Algebraic classifications for fragments of first-order logic 
and beyond

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Abstract

Complexity and decidability of logics is a major research area involving a huge 
range of different logical systems. This calls for a unified and systematic app-
roach for the field. We introduce a research program based on an algebraic 
approach to complexity classifications of fragments of first-order logic (FO) 
and beyond. Our base system GRA, or general relation algebra, is equiexpres-
sive with FO. It resembles cylindric algebra but employs a finite signature with 
only seven different operators. We provide a comprehensive classification of the 
decidability and complexity of the systems obtained by limiting the allowed sets 
of operators. We also give algebraic characterizations of the best known de-
cidable fragments of FO. Furthermore, to move beyond FO, we introduce the 
notion of a generalized operator and briefly study related systems.

1 Introduction

The failure of Hilbert’s program and the realization of the undecidability of first-
order logic FO put an end to the most prestigious plans of automating mathematical 
reasoning. However, research with more modest aims continued right away. Per-
haps the most direct descendant of Hilbert’s program was the work on the classical 
decision problem, i.e., the initiative to classify the quantifier prefix classes of FO ac-
cording to whether they are decidable or not. This major program was successfully 
completed in the 1980’s, see \[7\] for an overview.

Subsequent work has been more scattered but highly active. Currently, the 
state of the art on decidability and complexity of fragments of FO divides roughly 
into two branches: research on variants of \textit{two-variable logic} FO\textsuperscript{2} and the \textit{guarded fragment} GF. Two-variable logic FO\textsuperscript{2} is the fragment of FO where only two variable 
symbols \(x, y\) are allowed. It was proved decidable in \[35\] and NEXP\textsuperscript{TIME}-complete in \[13\]. The extension of FO\textsuperscript{2} with counting quantifiers, known as C\textsuperscript{2}, was proved 
decidable in \[14, 37\] and NEXP\textsuperscript{TIME}-complete in \[38\]. Research on variants of FO\textsuperscript{2} 
is currently very active. Recent work has focused on complexity issues in restriction 
to particular structure classes and also questions related to built-in relations, see, 
e.g., \([5, 9, 22, 23, 24, 31, 45]\) for a selection of recent contributions. See also \([15, 21]\) 
where the \textit{uniform one-dimensional fragment} \(U_1\) is defined. This system extends 
FO\textsuperscript{2} to a logic that allows an arbitrary number of variables but preserves most of 
the relevant metalogical properties, including the NEXP\textsuperscript{TIME}-completeness of the 
satisfiability problem. The article \([26]\) provides a survey on \(U_1\).
The guarded fragment GF was initially conceived as an extension of modal logic, being a system where quantification is similarly localized as in the Kripke semantics for modal logic. After its introduction in [1], it was soon proved $2\text{ExpTime}$-complete in [12]. The guarded fragment has proved successful in relation to applications, and it has been extended in several ways. The *loosely guarded* [6], *clique guarded* [11] and *packed* [32] fragments impose somewhat more liberal conditions than GF for keeping quantification localized, but the basic idea is the same. All these logics have the same $2\text{ExpTime}$-complete complexity as GF (see, e.g., [4]). The more recently introduced *guarded negation fragment* GNFO [4] is a very expressive extension of GF based on restricting the use of negation in the same way GF restricts quantification. The logic GNFO also extends the *unary negation fragment* UNFO [44] which is orthogonal to GF in expressive power. Despite indeed being quite expressive, GNFO shares the $2\text{ExpTime}$-completeness of GF, and so does UNFO.

Complexity of fragments of FO is important also in knowledge representation, especially in relation to *description logics* [2]. In this field, complexities are classified in great detail, operator by operator. The Description Logic Complexity Navigator at [http://www.cs.man.ac.uk/~ezolin/dl/](http://www.cs.man.ac.uk/~ezolin/dl/) provides an extensive and detailed taxonomy of the best known relatively recent results. Most description logics limit to vocabularies with at most binary relations, but there are notable exceptions, e.g., the $\text{ExpTime}$-complete logic DLR [8].

Somewhat less extensively studied decidable fragments of FO include the *Maslov class* [33]; the *fluted logic* [42], [39]; the *binding form* systems [34] and generalizations of prefix classes in, e.g., [46]. We should also mention the *monadic fragment* of FO—possibly the first non-trivial FO-fragment shown decidable [30]. Of these frameworks, research relating to [46] and work on fluted logic has recently been active. The systems of [46] are largely based on limiting how the variables of different atoms can overlap, while fluted logic restricts how variables can be permuted.

While the completion of the classical decision problem in the 1980s was a major achievement, that project concentrated only on a very limited picture of FO: prefix classes only. The restriction to prefix classes can be seen as a *strong limitation*, both from the theoretical and applied perspectives. The subsequent research trends—e.g., the work on the guarded fragment, FO$^2$ and description logics—of course lifted this limitation, leading to a more liberal theory with a wide range of applications from verification to database theory and knowledge representation. Indeed, as we have seen above, the current state of the art studies a huge number of different logical frameworks, tailored for different purposes. However, consequently the related research is scattered, and could surely benefit from a more systematic approach. The current article suggests a framework for such a systematic approach.

**Our contributions.** We introduce a research program for classifying complexity and decidability of fragments of FO (and beyond) within an algebraic framework. To this end, we define an algebraic system designed to enable a systematic and fine-grained approach to classifying first-order fragments. One of the key ideas is to identify a finite collection of operators to capture the expressive power of FO, so our algebra has a finite signature. In FO, there are essentially infinitely many quantifiers $\exists x_i$ due to the different variable symbols $x_i$, and this issue gives rise to the infinite signature of cylindric set algebras, which are the principal algebraic
formulation of FO. Basing our investigations on finite signatures leads to a highly controlled setting that directly elucidates how the expressive power of FO arises. This is achieved by listing a finite set of operators that the expressivity of FO is based on.

The principal system we introduce is built on the algebraic signature

\[(e, p, s, I, \neg, J, \exists)\].

The atomic terms of the related algebra are simply relation symbols (of any arity), and complex algebraic terms are built from atoms by applying the operators in the signature in the usual way. This defines an algebra over every relational structure \(\mathcal{M}\). The atomic terms \(R\) are interpreted as the corresponding relations \(R^{\mathcal{M}}\). The operators correspond to functions that modify relations into new relations over \(\mathcal{M}\) as follows.

1. \(\neg\) is the complementation operator.

2. \(J\) is the join operator.

3. \(\exists\) is the existential quantification (or projection) operator.

4. \(p\) is a cyclic permutation operator.

5. \(s\) a swap operator (swapping the last two elements of tuples).

6. \(I\) is an identification (or substitution) operator. This operator deletes tuples whose last two members are not identical and then projects away the last coordinate of the remaining tuples. The operator allows us to perform operations that in standard (non-algebraic) FO would correspond to variable substitutions.

7. \(e\) is the constant operator denoting the equality (or identity) relation over the domain of the model \(\mathcal{M}\).

We let \(\text{GRA}(e, p, s, I, \neg, J, \exists)\) refer to the system based on these operators, with GRA standing for general relation algebra. To simplify notation, we also let \(\text{GRA}\) stand for \(\text{GRA}(e, p, s, I, \neg, J, \exists)\) in the current article. Furthermore, by \(\text{GRA} \setminus f_1, \ldots, f_k\) we refer to \(\text{GRA}\) with the operators \(f_1, \ldots, f_k \in \{e, p, s, I, \neg, J, \exists\}\) removed.

We begin our study by proving that \(\text{GRA}\) and FO are equiexpressive. The next aim is to classify the decidability and complexity properties of the principal subsystems of \(\text{GRA}\). Firstly, \(\text{GRA} \setminus \neg\) is trivially decidable, every term being satisfiable. Nevertheless, \(\text{GRA} \setminus \neg\) is interesting as we show it can define precisely all conjunctive queries with equality. Then we establish that \(\text{GRA} \setminus \exists\) is NP-complete. We then show that satisfiability of \(\text{GRA} \setminus J\) can be checked by a finite automaton, and furthermore, we prove \(\text{GRA} \setminus I\) to be NP-complete. We thereby identify new decidable low-complexity fragments of FO. Including the discovery of the algebraic systems, we identify, e.g., the NP-complete fragment \(\mathcal{F}\) of FO based on the restriction that when forming conjunctions \(\varphi(x_1, \ldots, x_m) \land \psi(y_1, \ldots, y_n)\), the sets \(\{x_1, \ldots, x_m\}\) and \(\{y_1, \ldots, y_n\}\) of variables should be disjoint.

On the negative side, we show that \(\text{GRA}(p, I, \neg, J, \exists)\) is \(\Pi^P_1\)-complete, so removing either \(e\) or \(s\) (or both) from \(\text{GRA}\) does not lead to decidability. Thus we have
the following close to complete first classification: removing any of the operators $\neg$, $\exists$, $I$, $J$ gives a decidable system, while dropping $e$ or $s$ (or both) keeps the system undecidable. The only open case concerning the removal of a single operator is $\text{GRA} \setminus p$. We leave the study of the complexity and decidability of subsystems of $\text{GRA}$ there in this introductory article.

To push our program further, we define a general notion of a relation operator which essentially puts connectives and (generalized) quantifiers under the same umbrella notion. The definition can be seen as a slight generalization of the notion of a generalized quantifier due to Mostowski [36] and Lindström [29]. We then study variants of $\text{GRA}$ with different sets of relation operators.

In particular, we characterize the guarded fragment, two-variable logic and fluted logic by different algebras. The guarded fragment corresponds to the algebra $\text{GRA}(e, p, s, \setminus, \cap, \exists)$ where the symbol $\setminus$ denotes the relative complementation operator and $\cap$ is a suffix intersection operator. The suffix intersection is an operator that is similar to standard intersection but makes sense also when intersecting relations of different arities. Two-variable logic—over vocabularies with at most binary relation symbols—corresponds to $\text{GRA}(e, s, \neg, \cap, \exists)$, and fluted logic turns out to be $\text{GRA}(\neg, \cap, \exists)$.

Note that the algebras for fluted logic and two-variable logic are clearly rather intimately related (note that we do not impose restrictions on relation symbol arities for fluted logic). Also, since the guarded fragment is characterized by $\text{GRA}(e, p, s, \setminus, \cap, \exists)$, and since $\text{GRA}(e, s, \neg, \cap, \exists)$ and $\text{GRA}(e, p, s, \neg, \cap, \exists)$ can be shown equiexpressive over vocabularies with at most binary relations, we observe that also two-variable logic and the guarded fragment are very nicely and closely linked. These kinds of results demonstrate the explanatory power and potential usefulness of the idea of comparing FO-fragments under the same umbrella framework based on different kinds of finite signature algebras. Indeed, each finite operator set specifies what the building blocks of the related logic are—precisely. In this article we give characterizations mainly to some of the main decidable fragments of FO, but a natural future research direction involves also pushing these studies beyond first-order logic by using the notion of relation operator defined here. Also, finding different operator sets that are expressively complete for FO is a relevant question.

The contributions of this article can be summarized as follows.

1. The main objective is to introduce the program of systematically classifying the complexity and decidability properties of logics with different finite signature algebras. We believe it is useful to introduce the idea, as no previous analogous study on first-order logic exists.

2. Concretely, we provide a comprehensive classification of the principal subsystems of $\text{GRA}(e, p, s, I, \neg, J, \exists)$ (which itself characterizes FO). In each solved case, we also pinpoint the complexity of the system. In the process, we identify interesting new low-complexity fragments of FO.

3. We find algebraic characterizations for FO and some of its main decidable fragments such as FO$^2$, the guarded fragment and fluted logic. (Furthermore, we additionally find algebras that characterize conjunctive queries, equality-free FO, quantifier-free FO and the set of first-order atoms.) We also provide
a 2ExpTime-completeness result for the algebra of for the guarded fragment GF. This turns out to require quite intricate proof techniques and new notions (e.g., the notion of a term guard) to keep the translations between GF and the algebra polynomial. Showing NExpTime-completeness of the algebra for FO² turns out easier.

4. We define a general notion of a relation operator. This relates also to further directions for our future research summarized in the concluding section.

As yet further work, we also consider GRA(e, s, \setminus, \cap, \exists) and show it ExpTime-complete. This algebra can be seen as a new, decidable fragment of FO. Furthermore, we characterize the unary negation fragment by using a unary negation operator as well as suitable join operators.

Further notes on related work. We already extensively surveyed the related work concerning our program. We now give further related information on algebraic issues.

There are various algebraic approaches to FO, e.g., Tarski’s cylindric algebras, their semantic counterparts cylindric set algebras and the polyadic algebras of Halmos. The book [16] gives a comprehensive and relatively recent account of the these systems. Also, variants of Codd’s relational algebra [10] are important, although the main systems studied within the related database-theory-oriented setting are not equivalent to standard FO but instead relate to domain independent first-order logic. The closest approach to our system is Quine’s predicate functor logic [40], [41], [43]. This system comes in several variants, with different sets of operators used. But the spirit of the approach bears some similarity to the main algebraic system we study in this article. Variants of predicate functor logic can be naturally considered to be within the scope of our research program. Predicate functor logic has been studied very little, and we are not aware of any work relating to complexity theory that has appeared before our current work. The notable works within this framework include the complete axiomatizations given in [25] and [3]. Concerning further algebraic settings, Tarski’s relation algebra (see [16]) is also related to our work, but focuses on binary relations.

This paper is an extended, full version of the conference paper [20]. The article [19] is an early version of the current article. The paper [19] has already been followed-up by [17], where several natural extensions of so-called ordered logic, fluted logic and FO² were studied within the algebraic framework introduced below. The paper [18] is the first version of [19], containing many of the results below, but using a slightly different set of algebraic operators. The research program realized in the current article and its earlier versions was proposed in [27], which also discusses the FO-equivalence of the operators of [18] and suggests, for example, studying systems with limited permutations.

2 Preliminaries

Let A be an arbitrary set. As usual, a k-tuple over A is an element of A^k. When k = 0, we let ϵ denote the unique 0-tuple in A^0 = A^0. Note also that A^0 = B^0 = ∅ = {ϵ} for all sets A and B. Note also that ∅^k = ∅ for all positive integers k. If k is a
non-negative integer, then a $k$-ary AD-relation over a set $A$ is a pair $(R, k)$ where $R \subseteq A^k$ is a $k$-ary relation in the usual sense, i.e., simply a set of $k$-tuples. Here ‘AD’ stands for *arity definite*. We call $(\emptyset, k)$ the empty $k$-ary AD-relation. For a non-negative integer $k$, we let $\top_k$ (respectively, $\bot_k$) denote an operator that maps any set $A$ to the AD-relation $\top_k(A) := (A^k, k)$ (respectively, $\bot_k(A) := (\emptyset, k)$). We may write $\top_k$ for $\top_k(A)$ and simply $T_0$ for $\top_0(A)$, and we may write $\bot_k$ or $\bot_k$ for $\bot_k(A)$. We note that $\bot_0 = \top_0$ iff $k \neq 0$. When $T = (R, k)$ is a $k$-ary AD-relation, we let $\text{rel}(T)$ denote $R$ and write $\text{ar}(T) = k$ to refer to the arity of $T$. If $T$ is an AD-relation, $t \in T$ always means that $t \in \text{rel}(T)$.

The notion of a model is defined as usual in model theory, assuming model domains are never empty. For simplicity, we restrict attention to relational models, i.e., vocabularies of models do not contain function or constant symbols. We use the convention where the domain of a model $\mathfrak{A}$ is denoted by $A$, the domain of $\mathfrak{B}$ by $B$ et cetera. We let $\bar{\tau}$ denote the full relational vocabulary containing countably infinitely many relation symbols of each arity $k \geq 0$. We let $\text{VAR} = \{v_1, v_2, \ldots\}$ denote the countably infinite set of exactly all variables used in first-order logic FO. We also use metavariables (e.g., $x, y, z, x_1, x_2, \ldots$) to refer to symbols in $\text{VAR}$. The syntax of FO is built in the usual way, starting from the set of atoms consisting of *equality atoms* (i.e., atoms with the equality symbol $=$) and *relation atoms* $R(x_1, \ldots, x_n)$ where $R \in \bar{\tau}$. When writing relation atoms, we often write $Rx_1, \ldots, x_n$ instead of $R(x_1, \ldots, x_n)$ to simplify notation. By an FO-formula $\varphi(x_1, \ldots, x_k)$ we refer to a formula whose free variables are exactly $x_1, \ldots, x_k$. An FO-formula $\varphi$ (without a list of variables) may or may not have free variables. The set of free variables of $\varphi$ is denoted by $\text{Free}(\varphi)$.

