As pointed out, chiral non-commutative theories exist, and examples can be constructed via string theory. Gauge anomalies require the matter content of individual gauge group factors, including $U(1)$ factors, to be non-chiral. All “bad” mixed gauge anomalies, and also all “good” (e.g. for $\pi^0 \rightarrow \gamma\gamma$) ABJ type flavor anomalies, automatically vanish in non-commutative gauge theories. We interpret this as being analogous to string theory, and an example of UV/IR mixing: non-commutative gauge theories automatically contain “closed string,” Green-Schwarz fields, which cancel these anomalies.
1. Introduction

Quantum field theories on non-commutative spaces are a subject of much recent interest, initiated by the realization that such theories actually occur in the worldvolume of D-branes in a background $B_{NS}$ field. There are many fascinating interconnections between non-commutative field theories and string theory. See [1] for a recent review and references. There have even been speculations that perhaps nature is actually noncommutative on some short distance scale.

We will here consider aspects of chiral non-commutative gauge theories and anomaly cancellation. Non-commutative gauge theory anomalies have been analyzed in several works [2-5], with the result that the only possible anomaly-free matter content for a single non-commutative gauge group is non-chiral. This is perhaps not surprising, given that the gauge group is $U(N)$, with only fundamental, anti-fundamental, and adjoint representations allowed (e.g. $SU(5)$ with a $5 + \overline{10}$ is not allowed).

There are, however, consistent chiral non-commutative gauge theories, based on product groups $\prod_i U(N_i)$. Vanishing $\text{Tr} U(N_i)^3$ gauge anomalies constrain the matter content of any given $U(N_i)$ gauge group to be non-chiral. But, as with the standard model, the representations of $\prod_i U(N_i)$ can be chiral via some bi-fundamental chiral fermion matter representations $(\overline{\alpha}, \overline{\beta})$ for which there is no conjugate $(\overline{\alpha}, \overline{\beta})$ chiral fermion. Such chiral non-commutative gauge theories can be explicitly constructed via D-branes at orbifold singularities. As long as the orbifold acts trivially on $B_{NS}$, there is no obstruction to including a background $B_{NS}$ and making the world-volume gauge theory non-commutative. Thus one can consider general orbifolds, but not orientifolds.

A concrete example of a chiral non-commutative gauge theory which exists via a string theory construction is that of $N$ D3 branes at a $C^3/Z_3$ orbifold singularity. The world-volume gauge theory has $\mathcal{N} = 1$ supersymmetry with the quiver gauge group and chiral

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1 By “chiral theories,” we mean that there are chiral fermions in a representation of the gauge group which is not vector-like. Thus e.g. a 4d $U(N)$ theory with a chiral fermion in the adjoint representation will not be considered to be a chiral theory. Such fermions, in vector-like representations, can always be given a large mass $\text{Tr} m \psi_\alpha \psi^\alpha$ and decoupled from the theory, and thus never contribute to anomalies. One might worry about such decoupling arguments in the non-commutative context, but it was explicitly verified in noncommutative $U(N)$ (see e.g. [4]) that vector-like fermion representations indeed do not contribute to anomalies.
superfield matter content

\[ U(N)_1 \times U(N)_2 \times U(N)_3 \]

\[ X_i \quad (\Box, \Box, 1) \]
\[ Y_i \quad (1, \Box, \Box) \]
\[ Z_i \quad (\Box, 1, \Box). \] 

(1.1)

with \( i = 1, 2, 3 \). As will always be the case in what follows, the overall \( U(1) \) is such that the fundamental \( \Box \) has charge +1, and the anti-fundamental \( \Box \) has charge \(-1\). There is no obstruction to turning on a background \( B_{NS} \), so the above theory also must exist as a chiral non-commutative gauge theory. One can similarly construct more general orbifold examples, e.g. based on IIB D3 branes at a \( C^3/(Z_N \times Z_M) \) singularity, or even non-supersymmetric examples via general orbifolds. As long as \( B_{NS} \) is not projected out, one can turn on its expectation value and make the worldvolume theory noncommutative.

Commutative gauge theories can have both \( \text{Tr} U(N_i)^3 \) anomalies and also mixed anomalies, like \( \text{Tr} U(N_i) U(N_j)^2 \); these mixed anomalies can generally be non-zero, and they must be cancelled by a generalized Green-Schwarz mechanism in order to obtain a consistent theory. (E.g. this is the case in the commutative version of the theory (1.1).)

Note that the \( \text{Tr} U(N_i)^3 \) anomaly comes entirely from planar diagrams, whereas the mixed \( \text{Tr} U(N_i) U(N_j)^2 \) anomaly comes entirely from non-planar diagrams. This is simply because the bi-fundamentals running in the loop \((\Box_i, \Box_j)\) can be regarded as an open string with \( U(N_i) \) Chan-Paton factor at one end and \( U(N_j) \) at the other, so all \( U(N_i) \) currents couple to one edge of the annulus and all \( U(N_j) \) currents to the other.

Because the mixed anomalies come from non-planar diagrams, they’re significantly altered by the non-commutativity. A standard explanation for anomalies is that one cannot regulate UV divergent loop momentum integrals in a way which is compatible with the naive, non-anomalous Ward identity. For example, one would recover the naive Ward identity if not for the fact that one cannot freely shift the integrated loop momentum, as the integral is linearly divergent in the UV. But, in the non-commutative case, the loop-momentum dependent phase factors of non-planar diagrams do regulate the loop integral in a way which is compatible with the naive Ward identity. This can be seen explicitly in the anomaly analysis of [1] for a bifundamental chiral fermion: the momentum loop integral is finite, and thus the loop momentum shift, which leads to the naive Ward identities, is valid. Thus mixed anomalies automatically vanish in non-commutative gauge theories.

The automatic vanishing of the mixed \( \text{Tr} U(N_i) \text{Tr} U(N_j)^2 \) occurs even if the \( U(N_i) \) is a global symmetry (which can be regarded as the \( g_i \to 0 \) limit of a gauge symmetry). Thus
both the “bad” mixed gauge anomalies, which would have rendered the theory inconsistent, and the “good” (e.g. for $\pi^0 \to \gamma\gamma$) ABJ type flavor anomalies automatically vanish in the non-commutative theory.

