B 2 Majorana Qubits

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1 Introduction

Ideas of topology a branch of mathematics which studies whether two objects can be transformed into each other under continuous deformations have proven to provide very fruitful concepts also in physics. Examples in classical physics involve vortices in fluid dynamics, electrical charges in electrodynamics, and the classification of defects in crystals [1]. In quantum mechanics the prime example of a topological effect is the Aharonov-Bohm effect [2]. Soon after the discovery of the quantum Hall effect [3], it was realized that the new phase is not characterized by a broken symmetry but that the topology of the Bloch wavefunction in the Brillouin zone described by the Chern number changes when entering the quantum Hall regime [4]. For some time the quantum Hall effect was considered the only example of a topological phase in a noninteracting system, nowadays called topological insulator. More recently, a second example was devised theoretically [5, 6, 7] in the form of the quantum spin Hall effect and confirmed experimentally in HgTe quantum wells [8]. This sparked further research in this area and culminated in the classification of all topological insulators and superconductors [9, 10]. The subject of this lecture are states, called Majorana fermions, which appear at defects in topological superconductors with broken time-reversal and spin-rotation symmetry in one or two dimensions.

The outline of the lecture is as follows. We will first introduce the notion of Majorana fermions. We will then show how these states appear as zero energy solutions of the Bogoliubov-de Gennes equation describing a spinless $p$-wave superconductor in one dimension. We will describe their usefulness in term of quantum information applications as they encode quantum information in a protected fashion before we will finish off discussing a possible experimental implementation. There are by now several reviews on this subject, see Refs. [11, 12, 13], where further information on this subject can be found.

2 Majorana fermions

Majorana fermions are fermionic particles which are their own antiparticles. Why the corresponding context is quite natural for bosons—most bosons (phonon, photons, magnons, plasmons, ...) are their own antiparticles—this is a rather uncommon property for fermions. In fact so far no elementary fermionic particle has been experimentally confirmed to be its own antiparticle. In more mathematical terms, a Majorana operator $\gamma_1$ (an operator which creates a Majorana particle) is a Hermitian operator $\gamma_1 = \gamma_1^\dagger$ which anticommutes with other Majorana operators $\gamma_2$ and squares to one $\gamma_2^2 = 1$; summarizing, the Majorana operators form a Clifford algebra defined by the anticommutation relation

$$\{\gamma_k, \gamma_l\} = 2\delta_{kl}. \quad (1)$$

Given the fact that these particles do not exist as elementary particles, we would like to know how to construct them from conventional Dirac fermions created by the operator $c_k^\dagger$. In fact, it is an easy exercise in algebra to show that given a set of $N$ Dirac fermions defined by $c_k^\dagger$, $k = 1, \ldots, N$, with the canonical anticommutation relations $\{c_k, c_l^\dagger\} = \delta_{kl}$ and $\{c_k, c_l\} = 0$, we can construct $2N$ Majorana operators $\gamma_k$ via

$$\gamma_{2k-1} = c_k + c_k^\dagger, \quad \gamma_{2k} = i(c_k^\dagger - c_k). \quad (2)$$
Inverting the defining Eq. (2), we find an expression of the Dirac fermions in terms of the Majorana operators

\[ c_k = \frac{1}{2} (\gamma_{2k-1} + i\gamma_{2k}), \quad c_k^\dagger = \frac{1}{2} (\gamma_{2k-1} - i\gamma_{2k}). \] (3)

The Hilbert space of a single fermionic mode is two-dimensional: the mode is either filled or empty distinguished by the eigenvalue of the number operators \( n_k = c_k^\dagger c_k \) which has eigenvalues 0 or 1.\(^2\) An operator which will turn out to be important in the following discussion is the fermion parity operator \( P_k = 1 - 2n_k = (-1)^{n_k} \) which has the eigenvalue +1 if the number of fermions is even and −1 if the number of fermions is odd. In terms of the Majorana operators, the parity operator assumes the simple form

\[ P_k = -i\gamma_{2k-1}\gamma_{2k}. \] (4)

So far, the introduction of Majorana fermions was an algebraic trick to go from one set of complex operators \( c_k \) to an equivalent description in terms of the Hermitian operators \( \gamma_k \). Naturally, the question arises if these operators are ‘physical’ in the sense that they describe the excitations of a physical system/Hamiltonian. The answer to this question is (maybe surprisingly) yes: over a decade ago, Kitaev constructed a model which leads to Majorana fermions [14].

To appreciate the difficulty in constructing such a model as well as to understand the resolution, we dwell a bit on the requirement/hurdles to construct such a model. Starting from Dirac fermions, we see from (2) that the Majorana operators are superposition of electron and hole operators. We know that ordinary (many-body) quantum mechanics is invariant under global \( U(1) \) transformations \( c_k \mapsto U_\varphi c_k U_\varphi^\dagger = e^{i\varphi} c_k \) with \( \varphi \) an arbitrary phase. The reason for this is the conservation of the total particle number (or charge for that matter). However, it is easy to check that under the same transformation Majorana operators corresponding to the same fermionic mode mix with each other

\[ \gamma_{2k-1} \mapsto U_\varphi^\dagger \gamma_{2k-1} U_\varphi = \cos(\varphi) \gamma_{2k-1} - \sin(\varphi) \gamma_{2k}. \] (5)

Thus, if we were to construct a Hamiltonian which has \( \gamma_{2k-1} \) (or \( \gamma_{2k} \) for that matter) as an elementary, localized excitation, we will have to break the global \( U(1) \) invariance as it mixes the two different physical modes \( \gamma_{2k-1} \) and \( \gamma_{2k} \). In fact, the \( U(1) \) symmetry (given by the phase \( \varphi \)) is broken down to a \( \mathbb{Z}_2 \) symmetry (corresponding to \( \varphi = 0, \pi \)). This is exactly what happens in superconducting systems, so we should look a bit more closely into the theory of superconductivity.

### 2.1 Bogoliubov-de Gennes equation

Superconductivity is an ordering phenomena which happens in interacting system at low temperatures. Experimentally, the state is characterized by a vanishing of the resistance and, more importantly, by a perfect diamagnetic response called Meißner-Ochsenfeld effect [15, 16].

