SHEAR VISCOSITY OF NUCLEAR MATTER

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Shear viscosity \( \eta \) is calculated for the nuclear matter described as a system of interacting nucleons with the van der Waals (VDW) equation of state. The Boltzmann-Vlasov kinetic equation is solved in terms of the plane waves of the collective overdamped motion. In the frequent collision regime, the shear viscosity depends on the particle number density \( n \) through the mean-field parameter \( a \) which describes attractive forces in the VDW equation. The temperature region \( T = 15\pm 40 \text{ MeV} \), a ratio of the shear viscosity to the entropy density \( s \) is smaller than 1 at the nucleon number density \( n = (0.5 \div 1.5) n_0 \), where \( n_0 = 0.16 \text{ fm}^{-3} \) is the particle density of equilibrium nuclear matter at zero temperature. A minimum of the \( \eta/s \) ratio takes place somewhere in a vicinity of the critical point of the VDW system. Large values of \( \eta/s \ll 1 \) are however found in both the low density, \( n \ll n_0 \), and high density, \( n > 2n_0 \), regions. This makes the ideal hydrodynamic approach inapplicable for these densities.

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I. INTRODUCTION

The shear viscosity \( \eta \) and its ratio to the entropy density \( s \) became recently attractive (see, e.g., Refs. [1–3] and references therein) in connection with a development of the hydrodynamic approach to the relativistic nucleus-nucleus collisions. Chapman and Enskog (CE) obtained [4–7] the shear viscosity \( \eta \) in a gas of non-relativistic particles by using the Boltzmann kinetic equation (BKE) for the phase space distribution function \( f(\mathbf{r}, \mathbf{p}, t) \), where \( \mathbf{r} \) and \( \mathbf{p} \) are the particle coordinate and momentum, respectively, and \( t \) denotes the time variable. In the framework of the hydrodynamic (HD) approach the time variations of the distribution function \( f(\mathbf{r}, \mathbf{p}, t) \) were approximated by those of a local equilibrium (LE) distribution function \( f_{\text{LE}}(\mathbf{r}, \mathbf{p}, t) \),

\[
\delta f = f(\mathbf{r}, \mathbf{p}, t) - f_{\text{LE}}(\mathbf{r}, \mathbf{p}, t), \tag{1}
\]

in terms of particle number density \( n(\mathbf{r}, t) \), temperature \( T(\mathbf{r}, t) \), and collective velocity \( \mathbf{u}(\mathbf{r}, t) \), \( m \) is the particle mass. Note that quantities \( n(\mathbf{r}, t) \) and \( \mathbf{u}(\mathbf{r}, t) \) are defined as the zero and first moments of \( f(\mathbf{r}, \mathbf{p}, t) \) in the momentum space. In the CE approach, the collision term was taken in the well-known Boltzmann form. Small dynamical variations \( \delta f \) in the collisionless part of the BKE, in fact, were presented as \( \delta f_{\text{LE}} \), i.e., they were decomposed into terms proportional to \( \delta n, \delta T, \) and \( \delta \mathbf{u} \). In addition, using the standard closed system of HD equations to calculate \( \delta f = \delta f_{\text{LE}}(\mathbf{r}, \mathbf{p}, t) \) in the collisionless part of the BKE, one derives [4–6] the explicit expression for the shear viscosity \( \eta \). The BKE was solved within the frequent collision (FC) regime for which one can use a perturbation expansion in a small parameter \( \omega/\nu \) where \( \nu \) is the collision frequency and \( \omega \) measures the characteristic dynamical variations of the distribution function \( f(\mathbf{r}, \mathbf{p}, t) \) as function of the time \( t \). For a gas of elastic scattering balls with the diameter \( d \), at leading order of this parameter, \( \omega/\nu \), one finds [4]:

\[
\eta_{\text{CE}} = \frac{5}{16\sqrt{\pi}} \frac{\sqrt{\nu T}}{d^2}. \tag{2}
\]

The shear viscosity \( \eta_{\text{CE}} \) appears to be independent of the particle number density \( n \). An extension of Eq. (2) to the mixture of different hadron species was considered in Ref. [8]. Several investigations were devoted to go beyond the HD approach [4], see, e.g., Refs. [9–24].

In the present paper, we use the Boltzmann-Vlasov kinetic equation (BVKE) for a system of interacting nucleons with the van der Waals (VDW) equation of state. Therefore, both scattering of particles due to the hard-core repulsions and Vlasov self-consistent mean field, due to the VDW attractive interaction, are taken into account in solving the BVKE. For small dynamical variations of the distribution function with respect to the local equilibrium one, Eq. (1),

\[
\delta f(\mathbf{r}, \mathbf{p}, t) = f(\mathbf{r}, \mathbf{p}, t) - f_{\text{LE}}(\mathbf{r}, \mathbf{p}, t), \tag{3}
\]

the linearized BVKE will be solved in terms of the damping plane waves (DPW). We shall consider the strong (overdamped, see e.g. Refs. [14, 24]) attenuation of the DPW. As in Ref. [4], the collision term of the BVKE is assumed to be dominating as compared to all of its other (collisionless) parts in the FC regime (the opposite situation for a weak absorption of the DPW with large \( \omega/\tau \) will be studied separately). Our approach is based on the methods applied earlier for calculations of

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the viscosity of the Fermi liquids [9, 10, 12, 13, 24]. The shear viscosity is calculated analytically for the nuclear matter considered as a gas of interacting nucleons with the VDW equation of state.

The paper is organized as follows. In Sec. II we remind the basic properties of thermodynamically equilibrated systems with the VDW equation of state. In Sec. III we outlook the kinetic approach based on the BVKE and give general definitions of the viscosity coefficient. In Sec. IV, the solution to the BVKE and its perturbation expansion is presented in terms of the plane waves interacting nucleons. Finally, this section is devoted to the main results for the VDW viscosity. The obtained results are discussed in Sec. V and summarized in Sec. VI. Some details of our calculations can be found in Appendices A-C.

II. VDW EQUATION OF STATE

The VDW equation of state presents the system pressure $P$ in terms of the particle number density $n$ and temperature $T$ as [25]:

$$P(T, n) = \frac{n T}{1 - b n} - a n^2,$$  

where $a > 0$ and $b > 0$ are the VDW parameters that describe attractive and repulsive interactions, respectively. The first term in the right hand side of Eq. (4) contains the excluded volume correction ($b = 2\pi d^3/3$ with $d$ being the particle hard-core diameter), while the second term comes from the mean field description of attractive interactions.

