THE STRUCTURE OF THE $W^*$–TENSOR PRODUCT OVER A $W^*$–SUBALGEBRA AND ITS PREDUAL
($\sigma$–FINITE CASE)

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Abstract. Let $M$, $N$, $R$ be $W^*$–algebras, with $R$ unitally embedded in both $M$ and $N$. by using Reduction Theory, we extend the previous description of the $W^*$–tensor product $M\otimes_R N$ over the common $W^*$–subalgebra $R$ and its predual $(M\otimes_R N)_*$ to the $\sigma$–finite case.

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1. INTRODUCTION

The structure of $W^*$–tensor products over common subalgebras was investigated in [17]. The tensor product over subalgebras generalizes the usual $W^*$–tensor product $M\otimes N$, corresponding to the case when $R$ is the field of complex numbers. The main goal of the paper [17] is to prove the generalization of the celebrated Tomita Commutation Theorem to the case of tensor products over subalgebras in the full generality. If properly interpreted, the commutant assumes the form

$$(M\otimes_R N)' = M'\otimes R' N',$$

and reduces itself to Tomita Commutation Theorem when $R = \mathbb{C}$ acting on $\mathbb{C}$, see [17], Section 5.9.

In the case of $W^*$–tensor product $M\otimes N$, there is an explicit description of the predual as

$$(M\otimes N)_* \cong M_* \otimes N_*$$

(1.1)

where $\otimes$ denotes the operator–projective tensor product, see [6].$^1$ Notice that (1.1) has several applications to various fields, see e.g. [4, 8, 9, 18] and the reference cited therein.

$^1$For standard definitions and results relative to operator space theory, the reader is referred to [5] and the references cited therein.
In the present paper we extend to the $\sigma$–finite case, the results of the previous paper [10] relative to the structure of the $W^*$–tensor product of von Neumann algebras over a common $W^*$–subalgebra. This is done by using reduction theory. Namely, using the extremal decomposition of KMS states (see e.g. [1]) for $C^*$–dynamical systems, and the general properties of $W^*$–tensor products over common subalgebras, we describe the structure of such a tensor product, following the lines adopted in [14]. Also for this situation, $R$, $M$, $N$, and finally $M \otimes_R N$ admit a common decomposition over the centre of $R$. Theorem 1 and Corollary 1 of [10] can be generalized to $\sigma$–finite case, and describe the structure of the $W^*$–tensor product a common $W^*$–subalgebra, and its predual respectively. These results can be considered as a further step towards the fully understand of the general case.

2. ON THE TENSOR PRODUCT OVER SUBALGEBRAS

We start with the definition of the tensor product over a subalgebra, as well as some preliminary properties already considered in [17].

**Definition 1.** Let $I_A \in R \subset M \subset A$, $I_A \in R \subset N \subset A$ be inclusions of von Neumann algebras. The $W^*$–algebra $A$ is said to be the $W^*$–tensor product of $M$ and $N$ over their common $W^*$–subalgebra $R$ if

(i) $A = M \mathbin{\mathop{\bigvee}\limits_{\lambda}} N$,
(ii) for some normal representation $\pi$ of $A$,

$$\pi(M) \subset \pi(R) \mathbin{\mathop{\bigvee}\limits_{\lambda}} M_1 , \quad \pi(N) \subset \pi(R) \mathbin{\mathop{\bigvee}\limits_{\lambda}} N_1$$

for commuting type I von Neumann subalgebras $M_1, N_1 \subset \pi(R)'$ whose common centre $Z$ coincides with that of $R$.

In the situation described by Definition 1, we write $A = M \mathbin{\mathop{\bigotimes}\limits_{\lambda}} R N$ and identify $\pi(A)$ with $A$ itself. Any such a representation as that in Definition 1 is said to be a *splitting* one. Further, the type I von Neumann algebras $M_1, N_1$ given in (ii) can be chosen to be homogeneous, see [10], Proposition 1. Then, we can start without loss of generality, by the following inclusions of von Neumann algebras

$$I_1 \otimes R \subset M \subset B(H_1) \mathbin{\mathop{\bigotimes}\limits_{\lambda}} R ,$$

$$R \otimes I_2 \subset N \subset R \mathbin{\mathop{\bigotimes}\limits_{\lambda}} B(H_2) ,$$

where $R$ is acting on $B(H)$. The $W^*$–tensor product of $M$ and $N$ over $R$ is described with a slightly abuse of notation, by

$$M \mathbin{\mathop{\bigotimes}\limits_{\lambda}} R N = (M \otimes I_2) \mathbin{\mathop{\bigvee}\limits_{\lambda}} (I_1 \otimes N) .$$
Further,

\[ I_1 \otimes Z(R) \subset Z(M), \quad Z(R) \otimes I_2 \subset Z(N). \]

From now on, we suppose that \( A \) is \( \sigma \)-finite if it is not otherwise specified.

