INFRARED BEHAVIOUR
OF
SOFTLY BROKEN SQCD AND ITS DUAL

Andreas Karch (1), Tatsuo Kobayashi (2),†, Jisuke Kubo (3),∗ and George Zoupanos (1,4),∗∗

(1) Institut für Physik, Humboldt-Universität zu Berlin, D-10115 Berlin, Germany
(2) Department of Physics, High Energy Physics Division, University of Helsinki and Helsinki Institute of Physics, FIN-00014 Helsinki, Finland
(3) Max-Planck-Institut für Physik, Werner-Heisenberg-Institut D-80805 Munich, Germany
(4) Physics Department, Nat. Technical University, GR-157 80 Zografou, Athens, Greece.

Abstract

Applying the recently obtained results on the renormalization of soft supersymmetry-breaking parameters, we investigate the infrared behaviour of the softly broken supersymmetric QCD as well as its dual theory in the conformal window. Under general assumptions on β-functions, it is shown that the soft supersymmetry-breaking parameters asymptotically vanish in the infrared limit so that superconformal symmetry in softly broken supersymmetric QCD and in its dual theory revives at the infrared fixed point, provided the soft scalar masses satisfy certain renormalization group invariant relations. If these relations are not satisfied, there exist marginal operators in both theories that lead to the breaking of supersymmetry and also colour symmetry.

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*On leave from: Institute for Theoretical Physics, Kanazawa University, Kanazawa 920-1192, Japan
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Hunting renormalization group (RG) invariant relations among independent parameters in a given phenomenological model, especially supersymmetric one, has proved in recent years to be successful in predicting, free otherwise, parameters of the model [1]–[4]. These are ranging from the top quark mass, the spectrum of the superpartners to the lightest Higgs mass in \( N = 1 \) supersymmetric finite and non-finite Gauge-Yukawa unified models [1]–[4].

The basic idea is based on the principle of coupling reduction proposed in ref. [5]. Most of the practical applications of the principle, however, have been restricted to lower orders in perturbation theory, because the \( \beta \)-functions are mostly known only in lower orders. Parallel to this development, interesting progresses has been made on the renormalization properties of the soft supersymmetry-breaking (SSB) parameters [6]–[9]. In ref. [6] the spurion formalism in a softly-broken supersymmetric gauge theory has been used to derive a certain set of rules to relate the SSB parameters with those of the corresponding supersymmetric theory. Later it has been shown [8, 9] that these rules can be realized by differential operators acting on the anomalous dimensions \( \gamma_i \) and the gauge coupling \( \beta \)-function \( \beta_g \). As a result, the \( \beta \)-functions of the SSB parameters can be obtained from \( \gamma_i \) and \( \beta_g \). A consequence is that it has become possible to obtain a closed form for RG invariant relations among the SSB parameters [10]–[13]. Another consequence is that the asymptotic behaviour of the SSB parameters in the ultraviolet (UV) as well as in infrared (IR) limit can be studied by using \( \gamma_i \) and \( \beta_g \) only. If \( \gamma_i \) and \( \beta_g \) are exact, the results on the SSB parameters derived from them are exact, too.

The purpose of the present paper is to examine on the light of the above mentioned new development the asymptotic behaviour of softly broken supersymmetric QCD (SQCD) and its dual theory in the IR limit, especially in the regime for which according to Seiberg’s conjecture [14] there would exist a non-trivial IR fixed point if supersymmetry was exact. The existence of a stable IR fixed point in the space of the dimensionless couplings (gauge and Yukawa couplings) of a supersymmetric Yang-Mills theory implies that the \( \beta \)-functions and their derivatives of these couplings should have certain properties in the IR limit. We will use these properties to derive the asymptotic behaviour of the SSB parameters, and will be assuming that the above mentioned perturbative technique to obtain their RG functions is applicable to the problem we are concerned with. Moreover, we assume that the kinetic
term in the dual theory takes the canonical form. We will find that supersymmetry in SQCD and also in its dual theory restores itself at the IR fixed point, provided the soft scalar masses satisfy the RG invariant relations given by (30) for SQCD, and (41) and (46) for its dual theory, respectively. If these relations are not satisfied, there exist marginal operators which will break supersymmetry and local colour symmetry in the IR limit.

