Cooperative Parrondo’s Games on a Two-dimensional Lattice

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Abstract

Cooperative Parrondo’s games on a regular two dimensional lattice are analyzed based on the computer simulations and on the discrete-time Markov chain model with exact transition probabilities. The paradox appears in the vicinity of the probabilities characteristic of the ”voter model”, suggesting practical applications. As in the one-dimensional case, winning and the occurrence of the paradox depends on the number of players.

Key words: Parrondo’s games; Brownian motors, Flashing ratchets; Game theory

1 Introduction

Parrondo’s games, inspired by the flashing Brownian ratchet, represent coin flipping games leading to an apparently paradoxical property that alternating plays of two losing games can produce a winning outcome [1]. Two types of games, each played by only one player, are involved denoted as game A and game B. In the former only one biased coin is used while in the latter two biased coins are used with the player’s current capital determining the state dependent rule. On average, when played individually each game causes the player to lose. However, when two games are played in any combination, on average the player always wins. More specifically, in game A a biased coin with probability of landing head up is \( p_A < 1/2 \), and assuming that the initial capital is \( C = 0 \), after \( n \) plays the expected value of the capital is \( \langle C \rangle = n(2p_A - 1) < 0 \). The game B is played with coins \( B_0 \) and \( B_1 \) with probabilities of landing head up of \( p_0 \) and \( p_1 \) respectively. The coin \( B_0 \) is flipped when \( C \equiv 0 \pmod{3} \) and coin \( B_1 \) otherwise. The quantities of the Brownian ratchets may be made analogous to the variables figuring in Parrondo’s games. For example,
the displacement of the particles corresponds to the capital amount after a certain number of games. The potential of the Brownian ratchets (usually electrostatic) is analogous to the games rules, which define the shape of the potential. A comprehensive review of Parrondo’s paradox and references is given in [2].

Several variations of these games have been introduced extending the range of applicability of games possessing the same, apparently paradoxical property as the original ones. In [3], capital dependence in game B was replaced by recent history of wins and loses which in turn inspired the setting in [4] where the capital dependence in game B was replaced by spatial neighbor dependence. The latter games, termed cooperative, are played by \( N \) number of players as a contrast to the original games which are played by only one player. Each of \( N \) players, arranged in a circle (since periodic boundary conditions are assumed), owns a capital \( C_i(t), i = 1, ..., N \), which evolves by combined playing of games A and B. Game A is the same as in the original setup, namely it consists of repeatedly flipping a biased coin A so that the cumulative capital of all players decreases in time. Game B depends on the winning or losing state of the neighbors on both sides of the player whose turn is to play. It was shown that the outcome is the same as in the original games, namely that alternation of games A and B, which may be losing or fair when played individually, leads to a winning temporal evolution of the total capital of all players. Recognizing that the spatial dependent games may be played in an asynchronous manner (each player, chosen randomly or in any other way, plays when his turn comes) and in a synchronous manner (all players play at the same time), we have developed a theoretical, discrete-time Markov chain model of cooperative games with exact rules that allow computation of transition probabilities for arbitrary number of players [5], [6]. For these games, termed one-dimensional asynchronous and one-dimensional synchronous games, rigorous results were obtained for a small number of players \( (N \leq 12) \) since analytical expressions increase in complexity for larger \( N \). Exploration of paradoxical properties of cooperative Parrondo’s games naturally extends to the two-dimensional case which may prove to be of considerable practical importance, surpassing from that aspect the one-dimensional case. As will be shown in this exposition two-dimensional games share some properties with the one-dimensional ones, namely the probabilities for which the paradox occurs depend on the number of players. Moreover, the paradox occurs for a large number of sets of which a fairly large quantity exists in the vicinity of probability values corresponding to the well know voter model \(^1\).

\(^1\) The voter model is a simple mathematical model of opinion formation in which voters are located at the nodes of a network. Each voter has an opinion (in the simplest case either 0 or 1), and randomly chosen voter assumes an opinion of the majority of its neighbors.
The paper is organized in the following manner: Following a short presentation of essential rules of the games, we show how a probability transition matrix may be obtained from the corresponding matrix of the one dimensional case. Results of computer simulations for the asynchronous and synchronous cases are presented next and we conclude with suggestions for possible new directions and applications of these games.

