ON DIFFUSION PHENOMENA FOR THE LINEAR WAVE EQUATION WITH SPACE-DEPENDENT DAMPING

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Abstract. In this paper, we prove the diffusion phenomenon for the linear wave equation with space-dependent damping. We prove that the asymptotic profile of the solution is given by a solution of the corresponding heat equation in the $L^2$-sense.

1. Introduction

In this paper, we consider the asymptotic behavior of solutions to the wave equation with space-dependent damping:

$$\begin{cases}
  u_{tt} - \Delta u + a(x)u_t = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\
  (u, u_t)(0, x) = (u_0, u_1)(x), & x \in \mathbb{R}^n.
\end{cases}$$

Here $u = u(t, x)$ is real-valued unknown function and $a(x) = \langle x \rangle^{-\alpha} := (1 + |x|^2)^{-\alpha/2}$ with $0 \leq \alpha < 1$. For simplicity, we assume that

$$\begin{cases}
  (u_0, u_1) \in C_0^\infty(\mathbb{R}^n), \\
  \text{supp} (u_0, u_1) \subset \{x \in \mathbb{R}^n \mid |x| \leq L\}
\end{cases}$$

with some $L > 0$. We also consider the initial value problem of the corresponding heat equation started at the time $\tau \geq 0$:

$$\begin{cases}
  a(x)v_t - \Delta v = 0, & (t, x) \in (\tau, \infty) \times \mathbb{R}^n, \\
  v(\tau, x) = v_\tau(x), & x \in \mathbb{R}^n
\end{cases}$$

with initial data $v_\tau(x) \in C_0^\infty(\mathbb{R}^n)$.

When $a(x) = 1$, there are many literature on the asymptotic behavior of solutions to (1.1). In this case, it is well known that the asymptotic profile of the solution of (1.1) is given by the solution of (1.3) with the initial data $v_0 = u_0 + u_1$ in several senses (see [6, 7, 14, 15, 16, 26] and see also [1, 8, 21] for abstract setting). On the other hand, Wirth [24] considered the wave equation with time-dependent damping

$$u_{tt} - \Delta u + b(t)u_t = 0.$$

He proved that if the damping is effective, that is, roughly speaking, $tb(t) \to +\infty$ as $t \to +\infty$ and $b(t)^{-1} \notin L^1((0, \infty))$, then the solution is asymptotically equivalent to that of the corresponding heat equation

$$b(t)v_t - \Delta v = 0$$

(see also [25] for abstract setting). We also mention that Ikehata, Todorova and Yordanov [11] recently proved the diffusion phenomenon for strongly damped wave equations.

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Recently, Nishiyama [20] proved the diffusion phenomenon for abstract damped wave equations. His result includes space-dependent damping which does not decay near infinity. Due to the authors knowledge, there are no results on the asymptotic profile of solutions for decaying potential cases as (1.1). The difficulty is that we cannot use the Fourier transform for (1.1) as the previous results.

We also refer the reader to [2] [3] [4] [5] [9] [12] [17] [19] for the asymptotic profile for semilinear problems.

Todorova and Yordanov [23] obtained the following $L^2$-estimates for (1.1) and (1.3):

\[(1.4) \quad \|u(t, \cdot)\|_{L^2} \leq C(1 + t)^{-\frac{n-2\alpha}{2(2-\alpha)}} + \varepsilon, \]

\[(1.5) \quad \|v(t, \cdot)\|_{L^2} \leq C(1 + t)^{-\frac{n-2\alpha}{2(2-\alpha)}} + \varepsilon \]

with arbitrary small $\varepsilon > 0$ (see also [10] for the case $\alpha = 1$). It seems that the decay rate $\frac{n-2\alpha}{2(2-\alpha)}$ is optimal. Because, the function

$$G(t, x) = t^{-\frac{n-\alpha}{2-\alpha}} e^{-\frac{1}{2(2-\alpha)^2}t}$$

formally satisfies the equation

$$|x|^{-\alpha} v_t - \Delta v = 0$$

and

$$\|G(t, \cdot)\|_{L^2} = C t^{-\frac{n-2\alpha}{2(2-\alpha)}}$$

with some constant $C > 0$. Indeed,

$$\|G(t, \cdot)\|_{L^2}^2 = t^{-\frac{2(n-\alpha)}{2-\alpha}} \int_{\mathbb{R}^n} e^{-\frac{2|x|^2}{(2-\alpha)^2}t} dx$$

$$= Ct^{-\frac{2(n-\alpha)}{2-\alpha}} \int_0^\infty e^{-\frac{2|x|^2}{(2-\alpha)^2}r} r^{n-1} dr$$

$$= Ct^{-\frac{2(n-\alpha)}{2-\alpha}} \frac{n-1}{2-\alpha} \int_0^\infty e^{-\frac{2|x|^2}{(2-\alpha)^2}r} \left(\frac{r}{t}\right)^{n-1} dr \left(\frac{1}{t^{1/(2-\alpha)}}\right)$$

$$= Ct^{-\frac{n-2\alpha}{2-\alpha}}.$$ 

We denote the solution operator of (1.3) by $E(t - \tau)$, that is, $v(x, t) = E(t - \tau) v_\tau(x)$ gives the solution of (1.3). It is known that $E(t - \tau)$ is a 0-th order pseudodifferential operator having the symbol

$$e(t - \tau, x, \xi) = e^{-\frac{|\xi|^2}{a(t-\tau)}} + r_0(t - \tau, x, \xi)$$

with a remainder term $r_0$ (see Kumano-go [13]). Our main result is the following.

**Theorem 1.1.** Let $n \geq 1$ and let $u$ be a solution of (1.1) with initial data $(u_0, u_1)$ satisfying (1.2). Then we have

\[(1.6) \quad \left\|u(t, \cdot) - E(t) \left[u_0 + \frac{1}{a(t)} u_1\right](\cdot)\right\|_{L^2} = o(t^{-\frac{n-2\alpha}{2(2-\alpha)}})\]

as $t \to +\infty$.

**Remark 1.1.** Our proof needs the compactness of the support of the data. However, this assumption may be removed by using the energy concentration lemma (see Lemma 1.4); but we do not pursue that here.
The crucial point of the proof of Theorem 1 is the following weighted energy estimates for higher order derivatives of solutions to (1.1). Let

\[
\psi(t, x) = A \langle x \rangle^{2-\alpha} \frac{1}{1 + t}, \quad A := \frac{1}{(2 - \alpha)^2(2 + \delta)}
\]

with small \(\delta > 0\). We also put

\[
I_0 = \int_{\mathbb{R}^n} e^{2\psi(0, x)} (u_0(x))^2 + |\nabla u_0(x)|^2 + |u_1(x)|^2 \, dx,
\]

\[
I_1 = \int_{\mathbb{R}^n} e^{2\psi(0, x)} (u_{tt}(0, x))^2 + |\nabla u_{tt}(0, x)|^2 \, dx + I_0,
\]

\[
I_2 = \int_{\mathbb{R}^n} e^{2\psi(0, x)} (u_{ttt}(0, x))^2 + |\nabla u_{ttt}(0, x)|^2 \, dx + I_1
\]

and by inductively

\[
I_k = \int_{\mathbb{R}^n} e^{2\psi(0, x)} (|\partial_t^{k+1} u(0, x)|^2 + |\nabla \partial_t^k u(0, x)|^2) \, dx + I_{k-1}.
\]

Then, we can obtain weighted energy estimates for any order of derivatives:

**Theorem 1.2** (weighted energy estimates for higher order derivatives). For any small \(\varepsilon > 0\), there is some \(\delta > 0\) such that the following estimates hold: For any integer \(k \geq 0\), there exists some constant \(C > 0\) such that for a solution \(u\) of (1.1) with initial data satisfying (1.2), we have

\[
(1 + t)^{\frac{n-\alpha}{2} + 2k - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(t, x)} a(x)|\partial_t^k u(t, x)|^2 \, dx \leq CI_k,
\]

\[
(1 + t)^{\frac{n-\alpha}{2} + 2k + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(t, x)} |\nabla \partial_t^k u(t, x)|^2 \, dx \leq CI_k.
\]

