THE WEAK LEFSCHETZ PROPERTY OF A SPECIAL CLASS OF ARTINIAN ALGEBRAS OVER FIELDS OF POSITIVE CHARACTERISTIC

HASSAN HAGHIGHI AND SEPIDEH TASHVIGHI

Abstract. In this paper, we study the dependence of the weak Lefschetz property of algebras defined by a special class of monomials ideals in a polynomial ring with coefficient in a field, to the characteristic of the base field.

1. Introduction

Let $K$ be an arbitrary infinite field, and let $R = \mathbb{K}[x_1, \ldots, x_r]$ be the polynomial ring with standard grading. Let $I$ be a homogeneous ideal in $R$ such that $A = R/I$ is an Artinian algebra. This is equivalent to say that the radical of $I$ is equal to $(x_1, \ldots, x_r)$, or, $A$ can be written as $\bigoplus_{i=0}^{e-1} A_i$.

A significant property which a standard graded Artinian $K$-algebra may pose is the weak Lefschetz property (WLP for short). A standard graded $K$-algebra has the WLP, if there exists a linear form in $R$ such that for each $0 \leq i \leq e - 1$, the multiplication map $\times \ell : A_i \rightarrow A_{i+1}$ has maximal rank, i.e., it is injective or surjective. This property, not only depends to algebraic structure of the algebra $A$, but also depends on the characteristic of the base field $K$.

In addition to intrinsic significance of the WLP for a standard graded Artinian $K$-algebra, this property is closely related to some problems in other disciplines of mathematics. For example the presence of this property is related to existence of finite projective planes [3, 8], has connection with some special family of curves in Algebraic Geometry [12], it is associated with the problem of enumerating the plane partitions in combinatorics [4, 13].

Even though, this property has a simple definition, but establishing it for a general standard graded Artinian $K$-algebra, is not an easy task. As a consequences of this fact, classifying all standard graded $K$-algebra which pose this property would be a hard problem. This forces to look for this property in special classes of standard graded Artinian $K$-algebras. Among

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such algebras, those for which $I$ is a monomial ideal, are the most accessible and excellent ones.

When the characteristic of the base field $K$ is zero, Stanley [14] showed that the algebra $K[x_1, \ldots, x_r]/(x_1^{d_1}, \ldots, x_r^{d_r})$, where $d_i s$ are greater than 1, enjoys the SLP, and hence the WLP, but when the characteristic of the base field is positive, this result is no longer hold. It can be easily shown that whenever $\text{char } K = p$, and $r \geq 3$, the algebra $K[x_1, \ldots, x_r]/(x_1^{p^k}, \ldots, x_r^{p^k})$, does not have the WLP, while it poses this property for $r \leq 2$. In [9], it is shown that if the ideal $I = (x_1^{d_1}, \ldots, x_r^{d_r})$, where $d_i s$ are greater than 1, satisfies the condition $d_1 > [t/2]$, where $t = d_1 + d_2 + \cdots + d_r - n$, then $R/I$ has the WLP, regardless of the characteristic of $K$. Moreover, in [2, Proposition 3.5], it is shown that if the char $K = p > 0$ and $d_1 \leq [t/2]$, and if $d_2 \leq p \leq d_1$ or for some positive integer $m$, the condition $d_1 \leq p^m \leq [t/2]$ holds, then $R/I$ fails to have the WLP. In [1], by a geometric method, the WLP of the algebra $K[x, y, z]/(x^d, y^d, z^d)$, in terms of integer $d$ and the characteristics of the field $K$ is determined. In [13], it is proved that if $p = \text{char } K$ is a prime divisor of the number of plane partitions $M(a, b, c)$, then the $K$-algebra $A = K[x, y, z]/(x^{a+b} y^{a+c} z^{b+c})$ does not have the WLP. In [7, Theorem 3.8], a complete classification of the SLP for all monomial ideals $I$, for which $I$ is a complete intersection and $K > 0$, is given. In [4], by tools which have been developed in [11], the WLP of the monomial ideals of the form $I = (x^{1+\alpha} y^{1+\beta} z^{1+\gamma}, x^{\alpha} y^{\beta} z^{\gamma})$, known as almost complete intersection, are investigated. In particular, those characteristics that these type of ideals may fail to have the WLP are determined in terms of the exponents $t, \alpha, \beta, \gamma$.

