Degenerate Integrability of Spin Calogero-Moser Systems and the duality with the spin Ruijsenaars systems

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Abstract

It is shown that spin Calogero-Moser systems are completely integrable in a sense of degenerate integrability. Their Liouville tori have dimension less then half of the dimension of the phase space. It is also shown that rational spin Ruijsenaars systems are degenerately integrable and dual to spin Calogero-Moser systems in a sense that action-angle variables of one are angle-action variables of the other.

1 Introduction

Let \( \mathfrak{g} \) be a simple complex Lie algebra of rank \( r \). Fix a Borel subalgebra \( \mathfrak{b} \). Let \( \mathfrak{h} \) be the corresponding Cartan subalgebra of \( \mathfrak{g} \) and \( \Delta \) be the root system of \( \mathfrak{g} \). Choose an enumeration of simple roots of \( \Delta \). Denote by \( x_\alpha, x_{-\alpha}, p_i, \alpha \in \Delta_+, i = 1, \ldots, r = \text{rank}(\mathfrak{g}) \) the coordinates on \( \mathfrak{g}^* \) corresponding to the Chevalley basis \( H_i, X_\alpha \) in \( \mathfrak{g} \) (these coordinates correspond to positive roots, negative roots and simple roots considered as elements of the Cartan subalgebra respectively). The functions \( x_\alpha, x_{-\alpha}, p_i \) generate the Poisson algebra of polynomial functions on \( \mathfrak{g}^* \) with the following determining relations:

\[
\begin{align*}
\{p_i, p_j\} &= 0 \\
\{p_i, x_\alpha\} &= a_{ij} x_{\alpha_j} \\
\{p_i, x_{-\alpha}\} &= -a_{ij} x_{-\alpha_j} \\
\{x_\alpha, x_{-\alpha}\} &= \delta_{ij} p_i \\
\{x_\alpha, \ldots, x_{\alpha_i}, x_{-\alpha_i}, \ldots\} &= 0, i \neq j \\
\{x_{-\alpha}, \ldots, x_{-\alpha_i}, x_{-\alpha_j}, \ldots\} &= 0, i \neq j 
\end{align*}
\]
Let $\mathcal{O}$ be a regular co-adjoint orbit in $\mathfrak{g}^*$ (i.e., an orbit through a regular semisimple element). It has a natural symplectic structure. Since the co-adjoint action of $H$ is Hamiltonian the quotient space $\mathcal{O}/Ad^*_H$ is a Poisson manifold. It naturally decomposes into the product of Poisson manifolds $\mathfrak{h}^* \times \mathcal{O}'$ where $\mathcal{O}'$ is symplectic with the symplectic structure obtained via the Hamiltonian reduction (with respect to $0 \in \mathfrak{h}$) and the Poisson structure on $\mathfrak{h}^*$ is trivial.

Let $H_{\text{reg}}$ be the regular part of the Cartan subgroup in $G$. Spin Calogero-Moser corresponding to the orbit $\mathcal{O}$ is the Hamiltonian system on the symplectic manifold $T^*(H_{\text{reg}}) \times \mathcal{O}'$.

Let $\gamma_\alpha$ be a coordinate function on $H$ corresponding to the root $\alpha$ and $p_i$ be the coordinate function on $\mathfrak{h}^* \subset \mathcal{O}/Ad_H$ corresponding to the simple root $\alpha_i$. We have the following Poisson brackets between these coordinate functions:

$$\{p_i, p_j\} = 0, \quad \{\gamma_\alpha, \gamma_\beta\} = 0, \quad \{p_i, \gamma_\alpha\} = (\alpha, \alpha_i) \gamma_\alpha$$

where $(.,.)$ is the Killing form on $\mathfrak{g}$ restricted to $\mathfrak{h}$ and normalized as below.

The Hamiltonian of the spin Calogero-Moser system is the following function on $T^*(H_{\text{reg}}) \times \mathcal{O}'$:

$$H_{CM} = (p, p)/2 + \sum_{\alpha \in \Delta_+} (\alpha, \alpha) \mu_{\alpha} \mu_{-\alpha} / (\gamma_{\alpha/2} - \gamma^{-1}_{\alpha/2})^2$$

Here $(.,.)$ is the bilinear form on $\mathfrak{h}^*$ induced by the Killing form $(p, p) = \sum_{ij} p_i p_j (b^{-1})_{ij}$ where $b_{ij}$ is the symmetrized Cartan matrix, $p_i$ and $\gamma_\alpha$ are coordinates on $T^*H$ described above and the product $\mu_{\alpha} \mu_{-\alpha}$ is a function on $\mathcal{O}'$ which is the product of coordinate functions $\mu_\alpha$ and $\mu_{-\alpha}$ on $\mathcal{O}$ corresponding to roots $\alpha$ and $-\alpha$ respectively.

The normalizer $N(H) \subset G$ of $H$ acts by conjugations on $\mathcal{O}$. This induces the natural action of the Weyl group $W$ on the quotient $\mathcal{O}/Ad^*_H$. The natural Poisson structure on this space is $W$-invariant. This action of $W$ descends to the reduced space $\mathcal{O}'$. It is clear that the Hamiltonian of the spin Calogero-Moser system is invariant with respect to the diagonal action of $W$ on $T^*(H_{\text{reg}}) \times \mathcal{O}'$. Thus, effectively the phase space of the Calogero-Moser system is $(T^*(H_{\text{reg}}) \times \mathcal{O}')/W$. We will call it effective phase space. This variety is not smooth. Desingularization of this variety is an interesting problem, which was studied a lot in the case of ”usual” (non-spin) Calogero-Moser systems in [BW] [EG] (see also references therein).

The complex holomorphic Hamiltonian system described above has two natural real forms.
Compact real form correspond to $p_i$ being real, $|\gamma_\alpha| = 1$ and $\mu_\alpha = -\bar{\mu}_{-\alpha}$.

Split real form correspond to real $p_i$, $\mu_\alpha$ and $\mu_{-\alpha}$ and to $\gamma_{\alpha/2} > 0$. In this case the phase space is $(T^*\mathbb{R}_0^r \times \mathcal{O}^\prime)/W$. It is naturally isomorphic to $\Delta \times \mathbb{R}_0^r \times \mathcal{O}^\prime$, where $\Delta \subset \mathbb{R}^r$ is the fundamental domain of the Weyl group $W$.

