Scalar-tensor theories and asymmetric resonant cavities.

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Abstract. Recently published experimental results indicate the appearance of unusual forces on asymmetric, electromagnetic resonant cavities. It is argued here that a particular class of scalar-tensor theories of gravity could account for this effect.

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1. Introduction

Very recently, experimental results were published\[1\] on the measurement of forces on closed, asymmetric electromagnetic resonant cavities. These results add to previous claims in the same line by an independent researcher who did the first experiments\[2\]. Such claims were criticized by the scientific community mainly due to the proposed theoretical explanation, as Maxwell equations and Special Relativity clearly indicate that no force is possible without the emission of radiation from the cavity. On the other hand, it appears that General Relativity might allow for such kind of reaction-less propulsion, as exemplified and noted for the first time in\[3\], where the low velocity limit of some warp drive spacetimes was analyzed. As indicated there, negative energy densities are required to accomplish that and, notably, some scalar fields present this possibility\[4\]. Of course, in order to have measurable effects similar to those reported, the coupling of the scalar field to matter or other fields acting as its source should be sufficiently strong, and this is precisely what has been proposed in\[5\] for the case of the electromagnetic field as source of the scalar field to explain discordant measurements of Newton gravitational constant. It is then only natural to wonder whether that theory (or a similar one) may account for the forces reported in resonant cavities. Of course, all this is highly speculative, and more prosaic explanations for these forces should be considered first. We proceed on the assumption that all spurious effects were accounted for, and on the belief that the possibility presented here is worth exploring.

The theory put forward by Mbelek and Lachièze-Rey in\[5\] (see also \[6\]) represents a reduction to four dimensions of a Kaluza-Klein theory coupled to an external scalar $\psi$, which in turn couples to matter. It is the source term of $\psi$ which allows for a possible strong coupling of the Kaluza-Klein scalar $\phi$ to other fields, in particular to the electromagnetic field. The theory was applied in cosmological\[7\] and galactic situations\[8\] and, as mentioned, it was also used to investigate the possibility of the Earth’s magnetic field influencing the measurements of Newton gravitational constant. In all these applications of the theory only its weak-field limit was used, and as this limit is similar for a wide range of theories, we employ in this work a rather general scalar-tensor theory, which incorporates the additional external scalar $\psi$.

In the next sections the equations of the mentioned scalar-tensor theory are derived from its proposed action, along with the equation of motion of neutral matter. Some axisymmetric electromagnetic modes of a truncated conical cavity are then presented and used as source in the weak-field approximation of the equations, previously obtained, to determine the force on the cavity. It is found that a coupling of the same magnitude as used in \[5\] between the scalar $\phi$ and the electromagnetic field results in a correct magnitude and sign for the forces reported in asymmetric resonant cavities. As expected, the solution for the cavity presents negative energy densities (more precisely, it violates the weak energy condition\[9\]). The theory, however, does not seem to be completely satisfactory because in its linearized version it also predicts strong gravitational effects by the Earth’s magnetic field, which are clearly not observed. A possible resolution of
this problem is considered in the last section.

2. Scalar-tensor theory

We will consider a scalar-tensor theory of the Brans-Dicke type\[10\] with inclusion of a Bekenstein’s direct interaction of scalar and Maxwell fields\[11\], and with an additional external scalar field $\psi$ minimally coupled to gravity, and universally coupled to matter, with action given by (SI units are used)

$$S = -\frac{c^3}{16\pi G_0} \int \sqrt{-g} \phi R d\Omega + \frac{c^3}{16\pi G_0} \int \sqrt{-g} \frac{\omega(\phi)}{\phi} \nabla^\nu \phi \nabla_\nu \phi d\Omega$$

$$+ \frac{c^3}{16\pi G_0} \int \sqrt{-g} \phi \left[ \frac{1}{2} \nabla^\nu \psi \nabla_\nu \psi - U(\psi) - J\psi \right] d\Omega$$

$$- \frac{\varepsilon_0 c}{4} \int \sqrt{-g} \lambda(\phi) F_{\mu\nu} F^{\mu\nu} d\Omega - \frac{1}{c} \int \sqrt{-g} j^\nu A_\nu d\Omega$$

$$+ \frac{1}{c} \int L_{\text{mat}} \left[ \exp (\beta \psi) g_{\mu\nu} \right] d\Omega. \quad (1)$$

In order to have a non-dimensional scalar field $\phi$ of values around unity, in expression (1) the constant $G_0$ representing Newton gravitational constant is included, $c$ is the velocity of light in vacuum, and $\varepsilon_0$ is the vacuum permittivity. $L_{\text{mat}}$ is the lagrangian density of matter, which is assumed to couple to the scalar $\psi$. The other symbols are also conventional, $R$ is the Ricci scalar, and $g$ the determinant of the metric tensor $g_{\mu\nu}$. The Brans-Dicke parameters $\omega(\phi)$ is considered a function of $\phi$, as it usually results so in the reduction to four dimensions of multidimensional theories\[12\]. The function $\lambda(\phi)$ in the term of the action of the electromagnetic field is of the type appearing in Bekenstein’s theory and other effective theories\[7\], it does not intervene in the weak field approximation ultimately employed, but is included for completeness. The electromagnetic tensor is $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$, given in terms of the electromagnetic quadri-vector $A_\nu$, with sources given by the quadri-current $j^\nu$. $U$ and $J$ are, respectively, the potential and source of the field $\psi$. The source $J$ contains contributions from the matter, electromagnetic field and the scalar $\phi$. In order to build upon a concrete model we follow the proposal in\[5\] (convenient dimensional factors differing from those in\[5\] are employed here)

