Faster Rates for the Frank-Wolfe Method over Strongly-Convex Sets

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Abstract

The Frank-Wolfe method (a.k.a. conditional gradient algorithm) for smooth optimization has regained much interest in recent years in the context of large scale optimization and machine learning. A key advantage of the method is that it avoids projections - the computational bottleneck in many applications - replacing it by a linear optimization step. Despite this advantage, the convergence rates of the FW method fall behind standard gradient methods for most settings of interest. It is an active line of research to derive faster FW algorithms for various settings of convex optimization.

In this paper we consider the special case of optimization over strongly convex sets, for which we prove that the vanilla FW method converges at a rate of $\frac{1}{t^2}$. This gives a quadratic improvement in convergence rate compared to the general case, in which convergence is of the order $\frac{1}{t}$, and known to be tight.

We show that various balls induced by $p$-norms, Schatten norms and group norms are strongly convex on one hand and on the other hand, linear optimization over these sets is straightforward and admits a closed-form solution. This gives a family of applications, e.g. linear regression with regularization, for which the FW algorithm enjoys both worlds: a fast convergence rate and no projections.

1 Introduction

The Frank-Wolfe method, that was originally introduced in a paper by Frank and Wolfe from the 1950’s [4], is a first order method for the minimization of a smooth convex function over a convex set. Its main advantage in large-scale programs is that it is a first-order and projection-free method - i.e. the algorithm proceeds by iteratively solving a linear optimization problem and remaining inside the feasible domain. For matrix completion problems, metric learning, sparse PCA, structural SVM and other large-scale machine learning problems, this feature was demonstrated to be crucial for obtaining practical algorithms with provable guarantees [11, 13, 8, 7, 9, 18, 15].
Despite its empirical success, the main drawback of the method is its relatively slow convergence rate in comparison to optimal first order methods. The convergence rate of the method is on the order of $1/t$ where $t$ is the number of iterations, and this is known to be tight. In contrast, Nesterov’s accelerated gradient descent method gives a rate of $1/t^2$ for general convex smooth problems and a rate $e^{-O(t)}$ is known for smooth and strongly convex problems. The following question arises: are there projection free methods with convergence rates matching that of projected gradient-descent and its extensions?

Motivated by this question, in this work we advance the line of research for faster convergence rates of projection free methods. We prove that in case both the objective function and the feasible set are strongly convex, the vanilla Frank-Wolfe method converges at an accelerated rate of $1/t^2$.

We motivate the study of optimization over strongly convex sets by showing that various norms that serve as popular regularizes in machine learning problems, including $\ell_p$ norms, matrix Schatten norms and matrix group norms, give rise to strongly convex sets. We further show that linear optimization over these sets is straightforward to implement and admits a closed-form solution. An immediate application of these results is the problem of regularized linear regression, in which the regularization is taken to be one of the above norms.

1.1 Related Work

The Frank-Wolfe method for smooth optimization dates back to the original work of Frank and Wolfe [4] which presented an algorithm for minimizing a quadratic function over a polytope using only linear optimization steps over the feasible set. Recent results by Clarkson [2], Hazan [8] and Jaggi [10] extend the method to smooth convex optimization over the simplex, spectrahedron and arbitrary convex and compact sets respectively. The $1/t$ convergence rate of the method could not be improved in general, even if the objective function is strongly convex for instance, as shown in [2, 8, 10], even though it is known that in this case, the projected gradient method achieves an exponentially fast convergence rate.

Over the past years, several results tried to improve the convergence rate of the Frank-Wolfe method under various conditions. GuéLat and Marcotte [6] showed that in case the objective function is strongly convex and the feasible set is a polytope, then under restrictive assumptions on the location of optimal solution with respect to the boundary of the set, the FW method converges exponentially fast.

Recently, Garber and Hazan [5] gave the first natural linearly-converging FW variant without any restricting assumptions on the location of the optimum. They showed that a variant of the Frank Wolfe method with away steps converges exponentially fast in case the objective function is strongly convex and the feasible set is a polytope. In follow-up work, Jaggi and Lacoste-Julien [1] gave a refined analysis of an algorithm presented in [6] and showed that also converges exponentially fast in the same setting as the Garber-Hazan result.

In a different line of work, Migdalas and recently Lan [17, 14] considered the Frank-Wolfe algorithm with a stronger optimization oracle that is able to solve
quadratic problems over the feasible domain. They show that in case the objective function is strongly convex then exponentially fast convergence is attainable. However, in most settings of interest, an implementation of such a non-linear oracle is much more computationally expensive than the linear oracle, and the key benefit of the Frank-Wolfe method is lost.

Also relevant to our work is the classical work of Levitin and Polyak [16] who showed that in case the feasible set is strongly convex and under the restrictive assumption that the norm of the gradient of the objective function is lower bounded by a constant on any point in the feasible set, then an exponentially fast convergence rate is attained. Note that they do not assume that the objective function is strongly convex, hence the assumption on the norm of the gradients is crucial.

In this work we assume that the objective function is strongly convex and that the feasible set is also strongly convex. Under these two assumptions alone, in particular without assumptions on the location of the optimum or a lower bound on the norm of the gradients, we show that a vanilla Frank-Wolfe method converges in an accelerated rate of roughly $1/t^2$. The improved convergence rate is independent of the dimension.

2 Preliminaries

2.1 Smoothness and strong convexity

For the following definitions let $E$ be a finite vector space and $\| \cdot \|$ be a norm over $E$.