Now, suppose $(x_1, \ldots, x_k)$ is a tuple of pairwise distinct variables and consider a formula $\varphi(x_1, \ldots, x_k)$. Suppose, likewise, that also $(y_1, \ldots, y_k)$ is a tuple of pairwise distinct variables. Then we let $\varphi(y_1, \ldots, y_k)$ denote the formula obtained from $\varphi(x_1, \ldots, x_k)$ by simultaneously replacing each free variable $x_i$ by $y_i$ for each $i \leq k$ (and avoiding variable capture in the process by renaming bounded variables suitably, if necessary).

Let $k \geq 0$ and consider an FO-formula $\varphi(v_{i_1}, \ldots, v_{i_k})$ where $i_1 < \cdots < i_k$. The formula $\varphi(v_{i_1}, \ldots, v_{i_k})$ defines the AD-relation 
\[
\{(a_{1}, \ldots, a_k) \in A^k \mid \mathfrak{A} \models \varphi(a_1, \ldots, a_k)\}, k
\]
in the model $\mathfrak{A}$. Notice that we make crucial use of the linear ordering of the subindices of the variables $v_{i_1}, \ldots, v_{i_k}$ in $\text{VAR}$. We let $\varphi^\mathfrak{A}$ denote the AD-relation defined by $\varphi$ in $\mathfrak{A}$. Notice—to give an example—that $\varphi(v_1, v_2, v_3)$ and $\varphi(v_6, v_8, v_9)$ define the same AD-relation over any model. It is important to recall this phenomenon below. When using the six metavariables $x, y, z, u, v, w$, we henceforth always assume $(x, y, z, u, v, w) = (v_{i_1}, v_{i_2}, v_{i_3}, v_{i_4}, v_{i_5}, v_{i_6})$ for some indices $i_1 < i_2 < i_3 < i_4 < i_5 < i_6$. Now, to clarify a further technical issue, let us consider the formulas $R(v_1, v_2)$ and $R(v_1, v_1)$. Observe that while $R(v_1, v_2)$ defines a binary AD-relation, the second atom $R(v_1, v_1)$ defines a unary AD-relation since the only free variable in it is $v_1$.

Consider the formulas $\varphi := v_1 \neq v_1$ and $\psi := v_1 \neq v_1 \land v_2 \neq v_2$. Now $\varphi^\mathfrak{A}$ is the empty unary AD-relation and $\psi^\mathfrak{A}$ the empty binary AD-relation. The negated formulas $\lnot \varphi$ and $\lnot \psi$ then define the universal unary and binary AD-relations $(\lnot \varphi)^\mathfrak{A} = (A, 1)$ and $(\lnot \psi)^\mathfrak{A} = (A \times A, 2)$, respectively. These are two different
AD-relations, of course. This demonstrates the intuition behind the choice to consider AD-relations rather than ordinary relations: if \( \varphi \) and \( \psi \) both defined the ordinary empty relation \( \emptyset \) in \( A \), then the action of \( \neg \) in \( A \) on the input \( \emptyset \) would appear ambiguous. Ordinary relations suffice for most purposes of studying FO, but we need to be more careful in our detailed algebraic study.

A **conjunctive query** (CQ) is a formula \( \exists x_1 \ldots \exists x_k \psi \) where \( \psi \) is a conjunction of relation atoms \( R(y_1, \ldots, y_n) \). For example \( \exists y \exists z (Rxyz \land Syzuv) \) is a CQ with the free variables \( x, u, v \). Conjunctive queries are first class citizens in database theory. A **conjunctive query with equality** (CQE) is like a CQ but also allows equality atoms in addition to relation atoms \( R(y_1, \ldots, y_n) \). (We note that \( x = y \), for example, is a CQE that defines the identity relation undefinable by CQs.)

### 3 An algebra for first-order logic

In this section we define an algebra equiexpressive with FO. To this end, consider the algebraic signature \((e, p, s, I, \neg, J, \exists)\) where \( e \) is an algebraically nullary symbol (i.e., a constant), the symbols \( p, s, I, \neg, \exists \) have arity one, and \( J \) has arity two. Let \( \tau \) be a vocabulary, i.e., a set of relation symbols. The vocabulary \( \tau \) defines a set of **terms** (or \( \tau \)-**terms**) built by starting from the the symbols \( e \) and \( R \in \tau \) and composing terms by using the symbols \( p, s, I, \neg, \exists \) in the usual way. Thereby \( e \) and each \( R \in \tau \) are terms, and if \( T \) and \( T' \) are terms, then so are \( p(T) \), \( s(T) \), \( I(T) \), \( \neg(T) \), \( J(T, T') \), \( \exists(T) \). We often leave out brackets when using unary operators and write, for example, \( I pR \) instead of \( I(p(R)) \). Each term \( T \) is associated with an arity \( ar(T) \) which, as we will see later on, equals the arity of the AD-relation that \( T \) defines on a model. We define that \( ar(R) \) is the arity of the relation symbol \( R \), and we define \( ar(e) = 2 \); \( ar(pT) = ar(T) \); \( ar(sT) = ar(T) \); \( ar(\neg T) = ar(T) \); \( ar(J(T, T'))) = ar(T) + ar(T') \). Finally, for \( I \) and \( \exists \), we define and \( ar(IT) = ar(\exists T) = ar(T) - 1 \) if \( ar(T) \geq 1 \) and \( ar(IT) = ar(\exists T) = 0 \) when \( ar(T) = 0 \).

Given a model \( A \) of vocabulary \( \tau \), each \( \tau \)-term \( T \) defines an AD-relation \( T^A \) over \( A \). The arity of \( T^A \) will indeed be equal to the arity of \( T \). Consider terms \( T \) and \( S \) and assume we have defined AD-relations \( T^A \) and \( S^A \). Then the below conditions hold.

\[
R \quad \text{Let } R \text{ be a } k\text{-ary relation symbol in } \tau, \text{ so } R \text{ is a constant term in the algebra. We define}
\]
\[
R^A = \{(a_1, \ldots, a_k) | A \models R(a_1, \ldots, a_k) \}, k) \).
\]

\[
e \quad \text{We define } e^A = \{(a, a) | a \in A \}, 2 \). The constant \( e \) is called the equality constant.
\]

\[
p \quad \text{If } ar(T) = k \geq 2, \text{ we define}
\]
\[
(p(T))^A = \{(a_k, a_1, \ldots, a_{k-1}) | (a_1, \ldots, a_k) \in T^A \}, k) \),
\]

where \((a_k, a_1, \ldots, a_{k-1})\) is the \(k\)-tuple obtained from the \(k\)-tuple \((a_1, \ldots, a_k)\) by moving the last element \(a_k\) to the beginning of the tuple. If \( ar(T) \) is 1 or
0, we define \((p(\mathcal{T}))^A = \mathcal{T}^A\). We call \(p\) the permutation operator, or cyclic permutation operator.

\(s\) ) If \(ar(\mathcal{T}) = k \geq 2\), we define
\[
(s(\mathcal{T}))^A = \\{(a_1, \ldots, a_{k-2}, a_k, a_{k-1}) | (a_1, \ldots, a_k) \in \mathcal{T}^A, k\},
\]
where \((a_1, \ldots, a_{k-2}, a_k, a_{k-1})\) is the \(k\)-tuple that is obtained from the \(k\)-tuple \((a_1, \ldots, a_k)\) by swapping the last two elements \(a_{k-1}\) and \(a_k\) but keeping the other elements as they are. If \(ar(\mathcal{T})\) is \(1\) or \(0\), we define \((s(\mathcal{T}))^A = \mathcal{T}^A\). We refer to \(s\) as the swap operator.

\(I\) ) If \(ar(\mathcal{T}) = k \geq 2\), we let \((I(\mathcal{T}))^A\) be the AD-relation
\[
\{(a_1, \ldots, a_{k-1}) | (a_1, \ldots, a_{k-1}, a_k) \in \mathcal{T}^A, a_{k-1} = a_k, k-1\}.
\]
If \(ar(\mathcal{T})\) is \(1\) or \(0\), we define \((I(\mathcal{T}))^A = \mathcal{T}^A\). We refer to \(I\) as the identification operator, or substitution operator. Intuitively, it discards away all tuples except for those where the two last elements are identical, and then it projects away the last one of the last two identical elements of the remaining tuples.

\(\neg\) ) Let \(ar(\mathcal{T}) = k\). We define
\[
(\neg(\mathcal{T}))^A = \{(a_1, \ldots, a_k) | (a_1, \ldots, a_k) \in A^k \setminus \text{rel}(\mathcal{T}^A)\}, k\).
\]
Note in particular that if \(\mathcal{T}^A = (\varnothing, 0) = \bot^A\), then \((\neg(\mathcal{T}))^A = (\{\epsilon\}, 0) = \top^A\), and vice versa, if \(\mathcal{T}^A = \top^A\), then \((\neg(\mathcal{T}))^A = \bot^A\). We call \(\neg\) the complementation operator.

\(J\) ) Let \(ar(\mathcal{T}) = k\) and \(ar(S) = \ell\). We define \((J(\mathcal{T}, S))^A\) to be the AD-relation
\[
\{(a_1, \ldots, a_k, b_1, \ldots, b_\ell) | (a_1, \ldots, a_k) \in \mathcal{T}^A, (b_1, \ldots, b_\ell) \in S^A, k + \ell\}.
\]
Here we note that \(\epsilon\) is interpreted as the identity of concatenation, so if \(\text{rel}(\mathcal{T}^A) = \{\epsilon\}\), then \((J(\mathcal{T}, S))^A = (J(S, T))^A = S^A\) and \((J(\mathcal{T}, T))^A = (\{\epsilon\}, 0)\). We call \(J\) the join operator.

\(\exists\) ) If \(ar(\mathcal{T}) = k \geq 1\), we let \((\exists(\mathcal{T}))^A\) be the AD-relation
\[
\{(a_1, \ldots, a_{k-1}) | (a_1, \ldots, a_k) \in \mathcal{T}^A, \text{ for some } a_k \in A\}, k-1\}
\]
where \((a_1, \ldots, a_{k-1})\) is the \((k-1)\)-tuple obtained by removing (i.e., projecting away) the last element of \((a_1, \ldots, a_k)\). When \(ar(\mathcal{T}) = 0\), then \((\exists(\mathcal{T}))^A = \mathcal{T}^A\). We call \(\exists\) is the existence or projection operator.
We denote this algebra by $\text{GRA}(e, p, s, I, \neg, J, \exists)$ where $\text{GRA}$ stands for general relation algebra. A set $\{f_1, \ldots, f_k\}$ of operators defines the general relation algebra $\text{GRA}(f_1, \ldots, f_k)$; we shall define various such systems below. In this paper—only to simplify notation—we write $\text{GRA}$ for $\text{GRA}(e, p, s, I, \neg, J, \exists)$. We identify $\text{GRA}(f_1, \ldots, f_k)$ with the set of $\tau$-terms of this algebra, where $\tau$ is the full relational vocabulary. On the logic side, we similarly identify $\text{FO}$ with the set of $\tau$-formulas.

Let $G$ be some set of terms of some general relation algebra $\text{GRA}(f_1, \ldots, f_k)$. Formally, the satisfiability problem for $G$ takes as input a term $T \in G$ and returns 'yes' iff there exists a model $A$ such that $T^A$ is not the empty AD-relation of arity $\text{ar}(T)$.

An FO-formula $\varphi$ and term $T$ are equivalent if $\varphi^A = T^A$ for every $\tau$-model $A$ (where $\tau$ is an arbitrary vocabulary that is large enough so that $\varphi$ is a $\tau$-formula and $T$ a $\tau$-term). For example, the formula $R(v_1, v_2)$ is equivalent to $R$, while $R(v_2, v_1) \land (P(v_1) \lor \neg P(v_1))$ is equivalent to $sR$. Note that under our definition, $R(v_3, v_4)$ and $R(v_1, v_2)$ are both equivalent to the term $R$ while the formulas are not equivalent to each other. This causes no ambiguities as long as we use the terminology carefully. Also, $R(v_1, v_2) \land v_3 = v_2$ is not equivalent to the term $R$ as it defines a ternary rather than a binary relation. Furthermore, recall that in our setting, the formula $T(v_1, v_1, v_2)$ defines a binary relation and $v_8 = v_8$ a unary relation.

In the investigations below, it is useful to remember how the use of the operator $p$ is reflected to corresponding FO-formulas: if $\text{rel}(R^A) = \{(a, b, c, d)\} = \text{rel}((Rxyz)^A)$, then $\text{rel}((pR)^A) = \{(d, a, b, c)\} = \text{rel}((Ryzux)^A)$, so the tuple $(a, b, c, d)$ has its last element moved to the beginning of the tuple, while the formula $Rxyz$ has the first variable $x$ moved to the end of the tuple of variables. It is also useful to understand how the operator $I$ works. Now, if $\text{rel}(R^A) = \{(a, b, c, d)\} = \text{rel}((Rxyz)^A)$, then

$$\text{rel}((IR)^A) = \begin{cases} \{(a, b, c, c)\} & \text{if } c = d, \\ \emptyset & \text{otherwise,} \end{cases}$$

so clearly the term $IR$ is equivalent to $Rxyz$ which is obtained from $Rxyz$ by the variable substitution that replaces $u$ with $z$.

Let $S_1$ be a set of terms of our algebra and $S_2$ a set of FO-formulas. We call $S_1$ and $S_2$ equiexpressive if each $T \in S_1$ has an equivalent formula in $S_2$ and each $\varphi \in S_2$ an equivalent term in $S_1$. The sets $S_1$ and $S_2$ are called sententially equiexpressive if each sentence $\varphi \in S_2$ has an equivalent term in $S_1$ and each term $T \in S_1$ of arity 0 has an equivalent sentence in $S_2$.

Theorem 3.1. $\text{FO}$ and $\text{GRA}$ are equiexpressive.

Proof. Let us find an equivalent term for an FO-formula $\varphi$. Consider first the cases where $\varphi$ is one of the equality atoms $x = x$, $x = y$. Then the corresponding terms are, respectively, $Ie$ and $e$.

Assume then that $\varphi$ is $R(v_{i_1}, \ldots, v_{i_k})$ for $k \geq 0$. Suppose first that no variable symbol gets repeated in the tuple $(v_{i_1}, \ldots, v_{i_k})$ and that $i_1 < \cdots < i_k$. Then the term $R$ is equivalent to $\varphi$. We then consider the cases where $(v_{i_1}, \ldots, v_{i_k})$ may have repetitions and the variables may not be linearly ordered (i.e., $i_1 < \cdots < i_k$.
does not necessarily hold). We first observe that we can permute any relation in every possible way by using the operators $p$ and $s$; for the sake of completeness, we present here the following steps that prove this claim:

- Consider a tuple $(a_1, \ldots, a_i, \ldots, a_k)$ of the relation $R^3$ in a model $\mathfrak{A}$. Now, we can move the element $a_i$ an arbitrary number $n$ of steps to the left (while keeping the rest of the tuple otherwise in the same order) by doing the following:
  1. Repeatedly apply $p$ to the term $R$, making $a_i$ the rightmost element of the tuple.
  2. Apply then the composed function $ps$ (so $s$ first and then $p$) precisely $n$ times.
  3. Apply $p$ repeatedly to put the tuple into the ultimate desired order.

