Exact Solution of Induced Lattice Gauge Theory at Large $N$

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Abstract

We find the exact solution of a recently proposed model of the lattice gauge theory induced by heavy scalar field in adjoint representation at $N = \infty$ for arbitrary dimension $D$. The nonlinear integral equation for the gauge invariant density of eigenvalues of the vacuum average of the scalar field is derived. In the continuum limit, the density grows as $\phi^\alpha$ where $\alpha = 1 + \frac{1}{\pi} \arccos \frac{D}{3D-2}$. 
1 Introduction

Recently, the new lattice gauge model was suggested by Kazakov and myself\cite{1}. We observed, that the Yang-Mills term in the Action could be induced by massive scalar field in adjoint representation, in the same spirit as actual QCD is induced by some yet unknown quark and gluon constituents. The origin and properties of these constituents is irrelevant for the purposes of QCD, in virtue of renormalizability.

The basic property of induced gauge models is the scaling law for the physical mass scale $M$ as a function of the constituent mass $m_0$

$$M \propto (m_0 - m_c)^\gamma$$

(1)

where $m_c$ is some critical value, depending upon the regularization scheme, and the critical index $\gamma$ depends upon the universal characteristics of the constituent field, such as spin and number of components.

This scaling law comes about when one follows the running gauge coupling constant $\beta(r) = \frac{1}{g^2(r)}$ from the lattice spacing scales $r = r_0$ where we assume $\beta(r_0) = 0$ all the way to the hadronic distances $r_2 \sim \frac{m_0}{e}$, where again $\beta(r_2) \sim 0$ due to confinement. Along the way, $\beta(r)$ first goes up as $c_1 \ln \frac{r}{r_0}$ due to screening by the constituent field, until we reach the scales $r_1$ such that $r_1^2(m_0^2 - m_1^2) \sim 1$ where the screening turns off, leaving us with some large constant $\beta(r_1) = c_1 \ln \frac{r_1}{r_0}$. This constant serves as the bare coupling for the gauge theory, with effective ultraviolet cutoff at $r_1$. The rest of the story is well known: the $\beta(r)$ goes down as $\beta(r_1) - c_2 \ln \frac{r_2}{r_1}$ due to famous antiscreening from induced selfinteraction of the gauge field, all the way to the physical scale $r_2$. Comparing two parts of this $\Lambda$ shaped curve in the $\ln r, \beta$ plane we find $c_1 \ln \frac{r_2}{r_0} = c_2 \ln \frac{r_2}{r_1}$ which yields the scaling law (1).

The above arguments provide only a rough estimate of induced coupling. There are feedback effects from the hard gluons, as well, as some induced selfinteraction of the inducing field. If we go back to the cutoff scales from the scale $r_1$ with some unknown constant $\beta_1$ we would have to compute the renormalization of the effective $\phi^4$ theory coupled to gauge field. The peculiarity of this theory is that the scalar particle is very heavy, not like the ordinary Coleman-Weinberg model. Clearly, the effective $\phi^4$ coupling cannot be small, otherwise the screening of charge from scalar particle would not overcome the antiscreening from the gluons. Indeed, as we shall see in that last Section, it must be adjusted to some critical value, in order to get the scaling law.

We already know the example of the heavy particle theory which in a large $N$ limit becomes the string theory rather than particle theory. This is the same scalar field in adjoint representation of $SU(N)$ but without the gauge field. If we take the large $N$ limit, then we are left with planar graphs, which have singularity at some value of the coupling constant. So, if we adjust the scalar coupling to approach this large $N$ singularity, instead of adjusting the bare mass to get the small physical mass, then the planar graphs are known to condense to a fishnet with lattice scale cells, leading us to the string theory.

The problem with the latter model is well known, it does not seem to have a continuum limit at $D > 1$. The model we are considering now, is slightly different. If you look at the strong coupling expansion of the induced gauge theory, by integrating the scalars first, and
representing the effective gauge action as sum over the scalar loops, you find the sum over closed surfaces made of these loops glued together by sides. The typical size of the loop is $r_1$, in above notations. In the large $N$ limit we expect only the planar surfaces to survive, and produce some kind of string theory, with possible extra internal degrees of freedom at the world sheet.

