Heegaard Floer homology and several families of Brieskorn spheres

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Abstract

In [OS03b], Ozsváth and Szabó gave a combinatorial description for the Heegaard Floer homology of boundaries of certain negative-definite plumbings. Némethi constructed a remarkable algorithm in [Ném05] for executing these computations for almost-rational plumbings, and his work in [Ném07] gives a formula computing the invariants for the Brieskorn homology spheres $-\Sigma(p,q,pqn+1)$. Here we give a formula for $HF^+(\Sigma(p,q,pqn-1))$, generalizing the one for the $n=1$ case given in [BN11]. We also compute $HF^+$ for the families $-\Sigma(2,5,k)$ and $-\Sigma(2,7,k)$.

1 Introduction

The Heegaard Floer homology package was first introduced by Ozsváth and Szabó [OS04], and has proven to be a useful collection of tools for the study of manifolds of dimensions 3 and 4. In particular, to a closed, oriented 3-manifold $Y$ one can associate a graded $\mathbb{Z}[U]$-module $HF^+(Y)$, which is the richest of the flavors (i.e. carries the most data). Combinatorial techniques and cut-and-paste techniques have since been developed to ease computation of various Heegaard Floer homologies (the methods in [LOT08], [Juh06], and [SW10] compute the graded $\mathbb{Z}$-module $\hat{HF}$ and those in [MOT09] compute $HF^+$).

However, Ozsváth and Szabó offered in [OS03b] a combinatorial description of $HF^+$ for a 3-manifold which bounds a negative-definite plumbing graph. Némethi provided a nice combinatorial framework for this description in [Ném05] via a very concrete algorithm: a plumbing graph leads to a function $\tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$, which in turn induces an intermediate object known as a “graded root”; this gadget both determines $HF^+$ and carries some extra data relevant to singularity theory. For a very approachable “user’s guide” to Némethi’s algorithm, see [AK12].

The purpose of the present article is to write down formulae for the invariants $HF^+$ associated to some Brieskorn homology spheres; these include $-\Sigma(p,q,pqn-1)$, where $p,q,n \in \mathbb{N}$ with $p,q$ coprime (Theorem 1) and $-\Sigma(2,j,k)$ where $j,k \in \mathbb{N}$, $k$ is odd and coprime and $j \in \{5,7\}$ (Theorem 2). These computations attempt to further illustrate the usefulness of Némethi’s formula in allowing one to compute $HF^+$ for such infinite families of Seifert manifolds.

Recall that

$$S^3_{1/n}(T_{p,q}) = -\Sigma(p,q,pqn-1) \quad \text{and} \quad S^3_{-1/n}(T_{p,q}) = \Sigma(p,q,pqn+1),$$

where $T_{p,q}$ denotes the right-handed $(p,q)$ torus knot and $S^3_r(K)$ denotes $r$-framed surgery on the knot $K \subset S^3$. 
Negative Dehn surgeries on algebraic knots were extensively studied in [Ném07], and Némethi writes down a closed formula for \(HF^+\left(-S^3_{-1/n}(K)\right)\) in § 5.6.2 of that paper. The torus knot \(T_{p,q}\) is indeed algebraic, and Némethi’s formula gives that when \(n \geq 0\),

\[
\begin{align*}
HF_{\text{even}}^+\left(-S^3_{-1/n}(T_{p,q})\right) &= T_0^+ (\alpha_{g-1})^{\oplus n} \oplus \bigoplus_{i=1}^{n(g-1)} T_{\left\lfloor \frac{i}{n} \right\rfloor +1}\left(\left\lfloor \frac{i}{n} \right\rfloor n +i \right) (\alpha_{g-1+\left\lfloor \frac{i}{n} \right\rfloor})^{\oplus 2} , \\
HF_{\text{odd}}^+\left(-S^3_{-1/n}(T_{p,q})\right) &= 0, \quad \text{and} \quad d\left(-S^3_{-1/n}(T_{p,q})\right) = 0
\end{align*}
\]

Note that \(\{x\} := x - \lfloor x \rfloor\); definitions of other objects and notations involved can be found [2].

The case of \(+1\)-surgery on \(T_{p,q}\) was studied by Borodzik and Némethi in [BN11], and a formula for \(HF^+\) was given there. Presently, we extend that computation to \(+1/n\)-surgery for \(n \geq 1\) to provide the following formula.

**Theorem 1.** Let \(p, q \geq 0\) be coprime integers, and let \(T_{p,q}\) denote the torus right-handed \((p, q)\) torus knot. Then

\[
\begin{align*}
HF_{\text{even}}^+\left(S^3_{1/n}(T_{p,q})\right) &= T_{-2\alpha_{g-1}} (\alpha_{g-1})^{\oplus (n-1)} \oplus \bigoplus_{i=1}^{n(g-1)} T_{\left\lfloor \frac{i}{n} \right\rfloor +1}\left(\left\lfloor \frac{i}{n} \right\rfloor n +i -1\right) -2\alpha_{g-1+\left\lfloor \frac{i}{n} \right\rfloor} (\alpha_{g-1+\left\lfloor \frac{i}{n} \right\rfloor})^{\oplus 2} \\
HF_{\text{odd}}^+\left(S^3_{1/n}(T_{p,q})\right) &= 0, \quad \text{and} \quad d\left(S^3_{1/n}(T_{p,q})\right) = -2\alpha_{g-1}
\end{align*}
\]

Némethi’s method doesn’t rely on the surgery presentation given above, and may be applied to many other infinite families of Seifert manifolds. For the sake of illustration, we compute \(HF^+\) for all Brieskorn homology spheres of the form \(-\Sigma(2, 5, k)\) or \(-\Sigma(2, 7, k)\). This computation is provided by the following (along with Equation [1] and Theorem [1]), which is proven in [4]. The main technical input comes from Lemma [13], which is stated and proved in § 5.

**Theorem 2.** Fix \(n \in \mathbb{N}\), and let \(M\) be any of the Brieskorn homology spheres \(-\Sigma(2, 5, 10n \pm 3), -\Sigma(2, 7, 14n \pm 3), \text{ or } -\Sigma(2, 7, 14n \pm 5)\). Then \(HF^+(M)\) is as characterized by Table [4].

