Morita Contexts, Idempotents, and Hochschild Cohomology
— with Applications to Invariant Rings —

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Dedicated to the memory of Peter Slodowy

Abstract. We investigate how to compare Hochschild cohomology of algebras related by a Morita context. Interpreting a Morita context as a ring with distinguished idempotent, the key ingredient for such a comparison is shown to be the grade of the Morita defect, the quotient of the ring modulo the ideal generated by the idempotent. Along the way, we show that the grade of the stable endomorphism ring as a module over the endomorphism ring controls vanishing of higher groups of selfextensions, and explain the relation to various forms of the Generalized Nakayama Conjecture for Noetherian algebras. As applications of our approach we explore to what extent Hochschild cohomology of an invariant ring coincides with the invariants of the Hochschild cohomology.

Contents

Introduction 1
1. Morita Contexts 3
2. The Grade of the Stable Endomorphism Ring 6
3. Equivalences to the Generalized Nakayama Conjecture 12
4. The Comparison Homomorphism for Hochschild Cohomology 14
5. Hochschild Cohomology of Auslander Contexts 17
6. Invariant Rings: The General Case 20
7. Invariant Rings: The Commutative Case 23
References 27

Introduction

One of the basic features of Hochschild (co-)homology is its invariance under derived equivalence, see [25], in particular, it is invariant under Morita equivalence, a result originally established by Dennis-Igusa [18], see also [26]. This raises the question how Hochschild cohomology compares in general for algebras that are just ingredients of a Morita context.

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For example, when a finite group $G$ acts on an algebra $S$, then the invariant ring $R = S^G$ and the skew group algebra $S\#G$ are related by a Morita context whose other ingredients are two copies of $S$, one considered as $(R, S\#G)$–bimodule, the second as $(S\#G, R)$–bimodule. It is known, see [27, 33], how to relate the Hochschild (co-)homology of $S$ to that of $S\#G$, but the relation of either to that of $R$ seems less well understood.

However, the analogous question for tangent (or André-Quillen) cohomology, that is, the cohomology of the cotangent complex, has been addressed as then one of the first applications of the theory. Indeed, M. Schlessinger showed in [31] that the quotient of a rigid complex analytic singularity by a finite group is again rigid, provided the depth along the branch locus of the group action is at least equal to 3. His ideas show more generally that, for $S$ the commutative ring of the singularity and $R = S^G$, the tangent cohomology satisfies $T^i_R \cong (T^i_S)^G$ for each $i \leq g - 2$, where $g$ is the depth along the branch locus. Our main result in this direction, 7.5, shows this to remain true if tangent cohomology is replaced by Hochschild cohomology.

Our starting point, in Section 1, is to view a Morita context as a ring $C$ with a distinguished pair of complementary idempotents $e, e' = 1 - e$, relating the rings $A = eCe$ and $B = e'Ce'$ through the bimodules $M = eCe$ and $N = eCe'$. We define the associated Morita defects as $\overline{C} = C/CeC$ and $\overline{C}' = C/Ce'C$, noting that $C$ represents a Morita equivalence between $A$ and $B$ if, and only if, both defects vanish.

The key notion of the grade of a module and its basic properties are reviewed in Section 2. The crucial observation then is that $C$ is the classical Morita context defined by a ring $A$ and a module $M$, or what we call an Auslander context, if, and only if, the grade of the Morita defect $\overline{C}$ is at least 2.

The natural question as to the significance of that grade in general has a surprisingly satisfying answer: that grade, call it $g$, indicates precisely the vanishing of $\text{Ext}^i_A(M \oplus A, M \oplus A)$ for $1 \leq i \leq g - 2$, as we show in 2.13. This makes evident the relation to various formulations, mainly by M. Auslander and I. Reiten, of the Generalized Nakayama Conjecture that ask, in essence, for which classes of rings $A$ and classes of modules $M$ over it, vanishing of those Ext–groups for each $i \geq 1$ forces $M$ to be projective. This angle is explored in Section 3, where we view these conjectures as attempts to extend Morita theory. Informally, the key question becomes, whether, or when, infinite grade of a Morita defect forces its vanishing.

In Section 4, we finally turn our attention to Hochschild cohomology. After introducing the map that compares Hochschild cohomology of a Morita context with that of one of its components, we show it to be a homomorphism of graded algebras. We added details on this result after the referee of an earlier version of the paper brought the work of E. Green and Ø. Solberg [22] to our attention that deals with this comparison map for Morita contexts under some additional hypotheses. The referee also pointed to various recent papers, such as [15, 16, 22, 30], that set up exact sequences for Hochschild cohomology of triangular algebras, a degenerate case of Morita contexts, but generalizing Happel’s original sequence for one-point-extension algebras [23]. The occurring comparison maps of Hochschild cohomology there are special cases of the one studied here, as pointed out already in [22].

Our main application of the grade of a Morita defect to Hochschild cohomology in Section 5 shows coincidence of those groups for algebras in a Morita context almost up to the grade.
To apply the results of Section 5 efficiently, the question that remains is how to determine the grade of a Morita defect. We do not know a general answer. However, for invariants under a finite group acting on a commutative noetherian domain, we relate in Section 7 that grade to the depth of the Noether different of the ring over its subring of invariants, after setting the stage in Section 6 by reviewing in general the homological algebra of finite group actions and explaining the notion of the Noether or homological different as originally defined by Auslander-Goldman [6] in the noncommutative setting.

It is not until Section 7 that we deal with commutative algebra proper, whereas the sections leading up to it should be seen as advocacy that sometimes excursions into noncommutative algebra help to shed light on problems in commutative algebra.

As the reader might easily guess from the references used, the ideas here are greatly influenced by, if not taken straight from the work of M. Auslander and his collaborators. Indeed, temptation was great to head this article “Homological theory of idempotents”, not only paying homage to [7], but also retaining the title of the original talk given at the Joint AMS/SMF meeting in Lyon, July 2001, of which this is a vastly extended and detailed version.

I also wish to thank the referee for a careful reading.

1. Morita Contexts

We begin with the fairly obvious observation that a (generalized) Morita context or pre-equivalence is nothing but a ring with a distinguished idempotent. This point of view is at least implicit whenever the data of a Morita context are arranged as $(2 \times 2)$-matrices, see, for example, [29].

1.1. Given an idempotent $e = e^2$ in a ring $C$, let $e' = 1 - e$ denote its complementary idempotent. The resulting Pierce decomposition $C = e'Ce' \oplus eCe' \oplus Ce' \oplus eCe$ into a direct sum of abelian groups allows to display $C$ as an algebra of $(2 \times 2)$-matrices,

$$C \cong \begin{pmatrix} e'Ce' & eCe \\ eCe' & eCe \end{pmatrix}.$$  

1.2. The diagonal entries, $A = eCe$ and $B = e'Ce'$, are rings with multiplicative identity $1_A = e$ and $1_B = e'$ respectively, but they are usually not (unital) subrings of $C$. In terms of the rings $A$ and $B$, the component $M = e'Ce \subseteq C$ is a left $B$-, right $A$-module, whereas the component $N = eCe' \subseteq C$ is a right $B$-, left $A$-module. Emphasizing $A, B$ and the bimodules $M, N$, the displayed decomposition of $C$ becomes

$$C \cong \begin{pmatrix} e'Ce' & e'Ce \\ eCe' & eCe \end{pmatrix} = \begin{pmatrix} B & M \\ N & A \end{pmatrix}.$$

The multiplication on $C$ induces

- a homomorphism of $B$-bimodules
  
  \[ f : M \otimes_A N = e'Ce \otimes eCe' eCe' \to e'Ce' = B, \]

- a homomorphism of $A$-bimodules
  
  \[ g : N \otimes_B M = eCe' \otimes e'Ce' eCe' \to eCe = A, \]
and associativity of the multiplication on \( C \) combines the associativity conditions required of the Morita context \((A, B, M, N, f, g)\). Conversely, arranging a Morita context into a \((2 \times 2)\)-matrix as above defines a ring \( C \) with the distinguished pair of complementary idempotents \( e = 1_A, e' = 1_B \). We thus don’t reinvent the wheel, when we make the

**Definition 1.3.** A Morita context, between rings \( A \) and \( B \), consists of a ring \( C \) together with an idempotent \( e \in C \), such that \( A \cong eCe, B \cong e'C'e' \). A morphism of Morita contexts \((C_1, e_1) \to (C_2, e_2)\) is a ring homomorphism \( \varphi : C_1 \to C_2 \) such that \( \varphi(e_1) = e_2 \).

Morita contexts were originally introduced to characterize when two rings have equivalent module categories. The crucial role of the given idempotents leads us to the following notion.

**Definition 1.4.** The homomorphic images
\[
\overline{C} = C/(e) = C/CeC \quad \text{and} \quad \overline{C}' = C/(e') = C/Ce'C
\]
are the Morita defects of \( C \). When \( \overline{C} = 0 \), we call \( e \) a Morita idempotent.

Note the following simple fact.

**Lemma 1.5.** Let \((C, e)\) be a Morita context. A right module \( X \) over \( C \) is annihilated by \( CeC \), thus naturally a right \( C \)-module, if and only if \( Xe = 0 \). Furthermore, the following two conditions are equivalent:

1. \( e \) is a Morita idempotent.
2. A right \( C \)-module \( X \) satisfies \( Xe = 0 \) if and only if \( X = 0 \).

**Proof.** For the first claim, observe that evidently \( Xe = 0 \) \iff \( XCeC = 0 \). The claimed equivalence follows from \( Ce = 0 \). \( \square \)

The classical theorem on Morita equivalence is essentially a consequence of the following, fundamentally important, exact sequence of \( C \)-bimodules,
\[
0 \to \Omega_{C/A} \to Ce \otimes eCe \xrightarrow{\mu_e} eCe \to C \to C \to 0,
\]
where \( \mu_e \) is induced by the multiplication on \( C \), and \( \Omega_{C/A} \) is defined as the kernel of \( \mu_e \).

**Lemma 1.6.** Applying \( e'C \otimes_C - \otimes_C e'C' \) to the exact sequence \( \Box \) above results in the exact sequence of \( B = e'C'e' \)-bimodules
\[
0 \longrightarrow \Omega_{C/A} \longrightarrow e'Ce \otimes eCe \xrightarrow{\mu_e} eCe \xrightarrow{\mu_{e'C} \otimes \mu_{e'C'}} e'Ce' \longrightarrow C \longrightarrow \overline{C} \longrightarrow 0.
\]

In particular, \( \overline{C} \cong B/((e \cap B) \cap B) \), whence the Morita defect \( \overline{C} \) is a homomorphic image of \( B \), isomorphic to the cokernel of \( f \).

The \( C \)-bimodule \( \Omega_{C/A} \) is annihilated by \( e \) on both sides, thus naturally a \( \overline{C} \)-bimodule.

