Interaction of $N$ solitons in the massive Thirring model and optical gap system: the complex Toda chain model

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Abstract

Using the Karpman-Solov’ev quasiparticle approach for soliton-soliton interaction I show that the train propagation of $N$ well separated solitons of the massive Thirring model is described by the complex Toda chain with $N$ nodes. For the optical gap system a generalised (non-integrable) complex Toda chain is derived for description of the train propagation of well separated gap solitons. These results are in favor of the recently proposed conjecture of universality of the complex Toda chain.

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I. INTRODUCTION

Recently the complex Toda chain attracted much attention as a possible candidate for description of the pulse interactions in integrable and non-integrable nonlinear evolution equations \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10\]. For instance, it was shown that the complex Toda chain describes the soliton train propagation for all the nonlinear evolution equations associated with the NLS hierarchy \[9\]. Quite recently the complex Toda chain was derived for the modified NLS equation \[10\], an integrable generalisation of the NLS equation, which is associated with the quadratic bundle.

The complex Toda chain is an integrable generalisation of the well-known real Toda chain (see, for instance, Refs. \[3, 4\]). In Refs. \[3, 4, 7\] and \[10\] the comparison of the complex Toda chain predictions with the numerical solutions of the NLS and MNLS equations has been performed and a good agreement has been established for various choices of the initial parameters of the solitons in the train.

It is noted that the complex Toda chain arises as an approximation of the evolution equations describing the inter-pulse interaction in the train comprised of well separated solitons with nearly equal amplitudes and velocities. The exponent of the (negative) separation between the solitons serves as the small parameter for the asymptotic expansion and derivation of the complex Toda chain can be based either on the variational approach (see Ref. \[8\]) or on the adiabatic perturbation theory for solitons (see, for instance, Refs. \[4, 10\]). However, as noted in Ref. \[11\] the variational approach should be used with care. The approach based on the adiabatic perturbation theory is equivalent to the Karpman-Solov’ev quasiparticle method for the two-soliton interactions \[12\]. This approach was developed in Refs. \[3, 4, 6, 9, 10\].

If the nonlinear PDE is not integrable but possesses stable soliton solutions, then the train propagation of solitons is described by a generalised (non-integrable) complex Toda chain as it is pointed out in Ref. \[8\].

The complex Toda chain allows a rich class of asymptotic regimes of the soliton train propagation \[3, 8, 11\]: \(i\) asymptotically free propagation of solitons, \(ii\) \(N\)-soliton bound states with the possibility of a quasi-equidistant propagation, \(iii\) mixed asymptotic regimes when part of the solitons form bound state(s) and the rest separate from them, \(iv\) regimes corresponding to the degenerate and singular solutions of the complex Toda chain. The
rich variety of dynamical regimes of the complex Toda chain indicates that it is a good candidate for analytical study of the soliton trains. Here I should point out that only few simple regimes are exhibited by the real Toda chain [13, 14], thus it is essential to have the complex Toda chain in description of the soliton trains. Moreover, the phase space of the complex Toda chain with \( N \) nodes is \( 4N \)-dimensional, which is precisely the number of the real parameters in the train of \( N \) solitons.

In the present paper I consider the \( N \)-soliton train propagation governed by two intimately related nonlinear PDEs, one of which is integrable and the other is not: the massive Thirring model of the classical field theory [15, 16] and the optical gap system [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

For the massive Thirring model I show that the train propagation of well separated solitons with nearly equal amplitudes and rapidities is governed by the complex Toda chain. Moreover, I derive a non-integrable generalisation of the complex Toda chain, which describes the train propagation of well separated gap solitons with nearly equal amplitudes and velocities.

The gap soliton propagation through a grated optical fiber was manifested in recent experiments [31, 32, 33, 34]. The results of [34] are of particular interest: there the multiple gap soliton formation was observed.

Localised solutions in nonlinear media with periodic band gaps have a great potential for technological applications. One of the most important band gap structures in optics is given by an optical fiber with periodic index grating along the axis. From the Floquet-Bloch theory of wave propagation in periodic structures it is known that there are forbidden frequency bands or band gaps for linear waves. On the other hand, nonlinear wave propagation in such structures is possible for the central frequency of the wave packet lying in the band gap. Such nonlinear wave is usually called gap soliton.

The optical gap system was derived within the coupled mode approach for nonlinear wave propagation in optical fibers with grating (see, for instance, Ref. [28]). It reads

\[
\begin{align*}
\dot{E}_1 - E_1 X & = E_2 + \left( |E_2|^2 + \rho |E_1|^2 \right) E_1 = 0, \\
\dot{E}_2 + E_2 X & = E_1 + \left( |E_1|^2 + \rho |E_2|^2 \right) E_2 = 0,
\end{align*}
\]

where \( E_1 \) and \( E_2 \) are the slowly varying envelopes of two counter propagating waves coupled
through the Bragg scattering induced by the grating (the linear cross-coupling terms), the nonlinear terms account for the self- and cross-phase modulation effects. The parameter $\rho$ ($\rho > 0$) at the self-phase modulation term may range up to infinity \cite{35}, in which case the optical gap system models dynamics in the nonlinear dual-core asymmetric coupler \cite{27}. Setting $\rho = 0$ in the system \cite{1} one obtains the massive Thirring model.

The gap solitons in optical fibers were theoretically studied in many works \cite{17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28} (see also the latest review Ref. \cite{29}). Recently, the relevance of the system \cite{1} for description of the optical gap solitons was analytically and numerically validated \cite{30}. The general family of the gap solitons was derived in \cite{19} using the similarity with the massive Thirring model. Recently it was shown that the gap soliton becomes unstable when its amplitude grows above some fixed value \cite{36, 37}.

The soliton-like solutions, which are similar to the gap solitons, were found in nonlinear diatomic lattices \cite{38, 39, 40} and in the quadratic ($\chi^{(2)}$) materials with a spatially periodic linear susceptibility (grating) \cite{41, 42, 43, 44, 45, 46, 47, 48}. Also, it was shown \cite{40, 43} that, at some limit, the equations governing nonlinear wave propagation in quadratic media with grating and in diatomic lattices are similar to the system \cite{1} though the underlying physics is different.

In the next section, section 2, I state the main results of the paper on the soliton train propagation for the massive Thirring model and optical gap system. The details of the derivation are placed in the following sections: section 3 for the massive Thirring model and section 4 for the optical gap system. The last section contains discussion of the results and suggestions for further work.

II. MAIN RESULTS

Before formulate the main results I would like to remind some facts about the models under study. Let us begin with the massive Thirring model of the classical field theory \cite{15, 16}:

$$i (v_t - v_x) + u + |u|^2 v = 0,$$

(2)

$$i (u_t + u_x) + v + |v|^2 u = 0,$$
where \( u \) and \( v \) are complex variables, \( t \) and \( x \) are the time and space co-ordinates, respectively. The system (2) is Lorentz-invariant:

\[
x \rightarrow x - \tanh(y)t, \quad t \rightarrow t - \tanh(y)x \quad \frac{1}{(1 - \tanh^2 y)^{1/2}},
\]

with \( u \) and \( v \) transforming as components of the Lorentz spinor,

\[
u \rightarrow e^{-y/2}u, \quad v \rightarrow e^{y/2}v.
\]

In the relativistic kinematics the parameter \( y \) is called “rapidity”. (Rapidities of two consecutive Lorentz transformations simply add together.)