The entropy density $s$ and energy density $\varepsilon$ for the VDW system are calculated as [25]:

$$s(T, n) = \frac{5}{2} n + n \ln \left[ \frac{1 - b n}{n} g \left( \frac{m T}{2\pi} \right)^{3/2} \right],$$  

$$\varepsilon(T, n) = n \left[ \frac{3}{2} T - a n \right].$$

In Eq. (5) $m$ is the particle mass and $g$ is the degeneracy factor ($g = 4$ for nucleons; two spin and two isospin states). Note that the VDW entropy density (5) is independent of the attractive mean-field interaction parameter $a$ whereas the energy density (6) does not depend on the particle repulsion constant $b$.

The VDW equation of state contains the first order liquid-gas phase transition with a critical point [25]:

$$T_c = \frac{8a}{27b}, \quad n_c = \frac{1}{3b}, \quad P_c = \frac{a}{27b^2}. $$

To study the phase coexistence region which exists below the critical temperature, $T < T_c$, the VDW isotherms should be corrected by the well known Maxwell construction of equal areas.

The VDW equation of state was recently applied to a description of nuclear matter in Ref. [26]. In the present study we fix the VDW parameters for the system of interacting nucleons as $d = 1$ fm, i.e., $b \cong 2.1$ fm$^3$, and $a = 100$ MeV fm$^3$. This gives $n_c \cong n_0 = 0.16$ fm$^{-3}$ and $T_c \cong 14$ MeV ($n_0 = 0.16$ fm$^{-3}$ corresponds to the nucleon number density of the normal nuclear matter at zero temperature). In what follows we restrict our analysis of the kinetic properties of the VDW system of nucleons to $T > T_c$. In this region of the phase diagram the VDW equation of state describes a homogeneous one-phase system, and all criteria of the thermodynamical stability are satisfied. We do not consider too large temperatures by taking $T \leq 40$ MeV. This allows us to neglect a production of new particles (pions and baryonic resonances) in the system of interacting nucleons. In addition, this restriction guarantees a good accuracy of the non-relativistic approximation adopted in the present study. Note also that at $T \to 0$ the quantum statistics effects neglected in the present study should be taken into account (see Ref. [26]).

III. KINETIC APPROACH

For calculations of the shear viscosity, we start with small deviations $\delta f$, Eq. (3), which can be approximated by $\delta f(r, p, t) \approx f(r, p, t) - f_{GE}(p)$, where $f_{GE}(p)$ is the global equilibrium distribution function,

$$f_{GE}(p) = \frac{n}{(2m\pi T)^{3/2}} \exp \left(-\frac{p^2}{2mT}\right).$$

This equilibrium distribution function is taken in the Maxwell form with constant values of $T$ and $n$, and zero value of the collective velocity, $\mathbf{u} = 0$, in Eq. (1). For simplicity, we neglect a difference between the local, $f_{LE}$, and global, $f_{GE}$, equilibrium distribution functions up to second order terms in $\delta f$ by reasons discussed below. The BVKE linearized near the global equilibrium (8) reads:

$$\frac{\partial \delta f}{\partial t} + \frac{p}{m} \frac{\partial \delta f}{\partial r} - \frac{\partial f_{GE}}{\partial p} \frac{\partial \delta U}{\partial r} = \delta St. \quad (9)$$

The dynamical part of the attractive potential $\delta U$ from the VDW forces is defined self-consistently as

$$\delta U(r, t) = -a \int \mathbf{dp} \delta f(r, p, t).$$

In Eq. (9), the collision term $\delta St$ is taken in the standard Boltzmann form [4, 5],

$$\delta St = \frac{2\pi}{m} \int \mathbf{dp}_1 \mathbf{dp} \int \beta |\mathbf{p}_1 - \mathbf{p}| |\beta| \delta Q,$$

where

$$\delta Q \approx f_{GE}(p') \delta f(r, p'_1, t) + f_{GE}(p'_1) \delta f(r, p', t) - f_{GE}(p) \delta f(r, p_1, t) - f_{GE}(p_1) \delta f(r, p, t)$$

is the variation of $f(r, p', t)f(r, p'_1, t) - f(r, p, t)f(r, p_1, t)$ over $\delta f$. Note that the difference $f_{LE} - f_{GE}$ does not influence on $\delta St$, Eq. (11), via $\delta Q$, Eq. (12), in this linearized approximation. In Eq. (11), $\beta$ is the impact parameter for two-body collisions. Figure 1 shows the collision geometry for two hard-core
The velocity field $u$ is defined through the first $p$ moment of $\delta f(\mathbf{r}, p, t)$,

$$
u = \frac{\delta f(\mathbf{r}, p, t)}{n} \int d\mathbf{p} \frac{\mathbf{p}}{m} \delta f(\mathbf{r}, p, t).$$  \hspace{1cm} (19)$$

A difference $f_{LE} - f_{GE}$ as a scalar does not contribute into the dynamical variation of the non-diagonal linearized stress tensor $\delta \sigma_{\mu\nu}$ (14) being exactly equal to zero, that is the reason for the approximation used above to the definition (3) of $\delta f$. This is in contrast to calculations of other transport coefficients, e.g., the volume viscosity and thermal conductivity (Ref. [24]) which we are going to discuss in a separate publication.

IV. DISPERSION RELATION AND VISCOSITY

We suggest to calculate the shear viscosity $\eta$ by direct solving the BVKE (9) in terms of the plane wave representation for the dynamical distribution-function variations $\delta f(\mathbf{r}, p, t)$ in the following rather general form [10, 13, 24]:

$$
\delta f(\mathbf{r}, p, t) = \varphi(\hat{\mathbf{p}}) \exp(-i\omega t + i\mathbf{k}\mathbf{r}),  \hspace{1cm} (20)
$$

