A Characterization of a New Type of Strong Law of Large Numbers
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Abstract  Let 0 < p < 2 and 1 ≤ q < ∞. Let \{X_n; n ≥ 1\} be a sequence of independent copies of a real-valued random variable X and set \( S_n = X_1 + \cdots + X_n, \ n ≥ 1 \). We say X satisfies the \((p,q)\)-type \textit{strong law of large numbers} (and write \( X \in SLLN(p,q) \)) if \( \sum_{n=1}^{\infty} \frac{1}{n^p} \left( \frac{S_n}{n} \right)^q < \infty \) almost surely. This paper is devoted to a characterization of \( X \in SLLN(p,q) \). By applying results obtained from the new versions of the classical Lévy, Ottaviani, and Hoffmann-Jörgensen (1974) inequalities proved by Li and Rosalsky (2013) and by using techniques developed by Hechner and Heinkel (2010), we show that \( X \in SLLN(p,q) \) if and only if

\[
\begin{cases} 
\mathbb{E}X = 0 \quad \text{and} \quad \int_0^\infty \mathbb{P}^{q/p}(|X|^q > t) \, dt < \infty & \text{if } 1 < q < p < 2, \\
\mathbb{E}X = 0, \ \mathbb{E}|X|^p < \infty, \ \text{and} \ \sum_{n=1}^{\infty} \frac{\int_{\min\{u_n, n\}}^{n} \mathbb{P}(|X|^p > t) \, dt}{n} < \infty & \text{if } 1 > p = q < 2, \\
\mathbb{E}X = 0 \quad \text{and} \quad \mathbb{E}|X|^p < \infty & \text{if } 1 < p < 2 \text{ and } q > p, \\
\mathbb{E}X = 0, \ \sum_{n=1}^{\infty} \frac{\mathbb{E}I\{|X| ≤ n\}}{n} < \infty, \ \text{and} \\
\sum_{n=1}^{\infty} \frac{\int_{\min\{u_n, n\}}^{n} \mathbb{P}(|X|^p > t) \, dt}{n} < \infty & \text{if } q = p = 1, \\
\mathbb{E}X = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\mathbb{E}I\{|X| ≤ n\}^q}{n} < \infty & \text{if } p = 1 < q, \\
\mathbb{E}|X|^p < \infty & \text{if } 0 < p < 1 ≤ q,
\end{cases}
\]

where \( u_n = \inf \{t : \mathbb{P}(|X| > t) < \frac{1}{n}\} \), \ n ≥ 1. For \( q = 1 \), this equivalence has recently been discovered by Li, Qi, and Rosalsky (2011). Versions of above results in a Banach space setting are also presented.

Keywords  Kolmogorov-Marcinkiewicz-Zygmund strong law of large numbers · \((p,q)\)-type strong law of large numbers · Sums of i.i.d. random variables · Real separable Banach space · Rademacher type \( p \) Banach space · Stable type \( p \) Banach space

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Running Head: Strong law of large numbers

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1 Introduction

Throughout, let \((B, \| \cdot \|)\) be a real separable Banach space equipped with its Borel \(\sigma\)-algebra \(\mathcal{B}\) (= the \(\sigma\)-algebra generated by the class of open subsets of \(B\) determined by \(\| \cdot \|\)) and let \(\{X_n; \ n \geq 1\}\) be a sequence of independent copies of a \(B\)-valued random variable \(X\) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). As usual, let \(S_n = \sum_{k=1}^{n} X_k, \ n \geq 1\) denote their partial sums. If \(0 < p < 2\) and if \(X\) is a real-valued random variable (that is, if \(B = \mathbb{R}\)), then

\[
\lim_{n \to \infty} \frac{S_n}{n^{1/p}} = 0 \text{ almost surely (a.s.)}
\]

if and only if

\[
\mathbb{E}|X|^p < \infty \text{ where } \mathbb{E}X = 0 \text{ whenever } p \geq 1.
\]

This is the celebrated Kolmogorov-Marcinkiewicz-Zygmund strong law of large numbers (SLLN); see Kolmogoroff [8] for \(p = 1\) and Marcinkiewicz and Zygmund [13] for \(p \neq 1\).

The classical Kolmogorov SLLN in real separable Banach spaces was established by Mourier [17]. The extension of the Kolmogorov-Marcinkiewicz-Zygmund SLLN to \(B\)-valued random variables is independently due to Azlarov and Volodin [1] and de Acosta [3].

**Theorem 1.1.** (Azlarov and Volodin [1] and de Acosta [3]). Let \(0 < p < 2\) and let \(\{X_n; \ n \geq 1\}\) be a sequence of independent copies of a \(B\)-valued random variable \(X\). Then

\[
\lim_{n \to \infty} \frac{S_n}{n^{1/p}} = 0 \text{ a.s.}
\]

if and only if

\[
\mathbb{E}\|X\|^p < \infty \text{ and } \frac{S_n}{n^{1/p}} \to_\mathbb{P} 0.
\]

Let \(\{R_n; \ n \geq 1\}\) be a Rademacher sequence; that is, \(\{R_n; \ n \geq 1\}\) is a sequence of independent and identically distributed (i.i.d.) random variables with \(\mathbb{P}(R_1 = 1) = \mathbb{P}(R_1 = -1) = 1/2\). Let \(B^\infty = B \times B \times B \times \cdots\) and define

\[
\mathcal{C}(B) = \left\{(v_1, v_2, \ldots) \in B^\infty : \sum_{n=1}^{\infty} R_nv_n \text{ converges in probability}\right\}.
\]

Let \(1 \leq p \leq 2\). Then \(B\) is said to be of Rademacher type \(p\) if there exists a constant \(0 < C < \infty\) such that

\[
\mathbb{E}\left\|\sum_{n=1}^{\infty} R_nv_n\right\|^p \leq C \sum_{n=1}^{\infty} \|v_n\|^p \text{ for all } (v_1, v_2, \ldots) \in \mathcal{C}(B).
\]

Hoffmann-Jørgensen and Pisier [7] proved for \(1 \leq p \leq 2\) that \(B\) is of Rademacher type \(p\) if and only if there exists a constant \(0 < C < \infty\) such that

\[
\mathbb{E}\left\|\sum_{k=1}^{n} V_k\right\|^p \leq C \sum_{k=1}^{n} \mathbb{E}\|V_k\|^p
\]

for every finite collection \(\{V_1, ..., V_n\}\) of independent mean 0 \(B\)-valued random variables.

If \(B\) is of Rademacher type \(p\) for some \(p \in (1, 2]\), then it is of Rademacher type \(q\) for all \(q \in [1, p)\). Every real separable Banach spaces is of Rademacher type (at least) 1.

Let \(0 < p \leq 2\) and let \(\{\Theta_n; \ n \geq 1\}\) be a sequence of i.i.d. stable random variables each with characteristic function \(\psi(t) = \exp\{-|t|^p\}, \ -\infty < t < \infty\). Then \(B\) is said to be of stable type \(p\) if \(\sum_{n=1}^{\infty} \Theta_nv_n\) converges a.s. whenever \(\{v_n : \ n \geq 1\} \subseteq B\) with \(\sum_{n=1}^{\infty} \|v_n\|^p < \infty\). Equivalent
characterizations of a Banach space being of stable type \( p \), properties of stable type \( p \) Banach spaces, as well as various relationships between the conditions “Rademacher type \( p \)” and “stable type \( p \)” may be found in Maurey and Pisier [16], Woyczyński [20], Marcus and Woyczyński [15], Rosiński [19], Pisier [18], and Ledoux and Talagrand [9]. Some of these properties and relationships are summarized in Li, Qi, and Rosalsky [10].

De Acosta [3] also provided a remarkable characterization of Rademacher type \( p \) Banach spaces. Specifically, de Acosta [3] proved the following theorem.

**Theorem 1.2.** (de Acosta [3]). Let \( 1 \leq p < 2 \). Then the following two statements are equivalent:

(i) The Banach space \( B \) is of Rademacher type \( p \).

(ii) For every sequence \( \{X_n; n \geq 1\} \) of independent copies of a \( B \)-valued variable \( X \),

\[
\lim_{n \to \infty} \frac{S_n}{n^{1/p}} = 0 \quad \text{a.s. if and only if} \quad E\|X\|^p < \infty \quad \text{and} \quad EX = 0.
\]

At the origin of the current investigation are the following recent and striking result by Hechner and Heinkel [5] which is new even in the case where the Banach space \( B \) is the real line. The earliest investigation that we are aware of concerning the convergence of the series \( \sum_{n=1}^{\infty} \frac{1}{n} (E|S_n|/n^{1/p}) \) was carried out by Hechner [4] for the case where \( \{X_n; n \geq 1\} \) is a sequence of i.i.d. mean 0 real-valued random variables.

**Theorem 1.3.** (Hechner and Heinkel [5]). Suppose that \( B \) is of stable type \( p \) (1 < \( p \) < 2) and let \( \{X_n; n \geq 1\} \) be a sequence of independent copies of a \( B \)-valued variable \( X \) with \( EX = 0 \). Then

\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{E\|S_n\|}{n^{1/p}} \right) < \infty
\]

if and only if

\[
\int_0^{\infty} P^{1/p}(\|X\| > t)dt < \infty.
\]

Inspired by the above discovery by Hechner and Heinkel [5], Li, Qi, and Rosalsky [10] obtained sets of necessary and sufficient conditions for

\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|S_n\|}{n^{1/p}} \right) < \infty \quad \text{a.s.}
\]

for the three cases: 0 < \( p \) < 1, \( p = 1 \), 1 < \( p \) < 2. Moreover, Li, Qi, and Rosalsky [10] obtained necessary and sufficient conditions for

\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{E\|S_n\|}{n} \right) < \infty.
\]

Again, these results are new when \( B = \mathbb{R} \); see Theorem 2.5 of Li, Qi, and Rosalsky [10].