2 Features of the Games

2.1 Rules and mathematical notation

Each player (or a spin-like particle) may be in one of two states: state 0 ("loser") or state 1 ("winner"). The state of the ensemble of $M$ players, in one-dimensional case, may be represented as a binary string $s = (s_1, ..., s_M)$, $s_i \in (0, 1)$ of length $M$, or equivalently, as state $s$ in decimal notation. Periodic boundary conditions are assumed so that $s_{M+1} = s_1$. To each state $s$ corresponds a vector equivalent to a basis vector $|s\rangle$ in $P = 2^M$ dimensional state space

$$S_P = \{|s\rangle \mid s = 0, 1, ..., P - 1\}.$$

For example, when $M = 3$ and $P = 8$, state (011) is equivalent to the state 3, and the corresponding vector is $|3\rangle = (00010000)^T$, while state (111) is equivalent to state 7 and the corresponding vector is $|7\rangle = (00000001)^T$.

It is easy to generalize this representation to a two dimensional $M \times N$ array of players located at the nodes of a two-dimensional lattice, in which case the dimension of the state space becomes $P = 2^{M \times N}$. For example in the simplest configuration when there are 4 players arranged in a 2×2 lattice, the state space is 16-dimensional and the correspondences are illustrated by the following examples:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = (0 0 0 0), \text{ in decimal notation } 0; \quad \leftrightarrow \quad |0\rangle = (1000000000000000)^T$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = (1 0 1 0), \text{ in decimal notation } 10; \quad \leftrightarrow \quad |10\rangle = (0000000000100000)^T$$

Game A is the same as in the classical setup, while probabilities of winning in game B depend on the present state of the left and right neighbors (one-dimensional case) or the four nearest neighbors (two-dimensional case). For the two-dimensional case five possible configurations of neighboring players surrounding the player whose turn is to play the game is presented in Fig. 1.
Configuration of neighboring players may be denoted as an ordered set \((s_{j-1} \ s_{k-1} \ s_{j+1} \ s_{k+1})\), where the indices \(k - 1, \ k + 1, \ j - 1\) and \(j + 1\) denote the state of neighbors to the left, right, up and below respectively, of a player who is about to play the game. Hence, the probabilities corresponding to configurations presented in Fig. 1 are:

- \(p_{0}^{(B)}\) when \((s_{j-1} \ s_{k-1} \ s_{j+1} \ s_{k+1}) = (0 \ 0 \ 0 \ 0)\)
- \(p_{1}^{(B)}\) when \((s_{j-1} \ s_{k-1} \ s_{j+1} \ s_{k+1}) = (0 \ 0 \ 0 \ 1)\)
- \(p_{2}^{(B)}\) when \((s_{j-1} \ s_{k-1} \ s_{j+1} \ s_{k+1}) = (0 \ 0 \ 1 \ 1)\)
- \(p_{3}^{(B)}\) when \((s_{j-1} \ s_{k-1} \ s_{j+1} \ s_{k+1}) = (0 \ 1 \ 1 \ 1)\)
- \(p_{4}^{(B)}\) when \((s_{j-1} \ s_{k-1} \ s_{j+1} \ s_{k+1}) = (1 \ 1 \ 1 \ 1)\).

It should be remarked that \(p_{1}^{(B)}\) applies to any state with only one winner in the neighborhood, so that, from the aspect of the player who is about to play, the states \((0 \ 0 \ 0 \ 1)\), \((0 \ 0 \ 1 \ 0)\), \((0 \ 1 \ 0 \ 0)\) and \((1 \ 0 \ 0 \ 0)\) are equivalent (or four fold degenerate). States \((0 \ 0 \ 1 \ 1)\) and \((0 \ 1 \ 1 \ 1)\) are six and four fold degenerate respectively.

### 2.2 Evolution of Probabilities

Winning or losing in any particular game leaves a player in state 1 ("winner") or state 0 ("loser") respectively. This state remains in effect until he gets a random chance to play again in the case of asynchronous play or it changes in each round of synchronous games. Following a play by one of the players, the state of the ensemble has changed from a state \(s(t)\) at time \(t\) to state \(s(t+1)\) at time \(t + 1\). If the probability that an ensemble in state \(s(t)\) (or \(|s(t)\rangle\)) is \(\pi_s(t)\), then the probability distribution \(\pi(t)\) at time \(t\) is:

\[
|\pi(t)\rangle = \sum_{s=0}^{P} \pi_s(t) \ |s\rangle ,
\]

while the corresponding probability distribution evolution equation is

\[
|\pi(t+1)\rangle = T \ |\pi(t)\rangle ,
\]

where \(T\) is the probability transition matrix.