**Proof.** Applying \( C \otimes_C - \otimes_C C \) does not change the exact sequence \( \Box \). Now, as a right \( C \)-module, \( C = eC \oplus e'C \), the tensor product \( eC \otimes_C - \) is an exact functor, and \( eC \otimes_C \mu_e \) is clearly an isomorphism. In particular, \( e\Omega_{C/A} = 0 \). On the other
side, \( C = C e \oplus C e' \) as left \( C \)-modules, and the result follows by combining these observations. \( \square \)

**Remark 1.7.** One may show directly, cf. [7, Lemma 1.5, Prop. 4.6], that

\[ \Omega_{\mathbb{C}/A} \cong \text{Tor}^\mathbb{C}_2(\mathbb{C}, \mathbb{C}) \]

as \( \mathbb{C} \)-bimodules, and then the last claim in 1.6 becomes completely transparent.

1.8. Henceforth, without further specification, a “module” over some ring \( A \) will mean a unital right module, and the category formed by those will be denoted \( \textbf{Mod} A \). Accordingly, we write \( \text{Hom}_A \) to denote homomorphisms of right modules, \( \text{Hom}_{A^{op}} \) to denote those of left \( A \)-modules, where \( A^{op} \) is the opposite ring. Modules over commutative rings are considered symmetric bimodules, as usual.

Recall that a module \( X \) over a ring \( A \) is a generator if there exists an \( A \)-linear epimorphism \( X \rightarrow A \), equivalently, \( A \) is a direct summand of a finite direct sum of copies of \( X \). If \( X \) is any module, then clearly \( X \oplus A \) is a generator.

The following classical result, see [11, II. Thm. 3.4], characterizes a Morita idempotent, and lists its crucial properties for the associated Morita context. It is essentially an immediate consequence of 1.6.

**Proposition 1.9.** The following conditions are equivalent for an idempotent \( e \in C \).

1. \( e \in C \) is a Morita idempotent,
2. the restricted multiplication map
   \[ \mu_e : C e \otimes_{e C e} e C \rightarrow C \]
   is an epimorphism of \( C \)-bimodules, and in that case, it is an isomorphism.
3. the restricted multiplication map
   \[ f : N \otimes_A M = e' C e \otimes_{e C e} e C e' \rightarrow e' C e' = B \]
   is an epimorphism of \( B \)-bimodules, and in that case, it is an isomorphism.

Moreover, if \( e \in C \) is a Morita idempotent, then

(i) the \( B = e' C e' \)-modules* \( M = e C e' \) and \( N = e' C e \), as well as \( C e' \) and \( e' C \), are generators.
(ii) the \( A = e C e \)-modules \( M = e C e' \) and \( N = e' C e \), as well as \( e C \) and \( C e \), are finite projective.
(iii) the multiplication on \( C \), equivalently, the adjoints to \( g : M \otimes_B N \rightarrow A \), yield isomorphisms
   \[ M = e C e' \xrightarrow{\cong} \text{Hom}_{(e C e)}(e' C e, e C e) = \text{Hom}_A(N, A) \]
   of \( (B, A) \)-bimodules, and
   \[ N = e' C e \xrightarrow{\cong} \text{Hom}_{e C e \oplus}(e C e', e C e) = \text{Hom}_{A^{op}}(M, A) \]
   of \( (A, B) \)-bimodules.

*violating our general convention, here one of them is a right, the other a left module, as is obvious from the context...
(iv) the multiplication on $C$ yields ring isomorphisms
\[
B = e'Ce' \xrightarrow{\cong} \text{End}_{eCe}(e'Ce) = \text{End}_A(M)
\]
\[
B = e'Ce' \xrightarrow{\cong} \text{End}_{(eCe)\text{op}}(e'Ce')^{\text{op}} = \text{End}_{A^{\text{op}}}(N)^{\text{op}}.
\]

Classical Morita theory asserts that $(A, B, M, N, f, g)$ represents a Morita equivalence if and only if $f$ and $g$ are surjective. The latter condition is equivalent to the vanishing of both Morita defects by the preceding results. Indeed, we have the following.

**Corollary 1.10.** Let $(C, e)$ be a Morita context.

1. The rings $A = eCe$ and $C$ are Morita equivalent through the bimodules $Ce, eC$, if and only if $C = 0$.
2. The rings $B = e'Ce'$ and $C$ are Morita equivalent through the bimodules $Ce', e'C$, if and only if $C' = 0$.
3. The rings $A$ and $B$ are Morita equivalent via $(C, e)$ if and only if $C = 0$ and $C' = 0$.

**Proof.** It clearly suffices to prove (1). The data $(A, C, Ce, eC, \mu_e, e\mu_{Ce}: eC \otimes C Ce \to A)$ form a Morita context, in which the last map is surjective by definition of $A$. The map corresponding to $f$ is $\mu_e$, whence the result. □

We cannot resist rephrasing this as well in categorical terms, see, for example, [3, 5.3, 7.1], [20], as this formulation exhibits clearly the role of the Morita defect when comparing module categories.

**Theorem 1.11.** Let $(C, e)$ be a Morita context and set $A = eCe$ as before. Restriction of scalars embeds $\text{Mod} C$ as a Serre\(^1\) subcategory into $\text{Mod} C$, and the quotient category $\text{Mod} C/\text{Mod} C$ is equivalent to $\text{Mod} A$ under the functor $- \otimes_{C} Ce : \text{Mod} C \to \text{Mod} A$. There is thus an exact sequence of abelian categories
\[
0 \to \text{Mod} C \to \text{Mod} C \xrightarrow{- \otimes_{C} Ce} \text{Mod} A \to 0.
\]
Moreover, both the inclusion and the projection functor admit both a left and a right adjoint, and the adjoints, $- \otimes_{A} eC$ and $\text{Hom}_{A}(Ce, -)$, to $- \otimes_{C} Ce$ are fully faithful.

Clearly, $- \otimes_{C} Ce$ is an equivalence if and only if $\text{Mod} C = 0$, that is, $C = 0$. □

2. The Grade of the Stable Endomorphism Ring

The main theme here are the homological properties of the Morita defects. The key invariants that we exploit are their grades as right or left $C$–modules respectively.

**Definition 2.1.** Recall that for a module $X$ over a ring $A$, its grade is given by
\[
\text{grade}_A X = \inf \{ i \geq 0 \mid \text{Ext}_A^i(X, A) \neq 0 \} \in \mathbb{Z} \cup \{ \infty \}.
\]
If $X$ happens to be an $A$-bimodule, we will write $l\text{grade}_A(X)$ for its grade as left module.

We will use repeatedly the following simple fact.

\(^1\)This property is equivalent to $\overline{C}$ being defined by an idempotent ideal, see [3]!
We will call \( \text{End}_A(M) \) the **endomorphism ring** of \( M \) from \( A \) to \( A \), a finite free, equivalently a finite projective, \( A \)-module. The morphism of \( A \)-modules

\[
\tau_M(A) \subseteq A
\]

is naturally a homomorphic image of \( B = e'Ce' \), whence any \( C \)-module can be viewed naturally as a \( B \)-module.

**Lemma 2.3.** Assume \( \mathcal{C} = 0 \), so that \( B \) and \( C \) are Morita equivalent. In that case,

\[
\text{grade}_B \mathcal{C} = \text{grade}_C \mathcal{C}.
\]

**Proof.** By \([2.2]\), the functor \( \mathcal{C} \otimes_A - \) is an equivalence from \( \text{Mod}_C \) to \( \text{Mod}_B \). One has \( \mathcal{C} \otimes_C Ce' = \mathcal{C}e' = e'Ce' = B \oplus N \) as \( B \)-modules, with \( N \) finite projective by \([2.3]\) ii). Thus,

\[
\text{Ext}^i_B(\mathcal{C}, B) \cong \text{Ext}^i_B(\mathcal{C}, B \oplus N)
\]

and these groups vanish if, and only if, \( \text{Ext}^i_B(\mathcal{C}, B) \) vanishes. \( \square \)

The following example plays a key role.

**Example 2.4.** Let \( A \) be a ring and \( M \) a right \( A \)-module. The projection from \( M \oplus A \) onto \( A \) defines the idempotent \( e = (M \oplus A \to A \to M \oplus A) \) in the endomorphism ring \( C = \text{End}_A(M \oplus A) \) to yield a Morita context between \( A \) and \( B = \text{End}_A(M) \),

\[
C(A, M) \cong \begin{pmatrix} B = \text{End}_A(M) & M = \text{Hom}_A(M, A) \\ M^* = \text{Hom}_A(M, A) & A = \text{End}_A(A) \end{pmatrix}
\]

We will call \((C(A, M), e)\) the **Auslander context** defined by the pair \((A, M)\).

2.5. Let us recall the meaning and structure of the maps \( f \) and \( g \) for such an Auslander context, cf. \([II.4]\). The \( \text{End}_A(M) \)-bimodule homomorphism \( f : M \otimes_A M^* \to \text{End}_A(M) \) is given by \( f(m, \lambda)(m') = m \cdot \lambda(m') \). It is the **norm map** of the module \( M \), whose image consists of all endomorphisms that factor through a finite free, equivalently a finite projective, \( A \)-module. Its cokernel, isomorphic to the Morita defect \( \mathcal{C}(A, M) \), is by definition the **stable endomorphism ring** \( \text{End}_A(M) \) of \( M \) over \( A \), and fits into the exact sequence of \( \text{End}_A(M) \)-bimodules

\[
0 \to \Omega_{\text{End}_A(M \otimes A)/A} \to M \otimes_A M^* \xrightarrow{f} \text{End}_A(M) \to \text{End}_A(M) \to 0.
\]

The morphism of \( A \)-bimodules \( g : M^* \otimes_{\text{End}_A(M)} M \to A \) is simply the evaluation map, \( g(\lambda, m) = \lambda(m) \), and its image is, by definition, the (twosided) **trace ideal** \( \tau_M(A) \subseteq A \) of \( M \) in \( A \). Note that the trace ideal is all of \( A \), equivalently, \( g \) is bijective, if and only if \( M \) is an \( A \)-generator.

In other words, the vanishing of the Morita defects for an Auslander context has the following classical interpretation, see \([II.4]\):

**Proposition 2.6.** Let \( C = C(A, M) \) be an Auslander context as just described.
(1) \(\overline{C} = \text{End}_A(M)\) vanishes if and only if \(M\) is a finite projective \(A\)-module.
(2) \(\overline{C} = A/\tau_M(A)\) vanishes if and only if \(M\) is an \(A\)-generator. \(\square\)

2.7. Every Morita context \((C, e)\) defines several Auslander contexts. For example, \(Ce\) is a right \(A = eCe\)-module that splits as \(Ce = e'Ce \oplus eCe = M \oplus A\). The left \(C\)-module structure on the \((C, A)\)-bimodule \(Ce\) provides a ringhomomorphism

\[\alpha_C : C \to \text{Hom}_{eCe}(Ce, Ce) = \text{End}_A(M \oplus A),\]

in detail,

\[C = \begin{pmatrix} B & M \\ N & A \end{pmatrix} \xrightarrow{\alpha_C = \begin{pmatrix} \beta & \text{id}_M \\ g^* & \text{id}_A \end{pmatrix}} \begin{pmatrix} \text{End}_A(M) & M^* \\ A & A \end{pmatrix} = \text{End}_{eCe}(Ce),\]

where \(\beta : B \to \text{End}_A(M)\) defines the \(B\)-module structure on the \((B, A)\)-bimodule \(M\) and \(g^* : N \to M^* = \text{Hom}_A(M, A)\) is adjoint to \(g : N \otimes_B M \to A\). The idempotent \(e \in C\), corresponding to \(1_A\), maps to the idempotent corresponding to \(\text{id}_A\) in \(\text{End}_A(M \oplus A)\).

