On the reversal of radial SLE, I:
Commutation Relations in Annuli

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Abstract

We aim at finding the reversal of radial SLE and proving the reversibility of whole-plane SLE. For this purpose, we define annulus SLE($\kappa, \Lambda$) processes in doubly connected domains with one marked boundary point. We derive some partial differential equation for $\Lambda$, which is sufficient for the annulus SLE($\kappa, \Lambda$) process to satisfy commutation relation. If $\Lambda$ satisfies this PDE, then using a coupling technique, we are able to construct a global commutation coupling of two annulus SLE($\kappa, \Lambda$) processes. If more conditions are satisfied, the coupling exists in the degenerate case, which becomes a coupling of two whole-plane SLE$_\kappa$ processes. The reversibility of whole-plane SLE$_\kappa$ follows from this coupling together with the assumption that such annulus SLE($\kappa, \Lambda$) trace ends at the marked point. We then conclude that the limit of such annulus SLE($\kappa, \Lambda$) trace is the reversal of radial SLE$_\kappa$ trace. In the end, we derive some particular solutions to the PDE for $\Lambda$.

1 Introduction

The stochastic Loewner evolution (SLE) process introduced by Oded Schramm ([16]) describes some random fractal curves in plane domains that satisfy conformal invariance and Domain Markov Property. These two properties make SLE the most suitable candidates for the scaling limits of many two-dimensional lattice models at criticality. These models are proved or conjectured to converge to SLE with different parameters (e.g., [19] [11] [10] [17] [18] [20]). For basics of SLE, the reader may refer to [15] and [7].

There are several different versions of SLE, among which chordal SLE and radial SLE are most well-known. A chordal or radial SLE trace is a random fractal curve that grows in a simply connected plane domain from a boundary point. The difference is that chordal SLE trace ends at another boundary point, while radial SLE trace ends at an interior point. Their behaviors both depend on a parameter $\kappa > 0$. When $\kappa \in (0, 4]$, both traces are simple curves, and all points on the trace lie inside the domain except the end points.

A coupling technique was introduced in [25] to prove that, for $\kappa \in (0, 4]$, chordal SLE$_\kappa$ satisfies reversibility, which means that if $\beta$ is a chordal SLE$_\kappa$ trace in a domain $D$ from $a$ to $b$, then after a time-change, the reversal of $\beta$ has the distribution of a chordal SLE$_\kappa$ trace in $D$ from $b$ to $a$. We use the coupling technique to construct a coupling of two chordal SLE$_\kappa$ traces: one is from $a$ to $b$, the other is from $b$ to $a$, such that the two curves overlap with each other.
The technique was later used to prove the Duplantier’s duality conjecture ([26][27]), which says that, for $\kappa > 4$, the boundary of the hull generated by a chordal SLE$_\kappa$ trace looks locally like an SLE$_{16/\kappa}$ trace. It was also used to prove the reversibility of SLE($\kappa, \rho$) trace with degenerate force point when $\kappa \in (0, 4]$ and $\rho \geq \kappa/2 - 2$ ([28]).

Our goal now is to find the reversal of radial SLE$_\kappa$ traces for $\kappa \in (0, 4]$. Unlike the case of chordal SLE, the initial point and terminal point of a radial SLE are topologically different, so the reversal of a radial SLE trace can not be a radial SLE trace. However, we may consider whole-plane SLE instead, which is another kind of well-known SLE. It describes a random fractal curve in the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ that grows from one point in $\hat{\mathbb{C}}$ to another point in $\hat{\mathbb{C}}$. Whole-plane SLE is closely related to radial SLE. In the case of $\kappa \in (0, 4]$, a whole-plane SLE$_\kappa$ trace is a simple curve. Let $\beta$ be such a curve that starts from $a$ and ends at $b$. Let $T$ be a stopping time for $\beta$ such that $\beta(T)$ is neither $a$ nor $b$. Let $D_T$ be $\hat{\mathbb{C}}$ without the part of $\beta$ before $T$. Then $D_T$ is a simply connected domain, $b$ is an interior point of $D_T$, and $\beta(T)$ is a boundary point of $D_T$. If we condition on the part of $\beta$ before $T$, then the part of $\beta$ after $T$ has the distribution of a radial SLE$_\kappa$ curve in $D_T$ from $\beta(T)$ to $b$.

**Conjecture 1** The whole-plane SLE$_\kappa$ trace satisfies reversibility for $\kappa \in (0, 4]$.

The conjecture in the case $\kappa = 2$ has been proved in [29]. One result in [29] is: given any $z_1 \neq z_2 \in \mathbb{C}$, the loop-erased random walk (LERW) on the lattice $\delta \mathbb{Z}^2$ from $z_1^{(\delta)}$ to $z_2^{(\delta)}$, where $z_j^{(\delta)}$ is the vertex closest to $z_j$, $j = 1, 2$, converges to the whole-plane SLE$_2$ trace in $\hat{\mathbb{C}}$ from $z_1$ to $z_2$, as $\delta \to 0^+$. So the conjecture in the case $\kappa = 2$ follows from the reversibility of LERW ([6]). But we are still interested in some proof without lattice models.

It turns out that proving Conjecture 1 is closely related to finding the reversal of radial SLE. Assume that the conjecture holds for some $\kappa \in (0, 4]$. Then we have a coupling of two whole-plane SLE$_\kappa$ traces: $\beta_1$ and $\beta_2$, which overlap each other and have opposite directions. Fix $j \neq k \in \{1, 2\}$, i.e., $(j, k) = (1, 2)$ or $(2, 1)$. Let $T_k$ be a finite stopping time for $\beta_k$. Let $S_j$ be the first time that $\beta_j$ visits $\beta_k(T_k)$. If we condition on the part of $\beta_k$ before $T_k$, then from the property of whole-plane SLE$_\kappa$, the part of $\beta_k$ after $T_k$ is a radial SLE$_\kappa$ trace in the remaining domain from $\beta_k(T_k)$ to the end point of $\beta_k$. Since $\beta_j$ overlaps with $\beta_k$, so the part of $\beta_j$ before $S_j$ is the reversal of the above radial SLE$_\kappa$. On the other hand, we find that, to prove Conjecture 1 we need the information about the reversal of radial SLE$_\kappa$.

We can observe the following facts about the reversal curve, say $\beta$, of a radial SLE trace when $\kappa \in (0, 4]$. Such $\beta$ is a simple curve in a simply connected domain that grows from an interior point to a boundary point. After any non-degenerate initial part, the remaining domain is a doubly connected domain, and the rest part of $\beta$ grows in this doubly connected domain from one boundary point to another boundary point. This observation makes us believe that $\beta$ is the limit of some SLE-type processes defined for doubly connected domains. Moreover, if Conjecture 1 holds, such $\beta$ must satisfy some commutation relation. This means that in a doubly connected domain $D$ with two distinct boundary points $a$ and $b$, an annulus SLE trace started from $a$ with marked point $b$ commute with an annulus SLE trace started from $b$ with marked point $a$. 


In this paper, we use the annulus Loewner equation introduced in [22] together with some annulus drift function Λ to define the so-called annulus SLE(κ, Λ) process, which are used to describe the SLE process in a doubly connected domain with one marked boundary point other than the initial point. Then we define the disc SLE(κ, Λ) process as a natural limit of the above process. After this, we show that if the drift function Λ satisfies some partial differential equation involving elliptic functions (4.11), then we may construct a coupling of two annulus SLE(κ, Λ) processes that commute with each other.

We also find that, if Λ satisfies the above differential equation together with some condition about the behavior of Λ when the modulus tends to ∞, then the above coupling exists in the degenerate case: the two boundary components of the doubly connected domain shrink to two distinct points of ˆC, so we get a coupling of two whole-plane SLEκ traces. In addition, if we know that a disc SLE(κ, Λ) trace almost surely ends at the marked boundary point, then the above two whole-plane SLE traces overlap with each other, so Conjecture 1 is proved for the corresponding κ. We then immediately conclude that the reversal of a radial SLEκ trace is a disc SLE(κ, Λ) trace. This paper focuses on constructing the commutation coupling. Another paper in preparation ([30]) will discuss when the disc SLE(κ, Λ) trace ends at the marked point, and prove that Conjecture 1 at least holds for κ = 2, 3, 4.

The marked point and the initial point of an annulus SLE(κ, Λ) process could either lie on two different boundary components, or lie on the same boundary component. We study the first case for the proof of Conjecture 1. The second case is also interesting. We also derive some partial differential equation (4.58) about Λ which gives the commutation relation. The examples of such SLE include: a chordal SLE8/3 conditioned to avoid a hole (c.f. [10]); the scaling limits (if exist) of some lattice models in doubly connected domains such as loop-erased random walk (κ = 2, c.f. [11][24]), Gaussian free field contour line (κ = 4, c.f. [18]), uniform spanning tree Peano curve (κ = 8, c.f. [11]); and the critical Ising model (κ = 16/3, 3, c.f. [20]). We hope that the work in this paper will shed some light on the study of these processes.

The study about commutation relation of SLE in doubly connected domains continues the work in [5] by Julien Dubédat, who used some tools from Lie Algebra to derive commutation conditions of SLE in simply connected domains.

We may also use other Loewner equations, e.g., chordal Loewner equation, to define annulus SLE(κ, Λ) processes. In that case, Λ should be viewed as a pre-Schwarzian form of order 3 − κ/2, and the driving function for that Loewner equation is \( \sqrt{\kappa B(t)} \) plus a differentiable drift function, whose derivative is the expression of the form Λ in the corresponding boundary chart (c.f. [12]). We choose the annulus Loewner equation in this paper because the partial differential equation for the commutation relation is much simpler.

This paper is organized as follows. In Section 2 we review several Loewner equations and their covering equations. In Section 3 we define annulus SLE(κ, Λ) process and its limit case: disc SLE(κ, Λ) process. In Section 4 we prove that when Λ satisfies some partial differential equation then we have a commutation coupling of two annulus SLE(κ, Λ) processes. In Section 5 we construct a coupling of two whole-plane SLEκ processes as the limit of the coupling in the previous section. In the last section, for some special values of κ we derive some solutions of the partial differential equation for Λ given in Section 4.
2 Loewner Equations

2.1 Hulls and Loewner chains

Hulls and Loewner chains can be defined in any finitely connected plane domains (c.f. [24]). Here we only need these notation to be defined in simply or doubly connected domains. Throughout this paper, a simply connected domain is a plane domain that is conformally equivalent to a disc; and a doubly connected domain is a plane domain that is conformally equivalent to a non-degenerate annulus, so has finite modulus, which is denoted by mod(·). A relatively closed subset \( H \) of a simply connected domain \( D \) is called a hull of \( D \) if \( D \setminus H \) is also simply connected. Moreover, if \( z_0 \in D \setminus H \), then we say that \( H \) is a hull in \( D \) w.r.t. \( z_0 \). If \( D \) is a doubly connected domain, and \( C_2 \) is a boundary component, a relatively closed subset \( H \) of \( D \) is called a hull of \( D \) w.r.t \( C_2 \) if \( D \setminus H \) is a doubly connected domain that contains a neighborhood of \( C_2 \) in \( D \).

In this case, \( C_2 \) is also a boundary component of \( D \setminus H \).

If \( H \) is a hull in a simply connected domain \( D \) w.r.t. \( z_0 \in D \), then from Riemann mapping theorem, there is \( g \) that maps \( D \setminus H \) conformally onto \( D \), and fixes \( z_0 \). Such \( g \) is not unique, but \(|g'(z_0)|\) is determined by \( H \). Then \( \ln(|g'(z_0)|) \) is called the capacity of \( H \) in \( D \) w.r.t \( z_0 \), and is denoted by \( \text{cap}_{D;z_0}(H) \). If \( H \) is a hull in doubly connected domain \( D \), then \( D \setminus H \) is also a doubly connected domain. The difference of these two moduli, i.e., \( \text{mod}(D) - \text{mod}(D \setminus H) \), is called the capacity of \( H \) in \( D \), and is denoted by \( \text{cap}_D(H) \). In either of these two cases, the capacity of \( H \) is always \( \geq 0 \), and the equality holds iff \( H = \emptyset \); and if \( H_1 \subsetneq H_2 \) are two hulls then the capacity of \( H_1 \) is strictly less than the capacity of \( H_2 \).

Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \). So \( \mathbb{D} \) is a simply connected domain, \( 0 \in \mathbb{D} \), and \( \mathbb{T} = \partial \mathbb{D} \). For \( p > 0 \), let \( A_p = \{ z \in \mathbb{C} : e^{-p} < |z| < 1 \} \) and \( \mathbb{T}_p = \{ z \in \mathbb{C} : |z| = e^{-p} \} \). Then \( A_p \) is a doubly connected domain, \( \mathbb{T}_p \) and \( \mathbb{T} \) are two boundary components of \( A_p \), and \( \text{mod}(A_p) = p \). The following proposition relates the two different kinds of capacities.

**Proposition 2.1** Suppose \( z_0 \in \mathbb{T} \), \( I \) is an open arc on \( \mathbb{T} \) containing \( z_0 \), and \( \Omega \) is a neighborhood of \( I \) in \( \mathbb{D} \), \( j = 1, 2 \). Let \( W \) be a conformal map from \( \Omega \) into \( \mathbb{D} \), such that if \( z \to I \) in \( \Omega \), then \( W(z) \to \mathbb{T} \). From Schwarz reflection principle, \( W \) extends conformally across \( I \), and maps \( I \) into \( \mathbb{T} \). Especially, \( W \) is analytic at \( z_0 \). For any \( p, p_1, p_2 > 0 \), we have

\[
\lim_{H \to 0} \frac{\text{cap}_{A_p}(W(H))}{\text{cap}_{D;0}(H)} = |W'(z_0)|^2. \tag{2.1}
\]

\[
\lim_{H \to 0} \frac{\text{cap}_{A_{p_1}}(W(H))}{\text{cap}_{A_{p_2}}(H)} = |W'(z_0)|^2. \tag{2.2}
\]

Here \( H \to 0 \) means that \( H \) is a nonempty hull in \( \mathbb{D} \) w.r.t. \( 0 \), and \( \text{diam}(H \cup \{0\}) \to 0 \).

**Proof.** \([21]\) is Lemma 2.1 in [22], and \([2.2]\) follows easily from \([2.1]\). \( \square \)

A Loewner chain in a simply or doubly connected domain \( D \) is a family of hulls in \( D \), say \((L(t), 0 \leq t < T)\), where \( T \in (0, \infty) \), such that (i) \( L(0) = \emptyset \); (ii) \( L(t_1) \supseteq L(t_2) \) if \( t_1 < t_2 \); and
For a real interval $I$ and any fixed compact set $F \subset D \setminus L(t_0)$ with $\text{diam}(F) > 0$, the extremal length (c.f. [1]) of the family of curves in $D \setminus L(t)$ that disconnect $L(t + \varepsilon) \setminus L(t)$ from $F$ tends to 0 as $\varepsilon \to 0^+$, uniformly in $t \in [0, t_0]$. Moreover, if $D$ is a simply connected domain, and each $L(t)$ is a hull in $D$ w.r.t. $z_0 \in D$, then $(L(t))$ is called a Loewner chain in $D$ w.r.t. $z_0$; if $D$ is a doubly connected domain with a boundary component $C_2$, and each $L(t)$ is a hull in $D$ w.r.t. $C_2$, then $(L(t))$ is called a Loewner chain in $D$ w.r.t. $C_2$. Using conformal invariance of extremal length, one can easily see that Loewner chains are also conformally invariant. The idea of the definition of Loewner chain first appeared in [13].

Let $D$ be the Riemann sphere $\hat{\mathbb{C}}$ or a simply connected domain. An interior hull in $D$ is a compact subset of $D$, say $H$, such that $\text{diam}(H) > 0$ and $D \setminus H$ is connected. If $D = \hat{\mathbb{C}}$, then $D \setminus H$ is simply connected; if $D$ is simply connected, then $D \setminus H$ is doubly connected, and $\partial D$ is a boundary component of $D \setminus H$. Let $z_0 \in D$. An interior Loewner chain in $D$ started from $z_0$ is a family of interior hulls in $D$, say $(L(t), -\infty < t < T)$, where $T \in (-\infty, \infty]$, such that (i) $L(t_1) \subseteq L(t_2)$ if $t_1 < t_2$; (ii) $\bigcap_{t \in T} L(t) = \{z_0\}$; and (iii) for any $t_0 \in (-\infty, T)$, $L(t_0 + t) \setminus L(t_0), 0 \leq t < T - t_0)$ is a Loewner chain in $D \setminus L(t_0)$.

### 2.2 Radial Loewner equation

For a real interval $I$, we use $C(I)$ to denote the space of real continuous functions on $I$. For $T \in (0, \infty]$ and $\xi \in C([0, T))$, the radial Loewner equation driven by $\xi$ is

$$
\dot{g}(t, z) = g(t, z) \frac{e^{i\xi(t)} + g(t, z)}{e^{i\xi(t)} - g(t, z)}, \quad g(0, z) = z.
$$

(2.3)

Here we use the dot to denote the partial derivative w.r.t. the first variable: $t$. For $0 \leq t < T$, let $K(t)$ be the set of $z \in \mathbb{D}$ such that the solution $g(s, z)$ blows up before or at time $t$. We call $K(t)$ and $g(t, \cdot)$, $0 \leq t < T$, radial Loewner hulls and maps, respectively, driven by $\xi$. The following proposition is the main theorem in [13].

**Proposition 2.2** Suppose $K(t)$ and $g(t, \cdot)$, $0 \leq t < T$, are radial Loewner hulls and maps, respectively, driven by $\xi \in C([0, T))$. Then $(K(t), 0 \leq t < T)$ is a Loewner chain in $\mathbb{D}$ w.r.t. 0. For every $t \in [0, T)$, $g(t, \cdot)$ maps $\mathbb{D} \setminus K(t)$ conformally onto $\mathbb{D}$ with $g(t, 0) = 0$ and $g'(t, 0) = e^t$, so $\text{cap}_{\mathbb{D}, 0}(K(t)) = t$. Moreover, for every $t \in [0, T)$,

$$
\{e^{i\xi(t)}\} = \bigcap_{\varepsilon \in (0, T - t)} g(t, K(t + \varepsilon) \setminus K(t)).
$$

Let $e^t$ denote the map $z \mapsto e^{iz}$. Then $e^t$ is a covering map from $\mathbb{H}$ onto $\mathbb{D} \setminus \{0\}$, and from $\mathbb{R}$ onto $\mathbb{T}$. For $\xi \in C([0, T))$, the covering radial Loewner equation driven by $\xi$ is

$$
\tilde{g}(t, z) = \cot_2(\tilde{g}(t, z) - \xi(t)), \quad \tilde{g}(0, z) = z.
$$

(2.4)
Here \( \cot_2(z) := \cot(z/2) \). For \( 0 \leq t < T \), let \( \bar{K}(t) \) be the set of \( z \in \mathbb{H} \) such that the solution \( \bar{g}(s, z) \) blows up before or at time \( t \). We call \( \bar{K}(t) \) and \( \bar{g}(t, \cdot) \), \( 0 \leq t < T \), covering radial Loewner hulls and maps, respectively, driven by \( \xi \). Let \( K(t) \) and \( g(t, \cdot) \) be as in the last paragraph, then we have \( \bar{K}(t) = (e^t)^{-1}(K(t)) \) and \( e^t \circ \bar{g}(t, \cdot) = g(t, \cdot) \circ e^t \) for \( 0 \leq t < T \). Throughout this paper, we will use tilde to denote the covering Loewner maps or hulls.

Let \( B(t) \), \( 0 \leq t < \infty \), be a (standard) Brownian motion, i.e., \( B(0) = 0 \) and \( \mathbb{E}[B(1)^2] = 1 \). Let \( \kappa \geq 0 \). Then the radial Loewner hulls \( K(t) \), \( 0 \leq t < \infty \), driven by \( \xi(t) = \sqrt{\kappa}B(t) \), \( 0 \leq t < \infty \), are called the standard radial SLE\( \kappa \) hulls. Let \( g(t, \cdot) \) be the corresponding radial Loewner maps. From the existence of chordal SLE\( \kappa \) trace ([15]) and the equivalence between chordal SLE\( \kappa \) and radial SLE\( \kappa \) ([22]) we know that almost surely

\[
\beta(t) := \lim_{\mathbb{D} \ni z \to e^{i\xi(t)}} g(t, \cdot)^{-1}(z)
\]

exists for every \( 0 \leq t < \infty \), and \( \beta(t) \), \( 0 \leq t < \infty \), is a continuous curve in \( \overline{\mathbb{D}} \) such that \( \beta(0) = 1 \) and \( \lim_{t \to \infty} \beta(t) = 0 \) a.s.. Moreover, if \( \kappa \in (0, 4] \) then \( \beta \) is a simple curve, which intersects \( \mathcal{T} \) only at the initial point, which is \( e^{i\xi(0)} = 1 \), and \( K(t) = \beta((0, t]) \); if \( \kappa > 4 \) then \( \beta \) is not simple, and \( K(t) \) is such that \( \mathbb{D} \setminus K(t) \) is the connected component of \( \mathbb{D} \setminus \beta((0, t]) \) that contains 0. Such \( \beta \) is called a standard radial SLE\( \kappa \) trace.

The radial SLE in general simply connected domains are defined by conformal maps. Suppose \( D \) is a simply connected domain, \( a \) is a boundary point or prime end, and \( b \) is an interior point. Then there is \( W \) that maps \( \mathbb{D} \) conformally onto \( D \) such that \( W(1) = a \) and \( W(0) = b \). Let \( K(t) \) and \( \beta(t) \), \( 0 \leq t < \infty \), be the standard radial SLE\( \kappa \) hulls and trace. Then \( W(K(t)) \) and \( W(\beta(t)) \), \( 0 \leq t < \infty \), are called the radial SLE\( \kappa \) hulls and trace in \( D \) from \( a \) to \( b \).

### 2.3 Whole-plane Loewner equation

The whole-plane Loewner equation generates some interior Loewner chain in \( \hat{\mathbb{C}} \) started from 0. The following proposition is a special case of Proposition 4.21 in [7].

**Proposition 2.3** Suppose \( \xi \in C((-\infty, T)) \) for some \( T \in (-\infty, \infty] \). Then there is an interior Loewner chain \( K_I(t) \), \( -\infty < t < T \), in \( \hat{\mathbb{C}} \) started from 0, and a family of maps \( g_I(t, \cdot) \), \( -\infty < t < T \), such that, for each \( t \in (-\infty, T) \), \( g_I(t, \cdot) \) maps \( \hat{\mathbb{C}} \setminus K_I(t) \) conformally onto \( \hat{\mathbb{C}} \setminus \mathbb{D} \), fixes \( \infty \), and satisfies

\[
\dot{g}_I(t, z) = g_I(t, z) \frac{e^{i\xi(t)} + g_I(t, z)}{e^{i\xi(t)} - g_I(t, z)}; \tag{2.5}
\]

\[
\lim_{t \to -\infty} e^t g_I(t, z) = z, \quad z \in \mathbb{C} \setminus \{0\}. \tag{2.6}
\]

Moreover, for any \( t \in (-\infty, T) \), \( K_I(t) \) and \( g_I(t, \cdot) \) are determined by \( e^{i\xi(s)} \), \( -\infty < s \leq t \). We call \( K_I(t) \) and \( g_I(t, \cdot) \), \( -\infty < t < T \), the whole-plane Loewner hulls and maps driven by \( \xi \).

The whole-plane SLE is defined by choosing the driving function \( \xi(t) \) such that \( e^{i\xi(t)} \), \( -\infty < t < \infty \), is a Brownian motion on \( \mathcal{T} \) started from uniform distribution. The construction is as
follows. Fix $\kappa > 0$. Let $B_+(t)$ and $B_-(t)$, $t \geq 0$, be two independent Brownian motions. Let $x$ be a random variable, which is uniformly distributed on $[0, 2\pi)$, and is independent of $(B_+(t))$ and $(B_-(t))$. Let $B^{(\kappa)}(t) = x + \sqrt{\kappa}B_{\text{sign}}(t)(|t|)$ for $t \in \mathbb{R}$. Then $B^{(\kappa)}(t)$ satisfies the property that for any fixed $T \in \mathbb{R}$, $B^{(\kappa)}(T + t) - B^{(\kappa)}(T)$, $t \geq 0$, has the same distribution as $\sqrt{\kappa}B(t)$, $t \geq 0$, and is independent of $e^{ix}(B^{(\kappa)}(t))$, $-\infty < t \leq T$. The whole-plane Loewner hulls, $K_f(t)$, $-\infty < t < \infty$, driven by $\xi(t) = B^{(\kappa)}(t)$, are called the standard whole-plane SLE$_\kappa$ hulls. Let $g_f(t, \cdot)$ be the corresponding whole-plane Loewner maps. Recall that $g_f(t, \cdot)$ maps $\mathbb{C} \setminus K_f(t)$ conformally onto $\{|z| > 1\}$. It is known that a.s.

$$\beta_f(t) := \lim\limits_{|z|>1, z \rightarrow e^{\xi(t)}} g_f(t, \cdot)^{-1}(z)$$

exists for all $t \in \mathbb{R}$, and $\beta_f(t)$, $t \in \mathbb{R}$, is a continuous curve in $\mathbb{C}$, and satisfies $\lim_{t \rightarrow -\infty} \beta_f(t) = 0$ and $\lim_{t \rightarrow \infty} \beta_f(t) = \infty$. Such $\beta_f$ is called the standard whole-plane SLE$_\kappa$ trace. If $\kappa \leq 4$, $\beta_f$ is a simple curve, and $K_f(t) = \beta_f([-\infty, t])$ for each $t \in \mathbb{R}$; if $\kappa > 4$, $\beta_f$ is not simple, and $\hat{\mathbb{C}} \setminus K_f(t)$ is the component of $\hat{\mathbb{C}} \setminus \beta_f([-\infty, t])$ that contains $\infty$. Whole-plane SLE is related to radial SLE in the way that, if $T \in \mathbb{R}$ is fixed, then conditioned on $K_f(t)$, $-\infty < t \leq T$, the curve $\beta_f(T + t)$, $t \geq 0$, is the radial SLE$_{4\kappa}$ trace in $\hat{\mathbb{C}} \setminus K_f(T)$ from $\beta_f(T)$ to $\infty$.

The standard whole-plane SLE$_\kappa$ process is self conformally invariant. This means that if $\beta_f$ is a standard whole-plane SLE$_\kappa$ trace, and $W$ maps $\hat{\mathbb{C}}$ conformally onto itself, and fixes 0 and $\infty$ (so $W$ is of the form $z \mapsto Cz$ with $C \in \mathbb{C} \setminus \{0\}$), then $(W(\beta_f(t)))$ has the same distribution as $(\beta_f(t))$ with a possible time-change. This is also true if $W$ is a conjugate conformal map. If $z_1 \neq z_2 \in \hat{\mathbb{C}}$ are given, we may choose some $W$ that maps $\hat{\mathbb{C}}$ conformally or conjugate conformally onto itself such that $W(0) = z_1$ and $W(\infty) = z_2$. Then we define $W(\beta_f(t))$, $t \in \mathbb{R}$, to be the whole-plane SLE$_\kappa$ trace in $\hat{\mathbb{C}}$ from any $z_1$ to $z_2$. The definition up to a time-change does not depend on the choice of $W$.

In the above content, we use subscripts $I$ to emphasize that the whole-plane interior Loewner chain grows from 0. We will need the following inverted whole-plane Loewner chain, which grows from $\infty$. Let $I_0(z) = 1/z$. Then $I_0$ is a conjugate conformal map from $\hat{\mathbb{C}}$ onto itself, exchanges 0 and $\infty$, and maps $\hat{\mathbb{C}} \setminus \mathbb{D}$ onto $\mathbb{D}$. For $-\infty < t < T$, let $K(t) = I_0(K_f(t))$ and $g(t, \cdot) = I_0 \circ g_f(t, \cdot) \circ I_0$. Then $K(t)$, $-\infty < t < T$, is an interior Loewner chain in $\hat{\mathbb{C}}$ started from $\infty$. For each $t$, $g(t, \cdot)$ maps $\hat{\mathbb{C}} \setminus K(t)$ conformally onto $\mathbb{D}$, and fixes 0. Moreover, $g(t, \cdot)$ satisfies

$$\dot{g}(t, z) = g(t, z) \frac{e^{\xi(t)} + g(t, z)}{e^{\xi(t)} - g(t, z)}, \quad -\infty < t < T.$$ (2.8)

We call $K(t)$ and $g(t, \cdot)$ the inverted whole-plane Loewner hulls and maps driven by $\xi$.

**Lemma 2.1** For any $t \in (-\infty, T)$ and $\varepsilon \in (0, T - t)$, let $K_f(\varepsilon) = g(K(t + \varepsilon) \setminus K(t))$. Then $K_f(\varepsilon)$ is a hull in $\mathbb{D}$ w.r.t. 0, and $\text{cap}_{\mathbb{D}, 0}(K_f(\varepsilon)) = \varepsilon$. Moreover,

$$\{e^{\xi(t)}\} = \bigcap_{\varepsilon \in (0, T-t)} K_f(\varepsilon) = \bigcap_{\varepsilon \in (0, T-t)} g(t, K(t + \varepsilon) \setminus K(t)).$$
Proof. Fix $t \in (-\infty, T)$. For $0 \leq \varepsilon < T - t$, let $g_t(\varepsilon, \cdot) = g(t + \varepsilon, \cdot) \circ g(t, \cdot)^{-1}$. Then $g_t(\varepsilon, \cdot)$ maps $\mathbb{D} \setminus K_t(\varepsilon)$ conformally onto $\mathbb{D}$, $g_t(0, \cdot) = \text{id}$, and from (2.3), it satisfies
\[
\frac{\partial}{\partial \varepsilon} g_t(\varepsilon, z) = g_t(\varepsilon, z) \frac{e^{i\xi(t+\varepsilon)} + g_t(\varepsilon, z)}{e^{i\xi(t+\varepsilon)} - g_t(\varepsilon, z)}.
\]
Thus, $K_t(\varepsilon)$ and $g_t(\varepsilon, \cdot)$, $0 \leq \varepsilon < T - t$, are radial Loewner hulls and maps, respectively, driven by $\xi_t(\varepsilon) := \xi(t + \varepsilon)$, $0 \leq \varepsilon < T - t$. From Proposition 2.2 $K_t(\varepsilon)$ is a hull in $\mathbb{D}$ w.r.t. 0, cap$_{\mathbb{D}, 0}(K_t(\varepsilon)) = \varepsilon$, and
\[
\{e^{i\xi(t)}\} = \{e^{i\xi_t(0)}\} = \bigcap_{\varepsilon \in (0, T-t)} g_t(0, K_t(\varepsilon) \setminus K_t(0)) = \bigcap_{\varepsilon \in (0, T-t)} K_t(\varepsilon). \quad \square
\]

The covering whole-plane Loewner equation is defined as follows. Let $\xi \in C((\infty, T))$ for some $T \in (-\infty, \infty)$. Let $K_t(t)$ and $g_t(t, \cdot)$, $-\infty < t < T$, be the whole-plane Loewner hulls and maps driven by $\xi$. Let $\tilde{K}_t(t) = (e^i)^{-1}(K_t(t))$, $-\infty < t < T$. Suppose $\tilde{g}_t(t, \cdot)$, $-\infty < t < T$, satisfy that for each $t$, $\tilde{g}_t(t, \cdot)$ maps $\mathbb{C} \setminus \tilde{K}_t(t)$ conformally onto $-\mathbb{H}$, $e^i \circ \tilde{g}_t(t, \cdot) = g_t(t, \cdot) \circ e^i$, and the following differential equation holds:
\[
\tilde{g}_t(t, z) = \cot_2(\tilde{g}_t(t, z) - \xi(t));
\]
\[
\lim_{t \to -\infty} (\tilde{g}_t(t, z) - it) = z. \quad (2.9)
\]
Then we call $\tilde{K}_t(t)$ and $\tilde{g}_t(t, \cdot)$ the covering whole-plane Loewner hulls and maps driven by $\xi$. Such family of $\tilde{g}_t(t, \cdot)$ exists and is unique. In fact, for each $t \in (-\infty, T)$, we can find some $\tilde{g}_t(t, \cdot)$ that maps $\mathbb{C} \setminus \tilde{K}_t(t)$ conformally onto $-\mathbb{H}$ such that $e^i \circ \tilde{g}_t(t, \cdot) = g_t(t, \cdot) \circ e^i$. Such $\tilde{g}_t(t, \cdot)$ is not unique. Since $g_t(t, \cdot)$ is differentiable in $t$, so one may choose $\tilde{g}_t(t, \cdot)$ such that it is also differentiable in $t$. From (2.3) we conclude that (2.9) must hold. From (2.6) we conclude that $\lim_{t \to -\infty} (\tilde{g}_t(t, z) - it) = z + i2n\pi$ for some $n \in \mathbb{Z}$, and such $n$ is the same for every $z$. Now we replace $\tilde{g}_t(t, \cdot)$ by $\tilde{g}_t(t, \cdot) - i2n\pi$. Then (2.9) and (2.10) still hold. So we have the existence of $\tilde{g}_t(t, \cdot)$. The uniqueness follows from the same argument. Moreover, we see that for any $t \in (-\infty, T)$, $\tilde{g}_t(t, \cdot)$ is determined by $e^{i\xi(s)}$, $-\infty < s \leq t$.