Motivated by the results obtained by Li, Qi, and Rosalsky [10], we introduce a new type strong law of large numbers as follows.

**Definition 1.1.** Let 0 < \( p \) < 2 and 0 < \( q \) < \( \infty \). Let \( \{X_n; n \geq 1\} \) be a sequence of independent copies of a \( B \)-valued random variable \( X \). We say \( X \) satisfies the \((p, q)\)-type strong law of large numbers (and write \( X \in SLLN(p, q) \)) if

\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|S_n\|}{n^{1/p}} \right)^q < \infty \quad \text{a.s.}
\]
The following result was recently obtained by Li, Qi, and Rosalsky [11] who proved it by employing new versions of the classical Lévy, Ottaviani, and Hoffmann-Jørgensen [6] inequalities established by Li and Rosalsky [12] and by using some of techniques developed by Hechner and Heinkel [5]. Note that no conditions are imposed on the Banach space $B$. Theorem 1.4 will be used in the proofs of the main results of the current work.

**Theorem 1.4.** Let $0 < p < 2$ and $0 < q < \infty$. Let $\{X_n; n \geq 1\}$ be a sequence of independent copies of a $B$-valued random variable $X$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left( \left\| S_n \right\|^q \left( n^{1/p} \right)^q \right) < \infty \quad (1.1)$$

if and only if

$$X \in SLLN(p,q) \quad (1.2)$$

and

$$\left\{ \begin{array}{ll}
\int_0^\infty \mathbb{P}^{a/p} (\|X\|^q > t) \, dt < \infty & \text{if } 0 < q < p, \\
\mathbb{E} \|X\|^p \ln(1 + \|X\|) < \infty & \text{if } q = p, \\
\mathbb{E} \|X\|^q < \infty & \text{if } q > p.
\end{array} \right. \quad (1.3)$$

Furthermore, each of (1.1) and (1.2) implies that

$$\lim_{n \to \infty} \frac{S_n}{n^{1/p}} = 0 \quad a.s. \quad (1.4)$$

For $0 < q < p$, (1.1) and (1.2) are equivalent so that each of them implies that (1.3) and (1.4) hold.

**Remark 1.1.** Let $q = 1$. Then one can easily see that Theorems 2.1 and 2.2 of Li, Qi, and Rosalsky [10] follow from Theorem 1.4.

**Remark 1.2.** It follows from the conclusion (1.4) of Theorem 1.4 that, if (1.2) holds for some $q = q_1 > 0$ then (1.2) holds for all $q > q_1$.

The current work continues the investigations by Hechner and Heinkel [5] and Li, Qi, and Rosalsky [10] and [11]. More specifically:

(i) For $0 < p < 1$ and $p < q < \infty$ and without any conditions being imposed on the Banach space $B$ we obtain in Theorem 2.1 necessary and sufficient conditions for $X \in SLLN(p,q)$.

(ii) For $1 \leq q < \infty$ we obtain assuming the Banach space $B$ is of stable type $p$ where $1 < p < 2$ (Theorem 2.2) or $p = 1$ (Theorem 2.3) necessary and sufficient conditions for $X \in SLLN(p,q)$.

Theorems 1.4, 2.1, 2.2, and 2.3 are new results when $B = \mathbb{R}$ (Theorem 2.4).

When $B = \mathbb{R}$, necessary and sufficient conditions for $X \in SLLN(p,q)$ for the case where $0 < q < 1 \leq p < 2$ and for the case where $0 < q \leq p < 1$ remain open problems.

The plan of the paper is as follows. The main results are stated in Section 2 and they are proved in Section 3. In Section 4, three examples will be provided for illustrating the necessary and sufficient conditions obtained in this paper.
2 Statement of the main results

With the preliminaries accounted for, the main results may be stated.

**Theorem 2.1.** Let $0 < p < 1$ and $p < q < \infty$. Let $\{X_n; n \geq 1\}$ be a sequence of independent copies of a $B$-valued random variable $X$. Then we have the following two statements:

(a) $X \in SLLN(p,q)$ if and only if $E\|X\|^p < \infty$,

(b) $\sum_{n=1}^{\infty} \frac{1}{n} E \left( \left\| \frac{S_n}{n^{1/p}} \right\|^q \right) < \infty$ if and only if $E\|X\|^q < \infty$.

Let $X$ be a $B$-valued random variable. For each $n \geq 1$, we define the quantile $u_n$ of order $1 - \frac{1}{n}$ of $\|X\|$ as follows:

$$u_n = \inf \left\{ t : \mathbb{P}(\|X\| \leq t) > 1 - \frac{1}{n} \right\} = \inf \left\{ t : \mathbb{P}(\|X\| > t) < \frac{1}{n} \right\}.$$

If $E\|X\| < \infty$, then it is easy to show that

$$\lim_{n \to \infty} \frac{u_n}{n} = 0.$$

**Theorem 2.2.** Let $1 < p < 2$ and $1 \leq q < \infty$. Let $B$ be a Banach space of stable type $p$. Let $\{X_n; n \geq 1\}$ be a sequence of independent copies of a $B$-valued random variable $X$. Then

$$X \in SLLN(p,q)$$

(2.1)

if and only if

$$\begin{cases} \mathbb{E}X = 0 & \text{and} \\ \int_0^\infty \mathbb{P}(\|X\|^q > t) \, dt < \infty & \text{if } 1 \leq q < p, \\ \mathbb{E}\|X\|^p < \infty & \text{and} \sum_{n=1}^{\infty} \frac{\int_{\min\{u_n,1\}}^{u_n} \mathbb{P}(\|X\|^p > t) \, dt}{n} < \infty & \text{if } q = p, \\ \mathbb{E}\|X\|^p < \infty & & \text{if } q > p. \end{cases}$$

(2.2)

**Remark 2.1.** When $q = 1$ and $B$ is of stable $p$ where $1 < p < 2$, Corollary 2.1 of Li, Qi, and Rosalsky [10] follows immediately from Theorems 1.4 and 2.2; that is, (1.1), (1.2), and (2.2) are equivalent.

Note by Lemma 5.6 of Li, Qi, and Rosalsky [10] that

$$\sum_{n=1}^{\infty} \frac{\int_{\min\{u_n,1\}}^{u_n} \mathbb{P}(\|X\|^p > t) \, dt}{n} < \infty$$

whenever $E\|X\|^p \ln^\delta (1 + \|X\|) < \infty$ for some $\delta > 0$.

Thus, for the interesting case $q = p$, Theorem 2.2 yields the following result.
Corollary 2.1. Let $1 < p < 2$ and let $\{X_n; n \geq 1\}$ be a sequence of independent copies of a $\mathbb{B}$-valued random variable $X$. If $\mathbb{B}$ is of stable type $p$, then

$$X \in SLLN(p,p) \text{ whenever } \mathbb{E}X = 0 \text{ and } \mathbb{E}\|X\|^p \ln^\delta(1 + \|X\|) < \infty \text{ for some } \delta > 0.$$  

For the case where $1 < p < 2$ and $1 \leq q < \infty$, combining Theorems 1.4 and 2.2, we immediately obtain necessary and sufficient conditions for (1.1) to hold assuming that $\mathbb{B}$ is of stable type $p$.

Corollary 2.2. Let $1 < p < 2$ and $1 \leq q < \infty$. Let $X$ be a $\mathbb{B}$-valued random variable. If $\mathbb{B}$ is of stable type $p$, then (1.1) holds if and only if

$$\begin{align*}
\mathbb{E}X &= 0 \\
\int_0^{\infty} \mathbb{P}^{q/p} (\|X\|^q > t) dt &< \infty \quad \text{if } 1 \leq q < p, \\
\mathbb{E}\|X\|^p \ln(1 + \|X\|) &< \infty \quad \text{if } q = p, \\
\mathbb{E}\|X\|^q &< \infty \quad \text{if } q > p.
\end{align*}$$

Remark 2.2. For the case where $q = 1$, Corollary 2.2 above is Theorem 1.3 (i.e., Theorem 5 of Hechner and Heinkel [5]). Actually Corollary 2.2 for the case where $q = 1$ is somewhat stronger than Theorem 5 (necessity half) of Hechner and Heinkel [5] because $\mathbb{E}X = 0$ is an assumption in Theorem 5 of Hechner and Heinkel [5].

We now present necessary and sufficient conditions for (1.2) for the case where $p = 1$ and $1 \leq q < \infty$.

Theorem 2.3. Let $1 \leq q < \infty$ and let $\mathbb{B}$ be a Banach space of stable type 1. Let $\{X_n; n \geq 1\}$ be a sequence of independent copies of a $\mathbb{B}$-valued random variable $X$. Then

$$X \in SLLN(1,q)$$

if and only if

$$\begin{align*}
\mathbb{E}\|X\| &< \infty, \quad \mathbb{E}X = 0, \quad \text{and} \\
\sum_{n=1}^{\infty} \frac{||\mathbb{E}X I\{\|X\| \leq n\}||}{n} &< \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\int_{\min\{s_n,n\}}^{\infty} \mathbb{P}(\|X\| > t) dt}{n} < \infty \quad \text{if } q = 1, \\
\sum_{n=1}^{\infty} \frac{||\mathbb{E}X I\{\|X\| \leq n\}||^q}{n} &< \infty \quad \text{if } q > 1.
\end{align*}$$

Remark 2.3. For the case where $q = 1$, Theorem 2.3 is Theorem 2.3 of Li, Qi, and Rosalsky [10].

