STUDYING MAGNETOHYDRODYNAMIC TURBULENCE WITH SYNCHROTRON POLARIZATION DISPERSION

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ABSTRACT

We test a new technique for studying magnetohydrodynamic turbulence suggested by Lazarian & Pogosyan, using synthetic observations of synchrotron polarization. This paper focuses on a one-point statistics, which is termed polarization frequency analysis, that is characterized by the variance of polarized emission as a function of the square of the wavelength along a single line of sight. We adopt the ratio \( \eta \) of the standard deviation of the line-of-sight turbulent magnetic field to the line-of-sight mean magnetic field to depict the level of turbulence. When this ratio is large (\( \eta \gg 1 \)), which characterizes a region dominated by turbulent field, or small (\( \eta \lesssim 0.2 \)), which characterizes a region dominated by a mean field, we obtain the polarization variance \( \langle P^2 \rangle \propto \lambda^{-2} \) or \( \langle P^2 \rangle \propto \lambda^{-2} \end{eqnarray} \), respectively. At small \( \eta \), i.e., in the region dominated by the mean field, we successfully recover the turbulent spectral index from the polarization variance. We find that our simulations agree well with the theoretical prediction of Lazarian & Pogosyan. With existing and upcoming data cubes from the Low-Frequency Array for Radio Astronomy (LOFAR) and the Square Kilometer Array (SKA), this new technique can be applied to study the magnetic turbulence in the Milky Way and other galaxies.

Key words: magnetohydrodynamics (MHD) – radio continuum: ISM – turbulence

1. INTRODUCTION

Magnetohydrodynamic (MHD) turbulence is ubiquitous in astrophysical environments (evidence from electron densities: Armstrong et al. 1995; Chepurnov & Lazarian 2010; Burkhart & Lazarian 2012, from spectral lines: Larson 1981; Lazarian & Pogosyan 2000; Lazarian 2009; Chepurnov & Lazarian 2010; Chepurnov et al. 2015, and from synchrotron fluctuations: Cho & Lazarian 2010; Gaensler et al. 2011; Burkhart et al. 2012; Iacobelli et al. 2014, see also Elmegreen & Scalo 2004; McKee & Ostriker 2007 for reviews). It plays a critical role in key astrophysical processes, such as star formation, acceleration and propagation of cosmic rays, heat transport, and magnetic reconnection (see Cho et al. 2003; Mac Low & Klessen 2004; Lazarian et al. 2015). Significant progress has been achieved in understanding the theory of MHD turbulence (e.g., Goldreich & Sridhar 1995; Cho & Lazarian 2002, see also Brandenburg & Lazarian 2013; Beresnyak & Lazarian 2015 for recent reviews) and its implications, in particular the physics of turbulent reconnection (e.g., Lazarian & Vishniac 1999; Kowal et al. 2009; Lazarian et al. 2015 for a recent review). However, the theory of MHD turbulence is a developing field with many outstanding questions. For instance, even the spectral slope of Alfvenic turbulence is the subject of vigorous debate (e.g., see Beresnyak 2014 and references therein). The changes of the spectral slope with the physical conditions in plasmas are mostly unknown quantities that are difficult to study with the numerical resolution currently available. Studies of these features using observations are really advantageous.

The seminal theoretical work of Goldreich & Sridhar (1995) predicted that the spectrum of MHD turbulence follows the Kolmogorov spectrum with \( k^{-11/3} \) in terms of the three-dimensional (3D) spectrum. However, the spectrum of MHD turbulence may become different in some astrophysical environments. For example, the magnetic spectrum in terms of 3D spectrum is claimed to be \( k^{-1} \) in the case of viscosity-damped turbulence when the ratio of viscosity to resistivity is much larger than 1 (Cho et al. 2002, 2003; Lazarian et al. 2004). The recovery of the spectral index of magnetic turbulence is the focus of this paper. In addition, spectra of both density and velocity are also important for revealing information about MHD turbulence. The former can be obtained directly from maps of column density (e.g., Stutzki 1999). However, the study of the latter is relatively complex. New techniques to study the velocity spectrum, which are termed velocity channel analysis (VCA) and velocity coordinate spectrum (VCS), have been proposed (Lazarian & Pogosyan 2000, 2004, 2006, 2008), and successfully tested and applied to observations (Lazarian et al. 2001; Padoan et al. 2006, 2009; Chepurnov & Lazarian 2009; Chepurnov et al. 2010, 2015).

The overall spectral shape and spectral index of MHD turbulence can provide very valuable information, such as the sources and sinks of turbulence and the cascading processes of energy. Because of the complexity and uncertainty of MHD turbulence, it is important to have techniques to study turbulence from observations. There is no doubt that a correct understanding of MHD turbulence can help us to shed light on some intrinsic properties of astrophysical processes, and even challenge paradigms of some traditional models, e.g. the evolution of molecular clouds (see Mouschovias & Spitzer 1976; Stone et al. 1998; Lazarian 2014).

When nonthermal relativistic electrons propagate in turbulent magnetic fields they emit polarized synchrotron radiation, which carries information on the statistics of turbulent flows. Therefore, synchrotron fluctuations are a promising tool for investigating MHD turbulence. A theoretical description of the fluctuations of synchrotron intensity arising from magnetic turbulence is provided in Lazarian & Pogosyan (2012, henceforth LP12) and it was tested (Herron et al. 2016) by using synthetic observations.
The study of synchrotron polarization fluctuations together with the Faraday rotation measure (RM) was initiated in Burn (1966), and since then many studies have been dedicated to this subject (e.g., Brentjens & de Bruyn 2005; Frick et al. 2010, 2011; Beck et al. 2012); see Heal (2015) for a recent review.

