THE INTERPLAY OF REGULARIZING FACTORS IN THE
MODEL OF UPPER HYBRID OSCILLATIONS OF COLD
PLASMA

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Abstract. A one-dimensional nonlinear model of the so-called upper hybrid oscillations in a magnetically active plasma is investigated taking into account electron-ion collisions. It is known that both the presence of an external magnetic field of strength $B_0$ and a sufficiently large collisional factor $\nu$ help suppress the formation of a finite-dimensional singularity in a solution (breaking of oscillations). Nevertheless, the suppression mechanism is different: an external magnetic field increases the oscillation frequency, and collisions tend to stabilize the medium and suppress oscillations. In terms of the initial data and the coefficients $B_0$ and $\nu$, we establish a criterion for maintaining the global smoothness of the solution. Namely, for fixed $B_0$ and $\nu \geq 0$ one can precisely divide the initial data into two classes: one leads to stabilization to the equilibrium, and the other leads to the destruction of the solution in a finite time.

Next, we examine the nature of the stabilization. We show that for small $|B_0|$ an increase in the intensity factor first leads to a change in the oscillatory behavior of the solution to monotonic damping, which is then again replaced by oscillatory damping. At large values of $|B_0|$, the solution is characterized by oscillatory damping regardless of the value of the intensity factor $\nu$.

INTRODUCTION

The equations of hydrodynamics of "cold" plasma in the non-relativistic approximation take the form

$$
\frac{\partial n}{\partial t} + \text{div}(n \mathbf{V}) = 0, \quad \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = \frac{e}{m} \left( \mathbf{E} + \frac{1}{c} [\mathbf{V} \times \mathbf{B}] \right) - \nu_e \mathbf{V},
$$

$$
\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = -\frac{4\pi}{c} e n \mathbf{V} + \text{rot} \mathbf{B}, \quad \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = -\text{rot} \mathbf{E}, \quad \text{div} \mathbf{B} = 0,
$$

where $e, m$ — charge and mass of the electron (here the electron charge has a negative sign: $e < 0$), $c$ — speed of light; $n, \mathbf{V}$ — density and velocity of electrons; $\mathbf{E}, \mathbf{B}$ — vectors of electric and magnetic fields, the term $-\nu_e \mathbf{V}$ describes electron-ion collisions, which can be interpreted as a friction between particles, see, for example, [1]. System (1) is described in sufficient detail in textbooks and monographs (see, for example, [9]). Currently, the great attention is paid to a study of the cold plasma due to a possibility of acceleration of electrons in the wake wave of a powerful laser pulse [7], nevertheless theoretical results in this field are very scarce.

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It is commonly known that the plasma oscillations described by (1), tend to break. Mathematically, the breaking process means a blow up of solution, and the appearance of a delta-shape singularity of the electron density [6]. Among main interests is a study of possibility of existence of a smooth solution as long as possible.

To obtain exact mathematical results we use the model of upper hybrid oscillations of cold magnetoactive plasma (see, for example, [6]), a simplification that the physicists traditionally use to study the effect of an external magnetic field. The model was appeared, apparently, for the first time, in [5], further it was used in various contexts, e.g. [11], [12], [10]. To derive the model they expand the solution to (1) in a series with respect a small parameter and retain physically significant terms. In the Cartesian coordinates $x, y, z$ the solution has the following structure:

$V = (V_x(x,t), V_y(x,t), 0)$, $E(x,t) = (E_x(x,t), 0, 0)$, $B(x,t) = (0, 0, B_0)$, $b_0 \equiv \text{const}$.

