BMN operators with a scalar fermion pair and operator mixing in $\mathcal{N} = 4$ Super Yang-Mills Theory

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Abstract: The mixings between BMN operators with two scalar impurities and those with a scalar fermion pair are discussed to the lowest order at planar level. For this purpose, matrix model effective vertices are calculated to $O(g^3)$. All the mixing patterns are explicitly obtained.

Keywords: Gauge Symmetry, Supersymmetric gauge theory
1. Introduction

\( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory attracts a lot of interest after the discovery of AdS/CFT correspondence. During the early years, due to the BPS properties of the states from dimensional reduction of the supergravity, studies on the CFT side were mainly focused on the protected operators (see [1] for a review). The proposal of BMN correspondence [2] provides further evidence of AdS/CFT beyond supergravity. Operators \( \text{Tr}(Z^J) \) are BPS and correspond to the string vacuum states \( |0,p^+\rangle \) in the light-cone quantization of the string in a pp wave background, whereas operators like \( \text{Tr}(\phi_i Z^p \phi_j Z^{J-p}) \) are not protected and get mixed in order to remain the eigenstates of the one-loop dilatation operator. In the BMN limit these eigenstate operators (BMN operators) correspond to the string excitation states \( \alpha_n^{i\dagger} \alpha_n^j |0,p^+\rangle \) (see e.g. [3] for a pedagogical review). BMN operators involving vector [4–6] and fermions [7–10] are also examined. Exact BMN operators at finite \( N \) and \( J \) are obtained in [11] and extended to the whole superconformal multiplet incorporating BMN operators with various combinations of two impurities by using superconformal symmetry. Operator mixings beyond one loop in the scalar sector are also discussed in [12].

Unlike the mixings in the pure scalar sector, mixings between scalar and fermionic sectors are less studied. In [7], some low dimension operator mixings are discussed. Instanton contributions to their mixings are also studied in [13]. After mapping the operators into spin chains and the dilatation operator into a spin-chain Hamiltonian, a Bethe ansatz can be used to solve the mixing problem [14,15]. Integrability and the analytic Bethe ansatz
make it possible to obtain anomalous dimensions for operators with more than two impurities. For $\mathcal{N} = 4$ SUSY YM, this was first proposed in [14] for an $so(6)$ spin chain and further extended to the $psu(2,2|4)$ super spin chain in [16]. A subsector $su(2|3)$ dynamic spin chain which includes the mixing between fermion pairs and bosons is discussed along these lines in [17]. The coordinate-space Bethe ansatz in calculating the spin-chain $S$-matrix can also produce the eigenstates with multiple impurities [15, 18]. This is known as the perturbative asymptotic Bethe ansatz in this context. The $su(2|2)$ dynamic $S$-matrix of the $su(2|3)$ subsector was subsequently discussed using the nested Bethe ansatz [19]. In this subsector, the fermions form singlets of the $SU(3)$ subgroup of the $R$ symmetry.

In the present paper, the fermions we will discuss transform under the $SO(4) = SU(2) \times SU(2)$ subgroup of the $SU(4)_R$ symmetry as two fundamental representations for the two $SU(2)$’s respectively. As a first attempt, we only discuss the mixings between BMN operators with two impurities at planar level and the fermion pair form a Lorentz scalar. In particular, we will discuss the mixings between operators:

$$O_{-1}^{1,j} = \left(\frac{2\pi}{\sqrt{N}}\right)^{J+2} \text{Tr}(ZZ^{-j+1}) \quad (1.1)$$

$$O_{ij}^{p,j} = \left(\frac{2\pi}{\sqrt{N}}\right)^{J+2} \text{Tr}(\phi_i Z^p \phi_j Z^{-j-p}) \text{, for } i, j = 1, 2, 4, 5 \quad (1.2)$$

$$\Psi_{ij}^{p,j} = \left(\frac{2\pi}{\sqrt{N}}\right)^{J} \left(\frac{2^{1/4} \pi}{\sqrt{N}}\right)^2 \text{Tr}(\psi_i^\alpha Z^p \psi_j^\beta Z^{-j-p}) \text{, for } i, j = 1, 2, \quad (1.3)$$

$$\Psi_{ij}^{p,j} = \left(\frac{2\pi}{\sqrt{N}}\right)^{J} \left(\frac{2^{1/4} \pi}{\sqrt{N}}\right)^2 \text{Tr}(\bar{\psi}_{i,\dot{\alpha}} Z^p \bar{\psi}_{j,\dot{\beta}} Z^{-j-p}) \text{, for } i, j = 3, 4. \quad (1.4)$$

We have normalized the operators so that

$$\langle O_{ij}^{p,j}(x) \bar{O}_{ij}^{p,j}(0) \rangle = \frac{1}{x^{2(J+2)}},$$

$$\langle \Psi_{ij}^{p,j-1}(x) \bar{\Psi}_{ij}^{p,j-1}(0) \rangle = \frac{1}{x^{2(J+2)}} \quad (1.5)$$

for tree level planar diagrams. From the $SO(4)$ point of view, since scalars with two impurities can form single trace operators of $1 + 3 + \bar{3} + 9$ representations, operators like (1.3) with $i = 1, 2$ and $j = 3, 4$ which is in vector 4 representation can not be mixed with (1.2). Because the common representations of two scalars or two spinors are $1 + 3 + \bar{3}$, we expect that they only mix in these representations.

In [20], a method is developed to obtain the mixing between these operators using the property that the eigenstate operator should be annihilated by some supercharges. We will reproduce their results by directly diagonalizing the dilatation operator and give all the mixing patterns. Since we only consider two impurity cases, it is easy to do it without using Bethe ansatz. We will follow the method used in [21–25] to deduce the dilatation operator in the name of effective vertices. Even though the algebraic method was used to extend the scalar one-loop dilatation operator to the full $\mathcal{N} = 4$ SUSY YM [27], we will still deduce the one-loop effective vertices by feynman diagrams since it is not hard to calculate
the correlation functions with only scalar fermion pairs [9, 10] and we can formulate the effective vertices taking advantage of the scalar property of the fermion pair. We also need to calculate the effective vertices to $O(g^3)$ which is the lowest order at which the mixing between scalar sector and fermionic sector occurs.

In section 2, we review the matrix model effective vertices and their relation with dilatation operators and calculate the one-loop effective vertices with one scalar fermion pair. In section 3, we review the mixing of pure bosonic operators using the one-loop effective vertices and obtain the mixing of operators with one scalar fermion pair using the results from pure scalar operator mixing. In section 4, we perform a general order analysis about the form of the eigenstates. In section 5, we give the matrix model vertices to $O(g^3)$ with a pair of fermions. In section 6, we use the effective vertices to obtain the dilatation matrix to $O(g^3)$ and solve the eigenvector operators. The results are consistent with [20]. Section 7 is the discussion and outlook. Appendix A gives the conventions we use in our paper.

