ASYMPTOTIC ANALYSIS OF RUIN IN CEV MODEL

F. KLEBANER AND R. LIPTSER

Abstract. We give asymptotic analysis for probability of absorption
\( P(\tau_0 \leq T) \) on the interval \([0, T]\), where \( \tau_0 = \inf\{t : X_t = 0\} \) and \( X_t \) is a nonnegative diffusion process relative to Brownian motion \( B_t \),
\[
dX_t = \mu X_t \, dt + \sigma X_t^\gamma \, dB_t,
\]
\( X_0 = K > 0 \)
Diffusion parameter \( \sigma x^\gamma \), \( \gamma \in \left[\frac{1}{2}, 1\right) \) is not Lipschitz continuous and assures \( P(\tau_0 > T) > 0 \). Our main result:
\[
\lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P(\tau_0 \leq T) = -\frac{1}{2E M_T^2},
\]
where \( M_T = \int_0^T \sigma(1-\gamma)e^{-(1-\gamma)x} dB_s \). Moreover we describe the most likely path to absorption of the normed process \( \frac{x}{K} \) for \( K \to \infty \).

1. Introduction

In this paper, we analyze the Constant Elasticity of Variance Model (CEV), introduced by Cox 1996, \[1\] and applied to Option Pricing (see e.g. Delbaen, Shirakawa \[2\] and Lu, Hsu \[9\]). This model is given by the Itô equation with respect to a standard Brownian motion \( B_t \) and a positive initial condition \( X_0 = K > 0 \),
\[
dX_t = \mu X_t \, dt + \sigma X_t^\gamma \, dB_t, \tag{1.1}
\]
where \( \mu, \sigma \neq 0 \) are arbitrary constants and \( \gamma \in \left[\frac{1}{2}, 1\right) \). For \( \gamma = \frac{1}{2} \), this model is known as CIR (Cox, Ingersol and Ross) model. The diffusion coefficient \( \sigma x^\gamma \) is only Hölder continuous, yet the Itô equation \[1.1\] has unique strong solution\[1\]. In contrast to Black-Scholes model (\( \gamma = 1 \)) with \( X_t > 0 \) for any \( t > 0 \), for CEV model the process \( X_t \) is absorbed in zero at the time \( \tau_0 = \inf\{t : X_t = 0\} \) with \( P(\tau_0 < \infty) > 0 \) which can be interpreted as time of ruin.

In a proposed asymptotic analysis, as \( K \to \infty \), a crucial role plays the normed process \( x^K_t = \frac{X_t}{K} \), being the unique solution of the following Itô equation
\[
dx^K_t = \mu x^K_t \, dt + \frac{\sigma}{K^{(1-\gamma)}}(x^K_t)^\gamma dB_t, \tag{1.2}
\]
subject to the initial condition \( x^K_0 = 1 \), and a small diffusion parameter \( \frac{\sigma x^{\gamma}}{K^{1-\gamma}} \). We emphasize that the process \( x^K_t \) inherits the ruin time \( \tau_0 \).

The assumption \( \gamma < 1 \) implies that the diffusion in \[1.2\] has a small diffusion coefficient. This enables us to find a rough lower bound of \( P(\tau_0 \leq T) \) for any \( K > 0 \) (see Remark \[2\]). With \( K \to \infty \), this lower bound is best

\(^1\)Delbaen and Shirakawa, \[2\] - existence; Yamada-Watanabe - uniqueness (see e.g., Rogers and Williams, p. 265 \[10\] or \[9\] p.17 and Theorem 13.1)
possible on logarithmic scale. To this end we apply the Large Deviation Theory for asymptotic analysis of two families:

\[ \left\{ \left( x^K_t \right)_{t \in [0,T]} \right\}_{K \to \infty} \quad \text{and} \quad \left\{ \frac{1}{K^{1-\gamma}} M_T \right\}_{K \to \infty}, \]

where

\[ M_t = \int_0^t \sigma(1-\gamma)e^{-(1-\gamma)\mu s} dB_s. \]  

(1.3)

