Uniform convergence on a Bakhvalov-type mesh using the preconditioning approach:

Technical report

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Abstract

The linear singularly perturbed convection-diffusion problem in one dimension is considered and its discretization on a Bakhvalov-type mesh is analyzed. The preconditioning technique is used to obtain the pointwise convergence uniform in the perturbation parameter.

Keywords: singular perturbation, convection-diffusion, boundary-value problem, Bakhvalov-type mesh, finite differences, uniform convergence, preconditioning

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1 Introduction

The report is a supplement to [8].

2 The continuous problem

We consider the problem

$$L u := - \varepsilon u'' - b(x) u' + c(x) u = f(x), \quad x \in (0, 1), \quad u(0) = u(1) = 0,$$

(1)

with a small positive perturbation parameter $\varepsilon$ and $C^1[0, 1]$-functions $b, c,$ and $f,$ where $b$ and $c$ satisfy

$$b(x) \geq \beta > 0, \quad c(x) \geq 0 \quad \text{for } x \in I := [0, 1].$$

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It is well known, see [3, 5] for instance, that (1) has a unique solution \( u \) in \( C^3(I) \), which in general has a boundary layer near \( x = 0 \). Our goal is to find this solution numerically.

The solution \( u \) can be decomposed into the smooth and boundary-layer parts. We present here Linß’s [4, Theorem 3.48] version of such a decomposition:

\[
 u(x) = s(x) + y(x),
\]

\[
|s^{(k)}(x)| \leq C \left( 1 + \epsilon^2 - k \right), \quad |y^{(k)}(x)| \leq C \epsilon^{-\beta x/\epsilon},
\]

\( x \in I, \quad k = 0, 1, 2, 3 \).

Above and throughout the report, \( C \) denotes a generic positive constant which is independent of \( \epsilon \). For the construction of the function \( s \), see [4], since the details are not of interest here. As for \( y \), it is important to note that it solves the problem

\[
 L y(x) = 0, \quad x \in (0, 1), \quad y(0) = -s(0), \quad y(1) = 0,
\]

with a homogeneous differential equation. We shall use this fact later on in the report.

3 The discrete problem and condition number estimate

We first define a finite-difference discretization of the problem (1) on a general mesh \( I^N \) with mesh points \( x_i, i = 0, 1, \ldots, N \), such that \( 0 = x_0 < x_1 < \cdots < x_N = 1 \). Throughout the rest of the paper, the constants \( C \) are also independent of \( N \).

Let \( h_i = x_i - x_{i-1}, i = 1, 2, \ldots, N \), and \( h_i = (h_i + h_{i+1})/2, i = 1, 2, \ldots, N-1 \). Mesh functions on \( I^N \) are denoted by \( W^N, U^N \), etc. If \( g \) is a function defined on \( I \), we write \( g_i \) instead of \( g(x_i) \) and \( g^N \) for the corresponding mesh function. Any mesh function \( W^N \) is identified with an \((N+1)\)-dimensional column vector, \( W^N = [W^N_0, W^N_1, \ldots, W^N_N]^T \), and its maximum norm is given by

\[
 ||W^N|| = \max_{0 \leq i \leq N} |W^N_i|.
\]

For the matrix norm, which we also denote by \( \| \cdot \| \), we take the norm subordinate to the above maximum vector norm.

We discretize the problem (1) on \( I^N \) using the upwind finite-difference scheme:

\[
 L^N U^N_i := -\epsilon D^N U^N_i - b_i D^N U^N_i + c_i U^N_i = f_i, \quad i = 1, 2, \ldots, N - 1,
\]

\[
 U^N_N = 0,
\]

where

\[
 D^N W^N_i = \frac{1}{h_i} \left( \frac{W^N_{i+1} - W^N_i}{h_{i+1}} - \frac{W^N_i - W^N_{i-1}}{h_i} \right)
\]

and

\[
 D^N W^N_i = \frac{W^N_{i+1} - W^N_i}{h_{i+1}}.
\]

2
The linear system (4) can be written down in matrix form,

\[ A_N U^N = \hat{f}^N, \]  

where \( A_N = [a_{ij}] \) is a tridiagonal matrix with \( a_{00} = 1 \) and \( a_{NN} = 1 \) being the only nonzero elements in the 0th and Nth rows, respectively, and where \( \hat{f}^N = [0, f_1, f_2, \ldots, f_{N-1}, 0]^T \).