This automatic vanishing of non-planar anomalies is reminiscent of the situation in string theory, which originally led to the Green-Schwarz mechanism [6,7]. From the worldsheet viewpoint, an extra factor of $q^{P_N}/4$, with $P_N$ the total non-planar momentum running through the diagram, regulates the non-planar hexagon diagram; thus the mixed anomaly vanishes. On the other hand, a low-energy space-time analysis of the field theory would seem to give a non-zero mixed anomaly; the resolution of [6] is that string theory automatically cancels this mixed anomaly via a tree-level contribution from a closed string state ($B_{\mu\nu}$). In the closed string channel picture of the non-planar diagram, the $q^{P_N}/4$ factor can be considered as coming from poles of these closed string states.

We interpret the automatically vanishing mixed anomalies of non-commutative gauge theories as a UV/IR mixing effect, similar to string theory: these theories automatically contain “closed string” fields, with axionic coupling to the gauge fields, which implement a Green-Schwarz anomaly cancellation mechanism. The cancelling of the mixed flavor anomalies in non-commutative gauge theory is also analogous to string theory, which is believed to have no global symmetries; any symmetry must be a gauge symmetry, and therefore must be non-anomalous.

In the context of string theory constructions of chiral gauge theories via branes at orbifold singularities, the automatic anomaly cancellation is perhaps not so surprising. It is already known [8,9,10,11] in the commutative case that certain closed string modes (twisted RR fields) remain present, even in the decoupling limit, and cancel all mixed anomalies via a generalized Green-Schwarz mechanism; e.g. this happens for the $C^3/Z_3$ example (1.1). The situation in the noncommutative case, to be discussed, is fairly similar.

What is perhaps surprising is that one can apparently consider quantum field theories not constructed via string theory, and thus potentially sick due to uncancelled mixed anomalies. Making space noncommutative automatically cures these anomalies\(^2\). Our interpretation is that the noncommutative field theory, via UV/IR type effects, knows to include the necessary “closed string” Green-Schwarz modes also found in the string theory

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\(^2\) Some string theorists believe that noncommutative quantum field theories only exist as a limit of string theory. The automatic mixed anomaly cancellation perhaps bolsters this philosophy, with the cancellation regarded as being simply inherited from the underlying string theory.
constructions. We claim that these “closed string” fields $\chi_i$ couple to the noncommutative product $U(N_i)$ gauge theory as

$$\sum_{i=1}^{n} \int d^4 k \chi_i(k) O_{\chi_i}(-k),$$

(1.2)

with $O_{\chi_i}(-k)$ the gauge invariant analog of (the Fourier transform of) $\text{Tr} F_i \wedge F_i$, which involves an open Wilson loop and is thus non-local. If the $\chi_i(x)$ had constant expectation value, they would give the theta angle terms of the gauge groups, $\langle \chi_i(x) \rangle = \frac{\theta_i}{2\pi}$ (see e.g. [12]). However, the $\chi_i$ cannot be taken to be constant. The mixed anomalies are cancelled because $\chi_i(k)$ are not exactly gauge invariant. (The $\chi_i$ kinetic terms can be taken to vanish, so they become non-dynamical in the decoupling limit.)

Noncommutative gauge theories constructed via orbifolds were also considered in [13], in the context of analyzing UV/IR mixing of noncommutative field theories [14,15]. In line with the proposal of [14,15], it was argued in [13] that UV divergences associated with non-zero 1-loop beta functions, involve closed string modes which couple to the noncommutative gauge field kinetic terms. Other works, e.g. [16,17] have argued that the $p \circ p$ regulator [14,15] of non-planar diagrams are better thought of as an open string effect, having to do with stretched strings [18,19,20]. Of course, in the string theory, there’s some freedom as to whether effects are due to open or closed strings, thanks to the open-closed string channel duality.

In any case, the non-decoupled closed string modes discussed in [13] also come from the twisted closed string sector. We expect that these closed string modes $\phi_i$ come from twisted sector NS fields, and that they couple as

$$\sum_{i} \int d^4 k \phi_i(k) O_{\phi_i}(-k),$$

(1.3)

with $O_{\phi_i}(-k)$ the gauge invariant analog of $\text{Tr} F_{\mu \nu} F^{\mu \nu}$, which involves an open Wilson loop, and is thus not local. If the $\phi_i(x)$ had constant expectation value, they would give the effective gauge coupling of the gauge groups, $\langle \phi_i(x) \rangle = \frac{4\pi i}{g_i^2}$ (see e.g. [12]). As discussed in [13], when the gauge groups have non-vanishing 1-loop beta functions, the UV divergence can lead to IR poles and a Wilsonian interpretation requires including the fields $\phi_i$ in the theory. This is very similar to our discussion above and claim that the closed string fields $\chi_i$ of (1.2) are to be included in the theory.
In the commutative context, one can cancel all $U(1)$ anomalies via a generalized Green-Schwarz mechanism. This includes both mixed anomalies and also anomalies in a single $U(1)$ factor with chiral matter. The noncommutative Green-Schwarz mechanism of (1.2) is capable of cancelling all mixed anomalies, but not the $\text{Tr}U(1)^3$ anomalies of a single, chiral $U(1)$ gauge theory. We claim that, unlike the commutative version, $\text{Tr}U(1)^3$ anomalies of chiral noncommutative $U(1)$ theories can not be cured by the Green-Schwarz mechanism. Thus chiral $U(1)$ theories do not admit noncommutative generalizations. The lack of a Green-Schwarz anomaly cancellation mechanism for chiral non-commutative $U(1)$ could have been anticipated by its Morita equivalence to non-commutative $U(N > 1)$, as there is no Green-Schwarz cancellation mechanism for commutative $U(N > 1)^3$ anomalies.

One can also consider chiral gauge theories in other dimensions. For example, consider $N$ IIB D5 branes at a $C^2/Z_2$ singularity. The worldvolume gauge theory is a chiral 6d $\mathcal{N} = (1,0)$ theory with quiver gauge group and hypermultiplet matter content 

$$U(N)_1 \times U(N)_2$$

$$Q_1 \quad (\Box, \Box)$$

$$Q_2 \quad (\Box, \Box).$$

(1.4)

The theory has a 6d $\mathcal{N} = (1,0)$ tensor multiplet, which cancels the mixed anomalies [8], and a hypermultiplet which cancels [8] the anomalies associated with the $U(1)$ with charged matter. The $\text{Tr}SU(N_c)^4$ irreducible anomaly is proportional to $2N_c - N_f$ and thus consistent theories must always have $N_f = 2N_c$ for each $SU(N_c)$ factor, as is the case in (1.4). In the commutative context, $U(1)$ factors with charged matter are always anomalous, and thus generally get a mass in the process of anomaly cancellation described in [8]. In the non-commutative context, however, it is the full irreducible $\text{Tr}U(N_c)^4$ (rather than $\text{Tr}SU(N_c)^4$) which is proportional to $2N_c - N_f$, and this applies for all $N_c$, including $N_c = 1$. So the $N_f = 2N_c$ condition (needed to cancel twisted sector tadpoles [8,9]) found already in the commutative context ensures that the full $\text{Tr}U(N_c)^4$ anomalies cancel; no additional GS anomaly cancellation is needed for the $U(1)$ factors, and the $U(1)$ factors do not get massive or decouple in the non-commutative theory.