For the second family, Large Deviation Principle (LDP) is well known. For the first family, Freidlin-Wentzell’s LDP, [5], is anticipated even though the diffusion parameter is only Hölder continuous and singular at zero. For \( \gamma = \frac{1}{2} \) LDP is known from Donati-Martin et al., [3], with the speed rate \( \frac{1}{K} \) and the rate function of Freidlin-Wentzell’s type with a corresponding modification: \( J_T(u) = \frac{1}{2} \int_0^T \left( \frac{u - \mu u}{\sqrt{\sigma^2}} \right)^2 I_{\{u_t > 0\}} dt. \) We show that for \( \gamma \in (\frac{1}{2}, 1) \) LDP is also valid with the speed rate and the rate function depending on \( \gamma \). Combining both LDP’s we obtain the following asymptotic result:

There is a smooth nonnegative function \( u^*_T \), with \( u^*_T = 1 \) and absorbed at the time \( T \), \( u^*_T = 0 \), such that for any smooth nonnegative function \( u_t \), with \( u_0 = 1 \) and absorbed on the interval \([0, T]\), \( u_T = 0 \),

\[ \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P(\tau_0 \leq T) \to \]

\[ = \lim_{\delta \to 0} \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P \left( \sup_{t \in [0, T]} |x^K_t - u^*_t| \leq \delta \right) \]

\[ \geq \lim_{\delta \to 0} \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P \left( \sup_{t \in [0, T]} |x^K_t - u_t| \leq \delta \right). \]

The latter inequality gives us a motivation to consider \( u^*_T \) as the most likely path to absorption of the normed process \( x^K_t \).

Note that calculations for \( P(\tau_0 \leq T) \) on logarithmic scale requires a non-standard technique. The set \( \{ \tau_0 \leq T \} = \left\{ (x^K_t)_{t \in [0,T]} \in D \right\} \), where

\[ D = \{ u \in C_{[0,T]} : u_0 = 1; u_t = u_{\theta(u)} \wedge t, \theta(u) = \inf \{ t : u_t = 0 \} \leq T \}. \]

\( D \) is closed in the uniform metric \( (\varnothing) \) in the space \( C_{[0,T]} \) of continuous functions on \([0, T]\). Hence, the upper limit \( \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P(\tau_0 \leq T) \) is done according to the LDP technique. However, \( D \) has an empty interior. This fact prevents us to use the LDP technique for the lower bound \( \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P(\tau_0 \leq T) \). Nevertheless, we obtain this lower by using an inclusion \( \{ \tau_0 \leq T \} \supseteq \left\{ \frac{1}{K} M_T < -1 \right\} \), where \( M_T \) is defined in (1.3). The probability of the latter is easily computable, and gives a surprising result

\[ \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log \left\{ \frac{1}{K^{1-\gamma}} M_T < -1 \right\} = \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P(\tau_0 \leq T). \]

Of course,

\[ \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log \left\{ \frac{1}{K^{1-\gamma}} M_T < -1 \right\} \leq \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P(\tau_0 \leq T) \]

Footnote: For \( \gamma = \frac{1}{2} \) see also [2] and Rouault [11].
which together the above establish the desired limit. This trick is of independent interest and might be useful for establishing LDP in other problems.

2. Asymptotic of $P(\tau_0 \leq T)$ as $K \to \infty$ on logarithmic scale

The random process $M_t$ (see (2.1)) is a Gaussian martingale with the variation process $\langle M \rangle_t = EM_t^2$:

$$\langle M \rangle_t = \int_0^t \sigma^2(1 - \gamma)^2e^{-2(1-\gamma)\mu_s}ds.$$  \hspace{1cm} (2.1)

**Theorem 2.1.** For any $T > 0$,

$$\lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P(\tau_0 \leq T) = -\frac{1}{2\langle M \rangle_T}.$$  \hspace{1cm} (2.2)

**Proof.** To apply the Itô formula in a vicinity of $\tau_0$, let us define a stopping time $\tau_\varepsilon = \inf\{t \leq T : x^K_t = \varepsilon\}$, $\varepsilon > 0$. Now, by Itô’s formula, applied to $(x^K_t)^{1-\gamma}$, $t \leq \tau_\varepsilon < T$, we find that

$$(x^K_t)^{1-\gamma} = 1 + \int_0^t (1 - \gamma)\mu_s(x^K_s)^{1-\gamma}ds + \int_0^t (1 - \gamma)\sigma_s e^{-2(1-\gamma)\mu_s}dB_s$$  

and in turn,

$$(x^K_{\tau_\varepsilon})^{1-\gamma} = 1 + \int_0^{\tau_\varepsilon} (1 - \gamma)\mu_s(x^K_s)^{1-\gamma}ds + \int_0^{\tau_\varepsilon} (1 - \gamma)\sigma_s e^{-2(1-\gamma)\mu_s}dB_s$$  

