Cointegration and why it works for SHM

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Abstract. One of the most fundamental problems in Structural Health Monitoring (SHM) is that of projecting out operational and environmental variations from measured feature data. The reason for this is that algorithms used for SHM to detect changes in structural condition should not raise alarms if the structure of interest changes because of benign operational or environmental variations. This is sometimes called the data normalisation problem. Many solutions to this problem have been proposed over the years, but a new approach that uses cointegration, a concept from the field of econometrics, appears to provide a very promising solution. The theory of cointegration is mathematically complex and its use is based on the holding of a number of assumptions on the time series to which it is applied. An interesting observation that has emerged from its applications to SHM data is that the approach works very well even though the aforementioned assumptions do not hold in general. The objective of the current paper is to discuss how the cointegration assumptions break down individually in the context of SHM and to explain why this does not invalidate the application of the algorithm.

1. Introduction
One of the largest stumbling blocks to the implementation of structural health monitoring (SHM) in practice outside of laboratory conditions is the confounding influence of changing environmental and operational conditions on damage sensitive features. Recently the idea of cointegration, from the field of econometrics, has been introduced by the authors as a way to remove trends induced by changing environmental and operational conditions [1]. Cointegration is a property of nonstationary time series which admit to a stationary linear combination. The main ideas behind the use of cointegration for SHM will be discussed briefly here, but see [1] for more details.

To use cointegration in the way suggested by the authors, a nonstationary multivariate training set of feature data from a structure operating in its normal condition is employed. If the variables in this training set are cointegrated, then a linear combination of them may be found that is stationary. Following common practice in econometrics, the Johansen Procedure [2] is implemented to find the most stationary linear combination of the variables possible. The Johansen procedure is a maximum likelihood approach for finding stationary combinations (culminating in a cointegrating vector) of nonstationary variables whose first difference is stationary. Briefly, the variables of interest \( \{y_i\} \) are arranged in a vector error correction model form (VECM), that may be seen as a rearrangement of the more familiar vector autoregressive model (VAR). This VECM takes the form:

\[
\{\Delta y_i\} = [\Pi]\{y_{i-1}\} + \sum_{j=1}^{p-1} [B_j]\{\Delta y_{i-j}\} + \{\epsilon_i\}
\]  

(1)
where $y_i$ denotes an $n$-vector including all $n$ variables to be analysed, with the subscript $i$ relating to time, $i = 1, \ldots, N$, $p$ represents the model order, or the number of lags to be included in the model, and $\epsilon_i$ is a noise process. In this form, the matrix $[\Pi]$ contains the coefficients that will create the most stationary linear combination of the original variables, in other words, $[\Pi]$ contains the cointegrating vectors. Unfortunately, if the variables are cointegrated, $[\Pi]$ must be of reduced rank, which means that parameter estimates for (1) must be obtained using a reduced rank regression. The reduced rank regression involves the decomposition of the matrix $[\Pi]$, and parameter estimation through maximising the likelihood of observing the correct noise sequence $\{\epsilon_i\}$, however, as the process is rather in depth no further details are given here, instead readers are again referred to [1] or [2].

If a stationary linear combination can be found, the continuing stationarity of the combination (commonly referred to as a residual) when new data is projected onto it can be used as an indicator that the structure continues to operate in its normal condition. An example of data from the Tamar Bridge, monitored by the Vibration Engineering Section at the University of Sheffield, is included here for illustrative purposes.

Figure 1 shows the displacement measurements of nine points on the deck and towers of the Tamar Bridge over a three day period and over a two month period respectively, measured by a total positioning system (TPS). The daily and seasonal trends visible are induced mainly by the changing temperature. As the variables share common trends it is likely that they will submit to a stationary linear combination (i.e. they are cointegrated). Figure 2(a) shows the linear combination suggested by the Johansen Procedure when the data shown in Figure 1(a) is used as a training set. From this figure one can see that the temperature induced trend has successfully been removed. If the data over the longer time period is then projected onto this combination the residual remains stationary, as shown in Figure 2(b) indicating without influence from the temperature that the structure continues to respond in its normal condition, as defined by the training set.