In particular, we use the following estimates for the proof of Theorem 1.1

**Lemma 1.3.** For any small \(\varepsilon > 0\), there are some constants \(\delta > 0\) and \(C > 0\) such that the following estimates hold:

(i) For a solution \(u\) of (1.1), we have

\[
(1 + t)^{\frac{n-\alpha}{2} + 2 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(t, x)} a(x)u(t, x)^2 \, dx \leq CI_1,
\]

\[
(1 + t)^{\frac{n-\alpha}{2} + 3 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(t, x)} |\nabla u(t, x)|^2 \, dx \leq CI_1,
\]

\[
(1 + t)^{\frac{n-\alpha}{2} + 4 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(t, x)} |\nabla u_{tt}(t, x)|^2 \, dx \leq CI_2,
\]

\[
(1 + t)^{\frac{n-\alpha}{2} + 6 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(t, x)} a(x)u_{ttt}(t, x)^2 \, dx \leq CI_3.
\]
(ii) For a solution $v$ of (1.3), we have

$$\int_{\mathbb{R}^n} a(x) |v(t, x)|^2 dx \leq \int_{\mathbb{R}^n} a(x) |v_{\tau}(x)|^2 dx,$$

(1.14)

$$\leq \int_{\mathbb{R}^n} a(x)^{-1} |\nabla v_{\tau}(x)|^2 dx$$

$$+ C(1 + \tau)^{-\frac{n-\alpha}{2} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau, x)} |\nabla v_{\tau}(x)|^2 dx$$

$$+ C(1 + \tau)^{-\frac{n-\alpha}{2} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau, x)} a(x) v_{\tau}(x)^2 dx.$$  

(1.15)

We also use an energy concentration lemma.

**Lemma 1.4** (exponential decay outside parabolic regions). For any small $\varepsilon > 0$, there is some $\delta > 0$ such that the following holds: let

$$0 < \rho < 1 - \alpha, \quad 0 < \mu < 2\alpha$$

and

$$\Omega_\rho(t) := \{ x \in \mathbb{R}^n \mid \langle x \rangle^{2-\alpha} \geq (1 + t)^{1+\rho} \}.$$  

We also assume that $v$ is a solution of (1.3) with $\tau = 0$. Then we have

$$\int_{\Omega_\rho(t)} v(t, x)^2 dx \leq C(1 + t)^{\frac{n-\alpha}{2} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(0, x)} a(x)v_0(x)^2 dx,$$

(1.16)

where $C > 0$ is a constant depending on $\delta, \rho$ and $\mu$.

This paper is organized as follows. In the next section, we introduce a basic weighted energy method. This method was originally developed by Todorova and Yordanov [22] and refined by themselves [23] and Nishihara [18] to fit for the space-dependent damping. In Section 3, we prove Lemmas 1.3 and 1.4 and Theorem 1.2 by using the basic weighted energy estimates obtained in Section 2. In the final section, we give a proof of the main theorem.

Finally, we explain some notation used in this paper. First, we note that the letter $C$ indicates the generic constant, which may change from line to line. We also use the symbols $\lesssim$ and $\sim$. The relation $f \lesssim g$ means $f \leq Cg$ with some constant $C > 0$ and $f \sim g$ means $f \lesssim g$ and $g \lesssim f$. The bracket $\langle x \rangle := \sqrt{1 + |x|^2}$. We denote the usual $L^2$-norm by $\| \cdot \|_{L^2}$, that is,

$$\|f(\cdot)\|_{L^2} := \left( \int_{\mathbb{R}^n} |f(x)|^2 dx \right)^{1/2}.$$  

**2. Basic weighted energy estimates**

In this section, we give an estimate of a weighted $L^2$-norm

$$\int_{\mathbb{R}^n} e^{2\psi(t, x)} a(x) u(t, x)^2 dx$$

and a weighted energy

$$\int_{\mathbb{R}^n} e^{2\psi(t, x)} (u(t, x)^2 + |\nabla u(t, x)|^2) dx.$$
The following estimate was essentially already obtained by Nishihara [18]. However, for the sake of completeness, we shall give a proof in this section.

**Proposition 2.1** (Basic weighted energy estimates). For any small $\varepsilon > 0$, there is some $\delta > 0$ having the following property: let $u$ be a solution of (1.1) with initial data $(u_0, u_1)$ satisfying (1.2). Then we have

\[
(1 + t)^{\frac{n - \alpha}{2} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(t,x)}(u_t(t,x))^2 + |\nabla u(t,x)|^2)\, dx
+ (1 + t)^{\frac{n - \alpha}{2} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(t,x)}a(x)u(t,x)^2\, dx
+ \int_0^t \left\{ (1 + \tau)^{\frac{n - \alpha}{2} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)}(u_t(\tau,x))^2 + |\nabla u(\tau,x)|^2)(\tau,x)dx
+ (1 + \tau)^{\frac{n - \alpha}{2} - 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)}a(x)u(\tau,x)^2\, dx
+ (1 + \tau)^{\frac{n - \alpha}{2} - 1 + \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)}a(x)u_t(\tau,x)^2\, dx \right\} d\tau
\leq C\varepsilon_0.
\]

**Proof.** Form (1.7), it is easy to see that

\[
-\psi_t = \frac{1}{1 + t} \psi,
\]

\[
\nabla \psi = A \frac{(2 - \alpha)(x)^{-\alpha}x}{1 + t},
\]

\[
\Delta \psi = A(2 - \alpha)(n - \alpha)\frac{(x)^{-\alpha}}{1 + t} + A(2 - \alpha)\alpha \frac{(x)^{-2 - \alpha}}{1 + t}
\geq \frac{n - \alpha}{(2 - \alpha)(2 + \delta)} \frac{a(x)}{1 + t}
\]

\[
= \left( \frac{n - \alpha}{2(2 - \alpha)} - \delta_1 \right) \frac{a(x)}{1 + t}.
\]

Here and after, $\delta_i (i = 1, 2, \ldots)$ denote positive constants depending only on $\delta$ such that

\[
\delta_i \to 0^+ \quad \text{as} \quad \delta \to 0^+.
\]

We also have

\[
(-\psi_t)a(x) = A\frac{(x)^{2 - 2\alpha}}{(1 + t)^2}
\geq \frac{1}{(2 - \alpha)^2} A^2(2 - \alpha)^2 \frac{(x)^{-2\alpha}|x|^2}{(1 + t)^2}
= (2 + \delta)|\nabla \psi|^2.
\]
By multiplying (1.1) by $e^{2\psi}u_t$, it follows that

\begin{equation}
\frac{\partial}{\partial t} \left[ \frac{e^{2\psi}}{2} (u_t^2 + |\nabla u|^2) \right] - \nabla \cdot (e^{2\psi} u_t \nabla u) + e^{2\psi} \left( a(x) - \frac{|\nabla \psi|^2}{-\psi_t} - \psi_t \right) u_t^2 + \frac{e^{2\psi}}{-\psi_t} |\psi_t \nabla u - u_t \nabla \psi|^2 = 0.
\end{equation}

Using the Schwarz inequality and (2.4), we can calculate

\[
T_1 = \frac{e^{2\psi}}{-\psi_t} (\psi_t^2 |\nabla u|^2 - 2\psi_t u_t \nabla u \cdot \nabla \psi + u_t^2 |\nabla \psi|^2) \\
\geq \frac{e^{2\psi}}{-\psi_t} \left( \frac{1}{5} \psi_t^2 |\nabla u|^2 - \frac{1}{4} u_t^2 |\nabla \psi|^2 \right) \\
\geq e^{2\psi} \left( \frac{1}{5} (-\psi_t) |\nabla u|^2 - \frac{a(x)}{4(2+\delta)} u_t^2 \right).
\]

This inequality and (2.4) lead

\begin{equation}
\frac{\partial}{\partial t} \left[ \frac{e^{2\psi}}{2} (u_t^2 + |\nabla u|^2) \right] - \nabla \cdot (e^{2\psi} u_t \nabla u) + e^{2\psi} \left\{ \frac{1}{4} a(x) - \psi_t \right\} u_t^2 + \frac{e^{2\psi}}{5} |\nabla u|^2 \leq 0.
\end{equation}