In view of $\lim_{\varepsilon \to 0} M_{\tau_\varepsilon} = M_{\tau_0} \text{ a.s.}$ and the monotone convergence theorem

$$\lim_{\varepsilon \to 0} \int_0^{\tau_\varepsilon} \frac{\sigma^2}{2K^{2(1-\gamma)}} \gamma(1 - \gamma)(x^K_s)^{-1-\gamma}ds$$  

in both sides of the above equality $\lim_{\varepsilon \to 0}$ is applicable, that is, we have

$$0 \leq (x^K_{\tau_0})^{1-\gamma}e^{-2(1-\gamma)\mu_{\tau_0}}$$  

and

$$\int_0^{\tau_\varepsilon} \frac{\sigma^2}{2K^{2(1-\gamma)}} \gamma(1 - \gamma)(x^K_s)^{-1-\gamma}ds = 1 + \frac{1}{K^{1-\gamma}}M_{\tau_0}.$$  

(2.3)

(2.3) implies $1 + \frac{1}{K^{1-\gamma}}M_{\tau_0} \geq 0$. If $\omega \in \{\tau_0 > T\}$, then $1 + \frac{1}{K^{1-\gamma}}M_T(\omega) \geq 0$. In other words, $\{\tau_0 > T\} \subset \{1 + \frac{1}{K^{1-\gamma}}M_T \geq 0\}$, and so we obtain inclusion

$$\{\tau_0 \leq T\} \supset \left\{ \frac{1}{K^{1-\gamma}}M_T + 1 < 0 \right\}.$$  \hspace{1cm} (2.4)

It is well known that the families $\{\frac{1}{K^{1-\gamma}}M_T\}_{K \to \infty}$ obeys LDP in the metric space $(\mathbb{R}, \rho)$ ($\rho$ is the Euclidian metric) with the rate speed $\frac{1}{K^{2(1-\gamma)}}$ and the
rate function \( I(v) = \frac{v^2}{2(M)_T} \). In accordance with the large deviation theory,
\[
\lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P(\tau \leq T_0) \geq -\inf_{v \geq 1} I(v) = -\frac{1}{2(M)_T}, \tag{2.5}
\]
A verification of the upper bound
\[
\lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P(\tau \leq T_0) \leq -\frac{1}{2(M)_T} \tag{2.6}
\]
is more involved. We select a set
\[
D = \{ u \in C_r(0,T) : u_0 = 1; u_t = u_{\theta(u)\wedge T}, \theta(u)=\inf\{t:u_t=0\} \leq T \}
\]
which is closed in the uniform metric \( \rho \) related to the space \( C_r(0,T) \) of continuous functions on \( [0,T] \). Obviously, \( \{\tau_0 \leq T\} \subseteq \{(x^K_t)_{t\in[0,T]} \in D\} \), which suggests to find
\[
\lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P( (x^K_t)_{t\in[0,T]} \in D). \tag{2.7}
\]
The most convenient tool to this asymptotic analysis is LDP for family \( \{(x^K_t)_{t\in[0,T]} \}_{K \to \infty} \) having the speed rate \( \frac{1}{K^{1-\gamma}} \) (!) and the rate function
\[
J_T(u) = \begin{cases} \frac{1}{2\sigma^2} \int_0^{\theta(u) \wedge T} \left( \frac{\dot{u}_t - \mu u_t}{u_t} \right)^2 dt, & \text{if } u_{\theta(u) \wedge T} = 1 \\
\infty, & \text{otherwise} \end{cases}
\]
(Theorem A.1). In accordance to the large deviation theory
\[
\lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P( (x^K_t)_{t\in[0,T]} \in D) \leq -\inf_{u \in D} J_T(u),
\]
so that, it remains to prove
\[
\inf_{u \in D} J_T(u) = \frac{1}{2(M)_T}.
\]
A minimization procedure of \( J(u) \) in \( u \in D \) exclude from consideration functions \( u_t \) with \( du_t \ll dt \) and \( \int_0^{\theta(u) \wedge T} \left[ \frac{u_t - \mu u_t}{u_t} \right] dt = \infty \). This minimization is realized with a help of a specific deterministic control problem with a control action \( w_t \) and a controlled process \( u_t \), being the solution of differential equation
\[
\dot{u}_t = \mu u_t + \sigma u_t^{-\gamma} w_t, \quad t \leq \theta(u) \wedge T \tag{2.8}
\]
subject to the initial condition \( u_0 = 1 \). Obviously, the function \( u_t \) belongs to \( D \). The pair \( (w^*_t, \theta(u^*)) \), with \( u^*(t) \) related to \( w^*_t \), is said to be optimal if
\[
\int_0^{\theta(u^*) \wedge T} (w^*_t)^2 dt \leq \int_0^{\theta(u) \wedge T} w_t^2 dt
\]
for any pair \( (w_t, u_t) \) with \( \int_0^{\theta(u) \wedge T} w_t^2 dt < \infty \). Technically, it is convenient to use the following change of variables: \( v_t = u_t^{-\gamma} \) enables us to reduce the problem to a linear differential equation
\[
\dot{v}_t = \mu(1-\gamma) v_t + \sigma(1-\gamma) w_t \tag{2.9}
\]
instead of nonlinear one (2.8), subject to the initial condition \( v_0 = 1 \). We shall exploit also the property \( v_{\theta(u)} \begin{cases} = 0, & u_{\theta(u)} = 0 \\ > 0, & u_{\theta(u)} > 0. \end{cases} \) The explicit solution of equation (2.9), under the assumption \( \theta(u) \leq T \), implies:

\[
0 = v_{\theta(u)} e^{-\mu (1-\gamma)t} = \left[ 1 + \sigma(1-\gamma) \int_{[0,\theta(u) \wedge T]} e^{-\mu (1-\gamma)s} w_s \, ds \right]
\]

or, equivalently, the equality:

\[
- \frac{1}{\sigma(1-\gamma)} = \int_{[0,\theta(u) \wedge T]} e^{-\mu (1-\gamma)s} w_s \, ds \tag{2.10}
\]

that, due to the Cauchy-Schwarz inequality, can be transformed into the inequality:

\[
\int_{[0,\theta(u) \wedge T]} w_t^2 \, dt \geq \frac{2\mu}{\sigma^2(1-\gamma) [1 - e^{-2\mu(1-\gamma)\theta(u)}]} \geq \frac{2\mu}{\sigma^2(1-\gamma) [1 - e^{-2\mu(1-\gamma)T}]}.
\]

The choice of \( w^*_t \) is conditioned by two requirements:

1) (2.10) remains valid for \( w_t \) replaced by \( w^*_t \)

2) \( \int_{[0,\theta(u) \wedge T]} (w_t^*)^2 \, dt = \frac{2\mu}{\sigma^2(1-\gamma) [1 - e^{-2\mu(1-\gamma)\theta(u)}]} \).

Both requirements are satisfied for \( w^*_t = -\frac{1}{\sigma} \frac{2\mu}{1 - e^{-2\mu(1-\gamma)\theta(u)}} e^{\mu(1-\gamma)t} \). Hence, \( \frac{1}{2} \int_0^T (w^*_t)^2 \, dt = \frac{1}{2(M)T} \).

**Remark 1.** \( u_t = \frac{(M)T}{(M)T} \).

**Remark 2.** The fact that the random variable \( M_T \) is gaussian with parameters \((0, (M)_T)\) and (2.11) yield for any \( K > 0 \), \( P(\tau_0 \leq T) \geq P(M_T \leq -K^{1-\gamma}) \).

### 3. Most likely path to ruin of the normed process \( x^K_t \)

Since \( u^*_t \equiv (v^*_t)^{1/\gamma} \), where \( v^*_t \) solves the differential equation

\[
\dot{v}^*_t = \mu(1-\gamma)v^*_t + \sigma(1-\gamma)w^*_t
\]

with \( v^*_0 = 1 \), we find that

\[
w^*_t = e^{\mu t} \left[ 1 - \frac{2\mu(M)_T}{\sigma^2(1-\gamma)} \right]^{1/(1-\gamma)} \equiv e^{\mu t} \left[ 1 - \frac{e^{-2(1-\gamma)\mu t}}{e^{-2(1-\gamma)\mu T}} \right]^{1/(1-\gamma)}.
\]

