Coarse and Sharp Thresholds of Boolean Constraint Satisfaction Problems

Gabriel Istrate

Abstract

We study threshold properties of random constraint satisfaction problems under a probabilistic model due to Molloy [11]. We give a sufficient condition for the existence of a sharp threshold that leads (for boolean constraints) to a necessary and sufficient for the existence of a sharp threshold in the case where constraint templates are applied with equal probability, solving thus an open problem from [3].

1 Introduction

Classifying threshold properties of random constraint satisfaction problems is a problem that has been intensely studied in recent literature. The well-known Friedgut-Bourgain result [5] showed that 3-satisfiability has a sharp threshold, via a more general result on threshold properties of monotonic sets.

For random satisfiability problems (i.e. constraint satisfaction problems with a boolean domain) we have obtained [7] a classification of thresholds of such properties, under a random model that allows constraints of different lengths. The results from [7] intuitively show that satisfiability problems with a coarse threshold qualitatively "behave like random Horn satisfiability", a problem with a known coarse threshold [8]. A drawback of these results is that the classification is not “structural” (that is defined in terms of properties of the formula hypergraph), and thus does not offer suggestions for a suitable generalization to non-boolean domains.

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e-mail: istrate@lanl.gov, CCS-5, Basic and Applied Simulation Science, Los Alamos National Laboratory, Mail Stop M 997, Los Alamos, NM 87545, U.S.A.
Recently Molloy \[11\] has investigated a model of random constraint satisfaction that allows constraints of the same arity and unequal probabilities for the application of the given constraint templates (hence is only partly comparable with the results in \[7\]). He offers a nice structural condition that is necessary and sufficient for the existence of a weaker form of threshold property (he calls a \textit{transition}).

While Molloy’s model certainly has several remarkable features (such as the location of the transition in the region of formulas whose ratio between clauses and variables is constant), it was observed in \[11\] that the necessary and sufficient condition for the existence of a transition is \textit{not} sufficient for the existence of a \textit{sharp} threshold. The counterexample involves binary constraints on \{0, 1, 2, 3\} and a nonuniform probability distribution on the set of such templates. For the case of a uniform probability distribution he gave (Theorem 6) a sufficient condition for the existence of a sharp threshold.

In this note we strengthen this latter result: We give a more general sufficient condition for the existence of a sharp thresholds, that allows to obtain a precise classification of sharp/coarse thresholds for random \textit{satisfiability problems} (problems with a binary domain), confirming thus a conjecture due to Creignou and Daudé. The key to these results is a “monotonicity” property of the Friedgut-Bourgain method for proving the existence of a sharp threshold. We have first observed this property in \[7\], and it is, we believe, of independent interest.

2 Preliminaries

Throughout the paper we will assume familiarity with the general concepts of phase transitions in combinatorial problems (see e.g. \[10\]), random structures \[2\]. Some papers whose concepts and methods we use in detail (and we assume greater familiarity with) include \[5\], \[1\].

Consider a monotonically increasing problem \(A = (A_n)\). The two models used in the theory of random structures are:

- The \textit{constant probability model} \(\Gamma(n, p)\). A random sample from this model is obtained by independently setting to 1 with probability \(p\) each bit of the random string.

- The \textit{counting model} \(\Gamma(n, M)\). A random sample from this model is obtained by setting to 1 \(M\) different bits chosen uniformly at random from the \(n\) bits of the random sample.

As usual, for \(\epsilon > 0\) let \(p_\epsilon = p_\epsilon(n)\) define the canonical probability such that \(\Pr_{x \in \Gamma(n, p_\epsilon(n))} [x \in A] = \epsilon\). The probability that a random sample \(x\) satisfies prop-
property $A$ (i.e. $x \in A$) is a monotonically increasing function of $p$. Sharp thresholds are those for which this function has a “sudden jump” from value 0 to 1:

**Definition 1** Problem $A$ has a sharp threshold iff for every $0 < \epsilon < 1/2$, we have
\[
\lim_{n \to \infty} \frac{p_{1-\epsilon}(n) - p_{\epsilon}(n)}{p_{1/2}(n)} = 0.
\] $A$ has a coarse threshold if for some $\epsilon > 0$ it holds that
\[
\lim_{n \to \infty} \frac{p_{1-\epsilon}(n) - p_{\epsilon}(n)}{p_{1/2}(n)} > 0.
\]