Integrating over $\mathbb{R}^n$, we obtain

\begin{equation}
\frac{d}{dt} \int_{\mathbb{R}^n} e^{2\psi} \left( u_t^2 + |\nabla u|^2 \right) dx + \int_{\mathbb{R}^n} e^{2\psi} \left\{ \frac{1}{4} a(x) - \psi_t \right\} u_t^2 + \frac{e^{2\psi}}{5} |\nabla u|^2 \right) dx \leq 0.
\end{equation}

On the other hand, we multiply (1.1) by $e^{2\psi} u$ and have

\begin{equation}
\frac{\partial}{\partial t} \left[ e^{2\psi} \left( uu_t + \frac{a(x)}{2} u^2 \right) \right] - \nabla \cdot (e^{2\psi} u \nabla u) + e^{2\psi} \left\{ |\nabla u|^2 + (-\psi_t) a(x) u^2 + 2u \nabla \psi \cdot \nabla u - 2\psi_t uu_t + u_t^2 \right\} = 0.
\end{equation}

We can rewrite the term $e^{2\psi} T_2$ as

\[
e^{2\psi} T_2 = 4e^{2\psi} u \nabla \psi \cdot \nabla u - 2e^{2\psi} u \nabla \psi \cdot \nabla \psi - (e^{2\psi} u^2 \nabla \psi) + 2e^{2\psi} u^2 |\nabla \psi|^2 + e^{2\psi} (\Delta \psi) u^2.
\]

By substituting this and using (2.3), it follows from (2.8) that

\begin{equation}
\frac{\partial}{\partial t} \left[ e^{2\psi} \left( uu_t + \frac{a(x)}{2} u^2 \right) \right] - \nabla \cdot (e^{2\psi} (u \nabla u + u^2 \nabla \psi)) + e^{2\psi} \left\{ |\nabla u|^2 + 4u \nabla u \cdot \nabla \psi + ((-\psi_t) a(x) + 2|\nabla \psi|^2) u^2 \right\} \right) \\
+ \left( \frac{n - \alpha}{2} - 2\delta_1 \right) \frac{a(x)}{2(1+t)} u_t^2 - 2\psi_t uu_t - u_t^2 \right\} \leq 0.
\end{equation}
By the Schwarz inequality, we can estimate the term $T_3$ as

$$T_3 = |\nabla u|^2 + 4u \nabla u \cdot \nabla \psi$$

$$+ \left\{ \left( 1 - \frac{\delta}{3} \right) (-\psi_t)a(x) + 2|\nabla \psi|^2 \right\} u^2 + \frac{\delta}{3} (-\psi_t)a(x)u^2$$

$$\geq |\nabla u|^2 + 4u \nabla u \cdot \nabla \psi$$

$$+ \left( 4 + \frac{\delta}{3} - \frac{\delta^2}{3} \right) |\nabla \psi|^2 u^2 + \frac{\delta}{3} (-\psi_t)a(x)u^2$$

$$= \left( 1 - \frac{4}{4 + \delta_2} \right) |\nabla u|^2 + \delta_2 |\nabla \psi|^2 u^2$$

$$\geq \delta_3 (|\nabla u|^2 + |\nabla \psi|^2 u^2) + \frac{\delta}{3} (-\psi_t)a(x)u^2.$$

Substituting this into (2.9), we obtain

$$\frac{\partial}{\partial t} \left[ e^{2\psi} \left( uu_t + \frac{a(x)}{2} u^2 \right) \right] - \nabla \cdot \left( e^{2\psi} (u \nabla u + u^2 \nabla \psi) \right)$$

$$+ e^{2\psi} \delta_3 |\nabla u|^2$$

$$+ e^{2\psi} \left( \delta_3 |\nabla \psi|^2 + \frac{\delta}{3} (-\psi_t)a(x) + \left( \frac{n - \alpha}{2 - \alpha} - 2\delta_1 \right) \frac{a(x)}{2(1 + t)} \right) u^2$$

$$+ e^{2\psi} (-2\psi_t uu_t - u_t^2)$$

$$\leq 0.$$
By (2.11) + ν (2.10), we have

\[
\begin{align*}
(2.12) \frac{d}{dt} \left[ \int_{\mathbb{R}^n} e^{2\psi} \left\{ \frac{(t_0 + t)^\alpha}{2} (u_t^2 + |\nabla u|^2) + \nu \left( uu_t + \frac{a(x)}{2} u^2 \right) \right\} dx \right] \\
+ \int_{\mathbb{R}^n} e^{2\psi} \left( \frac{1}{4} - \nu - \frac{\alpha}{2} (t_0 + t)^{\alpha-1} + (t_0 + t)^{\alpha} (-\psi_t) \right) u_t^2 dx \\
+ \int_{\mathbb{R}^n} e^{2\psi} \left( \nu \delta_3 - \frac{\alpha}{2} (t_0 + t)^{\alpha-1} + (t_0 + t)^{\alpha} \frac{-\psi_t}{5} \right) |\nabla u|^2 dx \\
+ \nu \int_{\mathbb{R}^n} e^{2\psi} \left( \delta_3 |\nabla \psi|^2 + \frac{\delta}{3} (-\psi_t) a(x) + \left( \frac{n-\alpha}{2-\alpha} - 2\delta_1 \right) \frac{a(x)}{2(1+t)} \right) u^2 dx \\
+ \nu \int_{\mathbb{R}^n} e^{2\psi} 2(-\psi_t) uu_t dx \\
\leq 0.
\end{align*}
\]

By the Schwarz inequality, the last term of the left hand side of the above inequality can be estimated as

\[
|2(-\psi_t)uu_t| \leq \frac{\delta}{3} (-\psi_t)(t_0 + t)^{-\alpha} u^2 + \frac{3}{\delta} (-\psi_t)(t_0 + t)^{\alpha} u_t^2 \\
\leq \frac{\delta}{3} (-\psi_t) a(x) u^2 + \frac{3}{\delta} (-\psi_t)(t_0 + t)^{\alpha} u_t^2.
\]

Therefore, it follows from (2.12) that

\[
(2.13) \frac{d}{dt} \left[ \int_{\mathbb{R}^n} e^{2\psi} \left\{ \frac{(t_0 + t)^\alpha}{2} (u_t^2 + |\nabla u|^2) + \nu \left( uu_t + \frac{a(x)}{2} u^2 \right) \right\} dx \right] \\
+ \int_{\mathbb{R}^n} e^{2\psi} \left( \frac{1}{4} - \nu - \frac{\alpha}{2} (t_0 + t)^{\alpha-1} + \left( 1 - \frac{3\nu}{\delta} \right)(t_0 + t)^{\alpha} (-\psi_t) \right) u_t^2 dx \\
+ \int_{\mathbb{R}^n} e^{2\psi} \left( \nu \delta_3 - \frac{\alpha}{2} (t_0 + t)^{\alpha-1} + (t_0 + t)^{\alpha} \frac{-\psi_t}{5} \right) |\nabla u|^2 dx \\
+ \nu \int_{\mathbb{R}^n} e^{2\psi} \left( \delta_3 |\nabla \psi|^2 + \left( \frac{n-\alpha}{2-\alpha} - 2\delta_1 \right) \frac{a(x)}{2(1+t)} \right) u^2 dx \\
\leq 0.
\]