\(^2\)Note that \( n_k \) is idempotent as \( n_k^2 = c_k^\dagger c_k c_k^\dagger c_k = c_k^\dagger (1 - c_k^\dagger c_k) c_k = n_k \) which proves the fact that the eigenvalues of \( n_k \) are 0 or 1.
Starting from a model of interacting spinless electrons\textsuperscript{3}

\[ H = \int d^3r \left[ \frac{\hbar^2}{2m} \left| \nabla \psi(r) \right|^2 - \mu |\psi(r)|^2 \right] - \frac{1}{2} \int d^3r \, d^3r' \, \psi^\dagger(r') V(r - r') \psi(r') \psi^\dagger(r) \psi(r), \quad (6) \]

we employ the mean-field decoupling with the superconducting pair-potential \( \Delta(r - r') = V(r - r') \langle \psi(r') \psi(r) \rangle \) \textsuperscript{17} and arrive at the effective BCS mean-field Hamiltonian

\[ H_{\text{MF}} = \int d^3r \left[ \frac{\hbar^2}{2m} \left| \nabla \psi(r) \right|^2 - \mu |\psi(r)|^2 \right] - \frac{1}{2} \int d^3r \, d^3r' \, \psi^\dagger(r') \psi^\dagger(r') \Delta(r - r') \]

\[- \frac{1}{2} \int d^3r \, d^3r' \, \Delta^\ast(r - r') \psi(r') \psi(r) + \frac{1}{2} \int d^3r \, d^3r' \, \Delta(r - r') \frac{1}{V(r - r')} \quad (7)\]

In the resulting Hamiltonian, the \( U(1) \) degree of freedom \( \psi \mapsto e^{i\varphi} \psi \) for fixed ‘external field’ \( \Delta \) is broken down to a \( \mathbb{Z}_2 \) degree of freedom just as we wished.

Apart from an unimportant constant, the resulting Hamiltonian can be written as a quadratic form

\[ H_{\text{MF}} = \frac{1}{2} \int d^3r \, d^3r' \, \Psi^\dagger(r) h_{\text{BdG}}(r - r') \Psi(r') + \text{const.} \quad (8) \]

in Nambu space \( \Psi = (\psi^\dagger \, \psi) \) with the Bogoliubov-de Gennes Hamiltonian

\[ h_{\text{BdG}}(r - r') = \begin{pmatrix} \xi(p) & -\Delta(r - r') \\ \Delta^\ast(r - r') & -\xi(p) \end{pmatrix}, \quad (9) \]

where \( \xi(p) = \frac{p^2}{2m} - \mu = -\frac{\hbar^2}{2m} \nabla^2 - \mu \); note that \( h_{\text{BdG}} \) is Hermitian due to the fact that \( \Delta(-r) = -\Delta(r) \). The Hamiltonian is ‘diagonalized’ by a Bogoliubov transformation, i.e., by introducing new fermionic operators \( \beta_{E_n} \) which fulfill the canonical anticommutation relation and in terms of which the mean-field Hamiltonian assumes the form \textsuperscript{15,16}

\[ H_{\text{MF}} = \frac{1}{2} \sum_{E_n} E_n \beta^\dagger_{E_n} \beta_{E_n} + \text{const.}; \quad (10) \]

here, \( E_n \) are the eigenvalues of \( h_{\text{BdG}} \) and \( \beta_{E_n} = \int d^3r \, v^\dagger_{E_n}(r) \Psi(r) \) where \( v_{E_n}(r) \) are the associated eigenvectors.

Note that in getting to Eq. (9), we have apparently doubled the degrees of freedom. However, the resulting Bogoliubov-de Gennes Hamiltonian \( h_{\text{BdG}} \) enjoys an additional symmetry: in fact, the particle-hole symmetry operator \( C = r^x K \) with \( K \) the complex conjugation and \( r^x \) acting on the Nambu index anticommutes with the Hamiltonian \( \{ h_{\text{BdG}}, C \} = 0 \). This symmetry guarantees that for every eigenvector \( v_{E_n} \) of \( h_{\text{BdG}} \) to eigenvalue \( E_n \geq 0 \) there is an additional eigenvector \( v_{-E_n} = C v_{E_n} \) to eigenvalues \( -E_n \). Expressing this fact in the second quantized Bogoliubov operators, we have

\[ \beta^\dagger_{-E_n} = \int d^3r \, \Psi^\dagger(r) v_{-E_n}(r) = \int d^3r \, \Psi^\dagger(r) \tau^x v_{E_n}(r) = \beta_{E_n}. \quad (11) \]

\textsuperscript{3}As we want to end up with single unpaired Majorana fermions, we have to get rid of all possible degeneracies in particular the spin degeneracy.
In the end, combining the terms with $E_n$ and $-E_n$, we have $E_n(\beta_{E_n}^\dagger \beta_{E_n} - \beta_{-E_n}^\dagger \beta_{-E_n}) = 2E_n(\beta_{E_n}^\dagger \beta_{E_n} - \frac{1}{2})$ such that we only need to include the eigenvectors to positive eigenvalues in (10) on the expense of the factor $\frac{1}{2}$ in front of the sum.

Now, we are very close to our goal of realizing Majorana fermions starting from conventional Dirac fermions. Looking at Eq. (11), we see that if an eigenstate $n = 0$ of the Bogoliubov-de Gennes Hamiltonian has a vanishing eigenvalue, $E_0 = 0$, it is in fact a Majorana fermion with $\gamma_1 = \beta_0^\dagger$. Given this insight, we try to construct a physical situation where (9) incorporates such a zero mode. Here, I want to point out that there is a principle difference between the more general term Majorana fermion and the Majorana zero mode. Whereas Majorana fermion simply denotes a ‘real’ fermion, Majorana zero mode denotes Majorana fermions bound to zero energy at a topological defect in a superconductor/superfluid. While the statistics of the former is simply fermionic, the latter shows non-Abelian exchange statistics, see below.

### 2.2 Spinless $p$-wave superconducting nanowire

Following Kitaev [14], we would like to construct a simple model which shows Majorana zero modes. Thus, we consider a one-dimensional situation with $r = z$. The simplest choice of paring, $s$-wave pairing, with $\Delta(z) = \Delta \delta(z)$ is not allowed for spinless electrons as $\Delta(-z) \neq -\Delta(z)$. Thus, we take the next term in the gradient expansion into account, $p$-wave pairing, with $\Delta(z) = -i\Delta \lambda_F \delta'(z)$; here, we have introduced the (reduced) Fermi wavelength $\lambda_F = \hbar/\sqrt{2m\mu}$ such that $\Delta$ has the dimension of energy. Going over to momentum space, the Bogoliubov-de Gennes Hamiltonian can be written as

$$h_{\text{BdG}} = \xi(p) \tau^z - \Delta \frac{p}{p_F} \tau^x. \tag{12}$$

In the following, we will assume $\Delta > 0$ for convenience. It turns out that the model of the spinless $p$-wave superconducting wire (12) is closely related to the so-called Su, Schrieffer, Heeger model studied some time ago as a model for polyacetylene [19, 20]. We will not go into a detailed discussion of the similarities and difference of the two model, we just want to point out that Su et al. have found that their model in polyacetylene generates zero energy state of certain topological criteria are satisfied.