For satisfiability problems (whose complements are monotonically increasing) the constant probability model amounts to adding every constraint (among those allowed by the syntactic specification of the model) to the random formula independently with probability $p$. Related definitions can be given for the other two models for generating random structures, the counting model and the multiset model [2]. Under reasonable conditions [2] these models are equivalent, and we will liberally switch between them. In particular, for satisfiability problem $A$, and an instance $\Phi$ of $A$, $c_A(\Phi)$ will denote its constraint density, the ratio between the number of clauses and the number of variables of $\Phi$. To specify the random model in this latter cases we have to specify the constraint density as a function of $n$, the number of variables. We will use $c_A$ to denote the value of the constraint density $c_A(\Phi)$ (in the counting/multiset models) corresponding to taking $p = p_{1/2}$ in the constant probability model. $c_A$ is a function on $n$ that is believed to tend to a constant limit as $n \to \infty$. However, Friedgut’s proof [5] of a sharp threshold in $k$-SAT (and our results) leave this issue open.

**Definition 2** Let $\mathcal{D} = \{0, 1, \ldots, t - 1\}, t \geq 2$ be a fixed set. Consider the set of all $2^t - 1$ potential nonempty binary constraints on $k$ variables $X_1, \ldots, X_k$. We fix a set of constraints $C$ and define the random model $CSP(C)$. A random formula from $CSP_{n,p}(C)$ is specified by the following procedure:

- $n$ is the number of variables.
- for each $k$-tuple of ordered distinct variables $(x_1, \ldots, x_k)$ and each $C \in C$ add constraint $C(x_1, \ldots, x_k)$ independently with probability $p$.

When the number of variables $n$ is known (or unimportant) we will drop it from our notation, and write $CSP_p(C)$ instead.

**Remark 1** The model in Definition 2 differs from the model in [11] in two respects:

- It is a “constant probability”-type model (the one in [11] is a “counting”-type model).
More importantly, all templates $C \in \mathcal{C}$ are applied with the same probability (in [11] a probability $P$ on the set of all templates is considered, and templates are instantiated according to this distribution).

However, as discussed in [11] (Remark 2) one can map Molloy’s model onto a modified version of the constant probability model. In particular Definition 2 corresponds to this mapping when $P$ is the uniform distribution, which is why we chose this definition as the starting point.

**Definition 3** If $\Phi$ is a constraint satisfaction problem, the formula hypergraph of $\Phi$ is the hypergraph $H$ that has
- the set of variables that appear in $\Phi$ as vertices.
- the sets of variables that appear together in a constraint of $\Phi$ as edges.

$H$ is tree-like if it is a connected acyclic hypergraph, and is unicyclic if there exists an edge $e \in H$ such that $H \setminus e$ is tree-like.

### 3 Coarse and sharp thresholds of random generalized constraint satisfaction problems

In this section we study the sharpness of the threshold for random generalized constraint satisfaction problem defined by Molloy [11]. He defined a weaker type of threshold (he calls transition), and provided a necessary and sufficient condition for the existence of a transition, called very well-behavedness of the constraint set.

**Definition 4** A value $\delta$ is 0-bad if there exists some canonical variable $x_i$ and constraint $C \in \mathcal{C}$ such that $C \models (x_i \neq \delta)$. We say that $\delta$ is $j$-bad if there exists some constraint $C \in \mathcal{C}$ and variable $x_i$ such that $C \land (x_i = \delta)$ implies that some other variable $x_k$ of $C$ must be assigned a $j'$-bad value, for some $j' < j$. $\delta$ is bad if it is $j$-bad for some $j$ and good otherwise.

**Definition 5** A set of constraints $\mathcal{C}$ is well-behaved iff:

1. there exists at least one good value in $D$.
2. for every $\delta \in D$ there exists at least one constraint $C \in \mathcal{C}$ not satisfied by the assignment $(\delta, \delta, \ldots, \delta)$.

