GLOBAL STRUCTURE OF STEADY-STATES TO THE FULL CROSS-DIFFUSION LIMIT IN THE SHIGESADA-KAWASAKI-TERAMOTO MODEL

KOUSUKE KUTO†

Abstract. In a previous paper [8], the author studied the asymptotic behavior of coexistence steady-states to the Shigesada-Kawasaki-Teramoto model as both cross-diffusion coefficients tend to infinity at the same rate. As a result, he proved that the asymptotic behavior can be characterized by a limiting system that consists of a semilinear elliptic equation and an integral constraint. This paper studies the set of solutions of the limiting system. The first main result gives sufficient conditions for the existence/nonexistence of nonconstant solutions to the limiting system by a topological approach using the Leray-Schauder degree. The second main result exhibits a bifurcation diagram of nonconstant solutions to the one-dimensional limiting system by analysis of a weighted time-map and a nonlocal constraint.

1. Introduction

In this paper, we are concerned with the following Neumann problem of quasilinear elliptic equations:

\[
\begin{aligned}
\Delta [(d_1 + \alpha v)u] + f(u, v) &= 0 \quad \text{in } \Omega, \\
\Delta [(d_2 + \beta u)v] + g(u, v) &= 0 \quad \text{in } \Omega, \\
u \geq 0, \quad v \geq 0 &\quad \text{in } \Omega, \\
\partial_{\nu} u = \partial_{\nu} v &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]  

(1.1)

where

\[
f(u, v) := u(a_1 - b_1 u - c_1 v) \quad \text{and} \quad g(u, v) := v(a_2 - b_2 u - c_2 v).
\]  

(1.2)

The system (1.1) is derived from a diffusive Lotka-Volterra competition model, where the unknown functions \(u(x)\) and \(v(x)\) represent the stationary population densities of the competing species in the habitat \(\Omega\). Throughout this paper, \(\Omega\) is assumed to be a bounded domain in \(\mathbb{R}^N\) with a smooth boundary \(\partial \Omega\) if \(N \geq 2\); an interval if \(N = 1\). In (1.1), \(\Delta := \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}\) denotes the Laplacian; \(\nu(x)\) denotes the outward unit normal vector at \(x \in \partial \Omega\), and \(\partial_{\nu} u = \nu(x) \cdot \nabla u\) is the out-flux of \(u\). In the reaction terms \(f(u, v)\) and \(g(u, v)\), coefficients \(a_i\), \(b_i\) and \(c_i\) \((i = 1, 2)\) are positive constants; \(a_i\) denote the birth rates of the respective species, \(b_1\) and \(c_2\) denote the intra-specific competition coefficients; \(c_1\) and \(b_2\) denote the inter-specific competition coefficients. In the diffusion term pertaining to the Laplacian, \(d_i\) \((i = 1, 2)\) are positive constants; \(\alpha\) and \(\beta\) are nonnegative constants, \(d_1 \Delta u\) and \(d_2 \Delta v\) are linear diffusion terms describing a spatially random
movement of each species. Whereas $\Delta(uv)$ is a nonlinear diffusion term describing an interaction of diffusion caused by the population pressure resulting from interference between different species. The interaction term $\Delta(uv)$ is often called cross-diffusion (see a book by Okubo and Levin [20] for modelings of the biological diffusion). Such a Lotka-Volterra competition system with cross-diffusion (and some additional terms) was proposed by Shigesada, Kawasaki and Teramoto [23]. Beyond their bio-mathematical aim to realize segregation phenomena of two competing species observed in ecosystems, a lot of pure mathematicians have studied a class of Lotka-Volterra systems with cross-diffusion as a prototype of diffusive interactions. Such a class of Lotka-Volterra systems with cross-diffusion is referred to as the SKT model celebrating the authors of the pioneering paper [23]. We refer to book chapters by Jüngel [4], Ni [16], and Yamada [27, 28] as surveys for mathematical works relating to the SKT model.

Despite the long history of research on the SKT model, there are few papers on the global structure of the set of positive nonconstant solutions (such as $u > 0$ and $v > 0$ in $\Omega$) to (1.1) in case where $\alpha$ and $\beta$ are large. The purpose of this paper is to study the asymptotic behavior of positive nonconstant solutions of (1.1) in the full cross-diffusion limit as $\alpha, \beta \to \infty$ and $\alpha/\beta \to \gamma$ with some $\gamma > 0$.

In the unilateral cross-diffusion limit as $\alpha \to \infty$ with $\beta \geq 0$ fixed, Lou and Ni [12] established the $L^\infty(\Omega)$ a priori bound, which is independent of $\alpha$, for all solutions of (1.1) in case $N \leq 3$. Furthermore, they proved that the asymptotic behavior of solutions as $\alpha \to \infty$ (with fixed $\beta \geq 0$) can be characterized by a solution to either a limiting system of the first kind or a limiting system of the second kind. We refer to [6, 13, 14, 15, 24, 25, 26] and [7, 10] as papers on the limiting systems of the first and second kinds, respectively.

In the full cross-diffusion limit as $\alpha, \beta \to \infty$ and $\alpha/\beta \to \gamma > 0$, the author obtained the following $L^\infty(\Omega)$ a priori bound for all solutions of (1.1) without any restriction on $N$:

**Theorem 1.1** ([8]). For any small $\eta > 0$, there exists a positive constant $C = C(\eta, d_1, a_i, b_i, c_i)$ such that if $\alpha > 0$ and $\beta > 0$ satisfy $\eta \leq \alpha/\beta \leq 1/\eta$, then any solution $(u, v)$ of (1.1) satisfies

$$\max_{x \in \Omega} u(x) \leq C \quad \text{and} \quad \max_{x \in \Omega} v(x) \leq C.$$

By a combination of Theorem 1.1 and the elliptic regularity theory, the author obtained the following limiting systems which can characterize the asymptotic behavior of solutions of (1.1) in the full cross-diffusion limit. This result gives a rigorous justification of a formal observation by Kan-on [5] on the existence of the limiting systems.

**Theorem 1.2** ([8]). Suppose that $a_1/a_2 \neq b_1/b_2$ and $a_1/a_2 \neq c_1/c_2$. Let $\{u_n, v_n\}$ be any sequence of positive nonconstant solutions of (1.1) with $\alpha = \alpha_n \to \infty$, $\beta = \beta_n \to \infty$ and $\gamma_n := \alpha_n/\beta_n \to \gamma > 0$ as $n \to \infty$. Then either of the following two situations occurs, passing to a subsequence if necessary:

(i) there exist a positive function $u \in C^2(\overline{\Omega})$ and a positive number $\tau$ such that

$$\lim_{n \to \infty} (u_n, v_n) = \left( u, \frac{\tau}{u} \right) \quad \text{in} \quad C^1(\overline{\Omega}) \times C^1(\overline{\Omega}),$$

and

$$w := \delta u - \gamma \frac{\tau}{u} \quad \text{with} \quad \delta := \frac{d_1}{d_2} \quad \text{and} \quad d := \frac{d_2}{(1.3)}$$

satisfies a limiting system which consists of the semilinear elliptic equation

$$d \Delta w + f\left( u, \frac{\tau}{u} \right) - \gamma g\left( u, \frac{\tau}{u} \right) = 0 \quad \text{in} \quad \Omega,$$
subject the homogeneous Neumann boundary condition
\[ \partial_{\nu}w = 0 \quad \text{on } \partial \Omega \quad (1.4b) \]
and the integral constraint
\[ \int_{\Omega} f \left( \frac{u}{u}, \frac{\tau}{u} \right) = 0; \quad (1.4c) \]
(ii) there exist nonnegative functions \( u, v \in C(\overline{\Omega}) \) such that \( uv = 0 \) in \( \Omega \),
\[ \lim_{n \to \infty} (u_n, v_n) = (u, v) \quad \text{uniformly in } \Omega \]
and \( w_n := \delta u_n - \gamma \tau u_n^{-1} \) satisfies \( \lim_{n \to \infty} w_n = w \) in \( C^1(\overline{\Omega}) \) with some sign-changing function \( w \) satisfying
\[ \begin{aligned}
    d \Delta w + f \left( \frac{w_+}{\delta}, \frac{w_-}{\gamma} \right) - \gamma g \left( \frac{w_+}{\delta}, \frac{w_-}{\gamma} \right) &= 0 \quad \text{in } \Omega, \\
    \partial_{\nu}w &= 0 \quad \text{on } \partial \Omega, \\
    \int_{\Omega} f \left( \frac{w_+}{\delta}, \frac{w_-}{\gamma} \right) &= 0
\end{aligned} \quad (1.5) \]
and
\[ (u, v) = \left( \frac{w_+}{\delta}, \frac{w_-}{\gamma} \right), \]
where \( w_+ := \max\{w, 0\} \) and \( w_- := -\min\{w, 0\} \geq 0 \).

By Theorem 1.2, we know that a segregation of competing species can take place in the
sense that \( u_n(x)v_n(x) \) converges to a nonnegative constant \( \tau \) in the full cross-diffusion limit as \( \alpha_n, \beta_n \to \infty \) and \( \gamma_n = \alpha_n/\beta_n \to \gamma > 0 \). The second situation (ii) can be interpreted as the complete segregation in which territories of two competing species completely segregate each other because \( \tau = 0 \), while the first situation (i) can be interpreted as the incomplete segregation in which territories of two competing species do not completely segregate because \( \tau > 0 \). In [8], the author showed local bifurcation curves of nonconstant solutions of the limiting system (1.4) of the incomplete segregation. On the other hand, it was shown in [8] that the complete segregation (ii) cannot occur in the one-dimensional case.

This paper first show that the complete segregation (ii) cannot occur also in the higher dimensional case. Then it becomes important to derive information on the set of nonconstant solutions of the limiting system (1.4) in order to know the segregation mechanism of two competing species when cross-diffusion coefficients \( \alpha \) and \( \beta \) are sufficiently large. This paper will give some sufficient conditions for the existence/nonexistence of nonconstant solutions of the limiting system (1.4). In what follows, we usually regard \( (w, \tau) \) as a pair of unknowns with positive parameter \( d \). It should be noted that varying \( d \) corresponds to varying \( d_1 \) and \( d_2 \) keeping \( d_1/d_2 = \delta \) (see (1.3)), and moreover, \( (u, \tau) \) also can be regarded as a pair of unknowns by the relation in (1.3);
\[ w = \delta u - \gamma \tau u, \quad \text{that is,} \quad u = \frac{\sqrt{w^2 + 4\gamma \delta \tau + w}}{2\delta}. \]
Our strategy of the proof is as follows: First we use the Poincaré inequality and the maximum principle to show that the Neumann problem (1.4a)-(1.4b) does not admit any nonconstant solution if \( d > 0 \) or \( \tau > 0 \) is sufficiently large. Then, in the weak competition case \( c_1/c_2 < a_1/a_2 < b_1/b_2 \) or the strong competition case \( b_1/b_2 < a_1/a_2 < c_1/c_2 \), if \( d > 0 \) is large enough,
then (1.4a)-(1.4b) admits a unique positive solution \((u, \tau) = (u^*, u^*v^*)\), such as \(u > 0\) in \(\Omega\) and \(\tau > 0\), where \((u^*, v^*)\) corresponds to the positive constant solution

\[
(u^*, v^*) := \frac{1}{b_1c_2 - b_2c_1}(a_1c_2 - a_2c_1, b_1a_2 - b_2a_1)
\]

of (1.1). The index of an associated operator around \((w^*, \tau^*) := (\delta u^* - \gamma v^*, u^*v^*)\) will be calculated, and especially for the case where \(D(a_i, b_i, c_i, \gamma) > 0\) (see (3.5) for the definition of \(D(a_i, b_i, c_i, \gamma)\)), the homotopy invariance property of the Leray-Schauder degree will enables us to get sufficient intervals of \(d\) for the existence of nonconstant solutions of (1.4).

In particular, for the one-dimensional case, the global bifurcation structure of nonconstant solutions of (1.4) will be shown (see Figure 1). The procedure of the proof is as follows: First, the bifurcation structure of solutions to the Neumann problem (1.4a)-(1.4b) is obtained through the analysis of the time-map of the related initial value problem. We note that the time map contains a weighted singular integral and differs from the usual one. Next, we construct a bifurcation branch of the set of solutions to the limit system (1.4) by selecting functions in the set of solutions to (1.4a)-(1.4b) that satisfy the integral constraint (1.4c). For the selection, we adopt a topological method combining the singular perturbation of solutions of (1.4a)-(1.4b) and the global bifurcation theory. As a result, in the weak or strong competition case with \(D(a_i, b_i, c_i, \gamma) > 0\), for each \(j \in \mathbb{N}\), we shall construct a branch of solutions of (1.4) in which derivatives \(u'\) change the sign exactly \(j - 1\) times bifurcating from a pitchfork bifurcation point on the branch of the constant solution \(\{(d, u^*), \tau^*) : d > 0\}\) at some \(d = d(j) > 0\), and moreover, reaching a singular limit \((d, \tau) = (0, \tau_0)\) with some \(\tau_0 \geq 0\). Furthermore, it will be shown that \(\tau_0 > 0\) in the weak competition case, while \(\tau_0 = 0\) in the strong competition case.

The contents of this paper is as follows: In Section 2, the nonexistence of nontrivial solution of (1.5) will be proved. In Section 3, main results on the set of nonconstant solution of the limiting system (1.4) will be presented. In Section 4, we derive some a priori estimates of solutions to (1.4). In Section 5, we show sufficient conditions on the existence of nonconstant solutions of (1.4) in the multi-dimensional case. In Section 6, we construct the global bifurcation curves of nonconstant solutions of (1.4) in the one-dimensional case.

Throughout this paper, the usual norms of the functional spaces \(L^p(\Omega)\) for \(p \in [1, \infty)\) and \(L^\infty(\Omega)\) are denoted by

\[
\|u\|_p := \left(\int_{\Omega} |u(x)|^p \right)^{1/p} \quad \text{and} \quad \|u\|_\infty := \text{ess. sup}_{x \in \Omega} |u(x)|.
\]

Hence \(\|u\|_\infty = \max_{x \in \Omega} |u(x)|\) in a case when \(u \in C(\overline{\Omega})\). Furthermore, we denote by \(\{\Phi_j\}_{j=0}^\infty\) a complete orthonormal base in \(L^2(\Omega)\) consisting of eigenfunctions of \(-\Delta\) with the homogeneous Neumann boundary condition on \(\partial \Omega\), namely,

\[
\left\{
\begin{array}{ll}
-\Delta \Phi_j = \lambda_j \Phi_j & \text{in } \Omega, \\
\|\Phi_j\|_2 = 1, \\
\partial_n \Phi_j = 0 & \text{on } \partial \Omega,
\end{array}
\right.
\]

where

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \leq \cdots
\]

represent all eigenvalues counting multiplicity.
2. Nonexistence of nonconstant solutions for the complete segregation

Throughout this paper, the following maximum principle for elliptic equations will play an important role in the proofs.

Lemma 2.1 (e.g., [12]). Suppose that $h \in C(\overline{\Omega} \times \mathbb{R})$. Then the following properties (i) and (ii) hold true:

(i) If $U \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies
$$\Delta U + h(x, U) \geq 0 \text{ in } \Omega, \quad \partial_\nu U \leq 0 \text{ on } \partial \Omega,$$
and $U(x_0) = \max_{x \in \Omega} U(x)$, then $h(x_0, U(x_0)) \geq 0$.

(ii) If $U \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfies
$$\Delta U + h(x, U) \leq 0 \text{ in } \Omega, \quad \partial_\nu U \geq 0 \text{ on } \partial \Omega,$$
and $\overline{U}(x_0) = \min_{x \in \Omega} \overline{U}(x)$, then $h(x_0, \overline{U}(x_0)) \leq 0$.