Now we choose the parameters ν and t₀ such that

\[
\frac{1}{4} - \nu - \frac{\alpha}{2} (t_0 + t)^{\alpha-1} \geq c_0, \quad 1 - \frac{3\nu}{\delta} \geq c_0, \\
\nu \delta_3 - \frac{\alpha}{2} (t_0 + t)^{\alpha-1} \geq c_0, \quad \frac{1}{5} \geq c_0
\]

hold for some constant c₀ > 0. This is possible because we first determine ν sufficiently small depending on δ and then we choose t₀ sufficiently large depending
on $\nu$. Consequently, we obtain

$$(2.14) \quad \frac{d}{dt} \int_{\mathbb{R}^n} e^{2\psi} \left\{ \frac{(t_0 + t)^\alpha}{2} (u_t^2 + |\nabla u|^2) + \nu \left( uu_t + \frac{a(x)}{2} u^2 \right) \right\} \, dx$$

$$+ c_0 \int_{\mathbb{R}^n} e^{2\psi} (1 + (t_0 + t)^\alpha (-\psi_t)) (u_t^2 + |\nabla u|^2) \, dx$$

$$+ \nu \int_{\mathbb{R}^n} e^{2\psi} \left( \delta_3 |\nabla \psi|^2 + \left( \frac{n - \alpha}{2 - \alpha} - 2\delta_1 \right) \frac{a(x)}{2(1 + t)} \right) u^2 \, dx \leq 0.$$  

We put

$$\tilde{E}_1(t) = \int_{\mathbb{R}^n} e^{2\psi} \left\{ \frac{(t_0 + t)^\alpha}{2} (u_t^2 + |\nabla u|^2) + \nu \left( uu_t + \frac{a(x)}{2} u^2 \right) \right\} \, dx,$$

$$E_{1,\psi}(t) = \int_{\mathbb{R}^n} e^{2\psi} (1 + (t_0 + t)^\alpha (-\psi_t)) (u_t^2 + |\nabla u|^2) \, dx$$

$$\tilde{H}_1(t) = \nu \int_{\mathbb{R}^n} e^{2\psi} \left( \delta_3 |\nabla \psi|^2 + \left( \frac{n - \alpha}{2 - \alpha} - 2\delta_1 \right) \frac{a(x)}{2(1 + t)} \right) u^2 \, dx.$$

Then we can rewrite (2.14) as

$$(2.15) \quad \frac{d}{dt} \tilde{E}_1(t) + c_0 E_{1,\psi}(t) + \tilde{H}_1(t) \leq 0.$$  

Take arbitrary $\varepsilon > 0$ and we determine $\delta$ so that $\varepsilon = 3\delta_1$. By multiplying (2.15) by $(t_0 + t)^\frac{\alpha}{2 - \alpha} - \varepsilon$, we have

$$\frac{d}{dt} \left[ (t_0 + t)^\frac{\alpha}{2 - \alpha} - \varepsilon \tilde{E}_1(t) \right] - \left( \frac{n - \alpha}{2 - \alpha} - \varepsilon \right) (t_0 + t)^\frac{\alpha}{2 - \alpha} \tilde{E}_1(t)$$

$$+ c_0 (t_0 + t)^\frac{\alpha}{2 - \alpha} - \varepsilon E_{1,\psi}(t) + (t_0 + t)^\frac{\alpha}{2 - \alpha} - \varepsilon \tilde{H}_1(t) \leq 0.$$  

Since

$$|\nu uu_t| \leq \frac{\nu \delta_4}{2} a(x) u^2 + \frac{\nu}{2\delta_4} (t_0 + t)^\alpha u_t^2,$$

we estimate

$$\tilde{E}_1(t) \leq \int_{\mathbb{R}^n} e^{2\psi} \left( 1 + \frac{\nu}{\delta_4} \right) \frac{(t_0 + t)^\alpha}{2} u_t^2 + \frac{(t_0 + t)^\alpha}{2} |\nabla u|^2 + \nu (1 + \delta_4) \frac{a(x)}{2} u^2 \right) \, dx.$$  

Choosing $\delta_4$ sufficiently small and then $t_0$ sufficiently large so that

$$\left( \frac{n - \alpha}{2 - \alpha} - 2\delta_1 \right) - \left( \frac{n - \alpha}{2 - \alpha} - 3\delta_1 \right) (1 + \delta_4) \geq c_1,$$

$$c_0 - \frac{1}{2} \left( 1 + \frac{\nu}{\delta_4} \right) (t_0 + t)^{\alpha - 1} \geq c_1$$

with some $c_1 > 0$, we have

$$\frac{d}{dt} \left[ (t_0 + t)^\frac{\alpha}{2 - \alpha} - \varepsilon \tilde{E}_1(t) \right]$$

$$+ c_1 (t_0 + t)^\frac{\alpha}{2 - \alpha} - \varepsilon E_{1,\psi}(t) + (t_0 + t)^\frac{\alpha}{2 - \alpha} - \varepsilon \tilde{H}_1(t) \leq 0,$$
where
\[
H_1(t) = \nu \int_{\mathbb{R}^n} e^{2\psi} \left( \delta_3 |\nabla \psi|^2 + c_1 \frac{a(x)}{2(1+t)} \right) u^2 \, dx.
\]
Integrating over \([0,t]\), one can obtain
\[
(t_0 + t)^{\frac{n-\alpha}{2} - \epsilon} \tilde{E}_1(t) + \int_0^t (t_0 + \tau)^{\frac{n-\alpha}{2} - \epsilon} (c_1 E_{1,\psi}(\tau) + H_1(\tau)) d\tau \leq \tilde{E}_1(0).
\]
We also put
\[
E_1(t) := \int_{\mathbb{R}^n} e^{2\psi} \{ (t_0 + t)^{\alpha} (u_t^2 + |\nabla u|^2) + a(x) u^2 \} \, dx.
\]
Then it is easy to see that \( \tilde{E}_1(t) \sim E_1(t) \) and \( E_1(0) \lesssim I_0 \). From this, we have
\[
(2.16) \quad (t_0 + t)^{\frac{n-\alpha}{2} - \epsilon} E_1(t) + \int_0^t (t_0 + \tau)^{\frac{n-\alpha}{2} - \epsilon} (E_{1,\psi}(\tau) + H_1(\tau)) d\tau \leq C I_0.
\]
To reach the conclusion of the proposition, we multiply \((2.7)\) by \((t_0 + t)^{\frac{n-\alpha}{2} + 1 - \epsilon}\) and obtain
\[
\frac{d}{dt} \left[ (t_0 + t)^{\frac{n-\alpha}{2} + 1 - \epsilon} \int_{\mathbb{R}^n} \frac{e^{2\psi}}{2} (u_t^2 + |\nabla u|^2) \, dx \right] - \left( \frac{n-\alpha}{2 - \alpha} + 1 - \epsilon \right) (t_0 + t)^{\frac{n-\alpha}{2} - \epsilon} \int_{\mathbb{R}^n} \frac{e^{2\psi}}{2} (u_t^2 + |\nabla u|^2) \, dx
+ (t_0 + t)^{\frac{n-\alpha}{2} + 1 - \epsilon} \int_{\mathbb{R}^n} e^{2\psi} \left\{ \left( \frac{1}{4} a(x) - \psi_t \right) u_t^2 + \frac{-\psi}{5} |\nabla u|^2 \right\} \, dx \leq 0.
\]
By integrating over \([0,t]\), it holds that
\[
(2.17) \quad (t_0 + t)^{\frac{n-\alpha}{2} + 1 - \epsilon} \int_{\mathbb{R}^n} \frac{e^{2\psi}}{2} (u_t^2 + |\nabla u|^2) \, dx
- \left( \frac{n-\alpha}{2 - \alpha} + 1 - \epsilon \right) \int_0^t (t_0 + \tau)^{\frac{n-\alpha}{2} - \epsilon} \int_{\mathbb{R}^n} \frac{e^{2\psi}}{2} (u_t^2 + |\nabla u|^2) \, dx d\tau
+ \int_0^t (t_0 + \tau)^{\frac{n-\alpha}{2} + 1 - \epsilon} \int_{\mathbb{R}^n} e^{2\psi} \left\{ \left( \frac{1}{4} a(x) - \psi_t \right) u_t^2 + \frac{-\psi}{5} |\nabla u|^2 \right\} \, dx d\tau \leq C I_0.
\]
Taking \((2.16) + \eta \, (2.17)\) with small parameter \(\eta > 0\) satisfying
\[
1 - \frac{\eta}{2} \left( \frac{n-\alpha}{2 - \alpha} + 1 - \epsilon \right) > 0,
\]
we can see that
\[
(t_0 + t)^{\frac{n-2}{2-\alpha} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(t,x)}(u_t(t,x))^2 + |\nabla u(t,x)|^2 dx
+ (t_0 + t)^{\frac{n-2}{2-\alpha} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(t,x)}a(x)u(t,x)^2 dx
+ \int_0^t \left\{ (t_0 + \tau)^{\frac{n-2}{2-\alpha} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)}(u_t^2 + |\nabla u|^2)(\tau,x) d\tau dx
+ (t_0 + \tau)^{\frac{n-2}{2-\alpha} - 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)}\left(-\psi_t(\tau,x)\right)(u_t^2 + |\nabla u|^2)(\tau,x) d\tau dx
+ (t_0 + \tau)^{\frac{n-2}{2-\alpha} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)}a(x)u(\tau,x)^2 d\tau dx \right\} d\tau
\leq CI_0.
\]
Finally, we note that \((t_0 + t) \sim (1 + t)\) and obtain the conclusion. \(\square\)

