GENERATORS FOR COULOMB Branches OF QUIVER GAUGE THEORIES

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Abstract. We study the Coulomb branches of 3d \( N = 4 \) quiver gauge theories, focusing on the generators for their quantized coordinate rings. We show that there is a surjective map from a shifted Yangian onto the quantized Coulomb branch, once the deformation parameter is set to \( \hbar = 1 \). In finite ADE type, this extends to a surjection over \( \mathbb{C}[\hbar] \). We also show that these algebras are generated by the dressed minuscule monopole operators, for an arbitrary quiver (this is similar to the proof of [FT18] Theorem 4.29). Finally, we describe how the KLR Yangian algebra from [KTW+18] is related to Webster’s extended BFN category. This paper provides proofs for two results which were announced in [KTW+18].

1. Introduction

Let \( G \) be a connected reductive algebraic group over \( \mathbb{C} \), and \( N \) a complex representation of \( G \). Recently, Braverman, Finkelberg and Nakajima have given a mathematical construction of the Coulomb branch \( \mathcal{M}_C \) of the 3d \( \mathcal{N} = 4 \) gauge theory associated to \((G, N)\) [BFN16b], building upon [Nak15]. \( \mathcal{M}_C \) is an important space studied by physicists, which had escaped a rigorous mathematical definition in any generality. The theory of Coulomb branches has interesting physical and mathematical applications, for example in symplectic duality [Web16a], [BDGH16].

The BFN construction realizes \( \mathcal{M}_C \) as an affine algebraic variety, by constructing its coordinate ring as a certain convolution algebra. Their construction also naturally yields a deformation quantization \( \mathcal{A}^\text{ph} \) of the coordinate ring of \( \mathcal{M}_C \), over \( \mathbb{C}[\hbar] = H^*_C(\text{pt}) \). It is interesting to relate these algebras with other more familiar ones, as for example has been done for some Cherednik algebras [KN18, Web16a, BEF16]. To do so in general one needs to get a handle on generators, and possibly relations. An important source for generators are the monopole operators \( M_{\lambda, f} \) coming from physics; we note that the related monopole formula [CHZ14] was a primary motivation in Nakajima’s foundational work [Nak15].

In this paper we study the case of quiver gauge theories, and their connection with shifted Yangians, following [BFN16a]. For a simple quiver (that is, having neither loops nor multiple edges) with vertex set \( I \), we fix two \( I \)-graded vector spaces \( W = \bigoplus_{i \in I} W_i \) and \( V = \bigoplus_{i \in I} V_i \). To this data we associated a pair \((G, N)\) by

\[
G := \prod_{i \in I} \text{GL}(V_i) \tag{1}
\]

\[
N := \bigoplus_{i \rightarrow j} \text{Hom}_C(V_i, V_j) \oplus \bigoplus_i \text{Hom}_C(W_i, V_i) \tag{2}
\]

We think of \( G \) as an algebraic group over \( \mathbb{C} \), and \( N \) its representation in the natural way. If the underlying graph of the quiver is a Dynkin diagram of ADE type, then the Coulomb branch corresponding to \((G, N)\) is isomorphic to a generalized affine Grassmannian slice \( \overline{W}_\mu^\lambda \) by [BFN16a Theorem 3.10] (we ignore a diagram automorphism that was included there). This is a particularly satisfying result in that it provides a candidate definition for slices outside of finite type: as Coulomb branches.
Another approach to affine Grassmannian slices and their quantizations was introduced in work of Kamnitzer, Webster, Yakobi, and the author [KWWY14], via algebras $Y^\lambda_\mu$ called truncated shifted Yangians. This Yangian approach was related with Coulomb branches in [BDG17, Section 6.8], and further studied in [BFN16a, Appendix B]. In particular, in finite ADE type with $\mu$ dominant, an isomorphism of algebras was proved in [BFN16a, Corollary B.28]. In the present paper we extend this result:

**Theorem A.** Let $Q$ be any simple quiver, and $W, V$ as above.

(a) There is an isomorphism of $\mathbb{C}$–algebras $Y^\lambda_\mu \cong A_{sph}^{\mathbb{C}}(\hbar=1)$, where $Y^\lambda_\mu$ is a truncated shifted Yangian of type $g_Q$, the simply-laced Kac-Moody Lie algebra underlying $Q$.

(b) In finite ADE type, there is an isomorphism of $\mathbb{C}[\hbar]$–algebras $Y^\lambda_\mu \cong A_{sph}^{\mathbb{C}}$ for any $\mu$.

This theorem was announced as [KTW+18, Theorem 4.9], and its proof appears in Section 3.4.

In part (b), the algebra $Y^\lambda_\mu = \text{Rees} Y^\lambda_\mu$ is the Rees algebra with respect to a filtration defined explicitly in terms of the Yangian’s generators. We note that part (b) is strictly stronger than part (a) for general $\mu$, because gradings are not bounded below. In order to prove part (b), we rely on a geometric result: in finite ADE type, the Coulomb branch $M_C \cong W^\lambda_\mu$ is a closed subscheme of a certain infinite type scheme $W^\mu$ [BFN16a, Section 2(xi)]. The latter scheme is quantized by the shifted Yangian $Y_\mu$, as shown in [FKP+18], and it is this fact which lets us deduce (b) from (a). It seems reasonable to expect that some generalization of this argument might exist in general type.

Put differently, part (a) of the theorem simply says that certain dressed minuscule operators $M_{\omega_{i,1}, f}$ and $M_{\omega_{i,1}', f}$ generate $A_{sph}^{\mathbb{C}}$. One consequence of part (b) is that the coordinate ring of $M_C$ is Poisson generated by these elements, in finite ADE type.

In finite type A, Finkelberg and Tsymbaliuk have proved an analogue of part (b) for K-theoretic Coulomb branches and shifted quantum affine algebras [FT18]. Their proof overlaps partially with ours (see Proposition 3.1), but otherwise makes use of the very different machinery of shuffle algebras. We believe the rational version of their work proves part (b) of the above theorem in the finite type A case (i.e. for ordinary homological Coulomb branches). Conversely it should be possible to extend results from the present paper to the K-theoretic setting, with appropriate changes.

1.1. **Our approach.** We make use of a diagram of algebras

$$A_{sph} \longrightarrow A \leftarrow A^{ab}$$

They are all variants on the BFN construction. This is a generalization of the case of a pure gauge theory (meaning $N=0$), which may be more familiar, where the above may be identified with a diagram

$$\text{Toda}_h(G^\vee) \longrightarrow \mathbb{H}_h \leftarrow \mathbb{D}_h(T^\vee),$$

relating the quantized open Toda lattice for the Langlands dual group, the degenerate nil-DAHA $\mathbb{H}_h$, and the $\hbar$–differential operators on the dual torus $T^\vee$. These algebras are studied for example in [Gin18, Lon18].

In general, a useful analogy for $\mathfrak{g}$ to keep in mind comes from the study of quotient singularities: if $\Sigma$ is a finite group acting on a $\mathbb{C}$–algebra $A$, then we may consider

$$A^\Sigma \longrightarrow A^\# \Sigma \leftarrow A$$

This analogy is quite strong. $A$ is in many respects better behaved than $A_{sph}$, and it contains $A_{sph} \cong eAe$ as a spherical subalgebra. In fact, $A$ is a matrix algebra over $A_{sph}$ as shown by Webster

\footnote{With good reason: denoting $T \subset G$ a maximal torus and $T^\vee$ its dual, we can take $A = \mathbb{C}[T^\vee T^\vee]$ and $\Sigma$ the Weyl group. Then $\mathfrak{g}$ is a “classical approximation” of $\mathfrak{g}$, which neglects quantum corrections. See the introduction of [BDG17] for a discussion.}
We prove that, at least in the case of quiver gauge theories, the nilHecke algebra $\mathcal{NH}_G$ plays the role of the group algebra $\mathbb{C}[\Sigma]$:

**Theorem B** (Corollary 3.2 in the main text). For any quiver gauge theory, $A$ is generated by its subalgebras $A^{ab}$ and $\mathcal{NH}_G$.

This gives a positive answer to a conjecture of Dimofte and Garner [DG19, Section 3.5]. In a sense, it tells us that $A$ is some sort of smash product of $A^{ab}$ and $\mathcal{NH}_G$.

Along the way to proving this result, we explore other aspects of the relationship between the algebras $A^{sp}, A, A^{ab}$ which may be of independent interest (Sections 2.6 and 2.7). Ultimately, a main ingredient in the proof is the fact that $A^{sp}$ is generated by minuscules (Proposition 3.1). This result is known to experts [BFN16b, Remark 6.7], [BDG17, Section 4.3]. It is also proven during the course of the proof of [FT18, Theorem 4.29].

1.2. **Webster’s BFN category, cylindrical KLR diagrams, and Yangians.** In the final section of this paper we focus on the extended BFN category $\mathcal{B}$ introduced Webster [Web16a, Section 3], specialized to the situation of a quiver gauge theory $(G, N)$ as defined above. We consider a full subcategory $\mathcal{B}^v$ of $\mathcal{B}$, whose objects are labelled by certain tuples $i = (i_1, \ldots, i_n) \in I^v$, see Section 4.3. By repackaging [Web16a, Theorem 3.10], we describe how morphisms in this category can be encoded in terms of cylindrical KLR diagrams. This is similar to [Web16b], which covers the Jordan quiver case, as well as forthcoming work [Web].

One of the key players in [KTW+18] is the KLR Yangian algebra $\mathcal{Y}$, which is defined in terms of the same cylindrical KLR diagrams. We consider $\mathcal{Y}$ as an algebra over $\mathbb{C}[h]$. It contains idempotents $e(i)$ for each $i \in I^v$. For a certain choice $i_v \in I^v$ there is a primitive idempotent $e \in \mathcal{Y}$ which is a summand of $e(i_v)$.

**Theorem C** (Section 4.5 in the main text).

(a) There is an isomorphism of algebras

$$\bigoplus_{i,j \in I^v} e(j) e(i) \cong \bigoplus_{i,j \in I^v} \text{Hom}_{\mathcal{B}^v}(i,j)$$

(b) There is an isomorphism $Y^\lambda_\mu \cong e \mathcal{Y}|_{h=1} e$. In finite ADE type, this upgrades to an isomorphism $Y^\lambda_\mu \cong e \mathcal{Y} e$.

As discussed above, part (b) is the truly new content here, while part (a) is due to Webster. Part (b) completes the proof of [KTW+18, Theorem 4.19].

1.3. **Loops and multiple edges.** Although we focus on simple quivers, most of our results apply to quivers with multiple edges and/or loops, with minimal changes. Essentially, the exceptions to this rule are those statements about Yangians.