In this section, we show that the second situation (ii) of Theorem 1.2 never occurs in the full cross-diffusion limit.

Proposition 2.2. All solutions of (1.5) consist of three constant solutions: $w = \delta a_1 / b_1; w = 0$; $w = -\gamma a_2 / c_2$.

Proof. Let $w$ be any weak solution of (1.5). Then the elliptic regularity theory (e.g., [3]) ensures that $w \in W^{2,p}(\Omega)$ for any $p \in (1, \infty)$. Furthermore, the Sobolev embedding theorem leads to $w \in C^{1,\theta}(\overline{\Omega})$ for any $\theta \in (0, 1)$.

Suppose for contradiction that there exists a nonconstant solution $w$ of (1.5). Substituting (1.2) into (1.5), one can see that $w$ satisfies

$$\begin{aligned}
\left\{ d\Delta w + \frac{w_+}{\delta} \left( a_1 - \frac{b_1 w_+}{\delta} \right) - w_- \left( a_2 - \frac{c_2 w_-}{\gamma} \right) = 0 \text{ in } \Omega, \\
\partial_\nu w = 0 \quad \text{on } \partial \Omega
\right. 
\end{aligned}
$$

and the integral constraint

$$\int_\Omega w_+ \left( a_1 - \frac{b_1 w_+}{\delta} \right) = 0 \quad (2.2)$$

because $w_+ w_- = 0$ in $\Omega$. Suppose that $\Omega_+ := \{ x \in \Omega : w(x) > 0 \}$ is not empty. Let $x^* \in \Omega_+$ be a maximum point of $w$, namely, $w(x^*) = \max_{x \in \Omega} w(x)$. We observe that $w$ is of class $C^2$ in $\Omega_+$ and $w = 0$ on $\Omega \cap \partial \Omega_+$. Then applying (i) of Lemma 2.1 to (2.2), one can see

$$0 \leq \frac{w_+(x^*)}{\delta} \left( a_1 - \frac{b_1 w_+(x^*)}{\delta} \right) - w_-(x^*) \left( a_2 - \frac{c_2 w_-(x^*)}{\gamma} \right) = \frac{w_+(x^*)}{\delta} \left( a_1 - \frac{b_1 w_+(x^*)}{\delta} \right).$$

Then we know that $0 < w_+(x^*) \leq \delta a_1 / b_1$, and thereby, $0 < w_+(x) \leq \delta a_1 / b_1$ for any $x \in \Omega_+$. Hence it follows that

$$\frac{w_+(x)}{\delta} \left( a_1 - \frac{b_1 w_+(x^*)}{\delta} \right) \geq 0 \quad \text{for any } x \in \overline{\Omega}_+.$$

Since $w$ is not identically equal to zero or $\delta a_1 / b_1$ by our assumption, we see that

$$\int_\Omega w_+ \left( a_1 - \frac{b_1 w_+}{\delta} \right) = \int_{\Omega_+} w_+ \left( a_1 - \frac{b_1 w_+}{\delta} \right) > 0.$$

However, this contradicts (2.2). Now we can say that if $w$ is a nonconstant solution of (1.5), then $\Omega_+$ is empty.
Suppose that $\Omega_- := \{ x \in \Omega \mid w(x) < 0 \}$ is not empty. By the boundary condition and (2.2), we integrate the elliptic equation of (2.1) over $\Omega$ to obtain
\[
\int_{\Omega} w_\gamma \left( a_2 - \frac{c_2 w_-}{\gamma} \right) = 0. \tag{2.3}
\]
Let $x_* \in \Omega_-$ be a minimum point of $w$, namely, $w(x_*) = \min_{x \in \Omega} w(x)$. By a similar application of (ii) of Lemma 2.1 to (2.2), we see that
\[
w_\gamma(x_*) \left( a_2 - \frac{c_2 w_-(x_*)}{\gamma} \right) \geq 0,
\]
and then, $0 < w_\gamma(x) \leq \gamma a_2/c_2$ for any $x \in \Omega_-$. Since $w$ does not identically equal to zero or $-\gamma a_2/c_2$ by our assumption, then
\[
\int_{\Omega} w_\gamma \left( a_2 - \frac{c_2 w_-}{\gamma} \right) = \int_{\Omega_-} w_\gamma \left( a_2 - \frac{c_2 w_-}{\gamma} \right) > 0.
\]
This contradicts (2.3). Consequently, the proof by contradiction enables us to conclude that (1.5) possesses no nonconstant solution. We complete the proof of Proposition 2.2.

Proposition 2.2 implies that all solutions of (1.5) are corresponding to constant solutions of (1.1) such that $uv = 0$; $(u, v) = (0, 0)$, $(a_1/b_1, 0)$ and $(0, a_2/c_2)$. We recall that the situation (ii) of Theorem 1.2 asserts that $w$ is a sign-changing solution of (1.5). Then, together with Proposition 2.2, we obtain the following result:

**Corollary 2.3.** The situation (ii) of Theorem 1.2 cannot occur.

3. **Main results**

In this section, we state main results, which are concerned with the set of nonconstant solutions of the limiting system (1.4). For a simple expression of (1.4), we set
\[
u = u(w, \tau) := \sqrt{w^2 + 4\gamma \delta \tau + w}, \quad v = v(w, \tau) := \frac{\tau}{u} = \frac{\sqrt{w^2 + 4\gamma \delta \tau - w}}{2\gamma}. \tag{3.1}
\]
It is noted that (3.1) gives an injection from $(w, \tau) \in \mathbb{R} \times \mathbb{R}_+$ to $(u, v) \in \mathbb{R}_+ \times \mathbb{R}_+$, where $\mathbb{R}_+ := (0, \infty)$, and the inverse of $(u(w, \tau), v(w, \tau))$ is given by
\[
w(u, v) = \delta u - \gamma v, \quad \tau(u, v) = uv.
\]

Then the limiting system (1.4) is expressed as
\[
d \Delta w + f(u, v) - \gamma g(u, v) = 0 \quad \text{in } \Omega, \quad \tau > 0, \tag{3.2a}
\]
subject to the homogeneous Neumann boundary condition
\[
\partial_{\nu} w = 0 \quad \text{on } \partial \Omega \tag{3.2b}
\]
with the integral constraint
\[
\int_{\Omega} f(u, v) = 0. \tag{3.2c}
\]
Obviously, any solution $(w, \tau)$ of (3.2) also satisfies
\[
\int_{\Omega} g(u, v) = 0, \tag{3.3}
\]
which comes from the integration of (3.2a) over $\Omega$ using (3.2b) and (3.2c).
Here we note the set of constant solutions of (3.2). Hereafter \((A, B, C)\) will be denoted by

\[
A := \frac{a_1}{a_2}, \quad B := \frac{b_1}{b_2}, \quad C := \frac{c_1}{c_2}.
\]

In the weak competition case \(C < A < B\) or the strong competition case \(B < A < C\), (1.1) admits a unique positive constant solution \((u^*, v^*)\) represented as (1.6). Hence (3.1) induces that the limiting system (3.2) admits a unique constant solution satisfying \(\tau > 0\);

\[
(w^*, \tau^*) = (\delta u^* - \gamma v^*, u^* v^*)
\]

if \(C < A < B\) or \(B < A < C\). Our interest is the set of nonconstant solutions of (3.2). Hereafter \((w, \tau)\) a nonconstant solution of (3.2) when \(w \in C^2(\Omega)\) and \(\tau\) satisfy (3.2) and \(w(x)\) is a nonconstant function.

The first result is concerned with a priori estimate of all solutions of (3.2). In addition, the result asserts that a large region of \(d\) wipes out any nonconstant solution.

**Theorem 3.1.** There exists a positive constant \(C^* = C^*(a_i, b_i, c_i, \gamma, \delta)\) such that any solution \((w, \tau)\) of (3.2) satisfies

\[
\|w\|_\infty \leq C^* \quad \text{and} \quad \tau \leq \min\left\{ \frac{a_1^2}{4b_1 c_1}, \frac{a_2^2}{4b_2 c_2} \right\} (=: \overline{\tau}).
\]

Furthermore, there exists \(\overline{d} = \overline{d}(a_i, b_i, c_i, \gamma, \delta) > 0\) such that (3.2) does not admit any nonconstant solution if \(d > \overline{d}\).

The next result gives sufficient intervals of \(d\) for the existence of nonconstant solutions of (3.2) in the weak or strong competition case with a couple of additional conditions. To express an essential condition for the existence of nonconstant solutions, we introduce the following function:

\[
D(a_i, b_i, c_i, \gamma) := \gamma b_2(A - B)(B + C - 2A) + c_2(C - A)\{ A(B + C) - 2BC \}. \tag{3.5}
\]

**Theorem 3.2.** Assume the weak competition \(C < A < B\) or the strong competition \(B < A < C\). Suppose further that

\[
D(a_i, b_i, c_i, \gamma) > 0 \quad \text{and} \quad \frac{u^*}{v^*} \neq \frac{\gamma}{\delta}. \tag{3.6}
\]

Then there exists a sequence \(\{d^{(j)}\}\) with

\[
0 \leftarrow \cdots \leq d^{(j+1)} \leq d^{(j)} \leq \cdots \leq d^{(2)} \leq d^{(1)} \leq \overline{d}
\]

such that (3.2) admits at least one nonconstant solution in the following case (i) or (ii):

(i) \(C < A < B\) and \(d \in (d^{(j+1)}, d^{(j)})\) and \(j\) is odd;

(ii) \(B < A < C\) and \(d \in (d^{(j+1)}, d^{(j)})\) and \(j\) is even.

In particular, for the one-dimensional case \(\Omega = (0, 1)\), we show more detailed information on the set of nonconstant solutions of (3.2). To state the global bifurcation structure of nonconstant solutions of (3.2) with \(\Omega = (0, 1)\), we set

\[
S^+_j := \left\{ (d, u, \tau) \in \mathbb{R}_+ \times X : (d, u, \tau) \text{ satisfies (3.2) with } \Omega = (0, 1) \text{ and } (-1)^{i-1}u'(x) > 0 \text{ for } x \in ((i-1)/j, i/j) \text{ (}i = 1, 2, \ldots, j) \right\}
\]

and

\[
S^-_j := \left\{ (d, u, \tau) \in \mathbb{R}_+ \times X : (d, u, \tau) \text{ satisfies (3.2) with } \Omega = (0, 1) \text{ and } (-1)^{i-1}u'(x) < 0 \text{ for } x \in ((i-1)/j, i/j) \text{ (}i = 1, 2, \ldots, j) \right\}
\]
for each $j \in \mathbb{N}$, where $X := C^1(\overline{\Omega}) \times \mathbb{R}$. Our aim is to construct the global branch of nonconstant solutions, contained in $S^\pm_j$, that bifurcates from the positive constant solution $(u^*, \tau^*)$ at $d = d^{(j)}$ and reaches a singular limit as $d \searrow 0$ in the weak or strong competition case with $D(a_i, b_i, c_i, \gamma) > 0$. See also Figure 1.

**Theorem 3.3.** Assume the weak competition $C < A < B$ or the strong competition $B < A < C$. Assume further that $D(a_i, b_i, c_i, \gamma) > 0$. Then for each $j \in \mathbb{N}$, there exists a pair of connected sets $\Gamma^+_j \subset S^+_j$ and $\Gamma^-_j \subset S^-_j$ with the following properties:

(i) $\Gamma^+_j$ bifurcates from the positive constant solution branch $\{(d, u^*, \tau^*) : d > 0\}$ at $d = d^{(j)}$;
(ii) $\Gamma^+_j$ reaches a singular limit $(d, \tau) = (0, \tau_0)$ with some $\tau_0 \geq 0$. Furthermore, if $C < A < B$, then $\tau_0 > 0$, whereas if $B < A < C$, then $\tau_0 = 0$;
(iii) $\Gamma^-_j = \{(d, u(\bullet + 1/j), \tau) : (d, u, \tau) \in \Gamma^+_j\}$, where $u(\bullet + 1/j)$ is regarded as a periodic extension.
Here we should refer to a recent numerical result by Breden, Kuehn and Soresina [1], which numerically exhibits the bifurcation diagram of solutions of (1.1) with

\[(\alpha, \beta, a_1, a_2, b_1, b_2, c_1, c_2) = \left(100, 100, \frac{15}{2}, \frac{16}{7}, 4, 1, 6, 2\right)\]

regarding \(d = d_1 = d_2\) as a bifurcation parameter. It is easy to check that the above setting belongs to the weak competition and satisfies \(D(a_i, b_i, c_i, 1) = 17/64 > 0\). Comparing the bifurcation diagram [1, Figure 11] with Theorem 3.3, it can be seen that the set of solutions of the limit system (3.2) gives a good approximation of that of (1.1) when both cross-diffusion coefficients \(\alpha\) and \(\beta\) are sufficiently large.

4. A priori estimate

This section is devoted to the proof of Theorem 3.1. We begin with a priori estimate for any solution of (3.2):

**Lemma 4.1.** There exists a positive constant \(C^* = C^*(a_i, b_i, c_i, \gamma, \delta)\) such that any solution \((w, \tau)\) of (3.2) satisfies \(\|w\|_\infty \leq C^*\) and \(\tau \leq \tau\).

**Proof.** By (1.2), the integral constraint (3.2c) is equivalent to

\[\int_\Omega \left\{ u(a_1 - b_1 u) - c_1 \tau \right\} = 0.\]

By the nonnegativity of \(u\), one can see that

\[\tau = \frac{1}{c_1|\Omega|} \int_\Omega u(a_1 - b_1 u) \leq \frac{a_1^2}{4b_1c_1}.\] (4.1)

From (3.3), we can deduce that any solution \((w, \tau)\) of (1.4) satisfies

\[\tau \leq \frac{a_2^2}{4b_2c_2}\] (4.2)

by a similar way to get (4.1). Hence (4.1) and (4.2) give the required a priori estimate for the \(\tau\) component.

Setting \(v = \tau/u\) in the nonlinear term of (3.2a), we introduce a function \(h(u, \tau)\) as

\[h(u, \tau) := u(a_1 - b_1 u) - c_1 \tau - \frac{\gamma \tau}{u} \left( a_2 - b_2 u - \frac{c_2 \tau}{u} \right).\] (4.3)

It is noted that (3.2a) is equivalent to

\[d\Delta w + h(u, \tau) = 0 \quad \text{in } \Omega, \quad \tau > 0.\]

We remark that \(\lim_{u \searrow 0} h(u, \tau) = \infty\) and \(\lim_{u \to \infty} h(u, \tau) = -\infty\) for each \(\tau > 0\). Obviously \(h(u, \tau)\) has at least one, and at most three zeros on \(\{ u > 0 \}\) for each \(\tau > 0\). In what follows, the least zero and the greatest zero of \(h(u, \tau)\) \((u > 0)\) will be denoted by \(z_1(\tau)\) and \(\overline{\tau}(\tau)\), respectively. Our first task for the a priori estimate of \(\|w\|_\infty\) is to derive a lower bound of \(z_1(\tau)\) and an upper bound of \(\overline{\tau}(\tau)\).

