DAMPING OF KINETIC TRANSPORT EQUATION WITH DIFFUSE BOUNDARY CONDITION

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Abstract. We prove that exponential moments of a fluctuation of the pure transport equation decay pointwisely almost as fast as $t^{-3}$, when the diffuse boundary condition is imposed and the domain is any general convex subset of $\mathbb{R}^3$ with smooth boundary. The proof is based on a novel $L^1$-$L^\infty$ framework via stochastic cycles.

1. Introduction and the Result of this paper

An important and active research direction in the mathematical kinetic theory is on the asymptotic behavior of its solutions as $t \to \infty$ for both the collisional models (e.g. [7, 22, 15, 11, 10, 5]) and the collisionless models (e.g. [21, 2]). In this paper we are interested in a damping phenomenon induced by the mixing effect of a stochastic boundary in the simplest collisionless kinetic model. More precisely we consider a free transport equation in a bounded domain $\Omega \subset \mathbb{R}^3$, with the initial condition $F(t, x, v)|_{t=0} = F_0(x, v)$,

$$\partial_t F + v \cdot \nabla_x F = 0, \quad \text{for } (t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^3. \tag{1.1}$$

Throughout this paper we assume that the domain is smooth and convex ([3]): there exists a smooth function $\xi : \mathbb{R}^3 \to \mathbb{R}$ such that $\Omega = \{x \in \mathbb{R}^3 : \xi(x) < 0\}$ and $\sum_{i,j} \partial_i \partial_j \xi(x) \xi_i \xi_j \geq |\xi|^2$ for all $\xi \in \mathbb{R}^3$. The phase boundary $\gamma := \{(x, v) \in \partial \Omega \times \mathbb{R}^3\}$ can be decomposed into the outgoing boundary and incoming boundary: for the outward normal vector $n(x)$ at $x \in \partial \Omega$, $\gamma_\pm := \{(x, v) \in \partial \Omega \times \mathbb{R}^3, n(x) \cdot v \geq 0\}$. (1.2)

In this paper we consider an isothermal diffusive reflection boundary condition which is the simplest model among the family of stochastic boundary conditions (see [6] for the generalized models)

$$F(t, x, v) = c_{\mu} \mu(v) \int_{n(x) \cdot v_1 > 0} F(t, x, v_1)\{n(x) \cdot v_1\} dv_1, \quad \text{for } (t, x, v) \in \mathbb{R}_+ \times \gamma_-.$$ \tag{1.3}

For this model, $c_{\mu} \mu(v) = \frac{1}{(2\pi)^{d/2}}e^{-|v|^2/2}$ stands for the wall Maxwellian distribution of the wall temperature $= 1$, while we choose $c_{\mu} := \left( \int_{n(x) \cdot v_1 > 0} \mu(v_1)\{n(x) \cdot v_1\} dv_1 \right)^{-1} = \sqrt{2\pi}$. We set the total mass of the initial datum to be $\mathcal{M} \times |\Omega|$, for some $\mathcal{M} \geq 0$, so that

$$\int_{\Omega \times \mathbb{R}^3} F_0(x, v) dx dv = \int_{\Omega \times \mathbb{R}^3} \mathcal{M} \mu(v) dx dv. \tag{1.4}$$

The choice of $c_{\mu}$ formally guarantees a null flux condition of the boundary and the conservation of mass as well. These facts naturally lead us to study the asymptotic behavior of the fluctuation $f(t, x, v) = F(t, x, v) - \mathcal{M} \mu(v)$, which solves

$$\partial_t f + v \cdot \nabla_x f = 0, \quad \text{for } (t, x, v) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^3, \tag{1.5}$$

$$f(t, x, v)|_{t=0} = f_0(x, v) := F_0(x, v) - \mathcal{M} \mu(v), \quad \text{for } (x, v) \in \Omega \times \mathbb{R}^3, \tag{1.6}$$

$$f(t, x, v) = c_{\mu} \mu(v) \int_{n(x) \cdot v_1 > 0} f(t, x, v_1)\{n(x) \cdot v_1\} dv_1, \quad \text{for } (t, x, v) \in \mathbb{R}_+ \times \gamma_-.$$ \tag{1.7}

Due to its importance, the asymptotic behavior of the transport equation with the diffuse boundary condition has been actively studied recently in different aspects. We refer [18, 8] for more complete list of references. Among them we only overview some literatures which are directly relevant to our interest: a quantitative information on the asymptotic behavior of the moments of the fluctuation in $L^\infty$, $\sup_{x \in \Omega} \int_{\mathbb{R}^3} |v|^m |f(t, x, v)| dv$, as $t \to \infty$. It is worth to mention that a strong bound, such as a pointwise bound, of the moments is often essential in the study of nonlinear problems (e.g. Vlasov-Poisson-Boltzmann system in [16]). Perhaps the first quantitative study on the asymptotic behavior of the fluctuation can be found in [23], in which Yu provides a decay rate of moments of the fluctuation in $L^\infty$ when the boundary is an 1D slab using a probabilistic approach (of Markov chains of i.i.d. random variables). This approach has been successfully generalized to the multi-D cases of symmetric domains (a disk in 2D and a ball in 3D) in [17] with obtaining an optimal decay rate $t^{-d}$. The symmetry is an essential condition in their proof: under this condition the independent and identically distributed (i.i.d.) random variables can be formed; and a bound of derivatives of outgoing

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flux can be obtained (cf. see [9, 11] for the possible blow-up of derivatives in a general convex domain and non-convex domain respectively, and the formation of discontinuity in [12]). For a decay of the fluctuation in $L^1$, we refer [1] in the symmetric domains, [20] in a slab domains, and [13, 3] for general domains.

In this paper we contribute toward establishing a decay of exponential moments of the fluctuation in $L^\infty$ with an almost optimal rate $\frac{1}{\theta'}$ when the domain is a general convex (no need of symmetry) domain in 3D.

**Theorem 1.** Let $\Omega$ be smooth and convex. Assume (1.3) for any $\mathfrak{M} \geq 1$. Assume $\|e^{\theta|v|^2}f_0\|_{L^\infty_{x,v}} < \infty$ for $0 < \theta' < 1/2$, and $\|\varphi_4(t\tau)f_0\|_{L^1_{x,v}} < \infty$, with $\varphi_4(t\tau)$ defined through (1.10) and (2.19). There exists a unique solution $f(t,x,v) = F(t,x,v) - \mathfrak{M}\mu(v)$ to (1.10) and, $\sup_{t \geq 0} \|e^{\theta'|v|^2}f(t)\|_{L^\infty_{x,v}} \lesssim \|e^{\theta'|v|^2}f_0\|_{L^\infty_{x,v}}$, holds, and

$$\int_{\Omega \times \mathbb{R}^3} f(t,x,v) dv dx = \int_{\Omega \times \mathbb{R}^3} f_0(x,v) dv dx = 0, \text{ for all } t \geq 0. \quad (1.8)$$

Moreover, with a notation $\langle \cdot \rangle := e + |\cdot|$, $\sup_{x \in \Omega} \int_{\mathbb{R}^3} e^{\theta'|v|^2}|f(t,x,v)| dv \lesssim_{\theta'} \langle t \rangle^{-3}(\ln \langle t \rangle)^2$, for all $t \geq 0$ and $0 \leq \theta < \theta'$.

**Remark 1.** Without loss of generality we set $\mathfrak{M} = 1$ in the rest of the paper, for the sake of simplicity. Following the same proof of this paper it is straightforward to prove the result to a D-dimension for any $D \in \mathbb{N}$. However, for the sake of simplicity, we only consider the 3-dimension case in this paper.

**Remark 2.** Although in the proofs we use the convexity of the domain in many places, we believe that this assumption can be removed using more delicate study on the trajectory. In this paper we adopt the assumption to present our idea in a simpler manner.

**Notations.** We shall clarify some notations: $A \lesssim B$ if $A \leq CB$ for a constant $C > 0$ which is independent on $A, B$; $A \sim B$ if $A \lesssim B$ and $B \lesssim A$; $\| \cdot \|_{L^1_{x,v}}$, $\| \cdot \|_{L^\infty_{x,v}}$ or $\| \cdot \|_{\infty}$ for the norm of $L^1(\Omega \times \mathbb{R}^3)$; $\| \cdot \|_{L^\infty}$ for the norm of $L^\infty(\Omega \times \mathbb{R}^3)$; $\| f \|_{L^1_{x,v}}$ denotes $\int_{\Omega} |g(x,v)||n(x) \cdot v| dS_x dv$.

