Growing in time IDLA cluster is recurrent

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Abstract

We show that Internal Diffusion Limited Aggregation (IDLA) on \(\mathbb{Z}^d\) has near optimal Cheeger constant when the growing cluster is large enough. This implies, through a heat kernel lower bound derived previously in [11], that simple random walk evolving independently on growing in time IDLA cluster is recurrent when \(d \geq 3\).

Keywords: isoperimetry; internal diffusion limited aggregation; recurrence; random walk; random environment; changing environment.

AMS MSC 2010: Primary 60K37; 82C24, Secondary 60K35.

Submitted to ECP on January 16, 2019, final version accepted on April 16, 2019.
Supersedes arXiv:1809.11022v1.

1 Introduction

This work is concerned with proving recurrence of simple random walk on certain type of growing clusters. Specifically, we consider the Internal Diffusion Limited Aggregation (IDLA [14]) \(\{D_n\}_{n \in \mathbb{N}}\) on \(\mathbb{Z}^d\), \(d \geq 3\) in discrete time, that is, starting from \(D_0 = \{0\}\) consisting of the origin, one iteratively grows the cluster by adding to \(D_n\) the first vertex \(v_n \in \mathbb{Z}^d \setminus D_n\) that is hit by an independent particle injected from the origin performing discrete-time simple random walk (SRW), and set \(D_{n+1} = D_n \cup \{v_n\}\), for every \(n \in \mathbb{N}\). As such, \(D_n\) are random, finite connected vertex subsets of \(\mathbb{Z}^d\), monotonically increasing with \(|D_n| = n\), where we denote by \(|\cdot|\) the cardinality of a set. Being a canonical model of Laplacian growth, IDLA has been much studied, cf. [9, 14, 10, 2, 3, 12, 13] and references therein, with the well-known shape and fluctuation theorems stating, in particular, that for each \(d \geq 3\), there exists finite constant \(a = a(d)\), such that almost surely for all \(n = [\omega_d d^d]\) large enough, we have that \(B_{r-a\sqrt{\omega_d}} \subset D_n \subset B_{r+a\sqrt{\omega_d}}\), where \(\omega_d\) denotes the volume of a unit ball in \(\mathbb{R}^d\) and \(B_R = B_R \cap \mathbb{Z}^d\) for the origin-centered Euclidean ball \(B_R\) of radius \(R\). In this article, we view each \(D_n\) as a subgraph of \(\mathbb{Z}^d\) imposing an edge between any two vertices of \(\mathbb{Z}^d\)-distance 1 within the cluster, and the inclusion above holds in the sense of graphs.

Consider now an (autonomous) simple random walk \(\{X_n\}_{n \in \mathbb{N}}\) evolving independently on \(\{D_n\}_{n \in \mathbb{N}}\) starting from \(X_0 = 0\). That is, fixing a realization \(\omega\) of the IDLA cluster \(\{D_n^\omega\}_{n \in \mathbb{N}}\), at each step \(n\), \(X_n \in D_n^\omega\) chooses uniformly among all vertices in \(D_n^\omega\) having \(\mathbb{Z}^d\)-distance 1 from it and passes to \(X_{n+1}\), before domain changes from \(D_n^\omega\) to \(D_{n+1}^\omega\). There is at least one vertex available for \(X_n\) and at most \(2d\). Fixing \(\omega\), such random walk \(\{X_n\}_{n \in \mathbb{N}}\) is a time-inhomogeneous Markov chain, albeit at perpetual non-equilibrium, but we are interested in whether the walk is recurrent to the origin, and whether recurrence

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holds for almost every realization ω of its environment. Indeed, we have studied this problem within a more general set-up in [8], in particular showing that if one considers a continuous-time version of IDLA \( \{D_t\}_{t \geq 0} \), where particles are injected from the origin at an inhomogeneous Poisson rate, then on \( \mathbb{Z}^d \), \( d \geq 3 \), for a.e. \( \omega \), the (discrete-time) SRW \( \{X_t\}_{t \geq 0} \) returns to its starting point finitely many times with probability one as soon as \( \int_1^\infty |D_t|^{-1} dt \) is finite. However, it is only conjectured that for constant injection rate of particles, namely when \( |D_t| \) grows approximately linearly, w.p.1 \( \{X_t\}_{t \geq 0} \) visits its starting point infinitely often as \( \int_1^\infty |D_t|^{-1} dt \) then diverges. See Section 4 where we revisit this problem, for details. Similarly, the latter outcome is expected for the Theorem 2.1 ECP 24 is taken from \[13, \text{Theorem 0.1}\], \[12, \text{Theorem 1}\].

Further, we denote by \( B \) by imposing an edge between any \( n \) then we set \( D \) doing simple random walk from the origin till visiting the first vertex \( Y \), which we denote \( Y \). Hence, \( |D_n| = n \). While respecting its growth mechanism, we omit particle \( Y \)'s travel time and consider it settled on \( Z^d \) instantaneous upon injected, for every \( n \in \mathbb{N} \). Each \( D_n \) is then viewed as a subgraph of the lattice graph \( Z^d \), by imposing an edge between any \( x, y \in D_n \) of \( Z^d \)-distance 1, which we denote \( x \sim y \). Further, we denote by \( B(v, R) = B(v, R) \cap Z^d \), the graph obtained from the intersection of \( Z^d \) lattice graph and an open Euclidean ball of radius \( R > 0 \) centered at \( v \in \mathbb{Z}^d \), with \( B_R := B(0, R) \) in case the center is the origin.