The massive Thirring model is integrable by the inverse scattering transform method \[16\]. For instance, its one-soliton solution can be written as

\[
v = -i \sin(2\vartheta) \exp(-y/2 + i\Theta) \frac{1}{\cosh(z - i\vartheta)},
\]

\[
u = i \sin(2\vartheta) \exp(y/2 + i\Theta) \frac{1}{\cosh(z + i\vartheta)},
\]

where \( 0 < \vartheta < \pi/2 \) and

\[
z = \sin(2\vartheta)\cosh(y)(x_o(t) - x), \quad \Theta = -\cot(2\vartheta)\tanh(y)z + \delta(t).
\]

The soliton has four independent real parameters: \( \vartheta, y, x_o, \) and \( \delta \). The first two give the soliton amplitude and rapidity, while the rest two are the soliton position and central phase (the phase at \( x = x_o \)), respectively. The position and phase parameters depend on time:

\[
\frac{dx_o}{dt} = \tanh y, \quad \frac{d\delta}{dt} = -\cos(2\vartheta)\sech y.
\]

The first equation defines the soliton velocity: \( V = \tanh y \).

By a suitable Lorentz transformation the rapidity of a Thirring soliton can be put equal to zero and the soliton solution (3) reduces to the quiescent soliton of the massive Thirring model.

The following \textit{ansatz} is called the soliton train

\[
v = \sum_{\alpha=1}^{N} -i \sin(2\vartheta_\alpha) \exp(-y_\alpha/2 + i\Theta_\alpha) \frac{1}{\cosh(z_\alpha - i\vartheta_\alpha)},
\]

\[
u = \sum_{\alpha=1}^{N} i \sin(2\vartheta_\alpha) \exp(y_\alpha/2 + i\Theta_\alpha) \frac{1}{\cosh(z_\alpha + i\vartheta_\alpha)},
\]

The following \textit{ansatz} is called the soliton train
where \( z_\alpha \) and \( \Theta_\alpha \) are given by formulae similar to (4) and (5). It should be stressed that, for each soliton, all four soliton parameters in formula (6) are considered to be \( t \)-dependent.

The CTC for the MTM soliton train. Assume that the \( N \)-soliton train given by (6) consists of well separated pulses with nearly equal amplitudes \( \vartheta_\alpha \) and rapidities \( y_\alpha \), numerated by \( \alpha = 1, \ldots, N \) in such a way that \( x_{\alpha+1} - x_\alpha > 0 \) (here and below \( x_\alpha \) denotes the position parameter “\( x_\alpha \)” for the \( \alpha \)-th soliton). Mathematically these conditions are expressed as

\[
|\vartheta_\alpha - \bar{\vartheta}| \ll \bar{\vartheta}, \quad |y_\alpha - \bar{y}| \ll 1, \quad |x_\alpha - x_{\alpha \pm 1}| \gg 1, \\
|\sin(2\vartheta_\alpha) \cosh y_\alpha - \sin(2\bar{\vartheta}) \cosh \bar{y}| |x_\alpha - x_{\alpha \pm 1}| \ll 1. \quad (7)
\]

Here (and throughout the paper) \( \bar{\vartheta} \) and \( \bar{y} \) denote the averages:

\[
\bar{\vartheta} = \frac{1}{N} \sum_{\alpha=1}^{N} \vartheta_\alpha, \quad \bar{y} = \frac{1}{N} \sum_{\alpha=1}^{N} y_\alpha. \quad (8)
\]

Define the following new variables: a modified time

\[
\tau = \sin(2\bar{\vartheta}) \text{sech}(\bar{y}) t, \quad (9)
\]

an average phase

\[
\bar{\delta} = -\cos(2\bar{\vartheta}) \text{sech}(\bar{y}) t, \quad (10)
\]

and the following complex variable for each soliton

\[
q_\alpha = -\sin(2\bar{\vartheta}) \cosh(\bar{y}) x_\alpha - i[\delta_\alpha - \bar{\delta} - \cos(2\bar{\vartheta}) \sinh(\bar{y}) x_\alpha + \alpha \pi] + 2\alpha \ln[2\sin(2\bar{\vartheta})]. \quad (11)
\]

Then in the first order of the soliton overlap parameter \( \epsilon \),

\[
\epsilon \simeq \exp\left\{ -|\sin(2\vartheta_\alpha) \cosh(y_\alpha) x_\alpha - \sin(2\vartheta_{\alpha+1}) \cosh(y_{\alpha+1}) x_{\alpha+1}| \right\}. \quad (12)
\]

the following two statements are claimed:

1. the average values \( \bar{\vartheta} \) and \( \bar{y} \) do not depend on \( t \);

2. evolution of the quantities \( q_\alpha, \alpha = 1, \ldots, N \), is given by the complex Toda chain with \( N \) nodes:

\[
\frac{d^2 q_\alpha}{d\tau^2} = e^{q_{\alpha+1} - q_\alpha} - e^{q_{\alpha} - q_{\alpha-1}}, \quad \alpha = 1, \ldots, N, \quad (13)
\]

where \( \text{Re}\{q_0\} = \infty \) and \( \text{Re}\{q_{N+1}\} = -\infty \) (i. e., \( x_0 = -\infty \) and \( x_{N+1} = \infty \), see (11)).
The set of inequalities (7) is similar to the inequalities for the NLS soliton train in Ref. [4].

Now I will formulate similar result for the train propagation of pulses governed by the
optical gap system [28]:

\[ i(E_{1t} - E_{1x}) + E_2 + \left(|E_2|^2 + \rho|E_1|^2\right) E_1 = 0, \]
\[ i(E_{2t} + E_{2x}) + E_1 + \left(|E_1|^2 + \rho|E_2|^2\right) E_2 = 0, \] (14)

The soliton solution of the optical gap system (14) moving with the velocity \( V = \tanh y_o \) reads [19]

\[
\begin{pmatrix}
E_1(x, t) \\
E_2(x, t)
\end{pmatrix}
= \frac{e^{i\psi(x, t)}}{[1 + \rho \cosh(2y_o)]^{1/2}}
\begin{pmatrix}
v(x, t) \\
u(x, t)
\end{pmatrix}, \tag{15}
\]

where \( v \) and \( u \) have the form of a Thirring soliton, i.e., given by formulae (3)-(5) (with \( y \to y_o \)); the additional (nonlinear) phase \( \psi \) is

\[
\psi = -\frac{2\rho \sinh(2y_o)}{1 + \rho \cosh(2y_o)} \arctan(\tan \vartheta \tanh z),
\] (16)

with \( z \) as in equation (4).