**1.1. Novel $L^1$-$L^\infty$ framework via Stochastic Cycles.** In a broad sense, our argument of the $L^1$-$L^\infty$ framework to prove Theorem 1 bears some resemblance to the framework developed in the study of the Boltzmann equation [3, 11, 8]. A foundational idea of our novel $L^1$-$L^\infty$ framework over the whole paper is to transfer a velocity mixing from the diffusive reflection (1.7) to a spatial mixing through the transport operator. This idea is realized via the stochastic cycles:

**Definition 1 ([3, 11, 8]).** Define the backward exit time $t_b$ and the forward exit time $t_f$

$$t_b(x,v) := \sup \{ s \geq 0 : x - rv \in \Omega, \quad \forall r \in [0,s) \}, \quad x_b(x,v) := x - t_b(x,v) v, \quad (t_b, t_f)(x,v) := (t_b(x,v), -v). \quad (1.10)$$

We define the stochastic cycles: $t_1(t,x,v) := t - t_b(x,v), \quad x_1(x,v) := x - t_b(x,v) v, \quad t_k(t,x,v) := t - t_{k-1}(x,v) v, \quad x_k(t,x,v) := x - t_{k-1}(x,v) v, \quad x_k(x,v) := x - t_{k-1}(x,v) v, \quad k = 1, \ldots, \infty, \quad (1.11)$

where a free variable $v_j \in V_j := \{ v_j \in \mathbb{R}^3 : n(x_j) \cdot v_j > 0 \}$.

**Lemma 1 ([3, 11, 8]).** Suppose $f$ solve (1.7) and $t_* \leq t$. For $g(t,x,v) := g(t)s(w)\psi(t,x,v)$ with given $\psi(t), w(v)$,

$$g(t,x,v) = \mathbf{1}_{t_* \leq t} \left\{ \int_{t_*}^t \tilde{g}'(s)w(s)v f(s,x - (t-s)v,v) ds \right\}$$

$$+ \int_{\Pi_{j=1}^k} 1_{t_{i+1} \leq t \leq t_i} \left\{ \int_{t_i}^{t_{i+1}} w(v_1)\tilde{g}'(s)w(s,x - (t-s)v_1,v_1) ds \right\} d\Sigma_k,$$

$$+ \mathbf{1}_{t_* \geq t} g(t,v) \mathbf{1}_{t \geq t_*} d\Sigma,$$

where $d\Sigma = \sigma_0 \cdots \sigma_{i+1} = \frac{\sigma_{i+1} \cdots \sigma_1}{\sigma_{i+1} \cdots \sigma_1}$, with a probability measure $\sigma_j = c_{\mu}(v_j)\{n(x_j) \cdot v_j\} dv_j$ on $V_j$.

**1.2. Weighted $L^1$-estimates.** As the first part of our $L^1$-$L^\infty$ framework, we prove an $L^1$-decay of the fluctuation $f$ as $t \to \infty$ in Proposition 2 following the idea of aperiodic Ergodic theorem (e.g. [3, 19]). We prove a key lower bound with a unreachable defect, crucially using the stochastic formulation in Lemma 1 (see the precise statement in Lemma 9): for $f_0 \geq 0, t - t_* \gg 1$

$$f(t,x,v) \geq m(v)(\|f(t_*)\|_{L^1_{x,v}} - \|1_{t_* \geq t \geq t_*} f(t_*)\|_{L^1_{x,v}}) \text{ for some non-negative function } m. \quad (1.15)$$

This unreachable defect, stems from small velocity particles in the outgoing flux of the diffuse reflection (1.3), is intrinsic unless the wall Maxwellian $c_{\mu}(v)$ vanishes around $|v| = 0$. 

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Next we control the unreachable defect using the weighted $L^1$-estimates. Due to the invariance of $x_b$ and $x_f$ under $v \cdot \nabla_x$, which has been crucially used in construction of the distance function invariant under Vlasov operator in [5], a weight $\varphi(t_f)$ provides an effective dissipation $v \cdot \nabla_x \varphi(t_f) = -\varphi'(t_f)$ for $\varphi \geq 0$, as long as a byproduct term on $\gamma_v$ can be controlled. Inspired by the proof of an $L^1$-trace theorem of [9], we derive that

**Lemma 2.** Suppose $\varphi(\tau) \geq 0$, $\varphi' \geq 0$, and

$$\int_1^\infty \tau^{-5} \varphi(\tau) d\tau < \infty. \quad (1.16)$$

Suppose $f$ solves (1.3) and (1.7). Then there exists $C > 0$ such that for all $0 \leq t_s \leq t$,

$$\|\varphi(t_f)f(t)\|_{L^1_{x,v}} + \int_{t_s}^t \|\varphi'(t_f)f\|_{L^1_{x,v}} + \int_{t_s}^t \|\varphi(t_f)f\|_{L^1_{x,v}} - \frac{1}{4} \int_{t_s}^t \|f\|_{L^1_{x,v}} \leq \|\varphi(t_f)f(t_s)\|_{L^1_{x,v}} + C\|f(t_s)\|_{L^1_{x,v}}. \quad (1.17)$$

It is worth to inform beforehand that the exponent $-5$ in (1.16) will basically restrict the decay rate of Theorem 1. Some postulation on the wall Maxwellian such as $\mu(v)/|v|^r < \infty$ for some $r > 0$ in (1.3) or a similar assumption on the inflow boundary condition would provide faster decay.

Employing a function $\varphi$ with $\varphi' \to \infty$ as $\tau \to \infty$ (see $\varphi_1$ in (3.18)), an $L^1$-term majorizes the reachable defect of the lower bound (1.15) with a large factor $\varphi(4.12)$. Adding (1.15) and (1.17) with the proper ratio, suggested by the large factor, we establish the uniform estimates of the following energies (see $\varphi_i$’s in (3.18)), with $\|m\|_{L^1_{x,v}} \sim \delta_m t_0$ (see (3.13)),

$$\|g\| := \|f\|_{L^1_{x,v}} + \frac{4\delta_m t_0}{\varphi(1 - \frac{\tau_0^2}{4})}\|\varphi_{i-1}(t_f) f\|_{L^1_{x,v}} + \frac{4\delta_m t_0}{\varphi_i - 1(\frac{\tau_0}{4})}\|\varphi_i(t_f) f\|_{L^1_{x,v}}, \text{ for } i = 1, 4. \quad (1.18)$$

Finally we interpolate $\|\varphi_{i}(t_f) f\|_{L^1_{x,v}}$ by $\|\varphi_0(t_f) f\|_{L^1_{x,v}}$ and $\|\varphi_4(t_f) f\|_{L^1_{x,v}}$, and using the boundedness of $\|f\|_{L^1}$, we prove the $L^1$-decay result (see also the similar result in [3]):

**Proposition 2.** *Given the same assumptions of Lemma 1*,

$$\|f(t)\|_{L^1_{x,v}} \lesssim (\ln(t))^2 (t)^{-4}(\|e^{|v|^2} f_0\|_{L^\infty_{x,v}} + \|\varphi_4(t_f) f_0\|_1). \quad (1.19)$$

1.3. An $L^\infty$-estimate of Moments. We bootstrap the $L^1$-decay secured in Proposition 2 to the pointwise bound of the moments. Again, the crucial tool is the stochastic cycle representation in Lemma 1 for $t_s = 0$. In light of (1.19), we have a natural choice of $\varphi$ so that $\varphi'(t) \lesssim (\ln(t))^{-2}(t)^4$ (see (4.13)). We first establish the control of the time integration terms of (1.13):

**Lemma 3.** *For $i = 1, \ldots, k-1$, $w(v) = e^{|v|^2}$ for $\theta > 0$, and a differentiable $g(t)$, we have*

$$\left|\int_{\Pi_{1}^{k-1} \mathcal{V}_j} 1_{t_{k+1} < \sigma \leq t} \int_{t_i}^{t_{i+1}} w(v_i) g'(s) f(s, x_i - (t_i - s)v_i, v_i) ds d\Sigma_j\right| \lesssim t_i^{\theta} \left\|g(s)f(s)\right\|_{L^1_{x,v}} ds. \quad (1.20)$$

The key idea of the proof is using the change of variables $v_{i-1} \mapsto (x_b(x_{i-1}, v_{i-1}), t_b(x_{i-1}, v_{i-1}))$, which has been crucially used in evaluating the boundary singularity in [5]. By this change of variables we are able to convert the velocity integral of $d\sigma_{j-1}$ into an integration of the spatial variable $x_i - (t_i - s)v_i = x_b(x_{i-1}, v_{i-1}) - (t_i - t_b(x_{i-1}, v_{i-1}) - s)v_i$, while the singularity occurs from its Jacobian when $t_b(x_{i-1}, v_{i-1}) = 0$ (see Lemma 5). We remedy such singularity by applying the change of variables twice for $j = i - 1$ and $j = i - 2$; among the free variables $\{x_b(x_{i-1}, v_{i-1}), t_b(x_{i-1}, v_{i-1}), x_b(x_{i-2}, v_{i-2}), t_b(x_{i-2}, v_{i-2})\}$ we utilize $x_b(x_{i-1}, v_{i-1})$ and $t_b(x_{i-2}, v_{i-2})$ for the spatial variables $x_i - (t_i - s)v_i = x_b(x_{i-1}, v_{i-1}) - (t_i - t_b(x_{i-2}, v_{i-2}) - t_b(x_{i-1}, v_{i-1}) - s)v_i$, while we are able to appease singularity from the two change of variables using the integration of $x_b(x_{i-2}, v_{i-2})$ and $t_b(x_{i-1}, v_{i-1})$.

