NEUTRON STAR POPULATION DYNAMICS. I. MILLISECOND PULSARS

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ABSTRACT

We study the field millisecond pulsar (MSP) population to infer its intrinsic distribution in spin period and luminosity and to determine its spatial distribution within the Galaxy. Our likelihood analysis on data from extant surveys (22 pulsars with periods less than 20 ms) accounts for the following important selection effects: (1) the survey sensitivity as a function of direction, spin period, and sky coverage; (2) interstellar scintillation, which modulates the pulsed flux and causes a net increase in search volume of \( \sim 30\% \); and (3) errors in the pulsar distance scale.

Adopting power-law models (with cutoffs) for the intrinsic distributions, the analysis yields a minimum-period cutoff \( P_{\text{min}} > 0.65 \text{ ms} \) (99\% confidence), a period distribution proportional to \( P^{-2.0 \pm 0.3} \), and a pseudoluminosity distribution proportional to \( L_{\nu}^{-2.0 \pm 0.2} \) (where \( L_{\nu} \) is the product of the flux density and the square of the distance, for \( L_{\nu} \geq 1.1 \text{ mJy kpc}^2 \)).

We find that the column density of MSPs (uncorrected for beaming effects) is \( \sim 50^{+30}_{-20} \text{ kpc}^{-2} \) in the vicinity of the solar system. For a Gaussian model, the \( z \) scale height is \( 0.65^{+0.16}_{-0.12} \text{ kpc} \), corresponding to the local number density \( 29^{+11}_{-7} \text{ kpc}^{-3} \). (For an exponential model, the scale height becomes \( 0.50^{+0.19}_{-0.13} \text{ kpc} \), and the number density \( 44^{+7}_{-5} \text{ kpc}^{-3} \).) Estimates of the total number of MSPs in the disk of the Galaxy and for the associated birthrate are given. The contribution of a diffuse halo-like component (tracing the Galactic spheroid, the halo, or the globular cluster density profile) to the local number density of MSPs is limited to \( \leq 1\% \) of the midplane value.

We consider a kinematic model for the MSP spatial distribution in which objects in the disk are kicked once at birth and then orbit in a smooth Galactic potential, becoming dynamically well-mixed. The analysis yields a column density \( 49^{+7}_{-3} \text{ kpc}^{-2} \) (comparable to the above), a birth \( z \) kick velocity \( 52^{+4}_{-1} \text{ km s}^{-1} \), and a three-dimensional velocity dispersion of \( \sim 84 \text{ km s}^{-1} \). MSP velocities are smaller than those of young, long-period pulsars by about a factor of 5. The kinematic properties of the MSP population are discussed, including expected transverse motions, the occurrence of asymmetric drift, the shape of the velocity ellipsoid, and the \( z \) scale height at birth. If MSPs are long-lived, then a significant contribution to observed MSP \( z \) velocities is the result of diffusive processes that increase the scale height of old stellar populations; our best estimate of the one-dimensional velocity kick that is unique to MSP evolution is \( \sim 40 \text{ km s}^{-1} \) if such diffusion is taken into account.

The scale heights of millisecond pulsars and low-mass X-ray binaries are consistent, suggesting a common origin and that the primary channel for forming both classes of objects imparts only low velocities. Binaries involving a common envelope phase and a neutron star-forming supernova explosion can yield such objects, even with explosion asymmetries like those needed to provide the velocity distribution of isolated, nonspun-up radio pulsars.

Future searches for MSPs may be optimized using the model results. As an example, we give the expected number of detectable MSPs per beam area and the volumes of the Galaxy sampled per beam area for a hypothetical Green Bank Telescope all sky survey. Estimates for the volume that must be surveyed to find a pulsar faster than 1.5 ms are given. We also briefly discuss how selection effects associated with fast binaries influence our results.

Subject headings: pulsars: general — stars: kinematics — stars: neutron — stars: rotation — stars: statistics

1. INTRODUCTION

Millisecond pulsars (MSPs) differ from slower spin pulsars in important ways. First, their spin-down rates and derived surface magnetic fields are several orders of magnitude smaller. MSPs have implied fields of \( 10^{7.9-10.0} \text{ G} \), while pulsars with periods of order 1 s are characterized by magnetic fields of \( 10^{11-10^{13}} \text{ G} \). Closely related is observation that the characteristic spin-down times of MSPs, ranging from several tenths of a gigayear to ten or more gigayears, far exceed those of slower spin pulsars. Some

MSPs were born with periods near their present-day values and are, consequently, much younger than their spin-down times (Camilo, Thorstensen, & Kulkarni 1994). However, MSPs are thought to be active as radio pulsars for hundreds to thousands of times longer than strong-field pulsars. Taking active lifetimes into account, it appears that there may be comparable numbers of young pulsars and MSPs in the Galaxy, although the birthrate of MSPs is \( \sim 10^4 \) times smaller.

A second significant difference is that more than \( 3/2 \) of MSPs are in binary systems, while young, strong-field pulsars are largely solitary objects, a fact that has both theoretical and observational implications. Clearly, the evolutionary pathways that give rise to MSPs are related inte-
grally to the interaction of the binary stars (Alpar et al. 1982; Ruderman & Shaham 1983; for a general review, see Bhattacharya & van den Heuvel 1991). Moreover, searches for MSPs must confront the additional selection effects that mitigate against the detection of accelerated pulsars. As is shown below, detection of the fastest spinning pulsars is inhibited by any effects that smear out the pulse, such as dispersive propagation in the interstellar medium. Orbital motion, uncompensated for in the surveys we analyze here, also smears out pulses according to the change in velocity over the duration of the observation and is therefore most important for short-period pulsars in compact binaries (e.g., Johnston & Kulkarni 1991).

In this paper, we analyze the spatial distribution of MSPs. Our purpose is to derive the best estimates for MSP population parameters through careful consideration of survey sensitivities as a function of pulse period, dispersion measure, and other relevant factors. Our census essentially measures the local number density of MSPs and the falloff in number density above the Galactic plane, and it establishes upper limits on a diffuse, halo-like component of the MSP population. We analyze the implications of the spatial distribution for the kinematics of the MSP population, inferring the diffuse and impulsive velocity increments suffered. We compare predictions of the distribution of proper motions with extant observations, compare the spatial distribution of low-mass X-ray binaries (LMXBs) and MSPs, and describe the importance of our determination of the MSP kick velocity for binary evolutionary scenarios. We provide detailed analysis of the influence of selection effects on the discovery of short-period pulsars and pulsars in binaries.

Our results have immediate relevance in a number of respects. Recent MSP surveys have been conducted on the premise that the MSPs are essentially isotropically distributed around the Sun, at least to the depths that surveys probe. Our results establish the scale height and show that Arecibo-type surveys see beyond it.

The third way that MSPs differ from slow-spin pulsars is in their peculiar space motions. This is one of the main conclusions of the present paper. Kinematic evidence (e.g., Dewey, Cordes, & Wolszczan 1988; Cordes et al. 1990; Wolszczan 1994; Nice & Taylor 1995; Nicastro & Johnston 1995) suggests that MSPs are low-velocity objects, with typical transverse speeds \( \lesssim 100-200 \text{ km s}^{-1} \). Such velocities are much less than young pulsars, which have an average speed \( \sim 500 \text{ km s}^{-1} \) (Lyne & Lorimer 1994; Cordes & Chernoff 1997). Our results allow us to put more stringent, albeit statistical, limits on the MSP velocities than has been achieved hitherto.

The use of the spatial distribution of MSPs as an indirect means for determining their peculiar velocities is more robust than an analysis of proper-motion data. The primary reason is that the orbits of MSPs are perturbed significantly from circular motion around the Galactic center so that, given their ages (\( >0.1 \) Gyr), corrections for differential Galactic rotation cannot be made. Arnaud & Rothenflug (1981) applied a similar spatial analysis to young, high-field pulsars as a means for determining their velocities. (The methodology is correct, but their assumption that high-field pulsars form a steady, relaxed population is not. Today we know that \( \sim 25\%-30\% \) escape the Galaxy and that many radio pulsars shut off before traveling to the limiting distance for detection.)

Our approach differs in several ways from those taken by other authors. First, we use a likelihood analysis to provide the best estimates of the MSP population parameters, to account accurately for survey selection effects and distance errors, and to express clearly our physical assumptions. Second, we restrict our analysis to pulsars with spin periods less than 20 ms. We do so because it appears that these neutron stars (NS) are distinct from other pulsars that may have undergone accretion-driven spin-up but were left with longer periods (Bhattacharya & van den Heuvel 1991). We also consider them to be distinct from the higher mass NS-NS binaries, which have pulsars with longer pulse periods (B1913+16, \( P = 59 \text{ ms} \) [Taylor & Weisberg 1989]; B1534+12, \( P = 38 \text{ ms} \) [Wolszczan 1991]). Third, our approach includes the effects of interstellar scintillations, which modulate the pulsar flux density and influence the rate of detection in surveys.

The paper is organized as follows. In \$2\$, we discuss sensitivities of pulsar surveys and derive quantities that are needed in our analysis. A preliminary attack on the problem is given in \$3\$, where we present a \( V/V_{\max} \) analysis, analogous to that used on quasars and gamma-ray burst sources, in order to illustrate the uncertainties involved in pulsar surveys. We derive the survey likelihood function in \$4\$ and apply it in \$5\$ to a disk-only distribution of MSPs. In \$6\$, we apply the analysis to a disk model based on numerical integration of NS orbits in the Galactic potential. We consider a combined disk and diffuse halo-like model in \$7\$. Sections 8–11 present the implications of our results for the MSP birthrate, the origins of MSPs and their velocities, and the optimization of MSP surveys. In \$14\$, we summarize the paper.

