POLYNOMIALLY SPECTRUM-PRESERVING MAPS BETWEEN COMMUTATIVE BANACH ALGEBRAS

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Abstract. Let $A$ and $B$ be unital semi-simple commutative Banach algebras. In this paper we study two-variable polynomials $p$ which satisfy the following property: a map $T$ from $A$ onto $B$ such that the equality

$$\sigma(p(Tf, Tg)) = \sigma(p(f, g)), \quad f, g \in A$$

holds is an algebra isomorphism.

1. Introduction

The study of spectrum-preserving linear maps between Banach algebras dates back to Frobenius [3] who studied linear maps on matrix algebras which preserve the determinant. After over 100 years spectrum-preserving maps are studied for Banach algebras and the following conjecture seems to be still open: any spectrum-preserving linear map from a unital Banach algebra onto a unital semi-simple Banach algebra that preserves the unit is a Jordan morphism. The Gleason, Kahane and Želazko theorem [5, 11, 22] asserts that a unital linear functional defined on a Banach algebra is multiplicative if it is invertibility preserving and the theorem has inspired a number of papers on the subjects. For commutative Banach algebras it is a straightforward conclusion of the theorem of Gleason, Kahane and Želazko that a unital and spectrum-preserving linear map from a Banach algebra into a semi-simple commutative Banach algebra is a homomorphism. Thus the problems on spectrum-preserving linear maps mainly concerns with non-commutative Banach algebras and has seen much progress recently [1, 9, 15, 20].

Without assuming linearity, non-multiplicative and invertibility-preserving maps are almost arbitrary, and spectrum-preserving maps which

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are not linear nor multiplicative are also possible even in the case of commutative Banach algebras. On the other hand, spectrum-preserving maps on Banach algebras which are not assumed to be linear are studied by several authors [6, 7, 8, 10, 13, 16, 17, 18, 19] recently. In this paper we study linearity and multiplicativity of spectrum-preserving maps between commutative Banach algebras under additional assumptions.

Let $A$ and $B$ be unital Banach algebras. Suppose that $S$ is an algebra isomorphism from $A$ onto $B$. Then we have that the equality

$$\sigma(p(Tf)) = \sigma(p(f)), \quad f \in A$$

holds for every polynomial $p$, where $\sigma(\cdot)$ denotes the spectrum. But the converse does not hold in general. Suppose that $X$ is a compact Hausdorff space and $C(X)$ denotes the algebra of all complex-valued continuous functions on $X$. For each $f \in C(X)$, $\pi_f$ denotes a self homeomorphism on $X$. Put a map $T$ from $C(X)$ into itself by

$$Tf = f \circ \pi_f$$

for every $f \in C(X)$. Then $T$ may not be linear nor multiplicative while

$$\sigma(p(Tf)) = \sigma(p(f)), \quad f \in C(X)$$

holds for every polynomial. But the situation is very different for polynomials of two variables. In this paper we show that for certain two-variable polynomials $p(z, w)$ the following holds: a map $T$ from a unital semi-simple commutative Banach algebra $A$ onto another one $B$ is an algebra isomorphism if the equation

$$\sigma(p(Tf, Tg)) = \sigma(p(f, g)), \quad f, g \in A$$

holds.

2. PRELIMINARY

Let $X$ be a compact Hausdorff space. The algebra of all complex-valued continuous functions on $X$ is denoted by $C(X)$. For a subset $K$ of $X$ the uniform norm on $K$ is denoted by $\| \cdot \|_{\infty(K)}$. A uniform algebra on $X$ is a uniformly closed subalgebra of $C(X)$ which separates the points of $X$ and contains the constant functions. For a uniform algebra $A$ on $X$, $P(A)$ denotes the set of all peaking functions in $A$. The set of all weak peak points for $A$ is the Choquet boundary and denoted by $\text{Ch}(A)$. See [2, 4] for theory of uniform algebras. Let $A$ be a commutative Banach algebra. We denote the maximal ideal space of $A$ by $M_A$ and the Gelfand transformation of $f \in A$ is denoted by $\hat{f}$. 

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The spectral radius for \( f \in A \) is denoted by \( r(f) \) and the spectrum of \( f \) is denoted by \( \sigma(f) \). The complex number field is denoted by \( \mathbb{C} \).