In section 2, we review Green-Schwarz anomaly cancellation in commutative gauge theories. In section 3, we discuss noncommutative anomalies. In section 4 we discuss noncommutative anomaly cancellation and noncommutative analogs of the Green-Schwarz mechanism. In section 5 we comment on anomalies and Wess-Zumino terms in Higgsed noncommutative gauge theories, with an example similar to the standard-model.
2. Commutative gauge anomalies, and Green-Schwarz cancellation

The gauge anomaly of $d$ dimensional gauge theories is given by

$$
\delta_\lambda \log Z = 2\pi i \int [\text{Tr}_\rho \exp(\frac{if}{2\pi})]^{(1)},
$$

(2.1)

with $\rho$ the representations of the chiral fermions and $I_{d+2} = dI^{(0)}$ and $\delta I^{(0)} = dI^{(1)}$. In particular, in four spacetime dimensions the anomaly comes from

$$
\sim [\text{Tr}_\rho (F^3)^{(1)}] \sim \text{Tr}_\rho (\lambda d(AdA + \frac{1}{2}A^3)).
$$

(2.2)

This anomaly is proportional to $\text{Tr}_\rho (\{T^a, T^b\}T^c) \equiv A^{abc}$, and the theory is sick if this quantity is non-zero. For non-Abelian gauge groups, this sickness cannot be cured. (A cure is anomaly inflow of gauge charge [21,22] into extra dimensions, but our interest is in 4d theories which decouple from any higher dimensional bulk. So gauge theories whose anomalies cannot be cancelled purely in 4d will be considered as incurably sick.)

But abelian gauge theories with anomaly $A \neq 0$ can always be cured by a Green-Schwarz mechanism. As usual, the condition for Green-Schwarz anomaly cancellation is that the anomaly polynomial $\text{Tr}_\rho F^3$ should factorize. The anomalies involving $U(1)$ groups trivially factorize, because we can drop the Tr for the $U(1)$ factors.

For example, consider a theory with gauge group $U(1)$ and chiral fermions with charges $q_i$. The anomaly (2.2) can be written as

$$
\delta_\lambda \log Z \sim A \int \lambda d(AdA + \frac{1}{2}A^3) = A \int \lambda F \wedge F, \quad A \equiv \sum q_i^3.
$$

(2.3)

Note that generally

$$
\text{Tr} F \wedge F = d\text{Tr}(AdA + \frac{2}{3}A^3),
$$

(2.4)

but the $A^3$ term vanishes for commutative $U(1)$ gauge theory, so the difference between the $\frac{1}{2}$ coefficient in (2.2) versus the $\frac{2}{3}$ in (2.4) is immaterial. The $A^3$ term does not vanish in the non-commutative version of this theory, to be discussed in the following sections.

If $A \neq 0$, the theory is sick, but can be cured by introducing a scalar field “axion” $\chi$ with term in the action

$$
S_{GS} = \alpha \int \chi F \wedge F,
$$

(2.5)

which properly appears to be gauge invariant. However, if it happens that the gauge invariant field strength of $\chi$ is actually

$$
H = d\chi + \beta A,
$$

(2.6)
then $\chi$ cannot be gauge invariant, but must instead shift under gauge transformations,

$$
\delta_\lambda \chi = -\beta \lambda \quad \text{under} \quad A \to A + d\lambda.
$$

(2.7)

The coupling (2.5) then leads to tree-level anomaly contributions which cancels (2.3) provided the constants $\alpha$ and $\beta$ satisfy

$$
\alpha \beta = A.
$$

(2.8)

We can normalize $\chi$ so that $\alpha = 1$ and $\beta = A$. The gauge invariant $\chi$ kinetic term is (with $Z_\chi$ a normalization constant)

$$
-\frac{1}{2}Z_\chi (\partial_\mu \chi + \beta A_\mu)^2,
$$

(2.9)

which gives the photon a mass $m_\gamma^2 = \beta^2 Z_\chi$. The cross term in (2.9) can be written as

$$
-\beta Z_\chi B_2 \wedge F,
$$

(2.10)

where $B_2$ is the 2-form potential dual to $\chi$, $d\chi = *dB_2$ and we integrated by parts.

Consider next a $U(1) \times G$ gauge theory. The anomaly can contain terms

$$
\delta_\lambda \log Z \sim A_{111}[F^3_1]^{(1)} + A_{1GG}[F_1 F_G F_G]^{(1)} + A_{11G}[F_1 F_1 F_G]^{(1)} + A_{GGG}[F^3_G]^{(1)} ,
$$

(2.11)

with $A_{111} = \text{Tr}_\rho U(1)^3$ and $A_{11G} = \text{Tr}_\rho U(1)^2 G$ etc. Suppose that $G$ is non-abelian, with no $U(1)$ factor, so $A_{11G} = 0$, and that the $G$ matter content is chosen to be anomaly free, so $A_{GGG} = 0$. We can write the remaining anomaly in (2.11) as

$$
\delta_\lambda \log Z \sim \int \lambda_1 (A_{111} F_1 \wedge F_1 + A_{1GG} \text{Tr} F_G \wedge F_G),
$$

(2.12)

This anomaly can be cancelled via a field $\chi$ with coupling

$$
S_{GS} = \int \chi (\alpha_1 F_1 \wedge F_1 + \alpha_G \text{Tr} F_G \wedge F_G),
$$

(2.13)

provided $\chi$ has gauge invariant field strength $H = d\chi + \beta A_1$, so $\delta_\lambda \chi = -\beta \lambda_1$, and the coupling constants satisfy $\alpha_1 \beta = A_{111}$ and $\alpha_G \beta = A_{1GG}$.

The mixed anomaly $A_{1GG}$ corresponds to a net $U(1)$ charge of the fermion zero modes contained in the $G$ instanton’s ’t Hooft vertex. If $A_{1GG}$ is non-zero but cancelled by the Green-Schwarz mechanism discussed above, the $G$ instanton ’t Hooft vertex is actually $U(1)$ neutral, as the Green-Schwarz mechanism ensures that the $U(1)$ current is properly
conserved. The fermion zero modes still carry net $U(1)$ charge, but it is screened by the factor of $\exp(i\alpha_G \chi)$ which accompanies the instanton 't Hooft vertex because of (2.13).