Now, we recall the shape and fluctuation theorems for \( \{D_n\} \). The formulation below is taken from \[13, \text{Theorem 0.1}\], \[12, \text{Theorem 1}\].

**Theorem 2.1** ([14, 2, 3, 12, 13]). For \( d \geq 3 \), there exist some finite constant \( a = a(d) \)
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and integer \( n_0 \) such that for all \( n = \lfloor \omega d^d \rfloor \geq n_0 \), we have that

\[
P(S_n) \geq 1 - n^{-2}, \quad \text{where} \quad S_n := \{B_{r - a\sqrt{\log n}} \subset D_n \subset B_{r + a\sqrt{\log n}}\}.
\]  

(2.1)

For \( d = 2 \), the same holds with \( \log r \) replacing \( \sqrt{\log r} \) in \( S_n \).

We also require some notation for general graphs. To this end, given a graph \( G \), we write \( V(G) \) its vertex set, and \( E(G) \) its edge set. Throughout, we write \( |G| \) for \( |V(G)| \), for short. We now specify a notion of vertex boundary that we extensively use.

**Definition 2.2.** Given a graph \( G \), the inner vertex boundary of \( A \subset V(G) \) relative to \( G \) is defined to be

\[
\partial^G A := \{ x \in A : \exists y \in V(G) \setminus A \text{ such that } (x, y) \in E(G)\}.
\]

(2.2)

We omit the superscript \( G \) in \( \partial^G A \) in case \( G = \mathbb{Z}^d \).

Define the Cheeger constant (aka bottleneck ratio) of a finite graph \( G \) as

\[
\Phi(G) = \inf_{A \subset V(G), |A| \leq |G|/2} \left\{ \frac{|\partial^G A|}{|A|} \right\}.
\]

(2.3)

This quantity is a well-known gauge for the connectivity of a finite set, in particular governing the mixing rates for reversible Markov chains on the set, see [16]. Instead of the vertex boundary in (2.3), more commonly one uses the edge boundary (that is, the set of all edges with one end in \( A \), the other in \( V(G) \setminus A \)); in our setting, they differ by at most a factor of \((2d - 1)\), which is unimportant for us. In the sequel, we will be considering (2.3) for \( G = D_n \).

In the case of lattice balls \( B_R \subset \mathbb{Z}^d \), it is well known that there exists some positive constant \( c_d \), such that \( \Phi(B_R) \geq c_d |B_R|^{-1/d} \) for all \( R > 1 \), and such isoperimetry is optimal for finite subgraphs of \( \mathbb{Z}^d \). Given Theorem 2.1, it is pertinent to wonder if the isoperimetry of lattice balls is inherited by IDLA, and the main result of this article is the following.

**Theorem 2.3.** Fix \( d \geq 3 \). There exists some positive constant \( c = c(d) \), such that \( \mathbb{P}\text{-a.s. for all } n \text{ large enough, we have that} \)

\[
\Phi(D_n) \geq cn^{-1/d}(\log n)^{-3/2}.
\]

(2.4)

**Remark 2.4.** The result does not follow automatically from Theorem 2.1, since with high probability \( |D_n \setminus B_{r - a\sqrt{\log n}}| = O(n^{(d-1)/d}\sqrt{\log n}) \), and a-priori a cut in \( D_n \) can be as small as consisting of only one edge. The point here is to show that randomly produced cluster, here IDLA, is much better structured than deterministic ones under the same shape constraints.

Nevertheless, we believe that the logarithmic correction in (2.4) should not be there. For that reason, we do not state the result for \( d = 2 \).

Turning to the application to recurrence on \( \{D_n\} \) we have in mind, denote by \( \{X_n\}_{n \in \mathbb{N}} \) a lazy \textsc{sw} on \( \{D_n\} \) with \( X_0 = 0 \). Fixing a realization \( \omega \) of \( \{D_n\} \), its \( n \)-th step transition probability is defined to be

\[
P^{\omega}(X_{n+1} = y|X_n = x) = \begin{cases} (1 - \gamma_n^\omega(x)) \cdot |\{z \in D_n^\omega : x \sim z\}|^{-1}, & \text{if } x \sim y \in D_n^\omega, x \neq y \\ \gamma_n^\omega(x), & \text{if } x = y \in D_n^\omega \end{cases}
\]

(2.5)

for some \( \{\gamma_n^\omega(x)\} \) denoting the probabilities of staying put. In words, each step the random walk either stays put or chooses uniformly among neighbors in its current
cluster. Here, \( P^\omega \) denotes the quenched probability governing the random walk \( X \), and \( P \) is reserved for the randomness of \( \{D_n\} \). We will add the superscript \( \omega \) only when it is helpful to emphasize the randomness of the environment. We say that \( \{X_n\} \) is uniformly lazy with probability \( \gamma \in (0, 1) \), if \( \gamma_n^\omega(x) \geq \gamma \) for all \( \omega \), \( n \) and \( x \in D_n^\omega \). A special case of [11, Proposition 1.13] in our context is the following heat kernel lower bound for \( \{X_n\} \) on \( \{D_n\} \), where the uniform laziness is but a technical assumption.