3. A conclusion of a theorem of Kowalski and Słodkowski

Kowalski and Słodkowski [10] proved the following surprising generalization of a theorem of Gleason, Kahane and Želazko.

**Theorem 3.1.** Let \( A \) be a Banach algebra and \( \phi \) a complex-valued map defined on \( A \). Suppose that

\[
\varphi(f) - \varphi(g) \in \sigma(f - g)
\]

holds for every pair \( f \) and \( g \) in \( A \). Then \( \varphi - \varphi(0) \) is linear and multiplicative.

Applying the above theorem we see the following.

**Theorem 3.2.** Let \( A \) be a Banach algebra and \( B \) a semi-simple commutative Banach algebra, and \( p(z, w) = az + bw \ (ab \neq 0) \). Suppose that \( T \) is a (not necessarily linear) map from \( A \) into \( B \) which satisfies that the inclusion

\[
\sigma(p(Tf, Tg)) \subset \sigma(p(f, g))
\]

holds for every pair \( f \) and \( g \) in \( A \). Then we have the following.

1. If \( a + b \neq 0 \), then \( T \) is linear and multiplicative.
2. If \( a + b = 0 \), then \( T - T(0) \) is linear and multiplicative.

**Proof.** First we show that

\[
\sigma(Tf - Tg) \subset \sigma(f - g), \quad f, g \in A
\]

holds. Let \( f, g \in A \). Since \( a \neq 0 \), we have

\[
\sigma(Tf + \frac{b}{a} Tg) \subset \sigma(f + \frac{b}{a} g),
\]

so

\[
\sigma(T(-\frac{b}{a} g) + \frac{b}{a} Tg) \subset \sigma(-\frac{b}{a} g + \frac{b}{a} g) = \{0\}
\]

by putting \( f = -\frac{b}{a} g \). Thus the equality

\[
T(-\frac{b}{a} g) = -\frac{b}{a} Tg
\]

holds for every \( g \in A \) since \( B \) is semi-simple. It follows that

\[
\sigma(Tf - Tg) = \sigma(Tf - T(-\frac{b}{a}(-\frac{a}{b} g)) = \sigma(Tf + \frac{b}{a} T(-\frac{a}{b} g)) \subset \sigma(f - g)
\]

holds for every pair \( f \) and \( g \) in \( A \).
Put a map $S$ from $A$ into $B$ by $Sf = Tf - T(0)$. Then $S$ is surjective and
\[
\sigma(Sf - Sg) \subset \sigma(f - g)
\]
holds for every pair $f$ and $g$ in $A$. We show that $S$ is linear and multiplicative. Let $\phi \in M_B$ be chosen arbitrarily. Then
\[
\phi \circ S : A \to \mathbb{C},
\]
and
\[
\phi \circ S(0) = 0,
\]
and
\[
\phi \circ S(f) - \phi \circ S(g) = \phi(Sf - Sg) \in \sigma(Sf - Sg) \subset \sigma(f - g)
\]
holds for every pair $f$ and $g$ in $A$. Thus by a theorem of Kowalski and Slodkowski we have that $\phi \circ S$ is linear and multiplicative for every $\phi \in M_B$. Then conclusion follows immediately since $B$ is semi-simple.

We show that $T(0) = 0$ if $a + b \neq 0$. Putting $f = g = 0$ we have
\[
\sigma(aT(0) + bT(0)) \subset \sigma(a \cdot 0 + b \cdot 0) = \{0\}.
\]
Thus we have $T(0) = 0$ if $a + b \neq 0$. \hfill \Box

4. A theorem of Molnár and its generalizations

On the other hand Molnár [14] proved the following.

**Theorem 4.1.** (Molnár) Let $X$ be a first countable compact Hausdorff space. Suppose that $T$ is a map from $C(X)$ onto itself such that the equality
\[
\sigma(TfTg) = \sigma(fg)
\]
holds for every pair $f$ and $g$ in $C(X)$. Then there exist a continuous function $\eta : X \to \{-1, 1\}$ and a self-homeomorphism $\Phi$ on $X$ such that the equality
\[
Tf = \eta f \circ \Phi
\]
holds for every $f \in C(X)$. In particular, $T$ is an algebra isomorphism if $T1 = 1$.