Let \(p\) be a prime and \(K \subset S^3\) be a knot. Using a particular \(d\)-invariant for the \(p^n\)-fold cover of \(S^3\) branched along \(K\), Manolescu and Owens [MO07] (for \(p^n = 2\)) and Jabuka [Jab08] (for any prime \(p\) and any \(n \in \mathbb{Z}\)) define a concordance invariant \(\delta_{p^n}\mathbb{Z}\); see [2.7] for the definition of this invariant. Theorem [2] provides some new \(\delta_{p^n}\)-invariants for torus knots (although a few of the examples below appeared in [Jab08]). Recall from [Mil75] that when \(p, q, r \geq 0\) are pairwise coprime, in fact

\[-\Sigma(p, q, r) = \Sigma_p(T_{q,r}) = \Sigma_q(T_{p,r}) = \Sigma_r(T_{p,q}).\]

**Corollary 3.** For \(p, q \in \mathbb{N}\) coprime, let \(T_{p,q}\) denote the right-handed \((p, q)\)-torus knot.

(i) Let \(k = 10n \pm 3\), where \(n \in \mathbb{N}\). Then

\[
\delta_2(T_{5,k}) = \delta_5(T_{2,k}) = \begin{cases} 
4, & k = 10n + 3 \\
0, & k = 10n - 3
\end{cases}
\]

Moreover, whenever \(k\) is a prime power,

\[
\delta_k(T_{2,5}) = \begin{cases} 
-4, & k = 10n + 3 \quad (\text{e.g. } n = 1, 2, 4, 5, 7, \ldots) \\
0, & k = 10n - 3 \quad (\text{e.g. } n = 1, 2, 4, 5, 7, \ldots)
\end{cases}
\]

2
\(\begin{array}{|c|c|c|}
\hline
\text{manifold} & HF_{\text{red}}^+ \cong HF_{\text{even}}^+ & d \\
\hline
-\Sigma(2, 5, 10n - 1) & T_{-2}^+(1)^{\oplus(n-1)} \oplus \bigoplus_{i=0}^{n-1} T_{2i}^+(1)^{\oplus 2} & -2 \\
\hline
-\Sigma(2, 5, 10n + 1) & T_0^+(1)^{\oplus n} \oplus \bigoplus_{i=1}^{n} T_{2i}^+(1)^{\oplus 2} & 0 \\
\hline
-\Sigma(2, 5, 10n - 3) & T_0^+(1)^{\oplus(n-1)} \oplus \bigoplus_{i=0}^{n-1} T_{2i}^+(1)^{\oplus 2} & 0 \\
\hline
-\Sigma(2, 5, 10n + 3) & T_{-2}^+(1)^{\oplus n} \oplus \bigoplus_{i=0}^{n-1} T_{2i}^+(1)^{\oplus 2} & -2 \\
\hline
-\Sigma(2, 7, 14n - 1) & T_{-4}^+(2)^{\oplus(n-1)} \oplus \bigoplus_{i=0}^{n-1} T_{2i}^+(1)^{\oplus 2} \oplus \bigoplus_{i=0}^{n-1} T_{2n+4i}^+(1)^{\oplus 2} & -4 \\
\hline
-\Sigma(2, 7, 14n + 1) & T_0^+(2)^{\oplus(n)} \oplus \bigoplus_{i=1}^{n} T_{2i}^+(1)^{\oplus 2} \oplus \bigoplus_{i=1}^{n} T_{2n+4i}^+(1)^{\oplus 2} & 0 \\
\hline
-\Sigma(2, 7, 14n - 3) & T_{-2}^+(1)^{\oplus(2n-2)} \oplus \bigoplus_{i=0}^{n-1} T_{2i-2}^+(1)^{\oplus 2} \oplus \bigoplus_{i=0}^{n-1} T_{2n+4i-2}^+(1)^{\oplus 2} & -2 \\
\hline
-\Sigma(2, 7, 14n + 3) & T_0^+(1)^{\oplus(2n+1)} \oplus \bigoplus_{i=1}^{n} T_{2i}^+(1)^{\oplus 2} \oplus \bigoplus_{i=1}^{n} T_{2n+4i}^+(1)^{\oplus 2} & 0 \\
\hline
-\Sigma(2, 7, 14n - 5) & T_{-2}^+(1)^{\oplus(2n-3)} \oplus \bigoplus_{i=0}^{n-1} T_{2i-2}^+(1)^{\oplus 2} \oplus \bigoplus_{i=0}^{n-1} T_{2n+4i-2}^+(1)^{\oplus 2} & -2 \\
\hline
-\Sigma(2, 7, 14n + 5) & T_0^+(1)^{\oplus(2n+2)} \oplus \bigoplus_{i=1}^{n} T_{2i}^+(1)^{\oplus 2} \oplus \bigoplus_{i=1}^{n} T_{2n+4i}^+(1)^{\oplus 2} & 0 \\
\hline
\end{array}\)

Table 1: The structure of \(HF^+\) for lots of Brieskorn spheres, as provided by Theorems 1 and 2 and Bordzik and Némethi’s Equation 1.

(ii) Let \(k = 14n \pm 3\) or \(k = 14n \pm 5\), where \(n \in \mathbb{N}\). Then

\[
\delta_2(T_{7,k}) = \delta_7(T_{2,k}) = \begin{cases} 
-4, & k = 14n - 3 \text{ or } k = 14n - 5 \\
0, & k = 14n + 3 \text{ or } k = 14n + 5 
\end{cases}
\]

Moreover, whenever \(k\) is a prime power,

\[
\delta_k(T_{2,7}) = \begin{cases} 
-4, & k = 14n - 3 \text{ (e.g. } n = 1, 4, 5, 8, 10, \ldots) \\
0, & k = 14n + 3 \text{ (e.g. } n = 1, 2, 4, 5, 7, \ldots) \\
-4, & k = 14n + 5 \text{ (e.g. } n = 1, 3, 4, 6, 7, \ldots) \\
0, & k = 14n - 5 \text{ (e.g. } n = 2, 3, 6, 8, 11, \ldots) 
\end{cases}
\]

Remark 4. The families \(T_{5,k}\) and \(T_{7,k}\) provide several new infinite families of (non-alternating, of course) knots such that \(-\delta_2 \neq \sigma/2\) (c.f. [MO07]). For the reader’s convenience, we list those values here:
Conjecture 5. Let $p > 1$ be an odd integer and let $k$ be an integer with $\gcd(k, 2p) = 1$ and $k \neq \pm 1 \pmod{2p}$, then

$$d(-\Sigma(2, p, 2pn - k)) = \begin{cases} 0, & p \equiv 1 \pmod{4} \\ -2, & p \equiv 3 \pmod{4} \end{cases}$$

and

$$d(-\Sigma(2, p, 2pn + k)) = \begin{cases} 0, & p \equiv 3 \pmod{4} \\ -2, & p \equiv 1 \pmod{4} \end{cases}$$

Remark 6. Along with Theorem 1 and Equation 1, Conjecture 5 would determine the invariant $\delta_2$ for all torus knots $T_{p,q}$ with $p, q$ odd.