**Proposition 2.5** ([11]). Suppose \( \{X_n\} \) is uniformly lazy with probability \( \gamma \in (0, 1) \), and the following conditions on \( \{D_n\} \) are \( P \)-a.s. satisfied:

(i). There exist some deterministic sequence \( \{r(n)\} \), and a.s. finite random variable \( C(\omega) \) such that

\[
D_n \supset B_{r(n)}, \quad \forall n \geq C(\omega)
\]

and

\[
\lim_{n \to \infty} \left\{ \frac{|D_n - B_{r(n)}|}{|D_n|} \right\} = 0.
\]  

(ii).

\[
\liminf_{n \to \infty} \left\{ \frac{\sum_{j=[n/2]}^{n-1} \Phi(D_j)^2}{\log n} \right\} = \infty. \tag{2.6}
\]

Then, there exists some positive \( c_0 = c_{0}(d, \gamma) \) such that \( P \)-a.s. for all \( n \geq (2^d - 1)C(\omega) \),

\[
P^\omega(X_n = 0) \geq \frac{c_0}{|D_n|}
\]

The random variable \( C(\omega) \) is determined, thus condition (i) satisfied with \( r(n) = (n/\omega_d)^{1/d} \), by Borel-Cantelli applied to the events \( \{S_n^r\} \) of (2.1). Further, since \( |D_n| = n \), it follows from Theorem 2.3 that \( P \)-a.s. for all \( n \) large enough, we have

\[
\sum_{j=[n/2]}^{n-1} \Phi(D_j)^2 \geq c' n^{1-2/d-\epsilon},
\]

for some \( c' > 0 \) and any \( \epsilon \in (0, 1 - 2/d) \), thereby verifying (2.6) when \( d \geq 3 \).

We use here the a-priori weaker definition of recurrence.

**Definition 2.6.** We say that \( X \) is recurrent if it has infinite expected occupation time at the origin (hence at every other point), i.e. \( E \left[ \int_1^\infty 1 \{X_t = 0\} \, dt \right] = \infty \). Otherwise, it is called transient.

**Corollary 2.7.** For almost every realization \( \omega \) of the discrete-time IDLA cluster \( \{D_n\}_{n \in \mathbb{N}} \) on \( \mathbb{Z}^d \), \( d \geq 3 \), the uniformly lazy SRW \( \{X_n\}_{n \in \mathbb{N}} \) evolving independently on \( \{D_n^\omega\} \) is recurrent.

**Remark 2.8.** Our result does not cover \( d = 2 \), for which we conjecture recurrence. In fact, a more general conjecture currently open asserts that any sequence of growing subgraphs \( \{G_n\}_{n \in \mathbb{N}} \) of \( \mathbb{Z}^2 \) is recurrent for the independently evolving SRW \( \{X_n\}_{n \in \mathbb{N}} \) on it, see [8, Conjectures 1.10, 1.8] or [1, Conjectures 1.1, 7.1].

### 3 Proof

For large enough \( n = [\omega_d r^d] \), the IDLA cluster \( D_n \) concentrates on \( B_r \) with high probability by Theorem 2.1. We thus denote

\[
F_n := D_n \setminus B_{r - \sqrt{n \log r}}
\]

(the part of fluctuation, when such event occurs). Our main goal is to show the following.
Proposition 3.1. Fix $d \geq 3$. There exist some finite constant $C_\star = C_\star(a, d)$ and integer $n_1$ such that for all $n = \lceil \omega r^d \rceil \geq n_1$,
\[ \mathbb{P} \left( \exists A \subset \mathcal{V}(F_n) : \lvert A \rvert \geq C_\star \lvert \partial^{D_n} A \rvert (\log r)^{3/2} \right) \leq 3n^{-2}. \tag{3.1} \]

That this will be sufficient for proving Theorem 2.3 is due to a deterministic fact.

Lemma 3.2. Suppose a connected subgraph $G \subset \mathbb{Z}^d$ is such that $G \supset B_R$ for some $R > 1$. Suppose there exists some positive constant $c_1$ such that $|\partial^G A| \geq c_1 |A| (\log R)^{-3/2}$ holds for any $A \subset \mathcal{V}(G \setminus B_R)$. Then, there exists another positive constant $c_2 = c_2(c_1, d)$ such that $|\partial^G A| \geq c_2 |A| R^{-1} (\log R)^{-3/2}$, for any $A \subset \mathcal{V}(G)$.

