Aspects of Noncommutative Scalar/Tensor Duality

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We study the noncommutative massless Kalb-Ramond gauge field coupled to a dynamical $U(1)$ gauge field in the adjoint representation together with a compensating vector field. We derive the Seiberg-Witten map and obtain the corresponding mapped action to first order in $\theta$. The (emergent) gravity structure found in other situations is not present here. The off-shell dual scalar theory is derived and it does not coincide with the Seiberg-Witten mapped scalar theory. Dispersion relations are also discussed. The $p$-form generalization of the Seiberg-Witten map to order $\theta$ is also derived.

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I. INTRODUCTION

Field theories, specially gauge theories in noncommutative (NC) space-time have been actively studied in recent times. Maxwell and Yang-Mills gauge theories have been extensively analyzed \cite{1}. There also exists generalizations of massless vector fields to $p$-form theories which are interesting by themselves and also arise in several contexts in string theory, field theory and condensed matter. It is well-known that in 4 dimensions the 2-form theory, also known as the Kalb-Ramond (KR) theory, is dual to a massless scalar theory \cite{2}. Hence it is of interest to ask whether this feature remains true in a NC space-time as well. Dual field theories in NC space-time, using the Seiberg-Witten (SW) map \cite{3}, have been extensively studied. These include the Maxwell-Chern-Simons and the self-dual theories \cite{4}, Bose-Fermi equivalence \cite{5}, etc. In all these theories, the dual relation established in the commutative case does not carry forward to the NC space-time. This work is devoted to the study of the duality aspects of the abelian 2-form gauge theories in non-commutative space-time.

The $U(1)$ gauge theory in NC space-time has a non-abelian like structure, and hence the abelian KR in NC space-time can also be expected to have such a non-abelian like structure as well. The naive generalization of the abelian Lagrangian, viz $L = H \ast H$, (where $H = dB$ and $B$ is the 2-form field) will not lead to any NC correction as the star product can be removed from the quadratic piece in the action. As it is well known, a Yang-Mills generalization of the self-interacting KR theory does not exist and the closest non-abelian generalization of the 2-form theory have been proposed in \cite{6, 7}. In these models, KR fields transforming in the adjoint representation of the gauge group couple with non-abelian gauge fields.

We consider a massless KR field transforming in the adjoint representation of $U(1)$ (a possibility which does not exist in commutative space-time) coupled with the gauge field. The study of such a theory has also another motivation. The existence of several parallels between NC $U(1)$ gauge theory and commutative gravitational theory, led one of us to seek a direct relationship between them. It was observed that a massless scalar and vector fields coupled to NC $U(1)$ gauge field, after the SW mapping, has the same structure as that of fields coupled with gravity \cite{8}. This suggestion, that gravity can emerge from NC electromagnetism, has been further studied and extended in \cite{9, 10}. Hence it is natural to ask whether the 2-form field coupled to photons in NC space-time has such an emergent gravity coupling. First we discuss the NC KR theory. We recall the construction for the non-abelian case, where a topological coupling to a vector field with mass parameter was also considered \cite{11, 12} and show its extension to the NC case in section 2. Then, in section 3, we discuss the SW map and the emergent gravity picture. Section 4 is devoted to the analysis of the duality between both NC formulations while in section 5 we discuss the dispersion relations. Finally, in section 6 we present our conclusions and discuss the extension of our results to the case of a $p$-form field.

II. NC KALB-RAMOND THEORY

The NC action for a massless Kalb-Ramond 2-form field is given by \cite{11, 12}

$$
\int d^4x \left( \frac{1}{12} \hat{H}_{\mu\nu\lambda} \ast \hat{H}^{\mu\nu\lambda} - \frac{1}{4} \hat{F}_{\mu\nu} \ast \hat{F}^{\mu\nu} \right),
$$

(1)