**Definition 2.8.** We will say that \(C\) represents a (right) Auslander context, on the pair \((A, M)\), if \(\alpha_C\) is an isomorphism. In that case, we call \(e \in C\) an Auslander idempotent.

Auslander contexts can be characterized through the grade of the Morita defect \(\overline{C}\), as we now show.

**Proposition 2.9.** The canonical morphism of Morita contexts

\[\alpha_C : C \to \text{Hom}_{eCe}(Ce, Ce) = \text{End}_A(M \oplus A)\]

(1) is injective iff \(\text{Hom}_C(\overline{C}, C) = 0\) iff \(\text{grade}_C \overline{C} \geq 1\);
(2) is bijective iff \(\text{Ext}^i_C(\overline{C}, C) = 0\) for \(i = 0, 1\) iff \(\text{grade}_C \overline{C} \geq 2\).

In particular, \((C, e)\) is a (right) Auslander context if and only if \(\text{grade}_C \overline{C} \geq 2\).

**Proof.** The proof consists essentially of applying \(\text{Hom}_C(\,-,\, C)\) to the exact sequence \((\star)\). Indeed, remark first that with \(A = eCe\) one has

\[\text{Hom}_C(Ce \otimes_A eC, C) \cong \text{Hom}_A(Ce, \text{Hom}_C(eC, C)) \quad \text{by adjunction,}\]

\[\cong \text{Hom}_A(Ce, Ce) \quad \text{as} \quad \text{Hom}_C(eC, C) \cong Ce.\]

Moreover, with respect to these isomorphisms,

\[\alpha_C \cong \text{Hom}_C(\mu_e, C) : C \cong \text{Hom}_C(C, C) \to \text{Hom}_A(Ce, Ce) \cong \text{Hom}_C(Ce \otimes_A eC, C).\]

Now split the exact sequence \((\star)\) into two short exact sequences,

\[(\dagger) \quad 0 \to CeC \to C \to \overline{C} \to 0,\]

\[0 \to \Omega_{C/A} \to Ce \otimes_A eC \to CeC \to 0,\]
and apply $\operatorname{Hom}_C(-, C)$ to obtain the diagram with exact row and column

\[
\begin{array}{c}
\begin{array}{cccc}
0 & \operatorname{Hom}_C(C, C) & \to & C \\
\downarrow & \downarrow & \downarrow \\
0 & \operatorname{Hom}_C(CeC, C) & \to & \operatorname{Ext}^1_C(C, C) \\
\downarrow & \downarrow & \downarrow \\
\operatorname{Hom}_C(Ce \otimes_A eC, C) & \to & \operatorname{Hom}_C(\Omega_{C/A}, C)
\end{array}
\end{array}
\]

\[\vDash\]

The Ker-Coker-Lemma for the factorization of $\alpha_C$ displayed in this diagram shows immediately that $\alpha_C$ is injective iff $\operatorname{Hom}_C(C, C) = 0$, and that is the first claim. Moreover, if $\operatorname{Hom}_C(C, C) = 0$, then also $\operatorname{Hom}_C(\Omega_{C/A}, C) = 0$, by 2.2, as $(\Omega_{C/A})e = 0$ by 1.6. The second claim follows then again from the Ker-Coker-Lemma. □

**Corollary 2.10.** For a Morita context $(C, e)$, the following are equivalent:

1. $e$ is a Morita idempotent;
2. $M$ is $A$–projective, $\beta : B \to \operatorname{End}_A(M)$ is an isomorphism, and $g^a : N \to \operatorname{Hom}_A(M, A)$ is an isomorphism.
3. $N$ is $A$–projective and $\operatorname{grade}_C C \geq 2$.

**Proof.** If $e$ is a Morita idempotent, then $(C, e)$ is both a left and a right Auslander context by 2.9, and then 2.6 shows both that (1) \(\Rightarrow\) (2) and that (1) \(\Rightarrow\) (3). The isomorphisms in (2) exhibit $C$ as a right Auslander context on $(A, M)$, and the implication (2) \(\Rightarrow\) (1) is then 2.6 again. Finally, (3) simply states that aside from $N$ being $A$–projective, $C$ is isomorphic to the Auslander context $C(A^{op}, N)$ by the natural map $C \to \operatorname{End}_A eCe = \operatorname{End}_A(N \oplus A)^{op}$. It is thus dual to (2), hence equivalent to (1) as the latter statement is self dual. □

Next, we will show that the grade of the Morita defect for an Auslander context measures the vanishing of $\operatorname{Ext}^i_A(M \oplus A, M \oplus A)$. To this end, we use first the exact sequences (†) in the preceding proof to establish the following auxiliary results:

**Lemma 2.11.** If $\operatorname{grade}_C C = g \geq 2$, and $D$ is a finite projective $C$-module, then

(a) \[\operatorname{Ext}^i_C(CeC, D) \cong \begin{cases} 0 & \text{for } 1 \leq i < g - 1, \\ \operatorname{Ext}^g_C(C, D) & \text{for } i = g - 1. \end{cases}\]

(b) \[\operatorname{Ext}^i_C(CeC, D) \cong \operatorname{Ext}^i_C(Ce \otimes_A eC, D) \quad \text{for } 0 \leq i \leq g - 1.\]

In particular, \[\operatorname{Ext}^i_C(Ce \otimes_A eC, D) \cong \operatorname{Ext}^{i+1}_C(C, D) \quad \text{for } 1 \leq i \leq g - 1.\]

**Proof.** For each $i \geq 1$, the first exact sequence in (†) above yields isomorphisms

\[\operatorname{Ext}^i_C(CeC, D) \cong \operatorname{Ext}^{i+1}_C(C, D),\]

whereas the second one yields exact sequences

\[\operatorname{Ext}^{i-1}_C(\Omega_{C/A}, D) \to \operatorname{Ext}^i_C(CeC, D) \to \operatorname{Ext}^i_C(Ce \otimes_A eC, D) \to \operatorname{Ext}^i_C(\Omega_{C/A}, D)\]
Finally, use that $\text{Ext}^i_A(\Omega_{C/A}, D) = 0$ for $i < g$, by Lemma 2.12.

We also need the following.

**Lemma 2.12.** For each $j \geq 1$, the $C$–bimodule $\text{Tor}_j^C(eC, eC)$ is annihilated by $e$ on either side, thus naturally a $\mathbb{C}$–bimodule. Furthermore, $\text{Tor}_j^C(eC, eC) \cong \text{Tor}_j^C(e'C, e'C) = \text{Tor}_j^A(M, N)$.

**Proof.** Multiplying $eC$ with $e$ from the right factors as

$(-) e : eC \twoheadrightarrow eCe eC$.

Now we can establish the significance of the grade of the Morita defect for an Auslander context. Indeed, we have the following, more general result.

**Theorem 2.13.** Let $A$ be a ring and $M$ an $A$–generator. The grade $g$ of $\text{End}_A(M)$ as $\text{End}_A(M)$–module satisfies

$$g = 1 + \inf\{1 \leq i \leq \infty \mid \text{Ext}^i_A(M, M) \neq 0\} \geq 2.$$

Moreover, if $g$ is finite, then

$$\text{Ext}^g_{\text{End}_A(M)}(\text{End}_A(M), \text{End}_A(M)) \cong \text{Ext}^{g-1}_A(M, M).$$

**Proof.** Let $C = \text{End}_A(M \oplus A)$ be the Auslander context defined by $A$ and $M$, and note that $e'C$ is a finite projective $C$–module. Now consider the two spectral sequences with the same limit,

$$E_2^{i,j} = \text{Ext}^i_A(\text{Tor}^j_A(eC, eC), e'C) \Rightarrow \text{E}^{i+j},$$

$$E_2^{i,j} = \text{Ext}^i_A(eC, \text{Ext}^j_A(eC, e'C)) \Rightarrow \text{E}^{i+j}.$$  

The second one degenerates, with $E_2^{i,j} = 0$ for $j \neq 0$, as $eC$ is a projective $C$–module. Moreover,

$$\text{Hom}_C(eC, e'C) \cong e'C e \cong M$$

as $A$–modules, whence the limit term satisfies

$$\text{E}^* \cong \text{Ext}^*_A(eC, e'C) \cong \text{Ext}^*_A(A \oplus M, M).$$

By Lemma 2.12, the terms $\text{Tor}^j_A(eC, eC)$ are $\mathbb{C}$–modules for $j \geq 1$, hence $E_2^{i,j}$ vanishes for $j \geq 1$ and $i < g$ by Lemma 2.12. Accordingly, the edge homomorphisms $E_2^{i,0} \to \text{E}^i$ are isomorphisms for $i \leq g$, thus,

$$E_2^{i,0} = \text{Ext}^i_A(Ce \oplus A eC, e'C) \cong \text{Ext}^i_A(eC, e'C) \quad \text{for } i \leq g.$$  

As $C$ is an Auslander context, $g \geq 2$ by Lemma 2.11 and so

$$\text{Ext}^i_A(Ce \oplus A eC, e'C) \cong \text{Ext}^{i+1}_A(\mathbb{C}, e'C) \quad \text{for } 1 \leq i \leq g - 1,$$

by Lemma 2.11. Putting these isomorphisms together, we find for $1 \leq i < g - 1$ that

$$0 = \text{Ext}^{i+1}_A(\mathbb{C}, e'C) \cong \text{Ext}^i_A(Ce, e'C) = \text{Ext}^i_A(M \oplus A, M) \cong \text{Ext}^i_A(M, M),$$

whereas for $i = g - 1$, which is at least equal to 1, we get

$$\text{Ext}^g_A(\mathbb{C}, e'C) \cong \text{Ext}^{g-1}_A(Ce, e'C) \cong \text{Ext}^{g-1}_A(M, M).$$

To complete the proof, it remains to show that

$$\text{Ext}^g_A(\mathbb{C}, e'C) \cong \text{Ext}^g_{\text{End}_A(M)}(\text{End}_A(M), \text{End}_A(M)).$$
As $M$ is a generator, $C = 0$ by 2.6 and so $B$ and $C$ are Morita equivalent in view of 1.10. An explicit equivalence is given by $- \otimes_C e' : \text{Mod } C \to \text{Mod } B$, see 1.11, thus,
\[
\text{Ext}^g_C(C, e'C) \cong \text{Ext}^g_B(C \otimes_C e'C, e'C).
\]
To evaluate the right hand side, observe, as in the proof of 2.3, that $C \otimes_C e' = e'C = C \cong \text{End}_A(M)$ as $B$–(bi)module and that $e'C \otimes_C e' \cong e'C = B$. Thus, we get finally
\[
\text{Ext}^g_{\text{End}_A(M)}(\text{End}_A(M), \text{End}_A(M)) \cong \text{Ext}^g_C(C, e'C) \cong \text{Ext}^{g-1}_A(M, M)
\]
and the theorem is established.