## 2. Pulsar Surveys

### 2.1. Minimum Detectable Flux \( S_{\text{min}} \)

A pulsar survey has a minimum detectable flux density \( S_{\text{min}} \) that depends on radiometer noise, the pulse shape, and details of the Fourier analysis used to find pulsars. In Appendix A, we derive \( S_{\text{min}} \) for surveys, which includes pulse-broadening effects (from interstellar dispersion and scattering and from detector time constants) and the effects of flux variations from interstellar scintillations (Appendix B), which are strong for the MSP surveys.

For a given direction, the minimum detectable flux density is a function of pulse width and period and radiometer-noise level. The minimum detectable flux density is a function of direction (Galactic coordinates \( l, b \)), dispersion measure \( [\text{DM} = \int_0^d \text{d}x n_e(x), \text{where} n_e \text{ is the free-electron density and} D \text{ is the pulse distance}], \) radio frequency \( (\nu) \), bandwidth \( (\Delta \nu) \), and the number of channels \( (N_{\text{ch}}) \), as well as system temperature \( (T_{\text{sys}}) \), telescope gain \( (G) \), the intrinsic pulse duty cycle, and additional survey-dependent factors:

\[
S_{\text{min}} = S_{\text{min}}(l, b, P, \text{DM}, \nu, \Delta \nu, N_{\text{ch}}, T_{\text{sys}}, G, \ldots).
\]  

\( S_{\text{min}} \) depends on additional, unspecified parameters, especially those that describe orbital motion. In most of this paper, we ignore such motion; however, in \$11\$, we discuss survey biases against short-period binaries and their possible effects on our conclusions. In Figure 1, we show \( S_{\text{min}} \) plotted against period for several values of DM. These curves apply for drift-scan surveys made with the Arecibo telescope at 0.43 GHz toward the Galactic pole and at zero
zenith angle. We have used a numerical version of a 408 MHz survey (Haslam et al. 1982) to calculate the background sky temperature. We assume a Gaussian pulse shape with 3% intrinsic duty cycle. While this duty cycle is shorter than those of some MSPs, the duty cycle, and hence \( S_{\text{min}} \), is dominated by extrinsic pulse broadening from dispersion, scattering, and instrumentation. Pulse broadening from scattering is taken into account by estimating it from large-scale Galactic models for the electron density, as described by Cordes et al. (1991) and Taylor & Cordes (1993, hereafter TC).

The maximum distance to which a particular pulsar is detectable, \( D_{\text{max}} \), is given by

\[
D_{\text{max}} = \left( \frac{L_p}{S_{\text{min}}} \right)^{1/2}.
\]  

We use the “pseudoluminosity,” \( L_p \equiv SD^2 \), that is often adopted in population studies of pulsars. Although it is preferable to use a physical luminosity (i.e., expressed in units of ergs s\(^{-1}\)) in analyzing pulsar statistics (Chernoff & Cordes 1997), estimation of physical luminosities for MSPs is not yet possible because we do not understand radio beaming in MSPs to the same extent that we do for young, strong-field objects (Backer 1976; Rankin 1983, 1993; Lyne & Manchester 1988). Note that \( S_{\text{min}} \) depends implicitly on the dispersion measure in a given direction, which in turn depends on \( D_{\text{max}} \). For this reason, \( D_{\text{max}} \) must be found iteratively.

Figure 2 shows (solid lines) \( S_{\text{min}} \) plotted against distance for several values of pulse period and for several directions. These curves illustrate the strong dependence of \( S_{\text{min}} \) on direction and period. The dashed line indicates the inverse-square law variation of flux density for a source of 30 mJy at a distance of 1 kpc. The intersection points of the dashed and solid lines determine the maximum distances \( D_{\text{max}} \) that a pulsar can be seen for the different cases.

From \( D_{\text{max}} \), the total volume to which the survey is sensitive in a given beam area is

\[
V_{\text{max}} = \frac{1}{4} \Omega_b D_{\text{max}}^3.
\]  

Because of the strong period and luminosity dependence of \( D_{\text{max}} \) and hence \( V_{\text{max}} \), the latter quantity may be used to determine the period and luminosity distributions of MSPs.

### 2.2 MSP Surveys: Properties, Volumes, and Distances Sampled

We have applied the likelihood analysis to eight pulsar surveys that have been reported in the literature. Six use the Arecibo telescope, yielding 11 MSPs; the seventh is the Parkes southern hemisphere survey that yielded 10 MSPs, as reported by Manchester (1994) when the survey was about 75% complete. \(^2\)The eighth survey is the Jodrell Bank survey, a portion of which has been reported in the discovery of the MSP J1012+5307 (Nicastro et al. 1995). We have not included J0218+4232 (Navarro et al. 1995) because it was discovered in an aperture synthesis survey with selection criteria quite different from the periodicity searches of the surveys we consider. In a future analysis of the distributions of MSP orbital parameters, we will extend our analysis to synthesis surveys.

Table 1 lists the surveys we have used. The columns include the (1) survey number (an arbitrary choice based solely on the order in which we analyzed the surveys); (2)
observatory site for the survey; (3) survey frequency; (4) solid angle of interference-free observations; (5) number of MSPs found in the survey; (6) system temperature of the telescope, expressed in janskys, for observations at the zenith and toward the Galactic poles; (7) minimum flux density for detection of long-period pulsars at the zenith and toward the Galactic poles; and (8) survey reference number.

Table 2 lists the MSPs used in our analysis. Figure 3 shows the total volume searched in each of the eight programs as a function of spin period for a fixed luminosity of 16 mJy kpc². The Parkes survey by itself has searched the largest volume. The Arecibo surveys in aggregate cover a larger area on the sky but less sensitive at most periods, is better optimized to finding MSPs. We note that the Arecibo surveys at Arecibo (see references below) sample distances that are well beyond the scale height of the population, whereas surveys at Parkes, covering a larger area on the sky but less sensitive at most periods, is better optimized to finding MSPs. We note that the Arecibo search volumes that are devoid of MSPs provide the most stringent constraints on the scale height of MSPs, whereas

| Table 1 | MSP Survey Parameters |
|----------------------------------|------------|
| Survey Site | $v$ (GHz) | $\Omega_S$ (deg²) | $N_{\text{MSP}}$ (Jy) | $S_{\text{max}}$ (mJy) | Reference |
|-----------------|----------|----------------|----------------|----------------|-----------|
| (1) | (2) | (3) | (4) | (5) | (6) | (8) |
| 1 | A | 0.43 | 680 | 3 | 3 | 0.5 | 1 |
| 2 | A | 0.43 | 235 | 3 | 1.0 | 2 |
| 3 | A | 0.43 | 250 | 0 | 3 | 0.4 | 3 |
| 4 | A | 0.43 | 7 | 1 | 3 | 0.2 | 4 |
| 5 | A | 0.43 | 682 | 2 | 3 | 0.7 | 5 |
| 6 | A | 0.43 | 150 | 1 | 3 | 0.4 | 6 |
| 7 | P | 0.44 | 20,600 | 10 | 90 | 3.0 | 7 |
| 8 | J | 0.41 | 1650 | 1 | 70 | 3.1 | 8 |

*a A = Arecibo, J = Jodrell Bank, P = Parkes.

REFERENCES—(1) Camilo et al. 1996; (2) Nice et al. 1995; (3) Thorsett et al. 1993; (4) Lundgren et al. 1995; (5) Foster et al. 1995; (6) Manchester et al. 1996; (7) Wolszczan 1990; (8) Nicastro et al. 1995.

As we demonstrate in this paper, deep, high-latitude surveys at Arecibo (see references below) sample distances that are well beyond the scale height of the population, whereas surveys at Parkes, covering a larger area on the sky but less sensitive at most periods, is better optimized to finding MSPs. We note that the Arecibo search volumes that are devoid of MSPs provide the most stringent constraints on the scale height of MSPs, whereas

| Table 2 | Millisecond Pulsars Used |
|----------------------------------|----------------|
| MSP Name | $l$ (deg) | $b$ (deg) | $P$ (ms) | $\log AP$ (ms) | $S$ (mJy) | $\Delta S$ (mJy) | $D_L$ (kpc) | $D_V$ (kpc) | Reference |
|-----------------|----------|----------|--------|--------------|-------|-------------|---------|---------|-----------|
| J0034 - 0534    | 111.5    | -68.1    | 1.88   | -10.7        | 16    | 5           | 0.74    | 1.23    | 7        |
| J0437 - 4715    | 253.4    | -42.0    | 5.76   | -11.4        | 600   | 180         | 0.105   | 0.175   | 7        |
| J0613 - 0200    | 210.4    | -9.3     | 2.19   | -11.4        | 21    | 6           | 1.64    | 2.74    | 7        |
| J0711 - 6830    | 279.5    | -23.4    | 5.49   | -4.1         | 7     | 2           | 0.77    | 1.39    | 7        |
| J0751 + 1807    | 202.7    | 21.1     | 3.48   | -11.0        | 10    | 3           | 1.51    | 2.53    | 4        |
| J1012 + 5307    | 160.3    | 50.9     | 5.26   | -10.7        | 30    | 9           | 0.39    | 0.65    | 8        |
| J1045 - 4509    | 280.9    | 12.3     | 7.47   | -10.7        | 20    | 6           | 2.43    | 4.05    | 7        |
| B1257 + 12      | 311.3    | 75.4     | 6.22   | -12.7        | 20    | 6           | 0.47    | 0.77    | 6        |
| J1455 - 3330    | 330.7    | 22.6     | 7.99   | -10.2        | 13    | 4           | 0.56    | 0.93    | 7        |
| J1640 + 2224    | 41.1     | 38.3     | 3.15   | 12.3         | 23    | 4           | 0.88    | 1.48    | 5        |
| J1643 - 1224    | 5.7      | 21.2     | 4.62   | -10.5        | 75    | 23          | 4.84    | $\infty$| 7        |
| J1713 + 0747    | 28.8     | 25.2     | 4.57   | -12.1        | 36    | 10          | 0.8     | 1.6      | 5        |
| J1730 - 2304    | 3.1      | 6.0      | 8.12   | -10.5        | 43    | 13          | 0.38    | 0.64    | 7        |
| B1855 + 09      | 42.3     | 3.1      | 5.36   | -12.5        | 31    | 9           | 0.70    | 1.30    | 2        |
| B1937 + 21      | 57.5     | -0.3     | 1.56   | -12.7        | 240   | 72          | 3.60    | 15.7    | 2        |
| B1957 + 20      | 59.2     | -4.7     | 1.61   | -12.52       | 20    | 6           | 1.15    | 1.91    | 2        |
| J2019 + 2425    | 64.7     | -6.6     | 3.93   | -12.7        | 15    | 5.0         | 0.68    | 1.14    | 2        |
| J2124 - 3358    | 10.9     | 45.4     | 4.93   | -10.2        | 20    | 6           | 0.18    | 0.30    | 7        |
| J2145 - 0750    | 47.8     | -42.1    | 16.05  | 9.7          | 50    | 15          | 0.38    | 0.62    | 7        |
| J2317 + 1439    | 91.4     | -42.4    | 3.45   | -12.7        | 14    | 5           | 1.4     | 2.3     | 1        |
| J2322 + 2057    | 96.5     | -37.3    | 4.81   | -12.6        | 4     | 2           | 0.5     | 1.1     | 1        |
| J2229 + 2643    | 87.8     | -26.3    | 2.98   | -12.1        | 18    | 5           | 1.0     | 2.0     | 1        |