Here it should be pointed out that the gap soliton becomes unstable when the soliton
amplitude \( \vartheta \) grows above certain threshold (\( \vartheta_{thr} \approx \pi/4 \), see Ref. [36]). This scenario is also
possible for the train of gap solitons. This instability is the result of the soliton-radiation
interaction and is beyond the scope of the adiabatic approach. However, being interested in
\textit{stable} gap solitons, one can impose the condition \( \vartheta_\alpha < \vartheta_{thr} \), for all \( \alpha = 1, \ldots, N \).

The \textit{ansatz} I use for the train of \textit{well separated} gap solitons is given by application of
the transformation (15) to the train of well separated Thirring solitons (in this case \( y_o = \tilde{y} \)).
Due to the inequalities (7), the additional phase \( \psi_\alpha \) of each soliton in the train can be
approximated by formula (16) with \( \vartheta = \vartheta \) and \( z_\alpha = \sin(2\vartheta) \cosh(\tilde{y})(x_\alpha - x) \).

\textbf{The generalised CTC for the train of gap solitons.} Assume that the train of \( N \) gap
solitons consists of well separated pulses with nearly equal amplitudes \( \vartheta_\alpha \) and rapidities \( y_\alpha \)
numerated by \( \alpha = 1, \ldots, N \) in such a way that \( x_{\alpha+1} - x_\alpha > 0 \) and that the conditions (7) are
satisfied. Associate the following variables with each gap soliton

\[
Q_\alpha = -\sin(2\vartheta) \cosh(\tilde{y})x_\alpha - i\{\delta_\alpha - \delta - [\cos(2\vartheta) - \mu \sin(2\vartheta)(y_\alpha - \tilde{y})] \sinh(\tilde{y})x_\alpha + \alpha \pi \}
\]
\[+ 2\alpha \ln[2 \sin(2\vartheta)], \tag{17}\]
where

\[ \mu = \frac{4 \rho \tanh(2y)}{\rho + \text{sech}(2y)} \bar{\vartheta}. \]

Define the modified time \( \tau \) and average phase \( \bar{\vartheta} \) as in formulae (9) and (10). Then, in the first order of the soliton overlap parameter \( \epsilon \) (12), the following is claimed:

1. the average values \( \bar{\vartheta} \) and \( \bar{y} \) do not depend on \( t \);

2. evolution of the quantities \( Q_\alpha, \alpha = 1, \ldots, N \), is given by the following generalised complex Toda chain with \( N \) nodes:

\[ \frac{d^2 Q_\alpha}{d\tau^2} = (1 + A_\rho)
\left( e^{Q_{\alpha+1}-Q_\alpha} - e^{Q_\alpha-Q_{\alpha-1}} \right) + B_\rho
\left( e^{Q^*_{\alpha+1}-Q^*_\alpha} - e^{Q^*_\alpha-Q^*_{\alpha-1}} \right), \tag{18} \]

where \( \text{Re}\{Q_0\} = \infty \) and \( \text{Re}\{Q_{N+1}\} = -\infty \).

Equation (18) is valid for arbitrary values of the self-phase modulation parameter \( \rho \).

Here \( A_\rho \) and \( B_\rho \) are \( \rho \)-dependent coefficients:

\[ A_\rho = \frac{1}{2} \{ \nu - \kappa \mu + i[\kappa(1 + \nu) + \mu] \}, \]

\[ B_\rho = \frac{1}{2} \{ \nu + \kappa \mu - i[\kappa(1 + \nu) - \mu] \}, \tag{19} \]

with

\[ \kappa = \frac{\rho \tanh(2y)}{\rho + \text{sech}(2y)} \cdot \frac{4\bar{\vartheta} - \sin(4\vartheta)}{\sin^2(2\vartheta)}, \quad \nu = \frac{4\rho(2\bar{\vartheta} \cot(2\vartheta) - 1)}{\rho + \text{sech}(2y)}. \]

Setting \( \rho = 0 \) in (17) and (18) one obtains the complex Toda chain for the soliton train of the massive Thirring model.

III. THE CTC FOR THE MTM SOLITON TRAIN

I will use the adiabatic perturbation theory for derivation of the complex Toda chain. Recently the perturbation theory based on the Riemann-Hilbert problem was developed for the solitons of the massive Thirring model [49]. For instance, we have derived the following evolution equations for the soliton parameters in the adiabatic approximation:

\[ \frac{d\vartheta}{dt} = -\frac{1}{2 \cosh y} \int_{-\infty}^{\infty} \frac{dz}{\cosh(2z) + \cos(2\vartheta)} \text{Re} \left\{ e^z [r_{\perp}(k_1, z) + r^*_{\perp}(k_1^*, -z)] \right\}, \tag{20} \]
\[ \frac{dy}{dt} = \frac{1}{\cosh y} \int_{-\infty}^{\infty} \frac{dz}{\cosh(2z) + \cos(2\vartheta)} \Im \left\{ e^z \left[ r_\perp(k_1, z) + r_\perp^*(k_1^*, -z) \right] - 2r_\parallel(z) \right\}, \quad \text{(21)} \]

\[ \frac{dx_\sigma}{dt} = \tanh y - \frac{\Re\{J\}}{2 \sin^2(2\vartheta) \cosh^2 y}, \quad \text{(22)} \]

\[ \frac{d\delta}{dt} = -\cos(2\vartheta) \sech y - \frac{\Im\{J\} + \tanh y \cot(2\vartheta) \Re\{J\}}{2 \sin(2\vartheta) \cosh y}. \quad \text{(23)} \]

Here

\[ J = \int_{-\infty}^{\infty} \frac{dz}{\cosh(2z) + \cos(2\vartheta)} \left( 2\{2z[\cos(2\vartheta) + i \sin(2\vartheta) \tanh y] + \sinh(2z)\}r_\parallel(z) \right. \]

\[ + \left\{ e^z[\cos(2\vartheta)(1 - 2z) - 2iz \sin(2\vartheta) \tanh y] + e^{-z}\} \left[ r_\perp(k_1, z) + r_\perp^*(k_1^*, -z) \right] \right\}. \quad \text{(24)} \]

The functions \( r_\perp \) and \( r_\parallel \) are given by

\[ r_\perp(k_1, z, t) = \frac{ie^{-i\Theta}}{2} \left( k_1 \frac{\delta v}{\delta t} - k_1^* \frac{\delta u}{\delta t} \right), \quad \text{(25)} \]

\[ r_\parallel(z, t) = \frac{i}{4} \left( \frac{\delta |v|^2}{\delta t} - \frac{\delta |u|^2}{\delta t} \right), \quad \text{(26)} \]

where \( k_1 = -\exp\{-y/2 - i\vartheta\} \). The “variational” derivatives denote fictitious evolution of the \( u \) and \( v \) as if under the action of the perturbation only (in other words, either \( i\delta u/\delta t \) or \( i\delta v/\delta t \) is nothing but the perturbation added to the respective r.h.s. of the massive Thirring model (2)). Equations (21)-(26) can also be derived via the perturbation theory for the Thirring solitons developed in Ref. [50].