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2 Preliminaries

2.1 Seifert fibered integer homology spheres

Recall that the Seifert fibered space $\Sigma := \Sigma(e_0, (a_1, b_1), (a_2, b_2), \ldots, (a_m, b_m))$ bounds a plumbed negative-definite 4-manifold whose plumbing graph is star-shaped, consisting of $m$ “arms” emanating from a central vertex. The central vertex is labelled with weight $e_0$, and the $i$th arm is a chain of $n_i$ vertices with labels $-k_1, -k_2, \ldots, -k_{n_i}$ (ordered outward from the center), where

$$\frac{a_i}{b_i} = k_1 - \frac{1}{k_2 - \frac{1}{\ldots - \frac{1}{k_{n_i}}}}$$

Now define the quantities

$$e := e_0 + \sum_{i=1}^{m} \frac{b_i}{a_i} \quad \text{and} \quad \varepsilon := \frac{1}{e} \left( 2 - m + \sum_{i=1}^{m} \frac{1}{a_i} \right)$$

If we assume $e < 0$, then $\Sigma$ is an integer homology sphere if and only if $e = -1/(a_1 a_2 \ldots a_m)$, i.e.

$$-1 = e_0 a_1 a_2 \ldots a_m + \sum_{i=1}^{m} b_i \left( \frac{a_1 a_2 \ldots a_m}{a_i} \right)$$

This equation implies that the residue of $b_i$ modulo $a_i$ is determined by the $a_j$’s.
2.2 Torus knots

We review some notation related to the torus knot $T_{p,q}$ which appears in some formulae in the introduction. Let $S_{p,q} \subset \mathbb{Z}_{\geq 0}$ denote the semigroup

$$S_{p,q} := \{ ap + bq \mid (a, b) \in \mathbb{Z}_{\geq 0}^2 \}.$$ 

Now $\mathbb{Z}_{\geq 0} \setminus S_{p,q}$ is finite, and in fact

$$|\mathbb{Z}_{\geq 0} \setminus S_{p,q}| = \frac{(p-1)(q-1)}{2} = g_3(T_{p,q}) =: g, \text{ the 3-genus of } T_{p,q}.$$ 

For $i \geq 0$, we define a sequence of numbers $\alpha_i \in \mathbb{Z}_{\geq 0}$ via

$$\alpha_i := \# \{ s \in S_{p,q} \mid s > i \}.$$ 

One can verify that in fact $g = \alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_{2g-3} \geq \alpha_{2g-2} = 1$ and $\alpha_i = 0$ for $i > 2g - 2$.

2.3 Dedekind sums

Recall the “sawtooth function” $\langle \cdot \rangle : \mathbb{R} \to \mathbb{R}$, where

$$\langle x \rangle := \begin{cases} 0, & x \in \mathbb{Z} \\ x - \lfloor x \rfloor - \frac{1}{2}, & x \notin \mathbb{Z} \end{cases}$$

For $h, k \in \mathbb{Z} \setminus \{0\}$, one can define the classical Dedekind sum

$$s(h, k) := \sum_{i=1}^{k-1} \langle \frac{i}{k} \rangle \langle \frac{hi}{k} \rangle.$$ 

We’ll make use of a particular formula found in [Apo90] involving the Euclidean algorithm. Assume that $0 < h < k$ and that $r_0, r_1, \ldots, r_{n+1}$ are the remainders obtained when the Euclidean algorithm is applied to $h$ and $k$, i.e.

$$r_0 := k, \quad r_1 := h, \quad r_{j+1} \equiv r_{j-1} \pmod{r_j} \quad \text{(with } 1 \leq r_{j+1} < r_j) \text{, and } r_{n+1} = 1.$$ 

Then in fact the Dedekind sum can be computed via

$$s(h, k) = \frac{1}{12} \left( \sum_{j=1}^{n+1} (-1)^j \left( \frac{1 + r_j^2 + r_{j-1}^2}{r_j r_{j-1}} \right) \right) - \frac{1 + (-1)^n}{8} \quad (2)$$

2.4 Heegaard Floer homology

Let $Y$ be a rational homology 3-sphere, and fix $s \in \text{Spin}^c(Y)$. We study the $\mathbb{Q}$-graded Heegaard Floer homology groups $HF^+(Y, s)$, defined by Ozsváth and Szabó in [OS04]. Define the graded $\mathbb{Z}[U]$-modules

$$\mathcal{T}^+ := \frac{\mathbb{Z}[U, U^{-1}]}{U \cdot \mathbb{Z}[U]} \quad \text{and} \quad \mathcal{T}^+(n) := \frac{\mathbb{Z}(U^{-n+1}, U^{-n+2}, \ldots)}{U \cdot \mathbb{Z}[U]}, \quad \text{where } \deg(U^k) = -2k.$$
More generally, given a graded $\mathbb{Z}[U]$-module $M$ with $k$-homogeneous elements $M_k$ and some $d \in \mathbb{Q}$, let $M[d]$ denote the graded $\mathbb{Z}[U]$ module with $M[d]_{(k+d)} = M_k$. Then define the shifted modules $T_d^+ := T^+[d]$, $T_d^+(n) := T^+(n)[d]$. Recall that the $\mathbb{Q}$-graded Heegaard Floer groups decompose as

$$HF^+(Y,s) \cong T_{d(Y,s)}^+ \oplus HF_{\text{red}}^+(Y,s),$$

where the first summand is the image of the projection map $HF^\infty(Y,s) \to HF^+(Y,s)$ and second is its quotient. Note that the invariant $d(Y,s) \in \mathbb{Q}$ is the so-called \textbf{correction term} or \textbf{d-invariant} associated to the pair $(Y,s)$, first introduced in \cite{OS03}. Note that when $Y$ is an integer homology sphere, there is only one element in $\text{Spin}^c(Y)$; in this case, suppress the “$s$” and just write $d(Y)$.

Recall also that $HF^+$ is relatively $\mathbb{Z}$-graded and carries a well-defined absolute $\mathbb{Z}/2\mathbb{Z}$-grading. With respect to this grading, $HF_{\text{red}}^+$ further decomposes as

$$HF_{\text{red}}^+(Y) = HF_{\text{odd}}^+(Y) \oplus HF_{\text{even}}^+(Y).$$

\textbf{Remark 7.} Given $HF^+(-Y)$, it is straightforward to compute $HF^+(Y)$. Indeed, one should first use the long exact sequence

$$\ldots \to HF^-(-Y) \to HF^\infty(-Y) \to HF^+(-Y) \to \ldots$$

to recover $HF^-(-Y)$, and then use the fact that $HF^*_+(Y) \cong HF_{(-s-2)}^+(-Y)$.

\subsection{Némethi’s algorithm}

In \cite{Nem05}, Némethi describes a procedure for computing the Heegaard Floer homology for boundaries of negative-definite \textbf{almost-rational} plumbings. A plumbing is almost-rational if its graph contains \textit{at most one bad vertex}, i.e. a vertex $v$ with degree($v$) $> |\text{weight}(v)|$. Note that the star-shaped plumbing graph associated to a Seifert manifold can always be drawn such that no vertices are bad except possibly the central one. We’ll briefly describe the algorithm; see \cite{AK12} for a very concrete user’s guide.

