INDIFFERENCE PRICING OF PURE ENDOWMENTS VIA BSDES UNDER PARTIAL INFORMATION

CLAUDIA CECI, KATIA COLANERI, AND ALESSANDRA CRETAROLA

Abstract. In this paper we investigate the pricing problem of a pure endowment contract when the insurer has limited information on the mortality intensity of the policyholder. The payoff of this kind of policies depends on the residual lifetime of the insured as well as the trend of a portfolio traded in the financial market, where investments in a riskless asset, a risky asset and a longevity bond are allowed. We propose a modeling framework that takes into account mutual dependence between the financial and the insurance markets via an observable stochastic process, which affects the risky asset and the mortality index dynamics. Since the market is incomplete due to the presence of basis risk, in alternative to arbitrage pricing we use expected utility maximization under exponential preferences as evaluation approach, which leads to the so-called indifference price. Under partial information this methodology requires filtering techniques that can reduce the original control problem to an equivalent problem in complete information. Using stochastic dynamics techniques, we characterize the indifference price of the insurance derivative via the solutions of suitable backward stochastic differential equations.

Keywords: Pure endowment; partial information; backward stochastic differential equations; indifference pricing.

JEL Classification: C02; G12; G22.

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1. Introduction

A unit-linked life insurance contract is a long term insurance policy between a policyholder and an insurance company. These kinds of contract are hybrid financial products embodying banking, securities and insurance components. Indeed, the payoff depends on the insured remaining lifetime (insurance risk) and on the performance of the underlying stock or portfolio (financial risk). In this paper we focus on a pure endowment policy, which promises to pay an agreed amount if the policyholder is still alive on a specified future date.

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In these contracts benefits are random. This makes for instance, traditional valuation principles for pricing life insurance products with deterministic payoffs, inappropriate. Since the 70’s, it was clear that theory of financial valuation, suitably combined with mortality, was the right way forward, see, e.g. Brennan and Schwartz [12], Boyle and Schwartz [9], Aase and Persson [1]. In these papers the Black & Scholes pricing methodology is applied under the hypotheses of market completeness and independence between financial and insurance setting. Since then, many efforts have been done to relax the assumption of completeness and several approaches have been proposed, for instance in Møller [38], Ludkovski and Young [37], Bayraktar et al. [2], Delong [22], Blanchet-Scalliet et al. [8]. However the problem of incorporating some kind of dependence between the financial and the insurance market, which is empirically observed, has started to be addressed only recently.

The goal of this paper is to study the pricing problem of a pure endowment life insurance contract in a general modeling framework that takes into account *mutual dependence* between the financial and the insurance markets and *partial information* of the insurance company on the mortality intensity of the policyholder. Precisely, we consider a discounted financial market with a riskless asset, a risky asset and a longevity bond written on the mortality index of the same age cohort of population of the policyholder. We assume that the dynamics of the risky asset and the mortality index are governed by diffusion processes with coefficients that depend on the same *observable* stochastic process representing economic and environmental factors. The insurance company issues a pure endowment policy with maturity of $T$ years for an individual whose remaining lifetime is represented by a random time. The partial information scenario refers to the situation where the insurance company knows at any time if the policyholder is still alive but cannot directly observe her/his mortality intensity, which is influenced by an exogenous unobservable factor describing the social/health status of the individual.

The insurance contract can be treated as a contingent claim in the hybrid market model given by the financial securities and the insurance portfolio, and the goal is to study the pricing problem for the insurance company. In our setting mortality intensity of the population and of the policyholder do not coincide in general. This translates into the presence of a basis risk that, even in the context of complete information, does not permit perfect replication of the contract via self-financing strategies. In other words, the insurance company cannot perfectly hedge its exposure by investing in a hedging instrument (the longevity bond), which is based on the mortality of the whole population, rather than that of the insured, leaving a residual amount of risk. See e.g. Biagini et al. [5] for a deeper discussion on this issue. Therefore, in alternative to arbitrage pricing we use expected utility maximization under exponential preferences as evaluation approach, which leads to the so-called *indifference price* (see e.g. Henderson and Hobson [27] for a survey). This is a well-known technique for pricing derivatives in incomplete markets and has been successfully applied to value insurance derivatives under full information in e.g. Becherer [3], Ludkovski and Young [37], Delong [21], Eichler et al. [23], Liang and Lu [33]. However, to the best of our knowledge, applications of this methodology to insurance derivatives under partial information is an open problem.

In this paper we apply the indifference pricing methodology to price a pure endowment contract.
under partial information in an extended financial market where investment in a longevity bond is also possible. We formulate the problem for the insurance company that issues the contract in terms of two stochastic control problems, and we solve them by using a the backward stochastic differential equation (in short BSDE) approach. The partial information setting requires to first to apply filtering theory (see Appendix A) to transform the optimization problem into an equivalent problem in complete information involving only observable processes. Our main result is a characterization of the indifference price in terms of unique solution of BSDEs (with either quadratic-exponential or quadratic driver). To the best of our knowledge, this is the first time that the problem of evaluating an insurance claim via indifference pricing when there are restrictions on the available information in a general setting with mutual dependence between the financial and the insurance framework, is investigated.

The outline of the paper is as follows. Section 2 introduces the mathematical framework and describes the combined model allowing for a mutual dependence between the financial and the insurance markets and a limited information on the mortality intensity of the policyholders. The pricing problem formulation under partial information via utility indifference pricing can be found in Section 3. In Section 4 we study the resulting stochastic control problems following a BSDE approach and characterize the log-value process corresponding to the problem with (without, respectively) the pure endowment contract in terms of the solution to a quadratic-exponential (quadratic, respectively) BSDE. Finally, a characterization of the indifference price of the pure endowment policy is given in Section 5. We address the filtering problem in Appendix A. How to compute the longevity bond price process is shown in Appendix B. Technical results and proofs can be found in Appendix C.

2. Modeling framework

We consider the problem of an insurance company that issues a unit-linked life insurance contract. This type of contract has a relevant link with the financial market. Indeed, the value of the policy is determined by the performance of the underlying stock or portfolio. Moreover, it also depends on the remaining lifetime of the policyholder. Therefore, we construct a combined financial-insurance market model and treat the life insurance policy as a contingent claim. We will define the suitable modeling framework via the progressive enlargement of filtration approach, which allows for possible dependence between the financial market and the insurance portfolio.

We start by fixing a complete probability space $(\Omega, \mathcal{F}, P)$ endowed with a complete and right continuous filtration $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$, where $T > 0$ is a fixed and finite time horizon, such that $\mathcal{F} = \mathcal{F}_T$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$.

On this filtered probability space we consider a process $Z = \{Z_t, t \in [0,T]\}$ with cádlág trajectories and values in some set $\mathcal{Z}$ which will not be observable by the insurance company and denote by $\mathbb{F}^Z = \{\mathcal{F}^Z_t, t \in [0,T]\}$, with $\mathcal{F}^Z_t := \sigma\{Z_u, 0 \leq u \leq t\}$, for each $t \in [0,T]$, the natural filtration of $Z$. We may interpret the process $Z$ as an environmental process describing the social level/health status of an individual to be insured. We assume that the probability space supports three
\textbf{P}-independent standard $\mathbb{F}$-Brownian motions $W^j = \{W^j_t, \ t \in [0, T]\}$, with $W^j_0 = 0$, for each $j = 1, 2, 3$, which are also $\textbf{P}$-independent of the stochastic factor $Z$. Here, $W^j$, for $j = 1, 2, 3$, are supposed to drive the underlying financial market (see Subsection 2.2) and the mortality intensity defined on the same age cohort of the population, see (2.2). Now, denote by $\mathbb{F}^{W^j} = \{\mathcal{F}^j_t, \ t \in [0, T]\}$, with $\mathcal{F}^j_t := \sigma\{W^j_u, 0 \leq u \leq t\}$, $j = 1, 2, 3$, for every $t \in [0, T]$, the canonical filtrations of $W^j$, for every $j = 1, 2, 3$, respectively. In addition, set

$$\tilde{\mathcal{F}}_t := \mathcal{F}^{W^1}_t \vee \mathcal{F}^{W^2}_t \vee \mathcal{F}^{W^3}_t, \quad t \in [0, T]$$

(2.1)

and $\tilde{\mathbb{F}} = \{\tilde{\mathcal{F}}_t, \ t \in [0, T]\}$. We assume that the reference filtration $\mathbb{F}$ is given by

$$\mathbb{F} = \tilde{\mathbb{F}} \vee \mathbb{F}^Z,$$

completed by $\textbf{P}$-null sets, so that, it contains all knowledge of the financial-insurance market except for the information regarding the policyholder survival time.

\textbf{2.1. Construction of the death time and mortality intensities}. We consider an individual aged $l$ at time 0 to be insured. Let $\mu = \{\mu_t, \ t \in [0, T]\}$ be an $\mathbb{F}$-adapted process modeling the mortality intensity of an equivalent age cohort of the population. This process is observable and can be computed using publicly available data of the survivor index $S^\mu = \{S^\mu_t, \ t \in [0, T]\}$ given by

$$S^\mu_t := \exp\left(-\int_0^t \mu_s ds\right), \quad t \in [0, T].$$

We assume that $\mu$ evolves according to the following stochastic differential equation:

$$d\mu_t = b^\mu(t, \mu_t, Y_t)dt + \sigma^\mu(t, \mu_t, Y_t)dW^2_t, \quad \mu_0 \in \mathbb{R}^+,$$

(2.2)

where $Y = \{Y_t, \ t \in [0, T]\}$ is an observable stochastic process representing economic and environmental factors, satisfying

$$dY_t = b^Y(t, Y_t)dt + \sigma^Y(t, Y_t)dW^3_t, \quad Y_0 = y_0 \in \mathbb{R}.$$

(2.3)

Here, functions $b^\mu : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$, $b^Y : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$, $\sigma^\mu : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}^+$, and $\sigma^Y : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}^+$ are measurable and such that the system of equations (2.2)-(2.3) admits a unique strong solution, with $\mu_t \geq 0$, $\textbf{P}$-a.s., for all $t \in [0, T]$, see e.g. Øksendal [39].

\textbf{Remark 2.1}. As an example we consider a generalized Cox-Ingersoll-Ross model to represent the trend of the mortality intensity $\mu$, see , e.g. Dahl [20], Biffis [7]. In Biffis [7], mortality intensity of the sample population follows an affine dynamics with stochastic drift, given by

$$d\mu_t = a^\mu(Y_t - \mu_t) dt + \sigma^\mu \sqrt{\mu_t}dW^2_t, \quad \mu_0 \in \mathbb{R}^+,$$

where $a^\mu$, $\sigma^\mu \in \mathbb{R}^+$ and the process $Y$ satisfies

$$dY_t = a^Y(b^Y(t) - Y_t)dt + \sigma^Y \sqrt{Y_t - b^*(t)}dW^3_t, \quad Y_0 = y_0 \in \mathbb{R}^+,$$
for some nonnegative, bounded and continuous functions \( b^Y, b^* : [0, T] \rightarrow \mathbb{R}^+ \) and \( \sigma^Y \in \mathbb{R}^+ \). It is known that this model describes well mortality intensity and it is quite flexible to capture stylized features, such as fluctuations around a target mean (given here by \( b^Y \)), and \( \mathbb{P}\text{-a.s.} \) positivity of the intensity process \( \mu \) which is satisfied for instance when \( b^Y(t) \geq b^*(t) \) for every \( t \in [0, T] \) and \( y_0 \geq b^*(0) \). See Biffis [7] for a deeper discussion.

To describe the stochastic residual lifetime of the individual, we adopt the canonical construction of a random time in terms of a given hazard process, in analogy to reduced-form credit risk models. For this reason, we shall postulate that the underlying filtered probability space \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) is sufficiently rich to support a random variable \( \Theta \) having unit exponential distribution, \( \mathbb{P}\)-independent of \( \mathcal{F}_T \). Let \( \lambda : [0, T] \times \mathbb{R}^+ \times \mathcal{Z} \rightarrow (0, +\infty) \) be a positive function such that \( \mathbb{E} \left[ \int_0^T \lambda(s, \mu_s, Z_s) ds \right] < \infty \) and define the random time \( \tau : \Omega \rightarrow \mathbb{R}^+ \) by setting

\[
\tau := \inf \left\{ t \geq 0 : \int_0^t \lambda(s, \mu_s, Z_s) ds \geq \Theta \right\}.
\]

In this framework, \( \tau \) represents the remaining lifetime of an individual and \( \lambda \) is the \( \mathbb{F}\text{-mortality intensity} \) process. The associated \( \mathbb{F}\)-hazard process is given by \( \left\{ \int_0^t \lambda(s, \mu_s, Z_s) ds, t \in [0, T] \right\} \). Note that by the \( \mathbb{P}\)-independence assumption on \( \Theta \) and the \( \mathcal{F}_t\)-measurability of \( \int_0^t \lambda(s, \mu_s, Z_s) ds \), we get that

\[
\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P} \left( \int_0^t \lambda(s, \mu_s, Z_s) ds < \Theta | \mathcal{F}_t \right) = e^{-\int_0^t \lambda(s, \mu_s, Z_s) ds}, \quad t \in [0, T],
\]

and the following property of the canonical construction of the remaining lifetime \( \tau \) holds

\[
\mathbb{P}(\tau \leq t|\mathcal{F}_t) = \mathbb{P}(\tau \leq t|\mathcal{F}_T), \quad t \in [0, T],
\]

see, for instance, Bielecki and Rutkowski [6, Section 8.2.1].

**Remark 2.2.** It is intuitively clear that, in general the mortality rate of the insured \( \lambda \) is different from that of its age cohort, \( \mu \). In our model \( \lambda \) is a function of \( \mu \) as well as the unobservable process \( Z \). A possible choice could be

\[
\lambda(t, \mu_t, Z_t) = \mu_t \tilde{\lambda}(Z_t), \quad t \in [0, T],
\]

where \( \tilde{\lambda} \) is a strictly positive function of the environmental factor \( Z \), meaning that when \( \tilde{\lambda}(z) < 1 \) the risk of the policyholder is smaller than that of the reference population, and bigger if \( \tilde{\lambda}(z) > 1 \).

Since the random time \( \tau \) is not a stopping time with respect to filtration \( \mathbb{F} \), we introduce an enlarged filtration that makes \( \tau \) a stopping time. First, we define the death indicator process \( H = \{ H_t, t \in [0, T] \} \) associated to \( \tau \) as follows

\[
H_t := 1_{(\tau \leq t)}, \quad t \in [0, T],
\]

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and set \( F^H_t := \sigma \{ H_u, 0 \leq u \leq t \} \), for every \( t \in [0, T] \). Let \( G = \{ G_t, t \in [0, T] \} \) be the enlarged filtration given by

\[
G_t := F_t \lor F^H_t, \quad t \in [0, T].
\]

Then, \( G \) is the smallest filtration which contains \( F \) and such that \( \tau \) is a \( G \)-stopping time. Filtration \( G \) plays the role of the market full information: it contains all the knowledge about the insurance and the market portfolio.

As an immediate consequence of the canonical construction of the residual lifetime \( \tau \), we get that the so-called martingale invariance property between filtrations \( F \) and \( G \) holds, i.e. every \((F, P)\)-(local) martingale is also a \((G, P)\)-(local) martingale, see Brémaud and Yor [11]. Moreover, the process \( \{ H_t - \int_0^{t \wedge \tau} \lambda(s, \mu_s, Z_s)ds, \ t \in [0, T] \} \) is a \((G, P)\)-martingale and \( \tau \) is a totally inaccessible \( G \)-stopping time.