REFERENCES—(1) Camilo et al. 1996; (2) Nice et al. 1995; (3) Foster et al. 1995; (4) Lundgren et al. 1995; (5) Thorsett et al. 1993; (6) Wolszczan 1990; (7) Manchester et al. 1996; (8) Nicastro et al. 1995.
the totality of detected MSPs determines essentially the local MSP number density.

3. \( V/V_{\text{max}} \) FOR MILLISECOND PULSARS

The main method of analysis in this paper uses a likelihood function to determine intrinsic properties of the MSP population after accounting for survey selection effects embodied in \( S_{\text{min}} \) as calculated above. Here we motivate our discussion by applying a \( V/V_{\text{max}} \) analysis to MSP surveys. Let us consider the line of sight to a MSP discovered in a survey. Subsequent observations yield precise determinations of \( P, \text{DM}, l, \) and \( b \). The flux density is \( S_J \) at the time of discovery and \( S \) as a long time average. The flux density is time dependent, owing predominantly to refractive and diffractive interstellar scintillation (DISS; e.g., Rickett 1990; Kaspi & Stinebring 1992; Stinebring & Condon 1990; Cordes, Weisberg, & Boriakoff 1985). A distance estimate derives from the DM and the TC model for the interstellar electron density. The model and, in some cases, auxiliary measurements (timing parallax, neutral-hydrogen absorption, and association with supernova remnants; Frail & Weisberg 1990) yield a range of possible distances, \( (D_L, D_H) \).

The volume between us and a given pulsar in a beam of solid angle \( \Omega_b \) is

\[
V = \frac{1}{3} \Omega_b D^3. \tag{4}
\]

The ratio of \( V \) to \( V_{\text{max}} \) from equation (3) may then be written as

\[
\frac{V}{V_{\text{max}}} = \left( \frac{S_{\text{min}}}{S} \right)^{3/2}. \tag{5}
\]

Application of equation (5) involves subtleties that depend on whether the flux density reported for a given object is influenced by the random process associated with DISS. DISS generally increases the volume in which a pulsar can be detected (cf. Appendix B). Saturated DISS that is not quenched by time-bandwidth averaging modulates the flux density by a random variable drawn from an exponential probability density function (pdf). Although more often than not the modulation is less than unity, the net effect is to increase the volume by a factor \( 11/5/2 \sim 1.33 \).

Figure 4 shows \( V/V_{\text{max}} \) for field MSPs plotted against \( z = D \sin b \). Horizontal error bars reflect uncertainties in the measured flux density and distance errors, the latter also determining the vertical error bars on \( z \). In most cases, we have used the 400 MHz flux density reported by Taylor, Manchester, & Lyne (1993), which is usually an average of many observations and is influenced minimally by DISS. Flux calibrations are typically only about 20% accurate. However, weak pulsars can appear brighter than average because of DISS at the time of discovery, so the discovery flux \( S_J > S_{\text{min}} \) while \( S < S_{\text{min}} \). For these cases (J0034–0534 and J0711–6830), \( V/V_{\text{max}} \rightarrow 1 \). The intrinsically brightest MSP, B1937+21, is detectable to only \( 8 \) kpc despite its large luminosity \( L_p \sim 1000 \) mJy kpc\(^2\) because, at its low Galactic latitude, dispersion and scattering effects grow rapidly with distance. Consequently, we argue that the claimed upper distance limit, \( D_U \sim 15.7 \) kpc, is a factor of 2 too large. The moderate-latitude pulsar J1643–1224 is attributed with only a lower bound on its distance by the TC model, \( D > 4.8 \) kpc, because its DM cannot be accounted for by the model. We suspect that this pulsar's DM is enhanced by unmodeled ionized gas along the line of sight and that the distance is most likely less than 4.8 kpc.

In the absence of further data, however, we use the distance lower bound as is.

Apart from the pulsar with a questionable distance estimate (J1643–1224), Figure 4 shows that MSPs are to be found at only low values of \( z \), suggesting therefore that the scale height for MSPs is \( \sim 0.5–1 \) kpc. Properly estimating the scale height requires careful accounting of selection effects in MSP surveys, as we do in § 4. However, Arecibo surveys at high latitudes search to several kiloparsecs for typical luminosities. The absence of high-\( z \) pulsars is therefore especially striking. The Arecibo MSPs also tend to have small values of \( V/V_{\text{max}} \) as would be expected for surveys that search well beyond the scale height of the population.

4. LIKELIHOOD ANALYSIS

4.1. Observables, Assumptions, and Statistical Method

A survey for MSPs typically searches many beam areas for each MSP discovery. The spatial distribution of MSPs determines this yield, along with the survey sensitivity as a function of period, the period distribution, and the luminosity function. Here we derive the likelihood function for a survey, taking these factors into account. We take as observables the directions of all beam areas searched, the survey sensitivities in these directions, and the parameters that describe individual pulsars, including direction, period, flux density, and dispersion measure. We also use distance estimates based primarily on the electron density model of TC. Such distances are imprecise, and our method takes into account the large uncertainties in distance that translate into large uncertainties in implied luminosity.

Let us consider a telescope beam with solid angle \( \Omega_b \). The mean number of MSPs expected in the beam per unit
period, luminosity, and distance is

$$\frac{\partial^3 \langle N_p \rangle}{\partial P \partial L_p \partial D} = \Omega_0 D^2 n_p(D, l, b) f_{L_p}(L_p) f_P(P),$$  

(6)

where the number density of MSPs, $n_p(D, l, b)$, is an arbitrary function of position. We have assumed that the joint probability distribution of period $P$, luminosity $L_p$, and position is factorable. The physical assumption is that the distribution in space is independent of the distribution of intrinsic pulsar properties ($P$ and $L_p$) and, furthermore, that $P$ and $L_p$ are uncorrelated. We write the period pdf, $f_P(P)$, and luminosity pdf, $f_{L_p}(L_p)$, each with unit normalization.

With the small number of MSPs currently available, there is scant evidence that the factorization is or is not appropriate; however, in the future, our method can be applied easily to more complicated joint distributions if warranted. In particular, we defer to another paper exploration of the joint statistics of $L_p$ and $P$. Here we consider only disk and disk + diffuse models, and we take into account the variation of the telescope gain across its beam.

To calculate the mean number of MSPs expected per beam, we integrate equation (6) to obtain

$$\langle N_p \rangle = \Omega_0 \int dP f_P(P) \int dL_p f_{L_p}(L_p) \int_0^{D_{\text{max}}} dD dD^2 n_p(D, l, b).$$  

(7)

The volume searched per beam, averaged over $P$ and $L_p$, is

$$\delta V_s = \frac{\Omega_0}{3} \int dP f_P(P) \int dL_p f_{L_p}(L_p) D_{\text{max}}^3.$$

(8)

Survey sensitivities are implicit in $D_{\text{max}}$, as discussed in § 2. For detections, we take into account the constraints that exist, because of postdiscovery observations, on period, flux density, and distance: $P \pm \Delta P/2$, $S \pm \Delta S/2$, and $D \in (D_L, D_U)$. Integrating over the subvolume bounded by these constraints, the mean number of MSPs is

$$\langle N_p \rangle_D = \Omega_0 \int_{P \pm \Delta P/2} dP f_P(P) \int_{D_L}^{\min(D_L, D_{\text{max}})} dD \int_{S \pm \Delta S/2}^{S_{\pm D^2}} dS f_{L_p}(S^2D^2),$$

$$\times \int_{\min(D_L, D_{\text{max}}), (P_S, D)}^{\max(D_L, D_{\text{max}}), (P_S, D)} dD^2 n_p(D, l, b).$$  

(9)

where $S_{1, 2} = S \pm \Delta S/2$. For each known pulsar, $D_{\text{max}}$ is the maximum distance for the survey that could have detected each pulsar. Several MSPs in our analysis were first discovered in other surveys but were subsequently detected (rediscovered) in one of the eight surveys, and we analyze them accordingly. For a given beam, the Poisson probabilities for detecting zero or one MSP are

$$P_0 = e^{-\langle N_p \rangle},$$

$$P_1 = \langle N_p \rangle e^{-\langle N_p \rangle}.$$  

(10)

We construct the survey likelihood function as the product of nondetection (ND) and detection (D) factors:

$$\mathcal{L} = \mathcal{L}_\text{ND} \mathcal{L}_D,$$

(11)

where, for $N_b$ total beams searched, $M_p$ MSPs found, and assuming $\langle N_p \rangle \ll 1$,

$$\mathcal{L}_\text{ND} = \exp \left(-\sum_{j=1}^{N_b} \langle N_p \rangle_j \right),$$  

(12)

$$\mathcal{L}_D = \prod_{k=1}^{M_p} \langle N_p \rangle_{D_k}.$$  

(13)