2. One-loop matrix model effective vertices and dilatation operators

Let us recall the relation between matrix model vertices and dilatation operators in [25]. The one-loop two-point function can be written as

$$\langle O_\alpha(x) \bar{O}_\beta(0) \rangle = \frac{1}{|x|^{2(J+2)}} \left( S_{\alpha\bar{\beta}} + T_{\alpha\bar{\beta}} \log(|x\Lambda|^2) \right).$$

The dilatation operator matrix element can be expressed as

$$D_{\alpha\beta} = (J + 2) \delta_{\alpha\beta} + T_{\alpha\bar{\gamma}} (S^{-1})_{\gamma\beta}.$$  

The basic idea of [22] is to construct matrix model effective vertices $H$ to reproduce the one-loop $T$ matrix elements for scalar sector by

$$T_{\alpha\beta} = \langle O_\alpha H \bar{O}_\beta \rangle = H_\alpha^\gamma \langle O_\gamma O_\beta \rangle = H_\alpha^\gamma S_{\gamma\beta}$$

and

$$H = \frac{g^2}{2(4\pi^2)} (V_D + V_F + V_K)$$

with

$$V_D = \frac{1}{2} : \text{Tr} [\phi^+_m, \phi^-_m] [\phi^+_n, \phi^-_n] : + N : \text{Tr} \phi^+_m \phi^-_m : - : \text{Tr} (\phi^+_m) \text{Tr} (\phi^-_m) :$$

$$V_F = -\frac{1}{2} : \text{Tr} [\phi^+_m, \phi^+_n] [\phi^-_m, \phi^-_n] :$$

$$V_K = -\frac{1}{4} : \text{Tr} [\phi^+_m, \phi^-_n] [\phi^+_n, \phi^-_m] :$$

where repeated $m$ and $n$ sum over $1, \ldots, 6$. $\phi^+$ and $\phi^-$ denote the $\phi$ at $x$ and $0$ respectively and contractions are nonzero only between $\phi^+$ and $\phi^-$. 

$$\langle (\phi^+_i)_{ab} (\phi^+_j)_{cd} \rangle = \delta_{ij} \delta_{ac} \delta_{bd}, \quad \langle \phi^+_i \phi^+_j \rangle = (\phi^+_i \phi^+_j) = 0.$$
It can be proven that $V_D$ is cancelled out when acting on traces of scalars $\phi$. But we need to reexamine this result when considering operators with fermion pairs like $\Psi^{p,J}_{ij}$ later. From (2.1), we see that $H_{\alpha \gamma}$ are the matrix elements of one-loop Dilation operator when acting on the operators with traces of scalars. For the operators $\Psi^{p,J}_{ij}$ with a scalar fermion pair the foregoing discussion does not need to be modified except that the $H$ operator should include the fermionic sector. Notice that for states in (1.4), $S_{\alpha \beta}$ is diagonalized and normalized. So to change the operators into Matrix model operators we need to make the replacement:

$$2\pi \phi_i \rightarrow \phi_i.$$  \hspace{1cm} (2.9)

Since we are interested in the operators with only scalar fermion pairs, we need only to calculate the correlation functions with scalar fermion pairs in order to obtain the effective vertices. The $\sigma^\mu$ matrices in the fermion propagators of SUSY YM would be assembled into matrix model coefficients and will not appear in the correlation functions of two fermions in the matrix model. We can specify the contraction rules of the fermions in the matrix model:

$$\langle (\psi_\alpha)_{ab} (\bar{\psi}_{\dot{\alpha}})_{cd} \rangle = \epsilon_{\alpha \dot{\alpha}} \delta_{ad} \delta_{bc}.$$  \hspace{1cm} (2.10)

and require

$$\psi^\dagger_\alpha = \bar{\psi}_{\dot{\alpha}}, \quad (\psi^\alpha)^\dagger = -\bar{\psi}_{\dot{\alpha}}.$$  \hspace{1cm} (2.11)

The correspondence between the SUSY YM operators and matrix model operators is:

$$\left(2^{1/4} \pi\right) \psi \rightarrow \left(\frac{1}{2^{1/4}}\right) \psi.$$  \hspace{1cm} (2.12)

We need to calculate the one-loop diagrams in SUSY YM in figure 1 to get the effective vertices. One can consult [10] for a detailed calculation of these diagrams. Because we are considering operators with only one scalar fermion pair, we do not need to worry about the contractions between different scalar fermion pairs. For a connected diagram like:

$$\sim \frac{4ig^2}{(4\pi^2)^3} (\bar{\sigma}^\mu)^{\beta\alpha} (x - y)^\mu \ln\left(-(x - y)^{-2}\Lambda^{-2}\right) \frac{(x - y)^6}{(x - y)^6},$$  \hspace{1cm} (2.13)

we can add another free fermion propagator to combine with the fermion $su(2)$ indices to form a Lorentz scalar coefficient. As a result we include another trace of a fermion pair in

\textbf{Figure 1:} One-loop diagrams in calculation of the effective vertices. The scalars and fermions are denoted as solid and dashed lines respectively.
the matrix model vertices. The coefficients are obtained so that the \( T \) matrix is the same both from SUSY YM side and the matrix model side:

\[
H_\psi = \frac{g^2}{2(4\pi^2)} \left\{ N : \left( \text{Tr} \psi^{\alpha-} \bar{\psi}^{\beta+} - \frac{1}{N} \text{Tr} \psi^{\alpha-} \text{Tr} \bar{\psi}^{\beta+} \right) \text{Tr} \psi^{\alpha-} \bar{\psi}^{\beta+} : + (- \leftrightarrow -) \right. \\
- \frac{1}{4} : \text{Tr} \left( \{ \psi^{\beta A-}, \bar{\psi}^{\alpha A+} \} \{ \Z^+, \Z^- \} \right) \text{Tr} \psi^{\beta A-} \bar{\psi}^{\alpha A+} : + (+ \leftrightarrow -) \\
- \frac{1}{4} : \text{Tr} \left( \{ \psi^{\beta A+}, \bar{\psi}^{\alpha A-} \} \{ \Z^+, \Z^- \} \right) \text{Tr} \psi^{\beta A+} \bar{\psi}^{\alpha A-} : + (+ \leftrightarrow -) \\
+ \frac{1}{2} \sum_{i=1,2} : \text{Tr} \left( \{ \psi^{\beta A-}, \Z^+ \} \{ \Z^-, \bar{\psi}^{\alpha A+} \} \right) \text{Tr} \psi^{\beta A-} \Z^+ : + (+ \leftrightarrow -) \\
- \frac{1}{2} \sum_{i=3,4} : \text{Tr} \left( \{ \psi^{\beta A+}, \Z^- \} \{ \Z^+, \bar{\psi}^{\alpha A-} \} \right) \text{Tr} \psi^{\beta A+} \Z^- : + (+ \leftrightarrow -) \\
+ \frac{1}{4} \sum_{i=1,2} : \text{Tr} \left( \{ \psi^{\beta A-}, \Z^+ \} \{ \Z^-, \bar{\psi}^{\alpha A+} \} \right) \text{Tr} \psi^{\beta A-} \Z^+ : + (+ \leftrightarrow -) \\
- \frac{1}{4} \sum_{i=3,4} : \text{Tr} \left( \{ \psi^{\beta A+}, \Z^- \} \{ \Z^+, \bar{\psi}^{\alpha A-} \} \right) \text{Tr} \psi^{\beta A+} \Z^- : + (+ \leftrightarrow -) \\
\left. + \frac{1}{2} : \text{Tr} \left( \{ \psi^{\beta A+}, \bar{\psi}^{\alpha A-} \} \{ \psi^{\beta A+}, \bar{\psi}^{\alpha A-} \} \right) : + (+ \leftrightarrow -) \right\}, \quad (2.14)
\]

where \((+ \leftrightarrow -)\) means exchanging the + and − in the superscripts. We will use this notation throughout the paper.