where $\omega$ and $\mathbf{k}$ are a frequency and a wave vector of the DPW, respectively. As unknown yet, amplitudes $\varphi(\hat{\mathbf{p}})$ are functions of the momentum angle variable $\hat{\mathbf{p}} = \mathbf{p}/p$. It is naturally to find solutions of the BVKE as proportional to the static distribution function, $f_{GE}(p)$, specifying the dependence of $\delta f$ on the modulus of momentum $p$ because the derivative of $f_{GE}(p)$ (8) over momentum in Eq. (9) and the variations of the collision integral (11) are proportional to $f_{GE}(p)$. Then, one can reduce the problem for solving the BVKE (9) to a function of angles $\varphi(\hat{\mathbf{p}})$ which, however, depends on the unknown frequency $\omega$. (We shall leave out the argument $\omega$ in $\varphi(\hat{\mathbf{p}})$ for simplicity of the notations.) Note that any physical quantity, in particular the viscosity coefficient, is independent of the direction of the unit wave vector $\hat{k} = \mathbf{k}/k$ of the DPW spreading in infinite nuclear matter. Therefore, it is convenient to use the spherical phase-space coordinate system with the polar axis directed to this vector $\hat{k}$. The solution for the plane-wave distribution function $\delta f(\mathbf{r}, p, t)$, Eq. (20), or more precisely $\varphi(\hat{\mathbf{p}})$, and the frequency $\omega$ depends only on the wave vector length $k$. For convenience, one may write the frequency $\omega$ through the wave number $k$ and the dimensionless sound velocity $c$,

$$
\omega = kv = kv_{T}c,  \hspace{1cm} (21)
$$

where $v = v_{T}c$ is the DPW speed, and $c$ its dimensionless value given in units of the most probable thermal velocity $v_{T}$ of particles at a given temperature $T$, $v_{T} = \sqrt{2T/m}$.

The viscosity $\eta$ is related to an attenuation of the DPW (20) measured by the collision term $\delta St$ (11). Following Refs. [4–6, 10], one applies the perturbation expansion of the dynamical distribution-function variations $\delta f$ through their amplitudes $\varphi(\hat{\mathbf{p}})$,

$$
\varphi(\hat{\mathbf{p}}) = \varphi^{(0)}(\hat{\mathbf{p}}) + \epsilon \varphi^{(1)}(\hat{\mathbf{p}}) + \epsilon^{2} \varphi^{(2)}(\hat{\mathbf{p}}) + ...,  \hspace{1cm} (22)
$$
and similarly, for the frequency $\omega$,
\[
\omega = \omega^{(0)} + \epsilon \omega^{(1)} + \epsilon^2 \omega^{(2)} + \ldots,
\tag{23}
\]
in a small parameter,
\[
\epsilon = \omega/\nu = \omega \tau.
\tag{24}
\]
Here, $\tau$ is the relaxation time\(^1\) defined by the collision term through the time-dependent rate $\nu$ (collision frequency) of the damping of distribution function $\delta f$,
\[
\tau = 1/\nu.
\tag{25}
\]
In the perturbation expansions (22) and (23), the coefficients $\varphi^{(n)}(\hat{p})$ and $\omega^{(n)}$ are assumed to be independent of $\epsilon$. They can be found at each order of $\epsilon$ by using the BVKE within the standard method applied to these eigenfunction, $\varphi(\hat{p})$ and eigenvalue, $\omega$, problem as in Refs. [5, 10, 24, 28]. Substituting the plane-wave representation (20) for the distribution function $\delta f$ into the BVKE (9), for convenience, one can also expand $\varphi(\hat{p})$ in series over the spherical harmonics $Y_{\ell m}(\hat{p})$,
\[
\varphi(\hat{p}) = \sum_{\ell=0}^{\infty} \varphi_{\ell} Y_{\ell0}(\hat{p}),
\tag{26}
\]
where
\[
\varphi_{\ell} = \frac{\hat{p} \cdot \hat{k}}{p k}.
\tag{27}
\]
This reduces the integro-differential BVKE to much more simple linear algebraic equations (B.1) for the partial multipole amplitudes $\varphi_{\ell}$ at each order in $\epsilon$ [Eq. (24) and Appendix B].

As shown in Appendices A and B, in the FC regime, $|\epsilon| \ll 1$ [Eq. (24)], one can truncate the multipole expansion (26) over $\ell$ at $\ell = 2$ because of a good convergence in the small parameter $\epsilon$. At this leading approximation to viscosity calculations, for the collision term $\delta St$, Eq. (11), one obtains (Appendix A) the following simple expression:
\[
\delta St = -\nu \delta f_2(r, p, t),
\tag{27}
\]
where
\[
\nu \approx \frac{3 \nu \tau \sigma}{2} = \pi d^2,
\tag{28}
\]
$\sigma$ is the cross section for a two elastic hard-core sphere scattering, as introduced above,
\[
\delta f_2(r, p, t) = f_{GE}(p) \varphi_2 Y_{20}(\hat{p}) \exp(-i \omega t + i k r).
\tag{29}
\]
As shown in Appendix A, within the accuracy about 6%, this value agrees with its mean effective quantity $\nu_{av}$ (A.18), evaluated through the momentum average of the collision term, $\langle \delta St \rangle_{av}$, over particle momenta $p$ with the help of the Maxwell distribution function $f_{GE}(p)$ (8). Multiplying then the BVKE (9), with the quadrupole collisional term (27), by the spherical function $Y_{\ell 0}(\hat{p})$ ($L = 0, 1, 2, \ldots$), one can integrate the BVKE term by term over angles $\hat{p}$ of the momentum $p$. Thus, one obtains the linear homogeneous equations (B.1) with respect to coefficients $\varphi_\ell$ of the expansion (26) in the plane wave amplitudes $\varphi(\hat{p})$ at any order in $\epsilon$ in Eqs. (22) and (23). This system has non-trivial solutions in the quadrupole approximation $\ell \leq 2$, valid at the leading (linear in $\epsilon$) approximation in expansions (22) and (23). They obey the cubic dispersion equation for $c = \omega/(k \nu \tau)$ (expansion of $c$ is similar to Eq. (23), see also Appendix B),
\[
\begin{align*}
\text{det} A_2 &\equiv c^3 + i \gamma c^2 - c \left[ \frac{1}{15} + \frac{1}{3} (1 - F) \right] \\
&- \frac{i}{3} (1 - F) \gamma = 0,
\end{align*}
\tag{30}
\]
where $F$ is the dimensionless VDW interaction parameter,
\[
F = a n / T.
\tag{31}
\]
The truncated (at $\ell = 2$) $3 \times 3$ matrix $A_2^{(2)}(c)$ is given by Eq. (B.4). For convenience, we introduced also the dimensionless collisional rate (28):
\[
\gamma = \frac{\nu}{k \nu \tau} = \frac{\nu c}{\omega} = \frac{c}{\omega \tau}.
\tag{32}
\]
The FC perturbation parameter $\epsilon$, Eq. (24), can be expressed in terms of the $\gamma$ and $c$ as
\[
\epsilon = c / \gamma, \quad |c / \gamma| \ll 1.
\tag{33}
\]
The cubic dispersion equation (30) has still two limit solutions with respect to the complex velocity, $c = c_r + ic_i$ for real $k$ (or equivalently, a complex wave number $k = k_r + ik_i$ for a real velocity $c$, both related by the same $\omega = k \nu \tau = \omega_r + i \omega_i$, where low subscripts denote the real and imaginary parts) depending on the small parameter $\gamma$. One of them can be called as the underdamped (weakly damped) first sound mode for which the imaginary part of $c$, $c_i$, is much smaller than the real one $c_r$, $|c_i / c_r| \ll 1$, while in the opposite case $|c_i / c_r| \gg 1$, one has the overdamped motion. In the first underdamped sound case ($|c_i / c_r| \ll 1$), the collision term can be considered as small with respect to the collisionless left hand side (LHS) of the BVKE, $|\gamma / c| \sim 1 / |\omega \tau| \ll 1$ (the RC regime). For the overdamped motion this term is dominating as in the FC case studied within the HD approximation in Ref. [4] by the perturbation expansion as Eq. (22). In our DPW derivations below one can use also the expansion (23) over the same small parameter $|c_i / c_r| \sim |\omega \tau| \ll 1$ [Eqs. (32) and (33)]. In order to compare with the results for the shear viscosity of Ref. [4], we shall consider in more details the overdamped motion while the underdamped case with the detailed comparison to a small sound absorption in Ref. [27] will be considered in one of separate publications.