In the dimensionless quantities

$\rho = k_p x$, $\theta = \omega_p t$, $V_1 = \frac{V_x}{c}$, $V_2 = \frac{V_y}{c}$,

$E = -\frac{e}{m c \omega_p} E_x$, $N = \frac{n}{n_0}$, $B_0 = -\frac{e b_0}{m c \omega_p}$, $\nu = \frac{\nu_e}{\omega_p}$,

where $\omega_p = \left(\frac{4\pi e^2 n_0}{m}\right)^{1/2}$ is the plasma frequency, $n_0$ is the value of the unperturbed electron density, $k_p = \omega_p/c$, the result takes the form

$$\begin{align*}
\frac{\partial V_1}{\partial \theta} + V_1 \frac{\partial V_1}{\partial \rho} &= -E - B_0 V_2 - \nu V_1, \\
\frac{\partial V_2}{\partial \theta} + V_1 \frac{\partial V_2}{\partial \rho} &= B_0 V_1 - \nu V_2, \\
\frac{\partial E}{\partial \theta} + V_1 \frac{\partial E}{\partial \rho} &= V_1.
\end{align*}$$

(2)

The solutions of this system are called in [5] nonlinear zero-temperature Bernstein modes.

We consider (2) with the initial conditions

(3)  
$V_1(\rho, 0) = V_1^0(\rho)$, $V_2(\rho, 0) = V_2^0(\rho)$, $E(\rho, 0) = E^0(\rho)$, $\rho \in \mathbb{R}$.

Due to the Gauss theorem the electron density can be found as

(4)  
$N(\rho, \theta) = 1 - \frac{\partial E(\rho, \theta)}{\partial \rho}$.

System (2) is of hyperbolic type. It is well known that for such systems, there exists, locally in time, a unique solution to the Cauchy problem of the same class as the initial data. The blow up of a solution is associated with unbounded derivatives [4].

The Cauchy problem (2), (3) was studied in the following cases:

- $\nu = B_0 = 0$ in [15];
- $B_0 = 0, \nu > 0$ in [14], see also [10] for the relativistic case;
- $\nu = 0, B_0 \neq 0$ in [18].

In all these case we obtained the criterion of the singularity formation for the solution in terms of the initial data and parameters $\nu, B_0$. In other words, for any specific initial data we can say in advance whether the solution keeps the initial smoothness for all $\theta > 0$ or blows up in a finite time $\theta^* > 0$. Moreover, for the
case \( \nu > 0 \) the global smoothness of solution means a stabilization to the trivial equilibrium state \([16]\).

Both parameters \( \nu \) and \( B_0 \) act as regularizers in the following sense: if we fix an arbitrary initial data \([3]\), we can obtain the global smoothness of solution by increasing \( \nu \) or \( |B_0| \). For example, the criterion of the singularity formation for the case \( \nu = 0 \) is the following.

**Theorem 1.** \([18]\). For the existence of a \( C^1 \) – smooth \( \frac{2\pi}{\sqrt{1+B_0^2}} \) - periodic solution 
\((V_1(\theta, \rho), V_2(\theta, \rho), E(\theta, \rho))\) of the problem \([2]\) with \( \nu = 0 \), \([3]\), with \((V_1^0, V_2^0, E^0)\) \( \in \mathcal{C}^2(\mathbb{R}) \), it is necessary and sufficient that at any point \( \rho \in \mathbb{R} \)
\[
(5) \quad \left( (V_1^0)’ \right)^2 + 2 (E^0)’ + 2B_0 (V_2^0)’ - B_0^2 - 1 < 0.
\]

If the opposite inequality \([2]\) holds at least at one point \( \rho_0 \), then the derivatives of the solution turn to infinity in a finite time.

We can see that the frequency of oscillations increases with \( B_0 \) and the smooth solution never stabilizes to a constant state.

For \( B_0 = 0 \) we can draw on the plane \(( (E^0)’(\rho_0), (V_1^0)’(\rho_0)) \) a smooth curve dividing the plane into two domains: one of them corresponds to a globally smooth solution that stabilizes to the zero equilibrium if the point \(( (E^0)’(\rho_0), (V_1^0)’(\rho_0)) \) gets there for all \( \rho_0 \in \mathbb{R} \). If, for certain initial data, the point \(( (E^0)’(\rho_0), (V_1^0)’(\rho_0)) \) does not belong to this domain for at least one \( \rho_0 \), the corresponding solution blows up. The domain of smoothness expands with increasing \( \nu \), and the character of stabilization changes with \( \nu \): for \( \nu \in (0, 2) \) the damping is oscillatory, while for \( \nu \geq 2 \) damping is monotonic starting from a certain moment of time.