$$J = \beta_{\text{mat}} (\psi, \phi) \frac{8\pi G_0}{c^4} T_{\text{mat}} + \beta_{\text{EM}} (\psi, \phi) \frac{4\pi G_0 \varepsilon_0}{c^2} F_{\mu\nu} F^{\mu\nu} + \beta_\phi (\psi, \phi) T^\phi, \quad (2)$$

where $T_{\text{mat}}$ is the trace of the energy-momentum tensor of matter, (note that this tensor is defined with respect to $g_{\mu\nu}$, not $\exp(\beta \psi) g_{\mu\nu}$),

$$T_{\mu\nu}^{\text{mat}} = -\frac{2}{\sqrt{-g}} \frac{\delta L_{\text{mat}}}{\delta g^{\mu\nu}},$$

and $T^\phi$ is the trace of the tensor

$$T_{\mu\nu}^\phi = \nabla_\mu \nabla_\nu \phi - \nabla^\gamma \nabla_\gamma \phi g_{\mu\nu} + \frac{\omega(\phi)}{\phi} \left( \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} \nabla^\gamma \phi \nabla_\gamma \phi g_{\mu\nu} \right).$$
Variation of (1) with respect to $g^{\mu\nu}$ results in ($T_{\mu\nu}^{EM}$ is the usual electromagnetic energy tensor)

$$
\phi \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = \frac{8\pi G_0}{c^4} \left[ \lambda (\phi) T_{\mu\nu}^{EM} + T_{\mu\nu}^{mat} \right] + T_{\mu\nu}^\phi \\
+ \frac{\phi}{2} \left( \nabla_\mu \psi \nabla_\nu \psi - \frac{1}{2} \nabla^\gamma \psi \nabla_\gamma \psi g_{\mu\nu} \right) \\
+ \frac{\phi}{2} (U + J\psi) g_{\mu\nu}.
$$

(3)

Variation with respect to $\phi$ gives

$$
\phi R + 2\omega \nabla^\nu \nabla_\nu \phi = \left( \frac{\omega}{\phi} - \frac{d\omega}{d\phi} \right) \phi \nabla^\nu \phi \nabla_\nu \phi - \frac{4\pi G_0 \varepsilon_0}{c^2} \phi \frac{d\lambda}{d\phi} F_{\mu\nu} F^{\mu\nu} \\
- \frac{\partial J}{\partial \phi} \psi \phi + \phi \left[ \frac{1}{2} \nabla^\nu \psi \nabla_\nu \psi - U (\psi) - J \psi \right],
$$

which can be rewritten, using the contraction of (3) with $g_{\mu\nu}$ to replace $R$, as

$$
(2\omega + 3) \nabla^\nu \nabla_\nu \phi = - \frac{d\omega}{d\phi} \phi \nabla^\nu \phi \nabla_\nu \phi - \frac{4\pi G_0 \varepsilon_0}{c^2} \phi \frac{d\lambda}{d\phi} F_{\mu\nu} F^{\mu\nu} + \frac{8\pi G_0}{c^4} T^{mat} \\
+ \phi \left[ \frac{1}{2} \nabla^\nu \psi \nabla_\nu \psi - U (\psi) - J \psi \right] - \frac{\partial J}{\partial \phi} \psi \phi,
$$

(4)

where it was used that $T_{\mu\nu}^{EM} = T_{\mu\nu}^{EM} g_{\mu\nu} = 0$.

The non-homogeneous Maxwell equations are obtained by varying (1) with respect to $A_\nu$,

$$
\nabla_\mu \left\{ \lambda (\phi) F_{\mu\nu} \right\} = \mu_0 j^{\nu}.
$$

(5)

with $\mu_0$ the vacuum permeability.

The variation with respect to $\psi$ results in

$$
\nabla^\nu \nabla_\nu \psi + \frac{1}{\phi} \nabla^\nu \psi \nabla_\nu \phi = - \frac{\partial U}{\partial \psi} - J - \frac{\partial J}{\partial \psi} \psi + \frac{\beta}{\phi} \frac{8\pi G_0}{c^4} T^{mat}.
$$

(6)

Having included $G_0$, it is understood that $\phi$ takes values around its vacuum expectation value (VEV) $\phi_0 = 1$. The scalar $\psi$ is also dimensionless and of VEV $\psi_0$.

Finally, we consider the motion of neutral test particles, coupled to the scalar $\psi$ as indicated in (1), which is then obtained requiring that

$$
\delta \int mc \sqrt{\exp (\beta \psi) g_{\mu\nu} dx^{\mu} dx^{\nu}} = 0,
$$

to give

$$
\frac{Du^{\gamma}}{Ds} = \frac{\beta}{2} (g^{\gamma \nu} - u^{\gamma} u^{\nu}) \partial_\nu \psi.
$$

(7)

It is important to mention that, in order to derive the previous equations, we have followed the prescription of not varying the trace of the energy-momentum tensors nor $F_{\mu\nu} F^{\mu\nu}$ in the source term (2), but only its coefficients $\beta'$s, as done in [5]. There is no clear reason for doing so, but on the one hand, inconsistent equations result if the
mentioned variations are included. On the other hand, the exact source term may not depend explicitly on the tensors considered, and only after the equations are derived and substitutions made might it be expressible in terms of the say tensors.