Being interested in solutions of (12) with vanishing eigenvalue, we note first that the spectrum of the system $E_p = \sqrt{\xi(p)^2 + \Delta^2(p/p_F)^2}$ is fully gapped in the translationally invariant case with $\mu \neq 0$, see Fig. 1(a). In order to find a zero mode, we have to look at interfaces between different materials such that the parameters $\mu, \Delta, m$ become spatially dependent. In the simplest case, we look at an interface between vacuum for $z < 0$ (modeled by $\mu \to -\infty$) and a $p$-wave superconductor for $z > 0$, see Fig. 1(b). To simplify the discussion, we assume that only the states close to the Fermi surface are important and that the wire is long enough such that the second interface does not influence our discussion. Thus, we can write $\psi(z) = e^{ikFz}\psi_L(z) + \psi_R(z)$

4Here, it is important that there is only a single zero mode present. Having two modes $\beta_1$ and $\beta_2$ at zero energy, we can only conclude that $\beta_1 = \beta_2^\dagger$.

5However in the recent literature, the general term ‘Majorana fermion’ is often used to denote the more special term ‘Majorana zero mode’.

6The phase of $\Delta$ is in fact an unobservable quantity as only phase-differences are observable, e.g., via the Josephson effect [15][16][18].

7Physically this assumption means that the interface is not too abrupt in order not to scatter states with vastly different momentum into each other.
Fig. 1: (a) Plot of the energy-momentum relation of a p-wave superconducting nanowire. All energies are measured with respect to the chemical potential $\mu > 0$ which is indicated by the dashed line. Above the chemical the spectrum corresponds to an excitation spectrum; below the chemical potential it should be understood as an absorption spectrum. Due to the presence of the superconductivity the two spectra are equivalent when mirrored at the dashed line. The black line shows two parabolas which is the spectrum without superconductivity, i.e., $\Delta = 0$. The blue and red lines correspond to a superconducting pair-potential $\Delta = 0.2\mu$ and $\Delta = 0.4\mu$, respectively. The presence of the superconductor opens a gap of size $2\Delta$ at the chemical potential. Note that for any finite superconducting pairing $\Delta \neq 0$ the system is completely gapped. At $p = 0$ and $E_{p=0} = \pm \mu$ the bottom of the band is visible. Decreasing $\mu$ brings the two vertices of the black parabolas closer together until they cross for $\mu = 0$ which is the point of the topological phase transition. (b) Interface of a p-wave superconducting nanowire (gray cylinder) with vacuum. The interface is located at $x = 0$. To the right there is the nanowire with $\mu > 0$ the vacuum is modeled by $\mu \to -\infty$, i.e., a vanishing electron density. Due to the fact that the topological charge changes at the interface a single zero mode emerges which is a Majorana fermion depicted by the red sphere.

$e^{-ik_F z}\psi_R(z)$ where we assume $\psi_L$ and $\psi_R$ to be slowly varying on the scale $\lambda_F$. This ansatz effectively linearize the spectrum $\xi(p)$ around the Fermi points and leads to the Hamiltonian

$$h_{\text{BdG}} = v_F p \eta^z + \Delta \tau^y \eta^y$$

with $v_F = p_F/m$ and where $\eta^z$ acts on the left/right-moving basis. As the Hamiltonian commutes with $\tau^z \eta^z$, we can block diagonalize it in each of the eigenspaces $\tau^z \eta^z = \pm 1$ with the result of having two decoupled problems. A state $v_0$ at zero energy thus satisfies the differential equation $h_{\text{BdG}} v_0 = 0$ with $p = -i\hbar \partial_z$. The solutions are given by

$$(v_0^+ \tau^z \eta^z) e^{-z/\xi - i\pi \tau^z/4} = (v_0^- \tau^z \eta^z)$$

with the superconducting coherence length $\xi = \hbar v_F/\Delta$; note that we have retained only those solutions which are exponentially decaying away from the interface such that the wavefunction remain normalizable. Additionally, we have chosen the overall phase of the wavefunctions such that $C v_0^+ = v_0^-$. The system is terminated with vacuum for $z < 0$. In our description in terms of linearized spectrum, the boundary condition with vacuum is implemented by demanding a vanishing quasiparticle density at $z = 0$. The quasiparticle density operator is given by $\rho = \frac{1}{2}(1 + \eta^z)\delta(z)$. Thus,
we seek a solution of the form $v_0(z) = \alpha v_0^+ (z) + \beta v_0^- (z)$ with the requirement $0 \overset{!}{=} \langle v_0|\rho|v_0 \rangle \propto |\alpha + \beta|^2$. The condition is satisfied for $\alpha = -\beta$ yielding a single bound state of the form

$$
\begin{aligned}
v_0(z) = \alpha e^{-z/\xi} \left( e^{-i\pi/4} - e^{i\pi/4} \right) \tau \\
0 = \langle v_0|\rho|v_0 \rangle \propto |\alpha + \beta|^2.
\end{aligned}
$$

(15)

at zero energy located at the interface between the $p$-wave superconductor and vacuum. In order that the Bogoliubov operator $\gamma_1 = \beta_0$ associated to $v_0$ is a Majorana operator with $\beta_0 = \beta_0$, we need to have $\mathcal{C}v_0 = v_0$ which demands that $\alpha = \pm i \mathbb{R}$. The proper normalization of the Majorana operator to $\gamma_1 = 1$ is realized with $\alpha = i/\sqrt{2\xi}$ and the Majorana operator assumes the form

$$
\gamma_1 = \int_{z \geq 0} dz \psi_L^+(z) = \frac{1}{\sqrt{2\xi}} \int_{z \geq 0} dz \sin(k_F z) e^{-z/\xi} \left[ e^{-i\pi/4} \psi(z) + e^{i\pi/4} \psi^\dagger(z) \right].
$$

(16)

The analysis can be generalized to include the second interface of the $p$-wave superconducting nanowire with vacuum which yields a second Majorana operator $\gamma_2$. In fact, it is a physical constraint that on each connected piece of superconductor there is an even number of Majorana fermions. Two Majorana fermions together can host a conventional fermionic quasiparticle. Note that due to the superconducting condensate the fermion number is not conserved. However, the fermion parity is still conserved. Thus in the situation as in Fig. 1(b), the parity $\mathcal{P} = -i \gamma_1 \gamma_2$ encodes the parity of the total number of electrons in the combined system of the superconducting island and the nanowire.