As a first application, we have the following.

**Corollary 2.14.** Assume $M$ is an $A$–generator such that $\text{End}_A(M)$ is a right noetherian ring of finite global dimension, say, $\text{gldim } \text{End}_A(M) = d$. If $\text{Ext}^i_A(M, M) = 0$ for $i = 1, \ldots, d-1$, then $M$ is finite projective over $A$, and $A$ is of global dimension equal to $d$.

**Proof.** As $B = \text{End}_A(M)$ is assumed to be right noetherian of global dimension $d$, for any finitely generated $B$–module $Y$, vanishing of $\text{Ext}^i_B(Y, B)$ for $i \leq d$ implies $Y = 0$. Applied to $Y = \text{End}_A(M)$, the preceding theorem shows then that $\text{Ext}^i_A(M, M) = 0$ for $i = 1, \ldots, d-1$ implies $\text{End}_A(M) = 0$. However, this means that both Morita defects for $C = \text{End}_A(M \oplus A)$ vanish, whence that $M$ is a finite projective generator and that $\text{End}_A(M)$ is Morita equivalent to $A$.

Theorem 2.13 is formulated directly for the ring $A$ and its generator $M$, with the associated Auslander context hidden in the proof. To display that context more prominently, we make the following definition that is motivated by $[3, 8]$.

**Definition 2.15.** A Morita context $(C, e)$ is a Wedderburn context if the following conditions are satisfied.

1. The Morita defect $C'$ vanishes.
2. The Morita defect $C$ satisfies $\text{grade}_C C \geq 2$.

Recall, $[3, \text{Sect.8}]$, that a module $P$ over a ring $R$ is a Wedderburn projective if it is finite projective and the natural ring homomorphism
\[
R \to \text{End}_{\text{End}_R(P)}(\text{Hom}_R(P, R))
\]
is an isomorphism. With this notion, we can complement 2.13 as follows.

**Proposition 2.16.** For a Morita context $(C, e)$ the following conditions are equivalent.

1. $(C, e)$ is a Wedderburn context.
2. $(C, e)$ is a (right) Auslander context with $M = e'C$ an $A = eC$–generator.
3. $(C, e')$ is a (right) Auslander context with $N = eCe'$ a Wedderburn projective over $B = e'C$. Note that from this point of view, $C \cong B/\tau_N(B)$, the quotient of $B$ modulo the trace ideal of $N$. 

As a second application, we have the following.

**Corollary 2.17.** Assume $M$ is an $A$–generator such that $\text{End}_A(M)$ is a right noetherian ring of finite global dimension, say, $\text{gldim } \text{End}_A(M) = d$. If $\text{Ext}^i_A(M, M) = 0$ for $i = 1, \ldots, d-1$, then $M$ is finite projective over $A$, and $A$ is of global dimension equal to $d$.

**Proof.** As $B = \text{End}_A(M)$ is assumed to be right noetherian of global dimension $d$, for any finitely generated $B$–module $Y$, vanishing of $\text{Ext}^i_B(Y, B)$ for $i \leq d$ implies $Y = 0$. Applied to $Y = \text{End}_A(M)$, the preceding theorem shows then that $\text{Ext}^i_A(M, M) = 0$ for $i = 1, \ldots, d-1$ implies $\text{End}_A(M) = 0$. However, this means that both Morita defects for $C = \text{End}_A(M \oplus A)$ vanish, whence that $M$ is a finite projective generator and that $\text{End}_A(M)$ is Morita equivalent to $A$.
Proof. The equivalence \( \text{(1)} \iff \text{(2)} \) follows from 2.9 and 2.6.

If \((C, e)\) is a Wedderburn context, then both \((C, e)\) and \((C, e')\) are Auslander contexts by 2.9 and \(N\) is finite projective over \(B\) by 2.6. That \((C, e')\) is an Auslander context implies \(M \cong \text{Hom}_B(N, B)\) as \((B, A)\)-modules and \(A \cong \text{End}_B(N)\) as rings, see 2.7. That \((C, e)\) is an Auslander context yields, by 2.7 again, the first isomorphism in the chain

\[
B \cong \text{End}_A(M) \cong \text{End}_{\text{End}_B(N)}(\text{Hom}_B(N, B)),
\]

whereas the second simply substitutes what we just stated. Thus, \(N\) is a Wedderburn projective over \(B\) and \(\text{(1)} \Rightarrow \text{(3)}\) is established.

Finally, we show \(\text{(3)} \Rightarrow \text{(2)}\). As \(N\) is \(B\)-projective and \((C, e')\) is isomorphic to the Auslander context \(C(B, N)\) by assumption, \(\mathcal{C}' = 0\) by 2.6. Moreover, for any projective \(B\)-module \(N\), the \(\text{End}_B(N)\)-module \(\text{Hom}_B(N, B)\) is a generator, as is well known or follows also directly from 2.6 applied to the Morita context \(C(B, N)\). As \((C, e')\) is an Auslander context, the natural ring homomorphism \(A \rightarrow \text{End}_B(N)\) is an isomorphism, and the implication follows.

For the final comment in \(\text{(3)}\), observe that \(C \cong \text{Coker}(f : M \otimes_A N \rightarrow B)\) by 1.6 and that \(f\) can be identified with the evaluation map \(\text{Hom}_B(N, B) \otimes_{\text{End}_B(N)} N \rightarrow B\), with image \(\tau_N(B)\), as \((C, e')\) is an Auslander context.

Remark 2.17. The classical duality between \((\text{Rings, Generators})\) and \((\text{Rings, Wedderburn Projectives})\), see \([3, 8]\), appears here simply as the exchange \(e \rightleftharpoons e'\) of the complementary idempotents in the Wedderburn context \((C, e)\).

3. Equivalences to the Generalized Nakayama Conjecture

The result in 2.13 allows to shed further light onto the relation among various homological conjectures that have been originally formulated for Artin algebras and their modules; see \([35]\) for a detailed account of how little is yet known about these conjectures!

Auslander and Reiten always maintained that many of these conjectures should indeed be true for rings that are finite as modules over a noetherian commutative ring. In this section only, we call such rings \(\text{Noetherian algebras}\).

For example, they stated the following form of the Generalized Nakayama Conjecture, first for Artin algebras in \([8]\ p.70]\), later in the more general context here, see \([51]\ Introduction to Chap.V]\):

Conjecture (GNC). Let \(A\) be a Noetherian algebra. If \(M\) is a finitely generated \(A\)-generator, then

\[
\text{Ext}_A^i(M, M) = 0 \quad \text{for} \ i > 0 \quad \implies \ M \text{ is projective}.
\]

For Artin algebras, it was shown in the same paper \([8]\) that this conjecture is equivalent to:

Conjecture (GNC'). For any simple module \(S\) over an Artinian algebra \(A\), there is an integer \(n \geq 0\) such that \(\text{Ext}_A^n(S, A) \neq 0\).

This conjecture was strengthened for Artin Algebras by Colby and Fuller in \([17]\) to the Strong Nakayama Conjecture that we formulate here again for Noetherian algebras:
Conjecture (SNC). Let $A$ be a Noetherian algebra. If $M$ is a finitely generated $A$-module, then
\[
\text{Ext}_A^i(M, A) = 0 \quad \text{for } i \geq 0 \implies M = 0.
\]

Obviously, (SNC) ⇒ (GNC') for Artin algebras. In general, (SNC) implies immediately what one might call the Idempotent Nakayama Conjecture:

Conjecture (INC). If $(C, e)$ is a Wedderburn context with $C$ a Noetherian algebra, then
\[
\text{Ext}_C^i(C, C) = 0 \quad \text{for } i \geq 0 \implies C = 0,
\]
in other words, infinite grade of $C$ should force a Wedderburn context to be a Morita equivalence.

Indeed, the following formulation that we will show to be equivalent to (INC) in a moment, indicates that this conjecture is, on the face of it, considerably weaker than (SNC), if only for the reason that over many rings a Wedderburn projective module is automatically a generator.

Conjecture (INC'). If $N$ is a Wedderburn projective over a Noetherian algebra $B$, then
\[
\text{Ext}_B^i(B/\tau_N(B), B) = 0 \quad \text{for } i \geq 0 \implies N \text{ is a generator}.
\]

As a consequence of 2.13, we now show that also for Noetherian algebras the Strong Nakayama Conjecture implies the Generalized one, and that the two conjectures mentioned last are equivalent to (GNC).

Proposition 3.1. For Noetherian algebras and finitely generated modules over them, one has
\[
\text{(SNC)} \implies \text{(INC')} \iff \text{(INC)} \iff \text{(GNC)}.
\]

Proof. As already remarked, the first implication is obvious. For the equivalences, we only have to note that we stay within the class (Noetherian Algebras, Finitely Generated Modules) when we pass from a Morita context to its components, or associate to such a pair its Auslander context. Indeed, $C$ is a Noetherian algebra if and only if $A, B$ are Noetherian algebras and $M, N$ are finitely generated modules over each $A$ and $B$, see [29, 1.1.7], and if $A$ is a Noetherian algebra and $M$ a finitely generated $A$-module, then the Auslander context $\text{End}_A(M \oplus A)$ is clearly as well a Noetherian algebra. The equivalences (INC') ⇔ (INC) ⇔ (GNC) then follow from 2.13 and 2.16. □

Remarks 3.2. (1) The conjectures (INC'), (INC) trivially hold if the algebras $C, B$ involved are already commutative noetherian rings. However, there seems to be no real advantage gained in either (SNC) or (GNC) if one assumes that $A$ is already commutative. In this sense, the aforementioned conjectures truly belong to the realm of (slightly) noncommutative algebra.

(2) One could as well state the above conjectures for arbitrary rings and modules. However, R. Schulz [32] gave a counterexample to (GNC), even for a self injective artinian ring $A$. Of course, his construction relies heavily upon the fact that the ring in question is not finite over its centre.

(3) The Generalized Nakayama Conjecture for Artin Algebras implies the so-called Tachikawa conjectures for Artin Algebras; see again [35]. For a treatment of those conjectures for commutative noetherian rings, see [10].
4. The Comparison Homomorphism for Hochschild Cohomology

Let \((C, e)\) be a Morita context and \((A, B, M, N)\) the associated Pierce components of \(C\). Additionally, we assume from now on that \(C\) comes equipped with a \(K\)-algebra structure over some commutative ring \(K\). The rings \(A, B\) inherit then a \(K\)-algebra structure from \(C\), and \(M, N\) become symmetric \(K\)-modules.

The aim here is to show that there exists a canonical homomorphism of rings
\[
\chi_{C/A} : \text{HH}(C) \rightarrow \text{HH}(A).
\]

The existence of such a homomorphism was already observed in [22], though under several additional assumptions.

Before introducing the homomorphism, we first recall two general facts from homological algebra that will be used below.