3. Weighted energy estimates for higher order derivatives

In this section, we give a proof of Lemmas 1.3 and 1.4 and Theorem 1.2. We first prove (1.10). Differentiating (1.1) with respect to \(u\), we obtain

\[
(3.1) \quad u_{tt} - \Delta u_t + a(x)u_{tt} = 0.
\]
We apply the weighted energy method again. First, by Proposition 2.1 we have

\[
(3.2) \quad \int_0^t (t_0 + \tau)^{\frac{n-2}{2-\alpha} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)}a(x)u_t(\tau,x)^2 dx d\tau \leq CI_0.
\]
Multiplying (3.1) by \(e^{2\varphi}u_{tt}\) and \(e^{2\varphi}u_t\), and the same argument as the derivation of (2.13), we can obtain

\[
(3.3) \quad \frac{d}{dt} E_2(t) + c_0 E_{2,\psi}(t) + \dot{H}_2(t) \leq 0,
\]
where

\[
E_2(t) = \int_{\mathbb{R}^n} e^{2\psi} \left\{ \frac{(t_0 + t)^\alpha}{2} (u_{tt}^2 + |\nabla u_t|^2) \right\} dx,
E_{2,\psi}(t) = \int_{\mathbb{R}^n} e^{2\psi} (1 + (t_0 + t)^\alpha) (-\psi_t)(u_{tt}^2 + |\nabla u_t|^2) dx,
\]
\[
\dot{H}_2(t) = \nu \int_{\mathbb{R}^n} e^{2\psi} \left( \frac{n - \alpha}{2 - \alpha} - 2\delta_1 \right) \frac{a(x)}{2(1 + t)} u_t^2 dx.
\]}
Multiplying (3.3) by \((t_0 + t)\frac{n-a}{2} + 2\varepsilon\) and retaking \(t_0\) larger, we have
\[
\frac{d}{dt}[(t_0 + t)^{\frac{n-a}{2} + 2\varepsilon}\tilde{E}_2(t)] + c_1(t_0 + t)^{\frac{n-a}{2} + 2\varepsilon}E_{2,\psi}(t)
+ \nu\delta_3(t_0 + t)^{\frac{n-a}{2} + 2\varepsilon}\int_{\mathbb{R}^n} e^{2\psi}|\nabla\psi|^2 u_t^2 dx
\leq C(t_0 + t)^{\frac{n-a}{2} + 1+\varepsilon}\int_{\mathbb{R}^n} e^{2\psi}a(x)u_t^2 dx
\]
with some \(c_1 > 0\). By (3.2), integrating over \([0, t]\) and noting \(\tilde{E}_2(t) \sim E_2(t)\), where
\[
E_2(t) = \int_{\mathbb{R}^n} e^{2\psi}\left\{(t_0 + t)^{\alpha}(u_{tt}^2 + |\nabla u_t|^2) + a(x)u_t^2\right\} dx,
\]
it follows that
\[
(t_0 + t)^{\frac{n-a}{2} + 2\varepsilon}E_2(t)
+ \int_0^t (t_0 + \tau)^{\frac{n-a}{2} + 2\varepsilon}E_{2,\psi}(\tau)d\tau
+ \int_0^t (t_0 + \tau)^{\frac{n-a}{2} + 2\varepsilon}\int_{\mathbb{R}^n} e^{2\psi}|\nabla\psi|^2 u_t^2 dx d\tau
\leq CI_1.
\]
In particular, we can see that (1.10) holds. Furthermore, we obtain
\[
\int_0^t (t_0 + \tau)^{\frac{n-a}{2} + 2\varepsilon}\int_{\mathbb{R}^n} e^{2\psi}(u_{tt}^2 + |\nabla u_t|^2) dx d\tau \leq CI_1.
\]
Using this, we can prove (1.11). Indeed, by the same argument as proving (2.7), we have
\[
\frac{d}{dt} \int_{\mathbb{R}^n} \frac{e^{2\psi}}{2}(u_{tt}^2 + |\nabla u_t|^2) dx
+ \int_{\mathbb{R}^n} e^{2\psi}\left\{\left(\frac{1}{4}a(x) - \psi_t\right)u_{tt}^2 + \frac{-\psi_t}{5}|\nabla u_t|^2\right\} dx \leq 0.
\]
Multiplying (3.6) by \((t_0 + t)^{\frac{n-a}{2} + 3-\varepsilon}\), we obtain
\[
\frac{d}{dt} \left[(t_0 + t)^{\frac{n-a}{2} + 3-\varepsilon}\int_{\mathbb{R}^n} \frac{e^{2\psi}}{2}(u_{tt}^2 + |\nabla u_t|^2) dx\right]
+ (t_0 + t)^{\frac{n-a}{2} + 3-\varepsilon}\int_{\mathbb{R}^n} e^{2\psi}\left\{a(x)u_{tt}^2 + (-\psi_t)(u_{tt}^2 + |\nabla u_t|^2)\right\} dx
\leq C(t_0 + t)^{\frac{n-a}{2} + 2-\varepsilon}\int_{\mathbb{R}^n} e^{2\psi}(u_{tt}^2 + |\nabla u_t|^2) dx.
\]
Integration over the interval \([0, t]\) and the estimate (3.5) imply
\[
(t_0 + t)^{\frac{n-a}{2} + 3-\varepsilon}\int_{\mathbb{R}^n} e^{2\psi}(u_{tt}^2 + |\nabla u_t|^2) dx
+ \int_0^t (t_0 + \tau)^{\frac{n-a}{2} + 3-\varepsilon}\int_{\mathbb{R}^n} e^{2\psi}\left\{a(x)u_{tt}^2 + (-\psi_t)(u_{tt}^2 + |\nabla u_t|^2)\right\} dx d\tau
\leq CI_1.
\]
In particular, we have (3.12) and
\[ \int_0^t (t_0 + \tau)^{\frac{n-2}{2-\alpha}} \int_{\mathbb{R}^n} e^{2\psi} a(x) u_{tt}(\tau, x)^2 dx d\tau \leq CI_1, \]
which will be used to obtain (1.12) and (1.13).