In [10], the weak Lefschetz property of algebras defined by another class of monomial ideals which are in the form

$$I_{r, k, d} = (x_1^k, \ldots, x_r^k) + (\text{all squarefriemonomilas of degree } d)$$

are studied and the WLP of these type of ideals, whenever $d = 2$ and $d = 3$, are established ([10, Theorem 3.3]), and for general case is stated as the following conjecture (see [10, Conjecture 3.4.])

**Conjecture 1.1.** Consider the algebra $R/I_{r, k, d}$, where the ideal $I_{r, k, d}$, is defined as

$$(x_1^k, \ldots, x_r^k) + (\text{all square free monomials of degree } d).$$

Then

(a) If $d = 4$, then it has the WLP if and only if $k \mod 4$ is 2 or 3.
(b) If $d = 5$, then the WLP fails.
(c) If $d = 6$, then the WLP fails.
The proof of part (b) of [10, Theorem 3.3], and the above conjecture has motivated us to investigate the presence of the WLP for a class of monomial ideals which their generators are nearly similar to the generators of $I_{r,k,d}$. I.e., we consider the ideals of type

$$I = (x_1^{\alpha_1}, \ldots, x_r^{\alpha_r}) + (\text{ all square free monomials of degree } d),$$

where $2 \leq \alpha_i \leq 4$ for $1 \leq i \leq r$, and study the WLP behavior of $R/I$ with respect to small prime numbers.

Our main results are:

**Theorem A.** Let $I$ be as the above ideal, where $2 \leq \alpha_i \leq 3$ for all $d \geq 4$ and $1 \leq i \leq r$. Then $A = R/I$ does not have the WLP, whenever char $\mathbb{K} = 3$.

In particular, if all $\alpha_i$s, are equal to 2, or are equal to 3, then Theorem A, implies the part (a) of the conjecture [1] can not be true, while it confirms what is claimed in parts (b), (c) of [1] for a specific value.

**Theorem B.** Let $I$ be as the ideal, where $2 \leq \alpha_i \leq 4$ for all $d \geq 5$ and $1 \leq i \leq r$. Then $A = R/I$ does not have the WLP, whenever char $\mathbb{K} = 2$.

Another result of this paper is a little bit different from the other results. In fact, we determined all characteristics of the base field which a special type of ideals define an algebra without the WLP.

**Proposition 1.2.** Let

$$I' = (x_1^2, \ldots, x_r^2) + (\text{all squarefree monomials of degree } d)$$

be an ideal in $R$. Then $R/I'$ is a level algebra and $e = \text{SocleDegree}(I') = d - 1$. Moreover, $R/I'$ doesn’t have the WLP in char $\mathbb{K} = p$ whenever $p$ is a prime number less than $i + 2$, where $1 \leq i \leq \lceil r/2 \rceil$.

The method of proof of Theorem A and Theorem B, can be applied to prove the failure of the WLP for the following class of monomial ideals:

$$J = (x_1^{\alpha}, \ldots, x_r^{\alpha}, x_1^{\alpha-2}x_2^2, x_1^{\alpha-2}x_2^2, \ldots, x_r^{\alpha-2}x_r^2, x_r^{\alpha-2}x_r^2),$$

provided $r \geq 4, \alpha \geq 5$.