Rational spin Calogero-Moser systems (for $sl_n$) were introduced and studied in [GH84]. The system (2) and its elliptic generalization has been studied for $sl_n$ in [KBBT94] where its complete integrability has been established. It has been noticed [ABB96] and [LX00] that using dynamical $r$- matrices one can construct Poisson commuting Hamiltonians for spin Calogero-Moser system.

In this note we show that spin Calogero-Moser systems are completely integrable for all co-adjoint orbits. More precisely, they are degenerately integrable [N72]: Liouville tori have dimension $r = rank(\mathfrak{g})$, which is less then half of the dimension of the phase space (except for special orbits in $sl_n$ case when the system becomes the usual Calogero-Moser system).

Degenerate integrability occurs in other integrable system related to simple Lie algebras [GS97][R01]

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2 Degenerate integrability

The notion of degenerate integrability generalizes the "usual" Liouville integrability. Degenerately integrable system of rank $k$ on a $2n$-dimensional symplectic manifold is a Hamiltonian dynamical system whose trajectories are parallel to a co-isotropic fibration on the phase space with fibers of dimension $k < n$.

2.1

Let $C(\mathcal{M}_{2n})$ be the algebra of functions on a $2n$-dimensional symplectic manifold $\mathcal{M}_{2n}$. Assume that we have the following structure on it:

- $2n-k$ independent functions $J_1, \ldots, J_{2n-k}$ functions which generate Poisson subalgebra $C_J(\mathcal{M}_{2n})$. 
• $k$ independent functions $I_1, \ldots, I_k$ generating the center of the Poisson subalgebra $C_J(\mathcal{M}_{2n})$.

Geometrically, these data means that we have two Poisson projections:

$$\mathcal{M}_{2n} \xrightarrow{\psi} B^I_{2n-k} \xrightarrow{\pi} B^I_k$$

(3)

where $B^I$ and $B^J$ are Poisson varieties (level surfaces of $J_i$ and $I_i$ respectively), such that connected components of the preimage $\pi^{-1}(b) \subset B^I$ of a generic point $b \in B^I$ are symplectic leaves in $B^I$.

Let $H \in C(\mathcal{M}_{2n})$ be a function which is in the center of the Poisson subalgebra $C_J(\mathcal{M}_{2n})$ (in particular, it Poisson commutes with functions $I_i$). Let $\mathcal{M}(c_1, \ldots, c_{2n-k}) = \{x \in \mathcal{M} | J_i(x) = c_i\}$ be a level surface of functions $J_i$. We will call it generic relative to functions $I_1, \ldots, I_n$ if the form $dI_1 \wedge \cdots \wedge dI_k$ does not vanish identically on it. The following is true [N72]:

**Theorem 1**

1. Flow lines of $H$ are parallel to level surfaces of $J_i$.

2. Each connected component of a generic (relative to functions $I_i$) level surface has canonical affine structure generated by the flow lines of $I_1, \ldots, I_k$. If the connected component is compact, it is isomorphic to an $n$-dimensional torus.

3. Flow lines of $H$ are linear in this affine structure.

2.2

Let $b \in B^I$ be a generic point and $D$ be an open neighborhood of $b$. Choose the trivialization of $\pi$ over $D$:

$$\pi^{-1}(D) \simeq \pi^{-1}(b) \times D.$$ 

Choose a generic point $c \in B^I$. Let $c_0 \in \pi^{-1}(b)$ be such that with respect to this trivialization $c = (c_0, \{0\})$. Choose a neighborhood $\tilde{U} \subset \pi^{-1}(D)$ of $c$. The trivialization of $\pi$ gives an isomorphism $\tilde{U} \simeq D \times U$, where $U \subset \pi^{-1}(b)$ is a neighborhood of $c_0$. This neighborhood inherits natural symplectic structure which we will denote $\omega_U$.

Now we can trivialize $\psi$ over $\tilde{U}$. This gives an isomorphism

$$\psi^{-1}(\tilde{U}) \simeq U \times D \times T.$$ 

where $T$ is the fiber of $\psi$ over $c$. If $T$ is compact, it is isomorphic to a disjoint union of tori.

Functions $I_1, \ldots, I_k$ define a local coordinate system on $D$. Their Hamiltonian flows generate $k$ independent Hamiltonian vector fields on generic level surfaces of $\psi$. These flows define affine coordinates $\phi_1, \ldots, \phi_k$ on this level surfaces.
Theorem 2 There exists a trivialization $f : \psi^{-1}(\tilde{U}) \simeq U \times D \times T$ such that the symplectic form $\omega$ on $\mathcal{M}$ has the form

$$\omega = f^* \left( \sum_{i=1}^{k} dI_i \wedge d\phi_i + \omega_U \right)$$

where $I_i$ are coordinates on $D$, $\phi_i$ are coordinates on $T$ induced by the Hamiltonian flows of $I_i$ and $\omega_U$ is the symplectic form on the $U$ induced by the Poisson structure on $\pi^{-1}(b)$.

The coordinates $I_i, \phi_i$ are called action-angle variables for degenerate integrable systems. If the Hamiltonian of the system is a pull-back of a function on $B^I$, its trajectories lie in $T$ and are lines in coordinates $\phi_i$:

$$\phi_i(t) = \phi_i(0) + \omega_i(H,I)t$$

where $\omega_i = \frac{\partial H}{\partial I_i}$. One can replace real smooth manifolds by complex manifolds (complex algebraic) and Poisson structures by complex holomorphic (complex algebraic) structures.

3 The Poisson variety $T^*G//Ad_G$

3.1

Let $G$ be a simple Lie group and $(.,.)$ be a Killing form on the corresponding Lie algebra $\mathfrak{g}$. We will fix its choice throughout this paper. It fixes a linear isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$. Let $T^*G$ be the cotangent bundle to $G$. We can trivialize it by left translations as $T^*G \simeq \mathfrak{g}^* \times G$. We will fix this trivialization as well as the corresponding trivialization of the tangent bundle throughout the paper.