It is easy to see that \( A_N \) is an \( L \)-matrix, i.e., \( a_{ii} > 0 \) and \( a_{ij} \leq 0 \) if \( i \neq j \), for all \( i, j = 0, 1, \ldots, N \). The matrix \( A_N \) is also inverse monotone, which means that it is non-singular and that \( A_N^{-1} \geq 0 \) (inequalities involving matrices and vectors should be understood component-wise), and therefore an \( M \)-matrix (inverse monotone \( L \)-matrix). This can be proved using the following \( M \)-criterion, see [2] for instance.

**Theorem 1.** Let \( A \) be an \( L \)-matrix and let there exist a vector \( w \) such that \( w > 0 \) and \( Aw \geq \gamma \) for some positive constant \( \gamma \). \( A \) is then an \( M \)-matrix and it holds that \( \|A^{-1}\| \leq \gamma^{-1}\|w\| \).

To see that \( A_N \) is an \( M \)-matrix, just set \( w_i = 2 - x_i \), \( i = 0, 1, \ldots, N \) in Theorem 1 to get that \( A_N w \geq \min \{1, \beta\} \). This also implies that the discrete problem (5) is stable uniformly in \( \varepsilon \),

\[ \|A_N^{-1}\| \leq \frac{2}{\min \{1, \beta\}} \leq C. \]  

(6)

Of course, the system (5) has a unique solution \( U^N \).

### 4 A Bakhvalov-type mesh

A generalization of the Bakhvalov mesh [1] to a class of Bakhvalov-type meshes can be found in [9]. Here we take one of the Bakhvalov-type meshes from [9] for the discretization mesh \( I^N \). We refer to this mesh as as Vulanović-Bakhvalov mesh (VB-mesh). The points of the VB-mesh are generated by the function \( \lambda \) in the sense that \( x_i = \lambda(t_i) \), where \( t_i = i/N \). The mesh-generating function \( \lambda \) is defined as follows:

\[ \lambda(t) = \begin{cases} 
\psi(t), & t \in [0, \alpha], \\
\psi(\alpha) + \psi'(\alpha)(t - \alpha), & t \in [\alpha, 1], 
\end{cases} \]  

(7)

with \( 0 < q < 1 \) and \( \psi = a\varepsilon\phi \), where

\[ \phi(t) = \frac{t}{q-t} = \frac{q}{q-t} - 1, \quad t \in [0, \alpha]. \]

On the interval \( [\alpha, 1] \), \( \lambda \) is the tangent line from the point \((1,1)\) to \( \psi \), touching \( \psi \) at \((\alpha, \psi(\alpha))\). The point \( \alpha \) can be determined from the equation

\[ \psi(\alpha) + \psi'(\alpha)(1 - \alpha) = 1. \]

Since \( \phi'(t) = q/(q - t)^2 \), the above equation reduces to a quadratic one,

\[ a\varepsilon\alpha(q - \alpha) + a\varepsilon q(1 - \alpha) = (q - \alpha)^2, \]
which is easy to solve for $\alpha$:

$$\alpha = q - \frac{a\varepsilon q(1 - q + a\varepsilon)}{1 + a\varepsilon}.$$

We have to assume that $a\varepsilon < q$ (which is equivalent to $\psi'(0) < 1$) and then $\alpha > 0$. Note also that $\alpha < q$ and

$$q - \alpha = \zeta \sqrt{\varepsilon}, \quad \zeta \leq C, \quad \frac{1}{\zeta} \leq C. \quad (8)$$

Let $J$ be the index such that $t_{J-1} < \alpha \leq t_J$. Starting from the mesh point $x_J$, the mesh is uniform, with step size $H$. However, $x_J$ behaves differently from the transition point of the Shishkin mesh because

$$x_J \geq \psi(\alpha) = \frac{a\alpha\varepsilon}{\zeta}.$$

We note that the transition point $\psi(\alpha)$ is different also from the Bakhvalov-Shishkin of Vulanović-Shishkin meshes in the sense of [7].