Capital \(C(t)\) is a function of the ensemble of players which is incremented by 1 or decremented by 1 if one of the players wins or loses respectively. We introduce the vector of the capital \(|C\rangle\) whose components (their number is equal to the number of ensemble states) represent the capital generated by each ensemble state. Each component \(C_s\) of \(|C\rangle\) represents normalized capital
generated by that specific state which is equal to the sum of all winning and losing individual states in a given ensemble state. Hence, the state 0 generates capital −1, while the state 1 generates capital +1. Explicitly,

\[ C_s = \frac{1}{N} \sum_{i=1}^{N} (-1)^{s_i+1}, \]  

(3)

so that the elements of \(|C\rangle\) are average values of the capital generated by each ensemble state separately. For example for \(\bar{N} = M \times N = 4\), the vector of the capital is

\[ |C\rangle = \frac{1}{4} (-4 -2 -2 0 -2 0 2 -2 0 0 2 2 2 2 4)^T. \]  

(4)

The fifth component of \(|C\rangle\), for example, −2 corresponds to the state 4 = (0 1 0 0). The ensemble switching from any state \(s(t)\) to the state \(s(t + 1) = 4\), generates an average capital \(|C(t + 1)\rangle = \langle C | 4\rangle = -2/4 = -1/2\). Furthermore, an ensemble remaining in state 4 throughout its temporal evolution would on the average generate capital −1/2 in each turn of the game. Hence, the average capital generated by the ensemble is

\[ \langle C \rangle = \langle C | \pi \rangle. \]  

(5)

Denoting the probabilities of winning or losing in a certain game by \(P_{\text{win}}\) and \(P_{\text{lose}}\) it is easily noticed that

\[ P_{\text{win}} + P_{\text{lose}} = 1, \]
\[ P_{\text{win}} - P_{\text{lose}} = \langle C \rangle, \]

so that

\[ P_{\text{win}} = \frac{1}{2}(1 + \langle C \rangle). \]  

(6)

3 Probability Transition Matrix

3.1 Asynchronous case

Probability transition matrix for game A may be easily derived from the one corresponding to game B, so the later will be derived first. Since at each moment of time only one player plays and consequently changes the state, the change may be represented by a Hamming distance between the initial (i) and the final (f) state of player \(k\):

\[ d^H = \sum_{s=0}^{N} |i_k - f_k|, \]
where $N = M \times N$, the number of players. Since $d^H$ may be either 0 or 1, each case will be considered separately.

### 3.1.1 case 1: $d^H = 0$

If the ensemble is in state $s$, then the $k$-th player is in state $s_k$. State $s$ of the ensemble also defines the neighborhood $\eta_k : (s_{j-1} s_k s_{j+1} s_{k+1})$ of the $k$-th player, which in turn determines the probability of winning. The ensemble initially in state $i$ may switch to the state $f = i$ in one of $N$ different ways as a result of one of the players switching from state $i_k$ to $f_k = i_k$, so that the probability of transition is equal to the sum of probabilities of independent events

$$T_{fi} = w(i \rightarrow f) = \frac{1}{N} \sum_{k=1}^{N} w(i_k, f_k),$$

with

$$w(i_k, f_k) = \begin{cases} 1 - p^{(B)}_{i_k}, & \text{when } f_k = 0 \\ p^{(B)}_{i_k}, & \text{when } f_k = 1 \end{cases}$$

### 3.1.2 case 1: $d^H = 1$

The $k$-th player switches from state $i_k$ to state $f_k$ ($i_k \neq f_k$), with probability

$$T_{fi} = w(i \rightarrow f) = \frac{1}{N} w(i_k, f_k).$$

The size of the probability transition matrix becomes large even for the case of nine players arranged in a $3 \times 3$ lattice since the dimension of the state space is $2^9 = 512$, imposing a $512 \times 512$ matrix size. As in the one-dimensional case this matrix is sparse for the asynchronous game B while for synchronous game all the elements are non-zero.