The adjoint action of the Lie group $G$ on $G$ extends naturally on $T^*G$. Let $(x, \gamma) \in \mathfrak{g}^* \times G$ be a point in the cotangent bundle, then

$$h : x \mapsto \text{Ad}^*_h(x), \quad \gamma \mapsto h\gamma h^{-1}.$$ 

This action is Hamiltonian with the moment map $\mu : T^*G \rightarrow \mathfrak{g}^*$

$$\mu(x, \gamma) = x - \text{Ad}^*_\gamma(x)$$

Let $T^*G//Ad_G$ be the corresponding categorical quotient (the spectrum of the ring of $G$-invariant functions on $T^*G$). Since the action of $G$ is Hamiltonian we have

Proposition 1 The variety $T^*G//Ad_G$ has natural Poisson structure.
Symplectic leaves of $T^*G/\text{Ad}G$ can be described using the moment map. Let $\mathcal{O}_\mu$ be the coadjoint orbit in $\mathfrak{g}^*$ passing through generic semisimple point $\mu$. The symplectic leaf in $T^*G/\text{Ad}G$ passing through $[(x,\gamma)]$ is $\mu^{-1}(\mathcal{O}_{\mu(x,\gamma)})//\text{Ad}G$. Its dimension is
\[
\dim(S^{[(x,\gamma)]}) = \dim(T^*G/\text{Ad}G) - \dim(G_{\mu(x,\gamma)}) = \dim(\mathfrak{g}) - \dim(G_{\mu(x,\gamma)}) = \dim(\mathcal{O})
\]
where $G_a \subset G$ is the stabilizer of $a \in \mathfrak{g}^*$ with respect to the $\text{Ad}^*_G$-action. We will denote the symplectic leaf in $T^*G/\text{Ad}G$ corresponding to the co-adjoint orbit $\mathcal{O}$ by $S_\mathcal{O}$.

3.2

Here we will describe the effective phase space of the spin Calogero-Moser system corresponding to the coadjoint orbit $\mathcal{O}$ as an open dense subvariety in $S_\mathcal{O}$. Let $G_{\text{reg}}$ be the subset of regular elements in $G$.

Consider the intersection $\mu^{-1}(\mathcal{O})_{\text{reg}} = \mathfrak{g}^* \times G_{\text{reg}} \cap \mu^{-1}(\mathcal{O})$. It consists of points $(x,\gamma) \in \mathfrak{g}^* \times G_{\text{reg}}$ such that
\[
x - \text{Ad}_{\gamma}(x) = \mu \in \mathcal{O}.
\]
By the adjoint $G$-action we can always bring a regular $\gamma$ to the form $\gamma \in H_{\text{reg}} \subset G_{\text{reg}}$. Assume that $\gamma$ is such. Then for Chevalley coordinates of $x$ and $\mu$ we have:
\[
(1 - \gamma_\alpha)x_\alpha = \mu_\alpha, \quad \alpha \in \Delta
\]
\[
\mu_i = 0, \quad i = 1, \ldots, r.
\]
Cartan coordinates of $\mu$ vanish:
\[
\mu_i = 0, \quad i = 1, \ldots, r
\]
Thus, on $\mu^{-1}(\mathcal{O})_{\text{reg}}$ we can choose coordinates $\mu_\alpha, \alpha \in \Delta$, $\gamma_i$ and $p_i$, $i = 1, \ldots, r$. It is clear that $\mu^{-1}(\mathcal{O})_{\text{reg}} \subset \mu^{-1}(\mathcal{O})$ is invariant with respect to the adjoint $G$-action.

Define the regular part of the symplectic leaf $S_\mathcal{O} = \mu^{-1}(\mathcal{O})//\text{Ad}G$ as $(S_\mathcal{O})_{\text{reg}} = \mu^{-1}(\mathcal{O})_{\text{reg}}//\text{Ad}G$. It is clear that $(S_\mathcal{O})_{\text{reg}}$ is open dense in $S_\mathcal{O}$. Taking the corresponding cross-section of the $\text{Ad}G$-orbit in $T^*G$ we obtain the following statement.

**Theorem 3** We have the isomorphism of symplectic varieties
\[
(S_\mathcal{O})_{\text{reg}} \simeq (T^*H_{\text{reg}} \times \mathcal{O}')/W
\]
where the Weyl group acts diagonally on factors and its action on $\mathcal{O}'$ is described in the introduction.
Let us describe the coordinate ring $C(\mathcal{O}')$. As the $\text{Ad}_G$-module the ring $\text{Pol}(\mathfrak{g}^*) = \mathbb{C}[x_\alpha, p_i]$ of polynomial functions on $\mathfrak{g}^*$ has the weight decomposition with $\text{wt}(x_\alpha) = \alpha$, $\text{wt}(p_i) = 0$ and $\text{wt}(ab) = \text{wt}(a) + \text{wt}(b)$.

**Proposition 2**

- Polynomials of zero weight form the Poisson subalgebra $\text{Pol}_0(\mathfrak{g}^*) \subset \text{Pol}(\mathfrak{g}^*)$.
- Functions $\{p_i\}_{i=1}^r$ and $\text{Ad}_G^\ast$-invariant functions generate the center of this Poisson subalgebra.
- There is an isomorphism of Poisson algebra $\text{Pol}_0(\mathfrak{g}^*) \simeq \text{Pol}(\mathfrak{h}^*) \otimes \mathbb{C} A$, where $A$ is the commutative ring generated by monomials $x_{\beta_1} \ldots x_{\beta_n}$ with $\sum_i \beta_i = 0$, $\beta_i \in \Delta$. The Poisson structure on the ring $\text{Pol}(\mathfrak{h}^*)$ is trivial and the ring $A$ is isomorphic to $C(\mathcal{O}')$ as a Poisson algebra.

## 4 The integrability of spin Calogero-Moser systems

In this section we construct the pair of Poisson maps as in (3) in such a way that the Hamiltonians of spin Calogero- Moser systems (2) will be pull-backs from functions on the base $B_I$. According to the section 2.1 this will prove the integrability of the system.

### 4.1

Consider the map

$$\psi' : T^*G \simeq \mathfrak{g}^* \times G \rightarrow \mathfrak{g}^* \times \mathfrak{g}^*$$

$$\psi'(x, \gamma) = (x, -\text{Ad}_G^\ast(x))$$

This map is the product of two moment maps, $\mu_{L,R} : T^*G \rightarrow \mathfrak{g}^*$ $\mu_L(x, \gamma) = x$, $\mu_R(x, \gamma) = -\text{Ad}_G^\ast(x)$ for left and right action of $G$ on $T^*G$ respectively.

