Discriminating sterile neutrinos and unitarity violation with CP invariants

Heinrich Päś and Philipp Sicking

Fakultät für Physik, Technische Universität Dortmund, 44221 Dortmund, Germany

Abstract

We present a new method to analyze upcoming results in the search for CP violating neutrino oscillations. The CP violating amplitudes $A^{kj}_{\alpha\beta}$ provide parametrization independent observables, which will be accessible by experiments soon. The strong prediction of a unique $A^{kj}_{\alpha\beta}$ (the Jarlskog invariant) in case of the standard three neutrino model does not hold in models with new physics beyond the Standard Model. Nevertheless there are still correlations among the amplitudes depending on the specific model. Due to these correlations it is possible to reject specific new physics models by determining only 3 of the CP violating amplitudes.
I. INTRODUCTION

The experimental observation of neutrino oscillations and its interpretation as a consequence of neutrino masses provided the first manifestation of new physics beyond the Standard Model (SM). The first conclusive evidence of neutrino oscillation by SNO \cite{1, 2} and Super-Kamiokande \cite{3} was honored recently by the Nobel Prize of Physics in 2015. With the exception of some anomalies, almost all current data can be well explained by a model of three neutrinos with two mass squared differences, $\Delta m^2_{31}$ and $\Delta m^2_{21}$, three mixing angles $\theta_{12}$, $\theta_{23}$ and $\theta_{31}$, and one CP phase $\delta$ \cite{4}. All parameters are measured to a relatively high precision, except for the octant of $\theta_{23}$, the mass-ordering and the CP phase. Ongoing and upcoming neutrino experiments will narrow down the viable space for these parameters (see \cite{5} for a review). A first hint for a maximal $\delta = [-2.03, -0.49](\text{NH}), [-1.87, -0.98](\text{IH})$ at 90\% CL has been reported by T2K \cite{6, 7}.

This situation cannot be understood as a proof of the minimal three neutrino picture, though. As has been shown by several authors, new physics models can fake a signal at current experiments which look like satisfying the three neutrino paradigm \cite{8–12}.

Neutrino oscillation probabilities are described by introducing the mixing matrix $U$, parametrizing the transformation from neutrino mass to flavor eigenstates, $|\nu_\alpha\rangle = \sum_k U_{\alpha k} |\nu_k\rangle$:

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* e^{-i \frac{\Delta m_{kj}^2 L}{2E}}$$

$$= \delta_{\alpha\beta} - 4 \sum_{k>j}^N \text{Re} (U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*) \sin^2 \left( \frac{\Delta m_{kj}^2 L}{4E} \right)$$

$$+ 2 \sum_{k>j}^N \text{Im} (U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*) \sin \left( \frac{\Delta m_{kj}^2 L}{2E} \right),$$

where $A^{kj}_{\alpha\beta} = \text{Im}(U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*)$. For antineutrinos the last term switches its sign, so the CP violation $P_{\nu_\alpha \rightarrow \nu_\beta} - P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta}$ depends only on the CP violating amplitudes $A^{kj}_{\alpha\beta}$. Here, $N$ indicates the number of light neutrinos involved in the oscillation process. If all neutrino mass eigenstates involved in the oscillation process are light compared to the beam energy $E$, the mixing matrix $U$ is unitary. If, on the other hand, heavy flavors are integrated out, the resulting effective mixing matrix $U$ can be non-unitary. Note that in this case in addition to neutrino oscillations zero-distance-effects can arise, which we do not consider in this work.

A common approximative parametrization used in the literature is based on a series expans-
sion in \( \alpha = \frac{\Delta m^2_{31}}{\Delta m^2_{21}} \):

\[
P_{\nu_e \to \nu_\mu} \sim A \sin^2 \Delta + \alpha \sin \delta B \sin^3 \Delta + \alpha \cos \delta C \cos \Delta \sin^2 \Delta + \alpha^2 D \sin^2 \Delta
\]

with \( \Delta = \frac{\Delta m^2_{31} L}{4E} \) and \( A, B, C, D \) are functions of the standard mixing angles \[13\].