Acting on an operator \( \Psi_{ij}^{\mu} \), the terms in \((2.4)\) which contribute are only in \((2.3)\) and can be written as:

\[
V_D \sim \frac{1}{2} \text{Tr} \left( \{ \Z^+, \Z^- \} \{ \Z^+, \Z^- \} \right) - N \text{Tr} \Z^- \Z^+ + \text{Tr} \Z^- \text{Tr} \Z^+ \quad (2.15)
\]

Like in the bosonic case, we can prove that these terms cancel with the second and third line in \((2.14)\). To see this, contract one \( \Z^+ \) in the first term of \((2.15)\) with \( \Z \) in \( \text{Tr} \left( \psi^{\alpha-} \Z^p \psi^{\beta+} \right) \) and after some cancellations, the terms left are

\[
\frac{1}{2} \left\{ -\text{Tr} \left( \psi^{\alpha-} \Z^p \psi^{\beta+} \right) + \frac{1}{2} \text{Tr} \left( \psi^{\alpha-} \Z^p \Z^+ \psi^{\beta+} \right) \right. \\
- \text{Tr} \left( \psi^{\alpha-} \Z^p \psi^{\beta+} \right) + \frac{1}{2} \text{Tr} \left( \psi^{\alpha-} \Z^p \Z^+ \psi^{\beta+} \right) \right\} \\
= \frac{1}{2} \text{Tr} \left( \psi^{\alpha-} \Z^p \psi^{\beta+} \right) - \frac{1}{2} \text{Tr} \left( \psi^{\alpha-} \Z^p \psi^{\beta+} \right). \quad (2.16)
\]

Contracting the first term in the second line of \((2.14)\) with the same operator and keeping track of the minus sign when one fermionic operator moves over the other, one gets

\[
\frac{1}{2} \text{Tr} \left( \psi^{\alpha-} \Z^p \psi^{\beta+} \right) + \frac{1}{2} \text{Tr} \left( \psi^{\alpha-} \Z^p \psi^{\beta+} \right), \quad (2.17)
\]

which just cancels \((2.16)\). Also as in the bosonic case, contracting the \( \Z^+ \) which is used in the contraction in \((2.16)\) with the \( \Z^- \) inside the first term in \((2.15)\) cancels the other terms left.
3. One-loop operator mixing

We only need to discuss the operator mixing at planar level. The pure bosonic operator eigenstates are given in [11]. Since we confine ourselves to only two impurity cases, it is easier to directly diagonalize the \( H \) matrix than use a Bethe ansatz. Let us review the bosonic operator mixing first. \( O_{ij} \) can be decomposed into \( SO(4) \) representations:

\[
\begin{align*}
O_{1}^{-1,J} & = - \text{Tr}(\bar{Z}Z^{J+1}), \\
O_{1}^{p,J} & = \frac{1}{2} \sum_{i=1,2,4,5} O_{ii}^{p,J}, \\
O_{(ij)}^{p,J} & = \frac{1}{2} \left( O_{ij}^{p,J} + O_{ji}^{p,J} \right) - \frac{1}{2} O_{1}^{p,J}, \\
O_{[ij]}^{p,J} & = \frac{1}{2} \left( O_{ij}^{p,J} - O_{ji}^{p,J} \right). 
\end{align*}
\]

\( O_{1}^{-1,J} \) and \( O_{1}^{p,J} \) are \( SO(4) \) singlets. \( O_{(ij)}^{p,J} \) is in the 9 representation and \( O_{[ij]}^{p,J} \) is in the \( 3 + \bar{3} \) representation.

The \( H \) matrix for \( O_{1}^{p,J} \) is

\[
\begin{align*}
\text{for } J \text{ odd:} \\
\frac{g^2 N}{4\pi^2} \begin{pmatrix}
1 & -1 & -1 & -1 \\
-1 & 2 & 1 & 1 \\
-1 & 2 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
-1 & 2 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
\end{pmatrix}, \\
\text{for } J \text{ even:} \\
\frac{g^2 N}{4\pi^2} \begin{pmatrix}
1 & -1 & -1 & -1 \\
-1 & 2 & 1 & 1 \\
-1 & 2 & 1 & 1 \\
\vdots & \vdots & \vdots & \vdots \\
-1 & 2 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
\end{pmatrix}, \\
\end{align*}
\]

Eigenstates: \( n = 0, \ldots, (J+1)/2 \)
\( \left( 2 \cos \frac{\pi n \pi}{J+3}, \ldots, 2 \cos \frac{(2p+3)\pi n}{J+3}, \ldots, 2 \cos \frac{(J+2)\pi n}{J+3} \right)^T, \)

where all the unspecified matrix elements are zero. The eigenstates can be written in both cases as

\[
\tilde{O}_1^{J,n} = \sum_{p=0}^{J} \cos \frac{(2p+3)\pi n}{J+3} O_{1}^{p,J} + 2 \cos \frac{\pi n}{J+3} O_{1}^{-1,J}.
\]

The eigenvalue is

\[
\delta D_2 = \frac{g^2 N}{4\pi^2} 4 \sin^2 \frac{n\pi}{J+3}.
\]

In fact, we could take the coefficients before \( O_{1}^{p,J} \) to be \( \cos \frac{(2p+a)\pi n}{J+b} \) or \( \sin \frac{(2p+a)\pi n}{J+b} \) which satisfies the eigenvalue requirement except the last line and the first line in the matrix for \( J = \text{odd} \). The last line and the first line act as boundary conditions: the last line determines the coefficients to be \( \cos \frac{(2p+a)\pi n}{J+a} \) and if we require \( a, b \) to be independent of \( J \), the first line determines \( a = 3 \).
For the symmetric operator $O_{(ij)}$, the $H$ matrix is the same form as in the $O_1$ case except that $p$ starts from 0 not from $-1$. As a result, the eigenstates are:

$$\tilde{O}_{J,n}^{J,n} = \sum_{p=0}^{J} \cos \left( \frac{(2p + 1)\pi n}{J + 1} \right) O_{(ij)}^p,$$

and the eigenvalue is

$$\delta D_2 = \frac{g^2 N}{4\pi^2} 4 \sin^2 \frac{n\pi}{J + 1}.$$  

(3.5)  