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where
\[ \hat{H}_{\mu\nu\lambda} = D_\mu \hat{B}_{\nu\lambda} + \text{cyclic terms}, \]
\[ = D_\mu [\hat{B}_{\nu\lambda} - (D_\nu \hat{C}_\lambda - D_\lambda \hat{C}_\nu)] + \text{cyclic terms}, \]
\[ \hat{F}^{\mu\nu} = -i[D^\mu, D^\nu], = -i(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]), \]
\[ (2) \]
and the action of the covariant derivative is defined by
\[ D_\mu \hat{B}_{\nu\lambda} = \partial_\mu \hat{B}_{\nu\lambda} - i[\hat{A}_\mu, \hat{B}_{\nu\lambda}], \]
\[ (4) \]
The field equation for \( B^{\nu\lambda}, C_\lambda \) and \( A_\nu \) are given, respectively, by
\[ D^\mu \hat{H}_{\mu\nu\lambda} = 0, \]
\[ \hat{H}_{\mu\nu\lambda}; \hat{F}^{\mu\nu} = 0, \]
\[ D^\mu \hat{F}^{\mu\nu} + 2i[\hat{C}^{\nu\lambda}, D^\mu \hat{H}_{\mu\nu\lambda}] - i[\hat{B}^{\nu\lambda}, \hat{H}_{\mu\nu\lambda}] = 0. \]
\[ (5) \]
\[ (6) \]
\[ (7) \]
The generalization of the commutative KR theory to NC space-time requires the Maxwell term, without which the theory becomes trivial. This can be easily seen from the fact that the first term in Eqn. (7) will be absent when the \( F^{\mu\nu} \hat{F}^{\mu\nu} \) is not present in the Lagrangian. In that case, substituting Eqn. (5) into Eqn. (7) will result in that either \( B \) or \( H \) is a constant leading to the absence of any effect of noncommutativity in Eqn. (1). Also, the need for the C field becomes clear from the consistency of the field equations given above: the divergence of Eqn. (5) will vanish only if we use the equation Eqn. (6).

The local symmetries of Eqn (1) are:
1) NC U(1) gauge transformations:
\[ \delta \hat{B}_{\mu\nu} = i[\hat{\lambda}, \hat{B}_{\mu\nu}], \]
\[ \delta \hat{C}_\mu = i[\hat{\lambda}, \hat{C}_\mu] + \hat{\lambda}, \]
\[ \delta \hat{A}_\mu = D_\mu \hat{\lambda}, \]
\[ (8) \]
2) the local Kalb-Ramond transformations, with a vector parameter, given by
\[ \delta \hat{B}_{\mu\nu} = D_\mu \hat{\lambda}_\nu - D_\nu \hat{\lambda}_\mu, \]
\[ \delta \hat{C}_\mu = \hat{\lambda}_\mu, \]
\[ \delta \hat{A}_\mu = 0. \]
\[ (9) \]
This defines the NC U(1) KR theory.

### III. THE SW MAP FOR THE NC U(1) KR THEORY

In this section we derive the SW map for NC U(1) KR theory. Earlier the SW map for the non-abelian 2-form theories has been studied by using the closure of the algebra\[13\]. Here we derive the mapping by solving explicitly the SW equation. To derive the SW map we consider the complete gauge transformations of \( \hat{B}_{\mu\nu}, \hat{C}_\mu \) and \( \hat{A}_\mu \)
\[ \delta \hat{A}_\mu = D_\mu \hat{\lambda}, \]
\[ \delta \hat{B}_{\mu\nu} = i[\hat{\lambda}, \hat{B}_{\mu\nu}] + (D_\mu \hat{\lambda}_\nu - D_\nu \hat{\lambda}_\mu), \]
\[ \delta \hat{C}_\mu = i[\hat{\lambda}, \hat{C}_\mu] + \hat{\lambda}_\mu. \]
\[ (10) \]
The SW map is obtained by demanding that the gauge orbit of the NC gauge field gets mapped to that of commutative theory. In the case of the vector field, under the U(1) transformation, the SW map is well known\[3\] and the SW equation for \( A_\mu \) is
\[ \hat{A}(A, \theta) + \delta \hat{A}(A, \theta) = \hat{A}(A + \delta \lambda A, \theta). \]
\[ (11) \]
To first order in $\theta$ we write $\hat{A} = A + A'(A)$ where $A'$ is a function of $A$ and $\theta$. And for the gauge parameter we write $\hat{\lambda}(\lambda, A) = \lambda + \lambda'(\lambda, A)$. Then Eqn. (11) becomes

$$A'_\mu(A + \delta A) - A'_\mu(A) = \partial_\mu \lambda' + \theta^{\alpha\beta} \partial_\mu A_\mu \partial_\sigma \lambda.$$  

The solution for $A'$ and $\lambda'$ satisfying the above equation is given by

$$A' = -\frac{1}{2} \theta^{\alpha\beta} A_\alpha (\partial_\beta A_\mu + F_{\beta\mu}),$$

$$\lambda' = \frac{1}{2} \theta^{\alpha\beta} \partial_\alpha \lambda A_\beta.$$  

(13)

Since the U(1) gauge field is inert under the KR transformations the above equations retain their familiar form.