$\mathcal{C}$ is very well-behaved if, in addition to the previous two properties, satisfies the following property: any constraint formula from CSP($\mathcal{C}$) whose constraint hypergraph is a cycle has a satisfying assignment where no variable is assigned a bad value.
Molloy has proved that the condition that $C$ is very well behaved is necessary for $CSP(C)$ to have a sharp threshold. Unfortunately this condition is not also sufficient for the existence of a sharp threshold: there exist pathological examples of very-well behaved binary constraint satisfaction problems with a 4-ary domain that have a transition but do not have a sharp threshold:

**Example 1** Let $C$ consist of two binary constraints over domain $\{0, 1, 2, 3\}$. The first constraint $C_1(x, y)$, forbids the pair $(x, y)$ from taking values from the set $(0, 0), (1, 1), (2, 2), (3, 3)$. The second one, $C_2(x, y)$ forbids the situation when one of the variables takes a value from the set $\{0, 1\}$ while the other constraint takes a value from the set $\{2, 3\}$. Molloy’s model allows for constraint templates to be applied with nonuniform probabilities, in this case $P(C_1) = 1/3$. $P(C_2) = 2/3$.

Molloy’s counterexample involves non-uniform probabilities, so it would be tempting to conjecture that at least in the case of uniform probabilities very well-behavedness is necessary and sufficient for the existence of a sharp threshold of $CSP(C)$. Unfortunately not even this is true:

**Example 2** Let $K$ consist of three constraints of arity 4 over domain $\{0, 1, 2, 3\}$ defined as follows (with respect to the constraints in Example 1):

\[
K_1(x, y, z) = C_1(x, y),
\]
\[
K_2(x, y, z) = C_2(x, y),
\]

and
\[
K_3(x, y, z) = C_2(z, y),
\]
The constraints are applied with uniform probability. It is easy to see that $CSP(K)$ is equivalent to the constraint satisfaction problem from Molloy’s counterexample.

Nevertheless, in what follows we will obtain a necessary condition for the existence of a sharp threshold that completely solves the problem in the case of binary constraints applied with uniform probability.

**Definition 6** A set of constraints $C$ is extremely well-behaved iff:

1. $C$ is very-well behaved.

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¹we had an incorrect proof of this claim in an earlier version of the paper
²we thank an anonymous referee for this counterexample.
2. There exists a mapping $\Gamma$ from good values to constraints in $\mathcal{C}$ such that, for every good value $\delta$, constraint $\Gamma_\delta := \Gamma(\delta) \in \mathcal{C}$ satisfies

$$\Gamma_\delta(x_1, \ldots, x_k) \models (x_1 = \delta) \lor \ldots \lor (x_k = \delta).$$

(1)

**Theorem 1** Let $\mathcal{C}$ be a set of extremely well-behaved constraints. Then $\text{CSP}(\mathcal{C})$ has a sharp threshold.

The result of the theorem is incomparable with Molloy’s sufficient condition for the existence of a sharp threshold: our result does not imply the fact that 3-coloring has a sharp threshold [1], while his does. On the other hand his result is not strong enough to yield the Corollary below.

We now apply the previous result to the case of boolean constraint satisfaction (satisfiability) problems:

**Definition 7** Constraint $C_2$ is an implicate of $C_1$ iff every satisfying assignment for $C_1$ satisfies $C_2$.

**Definition 8** A boolean constraint $C$ strongly depends on a literal if it has an unit clause as an implicate.

**Definition 9** A boolean constraint $C$ strongly depends on a 2-XOR relation if \( \exists i, j \in \{1, \ldots, k\} \) such that constraint “\( x_i \neq x_j \)” is an implicate of $C$.

In this case we obtain the following explicit result, that totally settles the case of random satisfiability problems, thus solving the open problem from [3] and extending the results from [4], where the result was shown to hold under the additional restriction that all constraint templates in set $\mathcal{P}$ are symmetric.

**Corollary 1** Consider a generalized satisfiability problem $\text{SAT}(\mathcal{C})$ (that is not trivially satisfiable by the “all zeros” or “all ones” assignment).

1. if some constraint in $\mathcal{C}$ strongly depends on one component then $\text{SAT}(\mathcal{C})$ has a coarse threshold.