It is clear that this map is invariant with respect to the natural $G$-action and it induces the map

$$\psi : T^*G/\text{Ad}_G \rightarrow (\mathfrak{g}^* \times \mathfrak{g}^*)/\text{Ad}_G^\ast$$

which bring $\text{Ad}_G(x, \gamma)$ to $\text{Ad}_G^\ast(x, \text{Ad}_G^\ast(x))$ assuming that $G$ acts diagonally $\mathfrak{g}^* \times \mathfrak{g}^*$ and via co-adjoint action on each of the factors.

Consider projections

$$\tilde{\pi}_{1,2} : \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$$
to the first and second factor respectively. They are Poisson maps and they define Poisson projections

$$\pi_{1,2} : (g^* \times g^*)/\text{Ad}_G \to g^*/\text{Ad}_G^* \simeq h^*/W.$$  

where \( h \) is a Cartan subalgebra of \( g \). The Poisson structure on \( g^*/\text{Ad}_G^* \) is trivial. It is also clear that

$$\pi_1 \circ \psi = \pi_2 \circ \psi.$$

Denote the image of \( \psi \) in \( (g^* \times g^*)/\text{Ad}_G \) by \( \mathcal{P}_\psi \). Thus we have maps

$$T^*G/\text{Ad}_G \xrightarrow{\psi} \mathcal{P}_\psi \xrightarrow{\pi} g^*/\text{Ad}_G^* \simeq h^*/W$$

acting as \([ (x, \gamma) ] \mapsto [(x, \text{Ad}_G^*(x))] \mapsto [x] \). Here \([A]\) is the corresponding \( G \)-orbit passing through \( A \).

### 4.2

Let \( S_{[(x, \gamma)]} \subset T^*G/\text{Ad}_G \) be the symplectic leaf in \( T^*G/\text{Ad}_G \) through \([ (x, \gamma) ] \). As we have seen in section 3.2 for the regular part of this symplectic leaf we have the isomorphism of symplectic varieties \((S_{[(x, \gamma)]})_{\text{reg}} \simeq (T^*H \otimes \mathcal{O}_{\mu(x, \gamma)}/W \) where \( \mathcal{O}_{\mu(x, \gamma)} \) is the result of the Hamiltonian reduction of the co-adjoint orbit passing through \( \mu(x, \gamma) \) with respect to the co-adjoint action of \( H \) on this orbit.

Restrict the map \( \psi \) to the symplectic leaf. We will have

$$S_{[(x, \gamma)]} \xrightarrow{\psi} S_{\psi}^{[(x, \gamma)]} \xrightarrow{\pi} g^*/\text{Ad}_G^* \simeq h^*/W.$$  \hfill (4)

Here \( S_{\psi}^{[(x, \gamma)]} \) is the image of \( S_{[(x, \gamma)]} \) in \( (g^* \times g^*)/\text{Ad}_G^* \) with respect to the map \( \psi \).

**Theorem 4** Let \( x \in g^* \) be regular, then,

$$\dim(\psi^{-1}(\psi([x, \gamma]))) = r.$$  

**Proof.** Assume that \( \psi([x, \gamma]) = \psi([x', \gamma']) \), i.e., \( x = \text{Ad}_{y'}^*(x') \), \( \text{Ad}_y^*(x) = \text{Ad}_y^*(x') \) for some \( y \in G \). Therefore

$$x = \text{Ad}_{y^{-1}y'}^*(x'),$$

which means \( y^{-1}y'z = z \in Z_x \subset G \) where \( Z_x \) is the stabilizer of \( x \in g^* \). For \( \gamma' \) we have

$$\gamma' = y^{-1}\gamma y$$

which shows that the set of elements \([x', \gamma']\) such that \( \psi([x', \gamma']) = \psi([x, \gamma]) \) has the same dimension as \( \dim(Z_x) = r \).

Let \([h_x] \in h^*/W \) be the image of \( x \in g^* \) in \( g^*/\text{Ad}_G^* \simeq h^*/W \).
Theorem 5  For regular $x \in \mathfrak{g}^*$ the preimage $\pi^{-1}([h_x]) \subset S_{\psi}^{[(x,\gamma)]}$ is the symplectic leaf through $\psi([x,\gamma])$.

This theorem follows from the fact that the center of the Poisson algebra $C(T^*G)$ coincides with the pull-back image of the moment map of $Ad^*_G$-invariant functions on $\mathfrak{g}^*$.

Corollary 1  Symplectic leaves of $\mathfrak{g}^* \times \mathfrak{g}^*$ are direct products of coadjoint orbits. The diagonal action of $G$ is Hamiltonian with the moment map $\mu : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$, $\mu(x,y) = x + y$. Thus, each symplectic leaf of $(\mathfrak{g}^* \times \mathfrak{g}^*)/Ad_G$ is the intersection of $\mu^{-1}(a \text{ coadjoint orbit})$ with the product of two coadjoint orbits.

The image of $\psi$ consists of elements $(x, y) \in \mathfrak{g}^* \times \mathfrak{g}^*$ satisfying $Ad^*_G(x) = Ad^*_G(y)$. Thus, symplectic leaves of $\psi(T^*G)$ are intersections of preimages of coadjoint orbits with respect to the moment map and preimages of points in $\mathfrak{g}^*/Ad^*_G$ with respect to the map $\pi$. Thus $\pi^{-1}([h_x])$ is a symplectic leaf in $S_{\psi}^{[(x,\gamma)]}$.

Corollary 2  A Hamiltonian system on $(S_{[(x,\gamma)]})_{\text{reg}}$ generated by the pull-back of an $Ad_G$-invariant function on $\mathfrak{g}^*$ is integrable (degenerately). The dimension of its Liouville tori is $r = \text{rank}(G)$.

4.3

Let us derive the Hamiltonian (2) as the pull-back of a function on $\mathfrak{g}^*/Ad^*_G$ with respect to the maps in (3). Consider the function

$$H(x, \gamma) = \frac{1}{2}(x,x), x \in \mathfrak{g}^*, \gamma \in G$$

(5)

on $T^*G \simeq \mathfrak{g}^* \times G$ (on the cotangent bundle to $G$ trivialized by left translations).

Consider its pull-back in the algebra of functions on the symplectic leaf $S_{\mathcal{O}} = \mu^{-1}(\mathcal{O}/Ad_G)$.