On the other hand, in accordance with Theorem A.1 for \( u^* \), we have

\[
\lim_{\delta \to 0} \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P \left( \sup_{t \in [0,T]} |x^K_t - u^*_t| \leq \delta \right) = -J_T(u^*).
\]

At the same time for any \( u \in D \), Theorem A.1 provides

\[
\lim_{\delta \to 0} \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P \left( \sup_{t \in [0,T]} |x^K_t - u_t| \leq \delta \right) = -J_T(u) \leq - \inf \{ u \in D^o \leq -J_T(u^*) \}.
\]

Consequently, the function \( u^*_t \) can be considered as the most likely path to ruin of the normed process \( x^K_t \) on time interval \([0, T]\).
APPENDIX A. LDP for the family \( \{(x^K_t)_{t \in [0,T]}\}_{K \to \infty} \)

The family \( \{(x^K_t)_{t \in [0,T]}\}_{K \to \infty} \) is in Freidlin-Wentzell’s framework \([5]\). In our setting, we take into account that the random process \( x^K_t \) is absorbed at the stopping time \( \tau_n \), so that its paths belong to a subspace \( \mathbb{C}^{abc}_{[0,T]}(\mathbb{R}^+) \) of \( \mathbb{C}[0,T](\mathbb{R}^+) \) the space of continuous nonnegative functions \( u_t = u_{t \land \theta(u)} \), where \( \theta(u) = \inf\{ t \leq T : u_t = 0 \} \). The subspace \( \mathbb{C}^{abc}_{[0,T]}(\mathbb{R}^+) \) is closed in the uniform metric \( \rho \) and, consequently, it suffices to analyze the LDP in the metric space \( \mathbb{C}^{abc}_{[0,T]}(\mathbb{R}^+, \rho) \). The use of \( \mathbb{C}^{abc}_{[0,T]}(\mathbb{R}^+, \rho) \) instead of \( \mathbb{C}[0,T](\mathbb{R}^+) \) enables us to apply standard approach to LDP proof adding a few simplest details only.

**Theorem A.1.** The family \( \{(x^K_t)_{t \geq 0}\}_{K \to \infty} \) obeys LDP in the metric space \( \mathbb{C}^{abc}_{[0,T]}(\mathbb{R}^+, \rho) \) with the speed rate \( \frac{1}{K^{2(1-\gamma)}} \) and the rate function

\[
J_T(u) = \begin{cases} 
\frac{1}{2\sigma^2} \int_0^{\theta(u) \land T} \left( \frac{\dot{u}_t - \mu u_t}{u_t^\gamma} \right)^2 dt, \\
\infty, 
\end{cases}
\]

for \( \Delta = (0, \infty) \), \( \theta(u) = \infty \) holds if \( \dot{u}_t \equiv 0 \), \( u_t \equiv 0 \).

_Proof._ The family \( \{(x^K_t)_{t \geq 0}\}_{K \to \infty} \) is exponentially tight (see, e.g., theorems 1.3 and 3.1, Liptser and Puhalskii, \([8]\)), that is,

\[
\lim_{C \to \infty} \lim_{K \to 0} \frac{1}{K^{2(1-\gamma)}} \log P \left( \sup_{t \in [0,T]} x^K_t \geq C \right) = -\infty, \tag{A.1}
\]

\[
\lim_{\Delta \to 0} \lim_{K \to 0} \sup_{\theta(u) \leq T} \frac{1}{K^{2(1-\gamma)}} \log P \left( \sup_{t \in [0,\Delta]} |x^K_{\theta + t} - x^K_\theta| \geq \eta \right) = -\infty, \tag{A.2}
\]

where \( \eta \) is arbitrary number and \( \theta \) is stopping time relative to a corresponding filtration. In \( A.1 \), without loss generality \( x^K_t \) might be replaced by \( (x^K_t)^{1-\gamma} \) which makes possible, in accordance with \( 2.2 \), to use the inequality \( (x^K_t)^{1-\gamma} \leq e^{(1-\gamma)\mu t} \left[ 1 + \int_0^t e^{(1-\gamma)\mu s} (1 - \gamma)\sigma ds \right] \), making the proof transparent. Due to \( A.1 \), the condition from \( A.2 \) can be replaced by an easy provable condition (here \( \mathfrak{A}_C = \{ \sup_{t \leq T} : x^K_t \leq C \} \)):

\[
\lim_{\Delta \to 0} \lim_{K \to 0} \sup_{\theta(u) \leq T} \frac{1}{K^{2(1-\gamma)}} \log P \left( \sup_{[0,\Delta]} |x^K_{\theta + t} - x^K_\theta| \geq \eta, \mathfrak{A}_C \right) = -\infty, \forall C > 0.
\]