There exists a rather extensive literature [15]–[18] on duality of softly broken $N=1$ SQCD. Our work is related to those of refs. [17, 18], in which the asymptotic behaviour of the SSB parameters has been also considered. But their results are obtained mainly in the free magnetic theory. At the end of our paper we will briefly comment on the relation to their result.

Following the notation of ref. [11], we start with a generic discussion and consider an $N=1$ supersymmetric gauge theory with the superpotential

$$W(\Phi) = \frac{1}{6}Y^{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{2}\mu^{ij}\Phi_i\Phi_j,$$

and introduce the SSB part $L_{SSB}$:

$$L(\Phi, W) = -\left(\int d^2\theta\eta(h^{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{2}b^{ij}\Phi_i\Phi_j + \frac{1}{2}MW_\alpha^aW_{A\alpha}) + \text{h.c.}\right) - \int d^4\tilde{\theta}\tilde{\eta}\tilde{\Phi}(m^2)^j_i(e^{2gV})^k_i\Phi_k,$$

where $\eta = \theta^2$, $\tilde{\eta} = \tilde{\theta}^2$ are the external spurion superfields and $\theta$, $\tilde{\theta}$ are the usual grassmannian parameters, and $M$ is the gaugino mass. The $\beta$-functions of the $M$, $h$ and $m^2$ parameters can be computed from [8, 9]:

$$\beta_M = 2O\left(\frac{\beta_g}{g}\right),$$

$$(\beta_h)^{ijk} = \gamma_i h^{ijk} + \gamma^j h^{ikl} + \gamma^k h^{ijl} - 2\gamma^i_Y Y^i jk - 2\gamma^j_Y Y^i k - 2\gamma^k_Y Y^i j,$$

$$(\beta_{m^2})^j_i = \left[\Delta + X \frac{\partial}{\partial g}\right] \gamma^j_i,$$

$$O = \left(M g^2 \frac{\partial}{\partial g^2} - h_{lmn} \frac{\partial}{\partial Y_{lmn}}\right),$$

$$\Delta = 2O^* + 2|\tilde{M}|g^2 \frac{\partial}{\partial g^2} + \tilde{Y}_{lmn} \frac{\partial}{\partial Y_{lmn}} + \tilde{Y}_{lmn} \frac{\partial}{\partial Y_{lmn}},$$

where $(\gamma^i_j)_l = O \gamma^i_j$, $Y_{lmn} = (Y_{lmn})^*$, and $\tilde{Y}^{ijl} = (m^2)^i_j Y^{i j} + (m^2)^j_i Y^{i k} + (m^2)^k_i Y^{i j}$. Note that $X$ in eq. (3) is explicitly known only in the lowest order [19]. However, there
exists an indirect method (which is based on a RG invariance argument) to fix the exact form of $X$ [11, 12, 13], as we will see below. We do not consider the $B$ parameters in the following discussions, because they do not enter into the $\beta$-functions of the other quantities and moreover they are absent in SQCD as well as in its dual theory.

Jack et al. [11], generalizing the idea of Kazakov [10] who has treated a finite theory, have found 1 that if

$$h_{ijk} = -M(Y_{ijk})' \equiv -M \frac{dY_{ijk}(g)}{d\ln g},$$

$$m_i^2 = |M|^2 \{(1 + \tilde{X}(g))(g/\beta_g)(\gamma_i(g)) + \frac{1}{2}[(g/\beta_g)\gamma_i(g)]' \}$$

are satisfied, then the differential operators $\mathcal{O}$ and $\Delta$ ([6] and [7], respectively) can be written as

$$\mathcal{O} = \frac{M}{2} \frac{d}{d\ln g}, \quad \Delta + X \frac{\partial}{\partial g} = |M|^2 \left[ \frac{1}{2} \frac{d^2}{d(\ln g)^2} + (1 + \tilde{X}(g)) \frac{d}{d\ln g} \right],$$

where

$$g\tilde{X}(g) = \frac{1}{|M|^2} X(g, Y(g), Y^*(g), h(M, g), h^*(M, g), m^2(|M|^2, g)).$$