Consider the restriction of this function to $(S_{\mathcal{O}})_{\text{reg}}$. The ring of functions on $(S_{\mathcal{O}})$ is generated by coordinate functions $p_i$, $\gamma_i$ with $\gamma_\alpha \neq 0$ and zero weight monomials in $\mu_\alpha$. Projecting $(x, \gamma) \in T^*G$ on this intersection we have

$$x = \sum_i p_i H_i + \sum_{\alpha \in \Delta} (1 - \gamma_\alpha)^{-1} \mu_\alpha X_\alpha$$

Here $H_i, X_\alpha$ is the Chevalley basis in the Lie algebra $\mathfrak{g}$ (we assume the identification of $\mathfrak{g}$ with its dual by the Killing form). Substitute this expression into (3) and we obtain the Hamiltonian of the Calogero-Moser system. This, together with the previous section proves the integrability of the Calogero-Moser system corresponding to the co-adjoint orbit $\mathcal{O}$. 
4.4

Consider (here we essentially follow [KKS]) a co-adjoint orbit through a semisimple element of minimal nonzero dimension for $\mathfrak{g} = \mathfrak{sl}_n$. If $\mu_{ij}$ are matrix coordinates on $\mathfrak{sl}_n$ such orbit can be described by coordinates $\phi_i, \psi_i, i = 1, \ldots, n$ with

$$\mu_{ij} = \phi_i \psi_j - \delta_{ij} \frac{1}{n} < \phi, \psi >$$

where $< \phi, \psi > = \sum_{i=1}^{n} \phi_i, \psi_i$. All central functions on such orbits are polynomials in the quadratic Casimir

$$c = \sum_{ij} \mu_{ij} \mu_{ji} = < \phi, \psi >^2 - 2 \frac{< \phi, \psi >^2}{n} + \frac{1}{n} < \phi, \psi >^2 = \frac{n-1}{n} < \phi, \psi >^2.$$

The constraint $\mu_{ii} = 0$ gives $\phi_i \psi_i = \frac{< \phi, \psi >}{n}$ and therefore the reduced manifold $\mathcal{O}'$ is zero dimensional with

$$\mu_{ij} \mu_{ji} = < \phi_i, \psi_i > < \phi_j, \psi_j > = \frac{< \phi, \psi >^2}{n^2} = \frac{c}{n(n-1)}.$$ 

Thus, for such orbits in $\mathfrak{sl}_n^*$ the Hamiltonian of the spin Calogero-Moser system becomes the usual Calogero-Moser Hamiltonian

$$H = < p, p > /2 + \frac{c}{4n(n-1)} \sum_{i<j} \frac{1}{\sinh(q_i - q_j)^2}$$

where $p = (p_1, \ldots, p_n)$, $\{p_i, q_i\} = \delta_{ij}$ and $\gamma_{e_i - e_j} = \exp(2q_i - 2q_j)$.

The symmetric group $S_n$ (the Weyl group of $\mathfrak{sl}_n$) acts on $(p, q)$ naturally by permutations and the Hamiltonian of Calogero-Moser system is invariant with respect to this action.

5 Degenerate integrability of rational spin Ruijsenaars systems

A duality relation between spin Calogero-Moser systems and systems which we will call rational spin Ruijsenaars systems was observed in [95] (see also references therein). We will define rational spin Ruijsenaars systems below and then will give examples of such systems for $SL_n$. In this section we will show that rational spin Ruijsenaars systems are degenerately integrable in a very similar way to spin Calogero-Moser systems, i.e. we will construct a pair of Poisson projections as in (3) and we will show that Hamiltonians of rational spin Ruijsenaars systems are pull-backs from the base of the last projection.
The algebra of polynomial functions on $T^*G \simeq \mathfrak{g}^* \times G$ is the tensor product of $\text{Pol}(\mathfrak{g}^*) \otimes C(G)$ as a commutative algebra. Consider the following Poisson structure on $T^*G$.

- The subalgebras $\text{Pol}(\mathfrak{g}^*)$ and $C(G)$ are Poisson subalgebras.
- Poisson brackets between a linear function $X \in \mathfrak{g}$ on $\mathfrak{g}^*$ and $f \in C(G)$ is
  $$\{X, f\} = (L_X - R_X)f$$
  where $L_X$ and $R_X$ are left and right invariant vector fields on $G$ generated by $X$.

To distinguish this Poisson structure from the standard symplectic structure on the cotangent bundle to a manifold we will write $(T^*G, p)$ for it.

The adjoint action of the group $G$ on $(T^*G, p)$ is Poisson. There is a natural Poisson map $\tilde{\psi} : T^*G \to (T^*G, p)$ acting as $(x, \gamma) \to (x - \text{Ad}\gamma(x), \gamma)$.

**Remark 1** We have maps $\psi' : T^*G \to \mathfrak{g}^* \times \mathfrak{g}^*$ and $\tilde{\psi}' : T^*G \to \mathfrak{g}^* \times G$. It is interesting that we have natural isomorphisms $\text{ker}(\psi') \simeq \text{ker}(\tilde{\psi}') \simeq C$ where $C = \{(x, \gamma) \in T^*G | x = \text{Ad}_\gamma(x)\}$. This variety plays an important role in representation theory of real reductive groups [K] (and an unpublished paper by B. Kostant on a characteristic variety in $T^*G$).

The map $\psi'$ induces the Poisson isomorphism
$$\tilde{\psi} : T^*G//\text{Ad} G \to (T^*G, p)//\text{Ad} G .$$

We also have a natural projection
$$\tilde{\pi} : (T^*G, p)//\text{Ad} G \to G//\text{Ad} G .$$

This projection is Poisson with the trivial Poisson structure on the base.

Restricting $\tilde{\psi}$ to a symplectic leaf of $T^*G//\text{Ad} G$ we have
$$S \stackrel{\tilde{\psi}}{\hookrightarrow} \tilde{\psi}(S) \subset (T^*G, p)//\text{Ad} G \stackrel{\tilde{\pi}}{\to} G//\text{Ad} G .$$

**Proposition 3** Over generic point $\dim(\ker \tilde{\psi}) = r$.

**Proof.** Assume that $\tilde{\psi}([(x, \gamma)]) = \tilde{\psi}([(x', \gamma')])$, i.e., that there exists $y \in G$ such that
$$\text{Ad}^*y(\mu(x, \gamma)) = \mu(x', \gamma'), \quad \text{Ad}(\mathfrak{g}) = \mathfrak{g}' .$$
The first equation implies

\[ \text{Ad}_y^*(x) - \text{Ad}_y^*\text{Ad}_\gamma^*(x) = x' - \text{Ad}_{\gamma'}^*(x) \]

which, together with the second one, means

\[ x - x' = \text{Ad}_{\gamma'}^*(x - x') . \]

From here it follows that the variety of points \([(x, \gamma)]\) such that \(\tilde{\psi}^*([((x, \gamma)]) = \tilde{\psi}([x', \gamma])\) has dimension \(r\).