The log likelihood is

$$\Lambda \equiv \ln \mathcal{L} = -\sum_{j=1}^{N_b} \langle N_p \rangle_j + \sum_{k=1}^{M_p} \ln \langle N_p \rangle_{D_k}.$$  

(14)

The likelihood function may be simplified if we factor the pulsar number density into a constant $n_0$ times a shape factor:

$$n_p(D, l, b) = n_0 h(D, l, b),$$

(15)

where $h(D, l, b)$ is dimensionless and has a maximum of unity. Substituting, the likelihood function becomes

$$\Lambda(\theta, n_0) = M_p \ln n_0 - n_0 V_d + \sum_{k=1}^{M_p} \ln \delta V_p,$$

(16)

where the vector $\theta$ denotes the set of parameters other than $n_0$. We define the survey detection volume as the sum over beams,

$$V_d = \sum_{j=1}^{N_b} \delta V_{dj},$$

(17)

where (dropping beam labels)

$$\delta V_d = \frac{\partial \langle N_p \rangle}{\partial n_0}.$$  

(18)

and the constrained subvolume per discovered MSP is

$$\delta V_p = \frac{\partial \langle N_p \rangle_D}{\partial n_0}.$$  

(19)

The survey detection volume $V_d$ is the volume searched weighted by the dimensionless MSP space density, $h$. The expected number of MSPs in a survey is simply $n_0 V_d$. Equation (16) applies to a single-component density model, such as the disk distribution we consider in the next two sections. Multiple components require the alternative treatment of § 7.

Maximizing $\mathcal{L}$ with respect to $n_0$, we obtain the best-fit number density (for a specific set of parameters, $\theta$)

$$\hat{n}_0 = \frac{M_p}{V_d}.$$  

(20)

Substituting, the log likelihood becomes

$$\Lambda(\theta, \hat{n}_0) = M_p \ln \hat{n}_0 - M_p + \sum_{k=1}^{M_p} \ln \delta V_p.$$  

(21)

For $n_0 \neq \hat{n}_0$, the variation in the log likelihood is

$$\Lambda(\theta, n_0) - \Lambda(\theta, \hat{n}_0) = M_p \left[ \ln \frac{n_0}{\hat{n}_0} - \left( \frac{n_0 - \hat{n}_0}{\hat{n}_0} \right) \right]$$

$$\approx - \frac{M_p}{2} \left( \frac{n_0 - \hat{n}_0}{\hat{n}_0} \right)^2,$$

(22)

where the approximate, quadratic form holds for $|n_0 - \hat{n}_0|/\hat{n}_0 \ll 1$. 




We want to know the marginal distribution of each parameter. For a given parameter \( \theta_j \in \theta \), the marginal pdf is the normalized integral over all other parameters,

\[
f_{\theta_j}(\theta_j) = \frac{\int \ldots \int \, d\theta \, \Lambda^0[\theta, \theta_j] f_{\theta_0, \ldots, \theta_n}(\theta_0, \ldots, \theta_n)}{\int \ldots \int \, d\theta \, \Lambda^0[\theta, \theta_j] F_{\theta_0, \ldots, \theta_n}(\theta_0, \ldots, \theta_n)},
\]

(23)

where the integral subscript “exc. \( \theta_j \)” means that all parameters except the \( j \)th one are integrated over. The approximate form in equation (23) assumes a sharp peak about \( \bar{\theta}_0 \) and becomes an increasingly good approximation as \( M_p \) grows. The marginal pdf for \( n_0 \) is

\[
f_{n_0}(n_0) = \frac{1}{\Lambda^0[\theta, n_0]} \approx \left( \frac{M_p}{2\pi} \right)^{1/2} \frac{1}{\sigma_z} \exp \left\{ -\frac{(n_0 - \bar{n}_0)^2}{2\sigma_z^2} \right\}.
\]

(24)

For disk models, the areal, or column, density of MSPs is less model-dependent than the number density and scale height separately. The column density is \( N_0 \equiv n_0 \sigma_z \), where \( \eta \) is a dimensionless factor of order unity and \( \sigma_z \) is a scale height parameter. The pdf of \( N_0 \) is calculated by marginalizing \( \Lambda^0[\theta, n_0] \) over all parameters except \( n_0 \) and \( \sigma_z \). The resultant joint pdf \( f_{n_0, \sigma_z} \) is then integrated according to

\[
f_{N_0}(N_0) = \eta^{-1} \int d\sigma_z \sigma_z^{-1} f_{n_0, \sigma_z}(N_0/\eta \sigma_z, \sigma_z).
\]

(25)

For disk models considered below, \( \eta = 2 \) for an exponential in \( z \), and \( \eta = (2\pi)^{1/2} \) for a Gaussian in \( z \).

4.2. Telescope Gain

MSP surveys usually involve drift scans or sustained pointings toward specific sky positions. The telescope’s gain toward a given source varies over the analyzed portion of the drift scan and is a function of the source’s position relative to the beam center (see, e.g., Camilo, Nice, & Taylor 1996). We account for gain variations by replacing the beam solid angle \( \Omega_b \) in equation (7) with a sum over equal–solid-angle terms

\[
\Omega_b \rightarrow \sum_{m=1}^{n_y} \delta \Omega_b,
\]

(26)

where the telescope gain varies with \( m, G_m \). The minimum detectable flux density \( S_{\min} \) is therefore a function of \( m \). For some drift-scan surveys, we take into account that the data are analyzed in data blocks that overlap by some fraction (usually 50%).

For drift scans, \( G \) varies with time over the data set, and the offset from the beam center in declination is also taken into account. The sum in equation (26) becomes a sum over discrete steps in declination. For pointed (tracking) observations, we use actual pointing directions and break the beam into equal–solid-angle annuli about the beam center, which we sum over as in equation (26). We find that only a small number of subbeam elements is needed to account for the shape of the beam, e.g., \( n_y \sim 2 \) or 3.

4.3. Interstellar Scintillations

In Appendix B, we derive the effects of diffractive interstellar scintillations on flux densities and on the (pseudo) luminosity function. To use these results, we replace \( f_{L_p} \) in equation (7) with the corresponding “scintillated” luminosity function, \( f_{L_p'} \), as defined in Appendix B. We do so for surveys assuming that specific sky positions are observed only once. However, we use the unscintillated luminosity function in equation (10) because flux densities reported for the known pulsars are generally long-term averages of many independent measurements.

4.4. Comparison with Other Statistical Methods

Our statistical method differs substantially from other studies of the MSP population. A common approach to population studies, including pulsars and gamma-ray bursts, makes use of nonparametric estimators. The rationale is to try to draw inferences about certain properties of the population without assuming a specific class of models. In contrast, our likelihood analysis makes very specific assumptions about the class of models to be examined; for example, we have assumed a priori that all probability distributions are continuous. The differences between parametric and nonparametric treatments highlight some of the strengths and weaknesses of our approach.

It has been shown (Loredo & Wasserman 1995) that nonparametric estimators may be derived from a special maximum likelihood model solution. Since our parametric treatment is also based on a maximum likelihood analysis, it is straightforward to study the relationship between alternative methodologies by making a comparison of the assumptions made in the two searches. The special solution leading to the nonparametric estimators of interest comes from a search for a maximum likelihood solution among all functions and generalized distributions (i.e., delta functions) with equal a priori weight. This class of functions is so large that the most likely model is always one that exactly and precisely describes the observed data; thus, nonparametric estimators satisfy the rationale for which they are introduced. This contrasts with the parametric treatment adopted here for which the class of functions is (by comparison) extremely small. We liken nonparametric estimators to models with large numbers of free parameters.

In deciding what treatment to adopt, it is helpful to appeal to the Bayesian odds ratio to decide whether adding a new parameter to a model is justified by the better description of the data it may entail. Roughly speaking, each newly added parameter will improve the quality of the model’s description of the data. The odds ratio allows a quantitative decision to be made on whether to adopt the more complex model by weighing the improvement in the description against the additional freedom to fit arbitrary data sets. The situation for the MSPs is that the population is rather small, and we have anticipated (without any detailed investigation) that the odds ratio will favor models with relatively small numbers of parameters. We have therefore focused in this paper on parametric methods with small numbers of parameters.

An additional factor in our choice of parametric methods is that it is straightforward to include ancillary information about the population (e.g., continuity of the model), whereas in nonparametric approaches, such constraints are difficult to incorporate. Moreover, we find the parametric approach naturally allows the inference of population parameters of significant interest (e.g., cutoffs in the period distribution).

The main drawback of the parametric approach is that the results apply only to the particular set of models that the parameters can describe. If the real data were much...
better described by some completely different unstudied model, one would have no indication of that fact. In this paper, we have considered several plausible models, but these cannot begin to describe all the possibilities.

A number of pulsar population studies are based in whole or in part on such estimators (Vivekenand & Narayan 1981, hereafter VN; Phinney & Blandford 1981; Narayan 1987). To be a bit more descriptive, in the VN method, a scale factor is calculated for each object detected in a survey. The factor represents how many pulsars with the same period $P$ and luminosity $L_p$ exist in the Galaxy given the fraction of the Galaxy searched. In our notation, the scale factor is

$$\mathcal{S}(P, L_p) = \frac{\sum_{b}^{\text{all sky}} \Omega_{bj} \int_{0}^{\infty} dDD^2 n_p(D, l, b) \Omega_{bj} \int_{D_{\text{max}}}^{\infty} dDD^2 n_p(D, l, b)},$$

(27)

where $D_{\text{max}}$, as before, depends on many survey and pulsar parameters, including $P$ and $L_p$. The number of pulsars in the Galaxy is then calculated through a sum over detected pulsars as

$$N_{\text{gal}} \approx \sum_{i=1}^{N_{\text{MSP}}} \mathcal{S}(P_i, L_{p_i}).$$

(28)

The resultant total number of pulsars is a mean value similar in nature to the mean value of the number density, $n_0$, that we have calculated. One drawback is that the VN method estimates the number of pulsars in the Galaxy exactly like those actually detected. In other words, it explicitly includes contributions to the mean only at the periods and luminosities of the known pulsars. It is inherently discrete as compared with our likelihood method based on continuous distributions. Another drawback is that the method does not directly allow computation of confidence intervals. Finally, since the scale factors are calculated for the detected pulsars only, there is no means for estimating the cutoffs of the distributions of $P$ and $L_p$. Below, we compare our results on MSPs with those of Lorimer et al. (1995) and Bailes & Lorimer (1995, hereafter BL) with these issues in mind.