### 2.3 Topological charge

The appearance of the single zero mode which lead to the Majorana operator in the last section was not an accident. In fact, its presence follows from the general classification of topological insulators/superconductors that at interfaces between superconductors with different topological charge there will be a certain number of topological states which are protected and do not depend on microscopic details. In the classification of topological matter, one considers the question whether two Hamiltonians can be smoothly deformed into each other without closing the gap. In the classification of topological superconductors, one does not allow for arbitrary single-particle Hamiltonians but restricts the class of Hamiltonians in such a way that one only allows for Hermitian matrices which have a particle-hole symmetry and thus represent a Bogoliubov-de Gennes Hamiltonian of a mean-field problem of the form Eq. (8) [9, 10]. A topological charge is an integer which is the same for all Hamiltonians which can be deformed into each other and which differs for two Hamiltonian for which this cannot be achieved. The classification tells us that in the case of a one-dimensional superconductor with particle-hole symmetry $\mathcal{C}$ and $\mathcal{C}^2 = 1$ (as in our case), there are two distinct topological classes. We have found that at an interface between vacuum (read conventional insulator) with $\mu \to -\infty$ and a $p$-wave superconductor there is always a bound state [21, 22, 23]. This indicates that the two

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8We do not allow for $\alpha \in \mathbb{C}$ as the corresponding operator $\beta_0$ would not be Hermitian in this case. What is left is the possibility to choose the sign of $\alpha$ which exactly corresponds to the $\mathbb{Z}_2$ symmetry described above.

9Two electrons can be taken out of superconducting condensate by breaking up a Cooper-pair.
parts are in different topological classes; we denote the topological charge $Q$ of the trivial insulator by 0 and the one of the $p$-wave superconductor by 1. As the topological charge can only change when the gap closes and the gap of (12) only closes for $\mu = 0$, we find that

$$Q(h_{\text{BdG}}) = \begin{cases} 1, & \mu > 0, \\ 0, & \mu < 0. \end{cases}$$ (17)

Due to this reasoning, the Majorana mode is always present in the model as long as $\mu$ changes sign across the interface. For a more complete discussion of this fact in the simpler model of polyacetylene see Ref. [24].

3 Quantum computation with Majorana fermions

If we think about an implementation for a quantum computer, we are used to the example of a spin-$\frac{1}{2}$ particle which is a drosophila for a generic two-level system [25]. However, we can ask ourself the question whether we can also use the many-body Fock space for quantum computation purposes. We know that the occupation states $|n_1, n_2, \ldots, n_N\rangle$ with $n_j \in \{0,1\}$ form a basis for the $N$-mode fermionic Fock space generated by the creation operators $c^\dagger_j$, $j \in \{1, \ldots, N\}$, starting from the vacuum state denoted by $|0\rangle$. The Fock space has dimension $2^N$ (each mode can be either occupied or empty). Thus counting the degrees of freedom, we are tempted to conclude that a fermionic system with $N$-modes emulates $N$-qubits. In the next section, we will see that this naïve counting argument is not completely correct as it violates the so-called superselection rule. We will argue that quantum computation with noninteracting fermions is not complete and will show what is needed to make the setup complete. Then, we will show that Majorana fermions are in fact non-Abelian particles such that some gates can be performed in a parity-protected way.

3.1 Fermionic quantum computation

Expressing a Hamiltonian $H$ or in fact any physical observable $A$ which are bosonic operators in terms of fermionic creation and annihilation operators, we are bound to only include terms where an even number of fermion operators appear. The result is that the total fermion parity

$$P = \prod_k P_k = (-1)^{\sum_k n_k}$$

is strictly conserved in a closed system; the reason for this is the fact that

$$PAP = A$$

which follows from $Pc_jP = -c_j$ and the fact that each term in $A$ involves an even number of fermionic operators. Note that the superconducting Hamiltonian (7) conserves the total fermion parity even so the number of fermions is not conserved. Due to this constraint, we have the following superselection rule: given two states in a fermionic Fock space $|\psi_+\rangle$ and $|\psi_-\rangle$ with different fermion parity, $P|\psi_\pm\rangle = \pm|\psi_\pm\rangle$ we have

$$\langle \psi_- | A | \psi_+ \rangle = \langle \psi_- | PAP | \psi_+ \rangle = -\langle \psi_- | A | \psi_+ \rangle = 0$$ (19)

10From the correspondence principle, we know that for large quantum numbers the expectation values of operators for physical observables should behave like (real) numbers. Due to the anticommutation relation of fermionic operators, the correspondence principle for a potential fermionic observable would instead lead to anticommuting Grassmann numbers.
Fig. 2: Sketch of the parity Majorana qubit. Two Majorana fermions together form a single Dirac fermionic mode whose Hilbert space is two-dimensional as the mode can either be empty or filled, both states at the same energy. Four Majorana fermions thus form a four-dimensional Hilbert space of which due to the conservation of the total fermion parity only a two-dimensional subspace can be accessed. This degenerate two-dimensional subspace is the Majorana qubit. Gates on the qubit can be either performed by braiding or by coupling two Majorana fermions. As indicated in the figure, coupling $\gamma_3$ to $\gamma_4$ implements a $\bar{\sigma}^z$-operation whereas coupling $\gamma_2$ to $\gamma_3$ leads to a $\bar{\sigma}^x$-operation. Given the fact that the Majorana fermions are sufficiently far apart from each other and that the environment only acts locally on the system, these operations are not performed ‘accidentally’ by the environment and the Majorana qubit is protected from both sign flip and bit flip errors. As these protection originates from the conservation of the total fermion parity, the qubit is called parity-protected.

for all observables $A$. Thus, there is no point in making superpositions between states of different parity as there will be no effect on any observable. We can thus restrict ourselves to one superselection sector and keep the fermion parity fixed with either $P = +1$ or $P = -1$. The conclusion of this argument is that out of the $2^N$ states in a fermionic Fock space, only $2^{N-1}$ can be effectively used for quantum computation purposes.

A further restriction to quantum computation using fermions arises from the fact that noninteracting fermions subject to beam splitters, phase-shifters (delay lines), measurements of the state of a single electron (so-called fermionic linear optics) does in fact not lead to any entanglement [26]. In order to generate entanglement, we need to add parity measurement of two electrons which effectively involves interactions between different electrons [27].