It is clear that \(\dim(G/\text{Ad} G) \simeq H/W\) = \(r\) and therefore we have the pair of Poisson projections

\[ S \xrightarrow{\tilde{\psi}} \tilde{\psi}(S) \xrightarrow{\pi} G/\text{Ad} G \]

which satisfy conditions described in section 2.1. In particular \(\dim(S) = \dim(\tilde{\psi}(S)) + \dim(G/\text{Ad} G)\).

**Proposition 4** For generic orbit \([h] \in G/\text{Ad} G, \ \pi^{-1}([h])\) is a symplectic leaf in \(\tilde{\psi}(S)\).

**Proof.** The projection of a Hamiltonian flow line on \((T^*G, p)\) to \(g^*\) is parallel to \(\text{Ad}_G^*\) action and the projection of a Hamiltonian flow line to \(G\) is parallel to \(\text{Ad}_G\)-action. Therefore symplectic leaves in \((T^*G, p)\) are products \(O \times C\) where \(O \subset g^*\) is a coadjoint orbit, \(C \subset G\) is a conjugacy class. For the same reason symplectic leaves in \((T^*G, p)/\text{Ad} G\) are \(O \times_G C\) and therefore they project to points on \(G/\text{Ad} G\).

**Corollary 3** Any Hamiltonian flow on a symplectic leaf \(S \subset T^*G/\text{Ad} G\) which is generated by the pull-back of an \(\text{Ad}_G\)-invariant function on \(G\) is degenerately integrable.

Let \(\omega_i\) be a fundamental weight of \(g\). We will call functions \(H_i = \tilde{\psi}^* \circ \tilde{\pi}^*(\chi_\omega_i)\) Hamiltonians of rational spin Ruijsenaars systems because for rank = 1 adjoint orbit in \(SL_n\) it coincides with the rational Ruijsenaars systems [N95](see below). We just proved that all these Hamiltonians are degenerately integrable.

**5.2**

Consider the special case \(G = SL_n\) and the coadjoint orbit of \(rank = 1\). Assume \(x \in sl_n^*\) is semisimple and choose a representative \((h, g)\) of the \(SL_n\)-orbit through \((x, \gamma)\) in which \(h\)
diagonal. As in case of spin Calogero-Moser system let us express $g$ in terms of $x$ and the value of the moment map, i.e. let us solve the equation

$$(h_i - h_j)g_{ij} = \sum_{k=1}^{n} \mu_{ik} g_{kj}. \tag{6}$$

for the non-diagonal elements of $g$.

For $\mu_{ij}$ on this orbit we have:

$$\mu_{ij} = \phi_i \psi_j - \delta_{ij} \frac{1}{n} < \phi, \psi >$$

Then the equation (6) can be written as

$$(h_i - h_j)g_{ij} = \phi_i \sum_k \psi_k g_{kj} - \kappa g_{ij} \tag{7}$$

where $\kappa = < \phi, \psi > / n$. From this equation we obtain the identity

$$g_{ij} = \frac{1}{h_i - h_j + \kappa} \phi_i \sum_k \psi_k g_{kj} \tag{7}$$

This gives the system of equations for $\psi_i \phi_i$

$$\sum_{i=1}^{n} \frac{\phi_i \psi_i}{h_i - h_j + \kappa} = 1$$

and the identity

$$g_{ii} = \frac{\phi_i}{\kappa} \sum_{k=1}^{n} \psi_k g_{ki}$$

This, together with the equation (7) implies

$$g_{ij} = \frac{\phi_i \phi_j^{-1} \kappa g_{jj}}{h_i - h_j + \kappa}.$$ 

The first two elementary $G$-invariant functions of $g$ are

$$\text{tr}(g) = \sum_{i=1}^{n} g_{ii},$$

$$\text{tr}(g^2) = \kappa^2 \sum_{ij} g_{ii} g_{jj} \frac{1}{(h_i - h_j + \kappa)(h_j - h_i + \kappa)}.$$ 

The reduced Poisson brackets are $\{h_i, g_{jj}\} = \delta_{ij} g_{jj}$. The second function is the Hamiltonian of the rational Ruijsenaars system.
5.3

Let us show that in a neighborhood of the identity in $G$ rational spin Ruijsenaars systems degenerate to rational Calogero-Moser systems.

First, identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using the Killing form. Then assume $x \in \mathfrak{g}$ is regular and choose a representative $(h, g)$ of the $G$-orbit in $T^*G$ through $(x, \gamma)$ with $h \in \mathfrak{h}$. Assume that $g$ is in a neighborhood of $e \in G$ which is the image of the exponential map and let $g = \exp(\epsilon)$. Denote by $\epsilon_\alpha, \epsilon_i \alpha \in \Delta, i = 1, \ldots, r$ the Chevalley coordinates of $\epsilon$.

Coordinates $\epsilon_\alpha$ can be expressed as a power series in $\mu$ and $h$. Indeed, we have

$$h - \exp(ad_\epsilon)(h) = \mu$$

It is clear that this equation does not define the Cartan coordinates $\epsilon_i$. It is also clear that $\epsilon_\alpha$ can be expressed as a power series in monomials $\mu_{\beta_1} \ldots \mu_{\beta_i}$ with $\sum_i \beta_i = \alpha$ such that

$$\epsilon_\alpha = \frac{\mu_\alpha}{\alpha(h)} + O\left(\frac{h^2}{h}\right)$$

The Cartan part of $\mu$ can be expressed in terms of $\mu_\alpha$ and $h$.

Thus, any polynomial $Ad_G$ invariant function of $\exp(\epsilon)$ is a power series in $\mu$ consisting of the monomials $\mu_{\beta_1} \ldots \mu_{\beta_i}$ with $\sum_i \beta_i = 0$.