Motivated by the above theorems and others we may consider the following question: let $A$ and $B$ be Banach algebras and $p$ a polynomial of two variables. Suppose that $T$ is a map from $A$ into $B$ such that the inclusion
\[
\sigma(p(Tf, Tg)) \subset \sigma(p(f, g))
\]
holds for every pair $f$ and $g$ in $A$. Does it follow that $T$ is linear and multiplicative? A theorem of Kowalski and Slodkowski states that it is the case for $B = \mathbb{C}$ and $p(z - w) = z - w$. On the other hand there
several negative answers to the above too general question (see \cite{6}).
Even the polynomial $p$ need some restriction for a positive answer.

**Example 4.2.** Let $X$ be a compact Hausdorff space. For each $f \in C(X)$, put $\varepsilon_f = 1$ or $-1$. Then the map $T$ from $C(X)$ into itself defined by

$$Tf = \varepsilon_ff, \quad f \in C(X)$$

can be non-linear nor multiplicative but surjective. Put $p(z, w) = z^2 + w^2$. Then the equality

$$\sigma(p(Tf, Tg)) = \sigma(p(f, g)), \quad f, g \in C(X)$$

holds.

One of the reasonable questions may be as follows.

**Question.** Let $A$ and $B$ be unital semi-simple commutative Banach algebras. Characterize the two-variable polynomials $p$ which satisfy the following property: a map $T$ from $A$ onto $B$ such that the equality

$$\sigma(p(Tf, Tg)) = \sigma(p(f, g)), \quad f, g \in A$$

holds is an algebra isomorphism.

A theorem of Molnár gives a positive answer to the question, namely if $A = B = C(X)$, then $p(z, w) = zw$ is a desired polynomial. Theorem 3.2 states that for a Banach algebra $A$ and a semi-simple commutative Banach algebra $B$ $p(z, w) = az + bw$ is a desired polynomial. If a type of a theorem of Kowalski and Sładkowski for $p(z, w) = zw$ were true, positive results would follow for various Banach algebras with $p(z, w) = zw$. Unfortunately it is not the case; A modified theorem does not hold. On the other hand Molnár \cite{14} also proved a positive results for the Banach algebra of all bounded operators on an infinite-dimensional Hilbert space.

Rao and Roy \cite{18} generalized a theorem of Molnár for uniform algebras on the maximal ideal spaces and Hatori, Miura and Takagi \cite{7} generalized for semi-simple commutative Banach algebras. For the case of uniform algebras, Hatori, Miura and Takagi \cite{6} considered the equality of the range instead of that of the spectrum and show a generalization of a theorem of Molnár. Luttman and Tonev \cite{13} consider the equation for more smaller set; the peripheral range. Let $A$ be a uniform algebra on a compact Hausdorff space $X$. For $f \in A$, the peripheral range $Ran_\pi(f)$ for $f \in A$ is denoted by

$$Ran_\pi(f) = \{z \in f(X) : |z| = \|f\|_{\infty(X)}\}.$$
Note that the peripheral range for uniform algebras coincides with the peripheral spectrum $\sigma_\pi(f)$;
\[ \sigma_\pi(f) = \{ z \in \sigma(f) : |z| = r(f) \}, \]
where $r(f)$ is the spectral radius. Luttman and Tonev proved the following.

**Theorem 4.3.** *Luttman and Tonev* Let $A$ and $B$ be uniform algebras on compact Hausdorff spaces $X$ and $Y$ respectively. Suppose that $T$ is a map from $A$ onto $B$ such that the equality
\[ \text{Ran}_\pi(TfTg) = \text{Ran}_\pi(fg) \]
holds for every pair $f$ and $g$ in $A$. Then there exist a function $\eta : M_B \to \{-1, 1\}$ and a homeomorphism $\Phi$ from $M_B$ onto $M_A$ such that the equality
\[ \widehat{Tf}(y) = \eta(y)\hat{f} \circ \Phi(y), \quad y \in M_B \]
holds for every $f \in A$, where $\hat{\cdot}$ denotes the Gelfand transform. In particular, $T$ is an algebra isomorphism if $T1 = 1$.

5. Main results

**Theorem 5.1.** Let $A$ and $B$ be uniform algebras on compact Hausdorff spaces $X$ and $Y$ respectively. Let $p(z, w) = zw + az + bw + ab$ be a polynomial. Suppose that $T$ is a map from $A$ onto $B$ such that the equality
\[ \text{Ran}_\pi(p(Tf, Tg)) = \text{Ran}_\pi(p(f, g)) \]
holds for every pair $f$ and $g$ in $A$. Then we have the following.