More generally, to cancel all anomalies involving $U(1)_i$ factors of a general gauge theory, one needs the same number of $\chi_i$ fields as there are $U(1)_i$ factors in the gauge group.

We will be generally interested in $\prod_{i=1}^n U(N_i)$ gauge theories. Chiral fermions in the fundamental $\square_i$ contribute to (2.1) as $[\text{tr} \exp(iF_i^{3/2 \pi})]^{(1)}$, chiral fermions in the anti-fundamental $\overline{\square_i}$ contribute as $[\text{tr} \exp(-iF_i^{3/2 \pi})]^{(1)}$, chiral fermions in the adjoint do not contribute in 4d, and chiral fermions in the $(\square_i, \overline{\square_j})$ contribute as $[\text{tr} \exp(iF_i^{3/2 \pi}) \text{tr} \exp(-iF_j^{3/2 \pi})]^{(1)}$, where for all these one just keeps the 6-form term and tr is in the fundamental of the corresponding $U(N_i)$. The general anomaly thus obtained is

$$\delta_\lambda \log Z = \left[\sum_i A_{iii} \text{tr} F_i^3 + \sum_{ij} A_{ijj} \text{tr} F_i^1 \text{tr} F_j^2\right]^{(1)}. \quad (2.14)$$

(No $A_{ijk} \text{tr} F_i^1 \text{tr} F_j^2 \text{tr} F_k^1$ terms occur if the only mixed representations are bi-fundamentals, as is the case in theories with noncommutative generalizations.)

The $A_{iii}$ term in (2.14), which is the $\text{Tr} U(N_i)^3$ anomaly, cannot be cancelled if $N_i > 1$ (or for $N_i = 1$ in the non-commutative case to be discussed). We thus assume the matter content is chosen such that $A_{iii} = 0$. The remaining $A_{ijj}$ terms in (2.14) are the $\text{Tr} U(N_i) U(N_j)^2$ mixed anomalies. These anomalies can be cancelled via $n$ fields $\chi_i$ with couplings

$$S_{GS} = \sum_{j=1}^n \int \chi_j \text{tr} F_j \wedge F_j, \quad (2.15)$$

if, under gauge transformations, the $\chi_j$ shift as

$$\delta_\lambda \chi_j = -\sum_i A_{ijj} \text{tr} \lambda_i. \quad (2.16)$$

For example, for the theory (1.1), the gauge anomaly is

$$\delta_\lambda \log Z = \frac{3i}{8\pi^2} \left[\text{tr} F_1^1 \text{tr} F_2^2 - \text{tr} F_2^1 \text{tr} F_1^2 + \text{tr} F_2^2 \text{tr} F_3^2 - \text{tr} F_3^2 \text{tr} F_2^2 + \text{tr} F_3^2 \text{tr} F_1^2 - \text{tr} F_1^2 \text{tr} F_3^2\right]^{(1)}. \quad (2.17)$$

To cancel these anomalies, one needs three fields $\chi_i$, $i = 1, 2, 3$, with couplings

$$S_{GS} = \int \sum_{i=1}^3 \chi_i \text{tr} F_i \wedge F_i, \quad (2.18)$$
which shift under gauge transformations as

\[ \delta \chi_1 \sim (\text{tr} \lambda_2 - \text{tr} \lambda_3), \quad \delta \chi_2 \sim (\text{tr} \lambda_3 - \text{tr} \lambda_1), \quad \delta \chi_3 \sim (\text{tr} \lambda_1 - \text{tr} \lambda_2). \]  

(2.19)

In string theory constructions of commutative gauge theories via branes at orbifold singularities, the required Green-Schwarz fields \( \chi_i \) arise as Ramond-Ramond fields, coming from the closed string twisted sector \([8,11]\). They can be regarded as e.g. the IIB \( C_2 \) field reduced on the collapsed \( S^2 \)'s of the orbifold singularity. These are the same fields which yield the moduli for changing the field theory theta angles of the orbifold quiver gauge theory (see e.g. the discussion in \([12]\)).

3. Noncommutative gauge theories and their anomalies

Taking the \( x_2 \) and \( x_3 \) spatial directions non-commutative, \([x_2, x_3] = i \theta_{23}\), the gauge group is generally \( U(N) \), with the \( U(1) \) and \( SU(N) \) parts of \( U(N) \) necessarily coupled by the noncommutative gauge symmetry. The only representations which properly represent the noncommutative algebra are the singlet, fundamental, anti-fundamental, and adjoint representations of \( U(N) \). Writing the gauge transformation in terms of \( U = e^{i \lambda} \) (expanding to first order in \( \lambda \) to get the Lie algebra), the gauge field transforms as \( (d + i A) \to U \ast (d + i A) \ast U^{-1} \), a fundamental matter field \( \psi_f \) (scalar or fermion) transforms as \( \psi_f \to U \ast \psi_f \) an anti-fundamental as \( \tilde{\psi}_f \to \tilde{\psi}_f \ast U^{-1} \), and an adjoint as \( \psi_a \to U \ast \psi_a \ast U^{-1} \).

It is also possible to consider product gauge groups \( \prod_i U(N_i) \), in which case the only possible mixed representations are bifundamentals \( \psi_{ij} \) transforming as \((\square_i, \square_j)\), i.e. \( \psi_{ij} \to U_i \psi_{ij} U_j^{-1} \), with \( U_i \) and \( U_j \) the \( U(N_i) \) and \( U(N_j) \) gauge transformations. A bi-fundamental chiral fermion \( \psi_{ij,\alpha} \in (\square_i, \square_j) \) contributes to the \( U(N_i) \) current as \( J^{U(N_i)}_{\mu} = \sigma_\mu^{\alpha\dot{\alpha}} \psi_{ij,\alpha} \ast \psi_{ji,\dot{\alpha}}^\dagger \) and to the \( U(N_j) \) current as \( J^{U(N_j)}_{\mu} = \sigma_\mu^{\alpha\dot{\alpha}} \psi_{ji,\dot{\alpha}}^\dagger \ast \psi_{ij,\alpha} \), as these orderings properly transform as \( J^{U(N_i)}_{\mu} \to U_i J^{U(N_i)}_{\mu} U_i^{-1} \). Other mixed representations, including \((\square_i, \square_j)\), are not allowed, as they do not properly represent the group multiplication.