The resulting theory must be universal, as we expect it to fall into one of the fixed points. In four dimensions there is only one nontrivial fixed point known, the QCD. So, the crucial question is whether the solution of particular induced gauge model is nontrivial. This question is a dynamical one, and could be answered only by exact solution.

Our best bet here is the scalar model in adjoint representation. The unique property of this model is the possibility to diagonalize the scalar field by the gauge transformation

$$\Phi(x) = \Omega(x)\phi(x)\Omega^\dagger(x); \Omega^\dagger(x)\Omega(x) = 1$$

so that the number of scalar degrees of freedom drops down from $N^2 - 1$ components of the traceless Hermitean matrix $\Phi(x)$ to its $N - 1$ eigenvalues $\phi(x) = \text{diag}(\phi_a(x)), a = 1, \ldots N, \sum \phi_a = 0$. These gauge invariant eigenvalues serve as a master field: they freeze at a certain vacuum average, with fluctuations $\delta \phi \sim \frac{1}{N}$.

The active degrees of freedom in the large $N$ limit are given by the gauge fields $U_\mu(x)$, associated with the links of the lattice. The analysis of [1] shows that the usual scalar particle loops in external gauge field within the strong coupling (random walk) expansion can be reproduced by the $U(N)$ group integrations over the gauge field for the frozen master field. The summations over eigenvalues, distributed in the strong coupling expansion according to the Wigner semi circle law

$$\rho(\phi) \propto \sqrt{a^2 - \phi^2}$$

yields in the large $N$ limit the same results as Gaussian integrations over matrix field $\Phi(x)$.

If we accept this scenario, then the model can be dramatically simplified by integration over gauge fields independently at each link. The catch is that this integral [3]

$$I(\Phi, \Psi) = \int DU \exp \left( N \text{tr} \Phi U \Psi U^\dagger \right) \propto \det_{ij} \exp(N\phi_i\psi_j) \Delta(\phi)\Delta(\psi)$$

where

$$\Delta(\phi) = \prod_{i<j}(\phi_i - \phi_j)$$

is the Vandermonde determinant, becomes at $N \to \infty$ nontrivial statistical system in external field, given by eigenvalue distribution

$$\rho_\phi(\mu) = \frac{1}{N} \sum_a \delta(\mu - \phi_a); \rho_\psi(\nu) = \frac{1}{N} \sum_a \delta(\nu - \psi_a)$$

This external field problem should be solved together with the saddle point equation

$$\varphi \int d\mu \frac{\rho_\phi(\mu)}{\phi_a - \mu} = \frac{1}{2} U'(\phi_a) - D \lim_{N \to \infty} \frac{1}{N} \left[ \frac{\partial \ln I(\Phi, \Psi)}{\partial \phi_a} \right]_{\Psi = \Phi}$$
where $D = 4$ is dimension of space, which we shall keep arbitrary in the rest of this paper. We introduced the general scalar particle potential

$$U(\phi) = \frac{1}{2} m_0^2 \phi^2 + \frac{1}{4} \lambda \phi^4 + \ldots$$

(8)

rather than the pure mass term considered in [1]. The higher terms would be needed to get to the correct fixed point of our model. Note, that we defined the mass term with traditional $\frac{1}{2}$ factor, which was skipped in [1].

In the previous paper [1] existence of two phases of this system was conjectured. The so called weak coupling phase, with a gap in the eigenvalue distribution reduces to the two matrix model

$$I(\Phi, \Psi) \propto \int dN x dN y \Delta(x) \Delta(y) \exp \left( N \sum_i x_i y_i - V_1(x_i) - V_2(y_i) \right)$$

(9)

with potentials

$$V_\Phi(x) = \int d\mu \rho_\Phi(\mu) \ln(\mu - x); \quad V_\Psi(y) = \int d\nu \rho_\Psi(\nu) \ln(\nu - y)$$

(10)

which model was solved exactly at $N = \infty$.

In the strong coupling phase, where there is no gap, the same representation (9) is formally valid, but potentials are singular, which leads to drastic changes in the large $N$ limit. The index sums cannot be replaced by the integrals in equations of the orthogonal polynomial method, but the individual terms become singular and produce the necessary factor of $N$. We managed to generate several terms of strong coupling expansion from the two matrix model, but could not find the exact solution.