It has been further shown in ref. [11] that the unknown term $\tilde{X}$ ([11]) has to have the form

$$\tilde{X} = \frac{1}{2} \left( \ln(\beta_g/g) \right)' - 1$$

in order that the expression (9) is RG invariant. Therefore, eq.(9) becomes [11],

$$m_i^2 = \frac{1}{2} |M|^2 (g/\beta_g)(\gamma_i(g))'.$$

In ref [12], with the use of the Novikov-Shifman-Vainstein-Zakharov (NSVZ) $\beta$-function [20] for the gauge coupling [21]

$$\beta_g^{\text{NSVZ}} = \frac{g^3}{16\pi^2} \left[ \frac{\sum_i T(R_i)(1 - 2\gamma_i) - 3C(G)}{1 - g^2 C(G)/8\pi^2} \right],$$

1Under the assumption that $\gamma_i = \gamma_i \delta_{ij}$, $(m^2)_{ij} = m_i^2 \delta_{ij}$, and $Y_{ijk}(\partial/\partial Y_{ijk}) = Y_{ijk}(\partial/\partial Y_{ijk})$ on the space of the RG functions.

2The factor 2 of $\gamma_i$ should be compare with 1/2 which appears for $\beta_g^{\text{NSVZ}}$ in ref. [22], where a typographical error has been made. There is an indication of the presence of a correction term to $\beta_g^{\text{NSVZ}}$ in higher order [23].
we have shown that the sum rule  

\[ m_i^2 + m_j^2 + m_k^2 = |M|^2 \left\{ \frac{1}{1 - g^2 C(G)/(8\pi^2)} \frac{d\ln Y_{ijk}}{d\ln g} + \frac{1}{2} \frac{d^2 \ln Y_{ijk}}{d(\ln g)^2} \right\} \]

(15)

is RG invariant if the \( \tilde{X} \) on the subspace defined by \( Y = Y(g) \) and eq. (8) takes the form [12]

\[ \tilde{X} = \frac{-|M|^2 C(G) + \sum_i m_i^2 T(R_i)}{C(G) - 8\pi^2/g^2} \].

(16)

Eq. (16) is consistent with (12) and the explicit calculations [19], of course. It has been shown later [13] that the restriction of the constrained subspace can be removed. For SQCD (without Yukawa couplings), the \( \beta_{m^2} \) in the NSVZ scheme becomes [12]

\[ \beta_{m_i^2}^{\text{NSVZ}} = \left[ |M|^2 \left\{ \frac{1}{1 - g^2 C(G)/(8\pi^2)} \frac{d}{d\ln g} + \frac{1}{2} \frac{d^2}{d(\ln g)^2} \right\} \right] \gamma_i^{\text{NSVZ}} \].

(17)

Before we shall come to discuss the IR behaviour of softly broken SQCD and its dual theory, we would like to briefly recall the dynamics of SQCD and also Seiberg’s conjectures [14]. SQCD is based on the gauge group \( SU(N_c) \), which contains a gauge superfield \( W_\alpha \) and quark supermultiplets in the \((N_c + \overline{N}_c)\) representations of \( SU(N_c) \). Let us call the quark superfields \( Q_i^\alpha, \overline{Q}_i^\alpha \), where \( i = 1, \ldots, N_f \) is a flavour index and \( \alpha = 1, \ldots, N_c \) is a colour index. When these superfields are massless, the theory exhibits the following global symmetry holding at the quantum level \( SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R \), where \( SU(N_f)_L \) acts on the \( Q_i^\alpha \), \( SU(N_f)_R \) acts on the \( \overline{Q}_i^\alpha \), the \( U(1)_B \) denotes the baryon number, and the \( U(1)_R \) denotes the anomaly free \( R \)-symmetry. The superfields transform under the full symmetry group of the theory \( G = SU(N_c) \times [ SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R ] \) as follows:

\( Q : (N_c; N_f, 1; 1/N_c, (N_f - N_c)/N_f) \), \( \overline{Q} : (\overline{N}_c, 1, \overline{N}_f; -1/N_c, (N_f - N_c)/N_f) \), \( W_\alpha : (N_c^2 - 1; 1, 1; 0, 1) \).