Next we control (1.14) by establishing the following estimate:

**Lemma 4.** *There exists $C = C(\Omega) > 0$ (see (4.12) for the precise choice) such that*

$$\text{if } k \geq Ct \text{ then } \sup_{(x,v) \in \Omega \times \mathbb{R}^3} \left(\int_{\Pi_{k-1}^{k-1} \mathcal{V}_j} 1_{t_k(t, x, v, v_1, \ldots, v_{k-1}) \geq 0} d\sigma_1 \cdots d\sigma_{k-1}\right) \lesssim e^{-t}. \quad (1.21)$$

The similar results have been used in [3] [11] [8] but in this paper we improve the result (the choice of $k$, in particular) using sharper bound for the summation of combination from the Stirling’s formula.

In the rest of paper, we collect some basic preliminaries in Section 2; then study the weighted $L^1$-estimates and prove Proposition 2 in Section 3; and finally key results in the $L^\infty$-estimate of moments, and then Theorem 1 in Section 4.
2. Preliminaries

In this section we state basic preliminaries mainly collected from [5 8 9 10 11].

Lemma 5 (Lemma 9 in [4]). For \( x \in \partial \Omega \), consider a map
\[
v \in \{ v \in \mathbb{R}^3 : n(x) \cdot v > 0 \} \mapsto (x_b, t_b) := (x_b(x, v), t_b(x, v)) \in \partial \Omega \times \mathbb{R}^+.
\] (2.1)
Then the map (2.1) is bijective and has the change of variable formula as
\[
dv = |t_b|^{-1}n(x_b) \cdot (x - x_b)|dtdS_{x_b}.
\] (2.2)

2) Similarly we have a bijective map
\[
v \in \{ v \in \mathbb{R}^3 : n(x) \cdot v < 0 \} \mapsto (x_f, t_f) := (x_f(x, v), t_f(x, v)) \in \partial \Omega \times \mathbb{R}^+, \text{ with } dv = |t_f|^{-1}n(x_f) \cdot (x - x_f)|dtdS_{x_f}.
\] (2.3)

Lemma 6 (Lemma 3, Lemma 4 in [5]). For any \( g \),
\[
\int_{\gamma_+} \int_{0}^{t_+} g(x \mp sv, v)n(x) \cdot v|dsdv|dS_x = \int_{\gamma_\mp} \int_{0}^{t_\mp} g(y, v)n(y) \cdot v|dv|dS_y.
\] (2.4)

Here, for the sake of simplicity, we have abused the notations temporarily: \( t_- = t_b, x_- = x_b \) and \( t_+, x_+ = x_f \).

Lemma 7. Suppose \( f \) solve (1.6) and (1.7). For \( 0 \leq t_s \leq t \),
\[
\|f(t)\|_{L^1_{x,v}} \leq \|f(t_s)\|_{L^1_{x,v}} + O(\delta) \int_{t_s}^{t} |f(s)|_{L^1_{x,v}}.
\] (2.6)

Proof. The bound (2.6) is from \( \|f(t)\|_{L^1_{x,v}} \leq \|f(t_s)\|_{L^1_{x,v}} + \int_{t_s}^{t} \|f\|_{L^1_{x,v}} = \|f(t_s)\|_{L^1_{x,v}} \), and, due to the choice of \( c_\mu \) in (2.3)
\[
\int_{t_s}^{t} |f(s)|_{L^1_{x,v}} + \int_{t_s}^{t} \int_{\gamma_+} \int_{0}^{t_+} |f| - \int_{t_s}^{t} \int_{\gamma_-} \int_{0}^{t_-} \int_{\gamma_-} |f| = \|f(t_s)\|_{L^1_{x,v}} - \int_{t_s}^{t} \int_{t_s}^{t_+} \int_{t_0}^{t_+} |f| - \int_{t_s}^{t} \int_{t_s}^{t_-} \int_{t_0}^{t_-} \int_{t_0}^{t_-} |f| = 0.
\]

Next we work on (2.7) inspired by the proof of the \( L^1 \)-trace theorem in [10]. Choose \( \delta \in (0, t - t_s) \). For \( (x, v) \in \gamma_+ \),
\[
|f(s, x, v)| \leq 1_{\delta \leq x - (t - \delta) < t_b(x, v)} |f(t - \delta, x - (s - (t - \delta))v)| + 1_{x - (t - \delta) \geq t_b(x, v)} |f(s - t_b(x, v), x_b(x, v), v)|.
\] (2.8)

From (2.4) and (2.8), we have \( \int_{t_s}^{t} \int_{\gamma_+} |f| \leq \|f(t - \delta)\|_{L^1_{x,v}} \leq \|f(t_s)\|_{L^1_{x,v}} \). Now we consider (2.8)2. For \( y = x_b(x, v) \), we have \( 1_{\delta \leq x - (t - \delta) \geq t_b(x, v)} = 1_{\delta, \delta \geq t_b(y, v)} \leq 1_{t_b(y, v)} \) for \( s \in [t_s, t] \). From the above inequality, further using the Fubini’s theorem, (2.6), and (1.7) successively, we derive that
\[
\int_{t_s}^{t} \int_{\gamma_+} |f| = \int_{\partial \Omega} \int_{n(y) \cdot v < 0} \int_{t_s}^{t - \delta} |f(s, y, v)| |n(y) \cdot v| |dv|dS_y.
\]

From \( |n(y) \cdot v|/|v|^{\alpha} \lesssim t_f(y, v) \), we note that \( 1_{|n(y) \cdot v| \lesssim |v|^{\alpha}} \geq 1_{t_f(y, v)} \). For \( \vartheta \) being the angle between \( v \) and \( n(y) \),
\[
\int_{|n(y) \cdot v| \lesssim |v|^{\alpha}} \mu(v) \{ n(y) \cdot v \} dv \leq \int_{|n(y) \cdot v| \lesssim |v|^{\alpha}} \mu(v) |v|^{\alpha} dv , \text{ by setting } r = |v|.
\]

Then, from (2.9) and (2.10), we conclude (2.9) \( \lesssim O(\delta) \int_{t_s}^{t} \int_{\gamma_+} |f| \).
\qed
Lemma 8 ([11, 6]). For the bounded domain with a smooth boundary,
\[
\max\{|n(y) \cdot (y - z)|, |n(z) \cdot (y - z)|\} \lesssim |y - z|^2 \text{ for all } y, z \in \partial \Omega. \tag{2.11}
\]
If we further assume that the domain is convex then there exists \(C_\Omega > 0\) such that
\[
\min\{|n(y) \cdot (y - z)|, |n(z) \cdot (y - z)|\} \geq C_\Omega |y - z|^2 \text{ for all } y, z \in \partial \Omega. \tag{2.12}
\]

3. Weighted \(L^1\)-Estimates

The main purpose of this section to prove Proposition [2] which happens at the end of this section. We shall start it from settling one of the key cornerstones, Lemma [3] the lower bound with the unreachable defect.

