Integration in General Relativity

Andrew DeBenedictis

Dec. 03, 1995

Abstract

This paper presents a brief but comprehensive introduction to certain mathematical techniques in General Relativity. Familiar mathematical procedures are investigated taking into account the complications of introducing a non trivial space-time geometry. This transcript should be of use to the beginning student and assumes only very basic familiarity with tensor analysis and modern notation. This paper will also be of use to gravitational physicists as a quick reference.

Conventions
The following notation is used: The metric tensor, $g_{\mu\nu}$, has a signature of $+2$ and $g = |\text{det}(g_{\mu\nu})|$. Semi-colons denote covariant derivatives while commas represent ordinary derivatives.

1 Introduction

Say we have a tensor $T$ then, like the partial derivative, the covariant derivative can be thought of as a limiting value of a difference quotient. A complication arises from the fact that a tensor at two different points in space-time transform differently. That is, the tensor at point $r$ will transform in a different way than a tensor at point $r + dr$. In polar coordinates we are familiar with the fact that the length and direction of the basis vectors change as one goes from one point to another. If we have a vector written in terms of its basis as follows,

$$V = V^\alpha e_\alpha$$

(1)
the derivative with respect to co-ordinate $x^\beta$ would be:

$$\frac{\partial V^\alpha}{\partial x^\beta} e_\alpha + V^\alpha \frac{\partial e_\alpha}{\partial x^\beta}.$$  \hspace{1cm} (2)

We define the Christoffel symbol $\Gamma^\mu_{\alpha\beta}$ as representing the coefficient of the $\mu^{th}$ component of $\frac{\partial e_\alpha}{\partial x^\beta}$. The above derivative becomes (after relabelling dummy indices),

$$\left( V^\alpha_{,\beta} + V^\mu \Gamma^\alpha_{\mu\beta} \right) e_\alpha.$$  \hspace{1cm} (3)

When it comes to integration, we are performing the reversal of a partial differentiation and can therefore not just integrate a covariant derivative. Also, an integral over tensor components does not give a result which is a tensor whereas integration over a scalar does.

We can convert expressions such as $P^\nu_{;\nu}$ into an expression containing only partial derivatives as follows: First write out the expression $P^\nu_{;\nu}$ in terms of the Christoffel symbol

$$P^\nu_{;\nu} = P^\nu_{;\nu} + P^\lambda \Gamma^\nu_{\lambda\nu}.$$  \hspace{1cm} (4)

Now use the fact that

$$\Gamma^\nu_{\lambda\nu} = \frac{1}{2} g^{\nu\alpha} \left( g_{\nu\alpha,\lambda} + g_{\alpha\lambda,\nu} - g_{\lambda\nu,\alpha} \right)$$  \hspace{1cm} (5)

$$= \frac{1}{2} g^{\nu\alpha} \left( g_{\alpha\lambda,\nu} - g_{\lambda\nu,\alpha} \right) + \frac{1}{2} g^{\nu\alpha} g_{\nu\alpha,\lambda}.$$  \hspace{1cm} (5)

The first term in the last expression is equal to zero since it is $g^{\nu\alpha}$ multiplied by a tensor which is antisymmetric in $\nu, \alpha$. Therefore:

$$\Gamma^\nu_{\lambda\nu} = \frac{1}{2} g^{\nu\alpha} g_{\nu\alpha,\lambda}.$$  \hspace{1cm} (6)

Using the fact that $g_{,\lambda} = g g^{\nu\alpha} g_{\alpha\nu,\lambda}$ gives,

$$\Gamma^\nu_{\lambda\nu} = \frac{1}{2} g_{,\lambda} g^{-1}$$  \hspace{1cm} (7)

$$= \frac{(\sqrt{g})_{,\lambda}}{(\sqrt{g})} = (\ln \sqrt{g})_{,\lambda}.$$  \hspace{1cm} (7)