**Definition 1** (smooth function). We say that a function $f : E \to \mathbb{R}$ is $\beta$ smooth over a convex set $K \subset E$ with respect to $\| \cdot \|$ if for all $x, y \in K$ it holds that

$$ f(y) \leq f(x) + \nabla f(x) \cdot (y - x) + \frac{\beta}{2} \| x - y \|^2 $$

**Definition 2** (strongly convex function). We say that a function $f : E \to \mathbb{R}$ is $\alpha$-strongly convex over a convex set $K \subset E$ with respect to $\| \cdot \|$ if it satisfies the following two equivalent conditions,

1. $\forall x, y \in K : f(y) \geq f(x) + \nabla f(x) \cdot (y - x) + \frac{\alpha}{2} \| x - y \|^2$

2. $\forall x, y \in K, \gamma \in [0, 1] : f(\gamma x + (1 - \gamma)y) \leq \gamma f(x) + (1 - \gamma)f(y) - \frac{\alpha}{2} \gamma(1 - \gamma)\| x - y \|^2$

The above definition (part 1) combined with first order optimality conditions imply that for a $\alpha$-strongly convex function $f$, if $x^* = \arg \min_{x \in K} f(x)$, then for any $x \in K$

$$ f(x) - f(x^*) \geq \frac{\alpha}{2} \| x - x^* \|^2 $$ (1)
Definition 3 (strongly convex set). We say that a convex set $K \subset E$ is $\alpha$-strongly convex with respect to $\| \cdot \|$ if for any $x, y \in K$, any $\gamma \in [0, 1]$ and any vector $z \in E$ such that $\|z\| = 1$, it holds that

$$\gamma x + (1 - \gamma)y + \gamma(1 - \gamma)\frac{\alpha}{2}\|x - y\|^2 z \in K$$

That is, $K$ contains a ball of of radius $\gamma(1 - \gamma)\frac{\alpha}{2}\|x - y\|^2$ induced by the norm $\| \cdot \|$ centered at $\gamma x + (1 - \gamma)y$.

2.2 The Frank-Wolfe algorithm

The Frank-Wolfe algorithm, also known as the conditional gradient algorithm, is an algorithm for the minimization of a convex function $f : E \rightarrow \mathbb{R}$ which is assumed to be $\beta_f$-smooth with respect to a norm $\| \cdot \|$, over a convex and compact set $K \subset E$. The algorithm implicitly assumes that the convex set $K$ is given in terms of a linear optimization oracle $O_K : E \rightarrow K$ which given a linear objective $c \in E$ returns a point $x = O_K(c) \in K$ such that $x \in \arg \min_{y \in K} y \cdot c$. The algorithm is given below. The algorithm proceeds in iterations, taking on each iteration $t$ the new iterate $x_{t+1}$ to be a convex combination between the previous feasible iterate $x_t$ and a feasible point that minimizes the dot product with the gradient direction at $x_t$, which is generated by invoking the oracle $O_K$ with the input point $\nabla f(x_t)$. There are various ways to set the parameter that controls the convex combination $\eta_t$ in order to guarantee convergence of the method. The option that we choose here is the optimization of $\eta_t$ via a simple line search rule, which in straightforward and computationally cheap to implement.

**Algorithm 1** Frank-Wolfe Algorithm

1: Let $x_1$ be an arbitrary point in $K$
2: for $t = 1$... do
3: \[ p_t \leftarrow O_K(\nabla f(x_t)) \]
4: \[ \eta_t \leftarrow \arg \min_{\eta \in [0, 1]} \eta (p_t - x_t) \cdot \nabla f(x_t) + \eta^2 \beta_f^2 \|p_t - x_t\|^2 \]
5: \[ x_{t+1} \leftarrow x_t + \eta_t (p_t - x_t) \]
6: end for

The following theorem is taken from [10].

**Theorem 1.** Let $x^* \in \arg \min_{x \in K} f(x)$ and denote $D_K = \max_{x, y \in K} \|x - y\|$ (the diameter of the set with respect to $\| \cdot \|$). For every iteration $t$ of algorithm 1 it holds that

$$f(x_t) - f(x^*) \leq \frac{2 \beta_f D_K^2}{t+1} = O\left(\frac{1}{t}\right)$$

2.3 Our Results

In this work, we consider the case in which the function to optimize $f$, is not only $\beta_f$-smooth with respect to $\| \cdot \|$, but also $\alpha_f$-strongly convex with respect to $\| \cdot \|$. We further assume that the feasible set $K$ is $\alpha_K$-strongly convex with respect to $\| \cdot \|$. Under these two additional two assumptions we prove the following theorem.
Lemma 1. Let $x^* = \arg \min_{x \in \mathcal{K}} f(x)$ and let $M = \frac{\sqrt{f'' \mathcal{K}}}{8 \sqrt{2} \beta f}$. Denote $D_\mathcal{K} = \max_{x, y \in \mathcal{K}} \|x - y\|$. For every iteration $t$ of algorithm 1 it holds that

$$f(x_t) - f(x^*) \leq \frac{\max\{\beta f D^2_\mathcal{K}/2, 27M^{-2}\}}{t^2} = O\left(\frac{1}{t^2}\right)$$

3 Proof of Theorem 2

We denote the approximation error at iteration $t$ of the algorithm by $h_t$. That is $h_t = f(x_t) - f(x^*)$ where $x^* = \arg \min_{x \in \mathcal{K}} f(x)$.