2. if some constraint in $\mathcal{C}$ strongly depends on a 2XOR-relation then $\text{SAT}(\mathcal{C})$ has a coarse threshold.

3. in all other cases $\text{SAT}(\mathcal{C})$ has a sharp threshold.
**Proof.** The first two cases were proved by Creignou and Daudé in [3]. In the third case first, it is easy to see that both values 0 and 1 are good, since there is no 0-bad value (hence no bad value, otherwise $C$ would strongly depend on one variable).

Since constraints in $C$ are not 0/1-valid and their domain is boolean it follows that condition (1) in the definition of extreme well-behavedness is satisfied. So all it is left to show is that $C$ is very well behaved.

Indeed, consider a formula $\Phi$ whose associated hypergraph is a cycle. If $\Phi$ were unsatisfiable then it would be minimally unsatisfiable (since all acyclic formulas are satisfiable). But Theorem 4.3 in [3] prevents that from happening, since for all minimally unsatisfiable formulas $|\text{Var}(S)| \leq (k-1)|S| - 1$, whereas for a cycle $|\text{Var}(S)(k-1)|S|$.

\[ \square \]

### 4 Proof of Theorem 1

Our proof relies on the Friedgut-Bourgain criterion for the existence of a sharp threshold in any monotonic graph (or hypergraph) property $A$.

It is a well-known (and easy to see) fact that if property $A$ has a coarse threshold then there exists $0 < \epsilon < 1/2$, $p^* = p^*(n) \in [p_1 - \epsilon, p_\epsilon]$ and $C > 0$ such that

\[ p \cdot \frac{d\mu_p(A)}{dp}|_{p=p^*(n)} < C. \]  
(2)

Bourgain and Friedgut show\(^3\) that

**Proposition 1** Suppose $p = o(1)$ is such that condition (2) holds. Then there is $\delta = \delta(C) > 0$ such that either

\[ \mu_p(x \in \{0,1\}^n \mid x \text{ contains } x' \in A \text{ of size } |x'| \leq 10C) > \delta \]  
(3)

or there exists $x' \notin A$ of size $|x'| \leq 10C$ such that the conditional probability

\[ \mu_p(x \in A \mid x \supset x') > \mu_p(A) + \delta. \]  
(4)

**Observation 1** We will, in fact, need one property that is not directly guaranteed by the Bourgain-Friedgut result as stated in [5], but follows from an observation made in [6]. For a finite set of words $W$ define the filter generated by $W$, $F(W)$ as

\[ F(W) = \{x \mid (\exists y \in W) \text{ with } x \supseteq y\}. \]

\(^3\)In [5] the proposition is stated assuming for convenience that $p = p_{1/2}$, but this is not needed. We give here the general statement.
Friedgut observed (\cite{friedgut2002a}, remarks on pages 5-6 of that paper) that the statement of condition (4) can be strengthened in the sense that the set $W$ of “booster” sets $x'$ satisfies $\mu_p(F(W)) = \Omega(1)$. Returning now to the case of random constraint satisfaction problems, since the number of isomorphism types of formulas of bounded size is finite, there exists $\delta > 0$ and a satisfiable booster formula $F$ such that

1. condition (4) holds with $x' = F$.
2. Formula $F$ appears with probability $\Omega(1)$ as a subformula in a random formula in $CSP_p(C)$.

A standard observation is that in the second condition of Proposition 1 instead of conditioning on the presence of $x'$ as a subset of $x$ one can, instead, add it:

**Proposition 2** Suppose $p = o(1)$ is such that condition (2) holds. Then there is $\delta = \delta(C) > 0$ such that either

$$\mu_p(x \in \{0,1\}^n \mid x contains x' \in A of size |x'| \leq 10C) > \delta$$

or there exists $x' \not\in A$ of size $|x'| \leq 10C$ such that

$$\mu_p(x \cup x' \in A) > \mu_p(A) + \delta.$$  \hfill (5)

Finally, note that for random constraint satisfaction problems, because of the invariance of such problems under variable renaming, one only needs to add a random copy of $x'$. That is, the following version of Proposition 2 holds:

**Proposition 3** Suppose $A = CSP(C)$ and $p = o(1)$ is such that condition (2) holds. Then there is $\delta = \delta(C) > 0$ such that either

$$\mu_p(x \in \{0,1\}^n \mid x contains x' \in A of size |x'| \leq 10C) > \delta$$

or there exists $x' \not\in A$ of size $|x'| \leq 10C$ such that, if $\Xi$ denotes the formula obtained by creating a copy of $x'$ on a random tuple of variables, then

$$\mu_p(x \cup \Xi \in A) > \mu_p(A) + \delta.$$  \hfill (6)

To show that random $CSP(C)$ has a sharp threshold, we will reason by contradiction. Assuming this is not the case, one needs to prove that the two conditions in Proposition 2 do not hold.