2.2. The combined financial-insurance market model. We define a combined financial-insurance market on the filtered probability space \((\Omega, G, G, P)\), with \( G = G_T \), where the tradeable securities are given by a riskless asset, a risky asset and a longevity bond. We assume that the price process of a risk free asset is equal to 1 at any time, and that the risky asset has discounted price process \( S^1 = \{ S^1_t, \ t \in [0, T] \} \) given by the following geometric diffusion with coefficients affected by the economic and environmental factor \( Y \)

\[
dS^1_t = S^1_t \left( \mu^S(t, Y_t)dt + \sigma^S(t, Y_t)dW^1_t \right), \quad S^1_0 = s^1_0 \in \mathbb{R}^+.
\]

The longevity bond has discounted price process \( S^2 = \{ S^2_t, \ t \in [0, T] \} \) satisfying the following stochastic differential equation with coefficients depending on the equivalent age cohort mortality intensity \( \mu \) and the stochastic factor \( Y \)

\[
dS^2_t = S^2_t \left( \mu^B(t, \mu_t, Y_t)dt + c^B(t, \mu_t, Y_t)dW^2_t + d^B(t, \mu_t, Y_t)dW^3_t \right), \quad S^2_0 = s^2_0 \in \mathbb{R}^+.
\]

Here \( \mu^S : [0, T] \times \mathbb{R} \to \mathbb{R}, \sigma^S : [0, T] \times \mathbb{R} \to \mathbb{R}^+, \mu^B : [0, T] \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}, c^B : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \) and \( d^B : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \) are measurable functions such that the system of equations (2.2)-(2.3)-(2.4)-(2.5) admits a unique strong solution. The motivation for the dynamics of the longevity bond price process is given in Appendix B.

**Remark 2.3.** It is clear that the proposed modeling framework allows for mutual dependence between the financial and the insurance markets via the stochastic factor \( Y \), which affects all stochastic processes dynamics given in (2.2), (2.4) and (2.5).

Throughout the rest of the paper, we work under the following integrability conditions.
Assumption 2.4.
\[
\int_{0}^{T} \left\{ \mu^S(t, Y_t)^2 + \sigma^S(t, Y_t)^2 \right\} \, dt < \infty, \quad P - a.s.,
\]
\[
\int_{0}^{T} \left\{ \mu^B(t, \mu_t, Y_t)^2 + c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2 \right\} \, dt < \infty, \quad P - a.s.,
\]
\[
\int_{0}^{T} \left\{ \left( \frac{\mu^S(t, Y_t)}{\sigma^S(t, Y_t)} \right)^2 + \frac{\mu^B(t, \mu_t, Y_t)}{c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2} \right\} \, dt < \infty, \quad P - a.s.
\]

Note that the discounted asset price processes \( S^1 \) and \( S^2 \) are continuous \((\mathbb{F}, P)\)-semimartingales and also \((\mathbb{G}, P)\)-semimartingales. As a consequence the underlying financial-insurance market model is arbitrage-free.

2.3. Available information and filtering. We assume that the insurance company observes prices of the assets negotiated on the markets, \( S^1 \) and \( S^2 \) (since the processes \( \mu \) and \( Y \) are directly observable) and the death time of the insured \( \tau \), but it has not full information about the intensity mortality intensity \( \lambda \), which depends on \( Z \). Therefore, the available information to the insurer is given by \( \mathbb{G} = \{ \mathbb{G}_t, \ t \in [0, T] \} \) where
\[
\mathbb{G}_t := \mathbb{F}_t \vee \mathbb{F}^H_t, \quad t \in [0, T],
\]
(2.6)
where \( \mathbb{F} \) is defined in (2.1). Note that
\[
\mathbb{G} \subseteq \mathbb{G} = \mathbb{F} \vee \mathbb{F}^H = \mathbb{F} \vee \mathbb{F}^Z \vee \mathbb{F}^H,
\]
and we refer to \( \mathbb{G} \) as the available information to the insurance company. We assume throughout the paper that all filtrations satisfy the usual hypotheses of completeness and right-continuity.

If \( Z \) is a Markov process, the intensity of the mortality process \( H \) with respect the information flow can be characterized via a filtering approach. To this, we define the filter process by setting
\[
\pi_t(f) := \mathbb{E} \left[ f(Z_t) \bigg| \mathbb{G}_t \right], \quad t \in [0, T],
\]
for every measurable function \( f \) such that \( \mathbb{E} [|f(Z_t)|] < \infty \), for each \( t \in [0, T] \). It is known that \( \pi(f) \) is a probability measure-valued process with cádlág trajectories (see Kurtz and Ocone [32]), which provides the conditional law of \( Z \) given the information flow \( \mathbb{G} \). Then, the \( \mathbb{G} \)-predictable intensity of \( H \) is given by
\[
(1 - H_{t-})\pi_{t-}(\lambda), \quad t \in [0, T],
\]
(2.7)
where \( \pi_{t-} \) denotes the left version of \( \pi_t \) and \( \pi_t(\lambda) \) is short for \( \pi_t(\lambda(t, \mu_t, \cdot)) \). This means that the compensated process \( M^\tau = \{ M^\tau_t, \ t \in [0, T] \} \) defined as
\[
M^\tau_t := H_t - \int_{0}^{t \wedge \tau} \pi_s(\lambda) \, ds = H_t - \int_{0}^{t} (1 - H_{s-})\pi_{s-}(\lambda) \, ds, \quad t \in [0, T],
\]
(2.8)
turns out to be a \((\mathbb{G}, P)\)-martingale. The filtering problem is discussed in Appendix A for a general Markov process \( Z \).
We suppose that the insurance company issues a unit-linked life insurance policy. This is a long term insurance contract between the policyholder and insurance company whose benefits are linked to financial assets. In particular, we consider a pure endowment contract with maturity of $T$ years, which can be defined as follows.

**Definition 3.1.** A pure endowment contract with maturity $T$ is a life insurance policy where the sum insured is paid at time $T$ if the insured is still alive. The associated final value is given by the random variable

$$G_T := \xi I_{\{\tau > T\}},$$

where $\xi \in L^2(\tilde{\mathcal{F}}_T, \mathbb{P})$ represents the payoff of a European-type contingent claim with maturity $T$.

The goal of this paper is to price a pure endowment policy with payoff given by (3.1) in a partially observable setting, where the insurance company does not have access to the full information given by the filtration $\mathcal{G}$. In particular, the insurer is not allowed to observe the evolution of the stochastic factor $Z$, which therefore implies that his/her decisions are based on the observation filtration $\tilde{\mathcal{G}}$. Moreover, we recall that our general setting accounts for possible mutual dependence between the financial and the insurance framework, which is a desirable characteristic when dealing with mortality derivatives. Indeed, nowadays it is commonly recognized that, in the long term, demographic changes may affect the economy and vice-versa. Unit-linked life insurance contracts have previously been studied under partial information in Ceci et al. [18, 19], where the goal was to solve the hedging problem in an incomplete market via local risk-minimization. Precisely, in Ceci et al. [18] the independence between financial market and insurance portfolio was assumed, while in Ceci et al. [19] the authors considered a more general situation where mutual dependence between financial and insurance context is allowed.

Insurance-financial market models are typically incomplete due to the fact that mortality events are in general not hedgeable. This implies that insurance contracts may have different risk-neutral prices. One of the criteria that can be used to compute a fair price corresponds to identify insurer’s preferences towards the risk via a specific utility function and maximize the expected utility whether she/he holds the insurance claim or not. In other words, to characterize the utility indifference price, which is the price $p$ that makes the insurer indifferent between not selling the policy and selling the policy at price $p$ now and paying the benefits at maturity.

In this paper, we follow a utility indifference pricing approach assuming that the insurance company is endowed with an exponential utility function of the form

$$U(x) = -e^{-\alpha x}, \quad x \in \mathbb{R},$$

where $\alpha > 0$ is a given constant, representing a coefficient of absolute risk aversion. This form of the utility function is frequently assumed in the literature and allows for more explicit computations of the price. In the sequel, we also assume that

$$\mathbb{E}[e^{\alpha G_T}] < \infty.$$
Suppose that the insurer has initial wealth $x$ and she/he invests this amount in the money market account, in the risky asset and in the longevity bond, following a self-financing strategy.

Set $S = (S^1, S^2)^\top$ and let $\theta = (\theta_1^1, \theta_2) = \{ (\theta_1^1, \theta_1^2)^\top, t \in [0, T] \}$ be the amount of wealth invested in the risky asset and in the longevity bond respectively. Given an initial wealth $x_0 \in \mathbb{R}^+$ the portfolio value $X^\theta = \{ X_t^\theta, t \in [0, T] \}$ satisfies
\begin{equation}
\begin{aligned}
dX_t^\theta &= \frac{\theta_1^1 S^1(t, Y_t)}{S_t} dt + \theta_1^1 \sigma^S(t, Y_t) dW^1_t + \theta_1^2 \left[ c^B(t, \mu_t, Y_t) dW^2_t + d^B(t, \mu_t, Y_t) dW^3_t \right], \\
S_t &= (\theta_1^1 S^1(t, Y_t) + \theta_1^2 \mu^B(t, \mu_t, Y_t)) dt + \theta_1^1 \sigma^S(t, Y_t) dW^1_t + \theta_1^2 \left[ c^B(t, \mu_t, Y_t) dW^2_t + d^B(t, \mu_t, Y_t) dW^3_t \right],
\end{aligned}
\end{equation}
with $X_0^\theta = x_0 \in \mathbb{R}^+$.

In the case where she/he sells the insurance contract the information at her/his disposal is given by the filtration $\tilde{G}$ defined in (2.6), whereas in the case of pure investment by the filtration $\tilde{F}$ given in (2.1). The set of admissible strategies is defined below.

**Definition 3.2.** An admissible strategy is a self-financing portfolio identified by an $\tilde{F}$-predictable (or even $\tilde{G}$-predictable), $\mathbb{R}^2$-valued process $\theta = (\theta_1^1, \theta_2)^\top$ such that
\begin{align*}
\int_0^T \left\{ \left( \theta_1^1 \sigma^S(t, Y_t) \right)^2 + |\theta_1^1 \mu^S(t, Y_t)| \right\} dt < \infty, & \quad \mathbb{P} \text{-a.s.}, \\
\int_0^T \left\{ \left( \theta_1^2 \right)^2 \left[ c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2 \right] + |\theta_1^2 \mu^B(t, \mu_t, Y_t)| \right\} dt < \infty, & \quad \mathbb{P} \text{-a.s.}
\end{align*}
and
\begin{equation}
\mathbb{E} \left[ \sup_{s \in [t, T]} e^{-\alpha s} X_s^\theta \right] < \infty, \tag{3.3}
\end{equation}
for any $t \in [0, T]$ and $p > 1$.

We denote by $\mathcal{A}(\tilde{F})$ and $\mathcal{A}(\tilde{G})$, the set of admissible $\tilde{F}$-predictable and $\tilde{G}$-predictable strategies, respectively.

Note that, condition (3.3) ensures the validity of the Bellman optimality principle, see Lim and Quenez [34, Section 4]. A sufficient condition for (3.3) is
\begin{equation*}
X_t^\theta - X_s^\theta \geq -K, \quad s \leq t \leq T, \quad \mathbb{P} \text{-a.s.,}
\end{equation*}
for some positive constant $K$.

In order to characterize the indifference price, we introduce the optimal investment problems with and without the insurance derivative. First, suppose that, at time $t$, the insurer sells a pure endowment contract with payoff given by (3.1). Then, the goal of the insurer is the following.

**Problem 3.3.** To maximize the expected utility of her/his terminal wealth, i.e. to solve
\begin{equation*}
\sup_{\theta \in \mathcal{A}(\tilde{G})} \mathbb{E} \left[ -e^{-\alpha (X_T^\theta - G_T)} \right].
\end{equation*}
For every \((t,x) \in [0,T] \times \mathbb{R}^+\), the value process in a dynamic framework is given by
\[
\tilde{V}_t(x) := \essinf_{\theta \in \mathcal{A}(\mathcal{G})} \mathbb{E} \left[ e^{-\alpha (x + \int_t^T \theta_u \, dS_u - G_T)} \big| \mathcal{G}_t \right] = e^{-\alpha X_t^G},
\]
where \(\mathcal{A}(\mathcal{G})\) denotes the class of admissible \(\mathcal{G}\)-predictable controls on the interval \([t,T]\) and the process \(V^G = \{V^G_t, \ t \in [0,T]\}\) is given by
\[
V^G_t := \essinf_{\theta \in \mathcal{A}(\mathcal{G})} \mathbb{E} \left[ e^{-\alpha (\int_t^T \theta_u \, dS_u - G_T)} \big| \mathcal{G}_t \right], \ t \in [0,T].
\]

Hence, the solution to Problem 3.3 is given by
\[
\sup_{\theta \in \mathcal{A}(\bar{\mathcal{G}})} \mathbb{E} \left[ -e^{-\alpha (X_T^G - G_T)} \right] = -e^{-\alpha x_0} V^G_0.
\]

**Remark 3.4.** Since \(\theta = (\theta^1, \theta^2)^T = (0,0)^T\) is an admissible strategy we get that for every \(t \in [0,T]\), \(V^G_t \leq \mathbb{E} \left[ e^{\alpha G_T} \big| \mathcal{G}_t \right] \) \(\mathbb{P}\)-a.s. which implies that \(\sup_{t \in [0,T]} \mathbb{E} \left[ V^G_t \right] \leq \mathbb{E} \left[ e^{\alpha G_T} \right]\). Clearly, \(V^G_t \geq 0 \) \(\mathbb{P}\)-a.s. and if there exists an optimal strategy, then \(V^G_t > 0 \) \(\mathbb{P}\)-a.s..

**Remark 3.5.** We remark here that, since \(G_T = \mathbf{1}_{\{\tau > T\}}\) and the mortality intensity \(\lambda(t, \mu_t, Z_t^-) (1 - H_t^-)\) is not observable by the insurance company, we are dealing with a utility maximization problem in a partial information framework. The idea is to consider an equivalent control problem under full information where the unobservable intensity of \(\tau\) is replaced by its filtered estimate (see equation (2.7)).

Now, we consider the case where the insurer simply invests her/his wealth in the market, without writing the insurance derivative. Then, the objective is the following.

**Problem 3.6.** To maximize the expected utility of his/her terminal wealth, i.e. to solve
\[
\sup_{\theta \in \mathcal{A}(\bar{\mathcal{G}})} \mathbb{E} \left[ -e^{-\alpha X_T^G} \right].
\]

For every \((t,x) \in [0,T] \times \mathbb{R}^+\), the associated value process is given by
\[
\tilde{V}_t^0(x) := \essinf_{\theta \in \mathcal{A}(\mathcal{G})} \mathbb{E} \left[ e^{-\alpha (x + \int_t^T \theta_u \, dS_u)} \big| \mathcal{F}_t \right] = e^{-\alpha x} V^0_t,
\]
where \(\mathcal{A}(\mathcal{G})\) is the set of admissible \(\mathcal{G}\)-predictable controls on the interval \([t,T]\) and \(V^0 = \{V^0_t, t \in [0,T]\}\) is defined as
\[
V^0_t := \essinf_{\theta \in \mathcal{A}(\mathcal{G})} \mathbb{E} \left[ e^{-\alpha \int_t^T \theta_u \, dS_u} \big| \mathcal{F}_t \right] \ t \in [0,T].
\]

**Definition 3.7.** The utility indifference price or reservation price \(p^0\) of the insurer related to the pure endowment contract is defined at any time \(t \in [0,T]\) as the \(\mathcal{G}\)-adapted process implicit solution to the equation
\[
\tilde{V}_t(x + p^0_t) = \tilde{V}_t^0(x).
\]
This means that starting at time $t$ with capital $x$, the insurer has the same maximal utility from selling the insurance product for $p_t^\alpha$ at time $t$ and solely trading on $(t, T]$ without writing the contract.