This is a pity, since the induced QCD scenario corresponds to the strong coupling phase, where the distribution of eigenvalues is qualitatively the same as the semi-circle law. There are also some internal inconsistencies in the weak coupling solution [1]. In particular, there are no physical excitations in the vacuum, as the hopping term in the effective Lagrangean for the vacuum fluctuations vanishes in the large $N$ limit.

In the present paper we find the relevant strong coupling solution by completely a different method. The solution of the lattice model is described in Section 2, where we derive the lattice version of the equation for the density of eigenvalues of the scalar field. The critical phenomena are investigated in Section 3, where we find a remarkably simple local version of this equation and solve it at the critical point.

2 Exact Solution in the Large $N$ Limit

We are going to apply to the one link integral the good old Schwinger-Dyson equations. The traces of properly normalized derivatives with respect to $\Phi$ matrix

$$\langle \nabla \phi \rangle_{ij} \equiv \frac{1}{N} \frac{\partial}{\partial \Phi_{ji}}$$

(11)
act as traces of $\Psi$ matrix on our integral

$$\text{tr} \left( \nabla^n_{\Phi} - \Psi^n \right) I(\Phi, \Psi) = 0 \quad (12)$$

as it follows from the unitarity condition. Let us introduce the matrix function

$$F(\Phi) = \nabla_{\Phi} \ln I(\Phi, \Psi) \quad (13)$$

and rewrite above equation as follows

$$\text{tr} \left( (F(\Phi) + \nabla_{\Phi})^n \ast 1 - \Psi^n \right) = 0 \quad (14)$$

Now, summing the geometric series, we define the matrix

$$G_\lambda(\Phi) = (\lambda - F(\Phi) - \nabla_{\Phi})^{-1} \ast 1 \quad (15)$$

which satisfies the linear differential equation

$$\nabla_{\Phi} G_\lambda(\Phi) = -1 + (\lambda - F(\Phi)) G_\lambda(\Phi) \quad (16)$$

and extra condition

$$\text{tr} \left( G_\lambda(\Phi) + \frac{1}{\psi - \lambda} \right) = 0 \quad (17)$$

The differential equation allows us to express $G$ in terms of $F$ at fixed spectral parameter $\lambda$, after which the last condition would provide equation for $F$. One may readily check that these equations work in the strong coupling phase, where the function $F(\phi)$ expands in power series of its matrix argument. This expansion goes along with the expansion of $G_\lambda(\Phi)$ in inverse powers of $\lambda$, generating the moments relations (17). The expansion coefficients of $F(\Phi)$ are obtained one after another from the moments relations. We checked that the same coefficients arise in the orthogonal polynomial treatment of the two matrix model in the strong coupling phase. The first several terms of this expansion were presented in ref[1].

Now, let us take the large $N$ limit. The key observation is that at large $N$ one does not have to differentiate the traces, as that would not produce the necessary factor of $N$. Such factors are only produced by derivatives of the powers of the $\Phi$ matrix

$$\nabla_{\Phi} \Phi^n = \sum_{k=1}^{n} \Phi^{n-k} \frac{1}{N} \text{tr} \Phi^{k-1} \quad (18)$$

which terms remain finite in the large $N$ limit, when this formula can be written in an integral form

$$\nabla_{\Phi} \Phi^n = \int d\mu \rho_{\Phi}(\mu) \frac{\mu^n - \Phi^n}{\mu - \Phi} \quad (19)$$

This representation allows us to write the linear integral equation for function $G_\lambda(\nu)$

$$\int d\mu \rho_{\Phi}(\mu) \frac{G_\lambda(\mu) - G_\lambda(\nu)}{\mu - \nu} = -1 + (\lambda - F(\nu)) G_\lambda(\nu) \quad (20)$$
As for the extra condition (17), it can be rewritten as follows

$$\int d\mu \rho_\Phi(\mu) G_\lambda(\mu) = \int d\nu \frac{\rho_\Psi(\nu)}{\lambda - \nu} = V'_\Psi(\lambda)$$

(21)

At the moment we are interested in the spatially homogeneous master field, so that $\Psi = \Phi$, in which case we can omit the subscripts of the densities and potentials.