To better illustrate our results, we first shortly revisit the proof technique of theorem 1. The main observation to be made is the following:

$$f(x_{t+1}) - f(x^*) = f(x_t + h_t(p_t - x_t)) - f(x^*)$$

$$\leq f(x_t) - f(x^*) + h_t(p_t - x_t) \cdot \nabla f(x_t) + \frac{\eta_t^2 \beta f}{2} \|p_t - x_t\|^2 \quad \text{\beta-f-smoothness of f}$$

$$\leq f(x_t) - f(x^*) + h_t(x^* - x_t) \cdot \nabla f(x_t) + \frac{\eta_t^2 \beta f}{2} \|p_t - x_t\|^2 \quad \text{optimality of \(p_t\)}$$

$$\leq f(x_t) - f(x^*) - h_t(f(x_t) - f(x^*)) + \frac{\eta_t^2 \beta f}{2} \|p_t - x_t\|^2 \quad \text{convexity of f} \quad (2)$$

For the choice of $\eta_t$ to be roughly $1/t$, the convergence rate of $1/t$ stated in theorem 1 is obtained. This rate cannot be improved in general since while the so-called duality gap $(x_t - p_t) \cdot \nabla f(x_t)$ could be arbitrarily small (as small as $(x_t - x^*) \cdot \nabla f(x_t)$), the quantity $\|p_t - x_t\|$ may remain as large as the diameter of the set. Note that in case $f$ is strongly-convex, then according to (1) it holds that $x_t$ converges to $x^*$ and thus according to (2) it suffices to solve the inner linear optimization problem on the intersection of $\mathcal{K}$ and a small ball centered at $x_t$. As a result the quantity $\|p_t - x_t\|^2$ will be proportional to the approximation error at time $t$, and a linear convergence rate will be attained. However in general, under the linear oracle assumption, we have no way to solve the linear optimization problem over the intersection of $\mathcal{K}$ and a ball without greatly increasing the number of calls to the linear oracle, which is the most expensive step in many settings.

Our main observation is that in case the feasible set $\mathcal{K}$ is strongly convex, then while the quantity $\|p_t - x_t\|$ may still be much larger than $\|x^* - x_t\|$ (the distance to the optimum), in this case, the duality gap must also be large, which results in faster convergence. This observation is illustrated in figure 1 and given formally in lemma 1.

Lemma 1. On any iteration $t$ of algorithm 1 it holds that

$$(x_t - p_t) \cdot \nabla f(x_t) \geq \max \left(1, \frac{\sqrt{f'' \mathcal{K}} \|x_t - p_t\|^2}{4 \sqrt{D_{\mathcal{K}}} h_t}\right) h_t$$

Proof. By the optimality of the point $p_t$ we have that,

$$(p_t - x_t) \cdot \nabla f(x_t) \leq (x^* - x_t) \cdot \nabla f(x_t) \quad (3)$$
Figure 1: For strongly convex sets, as in the left picture, the duality gap (denoted \(dg\)) increases with \(\|p_t - x_t\|^2\), which accelerates the convergence. As shown in the picture on the right, this property clearly does not hold for arbitrary convex sets.

Denote \(c = \frac{1}{2}(x_t + p_t)\). Since \(f\) is strongly convex we have that \(\|x^* - x_t\| \leq \sqrt{\frac{2}{\alpha_f}} h_t\). Thus, using the strong convexity of the set \(K\) we have that the point \(w = c + \frac{\sqrt{\alpha_f \alpha_K} \|x_t - p_t\|^2}{8\sqrt{2h_t}}(x^* - x_t)\) is in \(K\). Again using the optimality of \(p_t\) we have that,

\[
(p_t - x_t) \cdot \nabla f(x_t) \leq (w - x_t) \cdot \nabla f(x_t) = \frac{1}{2}(p_t - x_t) \cdot \nabla f(x_t) + \frac{\sqrt{\alpha_f \alpha_K} \|x_t - p_t\|^2}{8\sqrt{2h_t}}(x^* - x_t) \cdot \nabla f(x_t)
\]

Rearranging we have,

\[
(p_t - x_t) \cdot \nabla f(x_t) \leq \frac{\sqrt{\alpha_f \alpha_K} \|x_t - p_t\|^2}{4\sqrt{2h_t}}(x^* - x_t) \cdot \nabla f(x_t)
\]  

(4)

The lemma follow from Combining (3), (4) and since by convexity of \(f\) we have that \((x_t - x^*) \cdot \nabla f(x_t) \geq f(x_t) - f(x^*) = h_t\).

Lemma 2. For any iteration \(t\) of the algorithm it holds that, \(h_{t+1} \leq h_t(1 - M \sqrt{h_t})\) for \(M = \frac{\sqrt{\alpha_f \alpha_K}}{8\sqrt{2}\beta_f}\).

Proof. By smoothness of \(f\) we have,

\[
f(x_{t+1}) \leq f(x_t) + \eta_t(p_t - x_t) \cdot \nabla f(x_t) + \frac{\beta_f}{2} \eta_t^2 \|p_t - x_t\|^2
\]

Subtracting \(f(x^*)\) from both sides we have,

\[
h_{t+1} \leq h_t + \eta_t(p_t - x_t) \cdot \nabla f(x_t) + \frac{\beta_f}{2} \eta_t^2 \|p_t - x_t\|^2
\]

We now consider two cases.
case 1: $\|x_t - p_t\|^2 \leq \frac{4\sqrt{m}}{\sqrt{\alpha_f} \alpha_K}$. Using lemma 1 we have,

$$h_{t+1} \leq (1 - \eta_t)h_t + \frac{\beta_f}{2} \eta_t^2 \frac{4\sqrt{2}h_t}{\sqrt{\alpha_f} \alpha_K}$$