3. Weak-field approximation

In the weak field approximation, for values of $g_{\mu\nu}$ around $\eta_{\mu\nu}$ taken as those of flat Minkowski space with signature (1,-1,-1,-1), so that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we have

$$R_{\mu\nu} - \frac{1}{2} R \eta_{\mu\nu} = \frac{1}{2} \left( -\eta^{\gamma\delta} \partial_\gamma \partial_\delta h_{\mu\nu} + \partial_{\gamma\mu} \partial_\nu^\gamma \eta_{\gamma\nu} - \eta_{\mu\nu} \partial_\rho \xi^{\rho} \right),$$

with

$$\xi_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu},$$

where

$$h \equiv \eta^{\gamma\delta} h_{\gamma\delta} = -\eta^{\gamma\delta} \xi_{\gamma\delta}. $$

The system (3)-(5) can then be written, to lowest order in $h_{\mu\nu}$ and in the perturbations around the VEV’s of $\phi$ and $\psi$, as

$$-\eta^{\gamma\delta} \partial_\gamma \xi_{\mu\nu} = \frac{16 \pi G_0}{c^4} T_{\mu\nu}^{\text{mat}} + 2 \left( \partial_{\mu\nu} \phi - \eta^{\gamma\delta} \partial_\gamma \xi_{\mu\nu} \right),$$

with the Lorentz gauge

$$\partial_{\gamma} \xi^\gamma_{\nu} = 0,$$

$$(2\omega_0 + 3) \eta^{\gamma\delta} \partial_\gamma \phi = \frac{8 \pi G_0}{c^4} T_{\mu\nu}^{\text{mat}} - \frac{\partial J}{\partial \phi} \bigg|_{\phi_0,\psi_0} \psi_0,$$ 

$$\partial_\mu F^{\mu\nu} = \mu_0 j^\nu,$$

$$\eta^{\gamma\delta} \partial_\gamma \psi = \beta \frac{8 \pi G_0}{c^4} T_{\mu\nu}^{\text{mat}} - \frac{\partial J}{\partial \psi} \bigg|_{\phi_0,\psi_0} \psi_0,$$

where $\omega_0 = \omega(\phi_0)$. It was used in these equations that, in order to recover the usual physics when the scalar fields are not excited, one must have $\lambda(\phi_0) = 1$, $U(\psi_0) = J(\psi_0,\phi_0) = 0$. Also, as according to [5] the contribution form the energy-momentum tensor of the electromagnetic field through the source $J$ is much larger than all its other contributions, the latter were neglected in the above equations.

For slow moving neutral masses, the equation (7) corresponds to the action of a specific force (per unit mass) (Latin indices correspond to the spatial coordinates)

$$f_i = -\frac{c^2}{4} \frac{\partial}{\partial x_i} \left( \xi_{00} + \xi_{kk} + 2 \beta \psi \right) + c \frac{\partial \xi_{0i}}{\partial t}.$$  

Introducing the D’Alembertian operator

$$\Box = \eta^{\gamma\delta} \partial_\gamma \partial_\delta = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2, $$
applying it to the force equation (13), and using Eq. (8), one easily obtains

$$\square f_i = \frac{4\pi G_0}{c^2} \frac{\partial}{\partial x_i} \left( T_{00}^{\text{mat}} + T_{kk}^{\text{mat}} \right) - \frac{16\pi G_0}{c^3} \frac{\partial T_{00}^{\text{mat}}}{\partial t}$$

$$+ \frac{c^2}{2} \frac{\partial}{\partial x_i} \left( \frac{\square \phi - \beta \square \psi - 2 \frac{\partial^2 \phi}{c^2 \partial t^2}}{\square \phi - \beta \square \psi - 2 \frac{\partial^2 \phi}{c^2 \partial t^2}} \right).$$

From Eqs. (10) and (12), and retaining only the most important component $T_{00}^{\text{mat}}$, this expression can be conveniently recast as

$$f_i = -\frac{\partial \chi}{\partial x_i},$$

where the ”gravitational potential” $\chi$ satisfies

$$\square \chi = -\frac{4\pi G_0}{c^2} T_{00}^{\text{mat}} + 4\pi G_0 \left( \frac{\beta^2 - 1}{2\omega_0 + 3} \right) T_{00}^{\text{mat}}$$

$$+ \frac{\partial^2 \phi}{\partial t^2} + \frac{c^2 \psi_0}{2} \left( \frac{1}{2\omega_0 + 3} \frac{\partial J}{\partial \phi} - \beta \frac{\partial J}{\partial \psi} \right)_{\phi_0, \psi_0}. \quad (14)$$