**Theorem C.** Let $J$ be as the above ideal in $R$. Let char $\mathbb{K} = 2$. Then $R/J$ does not have the WLP.

2. **Preliminaries**

Let $A = R/I = \bigoplus_{i=0}^e A_i$ be a standard graded Artinian $\mathbb{K}$ algebra. Then the function $h_i = \dim A_i$, for $0 \leq i \leq e$ is called the Hilbert function of $A$. Since $\dim A_i = 0$ for $i > e$, this function can be represented as an array $h = (h_0, h_1, \ldots, h_e)$, which is called the $h$-vector of $A$. If we denote the maximal ideal of $A$ by $m$, then the ideal

$$(0 :_A m) = \{ a \in | am = 0 \} = \mathfrak{u}_0 \oplus \mathfrak{u}_1 \oplus \cdots \oplus \mathfrak{u}_e,$$
is called the socle of \( A \). Since this ideal gathers the annihilators of \( \mathfrak{m} \), its structure is closely related to the WLP of the algebra \( A \). Since \( A_{e+1} = 0 \), it is clear that \( A_e \subset (0 :_A \mathfrak{m}) \). The integer \( e \) is called the socle degree of \( A \). Moreover, if \( \mathcal{U}_i = 0 \) for all \( i < e \), then \( A \) is called a level algebra.

**Definition 2.1.** Let \( \ell \) be a general linear form in \( R \). We say that the Artinian ring \( A \) has the weak Lefschetz property (WLP for short) if the homomorphism induced by multiplication by \( \ell \),
\[
\times \ell : A_i \longrightarrow A_{i+1},
\]
has the maximal rank for every \( i, 0 \leq i \leq e - 1 \) (i.e., it is injective or surjective). In this case, the linear form \( \ell \) is called the Lefschetz element for \( A \).

We say that \( A \) has the strong Lefschetz property (SLP) if
\[
\times \ell^d : A_i \longrightarrow A_{i+d}
\]
has the maximal rank for every \( i \) and \( d \), with \( 0 \leq i \leq e - 2 \) and \( 1 \leq d \leq e - 1 \).

If a standard graded \( K \)-algebra \( A \), has a Lefschetz element \( \ell \), then it can be shown that there is a Zariski open set in \( P^{r-1} = P(K[x_1, \ldots, x_r]) \) which parameterizes all Lefschetz elements of \( A \).

In [11, Proposition 2.2], it is shown if the field \( K \) is infinite, and the monomial Artinian \( I \) satisfies the WLP, then the linear form \( \ell = x_1 + \cdots + x_r \) would be a Lefschetz element too. This simplifies the checking for posing or failure of the WLP. Moreover, in [7, Proposition 4.3], the assumption of being infinite for \( K \) has been weaken. This will allows us to use finite fields to construct examples or counterexamples for posing of failure of the WLP.

**Proposition 2.2.** ([7, Proposition 4.3]) Let \( K \) be a field and let \( K' \) be an extension field of \( K \). Let \( I \subset K[x_1, \ldots, x_r] \) be a monomial ideal. Then the following are equivalent.

(a) \( A = K[x_1, \ldots, x_r]/I \) has the WLP.
(b) \( A' = A \otimes_K K' \) has the WLP.
(c) \( x_1 + \cdots + x_r \) is a weak Lefschetz element of \( A \).
(d) \( x_1 + \cdots + x_r \) is a weak Lefschetz element of \( A' \)

### 3. Main Results

In this section, we prove the main results of this paper. The Artinian algebras that we consider are defined by the ideals of the following form.

\[
I = (x_1^{\alpha_1}, \ldots, x_r^{\alpha_r}) + (\text{all square free monomials of degree } d),
\]
where \( 2 \leq \alpha_i \leq 4 \) and \( 2 \leq d \leq r \).
The main idea of the proof of the main results, is to construct a homogeneous polynomial \( f \in A = R/I \), such that for suitable indices \( i \), the map \( \times \ell : A_i \rightarrow A_{i+1} \) fails to be injective.

**Theorem 3.1. (Theorem A.)** Let \( I \) be as above ideal, where \( d \geq 4 \) and \( 2 \leq \alpha_i \leq 3 \) for all \( 1 \leq i \leq r \). If char \( K = 3 \), then \( A \) does not have the WLP.