1. If $a \neq b$, then $T$ is an algebra isomorphism. Thus there exists a homeomorphism from $M_B$ onto $M_A$ such that
\[ \widehat{Tf}(y) = \hat{f} \circ \Phi(y), \quad y \in M_B \]
holds for every $f \in A$.

2. If $a = b$, then there exist a continuous map $\eta : M_B \to \{-1, 1\}$ and a homeomorphism $\Phi$ from $M_B$ onto $M_A$ such that the equality
\[ \widehat{Tf}(y) = \eta(y)\hat{f} \circ \Phi(y) + a(\eta(y) - 1), \quad y \in M_B \]
holds for every $f \in A$.

The author does not know a similar result as Theorem 5.1 holds for $p(z, w) = zw + az + bw + ab$ ($ab \neq c$). In general for several polynomials a similar result as Theorem 5.1 does not hold. For example let $p(z, w) = z^2 + w^2$. Let $X$ be a disconnected compact Hausdorff space and $A =$
$B = C(X)$. For each $f \in A$, $\eta_f$ is a map from $X$ into $\{-1, 1\}$. Put a map $T$ from $A$ into $B$ by

$$T f = \eta f f, \quad f \in A.$$ 

Then we have

$$\text{Ran}_\pi(p(T f, T g)) = \text{Ran}_\pi(p(f, g))$$

holds for every pair $f$ and $g$ in $A$. On the other hand $T$ may be surjective but non-linear nor multiplicative according to the choice of $\eta_f$.

**Proof.** Put a map $S : A \to B$ by

$$S f = T(f - b) + b, \quad f \in A.$$ 

By a simple calculation we see that $S(A) = B$ and

$$\text{Ran}_\pi(S(f)(S(g) + c)) = \text{Ran}_\pi(f(g + c))$$

holds for every pair $f, g \in A$, where $c = a - b$.

If $a = b$, then by a theorem of Luttman and Tonev [13] we see that there is a continuous function $\eta : M_B \to \{-1, 1\}$ and a homeomorphism from $M_B$ onto $M_A$ such that

$$\widehat{S f}(y) = \eta(y) \hat{f} \circ \Phi(y), \quad y \in M_B$$

holds for every $f \in A$. It follows that

$$\widehat{T f}(y) = \eta(y) \hat{f} \circ \Phi(y) + a(\eta(y) - 1) \quad y \in M_B$$

holds for every $f \in A$.

Suppose that $a \neq b$. We show that $S$ is an isometric algebra isomorphism. First we show that $S$ is injective. To this end suppose that $S f = S g$. Then for every $h \in A$ we have

$$\text{Ran}_\pi(f h) = \text{Ran}_\pi(S(f)(S(h - c) + c)) = \text{Ran}_\pi(S(g)(S(h - c) + c)) = \text{Ran}_\pi(gh).$$

Then by a routine argument applying peaking function argument we see that $f = g$. By putting $g = -c$ and $f \in A$ with $S f = 1$ in the equation [5.1] we have

$$\{0\} = \text{Ran}_\pi(f(-c + c)) = \text{Ran}_\pi(S(-c) + c),$$

so we have $S(-c) = -c$. Let $\lambda$ be an arbitrary complex number. Then we have

$$\lambda \text{Ran}_\pi(-c f) = \text{Ran}_\pi(\lambda(-c)f) = \text{Ran}_\pi(S(\lambda(-c))(S(f-c)+c))$$

(5.3)
and
\( \lambda \text{Ran}_\pi(-cf) = \lambda \text{Ran}_\pi(S(-c)(S(f - c) + c)) \)

\( = \text{Ran}_\pi(\lambda S(-c)(S(f - c) + c)) = \text{Ran}_\pi((-\lambda c)(S(f - c) + c)) \)

since \( S(-c) = -c \). By a simple calculation

\( B = \{S(f - c) + c : f \in A\} \)

holds, and thus for every \( G \in B \) we have

\( \text{Ran}_\pi(-\lambda cG) = \text{Ran}_\pi(S(-\lambda c)G) \)

holds by the equations \ref{eq:5.3} and \ref{eq:5.4}. It follows that

\( -\lambda c = S(-\lambda c) \)

holds and so

\( \lambda = S(\lambda) \)

holds for every complex number \( \lambda \) since \( c \neq 0 \).