If $V_t^G > 0$ and $V_t^0 > 0$, $\mathbb{P}$-a.s., for every $t \in [0, T]$, we get that $p^\alpha$ does not depend on the initial capital $x$ and it is given by

$$p_t^\alpha := \frac{1}{\alpha} \log \left( \frac{V_t^G}{V_t^0} \right), \quad t \in [0, T].$$

Note that, since $G_T = 0$ if $\tau < T$, then

$$V_t^G 1_{\{\tau \leq t\}} = V_t^0 1_{\{\tau \leq t\}} \quad t \in [0, T].$$

Therefore, the indifference price $p^\alpha$ is given by

$$p_t^\alpha = \frac{1}{\alpha} \left( \log(V_t^G) - \log(V_t^0) \right) 1_{\{\tau > t\}} \quad t \in [0, T], \quad (3.6)$$

provided that $V_t^G$, $V_t^0 > 0$ on $\{\tau > t\}$.

To solve Problems 3.3 and 3.6 we follow a direct method based on the Bellman optimality principle (see, for instance, El Karoui [24]).

**Proposition 3.8** (Bellman optimality principle). The following properties hold true:

(i) The process $V_t^G$ is the largest $\tilde{\mathcal{G}}$-adapted process such that $\{e^{-\alpha X_t^\theta} V_t^G, \ t \in [0, T]\}$ is a $(\tilde{\mathcal{G}}, \mathbb{P})$-submartingale for any strategy $\theta \in \mathcal{A}(\tilde{\mathcal{G}})$ with $V_T^G = e^{\alpha G_T}$.

(ii) The strategy $\theta^* \in \mathcal{A}(\tilde{\mathcal{G}})$ is optimal for Problem 3.3 if and only if the process $\{e^{-\alpha X_t^{\theta^*}} V_t^G, \ t \in [0, T]\}$ is a $(\tilde{\mathcal{G}}, \mathbb{P})$-martingale.

**Proof.** See Lim and Quenez [34, Proposition 4.1].

The same result holds for Problem 3.6 by replacing $V_t^G$ with $V_t^0$, $\tilde{\mathcal{G}}$ with $\tilde{\mathcal{F}}$ and $V_T^G = e^{\alpha G_T}$ with $V_T^0 = 1$.

4. Optimization problems via BSDEs

The goal of this section is to characterize dynamically the value processes $V_t^G$ and $V_t^0$ given in (3.4) and (3.5) respectively, and corresponding to the stochastic control problems with and without the insurance derivative, by using a BSDE-based approach. The BSDE method works well in non-Markovian settings where the classical stochastic control approach based on the Hamilton-Jacobi-Bellman equation does not apply. Several papers (see, e.g. El Karoui et al. [25], Ceci and Gerardi [17], Lim and Quenez [35] and references therein) deal with stochastic optimization problems in Finance by means of BSDEs. Moreover, this approach is also well suited to solve stochastic control problems under partial information in presence of an infinite-dimensional filter process, which is the
situation in our paper, see e.g. Ceci [15, 14], where partially observed power utility maximization problems in financial market with jumps are solved by applying this approach.

First, we define some spaces that are used throughout the sequel:

- $L^2(W; \tilde{G})$ (respectively $L^2_{loc}(W; \tilde{G})$) is the set of $\mathbb{R}$-valued $\tilde{G}$-predictable processes $u = \{u_t, \ t \in [0,T]\}$ such that
  \[
  \mathbb{E}\left[\int_0^T |u_s|^2 ds\right] < \infty \quad \text{(respectively } \int_0^T |u_s|^2 ds < \infty \ \mathbb{P} \text{-a.s.}) \quad (4.1)
  \]
  Moreover, $L^2(W; \tilde{F})$ (respectively $L^2_{loc}(W; \tilde{F})$) is the set of $\mathbb{R}$-valued $\tilde{F}$-predictable processes $u = \{u_t, \ t \in [0,T]\}$ satisfying (4.1).

- $L^p(M^\tau)$ (respectively $L^p_{loc}(M^\tau)$) for $p = 1, 2$ is the set of all $\mathbb{R}$-valued $\tilde{G}$-predictable processes $\eta = \{\eta_t, \ t \in [0,T]\}$ such that
  \[
  \mathbb{E}\left[\int_0^T |\eta_s|^p (1 - H_s) \pi_s(\lambda) ds\right] < \infty
  \]
  \[
  \quad \text{(respectively } \int_0^T |\eta_s|^p (1 - H_s) \pi_s(\lambda) ds < \infty, \ \mathbb{P} \text{-a.s.}) \quad .
  \]

4.1. The problem with life insurance liabilities. We consider the value process $V^G$ in (3.4) and notice that it turns out to be a $(\tilde{G}, \mathbb{P})$-submartingale. This is a consequence of Proposition 3.8 (i), and the fact that the strategy $\theta = (\theta^1, \theta^2) = (0, 0)$ is admissible with associated wealth $X_t^{(0,0)} = x_0$, for each $t \in [0, T)$. Then, $V^G$ admits a unique Doob-Meyer decomposition

\[
\text{d}V^G_t = \text{d}m^Y_t + \text{d}A_t, \quad (4.2)
\]

where $m^V = \{m^V_t, \ t \in [0,T]\}$ is a $(\tilde{G}, \mathbb{P})$-local martingale and $A = \{A_t, \ t \in [0,T]\}$ is an increasing $\tilde{G}$-predictable process with $A_0 = 0$. By a representation result for $(\tilde{G}, \mathbb{P})$-local martingales (see Proposition C.1 in Appendix C), we get that

\[
\text{d}m^Y_t = R^1_t \text{d}W^1_t + R^2_t \text{d}W^2_t + R^3_t \text{d}W^3_t + R^4_t \text{d}M^\tau_t, \quad (4.3)
\]

where $R^1, R^2, R^3 \in L^2_{loc}(W; \tilde{G})$, $R^4 \in L^1_{loc}(M^\tau)$ and $M^\tau$ is given in (2.8).

The process $A$ is specified in Theorem 4.1 below, where the value process $V^G$ is characterized as the solution of a suitable BSDE. The proof of the theorem is postponed to Appendix C.

**Theorem 4.1.** If there exists an optimal strategy for Problem 3.3, the quintuplet of processes $(V^G, R^1, R^2, R^3, R^4)$ is a solution of the following BSDE

\[
V^G_t = e^{G_T} - \int_t^T R^1_s \text{d}W^1_s - \int_t^T R^2_s \text{d}W^2_s - \int_t^T R^3_s \text{d}W^3_s - \int_t^T R^4_s \text{d}M^\tau_s - \int_t^T \text{ess sup}_{\theta \in A(\tilde{G})} f_\alpha(s, R^1_s, R^2_s, R^3_s, V^G_s, \theta^1_s, \theta^2_s) ds, \quad (4.4)
\]
where
\[
\begin{align*}
f_\alpha(t, r^1, r^2, r^3, v, \theta^1, \theta^2) \\
= & \alpha \cdot v \left[ \theta^1_t \mu^S(t, Y_t) + \theta^2_t \mu^B(t, \mu, Y_t) \right] \\
& + \alpha \left[ r^1 \theta^1_t \sigma^S(t, Y_t) + r^2 \theta^2_t c^B(t, \mu, Y_t) + r^3 \theta^2_t d^B(t, \mu, Y_t) \right] \\
& \quad - \frac{1}{2} \alpha^2 \cdot v \left[ (\theta^1_t \sigma^S(t, Y_t))^2 + (\theta^2_t)^2 \left( c^B(t, \mu, Y_t)^2 + d^B(t, \mu, Y_t)^2 \right) \right],
\end{align*}
\]

the processes \( R^1, R^2, R^3, R^4 \) are the integrand appearing in (4.3) and \( \sigma^S, c^B \) and \( d^B \) are the functions introduced in (2.4) and (2.5). Moreover, the optimal strategy realizes the essential supremum in (4.4). If the control \( \theta^* = (\theta^{1*}, \theta^{2*})^\top \), given by
\[
\begin{align*}
\theta^{1*}_t &= \frac{\mu^S(t, Y_t)}{\alpha \sigma^S(t, Y_t)} + \frac{R^1_t}{\alpha V^G_t \sigma^S(t, Y_t)}, \\
\theta^{2*}_t &= \frac{\mu^B(t, \mu, Y_t)}{\alpha c^B(t, \mu, Y_t)^2 + d^B(t, \mu, Y_t)^2} + \frac{c^B(t, \mu, Y_t) R^2_t + d^B(t, \mu, Y_t) R^3_t}{\alpha V^G_t c^B(t, \mu, Y_t)^2 + d^B(t, \mu, Y_t)^2},
\end{align*}
\]

for every \( t \in [0, T] \), belongs to the class \( \mathcal{A}(\tilde{\mathbb{G}}) \), then it is an optimal strategy and
\[
\begin{align*}
f_\alpha(t, R^1_t, R^2_t, R^3_t, V^G_t, \theta^{1*}_t, \theta^{2*}_t) &= f(t, R^1_t, R^2_t, R^3_t, V^G_t) \\
&= \frac{1}{2} \frac{(V^G_t \mu^S(t, Y_t) + R^1_t \sigma^S(t, Y_t))^2}{V^G_t \sigma^S(t, Y_t)^2} \\
&\quad + \frac{1}{2} \frac{(V^G_t \mu^B(t, \mu, Y_t) + c^B(t, \mu, Y_t) R^2_t + d^B(t, \mu, Y_t) R^3_t)^2}{V^G_t c^B(t, \mu, Y_t)^2 + d^B(t, \mu, Y_t)^2}.
\end{align*}
\]

**Remark 4.2.** An existence and uniqueness result for the solution to equation \( (4.4) \) is proved, for instance, in Lim and Quenez \[34\] for the case where coefficients are bounded and strategies are valued in a compact set. Precisely, existence and uniqueness hold when the driver is Lipschitz, uniformly in \( \omega \).

From now on we make the following boundedness assumption on the payoff of the insurance policy.

**Assumption 4.3.** The random variable \( \xi \) in (3.1) is bounded, that is,
\[
|\xi| \leq k \quad \mathbb{P} - a.s.
\]

with \( k \) positive constant.

By Remark 3.4 and Assumption 4.3 we get that \( 0 \leq V^G_t \leq e^{\alpha k} \), for each \( t \in [0, T] \).

The next result is a verification theorem, which provides an explicit optimal strategy by means of the solution to the associated BSDE. The proof is provided in Appendix C.
Theorem 4.4 (Verification theorem). Let \((U^G, \gamma^1, \gamma^2, \gamma^3, \gamma^4)\), with \(U^G\) nonnegative and bounded, \(\gamma^i \in L^2(W; \mathcal{G}), i = 1, 2, 3, 4 \in L^1(M^r)\), be solution to the BSDE
\[
U_t^G = e^{\alpha G T} - \int_t^T \lambda_t^1 dW_t^1 - \int_t^T \lambda_t^2 dW_t^2 - \int_t^T \lambda_t^3 dW_t^3 - \int_t^T \lambda_t^4 dM_t^\gamma,
\]
where
\[
f(t, r^1, r^2, r^3, r^4) = \frac{1}{2} \frac{(v \mu^S(t, Y_t) + r^1 \sigma^S(t, Y_t))^2}{v \sigma^S(t, Y_t)^2}
+ \frac{1}{2} \frac{(\mu^B(t, \mu_t, Y_t) + c^B(t, \mu_t, Y_t)r^2 + d^B(t, \mu_t, Y_t)r^3)^2}{v[c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2]}.
\]
Then, \(U_t^G = V_t^G\), for every \(t \in [0, T]\), \(\mathbb{P}\)-a.s. Moreover, if \(\theta^* = (\theta_1^*, \theta_2^*)^\top \in \mathcal{A}(\mathcal{G})\) where
\[
\theta_1^* = \frac{\mu^S(t, Y_t)}{\alpha \sigma^S(t, Y_t)^2} + \frac{1}{\alpha \sigma^S(t, Y_t)^2} \lambda_t^1 U_t^G,
\]
\[
\theta_2^* = \frac{\mu^B(t, \mu_t, Y_t)}{\alpha [c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2]} + \frac{c^B(t, \mu_t, Y_t)\gamma^1_t + d^B(t, \mu_t, Y_t)\gamma^3_t}{\alpha U_t^G[c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2]}
\]
for every \(t \in [0, T]\) then, \(\theta^*\) is an optimal strategy.

In view of equation (3.6), in the sequel we will characterize the log-value process \(\log(V^G)\), provided that \(V^G > 0\) on \([0, T]\), in terms of a BSDE with quadratic-exponential driver, as shown in the next verification result.

Proposition 4.5. Let \((\tilde{U}^G, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3, \tilde{\gamma}^4)\), with \(\tilde{U}^G\) bounded, \(\tilde{\gamma}^i \in L^2(W; \mathcal{G}), i = 1, 2, 3, 4 \in L^1(M^r)\), be solution to the BSDE
\[
\tilde{U}_t^G = \alpha G T - \int_t^T \sum_{i=1}^3 \tilde{\gamma}_i^i dW_t^i - \int_t^T \tilde{\gamma}_s^i dM_t^\gamma - \int_t^T \tilde{f}(s, \tilde{\gamma}^1_s, \tilde{\gamma}^2_s, \tilde{\gamma}^3_s, \tilde{\gamma}^4_s) ds,
\]
where
\[
\tilde{f}(t, \tilde{\gamma}^1_t, \tilde{\gamma}^2_t, \tilde{\gamma}^3_t, \tilde{\gamma}^4_t) = -(e^{\tilde{\gamma}^4_t} - \tilde{\gamma}^4_t - 1)(1 - H_t)\pi_t(\lambda) + \frac{1}{2}((\tilde{\gamma}^1_t)^2 + (\tilde{\gamma}^2_t)^2 + (\tilde{\gamma}^3_t)^2)
+ \frac{1}{2} \left(\frac{\mu^S(t, Y_t)}{\sigma^S(t, Y_t)} + \tilde{\gamma}^1_t\right)^2 + \frac{1}{2} \frac{(\mu^B(t, \mu_t, Y_t) + c^B(t, \mu_t, Y_t)\tilde{\gamma}^2_t + d^B(t, \mu_t, Y_t)\tilde{\gamma}^3_t)^2}{c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2},
\]
for every \(t \in [0, T]\). Then, \(V_t^G = e^{\tilde{U}_t^G}\), for every \(t \in [0, T]\), \(\mathbb{P}\)-a.s. and if \(\theta^* = (\theta_1^*, \theta_2^*)^\top \in \mathcal{A}(\mathcal{G})\), with
\[
\theta_1^* = \frac{\mu^S(t, Y_t)}{\alpha \sigma^S(t, Y_t)^2} + \frac{\tilde{\gamma}^1_t}{\alpha \sigma^S(t, Y_t)^2}, \quad t \in [0, T],
\]
\[
\theta_2^* = \frac{\mu^B(t, \mu_t, Y_t) + c^B(t, \mu_t, Y_t)\tilde{\gamma}^2_t + d^B(t, \mu_t, Y_t)\tilde{\gamma}^3_t}{\alpha [c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2]}, \quad t \in [0, T],
\]
then, $\theta^*$ is an optimal strategy.