5. DISK MODEL FOR MILLISECOND PULSARS

5.1. Method

The simplest spatial model is a disk with constant scale height $\sigma_z$, so that the density is a function of $z$ only. We let $n_d(z) = n_d h_d(z)$ with $h_d(0) = 1$, where $n_d$ is the midplane density that corresponds to $n_0$ in §4.

The parameters to be solved for describe the period, luminosity, and $z$-distributions, $f_0(P), f_0(L_p)$, and $n_d(z)$. We have considered three models for $n_d(z)$: (1) a Gaussian function in $z$ with an rms value of $z$ given by $\sigma_z$; (2) an exponential model with $1/e$ scale height $\sigma_z$; and (3) a numerically derived distribution of NS orbits, neither Gaussian nor exponential in form, discussed in §6. For the luminosity and period pdfs, we adopt power-law functions, i.e., $f_{L_p} \propto L_p^{-\alpha_{L_p}}$ and $f_P \propto P^{-\alpha_P}$, with respective lower and upper cutoffs, $L_{p_1}, L_{p_2}$ and $P_{1}, P_{2}$.

The greatest computational effort goes into the calculation of $L_{\text{ND}}$ (eq. [12]). We computed it efficiently by summing the $D$ integral in equation (7) over the survey beam areas for a grid of $\sigma_z$, $P$, and $L_p$; as stated before, we use the scintillation-modified luminosity function in this computation. Next, we form the likelihood for different model parameters $a_{L_p}, a_P, L_{p_1}, L_{p_2}, P_{1}, P_{2}$ by calculating integrals over $P$ and $L_p$ with weights $f_P$ and $f_{L_p}$ (cf. eq. [7]).

We maximized $A$ by varying the parameters (or subsets of the parameters) over a grid. We kept the upper cutoffs on the period and luminosity distributions fixed at $P_2 = 20$ ms and $L_{p_2} = 16,000$ mJy kpc$^{-2}$. The period cutoff corresponds to the selection used to define the sample. Since the number of objects decreases rapidly as $P$ increases, the upper cutoff plays a very small role in any of the results below. The luminosity cutoff corresponds to the maximum possible luminosity in the observed sample. We also tested the effects of varying $L_{p_2}$ and found that the results are not sensitive to this parameter. Exclusion of B1937+21, the most luminous MSP, allows a much smaller value for $L_{p_2}$ to describe the remaining 21 pulsars in the sample, but none of the other results below are altered substantially.

5.2. Results

The five parameters ($P_1, L_{p_1}, a_P, a_{L_p}$, and $\sigma_z$) were varied over a grid to find the maximum $A$. We formed marginal pdfs according to equations (23) and (24). Results are summarized in Table 3. Figure 5 shows the marginal pdfs for each of the six parameters (the above-mentioned five and the number density, $n_d$). Using these pdfs, we calculated the confidence intervals on the parameters that are given in Table 3. The maxima are well-defined and easily located.

5.2.1. Minimum Period $P_1$ and Period Distribution Slope $a_P$

Naturally, $P_1$ must be less than or equal to the period of the shortest period MSP in our sample. When other parameters are held fixed, it is straightforward to show that $A$ must decrease as $P_1$ is made smaller. The best-fit, minimum

| Parameter | Gaussian in $z$ | Exponential in $z$ | Gaussian in $V$ | Units |
|-----------|----------------|-------------------|----------------|-------|
| $a_P$     | $2.0 \pm 0.33$ | $2.0 \pm 0.33$    | $2.0 \pm 0.33$ |       |
| $a_{L_p}$ | $2.0 \pm 0.2$  | $2.0 \pm 0.2$     | $2.1 \pm 0.2$  |       |
| $\sigma_z$| $0.65^{+0.16}_{-0.12}$ | $0.50^{+0.19}_{-0.13}$ |       | kpc   |
| $\sigma_P$| $\ldots$       | $\ldots$          | $0.1$         |       |
| $\sigma_{L_p}$| $\ldots$       | $\ldots$          | $52^{+17}_{-11}$ | km s$^{-1}$ |
| $n_d$     | $29^{+17}_{-11}$ | $44^{+25}_{-16}$  | $53^{+28}_{-18}$ | kpc$^{-3}$ |
| $N_{\text{MSP}}$| $42^{+29}_{-16}$ | $49^{+27}_{-18}$  | $49^{+27}_{-17}$ | kpc$^{-2}$ |
| $P_1$     | $>1.0$ (95%)   | $>1.0$ (95%)      | $>1.0$ (95%)   | ms    |
| $L_{p_1}$ | $>0.65$ (99%)  | $>0.65$ (99%)     | $>0.70$ (99%)  | mJy kpc$^2$ |

* Fixed parameter.

Note.—Confidence intervals, except where noted, are two-sided 68% intervals.
period lies only slightly below that of the most rapidly spinning known pulsar, B1937+21 (1.56 ms). However, the data allow $P_1 < 1.56$ ms at a reduced level of confidence. The results are given in Table 3 for both Gaussian and exponential models. The cutoff is greater than 1 ms at 95% confidence and greater than 0.65 ms at 99% confidence.

The period distribution falls off steeply with period, implying the existence of many objects at small $P$ ($dN/dP \propto f_p \propto P^{-2.0 \pm 0.33}$). It is well known that physical instabilities will act on neutron stars with very short rotation periods. Ignoring the magnetic field and assuming accretion from an inner edge of a Keplerian disk, Cook, Shapiro, & Teukolsky (1994a, 1994b) have shown that 1.4 $M_\odot$ neutron stars can be spun up to critical rotation periods (well under 1 ms) for a variety of equations of state without triggering radial instability, e.g., exceeding the maximum neutron star mass. (The results do not assure stability against nonradial modes and the associated gravitational wave emission.) Our overall fit for the period distribution suggests the existence of MSPs faster than those that are currently known (1.56 ms) in view of the fact that the theoretical stability analyses do not rule out such objects. Of course, there may be evolutionary reasons that such objects do not occur, and we discuss the significance of the cutoff $P_1$ next.

The specific value of $P_1$ depends, of course, on our assumption of a power-law distribution for $P$. We have not explored other mathematical forms, but reasonable alternatives include a power law that flattens for periods less than some critical period and cuts off at $P_1$, or a distribution that rises slowly from zero at $P_1$ and peaks at or near 1.56 ms and then follows a power-law form like that we have fitted. It is easy to see that such alternative period distributions will lead to smaller $P_1$ than we have derived. The reason is that they imply that a smaller volume has been surveyed for $P_1 < 1.56$ ms, so the allowed $P_1$ can be smaller. Therefore, our derived $P_1$ using the power-law distribution is a maximally allowed value and suggests, conservatively, that the period range for MSPs may extend to as small a value as 1 ms (95% confidence) or 0.65 ms (99% confidence).

Harding (1984) analyzed the slope of $f_p$, assuming a steady state flux with births balanced by pulsars crossing the Hubble line. She showed that if pulsars are born with a power-law distribution of $B \propto B^6$ and with initial period $P$ approximately proportional to $B$ (accretion spin-up models imply $B^{6/7}$), then the resultant $f_p \propto P^6$. Today it is known that the spin-down times for the observed MSPs are too long for a steady state to be attained. However, with similar assumptions, we find the same slope in the period range
Fig. 6.—Selected contour plots of the log likelihood for the exponential model plotted against pairs of parameters while holding the other four parameters fixed at values that yield the maximum likelihood. Contour spacings are unity in natural log units, and the first contour is a factor $1/e$ from the peak.

$[P_{\text{min}}(T_H), P_{\text{max}}]$ where $P_{\text{min}}(T_H)$ is the period reached after a Hubble time ($T_H$) by the minimum initial period object (minimum magnetic field), and $P_{\text{max}}$ is the longest period at birth. (Different slopes are found in other period subintervals. Additional discussion will be found in Chernoff & Cordes 1997a.) Thus, one possible interpretation of the steep period slope is that the field distribution is proportional to $B^{-2.0 \pm 0.33}$. However, it is difficult to derive robust constraints on the field distribution without knowing both the time dependence of magnetic fields during the spin-up process and the spin-down law for MSPs subsequent to the spin-up phase.

5.2.2. Scale Height $\sigma_z$

The inferred Gaussian scale (0.65 kpc) and exponential scale (0.50 kpc) are in rough agreement. The values indicate that the MSPs have a relatively small scale height, comparable to the oldest disk stars. Although the confidence inter-
vals overlap, the actual shape of the distribution plays some role in the value of the scale height parameter and motivates, in part, a more physical analysis based on the motion of objects in the Galactic potential (§6).

5.2.3. Minimum Luminosity \( L_{\text{min}} \) and Slope \( \alpha_{L\nu} \)

The luminosity pdf of our best fit, \( dN/dL_p \propto f_{L_p}(L_p) \propto L_p^{-2.0 \pm 0.2} \), is similar to that of long-period pulsars (e.g., Lyne, Manchester, & Taylor 1985). Total numbers are dominated by weak sources. The lower cutoff is \( L_p^{\text{crit}} = 1.1^{+0.4}_{-0.5} \) mJy kpc\(^{-2}\) and is determined largely by the absence of nearby sources. We have shown for long-period pulsars that \( f_{L_p} \) is influenced strongly by geometrical beaming effects, the distribution of true luminosities, the spin-down law, and a death line (Chernoff & Cordes 1997a). Because all four of these elements may differ between high-field pulsars and MSPs, we currently regard the similarity between long-period and MSP luminosity distributions as fortuitous.