Expliciting the matter contribution to $\chi$, using expression (2), one has

$$\square \chi = -\frac{4\pi G_0}{c^2} T_{00}^{\text{mat}}$$

$$+ 4\pi G_0 \left[ \beta \left( \beta - \psi_0 \frac{\partial \beta_{\text{mat}}}{\partial \psi} \right) + \frac{1}{2\omega_0 + 3} \left( \psi_0 \frac{\partial \beta_{\text{mat}}}{\partial \phi} - 1 \right) \right]_{\phi_0, \psi_0} T_{00}^{\text{mat}} + \ldots,$$

where the dots represent non-matter terms. The first term corresponds to Newton gravity, while the second term, if one takes $\beta = 0$, corresponds to the matter contribution through the scalar $\phi$, which is constrained by Solar System tests, requiring large values of $\omega_0$. An interesting conclusion (not to be explored further here) is that the inclusion of the external scalar $\psi$ could thus allow $\omega_0 \sim 1$ if $\beta$ is small enough (or, alternatively, if $\beta \simeq \psi_0 \frac{\partial \beta_{\text{mat}}}{\partial \psi}$), and

$$\psi_0 \left. \frac{\partial \beta_{\text{mat}}}{\partial \phi} \right|_{\phi_0, \psi_0} \simeq 1. \quad (15)$$

Note that the condition (15) does not invalidate the conclusions in [8] as only the term dependent on the matter velocity of the ”force” in the right hand side of (7) is used therein to explain the dynamics of rotating spiral galaxies.

Making explicit the equation of the scalar $\phi$, Eq. (10), with the expression of the source $J$, Eq. (2), one has

$$\square \phi = \frac{8\pi G_0}{(2\omega_0 + 3) c^4} \left( 1 - \psi_0 \left. \frac{\partial \beta_{\text{mat}}}{\partial \phi} \right|_{\phi_0, \psi_0} \right) T_{\text{mat}}$$

$$- \frac{8\pi G_0 \varepsilon_0}{(2\omega_0 + 3) c^2 \psi_0} \left. \frac{\partial \beta_{\text{EM}}}{\partial \phi} \right|_{\phi_0, \psi_0} \left( B^2 - E^2 / c^2 \right), \quad (16)$$

where it was used that, in terms of the modulus of the electric and magnetic vector fields, $E$ and $B$, respectively, one has

$$F_{\mu\nu} F^{\mu\nu} = 2 \left( B^2 - E^2 / c^2 \right),$$
and where the contribution from $\phi$ itself as its source was not considered because, even if it is present, in the weak-field approximation one has

$$T^\phi = -3\Box \phi,$$

and so its effect amounts to a redefinition of the rest of the coefficients in the equations for $\phi$ and $\psi$.

A point worth noting is that condition (15) refers so far to the motion of massive bodies, while the most stringent bounds on $\omega_0$ come from the propagation of electromagnetic waves near the Sun[13], not affected by the coupling of $\psi$ to matter. According to the expression (16) these bounds can also be accommodated, always with $\omega_0 \sim 1$, if the same condition (15) holds.

With all this, the contributions other than the matter to the potential $\chi$ can then be obtained from (14) as (we write $\chi = \chi_{\text{mat}} + \chi'$)

$$\Box \chi' = \frac{\partial^2 \phi}{\partial t^2} + 4\pi G_0 \varepsilon_0 \psi_0 \left( \frac{1}{2\omega_0 + 3} \frac{\partial \beta_{\text{EM}}}{\partial \phi} - \beta \frac{\partial \beta_{\text{EM}}}{\partial \psi} \right)_{\phi_0, \psi_0} (B^2 - E^2/c^2). (17)$$

In [5] it is argued that in order to explain discordant measurements of $G = G_0/\phi$ as due to the $\phi$ generated by the Earth’s magnetic field according to (16), one must have

$$\frac{8\pi G_0 \varepsilon_0}{(2\omega_0 + 3)c^2 \psi_0} \left. \frac{\partial \beta_{\text{EM}}}{\partial \phi} \right|_{\phi_0, \psi_0} = -(5.4 \pm 0.6) \times 10^{-8} \frac{A^2}{N^2}, \quad (18)$$

while the value of $\beta \partial J/\partial \psi|_{\phi_0, \psi_0}$ does not enter the equation of $\phi$ and is thus left unspecified. In the following we will evaluate the force predicted by (17) for the resonant electromagnetic field in a conical cavity, assuming that the coefficient in the brackets in (17) can be estimated from the value (18) alone.

4. Normal modes in a conical cavity

As done in [14] we consider a conical cavity with side walls corresponding to a truncated cone, with spherical sections as end caps. The cone axis is taken as the $z$ direction, the lateral wall corresponds to the spherical angle $\theta = \theta_0$ (half angle of the cone), and the spherical caps to the radii $r = r_1, 2$, with $r_2 > r_1$.

The resonant modes correspond to standing electromagnetic waves satisfying the vector wave equation ($\mathbf{F}$ stands for either the electric field $\mathbf{E}$, or the magnetic induction $\mathbf{B}$)

$$\frac{1}{c^2} \frac{\partial^2 \mathbf{F}}{\partial t^2} - \nabla^2 \mathbf{F} = 0.$$

The modes with rotational symmetry and $\mathbf{B}$ transverse to the $z$ direction $\mathbf{e}_z$ (called the TM modes) that satisfy this equation are (spherical coordinates are employed, with unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$) (see [14] and references therein for details)

$$\mathbf{B} = -CkR(r)Q'(\theta) \cos(\omega t) \mathbf{e}_\phi,$$

$$\mathbf{E}/c = C \left\{ \frac{R(r)}{r} n (n + 1) Q(\theta) \mathbf{e}_r \right\}, \quad (19)$$
\[ + \left[ \frac{R(r)}{r} + R'(r) \right] Q'(\theta) \mathbf{e}_\theta \right\} \sin(\omega t) \]  