Suppose, indeed, that the first condition was true: that is, with positive probability it is true that a random formula $\Phi \in CSP(C)$ contains some unsatisfiable
subformula $\Phi'$ of size at most $10C$. One can, therefore, assume that with positive probability $\Phi$ contains a \textit{minimally unsatisfiable formula} $\Phi'$ of size at most $10C$.

On the other hand, with high probability all subformulas of a random formula $\Phi$ of size at most $10C$ are either tree-like or unicyclic. But because of well-behavedness, all formulas in $CSP(\mathcal{C})$ that are tree-like or unicyclic are satisfiable. Therefore the first condition in Proposition \ref{prop:un_young} cannot be true.

Assume, now, that the second condition is true: there exists a satisfiable formula $F$ of size at most $10C$, such that conditioning on a random formula in $CSP_p(\mathcal{C})$ containing a copy of $F$ boosts the probability of unsatisfiability by a value bounded away from zero. Because of the two conditions in Observation \ref{obs:two_conditions} we infer that there exists a constant $\delta > 0$ such that adding $F$ to a random formula $\Phi \in CSP_p(\mathcal{C})$ boosts the probability of unsatisfiability of the resulting formula by at least $\delta$. As we discussed, we assume that $F$ occurs with probability $\Omega(1)$ in a random formula in $CSP_p(\mathcal{C})$. Therefore $F$ is tree-like or unicyclic (this argument was also used in \cite{ref1}).

\textbf{Definition 10} A unit constraint is a constraint (not necessarily part of the constraint set $\mathcal{C}$) specified by a condition $X = \delta$, with $X$ being a variable and $\delta \in \mathcal{D}$.

\textbf{Lemma 4.1} Every tree-like or unicyclic formula has a satisfying assignment $W$ consisting only of good values.

\textbf{Proof.} This is easily proved by induction on the number of clauses for tree-like formulas, even in a stronger form: if we set one of the variables to an arbitrary good value, we can still set the other variables to good values in such a way that we obtain a satisfying assignment.

For a unicyclic formula we first set the variables appearing in its unique cycle to good values so that all such constraints are satisfied (this is possible since $\mathcal{C}$ is very well-behaved). We are now left with several tendrils, tree-like formulas on disjoint set of variables, the root of each such formula (the node appearing in the cycle) being set to a fixed good value, which we can satisfy as in the first case. \hfill $\Box$

\textbf{Claim 1} If $\Xi$ satisfies condition \ref{cond:xi} then there exists another formula $G$ that is specified by a finite conjunction of unit constraints

$$G \equiv (X_1 = \delta_1) \land \ldots \land (X_p = \delta_p),$$

with all the values $\delta_1, \ldots, \delta_p \in \mathcal{D}$ being good values, and that also satisfies condition \ref{cond:xi}.
**Proof.** Formula $\Xi$ appears with constant probability in a random $CSP(F)$ formula with probability $p$ and has constant size. Therefore $\Xi$ is either tree-like or unicyclic. The result follows easily from Lemma 4.1 by replacing $F$ with formula $G$ consisting of the conjunction of unit constraints corresponding to a satisfying assignment of $\Xi$ with good values. Indeed, $G$ is tighter than $\Xi$, so adding a random copy of $G$ instead of a random copy of $\Xi$ can only increase the probability that the resulting formula is unsatisfiable. □

The key to refuting condition (8) is to show that, if it did hold then, for every monotonically increasing function $f(n)$ that tends to infinity, we can also increase the probability of unsatisfiability by a positive constant if, instead of conditioning on $x$ containing a copy of formula $F$ we add $f(n)$ random constraints from constraint set $C$. We first prove:

