$,

then there exists a nonzero, proper subsheaf $F$ of $E$ with multirank $(s_i)$ not identically zero such that

1. $F$ is generated by global sections,
2. $$\frac{\chi(F) - (gd(rd + 1) + 1)}{\sum s_id_i} > \frac{\chi(E)}{rd} + 1.$$

**Lemma 3.2.9** ([Pan96, Lemma 5.1.2]). Let $e > e_2(d, g, r)$ as in Lemma 3.2.8. There exists an integer $t_2(d, g, r, e) > t_0(d, g, r, e)$ such that for each $t > t_2(d, g, r, e)$ the following holds:

If $\xi \in \text{Quot}^{P,L}_{\mathbb{C}^n \otimes \mathcal{O}_C / \text{Spec} \mathbb{C}}$ corresponds to

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0,$$

where $E$ is a coherent sheaf satisfying

(i) $\psi : \mathbb{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$ is injective,
(ii) $h^1(C, E) \neq 0$,
(iii) the torsion subsheaf $\tau$ of $E$ satisfies $\chi(\tau) < gd(rd + 1) + 1$,
then there exists a nonzero subspace $W \subset \psi(C^n \otimes H^0(C, O_C))$ generating a nonzero, proper subsheaf $0 \to G \to E$ such that
$$\dim W > \frac{h^0(C, G \otimes L^t)}{\Phi(t)}.$$  

### 3.3 GIT-semistable Sheaves

In this section, we demonstrate that the classes of quotients destabilized above are in fact the only unstable quotients. The central result is the following

**Proposition 3.3.1** ([Pan96 Prop. 6.1.1]). There exist bounds $e_3(d, g, r) > r(g - 1)$ and $t_3(d, g, r, e) > t_3(d, g, r, e)$ such that for each pair $e > e_3(d, g, r)$, $t > t_3(d, g, r, e)$, the following holds:

If $\xi \in \text{Quot}_{C^n \otimes O_C / C / \text{Spec } C}^{\Phi, L}$ corresponds to a quotient

$$C^n \otimes O_C \to E \to 0,$$

where

$$\psi : C^n \otimes H^0(C, O_C) \to H^0(C, E)$$

is an isomorphism and $E$ is a slope-stable (slope-semistable), torsion-free sheaf, then $\xi$ is a GIT stable (semi-stable) point.

**Proof.** The proof follows by explicitly constructing a basis satisfying the Numerical Criterion for Grassmannians. The degree and tensor bounds are established in Lemma 3.3.2 and Lemma 3.3.3. 

**Lemma 3.3.2** ([Pan96 Lemma 6.2.1]). Let $q$ be an integer. Then there exists an integer $e_3(d, g, r, q)$ such that for each $e > e_3(d, g, r, q)$, the following holds:

If $E$ is a slope-semistable, torsion-free sheaf on $C$ with Hilbert polynomial $\Phi(t)$ and

$$0 \to F \to E$$

is a nonzero subsheaf with multirank $(s_i)$ satisfying $h^1(C, F) \neq 0$, then

$$\frac{\chi(F) + q}{\sum s_i d_i} < \frac{\chi(E)}{rd} - 1.$$  

**Lemma 3.3.3** ([Pan96 Lemma 6.2.2]). Let $e > e_3(d, g, r, b) > r(g - 1)$ be as described in Lemma 3.3.2. Then there exists an integer $t_3(d, g, r, e) > t_3(d, g, r, e)$ such that for each $t > t_3(d, g, r, e)$, the following holds:

If $\xi \in \text{Quot}_{C^n \otimes O_C / C / \text{Spec } C}^{\Phi, L}$ corresponds to a quotient

$$C^n \otimes O_C \to E \to 0,$$
where $E$ is a torsion-free, slope-semistable sheaf on $C$ and $0 \to F \to E$ is a nonzero, proper subsheaf generated by global sections, then

$$\frac{h^0(C, F)}{n} \leq \frac{h^0(C, F \otimes L^t)}{\Phi(t)}.$$ 

Moreover, if $E$ is slope-stable, then

$$\frac{h^0(C, F)}{n} < \frac{h^0(C, F \otimes L^t)}{\Phi(t)}.$$ 

### 3.4 Strict Slope-semistability

In this subsection, we demonstrate that strict slope-semistability implies strict GIT semistability. The following proposition is similar to [Pan96, Prop. 6.4.1], and the proofs are similar. We include greater detail here to emphasize where the weakened hypotheses on the singularity type of the curve require the most care.

**Proposition 3.4.1.** Let $T = T(C)$ be the singularity type of $C$. There exist bounds $e_4(d, g, r, T)$ and $t_4(d, g, r, e)$ such that for each pair $e > e_4(d, g, r, T)$ and $t > t_4(d, g, r, e)$, the following holds:

If $\xi \in \text{Quot}_{\mathbb{C}^n \otimes \mathcal{O}_C / \text{Spec} \, C}^{\Phi^L}(C, F)$ corresponds to a quotient

$$\mathbb{C}^n \otimes \mathcal{O}_C \to E \to 0$$

where

$$\mathbb{C}^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$$

is an isomorphism, and $E$ is torsion-free and strictly slope-semistable, then $\xi$ is GIT strictly semistable.