The CP violating term (proportional to \( \sin \delta \)) is suppressed by \( \alpha \) but the unitarity of \( U^{3 \times 3} \) is implicitly used to derive this formula. Various efforts exist in the literature to improve the above approximation for new, more exact or shorter parametrizations \[14–22\] or to include matter effects \[23–31\].

Here we rely on the exact expressions given in equation (2) instead, which is invariant under reparametrization. In particular the CP violating amplitudes \( A_{kj}^{\alpha \beta} \) are independent of the parametrization \[32, 33\] and can be determined in various extensions to the SM case. A specific feature which had already been pointed out by Jarlskog \[34, 35\] is that in the case of exactly three flavors and a unitary mixing matrix \( U \), all CP violating amplitudes \( A_{kj}^{\alpha \beta} \) have identical absolute values. This observation was first exploited in the quark sector where the famous CKM unitarity triangle provides a precise test for unitarity and therefore for the SM itself. Analyses of the lepton sector in terms of unitarity triangles have been worked out in \[36–39\], but the insights are limited in cases where the triangle does not close, since the source of unitary violation cannot be determined.

Inspired by previous work \[32, 33\] we take a closer look to sums and ratios of the CP violating amplitudes \( A_{kj}^{\alpha \beta} \) and find useful correlations among them. These correlations depend highly on the specific model and therefore provide a useful test for new physics in CP violating neutrino oscillations.

### II. ANALYTIC TREATMENT OF 3+1 \( \nu \)

A popular extension of the three neutrino model is to add an additional light sterile neutrino \[40, 41\]. This is motivated by the LSND \[42\], MiniBooNE \[43\], reactor \[44\] and gallium anomalies \[45\] but in conflict with a recent IceCube analysis \[46\]. In this model the mixing matrix \( U \) is now a \( 4 \times 4 \) unitary mixing matrix but the \( 3 \times 3 \) sub matrix is not unitary anymore. Although the resulting amplitudes are no longer unique, they are related due to the unitarity of the complete mixing matrix. By exploiting these relations in the context of the quark sector it has been shown for four flavors that all amplitudes can be reduced.
to only three independent CP violating amplitudes \[47\]. In the following we follow these arguments translated to the notation commonly used in neutrino physics.

In total there exist \(4 \times 4 \times 4 \times 4 = 256\) (\(\alpha, \beta \in \{e, \mu, \tau, s\}\) and \(k, j \in \{1, 2, 3, 4\}\)) different CP violating amplitudes \(A_{\alpha\beta}^{kj} = \text{Im}(U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*)\), whereas the number is strongly reduced by the fact that \(A_{\alpha\beta}^{kj} = 0\) for \(\alpha = \beta\) or \(k = j\) and due to symmetry, \(A_{\alpha\beta}^{kj} = A_{\beta\alpha}^{kj}\) and \(A_{\alpha\beta}^{kj} = A_{\alpha\beta}^{jk}\). Therefore it is sufficient to only consider \(A_{\alpha\beta}^{kj}\) where \(\alpha < \beta\) and \(k > j\). Note that the previous relations hold due to the definition of \(A_{\alpha\beta}^{kj}\) regardless of the underlying \(U\) and are not specific for the 3+1\(\nu\) model. This reduces the number of CP violating amplitudes to 36. These 36 amplitudes are not independent of each other and can be expressed via only nine amplitudes (see Appendix [A]). Again, these nine amplitudes can be expressed by three remaining amplitudes via the following expression

\[
\begin{pmatrix}
A_{e\mu}^{32} \\
A_{e\mu}^{43} \\
A_{\mu\tau}^{21} \\
A_{\mu\tau}^{32} \\
A_{\tau s}^{21} \\
A_{\tau s}^{32}
\end{pmatrix} = M^{-1}
\begin{pmatrix}
R_{e\mu}^{22} A_{e\mu}^{21} \\
R_{\mu\tau}^{43} A_{\mu\tau}^{43} \\
R_{\mu\tau}^{21} A_{\mu\tau}^{21} \\
R_{\tau s}^{32} A_{\tau s}^{32} \\
(R_{\tau \tau} + R_{e\mu}) A_{\mu\tau} \\
(R_{\mu \tau} + R_{e\mu}) A_{\mu\tau}
\end{pmatrix}, \tag{4}
\]