In the past, the disk-determined \( f_{L_p}(L_p) \) (slope and cutoff) has also been used to make inferences about the number of MSPs in globular clusters. On evolutionary grounds, many properties of disk and globular cluster MSPs might be expected to differ (e.g., distributions of luminosity, spin period, orbital period, and velocity). Since the nearest cluster is too distant to allow direct measurement of the luminosity function near \( L_p^{\text{crit}} \), usage of the disk-determined form is necessary for many purposes. Fruchter & Goss (1990) measured the radio flux from nearby globular clusters and estimated \( \sim 10^3 \) MSPs in the Galaxy's globular cluster system. Our best-fit luminosity distribution, with cutoffs, is consistent with the one they assumed and does not alter the size of this estimate. Likewise, estimates by Foster & Tavani (1992) and Johnston, Kulkarni, & Phinney (1992) of the shape of the luminosity pdf for MSPs in globular clusters are also consistent with our best-fit \( f_{L_p} \) for disk MSPs, although both groups were unable to determine the lower luminosity cutoff and hence the absolute normalization. Wijers & van Paradijs (1991) find far fewer globular cluster MSPs than do Fruchter & Goss or Johnston, Kulkarni & Phinney, even though they adopted a lower luminosity cutoff 3 times smaller than that of Fruchter & Goss; the difference is probably related to their assumed dependence of luminosity on spin period and spin-period derivative that was based on young, high-field pulsars. An analysis of globular cluster MSP populations should probably use a treatment similar to this paper's but applied to cluster-only data.

5.2.4. Correlations

Most of the derived parameters are uncorrelated. However, \( \sigma_p \) and \( P_1 \) are positively correlated as are \( \alpha_{L\nu} \) and \( L_{p1} \), while \( n_8 \) is negatively correlated with the lower cutoffs in period (\( P_1 \)) and luminosity (\( L_{p1} \)). Figure 6 shows contours of constant likelihood plotted against pairs of parameters while holding all other parameters fixed at values that yield the maximum likelihood.

6. DYNAMICAL MODELS

6.1. Birth Kick Determination

In §5, we assumed functional forms for the \( z \)-distribution of MSPs and fitted for the associated scale height parameters. These parameters describe the present-day MSP distribution without regard to the orbit about the Galaxy. We have constructed a dynamical model that connects “birth parameters” to today’s spatial distribution as follows. We model the birthrate density of MSPs

\[
\dot{n}(R, z) = g(R) \exp \left( -\frac{z^2}{2 \sigma_{z,b}^2} \right),
\]

where \( R \) is the (cylindrical) Galactocentric radius, \( z \) is the height above the plane, \( \sigma_{z,b} \) is a scale height parameter, and \( g(R) \) is a surface density function, taken to be either constant (“uniform model”) or exponential with scale length 3.5 kpc (“exponential model”).

The birth velocity is the circular rotation velocity plus a kick component. Note that our use of “kick” includes any momentum impulse imparted to the pulsar’s progenitor or companion, if in a binary. The angular distribution of the kick is isotropic, and the velocity magnitude has a distribution proportional to \( V^2 e^{-V^2/2 \sigma^2} \). After birth, the MSP trajectory is determined by integration of the orbit about the Galaxy in a simplified model of the gravitational potential (Paczynski 1990). We ignore the role of scattering from irregularities (e.g., GMCs, spiral density waves, massive black holes) in the calculated motion. First, we discuss the kinematic properties of the MSP population inferred from the smooth model, and next we assess the degree to which our conclusions may be modified by the diffusion of stellar orbits.

About four million orbits were integrated over time spans of \( 10^7 \text{ yr} \), sufficiently long that the derived vertical distribution was stationary and well-mixed. For specific birth parameters, the vertical distribution of MSPs in the vicinity of the Sun (e.g., in an annulus of Galactocentric radii from 7.5 to 9.5 kpc) was calculated by appropriately weighting and combining the results for the individual orbits.

The statistical analysis described in previous sections was carried out to determine the birth parameters (\( \sigma_V \) and the intrinsic pulsar population parameters) of the uniform model. The results (Table 3) give a peak value \( \sigma_V = 52 \pm 11 \) km s\(^{-1}\). The initial scale height is not well-determined by the data and was held fixed, \( \sigma_{z,b} = 0.1 \) kpc. The column density of MSPs was calculated from equation (25). The result is consistent with values obtained using the Gaussian and exponential spatial models. Figure 7 illustrates the density distribution versus \( z \), comparing the range of allowed exponential fits (§5) with the most likely dynamical model. The differences are subtle and suggest that the assumed exponential form should be an adequate local description for many purposes.

6.2. Kinematics of Today’s Population

Kinematic properties of the MSP population may be inferred from the dynamical model. For example, the distributions of parallel and perpendicular velocities relative to the LSR are derived easily from the orbital calculations. Figure 8 shows the distributions for all simulated objects within 1 kpc of the Sun for the most likely dynamical model. The expected transverse motions are small; approximately 99% of the MSPs have \( \Delta V_t < 150 \) km s\(^{-1}\). As MSP samples increase in number, detailed distributions like these will provide important additional constraints on modeling.

Next, we consider the velocity ellipsoid of MSPs. Let \( v_R \), \( v_\theta \), and \( v_z \) be the components of velocity in the cylindrical radial direction, in the tangential direction (parallel to the local circular velocity, e.g., \( I = 270^\circ \) at the solar position), and out of the plane, respectively. In the well-mixed state,
the most interesting nonzero moments are \( \langle v_R^2 \rangle \), \( \langle v_z^2 \rangle \), and \( \langle v_t \rangle \). Table 4 lists the first and second moments for a range of MSP birth models with \( \sigma_p = 20, 40, 60, 80, \) and 100 km s\(^{-1}\), for two scale heights \( \sigma_{z,b} = 0.05 \) and 0.15 kpc, and for the uniform and exponential surface density distributions. (All velocity moments are given in units of \( \sigma_p \).)

When the kick velocity is small compared with the rotation velocity, and when disk properties do not vary significantly over the range of radii sampled, the results of epicyclic theory are directly applicable. For local objects, \( \langle v_R^2 \rangle / \langle v_z^2 \rangle \sim B/(A - B) \) has the observed value 0.45 \( \pm \) 0.09 (for Oort constants \( A = 14.5 \pm 1.5 \) km s\(^{-1}\)).

### Table 4

**Velocity Moments for Nearby Pulsars**

| \( \sigma_p \) (km s\(^{-1}\)) | \( \sigma_p \) (kpc) | \( \bar{v}_R \) | \( \bar{v}_R \) | \( \bar{v}_t \) | \( \bar{v}_t \) | \( \bar{v}_z \) | \( \bar{v}_z \) |
|-----------------------------|-------------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Uniform Surface Density (independent of \( R \))                |                   |                |                |                |                |                |                |
| 20            | 0.05          | 0.02          | 1.28          | 0.00          | 0.86          | 0.00          | 0.70          |
|                | 0.15          | 0.01          | 1.28          | -0.00         | 0.86          | -0.00         | 0.79          |
| 40            | 0.05          | 0.00          | 1.26          | -0.11         | 0.80          | 0.00          | 0.66          |
|                | 0.15          | 0.01          | 1.26          | -0.11         | 0.80          | -0.01         | 0.69          |
| 60            | 0.05          | 0.00          | 1.18          | -0.22         | 0.77          | 0.00          | 0.64          |
|                | 0.15          | 0.00          | 1.18          | -0.22         | 0.77          | -0.00         | 0.65          |
| 80            | 0.05          | 0.01          | 1.09          | -0.29         | 0.75          | 0.00          | 0.61          |
|                | 0.15          | 0.00          | 1.09          | -0.30         | 0.75          | -0.01         | 0.61          |
| 100           | 0.05          | 0.01          | 1.01          | -0.35         | 0.74          | 0.00          | 0.58          |
|                | 0.15          | 0.00          | 1.00          | -0.35         | 0.74          | -0.01         | 0.58          |
| Exponential Surface Density (in \( R \))                      |                   |                |                |                |                |                |                |
| 20            | 0.05          | 0.03          | 1.24          | -0.17         | 0.83          | -0.01         | 0.69          |
|                | 0.15          | 0.01          | 1.23          | -0.17         | 0.83          | -0.01         | 0.79          |
| 40            | 0.05          | -0.01         | 1.17          | -0.31         | 0.78          | -0.00         | 0.63          |
|                | 0.15          | 0.01          | 1.17          | -0.31         | 0.78          | -0.00         | 0.66          |
| 60            | 0.05          | -0.01         | 1.09          | -0.41         | 0.73          | 0.00          | 0.58          |
|                | 0.15          | 0.00          | 1.09          | -0.41         | 0.74          | 0.00          | 0.60          |
| 80            | 0.05          | 0.00          | 1.03          | -0.47         | 0.69          | 0.00          | 0.55          |
|                | 0.15          | 0.01          | 1.02          | -0.47         | 0.70          | 0.00          | 0.56          |
| 100           | 0.05          | 0.01          | 0.97          | -0.50         | 0.66          | 0.00          | 0.51          |
|                | 0.15          | 0.00          | 0.96          | -0.50         | 0.66          | -0.01         | 0.52          |

**Notes.** Velocity moments for particles of Galactocentric radius \( 7.5 < R < 9.5 \) kpc and \( |z| < 3 \) kpc. Here \( \bar{v}_R \) means \( \langle v_R \rangle / \sigma_p \), \( \bar{v}_R \) means \( \langle (v_R - \langle v_R \rangle)^2 \rangle / \sigma_p \), and so on. The top section refers to a disk with constant birth density in \( R \); the bottom section refers to a disk with birth density varying with exponential scale length 3.5 kpc in \( R \). All moments given in units of \( \sigma_p \). Numerical accuracy of \( \pm 0.02 \) for all entries. All mixed second-order moments are zero to \( \pm 0.03 \).
kpc\(^{-1}\) and \(B = -12 \pm 3\) km s\(^{-1}\) kpc\(^{-1}\) [Binney & Tremaine 1987]). The model-calculated value of 0.45 at \(\sigma_v = 20\) km s\(^{-1}\) is in good agreement with the value inferred from the observed Oort constants. Here we will concentrate on the changes that occur as \(\sigma_v\) increases and that are indicative of some of the differences between the velocity distributions of MSPs and of disk stars. A global model is necessary since a local epicyclic treatment for the MSPs is not well-founded. For example, with a kick of 60 km s\(^{-1}\), particles observed at the local position could come from initial radii in the approximate range \((0.5 - 2.6)R_0\), where \(R_0\) is the Sun's distance from the Galactic center. Also, kicks of this size create vertical excursions of greater than 0.3 kpc, spoiling a harmonic approximation to the potential.