We construct normal faithful conditional expectations which are useful in the sequel.\(^2\)

**Proposition 1.** There are normal faithful conditional expectations

\[ \varepsilon_1 : A \mapsto M, \quad \varepsilon_2 : A \mapsto N. \]

Furthermore,

\[ \varepsilon_1 \circ \varepsilon_2 = \varepsilon_2 \circ \varepsilon_1. \quad (2.3) \]

**Proof.** Consider for \( \varphi \in L^1(H_2)_{+,1} \), the slice map of \( B(H_1) \overline{\otimes} R \overline{\otimes} B(H_2) \) onto \( B(H_1) \overline{\otimes} R \overline{\otimes} B(H_2) \),

\[ F^1_\varphi(x \otimes y) = \varphi(y)x \otimes I_2, \]

together with its restriction \( E^1_\varphi := F^1_\varphi \mid_A \). The set \( \{ E^1_\varphi \mid \varphi \in L^1(H_2)_{+,1} \} \) is a separating family of conditional expectations of \( A \) onto \( M \). As \( A \) is \( \sigma \)-finite, there exists a denumerable maximal family \( \{ E^1_{\varphi_k} \} \) with mutually orthogonal support–projections (see [15], 11.5). Define

\[ \varepsilon_1 := \sum_{k=1}^{\infty} \frac{1}{2^k} E^1_{\varphi_k}, \]

which is the searched conditional expectation. Starting from

\[ F^2_\psi(x \otimes y) = \psi(x) I_1 \otimes y, \]

construct \( \varepsilon_2 : A \mapsto N \) by a denumerable maximal family \( \{ E^2_{\psi_k} \} \) with mutually orthogonal support–projections, where \( \psi_k \in L^1(H_1)_{+,1} \). We have, for \( a \in A \),

\[ E^1_\varphi(E^2_\psi(a)) \equiv F^1_\varphi(F^2_\psi(a)) \]
\[ = F^2_\psi(F^1_\varphi(a)) \equiv E^2_\psi(E^1_\varphi(a)), \]

which leads to the assertion taking into account (2.3) for \( \varepsilon_1 \), and the analogous one for \( \varepsilon_2 \).\( \square \)

Define

\[ \epsilon := \varepsilon_1 \circ \varepsilon_2, \quad \epsilon_1 := \varepsilon_1 \upharpoonright_M, \quad \epsilon_2 := \varepsilon_2 \upharpoonright_N. \quad (2.4) \]

Consider the diagram

\[ \text{Diagram} \]

\[ \text{We sometimes identify } M, N, R \text{ with their isomorphic copies } M \otimes I_2, I_1 \otimes N \]
\[ \text{and } I_1 \otimes R \otimes I_2 \text{ in } A \equiv M \overline{\otimes}_R N. \]
where \( \varepsilon_j, j = 1, 2 \) are given in Proposition 1, and \( \epsilon, \varepsilon_j, j = 1, 2 \) are given in (2.4).

**Corollary 1.** The above diagram gives rise to a commuting square of normal faithful conditional expectations.\(^3\)

*Proof.* The proof immediately follows by (2.3) as \( M \otimes I_2 \wedge I_1 \otimes N = I_1 \otimes R \otimes I_2. \) \( \square \)

Now we pass to investigate the standard representation of \( M \otimes R N \) for general (non necessarily \( \sigma \)-finite) \( W^* \)-algebras.

By applying the considerations in the beginning of Section 6 of [17], we can describe \( M \otimes R N \) in the following way. Put

\[
\tilde{M} := M \wedge B(H_1) \otimes Z, \quad \tilde{N} := N \wedge Z \otimes B(H_2).
\]

Then,

\[
M \otimes R N = \tilde{M} \otimes I_2 \vee R \vee I_1 \otimes \tilde{N} \equiv \tilde{M} \otimes Z R \otimes Z \tilde{N}.
\]

**Theorem 1.** The standard representation of the \( W^* \)-tensor product over a \( W^* \)-subalgebra is a splitting representation.

