Strichartz estimates for Schrödinger equations with variable coefficients and potentials at most linear at spatial infinity

By Haruya Mizutani

(Received Oct. 25, 2011)

Abstract. In the present paper we consider Schrödinger equations with variable coefficients and potentials, where the principal part is a long-range perturbation of the flat Laplacian and potentials have at most linear growth at spatial infinity. We then prove local-in-time Strichartz estimates, outside a large compact set centered at origin, without loss of derivatives. Moreover we also prove global-in-space Strichartz estimates under the non-trapping condition on the Hamilton flow generated by the kinetic energy.

1. Introduction.

In this paper we study the so called (local-in-time) Strichartz estimates for the solutions to $d$-dimensional time-dependent Schrödinger equations

$$i\partial_t u(t) = Hu(t), \ t \in \mathbb{R}; \ u|_{t=0} = u_0 \in L^2(\mathbb{R}^d),$$

where $d \geq 1$ and $H$ is a Schrödinger operator with variable coefficients:

$$H = -\frac{1}{2} \sum_{j,k=1}^{d} \partial_{x_j} a^{jk}(x) \partial_{x_k} + V(x).$$

Throughout the paper we assume that $a^{jk}(x)$ and $V(x)$ are real-valued and smooth on $\mathbb{R}^d$, and $(a^{jk}(x))$ is a symmetric matrix satisfying $(a^{jk}(x)) \geq C \text{Id}$, $x \in \mathbb{R}^d$, with some $C > 0$. We also assume

**Assumption 1.** There exist constants $\mu, \nu \geq 0$ such that, for any $\alpha \in \mathbb{Z}_+^d$,

$$|\partial_x^\alpha (a^{jk}(x) - \delta_{jk})| \leq C_\alpha \langle x \rangle^{-\mu - |\alpha|}, \quad |\partial_x^\alpha V(x)| \leq C_\alpha \langle x \rangle^{2 - \nu - |\alpha|}, \quad x \in \mathbb{R}^d,$$

2010 Mathematics Subject Classification. Primary 35Q41; Secondary 81Q20.

Key Words and Phrases. Strichartz estimates, Schrödinger equation, unbounded potential.

This research was partly supported by GCOE ‘Fostering top leaders in mathematics’, Kyoto University.
with some $C_\alpha > 0$.

We may assume $\mu < 1$ and $\nu < 2$ without loss of generality. It is well known that $H$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ under Assumption 1, and we denote the unique self-adjoint extension on $L^2(\mathbb{R}^d)$ by the same symbol $H$. By the Stone theorem, the solution to (1.1) is given by $u(t) = e^{-itH}u_0$, where $e^{-itH}$ is a unique unitary group on $L^2(\mathbb{R}^d)$ generated by $H$ and called the propagator.

Let us recall the (global-in-time) Strichartz estimates for the free Schrödinger equation which state that

$$\|e^{it\Delta/2}u_0\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T\|u_0\|_{L^2(\mathbb{R}^d)}, \quad (1.2)$$

where $(p, q)$ satisfies the following admissible condition

$$2 \leq p, q \leq \infty, \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2}, \quad (d, p, q) \neq (2, 2, \infty). \quad (1.3)$$

For $d \geq 3$, $(p, q) = (2, 2d/(d-2))$ is called the endpoint. It is well known that these estimates are fundamental to study the local well-posedness of Cauchy problem of nonlinear Schrödinger equations (see, e.g., [6]). The estimates (1.2) were first proved by Strichartz [23] for a restricted pair of $(p, q)$ with $p = q = 2(d+2)/d$, and have been extensively generalized for $(p, q)$ satisfying (1.3) by [12], [15]. Moreover, in the flat case ($a^{jk} \equiv \delta_{jk}$), local-in-time Strichartz estimates

$$\|e^{itH}u_0\|_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T\|u_0\|_{L^2(\mathbb{R}^d)}, \quad (1.4)$$

have been extended to the case with potentials decaying at infinity [25] or increasing at most quadratically at infinity [26]. In particular, if $V(x)$ has at most quadratic growth at spatial infinity, i.e.,

$$V \in C^\infty(\mathbb{R}^d; \mathbb{R}), \quad |\partial_x^\alpha V(x)| \leq C_\alpha \text{ for } |\alpha| \geq 2,$$

then it was shown by Fujiwara [11] that the fundamental solution $E(t, x, y)$ of the propagator $e^{-itH}$ satisfies $|E(t, x, y)| \lesssim |t|^{-d/2}$ for all $x, y \in \mathbb{R}^d$ and $t \neq 0$ small enough. The estimates (1.4) are immediate consequences of this estimate and the $TT^*$-argument due to Ginibre-Velo [12] (see Keel-Tao [15] for the endpoint estimate). For the case with magnetic fields or singular potentials, we refer to Yajima [26], [27] and references therein.

On the other hand, local-in-time Strichartz estimates on manifolds have recently been proved by many authors under several conditions on the geometry.
Staffilani-Tataru [22], Robbiano-Zuily [18] and Bouclet-Tzvetkov [2] studied the case on the Euclidean space with the asymptotically flat metric under several settings. In particular, Bouclet-Tzvetkov [2] proved local-in-time Strichartz estimates without loss of derivatives under Assumption 1 with $\mu > 0$ and $\nu > 2$ and the non-trapping condition. Burq-Gérad-Tzvetkov [4] proved Strichartz estimates with a loss of derivative $1/p$ on any compact manifolds without boundaries. They also proved that the loss $1/p$ is optimal in the case of $M = S^d$. Hassell-Tao-Wunsch [13] and the author [17] considered the case of non-trapping asymptotically conic manifolds which are non-compact Riemannian manifolds with an asymptotically conic structure at infinity. Bouclet [1] studied the case of an asymptotically hyperbolic manifold. Burq-Guillarmou-Hassell [5] recently studied the case of asymptotically conic manifolds with hyperbolic trapped trajectories of sufficiently small fractal dimension. For global-in-time Strichartz estimates, we refer to [10], [8] and the references therein in the case with electromagnetic potentials, and to [3], [24], [16] in the case of Euclidean space with an asymptotically flat metric.

The main purpose of the paper is to handle a mixed case of above two situations. More precisely, we show that local-in-time Strichartz estimates for long-range perturbations still hold (without loss of derivatives) if we add unbounded potentials which have at most linear growth at spatial infinity (i.e., $\nu \geq 1$), at least excluding the endpoint $(p, q) = (2, 2d/(d-2))$. To the best knowledge of the author, our result may be a first example on the case where both of variable coefficients and unbounded potentials in the spatial variable $x$ are present.

To state the result, we recall the non-trapping condition. We denote by

$$H_0 = H - V = -\frac{1}{2} \sum_{j,k=1}^d \partial_{x_j} a^{jk}(x) \partial_{x_k}, \quad k(x, \xi) = \frac{1}{2} \sum_{j,k=1}^d a^{jk}(x) \xi_j \xi_k,$$

the principal part of $H$ and the kinetic energy, respectively, and also denote by $(y_0(t, x, \xi), \eta_0(t, x, \xi))$ the Hamilton flow generated by $k(x, \xi)$:

$$\dot{y}_0(t) = \partial_{\xi} k(y_0(t), \eta_0(t)), \quad \dot{\eta}_0(t) = -\partial_{x} k(y_0(t), \eta_0(t)); \quad (y_0(0), \eta_0(0)) = (x, \xi).$$

Note that the Hamiltonian vector field $H_k$, generated by $k$, is complete on $\mathbb{R}^{2d}$ since $(a^{jk})$ satisfies the uniform elliptic condition. Hence, $(y_0(t, x, \xi), \eta_0(t, x, \xi))$ exists for all $t \in \mathbb{R}$. We consider the following non-trapping condition:

For any $(x, \xi) \in T^*\mathbb{R}^d$ with $\xi \neq 0$, $|y_0(t, x, \xi)| \to +\infty$ as $t \to \pm \infty$. \hfill (1.5)

We now state our main result.
Theorem 1.1. \( (i) \) Suppose that \( H \) satisfies Assumption 1 with \( \mu > 0 \) and \( \nu \geq 1 \). Then, there exist \( R_0 > 0 \) large enough and \( \chi_0 \in C_0^\infty(\mathbb{R}^d) \) with \( \chi_0(x) = 1 \) for \( |x| < R_0 \) such that, for any \( T > 0 \) and \( (p, q) \) satisfying (1.3) and \( p \neq 2 \), there exists \( C_T > 0 \) such that

\[
\| (1 - \chi_0) e^{-itH} u_0 \|_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T \| u_0 \|_{L^2(\mathbb{R}^d)}. \tag{1.6}
\]

\( (ii) \) Suppose that \( H \) satisfies Assumption 1 with \( \mu, \nu \geq 0 \) and \( k(x, \xi) \) satisfies the non-trapping condition (1.5). Then, for any \( \chi \in C_0^\infty(\mathbb{R}^d) \), \( T > 0 \) and \( (p, q) \) satisfying (1.3) and \( p \neq 2 \), we have

\[
\| \chi e^{-itH} u_0 \|_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T \| u_0 \|_{L^2(\mathbb{R}^d)}. \tag{1.7}
\]

Moreover, combining with (1.6), we obtain global-in-space estimates

\[
\| e^{-itH} u_0 \|_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T \| u_0 \|_{L^2(\mathbb{R}^d)},
\]

provided that \( \mu > 0 \) and \( \nu \geq 1 \).

We here display the outline of the paper and explain the idea of the proof of Theorem 1.1. By the virtue of the Littlewood-Paley theory in terms of \( H_0 \), the proof of (1.6) can be reduced to that of following semi-classical Strichartz estimates:

\[
\| (1 - \chi_0) \psi(h^2H_0)e^{-itH} u_0 \|_{L^p([-T,T];L^q(\mathbb{R}^d))} \leq C_T \| u_0 \|_{L^2(\mathbb{R}^d)}, \quad 0 < h \ll 1,
\]

where \( \psi \in C_0^\infty(\mathbb{R}) \) with \( \text{supp} \psi \subset (0, \infty) \) and \( C_T > 0 \) is independent of \( h \). Moreover, there exists a smooth function \( a \in C^\infty(\mathbb{R}^{2d}) \) supported in a neighborhood of the support of \( (1 - \chi_0) \psi \circ k \) such that \( (1 - \chi_0) \varphi(h^2H_0) \) can be replaced with the semi-classical pseudodifferential operator \( a(x, hD) \). In Section 2, we collect some known results on the semi-classical pseudo-differential calculus and prove such a reduction to semi-classical estimates. Rescaling \( t \mapsto th \), we want to show dispersive estimates for \( e^{ithH} \) on a time scale of order \( h^{-1} \) to prove semi-classical Strichartz estimates. To prove dispersive estimates, we construct two kinds of parametrices, namely the Isozaki-Kitada and the WKB parametrices. Let \( a^\pm \in S(1, dx^2/\langle x \rangle^2 + d\xi^2/\langle \xi \rangle^2) \) be symbols supported in the following outgoing and incoming regions:

\[
\{ (x, \xi); |x| > R_0, \ |\xi|^2 \in J, \ \pm x \cdot \xi > -(1/2)|x||\xi| \},
\]
Strichartz estimates for Schrödinger equations

respective, where $J \Subset (0, \infty)$ is an open interval so that $\pi_\xi(\text{supp}\, \psi \circ k) \Subset J$ and $\pi_\xi$ is the projection onto the $\xi$-space. If $H$ is a long-range perturbation of $-(1/2)\Delta$, then the outgoing (resp. incoming) Isozaki-Kitada parametrix of $e^{-itH}a^\pm(x, hD)$ for $0 \leq t \leq h^{-1}$ (resp. $e^{-itH}a^-(x, hD)$ for $-h^{-1} \leq t \leq 0$) has been constructed by Robert [20] (see, also [2]). However, because of the unboundedness of $V$ with respect to $x$, it is difficult to construct such parametrices of $e^{-ithH}a^\pm(x, hD)$. To overcome this difficulty, we use a method due to Yajima-Zhang [29] as follows. We approximate $e^{-ithH}$ by $e^{-ithH_h}$, where $H_h = H - V + V_h$ and $V_h$ vanishes in the region $\{x; |x| \gg h^{-1}\}$. Suppose that $a^+$ (resp. $a^-$) is supported in the intersection of the outgoing (resp. incoming) region and $\{x; |x| < h^{-1}\}$. In Section 3, we construct the Isozaki-Kitada parametrix of $e^{-ithH_h}a^\pm(x, hD)$ for $0 \leq \pm t \leq h^{-1}$ and prove the following justification of the approximation: for any $N > 0$,

$$\sup_{0 \leq \pm t \leq h^{-1}} \left\| (e^{-ithH} - e^{-ithH_h})a^\pm(x, hD)f \right\|_{L^2} \leq C_N h^N \left\| f \right\|_{L^2}, \quad 0 < h \ll 1.$$ 

In Section 4, we discuss the WKB parametrix construction of $e^{-ithH}a(x, hD)$ on a time scale of order $h^{-1}$, where $a$ is supported in $\{(x, \xi); |x| > h^{-1}, |\xi|^2 \in I\}$. Such a parametrix construction is basically known for the potential perturbation case (see, e.g., [28]) and has been proved by the author for the case on asymptotically conic manifolds [17]. Combining these results studied in Sections 2, 3 and 4 with the Keel-Tao theorem [15], we prove semi-classical Strichartz estimates in Section 5. Section 5 is also devoted to the proof of (1.7). The proof of (1.7) heavily depends on local smoothing effects due to Doi [9] and the Christ-Kiselev lemma [7] and the method of the proof is similar as that in Robbiano-Zuily [18]. Appendix A is devoted to prove some technical inequalities on the Hamilton flow needed for constructing the WKB parametrix.