**Proof.** Let $0 \to F \to E$ be a nonzero proper semistabilizing subsheaf. Suppose $F$ is generated by global sections and $h^1(C, F) = 0$. Then we may directly apply the Numerical Criterion for Grassmannians to the linearized $SL_n$ action on

$$G(\Phi(t), (\mathbb{C}^n \otimes \text{Sym}^i(H^0(C, L)))^\Phi(t)).$$

The point $\xi$ corresponds to the quotient

$$\psi^t : \mathbb{C}^n \otimes \text{Sym}^i(H^0(C, L)) \to H^0(C, E \otimes L^t) \to 0.$$ 

Let $U \otimes H^0(C, \mathcal{O}_C)$ be the pre-image of $H^0(C, F)$. Let $\tilde{v} = (v_1, \ldots, v_n)$ be a basis of $\mathbb{C}^n$ such that $v_1, \ldots, v_{h^0(C, F)}$ is a basis for $U$. Define $w(v_i) = 0$ for $1 \leq i \leq h^0(C, F)$, and 1 otherwise. We must show that for any $\Phi(t)$-tuple $(a_1, \ldots, a_{\Phi(t)})$ of $\tilde{v}$-pure elements of $\mathbb{C}^n \otimes \text{Sym}^i(H^0(C, L))$ which projects to a basis of $H^0(C, E \otimes L^t)$,

$$\sum_{i=1}^n \frac{w(v_i)}{n} = \frac{\Phi(t)}{\Phi(t)} \sum_{j=1}^{\Phi(t)} w(a_j).$$
Because $\xi$ is semi-stable, we have
\[ \sum_{i=1}^{n} \frac{w(v_i)}{n} \geq \sum_{j=1}^{\Phi(t)} \frac{w(a_j)}{\Phi(t)}. \]

Thus, it suffices to demonstrate the reverse inequality. By a few algebraic manipulations, we reduce this to
\[ h^0(C, F) r d t \geq h^0(C, E) (\sum s_i d_i) t, \]
but in fact we have equality because $F$ is strictly slope semistable.

The proof is now finished, pending a demonstration that there is a nonzero, proper semistabilizing subsheaf of $E$ generated by global sections and whose first cohomology group vanishes. This is the content of Lemma 3.4.2.

The following lemma is similar to [Pan96, Lemma 6.4.1]. The key difference in our result is the bound $\delta$, which has been modified to hold for pseudo-stable curves.

**Lemma 3.4.2.** Let $T = T(C)$ be the singularity type of $C$. There exists an integer $e_4(d, g, r, T)$ such that for any $e > e_4(d, g, r, T)$ the following holds: if $E$ is any slope-semistable, torsion free sheaf on $C$ with Hilbert polynomial $\Phi(t)$, and $0 \to F \to E$ is a nonzero subsheaf with multirank $(s_i)$ satisfying
\[ \sum s_i d_i = \frac{\chi(F)}{rd}, \]
then

(i) $h^1(C, F) = 0$.

(ii) $F$ is generated by global sections.

**Proof.** Suppose $F$ is a nonzero subsheaf of $E$ satisfying the hypothesis. By Lemma 3.3.2 if $e > e_3(d, g, r, 0)$, $h^1(C, F) = 0$.

Let $x \in C$ be a point. We have the exact sequence
\[ 0 \to m_x F \to F \to F/m_x F \to 0. \]
There is a constant $\delta_x$ (see Corollary 2.2.4), depending only on $r, g, d$, and $T$ such that
\[ \dim_k F/m_x F < \delta_x. \]
Let $\delta = \max_{x \in C} \delta_x$. Note that the multirank of $m_x F$ is the same as $F$. We have
\[ \chi(m_x F) + \delta > \chi(F). \]

By hypothesis,
\[ \frac{\chi(m_x F) + \delta}{\sum s_i d_i} > \frac{\chi(E)}{rd}. \]

For $e > e_3(d, g, r, \delta)$, $h^1(C, m_x F) = 0$ by Lemma 3.3.2. In this case, $F_x$ is generated by global sections for every point $x$. Thus, $F$ is globally generated, so we may take $e_4(d, g, r, T) = e_3(d, g, r, \delta)$.
3.5 Proof of Theorem 3.0.5

The proof is complete, but the pieces must be assembled. First, we explicitly state the degree bound determined by results in the previous section.

**Definition 3.5.1.** Let $g \geq 2$, $r$ and $d$ be integers, and $T$ be a singularity type. Let

- $e_0(d, g, r, T) = r(g - 1)$
- $e_1(d, g, r, T) = (dg(rd + 1) + 1)(rd)^2 - r(1 - g)$ (see Lemma 3.2.5)
- $e_2(d, g, r, T) = rd(2(dg(rd + 1) + 1) + g + rd - 1) - r(1 - g)$ (see Lemma 3.2.8)
- $e_3(d, g, r, T) = (rd)^2(g + dg(rd + 1) + 3) - r(1 - g)$ (see Lemma 3.3.2)
- $e_4(d, g, r, T) = (rd)^2(g + \delta + 2) - r(1 - g)$ (see Proposition 3.4.1),

where

$$\delta = rd \max_{A \in T} \dim k A/c + \dim k A/A.$$

Define

$$E(d, g, r, T) := \max_i e_i(d, g, r, T).$$

(3.1)

Let $T$ denote the singularity type of pseudo-stable curves and define

$$E(g, r) := \max_i e_i(10(2g - 2), g, r, T).$$

(3.2)

Lastly, define

$$t(d, g, r, e) = \max_{i=0,...,5} t_i(d, g, r, e),$$

(3.3)

where for $i = 0, \ldots, 4$ the $t_i$ are given by Propositions 3.2.2, 3.2.4, 3.2.7, 3.3.1, and 3.4.1, and $t_5 = t(d, g, r, e)$.