3The lower-order sum rules for the soft scalar masses have been obtained in various theoretical analyses, including those in superstring models. See [22, 3, 4].

4Eqs. (8) and (13) ensure finiteness of the SSB sector in finite theories [13]. In ref. [23] duality in finite theories has been discussed.
According to Seiberg’s conjecture [14], SQCD for \( N_f > N_c + 1 \) is dual to, i.e. can be described in the IR limit by, a supersymmetric gauge theory based on the group \( SU(\tilde{N}_c) \) with \( \tilde{N}_c = N_f - N_c \), which couples to the elementary chiral superfields \( q^i, \bar{q}_i, \) \( i = 1, \ldots, N_f \), as well as to an elementary gauge-singlet superfield \( T^j_i \). The meson superfield \( T \) couples to \( q, \bar{q} \) via the superpotential

\[
\tilde{W} = Y q T \bar{q}, \tag{18}
\]

where \( Y \) is a Yukawa coupling. Without the superpotential the theory would have an additional global \( U(1) \) symmetry acting on \( T \). One can explicitly check that the superpotential preserves the anomaly free \( R \)-symmetry. Therefore, the newly constructed theory has the same global symmetry as the original SQCD. Seiberg [14] refers to the relation between this theory and the original as non-Abelian electric-magnetic duality.

Seiberg [14] has further conjectured on the existence of an IR fixed point in the \( \beta \)-function of SQCD. To recall his proposal, consider the NSVZ \( \beta \)-function [14] for SQCD:

\[
\beta(g) = -\frac{g^3}{16\pi^2} \left( \frac{3N_c - N_f + 2N_f \gamma(g^2)}{1 - N_c g^2/8\pi^2} \right), \quad \gamma(g^2) = -\frac{g^2}{16\pi^2} \frac{N_c^2 - 1}{N_c} + O(g^4). \tag{19}
\]

There is a non-trivial zero of the \( \beta \)-function for \( N_f = (3 - \epsilon)N_c, N_f, N_c >> 1 \). At order \( \epsilon \), the fixed point \( g^2_* \) is given by

\[
N_c g^2_* = \frac{8\pi^2}{3}\epsilon. \tag{20}
\]

In fact it was argued in [14] that such a fixed point exists in the range \( \frac{2}{3}N_c \leq N_f \leq 3N_c \). In SQCD the key observation is that the superconformal theory at the fixed point has a dual (magnetic) description in terms of a different gauge theory based on \( SU(N_f - N_c) \).

Now we come to our main result. As we will see, the RG invariant relations (8) and (13) are IR attractive, and they play a crucial role in investigating the behaviour of the SSB parameters near an IR fixed point. The sum rule (15) will be used in the dual theory to derive from the behaviour of the soft scalar masses near the fixed point (which is obtained by linearizing the problem) a condition which should be satisfied away from the fixed point

\[\text{Such a behaviour was already conjectured to hold in ordinary QCD in [25] (see also [26]). For a recent discussion on the status of this speculation, see [27] and for recent findings in lattice studies, see e.g. [28].}\]
in order to restore supersymmetry at the fixed point. A number of numerical analyses on the 
IR stability of the SSB parameters using lower order $\beta$-functions have been made previously 
\cite{24, 4}. Their results are suggestive for our problem. But we would like to emphasize that 
we will discuss the IR stability of the RG invariant relations (8) and (13) in the conformal 
window for which an IR stable fixed point in the space of the gauge coupling for SQCD, and 
in the space of the gauge and Yukawa couplings for the dual theory is supposed to exist. 
Our analysis does not rely on the explicit form of the $\beta$-functions for the SSB parameters.

