MONOTONICITY FORMULAS UNDER RESCALED RICCI FLOW

JUN-FANG LI

Abstract. In this short notes, we discuss monotonicity formulas under various rescaled versions of Ricci flow. The main result is Theorem 2.1.

1. Functionals $W_{ek}$ from rescaled Ricci flow point of view

This is the research notes when the author wrote [Li07]. In the first section, we discuss the relation between functionals $W_{ek}(g, f, \tau)$ and rescaled Ricci flow.

In Theorem 4.2 [Li07], we have defined functionals $F_k(g, f) = \int_M (kR + |\nabla f|^2) e^{-f} d\mu$ where $k \geq 1$ and derived the first variational formula under a coupled system (1.2) as following

\[
\frac{d}{dt} F_k(g_{ij}, f) = 2(k - 1) \int_M |Rc|^2 e^{-f} d\mu + 2 \int_M |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu \geq 0.
\]

In particular, this yields the following theorem.

**Theorem 1.1.** (Theorem 5.2 in [Li07]) On a compact Riemannian manifold $(M, g(t))$, where $g(t)$ satisfies the Ricci flow equation for $t \in [0, T)$, the lowest eigenvalue $\lambda_k$ of the operator $-4\Delta + kR$ is nondecreasing under the Ricci flow. The monotonicity is strict unless the metric is Ricci-flat.

**Remark 1.2.** Formula (1.1) previously was found by physicists independently [OSW05]. We thank professor E. Woolgar for giving us the reference.

Based on these observations, to classify expanding Ricci breathers, we have introduced a family of new functionals $W_{ek}$ which has monotonicity properties modeled on expanders. There is a closed relation between functional $F_k$ and $W_{ek}$ connected by rescaled Ricci flow. Actually, this was one of the motivations for us to introduce $W_{ek}$ (see Remark 6.2 in [Li07]).

It is well-known that there is a one-to-one correspondence between Ricci flow and rescaled Ricci flow. Suppose $g(\cdot, t)$ is a solution of Ricci flow equation

\[
\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.
\]

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For any given function \( s(t) \), if \( \varphi(t) = \frac{1}{1 - \frac{2}{n} \int_0^t s(t) dt} \), and \( \bar{t} = \int_0^t \varphi(t) dt \), then \( \bar{g}(\cdot, \bar{t}) = \varphi(t) g(\cdot, t) \) solves the rescaled Ricci flow equation

\[
\frac{\partial}{\partial \bar{t}} \bar{g}_{ij} = -2(\bar{R}_{ij} - \frac{s}{n} \bar{g}_{ij})
\]

and

\[
\frac{\partial g_{ij}}{\partial f} = -2R_{ij} \quad \frac{\partial f}{\partial \bar{t}} = -\Delta f + |\nabla f|^2 - R
\]

Furthermore, (1.1) is equivalent to

\[
\frac{d}{dt} \tilde{F}_k = -2s_n \tilde{F}_k + 2(k - 1) \int_M |\bar{R}c|^2 e^{-f} d\mu + 2 \int_M |\bar{R}_{ij} + \bar{\nabla}_i \bar{\nabla}_j f|^2 e^{-f} d\bar{\mu},
\]

where we use \( \tilde{F}_k \) to denote \( \mathcal{F}_k(\bar{g}_{ij}, \bar{f}) = \int_M (k\bar{R} + \bar{\Delta} f) e^{-f} d\bar{\mu} \).

Notice that this rescaled Ricci flow is a generalized version which includes Hamilton’s normalized Ricci flow as a special case.

When \( s = 0 \), this is the monotonicity formula under Ricci flow without rescale. When \( s < 0 \), we have the monotonicity under rescaled Ricci flow. By the same proof of Theorem 5.2 in [Li07], we have the following monotonicity formula for the lowest eigenvalues under the rescaled Ricci flow.

**Proposition 1.3.** On a compact Riemannian manifold \((M, g(t))\), if \( g(t) \) satisfies the rescaled Ricci flow equation (1.3) for \( t \in [0, T) \) with \( s \leq 0 \), then the lowest eigenvalue \( \lambda \) of the operator \(-4\Delta + kR \) \((k \geq 1)\) is nondecreasing under the rescaled Ricci flow.

Compare Proposition 1.3 with Theorem 1.1, the main difference is Proposition 1.3 fails to classify the steady state of the lowest eigenvalues which is crucial for applications of monotonicity formulas in general. Namely when the monotonicity is not strict, it yields no information. Thus one cannot apply this formula directly to classify steady or expanding breathers while the functionals \( \mathcal{W}_{ek} \) we previously introduced served well for this purpose.