To prove (1.12) and (1.13), we differentiate (3.1) again and have
\[ u_{tttt} - \Delta u_{tt} + a(x) u_{ttt} = 0. \]

Using (3.8) instead of (3.2) and by the same argument as above, we can prove instead of (3.4) that
\[ (t_0 + \tau)^{\frac{n-2}{2-\alpha}} + \epsilon E_3(t_0 + \tau)^{\frac{n-2}{2-\alpha}} + \epsilon \int_0^\tau (t_0 + \tau + \tau)^{\frac{n-2}{2-\alpha}} \int_{\mathbb{R}^n} e^{2\psi} \nabla u_{tt}^2 u_{ttt}^2 dx d\tau \leq CI_2, \]
where
\[ E_3(t) = \int_{\mathbb{R}^n} e^{2\psi} \left\{(t_0 + t)^2 (u_{tt}^2 + |\nabla u_{tt}|^2) + a(x) u_{tt}^2 \right\} dx, \]
\[ E_{3, \psi}(t) = \int_{\mathbb{R}^n} e^{2\psi} (1 + (t_0 + t)^2 (-\psi_t))(u_{tt}^2 + |\nabla u_{tt}|^2) dx. \]

In particular, we obtain (1.12). Moreover, by the same argument as deriving (3.7), one can obtain
\[ (t_0 + t)^{\frac{n-2}{2-\alpha}} + 5\epsilon E_3(t_0 + t)^{\frac{n-2}{2-\alpha}} + \epsilon \int_0^\tau (t_0 + \tau + \tau)^{\frac{n-2}{2-\alpha}} \int_{\mathbb{R}^n} e^{2\psi} \left\{a(x) u_{tt}^2 + (-\psi_t)(u_{tt}^2 + |\nabla u_{tt}|^2) \right\} dx d\tau \leq CI_3. \]

In particular, we have
\[ \int_0^t (t_0 + \tau)^{\frac{n-2}{2-\alpha}} + 5\epsilon \int_{\mathbb{R}^n} e^{2\psi} a(x) u_{tt}^2 dx d\tau \leq CI_3. \]

Using (3.12) instead of (3.8) again, we can prove (1.13). Furthermore, we can continue the argument starting at (3.8) and obtaining (3.12) as much as we want. Therefore, we can obtain the conclusion of Theorem 1.2.

Finally, we prove (1.14) and (1.15). Multiplying (1.3) by \( v \), we have
\[ \frac{\partial}{\partial t} \left[ \frac{a(x)}{2} v^2 \right] - \nabla \cdot (v \nabla v) + |\nabla v|^2 = 0. \]
Integrating over \( \mathbb{R}^n \), one can obtain
\[ \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^n} a(x) v(t, x)^2 dx + \int_{\mathbb{R}^n} |\nabla v(t, x)|^2 dx = 0. \]
Thus, we have
\[ \frac{1}{2} \int_{\mathbb{R}^n} a(x) v(t, x)^2 dx + \int_\tau^t \int_{\mathbb{R}^n} |\nabla v(s, x)|^2 dx ds = \frac{1}{2} \int_{\mathbb{R}^n} a(x) v_\tau(x)^2 dx, \]
which implies \(1.14\). To prove \(1.15\), we apply a similar argument to \(1.3\) as in the previous section. We first multiply \(1.3\) by \(e^{2\psi}v_t\) and have

\[
\frac{\partial}{\partial t} \left[ \frac{e^{2\psi}}{2} |\nabla v|^2 \right] - \nabla \cdot (e^{2\psi}v_t \nabla v) + e^{2\psi} \left( a(x) - \frac{|\nabla \psi|^2}{-\psi_t} \right) v_t^2 + \frac{e^{2\psi}}{-\psi_t} |\psi_t v - v_t \nabla \psi|^2 = 0. 
\]

By \(2.4\) and

\[
T_1 \geq e^{2\psi} \left( \frac{1}{5} (-\psi_t)|\nabla v|^2 - \frac{a(x)}{4(2 + \delta)} v_t^2 \right),
\]

it follows that

\[
\frac{\partial}{\partial t} \left[ \frac{e^{2\psi}}{2} |\nabla v|^2 \right] - \nabla \cdot (e^{2\psi}v_t \nabla v) + e^{2\psi} \left( \frac{1}{4} a(x) v_t^2 + \frac{1}{5} (-\psi_t)|\nabla v|^2 \right) \leq 0.
\]

Integrating over \(\mathbb{R}^n\), we obtain

\[
(3.13) \quad \frac{d}{dt} \int_{\mathbb{R}^n} \frac{e^{2\psi}}{2} |\nabla v|^2 dx + \int_{\mathbb{R}^n} e^{2\psi} \left( \frac{1}{4} a(x) v_t^2 + \frac{1}{5} (-\psi_t)|\nabla v|^2 \right) dx \leq 0.
\]

On the other hand, by multiplying \(1.3\) by \(e^{2\psi}v\), it follows that

\[
\frac{\partial}{\partial t} \left[ \frac{e^{2\psi} a(x)}{2} v^2 \right] - \nabla \cdot (e^{2\psi}v \nabla v) + e^{2\psi} \{ |\nabla v|^2 + 2v \nabla \psi \cdot v + (-\psi_t)a(x)v^2 \} = 0.
\]

By the same argument as the derivation of \(2.10\), we can see that

\[
(3.14) \quad \frac{d}{dt} \left[ \int_{\mathbb{R}^n} e^{2\psi} \frac{a(x)}{2} v^2 dx \right] + \int_{\mathbb{R}^n} e^{2\psi} \left( \delta_1 |\nabla v|^2 + |\nabla \psi|^2 v^2 \right) + \frac{\delta}{3} (-\psi_t) a(x)v^2 + \left( \frac{n - \alpha}{2 - \alpha} - 2\delta_1 \right) \frac{a(x)}{2(1 + t)} v^2 \right) dx \leq 0.
\]
Taking $\nu(1 + t)^{\frac{n-m}{n-\alpha} + 1 - \varepsilon}$ with small parameter $\nu > 0$, we obtain

$$\begin{align*}
\frac{d}{dt} \left[ \nu(1 + t)^{\frac{n-m}{n-\alpha} + 1 - \varepsilon} & \int_{\mathbb{R}^n} e^{2\psi}(n^2)v^2 dx + (1 + t)^{\frac{n-m}{n-\alpha} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi}a(x)\frac{v^2}{2} dx \right] \\
+ & \left( \delta_3 - \frac{\nu}{2} \left( n - \alpha \right) + 1 - \varepsilon \right) (1 + t)^{\frac{n-m}{n-\alpha} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi} |\nabla v|^2 dx \\
+ & \nu(1 + t)^{\frac{n-m}{n-\alpha} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi} \left( \frac{1}{4} a(x)v_t^2 + \frac{-\psi_t}{5} |\nabla v|^2 \right) dx \\
+ & (1 + t)^{\frac{n-m}{n-\alpha} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi} \left( \delta_3 |\nabla \psi|^2 v^2 + \frac{\delta}{3} (-\psi_t) a(x)v^2 \right) dx \\
+ & (\varepsilon - 2\delta_1)(1 + t)^{\frac{n-m}{n-\alpha} - 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi} a(x)\frac{v^2}{2} dx \\
\leq & 0.
\end{align*}$$

We determine $\delta$ so that $\varepsilon = 3\delta_1$ holds. Then we take $\nu$ sufficiently small. Integrating the above inequality over $[\tau, t]$, we have

$$(3.15) \quad (1 + t)^{\frac{n-m}{n-\alpha} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi}(n^2)v^2 dx + (1 + t)^{\frac{n-m}{n-\alpha} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi}a(x)\frac{v^2}{2} dx$$

$$\begin{align*}
+ & \int_{\tau}^{t} (1 + s)^{\frac{n-m}{n-\alpha} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi} |\nabla v|^2 dx ds \\
+ & \int_{\tau}^{t} (1 + s)^{\frac{n-m}{n-\alpha} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi} a(x)v_t^2 + (-\psi_t) |\nabla v|^2 dx ds \\
+ & \int_{\tau}^{t} (1 + s)^{\frac{n-m}{n-\alpha} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi} (|\nabla \psi|^2 v^2 + (-\psi_t) a(x)v^2) dx ds \\
+ & \int_{\tau}^{t} (1 + s)^{\frac{n-m}{n-\alpha} - 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi} a(x)\frac{v^2}{2} dx ds \\
\leq & C(1 + \tau)^{\frac{n-m}{n-\alpha} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau, x)} |\nabla v(x)|^2 dx \\
+ & C(1 + \tau)^{\frac{n-m}{n-\alpha} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau, x)} a(x)v(x)\frac{v^2}{2} dx.
\end{align*}$$