We define these spectral integrals in the principal value sense, so that both $F(\mu)$ and $G_\lambda(\mu)$ are real functions, when $\lambda, \mu$ vary at the real axis. At $\lambda \to \infty$ by construction $G_\lambda(\mu) \to \frac{1}{\lambda}$. We shall assume that the eigenvalues are distributed along the whole real axis, this assumption will be justified later.

The saddle point equation allows us to eliminate one of the unknown functions, by expressing $F(\nu)$ in terms of the potential

$$F(\nu) = \frac{\frac{1}{2} U'(\nu) - \Re V'(\nu)}{D}$$

(22)

When this expression is substituted back into (20), the terms combine nicely and we find the following singular integral equation

$$1 + \varphi \int d\mu \frac{\rho(\mu)G_\lambda(\mu)}{\mu - \nu} = (\lambda - R(\nu))G_\lambda(\nu); \quad R(\nu) = F(\nu) + \Re V'(\nu)$$

(23)

This equation could be regarded as the boundary value problem for analytic function

$$\mathcal{T}_\lambda(z) = 1 + \int d\mu \frac{\rho(\mu) G_\lambda(\mu)}{\mu - z}$$

(24)

in the upper half of complex $z$ plane, with usual analytic continuation to the whole plane by symmetry

$$\mathcal{T}_\lambda(\bar{z}) = \bar{\mathcal{T}}_\lambda(z)$$

(25)

In virtue of symmetry of the eigenvalue distribution we could have mapped the upper half plane to the cut plane by conformal transformation $\zeta = -z^2$ but it appears to be simpler to work in the upper half plane, keeping in mind the symmetry of boundary values at the real axis.

We shall assume the convergent integrals, in which case

$$\mathcal{T}_\lambda(\infty) = 1$$

(26)

The next term of asymptotic expansion at $z \to \infty$ is also known in virtue of an extra condition (21), namely

$$\mathcal{T}_\lambda(z) \to 1 - \frac{\Re V'(\lambda)}{z}$$

(27)

1This assumption, in fact will be valid only in the local limit, when the small vicinity of the origin in the eigenvalue distribution will be rescaled to the whole complex plane. In any particular lattice model, there might be finite support, but these effects are lattice artifacts, much in the same way, as the lattice momenta are distributed within the cell of inverse lattice. The eigenvalues have dimension of $m^{D-1}$, which means that for $D > 2$ the cutoff for the eigenvalues grows as the power of the momentum cutoff. In the local limit both cutoffs disappear.
The boundary problems of this kind can be solved in general. We express $G$ in terms of discontinuity

$$\Re T_\lambda(\nu_+) = \pi \rho(\mu) G_\lambda(\mu)$$

and find the following ratio of the complex conjugate boundary values of $T$ at $\nu_\pm$

$$\frac{T_\lambda(\nu_+)}{T_\lambda(\nu_-)} = \frac{\lambda - R(\nu) + i\pi \rho(\nu)}{\lambda - R(\nu) - i\pi \rho(\nu)}$$

The boundary problem has a unique solution

$$T_\lambda(z) = \exp \left( \int_{-\infty}^{\infty} \frac{d\nu}{\pi(\nu - z)} \arctan \frac{\pi \rho(\nu)}{\lambda - R(\nu)} \right)$$

The ambiguity of the choice of the branch of arctangent is removed by an extra requirement that at $\lambda \to \infty$

$$T_\lambda(z) \to 1 - \frac{V'(z)}{\lambda}$$

Comparing the first term at $z \to \infty$ with the asymptotics $[27]$ we find the following integral equation for the potential

$$\Re V'(\lambda) = \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \arctan \frac{\pi \rho(\nu)}{\lambda - R(\nu)}$$

$$\Im V'(\nu_+) = -\pi \rho(\nu)$$

$$R(\nu) = \frac{1}{2D} U'(\nu) + \frac{D - 1}{D} \Re V'(\nu)$$

We shall refer to this equation as the Master Field Equation, or MFE. In Appendix we study general properties of this equation, assuming finite support.

