Persistence of Hardy’s nonlocality in time

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Hardy’s nonlocality argument, which establishes incompatibility of quantum theory with local-realism, can also be used to reveal the time-nonlocal feature of quantum states. For spin-½ systems, the maximum probability of success of this argument is known to be 25%. We show that this maximum remains 25% for all finite-dimensional quantum systems with suitably chosen observables. This enables a test of the quantum properties of macroscopic systems in analogy to the method of Leggett and Garg.

INTRODUCTION

For testing the existence of superposition of macroscopically distinct quantum states, Leggett and Garg [1] put forward the notion of macrorealism. This notion rests on the classical paradigm [2, 3] that (i) physical properties of a macroscopic object exist independent of the act of observation and (ii) any observable can be measured non-invasively, i.e., the ideal measurement of an observable at any instant of time does not influence its subsequent evolution.

These original assumptions of [1], namely the assumptions of ‘macroscopic realism’ and ‘noninvasive measurability’, have been generalized to derive a temporal version of the Bell-CHSH inequality irrespective of whether the system under consideration is macroscopic or not [4, 5]. Unlike the original Bell-CHSH scenario [6] where correlations between measurement results from two distantly located physical systems are considered, temporal Bell-CHSH inequalities (or its generalizations) are derived by focusing on one and the same physical system and analyzing the correlations between measurement outcomes at two different times. These derivations are based on the following two assumptions: (i) Realism: The measurement results are determined by (possibly hidden) properties, which the particles carry prior to and independent of observation, and (ii) Locality in time: The result of a measurement performed at time t₂ is independent of any ideal measurement performed at some earlier or later time t₁ [4, 7].

These inequalities get violated in Quantum Mechanics and thereby give rise to the notion of entanglement in time which has been a topic of current research interest [8, 9, 10, 11, 12]. Interestingly, the original argument of Hardy, which establishes the incompatibility of Quantum Theory with the notion of local-realism [13, 14], can also be used to reveal this time-nonlocal feature of quantum states [8–10]. Recently, Hardy’s argument was studied in the case of two observable settings at each time of measurement [8, 11]. It was shown there that the maximum probability of success of this argument assumes 25% for a spin-½ particle [8, 9], the experimental verification of the above fact followed soon after in [10].

So far, only spin observables have been considered in the context of the temporal Hardy argument. We here study Hardy’s argument for arbitrary observables of the system and find that the maximum success probability of this argument remains 25% irrespective of the dimension of the system. In addition, we argue that the same success probability can be observed with higher dimensional spin observables. This is in sharp contrast with the findings of reference [8] where for spin observables it has been stated that the maximum probability of success of Hardy’s argument decreases with increase in spin value of the system involved. We also discuss the reason of this discrepancy.

TEMPORAL VERSION OF NONLOCALITY CONDITIONS FOR d-LEVEL SYSTEMS

Consider a single d level physical system on which an observer (Alice) chooses to measure one of two observables $A_{1}$ or $A_{2}$ at time $t_{1}$, whereas at a later time $t_{2}$, another observer (Bob) [13] measures either of the two observables $B_{1}$ and $B_{2}$. Let us refer to an experiment with this setting in what follows as a "Hardy experiment". Consider now the following set of conditions on the probabilities for Alice and Bob to obtain outcomes $a_{i}$ and $b_{j}$ when measuring observables $A_i$ and $B_j$ respectively: $i, j \in \{1, 2\}$ [16]:

\[
\begin{align*}
\text{prob}(A_{1}, a_{1} ; B_{1}, b_{1}) &= 0, \\
\text{prob}(A_{1}, \neg a_{1} ; B_{2}, b_{2}) &= 0, \\
\text{prob}(A_{2}, a_{2} ; B_{1}, \neg b_{1}) &= 0, \\
\text{prob}(A_{2}, a_{2} ; B_{2}, b_{2}) &> 0.
\end{align*}
\]

The first condition says that if Alice chooses to measure the observable $A_{1}$ and Bob chooses observable $B_{1}$, he will not obtain $b_{1}$ as measurement result whenever Alice has detected the measurement value $a_{1}$. The remaining equations can be analyzed in a similar manner ($\neg a_{i}$ denotes a
measurement with any result other than \( a_i \) and similarly \( \neg b_j \) denotes a measurement with any result other than \( b_j \). These four conditions together form the basis of the temporal version of Hardy’s argument for \( d \)-level physical systems. This version of Hardy’s argument makes use of the fact that not all of the conditions \( 1 \)–\( 4 \) can be simultaneously satisfied in a time-local realistic theory, but they can be in quantum mechanics.