The purpose of this article is to study the relationship between the regularizing factors \( \nu \) and \( B_0 \). It is natural to expect that a smooth solution is stabilized in the presence of both factors, and the high intensity of electron-ion collisions suppresses oscillations. However, as we are going to show, the nature of the stabilization is drastically different. Namely, for any fixed \( B_0 \) there is a limit of the oscillation frequency \( \Omega(B_0, \nu) \) as \( \nu \to \infty \), equal to \( B_0 \).

Moreover, for every fixed \( B_0 \), \( |B_0| \leq B_0 = \frac{\sqrt{2}}{2} \) there are two threshold values of the intensity factor of electron-ion collisions \( \nu_1(B_0) \) and \( \nu_2(B_0) \), \( 0 < \nu_1(B_0) < \nu_2(B_0) < \infty \) such that for \( \nu \in [\nu_1, \nu_2] \) the oscillations of the medium are completely suppressed. For other relationships between the parameters \( B_0 \) and \( \nu \), including \( |B_0| \geq B_0 \), stabilization is accompanied by oscillations.

As for the criterion for the formation of singularities, similar to the previous cases, it can also be obtained. However, it is technically much more complex due to the fact that there are five independent parameters. The analytical formulas obtained here are very cumbersome, which hampers the analysis of the solution. However, the use of numerical methods allows one to study the domain of smoothness of the solution when some of the parameters are fixed, and using the methods of computer algebra, one can study the limiting behavior of the oscillation frequency.

The paper is organized as follows. In Sec\([1]\) we obtain a matrix Riccati equation to describe the behavior of the space derivatives of the solution. Then we use the Radon lemma to linearize it and obtain an implicit criterion of the singularity formation. In Sec\([2]\) we propose an algorithm to find the domain of smoothness of the solution numerically and test it for the initial data corresponding to a standard laser pulse, traditionally used in numerical computations. In Sec\([3]\) we study the
character of stabilization of a smooth solution, in particular, oscillations, induced by a large intensity of collisions. In Sec 4 we discuss the results and their applicability in physics.

1. A CRITERION OF THE SINGULARITY FORMATION

System (2) along characteristics can be written as follows:

\[ (6) \frac{dV_1}{d\theta} = -E - B_0 V_2 - \nu V_1, \quad \frac{dV_2}{d\theta} = B_0 V_1 - \nu V_2, \quad \frac{dE}{d\theta} = V_1, \quad \frac{d\rho}{d\theta} = V_1. \]

It immediately implies that along every characteristic \( V_1^2(t) + V_2^2(t) + E^2(t) \leq V_1^2(0) + V_2^2(0) + E^2(0) \), therefore components of the solution remain bounded till the moment of a possible singularity formation.

Let us denote \( q_1 = \frac{dV_1}{d\rho}, q_2 = \frac{dV_2}{d\rho}, s = \frac{dE}{d\rho} \) and (2) with respect to \( \rho \) and obtain along characteristics \( \frac{d}{d\rho} = V_1 \) the following Cauchy problem:

\[ (7) \frac{dq_1}{d\rho} = -q_1^2 - s - B_0 q_2 - \nu q_1, \quad \frac{dq_2}{d\rho} = -q_1 q_2 + B_0 q_1 - \nu q_2, \quad \frac{ds}{d\rho} = q_1(1 - s). \]

\[ (8) q_1(\rho, 0) = q_1^0(\rho), \quad q_2(\rho, 0) = q_2^0(\rho), \quad s(\rho, 0) = s^0(\rho). \]

Note that (7) is split off from the (6) system and can be written as the matrix Riccati equation.

In what follows we need the following version of the Radon lemma (1927) [8], Theorem 3.1, see also [13].

**Theorem 2** (The Radon lemma). A matrix Riccati equation

\[ (9) \dot{W} = M_{21}(t) + M_{22}(t) W - W M_{11}(t) - W M_{12}(t) W, \]

\( W = W(t) \) is a matrix \((n \times m), M_{21} \) is a matrix \((n \times m), M_{22} \) is a matrix \((m \times n), M_{11} \) is a matrix \((n \times n), M_{12} \) is a matrix \((m \times n) \) is equivalent to the homogeneous linear matrix equation

\[ (10) \dot{Y} = M(t) Y, \quad M = \left( \begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array} \right), \]

\( Y = Y(t) \) is a matrix \((n \times (n + m)), M \) is a matrix \(((n + m) \times (n + m)) \) in the following sense.