Throughout the paper we use the following notations. For $A, B \geq 0$, $A \lesssim B$ means that there exists some universal constant $C > 0$ such that $A \leq CB$. We denote the set of multi-indices by $\mathbb{Z}^d_+$. For Banach spaces $X$ and $Y$, $\mathcal{L}(X, Y)$ denotes the Banach space of bounded operators from $X$ to $Y$, and we write $\mathcal{L}(X) := \mathcal{L}(X, X)$.

Acknowledgements. The author would like to thank Professor Shu Nakamura for valuable discussions and comments. He also thanks the referee for careful reading the manuscript and for giving useful comments.

2. Reduction to semi-classical estimates.

In this section we show that the estimate (1.6) follows from semi-classical Strichartz estimates. We first record known results on the pseudo-differential
calculus and the $L^p$-functional calculus. For $a \in C^\infty(\mathbb{R}^{2d})$ and $h \in (0, 1]$, we denote the semi-classical pseudo-differential operator ($h$-PDO for short) by $a(x, hD_x)$:

$$a(x, hD_x)u(x) = (2\pi h)^{-d} \int e^{i(x-y) \cdot \xi / h} a(x, \xi) u(y) dyd\xi, \quad u \in \mathcal{S}(\mathbb{R}^d),$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class. For the metric $g = dx^2 / (x^2 + \xi^2 / \langle \xi \rangle^2)$ on $T^*\mathbb{R}^d$, we consider Hörmander’s symbol class $\mathcal{S}(m, g)$ with a weighted function $m$, namely we write $a \in \mathcal{S}(m, g)$ if $a \in C^\infty(\mathbb{R}^{2d})$ and

$$\left| \partial^\alpha_x \partial^\beta_\xi a(x, \xi) \right| \leq C_{\alpha \beta} m(x, \xi) \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|}, \quad x, \xi \in \mathbb{R}^d.$$

Let $a \in \mathcal{S}(m_1, g)$, $b \in \mathcal{S}(m_2, g)$. For any $N = 0, 1, 2, \ldots$, the symbol of the composition $a(x, hD)b(x, hD)$, denoted by $a\sharp b$, has an asymptotic expansion

$$a\sharp b(x, \xi) = \sum_{|\alpha| \leq N} \frac{h^{|\alpha|}}{i^{|\alpha|} \alpha!} \partial^\alpha_x a(x, \xi) \cdot \partial^\alpha_\xi b(x, \xi) + h^{N+1} r_N(x, \xi) \quad (2.1)$$

with some $r_N \in \mathcal{S}(\langle x \rangle^{-N-1} \langle \xi \rangle^{-N-1} m_1 m_2, g)$. For $a \in \mathcal{S}(1, g)$, $a(x, hD_x)$ is extended to a bounded operator on $L^2(\mathbb{R}^d)$. Moreover, if $a \in \mathcal{S}(\langle \xi \rangle^{-N}, g)$ for some $N > d$, then $a(x, hD)$ satisfies

$$\|a(x, hD)\|_{L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))} \leq C_{qr} h^{-d(1/q - 1/r)}, \quad 1 \leq q \leq r \leq \infty, \quad h \in (0, 1], \quad (2.2)$$

where $C_{qr} > 0$ is independent of $h$. We follow the argument in [2]. We denote by $A_h(x, y)$ the distribution kernel of $a(x, hD)$:

$$A_h(x, y) = (2\pi h)^{-d} \int e^{i(x-y) \cdot \xi / h} a(x, \xi) d\xi.$$ 

Since $|a(x, \xi)| \leq C \langle \xi \rangle^{-N}$ with $N > d$, this integral is absolutely convergent and we can write $A_h(x, y) = (2\pi)^{-d/2} h^{-d} \hat{a}(x, (y-x) / h)$, where $\hat{a}$ is the Fourier transform of $a$ with respect to the second variable. In particular, we have

$$\sup_{x,y} |A_h(x, y)| \leq Ch^{-d}$$

which implies (2.2) for $(q, r) = (1, \infty)$. Since $|\hat{a}(x, \eta)| \leq C_d \langle \eta \rangle^{-d-1}$ with $C_d > 0$ independent of $x$, a direct calculation yields
Strichartz estimates for Schrödinger equations

\[ \sup_x \int |A_h(x, y)| dy + \sup_y \int |A_h(x, y)| dx \leq C \]

for some \( C > 0 \) independent of \( h \). The Schur lemma then implies (2.2) for \( q = r \).

Finally, for arbitrarily fixed \( 1 \leq q \leq r \leq \infty \), we have the \( L(L^1, L^{r/q}) \) bound by an interpolation between the \( L(L^1) \) and \( L(L^1, L^\infty) \) bounds. Interpolating between the \( L(L^1, L^{r/q}) \) and \( L(L^\infty) \) bounds, we obtain the \( L(L^q, L^r) \) bound.

We next consider the \( L^p \)-functional calculus. The following lemma, which was proved by [2, Proposition 2.5], tells us that, for any \( \varphi \in C_0^\infty(\mathbb{R}) \) with \( \text{supp} \varphi \subset (0, \infty) \), \( \varphi(h^2H_0) \) can be approximated in terms of the \( h \)-PDO.

**Lemma 2.1.** Let \( \varphi \in C_0^\infty(\mathbb{R}) \), \( \text{supp} \varphi \subset (0, \infty) \) and \( N \geq 0 \) a non-negative integer. Then there exist symbols \( a_j \in S(1, g) \), \( j = 0, 1, \ldots, N \), such that

(i) \( a_0(x, \xi) = \varphi(k(x, \xi)) \) and \( a_j(x, \xi) \) are supported in the support of \( \varphi(k(x, \xi)) \).

(ii) For every \( 1 \leq q \leq r \leq \infty \) there exists \( C_{qr} > 0 \) such that

\[
\left\| a_j(x, hD_x) \right\|_{L(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))} \leq C_{qr} h^{-d(1/q - 1/r)},
\]

uniformly with respect to \( h \in (0, 1] \).

(iii) There exists a constant \( N_0 \geq 0 \) such that, for all \( 1 \leq q \leq r \leq \infty \),

\[
\left\| \varphi(h^2H_0) - a(x, hD_x) \right\|_{L(L^q(\mathbb{R}^d), L^r(\mathbb{R}^d))} \leq C_{Nqr} h^{N - N_0 - d(1/q - 1/r)}
\]

uniformly with respect to \( h \in (0, 1] \), where \( a = \sum_{j=0}^N h^j a_j \).

**Remark 2.2.** We note that Assumption 1 implies a stronger bounds on \( a_j \):

\[
|\partial_x^\alpha \partial_\xi^\beta a_j(x, \xi)| \leq C_{\alpha\beta}(x)^{-j - |\alpha| (\xi)^{-|\beta|},
\]

though we do not use this estimate in the following argument.

We next recall the Littlewood-Paley decomposition in terms of \( \varphi(h^2H_0) \). Consider a 4-adic partition of unity with respect to \([1, \infty)\):\n
\[
\sum_{j=0}^{\infty} \varphi(2^{-2j} \lambda) = 1, \quad \lambda \in [1, \infty),
\]

where \( \varphi \in C_0^\infty(\mathbb{R}) \) with \( \text{supp} \varphi \subset [1/4, 4] \) and \( 0 \leq \varphi \leq 1 \).
Lemma 2.3. Let \( \chi \in C_0^\infty(\mathbb{R}^d) \). Then, for \( q \in [2, \infty) \) with \( 0 \leq d(1/2 - 1/q) \leq 1 \),

\[
\| (1 - \chi) f \|_{L^q(\mathbb{R}^d)} \lesssim \| f \|_{L^2(\mathbb{R}^d)} + \left( \sum_{j=0}^{\infty} \left\| (1 - \chi) \varphi(2^{-2j}H_0)f \right\|_{L^q(\mathbb{R}^d)}^2 \right)^{1/2}.
\]

This lemma can be proved similarly to the case of the Laplace-Beltrami operator on compact manifolds without boundaries (see [4, Corollary 2.3]). By using this lemma, we have the following:

Proposition 2.4. Let \( \chi_0 \) be as that in Theorem 1.1. Suppose that there exist \( h_0, \delta > 0 \) small enough such that, for any \( \psi \in C_0^\infty((0, \infty)) \) and any admissible pair \((p, q)\) with \( p > 2 \),

\[
\| (1 - \chi_0) \psi(h^2H_0)e^{-itH}u_0 \|_{L^p([-\delta, \delta]; L^q(\mathbb{R}^d))} \leq C \| u_0 \|_{L^2(\mathbb{R}^d)},
\]

uniformly with respect to \( h \in (0, h_0] \). Then, the statement of Theorem 1.1 (i) holds.

Proof. By Lemma 2.3 with \( f = e^{-itH}u_0 \), the Minkowski inequality and the unitarity of \( e^{-itH} \) on \( L^2(\mathbb{R}^d) \), we have

\[
\| (1 - \chi_0)e^{-itH}u_0 \|_{L^p([-\delta, \delta]; L^q(\mathbb{R}^d))} \leq \| u_0 \|_{L^2(\mathbb{R}^d)} + \left( \sum_{j=0}^{\infty} \| (1 - \chi_0)\varphi(2^{-2j}H_0)e^{-itH}u_0 \|_{L^p([-\delta, \delta]; L^q(\mathbb{R}^d))}^2 \right)^{1/2}.
\]

For \( 0 \leq j \leq [-\log h_0] + 1 \), we have the bound

\[
\sum_{j=0}^{[-\log h_0] + 1} \| (1 - \chi_0)\varphi(2^{-2j}H_0)e^{-itH}u_0 \|_{L^p([-\delta, \delta]; L^q(\mathbb{R}^d))}^2 \lesssim \sum_{j=0}^{[-\log h_0] + 1} \| \varphi(2^{-2j}H_0) \|_{L^p(\mathbb{R}^d)} \| e^{-itH}u_0 \|_{L^\infty([-\delta, \delta]; L^2(\mathbb{R}^d))} \lesssim ([-\log h_0] + 1)2^{([-\log h_0] + 1)d(1/2 - 1/q)} \| u_0 \|_{L^2(\mathbb{R}^d)}.
\]

Choosing \( \psi \in C_0^\infty(\mathbb{R}) \) with \( \psi \equiv 1 \) on \( \text{supp} \varphi \), the Duhamel formula implies
Strichartz estimates for Schrödinger equations

\[ \varphi(h^2 H_0)e^{-itH} = \psi(h^2 H_0)e^{-itH}\varphi(h^2 H_0) + \psi(h^2 H_0)i \int_0^t e^{-i(t-s)H}[V, \varphi(h^2 H_0)]e^{-isH} ds =: \psi(h^2 H_0)e^{-itH}\varphi(h^2 H_0) + R(t, h). \]

Since \([H, \varphi(h^2 H_0)] = [V, \varphi(h^2 H_0)] = O(h)\) on \(L^2(\mathbb{R}^d)\), \(R(t, h)\) satisfies

\[ \sup_{0 \leq t \leq 1} \| R(t, h) \|_{L^q(\mathbb{R}^d)} \leq \| \psi(h^2 H_0) \|_{L^q(\mathbb{R}^d)} \| [V, \varphi(h^2 H_0)] \|_{L^q(\mathbb{R}^d)} \]

\[ \lesssim h^{-d(1/2 - 1/q) + 1}. \tag{2.4} \]

We here note that \(\gamma := -d(1/2 - 1/q) + 1 = -2/p + 1 > 0\) since \(p > 2\). By (2.3), (2.4) with \(h = 2^{-j}\) and the almost orthogonality of \(\text{supp}\ \varphi(2^{-2j} \cdot)\), we obtain

\[ \sum_{j = [-\log h_0]}^{\infty} \left\| (1 - \chi_0) \varphi(2^{-2j} H_0)e^{-itH} u_0 \right\|_{L^p([-\delta, \delta]; L^q(\mathbb{R}^d))} \lesssim \sum_{j = [-\log h_0]}^{\infty} \left( \| \varphi(2^{-2j} H_0) u_0 \|_{L^2(\mathbb{R}^d)}^2 + 2^{-2\gamma j} \| u_0 \|_{L^2(\mathbb{R}^d)}^2 \right) \lesssim \| u_0 \|_{L^2(\mathbb{R}^d)}^2. \]

Combining with the bound for \(0 \leq j \leq [-\log h_0] + 1\), we have

\[ \| (1 - \chi_0)e^{-itH} u_0 \|_{L^p([-\delta, \delta]; L^q(\mathbb{R}^d))} \lesssim \| u_0 \|_{L^2(\mathbb{R}^d)}. \]

Splitting the time interval \([-T, T]\) into \([T/\delta] + 1\) intervals with size \(2\delta\), we obtain

\[ \| (1 - \chi_0)\psi(h^2 H_0)e^{-itH} u_0 \|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \leq \sum_{k = [-T/\delta]}^{[T/\delta] + 1} \| (1 - \chi_0)\psi(h^2 H_0)e^{-i(k+1)H} u_0 \|_{L^p([-\delta, \delta]; L^q(\mathbb{R}^d))} \]

\[ \leq C_T \| u_0 \|_{L^2(\mathbb{R}^d)}. \]

In the last inequality, we used the unitarity of \(e^{-i(k+1)H}\) on \(L^2(\mathbb{R}^d)\). \(\square\)
3. Isozaki-Kitada parametrix.