(20)

where \( C \) is a global constant. The functions \( R \) and \( Q \) are defined as

\[
Q(\theta) = P_n(\cos \theta),
\]

\[
R(r) = R_+(r) \cos \alpha + R_-(r) \sin \alpha,
\]

\[
R_\pm(r) = \frac{J_{\pm(n+1/2)}(kr)}{\sqrt{r}},
\]

where \( P_n \) is the Legendre polynomial of order \( n \), \( J_m \) the Bessel function of the first kind of order \( m \), and \( \alpha \) and \( k \) constants to be determined along with the order \( n \). By construction, the magnetic field satisfies the boundary condition of zero normal component at the metallic walls, while in order to have zero tangential components of the electric field at the walls, the order \( n \) of the Legendre polynomial must satisfy

\[ P_n(\cos \theta_0) = 0, \]

the wavenumber \( k \) the condition

\[
\left[ \frac{R_+}{r} + R'_+ \right]_{r_2} \left[ \frac{R_-}{r} + R'_- \right]_{r_1} = \left[ \frac{R_+}{r} + R'_+ \right]_{r_1} \left[ \frac{R_-}{r} + R'_- \right]_{r_2},
\]

and \( \alpha \)

\[
\tan \alpha = \frac{R_+(r_2)}{R'_+(r_2) - R_-(r_2)}.
\]

The resonant mode angular frequency is thus determined as \( \omega = kc \).

There exists a complementary set of modes with \( \mathbf{E} \) transverse to the \( z \) direction (TE modes), but for concreteness we study only the lowest frequency TM modes.

An important parameter is the quality factor of the cavity, \( Q_{\text{cav}} \), for each mode. It is conventionally defined as

\[ Q_{\text{cav}} \equiv \frac{\omega \langle U \rangle}{\langle W \rangle}, \]  

(21)

where \( \omega \) is the angular frequency of the mode, \( \langle U \rangle \) is the temporal average of its electromagnetic energy, and \( \langle W \rangle \) is the average dissipated power in the wall cavities. As the average electric energy is equal to the average magnetic energy in the cavity, and the loss power can obtained from the value of the magnetic field at the boundary, an explicit, practical expression of \( Q_{\text{cav}} \) can be obtained in terms of solely the magnetic field as[18]

\[ Q_{\text{cav}} = \frac{2}{\delta} \frac{\int \langle B^2 \rangle \, dV}{\int \langle B^2 \rangle \, dS}, \]

where the integrals are extended to the volume and the internal surface of the cavity, respectively, and \( \delta \) is the penetration length in the metal wall, of resistivity \( \eta \),

\[ \delta = \sqrt{\frac{2\eta}{\mu_0 \omega}}. \]  

(22)
From (19) one can thus write
\[ Q_{\text{cav}} = \frac{2}{\delta} \int \frac{[R(r) Q'(\theta)]^2}{2} dV. \]  

(23)

If the cavity is fed with an average electromagnetic power \( P \), in the permanent regime one has \( \langle W \rangle = P \), and so, from (19) and (21),
\[ \langle U \rangle = \int \frac{[R(r) Q'(\theta)]^2}{2} dV = \frac{Q_{\text{cav}} P}{\omega}, \]
which allows to determine the global constant \( C \), given the fed average power and the characteristics of the cavity for the considered mode.

5. Force on the cavity

In the permanent regime of the established resonant mode, sustained against decay by a continuous power input \( P \), the electromagnetic field (19)-(20) corresponds to
\[ B^2 - E^2/c^2 = F_B(r,\theta) \cos^2(\omega t) - F_E(r,\theta) \sin^2(\omega t) \]
\[ = \frac{1}{2} (F_B - F_E) + \frac{1}{2} (F_B + F_E) \cos(2\omega t), \]
where
\[ F_B(r,\theta) = C^2 k^2 [R(r) Q'(\theta)]^2, \]
\[ F_E(r,\theta) = C^2 \left\{ \left[ \frac{R(r)}{r} n (n + 1) Q(\theta) \right]^2 + \left[ \frac{R(r)}{r} + R'(r) \right]^2 Q^2(\theta) \right\}. \]

(25)

(26)

(27)

From (16), expression (25) then leads to a constant plus an harmonic in time contribution to \( \phi \), which, together with (25) in (17), result in \( \chi' \) also having a constant plus an harmonic part. The latter has a zero contribution to the time average of the force, and so we consider only the constant part, \( \chi_0' \), whose equation is, from (17),
\[ \nabla^2 \chi_0' = \kappa (F_B - F_E), \]
where, using (15),
\[ \kappa = -2\pi G_0 \varepsilon_0 \psi_0 \left( \frac{1}{2\omega_0 + 3} \frac{\partial \beta_{EM}}{\partial \phi} - \beta \frac{\partial \beta_{EM}}{\partial \psi} \right)_{\phi_0,\psi_0} \]
\[ \simeq -\frac{2\pi G_0 \varepsilon_0 \psi_0}{2\omega_0 + 3} \frac{\partial \beta_{EM}}{\partial \phi} \bigg|_{\phi_0,\psi_0} \approx 1.2 \times 10^9 \left( \frac{Am}{N^2} \right)^2. \]

(29)