Lemma 9. Suppose \(f\) solve \((2.1)\) with \((2.2)\). Assume \(f_0(x, v) \geq 0\) (no need of \((2.2)\)). For any \(T_0 > 1\) and \(N \in \mathbb{N}\) there exists \(m(x, v) \geq 0\), which only depends on \(\Omega\) and \(T_0\) (see \((3.11)\) for the precise form), such that
\[
f(NT_0, x, v) \geq m(x, v) \int_{\Omega} f((N - 1)T_0, x, v)dvdx - \int_{\Omega} 1_{t(x,v) \geq \frac{N}{T_0} f((N - 1)T_0, x, v)dvdx}. \tag{3.1}
\]

Proof. Step 1. Clearly we have \(f(t, x, v) \geq 0\) from the assumption \(f_0 \geq 0\). Together with \([1.12, 1.13]\) for \(t = NT_0, t_* = (N - 1)T_0, k = 3\), we can derive that
\[
f(NT_0, x, v) \geq m(x, v) \int_{\Omega} \int_{V_1} \int_{V_2} \int_{V_3} 1_{t(x,v) \geq \frac{N}{T_0} f((N - 1)T_0, x, v)dvdx}. \tag{3.2}
\]

Now applying Lemma [3] for \(v_1 \in V_1\) and \(v_2 \in V_2\) with \((2.1)\) and \((2.2)\), we derive that
\[
f(NT_0, x, v) \geq m(x, v) \int_{\Omega} \int_{V_1} \int_{V_2} \int_{V_3} 1_{t(x,v) \geq \frac{N}{T_0} f((N - 1)T_0, x, v)dvdx}. \tag{3.3}
\]

Step 2. In order to have a positive pointwise lower bound of the integrands of the first two lines of \((3.3)\), we will further restrict integration regimes. Note that \(x_1 = x_3(x, v)\) is given, and \(x_2, x_3\) are free variables. Now we restrict the range of \(x_2\) as, for \(\delta > 0\),
\[
A_2^\delta := \{x_2 \in \partial \Omega : |x_1 - x_2| > \delta \text{ and } |x_2 - x_3| > \delta\}. \tag{3.4}
\]

For two free variables \(t_{b,1}\) and \(t_{b,2}\) we use, only inside the proof of Lemma [3] two free variables
\[
t_+ = t_{b,1} + t_{b,2} \in [0, T_0 - t_b(x,v)] \quad \text{and} \quad t_- = t_{b,1} - t_{b,2} \in [-T_0 - t_b(x,v), T_0 - t_b(x,v)]. \tag{3.5}
\]

Note that the ranges come from \(t_3 \geq (N - 1)T_0\). Now we restrict the integral regimes of the new variables as
\[
\mathcal{T}_+^{T_0} := \left\{ t_+ \in [0, \infty) : T_0 - t_b(x,v) - \min \left( t_{b,1}, t_{b,2}, \frac{T_0}{4} \right) \leq t_+ \leq T_0 - t_b(x,v) \right\},\tag{3.6}
\]
\[
\mathcal{T}_-^{T_0} := \left\{ t_- \in \mathbb{R} : |t_-| \leq T_0 - t_b(x,v) - \min \left( t_{b,1}, t_{b,2}, \frac{T_0}{4} \right) \right\}.
\]

As a consequence of \((3.6)\) we will derive \((3.7)\) and \((3.8)\). Firstly, from \(t_b(x,v) \leq \frac{T_0}{4}\) in \((3.3)\) and \((3.5)\), we have
\[
\min \left( t_{b,1}, t_{b,2} \right) = \min \left( \frac{t_+ + t_-}{2}, \frac{t_+ - t_-}{2} \right) \geq \frac{1}{2} \left( T_0 - t_b(x,v) - \frac{T_0}{4} \right) \geq \frac{T_0}{8}, \tag{3.7}
\]
\[
\max \left( t_{b,1}, t_{b,2} \right) = \max \left( \frac{t_+ + t_-}{2}, \frac{t_+ - t_-}{2} \right) \leq T_0. \tag{3.8}
\]

Secondly, we prove \((3.8)\). Note that if \(t_+ \in \mathcal{T}_+^{T_0}\) then \((N - 1)T_0 \leq t_3 = NT_0 - t_b(x,v) - t_\leq (N - 1)T_0 + \min \left( t_{b,1}, t_{b,2}, \frac{T_0}{8} \right)\). This implies that, for \(y = X((N - 1)T_0; t_3, x_3, v_3) = x_3 - (t_3 - (N - 1)T_0) v_3\),
\[
\text{if } t_f(y, v_3) = t_3 - (N - 1)T_0 = T_0 - t_b(x,v) - t_+ \in \left[ 0, \frac{3T_0}{4} \right] \text{ then } y = X((N - 1)T_0; t_3, x_3, v_3), \tag{3.8}
\]
where we have use an observation \(t_f(y, v_3) \leq t_b(x_3, v_3)\) since \(x_3 = x_f(y, v_3)\).
Step 3. For (3.3), we adopt the new variables (3.5), and apply the restriction of integral regimes in (3.4) and (3.6). Recall (2.12) from the convexity of the domain. From (3.7) and (2.12), we derive that

\[ (3.3) \quad t_b(x,v) \geq \frac{C_0^4}{(2\pi)^2} \int_{\Omega} e^{-\frac{(x-x_{t_b})^2}{\sigma^2}} \int_{S} \gamma \cdot v \, dt \]

where \( \gamma = \frac{x_t - x_{t_b}}{t} \) with the initial data \( t_{b}(x,v) \). Therefore

\[ (3.9) = \int_{0}^{T_{b}} f((N-1)T_{0}, x, v) \, dt \]

where \( \gamma = \frac{x_t - x_{t_b}}{t} \) with the initial data \( t_{b}(x,v) \). Hence, we derive (3.12).

An immediate consequence of Lemma 3 as in [3], follows.

**Proposition 3.** Suppose \( f \) solve (1.5) and (1.7), and satisfy (1.8). Then for all \( T_{0} \geq 0 \), \( 0 < \delta < 1 \), and \( N \in \mathbb{N} \),

\[ \| f((N-1)T_{0}) \|_{L_{r,\gamma}^{1,\nu}} \leq (1 - \| m \|_{L_{r,\gamma}^{1,\nu}}) \| f((N-1)T_{0}) \|_{L_{r,\gamma}^{1,\nu}} + 2 \| m \|_{L_{r,\gamma}^{1,\nu}} \| 1_{x \geq \frac{\pi}{T}} f((N-1)T_{0}) \|_{L_{r,\gamma}^{1,\nu}}. \]

Here, with \( \Lambda_{r,\gamma}^{1,\nu} \) in (3.4),

\[ \| m \|_{L_{r,\gamma}^{1,\nu}} = \delta_{m,T_{0}} \sim (2\pi)^{-2} C_0^4 \delta T_{0}^{-9} \exp(-64\text{diam}(\Omega)^2 T_{0}^{-2}) \| \Lambda_{r,\gamma}^{1,\nu} \|_{\partial \Omega}. \]

**Proof.** Decompose

\[ f((N-1)T_{0}, x, v) = f_{N-1,+}(x, v) - f_{N-1,-}(x, v) \]

\[ := 1_{f((N-1)T_{0}, x, v) > 0} |f((N-1)T_{0}, x, v)| - 1_{f((N-1)T_{0}, x, v) < 0} |f((N-1)T_{0}, x, v)|. \]

Let \( f_{\pm}(s, x, v) \) solve (1.5) for \( s \in [(N-1)T_{0}, NT_{0}] \) with the initial data \( f_{N-1,+} \) and \( f_{N-1,-} \) at \( s = (N-1)T_{0} \), respectively. Now we apply Lemma 3 to each \( f_{\pm}(s, x, v) \) and conclude (3.11) for both \( f_{+} \) and \( f_{-} \) respectively. We also note that \( \int f((N-1)T_{0}, x, v) \, dv \) implies \( \int f((N-1)T_{0}, x, v) \, dv = 0 \) and \( \int f((N-1)T_{0}, x, v) \, dv = 0 \) implies \( \int f((N-1)T_{0}, x, v) \, dv = 0 \). Then we derive that

\[ f_{\pm}(NT_{0}, x, v) \geq m(x, v) \int f_{\pm}(NT_{0}, x, v) \, dv \geq m(x, v) \int 1_{t_{b}(x,v) \geq \frac{\pi}{T}} f_{\pm}(NT_{0}, x, v) \, dv \]

Then we deduce that

\[ |f(NT_{0}, x, v)| \leq |f_{+}(NT_{0}, x, v) - l(x, v)| + |f_{-}(NT_{0}, x, v) - l(x, v)| \leq f_{+}(NT_{0}, x, v) + f_{-}(NT_{0}, x, v) - 2l(x, v). \]

Note that \( f_{+}(NT_{0}, x, v) + f_{-}(NT_{0}, x, v) \) solves (1.5) with the initial datum \( f_{N-1,+} + f_{N-1,-} = |f((N-1)T_{0}, x, v)| \) at \( (N-1)T_{0} \). Then using (1.8) and taking an integration to (3.14) over \( \Omega \times \mathbb{R}^{3} \), we derive (3.12).
For (3.12) it suffices to bound \( \|1_{t_b(x,v)\leq T_b}c_{\mu}(v)\|_{L^1_{x,v}} \). From (2.14) and \( t_b(x-s,v) = t_b(x,v) - s \),
\[
\|1_{t_b(x,v)\leq T_b}c_{\mu}(v)\|_{L^1_{x,v}} &= \int_{\partial \Omega} \int_{n(x,v)>0} \int_{1_{t_b(x,v)}}^{T_b(x,v)} c_{\mu}(v)\{n(x) \cdot v\}dsdvds_x \\
&= \int_{\partial \Omega} \int_{n(x,v)>0} \left( 1_{t_b(x,v)\leq T_b} + \int_{1_{t_b(x,v)}}^{T_b(x,v)} ds \right) c_{\mu}(v)\{n(x) \cdot v\}dvds_x \\
&\sim T_0|\partial \Omega|.
\]
Combining the above bound with (3.11), we conclude (3.13).