By the optimal choice of $\eta_t$ we can set in particular $\eta_t = \frac{\sqrt{h_t \alpha_f \alpha_K}}{4\sqrt{2} \beta_f}$ and get,

$$h_{t+1} \leq (1 - \frac{\sqrt{\alpha_f \alpha_K}}{8\sqrt{2} \beta_f} \sqrt{h_t})h_t$$

case 2: $\|x_t - p_t\|^2 > \frac{4\sqrt{m}}{\sqrt{\alpha_f} \alpha_K}$. Using lemma 1 again we have,

$$h_{t+1} \leq (1 - \eta_t \frac{\sqrt{\alpha_f \alpha_K} \|x_t - p_t\|^2}{4\sqrt{2}h_t})h_t + \frac{\beta_f}{2} \eta_t^2 \|x_t - p_t\|^2$$

Again by the optimal choice of $\eta_t$ we can set $\eta_t = \frac{\sqrt{h_t \alpha_f \alpha_K}}{4\sqrt{2} \beta_f}$ and get,

$$h_{t+1} \leq (1 - \frac{\alpha_f \alpha_K^2 \|x_t - p_t\|^2}{64 \beta_f})h_t$$

Using our assumption that $\|x_t - p_t\|^2 > \frac{4\sqrt{m}}{\sqrt{\alpha_f} \alpha_K}$ we have as in the previous case,

$$h_{t+1} \leq (1 - \frac{\sqrt{\alpha_f \alpha_K}}{8\sqrt{2} \beta_f} \sqrt{h_t})h_t$$

We can now prove theorem 2.

**Proof.** Let $M = \frac{\sqrt{\alpha_f \alpha_K}}{8\sqrt{2} \beta_f}$ and $C = \max\{\beta_f D_K^2/2, 27M^{-2}\}$. We prove by induction that $h_t \leq C$. For the base case $t = 1$ we need to prove that $h_1 = f(x_1) - f(x^*) \leq C$. By $\beta_f$ smoothness of $f$ we have,

$$f(x_1) - f(x^*) = f(x^* + (x_1 - f(x^*)) - f(x^*) \leq f(x^*) - f(x^*) + \nabla f(x^*)(x_1 - x^*) + \frac{\beta_f}{2} \|x_1 - x^*\|^2$$

By the first order optimality condition we have that $\nabla f(x^*)(x_1 - x^*) \geq 0$. Thus,

$$f(x_1) - f(x^*) \leq \frac{\beta_f}{2} \|x_1 - x^*\|^2 \leq \frac{\beta_f D_K^2}{2}$$

Assume now that the induction holds for time $t \geq 1$, that is $h_t \leq \frac{C}{t^2}$. If $h_t \leq \frac{C}{3t^2}$ then for any $t \geq 2$ we have that,

$$h_{t+1} < h_t \leq \frac{C}{3t^2} \leq \frac{C}{(t+1)^2}$$
where the first inequality follows from lemma \[2\].

Otherwise, \( h_t > \frac{27}{M^2} \). By lemma \[2\] and the induction assumption we have,

\[
    h_{t+1} \leq h_t \left(1 - Mh_t^{1/2}\right) < \frac{C}{t^2} \left(1 - M\sqrt{\frac{C}{3t}}\right) = \frac{C}{(t + 1)^2} \frac{(t + 1)^2}{t^2} \left(1 - M\sqrt{\frac{C}{3t}}\right)
\]

Thus for \( C \geq \frac{27}{M^2} \) we have that,

\[
    h_{t+1} \leq \frac{C}{(t + 1)^2} \left(1 + \frac{3}{t}\right) \left(1 - \frac{3}{t}\right) < \frac{C}{(t + 1)^2}
\]

\[
\]

4 Examples of Strongly Convex Sets

In this section we explore convex sets for which theorem \[2\] is applicable. That is, convex sets which on one hand are strongly convex, and on the other hand such that admit a simple and efficient implementation of a linear optimization oracle. We show that various norms that give rise to natural regularization functions in machine learning problems, induce convex sets that fit both of the above requirements.

4.1 Characterization of strongly convex sets

The following lemma will be useful to prove that convex sets that are induced by certain norms are strongly convex. The proof is differed to the appendix.

**Lemma 3.** Let \( E \) be a finite vector space and let \( \|\cdot\| \) be a norm over \( E \) and assume that the function \( f(x) = \|x\|^2 \) is \( \alpha \)-strongly convex over \( E \) with respect to the norm \( \|\cdot\| \). Then for any \( r > 0 \), the set \( B_{\|\cdot\|}(r) = \{ x \in E \mid \|x\| \leq r \} \) is \( \frac{\alpha r}{2} \)-strongly convex with respect to \( \|\cdot\| \).

4.2 \( p \)-norm balls for \( p \in (1, 2] \)

Given a parameter \( p \geq 1 \), consider the \( p \)-norm ball of radius \( r \),

\[
    B_p(r) = \{ x \in \mathbb{R}^n \mid \|x\|_p \leq r \}
\]

The following lemma is proved in \[19\].

**Lemma 4.** Fix \( p \in (1, 2] \). The function \( \frac{1}{2}\|x\|_p^2 \) is \( (p - 1) \)-strongly-convex w.r.t. the norm \( \|\cdot\|_p \).

The following corollary is a consequence of combining lemm \[4\] and lemma \[3\]. The proof is differed to the appendix.
Corollary 1. Fix $p \in (1, 2]$. The set $\mathbb{B}_p(r)$ is $\frac{p-1}{r}$ strongly convex with respect to the norm $\| \cdot \|_p$ and $(\frac{p-1}{r})^{\frac{n}{2}}$ strongly convex with respect to the norm $\| \cdot \|_2$.

The following lemma establishes that linear optimization over $\mathbb{B}_p(r)$ admits a simple closed-form solution that can be computed in time that is linear in the number of non-zeros in the linear objective. The proof is differed to the appendix.