- Moving $a_i$ to the right is similar. Intuitively, we keep moving $a_i$ to the left and continue even when it has gone past the leftmost element of the original tuple. Formally, we can move $a_i$ by $n$ steps to the right by performing the above three steps so that in step 2, we apply the composed function $ps$ exactly $k - n - 1$ times.

This shows that we can move an arbitrary element anywhere in the tuple, and thereby it is clear that with $p$ and $s$ we can permute a relation in all possible ways.

Since we indeed can permute tuples without restrictions, we can also deal with the possible repetitions of variables in $R(v_{i_1}, \ldots, v_{i_k})$. Indeed, we can bring any two elements to the right end of a tuple and then use $I$. We discussed this phenomenon already above, but for extra clarity, we once more illustrate the issue by providing a related, concrete example. So let us consider the formula $R(v_1, v_2, v_1)$ (which defines a binary relation). We observe that $R(v_1, v_2, v_1)$ is equivalent to the term $pIpp(R)$, so we first use $p$ twice to permute $R$, then we use $I$ to identify coordinates, and finally we use $p$ once more.

So, to sum up, we permute tuples by $p$ and $s$ and we use $I$ for identifying variables. Therefore, using $p, s, I$, we can find an equivalent term for every quantifier-free formula $R(v_{i_1}, \ldots, v_{i_k})$.

Now suppose we have equivalent terms $S$ and $T$ for formulas $\varphi$ and $\psi$, respectively. We will discuss how to translate $\lnot \varphi$, $\varphi \land \psi$ and $\exists v_i \varphi$. Firstly, clearly $\lnot \varphi$ can be translated to $\lnot S$. Translating $\varphi \land \psi$ is done in two steps. Suppose $\varphi$ and $\psi$ have, respectively, the free variables $v_{i_1}, \ldots, v_{i_k}$ and $v_{j_1}, \ldots, v_{j_l}$. We first write the term $J(S, T)$ which is equivalent to $\chi(v_1, \ldots, v_{k+l}) := \varphi(v_1, \ldots, v_k) \land \psi(v_{k+1}, \ldots, v_{k+l})$; note here the new lists of variables. We then deal with the possible overlap in the original sets $\{v_{i_1}, \ldots, v_{i_k}\}$ and $\{v_{j_1}, \ldots, v_{j_l}\}$ of variables of $\varphi$ and $\psi$. This is done by repeatedly applying $p, s$ and $I$ to $J(S, T)$ in the very same way as used above when dealing with atomic formulas. Indeed, we above observed that we can arbitrarily permute relations and identify variables by using $p, s, I$.

Finally, translating $\exists v_i \varphi$ is easy. We first repeatedly apply $p$ to the term $S$ corresponding to $\varphi$ to bring the element to be projected away to the right end of the tuple. Then we use $\exists$. After this we again use $p$ repeatedly to put the term into the final wanted form.
Translating terms to equivalent FO-formulas is straightforward.

We obtain the following characterization of atomic formulas as a corollary.

**Corollary 3.2.** GRA(p, s, I) is equiexpressive with the set of relational FO-atoms, and GRA(e, p, s, I) is equiexpressive with the set of FO-atoms.

**Proof.** We observed in the proof of Theorem 3.1 that all relational FO-atoms can be expressed in terms of p, s, I. The converse fact that every term GRA(p, s, I) expresses a relational atom is justified by the following observations. Firstly, p and s just permute relational tuples. And secondly, I has the following effect: if T and R(v_{i1}, ..., v_{ik}) are equivalent, then so are I(T) and R(v'_{i1}, ..., v'_{ik}), where (v'_{i1}, ..., v'_{ik}) is obtained from (v_{i1}, ..., v_{ik}) by replacing the occurrences of the variable with the greatest subindex $i_j$ by the variable with the second greatest subindex.

The claim for GRA(e, p, s, I) is now also clear, recalling, in particular, that x = x and y = y are equivalent to the terms Ie and e, respectively.

The FO-equivalent algebra GRA = GRA(e, p, s, I, ¬, J, ∃) is only one of many interesting related systems. Defining alternative algebras equiexpressive with FO is surely a relevant option, but it is also interesting to consider weaker, stronger and orthogonal systems. We next give a general definition that enables classifying all such algebras in a systematic way. In the definition, AD_A is the set of all AD-relations (of every arity) over A. If T_1, ..., T_k are AD-relations over a set A, then (A, T_1, ..., T_k) is called an **AD-structure**. A bijection $g : A \rightarrow B$ is an isomorphism between AD-structures $(A, T_1, ..., T_k)$ and $(B, S_1, ..., S_k)$ if $ar(T_i) = ar(S_i)$ for each i and g is an ordinary isomorphism between $(A, rel(T_1), ..., rel(T_k))$ and $(B, rel(S_1), ..., rel(S_k))$.

**Definition 3.3.** A **k-ary relation operator** $f$ is a map that outputs, given an arbitrary set A, a k-ary function $f^A : (AD_A)^k \rightarrow AD_A$. The operator f is isomorphism invariant in the sense that if the AD-structures $(A, T_1, ..., T_k)$ and $(B, S_1, ..., S_k)$ are isomorphic via $g : A \rightarrow B$, then $(A, f^A(T_1, ..., T_k))$ and $(B, f^B(S_1, ..., S_k))$ are, likewise, isomorphic via g.

An **arity-regular relation operator** is a relation operator with the property that the arity of the output AD-relation depends only on the sequence of arities of the input AD-relations.

To illustrate the notion of a relation operator, let us consider some concrete examples. Suppose $T$ and $S$ are both of arity k. We define

- $(T \cup S)^\mathcal{A} = (rel(T^\mathcal{A}) \cup rel(S^\mathcal{A}), k)$,
- $(T \cap S)^\mathcal{A} = (rel(T^\mathcal{A}) \cap rel(S^\mathcal{A}), k)$,
- $(T \setminus S)^\mathcal{A} = (rel(T^\mathcal{A}) \setminus rel(S^\mathcal{A}), k)$,

and if $T$ and $S$ have different arities, then $\cap$ and $\cup$ return $(\emptyset, 0)$ and $\setminus$ returns $T^\mathcal{A}$. Suppose then that $T$ and $S$ have arities k and $\ell$, respectively. Calling $m := \max\{k, \ell\}$, we let
with inherently infinite signatures

reational algebras
e,p,s,

equivalent to GRA(FO contains only terms of arity at most two. Thus it is easy to translate the terms into

semijoin operator

to be sententially equiexpressive with GF. The system in [28] uses, e.g., a

cabularies with at most binary relations.

so intuitively, the tuples overlap on some suffix of \((a_1, \ldots, a_m)\); note here that when

k or \(\ell\) is zero, then \((a_{m+1}, a_m)\) denotes the empty tuple \(\epsilon\). Now for example the

formula \(R(x, y) \land P(y)\) is equivalent to \(R \land P\) and the formula \(R(x, y) \land P(x)\) to

\(s(sR \land P)\). We call \(\cap\) the \textbf{suffix intersection}.

In the next section we prove that the guarded fragment GF is sententially equivalent to GRA\((e, p, s, \cap, \exists)\). We note that in [28, 16], the authors define Codd-style relational algebras \textit{with inherently infinite signatures}, and then prove the algebras to be sententially equiexpressive with GF. The system in [28] uses, e.g., a \textit{semijoin operator}, which is essentially a join operation but employs also a conjunction of identity atoms as part of the input to it. The algebra of [16] employs, e.g., a ternary join operator where one of the inputs essentially acts as a guard. Both algebras have an implicit access to variables via the infinite signatures in the usual way of Codd-style systems.

Importantly, the proofs of the characterizations in [28, 16] differ considerably from our corresponding argument, the translations from algebra to logic being inherently exponential in [28] and [16]. We carefully develop techniques that allow us to give a polynomial translation from GRA\((e, p, s, \setminus, \cap, \exists)\) to GF, which in turn allow us to prove a \(2\text{ExpTime}\) upper bound for the satisfiability problem of the algebra GRA\((e, p, s, \setminus, \cap, \exists)\), the same as that for GF. Since we will also give a polynomial translation from GF to GRA\((e, p, s, \setminus, \cap, \exists)\), it follows that the satisfiability problem for the algebra is \(2\text{ExpTime}\)-complete. Furthermore, the algebra GRA\((e, p, s, \setminus, \cap, \exists)\) is a genuinely variable-free system that indeed has a finite signature and a simple set of operators.

In addition to the guarded fragment, we can also prove the following characterization for two-variable logic.

**Theorem 3.4.** \(\text{FO}^2\) and GRA\((e, s, \neg, \cap, \exists)\) are sententially equiexpressive over vocabularies with at most binary relations.

**Proof.** The algebra GRA\((e, s, \neg, \cap, \exists)\) with at most binary relation symbols clearly contains only terms of arity at most two. Thus it is easy to translate the terms into \(\text{FO}^2\).

We then consider the converse translation. We assume that \(\text{FO}^2\) is built using \(\neg, \land\) and \(\exists\) and treat other connectives and \(\forall\) as abbreviations in the usual way.

Now, let \(\varphi \in \text{FO}^2\) be a \textit{sentence} with at most binary relations, and let \(x\) and \(y\) be the two variables that occur in \(\varphi\). Note indeed that \(\varphi\) is a sentence, not an open formula. We first convert \(\varphi\) into a sentence that does not contain any subformulas of type \(\psi(x) \land \chi(y)\) (or of type \(\psi(x) \lor \chi(y)\)) as follows. Consider any subformula \(\exists x \eta(x, y)\) where \(\eta(x, y)\) is quantifier-free. Put \(\eta\) into disjunctive normal form and distribute \(\exists x\) over the disjunctions. Then distribute \(\exists x\) also over the over conjunctions as follows. Consider a conjunction \(\alpha_i(x, y) \land \beta_i(y) \land \gamma_i\) where each of \(\alpha_i, \beta_i, \gamma_i\) are conjunctions of literals; the formula \(\gamma_i\) contains the nullary relation symbols and \(\alpha_i(x, y)\) contains the literals of type \(\pi(x, y)\) and \(\pi'(x)\). We distribute \(\exists x\) into \(\alpha_i(x, y) \land \beta_i(y) \land \gamma_i\) so that we obtain the formula \(\exists x \alpha_i(x, y) \land \beta_i(y) \land \gamma_i\).
Thereby the formula $\exists x \eta(x, y)$ gets modified into the formula

$$\bigvee_{i=1}^{n} (\exists x \alpha_i(x, y) \land \beta_i(y) \land \gamma_i)$$

which is of the right form and does not have $x$ as a free variable. Next we can repeat this process for other existential quantifiers in the formula (by treating the subformulas with one free variable in the way that atoms with one free variable were treated in the translation step for $\eta(x, y)$ described above). Having started from the sentence $\varphi$, we ultimately get a sentence that does not have subformulas of the form $\psi(x) \land \chi(y)$ or of the form $\psi(x) \lor \chi(y)$ but is nevertheless equivalent to $\varphi$.

Next we translate an arbitrary sentence $\varphi \in \text{FO}^2$ that satisfies the above condition to an equivalent term. We let $v \in \{x, y\}$ denote a generic variable. Atoms of the form $P(v)$ (respectively $v = v$) translate to $P$ (respectively $\exists e$). Relation symbols of arity 0 translate to themselves and

1. $R(x, y)$ translates to $R$,
2. $R(y, x)$ translates to $sR$,
3. $R(v, v)$ translates to $\exists (R \cap e)$,
4. $x = y$ and $y = x$ translate to the term $e$.

Now suppose we have translated $\psi$ to $T$. Then $\neg \psi$ translates to $\neg T$. If $\psi$ has one free variable $v$, then $\exists v \psi$ translates to $\exists T$. If $\psi$ has two free variables, then we either translate $\exists v \psi$ to $\exists T$ when $v$ is $y$ and to $\exists sT$ when $v$ is $x$.

Consider now a formula $\psi \land \chi$ and suppose that we have translated $\psi$ to $T$ and $\chi$ to $S$. If at least one of $\psi$ and $\chi$ is a sentence, we translate $\psi \land \chi$ to $(T \land S)$. Otherwise, due to the form of the sentence $\varphi$ to be translated, we have $\text{Free}(\psi) \cap \text{Free}(\chi) \neq \emptyset$.

Now $\psi(x, y) \land \chi(x, y)$, $\psi(y) \land \chi(x, y)$, and $\psi(x, y) \land \chi(y)$ are all translated to $T \land S$, while $\psi(x, y) \land \chi(x, y)$ and $\psi(x) \land \chi(x, y)$ are translated to $s(sT \cap sS)$ and $s(T \cap sS)$, respectively.

We note that limiting our algebraic characterizations of GF and $\text{FO}^2$ to sentential equiexpressivity is a choice based on the relative elegance of the results. We shall give related characterizations without the limitation in the full version of the paper. We note, however, that sentential equiexpressivity suffices for the almost all practical scenarios.

Now, let $\text{GRA}_2(e, s, \neg, \cap, \exists)$ denote the terms of $\text{GRA}(e, s, \neg, \cap, \exists)$ that use at most binary relation symbols; there are no restrictions on term arity, although it is easy to see that at most binary terms arise. The proof of Theorem 3.4 gives a translation from $\text{FO}^2$-sentences (with at most binary symbols) to $\text{GRA}_2(e, s, \neg, \cap, \exists)$. However, that translation is not polynomial, and thus it is not immediately clear if we get a $\text{NEXPTime}$ lower bound for the the satisfiability problem of the system $\text{GRA}_2(e, s, \neg, \cap, \exists)$. Nevertheless, we can prove the following Theorem.

**Theorem 3.5.** *The satisfiability problem of $\text{GRA}_2(e, s, \neg, \cap, \exists)$ is $\text{NEXPTime}$-complete.*
Proof. The upper bound follows from the fact that \( \text{GRA}_2(e, s, \neg, \cap, \exists) \) translates easily into \( \text{FO}^2 \) in polynomial time and \( \text{FO}^2 \) is well known to have a \( \text{NexpTime} \)-complete satisfiability problem.

We then consider the lower bound. For this, we use fluted logic (FL), defined in the appendix (Definition 8.1). The proof of Proposition 8.2 in the appendix shows that the two-variable fragment of fluted logic over vocabularies with at most binary relation symbols translates in polynomial time into \( \text{GRA}_2(e, s, \neg, \cap, \exists) \). In [39], it is established that the two-variable fragment of fluted logic is \( \text{NexpTime} \)-complete, and at most binary relations suffice for the lower bound there.

We then briefly consider fluted logic (FL), mentioned in the above proof. The logic FL is a decidable fragment of FO that has recently received increased attention in the research on first-order fragments. Definition 8.1 in the appendix gives a formal definition of the system. For the history of FL, we recommend the introduction of the article [39].

Now, it is straightforward to show (see Proposition 8.2 in the appendix) that fluted logic is equiexpressive with \( \text{GRA}(-, \cap, \exists) \). By comparing the algebraic characterizations, we observe \( \text{FO}^2 \) and fluted logic are very interestingly and intimately related, and the full system \( \text{GRA}(e, s, \neg, \cap, \exists) \) obviously contains both fluted logic and \( \text{FO}^2 \). Note also the close relationship of these systems to the algebra

\[
\text{GRA}(e, p, s, I, \neg_1, J, J, \exists)
\]

for GF. These connections clearly demonstrate how the algebraic approach can elucidate the relationships between seemingly different kinds of fragments of FO. Indeed, \( \text{FO}^2 \), FL and GF seem much more closely related than one might first suspect. By the algebraic characterizations, the three logics become associated with three simple finite collections of operators (these being the corresponding algebraic signatures), and there are nice elucidating links between the collections.

