ABSOLUTELY CONTINUOUS SPECTRUM FOR PARABOLIC FLOWS/MAPS

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Abstract. We provide an abstract framework for the study of certain spectral properties of parabolic systems; specifically, we determine under which general conditions to expect the presence of absolutely continuous spectral measures. We use these general conditions to derive results for spectral properties of time-changes of unipotent flows on homogeneous spaces of semisimple groups regarding absolutely continuous spectrum as well as maximal spectral type; the time-changes of the horocycle flow are special cases of this general category of flows. In addition we use the general conditions to derive spectral results for twisted horocycle flows and to rederive certain spectral results for skew products over translations and Furstenberg transformations.

1. Introduction.

1.1. Motivation. Spectral theory of dynamical systems has long been studied [13]; of particular interest, is the notion of when to expect the presence of absolutely continuous spectral measures. Since absolutely continuous spectrum implies mixing, this property can be thought of as an indicator of how chaotic, or how far from orderly, a system is. In the hyperbolic setting, systems are characterized as having a correlation decay that is exponential. As a result, techniques derived from the existence of a spectral gap as well as probabilistic tools are available for the study of spectral properties, and therefore, it is in the hyperbolic setting where the existence of absolutely continuous spectrum predominantly occurs. Interestingly, certain parabolic systems also share this property despite having at most polynomial decay of correlations. This slower decay of correlations precludes the use of the tools available in the spectral study of hyperbolic systems, and consequently, spectral theory of smooth parabolic flows and smooth perturbations of well known parabolic flows has been much less studied. This work is devoted to creating an abstract framework for the study of certain spectral properties of parabolic systems. Specifically, we attempt to answer the question: under what general conditions can we expect the existence of absolutely continuous spectral measures?

1.2. Statement of results. In Theorem 2.1 we present general conditions under which we expect a skew-adjoint operator to have absolutely continuous spectrum. The proof of this theorem is a general application of the method in [5],

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in which the authors show that the Fourier Transform of the spectral measures of smooth coboundaries are square integrable. The method in [5] was inspired by Marcus’s proof of mixing of horocycle flows [14] which requires a specific form of tangent dynamics from which one can exploit shear of nearby trajectories. In addition, we rely on the bootstrap technique from [5] to estimate the decay of correlations of specific smooth coboundaries. Our choice of coboundaries depends upon a growth condition involving the commutator of the skew-adjoint operator and a certain auxiliary operator.

We use this general, functional analytic result to derive the following results for certain parabolic dynamical systems. Theorem 3.1 states that time-changes of unipotent flows on homogeneous spaces of semisimple groups have absolutely continuous spectrum. In the compact case, we also show that the maximal spectral type is Lebesgue, following the method in [5]. The time-changes of the horocycle flow are special cases of this general category of flows, and spectral properties of the time-changes of horocycle flows were shown in [5], [18], and [19]. In addition, we use the general conditions to prove Theorem 3.5 and Corollary 3.2 regarding spectral results for twisted horocycle flows, combining the horocycle time-change with a circle rotation. Lastly, we rederive certain spectral results for skew products over translations and Furstenberg transformations, originally shown in [19]. Our results are slightly weaker as we prove results for functions of class $C^2$ while the author in [19] considers functions of class $C^1$ with an added Dini Condition.

Remark 1. It is standard to consider spectral decomposition in the setting of self-adjoint operators. When we consider flows that are represented as strongly continuous one-parameter unitary groups, the generating operators (vector fields) are essentially skew-adjoint. Since multiplication by $i$ gives an essentially self-adjoint operator we make no distinction from the standard setting, and thus, directly apply the theory for self-adjoint operators.

2. Abstract conditions.

2.1. Preliminary assumptions. Suppose that a closed operator $X$ on a Hilbert Space $\mathcal{H}$ defined on a dense subspace $D$ such that $X(D) \subset D$, generates a strongly continuous, one parameter group $\{e^{sX}\}$.

Suppose also that $e^{tU}(D) \subset D$ is a strongly continuous, unitary group with infinitesimal generator $U$, and that the commutator

$$H(t) = e^{-tU} [X, e^{tU}]$$

is defined on $D$.

For $u \in D$, let

$$\frac{H(t)}{t^\beta} \frac{u \to \mathcal{H}}{t \to \infty} H u,$$

such that $H(D) \subset D$ and $\text{Ran}(H) = \{Hu : u \in D\} = \mathcal{H}$.

For $B_1$, $B_2$ bounded operators on $\mathcal{H}$ such that $B_2 : D \to D$, let

$$\| \langle e^{tU} f, f \rangle_{\mathcal{H}} \|_{L^2(\mathbb{R})} \leq \left\| \frac{1}{t^\sigma} \int_0^t \langle e^{sX} e^{tU} f, B_1 e^{sX} B_2 f \rangle_{\mathcal{H}} ds \right\|_{L^2(\mathbb{R})}.$$

Theorem 2.1. If for $\beta > \frac{1}{2}$, $H(t)$ and $H$ satisfy:

(i) $\frac{H(t)}{t^\beta} H^{-1}$ is defined on $\text{Ran}(H)$, extends by continuity to a bounded operator
Let \( f \) have used a similar term to derive criteria for strong mixing, which was based on Marcus’s shear mechanism in [14]. The authors in [20] and [17] have used a similar term to derive criteria for strong mixing.

**Remark 2.** Often in ergodic theory, \( \mathcal{H} \) is a subspace of a larger Hilbert space; for example, \( \mathcal{H} = L^2(M) \) the space of zero-average functions in \( L^2(M) \). While in this setting the theorem doesn’t give a result for purely absolutely continuous spectrum it implies the existence of an absolutely continuous component.

**Remark 3.** The utilization of the operator \( H(t) \) is suggested by the method in [5] which was based on Marcus’s shear mechanism in [14]. The authors in [20] and [17] have used a similar term to derive criteria for strong mixing.

**Proof.** Let \( f \in \text{Ran}(H) \) and let \( \hat{\mu}_f(t) = \int_{\mathbb{R}} e^{it\xi} d\mu_f(\xi) \) be the Fourier Transform of the spectral measure \( \mu_f \).

\[
\| \hat{\mu}_f(t) \|_{L^2(\mathbb{R})} = \| \langle e^{itU} f, f \rangle_{\mathcal{H}} \|_{L^2(\mathbb{R})} \leq \left\| \frac{1}{\sigma} \int_0^\sigma \langle e^{sX} e^{itU} f, B_1 e^{sX} B_2 f \rangle_{\mathcal{H}} ds \right\|_{L^2(\mathbb{R})}.
\]

For \( s \in [0, \sigma] \), we integrate by parts:

\[
B_1 e^{sX} B_2 f \quad \frac{d}{ds} (B_1 e^{sX} B_2 f) = B_1 e^{sX} X(B_2 f)
\]

\[
e^{sX} e^{itU} f \quad \int_0^\sigma e^{sX} e^{itU} f ds.
\]

\[
\frac{1}{\sigma} \int_0^\sigma \langle e^{sX} e^{itU} f, B_1 e^{sX} B_2 f \rangle_{\mathcal{H}} ds = \frac{1}{\sigma} \left\| \int_0^\sigma e^{sX} e^{itU} f ds, B_1 e^{sX} B_2 f \right\|_{\mathcal{H}}
\]

\[
- \frac{1}{\sigma} \int_0^\sigma \int_0^S e^{sX} e^{itU} f ds, B_1 e^{sX} X(B_2 f) \rangle_{\mathcal{H}} dS.
\]

From our assumptions, both \( B_1 e^{sX} B_2 f \) and \( B_1 e^{sX} X(B_2 f) \) are bounded in \( \mathcal{H} \). Thus, in order to show that \( \hat{\mu}_f(t) = O(\frac{1}{\sigma}) \), we need a bound (in \( t \)) for

\[
\left\| \int_0^\sigma e^{sX} e^{itU} f ds \right\|_{\mathcal{H}}.
\]

Suppose that conditions (i), (ii), and (iii) hold, and let \( f \) be a coboundary of the form \( f = Hg \), for \( g \in \text{Dom}(H) \):

\[
\int_0^\sigma e^{sX} e^{itU} f ds = \int_0^\sigma e^{sX} e^{itU} H g ds
\]

\[
= \int_0^\sigma e^{sX} e^{itU} \left( H - \frac{H(t)}{t^\beta} \right) g ds + \int_0^\sigma e^{sX} e^{itU} \frac{H(t)}{t^\beta} g ds.
\]
I. Let
\[
\tilde{H}(s, t) = e^{sX} e^{tU} \left( I - \frac{\tilde{H}(t)H^{-1}}{t^3} \right) e^{-tU} e^{-sX},
\]
where \(\tilde{H}(t)H^{-1} \frac{1}{t^3}\) is the bounded extension of \(\frac{H(t)H^{-1}}{t^3}\).

Since \(\{e^{sX}\}\) is strongly continuous, for \(f \in D\),
\[
s^{-\lim_{s \to 0}} e^{sX} f = I f.
\]

Thus,
\[
\sup_{s \in [0, \sigma]} \|e^{sX}\|_\text{op} \leq 1 + k(\sigma)
\]
where
\[
\lim_{\sigma \to 0} k(\sigma) = 0.
\]

Since \(\limsup_{t \to \infty} \| I - \frac{\tilde{H}(t)H^{-1}}{t^3} \|_\text{op} < 1\), then for large \(t\) and small enough \(\sigma\),
\[
\|\tilde{H}(s, t)\|_\mathcal{H} < C_1 < 1.
\]

We compute, for \(f = e^{sX}(e^{tU}(H u))\),
\[
\lim_{\Delta s \to 0} \frac{\tilde{H}(s + \Delta s, t)f - \tilde{H}(s, t)f}{\Delta s}
\]
\[
= \lim_{\Delta s \to 0} \left( \frac{1}{e^{sX} e^{tU} \left( \frac{\tilde{H}(t)H^{-1}}{t^3} \right) e^{-tU} e^{-sX} f} - \frac{1}{\Delta s} \right)
\]
\[
= \lim_{\Delta s \to 0} \left( \frac{e^{(s + \Delta s)X} e^{tU} \left( \frac{\tilde{H}(t)H^{-1}}{t^3} \right) e^{-tU} e^{-(s + \Delta s)X} f}{\Delta s} - \frac{1}{\Delta s} \right)
\]
\[
= \lim_{\Delta s \to 0} \left( \frac{e^{(s + \Delta s)X} e^{tU} \left( \frac{\tilde{H}(t)H^{-1}}{t^3} \right) e^{-tU} e^{-(s + \Delta s)X} f - e^{sX} e^{tU} \left( \frac{\tilde{H}(t)H^{-1}}{t^3} \right) e^{-tU} e^{-sX} f}{\Delta s} \right)
\]
\[
= \lim_{\Delta s \to 0} \left( \frac{(e^{(s + \Delta s)X}) e^{tU} \left( \frac{\tilde{H}(t)H^{-1}}{t^3} \right) e^{-tU} (e^{-sX} - e^{-(s + \Delta s)X}) f}{\Delta s} \right)
\]
\[
\begin{align*}
&+ \frac{(e^{(s+\Delta s)X} - e^{sX})e^{tU} \left( \frac{H(t)H^{-1}}{t^\beta} \right) e^{-tU} e^{-sX} f}{\Delta s} \\
&= \lim_{\Delta s \to 0} \left( \frac{e^{sX} e^{\Delta sX} e^{tU} \left( \frac{H(t)H^{-1}}{t^\beta} \right) e^{-tU} e^{-sX} (I - e^{-\Delta sX}) f}{\Delta s} \right) \\
&+ \frac{e^{sX} (e^{\Delta sX} - I) e^{tU} \left( \frac{H(t)H^{-1}}{t^\beta} \right) e^{-tU} e^{-sX} f}{\Delta s} \\
&\pm \lim_{\Delta s \to 0} e^{sX} e^{\Delta sX} e^{tU} \left( \frac{\tilde{H}(t)H^{-1}}{t^\beta} \right) e^{-tU} e^{-sX} X f \\
&= \lim_{\Delta s \to 0} e^{sX} e^{\Delta sX} e^{tU} \left( \frac{H(t)H^{-1}}{t^\beta} \right) e^{-tU} e^{-sX} \left( \frac{I - e^{-\Delta sX}}{\Delta s} - X \right) f \\
&+ \lim_{\Delta s \to 0} \frac{e^{sX} (e^{\Delta sX} - I)}{\Delta s} e^{tU} \left( \frac{H(t)H^{-1}}{t^\beta} \right) e^{-tU} e^{-sX} f. \\
\end{align*}
\]