We now give the estimate for the condition number of $A_N$ when the discrete problem [1] is formed on the VB-mesh as described above. The condition number is

$$\kappa(A_N) := \|A_N^{-1}\| \|A_N\|.$$

We estimate the upper bound for $\|A_N\|$ by examining the entries of the matrix $A_N$ directly,

$$\|A_N\| \leq CN^2 \varepsilon.$$

Combining this with (6), we get the following result.

**Theorem 2.** The condition number of $A_N$ on the VB-mesh satisfies the following sharp bound:

$$\kappa(A_N) \leq CN^2 \varepsilon.$$

### 5 Conditioning

Let $M = \text{diag} (m_0, m_1, \ldots, m_N)$ be a diagonal matrix with the entries

$$m_0 = 1, \quad m_i = \frac{h_i}{H}, \quad i = 1, 2, \ldots, N - 1, \quad \text{and} \quad m_N = 1.$$

In other words,

$$m_0 = 1, \quad m_i = \frac{h_i}{H}, \quad i = 1, 2, \ldots, J, \quad \text{and} \quad m_i = 1, i = J + 1, \ldots, N. \quad (9)$$

When the system [5] is multiplied by $M$, this is equivalent to multiplying the equations 1, 2, $\ldots$, $J$ of the discrete problem [1] by $h_i/H, i = 1, 2, \ldots, J$. The modified system is

$$\tilde{A}_N U^N = M f^N, \quad (10)$$
where \( \tilde{A}_N = MA_N \). Let the entries of \( \tilde{A}_N \) be denoted by \( \tilde{a}_{ij} \), the nonzero ones being

\[
l_i := \tilde{a}_{i-1,i} = \begin{cases} 
-\frac{\varepsilon}{h_i} & 1 \leq i \leq J - 1, \\
-\frac{\varepsilon}{h_J} & i = J, \\
-\frac{\varepsilon}{H^2} & J + 1 \leq i \leq N - 1,
\end{cases}
\]

\[
r_i := \tilde{a}_{i,i+1} = \begin{cases} 
-\frac{\varepsilon}{h_{i+1}} - \frac{b_i h_i}{h_{i+1} H}, & 1 \leq i \leq J - 1, \\
-\frac{\varepsilon}{H^2} - \frac{b_i h_i}{H^2}, & i = J, \\
-\frac{\varepsilon}{H^2} - \frac{b_i}{H}, & J + 1 \leq i \leq N - 1,
\end{cases}
\]

and

\[
d_i := \tilde{a}_{ii} = \begin{cases} 
1, & i = 0 \\
-\frac{r_i + h_i}{H^2} c_i, & 1 \leq i \leq J, \\
-\frac{r_i + c_i}{H}, & J + 1 \leq i \leq N - 1, \\
1, & i = N.
\end{cases}
\]

Unlike the Shishkin mesh, which is piece-wise uniform, the VB-mesh is graded in the fine part. Because of this, it is more difficulty to prove the uniform stability of the modified scheme. This is done in Lemma 2 below, but first we need some crucial estimates for the graded mesh defined by (7).

**Lemma 1.** For the mesh-generating function given in (7), the following estimates hold true:

\[
\varepsilon \left( \frac{h_{i+1} - h_i}{h_i h_{i+1}} \right) \leq \frac{2}{a}, \quad i = 1, 2, \ldots, J - 2, \quad (11)
\]

and

\[
\varepsilon \left( \frac{H - h_J}{h_J H} \right) \leq \frac{\zeta \sqrt{\varepsilon}}{a q}. \quad (12)
\]

**Proof.** For \( i \leq J - 2 \), we have

\[
h_i = x_i - x_{i-1} = a \varepsilon \left( \frac{q}{q - t_i} - \frac{q}{q - t_{i-1}} \right) = \frac{a \varepsilon q}{N(q - t_{i-1})(q - t_i)},
\]

\[
h_{i+1} = \frac{a \varepsilon q}{N(q - t_i)(q - t_{i+1})},
\]

and

\[
h_{i+1} - h_i = \frac{2a \varepsilon q}{N^2(q - t_{i-1})(q - t_i)(q - t_{i+1})}.
\]
Then (11) follows because
\[
\frac{\varepsilon(h_{i+1} - h_i)}{h_{i+1}} = \frac{2(q - t_i)}{aq} = \frac{2}{a} \left( 1 - \frac{t_i}{q} \right) \leq \frac{2}{a}.
\]