3.1. \( \text{Tr}U(N)^3 \) anomalies

These anomalies come entirely from planar diagrams. We briefly review the results for the non-commutative theories, as has been analyzed e.g. in \([8,9]\). Recall first the commutative case. Define \( \Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = (2\pi)^4 \delta(p_1 + p_2 + p_3) \Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1, p_2) \) to be
the Fourier transform of $\langle T J^{a_1}_{\mu_1}(x_1) J^{a_2}_{\mu_2}(x_2) J^{a_3}_{\mu_3}(x_3) \rangle$, with $J^a_\mu$ the left-handed chiral current. The commutative anomaly (2.1) comes from

$$p_3^{\mu_3} \Gamma^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3} (p_2, p_3) = \frac{1}{24\pi^2} \epsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_2^\sigma \text{Tr}_\rho (T^{a_1} T^{a_2} T^{a_3} + T^{a_1} T^{a_3} T^{a_2})$$ \hspace{1cm} (3.1)

with $\rho$ the representation of all left-handed chiral fermions. The two terms in (3.1) come from the two contributing diagrams, one where the cyclic ordering of external momenta is $(p_1, p_2, p_3)$ and the other where it’s $(p_2, p_1, p_3)$. The anomaly (3.1) vanishes if

$$\text{Tr}_\rho (\{T^{a_1}, T^{a_2}\} T^{a_3}) \equiv A^{abc} = 0.$$ \hspace{1cm} (3.2)

The same two planar diagrams contribute in the non-commutative case and thus the only modifications are in phase factors depending on the external momenta:

$$p_3^{\mu_3} \Gamma^{a_1 a_2 a_3}_{\mu_1 \mu_2 \mu_3} (p_2, p_3) = \frac{1}{24\pi^2} \epsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_2^\sigma \text{Tr}_\rho (e^{-\frac{i}{2} p_1 \times p_2} T^{a_1} T^{a_2} T^{a_3} + e^{\frac{i}{2} p_1 \times p_2} T^{a_1} T^{a_3} T^{a_2}),$$ \hspace{1cm} (3.3)

with $p_1 \times p_2 \equiv p_{1\mu} \Theta^{\mu\nu} p_{2\nu}$. As noted in [4], vanishing anomaly (3.3) appears to require a stronger condition than in the commutative context:

$$\text{Tr}_\rho T^{a_1} T^{a_2} T^{a_3} = 0.$$ \hspace{1cm} (3.4)

However, given that the only allowed matter representations are the fundamental, anti-fundamental, or adjoint representation, (3.2) and (3.4) actually imply exactly the same condition, that the fermion matter content of a single gauge group be vector-like (equal number of fundamental and anti-fundamental chiral fermions). Even for $U(2)$, the $U(1)^3$ part of (3.2), along with the restriction on the classically allowed representations, would suffice to give the same information as (3.4), that the chiral fermions must have equal numbers of $\Box$ and $\overline{\Box}$ chiral reps (which are not equivalent).

In the commutative context, the anomaly (3.1) leads to (2.1), which satisfies the Wess-Zumino consistency conditions. In the non-commutative context, the claim of [5] is that the anomalies can still be written via a generalization of the descent formalism, as

$$\delta_\lambda \log Z \sim \int \text{Tr}_\rho (\lambda \star d(A \star dA + \frac{1}{2} A \ast A \ast A)).$$ \hspace{1cm} (3.5)

A subtlety here is that in the usual descent formalism, $I_{d+2} = dI^{(0)}$ and $\delta I^{(0)} = dI^{(1)}$, the forms $I$ are local and involve traces over the gauge group, whereas in the non-commutative context the trace over the gauge group should include an integral over the noncommutative space. E.g. the usual descent formalism makes use of the cyclic property of the trace, which wouldn’t hold in the non-commutative context without including the integral over space; it is argued in [5] that, nevertheless, a descent formalism can be applied and leads to anomalies such as (3.5).
3.2. Mixed $\text{Tr} U(N_i)U(N_j)^2$ anomalies

These anomalies come entirely from non-planar diagrams, with $U(N_i)$ gauge fields on one edge of the annulus and $U(N_j)$ gauge fields on the other. Here the difference between the commutative vs non-commutative cases is much more radical than for the $\text{Tr} U(N)^3$ anomalies discussed in the previous subsection.

The additional phase factors of non-planar diagrams in the noncommutative theory depend on the loop momentum, regulating the integrals in a way which is compatible with the naive (non-anomalous) Ward identities. For example, the naive Ward identity of the commutative theory is violated because one cannot shift the momentum integration variable of the triangle diagram, as the integral is linearly divergent. But the additional phase factor of the non-planar diagrams in the non-commutative theory leads to momentum integrals such as, quoting [4],

$$
\int \frac{d^4q}{2\pi^4} e^{iq\cdot p} \frac{q^\mu}{q^2(q+p)^2} = -\frac{i}{8\pi^2} \int_0^1 dx \, x K_0(\sqrt{\tilde{p}^2(-p^2)x(1-x)})
$$

$$
+ \frac{1}{8\pi^2} \tilde{p}^\mu \int_0^1 dx \, \sqrt{\tilde{p}^2(-p^2)x(1-x)} K_1(\sqrt{\tilde{p}^2(-p^2)x(1-x)}),
$$

with $\tilde{p}^\mu \equiv \theta^{\mu\nu} p_\nu$ and $\tilde{p}^2 \equiv \tilde{p}^\mu \tilde{p}^\nu \eta_{\mu\nu} \equiv p\circ p$. Thus, for any non-zero non-commutativity $\theta^{\mu\nu}$, the momentum integrals are finite. One can therefore freely shift the integration variable and recover the naive Ward identities, as was done in [4].

The upshot is generally a radical difference from commutative theories. One could start with a commutative theory with non-zero mixed anomalies (including ABJ type flavor anomalies), turn on an arbitrarily small non-commutativity, and suddenly these anomalies magically disappear.

4. Noncommutative Green-Schwarz mechanism

We interpret the automatic vanishing of mixed anomalies in noncommutative gauge theories as an example of the UV/IR mixing of non-commutative theories: the low energy theory automatically contains additional “closed string” modes, even if they were apparently not included in the original Lagrangian. These fields lead to automatic Green-Schwarz cancellation of all mixed anomalies.