Next let \( f \in A \). Then

\( \text{Ran}_\pi(f) = \text{Ran}_\pi(S(1)(S(f - c) + c)) = \text{Ran}_\pi(S(f - c) + c). \)

We also see that

\( \text{Ran}_\pi(f) = \text{Ran}_\pi(S(f)(S(1 - c) + c) = \text{Ran}_\pi(Sf) \)

since \( S(1 - c) = 1 - c. \)

Next let \( P(A) \) be the set of all peaking functions in \( A \). Then we see that

\( S(P(A)) = P(B). \)

Let \( f \in P(A) \). Then \( Tf \in P(B) \) since

\( \{1\} = \text{Ran}_\pi(f) = \text{Ran}_\pi(Sf). \)

Note that \( f \) is a peaking function if and only if \( \text{Ran}_\pi(f) = \{1\} \). Thus we have that \( S(P(A)) \subset P(B) \) holds and the converse inclusion is proved in the same way since \( S \) is a bijection. We also see by a simple calculation that

\( S(P(A) - c) + c = P(B). \)

This does not prove Theorem \ref{thm:5.1} we can give the rest of the proof as in \cite{6}, so we only sketch the rest of the proof.

For \( f \in P(A) \), put

\( L_f = \{x \in X : f(x) = 1\}. \)

Let \( \text{Ch}(A) \) be the set of all weak peak points for \( A \). We denote for \( x \in \text{Ch}(A) \)

\( P_x(A) = \{f \in P(A) : f(x) = 1\}. \)
Claim 1. Let $f, g \in P(A)$. If $L_T f \subset L_T g$, then we have $L_f \subset L_g$.

We show a proof. In the same way as in the proof of Lemma 2.2 in [6] we see that for every pair $f$ and $g$ in $P(A)$ the inclusion $L_f \subset L_g$ holds if and only if $1 \in \text{Ran}_\pi(ug)$ holds for every $u \in P(A)$ with $1 \in \text{Ran}_\pi(fu)$. Applying this and the equation 5.6 we can prove Claim 1 in a way similar to the proof of Lemma 3.2 in [6].

Claim 2. For every $y \in \text{Ch}(B)$, there exists an $x \in \text{Ch}(A)$ such that $S^{-1}(P_y(B)) \subset P_x(A)$.

We show a proof. Let $f_1, \ldots, f_n$ be a finite number of functions in $S^{-1}(P_y(B))$. We show that

$$\bigcap_{j=1}^n L_{f_j} \neq \emptyset.$$

Since $Sf_j \in P_y(B)$ we see that

$$\prod_{j=1}^n Sf_j \in P_y(B).$$

Since $S(A) = B$, there exists a $g \in A$ with $Sg = \prod_{j=1}^n Sf_j$. Note that $g \in P(A)$ since $Sg \in P_y(B)$. We see that $L_{Sg} \subset L_{Sf_j}$ by the definition for every $j = 1, \ldots, n$. Then by Claim 1 we have that $L_g \subset L_{f_j}$ for every $j = 1, \ldots, n$, and so

$$L_g \subset \bigcap_{j=1}^n L_{f_j}.$$

It follows that $\bigcap_{j=1}^n L_{f_j} \neq \emptyset$ since $g \in P(A)$ and so $L_g \neq \emptyset$. By the finite intersection property we see that

$$L = \bigcap_{f \in S^{-1}(P_y(B))} L_f \neq \emptyset.$$

Since $L$ is a weak peak set for a uniform algebra $A$, there exists an $x \in L \cap \text{Ch}(A)$. It follows that

$$S^{-1}(P_y(B)) \subset P_x(A).$$

Claim 3. For every $y \in \text{Ch}(B)$, there exists a unique $x_y \in \text{Ch}(A)$ such that

$$S(P_{x_y}(A)) = P_y(B).$$

We show a proof. Since $S^{-1}$ is a map from $B$ onto $A$ and the equality

$$\text{Ran}_\pi(S^{-1}(F)(S^{-1}(G) + c)) = \text{Ran}_\pi(F(G + c)), \ F, G \in B$$

holds we can adapt a similar argument as in the proof of Claim 2 for $S^{-1}$ we see that for every $x \in \text{Ch}(A)$ there exists a $y' \in \text{Ch}(B)$ such that

$$S(P_x(A)) \subset P_{y'}(B).$$
Then by Claim 2 we see that for every $y \in \text{Ch}(B)$ there exists an $x \in \text{Ch}(A)$ and so $y' \in \text{Ch}(B)$ such that

$$P_y(B) \subset S(P_x(A)) \subset P_{y'}(B).$$

It follows that $y = y'$ and the uniqueness of $x$ for $y \in \text{Ch}(B)$. We have proved Claim 3.