The first limit,
\[
\lim_{\Delta s \to 0} e^{sX} e^{\Delta sX} e^{tU} \left( \frac{\tilde{H}(t)H^{-1}}{t^\beta} \right) e^{-tU} e^{-sX} (I - e^{-\Delta sX}) f = 0,
\]
follows from the preliminary assumption,
\[
\lim_{\Delta s \to 0} \left( \frac{e^{\Delta s} - I}{\Delta s} \right) = X.
\]
and
\[
\lim_{\Delta s \to 0} e^{\Delta sX} = I.
\]
The second and third limits follow from above,
\[
\lim_{\Delta s \to 0} e^{sX} e^{\Delta sX} e^{tU} \left( \frac{H(t)H^{-1}}{t^\beta} \right) e^{-tU} e^{-sX} X f \\
= e^{sX} e^{tU} \left( \frac{H(t)H^{-1}}{t^\beta} \right) e^{-tU} e^{-sX} X f
\]
and
\[
\lim_{\Delta s \to 0} \frac{e^{sX} (e^{\Delta sX} - I)}{\Delta s} e^{tU} \left( \frac{\tilde{H}(t)H^{-1}}{t^\beta} \right) e^{-tU} e^{-sX} f \\
= e^{sX} X e^{tU} \left( \frac{H(t)H^{-1}}{t^\beta} \right) e^{-tU} e^{-sX} f.
\]
Since \( f = e^{sX}(e^{tU}(Hu)) \),
\[
e^{sX}e^{tU}H(t)H^{-1}e^{-tU}e^{-sX}f
= e^{sX}e^{tU}H(t)H^{-1}e^{-tU}e^{-sX}(e^{tU}(Hu))
= e^{sX}e^{tU}H(t)u \in \mathcal{H}
\]
and
\[
e^{sX}e^{tU}H(t)H^{-1}e^{-tU}Xe^{-sX}f
= e^{sX}e^{tU}H(t)H^{-1}e^{-tU}Xe^{-sX}(e^{tU}(Hu))
= e^{sX}e^{tU}H(t)H^{-1}e^{-tU}Xe^{tU}(Hu) \in \mathcal{H}
\]
since \( e^{tU}(Hu) \subset D \) and \( \tilde{H}(t)H^{-1} \) is a bounded operator on \( \mathcal{H} \).

Since \( \text{Ran}(U) \) is dense in \( \mathcal{H} \) and \( e^{tU} \) and \( e^{sX} \) are bounded, invertible operators, \( e^{sX} \left( e^{tU} \left( \text{Ran}(H) \right) \right) \) is dense in \( \mathcal{H} \). Thus,
\[
\frac{\partial}{\partial s}H(s,t) = \frac{1}{t^3} e^{sX} \left[ X, e^{tU}H(t)H^{-1}Xe^{-tU} \right] e^{-sX}
\]
is defined on \( \mathcal{H} \).

Now we can rewrite
\[
\int_0^\sigma e^{sX}e^{tU} \left( \frac{H - H(t)}{t^3} \right) dgds = \int_0^\sigma e^{sX}e^{tU} \left( I - \frac{H(t)H^{-1}}{t^3} \right) Hgds
\]
and again consider the extension
\[
\int_0^\sigma e^{sX}e^{tU} \left( I - \frac{\tilde{H}(t)H^{-1}}{t^3} \right) Hgds = \int_0^\sigma \tilde{H}(s,t)e^{sX}e^{tU}f ds.
\]
Integration by parts gives
\[
\int_0^\sigma \tilde{H}(s,t)e^{sX}e^{tU}fds
= \tilde{H}(\sigma,t)\int_0^\sigma e^{sX}e^{tU}fds - \int_0^\sigma \frac{\partial \tilde{H}(S,t)}{\partial S} \left[ \int_0^S e^{sX}e^{tU}fds \right] dS.
\]
So now we must ensure that
\[
\frac{\partial \tilde{H}(s,t)}{\partial s} = \frac{1}{t^3} e^{sX} \left[ X, e^{tU}H(t)H^{-1}e^{-tU} \right] e^{-sX}
\]
is uniformly bounded in \( \| \cdot \|_{op} \). Let \( h \in \mathcal{H} \).
\[
\frac{1}{t^3} e^{sX} \left( \left[ X, e^{tU}H(t)H^{-1}e^{-tU} \right] \right) e^{-sX}h
= \frac{1}{t^3} e^{sX} \left( \left[ X, e^{tU}H(t)H^{-1}e^{-tU} \right] \right) + e^{tU} \left[ X, H(t)H^{-1} \right] e^{-tU}
+ e^{tU} \tilde{H}(t)H^{-1} \left[ X, e^{-tU} \right] \right) e^{-sX}h.
\]
Using the identity \( e^{-tU} \left[ X, e^{tU} \right] = - \left[ X, e^{-tU} \right] e^{tU} \) we can simplify and combine terms:
\[
\frac{1}{t^3} e^{sX}e^{tU} \left( H(t)H(t)H^{-1} + \left[ X, H(t)H^{-1} \right] - H(t)H^{-1}H(t) \right) e^{-tU}e^{-sX}h
\]
\[= e^{sx} e^{tu} \left[ X, \frac{H(t)H^{-1}}{t^\beta} \right] - \frac{H(t)H^{-1}}{t^\beta} \left[ H(t), H \right] H^{-1} \right) e^{-tu} e^{-sx} h \]

where \( \left[ X, \frac{H(t)H^{-1}}{t^\beta} \right] \) and \( \left[ H(t), H \right] H^{-1} \) are the bounded extensions of \( \left[ X, \frac{H(t)H^{-1}}{t^\beta} \right] \) and \( \left[ H(t), H \right] H^{-1} \).

Conditions (i), (ii), and (iii) imply that

\[
\left\| \frac{\partial \hat{H}(s, t)}{\partial s} \right\|_H \\
\leq C \left( \left\| \frac{H(t)H^{-1}}{t^\beta} \right\|_{op} \left\| [H(t), H] H^{-1} \right\|_{op} + \left\| \left[ X, \frac{H(t)H^{-1}}{t^\beta} \right] \right\|_{op} \right) \cdot \| h \|_H \\
\leq C_2 \cdot \| h \|_H
\]

for some constants \( C \) and \( C_2 \), and thus,

\[
\left\| \frac{\partial \hat{H}(s, t)}{\partial s} \right\|_{op} \leq C_2.
\]

II. For \( g \in \text{Dom}(H) \),

\[
\frac{1}{t^\beta} \int_0^\sigma e^{sx} e^{tu} H(t) g ds = \frac{1}{t^\beta} \int_0^\sigma e^{sx} e^{tu} X g ds - \frac{1}{t^\beta} \int_0^\sigma \frac{d}{ds} e^{sx} e^{tu} g ds
\]

which implies that

\[
\left\| \frac{1}{t^\beta} \int_0^\sigma e^{sx} e^{tu} H(t) g ds \right\|_H \leq C_3 \| X g \|_H + C_4 \frac{\| g \|_H}{t^\beta}.
\]

Finally, from I. and II.,

\[
\sup_{s \in [0, \sigma]} \left\| \int_0^s e^{sx} e^{tu} f ds \right\|_H \leq \sup_{s \in [0, \sigma]} \left( \left\| \frac{\partial \hat{H}(s, t)}{\partial s} \right\|_{op} \cdot \int_0^s e^{sx} e^{tu} f ds \right) \cdot \| f \|_H \\
+ \sigma \cdot \sup_{s \in [0, \sigma]} \left( \left\| \frac{\partial \hat{H}(s, t)}{\partial s} \right\|_{op} \cdot \int_0^s e^{sx} e^{tu} f ds \right) \cdot \left\| \int_0^s e^{sx} e^{tu} f ds \right\|_H \\
+ C_5 \| X g \|_H + C_4 \| g \|_H
\]

So for \( \sigma > 0 \), chosen such that \( 0 < C_1 + \sigma C_2 < 1 \), for all \( t \) sufficiently large,

\[
\sup_{s \in [0, \sigma]} \left\| \int_0^s e^{sx} e^{tu} f ds \right\|_H \leq C_3 \| X g \|_H + C_4 \| g \|_H \cdot \frac{1}{t^\beta} = O(\frac{1}{t^\beta}).
\]

Thus, since \( \hat{\mu}_f(t) \in L^2(\mathbb{R}) \), \( \mu_f \) is absolutely continuous. As this holds for \( f \in \text{Ran}(H) \), \( \mu_f \) is absolutely continuous for any \( f \in \mathcal{H} \).

**Remark 4.** The proof for the discrete case is the same aside from the replacement of the continuous parameter \( t \) and norm \( \| \cdot \|_{L^2(\mathbb{R})} \) by the discrete parameter \( n \) and norm \( \| \cdot \|_{\ell^2(\mathbb{Z})} \). The conclusion becomes

\[
\| (e^{nU} f, f)_{\mathcal{H}} \|_{\ell^2(\mathbb{Z})} = O(\frac{1}{n^\beta})
\]

for \( \beta > \frac{1}{2} \), and thus, \( \mu_f(n) \in \ell^2(\mathbb{Z}) \).
3. Applications to flows.

3.1. Time-changes of unipotent flows on homogeneous spaces of semisimple groups. As a direct consequence of Theorem 2.1, we derive a result for a specific category of generating operators.

Let $G$ be a semisimple Lie group and let the manifold $M = \Gamma \backslash G$ for some lattice $\Gamma$ in $G$ such that $M$ has finite area.

By the Jacobson-Morozov Theorem, any nilpotent element $U$ of the semisimple Lie algebra of $G$ is contained in a subalgebra isomorphic to $\mathfrak{sl}_2$. This means that this subalgebra contains an element $X$, such that $[U, X] = U$. Let $e^{tU}$ be a unitary operator of the Hilbert space $L^2(M, \text{vol})$. Thus, if the unipotent flow generated by $U$, $f \circ \phi^U_t = e^{tU}f$, $f \in L^2(M, \text{vol})$, is ergodic, then from Lemma 5.1 in [15], it has purely absolutely continuous spectrum on

$$L^2_0(M, \text{vol}) = \left\{ f \in L^2(M) \mid \int_M f \text{ vol} = 0 \right\}.$$ 

Let $\tau : M \times \mathbb{R} \to \mathbb{R}$ such that $\tau \in C^\infty(M, \mathbb{R})$ and

$$\tau(x, t + t') = \tau(x, t) + \tau(\phi^U_t(x), t').$$

Let $\alpha : M \to \mathbb{R}^+$, be the infinitesimal generator of $\tau$, such that $\alpha \in C^\infty(M)$ and

$$\int_M \alpha \text{ vol} = \int_M \text{ vol}_\alpha = 1$$

where $\text{vol}$ is the $\phi^U_t$-invariant volume form and $\text{vol}_\alpha$ is the $\phi^U_t$-invariant volume form. Now we consider a time-changed flow, $\{\phi^U_{\alpha t}\}$, generated by

$$U_{\alpha} = U/\alpha.$$ 

Let $e^{tU_{\alpha}}$ be a unitary operator on the Hilbert space $L^2(M, \text{vol}_\alpha)$. The following formulas hold on $D = C^\infty(M)$.

$$[X, U_{\alpha}] = G(\alpha)U_{\alpha} = \left(\frac{X\alpha}{\alpha} - 1\right)U_{\alpha} = H$$

$$e^{-tU_{\alpha}} [X, e^{tU_{\alpha}}] = G(\alpha, t)U_{\alpha} = \left(\frac{t}{t} \int_0^t \left(\frac{X\alpha}{\alpha} - 1\right) \circ \phi^U_{\alpha}(x) d\tau\right)U_{\alpha} = H(t).$$