Let on some interval \( J \in \mathbb{R} \) the matrix-function \( Y(t) = \left( \begin{array}{c} Q(t) \\ P(t) \end{array} \right) \) (\( Q \) is a matrix \((n \times n), P \) is a matrix \((n \times m) \) ) be a solution of (10) with the initial data

\[ (11) Y(0) = \left( \begin{array}{c} I \\ W_0 \end{array} \right) \]

(I is the identity matrix \((n \times n), W_0 \) is a constant matrix \((n \times m) \) ) and \( \det Q \neq 0 \) on \( J \). Then \( W(t) = P(t) Q^{-1}(t) \) is the solution of (6) with \( W(0) = W_0 \) on \( J \).

In our case

\[ (12) W = \left( \begin{array}{c} q_1 \\
q_2 \\
s \end{array} \right), M_{21} = \left( \begin{array}{ccc} 0 \\
0 \\
0 \end{array} \right), M_{22} = \left( \begin{array}{ccc} -\nu & -B_0 & -1 \\
B_0 & -\nu & 0 \\
1 & 0 & 0 \end{array} \right), \]

\( M_{11} = (0), M_{21} = (1 \ 0 \ 0) \).
Lemma 1. The eigenvalues of matrix \( M \), constructed by the rules of Theorem 2 for the blocks (12), are the following: \( \lambda_1 = 0, \lambda_i, i = 2, 3, 4 \) are roots of a 3rd order algebraic equation

\[
\lambda^3 + 2\nu\lambda^2 + (1 + B_0^2 + \nu^2)\lambda + \nu = 0.
\]

The real parts of \( \lambda_k, k = 2, 3, 4 \) are negative for \( \nu > 0 \).

Proof. The method for obtaining (13) is standard. We use the Routh table for polynomials to show that all roots of (13) are in the left half-plane. \( \square \)

Lemma 2. The solution to the Cauchy problem (10), (11) for the matrix \( M \), constructed by the rules of Theorem 2 for the blocks (12), is

\[
W(\theta) = \sum_{k=1}^{4} C_k v_k e^{\lambda_k \theta},
\]

where \( v_k = v_{kj}, k, j = 1, \ldots, 4 \) are the eigenvectors of \( M \) and \( C_k, k = 1, \ldots, 4 \) are solutions to the algebraic linear system

\[
W(0) = \sum_{k=1}^{4} C_k v_k.
\]

The function \( Q(\theta) \), a part of the solution, has the form

\[
Q(\theta) = \sum_{k=1}^{4} C_k v_{k1} e^{\lambda_k \theta}.
\]

The proof is standard. The function \( Q(\theta) \) depends on the initial data \( q_0^1, q_0^2, s^0 \) and parameters \( B_0, \nu \).

Theorem 3. For the existence of a \( C^1 \)-smooth solution \((V_1(\theta, \rho), V_2(\theta, \rho), E(\theta, \rho))\) of the problem (7), (8), where \((V_1^0, V_2^0, E^0) \in C^2(\mathbb{R}), \) it is necessary and sufficient that at any point \( \rho \in \mathbb{R} \) the function \( Q(\theta) \), given as (14), does not have roots at the half-axis \( \theta > 0 \). Otherwise, the solution blows up at the time

\[
T_c = \inf_{\theta > 0} \{ \rho_0 \in \mathbb{R} | Q(\theta, \rho_0) = 0 \}.
\]

Proof. If we solve the respective system (11), then we find the solution to (7) according to Theorem 2. The relation \( W(\theta) = P(\theta)Q^{-1}(\theta) \) implies that the derivatives of the solution to the Cauchy problem (7), (8) go to infinity in a finite time along the characteristic starting from the point \( \rho_0 \in \mathbb{R} \) if and only if there is \( \theta_* > 0 \) such that \( Q(\theta, \rho_0) = 0 \). The smallest root \( \theta_* > 0 \) over all \( \rho_0 \in \mathbb{R} \) corresponds to the blow up time (15). \( \square \)