Next we prove an important result, Lemma 2 which will be used frequently in this paper.

**Proof of Lemma 2.** Note that in the sense of distribution \( \partial_t + v \cdot \nabla_x)(\varphi(t\tau)|f|) = \varphi'(t\tau)v \cdot \nabla_x f|f| = -\varphi'(t\tau)|f| \). From this equation and (1.7), we derive that
\[
\|\varphi(t\tau)|f(t)|\|_{L^1_{x,v}} + \int_{t_\tau}^t \|\varphi'(t\tau)|f(s)|\|_{L^1_{x,v}} + \int_{t_\tau}^t \int_{\gamma_+} \varphi(t\tau)|f| \leq \|\varphi(t\tau)|f(t)|\|_{L^1_{x,v}}
\]
\[
+ \int_{t_\tau}^t \int_{\partial \Omega} \int_{n(x,v)<0} \varphi(t\tau)c_{\mu}(v)n(x) \cdot v|f| \int_{n(x,v)>0} |f(s,x,v)|\{n(x) \cdot v\}dvds_xds.
\]
We only need to consider (3.13) with the corresponding \( \varphi(t\tau) \). We prove the following claim: If (1.10) holds then \( \sup_{x \in \partial \Omega} \int_{n(x,v)<0} \varphi(t\tau)(x,v)c_{\mu}(v)n(x) \cdot v|dv| \lesssim 1 \). From the claim (2.7), we conclude (1.17), through, for \( C > 1, \)
\[
(3.13) \leq C \int_{t_\tau}^t \int_{\gamma_+} |f(s,x,v)|\{n(x) \cdot v\}dvds_xds \leq C \|f((N-1)T_0)|L^1_{x,v}| + \frac{1}{4} \int_{(N-1)T_0}^NT_0 |f(s)|L^1_{x,v}.
\]
For \( 0 < \delta \ll 1 \), we split \( \int_{n(x,v)<0} \varphi(t\tau)(x,v)c_{\mu}(v)n(x) \cdot v|dv| \) into two parts: integration over the regimes of \( t_\tau \leq \delta \) and \( t_\tau > \delta \) respectively. When \( t_\tau \leq \delta \), from (2.12), we derive that \( |n(x) \cdot v||dv|^2 \lesssim t_\tau \leq \delta \). Then we bound
\[
\int_{n(x,v)<0} 1_{t_\tau \leq \delta} \varphi(t\tau)(x,v)c_{\mu}(v)n(x) \cdot v|dv| \lesssim \delta \varphi(\delta) \int_{\mathbb{R}^3} |v|^2c_{\mu}(v)|dv| \lesssim 1.
\]
Now we focus on the integration over the regimes of \( t_\tau > \delta \). From (2.3) we derive that \( \int_{n,v<0} \varphi(t\tau)c_{\mu}(v)|n \cdot v|dv \) equals
\[
c_{\mu}A \int_{\partial \Omega} \int_{\delta} \varphi(t\tau)\mu\left(\frac{|x-x_\tau|}{t_\tau}\right)\frac{|n(x) \cdot (x-x_\tau)|^2}{|t_\tau|^3} dS_{x_\tau}dS_{x_\tau}.
\]
From (2.11) and (1.10), we derive that (3.17) \( \lesssim \int_{\delta} \varphi(t\tau)|x-x_\tau|^4e^{-\frac{|x-x_\tau|^2}{2t_\tau^2}} dS_{x_\tau}dt_\tau \lesssim \int_{\delta} \varphi(t\tau)|x-x_\tau|^4 dS_{x_\tau}dt_\tau \lesssim 1 \). Together with above bound and (3.16) we prove our claim.

We will use the following \( \delta \)'s inspired from 3.

**Definition.** For \( \delta > 0 \),
\[
\varphi_0(\tau) := (\ln(e + 1))^{-1}\ln(e + \ln(e + \tau)), \quad \varphi_1(\tau) := (e \ln(e + 1))^{-1}(e + \tau) \ln(e + \ln(e + \tau)), \quad \varphi_3(\tau) := e^{-3}(\tau + e)^3(\ln(\tau + e))^{-(1+\delta)}, \quad \varphi_4(\tau) := e^{-4}(\tau + e)^4(\ln(\tau + e))^{-(1+\delta)}.
\]
First, we check \( \varphi_i \) satisfies (1.10) for \( i = 0, 1, 2, 3, 4 \): for example, for \( \delta > 0 \),
\[
\int_{1}^{\infty} \tau^{-5}e^{-4}(\tau + e)^4(\ln(\tau + e))^{-(1+\delta)}d\tau \lesssim 1 + \int_{1}^{\infty} (\tau + e)^{-1}(\ln(\tau + e))^{-(1+\delta)}d\tau \lesssim \int_{1}^{\infty} \frac{1}{\tau^3}d\tau , \text{ with } s = \ln(\tau + e).
\]
Second, we notice that
\[
\varphi_i(0) = 1 \quad \text{for } i = 0, 1, 2, 3, 4.
\]
Finally, we check
\[
\varphi'_i(\tau) = (e \ln(e + 1))^{-1}\{\ln(e + \ln(e + \tau)) + (e + \ln(e + \tau))^{-1}\} \geq (e \ln(e + 1))^{-1}\varphi_0(\tau), \quad \varphi'_0(\tau) \geq 0,
\]
\[
\varphi''_i(\tau) = (4 - \frac{1+\delta}{\ln(\tau + e)})e^{-4}(\tau + e)^3(\ln(\tau + e))^{-(1+\delta)} \geq \varphi_3(\tau), \quad \varphi''_3(\tau) \geq 0.
\]

**Proposition 4.** Choose \( T_0 > 10 \) such that
\[
4C(2 + T_0)T_0^{-1}\left(\frac{3T_0}{4}\right)^{-1} \leq \frac{1}{2} \quad \text{for } i = 0, 3.
\]
For all \( N \in \mathbb{N} \) and \( i \in \{1, 4\} \), with \( (3.22)_* := (1 - \delta_{m,T_0}\left(1 - \frac{4C(2+T_0)}{T_0\phi_{i-1}(\frac{3T_0}{4})}\right))\),

\[
\|f(NT_0)\|_{L^1,v} + \frac{4\delta_m,T_0}{\phi_{i-1}(\frac{3T_0}{4})} \left\{ \|\varphi_{i-1}(t)f(NT_0)\|_{L^1,v} + \frac{1}{T_0}\|\varphi_i(t)f(NT_0)\|_{L^1,v} + \frac{1}{2T_0}\int_{(N-1)T_0}^{NT_0} |f|_{L^1_v} \right\} 
\leq (3.22)_* \times \|f((N-1)T_0)\|_{L^1_v} + \frac{4\delta_m,T_0}{\phi_{i-1}(\frac{3T_0}{4})} \left\{ \frac{3}{4}\|\varphi_{i-1}(t)f((N-1)T_0)\|_{L^1_v} + \frac{1}{T_0}\|\varphi_i(t)f((N-1)T_0)\|_{L^1_v} \right\}.
\]

(3.22)

Proof. As key steps we will repeatedly apply Lemma 2 with \( \varphi_i \)'s in (3.18). Applying Lemma 2 to \( f(t,x,v) \), solving (1.5) and (1.14), with \( \varphi_i \) for \( i = 0, 1, 4 \) in (3.13), and using (3.14), we derive that, for \( i = 1, 4 \),

\[
\|\varphi_{i-1}(t)f(NT_0)\|_{L^1_v} + \frac{3}{4}\int_{(N-1)T_0}^{NT_0} |f|_{L^1_v} \leq \left( \|\varphi_{i-1}(t)f(t_*)\|_{L^1_v} + C\|f(t_*)\|_{L^1_v} \right) \text{ for } (N-1)T_0 \leq t_* \leq NT_0,
\]