The ergodicity of $\phi^U_{t\alpha}$ gives us the following limit a.e.,

$$\lim_{t \to \infty} \frac{G(\alpha, t)}{t} = \lim_{t \to \infty} \frac{1}{t} \int_0^t \left(\frac{X\alpha}{\alpha} - 1\right) \circ \phi^U_{\alpha}(x) d\tau$$

$$= \lim_{t \to \infty} \left(\frac{1}{t} \int_0^t \frac{X\alpha}{\alpha} \circ \phi^U_{\alpha}(x) d\tau + \frac{1}{t} \int_0^t -1 \circ \phi^U_{\alpha}(x) d\tau\right)$$

$$= \lim_{t \to \infty} \left(\frac{1}{t} \int_0^t \frac{X\alpha}{\alpha} \circ \phi^U_{\alpha}(x) d\tau - 1\right)$$

$$= \int_M \frac{X\alpha}{\alpha} d\text{vol}_\alpha - 1 = 1.$$ 

From the Dominated Convergence Theorem, with dominating function $2\|G(\alpha)\|_{\infty}$, we have convergence in $L^2(M)$. Thus, for $u \in C^\infty(M)$,

$$\lim_{t \to \infty} \frac{G(\alpha, t)}{t} U_{\alpha} u = -U_{\alpha} u = Hu.$$
Lastly,
\[
\int_M e^{tU_\alpha} f \cdot \overline{f} \, d\text{vol}_\alpha = \int_M e^{tU_\alpha} f \cdot \overline{f} \, d\text{vol} = \int_M e^{sX} e^{tU_\alpha} f \cdot \overline{e^{sX} f} \, d\text{vol} = \int_M e^{sX} e^{tU_\alpha} f \cdot \overline{\frac{1}{\alpha} e^{sX} f} \, d\text{vol}_\alpha.
\]

So if we integrate both sides of
\[
\langle e^{tU_\alpha} f, f \rangle_{L^2(M,\text{vol}_\alpha)} = \langle e^{sX} e^{tU_\alpha} f, \frac{1}{\alpha} e^{sX} f \rangle_{L^2(M,\text{vol}_\alpha)}
\]
with respect to \(s\), we obtain the following equality
\[
\| \langle e^{tU_\alpha} f, f \rangle_{L^2(M,\text{vol}_\alpha)} \|_{L^2(\mathbb{R},dt)} = \| \frac{1}{\sigma} \int_0^\sigma \langle e^{sX} e^{tU_\alpha} f, \frac{1}{\alpha} e^{sX} f \rangle_{L^2(M,\text{vol}_\alpha)} \, ds \|_{L^2(\mathbb{R},dt)}.
\]

Thus, the preliminary assumptions for Theorem 2.1 are satisfied with \(B_1 = \frac{1}{\alpha} I\) and \(B_2 = \alpha I\).

**Theorem 3.1.**

**a.** Any smooth time-change of an ergodic flow on \(M\) generated by a nilpotent element of a semisimple Lie algebra has absolutely continuous spectrum on \(L^2_0(M,\text{vol}_\alpha)\) if \(\|X_\alpha\|_\infty < 1\).

**b.** Any smooth time-change of a uniquely ergodic flow on \(M\) generated by a nilpotent element of a semisimple Lie algebra has absolutely continuous spectrum on \(L^2_0(M,\text{vol}_\alpha)\).

**Remark 5.** The condition in part **a.** is equivalent to the condition employed by Kushnirenko [12] (Theorem 2) to prove mixing for the time-changes of the horocycle flow. As shown by Marcus, this condition is unnecessarily restrictive. In the compact setting, the authors in [5] prove spectral results using the implicit unique ergodicity instead of requiring such a condition. The author in [18] proves similar spectral results by imposing this Kushnirenko condition; the author later substitutes this condition by a utilization of unique ergodicity [19] in the compact case. In the noncompact setting it remains open as to whether or not spectral results can be derived without imposing a Kushnirenko-type condition.

**Proof.** We show that the conditions of Theorem 2.1 hold.

**a. (i)** Let \(f = U_\alpha g\) for \(g \in C^\infty(M)\).

\[
\left\| \frac{H(t)}{t} H^{-1} f \right\|_{L^2(M,\text{vol}_\alpha)} = \left\| \frac{G(\alpha,t)}{t} U_\alpha (-U_\alpha^{-1} f) \right\|_{L^2(M,\text{vol}_\alpha)} \\
= \left\| \frac{G(\alpha,t)}{t} f \right\|_{L^2(M,\text{vol}_\alpha)} \\
\leq 2 \|G(\alpha)\|_\infty \cdot \|f\|_{L^2(M,\text{vol}_\alpha)} \\
\leq 2 \|f\|_{L^2(M,\text{vol}_\alpha)}
\]

Since the above holds for \(f \in \text{Ran}(U_\alpha)\), \(\frac{H(t)}{t} H^{-1}\) extends to a bounded operator on \(\text{Ran}(U_\alpha) = L^2_0(M,\text{vol}_\alpha)\) with uniformly bounded norm in \(t\).
Also,
\[
\left\| (I - \frac{H(t)}{t}H^{-1})f \right\|_{L^2(M, vol_\alpha)} = \left\| \left(1 + \frac{G(\alpha, t)}{t} \right) f \right\|_{L^2(M, vol_\alpha)}
\]
\[
= \left\| \left(1 + \left(\frac{1}{t} \int_0^t \frac{X_\alpha}{\alpha} - 1 \right) \circ \phi^U_{\tau}(x) d\tau \right) \right\|_{L^2(M, vol_\alpha)}
\]
\[
= \left\| \frac{1}{t} \int_0^t \frac{X_\alpha}{\alpha} \circ \phi^U_{\tau}(x) d\tau \right\|_{L^2(M, vol_\alpha)}
\]
\[
\leq \left\| \frac{1}{t} \int_0^t \frac{X_\alpha}{\alpha} \circ \phi^U_{\tau}(x) d\tau \right\|_\infty \cdot \|f\|_{L^2(M, vol_\alpha)}
\]
\[
\leq \left\| \frac{X_\alpha}{\alpha} \right\|_\infty \cdot \|f\|_{L^2(M, vol_\alpha)}
\]
Since the above holds on Ran(U_\alpha) the following is true on Ran(U_\alpha) = L^2(M, vol_\alpha),
\[
\limsup_{t \to \infty} \left\| I - \frac{H(t)}{t}H^{-1} I \right\|_{op} = \limsup_{t \to \infty} \left\| I + \frac{G(\alpha, t)}{t} I \right\|_{op} < 1.
\]
(ii) In the following calculation we use that
\[
D\phi^U_{\tau}(X) = G(\alpha, t)U_\alpha \circ \phi^U_{\tau} + X \circ \phi^U_{\tau}
\]
where \( D\phi^U_{\tau} \) denotes the differential of the diffeomorphism \( \phi^U_{\tau} \).
\[
\left[ X, \frac{H(t)}{t}H^{-1} \right] = X \left( \frac{1}{t} \int_0^t \left( \frac{X_\alpha}{\alpha} - 1 \right) \circ \phi^U_{\tau} d\tau \right) - \left( \frac{1}{t} \int_0^t \left( \frac{X_\alpha}{\alpha} - 1 \right) \circ \phi^U_{\tau} d\tau \right) X
\]
\[
= \frac{1}{t} \int_0^t \left( D\phi^U_{\tau}(X) \circ \phi^U_{\tau} \right) \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^U_{\tau} d\tau
\]
\[
= \frac{1}{t} \int_0^t \left( G(\alpha, \tau)U_\alpha + X \right) \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^U_{\tau} d\tau
\]
\[
= \frac{1}{t} \int_0^t G(\alpha, \tau)U_\alpha \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^U_{\tau}(x) d\tau + \frac{1}{t} \int_0^t X \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^U_{\tau}(x) d\tau
\]
We integrate
\[
\frac{1}{t} \int_0^t G(\alpha, \tau) \frac{d}{d\tau} \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^U_{\tau}(x) d\tau
\]
by parts,
\[
\frac{1}{t} \int_0^t G(\alpha, \tau) \frac{d}{d\tau} \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^U_{\tau}(x) d\tau
\]
\[
= \frac{G(\alpha, t)}{t} \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^U_{\tau} - \frac{1}{t} \int_0^t \left( \frac{X_\alpha}{\alpha} - 1 \right) \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^U_{\tau}(x) d\tau,
\]
and obtain the bound,

\[
\left\| \frac{1}{t} \int_0^t \left( G(\alpha, \tau) U_\alpha + X \right) \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^{U_\alpha}_\tau \, d\tau \right\|_\infty \\
\leq \left\| \frac{G(\alpha, t)}{t} \left( \frac{X_\alpha}{\alpha} \right) \right\|_\infty \\
+ \left\| \frac{1}{t} \int_0^t \left( \frac{X_\alpha}{\alpha} - 1 \right) \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^{U_\alpha}_\tau (x) \, d\tau \right\|_\infty \\
+ \left\| \frac{1}{t} \int_0^t X \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^{U_\alpha}_\tau (x) \, d\tau \right\|_\infty \\
\leq 2 \cdot \left\| \frac{X_\alpha}{\alpha} - 1 \right\|_\infty \cdot \left\| \left( \frac{X_\alpha}{\alpha} \right) \right\|_\infty \\
+ \left\| X \left( \frac{X_\alpha}{\alpha} \right) \right\|_\infty \\
\leq 2(2) + C''_\alpha
\]

where \(C''_\alpha\) depends on the second derivative of \(\alpha\).

Since \(\left[ X, H(t) \right] \) is the multiplication operator given by

\[
\left( \frac{1}{t} \int_0^t (G(\alpha, \tau) U_\alpha + X) \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^{U_\alpha}_\tau \, d\tau \right) \cdot I,
\]

we obtain the following bound,

\[
\left\| \left[ X, \frac{H(t)}{t} \right] H^{-1} \right\|_{L^2(M, \text{vol}_\alpha)} f \leq \left\| \frac{1}{t} \int_0^t (G(\alpha, \tau) U_\alpha + X) \left( \frac{X_\alpha}{\alpha} \right) \circ \phi^{U_\alpha}_\tau \, d\tau \right\|_\infty \cdot \| f \|_{L^2(M, \text{vol}_\alpha)} \\
\leq (4 + C''_\alpha) \cdot \| f \|_{L^2(M, \text{vol}_\alpha)}.
\]

Thus, \(\left[ X, \frac{H(t)}{t} \right] H^{-1}\) extends to a bounded operator on \(\overline{\text{Ran}(U_\alpha)}\) with operator norm uniformly bounded in \(t\):

\[
\left\| \left[ X, \frac{H(t)}{t} \right] H^{-1} \right\|_{\text{op}} \leq 4 + C''_\alpha.
\]

(iii)

\[
\| [H(t), H] H^{-1} f \|_{L^2(M, \text{vol}_\alpha)} = \| [G(\alpha, t) U_\alpha, -U_\alpha] (-U^{-1}_\alpha f) \|_{L^2(M, \text{vol}_\alpha)} \\
= \left\| U_\alpha \left( \int_0^t \left( \frac{X_\alpha}{\alpha} - 1 \right) \circ \phi^{U_\alpha}_\tau \, d\tau \right) \cdot f \right\|_{L^2(M, \text{vol}_\alpha)} \\
= \left\| \int_0^t \frac{d}{d\tau} \left( \frac{X_\alpha}{\alpha} - 1 \right) \circ \phi^{U_\alpha}_\tau \, d\tau \cdot f \right\|_{L^2(M, \text{vol}_\alpha)} \\
\leq \| G(\alpha) \circ \phi^{U_\alpha}_\tau - G(\alpha) \|_\infty \cdot \| f \|_{L^2(M, \text{vol}_\alpha)} \\
\leq 2 \| G(\alpha) \|_\infty \cdot \| f \|_{L^2(M, \text{vol}_\alpha)} \\
\leq 2(2) \cdot \| f \|_{L^2(M, \text{vol}_\alpha)}
\]
The above holds on coboundaries of the form \( f = U_\alpha g \), so on \( \text{Ran}(U_\alpha) = L^2_0(M, \text{vol}_\alpha) \),
\[
\| [H(t), H]^{-1} \|_{\text{op}} \leq 4.
\]

Since conditions (i), (ii), and (iii) of Theorem 2.1. are satisfied on \( \text{Ran}(U_\alpha) \), the time-changed flow \( \{ \phi_{U_\alpha}^t \} \) has purely absolutely continuous spectrum on \( \text{Ran}(U_\alpha) \).

Since \( \{ \phi_{U_\alpha}^t \} \) is ergodic, \( \text{Ran}(U_\alpha) = L^2_0(M, \text{vol}_\alpha) \).

This concludes the proof of part a.

