The non-nil-invariance of TP

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1 Introduction

In [8], Hesselholt defined a spectrum TP(X), the periodic topological cyclic homology of a scheme X, using topological Hochschild homology and the Tate construction, which is a topological analogue of Connes-Tsygan periodic cyclic homology HP defined by Hochschild homology and the Tate construction. In [7, Theorem II.5.1], Goodwillie proved that for R an algebra over a field of characteristic 0 and I a nilpotent ideal of R, the quotient map R → R/I induces an isomorphism on HP. In this paper, we show that the analogous result for TP does not hold, that is to say, there is an algebra of positive characteristic and a nilpotent ideal such that the quotient map does not induce an isomorphism on TP, even rationally. More precisely, we prove the following result.

**Theorem 1.1.** Let p be a prime number and k ≥ 2 a natural number. Then the canonical map

$$\text{TP}_*(\mathbb{F}_p[x]/(x^k)) \rightarrow \text{TP}_*(\mathbb{F}_p)$$

is not an isomorphism. Moreover, if k is not a p-power, then the map is also not an isomorphism after inverting p.

In [8], Hesselholt gives a cohomological interpretation by TP of the Hasse-Weil zeta function of a scheme smooth and proper over a finite field inspired by [5] and [4]. Furthermore, in [1] and [2], it is proved that TP satisfies the K"unneth formula for smooth and proper dg-categories over a perfect field of positive characteristic. Therefore, this new cohomology theory TP may be considered to be an important cohomology theory for p-adic geometry and non-commutative geometry. Our result concerns a fundamental property of this theory. In Theorem 3.3, we evaluate the TP-group of \(\mathbb{F}_p[x]/(x^k)\) completely.
2 Periodic topological cyclic homology

Periodic topological cyclic homology TP is proposed in [8]. In this section, we briefly recall some notions from there. We let \( T \) denote the circle group. The following construction written in the higher categorical language can be found at [14, I.4].

Let \( E \) be a free \( T \)-CW-complex whose underlying space is contractible. Then we consider the following cofibration sequence of pointed \( T \)-spaces

\[
E_+ \rightarrow S^0 \rightarrow \tilde{E},
\]

where \( E_+ \) is the pointed space \( E \sqcup \{\infty\} \) and \( S^0 = \{0, \infty\} \), and the left map sends \( \infty \) to the base point \( \infty \in S^0 \) and all other points to \( 0 \in S^0 \).

Let \( X \) be a \( T \)-spectrum. Smashing the internal hom spectrum \([E_+, X]\) with the above diagram and taking fixed points of a subgroup \( C \subset T \), we have the following sequence called Tate cofibration sequence

\[
(E_+ \otimes [E_+, X])^C \rightarrow ([E_+, X])^C \rightarrow (\tilde{E} \otimes [E_+, X])^C.
\]

We write this sequence as

\[
(E_+ \otimes [E_+, X])^C = \begin{cases} 
\mathcal{H}.(C, X), & \text{if } C \not\subset T \\
\Sigma \mathcal{H}.(C, X), & \text{if } C = T,
\end{cases}
\]

\[
([E_+, X])^C = \mathcal{H}^T (C, X)
\]

\[
(\tilde{E} \otimes [E_+, X])^C = \mathcal{H}^T (C, X)
\]

Let \( X \) be a scheme. The topological periodic cyclic homology of \( X \) is the spectrum given by

\[
\text{TP}(X) = \mathcal{H}^T (T, \text{THH}(X)),
\]

where THH denotes the topological Hochschild homology of \( X \) defined in [6] and [3]. In the present paper, we will only consider affine schemes. For a commutative ring \( R \), there is a conditionally convergent spectral sequence [13, §4],

\[
E^2_{i,j} = S\{t, t^{-1}\} \otimes \text{THH}_j(R) \Rightarrow \text{TP}_{i+j}(R),
\]

where \( \text{deg}(t) = (-2,0) \).
3 Truncated polynomial algebras

Our main result is the following

**Theorem 3.1.** Let $p$ be a prime number and $k \geq 2$ a natural number. If $k$ is not a $p$-power, then the canonical map

$$\text{TP}_*(\mathbb{F}_p[x]/(x^k))[1/p] \to \text{TP}_*(\mathbb{F}_p)[1/p]$$

is not an isomorphism.

Before proving our main result, we recall from [12] and [9] some calculations concerning $\text{THH}(\mathbb{F}_p[x]/(x^k))$. The following also shown in [15, Paper B] in the higher categorical language.