3.2 Parity-protected quantum computation

We have seen in the last section that due to the parity-conservation, we need to have two fermionic modes to encode a single qubit. For concreteness, we will work in the even parity superselection sector and have the single logical qubit encodes as $|\bar{0}\rangle = |00\rangle$ and $|\bar{1}\rangle = |11\rangle$. Thinking about a possible implementation in terms of Majorana fermions, we encode each fermionic mode in a pair of Majorana fermions which are localized states sufficiently far separated from each other. As we have seen before, two segments of $p$-wave superconducting nanowires exactly implement this situation, see Fig. 2. We denote the Majorana fermions in the left segment as $\gamma_1$ and $\gamma_2$ and the one on the right segment as $\gamma_3$ and $\gamma_4$ correspondingly. The Majorana fermions are at zero energy thus the two states $|\bar{0}\rangle$ and $|\bar{1}\rangle$ are degenerate in energy. The parity of the number of electrons on the superconducting segments are given by $P_L = -i\gamma_1\gamma_2$ and $P_R = -i\gamma_3\gamma_4$. Due to the parity constraint, we have $P_L = P_R$ and the action
of both operators on the logical qubit emulates the $\sigma^z$ Pauli-operator,

$$\bar{\sigma}^z = -i\gamma_1\gamma_2 = -i\gamma_3\gamma_4$$  \hspace{1cm} (20)

In order to have a complete qubit, we are left with the task to find a logical $\bar{\sigma}^x$, an operator which anticommutes with $\bar{\sigma}^z$. It is easy to see that

$$\bar{\sigma}^x = -i\gamma_2\gamma_3 = -i\gamma_1\gamma_4$$  \hspace{1cm} (21)

anticommutes with $\bar{\sigma}^z$ due to the fact that the single Majorana fermions shared by both operators anticommute with each other. In the situation where all the Majorana fermions are sufficiently far separated from each other, either gate on the logical qubit is a nonlocal operator. Due to this nonlocality, it is highly unlikely that uncontrolled, random fluctuations in the environment will execute a gate thus acts as an error on the logical qubit. This protection of the Majorana qubit is called symmetry-protected topological order [28, 29] or simply parity-protection [30]. The decisive difference to full topological order, as it is for example present in Kitaev’s toric code [31], is the fact that logical Pauli operators are only required to be nonlocal as long as the parity symmetry is conserved. Having a reservoir tunneling single electrons on the superconducting island is a local process which violates the parity-conservation and immediately brings the Majorana qubit out of its computational subspace.

The requirement for operating the Majorana qubit successfully in a protected manner is that the environment does not provide single unpaired electrons. This sounds on the first sight very stringent. However, the physical implementation of the system does only involve superconductors where most of the electrons are paired up into Cooper pairs and where at temperature $T$ only a exponentially small fraction proportional to the Boltzmann factor $e^{-\Delta/T}$ remains unpaired. The storage time of quantum information in a Majorana qubit thus will increase exponentially when lowering the electron temperature.

### 3.3 Anyons

In $3 + 1$ dimensions, we are use to the dichotomy of bosons and fermions. In fact the spin-statistic theorem can be proven in the context of relativistic field theory which states that particles with integer spin are bosons whereas particles with half-integer spin are fermions. The origin of this distinction lies in the fact that the Hamiltonian of identical particles commutes with an arbitrary element of the symmetric group $S_N$ which exchanges the $N$ identical particles. Thus, it is possible to classify the eigenstates of the Hamiltonian in terms of irreducible representations of the permutation group. Any representation whose dimension is larger than one leads to a degeneracy, which is called exchange degeneracy as it originates simply from the fact that particles are indistinguishable. Now it is an experimental fact that exchange degeneracies do not exist; the absence of exchange degeneracy was first noted in the context of statistical mechanics where it manifests itself in an entropy which is not extensive and where it has been dubbed Gibbs paradox\footnote{The statistics of identical particles which transforms according to higher dimension representations of the permutation groups is called parastatistics. However, even if particles with parastatistics where to exist they would offer nothing new as a set of Klein transformations could be used to map particles with parastatistics as bosons or fermions with a set of internal quantum numbers (like spin, . . .). Later, we will see that such a mapping is not possible in $2 + 1$ dimension and that higher dimensional representations of the braid group are truly different from the one-dimensional representations.}.
Fig. 3: (a) Elementary operation of the braid group. The geometric representation of the braid group is in space-time; the horizontal axis is the spacial axis whereas the vertical one is temporal. Every element (called braid) of the braid group $B_N$ consists of $N$ strands. In our example, we have $N = 5$ strands which are numbered from 1 to 5. The generators of the group $B_k$ denote the braiding two strands counterclockwise (in the figure, $B_4$ braids strand 1 and 2 counterclockwise). The inverse operation braids the strand clockwise (in the figure, $B_4^{-1}$ braids strand 4 and 5 clockwise). Two elements of the group are equivalent if the corresponding braids can be smoothly deformed into each other without moving the endpoints denoted by the black dots. The Yang-Baxter equation $B_k B_{k+1} B_k = B_{k+1} B_k B_{k+1}$ provides an important relation between the generators $B_k$ and $B_{k+1}$ and is shown pictorially in (b). That the two braids are topological equivalent can be seen as follows: in both braids strand $k+2$ can be considered to lie in the very back and to end up at the initial position of strand $k$. Similarly strand $k$ lies in front and ends up at the initial position of strand $k+2$. The middle strand starts at $k+1$ and ends at the same place. The braids are equivalent as they can be deformed into each other by sliding the middle strand $k+1$ in between the two other strands from the left to the right.

In $2 + 1$ dimension, the relevant group is the braid group $B_N$ of $N$ strands as trajectory in space-time for exchanging two particles clock or counterclockwise are topologically distinct. We denote with $B_j$ the counter-clockwise exchange of strand $j$ and $j+1$ ($1 \leq j \leq N - 1$). Note that different from the symmetric group $B_j \neq B_j^{-1}$. The braid group fulfills the following relations

$$B_k B_l = B_l B_k, \quad |k - l| \geq 2 \quad \text{and} \quad B_k B_{k+1} B_k = B_{k+1} B_k B_{k+1}, \quad (22)$$

the latter is called Yang-Baxter equation. Different from the symmetric group $S_N$ the group order is infinity which makes the classification of all irreducible representation difficult.

The one-dimensional (unitary) representations of the braid group are simple to construct; representing the action of $B_j$ onto a wavefunction by $e^{i\theta_j}$ with $\theta_j \in [0, 2\pi)$, we immediately get from the Yang-Baxter equation that all the angles are equal, i.e., $\theta_j = \theta$. Note that for $\theta = 0$, we get the customary result for bosons that interchanging two particles does nothing to the wavefunction whereas for $\theta = \pi$ interchanging introduces a minus sign which is the result for fermions. In $2 + 1$ dimension, all angles in between 0 and $\pi$ are allowed and particles with $\theta \neq 0$ or $\pi$ are called (Abelian) anyons. As an example, we note that quasiparticles in the fractional quantum Hall effect at filling fraction $\nu = \frac{1}{n}$ with $n$ an odd integer are anyons with $\theta = \nu \pi$.