4.1. All unadorned tensor products, enveloping algebras, Bar resolutions, Hochschild cohomology groups, and the like are from now on taken over \(K\). For a \(K\)-algebra \(A\), its enveloping algebra is \(A^{ev} = A^{op} \otimes A\), so that an \(A\)-bimodule \(M\) whose underlying \(K\)-module is symmetric is the same as a right \(A^{ev}\)-module through \(m(a^{op} \otimes a') = ama'\). Let \(\mathbb{B}(A)\) denote the Bar resolution or acyclic Hochschild complex of \(A\) over \(K\), and set
\[
\text{HH}(A, M) = H(\text{Hom}_{A^{ev}}(\mathbb{B}(A), M)),
\]
the Hochschild cohomology of \(A\) over \(K\) with values in an \(A\)-bimodule, more precisely an \(A^{ev}\)-module, \(M\). We abbreviate further and set \(\text{HH}(A) = \text{HH}(A, A)\).

Note the following general fact on the multiplicative structure of Hochschild cohomology.

**Lemma 4.2.** Let \(R\) be a \(K\)-algebra. If \(X \rightarrow Y\) is a morphism of complexes of \(R^{ev}\)-modules that is a homotopy equivalence if considered as a morphism of complexes of \(K\)-modules, then the induced map of complexes
\[
\text{Hom}_{R^{ev}}(\mathbb{B}R, X) \rightarrow \text{Hom}_{R^{ev}}(\mathbb{B}R, Y)
\]
is a quasiisomorphism.

**Proof.** Taking the mapping cone over \(X \rightarrow Y\), it suffices to show that \(\text{Hom}_{R^{ev}}(\mathbb{B}R, Z)\) is acyclic as soon as \(Z\) is a complex of \(R^{ev}\)-modules that is contractible as complex of \(K\)-modules. Viewing \(\text{Hom}_{R^{ev}}(\mathbb{B}R, Z)\) as the total (product) complex associated to the double complex \(\text{Hom}_{R^{ev}}(\mathbb{B}R_n, Z_m)\), each “column”, that is, the complex for a fixed \(n\), is acyclic, even contractible, as the terms \(\mathbb{B}R_n\) are induced from \(K\)-modules. As \(\mathbb{B}R_n = 0\) for \(n < 0\), the double complex is concentrated in the “right” halfplane, whence the total complex remains acyclic.

In view of the fact just recalled, the augmentation of the Bar resolution yields an isomorphism in cohomology
\[
H(\text{Hom}_{R^{ev}}(\mathbb{B}R, \mathbb{B}R)) \rightarrow \text{HH}(R),
\]
and the composition or Yoneda product on \(\text{Hom}_{R^{ev}}(\mathbb{B}R, \mathbb{B}R)\) induces the natural ring structure on \(\text{HH}(R)\). It is the same as the one originally defined by M. Gershenhaber [21].

We will also use below the following basic property of the Yoneda product.
Lemma 4.3. Let $f : X \to Y$ be a morphism of complexes over some abelian category and assume that the map $\text{Hom}(X, f)$ is a quasiisomorphism. The composition

$$H(\text{Hom}(X, f))^{-1} \circ H(\text{Hom}(f, Y) : H(\text{Hom}(X, X)) \to H(\text{Hom}(Y, Y))$$

is then a ring homomorphism with respect to Yoneda product on source and target.

Proof. Indeed, the hypothesis simply means that for any morphism of complexes $g : Y \to Y$, there exists a morphism of complexes $h : X \to X$ that renders the following square commutative

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{h} & & \downarrow{g} \\
X & \xrightarrow{f} & Y
\end{array}
$$

and that such $h$ is unique up to homotopy. The indicated map in cohomology sends the cohomology class of $g$ to that of $h$, and uniqueness up to homotopy then guarantees that the image of a composition is homologous to the composition of the images. □

Now we return to Morita contexts, with notation as introduced before.

Proposition 4.4. Let $(C, e)$ be a Morita context. There is a canonical semi-split inclusion of complexes of $C$–bimodules

$$\tilde{\mu}_e : C e \otimes_A \mathbb{B}(A) \otimes_A eC \hookrightarrow \mathbb{B}(C)$$

that lifts the multiplication map $\mu_e : C e \otimes_A eC \to C$.

The adjoint to the inclusion $\tilde{\mu}_e$, given by

$$\tilde{\mu}_e^\sharp : \mathbb{B}(A) \hookrightarrow \text{Hom}_{C^{op}}(C e \otimes eC, \mathbb{B}(C)) \cong e\mathbb{B}(C)e,$$

is a semi-split inclusion and quasiisomorphism of complexes of $A^{op}$–modules that induces $\text{id}_A$ in homology. Moreover, $\tilde{\mu}_e^\sharp$ is a homotopy equivalence of complexes of right, as well as of left $A$–modules.

Proof. Clearly,

$$C e \otimes_A A^{op}(+2) \otimes_A eC \cong C e \otimes (eCe)^{op} \otimes eC$$

is a direct summand of the $C$–bimodule $C^{op}(+2)$, and the differentials are compatible. Moreover, the induced map in the zeroth homology is clearly $\mu_e$.

As for the second assertion, $C e \otimes eC \cong (e^{op} \otimes e)C^{op}$ is a finite, even cyclic, projective $C^{op}$–module, whence

$$\text{Hom}_{C^{op}}(C e \otimes eC, \mathbb{B}(C)) \cong \mathbb{B}(C) \otimes_{C^{op}} C^{op}(e^{op} \otimes e) \cong e\mathbb{B}(C)e$$

remains exact, resolving $C \otimes_{C^{op}} C^{op}(e^{op} \otimes e) \cong eCe = A$. Furthermore, $\tilde{\mu}_e^\sharp$ realizes $A^{oe}(+2) = (eCe)^{op}(+2)$ as the obvious direct summand of $eC \otimes C^{op} \otimes C e$. Direct inspection shows that with respect to these identifications, $\tilde{\mu}_e^\sharp$ induces $\text{id}_A$ in the only nontrivial homology group.

As for the final claim, we note that the right $A$–linear map sending an element $w$ to $e \otimes w \in eC \otimes C^{op} \otimes C e$ contracts the augmented complex $e\mathbb{B}(C)e \to A$. The other side is left to the reader. Both augmented complexes $\mathbb{B}A \to A$ and $e\mathbb{B}Ce \to A$ are thus contractible as complexes of one-sided modules, whence $\tilde{\mu}_e^\sharp$, lying over the identity on $A$, must be a homotopy equivalence. □
4.5. For every $C$–bimodule $M$, we set
\[
\mathbb{B}(C/A) := \text{Coker}(\tilde{\mu}_e : Ce \otimes_A \mathbb{B}(A) \otimes_A eC \to \mathbb{B}(C))
\]
\[
\text{HH}(C/A, M) := H(\text{Hom}_{C^{ev}}(\mathbb{B}(C/A), M))
\]
to obtain first a semi-split exact sequence of $C^{ev}$–modules
\[(*)
0 \longrightarrow Ce \otimes_A \mathbb{B}(A) \otimes_A eC \longrightarrow \mathbb{B}(C) \longrightarrow \mathbb{B}(C/A) \longrightarrow 0
\]
and then long exact cohomology sequences
\[
\cdots \to \text{HH}^i(C/A, M) \to \text{HH}^i(C, M) \to \text{HH}^i(A, eMe) \to \text{HH}^{i+1}(C/A, M) \to \cdots,
\]
where we have used that
\[
\text{Hom}_{C^{ev}}(Ce \otimes_A \mathbb{B}(A) \otimes_A eC, M) \cong \text{Hom}_{A^{ev}}(\mathbb{B}(A), eMe)
\]
by adjunction.

4.6. If we take $M = C$, then $eMe = A$ and we get a natural map
\[
\chi^{*}_{C/A} = \text{HH}^*(\mu_e, C) : \text{HH}^*(C) \to \text{HH}^*(A)
\]
relating the Hochschild cohomology of $A$ to that of $C$. According to the classical theorem of M. Gerstenhaber [21], Hochschild cohomology of any algebra carries naturally the structure of a graded commutative algebra — and much more structure besides. We now show that $\chi^{*}_{C/A}$ respects at least the product structure.

**Theorem 4.7.** The map $\chi^{*}_{C/A}$ is a homomorphism of graded $K$–algebras.

**Proof.** In view of our preparations, the claim follows essentially from the commutativity of the following diagram:
\[
\begin{array}{ccc}
\text{Hom}_{C^{ev}}(\mathbb{B}C, \mathbb{B}C) & \xleftarrow{\cong} & \text{Hom}_{C^{ev}}(Ce \otimes_A \mathbb{B}A \otimes_A eC, \mathbb{B}C) \\
e(-) & \xrightarrow{\text{adjunction}} & e(\text{adjunction}) \\
\text{Hom}_{A^{ev}}(e\mathbb{B}Ce, e\mathbb{B}Ce) & \xleftarrow{\cong} & \text{Hom}_{A^{ev}}(\mathbb{B}A, e\mathbb{B}Ce) \\
\end{array}
\]

where $\cong$ denotes an isomorphism of complexes, $\simeq$ a quasiisomorphism. Namely, commutativity of the square is straightforward from the definition of $\mu_e$ and the fact that $Ce \otimes_A (-) \otimes_A eC$ is left adjoint to $e(-)e \cong eC \otimes_C (-) \otimes_C Ce$. The composed map along the top and down the right side induces by definition $\chi^{*}_{C/A}$ in cohomology, taking into account once for $R = C$ and once for $R = A$. The vertical map at the left is visibly a ring homomorphism, and $\tilde{\mu}^*_e$ satisfies the conditions in 4.3 in view of 4.4.

Taken together, we get
\[
\chi^{*}_{C/A} = \left( (H(\tilde{\mu}^*_e \circ e)^{-1} \circ H(e(-)e)\right) \circ H(e(-)e),
\]
as a composition of two homomorphisms of graded rings. \qed
5. Hochschild Cohomology of Auslander Contexts

The starting point for the results in this section is the classical observation that Hochschild theory is Morita invariant. More precisely, by specializing the argument in [26, 1.2.7], one has the following.

**Proposition 5.1 (Morita Invariance of Hochschild Theory).** If the Morita defect $\mathcal{C}$ vanishes, then the inclusion $\mu$ of complexes of $\mathcal{C}^{ev}$–modules in $\mathcal{C}$ is a homotopy equivalence, equivalently, $\mathcal{B}(C/A)$ is a contractible complex of $\mathcal{C}^{ev}$–modules. In particular, for every $\mathcal{C}^{ev}$–module $\mathcal{M}$, the natural map $\text{HH}^i(C, \mathcal{M}) \to \text{HH}^i(A, e\mathcal{M})$ is an isomorphism.