with \(M^{-1}\) defined by the inverse of

\[
M = \begin{pmatrix}
-(R_{e\mu}^{22} + R_{e\mu}^{21}) & R_{e\mu}^{22} & 0 & 0 & 0 & 0 \\
0 & R_{\mu\tau}^{43} & 0 & -(R_{\mu\tau}^{43} + R_{\tau s}) & 0 & 0 \\
0 & 0 & -(R_{\mu\tau}^{21} + R_{e\mu}^{21}) & 0 & R_{\mu\tau}^{21} & 0 \\
0 & 0 & 0 & 0 & R_{\tau s}^{33} - (R_{\tau s}^{33} + R_{\tau s}^{43}) & \ldots \\
R_{\tau \tau} & 0 & 0 & 0 & -R_{\mu\tau}^{33} & -R_{\mu\tau}^{32} \\
0 & 0 & 0 & 0 & 0 & 0 - R_{\mu\tau}^{32}
\end{pmatrix}. \tag{5}
\]

The amplitudes \(R_{\alpha\beta}^{kj} = \text{Re}(U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^*)\) correspond to the CP conserving amplitudes in neutrino oscillations. These relations therefore provide a connection between the CP violating and the CP conserving processes.

To emphasize the differences between 3\(\nu\) and 3+1\(\nu\) we want to highlight following relations:

\[
A_{e\mu}^{31} = -A_{e\mu}^{32} + A_{e\mu}^{43} \tag{6}
\]

\[
A_{e\tau}^{21} = -A_{\mu\tau}^{32} + A_{\mu\tau}^{43} \tag{7}
\]

\[
A_{e\tau}^{31} = -A_{e\tau}^{32} + A_{e\tau}^{43} + A_{\tau s}^{32} \tag{8}
\]
The relations reduce to the 3ν case, if no mixing with the light neutrino takes place. This corresponds to vanishing non diagonal elements in the fourth line and column of $U$. Consequently, all amplitudes vanish if $\alpha \vee \beta = s$ or $k \vee j = 4$. Due to the expected smallness of mixing with sterile states, the deviations from uniform amplitudes in the $3 \times 3$ sector could be treated in a perturbation approach.

III. NUMERIC ANALYSIS OF STERILE NEUTRINOS AND NON-UNITARY SCENARIOS

The relations in the previous section rely on the unitarity of the resulting $3 + 1\nu$ model. In general these relations are, if possible, harder to find and more complicated. An easier approach is to use a numeric analysis of the correlations of the different amplitudes for different models. Therefore we pick random numbers for all parameters in the specific model (SM and BSM parameters) and generate the resulting mixing matrix $U$. To check if the generated combination of parameters satisfy current experimental bounds, we compare the entries of the $3 \times 3$ sub matrix of $U$ with the bounds presented in [48], where a global fit is performed without implying a unitarity of $U^{3\times3}$.

$$|U|^{3\times3}_{3\sigma} = \begin{pmatrix}
0.76 \rightarrow 0.85 & 0.50 \rightarrow 0.60 & 0.13 \rightarrow 0.16 \\
0.21 \rightarrow 0.54 & 0.42 \rightarrow 0.70 & 0.61 \rightarrow 0.79 \\
0.18 \rightarrow 0.58 & 0.38 \rightarrow 0.72 & 0.40 \rightarrow 0.78
\end{pmatrix} \quad (9)$$

For a viable combination of parameters all accessible amplitudes $A^{kj}_{\alpha i}$ are calculated and extracted. For each model we extracted 100,000 viable combinations. To show the correlation we performed a kernel density estimation for different combination of amplitudes, i.e. estimating the underlying probability density function by summing up Gaussian kernels placed on every data point.