The proof of (12) is more complicated due to the presence of \(h_J\). First, \(h_J = \gamma_1 + \gamma_2\), where \(\gamma_1 = x_\alpha - x_{J-1}\), \(\gamma_2 = x_J - x_\alpha\), and \(x_\alpha = \psi(\alpha)\). Since
\[
\gamma_2 = \psi'(\alpha)(t_J - \alpha)
\]
\[
= \frac{a\varepsilon q}{q - \alpha} \left( t_J - \alpha \right)
\]
and
\[
\gamma_1 = a\varepsilon (\phi(\alpha) - \phi(t_{J-1}))
\]
\[
= a\varepsilon \left( \frac{\alpha}{q - \alpha} - \frac{t_{J-1}}{q - t_{J-1}} \right)
\]
\[
= \frac{a\varepsilon q}{q - \alpha} \frac{\alpha - t_{J-1}}{q - t_{J-1}}.
\]
we have
\[
h_J = \frac{a\varepsilon q}{q - \alpha} \left[ \frac{t_J - \alpha}{q - \alpha} + \frac{\alpha - t_{J-1}}{q - t_{J-1}} \right]
\]
\[
= \frac{a\varepsilon q}{(q - \alpha)^2} \left[ t_J - \alpha + \frac{(q - \alpha)(\alpha - t_{J-1})}{q - t_{J-1}} \right]
\]
\[
= \frac{a\varepsilon q}{\zeta^2} \left[ t_J - \alpha + \zeta \sqrt{\varepsilon}(\alpha - t_{J-1}) \right].
\]
Moreover,
\[
\psi'(\alpha) = \frac{a\varepsilon q}{(q - \alpha)^2} \text{ and } H = x_{J+1} - x_J = \frac{\psi'(\alpha)}{\zeta},
\]
implying that
\[
H = \frac{a\varepsilon q}{N(q - \alpha)^2}.
\]
Therefore,
\[
H - h_J = \frac{a\varepsilon q}{q - \alpha} \left[ \frac{1}{N(q - \alpha)} - t_J - \frac{\alpha - t_{J-1}}{q - t_{J-1}} \right]
\]
\[
= \frac{a\varepsilon q}{q - \alpha} \left[ \alpha - t_{J-1} \frac{1}{q - \alpha} - \frac{\alpha - t_{J-1}}{q - t_{J-1}} \right]
\]
\[
= \frac{a\varepsilon q}{q - \alpha} (\alpha - t_{J-1}) \left[ \frac{1}{q - \alpha} - \frac{1}{q - t_{J-1}} \right]
\]
\[
= \frac{a\varepsilon q}{q - \alpha} (\alpha - t_{J-1}) \frac{(q - \alpha)(q - t_{J-1})}{q - t_{J-1}}
\]
\[
= \frac{a\varepsilon q}{(q - \alpha)^2} (\alpha - t_{J-1})^2.
\]
We now have
\[
\frac{\epsilon H - h_J}{h_J H} = \frac{a\varepsilon^2 q}{(q - \alpha)^2} \cdot \frac{(\alpha - t_{J-1})^2}{q - t_{J-1}} \cdot \frac{q - \alpha}{a\varepsilon q} \cdot \frac{1}{\frac{t_j - \alpha}{q - \alpha} + \frac{\alpha - t_{J-1}}{q - t_{J-1}}} \cdot \frac{(q - \alpha)^2 N}{a\varepsilon q}.
\]
\[
= \frac{(q - \alpha)N}{aq} \cdot \frac{(\alpha - t_{J-1})^2}{q - t_{J-1}} \cdot \frac{(q - \alpha)(q - t_{J-1})}{N^2 - \alpha^2 + 2\alpha t_{J-1} - t_{J-1}t_J}
\]
\[
= \frac{(q - \alpha)^2 N}{aq} \cdot \frac{(\alpha - t_{J-1})^2}{\omega} \leq \frac{\zeta^2 \varepsilon}{aqN} \cdot \frac{1}{\omega},
\]
where
\[
\omega := \frac{q}{N} - \alpha^2 + 2\alpha t_{J-1} - t_{J-1}t_J
\]
and where in the last step we used (5) and the fact that $0 \leq \alpha - t_{J-1} \leq 1/N$.
The denominator $\omega$ can be estimated as follows:
\[
\omega = \frac{q}{N} - (\alpha - t_{J-1})^2 - \frac{t_{J-1}}{N}
\]
\[
= \frac{\zeta \sqrt{\varepsilon} + \alpha}{N} - (\alpha - t_{J-1})^2 - \frac{t_{J-1}}{N}
\]
\[
= \frac{\zeta \sqrt{\varepsilon}}{N} + \frac{1}{N} (\alpha - t_{J-1}) - (\alpha - t_{J-1})^2
\]
\[
= \frac{\zeta \sqrt{\varepsilon}}{N} + (\alpha - t_{J-1}) (t_j - \alpha)
\]
\[
\geq \frac{\zeta \sqrt{\varepsilon}}{N}, \text{ since } (\alpha - t_{J-1}) (t_j - \alpha) \geq 0.
\]
Therefore,
\[
\frac{\epsilon H - h_J}{h_J H} \leq \frac{\zeta^2 \varepsilon}{aqN} \cdot \frac{N}{\zeta \sqrt{\varepsilon}} = \frac{\zeta \sqrt{\varepsilon}}{aq}.
\]
This completes the proof of (12).