**b.** Now we assume that the flow \( \{ \phi_{U_\alpha}^t \} \), and hence \( \{ \phi_{U_\alpha}^t \} \), are uniquely ergodic.

\[
\left\| \left( I - \frac{H(t)}{t} H^{-1} \right) f \right\|_{L^2(M, \text{vol}_\alpha)}
\]
\[
= \left\| \left( 1 + \frac{G(\alpha, t)}{t} \right) f \right\|_{L^2(M, \text{vol}_\alpha)}
\]
\[
= \left\| \left( 1 + \frac{1}{t} \int_0^t \left( \frac{X_\alpha}{\alpha} - 1 \right) \circ \phi_{U_\alpha}^\tau (x) d\tau \right) f \right\|_{L^2(M, \text{vol}_\alpha)}
\]
\[
= \left\| \left( \frac{1}{t} \int_0^t \frac{X_\alpha}{\alpha} \circ \phi_{U_\alpha}^\tau (x) d\tau \right) f \right\|_{L^2(M, \text{vol}_\alpha)}
\]
\[
\leq \left\| \left( \frac{1}{t} \int_0^t \frac{X_\alpha}{\alpha} \circ \phi_{U_\alpha}^\tau (x) d\tau \right) \right\|_{\infty} \cdot \| f \|_{L^2(M, \text{vol}_\alpha)}.
\]

If \( \{ \phi_{U_\alpha}^t \} \) is uniquely ergodic, then the following converges uniformly,
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \frac{X_\alpha}{\alpha} \circ \phi_{U_\alpha}^\tau (x) d\tau = \int_M \frac{X_\alpha}{\alpha} d\text{vol}_\alpha = 0,
\]
and thus,
\[
\limsup_{t \to \infty} \left\| \left( I + \frac{H(t)}{t} H^{-1} \right) f \right\|_{\infty} \leq \limsup_{t \to \infty} \left\| \left( \frac{1}{t} \int_0^t \frac{X_\alpha}{\alpha} \circ \phi_{U_\alpha}^\tau (x) d\tau \right) \right\|_{\infty}
\]
\[
= \left\| \int_M \frac{X_\alpha}{\alpha} d\text{vol}_\alpha \right\|_{\infty} = 0.
\]

Hence,
\[
\limsup_{t \to \infty} \left\| \left( I + \frac{H(t)}{t} H^{-1} \right) \right\|_{\infty} < 1
\]
is satisfied on \( \text{Ran}(U_\alpha) = L^2_0(M, \text{vol}_\alpha) \) without imposing any further conditions on \( \frac{X_\alpha}{\alpha} \). The remainder of the proof is the same as in a except that
\[
\left\| \frac{X_\alpha}{\alpha} \right\|_{\infty} \leq C'_\alpha < \infty
\]
where \( C'_\alpha \) is finite but not necessarily equal to 1.

**Theorem 3.2 (Maximal Spectral Type).** The maximal spectral type of the uniquely ergodic flow \( \{ \phi_{U_\alpha}^t \} \) is Lebesgue on the subspace \( \text{Ran}(U_\alpha) \).
Proof. We follow the method in [5].

Lemma 3.3. [5] Suppose that the maximal spectral type of \( \{ \phi_t^{U_\alpha} \} \) is not Lebesgue. Then there exists a smooth non-zero function \( \omega \in L^2(\mathbb{R}, dt) \) such that for all functions \( g \in C^\infty(M) \) the following holds:

\[
\int_{\mathbb{R}} \omega(t) \int_0^\sigma e^{sX} e^{tU_\alpha} U_\alpha g \, ds \, dt = 0
\]

Proof. Since the maximal spectral type is not Lebesgue, then there exists a compact set \( A \subset \mathbb{R} \) such that \( A \) has positive Lebesgue measure but measure 0 with respect to the maximal spectral type. So we let \( \omega \in L^2(\mathbb{R}) \) be the complex conjugate of the Fourier transform of the characteristic function \( \chi_A \) of the set \( A \subset \mathbb{R} \). For \( f, h \in \text{Ran}(U_\alpha) \), let \( \mu_{f,h} \) denote the joint spectral measure (which we know is absolutely continuous with respect to Lebesgue since \( f, h \in \text{Ran}(U_\alpha) \)). Thus,

\[
\int_{\mathbb{R}} \omega(t) \langle e^{tU_\alpha} f, h \rangle_{L^2(M, vol)} \, dt = \int_{\mathbb{R}} \chi_A(\xi) d\mu_{f,h}(\xi) = 0.
\]

In particular, when \( f = U_\alpha g \) we have

\[
\int_{\mathbb{R}} \omega(t) \int_{\sigma_0} e^{sX} e^{tU_\alpha} U_\alpha g \, ds \, dt = \langle \int_{\mathbb{R}} \omega(t) \int_0^\sigma e^{sX} e^{tU_\alpha} U_\alpha g \, ds \, dt, h \rangle_{L^2(M, vol)} = 0.
\]

Recall that satisfying conditions \((i)\) and \((ii)\) and \((iii)\) in Theorem 2.1 results in the bound

\[
\sup_{s \in [0, \sigma]} \left\| \int_0^\sigma e^{sX} e^{tU_\alpha} U_\alpha g \, ds \right\|_{L^2(\mathbb{R}, dt)} \leq C_\sigma(\alpha) \max \left\{ \|g\|_{L^2(M)}, \|Xg\|_{L^2(M)}, \|U_\alpha g\|_{L^2(M)} \right\}
\]

where \( \beta = 1 \) and \( C_\sigma(\alpha) \) is a constant that depends on the time-change function \( \alpha \) and parameter \( \sigma > 0 \).

Because

\[
\int_0^\sigma e^{sX} e^{tU_\alpha} U_\alpha g \, ds
\]

is bounded on \( M \), it follows that

\[
\int_{\mathbb{R}} \omega(t) \int_0^\sigma e^{sX} e^{tU_\alpha} U_\alpha g \, ds \, dt
\]

vanishes.

Lemma 3.4. [5] Let \( \omega \in L^2(\mathbb{R}, dt) \). If for some \( x \in M \) and for all \( g \in C^\infty(M) \),

\[
\int_{\mathbb{R}} \omega(t) \int_0^\sigma e^{sX} e^{tU_\alpha} U_\alpha g(x) \, ds \, dt = 0,
\]

then \( \omega \) vanishes identically.

Proof. Fix \( x \in M \) and \( \sigma > 0 \). Since \( U \) is contained in a subalgebra isomorphic to \( \mathfrak{sl}_2 \), there exists an element \( V \) such that \( [U, V] = 2X \) and \( [X, V] = -V \). For any \( T > 0, \rho > 0, \) and \( \frac{1}{2} \gamma > 0 \), let \( E_{\rho, \sigma}^T \) be the flow-box for the the flow \( \{ \phi_t^{U_\alpha} \} \) defined as follows:

\[
E_{\rho, \sigma}^T = (\phi_t^{U_\alpha} \circ \phi_s^X \circ \phi_r^V)(x), \text{ for all } (r, s, t) \in (-\gamma, \gamma) \times (-\rho, \rho) \times (-\sigma, \sigma).
\]
For any $\chi \in C_0^\infty(-1,1)$ and any $\psi \in C_0^\infty(-T,T)$, let
\[ \tilde{g}(r,s,t) := \chi(r)\chi(s)\psi(t). \]

Let $g \in C^\infty(M)$ such that $g = 0$ on $M \setminus \text{Im}(E_T^{T,\rho,\sigma})$ and
\[ g \circ E_T^{T,\rho,\sigma} = \begin{cases} 0 & \text{on } M \setminus \text{Im}(E_T^{T,\rho,\sigma}) \\ \tilde{g}(r,s,t) & \text{on } \text{Im}(E_T^{T,\rho,\sigma}) \end{cases}. \]

Let $T_{\rho,\sigma} > 0$ be defined as:
\[ T_{\rho,\sigma} := \min \left\{ |t| > T : \cup_{s \in [-\sigma,\sigma]} (\phi^{U,\alpha}_r \circ \phi^{X}_s)(x) \cap \text{Im}(E_T^{T,\rho,\sigma}) \neq \emptyset \right\}. \]

From unique ergodicity,
\[ \lim_{\rho \to 0^+} T_{\rho,\sigma} = +\infty. \]

The composition of the flow box with $U_\alpha g$ and $X g$ follow from the commutation relations:
\[ (U_\alpha g) \circ E_T^{T,\rho,\sigma} := \chi \left( \frac{r}{\rho} \right) \chi \left( \frac{s}{\sigma} \right) \frac{d\psi(t)}{dt}(t) \]
and
\[ (X g) \circ E_T^{T,\rho,\sigma} = \frac{1}{\sigma} \chi \left( \frac{r}{\rho} \right) \frac{d\psi(t)}{ds}(s) \psi(t) \]
\[- \left( \int_0^t \frac{X_\alpha}{\alpha} - 1 \circ \phi^{U,\alpha}_r \circ \phi^{X}_s \circ \phi^{U,\alpha}_r(x) d\tau \right) \chi \left( \frac{r}{\rho} \right) \chi \left( \frac{s}{\sigma} \right) \frac{d\psi(t)}{dt}(t). \]

From the assumptions of Lemma 3.3 and by integrating 1, we have
\[ \chi(0) \left( \int_0^\sigma \chi \left( \frac{s}{\sigma} \right) ds \right) \left( \int_{-T}^T \omega(t) \frac{d\psi(t)}{dt} dt \right) \]
\[ + \int_{\mathbb{R} \setminus [-T_{\rho,\sigma}, T_{\rho,\sigma}]} \omega(t) \int_0^\sigma e^{sX} e^{tU_\alpha} (U_\alpha g) ds dt = 0. \]

The bound $C_\sigma(\alpha)$ of
\[ \int_0^\sigma e^{sX} e^{tU_\alpha} U_\alpha g ds \]
derived for the spectral results, combined with 1 and 2, give us the following $L^2$ bound,
\[ \left\| \int_0^\sigma e^{sX} e^{tU_\alpha} U_\alpha g ds \right\|_{L^2(\mathbb{R}, dt)} \leq \frac{C_\sigma(\alpha)}{t} \max \{|g|, |X g|, |U_\alpha g|\} \]
\[ \leq \frac{C_\sigma(\alpha)}{t} \max \{1, T\} \times \max \left\{ \|\chi\|_{L^\infty(\mathbb{R})}, \|\chi'\|_{L^\infty(\mathbb{R})}, \|\psi\|_{L^\infty(\mathbb{R})}, \|\psi'\|_{L^\infty(\mathbb{R})} \right\}. \]

Since the above bound is uniform with respect to $\rho$, we can conclude that the following limit holds,
\[ \lim_{\rho \to 0^+} \int_{\mathbb{R} \setminus [-T_{\rho,\sigma}, T_{\rho,\sigma}]} \omega(t) \int_0^\sigma e^{sX} e^{tU_\alpha} (U_\alpha g) ds dt = 0. \]

Combining equation 2 with the limit result in 3 implies that
\[ \int_{\mathbb{R}} \omega(t) \frac{d\psi(t)}{dt} dt = 0 \]
and thus, $\omega \equiv 0$. \qed
3.2. Time changes of the horocycle flow - compact and finite area. On $M = \Gamma \setminus PSL(2, \mathbb{R})$, where $M$ is either compact or of finite area, we consider the basis
\[
\begin{bmatrix} U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, & V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, & X = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \end{bmatrix}
\]
of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$, where $U$ and $V$ are the generators of the positive and negative horocycle flows, $\{h_t^{U_h}\}$ and $\{h_t^{V_h}\}$ respectively, and $X$ is the generator of the geodesic flow, $\{\phi^X_t\}$. From [1], we know that $iU, iV, iX$ are essentially self-adjoint on $C^\infty(M)$, and thus, $U, V, X$ are essentially skew-adjoint on $C^\infty(M)$. It follows that time-changes of the horocycle flow are special cases of Theorem 3.1 (when $M$ is of finite volume, $\{h_t^{U_h}\}$ is ergodic, and when $M$ is compact, $\{h_t^{U_h}\}$ is uniquely ergodic). The following Corollary was already proved in [5], [18], [19] under slightly weaker regularity assumptions; in this paper we have not attempted to optimize the regularity.

**Corollary 3.1.**

a. Any smooth time-change $\{h_t^{U_h}\}$ of the horocycle flow on $M$ (finite volume) has absolutely continuous spectrum on $L^2(M, \text{vol})$ if $\| \frac{\partial \phi^X_t}{\partial \theta}\|_\infty < 1$.

b. Any smooth time-change $\{h_t^{U_h}\}$ of the horocycle flow on $M$ (compact) has Lebesgue spectrum on $L^2(M, \text{vol})$.