In a realistic theory, values are assigned to all the observables (whether or not they are actually measured) in such a manner that they agree with experimental observations. Consider a situation where a realist has been supplied with a table asking for the values of \( \hat{A}_1 \), \( \hat{A}_2 \), \( \hat{B}_1 \) and \( \hat{B}_2 \) in several runs of a Hardy experiment. In order to satisfy the last Hardy condition, he will have to assign \( a_2 \) for \( \hat{A}_2 \) and \( b_2 \) for \( \hat{B}_2 \) a few times. Out of these few times, he cannot choose neither values corresponding to \( \neg a_1 \) for observable \( \hat{A}_1 \) nor values belonging to \( \neg b_1 \) for observable \( \hat{B}_1 \) since these events have zero probabilities according to conditions \( 2 \) and \( 3 \), respectively. However, the alternative values these observables can assume, i.e., \( a_1 \) and \( b_1 \), also lead to a zero probability according to the first Hardy condition \( 1 \). Hence in a realistic theory, for any choice of values of the observables \( \hat{A}_1 \) and \( \hat{B}_1 \) satisfying conditions \( 2 \)–\( 4 \), the probability to obtain \( a_1 \) for \( \hat{A}_1 \) and \( b_1 \) for \( \hat{B}_1 \) cannot be vanishing in contradiction with condition \( 1 \). This is different for example in Quantum Mechanics where the probability distribution depends on the choice of observables if this choice includes operators that do not commute. Therefore, these conditions allow to distinguish whether a system can be described by a realistic theory or not.

**SATISFACTION OF TEMPORAL NONLOCALITY CONDITIONS IN QUANTUM MECHANICS**

To show that the conditions \( 1 \)–\( 3 \) can be simultaneously satisfied in Quantum Mechanics, we consider a quantum mechanical system in a pure state \( |\psi\rangle \) with the associated Hilbert space \( \mathcal{H} \). Moreover, we here restrict to projective measurements of observables \( \hat{A}_i \) (\( \hat{B}_i \)) that can be degenerate (so-called von Neumann–Lüders measurements \( 15 \)). A measurement of \( \hat{A}_i \) with result \( a_i \) projects an initial state \( |\psi\rangle \) onto the corresponding eigenspace specified by the projector \( \Pi_{a_i,A_i} \), i.e., the unnormalized state after projection reads \( |\psi'\rangle = \Pi_{a_i,A_i}|\psi\rangle \), while the probability for this result to occur is given by \( \langle \psi' | \psi' \rangle \) according to Born’s rule. A measurement with any result other than \( a_i \) projects onto a vector in the orthogonal complement given by the projector \( \Pi_{\neg a_i,A_i} = \mathbb{I} - \Pi_{a_i,A_i} \), where \( \mathbb{I} \) denotes the identity operator on Hilbert space \( \mathcal{H} \). Hence, Hardy’s conditions \( 1 \)–\( 3 \) can be expressed as expectation values of projectors as follows \( 19 \):

\[
\begin{align*}
\text{prob}(\hat{A}_1, a_1; \hat{B}_1, b_1) &= \langle \psi|\Pi_{a_1,A_1}\Pi_{b_1,B_1}\Pi_{a_1,A_1}|\psi\rangle = 0, \\
\text{prob}(\hat{A}_1, \neg a_1; \hat{B}_2, b_2) &= \langle \psi|\Pi_{\neg a_1,A_1}\Pi_{b_2,B_1}\Pi_{\neg a_1,A_1}|\psi\rangle = 0, \\
\text{prob}(\hat{A}_2, a_2; \hat{B}_1, \neg b_1) &= \langle \psi|\Pi_{a_2,A_2}\Pi_{\neg b_1,B_1}\Pi_{a_2,A_2}|\psi\rangle = 0, \\
\text{prob}(\hat{A}_2, a_2; \hat{B}_2, b_2) &= \langle \psi|\Pi_{a_2,A_2}\Pi_{b_2,B_2}\Pi_{a_2,A_2}|\psi\rangle > 0.
\end{align*}
\]

These conditions lead to an upper bound of 1/4 for the last expression, independent of the dimension of the system as we will prove now.