(3.6)

For the anti-symmetric operators $O_{(ij)}^p$, ($p = 0, 1, \ldots, [(J - 1)/2]$), the $H$ matrix is

for $J$ odd:

$$\begin{pmatrix}
2 & -1 & \cdots & -1 & 2 & -1 \\
-1 & 2 & \cdots & -1 & 2 & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & 2 & \cdots & -1 & 2 & -1 \\
-1 & 3 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},$$

Eigenstates: $n = 1, \cdots, (J + 1)/2$

$$\begin{pmatrix}
2 \sin \frac{2\pi n}{J + 2}, \ldots, 2 \sin \frac{(2p + 2)\pi n}{J + 2} \\
\cdots \\
2 \sin \frac{(J + 1)\pi n}{J + 2} \\
\end{pmatrix}^T,$$

(3.7)

for $J$ even:

$$\begin{pmatrix}
2 & -1 & \cdots & -1 & 2 & -1 \\
-1 & 2 & \cdots & -1 & 2 & -1 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
-1 & 2 & \cdots & -1 & 2 & -1 \\
-1 & 3 & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix},$$

Eigenstates: $n = 1, \cdots, J/2$

$$\begin{pmatrix}
2 \sin \frac{2\pi n}{J + 2}, \ldots, 2 \sin \frac{(2p + 2)\pi n}{J + 2} \\
\cdots \\
2 \sin \frac{J\pi n}{J + 2} \\
\end{pmatrix}^T,$$

The eigenstates can be written as

$$\tilde{O}^{J,n}_{[ij]} = \sum_{p=0}^{J} \sin \left( \frac{(2p + 2)\pi n}{J + 2} \right) O^{p,J}_{[ij]}$$

(3.8)

and the eigenvalue is

$$\delta D_2 = \frac{g^2 N}{4\pi^2} 4 \sin^2 \frac{n\pi}{J + 2}.$$  

(3.9)

For operators with a scalar fermion pair, it can be checked that the Hamiltonian can be recast as

$$H = -\frac{g^2 N}{2(4\pi^2)} \sum_{i=1}^{L} (1 - \Pi_{i,i+1})$$

(3.10)

where $\Pi_{i,i+1}$ is the graded permutation operator which exchanges the adjacent fields and picks up a minus sign if the two fields are fermions. $L$ is the total number of fields in the operator to be acted on. This equation may not be true for two or more pairs of fermions, because there could be changes like $\psi^\alpha \psi^\beta \psi^\gamma \psi^\delta \rightarrow \psi^\alpha \psi^\beta \psi^\gamma \psi^\delta$.

As in the pure bosonic cases, we can also define symmetric and antisymmetric combinations for $(i, j = 1, 2)$ or $(i, j = 3, 4)$:

$$\Psi_{(ij)}^{p,J} = \frac{1}{2} (\Psi_{ij}^{p,J} + \Psi_{ji}^{p,J}),$$

$$\Psi_{[ij]}^{p,J} = \frac{1}{2} (\Psi_{ij}^{p,J} - \Psi_{ji}^{p,J}).$$

(3.11)
\[ \Psi_{p,J}^{i,j} \] are in the 3 representation for \( i, j = 1, 2 \) or \( \bar{3} \) representation for \( i, j = 3, 4 \). We expect that \( \mathcal{O}_{[ij]}^{p,J} \) will be mixed with \( \Psi_{p,J}^{i,j} \) and \( \mathcal{O}_{[ij]}^{p,J} \) with \( \Psi_{p,J}^{i,j} \).

The \( H \) matrix for \( \Psi_{p,J}^{i,j} \) is the same as in \( \mathcal{O}_{[ij]}^{p,J} \) case, therefore the eigenstates could be obtained just by replacing \( \Psi \) with \( \Psi \) in (3.3) and the eigenvalues are the same. In a similar way, the \( \Psi_{[ij]}^{p,J} \) case is the same as \( \mathcal{O}_{[ij]}^{p,J} \) case.

From these diagonalization procedures, we see that for a matrix with \( -1, 2, -1 \) at the near diagonal and diagonal positions like in (3.2) and (3.7), we can use an ansatz like

\[
\Psi_{p,J}^{i,j} = \sum a_i b_j \sin \frac{2\pi n_i}{4\pi} \sin \frac{2\pi n_j}{4\pi}.
\]

Or, if we are only \( \Psi_{p,J}^{i,j} = 0 \) or \( \Psi_{p,J}^{i,j} = 0 \), we can use an ansatz like \( \cos \frac{(2p+a)n \pi}{J+b} \) or \( \sin \frac{(2p+a)n \pi}{J+b} \) as coefficients because they automatically satisfy the eigenvalue equations. Combining the coefficient before the matrix (3.2) and (3.7), the eigenvalue is \( \frac{2\pi}{4\pi} N \pi \). The variables \( a, b \) and sin or cos are determined from the boundary condition: the first line and the last line of the matrix and the requirement that \( a \) and \( b \) are independent of \( J \). This is much like solving an eigenvalue problem of a differential equation with boundary conditions. This insight is helpful when we solve the mixing between pure scalar operators and operators with fermions.

4. Order analysis

In general, the \( SO(4) \) singlet part of \( \mathcal{O}_{[ij]}^{p,J} \) and \( \Psi_{[ij]}^{p,J-1} \) should also be mixed with operator

\[
V_{p,J} = \text{Tr}(D_{0} ZZ^p D^0 ZZ^{J-p-2}) \tag{4.1}
\]

In paper [26], the authors use the method developed in [20] to discuss the mixings among these operators in which \( V_{p,J} \) appears only from \( O(g^2) \) terms. We will try to understand the general form of the eigenstate vector by order analysis in this section.

Let us look at the Dilatation matrix structure of these three kinds of operator system to \( O(g^4) \). For the combined operator basis \( (\ldots, \mathcal{O}_p, \ldots, \ldots, \Psi_{p,J}^{i,j}, \ldots; \ldots, \Psi_{p,J}^{i,j}, \ldots) \), the correction to the dilatation operator up to \( O(g^4) \) in matrix blocks is in this form:

\[
\delta D \sim \begin{pmatrix}
O(g^2) + O(g^4) & O(g^3) & O(g^4) \\
O(g^2) & O(g^2) + O(g^4) & O(g^3) \\
O(g^4) & O(g^3) & O(g^2) + O(g^4)
\end{pmatrix} \tag{4.2}
\]

In general, the eigenvector should look like this

\[
\sum A_p \mathcal{O}_{p,J} + \sum B_p \Psi_{p,J-1} + \sum C_p V_{p,J-2} \tag{4.3}
\]