We compare 4 different approaches of neutrino physics beyond the three neutrino paradigm:

(i) a model of one additional light sterile neutrino ($3 + 1\nu$), motivated by LSND [42], MiniBooNE- [43], gallium- [45] and reactor anomaly [44]. Typically the additional mass squared difference lies in the $\sim 1$ eV range [40] [41]. Due to the low mass the sterile state participates in the oscillation. The sterile neutrino does not interact via
SM gauge interactions with other SM particles. The mixing matrix is a 4 × 4 unitary matrix (see sec. II for more details).

(ii) a model of two additional light sterile neutrinos (3 + 2ν), similar to model (i) but with an extended parameter space (additional mixing angles and CP phases) due to the additional sterile state. The mixing matrix is a 5 × 5 unitary matrix.

(iii) a scenario of non-unitarity without additional constraints (NU). This scenario is realized by modifying the unitary matrix with a lower triangular matrix $\alpha$

$$U_{NU} = (I - \alpha)U^{3 \times 3} = \begin{pmatrix}
1 - \alpha_{ee} & 0 & 0 \\
\alpha_{e\mu} & 1 - \alpha_{\mu\mu} & 0 \\
\alpha_{\tau e} & \alpha_{\mu\tau} & 1 - \alpha_{\tau\tau}
\end{pmatrix} U^{3 \times 3} \quad (10)$$

where $|\eta_{\alpha\beta}| < 1$. The diagonal entries are real and the off-diagonal entries are complex parameters (see for instance [49–51]).

(iv) a scenario of non-unitarity where additional fermions trigger rare decays like $\mu \rightarrow e\gamma$. The corresponding constraints from rare decays and electroweak precision observables are presented in [52] (“minimal flavor violation” MUV, the non unitarity is parametrized as in scenario (iii))

$$\alpha_{ee} < 1.3 \cdot 10^{-3}, \quad |\alpha_{\mu e}| < 6.8 \cdot 10^{-4},$$

$$\alpha_{\mu\mu} < 2.0 \cdot 10^{-4}, \quad |\alpha_{\tau e}| < 2.7 \cdot 10^{-3}, \quad (11)$$

$$\alpha_{\tau\tau} < 2.8 \cdot 10^{-3}, \quad |\alpha_{\tau\mu}| < 1.2 \cdot 10^{-3}.$$ 

These constraints are used as priors in our numeric analysis. Many new physics models can influence neutrino oscillation in a way described by NU and MUV. For instance heavy right handed neutrinos introduced in seesaw models or non standard neutrino interaction (NSI) at production and detection can be described by the MUV and NU scenarios, respectively.

IV. RESULTS

The 95% CL of the generated kernel density estimates for oscillations of $\nu_\mu$ are shown in figures 1 and 2. We focus on these modes since the production of $\nu_\mu$ is well understood and
the modes are investigated by several current experiments. We do not consider amplitudes where sterile states are involved due to missing detection mechanisms. We also do not consider amplitudes with additional mass differences beyond the solar and atmospheric $\Delta m_{12}^2$ and $\Delta m_{23}^2$ since these are by now not known and current experiments are optimized for the known mass squared differences. As can be seen clearly for the scenarios with additional light neutrinos and non unitarity without constraints the corresponding parameter spaces allow for significant deviation from the SM prediction of uniform CP violating. The MUV scenario albeit provides only a comparatively small allowed region. The strong constraints for the unitary violating parameters $\alpha$ (see equation (11)) as priors strongly restrict deviations from the SM prediction. The allowed regions fulfill all current bounds and display the uncertainties in equation (9) and the not yet determined CP phase(s).

The differences between the $3 + 1\nu$- and $3 + 2\nu$-model are negligible. Due to invariance under re-parametrization the amplitudes in the $3 \times 3$ sub matrix do not change by rotations in the 4-5-Plane in case of a $3 + 2\nu$-model. To investigate a difference between $3 + 1\nu$ and $3 + 2\nu$ scenarios, amplitudes with sterile states or additional mass squared differences have to be taken into account which are not expected to be accessible experimentally in the near future.

Comparing the models with additional light neutrinos with the scenario of unconstrained non-unitarity one can find large deviations. The scenario of non unitarity provides viable parameter sets which are far outside the 95% CL of the models with additional light neutrinos.