Lemma 5. Fix $p \in (1, 2]$ and $r > 0$. Given a linear objective $c \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n$ such that $x_i = -\frac{r}{\|c\|_q^q} \text{sgn}(c_i)|c_i|^{q-1}$ where $q$ satisfies $1/q + 1/p = 1$, and $\text{sgn}(\cdot)$ is the sign function. Then $x = \arg\min_{y \in \mathbb{B}_p(r)} y^T c$

4.3 Schatten $p$-norm balls for $p \in (1, 2]$

Given a matrix $X \in \mathbb{R}^{m \times n}$ we denote by $\lambda(X)$ the vector of singular values of $X$ in descending order, that is $\lambda(X)_1 \geq \lambda(X)_2 \geq \ldots \lambda(X)_{\min(m,n)}$. The Schatten $p$-norm is given by,

$$\|X\|_{S(p)} = \|\lambda(X)\|_p = \left( \sum_{i=1}^{\min(m,n)} |\lambda(X)_i|^p \right)^{1/p}$$

Consider the Schatten $p$-norm ball of radius $r$,

$$\mathbb{B}_{S(p)}(r) = \{ X \in \mathbb{R}^{m \times n} \mid \|X\|_{S(p)} \leq r \}$$

The following lemma is taken from [12].

Lemma 6. Fix $p \in (1, 2]$. The function $\frac{1}{2} \|X\|_{S(p)}^2$ is $(p-1)$-strongly-convex w.r.t. the norm $\| \cdot \|_{S(p)}$.

The proof of the following corollary follows the exact same lines as the proof of corollary [4] by using lemma [6] instead of lemma [3].

Corollary 2. Fix $p \in (1, 2]$. The set $\mathbb{B}_{S(p)}(r)$ is $\alpha = \frac{p-1}{r}$ strongly convex with respect to the norm $\| \cdot \|_{S(p)}$ and $(\frac{p-1}{r})^{\frac{\min(m,n)}{2}}$ strongly convex with respect to the frobenius norm $\| \cdot \|_F$.

The following lemma establishes that linear optimization over $\mathbb{B}_{S(p)}(r)$ admits a simple closed-form solution given the singular value decomposition of the linear objective. The proof is differed to the appendix.

Lemma 7. Fix $p \in (1, 2]$ and $r > 0$. Given a linear objective $C \in \mathbb{R}^{m \times n}$. Let $C = U\Sigma V^T$ be the singular value decomposition of $C$. Let $\lambda$ be a vector such that $\lambda_i = -\frac{r}{\|\lambda(C)_i\|_q^q} \text{sgn}(\lambda(C)_i)|\lambda(C)_i|^{q-1}$ where $q$ satisfies $1/q + 1/p = 1$. Finally, let $X = U\text{Diag}(\lambda)V^T$ where $\text{Diag}(\lambda)$ is an $m \times n$ diagonal matrix with the vector $\lambda$ as the main diagonal. Then $X = \arg\min_{Y \in \mathbb{B}_{S(p)}(r)} Y \bullet C$, where $\bullet$ denotes the standard inner product for matrices.
4.4 Group (s,p)-norm balls

Given a matrix \( X \in \mathbb{R}^{m \times n} \) denote by \( X_i \in \mathbb{R}^n \) the \( i \)th row of \( X \). That is \( X = (X_1, X_2, ..., X_m)^\top \).

The \((s,p)\)-group norm of \( X \) is given by,
\[
\|X\|_{s,p} = \| (\|X_1\|_s, \|X_2\|_s, ..., \|X_m\|_s) \|_p
\]

We define the group \((s,p)\)-norm ball as follows:
\[
B_{s,p}(r) = \{ X \in \mathbb{R}^{m \times n} | \|X\|_{s,p} \leq r \}
\]

The proof of the following lemma is differed to the appendix.

Lemma 8. Let \( s, p \in (1, 2) \). The set \( B_{s,p}(r) \) is \((s-1)(p-1)/(s+p-2)r\) strongly convex with respect to the norm \( \| \cdot \|_{s,p} \) and \( n^{\frac{1}{s} - 1} m^{\frac{1}{p} - 1} \frac{(s-1)(p-1)}{(s+p-2)p} \) strongly convex with respect to the frobenius norm \( \| \cdot \|_F \).

The following lemma establishes that linear optimization over \( B_{s,p}(r) \) admits a simple closed-form solution that can be computed in time that is linear in the number of non-zeros in the linear objective. The proof is differed to the appendix.

Lemma 9. Fix \( s, p \in (1, 2) \) and \( r > 0 \). Given a linear objective \( C \in \mathbb{R}^{m \times n} \). Let \( X \in \mathbb{R}^{m \times n} \) be such that \( X_{i,j} = -r \frac{\|C\|_q - \|C_i\|_z}{z \|C\|_q} \frac{\text{sgn}(C_{i,j}) |C_{i,j}|^{z-1}}{} \) where \( z \) satisfies \( 1/s + 1/z = 1 \), \( q \) satisfies \( 1/p + 1/q = 1 \) and \( C_i \) denotes the \( i \)th row of \( C \). Then \( X = \arg \min_{Y \in B_{s,p}(r)} Y \cdot C \).

5 Application to Regularized Linear Regression

In this section we show the immediate applicability of our results to solving regularized linear regression problems when the regularization is taken to be a \( p \)-norm with \( p \in (1, 2) \) in the single-variate case, and the Schatten \( p \)-norm or group \((s,p)\)-norm in the multivariate case, with \( s, p \in (1, 2) \).