Particles whose wavefunctions transform according to higher dimensional irreducible representations of the braid group are called non-Abelian anyons. A necessary ingredient is a ground

\[\text{In } 3 + 1 \text{ dimension clock and counterclockwise depends on the observer (coordinate system) and thus the two exchanges are topologically equivalent.}\]
Fig. 4: Sketch of a Y-junction where three superconducting nanowires meet. Having a Y-junction is an essential ingredient to be able to braid the Majorana fermions as exchanging two particles is not possible in a strictly one-dimensional setting. The two Majorana fermions symbolized by red and green spheres are situated at the ends of the segment of the wire in the topological phase with $Q = 1$ indicated by the brown shading. The gray part of the nanowire is depleted such that it is topological trivial with $Q = 0$. Exchanging the red and green Majorana fermion is done in four steps indicated in the figure. In each step, one of the Majorana fermion is moved to another end of the Y-junction by swapping the corresponding segments of the wire between topological and trivial.

state degeneracy (which grows exponentially with the number of particles). The effect of $B_j$, the counterclockwise exchange two particles $j$ and $j + 1$, is then represented by a unitary matrix $U_j$ on the ground state manifold. As different unitary matrices do not commute, the representation is non-Abelian and thus the particles are called non-Abelian anyons. The usefulness of non-Abelian anyons for topological quantum computation relies on the fact that the degeneracy of the ground state manifold is protected and the gates implemented by the exchange of particles are exact (up to an unimportant global phase) [32]. If for a specific species of non-Abelian anyons for any given gate a braid can be found which approximates the gate with arbitrary accuracy, the non-Abelian anyons are called universal for quantum computation.

3.4 Majorana fermions as non-Abelian particles

We restrict ourself to the discussion of two Majorana fermions. Braiding is local, so it should only affect those particles which are being braided. We will see that the braid statistics can be simply deduced from the fact that the parity remains conserved [33]. First, we have to see that moving a Majorana fermions: braiding of the Majorana fermions the system has to stay in the ground state manifold. Thus everything has to be adiabatic. We change parameters in the Hamiltonian slowly in such a way that everything is slow with respect to the gap. The system then evolves according to the unitary evolution $U(t) = T \exp[-i \int_0^t dt' H(t')]$ where $T$ is the time ordering operator. The operators transform like

$$\gamma_k(t) = U(t)^\dagger \gamma_k U(t).$$  \hspace{1cm} (23)

In a first step, we want to show that we can in fact move a single Majorana fermion without changing its state. Having two adjacent Majorana fermions $\gamma_k$ and $\gamma_{k+1}$ moving the former a bit in the time $T$ results in new Majorana fermions $\gamma'_k = \gamma_k(T)$ and $\gamma'_{k+1} = \gamma_{k+1}(T)$. As fermion parity is conserved, we have

$$P = -i \gamma_k \gamma_{k+1} = -i \gamma'_k \gamma'_{k+1}$$  \hspace{1cm} (24)

13In two dimensions, Majorana fermions exist as bound state in the vortex of chiral $p$-wave superconductors. To determine the braid statistics in this case is a bit more involved [34,35].
Since nothing has happened to $\gamma_{k+1}$ (the evolution did not affect the right Majorana fermion), we have $\gamma'_{k+1} = \gamma_{k+1}$. Plugging this into Eq. (24), we have $\gamma'_k = \gamma_k$, i.e., we can move the Majorana fermion without perturbing it.

In order to obtain the braid group, we need to find the effect of $B_k$, i.e., exchanging the two Majorana fermions in the clockwise direction. This cannot be done in strictly one dimension but involves $Y$-junctions, see Fig. 4 and Refs. [36, 37, 38]. As above, we will again denote the operators after time $T$, that is after the braiding operators with a prime. As the positions of the Majorana fermions is switched after the time $T$, we have $\gamma'_{k+1} = \alpha_k \gamma_{k+1}$ and $\gamma'_{k} = \alpha_{k+1} \gamma_k$.

Due to the fact that the operators have to remain Majorana fermions, we need to require that $\alpha_k, \alpha_{k+1} \in \{\pm 1\}$. The conservation of the parity leads to the relation

$$P = -i \gamma_k \gamma_{k+1} = -i \gamma'_k \gamma'_{k+1} = -i \alpha_k \alpha_{k+1} \gamma_k \gamma_{k+1}$$

(25)

thus, we need that $\alpha_k \alpha_{k+1} = -1$: one of the Majorana fermions picks up a minus sign and one does not. Which of the Majorana fermions picks up a minus sign is a gauge choice and there are no physical effect; so we choose $\alpha_k = -1$ and $\alpha_{k+1} = 1$. Note that braiding the Majorana fermions in the counter-clockwise direction is the inverse operation and thus reverses the sign of both $\alpha_k$ and $\alpha_{k+1}$. It is easy to see that the operator

$$U(T) \equiv U_k = \exp \left( \frac{\pi}{4} \gamma_k \gamma_{k+1} \right) = \frac{1}{\sqrt{2}} (1 + \gamma_k \gamma_{k+1})$$

(26)

implements this unitary operation, where $' \equiv ' $ denotes the fact that the time-evolution $U(T)$ and $U_k$ are equivalent up to an unimportant overall phase. Furthermore, one can check explicitly that the $U_k$s obey the relations in Eqs. (22) and thus they are a representation of the braid group. We would like to explicitly find the gates generated by braiding the Majorana zero mode for the example of a single Majorana qubit, cf. Fig. 2. Straightforward calculation shows

$$U_1 = U_3 = e^{i \pi \gamma_1 \gamma_2 / 4} = e^{i \pi \bar{\sigma}_z / 4}, \quad U_2 = e^{i \pi \gamma_2 \gamma_3 / 4} = e^{i \pi \bar{\sigma}_x / 4},$$

(27)

which are elements of the Clifford group. In fact, it can be shown that in general only Clifford gates can be implemented via braiding. So neither the single qubit rotations are universal nor there is an entangling gate [4]. However in order to make the setup universal, we need to subjoin only a two qubit parity measurement (to generate entanglement) and a $\pi/8$-phase gate [39, 32, 15]. We will show a way to implement such an unprotected gate in Sec. 4.2.