**Proof.** According to Remark 3.2, \( h_2 \leq h_3 \). We show that the map \( \times \ell : A_2 \rightarrow A_3 \) cannot be injective. To prove our claim, let

\[
f = \sum_{1 \leq i < m \leq r} (-1)^j x_i^j x_m^{2-j}, \quad \text{where } j = 1, 2.
\]

It is clear that \( f \) is a nonzero element of \( A_2 \). Moreover, in \( A_3 \), the element \( f \times \ell \), consists of the following terms:

- for \( j = 1 \), \(-x_i^2 x_m, -x_i x_m x_t \), where \( t \neq i, m \), \( 1 \leq t \leq r \);
- for \( j = 2 \), \( x_i^2 x_m, x_i^2 x_k, x_i^3 \), where \( 1 \leq k < i < m \leq r \).

Notice that for those \( \alpha_i \)'s which are equal to two, the above terms cannot be appeared in \( f \times \ell \). Moreover, as can be seen from the above expressions, for \( j = 1 \) and \( j = 2 \), the first two terms of the first row are respectively, additive inverses of the first two terms of the second row. Hence, their sum in \( f \times \ell \) cancel each other.

Moreover, each term \(-x_i x_m^2 \) in the first row, is the additive inverse of a monomial \( x_i^2 x_k \), in the second row, and vice versa. Hence, their sum in \( f \times \ell \) is zero.

On the other hand, by our assumption, \( x_i^3 \) modulo \( I \) is zero. Hence, the only terms which will be remained in \( f \times \ell \), are in the form \(-x_i x_m x_t \), where \( t \neq i, m \) and \( 1 \leq t \leq r \). We count the occurrences of these monomials in \( f \times \ell \) via two methods.

**First method.** We know that the number of squarefree monomials of degree 3 in \( r \) variables is equal to \( \binom{r}{3} \). We label these monomials as \( a_1, a_2, \ldots, a_{\binom{r}{3}} \) and set \( M = a_1 + a_2 + \cdots + a_{\binom{r}{3}} \).

**Second method.** By definition of \( f \), the terms of \( f \times \ell \) are products of all \( x_i x_m \) and the different variables with \( x_i \) and \( x_m \) that vary in \( \{x_1, \ldots, x_r\} \). The possible number of such monomials is equal to

\[
\binom{r-2}{2} \binom{r}{3} = \frac{r(r-1)(r-2)2}{2}.
\]

We label these monomials as \( b_1, b_2, \ldots, b_{\binom{r}{3}} \).

By comparing the results of these two methods, we observe that the monomials \( x_i x_m x_t \) of \( f \times \ell \), have a coefficient equal to 3 in \( A_3 \), since \( b_1 + b_2 + \cdots + b_{\binom{r}{3}} = 3(a_1 + a_2 + \cdots + a_{\binom{r}{3}}) = 3M \). Therefore, \( f \times \ell = 0 \).
modulo $I$, while $f \neq 0$. This means the kernel of the map $\times \ell : A_2 \to A_3$ is nontrivial. Hence, $A$ does not have the WLP.

**Remark 3.2.** Let $I$ be as the above ideal, where $d \geq 4$ and $2 \leq \alpha_i \leq 3$ for all $1 \leq i \leq r$. The number of monomials of degree $i$ in $R_i$, is equal to $(i^{+}(r-1))$. Hence $h_2 = (2^{+}(r-1)) \leq h_3 = (3^{+}(r-1))$, but we should remove those monomials which are multiples of the generators of $I$. Let $m$ be the number of these degree 2 terms and $n = r - m$ be the number of these degree 3 terms. Then we have

$$h_2 = (2^{+}(r-1)) - m \text{ and } h_3 = (3^{+}(r-1)) - (m.r) - n$$

If $h_2 \leq h_3$, we are nothing to do and it is the case that we need for our argument, but if $h_3 < h_2$, then this would be the exceptional cases which should be avoided. Hence, if we evaluate both sides of this inequality, then $h_2 = (2^{+}(r-1)) - m > h_3 = (3^{+}(r-1)) - (m.r) - n = (3^{+}(r-1)) - (m.r) - (r-m)$, which implies $7r + 6rm > r^3 + 12m$. Note that $m$ and $n$ varies in $\{0, 1, \ldots, r\}$.

