The splitting algorithms by Ryu, by Malitsky-Tam, and by Campoy applied to normal cones of linear subspaces converge strongly to the projection onto the intersection

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Abstract

Finding a zero of a sum of maximally monotone operators is a fundamental problem in modern optimization and nonsmooth analysis. Assuming that the resolvents of the operators are available, this problem can be tackled with the Douglas-Rachford algorithm. However, when dealing with three or more operators, one must work in a product space with as many factors as there are operators. In groundbreaking recent work by Ryu and by Malitsky and Tam, it was shown that the number of factors can be reduced by one. A similar reduction was achieved recently by Campoy through a clever reformulation originally proposed by Kruger. All three splitting methods guarantee weak convergence to some solution of the underlying sum problem; strong convergence holds in the presence of uniform monotonicity.

In this paper, we provide a case study when the operators involved are normal cone operators of subspaces and the solution set is thus the intersection of the subspaces. Even though these operators lack strict convexity, we show that striking conclusions are available in this case: strong (instead of weak) convergence and the solution obtained is (not arbitrary but) the projection onto the intersection. To illustrate our results, we also perform numerical experiments.

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1 Introduction

Throughout the paper, we assume that $X$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let $A_1, \ldots, A_n$ be maximally monotone operators on $X$. (See, e.g., [8] for background on maximally monotone operators.) One central problem in modern optimization and nonsmooth analysis asks to find $x \in X$ such that $0 \in (A_1 + \cdots + A_n)x$. (1)

In general, solving (1) may be quite hard. Luckily, in many interesting cases, we have access to the firmly nonexpansive resolvents $J_{A_i} := (\text{Id} + A_i)^{-1}$ and their associated reflected resolvents $R_{A_i} = 2J_{A_i} - \text{Id}$ which opens the door to employ splitting algorithms to solve (1). The most famous instance is the Douglas-Rachford algorithm [18] whose importance for this problem was brought to light in the seminal paper by Lions and Mercier [24]. However, the Douglas-Rachford algorithm requires that $n = 2$; if $n \geq 3$, one may employ the Douglas-Rachford algorithm to a reformulation in the product space $X^n$ [16, Section 2.2]. In recent breakthrough work by Ryu [27], it was shown that for $n = 3$ one may formulate an algorithm that works in $X^2$ rather than $X^3$. We will refer to this method as Ryu’s algorithm. Very recently, Malitsky and Tam proposed in [25] an algorithm for a general $n \geq 3$ that is different from Ryu’s and that operates in $X^{n-1}$. (No algorithms exist in product spaces featuring fewer factors than $n-1$ factors in a certain technical sense. See also [13] for an extension of the Malitsky-Tam algorithm to handle linear operators.) We will review these algorithms as well as a recent (Douglas-Rachford based) algorithm introduced by Campoy [15] in Section 3 below. These three algorithms are known to produce some solution to (1) via a sequence that converges weakly. Strong convergence holds in the presence of uniform monotonicity.

The aim of this paper is provide a case study for the situation when the maximally monotone operators $A_i$ are normal cone operators of closed linear subspaces $U_i$ of $X$. These operators are not even strictly monotone. Our main results show that the splitting algorithms by Ryu, by Malitsky-Tam, and by Campoy actually produce a sequence that converges strongly! We are also able to identify the limit to be the projection onto the intersection $U_1 \cap \cdots \cap U_n$! The proofs of these results rely on the explicit identification of the fixed point set of the underlying operators. Moreover, a standard translation technique gives the same result for affine subspaces of $X$ provided their intersection is nonempty.

The paper is organized as follows. In Section 2, we collect various auxiliary results for later use. The known convergence results on Ryu splitting, on Malitsky-Tam splitting, and on Campoy splitting are reviewed in Section 3. Our main results are presented in Section 4. Matrix representations of the various operators involved are provided in Section 5. These are useful for our numerical experiments in Section 6. We investigate the case of three lines in the Euclidean plane in Section 7. Finally, we offer some concluding remarks in Section 8.

The notation employed in this paper is standard and follows largely [8]. When $z = x + y$ and $x \perp y$, then we also write $z = x \oplus y$ to stress this fact. Analogously for the Minkowski sum $Z = X + Y$, we write $Z = X \oplus Y$ as well as $P_Z = P_X \oplus P_Y$ for the associated projection provided that $X \perp Y$.

2 Auxiliary results

In this section, we collect useful properties of projection operators and results on iterating linear/affine nonexpansive operators. We start with projection operators.
2.1 Projections

Fact 2.1. Suppose $U$ and $V$ are nonempty closed convex subsets of $X$ such that $U \perp V$. Then $U \oplus V$ is a nonempty closed subset of $X$ and $P_{U \oplus V} = P_U \oplus P_V$.

Proof. See [8, Proposition 29.6].

Here is a well known illustration of Fact 2.1 which we will use repeatedly in the paper (sometimes without explicit mentioning).

Example 2.2. Suppose $U$ is a closed linear subspace of $X$. Then $P_U \perp = \text{Id} - P_U$.

Proof. The orthogonal complement $V := U^\perp$ satisfies $U \perp V$ and also $U + V = X$; thus $P_{U + V} = \text{Id}$ and the result follows.

Fact 2.3. (Anderson-Duffin) Suppose that $X$ is finite-dimensional and that $U, V$ are two linear subspaces of $X$. Then $P_{U \cap V} = 2P_U(P_U + P_V)^\dagger P_V$, where "$^\dagger$" denotes the Moore-Penrose inverse of a matrix.

Proof. See, e.g., [8, Corollary 25.38] or the original [1].

Corollary 2.4. Suppose that $X$ is finite-dimensional and that $U, V, W$ are three linear subspaces of $X$. Then

$$P_{U \cap V \cap W} = 4P_U(P_U + P_V)^\dagger P_V (2P_U(P_U + P_V)^\dagger P_V + P_W)^\dagger P_W.$$  

Proof. Use Fact 2.3 to find $P_{U \cap V}$, and then use Fact 2.3 again on $(U \cap V, W)$.

Corollary 2.5. Suppose that $X$ is finite-dimensional and that $U, V$ are two linear subspaces of $X$. Then

$$P_{U + V} = \text{Id} - 2P_{U^\perp}(P_{U^\perp} + P_{V^\perp})^\dagger P_{V^\perp}$$

$$= \text{Id} - 2(\text{Id} - P_U) (2 \text{Id} - P_U - P_V)^\dagger (\text{Id} - P_V).$$

Proof. Indeed, $U + V = (U^\perp \cap V^\perp)^\perp$ and so $P_{U + V} = \text{Id} - P_{U^\perp \cap V^\perp}$. Now apply Fact 2.3 to $(U^\perp, V^\perp)$ followed by Example 2.2.

Fact 2.6. Let $Y$ be a real Hilbert space, and let $A : X \to Y$ be a continuous linear operator with closed range. Then $P_{\text{ran}A} = AA^\dagger$.

Proof. See, e.g., [8, Proposition 3.30(ii)].

2.2 Linear (and affine) nonexpansive iterations

We now turn results on iterating linear or affine nonexpansive operators.

Fact 2.7. Let $L : X \to X$ be linear and nonexpansive, and let $x \in X$. Then

$$L^k x \to P_{\text{Fix}L}(x) \iff L^k x - L^{k+1} x \to 0.$$
Fact 2.8. Let $T: X \to X$ be averaged nonexpansive with $\text{Fix } T \neq \emptyset$. Then $(\forall x \in X) \ T^k x - T^{k+1} x \to 0$.

Proof. See Bruck and Reich’s paper [14] or [8, Corollary 5.16(ii)].

Corollary 2.9. Let $L: X \to X$ be linear and averaged nonexpansive. Then $(\forall x \in X) \ L^k x \to P_{\text{Fix } L}(x)$.

Proof. Because $0 \in \text{Fix } L$, we have $\text{Fix } L \neq \emptyset$. Now combine Fact 2.7 with Fact 2.8.

Fact 2.10. Let $L$ be a linear nonexpansive operator and let $b \in X$. Set $T: X \to X: x \mapsto Lx + b$ and suppose that $\text{Fix } T \neq \emptyset$. Then $b \in \text{ran } (\text{Id} - L)$, and for every $x \in X$ and $a \in (\text{Id} - L)^{-1} b$, the following hold:

(i) $b = a - La \in \text{ran } (\text{Id} - L)$.
(ii) $\text{Fix } T = a + \text{Fix } L$.
(iii) $P_{\text{Fix } T}(x) = P_{\text{Fix } L}(x) + P_{(\text{Fix } L)^\perp}(a)$.
(iv) $T^k x = L^k (x - a) + a$.
(v) $L^k x \to P_{\text{Fix } L} x \iff T^k x \to P_{\text{Fix } T} x$.

Proof. See [10, Lemma 3.2 and Theorem 3.3].

Remark 2.11. Consider Fact 2.10 and its notation. If $a \in (\text{Id} - L)^{-1} b$ then $P_{(\text{Fix } L)^\perp}(a)$ is likewise because $b = (\text{Id} - L)a = (\text{Id} - L)(P_{\text{Fix } L}(a) + P_{(\text{Fix } L)^\perp}(a)) = (\text{Id} - L)P_{(\text{Fix } L)^\perp}(a)$; moreover, using [20, Lemma 3.2.1], we see that

$$(\text{Id} - L)^\dagger b = (\text{Id} - L)^\dagger (\text{Id} - L)a = P_{(\text{ker } (\text{Id} - L))^\perp}(a) = P_{(\text{Fix } L)^\perp}(a),$$

where again “$\dagger$” denotes the Moore-Penrose inverse of a continuous linear operator (with possibly non-closed range). So given $b \in X$, we may concretely set

$a = (\text{Id} - L)^\dagger b \in (\text{Id} - L)^{-1} b$;

with this choice, (iii) turns into the even more pleasing identity

$P_{\text{Fix } T}(x) = P_{\text{Fix } L}(x) + a$.

3 Ryu, Malitsky-Tam, and Campoy splitting

In this section, we present the precise form of Ryu’s, the Malitsky-Tam, and Campoy’s algorithms and review known convergence results.