We can now write

$$P^\nu_{;\nu} = P^\nu_{;\nu} + P^\lambda \left( \frac{\sqrt{g}}{\sqrt{g}} \right)_{,\lambda}$$  \hspace{1cm} (8)

$$= \frac{1}{\sqrt{g}} \left( \sqrt{g} P^\nu_{;\nu} \right)_{,\nu}.$$(after relabelling dummy indices)
This result is useful because it allows us to apply Gauss’ law which we know applies to partial derivatives. Gauss’ law states that the volume integral of a divergence can be re-written as an integral over the boundary surface as follows:

$$\int P_\alpha dV = \oint P_\alpha \hat{n}_\alpha dS. \quad (9)$$

Where $\hat{n}_\alpha$ is the outward unit normal to the surface. In our case we need to integrate over proper volume and therefore must use proper surface area whose element is $\sqrt{g} d^3 x$. Therefore, in general relativity, Gauss’ law is generalized to

$$\int P^\alpha \sqrt{g} d^4 x = \int (\sqrt{g} P_\nu)_{\nu} d^4 x = \oint P^\nu \hat{n}_\nu \sqrt{g} d^3 x. \quad (10)$$

### 1.1 Tensor Densities

Ordinary tensors transform according to the following transformation law:

$$T'_{\nu} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} T^\alpha_{\beta}. \quad (11)$$

An object which transforms according to (11) is called a tensor density of weight zero. A tensor density $\mathcal{I}$ of weight $w$ transforms as follows:

$$\mathcal{I}'_{\nu} = \left| \frac{\partial x}{\partial x'} \right|^w \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \mathcal{I}^\alpha_{\beta}, \quad (12)$$

which is similar to (11) except for the Jacobian term raised to the power of $w$. We can convert tensor densities of weight $w$ to ordinary tensors by noting the transformation of the metric’s determinant.

$$g' = |g_{\gamma'\kappa'}| = \left| g_{\alpha\beta} A^\alpha_{\gamma} A^\beta_{\kappa} \right| \quad (13)$$

$$= \left| g_{\alpha\beta} \right| \left| \frac{\partial x^\alpha}{\partial x'^\gamma} \frac{\partial x^\beta}{\partial x'^\kappa} \right|$$

$$= \left| \frac{\partial x}{\partial x'} \right|^2 g.$$

Therefore we can write

$$(g')^{-w/2} \mathcal{I}'_{\nu} = \left| \frac{\partial x}{\partial x'} \right|^{-w} \left| \frac{\partial x}{\partial x'} \right|^w \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \mathcal{I}^\alpha_{\beta} \quad (14)$$

3
which transforms like an ordinary tensor (i.e. tensor density of weight zero). It is these types of tensor densities which we want to consider when integrating. For example, consider the volume element $d^4x' = \left| \frac{\partial x'}{\partial x} \right| d^4x$. The corresponding invariant volume element (the proper element) is $\sqrt{g}d^4x = \sqrt{g}d^4x$. We see that $d^4x$ has a weight of -1 since $\sqrt{g}$ has a weight of +1.

**The covariant derivative of a scalar density of arbitrary weight**

The scalar field of weight $w$ transforms as

$$\Phi' = \left| \frac{\partial x}{\partial x'} \right|^w \Phi. \quad (15)$$

Taking the derivative of this creature we get

$$\frac{\partial \Phi'}{\partial x'^{\iota}} = \left| \frac{\partial x}{\partial x'} \right|^w \frac{\partial \Phi}{\partial x^\alpha} \frac{\partial x'^{\alpha}}{\partial x'^{\iota}} + w \left| \frac{\partial x}{\partial x'} \right|^w \frac{\partial x'^{\iota}}{\partial x^\beta} \frac{\partial^2 x^\beta}{\partial x'^{\iota} \partial x'^{\alpha}} \Phi. \quad (16)$$

Noting that the transformation property of the Christoffel symbol is

$$\Gamma'^\alpha_{\alpha \iota} = \Gamma^\sigma_{\sigma \alpha} \frac{\partial x^\alpha}{\partial x'^{\iota}} + \frac{\partial^2 x^\sigma}{\partial x'^{\alpha} \partial x'^{\iota}} \frac{\partial x'^{\iota}}{\partial x^\sigma}. \quad (17)$$

This equation can be multiplied by $w\Phi'$ and subtracted from the previous one to get

$$\frac{\partial \Phi'}{\partial x'^{\iota}} - w\Phi'\Gamma'^\alpha_{\alpha \iota} = \left| \frac{\partial x}{\partial x'} \right|^w \left( \frac{\partial \Phi}{\partial x^\alpha} - w\Phi\Gamma^\sigma_{\sigma \alpha} \right) \frac{\partial x^\alpha}{\partial x'^{\iota}} \quad (18)$$

which displays the transformation properties of $\Phi$ and is its covariant derivative.■