To begin with, using the formulae (3)–(5) and the RG invariant solutions (8) and (13), we 
show that there always exists at least a trajectory in the space of the SSB parameters that 
approaches the origin if the gauge and Yukawa couplings approach a non-trivial fixed point. 
To this end, we note that if eq. (8) is satisfied, then the differential operator $\mathcal{O}$ becomes a 
total derivative operator as we see from eq. (10). Then eq. (3) becomes nothing but the 
Hisano-Shifman relation \cite{7}

$$M = M_0(\beta_g/g) ,$$

where $M_0$ is a RG invariant quantity. Since $\beta_g \to 0$ as $g \to g_* \neq 0$ by assumption, we 
find that $M \to 0$ as $g \to g_*$. Similarly, eq. (8) implies that $h^{ijk} = -M_0(\beta_g/g)(Y^{ijk})' \to 0$.

Finally, from eq. (13) we see that $m_i^2 = (1/2)|M_0|^2(\beta_g/g)(\gamma_i)' \to 0$ as $g \to g_*$. In what 
follows, we will investigate carefully whether the origin of the SSB parameters is a stable IR 
fixed point.

First we consider softly broken SQCD and examine the IR behaviour of the gaugino mass $M$ and soft scalar squared masses $m_Q^2$, $m_{\tilde{Q}}^2$. Since there is no Yukawa coupling in SQCD, 
the differential operator (3) becomes a total derivative operator, and we have

$$\beta_M = Mg \frac{d}{dg}(\beta_g/g) = M(\beta_g/g)' .$$

The conjecture that $g_*$ is a stable IR fixed point for SQCD implies that

$$\Gamma_M \equiv \frac{d\beta_g}{dg}|_* > 0 ,$$

where $|_*$ means an evaluation at the fixed point. We may assume that

$$|\beta_g| \quad \text{and} \quad \left| \frac{d\beta_g}{dg} \right| < \infty$$

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for the range of $g$ we are considering. Eq. (22) implies the Hisano-Shifman relation [7], and so if $M_0$ in the r.h. side of (21) is a non-vanishing constant, then the gaugino mass $M$ has to vanish at the fixed point, as we have seen above. Moreover, one sees from eqs. (22) and (23) that the fixed point $M_s = 0$ is a stable one, because $M \sim e^{\Gamma_{M,t}} \to 0$ as $t \to -\infty$, where in the lowest order approximation in the $\epsilon$ expansion we have $\Gamma_M = \epsilon^2/3$.

To discuss the IR behaviour of $m^2_{Q,\overline{Q}}$ on the same level as $M$, we have to go the NSVZ scheme, and use the $\beta$-function (17). First we would like to show that the RG invariant relation (13) is IR attractive. To this end, we note that for the $\beta^\text{NSVZ}_Q$ given in eq. (14), the conditions (23) and (24) are satisfied if

$$\Gamma_\gamma = \frac{1}{2} \frac{d}{dg} (\gamma_Q + \gamma_{\overline{Q}})|_\ast < 0 \text{ and } N_c - 8\pi^2/g^2 < 0$$

are satisfied, where we have used $\gamma_Q = \gamma_{\overline{Q}}$. Then we consider the behaviour of $m^2_Q$ and $m^2_{\overline{Q}}$ near the RG invariant relation (13),

$$m^2_{Q,\overline{Q}} = m^2_{(0)Q,\overline{Q}} + \delta m^2_{Q,\overline{Q}}, \quad m^2_{(0)Q,\overline{Q}} \equiv \frac{1}{2} |M|^2 (g/\beta^\text{NSVZ}_Q)(\gamma^\text{NSVZ}_Q)'$$

Linearizing the evolution equation near the RG invariant relation (13), we find that

$$\frac{d}{dt} \delta m^2_Q \simeq \frac{d}{dt} \delta m^2_{\overline{Q}} \simeq \frac{1}{2} \Gamma_m^2 (\delta m^2_Q + \delta m^2_{\overline{Q}}), \quad \Gamma_m^2 \equiv \frac{g_* N_f \Gamma_\gamma}{N_c - 8\pi^2/g^2}$$

where $\Gamma_\gamma$ is defined in eq. (23). Since $\Gamma_m^2$ is positive (see eq. (25)), we find

$$\delta m^2_Q - \delta m^2_{\overline{Q}} = \text{const.}, \quad \delta m^2_Q + \delta m^2_{\overline{Q}} \sim e^{\Gamma_m^2 t} \to 0 \text{ as } t \to -\infty.$$  