3.1 Ryu splitting

We start with Ryu’s algorithm. In this subsection,

$A, B, C$ are maximally monotone operators on $X,$
with resolvents \( J_A, J_B, J_C \), respectively. The problem of interest is to

\[
\text{find } x \in X \text{ such that } 0 \in (A + B + C)x, \tag{2}
\]

and we assume that (2) has a solution. The algorithm pioneered by Ryu [27] provides a method for finding a solution to (2). It proceeds as follows. Set\(^1\)

\[
M : X \times X \to X \times X \times X : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} J_A(x) \\ J_B(J_A(x) + y) \\ J_C(J_A(x) - x + J_B(J_A(x) + y) - y) \end{pmatrix}. \tag{3}
\]

Next, denote by \( Q_1 : X \times X \times X \to X : (x_1, x_2, x_3) \mapsto x_1 \) and similarly for \( Q_2 \) and \( Q_3 \). We also set \( \Delta := \{(x, x, x) \in X^3 \mid x \in X\} \). We are now ready to introduce the Ryu operator

\[
T := T_{\text{Ryu}} : X^2 \to X^2 : z \mapsto z + ((Q_3 - Q_1)Mz, (Q_3 - Q_2)Mz). \tag{4}
\]

Given a starting point \((x_0, y_0) \in X \times X\), the basic form of Ryu splitting generates a governing sequence via

\[
(\forall k \in \mathbb{N}) \quad (x_{k+1}, y_{k+1}) := (1 - \lambda)(x_k, y_k) + \lambda T(x_k, y_k). \tag{5}
\]

The following result records the basic convergence properties by Ryu [27], and recently improved by Aragón-Artacho, Campoy, and Tam [2].

**Fact 3.1. (Ryu and also Aragon-Artacho-Campoy-Tam)** The operator \( T_{\text{Ryu}} \) is nonexpansive with

\[
\text{Fix } T_{\text{Ryu}} = \{(x, y) \in X \times X \mid J_A(x) = J_B(J_A(x) + y) = J_C(R_A(x) - y) \} \tag{6}
\]

and

\[
\text{zer}(A + B + C) = J_A(Q_1 \text{ Fix } T_{\text{Ryu}}). \]

Suppose that \( 0 < \lambda < 1 \) and consider the sequence generated by (5). Then there exists \((\bar{x}, \bar{y}) \in X \times X\) such that

\[
(x_k, y_k) \to (\bar{x}, \bar{y}) \in \text{Fix } T_{\text{Ryu}},
\]

\[
M(x_k, y_k) \to M(\bar{x}, \bar{y}) \in \Delta,
\]

and

\[
((Q_3 - Q_1)M(x_k, y_k), (Q_3 - Q_2)M(x_k, y_k)) \to (0, 0). \tag{7}
\]

In particular,

\[
J_A(x_k) \to J_A \bar{x} \in \text{zer}(A + B + C).
\]

**Proof.** See [27] and [2]. \( \blacksquare \)

\(^1\)We will express vectors in product spaces both as column and as row vectors depending on which version is more readable.
3.2 Malitsky-Tam splitting

We now turn to the Malitsky-Tam algorithm. In this subsection, let $n \in \{3, 4, \ldots \}$ and let $A_1, A_2, \ldots, A_n$ be maximally monotone operators on $X$. The problem of interest is to

$$\text{find } x \in X \text{ such that } 0 \in (A_1 + A_2 + \cdots + A_n)x,$$

and we assume that (8) has a solution. The algorithm proposed by Malitsky and Tam [25] provides a method for finding a solution to (8). Now set

$$M : X^{n-1} \to X^n : \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}, \quad \text{where}$$

$$\forall i \in \{1, \ldots, n\} \quad x_i = \begin{cases} J_{A_i}(z_1), & \text{if } i = 1; \\ J_{A_i}(x_{i-1} + z_i - z_{i-1}), & \text{if } 2 \leq i \leq n - 1; \\ J_{A_n}(x_1 + x_{n-1} - z_{n-1}), & \text{if } i = n. \end{cases}$$

As before, we denote by $Q_1 : X^n \to X : (x_1, \ldots, x_n) \mapsto x_1$ and similarly for $Q_2, \ldots, Q_n$. We also set

$$\Delta := \{ (x, \ldots, x) \in X^n \mid x \in X \},$$

which is also known as the diagonal in $X^n$. We are now ready to introduce the Malitsky-Tam (MT) operator

$$T := T_{MT} : X^{n-1} \to X^{n-1} : z \mapsto z + \begin{pmatrix} (Q_2 - Q_1)Mz \\ (Q_3 - Q_2)Mz \\ \vdots \\ (Q_n - Q_{n-1})Mz \end{pmatrix}.$$ 

Given a starting point $z_0 \in X^{n-1}$, the basic form of MT splitting generates a governing sequence via

$$\forall k \in \mathbb{N} \quad z_{k+1} := (1 - \lambda)z_k + \lambda Tz_k.$$ 

The following result records the basic convergence.

**Fact 3.2. (Malitsky-Tam)** The operator $T_{MT}$ is nonexpansive with

$$\text{Fix } T_{MT} = \{ z \in X^{n-1} \mid Mz \in \Delta \},$$

$$\text{zer}(A_1 + \cdots + A_n) = J_{A_1}(Q_1 \text{ Fix } T_{MT}).$$

Suppose that $0 < \lambda < 1$ and consider the sequence generated by (12). Then there exists $\bar{z} \in X^{n-1}$ such that

$z_k \to \bar{z} \in \text{Fix } T_{MT},$

$Mz_k \to M\bar{z} \in \Delta,$

and

$$\forall (i, j) \in \{1, \ldots, n\}^2 \quad (Q_i - Q_j)Mz_k \to 0.$$ 

In particular,

$$Q_1Mz_k \to Q_1M\bar{z} \in \text{zer}(A_1 + \cdots + A_n).$$

**Proof.** See [25].

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\footnote{Again, we will express vectors in product spaces both as column and as row vectors depending on which version is more readable.}
3.3 Campoy splitting

Finally, we turn to Campoy’s algorithm [15]. Again, let \( n \in \{3, 4, \ldots\} \) and let \( A_1, A_2, \ldots, A_n \) be maximally monotone operators on \( X \). The problem of interest is to find \( x \in X \) such that

\[
0 \in (A_1 + A_2 + \cdots + A_n)x, \tag{15}
\]

and we assume again that \( \text{(15)} \) has a solution. Denote the diagonal in \( X^{n-1} \) by \( \Delta \), and define the embedding operator \( E \) by

\[
E: X \to X^{n-1}: x \mapsto (x, x, \ldots, x). \tag{16}
\]

Next, define two operators \( A \) and \( B \) on \( X^{n-1} \) by

\[
A: (x_1, \ldots, x_{n-1}) \mapsto \frac{1}{n-1} \left( A_n x_1, \ldots, A_n x_{n-1} \right) + N_\Delta (x_1, \ldots, x_{n-1}),
\]

\[
B: (x_1, \ldots, x_{n-1}) \mapsto \left( A_1 x_1, \ldots, A_{n-1} x_{n-1} \right).
\]

We note that this way of splitting is also contained in early works of Alex Kruger (see [22] and [23]) to whom we are grateful for making us aware of this connection. However, the relevant resolvents (see Fact 3.3) were only very recently computed by Campoy. We now define the Campoy operator by

\[
T := T_C: X^{n-1} \to X^{n-1}: z \mapsto R_B R_A z = z - 2J_A z + 2J_B R_A z. \tag{17}
\]

Given a starting point \( z_0 \in X^{n-1} \), the basic form of Campoy’s splitting algorithm generates a governing sequence via

\[
(\forall k \in \mathbb{N}) \quad z_{k+1} := (1 - \lambda)z_k + \lambda T z_k. \tag{18}
\]

The following result records basic properties and the convergence result.

**Fact 3.3. (Campoy)** The operators \( A \) and \( B \) are maximally monotone, with resolvents

\[
M := J_A: (x_1, \ldots, x_{n-1}) \mapsto E \left( J_{\frac{1}{n-1}A_n} \left( \frac{1}{n-1} \sum_{i=1}^{n-1} x_i \right) \right), \tag{19a}
\]

\[
J_B: (x_1, \ldots, x_{n-1}) \mapsto \left( J_{A_1 x_1}, \ldots, J_{A_{n-1} x_{n-1}} \right), \tag{19b}
\]

respectively. We also have

\[
\text{zer}(A + B) = E(\text{zer}(A_1 + \cdots + A_n)).
\]

The operator

\[
F_C := \frac{1}{2} \text{Id} + \frac{1}{2} T_C = \text{Id} - J_A + J_B R_A \tag{20}
\]

is the standard Douglas-Rachford (firmly nonexpansive) operator for finding a zero of \( A + B \) and its reflected version is the Campoy operator \( T_C = 2F_C - \text{Id} \) is therefore nonexpansive. Suppose that \( 0 < \lambda < 1 \) and consider the sequence generated by (18). Then there exists \( \overline{z} \in X^{n-1} \) such that

\[
z_k \rightarrow \overline{z} \in \text{Fix} T_C,
\]

\[
Mz_k = J_A z_k \rightarrow J_A \overline{z} \in \text{zer}(A + B).
\]

**Proof.** See [15, Theorem 3.3(i)–(iii) and Theorem 5.1], and [8, Theorem 26.11].
If one wishes to avoid the product space reformulation, then one may set
\[ p_k := \frac{1}{n-1} \sum_{i=1}^{n-1} z_{ki} \]
so that \( J_A(z_k) = E(p_k) \). Now for \( i \in \{1, \ldots, n-1\} \), we define
\[ x_{ki} := J_A(2p_k - z_{ki}). \]
It follows that
\[
\begin{pmatrix}
  x_{k,1} \\
  \vdots \\
  x_{k,n-1}
\end{pmatrix}
= J_B(R_A(z)).
\]
Now one can rewrite Campoy’s algorithm as follows (and this is his original formulation): Given a starting point \( (z_{0,1}, \ldots, z_{0,n-1}) \in X^{n-1} \) and \( k \in \mathbb{N} \), update
\[
(\forall i \in \{1, \ldots, n-1\})
\begin{align*}
  x_{ki} &= J_A(2p_k - z_{ki}), \\
  z_{k+1,i} &= z_{k,i} + \lambda(x_{k,i} - p_k),
\end{align*}
\]
which gives us the equivalence of (18) and (21c).