According to general perturbation theory of quantum mechanics, we first diagonalize the \( O(g^2) \) part of the matrix. This is done in previous section and in paper [11]. The three groups of eigenvectors correspond to \( A_p \neq 0, B_p = C_p = 0 \) or \( B_p \neq 0, A_p = C_p = 0 \) or \( C_p \neq 0, A_p = B_p = 0 \) and the eigenvalues are not degenerate. We will use \( \tilde{\mathcal{O}}_{J,n}, \tilde{\Psi}_{J-1,n} \) and \( \tilde{V}_{J-2,n} \) to denote these eigenstates. So the non-degenerate perturbation theory could be used to calculate the correction to the next order \( O(g^3) \). It is easy to see that there is no correction to the eigenvalues to this order since the correction terms are proportional to \( \langle \tilde{\mathcal{O}}_{J,n}|H(O(g^3))|\tilde{\mathcal{O}}_{J,n} \rangle, \langle \tilde{\Psi}_{J,n}|H(O(g^3))|\tilde{\Psi}_{J,n} \rangle \) or \( \langle \tilde{V}_{J,n}|H(O(g^3))|\tilde{V}_{J,n} \rangle \). If we are only
interested in the mixing between $O^{p,J}$ and $\Psi^{p,J-1}$, we can just take the upper left $2 \times 2$ matrix and diagonalize it. This is what we will do later. We will see that it is easier to do it without using the quantum mechanics perturbation theory result. We can then do the perturbation to $O(g^4)$. Just like the $O(g^3)$ case, the nondiagonal matrix blocks do not contribute to the eigenvalue but contribute to the mixing coefficients. This means that if we only consider the lowest order of the mixing coefficients between $O^{p,J}$ and $V^{p,J-2}$ we can just use the diagonal $O(g^2)$ blocks and nondiagonal $O(g^4)$ blocks. Combining $O(g^3)$ and $O(g^4)$ results, the three kinds of solutions for the coefficients should be $A_p \sim O(1)$, $B_p \sim O(g)$, $C_p \sim O(g^2)$ or $A_p \sim O(g)$, $B_p \sim O(1)$, $C_p \sim O(g)$ or $C_p \sim O(g^2)$, $A_p \sim O(g)$, $B_p \sim O(1)$. Notice that the finite $J$ condition plays an important role here, because in the infinite $J$ limit the three groups of the leading order eigenstates degenerate and the non-degenerate perturbation argument can not be used here. Just diagonalizing the upper left $2 \times 2$ block matrix would not be valid.

5. Effective vertices of $O(g^3)$

In order to study the mixing between $O^{p,J}$ and $\Psi^{p,J-1}$ we need to calculate dilatation operator changing fermions into bosons and vice versa. The lowest order nonzero diagrams are of $O(g^3)$:

\[
\begin{align*}
H(A) &= -\frac{g^3}{(2\pi)^3} \text{Tr} \left[ \{[\psi^{\alpha A-}, \Phi_{AB}^+], [\Phi_{CD}^+, \psi^{\alpha -}_\alpha] \} \Phi_{BC}^+ - \{[\bar{\psi}^{\tilde{\alpha} A-}, \Phi_{AB}^+], [\Phi_{CD}^+, \bar{\psi}^{\tilde{\alpha} -}_{\tilde{A}}] \} \Phi_{BC}^+ \right] \\
&\quad + (+ \leftrightarrow -) \\
H(B) &= -\frac{g^3}{8(2\pi)^3} \text{Tr} \left[ -\{[\psi^{\alpha A-}, \psi^{R -}_\alpha] \} \Phi_{CD}^+ + \{[\bar{\psi}^{\tilde{\alpha} A-}, \bar{\psi}^{\tilde{\alpha} -}_{\tilde{B}}] \} \Phi_{CD}^+ \right] \\
&\quad + (+ \leftrightarrow -) \\
\end{align*}
\]

Using (A.3), we can change $\Phi$ into $Z$ and $\bar{Z}$:

\[
\begin{align*}
H(A) &= -\frac{g^3}{(2\pi)^3} \text{Tr} \left[ -\frac{1}{8} \epsilon_{ijk} [\psi^{\alpha i -}, Z_j^+] (\{[Z_l^+, \psi^{k -}_j], Z_l^+] - \{[Z_l^+, \bar{Z}_l^+], \psi^{l -}_\alpha] + \{[\psi^{l -}_\alpha, Z_l^+], \bar{Z}_l^+] \}) + \frac{1}{4} [\psi^{\alpha 4 -}, Z_j^+] (\{[Z_l^+, \psi^{j -}_\alpha], Z_l^+] - \{[Z_l^+, \bar{Z}_l^+], \psi^{j -}_\alpha] - \frac{1}{8} \epsilon_{ijk} \{[\psi^{\alpha 4 -}, Z_l^+], [Z_j^+, \psi^{j -}_\alpha] \} Z_k^+ \\
&\quad + h.c. \right] \\
H(B) &= -\frac{g^3}{8(2\pi)^3} \text{Tr} \left[ -\frac{1}{4} \epsilon_{ijk} [\psi^{\alpha i -}, \psi^{j -}_\alpha] + \frac{1}{2} \{\bar{\psi}^{\tilde{i} -}_{\tilde{A} 4}, \bar{\psi}^{\tilde{j} -}_{\tilde{B} k} \} (\{[\bar{Z}_l^+, [\bar{Z}_l^+, Z_l^+] + \{[\bar{Z}_l^+, \bar{Z}_l^+], Z_l^+] \} \\
&\quad + h.c. \right] \\
\end{align*}
\]
Note that the last term of the second line in (5.3) is just the \( H_3 \) part for the subsector \( su(2|3) \) discussed in [17].

We only consider planar diagrams in this paper, so the terms we are interested in must have adjacent fermions and bosons, and only \( \Psi_{ij}^{p=0,J} \) has non-zero correlation function with \( O_{ij}^{p=0} \) or \( O_{ij}^{p=1} \) at planar level to \( O(g^3) \). The matrix model vertices must involve all the impurities in both operators in the two-point correlation function. So, using the cyclic property of the trace of operator string, for effective vertices like \( \text{Tr}(\psi_i^\dagger \psi_m^\dagger \phi_j^+ \hat{Z}^+) \) in which the two impurity bosonic fields are adjacent, we can change the position of \( Z \) or \( \hat{Z} \) from the end of the three bosonic fields to the beginning of them or vice versa without changing the two point correlation function, that is \( \text{Tr}(\psi_i^\dagger \psi_m^\dagger \phi_j^+ \hat{Z}^+) \leftrightarrow \text{Tr}(\psi_i^\dagger \psi_m \phi_j^+ \phi_j^+ \hat{Z}^+) \). Note that this operation only works for two impurity operators. By appropriately using this operation, we collect the terms we are interested in as:

\[
H_{(A)} + H_{(B)} \sim \frac{g^3}{4(2\pi)^3} \text{Tr} \left\{ \left( \sum_{l=1,2,4,5} \phi_l^\dagger \phi_l \right) Z^2 - \frac{1}{4} \left( \sum_{l=1,2,4,5} \phi_l^\dagger \phi_l \right)^2 \right\}
\]

where \( \phi_l^\dagger \phi_l = \frac{1}{2}(\phi_l \phi_l - \phi_l \phi_l) \). We can define:

\[
O_{ij,kl}^{p,J} = O_{ij,kl}^{p,J} - O_{ijkl}^{p,J}, \quad O_{ij,kl}^{p,J} = O_{ij,kl}^{p,J} + O_{ijkl}^{p,J},
\]

\[
\Psi_{ij,kl}^{p,J} = \frac{1}{2}(\Psi_{ij}^{p,J} + \Psi_{kl}^{p,J}), \quad \Psi_{ij,kl}^{p,J} = \frac{1}{2}(\Psi_{ij}^{p,J} - \Psi_{kl}^{p,J}).
\]