Let $\Gamma$ be the plumbing graph and let $X(\Gamma)$ denote the associated plumbed 4-manifold. The algorithm uses $\Gamma$ to induce a \textbf{computation sequence} of vectors in $H_2(X(\Gamma))$, which in turn provides a \textbf{tau function} $\tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$. One then constructs a reduced version $\tilde{\tau}$ of the function by throwing out all repetition in the sequence ($\tau(i)$), keeping only the local extrema, and re-indexing the result. The function $\tilde{\tau}$ generates a $\mathbb{Z}$-graded infinite tree called a \textbf{graded root}; this tree has an obvious $\mathbb{Z}[U]$-action and recovers the module $HF^+(-\partial X(\Gamma))$.

\textbf{Remark 8.} Given a particular plumbing graph, one would use the reduced function $\tilde{\tau}$ to draw the graded root in practice; however, notice that Equation 3 indeed involves the full tau function $\tau$, and that’s the one commonly used in the arguments here.

\section{Some formulas of Bordzik and Némethi}

Let $\Sigma$ be a Seifert-fibered homology sphere (as \S 2.1). In \cite{BN11}, Bordzik and Némethi characterize the $d$-invariant and the tau function for $\Sigma$ in terms of its Seifert invariants $(\varepsilon_0, (a_1, b_1), \ldots, (a_m, b_m))$. Proposition 2.2 of \cite{BN11} states that for each $k \in \mathbb{Z}_{\geq 0}$,

$$\tau(k) = \sum_{j=0}^{k-1} \Delta_j, \text{ where } \Delta_j := 1 - j\varepsilon_0 - \sum_{i=1}^m \left\lfloor \frac{jb_i}{a_i} \right\rfloor$$

(3)
There is an alternate form for $\Delta_j$ which is sometimes more useful. For any $b \in \mathbb{Z}$ and $a_1, \ldots, a_m \in \mathbb{Z}_{>0}$, let

$$\varepsilon_a(b) := \sum_{i=1}^{m} \varepsilon_{a_i}(b), \quad \text{where} \quad \varepsilon_{a_i}(b) := \begin{cases} 1, & a_i | b \\ 0, & \text{else} \end{cases}$$

Then we have that

$$\Delta_j = 1 - \frac{m}{2} + \frac{j}{a_1 \ldots a_m} + \varepsilon_a(j) + \sum_{i=1}^{m} \frac{j b_i}{a_i}$$

The $d$-invariant is then given by

$$d(\Sigma) = \frac{1}{4} \left( \varepsilon^2 e + e + 5 - 12 \sum_{i=1}^{m} s(b_i, a_i) \right) - 2 \min_{k \geq 0} \tau(k)$$

### 2.7 The concordance invariant $\delta_{p^n}$

Let $K \subset S^3$ be a knot, let $p$ be a prime, and let $n \in \mathbb{N}$. Then let $\Sigma_{p^n}(K)$ denote the $p^n$-fold branched cover of $S^3$ branched along the knot $K$. $\Sigma_{p^n}(K)$ is a rational homology sphere, and we let $s_0 \in Spin^c(\Sigma_{p^n}(K))$ denote the element induced by the unique spin-structure. Manolescu and Owens [MO07] (for $p^n = 2$) and Jabuka [Jab08] (for general $p^n$) define

$$\delta_{p^n}(K) := 2d(\Sigma_{p^n}(K), s_0) \in \mathbb{Z}.$$  

This number is an invariant of the smooth knot concordance class of $K$, and in fact provides a homomorphism from the smooth knot concordance group to $\mathbb{Z}$. Corollary 3 provides some new computations of $\delta_{p^n}$ for some torus knots.

### 3 The Brieskorn spheres $\Sigma(p, q, pqn - 1)$

Let $p, q > 0$ be coprime integers. Recall that the Brieskorn homology sphere $\Sigma(p, q, pqn - 1)$ is a Seifert fibered space with Seifert invariants $(e_0, (p, p'), (q, q'), (r, r'))$, where $e_0 = -2$, $r = pqn - 1$, $r' = pqn - n - 1$, and $p', q'$ uniquely determined by the restrictions

$$0 < p' < p, \quad 0 < q' < q, \quad pq' \equiv 1 \pmod{q}, \quad \text{and} \quad qp' \equiv 1 \pmod{p}.$$  

**Remark 9.** When $p = 2$, the above constraints imply that $p' = 1$ and $q' = (q+1)/2$. The reader can verify that these parameters lead to the plumbing graph found in Figure 1b in §5. Note that we don’t need the graphs to compute the Heegaard Floer groups, but rather only the Seifert invariants.

Némethi gives a formula for the function $\tau$ for a Seifert manifold in [Ném05], and in [BN11] uses it to compute $HF^+$ for $+1$-surgery on $T_{p,q}$. In order to extend his result to $+1/n$-surgery ($n \in \mathbb{N}$), we first characterize the function $\tau$ for this case.

**Lemma 10.** Let $S_{p,q}$ denote the semigroup of $\mathbb{Z}_{\geq 0}$ generated by $p$ and $q$. Additionally, let $N := n(2g - 1)$. Then the following hold.

(i) The function $\tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ attains its local maxima (resp. minima) at the points $M_i$ (resp. $m_i$), where

$$M_i := pqi + 1 \ (\text{for } 0 \leq i \leq N - 2) \quad \text{and} \quad m_i := pqi - \lfloor i/n \rfloor \ (\text{for } 0 \leq i \leq N - 1).$$
(ii) For $0 \leq i \leq N - 2$,
\[
\tau(M_i) - \tau(m_i) = \# \left\{ s \in S_{p,q} \mid s \leq \left\lfloor \frac{i}{n} \right\rfloor \right\} > 0 \quad \text{and}
\]
\[
\tau(M_i) - \tau(m_{i+1}) = \# \left\{ s \notin S_{p,q} \mid s \geq \left\lfloor \frac{i+1}{n} \right\rfloor + 1 \right\} > 0.
\]

(iii) The sequence $(m_i)$ satisfies
\[
\tau(m_{i+1}) - \tau(m_i) \begin{cases} 
\leq 0 & \text{for } i \in \{0, \ldots, \frac{N-n}{2} - 1\} \\
= 0 & \text{for } i \in \left\{\frac{N-n}{2}, \ldots, \frac{N+n}{2} - 2\right\} \\
\geq 0 & \text{for } i \in \left\{\frac{N+n}{2} - 1, \ldots, N - 2\right\}
\end{cases}
\]

and thus $\tau$ achieves its global minimum value at the points $m_i$ with $i \in \left\{\frac{N-n}{2}, \ldots, \frac{N+n}{2} - 1\right\}$.