For future use in Section 5, we bring the Campoy operator into a framework similar to the other algorithms. As before, we denote by \( Q_1 : X^{n-1} \to X : (x_1, \ldots, x_{n-1}) \mapsto x_1 \) and similarly for \( Q_2, \ldots, Q_n \). We recall from (19a) that
\[
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_{n-1}
\end{pmatrix}
= M
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_{n-1}
\end{pmatrix}
\]
and we set
\[
\begin{pmatrix}
  x_1 \\
  \vdots \\
  x_{n-1}
\end{pmatrix}
= S
\begin{pmatrix}
  z_1 \\
  \vdots \\
  z_{n-1}
\end{pmatrix},
\]
where
\[
(\forall i \in \{1, \ldots, n-1\})
x_i = J_A(2Q_iMz - z_i).
\]
We can then rewrite the Campoy operator in (17) as
\[
T = T_C : X^{n-1} \to X^{n-1} : z \mapsto z + 2Sz - 2Mz.
\]

## 4 Main results

We are now ready to tackle our main results. We shall find useful descriptions of the fixed point sets of the Ryu, the Malitsky-Tam, and the Campoy operators. These description will allow us to deduce strong convergence of the iterates to the projection onto the intersection.
4.1 Ryu splitting

In this subsection, we assume that $U, V, W$ are closed linear subspaces of $X$.

We set

\[ A := N_U, \quad B := N_V, \quad C := N_W. \]

Then

\[ Z := \text{zer}(A + B + C) = U \cap V \cap W. \]

Using linearity of the projection operators, the operator $M$ defined in (3) turns into

\[ M: X \times X \to X \times X \times X : (x, y) \mapsto \begin{pmatrix} P_{U}x \\ P_V P_{U}x + P_V y \\ P_W P_{U}x + P_W P_V P_{U}x - P_W x + P_W P_V y - P_W y \end{pmatrix}, \]

while the Ryu operator is still (see (4))

\[ T := T_{\text{Ryu}}: X^2 \to X^2 : z \mapsto z + ((Q_3 - Q_1)Mz, (Q_3 - Q_2)Mz). \]

We now determine the fixed point set of the Ryu operator.

**Lemma 4.1.** Let $(x, y) \in X \times X$. Then

\[ \text{Fix} T = (Z \times \{0\}) \oplus \left((U^\perp \times V^\perp) \cap (\Delta^\perp + (\{0\} \times W^\perp))\right), \]

where $\Delta := \{(x, x) \in X \times X \mid x \in X \}$. Consequently, setting

\[ E := (U^\perp \times V^\perp) \cap (\Delta^\perp + (\{0\} \times W^\perp)), \]

we have

\[ P_{\text{Fix} T}(x, y) = (P_Z x, 0) \oplus P_E (x, y) \in (P_Z x \oplus U^\perp) \times V^\perp. \]

**Proof.** Note that $(x, y) = (P_{W^\perp} y + (x - P_{W^\perp} y), P_{W^\perp} y + P_{W} y) = (P_{W^\perp} y, P_{W^\perp} y) + (x - P_{W^\perp} y, P_W y) \in \Delta + (X \times W)$. Hence $X \times X = \Delta + (X \times W)$ is closed; consequently, by, e.g., [8, Corollary 15.35],

\[ \Delta^\perp + (\{0\} \times W^\perp) \quad \text{is closed.} \]

Next, using (6), we have the equivalences

\[ (x, y) \in \text{Fix} T_{\text{Ryu}} \]
\[ \Leftrightarrow P_{U}x = P_V (P_{U}x + y) = P_W (R_U x - y) \]
\[ \Leftrightarrow P_{U}x \in Z \land y \in V^\perp \land P_{U}x = P_W (P_{U}x - P_{U^\perp} x - y) \]
\[ \Leftrightarrow x \in Z + U^\perp \land y \in V^\perp \land P_{U^\perp} x + y \in W^\perp. \]
Now define the linear operator \( S : X \times X \to X : (x, y) \mapsto x + y \). Hence
\[
\text{Fix } T_{\text{Ryu}} = \{ (x, y) \in (Z + U^\perp) \times V^\perp \mid P_{U^\perp}x + y \in W^\perp \} \\
= \{ (z + u^\perp, v^\perp) \mid z \in Z, u^\perp \in U^\perp, v^\perp \in V^\perp, u^\perp + v^\perp \in W^\perp \} \\
= (Z \times \{ 0 \}) \oplus ((U^\perp \times V^\perp) \cap S^{-1}(W^\perp)).
\]
On the other hand, \( S^{-1}(W^\perp) = (\{ 0 \} \times W^\perp) + \ker S = (\{ 0 \} \times W^\perp) + \Delta^\perp \) is closed by (28). Altogether,
\[
\text{Fix } T_{\text{Ryu}} = (Z \times \{ 0 \}) \oplus ((U^\perp \times V^\perp) \cap ((\{ 0 \} \times W^\perp) + \Delta^\perp)),
\]
i.e., (26) holds. Finally, (27) follows from Fact 2.1.

We are now ready for the main convergence result on Ryu’s algorithm.

**Theorem 4.2. (main result on Ryu splitting)** Given \( 0 < \lambda < 1 \) and \( (x_0, y_0) \in X \times X \), generate the sequence \((x_k, y_k)_{k \in \mathbb{N}}\) via
\[
(\forall k \in \mathbb{N}) \quad (x_{k+1}, y_{k+1}) := (1 - \lambda)(x_k, y_k) + \lambda T(x_k, y_k).
\]
Then
\[
M(x_k, y_k) \to (P_Z(x_0), P_Z(x_0), P_Z(x_0));
\]
in particular,
\[
P_{U^0}(x_k) \to P_Z(x_0).
\]

**Proof.** Set \( T_\lambda := (1 - \lambda) \text{Id} + \lambda T \) and observe that \((x_k, y_k)_{k \in \mathbb{N}} = (T_\lambda^k(x_0, y_0))_{k \in \mathbb{N}}\). Hence, by Corollary 2.9 and (27)
\[
(x_k, y_k) \to P_{\text{Fix } T_\lambda}(x_0, y_0) = P_{\text{Fix } T}(x_0, y_0) \\
= (P_Z x_0, 0) + P_E(x_0, y_0) \in (P_Z x_0 + U^\perp) \times V^\perp,
\]
where \( E \) is as in Lemma 4.1. Hence \( Q_1 M(x_k, y_k) = P_{U^0} x_k \to P_U(P_Z x_0) = P_Z x_0 \). Now (7) yields
\[
\lim_{k \to \infty} Q_1 M(x_k, y_k) = \lim_{k \to \infty} Q_2 M(x_k, y_k) = \lim_{k \to \infty} Q_3 M(x_k, y_k) = P_Z x_0,
\]
i.e., (29) and we’re done.

### 4.2 Malitsky-Tam splitting

Let \( n \in \{ 3, 4, \ldots \} \). In this subsection, we assume that \( U_1, \ldots, U_n \) are closed linear subspaces of \( X \). We set
\[
(\forall i \in \{ 1, 2, \ldots, n \}) \quad A_i := N_{U_i} \quad \text{and} \quad P_i := P_{U_i}.
\]
Then
\[
Z := \text{zer}(A_1 + \cdots + A_n) = U_1 \cap \cdots \cap U_n.
\]
The operator \( M \) defined in (9) turns into
\[
M : X^{n-1} \to X^n : \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix}, \quad \text{where}
\]
\[
(30a)
\]
\[^3\text{Recall (24) and (25) for the definitions of } M \text{ and } T.\]
where and the MT operator remains (see (11))

\[
\text{Lemma 4.3.} \quad \text{The fixed point set of the MT operator } T = T_{\text{MT}} \text{ is}
\]

\[
\text{Fix } T = \{ (z, \ldots, z) \in X^{n-1} \mid z \in Z \} \oplus E,
\]

where

\[
E := \text{ran } \Psi \cap (X^{n-2} \times U_1^+)
\]

\[
\subseteq U_1^+ \times \cdots \times (U_1^+ + \cdots + U_{n-2}^+) \times ((U_1^+ + \cdots + U_{n-1}^+) \cap U_n^+)
\]

and

\[
\Psi : U_1^+ \times \cdots \times U_{n-1}^+ \to X^{n-1}
\]

\[
(y_1, \ldots, y_{n-1}) \mapsto (y_1 + y_2 + \cdots + y_n - 1)
\]

is the continuous linear partial sum operator which has closed range.

Let \( z = (z_1, \ldots, z_{n-1}) \in X^{n-1} \), and set \( \bar{z} := (z_1 + z_2 + \cdots + z_{n-1})/(n - 1) \). Then

\[
P_{\text{Fix } T} z = (P_{\bar{z}} z, \ldots, P_{\bar{z}} z) \oplus P_{\bar{z}} z \in X^{n-1}
\]

and hence

\[
P_1(Q_1 P_{\text{Fix } T}) z = P_{\bar{z}} z.
\]

**Proof.** Assume temporarily that \( z \in \text{Fix } T \) and set \( x = Mz = (x_1, \ldots, x_n) \). Then \( \bar{x} := x_1 = \cdots = x_n \) and so \( \bar{x} \in Z \). Now \( P_1 z_1 = x_1 = \bar{x} \in Z \) and thus

\[
z_1 \in \bar{x} + U_1^+ \subseteq Z + U_1^+.
\]

Next, \( \bar{x} = x_2 = P_2(x_1 + z_2 - z_1) = P_2 x_1 + P_2(z_2 - z_1) = P_2 \bar{x} + P_2(z_2 - z_1) = \bar{x} \), which implies \( P_2(z_2 - z_1) = 0 \) and so \( z_2 - z_1 \in U_2^+ \). It follows that

\[
z_2 \in z_1 + U_2^+.
\]

Similarly, by considering \( x_3, \ldots, x_{n-1} \), we obtain

\[
z_3 \in z_2 + U_3^+, \ldots, z_{n-1} \in z_{n-2} + U_{n-1}^+.
\]

Finally, \( \bar{x} = x_n = P_n(x_1 + x_{n-1} - z_{n-1}) = P_n(\bar{x} + z_{n-1} - z_{n-1}) = 2 \bar{x} - P_n z_{n-1} \), which implies \( P_n z_{n-1} = \bar{x} \), i.e., \( z_{n-1} \in \bar{x} + U_n^+ \). Combining with (37), we see that \( z_{n-1} \) satisfies

\[
z_{n-1} \in (z_{n-2} + U_{n-1}^+) \cap (P_1 z_1 + U_1^+).
\]
To sum up, our \( z \in \text{Fix } T \) must satisfy

\[
\begin{align*}
    z_1 & \in Z + U_1^\perp \\
    z_2 & \in z_1 + U_2^\perp \\
    & \vdots \\
    z_{n-2} & \in z_{n-3} + U_{n-2}^\perp \\
    z_{n-1} & \in (z_{n-2} + U_{n-1}^\perp) \cap (P_1z_1 + U_n^\perp).
\end{align*}
\]

(38a) (38b) (38c) (38d) (38e)

We now show the converse. To this end, assume now that our \( z \) satisfies (38). Note that \( Z^\perp = U_1^\perp + \cdots + U_n^\perp \). Because \( z_1 \in Z + U_1^\perp \), there exists \( z \in Z \) and \( u_1^\perp \in U_1^\perp \) such that \( z_1 = z \oplus u_1^\perp \). Hence \( x_1 = P_1z_1 = P_1z = z \). Next, \( z_2 \in z_1 + U_2^\perp \), say \( z_2 = z_1 + u_2^\perp = z \oplus (u_1^\perp + u_2^\perp) \), where \( u_2^\perp \in U_2^\perp \). Then

\[
x_2 = P_2(x_1 + z_2 - z_1) = P_2(z + u_1^\perp) = P_2z = z.
\]

Similarly, there exists also \( u_3^\perp \in U_3^\perp, \ldots, u_{n-1}^\perp \in U_{n-1}^\perp \) such that \( x_3 = \cdots = x_{n-1} = z \) and \( z = z \oplus (u_1^\perp + \cdots + u_n^\perp) \) for \( 2 \leq i \leq n - 1 \). Finally, we also have \( z_{n-1} = z \oplus u_n^\perp \) for some \( u_n^\perp \in U_n^\perp \). Thus \( x_n = P_n(x_1 + x_{n-1} - z_{n-1}) = P_n(2z - (z + u_n^\perp)) = P_nz = z \).