In this section we assume Assumption 1 with $0 < \mu = \nu < 1/2$ without loss of generality, and construct the Isozaki-Kitada parametrix. Since the potential $V$ can grow at infinity, it is difficult to construct directly the Isozaki-Kitada parametrix for $e^{-itH}$ even though we restrict it in an outgoing or incoming region. To overcome this difficulty, we approximate $e^{-itH}$ as follows. Let $\rho \in C_0^\infty(\mathbb{R}^d)$ be a cut-off function such that $\rho(x) = 1$ if $|x| \leq 1$ and $\rho(x) = 0$ if $|x| \geq 2$. For a small constant $\varepsilon > 0$ and $h \in (0, 1]$, we define $H_h$ by

$$H_h = H_0 + V_h, \quad V_h = V(x)\rho(\varepsilon hx).$$

We note that, for any fixed $\varepsilon > 0$,

$$h^2|\partial_x^\alpha V_h(x)| \leq C_{\varepsilon, \alpha} h^2 \langle x \rangle^{2-\mu-|\alpha|} \leq C_{\varepsilon, \alpha} \langle x \rangle^{-\mu-|\alpha|}, \quad x \in \mathbb{R}^d,$$

where $C_{\varepsilon, \alpha}$ may be taken uniformly with respect to $h \in (0, 1]$. Such a type modification has been used to prove Strichartz estimates and local smoothing effects for Schrödinger equations with super-quadratic potentials (see, Yajima-Zhang [29, Section 4]).

For $R > 0$, an open interval $J \Subset (0, \infty)$ and $-1 < \sigma < 1$, we define the outgoing and incoming regions by

$$\Gamma^\pm(R, J, \sigma) := \{(x, \xi) \in \mathbb{R}^{2d}; |x| > R, |\xi| \in J, \pm \frac{x \cdot \xi}{|x||\xi|} > -\sigma\},$$

respectively. Since $H_0 + h^2 V_h$ is a long-range perturbation of $-\Delta/2$, we have the following theorem due to Robert [20] and Bouclet-Tzvetkov [2].

**Theorem 3.1.** Let $J, J_0, J_1$ and $J_2$ be relatively compact open intervals, $\sigma, \sigma_0, \sigma_1$ and $\sigma_2$ real numbers so that $J \Subset J_0 \Subset J_1 \Subset J_2 \Subset (0, \infty)$ and $-1 < \sigma < \sigma_0 < \sigma_1 < \sigma_2 < 1$. Fix arbitrarily $\varepsilon > 0$. Then there exist $R_0 > 0$ large enough and $h_0 > 0$ small enough such that the followings hold.

(i) There exist two families of smooth functions

$$\{S^+_h; h \in (0, h_0], R \geq R_0\}, \quad \{S^-_h; h \in (0, h_0], R \geq R_0\} \subset C^\infty(\mathbb{R}^{2d}; \mathbb{R})$$

satisfying the Eikonal equation associated to $k + h^2 V_h$:
Strichartz estimates for Schrödinger equations

\[ k(x, \partial_x S_h^\pm(x, \xi)) + h^2 V_h(x) = \frac{1}{2} |\xi|^2, \quad (x, \xi) \in \Gamma^\pm(R^{1/4}, J_2, \sigma_2), \quad h \in (0, h_0], \]

respectively, such that

\[ |\partial_\xi^\alpha \partial_\xi^\beta (S_h^\pm(x, \xi) - x \cdot \xi)| \leq C_{\alpha \beta} |x|^{1-\mu-|\alpha|}, \quad \alpha, \beta \in \mathbb{Z}_+^d, \quad x, \xi \in \mathbb{R}^d, \quad (3.1) \]

where \( C_{\alpha \beta} > 0 \) may be taken uniformly with respect to \( R \) and \( h \).

(ii) For every \( R \geq R_0, h \in (0, h_0] \) and \( N = 0, 1, \ldots \), we can find

\[ b^\pm_h = \sum_{j=0}^{N} h^j b^\pm_{h,j} \quad \text{with} \quad b^\pm_{h,j} \in S(1, g), \quad \text{supp} b^\pm_{h,j} \subset \Gamma^\pm(R^{1/3}, J_1, \sigma_1), \]

such that, for every \( a^\pm \in S(1, g) \) with \( \text{supp} a^\pm \subset \Gamma^\pm(R, J, \sigma) \), there exist

\[ c^\pm_h = \sum_{j=0}^{N} h^j c^\pm_{h,j} \quad \text{with} \quad c^\pm_{h,j} \in S(1, g), \quad \text{supp} c^\pm_{h,j} \subset \Gamma^\pm(R^{1/2}, J_0, \sigma_0), \]

such that, for all \( \pm t \geq 0 \),

\[ e^{-ithH_h} a^\pm(x, hD) = U(S_h^\pm, b^\pm_h)e^{ith\Delta/2}U(S_h^\pm, c^\pm_h)^* + Q^\pm_{IK}(t, h, N), \]

respectively, where \( U(S_h^\pm, w) \) are Fourier integral operators, with the phases \( S_h^\pm \) and the amplitude \( w \), defined by

\[ U(S_h^\pm, w)f(x) = \frac{1}{(2\pi h)^d} \int e^{i(S_h^\pm(x, \xi) - y \cdot \xi)/h} w(x, \xi)f(y)dyd\xi, \]

respectively. Moreover, for any \( s = 0, 1, 2, \ldots \), there exists \( C_{N,s} > 0 \) such that

\[ \| (h^2H_h + L)^s Q^\pm_{IK}(t, h, N) \|_{L^2(\mathbb{R}^d)} \leq C_{N,s}h^{N-1} \quad (3.2) \]

uniformly with respect to \( h \in (0, h_0] \) and \( 0 \leq \pm t \leq h^{-1} \), where \( L > 1 \), independent of \( h, t \) and \( x \), is a large constant so that \( h^2V_h + L \geq 1 \).

(iii) The distribution kernels \( K^\pm_{IK}(t, h, x, y) \) of \( U(S_h^\pm, b^\pm_h)e^{-ith\Delta/2}U(S_h^\pm, c^\pm_h)^* \) satisfy dispersive estimates:

\[ |K^\pm_{IK}(t, h, x, y)| \leq C|th|^{-d/2}, \quad 0 \leq \pm t \leq h^{-1}, \quad (3.3) \]
respectively, where $C > 0$ is independent of $h \in (0, h_0]$, $0 \leq t \leq h^{-1}$ and $x, \xi \in \mathbb{R}^d$.

**Proof.** This theorem is basically known, and we only check (3.2) for the outgoing case. For the detail of the proof, we refer to [20, Section 4] and [2, Section 3]. We also refer to the original paper by Isozaki-Kitada [14].

The remainder $Q_{IK}^+(t, h, N)$ consists of the following three parts:

\[- h^{N+1} e^{-ithH} q_1(h, x, hD),
\]

\[- ih^N \int_0^t e^{-i(t-\tau)hH} U^+ (S^+ h q_2(h)) e^{ir\Delta/2} U^+ (S^+ h, c^+_h)^* d\tau,
\]

\[- (i/h) \int_0^t e^{-i(t-\tau)hH} \tilde{Q}(\tau, h) d\tau,
\]

where $\{q_1(h, \cdot, \cdot), q_2(h, \cdot, \cdot); h \in (0, h_0]\} \subset \bigcap_{M=1}^{\infty} S((x)^{-N} (\xi)^{-M}, g)$ is a bounded set, and $Q(s, h)$ is an integral operator with a kernel $\tilde{q}(s, h, x, y)$ satisfying

\[|\partial_x^\alpha \partial_\xi^\beta \tilde{q}(\tau, h, x, y)| \leq C_{\alpha\beta} h^{M-|\alpha+\beta|} (1 + |\tau| + |x| + |y|)^{-M+|\alpha+\beta|}, \quad \tau \geq 0,
\]

for any $M \geq 0$. A standard $L^2$-boundedness of $h$-PDO and FIO then imply

\[\|(h^2 H_0 + 1)^s (q_1(h, x, hD) + U^+ (S^+ h, q_2(h)))\|_{L(L^2(\mathbb{R}^d))} \leq C_s,
\]

and a direct computation yields

\[\|(h^2 H_0 + 1)^s \tilde{Q}(\tau, h)\|_{L(L^2(\mathbb{R}^d))} \leq C_M h^M.
\]

On the other hand, we choose a constant $L > 0$ so large that $h^2 V_h + L \geq 1$. Since $h^2 V_h + L \lesssim 1$ by the definition of $V_h$, we have

\[\|(h^2 H_0 + L)^s (h^2 H_0 + 1)^{-s}\|_{L(L^2(\mathbb{R}^d))} \leq C_s, \quad s = 1, 2, \ldots.
\]

Then (3.2) follows from the above three estimates since $(h^2 H_0 + L)^s$ commutes with $e^{-ithH}$.

The following key lemma tells us that one can still construct the Isozaki-Kitada parametrix of the original propagator $e^{-ithH}$ if we restrict the support of initial data in the region $\{x; |x| < h^{-1}\}$. \qed
Lemma 3.2. Suppose that \( \{a^\pm_h\}_{h \in (0,1]} \) are bounded sets in \( S(1,g) \) and satisfy
\[
\text{supp } a^\pm_h \subset \Gamma^\pm (R,J,\sigma) \cap \{ x; |x| < h^{-1} \},
\]
respectively. Then for any \( M \geq 0, h \in (0,h_0] \) and \( 0 \leq \pm t \leq h^{-1} \), we have
\[
\left\| (e^{-ithH} - e^{-ithH_h})a^\pm_h(x,hD) \right\|_{L(L^2(\mathbb{R}^d))} \leq C_M h^M,
\]
where \( C_M > 0 \) is independent of \( h \) and \( t \).

Proof. We prove the lemma for the outgoing case only, and the proof of incoming case is completely analogous. We set \( A = a^+_h(x,hD) \) and \( W_h = V - V_h \). The Duhamel formula yields
\[
(e^{-ithH} - e^{-ithH_h})A
= -ih \int_0^t e^{-i(t-s)hH} W_h e^{-ishH_h} Ads
= -ih \int_0^t e^{-i(t-s)hH} e^{-ishH_h} W_h Ads
- h^2 \int_0^t e^{-i(t-s)hH} \int_0^s e^{-i(s-\tau)hH_h} [H_0, W_h] e^{-i\tau hH_h} Ad\tau ds.
\]
Since \( \text{supp } a^+_h(\cdot,\xi) \subset \{ x; |x| < h^{-1} \} \), we learn \( \text{supp } W_h \cap a^+_h(\cdot,\xi) = \emptyset \) if \( \varepsilon < 1 \). Combining with the asymptotic formula (2.1), we see that this support property implies
\[
\left\| W_h A \right\|_{L(L^2(\mathbb{R}^d))} \leq C_M h^M
\]
for any \( M \geq 0 \). A direct computation yields that \([H_0, W_h]\) is of the form
\[
\sum_{|\alpha|=0,1} a_\alpha(x) \partial^\alpha_x, \quad \text{supp } a_\alpha \subset \text{supp } W_h, \quad |\partial^\beta_x a_\alpha(x)| \leq C_{\alpha\beta} \langle x \rangle^{-\mu + |\alpha| - |\beta|}.
\]
The support properties of \( W_h \) and \( a^+_h \) again imply
\[
\left\| [H_0, W_h] A \right\|_{L(L^2(\mathbb{R}^d))} \leq C_M h^M \quad \text{for any } M \geq 0.
\]
We next consider \([H_h, [K, W_h]]\) which has the form
\[
\sum_{|\alpha|=1,2} b_\alpha(x) \partial^\alpha_x + W_1(x),
\]
where \(b_\alpha\) and \(W_1\) are supported in supp \(W_h\) and satisfy
\[
|\partial^\beta b_\alpha(x)| \leq C_{\alpha\beta} \langle x \rangle^{-2-\mu + |\alpha| - |\beta|}, \quad |\partial^\beta W_1(x)| \leq C_{\alpha\beta} \langle x \rangle^{2-2\mu}.
\]
Setting \(I_1 = \sum_{|\alpha|=1,2} b_\alpha(x) \partial^\alpha_x\) and \(N_\mu := [1/\mu] + 1\), we iterate this procedure \(N_\mu\) times with \(W_h\) replaced by \(W_1\). Then can be brought to a linear combination of the following forms (modulo \(O(h^M)\) on \(L^2(\mathbb{R}^d)\)):
\[
\int_{t \geq s_1 \geq \cdots \geq s_j \geq 0} e^{-i(t-s_1)hH} e^{-i(s_1-s_j)hH} I_{j/2} e^{-i s_j hH} Ad s_j \cdots d s_1
\]
for \(j = 2m, m = 1, 2, \ldots, N_\mu\), and
\[
\int_{t \geq s_1 \geq \cdots \geq s_{N_\mu} \geq 0} e^{-i(t-s_1)hH} e^{-i(s_1-s_{N_\mu})hH} W_{N_\mu} e^{-i s_{N_\mu} hH} A d s_{2N_\mu} \cdots d s_1,
\]
where \(I_k\) are second order differential operators with smooth and bounded coefficients, and \(W_{N_\mu}\) is a bounded function since \(2-2\mu N_\mu < 0\). Moreover, they are supported in \(\{x; |x| > (\varepsilon h)^{-1}\}\). Therefore, it is sufficient to show that, for any \(h \in (0, h_0]\), \(0 \leq \tau \leq h^{-1}\), \(\alpha \in \mathbb{Z}^d\) and \(M \geq 0\),
\[
\| (1 - \rho(\varepsilon hx)) \partial^\alpha_x e^{-i \tau hH} A \|_{L^2(\mathbb{R}^d)} \leq C_{M,\alpha} h^{M-|\alpha|}. \tag{3.4}
\]
We now apply Theorem 3.1 to \(e^{-i \tau hH} A\) and obtain
\[
e^{-i \tau hH} A = U(S^+_h, b^+_h) e^{i \tau h \Delta/2} U(S^+_h, c^+_h) + Q^+_h(t, h, N).
\]
Recall that the elliptic nature of \(H_0\) implies, for every \(s \geq 0\),
\[
\| (D)^s (h^2 H_0 + 1)^{-s/2} f \|_{L^2(\mathbb{R}^d)} \leq C h^{-s} \| f \|_{L^2(\mathbb{R}^d)},
\]
\[
\| (h^2 H_0 + 1)^{s/2} (h^2 H_h + L)^{-s/2} f \|_{L^2(\mathbb{R}^d)} \leq C \| f \|_{L^2(\mathbb{R}^d)},
\]
if \(L > 0\) so large that \(h^2 H_h + L \geq 1\). Combining these estimates with (3.2), the
Strichartz estimates for Schrödinger equations