Proof of (i). For each $i \in \mathbb{Z}_{\geq 0}$, define the numbers $M_i$ and $m_i$ via the expressions stated in the lemma (we'll show that $\tau$ attains its local extrema at some of these points, i.e. the ones with indices restricted as in the lemma). To this end, fix $j \in \mathbb{Z}_{\geq 0}$ and compute $\triangle_j := \tau(j+1) - \tau(j)$. We'll first assume that $M_i \leq j < m_i + 1$, and employ an analysis similar to that in [BN11]. In particular, recall that for any integer $s \in [0, pq)$,
\[
s \in S_{p,q} \iff s = \alpha p + \beta q \quad \text{for some } 0 \leq \alpha < q, 0 \leq \beta < p
\]
and
\[
s \notin S_{p,q} \iff s + pq = \alpha p + \beta q \quad \text{for some } 0 \leq \alpha < q, 0 \leq \beta < p.
\]

Following [BN11], one can use this fact along with Equation 4 to show that
\[
\triangle_j = \begin{cases} 
\frac{jn}{pqn-1} - i & \text{when } (i+1)pq - j \in S_{p,q} \\
\frac{jn}{pqn-1} - i - 1 & \text{when } (i+1)pq - j \notin S_{p,q}
\end{cases}
\]

First assume that $0 \leq i \leq N - 2$. In this case, whenever $(i+1)pq - j \in S_{p,q}$, one finds that
\[
\frac{jn}{pqn-1} - i = \begin{cases} 
0 & \text{when } M_i \leq j < m_{i+1} \\
1 & \text{when } m_{i+1} \leq j < M_{i+1}
\end{cases}
\]

On the other hand, whenever $(i+1)pq - j \notin S_{p,q}$,
\[
\frac{jn}{pqn-1} - i - 1 = \begin{cases} 
-1 & \text{when } M_i \leq j < m_{i+1} \\
0 & \text{when } m_{i+1} \leq j < M_{i+1}
\end{cases}
\]

Now when $i \geq N - 1$, we find that $\triangle_j \geq 0$ regardless of whether $(i+1)pq - j \in S_{p,q}$.

Proof of (ii). Equations [6] and [7] follow from equations [8] and [9] bearing in mind that
\[
m_i \leq j \leq M_i \iff pq - 1 \leq (i+1)pq - j \leq pq + \left\lfloor \frac{i}{n} \right\rfloor \quad \text{and}
\]
\[
M_i \leq j \leq m_{i+1} \iff \left\lfloor \frac{i+1}{n} \right\rfloor + 1 \leq (i+1)pq - j \leq pq - 1
\]

Both quantities are strictly positive because $2g - 1 \notin S_{p,q}$ and $0 \in S_{p,q}$.
Proof of (iii). Observe that \( k \in S_{p,q} \iff 2g - 1 - k \notin S_{p,q}. \) Along with equations 6 and 7, this implies that
\[
\tau(m_{i+1}) - \tau(m_i) = \# \{ k \notin S_{p,q} \mid k \geq 2g - 1 - \left\lfloor \frac{i}{n} \right\rfloor \} - \# \{ k \notin S_{p,q} \mid k \geq \left\lfloor \frac{i+1}{n} \right\rfloor + 1 \},
\]
and the result follows.

In fact, the graded root determined by the function \( \tau \) is highly symmetric; the following makes this more precise.

**Lemma 11.** Let \( nk \leq i < nk + n \) for some \( 0 \leq k \leq g - 2 \).

(i) “Branch lengths” are symmetric, i.e.
\[
\tau(M_{n(g-1)-1-i}) - \tau(m_{n(g-1)-1-i}) = \tau(M_{ng+i-1}) - \tau(m_{ng+i}) = \alpha_{g+k}
\]

(ii) “Leaf heights” are symmetric, i.e.
\[
2\tau(m_{n(g-1)-1-i}) = 2\tau(m_{ng+i}) = (k + 1)(2i - nk) - 2\alpha_{g+k} + C(n,g),
\]
where \( C(n,g) = g(n - ng + 2) \).

Moreover, there is a “bunch” of \( n \) leaves at the bottom level, i.e. for \( n(g-1) \leq i \leq ng - 2 \),
\[
\tau(M_i) - \tau(m_i) = \alpha_{g-1} \quad \text{and} \quad 2\tau(m_i) = -2\alpha_{g-1} + C(n,g).
\]

**Proof of (i).** Follows from equation 6.

**Proof of (ii).** We have that
\[
\tau(M_0) = 1 \quad \text{and} \quad \tau(M_{i+1}) - \tau(M_i) = \left\lfloor \frac{i+1}{n} \right\rfloor + 1 - g \quad \text{for} \quad 0 \leq i \leq N - 3,
\]
and so
\[
\tau(M_i) = 1 + \sum_{m=1}^{i} \left( \left\lfloor \frac{m}{n} \right\rfloor + 1 - g \right) \quad \text{for} \quad 1 \leq i \leq N - 2.
\]

The statement follows from direct computations using equation 10.

**Proof of Theorem 7.** For \( 0 \leq i \leq n(g - 1) \), let \( G_i \) be given by
\[
G_i := ([i/n]) \left( \{ (i-1)/n \} n + i - 1 \right) - 2\alpha_{g-1+[i/n]} + C(n,g).
\]

Notice that lemmas 10 and 11 give us enough information about the function \( \tau \) to conclude that for some constant shift \( S \),
\[
HF^+_{p,q} \left( S^3_{1/n} (T_{p,q}) \right) = T^+_{G_0+S} (\alpha_{g-1}) \oplus (n-1) \oplus \bigoplus_{i=1}^{n(g-1)} T^+_{G_i+S} (\alpha_{g-1+[i/n]}) \oplus 2
\]
and \( d \left( S^3_{1/n} (T_{p,q}) \right) = -2\alpha_{g-1} + C(n,g) + S. \)
We claim that in fact $S + C(n, g) = 0$, which would finish the proof. Recall that Moser showed in \cite{Mos71} that $S^3_{pq-1}(T_{p,q})$ is a lens space. In \cite{OS03}, Oszváth and Szabó computed $d$-invariants for surgeries on lens space knots (knots in $S^3$ for which there exist positive integer surgeries yielding lens spaces); in particular, for $n > 0$, the $d$-invariant of $1/n$ surgery on a lens space knot is independent of $n$. Along with Equation 1, this implies that

$$d \left( S^3_1 (T_{p,q}) \right) = d \left( S^3_1 (T_{p,q}) \right) = -2\alpha_{g-1}. \quad \Box$$