We give the pointed finite set $\Pi_k = \{0, 1, x, \ldots, x^{k-1}\}$ with the base point 0 the pointed commutative monoid structure, where 1 is the unit, $0 \cdot 1 = 0$, $x^i \cdot x^j = x^{i+j}$ and $x^k = 0$. We denote the cyclic bar construction of $\Pi_k$ by $N_{\bullet}^{\text{cy}}(\Pi_k)$. More precisely, the set of $l$-simplicies is

$$N_{\bullet}^{\text{cy}}(\Pi_k) = \Pi_k \land \cdots \land \Pi_k,$$

where there are $l + 1$ smash factors and the structure maps are given by

- $d_i(x_0 \land \cdots \land x_l) = x_0 \land \cdots \land x_i x_{i+1} \land \cdots \land x_l$, $0 \leq i < l$,
- $d_l(x_0 \land \cdots \land x_l) = x_0 \land x_1 \land \cdots \land x_{k-1}$,
- $s_i(x_0 \land \cdots \land x_l) = x_0 \land \cdots \land x_i \land 1 \land x_{i+1} \land \cdots \land x_l$, $0 \leq i \leq l$,
- $t_l(x_0 \land \cdots \land x_l) = x_l \land x_0 \land x_1 \land \cdots \land x_{l-1}$.

We let $N_{\bullet}^{\text{cy}}(\Pi_k)$ denote the geometric realization of $N_{\bullet}^{\text{cy}}(\Pi_k)$.

In [11, Theorem 7.1], it is proved that there is a natural equivalence of cyclotomic spectra

$$\text{THH}(\mathbb{F}_p[x]/(x^k)) \simeq \text{THH}(\mathbb{F}_p) \otimes N_{\bullet}^{\text{cy}}(\Pi_k).$$

(a)

For each positive integer $i$, we also have the cyclic subset

$$N_{\bullet}^{\text{cy}}(\Pi_k, i) \subset N_{\bullet}^{\text{cy}}(\Pi_k)$$

generated by the $(i-1)$-simplex $x \land \cdots \land x$ ($i$ factors), and denote the geometric realization by $N_{\bullet}^{\text{cy}}(\Pi_k, i)$. We also have the cyclic subset $N_{\bullet}^{\text{cy}}(\Pi_k, 0)$ generated by the 0-simplex 1 with the geometric realization $N_{\bullet}^{\text{cy}}(\Pi_k, 0)$. Thus we obtain the following wedge decomposition

$$\bigvee_{i \geq 0} N_{\bullet}^{\text{cy}}(\Pi_k, i) = N_{\bullet}^{\text{cy}}(\Pi_k).$$
We consider the complex $\mathbb{T}$-representation, where $d = \lfloor (i - 1)/k \rfloor$ is the integer part of $(i - 1)/k$ for $i \geq 1$,

$$\lambda_d = \mathbb{C}(1) \oplus \mathbb{C}(2) \oplus \cdots \oplus \mathbb{C}(d),$$

where $\mathbb{C}(i) = \mathbb{C}$ with the $\mathbb{T}$ action;

$$\mathbb{T} \times \mathbb{C}(i) \to \mathbb{C}(i)$$

defined by $(z, w) \mapsto z^i w$. Then we have the following by [12, theorem B], for $i \geq 1$ such that $i \notin k\mathbb{N}$, there is an equivalence

$$N^{cy}(\Pi_k, i) \simeq S^{\lambda_d} \wedge (\mathbb{T}/C_i)_+,$$

where $C_i$ is the $i$-th cyclic group.

Let $\text{THH}(\mathbb{F}_p[x]/(x^k), (x))$ denote the fiber of the canonical map

$$\text{THH}(\mathbb{F}_p[x]/(x^k)) \to \text{THH}(\mathbb{F}_p),$$

and we write

$$\text{TP}(\mathbb{F}_p[x]/(x^k), (x)) = \hat{H}(\mathbb{T}, \text{THH}(\mathbb{F}_p[x]/(x^k), (x))).$$

The non-triviality of $\text{TP}(\mathbb{F}_p[x]/(x^k), (x))$ shall imply that TP is not nil-invariant. In order to obtain the non-triviality, we use the following decomposition.

**Lemma 3.2.** There is a canonical equivalence

$$\text{TP}(\mathbb{F}_p[x]/(x^k), (x)) \simeq \prod_{i \geq 1} \hat{H}(\mathbb{T}, \text{THH}(\mathbb{F}_p) \otimes N^{cy}(\Pi_k, i)).$$

**Proof.** By (a) and the wedge decomposition of $N^{cy}(\Pi_k)$ above, we have

$$\Sigma H(\mathbb{T}, \text{THH}(\mathbb{F}_p[x]/(x^k), (x))) \simeq \bigoplus_{i \geq 1} \Sigma H(\mathbb{T}, \text{THH}(\mathbb{F}_p) \otimes N^{cy}(\Pi_k, i)),$$

since $\Sigma H(\mathbb{T}, -)$ preserves all homotopy colimits. Since the connectivity of $\Sigma H(\mathbb{T}, \text{THH}(\mathbb{F}_p) \otimes N^{cy}(\Pi_k, i))$ goes to $\infty$ as $i$ goes to $\infty$, we have

$$\bigoplus_{i \geq 1} \Sigma H(\mathbb{T}, \text{THH}(\mathbb{F}_p) \otimes N^{cy}(\Pi_k, i)) \simeq \prod_{i \geq 1} \Sigma H(\mathbb{T}, \text{THH}(\mathbb{F}_p) \otimes N^{cy}(\Pi_k, i)).$$
Similarly, since $H(\mathbb{T}, -)$ preserves all homotopy limits, we have

$$H(\mathbb{T}, \text{THH}(\mathbb{F}_p[x]/(x^k), (x))) \simeq \prod_{i \geq 1} H(\mathbb{T}, \text{THH}(\mathbb{F}_p) \otimes N^{cv}(\Pi_k, i)).$$

Lastly, since $\text{TP}(\mathbb{F}_p[x]/(x^k), (x))$ is the cofiber of the map

$$\Sigma H(\mathbb{T}, \text{THH}(\mathbb{F}_p[x]/(x^k), (x))) \to H(\mathbb{T}, \text{THH}(\mathbb{F}_p[x]/(x^k), (x))),$$

we get the desired equivalence.