From the point of view of rescaled Ricci flow, it is natural for us to introduce functionals \( \mathcal{W}_{ek} \). Using (1.5), we have

\[
\frac{d}{dt} \tilde{F}_k = \frac{2s}{n} \tilde{F}_k - \frac{2k}{n} \tilde{F}_k^2 + 2(k - 1) \int_M |\bar{R}c - \frac{s}{n} \bar{g}|^2 e^{-f} d\mu + 2 \int_M |\bar{R}_{ij} + \bar{\nabla}_i \bar{\nabla}_j f - \frac{s}{n} \bar{g}|^2 e^{-f} d\bar{\mu}.
\]
In the following, if we choose \( s = \text{constant} \), and define \( \tilde{W}_k \overset{\text{def}}{=} W_k(\bar{g}, \bar{f}) = e^{-\frac{2\pi t}{s}}(\bar{g} - ks) \), then
\[
(1.7) \quad \frac{d}{dt} \tilde{W}_k = e^{-2snt}[2(k-1)\int_M |Rc - \frac{s}{k}g|^2e^{-f}d\mu + 2\int_M |\tilde{R}_{ij} + \nabla_i\nabla_j\bar{f} - \frac{s}{n}g|^2e^{-\bar{f}}d\bar{\mu}].
\]

We can write out \( \tilde{t}(t) \) and \( \phi(t) \) explicitly. If we denote \( \tau(t) = -\frac{2n}{s}\phi \), then
\[
\tau = (-\frac{2n}{s} + t).
\]
Now back to the corresponding unrescaled Ricci flow system, we have established
\[
(1.8) \quad W_k(g, f, \tau) \overset{\text{def}}{=} \tau^2 \int_M [k(R + \frac{n}{2\tau}) + \Delta f]e^{-f}d\mu = W_k(\bar{g}, \bar{f})
\]
and
\[
(1.9) \quad \frac{d}{dt} W_k = \frac{d}{dt} \tilde{W}_k \cdot \frac{d}{dt} \tilde{t} = 2(k-1)\tau^2 \int_M |Rc - \frac{1}{2\tau}g|^2e^{-f}d\mu + 2\tau^2 \int_M |\tilde{R}_{ij} + \nabla_i\nabla_j\bar{f} - \frac{1}{2\tau}g|^2e^{-\bar{f}}d\bar{\mu}
\]
This is the same formula of \( W_{ek} \) that we obtained in [Li07]. The advantage of this formula is, clearly, we can read that the steady states of the above functionals are Einstein manifolds. One can further apply this property to classify expanding breathers, see [Li07]. As an application to lowest eigenvalues, we have

**Theorem 1.4.** On a compact Riemannian manifold \((M, g(t))\), where \( g(t) \) satisfies Ricci flow equation (1.2) for \( t \in [0, T) \), if \( \lambda \) denotes the lowest eigenvalue of the operator \(-4\Delta + kR \), then \( \tau^2(\lambda + k\frac{n}{2\tau}) \) is nondecreasing under Ricci flow with \( \frac{d\tau}{dt} = 1 \). The monotonicity is strict unless the metric is Einstein. (See Remark 2.3.)

Back to the rescaled Ricci flow, we have the twin theorem of the above.

**Theorem 1.5.** On a compact Riemannian manifold \((M, g(t))\), where \( g(t) \) satisfies the rescaled Ricci flow equation (1.3) for \( t \in [0, T) \) with \( s = \text{constant} \), if \( \lambda \) denotes the lowest eigenvalue of the operator \(-4\Delta + kR \), then \( e^{-\frac{2\pi s}{s}(\lambda - ks)} \) is nondecreasing under the rescaled Ricci flow. The monotonicity is strict unless the metric is Einstein. (See Remark 2.3.)

**Remark 1.6.** One can find applications of the above theorems in Kähler Ricci flow, in which case, one can choose \( s \) to be constants.

2. **Monotonicity of lowest eigenvalues under rescaled Ricci flow**

A direct consequence of (1.6) yields the following monotonicity property.