In the case when \(\gamma b_2 > c_1\), we observe

\[h(u, \tau) = u(a_1 - b_1 u) + (\gamma b_2 - c_1) \tau + \frac{\gamma \tau}{u^2} (c_2 \tau - a_2 u).\] (4.4)

It is easily verified that

\[u(a_1 - b_1 u) + (\gamma b_2 - c_1) \tau \begin{cases} > 0 & \text{for } u \in (0, p_1(\tau)\tau), \\ < 0 & \text{for } u \in (p_1(\tau)\tau, \infty) \end{cases}\] (4.5)
with
\[ p_1(\tau) := \frac{2(\gamma b_2 - c_1)}{\sqrt{a_1^2 + 4b_1(\gamma b_2 - c_1)\tau} - a_1}. \]

and
\[ \frac{\gamma \tau}{u^2}(c_2 \tau - a_2u) \begin{cases} > 0 & \text{for } u \in (0, c_2\tau/a_2), \\ < 0 & \text{for } u \in (c_2\tau/a_2, \infty). \end{cases} \quad (4.6) \]

It follows from (4.4)-(4.6) that, in case \( \gamma b_2 > c_1 \),
\[ \min \left\{ p_1(\tau)\tau, \frac{c_2}{a_2} \right\} \leq z_1(\tau) \leq \overline{z}(\tau) \leq \max \left\{ p_1(\tau)\tau, \frac{c_2}{a_2} \right\} \quad \text{for any } \tau > 0. \quad (4.7) \]

Obviously, if \( \gamma b_2 = c_1 \), then
\[ \min \left\{ \frac{a_1}{b_1}, \frac{c_2}{a_2} \right\} \leq z_1(\tau) \leq \overline{z}(\tau) \leq \max \left\{ \frac{a_1}{b_1}, \frac{c_2}{a_2} \right\} \quad \text{for any } \tau > 0. \quad (4.8) \]

In the case when \( \gamma b_2 < c_1 \), we observe that
\[ u(a_1 - b_1u) \begin{cases} > 0 & \text{for } u \in (0, a_1/b_1), \\ < 0 & \text{for } u \in (a_1/b_1, \infty). \end{cases} \quad (4.9) \]

and
\[ (\gamma b_2 - c_1)\tau + \frac{\gamma \tau}{u^2}(c_2 \tau - a_2u) \begin{cases} > 0 & \text{for } u \in (0, p_2(\tau)\tau), \\ < 0 & \text{for } u \in (p_2(\tau)\tau, \infty) \end{cases} \quad (4.10) \]

with
\[ p_2(\tau) := \frac{2\gamma c_2}{\sqrt{(\gamma a_2)^2 + 4(c_1 - \gamma b_2)c_2\tau + \gamma a_2}}. \]

In view of (4.4), we know from (4.9) and (4.10) that, in case \( \gamma b_2 < c_1 \),
\[ \min \left\{ \frac{a_1}{b_1}, p_2(\tau)\tau \right\} \leq z_1(\tau) \leq \overline{z}(\tau) \leq \max \left\{ \frac{a_1}{b_1}, p_2(\tau)\tau \right\} \quad \text{for any } \tau > 0. \quad (4.11) \]

Suppose that \((w, \tau)\) is any solution of (3.2). Let \( x_* \in \overline{\Omega} \) be a minimum point of \( w \), namely, \( w(x_*) = \min_{x \in \Omega} w(x) \). Then, the application of (ii) of Lemma 2.1 to (3.2a)-(3.2b) implies
\[ h(u(x_*), \tau) \leq 0, \]
which leads to
\[ z_1(\tau) \leq u(x_*) = \frac{2\gamma \tau}{\sqrt{w(x_*)^2 + 4\gamma \delta \tau} - w(x_*)}. \]

Therefore, we see that \( w_-(x) := -\min \{ w(x), 0 \} \) satisfies
\[ w_-(x_*) = \frac{|w(x_*)| - w(x_*)}{2} < \frac{\sqrt{w(x_*)^2 + 4\gamma \delta \tau} - w(x_*)}{2} \leq \frac{\gamma \tau}{z_1(\tau)}. \]

From (4.7), (4.8) and (4.11), we can find a positive constant \( C_1 = C_1(a_i, b_i, c_i, \gamma) \) such that
\[ w_-(x_*) \leq C_1 \quad \text{for any } \tau \in (0, \overline{\tau}). \quad (4.12) \]

Let \( x^* \in \overline{\Omega} \) be a maximum point of \( w \); \( w(x^*) = \max_{x \in \Omega} w(x) \). By applying (i) of Lemma 2.1 to (3.2a)-(3.2b), we see that
\[ h(u(x_*), \tau) \geq 0, \]
which yields
\[ u(x^*) = \frac{\sqrt{w(x^*)^2 + 4\gamma \delta \tau} + w(x^*)}{2\delta} \leq \overline{z}(\tau). \]
Then \( w_+(x) := \max\{ w(x), 0 \} \) satisfies
\[
 w_+(x^*) = \frac{|w(x^*)| + w(x^*)}{2} \leq \frac{\sqrt{w(x^*)^2 + 4\gamma \delta \tau + w(x^*)}}{2} = \delta u(x^*) \leq \delta \tau.
\]
From (4.7), (4.8) and (4.11), we can find a positive constant \( C_2 = C_2(a_i, b_i, c_i, \gamma, \delta) \) such that
\[
 w_+(x^*) \leq C_2 \text{ for any } \tau \in (0, \tau_0]. \tag{4.13}
\]
It follows from (4.12) and (4.13) that \( \| w \|_\infty \leq C^* := \max\{ C_1, C_2 \} \). The proof of Lemma 4.1 is complete.

With the aid of the elliptic regularity theory, Lemma 4.1 leads to the following a priori estimate of solutions of (3.2).

**Corollary 4.2.** For any \( \varepsilon > 0 \), there exists a positive constant \( C^*_1 = C^*_1(\varepsilon, a_i, b_i, c_i, \gamma, \delta) \) which is independent of \( \tau \) such that if \( d \geq \varepsilon \), then any solution \((w, \tau)\) of (3.2) satisfies \( \| w \|_{C^1(\Omega)} < C^*_1 \).

**Proof.** By the combination of Lemma 4.1 and the elliptic regularity theory, we find a positive constant \( C^*_0 = C^*_0(a_i, b_i, c_i, \gamma, \delta) \) such that any solution \((w, \tau)\) of (3.2) satisfies \( d \| w \|_{W^{2,p}} \leq C^*_0 \) for any \( p > 1 \). Hence the Sobolev embedding theorem ensures \( C^*_1 \) fulfilling the required estimate.

The next result asserts the nonexistence of nonconstant solutions of (3.2) when \( d \) is sufficiently large.

**Lemma 4.3.** There exists a positive constant \( d = d(a_i, b_i, c_i, \gamma, \delta) \) which is independent of \( \tau \) such that (3.2) does not have any nonconstant solution if \( d > d \).

**Proof.** For any nonconstant solution \((w, \tau)\) of (3.2), let \((u, v)\) be as in (3.1). Then it follows that
\[
\begin{cases}
-(-d\Delta u - \gamma \Delta v) = f(u, v) - \gamma g(u, v) & \text{in } \Omega, \\
\partial_v u = \partial_v v = 0 & \text{on } \partial \Omega.
\end{cases}
\]
By taking the \( L^2(\Omega) \) inner product of the elliptic equation with
\[
u - \overline{\nu} \text{ and } v - \overline{v}, \quad \text{where } \overline{\nu} := \frac{1}{|\Omega|} \int_{\Omega} u, \quad \overline{v} := \frac{1}{|\Omega|} \int_{\Omega} v,
\]
we have
\[
d \left( \| \nabla u \|_2^2 - \gamma \int_{\Omega} \nabla u \cdot \nabla v \right) = - \int_{\Omega} f(u, v)(u - \overline{u}) - \gamma \int_{\Omega} g(u, v)(u - \overline{u})
\]
and
\[
d \left( \delta \int_{\Omega} \nabla u \cdot \nabla v - \gamma \| \nabla v \|_2^2 \right) = - \int_{\Omega} f(u, v)(v - \overline{v}) - \gamma \int_{\Omega} g(u, v)(v - \overline{v}),
\]
respectively. Subtracting the second identity from the first one, we get
\[
d \left( \| \nabla u \|_2^2 - (\gamma + \delta) \int_{\Omega} \nabla u \cdot \nabla v + \gamma \| \nabla v \|_2^2 \right) = - \int_{\Omega} \{ f(u, v) - \gamma g(u, v) \} \{ (u - \overline{u}) - (v - \overline{v}) \}. \tag{4.14}
\]
Proof of Theorem 3.1. Theorem 3.1 follows from Lemmas 4.1 and 4.3.

Similarly, one can obtain
\[ \int_{\Omega} \{ f(u, v) - \gamma g(u, v) \} (v - \overline{v}) = \int_{\Omega} \{ a_1 - b_1 (u + \overline{u}) \} (u - \overline{u}) (v - \overline{v}) - \gamma \int_{\Omega} \{ a_2 - c_2 (v + \overline{v}) \} (v - \overline{v})^2. \]

Here we remark that (3.1) and Lemma 4.1 ensure a positive constant \( M = M(a_i, b_i, c_i, \gamma, \delta) \) such that
\[ \| a_1 - b_1 (u + \overline{u}) \|_\infty \leq \frac{M}{2} \quad \text{and} \quad \gamma \| a_2 - c_2 (v + \overline{v}) \|_\infty \leq \frac{M}{2}. \]

Substituting (4.15) and (4.16) into (4.14) and using (4.17) and the Schwarz inequality, we obtain
\[ d \left( \delta \| \nabla u \|_2^2 - (\gamma + \delta) \int_{\Omega} \nabla u \cdot \nabla v + \gamma \| \nabla v \|_2^2 \right) \leq M \left( \| \nabla u \|_2^2 + \| \nabla v \|_2^2 \right). \]

Here we recall the Poincaré-Wirtinger inequality;
\[ \lambda_1 \| U - \overline{U} \|_2^2 \leq \| \nabla U \|_2^2 \quad \text{for any} \ U \in H^1(\Omega), \]
where \( \lambda_1 \) represents the least positive eigenvalue of (1.7). Therefore, we obtain
\[ d \left( \delta \| \nabla u \|_2^2 - (\gamma + \delta) \int_{\Omega} \nabla u \cdot \nabla v + \gamma \| \nabla v \|_2^2 \right) \leq \frac{M}{\lambda_1} \left( \| \nabla u \|_2^2 + \| \nabla v \|_2^2 \right). \]

It follows from \( v = \tau / u \) that
\[ \int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} \nabla u \cdot \nabla \left( \frac{\tau}{u} \right) = -\tau \int_{\Omega} \left| \frac{\nabla u}{u} \right|^2 < 0. \]

Then we know from (4.18) that
\[ \left( d \delta - \frac{M}{\lambda_1} \right) \| \nabla u \|_2^2 + \left( d \gamma - \frac{M}{\lambda_1} \right) \| \nabla v \|_2^2 \leq 0, \]
which concludes
\[ d \leq \overline{d} := \max \left\{ \frac{M}{\gamma \lambda_1}, \frac{M}{\delta \lambda_1} \right\} \]
because neither \( u \) nor \( v = \tau / u \) is constant. Then we complete the proof of Lemma 4.3. \( \square \)

Proof of Theorem 3.7. Theorem 3.1 follows from Lemmas 4.1 and 4.3. \( \square \)
5. Existence of nonconstant solutions of (3.2) by the Leray-Schauder degree theory, we set up a functional space $X := C^1(\Omega) \times \mathbb{R}$ and

$$F(d, w, \tau) := \left[ \begin{array}{c} F^{(1)}(d, w, \tau) \\ F^{(2)}(d, w, \tau) \end{array} \right],$$

where

$$F^{(1)}(d, w, \tau) := (I - \Delta)^{-1}\left[w + \frac{f(u(w, \tau), v(w, \tau)) - \gamma g(u(w, \tau), v(w, \tau))}{d}\right],$$

$$F^{(2)}(d, w, \tau) := \frac{1}{c_1|\Omega|} \int_{\Omega} u(w, \tau)\{ a_1 - b_1 u(w, \tau) \}.$$

Here $(u(w, \tau), v(w, \tau))$ is as in (3.1) and $(I - \Delta)^{-1}$ is regarded as a composition of the inverse operator of $I - \Delta : W_0^{2,p}(\Omega) := \{ w \in W^{2,p}(\Omega) : \partial_{\nu} w = 0 \text{ on } \partial\Omega \} \rightarrow L^p(\Omega)$ with the domain restricted to $C^1(\Omega)$ and the compact embedding from $W^{2,p}(\Omega)$ into $C^1(\Omega)$ with $p > N$. Then, for any $d > 0$, each weak solution of (3.2) is corresponding to each fixed point of $F(d, \bullet, \bullet)$. To find fixed points with $\tau > 0$, we introduce a bounded set $S^\eta_M$ in $X$ as

$$S^\eta_M := \{ (w, \tau) \in X : \eta < \|w\|_{C^1(\Omega)} < M, \eta < \tau < M \}$$

for $0 < \eta < M$. In the following lemma, $C^*_1$ and $\overline{\eta}$ are positive constants obtained in Theorem 3.1.

Lemma 5.1. Assume $A \neq B$ and $A \neq C$. Furthermore, especially in the weak competition case $C < A < B$ or the strong competition case $B < A < C$, assume that $(u^*, v^*)$ in (1.6) satisfies $u^*/v^* \neq \gamma/\delta$. Then, for any small $\varepsilon > 0$, there exists a small $\eta = \eta(\varepsilon) > 0$ such that if $d \geq \varepsilon$, then any solution $(w, \tau)$ of (3.2) satisfies $(w, \tau) \notin \partial S^\eta_M$, where $M$ is any constant satisfying $M > C^*_1$ and $M > \overline{\eta}.$

Proof. It follows from Theorem 3.1 and Corollary 4.2 that any solution $(w, \tau)$ of (3.2) with $d \geq \varepsilon$ satisfies $(w, \tau) \in B_M := \{ (w, \tau) \in X : \|w\|_{C^1(\Omega)} < M, 0 \leq \tau < M \}$ if $M > C^*_1$ and $M > \overline{\eta}$. Obviously, semitrivial solutions $(w, \tau) = (\delta a_1/b_1, 0), (-\gamma a_2/c_2, 0)$ and the trivial solution $(w, \tau) = (0, 0)$ are not contained in the closure of $S^\eta_M$ because $\eta > 0$.

Then our task is to prove that for any small $\varepsilon > 0$, there exists $\eta = \eta(\varepsilon) > 0$ such that any solution $(w, \tau)$ of (3.2) with $d \geq \varepsilon$ satisfies $\|w\|_{C^1(\Omega)} > \eta$ and $\tau > \eta$. Suppose for contradiction that there exists $\hat{\varepsilon} > 0$ such that for any small $\eta > 0$, there exists some $d = \hat{d}(\eta) \geq \hat{\varepsilon}$ such that (3.2) with $d = \hat{d}(\eta)$ has a solution $(\hat{w}(\eta), \hat{\tau}(\eta))$ satisfying $\|\hat{w}(\eta)\|_{C^1(\Omega)} \leq \eta$ or $\tau \leq \eta$. By virtue of Lemma 4.3, we can choose a subsequence $(\hat{d}_n, w_n, \tau_n) \in [\hat{\varepsilon}, \hat{d}] \times B_M$ of $\{ (\hat{d}(\eta), \hat{w}(\eta), \hat{\tau}(\eta)) \}_{\eta > 0}$ such that

$$\|w_n\|_{C^1(\Omega)} \rightarrow 0$$

or $\tau_n \rightarrow 0$

and $\hat{d}_n \rightarrow d_\infty$ with some $d_\infty \in [\hat{\varepsilon}, \hat{d}]$ as $n \rightarrow \infty$.

Suppose that $\|w_n\|_{C^1(\Omega)} \rightarrow 0$ and $\limsup_{n \rightarrow \infty} \tau_n > 0$. We may assume $\tau_n \rightarrow \tau_0 > 0$ by passing to a subsequence if necessary. By (3.1), one can see that

$$u_n := \sqrt{w_n^2 + 4\gamma d \tau_n + w_n} \rightarrow \sqrt{\frac{\gamma \tau_0}{\delta}} =: u_\infty \text{ in } C^1(\Omega),$$

$$v_n := \sqrt{w_n^2 + 4\gamma d \tau_n - w_n} \rightarrow \sqrt{\frac{\delta \tau_0}{\gamma}} =: v_\infty \text{ in } C^1(\Omega).$$
Setting \( n \to \infty \) in (3.22) and (3.3), we get \( f(u_\infty, v_\infty) = g(u_\infty, v_\infty) = 0 \). Hence it follows that \((u_\infty, v_\infty) = (u^*, v^*)\). However, this is impossible under the assumption \( u^*/v^* \neq \gamma/\delta \).