The logic UNFO is a well-established decidable fragment of FO that enjoys many of the desirable properties that modal logics have [44]. Roughly speaking, its syntax is obtained from that of FO by restricting the use of negation only to formulas that have at most one free variable. To characterize UNFO, we will need to introduce two additional relation operators. Suppose that \( T \) and \( S \) are terms of arity \( k \) AND \( \ell \) respectively. We define \((J(T, S))^A\) to be equal to

\[
\{ (a_1, \ldots, a_k, b_1, \ldots, b_\ell) \mid (a_1, \ldots, a_k) \in T^A \text{ or } (b_1, \ldots, b_\ell) \in S^A \}, k + \ell\).
\]

Thus \( J \) is the dual of \( J \). If \( k \leq 1 \), we define \((\neg_1(T))^A = (\neg(T))^A\), and otherwise \((\neg_1(T))^A = \bot_0\). We call \( \neg_1 \) the one-dimensional negation. In Appendix 9 UNFO is equiexpressive with \( \text{GRA}(e, p, s, I, \neg_1, J, J, \exists) \).

We point out yet another natural fragment inspired by our algebraic approach, namely the algebra \( \text{GRA}(e, s, \backslash, \cap, \exists) \). This algebra is interesting because, for example, it contains the guarded \( \text{FO}^2 \) and the guarded FL on the level of sentences.

Theorem 3.6. The satisfiability problem of \( \text{GRA}(e, s, \backslash, \cap, \exists) \) is \( \text{ExpTime} \)-complete.

Proof. See Appendix [10].

Going beyond FO is a key future research direction. There are many relevant possibilities. Define the equicardinality operator \( H \) such that \((H(T, S))^A = \top_0\).
if \( \text{rel}(T^\mathcal{A}) \) and \( \text{rel}(S^\mathcal{A}) \) have the same (possibly infinite cardinal) number of tuples; else output \( \bot_0 \). No restrictions on the input relation arities are imposed. Adding related quantifiers (e.g., the H"artig quantifier) to quite weak FO-fragments is known to lead to undecidability. Nevertheless, interesting related decidable systems can be defined. Also, the transitive closure operator is interesting and relevant in its various possible forms.

4 An algebra for the guarded fragment

In this section we consider \( \text{GRA}(e,p,s,\setminus,\hat{\cap},\exists) \) and show that it is sententially equiexpressive with GF. Recall that GF is the logic that has all atoms \( R(x_1,\ldots,x_k) \), \( x = y \) and \( x = x \), is closed under \( \neg \) and \( \land \), but existential quantification is restricted to patterns \( \exists x_1 \ldots \exists x_k(\alpha \land \psi) \) where \( \alpha \) is an atomic formula (a guard) having (at least) all the free variables of \( \psi \in \text{GF} \).

Going beyond FO is a key future research direction. For the sake of brevity, we mention here only one of many relevant operators to be investigated. Define the equicardinality operator \( H \) such that \( (H(T,S))^\mathcal{A} = \top_0 \) if \( \text{rel}(T^\mathcal{A}) \) and \( \text{rel}(S^\mathcal{A}) \) have the same (possibly infinite cardinal) number of tuples; else output \( \bot_0 \). No restrictions on the input relation arities are imposed. Adding related quantifiers (e.g., the H"artig quantifier) to quite weak FO-fragments is known to lead to undecidability. Nevertheless, we have already found natural decidable systems containing the operator \( H \)—to be discussed in the full version—that can be defined via suitably chosen sets of operators used.

We start by defining the notion of a term guard of a term \( T \). Term guards are a central concept in our proof. The term guard of a term \( T \) of \( \text{GRA}(e,p,s,\setminus,\hat{\cap},\exists) \) is a tuple \( (S,(i_1,\ldots,i_k)) \), where \( k = \text{ar}(T) \leq \text{ar}(S) \), with the following properties.

1. \( S \) is either \( e \) or a relation symbol occurring in \( T \).
2. The tuple \( (i_1,\ldots,i_k) \) consists of pairwise distinct integers \( i_j \) such that \( 1 \leq i_j \leq \text{ar}(S) \).
3. For every model \( \mathcal{A} \) and every tuple \( (a_1,\ldots,a_k) \in T^\mathcal{A} \), there exists a tuple \( (b_1,\ldots,b_{\text{ar}(S)}) \in S^\mathcal{A} \) so that \( (a_1,\ldots,a_k) = (b_1,\ldots,b_k) \).

The intuition is that the term guard \( (S,(i_1,\ldots,i_k)) \) of \( T \) gives an atomic term \( S \) and a list \( (i_1,\ldots,i_k) \) of coordinate positions (of tuples of \( S^\mathcal{A} \)) that guard the tuples of \( T^\mathcal{A} \). The remaining \( m - k \) coordinate positions of the tuples of \( S^\mathcal{A} \) are intuitively non-guarding.

The following lemma will be used below when translating algebraic terms to formulas of the guarded fragment.

**Lemma 4.1.** Every term \( T \in \text{GRA}(e,p,s,\setminus,\hat{\cap},\exists) \) has a term guard. Furthermore the term guard can be computed from \( T \) in polynomial time.

**Proof.** We will define inductively a mapping which maps each term \( T \) of the system \( \text{GRA}(e,p,s,\setminus,\hat{\cap},\exists) \) to a term guard for \( T \). We start by defining that \( e \) will be mapped to \( (e,(1,2)) \) and that every relational symbol \( R \) will be mapped to \( (R,(1,\ldots,\text{ar}(R))) \).
Suppose then that we have mapped a $k$-ary term $T$ to the term guard
$$(\mathcal{S}, (i_1, \ldots, i_k)).$$
Using the term guard $$(\mathcal{S}, (i_1, \ldots, i_k))$$ as a starting point, we will construct term guards for the terms $pT$ and $sT$. Firstly, term guard for $pT$ will be
$$(\mathcal{S}, (i_1, \ldots, i_{k-1})), $$
where we have simply permuted the tuple $(i_1, \ldots, i_k)$ with $p$. (Note that if $k \leq 1$, then the permuted tuple is the same as the original tuple, as $p$ leaves tuples of length up to 1 untouched.) Similarly, the term guard for $sT$ will be
$$(\mathcal{S}, (i_1, \ldots, i_{k-2}, i_k, i_{k-1})), $$
where this time we have permuted the tuple $(i_1, \ldots, i_k)$ with $s$. (Again if $k \leq 1$, the permuted tuple is the original tuple.)

The other cases are similar. Recall the assumption that we have mapped a $k$-ary term $T$ to the term guard $$(\mathcal{S}, (i_1, \ldots, i_k))$$, and suppose further that an $\ell$-ary term $P$ has been mapped to the term guard $$(\mathcal{S}', (j_1, \ldots, j_\ell)).$$ If $k \geq \ell$, then $T \cap P$ will be mapped to $$(\mathcal{S}, (i_1, \ldots, i_k))$$, and if $k < \ell$, then $T \cap P$ will be mapped to $$(\mathcal{S}', (j_1, \ldots, j_k)).$$ Independently of how the arities $k$ and $\ell$ are related, $T \cap P$ will always be mapped to $$(\mathcal{S}, (i_1, \ldots, i_k)).$$ (Recall that if the arities of the terms $Q$ and $R$ differ, then by definition $Q \cap R$ is equivalent to $Q$. If $k \geq 1$, the term $\exists T$ will be mapped to $$(\mathcal{S}, (i_1, \ldots, i_{k-1})).$$ If $k = 0$, the term $\exists T$ simply maps to the same term guard as the term $T$.

This completes the definition of the mapping. Since the mapping is clearly computable in polynomial time, the claim follows.

We will also make use of the following lemma which implies that we can use $I$ in $\text{GRA}(e, p, s, \setminus, \cap, \exists)$.

**Lemma 4.2.** The operator $I$ can be expressed with $e$ and $\cap$ as follows. If $ar(T) > 1$, then $IT$ is equivalent to $\exists (T \cap e)$, and if $ar(T) \leq 1$, then $IT$ is equivalent to $T$.

**Proof.** Immediate.

We can now prove the following.

**Theorem 4.3.** $\text{GRA}(e, p, s, \setminus, \cap, \exists)$ and $\text{GF}$ are sententially equiexpressive.

**Proof.** We will first show that for every formula $\exists x_1 \ldots \exists x_k \psi$ of $\text{GF}$, there exists an equivalent term $T$ of $\text{GRA}(e, p, s, \setminus, \cap, \exists)$. Let us begin by showing this for a formula $\varphi := \exists x_1 \ldots \exists x_k \psi$ where $\psi$ is quantifier-free. We assume
$$\varphi = \exists x_1 \ldots \exists x_k (\alpha(y_1, \ldots, y_n) \land \beta(z_1, \ldots, z_m))$$
where $\alpha(y_1, \ldots, y_n)$ is an atom and we have
$$\{z_1, \ldots, z_m\} \subseteq \{y_1, \ldots, y_n\} \quad \text{and} \quad \{x_1, \ldots, x_k\} \subseteq \{y_1, \ldots, y_n\}.$$

Now consider a conjunction $\alpha \land \rho$ where $\alpha = \alpha(y_1, \ldots, y_n)$ is our guard atom and $\rho$ an arbitrary atom whose set of variables is a subset of $\{y_1, \ldots, y_n\}$. We call such a conjunction an $\alpha$-guarded atom. For each $\alpha$-guarded atom, we can find an equivalent term as follows. First, by Corollary [3.2] we can write a term equivalent to any atomic FO-formula using $e, p, s, I$; note that we can use $I$ in $\text{GRA}(e, p, s, \setminus, \cap, \exists)$ by Lemma [7.2]. Therefore we can find terms $\mathcal{T}_\alpha$ and $\mathcal{T}_\rho$ equivalent to $\alpha$ and $\rho$, respectively. Now,
the term $T \cap S$ is not to be equivalent to $\alpha \land \rho$, as the variables in $\alpha \land \rho$ can be unfavourably ordered instead of matching each other nicely. However—recalling that $p$ and $s$ can be composed to produce arbitrary permutations—we first permute $T_\alpha$ to match $T_\rho$ at the last coordinates of tuples, then we combine the terms with $\cap$, and finally we permute the obtained term to the final desired form. In this fashion we obtain a term for an arbitrary $\alpha$-guarded atom.

Now recall the formula $\alpha(y_1, \ldots, y_n) \land \beta(z_1, \ldots, z_m)$ from above. For each atom $\gamma$ in $\beta$, let $T_\alpha^\gamma$ denote the term equivalent to the $\alpha$-guarded atom $\alpha \land \gamma$ formed from $\gamma$. The formula $\beta$ is a Boolean combination composed from atoms by using $\neg$ and $\land$. We let $T_\beta$ denote the term obtained from $\beta$ by replacing each atom $\gamma$ by the term $T_\alpha^\gamma$, each $\land$ by $\cap$ and each $\neg$ by relative complementation with respect to $T_\alpha$ (i.e., formulas $\neg \eta$ become replaced by $T_\alpha \setminus \eta^*$ where $\eta^*$ is the translation of the formula $\eta$). It is easy to show that $T_\beta$ is equivalent to $\alpha(y_1, \ldots, y_n) \land \beta(z_1, \ldots, z_m)$. Thus we can clearly use $p$ and $\exists$ in a suitable way to the term $T_\beta$ to get a term equivalent to the formula $\varphi = \exists x_1 \ldots \exists x_k (\alpha(y_1, \ldots, y_n) \land \beta(z_1, \ldots, z_m))$.

Thus we managed to translate $\varphi$. To get the full translation, we mainly just keep repeating the procedure just described. The only difference is that above the formula $\beta(z_1, \ldots, z_m)$ was a Boolean combination of atoms, while now $\beta$ will also contain formulas of the form $\exists x_1 \ldots \exists x_r (\delta \land \eta)$ in addition to atoms. Proceeding by induction, we get a term equivalent to $\exists x_1 \ldots \exists x_r (\delta \land \eta)$ by the induction hypothesis, and otherwise we proceed exactly as described above. This concludes the argument for translating formulas to terms.

Let us then consider how to translate terms into equivalent formulas of GF. The proof proceeds by induction. Since GF is closed under Boolean operators, the only non-trivial case is the translation of the projection operator $\exists$. The hard part in this case is to ensure that we can translate $\exists$ so that the resulting formula has a suitable guarding pattern (with a suitable guard atom) and thereby belongs to GF.

So suppose that we have translated $T$ to $\psi(v_1, \ldots, v_k)$. By Lemma [41] we can find a term guard $(S, (i_1, \ldots, i_k))$ for $T$. By the definition of term guards, $S$ is $e$ or some relation symbol $R$. We let $m$ denote the arity of $S$, and we let $\alpha(v_1, \ldots, v_m)$ denote

1. the formula $R(v_1, \ldots, v_m)$ if $S$ is a relation symbol,
2. the formula $v_1 = v_2$ if $S = e$ and therefore $m = 2$.

Notice that $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, m\}$ by the definition of term guards.

Now define

$$
\chi(v_{i_1}, \ldots, v_{i_k}) := \exists \overline{r}(\alpha(v_1, \ldots, v_m) \land \psi(v_{i_1}, \ldots, v_{i_k})),
$$

where $\overline{r}$ lists those variables from $\{v_1, \ldots, v_m\}$ that are not included in $(v_{i_1}, \ldots, v_{i_k})$. Modifying $\chi(v_{i_1}, \ldots, v_{i_k})$ to the formula $\chi(v_1, \ldots, v_k)$ and recalling the definition of term guards, we now observe that $\exists T$ is equivalent to $\exists v_k \chi(v_1, \ldots, v_k)$.

It is easy to see that our translations in the above proof are computable in polynomial time. Since the satisfiability problem for GF is $2\text{ExpTime}$-complete, we have the following corollary.

**Corollary 4.4.** The satisfiability problem for $\text{GRA}(e, p, s, \setminus, \cap, \exists)$ is $2\text{ExpTime}$-complete.
5 Decidable fragments of GRA

In this section we identify subsystems of $\text{GRA} = \text{GRA}(e, p, s, I, \neg, J, \exists)$ with a decidable satisfiability problem. We concentrate on systems obtained by limiting to a subset of the operators involved. We show that removing any of the operators $\neg, \exists, J, I$ leads to decidability, and we also pinpoint the exact complexity of each system. As a by-product of the work, we make observations about conjunctive queries (CQs) and show NP-completeness of, e.g., $\text{GRA}(\neg, J, \exists)$ and $\text{GRA}(I, \neg, J)$. As a further by-product, we also give a characterization for quantifier-free FO.

Our first result concerns $\text{GRA}$ with the complementation operation $\neg$ removed. All negation-free fragments of FO are trivially decidable—every formula being satisfiable—and thus so is $\text{GRA}(e, p, s, I, \exists)$. Nevertheless, this system has the following very interesting property concerning conjunctive queries with equality, or CQEs.

**Proposition 5.1.** $\text{GRA}(e, p, s, I, \exists)$ is equiexpressive with the set of CQE{s}. Also, the system $\text{GRA}(p, s, I, J, \exists)$ is equiexpressive with the set of conjunctive queries (CQs).

**Proof.** Analyzing the proof that $\text{GRA}(e, p, s, I, \neg, J, \exists)$ and FO are equiexpressive, we see that $\text{GRA}(e, p, s, I, J, \exists)$ can express every formula built from relational atoms and equality atoms with conjunctions and existential quantification. Conversely, an easy induction on term structure establishes that every term of the system $\text{GRA}(e, p, s, I, J, \exists)$ is expressible by a CQE. The claim for $\text{GRA}(p, s, I, J, \exists)$ follows similarly, noting that equality atoms are used only to express the atoms $x = x$ and $x = y$ in the proof of Theorem 3.1.

We then study $\text{GRA}$ without $\exists$. We begin with the following expressivity characterization.

**Proposition 5.2.** $\text{GRA}(e, p, s, I, \neg, J)$ is equiexpressive with the set of quantifier-free FO-formulas.

**Proof.** The fact that every term of $\text{GRA}(e, p, s, I, \neg, J)$ translates to a quantifier-free formula is seen by induction on terms. The non-trivial case is the use of $I$. Here, similarly to what we already observed in the proof of Corollary 3.2, we note that if $T$ and $\varphi$ are equivalent, then so are $I(T)$ and the formula $\varphi'$ obtained from $\varphi$ by replacing the occurrences of the free variable with the greatest subindex by the variable with the second greatest subindex. (Here we must take care to avoid variable capture.)