**Remark 12.** One could alternately use Equations 2 and 5 to compute the above $d$-invariant.

## 4 The Brieskorn spheres $\Sigma(2, 5, k)$ and $\Sigma(2, 7, k)$

We’ll compute $HF^+$ for these manifolds using the formulae in \cite{OS03, 2.1} and the only inputs we’ll need are the Seifert invariants (though the interested reader can see Figures 2a-2f in \cite{2.6} for associated plumbing graphs). The discussion in \cite{2.1} tells us that we can write

$$\Sigma(2, 5, 10n - 3) = \Sigma(-1, (2, 1), (5, 1), (10n - 3, 3n - 1))$$
$$\Sigma(2, 5, 10n + 3) = \Sigma(-2, (2, 1), (5, 4), (10n + 3, 7n + 2))$$
$$\Sigma(2, 7, 14n - 5) = \Sigma(-2, (2, 1), (7, 5), (14n - 5, 11n - 4))$$
$$\Sigma(2, 7, 14n - 3) = \Sigma(-2, (2, 1), (7, 6), (14n - 3, 9n - 2))$$
$$\Sigma(2, 7, 14n + 3) = \Sigma(-1, (2, 1), (7, 1), (14n + 3, 5n + 1))$$
$$\Sigma(2, 7, 14n + 5) = \Sigma(-1, (2, 1), (7, 2), (14n + 5, 3n + 1))$$

The structure of $HF^+$ can be read off from the tau function. We state and prove Lemma 13 in \cite{5} which characterize $\tau$ for $\Sigma(2, 5, k)$ and $\Sigma(2, 7, k)$. With those results in mind, we have the necessary ingredients for proving Theorem 2.

**Proof of Theorem 2.** We prove the statement for $-\Sigma(2, 7, 14n + 3)$, and leave the proofs for the other five families as exercises.

First notice in Table 2 that the local extrema of the tau function occur symmetrically, with

$$\tau \left( m_{6n+1-i} \right) = \tau \left( m_i \right) = \begin{cases} -2i, & i \in [0, n] \\ -n - i, & i \in [n + 1, 2n] \end{cases}$$
and

$$\tau \left( M_{6n-i} \right) = \tau \left( M_i \right) = \begin{cases} 1 - 2i, & i \in [0, n] \\ 1 - n - i, & i \in [n + 1, 2n] \end{cases}$$

Lemma 13, along with this observation, implies that for some shift $S \in \mathbb{Z},$

$$HF^+_{red} \left( -\Sigma(2, 7, 14n + 3) \right) = T^+_{-6n-1,1} \oplus \bigoplus_{i=1}^{n} T^+_{-6n+2i+1} \oplus \bigoplus_{i=1}^{n} T^+_{-4n+2i+1} \oplus \bigoplus_{i=1}^{n} T^+_{-4n+4i+1} \oplus \bigoplus_{i=1}^{n} T^+_{-4n+4i+1}$$
and

$$d \left( -\Sigma(2, 7, 14n + 3) \right) = -6n + S.$$
Unfortunately, these manifolds aren’t surgeries on torus knots. Fortunately, it is straightforward to compute the $d$-invariants directly via Equation 5. Applying the Euclidean algorithm to $k = 14n + 3$ and $h = 5n + 1$, one obtains the remainder sequences

$$(14n+3, 5n+1, 4n+1, n, n-1, 1)$$

for $n > 2$,  

$$(31, 11, 9, 2, 1)$$

for $n = 2$,  and  

$$(17, 6, 5, 1)$$

for $n = 1$.

With these (and Maple) in hand, we find that

$$d (-\Sigma(2, 7, 14n + 3)) = -\left(\frac{1}{4}(-24n) - 2(-3n)\right) = 0$$

and so $S = 6n$. 

\[\square\]
5 Appendix

The following provides the structure of the tau functions for the Brieskorn spheres in Theorem 2.

**Lemma 13.** Fix \( n \in \mathbb{N} \). For the tau function of \( \Sigma(2, 5, 10n \pm 3) \) (resp. \( \Sigma(2, 7, 14n \pm 3) \), resp. \( \Sigma(2, 7, 14n \pm 5) \)), the following hold.

(i) The function \( \tau : \mathbb{Z}_{\geq 0} \to \mathbb{Z} \) attains its local maxima at the points \( M_i \) and local minima at the points \( m_i \), where these sequences are defined in Table 2 (resp 3, resp 4).

(ii) Changes in \( \tau \) between consecutive extrema and the values of the function at these extrema are as given in Table 2 (resp 3, resp 4).

| manifold | \( \Sigma(2, 5, 10n - 3) \) | \( \Sigma(2, 5, 10n + 3) \) |
|----------|--------------------------|--------------------------|
| \( M_i \) | \( 10i + 1, \quad i \in [0, 3n - 2] \) | \( 10i + 1, \quad i \in [0, 2n - 1] \\ 10i + 7, \quad i \in [2n, 3n - 1] \) |
| \( m_i \) | \( \begin{cases} 0, & i = 0 \\ 10i - 2, & i \in [1, n - 1]^\dagger \\ 10i - 8, & i \in [n, 3n - 1] \end{cases} \) | \( 10i, \quad i \in [0, 3n] \) |
| \( \tau(M_i) - \tau(m_i) \) | \( \begin{cases} 1, & i \in [0, 2n - 1] \\ 2, & i \in [2n, 3n - 2]^\dagger \end{cases} \) | \( \begin{cases} 1, & i \in [0, 2n - 1] \\ 2, & i \in [2n, 3n - 1] \end{cases} \) |
| \( \tau(M_i) - \tau(m_{i+1}) \) | \( \begin{cases} 2, & i \in [0, n - 2]^\dagger \\ 1, & i \in [n - 1, 3n - 2] \end{cases} \) | \( \begin{cases} 2, & i \in [0, n - 1] \\ 1, & i \in [n, 3n - 1] \end{cases} \) |
| \( \tau(M_i) \) | \( \begin{cases} 1 - i, & i \in [0, n - 1] \\ 2 - n, & i \in [n, 2n - 2]^\dagger \\ 3 - 3n + i, & i \in [2n - 1, 3n - 2] \end{cases} \) | \( \begin{cases} 1 - i, & i \in [0, n - 1] \\ 1 - n, & i \in [n, 2n - 1] \\ 2 - 3n + i, & i \in [2n, 3n - 1] \end{cases} \) |
| \( \tau(m_i) \) | \( \begin{cases} -i, & i \in [0, n - 1] \\ 2 - n, & i \in [n, 2n - 1] \\ 3 - 3n + i, & i \in [2n, 3n - 1] \end{cases} \) | \( \begin{cases} -i, & i \in [0, n - 1] \\ -n, & i \in [n, 2n] \\ -3n + i, & i \in [2n + 1, 3n] \end{cases} \) |

