Generalized Lagrange Coded Computing: A Flexible Computation-Communication Tradeoff

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Abstract—We consider the problem of evaluating arbitrary multivariate polynomials over a massive dataset, in a distributed computing system with a master node and multiple worker nodes. Generalized Lagrange Coded Computing (GLCC) codes are proposed to provide robustness against stragglers who do not return computation results in time, adversarial workers who deliberately modify results for their benefit, and information-theoretic security of the dataset amidst possible collusion of workers. GLCC codes are constructed by first partitioning the dataset into multiple groups, and then encoding the dataset using carefully designed interpolation polynomials, such that interference computation results across groups can be eliminated at the master. Particularly, GLCC codes include the state-of-the-art Lagrange Coded Computing (LCC) codes as a special case, and achieve a more flexible tradeoff between communication and computation overheads in optimizing system efficiency.

Index Terms—coded distributed computing, Lagrange polynomial interpolation, interference cancellation, security and privacy.

I. INTRODUCTION

As the era of Big Data advances, distributed computing has emerged as a natural approach to speed up computationally intensive operations by dividing and outsourcing the computation among many worker nodes that operate in parallel. However, scaling out the computation across distributed workers leads to several fundamental challenges, typically including additional communication overhead compared to centralized processing, and slow or delay-prone worker nodes that can prolong computation execution time, known as the straggler effect \cite{1,2}. Furthermore, distributed computing systems are also much more susceptible to adversarial workers that deliberately modify the computations for their benefit, and raise serious security concern when processing sensitive raw data in distributed worker nodes. Therefore, designing computation and communication protocols that are robust against straggler effect and adversarial workers, while providing security guarantee is of vital importance for distributed computing applications.

Coded distributed computing is an emerging research direction that develops information-theoretic methods to alleviate the straggler effect, provide robustness against adversarial workers, and protect data security, via carefully adopting the idea of error control codes to inject computation redundancy across distributed workers \cite{3}. Coding for distributed computation was earlier considered in \cite{4} for linear function computation. Subsequently, polynomial codes \cite{5-7} were introduced for mitigating straggler effect and providing security guarantee \cite{8-10} by leveraging the algebraic structure of polynomial functions, within the context of distributed matrix multiplication. Also, references \cite{2,11} showed that matrix multiplication can be converted into the problem of computing the element-wise products of two batches of submatrices based on the concept of bilinear complexity \cite{12,13}. In a recent work \cite{14}, multi-linear batch codes were developed to compute the evaluations of a multi-linear map function over a batch of data, with minimizing communication and computation overheads while mitigating the effect of stragglers. A fundamental tradeoff between computation and communication overheads was established in \cite{13} for the general distributed computing frameworks like MapReduce.

![Fig. 1. A distributed computing system consisting of a master and N workers for evaluating arbitrary multivariate polynomials on dataset X. Worker i computes response $\hat{Y}_i$ to the master on local data $X_i$. The master waits for the results from fastest K workers, A out of whom may be malicious, to recover computation results. Up to T workers may collude to infer about X.](image)

As illustrated in Fig. 1 we consider the problem of evaluating $Y_m = \phi(X_m)$ for each data point $X_m$ in a large dataset $X = (X_1, X_2, \ldots, X_M)$ using a distributed computing system including a master node and $N$ worker nodes, where $\phi$ is an arbitrary multivariate polynomial. To do that, the master sends some encoded data of $X$ to each worker, while keeping $X$ secure from any up to $T$ colluding workers. Then each worker generates a response using the encoded data received. Due to straggler effect, the master only waits for the responses from a subset of fastest $K$ workers to recover desired evaluations, in the presence of $A$ adversarial workers, where the minimum
number of successful computing workers that the master needs to wait for is referred to as recovery threshold. This problem covers many computations of interest in machine learning, including the aforementioned linear computation $[4]$, matrix multiplication $[7]$, $[8]$ and multi-linear map computation $[4]$. Notably, the problem of evaluating multivariate polynomial was first introduced in $[15]$, where Lagrange Coded Computing (LCC) codes are proposed, but it requires high recovery threshold and download overhead, albeit at a low upload and computation overheads.