The main term can be handled in terms of the non-stationary phase method as follows. The distribution kernel of the main term is given by

\[
(2\pi h)^{d} (1 - \rho(\varepsilon hx)) \frac{\partial^{\alpha}}{\partial \alpha} \int e^{i\Phi_{h}^{+}(\tau, x, y, \xi)/h} b_{h}^{+}(x, \xi) c_{h}^{+}(y, \xi) d\xi, \tag{3.5}
\]

where \( \Phi_{h}^{+}(\tau, x, y, \xi) = S_{h}^{+}(x, \xi) - (1/2)\tau|\xi|^{2} - S_{h}^{+}(y, \xi) \). We here claim that

\[
\text{supp } c_{h}^{+} \subset \{ (x, \xi) \in \mathbb{R}^{2d} ; a_{h}^{+}(x, \partial_{\xi} S_{h}^{+}(x, \xi)) \neq 0 \} \tag{3.6}
\]

This property follows from the construction of \( c_{h}^{+} = \sum_{j=0}^{N} h^{j} c_{h,j}^{+} \). We set

\[
\widetilde{S}_{h}^{+}(x, y, \xi) = \int_{0}^{1} \partial_{x} S_{h}^{+}(y + \theta(x - y), \xi) d\theta.
\]

Let \( \xi \mapsto [\widetilde{S}_{h}^{+}]^{-1}(x, y, \xi) \) be the inverse map of \( \xi \mapsto \widetilde{S}_{h}^{+}(x, y, \xi) \), and we denote their Jacobians by \( A_{1} = |\det \partial_{\xi} \widetilde{S}_{h}^{+}(x, y, \xi)| \) and \( A_{2} = |\det \partial_{\xi} [\widetilde{S}_{h}^{+}]^{-1}(x, y, \xi)| \). \( c_{h,j}^{+} \) then satisfy the following triangular system:

\[
c_{h,j}^{+}(x, \xi) = b_{h,0}^{+}(x, \xi)^{-1} \left( r_{h,j}^{+}(x, \widetilde{S}_{h}^{+}(x, y, \xi)) A_{1} \right) \bigg|_{y=x}, \quad j = 0, 1, \ldots, N,
\]

where \( r_{h,0}^{+} = a_{h}^{+}(x, \widetilde{S}_{h}^{+}(x, y, \xi)) \) and, for each \( j \geq 1 \), \( r_{h,j}^{+} \) is a linear combination of

\[
\frac{1}{i^{\alpha} \alpha!} \left( \partial_{\xi}^{\alpha} \partial_{y}^{\alpha} b_{h,k_{0}}^{+}(x, [\widetilde{S}_{h}^{+}]^{-1}(x, y, \xi)) c_{h,k_{1}}^{+}(y, [\widetilde{S}_{h}^{+}]^{-1}(x, y, \xi)) A_{2} \right) \bigg|_{y=x},
\]

where \( \alpha \in \mathbb{Z}_{+}^{d} \) and \( k_{0}, k_{1} = 0, 1, \ldots, j \) so that \( 0 \leq |\alpha| \leq j \), \( k_{0} + k_{1} = j - |\alpha| \) and \( k_{1} \leq j - 1 \). Therefore, we inductively obtain

\[
\text{supp } c_{h,0}^{+} \subset \text{supp } r_{0}^{+}|_{y=x}, \quad \text{supp } c_{h,j}^{+} \subset \text{supp } c_{h,j-1}^{+}(h), \quad j = 1, 2, \ldots, N,
\]

and (3.6) follows. In particular, \( c_{h}^{+} \) vanishes in the region \( \{ x; |x| \geq h^{-1} \} \). By using (3.1), we have
\[ \partial_\xi \Phi_+^+(\tau, x, y, \xi) = (x - y)(\text{Id} + O(R^{-\mu/3})) - \tau \xi, \]

which implies

\[ |\partial_\xi \Phi_+^+(\tau, x, y, \xi)| \geq \frac{|x|}{2} - |y| - |\tau \xi| \]
as long as \( R \geq 1 \) large enough. We now set \( \varepsilon = (2\sqrt{\sup J_2} + 2)^{-1} \). Since \( |x| > (\varepsilon h)^{-1} \), \( |y| < h^{-1} \) and \( |\xi|^2 \in J_2 \) on the support of the amplitude, we have

\[ |\partial_\xi \Phi_+^+(\tau, x, y, \xi)| \gtrsim (|x| + h^{-1}) > c(1 + |x| + |y| + |\tau|), \quad 0 \leq \tau \leq h^{-1}, \]

for some \( c > 0 \) independent of \( h \). Therefore, integrating by parts (3.5) with respect to \(-ih|\partial_\xi \Phi_+^+|^2(\partial_\xi \Phi_+^+ \cdot \partial_\xi\), we obtain

\[ \left| (2\pi h)^{-d} (1 - \rho(\varepsilon hx)) \partial_x^\alpha \partial_y^\beta \int e^{i \Phi_+^+(\tau, x, y, \xi)/h} b_+^+(x, \xi) c_0(x, \xi) d\xi \right| \]

\[ \leq C_{\alpha, \beta, M} h^{M-|\alpha+\beta|}(1 + |x| + |y| + |\tau|)^{-M}, \]

for all \( M \geq 0 \), \( 0 \leq \tau \leq h^{-1} \) and \( \alpha, \beta \in \mathbb{Z}^d_+ \). (3.4) follows from this inequality and the \( L^2 \)-boundedness of FIOs. \( \square \)

4. **WKB parametrix.**

In the previous section we proved that \( e^{-ithH} \) is well approximated in terms of an Isozaki-Kitada parametrix on a time scale of order \( h^{-1} \) if we localize the initial data in regions \( \Gamma^\pm(R, J, \sigma) \cap \{x; R < |x| < h^{-1}\} \). Therefore, it remains to control \( e^{-ithH} \) on a region \( \{x; |x| \gtrsim h^{-1}\} \). In this section we construct the WKB parametrix for \( e^{-ithH}a(x, hD) \), where \( a \in S(1, g) \) with \( \text{supp} a \subset \{(x, \xi) \in \mathbb{R}^{2d}; |x| \gtrsim h^{-1}, |\xi|^2 \in J\} \). In what follows we assume that \( H \) satisfies Assumption 1 with \( \mu \geq 0 \) and \( \nu = 1 \).

We first consider the phase function of the WKB parametrix, that is a solution to the time-dependent Hamilton-Jacobi equation generated by \( p_h(x, \xi) = k(x, \xi) + h^2V(x) \). For \( R > 0 \) and an open interval \( J \Subset (0, \infty) \), we set

\[ \Omega(R, J) := \{(x, \xi) \in \mathbb{R}^{2d}; |x| > R/2, |\xi|^2 \in J\}. \]

We note that \( \Omega(R_1, J_1) \subset \Omega(R_2, J_2) \) if \( R_1 > R_2 \) and \( J_1 \subset J_2 \).
Proposition 4.1. Choose arbitrarily an open interval \( J \subset (0, \infty) \). Then, there exist \( \delta_0 > 0 \) and \( h_0 > 0 \) small enough such that, for all \( h \in (0, h_0] \), \( 0 < R \leq h^{-1} \) and \( 0 < \delta \leq \delta_0 \), we can construct a family of smooth functions 
\[
\{ \Psi_h(t, x, \xi) \}_{h \in (0, h_0]} \subset C^\infty((−R, R) \times \mathbb{R}^{2d})
\]
such that \( \Psi_h(t, x, \xi) \) satisfies the Hamilton-Jacobi equation associated to \( p_h \):
\[
\begin{aligned}
\partial_t \Psi_h(t, x, \xi) &= -p_h(x, \partial_x \Psi_h(t, x, \xi)), \quad 0 < |t| < \delta R, \quad (x, \xi) \in \Omega(R, J), \\
\Psi_h(0, x, \xi) &= x \cdot \xi, \quad (x, \xi) \in \Omega(R, J).
\end{aligned}
\] (4.1)

Moreover, for all \( |t| \leq \delta R \) and \( \alpha, \beta \in \mathbb{Z}^d_+ \), \( \Psi_h(t, x, \xi) \) satisfies
\[
|\partial_x^\alpha \partial_\xi^\beta (\Psi_h(t, x, \xi) - x \cdot \xi)| \leq C_\delta R^{1-|\alpha|}, \quad x, \xi \in \mathbb{R}^d, \quad |\alpha + \beta| \geq 2,
\] (4.2)
\[
|\partial_x^\alpha \partial_\xi^\beta (\Psi_h(t, x, \xi) - x \cdot \xi + tp_h(x, \xi))| \leq C_{\alpha \beta} \delta R^{-|\alpha|}|t|, \quad x, \xi \in \mathbb{R}^d.
\] (4.3)

Proof. We give the proof in Appendix A. \( \square \)

We next define the corresponding FIO. Let \( 0 < R \leq h^{-1}, J \subset J_1 \subset (0, \infty) \) open intervals and \( \Psi_h \) defined by the previous proposition with \( R, J \) replaced by \( R/4, J_1 \), respectively. We suppose that \( \{a_h(t, \cdot, \cdot)\}_{h \in (0, h_0]} \) is bounded in \( S(1, g) \) and supported in \( \Omega(R, J) \), and consider the time-dependent FIO with the phase \( \Psi_h(t) \) and amplitude \( a_h(t) \), namely
\[
U(\Psi_h(t), a_h(t))u(x) = \frac{1}{(2\pi h)^d} \int e^{i(\Psi_h(t, x, \xi) - y \cdot \xi)/h} a_h(t, x, \xi)u(y)dyd\xi.
\]

Lemma 4.2. Let \( \Psi_h(t) \) and \( a_h(t) \) be as above. \( U(\Psi_h(t), a(t)) \) then is bounded on \( L^2(\mathbb{R}^d) \) uniformly with respect to \( R, h \) and \( t \):
\[
\sup_{t \in (0, h_0], 0 \leq t \leq \delta R} \|U(\Psi_h(t), a(t))\|_{L(L^2(\mathbb{R}^d))} \leq C.
\]

Proof. For \( |t| \leq \delta R \), we define the map \( \Xi(t, x, y, \xi) \) on \( \mathbb{R}^{3d} \) by
\[
\Xi(t, x, y, \xi) = \int_0^1 (\partial_x \Psi_h)(t, y + \lambda(x - y), \xi)d\lambda.
\]
By (4.2), \( \Xi(t, x, y, \xi) \) satisfies
\[ |\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma (\Xi(t, x, y, \xi) - \xi)| \leq C_{\alpha\beta\gamma} \delta R^{-|\alpha + \beta|}, \quad |t| \leq \delta R, \ x, y \in \mathbb{R}^d, \]

and the map \( \xi \mapsto \Xi(t, x, y, \xi) \) hence is a diffeomorphism from \( \mathbb{R}^d \) onto itself for all \( |t| \leq \delta R \) and \( x, y \in \mathbb{R}^d \), provided that \( \delta > 0 \) is small enough. Let \( \xi \mapsto (\Xi)^{-1}(t, x, y, \xi) \) be the corresponding inverse. \( (\Xi)^{-1} \) satisfies the same estimate as that for \( \Xi \):

\[ |\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma ((\Xi)^{-1}(t, x, y, \xi) - \xi)| \leq C_{\alpha\beta\gamma} \delta R^{-|\alpha + \beta|} \quad \text{on} \quad [-\delta R, \delta R] \times \mathbb{R}^d. \]

Using the change of variables \( \xi \mapsto (\Xi)^{-1}, U(\Psi_h(t), a(t))U(\Psi_h(t), a(t))^* \) can be regarded as a semi-classical PDO with a smooth and bounded amplitude

\[ a_h(t, x, (\Xi)^{-1}(t, x, y, \xi)) a_h(t, y, (\Xi)^{-1}(t, x, y, \xi)) |\det_\xi (\Xi)^{-1}(t, x, y, \xi)|. \]

Therefore, the \( L^2 \)-boundedness follows from the Calderón-Vaillancourt theorem. \( \square \)

We now state the main result in this section.

