\textbf{L}-invariant for Siegel-Hilbert forms

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We prove in some cases a formula for the Greenberg-Benois \textbf{L}-invariant of the spin, standard and adjoint Galois representations associated with Siegel-Hilbert modular forms. In order to simplify the calculation, we give a new definition of the \textbf{L}-invariant for a Galois representation $V$ of a number field $F \neq \mathbb{Q}$; we also check that it is compatible with Benois’ definition for $\text{Ind}_{\mathbb{Q}}^{F}(V)$.

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1 Introduction

Since the historical results of Kummer and Kubota-Leopold on congruences for Bernoulli numbers, people have been interested in studying the $p$-adic variation of special values of $L$-functions. More precisely, fix a motive $M$ over $\mathbb{Q}$. We suppose that $M$ is Deligne critical at $s = 0$ and that there exists a Deligne’s period $\Omega(M)$ such that $\frac{L(M,0)}{\Omega(M)}$ is algebraic. Fix a prime $p$ and two embeddings

$$\mathbb{C}_p \leftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}.$$ 

Let $V$ be the $p$-adic realization of $M$ and suppose that $V$ is semistable (à la Fontaine). Thanks to work of Coates and Perrin-Riou, we have now days precise conjectures on how the special values should behave $p$-adically; we fix a regular sub-module of $V$. This corresponds to the choice of a sub-$\langle \varphi, N \rangle$-module of $\mathcal{D}_{\text{st}}(V)$ which is a section of the exponential map

$$\mathcal{D}_{\text{st}}(V) \rightarrow \ell(V) \cong \frac{\mathcal{D}_{\text{st}}(V)}{\text{Fil}^0 \mathcal{D}_{\text{st}}(V)}.$$
Let $h$ be the valuation of the determinant of $\varphi$ on $D$. We can state the following conjecture

**Conjecture 1.1.** There exists a formal series $L_p^D(V, T) \in \mathbb{C}_p[[T]]$ who grows as $\log^h$ such that for all non-trivial, finite-order characters $\varepsilon : 1 + p\mathbb{Z}_p \to \mu_p$ we have

$$L_p^D(V, \varepsilon(1 + p) - 1) = C_\varepsilon(D) \frac{L(M, 0)}{\Omega(M)}.$$ 

Moreover, for $\varepsilon = 1$ we have

$$L_p(V, 0) = E(D) \frac{L(M, 0)}{\Omega(M)},$$

where $E(D)$ is an explicit product of Euler-type factors depending on $D$ and $(D_{\text{st}}(V)/D)^{N=0}$.

It may happen that one of the factor of $E(D)$ vanishes and then we say that trivial zeros appear. Since the seminal work of [MTTS86], people have been interested in describing the square of a (Hilbert) modular form by Hida, Mok and Benois and for symmetric power of Hilbert modular forms.

**Conjecture 1.2.** Let $t$ the number of vanishing factors of $E(D)$. Then

- $\text{ord}_{\varepsilon=0} L_p^D(V, (1 + p)^s - 1) = t$,

- $L_p(V, 0)^* = \mathcal{L}(V, D) E^*(D) \frac{L(M, 0)}{\Omega(M)}$.

Here $E^*(D)$ is the product of non-vanishing factors of $E(D)$ and $\mathcal{L}(V, D)$ is a number defined in purely Galois theoretical terms (see Section 3.1).

The error factor $\mathcal{L}(V, D)$ is quite mysterious. It has been calculated in only few cases for the symmetric square of a (Hilbert) modular form by Hida, Mok and Benois and for symmetric power of Hilbert modular forms by Hida and Jorza-Harron. Unless $V$ is an elliptic curve over $\mathbb{Q}$ with multiplicative reduction at $p$ we can not prove the non-vanishing of $\mathcal{L}(V, D)$.

The aim of this paper is to calculate it in some new cases; let $F$ be a totally real field where $p$ is unramified and $\pi$ be an automorphic representation of $\text{GSp}_{2g}/F$. We suppose that it has Iwahoric level at all $p | p$. We suppose moreover that $\pi_p$ is either Steinberg (see Definition 4.9) or spherical. We partition consequently the prime ideals of $F$ above $p$ in $S_{\text{Stb}} \cup S_{\text{Sph}}$.

We have conjecturally two Galois representations associated to $\pi$, namely the spinorial one $V_{\text{spin}}$ and the standard one $V_{\text{sta}}$. Let $V$ be one of these two representations. We choose for each prime $p$ of $F$ dividing $p$ a regular sub module $D_p$ of $D_{\text{st}}(V_{|G_{F_p}})$.

Consider a family of Siegel-Hilbert modular forms as in [Urbi11] passing through $\pi$. Let us denote by $\beta_p(\kappa)$ the eigenvalue of the normalized Hecke operators $U_{1, p}$ (see Definition 4.10) on this family. Let $S_{\text{Sph}, 1} = S_{\text{Sph}, 1}(V, D)$ be the subset of $S_{\text{Sph}}$ for which $(D_{\text{st}}(V_p)/D_p)^{N=0}$ does not contain the eigenvalue $1$. Conjecturally, it is empty for the spin representation. The eigenvalues $1$ always appears in $D_{\text{st}}(V_p)$ for $V$ the standard representation but it may appear in $D_p$ (this is already the case for the symmetric square of a modular form).

Let $t_{\text{Stb}}$ be the cardinality of $S_{\text{Stb}}$ and $t_{\text{Sph}}$ be the cardinality of $S_{\text{Sph}, 1}$. We define $f_p = [\mathbb{F}_p^{ur} : \mathbb{F}_p]$.

**Theorem 1.3.** Let $\pi$ be as above, of parallel weight $k$. Let $V = V_{\text{spin}}$ and suppose hypothesis $\text{LGp}$ of Section 4.2 then the expected number of trivial zero for $L_p^D(V(k - 1), T)$ is $t_{\text{Stb}}$ and

$$\mathcal{L}(V(k - 1), D) = \prod_{p \in S_{\text{Stb}}} \frac{1}{f_p} \frac{d \log \beta_p(k)}{dk} \bigg|_{k=1}.$$
Let $V = V_{\text{std}}$, then the conjectural number of trivial zero for $L_p^D(V, T)$ is $t_{S_{\text{th}}} + t_{S_{\text{ph}}}$ and

$$L(V, D) = \mathcal{L}(V, D)^{S_{\text{ph}}} \prod_{p \in S_{\text{th}}} \frac{1}{\mathcal{L}(V, D)^{S_{\text{th}}}} \frac{d \log \beta_p(k)}{d k} \bigg|_{k=\frac{1}{2}},$$

where $\mathcal{L}(V, D)^{S_{\text{ph}}}$ is a priori global factor. It is 1 if $t_{S_{\text{ph}}} = 0$.

In Section 4.2 we shall provide also a formula for the $\mathcal{L}$-invariant of $V_{\text{std}}(s)$ (min$(k - g - 1, g - 1) \geq s \geq 1$).

The proof of the theorem is not different from the one of [Ben10, Theorem 2] which in turn is similar to the original one of [GS93].

Let now $g = 2$. Let $t$ be the number of primes above $p$ in $F$. We consider the 2t-dimensional eigenvariety for $GSp_4/F$ with variables $k \in \{k_{p,1}, k_{p,2}\}_p$ (see Section 5) and let us denote by $F_{p,i}(k)$ ($i = 1, 2$) the first two graded pieces of $D_{\text{rig}}^1(V_{\text{spin}})$. The 10-dimensional Galois representation $\text{Ad}(V_{\text{spin}})$ has a natural regular sub-$(\phi, N)$-module induced by the $p$-refinement of $D_{\text{rig}}^1(V_{\text{spin}})$ and which we shall denote by $D_{\text{Ad}}$. With this choice of regular sub module, $\text{Ad}(V_{\text{spin}})$ presents 2t trivial zeros. In Section 5 we prove the following theorem:

**Theorem 1.4.** Let $\pi$ be an automorphic form of weight $k$ and suppose hypothesis $LGp$ of Section 4.2 is verified for $V_{\text{spin}}$, we have then

$$\mathcal{L}(\text{Ad}(V_{\text{spin}}(\pi)), D_{\text{Ad}}) = \prod_p \frac{2^{t_p}}{f_p^2} \det \begin{pmatrix} \frac{\partial \log \beta_p(k)}{\partial k_{p,j,1}} & \frac{\partial \log \beta_p(k)}{\partial k_{p,j,2}} \\ \frac{\partial \log \beta_p(k)}{\partial k_{p,j,1}} & \frac{\partial \log \beta_p(k)}{\partial k_{p,j,2}} \end{pmatrix}_{1 \leq i,j \leq t_{|k=\frac{1}{2}}}. $$

We remark that this theorem is the first to really go beyond $GL_2$ and its representations $\text{Sym}^n$.

The motivation for Theorem 1.3 lies in a generalization of [Ros13b] to Siegel forms. In loc. cit. we use Greenberg-Stevens method to prove a formula for the derivative of the symmetric square $L$-function and calculate the analytic $\mathcal{L}$-invariant and the same method of proof can be generalized to finite slope Siegel forms thanks to the overconvergent Maass-Shimura operators and overconvergent projectors of Z. Liu’s thesis. With some work, it could also be generalized to totally real field where $p$ where is inert, as already done for the symmetric square [Ros13a].

We hope to calculate the $\mathcal{L}$-invariant for $V_{\text{std}}$ and $\text{Ad}(V_{\text{spin}})$ for more general forms in a future work.