Very recently, a novel technique involving the statistical description of synchrotron polarization fluctuations was suggested to investigate the spectrum and anisotropy of MHD turbulence (Lazarian & Pogosyan 2016, henceforth LP16). This new technique considers the complex polarized intensity $P (\equiv Q + iU)$, where $Q$ and $U$ are Stokes parameters) as a function of the density of intrinsic polarized intensity $P_i$, and then integrates along the line of sight in physical space$^5$:

$$P(X, \lambda^2) = \int_0^L dz P_i(X, z) e^{2i\lambda b(X, z)},$$  \hspace{1cm} (1)

where $P(X, \lambda^2)$ is a function of the 2D spatial separation, $X$, of the direction of measurements and the square of wavelength $\lambda^2$, and $L$ is the extent of the turbulent source along the line of sight. $\Phi(X, z) \propto \int_0^L n_{th}(z')B(z')dz'$ is called the Faraday RM or Faraday depth, where $n_{th}$ is the density of thermal electrons and $B(z)$ is the component of the magnetic field along the line-of-sight direction. Based on an analytical method of statistical descriptions, they proposed several versions of the technique that can be employed to obtain spectral slopes and correlation scales of the underlying MHD turbulence, and the spectrum of fluctuations in Faraday rotation. Two main versions of the technique are polarization frequency analysis (PFA), which makes use of the change in the variance of polarization intensity (or its derivative) as a function of the square of the wavelength $\lambda^2$, and polarization spatial analysis (PSA), which makes use of the spatial correlations of the polarization intensity (or its derivative) at the same wavelength as a function of the spatial separation $X$. LP16 stresses that the two techniques have some analogies with the VCS and VCA techniques of these authors for turbulence studies using spectroscopic data. These techniques have been successfully applied to study turbulence with atomic hydrogen and CO data.

The purpose of this paper is to test the theoretical work of LP16 in order to open ways for applying this new technique to studies of MHD turbulence in the Milky Way, distant galaxies, and even clusters of galaxies. This work is timely because upcoming telescopes, such as the Low-Frequency Array for Radio Astronomy (LOFAR) and the Square Kilometer Array (SKA) operating in a wide wavelength range, can be used to study MHD turbulence; the origin and evolution of cosmic magnetism is a key science project of the SKA (Beck 2015). In this paper, we focus on testing the PFA theory; the test for PSA is carried out in Lee et al. (2016).

The next section is devoted to methods of statistical descriptions for studying MHD turbulence, and a brief description of numerical techniques is provided in Section 3. We present our results in Section 4, discuss the results of statistical tests in Section 5, and summarize our findings in Section 6.

\footnote{In addition, the Faraday dispersion function is defined as a Fourier transform of the polarization surface brightness with regard to the \( \lambda^2 \) variable in Appendix D of LP16, which is used to study MHD turbulence via the Fourier transform from wavelength \( \lambda \) space to physical space.}

\footnote{Nevertheless, we will need averaging over many lines of sight to reduce statistical noise (see Equation (5)).}

\section{2. STATISTICAL DESCRIPTION OF TURBULENCE}

MHD turbulence is an extremely complex chaotic nonlinear process and exhibits diverse properties on a microscopic level, but it allows a macroscopic treatment in a simple statistical description (Biskamp 2003), which can reveal regular features behind chaotic phenomena of turbulent magnetic fluctuations. In fact, the method of a statistical description of MHD turbulence is widely used. It is claimed as an adequate and concise way to shed light on important properties of interstellar turbulence (LP16).

In practice, the structure and correlation functions of (any) physical variable $P(X)$ have traditionally been employed for studying MHD turbulence. The correlation function is given by

$$\text{CF}(R) = \langle P(X_1)P(X_2) \rangle,$$  \hspace{1cm} (2)

where $\langle \ldots \rangle$ indicates an average over the entire volume of interest. For the structure function, there is a wider variety one can measure. In the case of isotropic MHD turbulence, a second-order scenario is commonly used, and written as

$$\text{SF}(R) = \langle (P(X_1) - P(X_2))^2 \rangle = 2[\text{CF}(0) - \text{CF}(R)].$$  \hspace{1cm} (3)

As shown in Equation (3), the second-order structure function is formally related to the correlation function. On the other hand, the power spectrum, $E(k)$, is obtained by a Fourier transform of $P(X)$ from physical space to wavenumber space, and given by

$$E(k) = \frac{1}{2} \{ P(k)P^*(k) \},$$  \hspace{1cm} (4)

where the symbol $^*$ denotes the complex conjugate. Generally, from a spatial point of view, the measures in this paragraph belong to two-point statistics.

Just as stressed in Section 1, we intend to test the PFA theory in LP16, which is essentially spectroscopic, requiring sufficient frequency resolution and coverage. This technique is to measure the correlation of polarization at different wavelengths along a fixed line of sight. In other words, this is a variance of polarization intensity, that is, a one-point statistics and a special case of the correlation function at $R = 0$. We now describe briefly the main theoretical results of LP16 related to the current work as follows.