We now set up the SW equations for the fields $B_{\mu\nu}$ and $C_\mu$ much as in the same way as given by Eqn. (12)

$$B'_{\mu\nu}(B + \delta B, A + \delta A, C + \delta C) - B'_{\mu\nu}(B, A, C) = (\partial_\mu A'_\nu - \partial_\nu A'_\mu) - \theta^{\alpha\beta} \partial_\mu \lambda A_{\beta \sigma} B_{\mu\nu},$$

$$C'_\mu(C + \delta C) - C'_\mu(C) = \lambda'_\mu - \theta^{\alpha\beta} \partial_\mu \lambda A_{\beta \sigma} C_\mu.$$  

(14)

The solution of the SW equations to first order in $\theta$ is

$$\dot{B}_{\mu\nu} = B_{\mu\nu} + B'_{\mu\nu}$$

$$= B_{\mu\nu} - \theta^{\alpha\beta} (A_\alpha \partial_\beta B_{\mu\nu} + \partial_\mu A_\alpha \partial_\sigma C_\nu - \partial_\nu A_\alpha \partial_\sigma C_\mu - \partial_\mu A_\nu \partial_\sigma A_\alpha C_\mu),$$

$$\dot{A}_\mu = \Lambda_\mu + A'_\mu$$

$$= \Lambda_\mu - \theta^{\alpha\beta} A_\alpha \partial_\beta \Lambda_\mu,$$

$$\dot{C}_\mu = C' + C'_\mu$$

$$= C_\mu - \theta^{\alpha\beta} A_\alpha \partial_\beta C_\mu.$$  

(15)

It is easy to verify the closure of the algebra

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}] B'_{\mu\nu} = 0, \quad [\delta_{\lambda}, \delta_{\lambda}] B'_{\mu\nu} = 0, \quad [\delta_{\Lambda_1}, \delta_{\Lambda_2}] C'_\mu = 0, \quad [\delta_{\Lambda}, \delta_{\lambda}] C'_\mu = 0$$  

(16)

The next step is to derive the SW mapped action using the solution above. The SW mapped field strength $\hat{H}_{\mu\nu\lambda}$ is given to first order in $\theta$ by

$$\hat{H}_{\mu\nu\lambda} = H_{\mu\nu\lambda}(B) + \theta^{\alpha\beta} (F_{\rho\mu} \partial_\alpha B_{\nu\lambda} + \text{Cyclic terms}) - \theta^{\alpha\beta} A_\rho \partial_\sigma H_{\mu\nu\lambda}.$$  

(17)

The gauge invariance of $\hat{H}_{\mu\nu\lambda}$ under KR transformation and covariance under U(1) transformations is evident.

Using Eqn. (17) in Eqn. (11) the SW mapped action to first order in $\theta$ is

$$S_{sw} = \int d^4 x [\frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \frac{1}{2} \theta^{\rho\sigma} H_{\mu\nu\lambda} F_{\rho\mu} \partial_\alpha [B^{\nu\lambda} - \partial_\nu C_\lambda + \partial_\lambda C_\nu]$$

$$+ \frac{1}{24} \theta^{\rho\sigma} H_{\mu\nu\lambda} F_{\sigma\rho} H^{\mu\nu\lambda}$$

$$- \frac{1}{4} [F_{\mu\nu} F^{\mu\nu} + 2 \theta^{\rho\sigma} F_{\rho\mu} F_{\sigma\nu} F^{\mu\nu} - \frac{1}{2} \theta^{\rho\sigma} F_{\rho\sigma} F^{\mu\nu} F_{\mu\nu}]).$$  

(18)