In Section 2 we recall the theory of $(\phi, \Gamma)$-module over a finite extension of $\mathbb{Q}_p$. It will be used in Section 3 to generalize the definition of the $\mathcal{L}$-invariant à la Greenberg-Benois to Galois representations $V$ over general number field $F$ (note that we do not suppose $p$ split or unramified). This definition does not require one to pass through $\text{Ind}_F^G(V)$ to calculate the $\mathcal{L}$-invariant which in turn simplifies explicit calculation. We shall check that this definition coincides with Benois’ definition for $\text{Ind}_F^G(V)$. We prove the above-mentioned theorems in Section 4 and inspired mainly by the methods of [Hid07].

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2 Some results on rank one \((\varphi, \Gamma)\)-module

Let \(L\) be a finite extension of \(\mathbb{Q}_p\). The aim of this section is to recall certain results concerning \((\varphi, \Gamma)\)-modules over the Robba ring \(\mathcal{R}_L\). Let \(L_0\) be the maximal unramified extension contained in \(L\). Let \(L'_0\) be the maximal unramified extension contained in \(L_{\infty} := L(\mu_{p,\infty})\) and \(L' = L \cdot L'_0\). Let \(e_L := [L(\mu_{p,\infty}) : L_0(\mu_{p,\infty})] = [\Gamma_{L_0} : \Gamma_L]\), where \(\Gamma_L := \text{Gal}(L_{\infty}/L)\). We define

\[
\mathcal{B}^{\delta}_{L, \text{rig}} = \left\{ f = \sum_{n \in \mathbb{Z}} a_n \pi_L^n | a_n \in L'_0, \text{ such that } f(X) = \sum_{n \in \mathbb{Z}} a_n X^n \text{ is holomorphic on } p^{-1/2} \leq |X|_p < 1 \right\},
\]

\[\mathcal{R}_L = \bigcup_t \mathcal{B}_{\text{K}_L, \text{rig}}^{\delta},\]

where \(\pi_L\) is a certain uniformizer coming from the theory of field of norm. We have an action of \(\varphi\) on \(\mathcal{R}_L\).

If \(L = L_0\), there is no ambiguity and we have:

\[\varphi(\pi_L) = (1 + \pi_L)^p - 1, \quad \varphi(a_n) = \varphi_{L'_0}(a_n).\]

Otherwise the action on \(\pi_L\) is more complicated. Similarly, we have a \(\Gamma_L\)-action. If \(L = L_0\) we have

\[\gamma(\pi_L) = (1 + \pi_L)^{\chi_{\text{cycl}}(\gamma)} - 1,\]

where \(\chi_{\text{cycl}}\) is the cyclotomic character. If \(L\) is ramified we also have an action of \(\Gamma_L\) on the coefficients given by

\[\gamma(a_n) = \sigma_{\gamma}(a_n),\]

where \(\sigma_{\gamma}\) is the image of \(\gamma\) via

\[\Gamma_L \to \Gamma_L / \Gamma_{L'} \xto{\cong} \text{Gal}(L'_0/L_0).\]

If \(a_n\) is fixed by \(\varphi\) and \(\Gamma_L\), then it is in \(\mathbb{Q}_p\). We have \(\text{rk}_{\mathcal{R}_L} \mathcal{R}_L = [L_{\infty} : \mathbb{Q}_{p,\infty}]\).

Let \(\delta : L^\times \to E^\times\) be a continuous character. We define \(\mathcal{R}_L(\delta)\) to be the rank one \((\varphi, \Gamma_L)\)-module with basis \(e_{\delta}\) for which \(\varphi(e_{\delta}) = \delta(\pi_L)e_{\delta}\) and \(\gamma(e_{\delta}) = \delta(\chi_{\text{cycl}}(\gamma)) e_{\delta}\).

We classify now the cohomology of such a \((\varphi, \Gamma_L)\)-modules. It will be useful to calculate it explicitly in terms of \(C_{\varphi, \gamma}\)-complexes [Ben11 §1.1.5]. We fix then a generator \(\gamma_L\) of \(\Gamma_L\); if clear from the context, we shall drop the subscript \(L\) and write simply \(\gamma\).

**Proposition 2.1.** We have \(H^0(\mathcal{R}_L(\delta)) = 0\) unless \(\delta(z) = \prod_{\tau} \tau(z)^{m_{\tau}}\) with \(m_{\tau} \leq 0\) for all \(\tau\); in this case we have \(H^0(\mathcal{R}_L(\delta)) \cong E\). We shall denote its basis by \(t^m \otimes \epsilon_{\delta}\), where

\[t^m = \oplus t^{m_{\tau}} \in \oplus_{\tau} B_{\text{dR}}^{t} \otimes_{L, \sigma} E.\]

If \(\delta(z) = \prod_{\tau} \tau(z)^{m_{\tau}}\) with \(m_{\tau} \leq 0\), then

\[\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p] + 1.\]

If \(\delta(z) = |N_{L/ \mathbb{Q}_p}(z)|_p \prod_{\tau} \tau(z)^{k_{\tau}}\) with \(k_{\tau} \geq 1\), then

\[\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p] + 1.\]
We introduce another extension. We define $y$.

The last fact is then a direct consequence.\[\text{Lemma 2.4.}\]

To recall the following lemma \[\text{Ben11 Lemma 1.4.3}\]

Recall that have a canonical duality \[\text{Liu08}\] given by cup product

We define $\Delta$.

Hence \[\text{Lemma 2.3.}\]

This allows us to define a canonical basis of $H^2(\mathcal{R}_L([N_L/Q_p(z)]_p \prod \tau(z)^{k\gamma})$.

We define $H^1_1(D)$ has the $H^1$ of the complex

and we have immediately \[\text{Nak09 Proposition 2.7}\]

Hence

\textbf{Lemma 2.3.} If $\delta(z) = \prod \tau(z)^{m\gamma}$ with $m\gamma \leq 0$, then

\[
\dim_E H^1_1(\mathcal{R}_L(\delta)) = 1.
\]

If $\delta(z) = [N_L/\mathbb{Q}_p(z)]_p \prod \tau(z)^{k\gamma}$ with $k\gamma \geq 1$, then

\[
\dim_E H^1_1(\mathcal{R}_L(\delta)) = d.
\]

We now want to calculate $H^1_1(\mathcal{R}_L(\delta))$ for $\delta(z) = \prod \tau(z)^{m\gamma}$ with $m\gamma \leq 0$. We recall the following lemma \[\text{Ben11 Lemma 1.4.3}\]

\textbf{Lemma 2.4.} The extension in $H^1(\mathcal{R}_L(\delta))$ corresponding to the couple $(a, b)$ is crystalline if and only if the equation $(1 - \gamma)x = b$ has a solution in $D \left[\frac{1}{\gamma}\right]$.

The following proposition in an immediate consequence of the above lemma \[\text{Ben11 Theorem 1.5.7 (i)}\]

\textbf{Proposition 2.5.} Let $e$ be a basis for $\mathcal{R}_L(\delta)$. Then $x_m = \text{cl}(t^m, 0)e$ is a basis of $H^1_1(\mathcal{R}_L(\delta))$.

\textbf{Remark 2.6.} If $\delta$ is the trivial character then $x_0$ corresponds (via class field theory) to the unramified $\mathbb{Z}_p$-extension of $\text{Hom}(G_L, E^\times) \cong H^1(G_L, E)$.

We have now to cut out a "canonical" one-dimensional subspace in $H^1(\mathcal{R}_L(\delta))$ which trivially intersects $H^1_1(\mathcal{R}_L(\delta))$ (and reduces to the cyclotomic $\mathbb{Z}_p$-extension in the sense of the previous remark).

We introduce another extension. We define $y_k = \frac{1}{2\pi i} \log(\chi_{cycl}(\gamma_L)) \text{cl}(0, t^m)e_\kappa$.

We explain why this cocycle is of interest for us. We can calculate cohomology of induced $(\varphi, \Gamma_{Q_p})$-module. Indeed, we consider now two $p$-adic fields $K$ and $L$, $L$ a finite extension of $K$. The main reference for this part is \[\text{Liu08}\]. Let $D$ be a $(\varphi, \Gamma_L)$-module, we define

\textbf{Proposition 2.5.} Let $e$ be a basis for $\mathcal{R}_L(\delta)$. Then $x_m = \text{cl}(t^m, 0)e$ is a basis of $H^1_1(\mathcal{R}_L(\delta))$.\]
\[ \text{Ind}^{G_L \subset K}(D) = \{ f : \Gamma_K \to D | f(hg) = hf(g) \forall h \in \Gamma_L \}. \]

It has rank \([L : K]\) [rk][R L](D) over \(R_K\); indeed \(R_L\) is a \(R_K\)-module of rank \([L : K]/[\Gamma_K / \Gamma_L]\). (The unramified part of \(L/K\) plus the ramified part which is disjoint by \(K_\infty\)). See after [Liu08 Theorem 2.2]. If \(D\) comes from a \(G_L\)-representation \(V\) we have

\[ \text{D}_{\text{rig}}^+(\text{Ind}^{G_L \subset K}_L(V)) = \text{Ind}^{G_L \subset K}_L(\text{D}^+_\text{rig}(V)). \]

We have then the equivalent of Shapiro’s lemma

\[ H^i(D) \cong H^i(\text{Ind}^{G_L \subset K}_L(D)). \]

Moreover, the aforementioned duality for \((\varphi, \Gamma)\)-modules is compatible with induction [Liu08 Theorem 2.2]. If \(D \cong \mathcal{R}_L(\delta)\) is free of rank one, then we have an explicit description of \(\text{Ind}^{G_L \subset K}_L(D)\). Let \(e_\infty = |\Gamma_K / \Gamma_L|\), we write \(\{\omega_i\}_{i=0}^{e_\infty - 1}\) for \((\Gamma_K / \Gamma_L)^\wedge\). The \(\text{Ind}^{G_L \subset K}_L(D)\) is the \(\mathcal{R}_L\)-span of \(f_i\), where \(f_i(g) = \omega_i(g) \delta(\chi_{Cycl}(g)) e_\delta\).