**Theorem 4.3.** Let \( J \supset J_0 \supset J_1 \subset (0, \infty) \) be open intervals. Then there exist \( \delta_0, h_0 > 0 \) small enough such that, for all \( h \in (0, h_0], 0 < R \leq h^{-1}, 0 < \delta \leq \delta_0, N \geq 0 \) and all symbol \( a \in S(1, g) \) with \( \text{supp} \ a \in \Omega(R, J) \), we can find a semi-classical symbol \( b_h(t, x, \xi) = \sum_{j=0}^N h^j b_{h,j}(t, x, \xi) \) with

\[ \{b_{h,j}(t, \cdot, \cdot); h \in (0, h_0], 0 < R \leq h^{-1}, |t| \leq \delta R \} \subset S(1, g) \]

and \( \text{supp} \ b_{h,j}(t, \cdot, \cdot) \subset \Omega(R/2, J_0) \) uniformly with respect to \( h \in (0, h_0] \) and \( |t| \leq \delta R \), such that \( e^{-ithH} a(x, hD_x) \) can be brought to the form

\[ e^{-ithH} a(x, hD_x) = U(\Psi_h(t), b_h(t)) + Q_{WKB}(t, h, N), \]

where \( U(\Psi_h(t), b_h(t)) \) is the Fourier integral operator with the phase function \( \Psi_h(t, x, \xi) \), defined in Proposition 4.1 with \( R, J \) replaced by \( R/4, J_1 \), respectively, and its distribution kernel satisfies the following bounds:

\[ |K_{WKB}(t, h, x, y)| \leq C |th|^{-d/2}, \quad h \in (0, h_0], \ 0 < |t| \leq \delta R, \ x, \xi \in \mathbb{R}^d. \quad (4.4) \]

Moreover the remainder \( Q_{WKB}(t, h, N) \) satisfies
\|Q_{\text{WKB}}(t, h, N)\|_{L^2(\mathbb{R}^d)} \leq C_N h^N |t|, \quad h \in (0, h_0), \ |t| \leq \delta R.

Here the constants $C, C_N > 0$ can be taken uniformly with respect to $h, t$ and $R$.

**Remark 4.4.** The essential point of Theorem 4.3 is to construct the parametrix on the time interval $|t| \leq \delta R$. When $|t| > 0$ is small and independent of $R$, such a parametrix construction is basically well known (see, e.g., [19]).

**Proof of Theorem 4.3.** We consider the case when $t \geq 0$ and the proof for $t < 0$ is similar.

**Construction of the amplitude.** The Duhamel formula yields

$$e^{-ithH}U(\Psi_h(0), b_h(0)) = U(\Psi_h(t), b_h(t)) + \frac{i}{h} \int_0^t e^{-i(t-s)hH}(hD_s + h^2H)U(\Psi_h(s), b_h(s))ds.$$ 

Therefore, it suffices to show that there exist $b_{h,j}$ with $b_{h,0}|_{t=0} = a$ and $b_{h,j}|_{t=0} = 0$ for $j \geq 1$ such that

$$\|(hD_s + h^2H)U(\Psi_h(s), b_h(s))\|_{L^2} \leq C_N h^{N+1}, \quad 0 \leq s \leq \delta R. \quad (4.5)$$

Let $k + k_1$ be the full symbol of $H_0$: $H_0 = k(x, D) + k_1(x, D)$, and define a smooth vector field $X_h(t)$ and a function $Y_h(t)$ by

$$X_h(t, x, \xi) := (\partial_\xi k)(x, \partial_\xi \Psi_h(t, x, \xi)), \quad Y_h(t, x, \xi) := -(H_0\Psi_h)(t, x, \xi).$$

Symbols $\{b_{h,j}\}$ can be constructed in terms of the method of characteristics as follows. For all $0 \leq s, t \leq \delta R$, we consider the flow $z_h(t, s, x, \xi)$ generated by $X_h(t)$, that is the solution to the following ODE:

$$\partial_t z_h(t, s, x, \xi) = X_h(z_h(t, s, x, \xi), \xi); \quad z_h(s, s) = x.$$ 

Choose $R', R''$ and two intervals $J'_0, J''_0$ so that

$$R/2 > R' > R'' > R/4, \quad J_0 \subseteq J'_0 \subseteq J''_0 \subseteq (0, \infty).$$

(4.3) and the same argument as that in the proof of Lemmas A.1 and A.2 imply that there exists $\delta_0, h_0 > 0$ small enough such that, for all $0 < \delta \leq \delta_0$, $h \in (0, h_0]$ $0 < R \leq h^{-1}$ and $0 \leq s, t \leq \delta R$, $z_h(t, s)$ is well defined on $\Omega(R'', J''_0)$ and satisfies
\[ |\partial_x^\alpha \partial_\xi^\beta (z_h(t, s, x, \xi) - x)| \leq C_{\alpha\beta} R^{1-|\alpha|}. \]  

(4.6)

In particular, \((z_h(t, s, x, \xi), \xi) \in \Omega(R', J')\) for \(0 \leq s, t \leq \delta R\) if \(\delta > 0\), depending only on \(J''\), is small enough. We now define \(\{b_{h,j}(t, x, \xi)\}_{0 \leq j \leq N}\) inductively by

\[
\begin{align*}
    b_{h,0}(t, x, \xi) &= a(z_h(0, t), \xi) \exp \left( \int_0^t \mathcal{Y}_h(s, z_h(s, t, x, \xi), \xi) ds \right), \\
    b_{h,j}(t, x, \xi) &= -\int_0^t (iH_0 b_{h,j-1})(s, z_h(s, t), \xi) \exp \left( \int_u^t \mathcal{Y}_h(u, z_h(u, t, x, \xi), \xi) du \right) ds.
\end{align*}
\]

Since \(\text{supp} \ a \in \Omega(R, J)\) and \(z_h(t, s, \Omega(R, J)) \subset \{x; |x| > R/2\}\) for all \(0 \leq s, t \leq \delta R\), \(b_{h,j}(t)\) are supported in \(\Omega(R/2, J_0)\). Thus, if we extend \(b_{h,j}\) on \(\mathbb{R}^d\) so that

\[ b_{h,j}(t, x, \xi) = 0, \quad (x, \xi) \notin \Omega(R/2, J_0), \]

then \(b_{h,j}\) is still smooth in \((x, \xi)\). By (4.3) and (4.6), we learn

\[ |\partial_x^\alpha \partial_\xi^\beta \mathcal{Y}_h(s, z_h(s, t, x, \xi), \xi)| \leq C \delta R^{-1-|\alpha|}, \quad 0 \leq s, t \leq \delta R. \]

\(<b_{h,j}(t, \cdot, \cdot); h \in (0, h_0], \ 0 < R \leq h^{-1}, \ t \in [0, \delta R], \ 0 \leq j \leq N>\) thus is a bounded set in \(S(1, g)\) and \(\text{supp} b_{h,j}(t, \cdot, \cdot) \subset \Omega(R/2, J_0)\) uniformly with respect to \(h \in (0, h_0]\) and \(0 \leq t \leq \delta R\). A standard Hamilton-Jacobi theory shows that \(b_{h,j}(t)\) satisfy the following transport equations:

\[
\begin{align*}
    \partial_t b_{h,0}(t) + \chi_h(t) \cdot \partial_x b_{h,0}(t) + \mathcal{Y}_h(t) b_{h,0}(t) &= 0, \\
    \partial_t b_{h,j}(t) + \chi_h(t) \cdot \partial_x b_{h,j}(t) + \mathcal{Y}_h(t) b_{h,j}(t) &= -iH_0 b_{h,j-1}(t), \quad j \geq 1,
\end{align*}
\]

(4.7)

with the initial condition \(b_{h,0}(0) = a, \ b_{h,j}(0) = 0, \ j = 1, 2, \ldots, N\). A direct computation then yields

\[
e^{-i\Psi_h(s, x, \xi)/h} (hD_s + h^2 H) \left( e^{i\Psi_h(s, x, \xi)/h} \sum_{j=0}^N h^j b_{h,j} \right) = O(h^{N+1}) \quad \text{in} \ S(1, g)
\]

which, combined with Lemma 4.2, implies (4.5).

**Dispersive estimates.** The distribution kernel of \(U(\Psi_h(t), b_h(t))\) is given by
Strichartz estimates for Schrödinger equations

\[ K_{\text{WKB}}(t, h, x, y) = \frac{1}{(2\pi h)^d} \int e^{i(h/2)(\Psi_h(t, x, \xi) - y \cdot \xi)} b_h(t, x, \xi) d\xi. \]

Since \( b_h(t, x, \xi) \) has a compact support with respect to \( \xi \),

\[ |K_{\text{WKB}}(t, h, x, y)| \leq C h^{-d} \leq C |th|^{-d/2} \quad \text{for} \quad 0 < t \leq h. \]

We hence assume \( h < t \) without loss of generality. Choose \( \chi \in S(1, g) \) so that \( 0 \leq \chi \leq 1 \), \( \chi \equiv 1 \) on \( \Omega(R/2, J_0) \) and \( \text{supp} \chi \subset \Omega(R/4, J_1) \), and set

\[ \psi_h(t, x, y, \xi) = \frac{(x - y)}{t} \cdot \xi - p_h(x, \xi) + \chi(x, \xi) \left( \frac{\Psi_h(t, x, \xi) - x \cdot \xi}{t} + p_h(x, \xi) \right). \]

By the definition, we obtain

\[ \psi_h(t, x, y, \xi) = \frac{\Psi_h(t, x, \xi) - y \cdot \xi}{t}, \quad t \in [h, \delta R], \ (x, \xi) \in \Omega(R/2, J_1), \ y \in \mathbb{R}^d, \]

and (4.3) implies

\[ | \partial_\alpha x \partial_\beta \xi \psi_h(t, x, y, \xi) | \leq C_{\alpha\beta} \quad \text{on} \quad [0, \delta R] \times \mathbb{R}^{2d}, \ |\alpha + \beta| \geq 2. \]

Moreover, \( \partial_\xi^2 \psi_h(t, x, y, \xi) \) can be brought to the form

\[ \partial_\xi^2 \psi_h(t, x, y, \xi) = -(a^{jk}(x))_{j,k} + Q_h(t, x, \xi), \]

where the error term \( Q_h(t, x, \xi) \) is a \( d \times d \)-matrix satisfying

\[ | \partial_\alpha x \partial_\beta \xi Q_h(t, x, \xi) | \leq C_{\alpha\beta} \delta h^{|\alpha|} \quad \text{on} \quad [0, \delta R] \times \mathbb{R}^{2d}. \]

Since \( (a^{jk}(x)) \) is uniformly elliptic, the stationary phase theorem implies

\[ |K_{\text{WKB}}(t, h, x, y)| \leq C h^{-d} |t/h|^{-d/2} = C |th|^{-d/2}, \quad 0 < t \leq \delta R, \]

provided that \( \delta > 0 \) is small enough. We complete the proof. \( \square \)