On the basis of Equation (1), the variance of polarization as a function of wavelength is written as

$$\langle P(X^2)P^*(X^2) \rangle = \langle P^2(X^2) \rangle = \int_0^L dz_1 \int_0^L dz_2 e^{2i\lambda b(X_1, z_1)} \langle P_i(z_1)P_i^*(z_2) e^{2i\lambda^2 (\Phi(z_1) - \Phi(z_2))} \rangle,$$  \hspace{1cm} (5)

where $\bar{\phi} = \langle n_{th}(z)B(z) \rangle$ is an average of the RM density. After making two assumptions, which are (i) that $\phi$ is a Gaussian quantity and (ii) that the correlation between the intrinsic polarization fluctuation $P_i$ and the Faraday RM can be neglected, Equation (5) can further factorized into

$$\langle P^2(X^2) \rangle = \int_0^L dz_1 \int_0^L dz_2 e^{2i\lambda b(X_1, z_1 - z_2)} \times \langle P_i(z_1)P_i^*(z_2) \rangle e^{-4\lambda^2 D_{\Delta b}(0, A(z_1 - z_2))},$$  \hspace{1cm} (6)

where $D_{\Delta b}(0, A(z_1 - z_2))$ is the correlation function of Faraday depth fluctuations, and $A(z_1 - z_2)$ is the leakage length of the leakage length.
where \( \langle P^2 \rangle \equiv \langle PP^* \rangle \) and \( D_{\lambda \phi} \) is the structure function of Faraday RM, which was intensively studied in Section 3.2 of LP16. LP16 investigated different regimes of magnetic turbulence by changing the ratio \( \sigma_p / \bar{\phi} \), which can appropriately characterize the transition of magnetic turbulence from the case when RM is dominantly random to the case when it is uniform; here, \( \sigma_p \) is the root mean square of Faraday RM density fluctuations.

In the case of \( \bar{\phi} > \sigma_p \), the variance of polarization intensity is subject to

\[
\langle P^2 (x^2) \rangle \propto \lambda^{-2 - 2m}
\]  

(7)

in the long-wavelength regime, where \( m \) is the characteristic scaling slope of magnetic fluctuations (for instance \( m = 2/3 \) would correspond to Kolmogorov scaling). Obviously, Equation (7) reveals asymptotically the slope of the transverse magnetic field. In the opposite case, \( \bar{\phi} < \sigma_p \), corresponding to the long-wavelength regime,

\[
\langle P^2 (x^2) \rangle \propto \lambda^{-2}
\]  

(8)

follows the universal law \( \lambda^{-2} \) of Faraday rotation scaling. The weighted derivative of Equation (8) would reflect the scaling slope of the transverse magnetic field. As described in LP16, however, it is not easy to recover properties of magnetic fluctuations from observational data cubes because of the effects of noise in the data.

Testing the above expressions is the main goal of our study. In addition, we also present statistical studies for both Faraday depolarization and the variance of the polarization derivative as a function of wavelength. These studies would reveal the statistics of magnetic fluctuations from synchrotron polarization and Faraday rotation.

3. NUMERICAL TECHNIQUE

3.1. Synthetic Data Generation

The technique we use to generate 3D synthetic data cubes for magnetic fields and the density of thermal electrons is presented here. We generate data (\( \tilde{B}(k) \) functions) in wavenumber space and then transform them into physical space. The magnetic field is assumed to be

\[
B(r) = \sum_{k_{\text{min}} \leq |k| \leq k_{\text{max}}} \tilde{B}(k)e^{ik \cdot r}
\]  

(9)

with the condition of the solenoidal vector field \( \nabla \cdot B = 0 \), where \( k \) is the wavevector and \( r \) is the position vector. The Fourier coefficient \( \tilde{B}(k) \) is defined as \( \tilde{B}_n(k) = [\tilde{B}_n(k)]e^{i\zeta} \), where the subscript \( n = 1, 2, 3 \) indicates three directions of Cartesian coordinates. The amplitude follows a power law: \( |\tilde{B}(k)| \propto k^{-\beta} \). The phase factor \( \zeta \) is randomly distributed. The generation of the data cube for the thermal electron density is similar but slightly easier because there is no constraint for a divergence-free condition.

Generating 3D data cubes requires large computational resources, which makes it difficult to achieve high numerical resolution. Hence, we also create both 1D and 2D synthetic data with higher resolutions. The procedure for generating the 1D and 2D data is similar to that for 3D data. That is, we generate Fourier coefficients of the form \( F(k) = A(k) \cos \omega + i \sin \omega \) in wavevector space and carry out a Fourier transform from \( k \)-space to physical space. Here \( k \) is either a 2D vector or a 1D scalar, \( \omega = 2\pi \tau \), and \( \tau \) is a random number between 0 and 1.

3.2. Radiation Aspects

We give here a brief discussion of numerical aspects for radiative processes. The distribution of the nonthermal relativistic electrons is assumed to be \( N(\varepsilon) \propto \varepsilon^{-\gamma} \), where \( p \) is the spectral index of the electrons and \( \varepsilon \) is their energy. The formulae for synchrotron radiation in Appendix A of LP16 are adopted to calculate Stokes parameters \( I, Q, \) and \( U \). As seen in these formulae, the Stokes parameters are proportional to \( |B_\perp|^4 \), where \( B_\perp \) is the perpendicular component of the magnetic field and the exponent \( \gamma \) is related to the electron index of the electron distribution \( p \) by the relation \( \gamma = (p + 1)/2 \). We stress that we are using the original formula for polarization intensity to test the theoretical work of LP16. In our work, related physical quantities are calculated in arbitrary units because they show only magnitude differences and do not change the results of our statistical investigation.

3.3. Observational Effects

To simulate realistic observations, we also consider the effects of a finite telescopic angular resolution and random noise. We convolve the original polarization intensity (i.e., Stokes \( Q \) and \( U \)) “maps” with a Gaussian kernel in order to study the influence of finite angular resolution. For the noise, we generate a map of Gaussian noise and add it to the original polarization intensity map. We adjust the level of noise by considering the signal-to-noise ratio. The resulting polarization maps (with noise) are smoothed with a Gaussian beam, whose full width at half-maximum (FWHM) is equal to 3 pixels. We fix this resolution because the usual surveys have a typical value of 3 pixels corresponding to the FWHM of the peak of the telescope beam, resulting in a standard deviation of \( \sigma = 3/2.35 \approx 1.3 \). To determine the signal-to-noise ratio of the final map, we also convolve the original polarization map with the same Gaussian beam (i.e., \( \sigma \approx 1.3 \)) and then subtract it from the final map.