We continue the proof of Theorem 5.1. Put a map $\phi : \text{Ch}(B) \rightarrow \text{Ch}(A)$ by $\phi(y) = x_y$. Then in a similar way as in the proof of Theorem in [6] we see that the equality

$$(S(f - c) + c)(y) = f \circ \phi(y), \quad y \in \text{Ch}(B)$$

holds for every $f \in A$. Substituting $f$ by $f - c$ we see that

$$S(f)(y) = f \circ \phi(y), \quad y \in \text{Ch}(B).$$

It follows that $S$ is an algebra isomorphism from $A$ onto $B$. Thus by the routine argument of commutative Banach algebras we see that there exist a homeomorphism $\Phi$ from $M_B$ onto $M_A$ such that the equality

$$\hat{S}(f)(y) = \hat{f} \circ \Phi(y), \quad y \in M_B$$

holds for every $f \in A$. Then by the definition of $S$ we see by a simple calculation that the equality

$$\hat{T}(y) = \hat{f} \circ \Phi(y), \quad y \in M_B$$

holds for every $f \in A$.

**Theorem 5.2.** Let $A$ be a unital semi-simple commutative Banach algebra and $B$ a unital commutative Banach algebra. Put $p(z,w) = zw + az + bw + c$, where $a, b$ and $c$ are coefficients. Suppose that $T$ is a map from $A$ onto $B$ such that the equality

$$\sigma(p(Tf, Tg)) = \sigma(p(f, g))$$

holds for every pair $f$ and $g$ in $A$. Then we have the following.

1. If $a \neq b$, then $T$ is an algebra isomorphism. Thus there exists a homeomorphism from $M_B$ onto $M_A$ such that the equality

$$\hat{T}(y) = \hat{f} \circ \Phi(y), \quad y \in M_B$$

holds for every $f \in A$.

2. If $a = b$, then there exist a map $\eta : M_B \rightarrow \{-1, 1\}$ and a homeomorphism $\Phi$ from $M_B$ onto $M_A$ such that the equality

$$\hat{T}(y) = \eta(y)\hat{f} \circ \Phi(y) + a(\eta(y) - 1), \quad y \in M_B$$

holds for every $f \in A$.

In any case we have that $B$ is semi-simple and $A$ is algebraically isomorphic to $B$. 

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Proof. We consider the case where $B$ is semi-simple. (The general case follows from the case where $B$ is semi-simple. Consider the Gelfand transform $\Gamma$ of $B$. Then the composition map $\Gamma \circ T$ is a map from $A$ onto the Gelfand transform $\hat{B}$ of $B$. Then by the first part we see that $\Gamma \circ T$ is injective, which will follow that $\Gamma$ is injective. Thus we see that $B$ is semi-simple and we can deduce the case where $B$ is semi-simple.)

Put a map $S : A \to B$ by

$$S(f) = T(f - b) + b, \quad f \in A.$$  

Then by a simple calculation we see that $S(A) = B$ and the equality

$$\sigma(f(g + c)) = \sigma(S(f)(g + c)), \quad f, g \in A$$

holds, where $c = a - b$.

If $a = b$, then by a proof of Theorem 3.2 in [7] there exist a continuous function $\eta : M_B \to \{-1, 1\}$ and a homeomorphism $\Phi$ from $M_B$ onto $M_A$ such that the equality

$$\hat{S}f(y) = \eta(y) \hat{f} \circ \Phi(y), \quad y \in M_B$$

holds for every $f \in A$. It follows that

$$\hat{T}f(y) = \eta(y) \hat{f} \circ \Phi(y) + a(\eta(y) - 1) \quad y \in M_B$$

holds for every $f \in A$.