We go back to the previous setting, where \(K = \mathbb{Q}_p\) (hence \(e_\infty = e_L\)). We have the following exact sequence

\[ 0 \to \mathcal{R}_{\mathbb{Q}_p}(z^{\Sigma m_r}) \to \text{Ind}^{G_L \subset K}_{L^*}(\mathcal{R}_L(\delta)) \to D' \to 0, \quad (2.7) \]

where we send \(e_{\Sigma m_r} \to e_\delta\) (the image is \(\mathcal{R}_{\mathbb{Q}_p}e_\delta\)). Note that \(z^{\Sigma m_r}\) is the restriction of \(\delta\) to \(\mathbb{Q}_p^*\). Note that \(H^0(D') = 0\) by calculation similar to [Col08 Proposition 2.1] or [Nak09 Proposition 2.14]. The long exact sequence in cohomology induces

\[ H^0(\mathcal{R}_{\mathbb{Q}_p}(z^{\Sigma m_r})) \cong H^0(\text{Ind}^{G_L \subset K}_{L^*}(\mathcal{R}_L(\delta))), \]

\[ H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{\Sigma m_r})) \hookrightarrow H^1(\text{Ind}^{G_L \subset K}_{L^*}(\mathcal{R}_L(\delta))). \]

The first morphism is the identity: the fixed basis \(t^{\Sigma m_r} e_{\Sigma m_r}\) is sent to the basis \(t^k e_\delta\) (because \(E \otimes_{\mathbb{Q}_p} L \cong \oplus_{\tau} E\).

The cocycle \(x_m\) is the image of \(x_{\Sigma m_r}\). The image of \(y_{\Sigma m_r} \in H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{\Sigma k_r}))\) is instead \(y_m\) (if \(\gamma_L = \gamma_{\mathbb{Q}_p}\)).

**Proposition 2.8.** Let \(D\) a \((\varphi, \Gamma)\)-module over \(\mathcal{R}_L\) with non-negative Hodge-Tate weight. Suppose that \(\mathcal{D}_\mathfrak{st}(D) = \mathcal{D}_\mathfrak{st}(D)^{\varphi = 1}\). Then \(D\) is crystalline and

\[ D \cong \oplus \mathcal{R}_L(\delta_i) \]

with \(\delta_i(z) = \prod_\tau \tau(z)^{k_i \cdot \tau}\).

**Proof.** We follow closely the proof [Ben11 Proposition 1.5.8]. As \(N\varphi = p\varphi N\) we obtain immediately that \(N = 0\), hence \(D\) is crystalline.

Let \(r\) be the rank of \(D\) over \(\mathcal{R}_L\). We write the Hodge-Tate weight as \((k_i)_{i=1}^r\) where \(k_i = (k_{i, r})_r\) and \(k_i \leq k_{i+1}\).

We prove it by induction; the case \(r = 1\) is clear.

For \(r = 2\) we can suppose \(k_1 = 0\) by twisting. Let \(\delta\) be defined by \(\prod_\tau \tau(z)^{k_\tau}\). So we have an extension of \(\mathcal{R}_L(\delta)\) by \(\mathcal{R}_L\). Let \(m_2\) be a lift to \(D\) of a basis of \(\mathcal{R}_1\). As \(\varphi = 1\) we have \(\varphi m_2 = m_2\). As the extension is crystalline we know that \(\gamma\) acts trivially too, hence the extension splits.

Suppose now \(r > 2\). Take \(v\) in \(\text{Fil}^{k_2} \mathcal{D}_\mathfrak{st}\). Define \(\delta_d\) by \(\delta_d(z) = \prod_\tau \tau(z)^{k_d \cdot \tau}\), we have

\[ 0 \to \mathcal{R}_L(\delta_d) \to D \to D' \to 0. \]

By inductive hypothesis \(D' \cong \oplus_{i=1}^{d-1} \mathcal{R}_L(\delta_i)\). We can write

\[ \text{Ext}(D', \mathcal{R}_L(\delta_d)) = \oplus_{i=1}^{d-1} \text{Ext}(\mathcal{R}_L(\delta_i), \mathcal{R}_L(\delta_d)) \]

and we are reduced to the case \(r = 2\) which has already been dealt. \(\square\)
In this section we generalize Greenberg-Benois definition of the $L$-dimensional and it is not immediate to find a suitable subspace. Inspired by Hida’s work for symmetric

Proposition 2.10. Suppose that $M$ is not of type $U_{m,k}$. Then we have $\dim_E(H^1(M)) = 2[L : \mathbb{Q}_p]r$ and $H^2(M) = H^0(M) = 0$. Moreover, if we write

$$0 \to H^0(M_1) \xrightarrow{\Delta_{1}} H^1(M_0) \xrightarrow{f_1} H^1(M) \xrightarrow{g_1} H^1(M_1) \xrightarrow{\Delta_{1}} H^2(M_0) \to 0$$

we have $H^1(M_0) = \text{Im}(\Delta_1) \oplus H^1_1(M_0)$, $\text{Im}(f_1) = H^1_1(M)$ and $H^1(M_1) = \text{Im}(g_1) \oplus H^1_1(M_1)$.

Proof. We have $H^0(M) = 0$. Note that $M^*(\chi_{\text{cycl}})$ is a module of the same type, hence $H^2(M) = H^0(M^*(\chi_{\text{cycl}})) = 0$. We can write

$$0 \to H^0(M_1) \to H^1(M_0) \xrightarrow{f_1} H^1(M) \xrightarrow{g_1} H^1(M_1) \to H^2(M_0) \to 0$$

and conclude by Proposition 2.1.

Note that $\dim_E H^1_1(M) = rd$ by (2.2). By hypothesis, we have that $\text{Im}(\Delta_1) \cap H^1_1(M_0) = 0$ and the first statement follows from dimension counting. The third statement follows from duality.

For the second statement $H^1_1(M_0)$ injects into $H^1_1(M)$. As both have the same dimension, we conclude. $\square$

Suppose now that $M_0 = \mathcal{R}_L([N_L/\mathbb{Q}_p](z)_{\mathbb{Z}^m_\tau})$ and $M_1 = \mathcal{R}_L(\prod_{\tau} z^{m_{\tau}})$, we give the following key proposition for the definition of the $\mathcal{L}$-invariant.

Lemma 2.11. The intersection of $T := \text{Im}(H^1(M))$ and $\text{Im}(H^1(\mathcal{R}_{\mathbb{Q}_p}(z_{\sum_{\tau} m_{\tau}})))$ in $\text{Im}(H^1(M_1))$ is one dimensional.

Proof. The intersection is non-empty as the sum of their dimension is $d + 2$ and $\text{Im}(H^1(M_1))$ has dimension $d + 1$. We have that $H^1_1(M_1)$ is contained in $\text{Im}(H^1(\mathcal{R}_{\mathbb{Q}_p}(z_{\sum_{\tau} m_{\tau}})))$ and by the previous proposition the former is not in the image of $g_1$ and we are done. $\square$

In particular, we deduce that $T$ surjects into $\text{Im}(H^1_1(\mathcal{R}_{\mathbb{Q}_p}(z_{\sum_{\tau} m_{\tau}})))$.

3 $\mathcal{L}$-invariant over number fields

Let $F$ be a number field. We consider a global Galois representation

$$V : G_F \to \text{GL}_n(E)$$

where $E$ is $p$-adic field. We suppose that it is unramified outside a finite number of places $S$ containing all the $p$-adic places. We suppose moreover that it is semistable at all places above $p$ (i.e. $D_{\text{ur}}(V_{|_{F_F}})$ is of rank $n$ over $F_p^{ur} \otimes_{\mathbb{Q}_p} E$, being $F_p^{ur}$ the maximal unramified extension of $\mathbb{Q}_p$ contained in $F_p^{ur}$).

In this section we generalize Greenberg-Benois definition of the $\mathcal{L}$-invariant to such a $V$ when it presents trivial zeros. Note that we do not require $p$ split or unramified in $F$.

Let $t$ be the number of trivial zeros. The classical definition by Greenberg [Gre01] defines the $\mathcal{L}$-invariant as the “slope” of a certain $t$-dimensional subspace of $H^1(G_{\mathbb{Q}_p}, Q_p)$ which is a $2t$-dimensional space with a canonical basis given by $\text{ord}_p$ and $\log_p$.

In our setting, the main obstacle is that the cohomology of the trivial $\varphi, \Gamma$-module $\mathcal{R}_{F_p}$ is no longer two-dimensional and it is not immediate to find a suitable subspace. Inspired by Hida’s work for symmetric
powers of Hilbert forms [Hid07], we consider the image of $H^1(\mathcal{R}_{\mathbb{Q}_p})$ inside $H^1(\mathcal{R}_{F_p})$.

If $t$ denotes the number of expected trivial zeros, we show that we can define, similarly to [Ben11], a $t$-dimensional subspace of $H^1(G_{\mathbb{Q}_p}, V)$ whose image in $H^1(\mathcal{R}_{\mathbb{Q}_p})$ has trivial intersection with the crystalline cocycle. This is enough to define the $L$-invariant; we further check that our definition is compatible with Benois'.