4. RESULTS

We first provide maps of Stokes parameters from a 3D data cube in Section 4.1. Then the results of one-point statistics are presented in detail in the following sections.

4.1. Basics

In this section we use a 3D synthetic data cube with 512 pixels along each side and zero mean magnetic field \( \langle B_i \rangle = 0 \), which simulates an MHD turbulent flow with an isotropic Kolmogorov scaling, \( k^{-11/3} \), to study the statistics of polarization fluctuations. In Figure 1, a map is plotted in the left panel to depict a cross section of the perpendicular component of magnetic field, \( B_\perp = \sqrt{B_1^2 + B_2^2} \). As shown on the map, the distribution of magnetic field is homogeneous and isotropic. Using two 3D data cubes for \( B_x \) and \( B_y \), we obtain maps of Stokes parameters \( Q \) (middle panel) and \( U \) (right panel) by integrating along the \( z \)-axis. It should be noticed that the units of magnetic field and the Stokes parameters, \( Q \) and \( U \), are arbitrary. The spectral index of relativistic electrons is fixed as \( p = 3 \), which gives \( B_\perp^2 = B_z^2 \). We find that both \( Q \) and \( U \) have positive and negative values. It seems that the Stokes \( Q \) has a
larger probability for positive values but $U$ for negative values. In general, a positive $Q$ implies that the electric field vector is preferentially aligned with the $x$-axis. Then a negative $Q$ means a preferential alignment with the $y$-axis. The Stokes $U$ parameter has a similar meaning, just with a 45° rotated reference system.

We used a zero mean magnetic field ($B_z = 0$ here, corresponding to a region dominated by turbulent magnetic field, to carry out basic simulations. In the next sections, we will consider nonzero mean magnetic fields to perform related simulations. In this work, we use the ratio $\eta$ of the standard deviation $\sigma_z$ of the line-of-sight turbulent magnetic field to the line-of-sight mean magnetic field $\langle B_z \rangle$ to characterize the level of magnetic turbulence. A large $\eta$ will indicate a region dominated by turbulent magnetic field and a small $\eta$ will stand for a region dominated by the mean field.

### 4.2. Variance of Polarization

In this section, we test in detail the analytical PFA technique of LP16 by using 3D, 2D, and 1D synthetic data “cubes,” with different numerical resolutions, different values of mean magnetic field along the $z$-axis, and various magnetic energy spectra of $k^{-4}$, $k^{-11/3}$, and $k^{-7/2}$.

#### 4.2.1. 3D Tests

Using the numerical technique introduced in Section 3, we could in principle obtain a 3D box with any numerical resolution along each side. However, in practice, the size of the 3D grid is limited by the available memory of the computer. The maximum numerical resolution for our 3D cubes is $4096^3$. Note, however, that in order to reduce memory requirements we save data for only $128^3$ evenly spaced lines of sight. Therefore, although we use the notation $128 \times 128 \times N$ for the numerical resolution for 3D data, the actual numerical resolution is equivalent to $N^3$.

Figure 2 shows the polarization variance as a function of the square of wavelength $\lambda^2$ in the 3D case. The initial settings are $\beta = 11/3$ and $\eta = \infty$.

From a numerical point of view, when integrating polarization intensity, we have to deal with the complex exponential function $e^{i\phi}\cos\phi$ (which can be transformed into cosine and sine functions) in Equation (1) from the effect of Faraday rotation. Using general numerical integral methods, such as a rectangular integral method or trapezoidal integral method, to carry out integrals for the exponential function, cosine or sine function, it is obvious that insufficient integral precision will result in a rough, even spurious result. Therefore, it is necessary to provide a small enough integral step in order to obtain a real result. In fact, we could also understand this behavior from another point of view. On the basis of Equation (6), we know that polarization variance is proportional to $e^{-4.35\lambda^2}$. When setting $\lambda = 1$, the polarization variance will only be a function of spatial distance $z$, which is related to numerical resolution. A low numerical resolution would result in a large integral value of the polarization variance and vice versa, which corresponds to the location of the plateau. If one sets a constant magnetic field and a constant density, it is easy to test that integrating $e^{-4.35\lambda^2}$ over the spatial distance $z$ with a low numerical resolution (equivalent to a large integral step) will present a plateau in the region of large $\lambda$. With increasing numerical resolution, the plateau naturally shifts toward larger $\lambda$. 

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**Figure 1.** The left panel presents a cross section of the synthetic magnetic field $B_z$ that has 512 pixels along each side and zero mean magnetic field. The maps of Stokes parameters $Q$ (middle panel) and $U$ (right panel) are obtained by invoking the magnetic field structures in the left panel and integrating spatial separation along the $z$-axis. The spectral index of relativistic electrons is $p = 3$, which links to $B_z$ by $\gamma = (p + 1)/2$. The units of the magnetic field strength and of the Stokes parameters $Q$ and $U$ are arbitrary.

**Figure 2.** Polarization variance as a function of the square of wavelength $\lambda^2$ in the 3D case. The initial settings are $\beta = 11/3$ and $\eta = \infty$. 