Let $\tilde{I}_0(z) = \overline{z}$. Then $\tilde{I}_0$ is a conjugate conformal map from $\mathbb{C}$ onto itself, maps $-\mathbb{H}$ onto $\mathbb{H}$, and satisfies $e^i \circ \tilde{I}_0 = I_0 \circ e^i$. Let $\tilde{g}(t, \cdot) = \tilde{I}_0 \circ \tilde{g}_t(t, \cdot) \circ \tilde{I}_0$ and $K(t) = \tilde{I}_0(K_t(t))$. Then $K(t) = (e^i)^{-1}(K(t))$ and $e^i \circ \tilde{g}(t, \cdot) = g(t, \cdot) \circ e^i$. We call $\hat{K}(t)$ and $\hat{g}(t, \cdot)$ the inverted covering whole-plane Loewner hulls and maps driven by $\xi$. Moreover, we have
\[
\hat{g}(t, z) = \cot_2(\tilde{g}(t, z) - \xi(t)). \quad (2.11)
\]

2.4 Annulus Loewner equation

Annulus Loewner equation is introduced in [22]. For $p > 0$, define
\[
S(p, z) = \lim_{M \to \infty} \sum_{k=-M}^{M} \frac{e^{2kp + z}}{e^{2kp} - z} = P.V. \sum_{n \text{ even}} e^{np + z} \sum_{n \text{ even}} e^{np - z},
\]
\[ H(p, z) = -iS(p, e^i(z)) = -i \text{ P. V.} \sum_{n \text{ even}} e^{np} + e^{iz}. \] (2.12)

Then \( H(p, \cdot) \) is a meromorphic function in \( \mathbb{C} \), whose poles are \( \{2m\pi + i2kp : m, k \in \mathbb{Z} \} \), which are all simple poles with residue 2. Moreover, \( H(p, \cdot) \) is an odd function; takes real values on \( \mathbb{R} \setminus \{\text{poles}\} \); \( \text{Im} H(p, \cdot) \equiv -1 \) on \( ip + \mathbb{R} \); \( H(p, z + 2\pi) = H(p, z) \) and \( H(p, z + i2p) = H(p, z) - 2i \) for any \( z \in \mathbb{C} \setminus \{\text{poles}\} \). It is possible to explicit this kernel using classical functions in [4]:

\[ H(p, z) = 2\zeta(z) - \frac{2\pi}{\pi} \zeta(\pi)z = \frac{1}{\pi^2} \frac{\partial_\nu \theta}{\theta} \left( \frac{z + \nu i}{2\pi} \right), \]

where \( \zeta \) is the Weierstrass zeta function with basic periods \((2\pi, i2p)\), and \( \theta = \theta(\nu, \tau) \) is the Jacobi’s theta function.

The power series expansion of \( H(p, \cdot) \) near 0 is

\[ H(p, z) = \frac{2}{z} + r(p)z + O(z^3), \] (2.13)

where

\[ r(p) = \sum_{k=1}^{\infty} \sinh(kp)^{-2} - \frac{1}{6}. \]

As \( p \to \infty \), \( S(p, z) \to \frac{1+z}{1-z} \), \( H(p, z) \to \cot_2(z) \), and \( r(p) \to -1/6 \). So we write \( S(\infty, z) = \frac{1+z}{1-z} \), \( H(\infty, z) = \cot_2(z) \), and \( r(\infty) = -1/6 \). Then \( r \) is continuous on \((0, \infty] \), and (2.13) still holds even if \( p = \infty \). In fact, we have \( r(p) - r(\infty) = O(e^{-p}) \) as \( p \to \infty \), so we may define \( R \) on \((0, \infty] \) such that

\[ R(p) = - \int_{p}^{\infty} (r(t) - r(\infty))dt. \] (2.14)

Then \( R \) is continuous on \((0, \infty] \), \( R(p) = O(e^{-p}) \) as \( p \to \infty \), and \( R'(p) = r(p) - r(\infty) \).

Fix \( p \in (0, \infty) \). Let \( \xi \in C([0, T]) \) where \( 0 < T \leq p \). The annulus Loewner equation of modulus \( p \) driven by \( \xi \) is

\[ \dot{g}(t, z) = g(t, z)S(p - t, g(t, z)/e^{i\xi(t)}), \quad g(0, z) = z. \] (2.15)

For \( 0 \leq t < T \), let \( K(t) \) denote the set of \( z \in \mathbb{A}_p \) such that the solution \( g(s, z) \) blows up before or at time \( t \). We call \( K(t) \) and \( g(t, \cdot) \), \( 0 \leq t < T \), the annulus Loewner hulls and maps of modulus \( p \) driven by \( \xi \). The following proposition is Proposition 2.1 in [22].

**Proposition 2.4** (i) Suppose \( K(t) \) and \( g(t, \cdot) \), \( 0 \leq t < T \), are annulus Loewner hulls and maps of modulus \( p \) driven by \( \xi \in C([0, T]) \). Then \( (K(t), 0 \leq t < T) \) is a Loewner chain in \( \mathbb{A}_p \) w.r.t. \( T_p \). For every \( t \in [0, T) \), \( g(t, \cdot) \) maps \( \mathbb{A}_p \setminus K(t) \) conformally onto \( \mathbb{A}_{p-t} \), and maps \( T_p \) onto \( T_{p-t} \), so \( \text{cap}_{\mathbb{A}_p}(K(t)) = t \). Moreover, we have

\[ \{e^{i\xi(t)}\} = \bigcap_{\varepsilon \in (0, T-t)} g(t, K(t + \varepsilon) \setminus K(t)), \quad t \in [0, T). \]
(ii) Let \((K(t), 0 \leq t < T)\) be a Loewner chain in \(\mathbb{A}_p\) w.r.t. \(T_p\). Let \(v(t) = \text{cap}_{\mathbb{A}_p}(K(t))\), \(0 \leq t < T\). Then \(v\) is a continuous increasing function that maps \([0,T]\) onto \([0,S]\) for some \(S \in (0,p]\). Let \(L(s) = K(v^{-1}(s)), 0 \leq s < S\). Then \(L(s), 0 \leq s < S\), are annulus Loewner hulls of modulus \(p\) driven by some \(\zeta \in C([0,S])\).

Let \(t \in [0,T)\) and \(\varepsilon \in [0,T - t]\). Let \(g_{t,\varepsilon} = g(t + \varepsilon, \cdot) \circ g(t, \cdot)^{-1}\). Then \(g_{t,\varepsilon}\) maps \(\mathbb{A}_{p-t} \setminus g(t, K(t+\varepsilon) \setminus K(t))\) conformally onto \(\mathbb{A}_{p-t-\varepsilon}\), and maps \(T_{p-t}\) onto \(T_{p-t-\varepsilon}\), so \(g(t, K(t+\varepsilon) \setminus K(t))\) is a hull in \(\mathbb{A}_{p-t}\) w.r.t. \(T_{p-t}\), and

\[
\text{cap}_{\mathbb{A}_{p-t}}(g(t, K(t + \varepsilon) \setminus K(t))) = \varepsilon.
\]

The covering annulus Loewner equation of modulus \(p\) driven by the above \(\xi\) is

\[
\hat{g}(t, z) = H(p - t, \bar{g}(t, z) - \xi(t)), \quad \bar{g}(0, z) = z.
\]

Let \(S_p = \{z \in \mathbb{C} : \text{Re} z < p\}\) and \(\mathbb{R}_p = ip + \mathbb{R}\). Then \(e^t\) is a covering map from \(S_p\) onto \(\mathbb{A}_p\), and from \(\mathbb{R}_p\) onto \(T_p\). For \(0 \leq t < T\), let \(\tilde{K}(t)\) denote the set of \(z \in S_p\) such that the solution \(\tilde{g}(s,z)\) blows up before or at time \(t\). Then for \(0 \leq t < T\), \(\tilde{g}(t, \cdot)\) maps \(S_p \setminus \tilde{K}(t)\) conformally onto \(S_{p-t}\), and maps \(\mathbb{R}_p\) onto \(\mathbb{R}_{p-t}\). We call \(\tilde{K}(t)\) and \(\tilde{g}(t, \cdot), 0 \leq t < T\), the covering annulus Loewner hulls and maps of modulus \(p\) driven by \(\xi\). Let \(K(t)\) and \(g(t, \cdot)\) be as before. Then we have \(\tilde{K}(t) = (e^t)^{-1}(K(t))\) and \(e^t \circ \tilde{g}(t, \cdot) = g(t, \cdot) \circ e^t\) for \(0 \leq t < T\).

Note that if \(p = \infty\) in (2.15) and (2.17), then we get (2.8) and (2.4). So we may view the radial Loewner equation as a limit of annulus Loewner equations as the modulus \(p \to \infty\).

Let \(S_I(p, z) = 1 + S(p, e^pz)\) and \(H_I(p, z) = -iS_I(p, e^{iz}) = -i + H(p, z - ip)\). It is easy to check:

\[
S_I(p, z) = P \cdot V \sum_{n \text{ odd}} e^{np + z} + z, \quad H_I(p, z) = -i P \cdot V \sum_{n \text{ odd}} e^{np + e^{iz}} + e^{-ip - e^{iz}}.
\]

So \(H_I(p, \cdot)\) is a meromorphic function in \(\mathbb{C}\) with poles \(\{2m\pi + i(2k + 1)p : m,k \in \mathbb{Z}\}\), which are all simple poles with residue 2; \(H_I(p, \cdot)\) is an odd function; takes real values on \(\mathbb{R}\); and \(H_I(p, z + 2\pi) = H_I(p, z)\), \(H_I(p, z + 2ip) = H_I(p, z) - 2i\) for any \(z \in \mathbb{C} \setminus \{\text{poles}\}\).

Let \(I_p(z) := e^{-p/z} + \bar{z}\). Then \(I_p\) and \(\bar{I}_p\) are conjugate conformal automorphisms of \(A_p\) and \(S_p\), respectively. Moreover, \(I_p\) exchanges \(T_p\) and \(\bar{T}_p\), \(\bar{I}_p\) exchanges \(R_p\) and \(\mathbb{R}\), and \(I_p \circ e^t = e^t \circ \bar{I}_p\). Let \(K_I(t) = I_p(K(t)), g_I(t, \cdot) = I_{p-t} \circ g(t, \cdot) \circ I_p, \bar{K}_I(t) = \bar{I}_p(\bar{K}(t))\), and \(\bar{g}_I(t, \cdot) = \bar{I}_{p-t} \circ \bar{g}(t, \cdot) \circ \bar{I}_p\). Then \(K_I(t)\) is a hull in \(A_p\) w.r.t. \(T\); \(g_I(t, \cdot)\) maps \(A_p \setminus K_I(t)\) conformally onto \(A_{p-t}\), and maps \(T\) onto \(T\); so \(\text{cap}_{\mathbb{A}_p}(K_I(t)) = t\). We have that \(K_I(t) = (e^t)^{-1}(K_I(t)), g_I(t, \cdot)\) maps \(S_p \setminus K_I(t)\) conformally onto \(S_{p-t}\), and maps \(\mathbb{R}\) onto \(\mathbb{R}\), and satisfies \(e^t \circ \bar{g}_I(t, \cdot) = g_I(t, \cdot) \circ e^t\). Moreover, \(g_I(t, \cdot)\) and \(\bar{g}_I(t, \cdot)\) satisfy the following equations:

\[
\hat{g}_I(t, z) = g(t, z)S_I(p - t, g_I(t, z) / e^{i\xi(t)}), \quad g(0, z) = z;
\]

\[
\hat{g}_I(t, z) = H_I(p - t, \bar{g}_I(t, z) - \xi(t)), \quad \bar{g}(0, z) = z.
\]
We call $K_I(t)$ and $g_I(t, \cdot)$ (resp. $\bar{K}_I(t)$ and $\bar{g}_I(t, \cdot)$) the inverted annulus (resp. inverted covering annulus) Loewner hulls and maps of modulus $p$ driven by $\xi$.

From (2.15), we see that if $\hat{\xi} \in C([0, T))$ satisfies $e^{\hat{\xi}(t)} = e^{\xi(t)}$ for $0 \leq t < T$, then the annulus Loewner maps and hulls of modulus $p$ driven by $\hat{\xi}$ agree with those driven by $\xi$. This is also true for covering annulus Loewner maps and hulls because the $H$ in (2.17) has period $2\pi$ in the second variable. From the definitions, similar results hold true for inverted and inverted covering annulus Loewner objects.

Let $B(t)$ be a Brownian motion, and $\kappa > 0$. Then the annulus Loewner hulls $K(t), 0 \leq t < p$, of modulus $p$ driven by $\xi(t) = \sqrt{r}B(t), 0 \leq t < p$, are called the standard annulus $\text{SLE}_\kappa$ hulls of modulus $p$. Let $g(t, \cdot)$ be the corresponding annulus Loewner maps. It is known (c.f. [22]) that the standard annulus $\text{SLE}_\kappa$ process is locally equivalent to the standard radial $\text{SLE}_\kappa$ process. So a.s.

$$\beta(t) := \lim_{h_p \to 3 - e^{\xi(t)}} g(t, \cdot)^{-1}(z)$$

exists for $0 \leq t < p$, and $\beta(t), 0 \leq t < p$, is a continuous curve in $\mathbb{A}_p \cup \mathbb{T}$. If $\kappa \in (0, 4]$, $\beta$ is a simple curve, which intersects $\mathbb{T}$ only at $\beta(0) = 1$, and $K(t) = \beta((0, t])$; if $\kappa > 4$, then $\beta$ is not simple, and $\mathbb{A}_p \setminus K(t)$ is the connected component of $\mathbb{A}_p \setminus \beta((0, t])$ that contains a neighborhood of $\mathbb{T}_p$. We call such $\beta$ the standard annulus $\text{SLE}_\kappa$ trace of modulus $p$.

### 2.5 Disc Loewner equation

The disc Loewner equation generates some interior Loewner chain in the unit disc $\mathbb{D}$ started from 0. The following proposition is a slight modification of some propositions in [22].

#### Proposition 2.5
(i) Let $\xi \in C((−∞, T))$ for some $T \in (−∞, 0]$. Then there is an interior Loewner chain $K_I(t), −∞ < t < T$, in $\mathbb{D}$ started from 0, and a family of maps $g_I(t, \cdot), −∞ < t < T, −∞ < s < t < T$, such that for each $−∞ < t < T$, $g_I(t, \cdot)$ maps $\mathbb{D} \setminus K(t)$ conformally onto $\mathbb{A}_{−1}$ (so mod($\mathbb{D} \setminus K(t)) = −t), maps \mathbb{T}$ onto $\mathbb{T}$, and satisfies

$$\dot{g}_I(t, z) = g_I(t, z)S_I(−t, g_I(t, z)/e^{i\xi(t)}), \quad −∞ < t < T; \quad \lim_{t \to −∞} g_I(t, z) = z, \quad \forall z \in \overline{\mathbb{D}} \setminus \{0\}. \quad (2.21)$$

Moreover, for any $t \in (−∞, T)$, $K_I(t)$ and $g_I(t, \cdot)$ are determined by $e^{i\xi(s)}$, $−∞ < s \leq t$. We call $K_I(t)$ and $g_I(t, \cdot), −∞ < t < T, the disc Loewner hulls and maps driven by $\xi$.

(ii) Suppose $K_I(t), −∞ < t < T, is an interior Loewner chain in $\mathbb{D}$ started from 0. Let $v(t) = −\text{mod}(\mathbb{D} \setminus K(t))$. Then $v$ is continuous and increasing on $(−∞, T)$, and maps $(−∞, T)$ onto $(−∞, S)$ for some $S \in \mathbb{R}$. Let $L_I(s) = K_I(v^{-1}(s)), −∞ < s < S$. Then $L_I(s), −∞ < s < S, are the disc Loewner hulls driven by some $\xi \in C((−∞, S))$.

#### Proof.
(i) Apply Proposition 4.1 in [22] to $\chi(t) = e^{-i\xi(t)}, −∞ < t < T$. Then we have an interior Loewner chain $K_−(t), −∞ < t < T, in \mathbb{D}$ started from 0, and a family of maps $g_−(t, \cdot), \ldots$
Lemma 2.2

For any \( -\infty < t < T \), such that each \( g_-(t, \cdot) \) maps \( \mathbb{D} \setminus K_-(t) \) conformally onto \( \mathbb{H}_{-t} \), maps \( \mathbb{T} \) onto \( \mathbb{T}_{-t} \), and satisfies

\[
\dot{g}_-(t, z) = g_-(t, z)S(-t, g_-(t, z) / e^{-i\xi(t)}), \quad -\infty < t < T; 
\]

\[
\lim_{t \to -\infty} e^t / g_-(t, z) = z, \quad \forall z \in \mathbb{D} \setminus \{0\}. 
\]

Recall that \( \tilde{I}_0(z) = \overline{z} \) and \( I_{-t}(z) = e^t / \overline{z} \). Let \( K_I(t) = I_0(K_-(t)) \) and \( g_I(t, \cdot) = I_0 \circ g_-(t, \cdot) \circ \tilde{I}_0 \). Then \( (K_I(t), -\infty < t < T) \) is also an interior Loewner chain in \( \mathbb{D} \) started from 0, and \( g_I(t, \cdot) \) maps \( \mathbb{D} \setminus K_I(t) \) conformally onto \( \mathbb{H}_{-t} \), and maps \( \mathbb{T} \) onto \( \mathbb{T}_{-t} \). Now (2.21) and (2.22) follow from (2.23) and (2.24), respectively. From Proposition 4.1 in [22], \( K_I(t) \) and \( g_I(t, \cdot) \) are determined by \( \chi(s) = 1 / e^{\xi(s)} \), \( -\infty < s \leq t \), so are determined by \( e^{\xi(s)} \), \( -\infty < s \leq t \). We call (i). Now (ii) follows from Proposition 4.2 in [22] and the argument in the proof of (i). \( \square \)

Here we also use subscripts \( I \) to emphasize that the disc interior Loewner chain grows from 0. We will need the following inverted disc Loewner chain, which grows from \( \infty \). For \( -\infty < t < T \), let \( K(t) = I_0(K_I(t)) \) and \( g(t, \cdot) = I_0 \circ g_I(t, \cdot) \circ I_0 \). Then \( K(t), -\infty < t < T \), is an interior Loewner chain in \( \mathbb{C} \setminus \mathbb{D} \) started from \( \infty \). For each \( t \), \( g(t, \cdot) \) maps \( \mathbb{C} \setminus \mathbb{D} \setminus K(t) \) conformally onto \( \mathbb{H}_{-t} \), and maps \( \mathbb{T} \) onto \( \mathbb{T}_{-t} \). Moreover, \( g(t, \cdot) \) satisfies

\[
\dot{g}(t, z) = g(t, z)S(-t, g(t, z) / e^{i\xi(t)}), \quad -\infty < t < T. 
\]

We call \( K(t) \) and \( g(t, \cdot) \), \( -\infty < t < T \), the inverted disc Loewner hulls and maps driven by \( \xi \).

Lemma 2.2

For any \( t \in (-\infty, T) \) and \( \varepsilon \in (0, T - t) \), let \( K_t(\varepsilon) = g(t, K(t + \varepsilon) \setminus K(t)) \). Then \( K_t(\varepsilon) \) is a hull in \( \mathbb{H}_{-t} \) w.r.t. \( \mathbb{T}_{-t} \), and \( \text{cap}_{\mathbb{H}_{-t}}(K_t(\varepsilon)) = \varepsilon \). Moreover,

\[
\{e^{i\xi(t)}\} = \bigcap_{\varepsilon \in (0, T - t)} K_t(\varepsilon) = \bigcap_{\varepsilon \in (0, T - t)} g(t, K(t + \varepsilon) \setminus K(t)).
\]

Proof. Fix \( t \in (-\infty, T) \). For \( 0 \leq \varepsilon < T - t \), let \( g_t(\varepsilon, \cdot) = g(t + \varepsilon, \cdot) \circ g(t, \cdot)^{-1} \). Then \( g_t(\varepsilon, \cdot) \) maps \( \mathbb{H}_{-t} \setminus K_t(\varepsilon) \) conformally onto \( \mathbb{H}_{-t-\varepsilon} \), \( g_t(0, \cdot) = \text{id} \), and from (2.25), it satisfies

\[
\frac{\partial}{\partial \varepsilon} g_t(\varepsilon, z) = g_t(\varepsilon, z)S(-t - \varepsilon, g_t(\varepsilon, z) / e^{i\xi(t+\varepsilon)}).
\]

Thus, \( K_t(\varepsilon) \) and \( g_t(\varepsilon, \cdot) \), \( 0 \leq \varepsilon < T - t \), are annulus Loewner hulls and maps of modulus \( -t \), respectively, driven by \( \xi_t(\varepsilon) := \xi(t + \varepsilon) \), \( 0 \leq \varepsilon < T - t \). From Proposition 2.4, \( K_t(\varepsilon) \) is a hull in \( \mathbb{H}_{-t} \) w.r.t. \( \mathbb{T}_{-t} \), \( \text{cap}_{\mathbb{H}_{-t}}(K_t(\varepsilon)) = \varepsilon \), and we have

\[
\{e^{i\xi(t)}\} = \{e^{i\xi_t(0)}\} = \bigcap_{\varepsilon \in (0, T - t)} g_t(0, K_t(\varepsilon) \setminus K_t(0)) = \bigcap_{\varepsilon \in (0, T - t)} K_t(\varepsilon). \quad \square
\]
The covering disc Loewner hulls and maps are defined as follows. Let \( \tilde{K}_t(t) = (e^t)^{-1}(K_t(t)) \), \(-\infty < t < T\). Suppose \( \tilde{g}_t(t, \cdot), -\infty < t < T\), satisfy that, for each \( t \), \( \tilde{g}_t(t, \cdot) \) maps \( \mathbb{H} \setminus \tilde{K}_t(t) \) conformally onto \( \mathbb{S}_{-t} \), and maps \( \mathbb{R} \) onto \( \mathbb{R} \), \( e^t \circ \tilde{g}_t(t, \cdot) = g_t(t, \cdot) \circ e^t \), and the followings hold:

\[
\tilde{g}_t(t, z) = H_t(-t, \tilde{g}_t(t, z) - \xi(t));
\]

\[
\lim_{t \to -\infty} \tilde{g}_t(t, z) = z.
\]

Such family of \( \tilde{g}_t(t, \cdot) \) exists and is unique. This follows from the same argument used to show the existence and uniqueness of covering whole-plane Loewner maps. Moreover, we see that for \( \xi \in \mathbb{C} \), the \( -\infty < s < t \). We call \( \tilde{K}_t(t) \) and \( \tilde{g}_t(t, \cdot) \) the covering disc Loewner hulls and maps driven by \( \xi \).

Let \( \tilde{K}(t) = \tilde{I}_0(\tilde{K}_t(t)) \) and \( \tilde{g}(t, \cdot) = \tilde{I}_{-t} \circ \tilde{g}_t(t, \cdot) \circ \tilde{I}_0 \). Then \( \tilde{g}(t, \cdot) \) maps \( -\mathbb{H} \setminus \tilde{K}(t) \) conformally onto \( \mathbb{S}_{-t} \), and maps \( \mathbb{R} \) onto \( \mathbb{R}_{-t} \), \( e^t \circ \tilde{g}(t, \cdot) = g(t, \cdot) \circ e^t \), and satisfies

\[
\tilde{g}(t, z) = H(-t, \tilde{g}(t, z) - \xi(t)).
\]

We call \( \tilde{K}(t) \) and \( \tilde{g}(t) \) the inverted covering disc Loewner hulls and maps driven by \( \xi \).

### 2.6 Some estimations

**Lemma 2.3** If \( p \geq |\text{Im} z| + \ln(4) \), then \( |H_f(p, z)| < 9e^{|\text{Im} z|}e^{-p} \). For any \( h \in \mathbb{N} \), if \( p \geq |\text{Im} z| + h + \ln(4) \), then \( |H_f^{(h)}(p, z)| < 25\sqrt{he}^{|\text{Im} z|}e^{-p} \), where \( H_f^{(h)}(p, z) \) is the \( h \)-th partial derivative of \( H_f \) about the second variable: \( z \).

**Proof.** From (2.18), we have

\[
H_f(p, z) = \frac{1}{2i} \sum_{n \text{ odd}} \frac{e^{np} + e^{iz}}{e^{np} - e^{iz} + e^{-np} + e^{iz}} = \sum_{n \text{ odd}} \frac{\sin(z)}{\cosh(np) - \cos(z)}. \tag{2.28}
\]

It is known that \( |\sin(z)|, |\cos(z)| \leq e^{|\text{Im} z|} \). If \( p \geq |\text{Im} z| + \ln(4) \), then for any \( n \in \mathbb{Z} \setminus \{0\} \), \( \cosh(np) \geq e^{np/2} \geq e^{p/2} \geq 2e^{|\text{Im} z|} \), which implies that

\[
|\cosh(np) - \cos(z)| \geq \cosh(np) - |\cos(z)| \geq \cosh(np) - e^{|\text{Im} z|} \geq \cosh(np)/2 \geq e^{|np|}/4.
\]

So from (2.28) we have

\[
|H_f(p, z)| \leq \sum_{n \text{ odd}} \frac{4e^{|\text{Im} z|}}{e^{np}} = 2 \sum_{k=0}^{\infty} \frac{4e^{|\text{Im} z|}}{e^{2k+1}p} = \frac{8e^{|\text{Im} z|}/p}{1 - e^{-2p}} < 9e^{|\text{Im} z|}/p,
\]

where the last inequality follows from \( e^{-2p} \leq e^{-2\ln(4)} = 1/16 \).

Now suppose \( h \in \mathbb{N} \) and \( p \geq |\text{Im} z| + h + \ln(4) \). Then for any \( w \in \mathbb{C} \) with \( |w - z| = h \), we have \( p \geq |\text{Im} w| + \ln(4) \), so \( |H_f(p, w)| < 9e^{|\text{Im} w|}e^{-p} \leq 9e^{|\text{Im} z|}/e^{1/2}e^{|\text{Im} z|}e^{-p} < 25\sqrt{he}^{|\text{Im} z|}e^{-p} \). From Cauchy’s integral formula and Stirling’s formula, we have

\[
|H_f^{(h)}(p, z)| \leq \frac{h!e^h}{h^h} |e^{\text{Im} z}|e^{-p} \leq 9\sqrt{2\pi he}^{1/12h}e^{|\text{Im} z|}e^{-p} < 25\sqrt{he}^{|\text{Im} z|}e^{-p}.
\]

\( \Box \)
Lemma 2.4 There are positive continuous functions $N_L(p)$ and $N_S(p)$ defined on $(0, \infty)$ that satisfies $N_L(p), N_S(p) = O(pe^{-p})$ as $p \to \infty$ and the following properties. Suppose $K$ is an interior hull in $\mathbb{D}$ containing 0, $g$ maps $\mathbb{D} \setminus K$ conformally onto $\mathbb{H}_0$ for some $p \in (0, \infty)$, and maps $\mathbb{T}$ onto $\mathbb{T}$, and $\tilde{g}$ is differentiable, and satisfies $e^{i} \circ \tilde{g} = g \circ e^{i}$ on $\mathbb{R}$. Then for any $x \in \mathbb{R}$, $|\ln(\tilde{g}'(x))| \leq N_L(p)$ and $|S\tilde{g}(x)| \leq N_S(p)$, where $S\tilde{g}(x)$ is the Schwarzian derivative of $\tilde{g}$ at $x$.

Proof. Let $P(p, z) = -\text{Re} S_f(p, z) - \ln |z|/p$ and $\tilde{P}(p, z) = P(p, e^{iz}) = \text{Im} H_f(p, z) + \text{Im} z/p$. Then $P(p, \cdot)$ vanishes on $\mathbb{T}$ and $\mathbb{T}_p \setminus \{e^{-p}\}$, and is harmonic inside $\mathbb{H}_p$. Moreover, when $z \in \mathbb{H}_p$ is near $e^{-p}$, $P(p, z)$ behaves like $-\text{Re}(\frac{z - \frac{e^{-p}}{p}}{z - \frac{1}{2}}) + O(1)$. Thus, $P(p, \cdot)$ is a renormalized Poisson kernel in $\mathbb{H}_p$ with the pole at $e^{-p}$. Since $\ln |g^{-1}|$ is negative and harmonic in $\mathbb{H}_p$, and vanishes on $\mathbb{T}$, so there is a positive measure $\mu_K$ supported by $[0, 2\pi)$ such that for any $z \in \mathbb{H}_p$,

$$
\ln |g^{-1}(z)| = -\int P(p, z/e^{ik})d\mu_K(k).
$$

Since $e^{i} \circ \tilde{g} = g \circ e^{i}$, so $\text{Im} \tilde{g}^{-1} = -\ln |g^{-1} \circ e^{i}|$. Thus, for any $z \in \mathbb{H}_p$,

$$
\text{Im} \tilde{g}^{-1}(z) = \int P(p, e^{iz}/e^{ik})d\mu_K(k) = \int \tilde{P}(p, z - \xi)d\mu_K(k).
$$

So for any $x \in \mathbb{R}$ and $h = 1, 2, 3$,

$$
(\tilde{g}^{-1})^{(h)}(x) = \int \frac{\partial^h}{\partial x^{h-1}\partial y} \tilde{P}(p, x - \xi)d\mu_K(k). \tag{2.29}
$$

Let

$$
m_p = \inf_{x \in \mathbb{R}} \frac{\partial}{\partial y} \tilde{P}(p, x), \quad M_p = \sup_{x \in \mathbb{R}} \frac{\partial}{\partial y} \tilde{P}(p, x), \quad M_p^{(h)} = \sup_{x \in \mathbb{R}} |\frac{\partial^h}{\partial \xi^{h-1}\partial y} \tilde{P}(p, x)|, \quad h = 2, 3.
$$

Since $\tilde{P}(p, \cdot)$ is positive in $\mathbb{H}_p$, vanishes on $\mathbb{R}$, and has period $2\pi$, so $0 < m_p < M_p < \infty$. So from (2.29), for any $x \in \mathbb{R}$, $m_p |\mu_K| \leq (\tilde{g}^{-1})'(x) \leq M_p |\mu_K|$. Since $g^{-1}$ maps $\mathbb{T}$ onto $\mathbb{T}$, so $\tilde{g}^{-1}(2\pi) = \tilde{g}^{-1}(0) + 2\pi$. Thus,

$$
2\pi = \int_0^{2\pi} \tilde{g}^{-1}(x)dx \in [2\pi m_p |\mu_K|, 2\pi M_p |\mu_K|].
$$

So we have $1/M_p \leq |\mu_K| \leq 1/m_p$. Thus, for any $x \in \mathbb{R}$, $m_p/M_p \leq (\tilde{g}^{-1})'(x) \leq M_p/m_p$ and $|(\tilde{g}^{-1})^{(h)}(x)| \leq M_p^{(h)}/m_p$, $h = 2, 3$. So we have

$$
|S\tilde{g}^{-1}(x)| = \left| \frac{(\tilde{g}^{-1})^{(3)}(x)}{(\tilde{g}^{-1})'(x)} - \frac{3}{2} \left( \frac{1}{(\tilde{g}^{-1})'(x)} \right) \right| \leq \frac{M_p^{(3)} M_p}{m_p^2} + \frac{3}{2} \left( \frac{M_p^{(2)} M_p}{m_p^2} \right)^2.
$$

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Since \( \tilde{g} \) maps \( \mathbb{R} \) onto \( \mathbb{R} \), so for any \( x \in \mathbb{R} \), \( m_p/M_p \leq \tilde{g}'(x) \leq M_p/m_p \), and

\[
|S\tilde{g}(x)| = \left| - \frac{S\tilde{g}^{-1}(\tilde{g}(x))}{(\tilde{g}^{-1})'(\tilde{g}(x))^2} \right| \leq \frac{M_p^{(3)}M_p^2}{m_p^3} + \frac{3(M_p^{(2)})^2M_p^4}{m_p^5}.
\]

Let \( N_L(p) = \ln(M_p/m_p) > 0 \) and \( N_S(p) = \frac{M_p^{(3)}M_p^2}{m_p^3} + \frac{3(M_p^{(2)})^2M_p^4}{m_p^5} \), then \( |\ln(\tilde{g}'(x))| \leq N_L(p) \) and \( |S\tilde{g}(x)| \leq N_S(p) \) for any \( x \in \mathbb{R} \). Since \( \tilde{P}(p,z) = \text{Im}H_I(p,z) + \text{Im}z/p \), so \( \frac{\partial}{\partial y}\tilde{P}(p,x) = H_I(p,x) + \frac{1}{p} \) and \( \frac{\partial^2}{\partial x^2 - \partial y^2}\tilde{P}(p,x) = H_I^{(h)}(p,x), h = 2,3 \). From Lemma [2.3] \( M_p, m_p = \frac{1}{p} + O(e^{-p}) \) and \( M_p^{(h)} = O(e^{-p}), h = 2,3 \), as \( p \to \infty \). So \( N_L(p), N_S(p) = O(pe^{-p}) \) as \( p \to \infty \). □

## 3 SLE with Marked Points

### 3.1 Annulus SLE processes with one marked point

**Definition 3.1** A crossing annulus drift function is a real valued continuous function defined on \( (0,\infty) \times \mathbb{R} \) which has period \( 2\pi \) and is continuously differentiable in the second variable. A chordal-type annulus drift function is a real valued continuous function defined on \( (0,\infty) \times (\mathbb{R} \setminus \{2\pi n : n \in \mathbb{N}\}) \) which has period \( 2\pi \) and is continuously differentiable in the second variable. If \( \Lambda \) is a crossing or chordal-type annulus drift function, and \( \Lambda_I(p,x) = -\Lambda(p,-x) \), then \( \Lambda_I \) is called the dual function of \( \Lambda \). If \( \Lambda_I = \Lambda \), then \( \Lambda \) is called symmetric.