(In the lowest order approximation in the $\epsilon$ expansion we have $\Gamma_m^2 = \epsilon^2/3$.) Therefore, if the difference $\delta m^2_Q - \delta m^2_{\overline{Q}}$ is non-zero at some point, then we obtain

$$\delta m^2_Q = -\delta m^2_{\overline{Q}}$$

in the IR limit. Since, however, $m^2_{(0),Q,\overline{Q}}$ (defined in (26)) vanish at the fixed point, we see that

$$m^2_Q = m^2_{\overline{Q}}$$

should be satisfied in order not to break the colour symmetry in the IR limit. Then from eq. (28) we may conclude that

$$m^2_Q = m^2_{\overline{Q}} \sim e^{\Gamma_m^2 t} \to 0 \text{ as } t \to -\infty.$$  

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To conclude, we have shown that superconformal symmetry revives at the IR fixed point if $m_Q^2 = m_Q$. Otherwise, the $SU(N_c)$ gauge symmetry and supersymmetry is broken.

The basic idea for treating the IR behaviour of the SSB parameters of the dual theory is the same as the case of SQCD, where we assume that the kinetic term in the dual theory takes the canonical form. A slight difference is that in this case there is a Yukawa coupling $Y$ in the theory, and hence a trilinear coupling $h$ in the softly broken case.\footnote{The Yukawa coupling $Y$ is presumably related to the gauge coupling $\tilde{g}$, because in SQCD there is no such freedom. In this connection, the reduction of $Y$ in favour of $\tilde{g}$ has been suggested for the dual theory in ref.\footnote{This is satisfied in the $\tilde{\epsilon}$ expansion.}}

Following Seiberg, we assume that there exists an IR fixed point in the space of $\tilde{g}$ and $Y$ (the gauge coupling for the dual theory is denoted by $\tilde{g}$, and the gaugino mass by $\tilde{M}$). The non-triviality of the fixed point of $\beta_Y$ implies

$$ (\gamma_q + \gamma_{\bar{q}} + \gamma_T)|_* = 0 . $$

(32)

Further, the stability of the IR fixed point requires, among other things, that

$$ \Gamma_{\tilde{M}} \equiv \frac{d \beta_{\tilde{g}}}{d \tilde{g}}|_* > 0 . $$

(33)

We in addition assume that \footnote{This is satisfied in the $\tilde{\epsilon}$ expansion.}

$$ 2 \Gamma_h \equiv \frac{\partial \beta_Y}{\partial Y}|_* = \frac{\partial}{\partial Y}[Y(\gamma_q + \gamma_{\bar{q}} + \gamma_T)]|_* = Y_* \frac{\partial}{\partial Y}(\gamma_q + \gamma_{\bar{q}} + \gamma_T)|_* > 0 . $$

(34)

As in the case of SQCD, we assume that $\beta_{\tilde{g}}, \beta_Y$ together with their derivatives with respect to $\tilde{g}$ and $Y$ in the space of $\tilde{g}$ and $Y$ we are interested in exist. For the NSVZ scheme, eq. \footnote{This is satisfied in the $\tilde{\epsilon}$ expansion.} means that

$$ \frac{d}{d \tilde{g}}(\gamma_q + \gamma_{\bar{q}})|_* < 0 , $$

(35)

where as in the case of SQCD $\tilde{N}_c - 8\pi^2/\tilde{g}^2 < 0$ is assumed.