The fact that quantifier-free FO translates into $\text{GRA}(e, p, s, I, \neg, J)$ is immediate by inspection of the proof of Theorem 3.1.

The translations of the proposition are clearly polynomial, whence we get the following corollary due to the folklore result that the the satisfiability problem for quantifier-free FO is NP-complete.

**Corollary 5.3.** The satisfiability problem for $\text{GRA}(e, p, s, I, \neg, J)$ is NP-complete.

To sharpen the lower bound result, we also prove the following.
Proposition 5.4. The satisfiability problem for $\text{GRA}(I, \neg, J)$ is NP-complete.

Proof. The upper bound follows from Corollary 5.3. For the lower bound, we give a reduction from SAT. So, let $\varphi$ be a propositional logic formula and $\{p_1, \ldots, p_n\}$ the set of proposition symbols of $\varphi$. Let $\{P_1, \ldots, P_n\}$ be a set of unary relation symbols. We translate $\varphi$ to an equisatisfiable term $T$ of $\text{GRA}(I, \neg, J)$ as follows. Every $p_i$ translates to $P_i$. If $\psi$ is translated to $T$, then $\neg \psi$ is translated to $\neg T$. If $\psi$ translates to $\mathcal{T}$ and $\theta$ to $\mathcal{P}$, then $\psi \land \theta$ is translated to $J(\mathcal{T}, \mathcal{P})$.

Let $\mathcal{T}(\varphi)$ be the resulting term obtained like this from $\varphi$, and note that $\mathcal{T}(\varphi)$ is not yet equisatisfiable with $\varphi$, as for example $p_1 \land \neg p_1$ translates to a term equivalent to the formula $P_1(v_1) \land \neg P_1(v_2)$. Thus we still need to express that all variables $v_i$ are equal. This is easy to do with repeated use of the operator $I$.

We then leave the investigation of the $\exists$-free system $\text{GRA}(e, p, s, I, \neg, J)$ and consider the join-free fragment of GRA. This system turns out to be interestingly tame, having a very low complexity.

Theorem 5.5. Satisfiability of $\text{GRA}(e, p, s, I, \neg, \exists)$ can be checked by a finite automaton, i.e., the set of satisfiable terms is in REG.

Proof. Consider first terms that are built up starting from $e$ and using $p, s, I, \neg$ and $\exists$. If neither $\exists$ nor $I$ occurs in such a term, it is clearly satisfiable, so we may consider the case where the term has $\exists$ or $I$ in it. Let $\mathcal{T}$ denote the term under investigation. Suppose $\mathcal{T}$ is of the form $f_1 \ldots f_n g_1 \ldots g_m(e)$ where $f_n \in \{\exists, I\}$ and $g_1, \ldots, g_m \in \{p, s, \neg\}$, so the subterm $g_1 \ldots g_m(e)$ is a term of $\text{GRA}(e, p, s, \neg)$.

Notice that on every model, the interpretation of $g_1 \ldots g_m(e)$ is the identity relation if the number of negations in $g_1 \ldots g_m$ is even, and otherwise it is the complement of the identity relation. Notice also that the complement of the identity relation is the empty binary relation iff the model $\mathfrak{A}$ in question has a singleton domain, i.e., $|A| = 1$.

First suppose that $f_n = I$. Now, on every model $\mathfrak{M}$, we have $(I g_1 \ldots g_m(e))^{\mathfrak{M}} = \top^M$ if $g_1 \ldots g_m$ has an odd number of negations, and otherwise $(I g_1 \ldots g_m(e))^{\mathfrak{M}} = \bot^M$. Therefore, if $g_1 \ldots g_n$ has an odd number of negations, the term

$$\mathcal{T} = f_1 \ldots f_n g_1 \ldots g_m(e)$$

is satisfiable iff the number of negations in $f_1 \ldots f_{n-1}$ is odd. If $g_1 \ldots g_n$ has an even number of negations, the term $\mathcal{T}$ is satisfiable iff the number of negations in $f_1 \ldots f_{n-1}$ is even.

Suppose then that $f_n = \exists$, and consider satisfiability on models $\mathfrak{M}$ with a singleton domain. Restricting to such models, the criteria for satisfiability are obtained in the same way as in the above case for $f_n = I$, as now $(\exists g_1 \ldots g_m(e))^{\mathfrak{M}} = \bot^M$ if $g_1 \ldots g_m$ has an odd number of negations, and otherwise $(\exists g_1 \ldots g_m(e))^{\mathfrak{M}} = \top^M$. Thus we consider satisfiability over models with at least two elements. On every such model $\mathfrak{M}$, we always have $(\exists g_1 \ldots g_m(e))^{\mathfrak{M}} = \top^N$. Therefore the term $\mathcal{T} = f_1 \ldots f_n g_1 \ldots g_m(e)$ with $f_n = \exists$ is satisfiable on a model with at least two elements iff the number of negations in $f_1 \ldots f_{n-1}$ is even. Bringing the cases for singleton and non-singleton domains together, we see that with $f_n = \exists$, the term $\mathcal{T}$ is satisfiable if (1) the number of negations in $f_1 \ldots f_{n-1}$ is even, or (2) both $g_1 \ldots g_m$ and $f_1 \ldots f_{n-1}$ have an odd number of negations.
Based on the above, it is easy to see how to construct a finite automaton for checking satisfiability of terms built from $e$ by applying $p, s, I, \neg, \exists$. To conclude the proof, we will show that all terms of type $h_1 \ldots h_n(R)$ of $\text{GRA}(e, p, s, I, \neg, \exists)$, where $R$ is $k$-ary relation symbol, are satisfiable.

Now, define two $\{R\}$-models $A$ and $B$, both having the same singleton domain $A = \{a\}$ but with $R^A = \top^k$ and $R^B = \bot^k$. In any model $\mathfrak{M}$ with a singleton domain, every $n$-ary term $T$ of $\text{GRA}(e, p, s, I, \neg, \exists)$, where $R$ is $k$-ary relation symbol, are satisfiable. To conclude the proof, we will show that all terms of type $h_1 \ldots h_n(R)$ of $\text{GRA}(e, p, s, I, \neg, \exists)$, where $R$ is $k$-ary relation symbol, are satisfiable.

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We will then study $\text{GRA}$ without $I$ and show that it is $\text{NP}$-complete. We begin by identifying a new decidable fragment $\mathcal{F}$ of $\text{FO}$. The newly identified logic $\mathcal{F}$ turns out to be an interesting, low complexity fragment of $\text{FO}$, as we will prove it $\text{NP}$-complete. The fragment $\mathcal{F}$ is defined as follows.

1. $R(x_1, \ldots, x_n) \in \mathcal{F}$ and $x_1 = x_2 \in \mathcal{F}$ for all relation symbols $R$ and all variables $x_1, \ldots, x_n$.
2. If $\varphi, \psi \in \mathcal{F}$ and $\text{Free}(\varphi) \cap \text{Free}(\psi) = \emptyset$, then $(\varphi \land \psi) \in \mathcal{F}$.
3. If $\varphi \in \mathcal{F}$, then $\neg \varphi \in \mathcal{F}$.
4. If $\varphi \in \mathcal{F}$, then $\exists x \varphi \in \mathcal{F}$ for any $x$.

We then give a series of lemmas, ultimately showing $\text{NP}$-completeness of the system $\text{GRA}(e, p, s, \neg, J, \exists)$. We first show that the satisfiability problem of $\mathcal{F}$ is complete for $\text{NP}$. The upper bound is based on a reduction to the satisfiability problem of the set of relational Herbrand sentences. These are $\text{FO}$-sentences of the form

$$Q_1 x_1 \ldots Q_n x_n \bigwedge_i \eta_i$$

where $Q_i \in \{\exists, \forall\}$ are quantifiers and each $\eta_i$ is a first-order literal. We note the fact that checking satisfiability of equality-free relational Herbrand sentences is known to be $\text{PTIME}$-complete, see Theorem 8.2.6 in [7]. However, there seems to be no explicit proof of the $\text{PTIME}$-completeness of case with equality in the literature, so we provide it in the appendix (see Lemma [11.2] and the auxiliary Lemma [11.1] before that).

**Lemma 5.6.** The satisfiability problem of $\mathcal{F}$ is $\text{NP}$-complete.

**Proof.** The lower bound follows by the fact that if $\varphi$ is a propositional logic formula, we obtain an equisatisfiable formula of $\mathcal{F}$ by replacing each proposition symbol $p_i$ by the sentence $\forall x P_i(x)$.

We thus consider the upper bound. Let $\chi \in \mathcal{F}$ be a formula. Start by transforming $\chi$ into negation normal form, thus obtaining a formula $\chi'$. Now note that in $\mathcal{F}$, the formula $\forall x (\varphi \lor \psi)$ is equivalent to either $(\varphi \lor \psi)$, $(\forall x \varphi \lor \psi)$ or $(\varphi \lor \forall x \psi)$ since $\text{Free}(\varphi) \cap \text{Free}(\psi) = \emptyset$. Similarly, $\exists x (\varphi \land \psi)$ is equivalent to $(\varphi \land \psi)$, $(\exists x \varphi \land \psi)$.

20
or \((\varphi \land \exists x \psi)\). Thus we can push all quantifiers past all connectives in the formula \(\chi'\) in polynomial time, getting a formula \(\chi''\).

Consider now the following elementary trick. Let \(C\) be the set of all conjunctions obtained from \(\chi''\) as follows: begin from the syntax tree of \(\chi''\) and keep eliminating disjunctions \(\lor\), always keeping one of the two disjuncts. Now \(\chi''\) is satisfiable iff some \(\beta \in C\) is satisfiable. Starting from \(\chi''\), we nondeterministically guess some \(\beta \in C\) (without constructing \(C\)).

Now, \(\beta\) is a conjunction of formulas \(Q_1 x_1 \ldots Q_k x_k \eta\) where \(Q_i \in \{\forall, \exists\}\) for each \(i\) and \(\eta\) is a literal. Putting \(\beta\) in prenex normal form, we get a relational Herbrand sentence. Lemma 11.2 in the appendix proves that satisfiability of relational Herbrand sentences is complete for \(\text{PTime}\).

Lemma 5.7. GRA\((e, p, s, \neg, J, \exists)\)-terms translate to equisatisfiable formulas of \(\mathcal{F}\) in polynomial time.

**Proof.** We use induction on the structure of terms \(T\) of GRA\((e, p, s, \neg, J, \exists)\). We translate every \(k\)-ary term to a formula \(\chi(v_1, \ldots, v_k)\), so the free variables are precisely \(v_1, \ldots, v_k\). For the base case we note that \(e\) is equivalent to \(v_1 = v_2\) and \(R\) to \(R(v_1, \ldots, v_k)\). Suppose then that \(T\) is equivalent to \(\varphi(v_1, \ldots, v_k)\). Then \(\neg T\) is equivalent to \(\neg \varphi(v_1, \ldots, v_k)\) and \(3T\) to \(3v_k \varphi(v_1, \ldots, v_k)\). We translate \(sT\) to the variant of \(\varphi(v_1, \ldots, v_k)\) that swaps \(v_{k-1}\) and \(v_k\) and \(pT\) to \(\varphi(v_2, \ldots, v_1, v_1)\). Finally, suppose that \(T\) translates to \(\varphi(v_1, \ldots, v_k)\) and \(P\) to \(\psi(v_1, \ldots, v_k)\). Now \(J(T, P)\) is translated to

\[
\varphi(v_1, \ldots, v_k) \land \psi(v_{k+1}, \ldots, v_{k+\ell}).
\]

This concludes the proof.

Lemma 5.8. The satisfiability problem of GRA\((\neg, J, \exists)\) is \(\text{NP}\)-hard.

**Proof.** We give a simple reduction from SAT (which is similar to the reduction in Lemma 5.3). Let \(\varphi\) be a formula of propositional logic. Let \(\{p_1, \ldots, p_n\}\) be the set of proposition symbols in \(\varphi\), and let \(\{P_1, \ldots, P_n\}\) be a set of unary relation symbols. Let \(\varphi^*\) be the formula obtained from \(\varphi\) by replacing each symbol \(p_i\) with \(\forall x P_i(x)\). It is easy to see that \(\varphi\) and \(\varphi^*\) are equisatisfiable. Finally, since \(\forall x P_i(x)\) is equivalent to \(\neg \exists \neg P_i\), we see that the sentence \(\varphi^*\) can be expressed in GRA\((\neg, J, \exists)\).

Thereby we have now finally proved the following result for GRA without \(I\).

**Corollary 5.9.** The satisfiability problem of GRA\((e, p, s, \neg, J, \exists)\) is \(\text{NP}\)-complete.

### 6 Undecidable fragments of GRA

In this section we identify undecidable subsystems of GRA. We begin with GRA without \(e\).

**Proposition 6.1.** The satisfiability problem of GRA\((p, s, I, \neg, J, \exists)\) is \(\Pi^1_1\)-complete.

**Proof.** Analyzing the proof that GRA corresponds to FO, we easily observe that GRA\((p, s, I, \neg, J, \exists)\) is equiexpressive with the equality-free fragment of FO (which is well known to be \(\Pi^1_1\)-complete).
We then consider GRA without \( s \). To this end, we will show that the satisfiability problem of GRA\((p, I, \neg, J, \exists)\) is \( \Pi^0_1 \)-complete. Let us begin by recalling the tiling problem for \( \mathbb{N} \times \mathbb{N} \). A tile is a function \( t : \{R, L, T, B\} \rightarrow C \) where \( C \) is a countably infinite set of colors. We let \( t_X \) denote \( t(X) \). Intuitively, \( t_R, t_L, t_T \) and \( t_B \) correspond to the colors of the right, left, top and bottom edges of a tile. Now, let \( T \) be a finite set of tiles. A \( T \)-tiling of \( \mathbb{N} \times \mathbb{N} \) is a function \( f : \mathbb{N} \times \mathbb{N} \rightarrow T \) such that for all \( i, j \in \mathbb{N} \), we have \( f(i, j) = t_R \) when \( f(i, j) = t \) and \( f(i + 1, j) = t' \), and similarly, \( f_T = t_B' \) when \( f(i, j) = t \) and \( f(i + 1, j) = t' \). Intuitively, the right color of each tile equals the left color of its right neighbour, and analogously for top and bottom colors. The tiling problem for the grid \( \mathbb{N} \times \mathbb{N} \) asks, with the input of a finite set \( T \) of tiles, if there exists a \( T \)-tiling of \( \mathbb{N} \times \mathbb{N} \). It is well known that this problem is \( \Pi^0_1 \)-complete. We will show that the satisfiability problem for GRA\((p, I, \neg, J, \exists)\) is undecidable by reducing the tiling problem to it.

Define the standard grid \( \mathfrak{G}_\mathbb{N} := (\mathbb{N} \times \mathbb{N}, R, U) \) where we have \( R = \{(i, j), (i + 1, j)\} \mid i, j \in \mathbb{N} \) and \( U = \{(i, j), (i, j + 1)\} \mid i, j \in \mathbb{N} \). If \( \mathfrak{G} \) is a structure of the vocabulary \( \{R, U\} \) with binary relation symbols \( R \) and \( U \), then \( \mathfrak{G} \) is grid-like if there is a homomorphism \( \tau : \mathfrak{G}_\mathbb{N} \rightarrow \mathfrak{G} \). Consider then the extended vocabulary \( \{R, U, L, D\} \) where \( L \) and \( D \) are binary. Define

\[
\varphi_{\text{reverse}} := \forall x \forall y (R(x, y) \leftrightarrow L(y, x)) \\
\land \forall x \forall y (U(x, y) \leftrightarrow D(y, x))
\]

\[
\varphi_{\text{successor}} := \forall x (\exists y R(x, y) \land \exists y U(x, y))
\]

\[
\varphi_{\text{cycle}} := \forall x \forall y \forall z \forall u [(L(y, x) \land U(x, z) \land R(z, u)) \rightarrow D(u, y)].
\]

Then define \( \Gamma := \varphi_{\text{reverse}} \land \varphi_{\text{successor}} \land \varphi_{\text{cycle}} \). The intended model of \( \Gamma \) is the standard grid \( \mathfrak{G}_\mathbb{N} \) extended with two binary relations, \( L \) pointing left and \( D \) pointing down.