From the point of view of the inverse scattering transform method the ansatz (3) contains not only the \( N \)-soliton solution but also the contribution of radiation as well. However, due to large separations between the solitons in the train, the radiation component is negligible. Thus, I can use the adiabatic perturbation theory for derivation of evolution equations for the soliton parameters \( \vartheta_\alpha, y_\alpha, x_\alpha, \) and \( \delta_\alpha \). For the same reason, it is sufficient to consider the interaction between the neighboring pulses only (detailed discussion can be found in Ref. [4]).

First, one should compute the perturbation functions defined in (25) and (26). Substitution of the ansatz (3) into the massive Thirring model (2) and expansion of the cubic terms leads to the following formulae for the perturbation-induced evolutions (I consider the
interaction between the neighboring pulses only)

\[
\frac{i}{\delta t} \delta v_\alpha = - \sum_{\beta = \alpha \pm 1} \left( |u_\alpha|^2 v_\beta + 2 \text{Re}\{u_\alpha u_\beta^*\} v_\alpha \right),
\]

(27)

\[
\frac{i}{\delta t} \delta u_\alpha = - \sum_{\beta = \alpha \pm 1} \left( |v_\alpha|^2 u_\beta + 2 \text{Re}\{v_\alpha v_\beta^*\} u_\alpha \right)
\]

and, consequently,

\[
\frac{i}{\delta t} |v_\alpha|^2 = \sum_{\beta = \alpha \pm 1} 2i|u_\alpha|^2 \text{Im}\{v_\alpha v_\beta^*\},
\]

(28)

\[
\frac{i}{\delta t} |u_\alpha|^2 = \sum_{\beta = \alpha \pm 1} 2i|v_\alpha|^2 \text{Im}\{u_\alpha u_\beta^*\}.
\]

Before giving the perturbation-induced evolution of the soliton parameters some remarks must be maid on the details of the approximation due to the inequalities (7). The r.h.s.’s in (27) contain the small parameter \( \epsilon \). I consider the soliton-soliton interaction in the first order with respect to \( \epsilon \). Hence, due to the presence of the small parameter, the differences between the \( \alpha \)-th soliton amplitude and rapidity and the average values of these quantities (given by (8)) are negligible in the terms accounting for the inter-soliton interaction.

Substitution of (27) and (28) into (25) and (26) and the result into (20) and (21) gives the following equations for the amplitudes and rapidities

\[
\frac{d\vartheta_\alpha}{dt} = \sum_{\beta = \alpha \pm 1} \frac{2\sin^3(2\bar{\vartheta})}{\cosh \bar{y}} e^{-|\Delta_{\alpha\beta}|} \sin \Psi_{\alpha\beta},
\]

(29)

\[
\frac{dy_\alpha}{dt} = \sum_{\beta = \alpha \pm 1} \frac{4\text{sgn}(\Delta_{\alpha\beta}) \sin^3(2\bar{\vartheta})}{\cosh \bar{y}} e^{-|\Delta_{\alpha\beta}|} \cos \Psi_{\alpha\beta},
\]

(30)

where

\[
\Delta_{\alpha\beta} = \sin(2\bar{\vartheta}) \cosh(\bar{y})(x_\beta - x_\alpha),
\]

\[
\Psi_{\alpha\beta} = \delta_\alpha - \delta_\beta - \cos(2\bar{\vartheta}) \sinh(\bar{y})(x_\alpha - x_\beta).
\]

Here \( \exp(-|\Delta_{\alpha\beta}|) = \mathcal{O}(\epsilon) \). It can be easily verified that equations (29) and (30) do not affect the average amplitude \( \bar{\vartheta} \) and rapidity \( \bar{y} \).

The evolution equations for \( x_\alpha \) and \( \delta_\alpha \) (see (22) and (23)) are comprised of two addends, which account for the unperturbed and perturbation-induced evolution (the latter contain the soliton overlap parameter \( \epsilon \)). Hence, one can neglect the terms accounting for
the perturbation-induced evolution of these parameters as compared to their unperturbed evolution:

\[ \frac{dx_\alpha}{dt} = \tanh y_\alpha, \quad \frac{d\delta_\alpha}{dt} = -\cos(2\vartheta_\alpha)\text{sech}y_\alpha. \]  

(31)

Now everything is ready for derivation of the complex Toda chain. Let us differentiate the following quantity \(-\Delta_{\alpha\beta} + i\Psi_{\alpha\beta} (\beta = \alpha \pm 1)\):

\[ \frac{d}{dt}(-\Delta_{\alpha\beta} + i\Psi_{\alpha\beta}) = \sin(2\bar{\vartheta})\text{sech}(\bar{y})[(y_\alpha + 2i\vartheta_\alpha) - (y_\beta + 2i\vartheta_\beta)], \]

where the second order terms in \((\vartheta_\alpha - \bar{\vartheta})\) and \((y_\alpha - \bar{y})\) are dropped. On the other hand, from equations (29) and (30) one derives

\[ \frac{d}{dt}(y_\alpha + 2i\vartheta_\alpha) = \sum_{\beta=\alpha+1} \frac{4\text{sgn}(\Delta_{\alpha\beta})\sin^3(2\bar{\vartheta})}{\cosh \bar{y}} \exp\{\text{sgn}(\Delta_{\alpha\beta})(-\Delta_{\alpha\beta} + i\Psi_{\alpha\beta})\}; \]

or using the numeration \(x_{\alpha+1} - x_\alpha > 0\), i.e. \(\text{sgn}(\Delta_{\alpha,\alpha+1}) > 0\), for the solitons in the train:

\[ \frac{d}{dt}(y_\alpha + 2i\vartheta_\alpha) = \frac{4\sin^3(2\bar{\vartheta})}{\cosh \bar{y}} \left( \exp \{-\Delta_{\alpha+1} + i\Psi_{\alpha+1}\} - \exp \{\Delta_{\alpha-1} - i\Psi_{\alpha-1}\} \right). \]  

(32)

Introduce an average phase

\[ \bar{\delta} = -\cos(2\bar{\vartheta})\text{sech}(\bar{y})t, \]

and the following complex variables associated with each soliton

\[ q_\alpha = -\sin(2\bar{\vartheta})\cosh(\bar{y})x_\alpha - i[\delta_\alpha - \bar{\delta} + \alpha\pi - \cos(2\bar{\vartheta})\sinh(\bar{y})x_\alpha] + 2\alpha \ln[2\sin(2\bar{\vartheta})]. \]

(The average phase in the formula for \(q_\alpha\) eliminates the constant phase gradient.) Then

\[ e^{\pm(q_{\alpha+1} - q_\alpha)} = -4\sin^2(2\bar{\vartheta}) \exp(\mp\Delta_{\alpha+1} \pm i\Psi_{\alpha+1}). \]  

(33)

