Wreath products and multipartite quantum systems

Vladimir V Kornyak
Laboratory of Information Technologies, Joint Institute for Nuclear Research
141980 Dubna, Russia
E-mail: vkornyak@gmail.com

Abstract. A natural symmetry group of a multicomponent quantum system is a special combination of a symmetry group acting within a single component (“local group”) and a group that permutes the components (“spatial symmetry group”). This combination is called the wreath product. Unitary representations of wreath products describe quantum evolutions of multipartite systems. It is known that any unitary representation of a finite group is contained in some permutation representation. We describe an algorithm for decomposing permutation representations of wreath products into irreducible components. This decomposition makes it possible to study the quantum behavior (entanglement, non-local correlations, etc.) of multipartite systems in invariant subspaces of the permutation Hilbert space.

1. Introduction
The Hilbert space of a multipartite quantum system is the tensor product of the Hilbert spaces of the components: \( \tilde{H} = \bigotimes_{x=1}^{N} H_x \). The states of the multipartite system that can be represented as a weighted sum of tensor products of the states of the components are called separable. States that are not separable are called entangled. The vast majority of the states of a typical multipartite system are entangled. The concept of entanglement is the basis of quantum informatics. Entanglement leads to such experimentally observable phenomena as nonlocal quantum correlations and quantum teleportation. Moreover, in recent years, the idea that physical space itself is not a fundamental entity, but emerges as an approximate phenomenological structure within “the Hilbert space of the Universe” as a result of some process of statistical selection has become increasingly popular. The idea of an emergent space is attractive, in particular, because of the possibility to reformulate the problem of reconciling quantum mechanics with gravity, where the main difficulties arise from the fact that quantum and space-time structures are considered to be equally fundamental. Typical approaches to identify geometric structures within a Hilbert space use the concept of entanglement (see, e.g., [1, 2]) and can be briefly described as follows. The quantum state in a large Hilbert space decomposes approximately into a tensor product of a large number of factors, which are interpreted as points (or bulks) of a geometric space. The distances between the points (metric) are determined by an entanglement measure — a typical example of which is quantum mutual information.

Let \( X \cong \{1, \ldots, N\} \) be a space with a finite set of points. To reproduce the usual property of space, homogeneity, we will assume that the group of spatial symmetries \( G = G(X) \) permutes transitively the components — indexed by the elements of \( X \) — of a multipartite system. In this case, the Hilbert space of the system can be written as

\[
\tilde{H} = H^\otimes N,
\]
where \( \mathcal{H} \) is a representative of the \( G \)-orbit \( \mathcal{H}_{xG} \). We will call \( \mathcal{H} \) a local Hilbert space.

A quantum description can be made constructive if continuous groups of unitary evolutions are replaced in quantum formalism by unitary representations of finite groups. It is known that any linear (which is always unitary for a simple general reason) representation of a finite group is a subrepresentation of some permutation representation. In particular, the so-called regular representation, that is, the permutation representation of the action of a group on its own elements, contains all possible irreducible representations of a given group. Thus, we can embed any constructive quantum model into a suitable invariant subspace of some permutation representation \[3, 4\]. If we can decompose a permutation representation of a group into irreducible components, then we can construct any representation of the group. Here we describe an algorithm for decomposing Hilbert space \([1]\) into invariant subspaces with respect to the natural symmetry group of a multipartite quantum system. In particular, the algorithm outputs a complete set of orthogonal irreducible invariant projectors to irreducible invariant subspaces.

2. Irreducible invariant projectors of wreath product representation

Let \( V \cong \{1, \ldots, M\} \) be a basis of the local Hilbert space \( \mathcal{H} \) on which a group of local symmetries \( F = F(V) \) acts. The sets \( X \) and \( V \) and the group \( F \) can be treated, respectively, as the base, the typical fiber and the structure group of a fiber bundle. A natural symmetry group that acts on the set \( V^X \) of the bundle sections and preserves the structure of the bundle is the wreath product of the groups \( F \) and \( G \). It is denoted as \( \widetilde{W} = F \wr G \) and has the structure of a semidirect product: \( \widetilde{W} \cong F^X \rtimes G \). The action of \( \widetilde{W} \) on \( V^X \) is defined as \( \widetilde{w}(x) = (f(x), g) = v(xg^{-1})f(xg^{-1}) \), where \( v \in V^X, f \in F^X, g \in G \). The right-action convention is used for all group actions.