Suppose \( \Lambda \) is a crossing annulus drift function. Let \( \kappa \geq 0 \) and \( p > 0 \), \( a \in T \) and \( b \in T_p \). Choose \( x_0, y_0 \in \mathbb{R} \) such that \( a = e^{ix_0} \) and \( b = e^{-p+i\pi y_0} \). Let \( B(t) \) be a Brownian motion. Let \( f(t) \) and \( q(t), 0 \leq t < p \), be the solution to the following system of ODEs:

\[
\begin{aligned}
f'(t) &= \Lambda(p-t,f(t)+\sqrt{\kappa}B(t)-q(t)), \\
q'(t) &= H_I(p-t,q(t)-f(t) - \sqrt{\kappa}B(t)), \quad f(0) = x_0, \\
q(0) &= y_0.
\end{aligned}
\]

(3.1)

From the condition of \( \Lambda \), the solution exists uniquely, and is adapted w.r.t. the filtration generated by \( B(t) \). Let \( \xi(t) = f(t) + \sqrt{\kappa}B(t), 0 \leq t < p \). Let \( \tilde{g}_I^{\xi}(t,\cdot) \), \( 0 \leq t < p \), be the inverted covering annulus Loewner maps of modulus \( p \) driven by \( \xi \). From [2.19], we have \( \tilde{g}_I^{\xi}(t,y_0) = q(t) \) for \( 0 \leq t < p \). So \( \xi(t), 0 \leq t < p \), satisfies the SDE:

\[
d\xi(t) = \sqrt{\kappa}dB(t) + \Lambda(p-t,\xi(t)-\tilde{g}_I^{\xi}(t,y_0))dt, \quad \xi(0) = x_0.
\]

(3.2)

**Definition 3.2** Let \( K(t), 0 \leq t < p \), be the annulus Loewner hulls and trace of modulus \( p \) driven by the above \( \xi \). Then we call \( K(t), 0 \leq t < p \), the (crossing) annulus \( \text{SLE}(\kappa,\Lambda) \) process in \( \mathbb{A}_p \) started from a \( \Lambda \) with marked point \( b \).

We will see that the above definition does not depend on the choices of \( x_0 \) and \( y_0 \). Suppose we have another pair \((\bar{x}_0,\bar{y}_0)\) such that \( a = e^{i\bar{x}_0} \) and \( b = e^{-p+i\pi \bar{y}_0} \). Then there are \( m, n \in \mathbb{Z} \) such that \( \bar{x}_0 - x_0 = 2m\pi \) and \( \bar{y}_0 - y_0 = 2n\pi \). Let \( \tilde{f}(t) = f(t) + 2m\pi \) and \( \tilde{q}(t) = q(t) + 2n\pi \). Since \( \Lambda \) and
Choose whether or not the initial point and marked point lie on the same boundary component. Here $\Lambda$ is a chordal-type or crossing annulus drift function depending on process in any doubly connected domain started from one boundary point with another marked point. Here $\Lambda$ is the dual function of $\Lambda$, then $\hat{\xi}(t) = f(t) + \sqrt{\kappa}B(t) = \xi(t) + 2n\pi$. Then $\hat{\xi}$ generates the same annulus Loewner process as $\xi$ since $e^{i\hat{\xi}(t)} = e^{i\xi(t)}$.

Now suppose $\Lambda$ is a chordal-type annulus drift function. Let $\kappa \geq 0$, $p > 0$, and $a, b \in \mathbb{T}$. Choose $x_0, y_0 \in \mathbb{R}$ such that $a = e^{ix_0}$ and $b = e^{iy_0}$. Let $B(t)$ be a Brownian motion. Let $f(t)$ and $q(t)$, $0 \leq t < T$, be the solution to the following system of ODEs:

$$
\begin{cases}
  f'(t) = \Lambda(p - t, f(t) + \sqrt{\kappa}B(t) - q(t)), & f(0) = x_0; \\
  q'(t) = H(p - t, q(t) - f(t) - \sqrt{\kappa}B(t)), & q(0) = y_0.
\end{cases}
$$

Suppose $T \in (0, p]$ is such that $[0, T)$ is the maximal interval of the solution. Let $\xi(t) = f(t) + \sqrt{\kappa}B(t)$, $0 \leq t < T$. Let $\tilde{g}(t, \cdot)$, $0 \leq t < p$, be the covering annulus Loewner maps of modulus $p$ driven by $\xi$. From (2.17), we have $\tilde{g}(t, y_0) = q(t)$ for $0 \leq t < T$. So $\xi(t)$, $0 \leq t < T$, satisfies the SDE:

$$
d\xi(t) = \sqrt{\kappa}dB(t) + \Lambda(p - t, \xi(t) - \tilde{g}(t, y_0))dt, \quad \xi(0) = x_0.
$$

**Definition 3.3** Let $K(t)$, $0 \leq t < T$, be the annulus Loewner hulls and trace of modulus $p$ driven by the above $\xi$. Then we call $K(t)$, $0 \leq t < T$, the (chordal-type) annulus SLE$(\kappa, \Lambda)$ process in $\mathcal{A}_p$ started from $a$ with marked point $b$.

For the same reason as before, the definition of the above chordal-type SLE$(\kappa, \Lambda)$ process does not depend on the choices of $x_0$ and $y_0$.

In Definition 3.2 and Definition 3.3, since $\xi$ a semi-martingale with $\langle \xi \rangle_t = \kappa t$, so from the existence of annulus SLE$_\kappa$ trace and Girsanov Theorem, $\beta(t)$ defined by (2.20) a.s. exists for $0 \leq t < p$ or $0 \leq t < T$, and has properties similar to the standard annulus SLE$_\kappa$ trace. We call such $\beta$ the annulus SLE$(\kappa, \Lambda)$ trace in $\mathcal{A}_p$ started from $a$ with marked point $b$.

The crossing or chordal-type annulus SLE$(\kappa, \Lambda)$ process is self conformally invariant. This means that if $\beta$ is an annulus SLE$(\kappa, \Lambda)$ trace in $\mathcal{A}_p$ started from $a$ with marked point $b$, and $W$ maps $\mathcal{A}_p$ conformally onto itself and fixes $\mathbb{T}$, then $W(\beta)$ is an annulus SLE$(\kappa, \Lambda)$ trace in $\mathcal{A}_p$ started from $W(a)$ with marked point $W(b)$. If this $W$ is a conjugate conformal map, and $\Lambda_I$ is the dual function of $\Lambda$, then $W(\beta)$ is an annulus SLE$(\kappa, \Lambda_I)$ trace in $\mathcal{A}_p$ started from $W(a)$ with marked point $W(b)$. So if $\Lambda$ is symmetric, then the annulus SLE$(\kappa, \Lambda)$ process is also self conjugate conformally invariant. Via conformal maps, we can then define annulus SLE$(\kappa, \Lambda)$ process in any doubly connected domain started from one boundary point with another marked boundary point. Here $\Lambda$ is a chordal-type or crossing annulus drift function depending on whether or not the initial point and marked point lie on the same boundary component.

**Lemma 3.1** Let $\xi$ be as in Definition 3.2. Let $\Lambda_I$ be the dual function of $\Lambda$. Then the inverted annulus Loewner hulls of modulus $p$ driven by $\xi$ have the distribution as annulus SLE$(\kappa, \Lambda_I)$ hulls in $\mathcal{A}_p$ started from $e^{-p}a$ with marked point $e^{p}b$.

**Proof.** This follows from the fact that $I_p(z) = e^{-p/\bar{z}}$ maps $\mathcal{A}_p$ conjugate conformally onto itself, maps $(a, b)$ to $(e^{-p}a, e^{p}b)$, and maps $K(t)$ onto $K_I(t)$. \(\square\)
3.2 Disc SLE processes with one marked point

**Definition 3.4** Let $\kappa > 0$, $b \in \mathbb{T}$, and $\Lambda$ be a crossing annulus drift function. Let $\Lambda_I$ be the dual function of $\Lambda$. Choose $y_0 \in \mathbb{R}$ such that $e^{i\theta_0} = b$. Suppose $\xi(t), -\infty < t < 0$, is a real valued continuous random process which satisfies that, for any $t_0 \in (-\infty, 0)$,

$$B_{t_0}(t) := \frac{1}{\sqrt{\kappa}} \left( \xi(t_0 + t) - \xi(t_0) - \int_{t_0}^{t_0 + t} \Lambda_I(-s, \xi(s) - \tilde{g}_t^\xi(s, y_0)) ds \right), \quad 0 \leq t < -t_0, \quad (3.3)$$

is a standard Brownian motion, and is independent of $e^{i\xi(t)}, -\infty < t \leq t_0$. Here $\tilde{g}_t^\xi(t, \cdot)$ are the covering disc Loewner maps driven by $\xi$. Then we call the disc Loewner hulls driven by $\xi$ the standard disc SLE($\kappa, \Lambda$) hulls in $\mathbb{D}$ started from $0$ with marked point $b$.

**Remark.** In fact, the conditions in Definition 3.3 are equivalent to that there exists some $S \in (-\infty, 0)$, such that for $t_0 \in (-\infty, S)$, $B_{t_0}$ satisfies the condition in the definition. To see this, suppose the condition holds for any $t_0 \in (-\infty, S)$, and we choose any $t_1 \in [S, 0)$. We will show that $B_{t_1}(t), 0 \leq t < -t_1$, is a Brownian motion independent of $e^{i\xi(t)}, -\infty < t \leq t_1$. Choose any $t_0 \in (-\infty, S)$. From the assumption, $B_{t_0}(t), 0 \leq t < -t_0$, is a Brownian motion independent of $e^{i\xi(t)}, -\infty < t \leq t_0$. Note that $B_{t_1}(t) = B_{t_0}(t_1 - t_0 + t) - B_{t_0}(t_1 - t_0)$. So $B_{t_1}(t), 0 \leq t < -t_1$, is a Brownian motion independent of both $e^{i\xi(t)}, -\infty < t \leq t_0$ and $B_{t_0}(t), 0 \leq t \leq t_1 - t_0$. From (3.3) and that $\Lambda$ has period $2\pi$ in the second variable, we see that $\xi(t_0 + t) - \xi(t_0)$ is adapted w.r.t. the filtration $(\mathcal{F}_t^{t_0})$, where $\mathcal{F}_t^{t_0} = \mathcal{F}(e^{i\xi(t_0)}, B_{t_0}(s), 0 \leq s \leq t)$. So $B_{t_1}(t), 0 \leq t < -t_1$, is independent of $e^{i\xi(t)}, -\infty < t \leq t_0$, and $\xi((t_0 + t) - \xi(t_0), 0 \leq t \leq t_1 - t_0$. Thus, $B_{t_1}(t), 0 \leq t < -t_1$, is independent of $e^{i\xi(t)}, -\infty < t \leq t_1$.

Now we consider the existence and uniqueness of the disc SLE($\kappa, \Lambda$) process. Let $\Phi_I = \Lambda_I + H_I$. Let $B(t)$ be a Brownian motion. For $t_0 \in (-\infty, 0)$ and $x_0 \in \mathbb{R}$, let $\tilde{X}_{t_0, x_0}(t), t_0 \leq t < 0$, be the solution of the SDE:

$$d\tilde{X}_{t_0, x_0}(t) = \sqrt{\kappa} dB(t - t_0) + \Phi_I(-t, \tilde{X}_{t_0, x_0}(t)) dt, \quad \tilde{X}_{t_0, x_0}(t_0) = x_0.$$ 

Then $(\tilde{X}_{t_0, x_0}(t))$ is a real valued Markov process. Since $\Phi_I$ has period $2\pi$ in the second variable, so for any $n \in \mathbb{Z}$, $\tilde{X}_{t_0, x_0 + 2n\pi}(t) = \tilde{X}_{t_0, x_0}(t) + 2n\pi$. For $t_0 \in (-\infty, 0)$ and $w_0 \in \mathbb{T}$, choose $x_0 \in \mathbb{R}$ such that $e^{ix_0} = w_0$, and define $X_{t_0, w_0}(t) = e^{i}(\tilde{X}_{t_0, x_0}(t))$. Then the definition does not depend on the choice of $x_0$, and $(X_{t_0, x_0}(t))$ is a $\mathbb{T}$-valued Markov process. Let $(P_{t_0, t})$ be its transition probabilities, i.e., for $t_0 \leq t_1 \leq t_2 \leq t_3$ and $w_0 \in \mathbb{T}$, and a measurable set $A \subset \mathbb{T}$,

$$P_{t_0, t_1}(w_0, A) = \mathbb{P}[X_{t_0, w_0}(t_1) \in A].$$

The family $\{P_{t_1, t_2} : t_1 \leq t_2 \leq t_3 \}$ is consistent in the sense that if $t_1 \leq t_2 \leq t_3$ then $P_{t_1, t_2} * P_{t_2, t_3} = P_{t_1, t_3}$. We claim that there is a family of distributions $\{\mu_t : t \in (-\infty, 0)\}$ on $\mathbb{T}$ such that for any $t_0 \leq t_1 \in (-\infty, 0)$, $\mu_{t_0} * P_{t_0, t_1} = \mu_{t_1}$.

To prove the above claim, consider $-m \leq -n \in \mathbb{N}$. Then $P_{-m, -n}(1, \cdot)$ is a probability measure on $\mathbb{T}$. Let $-n$ be fixed and $-m \to -\infty$. Since $\mathbb{T}$ is compact, so $(P_{-m, -n}(1, \cdot))_{m=n}^{\infty}$ has
From Lemma 2.3, the above improper integral converges, so \( q \) converges to some measure, say \( \mu_{-n} \) on \( \mathbb{T} \). For any \( -n_1 \leq -n_2 \leq -N \), if \( k \) is big enough, there exists \( m_k \leq -n_1 \), and we have \( P_{-m_k,-n_2}(1,\cdot) \rightarrow P_{-m_k,-n_2} \). Letting \( k \rightarrow \infty \), we get \( \mu_{-n_1} \ast P_{-n_1,-n_2} = \mu_{-n_2} \). Finally, for each \( t \in (-\infty,0) \) we choose \( n \in \mathbb{N} \) such that \( -n < t \) and define \( \mu_t = \mu_{-n} \ast P_{-n,t} \). Then the definition of \( \mu_t \) does not depend on the choice of \( -n \), and the family \( (\mu_t : t \in (-\infty,0)) \) satisfies the property in the above claim.

From Kolmogorov extension theorem, there is a continuous \( \mathbb{T} \)-valued Markov process \( (X(t) : -\infty < t < 0) \) with transition probability \( \{P_{t_1,t_2}\} \) such that \( X(t) \sim \mu_t \) for any \( t \in (-\infty,0) \). Choose a continuous real valued process \( (\tilde{X}(t) : -\infty < t < 0) \) such that \( X(t) = e^i(\tilde{X}(t)) \), \( -\infty < t < 0 \). Now define

\[
q(t) = y_0 - \int_{-\infty}^t \mathbf{H}_t(-s, \tilde{X}(s))ds, \quad -\infty < t < 0.
\]  

From Lemma 2.3, the above improper integral converges, so \( q(t) \) is well defined. Define

\[
\xi(t) = q(t) + \tilde{X}(t), \quad -\infty < t < 0.
\]  

Let \( \tilde{g}^\xi(t,\cdot) \), \( -\infty < t < 0 \), be the covering disc Loewner maps driven by \( \xi \). Then from (2.26) and (2.27), \( q(t) = \tilde{g}^\xi_t(t, y_0) \), \( -\infty < t < 0 \). For \( t_0 \in (-\infty,0) \), let \( B_{t_0}(t) \) be defined by (3.3). Then

\[
\sqrt{k}B_{t_0}(t) = \tilde{X}(t_0 + t) - \tilde{X}(t_0) - \int_{t_0}^{t_0+t} \Phi_I(-s, \tilde{X}(s))ds, \quad 0 \leq t < -t_0.
\]  

We claim that \( (B_{t_0}(t)) \) is a Brownian motion independent of \( e^i\tilde{X}(t) = X(t) \), \( -\infty < t \leq t_0 \). Then from (3.4) and (3.5), \( (B_{t_0}(t)) \) is also independent of \( e^i\xi(t) \), \( -\infty < t \leq t_0 \). So we have the existence of the disc SLE\( (\kappa, \Lambda) \) process.

To prove the above claim, we choose a Brownian motion \( \hat{B}_{t_0}(t) \), \( 0 \leq t < t_0 \), that is independent of \( \tilde{X}(t) \), \( -\infty < t \leq t_0 \). Define another process \( \tilde{Y}(t) \), \( -\infty < t < 0 \), such that \( \tilde{Y}(t) = \tilde{X}(t) \); for \( 0 \leq t < t_0 \), \( \tilde{Y}(t) \) is the solution of

\[
d\tilde{Y}(t) = -\sqrt{k}d\hat{B}_{t_0}(t - t_0) + \Phi_I(-t, \tilde{Y}(t))dt, \quad \tilde{Y}(t_0) = \tilde{X}(t_0).
\]  

Let \( Y(t) = e^i(\tilde{Y}(t)) \), \( -\infty < t < 0 \). From the transition probability we see that \( (Y(t)) \) has the same distribution as \( (X(t)) \). Now we condition on \( X(t) \), \( -\infty < t \leq t_0 \). Since \( (Y(t)/Y(t_0), t_0 \leq t < t_0) \) has the same distribution as \( (X(t)/X(t_0), t_0 \leq t < t_0) \), \( Y = e^i(\tilde{Y}) \), \( X = e^i(\tilde{X}) \), and both \( \tilde{Y}(t) \) and \( \tilde{X}(t) \) are continuous, so \( \tilde{Y}(t) = \tilde{X}(t) \leq t < t_0 \) has the same distribution as \( \tilde{X}(t) - \tilde{X}(t_0) : t_0 \leq t < t_0 \). Since \( \Phi_I \) has period \( 2\pi \) in the second variable, so \( \Phi_I(-t, \tilde{X}(t)) ) \) has the same distribution as \( \Phi_I(-t, X(t) : t_0 \leq t < t_0) \). Comparing (3.6) with (3.7), we conclude that conditioning on \( X(t) \), \( -\infty < t \leq t_0 \), \( (\hat{B}_{t_0}(t)) \) has the same distribution as \( (B_{t_0}(t)) \). Since \( (\hat{B}_{t_0}(t)) \) is independent of \( \tilde{X}(t) \), \( -\infty < t \leq t_0 \), so conditioning on \( X(t) \), \( -\infty < t \leq t_0 \), \( (B_{t_0}(t)) \) is a Brownian motion, which implies that \( (B_{t_0}(t)) \) is also a Brownian
motion under the same conditioning. Thus, \((B_{t_0}(t))\) is a Brownian motion independent of \(X(t)\), \(-\infty < t \leq t_0\). This finishes the proof of existence.

Now we discuss the uniqueness. Suppose \((\xi(t))\) satisfies the condition in Definition 3.4. Let \(\tilde{X}(t) = \xi(t) - g_1(t, y_0)\) and \(X(t) = e^t(\tilde{X}(t))\), \(-\infty < t < 0\). Then from a similar argument, we can conclude that \((X(t) : -\infty < t < 0)\) is a Markov process with transition probability \(\{P_{t_1, t_2}\}\). Since we have

\[
e^{i\xi(t)} = \frac{e^{i\xi_0}}{X(t)} \exp \left( \int_{-\infty}^{t} S_I(-s, X(s))ds \right), \quad -\infty < t < 0,
\]

and the disc Loewner hulls driven by \(\xi\) are determined by \((e^{i\xi(t)})\), so we suffice to show that the distribution of the Markov process with transition probability \(\{P_{t_1, t_2}\}\) is unique.

Suppose \((X(t))\) is a Markov process with transition probability \(\{P_{t_1, t_2}\}\). Let \(\nu_t\) denote the distribution of \(X(t)\), \(-\infty < t < 0\). It suffices to show that \(\nu_t = \mu_t\) for \(-\infty < t < 0\). Now for \(-m \leq -n \in -\mathbb{N}\), we have \(\nu_{-m} * P_{-m, -n} = \nu_{-n}\). Assume that for any \(w_0 \in \mathbb{T}\), as \(-m \to -\infty\), \(P_{-m, -n}(w_0, \cdot)\) converges weakly to a measure depending only on \(-n\), and the convergence is uniform in \(w_0 \in \mathbb{T}\). Then the limit measure must be \(\mu_{-n}\) because of the definition of \(\mu_{-n}\). Letting \(-m \to -\infty\) in the equality \(\nu_{-m} * P_{-m, -n} = \nu_{-n}\), we get \(\mu_{-n} = \nu_{-n}\) for any \(-n \in -\mathbb{N}\). Finally, for any \(t \in (-\infty, 0)\), we may choose \(-n \in -\mathbb{N}\) such that \(-n \leq t\). Then \(\nu_t = \nu_{-n} * P_{-n, t} = \mu_{-n} * P_{-n, t} = \mu_t\).

To get the uniqueness, we want that the Prohorov diameter of the set \(\{P_{t_1, -n}(w_0, \cdot) : t \leq t_0, w \in \mathbb{T}\}\) tends to 0 as \(t_0 \to -\infty\). Suppose \(t_1 \leq t_2 \in (-\infty, 0)\) and \(w_1, w_2 \in \mathbb{T}\). We need to know whether the Prohorov distance between \(P_{t_1, -n}(w_1, \cdot)\) and \(P_{t_2, -n}(w_2, \cdot)\) tends to 0 as \(t_1, t_2 \to -\infty\), uniform in \(w_1, w_2 \in \mathbb{T}\). For this purpose we may construct a coupling of two random variables \(A_1\) and \(A_2\) with these two distributions respectively, such that \(P[A_1 = A_2]\) tends to 1 as \(t_1, t_2 \to -\infty\). Now we run two independent Markov processes \((X_1(t)) : t_1 \leq t < 0\) and \((X_2(t)) : t_2 \leq t < 0\) started from \(X_j(t_j) = w_j, j = 1, 2\), such that they both have transition probability \(\{P_{t_1, t_2}\}\). Let \(A_j = X_j(-n), j = 1, 2\). Then \(A_j \sim P_{t_j, -n}(w_j, \cdot), j = 1, 2\). Let \(\tau\) be the first \(t\) such that \(X_1(t) = X_2(t);\) if such time does not exist, let \(\tau = 0\). Let \((\tilde{X}_1(t)) = (X_1(t))\). Define \(\tilde{X}_2\) such that \(\tilde{X}_2(t) = X_2(t)\) for \(t \leq \tau\) and \(\tilde{X}_2(t) = X_1(t)\) for \(t \geq \tau\). Then for \(j = 1, 2\), \((\tilde{X}_j(t))\) has the same distribution as \((X_j(t))\). Define \(A_j\) for \(\tilde{X}_j, j = 1, 2\), as well. Now we have \(\{A_1 = A_2\} = \{\tau \leq -n\}\). For the uniqueness, we want that \(P[\tau \leq -n] \to 1\) as \(t_1, t_2 \to -\infty\). For example, if \(\Lambda\) is uniformly bounded on \((-\infty, t_0] \times \mathbb{R}\) for any \(t_0 \in (-\infty, 0)\), then this holds. In general, we do not expect the uniqueness of disc SLE(\(\kappa, \Lambda\) process. We will not go into details of this discussion.

Let \(\xi\) be as in Definition 3.3. Let \(K_I(t)\) and \(g_I(t, \cdot), -\infty < t < 0\), be the disc Loewner hulls and maps driven by \(\xi\), respectively. Then a.s.

\[
\beta_I(t) := \lim_{A \to -\infty} \lim_{t \to -\infty} g_I(t, \cdot)^{-1}(z)
\]

exists for \(-\infty < t < 0\), and \(\beta_I(t), -\infty < t < 0\), is a continuous curve in \(\mathbb{D}\) with \(\lim_{t \to -\infty} \beta_I(t) = 0\). We call \(\beta_I\) the disc SLE(\(\kappa, \Lambda\)) trace in \(\mathbb{D}\) started from 0 with marked point \(b\). If \(\kappa \in (0, 4]\),
then $\beta_I$ is a simple curve and $K_I(t) = \beta_I([−∞, t])$; if $κ > 4$, then $\beta_I$ is not simple, and $C \setminus K_I(t)$ is the unbounded component of $C \setminus \beta_I([−∞, t])$.

The definition of disc SLE($κ, \Lambda$) process is self conformally invariant. If $β$ is a disc SLE($κ, \Lambda$) trace in $\mathbb{D}$ started from 0 with marked point $b$, and $W$ maps $\mathbb{D}$ conformally onto $\mathbb{D}$, and fixes 0, then $W(β)$ has the distribution as a disc SLE($κ, \Lambda$) trace in $\mathbb{D}$ started from 0 with marked point $W(b)$. If $W$ is a conjugate conformal map, then $W(β)$ is a disc SLE($κ, Λ_I$) trace. So via conformal maps, we can define SLE($κ, \Lambda$) hulls or trace in any simply connected domain started from an interior point with some marked boundary point.

The disc SLE($κ, \Lambda$) process is related to annulus SLE($κ, \Lambda$) process in the following way. Suppose $β_I(t)$, $−∞ < t < 0$, is a disc SLE($κ, \Lambda$) trace started from $a$ with marked point $b$. Let $K_I(t)$, $−∞ < t < 0$, be the corresponding hulls. Fix $t_0 \in (−∞, 0)$. If we condition on $K_I(t)$, $−∞ < t \leq t_0$, then $β_I(t_0 + t)$, $0 \leq t < −t_0$, is an annulus SLE($κ, \Lambda$) trace in $D \setminus K_I(t_0)$ started from $β_I(t_0)$ with marked point $b$. This follows from the two definitions and Lemma 3.1.

## 4 Coupling of Two Annulus SLE Processes

In this section, we will prove the following theorem. Recall that $I_p(z) = e^{-p/|z|}$.

**Theorem 4.1** Let $κ > 0$. Suppose $Λ$ is a $C^{1,2}$ differentiable crossing annulus drift function, and satisfies the following PDE:

$$\dot{Λ} = \frac{κ}{2} Λ'' + \left(3 - \frac{κ}{2}\right) H' Λ' + Λ H' Λ' + Λ Λ'$$

(4.1)

on $(0, ∞) \times \mathbb{R}$, where the dot denotes the partial derivative w.r.t. the first variable, and the primes denote the partial derivatives w.r.t. the second variable. Moreover, suppose that

$$\int_{−π}^{π} Λ(p, x) dx = 0, \quad 0 < p < ∞.$$  (4.2)

Let $Λ_1 = Λ$, and $Λ_2$ be the dual function of $Λ$. Then for any $p > 0$, $a_1, a_2 \in T$, there is a coupling of two processes $K_1(t)$ and $K_2(t)$, $0 \leq t < p$, such that for $j \neq k \in \{1, 2\}$, the followings hold.

(i) $K_j(t)$, $0 \leq t < p$, is an annulus SLE($κ, Λ_j$) process in $κ_p$ started from $a_j$ with marked point $a_{I,k} := I_p(a_k)$.

(ii) If $t_k < p$ is a stopping time for $(K_k(t))$, then conditioned on $K_k(t)$, $0 \leq t \leq t_k$, after a time-change, $K_j(t)$, $0 \leq t < T_j(t_k)$, is a stopped annulus SLE($κ, Λ_j$) process in $κ_p \setminus I_p(K_k(t_k))$ started from $a_j$ with marked point $β_{I,k} := I_p(β_k(t_k))$, where $β_k(t)$ is the trace that corresponds to $K_k(t)$, $0 \leq t < p$, and $T_j(t_k)$ is the maximal number in $(0, p]$ such that $K_j(t) \cap I_p(K_k(t_k)) = ∅$ for $0 \leq t < T_j(t_k)$.
Lemma 4.1 Fix $\kappa > 0$. Suppose $\Lambda$ is a crossing annulus drift function that satisfies (4.1). Then there is a positive $C^{1,2}$ differentiable function $\Gamma$ on $(0, \infty) \times \mathbb{R}$, which satisfies
\[
\frac{\Gamma'}{\Gamma} = \frac{\Lambda}{\kappa},
\]
\[
\dot{\Gamma} = \frac{\kappa}{2} \Gamma'' + H_I \Gamma' + \frac{6 - \kappa}{2\kappa} H_I' \Gamma.
\]
Moreover, if $\Lambda$ satisfies (4.2) then $\Gamma$ has period $2\pi$ in the second variable.

Proof. Define $\hat{\Gamma}$ on $(0, \infty) \times \mathbb{R}$ such that
\[
\hat{\Gamma}(t, x) = \exp \left( \int_0^x \frac{1}{\kappa} \Lambda(t, y) dy \right).
\]
Then we have $\frac{\hat{\Gamma}'}{\hat{\Gamma}} = \frac{\Lambda}{\kappa}$. So $\dot{\hat{\Gamma}} = \frac{\kappa}{2} \hat{\Gamma}'' + \left( 3 - \frac{\kappa}{2} \right) H_I'' + (\Lambda H_I)' + \frac{(\Lambda^2)'}{2}$.