*Proof.* After taking a possible ampliation, we can suppose that \( R \) is acting on \( L^2(Z) \otimes H \),\(^4\) and the standard representations of \( \tilde{M}, \tilde{N} \) and \( R \) can be obtained by induction on the Hilbert spaces \( H_1 \otimes L^2(Z) \),\(^4\)

\(\text{For the definition of a commuting square of conditional expectations see e.g. [11].}\)

\(\text{In the general commutative case, representing } Z \text{ by the GNS representation relative to a normal faithful weight } \varphi \text{ ([17], Section 10.14), we have } Z \sim L^\infty(\Gamma_\varphi, \mu_\varphi) \text{ where } \mu_\varphi \text{ is a positive Radon measure on the locally compact dense subspace } \Gamma_\varphi \text{ of the spectrum } \Omega \text{ of } M, \text{ see [19], Theorem III.1.18. Hence, } L^2(Z) \cong L^2(\Gamma_\varphi, \mu_\varphi).\)
$L^2(Z) \otimes H_2, L^2(Z) \otimes H$ where they are naturally acting. Let $e_1 \in \tilde{M}'$, $e_2 \in \tilde{N}'$, $e \in R'$ be the corresponding selfadjoint projections. Put

$$E := (e_1 \otimes I \otimes I_2)(I_1 \otimes e \otimes I_2)(I_1 \otimes I \otimes e_2).$$

Due to Theorem 4.7, the operator defined above is a selfadjoint projection. Furthermore, $e_1 \in B(H_1)\overline{\otimes} Z$, $e_2 \in Z \otimes B(H_2)$.

The proof follows as $(M \otimes R N)E$ is the standard representation of $M \otimes R N$ acting on $E(H_1 \otimes L^2(Z) \otimes H \otimes H_2)$, and the ampliations–inductions $(e_1(B(H_1)\overline{\otimes} Z)e_1 \otimes I \otimes I_2)E$, $(I_1 \otimes I \otimes e_2(Z \overline{\otimes} B(H_2))e_2)E$ of the reduced algebras $e_1(B(H_1)\overline{\otimes} Z)e_1$, $e_2(Z \overline{\otimes} B(H_2))e_2$ are the splitting type I algebras appearing in Definition 1.

We end by noticing that the standard representation is not homogeneous in general.

3. ON THE DECOMPOSITION OF VON NEUMANN ALGEBRAS ARISING FROM LEFT HILBERT ALGEBRAS AND THEIR PREDUAL

Let $\{\mathfrak{A}_\xi\}_{\xi \in \Omega}$ be a field of left Hilbert algebras defined on the finite measure space $(\Omega, \mu)$. We suppose that such a field of left Hilbert algebras satisfies Conditions (1.1)–(1.5) listed in [14]. In this situation, the field $\{\mathfrak{A}_\xi\}_{\xi \in \Omega}$ is said to be integrable. This is the case arising from left Hilbert algebras associated to GNS representations of KMS states for $C^*$-dynamical systems. To the field mentioned above, there is associated a Hilbert space $\mathcal{H}$ which is the direct integral of an integrable field $\{\mathcal{H}_\xi\}_{\xi \in \Omega}$ of Hilbert spaces, see e.g. [21].

Let $\mathcal{L}(\mathfrak{A})$ be the associated left von Neumann algebra together with the corresponding field $\{\mathcal{L}(\mathfrak{A}_\xi)\}_{\xi \in \Omega}$ of left von Neumann algebras. Consider a measurable vector field $\xi \mapsto T(\xi)$ such that $T(\xi) \in \mathcal{L}(\mathfrak{A}_\xi)'$ almost surely. It is readily seen that

$$T := \int_X T(\xi) \, d\mu(\xi)$$

(3.1)

defines an element of $\mathcal{L}(\mathfrak{A})'$. Conversely, any $T \in \mathcal{L}(\mathfrak{A})'$ admits a (essentially unique) decomposition as above. Indeed, denote by $\mathcal{Z}$ the algebra consisting of all diagonal operators on $\mathcal{H}$ (see e.g. [19, 20]). Any such $T$ is decomposable as $T \in \mathcal{Z}'$, see Theorem 1.7 of [21], and Proposition 2.1 of [20]. Define $\Theta := JTJ \in \mathcal{L}(\mathfrak{A})$, where $J$ is the canonical conjugation associated to $\mathfrak{A}$ (see e.g. [16], Section 10.1). By Proposition 1.3 and Theorem 1.4 of [14], $\Theta$ admits a unique natural
decomposition $\xi \mapsto \Theta(\xi)$. Taking into account (1.4) of [14], $\xi \mapsto J(\xi)\Theta(\xi)J(\xi)$ provides a natural decomposition of $T$ as in (3.1), which is essentially unique.