Proof. By Itô’s formula and equation (4.11), we get that

$$e^{\sigma G_t} - \tilde{e}^{\sigma G}_{t^*} = \int_t^T e^{\tilde{G}_s} \gamma_1^2 dW_s + \int_t^T e^{-\tilde{G}_s} \gamma_3^2 dW_s + \int_t^T e^{-\tilde{G}_s} \gamma_4^3 dW_s$$

$$+ \int_t^T \frac{1}{2} \gamma_1^2 \left( \tilde{G}_s \frac{\sigma^2(s,\mu_s, Y_s)}{\sigma^2(s,\mu_s, Y_s)} \right)^2 ds + \frac{1}{2} \int_t^T \gamma_1^2 \left( \tilde{G}_s \frac{\sigma^2(s,\mu_s, Y_s)}{\sigma^2(s,\mu_s, Y_s)} \right)^2 ds$$

$$+ \gamma_3^2 \int_t^T e^{\tilde{G}_s} \left( \tilde{G}_s - 1 \right) ds.$$

Now, we set for every $t \in [0,T]$,

$$U_t^G = e^{\tilde{G}_t}, \quad \gamma_1^1 = e^{\tilde{G}_t}, \quad \gamma_2^2 = e^{\tilde{G}_t}, \quad \gamma_3^3 = e^{\tilde{G}_t}, \quad \gamma_4^4 = e^{\tilde{G}_t} \left( \tilde{G}_t - 1 \right).$$

Then, the vector process $(U^G, \gamma_1^1, \gamma_2^2, \gamma_3^3, \gamma_4^4)$ solves BSDE (4.9) with $U^G$ nonnegative and bounded, $\gamma_i^i \in L^2(W; \mathbb{G}), i = 1, 2, 3, \text{ and } \gamma_4^4 \in L^1(M^\tau)$. Finally, by Theorem 4.4 it follows that $V_t^G = e^{\tilde{G}_t}$, for every $t \in [0,T], \text{ P-a.s.}$.

Remark 4.6. Note that $\theta^* = (\theta_1^{1*}, \theta_2^{2*})^\top$ defined by (4.12) satisfies the integrability condition $\int_0^T \left\{ (\theta_1^{1*})^2 + (\theta_2^{2*})^2 \right\} \sigma^2(t, Y_t) ds + (\theta_2^{2*})^2 \sigma^2(t, Y_t) ds < \infty, \text{ P-a.s.}$ Indeed, by Assumption 2.4, we get $\int_0^T (\theta_2^{2*})^2 \sigma^2(s, Y_s)^2 ds < \infty, \text{ P-a.s.},$ since $\tilde{G}_1 \in L^2(W; \mathbb{G})$ and

$$\int_0^T (\theta_2^{2*})^2 \sigma^2(s, Y_s)^2 ds = \int_0^T (\theta_2^{2*})^2 \sigma^2(s, Y_s)^2 ds$$

$$\leq \frac{2}{\alpha^2} \int_0^T \left( \frac{\mu^B(s, \mu_s, Y_s)^2}{\sigma^2(s, \mu_s, Y_s)^2} + \frac{c^B(s, \mu_s, Y_s)^2 + d^B(s, \mu_s, Y_s)^2}{\sigma^2(s, \mu_s, Y_s)^2} \right) ds$$

$$\leq \frac{4}{\alpha^2} \left\{ \int_0^T \mu^B(s, \mu_s, Y_s)^2 ds + \int_0^T \frac{c^B(s, \mu_s, Y_s)^2 + d^B(s, \mu_s, Y_s)^2}{\sigma^2(s, \mu_s, Y_s)^2} \gamma_2^2 ds \right\}$$

$$+ \int_0^T \frac{c^B(s, \mu_s, Y_s)^2 + d^B(s, \mu_s, Y_s)^2}{\sigma^2(s, \mu_s, Y_s)^2} (\gamma_3^3)^2 ds$$

$$\leq \frac{4}{\alpha^2} \left\{ \int_0^T \mu^B(s, \mu_s, Y_s)^2 ds + \int_0^T (\gamma_2^2)^2 ds + \int_0^T (\gamma_3^3)^2 ds \right\} < \infty, \text{ P-a.s.,}$$

where the last inequality holds since $\gamma_2^2, \gamma_3^3 \in L^2(W; \mathbb{G})$. Moreover, thanks to Assumption 2.4 and since $\tilde{G}_1, \gamma_2^2, \gamma_3^3 \in L^2(W; \mathbb{G})$, it is easy to check that the integrability condition $\int_0^T \left\{ (\theta_1^{1*})^2 \sigma(t, Y_t)^2 + (\theta_2^{2*})^2 \sigma(t, Y_t)^2 \right\} dt < \infty, \text{ P-a.s.,}$ is satisfied.

Remark 4.7. To ensure that $\theta^* = (\theta_1^{1*}, \theta_2^{2*})^\top \in A(\mathbb{G})$, one should also show that condition (3.3) holds. In a very general setting, this is not an easy task. However, this can be verified in some special cases. For instance, in a simpler financial-insurance market model where the risky asset
price process and the longevity bond price process are not affected by the stochastic factor \(Y\), and thus the financial and the insurance markets are independent, BSDE (4.11) reduces to

\[
\tilde{U}_t^G = \alpha G_T - \int_t^T \sum_{i=1}^2 \tilde{\gamma}_s^i dW_s^i - \int_t^T \tilde{\gamma}_s^i dM_s^i + \int_t^T g(s, \tilde{\gamma}_s^1, \tilde{\gamma}_s^2, \tilde{\gamma}_s^4) ds,
\]

where

\[
g(t, \tilde{\gamma}_t^1, \tilde{\gamma}_t^2, \tilde{\gamma}_t^4) = (e^{\tilde{\gamma}_t^1} - \tilde{\gamma}_t^4 - 1)(1 - H_t)\pi_t(\lambda)
\]

\[
+ \frac{1}{2} \left\{ \left( \frac{\mu^S(t)}{\sigma^S(t)} \right)^2 + \left( \frac{\mu^B(t, \mu_t)}{\sigma^B(t, \mu_t)} \right)^2 + 2\frac{\mu^S(t)}{\sigma^S(t)} + 2\frac{\mu^B(t, \mu_t)}{\sigma^B(t, \mu_t)} \tilde{\gamma}_t^2 \right\}.
\]

For this kind of equation, under the assumption that the function \(\lambda\) is bounded, existence and uniqueness of the solution \((\tilde{U}_t^G, \tilde{\gamma}_t^1, \tilde{\gamma}_t^2, \tilde{\gamma}_t^4)\) with \(\tilde{U}_t^G\) bounded, \(\tilde{\gamma}_i \in L^2(W; \tilde{G}), i = 1, 2, \tilde{\gamma}_4 \in L^1(M^t)\) is proved in Becherer [4, Theorem 3.5]. In this particular case, by Proposition 4.5 we get that

\[
\theta^* = (\theta^*_1, \theta^*_2) = \left( \frac{\mu^S(t)}{\alpha\sigma^S(t)^2} + \frac{\tilde{\gamma}_t^1}{\alpha\sigma^S(t)^2}, \frac{\mu^B(t, \mu_t)}{\alpha\sigma^B(t, \mu_t)^2} + \frac{\tilde{\gamma}_t^2}{\alpha\sigma^B(t, \mu_t)^2} \right)\]

for every \(t \in [0, T]\), belongs to \(\mathcal{A}(\tilde{G})\) and then it is an optimal investment strategy.

### 4.2. The pure investment problem.

First, note that Problem 3.6 corresponds to a special case of Problem 3.3, choosing \(G_T = 0\) and available information level given by \(\tilde{F}\). Therefore, we solve Problem 3.6 by applying similar techniques to those given in the previous subsection. Similarly to process \(V^G\), we observe that the value process \(V^0\) in (3.5) is an \((\tilde{F}, \mathbf{P})\)-submartingale, and therefore it admits a unique Doob-Meyer decomposition

\[
dV_t^0 = dm_t^0 + dA_t^0,
\]

where \(m_t = \{m_t, t \in [0, T]\}\) is a \((\tilde{F}, \mathbf{P})\)-local martingale with representation

\[
dm_t^0 = \Psi_t^1 dW_t^1 + \Psi_t^2 dW_t^2 + \Psi_t^3 dW_t^3,
\]

for \(\Psi^1, \Psi^2, \Psi^3 \in L^2_{loc}(\tilde{W}; \tilde{F}), \) and \(A^0 = \{A_t, t \in [0, T]\}\) is an increasing \(\tilde{F}\)-predictable process with \(A_0 = 0\). The following result holds for the value process \(V^0\).

**Theorem 4.8.** If there exists an optimal strategy for Problem 3.6, the quadruplet of processes \((V^0, \Psi^1, \Psi^2, \Psi^3)\) is a solution of the following BSDE

\[
V_t^0 = 1 - \int_t^T \sum_{i=1}^3 \Psi_s^i dW_s^i - \int_t^T \text{ess sup}_{\theta \in \mathcal{A}(\tilde{F})} f_0^0(s, \Psi_s^1, \Psi_s^2, \Psi_s^3, V_s^0, \theta_s^1, \theta_s^2) ds,
\]

where
where

\[
f^0_{\alpha}(t, \psi^1, \psi^2, \psi^3, v, \theta^1, \theta^2) = \alpha v \left[ \theta^1_0 \mu^S(t, Y_t) + \theta^1_0 \mu^B(t, \mu_t, Y_t) \right] \\
+ \alpha \left[ \psi^1 \theta^1_0 \sigma^S(t, Y_t) + \psi^2 \theta^1_0 \sigma^S(t, Y_t) + \psi^3 \theta^1_0 \sigma^S(t, Y_t) \right] \\
- \frac{1}{2} \alpha v \left[ (\theta^1_0 \sigma^S(t, Y_t))^2 + (\theta^2_0 \sigma^S(t, Y_t))^2 + (\theta^3_0 \sigma^S(t, Y_t))^2 \right],
\]

the processes $\Psi^1$, $\Psi^2$, $\Psi^3$ are the integrand appearing in (4.14) and $\sigma^S$, $c^B$ and $d^B$ are the functions introduced in (2.4) and (2.5).

Moreover, the optimal strategy realizes the essential supremum in (4.15). If the control $\theta^* = (\vartheta^{1,*}, \vartheta^{2,*})^\top$, given by

\[
\vartheta^{1,*}_t = \frac{\mu^S(t, Y_t)}{\alpha \sigma^S(t, Y_t)^2} + \frac{\Psi^1_t}{\alpha V^0_{\tau} \sigma^S(t, Y_t)^2}, \\
\vartheta^{2,*}_t = \frac{\mu^B(t, \mu_t, Y_t)}{\alpha [c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2]} + \frac{c^B(t, \mu_t, Y_t) \Psi^2_t + d^B(t, \mu_t, Y_t) \Psi^3_t}{\alpha V^0_{\tau} [c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2]},
\]

for every $t \in [0, T]$, belongs to the class $A(\mathbb{F})$, then it is an optimal strategy and

\[
f^0_{\alpha}(t, \Psi^1_t, \Psi^2_t, \Psi^3_t, V^0_t, \vartheta^{1,*}_t, \vartheta^{2,*}_t) = f^0(t, \Psi^1_t, \Psi^2_t, \Psi^3_t, V^0_t)
\]

\[
= \frac{1}{2} \left( V^0_{\tau} \mu^S(t, Y_t) + \Psi^1_t \sigma^S(t, Y_t) \right)^2 \\
\frac{1}{2} \left( V^0_{\tau} \mu^B(t, \mu_t, Y_t) + c^B(t, \mu_t, Y_t) \Psi^2_t + d^B(t, \mu_t, Y_t) \Psi^3_t \right)^2.
\]

The proof follows the same lines as Theorem 4.1.

We now provide a verification theorem, analogous to Theorem 4.4.

**Theorem 4.9 (Verification theorem).** Let $(U^0, \phi^1, \phi^2, \phi^3)$, with $U^0$ nonnegative and bounded, $\phi^i \in L^2(W; \mathbb{F})$, $i = 1, 2, 3$, be solution to the BSDE

\[
U^0_t = 1 - \int_t^T \sum_{i=1}^3 \phi^i_s dW^i_s - \int_t^T f^0(s, \phi^1_s, \phi^2_s, \phi^3_s, U^0_s) ds,
\]

where

\[
f^0(t, \psi^1, \psi^2, \psi^3, v) = \frac{1}{2} \left( v \mu^S(t, Y_t) + \psi^1 \sigma^S(t, Y_t) \right)^2 \\
+ \frac{1}{2} \left( v \mu^B(t, \mu_t, Y_t) + c^B(t, \mu_t, Y_t) \psi^2 + d^B(t, \mu_t, Y_t) \psi^3 \right)^2.
\]
Then, \( U^0_t = V^0_t \), for every \( t \in [0, T] \), \( P \)-a.s.. Moreover, if \( \vartheta^* = (\vartheta^1_t, \vartheta^2_t)^\top \in \mathcal{A}(\mathbb{F}) \) where

\[
\vartheta^1_t = \frac{\mu^S(t, Y_t)}{\alpha \sigma_S(t, Y_t)^2} + \frac{1}{\alpha \sigma_S(t, Y_t)^2} \frac{\phi^1_t}{U^0_t}, \\
\vartheta^2_t = \frac{\mu^B(t, \mu_t, Y_t)}{\alpha [\sigma^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2]} + \frac{c^B(t, \mu_t, Y_t) \phi^2_t + d^B(t, \mu_t, Y_t) \phi^3_t}{\alpha U^0_t [\sigma^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2]},
\]

for every \( t \in [0, T] \), then \( \vartheta^* \) is an optimal strategy.

Note that here the optimal strategy in (4.16)-(4.17) is given in feedback form. Similarly to Subsection 4.1, we look for a solution to the equation of the form \( U^0_t = e^\tilde{U}^0_t \). This allows us to formulate the following verification result for the log-value process \( \log(V^0) \).