It is known that, for a $\mathbb{T}$-spectrum $X$, there is a $\mathbb{T}$-equivalence

$$X \otimes (\mathbb{T}/C_i)_+ \simeq \Sigma[(\mathbb{T}/C_i)_+, X],$$

see for example [11, 8.1]. Hence, we have

$$\hat{H}(\mathbb{T}, \text{THH}(\mathbb{F}_p) \otimes (\mathbb{T}/C_i)_+) = (E \otimes [E_+, \text{THH}(\mathbb{F}_p) \otimes (\mathbb{T}/C_i)_+])^{\mathbb{T}}$$

$$\simeq \Sigma(E \otimes [E_+, (\mathbb{T}/C_i)_+, \text{THH}(\mathbb{F}_p)]]^{\mathbb{T}}$$

$$\simeq \Sigma(E \otimes [E_+, \text{THH}(\mathbb{F}_p)])^{\mathbb{T}}$$

$$\simeq \Sigma(E \otimes [E_+, \text{THH}(\mathbb{F}_p)])^{\mathbb{T}}$$

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Furthermore, we have an equivalence of spectra

$$\hat{H}(C_i, \text{THH}(\mathbb{F}_p) \otimes S^{\lambda_d}) \simeq \hat{H}(C_{p^{\nu_p(i)}}, \text{THH}(\mathbb{F}_p) \otimes S^{\lambda_d}),$$

where $\nu_p$ denotes the $p$-adic valuation.

Hesselholt and Madsen have calculated the homotopy groups of the above spectra [11, §9],

$$\pi_* \hat{H}(C_{p^n}, \text{THH}(\mathbb{F}_p) \otimes S^{\lambda_d}) \cong S_{\mathbb{Z}/p^n\mathbb{Z}}\{t, t^{-1}\},$$

where $t$ is the divided Bott element. More precisely, $\pi_* \hat{H}(C_{p^n}, \text{THH}(\mathbb{F}_p) \otimes S^{\lambda_d})$ is a free module of rank 1 over $\mathbb{Z}/p^n\mathbb{Z}[t, t^{-1}]$ on a generator of degree $2d$.

A preferred generator is specified in [10, Proposition 2.5]. Combining these and (b), we obtain for $i \notin k\mathbb{N}$ a canonical isomorphism

$$\pi_j \hat{H}(\mathbb{T}, \text{THH}(\mathbb{F}_p) \otimes N^{cv}(\Pi_k, i)) \cong \begin{cases} \mathbb{Z}/p^{\nu_p(i)}\mathbb{Z}, & j - \lambda_d + 1 \text{ even} \\ 0, & j - \lambda_d + 1 \text{ odd} \end{cases}$$
and note that $-\lambda_d + 1$ is always odd by definition. They have similarly showed that for $i \in k\mathbb{N}$, there is a canonical isomorphism

$$\pi_j \hat{H}(\mathbb{T}, \text{THH}(\mathbb{F}_p) \otimes N^{sy}(\Pi_k, i)) \cong \begin{cases} \mathbb{Z}/p^{v_p(k)}\mathbb{Z}, & j \text{ odd} \\ 0, & j \text{ even}. \end{cases}$$

From these, we obtain the following.

**Theorem 3.3.** If $j$ is an odd integer, then there is a canonical isomorphism

$$\text{TP}_j(\mathbb{F}_p[x]/(x^k), (x)) \cong \prod_{i \geq 1, i \in k\mathbb{N}} \mathbb{Z}/p^{v_p(i)}\mathbb{Z} \times \prod_{i \geq 1, i \in k\mathbb{N}} \mathbb{Z}/p^{v_p(i)}\mathbb{Z}.$$ 

If $j$ is an even integer, then

$$\text{TP}_j(\mathbb{F}_p[x]/(x^k), (x)) = 0.$$ 

Therefore, we get our main result by this theorem. Moreover, this concrete calculation gives us the following.

**Corollary 3.4.** If $k = p^r$ with a natural number $r \in \mathbb{N}$, the canonical map

$$\text{TP}_*(\mathbb{F}_p[x]/(x^{p^r}))[1/p] \to \text{TP}_*(\mathbb{F}_p)[1/p]$$

is an isomorphism.

Thus, in this specific case, the analogue of Goodwillie’s theorem for TP holds. In addition, by [14, Corollary 1.5] and the main theorem of [12], we get the following.

**Corollary 3.5.** Topological negative cyclic homology is not nil-invariant, even rationally.

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