Note that the magnetic field in the right-hand side of (17) is the total field, which includes the contribution from the Earth’s magnetic field. The latter, although of much smaller magnitude than that of the cavity cannot be neglected due to its large spatial scale. However, one has for the time average (denoted by \( \langle \ldots \rangle \))
\[ \langle B_{\text{Earth}}^2 + B_{\text{cavity}}^2 \rangle = B_{\text{Earth}}^2 + \langle B_{\text{cavity}}^2 \rangle + 2B_{\text{Earth}} \cdot \langle B_{\text{cavity}} \rangle \]
\[ = B_{\text{Earth}}^2 + \frac{F_B}{2}, \]
since \( \langle B_{cavity} \rangle = 0 \). In this way, the contribution from the magnetic fields of the Earth and of the cavity to the potential \( \chi'_0 \) can be separated, and that of the cavity alone is correctly described by Eq. (28).

Eq. (28) is solved taking into account that its right-hand side is zero outside the cavity, so that, using the axial symmetry, the solution of Poisson equation (28) is

\[
\chi'_0(r, \theta) = -\frac{\kappa}{\pi} \int \frac{F_B(r', \theta') - F_E(r', \theta')}{\sqrt{r^2 + r'^2 - 2rr'\cos(\theta' - \theta)}} \times K\left(-\frac{4rr'\sin \theta \sin \theta'}{r^2 + r'^2 - 2rr'\cos(\theta' - \theta)}\right) r'^2 \sin \theta' dr' d\theta' \tag{30}
\]

where \( K \) is the complete elliptic integral of the first kind, and the integral is extended to the interior of the cavity. Note that as the volume integral of the left-hand side of (28) is equal to zero, \( \nabla \chi'_0 \) decays rapidly outside the cavity.

Assuming a cavity with thin walls (but much thicker than the penetration depth \( \delta \), in order to the boundary conditions used to be correct) of mass surface density \( \sigma \), the force on the cavity is finally evaluated as

\[
\mathbf{F} = -\sigma \int \nabla \chi'_0 d\mathbf{S}, \tag{31}
\]

where the integral is extended to the internal surface of the cavity. Due to the axial symmetry the force has only a \( z \) component

\[
F_z = -\sigma \int \frac{\partial \chi'_0}{\partial z} d\mathbf{S} = -\sigma \int \nabla \chi'_0 \cdot \mathbf{e}_z d\mathbf{S},
\]

which is explicitly written as

\[
-\frac{F_z}{2\pi \sigma} = r_2^2 \int_0^{\theta_0} \left( \cos \theta \frac{\partial \chi'_0}{\partial r} - \frac{\sin \theta \partial \chi'_0}{r \partial \theta} \right) \sin \theta d\theta \bigg|_{r_2}^{r_1} + r_1^2 \int_0^{\theta_0} \left( \cos \theta \frac{\partial \chi'_0}{\partial r} - \frac{\sin \theta \partial \chi'_0}{r \partial \theta} \right) \sin \theta d\theta \bigg|_{r_1}^{r_2} + \sin \theta_0 \int_{r_1}^{r_2} \left( \cos \theta \frac{\partial \chi'_0}{\partial r} - \frac{\sin \theta \partial \chi'_0}{r \partial \theta} \right) \\ \bigg|_{\theta_0} rdr. \tag{32}
\]

There are no details in the literature as to the precise dimensions of the cavities used in the experiments, so that an example roughly similar to the overall dimension reported and with the proportions observed in the published photographs will be used. Assuming a wall of thickness 1 mm, and a copper mass density of \( 8.9 \times 10^3 \text{ kg/m}^3 \), we have \( \sigma = 8.9 \text{ kg/m}^2 \).

We further consider the copper cavity to have \( r_1 = 18 \text{ cm} \), \( r_2 = 36 \text{ cm} \), and \( \theta_0 = 22^\circ \). For this cavity, the lowest TM mode corresponds to the order \( n = 5.75632 \) of the Legendre polynomial, with a resonant frequency \( \nu = 1.05 \text{ GHz} \). For a resistivity \( \eta = 1.72 \times 10^{-8} \Omega \text{ m} \) the quality factor for this mode is \( Q_{cav} = 3.13 \times 10^4 \). The next two TM modes have the same order \( n = 5.75632 \), and resonant frequencies \( \nu = 2.05 \text{ GHz} \).
and $\nu = 2.76 \text{GHz}$, with quality factors $Q_{cav} = 3.11 \times 10^4$ and $Q_{cav} = 5.24 \times 10^4$, respectively.

For an average power $P = 1 \text{kW}$, the constant $C$ is evaluated for each mode using (24), and (26) and (27) used in (30) to obtain by numerical integration the values of $\chi'_{0}(r, \theta)$ needed in the numerical evaluation of (32).

Note that, from (24), the force on the cavity is proportional to the fed power, and to the quality factor $Q_{cav}$.