More precisely, the results in Section 2 are valid for arbitrary Coulomb branches, while the results in Section 3 are valid for arbitrary quivers, with the exception of Section 3.4. What is currently lacking is an appropriate generalization of shifted Yangians to this setting. Note that the Jordan quiver is related to the Yangian of $\hat{\mathfrak{gl}}(1)$ [KN18]. In general, it seems reasonable the appropriate generalization should come from shuffle or cohomological Hall algebras. Similarly, the results in Section 4 can be generalized – Webster’s work [Web16a] applies to arbitrary Coulomb branches – but in the present paper we have stuck to the setting of simple quivers.

Finkelberg and Goncharov study multiloop versions of the Jordan quiver, having $r$ loops instead of one. For certain dimension vectors they realize the Coulomb branch as a Slodowy slice for $\mathfrak{sp}(2r)$ [FG19], and they expect that the quantized Coulomb branch is isomorphic to the corresponding
finite $W$–algebra. It would be interesting to understand what the analogue of the shifted Yangian is in this case, and how it relates to $U(\mathfrak{sp}(2r))$.

1.4. Relation to other work. As was discussed above, the connection between Yangians and Coulomb branches was studied in [BDG17], [BN16]. In the case of the Jordan quiver, Kodera and Nakajima [KN18] prove that the quantized Coulomb branch is a subquotient of the Yangian of $\mathfrak{gl}(1)$. This is the appropriate analogue of Theorem A in this case. The Jordan case is also studied in [Web16b], [BFP16], and connected to Cherednik algebras.

There are several papers studying related questions in the physics literature, including [BDG17]. The work of Dimofte and Garner [DG19] is close to our approach. They study the (quantized) coordinate rings for Coulomb branches star-shaped quivers, and in particular also make use of divided difference operators. Hanany and Miketa study certain quivers of finite AD type [HM19], and also see what we believe are part of the Yangian’s relations. We are hopeful that the results and techniques of the present paper will prove useful in further physical research.

Finkelberg and Tsymbaliuk have developed the parallel situation of K-theoretic Coulomb branches for quiver gauge theories, and their relationship to shifted quantum affine algebras [FT17], [FT18], which are algebras they introduced. As mentioned above, they have recently proven the analog of Theorem A in finite type A, by making use of shuffle algebras.

Finally, Cautis and Williams [CW18] have studied some questions related to ours, but one categorical level higher: they study (perverse) coherent sheaves on $\mathcal{G}_{GL(n)}$. Put differently, they study the categorification of the (K-theoretic) Coulomb branch for pure $GL(n)$ gauge theory.

1.5. Some conventions. Let $G$ be a group and $H \subset G$ its subgroup. For $V$ a representation of $H$, we will denote the induced vector bundle over $G/H$ by $G \times^H V = (G \times V)/H$. It is the quotient by the $H$–action $h \cdot (g, v) = (gh^{-1}, hv)$.

1.5.1. Quiver gauge theories. As above, we will fix a quiver $Q$ with vertex set $I$. For brevity, we will write $i \to j$ if and only if there is an arrow from $i$ to $j$. We write $i \sim j$ if $i \to j$ or $j \to i$. We assume that $Q$ is simple: there is at most one arrow connecting $i$ and $j$, and there are no loops $i \to i$.

Fix $I$–graded vector spaces $W = \bigoplus_{i \in I} V_i$. Throughout the paper $(G, N)$ will be defined as in [1] and [2], although the results of Section 2 are valid for arbitrary $(G, N)$. We will also let $F \subseteq \prod_{i \in I} GL(W_i)$ be a maximal torus. This group acts on $N$, commuting with the action of $G$. Our choice of $F$ is a matter of convenience, and with minor changes our results hold for the full flavour symmetry group $F = N^\circ_{GL(N)}(G)/G$ as considered in [Web16b, Section 2].

Write write $w_i = \dim_C W_i$ and $v_i = \dim_C V_i$, and also $|V| = \sum_i v_i$. For each $i \in I$, we will fix an ordered basis $\{e_{i,r} : 1 \leq r \leq v_i\}$ for $V_i$. Let $T \subset G$ be the corresponding maximal torus consisting of diagonal matrices at each node, and $B, B^- \subset G$ be the Borel subgroups given by upper/lower triangular matrices at each node. We denote $g = \text{Lie } G$ and $t = \text{Lie } T$.

The Weyl group $\Sigma$ of $G$ is a product of symmetric groups $\Sigma = \prod_i \Sigma_{v_i}$. The coweight lattice of $G$ and $T$ is $Z^\vee = \bigoplus_{i \in I} Z^{\vee_i}$. For our choice of $B$, recall that a coweight $\lambda = (\lambda_{i,r})_{i \in I, 1 \leq r \leq v_i}$ is dominant iff $\lambda_{i,r} \geq \lambda_{i,s}$ for all $i \in I$ and $1 \leq r \leq s \leq v_i$. The extended affine Weyl group of $GL(V_i)$ is the semidirect product $\hat{\Sigma}_{v_i} = Z^{\vee_i} \rtimes \Sigma_{v_i}$, while the extended affine Weyl group associated to $G$ is the product $\hat{\Sigma} = \prod_{i \in I} \hat{\Sigma}_{v_i}$. For $\lambda \in Z^\vee$ we will sometimes write $z^\lambda \in \hat{\Sigma}$, or else just $\lambda \in \hat{\Sigma}$.

1.5.2. (Co)homology. Corresponding to our basis of $V$, there are coordinates on $t$ which we denote $w_{i,r}$. We identify the cohomology rings

$$H^*_T(pt) = \mathbb{C}[w_{i,r} : i \in I, 1 \leq r \leq v_i], \quad H^*_C(pt) = \mathbb{C}[h],$$

so that $h = c_1(\mathbb{C})$ for the weight 1 action of $\mathbb{C}^\times$ on $\mathbb{C}$. 

Throughout this paper, we implicitly use the notion of Borel-Moore homology of a placid ind-scheme with a dimension theory [Lon17] Section 3.9. We will always take \( \mathbb{C} \)-coefficients, although some results will continue to hold with coefficients in an arbitrary commutative ring. This is an appropriate formalism for all computations below, and one can verify compatibility of dimension theories as required. For our purposes, it suffices to know that there are versions of all usual operations on equivariant Borel-Moore homology (e.g. proper pushforward, flat pull-back, restriction with supports/ refined Gysin homomorphisms) which interact as expected (e.g. proper base change). In other words, one can more or less pretend to be working in the context of [Ful98].

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2. Quantized Coulomb branches

In this section we overview the theory of Coulomb branches from [BFN16b], as well some results from [Web16a]. We then study relationships between elements of \( A^{ab}, A \) and \( A^{ab} \).

For simplicity of notation, we will work throughout with \((G, N)\) a quiver gauge theory. However, modulo minor notational differences, all results in this section hold for arbitrary \((G, N)\).

2.1. The affine Grassmannian and affine flag variety. Consider the loop groups \( G[[z]] \subset G((z)) \), as well as the Iwahori subgroup \( I \subset G[[z]] \) defined by

\[
I := \{ g(z) \in G[[z]] : g(0) \in B^- \}
\]

(5)

Recall that the affine Grassmannian and affine flag variety for \( G \) are defined as the quotient spaces

\[
\text{Gr} := G((z))/G[[z]], \quad \mathcal{F} := G((z))/I
\]

These objects may be thought of as ind-schemes over \( \mathbb{C} \), see for example [Zhu16] for an overview. We will mostly work on the level of \( \mathbb{C} \)-points of \( \text{Gr} \) or \( \mathcal{F} \), and by abuse of notation will simply write \( G[[z]] \) rather than \( G(\mathbb{C}[[z]]) \), etc. We denote a point of \( \text{Gr} \) or \( \mathcal{F} \) by a class \([g]\), where \( g \in G((z)) \).

There is an embedding of \( \mathfrak{S} \) into \( \mathcal{F} \), and of \( \mathfrak{S}^\vee \) into \( \text{Gr} \). For \( G = \prod_i \text{GL}(V_i) \) we can even embed \( \mathfrak{S} \) into \( G((z)) \): to \( \lambda \in \mathbb{Z}^\vee \) we associate the point \( z^\lambda = \prod_i \text{diag}(z^{\lambda_{i,1}}, \ldots, z^{\lambda_{i,n_i}}) \), while any \( w \in \mathfrak{S} \) may be embedded as its permutation matrix \( w \in G \subset G((z)) \). In any case, we will only ever use their images in \( z^\lambda \in \text{Gr} \) and \( wz^\lambda \in \mathcal{F} \) for \( \lambda \in \mathbb{Z}^\vee, w \in \mathfrak{S} \).

There is a stratification of \( \text{Gr} \) by the \( G[[z]] \)-orbits \( \text{Gr}^\lambda = G[[z]]z^\lambda \) for \( \lambda \) dominant. Similarly, \( \mathcal{F} \) is stratified by the \( I \)-orbits \( \mathcal{F}^w = Iw \) (Schubert cells) for \( w \in \mathfrak{S} \). These satisfy the closure relations

\[
\overline{\text{Gr}^\lambda} = \bigsqcup_{\mu \leq \lambda} \text{Gr}^\mu, \quad \overline{\mathcal{F}^w} = \bigsqcup_{v \leq w} \mathcal{F}^v,
\]

where \( \leq \) denotes the affine Bruhat order. In particular, \( \mu \leq \lambda \) iff \( \lambda - \mu \) is a non-negative sum of simple coroots of \( G \).

\[\text{We should mention that the consideration of positive characteristic coefficients has led to several interesting results about Coulomb branches, for example in [Lon17] and [MW18].}\]
2.2. Quantized Coulomb branch algebras. We consider three variations on the quantized Coulomb branch algebra from [BFN16b]. To this end, consider the spaces

$$R^{sph} := \left\{ [g, s] \in G((z)) \times G[[z]] \langle N[[z]] : gs \in N[[z]] \right\},$$

$$R := \left\{ [g, s] \in G((z)) \times T Z N[[z]] : gs \in N[[z]] \right\},$$

$$R^{ab} := \left\{ ([l, n]) \in T((z)) \times T[[z]] N[[z]] : tn \in N[[z]] \right\}$$

Each may be considered as a placid ind-scheme of infinite type over $\mathbb{C}$. Following [BFN16b Section 2(ii)], we can define the equivariant Borel-Moore homology

$$A^{sph} := H^{(G[[z]] \times F) \times C^\times} (R^{sph}),$$

$$A := H^{(T \times F) \times C^\times} (R),$$

$$A^{ab} := H^{(T[[z]] \times F) \times C^\times} (R^{ab})$$

Note that in (9)–(11), the group $C^\times$ is acting by loop rotation on the spaces (6)–(8), respectively. Following [BFN16b], we will modify this action slightly by letting $C^\times$ also act on $N$ by weight $1/2$.

Explicitly, for a point $[g(z), s(z)]$ in any of these respective spaces, the action of $\tau \in C^\times$ is by

$$\tau \cdot [g(z), s(z)] = [g(\tau z), \tau^{1/2}s(\tau z)]$$

Meanwhile, the group $G[[z]] \times F$ acts on $R^{sph}$ by $(g_1, f) \cdot [g_2, s] = [g_1 g_2, f s]$, and similarly for $R, R^{ab}$.