### 3.1.3 An example

We illustrate the evaluation of $T_{fi}^{(B)}$ for the simple case of $3 \times 3$ lattice, in the case of transition from state 85 to state 95, i.e.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = (0 0 1 0 1 0 1 0 1) \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = (0 0 1 0 1 1 1 1 1).$$
Evaluation of expression 7 for this case yields:

\[
T_{95}^{85} = \frac{1}{9}[\text{prob}(0 \rightarrow 0) + \text{prob}(0 \rightarrow 0) + \text{prob}(1 \rightarrow 1) \\
+ \text{prob}(0 \rightarrow 0) + \text{prob}(1 \rightarrow 1) + \text{prob}(0 \rightarrow 1) \\
+ \text{prob}(1 \rightarrow 1) + \text{prob}(0 \rightarrow 1) + \text{prob}(1 \rightarrow 1)] \\
= [(1 - p_2) + (1 - p_2) + p_1 + (1 - p_2) + p_0 + \\
p_3 + p_1 + p_3 + p_2 \\
= \frac{1}{9}(3 + p_0 + 2p_1 - 2p_2 + 2p_3)].
\]

Other elements may be derived in a similar manner. The probability transition matrix for game A is easily obtained by replacing \(p_{\eta_k}^{(B)}\) with \(p^{(A)}\), for each \(\eta \in \{0, 1, 2, 3\}\).

### 3.2 Synchronous case

Since at each moment of time all players play simultaneously and consequently change each individual state, the probability of transition is

\[
T_{fi} = \prod_{k=1}^{N} w(i_k, f_k),
\]

where transition probabilities are

\[
w(i_k, f_k) = \begin{cases} 
1 - p_{\eta_k}^{(B)}, & \text{when } f_k = 0 \\
p_{\eta_k}^{(B)}, & \text{when } f_k = 1 
\end{cases}.
\]

#### 3.2.1 Combination of games A and B

Combination of games A and B may be played, as in the one-dimensional model, in two distinctive ways. First, players may simultaneously play game A or game B in any predetermined or random order, and this case is denoted by A+B. The other possibility is that each player may at each turn of the game chose randomly whether to play game A or game B, this case being denoted as A*B. In both cases the paradoxical result that games A+B or A*B may be winning while each game, A and B, individually losing persists, however for different sets of probabilities figuring in game B.
4 Results of simulations

4.1 Capital accumulation as a function of time

In Figs. 2 and 3 the averaged capital as a function of time in games A, B and A+B for the asynchronous and synchronous cases is shown, respectively. In the synchronous case, Fig. 3, the paradoxical property becomes visible in a very short amount of time, while the asynchronous case displays paradoxical property following a short period during which game A fluctuates around a fair outcome while game B experiences a transient in the positive direction.

4.2 Probability space defining the paradox

The results were obtained both by direct numerical simulation and calculations based on the analytically derived expressions for the probability transition matrix. As in the one-dimensional setting, the results show perfect agreement as illustrated in Fig. 4, where the averaged capital is shown as a function of probability \( p_1^{(B)} \) for the case asynchronously played games. In this diagram a A*B choice for the alternation of two games is applied. In all simulations the capital pertaining to the asynchronous games was averaged over 10 000 time steps and over 1000 runs (ensembles). For synchronous case, since all players play at each time step, the averaging was performed over 1000 time steps and over 200 runs (ensembles). In the case of calculations based on analytical expressions for the probability transition matrix, the capital was averaged as in the case of synchronous games. Simulations of the capital evolution in the five dimensional probability space show that the paradox occurs for a very large number of sets and in order to illustrate this, we concentrate on the appearance of the paradox in the vicinity of the voter model values, i.e. \( p_0^{(B)} = 0, p_1^{(B)} = 0.25, p_2^{(B)} = 0.5, p_3^{(B)} = 0.75 \) and \( p_4^{(B)} = 1 \). The obtained probability sets are displayed in Table 1.