Expanding the LHS of the truncated (quadrupole) dispersion equation (30) for $c$ in powers of $\epsilon$, see Eqs. (24) and (33), in the FC perturbation expansions (22) and (23), one can divide its all terms by $\gamma^3$. Then, one can neglect the relatively small cubic $[c / \gamma^3 \sim c^3]$ and quadratic ($\sim c^2$) terms as compared to the last two linear (in $\epsilon$) ones depending explicitly on the interaction

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\(^1\) We do no not use the standard $\tau$-approximation and introduce the relaxation time $\tau$ for sake of the convenience in comparison with other approaches.
parameter $\mathcal{F}$ [Eq. (31)]. At this leading order, one results in the explicit quadrupole solution for the velocity $c$ [Eq. (B.7)],

$$c = i c_i = -\frac{5i}{9} \frac{1 - \mathcal{F}}{1 - 5\mathcal{F}/9} \gamma .$$  \hspace{1cm} (34)

In order to get small corrections of the real sound velocity $c_r$, one has to take into account the quadratic and cubic in $c$ terms of the dispersion equation (30). The latter describes the sound attenuation as the exponential decrease of the DPW amplitude, $\delta f \propto \exp(-t/T)$ [Eq. (20)] with the damping time $T$,

$$T \approx \frac{6}{5\pi} \frac{1 - 5\mathcal{F}/9}{(1 - \mathcal{F})n_{rel}d^2}. \hspace{1cm} (35)$$

This time was obtained as the imaginary part of the complex frequency $\omega = -i/T$, through Eq. (34), formally introduced above (finally, all physical quantities will be determined by taking their real parts). Note also that the relaxation time $\tau$, Eqs. (25) and (28),

$$\tau = \frac{2}{\nu} \approx \frac{2}{3n_{rel} \pi d^2}, \hspace{1cm} (36)$$

differs from the damping time, $T$ [Eq. (35)]. In particular, this time $T$, being of the order of $\tau$, depends on the interaction constant $\mathcal{F}$.

Using the DPW solutions (20) for $\delta f$ of the BVKE (9), and Eqs. (C.3) for $U_{zz}$ and (C.5) for $\sigma_{zz}$, for the definition of the shear viscosity $\eta$, Eq. (17), one finds the FC expansion [(22) and (23)] of $\eta$ in powers of small $\epsilon$ (see Eq. (24), and Appendices C and B). As shown in Appendix C, the leading term of this FC shear viscosity $\eta$ at first order in $\epsilon$ is approximately a constant, independent of $\omega$ (or $k$), and proportional to $1/\nu$, i.e. to the relaxation time $\tau$ [Eq. (36)]. Thus, up to relatively high (second) order terms in the small parameter $\epsilon$ we arrive at

$$\eta = \frac{27\sqrt{3}}{80} \left(1 - \frac{5}{9} \mathcal{F}\right) mn_{rel}^2 \pi d^2. \hspace{1cm} (37)$$

In these derivations we used, at the leading first order in $\epsilon$, the quadrupole multipolarity truncation of rapidly converged series (26), see Eqs. (C.8) for the amplitudes $\varphi_i$ and (34) for the sound velocity $c$ ($\ell \leq 2$) within the dispersion equation (30). Note that the relationship $\eta \propto \tau$ is typical for the FC regime, in contrast to the RC one, $\eta \propto 1/\tau$, which should be expected for the perturbation expansion at leading order in the opposite small parameter $1/\epsilon$, see Refs. [10, 12, 13, 24]. Finally, with the quadrupole relaxation time $\tau$, Eq. (36), from Eq. (37) one explicitly obtains

$$\eta = \frac{9}{20\sqrt{2\pi}} \left(1 - \frac{5}{9} \mathcal{F}\right) \frac{\sqrt{mT}}{d^2}. \hspace{1cm} (38)$$

Note that the perturbation method for the eigenfunctions $\varphi(\vec{p})$ [or $\varphi_i$, Eq. (22) as in Ref. [4]], and in addition, eigenvalues $\omega$ allow us to obtain in a regular way high order corrections in $\epsilon$. In this way, one has to go beyond the quadrupole multipolarity ($\ell \leq 2$) approximation taking into account, consistently at a given $\epsilon$, higher order terms, $\ell > 2$, in expansion (26) for $\varphi(\vec{p})$.