**Proof of Theorem 3.0.5** Let $g \geq 2$, $r$, and $d$ be integers. Define $\Phi(t) = e + r(1 - g) + rd$, and let $n = \Phi(0)$. Let $(C, L)$ be a polarized reduced Gorenstein curve such that $\deg L = d$. Let $T$ be the singularity type of $C$, and let $E(d, g, r, T)$ be as in (3.1). Let $e > E(d, g, r, T)$ and $\xi \in \text{Quot}^{\Phi L}_{C^n \otimes \mathcal{O}_C/C/\text{Spec} C}$ corresponding to

$$C^n \otimes \mathcal{O}_C \to E \to 0,$$

and let

$$\psi : C^n \otimes H^0(C, \mathcal{O}_C) \to H^0(C, E)$$

be the function on global sections. Let $t(d, g, r, e)$ be as given by (3.3). By Propositions 3.2.2, 3.2.4, 3.2.7, and 3.2.4, $\xi$ is GIT unstable if any of the following are true:

- $\psi$ is not an isomorphism,
• $E$ is not torsion-free,
• $E$ is slope-unstable.

Now we turn to the reverse implication. By Proposition 3.3.1, if $E$ is torsion-free and slope-stable (slope-semistable) and $\psi$ is an isomorphism, $\xi$ is GIT stable (semi-stable).

Finally, by Proposition 3.4.1, if $\xi$ is GIT semi-stable then $\xi$ is stable if and only if $E$ is slope-stable. \hfill \Box

4 Construction of the universal GIT quotient
4.1 Construction of the GIT quotient over a fixed curve

In this subsection, we construct the fiberwise GIT quotient.

Definition 4.1.1. Let $(C, L)$ be a polarized curve of genus $g$ and let $d$ be the degree of $L$. Let $e$ and $r$ be integers such that $e > E(d, g, r, T(C))$ defined in (3.1). Define $\Phi(t) = e + r(1 - g) + rd$. Let

$$Q^r \subset \text{Quot}^{\Phi, L}_{C^n \otimes \mathcal{O}_C/C/\text{Spec} \mathbb{C}}$$

be the locus of sheaves of uniform rank $r$. By Lemma 4.3.1 $Q^r$ is both closed and open. Define

$$\overline{U}_{e,r}^L(C) := Q^r / \text{SL}_n.$$  

Observe that twisting by $L^a$ induces an isomorphism

$$\text{Quot}^{\Phi, L}_{C^n \otimes \mathcal{O}_C/C/\text{Spec} \mathbb{C}} \cong \text{Quot}^{(e + ard) + r(1 - g) + rd, L^a}_{C^n \otimes \mathcal{O}_C/C/\text{Spec} \mathbb{C}}.$$  

Thus, for any $e$, we may take the smallest integer $a$ with the property that $e + ard$ is greater than the $e_i$ and define

$$\overline{U}_{e,r}^L(C) := \overline{U}_{e + ard,r}^L(C).$$

The following theorem is similar to [Sim94, Thm. 1.21] which makes the same claim, but for $e \gg 0$. Our formulation includes an explicit lower bound on $e$ and includes only sheaves of uniform rank. The explicit bound on $e$ will allow us to ensure the description holds for families of curves.

Theorem 4.1.1. Let $(C, L)$ be a polarized curve of genus $g$ with $\deg L = d$. Then for all $r$ and $e > E(d, g, r, T(C))$, the scheme $\overline{U}_{e,r}^L(C)$ is a projective variety containing the set of aut-equivalence classes of slope-semistable vector bundles as an open subset.

Proof. The proof follows from the definition of $E$ and results above, and is roughly equivalent to the proof of Theorem 4.3.2 below. We leave the details to the reader. \hfill \Box

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4.2 GIT Quotients of Products

In this subsection, we study the properties of quotients of products. Let $G$ and $H$ be reductive groups. Let $(X,L_X)$ and $(Y,L_Y)$ be polarized schemes with linearized actions of $G$ on both $X$ and $Y$, and of $H$ on $Y$. Assume that the actions of $G$ and $H$ commute on $Y$. Now we have an induced action of $G \times H$ on the product $X \times Y$ given by

$$(g,h) \cdot (x,y) = (g \cdot x, g \cdot (h \cdot y)).$$

Moreover, we have many linearizations on $X \times Y$ corresponding to

$$L_X^{\otimes a} \boxtimes L_Y^{\otimes b} := \pi_X^* L_X^{\otimes a} \otimes \pi_Y^* L_Y^{\otimes b}$$

for all $(a,b) \in \mathbb{Z}_2^2$.

With several group actions under consideration, we fix some notation: the superscripts $S$ and $SS$ will indicate stability and semi-stability with respect to the product action of $G \times H$. Stability and semi-stability for $G$ alone will be indicated by superscripts $S_G$ and $SS_G$, and similarly for $H$ alone.