**Lemma 6.2.** Let \( \mathfrak{G} \) be a structure of the vocabulary \( \{R, U, L, D\} \). Suppose \( \mathfrak{G} \) satisfies \( \Gamma \). Then there exists a homomorphism from \( \mathfrak{G}_\mathbb{N} \) to \( \mathfrak{G} \upharpoonright \{R, U\} \), i.e., to the restriction of \( \mathfrak{G} \) to the vocabulary \( \{R, U\} \).

**Proof.** As \( \mathfrak{G} \) satisfies \( \varphi_{\text{reverse}} \) and \( \varphi_{\text{cycle}} \), it is easy to see that \( \mathfrak{G} \) satisfies the sentence

\[
\varphi_{\text{grid-like}} := \\
\forall x \forall y \forall z \forall u [R(x, y) \land U(x, z) \land R(z, u)) \rightarrow D(u, y)].
\]

Using this sentence and \( \varphi_{\text{successor}} \), it is easy to inductively construct a homomorphism from \( \mathfrak{G}_\mathbb{N} \) to \( \mathfrak{G} \upharpoonright \{R, U\} \).

The sentence \( \varphi_{\text{grid-like}} \) used above reveals the key trick in our argument towards proving undecidability of GRA\((p, I, \neg, J, \exists)\). The sentence \( \varphi_{\text{grid-like}} \) would be the natural choice for our argument rather than \( \varphi_{\text{cycle}} \). Indeed, we could replace \( \Gamma = \varphi_{\text{reverse}} \land \varphi_{\text{successor}} \land \varphi_{\text{cycle}} \) in the statement of Lemma 6.2 by \( \varphi_{\text{successor}} \land \varphi_{\text{grid-like}} \), as the proof of the lemma shows. But translating \( \varphi_{\text{grid-like}} \) to GRA\((p, I, \neg, J, \exists)\) becomes challenging due to the arrangement of the variables and the lack of \( s \) in the algebra. We solve this issue by using \( \varphi_{\text{cycle}} \) instead of \( \varphi_{\text{grid-like}} \). By extending the vocabulary, we can formulate \( \varphi_{\text{cycle}} \) so that the variables in it occur in a cyclic order. The proof of Theorem 6.4 below demonstrates that—indeed—by using this
cyclicity, we can express $\varphi_{cycle}$ in $\text{GRA}(p, I, \neg, J, \exists)$ even though it lacks the swap operator $s$.

Fix a set of tiles $T$. We simulate the tiles $t \in T$ by unary relation symbols $P_t$. Let $\varphi_T$ be the conjunction of the following four sentences (the second one could be dropped):

$$
\forall x \forall_{t \in T} P_t(x),
\forall x \neg(P_t(x) \land P_v(x)),
\forall x \forall y \neg(P_t(x) \land R(x, y) \land P_v(y)),
\forall x \forall y \neg(P_t(x) \land U(x, y) \land P_v(y)).
$$

Now $\varphi_T$ expresses that $N \times N$ is $T$-tilable:

**Lemma 6.3.** $N \times N$ is $T$-tilable iff $\varphi_T \land \Gamma$ is satisfiable.

**Proof.** Suppose there is a model $\mathfrak{B}$ so that $\mathfrak{B} \models \varphi_T \land \Gamma$. Therefore, by Lemma 6.2 there exists a homomorphism $\tau : \mathfrak{B}_N \rightarrow \mathfrak{B} \upharpoonright \{R, U\}$. Define a tiling $T$ of $N \times N$ by setting $T(i, j) = t$ if $\tau((i, j)) \in P_t$. Since $\mathfrak{B} \models \varphi_T$ and $\tau$ is homomorphism, the tiling is well-defined and correct.

Now suppose that there is a tiling $T$ of $N \times N$ using $T$. Thus we can expand $\mathfrak{B}_N = (N \times N, R, U)$ to $\mathfrak{B}_N^T = (N \times N, R, U, L, D, (P_t)_{t \in T})$ in the obvious way. Clearly $\mathfrak{B}_N^T \models \varphi_T \land \Gamma$. 

We are now ready to prove the following theorem.

**Theorem 6.4.** The satisfiability problem of $\text{GRA}(p, I, \neg, J, \exists)$ is $\Pi_1^0$-complete.

**Proof.** The upper bound follows by $\text{GRA}(p, I, \neg, J, \exists)$ being contained in FO. For the lower bound, we will establish that $\varphi_T$ and each sentence in $\Gamma$ can be expressed in $\text{GRA}(p, I, \neg, J, \exists)$.

Now, note that $\varphi_{inverse}$ is equivalent to the conjunction of the four sentences

$$
\forall x \forall y (R(x, y) \rightarrow L(y, x)) \land \forall x \forall y (L(y, x) \rightarrow R(x, y)),
\forall x \forall y (U(x, y) \rightarrow D(y, x)) \land \forall x \forall y (D(y, x) \rightarrow U(x, y)).
$$

Let us show how to express $\forall x \forall y (R(x, y) \rightarrow L(y, x))$ in $\text{GRA}(p, I, \neg, J, \exists)$; the other conjuncts are treated similarly. Consider the formula $R(x, y) \rightarrow L(y, x)$. To express this, consider first the formula $\psi := R(x, y) \rightarrow L(z, u)$ which can be expressed by the term $T = \neg J(R, \neg L)$. Now, to make $\psi$ equivalent to $R(x, y) \rightarrow L(y, x)$, we could first write $y = z \land x = u \land \psi$ and then existentially quantify $z$ and $u$ away. On the algebraic side, an essentially corresponding trick is done by transitioning from $T$ first to $Ip(T)$ and then to $IpIp(T)$ and finally reordering this by $p$, i.e., going to $pIpIp(T)$. This term is equivalent to $R(x, y) \rightarrow L(y, x)$. Therefore the sentence $\forall x \forall y (R(x, y) \rightarrow L(y, x))$ is equivalent to $\forall \forall p Ip Ip T$ where $\forall = \neg \exists$.

Consider then the formula $\varphi_{cycle} = \forall x \forall y \forall z \forall u ([L(y, x) \land U(x, z) \land R(z, u)] \rightarrow D(u, y))$. In the quantifier-free part, the variables occur in a cyclic fashion, but with repetitions. We first translate the repetition-free variant $(L(v_1, v_2) \land U(v_3, v_4) \land R(v_5, v_6)) \rightarrow D(v_7, v_8)$ by using $\neg$ and $J$, letting $T$ be the resulting term. Now we would need to modify $T$ so that the repetitions are taken into account. To introduce one repetition, first use $p$ on $T$ repeatedly to bring the involved coordinates to the
right end of tuples, and then use $I$. Here $p$ suffices (and $s$ is not needed) because $\varphi_{\text{cycle}}$ was designed so that the repeated variable occurrences are cyclically adjacent to each other in the variable ordering. Thus it is now easy to see that we can form a term $T'$ equivalent to $(L(v_1, v_2) \land U(v_2, v_3) \land R(v_3, v_4)) \rightarrow D(v_4, v_1)$, and $T'$ can easily be modified to a term for $\varphi_{\text{cycle}}$.

From subformulas of $\varphi_T$, consider the formula $\neg (P_l(x) \land R(x, y) \land P_l'(y))$. Here $\psi(x, y) := R(x, y) \land P_l'(y)$ is equivalent to $T := IJ(R, P_l')$ and $P_l(x) \land \psi(x, y)$ thus to $pIJ(pT, P_l)$. The rest of $\varphi_T$ and the other remaining formulas are now easy to translate.

The satisfiability problem of $\text{GRA}(e,p,I,\neg,J,\exists)$ is $\Pi^0_1$-complete.

7 Conclusion

The principal aim of the article has been to introduce our program that facilitates a systematic study of logics via algebras based on finite signatures.

After presenting GRA, we proved it equivalent to FO. We also provided algebraic characterizations for FO$^2$, GF and fluted logic and introduced a general notion of a relation operator. We then provided a comprehensive classification of the decidability of subsystems of GRA. Out of the cases obtained by removing one operator, only the case for GRA without $p$ was left open. In each solved case we also identified the related complexity.

Our work can be continued into many directions; the key is to identify relevant collections of relation operators—as defined above—and provide classifications for the thereby generated systems. This work can naturally involve systems that capture FO, but also stronger, weaker and orthogonal ones. In addition to decidability, complexity and expressive power, also completeness of equational theories (including the one for GRA) is an interesting research direction. Furthermore, model checking of different particular systems and model comparison games for general as well as particular sets of relation operators are relevant topics.

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References

[1] Hajnal Andréka, István Németi, and Johan van Benthem. Modal languages and bounded fragments of predicate logic. Journal of Philosophical Logic, 27(3):217–274, 1998.

[2] Franz Baader, Ralf Küsters, and Frank Wolter. Extensions to description logics. In Franz Baader, Diego Calvanese, Deborah L. McGuinness, Daniele Nardi, and Peter F. Patel-Schneider, editors, The Description Logic Handbook: Theory, Implementation, and Applications, pages 219–261. Cambridge University Press, 2003.
[3] John Bacon. The completeness of a predicate-functor logic. *Journal of Symbolic Logic*, 50:903–921, 1985.

[4] Vince Bárány, Balder ten Cate, and Luc Segoufin. Guarded negation. *Journal of the ACM*, 62(3):22:1–22:26, 2015.

[5] Saguy Benaim, Michael Benedikt, Witold Charatonik, Emanuel Kieronski, Rastislav Lenhardt, Filip Mazowiecki, and James Worrell. Complexity of two-variable logic on finite trees. *ACM Transactions on Computational Logic*, 17(4):32:1–32:38, 2016.

[6] Johan Van Benthem. Dynamic bits and pieces. ILLC research report, University of Amsterdam, 1997.

[7] Egon Börger, Erich Grädel, and Yuri Gurevich. *The Classical Decision Problem*. Perspectives in Mathematical Logic. Springer, 1997.

[8] Diego Calvanese, Giuseppe De Giacomo, and Maurizio Lenzerini. On the decidability of query containment under constraints. In Alberto O. Mendelzon and Jan Paredaens, editors, *Proceedings of the Seventeenth ACM SIGACT-SIGMOD-SIGART Symposium on Principles of Database Systems*, pages 149–158. ACM Press, 1998.

[9] Witold Charatonik and Piotr Witkowski. Two-variable logic with counting and trees. *ACM Transactions on Computational Logic*, 17(4):31:1–31:27, 2016.

[10] Edgar F. Codd. Relational completeness of data base sublanguages. *Research Report / RJ / IBM / San Jose, California*, RJ987, 1972.

[11] Erich Grädel. Invited talk: Decision procedures for guarded logics. In Harald Ganzinger, editor, *Automated Deduction - CADE-16, 16th International Conference on Automated Deduction, Proceedings*, volume 1632 of *Lecture Notes in Computer Science*, pages 31–51. Springer, 1999.

[12] Erich Grädel. On the restraining power of guards. *Journal of Symbolic Logic*, 64(4):1719–1742, 1999.

[13] Erich Grädel, Phokion Kolaitis, and Moshe Vardi. On the decision problem for two-variable first-order logic. *Bulletin of Symbolic Logic*, 3(1):53–69, 1997.

[14] Erich Grädel, Martin Otto, and Eric Rosen. Two-variable logic with counting is decidable. In *Proceedings, 12th Annual IEEE Symposium on Logic in Computer Science LICS*, pages 306–317. IEEE, 1997.

[15] Lauri Hella and Antti Kuusisto. One-dimensional fragment of first-order logic. In Rajeev Goré, Barteld P. Kooi, and Agi Kurucz, editors, *Invited and contributed papers from the tenth conference on "Advances in Modal Logic" AiML*, pages 274–293. College Publications, 2014.

[16] Robin Hirsch and Ian Hodkinson. *Relation algebras by games*. North Holland, 2002.
[17] Reijo Jaakkola. Ordered Fragments of First-Order Logic. In 46th International Symposium on Mathematical Foundations of Computer Science (MFCS 2021), volume 202 of Leibniz International Proceedings in Informatics (LIPIcs), pages 62:1–62:14, 2021.

[18] Reijo Jaakkola and Antti Kuusisto. Algebraic classifications for fragments of first-order logic and beyond. CoRR, abs/2005.01184v1, 2020.

[19] Reijo Jaakkola and Antti Kuusisto. Algebraic classifications for fragments of first-order logic and beyond. arXiv Preprint, arXiv:2005.01184v2, 2021.

[20] Reijo Jaakkola and Antti Kuusisto. Complexity classifications via algebraic logic. In Proceedings of Computer Science logic CSL 2023, to appear, 2023.

[21] Emanuel Kieronski and Antti Kuusisto. Complexity and expressivity of uniform one-dimensional fragment with equality. In Erzsébet Csuhaj-Varjú, Martin Dietzfelbinger, and Zoltán Ésik, editors, Mathematical Foundations of Computer Science 2014 - 39th International Symposium, MFCS, Proceedings, Part I, volume 8634 of LNCS, pages 365–376. Springer, 2014.

[22] Emanuel Kieronski, Jakub Michaliszyn, Ian Pratt-Hartmann, and Lidia Tendera. Two-variable first-order logic with equivalence closure. SIAM Journal on Computing, 43(3):1012–1063, 2014.

[23] Emanuel Kieronski, Ian Pratt-Hartmann, and Lidia Tendera. Equivalence closure in the two-variable guarded fragment. J. Log. Comput., 27(4):999–1021, 2017.

[24] Eryk Kopczynski and Tony Tan. Regular graphs and the spectra of two-variable logic with counting. SIAM J. Comput., 44(3):786–818, 2015.

[25] Steven T. Kuhn. An axiomatization of predicate functor logic. Notre Dame Journal of Formal Logic, 24:233–241, 1983.

[26] Antti Kuusisto. On the uniform one-dimensional fragment. In Maurizio Lenzerini and Rafael Peñaloza, editors, Proceedings of the 29th International Workshop on Description Logics, Cape Town, South Africa, April 22-25, 2016, volume 1577 of CEUR Workshop Proceedings. CEUR-WS.org, 2016.

[27] Antti Kuusisto. On games and computation. CoRR, abs/1910.14603, 2019.

[28] Dirk Leinders, Maarten Marx, Jerzy Tyszkiewicz, and Jan Van den Bussche. The semijoin algebra and the guarded fragment. Journal of Logic, Language and Information, 14(3):331–343, 2005.

[29] Per Lindström. First order predicate logic with generalized quantifiers. Theoria, 32(3):186–195, 1966.

[30] Leopold Löwenheim. Über möglichkeiten im relativkalkül. Matematische Annalen, 76(4):447–470, 1915.
[31] Amaldev Manuel and Thomas Zeume. Two-variable logic on 2-dimensional structures. In Simona Ronchi Della Rocca, editor, Computer Science Logic 2013 (CSL 2013), volume 23 of LIPIcs, pages 484–499. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2013.

[32] Maarten Marx. Tolerance logic. Journal of Logic, Language and Information, 10(3):353–374, 2001.

[33] S. Ju. Maslov. The inverse method for establishing deducibility for logical calculi. In V. P. Orevkov, editor, Logical and logical-mathematical calculus. Part I, Trudy Mat. Inst. Steklov, volume 98, pages 26–87. Public, 1968.

[34] Fabio Mogavero and Giuseppe Perelli. Binding forms in first-order logic. In Stephan Kreutzer, editor, 24th EACSL Annual Conference on Computer Science Logic, CSL, volume 41 of LIPIcs, pages 648–665, 2015.

[35] Michael Mortimer. Reasoning about strategies: On the model-checking problem. Mathematical Logic Quarterly, 21(1), 1975.