From (5.3) we see that \( O_{ij,kl}^{p,J} \) are mixed with \( \Psi_{ij,kl}^{p,J-1} \) and \( \Psi_{kl}^{p,J-1} \) and \( O_{ijkl}^{p,J} \) mixed with \( \Psi_{ijkl}^{p,J-1} \), \( O_{ij,kl}^{p,J-1} \) mixed with \( \Psi_{ij,kl}^{p,J-1} \), etc. We can list these mixed operators in the rows of table 1.

These mixing patterns are consistent with group analysis as follows: if we keep \( Z = 2\Phi_{34} \) and \( \hat{Z} = 2\Phi_{12} \) invariant, the \( SU(4)_R \) symmetry is broken to \( SO(4) = SU(2) \times SU(2) \) with \( \psi^{1,2} \) and \( \bar{\psi}_{3,4} \) as fundamental representations of the two \( SU(2) \)'s respectively. Using (A.3), (A.8) and omitting \( Z \), we can obtain:

\[
iO_{14,25}^{12} \sim \Phi_{14}^{14} \Phi_{23}^{14} - \Phi_{23}^{14} \Phi_{14}^{14} - \Phi_{24}^{14} \Phi_{13}^{14} - \Phi_{13}^{14} \Phi_{24}^{14},
\]

\[
o_{12,45}^{14} \sim \Phi_{14}^{14} \Phi_{23}^{14} - \Phi_{23}^{14} \Phi_{14}^{14} - \Phi_{24}^{14} \Phi_{13}^{14} + \Phi_{13}^{14} \Phi_{24}^{14},
\]

\[
iO_{24,15}^{14} \sim \Phi_{23}^{14} \Phi_{24}^{14} + \Phi_{24}^{14} \Phi_{13}^{14} - \Phi_{13}^{14} \Phi_{24}^{14} - \Phi_{24}^{14} \Phi_{23}^{14},
\]
from which we see that the indices 3 and 4 are antisymmetric and form a singlet under the second $SU(2)$. The indices 1 and 2 have precisely the same transformation property as the corresponding mixed operators in table 1. The same analysis can be performed on the other rows. In fact, $\Psi_{11}$, $\Psi_{22}$ and $\Psi_{(12)}$ form a 3 representation of $SO(4)$ and $\Psi_{33}$, $\Psi_{44}$ and $\Psi_{(34)}$ form a $\bar{3}$ representation. The corresponding scalars are also expected to form 3 and $\bar{3}$ representations. As a result, the operators $V_{p,J}$ are only mixed with the first line of table 1 but not with the other lines.

6. Mixing of operators

Let us look at the mixing between $O_1$, $\Psi_{[12]}$ and $\Psi_{[34]}$ first. Since there is no correlation between $\Psi_{[12]}$ and $\Psi_{[34]}$ up to $O(g^3)$, we can consider the mixing between $O_1$ and $\Psi_{[12]}$, and $O_1$ and $\Psi_{[34]}$ separately. Since the coefficients before $\bar{\psi}_1\bar{\psi}_2$ and $\psi^3\psi^4$ in the vertices differ only by a minus sign, the mixing coefficients for $\Psi_{[12]}$ and $\Psi_{[34]}$ with $O_1$ must differ only by a minus sign. For operator basis $(O_1^{-1,J}, O_1^{0,J}, \ldots, O_1^{[J/2],J}, \Psi_{[12]}^{-1,J}, \ldots, \Psi_{[12]}^{[J/2]-1,J-1})$, the $H$ matrix should combine the one-loop $H$ matrix for $O_1^{p,J}$ and $\Psi_{[12]}^{p,J-1}$ and some $O(g^3)$ crossing terms:

\[
\begin{pmatrix}
1 & -1 & -g\sqrt{N} \\
-1 & 2 & -1 \\
\text{\ldots} & \text{\ldots} & \text{\ldots} \\
g^2N & 2\sqrt{2}(2\pi) & g\sqrt{N} \\
\text{\ldots} & \text{\ldots} & \text{\ldots} \\
-1 & 2 & -1 \\
-1 & 1 & 2 \\
\end{pmatrix}
\]

(6.1)

After diagonalization, we require the non-diagonal elements to be $O(g^4)$. So we require the correction to the eigenvector $\tilde{O}_1$ to be $O(g)$. The upper right non-diagonal element
contribution to the eigenvalue is already $O(g^4)$. The upper left block of the $H$ matrix acting on the $O(1)$ part gives just the one-loop case which determines the eigenvalue to be $(g^2N/4\pi^2)4\sin^2(n\pi/(J+3))$. As stated at the end of section 3, we could make the ansatz for the coefficients before $\Psi_{[12]}^{p,J-1}$ to be $2gb\sin(2p+4n\pi)/(J+3)$. The upper left block of the $H$ matrix then determines $a = 4$. The eigenvalue equation for the coefficient before $\Psi_{[12]}^{0,J-1}$ acts as another boundary condition which determines $b$. We then obtain the eigenvector with $O(g)$ correction to $\tilde{O}_{1}^{p,J}$:

$$Q_{1,12}^{J,n} = \tilde{O}_{1}^{J,n} + \frac{\sqrt{2g}\sqrt{N}}{2\pi} \sin \frac{n\pi}{J+3} \sum_{p=0}^{J-1} \sin \frac{(2p+4)n\pi}{J+3} \Psi_{[12]}^{p,J-1},$$

for $n = 0, \ldots, [J/2] + 1$. (6.2)

We can also require the coefficients before $\Psi_{[12]}^{p,J-1}$ to be $O(1)$ and those before $\tilde{O}_{1}^{J}$ to be $O(g)$. This time the lower left nondiagonal element contribution is already $O(g^4)$, and the upper right nondiagonal elements contribute to two eigenvalue equations. As a result, the coefficient before $\tilde{O}_{1}^{1,J}$ should be considered as independent of the other coefficients: we set it to be $A_{-1}$ and for $p = 0, \ldots, (J-1)/2$ the coefficients before $\tilde{O}_{1}^{p,J}$ are $2b\cos(2p+4n\pi)/(J+1)$. As in previous case, the last line in the upper left block determines $a = 1$. Those two equations from the first two lines determine $A_{-1}$ to be zero and the value of $b$, thus the eigenvector in this case is

$$\Xi_{[12]}^{J-1,n} = \tilde{\Psi}_{[12]}^{J-1,n} - \frac{g\sqrt{N}}{\sqrt{2}(2\pi)} \sin \frac{n\pi}{J+1} \sum_{p=0}^{J} \cos \frac{(2p+1)n\pi}{J+1} \tilde{O}_{1}^{p,J},$$

for $n = 1, \ldots, [(J+1)/2]$. (6.3)

It is easy to check that for the $J = \text{even}$ case the eigenvectors can also be written in this form. Notice that the total number of eigenvectors in (6.2) and (6.3) is equal to the rank of the matrix and the eigenvalues are not degenerate. So these are all the eigenvectors of the $H$ matrix (6.1).