**Claim 2** Let $0 < \tau < 1$ be a constant and let $p$ be such that $\mu_p(CSP(C)) \geq \tau$. Assume that $r \geq 1$ and that $g_1, g_2, \ldots, g_r$ are good values such that, when $(X_1, X_2, \ldots, X_r)$ is a random $r$-tuple of different variables

$$\Pr(\Phi \text{ has a satisfying assignment with } X_1 = g_1, \ldots, X_r = g_r) \leq \frac{\tau}{2}. \quad (9)$$

Then there exists constant $m \geq 1$ (that only depends on $k, r, \tau$) such that, if $\eta$ denotes a formula from $CSP(C)$ obtained by adding, for each good value $x$, $m \cdot r \cdot 2^k$ random copies of $\Gamma(x)$, then

$$\Pr(\Phi \cup \eta \text{ is satisfiable}) \leq \frac{\tau}{2} \quad (10)$$

**Proof.** We will give a proof of Claim 2 that is very similar to that of the corresponding proof in [1], thus obtaining the desired contradiction.

For $i \in \{1, \ldots, r\}$ define $A_i$ to be the event that the formula $\Phi$ has no satisfying assignment with the first $i$ constraints $X_1 = g_1, \ldots, X_i = g_i$ holding. Also define $A_0$ to be the event that $\Phi$ is not satisfiable.

The hypothesis translates as the fact that both inequalities $Pr(A_0) \geq \tau$ and $Pr(\overline{A_r}) \leq \frac{\tau}{2}$ are true. Therefore

$$Pr(A_r|A_0) = \frac{Pr(A_r \wedge A_0)}{Pr(A_0)} \leq \frac{\tau/2}{\tau} = \frac{1}{2}. \quad (11)$$

Thus we have

$$\alpha_r := Pr[A_r|\overline{A_0}] = Pr[A_{r-1}|\overline{A_0}] + Pr[A_r|\overline{A_{r-1}} \wedge \overline{A_0}] \cdot Pr[\overline{A_{r-1}}|\overline{A_0}] \geq \frac{1}{2} \quad (11)$$
\[ Pr[A_r|\overline{A_{r-1}} \land \overline{A_0}] = Pr[A_r|\overline{A_{r-1}}] \] is the fraction of variables that have to receive values different from \( g \) if constraints \( X_1 = g_1, \ldots, X_{r-1} = g_{r-1} \) are added to \( \Phi \); let \( C_r \) be the set of such variables. If instead of the last constraint we add a random copy of the constraint \( \Gamma(g_r) \) we spoil satisfiability as well when all the variables appearing in the constraint are in the set \( C_r \). Denoting \( \lambda_r = (Pr[A_r|\overline{A_{r-1}}])^k \), this last event happens with probability \( \lambda_r/(1 - o(1)) \), so the probability that the resulting random formula is unsatisfiable is at least

\[
\beta_r \ := \ Pr[A_{r-1}|A_0] + \frac{\lambda_r}{1 - o(1)} \cdot Pr[A_{r-1}|A_0].
\]

Because of the convexity of the function \( f(x) = x^k \) and constraint [11] by applying Jensen’s inequality it follows that

\[
\frac{1}{2^k} \leq \alpha_r^k = (Pr[A_{r-1}|A_0] \cdot 1 + Pr[A_r|\overline{A_{r-1}}] \cdot Pr[A_{r-1}|A_0])^k \leq
\]

\[
Pr[A_{r-1}|A_0] \cdot 1^k + Pr[A_r|\overline{A_{r-1}}]^k \cdot Pr[A_{r-1}|\overline{A_0}] =
\]

\[
= (Pr[A_{r-1}|A_0] + \lambda_r \cdot Pr[A_{r-1}|A_0]) = \beta_r \cdot (1 + o(1)).
\]

Thus \( \beta_r \geq \frac{1}{2^k} \cdot (1 - o(1)) \). The conclusion of this argument is that adding one random copy of \( \Gamma(b_r) \) instead of the \( r \)-th constraint lowers the probability of unsatisfiability to no less than \( \frac{1}{2^k} \cdot (1 - o(1)) \). Adding the copy of the constraint before the first \( r - 1 \) constraints and repeating the argument recursively implies the fact that, if instead of adding the \( r \) constraints to \( \Phi \) we add \( r \) random copies of \( \Gamma(b_1), \ldots, \Gamma(b_r) \) the probability of unsatisfiability of the resulting formula, given that \( \Phi \) was satisfiable, is at least \( \gamma_r = \frac{1}{2^k (1 - o(1))} \). Since the values \( b_1, \ldots, b_r \) can repeat themselves, the same is true if we add \( r \) random copies of \( \Gamma(x) \) for every good value \( x \).