So for each $t \in (0, \infty)$ there is $C(t) \in \mathbb{R}$ such that
\[
\kappa \frac{\hat{\Gamma}}{\Gamma} = \frac{\kappa}{2} \Lambda' + H_I \Lambda + \left( 3 - \frac{\kappa}{2} \right) H_I' + \frac{\Lambda^2}{2} + C(t).
\]
So $C(t)$ is continuous. Since $\Lambda = \frac{\kappa}{2} \hat{\Gamma}$, so $\Lambda' = \frac{\kappa}{2} \hat{\Gamma}' - \kappa (\hat{\Gamma})^2$. From the above formula, we have
\[
\dot{\hat{\Gamma}} = \frac{\kappa}{2} \hat{\Gamma}'' + H_I \hat{\Gamma}' + \frac{6 - \kappa}{2\kappa} H_I' \hat{\Gamma} + \frac{1}{\kappa} C(t) \hat{\Gamma}.
\]
Let
\[
\Gamma(t, x) = \hat{\Gamma}(t, x) \exp \left( - \int_1^t \frac{1}{\kappa} C(s) ds \right).
\]
Then it is easy to see that $\Gamma$ satisfies (4.3) and (4.4). Finally, if $\Lambda$ satisfies (4.2) then from (4.3) we see that $\Gamma$ has period $2\pi$ in the second variable. □

4.1 Ensemble

Let $p > 0$, and $\xi_1, \xi_2 \in C([0, p])$. For $j = 1, 2$, let $K_j(t)$ and $g_j(t, \cdot)$ (resp. $K_{I,j}(t)$ and $g_{I,j}(t, \cdot)$), $0 \leq t < p$, be the annulus (resp. inverted annulus) Loewner hulls and maps of modulus $p$ driven by $\xi_j$. Let $\tilde{K}_j(t, \cdot)$, $\tilde{K}_{I,j}(t, \cdot)$, $\tilde{g}_j(t, \cdot)$ and $\tilde{g}_{I,j}(t, \cdot)$, $0 \leq t < p$, $j = 1, 2$, be the corresponding covering Loewner hulls and maps. Then $K_{I,j}(t) = I_p(K_j(t))$, $j = 1, 2$. Define
\[
D = \{(t_1, t_2) \in [0, p)^2 : K_1(t_1) \cap K_{I,2}(t_2) = \emptyset\} = \{(t_1, t_2) \in [0, p)^2 : K_{I,1}(t_1) \cap K_2(t_2) = \emptyset\}.
\]
For \((t_1, t_2) \in \mathcal{D}, \mathbb{A}_p \setminus K_1(t_1) \setminus K_2(t_2)\) and \(\mathbb{A}_p \setminus K_1(t_1) \setminus K_2(t_2)\) are doubly connected domains that have the same modulus, so we may define

\[
m(t_1, t_2) = \text{mod}(\mathbb{A}_p \setminus K_1(t_1) \setminus K_2(t_2)) = \text{mod}(\mathbb{A}_p \setminus K_1(t_1) \setminus K_2(t_2)).
\] (4.5)

Fix any \(j \neq k \in \{1, 2\}\) and \(t_k \in [0, p)\). Let \(T_j(t_k) = \sup\{t_j : K_j(t_j) \cap K_{I,k}(t_k) = \emptyset\}\). Then for any \(t_j < T_j(t_k)\), we have \((t_1, t_2) \in \mathcal{D}\). Moreover, as \(t_j \to T_j(t_k)\), the spherical distance between \(K_j(t_j)\) and \(K_{I,k}(t_k)\) tends to 0, so \(m(t_1, t_2) \to 0\).

From Proposition 2.4, \(K_j(t_j), 0 \leq t_j < p\) is a Loewner chain in \(\mathbb{A}_p\) w.r.t. \(T_p\). Since for \(0 \leq t_j < T_j(t_k)\), \(K_j(t_j)\) lies in \(\mathbb{A}_p \setminus K_{I,k}(t_k)\), so \(K_j(t_j), 0 \leq t_j < T_j(t_k)\), is also a Loewner chain in \(\mathbb{A}_p \setminus K_{I,k}(t_k)\). Since \(g_{t,k}(t, \cdot)\) maps \(\mathbb{A}_p \setminus K_{I,k}(t_k)\) conformally onto \(\mathbb{A}_{p-t_k}\), and maps \(\mathbb{T}\) onto \(\mathbb{T}\), so from conformal invariance of extremal length, \(K_{j,t_k}(t_j) := g_{t,k}(t_k, K_j(t_j)), 0 \leq t_j < T_j(t_k)\), is a Loewner chain in \(\mathbb{A}_{p-t_k}\) w.r.t. \(T_{p-t_k}\). Now we apply Proposition 2.4. Let

\[
v_{j,t_k}(t_j) = \text{cap}_{\mathbb{A}_{p-t_k}}(K_{j,t_k}(t_j)) = p - t_k - \text{mod}(\mathbb{A}_{p-t_k} \setminus g_{t,k}(t_k, K_j(t_j)))
\] (4.6)

Here the third “=” holds because \(g_{t,k}(t, \cdot)\) maps \(\mathbb{A}_p \setminus K_{I,k}(t_k) \setminus K_j(t_j)\) conformally onto \(\mathbb{A}_{p-t_k} \setminus g_{I,k}(t_k, K_j(t_j))\). Then \(v_{j,t_k}\) is continuous and increasing, and maps \([0, T_j(t_k))\) onto \([0, S_{j,t_k})\) for some \(S_{j,t_k} \in (0, p - t_k)\). Since \(\zeta_{j,t_k} \in C([0, p - t_k])\), there exists a \(1\)-annulus Loewner hulls of modulus \(p - t_k\) driven by some \(\zeta_{j,t_k} \in C([0, p - t_k])\). Let \(L_{j,t_k}(t)\) be the corresponding inverted annulus Loewner hulls. Let \(h_{j,t_k}(t, \cdot)\) and \(h_{I,j,t_k}(t, \cdot)\) be the corresponding annulus and inverted annulus Loewner maps. Let \(\tilde{L}_{j,t_k}(t), \tilde{L}_{I,j,t_k}(t), \tilde{h}_{j,t_k}(t, \cdot),\) and \(\tilde{h}_{I,j,t_k}(t, \cdot)\) be the corresponding covering hulls and maps.

For \(0 \leq t_j < T_j(t_k)\), let \(\xi_{j,t_k}(t_j), K_{I,j,t_k}(t_j), g_{j,t_k}(t_j, \cdot), g_{I,j,t_k}(t_j, \cdot), K_{j,t_k}(t_j), \tilde{K}_{I,j,t_k}(t_j), \tilde{g}_{j,t_k}(t_j, \cdot), \text{ and } \tilde{g}_{I,j,t_k}(t_j, \cdot)\) be the time-change of \(\zeta_{j,t_k}(t), L_{I,j,t_k}(t), h_{j,t_k}(t, \cdot), h_{I,j,t_k}(t, \cdot), L_{j,t_k}(t), L_{I,j,t_k}(t), h_{j,t_k}(t, \cdot),\) and \(h_{I,j,t_k}(t, \cdot)\), respectively, via the map \(v_{j,t_k}\). For example, this means that \(\xi_{j,t_k}(t_j) = \zeta_{j,t_k}(v_{j,t_k}(t_j))\) and \(g_{j,t_k}(t_j, \cdot) = h_{j,t_k}(v_{j,t_k}(t_j), \cdot)\).

From (2.16), for \(0 \leq t_j < p\) and \(\varepsilon \in (0, p - t_j)\), \(g_{j,t_k}(t_j, K_j(t_j + \varepsilon) \setminus K_j(t_j))\) is a hull in \(\mathbb{A}_{p-t_j}\) w.r.t. \(T_{p-t_j}\), and

\[
\text{cap}_{\mathbb{A}_{p-t_j}}(g_{j,t_k}(t_j, K_j(t_j + \varepsilon) \setminus K_j(t_j))) = \varepsilon.
\] (4.7)

From Proposition 2.4, we have

\[
\{e^{i\xi_{j,t_k}(t_j)}\} = \bigcap_{\varepsilon \in (0, p - t_j)} g_{j,t_k}(t_j, K_j(t_j + \varepsilon) \setminus K_j(t_j)), \quad 0 \leq t_j < p.
\] (4.8)

From (2.16) and (4.6), for \(0 \leq t_j < T_j(t_k)\) and \(\varepsilon \in (0, T_j(t_k) - t_j)\),

\[
g_{j,t_k}(t_j, K_{j,t_k}(t_j + \varepsilon) \setminus K_{j,t_k}(t_j)) = h_{j,t_k}(v_{j,t_k}, L_{j,t_k}(v_{j,t_k}(t_j + \varepsilon)) \setminus L_{j,t_k}(v_{j,t_k}(t_j)))
\]

is a hull in \(\mathbb{A}_{p-t_k-v_{j,t_k}(t_k)}\) w.r.t. \(T_{m(t_1, t_2)}\), and

\[
\text{cap}_{\mathbb{A}_{m(t_1, t_2)}}(g_{j,t_k}(t_j, K_{j,t_k}(t_j + \varepsilon) \setminus K_{j,t_k}(t_j))) = v_{j,t_k}(t_j + \varepsilon) - v_{j,t_k}(t_j).
\] (4.9)
From Proposition 2.4 we have
\[ \{e^{i\zeta_j,t_k(t_j)}\} = \bigcap_{\varepsilon \in (0, T_j(t_k) - t_j)} g_{j,t_k}(t_j, K_{j,t_k}(t_j + \varepsilon) \setminus K_{j,t_k}(t_j)), \quad 0 \leq t_j < p. \] (4.10)

For \(0 \leq t_j < T_j(t_k)\), let
\[ G_{I,k,t_k}(t_j, \cdot) = g_{j,t_k}(t_j, \cdot) \circ g_{I,k}(t_k, \cdot) \circ g_{j}(t_j, \cdot)^{-1}; \]
(4.11)
\[ \tilde{G}_{I,k,t_k}(t_j, \cdot) = \tilde{g}_{j,t_k}(t_j, \cdot) \circ \tilde{g}_{I,k}(t_k, \cdot) \circ \tilde{g}_{j}(t_j, \cdot)^{-1}. \] (4.12)
Then \( G_{I,k,t_k}(t_j, \cdot) \) maps \( \mathbb{A}_{p-t_j} \setminus g_j(t_j, K_{I,k}(t_k)) \) conformally onto \( \mathbb{A}_{m(t_j, t_2)} \), and maps \( \mathbb{T} \) onto \( \mathbb{T} \); \( e^i \circ \tilde{G}_{I,k,t_k}(t_j, \cdot) = G_{I,k,t_k}(t_j, \cdot) \circ e^i \), and \( \tilde{G}_{I,k,t_k}(t_j, \cdot) \) maps \( \mathbb{R} \) onto \( \mathbb{R} \). For any \( t_j \in [0, T_j(t_k)] \) and \( \varepsilon \in (0, T_j(t_k) - t_j) \), we have
\[ G_{I,k,t_k}(t_j, g_j(t_j, K_{j,t_k}(t_j + \varepsilon) \setminus K_{j,t_k}(t_j))). \] (4.13)
From (4.8), (4.10), and (4.13) we have \( e^{i\zeta_j,t_k(t_j)} = G_{I,k,t_k}(t_j, e^{i\zeta_j(t_j)}) = e^i \circ \tilde{G}_{I,k,t_k}(t_j, \zeta_j(t_j)). \)
So there is \( n \in \mathbb{Z} \) such that \( G_{I,k,t_k}(t_j, \zeta_j(t_j)) = \tilde{G}_{I,k,t_k}(t_j, \zeta_j(t_j)) + 2n\pi \) for \( 0 \leq t_j < p \). Since \( \zeta_j,t_k + 2n\pi \) generates the same annulus Loewner hulls as \( \zeta_j,t_k \), so we may choose \( \zeta_j,t_k \) such that
\[ \zeta_j,t_k(t_j) = \tilde{G}_{I,k,t_k}(t_j, \xi_j(t_j)), \quad 0 \leq t_j < T_j(t_k). \] (4.14)
From (4.7), (4.9), and Proposition 2.1 we have
\[ v_{j,t_k}(t_j) = |G_{I,k,t_k}(t_j, \zeta_j(t_j))|^2 = \tilde{G}_{I,k,t_k}(t_j, \xi_j(t_j))^2, \quad 0 \leq t_j < T_j(t_k). \] (4.15)
From (4.6) we have
\[ \partial_j m(t_1, t_2) = -\tilde{G}''_{I,k,t_k}(t_j, \xi_j(t_j))^2, \quad 0 \leq t_j < T_j(t_k). \] (4.16)
Since \( \tilde{g}_j(t, \cdot) \) are covering annulus Loewner maps of modulus \( p \) driven by \( \xi \), so it satisfies
\[ \hat{\tilde{g}}_{j}(t, z) = H(p - t_j, \tilde{g}_j(t, z) - \xi_j(t_j)), \quad 0 \leq t_j < p. \] (4.17)
Since \( \tilde{g}_{j,t_k}(t_j, \cdot) = \tilde{h}_{j,t_k}(v_{j,t_k}(t_j), \cdot) \), \( \tilde{h}_{j,t_k}(t, \cdot) \) are covering annulus Loewner maps of modulus \( p - t_k \) driven by \( \zeta_j,t_k \), and \( \xi_j,t_k(t_j) = \tilde{h}_{j,t_k}(v_{j,t_k}(t_j)) \), so from (4.6) and (4.15), we have
\[ \hat{\tilde{g}}_{j,t_k}(t_j, z) = \tilde{G}''_{I,k,t_k}(t_j, \xi_j(t_j))^2 H(m(t_1, t_2), \tilde{g}_{j,t_k}(t_j, z) - \xi_j,t_k(t_j)), \quad 0 \leq t_j < T_j(t_k). \] (4.18)
From (4.12) we see that, for any \( z \in \mathbb{S}_p \setminus \tilde{K}_{j}(t_j) \setminus \tilde{K}_{I,k}(t_k) \), we have
\[ \tilde{G}_{I,k,t_k}(t_j, \cdot) \circ \tilde{g}_{j}(t_j, z) = \tilde{g}_{j,t_k}(t_j, \cdot) \circ \tilde{g}_{I,k}(t_k, z). \] (4.19)
Differentiate (4.19) w.r.t. \( t_j \). From (4.14), (4.17), and (4.18), we get
\[ \hat{\tilde{G}}_{I,k,t_k}(t_j, \tilde{g}_j(t_j, z)) + \tilde{G}''_{I,k,t_k}(t_j, \tilde{g}_j(t_j, z)) H(p - t_j, \tilde{g}_j(t_j, z) - \xi_j(t_j)) \]
\[ \dot{G}_{I,k,t_k}(t_j, \xi_j(t_j))^2 \mathbf{H}(m(t_1, t_2), \tilde{G}_{I,k,t_k}(t_j, \bar{g}_j(t_j, z)) - \tilde{G}_{I,k,t_k}(t_j, \xi_j(t_j))). \]

Since \( g_j(t_j, \cdot) \) maps \( S_p \setminus \bar{K}_j(t_j) \setminus \tilde{K}_{I,k}(t_k) \) conformally onto \( S_{p-t_j} \setminus \bar{g}_j(t_j, \tilde{K}_{I,k}(t_k)) \), so from the above formula, we see that, for any \( w \in S_{p-t_j} \setminus \bar{g}_j(t_j, \tilde{K}_{I,k}(t_k)) \),

\[ \dot{G}_{I,k,t_k}(t_j, \xi_j(t_j))^2 \mathbf{H}(m, \tilde{G}_{I,k,t_k}(t_j, w) - \tilde{G}_{I,k,t_k}(t_j, \xi_j(t_j)))) - \tilde{G}_{I,k,t_k}(t_j, w) \mathbf{H}(p - t_j, w - \xi_j(t_j)). \]

Differentiate (4.20) w.r.t. \( w \). Then we have

\[ \dot{\dot{G}}_{I,k,t_k}(t_j, \xi_j(t_j)) = \frac{1}{2} \left( \frac{\tilde{G}_{I,k,t_k}(t_j, \xi_j(t_j))}{\tilde{G}_{I,k,t_k}(t_j, \xi_j(t_j))} \right)^2 - \frac{4}{3} \frac{\tilde{G}_{I,k,t_k}(t_j, \xi_j(t_j))}{\tilde{G}_{I,k,t_k}(t_j, \xi_j(t_j))} \]

\[ + \tilde{G}_{I,k,t_k}(t_j, \xi_j(t_j))^2 \mathbf{r}(m) - \mathbf{r}(p - t_j). \]  

**Remark.** It is in [8] that the ideas behind (4.14), (4.15), and (4.21) first appeared, which were there to show that SLE\(_6\) satisfies locality property. The first formula that is similar to (4.22) appeared in [10], which was used to show that SLE\(_{8/3}\) satisfies restriction property.

From the definition of inverted annulus Loewner maps, we see that \( h_{I,k,t_j}(t, \cdot) \) maps \( A_{p-t_j} \setminus L_{I,k,t_j}(t) \) conformally onto \( A_{p-t_j-t} \), and maps \( \mathbb{T} \) onto \( \mathbb{T} \). Since \( g_{I,k,t_j}(t, \cdot) = h_{I,k,t_j}(v_{k,t_j}(t_k), \cdot) \),

\[ K_{I,k,t_j}(t_k) = L_{I,k,t_j}(v_{k,t_j}(t_k)), \]

so from (4.6), both \( G_{I,k,t_k}(t_j, \cdot) \) and \( g_{I,k,t_k}(t_k, \cdot) \) map \( A_{p-t_j} \setminus K_{I,k,t_j}(t_k) \) conformally onto \( A_{m(t_1,t_2)} \), and maps \( \mathbb{T} \) onto \( \mathbb{T} \). So they differ by a multiplicative constant of modulus 1. Since \( G_{I,k,t_k}(t_j, \cdot) \circ e^i = e^i \circ G_{I,k,t_k}(t_j, \cdot) \) and \( g_{I,j,t_k}(t_j, \cdot) \circ e^i = e^i \circ g_{I,j,t_k}(t_j, \cdot) \), so there is \( C_k(t_1, t_2) \in \mathbb{R} \) such that

\[ \tilde{G}_{I,k,t_k}(t_j, \cdot) = \tilde{g}_{I,k,t_k}(t_k, \cdot) + C_k(t_1, t_2). \]  

Exchanging \( j \) and \( k \) in (4.23), we have \( C_j(t_1, t_2) \in \mathbb{R} \) such that

\[ \tilde{G}_{I,j,t_k}(t_k, \cdot) = \tilde{g}_{I,j,t_k}(t_j, \cdot) + C_j(t_1, t_2). \]
Since \( g_{j,t_k}(t_j, \cdot) \) and \( \tilde{g}_{I,j,t_k}(t_j, \cdot) \) are time-changes of \( h_{j,t_k}(t, \cdot) \) and \( \tilde{h}_{I,j,t_k}(t, \cdot) \) via \( v_{j,t_k} \), respectively, and \( h_{j,t_k}(t, \cdot) = I_{p-t_k} \circ h_{I,j,t_k}(t_j, \cdot) \circ I_{p-t_k} \), so from (4.14), we have
\[
\tilde{g}_{j,t_k}(t_j, \cdot) = I_{m(t_1,t_2)} \circ \tilde{g}_{I,j,t_k}(t_j, \cdot) \circ I_{p-t_k};
\] (4.25)
Similarly, we have
\[
\tilde{g}_{I,k,t_j}(t_k, \cdot) = I_{m(t_1,t_2)} \circ \tilde{g}_{k,t_j}(t_k, \cdot) \circ I_{p-t_j}.
\] (4.26)
Now let
\[
\tilde{G}_{j,t_j}(t_k, \cdot) = I_{m(t_1,t_2)} \circ \tilde{G}_{I,j,t_j}(t_k, \cdot) \circ I_{p-t_k}.
\] (4.27)
Then from (4.24), (4.25), and (4.27), we have
\[
\tilde{G}_{j,t_j}(t_k, \cdot) = \tilde{g}_{j,t_k}(t_j, \cdot) + C_j(t_1,t_2).
\] (4.28)
Exchanging \( j \) and \( k \) in (4.19), and using (4.26), (4.27), and that \( \tilde{g}_{j}(t_j, \cdot) = I_{p-t_j} \circ \tilde{g}_{I,j}(t_j, \cdot) \circ I_{p} \) and \( \tilde{g}_{I,k}(t_k, \cdot) = I_{p-t_k} \circ \tilde{g}_{k}(t_k, \cdot) \circ I_{p} \), we get
\[
\tilde{g}_{l,k,t_j}(t_k, \cdot) \circ \tilde{g}_{j}(t_j, \cdot) = \tilde{G}_{l,j,t_j}(t_k, \cdot) \circ \tilde{g}_{I,k}(t_k, \cdot).
\] (4.29)
Comparing (4.29) with (4.19), and using (4.23) and (4.28), we see that
\[
C_1(t_1,t_2) + C_2(t_1,t_2) = 0.
\] (4.30)
From (4.14), for \( (t_1,t_2) \in \mathcal{D} \), we may define
\[
X_j(t_1,t_2) = \xi_{j,t_k}(t_j) - \tilde{g}_{l,j,t_k}(t_j, \xi_k(t_k)) = \tilde{G}_{l,j,t_k}(t_j, \xi_j(t_j)) - \tilde{g}_{I,l,j,t_k}(t_j, \xi_k(t_k)).
\] (4.31)
From (4.23), (4.24) and (4.30) we have
\[
X_1(t_1,t_2) + X_2(t_1,t_2) = 0.
\] (4.32)
For \( (t_1,t_2) \in \mathcal{D} \), we define
\[
A_{j,h}(t_1,t_2) = \tilde{g}_{l,k,j}(t_k, \xi_j(t_k)), \quad h = 0, 1, 2, 3,
\] (4.33)
\[
A_{j,S}(t_1,t_2) = \frac{A_{j,2}(t_1,t_2)}{A_{j,1}(t_1,t_2)} - \frac{3}{2} \left( \frac{A_{j,2}(t_1,t_2)}{A_{j,1}(t_1,t_2)} \right)^2,
\] (4.34)
where the superscript “(h)” denotes the \( h \)-th partial derivative w.r.t. the second variable, so \( A_{j,S}(t_1,t_2) \) is the Schwarzian derivative of \( \tilde{g}_{l,k,j}(t_k, \cdot) \) at \( \xi_j(t_j) \). Since \( H''''(p, \cdot) \) is even, so from (4.32), for \( (t_1,t_2) \in \mathcal{D} \), we may define
\[
Q(t_1,t_2) = H''''(m(t_1,t_2), X_1(t_1,t_2)) = H''''(m(t_1,t_2), X_2(t_1,t_2)).
\] (4.35)
\[
F(t_1,t_2) = \exp \left( \int_0^{t_2} \int_0^{t_1} A_{1,1}(s_1,s_2)^2 A_{2,1}(s_1,s_2)^2 Q(s_1,s_2) ds_1 ds_2 \right).
\] (4.36)
We may rewrite (4.18) as

\[ \dot{g}_{I,j,t_k}(t_j, z) = A_{j,1}^2 H_I(m, g_{I,j,t_k}(t_j, z) - \xi_{j,t_k}(t_j)). \] (4.37)

Differentiate this formula w.r.t. \( z \) twice. We get

\[ \frac{\partial}{\partial t_j} \left( \frac{\dot{g}_{I,j,t_k}(t_j, z)}{g_{I,j,t_k}(t_j, z)} \right) = A_{j,1}^2 H_I'(m, g_{I,j,t_k}(t_j, z) - \xi_{j,t_k}(t_j)) \] (4.38)

\[ \frac{\partial^2}{\partial t_j^2} \left( \frac{\dot{g}_{I,j,t_k}(t_j, z)}{g_{I,j,t_k}(t_j, z)} \right) = A_{j,1}^2 H_I''(m, g_{I,j,t_k}(t_j, z) - \xi_{j,t_k}(t_j)) \frac{g_{I,j,t_k}(t_j, z)}{g_{I,j,t_k}(t_j, z)}. \] (4.39)

Let \( z = \xi_k(t_k) \) in (4.37), (4.38), and (4.39). Since \( H_I(p, \cdot) \) and \( H_I'(p, \cdot) \) are odd, and \( H_I'(p, \cdot) \) is even, so from (4.31), we have

\[ \partial_j A_{k,0} = -A_{j,1}^2 H_I(m, X_j) \partial t_j. \] (4.40)

\[ \frac{\partial_j A_{k,1}}{A_{k,1}} = A_{j,1}^2 H_I'(m, X_j) \partial t_j. \] (4.41)

\[ \frac{\partial}{\partial t_j} \left( \frac{A_{k,2}}{A_{k,1}} \right) = -A_{j,1}^2 H_I''(m, X_j) A_{k,1}. \] (4.42)

Differentiate (4.39) w.r.t. \( z \) again, and let \( z = \xi_k(t_k) \). Since \( H_I''(p, \cdot) \) is even, so we get

\[ \frac{\partial}{\partial t_j} \left( \frac{A_{k,3}}{A_{k,1}} \right) = A_{j,1}^2 \left[ H_I'''(m, X_j) A_{k,1}^2 - H_I''(m, X_j) A_{k,2} \right]. \] (4.43)

From (4.34), (4.35), (4.42), and (4.43), we have

\[ \partial_j A_{k,S} = A_{j,1}^2 A_{k,1}^2 Q. \] (4.44)

Since \( g_{I,j,t_k}(0, \cdot) = \bar{g}_{I,j,t_k}(0, \cdot) = id \), so when \( t_j = 0 \), we have \( A_{k,1} = 1 \), \( A_{k,2} = A_{k,3} = 0 \), hence
\[ A_{k,S} = 0. \] From (4.36) and (4.44), we see that for any \( k \in \{1, 2\} \),

\[ \frac{\partial_h F}{F} = A_{k,S}. \] (4.45)

4.2 Martingales in two time variables

Let \( a_1, a_2 \in \mathbb{T} \) be as in Theorem 4.1. Let \( a_I,j = I_p(a_j) \in \mathbb{T}_p \), \( j = 1, 2 \). Choose \( x_1, x_2 \in \mathbb{R} \) such that \( a_j = e^{ix_j} \), \( j = 1, 2 \). Let \( B_1(t) \) and \( B_2(t) \) be two independent Brownian motion, which generate filtrations \( (\mathcal{F}_t^1) \) and \( (\mathcal{F}_t^2) \), respectively. Let \( (\mathcal{F}_t^j) \) be the augmentation of \( (\mathcal{F}_t^j) \) w.r.t. the distribution of \( (B_j(t)) \), \( j = 1, 2 \). Let \( \Lambda \), \( \Lambda_1 \) and \( \Lambda_2 \) be as in Theorem 4.1. We adopt the notation in the last section. For \( j = 1, 2 \), let \( \xi_j(t_j) \), \( 0 \leq t_j < p \), be the solution to the SDE:

\[ d\xi_j(t_j) = \sqrt{\kappa} dB_j(t_j) + \Lambda_j(p - t_j, \xi_j(t_j) - g_{I,j}(t_j, x_{3-j})) dt_j, \quad \xi_j(0) = x_j. \] (4.46)

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Then for $j = 1, 2$, $(\xi_j(t))$ is an $(\mathcal{F}_t^j)$-adapted semi-martingale with $\langle \xi_j \rangle_t = \kappa t$; $(\xi_1(t))$ and $(\xi_2(t))$ are independent, and $K_j(t_j)$, $0 \leq t_j < p$, are annulus SLE$(\kappa, A_j)$ hulls in $a_p$ started from $a_j$ with marked point $a_{1,3-j}$.

As annulus Loewner objects driven by $\xi_j$, $(K_j(t_j))$, $(K_{I,j}(t_j))$, $(g_{I,j}(t_j, \cdot))$, $(\tilde{g}_j(t_j, \cdot))$, and $(\tilde{g}_{I,j}(t_j, \cdot))$ are all $(\mathcal{F}_t^j)$-adapted. Fix $j \neq k \in \{1, 2\}$. Since $(K_j(t_j))$ is $(\mathcal{F}_t^j)$-adapted, and $(g_{I,k}(t_k, \cdot))$ is $(\mathcal{F}_t^k)$-adapted, so $(K_{j,t_k}(t_j) = g_{I,k}(t_k, K_j(t_j))$ defined on $D$ is $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted. Since $\tilde{g}_{j,t_k}(t_j, \cdot)$ and $\tilde{g}_{I,j,t_k}(t_j, \cdot)\) are determined by $(K_{j,t_k}(s_j))$, $0 \leq s_j \leq t_j$, so they are $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted. From $(4.12)$, $(\tilde{G}_{I,k,t_k}(t_j, \cdot))$ is $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted. From $(4.14)$, $(\xi_j,t_k(t_j))$ is also $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted. From $(4.5)$, $(4.31)$, $(4.33)$, and $(4.34)$, we see that $(m)$, $(X_j)$, $(A_{j,h})$, $h = 0, 1, 2, 3$, and $(A_{j,s})$ are all $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted.

Fix $j \neq k \in \{1, 2\}$, and any $(\mathcal{F}_t^k)$-stopping time $t_k \in [0, p)$. Let $\mathcal{F}^{j,t_k}_t = \mathcal{F}_t^j \times \mathcal{F}_t^k$, $0 \leq t_j < p$. Then $(\mathcal{F}_t^{j,t_k})_{0 \leq t_j < p}$ is a filtration. Since $(B_j(t_j))$ is independent of $\mathcal{F}_t^k$, so it is also an $(\mathcal{F}_t^{j,t_k})$-Brownian motion. Thus, $(4.46)$ is an $(\mathcal{F}_t^{j,t_k})$-adapted SDE. From now on, we will apply Itô’s formula repeatedly, all SDE will be $(\mathcal{F}_t^{j,t_k})$-adapted, and $t_j$ ranges in $[0, T_j(t_k))$.

From $(4.21)$, $(4.31)$, $(4.33)$, and $(4.46)$, we see that $X_j$ satisfies

$$
\partial_t X_j = A_{j,1} \partial \xi_j(t_j) + \left(\frac{\kappa}{2} - 3\right) A_{j,2} \partial t_j + A_{j,1}^2 H^j(m, X_j) \partial t_j.
$$

Let $\Gamma$ be as in Lemma 4. Let $\Gamma_1 = \Gamma$ and $\Gamma_2(p, x) = \Gamma(p, -x)$. Since $\Gamma$ and $A_j$ satisfy $(4.3)$ and $(4.4)$, and $H^j(m, \cdot)$, is odd, so $\Gamma_j$ and $\Lambda_j$, $j = 1, 2$, also satisfy $(4.3)$ and $(4.4)$. From $(4.32)$, for $(t_1, t_2) \in D$, we may define

$$
Y(t_1, t_2) = \Gamma_1(m(t_1, t_2), X_1(t_1, t_2)) = \Gamma_2(m(t_1, t_2), X_2(t_1, t_2)).
$$

From $(4.3)$, $(4.4)$, $(4.16)$, $(4.47)$, and $(4.48)$, we have

$$
\frac{\partial Y}{Y} = \frac{1}{\kappa} A_j(m, X_j) A_{j,1} \partial \xi_j(t_j) + \frac{\kappa - 6}{2\kappa} \left( A_{j,1}^2 H^j(m, X_j) + \Lambda_j(m, X_j) A_{j,2} \right) \partial t_j.
$$

From $(4.22)$ we have

$$
\frac{\partial_j A_{j,1}}{A_{j,1}} = A_{j,2} \cdot \partial \xi_j(t_j) + \left( \frac{1}{2} \cdot \left( \frac{A_{j,2}}{A_{j,1}} \right)^2 + \left( \frac{\kappa}{2} - \frac{4}{3} \right) \cdot \frac{A_{j,3}}{A_{j,1}} \right) \partial t_j
$$

$$
+ A_{j,1}^2 r(m) \partial t_j - r(p - t_j) \partial t_j.
$$

Let

$$
\alpha = \frac{6 - \kappa}{2\kappa}, \quad c = \frac{(8 - 3\kappa)(\kappa - 6)}{2\kappa}.
$$

Actually, $c$ is the central charge for SLE$_\kappa$. Then we compute

$$
\frac{\partial_j A_{j,1}^\alpha}{A_{j,1}} = \alpha \cdot \frac{A_{j,2}}{A_{j,1}} \cdot \partial \xi_j(t_j) + \frac{c}{6} A_{j,3} \partial t_j + \alpha A_{j,1}^2 r(m) \partial t_j - \alpha r(p - t_j) \partial t_j.
$$

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Recall the definition of $R$ in (2.14). Define $\hat{M}$ on $\mathcal{D}$ such that

$$\hat{M} = \hat{A}_{1,1} R^\alpha_1 F^{-c/\theta} \exp(\alpha R(m) - \alpha R(p-t_1) - \alpha R(p-t_2) + \alpha R(p)).$$

Then $\hat{M}$ is positive. From (4.16), (4.41), (4.45), (4.49), (4.51), and that $R'(t) = r(t) - r(\infty)$, we have

$$\frac{\partial_j \hat{M}}{M} = \frac{6 - \kappa}{2\kappa} \frac{A_{j,2}}{A_{j,1}} \cdot \partial_j \xi_j(t_j) + \frac{1}{\kappa} A_j(m, X_j) \Lambda_j \partial_j \xi_j(t_j).$$

(4.53)

When $t_k = 0$, $A_{j,1} = 1$, $A_{j,2} = 0$, $m = p - t_j$ and $X_j = \xi_j(t_j) - \tilde{g}_{I,j}(t_j, x_k)$, so the right-hand side of (4.53) becomes $\frac{1}{\kappa} A_j(p - t_j, \xi_j(t_j) - \tilde{g}_{I,j}(t_j, x_k)) \partial_j \xi_j(t_j)$. Define $M$ on $\mathcal{D}$ such that

$$M(t_1, t_2) = \frac{\hat{M}(t_1, t_2) \hat{M}(0, 0)}{\hat{M}(t_1, 0) \hat{M}(0, t_2)}.$$

(4.54)

Then $M$ is also positive, and $M(t_1, 0) = M(0, t_2) = 1$ for $t_1, t_2 \in [0, p)$. From (4.16) and (4.53) we have

$$\frac{\partial M}{M} = \left[ \left( \frac{3 - \kappa}{2} \frac{A_{j,2}}{A_{j,1}} + A_j(m, X_j) \Lambda_j - A_j(p - t_j, \xi_j(t_j) - \tilde{g}_{I,j}(t_j, x_k)) \right) \frac{\partial B_j(t_j)}{\sqrt{\kappa}}. \right.$$

(4.55)

So when $t_k \in [0, p)$ is a fixed $(\mathcal{F}_t^k)$-stopping time, $M$ is a local martingale in $t_j$.