There is a convolution product $\ast$ on the above algebras, defined as in [BFN16b Section 3(iii)], or alternatively as in [BFN17 Appendix B(ii)(d)]. The Coulomb branch construction comes equipped with the following features:

**Theorem 2.1 (BFN16b).**

(a) $A^{sph}$ is an associative algebra under the convolution product $\ast$, with unit $1 \in A^{sph}$ the fundamental class of the fibre of $R^{sph} \to \text{Gr}$ over the identity point $[1] \in \text{Gr}$.

(b) The convolution product is $H^*_{G \times F \times C^\times}(pt)$–linear in the first variable with respect to cap product. In particular, there is an algebra embedding $H^*_{G \times F \times C^\times}(pt) \to A^{sph}$ defined by $\varphi \mapsto \varphi \cap 1$

(c) $A^{sph}$ is free as a left (or right) module over $H^*_{T \times F \times C^\times}(pt)$.

Analogous claims hold for $A$ and $A^{ab}$, where in parts (b), (c) the relevant algebra is $H^*_{G \times T \times F \times C^\times}(pt)$.

We call $A^{sph}$ the **quantized Coulomb branch algebra** associated to the pair $(G, N)$. Its base change $A^{sph}_{h=0}$ at the irrelevant ideal $H^*_{C^\times}(pt) \to \mathbb{C}$ is a commutative ring, and

$$\mathcal{M}_C := \text{Spec} A^{sph}_{h=0}$$

is called the **Coulomb branch** associated to $(G, N)$ [BFN16b Definition 3.13]. (More precisely, it is the **flavour deformation** of the Coulomb branch corresponding to $F$, see [BFN16b Section 3(viii)]).

Similarly, $A^{ab}$ is the quantized Coulomb algebra branch associated to $(T, N)$. The algebra $A$ is not strictly speaking a quantized Coulomb branch algebra, but rather is a matrix algebra over $A^{sph}$ (see Theorem 2.6). $A_{h=0}$ can thus be thought of as the endomorphism algebra of a (trivial) vector bundle over $\mathcal{M}_C$.

2.3. Filtrations and monopole operators. There is a natural map $R^{sph} \to \text{Gr}$ defined by $[g, s] \mapsto [g]$. Denote by $R^{sph}_{\leq \lambda}$ the preimage of $\overline{\text{Gr}^\lambda}$ under this map. By definition,

$$A^{sph} = \lim_{\to} H^*_{G[[z]] \times C^\times \times F} (R^{sph}_{\leq \lambda})$$

(12)
where this directed system is defined with respect to proper pushforward maps. This is an algebra filtration \[ [BFN16b] \text{Section 6(i)} \]. For \( \lambda \) minuscule (or equivalently, such that \( \text{Gr}^\lambda = \text{Gr}^\mu \)) there are well-defined classes
\[
M_{\lambda, f} := f \cap [R_{\leq \lambda}^{\text{sp}}] \in H^*_s(G[[z]] \times F) \times \mathbb{C}^\times (R_{\leq \lambda}^{\text{sp}})
\]
which we call **dressed minuscule monopole operators**. Here the “dressing” \( f \in H^*_T \times F \times \mathbb{C}^\times (pt) \Sigma^\lambda \), where \( \Sigma^\lambda \subset \Sigma \) is the stabilizer of \( \lambda \) and acts only on the \( H^*_T(pt) \) part. See [BFN16b \text{Section 6(ii)}] for more details.

**Remark 2.2.** It is predicted from physics that \( A^{\text{sp}} \) has a basis given by dressed monopole operators \( M_{\lambda, f} \) for all dominant \( \lambda \), not necessarily minuscule [BDG17]. Such elements are not canonically defined in general [BFN16b \text{Remark 6.5}], although they are well-defined in the associated graded with respect to the above filtration.

Similarly, there is a map \( R \to \mathcal{F} \), and we denote by \( R_{\leq w} \) the preimage of Schubert variety \( \mathcal{F}^w \). \( A \) is a filtered algebra, analogously to [I2], and in this case the fundamental classes \( [R_{\leq w}] \) form a basis for \( A \) over \( H^*_T \times F \times \mathbb{C}^\times (pt) \) as a left (or right) module.

Finally, there is a map \( R^\alpha \to \text{Gr}_T = T((z))/T[[z]] \), and for \( \lambda \in \mathbb{Z}^\gamma \) we define \( R^\lambda \) to be the preimage of the point \( z^\lambda \in \text{Gr}_T \). We will denote the corresponding fundamental class by \( r^\lambda := [R^\lambda] \). These elements form a basis for \( A^{\text{ab}} \) as a left (or right) module over \( H^*_T \times F \times \mathbb{C}^\times (pt) \).

There is an explicit formula \( r^\lambda r^\mu = A(\lambda, \mu) r_{\lambda+\mu} \) for their structure constants, given in [BFN16b \text{Theorem 4(iii)}].

### 2.4. Relationship between \( A^{\text{ab}} \) and \( A \)

Since there is a surjection \( I \to T[[z]] \) with pro-unipotent kernel, the inclusion \( R^\alpha \to R \) induces a pushforward map
\[
A^{\text{ab}} = H^*_s(T[[z]] \times F) \times \mathbb{C}^\times (R^\alpha) \to H^*_s(T[[z]] \times F) \times \mathbb{C}^\times (R) \cong H^*_s(T \times F) \times \mathbb{C}^\times (R) = A
\]
This is an injective algebra homomorphism, see [Web16a \text{Section 3}] (c.f. also [BFN16b \text{Section 5(iii)}]).

### 2.5. The nilHecke algebra

Recall that there is a natural action of the Weyl group \( \Sigma \) on \( H^*_T(pt) \), and it is a free module over its invariant subalgebra \( H^*_G(pt) = H^*_T(pt)^\Sigma \) of rank \( |\Sigma| \). The **nilHecke algebra** for \( G \) may be defined as
\[
N^\Sigma H^*_G := \text{End}_{H^*_G(pt)}(H^*_T(pt))
\]
See [Kum02 \text{Section XI}], [Dem73], or [Gim18 \text{Section 7.1}]. \( N^\Sigma H^*_G \) is a matrix algebra over \( H^*_G(pt) \) of rank \( |\Sigma| \), and is generated by \( H^*_T(pt) \) (acting on itself by multiplication) together with the divided difference operators \( \partial_{i,r} \) for \( i \in I \) and \( 1 \leq r < \nu_i \). These are the operators on \( H^*_T(pt) \) defined by
\[
\partial_{i,r} = \frac{1}{w_{i,r} - w_{i,r+1}} (1 - s_{i,r})
\]
where \( s_{i,r} \in \Sigma_{\nu_i} \) is a simple transposition. These operators satisfy the braid relations of \( \Sigma \), so in particular there is a well-defined element \( \partial_w \in N^\Sigma H^*_G \) for any \( w \in \Sigma \). There is an obvious inclusion \( C(\Sigma) \subset N^\Sigma H^*_G \), since \( s_{i,r} = (w_{i,r} - w_{i,r+1}) \partial_{i,r} + 1 \).

As is well-known, the nilHecke algebra may also be realized as the homology \( H^B_*(G/B^-) \), endowed with the convolution product \( \alpha * \beta := m_s(q^*)^{-1} p^*(\alpha \otimes \beta) \) defined using the diagram
\[
G/B^- \times G/B^- \xrightarrow{p} G \times G/B^- \xrightarrow{q} G \times \mathbb{B}^- \xrightarrow{m} G/B^- \quad (15)
\]
Analogously to [BFN16b \text{Section 3}], For each \( w \in \Sigma \) there is a Schubert variety \( X^w = B^- w B^- / B^- \subset G/B^- \). The element \( \partial_w \) corresponds to the fundamental class \( [X^w] \), and more generally \( \partial_w = [X^w] \)
for any \( w \in \Sigma \), as follows from the study of Bott-Samelson resolutions. The equality (14) expresses the localization theorem at \( T \)-fixed points. Note that localization can be thought of as an algebra homomorphism into the smash product \( \text{Frac}(H_T^*(pt))/\Sigma \).

For each simple reflection \( s_{i,r} \) there is also a fundamental class \([R_{\leq s_{i,r}}] \in A\), as in Section 2.3. By abuse of notation we will denote this class by \( \partial_{i,r} \), as is justified by the following:

**Lemma 2.3.** Together with \( H_T^*(pt) \subset A \), the cycles \( \partial_{i,r} = [R_{\leq s_{i,r}}] \in A \) generate a copy of the nilHecke algebra. Moreover, \( \partial_w = [R_{\leq w}] \) for any \( w \in \Sigma \).

**Proof.** There is an algebra embedding \( \mathbf{z}^* : A \hookrightarrow H_s^{(\mathcal{I} \times \mathcal{F}) \times \mathcal{C}}(\mathcal{F}) \), as in [BFN16b, Lemma 5.11]. For any element \( w \in \Sigma \) of the finite Weyl group, it is not hard to see that \( R_{\leq w} \) is the full preimage of \( \mathcal{F}_w \) under the natural map \( \mathbf{G}((z)) \times \mathcal{I} \mathbb{N}[[z]] \to \mathcal{F} \) (and in fact \( R_{\leq w} \cong \mathcal{F}_w \times \mathbb{N}[[z]] \)). By the construction of \( \mathbf{z}^* \) it follows that \( \mathbf{z}^*([R_{\leq w}]) = \underline{\mathcal{F}^w} \). The cycle \( \underline{\mathcal{F}^w} \) is the image of \( X^w \) under the natural embedding of the finite flag variety \( \mathbf{G}/B^- \subset \mathcal{F} \). This embedding is compatible with convolution, and therefore the elements \( \underline{\mathcal{F}^w} \) give a copy of the nilHecke algebra. Since \( \mathbf{z}^* \) is injective, this proves the claim. \( \square \)

Consequently there is a chain of embeddings \( \mathbb{C}[\Sigma] \subset \mathcal{N} \mathcal{H}_\mathbf{G} \subset A \), and so \( A \) contains the symmetrizing idempotent \( e \in \mathbb{C}[\Sigma] \). Geometrically, this can be written alternatively as

\[
e = \frac{1}{|\Sigma|} \text{eu}(T) \cap [R_{\leq w_0}] = (-1)^{t(w_0)} \frac{1}{|\Sigma|} [R_{\leq w_0}] \ast \Delta
\]

as follows from a standard nilHecke algebra calculation (e.g. by the localization theorem on \( \mathbf{G}/B \)). Here \( w_0 \in \Sigma \) is the longest element, \( T \) is the pull-back of the tangent bundle under \( R_{\leq w_0} \to \underline{\mathcal{F}^w} \cong \mathbf{G}/B \), and \( \Delta \in H_T^*(pt) \) is the product of the positive roots of \( \mathbf{G} \). Since \( e \) is a full idempotent of \( \mathcal{N} \mathcal{H}_\mathbf{G} \), it is also a full idempotent of \( A \).