Proof of Lemma 3.2. Fixing $A \subset \mathcal{V}(G)$ a vertex subset, we denote $A_1 = A \cap \mathcal{V}(B_R)$ and $A_2 = A \cap \mathcal{V}(B_R^c)$ such that $A = A_1 \uplus A_2$. We further denote $Q_2 := \{ x \in \partial^G A_2 : \exists y \in A_1 \text{ such that } x \sim y \}$ and $Q_1 := \{ x \in A_1 : \exists y \in A_2 \text{ such that } x \sim y \}$. Trivially $|Q_2| \leq (2d - 1) |Q_1|$. Recall that by the optimal isoperimetry satisfied by lattice balls, $\Phi(B_R) \geq c_d R^{-1}$ for some $c_d > 0$ and any $R > 1$, hence $|\partial^{B_R} A_1| \geq c_d R^{-1} |A_1|$. Further, we can decompose $\partial^G A$ as follows (where $\uplus$ denotes disjoint union)
\[ \partial^G A \supset \partial^{B_R} A_1 \uplus (\partial^G A_2 \setminus Q_2). \]

We separately discuss two cases. (a) If $|A_1| \geq |A_2|$, then clearly we have that
\[ |\partial^G A| \geq |\partial^{B_R} A_1| \geq c_d R^{-1} |A_1| \geq \frac{1}{2} c_d R^{-1} |A|. \]

(b) Now consider $|A_1| < |A_2|$. Since
\[ |\partial^{B_R} A_1| \geq c_d R^{-1} |A_1| \geq c_d R^{-1} |Q_1| \geq c_d (2d - 1)^{-1} R^{-1} |Q_2|, \]
we have that
\[ |\partial^G A| \geq |\partial^{B_R} A_1| + |\partial^G A_2 \setminus Q_2| \geq c_d (2d - 1)^{-1} R^{-1} |Q_2| + |\partial^G A_2 \setminus Q_2| \geq \frac{1}{2} c_1 c_d (2d - 1)^{-1} |A| R^{-1} (\log R)^{-3/2}. \]
Combining both cases proves the claim.

Our proof relies on partitioning a discrete sphere into subsets of comparable size and diameter. There may be more than one way to achieve this. Here we utilize a notion of almost regular partition of the Euclidean unit sphere $S^{d-1} := \{ x \in \mathbb{R}^d : \|x\|_2 = 1 \}$.

Lemma 3.3 ((7, Theorem 6.4.2)). For every integer $M$, there exists a partition $\{ T_1, T_2, \ldots, T_M \}$ consisting of closed subsets of $S^{d-1}$ that satisfies the following:
(i). $S^{d-1} = \bigcup_{j=1}^M T_j$ and $T_i \cap T_j = \emptyset$ whenever $i \neq j$, where $T_j$ denotes the interior of $T_j$.
(ii). $\sigma_d(T_j) = M^{-1} \sigma_d(S^{d-1})$, for any $j = 1, \ldots, M$, where $\sigma_d(\cdot)$ denotes surface area measure.
(iii). For some positive $c_p = c_p(d)$, the partition norm
\[ \max_{j=1}^M \max_{x, y \in T_j} \|x - y\|_2 \leq c_p M^{-\frac{1}{d+1}}. \]

Such area regular partition of $S^{d-1}$ naturally induces a partition of any discrete sphere.
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**Definition 3.4.** For a discrete sphere \(\partial B_R \subset \mathbb{Z}^d\), \(R > 1\), we call \(\{T_1, T_2, ..., T_M\}\) its area regular partition, if \(x \in T_j\) whenever \(x/\|x\|_2 \in T_j\), for every \(x \in \partial B_R\) and \(j = 1, ..., M\), where \(\{T_1, ..., T_M\}\) is an area regular partition of \(S^{d-1}\) in the sense of Lemma 3.3.

We define the notion of a cone in \(\mathbb{R}^d\) determined by the origin and a closed subset of \(S^{d-1}\).

**Definition 3.5.** We say that \(K = K(T) \subset \mathbb{R}^d\) is an infinite cone determined by a closed subset \(T \subset S^{d-1}\) and the origin, if

\[
K(T) := \{x \in \mathbb{R}^d : x/\|x\|_2 \in T\}.
\]

We then define the notion of a cell which is the intersection of a cone and an annulus in \(\mathbb{R}^d\).

**Definition 3.6.** We say that \(C = C(A, T) \subset \mathbb{R}^d\) is a cell determined by an annulus \(A = A(R, R') := B_R \setminus B_{R'},\) where \(R > R'\), and a closed subset \(T \subset S^{d-1}\), if \(C = A \cap K\) where \(K = K(T)\) is the infinite cone determined by \(T\) and the origin in the sense of Definition 3.5.

We also need a lemma for a specific hitting probability of (ordinary) SRW on \(\mathbb{Z}^d\). We defer its proof to the end of the section.

**Lemma 3.7.** Fix \(d \geq 3\). Let \(C \subset \mathbb{R}^d\) be a cell determined by the annulus \(A = A(R + \delta \sqrt{\log R}, R)\) and a closed subset \(T \subset S^{n-1}\) with \(\max_{x \in T} \|x - y\|_2 \leq c_p/R\), as in Definition 3.6, where \(R, \delta, c_p > 1\). We further denote \(C = C \cap \mathbb{Z}^d\) the discrete cell, and \(Y\) a SRW on \(\mathbb{Z}^d\). Then, there exist finite constants \(R_1 = R_1(d)\) and \(C' = C'(d, \delta, c_p)\) such that for any \(R > R_1\),

\[
\mathbb{P}_0(Y\text{ hits }C\text{ before }B^c_{R+\delta \sqrt{\log R}}) \leq C'_d \log R \left(\frac{R^d}{R^{d-1}}\right),
\]

where \(\mathbb{P}_v\) indicates the law of \(Y\) when starting from \(v \in \mathbb{Z}^d\).