Let $\mathcal{J}$ denote the set of Jordan curves in $\Lambda_p$ that separates $\mathcal{T}$ and $\mathcal{T}_p$. For $J \in \mathcal{J}$ and $j = 1, 2$, let $T_j(J)$ denote the smallest $t$ such that $K_j(t) \cap J \neq \emptyset$. Recall that $I_p(z) = e^{-p}/z$ and $K_{I,j}(t) = I_p(K_j(t))$, so $T_j(J)$ is also the smallest $t$ such that $K_{I,j}(t) \cap I_p(J) \neq \emptyset$. Let $\mathcal{J}^2$ denote the set of pairs $(J_1, J_2) \in \mathcal{J}^2$ such that $I_p(J_1) \cap J_2 = \emptyset$ and $I_p(J_1)$ is surrounded by $J_2$. This is equivalent to that $I_p(J_2) \cap J_1 = \emptyset$ and $I_p(J_2)$ is surrounded by $J_1$. Then for every $(J_1, J_2) \in \mathcal{JP}$, $K_{I,1}(t_1) \cap K_{I,2}(t_2) = \emptyset$ when $t_1 \leq T_1(J_1)$ and $t_2 \leq T_2(J_2)$, so $[0, T_1(J_1)] \times [0, T_2(J_2)] \subset \mathcal{D}$.

**Lemma 4.2** (Boundedness) Fix $(J_1, J_2) \in \mathcal{JP}$. Then $|\ln(M)|$ is bounded on $[0, T_1(J_1)] \times [0, T_2(J_2)]$ by a constant depending only on $J_1$ and $J_2$.

**Proof.** In this proof, we say a function is uniformly bounded if its values on $[0, T_1(J_1)] \times [0, T_2(J_2)]$ are bounded by a constant depending only on $J_1$ and $J_2$. From (4.52) and (4.54) and that $p - t_1 = m(t_1, 0)$, $p - t_2 = m(0, t_2)$, and $p = m(0, 0)$, we suffice to show that $\ln(A_{1,1})$, $\ln(A_{2,1})$, $\ln(F)$, $\ln(Y)$, and $R(m)$ are all uniformly bounded. From (4.16) we have $m \leq p$. Let $D(J_1, J_2)$ denote the doubly connected domain bounded by $I_0(J_1)$ and $J_2$. Let $p_0 > 0$ denote its modulus. For $(t_1, t_2) \in [0, T_1(J_1)] \times [0, T_2(J_2)]$, $D(J_1, J_2)$ disconnects $K_{I,1}(t_1)$ from $K_{I,2}(t_2)$, so we have $m(t_1, t_2) \geq p_0$. Thus, $m \in [p_0, p]$ on $[0, T_1(J_1)] \times [0, T_2(J_2)]$. Since $R$ is continuous on $(0, \infty)$, so $R(m)$ is uniformly bounded. From (4.48), $\ln(Y) = \ln(\Gamma(m, X_1))$. Since $\ln(\Gamma)$ is continuous on $(0, \infty) \times \mathbb{R}$ and has period $2\pi$ in the second variable, so $\ln(Y)$ is uniformly bounded. Similarly, $Q$ is uniformly bounded. From Lemma 2.4, $\ln(A_{j,1}) = O(m e^{-m})$, $j = 1, 2$. So $\ln(A_{1,1})$ and $\ln(A_{2,1})$ are uniformly bounded. Finally, from (4.36), $\ln(Y)$ is uniformly bounded. \qed
4.3 Local coupling

Let $\mu_j$ denote the distribution of $(\xi_j)$, $j = 1, 2$. Let $\mu = \mu_1 \times \mu_2$. Then $\mu$ is the joint distribution of $(\xi_1)$ and $(\xi_2)$ since $\xi_1$ and $\xi_2$ are independent. Fix $(J_1, J_2) \in \mathcal{J} \mathcal{P}$. From the local martingale property of $M$ and Lemma \[4.12\] we have $\mathbf{E}_\mu[M(T_1(J_1), T_2(J_2))] = M(0, 0) = 1$.

Define $\nu_{J_1, J_2}$ such that $d\nu_{J_1, J_2}/d\mu = M(T_1(J_1), T_2(J_2))$. Then $\nu_{J_1, J_2}$ is a probability measure. Let $\nu_1$ and $\nu_2$ be the two marginal measures of $\nu_{J_1, J_2}$. Then $d\nu_1/d\mu_1 = M(T_1(J_1), 0) = 1$ and $d\nu_2/d\mu_2 = M(0, T_2(J_2)) = 1$, so $\nu_j = \mu_j$, $j = 1, 2$. Suppose temporarily that the joint distribution of $(\xi_1)$ and $(\xi_2)$ is $\nu_{J_1, J_2}$ instead of $\mu$. Then the distribution of each $(\xi_j)$ is still $\mu_j$.

Fix an $(\mathcal{F}_t^2)$-stopping time $t_2 \leq T_2(J_2)$. From \[4.46\], \[4.55\], and Girsanov theorem (c.f. \[14\]), under the probability measure $\nu_{J_1, J_2}$, there is an $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)_{t_2 \geq 0}$-Brownian motion $\tilde{B}_{t_1, t_2}(t_1)$ such that $\xi_1(t_1)$, $0 \leq t_1 \leq T_1(J_1)$, satisfies the $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)_{t_1 \geq 0}$-adapted SDE:

$$d\xi_1(t_1) = \sqrt{\kappa}d\tilde{B}_{t_1, t_2}(t_1) + \left(3 - \frac{\kappa}{2}\right)\frac{A_{1, 2}}{A_{1, 1}}dt_1 + \Lambda_1(m, X_1)A_{1, 1}dt_1. \quad (4.56)$$

Note that $X_1(t_1, t_2) = \xi_1, t_2(t_1) - \tilde{g}_{t_1, t_2}(t_1, \xi_2(t_2))$. From \[4.14\] and \[4.21\], we have

$$d\xi_1, t_2(t_1) = A_{1, 1}\sqrt{\kappa}d\tilde{B}_{t_1, t_2}(t_1) + A_{1, 1}^2\Lambda_1(\mu, \xi_1, t_2(t_1) - \tilde{g}_{t_1, t_2}(t_1, \xi_2(t_2)))dt_1.$$

Recall that $\zeta_1, t_2(s_1) = \xi_1, t_2(v_1^{-1}_1(s_1))$ and $\tilde{h}_{t_1, t_2}(s_1, \cdot) = \tilde{g}_{t_1, t_2}(v_1^{-1}_1(s_1), \cdot)$. So from \[4.6\] and \[4.15\], there is another Brownian motion $\tilde{B}_{t_1, t_2}(s_1)$ such that for $0 \leq s_1 \leq v_1, t_2(T_1(J_1)),$

$$d\zeta_1, t_2(s_1) = \sqrt{\kappa}d\tilde{B}_{t_1, t_2}(s_1) + \Lambda_1(p - t_2 - s_1, \xi_1, t_2(s_1) - \tilde{h}_{t_1, t_2}(s_1, \xi_2(t_2)))ds_1. \quad (4.57)$$

Moreover, the initial values is $\zeta_1, t_2(0) = \xi_1, t_2(0) = \tilde{G}_{t_2}(0, x_1) = \tilde{g}_{t_2}(t_2, x_1)$. Since $L_{t_1, t_2}(t)$ and $\tilde{h}_{t_1, t_2}(t)$ are inverted annulus Loewner hulls and inverted covering annulus Loewner maps, respectively, of modulus $p - t_2$ driven by $\zeta_1, t_2(t)$, so from \[4.57\], conditioned on $\mathcal{F}_{t_2}^2$, $L_{t_1, t_2}(t)$, $0 \leq t \leq v_1, t_2(T_1(J_1))$, is a stopped annulus SLE$(\kappa, \Lambda_1)$ process in $\mathcal{A}_{p - t_2}$ started from $e^{i}(\tilde{g}_{t_2}(t_2, x_1)) = \tilde{g}_{t_2}(t_2, x_1)$ with marked point $I_{p - t_2} \circ e^{i}(\xi_2(t_2))$. Let $\beta_2$ be the trace that corresponds to $K_2(t)$, and $\beta_{I, 2} = I_{\rho} \circ \beta_2$. Then $g_{t_2}(t_2, \cdot)$ maps $\mathcal{A}_p \setminus K_{I, 2}(t_2)$ conformally onto $\mathcal{A}_{p - t_2}$ and maps $\beta_{I, 2}(t_2)$ to $I_{p - t_2} \circ e^{i}(\xi_2(t_2))$. Since $L_{t_1, t_2}(t_2, x_1) = K_{t_2}(t_1) = g_{t_2}(t_2, K_1(t_1))$, so conditioned on $\mathcal{F}_{t_2}^2$, after a time-change, $K_1(t_1)$, $0 \leq t_1 \leq T_1(J_1)$, is a stopped annulus SLE$(\kappa, \Lambda_1)$ process in $\mathcal{A}_p \setminus K_{I, 2}(t_2)$ started from $a_1$ with marked point $\beta_{I, 2}(t_2)$. Similarly, if $t_1$ is a fixed $(\mathcal{F}_{t_1}^1)$-stopping time with $t_1 \leq T_1(J_1)$, and $\beta_1$ is the trace that corresponds to $K_1(t_1)$, then conditioned on $\mathcal{F}_{t_1}^1$, after a time-change, $K_2(t_2)$, $0 \leq t_2 \leq T_2(J_2)$, is a stopped annulus SLE$(\kappa, \Lambda_2)$ process in $\mathcal{A}_p \setminus K_{I, 1}(t_1)$ started from $a_2$ with marked point $\beta_{I, 1}(t_1) := I_{p} \circ \beta_1(t_1)$.

4.4 Global coupling

To lift the local couplings to a global coupling, we need the following theorem.

**Theorem 4.2** Suppose $n \in \mathbb{N}$ and $(J_1^m, J_2^m) \in \mathcal{J} \mathcal{P}$, $1 \leq m \leq n$. There is a continuous function $M_*(t_1, t_2)$ defined on $[0, p]^2$ that satisfies the following properties.

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(i) \( M_\ast = M \) on \([0, T_1(J_1^m)] \times [0, T_2(J_2^m)]\) for \(1 \leq m \leq n\);

(ii) \( M_\ast(t, 0) = M_\ast(0, t) = 1\) for any \(t \in [0, p]\);

(iii) \( M(t_1, t_2) \in [C_1, C_2]\) for any \(t_1, t_2 \in [0, p]\), where \(C_2 > C_1 > 0\) are constants depending only on \(J_j^m\), \(j = 1, 2, 1 \leq m \leq n\);

(iv) fix \(j \neq k \in \{1, 2\}\) and any \((\mathcal{F}_k^j)\)-stopping time \(t_k \in [0, p]\), \(M_\ast\) is a bounded \((\mathcal{F}_j^1 \times \mathcal{F}_k^2)_{t \geq 0}\)-martingale in \(t_j\).

This theorem is similar to Theorem 6.1 in [25] and Theorem 4.5 in [26]. Their proofs are also similar. To save the length of the paper, we omit the most part of the proof, and only show how \(M_\ast\) is defined. Let \(S\) be a subset of \(\{m \in \mathbb{N} : m \leq n\}\) such that

\[
\bigcup_{m \in S} [0, T_1(J_1^m)] \times [0, T_2(J_2^m)] = \bigcup_{m=1}^{n} [0, T_1(J_1^m)] \times [0, T_2(J_2^m)]
\]

Moreover, we may assume that if another set \(S'\) also satisfies this property, then \(\sum_{m \in S^\prime} m < \sum_{m \in S} m\). Then \(S\) is uniquely determined. But it is a random set in general.

We may order the members in \(S\) by \(m_1, m_2, \ldots, m_k\) such that \(T_1(J_1^{m_1}) < T_1(J_1^{m_2}) < \cdots < T_1(J_1^{m_k})\), and \(T_2(J_2^{m_1}) > T_2(J_2^{m_2}) > \cdots > T_2(J_2^{m_k})\). Formally define \(T_1(J_1^{m_0}) = T_2(J_2^{m_0}) = 0\) and \(T_1(J_1^{m_k+1}) = T_2(J_2^{m_k}) = p\). Now the vertical lines \(\{x = T_1(J_1^{m_j})\}, 1 \leq j \leq k\), and horizontal lines \(\{y = T_2(J_2^{m_j})\}, 1 \leq j \leq k\), divide the square \([0, p]^2\) into \((k + 1)^2\) rectangles. We use \(R_{j_1,j_2}\) to denote the closed rectangle bounded by \(\{x = T_1(J_1^{m_{j_1}})\}, \{x = T_1(J_1^{m_{j_1-1}})\}, \{y = T_2(J_2^{m_{j_2}})\}, \{y = T_2(J_2^{m_{j_2-1}})\}\), \(1 \leq j_1, j_2 \leq k + 1\). Then

\[
\bigcup_{m=1}^{n} [0, T_1(J_1^m)] \times [0, T_2(J_2^m)] = \bigcup_{m \in S} [0, T_1(J_1^m)] \times [0, T_2(J_2^m)] = \bigcup_{j_1 < j_2} R_{j_1,j_2}
\]

We first define \(M_\ast\) on \(R_{j_1,j_2}\) for \(j_1 < j_2\) such that \(M_\ast = M\); and define it to be constant 1 on \(\{t_1 = 0\}\) and \(\{t_2 = 0\}\). Then we extend \(M_\ast\) to \([0, p]^2\) such that it is continuous, and its restriction to each \(R_{j_1,j_2}\) with \(j_1 \geq j_2\) is a product of two functions depending only on \(t_1\) and \(t_2\), respectively. Such extension can be done step by step, and is unique. Then \(M_\ast\) clearly satisfies (i) and (ii). Lemma 4.2 will be used to prove (iii). The local martingale property of \(M\) and the boundedness of \(M_\ast\) together yield (iv), the martingale property of \(M_\ast\).

Let \(\mathcal{J}_\ast\) be the set of \((J_1, J_2) \in \mathcal{J}\) such that for \(j = 1, 2\), \(J_j\) is a polygon curve whose vertices have rational coordinates. Then \(\mathcal{J}_\ast\) is countable. Let \((J_1^m, J_2^m), m \in \mathbb{N},\) be an enumeration of \(\mathcal{J}_\ast\). For each \(n \in \mathbb{N},\) let \(M_n^*(t_1, t_2)\) be the \(M_\ast(t_1, t_2)\) given by Theorem 4.2 for \((J_1^m, J_2^m), 1 \leq m \leq n),\) in the above enumeration.

For \(j = 1, 2\), let \(\xi_j(t), K_j(t), K_{1,j}(t), \beta_j(t),\) and \(\beta_{1,j}(t), 0 \leq t < p,\) be as in the last subsection. Recall that \(K_j(t), 0 \leq t < p,\) is an annulus \(\text{SLE}(\kappa, \Lambda_j)\) process in \(\Lambda_p\) started from \(a_j\) with marked point \(a_{1,3-j}\). For \(j = 1, 2\), let \(\mu_j\) denote the distribution of \(\xi_j(t), 0 \leq t < p.\)
We first suppose that $\xi_1$ and $\xi_2$ are independent, then $\mu := \mu_1 \times \mu_2$ is the joint distribution of $\xi_1$ and $\xi_2$, which is a probability measure on $C(\{0, p\})^2$. For each $n \in \mathbb{N}$, define $\nu^n$ such that $db^n = M^n_\nu(p, p)d\mu$. Then $\nu^n$ is also a probability measure on $C(\{0, p\})^2$. Let $\nu^n_1$ and $\nu^n_2$ be the marginal measure of $\nu^n$. Since $M^n_\nu(t_1, t_2) = 1$ when $t_1 = 0$ or $t_2 = 0$, so $\nu^n_2 = \nu_j$, $j = 1, 2$. If the joint distribution of $\xi_1$ and $\xi_2$ is $\nu^n$ instead of $\mu$, then the distributions of $\xi_1$ and $\xi_2$ are still $\mu_1$ and $\mu_2$, respectively; and from the discussion at the end of last subsection, we see that for any $1 \leq m \leq n$ and $j \neq k \in \{1, 2\}$, if $t_k$ is an ($F^k_t$)-stopping time with $t_k \leq T_k(J^m_k)$, then conditioned on $F^k_{t_k}$, $K_j(t)$, $0 \leq t \leq T_j(J^m_k)$, is a stopped annulus $\text{SLE}(\kappa, \Lambda_j)$ process in $\mathcal{A}_p \setminus K_{1, k}(t_k)$ started from $a_j$ with marked point $\beta_{1, k}(t_k)$. Now we are ready to finish the proof of Theorem 4.1.

A sketch of the proof of Theorem 4.1. Let $\mathcal{H}(\hat{C})$ denote the space of nonempty compact subsets of $\hat{C}$ (with spherical metric) endowed with the Hausdorff metric. We view $C(\{0, p\})$ as a subspace of $\mathcal{H}(\hat{C})$ by identifying each $\xi \in C(\{0, p\})$ with $G(\xi) \in \mathcal{H}(\hat{C})$, where $G(\xi)$ is the closure of $\{x + i\xi(x) : 0 \leq x < p\}$ in $\hat{C}$. Then $\text{Pr}(C(\{0, p\})^2)$ becomes a subspace of $\text{Pr}(\mathcal{H}(\hat{C})^2)$. Let $\nu \in \text{Pr}(\mathcal{H}(\hat{C})^2)$ be a subsequential limit of $(\nu^n)_{n \in \mathbb{N}}$. Let $\nu_1$ and $\nu_2$ be the marginals of $\nu$. Since for each $n \in \mathbb{N}$, $\nu^n_j = \mu_j$ for $j = 1, 2$, so $\nu_1 = \mu_1$ and $\nu_2 = \mu_2$. Since $\mu_1, \mu_2 \in \text{Pr}(C(\{0, p\}))$, so $\nu \in \text{Pr}(C(\{0, p\})^2))$. Now suppose the joint distribution of $\xi_1$ and $\xi_2$ is $\nu$ instead of $\mu$. Let $K_j(t)$, $\beta_j(t)$, $K_{1, j}(t)$ and $\beta_{1, j}(t)$ be as before. Since $\nu_1 = \mu_1$ and $\nu_2 = \mu_2$, so for $j = 1, 2$, $K_j(t)$, $0 \leq t < p$, is still an annulus $\text{SLE}(\kappa, \Lambda_j)$ process in $\mathcal{A}_p$ started from $a_j$ with marked point $a_{1, 3-j}$. So we have (i). From Theorem 4.2 (i) it is easy to see that for any $(J_1, J_2) \in \mathcal{J}_p$, the joint distribution of $(K_1(t) : 0 \leq t \leq T_1(J_1))$ and $(K_{1, 2}(t) : 0 \leq t \leq T_2(J_2))$ is absolutely continuous w.r.t. the product measure of these two distributions, and the Radon-Nikodym derivative is $M(T_1(J_1), T_2(J_2))$. Moveover, it is easy to check that, for $j \neq k \in \{1, 2\}$, if $t_k < p$ is an ($F^j_t$)-stopping time, then $T_j(t_k) = \sup_{m \in \mathbb{N}}\{T_j(J^m_k) : t_k \leq T_k(J^m_k)\}$. From the discussion before this proof, one can conclude that (ii) also holds. The reader may read Section 4.3 in [26] for the details of this argument. \hfill \Box

Remark. The condition (4.2) in Theorem 4.1 is used to guarantee that $\Gamma$ is uniformly bounded when the first variable is bounded away from 0 and $\infty$. If we do not assume this condition, then the statement of the theorem should be modified. For $j = 1, 2$, let $S_j$ be the biggest number in $(0, p]$ such that $T \not\subset K_j(t)$ for $0 \leq t < S_j$. In Theorem 4.1, the condition $t_k < p$ should be replaced by $t_k < S_k$; the range of $K_j(t)$ should be $0 \leq t < T_j(t_k) \land S_j$ instead of $0 \leq t < T_j(t_k)$; and others can be kept unchanged. In the case $\kappa \leq 4$, we have a.s. $S_J = p$, $j = 1, 2$, which implies that the theorem holds without modification if we do not assume (4.2).

Using the idea in the proof of Theorem 4.1, we can also prove the following theorem.

Theorem 4.3 Let $\kappa > 0$. Suppose $\Lambda$ is a $C^{1, 2}$ differentiable chordal-type annulus drift function, and satisfies the following PDE:

$$
\dot{\lambda} = \frac{\kappa}{2} \lambda'' + (3 - \frac{\kappa}{2}) \lambda' + \lambda \Phi' + \Phi \lambda' + \lambda \Phi'
$$

(4.58)
on \((0, \infty) \times (\mathbb{R} \setminus \{2n\pi : n \in \mathbb{N}\})\). Let \(\Lambda_1 = \Lambda\), and \(\Lambda_2\) be the dual function of \(\Lambda\). Then for any \(p > 0\) and \(a_1 \neq a_2 \in \mathbb{T}\), there is a coupling of two processes \(K_1(t), 0 \leq t < T_1\), and \(K_2(t), 0 \leq t < T_2\), such that for \(j \neq k \in \{1, 2\}\) the followings hold.

(i) \(K_j(t), 0 \leq t < T_j\), is an annulus \(\text{SLE}(\kappa, \Lambda_j)\) process in \(\mathbb{H}_p\) started from \(a_j\) with marked point \(a_k\).

(ii) If \(t_k \in [0, T_k)\) is a stopping time w.r.t. \((K_k(t))\), then conditioned on \(K_k(t), 0 \leq t \leq t_k\), after a time-change, \(K_j(t), 0 \leq t < T_j(t_k)\), is a stopped annulus \(\text{SLE}(\kappa, \Lambda_j)\) process in \(\mathbb{H}_p \setminus K_k(t_k)\) started from \(a_j\) with marked point \(\beta_k(t_k)\), where \(\beta_k\) is the trace that corresponds to \(K_k(t), 0 \leq t < T_k\), and \(T_j(t_k)\) is the maximal number in \([0, T_j]\) such that \(K_j(t) \cap K_k(t_k) = \emptyset\) for \(0 \leq t < T_j(t_k)\).

### 4.5 The limit cases

In some sense, Theorem 4.1 and Theorem 4.3 still hold in the limit case, i.e., when \(p = \infty\). In the next section, we will prove a theorem about constructing a coupling of two whole-plane \(\text{SLE}\) processes, which can be viewed as the limit case of Theorem 4.1. In this subsection, we will state without proof about the limit case of Theorem 4.1, which is Theorem 4.4 below about constructing a coupling of two radial \(\text{SLE}(\kappa, \Lambda)\) processes.

Suppose \(\Lambda\) is a \(C^1\) function on \(\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}\) with period \(2\pi\). Let \(\kappa > 0\) and \(B(t)\) be a Brownian motion. Let \(a \neq b \in \mathbb{T}\). Choose \(x_0, y_0 \in \mathbb{R}\) such that \(e^{ix_0} = a\) and \(e^{iy_0} = b\). Suppose \(\xi(t), 0 \leq t < T\), is the maximal solution that solves the SDE:

\[
d\xi(t) = \sqrt{\kappa}dB(t) + \Lambda(\xi(t) - \tilde{g}^{\xi}(t, y_0))dt, \quad \xi(0) = x_0,
\]

where \(\tilde{g}^{\xi}(t, \cdot)\) are covering radial Loewner maps driven by \(\xi\). Then we call the radial Loewner hulls driven by \(\xi\) the radial \(\text{SLE}(\kappa, \Lambda)\) process in \(\mathbb{D}\) started from \(a\) with marked points \(0\) and \(b\). Here \(0\) is a marked point because it is a special point in the radial Loewner equation. Via conformal maps, we can define radial \(\text{SLE}(\kappa, \Lambda)\) process in any simply connected domain \(D\) started from one boundary point with a pair of marked points: one is an interior point, the other is another boundary point. We have the following theorem.

**Theorem 4.4** Let \(\kappa > 0\). Suppose \(\Lambda\) satisfies

\[
0 = \frac{\kappa}{2} \Lambda'' + \left(3 - \frac{\kappa}{2}\right) \cot_2' + \Lambda' + \Lambda' \tag{4.59}
\]

on \((\mathbb{R} \setminus \{2n\pi : n \in \mathbb{N}\})\). Let \(\Lambda_1 = \Lambda\), and \(\Lambda_2(x) = -\Lambda(-x)\). Then for any \(a_1 \neq a_2 \in \mathbb{T}\), there is a coupling of two processes \(K_1(t), 0 \leq t < T_1\), and \(K_2(t), 0 \leq t < T_2\), such that for \(j \neq k \in \{1, 2\}\) the followings hold.

(i) \(K_j(t), 0 \leq t < T_j\), is a radial \(\text{SLE}(\kappa, \Lambda_j)\) process in \(\mathbb{D}\) started from \(a_j\) with marked points \(0\) and \(a_k\).
(ii) If \( t_k \in [0, T_k) \) is a stopping time w.r.t. \((K_k(t)), \) then conditioned on \( K_k(t), 0 \leq t \leq t_k, \) after a time-change, \( K_j(t), 0 \leq t < T_j(t_k), \) is a stopped radial SLE\( \kappa, \Lambda_j \) process in \( \mathbb{D} \setminus K_k(t_k) \) started from \( a_j \) with marked points 0 and \( \beta_k(t_k), \) where \( \beta_k \) is the trace that corresponds to \( K_k(t), 0 \leq t < T_k, \) and \( T_j(t_k) \) is the maximal number in \((0, T_j)\) such that \( K_j(t) \cap K_k(t_k) = \emptyset \) for \( 0 \leq t < T_j(t_k). \)

The covering radial Loewner equation is very similar to the strip Loewner equation (c.f. [2][21]). Let \( \xi \in C([0, T)) \) for some \( T \in (0, \infty]. \) The strip Loewner equation driven by \( \xi \) is

\[
g(t, z) = \coth_2(g(t, z) - \xi(t)), \quad g(0, z) = z,
\]

where \( \coth_2(z) = \coth(z/2) \). For \( 0 \leq t < T \), let \( K(t) \) be the set of \( z \in \mathbb{S}_\pi = \{z \in \mathbb{C} : \text{Im} \, z \in (0, \pi)\} \) such that solution \( g(s, z) \) blows up before or at time \( t. \) Then \( g(t, \cdot) \) maps \( \mathbb{S}_\pi \setminus K(t) \) conformally onto \( \mathbb{S}_\pi \), and fixes \( +\infty \) and \( -\infty \). We call \( K(t) \) and \( g(t, \cdot), 0 \leq t < p, \) the strip Loewner hulls and maps driven by \( \xi. \)

Let \( \kappa \geq 0 \) and \( B(t) \) be a Brownian motion. The standard strip SLE\( \kappa \) is defined by choosing \( \xi(t) = \sqrt{\kappa}B(t), 0 \leq t < \infty. \) The corresponding trace is defined by

\[
\beta(t) = \lim_{\mathbb{S}_\pi \ni z \to \xi(t)} g(t, \cdot)^{-1}(\xi(t)), \quad 0 \leq t < \infty.
\]

Such \( \beta \) is a continuous curve in \( \overline{\mathbb{S}_\pi} \) started from 0. The behavior of \( \beta \) depends on \( \kappa \) in the same way as radial and annulus SLE\( \kappa \) does.

Suppose \( \Lambda \) is a \( C^1 \) function on \((0, \infty). \) Let \( a > b \in \mathbb{R}. \) Suppose \( \xi(t), 0 \leq t < T, \) is the maximal solution of the SDE:

\[
d\xi(t) = \sqrt{\kappa}dB(t) + \Lambda(\xi(t) - g^\xi(t, b))dt, \quad \xi(0) = a,
\]

where \( g^\xi(t, \cdot) \) are strip Loewner maps driven by \( \xi. \) Then we call the strip Loewner hulls driven by \( \xi \) the strip SLE\( \kappa, \Lambda \) process in \( \mathbb{S}_\pi \) started from \( a \) with three marked points: \( b, +\infty \) and \(-\infty. \) If \( \Lambda \) is \( C^1 \) on \((-\infty, 0)\) and \( a < b \in \mathbb{R}, \) using the same equation we may define strip SLE\( \kappa, \Lambda \) process in \( \mathbb{S}_\pi \) started from \( a \) with marked points \( b, +\infty \) and \(-\infty. \) Via conformal maps, we can define strip SLE\( \kappa, \Lambda \) process in any simply connected domain \( D \) started from one boundary point with a triple of marked boundary points. The following theorem is similar to Theorem 4.4 and in some sense can be viewed as the limit of Theorem 4.3 when \( p \to 0. \)

**Theorem 4.5** Let \( \kappa > 0. \) Suppose \( \Lambda(x), 0 < x < \infty, \) satisfies

\[
0 = \frac{\kappa}{2} \Lambda'' + \left(3 - \frac{\kappa}{2}\right) \coth'' + \Lambda \coth' + \coth \Lambda' + \Lambda \Lambda'.
\]

Let \( \Lambda_1 = \Lambda, \) and \( \Lambda_2(x) = -\Lambda(-x). \) Then for any \( a_1 > a_2 \in \mathbb{R}, \) there is a coupling of two processes \( K_1(t), 0 \leq t < T_1, \) and \( K_2(t), 0 \leq t < T_2, \) such that for \( j \neq k \in \{1, 2\} \) the followings hold.
(i) $K_j(t)$, $0 \leq t < T_j$, is a strip SLE($\kappa, \Lambda_j$) process in $\mathbb{S}_\kappa$ started from $a_j$ with marked points $a_k$, $+\infty$ and $-\infty$.

(ii) If $t_k \in [0, T_k)$ is a stopping time w.r.t. $(K_k(t))$, then conditioned on $K_k(t)$, $0 \leq t \leq t_k$, after a time-change, $K_j(t)$, $0 \leq t < T_j(t_k)$, is a stopped strip SLE($\kappa, \Lambda_j$) process in $\mathbb{S}_\kappa \setminus K_k(t_k)$ started from $a_j$ with marked points $\beta_k(t_k)$, $+\infty$ and $-\infty$, where $\beta_k$ is the trace that corresponds to $K_k(t)$, $0 \leq t < T_k$, and $T_j(t_k)$ is the maximal number in $(0, T_j]$ such that $K_j(t) \cap K_k(t_k) = \emptyset$ for $0 \leq t < T_j(t_k)$.