In this paper we focus on establishing a flexible tradeoff among recovery threshold, communication and computation overheads for the problem above. We propose generalized LCC (GLCC) codes, which include LCC codes as a special case and are more efficient for optimizing system performance to speed up the evaluation computation. The key components of GLCC codes include 1) partitioning the dataset into multiple groups and then encoding the data in each of groups using Lagrange interpolate polynomial to create computational redundancy across the workers, 2) eliminating the interference between all the groups at the master by using additional polynomials as interference cancellation coefficients. Consequently, the responses at workers can be viewed as evaluations of a composition polynomial and accordingly the decoding is completed by interpolating the polynomial like LCC codes.

**Notation:** For a finite set $\mathcal{K}, |\mathcal{K}|$ denotes its cardinality. For any positive integers $m, n$ such that $m < n$, $[n]$ and $[m : n]$ denote the sets $\{1, 2, \ldots, n\}$ and $\{m, m+1, \ldots, n\}$, respectively. Define $Y_{\mathcal{K}}$ as $\{Y_{k_1}, \ldots, Y_{k_m}\}$ for any index set $\mathcal{K} = \{k_1, \ldots, k_m\} \subseteq [n]$.

### II. Problem Formulation

Consider the problem of evaluating a multivariate polynomial $\phi : \mathbb{U} \rightarrow \mathbb{V}$ of total degree at most $D$ over a dataset $X = (X_1, X_2, \ldots, X_M)$ of $M$ input data, where $\mathbb{U}$ and $\mathbb{V}$ are vector spaces of dimensions $\mathbb{U}$ and $\mathbb{V}$, respectively, over some finite field $\mathbb{F}$. We are interested in computing polynomial evaluations of the dataset, over a distributed computing system with a master node and $N$ worker nodes in a secure manner, in which the goal of the master is to compute $Y = (Y_1, Y_2, \ldots, Y_M)$ such that $Y_m \triangleq \phi(X_m)$ for all $m \in [M]$, while keeping the dataset secure from any colluding subset of up to $T < N$ workers, as shown in Fig. 1.

For this purpose, the master sends $G$ encoded input data to each worker, for some design parameter $G$. Specifically, for each $i \in [N]$, the master generates the coded data $\tilde{X}_i = (\tilde{X}_{i,1}, \tilde{X}_{i,2}, \ldots, \tilde{X}_{i,G})$ for worker $i$, where each $\tilde{X}_{i,g}, g \in [G]$, is a coded input in $\mathbb{U}$, computed from some encoding function $f_{i,g}$ over the dataset $X$ and some random data $Z$ generated privately at the master, i.e., for all $g \in [G]$ and $i \in [N]$.

$$\tilde{X}_{i,g} = f_{i,g}(X, Z).$$

Upon receiving $\tilde{X}_i$, each worker $i$ first computes the evaluations of the polynomial $\phi$ over the $G$ encoded inputs in $\tilde{X}_i$ respectively, and then generates a response $\tilde{Y}_i$ as some function $h_i$ of the computation results, i.e.,

$$\tilde{Y}_i = h_i(\phi(\tilde{X}_{i,1}), \phi(\tilde{X}_{i,2}), \ldots, \phi(\tilde{X}_{i,G})), \quad \forall i \in [N],$$

which is sent back to the master.

Due to heterogeneity of computing resources and unreliable network conditions in distributed computing systems, we assume the presence of some straggler workers who may fail to respond in time. We also consider a set of up to $A$ adversary workers who may provide arbitrarily erroneous responses. Note that the master has no prior knowledge of the identities of the stragglers and adversary workers. In order to speed up computation, the master only waits for the responses from a subset of fastest workers, and then decodes the desired evaluations from their responses.

We consider distributed computing protocols that satisfy the following two requirements:

- **Security Constraint:** Any $T$ colluding workers must not reveal any information about the dataset, i.e.,

$$I(X; \tilde{X}_T) = 0, \quad \forall T \subseteq [N], |T| = T. \quad (1)$$

- **Correctness Constraint:** For some design parameter $K \leq N$, the master must be able to correctly decode the desired evaluations from the collection of responses of any fastest $K$ workers, even when up to $A$ out of the $K$ workers are adversarial, i.e.,

$$H(Y|\tilde{Y}_K) = 0, \quad \forall K \subseteq [N], |K| = K.$$

The performance of a distributed computing protocol is measured by the following key quantities:

1. The **recovery threshold** $K$, which is the minimum number of workers the master needs to wait for in order to recover the desired function evaluations.
2. The **communication cost**, which is comprised of the normalized upload cost for the dataset and the normalized download cost from the workers, defined as

$$P_u \triangleq \frac{\sum_{i=1}^{N} H(\tilde{X}_i)}{U}, \quad P_d \triangleq \max_{K: K \subseteq [N], |K| = K} \frac{H(\tilde{Y}_K)}{V}. \quad (2)$$

3. The **computation complexity**, which includes the complexities of encoding, worker computation and decoding. The encoding complexity $C_e$ at the master is defined as the number of arithmetic operations required to generate the encoded data $\tilde{X}_{[N]}$, normalized by $U$. The complexity of worker computation $C_w$ is defined as the number of arithmetic operations required to compute the response $\tilde{Y}_i$, maximized over $i \in [N]$ and normalized by the complexity of evaluating the polynomial on a single input. Finally, the decoding complexity $C_d$ at the master is defined as the number of arithmetic operations required to decode the desired evaluations $Y$ from the responses of fastest workers in $K$, maximized over all $K \subseteq [N]$ with $|K| = K$ and normalized by $V$.

Given the above secure distributed computing framework, our goal in this paper is to design encoding, computing and
decoding functions that minimize the recovery threshold, the communication cost and the computation complexity.

III. MAIN RESULTS AND DISCUSSIONS

We propose a family of novel Generalized Lagrange Coded Computing (GLCC) codes for the problem of evaluating multivariate polynomial in the distributed computing system. We state our main results in the following theorem.

**Theorem 1.** For computing any multivariate polynomial \( \phi \) of total degree at most \( D \) over \( M \) input data, over a distributed computing system of \( N \) workers with \( T \)-colluding security constraint and up to \( A \) adversary workers, the following performance metrics are achievable as long as \( N \geq D(M/G + T - 1) + (M - M/G) + 2A + 1 \), for any positive integer \( G \) such that \( G|M \).

**Recovery Threshold:**
\[ K = D \left( \frac{M}{G} + T - 1 \right) + \left( M - \frac{M}{G} \right) + 2A + 1, \]

**Upload Cost:** \( P_u = GN \),

**Download Cost:** \( P_d = D \left( \frac{M}{G} + T - 1 \right) + \left( M - \frac{M}{G} \right) + 2A + 1, \)

**Encoding Complexity:** \( C_e = O \left( GN(\log N)^2 \log \log N \right) \),

**Worker Computation Complexity:** \( C_w = G \),

**Decoding Complexity:** \( C_d = O(K(\log K)^2 \log \log K) \).

Theorem 1 is formally proven in Section IV by presenting the proposed GLCC codes. Intuitively, the master first partitions the dataset into \( G \) disjoint groups. To keep the dataset secure from any \( T \) colluding workers, the master encodes the \( M/G \) data in each of groups along with \( T \) random noises using a Lagrange interpolate polynomial of degree \( M/G + T - 1 \), and then shares the evaluations of the polynomial at distinct points with workers. This creates computational redundancy across the distributed workers that is used for providing robustness against stragglers, adversaries, and data security. After receiving the securely encoded data for the \( G \) groups, each worker first evaluates the multivariate polynomial \( \phi \) over the coded inputs in each group, which can be viewed as evaluating the composition of a Lagrange polynomial of degree \( M/G + T - 1 \) with a multivariate polynomial of total degree \( D \). The worker then generates the response by further combining the results from all groups, using coefficients that are evaluations of carefully chosen polynomials of degree \( M - M/G \). Overall, the responses at workers can be viewed as evaluations of a composition polynomial of degree \( D(M/G + T - 1) + (M - M/G) \). The master can interpolate this polynomial from any \( D(M/G + T - 1) + (M - M/G) + 2A + 1 \) worker responses in the presence of \( A \) adversarial workers, and finally completes the computation by evaluating the polynomial.

**Remark 1.** Our proposed GLCC codes include the LCC codes [16] as a special case by setting \( G = 1 \). Allowing each worker to process \( G > 1 \) coded inputs, GLCC proposes to partition the inputs into \( G \) groups and perform Lagrange coding within each group, and then combine the group computation results in a way that interference across groups can be eliminated at the master. This grouping-and-elimination strategy effectively reduces the degree of interpolation polynomial in each group, and hence the degree of the overall polynomial interpolated at the master, without sacrificing any security guarantee. Thus, GLCC codes achieve more flexible tradeoff among recovery threshold, communication cost and computation complexity. As the number of partitioning groups \( G \) increases, GLCC codes can reduce the recovery threshold, download cost and decoding complexity by increasing the upload cost, encoding complexity and worker computation complexity. One can optimize the grouping parameter \( G \) to minimize the delay of the overall computation.