It is easy to see that $\tilde{A}_N$ is an $L$-matrix. The next lemma shows that $\tilde{A}_N$ is an $M$-matrix and that the modified discretization (10) is stable uniformly in $\varepsilon$.

Lemma 2. Let $\varepsilon$ be sufficiently small, independently of $N$, and let $a > 4/\beta$. Then the matrix $\tilde{A}_N$ of the system (10) satisfies
\[
\|\tilde{A}_N^{-1}\| \leq C.
\]

Proof. We want to construct a vector $v = [v_0, v_1, \ldots, v_N]^T$ such that
(a) $v_i \geq \delta$, $i = 0, 1, \ldots, N$, where $\delta$ is a positive constant independent of both $\varepsilon$ and $N$,
(b) $v_i \leq C$, $i = 0, 1, \ldots, N$,
(c) $\sigma_i := l_i v_{i-1} + d_i v_i + r_i v_{i+1} \geq \delta$, $i = 1, 2, \ldots, N - 1$.

Then, according to the $M$-criterion,
\[
\|\tilde{A}_N^{-1}\| \leq \delta^{-1} \|v\| \leq C.
\]
The following choice of the vector \( v \) is motivated by \([6, 11, 8]\):

\[
 v_i = \begin{cases} 
 \alpha - Hi + \lambda, & i \leq J - 1, \\
 \alpha - Hi + \frac{\lambda}{1 + \rho_j} (1 + \rho)^{J-i}, & i \geq J,
\end{cases}
\]

where \( \rho_j = \beta h_j / (2\varepsilon) \), \( \rho = \beta H / (2\varepsilon) \), and \( \alpha \) and \( \lambda \) are fixed positive constants. Since \( HN \leq C \), there exists a constant \( \alpha \) such that \( v_i \geq \alpha - Hi \geq \delta > 0 \), so the condition (a) is satisfied. Then, because of \( v_i \leq \alpha + \lambda \), the condition (b) holds true if we show that \( \lambda \leq C \). We do this next as we verify the condition (c).

When \( 1 \leq i \leq J - 2 \), we use (11) to get

\[
\sigma_i = (l_i + d_i + r_i)v_i + l_i H - r_i H
\]

\[
\geq h_i c_i v_i - \frac{\varepsilon}{b_i} + \frac{\varepsilon}{h_i + 1} + \frac{b_i h_i}{h_i + 1}
\]

\[
\geq - \frac{\varepsilon (h_{i+1} - h_i)}{h_{i+1} h_i} + \frac{b_i}{2} + \frac{b_i h_i}{h_{i+1} + 1}
\]

\[
\geq - \frac{2 + b_i}{\alpha} - \frac{\beta}{2} \geq \frac{2}{\alpha} =: \delta > 0.
\]

The constant \( \delta \) exists because of the assumption \( \alpha > 4/\beta \).