The above action is invariant under the usual commutative gauge transformations

$$\delta B_{\mu\nu} = \partial_{(\mu} A_{\nu)} , \quad \delta A_\mu = \partial_\nu C_\lambda \quad \text{and} \quad C_\mu = \Lambda_\mu.$$  

(19)

The presence of C field in the SW mapped action is essential to maintain the B field symmetry $\partial_{[\mu} A_{\nu]}$. When we take the commutative limit $\theta \to 0$, the field C will disappear as it happens in the commutative case. Next we must check the consistency of the field equations for $A_\mu$, $B_{\mu\nu}$ and $C_\lambda$ derived from the action above. The field equation for the field $A_\mu$ is

$$\partial_\mu F^{\mu\nu} + \theta^{\alpha\beta} F_\alpha^{\mu} (\partial_\beta F_\nu^{\mu} + \partial_\mu F_\beta^{\nu}) - \frac{1}{2} \theta^{\alpha\beta} \partial_\alpha H_{\nu\tau\lambda} \partial_\beta B^{\tau\lambda} = 0,$$  

(20)
while the equation for $B_{\rho\lambda}$ is
\[ \partial^\mu H_{\mu\nu\lambda} + \theta^{\rho\sigma} \partial^\nu (F_{\rho\mu} \partial_\sigma B_{\nu\lambda} + F_{\rho\nu} \partial_\sigma B_{\mu\lambda} + F_{\rho\lambda} \partial_\sigma B_{\mu\nu}) + \theta^{\rho\sigma} \partial_\lambda H_{\mu\nu\rho} F_{\rho}^\mu = 0, \] (21)
and the $C_\lambda$ field equation is
\[ \theta^{\rho\sigma} \partial_\rho F_{\mu\nu} \partial_\lambda H_{\mu\nu\lambda} = [F_{\mu\nu}, H_{\mu\nu\lambda}]_s = 0. \] (22)

Notice that field equation for $C_\lambda$ is needed for the consistency of the $B_{\mu\nu}$ field equation. The divergence of the B equation will vanish only if we use the C equation of motion. The divergence of both, the C and A field equations are zero identically.

It was shown by one of us [8] that, after the SW map, the coupling of a massless scalar field to a gauge field in a NC space-time has the same structure as a gravitational coupling. Hence it is of interest to inquire whether a similar phenomenon takes place here too. In order to do that we note that the SW mapped Lagrangian can be rewritten as
\[ S = \int d^4x \{ \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \frac{1}{12} h H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{12} h H_{\mu\nu\lambda} H^{\mu\nu\lambda} \}. \] (23)

Notice that at this point we are working with $B_{\mu\nu} = B_{\mu\nu} - \partial_\nu C_\mu$ and $H(B) = H(B)$. Next we consider the coupling of the Kalb-Ramond field to gravity
\[ S_{g,B} = \int d^4x \sqrt{-g} g^{\mu\nu} g^{\rho\sigma} g^{\lambda\sigma} H_{\mu\nu\lambda} H_{\rho\sigma\beta}, \] (24)
and the expansion of the metric $g_{\mu\nu}$ around flat space-time $g_{\mu\nu}$ as
\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \eta_{\mu\nu} h, \] (25)
where $h_{\mu\nu}$ is traceless. We then get
\[ S_{g,B} = \int d^4x \{ \frac{1}{12} h H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{12} h H_{\mu\nu\lambda} H^{\mu\nu\lambda} - \frac{1}{12} h H_{\mu\nu\lambda} [\partial_\alpha B^{\alpha\lambda} + 2\partial^\nu B_{\alpha\lambda}] \}. \] (26)

To find an expression for $h^{\alpha\mu}$ in terms of the gauge fields we should compare Eqn. (23) and Eqn. (26). But the presence of a antisymmetric part for the term in the square bracket in Eqn.(23) destroys this comparison. Thus the NC SW mapped KR theory does not have the structure of emergent gravity as for the NC scalar field. [8].