Notice that whenever $r$ becomes larger, the above inequality is no longer hold. Therefore, these exceptional cases happen whenever $m = r = 4$.

In [10, Theorem 4.3], it is proved that the Artinian algebra defined by the ideal in the general form

$$I_{r,r} = (x_1^r, \ldots, x_r^r, x_1 x_2 \cdots x_r),$$

fails to have the WLP. As a special case of the above theorem, we can state the following result.

**Corollary 3.3.** Let the ideal $I$ be as in Theorem 3.1. If $d = r$, then the algebra $R/I$ does not have the WLP.

Now we can state a similar result for the case $\text{char } K = 2$.

**Theorem 3.4. (Theorem B.)** Let $I$ be the ideal as in Theorem 3.1. Let $d \geq 5$ and $2 \leq \alpha_i \leq 4$ for all $1 \leq i \leq r$. Then $A$ does not have the WLP, whenever $\text{char } K = 2$.

**Proof.** According to Remark 3.1, $h_3 \leq h_4$. We show that the map $\times \ell : A_3 \to A_4$ can not be injective. To prove this, let $f \in A_3$ be as follow

$$f = \sum_{1 \leq i < m \leq r} x_i^j x_m^{3-j} \text{ where } j = 1, 2, 3.$$

Then terms of the polynomial $f \times \ell$ can be grouped together in terms of the value of $j$ and consists of

- for $j = 1$, $x_i^2 x_m^2$, $x_i x_m^3$, $x_i x_m^2 x_t$; $t \neq i, m, 1 \leq t \leq r$,
- for $j = 2$, $x_i^3 x_m$, $x_i^2 x_m^2$, $x_i x_m^3 x_t$; $t \neq i, m, 1 \leq t \leq r$,
- for $j = 3$, $x_i^4$, $x_i^3 x_m$, $x_i^2 x_k$; $1 \leq k < i < m \leq r$. 


As the above list shows, the terms $x_j^2x_m$ appear in $f \times \ell$ for $j = 1$ and $j = 2$. Hence, the sum of these terms in $f \times \ell$ has coefficient 2, which by our assumption, this sum would become zero. Moreover, the terms $x_j^3x_m$ exist in the rows $j = 2$ and $j = 3$. Hence, the coefficient of their sum is 2. Therefore, these terms would be killed in $f \times \ell$. As well as, each monomial $x_i^2x_m$, in the first row of the above list, is equal to a term $x_i^3x_k$, in the third row and vice versa. Hence, their sum would be zero in $f \times \ell$. On the other hand, $x_i^4$ is zero modulo $I$. Hence, it remains the terms $x_i^2x_m^2x_t$ and $x_i^2x_m^2x_t$ in $f \times \ell$. We count the number of occurrence of these terms in $f \times \ell$ via two ways.

First method. The number of distinct squarefree monomials of degree 3 in terms of $\{x_1, \ldots, x_r\}$ is equal to $\binom{r}{3}$, and since the exponent of one of the variables in each terms is equal to 2, the total number of these type of terms is equal to

$$3 \binom{r}{3} = \frac{r(r-1)(r-2)}{2}.$$  

We label these terms as $a_1, a_2, \ldots, a_3(\binom{r}{3})$, and set $M = a_1 + a_2 + \cdots + a_3(\binom{r}{3})$.

Second method. Each of one of the monomials $x_i^2x_m^2x_t$ and $x_i^2x_m^2x_t$ is obtained by multiplying $x_t \in \{x_1, \ldots, x_r\}$, where $t \neq i, m$, by $x_i^2x_m$ and $x_i^2x_m$ of $f$. Hence, the total number of these summands is equal to $2(r-2)\binom{r}{2} = r(r-1)(r-2)$. We label these monomials as $b_1, b_2, \ldots, b_2(r-2)\binom{r}{2}$.