Our plan, following [Pan96], is to shift the weight of the polarization almost entirely to $X$. This, we will show, reduces the stability condition for $G \times H$ on $X \times Y$ to the condition for stability $H$ on $Y$. The following key propositions, understood in the context of variation of GIT, makes this precise:

**Proposition 4.2.1.** Let $\pi_X : X \times Y \to X$ be the natural projection map. There exists a $k \gg 0$ such that with respect to the linearization $L_X^{\otimes k} \boxtimes L_Y$ on $X \times Y$, we have

$$\pi_X^{-1}(X^{S_G}) \subset (X \times Y)^{S_G}_{[k,1]} \subset (X \times Y)^{SS_G}_{[k,1]} \subset \pi_X^{-1}(X^{SS_G}).$$

**Proof.** This is a standard result in variation of GIT; see e.g. [Tha96, Lemma 4.1]. This particular formulation is equivalent to Propositions 7.1.1 and 7.1.2 in [Pan96].

**Proposition 4.2.2** ([Pan96 Prop. 8.2.1]). Let $Q \subset \pi_X^{-1}(X^{S_G})$ be a closed subscheme. Then for $k$ as in Proposition 4.2.1, we have

$$Q^{S_H}_{[k,1]} = Q^{S}_{[k,1]} \text{ and } Q^{SS_H}_{[k,1]} = Q^{SS}_{[k,1]}.$$

**Proof.** We sketch a variation of GIT argument here. An explicit proof in coordinates can also be found in [Pan96 Prop. 8.2.1]. We only prove the statement for stable loci; the semi-stable case is identical.

To begin, certainly, $Q^{S}_{[k,1]} \subset Q^{S_H}_{[k,1]}$, so it suffices to demonstrate the opposite inclusion. For this, let $\lambda$ be a one-parameter subgroup of $G \times H$, with components $\lambda_G$ and $\lambda_H$. Let $\mu$ denote the Hilbert-Mumford index and fix $(x,y) \in Q^{S}_{[k,1]}$. From local arguments, we have

$$\mu^{L_X^{\otimes a} \boxtimes L_Y}(x,y,\lambda) = k\mu^{L_X}(x,\lambda_G) + \mu^{L_Y}(y,\lambda_G) + \mu^{L_Y}(y,\lambda_H).$$ (4.1)
From the first inclusion of Proposition 4.2.1 and the assumption that \( Q \subset \pi^{-1}(X^S) \), it follows that the sum of the first two terms on the right-hand side of (4.1) is negative. From the assumption that \((x, y) \in Q^S_{[k, 1]}\), it follows that the last term is also negative. Therefore, \((x, y) \in Q^S_{[k, 1]}\).

4.3 Construction of the universal GIT quotient

We now specialize to the main case of interest in this paper, the construction of a universal moduli space of slope-semistable vector bundles over the moduli space of pseudo-stable curves. Recall that a projective curve is pseudo-stable if

- it is connected, reduced, and has only nodes and cusps as singularities;
- every subcurve of genus one meets the rest of the curve in at least two points;
- the canonical sheaf of the curve is ample.

The moduli functor \( \overline{M}^g \) associates to a scheme \( S \) the set of all \( \{ f : C \to S \} \), where \( f \) is a flat and proper morphism such that every geometric fiber is a pseudo-stable curve of genus \( g \), modulo isomorphism. There is a contraction of categorical moduli spaces \( M_g \to \overline{M}^g \) sending curves with an elliptic tail to cuspidal curves. Roughly speaking, curves with elliptic tails (elliptic components meeting the rest of the curve at one point) are replaced by the curves obtained by contracting the tail to a cusp \([HH09]\).

We require a lemma before detailing the construction of the moduli space. The lemma is similar to [Pan96, Lemma 8.1.1], with the difference that we work with pseudo-stable curves instead of Deligne-Mumford stable curves.

**Lemma 4.3.1.** Let \( g \geq 2 \) and \( r \) be integers. Define \( \Phi(t) = e + rt(1 - g) + 10(2g - 2)rt \) and let \( n = \Phi(0) \). Let \( H_g \) be the locus in the Hilbert scheme of \( 10 \)-canonically embedded curves of genus \( g \) pseudo-stable curves. Let \( U_H \) be the universal curve over \( H_g \), and \( \nu : U_H \to \mathbb{P}^N \) the universal projection. Define

\[
Q^r \subset \text{Quot}^{\Phi, \nu^* O_{\mathbb{P}^N}}(1)
\]

to be the subset corresponding to quotients

\[
\mathbb{C}^n \otimes O_C \to E \to 0,
\]

where \( E \) has uniform rank \( r \) on \( C \). Then the subscheme \( Q^r \) is open and closed in \( \text{Quot}^{\Phi, \nu^* O_{\mathbb{P}^N}}(1) \).

**Proof.** Let \( \kappa : C \to B \) be a projective, flat family of genus \( g \) pseudo-stable curves over an irreducible curve. Let \( E \) be a \( \kappa \)-flat coherent sheaf.

Suppose there exists a \( b^* \in B \) such that \( E_{b^*} \) has uniform rank \( r \) on \( C_{b^*} = C \). Let \( \{ C_i \} \) be the irreducible components of \( C \). The morphism \( \kappa \) is flat and
surjective of relative dimension 1, and so each \( C_i \) contains a component of \( C \). By the semi-continuity of
\[
 r(z) := \dim_k(z)(E \otimes k(z)),
\]
there is an open set \( U_i \subset C_i \) where \( r(z) \leq r \).