For \( i = J - 1 \), we have

\[
\sigma_{J-1} = \frac{h_{J-1}}{H} c_{J-1} v_{J-1} + l_{J-1} H - r_{J-1} H
\]

\[
+ \lambda l_{J-1} + \lambda d_{J-1} + r_{J-1} \frac{\lambda}{1 + \rho_j}
\]

\[
\geq \frac{\varepsilon}{h_{J-1}} + \frac{\varepsilon}{h_{j}} + \frac{b_{J-1} h_{J-1}}{h_{j}} - r_{J-1} \frac{\lambda \rho_j}{1 + \rho_j}
\]

\[
\geq \frac{\varepsilon}{h_{J-1}} + \frac{b_{J-1}}{2} - r_{J-1} \frac{\lambda \rho_j}{1 + \rho_j}
\]

\[
\geq \frac{\varepsilon}{h_{J-1}} + \frac{\beta}{2} + \frac{\varepsilon}{h_{J-1}} \frac{b_{J-1} h_{J-1}}{h_{j} H} \frac{\lambda \beta h_{j}}{2\varepsilon + \beta h_{j}}
\]

\[
\geq \frac{\beta}{2} - \frac{\varepsilon}{h_{J-1}} + \frac{\lambda \beta}{4H} \geq \frac{\beta}{2} \geq \delta
\]

with a suitable positive constant \( \lambda \). We can choose such \( \lambda \) because the estimates \( H \leq 2N^{-1} \) and \( q - t_{J-1} \leq q - t_{J-2} \leq 1 \) imply

\[
\frac{\lambda \beta}{4H} - \frac{\varepsilon}{h_{J-1}} = \frac{\lambda \beta}{4H} - \frac{N}{aq} (q - t_{J-1}) (q - t_{J-2}) \geq N \left( \frac{\lambda \beta}{8} - \frac{1}{aq} \right) \geq 0.
\]
For $i = J$, we get

$$
\sigma_J = \frac{h_J}{H} c_J v_J + l_J H - r_J H + \lambda \left[ l_J + \frac{d_J}{1 + \rho_J} + \frac{r_J}{(1 + \rho_J)(1 + \rho)} \right]
$$

$$
\geq -\frac{\varepsilon}{h_J} + \frac{\varepsilon}{H} + \frac{b_j h_J}{H} + \frac{\lambda}{1 + \rho_J}(1 + \rho) \left[ l_J(1 + \rho_J)(1 + \rho) + d_J(1 + \rho) + r_J \right]
$$

$$
\geq -\frac{\varepsilon}{H} - \frac{\varepsilon}{h_J} + \frac{b_J}{2} + \frac{\lambda}{1 + \rho_J}(1 + \rho) \left[ l_J(1 + \rho_J)(1 + \rho) + d_J(1 + \rho) + r_J \right]
$$

$$
\geq \frac{\beta}{2} - \frac{\varepsilon(H - h_J)}{h_J H} \geq \delta > 0.
$$

The above estimate holds true because (12) implies that

$$
\frac{\varepsilon(H - h_J)}{h_J H} \leq \frac{\zeta \sqrt{\varepsilon}}{\alpha q} \leq \frac{2}{a},
$$

when $\varepsilon$ is sufficiently small, and because we can show that

$$
[l_J(1 + \rho_J)(1 + \rho) + d_J(1 + \rho) + r_J] \geq 0.
$$

Indeed,

$$
l_J(1 + \rho_J)(1 + \rho) + d_J(1 + \rho) + r_J = l_J \rho_J + l_J \rho_J \rho - r_J \rho
$$

$$
= -\frac{\varepsilon}{h_J H} - \frac{\varepsilon}{H} + \left[ \frac{\beta h_J}{H} \right] \frac{\beta H}{2\varepsilon} + \frac{b_j h_J}{H} \frac{\beta H}{2\varepsilon}
$$

$$
= -\frac{\beta^2}{4\varepsilon} + \frac{\beta b_J h_J}{2H\varepsilon}
$$

$$
= -\frac{\beta^2}{4\varepsilon} + \frac{\beta b_J (h_J + H)}{4H\varepsilon}
$$

$$
\geq -\frac{\beta^2}{4\varepsilon} + \frac{\beta b_J}{4\varepsilon} \geq 0.
$$