Now we pass to the description of the predual $\mathcal{L}(\mathfrak{A})_*$ of $\mathcal{L}(\mathfrak{A})$ in the situation described above. This description parallels the analogous one concerning the separable situation (see e.g. [19], Section IV. 8), taking into account of appropriate changes.

Consider the subfield of $\prod_{\xi \in X}\mathcal{L}(\mathfrak{A}_\xi)_*$ consisting of elements $\xi \mapsto \varphi(\xi)$ such that

(i) the map $\xi \mapsto \varphi(\xi)(T(\xi))$ is measurable for each $T \in \mathcal{L}(\mathfrak{A})$,

\[ T = \int_X T(\xi) \, d\mu(\xi) \]

being the natural decomposition of $T$;

(ii) there exists an element $c_\varphi \in L^1(X,\mu)_+$ such that $\|\varphi(\xi)\| \leq c_\varphi(\xi)$ almost everywhere.\footnote{A natural decomposition for $T \in \mathcal{L}(\mathfrak{A})$ (resp $T \in \mathcal{L}(\mathfrak{A})'$) is a decomposition $\xi \mapsto T(\xi)$ of $T$ such that $T(\xi) \in \mathcal{L}(\mathfrak{A}_\xi)$ (resp $T(\xi) \in \mathcal{L}(\mathfrak{A}_\xi)'$) almost surely, see [14], Definition 1.2.}

**Proposition 2.** There is a one–to–one correspondence between elements $\varphi \in \mathcal{L}(\mathfrak{A})_*$ and elements $\xi \in X \mapsto \varphi(\xi) \in \mathcal{L}(\mathfrak{A}_\xi)_*$ satisfying (i), (ii) above.

**Proof.** Let $\xi \mapsto \varphi(\xi)$ be a measurable field of functional as above. Define

\[ \varphi(T) := \int_X \varphi(\xi)(T(\xi)) \, d\mu(\xi), \]

which is well defined by Proposition 1.3 of [14]. We get

\[ |\varphi(\xi)(T(\xi))| \leq \|\varphi(\xi)\|\|T\| \leq c_\varphi(\xi)\|T\|, \]

which means

\[ |\varphi(T)| \leq \int_X |\varphi(\xi)(T(\xi))| \, d\mu(\xi) \leq \left( \int_X c_\varphi(\xi) \, d\mu(\xi) \right)\|T\|, \]

that is $\|\varphi\| \leq \|c_\varphi\|$. It is readily seen by Dominated Convergence Theorem, that $\varphi$ is normal. Moreover, by considering the polar decomposition of normal functionals (see e.g. Theorem 5.16 of [16]), and Theorem 1.4 of [14], if $\varphi$ is the null functional, then $\varphi(\xi) = 0$ almost surely. The construction of a field of functional $\xi \mapsto \varphi(\xi)$ as above, starting from $\varphi \in \mathcal{L}(\mathfrak{A})_*$ follows the same line of the analogous construction in the separable situation, see [19], Proposition IV. 8. 34. \qed

\footnote{Notice that we cannot conclude, in non separable cases, that $\xi \mapsto \|\varphi(\xi)\|$ is measurable.}
Summarizing, we have the following terminology. Define

\[ M := \mathcal{L}(A), \quad M(\xi) := \mathcal{L}(A_\xi), \quad \xi \in X \]
\[ M' := \mathcal{L}(A'), \quad M'(\xi') := \mathcal{L}(A_\xi'), \quad \xi' \in X \]
\[ M_* := \mathcal{L}(A)_*, \quad M(\xi)_* := \mathcal{L}(A_\xi)_*, \quad \xi \in X. \]

We write

\[ M = \int_\mathcal{X} M(\xi) \, d\mu(\xi), \]
\[ M' = \int_\mathcal{X} M(\xi)' \, d\mu(\xi), \]
\[ M_* = \int_\mathcal{X} M(\xi)_* \, d\mu(\xi). \]

4. THE STRUCTURE OF THE TENSOR PRODUCT OVER A SUBALGEBRA AND ITS PREDUAL

We proved in Theorem 1, that the standard representation is a splitting one. In general, it is non homogeneous. We start by recalling the structure of such a standard representation of \( A \equiv M \otimes_R N \).