### 2.6. Relationship between \( A^{\text{sph}} \) and \( A \).

By [Web16a, Theorem 3.3] there is an isomorphism \( A^{\text{sph}} \cong eAe \), and in fact \( A \) is a matrix algebra over \( A^{\text{sph}} \) of rank \( |\Sigma| \) (c.f. also [BEF16, Section 4.2], [ET17, Corollary 3.8]). Our goal in this section is to give a slightly different perspective on this result, which is more explicit, inspired by results from [Sau18]. To this end, consider

\[
\mathcal{P} := H_s^{(\mathcal{I} \times \mathcal{F}) \times \mathcal{C}}(R^{\text{sph}}), \quad \mathcal{Q} := H_s^{(\mathbf{G}[[z]] \times \mathcal{F}) \times \mathcal{C}}(R)
\]

Using straightforward variations on the convolution diagram (3.2) in [BFN16b, and on the proof of associativity [BFN16b, Theorem 3.10], we have:

**Lemma 2.4.** There are convolution products

\[
A \otimes \mathcal{P} \longrightarrow \mathcal{P}, \quad \mathcal{P} \otimes A^{\text{sph}} \longrightarrow \mathcal{P},
\]

\[
Q \otimes A \longrightarrow Q, \quad A^{\text{sph}} \otimes Q \longrightarrow Q,
\]

\[
\mathcal{P} \otimes Q \longrightarrow A, \quad Q \otimes \mathcal{P} \longrightarrow A^{\text{sph}}
\]

All compositions of convolution products are associative, whenever defined.

In particular, convolution makes \( Q \) an \( A^{\text{sph}} \cdot A \)-bimodule and \( \mathcal{P} \) an \( A \cdot A^{\text{sph}} \)-bimodule, and defines bimodule homomorphisms \( Q \otimes A \mathcal{P} \longrightarrow A^{\text{sph}} \) and \( \mathcal{P} \otimes A Q \longrightarrow A^{\text{sph}} \). Now, consider the elements

\[
\text{Inc} := [R^{\text{sph}}_{\leq 1}] \in \mathcal{P}, \quad \text{Av} := \frac{1}{|\Sigma|} \text{eu}(T) \cap [R_{\leq w_0}] \in Q
\]

where the latter is defined as in (16) (note that this is a \( \mathbf{G}[[z]] \)-equivariant cycle). We will need some basic properties of these elements under the above convolution products.

**Lemma 2.5.** (a) For \( a \in A^{\text{sph}} \) (or \( a \in Q \)) the convolution \( \text{Inc} \ast a = \text{Forg}(a) \), where \( \text{Forg} \) is the natural homomorphism of restriction from \( \mathbf{G}[[z]] \) to \( \mathcal{I} \)-equivariance.
(b) For $b \in A$ (or $b \in P$) the convolution $b * \text{Inc} = \pi_* (b)$, where $\pi : R \to R^{sph}$ is the natural map $[g, s] \mapsto [g, s]$.

(c) $\text{Inc} * \text{Av} = e \in A$.

(d) $\text{Av} * \text{Inc} = 1 \in A^{sph}$ is the unit element.

Proof. Parts (a), (b) are similar to the proof that $1$ is the unit element in [BFN16b, Theorem 3.10]. See also parts (a), (b) of [BFN16b, Lemma 5.7]. Parts (c) and (d) follow immediately from parts (a) and (b), respectively.

We now come to the main result of this section:

**Theorem 2.6.**

(a) The map $A^{sph} \to A$, $a \mapsto \text{Inc} * a * \text{Av}$ gives an isomorphism of algebras $A^{sph} \cong e.Ae$. It sends $1 \mapsto e$.

(b) The map $P \to A$, $a \mapsto a * \text{Av}$ identifies $P \cong Ae$, compatibly with bimodule structures under (a). It sends $\text{Inc} \mapsto e$.

(c) The map $Q \to A$, $a \mapsto \text{Inc} * a$ identifies $Q \cong e.A$, compatibly with bimodule structures under (a). It sends $\text{Av} \mapsto e$.

Moreover $A$ is a matrix algebra over $A^{sph}$ of rank $|\Sigma|$, and is generated by its subalgebras $N\mathcal{H}_G$ and $A^{sph}$.

Proof. We prove part (a), with (b), (c) being similar. Using Lemma 2.5, it is easy to see that our map $A^{sph} \to A$ lands in $e.Ae$, and that its inverse $e.Ae \to A^{sph}$ is given by $b \mapsto \text{Av} * b * \text{Inc}$. Therefore it is an isomorphism.

Finally, since $N\mathcal{H}_G$ is a matrix algebra, with a complete set of minimal idempotents (all isomorphic to $e$), we get the final claim of the theorem.

As a consequence, since $e \in A$ is a full idempotent, we see that $A^{sph}$ is Morita equivalent to $A$. This Morita equivalence is witnessed by the bimodules $P, Q$.

**Remark 2.7.** As mentioned above, this result is a variation on [Web16a, Theorem 3.3]. Modulo addressing some issues of infinite-dimensionality, our two approaches should agree as a consequence of [Sau18].

**Remark 2.8.** Because of the embedding $\mathbb{C}[\Sigma] \subset N\mathcal{H}_G \subset A$, we get actions of $\Sigma$ on $A$ by left and right multiplication. Since $e \in A$ is the symmetrizing idempotent for $\Sigma$, we can rephrase the theorem: the left $\Sigma$–invariants of $A$ are isomorphic to $Q$, the right $\Sigma$–invariants to $P$, and the $\Sigma \times \Sigma$–invariants are isomorphic to $A^{sph}$. This is analogous to [Sau18, Proposition 1].

An essential property of the above isomorphism $A^{sph} \cong e.Ae$ is the following:

**Lemma 2.9.** Let $a \in A^{sph}$ and $b \in A$ be such that $\text{Forg}(a) = \pi_*(b)$. Then

$$\text{Inc} * a * \text{Av} = b * e$$

Proof. $\text{Inc} * a = \text{Forg}(a)$ by Lemma 2.5(a), which is an element of $P$. Meanwhile, $b * e \in A.e$. It suffices to verify that these elements correspond to one another under the isomorphism $P \cong A.e$ from part (b) of the previous theorem. Note that the inverse to this isomorphism is the map $g \mapsto g * \text{Inc}$. Applying this inverse map to $b * e$, by Lemma 2.5 we get

$$(b * e) * \text{Inc} = b * \text{Inc} = \pi_*(b) = \text{Forg}(a),$$

which proves the claim.
2.7. Generation by minuscules. Let \( \lambda \) be a minuscule coweight for \( G \). Then there is an isomorphism \( G/P \cong \text{Gr}^\lambda = \text{Gr}^{\lambda^\circ} \) defined by \( gP \mapsto gz^\lambda \), where \( P = P_\lambda \subset G \) is a parabolic subgroup (the stabilizer of \( z^\lambda \) in \( G \)). Note that since \( \lambda \) is dominant \( B^- \subset P \). The associated parabolic Weyl group \( \Sigma_\lambda \subset \Sigma \) is the stabilizer of \( \lambda \), and we will denote its longest element by \( w_\lambda \).

Recall from Section 2.3 that there are elements \( M_{\lambda,f} \in A^{\text{sph}, f} \), for \( f \in P^{\Sigma_\lambda} \). The following result generalizes [KTW18, Lemma 4.5]:

**Lemma 2.10.** Suppose that \( \lambda \) is a minuscule coweight. Under \( A^{\text{sph}} \cong eAe \), for any \( f \in P^{\Sigma_\lambda} \) we have

\[
M_{\lambda,f} \mapsto \partial_{w_0w_\lambda}fr_\lambda \cdot e
\]

**Proof.** Denote \( w = w_0w_\lambda \) for brevity. By Lemma 2.9 it suffices to prove that \( \pi_* (\partial_w r_\lambda) = \text{Forg}(M_{\lambda,f}) \). This reduces to a calculation for finite dimensional (partial) flag varieties. Consider the natural proper map \( \pi : G/B^- \rightarrow G/P \). Its restriction to the Schubert variety \( \pi : X^w \rightarrow G/P \) is a proper birational map, since the element \( w \) is the minimal length coset representative of \( w_0\Sigma_\lambda \) (by e.g. [Kum02, Corollary 7.1.18]). As in Section 2.5 we can think of \( \partial_w = [X^w] \), so we conclude that \( \pi_* (\partial_w) = [G/P] \). For any \( g \in H^*_G(G/P) \cong H^*_T(pt)^W_G \),

\[
g \cap [G/P] = g \cap \pi_* \partial_w = \pi_*(\pi^*(g) \cap \partial_w)
\]

by the projection formula. By the localization theorem at \( T \)-fixed points on \( G/B^- \), we can write \( \partial_w = \sum_{v \leq w} a_v v \) for some \( a_v \in \text{Frac} H^*_T(pt)^W_G \). We find that, as elements of \( N^\lambda H_G \),

\[
\pi^*(g) \cap \partial_w = \sum_{v \leq w} v(g)a_v v = \sum_{v \leq w} a_v v = \partial_w g
\]

Applying this discussion to \( g = f \cdot \text{eu}(z^\lambda N[[z]]/z^\lambda N[[z]] \cap N[[z]]) \) proves the claim, by [BFN16a, Proposition A.2]

We say that \( A^{\text{sph}} \) is generated by minuscules if it is generated as an algebra by \( H^*_G \times F \times C \times (pt) \) and the dressed minuscule monopole operators \( M_{\lambda,f} \).

**Corollary 2.11.** If \( A^{\text{sph}} \) is generated by minuscules, then \( A \) is generated by its subalgebras \( N^\lambda H_G \) and \( A^{\text{ab}} \).

**Proof.** Consider the subalgebra \( A' \subset A \) generated by \( N^\lambda H_G \) and \( A^{\text{ab}} \). To prove that \( A' = A \), by the final part of Theorem 2.6 it suffices to show that the images of generators of \( A^{\text{sph}} \) land in \( eA'e \) under the isomorphism \( A^{\text{sph}} \cong eAe \). Lemma 2.10 shows that this is true for the elements \( M_{\lambda,f} \). Since we assume \( A^{\text{sph}} \) is generated these elements, this proves the claim.