Suppose that \( \|w_n\|_{C^1(\overline{\Omega})} \to 0 \) and \( \tau_n \to 0 \). We set
\[
\bar{w}_n(x) := \frac{w_n(x)}{\|w_n\|_\infty}.
\] (5.2)
Substituting (3.2) and (3.1) into (3.2)-(3.3) and dividing the resulting expressions by \( \|w_n\|_\infty \), one can see that
\[
\begin{aligned}
\bar{w}_n + \frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n} + w_n}{2\|w_n\|_\infty} (a_1 - b_1 \frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n}}{2\delta} + c_1 \frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n} - w_n}{2\gamma}) \\
\frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n} - w_n}{2\|w_n\|_\infty} (a_2 - b_2 \frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n}}{2\delta} - c_2 \frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n} - w_n}{2\gamma}) &= 0 \quad \text{in } \Omega, \\
\partial_n \bar{w}_n &= 0 \quad \text{on } \partial \Omega
\end{aligned}
\] (5.3)
and
\[
\begin{aligned}
\int_\Omega \frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n} + w_n}{2\|w_n\|_\infty} (a_1 - b_1 \frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n}}{2\delta} + c_1 \frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n} - w_n}{2\gamma}) = 0, \\
\int_\Omega \frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n} - w_n}{2\|w_n\|_\infty} (a_2 - b_2 \frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n}}{2\delta} - c_2 \frac{\sqrt{w_n^2 + 4\gamma \delta \tau_n} - w_n}{2\gamma}) = 0.
\end{aligned}
\] (5.4)
By applying the elliptic regularity theory to (5.3), we find a function \( \bar{w} \in W^{2,p}(\Omega) \) for any \( p > 1 \) such that
\[
\lim_{n \to \infty} \bar{w}_n = \bar{w} \quad \text{weakly in } W^{2,p}(\Omega) \text{ and strongly in } C^1(\overline{\Omega})
\]
by passing to a subsequence. Hence it follows that \( \|\bar{w}\|_\infty = 1 \). Then we set \( n \to \infty \) in (5.4) to know
\[
\int_\Omega \bar{w}_n = \int_\Omega \bar{w}_n = 0,
\]
which leads to \( \bar{w} \equiv 0 \). However, this contradicts \( \|\bar{w}\|_\infty = 1 \).

Suppose that \( \limsup_{n \to \infty} \|w_n\|_{C^1(\overline{\Omega})} > 0 \) and \( \tau_n \to 0 \). Similarly, a usual compactness argument applying the elliptic regularity theory to (3.2) ensures a function \( w_\infty \in W^{2,p}(\Omega) \) for any \( p > 1 \) such that \( \lim_{n \to \infty} w_n = w_\infty \in C^1(\overline{\Omega}) \), \( \|w_\infty\|_{C^1(\overline{\Omega})} > 0 \), and \( w_\infty \) is weak solution of (3.2) with \( d = d_\infty \). By virtue of Proposition 2.2, we see that \( w_\infty = \delta a_1/b_1 \) or \( w_\infty = -\gamma a_2/c_2 \in \Omega \).

Suppose that \( w_\infty = \delta a_1/b_1 \in \Omega \). It follows that
\[
\lim_{n \to \infty} (u_n, v_n) = \left( \frac{a_1}{b_1}, 0 \right) \quad \text{uniformly in } \overline{\Omega},
\] (5.5)
where \((u_n, v_n)\) is defined by (5.2). From (5.3), we observe that
\[
\int_\Omega v_n (a_2 - b_2 u_n - c_2 v_n) = 0 \quad \text{for any } n \in \mathbb{N}.
\] (5.6)
Owing to the assumption \( A \neq B \), we know from (5.5) that
\[
a_2 - b_2 u_n - c_2 v_n > 0 \quad \text{or} \quad a_2 - b_2 u_n - c_2 v_n < 0 \quad \text{in } \Omega
\]
if \( n \) is sufficiently large. This obviously contradicts (5.6) because \( v_n > 0 \) in \( \Omega \) for any \( n \in \mathbb{N} \). By a similar way, we can see that \( w_\infty = -\gamma a_2/c_2 \) is also impossible by the assumption \( A \neq C \).

Consequently, we obtain the required assertion in Lemma 5.1 by the contradiction argument. \( \square \)
Lemma 5.2. Assume the weak competition $C < A < B$ or the strong competition $B < A < C$. Suppose further that $D(a, b, c, \gamma) > 0$. Then there exists a sequence $\{d^{(j)}\}_{j=1}^{\infty}$ with

$$0 \leq d^{(1)} \leq d^{(2)} \leq \cdots \leq d^{(j)} \leq \cdots \leq d^{(2)} \leq d^{(1)} \leq \bar{d}$$

such that if $C < A < B$, then

$$\text{ind}(I - F(d, \bullet, \bullet), (w^*, \tau^*)) = \begin{cases} 1 & \text{for } d \in (d^{(j+1)}, d^{(j)}) \text{ and } j \text{ is even}, \\ -1 & \text{for } d \in (d^{(j+1)}, d^{(j)}) \text{ and } j \text{ is odd}, \end{cases}$$

whereas, if $B < A < C$, then

$$\text{ind}(I - F(d, \bullet, \bullet), (w^*, \tau^*)) = \begin{cases} 1 & \text{for } d \in (d^{(j+1)}, d^{(j)}) \text{ and } j \text{ is odd}, \\ -1 & \text{for } d \in (d^{(j+1)}, d^{(j)}) \text{ and } j \text{ is even}. \end{cases}$$

Proof. For the sake of the calculation of $\text{ind}(I - F(d, \bullet, \bullet), (w^*, \tau^*))$, the following eigenvalue problem will be considered:

$$(I - L(d)) \begin{bmatrix} \phi \\ \xi \end{bmatrix} = \mu \begin{bmatrix} \phi \\ \xi \end{bmatrix},$$

where $L(d)$ is a linear compact operator from $X$ to $X$ defined by the linearized operator of $F(d, \bullet, \bullet)$ around $(w^*, \tau^*)$ as follows:

$$L(d) := F_{(w, \tau)}(d, w^*, \tau^*).$$

It follows from the index formula (see e.g., [18, Theorem 2.8.1]) that

$$\text{ind}(I - F(d, \bullet, \bullet), (w^*, \tau^*)) = (-1)^{\sigma(d)},$$

where $\sigma(d)$ is the number of negative eigenvalues (counting algebraic multiplicity) of $(5.8)$. Hereafter, each entry of $L(d)$ will be denoted by

$$L(d) = \begin{bmatrix} L_{11}(d) & L_{12}(d) \\ L_{21}(d) & L_{22}(d) \end{bmatrix} := \begin{bmatrix} F_{w}^{(1)}(d, w^*, \tau^*) & F_{\tau}^{(1)}(d, w^*, \tau^*) \\ F_{w}^{(2)}(d, w^*, \tau^*) & F_{\tau}^{(2)}(d, w^*, \tau^*) \end{bmatrix}.$$

It follows from $(5.11)$ that

$$L_{11}(d) = \left(1 + \frac{f_u^* u_w^* + f_v^* v_w^* - \gamma (g_u^* u_w^* + g_v^* v_w^*)}{d} \right)(I - \Delta)^{-1},$$

$$L_{12}(d) = \frac{f_u^* u_w^* + f_v^* v_w^* - \gamma (g_u^* u_w^* + g_v^* v_w^*)}{d}(I - \Delta)^{-1},$$

$$L_{21}(d) = \frac{(a_1 - 2b_1 u^*) u_w^*}{c_1 |\Omega|} \int_{\Omega} \bullet, \quad L_{22}(d) = \frac{(a_1 - 2b_1 u^*) u_w^*}{c_1},$$

Therefore, under assumptions in Lemma 5.1, the compact nonlinear map $F(d, \bullet, \bullet) : X \to X$ has no fixed point on $\partial S_{\eta,M}$ for any $d \geq \varepsilon$. Hence the homotopy invariance of the Leray-Schauder degree implies that

$$\text{deg}(I - F(d, \bullet, \bullet), S_{\eta,M}, 0)$$

is constant for any $d \geq \varepsilon$. (5.7)
where \( f_u^* := f_u(u^*, v^*), u_w^* := u_w(w^*, \tau^*) \) and other notations are defined by the same manner. Observing that \( f(u^*, v^*) = g(u^*, v^*) = 0 \) and \( \sqrt{w^2 + 4\gamma \delta \tau} = \delta u + \gamma v \), one can verify
\[
\begin{bmatrix}
  f_u^* & f_v^* \\
  g_u^* & g_v^*
\end{bmatrix} = - \begin{bmatrix}
  b_1 u^* & c_1 u^* \\
  b_2 v^* & c_2 v^*
\end{bmatrix}, \quad \begin{bmatrix}
  u_w^* & u_\tau^* \\
  v_w^* & v_\tau^*
\end{bmatrix} = \frac{1}{\delta u^* + \gamma v^*} \begin{bmatrix}
  u^* & \gamma \\
  -v^* & \delta
\end{bmatrix}
\]
by a straightforward calculation. Substituting these expressions into (5.10), we get
\[
L_{11}(d) = \left( 1 + \frac{\gamma b_2 + c_1}{c_1} u^* - \frac{b_1 (u^*)^2 - \gamma c_2 (v^*)^2}{(\delta u^* + \gamma v^*)d} \right) (I - \Delta)^{-1},
\]
\[
L_{12}(d) = - \frac{\gamma b_1 + \delta c_1}{c_1} u^* - \frac{\gamma (\beta b_2 + \delta c_2) v^*}{(\delta u^* + \gamma v^*)d} (I - \Delta)^{-1},
\]
\[
L_{21}(d) = \frac{c_1 u^*}{c_1 (\delta u^* + \gamma v^*)|\Omega|} \int_\Omega \cdot, \quad L_{22}(d) = \frac{\gamma (c_1 v^* - b_1 u^*)}{c_1 (\delta u^* + \gamma v^*)},
\]
where \( a_1 - 2b_1 u^* = c_1 v^* - b_1 u^* \) is used for expressions of \( L_{21}(d) \) and \( L_{22}(d) \). Therefore, we substitute (5.11) into (5.8) to see that the eigenvalue problem (5.8) is equivalent to
\[
\begin{cases}
  - (1 - \mu) \Delta \phi - \frac{(\gamma b_2 + c_1) u^* - \frac{b_1 (u^*)^2 - \gamma c_2 (v^*)^2}{(\delta u^* + \gamma v^*)d} \phi}{(\delta u^* + \gamma v^*)d} \\
  + \frac{(\gamma b_1 + \delta c_1) u^* - \gamma (\beta b_2 + \delta c_2) v^*}{(\delta u^* + \gamma v^*)d} \xi = \mu \phi \quad \text{in } \Omega,
\end{cases}
\]
\[
\begin{cases}
  \frac{b_1 (u^*)^2 - c_1 u^*}{(\delta u^* + \gamma v^*)d} \phi + \frac{(\gamma b_1 + \delta c_1) u^*}{c_1 (\delta u^* + \gamma v^*)} \xi = \mu \xi, \\
  \partial_\nu \phi = 0 \quad \text{on } \partial \Omega,
\end{cases}
\]
where \( \overline{\phi} := |\Omega|^{-1} \int_\Omega \phi \). To seek for nontrivial solutions of (5.12), we introduce the Fourier expansion of \( \phi \) as
\[
\phi(x) = \sum_{j=0}^{\infty} q_j \Phi_j(x),
\]
where \( \{\Phi_j\}_{j=0}^{\infty} \) is a complete orthonormal basis in \( L^2(\Omega) \) defined by (1.7). Substituting (5.13) into the first equation of (5.12), we obtain
\[
\sum_{j=0}^{\infty} \left( (1 - \mu) \lambda_j - \mu - \frac{(\gamma b_2 + c_1) u^* - \frac{b_1 (u^*)^2 - \gamma c_2 (v^*)^2}{(\delta u^* + \gamma v^*)d}}{(\delta u^* + \gamma v^*)d} \right) q_j \Phi_j
\]
\[
+ \frac{(\gamma b_1 + \delta c_1) u^* - \gamma (\beta b_2 + \delta c_2) v^*}{(\delta u^* + \gamma v^*)d} \xi = 0 \quad \text{in } \Omega.
\]
Integrating (5.14) over \( \Omega \), we see
\[
- \frac{(\gamma b_2 + c_1) u^* - b_1 (u^*)^2 - \gamma c_2 (v^*)^2}{(\delta u^* + \gamma v^*)d} q_0 |\Omega|^{-1/2} + \frac{(\gamma b_1 + \delta c_1) u^* - \gamma (\beta b_2 + \delta c_2) v^*}{(\delta u^* + \gamma v^*)d} \xi = \mu q_0 |\Omega|^{-1/2}.
\]
Here it is noted that \( \Phi_0 = |\Omega|^{-1/2} \) and \( \overline{\phi} = q_0 |\Omega|^{-1/2} \). Together with the second equation of (5.12), we obtain the following equation which \( \xi \) and the constant component of \( \phi \) satisfy
\[
M(d; a_i, b_i, c_i, \gamma, \delta) \begin{bmatrix} \overline{\phi} \\ \xi \end{bmatrix} = \mu \begin{bmatrix} \overline{\phi} \\ \xi \end{bmatrix},
\]
(5.15)
where
\[ M(d; a_i, b_i, c_i, \gamma, \delta) = \frac{1}{\delta u^* + \gamma v^*} \begin{bmatrix} \frac{(\gamma b_2 + c_1)\tau^* - b_1(u^*)^2 - \gamma c_2(v^*)^2}{d} & -\frac{(\gamma b_1 + \delta c_1)u^* - \gamma (\gamma b_2 + \delta c_2)v^*}{c_1} \\ -\frac{(\gamma b_1 + \delta c_1)u^* - \gamma (\gamma b_2 + \delta c_2)v^*}{c_1} & \frac{(\gamma b_1 + \delta c_1)u^*}{d} \end{bmatrix}. \]

In what follows, we denote by \( \mu_0^-(d) \) and \( \mu_0^+(d) \) eigenvalues of (5.15) satisfying
\[ \text{Re} \mu_0^-(d) \leq \text{Re} \mu_0^+(d) \quad \text{and} \quad \text{Im} \mu_0^-(d) \leq \text{Im} \mu_0^+(d). \]

By a straightforward computation, one can verify that
\[ \mu_0^-(d)\mu_0^+(d) = |M(d; a_i, b_i, c_i, \gamma, \delta)| = \frac{\gamma u^*v^*}{c_1(\delta u^* + \gamma v^*)^2d}(b_1c_2 - b_2c_1). \]

Hence it follows that
\[ \mu_0^-(d)\mu_0^+(d) \begin{cases} > 0 & \text{if } C < A < B, \\ < 0 & \text{if } B < A < C. \end{cases} \tag{5.16} \]

Here we set
\[ \sigma_0(d) := \text{the number of negatives of } \{\mu_0^-(d), \mu_0^+(d)\}. \]

By (5.16), one can see that, for any \( d > 0 \),
\[ \sigma_0(d) = \begin{cases} 0 \text{ or } 2 & \text{if } C < A < B, \\ 1 & \text{if } B < A < C. \end{cases} \tag{5.17} \]