Equation \( 5 \) can be rewritten as

\[
\langle \psi|\Pi_{a_1,A_1}\Pi_{b_1,B_1}\Pi_{a_2,A_2}|\psi\rangle = 0
\]

This implies \( \Pi_{b_1,B_1}\Pi_{a_2,A_2}|\psi\rangle = 0 \) which further leads to \( \Pi_{a_1,A_1}|\psi\rangle = 0 \) or \( \Pi_{a_2,A_2}|\psi\rangle = 0 \). Similarly from equation \( 6 \) one obtains \( \Pi_{\neg a_1,A_1}|\psi\rangle = 0 \) or \( \Pi_{\neg b_1,B_1}\Pi_{a_2,A_2}|\psi\rangle = 0 \). But \( \Pi_{a_2,A_2} |\psi\rangle = 0 \) contradicts the last of Hardy’s conditions \( 5 \), so we discard it.

Hence for a quantum mechanical state \( |\psi\rangle \) to exhibit Hardy’s time-nonlocality, at least one of the following sets of conditions must be simultaneously satisfied:

\[
\begin{align*}
\Pi_{a_1,A_1}|\psi\rangle &= 0, \quad \Pi_{\neg a_1,A_1}|\psi\rangle = 0, \\
\langle \psi|\Pi_{a_2,A_2}\Pi_{b_2,B_2}\Pi_{a_2,A_2}|\psi\rangle &> 0;
\end{align*}
\]

\[
\begin{align*}
\Pi_{a_1,A_1}|\psi\rangle &= 0, \quad \Pi_{b_2,B_2}\Pi_{\neg a_1,A_1}|\psi\rangle = 0, \\
\langle \psi|\Pi_{a_2,A_2}\Pi_{b_2,B_2}\Pi_{a_2,A_2}|\psi\rangle &> 0;
\end{align*}
\]

\[
\begin{align*}
\Pi_{\neg b_1,B_1}\Pi_{a_1,A_1}|\psi\rangle &= 0, \quad \Pi_{\neg a_1,A_1}|\psi\rangle = 0, \\
\langle \psi|\Pi_{a_2,A_2}\Pi_{b_2,B_2}\Pi_{a_2,A_2}|\psi\rangle &> 0;
\end{align*}
\]

\[
\begin{align*}
\Pi_{b_1,B_1}\Pi_{a_1,A_1}|\psi\rangle &= 0, \quad \Pi_{b_2,B_2}\Pi_{\neg a_1,A_1}|\psi\rangle = 0, \\
\langle \psi|\Pi_{a_2,A_2}\Pi_{b_2,B_2}\Pi_{a_2,A_2}|\psi\rangle &> 0.
\end{align*}
\]

We continue by demonstrating that conditions \( 6 \), \( 11 \), and \( 12 \) are inconsistent in themselves while conditions \( 10 \) imply a maximal success probability of 1/4 as claimed above.

The first two conditions of \( 6 \), namely, \( \Pi_{a_1,A_1}|\psi\rangle = 0 \) and \( \Pi_{a_1,A_1}|\psi\rangle = 0 \) cannot be simultaneously true.