We now prove the main proposition.

**Proof of Proposition 3.1.** For brevity, we write

\[
n' := \lfloor \omega_d (r - 2a \sqrt{\log r})^d \rfloor.
\]

First note that, under the event \(S_{n'}\) (see (2.1)) we have \(F_n \cap D_{n'} = \emptyset\) and \(D_{n'} \supset B_{r - 3a \sqrt{\log r}}\).

Further, since \(S_{n'}\) depends only on the first \(n'\) injected particles, conditioning on \(S_{n'}\) does not alter the independence of subsequent particles injections.

By Lemma 3.3 and Definition 3.4, there exists an area regular partition \(\{T_1, ..., T_M\}\) of \(S^{d-1}\) with \(M = M(n) := \lceil c'r^{d-1} \rceil\) for any \(c' > 0\) chosen fixed, and its induced partition \(\{T_1, ..., T_M\}\) of \(\partial B_{r - 3a \sqrt{\log r}}\) such that \(\|x - y\|_2 \leq c_p\) for some finite constant \(c_p = c_p(d, c') > 1\) and any \(x, y \in T_j, j = 1, ..., M\). Each partition block \(T_j, j = 1, ..., M\) and the annulus \(A(r + a \sqrt{\log r}, r - 3a \sqrt{\log r})\) determines a cell \(C_j = C_j(A, T_j)\) in the sense of Definition 3.6.

We then denote the discrete cells \(C_j := C_j \cap \mathbb{Z}^d\), \(j = 1, ..., M\).

Let us focus on each fixed cell \(C_j\), \(j \in \{1, ..., M\}\). We compute the probability that the \(m\)-th particle \(Y_m, m \in [n', n]\), upon being injected from the origin, hits \(C_j\) before \(D_m \setminus C_j\), conditional on \(S_{n'}\). To this end, we define the events

\[
E_{m}^{(j)} := \{Y_m \text{ hits }C_j\text{ before }D_m \setminus C_j\}, \quad n' < m \leq n, \quad j \in \{1, ..., M\}.
\]

For each \(m \in [n', n]\), on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\) we define \(Y^\text{ext}_m\) to be a SRW evolving on \(\mathbb{Z}^d\) obtained by continuing to run the IDLA particle \(Y_m\) even after it has exited \(D_m\) (and hence settled). Since \(D_m \subset D_n\) and \(D_m \setminus C_j \supset B_{r + a \sqrt{\log r}}\), for \(n' < m \leq n\), we have the following inclusion of events

\[
E_{m}^{(j)} \cap \{D_n \subset B_{r + a \sqrt{\log r}}\} \subset E_{m}^{(j)} \cap \{D_m \subset B_{r + a \sqrt{\log r}}\} \subset E_{m}^{(j)},
\]

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where

\[ \tilde{E}_m^{(j)} := \{ Y_m^{\text{EXT}} \text{ hits } C_j \text{ before } B_{r + \sqrt{\log r}}^c \}, \quad n' < m \leq n, \ j \in \{1, \ldots, M\}. \]

By Lemma 3.7, there exists some finite constant \( C' = C'(d, a, c_p) \) such that for any \( n' < m \leq n \) and \( j \in \{1, \ldots, M\} \),

\[ P(\tilde{E}_m^{(j)} \cap \{ D_n \subset B_{r + \sqrt{\log r}}^c \} \mid S_{n'}) \leq P(\tilde{E}_m^{(j)} \mid S_{n'}) \leq C' \frac{r}{n - 1}. \tag{3.5} \]

Since the particles trajectories \( \{ Y_m : n' < m \leq n \} \) are independent prior to settling, in view of (3.4), for each fixed \( j \in \{1, \ldots, M\} \), the events \( \{ E_m^{(j)} \cap \{ D_n \subset B_{r + \sqrt{\log r}}^c \} : n' < m \leq n \} \) (when conditioned on \( S_{n'} \)) are stochastically dominated from above by a sequence of independent Bernoulli’s with success probability at most the RHS of (3.5).

Since also \( n - n' = c_n(d, a)d^{-1} \sqrt{\log r} \) for some explicit constant \( c_n(d, a) \in (0, \infty) \), we have by Chernoff bound

\[ P\left( \sum_{m=n'+1}^{n} 1\{ E_m^{(j)} \} \geq 2C'c_s(\log r)^{3/2} \right) \cap \{ D_n \subset B_{r + \sqrt{\log r}}^c \} \mid S_{n'} \right) \leq e^{-C'c_s(\log r)^{3/2}}. \tag{3.6} \]

Since \( A \subset V(F_n) \) and \( F_n \cap D_n = \emptyset \), the particle \( Y_k \) that eventually settles on some \( x \in A \) must have index \( n' < k = k(x) \leq n \). Further, since \( \partial^{D_n} A \) is a vertex-cutset that separates \( A \) from the origin, such \( Y_k \) must pass through \( \partial^{D_n} A \) at least once before settling on \( x \in A \). Let us denote by \( v = v(x) \) the first vertex in \( \partial^{D_n} A \) that \( Y_k \) hits, before settling at \( x \). Since \( v \in V(F_n) \), it belongs to some cell \( C_{j(v)} \), where \( j(v) \in \{1, \ldots, M\} \).