The general solutions to (4.59) and (4.60) can be expressed in terms of hypergeometric function. Some particular solutions can be expressed in terms of trigonometric or hyperbolic functions. For example, $\Lambda_1(x) = (\kappa/2 - 3) \cot_2(x)$ and $\Lambda_1(x) = \cot_2(x)$ solve (4.59). The SLE($\kappa, \Lambda_1$) process is actually the usual chordal SLE$_\kappa$ process aimed at the marked boundary point, so the interior marked point does not play a role. The radial SLE($\kappa, \Lambda_2$) process is the chordal SLE$_\kappa$ process aimed at the marked boundary point conditioned to pass through the marked interior point (c.f. [3]). Two other particular solutions to (4.59) are $\Lambda_{3,4}(x) = (\kappa/4 - 1) \cot_2(x) \pm (\kappa/4 - 2) \csc(x/2)$. Similarly, $\Lambda_2(x) = (\kappa/2 - 3) \coth_2(x)$ and $\Lambda_2(x) = \coth_2(x)$ solve (4.60). The strip SLE($\kappa, \Lambda_5$) process is actually the usual chordal SLE$_\kappa$ process aimed at the marked boundary point other than $\pm \infty$. Two other particular solutions to (4.60) are $\Lambda_{7,8}(x) = (\kappa/4 - 1) \coth_2(x) \pm (\kappa/4 - 2) \csch(x/2)$.

4.6 The deterministic cases

Theorem 4.1, Theorem 4.3, Theorem 4.4, and Theorem 4.5 all hold when $\kappa = 0$. In that case, the processes are deterministic, we can not apply Girsanov Theorem, and the function $\Gamma$ in Lemma 4 does not exist. But the proofs turns out to be simpler. Now we consider Theorem 4.1 when $\kappa = 0$, for example. All formulas in Section 4.1, and (4.46) and (4.47) still hold.

For $\{j, k\} = \{1, 2\}$, define $U_k$ and $V_k$ on $\mathcal{D}$ such that

$$U_k = A_{k,1} \Lambda_k(m, x_k) + 3 A_{k,2} \Lambda_{k,1}, \quad V_k = \partial U_k / \partial t_j. \quad (4.61)$$

Fix $j \neq k \in \{1, 2\}$. If $t_j = 0$, then $A_{k,1} = 1$, $A_{k,2} = 0$, $m = p - t_k$, and $X_k = \xi_k(t_k) - \tilde{g}_{t_k}(t_k, x_j)$. From (4.46), $\xi_k$ satisfies $d\xi_k(t_k) = U_k|_{t_j=0} dt_k$. Similarly, $\xi_j$ satisfies $d\xi_j(t_j) = U_j|_{t_k=0} dt_j$. From (4.32), (4.47), and that $H_f(p, \cdot)$ is odd, $X_j$ and $X_k$ satisfies

$$\frac{\partial}{\partial t_j} X_k = - \frac{\partial}{\partial t_j} X_j = - A_{j,1} [U_j|_{t_k=0}] + 3 A_{j,2} + A_{j,1}^2 H_f(m, x_k). \quad (4.62)$$

From (4.16), (4.32), (4.41), (4.42), (4.61), (4.62), and that $H_f(p, \cdot)$ is even, $H_f'(p, \cdot)$ is odd, we have

$$V_k = A_{k,1} A_{j,1}^2 H_f'(m, x_k) \Lambda_k(m, x_k) - A_{k,1} A_{j,1}^2 \Lambda_k(m, x_k) + 3 A_{k,1} A_{j,1}^2 H_f'(m, x_k)$$
Theorem 5.1

Fix $-\infty < t < \lim_{\beta \to 0} \beta(t_k)$ by Lemma 4. Moreover, suppose that $\Lambda_2$.

In this section, we will prove the following Theorem. Recall that $\zeta$, $\xi$ from (4.40), (4.31), (4.33), and the above formula, we see that

\[ V_k = A_{k,1}A_{j,1}\Lambda_k(m, X_k)\{U_j - U_{j | t_k = 0}\}, \]

which implies that

\begin{align}
V_1(t_1, t_2) &= R(t_1, t_2) \int_{t_0}^{t_1} V_2(s_1, t_2)ds_1, \\
V_2(t_1, t_2) &= R(t_1, t_2) \int_{t_0}^{t_2} V_1(t_1, s_2)ds_2,
\end{align}

where $R := A_{1,1}A_{2,1}\Lambda_1'(m, X_1) = A_{1,1}A_{2,1}\Lambda_2'(m, X_2)$.

The only solution to (4.63) and (4.64) is $V_1 \equiv V_2 \equiv 0$. So we have $U_j = U_{j | t_k = 0}$. From (4.62) and that $H_I(p, \cdot)$ is odd, we have

\[ \frac{\partial}{\partial t_j} X_j = A_{j,1}U_j - 3A_{j,2} + A_{j,1}^2 H_I(m, X_j) = A_{j,1}^2 \Lambda_j(m, X_j) + A_{j,1}^2 H_I(m, X_j). \]

From (4.40), (4.31), (4.33), and the above formula, we see that

\[ d\xi_{j,t_k}(t_j) = A_{j,1}^2 \Lambda_j(m, X_j)dt_j. \]

Since $\xi_{j,t_k}(t_j) = \zeta_{j,t_k}(v_{j,t_k}(t_j))$ and $\tilde{g}_{I,j,t_k}(t_j, \cdot) = \tilde{h}_{I,j,t_k}(v_{j,t_k}(t_j), \cdot)$, so from (4.60), (4.15), (4.31), and the above formula, $\zeta_{j,t_k}(t)$ satisfies

\[ d\zeta_{j,t_k}(t) = \Lambda_j(p - t_k - t, \zeta_{j,t_k}(t) - \tilde{h}_{I,j,t_k}(t, \xi_{k}(t_k)))dt. \]

When $j = 1$ and $k = 2$, this equation agrees with (4.57) when $\kappa = 0$. Arguing as in Section 4.3, we can complete the proof of Theorem 4.1 in the case $\kappa = 0$.

5 Coupling in the Degenerated Case

In this section, we will prove the following Theorem. Recall that $I_0(z) = 1/\zeta$.

Theorem 5.1 Fix $\kappa > 0$. Suppose $\Lambda$ satisfies the conditions in Theorem 4.1. Let $\Lambda$ be given by Lemma 4. Moreover, suppose that $\lim_{p \to \infty} \Gamma(p, x) = 1$ uniformly in $x \in \mathbb{R}$. Let $\Lambda_1 = \Lambda$ and $\Lambda_2$ be the dual function of $\Lambda$. Then there is a coupling of two processes $K_{I,1}(t)$ and $K_{I,2}(t)$, $-\infty < t < \infty$, such that for $j \neq k \in \{1, 2\}$, the followings hold.

(i) $K_{I,j}(t)$, $-\infty < t < \infty$, are whole-plane SLE$_{\kappa}$ hulls in $\mathbb{C}$ from $0$ to $\infty$;

(ii) Let $t_k$ be a finite stopping time w.r.t. the filtration generated by $(K_{I,k}(t))$. Then conditioned on $K_{I,k}(s)$, $-\infty < s \leq t_k$, the process $K_{I,j}(t_j)$, $-\infty < t_j < T_j(t_k)$, has the distribution of a disc SLE($\kappa, \Lambda_j$) process in $\mathbb{C} \setminus I_0(K_{I,1}(t_k))$ started from 0 with marked point $\beta_k(t_k)$, where $T_j(t_k)$ is the maximal number in $\mathbb{R}$ such that $K_j(t) \cap I_0(K_{I,k}(t_k)) = 0$ for $-\infty < t < T_j(t_k)$, $\beta_k(t_k) = I_0(\beta_{I,k}(t_k))$, and $\beta_{I,k}(t)$ is the trace that corresponds to $(K_{I,k}(t))$. 

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5.1 Ensemble

The argument in this subsection is parallel to that in Section 4.1. Let \( \xi_1, \xi_2 \in C(\mathbb{R}) \). For \( j = 1, 2 \), let \( K_{I,j}(t) \) and \( g_{I,j}(t, \cdot) \) (resp. \( K_j(t) \) and \( g_j(t, \cdot) \)) be the whole-plane (resp. inverted whole-plane) Loewner hulls and maps driven by \( \xi_j(t), t \in \mathbb{R} \). Let \( \tilde{K}_{I,j}(t), \tilde{g}_{I,j}(t, \cdot), \tilde{K}_j(t), \) and \( \tilde{g}_j(t, \cdot) \) be the corresponding covering hulls, and maps. So we have \( K_j(t) = I_0(K_{I,j}(t)) \), \( g_j(t, \cdot) = I_0 \circ g_{I,j}(t, \cdot) \circ I_0 \) and \( \tilde{g}_j(t, \cdot) = \tilde{I}_0 \circ \tilde{g}_{I,j}(t, \cdot) \circ \tilde{I}_0 \). Define

\[
\mathcal{D} = \{(t_1, t_2) \in \mathbb{R}^2 : K_{I,1}(t_1) \cap K_{I,2}(t_2) = \emptyset\} = \{(t_1, t_2) \in \mathbb{R}^2 : K_1(t_1) \cap K_2(t_2) = \emptyset\}.
\]

For \( (t_1, t_2) \in \mathcal{D} \), \( \mathcal{C} \setminus K_{I,1}(t_1) \setminus K_{I,2}(t_2) \) and \( \mathcal{C} \setminus K_{I,1}(t_1) \setminus K_{I,2}(t_2) \) are doubly connected domains with the same modulus, so we may define

\[
m(t_1, t_2) = \text{mod}(\mathbb{C} \setminus K_{I,1}(t_1) \setminus K_{I,2}(t_2)) = \text{mod}(\mathbb{C} \setminus K_{I,1}(t_1) \setminus K_{I,2}(t_2)).
\] (5.1)

Fix any \( j \neq k \in \{1, 2\} \) and \( t_k \in \mathbb{R} \). Let \( T_j(t_k) = \sup\{t_j : K_{I,j}(t_j) \cap K_j(t_k) = \emptyset\} \). Then for any \( t_j < T_j(t_k) \), we have \( (t_1, t_2) \in \mathcal{D} \). Moreover, as \( t_j \to T_j(t_k)^- \), the spherical distance between \( K_{I,j}(t_j) \) and \( K_j(t_k) \) tends to 0, so \( m(t_1, t_2) \to 0 \).

From Proposition 2.3 \( K_{I,j}(t_j) \), \( -\infty < t_j < \infty \), is an interior Loewner chain in \( \mathcal{C} \) started from 0. Since for \( -\infty < t_j < T_j(t_k) \), \( K_{I,j}(t_j) \) lies in \( \mathcal{C} \setminus K_j(t_k) \), so \( K_{I,j}(t_j), -\infty < t_j < T_j(t_k) \), is also an interior Loewner chain in \( \mathcal{C} \setminus K_j(t_k) \). Let \( K_{I,j}(t_j) = g_j(t_k, K_{I,j}(t_j)) \). Recall that \( g_j(t_k, \cdot) \) maps \( \mathcal{C} \setminus K_j(t_k) \) conformally onto \( \mathbb{D} \), and fixes 0, so \( K_{I,j}(t_j), -\infty < t_j < T_j(t_k) \), is an interior Loewner chain in \( \mathbb{D} \) started from 0. Now we apply Proposition 2.3. For \( -\infty < t_j < T_j(t_k) \), let

\[
v_{j,t_k}(t_j) := -\text{mod}(\mathbb{D} \setminus K_{I,j}(t_j)) = -\text{mod}(\mathcal{C} \setminus K_j(t_k) \setminus K_{I,j}(t_j)) = -m(t_1, t_2).
\] (5.2)

Here the second “=” holds because \( g_j(t_k, \cdot) \) maps \( \mathcal{C} \setminus K_j(t_k) \setminus K_{I,j}(t_j) \) conformally onto \( \mathbb{D} \setminus K_{I,j,t_k}(t_j) \). Then \( v_{j,t_k} \) is continuous and increasing, and maps \( (-\infty, T_j(t_k)) \) onto \( (-\infty, S) \) for some \( S \in (-\infty, 0) \). Since \( m \to 0 \) as \( t_j \to T_j(t_k)^- \), \( S = 0 \). Let \( L_{I,j,t_k}(t) = L_{I,j,t_k}(v_{j,t_k}^{-1}(t)) \), \( -\infty < t < 0 \). Then \( L_{I,j,t_k}(t), -\infty < t < 0 \), are disc Loewner hulls driven by some \( \zeta_{j,t_k} \in C'((-\infty, 0)) \). Let \( L_{j,t_k}(t) \) be the corresponding inverted disc Loewner hulls. Let \( h_{I,j,t_k}(t, t_1) \) and \( h_{j,t_k}(t, t_1) \) be the corresponding disc and inverted disc Loewner maps. Let \( \tilde{L}_{I,j,t_k}(t), \tilde{L}_{j,t_k}(t), \tilde{h}_{I,j,t_k}(t, t_1), \) and \( \tilde{h}_{j,t_k}(t, t_1) \) be the corresponding covering Loewner hulls and maps.

For \( -\infty < t_j < T_j(t_k) \), let \( \zeta_{j,t_k}(t_j), K_{j,t_k}(t_j), g_{j,t_k}(t_j, \cdot), \tilde{g}_{j,t_k}(t_j, \cdot), K_{I,j,t_k}(t_j), \tilde{K}_{I,j,t_k}(t_j), \tilde{g}_{I,j,t_k}(t_j, \cdot), \) and \( \tilde{g}_{I,j,t_k}(t_j, \cdot) \) be the time-change of \( \zeta_{j,t_k}(t), L_{j,t_k}(t), h_{j,t_k}(t, \cdot), \tilde{h}_{j,t_k}(t, \cdot), \tilde{L}_{I,j,t_k}(t), \tilde{L}_{j,t_k}(t), \tilde{h}_{I,j,t_k}(t, \cdot), \) and \( \tilde{h}_{j,t_k}(t, \cdot) \), respectively, via \( v_{j,t_k} \). From Lemma 2.1 for each fixed \( t_j \in \mathbb{R} \) and \( \varepsilon > 0 \), \( g_j(t_j, K_j(t_j + \varepsilon) \setminus K_j(t_j)) \) is a hull in \( \mathbb{D} \) w.r.t. 0, and we have

\[
\text{cap}_{\mathbb{D},0}(g_j(t_j, K_j(t_j + \varepsilon) \setminus K_j(t_j))) = \varepsilon;
\] (5.3)

\[
\{e^{g(t_j)}\} = \bigcap_{\varepsilon > 0} g_j(t_j, K_j(t_j + \varepsilon) \setminus K_j(t_j)), -\infty < t_j < \infty.
\] (5.4)
From Lemma 2.2 for each fixed \( t_j \in (-\infty, T_j(t_k)) \) and \( \varepsilon \in (0, T_j(t_k) - t_j) \),

\[
g_{j,t_k}(t_j, K_{j,t_k}(t_j + \varepsilon) \setminus K_{j,t_k}(t_j)) = h_{j,t_k}(v_{j,t_k}(t_j), L_{j,t_k}(v_{j,t_k}(t_j + \varepsilon)) \setminus L_{j,t_k}(v_{j,t_k}(t_j)))
\]

is a hull in \( A_{-v_{j,t_k}(t_j)} = A_m(t_1,t_2) \) w.r.t. \( T_m(t_1,t_2) \), and we have

\[
\text{cap}_{A_m}(g_{j,t_k}(t_j, K_{j,t_k}(t_j + \varepsilon) \setminus K_{j,t_k}(t_j))) = v_{j,t_k}(t_j + \varepsilon) - v_{j,t_k}(t_j);
\] (5.5)

\[
\{e^{\xi_{j,t_k}(t_j)}\} = \bigcap_{\varepsilon \in (0, T_j(t_k) - t_j)} g_{j,t_k}(t_j, K_{j,t_k}(t_j + \varepsilon) \setminus K_{j,t_k}(t_j)), \quad -\infty < t_j < T_j(t_k). \tag{5.6}
\]

For \( -\infty < t_j < T_j(t_k) \), let \( G_{1,k,t_k}(t_j, \cdot) \) and \( \tilde{G}_{1,k,t_k}(t_j, \cdot) \) be defined by (4.11) and (4.12). Then \( G_{1,k,t_k}(t_j, \cdot) \) maps \( \mathbb{D} \setminus g_{j,t_k}(t_j, K_{1,k,t_k}(t_k)) \) conformally onto \( A_{-v_{j,t_k}(t_j)} \), and maps \( \mathbb{T} \) onto \( \mathbb{T} \). Moreover, we have (4.13). Arguing as in Section 4.1 using (5.4) and (5.6) and the fact that \( \zeta_{j,t_k} + 2\pi/n \) generates the same disc Loewner maps as \( \zeta_{j,t_k} \), we conclude that \( \tilde{\zeta}_{j,t_k} \) could be rechosen such that (4.14) holds. From (5.3), (5.5), (4.13), and Proposition 2.1 we can derive (4.15), which then implies (4.16).

Since \( \tilde{g}_j(t_j, \cdot) \) are the inverted covering whole-plane Loewner maps driven by \( \xi_j \), so from (2.11) and that \( H(\infty, z) = \cot_2(z) \), we see that \( \tilde{g}_j(t_j, \cdot) \) satisfies

\[
\tilde{g}_j(t_j, z) = H(\infty, \tilde{g}_j(t_j, z) - \xi_j(t_j)), \quad -\infty < t_j < \infty. \tag{5.7}
\]

Since \( \tilde{g}_{j,t_k}(t_j, \cdot) = \tilde{h}_{j,t_k}(v_{j,t_k}(t_j), \cdot), \tilde{h}_{I,j,t_k}(t, \cdot) \) are the inverted covering disc Loewner maps driven by \( \zeta_{j,t_k} \) and \( \xi_{j,t_k}(t_j) \), so from (4.15) and (5.2), \( \tilde{g}_{j,t_k}(t_j, \cdot) \) satisfies (4.18).

For \( j = 1, 2 \), let \( X_j \) be defined by (4.31); let \( A_{j,h}, h = 0, 1, 2, 3, \) and \( A_{j,S} \) be defined by (4.33) and (4.34). Arguing as in Section 4.1 but using (5.7) instead of (4.17), we see that (4.21), (4.22), (4.40), (4.41), and (4.42) still hold here, and (4.22) holds here with \( p - t_j \) replaced by \( \infty \). So we may define \( Q \) by (4.35). Then (4.44) still holds here.

From Lemma 2.3 we have

\[
Q(t_1, t_2) = O(e^{-m(t_1,t_2)}), \quad \text{as } m(t_1, t_2) \to \infty. \tag{5.8}
\]

From Lemma 2.4 we see that, for \( j = 1, 2 \),

\[
\ln(A_{j,1}(t_1, t_2), A_{j,S}(t_1, t_2) = O(m(t_1, t_2)e^{-m(t_1,t_2)}), \quad \text{as } m(t_1, t_2) \to \infty. \tag{5.9}
\]

The bounds in the two lemmas are uniform. This means that we have positive continuous functions \( f_1 \) and \( f_2 \) on \((0, \infty)\) such that \( f_1(x) = O(e^{-x}) \) and \( f_2(x) = O(xe^{-x}) \) as \( x \to \infty \), and \( |Q| \leq f_1(m), |\ln(A_{j,1})| \leq f_2(m), \) and \( |A_{j,S}| \leq f_2(m), j = 1, 2. \)

From [2], \( t_j \) is the whole-plane capacity of \( K_{I,j}(t_j) \), so \( \text{diam}(K_{I,j}(t_j)) \leq 4e^{t_j} \). Since \( 0 \in K_{I,j}(t_j) \), \( K_{I,j}(t_j) \subset \{|z| \leq 4e^{t_j/4}\} \), which implies \( K_j(t_j) \subset \{|z| \geq e^{-t_j/4}\} \). Thus, if \( t_1 + t_2 < -\ln(16) \), then the annulus \( \{e^{t_1} < |z| < e^{-t_2/4}\} \) separates \( K_{I,j}(t_1) \) and \( K_2(t_2) \). So we have

\[
\{(t_1, t_2) \in \mathbb{R}^2: t_1 + t_2 < -\ln(16)\} \subset \mathcal{D}; \tag{5.10}
\]
Here, and Lemma 2.4, we have
\[
A_{1,1}(t_1, t_2)^2 A_{2,1}(t_1, t_2)^2 Q(t_1, t_2) = O(e^{t_1 + t_2}), \quad t_1, t_2 \to -\infty.
\] (5.12)

From this estimation, we may define \( F(t_1, t_2) \) on \( \mathcal{D} \) such that
\[
F(t_1, t_2) = \exp \left( \int_{-\infty}^{t_2} \int_{-\infty}^{t_1} A_{1,1}(s_1, s_2)^2 A_{2,1}(s_1, s_2)^2 Q(s_1, s_2) ds_1 ds_2 \right).
\]

From (5.12), the two improper integrals both converge, and they are exchangeable. From Lemma 2.4, we have \( A_{k, s} \to 0 \) as \( t_j \to -\infty \). Thus, from (4.44) we see that (4.45) still holds here, and
\[
\ln(F(t_1, t_2)) = \int_{-\infty}^{t_1} A_{1,1}(s_1, t_2) ds_1 = \int_{-\infty}^{t_1} \frac{A_{1,S}(s_1, t_2)}{A_{1,1}(s_1, t_2)^2} A_{1,1}(s_1, t_2)^2 ds_1.
\] (5.13)

From (5.9) we have \( \frac{A_{1,S}}{A_{1,1}} = O(m e^{-m}) \). By changing variable \( x = x(s_1) = m(s_1, t_2) \) in (5.13) and using (4.16) we conclude that
\[
\ln(F(t_1, t_2)) = O(m(t_1, t_2)e^{-m(t_1, t_2)}), \quad \text{as} \quad m(t_1, t_2) \to \infty.
\] (5.14)

Again the bound is uniform, which means that we can find a continuous positive function \( f_3 \) on \((0, \infty)\) such that \( f_3(x) = O(xe^{-x})\) as \( x \to \infty \), and \( |\ln(F)| \leq f_3(m) \).

Let \( \Gamma, \Lambda, \Lambda_1 \) and \( \Lambda_2 \) be as in Theorem 5.1. Let \( \Lambda_{I,j} \) be the dual function of \( \Lambda_j \), \( j = 1, 2 \). So \( \Lambda_{I,2} = \Lambda \), and \( \Lambda_{I,1} \) is the dual function of \( \Lambda \). Let \( \Gamma_{I,2} = \Gamma \) and \( \Gamma_{I,1}(p, x) = \Gamma(p, -x) \). Since \( \Gamma \) and \( \Lambda \) satisfy (4.3) and (4.4), and \( H_I(p, \cdot) \) is odd, so \( \Gamma_{I,j} \) and \( \Lambda_{I,j}, j = 1, 2 \), also satisfy (4.3) and (4.4). From (4.32), for \((t_1, t_2) \in \mathcal{D}, \) we may define
\[
Y(t_1, t_2) = \Gamma_{I,1}(m(t_1, t_2), X_1(t_1, t_2)) = \Gamma_{I,2}(m(t_1, t_2), X_2(t_1, t_2)).
\] (5.15)

From the condition of \( \Gamma \), we see that
\[
\ln(Y(t_1, t_2)) = o(m(t_1, t_2)) \quad \text{as} \quad m(t_1, t_2) \to \infty.
\] (5.16)

### 5.2 Martingales in two time variables

In this section, we will construct \( M(t_1, t_2) \) on \( \mathcal{D} \), which is a local martingale in one variable, when the other variable is fixed. The argument here is parallel to that in Section 4.2. The difference is that the time variable here often runs in the intervals of the form \((-\infty, T)\) instead of \([0, T)\), and Itô’s formula may not be applied directly to these intervals. The way that we use to overcome this problem is to truncate the time interval. For example, suppose \( T \) is bounded below by \( t_0 \), to show that a random process \( N \) defined on \((-\infty, T)\) is a local martingale, we
suffice to show that $N$ is bounded near $-\infty$, and for any $t_0 \in (-\infty, r_0)$, $N_{t_0}(t) := N(t_0 + t)$, $0 \leq t < T - t_0$, is a local martingale. Then we can apply Itô's formula to each $N_{t_0}$.

Fix $\kappa > 0$. Recall the definition of $B^{(\kappa)}(t), t \in \mathbb{R}$, in Section 2.3. Let $(B^{(\kappa)}_1(t))$ and $(B^{(\kappa)}_2(t))$ be two independent copies of $(B^{(\kappa)}(t))$. Let $\xi_j(t) = B^{(\kappa)}_j(t), t \in \mathbb{R}, j = 1, 2$. We adopt the notation in the last section. Then $K_{1,j}(t)$ are whole-plane SLE$_\kappa$ hulls in $\hat{\mathbb{C}}$ from 0 to $\infty$.

For $j = 1, 2$, and $t \in \mathbb{R}$, let $\mathcal{F}_t^j$ be the completion of the $\sigma$-algebra generated by $e^i(\xi_j(s)), -\infty < s \leq t$, with respect to the distribution of $\xi_j$. Then the whole-plane Loewner objects driven by $\xi_j$ are all $(\mathcal{F}_t^j)$-adapted, because they are all determined by $(e^i(\xi_j(t))$, which is $(\mathcal{F}_t^j)$-adapted. But $(\xi_j(t))$ may not be $(\mathcal{F}_t^j)$-adapted. Thus, $K_{1,1,t_2}(t_1) = g_{2,t_2}(K_{1,1}(t_1))$ is $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted. Since $\tilde{g}_{t_1,t_2}(t_1, \cdot)$ is determined by $K_{1,1,t_2}(s_1), -\infty < s_1 \leq t_1$, so $(\tilde{g}_{t_1,t_2}(t_1, \cdot))$ is $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted. These $\tilde{g}_{t_1,t_2}(t_1, \cdot)$ satisfies $\tilde{g}_{t_1,t_2}(t_1, z + 2\pi) = \tilde{g}_{t_1,t_2}(t_1, z) + 2\pi$. So for $h = 1, 2, 3, \tilde{g}^{(h)}_{t_1,t_2}(t_1, \cdot)$ has period $2\pi$. Since $A_{1,h}(t_1, t_2) = \tilde{g}^{(h)}_{t_1,t_2}(t_2, \xi_1(t_1)), (e^i(\xi_1(t_1)))$ is $(\mathcal{F}_t^1)$-adapted, so $(A_{1,h})$ is $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted for $h = 1, 2, 3$, and so is $(A_{1,0})$. For the same reason, $(A_{2,h}), h = 1, 2, 3$, and $(A_{2,S})$ are all $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted. From (5.11) and (5.12), $G_{1,2,t_1}(t_1, \cdot)$ is $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted. From (5.14), $(e^i(\xi_{1,t_2}(t_1))) = (G_{1,2,t_1}(t_2, e^i(\xi_1(t_1))))$ is $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted. From (5.31) and (5.32), $(e^i(X_1))$ and $(e^i(X_2))$ are both $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted. But the images of these functions under the map $e^i$ all become $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)$-adapted.

Fix $r_1, r_2 \in \mathbb{R}$ with $r_1 + r_2 < -\ln(16)$. Fix an $(\mathcal{F}_t^j)$-stopping time $t_2 \in (-\infty, r_2]$. Since $r_1 + t_2 < r_1 + r_2 < -\ln(16)$, so from (5.10), $r_1 < T_{1,t_2}$ holds for sure. Let $\mathcal{F}_{t_1,t_2}^j = \mathcal{F}_t^j \times \mathcal{F}_t^2$. Then $(\mathcal{F}_{t_1,t_2}^j)_{-\infty < t_1, t_2 < \infty}$ is a filtration, and $T_{1,t_2}$ is an $(\mathcal{F}_{t_1,t_2}^j)$-stopping time. For $0 \leq t_1' < \infty$, let $\hat{B}_{1,r_1}(t_1') = \frac{1}{\sqrt{\kappa}}(B^{(\kappa)}_1(r_1 + t_1') - B^{(\kappa)}_1(r_1))$. Then $\hat{B}_{1,r_1}(t_1')$ is an $(\mathcal{F}_t^1 \times (t_1')_{t_1' > 0})$-Brownian motion. Let $\mathcal{F}_{t_1,t_2}^j = \mathcal{F}_{t_1,t_2}^{r_1,r_1'} = \mathcal{F}_{r_1,t_2}^{r_1',t_1} \times \mathcal{F}_t^2$. Then we get a filtration $(\mathcal{F}_{t_1'}^0)_{t_1' > 0}$. Since $(B^{(\kappa)}_1(t))$ is independent of $(\mathcal{F}_t^2)$, so is $(\hat{B}_{1,r_1}(t_1'))$. Thus, $\hat{B}_{1,r_1}(t_1')$ is an $(\mathcal{F}_{t_1'}^{r_1,t_2,1})$-Brownian motion.

Since $T_{1,t_2}$ is an $(\mathcal{F}_{t_1,t_2}^j)$-stopping time, and $T_{1,t_2} > r_1$ always holds, so $T_{1,r_1}(t_2) := T_{1,t_2} - r_1 > 0$ is an $(\mathcal{F}_{t_1',t_2,1})$-stopping time. For $0 \leq t_1' < T_{1,r_1}(t_2)$ and $j = 1, 2$, define $\xi_{1,1,t_1,r_1}(t_1'), m_{r_1}(t_1', t_2), A_{j,h,r_1}(t_1', t_2), h = 0, 1, 2, 3, A_{j,S,r_1}(t_1', t_2), X_{j,r_1}(t_1', t_2)$ and $Y_{r_1}(t_1', t_2)$ to be $\xi_1(r_1 + t_1'), \xi_1,m_{r_1}(r_1 + t_1'), A_{j,h,r_1}(r_1 + t_1'), A_{j,S}(r_1 + t_1', t_2), X_{j,r_1}(r_1 + t_1', t_2)$ and $Y(r_1 + t_1', t_2)$, respectively. Then $(m_{r_1}(t_1', t_2)), (A_{j,h,r_1}(t_1', t_2)), h = 1, 2, 3, (A_{j,S,r_1}(t_1', t_2))$, and $(Y_{r_1}(t_1', t_2))$ are all $(\mathcal{F}_{t_1'}^{r_1,t_2,1})$-adapted. For $0 \leq t_1' < T_{1,r_1}(t_2)$ and $j = 1, 2$, define

\begin{align}
\tilde{\xi}_{1,r_1}(t_1') &= \xi_{1,r_1}(t_1') - \xi_{1,r_1}(0); \\
\tilde{\xi}_{1,t_2,r_1}(t_1') &= \xi_{1,t_2,r_1}(t_1') - \xi_{1,t_2,r_1}(0); \\
\tilde{\xi}_{j,0,r_1}(t_1', t_2) &= A_{j,0,r_1}(t_1', t_2) - A_{j,0,r_1}(0, t_2);
\end{align}

(5.17) (5.18) (5.19)
\[ \hat{X}_{j,r}(t'_1, t_2) = X_{j,r}(t'_1, t_2) - X_{j,r}(0, t_2). \] (5.20)

Then \((\tilde{\xi}_{1, r_1}'), (\tilde{\xi}_{1, t_2, r_1}'), (\tilde{A}_{j, 0, r_1}('), t_2)'), and \((\tilde{X}_{j,r}('), t_2)')\) are all \((\mathcal{F}_{t_1}^{1, l_2, r_1})\)-adapted. And \(\tilde{\xi}_{1, r_1}(t'_1) = \sqrt{\kappa} \tilde{B}_{1, r_1}(t'_1)\). From now on, we will apply Ito’s formula repeatedly, all SDE will be \((\mathcal{F}_{t_1}^{1, l_2, r_1})\)-adapted, and \(t'_1\) ranges in \([0, T_{1, r_1}(t_2)]\).