The first condition of \( 11 \), namely \( \Pi_{a_1,A_1}|\psi\rangle = 0 \) implies that \( \Pi_{a_1,A_1}|\psi\rangle \) is a vector in the orthogonal complement of the image of \( \Pi_{b_1,B_1} \), i.e.,

\[
\Pi_{a_1,A_1} \leq \Pi_{b_1,B_1},
\]
defined by $\Pi_{-b_1, B_1} - \Pi_{-a_1, A_1}$ being a positive operator.  
Equation (12) can be rewritten as

$$\Pi_{b_1, B_1} \leq \Pi_{-a_1, A_1},$$

(14)

Similarly, from the third condition of (11), namely from 
$\Pi_{-b_1, B_1, \Pi_{-a_2, A_2}} = 0$, one obtains

$$\Pi_{a_2, A_2} \leq \Pi_{b_1, B_1},$$

(15)

Equations (14) and (15) together imply

$$\Pi_{a_2, A_2} \leq \Pi_{-a_1, A_1},$$

(16)

and hence

$$\langle \psi | (\Pi_{a_2, A_2} - \Pi_{-a_1, A_1}) | \psi \rangle \leq 0.$$  

(17)

Taking into account the second condition of (11), namely 
$\Pi_{-a_1, A_1} | \psi \rangle = 0$ equation (17) yields

$$\langle \psi | \Pi_{a_2, A_2} | \psi \rangle \leq 0.$$  

(18)

But, $\langle \psi | \Pi_{a_2, A_2} | \psi \rangle$ is the probability of obtaining $a_2$ as measurement result in a measurement of observable $\hat{A}_2$ on $| \psi \rangle$, so it can neither be negative, nor can it be equal to zero as this would imply that the probability in the last of Hardy’s conditions (11) vanishes.

Using a similar argument we show in Appendix A that the first three conditions of (11) imply 
$\langle \psi | \Pi_{a_2, A_2} \Pi_{b_2, B_2} \Pi_{a_2, A_2} | \psi \rangle \leq 0$ which contradicts the last condition in (11).

Thus we are left with the set (10) only. From its first condition, namely, from $\Pi_{a_1, A_1} | \psi \rangle = 0$, it follows that 
$\Pi_{-a_1, A_1} | \psi \rangle = | \psi \rangle$. The second condition of (10) then reads

$$\Pi_{b_2, B_2} | \psi \rangle = 0$$

which implies

$$| \psi \rangle \langle \psi | \leq \Pi_{-b_2, B_2};$$

(19)

Equation (19) can be rewritten as

$$\Pi_{b_2, B_2} \leq I - | \psi \rangle \langle \psi |$$

(20)

The nonzero probability in Hardy’s argument thus reads

$$\langle \psi | \Pi_{a_2, A_2} \Pi_{b_2, B_2} \Pi_{a_2, A_2} | \psi \rangle$$

$$\leq \langle \psi | \Pi_{a_2, A_2} (I - | \psi \rangle \langle \psi |) \Pi_{a_2, A_2} | \psi \rangle$$

$$= \langle \psi | \Pi_{a_2, A_2} \Pi_{a_2, A_2} | \psi \rangle - \langle \psi | \Pi_{a_2, A_2} | \psi \rangle \langle \psi | \Pi_{a_2, A_2} | \psi \rangle$$

$$= \langle \psi | \Pi_{a_2, A_2} | \psi \rangle - (\langle \psi | \Pi_{a_2, A_2} | \psi \rangle)^2$$

$$\leq \frac{1}{4}$$

(21)

as the maximum value of $\langle \psi | \Pi_{a_2, A_2} | \psi \rangle - (\langle \psi | \Pi_{a_2, A_2} | \psi \rangle)^2$ is $1/4$ and in which case $\langle \psi | \Pi_{a_2, A_2} | \psi \rangle = \frac{1}{2}$.  

The maximum probability is achieved, e.g., if the measurements are chosen such that

$$\Pi_{-b_2, B_2} = \Pi_{-a_1, A_1} = \sum_{i=1}^{n} | \psi_i \rangle \langle \psi_i | < I$$

(22)

for an orthonormal set of vectors $| \psi_i \rangle$ containing the initial state vector, say $| \psi \rangle = (| \varphi \rangle + | \varphi^\perp \rangle) / \sqrt{2} = | \psi_1 \rangle$ and

$$\Pi_{a_2, A_2} = \Pi_{b_1, B_1} = \sum_{i=1}^{n} | \phi_i \rangle \langle \phi_i | < I$$

(23)

for an orthonormal set of vectors $| \phi_i \rangle$ that span a subspace which comprises one component $| \varphi \rangle$ of the initial state but not the perpendicular one $| \varphi^\perp \rangle$. This guarantees the third condition in (10) while the first two are satisfied merely due to the choice (22). In addition, in order to yield the maximum probability (i.e., for equality in (21) to hold) the vectors $| \psi_i \rangle$ can be chosen such that $\langle \psi_i | \varphi \rangle = 0$ for $i = 2, 3 \ldots n$.