The MUV scenario provides only a small deviation from the SM due to the strong constraints from electroweak precision observables. The expected deviations are out of reach of current experiments. Therefore a sizable measured deviation from the SM has to have another source than the MUV scenario.

Hence the experimental measurement of the corresponding CP violating amplitudes can be a direct test for the three neutrino paradigm and can also discriminate between different SM extensions: If the experimental values will turn out to lie outside a viable region of $3 + 1\nu$, $3 + 2\nu$ or the MUV scenario these models can be ruled out consistently.

Similar plots have been fabricated for all combinations of amplitudes and yield similar results. Whether the best discriminators are provided by the sums or the ratios of amplitudes will turn out once experimental data will be available.
Figure 1. Kernel density estimates for the different scenarios: 3 + 1$\nu$ in red, 3 + 2$\nu$ in blue, Non-Unitarity in yellow and Minimal Unitarity Violation in green. Shown is the differences of the 3 different CP violating amplitudes in the $\nu_e \rightarrow \nu_\mu$-channel. The colored area corresponds to the 95% CL of the KDE. The three neutrino prediction corresponds to the point at (0, 0). Except for numerical effects, the areas for the 3 + 1$\nu$ and the 3 + 2$\nu$ model match each other. A significant deviation between NU and new sterile states can be observed. Due to the strong constraints for MUV, the viable regions are extremely small and deviations from three neutrino prediction will be hard to measure.
Figure 2. Kernel density estimates for the different scenarios: $3 + 1\nu$ in red, $3 + 2\nu$ in blue, Non-Unitarity in yellow and Minimal Unitarity Violation in green. Shown is the ratios of the 3 different CP violating amplitudes in the $\nu_\mu \rightarrow \nu_\tau$-channel. The colored area corresponds to the 95% CL of the KDE. The three neutrino prediction corresponds to the point at $(1, -1)$. Except for numerical effects, the areas for the $3 + 1\nu$ and the $3 + 2\nu$ model match each other. A significant deviation between NU and new sterile states can be observed. Due to the strong constraints for MUV, the viable regions are extremely small and deviations from three neutrino prediction will be hard to measure.

V. CONCLUSION

In this work we have developed a new method to test and discriminate the standard three neutrino paradigm and several extensions based on the study of various combinations of CP violating amplitudes $A_{\alpha\beta}^{kj}$. These amplitudes are easily accessible via oscillation experiments searching for CP violation. The amplitudes and the relations among them have been trans-
lated into the notation commonly used in the neutrino community. Moreover, the concept has been generalized to scenarios with five neutrinos and non-unitary mixing matrices. Powerful discriminators between different scenarios of physics beyond the SM can be exploited once experiments determine three different amplitudes. In this case it is possible to rule out not only the three neutrino paradigm but also models of additional sterile light neutrinos or the scenario of MUV in large regions of the respective parameter spaces. On the other hand, a determination of a unique amplitude would be in agreement with both the three neutrino model but also with specific parameter combinations of new physics models. Note, that these calculations rely on the vacuum values of neutrino properties. They are independent of specific mass differences. Matter effects are not included yet but will be addressed in future work. Thus a comparison between different experimental results involving matter effects requires some care.

VI. ACKNOWLEDGEMENT

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Appendix A: Analytic relations of CP violating amplitudes