### 3 Critical Point

Let us study the small vicinity of the origin in the MFE. The idea is, that the eigenvalues scale as positive power of mass, therefore, in the lattice units we are using, the continuum theory corresponds to infinitesimal vicinity of the origin. The normalization of the density is also the lattice artifact, as it follows from dimensional counting. The internal nonlinearities in the scaling region, rather then the global normalization, would fix the scale of the density at the origin.

With this picture in mind, let us restore the imaginary part of MFE from analyticity in the upper half plane

$$V'(z) = \int_{-\infty}^{\infty} \frac{d\nu}{\pi i} \ln \left( 1 + \frac{i\pi \rho(\nu)}{z - R(\nu)} \right)$$

In the lower half plane we would have $-i$ instead of $i$ in the same integral.$^2$

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$^2$Let us stress once again, that this formula is not valid for finite support of the eigenvalues. The correct formulas in this case are studied in Appendix, and this one can be obtained from the general formulas in the limit $a \to \infty$. 

6
To make it less surprising, let us consider \( \Omega = \frac{V'(z)}{z} \) as analytic function of \( \zeta = -z^2 \), which it is, since the distribution of eigenvalues is symmetric. Then the upper half plane would map onto the first sheet of the Riemann surface, the real \( z \) axis being mapped onto the cut from \(-\infty\) to 0 in the \( \zeta \) plane, with negative (positive) \( z \) corresponding to upper (lower) side.

Imaginary part of \( V'(z_+) \) is negative both for positive and negative \( z \), and the real part changes sign with \( z \), therefore the values of \( \Omega(\zeta) \) at the opposite sides of the cut are complex conjugate, as usual. The second sheet function, with \( i \to -i \) has the opposite sign of the imaginary part, so it matches the corresponding boundary values at the cut, as it should.

We went into these boring details, because it took us some time to turn away from the usual picture with finite cut in the \( z \) plane and no singularity at infinity. We are going to use the old \( z \) variable in the rest of the paper, but restrict it to the upper half plane.

Let us study the equation for the imaginary part \( -\pi \rho(\lambda) \) of \( V'(\lambda_+) \). The imaginary part of MFE reads

\[
\pi \rho(\lambda) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \ln \left( 1 + \frac{\pi^2 \rho^2(\nu)}{(R(\nu) - \lambda)^2} \right).
\]

Let us differentiate this equation, we find

\[
\rho'(\lambda) = -\varphi \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{\rho^2(\nu)}{(\lambda - R(\nu))((\lambda - R(\nu))^2 + \pi^2 \rho^2(\nu))}.
\]

Now, investigation of this equation reveals that critical phenomena take place when the renormalized scalar potential

\[
2u(\phi) = U(\phi) - D\phi^2 = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} g\phi^4
\]

(36)

goes to zero in the lattice units we are using, so that \( R(\nu) \to \nu^3 \).

The relevant eigenvalues would scale as some power of scalar mass \( \nu \sim m^\gamma \), and density would scale as \( \rho \sim m^{2+\gamma} \), the same as \( u'(\phi) \) where the critical index \( \gamma \) will be computed below.

The scaling solution comes about in a tricky way. There are two relevant regions of integration in (35). In the region of \( |\nu - \lambda| \sim m^\gamma \), we could neglect \( \rho \) in the denominator, which yields the standard principle value integral. Note, that the size of the gap \( \delta \nu \) which is needed to define the principle value integral, could be any number in the interval \( m^{2+\gamma} \ll \delta \nu \ll m^\gamma \), the result would not depend upon this gap then.