Altogether, \( z \in \text{Fix } T \). We have thus verified the description of \( \text{Fix } T \) announced in (32), using the convenient notation of the operator \( \Psi \) which is easily seen to have closed range.

Next, we observe that

\[
D := \{(z, \ldots, z) \in X^{n-1} \mid z \in Z\} = Z^{n-1} \cap \Delta,
\]

(39)

where \( \Delta \) is the diagonal in \( X^{n-1} \) which has projection \( P_\Delta(z_1, \ldots, z_n) = (\tilde{z}, \ldots, \tilde{z}) \) (see, e.g., [8, Proposition 26.4]). By convexity of \( Z \), we clearly have \( P_\Delta(Z^{n-1}) \subseteq Z^{n-1} \). Because \( Z^{n-1} \) is a closed linear subspace of \( X^{n-1} \), [17, Lemma 9.2] and (39) yield \( P_D = P_{Z^{n-1}}P_\Delta \) and therefore

\[
P_Dz = P_{Z^{n-1}}P_\Delta z = (P_Z\tilde{z}, \ldots, P_Z\tilde{z}).
\]

(40)

Combining (32), Fact 2.1, (39), and (40) yields (35).

Finally, observe that \( Q_1(P_\ell z) \in U_1^\perp \) by (33). Thus \( Q_1(P_{\text{Fix } T}z) \in P_Z\tilde{z} + U_1^\perp \) and (36) follows. \( \blacksquare \)

We are now ready for the main convergence result on the Malitsky-Tam algorithm.

**Theorem 4.4. (main result on Malitsky-Tam splitting)** Given \( 0 < \lambda < 1 \) and \( z_0 = (z_{0,1}, \ldots, z_{0,n-1}) \in X^{n-1} \), generate the sequence \( (z_k)_{k \in \mathbb{N}} \) via

\[
(\forall k \in \mathbb{N}) \quad z_{k+1} := (1 - \lambda)z_k + \lambda T z_k.
\]

Set

\[
p := \frac{1}{n-1} (z_{0,1} + \cdots + z_{0,n-1}).
\]

Then there exists \( \tilde{z} \in X^{n-1} \) such that

\[
z_k \to \tilde{z} \in \text{Fix } T,
\]

and

\[
Mz_k \to M\tilde{z} = (P_Z p, \ldots, P_Z p) \in X^n.
\]

(41)

\(^4\)Recall (30) and (31) for the definitions of \( M \) and \( T \).
In particular,
\[ P_1(Q_1z_k) = Q_1Mz_k \to P_Z(p) = \frac{1}{n-1}P_z(\sum_{i=1}^{n-1} x_i). \]  \hspace{1cm} (42)

Consequently, if \( x_0 \in X \) and \( z_0 = (x_0, \ldots, x_0) \in X^{n-1} \), then
\[ P_1Q_1z_k \to P_Zx_0. \]  \hspace{1cm} (43)

**Proof.** Set \( T_\lambda := (1-\lambda)\text{Id} + \lambda T \) and observe that \((z_k)_{k \in \mathbb{N}} = (T_\lambda^k z)_{k \in \mathbb{N}}\). Hence, by Corollary 2.9 and Lemma 4.3,
\[ z_k \to P_{\text{Fix } T_\lambda} z_0 = P_{\text{Fix } T} z_0 = (P_Zp, \ldots, P_Zp) + P_E(z_0), \]
where \( E \) is as in Lemma 4.3. Hence, using also (36),
\[ Q_1Mz_k = P_1Q_1z_k \]
\[ \to P_1Q_1((P_Zp, \ldots, P_Zp) + P_E(z_0)) \]
\[ = P_1(P_Zp + Q_1(P_E(z_0))) \]
\[ \in P_1(P_Zp + U^+) \]
\[ = \{P_1P_Zp\} \]
\[ = \{P_Zp\}, \]
i.e., \( Q_1Mz_k \to P_Zp \). Now (14) yields \( Q_iMz_k \to P_Zp \) for every \( i \in \{1, \ldots, n\} \). This yields (41) and (42).

The “Consequently” part is clear because when \( z_0 \) has this special form, then \( p = x_0 \). \( \blacksquare \)

### 4.3 Campoy splitting

Let \( n \in \{3,4,\ldots\} \). In this subsection, we assume that \( U_1, \ldots, U_n \) are closed linear subspaces of \( X \). We set
\[ (\forall i \in \{1,2,\ldots,n\}) \quad A_i := N_{U_i} \quad \text{and} \quad P_i := P_{U_i}. \]
Then
\[ Z := \text{zer}(A_1 + \cdots + A_n) = U_1 \cap \cdots \cap U_n. \]
By (19),
\[ J_A : (x_1, \ldots, x_{n-1}) \mapsto E\left(P_n\left(\frac{1}{n-1}\sum_{i=1}^{n-1} x_i\right)\right), \]  \hspace{1cm} (44a)
\[ J_B : (x_1, \ldots, x_{n-1}) \mapsto (P_1x_1, \ldots, P_{n-1}x_{n-1}). \]  \hspace{1cm} (44b)

Now recall from (16) that \( E : x \mapsto (x, \ldots, x) \) and denote by \( \Delta = \{(x, \ldots, x) \in X^{n-1} \mid x \in X\} \), which is the diagonal in \( X^{n-1} \). We are now ready for the following result.

**Lemma 4.5.** Set \( \bar{U} := E(U_n) = U_{n-1} \cap \Delta \subseteq X^{n-1} \) and \( \bar{V} := U_1 \times \cdots \times U_{n-1} \subseteq X^{n-1} \). Then for every \( z = (z_1, \ldots, z_{n-1}) \in X^{n-1} \) and \( \bar{z} = \frac{1}{n-1}\sum_{i=1}^{n-1} z_i \), we have
\[ J_A z = P_{\bar{U}_{n-1}}J_A z = (P_n z_1, \ldots, P_n \bar{z}) = P_{\bar{U}} z, \]  \hspace{1cm} (45a)
\[ J_B z = P_{U_1 \times \ldots \times U_{n-1}} z = \tilde{P}_\varphi z, \quad \text{(45b)} \]
\[ T = T_C = \text{Id} - 2 J_A + 2 J_B R_A, \quad \text{(45c)} \]

and
\[ A = N_\tilde{U} \quad \text{and} \quad B = N_\tilde{V}. \quad \text{(46)} \]

Moreover, \( \tilde{U} \cap \tilde{V} = Z_{n-1} \cap \Delta, \) \( P_{\text{Fix} T} = P_{\tilde{U} \cap \tilde{V}} \oplus P_{\tilde{U} \cap \tilde{V}^\perp}, \) and
\[ J_A P_{\text{Fix} T} z = P_{E(z)} z = E(P_Z z). \quad \text{(47)} \]

**Proof.** It is clear from (44a) that \( J_A z = P_{U_{n-1}^\perp} P_\Delta z. \) Note that \( P_{U_{n-1}^\perp} \subseteq \Delta. \) It thus follows from [17, Lemma 9.2] that
\[ P_\Delta P_{U_{n-1}^\perp} = P_{U_{n-1}^\perp \cap \Delta} = P_{\tilde{U}}. \quad \text{(48)} \]

and (45a) follows. The formula for (45b) is clear from (44b).

It is clear from (17) and (20) that \( F_C \) is the Douglas-Rachford operator for the pair \( (A, B), \) which has the same fixed point set as the Campoy operator \( T_C. \) In view of (46), this is the feasibility case applied to the pair of subspaces \( (\tilde{U}, \tilde{V}). \) Note that
\[ \tilde{U} \cap \tilde{V} = E(Z) = Z_{n-1} \cap \Delta. \quad \text{(49)} \]

By [6, Proposition 3.6], \( \text{Fix } T = (\tilde{U} \cap \tilde{V}) \oplus (\tilde{U}^\perp \cap \tilde{V}^\perp), \)
\[ P_{\text{Fix } T} = P_{\tilde{U} \cap \tilde{V}} \oplus P_{\tilde{U} \cap \tilde{V}^\perp}, \quad \text{(50)} \]

and
\[ J_A P_{\text{Fix } T} = P_{\tilde{U}} P_{\text{Fix } T} = P_{\tilde{U} \cap \tilde{V}} = P_{E(Z)} = P_{Z_{n-1} \cap \Delta} = P_{\tilde{U}}. \quad \text{(51)} \]

where the rightmost identity in (51) follows from the same argument as in the proof of (48).

**Theorem 4.6. (main result on Campoy splitting)** Given \( z_0 = (z_{0,0}, \ldots, z_{0,n-1}) \in X_{n-1} \) and \( 0 < \lambda < 1, \)
generate the sequence \((z_k)_{k \in \mathbb{N}}\) via\(^5\)
\[ (\forall k \in \mathbb{N}) \quad z_{k+1} := (1 - \lambda) z_k + \lambda T z_k. \]

Set
\[ z := \frac{1}{n-1} (z_{0,0} + \cdots + z_{0,n-1}). \]

Then
\[ z_k \rightarrow P_{\text{Fix } T} z_0 \in \text{Fix } T \quad \text{(52)} \]

and
\[ M z_k = I_A z_k \rightarrow I_A P_{\text{Fix } T} z_0 = (P_Z z, \ldots, P_Z z) \in X^n. \quad \text{(53)} \]

**Proof.** Because \( T \) is nonexpansive (see Fact 3.3), we see that (52) follows from Corollary 2.9. Finally, (53) follows from (47).