Thus the success probability of Hardy’s temporal nonlocality argument can go up to 25% in quantum theory irrespective of the dimension of the system.

This result differs from the result obtained in [8] because there apparently only a restricted set of observables and states of spin systems were considered. For example, for spin $s = 1$ (a three level system) reference [8] claims a maximal success probability of $1/16$. However, it can be checked (cf. Appendix-B) that for such a system, the following setting achieves a maximal success probability $1/4$ of Hardy’s argument in agreement with the upper bound shown above:

$$| \psi \rangle = \left( \begin{array}{c} 0 \\ -\sin \alpha \\ \cos \alpha \end{array} \right)$$

$$\hat{A}_1 = \hat{B}_2 = \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right) = \hat{S}_Z$$

$$\hat{A}_2 = \hat{B}_1 = \cos \alpha \hat{S}_Z - \sin \alpha \hat{S}_X$$

where $\alpha = \cos^{-1} (\sqrt{2} - 1)$, i.e., $\alpha \approx 65.53^\circ$ and

$$\hat{S}_X = \left( \begin{array}{ccc} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{array} \right).$$

From the set of conditions (10), it is clear that for a given observable setting, there can be more than one pure state exhibiting Hardy’s time-nonlocality. Hence a mixture of them will also exhibit this nonlocality. But, as the success probability of Hardy’s argument in this case is a convex sum of the success probabilities for individual pure states, hence for mixed states too, the maximum success probability of Hardy’s argument cannot go beyond $1/4$.  

CONCLUSION

In conclusion, we have shown that the maximum of the success probability appearing in the temporal version of Hardy’s argument is 25% irrespective of the dimension of the quantum mechanical system and the type of observables involved. For the special case of spin measurements, for spin-1 and spin-3/2 observables, we have shown in Appendix-B that this maximum can also be achieved. Moreover, we conjecture that this maximum can be observed for any spin system. Thus this temporal nonlocality persists as opposed to the idea that quantum systems with higher dimensional state space behave more classical which was put forward in [20] for the case of spatial nonlocality. Our result is at par with the findings for spatially separated systems where the success probability for Hardy’s argument is also independent of the dimension of systems’ Hilbert space [21, 22]. Moreover, contrary to the implications from [8], our result ensures the possibility to probe the existence of quantum superpositions for macroscopic systems by means of Hardy’s argument and thus independent of the Leggett-Garg inequality [1].

We have given a recipe to achieve a maximal success probability for Hardy’s argument for general (degenerate or non-degenerate) observables. Note, however that each of them can be replaced by a dichotomic, degenerate observable by only distinguishing the cases where a certain measurement result (say a) occurs from the cases where it does not (∼a), cp. conditions (1)-(4). This is in agreement with our result that the maximal success probability in d-dimensions is the same as for the qubit case which only features dichotomic observables. Thus the maximum value of the success probability in Hardy’s argument for a d-level system, will remain one and the same in the framework of all generalized time-nonlocal theories constrained only by ‘no signalling in time’ [23] where a measurement does not change the outcome statistics of a later measurement.