By comparing the results of these two methods of counting, one can observe that all monomials in the $f \times \ell$, are multiplied by two, since $b_1 + b_2 + \cdots + b_2(r-2)\binom{r}{2} = 2(a_1 + a_2 + \cdots + a_3(\binom{r}{3})) = 2M$. Hence, by our assumption, the sum is equal to 0, and the proof completes.

Remark 3.5. Let $I$ be as the above ideal, where $d \geq 5$ and $2 \leq \alpha_i \leq 4$ for all $1 \leq i \leq r$. Then an argument similar to the one used in the Remark 3.2 shows that $h_3 \leq h_4$.

Due to its importance, we state a special case of the above theorem as a corollary.

Corollary 3.6. Let the ideal $I$ be as in Theorem 3.4. If $d = r$, then the algebra $R/I$ does not have the WLP.

Contrary to the previous results which in characteristics 2 and 3, we showed that the algebras that we considered do not pose the WLP, in the next result, we determine all primes that the Artinian algebra defined a specific ideal may not pose the WLP.

Proposition 3.7. Let

$$I' = (x_1^2, \ldots, x_r^2) \in (all \ squarefree \ monomials \ of \ degree \ d)$$
be an ideal in $R$. Then $R/I'$ is a level algebra and $e = \text{SocleDegree}(I') = d - 1$. Moreover, $R/I'$ doesn't have the WLP in char $\mathbb{K} = p$ whenever $p$ is a prime number less than $i + 2$, where $1 \leq i \leq \lceil r/2 \rceil$.

**Proof.** By our assumption on $I$, each homogeneous component $A_i$ of $A$ is generated by squarefree monomials of degree $i$. Hence $h_i = \binom{r}{i}$. Therefore, $h_0 \leq h_1 \leq \cdots \leq h_{\lceil r/2 \rceil}$.

We show that the map $\times \ell : A_i \rightarrow A_{i+1}$ can not be injective for any $1 \leq i \leq \lceil r/2 \rceil$. Let

$$f = \sum x_{j_1}x_{j_2}\cdots x_{j_i},$$

where $1 \leq j_i \leq r - 1$. According to the definition of $f$, the nonzero terms of $f \times \ell$ are in the form:

$$x_{j_1}x_{j_2}\cdots x_{j_i}x_m$$

where $1 \leq m \leq r$ and $m$ is not in $\{j_1, \ldots, j_i\}$.

Now we want to count the occurrence of these terms in $A_{i+1}$.

*First method.* We know that the number of squarefree monomials of degree $i + 1$ in $R$ is equal to $\binom{r}{i+1}$. We label these terms as $a_1, a_2, \ldots, a_{\binom{r}{i+1}}$, and set $M = a_1 + a_2 + \cdots + a_{\binom{r}{i+1}}$.

*Second method.* By definition of $f$, these terms of $f \times \ell$ are products of all $x_{j_1}x_{j_2}\cdots x_{j_i}$ and the different variables with $x_m$ in $\{x_1, \ldots, x_r\} \setminus \{x_{j_1}, \ldots, x_{j_i}\}$. The number of possible such terms is equal to

$$\binom{r - i}{1} = \frac{(r-1)\cdots(r-i+1)}{i!}.$$ 

We label these terms as $b_1, b_2, \ldots, b_{\binom{r}{r-i}}$.

By comparing the results of these two methods of counting, we observe that the terms $x_{j_1}x_{j_2}\cdots x_{j_ix_m}$ of $f \times \ell$, have coefficient equal to $i + 1$ in $A_{i+1}$, because $b_1 + b_2 + \cdots + b_{\binom{r}{r-i}} = (i + 1)(a_1 + a_2 + \cdots + a_{\binom{r}{i+1}}) = (i + 1)M$.

Therefore, while $f \neq 0$, $f \times \ell = 0$ modulo $I$, whenever $p$ is a prime divisor of $i + 1$. Hence, $A$ does not have the WLP in these cases. \hfill \Box

The following corollary, not only shows that the WLP may fail for only finitely many prime numbers, but also it determines these primes exactly.