Now we consider the RG invariant relation \footnote{This is satisfied in the $\tilde{\epsilon}$ expansion.} and show that it is IR attractive. Defining

$$ h = h_0 + \delta h , \quad h_0 = -\tilde{M}Y' = -\tilde{M}\tilde{g} \frac{dY}{d\tilde{g}} $$

(36)
and linearizing the evolution equations, we find
\[
\frac{d\tilde{M}}{dt} \simeq \tilde{M}(\beta_{\tilde{g}}/\tilde{g})' - 2\delta h \frac{\partial}{\partial Y}(\beta_{\tilde{g}}/\tilde{g}) , \quad \frac{d\delta h}{dt} \simeq (1 + 2Y \frac{\partial}{\partial Y})[\gamma_q + \gamma_T + \gamma_T] \delta h .
\] (37)

So near the fixed point, \(\delta h\) behaves like \(\delta h \sim e^{\Gamma_h t} \to 0\) as \(t \to -\infty\), where \(\Gamma_h\) is defined in eq. (34) and is a positive number. Consequently, the gaugino mass behaves like \(\tilde{M} \sim C_1 e^{\Gamma_{\tilde{M}} t} + C_2 e^{\Gamma_{\tilde{h}} t} \to 0\) as \(t \to -\infty\), where \(C_1\) and \(C_2\) are integration constants, and \(\Gamma_{\tilde{M}}(>0)\) are defined in eq. (33). Therefore, we find that \(\tilde{M}_* = h_* = 0\) is a stable fixed point. In the lowest order approximation in the \(\tilde{c}\) expansion we have \(\Gamma_{\tilde{M}} = \tilde{c}^2/3\) and \(\Gamma_h = \tilde{c}/3\), where \(\tilde{c} = 3 - N_f/\tilde{N}_c\).

Next we consider \(m^2_{q,T}\). Since near the IR fixed point the RG invariant relation (8) (or \(h_0\) given in eq. (36)) is attractive, we may use \(h = h_0\) in the linearization procedure. We then go to the NSVZ scheme, consider a deviation from the RG invariant relation (13), and define
\[
m^2_i = m^2_{(0)i} + \delta m^2_i , \quad m^2_{(0)i} \equiv \frac{1}{2} |\tilde{M}|^2 (\tilde{g}/\beta_{\tilde{g}})^{NSVZ}(\gamma^NSVZ)^i ,
\] (38)
where \(i = q, \bar{q}, T\). We find
\[
\frac{d}{dt} \delta m^2_q \simeq \frac{d}{dt} \delta m^2_{\bar{q}} \simeq \frac{1}{2} \Gamma m^2_q (\delta m^2_q + \delta m^2_{\bar{q}}) , \quad \frac{d}{dt} \delta m^2_T \simeq \frac{1}{2} \Gamma m^2_T (\delta m^2_q + \delta m^2_{\bar{q}}) ,
\] (39)
where
\[
\Gamma m^2_q \equiv \frac{\tilde{g}_* N_f \Gamma_{\gamma_q}}{\tilde{N}_c - 8\pi^2/\tilde{g}_*^2} , \quad \Gamma m^2_{\bar{q}} \equiv \frac{\tilde{g}_* \Gamma_{\gamma_T}}{\tilde{N}_c - 8\pi^2/\tilde{g}_*^2} , \quad \Gamma_{\gamma_q,T} \equiv (d\gamma^NSVZ/d\tilde{g})|_* ,
\] (40)

and we have used \(\gamma^NSVZ_q = \gamma^NSVZ_T\). As one can easily find, there are two zero eigenvalues and one positive one (\(= \Gamma m^2_q\)) in the linearized problem (39). One of the two zero eigenvalues corresponds to the solution that the difference \(\delta m^2_q - \delta m^2_{\bar{q}}\) is constant independent of \(t\). So, as in the case of SQCD, we see that the dual colour symmetry is broken unless
\[
m^2_q = m^2_{\bar{q}}
\] (41)
is satisfied. The other zero eigenvalue expresses the fact that \(m^2_T\) may contain a piece which is constant independent of \(t\) in the IR limit. The presence of the constant part breaks supersymmetry at the IR fixed point. If the relation (30) in the softly broken SQCD is
satisfied (so that supersymmetry is recovered at the IR fixed point), we have to demand the dual theory, too, to be supersymmetric at the IR fixed point. Therefore, we have the unique solution to (39) which preserve supersymmetry at the fixed point:

$$\delta m_q^2 = \delta m_T^2 \sim e^\Gamma \rightarrow 0 \text{ as } t \rightarrow -\infty$$

with

$$\frac{\delta m_q^2}{\delta m_T^2} \simeq \frac{\Gamma_{m_q^2}}{\Gamma_{m_T^2}} = \frac{\Gamma_{\alpha}}{\Gamma_{\gamma}}. \tag{42}$$

where $\Gamma$’s are defined in eq. (40), and $\Gamma_{m_q^2} > 0$ because $\Gamma_{\alpha} < \Gamma_{\gamma}$. In the lowest order approximation in the $\tilde{\epsilon}$ expansion we have $\Gamma_{m_T^2} \sim \Gamma_{\alpha}/3$.