(3.23)

\[
\|\varphi_i(t)f(NT_0)\|_{L^1_v} + \int_{(N-1)T_0}^{NT_0} \|\varphi_{i-1}(t)f(t_*)\|_{L^1_v} + \frac{3}{4}|f|_{L^1_v} \leq \left( \|\varphi_i(t)f((N-1)T_0)\|_{L^1_v} + C\|f((N-1)T_0)\|_{L^1_v} \right) \text{ for } (N-1)T_0 \leq t_* \leq NT_0.
\]

(3.24)

From (3.20), (3.23) and (2.6), we derive that, for \( i = 1, 4 \),

\[
\int_{(N-1)T_0}^{NT_0} \|\varphi_{i-1}(t)f(t_*)\|_{L^1_v} \geq \int_{(N-1)T_0}^{NT_0} \|\varphi_{i-1}(t)f(t_*)\|_{L^1_v} dt_* = T_0\|\varphi_{i-1}(t_0)f(NT_0)\|_{L^1_v} - CT_0\|f((N-1)T_0)\|_{L^1_v}.
\]

From the above bound and (3.24), we conclude that, for \( i = 1, 4 \),

\[
\|\varphi_i(t)f(NT_0)\|_{L^1_v} + T_0\|\varphi_{i-1}(t_0)f(NT_0)\|_{L^1_v} + \frac{3}{4}\int_{(N-1)T_0}^{NT_0} |f|_{L^1_v} \leq \|\varphi_i(t_0)f((N-1)T_0)\|_{L^1_v} + C(T_0)\|f((N-1)T_0)\|_{L^1_v}.
\]

(3.25)

Now we combine (3.12) with (3.23) - (3.25). From (3.12), \( 1_{t_0 > \frac{3T_0}{4}} \leq (\varphi_{i-1}(\frac{3T_0}{4}))^{-1} \varphi_{i-1}(t_0) \), with \( \delta_{m,T_0} \) in (3.13),

\[
\|f(NT_0)\|_{L^1_v} \leq (1 - \delta_{m,T_0})\|f((N-1)T_0)\|_{L^1_v} + 2\delta_{m,T_0} \left( \varphi_{i-1}(\frac{3T_0}{4}) \right)^{-1} \|\varphi_{i-1}(t_0)f((N-1)T_0)\|_{L^1_v}.
\]

(3.26)

For \( i = 1, 4 \), from (3.20) and \( \frac{4\delta_m,T_0}{\phi_{i-1}(\frac{3T_0}{4})} \{ (3.23)_* \} \geq \{ (3.25)_* \} \), and \( T_0 > 0 \) in (3.21), we deduce (3.22). \( \square \)

Now we are well equipped to prove Proposition 2.

Proof of Proposition 2 Fix \( T_0 \) in (3.21) and recall norms of \( \|\cdot\|_1 \) and \( \|\cdot\|_4 \) in (3.18). From (3.22), for \( i = 1, 4 \),

\[
\|f(NT_0)\|_i \leq \|f((N-1)T_0)\|_i \leq \cdots \leq \|f(0)\|_i \quad \text{for all } N \in \mathbb{N}.
\]

(3.27)

Step 1. Since \( \varphi_1(t) / \varphi_2(t) \) is a decreasing function of \( \tau \gg 1 \), for \( M \gg 1 \), we have \( \varphi(t) = 1_{t \geq M} \varphi(t) + 1_{t < M} \varphi(t) = 1_{t \geq M} \varphi(M) \varphi(t) + 1_{t < M} M \varphi_0(t) \). From the above bound and (3.27) for \( i = 4 \), for \( M \gg 1 \), \( N \in \mathbb{N} \),

\[
\frac{1}{M}\|\varphi(t)f((N-1)T_0)\|_{L^1_v} \leq \frac{\varphi(M)}{M\varphi(M)} \|\varphi(t)f((N-1)T_0)\|_{L^1_v} + \|\varphi_0(t)f((N-1)T_0)\|_{L^1_v}.
\]

(3.28)

From (3.22) and (3.28), with \( (3.29)_* := \max \left\{ \left(1 - \delta_{m,T_0}\left(1 - \frac{4C(2+T_0)}{T_0\phi(M)(\frac{3T_0}{4})}\right)\right), \left(\frac{3}{4} + \frac{1}{M}\right), (1 - \frac{1}{M}) \right\} \),

\[
\|f(NT_0)\|_1 \leq (3.29)_* \|f((N-1)T_0)\|_1 + \frac{\varphi(M)}{M\varphi(M)} \|\varphi_0(f((N-1)T_0)\|_4.
\]

(3.29)

Step 2. Tentatively we make an assumption, which will be justified later behind (3.34),

\[
\left(1 + \frac{1}{M}\right)^{-1} \geq \max \left\{ \left(1 - \delta_{m,T_0}\left(1 - \frac{4C(2+T_0)}{T_0\phi(M)(\frac{3T_0}{4})}\right)\right), \left(\frac{3}{4} + \frac{1}{T_0}\right), (1 - \frac{1}{M}) \right\}.
\]

(3.30)

For \( t \geq 0 \), choose \( N_* \in \mathbb{N} \) such that \( t \in [(N_* - 1)T_0, N_*T_0] \). From (3.24) and (3.26), we derive that, for all \( N \leq N_* \),

\[
\|f(NT_0)\|_1 \leq \left(1 + \frac{1}{M}\right)^{-1} \|f((N-1)T_0)\|_1 + \Re, \quad \text{with } \Re := \frac{\varphi(M)}{M\varphi(M)} \|\varphi_0(f((N-1)T_0)\|_4.
\]

(3.31)

Now applying (3.31) successively, we conclude that \( \|f(t)\|_1 \leq \|f(N_0)\|_1 + \Re \leq \left(1 + \frac{1}{M}\right)^{-1} \|f((N_*-1)T_0)\|_1 + 2\Re \leq \cdots \leq \left(1 + \frac{1}{M}\right)^{-N} \|f(0)\|_1 + (2 + M)\Re \).
From \((1 + \frac{1}{M})^{-N} = ((1 + \frac{1}{M})^{-M})^\frac{N}{M} \leq e^{-\frac{N}{M}} \leq e^{-\frac{2\pi}{\varphi_4(M)}}\), \((2 + M)\Re \leq 2\varphi_4(M)\mu\varphi_4(M)\|f(0)\|_4\), we have

\[
\|f(t)\|_1 \leq \|f(0)\|_1 + \|f(t)\|_4 \leq \max \left\{e^{-\frac{2\pi}{\varphi_4(M)}}, \varphi_1(M)/\varphi_4(M) \right\} \|f(0)\|_1 + \|f(t)\|_4. \tag{3.32}
\]

Following an optimization trick (making \(|e^{-\frac{2\pi}{\varphi_4(M)} - \varphi_1(M)/\varphi_4(M)}| \ll 1\) as much as possible), choosing

\[
M = t[2T_0 \ln(10 + t^3)]^{-1},
\]

so that \(\max \left\{e^{-\frac{2\pi}{\varphi_4(M)}}, \varphi_1(M)/\varphi_4(M) \right\} \leq (\ln(t))^{-\frac{1}{2}t^{-3}}. \tag{3.34}
\]

Clearly such a choice assures our precondition \((3.30)\) for \(t \gg 1\). On the other hand it is straightforward to check \(|f(t)| \leq \|e^{\theta'}|f_0||_{L^\infty} \) from \((2.4)\) and \((2.2)\), while \(|\varphi_1(t)f_0||_{L^\infty} < \infty\) has been taken for granted from the postulation of Theorem \(1\) Applying \((1.17)\) with \(\varphi = \varphi_1, t_* = t/2, \) and \((2.6), (3.20),\) and then using \((3.34)\) and \((3.33)\), we finally prove \((1.19)\) via \(\frac{1}{2}\|f(t)\|_{L^\infty} \leq \int_0^t \|\varphi_1(t_f)\|_{L^\infty} \leq \|f(t/2)\|_1 \leq (\ln(t))^{-\frac{1}{2}t^{-3}}\|e^{\theta'}|f_0||_{L^\infty} + \|\varphi_4(t_f)f_0||_{L^\infty} \). \(\square\)

4. \(L^\infty\)-Estimates of Moments

We give proofs for Lemma \(8\) and Lemma \(9\)

**Proof of Lemma** \(8\). For \((4.2)\) it suffices to prove this upper bound for

\[
\int_{V_1} \cdots \int_{V_{i-3}} \int_{t_{i-1}}^{t_i} \int_{t_{i-2}}^{t_{i-1}} \int_{t_{i-3}}^{t_{i-2}} \cdots \int_{t_{i-3}}^{t_{i-3}} \left[ n(x) \cdot v_i \right] dv_i \, ds \, d\sigma_1 \cdots d\sigma_i. \tag{4.1}
\]