**Remark 2.** As mentioned in Introduction, the problem considered in this paper covers many computations of interest in machine learning, including linear computation [4], matrix multiplication [5], [8], [10] and multi-linear map computation [14]. Specifically, consider the problem of Secure Batch Matrix Multiplication (SBMM) in [10], [17], [18] where the goal of the master is to compute the element-wise product \( \{ A_m \cdot B_m : m \in [M]\} \) of two batches of massive matrices \( (A_1, \ldots, A_M) \) and \( (B_1, \ldots, B_M) \), while keeping the matrices secure from any colluding subset of workers. Our problem contains SBMM as a special case by setting the data \( X_m = (A_m, B_m) \) and the multivariate polynomial \( \phi(X_m) = A_m \cdot B_m \) for all \( m \in [M] \), and particularly GLCC codes achieve the same system performance as SBMM codes in [10], [17], [18] when the partitioning of data matrices is not considered. Moreover, the problem of evaluating general multivariate polynomial in this paper also includes as special case the problem of multi-linear map computation in [14], obtained by setting \( \phi = \phi(x_1, x_2, \ldots, x_D) \) to be a multi-linear map function with \( D \) variables \( (x_1, x_2, \ldots, x_D) \) satisfying \( \phi(x_1, \ldots, x_{n-1}, a x_n + b x'_{n+1}, \ldots, x_D) = a \phi(x_1, \ldots, x_{n-1}, x_n, x_{n+1}, \ldots, x_D) + b \phi(x_1, \ldots, x_{n-1}, x'_n, x_{n+1}, \ldots, x_D) \) for all \( n \in [D] \) and \( a, b \in \mathbb{F} \). The performance of our GLCC codes are identical as multi-linear batch codes [14] for this special case.

IV. GENERALIZED LAGRANGE CODED COMPUTING CODES

In this section, we describe the GLCC codes, and analyse its security guarantees, communication cost and computation complexity. This provides the proof for Theorem 1.

We start with a simple example to illustrate the key idea behind the proposed GLCC codes.

**A. Illustrative Example**

We consider the function \( \phi(X_1) = X_1^2 \) with system parameters \( M = 4, T = 1, A = 1 \), i.e., the master wishes to compute \( X_1^2, X_2^2, X_3^2, X_4^2 \) from the distributed system with 1-security constraint and robustness against 1 adversarial worker.

Let us set \( G = 2 \) and partition the \( M = 4 \) data into \( G \) = 2 groups, each containing 2 data inputs, i.e., \((X_1, X_2)\) and \((X_3, X_4)\). To ensure security, the master chooses two random noises \( Z_1, Z_2 \), and then encodes the two groups of data
\((X_1, X_2, Z_1)\) and \((X_3, X_4, Z_2)\) using Lagrange interpolation polynomials, as follows.

\[
f_1(x) = X_1 \cdot \frac{(x - 2)(x - 5)}{(1 - 2)(1 - 5)} + X_2 \cdot \frac{(x - 1)(x - 5)}{(2 - 1)(2 - 5)} + Z_1 \cdot \frac{(x - 1)(x - 2)}{(5 - 1)(5 - 2)}.
\]

\[
f_2(x) = X_3 \cdot \frac{(x - 4)(x - 6)}{(3 - 4)(3 - 6)} + X_4 \cdot \frac{(x - 3)(x - 6)}{(4 - 3)(4 - 6)} + Z_2 \cdot \frac{(x - 3)(x - 4)}{(6 - 3)(6 - 4)}.
\]

Let \(\{\alpha_i : i \in [N]\}\) be pairwise distinct elements from \(F\) such that \(\{\alpha_i : i \in [N]\} \cap \{0\} = \emptyset\). The master sends \(X_{i,1} = f_1(\alpha_i)\) and \(X_{i,2} = f_2(\alpha_i)\) to worker \(i\), who then responds with

\[
\tilde{Y}_i = \phi(\alpha) \cdot (\alpha - 3)(\alpha - 4) + \phi(\alpha) \cdot (\alpha - 1)(\alpha - 2),
\]

which is equivalent to the evaluation of the following composition polynomial \(h(x)\) at point \(x = \alpha_i\).