3.3. Twisted horocycle flows. Much work has been done on the spectral analysis of skew products on tori, for example, [3], [9], [10], [11]. We would like to consider a skew product for which the base dynamics are ergodic (in fact uniquely ergodic), but not an action on $S^1$. For such an example, we will examine the conditions under which the spectral properties persist or do not persist after we combine the horocycle time-change with a circle rotation. Our new space is $\tilde{M} = (\Gamma \setminus PSL(2, \mathbb{R})) \times S^1$ for $\Gamma$ a cocompact lattice. We define the following operators:
\[
\tilde{X} = (X, 0) \quad \text{where} \quad X \text{ is the generator of the geodesic flow.}
\]
\[
\tilde{V} = (V, 0) \quad \text{where} \quad V \text{ is the generator of the negative horocycle flow.}
\]
\[
\frac{d}{d\theta} = (0, \frac{d}{d\theta}) \quad \text{where} \quad \frac{d}{d\theta} \text{ is a rotation on } S^1.
\]

$W = (U, 0) + (0, \alpha \frac{d}{d\theta})$ where $U$ is the generator of the positive horocycle flow and $\alpha = \alpha(x), x \in \Gamma \setminus PSL(2, \mathbb{R})$, is the time change function as in 3.1.

**Proposition 1.** The flow $\{\phi^W_t\}$ is uniquely ergodic.

**Proof.** Consider the time-change $\{\phi^W_t\} = \frac{1}{2}W = \tilde{U}_\alpha \times \frac{d}{d\theta}$. Since $\{h_t^{U_h}\}$ is mixing [14], then it is weakly mixing, and thus $\{\phi^W_t\}$ is ergodic [4]. This implies the ergodicity of $\{\phi^W_t\}$. Since $\{\phi^W_t\}$ is ergodic and $\{h_t^U\}$ is uniquely ergodic [7], then from [6] (applied to flows), $\{\phi^W_t\}$ is uniquely ergodic.

We are interested in the spectrum of the flow $\{\phi^W_t\}$, so we compute the commutator with $\tilde{X}$,
\[
e^{-tW}[\tilde{X}, e^{tW}] = \left[tW + \left(\int_0^t (\tilde{X} - \alpha) \circ \phi^W_t(x) \, d\tau\right) \frac{d}{d\theta} = H(t)\right]
\]

For $u \in C^\infty(\tilde{M})$,
\[
\lim_{t \to \infty} \frac{H(t)}{t} u = \left(W - \frac{d}{d\theta}\right) u = Hu
\]
Since,

\[
\| \langle e^{iW} f, f \rangle \|_{L^2(\mathbb{R}, dt)} = \left\| \frac{1}{\sigma} \int_0^\sigma \langle e^{i\tilde{X}_s} e^{iW} f, e^{i\tilde{X}_f} f \rangle \, ds \right\|_{L^2(\mathbb{R}, dt)},
\]

the preliminary assumptions are satisfied with \( B_1 = B_2 = I \).

However, if we proceed with verifying the conditions of Theorem 2.1 for functions in the range of \( H \), we are unable to extend pointwise bounds in \( L^2(\hat{M}) \) to uniform bounds in the operator norm. Instead we modify our operators by introducing an operator \( P \), defined in such a way that it not only acts as a projection operator but also preserves regularity.

Let \( \chi \in C_0^\infty(\mathbb{R} \setminus \{0\}) \) such that the support of \( \chi \) is a compact subset of the spectrum of \( H \) away from 0. For \( f, g \in L^2(\hat{M}) \),

\[
\langle Pf, g \rangle_{L^2(\hat{M})} = \int_{\mathbb{R}} \hat{\chi}(t) \mu_f g(t) \, dt
\]

since \( H \) is a vector field, and thus,

\[
Pf = \int_{\mathbb{R}} \hat{\chi}(t) e^{itH} f \, dt.
\]

The decay of \( e^{itH} f = f \circ \phi^H_t \) is at most polynomial in \( t \), however, since \( \chi \in C_0^\infty(\mathbb{R} \setminus \{0\}) \), \( \hat{\chi} \in \mathcal{S}(\mathbb{R}) \), and thus, must decay faster than any power of \( \frac{1}{t} \). In this way, we guarantee that

\[
P : C^\infty(\hat{M}) \to C^\infty(\hat{M}),
\]

and we take

\[
D = P(C^\infty(\hat{M})).
\]

Now we introduce our modified operators.

Let

\[
\hat{X}_p = P \hat{X} P.
\]

Since \( P \) commutes with \( e^{itH} \), \( P \) commutes with \( H \). Thus, \( P \) commutes with \( W \) and \( e^{itW} \). Therefore,

\[
e^{-itW} \left[ \hat{X}_p, e^{itW} \right] = P e^{-itW} \left[ \hat{X}, e^{itW} \right] P = PH(t)P = H_P(t).
\]

For \( u \in C^\infty(\hat{M}) \),

\[
\lim_{t \to \infty} \frac{H_P(t)}{t} u = PHPu = HP^2u = H_Pu.
\]

Note that now \( H_P \) is a bounded, invertible operator. Let

\[
C_{H_P} = \| H_P \|_{op}
\]

\[
C_{H_P}^{-1} = \| H_P^{-1} \|_{op}
\]

\[
C_{P} = \| P \|_{op}
\]

\[
C_{P}^\prime = \| H_P \|_{op}
\]

\[
C_{\alpha} = \| \hat{X}\alpha - \alpha \|_{\infty}
\]

**Theorem 3.5.** The flow \( \{ \phi^W_t \} \) has absolutely continuous spectrum on \( \operatorname{Ran}(H) \).
Proof. We will verify the conditions of Theorem 2.1 on each subspace

\[ \mathcal{E}_n = \left\{ \frac{\hat{a}}{\hat{d}}u = \hat{u} : u \in L^2(\hat{M}) \right\}. \]

(i)

\[ \frac{H_P(t)}{t} = PW P + Pin \left( \frac{1}{t} \int_0^t (\hat{X} - \alpha) \circ \phi_{x}^W(x) \, d\tau \right) P \]

\[ = PHP + PinP + Pin \left( \frac{1}{t} \int_0^t (\hat{X} - \alpha) \circ \phi_{x}^W(x) \, d\tau \right) P \]

\[ = H_P + Pin \left( \frac{L(t)}{t} + 1 \right) P \]

for

\[ L(t) = \int_0^t (\hat{X} - \alpha) \circ \phi_{x}^W(x) \, d\tau. \]

Let \( f = H_P g, \)

\[ \left\| \frac{H_P(t)}{t} H_P^{-1} f \right\|_{L^2(\hat{M})} \]

\[ = \left\| H_P(t) g \right\|_{L^2(\hat{M})} \]

\[ = \left\| H_P g + inP \left( \frac{L(t)}{t} + 1 \right) P g \right\|_{L^2(\hat{M})} \]

\[ \leq \left\| H_P g \right\|_{L^2(\hat{M})} + \left\| inP \left( \frac{L(t)}{t} + 1 \right) P g \right\|_{L^2(\hat{M})} \]

\[ \leq C_{H_P} \| g \|_{L^2(\hat{M})} + nC_{P} \| \hat{X} - \alpha \|_{\infty} \| g \|_{L^2(\hat{M})} \]

\[ + nC_{P}^2 \| g \|_{L^2(\hat{M})} \]

\[ = (C_{H_P} + nC_{P}^2(C_{\alpha} + 1)) \| g \|_{L^2(\hat{M})}. \]

So,

\[ \left\| \frac{H_P(t)}{t} H_P^{-1} \right\|_{op} \leq C_{H_P} + nC_{P}^2(C_{\alpha} + 1). \]

Also,

\[ \left\| \left( I - \frac{H_P(t)}{t} H_P^{-1} \right) f \right\|_{L^2(\hat{M})} \]

\[ = \left\| \left( I - I - inP \left( \frac{L(t)}{t} + 1 \right) PH_P^{-1} \right) f \right\|_{L^2(\hat{M})} \]

\[ = \left\| inP \left( \frac{L(t)}{t} + 1 \right) P g \right\|_{L^2(\hat{M})} \]

\[ \leq nC_{P}^2 \left\| \frac{L(t)}{t} + 1 \right\|_{\infty} \| g \|_{L^2(\hat{M})}. \]

Since \( \{ \phi_{x}^W \} \) is uniquely ergodic, the following converges uniformly,

\[ \lim_{t \to \infty} \frac{L(t)}{t} + 1 = \lim_{t \to \infty} \frac{1}{t} \int_0^t (\hat{X} - \alpha) \circ \phi_{x}^W(x) \, d\tau + 1 = 0. \]

So,

\[ \limsup_{t \to \infty} \left\| I - \frac{H_P(t)}{t} H_P^{-1} \right\|_{op} \leq \limsup_{t \to \infty} n(C_{P})^2 \left\| \left( \frac{L(t)}{t} + 1 \right) \right\|_{\infty} = 0, \]
and hence,

$$\limsup_{t \to \infty} \left\| I - \frac{H_p(t)}{t} H_p^{-1} \right\|_{op} < 1.$$  

(ii)

$$\left[ \hat{X}_p, \frac{H_p(t)}{t} (H_p)^{-1} \right] = P \left\{ \hat{X}, P \right\} \left[ \frac{PH(t)}{t} PH_p^{-1} \right]_{a}$$

$$+ P^2 \left[ \hat{X}, P \right] \frac{H(t)}{t} PH_p^{-1} \right\}_{b}$$

$$+ P^3 \left[ \hat{X}, \frac{H(t)}{t} \right] PH_p^{-1} \right\}_{c}$$

$$+ P^3 \left[ \hat{X}, P \right] \left[ \frac{PH(t)}{t} \right] PH_p^{-1} \right\} \left[ \hat{X}, \frac{H(t)}{t} \right] PH_p^{-1} \right\}_{d}$$

$$+ P^3 \left[ \hat{X}, P \right] \left[ \frac{PH(t)}{t} \right] PH_p^{-1} \right\} \left[ \hat{X}, \frac{H(t)}{t} \right] PH_p^{-1} \right\} \left[ \hat{X}, \frac{H(t)}{t} \right] PH_p^{-1} \right\}_{e}.$$  

Before we bound terms a-e, we show bounds for the terms $\left[ \hat{X}, P \right]$ and $\left[ \hat{X}, \frac{H(t)}{t} \right] P$.

$$\left[ \hat{X}, P \right] = \int_\mathbb{R} \hat{\chi}(t) \left[ \hat{X}, e^{tH} \right] f dt$$

$$= \int_\mathbb{R} \hat{\chi}(t)e^{tW} e^{-tW} \left[ \hat{X}, e^{tH+\text{int} \text{nin} \text{in} t} \right] \left[ \hat{X}, e^{tW} \right] f dt$$

$$= \int_\mathbb{R} \hat{\chi}(t)e^{tW} e^{-tW} \left[ \hat{X}, e^{tW} \right] e^{-tW} f dt$$

$$= \int_\mathbb{R} \hat{\chi}(t)e^{tW} H(t) e^{-tW} f dt$$

$$= \int_\mathbb{R} \hat{\chi}(t)e^{tW} tW e^{-tW} f dt + \int_\mathbb{R} \hat{\chi}(t)e^{tW} (\text{inL}(t)) \left[ \hat{X}, e^{-tW} \right] f dt$$

$$= \int_\mathbb{R} \hat{\chi}(t)e^{-tW} \frac{d}{dt} \left( e^{tW} f \right) dt + \int_\mathbb{R} \hat{\chi}(t)e^{tW} \left( \text{inL}(t) \right) e^{-tW} f dt.$$  