Let \( I, J \) be the sets of cardinalities appearing in the decomposition in homogeneous parts ([19], Theorem V.1.27) of \( M_1, N_1 \) respectively appearing in the standard representation of \( A \). We have

\[ H = \bigoplus_{\alpha, \beta} H_\alpha \otimes H_{\alpha,\beta} \otimes K_\beta \]

where \( \dim(H_\alpha) = \alpha, \dim(K_\beta) = \beta \). Accordingly,

\[ R = \bigoplus_{\alpha, \beta} I_{H_\alpha} \otimes R_{\alpha,\beta} \otimes I_{K_\beta}, \quad Z = \bigoplus_{\alpha, \beta} I_{H_\alpha} \otimes Z_{\alpha,\beta} \otimes I_{K_\beta}, \]
\[ M = \bigoplus_{\alpha, \beta} M_{\alpha,\beta} \otimes I_{K_\beta}, \quad N = \bigoplus_{\alpha, \beta} I_{H_\alpha} \otimes N_{\alpha,\beta}, \quad (4.1) \]
\[ M_1 = \bigoplus_{\alpha, \beta} B(H_\alpha) \overline{\otimes} Z_{\alpha,\beta} \otimes I_{K_\beta}, \quad N_1 = \bigoplus_{\alpha, \beta} I_{H_\alpha} \otimes Z_{\alpha,\beta} \overline{\otimes} B(K_\beta). \]

Looking at any single summand, we have for every \( \alpha \in I, \beta \in J \),

\[ I_{H_\alpha} \otimes R_{\alpha,\beta} \subset M_{\alpha,\beta} \subset B(H_\alpha) \overline{\otimes} R_{\alpha,\beta}, \]
\[ R_{\alpha,\beta} \otimes I_{K_\beta} \subset N_{\alpha,\beta} \subset R_{\alpha,\beta} \overline{\otimes} B(K_\beta). \]
As $A$ is $\sigma$–finite, we have that $I$ and $J$ are finite or countable sets. In this situation
\[
M \otimes_R N = \bigoplus_{\alpha, \beta} \left( M_{\alpha, \beta} \otimes I_{K_{\beta}} \right) \bigvee \left( I_{H_{\alpha}} \otimes N_{\alpha, \beta} \right).
\] (4.2)

Let now $N \subseteq M$ be an inclusion of $\sigma$–finite von Neumann algebras such that there exists a normal faithful conditional expectation $E : M \mapsto N$ of $M$ onto $N$. Pick a normal faithful state $\psi := \varphi \circ E$. Consider the weakly dense $C^*$–subalgebra $M$ (resp. $N$) made of elements $T \in M$ (resp. $T \in N$) such that the function $\tau \mapsto \sigma_\tau^\psi(T)$ (resp. $\tau \mapsto \sigma_\tau^\varphi(T)$) is continuous w.r.t. the norm topology.

**Lemma 1.** We have $N = M \cap N$ and $E[\mathcal{M}$ is a conditional expectation of $\mathcal{M}$ onto $\mathcal{N}$.

*Proof.* By Takesaki Theorem (see e.g. [15], Section 10), it is enough to show that $E(\mathcal{M}) \subseteq N$. Let $M$ be directly represented in standard form on $\mathcal{H}$, such that the standard vector $\Psi \in \mathcal{H}$ gives rise the state $\psi$ on $M$. Let $\Delta$ be the modular operator relative to $\Psi$, and $P \in N'$ the projection inducing the standard representation of $N$ on $P\mathcal{H}$. Then $P$ reduces $\Delta$, and we have
\[
\sigma_\tau^\psi(E(T))\Psi = \sigma_\tau^\psi(E(T))\Omega = \Delta^i PT\Psi = P\Delta^i T\Psi = E(\sigma_\tau^\psi(T))\Psi,
\]
that is $\sigma_\tau^\psi \circ E = E \circ \sigma_\tau^\psi$. Namely, if $T$ is a regular element of $M$, $E(T)$ is a regular element of $N$. \qed

Fix a normal faithful state $\phi$ on $R$, and extend it to all of $A$ by using the conditional expectation $\epsilon$. Taking into account the commuting square given in (2.5), we easily have
\[
\phi \circ \epsilon[\mathcal{M}] = \phi \circ \epsilon_1, \quad \phi \circ \epsilon[\mathcal{N}] = \phi \circ \epsilon_2.
\]