V. DISCUSSION OF THE RESULTS

Eq. (38) for $\eta$ has the same classical HD dependence on the temperature $T$ and diameter $d$, $\eta \propto \sqrt{mT}/d^2$ [cf. with Eq. (2)], because of using the FC approximation as in both the molecular kinetic theory [7] and the CE approach [4]. In this approximation for the overdamped case (Appendix B) the dominating contribution into the viscosity yields from the collision term which mainly determines both the classical HD and our DPW solutions for the distribution function to the BVKE. Therefore, as expected, in the limit $a \to 0$ ($\mathcal{F} \ll 1$), one finds the number constant [in front of $\sqrt{mT}/d^2$, see Fig. 2 and Eq. (38)] that approximately coincides within the accuracy of 2% with the CE result (2). The difference between the HD (2) and overdamped DPW (38) viscosities in the zero interaction constant limit should be, indeed, small as compared to the leading collisional term.

Figure 2 shows the shear viscosity $\eta$, Eq. (38), for a few temperatures above the critical value $T_c$ [Eq. (7)]. From Fig. 2, one can clearly see that the shear viscosity $\eta$ differs significantly from the classical HD formula (2) by the particle density dependence. It appears through the VDW parameter $\mathcal{F}$, Eq. (31), due to accounting for dynamical variations of the mean-field interaction (10) in our derivations. As displayed in this figure, the significant effects originate by the Vlasov self-consistent attractive-interaction terms of the BVKE. In our approach this is achieved by using the perturbation expansion (23) for the frequency $\omega$ as solutions of the dispersion equation, in addition to Eq. (22). Note
VI. CONCLUSIONS

The shear viscosity of a nucleon gas is derived by solving the BVKE for the FC regime with taking into account the van der Waals interaction parameters for both the hard-elastic sphere scattering and attractive mean-field interaction. The viscosity $\eta$ depends on the particle density $n$ through the dynamical mean-field forces measured by the VDW parameter, $an/T$, which is positive for the attractive long-distance mean-field interaction. Therefore, the viscosity $\eta$ decreases with the interaction constant $a > 0$ through the VDW parameter $F$. The ratio of the FC viscosity to the entropy density, $\eta/s$, as function of the particle density $n$ and temperature $T$ is found to have a minimum which is essentially smaller than one. The viscosity is significantly smaller at this minimum which moves to smaller temperatures toward the critical temperature due to the long-distance interaction, as compared to the classical HD CE result. Our DPW viscosity calculations has the same overdamped behavior (strong attenuation) such that the collisional term is dominating above all of other parts of the BVKE.

Figures 3 and 4 show the ratio, $\eta/s$, of the viscosity $\eta$ to the entropy density $s$ [Eq. (5)] given by Eqs. (38) and (2), respectively, in the $n-T$ plane for temperatures $T$ above the critical value $T_c$, Eq. (7). As seen from comparison of the two overdamped viscosities in units of the entropy density in these Figures, the ratio $\eta/s$ takes form of a minimum with values $\eta/s \lesssim 1$ at densities $(0.5 \div 2)n_0$, somewhere in a vicinity of the critical point $(T_c, n_c)$. This minimum is significantly smaller and moves to smaller densities in our DPW calculations (Fig. 3) as compared to the CE ones (Fig. 4) though they are both smaller than 1. Note also a weak sensitivity of these properties depending of the size of the hard spheres $d$ around $d = 1$ fm for its deflections in about 20%. On the other hand, $\eta/s \gg 1$ both at small ($n \ll n_0$) and large ($n \gtrsim 2n_0$) particle density, that makes the ideal hydrodynamic approach inapplicable for these densities.

also that with increasing attractive interaction parameter $a$ ($a > 0$), one finds a linearly decreasing viscosity $\eta$ through the dimensionless parameter $F$.
Note that the viscosity coefficient can be considered as a response (Ref. [11, 24, 29, 30]) of the stress tensor $\sigma_{\mu\nu}$ for the shear pressure on the velocity derivative tensor $U_{\mu\nu}$, see e.g. Eq. (17). See also the Green’s–Kubo formula for the shear viscosity, as for the conductivity coefficient [11].

Our results might be interesting for the kinetic and hydrodynamic studies of nucleus-nucleus collisions at laboratory energies of a few hundreds MeV per nucleon. The ideal hydrodynamics can be fairly good approximation for a system of the interacting nucleons in the region of $n$ and $T$ that corresponds at least to $\eta/s \ll 1$. However, the classical HD approach for both the dilute nucleon gas with $n \ll n_0$ and the nuclear-dense matter with $n \gtrsim 2n_0$ seems to be rather questionable to use. As a different perturbation theory has to be used in expansions over small $\omega\tau$ for the FC and small $1/(\omega\tau)$ in the RC regime, we should expect very different dependencies of the viscosity (and other transport coefficients) on the particle density $n$ in these two opposite limits. For instance, the RC regime is important to study a weak absorption of the DPW in the gas system with small far-acting interactions, especially for ultrasonic absorption [31, 32]. Therefore, in the case when the contributions of collisions into the BVKE dynamics are changed from the dominant (small $\epsilon$) to almost collisionless process (small $1/\epsilon$) with increasing DPW frequency for a given collision frequency $\nu$, a transition from the FC to RC regime should be accounted beyond the classical HD approach. This can be realized for small $n/n_0$ and large $\eta/s$, in the corresponding $n-T$ regions of the phase diagram for analysis of the nucleus-nucleus collisions. Our approach can be applied to calculations of the thermal conductivity and diffusion coefficients in nuclear physics, also those and viscosity in nuclear astrophysics, and to study different phenomena in the electron-ion plasma.

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Appendix A: COLLISION TERM CALCULATIONS

We shall neglect approximately an influence of the effective potential $\delta U[n(r, t)]$ (10) of the long-range particle interaction during a two-particle collision of hard-core spherical particles of the gas in the FC regime. Using the multipole expansion (26) of the amplitude factor $\varphi(\hat{p})$, one can simplify the linearized collisional term, $\delta St$ [Eqs. (11) and (20)], in the BVKE (9),

$$
\delta St = \frac{d^2}{4m} \sum \chi_\ell \int dp_1 |p_1 - p| \int d\Omega_{p'} \times \{ f_{GE}(p') f_{GE}(p') [Y_{\ell 0}(p_1') + Y_{\ell 0}(\hat{p})] - f_{GE}(p_1) f_{GE}(p) [Y_{\ell 0}(\hat{p}_1) + Y_{\ell 0}(\hat{p})] \},
$$

(A.1)

where $f_{GE}(p)$ is the GE distribution function (8),

$$
\chi_\ell = \varphi_\ell \exp(-i\omega t + i\mathbf{k}\mathbf{r}),
$$

(A.2)