**Proposition 4.10.** Let \( (\tilde{U}^0, \tilde{\varphi}^1, \tilde{\varphi}^2, \tilde{\varphi}^3) \), with \( \tilde{U}^0 \) bounded, \( \tilde{\varphi}^j \in L^2(W; \mathbb{F}), i = 1, 2, 3 \), be a solution to the BSDE

\[
\tilde{U}^0_t = -\int_t^T \sum_{i=1}^3 \tilde{\varphi}^i_s dW^i_s - \int_t^T \frac{1}{2} \tilde{f}^0(s, \tilde{\phi}^1_s, \tilde{\varphi}^2_s, \tilde{\varphi}^3_s) ds,
\]

where

\[
\tilde{f}^0(t, \tilde{\psi}^1, \tilde{\psi}^2, \tilde{\psi}^3) = - \left( (\tilde{\psi}^1)^2 + (\tilde{\psi}^2)^2 + (\tilde{\psi}^3)^2 \right) + \left( \frac{\mu^S(t, Y_t)}{\sigma^S(t, Y_t)} + \tilde{\psi}^1 \right)^2
\]

\[
+ \left( \frac{\mu^B(t, \mu_t, Y_t) + c^B(t, \mu_t, Y_t) \tilde{\psi}^2 + d^B(t, \mu_t, Y_t) \tilde{\psi}^3}{(c^B(t, \mu_t, Y_t)^2 + (d^B(t, \mu_t, Y_t))^2)} \right).
\]

Then, \( V^0_t = e^{\tilde{U}^0_t} \), for every \( t \in [0, T] \), \( P \)-a.s. and if \( \vartheta^* = (\vartheta^1_t, \vartheta^2_t)^\top \in \mathcal{A}(\mathbb{F}) \), with

\[
\vartheta^1_t = \frac{\mu^S(t, Y_t)}{\alpha \sigma_S(t, Y_t)^2} + \tilde{\phi}^1_t, \\
\vartheta^2_t = \frac{\mu^B(t, \mu_t, Y_t) + c^B(t, \mu_t, Y_t) \tilde{\psi}^2 + d^B(t, \mu_t, Y_t) \tilde{\psi}^3}{\alpha [(c^B(t, \mu_t, Y_t))^2 + (d^B(t, \mu_t, Y_t))^2]},
\]

for every \( t \in [0, T] \), then \( \vartheta^* \) is an optimal strategy.

**Proof.** By applying Itô’s formula we get that

\[
1 - e^{\tilde{U}^0_t} = \int_t^T e^{\tilde{U}^0_s} \sum_{i=1}^3 \tilde{\varphi}^i_s dW^i_s + \frac{1}{2} \int_t^T e^{\tilde{U}^0_s} \left( \frac{\sigma^S(s, Y_s)}{\sigma^S(s, Y_s)} + \tilde{\varphi}^1_s \right)^2 ds
\]

\[
+ \frac{1}{2} \int_t^T e^{\tilde{U}^0_s} \left( \frac{\sigma^B(s, \mu_s, Y_s) + c^B(s, \mu_s, Y_s) \tilde{\psi}^2_s + d^B(s, \mu_s, Y_s) \tilde{\psi}^3_s}{(c^B(s, \mu_s, Y_s))^2 + (d^B(s, \mu_s, Y_s))^2} \right) ds.
\]

Then we set

\[
U^0_t = e^{\tilde{U}^0_t}, \quad \tilde{\varphi}^1_t = e^{\tilde{U}^0_t} \tilde{\phi}^1_t, \quad \tilde{\varphi}^2_t = e^{\tilde{U}^0_t} \tilde{\psi}^2_t, \quad \tilde{\varphi}^3_t = e^{\tilde{U}^0_t} \tilde{\psi}^3_t,
\]

for every \( t \in [0, T] \). The remainder of the proof follows the same lines as Proposition 4.5. \( \square \)
Lemma 5.2. Further details can be found in Appendix A.

on the time interval for some constants Assumption 5.1. with the solution of BSDE (5.1) given in Lemma 5.2 below.

According to Kharroubi et al. [31, Theorem 4.3] and Jeanblanc et al. [30, Proposition 4.1], we introduce a BSDE in the Brownian filtration \( \tilde{\mathcal{F}} \), stopped at \( \tau \), and establish an equivalence result with the solution of BSDE (5.1) given in Lemma 5.2 below. The following condition is in force throughout this section.

Assumption 5.1. The function \( \lambda \) is bounded, i.e.

\[ 0 < a \leq \lambda(t, \mu, z) \leq b, \quad \forall (t, \mu, z) \in [0, T] \times \mathbb{R}^+ \times \mathbb{Z}, \]

for some constants \( a, b \in \mathbb{R}^+ \).

Note that, under Assumption 5.1, condition (A.4) in Appendix A is satisfied. Then, we have that on the time interval \( \{t < \tau\} \) the process \( \pi(\lambda) \) coincides with the \( \tilde{\mathcal{F}} \)-adapted process \( \tilde{\pi}(\lambda) \) given by

\[
\tilde{\pi}_t(\lambda)(\omega) = \frac{E[\lambda(t, \mu_t(\omega), Z_t)e^{-\int_0^t \lambda(u, \mu_u(\omega), Z_u)du}]}{E[e^{-\int_0^t \lambda(u, \mu_u(\omega), Z_u)du}]}, \quad t \in [0, T].
\]

Further details can be found in Appendix A.

Lemma 5.2. Let \( \xi \) be a bounded \( \tilde{\mathcal{F}}_T \)-measurable random variable. Let \( (\tilde{\xi}, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3) \), where \( \tilde{\xi} \) is \( \tilde{\mathcal{F}} \)-adapted and bounded, and \( \tilde{\gamma}^i \in L^2(W; \tilde{\mathcal{F}}) \), \( i = 1, 2, 3 \), be a solution to the BSDE

\[
\tilde{U}_t = \alpha \xi - \int_t^T \sum_{i=1}^3 \tilde{\gamma}^i_s dW^i_s + \int_t^T \left( \frac{1}{2} \left( \frac{\mu^S(s, Y_s)}{\sigma^S(s, Y_s)} \right)^2 + \frac{\mu^S(s, Y_s) \tilde{\gamma}_s}{\sigma^S(s, Y_s)} \right) ds
+ \int_t^T \frac{1}{2} (\mu^B(s, \mu_s, Y_s))^2 + \mu^B(s, \mu_s, Y_s) (c^B(s, \mu_s, Y_s) \tilde{\gamma}^2_s + d^B(s, \mu_s, Y_s) \tilde{\gamma}^3_s) ds,
+ \int_t^T \left( c^B(s, \mu_s, Y_s)^2 + d^B(s, \mu_s, Y_s)^2 \right) ds,
+ \int_t^T \left( e^{-\tilde{U}_s} - 1 \right) \tilde{\pi}_s(\lambda) + \frac{d^B(s, \mu_s, Y_s) \tilde{\gamma}^2_s - c^B(s, \mu_s, Y_s) \tilde{\gamma}^3_s)^2}{2[c^B(s, \mu_s, Y_s)^2 + d^B(s, \mu_s, Y_s)^2]} \right) ds
\]

for every \( t \in [0, T] \), where \( \tilde{\pi}(\lambda) \) is given by (5.2). Then, \( (\tilde{U}^G, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3, \tilde{\gamma}^4) \) defined as

\[
\tilde{U}^G_t = \tilde{U}_t \mathbf{1}_{\{t < \tau\}}, \quad \tilde{\gamma}^i_t = \tilde{\gamma}^i_t \mathbf{1}_{\{t < \tau\}}, \quad \tilde{\gamma}^4_t = -\tilde{U}_t \mathbf{1}_{\{t < \tau\}},
\]

5. The Indifference Price of the Pure Endowment

In order to compute the indifference price given in (3.6), we are interested in the solution of BSDE (4.11) over the stochastic interval \([0, \tau \wedge T]\). Since \( G_T = \xi 1_{\{\tau > T\}} \), see Definition 3.1, this corresponds to consider the following BSDE with random time horizon

\[
\tilde{U}^G_t = \alpha \xi 1_{\{\tau > T\}} - \int_{t \wedge \tau}^T \sum_{i=1}^3 \tilde{\gamma}^i_s dW^i_s - \int_{t \wedge \tau}^T \tilde{\gamma}^1_s dM^1_s + \int_{t \wedge \tau}^T f(s, \tilde{\gamma}^1_s, \tilde{\gamma}^2_s, \tilde{\gamma}^3_s, \tilde{\gamma}^4_s) ds,
\]

for every \( t \in [0, T] \), which is equivalent to equation (4.11) over the stochastic time interval \([0, \tau \wedge T]\). According to Kharroubi et al. [31, Theorem 4.3] and Jeanblanc et al. [30, Proposition 4.1], the solution of BSDE (5.1) given in Lemma 5.2 below.

Remark 4.11. In equation (4.18) we deal with a BSDE with quadratic generator given by (4.19). For this type of equation, existence and uniqueness of the solution are provided, for instance, in Zhang [40, Theorem 7.3.3].
is a solution of the BSDE \((5.1)\), where \(\tilde{\gamma}^i \in L^2(W; \tilde{\mathbb{G}})\), \(i = 1, 2, 3\), \(\tilde{U}^G\) is \(\tilde{\mathbb{G}}\)-adapted and bounded, and \(\tilde{\gamma}^4 \in L^1(M^r)\).

**Proof.** To get the result, we apply the Itô product rule to \(\tilde{U}^G_t = \tilde{U}_t \mathbf{1}_{\{t < \tau\}} - \tilde{U}_t (1 - H_t)\) and observe that \(\tilde{U}^G_{T\wedge \tau} = \alpha \xi \mathbf{1}_{\{\tau > T\}}\). \(\Box\)

By Lemma 5.2, it is clear that existence and uniqueness of the solution of BSDE \((5.1)\) follows from existence and uniqueness of the solution of equation \((5.3)\), which is a quadratic-exponential BSDE, only driven by Brownian motions. Using this argument we get the following result.

**Proposition 5.3.** Let \(\xi\) be a bounded \(\tilde{\mathcal{F}}_T\)-measurable random variable. Then, there exists a unique solution \((\tilde{U}^G, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3, \tilde{\gamma}^4)\) to BSDE \((5.1)\) where \(\tilde{\gamma}^i \in L^2(W; \tilde{\mathbb{G}})\), \(i = 1, 2, 3\), \(\tilde{U}^G\) is \(\tilde{\mathbb{G}}\)-adapted and bounded, \(\tilde{\gamma}^4 \in L^1(M^r)\) such that

\[
\int_0^t \sum_{i=1}^3 \tilde{\gamma}^i_s dW^i_s + \int_0^t (e^{\tilde{\gamma}^4_s} - 1) dM^r_s
\]

is a BMO(\(\tilde{\mathbb{G}}\))-martingale.

**Proof.** Existence and uniqueness of the solution \((\tilde{U}, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3)\), with \(\tilde{U} \in L^2(W; \tilde{\mathbb{F}})\) bounded, and \(\tilde{\gamma}^i \in L^2(W; \tilde{\mathbb{F}})\), \(i = 1, 2, 3\), to equation \((5.3)\) follow from the same argument used in the proof of Jeanblanc et al. \([30, \text{Theorem 4.1}]\). Precisely, by Lemma C.3 in Appendix C and Assumption 5.1 hypotheses (H1) and (H2) in Jeanblanc et al. \([30, \text{Section 2.2}]\) hold. Moreover, the driver is of the form

\[
(e^{-u} - 1) \tilde{h}(\lambda) + \tilde{g}(t, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3),
\]

where \(g\) is a map from \([0, T] \times \mathbb{R} \times \mathbb{R}\) to \(\mathbb{R}\) defined as

\[
g(t, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3) = \frac{(d^B(t, \mu_t, Y_t)\tilde{\gamma}^2 - c^B(t, \mu_t, Y_t)\tilde{\gamma}^3)^2}{2[c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2]} + \frac{1}{2} \left( \frac{\mu^S(t, Y_t)}{\sigma^S(t, Y_t)} \right)^2 + \frac{\mu^S(t, Y_t)}{\sigma^S(t, Y_t)} \tilde{\gamma}^1 + \frac{1}{2} (\mu^B(t, \mu_t, Y_t))^2 + \mu^B(t, \mu_t, Y_t) \left( c^B(t, \mu_t, Y_t) \tilde{\gamma}^2 + d^B(t, \mu_t, Y_t) \tilde{\gamma}^3 \right) \frac{c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2}{c^B(t, \mu_t, Y_t)^2 + d^B(t, \mu_t, Y_t)^2}.
\]

For every \((\tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}\), \(g(\cdot, \tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3)\) is \(\tilde{\mathbb{F}}\)-progressively measurable. It is also easy to check that for \((\tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3) = (0, 0, 0)\), \(g(t, 0, 0, 0) = 0\), for every \(t \in [0, T]\), and that \(g\) is Lipschitz with respect to \(\tilde{\gamma}^1\), \(\tilde{\gamma}^2\) and \(\tilde{\gamma}^3\), which imply that Assumption 4.1 in Jeanblanc et al. \([30]\) holds. Then, by Lemma 5.2 we get existence of a solution to BSDE \((5.1)\) and Jeanblanc et al. \([30, \text{Lemma 4.1}]\) yields uniqueness. \(\Box\)

Finally, by gathering the results we have the following characterization of the indifference price process of the pure endowment introduced in \((3.1)\).
Proposition 5.4. Let $\xi$ be a bounded $\tilde{\mathcal{F}}_T$-measurable random variable. Let $\tilde{U}^0_t$ be the unique bounded and $\tilde{\mathbb{F}}$-adapted solution to equation (4.18) and let $\tilde{U}$ be the unique bounded $\tilde{\mathbb{F}}$-adapted solution to equation (5.3). Then, the indifference price $p^\alpha_t$ of the pure endowment is given by

$$p^\alpha_t = \frac{1}{\alpha} \left( \tilde{U}_t - \tilde{U}^0_t \right) \mathbf{1}_{\{\tau > t\}} \quad t \in [0, T].$$

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Appendix A. Filtering

Let $Z = \{Z_t, \ t \in [0, T]\}$ be a cádlág Markov stochastic process taking values in a locally compact and separable space $\mathcal{Z}$. Denote by $\mathcal{L}^Z$ its Markov generator and by $\mathcal{D} \subseteq C_b(\mathcal{Z})$ the domain of the generator, that is for every function $f \in \mathcal{D} \subseteq C_b(\mathcal{Z})$

$$f(Z_t) = f(z_0) + \int_0^t \mathcal{L}^Z f(Z_s) ds + M^Z_t, \quad t \in [0, T],$$

for some $(\mathbb{F}^Z, \mathbb{P})$-martingale $M^Z = \{M^Z_t, \ t \in [0, T]\}$, with $z_0 \in \mathcal{Z}$.

We make the following assumptions.

Assumption A.1.

(i) The martingale problem for the operator $\mathcal{L}^Z$, for any initial value $z_0 \in \mathcal{Z}$, is well posed on the space of cádlág trajectories with values in $\mathcal{Z}$, $D_Z[0, T]$;

(ii) $\mathcal{L}^Z f \in C_b(\mathcal{Z})$ for any $f \in \mathcal{D}$;

(iii) $\mathcal{D}$ is an algebra dense in $C_b(\mathcal{Z})$.