Consider the problem of single-variate linear regression in which the data set is given by pairs of feature vectors and numerical scalars \( \{x_i, y_i\}_{i=1}^m \subset \mathbb{R}^n \times \mathbb{R} \). Let \( p \in (1, 2) \). We consider the following \( p \)-norm regularized linear regression problem.

\[
\min_{w \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^m (y_i - w^\top x_i)^2 + \lambda \|w\|_p^2 \tag{5}
\]

where \( \lambda \) is the regularization parameter. Let us denote \( g(w) = \sum_{i=1}^m (y_i - w^\top x_i)^2 \). The hessian matrix of \( g(w) \) at any point \( w \) is given by \( \nabla^2 g(w) = \frac{1}{m} \sum_{i=1}^m x_i x_i^\top \). Assuming this matrix has full rank, we have that \( g(w) \) is both smooth and strongly convex w.r.t. the standard euclidean norm \( \| \cdot \|_2 \). The function \( \|w\|_p^2 \) is strongly convex but not smooth for \( p \in (1, 2) \). Hence algorithms for smooth minimization could not be directly applied to problem (5).
The unconstrained problem \((5)\) could be cast as the following constrained optimization problem,

\[
\min_{w \in \mathbb{B}_p(r)} \frac{1}{m} \sum_{i=1}^{m} (y_i - w^T x_i)^2
\tag{6}
\]

Where \(r\) is a parameter that controls the \(p\)-norm of the solution. The objective in \((6)\) is now both smooth and strongly convex with respect to \(\| \cdot \|_2\) whereas according to corollary \(1\), the feasible set is also strongly convex with respect to \(\| \cdot \|_2\), and thus the result of theorem \(2\) could be applied.

In the multi-variate case the data is given by pairs of feature vectors and target vectors \(\{x_i, y_i\}_{i=1}^{m} \subset \mathbb{R}^n \times \mathbb{R}^k\). We consider here two types of regularizes for the multi-variate linear regression problem which corresponds to the following two optimization problems.

**Schatten \(p\)-norm regularization**

\[
\min_{W \in \mathbb{R}^{k \times n}} \frac{1}{m} \sum_{i=1}^{m} \|y_i - W x_i\|_2^2 + \lambda \|W\|_{S(p)}^2
\]

**Group \((s,p)\)-norm regularization**

\[
\min_{W \in \mathbb{R}^{k \times n}} \frac{1}{m} \sum_{i=1}^{m} \|y_i - W x_i\|_2^2 + \lambda \|W\|_{s,p}^2
\]

As in the single-variate case, for \(s,p \in (1,2]\) both of the above objectives are strongly convex w.r.t the euclidean norm (under the assumption that the Hessian is full rank), but both are generally not smooth.

Transforming the above problems to a constrained formulation where the feasible set is given, as in the single-variate case, by a ball induced by the regularizing norm, leads to a problem of minimizing a smooth and strongly convex objective with respect to the frobenius norm \(\| \cdot \|_F\) over a strongly convex set which according to corollary \(2\)/ lemma \(8\) is also strongly convex with respect to \(\| \cdot \|_F\), and hence the accelerated rate stated in theorem \(2\) can be attained.

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A Proofs of Lemmas and Corollaries from Section 4

A.1 Proof of lemma 3

Proof. It suffices to show that given \( x, y \in \mathbb{E} \) such that \( f(x) \leq r^2, f(y) \leq r^2 \), a scalar \( \gamma \in [0, 1] \) and \( z \in \mathbb{E} \) such that \( \| z \| \leq \frac{\alpha}{4r} \gamma (1 - \gamma) \| x - y \|^2 \), it holds that, \( f(\gamma x + (1 - \gamma) y + z) \leq r^2 \).

By the definition of \( f \) and the triangle inequality for \( \| \cdot \| \) we have,

\[
\begin{align*}
\sqrt{f(\gamma x + (1 - \gamma) y + z)} &\leq \sqrt{f(\gamma x + (1 - \gamma) y) + \| z \|^2} \\
&= \left( f(\gamma x + (1 - \gamma) y) + \| z \|^2 \right)^{1/2} \\
&= \left( \sqrt{f(\gamma x + (1 - \gamma) y)} + \| z \| \right)^2. 
\end{align*}
\]

(7)

Since \( f \) is \( \alpha \) strongly convex with respect to \( \| \cdot \| \) we have that,

\[
\begin{align*}
f(\gamma x + (1 - \gamma) y) &\leq \gamma f(x) + (1 - \gamma) f(y) - \frac{\alpha}{2} \gamma (1 - \gamma) \| x - y \|^2 \\
&\leq r^2 - \frac{\alpha}{2} \gamma (1 - \gamma) \| x - y \|^2
\end{align*}
\]

The function \( g(t) = \sqrt{t} \) is concave, meaning \( \sqrt{a - b} = g(a-b) \leq g(a) - g'(a) \cdot b = \sqrt{a} - \frac{b}{2 \sqrt{a}}. \) Thus,

\[
\sqrt{f(\gamma x + (1 - \gamma) y)} \leq \sqrt{r^2 - \frac{\alpha}{2} \gamma (1 - \gamma) \| x - y \|^2} \leq r - \frac{\alpha \gamma (1 - \gamma) \| x - y \|^2}{4r}.
\]

Plugging back in (7) we have

\[
f(\gamma x + (1 - \gamma) y + z) \leq \left( r - \frac{\alpha \gamma (1 - \gamma) \| x - y \|^2}{4r} + \| z \| \right)^2
\]

By our assumption on \( \| z \| \) we have,

\[
f(\gamma x + (1 - \gamma) y + z) \leq \left( r - \frac{\alpha \gamma (1 - \gamma) \| x - y \|^2}{4r} + \frac{\alpha}{4r} \gamma (1 - \gamma) \| x - y \|^2 \right)^2 = r^2
\]

\[\square\]

A.2 Proof of corollary 1

Proof. The strong convexity of the set w.r.t. \( \| \cdot \|_p \) is an immediate consequence of lemma 3.