Suppose that $a \neq b$. Then by the same way as in the proof of Theorem 5.1 we see that $S(-c) = -c$ and the equality

$$S\lambda = \lambda$$

holds for every complex number $\lambda$.

Claim 1. For every $f \in A^{-1}$, the equality $S(f)(S(f^{-1} - c) + c) = 1$.

We show a proof. Since

$$\{1\} = \sigma(ff^{-1}) = \sigma(S(f)(S(f^{-1} - c) + c)$$

we have

$$S(f)(S(f^{-1} - c) + c) = 1$$

since $B$ is semi-simple. We denote the uniform closure of $\hat{A}$ in $C(M_A)$ by $\text{cl}(A)$, where $C(M_A)$ is the algebra of all complex-valued continuous functions on $M_A$. Note that the maximal ideal space of $\text{cl}(A)$ coincides with $M_A$. In the following the Gelfand transformation of $f$ in $A$ and $\text{cl}(A)$ is denoted also by $f$ for simplicity.

Claim 2. Let $\{f_m\}$ be a sequence in $A^{-1}$ and $f \in C(M_A)$ such that

$$\|f_m - f\|_{\infty(M_A)} \to 0$$
as $m \to \infty$. Then $\{Sf_m\}$ is a Cauchy sequence in $B$ with respect to the uniform norm on $M_B$ and the uniform limit $\lim Sf_m$ is an invertible function in $\text{cl}(B)$.

We show a proof of Claim 2. We may assume that there exists a positive integer $K$ with the inequality
\[
\frac{1}{K} < |f_m(x)| < K, \quad x \in M_A
\]
holds for every positive integer $m$. Note that
\[
\frac{1}{K} < |Sf_m(y)| < K, \quad y \in M_B
\]
holds for every positive integer $m$ since
\[
\sigma(f_m) = \sigma(Sf_m(S(1 - c) + c)) = \sigma(Sf_m)
\]
holds. Then by a simple calculation we see that for every positive $\varepsilon$, there exists a positive integer $N$ such that the inequality
\[
\left| \frac{f_n(x)}{f_m(x)} - 1 \right| < \varepsilon, \quad x \in M_A
\]
holds for every $m, n > N$. Since $Sf_m(S(f_m^{-1} - c) + c) = 1$ we see that
\[
\sigma(f_n f_m^{-1}) = \sigma(Sf_n(S(f_m^{-1} - c) + c) = \sigma(Sf_n(Sf_m^{-1}))
\]
so the inequality
\[
\left| \frac{Sf_n(y)}{Sf_m(y)} - 1 \right| < \varepsilon, \quad y \in M_B
\]
holds for every $m, n > N$. Thus we see that
\[
\|Sf_n - Sf_m\|_{\infty(M_B)} \leq \|Sf_n\|_{\infty(M_B)} \|\frac{Sf_n}{Sf_m} - 1\|_{\infty(M_B)} \leq K\varepsilon
\]
holds for every $m, n > N$, so $\{Sf_m\}$ is a Cauchy sequence with respect to the uniform norm and
\[
\frac{1}{K} \leq |\lim Sf_m| \leq K
\]
on $M_B$, so $\lim Sf_m$ is invertible in $\text{cl}(B)$ since the maximal ideal space of $\text{cl}(B)$ coincides with $M_B$. We have proved Claim 2.

**Claim 3.** Then map $S$ is extended to an injective map $\bar{S}$ from $A \cup (\text{cl}(A))^{-1}$ onto $B \cup (\text{cl}(B))^{-1}$ such that the equality
\[
\text{Ran}(\bar{S}f(\bar{S}g + c)) = \text{Ran}(f(g + c))
\]
holds for every pair $f$ and $g$ in $A \cup (\text{cl}(A))^{-1}$.