\[ A_{e\mu}^{21} = A_{e\mu}^{21}, \quad A_{e\mu}^{31} = -A_{e\mu}^{32} + A_{e\mu}^{33}, \]
\[ A_{e\mu}^{41} = -A_{e\mu}^{21} + A_{e\mu}^{32} - A_{e\mu}^{43}, \quad A_{e\mu}^{42} = A_{e\mu}^{21} - A_{e\mu}^{32}, \]
\[ A_{e\mu}^{43} = -A_{e\mu}^{21} + A_{e\mu}^{21}, \quad A_{e\mu}^{43} = A_{e\mu}^{43}, \]
\[ A_{e\mu}^{21} = -A_{e\mu}^{21} + A_{e\mu}^{21}, \quad A_{e\mu}^{21} = A_{e\mu}^{21}, \]
\[ A_{e\mu}^{31} = A_{e\mu}^{31} - A_{e\mu}^{32} + A_{e\mu}^{33} - A_{e\mu}^{32} - A_{e\mu}^{33}, \]
\[ A_{e\mu}^{41} = -A_{e\mu}^{32} + A_{e\mu}^{32}, \quad A_{e\mu}^{42} = A_{e\mu}^{32} + A_{e\mu}^{32}, \]
\[ A_{e\mu}^{43} = -A_{e\mu}^{32} + A_{e\mu}^{32}, \quad A_{e\mu}^{43} = A_{e\mu}^{32} - A_{e\mu}^{32}, \]
\[ A_{e\mu}^{21} = -A_{e\mu}^{21} + A_{e\mu}^{21} + A_{e\mu}^{21} + A_{e\mu}^{21}, \quad A_{e\mu}^{21} = A_{e\mu}^{21}, \]
\[ A_{e\mu}^{31} = -A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31}, \quad A_{e\mu}^{31} = A_{e\mu}^{31}, \]
\[ A_{e\mu}^{41} = -A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31}, \quad A_{e\mu}^{41} = A_{e\mu}^{31}, \]
\[ A_{e\mu}^{21} = -A_{e\mu}^{21} + A_{e\mu}^{21} + A_{e\mu}^{21} + A_{e\mu}^{21}, \quad A_{e\mu}^{21} = A_{e\mu}^{21}, \]
\[ A_{e\mu}^{31} = -A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31}, \quad A_{e\mu}^{31} = A_{e\mu}^{31}, \]
\[ A_{e\mu}^{41} = -A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31}, \quad A_{e\mu}^{41} = A_{e\mu}^{31}, \]
\[ A_{e\mu}^{21} = -A_{e\mu}^{21} + A_{e\mu}^{21} + A_{e\mu}^{21} + A_{e\mu}^{21}, \quad A_{e\mu}^{21} = A_{e\mu}^{21}, \]
\[ A_{e\mu}^{31} = -A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31}, \quad A_{e\mu}^{31} = A_{e\mu}^{31}, \]
\[ A_{e\mu}^{41} = -A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31} + A_{e\mu}^{31}, \quad A_{e\mu}^{41} = A_{e\mu}^{31}, \]

[1] Q. R. Ahmad et al. Direct evidence for neutrino flavor transformation from neutral current interactions in the Sudbury Neutrino Observatory. *Phys. Rev. Lett.*, 89:011301, 2002.
[2] N. Barros. Final results from SNO. *Nucl. Phys. Proc. Suppl.*, 237-238:107–110, 2013.
[3] Y. Fukuda et al. Evidence for oscillation of atmospheric neutrinos. *Phys. Rev. Lett.*, 81:1562–1567, 1998.
[4] Ivan Esteban, M. C. Gonzalez-Garcia, Michele Maltoni, Ivan Martinez-Soler, and Thomas Schwetz. Updated fit to three neutrino mixing: exploring the accelerator-reactor complemen-
[5] K. A. Olive et al. Review of Particle Physics. *Chin. Phys.*, C38:090001, 2014.

[6] K. Abe et al. Measurements of neutrino oscillation in appearance and disappearance channels by the T2K experiment with $6.6 \times 10^{20}$ protons on target. *Phys. Rev.*, D91(7):072010, 2015.

[7] Hirohisa Tanaka for the T2K collaboration. Status, recent results and plans for T2K. Presented at Neutrino 2016 conference, 2016.

[8] Srubabati Goswami and Toshihiko Ota. Testing non-unitarity of neutrino mixing matrices at neutrino factories. *Phys. Rev.*, D78:033012, 2008.

[9] O. G. Miranda, M. Tortola, and J. W. F. Valle. New ambiguity in probing CP violation in neutrino oscillations. *Phys. Rev. Lett.*, 117(6):061804, 2016.

[10] Shao-Feng Ge, Pedro Pasquini, M. Tortola, and J. W. F. Valle. Measuring the Leptonic CP Phase in Neutrino Oscillations with Non-Unitary Mixing. 2016.