Table 2: Features of the tau functions for the manifolds \( \Sigma(2, 5, 10n \pm 3) \). Cases marked with “\( ^\dagger \)” only appear when \( n > 1 \).
| manifold | $\Sigma(2, 7, 14n - 3)$ | $\Sigma(2, 7, 14n + 3)$ |
|----------|----------------------|----------------------|
| $M_i$    | 14$i + 1$, $i \in [0, 2n - 1]$ | 14$i + 1$, $i \in [0, 2n]$ |
|          | $14(n + \frac{i}{2}) - 5$, $i \in [2n, 4n - 2]$, $i$ even | $14(n + \frac{i}{2}) + 1$, $i \in [2n + 1, 4n]$ |
|          | $14(n + \frac{i-1}{2}) + 1$, $i \in [2n, 4n - 2]$, $i$ odd | $14(n + \frac{i-1}{2}) + 7$, $i \in [2n + 1, 4n]$, $i$ odd |
|          | $14(i - n) + 9$, $i \in [4n - 1, 6n - 3]$ | $14(i - n) + 1$, $i \in [4n + 1, 6n]$ |
| $m_i$    | 14$i$, $i \in [0, 2n - 1]$ | 14$i$, $i \in [0, 2n]$ |
|          | $14(n + \frac{i}{2}) - 8$, $i \in [2n, 4n - 1]$, $i$ even | $14(n + \frac{i}{2}) + 6$, $i \in [2n + 1, 4n + 1]$, $i$ even |
|          | $14(n + \frac{i-1}{2})$, $i \in [2n, 4n - 1]$, $i$ odd | $14(n + \frac{i-1}{2}) + 6$, $i \in [2n + 1, 4n + 1]$, $i$ odd |
|          | $14(i - n)$, $i \in [4n, 6n - 2]$ | $14(i - n) - 8$, $i \in [4n + 2, 6n + 1]$ |
| $\tau(M_i) - \tau(m_i)$ | 1, $i \in [0, 4n - 2]$ | 1, $i \in [0, 4n]$ |
|          | 2, $i \in [4n - 1, 5n - 2]$ | 2, $i \in [4n + 1, 5n]$ |
|          | 3, $i \in [5n - 1, 6n - 3]$ | 3, $i \in [5n + 1, 6n]$ |
| $\tau(M_i) - \tau(m_{i+1})$ | 3, $i \in [0, n - 2]$ | 3, $i \in [0, n - 1]$ |
|          | 2, $i \in [n - 1, 2n - 2]$ | 2, $i \in [n, 2n - 1]$ |
|          | 1, $i \in [2n - 1, 6n - 3]$ | 1, $i \in [2n, 6n]$ |
| $\tau(M_i)$ | 1 - 2$i$, $i \in [0, n - 1]$ | 1 - 2$i$, $i \in [0, n]$ |
|          | 2 - $n - i$, $i \in [n, 2n - 1]$ | 1 - $n - i$, $i \in [n + 1, 2n - 1]$ |
|          | 3 - 3$n$, $i \in [2n, 4n - 3]$ | 1 - 3$n$, $i \in [2n, 4n]$ |
|          | 5 - 7$n + i$, $i \in [4n - 2, 5n - 2]$ | 1 - 7$n + i$, $i \in [4n + 1, 5n]$ |
|          | 7 - 12$n + 2i$, $i \in [5n - 1, 6n - 3]$ | 1 - 12$n + 2i$, $i \in [5n + 1, 6n]$ |
| $\tau(m_i)$ | $-2i$, $i \in [0, n - 1]$ | $-2i$, $i \in [0, n]$ |
|          | 1 - $n - i$, $i \in [n, 2n - 1]$ | $-n - i$, $i \in [n + 1, 2n - 1]$ |
|          | 2 - 3$n$, $i \in [2n, 4n - 2]$ | $-3n$, $i \in [2n, 4n + 1]$ |
|          | 3 - 7$n + i$, $i \in [4n - 1, 5n - 2]$ | $-1 - 7$n + i$, $i \in [4n + 2, 5n]$ |
|          | 4 - 12$n + 2i$, $i \in [5n - 1, 6n - 2]$ | $-2 - 12$n + 2i$, $i \in [5n + 1, 6n + 1]$ |

Table 3: Features of the tau functions for the manifolds $\Sigma(2, 7, 14n \pm 3)$. Cases marked with “†” only appear when $n > 1$.  

| manifold | $\Sigma(2, 7, 14n - 5)$ | $\Sigma(2, 7, 14n + 5)$ |
|----------|-----------------|-----------------|
| $M_i$    |                  |                  |
| $14i + 1$, $i \in [0, 2n - 1]$ | $14i + 1$, $i \in [0, 2n]$ |
| $14(n + \frac{i}{2}) - 9$, $i \in [2n, 4n - 3], i$ even$^\dagger$ | $14(n + \frac{i}{2}) + 1$, $i \in [2n + 1, 4n + 1], i$ even$^\dagger$ |
| $14(n + \frac{i-1}{2}) + 1$, $i \in [2n, 4n - 3], i$ odd$^\dagger$ | $14(n + \frac{i-1}{2}) + 11$, $i \in [2n + 1, 4n + 1], i$ odd$^\dagger$ |
| $14(i - n) + 15$, $i \in [4n - 2, 6n - 4]$ | $14(i - n) + 1$, $i \in [4n + 2, 6n + 1]$ |
| $m_i$    |                  |                  |
| $14i$, $i \in [0, 2n - 1]$ | $14i$, $i \in [0, 2n]$ |
| $14(n + \frac{i}{2}) - 10$, $i \in [2n, 4n - 2], i$ even | $14(n + \frac{i}{2}) + 10$, $i \in [2n + 1, 4n + 2], i$ even |
| $14(n + \frac{i-1}{2})$, $i \in [2n, 4n - 2], i$ odd | $14(n + \frac{i-1}{2}) + 10$, $i \in [2n + 1, 4n + 2], i$ odd |
| $14(i - n) + 4$, $i \in [4n - 1, 6n - 3]$ | $14(i - n) - 8$, $i \in [4n + 3, 6n + 2]$ |
| $\tau(M_i) - \tau(m_i)$ |                  |                  |
| $1, i \in [0, 4n - 3]$ | $1, i \in [0, 4n + 1]$ |
| $2, i \in [4n - 2, 5n - 3]$ | $2, i \in [4n + 2, 5n + 1]$ |
| $3, i \in [5n - 2, 6n - 4]^\dagger$ | $3, i \in [5n + 2, 6n + 1]$ |
| $\tau(M_i) - \tau(m_{i+1})$ |                  |                  |
| $3, i \in [0, n - 2]^\dagger$ | $3, i \in [0, n - 1]$ |
| $2, i \in [n - 1, 2n - 2]$ | $2, i \in [n, 2n - 1]$ |
| $1, i \in [2n - 1, 6n - 4]$ | $1, i \in [2n, 6n + 1]$ |
| $\tau(M_i)$ |                  |                  |
| $1 - 2i$, $i \in [0, n - 1]$ | $1 - 2i$, $i \in [0, n]$ |
| $2 - n - i$, $i \in [n, 2n - 1]$ | $1 - n - i$, $i \in [n + 1, 2n - 1]$ |
| $3 - 3n$, $i \in [2n, 4n - 4]^\dagger$ | $1 - 3n$, $i \in [2n, 4n + 1]$ |
| $6 - 7n + i$, $i \in [4n - 3, 5n - 3]$ | $-7n + i$, $i \in [4n + 2, 5n + 1]$ |
| $9 - 12n + 2i$, $i \in [5n - 2, 6n - 4]^\dagger$ | $-1 - 12n + 2i$, $i \in [5n + 2, 6n + 1]$ |
| $\tau(m_i)$ |                  |                  |
| $-2i$, $i \in [0, n - 1]$ | $-2i$, $i \in [0, n]$ |
| $1 - n - i$, $i \in [n, 2n - 1]^\dagger$ | $-n - i$, $i \in [n + 1, 2n - 1]^\dagger$ |
| $2 - 3n$, $i \in [2n, 4n - 3]^\dagger$ | $-3n$, $i \in [2n, 4n + 2]$ |
| $4 - 7n + i$, $i \in [4n - 2, 5n - 2]$ | $-2 - 7n + i$, $i \in [4n + 3, 5n + 2]$ |
| $6 - 12n + 2i$, $i \in [5n - 1, 6n - 3]$ | $-4 - 12n + 2i$, $i \in [5n + 3, 6n + 2]$ |