5.2. What can we say in case the Morita defect does not vanish? This clearly comes down to understanding the cohomology groups $\text{HH}^i(C/A, \mathcal{M})$ introduced in 4.5. To be able to control those groups through the hypercohomology spectral sequence, we assume henceforth that $\mathcal{C}$ is projective as $K$–module. Under this assumption, each of the direct $K$–summands $A, B, M, N$ of $\mathcal{C}$ is a projective $K$–module as well, and the terms of the complexes in the exact sequence $(*)$ from 4.5 are projective $\mathcal{C}^{ev}$–modules.

Moreover, we can easily determine the homology groups of those complexes.

**Lemma 5.3.** The homology of $\mathcal{C} \otimes_A \mathcal{B}(A) \otimes_A e\mathcal{C}$ satisfies

$$H_j(C \otimes_A \mathcal{B}(A) \otimes_A e\mathcal{C}) = \begin{cases} \mathcal{C} & \text{if } j = 0, \\ \Omega_{C/A} & \text{if } j = 1, \\ \text{Tor}^A_{j-1}(\mathcal{C}, e\mathcal{C}) \cong \text{Tor}^A_{j-1}(M, N) & \text{if } j > 1. \end{cases}$$

**Proof.** The first statement is classical: As $A$ and $\mathcal{C}$ are projective $K$–modules, $\mathcal{C} \otimes_A \mathcal{B}(A)$ constitutes a projective resolution of $\mathcal{C}$ as $A$–module. The second statement follows from the first and the long exact homology sequence associated to $(*)$, as $\mathcal{B}(C)$ is a resolution of $C$, and the exact sequence of low order homology is canonically identified with the fundamental sequence in 1.6.

As the complex $\mathcal{B}(C/A)$ is bounded in the direction of the differential and consists of projective $\mathcal{C}^{ev}$–modules, the cohomology $\text{HH}(C/A)$ of $\text{Hom}_{\mathcal{C}^{ev}}(\mathcal{B}(C/A), C)$ can be calculated through the hypercohomology spectral sequence

$$E_2^{i,j} = \text{Ext}^i_{\mathcal{C}^{ev}}(\mathcal{H}_j, C) \implies E^{i+j} = \text{HH}^{i+j}(C/A).$$

The next result evaluates the low order terms of this spectral sequence relative to the grade of the Morita defect.
Lemma 5.4. If $(C, e)$ is a Morita context with grade$_C \overline{C} = g$, then

$$\text{HH}^i(C/A) \cong \begin{cases} 0 & \text{for } i < g, \\ \text{HH}^i(C, \text{Ext}^g_C(\overline{C}, C)) & \text{for } i = g. \end{cases}$$

Proof. First note that grade$_C \overline{H}_j \geq g$ by 5.3 and 5.2, and that thus $g = \inf\{j + \text{grade}_C \overline{H}_j \mid j \geq 0\}$ as $\overline{H}_0 = \overline{C}$. We now analyze the $E_2$-terms in the hypercohomology spectral sequence above using yet another spectral sequence, see [13, XVI.4(5)]:

If $X, Y$ are $(R, S)$-bimodules over $K$-projective $K$-algebras $R, S$, then each $\text{Ext}^g_S(X, Y) = \text{HH}^g(R, \text{Ext}^g_S(X, Y))$ is an $R$-bimodule and there is a spectral sequence

$$E_2^{p,q} = \text{HH}^p(R, \text{Ext}^q_S(X, Y)) \implies E^{p+q} = \text{Ext}^{p+q}_{R^e \otimes S}(X, Y).$$

Taking $R = S = C, X = \overline{H}_j, Y = C$, we get the implications

$$\text{Ext}^g_S(\overline{H}_j, C) = 0 \quad \text{for } q < \text{grade}_C \overline{H}_j$$

$$\implies \text{Ext}^g_{C^{op}}(\overline{H}_j, C) = 0 \quad \text{for } q < \text{grade}_C \overline{H}_j$$

$$\implies \text{HH}^i(C/A) = 0 \quad \text{for } i < g = \inf\{j + \text{grade}_C \overline{H}_j \mid j \geq 0\}.$$ 

As thus $E_2^{i,j} = 0$ for $i < g$, the edge homomorphism $E_2^{g,0} \to E_2^g$ is an isomorphism, whence

$$\text{Ext}^g_{C^{op}}(\overline{C}, C) = E_2^{g,0} \cong E^g = \text{HH}^g(C/A).$$

On the other hand, $E^{p,q}_2 = 0$ for $q < g$, shows that the edge homomorphism $\text{E}^g \to E_2^{0,q}$ is an isomorphism, whence for $\overline{H}_0 = \overline{C}$ we get

$$\text{Ext}^g_{C^{op}}(\overline{C}, C) \cong E^g \xrightarrow{\sim} E^{0,g}_2 = \text{HH}^0(C, \text{Ext}^g_C(\overline{C}, C)).$$

This analysis of the relative Hochschild cohomology groups $\text{HH}^i(C/A)$ translates immediately into the following result.

Theorem 5.5. If $(C, e)$ is a Morita context with grade$_C \overline{C} = g$, then the maps

$$\chi^{g}_{C/A} : \text{HH}^i(C) \to \text{HH}^i(A)$$

are isomorphisms for $j \leq g - 2$, and there is an exact sequence

$$0 \to \text{HH}^{g-1}(C) \xrightarrow{\chi^{g-1}} \text{HH}^{g-1}(A) \to \text{HH}^0(C, \text{Ext}^g_C(\overline{C}, C)) \to \text{HH}^g(C) \xrightarrow{\chi^g} \text{HH}^g(A).$$

We list a few consequences.

Corollary 5.6. If $(C, e)$ is an Auslander context, then $C$ and $A$ have isomorphic centres via $\chi^0_{C/A} : Z(C) \cong Z(A)$, and every outer derivation on $C$ is induced from a derivation on $A$. 

Corollary 5.7. If the Morita defect $\overline{C}$ is of infinite grade, then $\chi^{C/A} : \text{HH}(C) \to \text{HH}(A)$ is an isomorphism.

We now give an application to deformation theory. Recall that if the algebra $A$ is projective over $K$, then its infinitesimal first order deformations over $K$ are parametrized by $\text{HH}^2(A)$. Every formal deformation of $A$ over $K$ is trivial, that is, the algebra is rigid, if, and only if, $\text{HH}^2(A) = 0$. Similarly, if $M$ is an $A$-module,
then $\Ext^1_A(M, M)$ describes the infinitesimal first order deformations of $M$ as $A$–module. Again, every formal deformation of $M$ as $A$–module is trivial, and $M$ is then called rigid as $A$–module, if, and only if, this extension group vanishes. In light of these facts, we have the following application.

**Corollary 5.8.** Let $A$ be a $K$–algebra and $M$ a module over it, with $A$ and $M$ projective as $K$–modules. If $M$ is rigid as $A$–module and $A$ is rigid as $K$–algebra, then $\End_A(M \oplus A)$ is rigid too as a $K$–algebra.

**Proof.** By 

vanishing of $\Ext^1_A(M, M)$ means that $\text{grade}_C C \geq 3$ for the Auslander context $C = \End_A(M \oplus A)$. Consequently, $\HH^2(C)$ embeds into $\HH^2(A)$ by 5.5 and so $C$ is rigid along with $A$. □

Coming back to the general situation, in case the Morita defect $\overline{C'}$ vanishes, we get more precise information in form of a spectral sequence that relates directly the Hochschild cohomology of $B$ to that of $A$.

**Theorem 5.9.** Let $(C, e)$ be a Morita context with $g = \text{grade}_C C$. If $e'$ is a Morita idempotent, then there is a spectral sequence of graded $K$–algebras

$$E^{i,j}_2 = \HH^i(B, \Ext^j_A(M, M)) \implies E^{i+j}_\infty = \HH^{i+j}(A),$$

and $E^{i,j}_2 = 0$ for $1 \leq j < g - 1$. Moreover, there are natural $K$–algebra homomorphisms

$$\HH(B) \xrightarrow{\HH(B, \beta)} \HH(B, \End_A(M)) = E^0_2 \xrightarrow{\delta} E_2 = \HH(A),$$

induced by the ring homomorphism $\beta : B \to \End_A(M)$, as in 2.7 and the edge homomorphism $d$.

If furthermore $g \geq 2$, that is, if $(C, e)$ is a Wedderburn context, then $\HH^i(B) \cong E^{i,0}_2$ for each $i \geq 0$, and the edge homomorphisms provide isomorphisms

$$\HH^i(B) \cong \HH^i(A) \quad \text{for } i \leq g - 2.$$
homomorphisms and exact sequence of low order terms resulting from the partial degeneration of the spectral sequence above.

The spectral sequence in [6.9] has already been used in [13] to investigate the Hochschild cohomology of Artin algebras of finite representation type as well as the behaviour of Hochschild cohomology under pseudo-tilting.

6. Invariant Rings: The General Case

The aim of this section and the next is to apply our results so far to invariant rings with respect to finite groups. For background material on skew group rings see [29] and [9]. We first recall the notions of a separable ring homomorphism and of the Noether or homological different, see [6] [24].

**Definition 6.1.** A homomorphism \( f : R \rightarrow S \) of rings is separable if the epimorphism of \( S \)–bimodules \( \mu_f = \mu_{S/R} : S \otimes_R S \rightarrow S, s' \otimes s'' \mapsto s's'' \), splits.

6.2. A crucial measure of (non-)separability is the Noether different, defined by E. Noether for homomorphisms \( f \) between commutative rings and generalized as homological different by Auslander-Goldman in [6] for associative algebras. To define it, recall that for a ring \( S \) and an \( S \)–bimodule \( M \), the \( S \)–invariants in \( M \) are given by the subgroup

\[ \mathcal{M}^S = \{ m \in M \mid ms = sm \text{ for all } s \in S \}, \]

a (symmetric) module over the centre \( Z(S) := S^S \) of \( S \). If we view \( S \) as an algebra over some commutative ring, say over \( \mathbb{Z} \), then \( \mathcal{M}^S \cong \text{Hom}_{S_{\text{ev}}}(S, M) \cong \text{HH}^0(S, M) \).

**Definition 6.3.** Let \( f : R \rightarrow S \) be a ring homomorphism. The Noether different of \( S \) over \( R \), or rather of \( f \), is

\[ \theta(S/R) := \text{Im} \left( \mu_{S/R}^S : (S \otimes_R S)^S \rightarrow S^S \right) \subseteq Z(S), \]

an ideal in the centre of \( S \).

The following two classical facts on the Noether different, taken from [6], will be relevant in the next section:

6.4. The ring \( S \) is separable over \( R \) if, and only if, \( \theta(S/R) = Z(S) \).

6.5. If \( R \) is commutative noetherian, and if \( f \) turns \( S \) into a module–finite \( R \)–algebra, then for any prime ideal \( p \subset R \), the localization \( f_p : R_p \rightarrow S \otimes_R R_p \) is separable if, and only if, \( p \not\subset f^{-1}(\theta(S/R)) \). In other words, \( V(f^{-1}(\theta(S/R))) \subseteq \text{Spec } R \) is the locus over which \( f \) is not separable.

Now we turn our attention to group actions. Suppose given a finite group \( G \) acting through automorphisms on a ring \( S \). Denote by \( SG = S \# G \) the corresponding skew group ring, and by \( R = SG^G \) the ring of \( G \)–invariants in \( S \).