The set \( U = \cap_i \kappa(U_i) \subset \mathcal{B} \) is open, and has the property that for every \( b \in U \) the rank of \( E_b \) at the generic point of each irreducible component of \( C_b \) is at most \( r \). We will show that \( U = \mathcal{B} \) and conclude that \( E_b \) is of uniform rank for every \( b \in U \).

By way of contradiction, suppose that there exists a \( b' \in \mathcal{B} \) such that \( E_{b'} \) is not of uniform rank \( r \). Then again by semi-continuity, there is an \( i \) so that \( r(z) < r \) on an open \( W \subset C_i \). As \( E \) is flat over \( \mathcal{B} \), the Hilbert polynomial of \( E_{b'} \) is constant. In particular, the coefficient \( rd \) of \( t \) is constant. By Remark 2.1.6, \[
\sum_j r_j d_j = rd,
\]
where \( r_j \) is the generic rank on the \( j \)-th component of \( C_{b'} \). If \( E_{b'} \) is not of uniform rank, then some \( r_j \) is greater than \( r \) and some \( r_i \) is less than \( r \). Fixing the component \( C_i \) containing that component, we may appeal to upper semi-continuity and see that there is an open subset with rank bounded by \( r_i < r \).

But for any \( b \in U \cap \kappa(W) \), the multiranks of \( E_b \) is at most \( r \) on each component and strictly less than \( r \) on at least one component. By Lemma 2.1.1, \( E_b \) cannot have Hilbert polynomial \( \Phi(t) \), a contradiction.

Thus, there was no such \( b' \), and so for every \( b \in \mathcal{B} \), \( E_b \) has uniform rank \( r \), proving the lemma.

We will now construct the universal moduli space of vector bundles over \( \overline{M}_{g}^{ps} \), the moduli space of pseudo-stable curves. The key condition we need is that \( \overline{M}_{g}^{ps} \) contains no strictly semi-stable curves (see for instance [HH09, Section 2]). We will employ the 10-canonical bundle as our ample line bundle on each curve. Let \( H_g \) denote the locus in the appropriate Hilbert scheme of genus \( g \) degree \( d \) pseudo-stable curves. Embedding \( H_g \to \mathbb{P}^N \), induces an action and linearization of \( SL_{N+1} \) for which it is known (see [Sch91]) that \(
\overline{M}_{g}^{ps} \cong H_g//SL_{N+1}.
\)

Let \( U_H \) be the universal curve over \( H_g \). We have a natural inclusion \( U_H \to H_g \times \mathbb{P}^N \). Let \( \nu : U_H \to \mathbb{P}^N \) be the projection map. We will construct our moduli space using the relative Quot scheme taking the locus of uniform rank
\[
Q_{e,r} \subset \text{Quot}_{C^n \otimes \mathcal{O}_{U_H} / U_H, H_g}^{\nu^* \mathcal{O}_{\mathbb{P}^N}(1), \Phi},
\]
where \( \Phi \) is a Hilbert polynomial with respect to \( \nu^* \mathcal{O}_{\mathbb{P}^N}(1) \) ensuring that all parametrized sheaves have uniform rank \( r \) and degree \( e \). Note that for \( e > (rd)^2 (g+\delta+2) - r(1-g) \), where \( \delta \) measures the singularities of \( C \) (see Definition 3.5.1), all slope semi-stable sheaves appear in \( Q_{e,r} \) (see Lemma 3.4.2). The action of \( SL_{N+1} \) on \( H_g \) lifts naturally to an action on \( Q_{e,r} \). Moreover, \( Q_{e,r} \) is equipped with an action of \( SL_n \) by changing coordinates in \( \mathbb{C}^n \otimes \mathcal{O}_{U_H} \). These
two actions commute. Observe that there is a closed immersion respecting the group actions
\[ Q_{e,r} \subset H_g \times \text{Quot} \nu^*\mathcal{O}_H(1), \Phi \]
Denote the ample line bundles on the terms of the product by \( \mathcal{O}_{H_g}(1) \) and \( \mathcal{O}_{\text{Quot}}(1) \), respectively. Let
\[ \mathcal{L}_k = \mathcal{O}_{H_g}(k) \boxtimes \mathcal{O}_{\text{Quot}}(1). \]
Because \( \pi_{H_g}(Q_{e,r}) \) is in the \( SL_{N+1} \)-stable locus of \( H_g \), we may apply Proposition 4.2.2 and conclude that for \( k \gg 0 \), the \( SL_{N+1} \times SL_n \)-stable (semi-stable) locus of \( Q_{e,r} \) with the linearization \( \mathcal{L}_k \) is equal to the \( SL_n \)-stable (semi-stable) locus. Thus, taking sufficiently large \( k \), we define
\[ U_{e,r} := Q_{e,r} / / SL_{N+1} \times SL_n. \]
Note that for \( k \gg 0 \), the comments in the previous paragraph imply that the above definition is independent of \( k \). The morphism \( Q_{e,r}^{SS} \rightarrow H^S_g \rightarrow \overline{M}_g^{ps} \) is equivariant with respect to the group action, and so by the universal property of the GIT quotient, induces a morphism
\[ U_{e,r} \overset{\pi}{\rightarrow} \overline{M}_g^{ps}, \]
sending a curve and a sheaf to the underlying curve. For a fixed \( r \) and large \( e \), we have the following theorem.