\[
h(x) = \phi(f_1(x)) \cdot (x - 3)(x - 4) + \phi(f_2(x)) \cdot (x - 1)(x - 2).
\]

Here \(h(x)\) is a polynomial in \(x\) with degree \(6\). Hence \((Y_1, \ldots, Y_N) = (h(\alpha_1), \ldots, h(\alpha_N))\) forms an \((N, 7, 3)\) Reed-Solomon (RS) codeword, and accordingly the master can decode \(h(x)\) from the responses of any \(K\) workers employing an RS decoder, in presence of \(1\) adversary response. Having recovered \(h(x)\), the master evaluates it at points \(x = 1, 2, 3, 4\) to obtain

\[
h(1) = 6X_1^2, \quad h(2) = 2X_2^2, \quad h(3) = 2X_3^2, \quad h(4) = 6X_4^2.
\]

The master completes the desired evaluations \(X_1^2, X_2^2, X_3^2, X_4^2\) by eliminating the interference from constant terms.

\section*{B. General Description of GLCC}

Let \(G\) be any positive integer such that \(G | M\). We first divide the dataset \(X_1, \ldots, X_M\) evenly and arbitrarily into \(G\) groups, each containing \(L \cdot M / G\) data inputs. Without loss of generality, we can group the inputs such that the \(\ell\)-th input of group \(g\), denoted by \(X_{g,\ell}\), is given by

\[
X_{g,\ell} = X_{(g-1)L+\ell}, \quad \forall \ g \in [G], \ell \in [L]. \tag{3}
\]

We choose arbitrarily \(G(L + T) + N\) pairwise distinct elements from \(F\), denoted by \(\{\beta_{g,\ell,\alpha} : g \in [G], \ell \in [L + T], \alpha \in [N]\}\). To guarantee the security of data in group \(g\) for each \(g \in [G]\), the master samples independently and uniformly from \(U, T\) random variables \(Z_{g,L+1}, Z_{g,L+2}, \ldots, Z_{g,L+T}\), and constructs a polynomial \(f_g(x)\) of degree at most \(L + T - 1\) such that

\[
f_g(x) = \sum_{j=1}^{L} X_{g,j} \cdot \prod_{k \in [L+T] \setminus \{j\}} \frac{x - \beta_{g,k}}{\beta_{g,j} - \beta_{g,k}} \tag{4}
\]

By Lagrange interpolation rule and the degree restriction, we can exactly express \(f_g(x)\) as

\[
f_g(x) = \sum_{j=1}^{L} X_{g,j} \cdot \prod_{k \in [L+T] \setminus \{j\}} \frac{x - \beta_{g,k}}{\beta_{g,j} - \beta_{g,k}} \tag{5}
\]

For each \(i \in [N]\), a total of \(G\) encoded inputs \(\tilde{X}_{i,1}, \tilde{X}_{i,2}, \ldots, \tilde{X}_{i,G}\) are sent to worker \(i\), where for each \(g \in [G]\), \(\tilde{X}_{i,g}\) is generated by evaluating \(f_g(x)\) at point \(x = \alpha_i\), i.e., \(X_{i,g} = f_g(\alpha_i)\). Hence, the coded data \(\tilde{X}_i\) at worker \(i\) is given by

\[
\tilde{X}_i = (f_1(\alpha_i), f_2(\alpha_i), \ldots, f_G(\alpha_i)). \tag{6}
\]

The response \(\tilde{Y}_i\) returned by worker \(i\) to the master is computed as follow.

\[
\tilde{Y}_i = \sum_{g=0}^{G} \left( \phi(\alpha) \cdot \prod_{g' \in [G] \setminus \{g\}} \prod_{\ell \in [L]} (x - \beta_{g',\ell}) \right). \tag{7}
\]

Since \(\phi\) is a multivariate polynomial with total degree at most \(D\) and \(f_g(x)\) is a polynomial of degree \(L + T - 1\) for any \(g \in [G]\) by \(\dag\), the composite polynomial \(\phi(f_g(x))\) has degree at most \(D(L + T - 1)\). Accordingly, \(h(x)\) can be viewed as a polynomial in variable \(x\) with degree at most \(D(L + T - 1) + (G - 1)L\). Recall that \(\alpha_1, \ldots, \alpha_N\) are pairwise distinct elements in \(F\). Thus, \((\tilde{Y}_1, \ldots, \tilde{Y}_N) = (h(\alpha_1), \ldots, h(\alpha_N))\) forms an \((N, D(L + T - 1) + (G - 1)L + 1)\) RS codeword. The master can decode the polynomial \(h(x)\) from the responses of any \(D(L + T - 1) + (G - 1)L + 2A + 1\) workers by using RS decoding algorithms \(\cite{19, 20}\) in the presence of \(A\) adversary workers.