### 3.1 Definition of the $L$-invariant

We define local cohomological conditions $L_v$ in order to define a Selmer group; we denote by $G_v$ a fixed decomposition group at $v$ in $G_{F,S}$ and by $I_v$ the inertia. For $v \mid p$ we define

$$L_v := \operatorname{Ker}\left(H^1(G_v, V) \to H^1(I_v, V)\right).$$

If $v \mid p$ we define

$$L_v := H^1(F_v, V) = \operatorname{Ker}(H^1(G_v, V) \to H^1(G_v, V \otimes E \mathbf{B}_{\text{cris}})).$$

If $D^\dagger_{\text{rig}}(V)$ denotes the $(\varphi, \Gamma)$-module associated with $V$ we also have $L_p = H^1_t(D^\dagger_{\text{rig}}(V))$. We define then the Bloch-Kato Selmer group

$$H^1_t(V) := \operatorname{Ker}\left(H^1(G_{F,S}, V) \to \prod_{v \in S} \frac{H^1(D_v, V)}{L_v}\right).$$

We make the following additional hypotheses

- **C1** $H^1_t(V) = H^1_t(V^*(1)) = 0$,
- **C2** $H^0(G_{F,S}, V) = H^0(G_{F,S}, V^*(1)) = 0$,
- **C3** $\varphi$ on $D_{\text{st}}(V_{\mathbb{Q}_p})$ is semisimple at 1 $\in F_{\mathbb{p}}^{ur} \otimes_{\mathbb{Q}_p} E$ and $p^{-1} \in F_{\mathbb{p}}^{ur} \otimes_{\mathbb{Q}_p} E$ for all $p \mid p$,
- **C4** $D^\dagger_{\text{rig}}(V_{\mathbb{Q}_p})$ has no saturated sub-quotient of type $U_{m,k}$ for all $p \mid p$.

Note that if $V$ satisfies the previous four conditions, so does $V^*(1)$.

The first two conditions tell us that the Poitou-Tate sequence reduces to

$$H^1(G_{F,S}, V) \cong \bigoplus_{v \in S} \frac{H^1(D_v, V)}{H^1_t(V, F_v)} \quad (3.1)$$

For each $p \mid p$ we denote by $V_p$ the restriction to $G_{F_p}$ of $V$. We choose a regular sub-module $D_p \subset D_{\text{st}}(V_p)$ and define a filtration $(D_{p,i})$ of $D_{\text{st}}(V_p)$.

$$D_{p,i} = \begin{cases} 0 & i = -2, \\ (1 - p^{-1} \varphi)D_p + N(D_p^{\varphi=1}) & i = -1, \\ D_p & i = 0, \\ D_p + D_{\text{st}}(V_p)^{\varphi=1} \cap N^{-1}(D_p^{\varphi=p^{-1}}) & i = 1, \\ D_{\text{st}}(V_p) & i = 2. \end{cases} \quad (3.2)$$

We have that $D_{p,1}/D_{p,-1}$ coincides with the eigenvectors of $\varphi$ on $D_{\text{st}}(V_p)$ of eigenvalue 1 resp. $p^{-1}$ and which are in the kernel of $N$ resp. in the image of $N$.

This filtration induces a filtration on $D^\dagger_{\text{rig}}(V_p)$. Namely, we pose

$$F_i D^\dagger_{\text{rig}}(V_p) = D^\dagger_{\text{rig}}(V_p) \cap (D_{p,i} \otimes \mathcal{R}_{F_p, \log}[t^{-1}]).$$
We define
\[ W_p := F_1 D_{rig}^t(V_p)/F_{-1} D_{rig}^t(V_p). \]
We have
\[ W_p = W_{p,0} \bigoplus W_{p,1} \bigoplus M_p \]
where \( t_{p,0} = \dim_E H^0(W) = \text{rank}_E W_0, t_{p,1} = \dim_E H^0(W^*(1)) = \text{rank}_E W_1 \) and \( M \) sits in a non-split sequence
\[ 0 \to M_{p,0} \xrightarrow{f} M_p \xrightarrow{g} M_{p,1} \to 0 \]
such that \( \text{gr}^0(D_{rig}^t(V_p)) = W_{p,0} \oplus M_{p,0} \) and \( \text{gr}^1(D_{rig}^t(V_p)) = W_{p,1} \oplus M_{p,1}. \)
We can prove exactly in the same way as [Ben11, Proposition 2.1.7 (i)] that \( \text{C4} \) implies \( \text{rank}_{E_p} M_1 = \text{rank}_{E_p} M_0. \)
In order to define the \( L \)-invariant we shall follow verbatim Benois' construction. For sake of notation, we write \( D_p \) for \( D_{rig}^t(V_p). \) We obtain from [Ben11] Proposition 1.4.4 (i)
\[ H^1_t(\text{gr}^2(D_p^1)) = H^0(\text{gr}^2(D_p^1)) = 0. \]
We deduce the following isomorphism
\[ H^1_t(F_1 D_p^t) = H^1_t(D_p^1) = H^1_t(F_p, V). \] (3.3)
As the Hodge-Tate weights of \( F_{-1} D_p^t \) are \( < 0, \) we obtain from [Ben11] Proposition 1.5.3 (i) and Poiteau-Tate duality \( H^2(F_{-1} D_p^t) = 0. \) Using the long exact sequence associated to
\[ 0 \to F_{-1} D_p^t \to F_1 D_p^t \to W_p \to 0 \]
we see that
\[ \frac{H^1(W_p)}{H^1_t(W_p)} = \frac{H^1(F_{-1} D_p^t)}{H^1_t(F_p, V)}. \]
We suppose now
\[ C5 \) \( W_{p,0} = 0 \) for all \( p \mid p. \)
Write \( \text{gr}^1(D_p^1) = \bigoplus_{i=1}^{t_{p,1}+r_p} \mathcal{R}_{F_p}(\prod_{\tau_p} \tau_p(z)^{m_{i,\tau_p}}). \) We define the \( 2(t_{p,1} + r_p) \)-dimensional subspace obtained as the image of
\[ \text{Ind}_p := \text{Im} \left( H^1 \left( \bigoplus_{i=1}^{t_{p,1}+r_p} \mathcal{R}_{Q_p} \left( z^{\sum_{\tau_p} m_{i,\tau_p}} \right) \right) \right) \subset H^1(\text{gr}^1(D_p^1)). \] (3.4)
We define
\[ T_p = (H^1(F_1 D_p^t) \cap \text{Ind}_p)/H^1_t(F_p, V). \]
It has dimension \( t_{p,1} + r_p. \)
Write \( t = \sum_p t_{p,1} + r_p. \) We have a unique \( t \)-dimensional subspace \( H^1(D, V) \) of \( H^1(G_{F,S}, V) \) projecting via (3.1) to \( \oplus_p T_p. \) We have an isomorphism [Ben11] Proposition 1.5.9
\[ H^1 \left( \bigoplus_{i=1}^{t_{p,1}+r_p} \mathcal{R}_{Q_p} \left( z^{\sum_{\tau_p} m_{i,\tau_p}} \right) \right) = \mathcal{D}_{\text{cris}}(W_1 \oplus M_1) \oplus \mathcal{D}_{\text{cris}}(W_1 \oplus M_1) \]
We shall denote the two projections by $\iota_f$ and $\iota_c$.

A canonical basis is given by the above mentioned cocycles $x_m$ (resp. $y_m$) defined in (resp. right after) Proposition 2.5.

By abuse of notation, we still denote by $\iota_f$ resp. $\iota_c$ be the projection of $H^1(D, V)$ to $D_{\text{cris}}(W)$ via $\iota_f$ resp. $\iota_c$.

By the remark after Lemma 2.11 and the definition of $T_p$, we have that $H^1(D, V)$ surjects into $D_{\text{cris}}$ via $\rho_c$. Summing up, we can give the following definition;

**Definition 3.5.** The $\mathcal{L}$-invariant of the pair $(V, D)$ is

$$\mathcal{L}(D, V) := \det(\iota_f \circ \iota_c^{-1}),$$

where the determinant is calculated w.r.t. the basis $\{x_{mi}, y_{mi}\}_{1 \leq i, j \leq t}$.

**Remark 3.6.** There is no a priori reason for which $\mathcal{L}(D, V)$ should be non-zero.

In the case $W_p = M_p$ we see from the description of $H^1(F_1 D_p^1)$ that the space $T_p$ depends only on $V_{\iota_p}$ exactly as in the classical case.

### 3.2 Comparison with Benois’ definition

Fix a global field $F$ and let $\{p\}$ be the set of primes above $p$. Let $G_p$ denote a fixed decomposition group at $p$ in $G_Q$ and let $p_0$ be the corresponding place of $F$. Let $G_{p_0, F}$ be the decomposition group at $p_0$ in $G_F$. For each other place $p$ above $p$ in $F$, we have $G_p = \sigma_p G_p \sigma_p^{-1}$. We shall denote by $G_{p, F}$ the corresponding decomposition group in $G_F$. Consider a $p$-adic Galois representation

$$V : G_F \to \text{GL}_n(E).$$

We shall suppose $E$ big enough to contain the Galois closure of $F_p$, for all $p$. As before, we suppose $V$ semistable at all primes above $p$. We have then

$$\text{Ind}_F^G(V) \cong \bigoplus_p \sigma_p^{-1} \text{Ind}_{G_{p, F}}^{G_p} V_{\iota_p, F},$$

where $\sigma_p \in G_p \setminus \text{Hom}(F, \overline{Q})$.

Consider the $(\varphi, \Gamma)$-module

$$D^\dagger := D^\dagger_{\text{rig}} \left( \text{Ind}_F^G(V) \right).$$

We let $D$ be the regular $(\varphi, N)$-module of $D_{\text{st}}(D^\dagger)$ induced by $\{D_p\}_p$. As before we have a filtration $(F_i D^\dagger)$ on $D^\dagger$ induced by the filtration on $D$. We denote by $W$ the quotient $F_1 D^\dagger / F_{-1} D^\dagger$. Note that it is semistable. We write $W = W_0 \oplus M \oplus W_1$. We suppose C1-C5 of the previous section.