For future use in Section 5, we note that the operators defined in (22a) and (22b) turn into
\[ M : X_{n-1} \rightarrow X_{n-1} : z = \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} \mapsto \frac{1}{n-1} \begin{pmatrix} \sum_{i=1}^{n-1} P_i(z_i) \\ \vdots \\ \sum_{i=1}^{n-1} P_i(z_i) \end{pmatrix} \quad \text{and} \quad \text{(54a)} \]

\(^5\)Recall (45c) for the definition of \( T.\)
\[ S : X^{n-1} \to X^{n-1} : z = \begin{pmatrix} z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}, \text{ where} \]

\[ (\forall i \in \{1, \ldots, n\}) \quad x_i = P_i(2Q_iMz - z_i), \] (54c)

while the Campoy operator remains (see (23))

\[ T = T_C : X^{n-1} \to X^{n-1} : z \mapsto z + 2Sz - 2Mz. \] (55)

### 4.4 Extension to the consistent affine case

In this subsection, we comment on the behaviour of the above splitting algorithms in the consistent affine case. To this end, we shall assume that \( V_1, \ldots, V_n \) are closed affine subspaces of \( X \) with nonempty intersection:

\[ V := V_1 \cap V_2 \cap \cdots \cap V_n \neq \emptyset. \]

We repose the problem of finding a point in \( Z \) as

\[ \text{find } x \in X \text{ such that } 0 \in (A_1 + A_2 + \cdots + A_n)x, \]

where each \( A_i = N_{V_i} \). When we consider Ryu splitting, we also impose \( n = 3 \). Set \( U_i := V_i - V_i \), which is the parallel space of \( V_i \). Now let \( v \in V \). Then \( V_i = v + U_i \) and hence \( J_{N_{V_i}} = P_{V_i} = P_{v + U_i} \) satisfies \( P_{v + U_i} = v + P_{U_i}(x - v) = P_{U_i}x + P_{U_i}(v) \). Put differently, the resolvents from the affine problem are translations of the the resolvents from the corresponding linear problem which considers \( U_i \) instead of \( V_i \).

The construction of the operator \( T \in \{ T_{\text{Ryu}}, T_{\text{MT}}, T_C \} \) now shows that it is a translation of the corresponding operator from the linear problem. And finally \( T_{\lambda} = (1 - \lambda) \text{Id} + \lambda T \) is a translation of the corresponding operator from the linear problem which we denote by \( L_{\lambda} : L_{\lambda} = (1 - \lambda) \text{Id} + \lambda L \), where \( L \) is the Ryu operator, the Malitsky-Tam operator, or the Campoy operator of the parallel linear problem, and there exists \( b \in X^{n-1} \) such that

\[ T_{\lambda}(x) = L_{\lambda}(x) + b. \]

By Fact 2.10 (applied in \( X^{n-1} \)), there exists a vector \( a \in X^{n-1} \) such that

\[ (\forall k \in \mathbb{N}) \quad T_{\lambda}^k x = a + L_{\lambda}^k(x - a). \] (56)

In other words, the behaviour in the affine case is essentially the same as in the linear parallel case, appropriately shifted by the vector \( a \). Moreover, because \( L_{\lambda}^k \to P_{\text{Fix} L} \) in the parallel linear setting, we deduce from Fact 2.10 that

\[ T_{\lambda}^k \to P_{\text{Fix} T}. \]

By (56), the rate of convergence in the affine case are identical to the rate of convergence in the parallel linear case.

Each of our three algorithms under consideration features an operator \( M \) — see (24), (30), (54a) — for Ryu splitting, for Malitsky-Tam splitting, for Campoy splitting, respectively. In all cases, the convergence results established guarantee that

\[ MT_{\lambda}^k z_0 \to P_V z, \]
where \( \mathcal{M} \) returns the arithmetic average of the output of \( M \), where \( z_0 \) is the starting point, and where \( z \) is either the first component of \( z_0 \) (for Ryu splitting) or the arithmetic average of the components of \( z_0 \) (for Malitsky-Tam and for Campoy splitting); see (29), (42), and (53).

To sum up this subsection, we note that in the consistent affine case, Ryu’s, the Malitsky-Tam, and Campoy’s algorithm each exhibits the same pleasant convergence behaviour as their linear parallel counterparts!

It is, however, less clear how these algorithms behave when \( V = \emptyset \).

5 Matrix representation

In this section, we assume that \( X \) is finite-dimensional, say

\[ X = \mathbb{R}^d. \]

All three splitting algorithms considered in this paper are of the form

\[ T^\lambda_\delta \rightarrow P_{\text{Fix} T}, \quad \text{where } 0 < \lambda < 1 \text{ and } T_\lambda = (1 - \lambda) \text{Id} + \lambda T. \tag{57} \]

In anticipation of Section 6, we will also consider a fourth operator based on “POCS”, the (sequential) method of Projections Onto Convex Sets. Starting from Section 4, we have dealt with a special case where \( T \) is a linear operator; hence, so is \( T_\lambda \) and by [10, Corollary 2.8], the convergence of the iterates is linear because \( X \) is finite-dimensional. What can be said about this rate? By [7, Theorem 2.12(ii) and Theorem 2.18], a (sharp) lower bound for the rate of linear convergence is the spectral radius of \( T_\lambda - P_{\text{Fix} T} \), i.e.,

\[ \rho(T_\lambda - P_{\text{Fix} T}) := \max \left\{ \text{(possibly complex) eigenvalues of } T_\lambda - P_{\text{Fix} T} \right\}, \]

while an upper bound is the operator norm

\[ \|T_\lambda - P_{\text{Fix} T}\|. \]

The lower bound is optimal and close to the true rate of convergence, see [7, Theorem 2.12(i)]. Both spectral radius and operator norms of matrices are available in programming languages such as Julia [12] which features strong numerical linear algebra capabilities. In order to compute these bounds for the linear rates, we must provide matrix representations for \( T \) (which immediately gives rise to one for \( T_\lambda \)) and for \( P_{\text{Fix} T} \). In the previous sections, we casually switched back and forth between column and row vector representations for readability. In this section, we need to get the structure of the objects right. To visually stress this, we will use square brackets for vectors and matrices.

For the remainder of this section, we fix three linear subspaces \( U, V, W \) of \( \mathbb{R}^d \), with intersection

\[ Z := U \cap V \cap W. \]

We assume that the matrices \( P_U, P_V, P_W \) in \( \mathbb{R}^{d \times d} \) are available to us (and hence so are \( P_U^\perp, P_V^\perp, P_W^\perp \) and \( P_Z \), via Example 2.2 and Corollary 2.4, respectively).
5.1 Ryu splitting

In this subsection, we consider Ryu splitting. First, the block matrix representation of the operator $M$ occurring in Ryu splitting (see (24)) is

$$M = \begin{bmatrix} P_U & 0 \\ P_V P_U & P_V \\ P_W P_U + P_W P_V P_U - P_W & P_W P_V - P_W \end{bmatrix} \in \mathbb{R}^{3d \times 2d}. \quad (58)$$

Hence, using (25), we obtain the following matrix representation of the Ryu splitting operator $T = T_{\text{Ryu}}$:

$$T = \begin{bmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix} + \begin{bmatrix} - \text{Id} & 0 & \text{Id} \\ 0 & - \text{Id} & \text{Id} \end{bmatrix} \begin{bmatrix} P_U & 0 \\ P_V P_U & P_V \\ P_W P_U + P_W P_V P_U - P_W & P_W P_V - P_W \end{bmatrix} \quad (59a)$$

$$= \begin{bmatrix} \text{Id} - P_U + P_W P_U + P_W P_V P_U - P_W & P_W P_V - P_W \\ P_W P_U + P_W P_V P_U - P_W - P_V P_U & \text{Id} + P_W P_V - P_V - P_W \end{bmatrix} \in \mathbb{R}^{2d \times 2d}. \quad (59b)$$

Next, we set, as in Lemma 4.1,

$$\Delta = \{ [x,x]^{\top} \in \mathbb{R}^{2d} \mid x \in X \}, \quad (60a)$$

$$E = (U^\perp \times V^\perp) \cap (\Delta^\perp + (\{0\} \times W^\perp)) \quad (60b)$$

so that, by (27),

$$P_{\text{Fix}} T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} P_Z x \\ 0 \end{bmatrix} + P_E \begin{bmatrix} x \\ y \end{bmatrix}. \quad (61)$$

With the help of Corollary 2.4, we see that the first term, $[P_Z x, 0]^{\top}$, is obtained by applying the matrix

$$\begin{bmatrix} P_Z & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 P_U (P_U + P_V)^{\dagger} P_V (2 P_U (P_U + P_V)^{\dagger} P_V + P_W)^{\dagger} P_W \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2d \times 2d} \quad (62)$$

to $[x, y]^{\top}$. Let’s turn to $E$, which is an intersection of two linear subspaces. The projector of the left linear subspace making up this intersection, $U^\perp \times V^\perp$, has the matrix representation

$$P_{U^\perp \times V^\perp} = \begin{bmatrix} \text{Id} & 0 \\ 0 & \text{Id} - P_V \end{bmatrix}. \quad (63)$$

We now turn to the right linear subspace, $\Delta^\perp + (\{0\} \times W^\perp)$, which is a sum of two subspaces whose complements are $\Delta^\perp = \Delta$ and $(\{0\} \times W^\perp) = X \times W$, respectively. The projectors of the last two subspaces are

$$P_{\Delta} = \frac{1}{2} \begin{bmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix} \quad \text{and} \quad P_{X \times W} = \begin{bmatrix} \text{Id} & 0 \\ 0 & P_W \end{bmatrix},$$

respectively. Thus, Corollary 2.5 yields

$$P_{\Delta^\perp + (\{0\} \times W^\perp)} = \begin{bmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix} - 2 \cdot \frac{1}{2} \begin{bmatrix} \text{Id} & \text{Id} \\ \text{Id} & \text{Id} \end{bmatrix} \left( \frac{1}{2} \begin{bmatrix} \text{Id} & \text{Id} \\ \text{Id} & \text{Id} \end{bmatrix} + \begin{bmatrix} \text{Id} & 0 \\ 0 & P_W \end{bmatrix} \right)^{\dagger} \begin{bmatrix} \text{Id} & 0 \\ 0 & P_W \end{bmatrix} \quad (64a)$$

$$= \begin{bmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix} - 2 \cdot \frac{1}{2} \begin{bmatrix} \text{Id} & \text{Id} \\ \text{Id} & \text{Id} \end{bmatrix} \left( \frac{1}{2} \begin{bmatrix} \text{Id} & \text{Id} \\ \text{Id} & \text{Id} \end{bmatrix} + \begin{bmatrix} \text{Id} & 0 \\ 0 & P_W \end{bmatrix} \right)^{\dagger} \begin{bmatrix} \text{Id} & 0 \\ 0 & P_W \end{bmatrix} \quad (64b)$$

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To compute $P_E$, where $E$ is as in (60b), we combine (63), (64) under the umbrella of Fact 2.3 — the result does not seem to simplify so we don’t typeset it. Having $P_E$, we simply add it to (62) to obtain $P_{\text{Fix}T}$ because of (61).