We recall here that

$$\tilde{\mathbb{F}} = \mathbb{F}^{W^1} \lor \mathbb{F}^{W^2} \lor \mathbb{F}^{W^3}, \quad \mathbb{F} = \mathbb{F} \lor \mathbb{F}^{Z}, \quad \mathcal{G} = \mathbb{F} \lor \mathbb{F}^H, \quad \tilde{\mathcal{G}} = \tilde{\mathbb{F}} \lor \mathbb{F}^H,$$

where $W^j = \{W^j_t, \ t \in [0, T]\}, j = 1, 2, 3$, are $\mathbb{P}$-independent Brownian motions and $\mathbb{P}$-independent of $Z$. The process $H$ is the death indicator given by $H_t := \mathbf{1}_{\{\tau \leq t\}},$ for every $t \in [0, T]$, with $(\mathcal{G}, \mathbb{P})$-predictable intensity given by $\{(1 - H_t - \lambda(t, \mu_t, Z_{t-}), t \in [0, T]\}$. The goal of this section is to derive the dynamics of the filter $\pi = \{\pi_t, \ t \in [0, T]\}$ which provides the conditional distribution of the unobservable process $Z$, given the observation flow $\tilde{\mathcal{G}}$. In other terms, we will compute

$$\pi_t(f) = \mathbb{E} \left[ f(Z_t) \bigg| \tilde{\mathcal{G}}_t \right].$$

Electronic copy available at: https://ssrn.com/abstract=3317541
for every \( t \in [0, T] \) and for every \( f \in \mathcal{D} \). This is essential to compute the \((\tilde{G}, \mathbf{P})\)-predictable intensity of \( H \), given by \( \{(1-H_\tau)\pi_t(\cdot), \ t \in [0,T]\} \), where \( \pi_t(\lambda) \) indicates \( \pi_t(\lambda(t,\mu_t,\cdot)) \) (i.e. \( \pi_t(\lambda)(\omega) = \mathbb{E}\left[\lambda(t,\mu_t(\omega),Z_t)|\tilde{G}_t\right](\omega), \ \forall \omega \in \Omega \).

The following result characterizes the filter as the unique strong solution of the so-called Kushner-Stratonovich equation.

**Proposition A.2.** If Assumption A.1 holds, the function \( \lambda : [0,T] \times \mathbb{R}^+ \times \mathcal{Z} \to (0, +\infty) \) is continuous for every \( z \in \mathcal{Z} \) and \( \sup_{(t,\mu,z)\in[0,T] \times \mathbb{R}^+ \times \mathcal{Z}} \lambda(t,\mu,z) < \infty \), then the filter \( \pi = \{\pi_t, \ t \in [0,T]\} \) is the unique strong solution to the equation

\[
\pi_t(f) = f(z_0) + \int_0^t \pi_s(\mathcal{L}^Z f) ds + \int_0^t \frac{\pi_s(\lambda f) - \pi_s(\lambda)\pi_s(f)}{\pi_s(\lambda)} dM_s^\tau, \quad (A.2)
\]

for every \( t \in [0,T] \) and for all \( f \in \mathcal{D} \).

**Proof.** To prove the result we use the Innovation approach. Since \( W^1, W^2 \) and \( W^3 \) are \((\tilde{G}, \mathbf{P})\)-Brownian motions and the process \( M^\tau \) given in \((2.8)\) is a \((\tilde{G}, \mathbf{P})\)-jump martingale, we define the Innovation process by \((W^1, W^2, W^3, M^\tau)\).

For every function \( f \in \mathcal{D} \), by projecting equation \((A.1)\) on \( \tilde{G} \), we get

\[
\pi_t(f) = \mathbb{E}\left[f(Z_t)|\tilde{G}_t\right] = f(z_0) + \int_0^t \pi_s(\mathcal{L}^Z f) ds + M_t^{(1)}, \quad t \in [0,T],
\]

where \( M_t^{(1)} := \{M_t^{(1)}, \ t \in [0,T]\} \) is the \((\tilde{G}, \mathbf{P})\)-martingale given by \( M_t^{(1)} := \mathbb{E}\left[M_t^{\mathcal{L}^Z}|\tilde{G}_t\right] + \mathbb{E}\left[\int_0^t \mathcal{L}^Z f(Z_s)|\tilde{G}_t\right] - \int_0^t \pi_s(\mathcal{L}^Z f) ds \) (see, e.g. Brémaud [10, Chapter IV, Theorem T1]).

By the Martingale Representation Theorem (see, e.g. Lipster and Shiryaev [36, Theorem 3.34]) with respect to filtration \( \tilde{G} \) and probability measure \( \mathbf{P} \), there exist \( \tilde{G} \)-adapted processes \( \tilde{h}^i = \{\tilde{h}^i_t, \ t \in [0,T]\} \) and a \( \tilde{G} \)-predictable process \( \tilde{\varphi} = \{\tilde{\varphi}_t, \ t \in [0,T]\} \) satisfying

\[
\mathbb{E}\left[\int_0^T \left(\sum_{i=1}^3 (\tilde{h}^i_t)^2 + |\tilde{\varphi}_t|\lambda(t,\mu_t,Z_t)\right) dt\right] < \infty.
\]

and such that

\[
M_t^{(1)} = M_0^{(1)} + \sum_{i=1}^3 \int_0^t \tilde{h}^i_t dW^i_t + \int_0^t \tilde{\varphi}_s dM^\tau_s, \quad t \in [0,T].
\]

In order to identify the processes \( \tilde{h}^i \) for \( i = 1, 2, 3 \) and \( \tilde{\varphi} \), we observe that \( \mathbb{E}\left[f(Z_t)W^i_t|\tilde{G}_t\right] = \mathbb{E}\left[f(Z_t)|\tilde{G}_t\right] W^i_t \); then, by computing both quantities and comparing the finite variation parts we get that \( \tilde{h}^i_t = 0 \ \mathbf{P}\text{-a.s. for every } t \in [0,T] \). Moreover, it holds that for every process \( U = \{U_t, \ t \in [0,T]\} \) of the form \( U_t = \int_0^t C_s dH_s \) for some \((\tilde{G}, \mathbf{P})\)-predictable process \( C = \{C_t, \ t \in [0,T]\}, \)

\[
\mathbb{E}\left[f(Z_t)U_t|\tilde{G}_t\right] = \mathbb{E}\left[f(Z_t)|\tilde{G}_t\right] U_t. \quad \text{Then, by computing separately the right-hand side and the}
\]
We make a few remarks. Over the set and at time \( t \in [0, T] \),
and since the process \( C \) is arbitrary, we obtain that on the set \( \{ t < \tau \} \)
\[
\hat{\varphi}_t = \frac{\pi_{t^-}(f\lambda)}{\pi_{t^-}(\lambda)} - \pi_{t^-}(f), \quad \mathbb{P} - \text{a.s.}
\]
Therefore, \( M_t^{(1)} = M_0^{(1)} + \int_0^t \left( \frac{\pi_{s^-}(f\lambda)}{\pi_{s^-}(\lambda)} - \pi_{s^-}(f) \right) \, dM^\tau_s \), and plugging this expression in \( (\text{A.3}) \) we get the result.
Uniqueness can be proved as in Ceci and Colaneri \([16, \text{Theorem 3.3}]\), by applying the Filtered Martingale approach. We start by observing that for any \( f \in \mathcal{D} \) and any measurable function \( \phi \) on \( [0, 1] \), we have
\[
f(Z_t)\phi(H_t) = f(Z_0)\phi(H_0) + M_t^{f,\phi}
\]
for every \( t \in [0, T] \), where \( M_t^{f,\phi} = \{ M_t^{f,\phi}, \ t \in [0, T] \} \) is a \( (\mathbb{G}, \mathbb{P}) \)-martingale. Then, for any \( \mu \in \mathbb{R}^+ \) it follows that the pair \( (Z, H) \) solves the martingale problem for the operator \( \mathcal{L}^\mu \) defined by
\[
\mathcal{L}^\mu \psi(t, z, h) := \frac{\partial \psi}{\partial t}(t, z, h) + \mathcal{L}^Z \psi(t, z, h) + [\psi(t, x, h + 1) - \psi(t, z, h)](1 - h)\lambda(t, \mu, z)
\]
for every function \( \psi \) in the domain \( \mathcal{D}^\mu \) of \( \mathcal{L}^\mu \), where \( \mathcal{D}^\mu \) consists of all bounded functions \( \psi \) having continuous partial derivatives with respect to \( t \) and such that \( \psi(t, \cdot, h) \in \mathcal{D}, \forall (t, h) \in [0, T] \times [0, 1] \).
The pair \( (\mathcal{L}^\mu, \mathcal{D}^\mu) \) satisfies Assumption \( \text{A.1} \) where we replace \( \mathcal{Z} \) with \( [0, T] \times \mathcal{Z} \times [0, 1] \). By Kurtz and Ocone \([32, \text{Theorem 3.3}]\) we get that the Filtered Martingale Problem for the operator \( \mathcal{L}^\mu \) is well posed. Then, we can apply Ceci and Colaneri \([16, \text{Theorem 3.3}]\), which ensures strong uniqueness.

We make a few remarks. Over the set \( \{ t < \tau < T \} \) the filter solves a nonlinear equation given by
\[
\pi_t(f) = f(z_0) + \int_0^t (\pi_s(\mathcal{L}^Z f) \, ds - \pi_s(\lambda f) + \pi_s(\lambda)\pi_s(f)) \, ds,
\]
and at time \( \tau < T \) we get that
\[
\pi_\tau(f) = \pi_{\tau^-}(f) + \frac{\pi_{\tau^-}(\lambda f) - \pi_{\tau^-}(\lambda)\pi_{\tau^-}(f)}{\pi_{\tau^-}(\lambda)}.
\]
Finally, after the jump, that is over the set \( \{ \tau < t \leq T \} \), the filtering equation is linear, of the form
\[
\pi_t(f) = \pi_{\tau}(f) + \int_{\tau}^t \pi_s(\mathcal{L}^Z f) \, ds.
\]
In order to obtain an explicit expression for the filter, we apply a suitable change of probability measure, which allows to obtain a linear equation for the unnormalized filter, known in literature as the Zakai equation. To this aim we introduce the process \( L_t = \{ L_t, \ t \in [0, T] \} \) by

\[
L_t := \mathcal{E} \left( \int_0^T \frac{1 - \lambda(s, \mu_s, Z^-_s)}{\lambda(s, \mu_s, Z^-_s)} \left\{ dH_s - (1 - H^-_s) \lambda(s, \mu_s, Z^-_s) ds \right\}_t \right),
\]

for every \( t \in [0, T] \), where \( \mathcal{E} \) denotes the Doléans-Dade exponential. We assume that \( L \) is a \((\mathcal{G}, \mathcal{P})\)-martingale. This is implied, for instance, by the condition

\[
\mathbb{E} \left[ e^{\int_0^T \frac{(1 - \lambda(s, \mu_s, Z^-_s))^2}{\lambda(s, \mu_s, Z^-_s)} \left( 1 - H^-_s \right) ds} \right] < \infty,
\]

and satisfied, in particular, if the function \( \lambda \) is bounded from below and above. Then we define the probability measure \( \mathcal{Q} \) equivalent to \( \mathcal{P} \) by

\[
\frac{d\mathcal{Q}}{d\mathcal{P}} |_{\mathcal{G}_t} := L_t,
\]

for every \( t \in [0, T] \). By the Girsanov Theorem we have that

\[
\left\{ H_t - \int_0^t (1 - H^-_s) ds, \ t \in [0, T] \right\}
\]

is a \((\mathcal{G}, \mathcal{Q})\)-martingale, and the process \( \{ 1 - H^-_t, \ t \in [0, T] \} \) provides the \((\mathcal{G}, \mathcal{Q})\)-predictable intensity of \( H \). We introduce the unnormalized filter, which is the finite measure valued process \( \rho = \{ \rho_t, \ t \in [0, T] \} \) given by

\[
\rho_t(f) := \mathbb{E}^\mathcal{Q} \left[ L_t^{-1} f(Z_t) | \mathcal{G}_t \right], \quad t \in [0, T],
\]

for every bounded measurable function \( f \). By applying the Kallianpur-Striebel formula we get that

\[
\pi_t(f) = \mathbb{E} \left[ f(Z_t) | \mathcal{G}_t \right] = \frac{\rho_t(f)}{\rho_t(1)}, \quad t \in [0, T],
\]

for every bounded and measurable function \( f \), where \( \rho_t(1) := \mathbb{E}^\mathcal{Q} \left[ L_t^{-1} | \mathcal{G}_t \right] \). The dynamics of process \( \rho(1) \) can be easily computed by observing that the \((\mathcal{G}, \mathcal{P})\)-intensity of \( H \) is given by \( \{(1 - H^-_t) \pi_t - (\lambda), t \in [0, T] \} \) and the \((\mathcal{G}, \mathcal{Q})\)-intensity of \( H \) is \( \{1 - H^-_t, \ t \in [0, T] \} \), then we get that \( \rho(1) \) is an exponential martingale satisfying the following stochastic differential equation

\[
d\rho_t(1) = \rho_t(1)(\pi_t(\lambda) - 1)(dH_t - (1 - H^-_t)dt), \quad \rho_0(1) = 1.
\]

Then, by applying the product rule to \( \rho_t(f) = \pi_t(f) \rho_t(1) \) and using equation (A.2), we get that

\[
\rho_t(f) = f(z_0) + \int_0^t \rho_s(\mathcal{L}Z f) ds + \int_0^t \rho_s(1)(f(\lambda - 1))(dH_s - (1 - H^-_s)ds],
\]

for every \( t \in [0, T] \), where \( \rho_t(\lambda) \) indicates \( \rho_t(\lambda(t, \mu_t, \cdot)) \). Over the set \( \{ t < \tau < T \} \) this equation reduces to

\[
\rho_t(f) = f(z_0) + \int_0^t \left( \rho_s(\mathcal{L}Z f) - \rho_s(f(\lambda - 1)) \right) ds
\]

and the solution can be computed explicitly, as shown in the following proposition.
Proposition A.3. Let
\[ \tilde{\rho}_t(f)(\omega) := \mathbb{E}\left[f(Z_t) e^{-\int_0^t (\lambda(u, \mu_t(\omega), Z_u) - 1)du}\right], \quad t \in [0, \tau(\omega)). \]

Then, \( \tilde{\rho} \) solves equation (A.5) over \( \{t < \tau\} \).

Proof. For any fixed trajectory \( t \to \mu_t(\omega) \) of process \( \mu \), we set \( \gamma_t := e^{-\int_0^t (\lambda(s, \mu_t(\omega), Z_s) - 1)ds} \). By the product rule we get
\[ d(f(Z_t)\gamma_t) = \gamma_t[\mathcal{L}^Z f(Z_t) - f(Z_t)(\lambda(t, \mu_t(\omega), Z_t) - 1)]dt + \gamma_t dM_t^Z. \]
Now, taking expectation
\[ \mathbb{E}\left[f(z_0) + \int_0^t \mathbb{E}\left[\gamma_s[\mathcal{L}^Z f(Z_s) - f(Z_s)(\lambda(s, \mu_s(\omega), Z_s) - 1)]\right]dt. \]
Then, we get that \( \mathbb{E}\left[f(Z_t)e^{-\int_0^t (\lambda(s, \mu_s(\omega), Z_s) - 1)ds}\right] \) solves equation (A.5) for any fixed trajectory of the process \( \mu \) and this concludes the proof. \( \square \)

As a consequence of Proposition A.3, we have an explicit representation of the filter. Indeed, we set \( \tilde{\pi}_t(f) := \frac{\tilde{\rho}_t(f)}{\tilde{\rho}_t(1)} \) for \( t < \tau < T \), \( \tilde{\pi}_\tau(f) := \frac{\tilde{\pi}_\tau(\lambda f)}{\tilde{\pi}_\tau(\lambda)} \), for \( t = \tau < T \) and \( \tilde{\pi}_t(f) := \int_{\mathbb{R}} \psi_t^f(\tau, x)\tilde{\pi}_\tau(dx) \), for \( t > \tau \), where \( \psi_t^f(s, x) := \mathbb{E}[f(Z_t)|Z_s = x], t > s \). Then, by a direct computation we can show that \( \tilde{\pi} \) solves equation (A.2) and by uniqueness we get that \( \tilde{\pi} = \pi \). Define the \( \tilde{\mathbb{F}} \)-adapted process \( \hat{\pi} := \{\tilde{\pi}_t, t \in [0, T]\} \) by
\[ \hat{\pi}_t(f)(\omega) = \frac{\mathbb{E}\left[f(Z_t)e^{-\int_0^t \lambda(s, \mu_s(\omega), Z_s)ds}\right]}{\mathbb{E}\left[e^{-\int_0^t \lambda(s, \mu_s(\omega), Z_s)ds}\right]}, \quad t \in [0, T]. \]
Then, we get that on \( \{t < \tau\} \), the filter \( \pi \) coincide with the process \( \hat{\pi} \).