Taking the \( L^2(\Omega) \) inner product of (5.14) with \( \Phi_j \), we know that
\[ \left\{ (1 - \mu)\lambda_j - \frac{(\gamma b_2 + c_1)\tau^* - b_1(u^*)^2 - \gamma c_2(v^*)^2}{(\delta u^* + \gamma v^*)d} \right\} q_j = 0 \quad \text{for any } j \in \mathbb{N}. \]

Then the component of \( \Phi_j \) of (5.14) is nontrivial as \( q_j \neq 0 \) if
\[ \mu = \mu_j(d) := \frac{1}{\lambda_j + 1} \left( \lambda_j - \frac{(\gamma b_2 + c_1)\tau^* - b_1(u^*)^2 - \gamma c_2(v^*)^2}{(\delta u^* + \gamma v^*)d} \right). \tag{5.18} \]

Consequently, we deduce that all eigenvalues of \( I - L(d) \) consist of
\[ \{ \mu_0^-(d), \mu_0^+(d), \mu_1(d), \mu_2(d), \ldots, \mu_j(d), \ldots \}. \]

It follows from (5.18) that, if \((\gamma b_2 + c_1)\tau^* - b_1(u^*)^2 - \gamma c_2(v^*)^2 \leq 0\), then \( \mu_j(d) > 0 \) for any \( d > 0 \). On the other hand, if \((\gamma b_2 + c_1)\tau^* - b_1(u^*)^2 - \gamma c_2(v^*)^2 > 0\), then, for each \( j \in \mathbb{N} \), \( \mu_j(d) \) is monotone increasing with respect to \( d > 0 \) and satisfies
\[ \mu_j(d) \begin{cases} < 0 & \text{for } d \in (0, d^{(j)}), \\ = 0 & \text{for } d = d^{(j)}, \\ > 0 & \text{for } d \in (d^{(j)}, \infty), \end{cases} \tag{5.19} \]

where
\[ d^{(j)} := \frac{(\gamma b_2 + c_1)\tau^* - b_1(u^*)^2 - \gamma c_2(v^*)^2}{(\delta u^* + \gamma v^*)\lambda_j} > 0. \tag{5.20} \]

By substituting (1.6) into (5.20), one can verify that
\[ d^{(j)} = \frac{a_2^2b_2c_2}{(b_2c_1 - b_1c_2)^2(\delta u^* + \gamma v^*)\lambda_j}D(a_i, b_i, c_i, \gamma). \tag{5.21} \]
It should be noted that $D(a_i, b_i, c_i, \gamma)$ defined by (4.5) is independent of $j \in \mathbb{N}$. Hence $d^{(j)} > 0$ if and only if $D(a_i, b_i, c_i, \gamma) > 0$. Furthermore, if $D(a_i, b_i, c_i, \gamma) > 0$, then

$$0 < d^{(j+1)} \leq d^{(j)} \quad \text{for any } j \in \mathbb{N} \quad \text{and} \quad d^{(j)} = O(\lambda_j^{-1}) \quad \text{as } j \to \infty. \quad (5.22)$$

Together with (5.17) and (5.19), we can deduce that the number $\sigma(d)$ of negative eigenvalues of $L(d)$ satisfies that

$$\sigma(d) = \begin{cases} j & \text{or} \quad j + 2 \quad \text{if } C < A < B \quad \text{and} \quad d \in (d^{(j+1)}, d^{(j)}), \\ j + 1 & \text{if } B < A < C \quad \text{and} \quad d \in (d^{(j+1)}, d^{(j)}). \end{cases}$$

By virtue of (5.9), we establish the assertion of Lemma 5.2. \hfill \Box

**Proof of Theorem 3.2.** Under assumptions of Theorem 3.2, we know from Theorem 3.1 and Lemma 5.1 that if $d > \overline{d}$, then the following equation of unknowns $(w, \tau) \in S_{\eta,M}$;

$$(w, \tau) - F(d, w, \tau) = 0$$

has the unique solution $(w^*, \tau^*)$. Therefore, the well-known property of the Leray-Schauder degree implies that if $d > \overline{d}$, then

$$\deg(I - F(d, \bullet, \bullet), S_{\eta,M}, 0) = \text{ind}(I - F(d, \bullet, \bullet), (u^*, \tau^*)) = (-1)^{\sigma(d)}. \quad (5.23)$$

It follows from (5.19) and (5.22) that $\mu_j(d) > 0$ for any $j \in \mathbb{N}$ if $d > d^{(1)}$. Then, for any $d > d^{(1)}$, the number $\sigma(d)$ of negative eigenvalues of $I - L(d)$ is equal to that of negatives of \{ $\mu_0(d), \mu_0^+(d)$ \}, namely, $\sigma(d) = \sigma_0(d)$). It follows from (5.17) and (5.23) that if $d > \overline{d}$, then

$$\deg(I - F(d, \bullet, \bullet), S_{\eta,M}, 0) = (-1)^{\sigma_0(d)} = \begin{cases} 1 & \text{if } C < A < B, \\ -1 & \text{if } B < A < C. \end{cases} \quad (5.24)$$

Together with the homotopy invariance (5.17) of $\deg(I - F(d, \bullet, \bullet), S_{\eta,M}, 0)$, we see that (5.24) holds true for any $d \geq \varepsilon$.

Assume that $C < A < B$ in addition to (3.6). We shall show that (3.2) admits at least one nonconstant solution when $d \in (d^{(j+1)}, d^{(j)}) \cap [\varepsilon, \infty)$ and $j$ is odd. Suppose for contradiction that there is no nonconstant solution of (3.2) for some $\hat{d} \in (d^{(j+1)}, d^{(j)}) \cap [\varepsilon, \infty)$ with some odd $j$. Then $(u^*, \tau^*)$ is the only fixed point of $F(\hat{d}, \bullet, \bullet) : S_{\eta,M} \to X$. In this situation, we can use the index formula of the Leray-Schauder degree to see

$$\deg(I - F(\hat{d}, \bullet, \bullet), S_{\eta,M}, 0) = \text{ind}(I - F(\hat{d}, \bullet, \bullet), (u^*, \tau^*)) = -1,$$

where the last equality comes from Lemma 5.2. Obviously, this contradicts (5.24). By taking account for the arbitrary of $\varepsilon > 0$, we obtain the assertion in case (i) of Theorem 3.2. Also in the other case $B < A < C$ with (3.6), a similar argument using Lemma 5.2 proves the assertion in case (ii) of Theorem 3.2. Then we complete the proof of Theorem 3.2. \hfill \Box

6. **One-dimensional analysis of the full cross-diffusion limit**

This section focuses on the global bifurcation structure of nonconstant solutions of (3.2) in the one-dimensional case $\Omega = (0, 1)$. Then we shall consider the following nonlinear ordinary differential equation

$$du'' + h(u, \tau) = 0, \quad u > 0 \quad \text{in} \quad (0, 1), \quad \tau > 0, \quad (6.1a)$$

subject to the homogeneous Neumann boundary condition

$$w'(0) = w'(1) = 0 \quad (6.1b)$$
with the integral constraint
\[ \int_0^1 f \left( u, \frac{\tau}{u} \right) = 0, \] (6.1c)
where \( u = u(w, \tau) \) is defined by (3.1) and \( h(u, \tau) \) is the nonlinear term given as (4.3). It follows from (3.3) that
\[ \int_0^1 g \left( u, \frac{\tau}{u} \right) = 0. \] (6.2)

In this section, the prime symbol represents the derivative by \( x \). It will be shown that, for each fixed small \( \tau > 0 \), there exist three zeros of \( h(u, \tau) \) on \( \{ u > 0 \} \):

**Lemma 6.1.** If \( \tau > 0 \) is sufficiently small, then \( h(u, \tau) (u > 0) \) possesses three zeros \( 0 < z_1(\tau) < z_2(\tau) < z_3(\tau) \) such that
\[
\begin{align*}
& h(u, \tau) \begin{cases} > 0 & \text{for } u \in (0, z_1(\tau)) \cup (z_2(\tau), z_3(\tau)), \\ < 0 & \text{for } u \in (z_1(\tau), z_2(\tau)) \cup (z_3(\tau), \infty) \end{cases} \\
\end{align*}
\]
and
\[
\begin{align*}
& \lim_{\tau \downarrow 0} \frac{z_1(\tau)}{\tau} = \frac{c_2}{a_2}, \quad \lim_{\tau \downarrow 0} \frac{z_2(\tau)}{\sqrt{\tau}} = \sqrt{\frac{\gamma a_2}{a_1}}, \quad \lim_{\tau \downarrow 0} z_3(\tau) = \frac{a_1}{b_1}. \\
\end{align*}
\]

**Proof.** Obviously, the fundamental theorem of algebra ensures that \( h(u, \tau) \) has at most three zeros on \( \{ u > 0 \} \). By a straightforward calculation, one can verify
\[
\begin{align*}
& \lim_{\tau \downarrow 0} h(\kappa_1, \tau) = -\frac{\gamma}{\kappa_1} \left( \frac{a_2 - c_2}{\kappa_1} \right) \begin{cases} > 0 & \text{if } \kappa_1 \in (0, c_2/a_2), \\ = 0 & \text{if } \kappa_1 = c_2/a_2, \\ < 0 & \text{if } \kappa_1 \in (c_2/a_2, \infty) \end{cases} \\
\end{align*}
\]
and
\[
\begin{align*}
& \lim_{\tau \downarrow 0} \frac{h(\kappa_2, \tau)}{\sqrt{\tau}} = \kappa_2 a_1 - \frac{\gamma a_2}{\kappa_2} \begin{cases} < 0 & \text{if } \kappa_2 \in (0, \sqrt{\gamma a_2/a_1}), \\ = 0 & \text{if } \kappa_2 = \sqrt{\gamma a_2/a_1}, \\ > 0 & \text{if } \kappa_2 \in (\sqrt{\gamma a_2/a_1}, \infty) \end{cases} \\
\end{align*}
\]
and
\[
\begin{align*}
& \lim_{\tau \downarrow 0} h(u, \tau) = u(a_1 - b_1u) \begin{cases} > 0 & \text{if } u \in (0, a_1/b_1), \\ = 0 & \text{if } u = a_1/b_1, \\ < 0 & \text{if } u \in (a_1/b_1, \infty). \end{cases} \\
\end{align*}
\]
Therefore, we obtain the required assertion. \( \square \)

We introduce the set \( \mathcal{T} = \mathcal{T}(a_i, b_i, c_i, \gamma) \) such as
\[ \mathcal{T} := \{ \tau \in (0, \tau_0] : h(u, \tau) \text{ has three zeros } 0 < z_1(\tau) < z_2(\tau) < z_3(\tau) \}, \]
where \( \tau_0 \) is obtained in Theorem 3.1. Hence \( \mathcal{T} \) is not empty because any small \( \tau > 0 \) belongs to \( \mathcal{T} \) by Lemma 6.1. Our strategy of analysis of (6.1) is as follows: First we obtain solutions of the Neumann problem (6.1a)-(6.1b) without the integral constraint (6.1c). Next we construct the set of solutions of (6.1) by choosing functions satisfying (6.1c) in the set of solutions of (6.1a)-(6.1b). To this end, we first obtain the following existence of solutions of (6.1a)-(6.1b):

**Proposition 6.2.** Suppose that \( \tau \in \mathcal{T} \) and \( h_u(z_2(\tau), \tau) > 0 \). For each \( j \in \mathbb{N} \), if
\[
0 < d < \frac{h_u(z_2(\tau), \tau)}{(\delta + \frac{\gamma}{z_2(\tau)})(j\pi)^2},
\]
then...
then (6.1a)-(6.1b) admits at least two solutions \( w^+_j(x;d,\tau) \) and \( w^-_j(x;d,\tau) \) which satisfy
\[
(-1)^{i-1}(w^+_j)'(x;d,\tau) > 0 \quad \text{for any } x \in \left(\frac{i-1}{j}, \frac{i}{j}\right), \quad (i = 1, 2, \ldots, j) \tag{6.3}
\]
and
\[
(-1)^{i-1}(w^-_j)'(x;d,\tau) < 0 \quad \text{for any } x \in \left(\frac{i-1}{j}, \frac{i}{j}\right), \quad (i = 1, 2, \ldots, j). \tag{6.4}
\]

**Proof.** In order to find solutions of the Neumann problem (6.1a)-(6.1b) by the shooting method, we consider the associated initial-value problem
\[
\begin{cases}
dw'' + h(u,\tau) = 0, & x > 0, \\
w(0) = m > 0, & w'(0) = 0,
\end{cases}
\tag{6.5}
\]
where \( u \) is defined by (3.1). In the rising part of the proof, following the standard shooting method, we multiply the differential equation of (6.5) by \( w' \) as follows:
\[
dw'w'' + h(u,\tau)w' = 0. \tag{6.6}
\]
Noting here \( w = \delta u - \gamma \tau / u \), we substitute
\[
w' = \left(\delta + \frac{\gamma \tau}{u^2}\right)u' \tag{6.7}
\]
into the latter \( w \) in (6.6) to get
\[
\left(\frac{d}{2} w'(x)^2 + H(u(x),\tau)\right)' = 0,
\]
where
\[
H(u,\tau) = \int_{z_2(\tau)}^u h(s,\tau) \left(\delta + \frac{\gamma \tau}{s^2}\right) ds. \tag{6.8}
\]
Obviously, for each fixed \( \tau \in T \), the function \( H(u,\tau) \) \((u > 0)\) attains local maximums at \( u = z_1(\tau), z_3(\tau) \) and a local minimum at \( u = z_2(\tau) \). Then we obtain
\[
\frac{d}{2} w'(x)^2 + H(u(x),\tau) = H(m,\tau)
\]
for any \( x \) as long as the solution of (6.5) exists. Hence any solution \( w \) with monotone increasing for small \( x > 0 \) satisfies
\[
w'(x) = \sqrt{\frac{d}{2} \sqrt{H(m,\tau) - H(u,\tau)}}
\]
With (6.7), one can see
\[
u'(x) = \sqrt{\frac{d}{2} \frac{\sqrt{H(m,\tau) - H(u,\tau)}}{\delta + \frac{\gamma \tau}{u^2}}}
\]
Therefore, derivatives of inverse functions give
\[
\frac{dx}{du} = \sqrt{\frac{d}{2} \left(\frac{\delta}{\sqrt{H(m,\tau) - H(u,\tau)}} + \frac{\gamma \tau}{u^2 \sqrt{H(m,\tau) - H(u,\tau)}}\right)} \tag{6.9}
\]
for any \( x \) as long as the solution \( w(x) \) of (6.5) fulfills \( u'(x) > 0 \).

If necessary, we denote by \( w(x,m) \) and \( u(x,m) \) the solution \( w \) of (6.5) and \( u \) defined by (3.1) in order to specify the dependence on \( m \). By (6.7), increase/decrease of \( w(x,m) \) and \( u(x,m) \) matches.
The following shooting argument using the \((u, u')\) phase plane will be divided into two cases (i) and (ii);

(i) \(H(z_1(\tau), \tau) \leq H(z_3(\tau), \tau)\);  
(ii) \(H(z_1(\tau), \tau) > H(z_3(\tau), \tau)\).