**Theorem 4.3.2.** For all \( r, g \) there exists an integer \( E(r, g) \) (see (3.2)) such that for all \( e > E(r, g) \), the scheme \( U^{ps}_{e,r,g} \) is a projective variety containing the set of aut-equivalence classes of slope semi-stable vector bundles on a smooth curve of genus \( g \) as a dense open subset. The points in a fiber over a curve \( [C] \in \overline{M}_g^{ps} \) correspond to slope semi-stable vector bundles over \( C \) modulo aut-equivalence.

**Remark 4.3.1.** Let \( [C] \in \overline{M}_g^{ps} \). Then the fiber of \( \pi \) over \( [C] \) is isomorphic to \( U_{e,r}(C)/\text{Aut}(C) \). Considering the curve \( C_{g} \rightarrow \overline{M}_g \) over the moduli of smooth, automorphism free curves with the relative canonical polarization, Simpson’s construction gives a moduli space \( U_{e,r}(C_{g}/M^s_g) \) of slope semi-stable, pure, torsion-free sheaves, with fibers \( U_{e,r}(C) \). We will see (Theorem 5.0.8) that the pre-image of \( U_{e,r,g}^{ps} \) of \( M^s_g \) represents the same functor, and so is isomorphic to \( U_{e,r}(C^o/M^s_g) \). Thus, \( U_{e,r,g}^{ps} \) provides a compactification of \( U_{e,r}(C^o/M^s_g) \) with \( U_{e,r}(C^o/M^s_g) \) as a dense, open subset. Moreover, we establish in Proposition 4.3.6 that \( U_{e,r,g}^{ps} \) is irreducible, which immediately implies the density of \( U_{e,r}(C^o/M^s_g) \).

**Proof.** First, observe that by Lemma 4.3.1 the space \( U_{e,r,g}^{ps} \) is the GIT quotient of a closed subset of a projective scheme, and is therefore projective. By our construction above, Proposition 4.2.2 guarantees that the stable (semi-stable) loci are exactly described by the fiberwise stable (semi-stable) loci, described
in Theorem 3.0.5. The locus of slope semi-stable vector bundles on a smooth curve is open because it is the preimage under $\pi$ of an open subset of $M^ss_g$.

Next, we classify the orbit closures. Let $\xi \in Q^ss$ and suppose that $\bar{\xi} \in Q^ss$ lies in the orbit closure of $\xi$. It is immediate that $\pi(\bar{\xi})$ is in the orbit closure of $\pi(\xi)$. Thus, if $\xi$ corresponds to $(C, F)$ and $\bar{\xi}$ corresponds to $(\overline{C}, \overline{F})$, we see that $C$ and $\overline{C}$ are projectively equivalent. The $SL_{N+1}$-orbit closure of $\bar{\xi}$ consists of the images of $\overline{F}$ under projective automorphisms of $\overline{C}$. On the other hand, the $SL_n$-orbit closure of $\xi$ is known (e.g., [Sim94, Thm. 1.21]) to consist of sheaves $E$ which are aut-equivalent to $F$.

We will demonstrate that these two orbit closures intersect, which will prove that $\xi$ and $\bar{\xi}$ are aut-equivalent. Consider a path

$$\gamma = (\gamma_1, \gamma_2) : \Delta \setminus \{p\} \to SL_{N+1} \times SL_n,$$

such that

$$\lim_{z \to p} \gamma(z) \cdot \xi = \bar{\xi}.$$

Composing the path with the group action induces

$$\mu : \Delta \setminus \{p\} \to Q, \quad \mu(z) := \gamma_2(z) \cdot \xi.$$

As $Q$ is projective, $\mu$ extends to $\Delta$. Notice that $\mu(p)$ is in the $SL_n$-orbit closure of $\xi$. If we can demonstrate that $\mu(p)$ is also in the $SL_{N+1}$-orbit closure of $\bar{\xi}$, then we are done. We have

$$\lim_{z \to p} \gamma_1(z) \cdot \mu(p) = \lim_{z \to p} \gamma_1(z) \cdot \lim_{z \to p} (\gamma_2(z) \cdot \xi) = \lim_{z \to p} (\gamma_1(z) \cdot \gamma_2(z) \cdot \xi) = \bar{\xi}.$$

Now, we establish the irreducibility of $U^ss_{e,r,g}$. The following lemmas lay the groundwork for a deformation-theoretic argument.

**Lemma 4.3.3** (c.f. [Pan96, Lemma 9.1.1]). Let $\mu : C \to S$ be a flat family of pseudo-stable, genus $g \geq 2$ curves. Let $E$ be a $\mu$-flat coherent sheaf on $C$. The condition that $E_s$ is a slope-semistable torsion-free sheaf of uniform rank on $C_s$ is open on $S$.

**Proof.** Suppose $E_{s_0}$ is a slope-semistable sheaf of uniform rank $r$ on $C_{s_0}$ for some $s_0 \in S$. There exists an integer $m$ such that

1. $h^1(E_s \otimes \omega_{C_s}^m, C_s) = 0$ for all $s \in S$.
2. $E_s \otimes \omega_{C_s}^m$ is generated by global sections for all $s \in S$.
3. $\deg(E_{s_0} \otimes \omega_{C_{s_0}}^m) > E(g, r)$.