For the lowest TM mode ($\nu = 1.05 \text{GHz}$) the value obtained is $F_z = 7.7 \text{N}$, while for the next two TM modes, with $\nu = 2.05 \text{GHz}$ and $\nu = 2.76 \text{GHz}$, we obtained $F_z = -1.4 \text{N}$ and $F_z = -0.9 \text{N}$, respectively. The values reported in [1] are not easy to compare with as the power of the microwave source is distributed over a rather wide range of frequencies, so that the actual power into the resonant mode is not precisely defined. Using a spectrum analysis of the power source the authors evaluate, for instance, that when $F_z = -0.3 \text{N}$ the actual power into the resonant mode is $P = 0.12 \text{kW}$, which would correspond to $F_z = -2.5 \text{N}$ at $P = 1 \text{kW}$. The last two modes considered are closer to the reported value of the resonance, $\nu = 2.45 \text{GHz}$, and give theoretical results with the correct sign and similar magnitude. As according to the model the force is proportional to the thickness of the wall, depends also on the precise geometry of the cavity (neither of them reported in the literature), and as the value (29) is only an estimation, since the contribution from $\beta \partial_{EM}/\partial \psi|_{\phi_0, \psi_0}$ cannot be ascertained independently, the results seem consistent with the measured force being due to the studied effect.

Note that the lowest mode ($\nu = 1.05 \text{GHz}$) leads to a force much larger in magnitude and of opposite direction to that of the next two modes. This and other dependencies of the predicted force, as the proportionality to the cavity wall thickness (within certain limits as $\nabla \chi'_{0}$ decays rapidly outside the cavity), can be explored experimentally with relative ease to test the theory.

Finally, it is worth noting that the weak energy condition (WEC) [9] is violated for the cavity, as is the case in other models of propellant-less drive [3]. In effect, from (8), the WEC is written for the cavity

$$\left(\partial_{\mu\nu}\phi - \eta^{\gamma\delta}\partial_{\gamma\delta}\phi\eta_{\mu\nu}\right)U^\mu U^\nu \geq 0,$$

for any time-like four-vector $U^\mu$. By taking $U^\mu = (1, 0, 0, 0)$ one has the particular WEC $\nabla^2 \phi \geq 0$, which is seen from (16) and (25) to be violated at different times and regions inside the cavity.

6. Discussion

It was shown that the weak field approximation of a rather general scalar-tensor theory of gravity, which includes an additional scalar with strong coupling to the electromagnetic field, as proposed in [5], could account for the forces reported on asymmetric resonant
cavities. Although highly speculative, it is interesting that this was done using the same coupling coefficient adjusted by [5] to explain discordant measurements of Newton gravitational constant. It is also of interest that the inclusion of the external scalar $\psi$ can help to reconcile the Solar System tests with values of the Brans-Dicke parameter $\omega$ close to unity (see relation (15)). The weakest part of the theory seems to be that there is no clear way of preventing large gravitational effects due to the magnetic field of the Earth, as predicted by Eq. (17). A possible solution can be sought in non-linear effects, such as those due to the second terms in the left-hand sides of (11) and (16). In effect, their inclusion would modify (14) to

$$\Box \chi = (\Box \chi)_{\text{original}} + \frac{c^2}{2} \left( \omega_0' \partial^\nu \phi \partial_\nu \phi - \frac{1}{2} \partial^\nu \psi \partial_\nu \psi - \beta \partial^\nu \phi \partial_\nu \psi \right),$$

where $\omega_0' \equiv \left( \frac{d\omega}{d\phi} \right)_{\phi_0,\psi_0}$. For the stationary case of the Earth's magnetic field one would then have

$$\nabla^2 \chi = \frac{4\pi G_0}{c^2} T_{\text{mat}}^{\mu \nu} - \frac{4\pi G_0}{c^2} \left( \beta^2 - \frac{1}{2\omega_0 + 3} \right) T_{\text{mat}}^{\mu \nu}$$

$$- \frac{c^2 \psi_0}{2} \left( \frac{1}{2\omega_0 + 3} \frac{\partial J}{\partial \phi} - \beta \frac{\partial J}{\partial \psi} \right)_{\phi_0,\psi_0}$$

$$+ \frac{c^2}{2} \left( \omega_0' \nabla \phi \cdot \nabla \phi - \frac{1}{2} \nabla \psi \cdot \nabla \psi - \beta \nabla \phi \cdot \nabla \psi \right).$$

(34)

If the terms $\nabla \phi \cdot \nabla \phi$, $\nabla \psi \cdot \nabla \psi$ and $\nabla \phi \cdot \nabla \psi$ were to dominate over $\nabla^2 \phi$ and $\nabla^2 \psi$, respectively, the equations (11) and (16) would result in

$$\omega_0' \nabla \phi \cdot \nabla \phi - \frac{1}{2} \nabla \psi \cdot \nabla \psi \simeq - \frac{8\pi G_0}{c^4} T_{\text{mat}}^{\mu \nu} + \frac{\partial J}{\partial \phi} \bigg|_{\phi_0,\psi_0} \psi_0,$$

$$\nabla \phi \cdot \nabla \psi \simeq - \beta \frac{8\pi G_0}{c^4} T_{\text{mat}}^{\mu \nu} + \frac{\partial J}{\partial \psi} \bigg|_{\phi_0,\psi_0} \psi_0,$$

which clearly cancel the terms in (34) leading to large values of the force. Put more plainly, $\nabla^2 \phi$ and $\nabla^2 \psi$ are sources of the potential $\chi'$ and so situations where they are small or even zero would reduce the gravitational effect of the electromagnetic field. Note that in the case of a static magnetic field outside its sources one can write $B = \nabla \Psi$, with $\nabla^2 \Psi = 0$, so it is possible that equations like (11) and (16) for the static case

$$(2\omega_0 + 3) \nabla^2 \phi + \omega_0' \nabla \phi \cdot \nabla \phi - \frac{1}{2} \nabla \psi \cdot \nabla \psi \propto B^2 = \nabla \Psi \cdot \nabla \Psi,$$

$$\nabla^2 \psi + \nabla \phi \cdot \nabla \psi \propto B^2 = \nabla \Psi \cdot \nabla \Psi,$$

have the solutions $\nabla \phi \propto \nabla \psi \propto \nabla \Psi$, and $\nabla^2 \phi = \nabla^2 \psi = 0$, which is in general not possible for the case of the cavity, for which the constant part of $B^2 - E^2 / c^2$ cannot be written as $\nabla \Psi \cdot \nabla \Psi$.