[36] Andrzej Mostowski. On a generalization of quantifiers. Fundamenta Mathematicae, 44(1):12–36, 1957.

[37] Leszek Pacholski, Wieslaw Szwast, and Lidia Tendera. Complexity of two-variable logic with counting. In Proceedings, 12th Annual IEEE Symposium on Logic in Computer Science LICS, pages 318–327. IEEE, 1997.

[38] Ian Pratt-Hartmann. Complexity of the two-variable fragment with counting quantifiers. Journal of Logic, Language and Information, 14(3):369–395, 2005.

[39] Ian Pratt-Hartmann, Wieslaw Szwast, and Lidia Tendera. The fluted fragment revisited. Journal of Symbolic Logic, 84(3):1020–1048, 2019.

[40] Willard Van Quine. Toward a calculus of concepts. The Journal of Symbolic Logic, I:2–25, 1936.

[41] Willard Van Quine. Variables explained away. In Proceedings of the American Philosophical Society, 1960.

[42] Willard Van Quine. On the limits of decision. In Proceedings of the 14th International Congress of Philosophy, volume III, pages 57–62. University of Vienna, 1969.

[43] Willard Van Quine. Algebraic logic and predicate functors. In Logic and Art, pages 214–238. Bobbs-Merrill, Indianapolis, Indiana, 1972.

[44] Luc Segoufin and Balder ten Cate. Unary negation. Logical Methods in Computer Science, 9(3), 2013.

[45] Szymon Torunczyk and Thomas Zeume. Register automata with extrema constraints, and an application to two-variable logic. In Holger Hermanns, Lijun Zhang, Naoki Kobayashi, and Dale Miller, editors, LICS ’20: 35th Annual ACM/IEEE Symposium on Logic in Computer Science, pages 873–885. ACM, 2020.
8 Appendix: An algebra for fluted logic

Definition 8.1. Here we provide the definition of fluted logic (FL) as given in [39]. Fix the infinite sequence $v_{\omega} = (v_1, v_2, \ldots)$ of variables. For every $k \in \mathbb{N}$, we define sets $FL^k$ as follows.

1. Let $R$ be an $n$-ary relation symbol and consider the subsequence $(v_{k-n+1}, \ldots, v_k)$ of $v_{\omega}$ containing precisely $n$ variables. Then $R(v_{k-n+1}, \ldots, v_k) \in FL^k$.
2. For every $\varphi, \psi \in FL^k$, we have that $\neg \varphi, (\varphi \land \psi) \in FL^k$.
3. If $\varphi \in FL^{k+1}$, then $\exists v_{k+1} \varphi \in FL^k$.

Finally, we define the set of fluted formulas to be $FL := \bigcup_k FL^k$.

Proposition 8.2. $FL$ and $GRA(\neg, \land, \exists)$ are equiexpressive.

Proof. We first translate formulas to algebraic terms. Formulas of the form $R(v_{k-n+1}, \ldots, v_k)$ are translated to $R$. Note that when if $R$ has arity 0, then $R(v_{k-0+1}, v_k)$ of course denotes the formula $R$ (which translates to the term $R$). Suppose then that $\neg \varphi, (\varphi \land \psi) \in FL^k$ and that we have translated $\varphi$ to $T$ and $\psi$ to $S$. We translate $\neg \varphi$ to $\neg T$. Now, observe that if $\alpha \in FL^k$, then the free variables of $\alpha$ form some suffix of the sequence $(v_1, \ldots, v_k)$. Thus we can translate $(\varphi \land \psi)$ to $(T \land S)$. Finally, if $\exists v_{k+1} \varphi \in FL^k$ and $\varphi$ translates to $T$, then we can translate $\exists v_{k+1} \varphi$ to $\exists T$.

We then translate algebraic terms into fluted logic. An easy way to describe the translation is by giving a family of translations $f_{v_m, \ldots, v_k}$ where $(v_m, \ldots, v_k)$ is a suffix of $(v_1, \ldots, v_k)$. (It is also possible that $f_{v_m, \ldots, v_k} = f_{v_{k+1}, v_k}$ which happens precisely when translating a term of arity zero.) The translations are as follows.

1. $f_{v_m, \ldots, v_k}(R) := R(v_m, \ldots, v_k)$ for $ar(R) = k - m + 1$. (When $ar(R) = 0$, then $R$ translates to $R$.)
2. $f_{v_m, \ldots, v_k}(\neg T) := \neg f_{v_m, \ldots, v_k}(T)$ for $ar(T) = k - m + 1$.
3. $f_{v_m, \ldots, v_k}(T \land S) := f_{v_m, \ldots, v_k}(T) \land f_{v_m, \ldots, v_k}(S)$ for $ar(T \land S) = k - m + 1$; $ar(T) = k - n + 1$; and $ar(S) = k - \ell + 1$.
4. $f_{v_m, \ldots, v_k}(\exists T) = \exists v_{k+1} f_{v_m, \ldots, v_{k+1}}(T)$ for $ar(\exists T) = k - m + 1$.

This concludes the proof.
9 Appendix: An algebra for unary negation fragment

We will start by formally defining the syntax of unary negation fragment.

**Definition 9.1.** We define unary negation fragment (UNFO) as the set of formulas generated by the following grammar

\[ \varphi ::= x = y \mid R(x_1, \ldots, x_k) \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \exists x \varphi \mid \neg \varphi(x), \]

where \( R \) is a relation symbol and the formula \( \varphi(x) \) has at most one free variable.

Using the technique of Theorem 3.1, one can easily show the following.

**Proposition 9.2.** UNFO and \( \text{GRA}(e, p, s, I, \neg, J, \exists) \) are equiexpressive.

**Proof.** Clearly terms of \( \text{GRA}(e, p, s, I, \neg, J, \exists) \) can be translated to UNFO. For the converse direction, one can imitate the proof of Theorem 3.1. Using \( e, p, s \) and \( I \) we can again translate arbitrary atomic formulas into equivalent terms. Conjunctions and disjunctions can be translated using \( J \) and \( J \) respectively together with \( p, s \) and \( I \). Existential quantifiers can be translated using \( p \) and \( \exists \). Finally, the unary negation can be translated by just using \( \neg \). \( \square \)

10 Appendix: Proof of Theorem 3.6

Here we prove Theorem 3.6. That is, we show that the satisfiability problem of \( \text{GRA}(e, s, \setminus, \land, \exists) \) is \text{ExpTime}-complete by designing an alternating polynomial space Turing machine which solves it (the lower bound being clear). Before starting, we first go through some useful notation that we are going to employ. We will use \( \exists^n \) to denote a sequence of length \( n \) that consists only of the symbol \( \exists \), and we will use \( \forall^n \) to denote \( \neg \exists^n \). Furthermore, we will use \( \cup \) and \( \dot{\cup} \) to denote the dual operators of \( \cap \) and \( \dot{\cap} \), which are clearly definable using \( \cap, \dot{\cap} \) and \( \neg \). Finally, given two \( k \)-ary terms \( T \) and \( P \), we will use \( \models \) to denote that \( T \) entails \( P \), i.e., for every model \( A \) and \( a \in A \), we have that if \( a \in T_A \) then \( a \in P_A \).

We begin by introducing a suitable version of Scott normal form for our algebra.

**Definition 10.1.** We say that a 0-ary term \( T \in \text{GRA}(e, s, I, \setminus, \land, \exists) \) is in normal form, if it has the following form

\[ n_1 \exists \alpha_i \cap \prod_{j=1}^{n_2} \forall \lambda_j \cap \bigcap_{i=1}^{n_3} \forall \gamma_i (\neg \alpha_i \cup \exists \beta_i) \cap \bigcap_{j=1}^{n_4} \neg \beta_j \cup (\gamma_j \cup \forall (\neg \delta_j \cup S_j)), \]

where \( \alpha_j \) are relation symbols, \( \beta_j \) are either \( I e \) or a term of \( \text{GRA}(s, I) \), \( \kappa_i, \lambda_j, \gamma_j, \delta_j \) are terms of \( \text{GRA}(e, s, I) \) and \( \beta \) and \( S \) are terms of \( \text{GRA}(e, s, I) \). The terms \( \forall \lambda_j \) are called **unary universal requirements** while the terms \( \forall \gamma_j (\neg \beta_j \cup (\gamma_j \cup \forall (\neg \delta_j \cup S_j))) \) are called **polyadic universal requirements**.

We will require the following easy technical lemma, which is essentially a simplified version of Lemma 4.1.

**Lemma 10.2.** Let \( T \in \text{GRA}(e, s, I, \setminus, \land, \exists) \) be a \( k \)-ary term. Then there exists a \( k \)-ary term \( \alpha \in \text{GRA}(e, s, I) \) such that \( T \) is equivalent with \( \alpha \cap T \).
Proof. We prove the existence of \( \alpha \) using induction. If \( T \) is \( R \), for some relation symbol \( R \), or \( e \), then we can set \( \alpha := T \). Suppose then that the claim holds for \( T \) and \( P \), i.e., there exists \( \alpha, \beta \in \text{GRA}(e, s) \) such that \( T \) and \( P \) are equivalent with \( \alpha \cap T \) and \( \beta \cap P \) respectively. For \( sT \) we can choose the term \( sa \). For \( IT \) we can choose the term \( Ia \). For \( (T \setminus P) \) we can take \( \alpha \) (even if \( ar(T) \neq ar(P) \)). Finally, if \( ar(T) \geq ar(P) \), then for \( (T \cap P) \) we can choose \( \alpha \), and otherwise we can choose \( \beta \). 

The following lemma is fairly standard, but proving it in our setting seems to require a quite involved argument.

**Lemma 10.3.** There is a polynomial time nondeterministic procedure, taking as its input a 0-ary term \( T_1 \in \text{GRA}(e, s, \setminus, \cap, \exists) \) and producing a 0-ary term \( T_2 \in \text{GRA}(e, s, I, \setminus, \neg, \cap, \exists) \) in normal form (possibly over an extended signature) such that the following two conditions hold.

1. If \( T_1^{\exists} \) is non-empty, for some model \( A \), then there exists a run of the procedure which produces a term \( T_2 \) so that \( T_2^{\exists} \) is non-empty for some expansion of \( A \).
2. If the procedure has a run which produces \( T_2 \), then for every model \( A \) for which \( T_2^{\exists} \) is non-empty, we have that also \( T_1^{\exists} \) is non-empty.

Proof. Choose an innermost subterm of \( T \) which is of the form \( \exists P \), where \( P \in \text{GRA}(e, s, \setminus, \cap) \). If \( P \) is unary, then we will guess a truth value, and replace \( \exists P \) with either \( \bot \) or \( T \) according to this guess. If the resulting term is \( T' \), then \( T \) is equi-satisfiable with either \( T' \cap \exists P \) or \( T' \cap \neg \exists P \).

Consider then the case where the arity of \( P \) is strictly greater than one. Let \( \alpha \in \text{GRA}(e, s) \) be the term guaranteed by Lemma 10.2. If \( \alpha \) is equivalent with \( e \), then we will replace \( P \) with \( IP \) in \( T \).

Suppose then that \( \alpha \) is not equivalent with \( e \), i.e., it is a term of \( \text{GRA}(s, I) \). Suppose that \( \exists P \) is an \( n \)-ary term. We now replace \( \exists P \) with a fresh \( n \)-ary relation \( Q \). If \( T' \) denotes the resulting term, then \( T \) is equi-satisfiable with \( T' \cap \forall^{\exists}(Q \cup \exists(\alpha \cap P)) \cap \forall^{\exists}(Q \cup \forall(\neg \alpha \cup \neg P)) \).

Note that \( \forall^{\exists}(Q \cup \forall(\neg \alpha \cup \neg P)) \) is not yet in normal form.

Since \( T' \) is a 0-ary term, there must be a subterm \( \exists S \) of \( T' \) such that \( Q \) occurs as a subterm of \( S \). Now we will repeat the rewriting process that we did on \( \exists P \) on all of the (proper) subterms of \( S \) which are of the form \( \exists R \). If \( S' \) denotes the resulting term, then we will use \( T'' \) to denote the term obtained from \( T' \) by replacing \( S \) with \( S' \). Note that \( S' \in \text{GRA}(e, s, I, \setminus, \cap) \). Thus, using Lemma 10.2 again, we can deduce that it has a guard \( \beta \in \text{GRA}(e, s, I) \) such that \( S' \) is equivalent with \( \beta \cap S' \).

We have now two cases based on whether or not \( \beta \) is equivalent with \( e \). If \( \beta \) is not equivalent with \( e \), and hence is a member of \( \text{GRA}(s, I) \), then we first replace \( \exists S' \) with \( \exists(\beta \cap S') \), after which for every fresh relation symbol \( Q \), that was introduced during the previous rewriting step, we replace the corresponding universal requirements
\[
\forall^n(Q \cup \forall(\neg \delta \cup \neg R))
\]
with the following universal requirements
\[
\forall^n(\neg \beta \cup (Q \cup \forall(\neg \delta \cup \neg R))),
\]
which are in normal form.

Consider then the case where \( \beta \) is equivalent with \( \varepsilon \). First, we replace \( \exists S' \) with the term \( IS' \). Then, for every fresh relation symbol \( Q \) that was introduced during the previous rewriting step, we replace the corresponding universal requirements

\[
\forall^n(Q \cup \forall(\neg \delta \cup \neg R))
\]

with the following universal requirements

\[
\forall^{n-1}(IQ \cup \forall(\varepsilon \cup P)) \cap \forall^n(\neg P \cup \forall(\neg \delta \cup \neg R)),
\]

where \( P \) is a fresh binary relation symbol. We note that the term \( \forall^{n-1}(IQ \cup \forall(\varepsilon \cup P)) \) is still not in normal form. However, if \( IQ \) is a unary term, then we can rewrite it as a term in normal form by employing \( Ie \) as a guard.

Suppose then that \( IQ \) is at least a binary relation. Using again the fact \( T'' \) is 0-ary, \( IS' \) must occur as a subterm in a term of the form \( \exists S'' \). Thus we can repeat the rewriting process that we did on \( \exists S \) on \( \exists S'' \) until we either reach a term \( \beta \) in GRA\((s, I)\) that we can use as a guard or we reach a situation where \( Ie \) can be used as a guard.

Having done the previous rewriting, we reach a term of the form \( T_1 \cap T_2 \), where \( T_2 \) is in normal form. If \( T_1 \) is in normal form, then we are done. Otherwise we repeat the whole rewriting process on \( T_1 \).

Before we present the promised alternating polynomial space Turing machine for solving the satisfiability problem of \( \text{GRA}(e, s, \emptyset, \top, \exists) \), we will need to introduce several technical definitions. Let \( \tau \) be a relational vocabulary. A \( k \)-ary \((s, I)\)-atom over \( \tau \) is an \( \ell \)-ary term in \( \text{GRA}(s, I) \), where \( \ell \leq k \). A \( k \)-ary \((s, I)\)-literal over \( \tau \) is a term in \( \text{GRA}(s, I, \neg) \) which is either of the form \( \alpha \) or \( \neg \alpha \), where \( \alpha \) is a \( k \)-ary \((s, I)\)-atom over \( \tau \). For \( k \geq 2 \), we define \( k \)-ary \((s, I)\)-table \( \tau \) as a maximal set \( \rho \) of \( k \)-ary \((s, I)\)-literals of arity at least two over \( \tau \) for which \( \bigcap_{\alpha \in \rho} \alpha \) is satisfiable. We will identify tables \( \rho \) with terms \( \bigcap_{\alpha \in \rho} \alpha \). 1-type is a maximally consistent set of unary \((s, I)\)-literals over \( \tau \). We will identify 1-types \( \pi \) with terms \( \bigcap_{\alpha \in \pi} \alpha \). For \( k \geq 2 \), a \( k \)-type over \( \tau \) is a triple \((\pi_1, \pi_2, \rho)\), where \( \pi_1, \pi_2 \) are 1-types over \( \tau \) and \( \rho \) is a \( k \)-ary \((s, I)\)-table over \( \tau \), such that \( \pi_1 \cap s(\pi_2 \cap \rho) \) is satisfiable. We will identify types \((\pi_1, \pi_2, \rho)\) with terms \( \pi_1 \cap s(\pi_2 \cap \rho) \). For notational ease, we will sometimes use \((\pi_1, \pi_2, \rho)\) to also denote the 1-type \( \pi_2 \).