In \([5]\), we proposed an algorithm for decomposing representations of finite groups based on the construction of a complete set of mutually orthogonal irreducible invariant projectors. These projectors are special elements of the centralizer algebra, which is defined as the algebra of matrices that commute with all matrices of the representation. The dimension of the centralizer algebra is called the rank of the representation. The computer implementation of the algorithm in \([5]\) proved to be very effective in problems with low ranks. In particular, the program coped with many high dimensional representations of simple groups and their “small” extensions (such representations typically have low ranks), presented in the ATLAS \([7]\), in the computationally difficult case of characteristic zero \([6]\).

The permutation representation \( \tilde{P} \) of the wreath product is a representation of \( \widetilde{W} \) by \((0, 1)\)-matrices of the size \( M^N \times M^N \) that have the form \( \tilde{P}(\tilde{w})_{u, v} = \delta_{u, \tilde{w}v} \), where \( \tilde{w} \in \widetilde{W}; u, v \in V^X \); \( \delta \) is the Kronecker delta. To obtain all irreducible components of the representation \( \tilde{P} \) in the Hilbert space \([1]\) it is sufficient to assume that the base field of this space is some abelian extension of the field of rational numbers \( \mathbb{Q} \) that is a splitting field for the local group \( F \).

Unfortunately, to split the representation \( \tilde{P} \) we cannot apply the algorithm \([6]\) because wreath products are far from simple groups and their representations have too high ranks. However, it is possible to express the irreducible invariant projectors for the wreath product representation in terms of the projectors for the local group representation.

Let \( B_1, \ldots, B_L \) be the complete set of mutually orthogonal irreducible invariant projectors for the permutation representation of the local group \( F \). Denote by \( T^X \) the set of all maps from \( X \) to \( L \), where \( L = \{1, \ldots, L\} \). The action of \( g \in G \) on the map \( \ell = [\ell_1, \ldots, \ell_N] \in T^X \) is defined as \( \ell g = [\ell_{1g}, \ldots, \ell_{Ng}] \). Then we can prove the following

**Proposition.** The irreducible invariant projector for the wreath product representation \( \tilde{P} \) is

\[
\tilde{B}_k = \sum_{\ell \in kG} B_{\ell_1} \otimes \cdots \otimes B_{\ell_N} ,
\]  

(2)
where \( kG \) denotes the \( G \)-orbit of the map \( k \) on the set \( L^X \).

The easily verifiable completeness condition \( \sum_{i=1}^{K} \tilde{B}_{k(i)} = I_{MN} \) holds. Here \( K \) is the number of irreducible components of the wreath product representation, \( k^{(i)} \) denotes some numbering of the orbits of \( G \) on \( L^X \), \( I_{MN} \) is the identity matrix in the Hilbert space \( \mathbb{H} \).

To compute the projectors \( \tilde{B} \), we wrote a program in C. The input data for the program are the generators of the spatial and local groups as well as the complete set of irreducible invariant projectors of the local group (obtained, e.g., by the program described in [6]).

### 3. Examples of calculations

We give here three examples of applying the program to wreath products constructed from the proper symmetry groups of the octahedron, icosahedron, and dodecahedron. The calculations were performed on a PC with 3.30GHz CPU and 16GB RAM. Irreducible representations are denoted below by their dimensions in bold, and permutation representations by dimensions in bold with underlining. The space \( X \) in all cases is the icosahedron, whereas the local basis \( V \) runs through all three Platonic solids. For the simplest example \((V = \text{octahedron})\), we give a more detailed (with inevitable gaps) output and only brief information in other cases.