There is a sort of converse to Lemma 2.10 which we will need later: for \( \mu \) in the Weyl orbit of a dominant minuscule coweight \( \lambda \), we can express \( r_\mu \in A^{\text{ab}} \subset A \) using only \( N^\lambda H_G \) and the dressed monopole operators \( M_{\lambda,f} \). To prove this, we will use the following basic fact about nilHecke algebras:

**Lemma 2.12.** There exist bases \( \{x_w\}_{w \in \Sigma} \) and \( \{y_w\}_{w \in \Sigma} \) for \( H^*_T(pt) \) over \( H^*_G(pt) \), such that the following identity holds in \( N^\lambda H_G \):

\[
1 = \sum_{w \in \Sigma} x_w \partial_{w_0} y_w
\]

\[\footnote{Since \( \pi_* (\partial_w) = [G/P] \), these coefficients have the property that, for any fixed coset \( xW_\lambda \), the sum}

\[
\sum_{v \leq w, vW_\lambda = xW_\lambda} a_v = \frac{1}{\text{eu}(T,G/P)}
\]
Proof. Consider the bilinear pairing \(H_*(pt) \otimes H_*(pt) \to H_*(G(pt))\) defined by \(f \otimes g \mapsto \partial_{w_0}(fg)\). Our claim is equivalent to the non-degeneracy of this pairing, i.e. that dual bases exist so that \(\partial_{w_0}(x_w y_v) = \delta_{w,v}\). This pairing can be identified with the \(G\)-equivariant Poincaré pairing for \(G/B\), see e.g. [HS08] Section 3]. Therefore, non-degeneracy follows from equivariant Poincaré duality [Bri00 Proposition 1].

Proposition 2.13. Let \(\lambda\) be dominant minuscule, and \(\mu \in \Sigma \lambda\). Then there exist elements \(x_p, y_p \in H^*_T \times C^*(pt)\) and \(f_p \in H^*_T \times F \times C^*(pt)\) for \(p\) running in some finite index set, such that

\[ r_{\mu} = \sum_p x_p (\mathrm{Inc} * M_{\lambda, f_p} * \partial_{w_0}) y_p \]

as elements of \(A\).

Proof. In Section 2.4 we used the embedding \(\mathcal{R}^{ab} \subset \mathcal{R}\), but we note that there is a similar embedding \(\mathcal{R}^{ab} \subset \mathcal{R}^{sph}\). Under the latter \(\mathcal{R}^{ab} \subset \mathcal{R}^{sph}\), so there is a corresponding fundamental class

\[ r_{\mu} \in H^*_T(\mathbb{T}[[z]] \times F \times C^*(R_{\leq \lambda}^{\text{sph}})) \equiv H^*_T(pt) \otimes H^*_G(pt) \to H^*_G([z]] \times F \times C^*(R_{\leq \lambda}^{\text{sph}})\]

where the isomorphism is [BFN16b] Lemma 5.3]. Therefore, there exist elements \(g_i \in H^*_T(pt)\) and \(f_i \in H^*_T \times F \times C^*(pt)\) such that \(r_{\mu} = \sum_i g_i \cdot \text{Forg}(M_{\lambda, f_i})\). It is clear that the restriction of \(\pi : \mathcal{R} \to \mathcal{R}^{sph}\) to \(\mathcal{R}^{ab}\) is an isomorphism, so \(\pi_* (r_{\mu}) = r_{\mu}^{\text{sph}}\). We can think of \(r_{\mu}^{\text{sph}} \in \mathcal{P}\), and as in Lemma 2.9 we deduce that in \(A\)

\[ r_{\mu} * e = r_{\mu}^{\text{sph}} * \partial_{w_0} = \sum_i g_i (\text{Inc} * M_{\lambda, f_i} * \partial_{w_0}) \]

Now, by [BFN16b] Lemma 3.21], for any \(x \in H^*_T \times C^*(pt)\) we have \(\partial_{w_0} = \partial_{w_0}\). Putting this all together, we have

\[ \sum_{w,p} \mu_{\mu}(x_w) g_i (\text{Inc} * M_{\lambda, f_i} * \partial_{w_0}) = \sum_{w} \mu_{\mu}(x_w) r_{\mu} e \partial_{w_0} y_w = \sum_{w} r_{\mu} x_w \partial_{w_0} y_w = r_{\mu} \]

where the final equality is Lemma 2.12. The left-hand side is of the claimed form, so we are done.

3. Generators in the quiver gauge theory case

In this section we focus on the quiver gauge theory case, and apply the results developed in the previous section.

3.1. Generation by minuscules. Let us fix some notation. For each \(i \in I\) and \(1 \leq r \leq v_i\), denote by \(\varepsilon_{i,r}\) the coweight \((0, \ldots, 1, \ldots, 0)\) of \(GL(V_i)\) (with 1 in its \(r\)th component). Following the notation of [BFN16a] Appendix A(ii)], recall that the minuscule coweights of \(GL(V_i)\) are

\[ \varpi_{i,n} = \sum_{r=1}^n \varepsilon_{i,r} = (1, \ldots, 1, 0, \ldots, 0), \]

\[ \varpi_{i,n}^* = -\sum_{r=1}^n \varepsilon_{i,n-r+1} = (0, \ldots, 0, -1, \ldots, -1) \]

for \(1 \leq n \leq v_i\). The minuscule coweights for the group \(G = \prod_{i \in I} GL(V_i)\) correspond to a choice of a minuscule coweight (or zero) of \(GL(V_i)\) at each vertex \(i \in I\).
The next result is known to experts [BFN16b, Remark 6.7], [BDG17, Section 4.3]. It is proven during the course of the proof of [ET18, Theorem 4.29], but for the convenience of the reader we include a proof.

**Proposition 3.1.** For any quiver gauge theory \( A^{sv} \) is generated by minuscules.

**Proof.** We will apply [BFN16b, Proposition 6.8], whose proof provides a procedure to find a generating set for \( A^{sv} \). Consider the dominant Weyl chamber for \( G = \prod_i GL(m_i) \), and inside it the hyperplane arrangement given by the weights of \( N \). For each chamber of this arrangement, we find generators for its semigroup of integral points, and for each such generator \( \lambda \), we consider the monopole operators \( M_{\lambda,f} \) (more precisely, we must choose some lifts from the associated graded, see Remark 2.2). Taking all of these elements, together with \( H^*_G F \times \mathbb{C}^\times (pt) \) we get a generating set for \( A^{sv} \).

Note that if we refine the hyperplane arrangement by adding more hyperplanes, the above procedure applied to the refined arrangement will still produce a set of generators for \( A^{sv} \) (possibly with redundancies). In particular, for a quiver gauge theory observe that we may refine the arrangement coming from \( N \) to include the hyperplanes \( w_{j,s} - w_{i,r} = 0 \) and \( w_{i,r} = 0 \) for all \( i, j, r, s \) (i.e. even if \( i \neq j \)). Denote a point \( \eta = (\eta_{i,r}) \in \mathbb{R}_+^s \). Then each chamber of the refined arrangement corresponds to an ordering (as real numbers)

\[
\eta_{i,j} > \eta_{i,j'2} > \ldots > \eta_{i,p,r} > 0 > \eta_{i,p+1,r} > \ldots > \eta_{i,|v|,r} \mid v \mid \]

where we denote \( |v| = \sum_i v_i \). This chamber roughly looks like a Weyl chamber for \( GL(|v|) \); it is easy to see that its semigroup of integral points is generated by the set

\[
\left\{ \sum_{a=1}^{s} \varepsilon_{i_a,r_a} : 1 \leq s \leq p \right\} \cup \left\{ -\sum_{a=s}^{|v|} \varepsilon_{i_a,r_a} : p + 1 \leq s \leq |v| \right\}
\]

To complete the proof, we simply observe that every element of this set is a minuscule coweight for \( G \).

\( \square \)

As an immediate consequence of this proposition and Corollary 2.11, we have:

**Corollary 3.2.** For any quiver gauge theory, \( A \) is generated by its subalgebras \( A^{ab} \) and \( \mathcal{N} \mathcal{H}_G \).

**Remark 3.3.** Proposition 3.1 and Corollary 3.2 give honest algebra generators, i.e. no Poisson brackets or division by \( h \) are used. In particular, both results remain true at \( h = 0 \).

**Remark 3.4.** Suppose that \( G = \prod_i GL(V_i) \), and that \( N \) is a representation of \( G \) such that its weights are all multiples of \( w_{j,s} - w_{i,r} \) or \( w_{i,r} \). Then it is clear that Proposition 3.1 and Corollary 3.2 still apply for \((G,N)\). But as far as the author is aware, the only \( N \) with this property come from quivers (allowing loops and multiple edges).

### 3.2. Generators for \( A^{ab}_{h=1} \)

Let \( \lambda = (\lambda_{i,r}) \) be any coweight for \( G \). If \( \lambda_{i,r} \geq 0 \) for some \( i, r \), then applying [BFN16b, Section 4(iii)] we find that in \( A^{ab} \) we have

\[
r_{\varepsilon_{i,r}} r_{\lambda} = \left( \prod_{i \rightarrow j} \prod_{1 \leq s \leq v_j \atop \lambda_{i,r} < \lambda_{j,s}} (w_{j,s} - w_{i,r} - \frac{1}{2} h) \right) \left( \prod_{i \rightarrow j} \prod_{1 \leq s \leq v_j \atop \lambda_{i,r} < \lambda_{j,s}} (w_{i,r} - w_{j,s} + \frac{1}{2} h) \right) r_{\varepsilon_{i,r} + \lambda}
\]

There is a similar expression for the product \( r_{-\varepsilon_{i,r}} r_{\lambda} \) when \( \lambda_{i,r} \leq 0 \). Notice that if there are no pairs \((j, s)\) such that \( \lambda_{i,r} < \lambda_{j,s} \), then the above product is simply \( r_{\varepsilon_{i,r} + \lambda} \). The proof below is in some sense a generalization of this observation.
Proposition 3.5. $A^{ab}_{h=1}$ is generated by $H^*_T \times F(pt)$ and the elements $r_{\pm \varepsilon_{i,r}}$.

Proof. It is sufficient to prove that any $r_\lambda$ can be expressed in terms of these elements. We proceed by induction on $|\lambda| := \sum_{i,r} |\lambda_{i,r}|$. When $|\lambda| = 0$ the statement is obviously true, so take $\lambda$ some arbitrary coweight with $|\lambda| > 0$.

Suppose that there exists some $\lambda_{i,r} > 0$. Then we will express $r_\lambda$ as a linear combination of the products $r_{\varepsilon_{i,r}} r_{\lambda - \varepsilon_{i,r}}$, where $\lambda_{i,r} > 0$, thought of as elements of $A^{ab}_{h=1}$ viewed as a left module over $H^*_T(pt)$. For $\lambda_{i,r} > 0$, by equation (19) we have

$$r_{\varepsilon_{i,r}} r_{\lambda - \varepsilon_{i,r}} = \pm \left( \prod_{j,s} \left( w_{i,r} - w_{j,s} + \frac{1}{2} \right) \right) r_\lambda$$

Consider the set $S$ of all pairs $(i, r)$ such that $\lambda_{i,r}$ is maximal. If $(i, r) \in S$, then the polynomial on the right-hand side of (20) contains only factors where $(j, s) \in S$. Taken together over all $(i, r) \in S$, these polynomials therefore define an ideal $I$ inside the subring

$$\mathbb{C}[w_{i,r} : (i, r) \in S] \subseteq H^*_T(pt),$$

We claim that the vanishing locus $V(I)$ has no $\mathbb{C}$–points. Assuming this claim for the moment, it follows that $1 \in I$. Using (20), we can therefore express $r_\lambda$ as a (left) linear combination over $\mathbb{C}[w_{i,r} : (i, r) \in S]$ of the elements $r_{\varepsilon_{i,r}} r_{\lambda - \varepsilon_{i,r}}$, as desired. Now, to prove that $V(I)$ has no points, let us assume that $x = (x_{i,r}) \in V(I)$ is a $\mathbb{C}$–point. For every $(i, r) \in S$, since the polynomial on the right-hand side of (20) must vanish, we must have

$$x_{i,r} \in \left\{ x_{j,s} - \frac{1}{2} : (j, s) \in S, i \sim j, 1 \leq s \leq v_j \right\}$$

As $x$ is a finite list of complex numbers some $x_{i,r}$ has maximal real part, so (21) is impossible. Hence $V(I)$ is empty.