$\varphi_\ell$ is the $\ell$ coefficient in the expansion (26) for amplitudes $\varphi(\hat{p})$. One finds the relationship between the impact parameter $\beta$ in the center-of-mass coordinate system, see Fig. 1, and the scattering angle $\theta_{p'}$ (and $\beta d\beta$ to $d\Omega_{p'}$),

$$
\beta = \cos(\theta_{p'}/2)d, \quad \beta d\beta = \frac{d^2}{8\pi} d\Omega_{p'} = \frac{d^2}{8\pi} \sin \theta_{p'} d\theta_{p'} d\varphi_{p'}. \quad \text{(A.3)}
$$

The Boltzmann collision term (A.1) is defined in such a way that its zero and first $p$ moments have to be zero because of the particle number conservation (related to the continuity equation), and momentum conservation

$$
\mathbf{p} + \mathbf{p}_1 = \mathbf{p}' + \mathbf{p}_1', \quad \text{(A.4)}
$$

(associated with the momentum continuity equation) during a two-body collision. We take also into account that the distribution function (8) is located within a small momentum interval $(2mT)^{1/2}$. Within this range the momentum vectors are approximately changed only by their direction angles,

$$
\hat{p} + \hat{p}_1 \approx \hat{p}' + \hat{p}_1', \quad \text{(A.5)}
$$

and one can use also the kinetic-energy conservation equation,

$$
p^2 + p_1^2 = p'^2 + p_1'^2. \quad \text{(A.6)}
$$

Substituting Eq. (8) for the static distribution function, $f_{GE}(p)$, and using the conservation equations (A.4)–(A.6) in Eq. (A.1), one finds $f_{GE}(p') f_{GE}(p') = f_{GE}(p_1) f_{GE}(p)$. Therefore, from Eq. (A.1), one obtains

$$
\delta St = \frac{d^2}{4m} \int dp_1 |p_1 - p| f_{GE}(p_1) \times \int d\Omega_{p'} [Y_{\ell 0}(p_1') + Y_{\ell 0}(\hat{p}) - Y_{\ell 0}(\hat{p}_1) - Y_{\ell 0}(\hat{p})]. \quad \text{(A.7)}
$$

Thus, the collisional term (A.1) ensures all necessary (particle number, momentum and energy) conservation laws. In particular, one can check that there is no $\ell = 0$ and 1 terms in the sum over $\ell$ of Eq. (A.7). By that reason, because of zero two first moments of the collision term $\delta St$ (A.7), there is no contributions from (A.7) into the continuity equation [zero p moment of the Boltzmann equation (9)], and explicitly, into the momentum equation [the first p moment of (9)]. This term $\delta St$ will effect only on the momentum flux tensor $\delta \Pi_{\mu\nu}$ (13) through the solutions (20) for the distribution function $\delta f$ [see, e.g., Eq. (14)] in terms of the viscosity coefficients [Eq. (17)].

For the integration over $p_1$ in Eq. (A.7), it is convenient to use the system of the center of mass for a given two-body collision, with the symmetry $z$ axis directed along the relative motion of projectile particle having the reduced mass (Fig. 1). We transform the integral over $p_1$ to the relative momentum $\mathbf{q} = p_1 - p$. Then, using the spherical-coordinate system with the symmetry $z$ axis directed along the vector $\mathbf{q}$ of the relative motion of projectile particle (Fig. 1), $d\mathbf{q} = q^2 dq d\theta_\phi d\varphi_\phi$ and $(x_\phi = \cos \theta_\phi)$, one finds from (A.7)

$$
\delta St_L = \int d\Omega_p Y_{\ell 0}(\hat{p}) \delta St(p)
$$

$$
= \frac{\pi^2 d^2}{4m} f_{GE}(p) f_{GE}(p) \sum \chi_\ell \sqrt{(2L + 1)(2\ell + 1)}
$$

$$
\times \int_1^- dx P_L(x) \int_1^1 x dq \int_0^\infty q^2 dq \exp \left[ - \frac{q^2 + 2pqxq}{2mT} \right]
$$

$$
\times \int_1^- dx' [P_L(x') + P_L(x') - P_L(x) - P_L(x)]. \quad \text{(A.8)}
$$

Here, $P_L(\cos \theta_p) = (4\pi/(2\ell + 1))^{1/2} Y_{\ell 0}(\hat{p})$ is the Legendre polynomial of $\ell$ order, and several transformations of the angle coordinates are performed,

$$
x = \cos \theta_p, \quad x_1 = \cos \theta_{p_1}, \quad x' = \cos \theta_{p'}, \quad x_1' = \cos \theta_{p_1}'. \quad \text{(A.9)}
$$

Note that the integration over the azimuthal angle of the relative momentum was taken from zero to $\pi$ [9]. For the integration over the modulus of the relative momentum $q$, for the fixed $x$ and $x'$, one can change the angle variables to functions of the relative $x_q$,

$$
x_1' = x_1 + x - x', \quad x_1 = z\sqrt{2mT} x_q/p + x. \quad \text{(A.10)}
$$

For the fixed $x$ and $x'$ we integrate first analytically over the variable $z = q/p_T$, where $p_T = \sqrt{2mT}$, and then, over $x_q$. Integrating then, e.g. the $\ell = 2$ term, $\delta St_2$ of
Eq. (A.8), explicitly over remaining angles $x$ and $x'$, one obtains

$$\delta St_2 = \frac{5\pi^2 d^2}{2m} T^4 \chi_2 f_{GE}(p) I_{St}(p), \quad (A.11)$$

where

$$I_{St}(p) = f_{GE}(p) \int_{-1}^{1} dx P_2(x) \int_{-1}^{1} dx' \int_{-1}^{1} dx_q \times \int_{0}^{\infty} z^3 dz \exp \left[-\left(z^2 + 2pxz/p_T\right)\right] \times \left[P_2\left(z\sqrt{2mT} x_q/p + 2x - x'\right) + P_2\left(x'\right) - P_2\left(z\sqrt{2mT} x_q/p + x\right) - P_2\left(x\right)\right] \frac{2n(2p^2 + p_T^2)}{5\pi p p_T^2} \text{erf}\left(\frac{p}{p_T}\right) + \frac{4n}{5\pi^3/2p_T^2} \times \exp\left(-\frac{p_T^2}{p^2}\right) \frac{4n}{5\pi p_T^2} J_{St}(p/p_T), \quad (A.12)$$

with the error function $\text{erf}(y) = 2 \int_{0}^{y} dz \exp(-z^2)/\sqrt{\pi}$,

$$J_{St}(y) = \frac{y^{2+1/2}}{y} \text{erf}(y) + \frac{1}{\sqrt{\pi}} \exp\left(-y^2\right), \quad y = p/p_T. \quad (A.13)$$