For \(0 \leq t'_1 < T_{1, r_1}(t_2)\) and \(x \in \mathbb{R}\), let

\[ H_{t_2, r_1}(t'_1, x) = \tilde{G}_{I, 2, t_2}(r_1 + t'_1, \xi_1(r_1) + x) - \tilde{G}_{I, 2, t_2}(r_1, \xi_1(r_1)). \]

From (4.14), (5.17), and (5.18), we have

\[ \tilde{\xi}_{1, t_2, r_1}(t'_1) = H_{t_2, r_1}(t'_1, \tilde{\xi}_{1, r_1}(t'_1)). \] (5.21)

Since \(\tilde{G}_{I, 2, t_2}(t_1, \cdot)\) is \(\mathcal{F}_{t_1}^{1} \times \mathcal{F}_{t_2}^{2}\)-measurable, satisfies that \(\tilde{G}_{I, 2, t_2}(t_1, z + 2\pi) = \tilde{G}_{I, 2, t_2}(t_1, z) + 2\pi\), and \(e^{i}(\xi_1(r_1))\) is \(\mathcal{F}_{r_1}^{1}\)-measurable, so \((H_{t_2, r_1}(t'_1, \cdot))\) is \((\mathcal{F}_{t_1}^{1, l_2, r_1})\)-adapted. It is clear that

\[ H_{t_2, r_1}^{(h)}(t'_1, x) = \tilde{G}_{I, 2, t_2}^{(h)}(r_1 + t'_1, \xi_1(r_1) + x), \quad h = 1, 2; \] (5.22)

\[ \dot{H}_{t_2, r_1}(t'_1, x) = \tilde{G}_{I, 2, t_2}(r_1 + t'_1, \xi_1(r_1) + x). \] (5.23)

From (4.21), (4.33), (5.21), (5.22), and (5.23), we have

\[ d\tilde{\xi}_{1, t_2, r_1}(t'_1) = A_{1,1, r_1}(t'_1, t_2) d\tilde{\xi}_{1, r_1}(t'_1) + \left(\frac{\kappa}{2} - 3\right) A_{1,2, r_1}(t'_1, t_2) dt'_1. \] (5.24)

From (4.40) and (5.19), we have

\[ \partial_1 \tilde{A}_{2,0, r_1}(t'_1, t_2) = - A_{2,1, r_1}^2 H_{I, m_{r_1}, X_{1}}(t'_1, t_2). \] (5.25)

From (4.31), (4.33), (5.18), (5.49), and (5.20), \(\tilde{X}_{1, r_1}(t'_1, t_2) = \tilde{\xi}_{1, t_2, r_1}(t'_1, t_2) - \tilde{A}_{2,0, r_1}(t'_1, t_2)\). So from (5.24) and (5.25), \(\tilde{X}_{1, r_1}\) satisfies

\[ \partial_1 \tilde{X}_{1, r_1} = A_{1,1, r_1} \partial \tilde{\xi}_{1, r_1}(t'_1) + \left(\frac{\kappa}{2} - 3\right) A_{1,2, r_1} \partial t'_1 + A_{2,1, r_1}^2 H_{I, m_{r_1}, X_{1, r_1}}(t'_1, t_2) \partial t'_1. \] (5.26)

Let \(j = 1\) and \(k = 2\) in (4.41). Then we obtain

\[ \frac{\partial_1 A_{2,1, r_1}}{A_{2,1, r_1}} = \frac{A_{1,1, r_1}^2 H_{I, m_{r_1}, X_{1, r_1}}(t'_1, t_2)}{A_{1,1, r_1}} \partial t'_1. \] (5.27)

Let \(\alpha\) and \(c\) be as in (4.50). Using (4.22) with \(p - t_j\) replaced by \(\infty\), we compute

\[ \frac{\partial_1 A_{1,1, r_1}^p}{A_{1,1, r_1}^p} = \alpha \cdot \frac{A_{1,2, r_1}}{A_{1,1, r_1}} \cdot \partial \tilde{\xi}_{1, r_1}(t'_1) + \frac{c}{6} A_{1,8, r_1} \partial t'_1 + \alpha A_{2,1, r_1}^p r_{m_{r_1}}(t'_1) - \alpha r(\infty) \partial t'_1, \] (5.28)

which is similar to (4.51) when \(j = 1\) and \(k = 2\).
For $0 \leq t'_1 < T_{1,r_1}(t_2)$ and $x \in \mathbb{R}$, let
\[ f_{t_2,r_1}(t'_1, x) = \Gamma_{I,1}(m_{r_1}(t'_1, t_2), X_1(r_1, t_2) + x). \]  
(5.29)

From (5.15) and (5.20), we have
\[ Y_{r_1}(t'_1, t_2) = f_{t_2,r_1}(t'_1, \tilde{X}_1, r_1(t'_1, t_2)). \]  
(5.30)

Since $\Gamma_{I,1}$ has period $2\pi$ in the second variable, and $(e^i(X_1(r_1, t_2)))$ is $\mathcal{F}_{t_1}^1 \times \mathcal{F}_{t_2}^2$-measurable, so $(f_{t_2,r_1}(t'_1, \cdot))$ is $(\mathcal{F}_{t_1}^{1,t_2,r_1})$-adapted. From (4.16) we have
\[ \hat{f}_{t_2,r_1}(t'_1, x) = -A_{1,1,r_1}(t'_1, t_2)^2 \hat{\Gamma}_{I,1}(m_{r_1}(t'_1, t_2), X_1(r_1, t_2) + x). \]  
(5.31)

Define $M$ on $\mathcal{D}$ such that
\[ M = A_{1,1}^{\alpha} A_{2,1}^{\alpha} F^{-c/6} Y \exp(\alpha R(m)). \]  
(5.34)

For $0 \leq t'_1 < T_1(t_2) - r_1$, let $F_{r_1}(t'_1, t_2) = F(r_1 + t'_1, t_2)$ and $M_{r_1}(t'_1, t_2) = M(r_1 + t'_1, t_2)$. Then
\[ M_{r_1} = A_{1,1}^{\alpha} A_{2,1}^{\alpha} F_{r_1}^{-c/6} Y \exp(\alpha R(m)). \]  
(5.35)

From (4.15), (5.7), (5.27), (5.28), (5.30), (5.31), and (5.32), we have
\[ \frac{\partial_1 Y_{r_1}}{Y_{r_1}} = \frac{1}{k} A_{1,1,r_1}(m_{r_1}, X_1, r_1) A_{1,1} A_{1,2,r_1} \frac{\partial_1 \tilde{\xi}_{1,1}(t'_1)}{\partial_1 \tilde{\xi}_{1,1}(t'_1)}. \]  
(5.33)

This is similar to (4.49) when $j = 1$ and $k = 2$.

Define $M$ on $\mathcal{D}$ such that
\[ M = A_{1,1}^{\alpha} A_{2,1}^{\alpha} F^{-c/6} Y \exp(\alpha R(m)). \]  
(5.34)

For $0 \leq t'_1 < T_1(t_2) - r_1$, let $F_{r_1}(t'_1, t_2) = F(r_1 + t'_1, t_2)$ and $M_{r_1}(t'_1, t_2) = M(r_1 + t'_1, t_2)$. Then
\[ M_{r_1} = A_{1,1}^{\alpha} A_{2,1}^{\alpha} F_{r_1}^{-c/6} Y \exp(\alpha R(m)). \]  
(5.35)

From (4.15), (4.45), (5.26), (5.27), (5.30), (5.31), and using $R'(t) = r(t) - r(\infty)$ and $\tilde{\xi}_{1,1}(t'_1) = \sqrt{k} B_{1,r_1}(t'_1)$, we compute
\[ \frac{\partial_1 M_{r_1}}{M_{r_1}} = \left[ \left(3 - \frac{\kappa}{2}\right) A_{1,2,r_1} + A_{1,1}(m_{r_1}, X_1, r_1) A_{1,1} \right] \frac{\partial_1 \tilde{B}_{1,r_1}(t'_1)}{\sqrt{k}}. \]  
(5.35)

Since $\tilde{B}_{1,r_1}(t'_1)$ is an $(\mathcal{F}_{t_1}^{1,t_2,r_1})$-Brownian motion, so $(M_{r_1}(t'_1, t_2) = M(r_1 + t'_1, t_2), 0 \leq t'_1 < T_1(t_2) - r_1)$ is a local martingale. Thus, $(M(t_1, t_2) : r_1 \leq t_1 < T_1(t_2))$ is a local martingale. This holds for any $r_1 \in (-\infty, -\ln(16) - r_2)$, if $r_2 \in \mathbb{R}$ is an upper bound of $t_2$. Thus, for any fixed $(\mathcal{F}_{t_2}^2)$-stopping time $t_2$, which is uniformly bounded above, $(M(t_1, t_2) : -\infty < t_1 < T_1(t_2))$ is a local martingale. Since the definition of $M$ is symmetric in $t_1$ and $t_2$, so for any fixed $(\mathcal{F}_{t_1}^1)$-stopping time $t_1$ which is uniformly bounded above, $(M(t_1, t_2) : -\infty < t_2 < T_2(t_1))$ is a local martingale.
From (5.9), (5.14), (5.16), and that $R(p) = O(e^{-p})$ as $p \to \infty$, we see that there is a positive continuous function $f$ on $(0, \infty)$ that satisfies $\lim_{x \to \infty} f(x) = 0$ such that

$$|\ln(M(t_1, t_2))| \leq f(m(t_1, t_2)).$$

(5.36)

Let $\mathcal{J}$ denote the set of Jordan curves in $\mathbb{C} \setminus \{0\}$ that surround 0. For $J \in \mathcal{J}$ and $j = 1, 2$, let $T_j(J)$ denote the smallest $t$ such that $K_j(t) \cap J \neq \emptyset$. Recall that $I_0(\mathbb{C}) = 1/\mathbb{C}$ and $K_j(t) = I_0(K_{I,j}(t))$. So $T_j(J)$ is also the the smallest $t$ such that $K_{I,j}(t) \cap I_0(J) \neq \emptyset$. Let $H_J$ denote the closure of the domain bounded by $I_0(J)$. Then $H_J$ is an interior hull in $\mathbb{C}$. Let $c_j$ denote the whole-plane capacity of $H_J$, and $d_j = \text{dist}(0, I_0(J))$. If $K_{I,j}(t) \subset H_J$, then $t \leq c_j$ as $t$ is the whole-plane capacity of $K_{I,j}(t)$. We have seen that $K_{I,j}(t) \subset \{|z| \leq 4e^t\}$. Thus, if $K_{I,j}(t) \cap J \neq \emptyset$, then $t \geq \ln(d_j/4)$. So we have

$$\ln(d_j/4) \leq T_j(J) \leq c_j. \quad (5.37)$$

Let $\mathcal{J}^2$ denote the set of pairs $(J_1, J_2) \in \mathcal{J}^2$ such that $I_0(J_1) \cap J_2 = \emptyset$ and $I_0(J_1)$ is surrounded by $J_2$. This is equivalent to that $I_0(J_2) \cap J_1 = \emptyset$ and $I_0(J_2)$ is surrounded by $J_1$. Then for every $(J_1, J_2) \in \mathcal{J}^2$, $K_{I,1}(t_1) \cap K_{I,2}(t_2) = \emptyset$ when $t_1 \leq T_1(J_1)$ and $t_2 \leq T_2(J_2)$, so $(-\infty, T_1(J_1)) \times (-\infty, T_2(J_2)) \subset D$.

**Lemma 5.1** (Boundedness) (i) Fix $(J_1, J_2) \in \mathcal{J}^2$. Then $|\ln(M)|$ is bounded on $(-\infty, T_1(J_1))] \times (-\infty, T_2(J_2))]$ by a constant depending only on $J_1$ and $J_2$. (ii) Fix any $j \neq k \in \{1, 2\}$ and $T_k \in \mathbb{R}$. Then $M \to 1$ as $t_j \to -\infty$ uniformly in $t_k \in (-\infty, T_k]$.

**Proof.** (i) From (5.36) we suffice to show that $m$ is bounded away from 0 on $(-\infty, T_1(J_1))] \times (-\infty, T_2(J_2))]$. Let $D(J_1, J_2)$ denote the doubly connected domain bounded by $I_0(J_1)$ and $J_2$. Let $p_0 > 0$ denote its modulus. For $(t_1, t_2) \in (-\infty, T_1(J_1))] \times (-\infty, T_2(J_2))]$, $D(J_1, J_2)$ disconnects $K_{I,1}(t_1)$ from $K_{I,2}(t_2)$, so from (5.11) we have $m(t_1, t_2) \geq p_0$. (ii) This follows from (5.10), (5.11), and (5.36). □

Now we define $\tilde{D} = D \cup \{-\infty\} \times [-\infty, \infty) \cup (-\infty, \infty) \times \{-\infty\}$, and extend $M$ to $\tilde{D}$ such that $M = 1$ if $t_1$ or $t_2$ equals to $-\infty$. From Lemma 5.1 we see that $M$ is positive and continuous on $\tilde{D}$. So we can conclude that for any fixed $j \neq k \in \{1, 2\}$ and any fixed $(\mathcal{F}_k)$-stopping time $t_k$ which is uniformly bounded above, $M$ is a local martingale in $t_j \in [-\infty, T_j(t_k)]$.

### 5.3 Local coupling

Let $\mu_j$ denote the distribution of $(\xi_j)$, $j = 1, 2$. Let $\mu = \mu_1 \times \mu_2$. Then $\mu$ is the joint distribution of $(\xi_1)$ and $(\xi_2)$ since $\xi_1$ and $\xi_2$ are independent. Fix $(J_1, J_2) \in \mathcal{J}^2$. From the local martingale property of $M$ and Lemma 5.1, we have $E_{\mu}[M(T_1(J_1), T_2(J_2))] = M(-\infty, -\infty) = 1$. Define $\nu_{J_1, J_2}$ such that $d\nu_{J_1, J_2} = M(T_1(J_1), T_2(J_2))d\mu$. Then $\nu_{J_1, J_2}$ is a probability measure. Let $\nu_1$ and $\nu_2$ be the two marginal measures of $\nu_{J_1, J_2}$. Then $d\nu_1/d\mu_1 = M(T_1(J_1), -\infty) = 1$ and $d\nu_2/d\mu_2 = M(-\infty, T_2(J_2)) = 1$, so $\nu_1 = \mu_1$, $j = 1, 2$. Suppose temporarily that the distribution of $(\xi_1, \xi_2)$ is $\nu_{J_1, J_2}$ instead of $\mu$. Then the distribution of each $(\xi_j)$ is still $\mu_j$. 

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Now fix an \((\mathcal{F}_t^2)\)-stopping time \(t_2 \in (-\infty, T_2(J_2))\). Fix \(r_1 \in (-\infty, \ln(d_{J_1}/4))\). From (5.37) we have \(r_1 < T_1(J_1)\). Let \(T_{1,r_1}(J_1) = T_1(J_1) - r_1 > 0\). For \(0 \leq t'_1 \leq T_{1,r_1}(J_1)\), define

\[
\tilde{B}_{1,t_2,r_1}(t'_1) = \tilde{B}_{1,r_1}(t'_1) - \frac{1}{\sqrt{\kappa}} \int_0^{t'_1} \left[ \left( 3 - \frac{\kappa}{2} \right) \frac{A_{1,2,r_1}}{A_{1,1,r_1}} + \Lambda_{1,1}(m_{r_1}, X_{1,r_1}) A_{1,1,r_1} \right] (s'_1, t_2) \, ds'_1.
\]

Since \(\tilde{B}_{1,r_1}(t'_1)\) is an \((\mathcal{F}_{t'_1}^{1,t_2,r_1})\)-Brownian motion under \(\mu\), so from (5.35) and Girsanov theorem, \(\tilde{B}_{1,t_2,r_1}(t'_1), 0 \leq t'_1 \leq T_{1,r_1}(J_1)\), is an \((\mathcal{F}_{t'_1}^{1,t_2,r_1})\)-Brownian motion under \(\nu_{J_1,J_2}\). Since \(\tilde{\xi}_{1,r_1}(t'_1) = \sqrt{\kappa \tilde{B}_{1,r_1}(t'_1)}\), so \(\tilde{\xi}_{1,r_1}(t'_1), 0 \leq t'_1 \leq T_{1,r_1}(J_1)\), satisfies the SDE:

\[
d\tilde{\xi}_{1,r_1}(t'_1) = \sqrt{\kappa} d\tilde{B}_{1,t_2,r_1}(t'_1) + \left[ \left( 3 - \frac{\kappa}{2} \right) \frac{A_{1,2,r_1}}{A_{1,1,r_1}} + \Lambda_{1,1}(m_{r_1}, X_{1,r_1}) A_{1,1,r_1} \right] (t'_1, t_2) \, dt'_1.
\]

From (5.23) and the above formula, we see that \(\tilde{\xi}_{1,t_2,r_1}(t'_1), 0 \leq t'_1 \leq T_{1,r_1}(J_1)\), satisfies

\[
d\tilde{\xi}_{1,t_2,r_1}(t'_1) = \sqrt{\kappa} A_{1,1,r_1}(t'_1, t_2) d\tilde{B}_{1,t_2,r_1}(t'_1) + \left[ \Lambda_{1,1}(m_{r_1}, X_{1,r_1}) A_{1,1,r_1} \right] (t'_1, t_2) \, dt'_1. \tag{5.38}
\]

Let \(r''_1 = v_{1,t_2}(r_1)\), and

\[
\tilde{v}_{1,t_2,r_1}(t'_1) = v_{1,t_2}(r_1 + t'_1) - r''_1, \quad 0 \leq t'_1 < T_{1,r_1}(t_2). \tag{5.39}
\]

From (5.2) we see that \((\tilde{v}_{1,t_2,r_1})\) is also \((\mathcal{F}_{t'_1}^{1,t_2,r_1})\)-adapted. From (4.15), we have

\[
\tilde{v}_{1,t_2,r_1}(t'_1) = A_{1,1,r_1}(t'_1, t_2)^2, \quad 0 \leq t'_1 < T_{1,r_1}(t_2). \tag{5.40}
\]

Since \(\tilde{v}_{1,t_2,r_1}\) is continuous, increasing, and maps \([0, T_{1,r_1}(t_2)]\) onto \([0, -r''_1]\), so \(\tilde{v}_{1,t_2,r_1}^{-1}\) is well defined on \([0, -r''_1]\). We now extend \(\tilde{v}_{1,t_2,r_1}^{-1}\) to \([0, \infty)\) such that if \(t \geq -r''_1\) then \(\tilde{v}_{1,t_2,r_1}^{-1}(t) = \infty\).

Since \((\tilde{v}_{1,t_2,r_1}(t'_1))\) is \((\mathcal{F}_{t'_1}^{1,t_2,r_1})\)-adapted, and \(T_{1,r_1}(t_2)\) is an \((\mathcal{F}_{t'_1}^{1,t_2,r_1})\)-stopping time, so for each \(t \in [0, \infty), \tilde{v}_{1,t_2,r_1}^{-1}(t)\) is an \((\mathcal{F}_{t'_1}^{1,t_2,r_1})\)-stopping time. Since \(\tilde{v}_{1,t_2,r_1}^{-1}(t)\) increases in \(t\), so we have a new filtration \((\mathcal{F}_{\tilde{v}_{1,t_2,r_1}(t)}^{1,t_2,r_1})\)\(_{t \geq 0}\).

For \(-\infty < s < 0\) and \(0 \leq t < -s\), let

\[
\tilde{\xi}_{1,t_2,s}(t) = \xi_{1,t_2}(s + t) - \xi_{1,t_2}(s). \tag{5.41}
\]

Recall that \(\xi_{1,t_2}(t_1) = \xi_{1,t_2}(v_{1,t_2}(t_1))\). From (5.18) and (5.39), we have

\[
\tilde{\xi}_{1,t_2,r_1}(t) = \xi_{1,t_2,r_1}(\tilde{v}_{1,t_2,r_1}^{-1}(t)), \quad 0 \leq t < -r''_1. \tag{5.42}
\]

Thus, \(\tilde{\xi}_{1,t_2,r_1}^{v''}(t)\) is \((\mathcal{F}_{\tilde{v}_{1,t_2,r_1}(t)}^{1,t_2,r_1})\)-adapted. From (5.38) and (5.40), there is an \((\mathcal{F}_{\tilde{v}_{1,t_2,r_1}(t)}^{1,t_2,r_1})\)-Brownian motion \(B(t)\) such that \(\tilde{\xi}_{1,t_2,r_1}(t), 0 \leq t \leq \tilde{v}_{1,t_2,r_1}(T_{1,r_1}(J_1))\), satisfies the \((\mathcal{F}_{\tilde{v}_{1,t_2,r_1}(t)}^{1,t_2,r_1})\)-adapted SDE:

\[
d\tilde{\xi}_{1,t_2,r_1}^{v''}(t) = \sqrt{\kappa} dB(t) + \Lambda_{1,1}(m_{r_1}(\tilde{v}_{1,t_2,r_1}^{-1}(t), t_2), X_{1,r_1}(\tilde{v}_{1,t_2,r_1}^{-1}(t), t_2)) \, dt. \tag{5.43}
\]
Let \( t \in [0, -r_1^v) \). From (5.39), we have
\[
v_{1,t_2}(r_1 + \hat{v}_{1,t_2,r_1}^{-1}(t)) = r_1^v + t. \tag{5.44}
\]
From (5.2) and (5.44), we have
\[
m_{r_1}(\hat{v}_{1,t_2,r_1}^{-1}(t), t_2) = -v_{1,t_2}(r_1 + \hat{v}_{1,t_2,r_1}^{-1}(t)) = -r_1^v - t. \tag{5.45}
\]
Since \( \xi_{1,t_2}(t_1) = \zeta_{1,t_2}(v_{1,t_2}(t_1)) \) and \( \tilde{g}_{I,1,t_2}(t_1, \cdot) = \tilde{h}_{I,1,t_2}(v_{1,t_2}(t_1), \cdot) \), so from (4.31), and (5.44), we have
\[
X_{1,r_1}(\hat{v}_{1,t_2,r_1}^{-1}(t), t_2) = X_1(r_1 + \hat{v}_{1,t_2,r_1}^{-1}(t), t_2) = \zeta_{1,t_2}(r_1^v + t) - \tilde{h}_{I,1,t_2}(r_1^v + t, \xi_2(t_2)). \tag{5.46}
\]
So from (5.43), (5.45), and (5.46), we see that \( \hat{v}_{1,t_2,r_1}^{-1}(t), 0 \leq t \leq \hat{v}_{1,t_2,r_1}(T_{1,r_1}(J_1)) \), satisfies
\[
d\hat{v}_{1,t_2,r_1}(t) = \sqrt{\kappa B(t)} + \Lambda_{I,1}(-r_1^v - t, \zeta_{1,t_2}(r_1^v + t) - \tilde{h}_{I,1,t_2}(r_1^v + t, \xi_2(t_2)))dt. \tag{5.47}
\]
Since \( t_2 \leq T_2(J_2) \), so \( I_0(J_1) \) is a Jordan curve in \( \hat{C} \setminus K_2(t_2) \setminus \{0\} \) surrounding 0. Since \( g_2(t_2, \cdot) \) maps \( \hat{C} \setminus K_2(t_2) \) conformally onto \( \mathbb{D} \), and fixes 0, so \( g_2(t_2,I_0(J_1)) \) is a Jordan curve in \( \mathbb{D} \setminus \{0\} \) surrounding 0. Let
\[
d(J_1,t_2) = dist(0,g_2(t_2,I_0(J_1))), \quad m(J_1,t_2) = \mod(\mathbb{D} \setminus [0,d(J_1,t_2)]).
\]
From an argument using extremal length (c.f. [1]), if \( L \) is an interior hull in \( \mathbb{D} \) such that \( 0 \in L \) and \( L \cap \{ |z| = d(J_1,t_2) \} \neq \emptyset \), then \( \mod(\mathbb{D} \setminus L) < m(J_1,t_2) \). Since \( K_{I,1}(T_1(J_1)) \cap I_0(J_1) \neq \emptyset \) and \( K_{I,1,t_2}(T_1(J_1)) = g_2(t_2,K_{I,1}(T_1(J_1))) \), so \( K_{I,1,t_2}(T_1(J_1)) \cap g_2(t_2,I_0(J_1)) \neq \emptyset \). So \( K_{I,1,t_2}(T_1(J_1)) \) satisfies the property of \( L \). Thus, from (5.2) we have \( v_{1,t_2}(T_1(J_1)) \geq -m(J_1,t_2) \).

Fix \( t_0 \in (-\infty, -m(J_1,t_2)) \). Then \( t_0 < v_{1,t_2}(T_1(J_1)) \) holds for sure. From (5.37), we have \( T_2(J_2) < c_{J_2} \). Choose \( s_0 \in (-\infty, t_0) \) and \( r_1 \in (-\infty, s_0 - c_{J_2} - ln(16)) \). Since \( t_2 \leq T_2(J_2) \), so from (5.2) and (5.11), \( r_1^v = v_{1,t_2}(r_1) < s_0 \) always holds. Fix any \( s \in [s_0,t_0] \). Then \( s - r_1^v \) is a positive random variable. We claim that \( s - r_1^v \) is an \( (F_{t_1}^{1,t_2,r_1}) \)-stopping time. Now we prove this claim. Fix any \( a, b \in [0, \infty) \). Since \( v_{1,t_2}(a) \) is an \( (F_{t_1}^{1,t_2,r_1}) \)-stopping time, so \( \{v_{1,t_2}(a) \leq b \} \in F_{t_1}^{1,t_2,r_1} \). On the other hand, since \( v_{1,t_2}(t) \) is \( (F_{t_1}^{1,t_2}) \)-adapted, so
\[
\{ s - r_1^v \leq a \} = \{ v_{1,t_2}(r_1) \leq a + s \} \in F_{r_1}^{1,t_2} = F_{0}^{2,t_2,r_1} \subset F_{t_1}^{1,t_2,r_1}.
\]
So \( \{ s - r_1^v \leq a \} \cap \{ v_{1,t_2}(a) \leq b \} \in F_{t_1}^{1,t_2,r_1} \) always holds. Since this holds for all \( b \in [0, \infty) \), so \( \{ s - r_1^v \leq a \} \in F_{t_1}^{1,t_2,r_1} \). Since this holds for all \( a \in [0, \infty) \), so \( s - r_1^v \) is an \( (F_{t_1}^{1,t_2,r_1}) \)-stopping time. Thus, the claim is proved. For \( s \in [s_0,t_0] \), let \( G_s \) denote the \( \sigma \)-algebra obtained from the filtration \( (F_{t_1}^{1,t_2,r_1}) \) and its stopping time \( s - r_1^v \). Then \( (G_s)_{s_0 \leq s \leq t_0} \) is an filtration.
Let \( \widehat{B}_{t_0}(t) = B(t_0 - r_1^w + t) - B(t_0 - r_1^w) \), \( t \geq 0 \). Since \( B(t) \) is an \((\mathcal{F}_{t_1}^{1,t_2,r_1})^{v_{1,t_2,r_1}(t)}\)-Brownian motion, so \( \widehat{B}_{t_0}(t) \) is a Brownian motion independent of \( \mathcal{G}_{t_0} \). From (5.41) and (5.47), we see that for \( 0 \leq t \leq v_{1,t_2}(T_1(J_1)) - t_0 \),
\[
\sqrt{\kappa} \widehat{B}_{t_0}(t) = \zeta_{1,t_2}(t_0 + t) - \zeta_{1,t_2}(t_0) - \int_{t_0}^{t_0 + t} \Lambda_{1,1}(-s, \zeta_{1,t_2}(s) - \overline{h}_{1,1,t_2}(s, \xi_2(t_2)))ds. \tag{5.48}
\]
From (5.44) and that \( \xi_{1,t_2}(t_1) = \zeta_{1,t_2}(v_{1,t_2}(t_1)) \), we have \( \xi_{1,t_2}(r_1 + \overline{v}_{1,t_2,r_1}(t)) = \zeta_{1,t_2}(r_1^w + t) \). Since \((e^i(\xi_{1,t_2}(r_1 + t_1)))\) is \((\mathcal{F}_{t_1}^{1,t_2,r_1})\)-adapted, so \((e^i(\zeta_{1,t_2}(r_1^w + t)))\) is \((\mathcal{F}_{t_1}^{1,t_2,r_1})\)-adapted. Fix \( s \in [s_0, t_0] \). Since \( s - r_1^w \) is an \((\mathcal{F}_{t_1}^{1,t_2,r_1})\)-stopping time, and \( \mathcal{G}_s \) is the \( \sigma \)-algebra obtained from the above filtration and its stopping time \( s - r_1^w \), so \((e^i(\xi_{1,t_2}(s))) = (e^i(\zeta_{1,t_2}(r_1^w + (s - r_1^w)))\) is \( \mathcal{G}_s \)-measurable, \( s \in [s_0, t_0] \). Since \( \widehat{B}_{t_0}(t) \) is independent of \( \mathcal{G}_{t_0} \), and \( \mathcal{G}_s \subset \mathcal{G}_{t_0} \) for \( s \in [s_0, t_0] \), so \( \widehat{B}_{t_0}(t) \) is independent of \((e^i(\xi_{1,t_2}(s)))\) \( \forall s \leq s_0 \). Since this holds for any \( s_0 \in (-\infty, t_0) \), so \( \widehat{B}_{t_0}(t) \) is independent of \((e^i(\xi_{1,t_2}(s)))\) \( \forall s < s_0 \). Since this holds for any \( t_0 \in (-\infty, -m(J_1, t_2)) \), and \( \overline{h}_{1,1,t_2}(t, \cdot) \) are covering disc Loewner maps driven by \( \zeta_{1,t_2}(t) \), so from (5.48) and the remark after Definition 3.3, conditioned on \( \mathcal{F}_{t_1}^{2} \), \( \zeta_{1,t_2} \) is the driving function of a stopped disc \( \text{SLE}(\kappa, \Lambda_1) \) process in \( \mathbb{D} \) started from 0 with marked point \( e^{i\xi_2(t_2)} \). Since \( L_{1,1,t_2}(t) \) are disc Loewner hulls driven by \( \zeta_{1,t_2} \), \( g_2(t_2, \cdot) \) maps \( \hat{\mathbb{C}} \setminus K_2(t_2) \) conformally onto \( \mathbb{D} \), fixes 0, and \( g_2(t_2, K_{1,1}(t_1)) = L_{1,1,t_2}(v_{1,t_2,r_1}(t_2)) \), so conditioned on \( \mathcal{F}_{t_1}^{2} \), after a time-change, \( K_{1,1}(t_1), \ -\infty < t_1 \leq T_1(J_1) \), is a stopped disc \( \text{SLE}(\kappa, \Lambda_1) \) process in \( \hat{\mathbb{C}} \setminus K_2(t_2) \) started from 0 with marked point \( g_2(t_2, \cdot)^{-1}(e^{i\xi_2(t_2)}) \). Now if \( \beta_{1,2}(t), \ t \in \mathbb{R} \), is the whole-plane \( \text{SLE}_\kappa \) trace that corresponds to \( K_{1,2}(t) \), then the marked point is \( g_2(t_2, \cdot)^{-1}(e^{i\xi_2(t_2)}) = I_0(\beta_{1,2}(t_2)) = \beta_2(t_2) \). Using an a symmetric argument, we see that if \( t_1 \) is any \((\mathcal{F}_{t_1}^{2})\)-stopping time and \( t_1 \leq T_1(J_1) \), then conditioned on \( \mathcal{F}_{t_1}^{2} \), after a time-change, \( K_{1,2}(t_2), \ -\infty < t_2 \leq T_2(J_2) \), is a stopped disc \( \text{SLE}(\kappa, \Lambda_2) \) process in \( \hat{\mathbb{C}} \setminus K_1(t_1) \) started from 0 with marked point \( \beta_1(t_1) = I_0(\beta_{1,1}(t_1)) \).

### 5.4 Global coupling

The proof of Theorem 5.1 can be now completed using the theorem below and the argument in Section 4.3.

**Theorem 5.2** Suppose \( n \in \mathbb{N} \) and \((J_1^m, J_2^m) \in \mathbb{J}_P, \ 1 \leq m \leq n \). There is a continuous function \( M_*(t_1, t_2) \) defined on \([-\infty, \infty]^2\) that satisfies the following properties:

1. \( M_* = M \) on \([-\infty, T_1(J_1^m)] \times [-\infty, T_2(J_2^m)] \) for \( 1 \leq m \leq n \);

2. \( M_*(t, -\infty) = M_*(\infty, t) = 1 \) for any \( t \in (-\infty, \infty) \);

3. \( M(t_1, t_2) \in [C_1, C_2] \) for any \( t_1, t_2 \in [-\infty, \infty] \), where \( C_2 > C_1 > 0 \) are constants depending only on \( J_j^m, \ j = 1, 2, \ 1 \leq m \leq n \);
(iv) for any fix $j \neq k \in \{1, 2\}$ and any any $(\mathcal{F}_t^k)$-stopping time $t_k \in [-\infty, \infty]$, $M_\ast$ is a bounded $(\mathcal{F}_t^j \times \mathcal{F}_t^k)_{-\infty \leq t \leq \infty}$-martingale in $t_j$.

This theorem is similar to Theorem 4.2, so may be proved using the idea in the proofs of Theorem 6.1 in [28] and Theorem 4.5 in [29]. The function $M_\ast$ could be constructed in the same way as in the discussion after Theorem 4.2 with $[0, \cdot]$ replaced by $[-\infty, \cdot]$. Lemma 5.1 will be used here to prove (iii). The local martingale property of $M$ and the boundedness of $M_\ast$ yield (iv), the martingale property of $M_\ast$.