Differentiating \(q_\alpha\) and neglecting the second order terms I get

\[ \frac{dq_\alpha}{dt} = -\sin(2\bar{\vartheta}) \cosh \bar{y} \tanh y_\alpha - i[\sin(2\bar{\vartheta})\text{sech}(\bar{y})(2\vartheta_\alpha - 2\bar{\vartheta}) + \cos(2\bar{\vartheta})\text{sech} \bar{y} \tanh(\bar{y})(y_\alpha - \bar{y})]

- \cos(2\bar{\vartheta}) \sinh \bar{y} \tanh y_\alpha] \]

(here the average phase \(\bar{\delta}\) plays an important role for the expansion over \(\vartheta_\alpha - \bar{\vartheta}\)). The second differentiation (with removal of the second order terms) gives:

\[ \frac{d^2q_\alpha}{dt^2} = -\sin(2\bar{\vartheta})\text{sech}(\bar{y}) \left( \frac{dy_\alpha}{dt} + 2i\frac{d\vartheta_\alpha}{dt} \right). \]  

(34)
At the same time, as it follows from (32) and (33),
\[
\frac{d}{dt}(−y_α - i(2∂_α - 2\bar{∂})) = \sin(2\bar{∂})\text{sech}(\bar{y}) \left(e^{q_{α+1}-q_α} - e^{q_α-q_{α-1}}\right).
\]

(35)

Therefore, what is left is to introduce a new time variable
\[
τ = \sin(2\bar{∂})\text{sech}(\bar{y})t.
\]

Then equations (34) and (35) give the complex Toda chain for the train of Thirring solitons:
\[
\frac{d^2 q_α}{dτ^2} = e^{q_{α+1}-q_α} - e^{q_α-q_{α-1}}, \quad α = 1, ..., N.
\]

Here it is assumed that Re\{q_0\} = ∞ and Re\{q_{N+1}\} = −∞ (i.e., x₀ = −∞ and x_{N+1} = ∞).

IV. THE GENERALISED CTC FOR THE TRAIN OF GAP SOLITONS

In derivation of the complex Toda chain for the optical gap system I will use a one-to-one

mapping, found recently in Ref. [49], between the optical gap system (1) and the following
generalisation of the massive Thirring model, the \(γ\)-system for short,

\[
i(\mathcal{V}_t - \mathcal{V}_x) + \mathcal{U} + |\mathcal{U}|^2\mathcal{V} + γ_+ \left(|\mathcal{V}|^2 - |\mathcal{U}|^2\right)\mathcal{V} = 0,
\]

(36)

\[
i(\mathcal{U}_t + \mathcal{U}_x) + \mathcal{V} + |\mathcal{V}|^2\mathcal{U} + γ_- \left(|\mathcal{U}|^2 - |\mathcal{V}|^2\right)\mathcal{U} = 0.
\]

The transformation relating the two systems is as follows. Let \(\mathcal{U}(x,t)\) and \(\mathcal{V}(x,t)\) be a

solution of the \(γ\)-system (36) with the following \(γ_±\):

\[
γ_± = \frac{ρe^{±2y_o}}{1 + ρ \cosh(2y_o)},
\]

(37)

then

\[
\begin{pmatrix}
E_1(X, T) \\
E_2(X, T)
\end{pmatrix}
= \frac{e^{iψ(x,t)}}{[1 + ρ \cosh(2y_o)]^{1/2}} \begin{pmatrix}
e^{−y_o/2}\mathcal{V}(x,t) \\
e^{y_o/2}\mathcal{U}(x,t)
\end{pmatrix},
\]

(38)

where

\[
x = \frac{X - \tanh(y_o)T}{[1 - \tanh^2(y_o)]^{1/2}}, \quad t = \frac{T - \tanh(y_o)X}{[1 - \tanh^2(y_o)]^{1/2}}.
\]

(39)

is a solution to the optical gap system (1) with the phase \(ψ\) given by the following system of equations (in the light-cone variables \(η = (t + x)/2\) and \(ξ = (t - x)/2\))

\[
\frac{∂ψ}{∂η} = \frac{1}{2}(γ_+ - γ_-)|\mathcal{V}|^2, \quad \frac{∂ψ}{∂ξ} = -\frac{1}{2}(γ_+ - γ_-)|\mathcal{U}|^2.
\]

(40)
(Note that the conservation of the number of particles, i. e.,
\[ \frac{\partial}{\partial \xi} |V|^2 + \frac{\partial}{\partial \eta} |U|^2 = 0, \]
ensures the compatibility of the equations for the phase \( \psi \).)

Note that the co-ordinates are related via a Lorentz transformation. The mapping (38) can be verified by direct substitution into (1) via simple calculations with the use of (37), (39), and (40).

The presented mapping is valid for arbitrary solutions of the optical gap system. However, the importance of the mapping (37)-(40) stems from the fact that by choosing the quiescent Thirring soliton,
\[ V = -i \sin(2 \vartheta) e^{i \delta} \cosh(z - i \vartheta), \quad U = i \sin(2 \vartheta) e^{i \delta} \cosh(z + i \vartheta), \]

\[ z = -\sin(2 \vartheta)(x - x_o), \quad \frac{d \delta}{dt} = -\cos(2 \vartheta), \]

which is a solution to the system (36) due to \(|V| = |U|\), one can recover the optical gap soliton moving with any given velocity \( V = \tanh y_o \). In this case one obtains formula (16) for the additional phase phase \( \psi \). Although the optical gap system is not Lorentz-invariant, still it makes sense to call \( y_o \) “rapidity” of the gap soliton due to the transformation (37)-(40).

A train of \( N \) well separated gap solitons moving with arbitrary average rapidity \( y_o \) can always be represented via the transformation (37)-(40) as a train of \( N \) well separated almost quiescent Thirring solitons. Indeed, application of the mapping (37)-(40) to equation (3) with \( V = v \) and \( U = u \), under the conditions \(|y_\alpha| \ll 1 \) and \( \bar{\gamma} = 0 \), yields the train of \( N \) well separated gap solitons with nearly equal amplitudes and rapidities, where the average rapidity is equal to the given arbitrary value \( y_o \). Moreover, if one neglects the terms of order \( O(\epsilon) \), then the additional phase \( \psi_\alpha \) of each gap soliton in the train is determined by equation (16) with evident changes: \( \vartheta \to \bar{\vartheta}, \ z \to z_\alpha \) and \( x_o \to x_\alpha \).

Convenience of the \( \gamma \)-system (30) for the analytical study of gap solitons is based on the two following facts. Firstly, the quiescent Thirring soliton satisfies \(|U| = |V|\). Hence the last terms in the \( \gamma \)-system are small if the solution under study is close to the quiescent Thirring soliton, or, in terms of the optical gap system, the solution is close to the gap soliton. Secondly, the parameters \( \gamma_{\pm} \) are bounded for all values of \( y_o \) and \( \rho \) (including \( \rho = \infty \)). Therefore the use of the equivalent \( \gamma \)-system allows one to apply the perturbation theory.
developed for the train of almost quiescent Thirring solitons to the train of gap solitons for arbitrary values of the self-phase modulation parameter $\rho$.