A similar argument applies to the case when there exists $\lambda_{i,r} < 0$. Putting these two cases together, we see that any $r_\lambda$ is expressible over $H^*_T(pt)$ as a linear combination of summands $r_{\varepsilon_{i,r}} r_{\lambda - \varepsilon_{i,r}}$. As $|\lambda| - |\varepsilon_{i,k}| < |\lambda|$, by induction on $|\lambda|$ the result follows.

Because of Corollary 3.6 we can conclude that:

Corollary 3.6. As an algebra over $H^*_F(pt)$, $A_{h=1}$ is generated by $N'H_G$ and the elements $r_{\pm \varepsilon_{i,r}}$.

3.3. Generators for $A^{sph}_{h=1}$. We now come to a main result of this paper:

Theorem 3.7. $A^{sph}_{h=1}$ is generated by $H^*_G \times F(pt)$ and the dressed minuscule monopole operators $M_{\varpi_{i,1},f}$, $M_{\varpi^{\ast}_{i,1},f}$.

Proof. Consider the subalgebra $S^{sph} \subset A_{h=1}$ generated by these elements, and also the corresponding subalgebra $S \subset A_{h=1}$ generated by $S^{sph}$ and $N'H_G$. It is enough to prove that $S = A_{h=1}$. Because of Corollary 3.6 it further suffices to show that $r_{\pm \varepsilon_{i,r}} \in S$ for all $i, r$. Since $\varepsilon_{i,r} \in \Sigma \varpi_{i,1}$ and $-\varepsilon_{i,r} \in \Sigma \varpi^{\ast}_{i,1}$, this follows from Proposition 2.13.

We note that all of our $h = 1$ results above are also true over $\mathbb{C}[h, h^{-1}]$, with slightly modified proofs. In general they will fail over $\mathbb{C}[h]$; one must give more generators.

Conjecture 3.8. Consider the Poisson bracket $\{ a, b \} = \frac{1}{h}[a, b]$ on $A^{sph}$. Then $A^{sph}$ is Poisson generated by $H^*_G \times F \times \mathbb{C} \times (pt)$ and the dressed minuscule monopole operators $M_{\varpi_{i,1},f}$, $M_{\varpi^{\ast}_{i,1},f}$.
3.4. Relationship to shifted Yangians. Suppose that our quiver $Q$ is an orientation of a Dynkin diagram of finite ADE type, with corresponding simple Lie algebra $\mathfrak{g}_Q$. Given $I$–graded vector spaces $W, V$, we can define a pair of coweights $\lambda, \mu$ for $\mathfrak{g}_Q$, as in [BFN16a Section 3(iii)]. By [BFN16a Theorem B.18] specialized at $h = 1$, Theorem 3.7 translates directly into the statement that there is a surjection

$$Y_\mu[z_1, \ldots, z_N] \twoheadrightarrow \mathcal{A}_{h=1}^{\text{sp}}$$  \hspace{1cm} (22)

Recall that in the notation of [BFN16a] we identify $H^*_F(pt) = \mathbb{C}[z_1, \ldots, z_N]$. Meanwhile, $Y_\mu[z_1, \ldots, z_N]$ is a shifted Yangian for $\mathfrak{g}_Q$. We will not give its precise definition here, and instead refer the reader to [BFN16a Definition B.2] or [FT18 Sections 2(vi)–(vii)]. For us, it suffices to know that its generators map to certain classes $M_{\omega_{i,1}}^*, M_{\omega_{i,1}^*}^*$, or to elements of $H^*_{G \times F}(pt)$. Note that the fact that these elements satisfy the Yangian relations is a rank 2 calculation, done in [BFN16a Appendix B].

As in [KTW18 Section 4], one can appropriately define $Y_\mu[z_1, \ldots, z_N]$ and the homomorphism (22) in arbitrary simply-laced Kac-Moody type.

We similarly conclude:

**Corollary 3.9.** For any simple quiver $Q$, there is a surjective homomorphism of $\mathbb{C}[z_1, \ldots, z_N]$–algebras

$$Y_\mu[z_1, \ldots, z_N] \twoheadrightarrow \mathcal{A}_{h=1}^{\text{sp}},$$

where $Y_\mu[z_1, \ldots, z_N]$ is an appropriately defined shifted Yangian for the simply-laced Kac-Moody Lie algebra $\mathfrak{g}_Q$.

The above discussion may be rephrased as an isomorphism $Y_\mu^\lambda \cong \mathcal{A}_{h=1}^{\text{sp}}$, where $Y_\mu^\lambda$ is the truncated shifted Yangian [KTW18 Section 4.2]. Essentially, $Y_\mu^\lambda$ is defined as the image of $Y_\mu[z_1, \ldots, z_N] \rightarrow \mathcal{A}_{h=1}^{\text{sp}}$.

Following [KTW18 Section 4.3], we can also define the flag Yangian $FY_\mu^\lambda$, which is a matrix algebra over $Y_\mu^\lambda$ built using $N\mathcal{H}_G$. Using Theorem 2.6 it is easy to see that the above isomorphism extends is an isomorphism $FY_\mu^\lambda \cong \mathcal{A}_{h=1}$, compatibly with the inclusions of the spherical subalgebras $Y_\mu^\lambda$ and $\mathcal{A}_{h=1}^{\text{sp}}$. Put differently, the embedding $Y_\mu^\lambda \hookrightarrow FY_\mu^\lambda$ is defined in terms of their faithful actions on the polynomial rings

$$P^\Sigma = H_{G \times F}^*(pt) \subset P = H_{T \times F}^*(pt)$$

Elements of $Y_\mu^\lambda$ act on $P^\Sigma$, and we extend this to an action on $P$ by projection/inclusion from $P^\Sigma$. The element Av corresponds to projection onto $\Sigma$–invariants, while $\text{inc}$ corresponds to their inclusion, matching up with $\mathcal{A}_{h=1}^{\text{sp}} \hookrightarrow \mathcal{A}_{h=1}$.

In finite ADE type the algebra $Y_\mu[z_1, \ldots, z_N]$ carries a filtration defined explicitly in terms of its PBW generators, such that (22) is compatible with the canonical filtration on $\mathcal{A}_{h=1}^{\text{sp}}$ induced by the grading on $\mathcal{A}_{h=1}^{\text{sp}}$, see [BFN16a Section B(vii)]. As argued in [KTW18 Corollary 4.10], it follows from Corollary 3.9 that:

**Corollary 3.10.** In finite ADE type, the canonical filtration on $\mathcal{A}_{h=1}^{\text{sp}}$ agrees with the quotient filtration from $Y_\mu[z_1, \ldots, z_N] \rightarrow \mathcal{A}_{h=1}^{\text{sp}}$. In other words, there is a surjection of $\mathbb{C}[h, z_1, \ldots, z_N]$–algebras

$$Y_\mu[z_1, \ldots, z_N] := \text{Rees} Y_\mu[z_1, \ldots, z_N] \twoheadrightarrow \mathcal{A}_{h=1}^{\text{sp}}$$

and an isomorphism $Y_\mu^\lambda := \text{Rees} Y_\mu^\lambda \cong \mathcal{A}_{h=1}^{\text{sp}}$.

---

4As explained there, the conventions [KTW18] are to take $h = 2$. We may ignore this distinction, since it is essentially a cosmetic difference.
This result generalizes [BFN16a, Corollary B.28], which covers the case where $\mu$ is dominant. As a consequence, it follows that Conjecture 3.8 is true in the finite ADE case, since the Yangian is Poisson generated by its Chevalley-like generators.

**Remark 3.11.** The proof of this corollary uses the fact that $\text{gr} Y_{\mu} \to A_{h=0}^{\text{sph}}$, which comes from geometry: there is a closed embedding $W_{u^*} = \text{Spec} A_{h=0}^{\text{sph}} \subset W_{\mu^*} = \text{Spec} \text{gr} Y_{\mu}$. See [FKP+18, Section 5]. Note that, on its own, this embedding at $h = 0$ is not strong enough to deduce the above result, since in general gradings are unbounded below (as opposed to the dominant case [BFN16a, Corollary B.28]).

### 4. Coulomb branch categories

In this section, following Webster [Web16a, Section 3], we define a category such that the algebra $A$ appears as an endomorphism algebra. Note that there are many variations on this category, and we have chosen rather plain conventions, in order to simplify our goal of relating this category with the KLR Yangian algebra $R$ from [KTW+18, Section 4.4].

#### 4.1. Webster’s extended BFN category.

Denote by $\mathfrak{t}_R \subset \mathfrak{t} = \text{Lie } T$ the set of diagonal matrices with entries in $R$. For a point $\eta = (\eta_i,r) \in \mathfrak{t}_R$, we can associate linear operators on $g(\langle \rangle)$ and $N(\langle \rangle)$, defined by the (adjoint) action of $\eta + z \frac{\partial}{\partial z}$. The eigenvalues of these operators are in $R$. We define $I_{\eta} \subset G(\langle \rangle)$ to be the Iwahori subgroup whose Lie algebra is the sum of the non-negative eigenspaces. Similarly, we define $U_{\eta} \subset N(\langle \rangle)$ to be sum of the non-negative degree eigenspaces. It is easy to see that $I_{\eta}$ preserves the subspace $U_{\eta}$.

For generic $\eta$, note that $I_{\eta}$ only depends on which alcove $\eta$ lies in, while $U_{\eta}$ only depends on $\eta$’s cell in a hyperplane arrangement determined by the weights of $N$. Following [BFN16b, Definition 5.2] let us define a generalized root for the pair $(G,N)$ to be either (i) a root of $\text{Lie } G$ or (ii) a weight of $N$. Denote by $t^0_R \subset t_R$ the complement of all integer shifts of generalized root hyperplanes. Explicitly, these are the hyperplanes

$$w_{i,r} - w_{i,s} = n, \quad \text{for } i \in I, \text{ all } r \neq s, \text{ and } n \in \mathbb{Z} \quad (23)$$

$$w_{i,r} = n, \quad \text{for } i \in I, \text{ all } r, \text{ and } n \in \mathbb{Z} \quad (24)$$

$$w_{i,r} - w_{j,s} = n, \quad \text{for } i \sim j, \text{ all } r,s, \text{ and } n \in \mathbb{Z} \quad (25)$$

Their complement $t^0_R$ is a disjoint union of cells. By the above, the pair $(I_{\eta}, U_{\eta})$ depends only on the cell of $\eta$ in $t^0_R$.