For any \(g \in [G], \ell \in [L]\), evaluating \(h(x)\) at \(x = \beta_{g,\ell}\) yields

\[
h(\beta_{g,\ell}) = \sum_{j=1}^{G} \left( \phi(f_j(\beta_{g,\ell})) \cdot \prod_{g' \in [G] \setminus \{g\}} \prod_{\ell' \in [L]} (\beta_{g',\ell'} - \beta_{g,\ell'}) \right) \tag{8}\]

\[
= \phi(f_g(\beta_{g,\ell})) \cdot \prod_{g' \in [G] \setminus \{g\}} \prod_{\ell' \in [L]} (\beta_{g',\ell'} - \beta_{g,\ell'}) \tag{9}\]

where \((a)\) is due to \(\prod_{g' \in [G] \setminus \{g\}} \prod_{\ell' \in [L]} (\beta_{g',\ell'} - \beta_{g,\ell'}) = 0\) for all \(j \in [G] \setminus \{g\}\) and \((b)\) follows by \(\dag\). Note from \(\dag\) that, the evaluation \(Y_{(g-1)L+\ell} = \phi(X_{(g-1)L+\ell}) = \phi(X_{g,\ell})\) can be obtained by eliminating the non-zero constant term \(\prod_{g' \in [G] \setminus \{g\}} \prod_{\ell' \in [L]} (\beta_{g',\ell'} - \beta_{g,\ell'})\) in \(h(\beta_{g,\ell})\). Finally, the master completes the desired computation \(Y = (Y_1, Y_2, \ldots, Y_M)\) after traversing all \(g \in [G], \ell \in [L]\).

To this end, we have demonstrated that for a given parameter \(G\), the proposed GLCC code achieves a recovery threshold of \(K = D(L + T - 1) + (G - 1)L + 2A + 1\) as long as \(N \geq K\).
C. Security, Communication and Computation Overheads

In this subsection, we prove the security of the proposed GLCC codes and analyze its communication cost and computation complexity.

Lemma 1 (Secret Sharing [21]). Given any positive integers $N,L,T$, let $W_1,\ldots,W_L \in \mathbb{U}$ be $L$ secrets and $Z_1,\ldots,Z_T$ be $T$ random noises distributed independently and uniformly on $\mathbb{U}$. Let $\alpha_1,\ldots,\alpha_N$ be $N$ distinct elements in $\mathbb{F}$. Denote

$$\varphi(x) = W_1 r_1(x) + \ldots + W_L r_L(x) + Z_1 c_1(x) + \ldots + Z_T c_T(x),$$

where $r_1(x),\ldots,r_L(x), c_1(x),\ldots,c_T(x) \in \mathbb{F}[x]$ are arbitrary functions of $x$. If the matrix

$$C = \begin{bmatrix} c_1(\alpha_i) & \ldots & c_T(\alpha_i) \\ \vdots & \ddots & \vdots \\ c_1(\alpha_u) & \ldots & c_T(\alpha_u) \end{bmatrix}_{T \times T}$$

is non-singular over $\mathbb{F}$ for any $T = \{i_1,\ldots,i_T\} \subseteq [N]$ with $|T| = T$, then the $T$ values $(\varphi(\alpha_{i_1}),\ldots,\varphi(\alpha_{i_T}))$ cannot leak any information about the secrets $W_1,\ldots,W_L$, i.e., $I(W_1,\ldots,W_L;\varphi(\alpha_{i_1}),\ldots,\varphi(\alpha_{i_T})) = 0$.

Lemma 2 (Polynomial Evaluation and Interpolation [22]). The evaluation of a $k$-th degree polynomial at $k+1$ arbitrary points can be done in $O(k(\log k)^2 \log k)$ arithmetic operations, and consequently, its dual problem, interpolation of a $k$-th degree polynomial from $k+1$ arbitrary points can also be performed in $O(k(\log k)^2 \log k)$ arithmetic operations.