Moreover,
\[
[\mathfrak{A}_{\phi \circ \epsilon}] = [\mathfrak{M}_{\phi \circ \epsilon_1} \cup \mathfrak{M}_{\phi \circ \epsilon_2}],
\]
\[
[\mathfrak{R}_\phi] = [\mathfrak{M}_{\phi \circ \epsilon_1} \cap \mathfrak{M}_{\phi \circ \epsilon_2}],
\] (4.3)
where $\mathfrak{A}_{\phi \circ \epsilon}$, $\mathfrak{M}_{\phi \circ \epsilon_1}$, $\mathfrak{M}_{\phi \circ \epsilon_2}$, $\mathfrak{R}_\phi$ are the left Hilbert algebras with (the same) unity relative to the states $\phi \circ \epsilon$, $\phi \circ \epsilon_1$, $\phi \circ \epsilon_2$, $\phi$, respectively, $[\cdot]$ denotes the closed generated subspace, and $\mathfrak{A}_{\phi \circ \epsilon}$ is a dense subspace of the Hilbert space of the standard representation of $A$.

Consider the $C^*$–subalgebras $\mathcal{A}$, $\mathcal{M}$, $\mathcal{N}$, $\mathcal{R}$ of regular elements of $A$, $M$, $N$, $R$ w.r.t. the modular group.
Proposition 3. We have

\[ M \boxtimes_R N \equiv \pi_{\varphi}(A)'' = (\pi_{\varphi_1}(M)')'' \otimes (\pi_\delta(R)')'' \]

Proof. The assertion immediately follows by Lemma 1, and the previous considerations. \qed

Now we are ready to describe the structure of the \( W^* \)-tensor product over a \( W^* \)-subalgebra and its predual. In order to give a more readable description, we treat each homogeneous component separately.

Fix a sequence \( \{ \varphi_{\alpha\beta} \} \) of normal states on \( R_{\alpha\beta} \), one for each homogeneous component in (4.1). By restricting ourselves to each homogeneous component, we consider separately \( A_{\alpha\beta} := M_{\alpha\beta} \otimes R_{\alpha\beta} N_{\alpha\beta} \). In this situation, \( A_{\alpha\beta} \) has the form (2.2), for inclusion of algebras as in (2.1).

Let \( \varphi_{\alpha\beta} \circ \epsilon \), \( \varphi_{\alpha\beta} \circ \epsilon_1 \) and \( \varphi_{\alpha\beta} \circ \epsilon_2 \) be the corresponding extensions to \( A_{\alpha\beta} \), \( M_{\alpha\beta} \) and \( N_{\alpha\beta} \) respectively. Consider the \( C^* \)-dynamical systems \((A_{\alpha\beta}, \alpha_{\phi_{\alpha\beta}}, \varphi_{\alpha\beta} \circ \epsilon)\), \((M_{\alpha\beta}, \alpha_{\phi_{\alpha\beta}} \circ \epsilon_1, \varphi_{\alpha\beta} \circ \epsilon_1)\), \((N_{\alpha\beta}, \alpha_{\phi_{\alpha\beta}} \circ \epsilon_2, \varphi_{\alpha\beta} \circ \epsilon_2)\), \((R_{\alpha\beta}, \alpha_{\phi_{\alpha\beta}}, \varphi_{\alpha\beta})\), where the \( \alpha^\# \) are the restriction of the modular groups \( \sigma^\# \) to the corresponding regular elements. Consider the extremal decomposition of the \( \alpha_{\phi_{\alpha\beta}} \)-KMS state \( \varphi_{\alpha\beta} \). Let \( \mu_{\alpha\beta} \) be the maximal measure on the compact convex set \( K(R_{\alpha\beta}, \alpha_{\phi_{\alpha\beta}}) \) corresponding to \( \varphi_{\alpha\beta} \). It is well known that \( \mu_{\alpha\beta} \) coincides with the central measure of \( \varphi_{\alpha\beta} \) on \( K(R_{\alpha\beta}, \alpha_{\phi_{\alpha\beta}}) \), see e.g. [2]. Moreover, the measure \( \mu_{\alpha\beta} \) is pseudo-supported on the extreme point \( \partial K(R_{\alpha\beta}, \alpha_{\phi_{\alpha\beta}}) \).

Take

\[ \Omega_{\alpha\beta} := \partial K(R_{\alpha\beta}, \alpha_{\phi_{\alpha\beta}}) \]

and define on \( M \cap \Omega_{\alpha\beta} \),

\[ \tilde{\nu}_{\alpha\beta}(M \cap \Omega_{\alpha\beta}) := \mu_{\alpha\beta}(M) \]

where \( M \) is a Baire measurable set of \( K(R_{\alpha\beta}, \alpha_{\phi_{\alpha\beta}}) \). Let \( \nu_{\alpha\beta} \) be the completion of the probability measure \( \tilde{\nu}_{\alpha\beta} \).