APPENDIX B. LONGEVITY BOND PRICE

We start from a filtered probability space \((\Omega, \tilde{\mathbb{F}}, \tilde{\mathbb{P}}, \mathbb{Q})\), where \( \mathbb{Q} \) is a risk neutral measure equivalent to \( \mathbb{P} \). The objective of this section is to characterize the fair price of the longevity bond under the measure \( \mathbb{Q} \) and get the \( \mathbb{P} \)-price dynamics via change of measure. Let \( W^{1, \mathbb{Q}} = \{W^{1, \mathbb{Q}}, t \in [0, T]\} \), \( W^{2, \mathbb{Q}} = \{W^{2, \mathbb{Q}}, t \in [0, T]\} \), \( W^{3, \mathbb{Q}} = \{W^{3, \mathbb{Q}}, t \in [0, T]\} \) be \( \mathbb{Q} \)-independent Brownian motions and define the density process \( L^\mathbb{P} = \{L_t^\mathbb{P}, t \in [0, T]\} \) of \( \mathbb{P} \) with respect to \( \mathbb{Q} \) by
\[ L_t^\mathbb{P} := \frac{d\mathbb{P}}{d\mathbb{Q}}|_{\tilde{\mathbb{F}}_t} = \mathcal{E}\left(\int_0^t \mu^S(u, Y_u) dW_u^{1, \mathbb{Q}} - \int_0^t \sigma^S(u, Y_u) dW_u^{2, \mathbb{Q}} - \int_0^t \alpha^Y(u, \mu_u, Y_u) dW_u^{1, \mathbb{Q}}\right), \]
for every \( t \in [0, T] \), where functions \( \mu^S, \sigma^S, \alpha^\mu : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \) and \( \alpha^Y : [0, T] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \) are measurable and such that \( L^\mathbb{P} \) is an \((\tilde{\mathbb{F}}, \mathbb{Q})\)-martingale. By applying Girsanov theorem we get
that processes \( W^1 = \{ W^1, \, t \in [0, T] \} \), \( W^2 = \{ W^2, \, t \in [0, T] \} \), \( W^3 = \{ W^3, \, t \in [0, T] \} \) respectively defined by

\[
W^1_t := W^1_t^Q - \int_0^t \frac{\mu^S(u, Y_u)}{\sigma^S(u, Y_u)} \, du, \quad t \in [0, T],
\]
\[
W^2_t := W^2_t^Q + \int_0^t \alpha^\mu(u, \mu_u, Y_u) \, du, \quad t \in [0, T],
\]
\[
W^3_t := W^3_t^Q + \int_0^t \alpha^Y(u, \mu_u, Y_u) \, du, \quad t \in [0, T],
\]

are \( P \)-independent \( \bar{\mathbb{F}} \)-Brownian motions. Following Cairns et al. [13], a longevity bond is defined as a zero-coupon bond that pays out the value of the survivor or longevity index at time \( T \). Then, its discounted price process \( S^2 \) at any time \( t \) is given by

\[
S^2_t = \mathbb{E}^Q \left[ S^2_T^Q \mid \bar{\mathcal{F}}_t \right] = \mathbb{E}^Q \left[ \exp \left( - \int_0^T \mu_s \, ds \right) \right] \bar{F}_t , \quad t \in [0, T].
\]

(B.1)

We write the dynamics of the pair \((\mu, Y)\) w.r.t. \( Q \) as

\[
d\mu_t = (b^\mu(t, \mu_t, Y_t) + \alpha^\mu(t, \mu_t, Y_t)) \, dt + \sigma^\mu(t, \mu_t, Y_t) \, dW^2_t^Q, \quad \mu_0 \in \mathbb{R}^+,
\]
\[
dY_t = (b^Y(t, Y_t) + \alpha^Y(t, \mu_t, Y_t)) \, dt + \sigma^Y(t, Y_t) \, dW^3_t^Q, \quad Y_0 = y_0 \in \mathbb{R},
\]

Since the pair \((\mu, Y)\) is an \((\bar{\mathbb{F}}, Q)\)-Markov process with infinitesimal generator \( \mathcal{L}^{\mu,Y} \) under \( Q \), setting

\[
F(t, \mu, y) := \mathbb{E}^Q \left[ \exp \left( - \int_0^T \mu_s \, ds \right) \right] \mu_t = \mu, \, Y_t = y ,
\]

we get that relation \([B.1]\) can be written as

\[
S^2_t = e^{-\int_0^t \mu_s \, ds} F(t, \mu_t, Y_t), \quad t \in [0, T].
\]

(B.3)

The function \( F \) is strictly positive and bounded from above by 1. If the function \( F \) is regular, that is, \( F \in C^{1,2}_b([0, T] \times \mathbb{R}^+ \times \mathbb{R}) \), then it can be characterized via the Feynman-Kac formula as the solution of the boundary problem

\[
\begin{cases}
\frac{\partial F}{\partial t}(t, \mu, y) + \mathcal{L}^{\mu,Y} F(t, \mu, y) - \mu F(t, \mu, y) = 0, \quad (t, \mu, y) \in [0, T] \times \mathbb{R}^+ \times \mathbb{R}, \\
F(T, \mu, y) = 1, \quad (\mu, y) \in \mathbb{R}^+ \times \mathbb{R}.
\end{cases}
\]

(B.4)

To ensure that Feynman-Kac formula \([B.2]\) applies, we make the following set of assumptions.

**Assumption B.1.** Functions \( b^{\mu}, \, b^Y, \, \alpha^{\mu}, \alpha^Y, \sigma^{\mu}, \sigma^Y \) are continuous in all variables and satisfy sublinear growth-conditions on \((\mu, y) \in \mathbb{R}^+ \times \mathbb{R}, \) uniformly in \( t \in [0, T] \). Moreover, \( b^{\mu}, \, b^Y, \, \alpha^{\mu}, \alpha^Y, \) and \((\sigma^{\mu})^2, \) \((\sigma^Y)^2\) are Lipschitz continuous on \((\mu, y) \in \mathbb{R}^+ \times \mathbb{R}, \) uniformly in \( t \in [0, T], \) and \( \sigma^{\mu}, \sigma^Y \) are bounded from below.
The following result shows existence and uniqueness for the solution of the boundary problem (B.4).

**Proposition B.2.** Under Assumption B.1, there exists a unique classical solution \(F\) to the boundary problem (B.4) and the Feynman-Kac representation (B.2) holds.

**Proof.** The result follows from Heath and Schweizer [26, Theorem 1]. Indeed, condition \((A2)\) in Heath and Schweizer [26, Theorem 1] is a consequence of sublinear growth-condition and Lipschitz continuity of the coefficients and by Heath and Schweizer [26, Lemma 2], \(F\) given in (B.2) is continuous in \([0,T] \times \mathbb{R}^+ \times \mathbb{R}\).

Now, we can apply Itô’s formula to \(S^2\) given in (B.3) and since \(F\) is solution of (B.4), we get that \(S^2\) solves
\[
d S^2_t = S^2_t \left( c^B(t, \mu_t, Y_t) d W^2_t + d^B(t, \mu_t, Y_t) d W^3_t \right), \quad S^2_0 = s^2_0 \in \mathbb{R}^+,
\]
where we have set
\[
c^B(t, \mu, y) := \frac{\sigma^\mu(t, \mu, y) \partial F}{F(t, \mu, y)}(t, \mu, y), \quad d^B(t, \mu, y) := \frac{\sigma^Y(t, y) \partial F}{F(t, \mu, y)}(t, \mu, y),
\]
for every \((t, \mu, y) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}\). Finally, the \(\mathbb{P}\)-dynamics of the process \(S^2\) is given by (2.5) where we have set \(\mu^B(t, \mu_t, Y_t) := c^B(t, \mu_t, Y_t) \alpha^\mu(t, \mu_t, Y_t) + d^B(t, \mu_t, Y_t) \alpha^Y(t, \mu_t, Y_t)\).

**Appendix C. Technical results and proofs**

This section contains a few technical results and proofs that are used in the body of the paper.

The first proposition is a representation result for \((\tilde{G}, \mathbb{P})\)-martingales which is proved, e.g., in Jeanblanc et al. [29].

**Proposition C.1.** Any \((\tilde{G}, \mathbb{P})\)-local martingale \(N = \{N_t, \ t \in [0,T]\}\) has the following representation:
\[
N_t = N_0 + \int_0^t a^1_s d W^1_s + \int_0^t a^2_s d W^2_s + \int_0^t a^3_s d W^3_s + \int_0^t b_s d M^\tau_s, \quad t \in [0,T], \ \mathbb{P} - a.s., \quad (C.1)
\]
where \(a^1, a^2, a^3 \in L^2_{loc}(W; \tilde{G})\) and \(b \in L^1_{loc}(M^\tau)\). If \(N\) is a square integrable \((\tilde{G}, \mathbb{P})\)-martingale, each term on the right-hand side of the representation \((C.1)\) is square integrable.

In the sequel we provide the proof of Theorem 4.1.

**Proof of Theorem 4.1.** Let \(\theta \in \mathcal{A}(\tilde{G})\) and apply the Itô product rule to \(e^{-\alpha X^\theta_t} V^G_t\) we have
\[
d \left( e^{-\alpha X^\theta_t} V^G_t \right) = e^{-\alpha X^\theta_t} d V^G_t + V^G_t \left( d \left( e^{-\alpha X^\theta_t} \right) + d \langle e^{-\alpha X^\theta}, V^G \rangle_t \right). \quad (C.2)
\]
Plugging the dynamics of the wealth given by equation (3.2) and that of $V^G$ specified by equations (4.2) and (4.3), into (C.2) we get that
\[
d\left(e^{-\alpha X_t^G} V^G_t\right) = e^{-\alpha X_t^G} \left\{ dm^1_t - \alpha V^G_t \left[ \sigma^S(t, Y_t) dW^1_t + \theta^2_t e^B(t, \mu_t, Y_t) dW^2_t + \theta^2_t d^B(t, \mu_t, Y_t) dW^3_t \right] \right\} \\
+ \frac{1}{2} \alpha^2 V^G_t \left[ (\theta^2_t \sigma^S(t, Y_t))^2 + (\theta^2_t e^B(t, \mu_t, Y_t))^2 + d^B(t, \mu_t, Y_t)^2 \right] dt \\
- \alpha \left[ R^2_t \theta^2_t \sigma^S(t, Y_t) + \theta^2_t (R^2_t e^B(t, \mu_t, Y_t) + R^3_t d^B(t, \mu_t, Y_t)) \right] dt
\]
where the function $f_\alpha$ is given in (4.5) and $M^{V, \theta} = \{ M^{V, \theta}_t, t \in [0, T] \}$ is the $(\tilde{G}, \mathbf{P})$-local martingale given by
\[
M^{V, \theta}_t := M^{V, \theta}_0 + \int_0^t e^{-\alpha X_u^G} \left( R^1_u - \alpha V^G_{u-} \theta^1_u \sigma^S(u, Y_u) \right) dW^1_u \\
+ \int_0^t e^{-\alpha X_u^G} \left( R^2_u - \alpha V^G_{u-} \theta^2_u e^B(u, \mu_u, Y_u) \right) dW^2_u \\
+ \int_0^t e^{-\alpha X_u^G} \left( R^3_u - \alpha V^G_{u-} \theta^2_u d^B(u, \mu_u, Y_u) \right) dW^3_u + \int_0^t e^{-\alpha X_u^G} R^4_u dM^\tau_u,
\]
for every $t \in [0, T]$. By Proposition 3.8(i), for every $\theta \in \mathcal{A}(\tilde{G})$, the process \{ $e^{-\alpha X^G_t} V^G_t$, $t \in [0, T]$ \} is a $(\tilde{G}, \mathbf{P})$-submartingale. It follows that $dA_t = f_\alpha(t, R^1_t, R^2_t, R^3_t, V^G_t, \theta^1_t, \theta^2_t) dt \geq 0$, which yields
\[
dA_t \geq \text{ess sup}_{\theta \in A(\tilde{G})} f_\alpha(t, R^1_t, R^2_t, R^3_t, V^G_t, \theta^1_t, \theta^2_t) dt.
\]
In addition, by Proposition 3.8(ii), we get that if $\theta^* = (\theta^{1*}_t, \theta^{2*}_t)^\top \in \mathcal{A}(\tilde{G})$, then it is an optimal control if and only if the associated process \{ $e^{-\alpha X^G_t} V^G_t$, $t \in [0, T]$ \} is a $(\tilde{G}, \mathbf{P})$-martingale. The latter holds if and only if
\[
dA_t = f_\alpha(t, R^1_t, R^2_t, R^3_t, V^G_t, \theta^{1*}_t, \theta^{2*}_t) dt = \text{ess sup}_{\theta \in A(\tilde{G})} f_\alpha(t, R^1_t, R^2_t, R^3_t, V^G_t, \theta^1_t, \theta^2_t) dt.
\]
Finally, using the first-order necessary condition for optimality, we have that the admissible strategy $\theta^* = (\theta^{1*}_t, \theta^{2*}_t)^\top$ given in (4.6)-(4.7) is the only stationary point. Moreover, since the Hessian of function $f_\alpha$ is positive definite when evaluated at $\theta^*$, it realizes the essential supremum in (4.4). Then, deriving (4.8) is straightforward.

The following proposition is a general verification result.

**Proposition C.2.** If there exists a $\tilde{G}$-adapted process $D = \{ D_t, t \in [0, T] \}$ such that
\[
(i) \quad D_T = e^{\alpha G_T};
\]
 Electronic copy available at: https://ssrn.com/abstract=3317541
(ii) \( \{ e^{-\alpha x_t^\theta} D_t, \ t \in [0,T]\} \) is a \((\bar{\mathcal{G}}, \mathbb{P})\)-submartingale for any \( \theta \in \mathcal{A}(\bar{\mathcal{G}}) \) and a \((\bar{\mathcal{G}}, \mathbb{P})\)-martingale for some \( \theta^* \in \mathcal{A}(\bar{\mathcal{G}}) \),

then, \( V_t^G = D_t \), for every \( t \in [0,T] \), \( \mathbb{P} \)-a.s. and \( \theta^* \) is an optimal investment strategy for Problem 3.3.