There is another, much smaller region, that of \( |\nu - \lambda| \sim m^{\gamma+2} \ll \delta \nu \) where we cannot neglect \( \rho \) but we can expand \( u'(\nu) \) and \( \rho(\nu) \) in Taylor series near \( \nu = \lambda \) after which integration reduces to the residue in complex pole. The integral converges in this region, so that the boundary \( \delta \nu \) between regions drops from the answer.

\[3\text{The remarkable fact, that in spite of all the strong fluctuations of the link variables, which were very far from continuum, we still get the same critical value } m_c^2 = 2D \text{ of the bare mass, as in asymptotically free gauge theory, would be discussed later.}\]
Collecting the terms, we find, that the term $\rho'(\lambda)$ cancels, so that we are left with the following integrodifferential equation

$$
(\rho(\lambda)r(\lambda))'' = \text{regular terms} - \varphi \int_{-\infty}^{\infty} d\nu \frac{\rho^2(\nu)}{(\lambda - \nu)^3}
$$

$$
r(\lambda) = u'(\lambda) + \frac{D-1}{D} \varphi \int_{-\infty}^{\infty} d\mu \frac{\rho(\mu)}{\lambda - \mu}
$$

The regular terms represent the contributions from the cutoff scales. As usual in the field theory, such contributions provide the contact terms, without singularities. Note, that this equation could be integrated twice in $\nu$ at the expense of redefinition of the regular terms. On the left side we would obtain $\rho(\lambda)r(\lambda)$, and on the right the denominator $(\lambda - \nu)^3$ would be replaced by $2(\lambda - \nu)$.

This convenient method of elimination of regular terms is to introduce two analytic functions

$$
P(z) = \frac{D}{1-D} u'(z) - V'(z)
$$

$$
Q(z) = \text{regular terms} - \pi \int_{-\infty}^{\infty} d\mu \frac{\rho^2(\mu)}{z - \mu}
$$

and note that at $\Im z \to +0$ by construction

$$
\Im Q = (\Im P)^2.
$$

On the other hand, in virtue of the above equation for density

$$
\Re Q = \frac{1-D}{D} \Im (P^2)
$$

These two equations should be supplied with the symmetry conditions, namely, imaginary parts of these two functions at $\Im z \to +0$ must be even functions of $z$, whereas real parts must be odd. The derivative of the real part of $P(z)$ at the origin plays the role of effective scalar mass $m^2$.

This nonlinear boundary problem, which we call Local Master Field Equation, (LMFE) represents the main result of this work. At first glance it looks too simple to have nontrivial solution, but this is not true, as we found by analytical and numerical studies.

We doubt that there could exist an exact solution in the general case, but precisely at the critical point, corresponding to the vanishing mass, such solution can be found in virtue of scale invariance. Let us take the powerlike Anzatz

$$
P(z) = p z^\alpha, \; Q(z) = q z^{2\alpha}, \; \alpha = 1 + \frac{2}{\gamma}
$$

The symmetry conditions would be satisfied provided

$$
p = iv_0 \exp \left(-\frac{1}{2}i\pi \alpha\right) ; \; q = iv_0 \exp (-i\pi \alpha)
$$

8
where \( p_0 \) and \( q_0 \) are real numbers. Now, the powers of \( z \) would cancel in LMFE, so that we may check it at \( z = 1 \) where

\[
\Im P = p_0 \cos \frac{1}{2} \pi \alpha; \quad \Im P^2 = p_0^2 \sin \pi \alpha; \quad \Im Q = q_0 \cos \pi \alpha; \quad \Re Q = q_0 \sin \pi \alpha
\]  

(44)

Comparing this with LMFE, we find

\[
\cos \pi \alpha = -\frac{D}{3D - 2}; \quad \cos \frac{2\pi}{\gamma} = \frac{D}{3D - 2}; \quad q_0 = \frac{1 - D}{D} p_0^2
\]

(45)

The index \( \gamma(D) \) takes special values \( \gamma(0) = 4, \gamma(1) = \infty, \gamma(2) = 6 \) at low dimension, after which it smoothly goes to its asymptotic value \( \gamma(\infty) = 5 \) taking only transcendental values at integer dimensions. In particular, \( \gamma(4) = 5.41991, \alpha(4) = 1.36901 \). Note, that unlike the singularity at \( D = 1 \) which was manifest in initial model, triviality of the theory at \( D = 2 \) comes about in a very nontrivial way. In our opinion, this is an independent evidence for equivalence of this model to the gauge theory.

The general scaling solution

\[
\rho(\phi) = m^{2+\gamma} f \left( \frac{\phi}{m^{\gamma}} \right)
\]

(46)

starts from some constant at \( \phi = 0 \), then grows, approaching the above powerlike asymptotics, \( \rho \propto \phi^\alpha \) at \( \phi \gg m^{\gamma} \), so that the mass cancels.