IV. DUALITY BETWEEN THE KR AND SCALAR FIELDS

It is well known that in four dimensions the massless KR theory is dual to a massless scalar theory [2]. In this section we will search for the dual field theory of the SW mapped KR theory. We will apply the method developed in [14], which is an off-shell dual formulation. We recast the action using an auxiliary field $A_{\alpha\beta\gamma} = \partial_\alpha B_{\beta\gamma}$ enforced through a Lagrange multiplier $\phi$. To implement the duality transformation we start with the first order Lagrangian
\[ S_{sw} = \int d^4x \left\{ \frac{1}{4} (A_{\mu\nu\lambda} + 2A_{\lambda\mu\nu}) A^{\mu\nu\lambda} + \frac{1}{2} \theta^{\rho\sigma} (A_{\mu\nu\lambda} + 2A_{\lambda\mu\nu}) F_{\rho\sigma} (A^{\sigma\nu\lambda} - \partial_\sigma (\partial^\nu C^\lambda - \partial^\lambda C^\nu)) + \frac{1}{8} \theta^{\rho\sigma} (A_{\mu\nu\lambda} + 2A_{\lambda\mu\nu}) A^{\mu\nu\lambda} F_{\rho\sigma} + \frac{1}{3} \phi^{\mu\nu\lambda} (A_{\mu\nu\lambda} - \partial_\mu B_{\nu\lambda}) + S_{M_{\text{maxwell}}} \right\}. \] (27)

Eliminating $\phi$ by using its field equation gives back the original theory. If instead we eliminate the original field B we find a constraint on $\phi$, i.e, $\partial_\mu \phi^{\mu\nu\lambda} = 0$, which can be solved as $\phi^{\mu\nu\lambda} = \epsilon^{\mu\nu\lambda\gamma} \partial_\gamma \Phi$. The field equation for A is then given by
\[ A_{\alpha\beta\gamma}(1 + \frac{1}{2} \theta^{\rho\sigma} F_{\rho\sigma}) + \theta^{\rho\sigma} (F_{\rho\alpha} A_{\sigma\beta\gamma} + F_{\rho\gamma} A_{\sigma,\alpha\beta} + F_{\rho\beta} A_{\sigma,\gamma\alpha}) + \theta^{\rho\sigma} F_{\rho}^{\alpha\beta} A_{\gamma,\mu\beta} + A_{\gamma,\mu\beta} + \theta^{\rho\sigma} [F_{\rho\sigma} \partial_\gamma (\partial_\beta C_\gamma)] + F_{\rho\beta} \partial_\gamma (\partial_\alpha C_\gamma) + F_{\rho\gamma} \partial_\delta (\partial_\beta C_\delta) + \frac{1}{3} \epsilon_{\alpha\beta\gamma} \partial^\gamma \Phi = 0. \] (28)
Now we can solve it perturbatively in $\theta$ to get

$$A_{[\alpha\beta\gamma]} = \frac{1}{3} \epsilon_{\alpha\beta\gamma\lambda} \partial^\lambda \Phi + \theta^{\rho\sigma} \frac{1}{3} [F_{\rho\mu\epsilon_{\beta\gamma\lambda}} \partial^\lambda \Phi + F_{\rho\nu} \epsilon_{\sigma\beta\gamma\lambda} \partial^\lambda \Phi + F_{\rho\lambda} \epsilon_{\sigma\gamma\alpha\lambda} \partial^\lambda \Phi]$$

$$+ \frac{1}{9} \theta^{\rho\sigma} F_{\rho\mu} \epsilon_{\mu\beta\gamma\lambda} \partial^\lambda \Phi + (\alpha \to \beta \to \gamma)] + \theta^{\rho\sigma} \frac{1}{3!} \epsilon_{\alpha\beta\gamma\lambda} \partial^\lambda \Phi F_{\sigma\rho}$$

$$+ \theta^{\rho\sigma} [F_{\rho\sigma} \partial_\sigma (\partial_\beta C_\gamma - \partial_\gamma C_\beta) + F_{\rho\beta} \partial_\sigma (\partial_\alpha C_\eta - \partial_\alpha C_\eta) + F_{\rho\gamma} \partial_\sigma (\partial_\alpha C_\beta - \partial_\beta C_\alpha)] \tag{29}$$