**Lemma 3.7.** Let $M$ be as in (C4). We have

$$0 \to \text{Ind}(M_0) \to \text{Ind}(M) \to \text{Ind}(M_1) \to 0.$$

We can now compare our definition of $\mathcal{L}$-invariant with Benois’.

**Proposition 3.8.** We have a commutative diagram

$$
\begin{array}{cccccc}
H^1(G_{Q, S}, \text{Ind}(V)) & \longrightarrow & H^1(\text{Ind}(D), \text{Ind}(V)) & \overset{\text{Res}_{G_p}}{\longrightarrow} & H^1(F, D^\dagger_{\text{rig}}(\text{Ind}(V))) & \overset{\text{Res}_{G_p}}{\longrightarrow} & H^1(F, D^\dagger_{G_p, \text{Ind}(V)}) \cong H^1(F, D^\dagger_{G_{p, F}, \text{Ind}(V)}) \\
H^1(G_{F, S}, V) & \longrightarrow & H^1(D, V) & \overset{\oplus \text{Res}_p}{\longrightarrow} & \prod_p T_p \end{array}
$$

whose vertical arrows are isomorphism.
We are left to show that $H^1(F, D^!(\text{Ind}(V)))$ is sent by $\text{Res}_p$ into $(H^1(F, D^!_p) \cap \text{Inv}_p)$ and we shall conclude by dimension counting.

We have then an injection

$$F_1 D^!(\text{Ind}(V)) \hookrightarrow \oplus_p \text{Ind}(F_1(D^!_{\text{rig}}(V_p))).$$

Then clearly the image of $\iota_p$ lands in $H^1(F, D^!_p)$. But we have also the injection

$$\text{gr}^1(D^!_{\text{rig}}(\text{Ind}(V))) \hookrightarrow \oplus_p \text{Ind}(\text{gr}^1(D^!_{\text{rig}}(V_p)))$$

induced by (27). Then the image of $\iota_p$ lands in $\text{Inv}_p$ and we are done. \hfill \square

**Proposition 3.9.** We have $L(V) = L(\text{Ind}^G_P(V))$.

**Proof.** After the previous proposition, what we have to do is to notice that the cocycle $x_{\Sigma \iota r}$ (resp. $y_{\Sigma \iota r}$) is identified with $x_m$ (resp. $y_m$). \hfill \square

### 4 Siegel-Hilbert modular forms, the local case

The calculation of the $L$-invariant requires to produce explicit cocycles in $H^1(D, V)$; when $V$ appears in $\text{Ad}(V')$ for a certain representation $V'$ we can sometimes use the method of Mazur and Tilouine [MT90] to produce these cocycles. This has been done in many case for the symmetric square [Hid04, Mok12] and generalized to symmetric powers of the Galois representation associated with Hilbert modular forms in [Hid07, HJ13]. The main limit of this approach is that for most of the representations $V$ is this computationally heavy to make it appear as the quotient of an adjoint representations.

In the case $D^!_{\text{rig}}(V) = W = M$ the situation is way simpler; if $t = 1$ it has been proved in [Ben10] that to produce the cocycle in $H^1(V, D)$ it is enough to find deformations of $V|_{Q_p}$.

We shall generalized the construction of Benois to our situation in the case $W_p = M_p$ and $r_p = 1$. This will allow us to give a complete formula for the L-invariant of the Galois representations associated with a Siegel-Hilbert modular form which is Steinberg at all primes above $p$.

#### 4.1 The case $t_p = r_p = 1$

We suppose now that $W_p = M_p$ and $r_p = 1$. For sake of notation, in this section we shall drop the index $p$.

All that we have to do is to check that the calculation of [Ben11, Theorem 2] works in our setting.

We write as before

$$0 \to M_0 \to M \to M_1 \to 0$$

and, only in this subsection, we shall write $\delta$ for the character defining $M_0$ and $\psi$ for the character defining $M_1$. We have $\delta(z) = |N_{L/\mathbb{Q}_p}(z)|_p \prod_{\tau} \tau(z)^{k_{\tau}}$ with $k_{\tau} \geq 1$ and $\psi(z) = \prod_{\tau} \tau(z)^{m_{\tau}}$ with $m_{\tau} \leq 0$. We consider an infinitesimal deformation

$$0 \to M_{0,A} \to M_A \to M_{1,A} \to 0,$$

over $A = E[T]/(T^2)$. We suppose that $M_{0,A}$ (resp. $M_{1,A}$) is an infinitesimal deformation of $M_0$ (resp. $M_1$).

We shall write $\delta_A$ and $\psi_A$ for the corresponding one-dimensional character.
Theorem 4.1. Suppose that \( d \log(\delta_A \psi^{-1}_A(\chi_{\text{cycl}}(\gamma_{Q_p})))|_{\tau = 0} \neq 0 \); then

\[
\mathcal{L}(M, M_0) = -\log(\chi_{\text{cycl}}(\gamma_{Q_p})) \cdot \frac{d \log(\delta_A \psi^{-1}_A(p)|_{\tau = 0})}{d \log(\delta_A \psi^{-1}_A(\chi_{\text{cycl}}(\gamma_{Q_p})))|_{\tau = 0}}
\]

Proof. Recall the definition of Ind in \([\text{Ben}10, \text{§}3.2]\). We have a vector \( v = ax_m + by_m \) in \( H^1(F_1 \mathbf{D}^\dagger) \cap \text{Ind} \). By definition \( \mathcal{L}(M) = ab^{-1} \). The extension \( M_{1,A} \) provides us with connecting morphisms \( B_j^* : H^i(M) \to H^{i+1}(M_j) \). We have by definition

\[
B_j^0(t^{-m}e_m) = \text{cl}(d \log(\delta_A)(\pi_L)t^{-m}e_m, d \log(\delta_A)(\chi_{\text{cycl}}(\gamma)))t^{-m}e_m)
\]
(4.2)

\[
= d \log(\delta_A)(\pi_L)x_m + d \log(\delta_A)(\chi_{\text{cycl}}(\gamma))y_m.
\]
(4.3)

As in \([\text{Ben}10, \text{§}3.2]\) we consider the dual extension

\[
0 \to M_1^*(\chi_{\text{cycl}}) \to M^*(\chi_{\text{cycl}}) \to M_1^*(\chi_{\text{cycl}}) \to 0,
\]
and we shall denote with \( \ast \) the corresponding map in the long exact sequence of cohomology. We have hence \( \ker(\Delta_1) \perp \text{Im}(\Delta_0^*) \) under duality, and a map

\[
H^1(M_1^*) \to H^1(R_{Q_p}(|z|^{-1-\sum m_r})).
\]

By duality again, we deduce that the image of \( \Delta_0^* \) inside the target of the above arrow is

\[
a\alpha_1 - \sum m_r + b\beta_1 - \sum m_r,
\]
where \( \alpha_1 - \sum m_r \) (resp. \( \beta_1 - \sum m_r \)) is the dual of \( x \sum m_r \) (resp. \( y \sum m_r \)). We consider now the map

\[
B_1^1 : H^1(M_1^*(\chi_{\text{cycl}})) \to H^2(M_1^*(\chi_{\text{cycl}})) = H^2(R_{Q_p}(|z|^{-1-\sum m_r})),
\]
where the identity is the dual of the identity induced by \([\text{Ben}10, \text{§}3.2]\).

We can use \([\text{Ben}10, \text{Proposition 2.4}]\) to see that after the above identification of \( H^2 \) with \( E \) we have

\[
B_1^1(\alpha_1 - \sum m_r) = c \log_p(\chi_{\text{cycl}}(\gamma_{Q_p}))^{-1}d \log_p(\delta_A)(\chi_{\text{cycl}}(\gamma_{Q_p}))|_{\tau = 0},
\]
(4.4)

\[
B_1^1(\beta_1 - \sum m_r) = cd \log_p(\delta_A)(p)|_{\tau = 0},
\]
(4.5)

where \( c \in E^* \). We consider the following anti-commutative diagram

\[
\begin{array}{ccc}
H^0(M_0^*(\chi_{\text{cycl}})) & \xrightarrow{\Delta_0^*} & H^1(M_1^*(\chi_{\text{cycl}})) \\
| & B_1^1 & | \\
H^1(M_0^*(\chi_{\text{cycl}})) & \xrightarrow{\Delta_1^*} & H^2(M_1^*(\chi_{\text{cycl}}))
\end{array}
\]

which implies

\[
B_1^1 \Delta_0^* = -\Delta_1^* B_0^1.
\]

We calculate this identity on \( t^{1-k} \). Applying \([\text{Ben}10, \text{§}2.2]\) to \( \psi^{-1}_A \chi_{\text{cycl}}(\gamma_{Q_p}) \), \([\text{Ben}10, \text{§}2.2]\) to \( \delta_A^{-1} \chi_{\text{cycl}}(\gamma_{Q_p}) \) and using \([\text{Ben}10, (3.6)]\) which says

\[
\Delta_1^* B_0^1(t^{1-k}) = c(b \log_p(\delta_A)(p) + a d \log_p(\delta_A)(\chi_{\text{cycl}}(\gamma)))
\]

and

\[
\Delta_1^* B_0^1(t^{1-k}) = c(b \log_p(\delta_A)(p) + a d \log_p(\delta_A)(\chi_{\text{cycl}}(\gamma)))
\]

which implies

\[
B_1^1 \Delta_0^* = -\Delta_1^* B_0^1.
\]
we get
\[
b^{-1}a = - \log_p(\chi_{cycl}(\gamma Q_p)) - \frac{d \log_p(\delta A^{-1})(p)|_{T=0}}{d \log_p(\delta A^{-1})(\chi_{cycl}(\gamma Q_p))|_{T=0}}.
\]

Remark 4.6. In particular, this theorem proves that this definition of $L$-invariant is compatible with the Coleman or Fontaine-Mazur ones [Pot14, Zha14].