[11] André de Gouvêa and Kevin J. Kelly. False Signals of CP-Invariance Violation at DUNE. 2016.

[12] Debajyoti Dutta and Pomita Ghoshal. Probing CP violation with T2K, NOνA and DUNE in the presence of non-unitarity. *JHEP*, 09:110, 2016.

[13] Martin Freund. Analytic approximations for three neutrino oscillation parameters and probabilities in matter. *Phys. Rev.*, D64:053003, 2001.

[14] M. C. Gonzalez-Garcia, Y. Grossman, A. Gusso, and Y. Nir. New CP violation in neutrino oscillations. *Phys. Rev.*, D64:096006, 2001.

[15] Virendra Gupta and Xiao-Gang He. A New parametrization of the neutrino mixing matrix for neutrino oscillations. *Phys. Rev.*, D64:117301, 2001.

[16] Wan-lei Guo and Zhi-zhong Xing. Rephasing invariants of CP and T violation in the four neutrino mixing models. *Phys. Rev.*, D65:073020, 2002.

[17] Massimo Blasone, Antonio Capolupo, and Giuseppe Vitiello. Quantum field theory of three flavor neutrino mixing and oscillations with CP violation. *Phys. Rev.*, D66:025033, 2002.

[18] Werner Rodejohann. A Parametrization for the neutrino mixing matrix. *Phys. Rev.*, D69:033005, 2004.

[19] Nan Li and Bo-Qiang Ma. A New parametrization of the neutrino mixing matrix. *Phys. Lett.*, B600:248–254, 2004.
[20] G. Dattoli and K. V. Zhukovsky. Neutrino mixing and the exponential form of the Pontecorvo-Maki-Nakagawa-Sakata matrix. *Eur. Phys. J.*, C55:547–552, 2008.

[21] W. Rodejohann and J. W. F. Valle. Symmetrical Parametrizations of the Lepton Mixing Matrix. *Phys. Rev.*, D84:073011, 2011.

[22] Melin Huang, Dawei Liu, Jen-Chieh Peng, S. D. Reitzner, and Wei-Chun Tsai. Dependence of Neutrino Mixing Angles and CP-violating Phase on Mixing Matrix Parametrizations. 2011.

[23] K. Kimura, A. Takamura, and H. Yokomakura. Exact formula of probability and CP violation for neutrino oscillations in matter. *Phys. Lett.*, B537:86–94, 2002.

[24] Evgeny K. Akhmedov, Robert Johansson, Manfred Lindner, Tommy Ohlsson, and Thomas Schwetz. Series expansions for three flavor neutrino oscillation probabilities in matter. *JHEP*, 04:078, 2004.

[25] Ye-Ling Zhou. The Kobayashi-Maskawa Parametrization of Lepton Flavor Mixing and Its Application to Neutrino Oscillations in Matter. *Phys. Rev.*, D84:113012, 2011.

[26] L. J. Flores and O. G. Miranda. Constant matter neutrino oscillations in a parametrization-free formulation. *Phys. Rev.*, D93(3):033009, 2016.

[27] Peter B. Denton, Hisakazu Minakata, and Stephen J. Parke. Compact Perturbative Expressions For Neutrino Oscillations in Matter. *JHEP*, 06:051, 2016.

[28] Mikkel B. Johnson and Leonard S. Kisslinger. Simplified Theory of Neutrino Oscillations in Matter with CP violation. 2016.

[29] Mikkel B. Johnson, Ernest M. Henley, and Leonard S. Kisslinger. Analytical Theory of Neutrino Oscillations in Matter with CP violation. *Phys. Rev.*, D91(7):076005, 2015.

[30] L. J. Flores and O. G. Miranda. Matter neutrino oscillations, an approximation in a parametrization-free framework. *J. Phys. Conf. Ser.*, 761(1):012041, 2016.

[31] Hisakazu Minakata and Stephen J Parke. Simple and Compact Expressions for Neutrino Oscillation Probabilities in Matter. *JHEP*, 01:180, 2016.

[32] D. J. Wagner and Thomas J. Weiler. Boxing with neutrino oscillations. *Phys. Rev.*, D59:113007, 1999.