Hence, derivation of the complex Toda chain for the gap soliton train can be done in much the same way as the derivation of the complex Toda chain for the train of Thirring solitons (more precisely, almost quiescent Thirring solitons). The only difference is that there are additional small perturbations given by the terms with $\gamma_\pm$ in (36), the $\gamma$-terms for short. Let us first calculate their contribution to the evolution of the soliton parameters and then calculate evolution of $q_\alpha$, defined in a similar was as in the previous section, with account of these terms as well.

Below I will take into account that gap soliton train is transformed by the mapping (37)-(40) with $y_\alpha = \bar{y}$ to the train of almost quiescent Thirring solitons (in the variables $V$ and $U$). For instance, the latter train has $\bar{y} = 0$ and $|y_\alpha| \ll 1$. Consider the $\alpha$-th soliton in such train. From (36) one gets expanding the cubic terms:

$$i \frac{\delta V_\alpha}{\delta t} = - \sum_{\beta=\alpha \mp 1} \left( |U_\alpha|^2 V_\beta + 2 \text{Re} \{ U_\alpha U_\beta^* \} V_\alpha \right) - \gamma_+ \left( |V_\alpha|^2 - |U_\alpha|^2 \right) V_\alpha$$

$$- \sum_{\beta=\alpha \pm 1} \gamma_- \left( 2 |V_\alpha|^2 V_\beta + V_\alpha^2 V_\beta^* - |U_\alpha|^2 V_\beta - 2 \text{Re} \{ U_\alpha U_\beta^* \} V_\alpha \right).$$

Let us separate the r.h.s. into two parts. The first part is just the same as in the case of the massive Thirring solitons, while the second, given by the following formula

$$i \left( \frac{\delta V_\alpha}{\delta t} \right)_\gamma = - \gamma_- \left( |V_\alpha|^2 - |U_\alpha|^2 \right) V_\alpha$$

$$- \sum_{\beta=\alpha \pm 1} \gamma_- \left( 2 |V_\alpha|^2 v_\beta + V_\alpha^2 v_\beta^* - |U_\alpha|^2 v_\beta - 2 \text{Re} \{ U_\alpha U_\beta^* \} V_\alpha \right),$$

is due to the $\gamma$-perturbation and is specific for the gap soliton train only. I will detailly consider only the $\gamma$-perturbation since the first part is accounted just in the same way as for the train of almost quiescent Thirring solitons with the same resulting formulae.

In formula (41) the first term on the r.h.s. is due to the self-interaction of the gap soliton (due to the $\gamma$-terms in (36), if the rapidity $y_\alpha \neq 0$), the rest account for the inter-soliton interaction in the train. The latter terms contain the small parameter $\epsilon$ defined in (12).

I will use the same approximation which has been used for the derivation of the complex Toda chain for the massive Thirring model. Additionally one can neglect the difference between the modules of $V_\alpha$ and $U_\alpha$ when calculating the contribution from the inter-soliton
interaction terms. This is because the inter-soliton interaction terms already contain the small parameter $\epsilon$ and $|y_\alpha| \ll 1$, thus one can put $y_\alpha = 0$ there. Then, the “variational” derivatives simplify considerably:

$$
i \left( \frac{\delta V_\alpha}{\delta t} \right)_\gamma = -\gamma_- \left( |V_\alpha|^2 - |U_\alpha|^2 \right) V_\alpha - \sum_{\beta=\alpha+1} 4 \gamma_- \sin(\Psi_{\alpha\beta}) \text{Im} \{V_{\alpha}\overline{U}_{\beta}\} V_\alpha, \quad (42)$$

$$
i \left( \frac{\delta U_\alpha}{\delta t} \right)_\gamma = -\gamma_+ \left( |U_\alpha|^2 - |V_\alpha|^2 \right) V_\alpha - \sum_{\beta=\alpha+1} 4 \gamma_+ \sin(\Psi_{\alpha\beta}) \text{Im} \{U_{\alpha}\overline{V}_{\beta}\} U_\alpha, \quad (43)$$

where $V_{\alpha\alpha} = \exp\{-i\delta_\alpha\} V_\alpha, U_{\alpha\alpha} = \exp\{-i\delta_\alpha\} U_\alpha,$ and $\Psi_{\alpha\beta} = \delta_\alpha - \delta_\beta$. Formula (43) obtains from (42) by the evident substitution $V \to U, U \to V$ and $\gamma_- \to \gamma_+$. Note that from formulae (42) and (43) it follows that

$$
\left( \frac{\delta |V_\alpha|^2}{\delta t} \right)_\gamma = 0, \quad \left( \frac{\delta |U_\alpha|^2}{\delta t} \right)_\gamma = 0.
$$

Consider first the contribution to evolution of the soliton parameters coming from the inter-soliton interaction $\gamma$-terms, i.e., the first terms on the r.h.s.’s in formulae (42) and (43). First, the perturbation functions given in (25) and (26) must be calculated. The inter-soliton interaction terms give the following contributions to the necessary functions: $r_\perp(z_\alpha) = 0$ and

$$
r_\parallel(k_\alpha, z_\alpha) + r_\parallel^*(k_\alpha^*, -z_\alpha) = - \sum_{\beta=\alpha+1} 8i \sin^4(2\bar{\vartheta}) e^{-|\Delta_{\alpha\beta}|} \sin(\Psi_{\alpha\beta}) \sinh(2z_\alpha)
$$

$$
\times \left[ (\gamma_+ - \gamma_-) e^{2y_\alpha} + (\gamma_+ e^{2i\bar{\vartheta}} - \gamma_- e^{-2i\bar{\vartheta}}) e^{-z_\alpha} \right], \quad (44)
$$

where $k_\alpha = -\exp\{-y_\alpha - i\vartheta_\alpha\}$ and $\Delta_{\alpha\beta} = \sin(2\bar{\vartheta})(x_\beta - x_\alpha)$. (On the r.h.s. of (44) the terms of the second order in $\vartheta_\alpha - \bar{\vartheta}$ are neglected due to the small multiplier $\exp(-|\Delta_{\alpha\beta}|) = \mathcal{O}(\epsilon)$.) Now, substitution of the expression (44) into (20) and (21) leads to the following contributions to evolution equations for $\vartheta_\alpha$ and $y_\alpha$:

$$
\left( \frac{d\vartheta_\alpha}{dt} \right)_\gamma = 0,
$$

$$
\left( \frac{dy_\alpha}{dt} \right)_\gamma = - \sum_{\beta=\alpha+1} 4\kappa \sin^3(2\bar{\vartheta}) e^{-|\Delta_{\alpha\beta}|} \sin \Psi_{\alpha\beta},
$$

where

$$
\kappa = \frac{\rho \tanh(2\bar{\vartheta})}{\rho + \text{sech}(2\bar{\vartheta})} \frac{4\bar{\vartheta} - \sin(4\bar{\vartheta})}{\sin^2(2\bar{\vartheta})}.
$$
What concerns the other two parameters $x_\alpha$ and $\delta_\alpha$, similar as in section 3, the inter-soliton interaction is of order $\mathcal{O}(\epsilon)$ and its contribution to evolution of the soliton position and phase can be neglected as compared to their unperturbed evolution.