Using this we can eliminate the field $A_{\mu\nu\lambda}$ from the action Eqn. (24) to get the dual Lagrangian

$$L_D = \frac{1}{2} \partial_\lambda \Phi \partial^\lambda \Phi + \frac{5}{36} \theta^{\rho\sigma} F_{\rho\sigma} \partial_\lambda \partial^\lambda \Phi + \frac{1}{6} \theta^{\rho\sigma} F_{\rho\lambda} \partial_\sigma \partial^\lambda \Phi$$

$$+ \frac{1}{3} \theta^{\rho\sigma} F_{\rho\mu} \partial_\sigma (\partial^\beta C^\gamma - \partial^\gamma C^\beta) \epsilon_{\alpha\beta\gamma\lambda} \partial^\lambda \Phi + \text{Maxwell’s term.} \tag{30}$$

Now from this dual Lagrangian the field equations for $\Phi$, $A$ and $C$ to order $\theta$ are given by

$$- \partial_\lambda \partial^\lambda \Phi - \frac{7}{36} \theta^{\rho\sigma} \partial_\lambda F_{\alpha\beta} \partial^\lambda \Phi + \frac{1}{3} \theta^{\rho\sigma} F_{\rho\lambda} \partial_\sigma \partial^\lambda \Phi + \frac{2}{3} \theta^{\rho\sigma} \epsilon_{\alpha\beta\gamma\lambda} \partial_\rho \partial^\lambda \Phi = 0, \tag{31}$$

$$\partial_\mu F^{\mu\sigma} + \theta^{\rho\sigma} F^{\mu}_{\rho\sigma} (\partial_\beta F^{\mu}_{\beta\sigma} + \partial_\sigma F^{\mu}_{\beta\sigma}) - \frac{5}{18} \theta^{\rho\sigma} \partial_\lambda (\partial_\mu \Phi \partial^\lambda \Phi) + \frac{1}{6} \theta^{\rho\sigma} \partial_\lambda (\partial_\mu \Phi \partial^\lambda \Phi) + \frac{2}{3} \theta^{\rho\sigma} \partial_\mu \partial_\rho C_\gamma \epsilon_{\alpha\beta\gamma\lambda} \partial^\lambda \Phi = 0, \tag{32}$$

$$[F_{\mu\nu}, \partial_\sigma \tilde{\Phi}]_+ = 0. \tag{33}$$

The $C$ field equation on both sides of the dual theory (i.e., Eqn. (22) and Eqn. (23)) is consistent with the familiar dual relation $\partial_\mu \Phi = \frac{1}{\sqrt{2}} \epsilon_{\mu\nu\lambda\sigma} H^{\nu\lambda\sigma}$

In [3] the SW mapped scalar theory is given by

$$S_\varphi = \frac{1}{2} \int d^4 x \left[ \partial^\mu \varphi \partial_\mu \varphi + 2 \theta^{\mu\alpha} F_{\alpha} \nu \left( - \partial_\mu \varphi \partial_\nu \varphi + \frac{1}{4} \eta_{\mu\nu} \partial^\rho \varphi \partial_\rho \varphi \right) \right]. \tag{34}$$

Notice that the structure of the dual theory Eqn. (30) is different from the above one and also that it does not have the form of a gravity coupling.

**V. DISPERSION RELATION**

In this section we will look for the solutions to the field equations in order to derive the dispersion relations. Dispersion relations in Maxwell equations case have been well studied [13]. The field equations are given by (20) and (21). There is a trivial solution where $F$ and $B$ are both constants. If $B$ is constant then the dispersion relation for $F$ remains the same as that of the SW mapped Maxwell case alone. Next, we look for a transverse plane wave solution for $A_\mu$ and $B_\mu$. To this end we define $A_\mu = A_\mu (kx)$ and $B_\nu = B_\nu (kx)$, with $k$ being a wave vector. Using $\partial_\alpha A_\mu = k_\alpha A'_\mu$ where the prime denotes differentiation w.r.t to $kx$. Then the field equation for $A_\mu$ becomes

$$k^2 A'_\nu - k_\mu k_\nu A^{\mu} + \theta^{\rho\sigma} (k_\alpha A^{\alpha}_\mu - k_\mu A'_\alpha) + k_\beta k_\nu A'_\mu - k_\mu k_\nu A'_\beta = 0.$$

$$- \frac{\theta^{\rho\sigma}}{2} k_\alpha (k_\lambda B^{\lambda}_\nu + k_\lambda B'^{\lambda}_\nu + k_\nu B^{\nu}_\lambda) k_\beta B'^{\sigma\lambda} = 0. \tag{35}$$