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### Footnote 7

From a numerical point of view, when integrating polarization intensity, we have to deal with the complex exponential function $e^{i\phi}\cos\phi$ (which can be transformed into cosine and sine functions) in Equation (1) from the effect of Faraday rotation. Using general numerical integral methods, such as a rectangular integral method or trapezoidal integral method, to carry out integrals for the exponential function, cosine or sine function, it is obvious that insufficient integral precision will result in a rough, even spurious result. Therefore, it is necessary to provide a small enough integral step in order to obtain a real result. In fact, we could also understand this behavior from another point of view. On the basis of Equation (6), we know that polarization variance is proportional to $e^{-4.35\lambda^2}$. When setting $\lambda = 1$, the polarization variance will only be a function of spatial distance $z$, which is related to numerical resolution. A low numerical resolution would result in a large integral value of the polarization variance and vice versa, which corresponds to the location of the plateau. If one sets a constant magnetic field and a constant density, it is easy to test that integrating $e^{-4.35\lambda^2}$ over the spatial distance $z$ with a low numerical resolution (equivalent to a large integral step) will present a plateau in the region of large $\lambda$. With increasing numerical resolution, the plateau naturally shifts toward larger $\lambda$. 

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The transition of polarization variance from the stochastic RM-dominant regime for \( \eta = \infty \) to the uniform RM-dominant one for \( \eta = 0.080 \) in the 3D case. The initial setting corresponds to Kolmogorov scaling of \( \beta = 11/3 \). The 3D data cube with the highest resolution is used.

High numerical resolution, this unreal phenomenon would be eliminated. Besides, with increasing pixel size, the slope of the curve between the two plateaus approaches \(-1\) (i.e., the curve becomes compatible with \( \lambda^{-2} \)). This result agrees with the theoretical prediction in Equation (8). Hereinafter, we shall use 2D and 1D synthetic data to confirm this point.

Using the 3D data cube with the highest resolution (i.e., the one with \( 128 \times 128 \times 4094 \) resolution in Figure 2), we study in Figure 3 the scenario of how the change in the mean magnetic field along the \( z \)-axis (i.e., the line of sight) affects the polarization variance. The zero mean magnetic field characterizes a stochastic RM regime where the power-law index approximates to 1.95 (i.e., \( \langle \beta^2 \rangle \propto \lambda^{-1.95} \)), and the low value of \( \eta = 0.040 \) would correspond to a uniform RM-dominant regime where the power-law index approximates to 3.33, which is in good agreement with the theoretical prediction, i.e., \( \eta = 2m = 10/3 \) with \( m = 2/3 \) for Kolmogorov scaling. The transition from the stochastic RM-dominant regime to the uniform RM-dominant one is finished at \( \eta \approx 0.080 \).

We explore the behavior of polarization variance by varying the MHD scaling index in Figure 4. Three different spectral indices \( \beta = 4, 11/3, \) and \( 7/2 \) are used to test both random and uniform RM-dominant regions. As shown in Figure 3, \( \eta = \infty \) or a low \( \eta \) value can just characterize these two turbulent regimes. We hence fix \( \eta = \infty \) and 0.080 for different scaling indices, respectively. It is interesting that we find that in the random RM-dominant regime MHD turbulence for different types always reveals the same asymptotic \( \lambda^{-1.95} \) behavior in the long-wavelength region, which is close to the predicted scaling relation, \( \lambda^{-2} \). Moreover, in the uniform RM-dominant regime it follows a certain relation, \( \lambda^{-2-2m} \), in the long-wavelength regime. Specifically, \( \lambda^{-3.99} \) for \( \beta = 4, \lambda^{-3.25} \) for \( \beta = 11/3, \) and \( \lambda^{-2.85} \) for \( \beta = 7/2, \) corresponding to the theoretical prediction of LP16, \( \lambda^{-2-2m} = \lambda^{-4} \) for \( m = 1, \) \( \lambda^{-2-2m} = \lambda^{-10/3} \) for \( m = 2/3, \) and \( \lambda^{-2-2m} = \lambda^{-3} \) for \( m = 1/2, \) respectively.

We feel that we are unsatisfied with these 3D tests due to the limitations of numerical resolution and the statistical sample. Hence, we shall work with higher-resolution 2D and 1D synthetic data in the following sections.

4.2.2. 2D Tests

We first test the influence of numerical resolution on the polarization variance by using 2D data on a square grid with the same numerical resolution along each direction. The results are plotted in Figure 5. Similar to Figure 2, the overall spectral shape presents two plateaus as well. The first plateau in the short-wavelength regime shows a slight change in magnitude due to changes of numerical resolution. For the highest numerical resolution shown, the power-law index in the long-wavelength region approaches 1.98 (i.e., \( \langle \beta^2 \rangle \propto \lambda^{-1.98} \)), which is in good agreement with the theoretical value of 2 (see also Equation (8)).

We present the effect of the mean magnetic field along the line of sight on the variance of polarization fluctuations, especially the transition from the stochastic RM-dominant regime to uniform RM-dominant one, in Figure 6. The case of zero mean magnetic field characterizes the situation in which RM is predominantly stochastic, and the cases with strong mean magnetic field (along the \( z \)-axis) correspond to the situation in which Faraday rotation is mainly uniform. It can be seen that the transition from the stochastic RM-dominant regime to the uniform RM-dominant regime occurs at \( \eta \approx 0.434 \). For \( \eta \ll 0.231 \), the power-law index is practically the same, i.e., \( \lambda^{-3.36} \), only presenting a shift of the curve toward shorter wavelengths. It is obvious that the result matches the theoretical prediction of LP16, \( \lambda^{-2-2m} = \lambda^{-10/3} \) for a Kolmogorov scaling slope of \( \beta = 11/3 \).