4 Implementations

4.1 Semiconducting nanowires

Last year at TU Delft evidence of Majorana fermions has been found in semiconducting InSb nanowires coupled to a NbTiN superconductor in the presence of a magnetic field of around 100 mT [40]. The experiment realized an idea of Refs. [41, 42] which extended earlier ideas for quantum wells to nanowires [43, 44]. In this chapter, we provide a brief introduction into...
the physics of semiconducting nanowires and motivate the connection to Majorana fermions in spinless \( p \)-wave nanowires. The simplest model for conduction electrons a semiconducting nanowire in a magnetic field \( B \) pointed along the nanowire direction is given by

\[
H_{NW} = \int dz \left[ \frac{\hbar^2}{2m} |\psi'(z)|^2 - \mu |\psi(z)|^2 + \frac{\alpha_{so}}{\hbar} \sigma^y p \psi(z) - \frac{g \mu_B B}{2} \sigma^z \psi(z) \right]
\]  

(28)

with the field field operator \( \Psi^T = (\psi_\uparrow \ \psi_\downarrow) \). Here, we have assumed that the spin-orbit characterized by \( \alpha_{so} \) is of Rashba-type due to electric fields perpendicular to the substrate as Dresselhaus terms are absent for experimentally-relevant zincblende nanowires grown along the 111 crystal direction. For InSb nanowires the parameters are given by \( m \approx 0.015 m_e \), \( g \approx 50 \), and \( \alpha \approx 0.2 \, \text{eV} \). In writing down the Hamiltonian \( (28) \), we have silently assumed that only a single mode is relevant in the nanowire whereas in the experiment there are most likely of the order of 5 modes contributing. This difference, even though important for a detailed description of the experimental findings, is not relevant for the generic discussion we intend to provide. In the same spirit, we have neglected any orbital effects of the magnetic field on the nanowire.

The proximity to an \( s \)-wave superconductor introduces pair-correlations in the nanowire. This can be expressed with an additional term in the Hamiltonian of the form

\[
H_{SC} = \Delta \int dz \left[ \psi_\uparrow(z) \psi_\uparrow(z) + \psi_\downarrow(z) \psi_\downarrow(z) \right]
\]  

(29)

where \( \Delta / \hbar > 0 \) is the rate at which Cooper pairs are injected into the nanowire. For reasonable strong superconductors and clean interfaces between the superconductor and the nanowire we may expect \( \Delta \approx 1 \, \text{K} \).

Following Ref. [44], we want to show that for strong magnetic fields such that the Zeeman energy \( E_Z = \frac{1}{2} g \mu_B B \) is larger than \( \Delta \) and \( m \alpha_{so}^2 / \hbar^2 \) the system is equivalent to the spinless nanowire discussed in Sec. [2.2]. For such large magnetic fields basically only electrons with spin-up are present. We thus write \( \psi(z) = \psi_\uparrow(z) \). As the spin-orbit term in \( (28) \) does not commute with the Zeeman term, it admixes the spin-down component. In lowest order perturbation theory, we thus have

\[
\psi_\downarrow(z) = \frac{\alpha_{so} \sigma^y p}{2 \hbar E_Z} \psi(z) = \frac{\alpha_{so}}{2 \hbar E_Z} \psi'(z)
\]  

(30)

Plugging this expression into the full Hamiltonian \( H_{NW} + H_{SC} \), we obtain to lowest order the effective Hamiltonian

\[
H_{eff} = \int dz \left[ \frac{\hbar^2}{2m} |\psi'(z)|^2 - \mu_{eff} |\psi(z)|^2 - \Delta_{eff}[\psi_\uparrow(z) \psi_\downarrow(z) - \psi(z) \psi'(z)] \right]
\]  

(31)

with \( \mu_{eff} = \mu + E_Z \) and \( \Delta_{eff} = \Delta_{so} / 2 \hbar E_Z \). The expression \( (31) \) coincides with the BCS mean-field Hamiltonian \( H_{MF} \) for spinless electrons Eq. \( (7) \) with an effective \( p \)-wave pairing \( \Delta(z) = \Delta_{eff} \delta'(z) \). We have seen in Sec. [2.3] that this system hosts Majorana fermions at its ends provided that \( \mu_{eff} > 0 \).

We know however that the topological phase is stable and can only be removed by closing the bulk gap of the nanowire. The condition for the closing of the gap of \( H_{NW} + H_{SC} \) is a zero eigenvalue of the Bogoliubov-de Gennes Hamiltonian

\[
\hbar_{BdG}(p) = \xi(p) \tau^z + \frac{\alpha_{so} p}{\hbar} \sigma^y \tau^z - E_Z \sigma^z + \Delta \tau^x
\]  

(32)
subject to periodic boundary conditions. Due to the presence of the spin-orbit term proportional to $\alpha_{so}$ the gap can only be closed for $p = 0$. At $p = 0$, the energy eigenvalues of $h_{\text{BdG}}$ are given by

$$E_{p=0} = \pm \sqrt{\mu^2 + \Delta^2 - E_Z}.$$  \hspace{1cm} (33)

We have seen before that for large magnetic field with $E_Z \gg \Delta, |\mu|$ the system essentially implement a spinless $p$-wave nanowire in the topological phase with Majorana fermions at the end. This phase extends in the full model up to the point where $E_{p=0} = 0$, i.e., the gap closes. As a consequence, we obtain an expression for the topological charge

$$Q(h_{\text{BdG}}) = \begin{cases} 1, & |\mu| < \mu_c, \\ 0, & |\mu| > \mu_c, \end{cases} \quad \text{with} \quad \mu_c = \begin{cases} \sqrt{E_Z^2 - \Delta^2}, & E_Z > \Delta, \\ 0, & E_Z < \Delta. \end{cases}$$  \hspace{1cm} (34)

In order for the experiments to be in the regime where there are Majorana fermions (indicated by $Q = 1$), we thus need $E_Z > \Delta$. This is only possible due to the large $g$-factor of the InSb nanowire with respect to the superconductor with $g_{\text{SC}} \approx 2$ as the superconducting state is destroyed latest at a magnetic field such that $\frac{1}{2}g_{\text{SC}} \mu_B B = (g_{\text{SC}}/g) E_Z \lesssim \Delta$ which is the so-called Pauli limit.

### 4.2 Hybrid structures

As discussed in details above, the parity protection of the Majorana qubit is potentially useful as it allows quantum computation where the qubit as well as the gates are topological protected. However, we have also seen that the braiding operation do not allow for universal quantum computation. In particular, a $\pi/8$-phase gate and a parity measurement of two qubits are missing. In this section, we want to show how hybrid structures involving superconducting qubits besides the Majorana qubits offer a way to make the Majorana qubits universal for quantum computation.

It is important to note that in order to be able to measure the state of the Majorana qubit its parity-protection has to be removed. This can in principle be achieve by fusing two Majorana fermions, i.e., make their wavefunctions overlapping, and measuring the resulting ground state energy of the system. However, as we need additionally a measurement of the parity of two qubits (without determining the state of each of the qubits) interactions, terms which couple four Majorana fermions, are indispensable \([32]\).