This reduces to

$$k^2 (1 - 2 \theta^{\rho\beta} k_\alpha A'_\beta) A'_\nu = 0 \tag{36}$$

by use of the transversality conditions $k_\alpha A'_\alpha = k_\alpha B'^{\gamma} = 0$. This produces the usual dispersion relation $k^2 = 0$. Now we have to check the consistency with B field equation. It is given by
\[ k^\mu (k_\mu \tilde{B}_{\nu\lambda} + k_\nu \tilde{B}_{\lambda\mu} + k_\lambda \tilde{B}_{\mu\nu}) + \theta^{\rho\sigma} k^\mu [(k_\rho A_\mu - k_\mu A_\rho) k_\sigma \tilde{B}_{\nu\lambda}] + (k_\rho A_\nu - k_\nu A_\rho) k_\sigma \tilde{B}_{\lambda\mu} + (k_\rho A_\lambda - k_\lambda A_\rho) k_\sigma \tilde{B}_{\mu\nu}] + \theta^{\rho\sigma} k_\sigma (k_\mu \tilde{B}_{\nu\lambda} + k_\nu \tilde{B}_{\lambda\mu} + k_\lambda \tilde{B}_{\mu\nu}) (k_\rho \tilde{A}^\mu - k_\mu \tilde{A}^\rho) = 0. \] (37)

The transversality condition of \( B \) will reduce this further to
\[ k^2 (1 - 2\theta^{\alpha\beta} A_\alpha') B_{\nu\lambda} = 0, \] (38)
so that we have the usual dispersion relation \( k^2 = 0 \). Next we look for a solution where \( F \) is constant and \( H \) is a plane wave. From Eqn. (21) and using the transversality condition for \( B \), we get
\[ \tilde{k}_\lambda H^{\lambda\mu\nu} + \theta^{\alpha\beta} F_{\alpha\lambda} k_\beta k^{\lambda} B_{\mu\nu} = 0, \] (39)
where we have defined \( \tilde{k}_\mu = k_\mu + \theta^{\alpha\beta} F_{\alpha\mu} k_\beta \). The Bianchi identity (BI) for \( H \) is given by
\[ k_\mu \tilde{H}'_{\nu\lambda\rho} - k_\nu \tilde{H}'_{\mu\lambda\rho} + k_\lambda \tilde{H}'_{\mu\nu\rho} - k_\rho \tilde{H}'_{\lambda\mu\nu} = 0. \] (40)
Now we contract the BI with \( \tilde{k}^\mu \). The \( \tilde{B}' \) terms can be collected together as \( \tilde{H}' \) and we get
\[ (k \tilde{k} + \theta^{\alpha\beta} F_{\alpha\mu} k_\beta k^{\alpha}) \tilde{H}'_{\nu\lambda\rho} = 0, \] (41)
so the dispersion relation is
\[ k^2 = -2\theta^{\alpha\beta} F_{\alpha\lambda} k_\beta k^{\lambda}. \] (42)

Now adopting the same conventions as in [8] for the decomposition into transversal and longitudinal components this will lead to
\[ \frac{k^2}{\omega^2} = 1 - 2\theta_{TT} \cdot (\tilde{B}_T - \frac{\tilde{k}}{\omega} \times \tilde{E}_T). \] (43)

A similar relation was obtained in [8] for the scalar field. Though the dual equivalence between KR and scalar fields at the level of the partition function is absent, at the level of the dispersion relations they agree.

### VI. CONCLUSION

In this work we have studied the scalar/tensor duality in NC space-time. The SW equation was set up and solved. The solution given in [15] is not unique. There exists one more solution below:

\[
\begin{align*}
\tilde{B}_{\mu\nu} & = B_{\mu\nu} + B''_{\mu\nu} \\
& = B_{\mu\nu} - \theta^{\rho\sigma} (\partial_\rho A_\mu B_{\sigma\nu} - \partial_\sigma A_\mu B_{\rho\nu} + A_\rho \partial_\sigma B_{\mu\nu} - A_\sigma \partial_\rho B_{\mu\nu}) \\
& \quad - \partial_\rho A_\mu B_{\sigma\nu} + \partial_\sigma A_\mu B_{\rho\nu} + \partial_\rho \partial_\sigma A_\mu C_{\rho\nu} - \partial_\rho \partial_\sigma A_\rho C_{\mu\nu} \\
\tilde{C}_\mu & = C_\mu + C''_\mu \\
& = C_\mu - \theta^{\rho\sigma} [A_\rho \partial_\sigma C_\mu - \partial_\rho A_\mu C_\sigma + \partial_\mu A_\rho C_\sigma], \\
\tilde{A}_\mu & = A_\mu + A''_\mu \\
& = A_\mu - \theta^{\rho\sigma} [A_\rho \partial_\sigma A_\mu - \partial_\rho A_\mu A_\sigma + \partial_\mu A_\rho A_\sigma].
\end{align*}
\] (44)