The remainder of the paper aims to give a general discussion about why cointegration is applicable to engineering data. In the following discussion the authors will argue that measured responses from healthy structures exhibit nonstationary behaviour over relatively short time.
Figure 2: Cointegrated residual of variables (a) for three day training period, (b) over two months.

periods, but should generally be stationary in the long term. As one would wish to detect any occurrence of structural degradation swiftly, these shorter time periods are of great interest to SHM. The discussion below will argue that such nonstationarity exhibited by structural response variables may be well represented by econometric theory, which models nonstationarity with unit root processes or with a deterministic time trend (or both). While under the time periods of interest such models of nonstationarity are valid, and therefore, too, the applied theory of cointegration, the acceptance of a unit root generating process for a structural response is not a comfortable one. The authors will argue that so long as the econometric models fit the data well, the philosophical question of whether a unit root assumption is valid is not really of interest. Indeed, to the engineer, so long as the combination of variables found via the Johansen procedure, or any other cointegrating method, does the job of removing confounding influences, the underlying assumptions of the process used to arrive at this combination may not be of any interest. None the less, a good understanding of such assumptions may help to avoid any suboptimal use of the sophisticated theory imported from econometrics.

2. The assumptions behind cointegration applied to engineering data

In previous works by the authors [1, 3], cointegration has been adopted from econometrics for the purposes of removing environmental and operational trends in damage sensitive features. The theory applied in these works, and in the example above, assumes that the variables of interest are nonstationary and, for the Johansen procedure, integrated of order one (nonstationary with first difference stationary, i.e. a unit root process). A unit root process is one whose characteristic equation has at least one root equal to one; it is these roots that will determine the stability/stationarity of a time series. For an auto-regressive model (AR) of order $p$:

$$y_i = a_1 y_{i-1} + a_2 y_{i-2} + \cdots + a_p y_{i-p} + \varepsilon_i$$

where $\varepsilon_i$ can be considered to be a Gaussian white noise process driving the model, its characteristic equation is as follows:
where \( z \) are the roots of the characteristic equation. This process will remain stable as long as \( |z_i| > 1 \). If any \( |z_i| < 1 \) the process will behave explosively. Now if any \( z_i = 1 \), (unit root), then the process has marginal stability. This section aims to discuss whether the unit root assumption is valid in the context of an engineering application and whether structural response can be nonstationary in the long term.

A nonstationary time series is defined as one whose statistics change with time. A weakly stationary process has constant mean and variance, and an autocovariance that depends only on the lag length considered. In the presence of nonstationarity standard regression techniques and inferences made on such regressions have been found to be unreliable, and since Yule’s seminal paper [4], much research has been carried out on nonstationary processes.

Of the variables of interest to SHM, many of them appear to exhibit nonstationarity. The most relevant example to this paper is the easterly displacements of the deck of the Tamar bridge, as plotted in Figure 1. Studying Figure 1, the time series certainly appear nonstationary over the time of available measurements.

To an engineer, when considering a dynamic response the usual modelling approach would be to employ the classic second order differential equation of motion. The nonstationarity of a response is then explained by the changing physical parameters such as mass and stiffness, or a change in the nature of the excitation. In the context of applying cointegration theory, the interest is in modelling the entire process (response) with fixed parameters that cannot change. For this task, the most comfortable way of thinking about a structural response is probably as a function of a number of external conditions which are themselves fluctuating. One might expect that if all external conditions could be accounted for, a structural response would be stationary around the trends introduced by these external conditions. Unfortunately, it may be very infeasible to account for all external conditions driving the response of a structure, as it certainly is in the field of econometrics where relations between variables of interest are uncertain. This is where an AR type model can be useful and why they are regularly applied to econometric time series, as one no longer needs information about each driving factor to be able to describe a dynamic process. The econometric cointegration theory commonly used was developed in the context of such models, and so to utilise this theory one must adopt them too.

In econometrics and for the cointegration procedure, an error correction model (ECM) is used to model a process variable, \( y_i \) of interest:

\[
\triangle y_i = \rho y_{i-1} + \sum_{j=1}^{p-1} b_j \triangle y_{i-j} + \mu + \nu t + \varepsilon_i, \tag{4}
\]

When considering such models, nonstationarity of a time series can be prescribed to two different mechanisms; either a deterministic trend \( \nu t \) or a unit root. Where a deterministic trend is the cause of nonstationarity, econometricians refer to the time series as a trend stationary process; fluctuations around this deterministic trend are stationary in nature. Where a unit root is the cause of nonstationarity, the time series is referred to as difference stationary processes, due to the fact that difference operations will render the series stationary.