4.2 Calculation of the $L$-invariant for Steinberg forms

We fix a totally real field $F$. Let $I$ be the set of real embeddings. Fix two embeddings $\mathbb{C} \leftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$
as before. We partition $I = \sqcup I_p$ according to the $p$-adic place which each embedding induces. We shall denote by $q_p$ the residual cardinality for each prime ideal $p$. We consider an irreducible representation $\pi$ of $GSp_{2g}/F$ algebraic of weight $k = (k_1, \ldots, k_{g+1})$ (where $(k_1, \ldots, k_{g+1})$ is a parallel weight for $Res^{-1}_F(G_m)$). With $k_{r,1} \leq k_{r,2} \leq \cdots \leq k_{r,g}$. If $k_{r,1} \geq g+1$, then the weight is cohomological. The cohomological weight of $\pi$ is then
\[
(\mu_\tau)_\tau = (k_\tau)_\tau - (g+1, \ldots, g+1; 0)_\tau.
\]

For parallel weight $k$, we shall choose $k_0 = gk$.

We describe now the conjectural Galois representation associated with $\pi$. We have a spin Galois representation $V_{spin}$ (whose image is contained in $GL_{2g}$) and a standard Galois representation $V_{sta}$ (whose image is contained in $GL_{2g+1}$) given respectively by the spinorial and the standard representation of $GSpin_{2g+1} = L(GSp_{2g})$.

Thanks to the work of Scholze [Sch13] we dispose now of the standard Galois representation (see for example [HJ13, Theorem 18]). We also know the existence of the spin representation in many cases [KST14].

We recall now some expected properties of these Galois representations. Our main reference is [HJ13, §3.3]. We will make the following assumption on $\pi$ at $p$;

for each $p \mid p$ either $\pi_p$ is spherical or Steinberg.

We explain what we mean by Steinberg. Consider the Satake parameters at $p$, normalized as in [BS00, Corollary 3.2], $(\alpha_p, \ldots, \alpha_p, g)$. We have the following theorem on Iwahori spherical representation of $GSp_{2g}(F_p)$ [Tad94, Theorem 7.9]

**Theorem 4.7.** Let $\alpha_1, \ldots, \alpha_g, \alpha$ be $g+1$ character of $F_p^\times$. Let $B_{GSp_{2g}}$ be the Borel subgroup of $Sp_{2g}(F_p)$. Then $Ind_{B_{GSp_{2g}}}(\alpha_1 \times \cdots \times \alpha_g \times \alpha)$ is not irreducible if and only if one of the following conditions is satisfied:

i) There exist at least three indexes $i$ such that $\alpha_i$ has exact order two and the $\alpha_i$'s are mutually distinct;

ii) There exists $i$ such that $\alpha_i = |N( \ )|_p^\pm 1$;

iii) There exist $i$ and $j$ such that $\alpha_i = |N( \ )|_p^\pm 1 \alpha_j^\pm 1$.

**Remark 4.8.** As shown in [HJ13, Lemma 19], such a points are contained in a proper subset of the Hecke eigenvariety for $GSp_{2g}$.
Definition 4.9. We say that $\pi_p$ is Steinberg if $\alpha_i = |N( )|^i p^{-1} \alpha_1$.

If $\pi_p$ is Steinberg at $p$, then $\alpha_{p,i}(\varpi_p) = q_p^i \alpha_{p,1}(\varpi_p)$.

Trivial zeros appear also for automorphic forms which are only partially Steinberg at $p$ and can be dealt exactly at the same way as the parallel one but for the sake of notation we prefer not to deal with them.

To each $g+1$ non-zero elements $(t_1, \ldots, t_g; t_0) \in (A^\times)^{g+1}$ we associate the diagonal matrix

$$u(t_1, \ldots, t_g; t_0) := (t_1, \ldots, t_g, t_0^{-1}_g, \ldots, t_0^{-1}_1)$$

of $GSp_{2g}(A)$.

For $1 \leq i \leq g-1$ we denote by $u_{p,i}$ the diagonal matrix associated with $(1, \ldots, 1, \varpi_p^{-1}, \ldots, \varpi_p^{-1}; \varpi_p^{-2})$, where $\varpi_p$ appears $i$ times; we also denote by $u_{p,0}$ the diagonal matrix corresponding to $(1, \ldots, 1; \varpi_p^{-1})$.

Definition 4.10. The Hecke operators $U_{p,i}$, for $1 \leq i \leq g$ are defined as the double coset operator $[Iuw_{p,g-i}lw]$.

We have that $U_{p,g}$ is the “classical” $U_p$ operator [BS00 §0]. We shall say then that $\pi$ is of finite slope for $U_{p,g}$ if $U_{p,g}$ has eigenvalue $\alpha_{p,0} \neq 0$ on $\pi_p$.

We are interested to study the possible $p$-stabilization of $\pi$ (i.e. Iwahori fixed vectors). If $\pi_p$ is unramified at $p$, we have then $2^g!$ choices (see [HL13 Lemma 16] or [BS00 Proposition 9.1]). If $\pi_p$ is Steinberg, we have instead only one possible choice, as the monodromy $N$ has maximal rank.

Suppose that we can lift $\pi$ to an automorphic representation $\pi^{(2^g)}$ of $GL_{2g}$. We suppose also that we can lift $\pi$ to an automorphic representation $\pi^{(2g+1)}$ of $GL_{2g+1}$.

Let $V = V_{\text{spin}}$ (resp. $V_{\text{sta}}$) be the Galois representation associated with $\pi^{(2^g)}$ (resp. $\pi^{(2g+1)}$). We make the following assumption

**LGp**: $V$ is semistable at all $p | p$ and strong local-global compatibility at $l = p$ holds.

These hypotheses are conjectured to be always true for $f$ as above. Arthur’s transfer from $GSp_{2g}$ to $GL_{2g+1}$ has been proven in [Xu] (note that it is now unconditional [MW]) and for $V = V_{\text{sta}}$ this hypothesis is then verified thanks to [Car13 Theorem 1.1]. These hypotheses are also satisfied in many cases for $V = V_{\text{spin}}$ in genus 2 (see [AS06] [PSS14]).

Roughly speaking, we require that

$$\text{WD}(V_{F_p})^{ss} \cong \tau_n^{-1} \pi_p^{(n)}$$

where $\text{WD}(V_{F_p})$ is the Weil-Deligne representation associated with $V_{F_p}$ à la Berger, $\pi_p^{(n)}$ is the component at $p$ of $\pi^{(n)}$, and $\tau_n$ is the local Langlands correspondence for $GL_n(F_p)$ geometrically normalized ($n = 2g+1$ when $V$ is the standard representation and $n = 2g$ when $V$ is the spinorial representation).

When $\pi_p$ is an irreducible quotient of $\text{Ind}_B^{GSp_{2g}}(\alpha_{p,1} \otimes \cdots \otimes \alpha_{p,g})$ we have that the Frobenius eigenvalues on $\text{WD}(V_{\text{spin}}; F_p)^{ss}$ are the $2^g$ numbers

$$\left( \begin{array}{c} \alpha_{p,0} \\ \prod_{0 \leq r \leq g} \alpha_{p,i_r} \varpi_p \cdots \alpha_{p,i_1} \varpi_p \end{array} \right)$$

\[ 1 \leq i_1 < \cdots < i_r \leq g \]
The ones on $\text{WD}(V_{\text{sta} | F_p})^s$ are
\[ (\alpha_{p,1}^{-1}(w_p), \ldots, \alpha_{p,g}^{-1}(w_p), 1, \alpha_{p,1}(w_p), \ldots), \alpha_{p,g}(w_p)) \] .

Moreover, the monodromy operator should have maximal rank (i.e. one-dimensional kernel) if we are Steinberg or be trivial otherwise. (This is also a consequence of the weight-mondromy conjecture for $V$.) Let $p$ be a $p$-adic place of $V$ and let $\tau$ be a complex place in $I_p$. The Hodge-Tate weights of $V_{\text{spin} | F_p}$ at $\tau$ are then
\[ \left( \frac{k_0}{2} + \frac{1}{2} \sum_{i=1}^{g} \varepsilon(i)(k_{\tau,i} - i) \right) \varepsilon, \]
where $\varepsilon$ ranges among the $2^g$ maps from $\{1, \ldots, g\}$ to $\{\pm 1\}$. The one of $V_{\text{sta} | F_p}$ are $(1 - k_{\tau,g}, \ldots, g - k_{\tau,1}, 0, k_{\tau,1} - g, \ldots, k_{\tau,g} - 1)$. Thanks to work of Tilouine-Urban [TU99], Urban [Urb11], Andreatta-Iovita-Pilloni [AIP12] we have families of Siegel modular forms;

**Theorem 4.11.** Let $W = \text{Hom}_{\text{cont}} \left( Z_p^x \times ((O_F \otimes \mathbb{Z} \mathbb{Z})^x)^g, C_p^x \right)$ be the weight space. There exist an affinoid neighborhood $U$ of $\kappa_0 = ((z, (z_i)_{i=1}^g) \mapsto z^{\kappa_0} \prod_{i \in I} \tau(z_i)^{\kappa_{r,i}})$ in $W$, an equidimensional rigid variety $X = X_{\varepsilon}$ of dimension $dg + 1$, a finite surjective map $w : X \to U$, a character $\Theta : H^{N} \to O(X)$, and a point $x$ in $X$ above $k$ such that $x \circ \Theta$ corresponds to the Hecke eigensystem of $\pi$. Moreover, there exists a dense set of points $x$ of $X$ coming from classical cuspidal Siegel-Hilbert automorphic forms of weight $(k_{\tau,i}; k_0)$ which are regular and spherical at $p$.