\subsection{5.2 Malitsky-Tam splitting}

In this subsection, we turn to Malitsky-Tam splitting for the current setup — this corresponds to Section 4.2 with $n = 3$ and where we identify $(U_1, U_2, U_3)$ with $(U, V, W)$. The block matrix representation of $M$ from (30) is

$$M = \begin{bmatrix} P_U & 0 & 0 \\ -P_V (\text{Id} - P_U) & P_V \\ P_W (P_U + P_V P_U - P_V) & -P_W (\text{Id} - P_V) \end{bmatrix} \in \mathbb{R}^{3d \times 2d}. \quad (65)$$

Thus, using (31), we obtain the following matrix representation of the Malitsky-Tam splitting operator $T = T_{MT}$:

$$T = \begin{bmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{bmatrix} + \begin{bmatrix} -\text{Id} & 0 \\ 0 & -\text{Id} \end{bmatrix} \begin{bmatrix} P_U & 0 \\ -P_V (\text{Id} - P_U) & P_V \\ P_W (P_U + P_V P_U - P_V) & -P_W (\text{Id} - P_V) \end{bmatrix} \in \mathbb{R}^{2d \times 2d}. \quad (66a)$$

$$= \begin{bmatrix} \text{Id} - P_U - P_V (\text{Id} - P_U) & P_V \\ P_V (\text{Id} - P_U) + P_W (P_U + P_V P_U - P_V) & \text{Id} - P_V - P_W (\text{Id} - P_V) \end{bmatrix} \quad (66b)$$

$$= \begin{bmatrix} (\text{Id} - P_V) (\text{Id} - P_U) & P_V \\ (\text{Id} - P_W) P_V (\text{Id} - P_U) + P_W P_U & (\text{Id} - P_W) (\text{Id} - P_U) \end{bmatrix} \in \mathbb{R}^{2d \times 2d}. \quad (66c)$$

Next, in view of (35), we have

$$P_{\text{Fix}T} = \frac{1}{2} \begin{bmatrix} P_Z & P_Z \\ P_Z & P_Z \end{bmatrix} + P_E, \quad (67)$$

where (see (33) and (34))

$$E = \text{ran} \Psi \cap (X \times W^\perp) \quad (68)$$

and

$$\Psi: U^\perp \times V^\perp \to X^2: \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \mapsto \begin{bmatrix} y_1 \\ y_1 + y_2 \end{bmatrix}. \quad$$

We first note that

$$\text{ran} \Psi = \text{ran} \begin{bmatrix} \text{Id} & 0 \\ \text{Id} & \text{Id} \end{bmatrix} \begin{bmatrix} P_{U^\perp} & 0 \\ 0 & P_{V^\perp} \end{bmatrix} = \text{ran} \begin{bmatrix} P_{U^\perp} & 0 \\ P_{U^\perp} & P_{V^\perp} \end{bmatrix}. \quad$$

We thus obtain from Fact 2.6 that

$$P_{\text{ran} \Psi} = \begin{bmatrix} P_{U^\perp} & 0 \\ P_{U^\perp} & P_{V^\perp} \end{bmatrix} \begin{bmatrix} P_{U^\perp} & 0 \\ 0 & P_{V^\perp} \end{bmatrix} \in \mathbb{R}^{2d \times 2d}. \quad (69)$$
On the other hand,

\[ P_{X \times W^\perp} = \begin{bmatrix} \Id & 0 \\ 0 & P_W \end{bmatrix}. \]  

(70)

In view of (68) and Fact 2.3, we obtain

\[ P_E = 2P_{\text{ran } \Psi} \left( P_{\text{ran } \Psi} + P_{X \times W^\perp} \right)^\dagger P_{X \times W^\perp}. \]  

(71)

We could now use our formulas (69) and (70) for \( P_{\text{ran } \Psi} \) and \( P_{X \times W^\perp} \) to obtain a more explicit formula for \( P_E \) — but we refrain from doing so as the expressions become unwieldy. Finally, plugging the formula for \( P_Z \) from Corollary 2.4 into (67) as well as plugging (71) into (67) yields a formula for \( P_{\text{Fix } T} \).

### 5.3 Campoy splitting

In this subsection, we look at the Campoy splitting for the current setup — this corresponds to Section 4.2 with \( n = 3 \) and where we identify \((U_1, U_2, U_3)\) with \((U, V, W)\).

Using the linearity of \( P_W \), we see that the block matrix representation of \( M \) from (54a) is

\[ M = \begin{bmatrix} P_W & P_W \\ P_W & P_W \end{bmatrix}. \]  

(72)

We then write \( S \) from (54b) as

\[ S = \begin{bmatrix} P_U & 0 \\ 0 & P_V \end{bmatrix} \left( 2M - \begin{bmatrix} \Id & 0 \\ 0 & \Id \end{bmatrix} \right) \]

\[ = \begin{bmatrix} P_U & 0 \\ 0 & P_V \end{bmatrix} \begin{bmatrix} P_W - \Id & P_W \\ P_W & P_W - \Id \end{bmatrix} \]

\[ = \begin{bmatrix} P_UP_W - P_U & P_UP_W \\ P_V P_W - P_V & P_V P_W - P_V \end{bmatrix}. \]

Therefore, using (55), we see that the Campoy operator \( T = T_C \) is expressed as

\[ T = \begin{bmatrix} \Id & 0 \\ 0 & \Id \end{bmatrix} + 2S - 2M \]  

(73a)

\[ = \begin{bmatrix} \Id & 0 \\ 0 & \Id \end{bmatrix} + 2 \begin{bmatrix} P_UP_W - P_U & P_UP_W \\ P_V P_W - P_V & P_V P_W - P_V \end{bmatrix} - \begin{bmatrix} P_W & P_W \\ P_W & P_W \end{bmatrix} \]  

(73b)

\[ = \begin{bmatrix} \Id + 2P_UP_W - 2P_U - P_W & 2P_UP_W - P_W \\ 2P_V P_W - P_V & \Id + 2P_V P_W - 2P_V - P_W \end{bmatrix}. \]  

(73c)

We set, as in Lemma 4.5,

\( \tilde{U} := W^2 \cap \Delta \) and \( \tilde{V} := U \times V \),

where \( \Delta \) is the diagonal in \( X^2 \), and we obtain from (45)

\[ P_{\tilde{U}} = P_{W^2}P_{\Delta} = \begin{bmatrix} P_W & 0 \\ 0 & P_W \end{bmatrix} \frac{1}{2} \begin{bmatrix} \Id & \Id \\ \Id & \Id \end{bmatrix} = \frac{1}{2} \begin{bmatrix} P_W & P_W \\ P_W & P_W \end{bmatrix} \quad \text{and} \quad P_{\tilde{V}} = \begin{bmatrix} P_U & 0 \\ 0 & P_V \end{bmatrix}. \]  

(74)

We now use these projection formulas along with Lemma 4.5, Fact 2.3, and Example 2.2 to obtain

\[ P_{\text{Fix } T} = P_{\tilde{U} \cap \tilde{V}} + P_{\tilde{U} \cap \tilde{V}^\perp} \]  

(75a)

\[ = 2P_{\tilde{U}}(P_{\tilde{U}} + P_{\tilde{V}})^\dagger P_{\tilde{V}} + 2(\Id - P_{\tilde{U}})(2\Id - P_{\tilde{U}} - P_{\tilde{V}})^\dagger(\Id - P_{\tilde{V}}). \]  

(75b)

If desired, one may express \( P_{\text{Fix } T} \) in terms of \( P_{\tilde{U}}, P_{\tilde{V}}, P_W \) by substituting (74) into (75); however, due to limited space, we refrain from listing the outcome.
5.4 POCS

When applied to linear subspaces, it is known that the method of projections onto convex sets (POCS) converges to the projection onto the intersection. Hence POCS is a natural (sequential) algorithm to compare the above splitting methods to. The standard POCS operator is \( P_W P_V P_U \). We now derive a corresponding operator \( T \) based on the notion of averaged operators (see, e.g., [8, Section 4.5] for more on this notion). Observe first that each projector is \( \frac{1}{2} \)-averaged ([8, Remark 4.34(iii)]). Next, using [8, Proposition 4.46], the composition \( P_W P_V P_U \) is \( \frac{3}{4} \)-averaged. We therefore set

\[
T := \frac{4}{3} P_W P_V P_U - \frac{1}{3} \text{Id}.
\]

Then

\[
\text{Fix } T = U \cap V \cap W = Z
\]

so \( P_{\text{Fix } T} = P_Z \) and \( P_W P_V P_U = T_{3/4} \). Because \( T \) is nonexpansive, (57) yields that \( T^k \to P_Z \) when \( 0 < \lambda < 1 \).

Remark 5.1. (sequential vs parallel) Consider the much simpler case of comparing alternating projections (unrelaxed POCS for two subspaces) to parallel projections. Combining works by Kayalar and Weinert [21, Theorem 2] (see also [17, Theorem 9.31]) and by Badea, Grivaux, and Müller [3, Proposition 3.7] yields

\[
\|\frac{1}{2}(P_U + P_V) - P_{U \cap V}\| = \frac{1}{2} \|P_V P_U - P_{U \cap V}\| + \frac{1}{2},
\]

which shows that alternating projections perform better than parallel projections. Unfortunately, the situation is less clear for 3 or more subspaces – we refer the reader to [3, Theorem 4.4.(i)] where an estimate for the unrelaxed POCS operator is presented. However, the case of two subspaces indicates that it would not come as a surprise if a sequential methods outperforms a parallel one.