Consequently, such \( Y_k \) in order to settle on \( x \in A \) must hit \( C_{j(v)} \) before \( D_k \setminus C_{j(v)} \) (since otherwise \( Y_k \) already settles on the first vertex it encounters in \( D_k \setminus C_{j(v)} \) and will not hit \( v \) hence will not settle on \( x \)).

With \( A \subset V(F_n) \), \( \partial^{D_n} A \) cannot intersect more than \( 2|\partial^{D_n} A \cap M \) number of cells, as the cells are vertex-disjoint except possibly at their boundaries. Further, every \( x \in A \) is settled by some particle \( Y_{k(x)} \). Thus, the event \( \{ \exists A \subset V(F_n) : |A| \geq 4C'c_s|\partial^{D_n} A|(|\log r|)^{3/2} \} \) implies, by pigeonhole principle, that there exists at least one cell, say \( C_j, \) for which the event \( \{ \sum_{m=n'+1}^{n} 1\{ E_m^{(j)} \} \geq 2C'c_s(\log r)^{3/2} \} \) occurs. Due to in total \( M = \lceil c' r^{d-1} \rceil \) number of cells, we obtain (by (3.6)) and union bound that

\[ P(\exists A \subset V(F_n) : |A| \geq 4C'c_s|\partial^{D_n} A|(|\log r|)^{3/2} \cap \{ D_n \subset B_{r + \sqrt{\log r}}^c \} \mid S_{n'}) \]

\[ = P\left( \exists j^* \in \{1, \ldots, M\} : \sum_{m=n'+1}^{n} 1\{ E_m^{(j^*)} \} \geq 2C'c_s(\log r)^{3/2} \right) \cap \{ D_n \subset B_{r + \sqrt{\log r}}^c \} \mid S_{n'} \right) \]

\[ \leq \sum_{j=1}^{M} P\left( \sum_{m=n'+1}^{n} 1\{ E_m^{(j)} \} \geq 2C'c_s(\log r)^{3/2} \right) \cap \{ D_n \subset B_{r + \sqrt{\log r}}^c \} \mid S_{n'} \right) \]

\[ \leq c'r^{d-1}e^{-C'c_s(\log r)^{3/2}}. \]

Noticing that \( \{ D_n \subset B_{r + \sqrt{\log r}}^c \}^c \subset S_n^c \), we can further bound

\[ P(\exists A \subset V(F_n) : |A| \geq 4C'c_s|\partial^{D_n} A|(|\log r|)^{3/2}) \]

\[ \leq P(S_n^c \cup S_n^c) + P(\exists A \subset F_n : |A| \geq 4C'c_s|\partial^{D_n} A|(|\log r|)^{3/2} \cap \{ D_n \subset B_{r + \sqrt{\log r}}^c \} \mid S_{n'}). \]

Upon taking \( n \) large enough such that \( n' \geq n_0 \), by Theorem 2.1 we have that

\[ P(\exists A \subset V(F_n) : |A| \geq 4C'c_s|\partial^{D_n} A|(|\log r|)^{3/2}) \leq 2n^{-2} + c'r^{d-1}e^{-C'c_s(\log r)^{3/2}}. \]

which can be made smaller than \( 3n^{-2} \) upon taking \( n \geq n_1 \) for some large enough \( n_1 \).

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Proof of Theorem 2.3. By Proposition 3.1 and Theorem 2.1, there exist positive constant $c_1 = c_1(a, d) = 1/C^*$ and finite integer $n_1$, such that for all $n = [\omega_d r^d] \geq n_1$,

$$
P\left(\{\forall A \subset \mathcal{Y}(F_n) : |\partial B_n A| \geq c_1 |A| (\log r)^{-3/2}\} \cap S_n\right) \geq 1 - 4n^{-2}.
$$

Combining with Lemma 3.2, there exists another positive constant $c_2 = c_2(c_1, d)$ such that

$$
P\left(\{\forall A \subset \mathcal{Y}(D_n) : |\partial B_n A| \geq c_2 |A| r^{-1} (\log r)^{-3/2}\} \cap D_n\right) \geq 1 - 4n^{-2}.
$$

By Borel-Cantelli, the left event occurs a.s. for all large enough $n$, namely (2.4) holds. 

We are left to prove Lemma 3.7. First, we clarify some additional notations. Since we work here with sw $Y$ on $\mathbb{Z}^d$, without incurring ambiguity we identity any subgraph with its vertex set. Further, we denote by $\tau (A)$ and $H_A (\cdot, \cdot)$, respectively, the first exit time (discounting time 0) and exit distribution of $A \subset \mathbb{Z}^d$ by a sw:

$$
\tau (A) := \inf \{k \geq 1 : Y_k \in A^c\}, \quad H_A (x, y) := P(Y_\tau (A) = y | Y_0 = x).
$$