Table 4 shows how the basic moments change as \(\sigma_v\) increases. We briefly note the most important conclusions: (1) A clearly noticeable effect is the occurrence of a nonzero tangential motion measured with respect to the local circular velocity ("asymmetric drift"). For \(\sigma_v = 60\) km s\(^{-1}\), the magnitude is \(\sim 13\) km s\(^{-1}\) in the uniform model (\(\sim 25\) km s\(^{-1}\) in the exponential model), an effect that is potentially detectable in a relatively small sample of objects with well-determined velocities. (2) The velocity ellipsoid [with axial ratios \((\langle v_\theta^2 \rangle)^{1/2}:\langle v_\phi^2 \rangle^{1/2}:\langle v_z^2 \rangle^{1/2}\)] becomes rounder as the magnitude of the kick grows. (3) The birth distribution in Galactocentric radius affects the value of all the nonzero velocity moments, including the shape of the velocity ellipsoid and the magnitude of the asymmetric drift. (4) The imprint of the birth scale height is essentially absent for objects with \(\sigma_v \gtrsim 60\) km s\(^{-1}\).

Determinations of asymmetric drift would provide valuable information on the birth locations of MSPs. Proper estimation of the effect will require more field MSPs than are currently known and careful treatment of distance errors. We defer a detailed discussion to another paper.

### 6.3. Orbital Diffusion

The model calculations presented above assume a regular background potential. Older stars are well known to have larger velocity dispersions, presumably from interaction with small-scale fluctuations in the gravitational field, but the actual physical source of the irregular field is not well understood [Wielen 1977]. The oldest stars, K and M giants of age \(9 \times 10^9\) yr, reach total dispersions of 77 km s\(^{-1}\); for comparison, using interpolated values for the uniform model in Table 4, we estimate that the best-fit model for the MSPs (\(\sigma_v = 53\) km s\(^{-1}\)) implies a total dispersion of 84 km s\(^{-1}\). In fact, the MSPs suffer comparable energy input from kicks and from diffusion. The key assumption is that the MSP population includes members with ages ranging uniformly up to the age of the Galaxy, so that the average effect of diffusion will be less than it is for the oldest stars. Using the velocity dispersion data of K and M giants with ages (0.3–9) \(\times 10^9\) [Wielen 1977], averaging uniformly in time, we infer that the rms dispersion is \(\sim 50\) km s\(^{-1}\). We suggest that the residual dispersion of 67.5 km s\(^{-1}\) [i.e., \((84^2 - 50^2)^{1/2}\)] is due to kick(s) unique to MSP evolution. This three-dimensional dispersion would then correspond to a one-dimensional kick of \(\sim 39\) km s\(^{-1}\).

### 6.4. Conclusions

The best-fit uniform model implies \(\sigma_v = 53\) km s\(^{-1}\); this is an upper limit because gravitational scattering processes are ignored in its estimate; the scale of the kick is \(\sim 40\) km s\(^{-1}\), assuming MSPs are long-lived and born at a uniform rate. If MSPs are visible for less than a Hubble time, the kick size will increase; if most of today's MSPs were formed early in the Galaxy's life, the kick size will decrease.

### 7. DISK + DIFFUSE MODEL

#### 7.1. Method

The MSP distribution may be more complex than a single-disk component with small scale height. For example, there may exist a population of MSPs that fill a halo-like region around the disk.

If MSPs are distributed in two components, the log likelihood becomes

\[
\Lambda(\theta, n_d, n_h) = -(n_d V_d + n_h V_h) + \sum_{k=1}^{M_p} \ln (n_d \delta V_d^{(k)} + n_h \delta V_h^{(k)}) .
\]

(30)

Here we label disk quantities with "d," while "h" denotes diffuse (halo-like) contributions; we suppress the dependences of the volumes on other parameters. Maximizing \(\Lambda\) with respect to \(n_d\) and \(n_h\), we find that the best-fit number densities \(\hat{n}_d\) and \(\hat{n}_h\) satisfy

\[
\hat{n}_d V_d + \hat{n}_h V_h = M_p .
\]

(31)

Also, if we take the \(n_h = 0\) case as a fiducial solution, which is our result in § 5 for the disk-only model, the log likelihood for \(n_h \neq 0\) may be expanded as

\[
\Lambda(\theta, n_d, n_h) = \Lambda(\theta, n_d, 0) - n_h V_h + \sum_{k=1}^{M_p} \ln \left(1 + \frac{n_h \delta V_h^{(k)}}{n_d \delta V_d^{(k)}} \right) .
\]

(32)

#### 7.2. Diffuse Models

One might expect a diffuse distribution of MSPs for any of several reasons. (1) The probability distribution of birth velocities may extend to values much larger than typically allowed by the assumed Gaussian or exponential forms. The high-velocity MSPs would oscillate to higher z-distances or escape the Galaxy altogether. (2) MSPs born in globular clusters may be ejected by dynamical interactions or when a cluster is tidally dissolved. Such objects would have a spatial distribution like the parent systems, assuming the ejection velocities were small compared with the rotation velocity. (3) Spheroid stars may evolve and produce long-lived MSPs just like disk stars (e.g., by accretion-induced spin-up). Such objects would have a spatial distribution like the Population II spheroid. (4) If the formation of the Galaxy involved hierarchical merging of smaller objects containing disklike structures, their MSPs will be cannibalized. Such objects might follow the dark matter halo distribution.

Without further considering the merits of these basic scenarios, we will adopt several geometrical distributions for the putative diffuse population and place upper limits on the number densities. Let us consider a density model for MSPs of the form

\[
n_h(r) = n_h [1 + (r/r_h)^2]^{-s_h/2} ,
\]

(33)

where \(r\) is the radius from the center, \(r_h\) is the characteristic radius, and \(s_h\) is the power-law index. Taking \(s_h = 0\) gives a uniform density halo, our reference model (in practice, all
distributions are truncated at 50 kpc). Taking \( s_h = 2 \) and \( r_h = 5 \) kpc gives an isothermal distribution with large core. Taking \( s_h = 3.5 \) and \( r_h = 1 \) kpc gives the observed globular cluster distribution (Thomas 1989).

Using these models, we calculate the diffuse pulsar density by integrating equation (7), and we evaluate the halo volume factors \( V_h \) and \( \delta V_p \) (eqs. [17] and [19]). We combine these with the analogous disk quantities and examine a grid in \( n_d \) and \( n_h \) to find the distribution of likelihood values.

7.3. Results

For our reference model, we find that the pure disk model is favored by a huge factor that implies that an upper bound on the diffuse density from the fitting is \( n_h \lesssim 0.4 \) kpc\(^{-3} \) (90% confidence; cf. Table 5). Figure 9 shows likelihood contours for the disk and diffuse densities, with a maximum at \( n_h = 0 \). The marginalized densities are shown in Figure 10 for the uniform density model.

For the other two models, we have expressed the results in terms of limits on the density parameter \( n_h \) (the value at the center of the Galaxy) and, equivalently, on \( n_d(R_0) \), where \( R_0 \) is the Sun’s Galactocentric radius. Although we have made calculations explicitly for the nonuniform density models, the local values are close to the reference model values. This follows because, on the Galactic scale, most surveys probe regions near the solar system.

7.4. Disk, Spheroid, Halo, and Globular Cluster Contributions

The local column density of MSPs, \( N_d \sim 50^{+30}_{-20} \) kpc\(^{-2} \), may be combined with the disk surface mass density \((\sim 66 \pm 8 \, M_\odot \text{ pc}^{-2})\) for Oort K giants; Bahcall 1984) to infer that the number of MSPs per unit disk mass \((dN/dM)_\text{disk} \approx 7.6^{+4.4}_{-2.3} \times 10^{-7} \, M_\odot^{-1} \) (the range reflects only the uncertainty in \( N_d \)). The total number of MSPs in the Galactic disk scaled to the disk mass \( M_\text{disk} \approx 3.0^{+1.8}_{-1.2} \times 10^4 \, M_\odot \) (for example, \( M_\text{disk} = 3.7 \times 10^{10} \, M_\odot \) [Bahcall & Soneira 1982] by one estimate, \( 3.5-4.6 \times 10^{10} \, M_\odot \) [Caldwell & Ostriker 1981] by another). The total does not include a correction for beaming.

Estimates of the local spheroid mass density are uncertain, e.g., \( \rho_\text{sph} = 1.88 \times 10^{-4} \, M_\odot \text{ pc}^{-3} \) (Bahcall, Schmidt, & Soneira 1982) or \( \rho_\text{sph} = (1.11-1.25) \times 10^{-3} \, M_\odot \text{ pc}^{-3} \) (Caldwell & Ostriker 1981). If \((dN/dM)_\text{disk} = (dN/dM)_\text{sph}\), then the spheroid makes a contribution to the MSP number density \( n_\text{sph} \approx 0.14 \) kpc\(^{-3} \) or \( 0.84-0.95 \) kpc\(^{-3} \), respectively. The upper limit we have derived for a uniform density model, \( n_h \lesssim 0.42 \) kpc\(^{-3} \), is marginally consistent. Future observations should be able to constrain contributions to the MSP population from Population II progenitors more strongly.