5. Proof of Theorem 1.1.

In this section we complete the proof of Theorem 1.1.
Proof of Theorem 1.1 (i). Let \( \chi_0 \in C^\infty_c(\mathbb{R}^d) \) with \( \chi_0 \equiv 1 \) on \( \{|x| < R_0\} \) and \( \psi \in C^\infty_c((0, \infty)) \). A partition of unity argument and Lemma 2.1 show that there exist \( a^\pm \in S(1, g) \) with \( \text{supp} a^\pm \subset \Gamma^\pm(R_0, J, 1/2) \) such that \((1 - \chi_0)\psi(h^2 H_0)\) is approximated in terms of \( a^\pm(x, hD) \):

\[
(1 - \chi_0)\psi(h^2 H_0) = a^+(x, hD) + a^-(x, hD) + Q_0(h),
\]

where \( J \subseteq (0, \infty) \) is an open interval with \( \pi(t) \ni h(\psi \circ k) \subseteq J \), and \( Q_0(h) \) satisfies

\[
\sup_{h \in (0, 1]} \|Q_0(h)\|_{\mathcal{L}(L^2(\mathbb{R}^d), L^q(\mathbb{R}^d))} \leq C_q,
\]

for any \( q \geq 2 \). Let \( b \in C^\infty_c(\mathbb{R}^d; \mathbb{R}) \) be a cut-off function such that \( b \equiv 1 \) on a neighborhood of \( J \). By the asymptotic formula (2.1), we can write

\[
a^\pm(x, hD)^* = b(hD)a^\pm(x, hD)^* + Q_1(h)
\]

where \( Q_1(h) \) satisfies the same \( \mathcal{L}(L^2, L^p) \)-estimate as that of \( Q_0(h) \). Therefore,

\[
\|(Q_0(h) + Q_1(h))e^{-itH}u_0\|_{L^p([-\delta, \delta]; L^q(\mathbb{R}^d))} \leq C\|u_0\|_{L^2(\mathbb{R}^d)}, \quad h \in (0, 1], \quad (5.2)
\]

for any \( p, q \geq 2 \). Next, we shall prove the following estimate for the main terms:

\[
\|b(hD)a^\pm(x, hD)^* e^{-i(t-s)H_{a^\pm}(x, hD)}b(hD)\|_{\mathcal{L}(L^2(\mathbb{R}^d), L^\infty(\mathbb{R}^d))} \leq C|t - s|^{-d/2}
\]

for \( 0 < |t - s| \leq \delta \). We first consider the outgoing case. Let us fix \( N > 1 \) so large that \( N \geq 2d + 1 \). After rescaling \( t - s \mapsto (t - s)h \) and choosing \( R_0 > 1 \) large enough, we apply Theorem 3.1 with \( R = R_0 \), Lemma 3.2 and Theorem 4.3 with \( R = h^{-1} \) to \( e^{-i(t-s)H_{a^\pm}(x, hD)} \). Then, we can write

\[
e^{-i(t-s)H_{a^\pm}(x, hD)} = U(S_h^+, b_h^+) e^{i(t-s)\Delta/2} U(S_h^+, c_h^+) + U(\Psi_h(t-s), b_h(t-s)) + Q_2(t-s, h),
\]

where the distribution kernels of main terms satisfy dispersive estimates

\[
|K_{\text{WKB}}(t-s, h, x, y)| + |K_{\text{WKB}}(t-s, h, x, y)| \leq C|(t-s)h|^{-d/2}, \quad (5.3)
\]
uniformly with respect to $h \in (0, h_0]$, $0 < t - s \leq \delta h^{-1}$ and $x, y \in \mathbb{R}^d$. Let $A(h, x, y)$ and $B(h, x, y)$ be the distribution kernels of $a(x, hD)^*$ and $b(hD)$, respectively. They clearly satisfy

$$\sup_x \int (|A(h, x, y)| + |B(h, x, y)|) dy + \sup_y \int (|A(h, x, y)| + |B(h, x, y)|) dx \leq C$$

uniformly in $h \in (0, 1]$. By using this estimate and (5.3), we see that the distribution kernel of

$$b(hD)a^+(x, hD)^*(e^{-i(t-s)h}a^+(x, hD) - Q_2^{-}(t-s, h))b(hD)$$

satisfies the same dispersive estimates as (5.3) for $0 < t - s \leq \delta h^{-1}$. On the other hand, $Q_2^{-}(t-s, h)$ satisfy

$$\left\| Q_2^{-}(t-s, h) \right\|_{L^2(\mathbb{R}^d)} \leq C_N h^N, \quad h \in (0, h_0], \quad 0 \leq t-s \leq \delta h^{-1}.$$  

We here recall that $a^+(x, hD)^*$ is uniformly bounded on $L^2(\mathbb{R}^d)$ in $h \in (0, 1]$ and $b(hD)$ satisfies

$$\left\| b(hD) \right\|_{L(L^2(\mathbb{R}^d), H^s(\mathbb{R}^d))} \leq \left\| \langle D \rangle^s \langle hD \rangle^{-s} \right\|_{L(L^2(\mathbb{R}^d))} \left\| \langle hD \rangle^s b(hD) \right\|_{L(L^2(\mathbb{R}^d))} \left\| \langle hD \rangle^{-s} \langle D \rangle^s \right\|_{L(L^2(\mathbb{R}^d))} \leq C_s h^{-2s}.$$  

$b(hD)a^+(x, hD)^*Q_2^{-}(t-s, h)b(hD)$ hence is a bounded operator in $L(H^{-s}, H^s)$ for some $s > d/2$ and has the uniformly bounded distribution kernel $\tilde{Q}_2^+(t-s, h, x, y)$ with respect to $h \in (0, h_0]$ and $0 \leq t-s \leq \delta h^{-1}$. Therefore,

$$\left| \tilde{Q}_2^+(t-s, h, x, y) \right| \leq 1 \leq |(t-s)h|^{-d/2}, \quad h \in (0, h_0], \quad 0 < t-s \leq \delta h^{-1}.$$  

The corresponding estimates for the incoming case also hold for $0 \leq -(t-s) \leq \delta h^{-1}$. Therefore, $b(hD)a^+(x, hD)^*e^{i(t-s)h}a^+(x, hD)b(hD)$ have distribution kernels $K^\pm(t-s, h, x, y)$ satisfying

$$\left| K^\pm(t-s, h, x, y) \right| \leq C |(t-s)h|^{-d/2} \quad (5.4)$$

uniformly with respect to $h \in (0, h_0]$, $0 \leq \pm(t-s) \leq \delta h^{-1}$ and $x, y \in \mathbb{R}^d$, respectively.
We here use a simple trick due to Bouclet-Tzvetkov [2, Lemma 4.3]. If we set
\[ U^\pm(t, h) = b(hD) a^\pm(x, hD) e^{-ithH} a^\pm(x, hD) b(hD), \]
then
\[ U^\pm(s - t, h) = U^\pm(t - s, h), \]
and hence
\[ K^\pm(s - t, h, x, y) = K^\pm(t - s, h, y, x). \]
Therefore, the estimates (5.4) also hold for \( 0 < \mp(t - s) \leq \delta h^{-1} \) and \( x, y \in \mathbb{R}^d \). Rescaling \( (t - s) h \mapsto t - s \), we obtain the estimate (5.2).

Finally, since the \( \mathcal{L}(L^2) \)-boundedness of \( a^\pm(x, hD) e^{-ithH} \) is obvious, (5.1), (5.2) and the Keel-Tao theorem [15] imply the desired semi-classical Strichartz estimates:
\[
\sup_{h \in (0, h_0]} \| (1 - \chi_0) \psi_0 h^2 H_0 e^{-ithH} u_0 \|_{L^p([-\delta, \delta]; L^q(\mathbb{R}^d))} \leq C \| u_0 \|_{L^2(\mathbb{R}^d)}.
\]
By the virtue of Proposition 2.4, we complete the proof. \( \square \)

We next give the proof of (ii). Suppose that \( H \) satisfies Assumption 1 with \( \mu, \nu \geq 0 \). We first recall the local smoothing effects for Schrödinger operators with at most quadratic potentials proved by Doi [9]. For any \( s \in \mathbb{R} \), we set
\[ B^s := \{ f \in L^2(\mathbb{R}^d); \langle x \rangle^s f \in L^2(\mathbb{R}^d), \langle D \rangle^s f \in L^2(\mathbb{R}^d) \}, \]
and define a symbol \( e_s \) by
\[ e_s(x, \xi) := (k(x, \xi) + |x|^2 + L(s))^{s/2} \in S((1 + |x| + |\xi|)^s, g). \]
We denote by \( E_s \) its Weyl quantization:
\[ E_s f(x) = \frac{1}{2\pi} \int e^{i(x-y)\cdot \xi} e_s \left( \frac{x+y}{2}, \xi \right) f(y) dy d\xi. \]
Here \( L(s) > 1 \) is a large constant depending on \( s \). Then, for any \( s \in \mathbb{R} \), there exists \( L(s) > 0 \) such that \( E_s \) is a homeomorphism from \( B^{r+s} \) to \( B^r \) for all \( r \in \mathbb{R} \), and \( (E_s)^{-1} \) is still a Weyl quantization of a symbol in \( S((1 + |x| + |\xi|)^{-s}, g) \).

**Lemma 5.1 (The local smoothing effects [9])**. Suppose that the kinetic energy \( k(x, \xi) \) satisfies the non-trapping condition (1.5). Then, for any \( T > 0 \) and \( \sigma > 0 \), there exists \( C_{T, \sigma} > 0 \) such that
\[ \| \langle \xi \rangle^{-1/2 - \sigma} E_{1/2} u \|_{L^2([-T, T]; L^2(\mathbb{R}^d))} \leq C_{T, \sigma} \| u_0 \|_{L^2}, \quad (5.5) \]
where \( u = e^{-ithH} u_0 \).
Remark 5.2. Let $\chi \in C_0^\infty(\mathbb{R}^d)$. (5.5) implies a usual local smoothing effect:

$$\|\langle D \rangle^{1/2} \chi u\|_{L^2([-T,T];L^2(\mathbb{R}^d))} \leq C_T \|u_0\|_{L^2(\mathbb{R}^d)}.$$  

(5.6)

Indeed, let $\chi_1 \in C_0^\infty(\mathbb{R}^d)$ be such that $\chi_1 \equiv 1$ on supp $\chi$. We split $\langle D \rangle^{1/2} \chi$ as follows:

$$\langle D \rangle^{1/2} \chi = \chi_1 \langle D \rangle^{1/2} + \chi_1 [\chi, D] \chi,$$

$$\chi_1 \langle D \rangle^{1/2} \chi = \chi_1 \langle D \rangle^{1/2} (H^{1/2} - E^{1/2})^{-1} E^{1/2} \chi + \chi_1 \langle D \rangle^{1/2} (E^{1/2})^{-1} [E^{1/2}, \chi_1] \chi.$$

By a standard symbolic calculus, $[\langle D \rangle^{1/2}, \chi_1] \chi$, $\chi_1 \langle D \rangle^{1/2} (H^{1/2} - E^{1/2})^{-1} E^{1/2} \chi$ and $[E^{1/2}, \chi_1] \chi$ are bounded on $L^2(\mathbb{R}^d)$ since $\chi_1$ has a compact support. Therefore, Lemma 5.1 implies

$$\|\langle D \rangle^{1/2} \chi u\|_{L^2([-T,T];L^2(\mathbb{R}^d))} \leq C \|\chi_1 E^{1/2} \chi u\|_{L^2([-T,T];L^2(\mathbb{R}^d))} + C_T \|u\|_{L^2(\mathbb{R}^d)} \leq C_T \|u_0\|_{L^2(\mathbb{R}^d)}.$$

Proof of Theorem 1.1 (ii). We consider the case when $0 \leq t \leq T$ only, and the proof for the negative time is similar. We mimic the argument in [18, Section II. 2]. A direct computation yields

$$(i\partial_t + \Delta) \chi u = \Delta \chi u + \chi Hu$$

$$= \chi_1 (H + \Delta) \chi_1 \chi u + (\chi_1 [\chi, H] + [\Delta, \chi_1] \chi) u.$$  

We define a self-adjoint operator by $\tilde{H} := -\Delta + \chi_1 (\Delta + H) \chi_1$, and set

$$\tilde{U}(t) := e^{-it\tilde{H}}, \quad F := (\chi_1 [\chi, H] + [\Delta, \chi_1] \chi) u.$$  

We here note that if $H_0$ satisfies the non-trapping condition then so does the principal part of $\tilde{H}$. By the Duhamel formula, we can write

$$\chi u = \tilde{U}(t) \chi u_0 + \int_0^t \tilde{U}(t-s) F(s) ds.$$  

Since $\chi_1 (H + \Delta) \chi_1$ is a compactly supported smooth perturbation, it was proved
by Staffilani-Tataru [22] that \( \bar{U}(t) \) is bounded from \( L^2(\mathbb{R}^d) \) to \( L^2([0, T]; H^{1/2}_{\text{loc}}(\mathbb{R}^d)) \), and that its adjoint

\[
\bar{U}^* f = \int_0^T U(-s)f(s, \cdot)ds
\]

is bounded from \( L^2([0, T]; H^{-1/2}_{\text{loc}}(\mathbb{R}^d)) \) to \( L^2(\mathbb{R}^d) \). Moreover, \( \bar{U}(t) \) satisfies Strichartz estimates (for any admissible pair \((p, q)\)):

\[
\|\bar{U}(t)v\|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \leq C_T \|v\|_{L^2}.
\]

Therefore, we have

\[
\left\| \int_0^T \bar{U}(t - s)F(s)ds \right\|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \leq C_T \|\bar{U}^* F\|_{L^2(\mathbb{R}^d)}
\]

\[
\leq C_T \|\langle D \rangle^{-1/2} F\|_{L^2([-T, T]; L^2(\mathbb{R}^d))}
\]

since \( F \) has a compact support with respect to \( x \). The Christ-Kiselev lemma (see [7], [21]) then implies

\[
\left\| \int_0^t \bar{U}(t - s)F(s)ds \right\|_{L^p([-T, T]; L^q(\mathbb{R}^d))} \leq C_T \|\langle D \rangle^{-1/2} F\|_{L^2([-T, T]; L^2(\mathbb{R}^d))},
\]

provided that \( p > 2 \). We split \( F \) as

\[
F = ([\chi, H]\chi_1 + [\Delta, \chi_1]\chi)u + [\chi_1, [\chi, H]]u =: F_1 + F_2.
\]

Since \([\chi, H]\) is a first order differential operator with bounded coefficients, we see that \([\chi_1, [\chi, H]]\) is bounded on \( L^2(\mathbb{R}^d) \), and \( \|\langle D \rangle^{-1/2} F_2\|_{L^2([-T, T]; L^2(\mathbb{R}^d))} \) is dominated by \( C_T \|u_0\|_{L^2(\mathbb{R}^d)} \). We now use (5.6) and obtain

\[
\|\langle D \rangle^{-1/2} F_1\|_{L^2([-T, T]; L^2(\mathbb{R}^d))} \leq C \|\chi_1 u\|_{L^2([-T, T]; H^{-1/2}(\mathbb{R}^d))}
\]

\[
\leq C \|\chi_1 u\|_{L^2([-T, T]; L^2(\mathbb{R}^d))}
\]

\[
\leq C_T \|u_0\|_{L^2},
\]

which completes the proof. \(\square\)
A. Proof of Proposition 4.1.