**Corollary 3.8.** Let $I'$ be an ideal as in Proposition 3.7. Then $R/I'$ has the WLP whenever char $\mathbb{K} = p$ is not a prime number less than $i + 2$, where $1 \leq i \leq \lceil r/2 \rceil$.

**Proof.** Two cases may arise. In the first case, which $i$ varies in the range $1 \leq i \leq d - 1 < \lceil r/2 \rceil$, we only need to prove the injectivity of the map
×ℓ : Ai → Ai+1 for every i. But this follows from Proposition 3.7, since all maps \( \times\ell : A_i \rightarrow A_{i+1} \), are injective.

In the other case, i varies in the range \([r/2] \leq i \leq d - 1\), and it is enough to prove that all maps \( \times\ell : A_i \rightarrow A_{i+1} \), are surjective. By Proposition 2.1 it is enough to do it for \( i = [r/2] \). Let \( x_j x_j \cdots x_{j+1} \) be an element of \( A_{i+1} \). Then it is clear that, it is the image of \( f = \sum_{m=1}^i x_j \cdots x_{jm-1} \) under the multiplication map by \( \ell \).

With a method similar to the proofs of Theorems 3.1 and 3.4, we can prove the failure of the WLP for another special class of monomial ideals.

**Theorem 3.9.** (Theorem C.) Let \( J \) be the following ideal of \( R \) for which \( r \geq 4 \) and \( \alpha \geq 5 \).

\[
J = (x_1^\alpha, x_r^\alpha, x_1^\alpha x_2^\alpha, x_2^\alpha x_2^\alpha, \ldots, x_{r-1}^\alpha x_r^\alpha, x_r^\alpha) .
\]

If \( \text{char } \mathbb{K} = 2 \), then \( R/J \) does not have the WLP.

**Proof.** According to Remark 3.10, \( h_{\alpha-1} \leq h_{\alpha} \). We show that the map \( \times\ell : A_{\alpha-1} \rightarrow A_{\alpha} \) is not injective. Let \( f \in A_{\alpha-1} \) be as follow

\[
f = \sum_{1 \leq i < m \leq r} x_i^{\alpha-j} x_m^{j-1} \text{ where } j = 1, 2, \alpha - 1.
\]

Then terms of \( f \times \ell \), with respect to different values of \( j \), can be grouped together as follows.

for \( j = 1 \), \( x_i^\alpha, x_i^{\alpha-1} x_m, x_k x_i^{\alpha-1} \), where \( 1 \leq k < i < m \leq r \);
for \( j = 2 \), \( x_i^{\alpha-1} x_m, x_i^{\alpha-2} x_m, x_i^{\alpha-2} x_m x_t, \) where \( t \neq i, m, 1 \leq t \leq r \);
for \( j = \alpha - 1 \), \( x_i x_m^{\alpha-1}, x_i^2 x_m^{\alpha-2}, x_i x_m^{\alpha-2} x_t, \) where \( t \neq i, m, 1 \leq t \leq r \).

The monomials \( x_i^\alpha, x_i^2 x_m^{\alpha-2}, \) and \( x_i^{\alpha-2} x_m^2 \) are zero modulo \( I \). As the above list shows, the monomials \( x_i^{\alpha-1} x_m \) appear for \( j = 1 \) and \( j = 2 \). Hence, the sum of these monomials in \( f \times \ell \) has a coefficient equal to 2, which by our assumption, this sum would become zero. As well as, whenever \( j = \alpha - 1 \), each monomial \( x_i x_m^{\alpha-1} \), where \( 1 \leq i < j \leq r \), is equal to a monomial \( x_k x_i^{\alpha-1} \) with \( 1 \leq k < i < m \leq r \) and \( j = \alpha - 1 \), and vice versa. Hence, their sum in \( f \times \ell \) is a multiple of 2. Finally, for \( j = 2 \) and \( j = \alpha - 1 \) the monomials \( x_i^{\alpha-2} x_m x_t \) and \( x_i x_m^{\alpha-2} x_t \) remain in \( f \times \ell \). We count the number of occurrence of these monomials in \( f \times \ell \) in two different ways.