In Figure 7 we compare the polarization variance for different types of MHD turbulence as a function of the square of wavelength in two different regimes of Faraday dispersion, using the mean magnetic field strengths of 0 and 20 to represent these two regimes. We use three different spectral indices for the magnetic spectrum: \( \beta = 4, 11/3, \) and \( 7/2 \). We find that in the stochastic RM regime, different MHD turbulence presents the same slope approximating to 1.98 in the long-wavelength region. In the uniform RM regime, the power-law slopes show variations. The measured power-law slopes are \( \lambda^{-3.95} \) for \( \beta = 4, \lambda^{-3.28} \) for \( \beta = 11/3, \) and \( \lambda^{-2.97} \) for \( \beta = 7/2, \) which are in good agreement with the theoretical prediction of LP16, \( \lambda^{-2-2m} = \lambda^{-4} \) for \( m = 1, \) \( \lambda^{-2-2m} = \lambda^{-10/3} \) for \( m = 2/3, \) and \( \lambda^{-2-2m} = \lambda^{-3} \) for \( m = 1/2, \) respectively. These differences can allow us to estimate the spectral slope of MHD turbulence in the emitting region.

4.2.3. 1D Tests

The numerical technique to generate 1D synthetic data is similar to that for 2D data, but we generate data only on a line along the line of sight. We find in our numerical simulations that the resulting distributions of the variance of polarization present a significant fluctuation. As a result, it is difficult to measure an exact power-law index. To reduce random fluctuations, we increase the number of 1D data sets to about 2000 by altering the seeds for random numbers and then average over them. Here, each random seed is equivalent to an individual line-of-sight direction and the averaging procedure plays the role of obtaining statistics on the whole sky plane of interest perpendicular to the line of sight.

Similar to the study of 3D and 2D cases, we first test the effect of numerical resolution on the polarization variance. The polarization variance as a function of the square of wavelength \( \lambda^2 \) is plotted in Figure 8. We only plot the cases with zero mean magnetic field, and the different curves in the figure correspond to different numerical resolutions. The plateau in the long-wavelength region, which is most pronounced at the lowest numerical resolution, i.e., 256 grid points, shifts toward longer
wavelengths with increasing numerical resolution. Henceforth, we carry out numerical studies with 218 grid points. In this case, we obtain a constant power-law index of 1.99, which agrees well with the theoretical prediction of LP16 given in Equation (8).

Figure 9 shows the transition of the variance of polarization fluctuations from the case with zero mean magnetic field, which corresponds to the situation in which RM is predominantly stochastic, to a case with a strong mean magnetic field, which corresponds to the situation in which RM is mainly due to the uniform magnetic field. As shown in this figure, the transition from the stochastic RM-dominant regime to the uniform RM-dominant regime occurs at $h \approx 0.34$ and is completed at $h \approx 0.206$. For larger $\eta$, the power-law index is practically the same, i.e., $\lambda = 3.34$. It is obvious that the result matches well the theoretical prediction of LP16, $\lambda = -2.2m$, for a Kolmogorov scaling slope of $b = 1.3$.

We present the studies for different MHD spectral indices in Figure 10. Two representative extremal magnetic field strengths are used to characterize two different regimes of MHD turbulence, that is, $\bar{f}_s > f$ for $h = 0.10$ and $\bar{f}_s < f$ for $h = \infty$. In the case of $\bar{f}_s > f$, i.e., the stochastic RM-dominant regime, the resulting slope does not change with the spectral index of MHD turbulence in the long-wavelength region, and there is only a slight influence on the amplitude of variance in the short-wavelength region, where the RM effect is negligible. In the opposite case of $\bar{f}_s < f$, i.e., the uniform RM-dominant regime, the power-law index shows significant changes. It is very interesting that these changes still follow the relation of $\lambda = -2m$, that is, $\lambda = 3.96$ for $\beta = 4$, $\lambda = 3.34$ for $\beta = 11/3$, and $\lambda = 2.96$ for $\beta = 7/2$, which conforms well with the theoretical prediction of LP16, $\lambda = -2m = \lambda^{-10/3}$ for $m = 1$, $\lambda = 2m = \lambda^{-4}$ for $m = 2/3$, and $\lambda = 2m = \lambda^{-3}$ for $m = 1/2$, respectively.

### 4.3. Influence of Mean Magnetic Field Distributions on Polarization Variance

In the previous sections, we assumed that the direction of mean magnetic field is oriented along the line of sight, i.e., the $z$-axis direction. However, it may be oriented in other directions.
in a real astrophysical environment. In order to explore this influence on polarization variance, we consider that the existence of a mean magnetic field ⟨B⟩ is confined in the x-z plane for simplicity, in which the angle subtended by the field direction and the line of sight (z-axis) is labeled as θ. Following the methods given in Section 3, we first generate 3D and 2D data cubes B_o, B_x0, B_z0, under an initial condition with zero mean magnetic field. We then project the mean field set in the x-z plane onto the x- and z-axis directions. As a result, data cubes are given as follows: B_x = B_o + ⟨B⟩ sinθ, B_y = B_o, and B_z = B_o + ⟨B⟩ cosθ.

In Figure 11, we study the influence of the direction of mean magnetic field on the polarization variance in the 3D case. The parameters are the turbulence index β = 11/3 and the mean magnetic field strength of adjustable orientation ⟨B⟩ = 6. Accordingly, the ratio η would be defined as η = σ_2/⟨B⟩cosθ.

As shown in this figure, two extreme scenarios, θ = 0° and 90°, correspond to the case when the direction of mean field is along the z-axis and the case when it is along the x-axis, respectively. The former is similar to the case of η = 0.040 presented in Figure 3, but where the mean magnetic field is generated by setting corresponding initial conditions. The latter is similar to the case of η = ∞, but here there is a nonzero mean magnetic field along the x-axis direction. The other θ values correspond to the transition between the two. It can be seen that they would result in different mean field levels in the z-axis, as characterized in η.