Assume Assumption 1 with \( \mu, \nu \geq 0 \). We here give the detail of the proof of Proposition 4.1. We first study the corresponding classical mechanics. Let \( h \in (0, 1] \) and consider the Hamilton flow \((X_h(t), \Xi_h(t)) = (X_h(t, x, \xi), \Xi_h(t, x, \xi))\) generated by the semi-classical total energy

\[
p_h(x, \xi) = k(x, \xi) + h^2 V(x),
\]

i.e., \((X_h(t), \Xi_h(t))\) is the solution to the Hamilton equations

\[
\begin{align*}
\dot{X}_{h,j}(t) &= \sum_k a^{jk}(X_h(t))\Xi_{h,k}(t), \\
\dot{\Xi}_{h,j}(t) &= -\frac{1}{2} \sum_{k,l} \frac{\partial a^{kl}}{\partial x_j}(X_h(t))\Xi_{h,k}(t)\Xi_{h,l}(t) - h^2 \frac{\partial V}{\partial x_j}(X_h(t)),
\end{align*}
\]

with the initial condition \((X_h(0), \Xi_h(0)) = (x, \xi)\), where \( \dot{f} = \partial_t f \). We first prepare an a priori bound of the flow.

**Lemma A.1.** For all \( h \in (0, 1], |t| \lesssim h^{-1} \) and \((x, \xi) \in \mathbb{R}^{2d}\),

\[
|X_h(t) - x| \lesssim (|\xi| + h(x)^{1-\nu/2})|t|, \quad |\Xi_h(t)| \lesssim |\xi| + h(x)^{1-\nu/2}.
\]

**Proof.** We consider the case \( t \geq 0 \). The proof for the case \( t < 0 \) is analogous. Since the Hamilton flow conserves the total energy, namely

\[
p_h(x, \xi) = p_h(X_h(t), \Xi_h(t)) \quad \text{for all} \quad t \in \mathbb{R},
\]

we have

\[
|\Xi_h(t)| \lesssim \sqrt{p_0(X_h(t), \Xi_h(t))} \\
\lesssim \sqrt{p_h(x, \xi) - h^2 V(X_h(t))} \\
\lesssim |\xi| + h(x)^{1-\nu/2} + h(X_h(t))^{1-\nu/2}.
\]

Applying the above inequality to the Hamilton equation, we have

\[
|\dot{X}_h(t)| \lesssim |\Xi_h(t)| \lesssim |\xi| + h(x)^{1-\nu/2} + h|X_h(t) - x|.
\]
Integrating with respect to $t$ and using Gronwall’s inequality, we obtain the assertion since $e^{th} \lesssim |t|$ for $|t| \lesssim h^{-1}$. \hfill \square

Let $J \Subset (0, \infty)$ be an open interval. For sufficiently small $\delta > 0$ and for all $0 < R \leq h^{-1}$, the above lemma implies

$$|x|/2 \leq |X_h(t, x, \xi)| \leq 2|x|$$

(A.1)

uniformly with respect to $h \in (0, 1]$, $|t| \leq \delta R$ and $(x, \xi) \in \Omega(R, J)$. By using this inequality, we have the following:

**Lemma A.2.** Let $J, \delta$ be as above. Then, for $h \in (0, 1]$, $0 < R \leq h^{-1}$, $|t| \leq \delta R$ and $(x, \xi) \in \Omega(R, J)$, $X_h(t, x, \xi)$ and $\Xi_h(t, x, \xi)$ satisfy

$$\begin{align*}
|X_h(t) - x| &\leq C(1 + \delta h|x|^{1-\nu})|t|, \\
|\Xi_h(t) - \xi| &\leq C(\langle x \rangle^{-1} + h^2|x|^{1-\nu})|t|,
\end{align*}$$

(A.2)

and, for $|\alpha + \beta| = 1$,

$$\begin{align*}
|\partial^\alpha_x \partial^\beta_\xi (X_h(t) - x)| &\leq C_{\alpha \beta} \langle x \rangle^{-|\alpha|} + h|\alpha| \langle x \rangle^{-|\alpha|\nu/2}|t|, \\
|\partial^\alpha_x \partial^\beta_\xi (\Xi_h(t) - \xi)| &\leq C_{\alpha \beta} \langle x \rangle^{-1-|\alpha|} + h^{1+|\alpha|} \langle x \rangle^{-(1+|\alpha|\nu/2)}|t|,
\end{align*}$$

(A.3)

and, for $|\alpha + \beta| \geq 2$,

$$\begin{align*}
|\partial^\alpha_x \partial^\beta_\xi (X_h(t) - x)| &\leq C_{\alpha \beta} \delta h^{|\alpha|} \langle x \rangle^{-1} R|t|, \\
|\partial^\alpha_x \partial^\beta_\xi (\Xi_h(t) - \xi)| &\leq C_{\alpha \beta} h^{|\alpha|} \langle x \rangle^{-1}|t|.
\end{align*}$$

(A.4)

Moreover $C, C_{\alpha \beta} > 0$ may be taken uniformly with respect to $R, h$ and $t$.

**Proof.** We only prove the case when $t \geq 0$, the proof for the case $t \leq 0$ is similar. Applying Lemma A.1 and (A.1) to the Hamilton equation, we have

$$|\dot{\Xi}^h(t)| \lesssim \langle X_h(t) \rangle^{-1} |\Xi_h(t)|^2 + h^2 \langle X_h(t) \rangle^{1-\nu}$$

$$\lesssim \langle x \rangle^{-1}(1 + h^2 \langle x \rangle^{2-\nu}) + h^2 \langle x \rangle^{1-\nu}$$

$$\lesssim \langle x \rangle^{-1} + h^2 \langle x \rangle^{1-\nu},$$

$$|\dot{X}^h(t)| \lesssim |\Xi_h(t)| \lesssim 1 + \delta h \langle x \rangle^{1-\nu},$$
Using Gronwall’s inequality, we have (A.3) since

\[ Q \]

cases are similar, and for higher derivatives follow from an induction on

\[ (A.2) \]

then imply

\[ \text{Define a weight function } w_h(x) = \langle x \rangle^{-1} + h(x)^{-\nu/2}. \]

A direct computation and (A.2) then imply

\[
\begin{align*}
|((\partial_\xi^{\alpha} \partial_x^{\beta} p_h) (X_h(t), \Xi_h(t)))| & \leq C_{\alpha, \beta} w_h(x)^{|\alpha|}, \quad |\alpha + \beta| = 2, \\
|((\partial_\xi^{\alpha} \partial_x^{\beta} p_h) (X_h(t), \Xi_h(t)))| & \leq C_{\alpha, \beta} \langle x \rangle^{2-|\alpha+\beta|} w_h(x)^{|\alpha|-1}, \quad |\alpha + \beta| \geq 3,
\end{align*}
\]

for all \(|t| \leq \delta R\) and \((x, \xi) \in \Omega(R, J)\), and \(\partial_x^{\beta} p_h \equiv 0\) on \(\mathbb{R}^{2d}\) for \(|\beta| \geq 3\). By integrating (A.5) with respect to \(t\), we have

\[
w_h(x) |\partial_\xi^{\alpha} \partial_x^{\beta} (X_h(t) - x)| + |\partial_\xi^{\alpha} \partial_x^{\beta} (\Xi_h(t) - \xi)| \\
\lesssim \int_0^t \left( w_h(x) |\partial_\xi^{\alpha} \partial_x^{\beta} (X_h(t) - x)| + |\partial_\xi^{\alpha} \partial_x^{\beta} (\Xi_h(t) - \xi)| \right) + w_h(x)^{1+|\alpha|} d\tau.
\]

Using Gronwall’s inequality, we have (A.3) since \(|t| \leq \delta R\).

For \(|\alpha + \beta| \geq 2\), we shall prove the estimate for \(\partial_\xi^{\alpha} X_h(t)\) only. Proofs for other cases are similar, and for higher derivatives follow from an induction on \(|\alpha + \beta|\).

By the Hamilton equation and (A.3), we learn

\[
\partial_\xi^{\alpha} X_h = \partial_x \partial_\xi p_h(X_h, \Xi_h) \partial_\xi^{\alpha} X_h + \partial_\xi^{\alpha} p_h(X_h, \Xi_h) \partial_\xi^{\alpha} \Xi_h + Q(h, x, \xi)
\]

where \(Q(h, x, \xi)\) satisfies

\[
Q(h, x, \xi) \leq C \sum_{|\alpha + \beta| = 3, |\beta| = 1, 2} \langle (\partial_x \partial_\xi^{\alpha} p_h (X_h, \Xi_h) \partial_\xi^{\beta} p_h (X_h, \Xi_h) \partial_\xi^{\alpha} \Xi_h) \rangle |\partial_\xi^{\beta} \Xi_h| |\partial_\xi^{\alpha} \Xi_h| |\partial_\xi^{\beta} \Xi_h| |\partial_\xi^{\alpha} \Xi_h|
\]

\[
\leq C \langle x \rangle^{-1} \sum_{|\alpha| = 1, 2, 3} w_h(x)^{|\alpha|-1} |t|^{|\alpha|}
\]

\[
\leq C \delta x^{-1} R.
\]
We similarly obtain

$$\partial^2_{\xi_1} \Xi_h = -\partial^2_{x_1} p_h(X_h, \Xi_h) \partial^2_{\xi_1} X_h - \partial_{\xi} \partial_{x_1} p_h(X_h, \Xi_h) \partial^2_{\xi_1} \Xi_h + O(\langle x \rangle^{-1}),$$

and these estimates and Gronwall’s inequality imply

$$\left| \partial^2_{\xi_1} X_h(t) \right| + \left| \partial^2_{\xi_1} \Xi_h(t) \right| \lesssim \int_0^t w_h(x) \left( \langle \delta R \rangle^{-1} \left| \partial^2_{\xi_1} X_h(t) \right| + \left| \partial^2_{\xi_1} \Xi_h(t) \right| \right) + \langle x \rangle^{-1} d\tau$$

for $0 \leq t \leq \delta R$. We hence have the assertion. \( \square \)

**Remark A.3.** If $\nu \geq 1$, then Lemma A.2 implies that for any $\alpha, \beta \in \mathbb{Z}_+^d$, there exists $C_{\alpha\beta}$ such that

$$\left| \partial^\alpha_x \partial^\beta_\xi (X_h(t) - x) \right| \leq C_{\alpha\beta} \delta R^{1-|\alpha|}, \quad \left| \partial^\alpha_x \partial^\beta_\xi (\Xi_h(t) - \xi) \right| \leq C_{\alpha\beta} \delta R^{-|\alpha|}, \quad (A.6)$$

uniformly with respect to $h \in (0, 1]$, $0 < R \leq h^{-1}$, $|t| \leq \delta R$ and $(x, \xi) \in \Omega(R, J)$.

**Lemma A.4.** Suppose that $\nu = 1$ and let $J_1 \Subset J'_1 \Subset (0, \infty)$ be open intervals. Then there exists $\delta > 0$ small enough such that, for any fixed $|t| \leq \delta R$, the map $g_h(t) : (x, \xi) \mapsto (X_h(t, x, \xi), \xi)$ is a diffeomorphism from $\Omega(R/2, J'_1)$ onto its range. Moreover, we have

$$\Omega(R, J_1) \subset g_h(t, \Omega(R/2, J'_1)), \quad |t| \leq \delta R. \quad (A.7)$$

**Proof.** We choose $J''_1$ so that $J_1 \Subset J''_1 \Subset (0, \infty)$. Choosing $\chi \in S(1, g)$ such that

$$0 \leq \chi \leq 1, \quad \text{supp } \chi \subset \Omega(R/3, J''), \quad \chi \equiv 1 \text{ on } \Omega(R/2, J'_1),$$

we define $X_h^\chi(t, x, \xi) := (1 - \chi(x, \xi)) x + \chi(x, \xi) X_h(t, x, \xi)$ and set

$$g_h^\chi(t, x, \xi) = (X_h^\chi(t, x, \xi), \xi).$$

We also define $(z, \xi) \mapsto \tilde{g}_h^\chi(t, z, \xi)$ by
\[ \tilde{g}_h^\chi(t, z, \xi) = (\tilde{X}_h^\chi(t, z, \xi), \xi) := (X_h^\chi(t, Rz, \xi)/R, \xi). \]

By (A.6), there exists \( \delta > 0 \) so small that, for \( |t| \leq \delta R, (z, \xi) \in \mathbb{R}^{2d}, \)
\[
|\frac{\partial}{\partial z} \frac{\partial}{\partial \xi}(\tilde{X}_h^\chi(t, z, \xi) - z)| \lesssim \delta R^{-|\alpha|}, \quad |\frac{\partial}{\partial z} \frac{\partial}{\partial \xi}(J(\tilde{g}_h^\chi)(t, z, \xi) - \text{Id})| \leq C_{\alpha\beta} \delta < \frac{1}{2},
\]
where \( J(\tilde{g}_h^\chi) \) is the Jacobi matrix with respect to \((z, \xi)\). The Hadamard global inverse mapping theorem then shows that \( \tilde{g}_h^\chi(t) \) is a diffeomorphism from \( \mathbb{R}^d \) onto itself if \( |t| \leq \delta R \). By definition, \( g_h(t) \) is a diffeomorphism from \( \Omega(R/2, J'_1) \) onto its range.