The first term we integrate by parts:

$$\int_\mathbb{R} \hat{\chi}(t)e^{-tW} \frac{d}{dt} \left( e^{tW} f \right) dt = \hat{\chi}(t)e^{-tW} \frac{d}{dt} \left( e^{tW} f \right) \bigg|_{-\infty}^{\infty}$$

$$+ \int_\mathbb{R} \left( \hat{\chi}'(t)e^{-tW} + \hat{\chi}(t)e^{-tW} - \text{in} \hat{\chi}(t)e^{-tW} \right) e^{tW} f dt$$

$$= \int_\mathbb{R} \left( \hat{\chi}'(t)e^{-tW} + \hat{\chi}(t)e^{-tW} - \text{in} \hat{\chi}(t)e^{-tW} \right) e^{tW} f dt.$$
So,

\[
\left\| \int_{\mathbb{R}} \left( \hat{\chi}'(t)e^{-itn t} + \hat{\chi}(t)e^{-itn t} - i n\hat{\chi}(t)e^{-itn t} \right) e^{itW} f \, dt \right\|_{L^2(\tilde{M})} \leq \left( \int_{\mathbb{R}} \left| \hat{\chi}'(t) \right| dt + \int_{\mathbb{R}} \left| \hat{\chi}(t) \right| dt + \int_{\mathbb{R}} \left| n\hat{\chi}(t) \right| dt \right) \| f \|_{L^2(\tilde{M})} \leq C_1 \| f \|_{L^2(\tilde{M})}.
\]

The boundedness of the second term follows immediately,

\[
\left\| \int_{\mathbb{R}} \hat{\chi}(t)e^{itL(t)}e^{-itn t} f \, dt \right\|_{L^2(\tilde{M})} \leq \int_{\mathbb{R}} \left| \hat{\chi}(t)(inL(t)) \right| dt \| f \|_{L^2(\tilde{M})} \leq C_2 \| f \|_{L^2(\tilde{M})}.
\]

Thus,

\[
\left\| \left[ \hat{X}, P \right] f \right\|_{L^2(\tilde{M})} \leq (C_1 + C_2) \| f \|_{L^2(\tilde{M})} = C \| f \|_{L^2(\tilde{M})},
\]

and hence,

\[
\left\| \left[ \hat{X}, P \right] \right\|_{\text{op}} \leq C.
\]

Also,

\[
\left[ \hat{X}, \frac{H(t)}{t} \right] P = \left[ \hat{X}, W \right] P + \left[ \hat{X}, \frac{1}{t} \int_{0}^{t} (\hat{X} - \alpha) \circ \phi_\tau^W(x) \, d\tau \right] P
\]

\[
= (\hat{U} + \hat{X} \alpha \in) P + \left[ \hat{X}, \frac{1}{t} \int_{0}^{t} (\hat{X} - \alpha) \circ \phi_\tau^W(x) \, d\tau \in \right] P
\]

\[
= (\hat{U} + (\alpha - 1)i \in - (\alpha - 1)\in + \hat{X} \alpha \in) P
\]

\[
+ \left[ \hat{X}, \frac{1}{t} \int_{0}^{t} (\hat{X} - \alpha) \circ \phi_\tau^W(x) \, d\tau \in \right] P
\]

\[
= HP + (\hat{X} \alpha - \alpha + 1) \in P + \left[ \hat{X}, \frac{1}{t} \int_{0}^{t} (\hat{X} - \alpha) \circ \phi_\tau^W(x) \, d\tau \in \right] P.
\]

Now we bound the following term,

\[
\left[ \hat{X}, \frac{1}{t} \int_{0}^{t} (\hat{X} - \alpha) \circ \phi_\tau^W(x) \, d\tau \right] P
\]

\[
= \hat{X} \left( \frac{1}{t} \int_{0}^{t} (\hat{X} - \alpha) \circ \phi_\tau^W(x) \, d\tau \right) \in
\]

\[
- \left( \frac{1}{t} \int_{0}^{t} (\hat{X} - \alpha) \phi_\tau^W(x) \, d\tau \right) \hat{X} \in
\]

\[
= \frac{1}{t} \int_{0}^{t} (D\phi_\tau^W(\hat{X}) \circ \phi^-_\tau^W(x))(\hat{X} - \alpha) \circ \phi_\tau^W(x) \, d\tau \in.
\]

Since \( \hat{X} \alpha - \alpha \) is a function on \( M \),

\[
\frac{1}{t} \int_{0}^{t} (D\phi_\tau^W(\hat{X}) \circ \phi^-_\tau^W(x))(\hat{X} - \alpha) \circ \phi_\tau^W(x) \, d\tau \in
\]

\[
= \frac{1}{t} \int_{0}^{t} (D\phi_\tau^U(X) \circ \phi^-_\tau^U(x))(X - \alpha) \circ \phi_\tau^U(x) \, d\tau \in
\]
\[ = \frac{1}{t} \int_0^t (X + \tau U)(X\alpha - \alpha) \circ \phi^U_t (x) \, d\tau \text{ in } \]

We consider the \( L^\infty(M) \) norm and let

\[ \|X\alpha\|_\infty = C_{\alpha,M} < \infty \]
\[ \|X\alpha - \alpha\|_\infty = C'_{\alpha,M} < \infty \]

and

\[ \|X(X\alpha)\|_\infty = C''_{\alpha,M} < \infty. \]

Hence,

\[
\left\| \frac{1}{t} \int_0^t (X + \tau U)(X\alpha - \alpha) \circ \phi^U_t (x) \, d\tau \right\|_\infty \leq \left\| \frac{1}{t} \int_0^t X(X\alpha - \alpha) \circ \phi^U_t (x) \, d\tau \right\|_\infty + \left\| \frac{1}{t} \int_0^t \tau U(X\alpha - \alpha) \circ \phi^U_t (x) \, d\tau \right\|_\infty \\
\leq n \|X(X\alpha - \alpha)\|_\infty + n \|(X\alpha - \alpha) \circ \phi^U_t - (X\alpha - \alpha)\|_\infty \\
\leq n \|X(X\alpha - \alpha)\|_\infty + 2n \|X\alpha - \alpha\|_\infty \\
\leq n \|X(X\alpha)\|_\infty + n \|X\|_\infty + 2n \|X\alpha - \alpha\|_\infty \\
\leq nC''_{\alpha,M} + nC_{\alpha,M} + 2nC'_{\alpha,M}. \]

Thus, \( \left[ \hat{X}, \frac{H(t)}{t} \right] P \) extends to a bounded operator on \( \overline{\text{Ran}(H_p)} \) with operator norm uniformly bounded in \( t \):

\[
\left\| \left[ \hat{X}, \frac{H(t)}{t} \right] P \right\|_{\text{op}} \leq \left( C_H^p + n(C_\alpha + 1)C_P^1 \right) + n \left( C''_{\alpha,M} + C_{\alpha,M} + 2C'_{\alpha,M} \right) C_P^1. \]

\( a: \)

\[
\left\| P \left[ \hat{X}, P \right] \frac{H(t)}{t} PH_P^{-1} \right\|_{\text{op}} \leq C_P^1 \cdot C \cdot \left\| \frac{H(t)}{t} PH_P^{-1} \right\|_{\text{op}} \leq C_P^1 \cdot C \cdot \left( C_H + nC_\alpha \right) \cdot \left( C_H + nC_\alpha + 1 \right). \]

\( b: \)

\[
\left\| P^2 \left[ \hat{X}, P \right] \frac{H(t)}{t} PH_P^{-1} \right\|_{\text{op}} \leq P^2 \left[ \hat{X}, P \right] \left( H + \left( \frac{L(t)}{t} + 1 \right) \text{in} \right) PH_P^{-1} \right\|_{\text{op}} \leq C_P^2 \cdot C \cdot \left\| HP \right\|_{\text{op}} + n \left( \left\| \hat{X} - \alpha \right\|_\infty + 1 \right) C_P^1 \\
\leq C_P^2 \cdot C \cdot \left( C_H + n(C_\alpha + 1)C_P^1 \right) C_{H_P}^{-1} \]

\( c: \)

\[
\left\| P^3 \left[ \hat{X}, \frac{H(t)}{t} \right] PH_P^{-1} \right\|_{\text{op}} \leq C_P^3 \left( \left( C_H^p + n\left( C_\alpha + 1 \right)C_P^1 \right) + n(C''_{\alpha,M} + C_{\alpha,M} + 2C'_{\alpha,M})C_P^1 \right) C_{H_P}^{-1} \]

\( d: \)

\[
\left\| P^3 \frac{H(t)}{t} \left[ \hat{X}, P \right] H_P^{-1} \right\|_{\text{op}} \leq C_P^2 \cdot \left( C_H^p + nC_P^1 \right) \cdot C \cdot C_{H_P}^{-1}. \]
integrals involving the spectral projector as 
\[ \chi \in \text{Dom}(H) \]
and since \( f \in \text{Dom}(H) \), we can express the following in terms of integrals involving the spectral projector as

\[
Hf = \int_{\mathbb{R}} x \, dE(x) f,
\]

and since \( f \in \text{Dom}(H) \),
\[
\int_{\mathbb{R}} x^2 \, dE(x) f < +\infty.
\]

Let \( \chi \) be such that
\[
\chi(x) = 1 \quad \text{for } x \in (-K, -\epsilon) \cup (\epsilon, K) = I_{\epsilon,K}
\]
and \( \text{supp}(\chi) \) vanishes outside of \( I_{\epsilon,K} \). Since on \( I_{\epsilon,K} \),
\[
H_P f = Hf,
\]
we consider
\[
H_P f - Hf
\]
on \( \mathbb{R} \setminus I_{\epsilon,K} \), i.e.,
\[
\int_{\mathbb{R} \setminus I_{\epsilon,K}} x(\chi(x) - 1) \, dE(x) f.
\]
For $|x| \leq \epsilon$,
\[
\lim_{\epsilon \to 0} \left\| \int_{|x| \leq \epsilon} x (\chi(x) - 1) \, dE(x) f \right\|^2_{L^2(\mathbb{R})} = \lim_{\epsilon \to 0} \int_{|x| \leq \epsilon} |x(\chi(x) - 1)|^2 \, dE(x) f \\
\leq \lim_{\epsilon \to 0} 4\epsilon^2 \int_{|x| \leq \epsilon} dE(x) f \\
\leq \lim_{\epsilon \to 0} 4\epsilon^2 \|f\|_{L^2(\hat{M})} = 0.
\]

For $|x| \geq K$,
\[
\lim_{K \to \infty} \left\| \int_{|x| \geq K} x (\chi(x) - 1) \, dE(x) f \right\|^2_{L^2(\mathbb{R})} = \lim_{K \to \infty} \int_{|x| \geq K} |x(\chi(x) - 1)|^2 \, dE(x) f \\
\leq \lim_{K \to \infty} \int_{|x| \geq K} 4x^2 \, dE(x) f = 0
\]
since
\[
\int_{\mathbb{R}} x^2 \, dE(x) f < +\infty.
\]
Thus,
\[
\inf_{\chi \in C_0^\infty(\mathbb{R} \setminus \{0\})} \|H_P f - H f\|_{L^2(\hat{M})} = 0.
\]
So, for any $H f$, there exists a sequence $\{H_{P_n}\}$ such that
\[
H_{P_n} \to H f,
\]
and thus,
\[
\text{Ran}(H) = \overline{\bigcup_{P} \text{Ran}(H_P)}.
\]
Consequently, for every $f \in \overline{\text{Ran}(H)}$, $\mu_f$ is absolutely continuous.

The characteristics of $\overline{\text{Ran}(H)}$ are linked to the properties of the cocycle
\[
a(x,t) = \int_0^t (\alpha - 1) \circ h_s^U(x) \, ds
\]
since
\[
\phi_t^H(x, \theta) = \left(h_t^U(x), \theta + \int_0^t (\alpha - 1) \circ h_s^U(x) \, ds \right).
\]
For $\pi$ a projection from $\mathbb{R}$ to $S^1$, let
\[
\hat{a} = \pi(a).
\]
Let
\[
k_0 = \min \left\{ k : k \hat{a}(x, t) = g \circ h_t^U(x) - g(x) \right\},
\]
for $k \in \mathbb{Z}^+$ and $g : M \to S^1$.

From [3] we have the following cases.

If $k_0 = \infty$ then $\{\phi_t^H\}$ has purely absolutely continuous spectrum on $L^2_0(\hat{M})$; this is the case when $\{\phi_t^H\}$ is ergodic, and thus,
\[
\overline{\text{Ran}(H)} = L^2_0(\hat{M}).
\]
When, $k_0 < \infty$, we have a nontrivial pure point component of the spectrum. To show this, we consider the subspaces given by

$$E_{nk_0} = \{ f(x)e^{ink_0\theta} \} \subset L^2(M)$$

for $f \in L^2(M)$ and $n \in \mathbb{Z}$.