The ratio $\delta m_q^2/\delta m_T^2$ being constant independent of $t$ suggests the existence of a RG invariant relation. In fact (42) is the consequence of the RG invariant sum rule (15). To see this, we insert $m_i^2$ ($i = q, \bar{q}, T$) into the sum rule (15), and find that the sum rule reduces to

$$2\delta m_q^2 + \delta m_T^2 = \frac{1}{C(G) - 8\pi^2/\tilde{\epsilon}^2} \left\{ \left[ (\tilde{\epsilon}/\beta_{\tilde{\epsilon}}^{\text{NSVZ}})(2N_f \gamma_q^{\text{NSVZ}} + \gamma_T^{\text{NSVZ}}) \right] \delta m_q^2 \right\}. \tag{43}$$

In the IR limit, the quantity in [ ] on the r.h.s. side contains an expression $0/0$. To obtain the correct limit, we compute

$$\frac{(d/d\tilde{\epsilon})(2N_f \gamma_q^{\text{NSVZ}} + \gamma_T^{\text{NSVZ}})}{(d/d\tilde{\epsilon})(\beta_{\tilde{\epsilon}}^{\text{NSVZ}}/\tilde{\epsilon})} \tag{44}$$

at the fixed point. We find that the expression (44) at the fixed point can be written as

$$(C(G) - 8\pi^2/\tilde{\epsilon}^2) \left( 2 + \frac{\Gamma_{\gamma}}{\Gamma_{\alpha}} \right), \tag{45}$$

implying that the sum rule (43) exactly becomes (42). Therefore, the soft scalar masses away from the IR fixed point have to satisfy the sum rule

$$m_q^2 + m_T^2 = |\tilde{M}|^2 \left\{ \frac{1}{1 - \tilde{\epsilon}^2 C(G)/(8\pi^2)} \frac{d\ln Y}{d\ln \tilde{\epsilon}} + \frac{1}{2} \frac{d^2\ln Y}{d(\ln \tilde{\epsilon})^2} \right\} + \frac{(N_f/2)(m_q^2 + m_T^2)}{C(G) - 8\pi^2/\tilde{\epsilon}^2} \frac{d\ln Y}{d\ln \tilde{\epsilon}} \tag{46}$$

and also (41) so that all the soft scalar masses asymptotically vanish in the IR limit. As a result, superconformal symmetry in the dual theory, too, revives at the IR fixed point. If eq. (30) for SQCD, and eqs. (41) and (46) for the dual theory are not satisfied, there will
be marginal operators that break supersymmetry as well as the local gauge symmetries in
the IR limit.

As we have mentioned at the beginning, our work is related to those of refs. [17, 18]. In
ref. [17, 18], the relations of the SSB parameters in the electric and magnetic sides outside of
the conformal window have been derived. Interestingly, it was found [18] that the magnetic
soft scalar masses vanish at the boundary between the free magnetic and conformal window,
which is consistent with our finding. Note that the sum rule (46) at the lowest order in
\( \tilde{g} \) becomes
\[
m_q^2 + m_T^2 = |\tilde{M}|^2.
\]
It is interesting to compare this with the sum rule
\[
m_q^2 + m_T^2 = 0
\]
on the conformal window in ref. [17, 18], from which it
has been concluded that the vacuum structure with the soft scalar masses differs from that
without them. This should be contrasted to our result that in the conformal window the
vacuum structure remains the same as long as the relations (30), (41) and (46) are satisfied,
supporting the duality hypothesis in the presence of the SSB terms in the conformal window.

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