6.6. Identifying the unit element \( e \in G \) with the multiplicative identity in \( SG \), the canonical ring homomorphism \( S \rightarrow SG \) can be thought of as multiplication with \( e \) on either right or left.

Set \( f = \sum_{g \in G} g \in SG \) and note that \( fg = f = gf \in SG \) for every \( g \in G \). In particular, \( f^2 = |G|f \), whence \( f \) is not quite an idempotent, and the \( SG \)–modules \( S\overline{G} \) or \( fSG \) are not quite projective. However, the compositions

\[
\begin{array}{ccc}
S & \xrightarrow{(-)e} & SG \\
S & \xrightarrow{(-)} & SGf \\
S & \xrightarrow{(-)} & fSG
\end{array}
\]
are always bijections, thus, \( SGf = Sf, fSG = fS \), and in this way \( S \) can be viewed both as a left or right \( SG \)-module. The subring \( R \subseteq S \subseteq SG \) commutes with \( f \), and so \( S \cong Sf \) becomes an \((SG, R)\)-bimodule, or, through \( S \cong fS \), an \((R, SG)\)-bimodule. The multiplication map on \( fS \otimes_{SG} Sf \) takes its values in \( R \subseteq SG \), and taken together, these data define a Morita context

\[
(C, e) = \begin{pmatrix} SG & Sf \\ fS & R \end{pmatrix}.
\]

6.7. The following facts are well known and easy to establish:

- The Morita defect \( C' \) is isomorphic to \( R/\text{tr}_G(S) \), where \( \text{tr}_G : S \to R \), \( \text{tr}_G(s) = \sum g(s) \), is the \( R \)-linear trace map from \( S \) to \( R \). As \( |G|r = \text{tr}_G(r) \) for every \( r \in R \), one has \( |G|C' = 0 \).
- The Morita defect \( C \) is isomorphic to \( \overline{SG} := SG/(SG)f(SG) = SG/SfS \).
- The Morita context \( (C, e') \) is an Auslander context, that is,

\[
C \cong \text{End}_{C'Ce'}(C'e') \cong \text{End}_{SG}(SG \oplus fS),
\]

or, in detail,

\[
C \cong \begin{pmatrix} SG & Sf \cong \text{Hom}_{SG}(fS, SG) \\ fS & R \cong \text{End}_{SG}(fS) \end{pmatrix}.
\]

In view of 6.8, one has thus always \( \text{grade}_C C' \geq 2 \). The Morita defect \( C' \) vanishes if and only if the trace map \( \text{tr}_G \) is surjective, and then \( fS \) is a projective \( SG \)-module. If these conditions are satisfied, then \( SG \) and \( C \) are Morita equivalent, see 1.10. In view of \( |G|C' = 0 \), these conditions are satisfied as soon as \( |G| \) is invertible in \( S \).

To compare now \( R \) with \( C \) or \( SG \), the crucial question becomes: What is the grade of the Morita context \( C' \)? We first consider when \( C' \) vanishes, equivalently, when \((C, e')\) is a Wedderburn context.

**Lemma 6.8.** If the Morita defect \( C' \) vanishes, then the inclusion \( R \hookrightarrow S \) is a separable ring homomorphism.

**Proof.** By 6.8 \( C' = 0 \) if and only if the restricted multiplication map

\[(*) \quad \mu_f : Sf \otimes_R fS = e'Ce \oplus_{eCe} eCe' \to e'Ce' = SG \]

is a bijection, necessarily of \( SG \)-bimodules. The \( S \)-bilinear map \( \pi : SG \to S \) that sends any \( g \in G \setminus \{\epsilon\} \) to 0 and \( \epsilon \) to 1 is clearly a split epimorphism of \( S \)-bimodules. The composition of these two maps is easily identified with the multiplication map \( \mu_{S/R} : S \otimes_R S \to S \).

6.9. Before we investigate the converse of the preceding result, we mention the following case straight out of [29 7.8]: If \( G \) acts through outer automorphisms on \( S \) and if \( S \) is a simple ring, then \( SG \) is simple too, whence \( (SG)f(SG) = SG \), that is, \( C = 0 \). This leads to the following well known fact.

\[\text{that is, there are no nontrivial two-sided ideals},\]
PROPOSITION 6.10. If $G$ acts on a simple ring $S$ through outer automorphisms, then $(C, e')$ is a Wedderburn context and $R$ is Morita equivalent to $C$. If, furthermore, the trace map is surjective, then $R, C$ and $SG$ are Morita equivalent. If in the latter situation $C$ is a $K$–algebra that is projective as $K$–module, then
\[ HH(R) \cong HH(C) \cong HH(SG). \]

REM 6.11. This result has been used recently in [11] to determine the Hochschild cohomology of invariant subrings of the Weyl algebra with respect to finite group actions.

We now exhibit a case, where separability of $S$ over $R$ forces vanishing of the Morita defect $\underline{C}'$. This case will cover most geometrically relevant situations. It is partly motivated by Yoshino’s treatment of quotient surface singularities in [36, 10.8].

DEFINITION 6.12. A finite group $G$ acts infinitesimally through outer automorphisms on $S$, if the natural inclusion of $S$–bimodules $Se \to SG$ induces a bijection on the subgroups of $S$–invariants, $(-)\epsilon : S^g \cong SG^g$. Equivalently, for every $g \neq \epsilon$ and for every $s' \neq 0 \in S$, there exists an $s \in S$ such that
\[ ss' - s'g(s) \neq 0. \]

Note the following simple fact.

LEMMA 6.13. If the finite group $G$ acts infinitesimally through outer automorphisms on $S$, then the centre of $SG$ satisfies
\[ Z(SG) \cong Z(S)^G. \]

In particular, if $S$ is commutative, then the centre of $SG$ equals $R$.

PROOF. Calculate $(SG)^{SG}$ as $((SG)^S)^G$. \hfill \Box

The following example provides a stock of such group actions and explains their ubiquity in geometry.

EXAMPLE 6.14. If a finite group $G$ acts faithfully on a commutative domain $S$, then it acts infinitesimally through outer automorphisms. Indeed, $ss' - s'g(s) = (\epsilon - g)(s) \cdot s' = 0$ for some $g \neq \epsilon$, some $s' \neq 0$, and all $s \in S$ iff $\epsilon - g = 0$ on $S$ iff the action is not faithful.

PROPOSITION 6.15. Assume the finite group $G$ acts infinitesimally through outer automorphisms on some ring $S$. The inclusion $R \to S$ is then a separable ring homomorphism if and only if $\underline{C} = 0$ if and only if $(C, e')$ is a Wedderburn context.

PROOF. By [BS] we only need to consider the case when $S$ is separable over $R$, whence when there exists an $S$–bilinear splitting $\zeta : S \to S \otimes_R S \cong Sf \otimes_R fS$ of the multiplication map on $S$. Let $\hat{\zeta} : S \to SG$ be the composition of $\zeta$ with the multiplication map $\mu_f$ on $SG$ as in \( \square \). To establish that \( \square \) is surjective, it clearly suffices to show $\hat{\zeta}(1) = 1$. Indeed, on the one hand, $s\hat{\zeta}(1) = \hat{\zeta}(s) = \hat{\zeta}(1)s$ for each $s \in S$, as $\hat{\zeta}$ is by construction $S$–bilinear. The assumption on the group action then implies that $\hat{\zeta}(1) \in Se \subset SG$. On the other hand, if $\zeta(1) = \sum_i s_i \otimes s_i''$, then $\zeta(1) = \sum_i s_i'g(s_i'')g$. Taken together, we get thus $\sum_i s_i'g(s_i'') = 0$ for $g \neq \epsilon$, and so $\hat{\zeta}(1) = \sum_i s_i's_i'' = 1$. \hfill \Box
Now we apply 5.9 of the preceding section to obtain the following result.

**Theorem 6.16.** If $G$ acts through $K$--linear automorphisms on the $K$--projective $K$--algebra $S$, and if the trace map $tr_G$ is surjective, then there is a spectral sequence

$$E_2^{i,j} = HH^i(SG, Ext^j_R(S, S)) \implies HH^{i+j}(R)$$

and natural homomorphisms $HH^i(SG) \to HH^i(S, End_R(S)) = E_2^{i,0} \to HH^i(R)$ that are isomorphisms for $i \leq \text{grade}_{SG} SG - 2$. □

Moreover, it is known how to calculate the $E_2$--terms of the preceding spectral sequence in terms of Hochschild cohomology of $S$, see [27, 33].

**Theorem 6.17.** For any $SG$--bimodule $M$, there is a spectral sequence

$$'E_2^{i,j} = H^i(G, HH^j(S, M)) \implies HH^{i+j}(SG, M).$$

In particular, if $|G|$ is invertible in $K$, then $'E_2^{i,j} = 0$ for $j \neq 0$, and the spectral sequence degenerates to yield

$$HH^i(SG, M) \cong HH^i(S, M)^G.$$ □

In case $G$ acts infinitesimally through outer automorphisms, we can combine the two results into the following.

**Corollary 6.18.** If $G$ acts infinitesimally through outer $K$--linear automorphisms on the $K$--projective $K$--algebra $S$, and if $|G|$ is invertible in $K$, then there is a spectral sequence

$$HH^i(S, Ext^j_R(S, S))^G \implies HH^{i+j}(R)$$

with natural homomorphisms

$$HH^i(S, SG)^G \cong HH^i(S)^G \oplus \bigoplus_{g \neq e} HH^i(S, Sg)^G \to HH^i(R),$$

and these homomorphisms are isomorphisms for $i \leq \text{grade}_{SG} SG - 2$. □

The remaining mysterious quantity is thus $\text{grade}_{SG} SG$. In case of commutative rings, it can be bounded from below by the depth of the Noether different, the key point of the next, and last, section.

### 7. Invariant Rings: The Commutative Case

From now on, $S$ will be a commutative ring, and we will assume that $G$ acts infinitesimally through outer automorphisms, a condition that holds for faithful actions on commutative domains by [4,14]. In this case, the Noether different of $S$ over its subring of invariants is contained in the $S$--annihilator of the Morita defect $\overline{G} \cong SG$ as we now show.

**Proposition 7.1.** The kernel of the composed ring homomorphism $S \to SG \to SG$ contains the Noether different $\theta(S/R)$, thus, equivalently, $\theta(S/R) \subseteq \text{Ann}_S SG$.