By Lemmas 5.5 and 5.6 of Li, Qi, and Rosalsky [10], (2.4) holds whenever $\mathbb{E}X = 0$ and $\mathbb{E}\|X\| \ln(1 + \|X\|) < \infty$. Combining Theorems 1.4 and 2.3, we immediately have the following result.
Corollary 2.3. Let $1 \leq q < \infty$ and let $B$ be a Banach space of stable type 1. Let $\{X_n; n \geq 1\}$ be a sequence of independent copies of a $B$-valued random variable $X$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{\|S_n\|}{n}\right)^q < \infty$$

if and only if

$$\begin{cases} 
\mathbb{E}X = 0 \text{ and } \\
\mathbb{E}\|X\| \ln(1 + \|X\|) < \infty & \text{if } q = 1, \\
\mathbb{E}\|X\|^q < \infty & \text{if } q > 1.
\end{cases}$$

As a summary of our Theorems 1.4 and 2.1-2.3 and Corollaries 2.2 and 2.3, we now present the following theorem for a real-valued random variable $X$. For $q = 1$, the equivalence of (i) and (ii) has recently been obtained by Li, Qi, and Rosalsky [10], and for $1 = q < p < 2$, the equivalence of (iii) and (iv) is due to Hechner and Heinkel [5] (see Theorem 1.3 above) assuming that $\mathbb{E}X = 0$ for the implication ((iii) $\Rightarrow$ (iv)).

Theorem 2.4. Let $0 < p < 2$ and $1 \leq q < \infty$. Let $\{X_n; n \geq 1\}$ be a sequence of independent copies of a real-valued random variable $X$. The following two statements are equivalent:

(i) $X \in SLLN(p, q)$,

$$\begin{cases} 
\mathbb{E}X = 0 \text{ and } \int_0^{\infty} \mathbb{P}^{q/p}(|X|^q > t) \, dt < \infty & \text{if } 1 \leq q < p < 2, \\
\mathbb{E}X = 0, \mathbb{E}|X|^p < \infty, \text{ and } \sum_{n=1}^{\infty} \int_{\min\{u_n, n\}}^{\infty} \mathbb{P}(|X|^p > t) \, dt \frac{n}{n} < \infty & \text{if } 1 < q = p < 2, \\
\mathbb{E}X = 0 \text{ and } \mathbb{E}|X|^p < \infty & \text{if } 1 < p < 2 \text{ and } q > p,
\end{cases}$$

(ii) $\sum_{n=1}^{\infty} \mathbb{E}X \{ |X| \leq n \} < \infty$, and

$$\begin{cases} 
\sum_{n=1}^{\infty} \int_{\min\{u_n, n\}}^{\infty} \mathbb{P}(|X| > t) \, dt \frac{n}{n} < \infty & \text{if } q = p = 1, \\
\mathbb{E}X = 0 \text{ and } \sum_{n=1}^{\infty} \mathbb{E}X \{ |X| \leq n \}^q < \infty & \text{if } p = 1 < q, \\
\mathbb{E}|X|^p < \infty & \text{if } 0 < p < 1 \leq q.
\end{cases}$$

The following two statements are equivalent:

(iii) $$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E}\left(\frac{|S_n|}{n^{1/p}}\right)^q < \infty,$$
3 Proofs of Theorems 2.1 - 2.3

In this section we denote by $C_k$ positive constants the precise values of which do not matter.

First we introduce some notation. Let $(a_k)_{1 \leq k \leq n}$ be a finite sequence of real numbers and $(a^*_k)_{1 \leq k \leq n}$ the nonincreasing rearrangement of the sequence $(|a_k|)_{1 \leq k \leq n}$. For a given $r \geq 1$,

$$
\|(a_k)_{1 \leq k \leq n}\|_{r, \infty} = \sup_{1 \leq k \leq n} k^{1/r} a^*_k
$$

is called the weak-$\ell_r$ norm of the sequence $(a_k)_{1 \leq k \leq n}$. Let $V_k, 1 \leq k \leq n$ be independent real-valued random variables. Then the remarkable Marcus-Pisier [14] inequality asserts that for all $r \geq 1$,

$$
\mathbb{P}\left(\|(V_k)_{1 \leq k \leq n}\|_{r, \infty} > u\right) \leq \frac{2e}{u^r} \sup_{t > 0} \left( t^r \sum_{k=1}^n \mathbb{P}\left(|V_k| > t\right) \right) \quad \forall \ u > 0. \quad (3.1)
$$

The original Marcus-Pisier [14] inequality involved the constant $262$ instead of $2e$. The improved constant is due to J. Zinn (see Pisier [18, Lemma 4.11]).

Let $X$ be a $\mathcal{B}$-valued random variable. For each $n \geq 1$, let the quantile $u_n$ of order $1 - \frac{1}{n}$ of $\|X\|$ be defined as in Section 2. We then see that for every $q > 0$,

$$
\inf \left\{ t : \mathbb{P}(\|X\|^q \leq t) > 1 - \frac{1}{n} \right\} = \inf \left\{ t : \mathbb{P}(\|X\|^{q} > t) < \frac{1}{n} \right\} = u_n^q;
$$

i.e., $u_n^q$ is the quantile of order $1 - \frac{1}{n}$ of $\|X\|^q$. Let $\{X_n; \ n \geq 1\}$ be a sequence of independent copies of $\mathcal{B}$-valued variable $X$. Write, for $n \geq 1$,

$$
S^{(1)}_n = \sum_{k=1}^n X_k I\{\|X_k\|^p \leq k\}, \quad S^{(2)}_n = S_n - S^{(1)}_n = \sum_{k=1}^n X_k I\{\|X_k\|^p > k\},
$$

$$
U_n = \sum_{k=1}^n X_k I\{\|X_k\|^p \leq n\}, \quad U^{(1)}_n = \sum_{k=1}^n X_k I\{\|X_k\| \leq u_n\}, \quad \text{and} \quad U^{(2)}_n = U_n - U^{(1)}_n, \quad n \geq 1.
$$

Motivated by Lemma 1 of Hechner and Heinkel [5] and its proof, we establish the following result.

**Lemma 3.1.** Let $1 < p < 2$ and $1 \leq q < p$. Let $\mathcal{B}$ be a Banach space of stable type $p$. Then there exists a universal constant $c(p, q) > 0$ such that, for every finite sequence $V_k, 1 \leq k \leq n$ of independent $\mathcal{B}$-valued random variables with $\max_{1 \leq k \leq n} \mathbb{E}\|V_k\|^q < \infty$,

$$
\mathbb{E}\left\|\sum_{k=1}^n (V_k - \mathbb{E}V_k)\right\|_q^q \leq c(p, q) \left( \sup_{t > 0} t^{p/q}\sum_{k=1}^n \mathbb{P}\left(\|V_k\|^q > t\right) \right)^{q/p}. \quad (3.2)
$$

**Remark 3.1.** Clearly, if $q = 1$, then Lemma 3.1 is Lemma 1 of Hechner and Heinkel [5].
Proof of Lemma 3.1 Let \( \{V'_k; 1 \leq k \leq n\} \) be an independent copy of \( \{V_k; 1 \leq k \leq n\} \) and let \( \{R_k; 1 \leq k \leq n\} \) be a Rademacher sequence independent of \( \{V_k, V'_k; 1 \leq k \leq n\} \). Since \( q \geq 1 \), \( g(x) = x^p \), \( x \in [0, \infty) \) is a convex nonnegative function. Applying (2.5) of Ledoux and Talagrand [9, p. 46], we have that

\[
E \left\| \sum_{k=1}^{n} (V_k - EV_k) \right\|^q \leq E \left\| \sum_{k=1}^{n} (V_k - V'_k) \right\|^q = E \left\| \sum_{k=1}^{n} R_k (V_k - V'_k) \right\|^q \leq 2^{q-1} E \left\| \sum_{k=1}^{n} R_k V_k \right\|^q. \tag{3.3}
\]

Since \( B \) is of stable type \( p \) with \( 1 \leq p < 2 \), the Maurey-Pisier [16] theorem asserts that it is also of stable type \( r \) for some \( r > p \). Let \( (A^*_k)_{1 \leq k \leq n} \) be the nonincreasing rearrangement of \( (\|V_k\|)_{1 \leq k \leq n} \). Note that \( r/q > 1 \), \( p/q > 1 \) (since \( 1 \leq q < p < r \)), and \( B \) is also of Rademacher type \( r \). We thus have that

\[
E \left\| \sum_{k=1}^{n} R_k V_k \right\|^q = E \left( E \left( \left\| \sum_{k=1}^{n} R_k V_k \right\|^r \right)^{q/r} \left\| V_1, ..., V_n \right\|^r \right)^{1/(r/q)} \leq C_1 E \left\{ \sum_{k=1}^{n} \|V_k\|^r \right\}^{q/r} \tag{3.4}
\]

\[
= C_1 E \left\{ \sum_{k=1}^{n} \left( k^{r/p} (A^*_k)^r \right) k^{-r/p} \right\}^{q/r} \leq C_1 E \left( \sup_{1 \leq k \leq n} k^{q/p} (A^*_k)^q \right) \left( \sum_{k=1}^{n} k^{-r/p} \right)^{q/r} = C_2 E \left\{ \left( \sup_{1 \leq k \leq n} k^{q/p} (A^*_k)^q \right) \left( \sum_{k=1}^{n} k^{-r/p} \right) \right\} \]

Write \( \Delta = \sup_{t>0} t^{p/q} \sum_{k=1}^{n} P(\|V_k\|^q > t) \). Using the Marcus-Pisier [14] inequality (3.1), we have that

\[
E \left\{ \left( \sup_{1 \leq k \leq n} \|V_k\|^q \right) \right\}^{p/q, \infty} = \left( \int_0^{\Delta^{q/p}} + \int_{\Delta^{q/p}}^{\infty} \right) P \left( \left( \sup_{1 \leq k \leq n} \|V_k\|^q \right) > t \right) dt \leq \Delta^{q/p} + \int_{\Delta^{q/p}}^{\infty} \frac{2e\Delta}{t^{p/q}} dt \tag{3.5}
\]

\[
= \left( 1 + \frac{2eq}{p - q} \right) \Delta^{q/p}.
\]

Now (3.2) follows from (3.3), (3.4), and (3.5). \( \square \)

The following nice result is Proposition 3 of Hechner and Heinkel [5].