Now let us consider the contribution to evolution of the soliton parameters coming from the self-interaction $\gamma$-terms, i.e., the first terms in (12) and (13). I get the following contributions to the functions in (25) and (26):

$$r_\parallel(z_\alpha) = 0, \quad r_\perp(k_\alpha, z_\alpha) + r_\perp^*(k_\alpha^*, -z_\alpha) = 0, \quad (45)$$

$$r_\perp(k_\alpha, z_\alpha) - r_\perp^*(k_\alpha^*, -z_\alpha) = \frac{4i \sin^3(2\bar{\vartheta}_\alpha)y_\alpha}{[\cosh(2z_\alpha) + \cos(2\bar{\vartheta}_\alpha)]^2} \left[ (\gamma_- - \gamma_+)e^{z_\alpha} + (\gamma_- e^{-2i\vartheta_\alpha} - \gamma_+ e^{2i\vartheta_\alpha})e^{-z_\alpha} \right] \quad (46)$$

(here it is taken into account that terms of the second order in $y_\alpha$ are negligible). From (15) it follows that the contributions from the self-interaction $\gamma$-terms to evolution equations for the amplitude $\vartheta_\alpha$ and rapidity $y_\alpha$ vanish, while substitution of (45) and (46) into (24) gives due to $|y_\alpha| \ll 1$:

$$J_\alpha = 2 \left\{ (\gamma_- + \gamma_+)[2 \sin^2(2\bar{\vartheta}_\alpha) - 2\bar{\vartheta}_\alpha \sin(4\vartheta_\alpha)] + i(\gamma_- - \gamma_+)2\bar{\vartheta}_\alpha \sin^2(2\vartheta_\alpha) \right\} y_\alpha.$$  

Thus the contributions from the self-interaction $\gamma$-terms to evolution of $x_\alpha$ and $\delta_\alpha$ are determined by the following coefficients

$$\nu_\alpha = -\frac{\text{Re}\{J_\alpha\}}{2 \sin^2(2\bar{\vartheta}_\alpha)y_\alpha} = (\gamma_- + \gamma_+)[4\vartheta_\alpha \cot(2\vartheta_\alpha) - 2],$$

$$\mu_\alpha = -\frac{\text{Im}\{J_\alpha\}}{2 \sin^2(2\bar{\vartheta}_\alpha)y_\alpha} = (\gamma_+ - \gamma_-)2\vartheta_\alpha.$$  

I can take just the average values of these coefficients (denoted below as $\nu$ and $\mu$), because the following combinations $\nu_\alpha y_\alpha$ and $\mu_\alpha y_\alpha$ will enter the evolution equations for the soliton parameters and $|y_\alpha| \ll 1$. In other words, one can throw away the terms of the second order in $\vartheta_\alpha - \bar{\vartheta}$ and $y_\alpha$. Taking into account the definition of the $\gamma_\pm$, where $y_o = \bar{y}$, I obtain:

$$\nu = \frac{4\rho (2\bar{\vartheta} \cot(2\bar{\vartheta}) - 1)}{\rho + \text{sech}(2\bar{y})}, \quad \mu = \frac{4\rho \tanh(2\bar{y})}{\rho + \text{sech}(2\bar{y})}.$$  

Let us collect all the contributions, i.e., the terms same as for the Thirring soliton train (see equations (29)-(31)) and those accounting for the $\gamma$-terms, and write down the corresponding evolution equations for the parameters of the $\alpha$-th gap soliton. They read:

$$\frac{d\vartheta_\alpha}{dt} = \sum_{\beta=\alpha\mp 1} 2 \sin^3(2\bar{\vartheta})e^{-|\Delta_{\alpha\beta}|} \sin \Psi_{\alpha\beta}, \quad (47)$$
\[
\frac{dy_\alpha}{dt} = \sum_{\beta=\alpha+1} 4 \sin^3(2\bar{\vartheta}) e^{-|\Delta_{\alpha\beta}|} \text{sgn}(\Delta_{\alpha\beta}) \cos \Psi_{\alpha\beta} - \kappa \sin \Psi_{\alpha\beta}, \tag{48}
\]

\[
\frac{d\delta_\alpha}{dt} = -\cos(2\bar{\vartheta}_\alpha) + \mu \sin(2\bar{\vartheta}) y_\alpha, \tag{49}
\]

\[
\frac{dx_\alpha}{dt} = (1 + \nu) y_\alpha. \tag{50}
\]

Here

\[
\Delta_{\alpha\beta} = \sin(2\bar{\vartheta})(x_\beta - x_\alpha), \quad \Psi_{\alpha\beta} = \delta_\alpha - \delta_\beta.
\]

It is easy to see that the averages \(\bar{\vartheta}\) and \(\bar{y}\) are not affected by equations (47) and (18). Equations (47)-(50) are similar to those for the (almost quiescent) Thirring solitons, however there are additional terms in the evolution equations for \(y_\alpha, \delta_\alpha,\) and \(x_\alpha\).

Let us now derive the generalised complex Toda chain corresponding to equations (47)-(50). As the derivation is quite similar to that for the quiescent Thirring solitons I will skip some details. As in the case of the massive Thirring model, introduce the modified time

\[
\tau = \sin(2\bar{\vartheta})t,
\]

an average phase

\[
\bar{\delta} = -\cos(2\bar{\vartheta})t,
\]

and the complex variables \(q_\alpha\) for each soliton:

\[
q_\alpha = -\sin(2\bar{\vartheta})x_\alpha - i(\delta_\alpha - \bar{\delta} + \alpha \pi) + 2\alpha \ln[2 \sin(2\bar{\vartheta})]. \tag{51}
\]

Differentiating \(q_\alpha\) and throwing away the second order terms one obtains

\[
\frac{dq_\alpha}{d\tau} = -\{(1 + \nu + i\mu) y_\alpha + i[2\bar{\vartheta}_\alpha - 2\bar{\vartheta}]\}.
\]