Finally, when $J + 1 \leq i \leq N - 1$, we have

$$
\sigma_i = c_i v_i + l_i H - r_i H + \frac{l_i}{1 + \rho_J} \left[ \frac{\lambda}{(1 + \rho)^{i-1 - J}} - \frac{\lambda}{(1 + \rho)^{i-J}} \right]
$$

$$
+ \frac{r_i}{1 + \rho_J} \left[ \frac{\lambda}{(1 + \rho)^{i-1 - J}} - \frac{\lambda}{(1 + \rho)^{i-J}} \right]
$$

$$
\geq b_i + \frac{\rho(1 + \rho)|l_i - \rho r_i|}{(1 + \rho_J)(1 + \rho)^{i+1-1 - J} \lambda}
$$

$$
\geq \frac{\beta}{2} + \frac{(l_i - r_i + l_i \rho) \rho}{(1 + \rho_J)(1 + \rho)^{i+1-1 - J} \lambda}
$$

$$
= \frac{\beta}{2} + \frac{(b_i - \frac{\beta}{H})}{\lambda \rho(1 + \rho)^{i-1}} \frac{\rho(1 + \rho)^{i-1-1}}{1 + \rho_J}
$$

$$
\geq \frac{\beta}{2} > \delta.
$$
By examining the elements of the matrix $\tilde{A}_N$, we see that
$$\|\tilde{A}_N\| \leq CN^2.$$ When we combined this with Lemma 2, we get the following result.

**Theorem 3.** The matrix $\tilde{A}_N$ of the system (10) satisfies
$$\kappa(\tilde{A}_N) \leq CN^2.$$

### 6 Uniform convergence

Let $\tau_i, i = 1, 2, \ldots, N - 1$, be the consistency error of the finite-difference operator $\mathcal{L}_N$,
$$\tau_i = \mathcal{L}_N u_i - f_i.$$

We have
$$\tau_i = \tau_i[u] := \mathcal{L}_N u_i - (\mathcal{L}u)_i$$
and by Taylor’s expansion we get that
$$|\tau_i[u]| \leq Ch_i^{i+1}(\varepsilon\|u''''\|_i + \|u''\|_i),$$
(13)
where $\|g\|_i := \max_{x_{i-1} \leq x \leq x_{i+1}} |g(x)|$ for any $C(I)$-function $g$. Let us define
$$\tilde{\tau}_i[u] = \begin{cases} h_i \tau_i[u], & 1 \leq i \leq J, \\ \tau_i[u], & J + 1 \leq i \leq N - 1. \end{cases}$$

**Lemma 3.** The following estimate holds true for all $i = 1, 2, \ldots, N - 1$:
$$|\tilde{\tau}_i[u]| \leq CN^{-1}.$$ 

**Proof.** We use the decomposition (2) and estimates (3). For the smooth part of the solution, it is easy to show that $|\tilde{\tau}[s]| \leq CN^{-1}$. Then we need to show that
$$|\tilde{\tau}_i[y]| \leq CN^{-1}.$$

**Case 1.** Let $i \geq J + 1$, i.e. $t_{i-1} \geq t_J \geq \alpha$. Then we have
$$|\tilde{\tau}_i[y]| = |\tau_i[y]| \leq Ch_{i+1} (\varepsilon\|u''''\|_i + \|u''\|_i)$$
$$\leq CN^{-1} \lambda(t_{i+1}) e^{-2\varepsilon^{-3\lambda(t_{i-1})}/\varepsilon}$$
$$\leq CN^{-1} \lambda(t_{i+1}) e^{-2\varepsilon^{-3\lambda(\alpha)/\varepsilon}}$$
$$\leq CN^{-1} e^{-2\varepsilon^{-3\lambda(\alpha)/\varepsilon}}$$
$$\leq CN^{-1},$$
where we have used the fact that $e^{-2\varepsilon^{-3\lambda(\alpha)/\varepsilon}} \leq C$. 