APPENDIX A

The first condition of (12) implies

\[ \Pi_{a_1, A_1} \leq \Pi_{\sim b_1, B_1} \]  

(A-1)

which can be rewritten as

\[ \Pi_{b_1, B_1} \leq \Pi_{\sim a_1, A_1} \]  

(A-2)

From the second condition of (12), it follows that

\[ \Pi_{\sim a_1, A_1} \leq \Pi_{\sim b_2, B_2} \]  

(A-3)

Equations (A-2) and (A-3) together imply

\[ \Pi_{b_1, B_1} \leq \Pi_{\sim b_2, B_2} \]  

(A-4)

The third condition of (12) gives

\[ \Pi_{a_2, A_2} \leq \Pi_{b_1, B_1} \]  

(A-5)

From (A-1) and (A-5), one obtains

\[ \Pi_{a_2, A_2} \leq \Pi_{\sim b_2, B_2} \]  

(A-6)

which gives

\[ \langle \psi | \Pi_{a_2, A_2} (\Pi_{\sim b_2, B_2} - \Pi_{a_2, A_2}) \Pi_{a_2, A_2} | \psi \rangle \geq 0 \]

\[ \Leftrightarrow \langle \psi | \Pi_{a_2, A_2} (\Pi_{\sim b_2, B_2} - \Pi_{a_2, A_2}) | \psi \rangle - \langle \psi | \Pi_{a_2, A_2} | \psi \rangle \geq 0 \]

\[ \Leftrightarrow \langle \psi | \Pi_{a_2, A_2} | \psi \rangle \geq 0 \]

\[ \Leftrightarrow \langle \psi | \Pi_{a_2, A_2} \Pi_{\sim b_2, B_2} \Pi_{a_2, A_2} | \psi \rangle \leq 0 \]

APPENDIX B

Case of spin-1 systems

Hardy’s time-nonlocality conditions for a three level system as given in [8]:

\[
\begin{align*}
\text{prob}(\hat{A}_1 = +1, \hat{B}_1 = +1) &= 0, \\
\text{prob}(\hat{A}_1 = 0, \hat{B}_2 = +1) &= 0, \\
\text{prob}(\hat{A}_1 = -1, \hat{B}_2 = +1) &= 0, \\
\text{prob}(\hat{A}_2 = +1, \hat{B}_1 = 0) &= 0, \\
\text{prob}(\hat{A}_2 = +1, \hat{B}_1 = -1) &= 0, \\
\text{prob}(\hat{A}_2 = +1, \hat{B}_2 = +1) &= q > 0.
\end{align*}
\]

(B-1)

As mentioned in Section III, the following observable-state setting achieves maximum success probability 1/4 of Hardy’s time-nonlocal argument:

\[ |\psi\rangle = -\sin \alpha |\hat{S}_Z = 0\rangle + \cos \alpha |\hat{S}_Z = -1\rangle \]

\[ \hat{A}_1 = \hat{B}_2 = \hat{S}_Z \]

\[ \hat{A}_2 = \hat{B}_1 = \cos \alpha \hat{S}_Z - \sin \alpha \hat{S}_X \]

where \( \alpha = \cos^{-1}(\sqrt{2} - 1), \) i.e., \( \alpha \approx 65.53° \),

\[ \hat{S}_Z = \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \]

\[ \hat{S}_X = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \]

and \( \Pi_{a_1, A_1}, \Pi_{a_2, A_2}, \Pi_{b_1, B_1}, \Pi_{b_2, B_2} \) are given as follows:

\[ \Pi_{a_1, A_1} = |S_Z = +1\rangle \langle S_Z = +1| = \Pi_{b_2, B_2} \]

\[ \Pi_{a_2, A_2} = |\hat{A}_2 = +1\rangle \langle \hat{A}_2 = +1| = \Pi_{b_1, B_1} \]
where

\[ |\hat{A}_2 = +1\rangle = \cos^2 \frac{\theta}{2} |\hat{S}_Z = +1\rangle + \sin^2 \frac{\theta}{2} |\hat{S}_Z = -1\rangle. \]

It can be checked that \( \Pi_{+1,A_2}(|\psi\rangle) \) is of the form:

\[ \Pi_{+1,A_2}(|\psi\rangle) = \frac{1}{2} |\psi\rangle + \frac{1}{2} |\phi\rangle, \]  

(B-2)

where \(|\phi\rangle = |\hat{S}_Z = +1\rangle\), which is orthogonal to \(|\psi\rangle\).