The most natural choice for an interaction is the Coulomb interaction due to the charge of the electrons which form the basis for the Majorana fermions \([45, 30]\). Starting with a Majorana qubit involving four Majorana fermions on a superconducting island, cf. Fig.2 the protection of the qubit can be lifted by ‘cutting’ the superconducting islands into two pieces each of them having two Majorana fermions present. The state of the Majorana qubit is then encoded in the fact whether there is an even or odd number of electrons on each of the parts, see Fig.5. The ‘cutting’ is of course not meant literally. In fact having from the beginning two separate superconducting islands coupled with a Josephson coupling with strength $E_J$ (originating from Cooper pairs tunneling between the two superconductors) and a capacitive coupling with strength $E_C$ (originating from the charging energy between the two superconducting islands). In the regime $E_J \gg E_C$, the two superconducting islands behave essentially as if they where a single piece of superconductor and the qubit is protected. Lowering the ratio $E_J/E_C$ increases the charge sensitivity (the difference in energy between the islands having even and
Fig. 5: (a) In order to measure and/or manipulate the Majorana qubit the topological protection has to be removed. This can be achieved by ‘cutting’ the superconducting island into two parts, a process which is implemented in reality by having from the beginning two superconducting islands which are coupled by a capacitance with energy $E_C$ and a Josephson junction with energy $E_J$ (bottom). In the regime where $E_J \gg E_C$, superconducting correlations between the two islands are strong and thus the two islands essentially behave like a single superconducting islands with a Majorana qubit consisting of four Majorana fermions. (b) Lowering the ratio $E_J/E_C$ the charge sensitivity $\delta$ which describes the energy difference between the two logical states of the Majorana qubit increases exponentially and readout and manipulation becomes possible.

odd number of electrons) and thereby lifts the parity-protection. In fact, the difference in energy of the two states of the Majorana qubit is exponentially depending on the ratio $E_J/E_C$ as $\delta \propto \exp\left(-\sqrt{8E_J/E_C}\right)$, see Ref. [46]. The protected qubit discussed so far is given for large $E_J/E_C$ where $\delta$ essentially vanishes.

In the concrete implementation proposed in [30], the read-out of the Majorana qubit is achieved by distribution the four Majorana fermions which form the logical qubits on the two plates of the capacitor in a Cooper pair box, see Fig. 6. By varying the flux $\Phi$ through a split Josephson junction, the Josephson energy $E_J \propto \cos(e\Phi/\hbar)$ becomes tunable. As the charge sensitivity depends exponentially on the ratio $E_J/E_C$, a variation of the charge sensitivity (the degree of protection of the qubit) over two orders of magnitude has been achieved in the transmon qubit design of the Yale group [47]. Due to this exponential dependence it becomes possible to turn the protection of the qubit on and off at will. The fact that there is a tunable energy difference $\delta$ of the two states of the Majorana qubit directly leads to the possibility of implementing an arbitrary phase gate. Starting and ending with a qubit in the protected state with $\delta = 0$, we can make $\delta$ finite for some time $\tau$ such that $\delta \tau/\hbar = 2\phi$ which implement the phase gate $\exp(i\phi\sigma_z)$. Especially for $\phi = \frac{\pi}{8}$, we obtain the missing $\frac{\pi}{8}$ phase gate.

Similarly, for the readout of the Majorana qubit we use the fact that due to the same interaction the energy difference $\Delta\varepsilon_{\bar{\sigma}_z} = \Delta\varepsilon + \delta \bar{\sigma}_z$ between the ground state and the first excited state of the transmon qubit depends on the state of the Majorana qubit $\bar{\sigma}_z = \pm 1$. The state of the transmon qubit can be read out by sending a microwave probe beam through the transmission

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16 The transmon is simply a Cooper pair box with $E_J \gtrsim E_C$ which is placed in a transmission line resonator for read-out, hence the name.
Fig. 6: Read out of a parity qubit in a Cooper pair box. Two superconducting islands (blue), connected by a split Josephson junction (crosses) form the Cooper pair box. The topological Majorana qubit is formed by four Majorana fermions (red spheres), at the end points of two undepleted segments of a semiconductor nanowire (striped ribbon indicates the depleted region). A magnetic flux $\Phi$ enclosed by the Josephson junction controls the charge sensitivity of the Cooper pair box. To read out the topological qubit, two of the four Majorana fermions that encode the logical qubit are moved from one island to the other. Depending on the quasiparticle parity, the resonance frequency in a superconducting transmission line enclosing the Cooper pair box (green) is shifted upwards or downwards by the amount which is exponentially small in $E_J/E_C$.

The resonance frequency $\omega_{\text{res}}$ of the cavity is given by

$$\omega_{\text{res}} = \omega_0 - \frac{g^2}{\omega_0 - \Delta\varepsilon_{\bar{\sigma}_z}/\hbar}$$  \hspace{1cm} (35)$$

provided the fact that the qubit is initially in its ground state and the cavity frequency $\omega_0$ is far detuned from $\Delta\varepsilon/\hbar$; here, $\omega_0$ is the bare resonance frequency of the cavity and $g$ is the Jaynes-Cummings coupling between the cavity mode and the transmon qubit. The small shift of the cavity frequency due to the dependence of $\omega_{\text{res}}$ on the state $\bar{\sigma}_z$ of the Majorana qubit can be measured sensitively as a phase shift of the transmitted microwaves.

In the case where there are more than two Majorana fermions per superconducting island, the transmon qubit couples directly to parity of the number of electrons on each of the island. Thus, a joint parity measurement on two Majorana qubits can likewise be performed by moving four out of the eight Majorana fermions to the other island.

Summarizing, the hybrid design of a coupled transmon and Majorana qubit retains the full topological protection with exponential accuracy in the off-state ($E_J/E_C \gg 1$). This hybrid device can be used to implement a phase gate on the Majorana qubit as well as to jointly read out sets of Majorana qubits. Together with braiding and single qubit readout, these are the operations required for a universal quantum computer.
5 Conclusion

We have shown that Majorana fermions can be perceived as ‘half’ and ordinary Dirac fermion. Interestingly, these particles emerge as end states at zero energy in superconducting $p$-wave nanowires independent of any microscopic details, a fact which can be traced back to the change of the topological charge from the nanowire to the surrounding vacuum. Four Majorana fermions encode a single qubit. Keeping the fermion parity in the system conserved, gates on the qubit can only be performed by operators which involve two Majorana fermions. As the Majorana fermions are spatially separated, random (local) noise due to environmental fluctuations will not couple to the Majorana qubit and thus will not lead to decoherence of the Majorana qubit. Most importantly, we have seen that Majorana fermions are non-Abelian particles which implies that one can perform protected gates just by braiding the particles around each other. Alas, the operations generated by braiding the particles are not enough to make the system a universal quantum computer. For that reason, we have shown how the coupling of a Majorana qubit to a superconducting transmon qubit can be used both for performing measurements of the Majorana qubit and implementing the missing gates. Finally, we have discussed the essential ingredients (semiconducting nanowire with strong spin-orbit interaction and large $g$-factor coupled to an $s$-wave superconductor) for implementing these ideas in a laboratory without the need for the elusive $p$-wave superconductor.

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