In particular, for any irreducible finite dimensional representation $V$ we have

$$tr_V(g) = dim(V) + \frac{c_2(V)dim(V)}{dim(\mathfrak{g})} \left(\frac{1}{2}(\epsilon^{(c)}, \epsilon^{(c)}) + \sum_{\alpha \in \Delta_+} (\alpha, \alpha) \frac{\mu_\alpha \mu_{-\alpha}}{\alpha(h)}\right) + \ldots$$

where $\epsilon^{(c)} = \sum_{i=1}^r \epsilon_i H_i$ is the Cartan part of $\epsilon$ and $(.,.)_h$ is the restriction of the Killing form to $\mathfrak{h}$. The second term in this expression is the Hamiltonian of the rational spin Calogero-Moser system.

Therefore, in a small neighborhood of the identity in $G$ spin rational Ruijsenaars system degenerates to the rational Calogero-Moser system.

6 Action-angle variables and the duality between rational spin Ruijsenaars and spin Calogero-Moser systems

6.1 Action-angle variables for spin Calogero-Moser systems
6.1.1

The degenerate integrability of spin Calogero-Moser systems was described earlier by two projections:

\[ S \subset T^*G//Ad_G \xrightarrow{\psi} (\mathfrak{g}^* \times \mathfrak{g}^*)/Ad_G \xrightarrow{\pi} \mathfrak{g}^*/Ad^*_G \simeq \mathfrak{h}^*/W \]

where \( S \) is the phase space of the system, which is a subvariety in a symplectic leaf of \( T^*G//Ad_G \).

Action variables for a spin Calogero-Moser system are affine coordinates on \( \text{Spec}(C_G(\mathfrak{g}^*)) \simeq \mathfrak{h}^*/W \) (the base of the projection \( \pi \)). Any linear basis in \( \mathfrak{h}^* \) determines such coordinate system. For convenience we will choose these functions to be dual to simple roots \( \alpha_i \in \mathfrak{h}^* \).

Angle variables, by definition, are affine coordinates on a fiber over generic point of the projection \( \psi \) which are generated by Hamiltonian flows of pull-backs of action variables. In these coordinates the Hamiltonian flow generated by the pull-back of a function on \( \mathfrak{h}^*/W \) is linear.

Denote by \( F_{(x,\gamma)} \) the fiber of \( \psi \) which contains the orbit \( Ad_G(x, \gamma) \). The following is clear.

**Lemma 1** We have:

\[ F_{(x,\gamma)} = Ad_G(x, Z_x \gamma) \]

where \( Z_x = \{ g \in G | Ad^*_g(x) = x \} \) is the stabilizer of the point \( x \).

The Hamiltonian flow generated by the pull-back of the function \( f \in C_G(\mathfrak{g}) \) on \( T^*G \) through the point \( (x, \gamma) \) has the form

\[ (x_t, \gamma_t) = (x, \exp(tdf(x)\gamma) \tag{8} \]

where \( df(x) \in \mathfrak{g} \) is the differential of \( f \) at \( x \in \mathfrak{g}^* \). It is clear that \( F_{(x,\gamma)} \) is invariant with respect to (8). Points in \( F_{(x,\gamma)} \) are parameterized by \( Z_x \) as \( (x, z\gamma) \) where \( z \in Z_x \). The Hamiltonian flow through such point generated by the pull-back \( f \) of a function from \( C_G(\mathfrak{g}^*) \) is \( z_t = \exp(tdf(x)z) \).

For regular \( x \in \mathfrak{g}^* \) we have the isomorphism of Lie groups \( Z_x \simeq H \). Thus, the same linear coordinate system on \( \mathfrak{h} \) which gives the action variables, after exponentiating gives angle variables on the fiber \( F_{(x,\gamma)} \) for regular \( x \).

6.1.2

Now let us describe how angle variables constructed above can be expressed in terms of coordinates \( p_i, \gamma_\alpha, \mu_\alpha \). Consider the space of \( Ad_G \)-orbits in \( T^*G//Ad_G \) through points \( (x, \gamma) \)
with regular $\gamma$. For the coordinate description of Hamiltonians of spin Calogero-Moser systems we have chosen the cross-section $\{(x, \gamma)| x \in \mathfrak{g}, \gamma \in H_{reg}\}$.

Let us identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the Killing form, then in coordinates we have used in section 4, for $(x, \gamma)$ we have:

$$x = \sum_{i=1}^{r} p_i H_i + \sum_{\alpha \in \Delta} \frac{\mu_\alpha}{1 - \gamma_\alpha} X_\alpha, \quad \gamma = (h_{\alpha_1}, \ldots, h_{\alpha_r})$$

If $x$ is semisimple there exists $x_0 \in \mathfrak{h}$ and $u \in G$ such that

$$x = ux_0 u^{-1} \quad (9)$$

Assume that the element $u^{-1}\gamma u$ has the Gaussian decomposition:

$$u^{-1}\gamma u = b_+ b_- b_0 \quad (10)$$

where $b_\pm \in N_\pm$, $b_0 \in H$.

It is clear that the decomposition (9) is defined modulo the action

$$v : (u, x_0) \to (uv, v^{-1} x_0 v) \quad (11)$$

of $N(H)$ on pairs $(u, x_0)$. It is also clear that

- $b_0$ depends only on $\text{Ad}_{N(H)}(x, \gamma)$,
- the action (11) brings $b_0$ to $vb_0v^{-1}$,
- the Hamiltonian flow generated by the pull-back of $f \in C_G(\mathfrak{g}^*) \simeq C_W(\mathfrak{h}^*)$ evolve $b_0$ as $b_0^t = b_0 \exp(tdf(x_0))$. Here $df$ is the differential of $f \in C_W(\mathfrak{h}^*)$ regarded as an element in $\mathfrak{h}$.

Thus, $b_0$ are angle variables for spin Calogero-Moser system and the equations (9) and (10) describe angle variables in terms of initial $(x, \gamma, \mu)$ coordinates.

### 6.2 Action-angle variables for Rational Ruijsenaars systems

The degenerate integrability of rational spin Ruijsenaars systems is described by two projections

$$S \subset T^*G//\text{Ad}_G \xrightarrow{\tilde{\psi}} \tilde{\psi}(S) \subset (T^*G, p)//\text{Ad}_G \xrightarrow{\tilde{\pi}} G//\text{Ad}_G$$
where $S$ is a symplectic leaf of $T^*G/\text{Ad}_G$ which is the phase space of the system.