#### 3.1. Octahedron

- For the vertex numbering as in Figure 1, the proper symmetry group \( G_O \cong S_4 \) of the octahedron can be generated, e.g., by the two permutations \((1, 3, 5), (2, 4, 6)\) and \((1, 2, 4, 5)\).
- The six-dimensional permutation representation of \( G_O, 6 \), has rank 3 with the following basis of the centralizer algebra

\[
A_1 = I_6, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} Y & Y \\ Y & Y \end{pmatrix}, \quad \text{where } Y = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{3}
\]

- The decomposition into irreducible components has the form \( 6 \cong 1 \oplus 2 \oplus 3 \).
- The complete set of irreducible orthogonal projectors expressed in basis \( \{3\} \) is

\[
B_1 = \frac{1}{6} (A_1 + A_2 + A_3), \quad B_2 = \frac{1}{3} \left( A_1 + A_2 - \frac{1}{2} A_3 \right), \quad B_3 = \frac{1}{2} (A_1 - A_2). \tag{4}
\]

![Figure 1. Octahedron.](image)
3.2. Icosahedron

- The permutations \((1, 7)(2, 8)(3, 12)(4, 11)(5, 10)(6, 9)\) and \((2, 3, 4, 5, 6)(8, 9, 10, 11, 12)\) generate the proper symmetry group \(G_I \cong A_5\) of the icosahedron.
- The four symmetric \(12 \times 12\) matrices \(I, A_2, A_3\) and \(A_4\) form the basis of the centralizer algebra of the permutation representation \(12\) of the group \(G_I\).
- The irreducible decomposition has the form \(12 \cong 1 \oplus 3 \oplus 3' \oplus 5\), where \(3\) and \(3'\) denote two nonequivalent three-dimensional irreducible representations of the group \(A_5\).
- The complete set of irreducible orthogonal projectors is

\[
B_1 = \frac{1}{12} (A_1 + A_2 + A_3 + A_4), \quad B_3 = \frac{1}{4} \left( A_1 - A_2 + \frac{\sqrt{5}}{5} A_3 - \frac{\sqrt{5}}{5} A_4 \right),
\]

\[
B_{3'} = \frac{1}{4} \left( A_1 - A_2 - \frac{\sqrt{5}}{5} A_3 + \frac{\sqrt{5}}{5} A_4 \right), \quad B_5 = \frac{5}{12} \left( A_1 + A_2 - \frac{1}{5} A_3 - \frac{1}{5} A_4 \right).
\]

3.3. Dodecahedron

- These matrices are too cumbersome to be given here, but can be easily computed with a simple procedure.
• The generators of the proper symmetry group $G_D \cong A_5$ of the dodecahedron are, e.g., $(2, 3, 4)(5, 7, 9)(6, 8, 10)(11, 13, 15)(12, 14, 16)(17, 18, 19)$ and $(1, 2, 5, 6, 3)(4, 10, 12, 13, 7)(8, 9, 11, 18, 14)(15, 16, 17, 20, 19)$.

• The rank of the permutation representation $20$ of the group $G_D$ is equal to 8. The centralizer algebra basis for the representation $20$ consists of eight $20 \times 20$ matrices:

\[
A_1 = \mathbb{1}_{20}, A_2, A_3, A_4, A_5, A_6, A_7, A_8 = A_I^T.
\]

• The irreducible decomposition of $20$, which has the form $20 \cong 1 \oplus 3 \oplus 3' \oplus 4 \oplus 4 \oplus 5$, contains two copies of the four-dimensional irreducible representation of the group $A_5$.

• The complete set of irreducible orthogonal projectors is

\[
B_1 = \frac{1}{20} \left( A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 \right),
\]

\[
B_3 = \frac{3}{20} \left( A_1 - A_2 + \frac{\sqrt{5}}{3} A_3 - \frac{\sqrt{5}}{3} A_4 - \frac{A_5}{3} + \frac{A_6}{3} + \frac{A_7}{3} + \frac{A_8}{3} \right),
\]

\[
B_3' = \frac{3}{20} \left( A_1 - A_2 - \frac{\sqrt{5}}{3} A_3 + \frac{\sqrt{5}}{3} A_4 - \frac{A_5}{3} - \frac{A_6}{3} + \frac{A_7}{3} + \frac{A_8}{3} \right),
\]

\[
B_4^{(1)} = \frac{1}{5} \left( A_1 - A_3 - \frac{A_4}{3} + \frac{2A_6}{3} - \frac{1}{6} - \frac{i\sqrt{11}}{6} A_7 - \frac{1 + i\sqrt{11}}{6} A_8 \right),
\]

\[
B_4^{(2)} = \frac{1}{5} \left( A_1 - A_3 - \frac{A_4}{3} + \frac{2A_6}{3} - \frac{1}{6} + \frac{i\sqrt{11}}{6} A_7 - \frac{1 - i\sqrt{11}}{6} A_8 \right),
\]

\[
B_5 = \frac{1}{4} \left( A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8 \right).
\]