The above result can be used to find the covariant derivative of a tensor density of arbitrary weight. Let $\gamma^{\mu}_{\nu}$ be a contravariant tensor of weight $w$ which we want to take the covariant derivative of. This can be written as follows,

$$\gamma^{\mu}_{\nu} = \left( \sqrt{g}^{-w} \sqrt{g}^{w} \gamma^{\mu}_{\nu} \right)_{;\rho}$$

$$= \left( \sqrt{g}^{-w} \right)_{;\rho} \sqrt{g}^{-w} \gamma^{\mu}_{\nu} + \sqrt{g}^{-w} \left( \sqrt{g}^{-w} \gamma^{\mu}_{\nu} \right)_{;\rho}. \quad (19)$$

The first term in the last expression is equal to zero from (18) and noting (7). The second term is a covariant derivative of a tensor of weight zero.


multiplied by the factor $\sqrt{g^w}$. The expression therefore equals
\[
\sqrt{g^w} \left( (\sqrt{g^{-w}} \varphi^\mu )_{,\rho} + (\sqrt{g^{-w}} \varphi^\lambda \Gamma^\mu_{\lambda\rho}) \right)
\]
\[
= \varphi^\mu_{,\rho} + \varphi^\lambda \Gamma^\mu_{\lambda\rho} - w \frac{\sqrt{g}}{\sqrt{g}} \varphi^\mu.
\]

This argument can be extended to give the covariant derivative of an arbitrary rank tensor of arbitrary weight $w$,
\[
T^{\alpha_1 \alpha_2 \ldots}_{\beta_1 \beta_2 \ldots \rho} = T^{\alpha_1 \alpha_2 \ldots}_{\beta_1 \beta_2 \ldots \rho} + T^{\mu \alpha_2 \ldots \Gamma_{\alpha_1}}_{\beta_1 \beta_2 \ldots \mu \rho} + \ldots
\]
\[
- T^{\alpha_1 \alpha_2 \ldots \Gamma_{\nu}}_{\nu \beta_1 \beta_2 \ldots \beta_1 \rho} - \ldots - w \frac{\sqrt{g}}{\sqrt{g}} T^{\alpha_1 \alpha_2 \ldots}_{\beta_1 \beta_2 \ldots \rho}.
\]

### 1.2 Integrals of Second Rank Tensors

Second rank tensors are most easily handled if they are antisymmetric. Consider an antisymmetric second rank tensor $F^{\alpha \beta}$. We can take the following covariant derivative:
\[
F^{\alpha \beta}_{;\beta} = F^{\alpha \beta}_{,\beta} + F^{\alpha \mu \Gamma^{\beta}_{\mu \beta}} + F^{\mu \beta \Gamma^{\alpha}_{\mu \beta}}.
\]

Since $F^{\alpha \beta}$ is antisymmetric and the Christoffel symbols are symmetric the $F^{\mu \beta \Gamma^{\alpha}_{\mu \beta}}$ term vanishes leaving:
\[
F^{\alpha \beta}_{;\beta} = F^{\alpha \beta}_{,\beta} + F^{\alpha \mu \Gamma^{\beta}_{\mu \beta}}.
\]

As before, we write $\Gamma^{\beta}_{\mu \beta} = \frac{(\sqrt{g})_{,\beta}}{\sqrt{g}}$ giving (after relabelling dummy indices)
\[
F^{\alpha \beta}_{;\beta} = F^{\alpha \beta}_{,\beta} + F^{\alpha \beta} (\sqrt{g}_{,\beta})_{,\beta}.
\]

Therefore, similar to the vector case
\[
F^{\alpha \beta}_{;\beta} = \frac{1}{\sqrt{g}} (\sqrt{g} F^{\alpha \beta})_{,\beta}.
\]
1.3 Killing Vectors

We can exploit symmetries in the space-time to aid us in integration of second rank tensors. For example, does the metric change at all under a translation from the point $x = x^\mu$ to $\tilde{x} = x^\mu + \epsilon k^\mu(x)$? This change is measured by the Lie derivative of the metric along $k$,

$$\mathcal{L}_k g_{\mu\nu} = \lim_{\epsilon \to 0} \frac{g_{\mu\nu}(x) - g_{\mu\nu}(\tilde{x})}{\epsilon}. \quad (26)$$

If the metric does not change under transport in the $k$ direction then the Lie derivative vanishes. This condition implies the Killing equation

$$k_{\nu;\mu} + k_{\mu;\nu} = 0. \quad (27)$$

The solutions (if any) to this equation are called killing vectors.