We next prove (A.7). Since \( g_h(t) = g_h^\chi(t) \) and \( g_h^\chi(t) \) is bijective on \( \Omega(R/2, J'_1) \), it suffices to check that
\[ \Omega(R, J_1)^c \supset g_h^\chi(t, \Omega(R/2, J'_1)^c). \]
Suppose that \((x, \xi) \in \Omega(R/2, J'_1)^c\). If \((x, \xi) \in \Omega(R/3, J''_1)^c\), then
\[ g_h^\chi(t, x, \xi) = (x, \xi) \in \Omega(R/3, J''_1)^c \subset \Omega(R, J_1)^c. \]
Suppose that \((x, \xi) \in \Omega(R/3, J''_1) \setminus \Omega(R/2, J'_1)\). By (A.2) and the support property of \( \chi \), we have
\[ |X_h^\chi(t)| \leq |x| + |\chi(X_h(t) - x)| \leq R/2 + C\delta R \]
for some \( C > 0 \) independent of \( R \) and \( h \). Choosing \( \delta \) satisfying \( 1/2 + C\delta < 1 \), we obtain \( g_h^\chi(t, x, \xi) \in \Omega(R, J_1)^c \). \( \square \)

Let \( \Omega(R, J_1) \ni (x, \xi) \mapsto (Y_h(t, x, \xi), \xi) \) be the inverse of \( \Omega(R/2, J'_1) \ni (x, \xi) \mapsto (X_h(t, x, \xi), \xi) \).

**Lemma A.5.** Let \( \delta, J_1 \) as above and \( \nu = 1 \). Then, for all \( h \in (0, 1], 0 < R \leq h^{-1}, 0 < |t| \leq \delta R \) and \((x, \xi) \in \Omega(R, J_1)\), we have
\[
|\frac{\partial}{\partial \xi}(Y_h(t, x, \xi) - x)| \leq C_{\alpha\beta} \delta R^{1-|\alpha|},
\]
\[
|\frac{\partial}{\partial \xi}(\Xi_h(t, Y_h(t, x, \xi)) - \xi)| \leq C_{\alpha\beta} \delta R^{-|\alpha|}.
\]

**Proof.** We prove the inequalities for \( Y_h \) only. Proofs for \( \Xi_h(t, Y_h(t, x, \xi), \xi) \) are similar. Since \((Y_h(t, x, \xi), \xi) \in \Omega(R/2, J'_1)\),
\[ |Y_h(t, x, \xi) - x| = |X_h(0, Y_h(t, x, \xi), \xi) - X_h(t, Y_h(t, x, \xi), \xi)| \]
\[ \leq \sup_{(x, \xi) \in \Omega(R/2, J'_1)} |X_h(t, x, \xi) - x| \]
\[ \lesssim \delta R. \]

Next, let \( \alpha, \beta \in \mathbb{Z}_d^+ \) with \( |\alpha + \beta| = 1 \) and apply \( \partial_\alpha x \partial_\beta \xi \) to the equality

\[ x = X_h(t, Y_h(t, x, \xi), \xi). \]

We then have the following equality

\[ A(t, Z_h(t))\partial_\alpha^\alpha \partial_\beta^\beta (Y_h(t, x, \xi) - x) = \partial_\alpha^\alpha \partial_\beta^\beta (y - X_h(t, y, \eta)) \big|_{(y, \eta) = Z_h(t)} \]

where \( Z_h(t, x, \xi) = (Y_h(t, x, \xi), \xi) \) and \( A(t, Z) = (\partial_x X_h(t, Z) \right) \). By (A.2) and a similar argument as that in the proof of Lemma A.4, we learn that \( A(Z_h(t)) \) is invertible, and that \( A(Z_h(t)) \) and \( A(Z_h(t))^{-1} \) are uniformly bounded with respect to \( h \in (0, 1], |t| \leq \delta R \) and \( (x, \xi) \in \Omega(R, J_1) \). Therefore,

\[ |\partial_\alpha^\alpha \partial_\beta^\beta (Y_h(t, x, \xi) - x)| \leq \sup_{(x, \xi) \in \Omega(R/2, J'_1)} |\partial_\alpha^\alpha \partial_\beta^\beta (y - X_h(t, y, \eta))| \]
\[ \leq C_{\alpha, \beta} \delta R^{1 - |\alpha|}. \]

The proof for higher derivatives is obtained by an induction on \( |\alpha + \beta| \), and we omit the details.

**Proof of Proposition 4.1.** We consider the case when \( t \geq 0 \), and the proof for \( t \leq 0 \) is similar. Choosing \( J \in J_1 \subseteq (0, \infty) \), we define the action integral \( \tilde{\Psi}_h(t, x, \xi) \) on \( [0, \delta R] \times \Omega(R/2, J_1) \) by

\[ \tilde{\Psi}_h(t, x, \xi) := x \cdot \xi + \int_0^t L_h(X_h(s, Y_h(t, x, \xi), \xi), \Xi_h(s, Y_h(t, x, \xi), \xi)) ds, \]

where \( L_h(x, \xi) = \xi \cdot \partial_\xi p_h(x, \xi) - p_h(x, \xi) \) is the Lagrangian associated to \( p_h \) and \( Y_h \) is defined by the above argument with \( R > 0 \) replaced by \( R/2 \). The smoothness property of \( \tilde{\Psi}_h \) follows from corresponding properties of \( X_h, \Xi_h \) and \( Y_h \). By the standard Hamilton-Jacobi theory, \( \tilde{\Psi}_h(t, x, \xi) \) solves the Hamilton-Jacobi equation (4.1) on \( \Omega(R/2, J_1) \) and satisfies
\[\partial_x \tilde{\Psi}_h(t, x, \xi) = \Xi_h(t, Y_h(t, x, \xi), \xi), \quad \partial_\xi \tilde{\Psi}_h(t, x, \xi) = Y_h(t, x, \xi).\]

In particular, we obtain the following energy conservation law:

\[p_h(x, \partial_x \tilde{\Psi}_h(t, x, \xi)) = p_h(Y_h(t, x, \xi), \xi).\]

This energy conservation and Lemma A.5 imply

\[|p_h(\partial_x \tilde{\Psi}_h(t, x, \xi) - p_h(x, \xi)|
\leq |Y_h(t, x, \xi) - x| \int_0^1 |\partial_x p_h(\lambda x + (1 - \lambda)Y_h(t, x, \xi)|d\lambda
\leq C\delta R((x)^{-1} + h^2)
\leq C\delta.\]

By using Lemma A.5, we also obtain

\[|\partial_\alpha \partial_\beta (p_h(x, \partial_x \tilde{\Psi}_h(t, x, \xi)) - p_h(x, \xi))| \leq C_{\alpha\beta}\delta R^{-|\alpha|}, \quad \alpha, \beta \in \mathbb{Z}^d_+.\]

Therefore,

\[|\partial_\alpha \partial_\beta (\tilde{\Psi}_h(t, x, \xi) - x \cdot \xi + tp_h(x, \xi))| \leq C_{\alpha\beta}\delta R^{-|\alpha|}|t|.\]

Choosing a cut-off function \(\chi \in S(1, g)\) so that \(0 \leq \chi \leq 1\), \(\chi \equiv 1\) on \(\Omega(R, J)\) and \(\text{supp} \chi \subset \Omega(R/2, J_1)\), we define

\[\Psi_h(t, x, \xi) := x \cdot \xi - tp_h(x, \xi) + \chi(x, \xi)(\tilde{\Psi}_h(t, x, \xi) - x \cdot \xi + tp_h(x, \xi)).\]

Clearly, \(\Psi_h(t, x, \xi)\) satisfies the statement of Proposition 4.1. \(\square\)

References

[1] J.-M. Bouclet, Strichartz estimates on asymptotically hyperbolic manifolds, Anal. PDE., 4 (2011), 1–84.
[2] J.-M. Bouclet and N. Tzvetkov, Strichartz estimates for long range perturbations, Amer. J. Math., 129 (2007), 1565–1609.
[3] J.-M. Bouclet and N. Tzvetkov, On global Strichartz estimates for non trapping metrics, J. Funct. Anal., 254 (2008), 1661–1682.
[4] N. Burq, P. Gérard and N. Tzvetkov, Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds, Amer. J. Math., 126 (2004), 569–605.
[5] N. Burq, C. Guillarmou and A. Hassell, Strichartz estimates without loss on manifolds with hyperbolic trapped geodesics, Geom. Funct. Anal., 20 (2010), 627–656.
[6] T. Cazenave, Semilinear Schrödinger Equations, Courant. Lect. Notes Math., 10, Amer. Math. Soc., Providence, RI, 2003.
[7] M. Christ and A. Kiselev, Maximal functions associated to filtrations, J. Funct. Anal., 179 (2001), 409–425.
[8] P. D'Ancona, L. Fanelli, L. Vega and N. Visciglia, Endpoint Strichartz estimates for the magnetic Schrödinger equation, J. Funct. Anal., 258 (2010), 3227–3240.
[9] S. Doi, Smoothness of solutions for Schrödinger equations with unbounded potentials, Publ. Res. Inst. Math. Sci., 41 (2005), 175–221.
[10] M. B. Erdoğan, M. Goldberg and W. Schlag, Strichartz and smoothing estimates for Schrödinger operators with almost critical magnetic potentials in three and higher dimensions, Forum Math., 21 (2009), 687–722.
[11] D. Fujiwara, Remarks on convergence of the Feynman path integrals, Duke Math. J., 47 (1980), 559–600.
[12] J. Ginibre and G. Velo, The global Cauchy problem for the nonlinear Schrödinger equation revisited, Ann. Inst. H. Poincaré Anal. Non Linéaire, 2 (1985), 309–327.
[13] A. Hassell, T. Tao and J. Wunsch, Sharp Strichartz estimates on nontrapping asymptotically conic manifolds, Amer. J. Math., 128 (2006), 963–1024.
[14] H. Isozaki and H. Kitada, Modified wave operators with time independent modifiers, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 32 (1985), 77–104.
[15] M. Keel and T. Tao, Endpoint Strichartz estimates, Amer. J. Math., 120 (1998), 955–980.
[16] J. Marzuola, J. Metcalfe and D. Tataru, Strichartz estimates and local smoothing estimates for asymptotically flat Schrödinger equations, J. Funct. Anal., 255 (2008), 1497–1553.
[17] H. Mizutani, Strichartz estimates for Schrödinger equations on scattering manifolds, Comm. Partial Differential Equations, 37 (2012), 169–224.
[18] L. Robbiano and C. Zuily, Strichartz Estimates for Schrödinger Equations with Variable Coefficients, Mém. Soc. Math. Fr. (N.S.), 101–102, Soc. Math. France, 2005.
[19] D. Robert, Autour de l’Approximation Semi-Classique, Progr. Math., 68, Birkhäuser, Basel, 1987.
[20] D. Robert, Relative time-delay for perturbations of elliptic operators and semiclassical asymptotics, J. Funct. Anal., 126 (1994), 36–82.
[21] H. F. Smith and C. D. Sogge, Global Strichartz estimates for nontrapping perturbations of the Laplacian, Comm. Partial Differential Equations, 25 (2000), 2171–2183.
[22] G. Staffilani and D. Tataru, Strichartz estimates for a Schrödinger operator with nonsmooth coefficients, Comm. Partial Differential Equations, 27 (2002), 1337–1372.
[23] R. S. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, Duke Math. J., 44 (1977), 705–714.
[24] D. Tataru, Parametrices and dispersive estimates for Schrödinger operators with variable coefficients, Amer. J. Math., 130 (2008), 571–634.
[25] K. Yajima, Existence of solutions for Schrödinger evolution equations, Comm. Math. Phys., 110 (1987), 415–426.
[26] K. Yajima, Schrödinger evolution equation with magnetic fields, J. Analyse Math., 56 (1991), 29–76.
[27] K. Yajima, Boundedness and continuity of the fundamental solution of the time dependent Schrödinger equation with singular potentials, Tohoku Math. J., 50 (1998), 577–595.
[28] K. Yajima, On the behaviour at infinity of the fundamental solution of time dependent Schrödinger equation, Rev. Math. Phys., 13 (2001), 891–920.
K. Yajima and G. Zhang, Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity, *J. Differential Equations*, **202** (2004), 81–110.

Haruya Mizutani
Department of Mathematics
Gakushuin University
1-5-1 Mejiro, Toshima-ku
Tokyo 171-8588, Japan
E-mail: haruya@math.gakushuin.ac.jp