Differentiation of this formula gives

\[
\frac{d^2q_\alpha}{d\tau^2} = -\sum_{\beta=\alpha+1} 4 \text{sgn}(\Delta_{\alpha\beta}) \sin^2(2\bar{\vartheta}) \exp \{\text{sgn}(\Delta_{\alpha\beta})(-\Delta_{\alpha\beta} + i\Psi_{\alpha\beta})\}
\]

\[-(\nu + i\mu) \text{Re} \sum_{\beta=\alpha+1} 4 \text{sgn}(\Delta_{\alpha\beta}) \sin^2(2\bar{\vartheta}) \exp \{\text{sgn}(\Delta_{\alpha\beta})(-\Delta_{\alpha\beta} + i\Psi_{\alpha\beta})\}
\]

\[+\kappa(1 + \nu + i\mu) \text{Im} \sum_{\beta=\alpha+1} 4 \text{sgn}(\Delta_{\alpha\beta}) \sin^2(2\bar{\vartheta}) \exp \{\text{sgn}(\Delta_{\alpha\beta})(-\Delta_{\alpha\beta} + i\Psi_{\alpha\beta})\}.\]
Taking into account the numeration of the solitons in the train, which is given by \(x_{\alpha+1} - x_\alpha > 0\) or \(\Delta_{\alpha\alpha+1} > 0\), and the following identity

\[
4 \sin^2(2\bar{\vartheta}) \exp\{\pm(-\Delta_{\alpha\alpha+1} + i\Psi_{\alpha\alpha+1})\} = -e^{\pm(q_\alpha+1 - q_\alpha)},
\]

I obtain a generalised complex Toda chain for the train of \(N\) well separated gap solitons with nearly equal amplitudes and rapidities:

\[
\frac{d^2q_\alpha}{d\tau^2} = (1 + A_\rho)(e^{q_{\alpha+1} - q_\alpha} - e^{q_\alpha - q_{\alpha-1}}) + B_\rho(e^{q_{\alpha+1}^* - q_\alpha^*} - e^{q_\alpha^* - q_{\alpha-1}^*}),
\]  
(52)

where \(A_\rho\) and \(B_\rho\) are \(\rho\)-dependent coefficients:

\[
A_\rho = \frac{1}{2}\{\nu - \kappa\mu + i[\kappa(1 + \nu) + \mu]\},
\]

\[
B_\rho = \frac{1}{2}\{\nu + \kappa\mu - i[\kappa(1 + \nu) - \mu]\}.
\]

As usual, \(\text{Re}\{q_0\} = \infty\) and \(\text{Re}\{q_{N+1}\} = -\infty\).

Though in equation (52) and in the definition of \(q_\alpha\) (51) I still have the variables \(\tau, x_\alpha, \delta_\alpha,\) and \(\bar{\delta}\) defined through the co-ordinates \(x\) and \(t\) (see formula (39)), it is easy to reverse to the co-ordinates \(X\) and \(T\) of the optical gap system (1). Indeed, to this end one should use the transformation (39) (with \(y_0 = \bar{y}\)) for the position \(x_\alpha\) and the central phase \(\delta_\alpha\) of the gap soliton (the phase at \(X = X_\alpha\)):

\[
x_\alpha = \cosh(\bar{y})[X_\alpha - \tanh(\bar{y})T],
\]  
(53)

\[
\delta_\alpha = [-\cos(2\vartheta_\alpha) + \mu \sin(2\bar{\vartheta})y_\alpha]t = [-\cos(2\vartheta_\alpha) + \mu \sin(2\bar{\vartheta})y_\alpha]\cosh(\bar{y})[T - \tanh(\bar{y})X_\alpha].
\]  
(54)

Also one must use the time transformation \(dT = \cosh(\bar{y})dt\) in the definition of \(\tau\) and the average phase \(\bar{\delta}\):

\[
\tau = \sin(2\bar{\vartheta})\text{sech}(\bar{y})T, \quad \bar{\delta} = -\cos(2\bar{\vartheta})\text{sech}(\bar{y})T.
\]

Now it is evident that, if equation (53) is used in the definition of \(q_\alpha\) (51), the term linear in \(T\) will not give contribution neither to the difference \(q_\alpha - q_\beta\) nor to the second derivative of \(q_\alpha\), hence it can be dropped. Further, notice that from the r.h.s. of equation (54) only the term linear in \(T\) will appear in \(q_\alpha\) (51), if one simply changes the time: \(t \rightarrow \text{sech}(\bar{y})T\). Hence, the term proportional to \(X_\alpha\) in (54) must be subtructed from the central phase \(\delta_\alpha\). In doing so, one can neglect the difference between \(\vartheta_\alpha\) and \(\bar{\vartheta}\) due to the inequalities (7) and that evolution of \(\vartheta_\alpha\) is of order \(O(\epsilon)\) (i.e., we throw away the second order terms from

18
the second derivative of $q_\alpha$). Thus we have arrived precisely at the quantity $Q_\alpha$ given by equation (17), where the shift of the soliton rapidities is taken into account: $y_\alpha \to \bar{y} + y_\alpha$. Therefore the result of section 2 is proven.

V. COMMENTS

The complex Toda chain model proves to be an universal model for the adiabatic description of the train interaction/propagation of solitons in nonlinear PDEs. Indeed, it was shown to describe the train propagation of pulses in the nonlinear PDEs of the whole NLS hierarchy [9] (i.e., the PDEs associated with the familiar Zakharov-Shabat spectral problem [51, 52]). More recently, the complex Toda chain was derived for the soliton train of the modified NLS equation [10]. This PDE is associated with the quadratic bundle, also known as the Wadati-Konno-Ichikawa spectral problem [53].

In this paper, the complex Toda chain is shown to describe the soliton train propagation in the massive Thirring model. Note that, as it is mentioned in Ref. [10], the massive Thirring model is just another representative of the modified NLS hierarchy. Thus the complex Toda chain arises in the adiabatic description of the soliton trains in the hierarchy of nonlinear integrable PDEs associated with the quadratic bundle as well. This is in favor of the universality of the complex Toda chain.

In construction of the perturbation theory for the massive Thirring model we have used the associated Riemann-Hilbert problem [49]. The usage of the Riemann-Hilbert problem allows one to develop the perturbation theory in an unified way for the entire hierarchy (see, for instance, Ref. [54], where this was done for the vector NLS hierarchy). Moreover, the perturbation-induced evolution equations for the spectral data have one and the same form for all integrable PDEs (one can compare the results of Refs. [55, 56]). This gives a possibility to prove the universality of the complex Toda chain using the approach based on the Riemann-Hilbert problem. This is one of the directions for future work.

In view of recent experimental observation [34] of the multiple gap soliton formation in optical fibers with index grating, it is important to have an analytical approach for description of interaction of optical gap solitons. In this paper an analytical approach is developed for the train interaction/propagation of gap solitons: the adiabatic propagation of $N$ gap solitons with nearly equal amplitudes and velocities is governed by a generalised
complex Toda chain with $N$ nodes.

Here I should point out that, due to non-integrability of the optical gap system, the train of gap solitons may become unstable. This instability is the result of the soliton-radiation interaction and is beyond the adiabatic approximation. However, the gap soliton is stable if the soliton amplitude lies below the instability threshold (see for details Ref. [30]). In that case, the generalised complex Toda chain (18) can be applied. Though in accordance with the Ref. [3] the generalised complex Toda chain is not integrable, it is just a finite dimensional dynamical system and can be investigated by the standard techniques. Moreover, in accordance with discussion of Ref. [4], one can systematically include various additional perturbations of the optical gap system into the complex Toda chain. This is the direction for further work.

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