For \( \theta(u) > T \), the proof of local LDP:

\[
\lim_{\delta \to 0} \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log P \left( \sup_{t \in [0,T]} x^K_t - u_t \leq \delta \right) = -J_T(u)
\]

does not differ from standard one and is omitted. The case of \( J_T(u) = -\infty \), including \( u_0 \neq 1 \), \( du_t \ll dt \), is analyzed in a standard way and is omitted too.

The analysis of “\( u_0 = 1 \), \( du_t = \dot{u}_t dt \), \( \int_0^{\theta(u) \land T} \left( \frac{\dot{u}_s - \mu u_s}{u_s^\gamma} \right)^2 ds < \infty, \theta(u) \leq T \)” is based on the following result.

**Proposition A.1.** [Dupuis, Ellis \([4]\), A.6.3.1] For any absolutely continuous function \( u = (u_t)_{t \in [0,T]} \), mapping \([0,T]\) into \( \mathbb{R} \), and any \( \alpha \in \mathbb{R} \)

\[
\int_0^T I_{\{u_t = \alpha, \alpha \neq 0\}} dt = 0.
\]
Local LDP upper bound. Set \( u^n_t = \frac{1}{n} \vee u_t \) and notice that \( \theta(u^n) > T \). Moreover, \( u^n_0 = 1 \), \( du^n_t = \dot{u}^n_t dt \) and, due to Proposition A.1, \( \dot{u}^n_t = \dot{u}_t \mathbf{1}_{\{u_t > \frac{1}{n}\}} ds \) and also \( \int_0^{\theta(u^n) \wedge T} \left( \frac{u^n_t - \mu t^n}{u^n_t} \right)^2 ds < \infty \).

Since \( \tau^n = \inf \{ t : u_t \leq \frac{1}{n} \} \to \theta(u) \), \( n \to \infty \), we find that

\[
\lim_{\delta \to 0} \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log \mathbb{P} \left( \sup_{t \in [0,T]} |x^K_t - u_t| \leq \delta \right) \\
\leq \lim_{\delta \to 0} \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log \mathbb{P} \left( \sup_{t \in [0,\tau^n \wedge T]} |x^K_t - u_t| \leq \delta \right) \\
\leq -\frac{1}{2\sigma^2} \int_0^{\tau^n \wedge T} \left( \frac{\dot{u}^n_t - \mu u^n_t}{u^n_t} \right)^2 dt = -\frac{1}{2\sigma^2} \int_0^{\tau^n \wedge T} \left( \frac{\dot{u}_t - \mu u_t}{u_t} \right)^2 dt.
\]

Local LDP lower bound. With \( \phi > \delta > 0 \), write

\[
\left\{ \sup_{t \in [0,T]} |x^K_t - u^n_t| \leq \delta \right\} \\
= \left\{ \sup_{t \in [0,T]} |x^K_t - u^n_t| \leq \delta \right\} \cap \left\{ \sup_{t \in [0,T]} |u^n_t - u_t| \leq \phi \right\} \\
\cup \left\{ \sup_{t \in [0,T]} |x^K_t - u^n_t| \leq \delta \right\} \cap \left\{ \sup_{t \in [0,T]} |u^n_t - u_t| > \phi \right\} \\
\cup \left\{ \sup_{t \in [0,T]} |x^K_t - u_t| \leq \phi + \delta \right\} \cap \left\{ \sup_{t \in [0,T]} |u^n_t - u_t| > \phi \right\} \\
\cup \left\{ \sup_{t \in [0,T]} |x^K_t - u_t| \leq 2\delta \right\} \cup \left\{ \sup_{t \in [0,T]} |u^n_t - u_t| > \phi \right\}.
\]