In order to reduce the BVKE to the perturbation eigenvalue problem [Eqs. (22) and (23)] for the eigenfunctions $\varphi(\vec{p})$ and eigenvalues $c = \omega/(k T)$ as solutions of the linear homogeneous equations for $\varphi(\vec{p})$, and dispersion equation for $c$ (Appendix B), we may derive now the accurate constant (independent of $y$) approximations to the function $J_{St}(y)$ [Eqs. (A.13)]. Using these approximations, one obtains Eq. (27) with (28) for the collision term $\delta St$ (A.8). Indeed, we may note that for the derivation of such approximations the collision term $St$ [Eq. (A.7)] can be considered through all of its $p$ moments. They are integrals over the modulus $p$, which are taken up to the constant from the product of $J_{St}(p)$ [Eq. A.13] and the power $p^\lambda$ at $\lambda \geq 2$, in addition to the Maxwell distribution function $f_{GE}(p)$,

$$\int_{0}^{\infty} dp \ p^\lambda \delta St \propto \chi_2 \int_{0}^{\infty} dy \ y^\lambda J_{St}(y) f_{GE}(yp_T). \quad (A.14)$$

Figure 5 shows a fast convergence of the product $y^\lambda J_{St}(y)$ [Eq. (A.13)] of the integrand in Eq. (A.14) to its asymptotics at large $y$ in powers of $1/y$ taking enough many terms,

$$J_{St}(y) = y + \frac{1}{2y} + O\left(\frac{1}{y^2}\right), \quad y \gg 1, \quad (A.15)$$

up to 4th order terms for all $y$ values due to the power factor $p^\lambda (\lambda \geq 2)$. Evaluating a smooth asymptotical function $J_{St}(y)$ (A.15) with respect to the Maxwell distribution function $f_{GE}(yp_T)$ at the maximum contribution into the integrals (A.14) at $y \approx 1$ ($p \approx p_T$), one obtains approximately the damping rate $\nu$ of the collision term (28):

$$\nu = nv_T \sigma J_{St}(1) \approx \frac{3\pi T v_T d^2}{2}. \quad (A.16)$$

Note that the second exponent term in Eq. (A.13) for $J_{St}$ was exactly canceled by the second term of the error function expansion, that leads to a good relative accuracy (about 6%) after neglecting terms of the order of $1/y^4$ in asymptotics (A.15).

This accuracy can be checked by comparison of (A.16) with calculations of the exact function $J_{St}(y)$, and its average $\langle J_{St}(y) \rangle_{av}$ over $y$ with the static distribution $f_{GE}(8)$,

$$\langle J_{St}(y) \rangle_{av} = \int_{0}^{\infty} dy J_{St}(y) f_{GE}(y). \quad (A.17)$$

Calculating $\nu$ traditionally [9, 15, 23] through the averaged value (A.17) of the collision term [or $J_{St}(y)$ (A.13)] over all momenta $p$ (or $y$), one obtains

$$\nu \approx \langle \nu \rangle_{av} = nv_T \sigma \langle J_{St}(y) \rangle_{av} \approx \sqrt{8/\pi} \frac{T v_T d^2}{2}. \quad (A.18)$$

Thus, both approximations for $\nu$, Eqs. (28) and (A.18), are almost the same within a good relative precision mentioned above.

### Appendix B: DERIVATIONS OF DISPERSION EQUATION

In order to derive the dispersion equation (30) for the ratio $c = \omega/(k T)$ with respect to $c$ in the FC regime, one may specify a small perturbation parameter $\epsilon$, Eq. (24), in perturbation expansion for $\varphi(\vec{p})$, Eqs. (22) and (23). Then, in the FC regime (small $c$), one can truncate the expansion of $\varphi(\vec{p})$ (26) over spherical functions $Y_{\ell l}(\vec{p})$ in the plane-wave distribution function $\delta f$ (20) at the quadrupole value of $\ell$, $\ell \leq 2$, because of a fast convergence of the sum (26) over $\ell$ [10]. Substituting the plane wave solution (20) with the multipole expansion (26) for $\varphi(\vec{p})$ in $\delta f$ into the BVKE (9), after simple
Proceeding with algebraic transformations, one finally arrives (within the same approximations used in Appendix A) at the following linear equations \((L = 0, 1, 2, \ldots)\) for \(\varphi_\ell\):

\[
\sum_{\ell} A_{\ell\ell}(c) \varphi_\ell = 0, \quad c = \omega/(kv) , \tag{B.1}
\]

where

\[
A_{\ell\ell}(c) = c\delta_{\ell\ell} - C_{\ell\ell;1} + \frac{c}{\sqrt{3}} \delta_{\ell1}\delta_{00} + i\gamma \delta_{\ell\ell}(1-\delta_{00})(1-\delta_{11}) , \tag{B.2}
\]

\[
C_{\ell\ell;1} = \sqrt{\frac{4\pi}{3}} \int d\Omega_p \, Y_{L0}(\hat{p}) \, Y_{10}(\hat{p}) = \frac{\sqrt{2\ell+1}}{2\ell+1} \left( C_{L0}^{L0;10} \right)^2 , \tag{B.3}
\]