**Theorem 2.1.** On a compact Riemannian manifold \((M^n, g(t))\), where \( g(t) \) satisfies the rescaled Ricci flow equation (1.3) for \( t \in [0, T) \), we denote \( \lambda(t) \) to be the lowest eigenvalue of the operator \(-4\Delta + kR \) \((k \geq 1)\) at time \( t \). Assume that there exists a function \( \varphi(x, t) \in C^\infty(M^n \times [0, T)) \), such that \( s(t) = \)
\[ \frac{1}{k} \int_M (kR + |\nabla \varphi|^2)e^{-\varphi}d\mu, \] and also \( \lambda(t) \) is a \( C^1 \) family of \( t \), then \( \lambda \) is nondecreasing under the rescaled Ricci flow, provided \( s \leq 0 \). The monotonicity is strict unless the metric is Einstein.

**Proof.** To simplify notations, we remove all the \( \bar{\ } \) in (1.6). Under the hypothesis of the theorem, proceed as before, we have

\[
\frac{d\lambda}{dt} = \frac{2s}{n}(\lambda - ks) + 2(k - 1) \int_M |Rc - \frac{s}{n}g|^2 e^{-f}d\mu + 2 \int_M |R_{ij} + \nabla_i \nabla_j f - \frac{s}{n}g|^2 e^{-f}d\mu,
\]

where \( e^{-f} \) is the eigenfunction of \( \lambda \). We know \( s \leq 0 \), and by definitions of \( \lambda \), \( s \), we have \( \lambda \leq ks \) and \( \frac{2s}{n}(\lambda - ks) \geq 0 \). Hence,

\[
\frac{d\lambda}{dt} \geq 2(k - 1) \int_M |Rc - \frac{s}{n}g|^2 e^{-f}d\mu + 2 \int_M |R_{ij} + \nabla_i \nabla_j f - \frac{s}{n}g|^2 e^{-f}d\mu.
\]

This completes the proof of the theorem. \( \square \)

**Remark 2.2.** (1.6) can be obtained by direct computations instead of using rescaling of the metrics.

**Remark 2.3.** In the proof of Theorems 1.4, 1.5, and 2.1, when \( k = 1 \), one actually first prove that it is a compact gradient expanding (or steady) Ricci soliton, then by the known classification theorems, we know it must be Einstein.

In case of Hamilton’s normalized Ricci flow, where \( s = \frac{\int_M Rd\mu}{\int_M d\mu} \) is the average total scalar curvature, and one can choose \( \varphi(t) = \ln Vol(M^n) \), we have

**Corollary 2.4.** On a compact Riemannian manifold \((M^n, g(t))\), where \( g(t) \) satisfies the normalized Ricci flow equation of Hamilton for \( t \in [0, T) \), if \( \lambda(t) \) denotes the lowest eigenvalue of the operator \(-4\Delta + kR \) (\( k \geq 1 \)) at time \( t \), assume \( \lambda(t) \) is a \( C^1 \) family of \( t \), then \( \lambda \) is nondecreasing under the normalized Ricci flow, provided the average total scalar curvature is nonpositive. The monotonicity is strict unless the metric is Einstein.

**Remark 2.5.** The above monotonicity property of lowest eigenvalues under normalized Ricci flow is dimensionless and work for all \( k \geq 1 \) and the case of \( k > 1 \) can be used to classify compact steady or expanding Ricci breathers directly, which, in fact, yields another proof for Corollary 8.1 in [Li07]. See references of related results in [Ha88], [Iv93], [Pe02], [GROUP07], [Ca07a] and [Li07].

**Remark 2.6.** Similar results of Corollary 2.4 appeared recently in [Ca07b].

In the special case \( k = 1 \), recall Perelman’s \( \bar{\lambda} = \lambda V \frac{\varphi}{V} \) invariant [Pe02]. Since normalized Ricci flow preserves the volume and \( \bar{\lambda} \) is a scale invariant, the monotonicity of lowest eigenvalues under normalized Ricci flow is equivalent to the monotonicity of \( \bar{\lambda} \) under Ricci flow. Naturally, one could extend
this definition and introduce a new scale invariant constant \( \bar{\lambda}_k \overset{\text{def}}{=} \lambda_k V^2/n \) which also has monotonicity properties along (normalized) Ricci flow and vanishes if and only if on an expanding Einstein manifold (instead of expanding gradient solitons in the \( k = 1 \) case).

**Remark 2.7.** There is some fundamental relation between \( \bar{\lambda} \) and the Yamabe constant \( Y \) discussed in [AIL07] and the references therein. Similar results can also be obtained for \( \lambda_k \).

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**Department of Mathematics, McGill University, Montreal, Quebec. H3A 2K6, Canada.**

*E-mail address: jli@math.mcgill.ca*