### 3.4. Computer outputs

#### 3.4.1. Wreath product $S_4$\texttt{(octahedron)} $\wr A_5$\texttt{(icosahedron)}

| Representation dimension: 2176782336 | Rank: 122776 |
|--------------------------------------|-----------|
| Number of different suborbit lengths: 46 |
| Wreath suborbit lengths: \(1^{35}, 2^{249}, 3^{11}, 4^{258}, 5^{16}, 6^{203}, 8^{1442}, 16^{1480}, 24^{2418}, 32^{2043}, 48^{413}, 64^{5092}, 80^{211}, 96^{6482}, 128^{6629}, 256^{8858}, 384^{10735}, 512^{7237}, 768^{67}, 1024^{9901}, 1280^{538}, 1536^{12006}, 2048^{5611}, 4096^{8209}, 6144^{1603}, 8192^{3093}, 12288^{39}, 16384^{4795}, 20480^{16}, 24576^{5558}, 32768^{1225}, 65536^{2171}, 98304^{2389}, 131072^{310}, 196608^{15}, 262144^{674}, 327680^{14}, 393216^{743}, 524288^{77}, 1048576^{146}, 1572864^{177}, 2097152^{18}, 4194304^{16}, 5242880^{2}, 6291456^{24}, 16777216^{7}. |
| Checksum = 2176782336 | Maximum multiplicity = 12006 |
Wreath invariant basis forms:
\[ \tilde{A}_1 = A_1^{S_{12}} \]
\[ \tilde{A}_2 = A_2^{S_{5}} \otimes A_2 \otimes A_1^{S_{2}} \otimes A_2 \otimes A_1^{S_{3}} \]
\[ \tilde{A}_3 = A_1^{S_{4}} \otimes A_2^{S_{2}} \otimes A_1^{S_{2}} \otimes A_2^{S_{2}} \otimes A_1^{S_{2}} \]
\[ \vdots \]
\[ \tilde{A}_{61387} = A_2^{S_{3}} \otimes A_1 \otimes A_3 \otimes A_1^{S_{2}} \otimes A_3^{S_{2}} \otimes A_1^{S_{2}} \otimes A_3 \]
\[ \quad + A_2 \otimes A_1 \otimes A_2^{S_{2}} \otimes A_1 \otimes A_3 \otimes A_2 \otimes A_3^{S_{3}} \otimes A_2 \]
\[ \quad + A_1 \otimes A_2^{S_{2}} \otimes A_3^{S_{3}} \otimes A_2 \otimes A_1 \otimes A_3 \otimes A_2^{S_{2}} \otimes A_1 \]
\[ \quad + A_1 \otimes A_3^{S_{2}} \otimes A_2^{S_{2}} \otimes A_3 \otimes A_2^{S_{2}} \otimes A_1 \otimes A_3 \otimes A_1 \otimes A_2 \]
\[ \tilde{A}_{61388} = A_2 \otimes A_1 \otimes A_2^{S_{2}} \otimes A_1 \otimes A_3 \otimes A_1^{S_{2}} \otimes A_3 \]
\[ \quad + A_2^{S_{2}} \otimes A_3 \otimes A_2^{S_{2}} \otimes A_1^{S_{2}} \otimes A_3^{S_{2}} \otimes A_2 \otimes A_3 \]
\[ \quad + A_1 \otimes A_3^{S_{2}} \otimes A_2^{S_{2}} \otimes A_3 \otimes A_2^{S_{2}} \otimes A_1 \otimes A_2 \otimes A_3 \]
\[ \quad + A_1 \otimes A_2 \otimes A_3^{S_{3}} \otimes A_2^{S_{2}} \otimes A_1 \otimes A_3 \otimes A_1 \otimes A_2^{S_{2}} \]
\[ \vdots \]
\[ \tilde{A}_{122774} = A_3^{S_{2}} \otimes A_2 \otimes A_3^{S_{9}} + A_3^{S_{3}} \otimes A_2 \otimes A_3^{S_{8}} + A_3^{S_{10}} \otimes A_2 \otimes A_3 + A_3^{S_{11}} \otimes A_2 \]
\[ \tilde{A}_{122775} = A_3^{S_{4}} \otimes A_2 \otimes A_3^{S_{7}} + A_3^{S_{5}} \otimes A_2 \otimes A_3^{S_{6}} + A_3^{S_{8}} \otimes A_2 \otimes A_3^{S_{3}} \]
\[ \quad + A_3^{S_{9}} \otimes A_2 \otimes A_3^{S_{2}} \]
\[ \tilde{A}_{122776} = A_2 \otimes A_3^{S_{11}} + A_3 \otimes A_2 \otimes A_3^{S_{10}} + A_3^{S_{6}} \otimes A_2 \otimes A_3^{S_{5}} + A_3^{S_{7}} \otimes A_2 \otimes A_3^{S_{4}} \]