In case (i), for any \((z_1(\tau), z_2(\tau))\), there exists \(M(m, \tau) \in (z_2(\tau), z_3(\tau))\) such that

\[ H(m, \tau) = H(M(m, \tau), \tau) \quad \text{and} \quad M(m, \tau) \nearrow z_2(\tau) \]

Here we set

\[ X(m, \tau) := \sup\{ \bar{x} > 0 : u'(x, m) > 0 \text{ for any } x \in (0, \bar{x}) \} \quad (6.10) \]

for \(\tau \in \mathcal{T}\). From (6.9), a standard analysis using the \((u, u')\) phase plane enables us to see that \(X(m, \tau)\) is well-defined and finite if and only if \(m \in (z_1(\tau), z_2(\tau))\). In this case, \(u(x, m)\) is monotone increasing for \(x \in (0, X(m, \tau))\) with \(u(X(m, \tau), \tau) = M(\tau, m)\). Actually, integrating (6.9) by \(u\) over \((m, M(m, \tau))\), we get

\[ X(m, \tau) = \sqrt{\frac{d}{2}} \{ \delta I(m, \tau) + \gamma \tau J(m, \tau) \} \quad (6.11) \]

for any \((m, \tau) \in (z_1(\tau), z_2(\tau)) \times \mathcal{T}\), where

\[ I(m, \tau) := \int_{m}^{M(m, \tau)} \frac{du}{\sqrt{H(m, \tau) - H(u, \tau)}} \]

and

\[ J(m, \tau) := \int_{m}^{M(m, \tau)} \frac{du}{u^2 \sqrt{H(m, \tau) - H(u, \tau)}} \]

By the change of variables \(u = m + (M(m, \tau) - m)\theta\), one can see

\[ I(m, \tau) = (M(m, \tau) - m) \int_{0}^{1} \frac{d\theta}{\sqrt{H(m, \tau) - H(m + (M(m, \tau) - m)\theta, \tau)}} \]

and

\[ J(m, \tau) = (M(m, \tau) - m) \int_{0}^{1} \frac{d\theta}{\{m + (M(m, \tau) - m)\theta\}^2 \sqrt{H(m, \tau) - H(m + (M(m, \tau) - m)\theta, \tau)}} \]

(6.12)

(6.13)

In order to derive the asymptotic behavior of \(I(m, \tau)\) and \(J(m, \tau)\) as \(m \nearrow z_2(\tau)\), we expand \(H(m, \tau)\) and \(H(m + (M(m, \tau) - m)\theta, \tau)\) into Taylor’s series around \(z_2(\tau)\) as follows:

\[ H(m, \tau) = H(m + (M(m, \tau) - m)\theta, \tau) \]

\[ = H(z_2(\tau), \tau) - \{ H(m + (M(m, \tau) - m)\theta, \tau) - H(z_2(\tau), \tau) \} \]

\[ = H_u(z_2(\tau), \tau)(m - z_2(\tau)) + \frac{H_{uu}(z_2(\tau), \tau)}{2}(m - z_2(\tau))^2 + o((m - z_2(\tau))^2) \]

\[ - H_u(z_2(\tau), \tau)\{ m + (M(m, \tau) - m)\theta - z_2(\tau) \} - \frac{H_{uu}(z_2(\tau), \tau)}{2}\{ m + (M(m, \tau) - m)\theta - z_2(\tau) \}^2 \]

\[ + o\{ m + (M(m, \tau) - m)\theta - z_2(\tau) \}^2 \].

Here we recall (6.8) to note

\[ H_u(z_2(\tau), \tau) = 0 \quad \text{and} \quad H_{uu}(z_2(\tau), \tau) \]

\[ = h_u(z_2(\tau), \tau)\left( \delta + \frac{\gamma \tau}{z_2(\tau)^2} \right) \].
Then it follows that
\[
H(m, \tau) - H(m + (M(m, \tau) - m)\theta, \tau) \\
= \frac{h_u(z_2(\tau), \tau)}{2} \left( \frac{\delta + \gamma \tau}{z_2(\tau)^2} \right) (M(m, \tau) - m)^2 \left( \frac{2(z_2(\tau) - m)}{M(m, \tau) - m - \theta} \right) \theta \\
+ o((m - z_2(\tau))^2) + o((m + (M(m, \tau) - m)\theta - z_2(\tau))^2)
\]
\] (6.14)
as \(m \nearrow z_2(\tau)\). Here we shall show
\[
\lim_{m \nearrow z_2(\tau)} \frac{2(z_2(\tau) - m)}{M(m, \tau) - m} = 1.
\] (6.15)
Differentiating \(H(m, \tau) = H(M(m, \tau), \tau)\) by \(m\), we see \(H_u(m, \tau) = H_u(M(m, \tau), \tau)M_m(m, \tau)\) for any \((m, \tau) \in (z_1(\tau), z_2(\tau)) \times T\). By Hôpital's rule, it is easy to check that
\[
\lim_{m \nearrow z_2(\tau)} M_m(m, \tau) = \lim_{m \nearrow z_2(\tau)} H_u(m, \tau) = \lim_{m \nearrow z_2(\tau)} \frac{h_u(m, \tau)}{h_u(M(m, \tau), \tau)M_m(m, \tau)}.
\]
Therefore, we obtain
\[
\lim_{m \nearrow z_2(\tau)} M_m(m, \tau)^2 = \lim_{m \nearrow z_2(\tau)} \frac{h_u(m, \tau)}{h_u(M(m, \tau), \tau)} = 1
\]
because \(h_u(z_2(\tau), \tau) > 0\). By the fact that \(M(m, \tau)\) is monotone decreasing for \(m \in (z_1(\tau), z_2(\tau))\) with each fixed \(\tau \in T\), we know that \(M_m(m, \tau) \to -1\) as \(m \nearrow z_2(\tau)\). Therefore, we obtain (6.15) by using Hôpital’s rule as follows:
\[
\lim_{m \nearrow z_2(\tau)} \frac{2(z_2(\tau) - m)}{M(m, \tau) - m} = \lim_{m \nearrow z_2(\tau)} \frac{-2}{M_m(m, \tau) - 1} = 1.
\]
Substituting (6.14) into (6.12) and (6.13), and then, setting \(m \nearrow z_2(\tau)\), we know from (6.13) that
\[
\lim_{m \nearrow z_2(\tau)} I(m, \tau) = \sqrt{\frac{2}{h_u(z_2(\tau), \tau)(\delta + \frac{\gamma \tau}{z_2(\tau)^2})}} \int_0^1 \frac{d\theta}{\sqrt{\theta(1 - \theta)}} = \sqrt{\frac{2}{h_u(z_2(\tau), \tau)(\delta + \frac{\gamma \tau}{z_2(\tau)^2})}} \pi
\]
and
\[
\lim_{m \nearrow z_2(\tau)} J(m, \tau) = \frac{1}{z_2(\tau)^2} \sqrt{\frac{2}{h_u(z_2(\tau), \tau)(\delta + \frac{\gamma \tau}{z_2(\tau)^2})}} \pi.
\]
Consequently, we set \(m \nearrow z_2(\tau)\) in (6.11) to get
\[
\lim_{m \nearrow z_2(\tau)} X(m, \tau) = \sqrt{\frac{d(\delta + \frac{\gamma \tau}{z_2(\tau)^2})}{h_u(z_2(\tau), \tau)}} \pi.
\] (6.16)
In case (i), for the derivation of the asymptotic behavior of \(X(m, \tau)\) as \(m \searrow z_1(\tau)\), we use the Taylor expansion of \(\theta \mapsto H(m + (M(m, \tau) - m)\theta, \tau)\) around \(\theta = 0\) to observe
\[
H(m, \tau) - H(m + (M(m, \tau) - m)\theta, \tau) \\
= -h(m, \tau) \left( \delta + \frac{\gamma \tau}{m^2} \right) (M(m, \tau) - m) \theta \\
- \frac{1}{2} \left[ h_u(m, \tau) \left( \delta + \frac{\gamma \tau}{m^2} \right) - h(m, \tau) \frac{2\gamma \tau}{m^3} \right] (M(m, \tau) - m)^2 \theta^2 + o((M(m, \tau) - m)^2 \theta^2)
\]
and
for each \( m \in (z_1(\tau), z_2(\tau)) \) as \( \theta \searrow 0 \). Here we assume \( h_u(z_1(\tau), \tau) < 0 \). Then for any \( \tau \in T \), there exist \( k_i(m, \tau) > 0 \) (\( i = 1, 2 \)) with \( \lim_{m \searrow z_1(\tau)} k_1(m, \tau) = 0 \) and \( \lim_{m \searrow z_1(\tau)} k_2(m, \tau) > 0 \) such that if \( \theta > 0 \) is sufficiently small and \( m \in (z_1(\tau), z_2(\tau)) \) is sufficiently close to \( z_1(\tau) \), then
\[
H(m, \tau) - H(m + (M(m, \tau) - m)\theta, \tau) \leq k_1(m, \tau)\theta + k_2(m, \tau)\theta^2.
\]
Therefore, there exists a small \( \epsilon > 0 \) such that
\[
I(m, \tau) \geq (M(m, \tau) - m) \int_0^\epsilon \frac{d\theta}{\sqrt{k_1(m, \tau)\theta + k_2(m, \tau)\theta^2}}
\]
and
\[
J(m, \tau) \geq \frac{M(m, \tau) - m}{z_2(\tau)^2} \int_0^\epsilon \frac{d\theta}{\sqrt{k_1(m, \tau)\theta + k_2(m, \tau)\theta^2}}
\]
if \( m \in (z_1(\tau), z_2(\tau)) \) is sufficiently close to \( z_1(\tau) \). Consequently, we know from (6.11) that, in case (i),
\[
\lim_{m \searrow z_1(\tau)} X(m, \tau) = \infty.
\]
It is easy to check that (6.17) holds true even when \( h_u(z_1(\tau), \tau) = 0 \) by observing the higher order expansion of \( \theta \mapsto H(m + (M(m, \tau) - m)\theta, \tau) \).

Next we consider the other case (ii); \( H(z_1(\tau), \tau) > H(z_3(\tau), \tau) \). For such \( \tau \in T \), there exists a unique \( m(\tau) \in (z_1(\tau), z_2(\tau)) \) such that \( H(m(\tau), \tau) = H(z_3(\tau), \tau) \). In case (ii), \( X(m, \tau) \) in (6.10) is well-defined and finite if and only if \( m \in (m(\tau), z_2(\tau)) \), and moreover, it is represented as (6.11). By a similar manner as in case (i), one can verify that
\[
\lim_{m \searrow z_1(\tau)} X(m, \tau) = \infty \quad \text{and} \quad \lim_{m \nearrow z_2(\tau)} X(m, \tau) = \frac{\sqrt{d(\delta + \gamma z_2(\tau)^2)/h_u(z_2(\tau), \tau)}}{\pi}.
\]
in case (ii).

Therefore, in both cases (i) and (ii), the algebraic equation \( X(m, \tau) = 1/j \) admits at least one root \( m = m_j(\tau) \in (z_1(\tau), z_2(\tau)) \) or \( (m(\tau), z_2(\tau)) \) provided
\[
\lim_{m \nearrow z_2(\tau)} X(m, \tau) = \frac{\sqrt{d(\delta + \gamma z_2(\tau)^2)/h_u(z_2(\tau), \tau)}}{\pi} < 1/j;
\]
that is, \( 0 < d < \frac{h_u(z_2(\tau), \tau)}{(\delta + \gamma z_2(\tau)^2)(j\pi)^2} \).

For such \( d \), the solution \( w(x, m_j(\tau)) \) of the initial-value problem (6.5) satisfies \( w'(x, m_j(\tau)) > 0 \) for \( x \in (0, 1/j) \); \( w'(1/j, m_j(\tau)) = 0 \); \( w'(x, m_j(\tau)) < 0 \) for \( x \in (1/j, 2/j) \); \( w(2/j, m_j(\tau)) = w(0, m_j(\tau)) \); \( w'(2/j, m_j(\tau)) = 0 \), and moreover, oscillates periodically for \( x > 0 \). Then
\[
w_j^+(x; d, \tau) := w(x, m_j(\tau))
\]
becomes a solution of the Neumann problem (6.1a)-(6.1b) and it satisfies (6.3). Furthermore, \( w_j^-(x; d, \tau) := w(x + 1/j, m_j(\tau)) \) is also a solution of (6.1a)-(6.1b) and satisfies (6.3). The proof of Proposition 6.2 is complete.

**Remark 6.3.** It is possible to prove that if \( h_u(z_2(\tau), \tau) = 0 \), then \( \lim_{m \searrow z_2(\tau)} X(m, \tau) = \infty \). Together with \( \lim_{m \searrow z_1(\tau)} X(m, \tau) = \infty \) or \( \lim_{m \searrow \infty} X(m, \tau) = \infty \), for each \( j \in \mathbb{N} \), there exists \( \tilde{d}(j) > 0 \) such that if \( d \in (0, \tilde{d}(j)) \), then (6.1a)-(6.1b) has at least four solutions \( \overline{w}_j^+(x; d, \tau), \overline{w}_j^-(x; d, \tau), \overline{w}_j^+(x; d, \tau) \) and \( \overline{w}_j^-(x; d, \tau) \), where \( \overline{w}_j^+(x; d, \tau) \) and \( \overline{w}_j^-(x; d, \tau) \) satisfy (6.3); \( \overline{w}^+_j(x; d, \tau) \) and \( \overline{w}^-_j(x; d, \tau) \) satisfy (6.4).

**Remark 6.4.** If \( \tau \not\in T \), then (6.1) does not admit any nonconstant solution. Actually, a standard phase plane analysis implies that any solution of (6.5) cannot satisfy \( w'(0) = 0 \) in case the number of zeros of \( h(u, \tau) \) is less than three.
Concerning the Neumann problem \((6.1a)-(6.1b)\) (without \((6.1c)\)), we discuss the singular limit as \(d \searrow 0\) of solutions obtained in Proposition \(6.2\). It is possible to verify that, for \(n \geq 2\), each \(w_n^+(x; d, \tau)\) can be constructed by connecting suitable rescaled or reflected pieces of \(w_1^+(x; d, \tau)\). Then we study the singular limit as \(d \searrow 0\) of \(w_1^+(x; d, \tau)\). Hereafter we use another notation \(\tilde{h}(w, \tau)\) of the nonlinear term of \((6.1a)\) by substituting \(u(w, \tau)\) defined by \((3.1)\) into \(h(u, \tau)\) such as \(\tilde{h}(w, \tau) := h(u(w, \tau), \tau)\). Then \((6.1a)-(6.1b)\) can be represented as

\[
\begin{aligned}
dw'' + \tilde{h}(w, \tau) &= 0 \quad \text{in } (0, 1), \quad \tau > 0, \\
w'(0) = w'(1) &= 0.
\end{aligned}
\]  

(6.18)

In view of Lemma 6.1, we define \(\xi_i(\tau) (i = 1, 2, 3)\) by

\[
\begin{equation}
z_i(\tau) = u(\xi_i(\tau), \tau), \quad \text{conversely,} \quad \xi_i(\tau) := \delta z_i(\tau) - \frac{\gamma \tau}{z_i(\tau)}.
\end{equation}
\]

From the monotone increasing relation of \(w \mapsto u(w, \tau)\) by \((6.7)\), we know from Lemma 6.1 that if \(\tau \in \mathcal{T}\), then

\[
\tilde{h}(w, \tau) \begin{cases} 
> 0 & \text{for } w \in (-\infty, \xi_1(\tau)) \cup (\xi_2(\tau), \xi_3(\tau)), \\
< 0 & \text{for } w \in (\xi_1(\tau), \xi_2(\tau)) \cup (\xi_3(\tau), \infty).
\end{cases}
\]

For such a bistable nonlinear term, we set

\[
\tilde{H}(w, \tau) := \int_{\xi_2(\tau)}^{w} \tilde{h}(s, \tau) ds.
\]  

(6.19)

Concerning the Neumann problem of ordinary differential equations with a class of bistable nonlinearities such as \(\tilde{h}(w, \tau)\), it is well known that the singular limit of solutions as \(d \searrow 0\) crucially depends on the sign of \(\tilde{H}(\xi_3(\tau), \tau) - \tilde{H}(\xi_1(\tau), \tau)\) as follows (see e.g., Nishiura [19, Lemma 3.1], Shi [22, Proposition 2.6]):

(i) If \(\tilde{H}(\xi_1(\tau), \tau) < \tilde{H}(\xi_3(\tau), \tau)\), then

\[
\lim_{d \searrow 0} w_1^+(x; d, \tau) = \begin{cases} 
\xi_1(\tau) & \text{for } x \in [0, 1), \\
\tilde{\eta}(\tau) & \text{for } x = 1,
\end{cases}
\]

where \(\tilde{\eta}(\tau) \in (\xi_2(\tau), \xi_3(\tau))\) is defined by \(\int_{\xi_1(\tau)}^{\tilde{\eta}(\tau)} \tilde{h}(s, \tau) ds = 0\).