To motivate this claim, consider a general $\prod_{i=1}^n U(N_i)$ gauge theory whose chiral fermion matter content has non-zero mixed $\text{Tr}_\rho U(N_i)U(N_j)^2$ anomalies $A_{i,jj} \neq 0$ in the
commutative context. An example is (1.1), with anomaly (2.17). This means that the $U(N_j)$ instanton 't Hooft vertex $O_{zm}^{j}$, associated with the $U(N_j)$ instanton’s fermion zero modes, is not a $U(N_i)$ singlet. If $U(N_i)$ is a gauge symmetry, the net $U(N_i)$ charge of the $U(N_j)$ instanton must be screened, by the charged Green-Schwarz field $e^{i\chi_j}$, in order to not spoil $U(N_j)$ gauge invariance.

In the noncommutative version of the theory, one seemingly gets no mixed anomaly, and thus no need for a GS mechanism - but we claim that the proper interpretation is rather that noncommutative field theory automatically implements a noncommutative analog of the GS mechanism. Indeed, we expect the noncommutative $U(N_j)$ instanton to have the same fermion zero modes as in the commutative theory, and thus the same factor $O_{zm}^{j}$, with net $U(N_i)$ charge, requiring screening by some GS fields $e^{i\chi_j}$.

We claim that the zero mixed anomaly should be interpreted as a GS cancellation of a non-zero anomaly

$$\delta_{\lambda} \log Z = \sum_{ij} \int d^4k A_{ij}(k) O_{\chi_j}(-k), \quad (4.1)$$

which is the natural, non-commutative generalization of the commutative mixed anomalies. Here $O_{\chi_j}(-k)$ is the non-local, gauge invariant, analog of $\text{Tr} F_j \wedge F_j$ which involves an open Wilson-loop:

$$O_{\chi_j}(-k) = \text{Tr} \int d^4x L_* \left[ \sqrt{\det(1 - \theta \hat{F}_j)} \left( \frac{1}{\lambda_1(k)} \right) W(x, C_k) \right] \star e^{-ik \cdot x}, \quad (4.2)$$

where we follow the notation of [23] and the operator (4.2) appeared as the operator $Q_4(-k)$ in [24]. The pre-cancelled anomaly (4.1) contains the information about the $U(N_i)$ charge of the $U(N_j)$ instanton’s fermion zero modes (see e.g. the discussion of index theorems in [24]). It is natural for the non-local operator $O_{\chi_j}(-k)$ to appear in the noncommutative mixed anomaly (4.1), as the mixed anomalies have non-zero non-planar momentum and thus involve stretched string factors as in [16, 27].

The $\lambda_i(k)$ appearing in (4.1) is defined as follows. Following the discussion in [24], we define $A_{i\mu}(k)$ by

$$(dA_j)(k) = \text{Tr} \int d^4x L_* \left[ \sqrt{\det(1 - \theta \hat{F}_j)} \left( \frac{1}{\lambda_1(k)} \right) W(x, C_k) \right] \star e^{-ik \cdot x}; \quad (4.3)$$

this is the Seiberg-Witten [26] map solution, conjectured in [23] and proven in [24, 27, 28] by considering Ramond-Ramond couplings (of the same kind as we’re considering): the
$A_{i\mu}(k)$ defined by (4.3) is simply the commutative $U(1)_i \subset U(N_i)$. Under a general noncommutative $U(N_j)$ gauge transformation $\delta \hat{A}_{j\mu} = \partial_\mu \hat{\lambda}_j + i \hat{\lambda}_j \ast \hat{A}_{j\mu} - i \hat{A}_{j\mu} \ast \hat{\lambda}_j$, the $A_{j\mu}$ defined by (4.3) should then transform as $A_{j\mu}(k) \rightarrow A_{j\mu}(k) + (\partial_\mu \lambda_j)(k)$. This defines $\lambda_j(k)$, the gauge variation of the commutative $U(1)_i \subset U(N_i)$, which appears in the proposed (pre-cancelled) anomaly (4.1).

According to the Seiberg-Witten [26] map, the non-commutative theory can be mapped to a commutative version of the theory, deformed by higher dimension (generally non-local) operators. Because anomalies are generally independent of such deformations, it is natural to expect the anomalies of the noncommutative theory to be the same as that of the commutative theory. This agrees with the proposed pre-cancelled anomaly (4.1): it is just the same mixed anomaly of the commutative theory, written in the non-commutative variables via the Seiberg-Witten map.

We propose that the anomaly (4.1) is automatically cancelled by the following non-commutative generalization of the Green-Schwarz mechanism. For our general $\prod_{i=1}^n U(N_i)$ gauge theory, one must introduce $n$ “closed string” GS fields $\chi_j$, which are almost gauge invariant, with coupling

$$S_{GS} = \sum_{j=1}^n \int d^4k \chi_j(k) O_{\chi_j}(-k),$$

(4.4)

with the $O_{\chi_j}(-k)$ the gauge invariant operators (4.2). Now suppose that the gauge invariant field conjugate to $\chi_j$ is

$$H_{j\mu}(k) = (\partial_\mu \chi_j)(k) + \sum_i A_{ij\mu}(k),$$

(4.5)

with $A_{ij\mu}(k)$ the commutative $U(1)_i \subset U(N_i)$ vector potential defined via (4.3). Then, the $\chi_j(k)$ cannot be precisely gauge invariant, rather:

$$\delta \lambda \chi_j(k) = - \sum_i A_{ij\mu}(k).$$

(4.6)

Combining (4.6) and (4.4) cancels the anomaly (4.1).

The coupling (4.4) implies that the noncommutative $U(N_j)$ instanton’s ’t Hooft vertex is $e^{i\chi_j(k)} O_{\chi_j}^{zm}(-k)$ and, via (4.4), the $e^{i\chi_j(k)}$ factor carries the appropriate charge under the commutative $U(1)_i \subset U(N_i)$ to cancel off that of the zero mode factor $O_{\chi_j}^{zm}(-k)$.

Given (4.5), the possible kinetic terms of the “closed string” modes $\chi_i$ are

$$-\frac{1}{2} \int d^4k \sum_{jj'} Z_{jj'} H_{j\mu}(k) H_{j'}^\mu(-k),$$

(4.7)
for some normalization constants $Z_{jj'}$ (which could vanish in the decoupling limit). Despite the fact that the $U(1)$ cannot be decoupled from the $U(N)$ in noncommutative theories, the gauge invariant terms (4.7) contains non-local photon mass terms like $A_{i\mu}(k)A_{i'}^{\mu}(-k)$. Via the Seiberg-Witten map, these are just the standard, commutative photon mass terms of the GS mechanism for the overall $U(1)_i \subset U(N_i)$.