Figure 12 shows the influence of the direction of mean magnetic field on the polarization variance in the 2D case. The parameters are the turbulence index β = 11/3 and the mean magnetic field of adjustable orientation ⟨B⟩ = 20. The related descriptions are similar to those given for Figure 11. We find in the 3D and 2D cases that for large η the polarization variance follows the usual relation ⟨P^2⟩ ∝ λ^−2, and for small η we have ⟨P^2⟩ ∝ λ^−2−2m = λ^−10/3. The resulting simulations are in agreement with the theoretical prediction of LP16.

4.4. Influence of Electron Spectral Index on Polarization Variance

We explore here how the power-law spectral slope of the cosmic-ray energy distribution affects the variance of polarization. Our results for variance of polarization (Section 4.2) demonstrate that the 1D result is best, so we perform this exploration based on 1D synthetic observations. The results are presented in Figure 13 for the case of Kolmogorov turbulence with β = 11/3. The left panel plots polarization variance as a function of the square of wavelength, in which different curves correspond to γ ∈ [1, 4] in increments of Δγ = 0.5. It can be seen that the amplitude of polarization variance increases with γ, but the overall shapes of the curves seem to be similar.

The right panels present the spectral index of the polarization variance as a function of γ and the resulting sum of the squared residuals of the linear fitting. The range of λ^2 used for fitting
corresponds to the wavelength range between two vertical dotted lines in the left panel, which coincides with the wavelength region for obtaining the prediction of the power-law index of polarization variance, i.e., the longer wavelength region. As shown in the upper right panel, the measured power-law indices show very small variations around the mean value of $m \approx 0.66$. As a result, variations in cosmic-ray index have no influence on the power-law index of polarization variance.

4.5. Observational Influences on Polarization Variance

The technique for adding observational effects, i.e., effects of both finite angular resolution and noise, introduced in Section 3, has a universality for relevant studies. For instance, it can be used to study the influence of observations on the structure function and power spectrum. We focus here on the study of observational influences on the variance of polarization.

Using a 3D synthetic data cube with 512 pixels (i.e., grid points) along each side, we study the influence of both angular resolution and noise on the power-law slope of the polarization variance. At each wavelength, the maps of Stokes $I$, $Q$, and $U$ are produced, which are called the idealized (original) images. Provided that one assumes that 1 pixel corresponds to 0.05 pc and the distance of the emission region is 2 kpc away, one would obtain an angular distance of 5 arcsec for 1 pixel. With a Gaussian kernel, the original maps can be convolved to an expected pixel that corresponds to the standard deviation of the radio telescope beam. Furthermore, Gaussian noise with standard deviation equal to a fraction of the mean synchrotron intensity is added to the original images, and then the new images are smoothed to FWHM = 3 pixels. Using these images, we calculate observed power-law slopes of the polarization variance. However, we find that the resulting slopes deviate from the original forms, $\lambda^{-2}$ and $\lambda^{-2-2m}$. The reason is that the 3D box with 512 pixels along each side does not have enough resolution. Therefore, we switch to 2D data and significantly enhance the numerical resolution.

The technique for studying observational effects in the 2D case is similar to that in 3D. The only difference is that the observed maps are one-dimensional. The influence of numerical resolution on the smoothing processes is presented in Figure 14. The original maps have been smoothed with an effective Gaussian beam of FWHM = 3 pixels. As shown in the figure, the curves for the smoothed maps get closer to those of the original maps as numerical resolution increases. We hence use the data with the maximum numerical resolution we can provide to carry out the following research.

The influence of Gaussian noise levels on $\langle P^2(\lambda^2) \rangle$ in the stochastic and uniform RM-dominant regimes is plotted in Figure 15, for a Kolmogorov magnetic spectrum with $\beta = 11/3$. The standard deviation of Gaussian noise is parameterized by $\Delta \sigma$, which is a fraction of the mean synchrotron intensity. The maps after adding Gaussian noise have been convolved to an effective Gaussian beam of FWHM = 3 pixels at different noise levels. We find that different noise levels indeed produce different results. The Gaussian noise makes the polarization variance curve upwards in the long-wavelength region. However, as seen in Figure 14, slopes corresponding to smoothed maps would deviate downward from the original slope. As a result, from the narrow point of view of the wavelength, adding noise could make the spectrum closer to the original slope. Thus, a range of wavelengths in observations are needed in order to recover the real properties of MHD turbulence.

The resulting $\langle P^2(\lambda^2) \rangle$ with both fixed angular resolution and fixed noise level are given in Figure 16, corresponding to stochastic (left panel) and uniform (right panel) RM-dominant regimes. It is obvious that a steep magnetic spectrum makes it easier to recover the universal scaling $\lambda^{-2}$, and $\lambda^{-2-2m}$ at the
same angular resolution. Besides, to recover the $\lambda^{-2-2m}$ slope, a higher signal-to-noise ratio is needed when comparing the left and right panels. Using the method given in Section 3, we deduce what the signal-to-noise ratio at individual wavelengths is. For instance, the signal-to-noise ratios at $\lambda^2 \sim 40$, corresponding to noise levels of $\Delta \sigma = 0.1\%$, $0.5\%$, $1\%$, and $2\%$, are $158$, $32$, $16$, and $8$, respectively. The noise has a small influence on the polarization variance in the region of slightly short wavelengths, which is an ideal region of wavelengths for recovering the properties of magnetic turbulence. The uniform RM-dominant regime is important for us to reveal the properties of magnetic turbulence from the polarization variance, because the $\lambda^{-2-2m}$ scaling is associated with different types of MHD turbulence by the spectral index $m$.