It is enough to prove the lemma for $E \otimes \omega_{C/S}^m$. Let $f$ be the Hilbert polynomial of $E \otimes \omega_{C/S}^m$. By the proof of Theorem 5.0.8 there exists an open $W \subset S$ containing $s_0$ and a morphism

$$\phi : W \to Q$$

such that $E \otimes \omega_{C/S}^m$ is isomorphic to the pullback of the universal quotient. Because $\phi(s_0) \in Q^ss_{[e,1]}$, which is open, the lemma is proven. 

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The following lemma is similar to [Pan96, Lemma 9.2.2], but here we treat cuspidal singularities.

**Lemma 4.3.4.** Let $S \subset \text{Spec} \,(\mathbb{C}[x, y, t])$ be the subscheme defined by the ideal $(y^2 - x^3 - 2t^6)$. Let $\mu : S \to \text{Spec} \,(\mathbb{C}[t])$. Let $\zeta = (0, 0, 0) \in S$. There exists a $\mu$-flat sheaf $I$ on $S$ such that $I_{\mu \not= 0}$ is locally free and $I_0 \cong m_\zeta$, where $m_\zeta$ is the maximal ideal defining $\zeta$ on $S_0 := \mu^{-1}(0)$.

**Proof.** There exists a section $L$ of $\mu$ defined by the ideal $(x - t^2, y - t^3)$. Let $I$ be the coherent sheaf corresponding to this ideal. We have the exact sequence

$$0 \to E \to O_S \to O_L \to 0. \quad (4.2)$$

Because $O_S$ is torsion-free over $\mathbb{C}[t]$, so is $E$. Thus, $E$ is $\mu$-flat because $\mathbb{C}[t]$ is a Dedekind domain. Moreover, $O_L$ is $\mu$-flat, and so (4.2) is exact after restriction to $\zeta$. Thus, $E_0 \cong m_\zeta$. 

The following lemma extends [Pan96, Lemma 9.2.3] to pseudo-stable curves.

**Lemma 4.3.5.** Let $C$ be a pseudo-stable curve of genus $g \geq 2$. Let $E$ be a slope-semistable torsion-free sheaf of uniform rank $r$ on $C$. Then there exists a family $\mu : C \to \Delta_0$ and a $\mu$-flat coherent sheaf $E$ on $C$ such that:

1. $\Delta_0$ is a pointed curve
2. $C_0 \cong C$, and for every $t \not= 0$, $C_t$ is a complete, nonsingular, irreducible genus $g$ curve.
3. $E_0 \cong E$, and for every $t \not= 0$, $E_t$ is a slope-semistable torsion-free sheaf of rank $r$.

**Proof.** Let $z \in C$ be a singular point. Because $E$ is torsion-free of uniform rank $r$, we have

$$E_z \cong O_z^{\oplus a_z} \oplus m_z^{\oplus r-a_z}.$$ 

This follows when $C$ has a node at $z$ from Propositions (2) and (3) of chapter (8) of [Ses82]. When $C$ has a cusp at $z$, the statement follows from the main theorem of [ARM12]. The structure of $E_z$, along with Lemma 4.3.4, allows us to describe a deformation over the disc of $(C_z, E_z)$ which smooths $C$ at $z$. This in turn induces deformations over associated Artin rings $A$ for the local deformation functor $Def^{\text{loc}}(C, E)$. As established in [FGvS97 Section A.], the morphism

$$Def(C, E) \xrightarrow{\text{loc}} Def^{\text{loc}}(C, E)$$

is smooth. Because $\text{loc}$ is smooth, we may lift the formal deformation over $A$ to a formal deformation over $A$ for $Def(C, E)$. Lastly, we need only check that the global formal deformation is algebraizable. The proof that the global deformation is effective is identical to the proof of [Ser06, Thm. 2.5.13]. The global formal deformation is then algebraizable by a special case of Artin’s algebraization theorem ([Ser06, Thm. 2.5.14]). Alternatively, a direct gluing argument may be carried out to explicitly construct a global deformation from the local deformation, as in [Pan96, Lemma 9.2.3].
Proposition 4.3.6 (c.f. [Pan96, Prop. 9.2.1]). $\overline{U}_{e,r,g}^{ps}$ is an irreducible variety.

Proof. Consider $\pi_{SS}: Q^{SS} \to H_g$. By [Ses82, Prop. 24], the scheme

$$\pi_{SS}^{-1}([C])$$

is irreducible for each nonsingular curve $C$, $[C] \in H_g$. Because the locus $H^0_g \subset H_g$ of nonsingular curves is irreducible, $\pi_{SS}^{-1}(H^0_g)$ is irreducible. By Lemma 4.3.5, $\pi_{SS}^{-1}(H^0_g)$ is dense in $Q^{SS}$. There is a surjection

$$Q^{SS} \to \overline{U}_{e,r,g}^{ps},$$

whence we conclude $\overline{U}_{e,r,g}^{ps}$ is irreducible.

5 Moduli Spaces of Slope Semi-stable Sheaves

Lastly, we describe a moduli functor of sheaves for which $\overline{U}_{e,r,g}^{ps}$ is the categorial moduli space. By this we mean that $\overline{U}_{e,r,g}^{ps}$ is initial for maps from the functor to schemes.