Lemma 3 (Decoding Reed-Solomon Codes [19, 20]). Decoding Reed-Solomon codes of dimension $k$ with $a$ erroneous symbols over arbitrary finite fields can be done in $O(k(\log k)^2 \log k)$ arithmetic operations if its minimum distance $d$ satisfies $d > 2a$.

Security: Let $T = \{i_1,\ldots,i_T\}$ be any $T$ indices out of the $N$ workers. From (5) and (6), the encoded data $f_g(\alpha_{i_1}),\ldots,f_g(\alpha_{i_T})$ sent to the workers in $T$ are protected by $T$ independent and uniform random noises $Z_{g,L+1},\ldots,Z_{g,L+T}$ for each $g \in [G]$, as shown below.

$$[f_g(\alpha_{i_1}),\ldots,f_g(\alpha_{i_T})]^T = [u_g(\alpha_{i_1}),\ldots,u_g(\alpha_{i_T})]^T +$$

$$\begin{bmatrix} c_{g,L+1}(\alpha_{i_1}) & \ldots & c_{g,L+T}(\alpha_{i_1}) \\ \vdots & \ddots & \vdots \\ c_{g,L+1}(\alpha_{i_T}) & \ldots & c_{g,L+T}(\alpha_{i_T}) \end{bmatrix}_{T \times S}^{C_g^T},$$

where $u_g(x) = \sum_{j=1}^{L} X_{g,j} \cdot \prod_{k \in [L+T] \setminus \{j\}} x_{g,j} + \frac{1}{\beta_{g,j}}$ and $c_{g,j}(x) = \prod_{k \in [L+T] \setminus \{j\}} x_{g,j} - \frac{1}{\beta_{g,j}}$ for all $j \in [L+1:L+T]$. Recall that $\beta_{g,j},\alpha_i, g \in [G], \ell \in [L+T], i \in [N]$ are pairwise distinct elements in $\mathbb{F}$. Thus, the matrix $C_g^T$ is non-singular over $\mathbb{F}$ for any $T$ and $g \in [G]$ [19]. Furthermore,

$$I(\{X-g\} | X_T)$$

$\equiv 0,$

where (a) is due to (3) and (6); (b) follows by Lemma 1 and the fact that $Z_{g,L+1},\ldots,Z_{g,L+T}$ in (10) are i.i.d. uniform random noises on $\mathbb{U}$ and are generated independently across all $g \in [G]$. This proves that the GLCC codes satisfy the security constraint in (1).

Communication Cost: The master shares $G$ encoded data of sizes $U$ to each worker by (6), and downloads a response of size $V$ from each of responsive workers by (7). Thus, from (2), the upload and download cost are given by $P_u = GN/U = GN$ and $P_d = KV/V = K$, respectively.

Computation Complexity: The encoding process for the dataset can be viewed as evaluating $G$ polynomials of degree $L + T = 1 < N$ at $N$ points by (5) and (6), which incurs a normalized complexity of $O(GN(\log N)^2 \log N)$ by Lemma 2. To generate the response (7), the computation mainly includes evaluating the multivariate polynomial $\phi$ over $G$ data inputs, which incurs a normalized complexity of $O$. Note that the terms $\prod_{g' \in [G]}(\alpha_{g'} - \beta_{g',k}), g \in [G]$ in (7) are constant independent of the dataset and can be pre-computed at the master, and thus the complexity of eliminating these terms is also negligible.

V. CONCLUDING REMARKS

In this paper, we presented GLCC codes for the problem of evaluating arbitrary multivariate polynomial in a distributed computing system. Our GLCC codes include the state-of-the-art LCC codes as a special case, and achieve a more flexible tradeoff among recovery threshold, communication cost and complexity complexity, which can be leveraged to optimize the computation latency of the system.

LCC codes have been demonstrated to accelerate training distributed linear regression [16] and logistic regression models [23]. Moreover, as shown in Introduction, matrix multiplication can be converted into the problem of computing the element-wise products of two batches of sub-matrices based on the concept of bilinear complexity [12, 13]. Since GLCC codes include LCC codes as a special case, applying GLCC codes to further accelerate system performance of training linear regression [16] and logistic regression models [23], and distributed matrix multiplication [7–10] are interesting for future research.

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