In particular, it follows that

$$(3.16) \quad \int_{\tau}^{t} (1 + s)^{\frac{n-m}{n-\alpha} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(s, x)} a(s, x)\frac{v_t^2}{2} dx ds$$

$$\begin{align*}
\leq & C(1 + \tau)^{\frac{n-m}{n-\alpha} + 1 - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau, x)} |\nabla v(\tau)|^2 dx \\
+ & C(1 + \tau)^{\frac{n-m}{n-\alpha} - \varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau, x)} a(\tau)x v(\tau)\frac{v^2}{2} dx.
\end{align*}$$

Using this estimate, we prove (1.15). We differentiate (1.3) with respect to $t$ and have

$$a(x)v_{tt} - \Delta v_t = 0.$$ 

Multiplying this by $v_t$, we obtain

$$\frac{\partial}{\partial t} \left[ \frac{a(x)}{2} v_t^2 \right] - \nabla \cdot (v_t \nabla v_t) + |\nabla v_t|^2 = 0.$$
Integrating over $\mathbb{R}^n$, one can see that
\[
\frac{d}{dt} \int_{\mathbb{R}^n} a(x)v_t(t,x)^2 \, dx \leq 0.
\]

Moreover, taking account into (3.16), multiplying this by $(1 + t - \tau)^{\frac{n-\alpha}{2}+2-\varepsilon}$ with $0 \leq \tau \leq t$, we have
\[
\frac{d}{dt} \left[ (1 + t - \tau)^{\frac{n-\alpha}{2}+2-\varepsilon} \int_{\mathbb{R}^n} a(x)v_t(t,x)^2 \, dx \right] \leq C(1 + t - \tau)^{\frac{n-\alpha}{2}+1-\varepsilon} \int_{\mathbb{R}^n} a(x)v_t(t,x)^2 \, dx
\]
\[
\leq C(1 + t)^{\frac{n-\alpha}{2}+1-\varepsilon} \int_{\mathbb{R}^n} a(x)v_t(t,x)^2 \, dx.
\]

Integrating over $[\tau, t]$, and using (3.16), we can conclude that
\[
(1 + t - \tau)^{\frac{n-\alpha}{2}+2-\varepsilon} \int_{\mathbb{R}^n} a(x)v_t(t,x)^2 \, dx
\]
\[
\leq \int_{\mathbb{R}^n} a(x)v_t(\tau,x)^2 \, dx
\]
\[
+ C \int_{\tau}^t (1 + s)^{\frac{n-\alpha}{2}+1-\varepsilon} \int_{\mathbb{R}^n} a(x)v_t(s,x)^2 \, dx \, ds
\]
\[
\leq \int_{\mathbb{R}^n} a(x)^{-1} |\Delta v_\tau(x)|^2 \, dx
\]
\[
+ C(1 + \tau)^{\frac{n-\alpha}{2}+1-\varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)}|\nabla v_\tau(x)|^2 \, dx
\]
\[
+ C(1 + \tau)^{\frac{n-\alpha}{2}-\varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)}a(x)v_\tau(x)^2 \, dx.
\]

Here we note that
\[a(x)v_\tau(\tau,x) = \Delta v_\tau(x),\]
since $v$ satisfies (1.3). Thus, we obtain (1.15).

**Proof of Lemma 1.4.** By (3.14), we have
\[
\frac{d}{dt} \left[ \int_{\mathbb{R}^n} e^{2\psi \frac{a(x)}{2}} v^2 \, dx \right] \leq 0.
\]

This shows
\[
\int_{\mathbb{R}^n} e^{2\psi(t,x)} \frac{a(x)}{2} v(t,x)^2 \, dx \leq \int_{\mathbb{R}^n} e^{2\psi(0,x)} \frac{a(x)}{2} v(0,x)^2 \, dx.
\]

Let $0 < \rho < 1 - \alpha$, $0 < \mu < 2A$ and
\[
\Omega_\rho(t) := \{ x \in \mathbb{R}^n \mid \langle x \rangle^{2-\alpha} \geq (1 + t)^{1+\rho} \}.
\]

A simple calculation implies
\[
e^{2\psi} a(x) \geq c(1 + t)^{-\frac{n-\alpha}{n-\alpha}} e^{(2A - \mu)\langle x \rangle^{2-\alpha}}.
\]

By noting that
\[
\frac{\langle x \rangle^{2-\alpha}}{1 + t} \geq (1 + t)^\rho
\]
we obtain (4.2) \( \frac{\partial E}{\partial t} \) and it suffices to prove that the each term of the right hand side are where in the \( L_a \) on \( \Omega \)

By the integration by parts, we have

This proves Lemma 1.4.

Thus, we obtain

This proves Lemma 1.4.

4. PROOF OF THE MAIN THEOREM

We can write

By the integration by parts, we have

where \( \frac{\partial E(t)}{\partial t} \) is the pseudodifferential operator with the symbol \( \frac{\partial E}{\partial t}(t, x, \xi) \), that is, \( \frac{\partial E(t)}{\partial t}v \) denotes the derivative of \( E(t) v \) with respect to \( t \). Therefore, we obtain

and it suffices to prove that the each term of the right hand side are \( O(t^{-\frac{3n}{2}+\alpha}}) \) in the \( L^2 \)-sense. First, by the finite propagation speed property, we have \( u(t, x) = \chi(t, x)u_0(x) \) with the characteristic function \( \chi(t, x) \) of the region \( \{ (t, x) \in (0, \infty) \times R^n \mid |x| < t + L \} \). Moreover, by Lemma 1.4 the \( L^2 \)-norm of \( (1 - \chi(t, x)) E(t) [u_0 + a^{-1} u_1] \) decays exponentially. Thus, by multiplying \( \chi(t, x) \) by \( \chi(t, x) \), it suffices to estimate the terms

\[
K_1 := \chi(t, x) \int_{t/2}^{t} E(t - \tau)[a(\cdot)^{-1} u_{tt}(\cdot, \cdot)]d\tau, \quad K_2 := \chi(t, x) E(t/2)[a^{-1} u(t/2)]
\]

\[
(1 + t)^{-\frac{1}{n}} \int_{\Omega_\rho(t)} e^{(2A - \mu)(1 + t)y} v(t, x) dx
\]

\[
\leq C(1 + t)^{-\frac{1}{n}} \int_{\Omega_\rho(t)} e^{(2A - \mu)(1 + t)y} v(t, x) dx
\]

\[
\leq C \int_{R^n} e^{2\psi(t, x)a(x)} v(t, x) dx
\]

\[
\leq C \int_{R^n} e^{2\psi(0, x)a(x)} v(0, x) dx.
\]

Thus, we obtain

\[
\int_{\Omega_\rho(t)} v(t, x)^2 dx \leq C(1 + t)^{-\frac{n}{2}} e^{-(2A - \mu)(1 + t)^\alpha} \int_{R^n} e^{2\psi(0, x)a(x)} v(0, x)^2 dx.
\]

\[\square\]
and
\[ K_3 := \chi(t, x) \int_0^{t/2} \frac{\partial E}{\partial t}(t - \tau)|a^{-1}u_t(\tau)| d\tau. \]

We first estimate \( K_1 \). By (1.14) and (1.12), we have
\[
\|K_1\|_{L^2} = \left\| \chi(t) \int_{t/2}^t E(t - \tau)|a^{-1}u_t(\tau)| d\tau \right\|_{L^2}
\leq (1 + t)^{\alpha/2} \left\| \sqrt{a}E(t - \tau)|a^{-1}u_t(\tau)| \right\|_{L^2} d\tau
\leq (1 + t)^{\alpha/2} \left\| \sqrt{a}u_t(\tau) \right\|_{L^2} d\tau
\leq (1 + t)^{3\alpha/2} \left\| \sqrt{a}u_t(\tau) \right\|_{L^2} d\tau
\leq (1 + t)^{3\alpha/2} \frac{2}{\alpha - 1} \alpha^{1+\varepsilon/2} = o(t^{-\frac{n-2\alpha}{2(2-\alpha)}}),
\]
provided that \( \varepsilon > 0 \) is taken sufficiently small. Because, it is true that
\[
3 \frac{\alpha}{2} - \frac{n - \alpha}{2(2 - \alpha)} - 1 < - \frac{n - 2\alpha}{2(2 - \alpha)}
\]
holds if \( 0 \leq \alpha < 1 \).