In what follows we need explicit expressions for \( \lambda_k \) and \( v_k \). To write them we introduce the following constants:

\[
K = B_0^6 + (2\nu^2 + 3)B_0^4 + (\nu^4 - 5\nu^2 + 3)B_0^2 + \left(1 - \frac{3}{4}\nu^2\right),
\]

\[
K_1 = \left(8\nu^3 + 36(2B_0^2 - 1)\nu + 24\sqrt{3K}\right)^\frac{1}{3}, \quad K_2 = \frac{1}{9}(3 + 3B_0^2 - \nu^2).
\]

Then the eigenvalues can be written as

\[
\lambda_2 = \frac{1}{6} K_1 - \frac{K_2}{K_1} - \frac{2}{3}\nu, \quad \lambda_{3,4} = \frac{1}{12} K_1 + \frac{3K_2}{K_1} - \frac{2}{3}\nu \pm i \frac{\sqrt{3}}{2} \left(\frac{1}{6} K_1 + \frac{K_2}{K_1}\right).
\]
The respective eigenvectors are

\[
\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v}_k = \begin{pmatrix} \frac{\nu}{\lambda_k} \\ \frac{\nu}{\lambda_k} \phi_k \\ \phi_k \end{pmatrix}, \quad k = 2, 3, 4.
\]

2. Global smoothness induced by the regularizing factors for a standard pulse

We call the "standard pulse" a set of initial data that looks like:

\[ V_1^0(\rho) = V_2^0(\rho) = 0, \quad E_0(\rho) = k \rho \exp\left\{ -\frac{\rho^2}{\sigma} \right\}, \quad k = \left( \frac{a_s}{\rho_\ast} \right)^2, \quad \sigma = \frac{\rho_\ast^2}{2}. \]

The data (18) was chosen in the assumption that the oscillations are excited by a laser pulse with a frequency of \( \omega_l \) (\( \omega_l \gg \omega_p \)) when it is focused in a line, which can be achieved by using a cylindrical lens (19). The parameters \( \rho_\ast \) and \( a_s \) characterize the scale of the localization region and the maximum value \( E_{\max} = a_s^2/(\rho_\ast 2\sqrt{\varepsilon}) \approx 0.3a_s^2/\rho_\ast \) of the electric field, respectively. The specific values of \( \rho_\ast \) and \( a_s \) are not important here. This kind of data are traditionally used for numerical simulation of different phenomena in a rarefied plasma, see e.g. (3).

Although the way of applying Theorem 3 is formally clear, the technical side is fraught with great difficulties. Indeed, we need to solve the transcendental equation \( Q(\theta) = 0 \) with very cumbersome coefficients, which depends on the eigenvectors of \( M \). Nevertheless, for any fixed set of parameters \( q_1^0, q_2^0, s^0, B_0, \nu \) we use a numerical procedure to study the effect of each particular parameter.

To study the blow-up for the data (18) we choose the most "dangerous" point \( \rho_0 = 0 \), where the derivative \( (E_0)' \) has a maximum. There \( q_1^0 = q_2^0 = 0, s^0 = k \), this reduces the number of free parameters to three and simplifies the analysis.

First, we picture on the plane \((\nu, B_0)\) the set of parameters that leads to a finite time blow-up for a fixed \( s^0 = k \). If \( k < \frac{1}{2} \), then this set is empty. This follows, for example, from Theorem 3 for the case \( \nu = 0 \) and from the estimate of the critical \( k \) from (14), \( \nu \in [0, 2) \), \( B_0 = 0 \),

\[ k_{cr} = \frac{1}{1 + \exp\left( -\frac{\rho_\ast^2}{\sqrt{4\varepsilon}} \right)}. \]

We note also that the nonnegativity of density dictates the requirement \( k < 1 \) (see (11)). Thus, we choose for the numerical illustrations the interval \( k = s^0 \in (\frac{1}{2}, 1) \). To estimate the sign of minimum of \( Q(\theta) \) for \( \theta > 0 \) we used Newton’s method built into the Optimization package of MAPLE. Fig.1, presents the "blow-up" domains for \( k = 0.6, k = 0.7 \) and \( k = 0.8 \) and shows that this domain expands with increasing \( k \). The limits of \( \nu \) as \( B_0 \to 0 \) on the boundary of the domain correspond to (19).