Consider the integrable field \( \{ \mathcal{N}_\varphi \}_{\varphi \in \Omega_{\alpha\beta}} \) of left Hilbert algebras whose elements have the form

\[ \varphi \in \Omega_{\alpha\beta} \mapsto \pi_\varphi(T)\Psi_\varphi, \quad T \in R_{\alpha\beta}, \]

where \((\pi_\varphi, H_\varphi, \Psi_\varphi)\) is the GNS representation of the modular state \( \varphi \) on \( R_{\alpha\beta} \).

Let \( \{ \mathcal{M}_{\varphi_{\alpha\beta}} \}_{\varphi \in \Omega_{\alpha\beta}} \), \( \{ \mathcal{M}_{\varphi_{\alpha\beta}} \}_{\varphi \in \Omega_{\alpha\beta}} \), \( \{ \mathcal{A}_{\varphi \alpha \beta} \}_{\varphi \in \Omega_{\alpha\beta}} \) be the integrable fields of left Hilbert algebras analogously obtained starting from the states

\footnote{We pursue such a strategy in order to give a more readable description of the structure of \( M \boxtimes_R N \) and its predual, see below.}
\[ \varphi \circ \epsilon_1 \in K(M_{\alpha \beta}, \alpha^{\phi \alpha \beta \circ \epsilon_1}) \quad \varphi \circ \epsilon_2 \in K(N_{\alpha \beta}, \alpha^{\phi \alpha \beta \circ \epsilon_2}) \quad \varphi \circ \epsilon \in K(A_{\alpha \beta}, \alpha^{\phi \alpha \beta \circ \epsilon}) \]

respectively.\footnote{Notice that the pull–back measures \( \epsilon_1^*(\nu_{\alpha \beta}), \epsilon_2^*(\nu_{\alpha \beta}) \) are precisely the orthogonal measures on \( K(M_{\alpha \beta}, \alpha^{\phi \alpha \beta \circ \epsilon_1}), K(N_{\alpha \beta}, \alpha^{\phi \alpha \beta \circ \epsilon_2}), K(A_{\alpha \beta}, \alpha^{\phi \alpha \beta \circ \epsilon}) \) corresponding to the abelian algebra \( Z_{\alpha \beta} \sim I_\alpha \otimes Z_{\alpha \beta} \otimes I_\beta \) which is a common subalgebra of \( Z(M_{\alpha \beta}) \sim Z(M_{\alpha \beta}) \otimes I_3, Z(N_{\alpha \beta}) \sim I_3 \otimes Z(M_{\alpha \beta}), Z(A_{\alpha \beta}) \) respectively, see e.g. [1, 2]. They provide the subcentral disintegration of the KMS states \( \phi_{\alpha \beta} \circ \epsilon_1, \phi_{\alpha \beta} \circ \epsilon_2, \phi_{\alpha \beta} \circ \epsilon \) respectively.}

Define

\[ \Omega := \bigcup_{\alpha \beta}^\circ \Omega_{\alpha \beta}, \quad \nu := \sum_{\alpha \beta} \nu_{\alpha \beta}, \] (4.4)

where \( \bigcup^\circ \) stands for the disjoint union, and \( \nu_{\alpha \beta} \) is understood as a measure on all of \( \Omega \) with support \( \Omega_{\alpha \beta} \).

**Proposition 4.** We have for the tensor product,

\[ M \otimes_R N = \int_{\Omega} M(\varphi) \otimes_{R(\varphi)} N(\varphi) \, d\nu(\varphi) \]

where \( \{ R(\varphi) \}_{\varphi \in \Omega} \) is the measurable field relative to the factor decomposition of \( R \), and \( \{ M(\varphi) \}_{\varphi \in \Omega}, \{ N(\varphi) \}_{\varphi \in \Omega} \) to the subcentral decomposition of \( M, N \) w.r.t. \( \bigoplus_{\alpha \beta} I_\alpha \otimes Z_{\alpha \beta} \subset Z(M), \bigoplus_{\alpha \beta} Z_{\alpha \beta} \otimes I_\beta \subset Z(N) \) respectively.