**Case of spin-3/2 systems**

Hardy’s time-nonlocality conditions for a three level system as given in \[S\]:

\[
\begin{align*}
\text{prob}(A_1 = +\frac{1}{2}, B_1 = +\frac{1}{2}) &= 0, \\
\text{prob}(A_1 = +\frac{1}{2}, B_2 = +\frac{1}{2}) &= 0, \\
\text{prob}(A_1 = -\frac{1}{2}, B_2 = +\frac{3}{2}) &= 0, \\
\text{prob}(A_1 = -\frac{1}{2}, B_2 = +\frac{3}{2}) &= 0, \\
\text{prob}(A_2 = +\frac{3}{2}, B_1 = -\frac{1}{2}) &= 0, \\
\text{prob}(A_2 = +\frac{3}{2}, B_2 = +\frac{3}{2}) &= 0, \\
\text{prob}(A_2 = +\frac{3}{2}, B_2 = +\frac{3}{2}) &= q > 0.
\end{align*}
\]

(B-3)

The following setting achieves a maximal success probability 1/4 of Hardy’s argument:

\[
|\psi\rangle = \sqrt{3} \tan \frac{\theta}{2} |\hat{S}_Z = +\frac{1}{2}\rangle + \sqrt{3} \tan^2 \frac{\theta}{2} |\hat{S}_Z = -\frac{1}{2}\rangle + \tan^3 \frac{\theta}{2} |\hat{S}_Z = -\frac{3}{2}\rangle.
\]

\[
\hat{A}_1 = \hat{B}_2 = \begin{pmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & -\frac{3}{2} \end{pmatrix} = \hat{S}_Z
\]

\[
\hat{A}_2 = \hat{B}_1 = \cos \theta \hat{S}_Z + \sin \theta \hat{S}_X
\]

where

\[
\hat{S}_X = \frac{1}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 2 \\ 0 & 2 & \sqrt{3} \end{pmatrix}
\]

and

\[
\cot^6 \frac{\theta}{2} - 3 \cot^4 \frac{\theta}{2} - 3 \cot^2 \frac{\theta}{2} - 1 = 0
\]

(B-4)

\[
i.e., \quad \theta = 2 \cot^{-1} \frac{1}{\sqrt{1 + \frac{3}{2} + \frac{3}{2}}}.
\]

The projectors \( \Pi_{a_1,A_1}, \Pi_{a_2,A_2}, \Pi_{b_1,B_1}, \Pi_{b_2,B_2} \) in this case are given as

\[
\Pi_{a_1,A_1} = |\hat{S}_Z = \frac{3}{2}\rangle \langle \hat{S}_Z = \frac{3}{2}| = \Pi_{b_2,B_2}
\]

\[
\Pi_{a_2,A_2} = |\hat{A}_2 = \frac{3}{2}\rangle \langle \hat{A}_2 = \frac{3}{2}| = \Pi_{b_1,B_1}
\]

General spin

We conjecture that the maximum of the success probability appearing in the temporal version of Hardy’s argument is 25% for any spin-s system. We write below the state and observable setting which may achieve this maximum:

\[
\hat{A}_1 = \hat{B}_2; \hat{A}_2 = \hat{B}_1, |\psi\rangle \perp |\hat{A}_1 = +s\rangle,
\]

\[
\Pi_{+s,A_2}(|\psi\rangle) = \frac{1}{2} \{ |\psi\rangle + e^{i\theta} |\phi\rangle \}
\]

where \(|\phi\rangle \in \Pi_{+s,B_2} H\).

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Ideal measurement here means a measurement with minimal state change, for example with respect to the fidelity as a measure of state change. Although not used in the assumptions of the original derivation of the temporal Bell inequality [4], without the restriction to ideal measurements, classical systems would violate the inequality too if invasive measurements are used. Hence the inequality could not be used to distinguish classical from quantum behavior.

Alice and Bob may be one and the same observer, but the randomness in the choice of $\hat{A}_1$ or $\hat{A}_2$ and that of $\hat{B}_1$ or $\hat{B}_2$ must be independent.

These conditions, in a different context, were first introduced in [17] to show the spatial nonlocal feature of two spin-$s$ systems.

Without loss of generality, we assume that the state of the system does not evolve with time between two successive measurements.

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