The next numerical experiment that we can perform is to fix \( \nu \) and \( B_0 \) and to find a domain \( \mathcal{D} \), consisting of points \( q_1^0, q_2^0, s^0 \), corresponding to a globally in time smooth solution (2), (18), provided at every point \( \rho \in \mathbb{R} \) the derivatives of the initial data fall in this domain. In other words, we try to picture an analog of the plain domain found in (14) for a fixed \( \nu \). However, the considered domain \( \mathcal{D} \) is three-dimensional,
and the result of calculations is difficult to present in a convenient form. Therefore one can draw a section of this domain, for example, with \( q_0^2 \) = const. We do not present the result of this computation here, just note that the "blow up" domain, the complement of the domain \( D \) to the entire space, expands with increasing \( q_0^2 \).

3. Character of stabilization

In this section, we establish certain properties of smooth solutions.

**Theorem 4.** Let the initial data (3) are such that the solution of the Cauchy problem (2), (3), is globally \( C^1 \) - smooth.

1. For any fixed \( B_0 \) there exists \( \lim_{\nu \to \infty} \Im \lambda_3(B_0, \nu) = B_0 \);
2. The motion is oscillatory if and only if \( K > 0 \), given as (16), is positive. If, in addition, \( \nu > 0 \), then the solution decays to the trivial steady state. If \( K \leq 0 \), then the solution decays to the trivial steady state monotonically starting from a sufficiently large \( \theta \).

**Proof.**
1. The first properties follows from the expansion
   \[
   \lambda_3(B_0, \nu) = -\nu + iB_0 + O\left(\frac{1}{\nu}\right), \quad \nu \to \infty.
   \]
   It implies that the oscillations cannot be suppressed by a large frequency of collisions.

2. If the function \( Q(\theta) \) does not have roots for \( \theta > 0 \) for all \( \rho_0 \in \mathbb{R} \), then taking into account Lemma 2 we conclude that the solution to (2), (3) stabilizes to the trivial state provided \( e^{\nu^k} \to 0, \theta \to \infty, \) \( k = 2, 3, 4 \).

   The stabilization is monotonically starting from a certain moment of time if and only if \( \Im \lambda_k = 0 \) for all \( k = 2, 3, 4 \).

   If \( K \) is nonnegative, then \( \Im \lambda_2 = 0 \), and \( \Im \lambda_{3,4} \neq 0 \) provided \( K_1^2 + 36K_2 \neq 0 \). It can be readily checked that \( K_1^2 + 36K_2 = 0 \) implies \( K = 0 \). We take into account that \( K_1 \) and \( K_2 \), given as (17), both vanish at the hyperbola \( 3B_0^2 - \nu^2 + 9 = 0 \), so the ratio \( \frac{K_2}{K_1} \) is finite.
If $K < 0$, then it suffices to prove that $\Im \lambda_3 = 0$. Since $|\Im \lambda_3| = |\Im \lambda_4|$, Lemma 1 implies that $\lambda_k$, $k = 2, 3, 4$, are real and negative. First of all we notice that if $K < 0$, then necessarily $K_1 = a + ib$, $a, b \in \mathbb{R}$, $ab \neq 0$. Then it is easy to compute that

\[ \Im \lambda_3 = \frac{\sqrt{3a - b}}{12(a^2 + b^2)} (36K_2 + a^2 + b^2). \]

Since $K_1 = (A \pm iB)^\frac{1}{2}$, $A = 8\nu^4 + 36(2B_3^2 - 1)\nu$, $B = 24\sqrt{-3K}$, we can find $a$ and $b$, then substitute the result in (20) and check directly that $\Im \lambda_3 = 0$. This is a rather cumbersome computation, however, it can be done using computer algebra.