For fixed \( \phi \), there exists a number \( n_\phi > \frac{1}{\phi} \) such that for any \( n \geq n_\phi \) the set \( \left\{ \sup_{t \in [0,T]} |u^n_t - u_t| > \phi \right\} = \emptyset \). Therefore, for sufficiently large numbers \( n \),

\[
\mathbb{P} \left( \sup_{t \in [0,T]} |x^K_t - u_t| \leq 2\phi \right) \geq \mathbb{P} \left( \sup_{t \in [0,T]} |x^K_t - u^n_t| \leq \delta \right).
\]

Hence and by Proposition A.1, a chain of lower bounds holds,

\[
\lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log \mathbb{P} \left( \sup_{t \in [0,T]} |x^K_t - u_t| \leq 2\phi \right) \\
\geq \lim_{\delta \to 0} \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log \mathbb{P} \left( \sup_{t \in [0,T]} |x^K_t - u^n_t| \leq \delta \right) \\
\geq -\frac{1}{2\sigma^2} \int_0^{\tau^n \wedge T} \left( \frac{\dot{u}_s - \mu u_s}{u_s} \right)^2 ds - \frac{1}{2\sigma^2} \int_0^{\tau^n \wedge T} \frac{\mu^2}{n^{2(1-\gamma)}} \\
\rightarrow_{n \to \infty} -\frac{1}{2\sigma^2} \int_0^{\theta(u) \wedge T} \left( \frac{\dot{u}_s - \mu u_s}{u_s} \right)^2 ds,
\]
providing
\[
\lim_{\phi \to 0} \lim_{K \to \infty} \frac{1}{K^{2(1-\gamma)}} \log \mathbb{P}\left( \sup_{t \in [0,T]} |x^K_t - u_t| \leq 2\phi \right) \leq -\frac{1}{2\sigma^2} \int_0^{\theta(u) \wedge T} \left( \frac{\dot{u}_s - \mu u_s}{u_s^2} \right)^2 ds.
\]
\[
\square
\]

References

[1] Cox, J. C. The Constant Elasticity of Variance Option. Pricing Model. *The Journal of Portfolio Management*. 23 (1997), no. 2, 15–17.

[2] Delbaen, F. and Shirakawa, H. A Note of Option Pricing for Constant Elasticity of Variance Model. available at: www.math.ethz.ch

[3] Donati-Martin, C.; Rouault, A.; Yor, M.; Zani, M. Large deviations for squares of Bessel and Ornstein-Uhlenbeck processes. *Probab. Theory Related Fields*. 129 (2004), no. 2, 261–289.

[4] Dupuis Paul; Ellis Richard S. A weak convergence approach to the theory of large deviations. Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1997.

[5] Freidlin, M. I.; Wentzell, A. D. Random perturbations of dynamical systems. Translated from the Russian by Joseph Szücs. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 260. Springer-Verlag, New York, 1984.

[6] Klebaner, Fima C. Introduction to stochastic calculus with applications. Second edition. *Imperial College Press*, London, 2005.

[7] Klebaner, F. C.; Liptser, R. Likely path to extinction in simple branching models with large initial population. *J. Appl. Math. Stoch. Anal.* (2006), Art. ID 60376.

[8] Liptser, Robert Sh.; Pukhalskii, Anatoliĭ A. Limit theorems on large deviations for semimartingales. *Stochastics Stochastics Rep.* 38 (1992), no. 4, 201–249.

[9] Lu, R.; Hsu, Yi-Hwa. Valuation of Standard Options under the Constant Elasticity of Variance Model. *International Journal of Business and Economics*. 4 (2005), no. 2, 157-165.

[10] Rogers, L. C. G.; Williams, David. Diffusions, Markov processes, and martingales. Vol. 2. Itô calculus. Reprint of the second (1994) edition. Cambridge Mathematical Library. *Cambridge University Press*, Cambridge. 2000.

[11] Rouault, Alain. Large deviations and branching processes. Proceedings of the 9th International Summer School on Probability Theory and Mathematical Statistics (Sofopol, 1997). *Pliska Stud. Math. Bulgar.*, 13 (2000), 15–38.

School of Mathematical Sciences,, Building 28M, Monash University,, Clayton Campus, Victoria 3800,, Australia

E-mail address: fima.klebaner@sci.monash.edu.au

Department of Electrical Engineering Systems, Tel Aviv University, 69978 Tel Aviv, Israel

E-mail address: liptser@eng.tau.ac.il; rliptser@gmail.com