We can estimate \( K_2 \) by a similar way. Using (1.14), (1.10) and (4.3), we obtain
\[
\|K_2\|_{L^2} = \left\| \chi(t) \frac{1}{\sqrt{a}} \int_{t/2}^t \sqrt{a}E(t/2)|a^{-1}u_t(t/2)| d\tau \right\|_{L^2}
\leq (1 + t)^{\alpha/2} \left\| \sqrt{a}E(t/2)|a^{-1}u_t(t/2)| \right\|_{L^2}
\leq (1 + t)^{\alpha/2} \left\| \sqrt{a}u_t(t/2) \right\|_{L^2}
\leq (1 + t)^{3\alpha/2} \left\| \sqrt{a}u_t(t/2) \right\|_{L^2}
\leq (1 + t)^{3\alpha/2} \frac{2}{\alpha - 1} \alpha^{1+\varepsilon/2} = o(t^{-\frac{n-2\alpha}{2(2-\alpha)}}).
\]

Finally, we estimate \( K_3 \). By (1.15), we have
\[
\|K_3\|_{L^2} = \left\| \chi(t) \int_0^{t/2} \frac{\partial E}{\partial t}(t - \tau)|a^{-1}u_t(\tau)| d\tau \right\|_{L^2}
\leq (1 + t)^{\alpha/2} \left\| \sqrt{a}\frac{\partial E}{\partial t}(t - \tau)|a^{-1}u_t(\tau)| \right\|_{L^2} d\tau
\leq (1 + t)^{\alpha/2} \left\| (1 + t - \tau)^{-\frac{n-\alpha}{2(2-\alpha)}}(J_1 + J_2 + J_3) \right\|_{L^2}
\leq (1 + t)^{\alpha/2} \frac{2}{\alpha - 1} \alpha^{1+\varepsilon/2} \int_0^{t/2} (J_1 + J_2 + J_3) d\tau,
\]
where

\[ J_1^2 = \int_{\mathbb{R}^n} a(x)^{-1} |\Delta(a(x)^{-1}u_t(\tau, x))|^2 \, dx, \]

\[ J_2^2 = (1 + \tau)^{\frac{n-2+n}{2}+1-\varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)} |\nabla(a(x)^{-1}u_t(\tau, x))|^2 \, dx \]

and

\[ J_3^2 = (1 + \tau)^{\frac{n-2-n}{2}-\varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)} a(x)|a(x)^{-1}u_t(\tau, x)|^2 \, dx. \]

By noting

\[ (4.4) \quad \alpha - \frac{n-\alpha}{2(2-\alpha)} - \frac{1}{2} < -\frac{n-2\alpha}{2(2-\alpha)} \]

if \( 0 \leq \alpha < 1 \), it is only necessary to prove

\[ (4.5) \quad \int_0^{t/2} J_k \, d\tau \leq C (1 + t)^{\frac{\alpha+1}{2}} \]

for \( k = 1, 2, 3 \) with some constant \( C > 0 \).

Now we prove (4.5). We first estimate \( J_3 \). By (1.10) and the finite propagation speed property again, we can estimate

\[ J_3^2 \lesssim (1 + \tau)^{\frac{n-2-n}{2}-\varepsilon} (1 + \tau)^2 \int_{\mathbb{R}^n} e^{2\psi(\tau,x)} a(x)u_t(\tau, x)^2 \, dx \]

\[ \lesssim (1 + \tau)^{2\alpha-2}. \]

By a simple calculation, we can see that

\[ \int_0^{t/2} J_3 \, d\tau \lesssim \int_0^{t/2} (1 + \tau)^{\alpha-1} \, d\tau \lesssim (1 + t)^{\frac{\alpha+1}{2}}. \]

Next, we estimate \( J_2 \). Noting

\[ \nabla(a^{-1}u_t) = \nabla(a^{-1})u_t + a^{-1}\nabla u_t \]

and \( |\nabla(a^{-1})| \lesssim \langle x \rangle^{\alpha-1} \), we have

\[ J_2^2 \lesssim (1 + \tau)^{\frac{n-2+n}{2}+1-\varepsilon} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)} (|\langle x \rangle^\alpha u_t(\tau, x)|^2 + |\langle x \rangle^\alpha \nabla u_t(\tau, x)|^2) \, dx. \]

By (1.10) and (1.12), we obtain

\[ \int_{\mathbb{R}^n} e^{2\psi(\tau,x)} (|\langle x \rangle^\alpha u_t(\tau, x)|^2 \, dx \lesssim (1 + \tau)^\alpha \int_{\mathbb{R}^n} e^{2\psi(\tau,x)} a(x)u_t(\tau, x)^2 \, dx \]

\[ \lesssim (1 + \tau)^{\alpha-\frac{n-2-n}{2}+2+\varepsilon} \]

and

\[ \int_{\mathbb{R}^n} e^{2\psi(\tau,x)} |\langle x \rangle^\alpha \nabla u_t(\tau, x)|^2 \, dx \lesssim (1 + \tau)^{2\alpha} \int_{\mathbb{R}^n} e^{2\psi(\tau,x)} |\nabla u_t(\tau, x)|^2 \, dx \]

\[ \lesssim (1 + \tau)^{2\alpha-\frac{n-2-n}{2}-3+\varepsilon}. \]

Therefore, it holds that

\[ \int_0^{t/2} J_2 \, d\tau \lesssim (1 + t)^{\frac{\alpha+1}{2}}. \]

Finally, we estimate \( J_1 \). Noting

\[ \Delta(a^{-1}u_t) = \Delta(a^{-1})u_t + 2\nabla(a^{-1}) \cdot \nabla u_t + a^{-1} \Delta u_t, \]
we further divide $J_1$ into three parts:

$$J_1^2 \lesssim \int_{\mathbb{R}^n} a^{-1} |\Delta(a^{-1})u_t|^2 \, dx + \int_{\mathbb{R}^n} a^{-1} |\nabla(a^{-1})|^2 |\nabla u_t|^2 \, dx + \int_{\mathbb{R}^n} a^{-1} |a^{-1} \Delta u_t|^2 \, dx$$

$$\equiv J_{11}^2 + J_{12}^2 + J_{13}^2.$$

By noting $|\Delta(a^{-1})| \lesssim \langle x \rangle^{\alpha - 2}$ and (1.10), we have

$$J_{11}^2 \lesssim \int_{\mathbb{R}^n} \langle x \rangle^{4\alpha - 4} a(x) u_t(\tau, x)^2 \, dx$$

$$\lesssim (1 + \tau)^{-\frac{n-\alpha}{2}} - 2 + \varepsilon.$$

Therefore, we immediately obtain

$$\int_0^{t/2} J_{11} \, d\tau \lesssim 1,$$

provided that $\varepsilon$ is sufficiently small.

Next, we estimate $J_{12}$.

$$J_{12}^2 \lesssim (1 + \tau)^{\alpha} \int_{\mathbb{R}^n} |\nabla u_t(\tau, x)|^2 \, dx \lesssim (1 + \tau)^{\alpha - \frac{n-\alpha}{2} - 3 + \varepsilon}$$

and hence

$$\int_0^{t/2} J_{12} \, d\tau \lesssim 1.$$

Since $u_t$ also satisfies (1.1), we can rewrite

$$\Delta u_t = u_{ttt} - au_{tt}.$$

Therefore, we have

$$J_{13}^2 \lesssim \int_{\mathbb{R}^n} a(x)^{-4} a(x) u_{ttt}(\tau, x)^2 \, dx + \int_{\mathbb{R}^n} a(x)^{-2} a(x) u_{tt}(\tau, x)^2 \, dx$$

$$\lesssim (1 + \tau)^{4\alpha - \frac{n-\alpha}{2} - 6 + \varepsilon} + (1 + \tau)^{2\alpha - \frac{n-\alpha}{2} - 4 + \varepsilon}.$$

These estimates imply

$$\int_0^{t/2} J_1 \, d\tau \lesssim 1.$$

This completes the proof.

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