**Proof.** Set $a = \text{Ker}(S \to SG)$ and note that $a = S \cap \text{Ker}(SG \to SG)$. With $X = S \times_{SG} (Sf \otimes_R fS)$ the indicated fibre product, the image of the first projection
$p_1 : X \to S$ is the ideal $\mathfrak{a}$, as the rows in the commutative diagram of $S$–bimodules

\[
\begin{array}{ccc}
S \otimes_R S & \xrightarrow{\mu_{S/R}} & S \\
\parallel & & \parallel \\
Sf \otimes_R fS & \xrightarrow{\mu_{SG}} & SG \\
\parallel & & \parallel \\
X & \xrightarrow{p_1} & S \\
\end{array}
\]

are exact. Note that $\pi(g) = \delta_{g,\epsilon}$ and that $\pi$ retracts the inclusion $S \subseteq SG$. As $S$ is commutative, $S = S^S$, and as $G$ acts infinitesimally through outer automorphisms, $S = S^S = (SG)^S$. Taking $S$–invariants preserves fibre products, whence applying $(\ )^S$ to the first two columns returns

\[
\begin{array}{ccc}
(S \otimes_R S)^S & \xrightarrow{\mu_{S/R}^S} & S \\
\parallel & & \parallel \\
(Sf \otimes_R fS)^S & \xrightarrow{\mu_{SG}^S} & (SG)^S \\
\parallel & & \parallel \\
X^S & \xrightarrow{p_2^S} & S \\
\end{array}
\]

By definition, $\mu_{S/R}^S$ has $\theta(S/R)$ as its image, and the image of $p_1^S : X^S \to S$ is contained in $\mathfrak{a}^S = \mathfrak{a} \subseteq S$.

**Corollary 7.2.** If the ring $S$ is noetherian, then

\[
\text{grade}_{SG} SG \geq \text{depth}(\theta(S/R), S).
\]

**Proof.** Deriving $\text{Hom}_{SG}(X, Y) = (\text{Hom}_S(X, Y))^G$, for $SG$–bimodules $X, Y$, yields a spectral sequence

\[
E_2^{i,j} = H^i(G, \text{Ext}_S^j(X, Y)) \Rightarrow \text{Ext}_{SG}^{i+j}(X, Y).
\]

For $X = SG, Y = SG$, one obtains $\text{Ext}_S^j(SG, SG) \cong \oplus_g \text{Ext}_S(SG, S)g$. Now, as is well known, $\text{Ext}_S^j(SG, S) = 0$ for $j < \text{depth}(\text{Ann}_S SG, S)$, see, e.g. [19, 18.4], thus, $E_2^{i,j} = 0$ for those $j$. Hence, $\text{Ext}_{SG}^j(SG, SG) = 0$ for $j < \text{depth}(\text{Ann}_S SG, S)$, that is, $\text{grade}_{SG} SG \geq \text{depth}(\text{Ann}_S SG, S)$. As we just saw, $\theta(S/R) \subseteq \text{Ann}_S SG$, whence $\text{depth}(\text{Ann}_S SG, S) \geq \text{depth}(\theta(S/R), S)$ and the claim follows.

We show next that the depth of $\theta(S/R)$ on $S$ controls as well vanishing of the groups $\text{HH}^i(S, Sg)$, $g \neq \epsilon$, that occurred in 6.15.

**Lemma 7.3.** Suppose that $S$ carries a $K$–algebra structure such that $G$ acts through $K$–linear isomorphisms. In that case, $Sg$ is naturally an $S^\text{ev}$–module for each $g \in G$, and there is an inclusion of $S$–ideals

\[
\theta(S/R) \subseteq \frac{\text{Ann}_{S^\text{ev}}(Sg) + \text{Ann}_{S^\text{ev}}(S)}{\text{Ann}_{S^\text{ev}}(S)} \subseteq \frac{S^\text{ev}}{\text{Ann}_{S^\text{ev}}(S)} = S.
\]

In particular, if $S^\text{ev}$ is noetherian, then $\text{Ext}_{S^\text{ev}}^i(S, Sg) = 0$ for each $g \neq \epsilon$ and $i < \text{depth}(\theta(S/R), S)$. 

PROOF. The proof is similar to that of 7.1. First note that \( K \subseteq R \) as \( G \) acts through \( K \)-linear automorphisms. Each \( Sg \) is a cyclic \( S^{ev} \)-module, generated by \( g \), and the resulting epimorphism of \( S \)-bimodules \( S^{ev} \rightarrow Sg \) factors through the canonical surjective ring homomorphism \( S^{ev} \rightarrow S \otimes_R S \). In view of this, we may replace \( \text{Ann}_{S^{ev}} \) in the claim everywhere by \( \text{Ann}_{S \otimes_R S} \). Setting \( I_g := \text{Ann}_{S \otimes_R S}(Sg) \) and \( i_g : I_g \hookrightarrow S \otimes_R S \), we get the following diagram with exact row and column

\[
\begin{array}{ccc}
0 & \rightarrow & I_g \\
\downarrow & & \downarrow i_g \\
0 & \rightarrow & S \otimes_R S & \rightarrow & Sg & \rightarrow & 0 \\
\downarrow & & \downarrow \mu_{S/R} & & \downarrow \mu_{S/R} & & \downarrow \\
0 & & S & & \mu_{S/R} & & S \otimes_R S & & I_g \\
\end{array}
\]

that shows the image of \( \mu_{S/R}i_g \) to be \( (\text{Ann}_{S^{ev}}(Sg) + \text{Ann}_{S^{ev}}(S)) / \text{Ann}_{S^{ev}}(S) \). As \( G \) acts infinitesimally through outer automorphisms, \( (Sg)^S = 0 \) for \( g \neq \epsilon \), and so \( (I_g)^S \cong (S \otimes_R S)^S \) maps to \( \theta(S/R) = \text{Im}(\mu_{S/R}i_g) \) in \( S \), whence the latter ideal is contained in the image of \( I_g \) in \( S \). \( \square \)

7.4. Before formulating the key result in this section, we summarize the necessary assumptions:

- \( S \) is a noetherian \( K \)-algebra that is projective as a \( K \)-module and such that \( S^{ev} \) is still a noetherian ring; for example, \( S \) is a finitely generated algebra over a field \( K \);
- the finite group \( G \) acts infinitesimally through outer \( K \)-linear automorphisms on \( S \); for example, \( S = K[V]/I \) is a domain, generated over \( K \) by the finite dimensional vector space \( V \), and \( G \subseteq GL(V) \);
- the invariant ring \( R \) is again noetherian and \( S \) is a finitely generated \( R \)-module.
- the order of \( G \) is invertible in \( K \).

Now we can state the main application of our results to invariant rings. It is clearly the analogue of Schlessinger’s theorem for tangent cohomology of invariant rings mentioned in the introduction.

THEOREM 7.5. Under the assumptions of 7.4, one has

\[ g = \text{grade}_{SG} S G \geq c = \text{depth}(\theta(S/R), S), \]

and

\[ \text{Ext}^j_R(S, S) = 0 \quad \text{for } 1 \leq j \leq g - 2. \]

There are natural isomorphisms

\[ \text{HH}^i(S)^G \cong \text{HH}^i(R) \quad \text{for } i \leq c - 2, \]
as well as an exact sequence

\[ 0 \to \HH^{c-1}(S)^G \to \HH^{c-1}(R) \to \HH^0(SG, \Ext^{c-1}_R(S, S)) \to \]
\[ \to \HH^c(S, SG)^G \to \HH^c(R). \]

The groups $\HH^i(R)$ appear in general as limit of a spectral sequence

\[ E_2^{i,j} = \HH^i(S, \Ext^j_R(S, S))^G \implies \HH^{i+j}(R). \]

**Proof.** The inequality $c \leq g$ is \[7.2\] For the claim on $\Ext^j_R(S, S)$, we may assume that $g \geq 2$, and in that case the Morita context $(C, e')$ from \[6.7\] is a Wedderburn context, as $|G|$ invertible in $K$ forces $C' = 0$. The vanishing result then follows from \[2.8\] and \[2.13\].

Further, $\HH^i(S, Sg) \cong \Ext^i_{SG}(S, Sg)$ for each $i$, as $S$ is projective over $K$, and those groups vanish for $i < c$ by \[7.3\] Combining these facts, \[6.18\] now yields the remaining assertions. □

**Corollary 7.6.** Aside from the general assumptions in \[7.4\] suppose furthermore that $G$ acts separably outside isolated fixed points on a domain $S$ that is smooth and finitely generated over a field $K$. In that case, either $R$ is regular or $c = \depth(\theta(S/R), S) = \dim S = g = \grade_{SG} SG$, and one has

\[ \HH^i(R) \cong \HH^i(S)^G \quad \text{for } i \leq \dim S - 2, \]
\[ \HH^{\dim S - 1}(R) \cong \HH^{\dim S - 1}(S)^G \oplus (\text{finite length } R-\text{module}) \]

as $R$–modules. If $\dim S = 2$, the finite length module is zero.

**Proof.** As $S$ is supposed to be smooth over $K$, the global dimension of $SG$ equals the Krull dimension of $S$, and so \[2.14\] shows $g \leq \dim S$, unless $R$ is regular. The ring $S$ is in particular Cohen-Macaulay, and then $c = \depth(\theta(S/R), S) = \dim S$ by \[6.5\] as all minimal primes lying over $\theta(S/R)$ are by assumption maximal. Combined with the inequality $c \leq g$, the claimed equalities follow.

For the remainder, we only need to consider the case $g = c \geq 2$. As $S$ is smooth over $K$, its Hochschild cohomology satisfies

\[ \HH^i(S) \cong \Hom_S(\Omega^i_{S/K}, S), \]

by the Hochschild-Kostant-Rosenberg Theorem, see \[26\], 3.4.4, and so $\HH^i(S)$ is a projective $S$–module. In particular, its direct $R$–summand $\HH^i(S)^G$ is of maximal depth as $R$–module, whereas

\[ \Ext^{\dim S - 1}_R(S, S) \cong \Ext^{\dim S}_{SG}(SG, SG) \]

is annihilated by $\theta(S/R)$ as $S$–module, and thus of finite length. Accordingly, the image $X$ of $\HH^{\dim S - 1}(R)$ in $\HH^0(SG, \Ext^{\dim S - 1}_R(S, S))$, is of finite length, and the resulting short exact sequence

\[ 0 \to \HH^{\dim S - 1}(S, S)^G \to \HH^{\dim S - 1}(R) \to X \to 0 \]

splits.

Finally, note that for $\dim S = 2$ the $R$–module $\HH^1(R) = \Der_K(R)$ is reflexive, thus cannot contain a nontrivial submodule of finite length. □
We finish with the following example, due to M. Auslander [4], that motivated our investigation of Morita contexts and Hochschild cohomology.

**Example 7.7.** Assume $S$ is the ring of formal power series in $n$ variables over a field $K$, and suppose $G$ is a subgroup of $GL(n, K)$, with $|G| \in K^*$, that acts through linear automorphisms on $S$. The group $G$ contains no pseudo-reflections if, and only if, the branch locus is of codimension at least 2. In that case, the natural map $SG \to \text{End}_R(S)$ is an isomorphism of rings, that is, $SG$ is an Auslander context, as $S$ contains $R$ as an $R$–module direct summand. If $n = 2$, then $S$ is indeed a generator of the category of reflexive $R$–modules, and thus $SG$ is isomorphic to an Auslander algebra of the reflexive $R$–modules. See, for example, [12] for further information on this example and an investigation of how the Hochschild cohomology of the stable Auslander algebra or Morita defect $SG$ relates to that of $SG$.

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