Table 4: Features of the tau functions for the manifolds $\Sigma(2, 7, 14n \pm 5)$. Cases marked with “$^\dagger$” only appear when $n > 1$. 

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Proof of (i). We give the proof of the Lemma for \( \Sigma(2, 7, 14n + 3) \) and leave the arguments for the other five families as exercises (as they work analogously).

As in the proof of Lemma 10, first define the sequences \( M_i \) and \( m_i \) via the expressions given in Table 2 (we also define \( M_{6n+1} := 70n + 15 \), although this won’t end up being the location of a local extremum). We’ll then compute \( \Delta_j = \tau(j + 1) - \tau(j) \) in each of four cases:

Case 1: Let \( M_i \leq j < M_{i+1} \), where \( 0 \leq i \leq 2n - 1 \). Now we can write \( j = 14i + 1 + k \), where \( 0 \leq k \leq 13 \). Equation 3 indicates that

\[
\Delta_j = 1 + j - \left( \left\lfloor \frac{j}{2} \right\rfloor + \left\lfloor \frac{j}{7} \right\rfloor + \left\lfloor \frac{j(5n + 1)}{14n + 3} \right\rfloor \right) = k + 2 - \left( \left\lfloor \frac{k + 1}{2} \right\rfloor + \left\lfloor \frac{k + 1}{7} \right\rfloor + \left\lfloor \frac{(k + 1)(5n + 1) - i}{14n + 3} \right\rfloor \right)
\]

Clearly \( \Delta_0 = 1 \). Notice that

\[
\left( \left\lfloor \frac{k + 1}{2} \right\rfloor \right)_{k=0}^{13} = (1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7),
\]

\[
\left( \left\lfloor \frac{k + 1}{7} \right\rfloor \right)_{k=0}^{13} = (1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2).
\]

Now in this case

\[(k + 1)(5n + 1) + 1 - 2n \leq (k + 1)(5n + 1) - i \leq (k + 1)(5n + 1),\]

and one can directly show that

\[
\left( \left\lfloor \frac{(k + 1)(5n + 1) - i}{14n + 3} \right\rfloor \right)_{k=0}^{13} = (1, 1, s, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 5), \text{ where } s := \begin{cases} 2, & i \in [0, n - 1] \\ 1, & i \in [n, 2n - 1] \end{cases}
\]

In light of this, Equation 4 gives that

\[
\left( \Delta_j \right)_{j=M_i}^{M_{i+1}-1} = (-1, 0, t, 0, 0, 0, 0, -1, 0, 0, 0, 0, 1) \text{ where } t := \begin{cases} -1, & i \in [0, n - 1] \\ 0, & i \in [n, 2n - 1] \end{cases}
\]

Case 2: Let \( M_i \leq j < M_{i+1} \), where \( 4n \leq i \leq 6n + 1 \). Now we write \( j = 14(i - n) + 1 + k \), where \( 0 \leq k \leq 13 \) and

\[(k + 1)(5n + 1) - 5n \leq (k + 1)(5n + 1) - (i - n) \leq (k + 1)(5n + 1) - 3n.\]

Via an analysis similar to that in the previous case, one obtains that

\[
\left( \Delta_j \right)_{j=M_i}^{M_{i+1}-1} = (-1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, t, 0, 1) \text{ where } t := \begin{cases} 0, & i \in [4n, 5n - 1] \\ 1, & i \in [5n, 6n] \end{cases}
\]

Case 3: Let \( M_i \leq j < M_{i+1} \), where \( 2n + 1 \leq i \leq 4n - 1 \) and \( i \) is odd. Now \( M_{i+1} - M_i = 8 \), so we write \( j = 14(n + \frac{i-1}{2}) + 7 + k \) with \( 0 \leq k \leq 7 \). One then finds that

\[
\left( \Delta_j \right)_{j=M_i}^{M_{i+1}-1} = (-1, 0, t, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0, 0, 0, 0, 1) \text{ where } t := \begin{cases} -1, & i \in [0, n - 1] \\ 0, & i \in [n, 2n - 1] \end{cases}
\]
Case 4: Let $M_i \leq j < M_{i+1}$, where $2n + 1 \leq i \leq 4n - 1$ and $i$ is even. Now $M_{i+1} - M_i = 6$, so we write $j = 14(n + \frac{i}{2}) + 1 + k$ with $0 \leq k \leq 5$. One then finds that
\[
\left( \Delta_j \right)_{j=M_i}^{M_{i+1}-1} = (-1, 0, 0, 0, 0, 1)
\]  
Equations 12-15 imply that $\tau$ is indeed (non-strictly) decreasing on $[M_i, m_{i+1}]$ and (non-strictly) increasing on $[m_i, M_i]$ for all $i \in [0, 6n]$. It’s not hard to show that if $j \geq m_{6n+1}$ that $\Delta_j \geq 0$.

Proof of (ii). Keeping in mind the expressions for the $M_i$ and $m_i$, one can use Equations 12-15 to obtain $\tau(M_i) - \tau(m_i)$ and $\tau(M_i) - \tau(m_{i+1})$; these in turn give the expressions for $\tau(M_i)$ and $\tau(m_i)$.

Figure 1: Plumbing graphs for the Brieskorn homology spheres $\Sigma(2, q, 2qn \pm 1)$; unlabelled vertices have weight -2. For $\Sigma(2, q, 2qn - 1)$, the graph shown is valid for $n > 1$; when $n = 1$, the rightmost arm only has the first $2q - 2$ vertices.
Figure 2: Plumbing graphs for the Brieskorn homology spheres $\Sigma(2,5,k)$ and $\Sigma(2,7,k)$. For $\Sigma(2,5,10n-3)$, $\Sigma(2,7,14n-3)$, and $\Sigma(2,7,14n-5)$, the graph shown is valid for $n > 1$ only; when $n = 1$, both the “$(n-2)$-tail” and the next vertex inward are missing from the rightmost arm.

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