Wreath product decomposition is multiplicity free

Number of irreducible components: 122776
Number of different dimensions: 134

Irreducible dimensions:
1, 46, 63, 86, 93, 1215, 1632, 187, 20, 2470, 3241, 3686, 45, 48191, 54, 26, 6484, 72208, 804, 817, 96412, 108223,
128114, 144413, 16234, 1808, 192204, 216226, 2434, 256104, 2881804, 3207, 324304, 384772, 4054, 432517,
48699, 512756, 5762508, 6481909, 72017, 7299, 768705, 8644303, 972818, 102451, 11522562, 12803, 12964455,
1458141, 1536479, 162016, 17285322, 19442712, 204820, 21874, 230413935, 25926708, 288014, 2916961, 3072223,
34564623, 36457, 38884949, 40964, 4374136, 4608004, 5120, 51846924, 58322745, 641459, 648018, 65619,
69122719, 77766966, 81923, 8748522, 9216329, 103684760, 1152010, 116644696, 1228819, 1312298, 138241011,
1458015, 155525781, 174961999, 1843243, 196835, 20736205, 233284826, 2592016, 26244511, 27648260,
31104264, 328053, 349922775, 368643, 3936655, 41472534, 46080, 466563012, 524881023, 5529615, 5832019,
590495, 6220857, 699847431, 78732242, 8294448, 933124038, 103680, 1049761079, 11809827, 124416102,
1312208, 139968051, 157464350, 168624300, 209952568, 2332807, 23619684, 279936148, 295245, 314928254,
3542946, 419904116, 4723970, 5248803, 531441, 62985662, 70858815, 94478426, 1180980, 14171709.
Checksum = 2176782336 Maximum number of equal dimensions = 6966
Wreath irreducible projectors:

\[
\tilde{B}_1 = B_1^{\otimes 12} \\
\tilde{B}_2 = B_1^{\otimes 3} \otimes B_2 \otimes B_1^{\otimes 6} \otimes B_2 \otimes B_1 \\
\tilde{B}_3 = B_1^{\otimes 9} \otimes B_2 \otimes B_1^{\otimes 2} + B_1^{\otimes 4} \otimes B_2 \otimes B_1^{\otimes 7} \\
\vdots \\
\tilde{B}_{61387} = B_2 \otimes B_3 \otimes B_1 \otimes B_2^{\otimes 2} \otimes B_1^{\otimes 3} \otimes B_3^{\otimes 2} \otimes B_2 \otimes B_3 \\
+ B_3 \otimes B_2 \otimes B_1 \otimes B_2 \otimes B_1 \otimes B_3 \otimes B_1 \otimes B_2 \otimes B_3 \otimes B_3 \otimes B_2 \\
+ B_1^{\otimes 2} \otimes B_3 \otimes B_2 \otimes B_3^{\otimes 2} \otimes B_2 \otimes B_3 \otimes B_1 \otimes B_2 \otimes B_1 \\
+ B_1 \otimes B_2^{\otimes 2} \otimes B_3 \otimes B_1 \otimes B_3^{\otimes 2} \otimes B_2 \otimes B_3 \otimes B_1 \otimes B_2 \otimes B_1 \\
\tilde{B}_{61388} = B_2^{\otimes 2} \otimes B_3 \otimes B_1^{\otimes 2} \otimes B_3 \otimes B_2 \otimes B_3 \otimes B_3 \otimes B_2 \otimes B_3 \\
+ B_1 \otimes B_2 \otimes B_3 \otimes B_2^{\otimes 2} \otimes B_3 \otimes B_2^{\otimes 2} \otimes B_3 \otimes B_3 \otimes B_3 \otimes B_3 \\
\tilde{B}_{61389} = B_2^{\otimes 2} \otimes B_3^{\otimes 3} \otimes B_1 \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_1 \otimes B_2^{\otimes 2} \\
+ B_2 \otimes B_3 \otimes B_2 \otimes B_3 \otimes B_1 \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_3 \otimes B_2 \\
+ B_1 \otimes B_2 \otimes B_3 \otimes B_1 \otimes B_3 \otimes B_2 \otimes B_1^{\otimes 3} \otimes B_2 \otimes B_3 \\
+ B_3 \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_1^{\otimes 2} \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_3 \\
+ B_2 \otimes B_3^{\otimes 3} \otimes B_1 \otimes B_2 \otimes B_1^{\otimes 3} \otimes B_2 \\
+ B_3 \otimes B_3^{\otimes 2} \otimes B_3 \otimes B_3 \otimes B_3 \otimes B_1 \otimes B_3 \otimes B_3 \otimes B_2 \\
\vdots \\
\tilde{B}_{122774} = B_3^{\otimes 2} \otimes B_2 \otimes B_3^{\otimes 9} + B_3^{\otimes 3} \otimes B_2 \otimes B_3^{\otimes 8} + B_3^{\otimes 10} \otimes B_2 \otimes B_3 + B_3^{\otimes 11} \otimes B_2 \\
\tilde{B}_{122775} = B_3^{\otimes 2} \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_3^{\otimes 7} + B_3^{\otimes 2} \otimes B_2 \otimes B_3^{\otimes 2} \otimes B_2 \otimes B_3^{\otimes 6} \\
+ B_3^{\otimes 3} \otimes B_2 \otimes B_3^{\otimes 4} \otimes B_2 \otimes B_3^{\otimes 3} + B_3^{\otimes 5} \otimes B_2 \otimes B_3^{\otimes 4} \otimes B_2 \otimes B_3 \\
+ B_3^{\otimes 8} \otimes B_2 \otimes B_3^{\otimes 2} \otimes B_2 + B_3^{\otimes 9} \otimes B_2 \otimes B_3 \otimes B_2 \\
\tilde{B}_{122776} = B_3^{\otimes 3} \otimes B_2^{\otimes 2} \otimes B_3^{\otimes 7} + B_2 \otimes B_3^{\otimes 4} \otimes B_2 \otimes B_3^{\otimes 6} \\
+ B_3 \otimes B_2 \otimes B_3^{\otimes 3} \otimes B_2 \otimes B_3^{\otimes 6} + B_3^{\otimes 6} \otimes B_2 \otimes B_3 \otimes B_2 \otimes B_3^{\otimes 3} \\
+ B_3^{\otimes 7} \otimes B_2^{\otimes 2} \otimes B_3^{\otimes 3} + B_3^{\otimes 9} \otimes B_2^{\otimes 2} \otimes B_3
\]

Time: 0.58 sec  
Maximum number of tensor monomials: 531441

3.4.2. Wreath product \(A_5(\text{icosahedron}) \triangleleft A_5(\text{icosahedron})\)

Representation dimension: 8916100448256  
Rank: 3875157  
Wreath product decomposition is multiplicity free  
Number of irreducible components: 3875157  
Number of different dimensions: 261  
Time: 7.35 sec  
Maximum number of tensor monomials: 16777216