Given a \( k \)-type \((\pi_1, \pi_2, \rho)\) and a \((k + 1)\)-type \((\pi'_1, \pi'_2, \rho')\), we say that the latter is an extension of the former, if \( \pi'_1 = \pi_2 \). Consider a \( k \)-type \((\pi_1, \pi_2, \rho)\) over \( \tau \). A reduct of \((\pi_1, \pi_2, \rho)\) is an \((\ell + p)\)-type \((\pi'_1, \pi'_2, \rho')\), such that one the following conditions holds:

1. \( \rho' \) consists of the set of all \( p \)-ary \((s, I)\)-literals \( \alpha \) for which there exists an unary term in \( \pi_2 \) which is equivalent with \( I^p \alpha \), and \( \pi'_1 = \pi'_2 = \pi_2 \).
2. \( \rho' \) consists of the set of all \((\ell + p)\)-ary \((s, I)\)-literals \( \alpha \) for which there exists an \( \ell \)-ary term in \( \rho \) which is equivalent with \( I^p \alpha \), and if \( p = 0 \), then \( \pi'_1 = \pi_1 \) and \( \pi'_2 = \pi_2 \), and if \( p > 0 \), then \( \pi'_1 = \pi'_2 = \pi_2 \).
3. \( \rho' \) consists of the set of all \((\ell + p)\)-ary \((s, I)\)-literals \( \alpha \) for which there exists an \( \ell \)-ary term in \( \rho \) which is equivalent with \( s^p \alpha \), and if \( p = 0 \), then \( \pi'_1 = \pi_2 \) and \( \pi'_2 = \pi_1 \), and if \( p > 0 \), then \( \pi'_1 = \pi'_2 = \pi_1 \).
Both in the first and in the second case we say that $(\pi_1', \pi_2', \rho')$ is a simple reduct of $(\pi_1, \pi_2, \rho)$.

Observe that so far types do not impose any equality constraints. To take these constraints into account, we will introduce two useful pieces of notation. First, if $\xi$ is a $k$-type and $P \in \text{GRA}(e, s, I, \setminus, \cap)$ is a $k$-ary term, then we write $\xi \models_\gamma P$, if $\neg(e \cap (\xi \cap \gamma P))$ is not satisfiable. Similarly, we write $\xi \not\models P$, if $\neg(e \cap (\xi \cap \gamma P))$ is not satisfiable.

Consider a unary universal requirement $\forall \lambda_j$. We say that a type $\xi = (\pi_1, \pi_2, \rho)$ violates this universal requirement, if $\pi_2 \cup \lambda_j$ is not satisfiable. Consider then a polyadic universal requirement

$$\forall m_j (\neg \beta_j \cup (\gamma_j \cup \forall (\neg \delta_j \cup S_j)))$$

and let $\xi$ be a type and let $\xi'$ be its extension. We say that the pair $(\xi, \xi')$ violates this universal requirement, if one of the following conditions holds.

1. $ar(\xi') = ar(\beta_j)$, $\xi \models \beta_j \cap \neg \gamma_j$ and $\xi' \models \delta_j$, but $\xi' \not\models P_i$.

2. There exists a $ar(\beta_j)$-type $\xi''$, which is a reduct of $\xi'$, and a $(ar(\beta_j) + 1)$-ary simple reduct $\xi'''$ of $\xi'$ such that $\xi'' \models \beta_j \cap \neg \lambda_j$ and $\xi''' \models \delta_j$, but $\xi''' \not\models S_j$.

The following estimate will play a crucial role when we will analyze the space requirements of our algorithm.

**Lemma 10.4.** Let $\tau$ be a relational vocabulary of size $m$. Now the number of non-equivalent $(s, I)$-atoms over $\tau$ is at most $2mK$, where $K = \max\{ar(R) \mid R \in \tau\}$. Furthermore, the number of non-equivalent types over $\tau$ is bounded above by $2^{m(2K+2)}$.

**Proof.** We start by noting that for every term $T$ we have that $ssT$ is equivalent with $T$ and $IsT$ is equivalent with $IT$. In particular, every $(s, I)$-atom over $\tau$ is (up to equivalence) either of the form $I^nR$ or $sI^nR$, for some $k$-ary $R \in \tau$, where $k \geq n$. For a fixed $k$-ary relation $R$ the number of such terms is bounded above by $2k$ and hence the number of $(s, I)$-atoms is bounded above by $2mK$. The furthermore part follows from the fact that every 1-type and every table is uniquely determined via the subset of $(s, I)$-atoms that it contains. \hfill \square

Let $\mathcal{A}$ be a $\tau$-model of vocabulary $\tau$ and let $(a_1, \ldots, a_k) \in A^k$. We say that $(a_1, \ldots, a_k)$ realizes a $k$-type $(\pi_1, \pi_2, \rho)$, if $a_{k-1} \in \pi_1^\mathcal{A}, a_k \in \pi_2^\mathcal{A}$ and $(a_1, \ldots, a_k) \in \rho^\mathcal{A}$. Notice that each $k$-tuple realizes a unique type. The type realized by a tuple $(a_1, \ldots, a_k)$ in a model $\mathcal{A}$ is denoted by $\text{tp}_\mathcal{A}(a_1, \ldots, a_k)$.

Now we are finally in a position where we can define the promised alternating polynomial space Turing machine for solving the satisfiability problem of the system $\text{GRA}(e, s, \setminus, \cap, \exists)$. As its input the algorithm receives a 0-ary term $T$ in normal form; let $\tau$ denote the vocabulary of $T$. We set $N_* := 2^{\mid \varphi \mid (2^{|\varphi|+2})}$. To simplify the presentation, we will assume that $n_1 = 1$.

**Existentially guess** a 1-type $\xi$ over $\tau$.

If $\xi \not\models \kappa_1$ then reject

If $\xi$ violates any unary universal requirements of $T$ then reject

32
For $1 \leq c \leq N_\tau + 1$ do

Universally choose $1 \leq i \leq n_\exists$ and a reduct $\xi'$ of $\xi$ so that $\alpha_i \in \xi'$
Set $\xi''$ to be the $(\text{ar}(P_i))$-ary simple reduct of $\xi'$
If $\xi'' \models P_i$ then skip
Existentially guess an extension $\xi'''$ of $\xi''$
If $\xi''' \not\models P_i$ then reject
If $(\xi', \xi''')$ violates any universal requirements of $\mathcal{T}$ then reject
Set $\xi := \xi'''$

Accept

It is straightforward to verify that the above algorithm is complete, i.e., if the input term is satisfiable, then the algorithm will accept. The following lemma shows that the algorithm is also sound.

Lemma 10.5. If the above algorithm accepts a given 0-ary term $\mathcal{T}$ in normal form, then the term is satisfiable.

Proof. Suppose that the existential player $\exists$ has a winning strategy $\sigma$ in the game played over the configuration graph of the above algorithm when it receives the 0-ary $\mathcal{T}$ in normal form as its input. Without loss of generality we can clearly assume that the extensions that $\sigma$ recommends to $\exists$ to select depend only on the current type $\xi$ and not on the value of $c$.

Now let $M$ denote the set of all $(s,I)$-types that $\sigma$ instructs $\exists$ to choose in different positions of the game. Since the for-loop is executed for $N_\tau + 1$ steps, where $N_\tau$ is an upper bound on the number of types over $\tau$, and the extensions that $\sigma$ recommends to $\exists$ do not depend on the value of $c$, every type in $M$ occurs as the current type $\xi$ in the for-loop at least once.

Using $\sigma$ and $M$, we will construct a tree-like model for $\mathcal{T}$ in a step-by-step manner. More formally, we will construct a sequence of models

$$A_0 \leq A_1 \leq \ldots$$

where $A \leq B$ means that $A$ is a submodel of $B$, in such a way that their union $A$ has the property that $\mathcal{T}A$ is non-empty. During the construction we will maintain the invariant that all the types of live tuples of these models, by which we mean either singletons or tuples that belong to the interpretation of some term from $\text{GRA}(s,I)$, are from $M$.

We start the construction of this sequence as follows. Let $\xi$ denote the first 1-type that $\sigma$ instructs $\exists$ to choose. As the first model $A_0$ we will take a model whose domain consists of a single element $a$ which satisfies the property that $\text{tp}_{A_0}[a] = \xi$. Since $\sigma$ is a winning strategy for $\exists$, $\xi$ does not violate any unary universal requirements of $\mathcal{T}$.

Suppose then that we have already constructed $A_n$ and we need to construct $A_{n+1}$. To achieve this, we need to provide witnesses for $n$-fresh tuples of $A_n$. Let $(a_1, \ldots, a_k)$ be an arbitrary $n$-fresh tuple of $A_n$ and let $\xi$ denote the type that it realizes in $A_n$ and let $1 \leq i \leq n_\exists$. We say that $(a_1, \ldots, a_k)$ is an $i$-defect, if there
exists a reduct $\xi'$ of $\xi$ so that $\alpha \in \xi'$, but $\xi'' \not\models \mathcal{P}_i$, where $\xi''$ is the $ar(\mathcal{P}_i)$-ary simple reduct of $\xi'$. Our goal is to provide witnesses for $i$-defects.

Suppose that $(a_1, \ldots, a_k)$ is an $i$-defect. If $\xi$ denotes its type, then we let $\xi'$ denote a reduct of $\xi$ for which $\alpha \in \xi'$. Now there exists $\ell \leq k$ and $p \geq 0$ such that either $(a_\ell, \ldots, a_{k-1}, b, a_k)$ or $(a_\ell, \ldots, a_1, a_k)$, where the suffix $(a_k, \ldots, a_k)$ contains $p+1$ elements, realizes $\xi'$. Suppose that the first tuple realizes $\xi'$ (the other case being entirely analogous). Let $\sigma$ be the type that $\exists$ instructs player $\exists$ to choose. We now extend the model $\mathfrak{A}_n$ by adding a fresh element $b$ and by specifying that

$$\text{tp}_{\mathfrak{A}_{n+1}}(a_\ell, \ldots, a_{k-1}, a_k, \ldots, a_k, b) = \xi''.$$ 

Notice that since $\xi''$ is an extension of $\xi'$, the element $a_k$ has the same 1-type both in $\xi'$ and in $\xi''$. Since $\sigma$ is a winning strategy, we have that $(a_\ell, \ldots, a_{k-1}, a_k, \ldots, a_k, b) \in \mathcal{P}_i$. Furthermore, since $\sigma$ is a winning strategy, this assignment does not violate any of the universal requirements of $T$.

Having provided witnesses for all the $i$-defects, we note that the resulting model $\mathfrak{A}_{n+1}$ is still incomplete, since there might be single elements for which we have not yet specified their 1-types (recall that by definition the types that we are using intuitively only specify the 1-types of the last two elements). To each such element we will simply assign the 1-type that we assigned to the single element in $\mathfrak{A}_0$. Having done this, there might still be tuples $(a_1, \ldots, a_k)$ and relation symbols $R \in \tau$ such that we have not yet specified whether or not $(a_1, \ldots, a_k)$ belongs to the interpretation of $R$. We complete the structure $\mathfrak{A}_{n+1}$ in a minimal way by specifying that in every such case the tuple does not belong to the interpretation of the corresponding relation symbol. The resulting model will be selected as the model $\mathfrak{A}_{n+1}$.

To conclude, we note that since the binary encoding of $N_\tau$ is of size at most polynomial with respect to $|T|$ and all the types over $\tau$ are also of size at most polynomial with respect to $|T|$, our algorithm clearly uses only polynomial space. This completes the proof of Theorem 3.6.

11 Appendix: Relational Herbrand formulas

**Lemma 11.1.** Checking whether a relational Herbrand sentence has a model of domain size one can be done in polynomial time.

**Proof.** Over models of size one, every relational Herbrand sentence is equivalent to a sentence of the form $\forall x \bigwedge \eta_i$ where each $\eta_i$ is either an atomic formula or a negation of such. If the sentence contains a formula $\neg x = x$, then it is unsatisfiable. Otherwise such a sentence is satisfiable iff it does not contain two complementary formulas $\eta_i$ and $\neg \eta_i$, which can be easily verified in polynomial time. \(\square\)

**Lemma 11.2.** The satisfiability problem for relational Herbrand sentences is PTime-complete.

**Proof.** The satisfiability problem for equality-free relational Herbrand formulas is PTime-complete by Theorem 8.2.6 of [7], so the lower bound follows from that result. Thus we only need to discuss the upper bound here. We will show this by a
reduction to equality-free Herbrand sentences, essentially by showing how equalities can be eliminated.

Let \( \varphi \) be a relational Herbrand sentence. If one of the conjuncts in \( \varphi \) is a literal of type \( \neg x = x \), then \( \varphi \) is unsatisfiable. And if one of the literals is \( x = x \), we can remove it and thereby obtain an equisatisfiable sentence. (In the extreme special case where all the conjuncts of \( \varphi \) are equalities of type \( x = x \), we cannot remove all of them, but in this case \( \varphi \) is clearly satisfiable.) Therefore we may assume that each equality and negated equality in \( \varphi \) has two different variables in it.

We also assume that every variable in \( \varphi \) is quantified precisely once. Moreover, for notational convenience, we assume that if \( \varphi \) contains an atomic formula \( x_i = x_j \) (or a negation of such), then the quantifier binding \( x_j \) appears in the scope of the quantifier binding \( x_i \). If \( x_i \) is a variable appearing in \( \varphi \), then we say that it is existentially (respectively, universally) quantified if the quantifier binding it is existential (respectively, universal).

We then show how to eliminate and deal with equalities and negated equalities from \( \varphi \) by considering a number of cases.

Suppose first that \( \varphi \) contains a formula \( \neg x_i = x_j \), so that one of the following cases holds.

1. \( x_i \) is existentially and \( x_j \) universally quantified.
2. \( x_i \) and \( x_j \) are both universally quantified.

Then clearly \( \varphi \) is not satisfiable, and we can reject it. Suppose then that \( \varphi \) contains a formula \( x_i = x_j \) so that one of the two conditions below holds.

1. \( x_i \) is existentially and \( x_j \) universally quantified.
2. \( x_i \) and \( x_j \) are both universally quantified.

Then \( \varphi \) is clearly satisfiable iff it has a model of size one, which can be checked in polynomial time by lemma 11.1.

In the remaining cases, we can eliminate equalities and negated equalities from \( \varphi \) as follows. First suppose that \( \varphi \) contains at least one instance of a formula \( \neg x_i = x_j \) so that either

1. \( x_i \) is universally and \( x_j \) existentially quantified, or
2. \( x_i \) and \( x_j \) are both existentially quantified.

Then we introduce a fresh binary relation symbol \( E \) and replace all such formulas \( \neg x_i = x_j \) with the formula \( \neg E(x_i, x_j) \). If \( \varphi^* \) denotes the resulting formula, then clearly \( \varphi \) is equisatisfiable with \( \forall x E(x, x) \land \varphi^* \), which can be easily transformed to a relational Herbrand sentence.

Suppose then that \( \varphi \) contains an instance of a formula \( x_i = x_j \) so that

1. \( x_i \) is universally and \( x_j \) existentially quantified, or
2. \( x_i \) and \( x_j \) are both existentially quantified.
In both cases we can remove the existential quantifier which is binding the variable $x_j$ from the sentence $\varphi$ and replace $x_j$ with $x_i$ in every atomic formula that appears in $\varphi$.

Thus we have essentially shown, for all types of equalities and negated equalities, how they can be eliminated from $\varphi$ so that an equisatisfiable formula is obtained. Since the resulting sentence belongs to the Herbrand fragment without equality, its satisfiability can be checked in polynomial time. \qed