(ii) If \(\tilde{H}(\xi_1(\tau), \tau) = \tilde{H}(\xi_3(\tau), \tau)\), then

\[
\lim_{d \searrow 0} w_1^+(x; d, \tau) = \begin{cases} 
\xi_1(\tau) & \text{for } x \in [0, 1/2), \\
(\xi_1(\tau) + \xi_3(\tau))/2 & \text{for } x = 1/2, \\
\xi_3(\tau) & \text{for } x \in (1/2, 1].
\end{cases}
\]

(iii) If \(\tilde{H}(\xi_1(\tau), \tau) > \tilde{H}(\xi_3(\tau), \tau)\), then

\[
\lim_{d \searrow 0} w_1^+(x; d, \tau) = \begin{cases} 
\tilde{\zeta}(\tau) & \text{for } x = 0, \\
\xi_3(\tau) & \text{for } x \in (0, 1],
\end{cases}
\]

where \(\tilde{\zeta}(\tau) \in (\xi_1(\tau), \xi_2(\tau))\) is defined by \(\int_{\xi_1(\tau)}^{\tilde{\zeta}(\tau)} \tilde{h}(s, \tau) ds = 0\).

Here we note that the change of variables \(w = \delta u - \gamma \tau / u\) links \((6.18)\) with \((6.19)\) in the sense of \(\tilde{H}(w, \tau) = H(u, \tau)\). Then by the change of variables, the above (i)-(iii) give the following
Lemma 6.6. Suppose that \( \tau \in \mathcal{T} \). The function \( u_1^+(x; d, \tau) \) satisfies either one of the following (i)-(iii) depending on the sign of \( u > 0 \):

(i) If \( H(z_1(\tau), \tau) < H(z_3(\tau), \tau) \), then

\[
\lim_{d \searrow 0} u_1^+(x; d, \tau) = \begin{cases} 
z_1(\tau) & \text{for } x \in [0, 1), \\
\eta(\tau) & \text{for } x = 1,
\end{cases}
\]

where \( \eta(\tau) \in (z_2(\tau), z_3(\tau)) \) is defined by \( \int_{z_1(\tau)}^{z(\tau)} h(s, \tau)(\delta + \frac{\eta + \tau}{\eta}) ds = 0 \).

(ii) If \( H(z_1(\tau), \tau) = H(z_3(\tau), \tau) \), then

\[
\lim_{d \searrow 0} u_1^+(x; d, \tau) = \begin{cases} 
z_1(\tau) & \text{for } x \in [0, 1/2), \\
(z_1(\tau) + z_3(\tau))/2 & \text{for } x = 1/2, \\
z_3(\tau) & \text{for } x \in (1/2, 1].
\end{cases}
\]

(iii) If \( H(z_1(\tau), \tau) > H(z_3(\tau), \tau) \), then

\[
\lim_{d \searrow 0} u_1^+(x; d, \tau) = \begin{cases} 
\z(\tau) & \text{for } x = 0, \\
z_3(\tau) & \text{for } x \in (0, 1],
\end{cases}
\]

where \( \z(\tau) \in (z_1(\tau), z_2(\tau)) \) is defined by \( \int_{z_1(\tau)}^{z_3(\tau)} h(s, \tau)(\delta + \frac{\tau + \z}{\tau}) ds = 0 \).

Our next task is to construct the set of nonconstant solutions of (6.1) by choosing functions in the set of solutions of (6.1a)-(6.1b) to match (6.1c). The following lemma will be useful to catch up with the global bifurcation branch of nonconstant solutions of (6.1).

Lemma 6.6. Suppose that \( \tau > 0 \) is sufficiently small. Then the following properties hold:

(i) If \( B < A < C \), then

\[
f\left(z_1(\tau), \frac{\tau}{z_1(\tau)}\right) < 0, \quad f\left(z_2(\tau), \frac{\tau}{z_2(\tau)}\right) > 0 \quad \text{and} \quad f\left(z_3(\tau), \frac{\tau}{z_3(\tau)}\right) < 0.
\]

(ii) If \( C < A < B \), then

\[
f\left(z_1(\tau), \frac{\tau}{z_1(\tau)}\right) > 0, \quad f\left(z_2(\tau), \frac{\tau}{z_2(\tau)}\right) > 0 \quad \text{and} \quad f\left(z_3(\tau), \frac{\tau}{z_3(\tau)}\right) > 0.
\]

Proof. In view of (4.3), we note that three zeros of \( h(u, \tau) \) are corresponding to three intersections of \( u \mapsto f(u, \tau/u) \) and \( u \mapsto g(u, \tau/u) \) on \( \{ u > 0 \} \). Thus we summarize the profiles of

\[
f\left(u, \frac{\tau}{u}\right) = u(a_1 - b_1u) - c_1\tau \quad \text{and} \quad g\left(u, \frac{\tau}{u}\right) = \frac{\tau}{u}(a_2 - \frac{c_2\tau}{u}) - b_2\tau
\]

for \( u > 0 \) with each fixed small \( \tau > 0 \). It is noted that

\[
\lim_{u \searrow 0} f\left(u, \frac{\tau}{u}\right) = -c_1\tau, \quad \lim_{u \rightarrow \infty} f\left(u, \frac{\tau}{u}\right) = -\infty,
\]

\[
\lim_{u \searrow 0} g\left(u, \frac{\tau}{u}\right) = -\infty, \quad \lim_{u \rightarrow \infty} g\left(u, \frac{\tau}{u}\right) = -b_2\tau.
\]
Clearly, \( u \mapsto f(u, \tau/u) \) has two zeros
\[
0 < Z_1(f, \tau) := \frac{a_1 - \sqrt{a_1^2 - 4b_1c_1\tau}}{2b_1} < Z_2(f, \tau) := \frac{a_1 + \sqrt{a_1^2 - 4b_1c_1\tau}}{2b_1}
\]
if \( \tau \in (0, a_1^2/4b_1c_1) \), whereas \( u \mapsto g(u, \tau/u) \) has two zeros
\[
0 < Z_1(g, \tau) := \frac{a_2 - \sqrt{a_2^2 - 4b_2c_2\tau}}{2b_2} < Z_2(g, \tau) := \frac{a_2 + \sqrt{a_2^2 - 4b_2c_2\tau}}{2b_2}
\]
if \( \tau \in (0, a_2^2/4b_2c_2) \). It follows from
\[
\lim_{\tau \searrow 0} \frac{Z_1(f, \tau)}{\tau} = \frac{c_1}{a_1} \quad \text{and} \quad \lim_{\tau \searrow 0} \frac{Z_1(g, \tau)}{\tau} = \frac{c_2}{a_2}
\]
that
\[
\begin{cases}
Z_1(g, \tau) < Z_1(f, \tau) & \text{if } A < C, \\
Z_1(f, \tau) < Z_1(g, \tau) & \text{if } C < A.
\end{cases}
\]
(6.21)
for sufficiently small \( \tau > 0 \). Furthermore, it follows from
\[
\lim_{\tau \searrow 0} Z_2(f, \tau) = \frac{a_1}{b_1} \quad \text{and} \quad \lim_{\tau \searrow 0} Z_2(g, \tau) = \frac{a_2}{b_2}
\]
that
\[
\begin{cases}
(Z_1(f, \tau) < Z_2(f, \tau) \text{ if } B < A, \\
(Z_1(g, \tau) < Z_2(g, \tau) \text{ if } A < B.
\end{cases}
\]
(6.22)
for sufficiently small \( \tau > 0 \). Therefore, we know from (6.21) and (6.22) that
\[
\begin{cases}
Z_1(g, \tau) < Z_1(f, \tau) < Z_2(g, \tau) < Z_2(f, \tau) & \text{if } B < A < C, \\
Z_1(f, \tau) < Z_1(g, \tau) < Z_2(f, \tau) < Z_2(g, \tau) & \text{if } C < A < B
\end{cases}
\]
(6.23)
for sufficiently small \( \tau > 0 \).

In the strong competition case \( B < A < C \), we know from (4.3) and (6.23) that three zeros \( z_j(\tau) (j = 1, 2, 3) \) of \( h(u, \tau) \) are located as
\[
0 < z_1(\tau) < Z_1(g, \tau) < Z_1(f, \tau) < z_2(\tau) < Z_2(g, \tau) < Z_2(f, \tau) < z_3(\tau)
\]
if \( \tau > 0 \) is sufficiently small. Hence the intermediate value theorem yields the assertion (i).

Next we consider the weak competition case \( C < A < B \). By virtue of (6.23), the intermediate value theorem ensures that the number of zeros of \( h(u, \tau) \) in \( u \in (Z_1(g, \tau), Z_2(f, \tau)) \) is one or three if \( \tau > 0 \) is sufficiently small. Here we recall Lemma 6.1 to note
\[
\frac{1}{\tau} f\left( z_1(\tau), \frac{\tau}{z_1(\tau)} \right) = \frac{z_1(\tau)}{\tau} \left( a_1 - b_1z_1(\tau) \right) - c_1 \rightarrow \frac{c_2}{a_2}a_1 - c_1 = c_2(A - C) > 0,
\]
\[
\frac{1}{\sqrt{\tau}} f\left( z_2(\tau), \frac{\tau}{z_2(\tau)} \right) = \frac{z_2(\tau)}{\sqrt{\tau}} \left( a_1 - b_1z_2(\tau) \right) - c_1 \sqrt{\tau} \rightarrow \sqrt{\gamma a_1a_2} > 0
\]
as \( \tau \searrow 0 \). Then \( f(z_i(\tau), \tau/z_i(\tau)) > 0 \) for \( i = 1, 2 \) if \( \tau > 0 \) is sufficiently small. Hence the number of zeros of \( h(u, \tau) \) in \( u \in (Z_1(g, \tau), Z_2(f, \tau)) \) is three, thereby, \( f(z_i(\tau), \tau/z_i(\tau)) > 0 \) for \( i = 1, 2, 3 \). Consequently, the assertion (ii) follows.

The next lemma gives infinitely many pieces of local bifurcation curves that bifurcate from \((d^{(j)}, u^*, \tau^*)\) for every \( j \in \mathbb{N} \), where \( d^{(j)} \) is the positive number defined by (5.24).
Lemma 6.7. Suppose that $B < A < C$ or $C < A < B$. Suppose further that $D(a_i, b_i, c_i, \gamma) > 0$. For each $j \in \mathbb{N}$, there exists a local curve

$$\Gamma_{j,\varepsilon} = \{ (d(s), u(s), \tau(s)) : -\varepsilon < s < \varepsilon \} \subset \mathbb{R}_+ \times X$$

such that

$$\Gamma_{j,\varepsilon}^+ := \{ (d(s), u(s), \tau(s)) : 0 < s < \varepsilon \} \subset S_j^+,$$

$$\Gamma_{j,\varepsilon}^- := \{ (d(s), u(s), \tau(s)) : -\varepsilon < s < 0 \} \subset S_j^-,$$

$$\lim_{s \to 0^+} (d(s), u(s), \tau(s)) = (d(j), u^*, \tau^*) \text{ in } \mathbb{R} \times X.$$

Proof. By a similar manner as the proof of [3] Theorem 4.1, we can construct the required local curve $\Gamma_{j,\varepsilon}$ which forms a piece of a bifurcation curve of solutions of (6.1) bifurcating from the pitchfork bifurcation point $(d(j), u^*, \tau^*)$.

Actually, in view of the proof of Lemma 6.2, one can recall that the operator $F(d, u, \tau)$ in (6.1) associated with (6.1) is degenerate, in the sense that the operator $I - L(d)$ has a zero eigenvalue, if and only if $d = d(j)$ with some $j \in \mathbb{N}$. Additional conditions for use of the local bifurcation theorem [2] Theorem 1.7 by Crandall and Rabinowitz can be verified by a similar argument to the proof of [3] Theorem 4.1. \qed

By virtue of Lemma 6.7, we set

$$\overline{T} := \sup \{ T : h(u, \tau) \text{ has three zeros } 0 < z_1(\tau) < z_2(\tau) < z_3(\tau) \text{ for any } \tau \in (0, T) \},$$

and

$$\overline{\tau} = \min \{ \overline{T}, \overline{\tau} \},$$

where $\overline{\tau}$ is defined by Theorem 3.1.

Lemma 6.8. Suppose that $B < A < C$ or $C < A < B$. Suppose further that $D(a_i, b_i, c_i, \gamma) > 0$. Then it holds that $\tau^* \in \mathcal{T}$ and $u^* = z_2(\tau^*)$. Furthermore, the following (i) and (ii) hold true:

(i) If $B < A < C$ and $D(a_i, b_i, c_i, \gamma) > 0$, then $\tau^* \in (0, \overline{\tau})$.

(ii) If $C < A < B$ and $D(a_i, b_i, c_i, \gamma) > 0$, then $\tau^* \notin (0, \overline{\tau})$.

Remark 6.9. By virtue of $\tau^* \in \mathcal{T}$, the assertion (ii) of Lemma 6.8 implies that $\mathcal{T}$ is not connected in case where $C < A < B$ and $D(a_i, b_i, c_i, \gamma) > 0$. An example of profiles of $h(u, \tau)$ in case $B < A < C$ and $D(a_i, b_i, c_i, \gamma) > 0$ is shown in Figure 2. In the same setting as [11] Figure 11 for the case $C < A < C$ and $D(a_i, b_i, c_i, \gamma) > 0$, profiles of $h(u, \tau)$ are shown in Figure 3. In view of Figure 3, one can see that $\mathcal{T}$ is not connected in this case.

Proof of Lemma 6.8. Suppose that $B < A < C$ or $C < A < B$. We recall that (6.1) has a unique positive constant solution $(u, \tau) = (u^*, \tau^*)$ ($w^* = \delta u^* - \gamma \tau^*/u^*$) for any $d > 0$, that is,

$$f(u^*, v^*) = g(u^*, v^*) = 0 \text{ with } v^* = \frac{\tau^*}{u^*}.$$ 

By a straightforward calculation, one can verify that $D(a_i, b_i, c_i, \gamma) > 0$ is equivalent to

$$h_u(u^*, \tau^*) > 0.$$ 

Taking account for facts that $\lim_{u \to 0} h(u, \tau) = \infty$, $h(u^*, \tau^*) = 0$ and $\lim_{u \to \infty} h(u, \tau) = -\infty$, we see that

$$u^* = z_2(\tau^*),$$

thereby,

$$f\left(z_2(\tau^*), \frac{\tau^*}{z_2(\tau^*)}\right) = g\left(z_2(\tau^*), \frac{\tau^*}{z_2(\tau^*)}\right) = 0.$$ 

(6.24)

(6.25)
Figure 2. Profiles of $h(u, \tau)$ with $(a_1, a_2, b_1, b_2, c_1, c_2, \gamma) = (1, 1, 1, 2, 1, 1)$

(a) $\tau = \tau^* = 1/9$
(b) $\tau = 1/3$

Figure 3. Profiles of $h(u, \tau)$ with $(a_1, a_2, b_1, b_2, c_1, c_2, \gamma) = (15/2, 16/7, 4, 1, 6, 2, 1)$

(a) $\tau = 1/20$
(b) $\tau = 1/3$
(c) $\tau = \tau^* = 207/392$
(d) $\tau = 1$
Furthermore, we obtain $\tau^* \in \mathcal{T}$. Actually, if $\tau^* \notin \mathcal{T}$, then $\tau^* > \tau$ because $h(u, \tau^*)$ has three zeros on $\{u > 0\}$. However, (4.11) and (4.12) ensure $\tau^* \leq \tau$. This is a contradiction.