Recall that Green's function on $\mathbb{Z}^d$, $d \geq 3$, is harmonic except at the origin, and satisfies the asymptotics as $\|x\|_2 \to \infty$

$$
G(x) = E_x \left[ \sum_{k=0}^{\infty} 1\{Y_k = 0\} \right] = C_d \|x\|_2^{2-d} + O(\|x\|_2^{1-d}), \quad \text{where} \quad C_d := \frac{2}{(d-2)\omega_d}. \quad (3.7)
$$

One also defines Green's function restricted to $A \subset \mathbb{Z}^d$ as the symmetric function

$$
G_A (x, y) = E_x \left[ \sum_{k=0}^{\tau_A (x)} 1\{Y_k = y\} \right].
$$

Proof of Lemma 3.7. The proof is similar to [15, Lemma 6.3.7]. By scaling relations, the property of $T \subset S^{d-1}$ transfers to the fact that any two vertices on the lower base of the discrete cell $C$ (on $\partial B_R$) satisfies $\|x - y\|_2 \leq c_p$. In particular, the lower base of the cell is contained in a spherical cap of radius $c_p$ on $\partial B_R$. Denoting $K := B_{R + \delta \sqrt{\log R}} \setminus C$, we have that $\partial B_R \subset K \subset B_{R + \delta \sqrt{\log R}}$.

It is a consequence of last exit decomposition, see [15, Lemma 6.3.6], that for any $x \in \partial (K^c)$, i.e. the outer vertex boundary of $K$,

$$
H_K (0, x) = \sum_{z \in \partial B_{R/2}} G_K (z, 0) P_x (Y_{\tau (K \setminus B_{R/2})} = z) = \sum_{z \in \partial B_{R/2}} G_K (z, 0) P_x (Y_{\tau (K \setminus B_{R/2})} = z).
$$

(3.8)

Uniformly for all $z \in \partial B_{R/2}$, by [15, Proposition 6.3.5] we have that

$$
G_K (z, 0) \leq G_{B_{R + \delta \sqrt{\log R}}} (z, 0) \leq C_d \left( (R/2)^{2-d} - (R + \delta \sqrt{\log R})^{2-d} \right) + O(R^{1-d}) \leq C_d \delta^{2-d} R^{2-d}.
$$

The first inequality is due to both vertices $0, z \in K \subset B_{R + \delta \sqrt{\log R}}$, and the last inequality holds provided we take $R > R_1$ for some large enough $R_1 = R_1 (d)$. Further, for any $x \in \partial (K^c)$,

$$
\sum_{z \in \partial B_{R/2}} P_x (Y_{\tau (K \setminus B_{R/2})} = z) = \sum_{y \in \partial B_{R/2}} \max_{y \in \partial B_n} \left\{ P_y (Y \text{ hits } B_{R/2} \text{ before } K^c) \right\},
$$

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since starting from $K^c$, in order to hit $B_{R/2}$, $Y$ must first pass through $\partial B_R$. Since $\{G_{(Y)}\}_{k\in\mathbb{N}}$ is a martingale provided $Y$ is bounded away from 0, by optional stopping theorem and (3.7), the latter probability is bounded by

$$\max_{y\in\partial B_R} \left\{ P_y(Y \text{ hits } B_{R/2} \text{ before } B_{R+\delta\log R}) \right\} \leq 2 \frac{R^{2-d} - (R + \delta\sqrt{\log R})^{2-d}}{(R/2)^{2-d} - (R + \delta \log R)^{2-d}} \leq 2^{d-1}(d-2)\delta \sqrt{\log R}/R,$$

when $R > R_1$, upon increasing $R_1$ if necessary. Plugging these estimates into (3.8), uniformly for all $x \in \partial (K^c)$, we have that $H_R(0, x) \leq C_4 d^{d-1}(d-2)\delta R^{1-d}\sqrt{\log R}$.

Now, since $\partial C \subset \partial (K^c)$, the w.s. of (3.2) is the same as $\sum_{x \in \partial C} H_R(0, x)$. The cardinality of $\partial C$ is at most $c(d)e_p^{d-1}\delta \sqrt{\log R}$ for some dimensional constant $c(d)$. We thus get the desired bound of the form $C'R^{1-d}\log R$ for some $C' = C'(d, \delta, c_p)$ finite. \hfill \Box

4 Extensions to continuous-time IDLA

We consider simple random walk $\{X_{t}\}_{t \geq 0}$ (2.5) evolving independently on a continuous-time analogue of IDLA $\{D_t\}_{t \geq 0}$ on $\mathbb{Z}^d$, $d \geq 3$, formed by particles injected from the origin at an inhomogeneous Poisson rate $\lambda(t) > 0$. As a result, one can achieve quite general growth rate for the cluster by tuning the particles injection rate. Such continuous-time extension of the classical model appears e.g. in [14, 10].

More specifically, let $\{N(t)\}_{t \geq 0}$ be an inhomogeneous Poisson process with mean $m(t) := \int_0^t \lambda(s)ds$, satisfying suitable conditions as specified in (4.1). Similarly to the discrete-time case, starting at $D_0 = \{0\}$, at each Poisson arrival time, a particle is injected from the origin, performing $\mbox{smw}$ until settling on the first unoccupied vertex, and then that vertex is added to the cluster. Clearly, by time $t$, a total of $N(t)$ particles have been injected. We still omit the travel time of the particles and consider them settled instantaneously, therefore $\lvert D_t \rvert = N(t)$. Given $\{D_t\}_{t \geq 0}$ we perform (lazy) $\mbox{smw}$ $\{X_{t}\}_{t \geq 0}$ independently on the cluster, jumping at integer times, as specified in (2.5) with $n = \lfloor t \rfloor$.