\(C_{L0}^{L0;10}\) is the Clebsh-Gordan coefficients [33], and \(\gamma\) is given by Eq. (32) (Appendix A). We multiplied the BVKE (9) by \(Y_{L0}(\hat{p})\), and integrated term by term over angles \(d\Omega_p\) of the unit momentum vector \(\hat{p}\) in the spherical coordinate system with the polar \(z\) axis along the unit wave vector \(\hat{k}\). The integrals can be calculated explicitly by using the orthogonal properties of spherical functions and Clebsh-Gordan techniques for calculations of a few spherical function products in the integrand. The matrix \(A_{\ell\ell}\) has a simple structure. At the diagonal, one finds non-zero values \(A_{\ell\ell}\) depending on the sound velocity \(c\). There are also two \(L = \ell \pm 1\) lines, parallel to the diagonal, above and below it, with the non-zero number coefficients, depending on the Clebsh-Gordan coefficients through Eq. (B.3). They are independent of the velocity \(c\) and the dimensionless collisional rate \(\gamma\). Other matrix elements are zero. The isotropic mean field \(\delta U\) (10) influences, through the interaction constant \(F\) (31), on only one matrix element, \(A_{00} = F/\sqrt{3} - C_{01;0}\). The damping rate constant \(\gamma\) related to the collision integral, Eq. (27), are placed only in the main diagonal \(A_{\ell\ell}\) at \(\ell \geq 2\), \(A_{\ell\ell} = c + i\gamma\) because of the conservation conditions, as explained above (Appendix A). For the FC regime, because of large \(\gamma\), one notes the convergence of the coefficients \(\varphi_\ell\) of the expansion in multipolarities (26): Any \(\varphi_\ell\) at \(\ell \geq 2\) is smaller than \(\varphi_{\ell-1}\) by factor \(1/(c+i\gamma)\) [10, 13]. See more explicit expressions for ratios of the amplitudes \(\varphi_\ell\) in Appendix C, Eq. (C.8), in the case of the quadrupole truncation of the characteristic matrix \(A\). Truncating this matrix at the quadrupole value \(\ell \leq 2\) and \(L \leq 2\), one obtains the following simple 3x3 matrix:

\[
A^{(2)} = \begin{pmatrix}
\frac{c}{\sqrt{3}} & -C_{11;1} & 0 \\
-C_{11;1} & \frac{c}{\sqrt{3}} & -C_{21;1} \\
0 & C_{21;1} & c + i\gamma
\end{pmatrix} , \tag{B.4}
\]

with \(C_{01;1} = C_{11;0} = 1/\sqrt{3}\), and \(C_{21;1} = C_{11;2} = 2/\sqrt{15}\). Accounting for Eq. (32) for \(\gamma\), and explicit expressions for these constants \(C_{\ell\ell;1}\) (B.3), in the quadrupole FC case, one obtains the condition of existence of non-zero solutions \((\det A^{(2)}(c) = 0)\) of linear equations (B.1), that is the cubic equation (30) with respect to \(c\).

Substituting \(c = c_r + ic_i\) into the dispersion equation (30), one can use the overdamped conditions within the FC regime,

\[
|c/c| = |\omega\tau| \ll 1, \quad |c_r/c_i| \ll 1 . \tag{B.5}
\]

Then, at leading order one obtains (for \(\gamma \neq 0\))

\[
-iF_1 \frac{c_i}{\gamma} - F_1 c_r - iF_2 = 0 , \tag{B.6}
\]

where \(F_1 = 3/5 - F/3\) and \(F_2 = (1 - F)/3\), and \(\gamma\) is given by Eq. (32). Separating real and imaginary parts, at leading order within the conditions (B.5), one finds the overdamped solution:

\[
c_r = 0, \quad c_i = -\frac{F_2}{F_1} \gamma , \tag{B.7}
\]

that is identical to Eq. (34).

**Appendix C: Moments of the Distribution Function and Viscosity**

For the shear viscosity \(\eta\) [Eq. (17)], one has to calculate the matrices \(U_{\mu\nu}\) [Eq. (18)] and \(\delta\sigma_{\mu\nu}\) [Eq. (14)]. Taking the polar axis of the spherical coordinate system in the momentum space along the unit wave vector \(\hat{k} = k/k\), we note that these matrices are symmetric with zero non-diagonal terms, and

\[
U_{xx} = U_{yy} = -\frac{1}{2} U_{zz} , \tag{C.1}
\]

\[
\sigma_{xx} = \sigma_{yy} = -\frac{1}{2} \sigma_{zz} . \tag{C.2}
\]

We find easy these relations using the symmetry arguments and properties of the integrals of the plane wave solution (20) for \(df\) over the angles \(d\Omega_p\) of vector \(p\). Therefore, from Eqs. (18), (14), (20) and (26) one has to obtain only the simplest \(zz\) components,

\[
U_{zz} = \frac{2}{3} \frac{\partial u_z}{\partial z} - \frac{2}{3} \nabla u = \frac{2}{3i} k \bar{u}_z \exp(-i\omega t + ikr) ,
\]

\[
\sigma_{zz} = \bar{\sigma}_{zz} \exp(-i\omega t + ikr) , \tag{C.3}
\]

where

\[
\bar{u}_z = \int \frac{dp}{nm} \, p_z f_{GE}(p) \varphi(\hat{p}) = \frac{v_r}{\pi \sqrt{3}} \varphi_1 ,
\]

\[
\varphi_\ell = \int d\Omega_p \, Y_{L0}(\hat{p}) \varphi(\hat{p}) , \tag{C.4}
\]

and

\[
\bar{\sigma}_{zz} = -\int \frac{dp}{3m} \, \left( 3p_z^2 - p^2 \right) f_{GE}(p) \varphi(\hat{p})
\]

\[
= -\frac{2}{3m} \frac{4\pi}{5} \int p^4 dp \, f_{GE}(p) \int d\Omega_p \, Y_{20}(\hat{p}) \varphi(\hat{p})
\]

\[
= -\frac{nT}{\sqrt{5\pi}} \varphi_2 , \tag{C.5}
\]
We calculated explicitly the Gaussian-like integrals over \( p \) using the static distribution function \( f_{GE} \), Eq. (8),

\[
I_\lambda = \int_0^\infty dp \lambda^2 f_{GE}(\lambda) = \frac{n \lambda^2}{2 \pi^{3/2}} \Gamma \left( \frac{\lambda + 1}{2} \right), \tag{C.6}
\]

where \( \Gamma(x) \) is the Gamma function. Using the orthogonal properties of the spherical functions and Eqs. (C.3), (C.4) and (C.5), from Eq. (17), one arrives at

\[
\phi_0 = \frac{1}{3} \sqrt{c},
\]

\[
\phi_1 = \frac{2}{\sqrt{15}} (c + i\gamma). \tag{C.8}
\]

With these expressions, from Eq. (C.7) one obtains

\[
\eta = \frac{3\sqrt{\pi}}{10} \frac{1}{1 + c/(i\gamma)} \frac{nT}{\nu} = \frac{1}{1 - \sqrt{\frac{mT}{d^2}}}.
\]

Substituting the overdamped solution for the sound velocity [Eq. (34)], from Eq. (C.9) one obtains Eq. (38).