4.6. Depolarization and Variance of Polarization Derivative

LP16 provides additional methods for extracting statistical information on polarization fluctuations. As examples, in this section we present studies of the degree of polarization, and the variance of polarization derivative with respect to the square of wavelength $(\langle dP/d\lambda^2 \rangle^2)$, based on 1D synthetic data.

Figure 17 shows the variance of polarization derivative with respect to $\lambda^2$ for different magnetic spectral indices $\beta = 4$, $11/3$, and $7/2$. We use two extremal values of mean magnetic field along the line of sight, which characterize different regimes of Faraday dispersion. In the case of $\eta = \infty$, the resulting scaling follows $\lambda^{-5.06}$ in the long-wavelength region. In the case of $\eta = 0.103$, the power-law indices are $\lambda^{-8.00}$ for $\beta = 4$, $\lambda^{-7.33}$ for $\beta = 11/3$, and $\lambda^{-7.00}$ for $\beta = 7/2$. Although there is no theoretical prediction, we believe that this method can surely recover the scaling properties of MHD turbulence.

In Figure 18 we present the degree of polarization of linearly polarized synchrotron emission as a function of the square of wavelength $\lambda^2$. As shown in the figure, the polarization for a steep magnetic spectral index is stronger than that for a shallow spectral index. Correspondingly, the depolarization of Faraday dispersion is also stronger for a steep slope than for a shallow one. The increase in the mean magnetic field along the line of sight results in a shift of effective Faraday dispersion toward much shorter wavelengths. It is easy to understand this point from the relation $\chi \propto \lambda^2 (B_\parallel + B_\perp)$, in which $\chi$ is the Faraday rotation angle of the polarized direction. When $\chi$ is invariant, the increase in the magnetic field will result in a decrease in $\lambda^2$. The random fluctuations in the degree of polarization caused by Faraday dispersion become significantly larger in a large mean field than zero mean field, particularly at $\lambda^2 \approx 0.003$.

5. DISCUSSION

We have proven that by using the dispersion of synchrotron polarization one can obtain the statistics of the magnetic field and the underlying statistics of Faraday rotation. In the case of negligible Faraday rotation, the statistics of synchrotron polarization provide the spectral properties of the perpendicular component of the magnetic fields (LP12). But as described in LP16, due to the presence of an extra imaginary part of the polarization intensity, resulting statistical studies can deliver more valuable
The variance of polarization derivative with respect to $\lambda^2$ for different MHD scaling indices, as a function of the square of wavelength $\lambda^2$ in the 1D case. $\eta = \infty$ and 0.103 correspond to different regions of Faraday dispersion. The 1D data with the highest resolution are used.

Figure 18. Degree of polarization of linearly polarized synchrotron emission as a function of the square of wavelength $\lambda^2$ in the 1D case. The potential scaling indices of MHD turbulence are set as $\beta = 4, 11/3,$ and $7/2$. $\eta = \infty$ and 0.103 correspond to different regions of Faraday dispersion. The 1D data with the highest resolution are used.

Figure 17. The variance of polarization derivative with respect to $\lambda^2$ for different MHD scaling indices, as a function of the square of wavelength $\lambda^2$ in the 1D case. $\eta = \infty$ and 0.103 correspond to different regions of Faraday dispersion. The 1D data with the highest resolution are used.

The importance of this work is obvious from yet another perspective. It is advantageous to use synergies of different techniques to study MHD turbulence. The VCA and VCS techniques mentioned in Section 1 can be used to obtain statistics of velocity fluctuations. In addition, we should stress that a spectrum is not the only property of turbulence. For instance, some new techniques, based on the measurements of the kurtosis, skewness, and genus of the gradients of the polarized synchrotron emission were recently applied to understand the Mach number of turbulence (Gaensler et al. 2011; Zhang et al. 2016;...
Burkhart et al. 2012). Other new techniques for obtaining the statistics of density have recently been developed using the analysis of moments of the density probabilities (Kowal et al. 2007; Burkhart et al. 2009, 2010). Tsallis statistics (Esquivel & Lazarian 2010; Tofflemire et al. 2011), bispectra, and genus (Lazarian 1999; Chepurnov et al. 2008; Burkhart et al. 2009). Using the new PFA technique, one can obtain the statistics of magnetic fields and Faraday rotation. This information is complementary for studies of turbulence.

Our work is intended to open an avenue toward analyzing the large number of exciting and upcoming radio data cubes from LOFAR and SKA, with the aim of applying the new technique to the study of the Milky Way and other galaxies.

6. SUMMARY

In this paper, we have used synthetic observations to test the analytical predictions of the new statistical techniques suggested in LP16. We found that our numerical results are in good agreement with the study of LP16. Based on 3D, 2D, and 1D synthetic observations, the numerical results we have obtained are briefly summarized as follows.

1. The studies of the dispersion polarization process of synchrotron radiation demonstrated that for the region dominated by stochastic RM fluctuations, corresponding to the ratio \( \eta \gg 1 \), the variance of polarization gives the universal spectrum \( \lambda^{-2} \). In a region with the dominant RM effect arising from the regular magnetic field \( \eta \lesssim 0.2 \), the variance of polarization follows \( \lambda^{-2-2m} \). Thus the variance reflects statistics of the magnetic field component perpendicular to the line of sight.

2. The studies of the dispersion of polarization derivative with regard to \( \lambda^2 \) show a power-law relation, which should reflect the fluctuation statistics of Faraday rotation.

3. The spectral index of relativistic electrons does not change the slope of the PFA measure, and thus does not prevent us from extracting the spectral properties of magnetic turbulence.

4. The PFA technique can be used practically, because we showed that the effects of angular resolution and inevitable observational noise do not present an obstacle to the recovery of the underlying spectra of turbulence.

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Zhang et al.

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