**Remark 4.1.** As in [Web16a], we could incorporate non-integral shifts above. For example, this is desirable in situations where some weight of $N$ coincides with a root of $G$. This does not occur for a quiver without loops, so we omit this complication.

For any $\eta, \eta' \in t^0_R$, we define analogues of (7) and (10),

$$\eta' R_\eta := \{ [g,s] \in G(\langle \rangle) \times \mathfrak{t}_\eta U_{\eta} : gs \in U_{\eta'} \}, \quad (26)$$

$$\eta' A_\eta := H_s(\mathcal{I}_{\eta'} \times F) \times \mathbb{C}^\times_{(\eta' R_\eta)} \quad (27)$$

The group action and Borel-Moore homology are defined as described in Section 2.2. An analogue of Theorem 2.1 holds, and in particular there is a convolution product

$$\eta'' A_{\eta'} \otimes \eta' A_\eta \to \eta'' A_\eta \quad (28)$$

Following [Web16a, Definition 3.5], we take:
Definition 4.2. The extended BFN category \( \mathcal{B} \) is the category whose objects are elements \( \eta \in t^*_\mathbb{R} \), and such that

\[
\text{Hom}_{\mathcal{B}}(\eta, \eta') := \eta' A_\eta,
\]

with composition of morphisms given by the convolution product \( \mathcal{I} \).

There is an action of the affine Weyl group \( \hat{\Sigma} = \Sigma \times \mathbb{Z}^\vee \) on \( t^\circ \); the finite Weyl group \( \Sigma \) acts via its standard permutation action on \( t \), while the lattice \( \mathbb{Z}^\vee \) acts by \( \lambda \cdot \eta = \eta - \lambda \). With these conventions, it is not hard to see that \( \pi \) is of one of two types: (a) If the hyperplane \( H_k \) corresponds to a weight of \( G \) (case \( 23 \)) then \( I^+_k = I^-_k \), while \( U^+_k \) differ by one dimension. We let

\[
r(\eta^+_k, \eta^-_k) = [\pi^{-1}(1)] = \text{Hom}_{\mathcal{B}}(\eta^-_k, \eta^+_k)
\]

be the fiber over the unit point of \( \mathcal{B} \), under the map \( \pi : \eta^+_k \eta^-_k \rightarrow \mathcal{B} \) sending \([g, s] \mapsto [g] \) as usual.

(b) If the hyperplane \( H \) correspond to a root of \( G \) (case \( 23 \)) then \( U^+_k = U^-_k \), while the Iwahori \( I^\pm_k \) differ by a single (affine) root \( \alpha \). In this case we let

\[
u_\alpha = [\pi^{-1}(I^+_k I^-_k / I^\pm_k)] \in \text{Hom}_{\mathcal{B}}(\eta^-_k, \eta^+_k)
\]

Note that \( \frac{I^+_k I^-_k / I^-_k}{I^\pm_k} \cong \mathbb{P}^1 \) by our assumption.

In this way, we have a morphism \( x_k \) for each \( H_k \). Finally, we define the morphism

\[
x_\pi = x_k x_{k-1} \cdots x_1 \in \text{Hom}(\pi(0), \pi(1)) \quad (30)
\]

This agrees with \( \text{Webster 16a, Definition 3.11} \), because of \( \text{Webster 16a, Relation (3.4e)} \).

Besides the elements \( x_\pi \), for any \( \eta \in \hat{\Sigma} \) there is an isomorphism

\[
y_\eta = [\pi^{-1}(\eta)] \in \text{Hom}_{\mathcal{B}}(\eta, \eta)\eta)
\]

where we think of \( \eta \in \mathcal{B} \) as in Section 2.4.

Webster gives a complete set of relations between the above elements \( \text{Webster 16a, Theorem 3.10} \), and also gives bases for any \( \text{Hom}_{\mathcal{B}}(\eta, \eta') \) \( \text{Webster 16a, Corollary 3.12} \). For our present purposes it will suffice to know that

\[
y_\eta x_\pi = x_w \eta y_w \quad (31)
\]

where \( \hat{\Sigma} \) acts on the set of paths pointwise, via its action on \( t \).
Remark 4.3. [Web16a, Theorem 3.10] also provides a faithful action of \( B \) where each \( \eta \in t_0^1 \) is assigned a copy of the polynomial ring \( \mathbb{H}_{1 \times t_R}^* \), which is helpful to keep in mind. Roughly speaking, acting on this polynomial representation \( (i) r(\eta) \) is multiplication by a linear polynomial, \( (ii) u_\alpha \) is a divided difference operator in the affine root \( \alpha \), and \( (iii) \gamma_w \) acts by \( w \).

4.3. A subcategory. Recall that we denote \( |v| = \sum_i v_i \). Consider the set
\[
I^v := \left\{ i = (i_1, \ldots, i_m) \in I^{|v|} : \# \{k : i_k = i\} = v_i \text{ for all } i \right\}
\]
An element \( i \in I^v \) is equivalent to a total order \( <_1 \) on the set of pairs
\[
\{(i, r) : i \in I, 1 \leq r \leq v_i \}
\]
which restricts to the order \( (i, 1) <_1 (i, 2) <_1 \ldots <_1 (i, v_i) \) for each \( i \in I \).

Example 4.4. If \( I = \{1, 2\} \) and \( v_1 = 3, v_2 = 2 \), then \( i = (1, 2, 1, 2, 1) \in I^v \). It corresponds to the total order
\[
(1, 1) <_1 (2, 1) <_1 (1, 2) <_1 (2, 2) <_1 (1, 3)
\]
For each \( i \), choose \( \eta = (\eta_{i,r})_{i \in I, 1 \leq r \leq v_i} \in t_0^1 \) with component \( 0 < \eta_{i,r} < 1 \), and such that
\[
(i, r) <_1 (j, s) \text{ implies } \eta_{i,r} < \eta_{j,s}.
\]
Then
\[
U_{\eta} = \bigoplus_{i \rightarrow j} \left( \bigoplus_{(i, r) <_1 (j, s)} \text{Hom}_C(V_{i,r}, V_{j,s})[[z]] \oplus \bigoplus_{(i, r) >_1 (j, s)} \text{Hom}_C(V_{i,r}, V_{j,s})[[z]] \right) \oplus \bigoplus_i \text{Hom}_C(W_i, V_i)[[z]]
\]
where we denote the lines \( V_{i,r} = C e_{i,r} \subset V_i \). Meanwhile, for any \( i \in I^v \) the Iwahori \( I_{\eta} = I \) is always our usual one \( (5) \).

Definition 4.5. The category \( B^v \) is the full subcategory of \( B \) having objects \( I^v \) (under \( i \mapsto \eta_i \)).

The categories \( B \) and \( B^v \) are closely related to the algebra \( A \) from (10). To see this, first fix any \( i_v \in I^v \) such that \( i \rightarrow j \) implies \( (i, r) <_{i_v} (j, s) \) for all \( r, s \). For example, by choosing a total order \( i_1 < i_2 < \ldots \) on \( I \) such that \( i \rightarrow j \) implies \( i < j \), we could define \( i_v \) by
\[
i_v = (i_1, \ldots, i_1, i_2, \ldots, i_2, \ldots)
\]
Remark 4.6. It is possible that no such total order exists, since \( Q \) might have an oriented cycle. But by [BFN16b, Section 6(viii)], if we reorient an arrow of \( Q \) then \( A^{\text{spf}} \) is unchanged up to isomorphism. Thus we can safely assume that \( Q \) is acyclic, so that a total order does exist.

A similar element (denoted \( i_m \)) is considered in [KTW+18, Section 4.5], but with additional requirements in its definition since there the Dynkin diagram \( I \) is assumed bipartite. (We do not make this bipartiteness assumption here, as it is not relevant to our study.)

From the explicit description \( (33) \) we see that \( U_{i_v} = \mathbb{N}[[z]] \). Since \( I_{\eta} = I \) for all \( i \in I^v \), there is an equality \( R = \gamma_i R_{i_v} \) and it is \( (I \times F) \times \mathbb{C}^\times \)-equivariant. Since the convolution products are also defined in the same way, it follows that

Lemma 4.7. For any choice of \( i_v \) as above, there is an equality of algebras \( A = \text{End}_{B^v}(i_v) \).

\(^5\text{For simplicity, we will choose the } \eta_{i,r} \text{ to partition the interval } [0,1] \text{ into equal parts.} \)
4.4. Cylindrical diagrams and morphisms. For the category $\mathcal{B}^\vee$, we can encode the paths from Section 4.2 using the cylindrical KLR diagrams from [KTW+18 Section 4.4]. Recall that a cylindrical KLR diagram $D$ consists of a finite number of curves in a cylinder $\mathbb{R}/\mathbb{Z} \times [0,1]$, locally of the form

\[ \hspace{1cm} \]

which meet the “bottom circle” $t = 0$ and “top circle” $t = 1$ at distinct points with $x \neq 0$, where $(x,t)$ are coordinates on $\mathbb{R}/\mathbb{Z} \times [0,1]$. Each curve carries a label $i \in I$. See the left-hand side of (35) for an example (which also depicts the coordinate axis $(x,t)$), and [KTW+18] for more details.

At the bottom of $D$, the sequence of labels of the curves defines a sequence $\text{bot}(D) \in I^m$ in order of increasing $x$. Similarly we get $\text{top}(D) \in I^m$. Also, at the top or bottom of $D$, the list of $x$–coordinates of the curves encode a point of $t^*_\mathbb{Z}$, so that the coordinate of the $r$th curve labelled $i$ is $\eta_{i,r}$. We will restrict our attention to those diagrams where $\text{bot}(D), \text{top}(D) \in I^\vee$, and such that the $x$–coordinates of the curves at the top and bottom are precisely given by $\eta_{\text{bot}(D)}$ and $\eta_{\text{top}(D)}$, respectively.

For the time being, we focus on diagrams which carry no dots. Looking at the bottom of $D$, consider the $r$th curve having label $i$. It may be lifted uniquely to a path $\pi_{i,r} : [0,1] \to \mathbb{R}$ in the universal cover of $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$, with the property $\pi_{i,r}(0) = \eta_{i,r}$. These lifts define an unrolled path $\pi = (\pi_{i,r}) : [0,1] \to t^*_\mathbb{Z}$, associated to $D$.

**Lemma 4.8.** For $D, \pi$ as above, there exists a unique $w \in \hat{W}$ such that $\pi(1) = w \cdot \eta_{\text{top}(D)}$.