The coupling (4.4) is of the same general form argued in [27,28,24] to appear in the $SWZ$ Wess-Zumino coupling of closed string RR fields to the worldvolume gauge theory. Indeed, we expect that the $\chi_i$ arise in precisely this way, from closed string twisted sector RR fields. These can be regarded, following [12], as the RR field $C_2$ reduced on the collapsed $S^2$s of the orbifold. As discussed in [27,28,24], the $SWZ$ terms yield the solutions of the Seiberg-Witten map [26] between noncommutative and commutative gauge theories, as $SWZ$ is actually independent of the interpolator $\Phi$ of [26]. Using the Seiberg-Witten map (assuming it exists for our theories), (4.1) is simply the same anomaly of the commutative theory, written in the noncommutative variables. Similarly, our claimed non-commutative GS mechanism (4.4) and (4.6) is essentially the same as the usual commutative GS mechanism, just written in the noncommutative variables.

4.1. Deconstructing the zero mixed anomaly

D-brane worldvolume anomalies can be analyzed in analogy with the discussion in [3]. Consider, in particular, D3 brane worldvolume anomalies when at orbifold singularities, such as the $C^3/Z_3$ example. The mixed $\text{Tr} U(N_i)U(N_j)^2$ anomaly comes from the amplitude with a $U(N_i)$ gauge field on one boundary of the annulus and two $U(N_j)$ gauge fields on the other boundary. There is an anomaly if this amplitude does not vanish when the $U(N_i)$ polarization is set equal to its momentum $k_{3\mu}$. We regulate the amplitude as in [6] by introducing a Pauli-Villars field of mass $M$ and then take $M \to \infty$. The anomalous amplitude is then proportional to

$$A_{ST} = \lim_{M \to \infty} A_{FT} M^2 \int_0^\infty dt e^{-M^2 t} e^{-\frac{\alpha'}{4} k_{3\mu} g^{\mu\nu} k_{3\nu}} \int_0^1 d\nu_1 e^{-\frac{\alpha'}{4} (k_1 \times k_2)[2\nu_1 + \epsilon(\nu_1)]}, \quad (4.8)$$

where $A_{FT} \equiv \text{Tr}_\rho(U(N_i)U(N_j)^2) k_{1\mu} k_{2\nu} \epsilon^{a\rho}_{1\mu} \epsilon^{a\sigma}_{2\nu} \epsilon^{\mu\nu\rho\sigma}$ is the standard commutative field theory amplitude for the anomaly. In the notation of [6], $q = \exp(-2\pi/t)$ and (4.8) contains the characteristic non-planar factor $q^{-s/4}$, where $s \equiv -\alpha' k_{3\mu} g^{\mu\nu} k_{3\nu}$ is the non-planar $s$-channel invariant momentum (with $s < 0$ as defined here). Note that $s$ involves only the closed string metric $g^{\mu\nu}$ (rather than the open string metric $G^{\mu\nu}$) for any non-commutativity.
Setting $\theta^\mu{}^\nu = 0$ and $\alpha' = 0$, the amplitude (4.8) gives $A_{ST} = A_{FT}$. For string theory, with $\alpha' \neq 0$, (4.8) gets a factor $\sim \lim_{M \to \infty} e^{-\alpha' M^2} \to 0$, thus there is no anomaly. This is the usual statement that string theory cancels this mixed anomaly via closed string modes (twisted sector RR fields) and the Green-Schwarz mechanism. The factor $q^{-s/4}$ corresponds to the closed string poles in the non-planar $s$-channel.

Now in the noncommutative field theory limit we take $\alpha' \to 0$ and adjust the closed string metric $g^{\mu\nu}$ so that the open string $G^{\mu\nu} = \eta^{\mu\nu}$ is the identity; this gives $\alpha' p_\mu g^{\mu\nu} q_\nu \to p_\mu (\theta^2)^{\mu\nu} q_\nu \equiv p \circ q$ [14]. Thus the anomaly amplitude (4.8) becomes

\[ A_{NCFT} = \lim_{M \to \infty} A_{FT} M^2 \int_0^\infty dt e^{-M^2 t} e^{-\frac{1}{2} k_3 \circ k_3} \int_0^1 d\nu_{1,2} e^{-\frac{i}{2} (k_1 \times k_2)[2\nu_{12}+\epsilon(\nu_{12})]} , \quad (4.9) \]

which still gives zero for the anomaly, $A_{NCFT} \to 0$ in the $M \to \infty$ limit, a result which we have already discussed. Since the vanishing anomaly comes from the same $q^{-s/4}$ closed string factor as in the commutative string theory version of the theory, it naturally has the same interpretation: a non-zero anomaly is cancelled by closed string twisted sector RR states $\chi_j$.

We would now like to deconstruct the zero anomaly of (4.9) to see that it corresponds to an automatic GS cancellation of the proposed anomaly (4.1). Suppose we drop the "closed string" factor $q^{-s/4} \equiv e^{-\frac{1}{2} k_3 \circ k_3}$ in (4.9), which is responsible for the anomaly cancellation. Then the remaining parts of the anomaly in (4.9) would give a non-zero anomaly, which differs from the commutative result by the extra phase factors coming from the $\nu_{1,2}$ integrals. These phase factors lead to $\ast'$ products as in [29,25], and the resulting anomaly is of the form

\[ \partial^\mu \text{tr} J^U(N_i) \propto \text{tr} \hat{F}^U(N_j) \wedge \ast' \hat{F}^U(N_j) ; \]

which is a leading term corresponding to the claimed mixed anomaly (4.1). It should be possible to obtain the full gauge invariant operator corresponding to (4.1) from a one-loop computation in some properly interpreted sense (presumably, properly interpreted, the Adler-Bardeen theorem applies).
4.2. $U(1)^3$ anomalies cannot be cancelled

In the commutative case, $U(1)$ is qualitatively different from $U(N > 1)$, in that non-zero $\text{Tr} U(1)^3$ anomalies of a chiral $U(1)$ theory can be cancelled via the GS mechanism as in (2.5), whereas $\text{Tr} U(N > 1)^3$ anomalies cannot (because of the $\text{Tr} SU(N)^3$ part). In noncommutative theories, $U(1)$ is Morita equivalent to $U(N > 1)$, so one should expect that GS anomaly cancelation either works or fails for cancelling $\text{Tr} U(N)^3$ anomalies for all $N$. We claim that, unlike the commutative case, there is no noncommutative GS mechanism to cancel nonzero $\text{Tr} U(1)^3$ anomalies.