For the mixing with $\Psi_{[34]}$, only the nondiagonal blocks in (6.1) change signs. As a result the coefficients of $O(g)$ of the eigenvector change signs. So combined with (6.2), we obtain the $O(g)$ correction to the eigenvector $\tilde{O}_{1}^{p,J}$

$$Q_{1}^{J,n} = \tilde{O}_{1}^{J,n} + \frac{\sqrt{2g}\sqrt{N}}{2\pi} \sin \frac{n\pi}{J+3} \sum_{p=0}^{J-1} \sin \frac{(2p+4)n\pi}{J+3} \left( \Psi_{[12]}^{p,J-1} - \Psi_{[34]}^{p,J-1} \right)$$

This is consistent with equation (3.17) in [20]. (6.3) is consistent with the first two terms in equation (2.12) in [26]. To obtain the full $O(g)$ correction, one must also consider the mixing matrix elements between $\Psi_{[12]}^{p,J}$ and $D_{\mu}ZZ^{p}D^{\mu}ZZ^{J-1-p-1}$.

As for the mixing between $\tilde{O}_{1}^{p,J}$ and $\Psi_{[12]}^{p,J-1}$, we can also have the $H$ matrix for
operators basis \( \left( iO_{[14,25]}^{0,J}, \ldots, iO_{[14,25]}^{\lfloor (J-1)/2 \rfloor,J}, \Psi_{[14,25]}^{0,J-1}, \ldots, \Psi_{[14,25]}^{\lfloor (J-1)/2 \rfloor,J-1} \right) \):

for \( J \) odd:

\[
\begin{pmatrix}
2 & -1 & \frac{g\sqrt{N}}{2\sqrt{2}(2\pi)} \\
-1 & 2 & -1 & \frac{g\sqrt{N}}{2\sqrt{2}(2\pi)} \\
\vdots & \ldots & \vdots & \ldots \\
-1 & 2 & -1 & 1 & -1 \\
-1 & 2 & -1 & \ldots & \ldots \\
\end{pmatrix}
\]

\[
\frac{g^2N}{4\pi^2} \begin{pmatrix}
\sqrt{2gN} & -gN \\
-\sqrt{2gN} & -gN/\sqrt{2(2\pi)} \\
\end{pmatrix}
\]

(6.5)

(6.6)

It is easy to obtain the eigenvectors as before:

\[
Q_{[14,25]}^{J,n} = i \sum_{p=0}^{J} \sin \left( \frac{2p + 2}{J + 2} n\pi \right) O_{[14,25]}^{p,J} - \frac{g\sqrt{N}}{\sqrt{2}(2\pi)} \sin \frac{n\pi}{J + 2} \sum_{p=0}^{J-1} \cos \left( \frac{2p + 3}{J + 2} n\pi \right) \Psi_{p,J-1}^{(12)},
\]

\[
\Xi_{(12)}^{J,n} = \tilde{\Psi}_{(12)}^{J,n} + \frac{i g\sqrt{N}}{\sqrt{2}(2\pi)} \left( -\cos \frac{n\pi}{J + 1} O_{[14,25]}^{0,J-1} + \sin \frac{n\pi}{J + 1} \sum_{p=1}^{J} \sin \frac{2p + 2}{J + 2} n\pi \right) \Psi_{p,J}^{(12)}.
\]

(6.7)

As in previous case the coefficient before the \( O^{0,J+1} \) term should be considered independently. This term corresponds to the non-asymptotic term in (3.1) of [19]. The other operator mixing cases in table \[\text{II}\] can be easily obtained by replacing \( iO_{[14,25]}^{p,J} \) and \( \Psi_{(12)}^{p,J} \) by the corresponding operators in the same column in table \[\text{II}\], because they produce the same \( H \) matrix. As we mentioned in the last section, these operators are not mixed with \( D_{\mu}ZZ \ldots D_{\mu}ZZ \ldots Z \).

7. Discussion

We have discussed the mixing between single trace BMN operators in the pure scalar sector and operators with one scalar fermion pair. A kind of matrix model effective vertices up to \( O(g^3) \) are deduced during the discussion. These effective vertices can also be used to discuss non-planar diagrams. All the mixing patterns at planar level are reflected in the vertices \[\text{III}\]. The \( O(g^3) \) matrix elements in the eigenvalue problem act as a boundary condition to the \( O(g^2) \) matrix. The results are consistent with [20] and [26]. It is worth pointing out that even though the correlation functions between operators with nonadjacent fermions and pure scalar operators are zero up to \( O(g^3) \) at planar level, they are mixed because the mixings between operators with adjacent fermions and those with nonadjacent fermions are needed to diagonalize the one-loop dilatation operator.
Since we discuss only operators with one fermion pair, the $O(g^2)$ effective vertices do not include the interactions between two fermion pairs like:

\[
\begin{array}{c}
\hat{\psi} \\
\downarrow \\
\hat{D}_a \\
\uparrow \\
\hat{\psi}
\end{array}
\]
If one wants to discuss operators with two or more scalar fermion pairs, one has to include the terms from this diagram which can be calculated as in [9]. In general, this term could also be obtained by using the results in [27]. The $O(g^3)$ terms (5.3) and (5.4) can be directly used in more than two fermion pair cases. It is an interesting problem to discuss the integrability of this kind of dynamic spin chain to $O(g^3)$. Unlike the $SU(2|3)$ spin chain in [17, 19, 29], the fermions are not invariant under the $SU(3)$ subgroup of $SU(4)_R$. To be precise, they transform under $SU(2) \times SU(2)$ subgroup. Understanding how to use the coordinate space Bethe ansatz in this kind of dynamic spin chain could be a further direction for research.

One can also discuss the mixing between the $SO(4)$ singlet parts of $\Psi^{p,J}$ and $V^{p,J}$ along these lines. This requires calculating the two point correlation function to $O(g^3)$ and we could expect that the $O(g^3)$ matrix elements act as boundary conditions, just as in the cases we discussed here. One can also go further, calculating the dilatation operator matrix between $O^{p,J}$ and $V^{p,J}$ to $O(g^4)$, and we still expect that the $O(g^4)$ matrix elements act as boundary conditions. These will produce the full equations (2.1) and (2.12) in [26].