6 Numerical experiments

We now describe several experiments to evaluate the performance of the algorithms described in Section 5. Each instance of an experiment involves three subspaces \( U_i \) of dimension \( d_i \) for \( i \in \{1, 2, 3\} \) in \( X = \mathbb{R}^d \). By [26, equation (4.419) on page 205],

\[
\dim(U_1 + U_2) = d_1 + d_2 - \dim(U_1 \cap U_2).
\]

Hence

\[
\dim(U_1 \cap U_2) = d_1 + d_2 - d \geq d_1 + d_2 - d.
\]

Thus \( \dim(U_1 \cap U_2) \geq 1 \) whenever

\[
d_1 + d_2 \geq d + 1.
\]

Similarly,

\[
\dim(Z) \geq \dim(U_1 \cap U_2) + d_3 - d \geq d_1 + d_2 - d + d_3 - d = d_1 + d_2 + d_3 - 2d.
\]

Along with (79), a sensible choice for \( d_i \) satisfies

\[
d_i \geq 1 + \lceil 2d/3 \rceil
\]

because then \( d_1 + d_2 \geq 2 + 2\lceil 2d/3 \rceil \geq 2 + 4d/3 > 2 + d \). Hence \( d_1 + d_2 \geq 3 + d \) and \( d_1 + d_2 + d_3 \geq 3 + 3\lceil 2d/3 \rceil \geq 3 + 2d \). The smallest \( d \) that gives proper subspaces is \( d = 6 \), for which \( d_1 = d_2 = d_3 = 5 \) satisfy the above conditions.

We now describe our set of three numerical experiments designed to observe different aspects of the algorithms.
6.1 Experiment 1: Bounds on the rates of linear convergence

As shown in Section 5, we have lower and upper bounds on the rate of linear convergence of the operator $T_{\lambda}$. We conduct this experiment to observe how these bounds change as we vary $\lambda$. To this end, we generate 1000 instances of triples of linear subspaces $(U_1, U_2, U_3)$. This was done by randomly generating triples of three matrices $(B_1, B_2, B_3)$ each drawn from $\mathbb{R}^{6 \times 5}$. These were used to define the range spaces of these subspaces, which in turn gave us the projection onto $U_i$ (using, e.g., [8, Proposition 3.30(ii)]) as

$$P_{U_i} = B_i B_i^\dagger.$$ 

For each instance, algorithm, and $\lambda \in \{0.01 \cdot k \mid k \in \{1, 2, \ldots, 110\}\}$, we obtain the operators $T_{\lambda}$ and $P_{\text{Fix} T}$ as outlined in Section 5 and compute the spectral radius and operator norm of $T_{\lambda} - P_{\text{Fix} T}$. Note that the convergence of the algorithms is only guaranteed for $\lambda \in \{0.01 \cdot k \mid k \in \{1, 2, \ldots, 99\}\}$, but we have plotted beyond this range to observe the behaviour of the algorithms. Figure 1 reports the median of the spectral radii and operator norms for each $\lambda$.

![Figure 1](attachment:image.png)

**Figure 1**: Experiment 1: The median (solid line) and range (shaded region) of the spectral radii and operator norms

For the spectral radius, while Ryu shows a steady decline in the median value for $\lambda < 1$, Malitsky-Tam, Campoy, and POCS start increasing well before $\lambda = 1$. The maximum value, which can be seen by the range of these values for each algorithm, is some value less than, but close to 1 for all $\lambda < 1$. The operator norm plot, which provides the upper bound for the convergence rates, also stays below 1 for $\lambda < 1$. For MT, the sharp edge in the spectral norm appears as the spectral radius involves the maximum of different spectral values. By visual inspection, we see that for Campoy the lower and upper bounds coincide — we weren’t aware of this beforehand and are now able to provide a rigorous proof in Remark 6.1 below.

**Remark 6.1. (relaxed Douglas-Rachford operator)** Suppose that $T = T_C$ is the Campoy operator which implies that $T_{\lambda} = (1 - \lambda) \text{Id} + \lambda T$ is the relaxed Douglas-Rachford operator for finding a zero of the sum of two normal cone operators of two linear subspaces. By [7, Theorem 3.10(i)], the matrix $T_{\lambda}$ is normal, i.e.,
\( T_\lambda T_\lambda^* = T_\lambda^* T_\lambda \). Moreover, [6, Proposition 3.6(i)] yields \( \text{Fix } T = \text{Fix } T_\lambda = \text{Fix } T_\lambda^* = \text{Fix } T^* \). Set \( P := P_{\text{Fix } T} = P_{\text{Fix } T^*} \) for brevity. Then \( P = P^* \) and \( T_\lambda P = P = T_\lambda^* P \). Taking the transpose yields \( P T_\lambda^* = P T_\lambda \). Thus

\[
(T_\lambda - P)(T_\lambda - P)^* = T_\lambda T_\lambda^* - T_\lambda P - P T_\lambda^* + P
\]

\[
= T_\lambda T_\lambda^* - P
\]

\[
= T_\lambda^* T_\lambda - P
\]

\[
= T_\lambda^* T_\lambda - T_\lambda^* P - P T_\lambda + P
\]

\[
= (T_\lambda - P)^*(T_\lambda - P),
\]

i.e., \( T_\lambda - P \) is normal as well. Now [19, page 431f] implies that the matrix \( T_\lambda - P \) is radial, i.e., its spectral radius coincides with its operator norm. This explains that the two (green) curves for the Campoy algorithm in Figure 1 are identical.

### 6.2 Experiment 2: Number of iterations to achieve prescribed accuracy

Because we know the limit points of the governing as well as shadow sequences, we investigate how varying \( \lambda \) affects the number of iterations required to approximate the limit to a given accuracy. For Experiment 2, we fix 100 instances of triples of subspaces \( (U_1, U_2, U_3) \). We also fix 100 different starting points in \( \mathbb{R}^6 \). For each instance of the subspaces, starting point \( z_0 \) and \( \lambda \in \{0.01 \cdot k \mid k \in \{1, 2, \ldots, 199\}\} \), we obtain the number of iterations (up to a maximum of \( 10^4 \) iterations) required to achieve \( \varepsilon = 10^{-6} \) accuracy.

For the governing sequence, the limit \( P_{\text{Fix } T}(z_0) \) is used to determine the stopping condition. Figure 2 reports the median number of iterations required for each \( \lambda \) to achieve the given accuracy.

![Convergence of the governing and shadow sequences](image)

**Figure 2**: Experiment 2: The median (solid line) and range (shaded region) of the number of iterations for the governing and shadow sequences
For the shadow sequence, we compute the median number of iterations required to achieve \(\varepsilon = 10^{-6}\) accuracy for the sequence \((\mathbf{M}z_k)_{k \in \mathbb{N}}\) with respect to its limit \(P_z z_0\), where \(z_0\) is the average of the components of \(z_0\). See Figure 2 for results. For POCS, we pick the governing sequence equal to the shadow sequence.

For Ryu and MT in both experiments, increasing values of \(\lambda\) result in a decreasing number of median iterations required for \(\lambda < 1\). For Campoy, the median iterations reach the minimum at \(\lambda \approx 0.5\) and keep increasing after that. As is evident from the maximum number of iterations required for a fixed \(\lambda\), the shadow sequence converges before the governing sequence for larger values of \(\lambda\) in \([0, 1]\). One can also see that Ryu consistently requires fewer median iterations for both the governing and the shadow sequence to achieve the same accuracy as MT for a fixed lambda. However, Campoy performs better than Ryu for small values of \(\lambda\), and beats MT for a larger range of \(\lambda \in [0, 0.5]\). POCS naturally performed better for all values of \(\lambda\). Its behaviour for values of \(\lambda\) beyond 1 is similar to Ryu.

### 6.3 Experiment 3: Convergence plots of shadow sequences

In this experiment, we measure the distance of the terms of the governing (and shadow) sequence from its limit point, to observe how the iterates of the algorithms approach the solution. Guided by Figure 2, we pick the \(\lambda\) for which the median iterates are the least: \(\lambda = 0.99\) for Ryu, \(\lambda = 0.97\) for MT, and \(\lambda = 0.57\) for Campoy. Similar to the setup of Experiment 2, we fix 100 starting points and 100 triples of subspaces \((U_1, U_2, U_3)\). We then run the algorithms for 150 iterations for each starting point and each set of subspaces, and we measure the distance of the iterates \(\mathbf{M}z_k\) to its limit \(P_z z_0\). Figure 3 reports the median of these values for each iteration counter \(k \in \{1, \ldots, 150\}\). Again note that for POCS, the governing sequence coincides with the shadow sequence. As can be seen in Figure 3 for both governing and shadow sequences, Ryu converges faster to the solution compared to MT and Campoy. Ryu and MT show faint “rippling” as is well known to occur for the Douglas-Rachford algorithm. Campoy, already being a DR algorithm, has more prominent ripples. As seen in the previous experiment, POCS outperforms the rest of the algorithms for its optimal value of \(\lambda\).

### 6.4 Discussion

We tested the performance of these algorithms in the specific case of 3 subspaces in \(\mathbb{R}^6\) through different experiments. The performance of Ryu is better than that of MT and Campoy; however, MT and Campoy have the upper hand when it comes to extending to more subspaces since Ryu is only designed for the 3 subspaces case. Being a sequential algorithm operating solely in the original space (rather than a product space), POCS is faster than the three others in terms of convergence — but the scope of these algorithms is not restricted to projection operators.