Consider a noncommutative chiral $U(N)$ theory, with non-vanishing $\text{Tr} U(N)^3$ anomaly coming from the planar diagrams with three $U(N)$ gauge fields on a single boundary of the annulus. Because these diagrams are planar, they differ from the commutative theory only by the phase factors depending on external momenta, corresponding to the replacement of ordinary products with $\star$ products, leading to (3.5). The question is whether or not these loop diagrams can be cancelled by a tree-level diagram involving a field $\chi$ with a noncommutative analog of the interaction (2.5).

But the only plausible noncommutative analog of the GS mechanism is to have the GS field $\chi$ be a gauge singlet (up to its anomalous shift). In particular, $\chi$ should not be an adjoint field under the gauge group, so the only plausible noncommutative analog of (2.4) is as in (4.4):

$$S_{GS} = \int d^4k \chi(k) \mathcal{O}_\chi(-k),$$

with $\mathcal{O}_\chi(-k)$ the gauge invariant version of $\text{Tr} F \wedge F$, defined as in (4.2). This could only cancel an anomaly like

$$\delta_\lambda \log Z \sim \int \lambda(k) \mathcal{O}_\chi(-k),$$

which is a possible, but incorrect, noncommutative analog of (2.3). It must be incorrect because the $\text{Tr} U(N)^3$ anomaly comes from a planar diagram and yields the anomaly discussed above. Anomalies like (4.11), involving the gauge invariant operators $\mathcal{O}_\chi(-k)$, can only come from non-planar diagrams. We can explicitly see that (4.11) does not agree with (3.3) by noting that the leading order term in $\mathcal{O}_\chi$ is $F \wedge \star F$, and $d[A \star dA - \frac{1}{2} A \star A \star A] \neq F \star F$.

Our conclusion is that non-zero non-commutative $\text{Tr} U(1)^3$ anomalies cannot be cancelled by any obvious generalization of the Green-Schwarz type mechanism. The only way then to avoid $\text{Tr} U(1)^3$ anomalies is to have non-chiral fermion representations, as is also the case for $U(N > 1)$ already in the commutative theory.
This conclusion would be problematic if one could find a string theory construction which yields a chiral $U(1)$ gauge theory, with non-zero $\text{Tr}U(1)^3$ anomaly cancelled by the commutative GS mechanism, which admits turning on $B_{NS} \neq 0$. Fortunately, consistent string theory constructions, which admit turning on $B_{NS} \neq 0$, do not need to make use of the commutative GS mechanism for cancelling $\text{Tr}U(1)^3$ anomalies, as the individual $U(1)$ groups are generally nonchiral, with $\text{Tr}U(1)^3 = 0$.

5. Comments on noncommutative Higgsed theories, like the noncommutative standard model

Consider a noncommutative analog of the standard model (ignoring hypercharge), with gauge group $U(3) \times U(2)$ and with each generation’s chiral fermion matter content as in

$$U(3) \times U(2)$$

$$\begin{align*}
q & : (\underline{3}, \overline{1}) \\
\bar{u} & : (\underline{1}, 1) \\
\bar{d} & : (\underline{1}, 1) \\
l & : (1, \overline{3})
\end{align*}$$

(5.1)

The anomaly is given by descent from the 6-form term in

$$\text{tr} \exp(F_3/2\pi)\text{tr} \exp(-F_2/2\pi) + 2\text{tr} \exp(-F_3/2\pi) + \text{tr} \exp(F_2/2\pi),$$

(5.2)

i.e.

$$\delta_{\lambda} \log Z = \frac{i}{8\pi^2} [\text{tr}F_3\text{tr}F_2^2 - \text{tr}F_2\text{tr}F_3^2 - \frac{2}{3}\text{tr}F_2^3](1).$$

(5.3)

In the commutative case these anomalies all involve only $U(1)$ factors (as $SU(2)$ has no cubic Casimir) and can be cancelled by introducing two GS fields $\chi_3$ and $\chi_2$.

In the non-commutative case, the mixed anomalies in (5.2) are cancelled automatically, but the $\text{Tr}U(2)^3$ anomaly in (5.2) cannot be GS cancelled, as discussed above. The theory with the matter content (5.1) would be incurably sick, because its $U(2)$ matter content is not vector-like. However, because the $U(2)$ is broken anyway by the Higgs mechanism, it is possible to include additional $U(2)$ chiral fermions such that the total matter content is vector-like. E.g. in the supersymmetric analog of the above theory, one has Higgs superfields $H_u$ and $H_d$ in the $\underline{1}$ of $U(2)$, with Yukawa couplings like $uqH_u$. Adding these two fields to those of a single generation of (5.1) makes the fermion content vector-like. Even with additional generations, one could always add additional $U(2)$ fundamental
matter to make the net matter content vector-like. These additional chiral fields can be
get large masses via making their expectation values or Yukawa couplings large.

Chiral fermions which can contribute to the anomalies can only be massive in Higgsed
theories, with the masses arising via Yukawa couplings. As usual, integrating out such
massive chiral fermions of Higgsed theories leads to Wess-Zumino-Witten terms, which is
how their contribution to anomalies shows up in the low energy effective field theory \[30\].
In the noncommutative theory, the relevant diagrams are planar, so the same WZW terms
are generated (e.g. the lengthy one in eqn. (24) of \[30\]), with the only difference being
that all products are replaced with \(*\) products.

**Note added:** A lattice construction of noncommutative theories with adjoint chiral
fermions, for arbitrary even dimensions, was recently reported in \[31\]. In \(d = 2 \mod 4\)
dimensions, the adjoint chiral fermion is not vector-like and would contribute to anomalies
in the commutative context. Surprisingly, no anomaly contribution was found in the lattice
construction of \[31\] for any \(d\) (though such anomalies generally do not vanish in the contin-
uum version of these noncommutative theories, as they come from planar diagrams). This
is perhaps suggestive of a lattice noncommutative cancellation mechanism similar to that
discussed here, and we thank one of the authors of \[31\] for bringing this to our attention.
However, because the missing anomalies are generally irreducible planar anomalies, they
cannot be cancelled by the GS mechanism or any known counter-term. So it’s unclear how
the lattice noncommutative theory is managing to cancel the anomaly.

**Acknowledgements**

We are grateful to S. Hellerman, A. Manohar, J. Michelson, S.-J. Rey, J. Schwarz and
W. Taylor for useful discussions. This work is supported by DOE-FG03-97ER40546.
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