**Proof.** We first consider the homogeneous case. Taking into account (4.3) and Proposition 3, we have the inclusions

\[ I_1 \otimes \mathcal{L}(\mathfrak{A}_\phi) \subset \mathcal{L}(\mathfrak{M}_{\phi \circ \epsilon_1}) \subset B(H_1) \otimes \mathcal{L}(\mathfrak{A}_\phi), \]

\[ \mathcal{L}(\mathfrak{A}_\phi) \otimes I_2 \subset \mathcal{L}(\mathfrak{M}_{\phi \circ \epsilon_2}) \subset \mathcal{L}(\mathfrak{A}_\phi) \otimes B(H_2), \]

\[ M \otimes_R N \equiv \mathcal{L}(\mathfrak{M}_{\phi \circ \epsilon_1}) \otimes I_2 \bigvee (I_1 \otimes \mathcal{L}(\mathfrak{M}_{\phi \circ \epsilon_2})). \]

Taking into account the above considerations together with the structure of the commutant of the left von Neumann algebra of a left Hilbert algebra (see Section 3), we obtain by applying the results of [14] and the Commutator Theorem 5.9 of [17],

\[ M \otimes_R N \equiv \mathcal{L}(\mathfrak{M}_{\phi \circ \epsilon}) = \int_{\Omega} \mathcal{L}(\mathfrak{M}_{\phi \circ \epsilon}) \, d\nu(\varphi) \]

\[ = \int_{\Omega} (\mathcal{L}(\mathfrak{M}_{\phi \circ \epsilon_1}) \otimes I_2) \bigvee (I_1 \otimes \mathcal{L}(\mathfrak{M}_{\phi \circ \epsilon_2})) \, d\nu(\varphi) \]

\[ \equiv \int_{\Omega} M(\varphi) \otimes_{R(\varphi)} N(\varphi) \, d\nu(\varphi). \]
Here, $M(\varphi) := L(M_{\varphi \rho_1})$, $N(\varphi) := L(N_{\varphi \rho_2})$, $R(\varphi) := L(R_{\varphi})$. The proof follows summing up all the homogeneous components appearing in the standard representation of $M \bar{\otimes} R N$. $\square$

We are ready to describe the structure of $M \bar{\otimes} R N$ and its predual in the $\sigma$–finite case.

**Theorem 2.** Let $\{M(\varphi)\}_{\varphi \in \Omega}$, $\{N(\varphi)\}_{\varphi \in \Omega}$, $\{R(\varphi)\}_{\varphi \in \Omega}$ be the measurable fields of von Neumann algebras appearing in the decompositions of $M$, $N$, $R$ given in Proposition 4, respectively. Define $\{\tilde{M}(\varphi)\}_{\varphi \in \Omega}$ and $\{\tilde{N}(\varphi)\}_{\varphi \in \Omega}$ as the measurable fields of von Neumann algebras such that

\[
\tilde{M}(\varphi) \otimes I(\varphi) = M(\varphi) \bigwedge (B(H_{\alpha \beta}) \otimes I(\varphi)),
\]

\[
I(\varphi) \otimes \tilde{N}(\varphi) = N(\varphi) \bigwedge (I(\varphi) \otimes B(H_{\alpha \beta}))
\]

whenever $\varphi \in \Omega_{\alpha \beta}$.

Then we have

\[
M \bar{\otimes} R N = \int_{\Omega} \tilde{M}(\varphi) \bar{\otimes} R(\varphi) \bar{\otimes} \tilde{N}(\varphi) \, d\nu(\varphi).
\]

**Proof.** Looking at each fiber, we have by Lemma 1 of [10],

\[
M(\varphi) \bar{\otimes} R(\varphi) N(\varphi) = \tilde{M}(\varphi) \bar{\otimes} R(\varphi) \bar{\otimes} R(\varphi) R(\varphi) \bar{\otimes} \tilde{N}(\varphi),
\]

where the last equality follows as all the $R(\varphi)$ are factors. The proof follows as $R(\varphi) \bar{\otimes} R(\varphi) R(\varphi)$ coincides with $R(\varphi)$. $\square$

As an immediate corollary we have the structure of the predual $(M \bar{\otimes} R N)^*$, taking into account the description given in Section 3, of the predual of a direct integral of left von Neumann algebras arising from an integrable field of left Hilbert algebras.

**Corollary 2.** In the situation of Theorem 2, we have

\[
(M \bar{\otimes} R N)^* = \int_{\Omega} \tilde{M}(\varphi)^* \hat{\otimes} R(\varphi)^* \hat{\otimes} \tilde{N}(\varphi)^* \, d\nu(\varphi)
\]

(4.5)

where $\hat{\otimes}$ denotes the operator–projective tensor product between operator spaces given in [6].

**Proof.** The proof directly follows from Theorem 2, taking into account Theorem 3.2 of [7]. $\square$
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