In order to prove the assertion (i), we assume that $B < A < C$ and $D(a_i, b_i, c_i, \gamma) > 0$. In view of (i) of Lemma 6.6, we recall that

$$f\left(\frac{z_1(\tau)}{z_1^*(\tau)}\right) < 0 \quad \text{and} \quad f\left(\frac{z_3(\tau)}{z_3^*(\tau)}\right) < 0$$

and

$$f\left(\frac{z_2(\tau)}{z_2^*(\tau)}\right) > 0$$

for sufficiently small $\tau > 0$. We shall show that (6.26) holds true for any $\tau \in (0, \tilde{\tau})$. Suppose for contradiction that there exists $\tau_0 \in (0, \tilde{\tau})$ such that

$$f\left(\frac{z_{i_0}(\tau_0)}{z_{i_0}(\tau_0)}\right) = 0 \quad \text{with some } i_0 \in \{1, 3\}.$$  

Furthermore, $h(z_{i_0}(\tau_0), \tau_0) = 0$ implies

$$f\left(\frac{z_{i_0}(\tau_0)}{z_{i_0}(\tau_0)}\right) = g\left(\frac{\tau_0}{z_{i_0}(\tau_0)}\right) = 0.$$  

Hence it follows that $(z_{i_0}(\tau_0), \tau_0) = (u^*, \tau^*)$. Together with (6.24), we see that $z_2(\tau^*) = z_{i_0}(\tau^*)$, thereby, the degeneracy of zeros of $h(u, \tau)$ occurs at $\tau = \tau^*$. However, this contradicts the fact $\tau^* \in \mathcal{T}$. Therefore, by taking account for the uniqueness of positive roots of $f(u, \tau/u) = g(u, \tau/u) = 0$, we know from (6.25) and (6.27) that

$$f\left(\frac{z_2(\tau)}{z_2(\tau)}\right) \begin{cases} > 0 & \text{for } \tau \in (0, \tau^*), \\ = 0 & \text{for } \tau = \tau^*, \\ < 0 & \text{for } \tau \in (\tau^*, \infty) \cap \mathcal{T}, \end{cases}$$

and moreover, (6.26) holds true for any $\tau \in (0, \tilde{\tau})$. Together with $\tau^* < \tilde{\tau}$, we obtain $\tau^* < \tilde{\tau}$.

For the proof of the assertion (ii), we assume that $C < A < B$ and $D(a_i, b_i, c_i, \gamma) > 0$. From (ii) of Lemma 6.6 we recall that

$$f\left(\frac{z_1(\tau)}{z_1(\tau)}\right) > 0, \quad f\left(\frac{z_2(\tau)}{z_2(\tau)}\right) > 0, \quad f\left(\frac{z_3(\tau)}{z_3(\tau)}\right) > 0$$

for sufficiently small $\tau > 0$. Suppose for contradiction that $\tau^* \in (0, \tilde{\tau})$. Then by (6.25) and the continuity of $\tau \mapsto f(z_1(\tau), \tau / z_1(\tau))$, there exists $\tau_0 \in (0, \tau^*)$ satisfying (6.28). As in the argument above, we are led to $z_2(\tau^*) = z_{i_0}(\tau^*)$ with some $i_0 \in \{1, 3\}$. Again this contradicts the fact $\tau^* \in \mathcal{T}$. Therefore, we can deduce that $\tau^* \not\in (0, \tilde{\tau})$ in case that $C < A < B$ and $D(a_i, b_i, c_i, \gamma) > 0$. The proof of Lemma 6.8 is complete. \qed

**Proof of Theorem 3.3.** In what follows, we denote by $\Gamma^\pm_j$ the connected component of $\{(d, u, \tau) \in \mathbb{R}_+ \times X : (d, u, \tau) \text{ satisfies } (6.11)\}$ which contains $\Gamma^\pm_{j, \pm}$ obtained in Lemma 6.7. We recall Remark 6.4 to note that any $(d, u, \tau) \in \Gamma^\pm_j$ satisfies $\tau \in \mathcal{T}$.

We first show $\Gamma^+_1 \subset S_1^+$, that is, what $\Gamma^+_1$ does not connects with any other $\Gamma^\pm_j$. If not, there exist a solution $(\hat{d}, \hat{u}, \hat{\tau})$ (not $(d(1), u^*, \tau^*)$) of (6.1) and a sequence $\{(d_{2,n}, u_n, \tau_n)\} \subset \Gamma^+_1$ such that $\lim_{n \to \infty}(d_{2,n}, u_n, \tau_n) = (\hat{d}, \hat{u}, \hat{\tau})$ in $\mathbb{R} \times X$ and $\hat{u}(x)$ has a degenerate critical point,
that is, \( \hat{u}''(x_0) = \hat{u}'(x_0) = 0 \) with some \( x_0 \in [0,1] \). Differentiating (6.1b) by \( x \), we see that
\[
\hat{w} := \delta \hat{u} - \gamma \hat{\tau} / \hat{u}
\]
satisfies
\[
\begin{align*}
&\hat{d}(\hat{w})'' + h_u(\hat{u}, \hat{\tau})\hat{u}' = 0, \quad 0 < x < 1, \\
&\hat{w}'(x_0) = \hat{w}'(x_0) = 0,
\end{align*}
\]
where the initial condition at \( x_0 \) comes from (6.4). By the uniqueness of solutions of this initial value problem, one can see that \( \hat{w}' = 0 \), thereby, \( \hat{u} \) is a nonnegative constant. If \( \hat{\tau} > 0 \), then \( (\hat{u}, \hat{\tau}) = (u^*, \tau^*) \). In view of Lemma 6.7, we recall that the branch of monotone solutions of (6.1) bifurcates from the constant solution \( (u^*, \tau^*) \) only at \( d = d^{(1)} \). This fact implies \( \hat{d} = d^{(1)} \). However, it contradicts the assumption. If \( \hat{\tau} = 0 \), then (6.18) ensures that \( w_n = \delta u_n - \gamma \tau_n / u_n \to \hat{w} \) in \( C^1(\bar{\Omega}) \) and \( (\hat{d}, \hat{w}) \) satisfies (1.3). Then, Proposition 2.2 leads to \( \hat{w} = \delta a_1/b_1 \) or \( \hat{w} = 0 \) or \( \hat{w} = -\gamma a_2/c_2 \). However, one can verify that all of them are impossible following the argument below (5.2) in the proof of Lemma 5.1. Consequently, \( \hat{\tau} = 0 \) is also impossible. Therefore, we obtain \( \Gamma \_1^+ \subset S^+_j \) and \( \Gamma \_1^- \subset S^-_j \) for each \( j \in \mathbb{N} \).

According to the global bifurcation theorem [21] by Rabinowitz (see also the unilateral global bifurcation theorem by Lopéz-Goméz [11, Theorem 6.4.3]), we can deduce that \( \Gamma \_1^+ \) reaches a singular limit \( d \searrow 0 \) or a state with some \( d > 0 \) and \( \tau = 0 \) because \( \Gamma \_1^+ \) cannot attain any bifurcation point \( (d^{(j)}, u^*, \tau^*) \), the \( (d, \tau) \) component of \( \Gamma \_1^+ \) is uniformly bounded by Theorem 4.1 and the \( u \) component of \( \Gamma \_1^+ \) cannot blow up in \( C^1([0,1]) \) at any positive \( d > 0 \) by Corollary 4.2. Then, we take any sequence \( \{(u_n, \tau_n, d_{2,n})\} \subset \Gamma \_1^+ \) satisfying \( \lim_{n \to \infty} d_{2,n} \tau_n = 0 \). It follows from Theorem 3.1 that \( \tau_n \in (0, \bar{\tau}) \) for any \( n \in \mathbb{N} \). Then we may assume that \( \lim_{n \to \infty} \tau_n = \tau_0 \) for some \( \tau_0 \in (0, \bar{\tau}) \) by passing to a subsequence if necessary. We shall show that either of the following is true:

(I) \( \tau_0 = 0 \);

(II) \( \tau_0 > 0 \) and \( H(z_1(\tau_0), \tau_0) = H(z_3(\tau_0), \tau_0) \).

Suppose for contradiction that \( \tau_0 > 0 \) and \( H(z_1(\tau_0), \tau_0) < H(z_3(\tau_0), \tau_0) \). Hence it follows that \( \lim_{n \to \infty} d_{2,n} = 0 \). Then we can verify
\[
\lim_{n \to \infty} u_n(x) = \begin{cases} z_1(\tau_0) & \text{for } x \in [0,1), \\
\eta(\tau_0) & \text{for } x = 1
\end{cases}
\]
by a slight modification of the usual scaling procedure to prove (i) of Lemma 6.5. Here we remark that \( \{u_n\} \) satisfies the integral constraint (6.1c) as well as (6.2):
\[
\int_0^1 f \left( u_n, \frac{\tau_n}{u_n} \right) = \int_0^1 g \left( u_n, \frac{\tau_n}{u_n} \right) = 0 \quad \text{for any } n \in \mathbb{N}.
\]
By the Lebesgue dominated convergence theorem, we set \( n \to \infty \) in (6.29) to get
\[
\left( z_1(\tau_0), \frac{\tau_0}{z_1(\tau_0)} \right) = g \left( z_1(\tau_0), \frac{\tau_0}{z_1(\tau_0)} \right) = 0.
\]
Therefore, the positivity of \( \tau_0 \) leads to \( (z_1(\tau_0), \tau_0) = (u^*, \tau^*) \). Together with Lemma 6.8, we see that \( z_1(\tau^*) = u^* = z_2(\tau^*) \). Obviously, this contradicts the fact that \( \tau^* \in \mathcal{T} \).

Suppose for contradiction that \( \tau_0 > 0 \) and \( H(z_1(\tau_0), \tau_0) > H(z_3(\tau_0), \tau_0) \). Then we know from (iii) of Lemma 6.5 that
\[
\lim_{k \to \infty} u_k(x) = \begin{cases} \zeta(\tau_0) & \text{for } x = 0, \\
z_3(\tau_0) & \text{for } x \in (0,1].
\end{cases}
\]
In a similar manner, we set \( n \to \infty \) in (5.24) to see \( z_3(\tau^*) = u^* = z_2(\tau^*) \). However, it is impossible that \( \tau^* \in \mathcal{T} \). Consequently, the contradiction argument enables us to conclude that (I) or (II) holds true.

Next we shall show \( \tau_0 = 0 \) in the case when \( B < A < C \) and \( D(a_i, b_i, c_i, \gamma) > 0 \). It follows from (i) of Lemma 6.8 that \( \tau^* \in (0, \bar{\tau}) \). By virtue of Lemma 6.7, if \( (d, u, \tau) \in \Gamma_j^\pm \) is sufficiently close to the bifurcation point \( (d^j, u^*, \tau^*) \), then \( \tau \in (0, \bar{\tau}) \). By taking account for the continuity of \( \Gamma_j^\pm \), we see that \( \tau \in \mathcal{T} \) and \( \tau \in (0, \bar{\tau}) \) as long as \( (d, u, \tau) \in \Gamma_j^\pm \). Hence it follows that \( \tau_0 \in [0, \bar{\tau}] \). Suppose for contradiction that \( \tau_0 > 0 \). Then, as we have already shown, \( H(z_1(\tau_0), \tau_0) = H(z_3(\tau_0), \tau_0) \) follows. In such a case, we remark that a slight delicate procedure is required to derive the singular limiting behavior of \( \{u_n\} \) as \( n \to \infty \). Following the argument in the proof of [9] Proposition 6.7, for instance, we can deduce that

\[
\lim_{n \to \infty} u_n(x) = \begin{cases} 
 z_1(\tau) & \text{for } x \in [0, \ell), \\
 (z_1(\tau) + z_3(\tau))/2 & \text{for } x = \ell, \\
 z_3(\tau) & \text{for } x \in (\ell, 1] 
\end{cases}
\]

with some \( \ell \in (0, 1) \). Owing to the Lebesgue dominated convergence theorem, we set \( n \to \infty \) in \( \int_0^1 f(u_n, \tau_n/u_n) = 0 \) to obtain

\[
\ell f\left(z_1(\tau_0), \frac{\tau_0}{z_1(\tau_0)}\right) + (1-\ell)f\left(z_3(\tau_0), \frac{\tau_0}{z_3(\tau_0)}\right) = 0, \quad (6.30)
\]

If \( B < A < C \) and \( D(a_i, b_i, c_i, \gamma) > 0 \), (6.30) is impossible because \( f(z_1(\tau_0), \tau_0/z_1(\tau_0)) < 0 \) and \( f(z_3(\tau_0), \tau_0/z_3(\tau_0)) < 0 \) as in the proof of Lemma 6.8 below (6.27). Consequently, this contradiction excludes the situation (II). Therefore, it holds true that \( \tau_0 = 0 \) in case where \( B < A < C \) and \( D(a_i, b_i, c_i, \gamma) > 0 \). In this case, we shall show \( \lim_{n \to \infty} d_{2,n} = 0 \). It follows from Theorem 3.1 that \( d_{2,n} \in (0, \bar{\mathcal{D}}] \) for any \( n \in \mathbb{N} \). Therefore, we may assume that \( \lim_{n \to \infty} d_{2,n} = d_0 \) with some \( d_0 \in [0, \bar{\mathcal{D}}] \) by passing to a subsequence if necessary. From Lemma 4.1 and (6.20), we see that \( \|u_n\|_{\infty} \) is uniformly bounded with respect to \( n \in \mathbb{N} \). Then by (1.4a), \( w_n := \delta u_n - \gamma r_n/u_n \) satisfies \( \lim_{n \to \infty} w_n = w_0 \) with some \( w_0 \in C(\bar{\Omega}) \). It follows from Proposition 2.2 that if \( d_0 > 0 \), then \( w_0 = \delta a_1/b_1 \) or \( w_0 = 0 \) or \( w_0 = -\gamma a_2/c_2 \) in \( \Omega \). However, repeating the argument below (5.2) in the proof of Lemma 5.1 we see that the above three situations cannot occur. Then, we can deduce that \( d_0 = 0 \). Consequently, we know that \( \lim_{n \to \infty} (d_{2,n}, \tau_n) = (0, 0) \) in case where \( B < A < C \) and \( D(a_i, b_i, c_i, \gamma) > 0 \). Obviously, the uniqueness of limits ensures that the full sequence \( \{(d_{2,n}, \tau_n)\} \) itself converges to \( (0, 0) \). Furthermore, one can verify that all \( \Gamma_j^\pm \) approach \( (d, \tau) = (0, 0) \) in a similar manner.

Next we consider the case when \( C < A < B \) and \( D(a_i, b_i, c_i, \gamma) > 0 \). It follows from Remark 6.9 that \( \mathcal{T} \) is not connected and \( \tau^* \notin (0, \bar{\tau}) \). This fact implies that the projection of \( \Gamma_1^+ \) on the \( \tau \) axis does not intersect with \( (0, \bar{\tau}) \). Then the situation (I); \( \tau_0 = 0 \) cannot occur, and then, (II) necessarily occurs. Consequently, we deduce that \( \lim_{n \to \infty} (d_{2,n}, \tau_n) = (0, \tau_0) \) with some \( \tau_0 \in (\bar{\tau}, \bar{\mathcal{T}}) \) satisfying \( H(z_1(\tau_0), \tau_0) = H(z_3(\tau_0), \tau_0) \). Hence it follows that \( \Gamma_1^+ \) approaches \( (d, \tau) = (0, \tau_0) \). It is possible to check that each \( \Gamma_j^\pm \) also attains \( (d, \tau) = (0, \tau_0) \). The proof of Theorem 6.3 is complete.

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