The operator

$$H_t : f(x) \to f \circ h_t^U(x) e^{ink_0\hat{\alpha}(x,t)}$$

is unitarily equivalent to the restriction of the unitary group $\{ \phi_t^H \}$ to $E_{nk_0}$.

Let

$$S_t : f(x) \to f \circ h_t^U(x)$$

and

$$V_t : f(x) \to f(x) e^{-ing(x)}.$$

We compute $H_t \circ V_t$:

$$(H_t \circ V_t)(f)(x) = f \circ h_t^U(x) e^{-ingh_t^U(x)} e^{ink_0\hat{\alpha}(x,t)}$$

$$= f \circ h_t^U(x) e^{-ingh_t^U(x)} e^{ingh_t^U(x)-ing(x)}$$

$$= f \circ h_t^U(x) e^{-ing(x)}$$

$$= (V_t \circ S_t)(f)(x).$$

Hence, on $E_{nk_0}$, $H_t$ is unitarily equivalent to $S_t$. Since $e^{ink_0\theta}$ is an invariant function for $S_t$, $H_t$ has an eigenfunction in $E_{nk_0}$ for every $n$. Thus, the spectrum on $E_{nk_0}$ has an absolutely continuous component as well as an infinite dimensional pure point component. This leads to the following Corollary.

**Corollary 3.2.** The spectrum of $\{ \phi_t^W \}$ on $C^\perp$, the orthocomplement of the constants, has a pure point component (possibly trivial) and an absolutely continuous component, but no singularly continuous component.

4. **Applications to maps.** The author in [19] uses the Mourre Estimate [2], [16] to prove the following spectral results. Here we rederive similar results by showing that the conditions of Theorem 2.1 are satisfied; our results are slightly weaker as we prove results for functions of class $C^2$ while the author in [19] considers functions of class $C^1$ with an added Dini Condition. We use the notation and description from [19].

4.1. **Skew products over translations.** Let $X$ be a compact metric abelian Lie group with normalized Haar measure $\mu$. Let $\{ F_t \}$ be a uniquely ergodic [6] translation flow (we assume that $F_1$ is ergodic),

$$F_t = y_t x$$

with vector field $Y$.

The associated operators $\{ V_t \}$ are given by

$$V_t \psi = \psi \circ F_t$$

with generator $P = -i\mathcal{L}_Y$.

Let $G$ be a compact metric abelian group. Let $\phi : X \to G$ such that $\phi$ can be written as $\phi = \xi \eta$ where $\xi$ is a group homomorphism and $\eta$ satisfies

$$\sup_{t>0} \left\| \frac{\mathcal{L}_Y(\chi \circ \eta) \circ F_t - \mathcal{L}_Y(\chi \circ \eta)}{t} \right\|_\infty < \infty$$

and

$$\chi \circ \eta = e^{i\tilde{\eta}_\chi}$$

for $\tilde{\eta}_\chi \in Dom(P)$ a real-valued function determined by $\chi$ and $\eta$. 
The skew product, $T : X \times G \rightarrow X \times G$, is defined by 
$$T(x, z) = (y_1 x, \phi(x) z)$$
with corresponding unitary operator
$$W \Psi = \Psi \circ T.$$
Let $\hat{G}$ be the character group of $G$. The decomposition $L^2(X \times G) = \bigoplus_{\chi \in \hat{G}} L_\chi$, $L_\chi = \{ \varphi \otimes \chi : \varphi \in L^2(X) \}$ and the restriction of $W$ to the subspaces $L_\chi$ allow us to study the spectrum of convenient, unitarily equivalent operators to $W|_{L_\chi}$, namely,
$$U_\chi \psi = (\chi \circ \psi)V_1 \psi$$
for $\chi \circ \xi \neq 1$. (Here $U_\chi$ takes the place of $e^{iU}$ as given in the conditions.)

We will choose to take the commutator with $P$; it follows from [1] that $P$ is essentially self-adjoint on $D = C^\infty(X)$.

$$[P, U_\chi] = [P, (\chi \circ \phi)V_1] = [P, (\chi \circ \phi)I] V_1 = -i (\xi_0 + \chi \circ \phi) V_1$$

Thus,

$$U_\chi^{-n}[P, U_\chi^n] = \sum_{k=1}^{n} U_\chi^{-k} G U_\chi^{k}$$

Note that

$$\frac{-i \mathcal{L}_Y(\chi \circ \eta)}{(\chi \circ \eta)} = \frac{i \mathcal{L}_Y(e^{i\tilde{\eta}_x})}{e^{i\tilde{\eta}_x}} = -i e^{i\tilde{\eta}_x} \cdot \mathcal{L}_Y(\tilde{\eta}_x) = -i \mathcal{L}_Y(\tilde{\eta}_x) = P \tilde{\eta}_x.$$

From unique ergodicity we get the following convergence

$$\lim_{n \to \infty} \frac{H(n)}{n} u = \left( -i \xi_0 + \int_{X} P \tilde{\eta}_x \, d\mu \right) u = \left( -i \xi_0 u = H u \right)$$

uniformly in $n$ for $u \in L_\chi$. 

Since
\[ \frac{H(n)}{n} H^{-1} f = \left( \frac{1}{n} \sum_{k=1}^{n} \left( -i \xi_0 - i \mathcal{L}_Y (\chi \circ \eta) \circ F_{-k} \right) \cdot \frac{i}{\xi_0} f \right) = f + \left( \frac{1}{n} \sum_{k=1}^{n} P \tilde{\eta}_\chi \circ F_{-k} \right) \cdot \frac{i}{\xi_0} f. \]
So,
\[ \left\| \frac{H(n)}{n} H^{-1} f \right\| \leq \left( 1 + \frac{\| P \tilde{\eta}_\chi \|_{L_\chi}}{|\xi_0|} \right) \cdot \| f \|_{L_\chi}. \]
Since \( \tilde{\eta}_\chi \in \text{Dom}(P) \),
\[ \| P \tilde{\eta}_\chi \|_{L_\chi} \leq C_1. \]
Thus, \( \frac{H(n)}{n} H^{-1} \) is a bounded operator with uniformly bounded norm in \( n \),
\[ \left\| \frac{H(n) H^{-1}}{n} \right\|_{op} \leq 1 + \frac{C_1}{|\xi_0|}. \]
Also,
\[ \left\| \left( I - \frac{H(n)}{n} H^{-1} \right) f \right\|_{L_\chi} = \left\| \left( 1 - \left( 1 + \left( \frac{1}{n} \sum_{k=1}^{n} P \tilde{\eta}_\chi \circ F_{-k} \right) \cdot \frac{i}{\xi_0} \right) \right) f \right\|_{L_\chi} \]
\[ = \left\| \frac{1}{n} \sum_{k=1}^{n} P \tilde{\eta}_\chi \circ F_{-k} \cdot \frac{i}{\xi_0} \right\|_{L_\chi} \cdot \| f \|_{L_\chi}. \]
As a result of unique ergodicity, the following converges uniformly,
\[ \lim_{n \to \infty} \left( \frac{1}{n} \sum_{k=1}^{n} P \tilde{\eta}_\chi \circ F_{-k} \right) \cdot \frac{i}{\xi_0} = \frac{i}{\xi_0} \int_X P \tilde{\eta}_\chi \, d\mu = 0, \]
and thus,
\[ \limsup_{n \to \infty} \left\| I - \frac{H(n)}{n} H^{-1} \right\|_{op} \leq \limsup_{n \to \infty} \left\| \left( \frac{1}{n} \sum_{k=1}^{n} P \tilde{\eta}_\chi \circ F_{-k} \right) \cdot \frac{i}{\xi_0} \right\|_{\infty} \]
\[ = \left\| \frac{i}{\xi_0} \int_X P \tilde{\eta}_\chi \, d\mu \right\|_{\infty} \]
\[ = 0. \]
Hence,
\[ \limsup_{n \to \infty} \left\| I + \frac{H(n)}{n} H^{-1} \right\|_{op} < 1. \]
Thus, condition (ii)

\[
\left[ P, \frac{H(n)}{n} H^{-1} \right] = \left[ P, I + \left( \frac{1}{n} \sum_{k=1}^{n} P\hat{\eta}_x \circ F_{-k} \right) \right] \cdot \frac{i}{\xi_0} I
\]

\[
= \left[ P, \left( \frac{1}{n} \sum_{k=1}^{n} P\hat{\eta}_x \circ F_{-k} \right) \right] \cdot \frac{i}{\xi_0} I
\]

\[
= \left( \frac{1}{n} \sum_{k=1}^{n} P(P\hat{\eta}_x) \circ F_{-k} \right) \cdot \frac{i}{\xi_0} I.
\]

Since \( \text{sup}_{t>0} \left\| \mathcal{L}_t(\chi \circ F_{-t} \mathcal{L}_t(\chi \circ F_{-t})) \right\|_{\infty} < \infty \), \( P(P\hat{\eta}_x) \) is bounded in \( L_\chi \) and

\[
\left\| \left[ P, \frac{H(n)}{n} H^{-1} \right] f \right\|_{L_\chi} \leq \left\| P(P\hat{\eta}_x) \right\|_{L_\chi} \cdot \left\| f \right\|_{L_\chi} \leq \frac{C_2}{\left| \xi_0 \right|} \left\| f \right\|_{L_\chi}.
\]

Thus, \( \left[ P, \frac{H(n)}{n} H^{-1} \right] \) extends to a bounded operator on \( L_\chi \) with uniformly bounded norm in \( n \),

\[
\left\| \left[ P, \frac{H(n)}{n} H^{-1} \right] \right\|_{\text{op}} \leq \frac{C_2}{\left| \xi_0 \right|}.
\]

(iii) Since the operator \( H \) is just multiplication by the constant \(-i\xi_0\),

\[
[H(n), H] H^{-1} = 0.
\]

Thus, condition (iii) is immediately satisfied.

Since conditions (i), (ii), and (iii) of Theorem 2.1 are satisfied on each \( L_\chi \), we have shown that the operator \( U_\chi \) has purely absolutely continuous spectrum on \( L_\chi \). Thus, \( W \) has purely absolutely continuous spectrum when restricted to the subspace \( \bigoplus_{\chi \in G \times \xi \neq 1} L_\chi \).

In addition, from the purity law in [8] extended to translations, the maximal spectral type is either purely Lebesgue, purely singularly continuous, or purely discrete with respect to \( \mu \) (the Haar measure). Since we know that the spectrum is absolutely continuous from above, we have proved the following theorem.

**Theorem 4.1.** The operator \( U_\chi \) has Lebesgue spectrum on \( L_\chi \). Thus, \( W \) has countable Lebesgue spectrum when restricted to the subspace \( \bigoplus_{\chi \in G \times \xi \neq 1} L_\chi \).

Similar results with less restrictive assumptions on \( \eta \) have been derived in [11] and [19].

### 4.2. Furstenberg transformations

Let \( \mu_n \) be the normalized Haar measure on \( T^n \simeq \mathbb{R}^n / \mathbb{Z}^n \) and \( \mathcal{H}_n = L^2(T^n, \mu_n) \). Let \( T_d : T^d \to T^d, d \geq 2 \), be the uniquely ergodic map [6]

\[
T_d(x_1, x_2, \ldots, x_d) = (x_1 + y, x_2 + b_{2,1}x_1 + h_1(x_1), \ldots, x_d + b_{d,1}x_1 + \cdots + b_{d,d-1}x_{d-1} + h_{d-1}(x_1, x_2, \ldots, x_{d-1}))
\]

\[
\mod Z^d
\]

for \( y \in \mathbb{R} \setminus \mathbb{Q}, b_{j,k} \in \mathbb{Z}, b_{l,l-1} \neq 0 \), and \( l \in \{2, \ldots, d\} \). (For \( n = 2 \), we get the skew product in 4.1.) Let each \( h_j : T^j \to \mathbb{R} \) satisfy a uniform Lipschitz condition in \( x_j \) and be in \( C^2(T^j) \). What follows is very similar to the case of the skew products over translations. We begin by considering the operator

\[
W_d : \mathcal{H}_d \to \mathcal{H}_d.
\]

The space \( \mathcal{H}_d \) can be decomposed into
Hence, \( \mathcal{H}_d = \mathcal{H}_1 \bigoplus_{j \in \{2, \ldots, d\}, k \in \mathbb{Z} \setminus \{0\}} \mathcal{H}_{j,k} \)
for \( \mathcal{H}_{i,k} = \text{Span} \{ \eta \otimes \chi_k \eta \in \mathcal{H}_{j-1} \} \) and \( \chi_k(x_j) = e^{2\pi ikx_j} \in \mathbb{T} \).