**Lemma 3.2.** (Hechner and Heinkel [5]). Let \( p > 1 \) and let \( \{X_n; n \geq 1\} \) be a sequence of independent copies of a \( B \)-valued random variable \( X \). Write

\[
u_n = \inf \left\{ t : P(\|X\| > t) < \frac{1}{n} \right\}, \quad n \geq 1.
\]
Then the following three statements are equivalent:

(i) \[ \int_0^\infty \mathbb{P}^{1/p}(\|X\| > t)dt < \infty; \]

(ii) \[ \sum_{n=1}^{\infty} \frac{u_n}{n^{1+1/p}} < \infty; \]

(iii) \[ \sum_{n=1}^{\infty} \frac{1}{n^{1+1/p}} \mathbb{E}\left(\max_{1 \leq k \leq n} \|X_k\|\right) < \infty. \]

The next lemma and its proof are similar to Lemma 3 of Hechner and Heinkel [5] and its proof, respectively.

**Lemma 3.3.** Let \(1 \leq q < p < 2\). Let \(X\) be a \(B\)-valued random variable with

\[ \int_0^\infty \mathbb{P}^{q/p}(\|X\|^q > t)\ dt < \infty. \quad (3.6) \]

If \(B\) is a Banach space of Rademacher type \(q\), then

\[ \sum_{n=1}^{\infty} \mathbb{E}\left(\left\|\left(S_n - U_n^{(1)}\right) - \mathbb{E}\left(S_n - U_n^{(1)}\right)\right\|^q_{n^{1+q/p}}\right) < \infty. \quad (3.7) \]

**Proof** Let \(f_q(t) = \mathbb{P}(\|X\|^q > t), \ t \geq 0\). Since \(B\) is a Banach space of Rademacher type \(q\) and

\[ \left(\left(S_n - U_n^{(1)}\right) - \mathbb{E}\left(S_n - U_n^{(1)}\right)\right) = \sum_{k=1}^{n} (X_k I\{\|X_k\| > u_n\} - \mathbb{E}X I\{\|X\| > u_n\}) , \ n \geq 1, \]

we have that

\[ \mathbb{E}\left\|\left(S_n - U_n^{(1)}\right) - \mathbb{E}\left(S_n - U_n^{(1)}\right)\right\|^q_{n^{1+q/p}} \leq C_3 n \mathbb{E}\|X I\{\|X\| > u_n\} - \mathbb{E}X I\{\|X\| > u_n\}\|^q \]

\[ \leq C_4 n \mathbb{E}(\|X\|^q I\{\|X\|^q > u_n^q\}\}

\[ = C_4 \left( nu_n^q \mathbb{P}(\|X\|^q > u_n^q) + n \int_{u_n^q}^{\infty} f_q(t) dt \right) \]

\[ \leq C_4 \left( u_n^q + n \int_{u_n^q}^{\infty} f_q(t) dt \right). \quad (3.8) \]

Set \(p_1 = p/q\), \(Y = \|X\|^q\), and \(u_{n,q} = u_n^q\), \(n \geq 1\). Noting that \(p_1 > 1\) (since \(1 \leq q < p < 2\)), by Lemma 3.2 (i.e., Proposition 3 of Hechner and Heinkel [5]), it follows from (3.6) that

\[ \sum_{n=1}^{\infty} \frac{u_n^q}{n^{1+q/p}} = \sum_{n=1}^{\infty} \frac{u_{n,q}}{n^{1+1/p_1}} < \infty. \quad (3.9) \]
Also (3.6) implies that

\[
\sum_{n=1}^{\infty} \frac{1}{n^{q/p}} \int_{u_n^q}^{\infty} f_q(t) dt = \sum_{n=1}^{\infty} \frac{1}{n^{1/p_1}} \int_{u_{n,q}}^{\infty} f_q(t) dt
\]

\[
= \sum_{n=1}^{\infty} \frac{n^{-1/p_1}}{n^{1/p_1}} \sum_{j=n}^{\infty} \int_{u_{j,q}}^{u_{j+1,q}} f_q(t) dt
\]

\[
= \sum_{j=1}^{\infty} \left( \int_{u_{j,q}}^{u_{j+1,q}} f_q(t) dt \right) \sum_{n=1}^{j} n^{-1/p_1}
\]

\[
\leq C_5 \sum_{j=1}^{\infty} \left( \int_{u_{j,q}}^{u_{j+1,q}} f_q(t) dt \right) \frac{j^{1-1/p_1}}{j^{1-1/p_1}}
\]

\[
\leq C_5 \int_{0}^{\infty} f_q^{1/p_1}(t) dt
\]

\[
= C_5 \int_{0}^{\infty} \mathbb{P}^{q/p} (\|X\| > t) dt
\]

\[
< \infty.
\]  

The conclusion (3.7) follows from (3.8), (3.9), and (3.10). \(\square\)

The proof of the next lemma is similar to that of Lemma 4 of Hechner and Heinkel [5] and Lemma 5.3 of Li, Qi, and Rosalsky [10] and it contains a nice application of Lemma 3.1 above.

**Lemma 3.4.** Let \(1 \leq q \leq p < 2\). Let \(X\) be a \(B\)-valued random variable with (3.6). If \(B\) is a Banach space of stable type \(p\), then

\[
\sum_{n=1}^{\infty} \mathbb{E} \left\| \frac{U_n^{(1)} - \mathbb{E} U_n^{(1)}}{n^{1+q/p}} \right\|^p \leq \infty.
\]  

(3.11)

**Remark 3.2.** Note that

\[
\mathbb{E} \|X\|^q = \int_{0}^{\infty} \mathbb{P} (\|X\| > t) dt.
\]

Thus for \(q = p\), (3.6) holds if and only if \(\mathbb{E} \|X\|^p < \infty\). By Lemma 3.4, if \(B\) is a Banach space of stable type \(p \in [1, 2)\), then

\[
\sum_{n=1}^{\infty} \mathbb{E} \left\| \frac{U_n^{(1)} - \mathbb{E} U_n^{(1)}}{n^2} \right\|^p \leq \infty
\]  

(3.12)

whenever \(\mathbb{E} \|X\|^p < \infty\).

**Proof of Lemma 3.4** Since \(B\) is of stable type \(p\), the Maurey-Pisier [16] theorem asserts that it is also of stable type \(r\) for some \(r > p\). Applying Lemma 3.1, there exists a universal constant \(0 < c(r, q) < \infty\)
such that

\[
E \left\| U_n^{(1)} - \mathbb{E}U_n^{(1)} \right\|^q \leq c(r, q) \left( \sup_{t > 0} t^{r/q} \sum_{k=1}^n \mathbb{P} \left( \|X_k\|^q I\{\|X_k\| \leq u_n \} > t \right) \right)^{q/r}
\]

\[
\leq c(r, q) \left( n \sup_{0 \leq t \leq u_n^q} t^{r/q} \mathbb{P} \left( \|X\|^q > t \right) \right)^{q/r}, \quad n \geq 1.
\]

It is easy to see that for all \( x > 0 \),

\[
\left( \int_0^x \mathbb{P}^{q/r} (\|X\|^q > t) \ dt \right)^{r/q} \geq \left( \int_0^x \mathbb{P}^{q/r} (\|X\|^q > x) \ dt \right)^{r/q}
\]

\[
= x^{r/q} \mathbb{P} (\|X\|^q > x).
\]

We thus have that

\[
E \left\| U_n^{(1)} - \mathbb{E}U_n^{(1)} \right\|^q \leq c(r, q) \left( n \sup_{0 \leq t \leq u_n^q} t^{r/q} \mathbb{P} \left( \|X\|^q > t \right) \right)^{q/r}
\]

\[
\leq c(r, q)n^{q/r} \int_0^{u_n^q} \mathbb{P}^{q/r} (\|X\|^q > t) \ dt, \quad n \geq 1.
\]

Let \( u_0 = 0 \) and note that \( \mathbb{P} (\|X\|^q > t) \geq 1/k \) for \( t \in [u_{k-1}^q, u_k^q) \), \( k \geq 1 \). It follows that

\[
\sum_{n=1}^{\infty} \frac{\mathbb{E} \left\| U_n^{(1)} - \mathbb{E}U_n^{(1)} \right\|^q}{n^{1+q/p}} \leq c(r, q) \sum_{n=1}^{\infty} \frac{1}{n^{1+q/p-q/r}} \int_0^{u_n^q} \mathbb{P}^{q/r} (\|X\|^q > t) \ dt
\]

\[
= c(r, q) \sum_{n=1}^{\infty} \frac{1}{n^{1+q/p-q/r}} \sum_{k=1}^n \int_{u_{k-1}^q}^{u_k^q} \mathbb{P}^{q/r} (\|X\|^q > t) \ dt
\]

\[
= c(r, q) \sum_{k=1}^{\infty} \left( \sum_{n=k}^{\infty} \frac{1}{n^{1+q/p-q/r}} \right) \int_{u_{k-1}^q}^{u_k^q} \mathbb{P}^{q/r} (\|X\|^q > t) \ dt
\]

\[
\leq C_6 \sum_{k=1}^{\infty} \frac{1}{k^{q/p-q/r}} \int_{u_{k-1}^q}^{u_k^q} \mathbb{P}^{q/r} (\|X\|^q > t) \ dt
\]

\[
\leq C_6 \int_0^{\infty} \mathbb{P}^{q/p} (\|X\|^q > t) \ dt < \infty
\]

proving (3.11) and completing the proof of Lemma 3.4. \( \square \)

**Lemma 3.5.** Let \( 1 \leq p < 2 \) and let \( X \) be a \( \mathbf{B} \)-valued random variable with \( E\|X\|^p < \infty \). Then

\[
\sum_{n=1}^{\infty} \frac{1}{n^{2}} \left( \sum_{k=1}^{n} E\|X\| I\{k < \|X\|^p \leq n\} \right)^p < \infty,
\]

(3.13)
\[
\sum_{n=1}^{\infty} \frac{u_n^p}{n^2} < \infty,
\]

(3.14)

and for every \( \delta > 0 \),

\[
\sum_{n=1}^{\infty} \frac{\mathbb{E}\|X\|^{p+\delta} \mathbb{I}\{\|X\|^{p} \leq n\}}{n^{1+\delta/p}} < \infty.
\]

(3.15)

Furthermore, if \( p > 1 \) then

\[
\sum_{n=1}^{\infty} \frac{(\mathbb{E}\|X\| \mathbb{I}\{\|X\|^{p} > n\})^p}{n^{2-p}} < \infty.
\]

(3.16)

Remark 3.3. For \( p = 1 \), (3.13) and (3.14) together are Lemma 5.1 of Li, Qi, and Rosalsky [10].