**Time-like Killing vector of a spherically symmetric space-time:**

Consider the following line element:

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2 \quad (28)$$

where $\alpha$ and $a$ are functions of the coordinates. The metric will be stationary if the metric is time independent in some coordinate system. That is,

$$\frac{\partial g^{\mu\nu}}{\partial x^0} = 0. \quad (29)$$

Where $x^0$ is a time-like coordinate. We write out the full expression for the Lie derivative of the metric

$$\mathcal{L}_k g_{\mu\nu} = k^\gamma g_{\mu\nu,\gamma} + g_{\mu\gamma} k^\gamma_{\nu} + g_{\nu\gamma} k^\gamma_{\mu}. \quad (30)$$

Setting this equal to zero, a time like solution satisfying this equation is the vector field

$$k^\alpha = \delta_0^\alpha. \quad (31)$$

Substituting this into (30) we get

$$\mathcal{L}_k g_{\mu\nu} = \delta_0^\alpha g_{\mu\nu,\gamma} \quad (32)$$

which equals zero from (29). Therefore $\delta_0^\alpha$ is a killing vector field for the stationary spherically symmetric space-time.■
Consider the conservation law

\[ T_{\mu\nu} = 0 \]  \hspace{1cm} (33)

Where \( T \) is the stress energy tensor. We cannot integrate over this as we did in the previous section since we would not be integrating over a scalar (due to the presence of a free index). Therefore in general there is no Gauss’ law for tensor fields of rank two or higher. If we can find a killing vector field in the space we can use the Killing equation to form the following equation:

\[(k_\mu T^{\mu\nu})_{;\nu} = k_\mu T^{\mu\nu} + k_\mu T^{\mu\nu}_{;\nu} = 0 \]  \hspace{1cm} (34)

(note that the second term equals zero from (27) and therefore the first term equals zero as well). We then proceed as follows:

\[(k_\mu T^{\mu\nu})_{;\nu} = 0 = J^{\nu}_{;\nu} \]  \hspace{1cm} (35)

to which we can apply Gauss’ law as before.

\[ \int (\sqrt{g} J^{\nu})_{;\nu} d^4x = \oint J^{\nu} \hat{n}_\nu \sqrt{g} d^3x \]  \hspace{1cm} (36)

**The Energy of a Scalar Field:**

The Einstein field equations can be written in mixed form as

\[ 8\pi \left( T^{\mu}_{\nu} - \frac{1}{2} \delta^{\mu}_{\nu} T \right) = R^{\mu}_{\nu}. \]  \hspace{1cm} (37)

If we choose a time-like killing vector \( k_\mu (t) = \frac{\partial}{\partial t} \), we can form a mass integral of the form

\[ M = - \frac{1}{16\pi} \int (T^{\mu}_{\nu} - \delta^{\mu}_{\nu} T) k^{\nu}_{(t)} dS_{\mu}. \]  \hspace{1cm} (38)

Where \( dS_{\mu} = \hat{n}_\mu \sqrt{g} d^3x \) has the following components:

\[ \hat{n}_\mu d^3x = (dx^1 dx^2 dx^3, dx^0 dx^2 dx^3, dx^0 dx^1 dx^3, dx^0 dx^1 dx^2) \].  \hspace{1cm} (39)

Equation (38) can be integrated to give the energy of the scalar field by noting that the scalar field has the same stress-energy tensor as a pressure=density perfect fluid. The stress-energy tensor of the real scalar field can also be written as

\[ T^{\alpha\beta} = (1/4\pi) \left( \phi^{;\alpha} \phi^{;\beta} - \frac{1}{2} g^{\alpha\beta} \phi^{;\gamma} \phi^{;\gamma} \right). \]  \hspace{1cm} (40)