**First method.** The number of distinct monomials in three distinct variables of \( \{x_1, \ldots, x_r\} \) is equal to \( \binom{r}{3} \), and since the exponent of only one of its variables is equal to \( \alpha - 2 \), hence the total number of these elements are equal to

\[
3 \binom{r}{3} = \frac{r(r-1)(r-2)}{2}.
\]
We label these monomials as $a_1, a_2, \ldots, a_{3\binom{r}{2}}$ and set $M = a_1 + a_2 + \cdots + a_{3\binom{r}{2}}$.

**Second method.** Each of one of the monomials $x_i^{\alpha-2}x_mx_t$ and $x_i x_m^{\alpha-2}x_t$ are obtained by multiplying $x_t \in \{x_1, \ldots, x_r\}$, where $t \neq i, m$, to terms $x_i^{\alpha-2}x_m$ and $x_i x_m^{\alpha-2}$ of $f$. Hence, the number of these monomials is equal to $2(r-2)\binom{r}{2} = r(r-1)(r-2)$. We label these terms as $b_1, b_2, \ldots, b_{2(r-2)\binom{r}{2}}$.

By comparing the results of these two methods of counting, one can observe that all monomials in $f \times \ell$, have a coefficient equal to two, since $b_1 + b_2 + \cdots + b_{2(r-2)\binom{r}{2}} = 2(a_1 + a_2 + \cdots + a_{3\binom{r}{2}}) = 2M$. Therefore, $f \times \ell = 0$, and the weak Lefschetz property does not hold for this algebra. \[\Box\]

**Remark 3.10.** Let $J$ be as the above ideal and $A = \bigoplus_{i=0}^{e} A_i$ with $h_i = \dim A_i$. In the above argument, we need to have $h_{\alpha-1} \leq h_{\alpha}$. This inequality does not always hold. In fact, from the structures of $A_{\alpha-1}$ and $A_{\alpha}$, we can deduce

$$h_{\alpha-1} = \binom{\alpha + r - 2}{r - 1} \leq h_{\alpha} = \binom{\alpha + r - 1}{r - 1} - r - 2 \binom{r}{2},$$

where for calculating $h_{\alpha}$, the total number of $x_i^\alpha s$, which is equal to $r$, and the total number of monomials in the forms $x_i^{\alpha-2}x_j^2$ and $x_i^2x_j^{\alpha-2}$, which is equal to $2\binom{r}{2}$, should be subtracted. The above inequality implies $r^2 \leq \frac{(\alpha+r-2)!}{(r-2)!\alpha!}$. But this inequality holds if $r \geq 4$ and $\alpha \geq 5$.

According to the method of proofs of the above theorems, we are able to determine many examples of monomial Artinian algebras without the WLP. On the other hand, it is possible to specify some classes of them with the WLP.

**Example 3.11.** Consider the ideal

$$I = (x_1^4, x_2^4, x_3^3, x_4^3, x_5, x_1x_2x_3x_4x_5),$$

in $\mathbb{K}[x_1, \ldots, x_5]$. The $h$-vector of $R/I$ is $(1, 5, 14, 28, 43, 52, 49, 35, 18, 6, 1)$. Then by Theorem 3.1, $A_3 \rightarrow A_4$ is not injective. Therefore, it doesn’t have the WLP whenever the characteristic of $\mathbb{K}$ is two.

**Example 3.12.** Let $I = (x_1^5, x_2^5, x_3^5, x_4^5, x_5^5, x_1^3x_2^2, x_1^2x_2^3, x_1 x_2^4, x_1^3x_2^3, x_1^2x_3^3)$ be an ideal in $\mathbb{K}[x_1, x_2, x_3, x_4]$. Its $h$ vector is $(1, 4, 10, 20, 35, 40, 26, 8, 1)$. Then by Theorem 3.9, $A_4 \rightarrow A_5$ is not injective, so it doesn’t have the WLP whenever the characteristic of $\mathbb{K}$ is two.

**References**

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