**Step 1.** Applying Lemma \(8\) \((2.1), (2.2)\) with \(x = x_j, v = v_j, \) we derive the change of variables, for \(j \geq 1, \) \(v_j \in V_j \mapsto (x_j(x_j, v_j), t_j(x_j, v_j)) \in \partial \Omega \times [0, t_j], \) with \(dv_j = |t_{b,j}^{-1} |x_j(x_j + 1) - x_j(x_j + 1)| dt_{b,j} \cdot dx_j.\)

Applying above change of variables twice, we derive that \((4.1)\) equals

\[
\int_{V_1} \cdots \int_{V_{i-3}} \int_{t_{i-2}}^{t_{i-1}} \int_{t_{i-2}}^{t_{i-1}} \int_{t_{i-3}}^{t_{i-2}} \cdots \int_{t_{i-3}}^{t_{i-3}} n(x) \cdot v_i \, |t_{b,i}^{-1}| \, ds \, d\sigma_1 \cdots d\sigma_i. \tag{4.2}
\]

with \(t_{i-2}, x_{i-2}\) defined in \((4.1),\) and \(t_{i-1} = t_{i-2} - t_{b,i-2}.\) Using \((4.2)\), we re-express the above integration as

\[
\int_{V_1} \cdots \int_{V_{i-3}} \int_{t_{i-2}}^{t_{i-1}} \int_{t_{i-2}}^{t_{i-1}} \int_{t_{i-3}}^{t_{i-2}} \cdots \int_{t_{i-3}}^{t_{i-3}} n(x) \cdot v_i \, |t_{b,i}^{-1}| \, ds \, d\sigma_1 \cdots d\sigma_i. \tag{4.3}
\]

**Step 2.** We claim that

\[
4.3. \quad \frac{1}{2} \int_{t_{b,i-1}} \cdots \int_{t_{b,i-3}} (t_{b,i-2})^{-5} + \frac{1}{2} \int_{t_{b,i-1}} \cdots \int_{t_{b,i-3}} (t_{b,i-2})^{-5}. \tag{4.4}
\]

We split the cases: \textit{Case 1:} \(t_{b,i-1} \leq t_{b,i-2}.\) Using \(|x_{i-2} - x_{i-1}| \leq 1,\) we bound

\[
c_{\mu} \left(\begin{array}{c}
|x_{i-2} - x_{i-1}| \\
|t_{b,i-2}|
\end{array}\right)^4 \left(\begin{array}{c}
|x_{i-2} - x_{i-1}| \\
|t_{b,i-2}|
\end{array}\right)^5 \leq \frac{1}{|t_{b,i-2}|} + \frac{1}{|t_{b,i-2}|^5}. \tag{4.5}
\]

\[
c_{\mu} \left(\begin{array}{c}
|x_{i-1} - x_i| \\
|t_{b,i-1}|
\end{array}\right)^4 \left(\begin{array}{c}
|x_{i-1} - x_i| \\
|t_{b,i-1}|
\end{array}\right)^5 \leq \mu \left(\begin{array}{c}
|x_{i-1} - x_i| \\
|t_{b,i-1}|
\end{array}\right) \left(\begin{array}{c}
1 \\
|t_{b,i-1}|
\end{array}\right) + \frac{1}{|t_{b,i-1}|^5}. \tag{4.6}
\]

We employ a change of variables, for \(x_i \in \partial \Omega \) and \(t_{b,i-1} \geq 0, \) \(x_i \in \partial \Omega \mapsto z := \frac{1}{t_{b,i-1}}(x_i - x_{i-1}) \in \mathcal{S}_x, t_{b,i-1},\) where the image \(\mathcal{S}_x, t_{b,i-1}\) of the map is a two dimensional smooth hypersurface. Using the local chart of \(\partial \Omega\) we have
$$dS_{x_{i-1}} \lesssim |t_{b,i-1}|^2 dS_z.$$ From this change of variables and (11.5), (11.6), we conclude that

$$1_{t_{b,i-1} \leq t_{b,i-2}} \lesssim 1_{t_{b,i-1} \leq t_{b,i-2}} \int_{\Omega} \mu^\frac{1}{t_{b,i-1}-t_{b,i-2}} dS_z$$

$$\lesssim 1_{t_{b,i-1} \leq t_{b,i-2}} \left\{ 1_{t_{b,i-1} \leq \frac{1}{t_{b,i-2}}} + 1_{t_{b,i-1} \geq \frac{1}{t_{b,i-2}}} \right\} \int_{\Omega} \mu^\frac{1}{t_{b,i-1}-t_{b,i-2}} dS_z$$

$$\lesssim 1_{t_{b,i-1} \leq t_{b,i-2}} \left\{ 1_{t_{b,i-1} \leq \frac{1}{t_{b,i-2}}} + 1_{t_{b,i-1} \geq \frac{1}{t_{b,i-2}}} \right\}$$

$$\lesssim 1_{t_{b,i-1} \leq t_{b,i-2}} \left\{ 1_{t_{b,i-1} \leq \frac{1}{t_{b,i-2}}} + 1_{t_{b,i-1} \geq \frac{1}{t_{b,i-2}}} \right\}$$

(4.7)

**Case 2:** $t_{b,i-1} \geq t_{b,i-2}$. We change the role of $i-1$ and $i-2$ and follow the argument of the previous case. Using $|x_{i-1} - x_i| \lesssim t_{\Omega}$, we bound $c \mu \left( \frac{|x_{i-2} - x_{i-1}|}{t_{b,i-1}} \right) \left( \frac{|x_{i-1} - x_i|}{t_{b,i-2}} \right) \lesssim 1_{t_{b,i-2} \leq \frac{1}{t_{b,i-2}}} + 1_{t_{b,i-1} \geq \frac{1}{t_{b,i-2}}}$. We employ a change of variables, for $x_{i-1} \in \partial \Omega$ and $t_{b,i-2} \geq 0$, $x_{i-2} \in \partial \Omega \mapsto z := \frac{1}{t_{b,i-2}}(x_{i-2} - x_{i-1}) \in \mathcal{S}_{x_{i-2},t_{b,i-2}}$, with $dS_{z} \lesssim |t_{b,i-2}|^2 dS_z$. Then we can conclude $1_{t_{b,i-1} \geq t_{b,i-2}} \lesssim 1_{t_{b,i-1} \leq t_{b,i-2}} + 1_{t_{b,i-1} \geq t_{b,i-2}}$. Clearly this bound and (4.7) imply (4.4).

**Step 3.** Now we use (4.1) to (4.3). Then we have

$$1_{t_{b,i-1} \geq t_{b,i-2}} \lesssim \int_{V_1} d\sigma_1 \cdots \int_{V_{i-3}} d\sigma_{i-3} \int_0^{t_{b,i-2}} dt_{b,i-1} (t_{b,i-1})^{-5} \int_0^{\min \{ t_{b,i-2} - t_{b,i-1}, t_{b,i-1} \}} dt_{b,i-2} \int_{\partial \Omega} dS_{z_i} \int_{t_{b,i-2}}^{t_{b,i-1}} dt_{b,i-2}$$

$$+ \int_{V_1} d\sigma_1 \cdots \int_{V_{i-3}} d\sigma_{i-3} \int_0^{t_{b,i-2}} dt_{b,i-1} (t_{b,i-1})^{-5} \int_0^{\max \{ t_{b,i-2} - t_{b,i-1}, t_{b,i-2} \}} dt_{b,i-2} \int_{\partial \Omega} dS_{z_i} \int_{t_{b,i-2}}^{t_{b,i-1}} dt_{b,i-2}$$

(4.8)

(4.9)

We first consider (4.3). We employ the following change of variables $(x_i, t_{b,i-1}) \mapsto y = x_i - (t_{i-2} - t_{b,i-2} - t_{b,i-1} - s)v_i \in \Omega$, with $|n(x_i) \cdot v_i| dS_z dt_{b,i-2} = dy$. Applying this change of variables we derive that

$$|t_{b,i-1} \geq t_{b,i-2}| \lesssim \int_{V_1} d\sigma_1 \cdots \int_{V_{i-3}} d\sigma_{i-3} \int_0^{t_{b,i-2}} dt_{b,i-1} (t_{b,i-1})^{-5} \int_0^{\min \{ t_{b,i-2} - t_{b,i-1}, t_{b,i-1} \}} dt_{b,i-2} \int_{\Omega \times \mathbb{R}^3} g(s) f(s, y, v_i) dv_i dy ds \lesssim \int_0^{t_{b,i-1}} ||f(s)||_{L^1_v} ds$$

A bound of (4.4) can be derived, using the change of variables $(x_i, t_{b,i-1}) \mapsto y = x_i - (t_{i-2} - t_{b,i-2} - t_{b,i-1} - s)v_i \in \Omega$ with $|n(x_i) \cdot v_i| dS_z dt_{b,i-2} = dy$.