This is the magnified vicinity of the origin in complete spectrum of eigenvalues; apparently, the density reaches the maximum, and then starts decreasing outside the scaling region. The density at the origin goes to zero, so that at negative \( m^2 \) the gap would appear in the distribution (clearly, one cannot continue our irrational powers of \( m \) to negative \( m^2 \)). In the forthcoming publication[2] we solve this equation numerically at various \( D \).

As discussed in the previous paper[1], the scaling laws in this model translate into logarithmic laws of original theory: for the effective induced gauge coupling of the form

\[
\frac{1}{g_0^2} \sim b N \ln \frac{\Lambda^2}{m_0^2 - 2D}
\]

(47)

the physical mass scale would be

\[
M^2 \sim (m_0^2 - 2D) \exp \left( -\frac{48\pi^2}{11N g_0^2} \right) \sim (m_0^2 - 2D) \left( \frac{m_0^2 - 2D}{\Lambda^2} \right)^{\frac{48\pi^2}{11}}
\]

(48)

Should we identify this scale with the scale of our field \( (m_0^2 - 2D)^{2\gamma} \) we would conclude that

\[
b = \frac{11}{48\pi^2} \left( \frac{4\pi}{\arccos \frac{D}{2}} - 1 \right)
\]

(49)

but this seems to be too long a shot: one should investigate the physical mass spectrum before drawing any conclusions.
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A Investigation of the Master Field Equation

Let us study the Master Field Equation (33) assuming the finite support with the following Ansatz for the normalized spectral density

$$\nu = a \cos(\alpha), \ 0 < \alpha < \pi, \ \rho(\nu) = \frac{1}{\pi} \sum_{k=1}^{\infty} \xi_k \frac{1 - \cos(2k\alpha)}{a \sin(\alpha)}, \ \sum_{k=1}^{\infty} \xi_k = 1$$

This is just the Fourier expansion, with correct symmetry properties. The corresponding Fourier expansion for the real part of potential can be found by elementary integration

$$\Re V'(a \cos(\beta)) = \varphi \int_{-a}^{a} d\nu \frac{\rho(\nu)}{a \cos(\beta) - \nu} = \sum_{k=1}^{\infty} \xi_k \frac{\sin(2k\beta)}{a \sin(\beta)}$$

and the potential as analytic function

$$V'(z) = \frac{1}{\sqrt{z^2 - a^2}} \left( 1 - \sum_{k=1}^{\infty} \xi_k \left( \frac{z - \sqrt{z^2 - a^2}}{a} \right)^{2k} \right)$$

Substituting in (33) the previous formula, and Fourier transforming, we find

$$\xi_k = \frac{a}{2\pi^2} \int_0^{\pi} d\alpha \sin(\alpha) \int_{-a}^{a} d\lambda \ln \left( \frac{\lambda - R - i\pi \rho}{\lambda - R + i\pi \rho} \right) \left( \frac{\lambda - \sqrt{\lambda^2 - a^2}}{a} \right)^{2k} \{ \rho \to 0 \}$$

where we skipped for brevity the arguments of $R, \rho$. The contour encircles anticlockwise the interval $-a, a$. The last factor takes complex conjugate values $\exp(\pm 2ik\beta)$ at $\lambda = a \cos(\beta)$, which yields correct integrand after taking discontinuity. The term with $\rho = 0$ is substructed, since this is how the branch of arctangent was defined.
The $\lambda$ integrand has only two singularities outside the contour, and decreases as integer negative power of $\lambda$, so that there is no residue at infinity. The discontinuity along the cut from $R - i\rho$ to $R + i\rho$ is equal to $\pi$, and this integral is elementary. As a result we find

$$\xi_k = \frac{a^2}{2\pi} \Im \int_0^\pi d\alpha \sin \alpha \left( \frac{(C - \sqrt{C^2 - 1})^{2k - 1}}{2k - 1} - \frac{(C - \sqrt{C^2 - 1})^{2k + 1}}{2k + 1} \right) - \{C \to \Re C\} \quad (54)$$

where

$$C \equiv \frac{R(\alpha) + i\pi \rho(\alpha)}{a} = \frac{m_0^2}{2D} \cos \alpha + \frac{D - 1}{aD} \Re V'(a \cos \alpha) - \frac{i}{a} \Im V'(a \cos \alpha + i0) \quad (55)$$

We restricted ourselves here to pure mass term, without interaction terms in the scalar potential.