Case 2. Let \( i \leq J \), i.e. \( t_{i-1} < \alpha \), and at the same time, let \( t_{i-1} \leq q - 3/N \). Note that, when \( t_{i-1} \leq q - 3/N \), we have

\[
t_{i+1} \leq q - 1/N < q \quad \text{and} \quad q - t_{i+1} \geq \frac{1}{3}(q - t_{i-1}).
\]

This is because

\[
q - t_{i-1} \geq \frac{3}{N} \quad \Rightarrow \quad \frac{2}{3}(q - t_{i-1}) \geq \frac{2}{N},
\]

which gives

\[
q - t_{i+1} = q - t_{i-1} - \frac{2}{N} = \frac{1}{3}(q - t_{i-1}) + \frac{2}{3}(q - t_{i-1}) - \frac{2}{N} \geq \frac{1}{3}(q - t_{i-1}).
\]

Therefore,

\[
|\bar{\tau}_i[y]| = \frac{h_i}{H} |\tau_i[y]| \leq \frac{h_i}{H} C H \epsilon_i (\epsilon \|y''\|_i + \|y''\|_i) \\
\leq CN^{-1} [\lambda(t_{i+1})]^{2} \epsilon^{-2} e^{-\beta \lambda(t_{i+1})/\epsilon} \\
\leq CN^{-1} [\phi'(t_{i+1})]^{2} e^{-a\beta \phi(t_{i+1})} \\
\leq C \epsilon^{-1} N^{-1} (q - t_{i+1})^{-4} e^{-a \beta \phi(q - t_{i+1}-1)} \\
\leq CN^{-1}(q - t_{i-1})^{-4} e^{-a \beta \phi(q - t_{i-1})} \\
\leq CN^{-1},
\]

because \( (q - t_{i-1})^{-4} e^{-a \beta \phi(q - t_{i-1})} \leq C \).

Case 3. In the last case, we consider the remaining possibility, \( q - 3/N < t_{i-1} < \alpha \). We use the fact that \( \mathcal{L} y = 0 \) to work with

\[
|\bar{\tau}_i[y]| = \frac{h_i}{H} |\tau_i[y]| \leq \frac{h_i}{H} (P_i + Q_i + R_i),
\]

where

\[
P_i = \epsilon |D'' y_i|, \quad Q_i = b_i |D' y_i|, \quad \text{and} \quad R_i = c_i |y_i|.
\]

We now follow closely the technique in \([10, \text{Lemma 5}]\), (see also \([11, \S]\)), to get

\[
\frac{h_i}{H} (P_i + Q_i + R_i) \leq C \left[ \frac{h_i}{H} \left( \frac{1}{h_i} \epsilon \cdot 2 \|y''\|_i \right) + \frac{h_i}{H} \left( \frac{1}{h_{i+1}} \|y''\|_i \right) + e^{-\beta \lambda(t_i)/\epsilon} \right] \\
\leq CN e^{-\beta \lambda(t_{i-1})/\epsilon} \\
\leq CN e^{-a \beta \phi(t_{i-1})} \\
\leq CN e^{-a \beta \phi(q - 3/N)} \\
\leq CN e^{-a \beta (qN/3 - 1)} \\
\leq CN^{-1}.
\]

Remark 1. The technique used in the above proof is based on \([9]\), where the same approach is successfully applied to reaction-diffusion problems. This approach is originally due to Bakhvalov \([1]\). The technique works here for convection-diffusion problems \([1]\) because an extra \( \epsilon \)-factor is obtained from the preconditioner \([\eta]\).
When Lemmas 2 and 3 are combined, which amounts to the use of the consistency-stability principle, we obtain the following result.

**Theorem 4.** Let $\varepsilon$ be sufficiently small, independently of $N$, and let $a > 4/\beta$. Then the solution $U^N$ of the discrete problem (5) on the VB-mesh satisfies
\[
\|U^N - u^N\| \leq CN^{-1},
\]
where $u$ is the solution of the continuous problem (1).

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