So, from our diagram $D$ we get a pair $(\pi, w)$. Denote the corresponding morphisms

\[ \begin{align*}
\tau_D := \tau_\pi \in \text{Hom}_{\mathcal{B}}(\eta_{\text{bot}(D)}, w \cdot \eta_{\text{top}(D)}), \\
y_D := y_w \in \text{Hom}_{\mathcal{B}}(\eta_{\text{top}(D)}, w \cdot \eta_{\text{top}(D)})
\end{align*} \]

Their composition gives a morphism in $\mathcal{B}^\vee$:

\[ \phi_D := y_D^{-1}\tau_D \in \text{Hom}_{\mathcal{B}^\vee}(\text{bot}(D), \text{top}(D)) \]

**Remark 4.9.** The reader may have noticed that we have secretly slightly generalized our notion of path, since cylindrical KLR diagrams might have several crossings at the same “time” $t$:

\[ \hspace{1cm} \]

Here, the corresponding path $\pi$ passes through an intersection of three hyperplanes. Thankfully, the discussion below shows that we can isotope $D$, without changing the corresponding morphisms in $\mathcal{B}$. So the reader may either accept these more general paths (which may pass through intersections
of any number of hyperplanes, so long as the hyperplanes are in disjoint sets of variables), or else restrict to those \( D \) where crossings never occur at the same height.

Cylindrical KLR diagrams \( D', D'' \) can be multiplied if \( \text{bot}(D') = \text{top}(D'') \): we define \( D' \cdot D'' \) by stacking \( D' \) on top of \( D'' \). We may also consider isotopies of cylindrical KLR diagrams, with the extra requirements that (a) all isotopies fix the top and bottom circles, and (b) they do not create or break crossings, including of the “seam” \( x = 0 \).

**Proposition 4.10.** For any cylindrical KLR diagram \( D \) without dots,

\( \varphi_D \) only depends on \( D \) up to isotopy, defined as above. In particular, \( \varphi_D \) only depends on \( D \) up to isotopy. 

(b) If \( D = D' \cdot D'' \), then \( \varphi_D = \varphi_{D'} \circ \varphi_{D''} \).

**Proof.** Both parts are consequences of [Web16a, Theorem 3.10]. In particular part (b) follows from (31).

Consider then part (a). The claim about \( y_D \) is well-known, and follows from the relations of \( \hat{\Sigma} \). For \( \tau_D \), first suppose that \( D \) has no crossings at the same time \( t \), and consider an isotopy \( D \to D' \) which preserves this property. Then the corresponding paths pass through precisely the same sequence of cells and hyperplanes in \( t \cdot \mathbb{R} \), so by the definition \( (30 \) we have that \( \tau_D = \tau_{D'} \). It remains to check that subdiagrams can “slide” past each other in time, if they are formed by disjoint sets of strands:

\[
\begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{array}
\end{array}
\end{array}
\quad \text{def}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{array}
\end{array}
\end{array}
\]

(37)

This is a consequence of [Web16a, Theorem 3.10], for a very simple reason: in the faithful polynomial representation on \( \mathcal{B} \) given there, crossings correspond to divided difference operators, difference operators, or multiplication by polynomials. Moreover, the set of variables on which a crossing acts is encoded by its strands. So in particular, the operations corresponding to our two subdiagrams take place in distinct sets of variables, and therefore commute.

The simplest diagrams are those with a single crossing, either between two strands or between a strand and the “seam” of \( \mathbb{R}/\mathbb{Z} \). Following the notation of [KTW+18, Section 4.4], we denote these by

\[
\psi_k(i) = \begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{array},
\quad
\sigma_+(i) = \begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{array},
\quad
\sigma_-(i) = \begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{array}
\]

The proposition shows that any \( \varphi_D \) (with no dots) can be decomposed as a product of \( \varphi \)'s corresponding to these simpler diagrams.

We have thus far neglected the dots that a diagram \( D \) might carry. Consider the diagrams with straight vertical lines:

\[
e(i) = \begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{array},
\quad
z_k(i) = \begin{array}{c}
\begin{array}{c}
\ldots \\
\ldots \\
\end{array}
\end{array}
\]

To these diagrams, we associate the morphisms \( 1 \) and \( w_{i,r} \) in \( H^*_T(pt) \subset \text{End}_{\mathcal{D}}(i) \), respectively, where as usual \( (i, r) \) is determined by \( i_k \) from \( i \). Appealing to [Web16a, Theorem 3.10] again, we claim that
these morphisms again isotope as expected, both with each other and with the $\varphi_D$ defined above. We can therefore define

$$\varphi_D \in \text{Hom}_{\mathcal{B}}(\text{bot}(D), \text{top}(D))$$  \hspace{1cm} (38)

for an arbitrary cylindrical KLR diagram $D$, and this is invariant under isotopy of $D$.

4.5. Comparison with the algebra $\mathfrak{Y}$. The KLR Yangian algebra $\mathfrak{Y}$ is defined in [KTW+18 Section 4.4], as a sort of KLR algebra on a cylinder. $\mathfrak{Y}$ is spanned by cylindrical KLR diagrams up to isotopy, modulo certain relations, and multiplication is given by stacking of diagrams. For any $i \in I^h$ there is an idempotent $\epsilon(i) \in \mathfrak{Y}$, which is the diagram with straight vertical lines as above. Note that in $\mathfrak{Y}$ the number of strands can be arbitrary, but we will restrict our attention to those diagrams with strands labelled by $I^v$.

To be more precise, in this paper we will consider $\mathfrak{Y}$ as an algebra over $\mathbb{C}$, as a sort of KLR algebra on a cylinder. $\mathfrak{Y}$ is spanned by cylindrical KLR diagrams up to isotopy, modulo certain relations, and multiplication is given by stacking of diagrams. For any $i \in I^h$ there is an idempotent $\epsilon(i) \in \mathfrak{Y}$, which is the diagram with straight vertical lines as above. Note that in $\mathfrak{Y}$ the number of strands can be arbitrary, but we will restrict our attention to those diagrams with strands labelled by $I^v$.

We recover the algebra from $\mathfrak{KTW}^+$ by taking $h = 2$, and specializing $z_1, \ldots, z_N$ to complex numbers. The latter can be encoded by an $I$–tuple of multisets, denoted $R$ in [KTW+18].

**Theorem 4.11.** The map $D \mapsto \varphi_D$ defines an isomorphism of $H_{F \times C^+}^*(pt)$–algebras

$$\bigoplus_{i,j \in I^v} \epsilon(j) \hat{\epsilon}(i) \xrightarrow{\sim} \bigoplus_{i,j \in I^v} \text{Hom}_{\mathcal{B}}(i,j)$$

**Proof.** We can verify that this map is well-defined by comparing the polynomial representations of $\mathfrak{Y}$ and $\mathcal{B}$, defined in [KTW+18 Theorem 4.17] and [Web16a Theorem 3.10], respectively. Note that in the conventions of [KTW+18], for each $I$ we should identify

$$H_T^*(pt) = \mathbb{C}[w_{i,r} : i \in I, 1 \leq r \leq v_i] \cong \mathbb{C}[Z_1(i), \ldots, Z_{|v|}(i)]$$

where $Z_1(i), \ldots, Z_{|v|}(i)$ map to the elements $w_{i,r}$ in increasing order under $<_I$. Accounting for this convention change, we easily check that the action of the elements $\phi_k(i), \sigma_k(i), \epsilon(i)$ and $z_k(i)$ is the same whether thought of in $\mathfrak{Y}$ or in $\mathcal{B}$. This shows that the homomorphism is well-defined.

To show that this map is an isomorphism, we look at bases. For each $w \in \hat{\Sigma}$, we fix a minimal length path $\pi$ from $\eta$ to $w\eta$, and consider the morphism $y_w^{-1} \tau_{\pi} \in \text{Hom}_{\mathcal{B}}(i,j)$. On the one hand, by [Web16a Corollary 3.12] these elements (over all $w \in \hat{\Sigma}$) give a basis for $\text{Hom}_{\mathcal{B}}(i,j)$ as a left (or right) module over $H_{T \times F \times C^+}^*(pt)$. The path $\pi$ can also be “rolled” into a diagram $D$, and Lemma 4.10 shows that $\varphi_D = y_w^{-1} \tau_{\pi}$. On the other hand, these same diagrams $D$ (over all $w \in \hat{\Sigma}$) form a basis for $\text{Hom}_{\mathcal{B}}(i,j)$, as shown during the proof of [KTW+18 Theorem 4.17]. By definition, this basis for $\epsilon(j) \hat{\epsilon}(i)$ maps bijectively onto the basis for $\text{Hom}_{\mathcal{B}}(i,j)$, and therefore the map is an isomorphism. \hfill \Box

In [KTW+18 Theorem 4.19], it was proven that there is a map $Y^\lambda_\mu \to \epsilon \mathfrak{Y}_{h=1} e$, and claimed that this map is an isomorphism. We can complete the proof of this result:

**Corollary 4.12.** The map $Y^\lambda_\mu \xrightarrow{\sim} \epsilon \mathfrak{Y}_{h=1} e$ is an isomorphism. In finite ADE type, this upgrades to an isomorphism $Y^\lambda_\mu \xrightarrow{\sim} \epsilon \mathfrak{Y}$. 
Proof. We claim that there is a commutative diagram

\[
\begin{array}{ccc}
Y^\lambda_\mu & \longrightarrow & e H_{h=1} e \\
\downarrow & & \downarrow \\
A_{h=1}^{sph} & \sim & e \text{End}_{\mathscr{A}^v}(i_v)_{h=1} e
\end{array}
\]

(39)

The top arrow is [KTW+18 Theorem 4.19], the left side isomorphism is from Section 3.4, and the right side isomorphism is from the previous theorem. Finally, the bottom comes from the isomorphism \( A_{sph} \cong e A e \) of Theorem 2.6, plus the equality \( A = \text{End}_{\mathscr{A}^v}(i_v) \) of Lemma 4.7.

To check commutativity, we can argue directly in terms of the images of the generators of the Yangian. The element \( F(r)_i \in Y^\lambda_\mu \) maps to the dressed monopole operator \( M \bullet, 1, f \in A_{h=1}^{sph} \) where \( f = z_r^{-1} \). By Lemma 2.10, this monopole operator maps to the endomorphism

\[
\partial_{i,v_i} \cdots \partial_{i,2} z_r^{-1} r^\lambda_\mu e \in e A_{h=1} e = e \text{End}_{\mathscr{A}^v}(i_v)_{h=1} e
\]

Indeed, the stabilizer \( W_\lambda = W_{i,v_i} \) of \((1,0, \ldots, 0)\) is the symmetric group on \( \{2, \ldots, v_i\} \), and it is easy to see that \( w_0 w_\lambda = s_i v_i \cdots s_i \). As argued in the proof of [KTW+18 Theorem 4.19], this endomorphism corresponds to a cylindrical KLR diagram encoding the image of \( F_i^{(r)} \) in \( H \), which is more or less (ignoring dots, the idempotent \( e \), and a sign)

At the bottom, the order on the curve labels \( i \) is determined by \( i_v \), defined as in (34). Thus the image of \( F_i^{(r)} \) is the same under either path in (39). Similar arguments hold for the other generators of \( Y^\lambda_\mu \), so the diagram is commutative.

\[\square\]

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