The restriction of \( W_d \), \( W_d \vert_{\mathcal{H}_{j,k}} \) is unitarily equivalent to the operator

\[ U_{j,k} \eta = e^{2\pi ik\phi} W_{j-1} \eta \]

for

\[ \eta \in \mathcal{H}_{j-1} \]

and

\( \phi_j(x_1, x_2, \ldots, x_{j-1}) = b_j_1 x_1 + \cdots + b_{j,j-1} x_{j-1} + h_{j-1}(x_1, x_2, \ldots, x_{j-1}). \)

We will choose to take the commutator with \( P_{j-1} = -i\partial_{j-1} \), the essentially self-adjoint \([1]\) generator of the translation group \( \{ \mathcal{V}_{t,j-1} \}_{t \in \mathbb{R}} \) in \( \mathcal{H}_{j-1} \). The following formulas hold on \( D = C^\infty(T^{j-1}). \)

\[ [P_{j-1}, U_{j,k}] = [P_{j-1}, e^{2\pi i k\phi_j} I] W_{j-1} \]

\[ = -i\partial_{j-1}(e^{2\pi i k\phi_j}) W_{j-1} \]

\[ = (2\pi kb_{j,j-1} + 2\pi k\partial_{j-1}h_{j-1}) e^{2\pi i k\phi} W_{j-1}. \]

So,

\[ [P_{j-1}, U_{j,k}] = (2\pi kb_{j,j-1} + 2\pi k\partial_{j-1}h_{j-1}) U_{j,k} = G U_{j,k}. \]

Thus,

\[ U_{j,k}^{-n} [P_{j,k}, U_{j,k}^n] = \sum_{l=1}^{n} U_{j,k}^{-l} G U_{j,k}^l = \sum_{l=1}^{n} G \circ T_{j-1}^{-l} = H(n). \]

From unique ergodicity we get the following convergence

\[ \lim_{n \to \infty} \frac{H(n)}{n} u = 2\pi kb_{j,j-1} + 2\pi k \int_{T_{j-1}} \partial_{j-1}h_{j-1} d\mu = \frac{2\pi kb_{j,j-1} u = Hu}{2\pi kb_{j,j-1}I} \]

uniformly in \( n \) for \( u \in D = \mathcal{H}_{j-1} \).

Since

\[ \| U_{j,k}^n f, f \|_{\mathcal{H}_{j-1}} \|_{L^2(\mathbb{T})} = \left\| \frac{1}{\sigma} \int_0^\sigma \langle e^{sP_{j-1}} U_{j,k}^n f, e^{sP_{j-1}} f \rangle_{\mathcal{H}_{j-1}} ds \|_{L^2(\mathbb{T})} \right\|, \]

the preliminary assumptions for Theorem 2.1 are satisfied for \( B_1 = B_2 = I \). Now we proceed by showing that the conditions of Theorem 2.1 hold.

(i) It is unnecessary to consider coboundaries since both \( H = 2\pi kb_{j,j-1}I \) and \( H^{-1} = \frac{1}{2\pi kb_{j,j-1}}I \) are constants. Instead we take any \( f \in \mathcal{H}_{j-1}. \)

\[ H(n) n^{-1} f = \left( \frac{1}{n} \sum_{l=1}^{n} (2\pi kb_{j,j-1} + 2\pi k\partial_{j-1}h_{j-1}) \circ T_{j-1}^{-l} \right) f = \frac{1}{2\pi kb_{j,j-1}} f. \]

Hence,

\[ \| H(n) n^{-1} f \|_{\mathcal{H}_{j-1}} \leq \left( 1 + \frac{\| 2\pi k\partial_{j-1}h_{j-1} \|_{\mathcal{H}_{j-1}}}{2\pi kb_{j,j-1}} \right) \| f \|_{\mathcal{H}_{j-1}}. \]

Since \( h_{j-1} \) satisfies a uniform Lipschitz condition in \( x_{j-1}, \)

\[ \| \partial_{j-1}h_{j-1} \|_{\mathcal{H}_{j-1}} \leq C_1. \]
Hence, and thus, since

\[ \| H(n) H^{-1} \|_{op} \leq 1 + \frac{|2\pi k| \| \partial_{\omega_{j-1}} h_{j-1} \|_{\mathcal{H}_{j-1}}}{2\pi k b_{j,j-1}} \leq \frac{C_1}{b_{j,j-1}}. \]

Also,

\[
\left\| \left( I - \frac{H(n)}{n} H^{-1} \right) f \right\|_{\mathcal{H}_{j-1}} = \left\| \left( 1 - \left( 1 + \frac{1}{n} \sum_{l=1}^{n} (2\pi k \partial_{\omega_{j-1}} h_{j-1}) \circ T_{j-1}^{-l} \right) \frac{1}{2\pi k b_{j,j-1}} \right) f \right\|_{\mathcal{H}_{j-1}} \\
\leq \left\| \frac{1}{n} \sum_{l=1}^{n} (2\pi k \partial_{\omega_{j-1}} h_{j-1}) \circ T_{j-1}^{-l} \right\|_{\mathcal{H}_{j-1}} \frac{1}{2\pi k b_{j,j-1}} \| f \|_{\mathcal{H}_{j-1}}.
\]

As a result of unique ergodicity, the following converges uniformly,

\[
\lim_{n \to \infty} \left( \frac{1}{n} \sum_{l=1}^{n} (2\pi k \partial_{\omega_{j-1}} h_{j-1}) \circ T_{j-1}^{-l} \right) = \frac{1}{2\pi k b_{j,j-1}} \int_{\mathcal{T}^1} \partial_{\omega_{j-1}} h_{j-1} \, d\mu = 0,
\]

and thus,

\[
\limsup_{n \to \infty} \left\| I - \frac{H(n)}{n} H^{-1} \right\|_{op} \leq \limsup_{n \to \infty} \left\| \left( \frac{1}{n} \sum_{l=1}^{n} (2\pi k \partial_{\omega_{j-1}} h_{j-1}) \circ T_{j-1}^{-l} \right) \frac{1}{2\pi k b_{j,j-1}} \right\|_{\mathcal{H}_{j-1}} = 0
\]

Hence,

\[
\limsup_{n \to \infty} \left\| I + \frac{H(n)}{n} H^{-1} \right\|_{op} < 1.
\]

(ii)

\[
\left[ P_{j-1}, \frac{H(n)}{n} H^{-1} \right] = \left[ P_{j-1}, I + \left( \frac{1}{n} \sum_{k=1}^{n} (2\pi k \partial_{\omega_{j-1}} h_{j-1}) \circ T_{j-1}^{-l} \right) \frac{1}{2\pi k b_{j,j-1}} \right] = \left[ P_{j-1}, \left( \frac{1}{n} \sum_{k=1}^{n} (2\pi k \partial_{\omega_{j-1}} h_{j-1}) \circ T_{j-1}^{-l} \right) \frac{1}{2\pi k b_{j,j-1}} \right] = \left( \frac{1}{n} \sum_{k=1}^{n} (2\pi k \partial_{\omega_{j-1}} h_{j-1}) \circ T_{j-1}^{-l} \right) \frac{1}{2\pi k b_{j,j-1}} I.
\]

Since \( h_{j-1} \in C^{2} (\mathcal{T}^{1}) \), \( \partial_{\omega_{j-1}} (\partial_{\omega_{j-1}} h_{j-1}) \) is bounded in \( \mathcal{H}_{j-1} \),

\[
\| [P, \frac{H(n)}{n} H^{-1}] f \|_{\mathcal{H}_{j-1}} \leq \frac{|2\pi k| \| \partial_{\omega_{j-1}} (\partial_{\omega_{j-1}} h_{j-1}) \|_{\mathcal{H}_{j-1}}}{|2\pi k b_{j,j-1}|} \| f \|_{\mathcal{H}_{j-1}} \leq \frac{C_2}{b_{j,j-1}} \| f \|_{\mathcal{H}_{j-1}}.
\]
Thus, \([P, \frac{H(n)}{n} H^{-1}]\) extends to a bounded operator on \(H_{j-1}\) with uniformly bounded norm in \(n\),
\[
\left\| [P, \frac{H(n)}{n} H^{-1}] \right\|_{op} \leq \frac{C_2}{|j_{j-1}|}.
\]

(iii) Since the operator \(H\) is just multiplication by \(2\pi k b_{j,j-1}\),
\[
[H(n), H]^{-1} = 0.
\]

Thus, condition (iii) is immediately satisfied.

Since conditions (i), (ii), and (iii) of Theorem 2.1 are satisfied, the operator \(U_{j,k}\) has purely absolutely continuous spectrum on each \(H_{j,k}\). Thus, \(W_d\) has purely absolutely continuous spectrum on the orthocomplement of \(H_1\). Applying the the purity law in [8], we derive, with slightly stronger regularity assumptions, a similar result to the ones found in [11] and [19].

**Theorem 4.2.** \(W_d\) has countable Lebesgue spectrum on the orthocomplement of \(H_1\).

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**REFERENCES**

[1] W. O Amrein, *Hilbert Space Methods in Quantum Mechanics*, Fundamental Sciences, EPFL Press Lausanne, 2009.

[2] W. O. Amrein, A. Boutet de Monvel and V. Georgescu, *\(C_0\)-groups, Commutator Methods and Spectral Theory of N-body Hamiltonians*, Progress in Math Birkhäuser, Basel, 1996.

[3] H. Anzai, *Ergodic skew product transformations on the torus*, Osaka Journal of Mathematics, 3 (1951), 83–99.

[4] J. Brown, *Ergodic Theory and Topological Dynamics*, Academic Press, 1976.

[5] G. Forni and C. Ulcigrai, *Time-changes of horocycle flows*, Journal of Modern Dynamics, 6 (2012), 251–273.

[6] H. Furstenberg, *Strict Ergodicity and transformation of the torus*, American Journal of Mathematics, 83 (1961), 573–601.

[7] H. Furstenberg, *The unique ergodicity of the horocycle flow*, Recent advances in topological dynamics (Proc. Conf., Yale Univ., New Haven, Conn., 1972; in honor of Gustav Arnold Hedlund), Lecture Notes in Mathematics, Springer, Berlin, 318 (1972), 95–115.

[8] H. Helson, *Cocyles on the circle*, Journal of Operator Theory, 16 (1986), 89–199.

[9] A. Iwanik, *Anzai skew products with Lebesgue with Lebesgue component of infinite multiplicity*, Bulletin of the London Mathematical Society, 29 (1997), 195–199.

[10] A. Iwanik, *Spectral properties of skew-product diffeomorphisms of tori*, Colloquium Mathematicum, 72 (1997), 223–235.

[11] A. Iwanik, M. Lemańczyk and D. Rudolph, *Absolutely continuous cocycles over irrational rotations*, Israel Journal of Mathematics, 83 (1993), 73–95.

[12] A. G. Kushnirenko, *Spectral properties of certain dynamical systems with polynomial dispersal*, (Russian) Vestnik Moskov. Univ. Ser. I Mat. Meh., 29 (1974), 101–108.

[13] M. Lemańczyk, *Spectral theory of dynamical systems*, Encyclopedia of Complexity and Systems Science, (2009), 8553–8575.

[14] B. Marcus, *Ergodic properties of horocycle flows for surfaces of negative curvature*, Annals of Mathematics, Second Series 105 (1977), 81–105.

[15] C. C. Moore, *Ergodicity of flows on homogeneous spaces*, American Journal of Mathematics, 88 (1966), 154–178.

[16] E. Mourre, *Absence of singular continuous spectrum for certain selfadjoint operators*, Communications in Mathematical Physics, 78 (1980/81), 391–408.
[17] S. Richard and R. Tiedra de Aldecoa, Commutator criteria for strong mixing II, preprint, arXiv:1510.00201

[18] R. Tiedra de Aldecoa, Spectral analysis of time-changes of the horocycle flow, *Journal of Modern Dynamics*, 6 (2012), 275–285.

[19] R. Tiedra de Aldecoa, Commutator methods for the spectral analysis of uniquely ergodic dynamical systems, *Ergodic Theory and Dynamical Systems*, 35 (2015), 944–967.

[20] R. Tiedra de Aldecoa, Commutator criteria for strong mixing, *Ergodic Theory and Dynamical Systems*, 37 (2017), 308–323.

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