### 7 Three simple lines in the Euclidean plane

In this section, we consider a very simple situation: Three lines passing through the origin, where the second line bisects the angle between the first and the third. We follow the set up in [6, Section 5]. Suppose that \(X = \mathbb{R}^2\), and set \(e_0 := [1, 0]^T\), \(e_{\pi/2} := [0, 1]^T\), and also

\[e_{\theta} := \cos(\theta)e_0 + \sin(\theta)e_{\pi/2}.\]
Define the counterclockwise rotator
\[ R_\theta := \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \]
and now set
\[ U := \Re e_0, \quad V := \Re e_\theta = R_\theta(U), \quad W := \Re e_{2\theta} = R_{2\theta}(U), \]
where \( \theta \in ]0, \pi/2[ \). It is clear that \( Z = U \cap V \cap W = \{0\} = U \cap V = V \cap W = U \cap W \) and that the Friedrichs angle between \( U \) and \( V \), and between \( V \) and \( W \), is \( \theta \) while the Friedrichs angle between \( U \) and \( W \) is \( 2\theta \). The projectors associated with \( U, V, W \) are
\[ P_U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \]
\[ P_V = \begin{bmatrix} \cos^2(\theta) & \sin(\theta) \cos(\theta) \\ \sin(\theta) \cos(\theta) & \sin^2(\theta) \end{bmatrix}, \]
\[ P_W = \begin{bmatrix} \cos^2(2\theta) & \sin(2\theta) \cos(2\theta) \\ \sin(2\theta) \cos(2\theta) & \sin^2(2\theta) \end{bmatrix}. \]

In the next few subsections, we discuss what the relevant splitting operators turn into, while in the last subsection we summarize our findings. The overall strategy is clear: Find a formula for the splitting operator \( T \) and then for \( \text{Fix} \ T \). Consider \( ((1 - \lambda) \text{Id} + \lambda T) - P_{\text{Fix} \ T} \) and determine its largest absolute eigenvalue and its largest spectral value to find the spectral radius and the operator norm. It turns out that this is easier said than done — even with the help of SageMath [28].
7.1 Ryu splitting

The underlying $T = T_{\text{Ryu}}$ (see (59)) turns out to be

$$T = \begin{bmatrix}
2\cos^2(\theta) - \cos^2(\theta) & -2(2\cos^3(\theta) - \cos^3(\theta)) \sin(\theta) & -2\sin^2(\theta) + \sin^2(\theta) & -2\cos^3(\theta) - \cos^3(\theta) \sin(\theta) \\
2\cos^3(\theta) \sin(\theta) & 2\cos^3(\theta) \sin^2(\theta) + 1 & 2\cos^3(\theta) \sin(\theta) & 2\cos^3(\theta) - 2\cos^3(\theta) \sin(\theta) \\
2\cos^4(\theta) - 2\cos^2(\theta) & -2(2\cos^3(\theta) - \cos^3(\theta)) \sin(\theta) & -2\cos^4(\theta) + 2\cos^2(\theta) & -2\cos^3(\theta) \sin(\theta) \\
(2\cos^3(\theta) - \cos(\theta)) \sin(\theta) & -4\cos^2(\theta) \sin^2(\theta) & -(2\cos^3(\theta) - \cos(\theta)) \sin(\theta) & 2\cos^4(\theta) - \cos^2(\theta)
\end{bmatrix}.
$$

Following the procedure of Section 5.1, one eventually arrives at

$$P_{\text{Fix}} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & -\frac{1}{4\sin^2(\theta) - 5} & \frac{2\cos^2(\theta) \sin(\theta)}{4\cos(\theta) + 1} & \frac{-2(\cos^4(\theta) - \cos^2(\theta))}{4\sin^2(\theta) - 5} \\
0 & \frac{2\cos^2(\theta) \sin(\theta)}{4\cos(\theta) + 1} & \frac{4(\sin^4(\theta) - \sin^2(\theta))}{4\sin^2(\theta) - 5} & \frac{4(\cos^4(\theta) - \cos^2(\theta))}{4\cos(\theta) + 1} \\
0 & \frac{-2(\cos^4(\theta) - \cos^2(\theta))}{4\sin^2(\theta) - 5} & \frac{4(\sin^4(\theta) - \sin^2(\theta))}{4\sin^2(\theta) - 5} & \frac{-4\cos^4(\theta) - \cos^2(\theta)}{4\sin^2(\theta) - 5}
\end{bmatrix}.
$$

Now set $T_\lambda := (1 - \lambda) \text{Id} + \lambda T$. We are again interested in the spectral radius and the operator norm of $T_\lambda - P_{\text{Fix}}$. Unfortunately, even with the help of SageMath [28], we were not able to compute these quantities. In fact, even the assignment $\lambda = \frac{1}{2}$ did not lead to manageable expressions.

7.2 Malitsky-Tam splitting

Using (66), we find that $T = T_{\text{MT}}$ is

$$T = \begin{bmatrix}
0 & -\cos(\theta) \sin(\theta) & \cos^2(\theta) & \cos(\theta) \sin(\theta) \\
0 & \cos^2(\theta) & \cos(\theta) \sin(\theta) & \sin^2(\theta) \\
4\cos^4(\theta) - 4\cos^2(\theta) + 1 & 2\cos(\theta) \sin^2(\theta) & 0 & -2(2\cos^2(\theta) - \cos(\theta)) \sin(\theta) \\
2(2\cos^3(\theta) - \cos(\theta)) \sin(\theta) & 2\sin^4(\theta) - \sin^2(\theta) & 0 & -4\cos^2(\theta) \sin^2(\theta) + 1
\end{bmatrix}.
$$

Following the recipe outlined in Section 5.2 yields

$$P_{\text{Fix}} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{\cos(\theta) \sin(\theta)}{\cos(\theta) + 1} & \frac{\sin^2(\theta) - \frac{1}{2}}{\cos(\theta) + 1} \\
0 & \frac{\cos(\theta) \sin(\theta)}{\cos(\theta) + 1} & \frac{-2\sin^4(\theta) + 2\sin^2(\theta)}{\cos(\theta) + 1} & \frac{2\cos(\theta) \sin^2(\theta) - \cos(\theta) \sin(\theta)}{\cos(\theta) + 1} \\
0 & \frac{\sin^2(\theta) - \frac{1}{2}}{\cos(\theta) + 1} & \frac{2\cos(\theta) \sin^3(\theta) - \cos(\theta) \sin(\theta)}{\cos(\theta) + 1} & \frac{2\sin^4(\theta) - 2\sin^2(\theta) + \frac{1}{2}}{\cos(\theta) + 1}
\end{bmatrix}.
$$

Now set $T_\lambda := (1 - \lambda) \text{Id} + \lambda T$. We are again interested in the spectral radius and the operator norm of $T_\lambda - P_{\text{Fix}}$. Unfortunately, even with the help of SageMath [28], we were not able to compute eigenvalues or the operator norm, even when we specialized to $\lambda = \frac{1}{2}$.

7.3 Campoy splitting

Using (73), we compute the Campoy operator $T = T_C$ to be

$$T = \begin{bmatrix}
-\sin^2(2\theta) & \sin(2\theta) \cos(2\theta) & \cos^2(2\theta) & \sin(2\theta) \cos(2\theta) \\
-\sin(2\theta) \cos(2\theta) & \cos^2(2\theta) & -\sin(2\theta) \cos(2\theta) & -\sin^2(2\theta) \\
\cos(2\theta) & \sin(2\theta) & 0 & 0 \\
0 & 0 & -\sin(2\theta) & \cos(2\theta)
\end{bmatrix}.
$$
The operator $T$ is actually an isometry (as can be checked directly or after applying [8, Proposition 3.5]); unfortunately, we could not determine $\text{Fix}_T$ because the computation of the symbolic computation of Moore-Penrose inverses in (75) turned out to be too complicated even when using SageMath [28].

7.4 POCS

The method of Projections onto Convex Sets (POCS) is known to converge to the projection onto the intersection when applied to linear subspaces. In contrast to the parallel splitting methods discussed above, POCS is sequential in nature not requiring any product space. First, we note that

$$P_W P_V P_U = \begin{bmatrix} 2 \cos^4(\theta) - \cos^2(\theta) & 0 \\ 2 \cos^3(\theta) \sin(\theta) & 0 \end{bmatrix} = \cos^2(\theta) \begin{bmatrix} 2 \cos^2(\theta) - 1 & 0 \\ 2 \cos(\theta) \sin(\theta) & 0 \end{bmatrix}$$

Next, the reflected POCS operator $T$ (see (76)) simplifies to

$$T := \frac{4}{3} P_W P_V P_U - \frac{1}{3} \text{Id} = \frac{1}{3} \begin{bmatrix} 8 \cos^4(\theta) - 4 \cos^2(\theta) - 1 & 0 \\ 8 \cos^3(\theta) \sin(\theta) & -1 \end{bmatrix}.$$

Note that $\text{Fix} T = \{0\}$ and hence $P_{\text{Fix} T} = 0$. Thus Next, we set as before $T_\lambda := (1 - \lambda) \text{Id} + \lambda T$; hence,

$$T_\lambda - P_{\text{Fix} T} = (1 - \lambda) \text{Id} + \lambda T.$$

After simplification, the two eigenvalues of $T_\lambda - P_{\text{Fix} T} = T_\lambda$ are seen to be real and equal to $1 - \frac{4}{3} \lambda$ and $1 + \frac{4}{3} \lambda (2 \sin^4(\theta) - 3 \sin^2(\theta))$. It follows that the spectral radius of $T_\lambda - P_{\text{Fix} T}$ is equal to

$$\max \{|1 - \frac{4}{3} \lambda|, |1 + \frac{4}{3} \lambda (2 \sin^4(\theta) - 3 \sin^2(\theta))|\}.$$

The eigenvalues of $(T_\lambda - P_{\text{Fix} T})^*(T_\lambda - P_{\text{Fix} T}) = T_\lambda^* T_\lambda$ are available but too complicated to list here. To make progress analytically, we assume that

$$\lambda = \frac{4}{3},$$

i.e., we consider the original POCS operator $P_W P_V P_U$. The spectral radius of $T_\lambda - P_{\text{Fix} T}$ simplifies then to

$$1 + 2 \sin^4(\theta) - 3 \sin^2(\theta) = \cos^2(\theta) \cos(2\theta)$$

while the operator norm of $T_\lambda - P_{\text{Fix} T}$ is

$$\cos^2(\theta).$$

7.5 Discussion

It is tempting to ask whether the splitting methods by Ryu, by Malitsky-Tam, and by Campoy allow a complete analysis like the one available for Douglas–Rachford splitting for two subspaces as carried out in [6]. Unfortunately, a symbolic analysis of these methods seems much harder even for the shockingly simple case considered in this section. Thus at present, we are rather pessimistic about the prospect of obtaining pleasant convergence rate for this new breed of splitting methods — but we hope that the reader will prove us wrong!
8 Conclusion

In this paper, we investigated the recent splitting methods by Ryu, by Malitsky-Tam, and by Campoy in the context of normal cone operators for subspaces. We discovered and proved that all three algorithms find not just some solution but in fact the projection of the starting point onto the intersection of the subspaces. Moreover, convergence of the iterates is strong even in infinite-dimensional settings. Our numerical experiments illustrated that Ryu’s method seems to converge faster although neither Malitsky-Tam splitting nor Campoy splitting is limited in its applicability to just 3 subspaces.

Two natural avenues for future research are the following. Firstly, when $X$ is finite-dimensional, we know that the convergence rate of the iterates is linear. While we illustrated this linear convergence numerically in this paper, it is open whether there are natural bounds for the linear rates in terms of some version of angle between the subspaces involved. For the prototypical Douglas-Rachford splitting framework, this was carried out in [6] and [7] in terms of the Friedrichs angle. Secondly, what can be said in the inconsistent affine case? Again, the Douglas-Rachford algorithm may serve as a guide to what the expected results and complications might be; see, e.g., [11]. However, these topics for further research appear to be quite hard.

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