Using Theorem 5.1 and the idea in the proof of reversibility of chordal SLE$_\kappa$ when $\kappa \in (0, 4]$, we can conclude the following Theorem.

**Theorem 5.3** Let $\kappa \in (0, 4]$. Suppose $\Lambda$ satisfies the condition in Theorem 5.1 and the following condition: if $\beta(t)$, $-\infty < t < 0$, is a disc SLE$(\kappa, \Lambda)$ trace in $\mathbb{D}$ started from 0 with marked point $b$, then a.s. $\lim_{t \to 0} \beta(t) = b$. Then the whole-plane SLE$_\kappa$ trace is reversible, and the disc SLE$(\kappa, \Lambda)$ trace is the reversal of radial SLE$_\kappa$ trace. In other words, if $\beta(t)$ is a whole-plane SLE$_\kappa$ trace in $\hat{\mathbb{C}}$ from $a$ to $b$, then after a time-change, the reversal of $\beta$ has the distribution of a whole-plane SLE$_\kappa$ trace in $\hat{\mathbb{C}}$ from $b$ to $a$; if $\beta(t)$ is a radial SLE$_\kappa$ trace in some simply connected domain $D$ from $a$ to $b$, then after a time-change, the reversal of $\beta$ has the distribution of a disc SLE$(\kappa, \Lambda)$ trace in $D$ started from $b$ with marked point $a$.

**Proof.** Let two standard whole-plane SLE$_\kappa$ processes $(K_{I,1}(t_1))$ and $(K_{I,2}(t_2))$ be coupled according to Theorem 5.1. Let $\beta_{I,1}$ and $\beta_{I,2}$ be the corresponding traces. Then for any $t_2 \in \mathbb{R}$, conditioned on $K_2(t_2)$, after a time-change, $\beta_{I,1}(t_1), -\infty < t_1 < T_1(t_2)$, is a stopped disc SLE$(\kappa, \Lambda_1)$ trace in $\hat{\mathbb{C}} \setminus K_2(t_2)$ started from 0 with marked point $\beta_2(t_2) = I_0(\beta_{I,2}(t_2))$. Now $T_1(t_2)$ is the first time that $\beta_{I,1}(t)$ intersects $K_2(t_2) = \beta_2([-\infty, t_2])$. So from the property of $\Lambda_1 = \Lambda$, we see that a.s. $\beta_{I,1}(T_1(t_2)) = \beta_2(t_2)$. So a.s. for any $t_2 \in \mathbb{Q}$, $\beta_{I,1}(T_1(t_2)) = \beta_2(t_2)$. Since $\{\beta_2(t_2) : t_2 \in \mathbb{Q}\}$ is dense in $\beta_2$, and $\beta_{I,1}$ is continuous, so a.s. $\beta_2 \subset \beta_{I,1}$. Since both $\beta_{I,1}$ and $\beta_2$ are simple curves, and have end points 0 and $\infty$, so a.s. $\beta_2 = \beta_{I,1}$. Thus, a.s. $\beta_2$ is a time-change of the reversal of $\beta_{I,1}$. Recall that $\beta_{I,1}$ and $\beta_{I,2}$ are whole-plane SLE$_\kappa$ traces in $\hat{\mathbb{C}}$ from 0 to $\infty$. Since $I_0$ maps $\hat{\mathbb{C}}$ conjugate conformally to itself, and exchanges 0 and $\infty$, so $\beta_2 = I_0(\beta_{I,2})$ is a whole-plane SLE$_\kappa$ traces in $\hat{\mathbb{C}}$ from $\infty$ to 0. So the reversibility of whole-plane SLE$_\kappa$ trace holds for $a = 0$ and $b = \infty$. The conclusion in the general case follows from conformal invariance.

Fix some $t_2 \in \mathbb{R}$. We know that $\beta_{I,1}(t), -\infty < t < T_1(t_2)$, is a time-change of the reversal of $\beta_2(t_2 + t), 0 \leq t < \infty$. If we condition on $K_2(t_2), -\infty < t \leq t_2$, then $\beta_2(t_2 + t), 0 \leq t < \infty$, is a radial SLE$_\kappa$ trace in $\hat{\mathbb{C}} \setminus K_2(t_2)$ from $\beta_2(t_2)$ to 0; and $\beta_{I,1}(t), -\infty < t < T_1(t_2)$, is a disc SLE$(\kappa, \Lambda)$ trace in $\hat{\mathbb{C}} \setminus K_2(t_2)$ started from 0 with marked point $\beta_2(t_2)$. So our conclusion about the reversal of radial SLE$_\kappa$ trace holds for $D = \hat{\mathbb{C}} \setminus K_2(t_2), a = \beta_2(t_2)$ and $b = 0$. The conclusion in the general cases follows from conformal invariance. \(\square\)
6 Some Particular Solutions

In this section, for some special values of $\kappa$, we will find solutions to PDE (4.1) and (4.58), which can be expressed in terms of $H$ and $H_I$. In the next paper [30] we will show that the $\Lambda_{2,\pi}$ in Proposition 6.1 and the $\Lambda_1$ in Proposition 6.2 and the $\Lambda_1$ in Proposition 6.3 satisfy the condition for $\Lambda$ in Theorem 5.3 in the case $\kappa = 4, \kappa = 2$, and $\kappa = 3$, respectively. So Conjecture [1] holds at least for $\kappa = 2, 3, 4$.

From Lemma 3.1 in [22], we see that $H$ satisfies
\[ \dot{H} = H'' + H'H. \] (6.1)
Since $H_I(p, z) = H(z - ip) - i$, it is easy to check that
\[ \dot{H}_I = H''_I + H'_IH_I. \] (6.2)
From (6.1) and (6.2), it is easy to check that $\Lambda$ satisfies (4.1) if and only if $\Phi := \Lambda + H_I$ satisfies
\[ \dot{\Phi} = \frac{\kappa}{2} \Phi'' + \Phi'\Phi + (4 - \kappa)\dot{H}_I'' \] (6.3)
on $(0, \infty) \times \mathbb{R}$; and $\Lambda$ satisfies (4.58) if and only if $\Phi := \Lambda + H$ satisfies
\[ \dot{\Phi} = \frac{\kappa}{2} \Phi'' + \Phi'\Phi + (4 - \kappa)\dot{H}'' \] (6.4)
on $(0, \infty) \times (\mathbb{R} \setminus \{\text{poles}\})$. In the case $\kappa = 4$, (6.3) and (6.4) both become
\[ \dot{\Phi} = 2\Phi'' + \Phi'\Phi. \] (6.5)

We first suppose that $\Phi$ depends only on the second variable, i.e., $\Phi(p, x) = f(x)$. Then (6.5) becomes an ODE: $2f'' + f'f = 0$. The solutions include: $f(x) = C, f(x) = 4/(x - C), f(x) = 4C_2\tanh(C_2(x - C_1)), f(x) = 4C_2\coth(C_2(x - C_1))$, and $f(x) = 4C_2\coth(C_2(x - C_1))$, where $C, C_1$, and $C_2$ are real constants. Among these functions, only $f(x) = C$ and $f(x) = 2\coth(x/2)$ have period $2\pi$, and have no poles other than $2n\pi, n \in \mathbb{Z}$. From (6.1) and (6.2), we find that $\Phi(p, x) = 2H(2p, x - C)$ and $\Phi(p, x) = 2H_I(2p, x - C)$ also solve (6.3). Thus, we have the following proposition.

**Proposition 6.1** Suppose $\kappa = 4$. The following functions are crossing annulus drift functions that solve (4.1): $\Lambda_{1, C}(p, x) = -H_I(p, x) + C$ and $\Lambda_{2, C}(p, x) = -H_I(p, x) + 2H_I(2p, x - C)$; the following functions are chordal-type crossing annulus drift functions that solve (4.58): $\Lambda_{3, C}(p, x) = -H(p, x) + C, \Lambda_{4, p, x} = -H(p, x) + 2\cot(x/2), \Lambda_{5, p, x} = -H(p, x) + 2H(2p, x)$, and $\Lambda_{6, C}(p, x) = -H(p, x) + 2H_I(2p, x - C)$, where $C$ is a real constant.

From the proof of Lemma 1 we see that, if there are a non-vanishing $C^{1,2}$ differentiable function $\hat{\Gamma}$ on $(0, \infty) \times \mathbb{R}$, and a continuous function $C$ on $(0, \infty) \times \mathbb{R}$ which depends only on the first variable, i.e., $C(p, x) = C(p)$, such that the following equation holds:
\[ \ddot{\hat{\Gamma}} = \frac{\kappa}{2} \hat{\Gamma}'' + H_I\hat{\Gamma}' + \frac{6 - \kappa}{2\kappa} H_I\hat{\Gamma}' + C\hat{\Gamma}, \] (6.6)
then $\Lambda := \kappa \hat{\Gamma}^\prime / \hat{\Gamma}$ solves (4.1). Similarly, if a non-vanishing $C^{1,2}$ differentiable function $\hat{\Gamma}$ on $(0, \infty) \times (\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\})$ satisfies
\[
\dot{\Gamma} = \frac{\kappa}{2} \Gamma'' + H\Gamma' + \frac{6 - \kappa}{2\kappa} H' \Gamma + C \Gamma
\] (6.7)
for some function $C$ depending only on the first variable, then $\Lambda := \kappa \hat{\Gamma}^\prime / \hat{\Gamma}$ solves (4.58).

Note that if $\Theta$ on $(0, \infty) \times \mathbb{R}$ satisfies
\[
\dot{\Theta} = \Theta'' + \Theta' H + C \Theta
\] (6.8)
for some function $C$ depending only on the first variable, then $\hat{\Gamma} := \Theta^\prime$ satisfies
\[
\dot{\hat{\Gamma}} = \hat{\Gamma}'' + H' \hat{\Gamma}' + H' \hat{\Gamma} + C \hat{\Gamma},
\] which is equation (6.6) when $\kappa = 2$. Thus, if $\Theta$ on $(0, \infty) \times (\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\})$ satisfies
\[
\dot{\Theta} = \Theta'' + \Theta' H + C \Theta
\] (6.9)
for some function $C$ depending only on the first variable, and $\Theta'$ does not vanish anywhere, then $\Lambda := 2\Theta'' / \Theta'$ solves (4.1) when $\kappa = 2$. Similarly, if $\Theta$ on $(0, \infty) \times (\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\})$ satisfies
\[
\dot{\Theta} = \Theta'' + \Theta' H + C \Theta
\] (6.8)
for some function $C$ depending only on the first variable, and $\Theta'$ does not vanish anywhere, then $\Lambda := 2\Theta'' / \Theta'$ solves (4.58) when $\kappa = 2$.

From (6.2) we see that $\Theta_1 = H_I$ solves (6.8) with $C \equiv 0$. From (6.1) we see that $\Theta_2 = H$ solves (6.9) with $C \equiv 0$. It is also easy to check that $\Theta_3(p, x) = p H_I(p, x) + x$ solves (6.8) with $C = 0$; and $\Theta_4(p, x) = p H(p, x) + x$ solves (6.9) with $C = 0$. It is clear that, for $j = 1, 2, 3, 4$, $\Theta_j(p, \cdot)$ has period $2\pi$. Now we consider the signs of $\Theta_j'$. Since $\Theta_1 = H_I'$, and $H_I(p, \cdot)$ is differentiable on $\mathbb{R}$ with period $2\pi$, so we can not expect that $\Theta_1(p, \cdot)$ does not vanish anywhere on $\mathbb{R}$. For the signs of $\Theta_j'$, $j = 2, 3, 4$, we have the following lemma.

**Lemma 6.1** For any $p \in (0, \infty)$ and $x \in \mathbb{R}$, $\Theta_3(p, x) > 0$. For any $p \in (0, \infty)$ and $x \in \mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$, $\Theta_2(p, x) < 0$, and $\Theta_4(p, x) < 0$.

**Proof.** From (2.12) we have
\[
H'(z) = \sum_{n \text{ even}} \frac{2 e^{np} e^{iz}}{(e^{np} - e^{iz})^2}
\] (6.10)
From (3) in [23], we have
\[
H(p, z) = i \pi H\left(\frac{\pi}{p}, \frac{\pi}{p} + \frac{z}{p}\right) - \frac{z}{p}
\] (6.11)
Since $H_I(p, z) = -i + H(p, z - ip)$, so we have
\[
H_I(p, z) = i \pi H\left(\frac{\pi}{p}, \frac{\pi}{p} + \frac{z}{p}\right) - \frac{z}{p}
\] (6.12)
From (6.11) and (6.12) we have

\[
H'(p, z) = -\pi^2 p^2 H'\left(\frac{\pi^2}{p}, -\frac{\pi}{p} \right) - \frac{1}{p};
\]  

(6.13)

\[
H'_1(p, z) = -\pi^2 p^2 H'\left(\frac{\pi^2}{p}, \pi + i \frac{\pi}{p} \right) - \frac{1}{p}.
\]  

(6.14)

From (6.10) and (6.14), for any \( p \in (0, \infty) \) and \( x \in \mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\} \), we have

\[
\Theta'_4(p, x) = pH'(p, x) + 1 = -\pi^2 p H'\left(\frac{\pi^2}{p}, i \frac{\pi}{p} \right) = -\pi^2 p \sum_{n \text{ even}} \frac{2 e^{n \frac{\pi^2}{p}} e^{-\frac{\pi x}{p}}}{e^{n \frac{\pi x}{p} + e^{\frac{\pi x}{2} p}} < 0;}
\]

\[
\Theta_2(p, x) = (\Theta'_4(p, x) - 1)/p < 0.
\]

From (6.10) and (6.14), for any \( p \in (0, \infty) \) and \( x \in \mathbb{R} \), we have

\[
\Theta_5(p, x) = pH'_1(p, x) + 1 = -\pi^2 p H'\left(\frac{\pi^2}{p}, \pi + i \frac{\pi}{p} \right) = \pi^2 p \sum_{n \text{ even}} \frac{2 e^{n \frac{\pi^2}{p}} e^{-\frac{\pi x}{p}}}{e^{n \frac{\pi x}{p} + e^{\frac{\pi x}{2} p}} > 0. \]

(6.15)

We will find more solutions of (6.9). Define

\[
\Theta_5(p, z) = H(2p, z) - H_I(2p, z);
\]

\[
\Theta_6(p, z) = \frac{1}{2} H\left(\frac{p}{2}, \frac{z}{2}\right) - \frac{1}{2} H\left(\frac{p}{2}, \frac{z}{2} + \pi\right);
\]

\[
\Theta_7(p, z) = \frac{1}{2} H\left(\frac{p}{2}, \frac{z}{2}\right) - \frac{1}{2} H\left(\frac{p}{2}, \frac{z}{2} - \pi\right) - \frac{1}{2} H\left(\frac{p}{2}, \frac{z}{2} + \pi\right) + \frac{1}{2} H_I\left(\frac{p}{2}, \frac{z}{2} + \pi\right).
\]

Note that \( \Theta_5(p, \cdot) = (\Theta_4(2p, \cdot) - \Theta_3(2p, \cdot))/2p \), so from Lemma 6.1 we have \( \Theta_5(p, \cdot) < 0 \) on \( \mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\} \). From an earlier discussion, we can conclude the following proposition.

**Proposition 6.2** Suppose \( \kappa = 2 \). Then \( \Lambda_1 = 2\Theta''_3/\Theta'_3 \) is a crossing annulus drift function that solves (4.4); \( \Lambda_2 = 2\Theta''_2/\Theta'_2 \), \( \Lambda_3 = 2\Theta''_4/\Theta'_4 \), and \( \Lambda_4 = 2\Theta''_5/\Theta'_5 \) are chordal-type annulus drift functions that solve (4.58).

Fix some \( p > 0 \). Let \( L_p = \{2n\pi + i2kp : n, k \in \mathbb{Z}\} \). Let \( F_{5,p} \) denote the set of odd analytic functions \( f \) on \( \mathbb{C} \setminus L_p \) such that each \( z \in L_p \) is a simple pole of \( f \), \( 2\pi \) is a period of \( f \), and \( i2p \) is an inverse period of \( f \), i.e., \( f(z + i2p) = -f(z) \). Let \( F_{6,p} \) denote the set of odd analytic functions \( f \) on \( \mathbb{C} \setminus L_p \) such that each \( z \in L_p \) is a simple pole of \( f \), \( 2\pi \) is a period of \( f \), and \( i2p \) is a period of \( f \). Let \( F_{7,p} \) denote the set of odd analytic functions \( f \) on \( \mathbb{C} \setminus L_p \) such that each \( z \in L_p \) is a simple pole of \( f \), and both \( 2\pi \) and \( i2p \) are inverse periods of \( f \). From the properties of \( H \) and \( H_I \), it is easy to check that \( \Theta_j(p, \cdot) \in F_{j,p} \), \( j = 5, 6, 7 \).
So for reflection function about $\Theta$ spanned by $\Theta$ vertical lines $n\pi$ conformally onto $\mathbb{C}$. From the periodicity of $g$, every $z \in L_p$ is a removable pole of $g$. So $g$ must be a constant. Since $g$ is odd, so the constant is 0. Thus, $f = C\Theta(p, \cdot)$. So $F_{5,p}$ is the linear space spanned by $\Theta(p, \cdot)$. Similarly, $F_{j,p}$ is the linear space spanned by $\Theta_j(p, \cdot)$, $j = 6.7$. We have the following lemma.

**Lemma 6.2** For $j = 5, 6, 7$, $\Theta_j$ solves (6.9) for some function $C$ depending only on $p$.

**Proof.** For $j = 5, 6, 7$, Define

$$J_j = \Theta_j - \Theta_j'' / \Theta_j', \quad C_j(p) = \frac{1}{2} \text{Res}_{z=0} J_j(p, \cdot).$$

We first consider the case that $j = 5$. Fix $p > 0$. Since 0 is a simple pole of $\Theta_5(p, \cdot)$, so from (2.13), it is easy to conclude that 0 is also a simple pole of $J_5(p, \cdot)$. From that $\Theta_5(p, \cdot) \in F_{5,p}$, and that $H(p, \cdot)$ has period 2$\pi$, and $H(p, z+2\pi) = H(p, z) - 2i$, it is easy to check that $J_5(p, \cdot) \in F_{5,p}$ as well. So $J_5(p, \cdot) = C_5(p)\Theta_5(p, \cdot)$. Thus, $\Theta_5$ solves (6.8) with $C = C_5$. Similarly, $\Theta_6$ and $\Theta_7$ solve (6.8) with $C = C_6$ and $C = C_7$, respectively. \(\square\)

For the signs of $\Theta_j$, $j = 5, 6, 7$, we make the following observations. Consider a conformal map $W$ from the rectangle $\{x+iy : 0 < x < \pi, 0 < y < p\}$ onto the forth quadrant $\{x+iy : x > 0, y < 0\}$ such that $W(0) = \infty$ and $W(\pi) = 0$. Then $W(\pi + ip)$ and $W(ip)$ are pure imaginary, and $0 > \text{Im } W(\pi + ip) > \text{Im } W(ip)$. Applying Schwarz reflection principle to reflections about vertical lines $n\pi + i\mathbb{R}, n \in \mathbb{Z}$, we may extend $W$ to an analytic function on $\mathbb{S}_p$. Note that the reflection function about $i\mathbb{R}$ is $z \mapsto -\overline{z}$, and the reflection function about $\pi + i\mathbb{R}$ is $z \mapsto 2\pi - \overline{z}$. So now $W$ satisfies $W(-\overline{z}) = -\overline{W(z)}$ and $W(2\pi - \overline{z}) = -\overline{W(z)}$. Thus, $W$ has period $2\pi$, takes real values on $\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$, and takes pure imaginary values on $\mathbb{R}_p$.

Applying Schwarz reflection principle to reflections about horizontal lines $\mathbb{R}_k$, $k \in \mathbb{Z}$, we may now extend $W$ to an analytic function on $\mathbb{C} \setminus L_p$. Since $W$ takes real values on $\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$, and the reflection function about $\mathbb{R}$ is $z \mapsto -\overline{z}$, so $W$ satisfies $W(\overline{z}) = W(z)$. Since $W$ takes pure imaginary values on $\mathbb{R}_p$, and reflection functions about $\mathbb{R}$ and $i\mathbb{R}$ are $z \mapsto i2p + \overline{z}$ and $z \mapsto -\overline{z}$, respectively, so $W$ satisfies $W(i2p + \overline{z}) = -\overline{W(z)}$. Then we can see that $i2p$ is an inverse period of $W$. Since $W$ satisfies $W(-\overline{z}) = -\overline{W(z)}$ and $W(2\pi - \overline{z}) = -\overline{W(z)}$ on $\mathbb{S}_p$, so $W$ still satisfies these equalities on $\mathbb{C} \setminus L_p$. Thus, $2\pi$ is a period of $W$. From $W(\overline{z}) = W(z)$ and $W(-\overline{z}) = -\overline{W(z)}$ we see that $W$ is odd. So $W \in F_{5,p}$. Thus, $W = C_W\Theta_5(p, \cdot)$, where $C_W = \text{Res}_{z=0} W(z)/2$. From the value of $W$ in $\{x+iy : 0 < x < \pi, 0 < y < p\}$, we see that $C_W > 0$. WLOG, we may assume that $C_W = 1$, so $W = \Theta_5(p, \cdot)$. Since $\Theta_5(p, \cdot) = W$ maps $[ip, \pi + ip]$ onto a closed interval $I \subset \{iy : y < 0\}$, so after reflection, $\Theta_5(p, \cdot)$ maps $\mathbb{R}_p$ onto $I$. Thus, $\Theta_5(p, \cdot)$ takes pure imaginary values on $\mathbb{R}_p$, and $\text{Im } \Theta_5(p, \cdot) < 0$ on $\mathbb{R}_p$.

Similarly, we can conclude that both $\Theta_6(p, \cdot)$ and $\Theta_7(p, \cdot)$ map $\{x+iy : 0 < x < \pi, 0 < y < p\}$ conformally onto $\{x+iy : x > 0, y < 0\}$, and satisfy $\Theta_6(p, 0) = \infty$, $0 = \Theta_6(p, ip) < \Theta_6(p, \pi + ip)$, $\Theta_7(p, 0) = \infty$, $0 = \Theta_7(p, \pi + ip) < \Theta_7(p, \pi)$, $\Theta_7(p, ip) \in i\mathbb{R}$, and $\text{Im } \Theta_7(p, ip) < 0$. So for $j = 6, 7$, $\Theta_j(p, \cdot) > 0$ on $(0, \pi]$. After reflection, we have $\Theta_j(p, \cdot) > 0$ on $(0, 2\pi)$. Since
$2\pi$ is an inverse period of $\Theta_j(p, \cdot)$, so for any $n \in \mathbb{Z}$, $\Theta_j(p, \cdot) > 0$ on $(4n\pi, (4n + 2)\pi)$ and $\Theta_j(p, \cdot) < 0$ on $((4n - 2)\pi, 4n\pi)$. In summary, we have the following lemma.

**Lemma 6.3** For any $p > 0$, $i\Theta_5(p, \cdot) > 0$ on $\mathbb{R}_+$; for $j = 6, 7$ and $n \in \mathbb{Z}$, $\Theta_j(p, \cdot) > 0$ on $(4n\pi, (4n + 2)\pi)$ and $\Theta_j(p, \cdot) < 0$ on $((4n - 2)\pi, 4n\pi)$.

Now we consider the case that $\kappa = 3$. Let $\hat{\Gamma}_1 = \Theta_5$, $\hat{\Gamma}_2 = \Theta_6$, $\hat{\Gamma}_3 = \Theta_7$, and $\hat{\Gamma}_j(p, z) = \hat{\Gamma}_{j-3}(p, z + ip)$, $j = 4, 5, 6$. We then have the following lemma.

**Lemma 6.4** Let $\kappa = 3$. Then $\hat{\Gamma}_1, \hat{\Gamma}_2, \hat{\Gamma}_3$ solve (6.7) for some function $C$ depending only on $p$; and $\hat{\Gamma}_4, \hat{\Gamma}_5, \hat{\Gamma}_6$ solve (6.7) for some function $C$ depending only on $p$.

**Proof.** This is similar to Lemma 6.2. For $j = 1, 2, 3$, define

$$J_j = \hat{\Gamma}_j - \frac{3}{2}\hat{\Gamma}''_j - \frac{1}{2}H\hat{\Gamma}_j, \quad C_j(p) = \frac{1}{2}\text{Res}_{z=0} J_j(p, \cdot).$$

For $j = 1$, we can conclude that $J_1(p, \cdot) \in F_{5,p}$ for any $p > 0$. So $J_1(p, \cdot) = C_1(p)\hat{\Gamma}_1(p, \cdot)$. Thus, $\hat{\Gamma}_1$ solves (6.7) with $C = C_1$. Similarly, for $j = 2, 3$, $\hat{\Gamma}_j$ solves (6.7) with $C = C_j$. For $j = 4, 5, 6$, from the definition of $\hat{\Gamma}_j$, that $\hat{\Gamma}_j - 3$ solves (6.7), and that $H\hat{\Gamma}(p, z) = \hat{H}(p, z - ip) - i$, it is easy to check that $\hat{\Gamma}_j$ solves (6.7) with $C = C_{j-3}$. \[\square\]

From Lemma 6.3, $\hat{\Gamma}_4$ is pure imaginary and does not vanish on $\mathbb{R}$, so $\hat{\Gamma}''_4/\hat{\Gamma}_4$ takes real values on $\mathbb{R}$; $\hat{\Gamma}_2$ and $\hat{\Gamma}_3$ are real valued and does not vanish on $\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$, so $\hat{\Gamma}''_2/\hat{\Gamma}_2$ and $\hat{\Gamma}''_3/\hat{\Gamma}_3$ take real values on $\mathbb{R} \setminus \{2n\pi : n \in \mathbb{Z}\}$. Since $2\pi$ is an inverse period of $\hat{\Gamma}_2$ and $\hat{\Gamma}_3$, so $2\pi$ is a period of $\hat{\Gamma}''_2/\hat{\Gamma}_2$ and $\hat{\Gamma}''_3/\hat{\Gamma}_3$. From an earlier discussion, we can conclude the following proposition.

**Proposition 6.3** Suppose $\kappa = 3$. Then $\Lambda_1 = 3\hat{\Gamma}'_4/\hat{\Gamma}_4$ is a crossing annulus drift function that solves (4.1); $\Lambda_2 = 3\hat{\Gamma}''_2/\hat{\Gamma}_2$ and $\Lambda_3 = 3\hat{\Gamma}''_3/\hat{\Gamma}_3$ are chordal-type annulus drift functions that solve (4.5).

Now we consider the solutions for some other values of $\kappa$. Let $H_2(p, z) = H(p, z/2)$. From (6.1), we have

$$H_2 = 4H_2 + 2H_2H_2. \quad (6.15)$$

Let $G = H - 2H_2$. Then for each $p > 0$, $G(p, \cdot)$ is an odd analytic function on $C \setminus L_p$, and each $z \in L_p$ is a simple pole of $G$. From $H(p, z + 2\pi) = H(p, z)$ and $H(p, z + 2ip) = H(p, z) - 2i$ we see that both $4\pi$ and $i4p$ are periods of $G(p, \cdot)$. Fix some $p > 0$, define

$$J(z) = \frac{G(p, z)^2}{2} - 2G'(p, z) + 3H'(p, z).$$

Then $J$ is an even analytic function on $C \setminus L_p$, and has periods $4\pi$ and $i4p$. Fix any $z_0 = 2n_0\pi + i2k_0p \in L_p$ for some $n_0, k_0 \in \mathbb{Z}$. Then $2z_0$ is a period of $J$, so $J_{z_0}(z) := J(z - z_0)$ is
Thus, \( \text{Res}_{z=2\nu} J(z) = 0 \). The degree of \( z_0 \) as a pole of \( J \) is at most 2. The principal part of \( J \) at \( z_0 \) is \( \frac{C(z_0)}{z-z_0} \) for some \( C(z_0) \in \mathbb{C} \). The principal part of \( G(p, \cdot) \) at \( z_0 \) is \( \frac{C(z_0)}{z-z_0} \) depending only on \( p \), and the principal part of \( H(p, \cdot) \) at \( z_0 \) is always \( \frac{2}{z-z_0} \). We can compute that either \( C(z_0) = \frac{(-6)^2}{2} - 2(6) + 3(-2) = 0 \) or \( C(z_0) = \frac{2^2}{2} - 2(-2) + 3(-2) = 0 \). Thus, every \( z_0 \in L_p \) is a removable pole of \( J \), which together with the periods \( 4\pi \) and \( i4p \) implies that \( J \) is a constant depending only on \( p \). Differentiating \( J \) w.r.t. \( z \), we conclude that

\[
2G'' = G'G + 3H''.
\]

(6.16)

From \( G = H - 2H_2 \) we have \( 2H_2 = H - G \). So from (6.15) and (6.16), we have

\[
\dot{H} - \dot{G} = 8H''' + 4H'G_2 = 4H'' - 4G'' + (H' - G')(H - G)
\]

\[
= 4H'' - 2(G'G + 3H'') + (H' - G')(H - G) = -2H'' - G'G + H'\dot{H} - G'H - H'G.
\]

From the above formula and (6.1), we have

\[
\dot{G} = 3H'' + G'G + H'G + G'H.
\]

(6.17)

Thus, \( G \) solves (4.58) when \( \kappa = 0 \). Note that \( H_I(p, z/2) \) also satisfies (6.15). Let \( G_I(p, z) := H(p, z) - H_I(p, z/2) \). Then \( G_I(p, \cdot) \) is also an odd analytic function on \( \mathbb{C} \setminus L_p \) and have periods \( 4\pi \) and \( i4p \). The principal part of \( G_I(p, \cdot) \) at every \( z_0 \in L_p \) is also either \( \frac{6}{z-z_0} \) or \( \frac{2}{z-z_0} \). Using a similar argument, we conclude that \( G_I \) also solves (4.58) when \( \kappa = 0 \).

Now let \( F = -G/3 \). Then \( G = -3F \). From (6.16) we have

\[
0 = 2F'' + 3F'F + \dot{H}''.
\]

(6.18)

From (6.17) we have

\[
\dot{F} = -H'' - 3F'F + H'F + F'\dot{H}.
\]

(6.19)

From \( \frac{4}{3} = (6.18) + (6.19) \), we get

\[
\dot{F} = \frac{8}{3}F'' + \frac{1}{3}H'' + H'F + F'\dot{H} + F'F.
\]

\[
\dot{F} = \frac{16}{3}F''.
\]

Thus, \( F \) solves (4.58) when \( \kappa = \frac{16}{3} \). Similarly, \( F_I = -G_I/3 \) also satisfies (4.58) when \( \kappa = \frac{16}{3} \). Note that \( F, G, F_I, G_I \) all have period \( 4\pi \) instead of \( 2\pi \) in the second variable.

Suppose for \( p > 0 \) and \( 1 \leq j \leq 8 \), \( \Lambda_j(p, \cdot) \) are functions defined on \( \mathbb{R} \setminus \{2n\pi\} \) with period \( 2\pi \) such that, \( \Lambda_1(p, \cdot), \Lambda_3(p, \cdot), \Lambda_5(p, \cdot), \) and \( \Lambda_7(p, \cdot) \) are the extensions of the restrictions of \( G(p, \cdot), G_I(p, \cdot), F(p, \cdot), \) and \( F_I(p, \cdot) \) to \( (0, 2\pi) \), respectively; and \( \Lambda_2(p, \cdot), \Lambda_4(p, \cdot), \Lambda_6(p, \cdot), \) and \( \Lambda_8(p, \cdot) \) are the extensions of the restrictions of \( G(p, \cdot), G_I(p, \cdot), F(p, \cdot), \) and \( F_I(p, \cdot) \) to \( (-2\pi, 0) \), respectively. Since \( H \) has period \( 2\pi \) in the second variable, so we have the following proposition.

**Proposition 6.4** When \( \kappa = 0 \), \( \Lambda_1, \Lambda_2, \Lambda_3, \) and \( \Lambda_4 \) are chordal-type annulus drift functions that solve (4.58). When \( \kappa = \frac{16}{3}, \Lambda_5, \Lambda_6, \Lambda_7, \) and \( \Lambda_8 \) are chordal-type annulus drift functions that solve (4.58).
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