Definition 5.0.2. Let $(C, L)$ be a polarized curve. The functor $\overline{U}_{e,r}(C)$ associates to each scheme $S$ the set of equivalence classes of sheaves $F$ on $S \times C$, flat over $S$, such that for each $s \in S$, $F_s$ is slope-semistable and torsion-free of uniform rank $r$ and degree $e$.

Definition 5.0.3. The functor $\overline{U}_{e,r,g}^{ps}$ associates to each scheme $S$ the following set of equivalence classes of the data:

- A family of pseudostable curves $\mu: C \to S$
- A coherent sheaf $F$ on $C$, flat over $S$, such that fiberwise $F$ is slope-semistable, pure of uniform rank $r$ and degree $e$.

Two such data sets, $(\mu, C, F)$ and $(\mu', C', F')$, are equivalent if there exists an $S$-isomorphism $\phi: C \to C'$ and a line bundle $L$ on $S$ such that $F \cong \phi^* F' \otimes \mu^* L$.

Remark 5.0.4. There is an isomorphism of functors

$$\overline{U}_{e,r,g}^{ps} \to \overline{U}_{e,\pm(2g-2),r,g}^{ps},$$

and similarly we have

$$\overline{U}_{e,r}(C) \to \overline{U}_{e,\pm(2g-2),r}(C).$$

Theorem 5.0.7. Let $(C, L)$ be a polarized curve. For any $e, r$ the scheme $\overline{U}_{e,r}(C)$ is the categorical moduli space for $\overline{U}_{e,r}(C)$. 
Proof. The proof is essentially identical to that of Theorem 5.0.8 and so for brevity, we leave the details in this case to the reader. 

Remark 5.0.5. The functor $\overline{U}_{e,r}(C)$ is the subfunctor of the moduli problem described in [Sim94, p. 9] consisting of sheaves of uniform rank. Theorem 5.0.7 thus implies that Theorem 1.21 of [Sim94] applies to $\overline{U}_{e,r}(C)$ as a subspace of the Simpson moduli space. In particular,

1. $\overline{U}_{e,r}(C)$ is projective;
2. The points of $\overline{U}_{e,r}(C)$ represent the equivalence classes of semi-stable sheaves under Jordan equivalence.

Theorem 5.0.8. For any $e, r, g$ with $r \geq 1$ and $g \geq 2$, the scheme $\overline{U}_{e,r}$ is the categorical moduli space for the functor $U_{e,r}$.

Proof. First, note that by our definitions and the isomorphism (5.1), it suffices to prove the claim for $e > E(g,r)$. Now, we construct a natural transformation $\phi : \overline{U} \to \text{Hom}(-, \overline{U})$.

Let $e > E(g,r)$. For a scheme $S$, let $(\mu, C, F) \in \overline{U}(S)$. The sheaf $\mu_*(\omega^{10}_{C/S})$ is locally free of rank $N + 1 := 10(2g - 2) - g + 1$. Additionally, as we have taken $e$ sufficiently large, we have shown in Lemma 3.4.2 (taking $F = E = F$) that $H^0(C_s, F_s) = \chi(F_s) = n$ for all $s \in S$, and thus $\mu_* F$ is locally free of rank $n$. Let $\{W_i\}$ be an open cover of $S$ trivializing both $\mu_*(\omega^{10}_{C/S}|_{W_i})$ and $\mu_* F|_{W_i}$:

$$\alpha_i : C^{N+1} \otimes \mathcal{O}_{W_i} \xrightarrow{\cong} \mu_*(\omega^{10}_{C/S}|_{W_i}),$$

$$\beta_i : C^n \otimes \mathcal{O}_{W_i} \xrightarrow{\cong} \mu_* F|_{W_i}.$$

If $V_i = \mu^{-1}(W_i)$, then pulling back we obtain compositions

$$C^{N+1} \otimes \mathcal{O}_{V_i} \xrightarrow{\cong} \mu^*(\mu_*(\omega^{10}_{C/S}|_{V_i})) \to \omega^{10}_{C/S}|_{V_i},$$

$$C^n \otimes \mathcal{O}_{V_i} \xrightarrow{\cong} \mu^*(\mu_* F|_{V_i}) \to F|_{V_i}.$$

The second morphisms, and hence the compositions, are surjective because both $\omega^{10}_{C/S}$ and $F_s$ are globally generated; the former because $g \geq 2$ and the latter by Lemma 3.4.2. Moreover, by construction, the induced maps on global sections are surjective as well. A dimension count shows that they are isomorphisms.

By the universal property of $Q$, we obtain a morphism $\phi_i : W_i \to Q$. Theorem 3.0.3 tells us that we may uniformly select a lower bound on $t$ depending only on $d, g, r, e$ such that for larger $t$, $\phi_i(W_i) \subset Q^{SS}$. This is due to what we have established about the fiberwise behavior of $F$:

- $F$ is fiberwise slope-semistable and torsion-free of uniform rank
- the fiberwise presentation of $F$ induces an isomorphism on global sections.
As $\phi_i|_{W_i \cap W_j}$ differs from $\phi_j|_{W_i \cap W_j}$ precisely by the trivializations defined above, we obtain a well-defined morphism

$$S \to \mathcal{U}.$$ 

The naturality of the universal property of $Q$ implies that the defined $\phi$ is also natural.

The proof is complete pending the universality of $\phi$. This is, however, a straightforward diagram chase and is left to the reader.

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