A few remarks on the method in [20]: in [20], the authors use the supercurrents deduced from dimensional reduction of the $D = 10$ SUSY YM. But if one discusses the quantum properties of $N = 4$ SUSY YM, one must include the gauge fixing and ghosts in the quantized Lagrangian. One would expect that this should modify the supercurrents, which could be different from those in [20] in the gauge sector and ghost sector. The problem is more explicit in $N = 1$ SUSY YM theory. We can do gauge fixing and add ghosts in superfield formalism without breaking $N = 1$ supersymmetry. Then we can have a supercurrent involving gauge fixing and ghosts. On the other hand, for the original lagrangian without ghosts and gauge fixing, we also have a supercurrent. Which one should be chosen to calculate the higher order correction to the SUSY variation of a composite operator? The method in [20] is equivalent to choosing the supercurrent without ghosts. Since all the perturbation calculations are based on the Lagrangian after quantization, the more consistent choice is to use the former supercurrent. It turns out that in [20] when they only discuss the scalar and fermionic sectors, using those supercurrents from ten-dimensional SUSY YM is fine. We can see intuitively why this is not a problem. After careful scrutiny, we can conclude that their method is consistent using either of these two currents. This is because after the Faddeev-Popov quantization the partition function is essentially proportional to the path integral using the classical action divided by the volume integration of the gauge redundancy:

\[
\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_l(x_l) \rangle_{F-P} = \frac{C \int [\Pi_{\alpha,x} d\phi_{\alpha}(x)] [\mathcal{O}_1(x_1) \ldots \mathcal{O}_l(x_l)] e^{iS_c}}{\int \rho(\Lambda(x))}, \quad (7.1)
\]
This is equation (15.5.20) of [30]. Operators $O_i$ are supposed to be gauge invariant. The subscript F-P means the path integral after Faddeev-Popov quantization and subscript $c$ means the classical one. $\rho[\Lambda(x)]$ is the invariant (Haar) measure on the space of group parameters. $\phi_n$ here denotes generally all the fields to be integrated. Note that the denominator and the constant $C$ are gauge invariant. We could deduce the Ward Identity using this path integral with classical action and get back to Faddeev-Popov ghosts later.

Then the current in the Ward Identity is the classical one without ghosts:

\[
\partial_\mu (j^\mu_c(x)O_1(x_1) \ldots O_l(x_l))_{F-P} = -i \sum_i \langle O_1(x_1) \ldots \delta O_i(x_i) \delta^4(x - x_i) \ldots O_l(x_l) \rangle_{F-P} \tag{7.2}
\]

Notice that the infinite integral in the denominator and numerator in (7.2) should be understood as being put on a finite lattice and then taking the infinite limit in front of the fraction. Since the Faddeev-Popov integral is well-defined, the limit is well-defined. This can be done in every line of (7.2). In their method, the Ward Identity is used on some operators which are not gauge invariant to deduce the SUSY transformation property of one operator. Then this SUSY transformation is used to obtain the SUSY variations of some gauge invariant operators. This is equivalent to using the Ward Identity on the gauge invariant operators from the beginning, thus (7.2) can be applied.

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A. $\mathcal{N} = 4$ SYM convention

The conventions we use in this paper follow closely the paper [20] except for some normalization differences:

\[
\text{Tr}(T^a T^b) = \delta^{ab}, \quad [T^a, T^b] = \sqrt{2} i f^{abc} T^c, \quad (T^a)_j^i (T^a)_k^l = \delta^i_j \delta^k_l \quad (A.1)
\]
We choose Minkowski metric $g_{\mu\nu} = diag(+, -, -, -)$. The $\mathcal{N} = 4$ Yang-Mills Lagrangian is

$$L = \text{Tr} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi_{AB} D^\mu \Phi^{AB} + i\psi^\alpha A_\sigma \sigma^{\alpha\dot{\alpha}} (D_\mu \bar{\psi}^{\dot{\alpha}}) + \frac{1}{2} g^2 [\Phi_{AB}, \Phi_{CD}] [\Phi_{AB}, \Phi_{CD}] - g \left( [\psi^\alpha A, \Phi_{AB}] \psi^{\dot{B}} - [\bar{\psi}^{\dot{A}} A, \Phi^{AB}] \bar{\psi}^B \right) \right]$$ (A.2)

For scalars, we use $\Phi_{ij}, Z_i, \bar{Z}^i$ and $\phi_i$ in different contexts:

$$\begin{cases} 
\Phi_{jk} = \frac{1}{2} \epsilon^{jki} Z_i \\
\Phi_{jk} = \frac{1}{2} \epsilon^{jki} \bar{Z}^i \\
\Phi_{i4} = \frac{1}{2} \bar{Z}^i \\
\Phi_{i4} = \frac{1}{2} Z_i 
\end{cases} \quad \text{for } i, j, k = 1, 2, 3$$ (A.3)

where

$$Z_i = \frac{1}{\sqrt{2}} (\phi_i + i\phi_{i+3}), \quad \bar{Z}^i = \frac{1}{\sqrt{2}} (\phi_i - i\phi_{i+3}).$$ (A.4)

There are some useful relations:

$$\begin{align*}
\bar{Z}^a Z^b + Z^a \bar{Z}^b &= \phi^a \phi^b + \phi_i^{a+3} \phi_j^{b+3} \\
Z^a \bar{Z}^b - \bar{Z}^a Z^b &= -i(\phi_i^{a} \phi_j^{b+3} - \phi_i^{a+3} \phi_j^{b}) \\
Z^a \bar{Z}^b + \bar{Z}^a Z^b &= \phi_i^{a} \phi_j^{b} - \phi_i^{a+3} \phi_j^{b+3} \\
Z^a \bar{Z}^b - \bar{Z}^a Z^b &= i(\phi_i^{a} \phi_j^{b+3} + \phi_i^{a+3} \phi_j^{b})
\end{align*}$$

The propagators are:

$$\begin{align*}
\langle A^a_\mu (x) A^b_\nu (y) \rangle &= -\delta^{ab} g_{\mu\nu} \Delta_{xy}, \\
\langle \psi^\alpha_{\dot{A}} (x) \bar{\psi}^\dot{B}_{\beta} (y) \rangle &= i\delta^{ab} \delta^\dot{A} A \sigma^{\alpha\dot{\alpha}} \partial_\mu \Delta_{xy}, \\
\langle \Phi_{AB} (x) \Phi_{CD} (y) \rangle &= \frac{1}{4} \delta^{ab} \epsilon_{ABCD} \Delta_{xy}, \\
\langle Z^a (x) \bar{Z}^b (y) \rangle &= \delta^{ab} \Delta_{xy}, \\
\langle \bar{Z}^a (x) Z^b (y) \rangle &= \langle Z^a (x) \bar{Z}^b (y) \rangle = 0,
\end{align*}$$ (A.9)

where

$$\Delta = -\frac{1}{4\pi^2 (x - y)^2}.$$ (A.10)

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