Since \( \mathbb{B}_p(r) \) is \( \alpha = (p - 1)/r \) strongly convex w.r.t. the norm \( \| \cdot \|_p \), we have that given \( x, y \in \mathbb{B}_p(r), \gamma \in [0, 1] \) and \( z \in \mathbb{R}^n \) such that \( \| z \|_p \leq 1 \) it holds that,

\[
\gamma x + (1 - \gamma) y + \frac{\alpha}{2} \gamma (1 - \gamma) \| x - y \|_p^2 z \in \mathbb{B}_p(r)
\]
For any $p \in (1, 2]$ and vector $v \in \mathbb{R}^n$ it holds that,
\[
\|v\|_2 \leq \|v\|_p \leq n^{\frac{1}{p} - \frac{1}{2}} \|v\|_2
\] (8)

Given a vector $z' \in \mathbb{R}^n$ such that $\|z'\|_F \leq 1$ we have that,
\[
\left\| \frac{\alpha}{2} \gamma (1 - \gamma) \|x - y\|_2^2 z' \right\|_p = \frac{\alpha}{2} \gamma (1 - \gamma) \|x - y\|_2^2 \|z'\|_p
\]

Using (8) we have,
\[
\left\| \frac{\alpha}{2} \gamma (1 - \gamma) \|x - y\|_2^2 z' \right\|_p \leq \frac{\alpha}{2} \gamma (1 - \gamma) \|x - y\|_2^2 n^{\frac{1}{p} - \frac{1}{2}} \|z'\|_2
\]
\[
\leq \frac{\alpha n^{\frac{1}{p} - \frac{1}{2}}}{2} \gamma (1 - \gamma) \|x - y\|_2^2
\]

Hence, $\mathbb{B}_p(r)$ is $\alpha n^{\frac{1}{p} - \frac{1}{p^*}} = \frac{(p-1)n^{\frac{1}{p} - \frac{1}{p}}}{p}$ strongly convex with respect to $\|\cdot\|_2$. □

A.3 Proof of lemma 5

Proof. Since $\|\cdot\|_p$ and $\|\cdot\|_q$ are dual norms, we have using Holder’s inequality that for all $x \in \mathbb{B}_p(r)$,
\[
x^T c \geq -\|x\|_p \|c\|_q \geq -r \|c\|_q
\]

Thus choosing $x_i = -\frac{r}{\|c\|_q} \text{sgn}(c_i) |c_i|^{q-1}$ we have that,
\[
x^T c = -\sum_{i=1}^n \frac{r}{\|c\|_q} \text{sgn}(c_i) |c_i|^{q-1} = -\sum_{i=1}^n \frac{r}{\|c\|_q^{q-1}} |c_i|^q = -r \|c\|_q
\]

Moreover,
\[
\|x\|_p^p = \frac{r^p}{(\|c\|_q^{q-1})^p} \sum_{i=1}^n (|c_i|^{q-1})^p
\]

Since $p = q/(q - 1)$ we have that,
\[
\|x\|_p^p = \frac{r^p}{\|c\|_q^q} \sum_{i=1}^n |c_i|^q = r^p
\]

Thus we have that $x \in \mathbb{B}_p(r)$. □
A.4 Proof of lemma 7

Proof. Since \( \| \cdot \|_{S(p)} \) and \( \| \cdot \|_{S(q)} \) are dual norms we from Holder’s inequality that for all \( X \in \mathbb{B}_{S(p)}(r) \),

\[
X \cdot C \geq -\|X\|_{S(p)} \|C\|_{S(q)} \geq -r\|C\|_{S(q)} = r\|\lambda(C)\|_q
\]

By our choice of \( X \) we have that,

\[
X \cdot C = \text{Tr}(X^T C) = \text{Tr}(V \text{Diag}(\lambda)^T U^T \Sigma V^T) = \text{Tr}(V \text{Diag}(\lambda)^T \Sigma)
\]

\[
= \sum_{i=1}^{\min(m,n)} - \frac{r}{\|\lambda(C)\|_q^{q-1}} \text{sgn}(\lambda(C)_i) |\lambda(C)_i|^{q-1} \cdot \lambda(C)_i = - \frac{r}{\|\lambda(C)\|_q^{q-1}} \sum_{i=1}^{\min(m,n)} |\lambda(C)_i|^q
\]

Moreover,

\[
\|X\|_{S(p)}^p = \|\lambda(X)\|_p^p = \frac{r^p}{\|\lambda(C)\|_q^{q-1}} \sum_{i=1}^{n} (|\lambda(C)_i|^{q-1})^p
\]

Since \( p = q/(q-1) \) we have that,

\[
\|X\|_{S(p)}^p = \frac{r^p}{\|\lambda(C)\|_q^{q-1}} \sum_{i=1}^{n} |\lambda(C)_i|^q = r^p
\]

Thus we have that \( X \in \mathbb{B}_{S(p)}(r) \). \qed

A.5 Proof of lemma 8

The following lemma will be of use in the proof.