Suppose now that we add \( r \cdot m \cdot 2^{k'} \) copies of each \( \Gamma(x) \) instead (that is, we repeat the random experiment \( m \cdot 2^{k'} \) times, for some integer \( m \geq 1 \)). By doing so the probability that none of the experiments will make the resulting formula unsatisfiable is at most \( (1 - \gamma_r)^m \cdot 2^{k'} \). For some constant \( m \) this is going to be at most \( 1 - \frac{x}{2} \). This means that

\[
Pr(\Phi \cup \eta \text{ is satisfiable}) \leq \frac{\tau}{2}
\]

(13)

\[ \Box \]

Condition [11] can be refuted directly, leading to a contradiction. This is done e.g. by Lemma 3.1 in [11]. For convenience, we now restate this result:
Lemma 4.2 For a monotone property $A$ let

$$\mu(p) = Pr[G \in \Gamma(n, p) \text{ has property } A],$$

$$\mu^+(p, M) = Pr[G_1 \cup G_2 \mid G_1 \in \Gamma(n, p), G_2 \in \Gamma(n, M) \text{ has property } A].$$

Let $A = A(n) \subseteq \{0, 1\}^n$ be a monotone property and $M = M(n)$ such that $M = o(\sqrt{np})$. Then:

$$|\mu(p) - \mu^+(p, M)| = o(1).$$

Now it is easy to obtain a contradiction: consider a random formula $\eta$ with $f(n)$ clauses, for some $f(n) \to \infty$. It is easy to show that the probability that $\eta$ contains, for some good value $x$, less than $M$ copies of constraint $\Gamma(x)$ is $o(1)$. So adding $\eta$ instead of the random formula in Claim 2 only increases the probability of the resulting formula being unsatisfiable. But then the conclusion of Lemma 4.2 directly contradicts that of Claim 2.

5 Conclusions

What made the proof work? Its main steps are, of course, quite similar to the proofs of similar results in [5, 11, 7, 11]. One element we want to highlight is a certain monotonicity property used in the proof of Claim 2. Since it already proved useful in obtaining further insights on classifying threshold properties of random constraint satisfaction problems [7, 9], and can reasonably be expected to help in obtaining a complete classification: In Claim 2 we have only used the fact that for every good value $\delta_1$ there exists a constraint $C_{\delta_1}(x_1, \ldots, X_k)$ in the image of map $\Gamma$ that implies the constraint $(X_1 = \delta_1) \lor \ldots \lor (X_k = \delta_1)$. This means that as long as the other steps of the proof continued to work we could have proved the Claim if the constraint $C_{\delta_1}$ was exactly the constraint $(X_1 = \delta_1) \lor \ldots \lor (X_k = \delta_1)$. This provides a general strategy for proving sharp threshold results:

1. identify a “base case” $B$ for which the analog of Claim 2 can be proved.

2. If we are given a set of constraints $C$ that are tighter than the ones of the base case $B$, to prove that $CSP(C)$ has a sharp threshold it is enough to verify that all the other steps of the proof still hold. The analog of Claim 2 will now follow from the corresponding claim for the base case $B$.

\footnote{Achlioptas and Friedgut assume $A$ to be a monotone graph property, but this fact is not used anywhere in their proof.}
To sum up, we have proved a sufficient condition for the existence of a sharp threshold for random constraint satisfaction problems that completely solves the boolean case. Example showed, however, that the results could not be extended to all very-well behaved sets of constraints. Obtaining a necessary and sufficient condition for the existence of a sharp threshold in random constraint satisfaction problems is an interesting open problem.

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