**Remark 4.12.** Assuming Leopoldt conjecture, the multiplicative group appearing in the definition of $W$ is, up to a finite subgroup, $((O_F \otimes \mathbb{Z} \mathbb{Z})^x)^g / O_F^x$ (i.e. the $\mathbb{Z}_p$-points of the torus of $\text{Res}_{F}^{\mathbb{Q}}(\text{GSp}_{2g})$ modulo the $\mathbb{Z}_p$-points of the center).

This allows us to define two pseudo-representations $R_\tau : G_{\mathbb{Q}} \to O(X)$, for $\tau = \text{spin}, \text{sta}$, interpolating the trace of the representations associated with classical Siegel forms [BC09, Proposition 7.5.4]. Suppose now that $V_\tau$ is absolutely irreducible (this is conjectured to hold when $\pi$ is Steinberg at least at one prime); we have then, shrinking $U$ around $k$ if necessary, a big Galois representation $\rho_\tau$ with value in $G_{\mathbb{Q}}(O(X))$ such that $\text{Tr}(\rho_\tau) = R_\tau$ [BC09 page 214].

For $1 \leq j < g$ we define $\lambda_\tau(u_{p,g-j}) = \sum_{r \in I_p} k_{r,1} + \cdots + k_{r,j} - k_0$ and $\lambda_\tau(u_{p,0}) = \sum_{r \in I_p} (k_{r,1} + \cdots + k_{r,g} - k_0)/2$.

We have analytic functions $\beta_{\rho_{\tau,i}} : \Theta(U_{p,i})/\lambda_\tau(u_{p,g-j}) \in O(X)$. We proceed now as in [HJ13]. We recall the following theorem [Lin13 Theorem 0.3.4]:

**Theorem 4.13.** Let $\rho : G_{F_p} \to G_{\mathbb{Q}}(O(X))$ be a continuous representation. Suppose that there exist $\kappa_1(x), \ldots, \kappa_n(x)$ in $F_p \otimes_{\mathbb{Q}} O(X)$, $F_1(x), \ldots, F_n(x)$ in $O(X)$, and a Zariski dense set of points $Z \subset X$ such that

- for any $x$ in $X$, the Hodge-Tate weights of $\rho_\tau$ are $\kappa_1(x), \ldots, \kappa_n(x)$;
- for any $z$ in $Z$, $\rho_\tau$ is crystalline;
- for any $z$ in $Z$, $\kappa_{r,1}(z) < \ldots < \kappa_{r,n}(z)$, for all $\tau \in I_p$;
- for any $z$ in $Z$, the eigenvalues of $\varphi^{I_p}$ on $D_{\text{cris}}(V_\tau)$ are $\prod_{r \in I_p} \tau(w_p)^{\kappa_{r,1}(z)} F_1(z), \ldots, \prod_{r \in I_p} \tau(w_p)^{\kappa_{r,n}(z)} F_n(z)$;
- for any $C$ in $\mathbb{R}$, defines $Z_C \subset Z$ as the set of points $z$ such that for all $I, J \subset \{1, \ldots, n\}$ such that $|\sum_{i \in I} \kappa_{r,i}(z) - \sum_{j \in J} \kappa_{r,j}(z)| > C$ for all $\tau \in I_p$. We require that for all $z \in Z$ and $C \in \mathbb{R}$, $Z_C$ accumulates at $z$. 

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Consider now the representation $\pi_i : \mathcal{O}_F^\times \to \mathcal{O}(X)^\times$ such that $\chi_i(u) = \prod_{\tau} \tau(u)^{c_{\tau,i}u_i}$.

Then, for all $x$ in $X$ non-critical and regular ($\kappa_i(x) < \ldots < \kappa_n(x)$ and the eigenvalues of $\varphi$ on $\mathcal{D}_{\text{cris}}(V_x)$ are distinct for all $i$) there exists a neighborhood $U$ of $x$ such that $\rho_U$ is trianguline and its graded pieces are $\mathcal{R}_U(\chi_i)$.

We can apply this theorem and show that the $(\varphi, \Gamma)$-module associated with $\rho_{\tau|G_{\text{rig}}}$ is trianguline. We explicit now the triangulation, given in $[\text{HJ13}, \S3.3]$.

As seen before, a $p$-stabilization of $\mathfrak{p}$ corresponds to a permutation $\nu$ and a map $\varepsilon$.

The eigenvalues of $\varphi$ are given by

$$
\prod_{\tau \in I_p} \tau(\varphi_p)^{c_{\tau,i}+\mu_{r,1}} \beta_{p,1},
\prod_{\tau \in I_p} \tau(\varphi_p)^{c_{\tau,i}+\mu_{r,i}} \beta_{p,-i},
\prod_{\tau \in I_p} \tau(\varphi_p)^{c_{\tau,i}+\mu_{r,g}} \beta_{p,g-1} \beta_{p,g}^2
$$

where $c_i$’s are a positive integer independent of the weight.

We define the following characters of $F_p$ with value in $\mathcal{O}(X)$:

$$
\chi_{p,1}(\varphi_p) = \beta_{p,1},
\chi_{p,i}(\varphi_p) = \beta_{p,-i} \beta_{p,i},
\chi_{p,g}(\varphi_p) = \beta_{p,g-1} \beta_{p,g}^2,
$$

and $\chi_{p,1}(u) = \prod_{\tau \in I_p} \tau(u)^{c_{\tau,i}+\mu_{r,i}}$.

From $[\text{HJ13}, \text{Lemma 19}]$ we have that the graded pieces of $\mathcal{D}_{\text{rig}}(V_{\text{sta}}|_{\mathfrak{p}})$ are then given by the characters $\chi_{p,g}, \ldots, 1, \ldots, \chi_{p,g}$.

Concerning $V_{\text{spin}}$, we number the subsets of $\{1, \ldots, g\}$ as $I_1, I_2, \ldots, I_{2g}$. Each $I_j$ correspond to a map $\varepsilon_j : \{1, \ldots, g\} \to \pm 1$.

We have then the graded pieces $\delta_{p,j}$ are given by the characters

$$
\delta_{p,\varepsilon_j}(u) = \prod_{\tau \in I_p} \tau(u)^{d_{\tau,i} + \frac{\kappa_{\tau,i} + \sum_{i \in \varepsilon_j}(i)k_{r,i}}{2}},
\delta_{p,\varepsilon_j}(\varphi_p) = \beta_{p,g} \prod_{i \in I_j} \chi_{p,i}(\varphi_p).
$$

Let $V$ be either $V_{\text{sta}}$ or $V_{\text{spin}}$. If $\pi_{\mathfrak{p}}$ is Steinberg, there is only one choice of a regular $(\varphi, N)$-sub-module $D_{\mathfrak{p}}$ of $\mathcal{D}_{\text{rig}}(V_{\text{spin}}|_{\mathfrak{p}})$, where $V$ is one of the two representations associated with $\pi$ described above. If the form is not Steinberg at $\mathfrak{p}$ many different regular sub-module can be chosen.

In any case, we expect (and we shall assume in the follow) that there is at most one trivial zero for each $\mathfrak{p}$.

Consider now the representation $\pi$ of parallel weight $k$ (i.e. associated with $N_{F/L}(\det \mathcal{E}), k \in \mathbb{Z}$) as in the introduction.

We give a preliminary proposition on the factorization of the $\mathcal{L}$-invariant. Recall the set $S_{\text{spin}}^{1}$ and $S_{\text{sta}}^{\mathfrak{p}}$ defined in the introduction, we have the following:

\begin{itemize}
  \item for $1 \leq i \leq n$ there exist $\chi_i : \mathcal{O}_F^\times \to \mathcal{O}(X)^\times$ such that $\chi_i(u) = \prod_{\tau} \tau(u)^{c_{\tau,i}u_i}$. 
\end{itemize}
Proposition 4.14. We have the following factorization
\[ L(V, D) = L(V, D)^{\text{Sph}} \prod_{p \in S^{\text{Stb}}} L(V, D)_p, \]
where \( L(V, D)^{\text{Sph}} \) comes from the prime in \( S^{\text{Sph}} \) and the factors \( L(V, D)_p \) are local.

Proof. We follow [Hid07, §1.3]. In the notation of Section 3 we write \( W_1 = \oplus_{p \in S^{\text{Stb}}} W_{p, 1} \) and \( M_1 = \oplus_{p \in S^{\text{Stb}}, i} M_{p, 1} \). We are left to show that the endomorphism \( \iota_f \circ \iota_c^{-1} \) of \( D_{\text{cris}}(W_1 \oplus M_1) \cong E^2 \) keeps stable \( D_{\text{cris}}(M_1) \) and on the quotient it respects the direct sum decomposition \( \oplus_{i \in S^{\text{Stb}}, c} D_{\text{cris}}(W_{p, 1}). \)

Consider a prime \( p_0 \in S^{\text{Stb}} \) and a cocycle \( c \in H^1(V, D) \) such that \( \text{res}_{p_0}(c) = 0 \) for all \( p \neq p_0 \). This means that \( \text{res}_{p_0}(c) = 0 \in H^1(F_p, V) = H^1(F_p, M_p) \) (by (3.3)). Hence by Proposition 2.11 (which holds only for \( p \) in \( S^{\text{Sph}} \)) we have \( \iota_{c, p_0}(c) = \iota_{c, p}(c) = 0 \).