Proof of Lemma 3.5 Since \( u_n^p \) is the quantile of order \( 1 - \frac{1}{n} \) of \( \|X\|^p \), (3.14) immediately follows from the second half of Lemma 5.1 of Li, Qi, and Rosalsky [10].

The proof of (3.15) is easy and we leave it to the reader.

We now show that \( \mathbb{E}\|X\|^p < \infty \) implies (3.13). For \( n \geq 2 \), let

\[
\Lambda_n = \sum_{k=2}^{n} k^p \mathbb{P}(k - 1 < \|X\|^p \leq k), \quad \lambda_{n,j} = \frac{j^p \mathbb{P}(j - 1 < \|X\|^p \leq j)}{\Lambda_n}, \quad 2 \leq j \leq n.
\]

Clearly

\[
\lambda_{n,j} \geq 0, \quad 2 \leq j \leq n, \quad \sum_{j=2}^{n} \lambda_{n,j} = 1,
\]

and

\[
\Lambda_n \leq \mathbb{E}\|X\|^p + 1 < \infty, \quad n \geq 2.
\]

Note that the function \( \phi(t) = t^p \) is convex on \([0, \infty)\) and

\[
\|X\| \sum_{k=1}^{n} \mathbb{I}\{k < \|X\|^p \leq n\} = \sum_{k=1}^{n} \sum_{j=k+1}^{n} \|X\| \mathbb{I}\{j - 1 < \|X\|^p \leq j\}
\]

\[
\leq \sum_{k=1}^{n} \sum_{j=k+1}^{n} j^{1/p} \mathbb{I}\{j - 1 < \|X\|^p \leq j\}
\]

\[
= \sum_{j=2}^{n} \sum_{k=1}^{j-1} j^{1/p} \mathbb{I}\{j - 1 < \|X\|^p \leq j\}
\]

\[
\leq \sum_{j=2}^{n} j^{1+1/p} \mathbb{I}\{j - 1 < \|X\|^p \leq j\}, \quad n \geq 2.
\]
We thus have that
\[
\left( \sum_{k=1}^{n} \mathbb{E} \|X\| I \{ k < \|X\|^p \leq n \} \right)^p = \left( \mathbb{E} \left( \left( \sum_{k=1}^{n} I \{ k < \|X\|^p \leq n \} \right)^p \right) \right).
\]
\[
\leq \left( \mathbb{E} \left( \sum_{j=2}^{n} j^{1/p} I \{ j - 1 < \|X\|^p \leq j \} \right)^p \right)
\]
\[
= \left( \sum_{j=2}^{n} j^{1/p} \mathbb{P} ( j - 1 < \|X\|^p \leq j ) \right)^p
\]
\[
= \Lambda_n^p \left( \sum_{j=2}^{n} j^{1/p} \lambda_{n,j} \right)^p
\]
\[
\leq \Lambda_n^p \sum_{j=2}^{n} \lambda_{n,j} \left( j^{1/p} \right)^p
\]
\[
= \Lambda_n^{p-1} \sum_{j=2}^{n} j^2 \mathbb{P} ( j - 1 < \|X\|^p \leq j )
\]
\[
\leq C_7 \sum_{j=2}^{n} j^2 \mathbb{P} ( j - 1 < \|X\|^p \leq j ) , \ n \geq 2.
\]

It now is easy to see that
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{j=2}^{n} j^2 \mathbb{P} ( j - 1 < \|X\|^p \leq j ) = \sum_{j=2}^{\infty} \left( \sum_{n=j}^{\infty} \frac{1}{n^2} \right) j^2 \mathbb{P} ( j - 1 < \|X\|^p \leq j )
\]
\[
\leq C_8 \sum_{j=2}^{\infty} j \mathbb{P} ( j - 1 < \|X\|^p \leq j )
\]
\[
\leq C_8 ( \mathbb{E} \|X\|^p + 1 ) < \infty
\]
thereby proving (3.13).

We now prove (3.16). Note that for \( n \geq 1, \)
\[
\mathbb{E} \|X\| I \{ \|X\|^p > n \} \leq \sum_{j=n+1}^{\infty} j^{1/p} \mathbb{P} ( j - 1 < \|X\|^p \leq j )
\]
\[
= \sum_{j=n+1}^{\infty} j^{1/p-1} ( j \mathbb{P} ( j - 1 < \|X\|^p \leq j ) )
\]
and
\[
\sum_{j=n}^{\infty} j \mathbb{P} ( j - 1 < \|X\|^p \leq j ) \leq \mathbb{E} \|X\|^p + 1.
\]
Thus, by the same arguments used in proving (3.13), we have that
\[
(\mathbb{E}\|X\| I \{\|X\|^p > n\})^p \leq C_9 \sum_{j=n}^{\infty} j^{1-p} (j \mathbb{P} (j - 1 < \|X\|^p \leq j)), \quad n \geq 1.
\]
Since \( p > 1 \), we get that
\[
\sum_{n=1}^{\infty} \frac{(\mathbb{E}\|X\| I \{\|X\|^p > n\})^p}{n^{2-p}} \leq C_9 \sum_{n=1}^{\infty} \frac{1}{n^{2-p}} \sum_{j=n}^{\infty} j^{1-p} (j \mathbb{P} (j - 1 < \|X\|^p \leq j))
\]
\[
= C_9 \sum_{j=1}^{\infty} \left( \sum_{n=1}^{j} n^{p-2} \right) j^{2-p} \mathbb{P} (j - 1 < \|X\|^p \leq j)
\]
\[
\leq C_{10} \sum_{j=1}^{\infty} j \mathbb{P} (j - 1 < \|X\|^p \leq j)
\]
\[
\leq C_{10} (\mathbb{E}\|X\|^p + 1) < \infty
\]
proving (3.16). \( \square \)

The following recent result of Li, Qi, and Rosalsky [11] is used in the proof of Theorem 2.3. It was proved by applying the new versions of the classical Lévy and classical Hoffmann-Jørgensen [6] inequalities established by Li and Rosalsky [12].

**Theorem 3.1.** (Li, Qi, and Rosalsky [11]). Let \( q > 0 \) and let \( \{a_n; \ n \geq 1\} \) be a sequence of nonnegative real numbers such that \( \sum_{n=1}^{\infty} a_n < \infty \). Let \( \{V_k; \ k \geq 1\} \) be a sequence of independent symmetric \( B \)-valued random variables. Write
\[
b_n = \sum_{k=n}^{\infty} a_k, \quad n \geq 1
\]
and
\[
\alpha = \begin{cases} 2^{1-q}, & \text{if } 0 < q \leq 1 \\ 1, & \text{if } q > 1. \end{cases} \quad \text{and} \quad \beta = \begin{cases} 1, & \text{if } 0 < q \leq 1 \\ 2^{q-1}, & \text{if } q > 1. \end{cases}
\]
Then, for all nonnegative real numbers \( s, t, \) and \( u \), we have that
\[
\mathbb{P} \left( \sup_{n \geq 1} b_n \|V_n\|^q > t \right) \leq 2\mathbb{P} \left( \sum_{n=1}^{\infty} a_n \left\| \sum_{i=1}^{n} V_i \right\|^q > \frac{t}{\alpha} \right)
\]
and
\[
\mathbb{P} \left( \sum_{n=1}^{\infty} a_k \left\| \sum_{i=1}^{n} V_i \right\|^q > s + t + u \right)
\]
\[
\leq \mathbb{P} \left( \sup_{n \geq 1} b_n \|V_n\|^q > \frac{s}{\beta^q} \right) + 4\mathbb{P} \left( \sum_{n=1}^{\infty} a_n \left\| \sum_{i=1}^{n} V_i \right\|^q > \frac{u}{\alpha \beta} \right) \mathbb{P} \left( \sum_{n=1}^{\infty} a_n \left\| \sum_{i=1}^{n} V_i \right\|^q > \frac{t}{\alpha \beta^q} \right).
\]
Furthermore, we have that
\[
\mathbb{E} \left( \sup_{n \geq 1} b_n \|V_n\|^q \right) \leq 2\alpha \mathbb{E} \left( \sum_{n=1}^{\infty} a_k \left\| \sum_{i=1}^{n} V_i \right\|^q \right).
\]
and
\[ \mathbb{E} \left( \sum_{n=1}^{\infty} a_k \left\| \sum_{i=1}^{n} V_i \right\|^q \right) \leq 6 (\alpha + \beta)^3 \mathbb{E} \left( \sup_{n \geq 1} b_n \left\| V_n \right\|^q \right) + 6 (\alpha + \beta)^3 t_0, \]

where
\[ t_0 = \inf \left\{ t > 0; \mathbb{P} \left( \sum_{n=1}^{\infty} a_n \left\| \sum_{i=1}^{n} V_i \right\|^q > t \right) \leq 24^{-1} (\alpha + \beta)^{-3} \right\}. \]