**Proof.** Let \( D \) be a \( \bar{\mathcal{G}} \)-adapted process satisfying conditions (i) and (ii). Then

\[
\mathbb{E} \left[ e^{-\alpha(x_t^\theta - G_T)} \mid \bar{\mathcal{G}}_t \right] = \mathbb{E} \left[ e^{-\alpha x^\theta_T} D_T \mid \bar{\mathcal{G}}_t \right] \geq e^{-\alpha x^\theta_t} D_t,
\]

for any \( \theta \in \mathcal{A}(\bar{\mathcal{G}}) \), and the equality holds for \( \theta = \theta^* \). Hence

\[
\mathbb{E} \left[ e^{-\alpha(t^\theta \theta^\top \frac{ds_u}{su} - G_T)} \mid \bar{\mathcal{G}}_t \right] \geq D_t = \mathbb{E} \left[ e^{-\alpha(t^\theta \theta^\top \frac{ds_u}{su} - G_T)} \mid \bar{\mathcal{G}}_t \right], \quad \theta \in \mathcal{A}(\bar{\mathcal{G}}).
\]

This implies

\[
\text{ess inf}_{\theta \in \mathcal{A}(\bar{\mathcal{G}})} \mathbb{E} \left[ e^{-\alpha(t^\theta \theta^\top \frac{ds_u}{su} - G_T)} \mid \bar{\mathcal{G}}_t \right] = D_t = \mathbb{E} \left[ e^{-\alpha(t^\theta \theta^\top \frac{ds_u}{su} - G_T)} \mid \bar{\mathcal{G}}_t \right],
\]

which concludes the proof. \( \square \)

In the next lines we give the proof of Theorem 4.4.

**Proof of Theorem 4.4.** For any \( \theta \in \mathcal{A}(\bar{\mathcal{G}}) \), we apply the Itô product rule to compute \( e^{-\alpha x^\theta_t} U_t^G \), for every \( t \in [0,T] \):

\[
d \left( e^{-\alpha x^\theta_t} U_t^G \right) = e^{-\alpha x^\theta_t} dU_t^G + U_t^G d \left( e^{-\alpha x^\theta_t} \right) + d \langle e^{-\alpha x^\theta}, U^G \rangle_t.
\]

By (4.9), we get that

\[
d \left( e^{-\alpha x^\theta_t} U_t^G \right) = dM_t^{U,\theta} + e^{-\alpha x^\theta_t} \left\{ f(t, \gamma^1_t, \gamma^2_t, \gamma^3_t, U_t^G) - f_a(t, \gamma^1_t, \gamma^2_t, \gamma^3_t, U_t^G, \theta^1_t, \theta^2_t) \right\} dt,
\]

(C.3)

where the functions \( f \) and \( f_a \) are given in (4.10) and (4.5) respectively, and the process \( M_t^{U,\theta} = \{ M_t^{U,\theta}, \ t \in [0,T] \} \) defined by

\[
M_t^{U,\theta} := M_0^{U,\theta} + \int_0^t e^{-\alpha x^\theta_u} \left( \gamma^1_u - \alpha U^G_u \theta^1_u \sigma^S(u, Y_u) \right) dW^1_u + \int_0^t e^{-\alpha x^\theta_u} \left( \gamma^2_u - \alpha U^G_u \theta^2_u \sigma^B(u, \mu_u, Y_u) \right) dW^2_u + \int_0^t e^{-\alpha x^\theta_u} \left( \gamma^3_u - \alpha U^G_u \theta^3_u \sigma^B(u, \mu_u, Y_u) \right) dW^3_u + \int_0^t e^{-\alpha x^\theta_u} \gamma^4_u dM^\tau_u,
\]

(C.4)

for every \( t \in [0,T] \), is a \((\bar{\mathcal{G}}, \mathbb{P})\)-local martingale. Let us observe that for any \( \theta \in \mathcal{A}(\bar{\mathcal{G}}) \),

\[
f(t, \gamma^1_t, \gamma^2_t, \gamma^3_t, U_t^G) = \text{ess sup}_{\theta \in \mathcal{A}(\bar{\mathcal{G}})} f_a(t, \gamma^1_t, \gamma^2_t, \gamma^3_t, U_t^G, \theta^1_t, \theta^2_t) \geq f_a(t, \gamma^1_t, \gamma^2_t, \gamma^3_t, U_t^G, \theta^1_t, \theta^2_t).
\]
Hence, $d\tilde{A}_t^\theta := e^{-\alpha X_t^\theta} \{ f(t, \gamma_1^t, \gamma_2^t, \gamma_3^t, U_t^G) - f_\alpha(t, \gamma_1^t, \gamma_2^t, \gamma_3^t, U_t^G, \theta_t^1, \theta_t^2) \} \, dt$ is an increasing process. Since $M_{U,\theta}^{t,u}$ given in (C.4) is a $(\tilde{G}, P)$-local martingale, by localization and by using (C.3) and (3.3), we get
\[
\mathbb{E} \left[ \tilde{A}_T^\theta \right] \leq \mathbb{E} \left[ \sup_{s \in [0,T]} e^{-\alpha X_s^\theta} \right] e^{\alpha k} < \infty
\]
and $\mathbb{E} \left[ \sup_{t \in [0,T]} e^{-\alpha X_t^\theta} U_t^G \right] < \infty$. Hence, $\mathbb{E} \left[ \sup_{t \in [0,T]} | M_{U,\theta}^{t,u} | \right] < \infty$, which implies that for every $\theta \in \mathcal{A}(\tilde{G})$, $M_{U,\theta}$ is a $(\tilde{G}, P)$-martingale. Finally, the thesis follows by Proposition C.2 $\blacksquare$

The following result is useful in the proof of Proposition 5.3.

**Lemma C.3.** In the modeling framework outlined in Section 2, the so-called density hypothesis holds with respect to filtration $\tilde{F}$. Precisely, for every $t \in [0,T]$, there exists a function $\tilde{\beta}(t, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, such that $(t,u) \mapsto \tilde{\beta}(t,u)$ is $\tilde{F}_t \otimes \mathcal{B}(0,\infty)$-measurable and such that
\[
P(\tau > s | \tilde{F}_t) = \int_s^\infty \tilde{\beta}(t,u) du, \quad s \in \mathbb{R}^+,
\]
and $\tilde{\beta}(t,u) 1_{\{t \geq u\}} = \tilde{\beta}(u,u) 1_{\{t \geq u\}}$.

**Proof.** Firstly, note that the Cox construction for the random time $\tau$ describing the residual life time of the policyholder, ensures that the density hypothesis is fulfilled with respect to the filtration $\tilde{F}$, see Jeanblanc and Le Cam [28, Section 5]. That is, there exists a map $\beta(t, \cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, such that $(t,u) \mapsto \beta(t,u)$ is $\mathcal{F}_t \otimes \mathcal{B}(0,\infty)$-measurable and such that
\[
P(\tau > s | \mathcal{F}_t) = \int_s^\infty \beta(t,u) du, \quad s \in \mathbb{R}^+,
\]
(C.5)
and $\beta(t,u) 1_{\{t \geq u\}} = \beta(u,u) 1_{\{t \geq u\}}$. Precisely, in our setting
\[
\beta(t,u) = \mathbb{E} \left[ \lambda(u, \mu_u, Z_u) e^{-\int_0^u \lambda(r, \mu_r, Z_r) dr} \bigg| \mathcal{F}_t \right].
\]
Conditioning in (C.5) with respect to $\tilde{F}_t \subseteq \mathcal{F}_t$ and applying the Fubini theorem yield
\[
P(\tau > s | \tilde{F}_t) = \mathbb{E} \left[ \int_s^\infty \beta(t,u) du \bigg| \tilde{F}_t \right] = \int_s^\infty \mathbb{E} \left[ \lambda(u, \mu_u, Z_u) e^{-\int_0^u \lambda(r, \mu_r, Z_r) dr} \bigg| \tilde{F}_t \right] du = \int_s^\infty \tilde{\beta}(t,u) du,
\]
where $\tilde{\beta}(t,u) := \mathbb{E} \left[ \lambda(u, \mu_u, Z_u) e^{-\int_0^u \lambda(r, \mu_r, Z_r) dr} \bigg| \tilde{F}_t \right]$. Note that, since $\mu$ is $\tilde{F}$-adapted and the process $Z$ is independent of $\tilde{F}$, for any $t \geq u$
\[
\tilde{\beta}(u,u)(\omega) = \mathbb{E} \left[ \lambda(u, \mu_u(\omega), Z_u) e^{-\int_0^u \lambda(r, \mu_r(\omega), Z_r) dr} \right] = \tilde{\beta}(t,u)(\omega), \quad \forall \omega \in \Omega.
\]
and this concludes the proof. $\blacksquare$
REFERENCES

[1] Aase, K. and Persson, S.: Pricing of unit-linked life insurance policies. Scand. Actuar. J., 1994(1):26–52, (1994).
[2] Bayraktar, E., Milevsky,M. A., Promislow, S. D. and Young, V.: Valuation of mortality risk via the instantaneous sharpe ratio: applications to life annuities. J. Econom. Dynam. Control, 33(3):676–691, (2009).
[3] Becherer, D.: Rational hedging and valuation of integrated risks under constant absolute risk aversion. Insurance Math. Econom., 33(1):1–28, (2003).
[4] Becherer, D.: Bounded solutions to backward SDEs with jumps for utility optimization and indifference hedging. Ann. Appl. Probab., 16(4):2027–2054, (2006).
[5] Biagini, F., Rheinländer, T. and Schreiber, I.: Risk-minimization for life insurance liabilities with basis risk. Math. Financ. Econ., 10(2):151–178, (2016).
[6] Bielecki, T.R. and Rutkowski, M.: Credit Risk: Modeling, Valuation and Hedging. Springer Finance. Springer-Verlag Berlin, Heidelberg, New York, (2004).
[7] Biffis, E.: Affine processes for dynamic mortality and actuarial valuations. Insurance Math. Econom., 37(3):443–468, (2005).
[8] Blanchet-Scalliet, C., Dorobantu, D., and Salhi, Y.: A model-point approach to indifference pricing of life insurance portfolios with dependent lives. Methodol. Comput. Appl. Probab., pages 1–26, (2017).
[9] Boyle,P. and Schwartz, E.: Equilibrium prices of guarantees under equity-linked contracts. J. Risk Insur., pages 639–660, (1977).
[10] Brémaud, P.: Point processes and queues. Springer-Verlag, Halsted Press, (1981).
[11] Brémaud, P. and Yor, M.: Changes of filtrations and of probability measures. Z. Wahrscheinlichkeit, 45(4):269–295, (1978).
[12] Brennan, M. and Schwartz, E.: The pricing of equity-linked life insurance policies with an asset value guarantee. J. Financ. Econ., 3(3):195–213, (1976).
[13] Cairns, A., Blake, D. and Dowd, K.: Pricing death: Frameworks for the valuation and securitization of mortality risk. Astin Bull., 36(01):79–120, (2006).
[14] Ceci, C.: Optimal investment problems with marked point stock dynamics. In F. Russo R. C. Dalang, M. Dozzi, editor, Progress in Probability, volume 63, pages 385–412. Birkhäuser Verlag Basel/Switzerland., (2011).
[15] Ceci, C.: Utility maximization with intermediate consumption under restricted information for jump market models. Int. J. Theor. Appl. Finance, 15(06):1250040 (34 pages), (2012).
[16] Ceci, C. and K. Colaneri, K.: Nonlinear filtering for jump diffusion observations. Adv. Appl. Probab., 44(03):678–701, (2012).
[17] Ceci, C. and Gerardi, A.: Utility indifference valuation for jump risky assets. Decis. Econ. Finance, 34(2):85–120, (2011).
[18] Ceci, C. and K. Colaneri, K. and Cretarola, A.: Hedging of unit-linked life insurance contracts with unobservable mortality hazard rate via local risk-minimization. Insurance Math. Econom., 60:47–60, (2015).
[19] Ceci, C. and K. Colaneri, K. and Cretarola, A.: Unit-linked life insurance policies: Optimal hedging in partially observable market models. Insurance Math. Econom., 76:149–163, (2017).
[20] Dahl, M.: Stochastic mortality in life insurance: market reserves and mortality-linked insurance contracts. Insurance Math. Econom., 35(1):113–136, (2004).
[21] Delong, L.: Indifference pricing of a life insurance portfolio with systematic mortality risk in a market with an asset driven by a Lévy process. Scand. Actuar. J., 2009(1):1–26, (2009).
[22] Delong, L.: No-good-deal, local mean-variance and ambiguity risk pricing and hedging for an insurance payment process. Astin Bull., 42(1):203–232, (2012).
[23] Eichler, A., Leobacher, G. and Szölgyenyi, M.: Utility indifference pricing of insurance catastrophe derivatives. European Actuarial Journal, 7(2):515–534, (2017).
[24] El Karoui, N.: Les Aspects Probabilistes du Controle Stochastique. In P.L. Hennequin, editor, Ecole d’Eté de Probabilités de Saint-Flour IX-1979, volume 876 of Lecture Notes in Mathematics, pages 73–238. Springer, Berlin, Heidelberg, (1981).
[25] El Karoui, N., Peng, S. and Quenez, M.C.: Backward stochastic differential equations in finance. Math. Finance, 7(1):1–71, (1997).
[26] Heath, D. and Schweizer, M.: Martingales versus PDEs in finance: an equivalence result with examples. J. Appl. Probab., 37(04):947–957, (2000).
[27] Henderson, V. and Hobson, D.: Utility indifference pricing: An overview. In R. Carmona, editor, Indifference Pricing: Theory and Applications, chapter 2, pages 44–73. Princeton University Press, (2009).
[28] Jeanblanc, M. and Le Cam, Y.: Progressive enlargement of filtrations with initial times. Stochastic Process. Appl., 119(8):2523–2543, (2009).
[29] Jeanblanc, M., Yor, M. and Chesney, M.: Mathematical Methods for Financial Markets. Springer Science & Business Media, (2009).
[30] Jeanblanc, M., Mastrolia, T., Possamaï, D. and Réveillac, A.: Utility maximization with random horizon: a bsde approach. Int. J. Theor. Appl. Finance, 18(07):1550045, (2015).
[31] Kharroubi, I., Lim, T. and Ngoupeyou, A.: Mean-variance hedging on uncertain time horizon in a market with a jump. Appl. Math. Optim., 68(3):413–444, (2013).
[32] Kurtz, T. G. and Ocone, D. L.: Unique characterization of conditional distributions in nonlinear filtering. Ann. Appl. Probab., pages 80–107, (1988).
[33] Liang, X. and Lu, Y. Indifference pricing of a life insurance portfolio with risky asset driven by a shot-noise process. Insurance Math. Econom., 77:119–132, (2017).
[34] Lim, T. and Quenez, M.C.: Exponential utility maximization in an incomplete market with defaults. Electron. J. Probab., 16(53):1434–1464, (2011).
[35] Lim, T. and Quenez M.C.: Portfolio optimization in a default model under full/partial information. Probab. Engrg. Inform. Sci., 29(4):565–587, (2015).
[36] Lipster, R.S. and A.N. Shiryaev, A.N.: Conditionally Gaussian sequences: Filtering and related problems. In Statistics of Random Processes II: Applications, chapter 13, pages 55–97. Springer Verlag, Berlin, (2001).
[37] Ludkovski, M. and Young, V.R.: Indifference pricing of pure endowments and life annuities under stochastic hazard and interest rates. Insurance Math. Econom., 42(1):14–30, (2008).

[38] Møller, T.: Indifference pricing of insurance contracts in a product space model: applications. Insurance Math. Econom., 32(2):295–315, (2003).

[39] Øksendal, B.: Stochastic Differential Equations: an Introduction with Applications. Springer Science & Business Media, (2013).

[40] Zhang, J.: Backward Stochastic Differential Equations: From Linear to Fully Nonlinear Theory, volume 86. Springer, (2017).