(This omission of particles travel time is inconsequential when $d \geq 3$, as explained already in [14, Section 6] and [10]. That is, our conclusions are unaltered if we allow particles to move simultaneously with each other and with the autonomous random walk $X$. When $d = 2$, there is indeed a difference between the two formalisms.)

Since both Theorem 2.1 and Theorem 2.3 are properties of fixed large cluster, irrespective of whether the IDLA emission is time-continuous or discrete, we have the following result.

**Corollary 4.1.** Fix $d \geq 3$. Assume that there exist some constants $0 \leq \alpha < d/2$ and $\beta > 0$, such that the following hold for all $t$ large enough

$$\beta \log t \leq m(t) \leq t^\alpha. \tag{4.1}$$

Then, for almost every realization $\omega$ of the continuous-time IDLA cluster $\{D_t\}_{t \geq 0}$ on $\mathbb{Z}^d$, the uniformly lazy $\mbox{smw}$ $\{X_{t}\}_{t \geq 0}$ evolving independently on $\{D_t\}_{t \geq 0}$ is recurrent as soon as $\int_1^\infty \lvert D_t \rvert^{-1} dt = \infty$. Further, $X$ is recurrent for a.e. realization $\omega$ as soon as $\int_1^\infty m(t)^{-1} dt = \infty$.

**Remark 4.2.** Transience under $\int_1^\infty m(t)^{-1} dt < \infty$ is proved in [8, Corollary 1.6, Remark 1.7] for the a-priori weaker definition of transience as “almost surely finitely many returns” (instead of finite expected occupation time). The method used is different from this article.

**Proof of Corollary 4.1.** Since $N(t)$ is Poisson distributed with mean $m(t) = \int_0^t \lambda(s)ds$, for
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any \( \eta > 0 \) and all \( t > 1 \) we have that

\[
P(N(t) > (1 + \eta)m(t)) \leq \exp \left\{ -\frac{\eta^2 m(t)}{2(1 + \eta)} \right\} \leq \exp \left\{ -\frac{\beta \eta^2 \log t}{2(1 + \eta)} \right\}. \tag{4.2}
\]

Given \( \beta > 0 \), there exists \( \eta = \eta_3 < \infty \) so that \( \beta \eta^2 / (2(1 + \eta)) > 1 \), which ensures that \( \text{rhs} \) of (4.2) is summable and \( \mathbb{P} \)-a.s. the \( \text{rhs} \) events of (4.2) at integer times \( \lfloor t \rfloor \) occurs only finitely often, by Borel-Cantelli. Since \( m(t) \to \infty \) as \( t \to \infty \), the condition (i) of Proposition 2.5 is satisfied when \( t \) is large enough, hence to apply that result it remains to verify again condition (ii) therein. Since \( \omega \), \( \omega \), for all \( \lfloor t \rfloor \) large enough

\[
|D_{\lfloor t \rfloor}| = N^\omega(\lfloor t \rfloor) \leq (1 + \eta_3)m(\lfloor t \rfloor), \tag{4.1}
\]

by Theorem 2.3 and (4.1) we have that

\[
\sum_{k=\lfloor t/2 \rfloor}^{\lfloor t \rfloor-1} \Phi^\omega(D_k)^2 \geq \sum_{k=\lfloor t/2 \rfloor}^{\lfloor t \rfloor-1} N^\omega(k)^{-2/d}(\log N^\omega(k))^{-3/2} \geq c(d, \beta) \sum_{k=\lfloor t/2 \rfloor}^{\lfloor t \rfloor-1} m(k)^{-2/d}(\log m(k))^{-3/2} \geq c(d, \beta, \alpha) \sum_{k=\lfloor t/2 \rfloor}^{\lfloor t \rfloor-1} k^{-2\alpha/d}(\log k)^{-3/2},
\]

where the constant changes from line to line. With \( 0 \leq \alpha < d/2 \), there exists \( \epsilon = \epsilon(d, \alpha) > 0 \) such that \( 1 - 2\alpha/d - \epsilon > 0 \) and thus

\[
\sum_{k=\lfloor t/2 \rfloor}^{\lfloor t \rfloor-1} \Phi^\omega(D_k)^2 \geq c(d, \beta, \alpha) t^{1-2\alpha/d-\epsilon} \gg \log t.
\]

Therefore, Proposition 2.5 applies to give for some \( c_0 = c_0(d, \gamma) > 0 \), a.e. \( \omega \), and all \( t \) sufficiently large

\[
P^\omega(X_1 = 0) \geq \frac{c_0}{|D_{\lfloor t \rfloor}|} \geq \frac{c_0}{(1 + \eta_3)m(\lfloor t \rfloor)} \geq \frac{c_0}{(1 + \eta_3)m(\lfloor t \rfloor)}.
\]

This immediately implies the results. \( \square \)

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**Acknowledgments.** I thank Eviatar Procaccia for pivotal comments that led to this work, and Amir Dembo for fruitful discussions.
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