As a further illustration, the probability space spanned by \( p_1^{(B)} \) and \( p_3^{(B)} \) and the corresponding paradoxical range are shown in Fig. 4. Figs. 6, 7, 8, 9 and 10 further illustrate the paradoxical property of the games through capital evolution as a function of each of the probabilities determining the state of the player who is about to play game B. In each of the figures one probability was varied while the remaining four were kept constant. The region of the paradox is enlarged and shown in the right diagram.
| \( p_0(B) \) | \( p_1(B) \) | \( p_2(B) \) | \( p_3(B) \) | \( p_4(B) \) |
|--------|--------|--------|--------|--------|
| 0      | 0.15   | 0.6    | 0.75   | 0.95   |
| 0      | 0.15   | 0.6    | 0.8    | 0.8    |
| 0      | 0.2    | 0.55   | 0.8    | 0.65   |
| 0      | 0.2    | 0.6    | 0.8    | 0.85   |
| 0      | 0.2    | 0.6    | 0.7    | 0.95   |
| 0      | 0.25   | 0.6    | 0.65   | 0.8    |
| 0      | 0.25   | 0.6    | 0.7    | 0.8    |
| 0.05   | 0.15   | 0.6    | 0.75   | 0.9    |
| 0.05   | 0.15   | 0.55   | 0.8    | 0.8    |
| 0.1    | 0.2    | 0.6    | 0.7    | 0.8    |

Table 1
Probabilities featured in game B leading to the paradox in the vicinity of the voter model values

4.3 Capital accumulation and lattice size

In the next two figures we compare the way the capital, as a function of probability \( p_2(B) \), depends on the size of the lattice. In Fig. 11, this dependence is presented for the asynchronous game B. In the cases where periodic boundary conditions have large influence (small lattice sizes, e.g. \( 3 \times 3 \)) the capital increases linearly and in a noticeable manner. The same behavior is seen for large lattice sizes (e.g. \( 100 \times 100 \) and larger) where periodic boundary conditions do not play an important role. However for intermediate lattice sizes (from \( 10 \times 10 \) up to \( 50 \times 50 \)) the capital increases substantially only in the (approximate) range \( 0.5 \leq p_2(B) \leq 0.7 \). For values on both sides of this range capital increases very slowly.

For synchronous games this dependence on the lattice size is not distinct, so that the capital increases, as a function of probability \( p_2(B) \), in almost the same manner and up to the same value for all lattice sizes (Fig. 12).

It is interesting that asynchronous game B played at intermediate lattice sizes leads to the same functional dependence of the capital on the probability \( p_2(B) \) as in synchronous games.
4.4 Percolation phase transition

The dynamics of the games suggests that one should investigate whether there is a connection between the paradoxical property of the games with the percolation phase transition. A typical snapshot of the configuration of "winners" and "losers" which arises during the course of the game B is shown in Fig. 13. In Fig. 14 we superpose two graphs. The first one shows the capital dependence in games B, A+B and A*B on the probability $p(A^B)$, so that the area of the paradox ($C(B) < 0$ and $C(A+B) > 0$ or $C(A*B) < 0$) is clearly visible. The second graph illustrates the probability of appearance of the spanning cluster in game B as a function of $p(B)$. The percolation phase transition occurs at the value of $p(B) \approx 0.64$ very close to the range of $p(B)$ values for which the paradox occurs (approximately $0.59 \leq p(B) \leq 0.62$) however it is clear that the paradox is not present at $p(B)$. The simulation presented here as a paradigm of the typical property of the games, was performed for the 50 × 50 lattice size.

5 Conclusion

Following a discrete-time Markov chain model of one-dimensional cooperative Parrondo’s games introduced previously, we analyzed the two-dimensional setting characterized by players arranged at a two-dimensional regular lattice. It was shown that the alternation of two losing games, on average, leads to a winning outcome and the number of probability sets for which this apparently paradoxical property occurs is very large. In particular, there is a number of probability sets very close to the values characteristic of the voter model, suggesting possible applications in the social or economic framework. Moreover, asynchronous games displaying the paradoxical property are sensitive to the lattice size (number of players), while this is not the case with the synchronous games. For medium sized lattices (between 10 ×10 to 50 ×50), the capital evolution as a function of one of the probabilities defining game B behaves in a similar manner for asynchronous and synchronous games. Percolation phase transition for game B occurs at the probability value very close to the probability values for which the paradox occurs, however they seem to be unrelated phenomena. No fronts or patterns were observed at the paradoxical probability values.

An interesting extension of this work pertains to the games where players are arranged on a network nodes and where links with the neighbors may be rewired according to specific rules[7]. In this setting an addition of rewiring probability in game B introduces interesting novel features of the dynamics
along with the existence of the paradox.

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