**Lemma 3.6.** (Li, Qi, and Rosalsky [11]). Let \((\mathbb{E}, G)\) be a measurable linear space and \(g : \mathbb{E} \to [0, \infty)\) be a measurable even function such that for all \(x, y \in \mathbb{E}\),
\[ g(x + y) \leq \beta (g(x) + g(y)), \]
where \(1 \leq \beta < \infty\) is a constant, depending only on the function \(g\). If \(V\) is an \(\mathbb{E}\)-valued random variable and \(\hat{V}\) is a symmetrized version of \(V\) (i.e., \(\hat{V} = V - \mathbb{E}V\) where \(\mathbb{E}V\) is an independent copy of \(V\)), then for all \(t \geq 0\), we have that
\[ \mathbb{P}(g(V) \leq t) \mathbb{E}g(V) \leq \beta \mathbb{E}g(\hat{V}) + \beta t \]
and
\[ \mathbb{E}g(\hat{V}) \leq 2\beta \mathbb{E}g(V). \]

Moreover, if
\[ g(V) < \infty \text{ a.s.,} \]
then
\[ \mathbb{E}g(V) < \infty \text{ if and only if } \mathbb{E}g(\hat{V}) < \infty. \]

**Lemma 3.7.** Let \(1 < p < 2\) and let \( \{X_n; n \geq 1\} \) be a sequence of independent copies of a \(\mathbb{B}\)-valued random variable \(X\) with \(\mathbb{E}X = 0\) and \(\mathbb{E}\|X\|^p < \infty\). If \(\mathbb{B}\) is a Banach space of Rademacher type \(p\), then
\[ \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left( \left\| S_n^{(1)} - \mathbb{E} S_n^{(1)} \right\| n \right|^p \) < \infty \text{ if and only if } \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left( \left\| U_n - \mathbb{E} U_n \right\|^p \right) < \infty. \] (3.17)

**Proof** Note that
\[ \left( S_n^{(1)} - \mathbb{E} S_n^{(1)} \right) - (U_n - \mathbb{E}U_n) = \sum_{k=1}^{n} (X_k I \{ k < \| X_k \|^p \leq n \} - \mathbb{E} \{ k < \| X \|^p \leq n \}), \quad n \geq 1. \]

Then since \(\mathbb{B}\) is a Banach space of Rademacher type \(p\), we have that
\[ \mathbb{E} \left( \left\| S_n^{(1)} - \mathbb{E} S_n^{(1)} \right\| \right)^p \leq C_{11} \sum_{k=1}^{n} \mathbb{E} \left( \| X \|^p \right) I \{ k < \| X \|^p \leq n \} \}
\[ < C_{12} \sum_{k=1}^{n} \mathbb{E} \left( \| X \|^p \right) I \{ k < \| X \|^p \leq n \}, \quad n \geq 1. \]

Let \(Y = \| X \|^p\). Then it follows from \(\mathbb{E}Y < \infty\) (since \(\mathbb{E}\|X\|^p < \infty\)) and the first conclusion of Lemma 3.5 (i.e., (3.13)) that
\[ \sum_{n=1}^{\infty} \frac{1}{n} \mathbb{E} \left( \left\| S_n^{(1)} - \mathbb{E} S_n^{(1)} \right\| \right)^p \leq C_{12} \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n} \mathbb{E}Y I \{ k < Y \leq n \} < \infty, \]
which yields (3.17). \[\square\]

Proof of Theorem 2.1 To prove Theorem 2.1, we make the following simple observation. Let $0 < p < q \leq 1$. Let $\{X_n; n \geq 1\}$ be a sequence of independent copies of a $B$-valued random variable $X$ with $\mathbb{E}\|X\|^p < \infty$. Set $\rho_1 = p/q$, $Y = \|X\|^q$, $Y_n = \|X_n\|^q$, $n \geq 1$. Then $0 < \rho_1 < 1$ and $\mathbb{E}Y^{\rho_1} < \infty$, and

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \left(\|S_n\|^q\right)^{\rho_1} \leq \sum_{n=1}^{\infty} \frac{\|X_k\|^q}{n^{1+q/p}} = \sum_{n=k}^{\infty} \left(\frac{1}{n^{1+q/p}}\right) \|X_k\|^q \leq \sum_{k=1}^{\infty} k^{-\rho_1/p} Y_k < \infty \ a.s.$$

(see Theorem 5.1.3 in Chow and Teicher [2, p. 118]). Theorem 2.1 follows immediately from this observation together with Theorem 1.4 and Remark 1.2. \[\square\]

Proof of Theorem 2.2 (Sufficiency) Firstly we consider the case where $1 \leq q < p < 2$. Since $\mathbb{E}X = 0$, we see that

$$S_n = (U_n^{(1)} - \mathbb{E}U_n^{(1)}) + \left((S_n - U_n^{(1)}) - \mathbb{E}(S_n - U_n^{(1)})\right), \ n \geq 1$$

so that, by Lemmas 3.3 and 3.4, (2.2) ensures (1.1) which implies (2.1).

Secondly we consider the case where $1 < p < q$. Since $B$ is of stable type $p$, the Maurey-Pisier [16] theorem asserts that it is also of stable type $p + \delta$ for some $0 < \delta < q - p$. By Remark 1.2, (2.1) holds if we can show that

$$X \in SLLN(p, p + \delta); \ i.e., \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} \left(\|S_n\|^p\right)^{p+\delta} < \infty \ a.s.$$ (3.18)

Since $\mathbb{E}X = 0$, we have that

$$S_n = \sum_{k=1}^{n} X_k I\{\|X_k\|^p \leq n\} + \sum_{k=1}^{n} X_k I\{\|X_k\|^p > n\}$$

$$= (U_n - \mathbb{E}U_n) - n\mathbb{E}XI\{\|X\|^p > n\} + \sum_{k=1}^{n} X_k I\{\|X_k\|^p > n\}, \ n \geq 1.$$ (3.19)

It is easy to see that

$$\left\{\max_{1 \leq k \leq n} \|X_k\|^p > n \ i.o.(n)\right\} = \{\|X_n\|^p > n \ i.o.(n)\}.$$ $$\text{Since}\ \{X_n; n \geq 1\} \text{is a sequence of independent copies of a B-valued random variable X with E}\|X\|^p < \infty,$$ $$\text{it follows from the Borel-Cantelli lemma that}$$

$$\mathbb{P} (\|X_n\|^p > n \ i.o.(n)) = 0$$

and hence

$$\mathbb{P} \left(\max_{1 \leq k \leq n} \|X_k\|^p > n \ i.o.(n)\right) = 0,$$ (3.20)

which ensures that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{n^{1/p}}{\|S_n\|^p\{\|X\|^p > n\}^{p+\delta}}\right)^{p+\delta} < \infty \ a.s.$$ (3.21)

Note that $1 < p < 2$ and $\mathbb{E}\|X\|^p < \infty$ imply that

$$n\mathbb{E}\|X\|^pI\{\|X\|^p > n\} \leq n^{1/p} \mathbb{E}\|X\|^p = \mathbb{E}\|X\|^p, \ n \geq 1.$$
Thus, by (3.16) of Lemma 3.5, we have that
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|n\mathbb{E}X\{\|X\|^p > n\}\|}{n^{1/p}} \right)^{p+\delta} 
\]
\[
= \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|n\mathbb{E}X\{\|X\|^p > n\}\|}{n^{1/p}} \right)^p \left( \frac{\|n\mathbb{E}X\{\|X\|^p > n\}\|}{n^{1/p}} \right)^{\delta} 
\]
\[
\leq (\mathbb{E}\|X\|^p)^{\delta} \sum_{n=1}^{\infty} \left( \mathbb{E}\|X\|I\{\|X\|^p > n\} \right)^p \frac{n^{1/p}}{n^{2-p}} 
\]
\[
< \infty. 
\]
Since \(B\) is also of Rademacher type \(p + \delta\), we get that
\[
\mathbb{E}\|U_n - \mathbb{E}U_n\|^{p+\delta} \leq C_{13}n\mathbb{E}\|X\|I\{\|X\|^p > n\} \leq C_{14}\mathbb{E}\|X\|^{p+\delta} \mathbb{E}\|X\|^p I\{\|X\|^p > n\}, \quad n \geq 1. 
\]
Thus, by (3.15) of Lemma 3.5, we have that
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|U_n - \mathbb{E}U_n\|}{n^{1/p}} \right)^{p+\delta} \leq C_{14} \sum_{n=1}^{\infty} \frac{\mathbb{E}\|X\|^{p+\delta} I\{\|X\|^p > n\}}{n^{1+\delta/p}} \]
and hence
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|U_n - \mathbb{E}U_n\|}{n^{1/p}} \right)^{p+\delta} < \infty \quad \text{a.s.,} 
\]
which, together with (3.19), (3.21), and (3.22), ensures (3.18).
Lastly we consider the case where \(1 < q = p < 2\). Since \(\mathbb{E}\|X\|^p < \infty\), we have that
\[
\lim_{n \to \infty} \frac{u_n^p}{n} = 0; \quad \text{i.e.,} \quad \lim_{n \to \infty} \frac{u_n}{n^{1/p}} = 0. 
\]
Hence we can assume, without loss of generality, that \(u_n < n^{1/p}\) for all \(n \geq 1\). Since \(\mathbb{E}X = 0\), we have that, for \(n \geq 1\),
\[
S_n = \sum_{k=1}^{n} X_k I\{\|X_k\| \leq u_n\} + \sum_{k=1}^{n} X_k I\{u_n < \|X_k\| \leq n^{1/p}\} + \sum_{k=1}^{n} X_k I\{\|X_k\| > n^{1/p}\} 
\]
\[
= \left( U_n^{(1)} - \mathbb{E}U_n^{(1)} \right) + \left( U_n^{(2)} - \mathbb{E}U_n^{(2)} \right) - n\mathbb{E}X I\{\|X\|^p > n\} + \sum_{k=1}^{n} X_k I\{\|X_k\|^p > n\}. 
\]
Since (3.20) follows from \(\mathbb{E}\|X\|^p < \infty\), we see that
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|\sum_{k=1}^{n} X_k I\{\|X_k\|^p > n\}\|}{n^{1/p}} \right)^{p} < \infty \quad \text{a.s.} 
\]
(3.24)
Since \(p > 1\) and \(\mathbb{E}\|X\|^p < \infty\), it follows from (3.16) of Lemma 3.5 that
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|n\mathbb{E}X I\{\|X\|^p > n\}\|}{n^{1/p}} \right)^{p} \leq \sum_{n=1}^{\infty} \frac{\mathbb{E}\|X\| I\{\|X\|^p > n\}}{n^{2-p}} \]
\[
< \infty. 
\]
Since $\mathbb{E}\|X\|^p < \infty$ and $B$ is a Banach space of stable type $p \in (1, 2)$, by Remark 3.2, (3.12) holds, which ensures that
\[
\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|U_n^{(1)} - \mathbb{E}U_n^{(1)}\|}{n^{1/p}} \right)^p < \infty \quad \text{a.s.} \quad (3.26)
\]
Since $B$ is also a Banach space of Rademacher type $p$, we have that, for all $n \geq 1$,
\[
\begin{align*}
\mathbb{E}\|U_n^{(2)} - \mathbb{E}U_n^{(2)}\|_p^p & \leq C_{15} \sum_{k=1}^{n} \mathbb{E}\left\|X_k I \left\{u_n < \|X_k\| \leq n^{1/p}\right\} - \mathbb{E}\left( X I \left\{u_n < \|X\| \leq n^{1/p}\right\}\right\}\right\|_p^p \\
& \leq 2C_{15} n \mathbb{E}\left(\|X\| I \left\{u_n^p < \|X\| \leq n\right\}\right) \\
& = 2C_{15} n \int_{u_n^p}^{n} t d\mathbb{P}(\|X\| \leq t) \\
& \leq 2C_{15} n u_n^p \mathbb{P}(\|X\| > u_n^p) + 2C_{15} n \int_{u_n^p}^{n} \mathbb{P}(\|X\| > t) dt \\
& \leq 2C_{15} u_n^p + 2C_{15} n \int_{u_n^p}^{n} \mathbb{P}(\|X\| > t) dt.
\end{align*}
\]
Now (3.14) holds by Lemma 3.5. Thus it follows from (3.14) and (2.2) that
\[
\sum_{n=1}^{\infty} \frac{\mathbb{E}\|U_n^{(2)} - \mathbb{E}U_n^{(2)}\|_p^p}{n^2} < \infty,
\]
which ensures that
\[
\sum_{n=1}^{\infty} \frac{\|U_n^{(2)} - \mathbb{E}U_n^{(2)}\|_p^p}{n^2} < \infty \quad \text{a.s.} \quad (3.27)
\]
Combining (3.23)-(3.27), we conclude that (2.1) holds for $q = p$. The proof of the sufficiency half of Theorem 2.2 is complete. □