3.4.3. Wreath product \(A_5(\text{dodecahedron}) \triangleleft A_5(\text{icosahedron})\)

Representation dimension: 409600000000000  
Rank: > 502985717
Wreath product decomposition has non-trivial multiplicities
Number of irreducible components: 502 985 717
Number of different dimensions: 1065
Time: 26 min 46.16 sec
Maximum number of tensor monomials: 2176782336

4. Remark on simulation of multiparticle quantum systems
Projection operators (2) obtained by our program are intended for calculations in models of multipartite quantum systems. These operators are matrices of huge dimensions, for example, about four quadrillions for the representation in section 3.4.3. Obviously, explicit computation of such matrices is impossible and actually not necessary. The presentation of projectors for wreath products in the form of tensor polynomials allows us to reduce the calculation of quantum correlations to a sequence of manipulations with small matrices of local projectors.

Consider, for example, the computation of a scalar product in an invariant subspace. Let $|m_1\rangle \otimes \cdots \otimes |m_N\rangle$ be an orthonormal basis in the local Hilbert space $H$. Then the orthonormal basis in the Hilbert space $\tilde{H} = H^\otimes N$ of the wreath product is formed by elements of the form $|m_1\rangle \otimes \cdots \otimes |m_N\rangle$, where $m = [m_1, \ldots, m_N] \in M^N$, $M = |V|$, $N = |X|$, $M = \{1, \ldots, M\}$ and $N = \{1, \ldots, N\}$. General vectors in the wreath product Hilbert space can be written as follows:

$$\tilde{\Phi} = \sum_{m \in V \times N} \varphi_m |m_1\rangle \otimes \cdots \otimes |m_N\rangle \text{ and } \tilde{\Psi} = \sum_{n \in V \times N} \psi_n |n_1\rangle \otimes \cdots \otimes |n_N\rangle,$$

where $\varphi_m$ and $\psi_n$ are arbitrary scalars from the base field $F$.

The scalar product of these vectors in the invariant subspace that is defined by projector (2) has the form

$$\langle \tilde{\Phi} | \tilde{B}_k | \tilde{\Psi} \rangle = \sum_{m \in V \times N} \sum_{n \in V \times N} \sum_{\ell \in kG} \varphi_m \psi_n A_{m,n,\ell},$$

where

$$A_{m,n,\ell} = \langle m_1 | \otimes \cdots \otimes \langle m_N | B_{\ell_1} \otimes \cdots \otimes B_{\ell_N} |n_1\rangle \otimes \cdots \otimes |n_N\rangle,$$

(5)

$$= \langle m_1 | B_{\ell_1} | n_1\rangle \cdots \langle m_N | B_{\ell_N} | n_N\rangle \equiv (B_{\ell_1})_{m_1n_1} \cdots (B_{\ell_N})_{m_Nn_N}.$$

(6)

The last expression (6), which is simply the product of $N$ scalars, is obtained by applying to (5) a general property of the tensor product — the identity $(A \otimes B) (C \otimes D) = (AC) \otimes (BD)$.

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References
[1] Van Raamsdonk M 2010 Building up spacetime from quantum entanglement Gen. Relativ. Grav. 42, 2323–9
[2] Cao C, Carroll S M and Michalakis S 2017 Space from Hilbert space: Recovering geometry from bulk entanglement Phys. Rev. D 95, 024031
[3] Kornyak V V 2018 Modeling Quantum Behavior in the Framework of Permutation Groups [EPJ Web of Conferences 173 01007]
[4] Kornyak V V 2018 Quantum models based on finite groups [J. Phys.: Conf. Series 965 012023]
[5] Meldrum J D P 1995 Wreath Products of Groups and Semigroups (Longman/Wiley)
[6] Kornyak V V 2019 A new algorithm for irreducible decomposition of representations of finite groups [J. Phys.: Conf. Ser. 1194 012060]
[7] Wilson R et al. Atlas of finite group representations

2 We can restrict ourselves to natural $\varphi_m$ and $\psi_n$, if we adhere to the ideology of the approach [3, 4].