Note that since we are extracting the 3rd order roots of complex numbers, we must account for the corresponding branch of this multivalued function. □

**Corollary 1.** Let the solution of (2), (3) is globally smooth in time. For every fixed $B_0$, $|B_0| \leq \bar{B}_0 = \sqrt{2}$ there are two threshold values of the factor of the intensity of collisions, $\nu_1(B_0)$ and $\nu_2(B_0)$, $\frac{2}{3}\sqrt{6} < \nu_1(B_0) < \nu_2(B_0) < +\infty$ such that for $\nu \in [\nu_1, \nu_2]$ the oscillations of the medium are completely suppressed. For other relations between the parameters $|B_0|$ and $\nu > 0$, including $B_0 \geq \bar{B}_0$, the stabilization is accompanied by oscillations.

For the proof we have to study the curve $K(B_0, \nu) = 0$ on the plane $(B_0, \nu)$. We resolve $K = 0$ with respect to $\nu$ and get two branches $\nu = \nu_{\pm}(B_0)$ with the following properties:

\[ \nu_{\pm} = \frac{1}{4} \sqrt{2 + 40B_0^2 - 16B_0^4 \pm 2\sqrt{1 - 24B_0^2 + 192B_0^4 - 512B_0^6}}. \]

Thus,

\[ B_0 \in [-\bar{B}_0, \bar{B}_0], \quad \bar{B}_0 = \sqrt{2} \approx 0.353 \ldots, \quad \nu_{\pm}(\bar{B}_0) = \frac{3}{4}\sqrt{6} \approx 1.837 \ldots, \]

the limit values are

\[ \lim_{B_0 \to 0} \nu_+ = 2, \quad \lim_{B_0 \to 0} \nu_- = +\infty. \]

The points $K(B_0, \nu) = 0$ for $B_0 \geq 0$ are presented in Fig.2, the picture is symmetric with respect to $B_0 = 0$. Thus, if we fix any $B_0 \in [-\bar{B}_0, \bar{B}_0]$ and begin to increase $\nu$, we start in the domain $K > 0$, then at $\nu_1 = \nu_-(\bar{B}_0)$ we fall in the domain $K \leq 0$ and then leave this domain at $\nu_2 = \nu_+(\bar{B}_0)$. This observation proves the corollary.

We note that the results is in compliance with the case $B_0 = 0$, studied in [14], where it was shown that the monotonic decay takes place for $\nu \geq 2$. □

**Remark 1.** The following expansions also make it possible to trace the effect of low collision intensity on the character of oscillations:

\[ \lambda_2(B_0, \nu) = -\frac{1}{B_0^2 + 1}\nu + O(\nu^2), \quad \nu \to 0, \]

\[ \lambda_{3,4}(B_0, \nu) = \pm i \sqrt{B_0^2 + 1} - \frac{1}{2} \frac{2B_0^2 + 1}{B_0^2 + 1}\nu + O(\nu^2). \]
We see that in the first approximation, the low intensity of collisions does not change the frequency of the oscillations, but causes their exponential decay. Generally speaking, the rate of this damping does not decrease at large values of $|B_0|$.

**Remark 2.** It should be noticed that the oscillations induced by friction are known in mechanical systems (see, e.g. [2]).

4. **Discussion**

We performed a mathematical analysis of equations of upper hybrid oscillations [2] for all allowed values of the constant parameters $\nu$ and $B_0$. Nevertheless, it should be noted that the most interesting phenomena, such as the persistence of oscillations at large values of $\nu$, can hardly be observed experimentally, since the physical values of $\nu$ are of the order $10^{-2}$ (see [14] for details). For small values of $\nu$, the effect of the external magnetic field is very close to the case $\nu = 0$ considered in [13]: basically, increasing the magnetic field prevents blow-up (except for specially adapted initial data). It should also be noted that the most powerful regularizing factor is the dependence of the intensity of electron-ion collisions on the density. An analysis of this situation in the absence of a magnetic field was carried out in [17]. It is natural to expect that, in the presence of a magnetic field, the effect of suppressing the formation of singularities for all smooth initial data will be retained.

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