THE ORIGIN OF A METRIC

B.G.Sidharth
B.M.Birla Science Centre, Hyderabad, 500463, India

Abstract

In the context of earlier work, we investigate the emergence of a "distance" in the physical world. For this we consider a Cantor ternary like process, but much more general: properties like perfectness and disconnectedness are not invoked, but instead we deal with Borel sets. An interesting case from a physical point of view is considered: when the process is truncated.

1 The Origin of a Metric

We first make a few preliminary remarks. When we talk of a metric or the distance between two "points" or "particles", a concept that is implicit is that of topological "nearness" - we require an underpinning of a suitably large number of "open" sets. Let us now abandon the absolute or background space time and consider, for simplicity, a universe (or set) that consists solely of two particles. The question of the distance between these particles (quite apart from the question of the observer) becomes meaningless. Indeed, this is so for a universe consisting of a finite number of particles. For, we could isolate any two of them, and the distance between them would have no meaning. We can intuitively appreciate that we would in fact need distances of intermediate or more generally, other points.

In earlier work, motivated by physical considerations we had considered a series of nested sets or neighbourhoods which were countable and also whose union was a complete Hausdorff space. The Urysohn Theorem was then invoked and it was shown that the space of the subsets was metrizable. The argument went something like this.
In the light of the above remarks, the concepts of open sets, connectedness and the like reenter in which case such an isolation of two points would not be possible.

More formally let us define a neighbourhood of a particle (or point or element) A of a set of particles as a subset which contains A and at least one other distinct element. Now, given two particles (or points) A and B, let us consider a neighbourhood containing both of them, n(A, B) say. We require a non empty set containing at least one of A and B and at least one other particle C, such that n(A, B) ⊂ n(A, C), and so on. Strictly, this "nested" sequence should not terminate. For, if it does, then we end up with a set n(A, P) consisting of two isolated "particles" or points, and the "distance" d(A, P) is meaningless.

We now assume the following property[2]: Given two distinct elements (or even subsets) A and B, there is a neighbourhood N_{A1} such that A belongs to N_{A1}, B does not belong to N_{A1} and also given any N_{A1}, there exists a neighbourhood N_{A_{1/2}} such that A ⊂ N_{A_{1/2}} ⊂ N_{A1}, that is there exists an infinite topological closeness.

From here, as in the derivation of Urysohn’s lemma[1], we could define a mapping f such that f(A) = 0 and f(B) = 1 and which takes on all intermediate values. We could now define a metric, d(A, B) = |f(A) − f(B)|. We could easily verify that this satisfies the properties of a metric.

With the same motivation we will now deduce a similar result, but with different conditions. In the sequel, by a subset we will mean a proper subset, which is also non null, unless specifically mentioned to be so. We will also consider Borel sets, that is the set itself (and its subsets) has a countable covering with subsets. We then follow a pattern similar to that of a Cantor ternary set [1, 3]. So starting with the set N we consider a subset N_1 which is one of the members of the covering of N and iterate this process so that N_{12} denotes a subset belonging to the covering of N_1 and so on.

We note that each element of N would be contained in one of the series of subsets of a sub cover. For, if we consider the case where the element p belongs to some N_{12...m} but not to N_{1,2,3...m+1}, this would be impossible because the latter form a cover of the former. In any case as in the derivation of the Cantor set, we can put the above countable series of sub sets of sub covers in a one to one correspondence with suitable sub intervals of a real interval (a, b).
Case I

If \( N_{1,2,3,\ldots,m} \to \) an element of the set \( N \) as \( m \to \infty \), that is if the set is closed, we would be establishing a one to one relationship with points on the interval \((a,b)\) and hence could use the metric of this latter interval, as seen earlier.

Case II

It is interesting to consider the case where in the above iterative countable process, the limit does not tend to an element of the set \( N \), that is set \( N \) is not closed and has what we may call singular points. We could still truncate the process at \( N_{1,2,3,\ldots,m} \) for some \( m > R \) arbitrary and establish a one to one relationship between such truncated subsets and arbitrarily small intervals in \( a,b \). We could still speak of a metric or distance between two such arbitrarily small intervals.

This case is of interest because of recent work which describes elementary particles as, what may be called Quantum Mechanical Kerr-Newman Black Holes or vortices, where we have a length of the order of the Compton wavelength (that is \( 10^{-12} \text{cms} \) or less), within which spacetime as we know it breaks down. Such cut offs lead to a non commutative geometry and what may be called fuzzy spaces\[4\],\[5\],\[6\],\[7\], \[8\],\[9\].(We note that the centre of the vortex is a singular point). In any case, the number of particles in the universe is of the order \( 10^{80} \), which approximates infinity from a physicist’s point of view.

Remarks

Interestingly, we usually consider two types of infinite sets - those with cardinal number \( n \) corresponding to countable infinities, and those with cardinal number \( c \) corresponding to a continuum, there being nothing inbetween. This is the well known but unproven Continuum Hypotheses.

What we have shown with the above process is that it is possible to conceive an intermediate possibility with a cardinal number \( n^p, p > 1 \).

We also note the similarity with transfinite Cantor sets. But in this latter case three properties are important: the set must be closed i.e. it must contain all its limit points, perfect i.e. in addition each of its points must be a limit point and disconnected i.e. it contains no nonnull open intervals. Only the first was invoked in Case I.

Finally we will remark on an origin for spatial dimensions, in the context of the above elementary particle model and an associated consistent cosmology\[10\], in which given \( n \) particles, \( \sqrt{n} \) would be fluctuationally created from a background Quantum vaccuum within the undefined time inter-
val at the Compton scale referred to above. If there are $n$ points, these could be lined up as a single dimension. But if with each point other points are associated, not belonging to the original set then this would define another dimension. That is the case with the fluctuationally created particles: There would now be not $n$ but $n^{3/2}$ particles or points. Further with each of the fluctuationally created particles, there would be $n^{1/4}$ further fluctuationally created particles\[11\]. So in effect we would have to deal with not $n^{3/2}$ but $n^{7/4}$ points and so on so that finally with the $n$th point or particle we would have $n^2$ points. The total number of points would therefore be $\Sigma n^2$ or $\sim n^3$ giving us the three spatial dimensions.

References

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