Summing up the equation for $\xi_k$ from $k = 1$ to $\infty$, using normalization condition (50) on the left, and summing up the series on the right, we find the following equation for $a$

$$\frac{\pi}{a^2} = \int_0^\pi d\alpha \sin \alpha \Im \left( C - \sqrt{C^2 - 1} + \sqrt{\Re C^2 - 1} \right) \quad (56)$$

But the same normalization condition yields the simpler equation

$$\frac{\pi}{a^2} = \int_0^\pi d\alpha \sin \alpha \Im C \quad (57)$$

which means that contribution from the square roots should cancel. This relation can be written as follows

$$0 = \oint \frac{d\omega}{2i\omega} \frac{1 - \omega^2}{2i\omega} \left( \sqrt{C^2 - 1} - \sqrt{\Re C^2 - 1} \right) \quad ; \quad \omega = e^{i\alpha} \quad (58)$$

The explicit expression for the analytic potential (52) follows from above representation

$$V'(z) = \frac{2}{a^2} \left( z - \sqrt{z^2 - a^2} \right) - a \mathcal{U} \left( \frac{z - \sqrt{z^2 - a^2}}{a} \right) \quad (59)$$

where

$$\mathcal{U}(\zeta) = \int_0^{2\pi} \frac{d\alpha}{2\pi i} \sin \alpha \left( 2\zeta \left( C - \sqrt{C^2 - 1} \right) - \ln \frac{C + \sqrt{C^2 - 1} + \zeta}{C + \sqrt{C^2 - 1} - \zeta} \right) - \{C \to \Re C\} \quad (60)$$

Now, using (54) we can write down the integral equation for $C(\alpha)$.

$$C(\alpha) = \frac{a^2 m_0^2}{4a^2 D} \omega + \frac{a^2 m_0^2 + 4D - 2}{2a^2 D\omega} + \frac{2D - 1}{2D} \mathcal{U} \left( \frac{1}{\omega} \right) - \frac{1}{2D} \mathcal{U}(\omega) \quad (61)$$

which should be solved together with (58).
The semi circle law at \( m_0^2 \to \infty \) corresponds to the following

\[
a^2 \to \frac{4}{m_0^2}; \quad C(\alpha) \to -\frac{m_0^2}{\omega}
\]  

(62)

since \( U(\zeta) \to 0 \) as \( C^{-3} \) at large \( C \). The function \( C(\omega) \) becomes analytic at infinity, which allows us to expand the contour and in (58) to infinity, where expansion goes in even inverse powers of \( \omega \), so that there is no residue. As for the integral with the real part, here we cannot expand the contour, but the integral is small by itself, as the imaginary part of this square root does not vanish only in the small region of angles near \( \frac{1}{2}\pi \).

We solved these equations at \( D = 4 \) numerically by iterations, taking as unknowns the values of \( C \) at \( M = 4096 \) of points in the first quadrant

\[
C_m = C \left( \alpha = \frac{\pi m}{2M + 1} \right); \quad 0 < m \leq M; \quad C_{2M+1-m} = -\bar{C}_m
\]  

(63)

The integral was approximated by sum over these equidistant points, which corresponds to the Gauss-Tchebyshev method.

As expected, at large \( m_0^2 \) we reproduced the semi circle, moreover, it was difficult to get anything else. The nontrivial term \( U \) in our equation was practically negligible, unless we adjusted the bare mass so, that the derivative \( \Re V''(0) \) vanished. Then we started seeing interesting phenomena, but in order to compare them with our local equations, much better resolution would be needed. According to our analysis in the Section 3, there are two different small scales involved, and there are strong cancellations. In such situation, the approximation of the integral by sum over equidistant points is no longer valid. The corresponding numerical solution is studied in the forthcoming work[2].