The action variables of a rational spin Ruijsenaars system are affine coordinates on $H/W$. Such coordinates are determined by the choice of a linear basis in the Lie algebra $\mathfrak{h} = \text{Lie}(H)$.

By definition of angle variables, they are affine coordinates on fibers over generic points of the projection $\tilde{\psi}$ which are generated by Hamiltonian flows of pull-backs of action variables.

Denote by $\tilde{F}(x, \gamma)$ the fiber of $\tilde{\psi}$ which contains the orbit $\text{Ad}_G(x, \gamma)$.

**Proposition 5** We have:

$$\tilde{F}(x, \gamma) = \text{Ad}_G(x + C_\gamma, \gamma)$$

where $C_\gamma = \{x \in \mathfrak{h}^*|\text{Ad}_\gamma^*(x) = x\}$.

The Hamiltonian flow generated by the function $f$ which is the pull-back of a function on $G/\text{Ad}_G$ has the form:

$$(x_t, \gamma_t) = (x + tf(\gamma), \gamma)$$

For simple $\gamma \in G$ we have the isomorphism of vector spaces $C_\gamma \simeq \mathfrak{h}^*$. Thus if we choose action variables by fixing a linear basis on $\mathfrak{h}$, the dual basis in $C_\gamma \simeq \mathfrak{h}^*$ defines angle variables for rational spin Ruijsenaars system.

**6.3 The duality between the two systems**

Projections $\psi$ and $\tilde{\psi}$ are dual in a sense that their fibers meet exactly at one point.

**Proposition 6** Let $F(x, \gamma)$ and $\tilde{F}(x, \gamma)$ be fibers of $\psi$ and $\tilde{\psi}$ respectively which contain the orbit $\text{Ad}_G(x, \gamma)$. Then

$$F(x, \gamma) \cap \tilde{F}(x, \gamma) = \text{Ad}_G(x, \gamma)$$

Indeed, $F(x, \gamma) = \{\text{Ad}_G(x, z\gamma)|z \in \mathbb{Z}_x\}$ and $\tilde{F}(x, \gamma) = \{\text{Ad}_G(x + c, \gamma)|c \in C_\gamma\}$. It is clear the intersection of these two sets consists of one point $\text{Ad}_G(x, \gamma)$.

Consider projections $p = \pi \circ \psi$ and $\tilde{p} = \tilde{\pi} \circ \tilde{\psi}$:

$$\begin{array}{ccc}
\mathfrak{g}^*/\text{Ad}_G & \xrightarrow{\tilde{p}} & G/\text{Ad}_G \\
\downarrow p & & \\
S_O & \xrightarrow{\tilde{\pi}} & G/\text{Ad}_G \\
\end{array}$$

It is clear that for generic $(x, \gamma) \in T^*G$ there are birational isomorphisms

$$p(\tilde{F}(x, \gamma)) = \{\text{Ad}_G^*(x + c)|c \in C_\gamma\} \simeq \mathfrak{h}^*/W$$

$$\tilde{p}(F(x, \gamma)) = \{\text{Ad}_G z \gamma|z \in \mathbb{Z}_x\} \simeq H/W$$

and therefore angle variables for rational a spin Ruijsenaars system are action variables for the corresponding spin Calogero-Moser system and vice versa. In this sense the two systems are dual to each other.
6.4 Self-duality for rational rational spin Calogero-Moser system

As it was pointed out in the section 5.3 rational spin Ruijenaars system degenerates to rational Calogero-Moser system in a neighborhood of the identity in $G$. The duality between two systems described in the previous section becomes the self-duality for spin Calogero-Moser system.

Degenerate integrability of rational spin Calogero-Moser systems follows from the results of the previous sections. Let us make a short summary of what how it works. In a small neighborhood of $e \in G$ the cotangent bundle $T^*G$ can be replaced by $T^*_g = g^* \oplus g$. The construction goes as follows:

- The group $G$ acts naturally on $T^*_g = g^* \oplus g$ and this action is Hamiltonian.

- The variety $T^*_g//G$ is Poisson.

- Symplectic leaves of $T^*_g//G$ are $\mu^{-1}(O)//G$ where $O$ is a coadjoint orbit and $\mu : T^*_g \rightarrow g^*$ is the moment map for the adjoint action of $G$ on $T^*_g$.

- Projections $\psi : T^*_g//G \rightarrow (g^* \oplus g^*)//Ad_G$, $\psi(G(x,a)) = G(x, ad^*_a(x))$ and $\tilde{\psi} : T^*_g//G \rightarrow (g \oplus g^*)//G$, $\tilde{\psi}(G(x,a)) = G(a, ad^*_a(x))$ are Poisson.

- The images of $\psi$ and $\tilde{\psi}$ are isomorphic as Poisson manifolds.

- Let $\pi : (g^* \oplus g^*)//G \rightarrow g^*/Ad_G$ and $\bar{\pi} : (g \oplus g^*)//G \rightarrow g//G$ be projections to the first summand.

Similarly to how it was done above for spin Calogero-Moser systems it is easy to verify that pairs of projections $(\psi, \pi)$ and $(\tilde{\psi}, \bar{\pi})$ describe degenerate integrability of spin Calogero-Moser systems.

The duality described in the previous section now becomes the duality between one copy of rational Calogero-Moser system with another copy of this system.

7 Conclusion

As we see now, a spin Calogero-Moser system corresponding to the coadjoint orbit $O$ defines an integrable system on the symplectic manifold $S_O$ which can be regarded as a desingularization of $(T^*H_{reg} \times O)/W$.

If $O' = \{ \text{point} \}$ (this occur for $G = SL_n$ and for a non-nilpotent coadjoint orbit of dimension $n - 1$) the variety $S_O$ is the Hilbert scheme of a point. This variety and its generalizations were studied in [BW] and [EG]. It seems that less is known about varieties $S_O$ for more complicated orbits.
The results of this paper can be easily generalized to "trigonometric" spin Ruijsenaars system (spin Macdonald-Ruijsenaars) it will be done in the next paper where we also will show how to quantize such systems using harmonic analysis on simple Lie groups and corresponding quantum groups. Phase spaces of such systems are symplectic leaves of the moduli space of flat $G$-connections over punctured torus. These systems are self-dual and the duality map can be regarded as the action of one of the generators of the modular group of a punctured torus.

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