We have also \( \iota_{c, p}(c) = 0 \) for all primes \( p \neq p_0 \) as \( H^1_{\text{c}} \) is the direct sum complement of \( H^1_\ast \) (see [Ben11, Proposition 1.5.9]).

The proposition then follows from standard linear algebra as in [Hid07, Corollary 1.9].

Remark 4.15. A key ingredient in the proof of the factorization at Steinberg places is that each prime ideal brings a single trivial zero.

We consider now the case \( V = V_{\text{sta}} \). We have a contribution to trivial zeros from the \( \pi_p \)'s which are Steinberg and possibly from the \( \pi_p \) which are spherical. In particular, if we choose the regular sub-module coming from an ordinary filtration, we always have a trivial zero coming from each place.

For all \( 1 \leq s \leq \min(k - g - 1, g - 1) \) we have also \( c_{\text{Sph}} \) trivial zeros for \( V(s) \).

Theorem 4.16. For \( \pi_p \) Steinberg we have
\[ L(V, D)_p = -\frac{1}{f_p} \frac{d \log_p \beta_{p, 1}(k)}{dk} |_{k = \Delta}, \]
where \( k \) is the parallel weight variable.

For \( 1 \leq s \leq \min(k - g - 1, g - 2) \) we also have
\[ L(V(s), D(s))_p = -\frac{1}{f_p} \frac{d \log_p (\beta_{p, s-1, 1}^{-1})}{dk} |_{k = \Delta}, \]
and if \( g - 1 \leq k - g - 1 \) we have
\[ L(V(g - 1), D(g - 1))_p = -\frac{1}{f_p} \frac{d \log_p (\beta_{p, g-1, 2}^{-2})}{dk} |_{k = \Delta}. \]

Proof. We apply Theorem 4.11 for the \( f_p \)-th root of \( \chi_{\Gamma, 1}(\omega_p) \) and we note that we can specialize to a parallel family, so that no contribution from the denominator appears. The \( \log_p(u) \) disappear because of the change of variable \( T \mapsto u^k - 1 \) (\( u \) any topological generator of \( \mathbb{Z}_p^\ast \)).

Remark 4.17. The presence of \( f_p \) in the denominator is explained in term of \( L \)-function and Euler factors at \( p \) in [Hid09].

From now on, \( V = V_{\text{spin}}(k - 1) \) \( (s = k - 1 \) is the only critical integer); if \( \pi_p \) is spherical it should not give any trivial zeros (as the corresponding \( p \)-adic representation is conjectured to be crystalline and consequently the \( \beta_i \)'s are Weil numbers of non-zero weight).

So we are left to see what happen at the primes Steinberg at \( p \). Twisting by \( \beta_{p, g} \) the triangulated \((\varphi, \Gamma)\)-module of \( \rho_{\text{spin}} \) we are in the hypothesis of Theorem 4.13 and we have
Theorem 4.18. For \( \pi_p \) Steinberg we have
\[
\mathcal{L}(V,D)_p = -\frac{1}{f_p} \frac{d \log \beta_{p,1}(k)}{dk} \bigg|_{k=\frac{1}{2}},
\]
where \( k \) is a parallel weight variable.

5 The case of the adjoint representation

We prove Theorem 4.11 of the introduction. We consider only the case \( g = 2 \). Fix an automorphic representation \( \pi \) of weight \( \underline{k} = (k_{r,1}, \ldots, k_{r,g}; \underline{b}) \) and let \( V = V_{\text{spin}} \) be the spin representation associated with \( \pi \). Let \( \rho = \rho_{\text{spin}} \) be the corresponding big Galois representation.

We specialize the eigenvariety \( X \) of Theorem 4.11 to the subspace of the weight space given by the equations \( k_{r,i} = k_{r',i} \) if \( \tau \) and \( \tau' \) induce the same \( p \)-adic place \( p \) and \( k_0 = \underline{b} \). We shall denote the new variable by \( k_{p,i} \) and this eigenvariety by \( X' \). For simplicity, we rewrite the graded pieces of \( V \) as

\[
\delta_{p,1}(w_p) = F_{p,1}(k), \quad \delta_{p,1}(u) = N_{F_p/Q_p}(u)^{k_{p,1}+k_{p,2}-1},
\]

\[
\delta_{p,2}(w_p) = F_{p,2}(k), \quad \delta_{p,2}(u) = N_{F_p/Q_p}(u)^{k_{p,2}+k_{p,3}-1},
\]

\[
\delta_{p,3}(w_p) = F_{p,3}(k), \quad \delta_{p,3}(u) = N_{F_p/Q_p}(u)^{k_{p,3}+k_{p,4}-1},
\]

\[
\delta_{p,4}(w_p) = F_{p,4}(k), \quad \delta_{p,4}(u) = N_{F_p/Q_p}(u)^{k_{p,4}+k_{p,5}-1},
\]

where \( k = (k_{p,1}, k_{p,2}; k_0)_p \).

The representation space of \( \text{Ad}(V) \) is given by the matrices

\[
\mathfrak{S}_{p} = \{ X \in \mathfrak{L}_4 | X J' + JX = 0 \}.
\]

The \( p \)-stabilization on \( V \) induces a natural \( p \)-stabilization and consequently a regular sub-module \( D_{\text{Ad}} \) on \( \text{Ad}(V_{\text{spin}}) \). We have

\[
D_{\text{Ad}^{-1}} = \{ \text{nilpotent } X \},
\]

\[
D_{\text{Ad}^0} = \{ \text{unipotent } X \}.
\]

The basis for the space \( D_{\text{Ad}^0}/D_{\text{Ad}^{-1}} \) is given by the two diagonal matrices \( d_1 = [-1, 0, 0, 1] \) and \( d_2 = [0, -1, 0, 0] \). We shall denote by \( d_{p,i} \) these matrices when seen as a vector for \( \text{Ad}(V_p) \).

Proposition 5.1. Suppose that \( C1-C4 \) holds for \( V \). Suppose that the classical \( E \)-point \( x \) in the eigenvariety \( X' \) corresponding to \( \pi \) is étale above the weight space. Then, the space \( \mathcal{L}(D_{\text{Ad}}, V) \) is generated by the image of \( \left( \frac{d \log \delta_{p,1}}{d k^p}, \delta_{p,1} \right)_{p', j=1, 2} \).

Proof. The proof is standard and goes back to [MT90], so we shall only sketch it. Let \( A = E[T]/(T^2) \). Consider an infinitesimal deformation of \( \rho \) given by

\[
\rho_A = V \oplus \rho',
\]

note that \( \rho' \) can be written as the first order truncation of \( \frac{\partial \rho}{\partial v} \), where \( v \) is any direction in the weight space. From \( \rho_A \) we can construct a cocyle \( c_{x,A} \) defined by

\[
G_F \ni \sigma \mapsto \rho'(\sigma)V^{-1}(\sigma).
\]
It is easy to check that this defines a cocycle with values in $V \otimes V^*$. Moreover its image lands in $\text{Ad}(V) \subset V \otimes V^*$ as the determinant is fixed (by our choice of the Hodge-Tate weight on $X'$). Writing explicitly the matrix for the $(\varphi, \Gamma)$-module associated with $\rho_A$ we obtain

$$
\begin{pmatrix}
\frac{\partial \delta_{p,1}}{\partial v} & * & * & \\
\frac{\partial \delta_{p,2}}{\partial v} & \frac{\partial \delta_{p,3}}{\partial v} & * & \\
\frac{\partial \delta_{p,4}}{\partial v} & * & * & \\
\end{pmatrix}
\begin{pmatrix}
\delta_{p,1}^{-1} & * & * \\
\delta_{p,2}^{-1} & \delta_{p,2} & * \\
\delta_{p,3}^{-1} & * & \delta_{p,4}^{-1} \\
\end{pmatrix}
|_{k=\wedge}
$$

In particular, they are upper triangular and their projection via $\iota_f$ onto the vector $d_{p,1}$ is $\frac{d \log F_p(t_i(k))}{d v}|_{k=\wedge}$. Similarly for $d_{p,2}$.

We also have that the projection via $\iota_c$ onto $d_{p,1}$ is $-\frac{\partial (k_{p,1}+k_{p,2})/2}{\partial v}|_{k=\wedge}$. This cocycle lies $H^1(G_{F,S}, V \otimes V^*)$ by construction of $\rho$.

As $\left\{\frac{\partial}{\partial \delta_{p,i}}\right\}_{p,i=1,2}$ is a base of the tangent space at $x$ in $X'$ we are done.

We can now prove Theorem 1.4 which we recall now;

**Theorem 5.2.** We have

$$
\mathcal{L}(\text{Ad}(V_{\text{spin}}), \text{D}_{\text{Ad}}) = \prod_p \frac{2}{f_p^2} \det\left(\begin{array}{cc}
\frac{\partial \log F_p(t_i(k))}{\partial k_{p,1}} & \frac{\partial \log F_p(t_j(k))}{\partial k_{p,1}} \\
\frac{\partial \log F_p(t_i(k))}{\partial k_{p,2}} & \frac{\partial \log F_p(t_j(k))}{\partial k_{p,2}}
\end{array}\right)_{1 \leq i,j \leq t} |_{k=\wedge}
$$

**Proof.** Once we have Proposition 5.1, we just have to follow the proof of [Hid06, Theorem 3.73]. The matrix of $\iota_i$ is exactly what appears in the Theorem, while the matrix of $\iota_c$ can be directly calculated using the formula $\frac{d \log(u^{t_k+u})}{d v} = \pm \delta_{p,1} \delta_{i,j}$ (where $\delta_{a,b}$ here is Kronecker delta) and gives a contribution of $2^{-1}$ for each prime ideal $p$. 

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