Now considering structural response variables that exhibit nonstationarity, any term in a descriptive model that continually increases or decreases in time is unlikely to mimic the behaviour of a stable structural response. This implies that to continue using an autoregressive type model, the nonstationarity of the variable must be described by a unit root process. Many debates surround the idea of unit root modelling and its suitability to real life applications, within and outside economics (for example, there is currently much debate around using unit root processes to model the Earth’s temperature change (see for example [5, 6, 7])).
The idea of a variance that increases with time, which is inherent to a unit root process, certainly sits uncomfortably when considering a structural response over time. However, while it may be that unit root processes do not ideally suit the dynamics of a structural response, the framework in which the cointegration theory has been borrowed from has been established on these types of models, and for the present it seems sensible to utilise what developed theory one can. If the ECM model structure can describe the variables of interest in SHM well enough, then the philosophical question on unit roots becomes unimportant, and the theory in the previous section can be applied without further ado. As, for the present, the interest is not in accurate forecasting of a structural response variable a potential model misspecification is less important.

For the purposes of utilising the cointegration theory, the authors suggest that so long as the variables of interest to an engineer follow a similar behaviour to a unit root process over a given time interval, then cointegration theory is applicable. In this case, the theory can happily be applied to process variables which are nonstationary with first difference stationary, which at the very least may be roughly checked visually with ease.

Returning to the previous example, the first difference of the easterly deck displacements at one point on the deck of the Tamar Bridge are plotted in Figure 3. Studying this figure the first difference appears to be a stationary process. As the variable is nonstationary with first difference stationary, the Johansen procedure is applicable.

In fact, it is only necessary that the first difference be effectively stationary i.e. that the longer time scale structure in the difference be highly suppressed relative to the shorter time scale behaviour characteristic of the structural dynamics. A relatively simple example can be used to argue the case, certainly for civil engineering structures. Suppose that the signal of interest has the form,

\[ y(t) = A_s \sin(\omega_s t) + A_d \sin(\omega_d t) + A_y \sin(\omega_y t) \]  

where \( \omega_s \) is a characteristic frequency of the structural dynamics, \( \omega_d \) is a daily frequency perhaps associated with daily temperature variations and \( \omega_y \) is a frequency of one year and is associated with seasonal variations. Differentiating (5) trivially yields,

\[ \dot{y}(t) = A_s \omega_s \cos(\omega_s t) + \omega_d A_d \cos(\omega_d t) + \omega_y A_y \cos(\omega_y t) \]  

Now supposing \( \omega_s \) to be of the order of 6.82 rad/s (1 Hz) for a typical civil structure, one has \( \omega_d = 7.27 \times 10^{-5} \) rad/s and \( \omega_y = 2 \times 10^{-7} \) rad/s. One obtains from (6),

\[ \dot{y}(t) = 6.82A_s \cos(\omega_s t) + 7.27 \times 10^{-5}A_d \cos(\omega_d t) + 2 \times 10^{-7}A_y \cos(\omega_y t) \]  

and one sees that, even if the daily (trend) component of the original signal is 1000 times bigger than the structural component, in the differenced signal the trend component represents only about 1% of the total. This analysis seems to beg the question as to why one wishes to use cointegration at all rather than simply work with differenced signals. There are two good reasons why one should not differentiate in general:

(i) It is usually a bad idea to numerically differentiate experimental data as this will greatly amplify any high frequency noise components. In contrast, cointegration will at worst generate a weighted average of the noises.

(ii) Damage itself will often manifest as a trend in the data and differentiation would remove this also. In contrast, the appearance of a new trend will still (usually) destroy the balance imposed by the cointegrating vector and leave the damage exposed.

The authors believe that, although SHM variables may exhibit nonstationary behaviour over observable windows, in the nature of engineering, these response variables should be stationary.
when considering long periods of time. By design, engineering variables from stable healthy structures should be describable by a long run mean and a variance. In the long term one can therefore describe them as stationary, which enables one to use the regression techniques commonly applied in attempt to model a features dependence on an external condition, for example [8, 9]. The interests of SHM, however, are focused on response variables over shorter time periods, as any useful detection of structural degradation should be swift. One must therefore rely upon nonstationary theory and, in this work, cointegration. In the circumstance where a set of variables of interest include stationary and nonstationary measurements, the inclusion of the stationary variables doesn’t invalidate the Johansen Procedure.

3. Conclusions
Econometricians understand nonstationarity in terms of trend stationary and difference stationary processes. This paper has argued that although neither approach may be philosophically suited for SHM feature variables, these feature variables do mimic the behaviour of difference stationary processes, for which the Johansen procedure was developed. With this in mind, no problems should arise with applying the theory presented in the previous chapter to SHM data.

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