Proof of Theorem 2.2 (Necessity) For the case where $q \neq p$, by Theorem 1.4, we see that (2.2) follows immediately from (2.1).

We now consider the case where $q = p$. By Theorem 1.4, (2.1) implies that $\mathbb{E}X = 0$ and $\mathbb{E}\|X\|^p < \infty$. Hence we can assume, without loss of generality, that $u_n^p < n$ for all $n \geq 1$. We thus only need to show that (2.1) (with $q = p$) implies that
\[
\sum_{n=1}^{\infty} \frac{\int_{u_n^p}^{n} \mathbb{P}(\|X\| > t) dt}{n} < \infty. \quad (3.28)
\]
To see this, let $\{X', X'_n; n \geq 1\}$ be an independent copy of $\{X, X_n; n \geq 1\}$. Let
\[
V_n = \left( X_n I \left\{ \|X_n\|^p \leq n \right\} - X'_n I \left\{ \|X'_n\|^p \leq n \right\} \right) \quad \text{and} \quad \hat{S}_n^{(1)} = \sum_{k=1}^{n} V_k, \quad n \geq 1.
\]
Then $\{V_n; n \geq 1\}$ is a sequence of independent symmetric $B$-valued random variables. By the Borel-Cantelli lemma, it follows from $\mathbb{E}\|X\|^p < \infty$ that
\[
\mathbb{P}(\|X_n\|^p > n \text{ i.o.}(n)) = 0,
\]
which ensures that
\[ S_n^{(2)} = \sum_{k=1}^{n} X_k I \{ \|X_k\|^p > k \} = O(1) \text{ a.s. as } n \to \infty \]
and hence
\[ \sum_{n=1}^\infty \frac{1}{n} \left( \frac{\|S_n^{(2)}\|}{n^{1/p}} \right)^p = \sum_{n=1}^\infty \frac{\|S_n^{(2)}\|^p}{n^2} < \infty \text{ a.s.} \]
Note that
\[ \|S_n^{(1)}\| \leq \|S_n\| + \|S_n^{(2)}\|, \quad n \geq 1. \]
It thus follows from (2.1) (with \( q = p \)) that
\[ \sum_{n=1}^\infty \frac{\|S_n^{(1)}\|^p}{n^2} < \infty \text{ a.s.} \quad (3.29) \]
and hence
\[ \sum_{n=1}^\infty \frac{\|\hat{S}_n^{(1)}\|^p}{n^2} < \infty \text{ a.s.} \quad (3.30) \]
Let \( a_n = 1/n^2, \quad n \geq 1. \) Then
\[ b_n = \sum_{k=n}^\infty a_k = \sum_{k=n}^\infty \frac{1}{n^2} \leq \frac{2}{n}, \quad n \geq 1 \]
and hence
\[ \sup_{n \geq 1} b_n \|V_n\|^p \leq \sup_{n \geq 1} \frac{2}{n} \left( 2n^{1/p} \right)^p = 2^{p+1} \text{ a.s.} \]
We thus have that
\[ \mathbb{E} \left( \sup_{n \geq 1} b_n \|V_n\|^p \right) < \infty. \quad (3.31) \]
By Theorem 3.1, we conclude from (3.30) and (3.31) that
\[ \sum_{n=1}^\infty \mathbb{E}\frac{\|\hat{S}_n^{(1)}\|^p}{n^2} < \infty; \]
that is,
\[ \mathbb{E} \left( \sum_{n=1}^\infty \frac{\|\hat{S}_n^{(1)}\|^p}{n^2} \right) < \infty. \quad (3.32) \]
By Lemma 3.6, it follows from (3.29) and (3.32) that
\[ \mathbb{E} \left( \sum_{n=1}^\infty \frac{\|S_n^{(1)}\|^p}{n^2} \right) < \infty; \]
that is,
\[ \sum_{n=1}^\infty \mathbb{E}\frac{\|S_n^{(1)}\|^p}{n^2} < \infty. \quad (3.33) \]
Since \( 1 < p < 2, \) applying (2.5) of Ledoux and Talagrand [9, p. 46], (3.33) ensures that
\[ \sum_{n=1}^\infty \frac{\|\mathbb{E}S_n^{(1)}\|^p}{n^2} < \infty. \]
which, together with (3.33), gives
\[ \sum_{n=1}^{\infty} \frac{E\|S_n^{(1)} - ES_n^{(1)}\|^p}{n^2} < \infty. \]

By Lemma 3.7, this is equivalent to
\[ \sum_{n=1}^{\infty} \frac{E\|U_n - EU_n\|^p}{n^2} < \infty. \] (3.34)

Since \( \|U_n^{(2)} - EU_n^{(2)}\| \leq \|U_n - EU_n\| + \|U_n^{(1)} - EU_n^{(1)}\|, \) \( n \geq 1 \) and \( B \) is of stable type \( p \) where \( 1 < p < 2, \) it follows from Remark 3.2 and (3.34) that
\[ \sum_{n=1}^{\infty} \frac{E\|U_n^{(2)} - EU_n^{(2)}\|^p}{n^2} < \infty. \] (3.35)

By Lemma 3.1 (ii) of Li, Qi, and Rosalsky [10],
\[ E \max_{1 \leq k \leq n} \|X_k I\{u_n^p < \|X_k\|^p \leq n\} - E\|X\|^p \leq n\|X_k\|^p \| \]
\[ = E \left( \max_{1 \leq k \leq n} \|X_k I\{u_n^p < \|X_k\|^p \leq n\} - E\|X\|^p \leq n\|X_k\|^p \| \right)^p \]
\[ \leq 2^{p+1} E\|U_n^{(2)} - EU_n^{(2)}\|^p, \ n \geq 1. \]

It thus follows from (3.35) and \( E\|X\|^p < \infty \) that
\[ \sum_{n=1}^{\infty} \frac{E\max_{1 \leq k \leq n} \|X_k I\{u_n^p < \|X_k\|^p \leq n\}\|^p}{n^2} \]
\[ \leq 2^{2p} \sum_{n=1}^{\infty} \frac{E\|U_n^{(2)} - EU_n^{(2)}\|^p}{n^2} + 2^{p-1} \sum_{n=1}^{\infty} \frac{E\|X\|^p}{n^2} \]
\[ < \infty, \]

and hence, by Lemma 5.4 of Li, Qi, and Rosalsky [10], noting that \( P(\|X\|^p > u_n^p) \leq n^{-1}, \ n \geq 1, \) we get that
\[ \sum_{n=1}^{\infty} \frac{E\|X\|^p I\{u_n^p < \|X\|^p \leq n\}}{n} < \infty. \] (3.36)

Using partial integration, one can easily see that
\[ \left| E\|X\|^p I\{u_n^p < \|X\|^p \leq n\} - \int_{u_n^p}^{n} P(\|X\|^p > t) \, dt \right| \leq \frac{u_n^p}{n} + nP(\|X\|^p > n), \ n \geq 1. \] (3.37)