Both solutions, however, are related by the field redefinition

\[
\begin{align*}
B''_{\mu\nu} & = B_{\mu\nu} + \theta^{\alpha\beta} (F_{\alpha\mu} B_{\beta\nu} - F_{\alpha\nu} \partial_\beta C_\nu) - C_\alpha \partial_\beta F_{\mu\nu}, \\
C''_\mu & = C_\mu - \theta^{\alpha\beta} C_\alpha F_{\beta\mu}.
\end{align*}
\] (45)

Thus the SW map is unique up to field redefinitions. This happens because more fields are available so that non-trivial field redefinitions can be performed. The SW map for the primed fields are the ones in Eqn. [15].
By generalizing the gauge transformation of the 2-form theory Eqns. 8 and 9 to a p-form, with the C field a p-1 form in d dimensions, coupled with the U(1) gauge field, the SW solution can be generalized as

\[ B'(B + \delta B, A + \delta A, C + \delta C) = B'(B, A, C) = dA' + \theta^{\rho\sigma} \partial_{\rho} A\theta_{\sigma} A - \theta^{\rho\sigma} \partial_{\rho} \lambda \partial_{\sigma} B \]

\[ C'(C + \delta C) = C'(C) = \Lambda' - \theta^{\rho\sigma} \partial_{\rho} \lambda \partial_{\sigma} C. \]  

(46)

The solution for \( B', C' \) and \( \Lambda' \) are,

\[ B' = -\theta^{\rho\sigma} [A_{\rho} \partial_{\sigma} B + dA_{\rho} \partial_{\sigma} C] + \theta^{\rho\sigma} \partial_{\rho} A \partial_{\sigma} C, \]

\[ C' = -\theta^{\rho\sigma} A_{\rho} \partial_{\sigma} C, \]

\[ \Lambda' = -\theta^{\rho\sigma} A_{\rho} \partial_{\sigma} \Lambda. \]  

(47)

Finally the p+1 form \( \hat{H} \) is,

\[ \hat{H} = -\theta^{\rho\sigma} dA_{\rho} \partial_{\sigma} B - \theta^{\rho\sigma} A_{\rho} \partial_{\sigma} dB + \theta^{\rho\sigma} \partial_{\rho} A \partial_{\sigma} B. \]  

(48)

The role of the C-field is interesting since it ensures the consistency of the KR equation, and also provides the KR symmetry of action. Unlike the scalar and Maxwell’s case the SW mapped KR action does not have a structure which could be interpreted as a gravitational coupling. This may mean that the emergent gravity picture of the NC Maxwell theory may not be viable for other gauge fields since the simplest coupling to a Riemannian geometry was not obtained. This, however, does not preclude the coupling to non-Riemannian geometries but this is still under scrutiny.

The dual scalar theory obtained from SW mapped KR theory through Fradkin’s method is derived. The C field is present in it and interestingly enough its field equation is in accordance with the well known commutative relation \( \partial_{\mu} \phi \rightarrow \hat{H}_{\mu} \) apart for \( O(\theta) \) correction. The dual scalar theory also does not have the structure of a gravity coupling, consistent with the absence of a similar structure for the KR theory. Thus the duality between the scalar and the tensor field, present in the commutative case, is not preserved in the NC case. It is in agreement with other studies on dualities where similar conclusions were reached. The dispersion relation for the plane wave solution with \( F \) tensor theory, present in the commutative case, is not preserved in the NC case. It is in agreement with other studies consistent with the absence of a similar structure for the KR theory. Thus the duality between the scalar and the tensor field, present in the commutative case, is not preserved in the NC case. It is in agreement with other studies.

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