If this is what happens in the case of the Earth’s magnetic field, it would seem to invalidate the derivations in [5], where the solution with $\nabla^2 \phi \neq 0$ was used. However, it can be shown that if $\omega_0' / (2\omega_0 + 3) \sim 1$, both solutions are numerically similar.

Note that if this is possible for a static magnetic field, it could possibly not be the case for a static electric field outside its sources, which also satisfies $E = \nabla \Psi$, with
\( \nabla^2 \Psi = 0 \), because the difference of sign would not allow real solutions. However, it can be readily shown that also the electrostatic case have solutions of the type considered if \( \omega'_0 > 0 \). It is so expected that static magnetic and electric fields show no unusual gravitational effects, while non-stationary electromagnetic fields do. Along these lines note finally that \( \partial^2 \phi / \partial t^2 \) is also a source of the potential \( \chi' \), which would contribute in transient situations.

**Appendix: Modified Maxwell equations**

The generalized Maxwell equations, Eq. (5), are

\[
\nabla_{\nu} \left[ \lambda (\phi) F^{\mu \nu} \right] = -\mu_0 j^\mu.
\]

Since

\[
\nabla_{\nu} \left[ \lambda (\phi) F^{\mu \nu} \right] = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\nu} \left[ \sqrt{-g} \lambda (\phi) F^{\mu \nu} \right] = \lambda (\phi) \left[ \frac{\partial F^{\mu \nu}}{\partial x^\nu} + F^{\mu \nu} \frac{\partial}{\partial x^\nu} \ln (\lambda \sqrt{-g}) \right],
\]

one has

\[
\frac{\partial F^{\mu \nu}}{\partial x^\nu} = -\frac{\mu_0}{\lambda (\phi)} j^\mu - F^{\mu \nu} \frac{\partial}{\partial x^\nu} \ln (\lambda \sqrt{-g}).
\]

Keeping up to first order in the excitations of the scalars one has

\[
\frac{\partial F^{\mu \nu}}{\partial x^\nu} = -\mu_0 \left[ 1 - \lambda'_0 (\phi - \phi_0) \right] j^\mu - F^{\mu \nu} \frac{\partial}{\partial x^\nu} \left( \lambda'_0 \phi - \hbar / 2 \right),
\]

with

\[
\lambda'_0 \equiv \left. \frac{d \lambda}{d \phi} \right|_{\phi_0}.
\]

The equations for \( \phi \) and \( \hbar \) are at the same order of approximation (with only electromagnetic sources)

\[
\Box \phi = -\frac{8 \pi G_0 \varepsilon_0}{(2 \omega_0 + 3) c^2} \psi_0 \frac{\partial \beta_{EM}}{\partial \phi} \bigg|_{\phi_0, \psi_0} \left( B^2 - E^2 / c^2 \right),
\]

\[
\Box \hbar = \eta^{\mu \nu} \Box h_{\mu \nu} = 6 \Box \phi,
\]

so that, if one defines \( \Theta \equiv \lambda'_0 (\phi - \hbar / 2) \), its equation is

\[
\Box \Theta = -\frac{8 \pi G_0 \varepsilon_0}{(2 \omega_0 + 3) c^2} \psi_0 \frac{\partial \beta_{EM}}{\partial \phi} \bigg|_{\phi_0, \psi_0} \left( B^2 - E^2 / c^2 \right),
\]

while the Maxwell equations can be written in usual vector form as (neglecting the order one correction \( \lambda'_0 (\phi - \phi_0) \) to the four-current \( j^\mu \))

\[
\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} - \nabla \Theta \cdot \mathbf{E},
\]

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{B} = 0,
\]

\[
\nabla \times \mathbf{B} = \mu_0 j + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c^2} \frac{\partial \Theta}{\partial t} \mathbf{E} - \nabla \Theta \times \mathbf{B}.
\]
These equations, together with Eq. (35), form a complete set of for the electromagnetic field.

It is interesting that in[5] the function $\lambda(\phi)$ obtained from reduction to four dimensions of the particular Kaluza-Klein theory employed is $\lambda(\phi) = \phi^3$, so that $\lambda_0' = 3$, which according to Eq. (35) results in $\Theta = 0$ (the value of $\Theta$ is assumed zero at spatial infinity). Up to first order the Maxwell equations are so the usual ones in this case. For different theories, however, $\lambda_0' \neq 3$, and the Maxwell equations can be modified. Note that even the case with no direct coupling, $\lambda(\phi) = 1$, leads to modified Maxwell equations, which comes directly form the modification of the metric (relative to Minkowski space) due to the scalar field.

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