**Proof of Lemma 4.**

**Step 1.** Define $\mathcal{V}_j^\delta := \{ v_i \in V_j : |n(x_i) \cdot v_i|/|v_i|^2 < \delta \}$. From (2.10), we have $\int_{\mathcal{V}_j^\delta} d\sigma_j \leq C \delta^2$. On the other hand, from (2.12), we have $t_{b}(x_i, v_i) \geq C_{t} |n(x_i) \cdot v_i|/|v_i|^2$. Therefore if $v_i \in V_j \setminus \mathcal{V}_j^\delta$, we have $t_{b}(x_i, v_i) \geq C_{t} \delta$.

If $t_k(t, x, v, v_1, \ldots, v_{k-1}) \geq 0$, we conclude such $v_i \in V_j \setminus \mathcal{V}_j^\delta$ can exist at most $\frac{M}{C_{t} \delta} + 1$ times. Denote the combination

$$\left( M \right) \frac{(M+1)\cdots(M+N-1)}{N(M)} \frac{M!}{N!}$$

for $M, N \in \mathbb{N}$ and $M \geq N$. For $0 < \delta \ll 1$, we have

$$\int_{\prod_{i=1}^{k} V_j} 1_{t_k(t, x, v, v_1, \ldots, v_{k-1}) \geq 0} d\sigma_{k-1} \cdots d\sigma_1 \leq \sum_{m=0}^{[\frac{k}{t_{b,i-1}}] + 1} \binom{k}{m} \int_{V_j^k} d\sigma_t t_{b,i-1}^{k-m} \leq \left( C \delta^2 \right)^{k-\frac{1}{2}t_{b,i-1}} \sum_{m=0}^{[\frac{k}{t_{b,i-1}}] + 1} \binom{k}{m}.$$ (4.10)

**Step 2.** Recall the Stirling’s formula $\sqrt{2\pi}k^{k+\frac{1}{2}}e^{-k} \leq k! \leq k^{k+\frac{1}{2}}e^{-k+\frac{1}{2}}$ (e.g. [4]). Using this bound and $(1 + \frac{a}{a-1})^{a-1} \leq e$, we have, for $a \geq 2$, $\left( \frac{k}{a} \right)^{\frac{k}{a}} \leq \left( a \left( 1 - \frac{a}{a-1} \right) \right)^{a-1} \frac{(a^a - a^a) + 1}{a^a - 1} \leq \frac{e^{\frac{a}{a-1}}}{\frac{a}{a-1}} \frac{a^{\frac{a}{a-1}}}{\frac{a}{a-1}}$. Hence,

$$\frac{\binom{k}{i}}{\sum_{i=1}^{[\frac{k}{a}]} \frac{k}{a} \binom{k}{i}} \leq \frac{e}{\pi^2} \frac{a^{\frac{a}{a-1}}}{\frac{a}{a-1}}$$

(4.11)

**Step 3.** Now we estimate (4.11). Fix $0 < \delta \ll 1$ which is independent of $t$, choose

$$a \in \mathbb{N} \text{ such that } (\delta^2 e a)^{e a} \leq e^{-2}, \text{ and set } k := a \left(\frac{t}{C_{t} \delta} + 1\right).$$ (4.12)

Using (4.11), we derive (4.10), $\lesssim \sqrt{\frac{t}{C_{t} \delta}} + 1 \left( e^{-\frac{k}{C_{t} \delta}} + 1 \right)^{t \frac{k^{t} \frac{1}{a-1} + 1}{\sum_{m=0}^{[\frac{k}{t_{b,i-1}}] + 1}} \lesssim \sqrt{\frac{t}{C_{t} \delta}} + 1 \left( e^{-\frac{k}{C_{t} \delta}} + 1 \right)^{t \frac{k^{t} \frac{1}{a-1} + 1}}$ and hence (4.10) is bounded by $(\delta^2 e a)^{e a} \sum_{m=0}^{[\frac{k}{t_{b,i-1}}] + 1} \left( e^{-\frac{k}{C_{t} \delta}} + 1 \right)^{t \frac{k^{t} \frac{1}{a-1} + 1}}$. This completes the proof. \(\blacksquare\)

Equipped with Proposition 2 and Lemma 3, we present a proof of the main theorem:
Proof of Theorem 1.4. Let $w(v) := e^{\theta |v|^2}$, $w'(v) := e^{\theta' |v|^2}$ for $0 < \theta < \theta' < 1/2$. It is standard (11) to construct a unique solution of $f$ to (1.3)-(1.7) and prove its bound $\|w'(f)\|_{L^\infty_{x,v}} \lesssim \|w'(f(0))\|_{L^\infty_{x,v}}$. Choose $k = \mathcal{C}$ as in Lemma 1. To utilize the $L^1$-decay of (1.19), we set
\[
\varrho(t) := (\ln(t))^{-2}(t)^5.
\]
Clearly we have $\varrho'(t) \lesssim (\ln(t))^{-2}(t)^4$ for $t \gg 1$. Applying Lemma 1 with the prescribed $w(v), \varrho(t)$, and $t_* = 0$, we obtain the corresponding expansion of $g = pw'f$ as (1.12)-(1.14). We will estimate (1.12), (1.13), and (1.14) term by term. For (1.12), using the change of variables $v \to y = x - tv \in \Omega$ with $dv = t^{-3}dy$,
\[
\int_{\mathbb{R}^d} (1.12) dv \lesssim \left( \int_{x-tv \in \Omega} \frac{w(v)}{w'(v)} dv \right) \left\{ 1 + \int_0^t \varrho'(s) ds \right\} \sup_{0 \leq s \leq t} \|w'(f(s))\|_{L^\infty_{x,v}} \lesssim \langle t \rangle^{-3} \varrho(t) \|w'(f(0))\|_{L^\infty_{x,v}}.
\]
Next, again from the pointwise bound, we bound the contribution of the first term of (1.13) in $\int_{\mathbb{R}^d} (1.13) dv$ by
\[
\begin{aligned}
&k \left( \sum_i \int_{\prod_j \mathbb{R}^d} 1_{|t_{i+1} < \varrho(t) \leq t_i} d\mathcal{C} \int_{\mathbb{R}^d} w'(f(0)) \right) \lesssim k \left( \sum_i \int_{\prod_j \mathbb{R}^d} 1_{|t_{i+1} < \varrho(t) \leq t_i} d\mathcal{C} \right) \sup_{0 \leq s \leq t} \|w'(f(s))\|_{L^\infty_{x,v}} \lesssim k \|w'(f(0))\|_{L^\infty_{x,v}}.
\end{aligned}
\]
Now we consider the contribution of the second term of (1.13) in $\int_{\mathbb{R}^d} (1.13) dv$, which turns out the fastest growing term as $t \to \infty$. From Proposition 2 and Lemma 3 we bound it by
\[
\begin{aligned}
k \times \sup_i (1.20) \lesssim k \int_0^t \|\ln(s)\|^2 f(s) \|_{L^\infty_{x,v}} ds \lesssim k t \times \{ \|w'(f(0))\|_{L^\infty_{x,v}} + \|\varphi_4(tf) f(0)\|_1 \}.
\end{aligned}
\]
Lastly we bound (1.14), using Lemma 1 by
\[
\int_{\mathbb{R}^d} (1.14) dv \lesssim \sup_{(x,v) \in \Omega \times \mathbb{R}^d} \left( \prod_{j=1}^{k-1} 1_{|\{ t_j \leq x,v,v_1,\ldots,v_{k-1} \} \geq \varrho d\mathcal{C}_1 \cdots d\mathcal{C}_{k-1}} \right) \sup_{t_k \in [0,t]} \|w'(f(t_k))\| \lesssim e^{-t} \|w'(f(0))\|_\infty.
\]
Collecting estimates from (4.14)-(4.17) and using $k \sim t$, we conclude $\sup_{x \in \Omega} \int_{\mathbb{R}^d} \varrho(t) w(v) [f(t,x,v)] dv$ beig bounded by $\max\{ (t)^{-3} \varrho(t), kt,e^{-t} \} \lesssim (t)^2$. This, together with (1.13), proves (1.9).

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