Since \( E\|X\|^p < \infty, \) we have
\[ \sum_{n=1}^{\infty} \frac{n P(\|X\|^p > n)}{n} = \sum_{n=1}^{n} P(\|X\|^p > n) < \infty, \] (3.38)
and, by Lemma 3.5, (3.14) holds. We thus see that (3.28) follows from (3.36), (3.37), (3.38), and (3.14) thereby completing the proof of the necessity half of Theorem 2.2. □

Proof of Theorem 2.3 We only need to consider the case where $q > 1$ since for the case where $q = 1$, Theorem 2.3 is Theorem 2.3 of Li, Qi, and Rosalsky [10]. Note that

$$\left\{ \max_{1 \leq k \leq n} \|X_k\| > n \text{ i.o.}(n) \right\} = \{\|X_n\| > n \text{ i.o.}(n)\}$$

and for $p = 1$,

$$U_n = \sum_{k=1}^{n} X_k I\{\|X_k\| \leq n\}, \; n \geq 1.$$ 

By the Borel-Cantelli lemma, it thus follows from $E\|X\| < \infty$ that

$$P\left( \max_{1 \leq k \leq n} \|X_k\| > n \text{ i.o.}(n) \right) = 0$$

and hence

$$P(S_n - U_n \neq 0 \text{ i.o.}(n)) = 0,$$

which ensures that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|S_n - U_n\|}{n} \right)^q < \infty \; \text{a.s.} \quad (3.39)$$

and by the Mourier [17] SLLN, it follows from (2.4) that

$$\lim_{n \to \infty} \frac{U_n - nE(XI\{\|X\| \leq n\})}{n} = \lim_{n \to \infty} \left( \frac{S_n}{n} - E(XI\{\|X\| \leq n\}) \right) - \lim_{n \to \infty} \frac{S_n - U_n}{n} = 0 \; \text{a.s.} \quad (3.40)$$

We now show that

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|U_n - nE(XI\{\|X\| \leq n\})\|}{n} \right)^q < \infty \; \text{a.s.} \quad (3.41)$$

Since $B$ is of stable type 1, the Maurey-Pisier [16] theorem asserts that it is also of stable type $1 + \delta$ for some $0 < \delta < q - 1$ and hence

$$E\|U_n - nE(XI\{\|X\| \leq n\})\|^{1+\delta} \leq C_{16} E\left(\|X\|^{1+\delta}I\{\|X\| \leq n\}\right), \; n \geq 1.$$ 

Thus, by (3.15) (with $p = 1$) of Lemma 3.5, we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{n} E\left( \frac{\|U_n - nE(XI\{\|X\| \leq n\})\|}{n} \right)^{1+\delta} < \infty$$

and hence

$$\sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{\|U_n - nE(XI\{\|X\| \leq n\})\|}{n} \right)^{1+\delta} < \infty \; \text{a.s.,}$$

which, together with (3.40), ensures that (3.41) holds since $q > 1 + \delta$. Note that

$$S_n = (S_n - U_n) + (U_n - nE(XI\{\|X\| \leq n\})) + nE(XI\{\|X\| \leq n\}), \; n \geq 1.$$ 

We thus see that (2.3) (with $q > 1$) follows from (3.39), (3.41), and the second half of (2.4) (with $q > 1$).
Conversely, by Theorem 1.4 and the Mourier [17] SLLN, it follows from (2.3) that \( \mathbb{E}X = 0 \) and \( \mathbb{E}\|X\| < \infty \) and hence (3.39) and (3.41) (since \( B \) is of stable type 1) hold. Note that

\[
n \mathbb{E}(X \{ \|X\| \leq n \}) = S_n - (S_n - U_n) - (U_n - n \mathbb{E}(X \{ \|X\| \leq n \})), \quad n \geq 1.
\]

It thus follows from (2.3), (3.39), and (3.41) that

\[
\sum_{n=1}^{\infty} \frac{\mathbb{E}(X \{ \|X\| \leq n \|}^q}{n^q} = \sum_{n=1}^{\infty} \left( \frac{n \mathbb{E}(X \{ \|X\| \leq n \})}{n} \right)^q < \infty
\]

and hence (2.4) holds (with \( q > 1 \)). The proof of Theorem 2.3 is complete. \( \square \)

4 Three Examples

Li, Qi, and Rosalsky [10] provided three examples (see, Examples 5.1, 5.2, and 5.3 of Li, Qi, and Rosalsky [10]) for illustrating the necessary and sufficient conditions that they obtained for (2.3) for the case where \( q = 1 \). In this section we provide three examples to illustrate our Theorems 1.4, 2.2, and 2.3.

Example 4.1. Let \( 1 < r < p < 2 \) and let \( X \) be a real-valued symmetric random variable such that

\[
\mathbb{P}(X = 0) = b \quad \text{and} \quad \mathbb{P}(|X| > t) = \int_t^{\infty} \frac{1}{x^{p+1} \ln t} dt, \quad t \geq e,
\]

where \( b = 1 - \int_{e}^{\infty} \frac{1}{x^{p+1} \ln x} dx \). Then

\[
\mathbb{P}(|X| > t) \sim \frac{1}{pt^{p \ln t}} \quad \text{as} \quad t \to \infty
\]

and hence, for \( 1 \leq q < p \),

\[
\mathbb{P}^{q/p}(|X|^q > t) = \mathbb{P}^{q/p}(|X| > t^{1/q}) \sim (q^{r/p})^{q/p} t^{-1/(q^{r/p})} \quad \text{as} \quad t \to \infty.
\]

We then see that

\[
\int_0^{\infty} \mathbb{P}^{q/p}(|X|^q > t) \, dt \begin{cases} < \infty & \text{if} \ p/r < q < p, \\ = \infty & \text{if} \ 1 \leq q \leq p/r. \end{cases}
\]

It is also easy to check that

\[
\mathbb{E}|X|^p \ln(1 + |X|) = \infty \quad \text{and} \quad \mathbb{E}|X|^q = \infty \quad \text{for all} \ q > p.
\]

By Theorem 2.2 and Remark 1.2, for this example, \( X \in \text{SLLN}(p,q) \) if and only if \( p/r < q < \infty \). However, by Corollary 2.2, (1.1) holds if and only if \( p/r < q < p \). This means that, if (1.1) holds for some \( q = q_1 > 0 \), one cannot conclude that (1.1) holds for either \( 0 < q < q_1 \) or \( q > q_1 \).

Example 4.2. Let \( 1 < p < 2 \) and let \( X \) be a real-valued symmetric random variable with density function

\[
f(x) = \frac{b}{|x|^{p+1} (\ln |x|) (\ln \ln |x|)^2} I\{|x| > 3\},
\]

where \( 0 < b < \infty \) is such that \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \). Clearly, we have that

\[
\mathbb{E}X = 0 \quad \text{and} \quad \mathbb{E}|X|^p < \infty.
\]
Since
\[ P(|X| > x) \sim \frac{2b/p}{x^p(\ln x)(\ln \ln x)^2} \quad \text{as} \quad x \to \infty, \]
we see that
\[ u_n \sim \frac{(2bn)^{1/p}}{(\ln n)^{1/p}(\ln \ln n)^{2/p}} \quad \text{as} \quad n \to \infty \]
and hence, for all sufficiently large \( n \),
\[ \int_{u_n^n}^{\infty} P(|X|^p > t) \, dt = \int_{u_n^n}^{\infty} P\left(|X| > t^{1/p}\right) \, dt \]
\[ \geq \int_{u_n^n}^{\infty} \frac{b}{(\ln t)(\ln \ln t)^2} \, dt \]
\[ \geq \frac{b}{(\ln n)(\ln \ln n)^2} \int_{u_n^n}^{\infty} \frac{1}{t} \, dt \]
\[ \sim \frac{b}{(\ln n)(\ln \ln n)} \quad \text{as} \quad n \to \infty. \]

Note that
\[ \sum_{n=3}^{\infty} \frac{b}{n(\ln n)(\ln \ln n)} = \infty \]
and so
\[ \sum_{n=1}^{\infty} \frac{\int_{\min\{u_n^n, n\}}^{\infty} P(|X|^p > t) \, dt}{n} = \infty. \]

By Theorem 2.2 and Remark 1.2, we thus conclude that \( X \notin SLLN(p, q) \) for this example for all \( 0 < q \leq p \).

Let \( 1 < p < 2 \) and let \( \{X_n; n \geq 1\} \) be a sequence of independent copies of a symmetric real-valued random variable \( X \). Then, by either Theorem 2.2 or Theorem 2.4, the following three statements are equivalent:

(i) \( \mathbb{E}X = 0 \) and \( \mathbb{E}|X|^p < \infty; \)
(ii) \( X \in SLLN(p, q) \) for some \( q > p; \)
(iii) \( X \in SLLN(p, q) \) for all \( q > p. \)

However, the following example says that this is not true when \( p = 1 \).

Example 4.3. Let \( X \) be a real-valued random variable such that
\[ P\left(X = -\frac{1}{1-a}\right) = 1 - a \quad \text{and} \quad P(X > x) = \int_{x}^{\infty} \frac{1}{t^2(\ln t)(\ln \ln t)^2} \, dt, \quad x \geq e^e \]
where \( a = \int_{e}^{\infty} \frac{1}{t^2(\ln t)(\ln \ln t)^2} \, dt. \) Then \( \mathbb{E}X = 0, \mathbb{E}|X| < \infty, \)
and, for all sufficiently large \( n \),
\[ \mathbb{E}XI\{|X| \leq n\} = -\mathbb{E}IX\{|X| > n\} = -\int_{n}^{\infty} \frac{1}{t(\ln t)(\ln \ln t)^2} \, dt = -\frac{1}{\ln \ln n}. \]
Note that
\[ \sum_{n=2}^{\infty} \frac{1}{n(\ln \ln n)^q} = \infty \quad \text{for all } q > 1. \]

Thus for this example, by either Theorem 2.3 or Theorem 2.4, \( X \notin SLLN(1,q) \) for all \( q > 1 \).

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