[33] Thomas J. Weiler and D. J. Wagner. Invariant box parameterization of neutrino oscillations. *AIP Conf. Proc.*, 444:46–58, 1998.

[34] C. Jarlskog. Commutator of the quark mass matrices in the standard electroweak model and a measure of maximal CP nonconservation. *Phys. Rev. Lett.*, 55:1039–1042, Sep 1985.
[35] C. Jarlskog. Flavor projection operators and applications to CP violation with any number of families. *Phys. Rev. D*, 36:2128–2136, Oct 1987.

[36] Zhi-zhong Xing and He Zhang. Reconstruction of the neutrino mixing matrix and leptonic unitarity triangles from long-baseline neutrino oscillations. *Phys. Lett.*, B618:131–140, 2005.

[37] James D. Bjorken, P. F. Harrison, and W. G. Scott. Simplified unitarity triangles for the lepton sector. *Phys. Rev.*, D74:073012, 2006.

[38] Zhi-Zhong Xing. Implications of the Daya Bay observation of $\theta_{13}$ on the leptonic flavor mixing structure and CP violation. *Chin. Phys.*, C36:281–297, 2012.

[39] Hong-Jian He and Xun-Jie Xu. Connecting Leptonic Unitarity Triangle to Neutrino Oscillation with CP Violation in Vacuum and in Matter. 2016.

[40] Joachim Kopp, Pedro A. N. Machado, Michele Maltoni, and Thomas Schwetz. Sterile Neutrino Oscillations: The Global Picture. *JHEP*, 05:050, 2013.

[41] K. N. Abazajian et al. Light Sterile Neutrinos: A White Paper. 2012.

[42] A. Aguilar-Arevalo et al. Evidence for neutrino oscillations from the observation of anti-neutrino(electron) appearance in a anti-neutrino(muon) beam. *Phys. Rev.*, D64:112007, 2001.

[43] A. A. Aguilar-Arevalo et al. Event Excess in the MiniBooNE Search for $\bar{\nu}_{\mu} \rightarrow \bar{\nu}_{e}$ Oscillations. *Phys. Rev. Lett.*, 105:181801, 2010.

[44] G. Mention, M. Fechner, Th. Lasserre, Th. A. Mueller, D. Lhuillier, M. Cribier, and A. Le-tourneau. The Reactor Antineutrino Anomaly. *Phys. Rev.*, D83:073006, 2011.

[45] Carlo Giunti and Marco Laveder. Statistical Significance of the Gallium Anomaly. *Phys. Rev.*, C83:065504, 2011.

[46] M. G. Aartsen et al. Searches for Sterile Neutrinos with the IceCube Detector. *Phys. Rev. Lett.*, 117(7):071801, 2016.

[47] T. Suzuki. Some formulas for invariant phases of unitary matrices by Jarlskog. *Journal of Mathematical Physics*, 50(12):123526–123526, December 2009.

[48] Stephen Parke and Mark Ross-Lonergan. Unitarity and the three flavor neutrino mixing matrix. *Phys. Rev.*, D93(11):113009, 2016.

[49] F. J. Escrihuela, D. V. Forero, O. G. Miranda, M. Tortola, and J. W. F. Valle. On the description of nonunitary neutrino mixing. *Phys. Rev.*, D92(5):053009, 2015. [Erratum: Phys. Rev.D93,no.11,119905(2016)].
[50] E. Fernandez-Martinez, M. B. Gavela, J. Lopez-Pavon, and O. Yasuda. CP-violation from non-unitary leptonic mixing. *Phys. Lett.*, B649:427–435, 2007.

[51] S. Antusch, C. Biggio, E. Fernandez-Martinez, M. B. Gavela, and J. Lopez-Pavon. Unitarity of the Leptonic Mixing Matrix. *JHEP*, 10:084, 2006.

[52] Mattias Blennow, Pilar Coloma, Enrique Fernandez-Martinez, Josu Hernandez-Garcia, and Jacobo Lopez-Pavon. Non-Unitarity, sterile neutrinos, and Non-Standard neutrino Interactions. 2016.