We show a proof. Let $f \in (\text{cl}(A))^{-1}$. Note that
\[
(\text{cl}(A))^{-1} = \{f \in \text{cl}(A) : 0 \not\in f(M_A)\}.
\]
since the maximal ideal space of $\text{cl}(A)$ coincides with $M_A$. Then there exists a sequence $\{f_m\}$ in $A$ with
\[ \|f_m - f\|_{\infty(M_A)} \to 0 \]
as $m \to \infty$. We may assume that $f_m \in A^{-1}$. Then by Claim 2 we see that the uniform limit $\lim Sf_m$ exists and it is easy to see that the limit does not depend on the choice of a sequence $\{f_m\}$ which converges to $f$. Put $\bar{S}f = \lim Sf_m$. Then by Claim 2 we see that $\bar{S}f \in (\text{cl}(B))^{-1}$. In this way we can define $\bar{S}$ from $A \cup (\text{cl}(A))^{-1}$ into $B \cup (\text{cl}(B))^{-1}$. By some calculation we see that
\[ \text{Ran}(\bar{S}(f(g + c))) = \text{Ran}(f(g + c)), \quad f, g \in A \cup (\text{cl}(A))^{-1} \]
holds. We also see in the same way as in the proof of Claims 3 and 4 in [6] that $\bar{S}$ is a bijection.

This does not prove the theorem, but the rest of the proof is similar to that of a proof of Theorem 3.2 applying a similar way as in the proof of Theorem [5,11] We omit a precise proof.  \[ \square \]

References

[1] B. Aupetit, Spectrum-preserving linear mapping between Banach algebras or Jordan-Banach algebras, Jour. London Math. Soc., 62(2000)917–924
[2] A. Browder, “Introduction to Function Algebras”, W.A. Benjamin, 1969.
[3] G. Frobenius, Über die Darstellung der endlichen Gruppen durch lineare Substitutionen, Sitzungsber. Deutsch. Akad. Wiss., Berlin, (1897)994–1015
[4] T. W. Gamelin, “Uniform Algebras 2nd ed.”, Chelsea Publishing Company, 1984.
[5] A. M. Gleason, A characterization of maximal ideals, J. Analyse Math., 19(1967)171–172
[6] O. Hatori, T. Miura and H. Takagi, Characterizations of isometric isomorphisms between uniform algebras via non-linear range-preserving properties, Proc. Amer. Math. Soc., to appear
[7] O. Hatori, T. Miura and H. Takagi, Unital and multiplicatively spectrum-preserving surjections between semi-simple commutative Banach algebras are linear and multiplicative, Jour. Math. Anal. Appl., to appear
[8] S. H. Hochwahld, Multiplicative maps on matrices that preserves the spectrum, Linear Algebra and Its Appl., 212/213, 339–351
[9] A. A. Jafarian and A. R. Sourour, Spectrum-preserving linear maps, Jour. Funct. Anal., 66(1987)255-261
[10] S. Kowalski and Z. Slodkowski, A characterization of maximal ideals in commutative Banach algebras, Studia Math., 67(1980)215–223
[11] J.-P. Kahane and W. Żelazko, A characterization of maximal ideals in commutative Banach algebras, Studia Math., 29(1968)339–343
[12] S. Lambert, Algebra isomorphisms and norm multiplicativity, talk at the Fifth Conference on Function Spaces at SIUE 2006
[13] A. Luttman and T. Tonev, Algebra isomorphisms and $\text{RAN}_{\pi}$-multiplicativity, Proc. Amer. Math. Soc., to appear
[14] L. Molnár, *Some characterizations of the automorphisms of $B(H)$ and $C(X)$*, Proc. Amer. Math. Soc., 130 (2002), 111–120
[15] M. Neal, *Spectrum preserving linear maps on JBW*-triples* Arch. Math., 79 (2002), 258–267
[16] T. Petek and P. Šemrl, *Characterization of Jordan Homomorphisms on $M_n$ using preserver properties*, Linear Algebra and Its Appl., 269 (1998), 33–46
[17] T. Ransford, *A Cartan theorem for Banach algebras*, Proc. Amer. Math. Soc., 124 (1996), 243–247
[18] N. V. Rao and A. K. Roy, *Multiplicatively spectrum-preserving maps of function algebras*, Proc. Amer. Math. Soc., 133 (2005), 1135–1142
[19] N. V. Rao and A. K. Roy *Multiplicatively spectrum-preserving maps of function algebras*.II, Proc. Edin. Math. Soc., 48 (2005), 219–229
[20] A. R. Sourour, *Invertibility preserving linear maps on $L(X)$*, Trans. Amer. Math. Soc., 348 (1996)13–30
[21] W. Żelazko, *A characterization of multiplicative linear functionals in complex Banach algebras*, Studia Math., 30 (1968)83–85
[22] W. Żelazko, “Banach Algebras”, Elsevier, 1973

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