Lemma 10. For any matrix \( A \in \mathbb{R}^{m \times n} \) and \( s, p \in (1, 2] \) it holds that,

\[
\|A\|_F \leq \|A\|_{s, p} \leq n^{\frac{1}{2} - \frac{1}{2}} m^{\frac{1}{2} - \frac{1}{2}} \|A\|_F
\]

Proof. For any vector \( v \in \mathbb{R}^n \) and \( p \in (1, 2] \) it holds that,

\[
\|v\|_2 \leq \|v\|_p \leq n^{\frac{1}{2} - \frac{1}{2}} \|v\|_2 \tag{9}
\]

Denote by \( A_i \) the \( i \)th row of \( A \). For any \( i \in [m] \) and \( p \in (1, 2] \) it holds that,

\[
\|A_i\|_2 \leq \|A_i\|_p \leq n^{\frac{1}{2} - \frac{1}{2}} \|A_i\|_2 \tag{10}
\]

Note that by definition \( \| \cdot \|_F \equiv \| \cdot \|_{2, 2} \). Applying (9) and (10) we have,

\[
\|A\|_F = \|A\|_{2, 2} = \|(\|A_1\|_2, \|A_2\|_2, \ldots, \|A_m\|_2)\|_2 \leq \|(\|A_1\|_s, \|A_2\|_s, \ldots, \|A_m\|_s)\|_p \leq n^{\frac{1}{2} - \frac{1}{2}} m^{\frac{1}{2} - \frac{1}{2}} \|(\|A_1\|_2, \|A_2\|_2, \ldots, \|A_m\|_2)\|_2 = n^{\frac{1}{2} - \frac{1}{2}} m^{\frac{1}{2} - \frac{1}{2}} \|A\|_F
\]

\qed
We can now prove lemma 8.

Proof. Let $z, q$ be such that $1/z + 1/s = 1$ and $1/q + 1/p = 1$. Note that $z, q \in [2, \infty)$. The norm $\| \cdot \|_{z,q}$ is the dual norm to $\| \cdot \|_{s,p}$ (see [12] for instance).

According to lemma 4, the functions $\|x\|_s^2$ and $\|x\|_p^2$ are $\alpha_s = 2(s - 1)$ strongly convex w.r.t. $\| \cdot \|_p$ and $\alpha_p = 2(p - 1)$ strongly convex w.r.t. $\| \cdot \|_q$ respectively. Hence by the strong convexity/strong/smoothness duality (see theorem 3 in [12]) we have that the functions $\|x\|_z^2$ and $\|x\|_q^2$ are $\alpha_s^{-1}$ smooth w.r.t. $\| \cdot \|_z$ and $\alpha_p^{-1}$ smooth w.r.t. $\| \cdot \|_q$ respectively.

By theorem 13 in [12] we have that the function $\|x\|_z^2$ is ($\alpha_s^{-1} + \alpha_p^{-1}$) smooth with respect to the norm $\| \cdot \|_{z,q}$. Again using the strong convexity/smoothness duality we have that $\|x\|_s^2$ is ($\alpha_p^{-1} + \alpha_s^{-1}$) strongly convex with respect to the norm $\| \cdot \|_{s,p}$. The first part of the lemma now follows from applying lemma 3.

A.6 Proof of lemma 9

Proof. Since by choice of $z, q$ it holds that $\| \cdot \|_{s,p}, \| \cdot \|_{z,q}$ are dual norms, we have by Holder’s inequality that,

$$X \cdot C \geq -\|X\|_{s,p}\|C\|_{z,q} \geq -r\|C\|_{z,q}$$
Thus choosing $X_{i,j} = -\frac{r}{\|C\|_{z,q}^s \|C_i\|_{z-q}^s} \text{sgn}(C_{i,j}) |C_{i,j}|^{z-1}$ we have that, 

$$X \cdot C = \sum_{i \in [m], j \in [n]} X_{i,j}C_{i,j} = \sum_{i \in [m], j \in [n]} -\frac{r}{\|C\|_{z,q}^s \|C_i\|_{z-q}^s} \text{sgn}(C_{i,j}) |C_{i,j}|^{z-1} \cdot C_{i,j}$$

$$= \sum_{i \in [m], j \in [n]} -\frac{r}{\|C\|_{z,q}^s \|C_i\|_{z-q}^s} |C_{i,j}|^z = \sum_{i \in [m]} -\frac{r}{\|C\|_{z,q}^s \|C_i\|_{z-q}^s} \|C_i\|^z = -\frac{r}{\|C\|_{z,q}^s} |C|^q = -r|C|_{z,q}$$

Moreover, for all $i \in [m]$ it holds that, 

$$\|X_i\|_s = \sum_{j=1}^{n} |X_{i,j}|^s = \frac{r^s}{\|C\|_{z,q}^s \|C_i\|_{z-q}^s} \sum_{j=1}^{n} |C_{i,j}|^{s(z-1)}$$

Since $s = z/(z - 1)$ we have, 

$$\|X_i\|_s = \frac{r^s}{\|C\|_{z,q}^s \|C_i\|_{z-q}^s} |C_i|^z = \frac{\|C_i\|^{s(z-1)}_{z-q}}{\|C\|_{z,q}^s} \cdot r^s$$

Using $z = s/(s - 1)$ we have, 

$$\|X_i\|_s = \frac{\|C_i\|_{z}^{s(q-1)}}{\|C\|_{z,q}^{s(q-1)}} r^s$$

Thus, 

$$\|X_i\|_s = \left( \frac{\|C_i\|_{z}}{\|C\|_{z,q}} \right)^{q-1} r$$

Finally we have, 

$$\|X\|_{s,p}^p = \sum_{i \in [m]} \|X_i\|_s^p = \sum_{i \in [m]} \left( \frac{\|C_i\|_{z}}{\|C\|_{z,q}} \right)^{p(q-1)} r = \sum_{i \in [m]} \left( \frac{\|C_i\|_{z}}{\|C\|_{z,q}} \right)^q r$$

$$= \frac{r}{\|C\|_{z,q}^s} \sum_{i \in [m]} \|C_i\|_{z}^q = r$$