Estimates of the dark matter halo density are \( \rho_\text{halo} = 9 \times 10^{-3} \, M_\odot \text{ pc}^{-3} \) (Bahcall et al. 1982) or \( \rho_\text{halo} = (5.9-10.2) \times 10^{-3} \, M_\odot \text{ pc}^{-3} \) (Caldwell & Ostriker 1981). If the dark matter halo component satisfies \((dN/dM)_\text{halo} = (dN/dM)_\text{disk}\), then its contribution is \( n_\text{halo} = 6.8 \) kpc\(^{-3} \) or \( 4.5-7.7 \) kpc\(^{-3} \), respectively. Our limit on \( n_h \) implies \((dN/dM)_\text{halo}/(dN/dM)_\text{disk} \lesssim 0.06 \) or \( 0.05-0.09 \), respectively.

Today’s globular clusters are known to have a significant enhancement of MSPs relative to the disk. With considerable uncertainty, Phinney & Kulkarni (1994) estimate \((dN/dM)_\text{gc} \approx 50(dN/dM)_\text{disk}\). If half of the original globular cluster system has been destroyed (e.g., a total mass \( M \approx 5 \times 10^7 \, M_\odot \)), if the MSP content was similarly enhanced, and if these MSPs orbit like the observed clusters, then the contribution to the local mass density is

| TABLE 5 |
| --- |
| **Diffuse Only** | **Disk + Diffuse** |
| Model | \( \hat{n}_h \) (kpc\(^{-3} \)) | \( \mathcal{L}(0, \hat{\phi}_h) / \mathcal{L}(-\hat{\phi}_o, 0) \) | \( \hat{n}_d \) (kpc\(^{-3} \)) | \( \hat{n}_s \) (kpc\(^{-3} \)) | \( \hat{n}_h(R_0) \) (kpc\(^{-3} \)) |
| Uniform density | 1.5 | \( 10^{-21.3} \) | 38 | <0.42 (90%) | ... |
| \( r_s = 1 \) kpc, \( s_h = 7/2 \ldots \) | 4080 | \( 10^{-16.4} \) | 38 | <0.84 (99%) | ... |
| \( r_s = 5 \) kpc, \( s_h = 2 \ldots \) | 10.8 | \( 10^{-15.1} \) | 38 | <1.520 (90%) | <0.83 |
| | | | | <3.3 (90%) | <0.31 |
| | | | | <6.7 (99%) | <0.62 |
Fig. 10.—Marginal probability density functions for the disk and diffuse MSP densities, \( n_d \) and \( n_h \), respectively.

1.1 \times 10^{-3} \, M_\odot \, \text{kpc}^{-3}, and the MSP number density is

4.2 \times 10^{-2} \, \text{pc}^{-2}. Our limit on \( n_h \) does not provide a strong constraint.

7.5. Conclusions

The observations place an upper limit on a diffuse halo-like contribution to the MSP density that is roughly 1% of the MSP disk density at midplane.

8. SPACE VELOCITIES OF MSPs

Our results indicate that millisecond pulsars are a low-velocity population, at least when compared with young, high-field pulsars. We have found that the three-dimensional rms velocity of MSPs in the Galactic disk is \( \sim 84 \, \text{km s}^{-1} \), about a factor of 5–7 lower than that of young, strong-field pulsars (Lyne & Lorimer 1994; Cordes & Chernoff 1997). We have reached this conclusion by determining the spatial distribution of MSPs, by excluding the existence of a significant nondisk population, and by modeling the motion of objects in the gravitational potential of the Galaxy.

8.1. Comparison with Proper-Motion Data

We may compare our results with direct measurements of proper motion using interferometric and pulse-timing methods; the indirect method of interstellar scintillation has also yielded determinations of MSP transverse speeds. To date, there are timing proper motions on eight MSPs: J0437−4715 (Bell et al. 1995), B1257+12 (Wolszczan 1994), J1713+0747 (Camilo, Foster, & Wolszczan 1994), B1855+09 and B1937+21 (Kaspi, Taylor, & Ryba 1994), B1957+20 (Arzoumanian, Fruchter, & Taylor 1994), and J2019+2425 and J2322+2057 (Nice & Taylor 1995). There are also scintillation speeds on some of these and other pulsars: B1855+09 (Dewey et al. 1988), B1937+21 (Cordes et al. 1990), and J0437−4715, J1455−3330, J1730−2304, and 2145−0750 (Nicastro & Johnston 1995). These MSPs have transverse speeds that are less than 100 km s\(^{-1}\), except for B1257+12, which has a speed of 285 km s\(^{-1}\) at its nominal distance of 0.62 kpc, and B1957+20, which has \( V_z \sim 173 \, \text{km s}^{-1} \) at a distance of 1.2 kpc (Aldcroft, Romani, & Cordes 1992). For the most part, these objects are consistent with our determination of the three-dimensional rms velocity based on the locations of 22 MSPs and the absence of MSPs in substantial portions of the volumes searched in high-latitude surveys. However, the estimated transverse speed for B1257+12 is inconsistent with the overall distributions in \( z \) and velocity that we have derived, even though it was included in the fitting. One possibility is that its distance is overestimated, perhaps by as much as a factor of 2, an amount sufficient to bring it into consistency with the statistical distribution. It is also possible that there are several evolutionary paths for producing MSPs (cf. § 10), most of which produce low-velocity MSPs and others that create rarer, faster MSPs.

Further study of larger samples of MSP proper motions will result from a combination of new surveys, which will discover large numbers of MSPs (cf. § 12), and the use of timing and VLBI techniques. Use of the VLBA in conjunction with the Arecibo telescope and the Green Bank Telescope (GBT) should allow measurement of proper motions for dim and slow MSPs out to a few kiloparsecs.

9. BIRTHRATES OF DISK AND HALO MSPs

For the disk-only model of § 5, we have found the column density of MSPs with \( P_1 = 1.56 \, \text{ms}, L_p = 1.1 \, \text{mJy kpc}^2 \) for a plane-parallel model in \( z \) to be \( N_d \approx 50^{+30}_{-20} \, \text{kpc}^{-3} \) (Table 3). The implied number of MSPs in a disk of radius \( R_d \) with \( P > P_1 \) and \( L_p > L_p^1 \) is

\[
N_{\text{MSP}}(> P, > L_p) \approx 1.6^{+0.9}_{-0.6} \times 10^4 \left( \frac{R_d}{10 \, \text{kpc}} \right)^2
\times \left( \frac{P}{1.56 \, \text{ms}} \right)^{-1 \pm 0.33}
\times \left( \frac{L_p}{1.1 \, \text{mJy kpc}^2} \right)^{-1 \pm 0.2}, \tag{34}
\]

where the upper and lower values denote the 68% interval. Extrapolation on a per mass basis from the local disk surface density to a total disk mass, \( M_{\text{disk}} \), implies

\[
N_{\text{MSP}}(> P, > L_p) \approx 3.0^{+1.5}_{-1.2} \times 10^4 \left( \frac{M_{\text{disk}}}{4 \times 10^{10} \, M_\odot} \right)
\times \left( \frac{P}{1.56 \, \text{ms}} \right)^{-1 \pm 0.33}
\times \left( \frac{L_p}{1.1 \, \text{mJy kpc}^2} \right)^{-1 \pm 0.2}. \tag{35}
\]

These estimates do not include any correction for pulse beaming, whose influence is highly uncertain for MSPs. Estimates for this correction range from 1 to 3 (e.g., BL). The totals are sensitive to the cutoff at small periods and at small luminosities.

The corresponding birthrate for MSPs, if constant over a Galactic age \( 10^{10} \, \text{yr} \), for the uniform disk,

\[
N_{\text{MSP}}(> P_1) = 1.6^{+0.9}_{-0.6} \times 10^{-6} \, \text{yr}^{-1} \left( \frac{R_d}{10 \, \text{kpc}} \right)^2, \tag{36}
\]

and for the extrapolated surface density is

\[
N_{\text{MSP}}(> P_1) = 3.0^{+1.5}_{-1.2} \times 10^{-6} \, \text{yr}^{-1} \left( \frac{M_{\text{disk}}}{4 \times 10^{10} \, M_\odot} \right). \tag{37}
\]
From our constraints on diffuse populations of MSPs, we conclude that, in the vicinity of the Sun, the MSP birthrate per unit volume is 100 times less than that from the disk.

9.1. Comparison with Other MSP Population Studies

Our estimates may be compared with those derived by BL, Lorimer (1995), and Lorimer et al. (1995), who used the VN scale-factor method to determine the number of MSPs in the Galaxy and the associated luminosity function. In their analyses, specific spatial distributions for the MSPs were adopted to derive the scale factors. BL assumed two different scale heights (0.3 and 0.6 kpc) along with a fixed radial distribution to estimate $10^{4.4}$ and $10^{6.6}$ MSPs, respectively, for $L_p > 2.5$ mJy kpc$^{-2}$ and if all MSPs beam toward us.

Lorimer et al. (1995) use the radial distribution of Lorimer et al. (1993) (a Gaussian with radial scale of 4.8 kpc) and a Maxwellian velocity distribution with rms velocity $3^{1/2} \times 100$ km s$^{-1}$ to estimate $(1.3 \pm 0.2) \times 10^{4}$ MSPs in the Galaxy that are beamed toward us with $L_p > 10$ mJy kpc$^2$.

Lorimer (1995) deduced lower bounds on the scale height and mean three-dimensional space velocity for MSPs of 0.5 kpc and 80 km s$^{-1}$, respectively. These bounds are consistent with our determinations.

The numbers of pulsars derived by BL and Lorimer et al. (1995) are greater than the estimate in equation (35) by a factor $\sim 2$–3 for a luminosity cutoff of 2.5 mJy kpc$^{-2}$. Since most of the MSPs known are near the Sun (within 2 kpc), an extrapolation to the whole Galaxy is necessary. The radial distributions used by BL and Lorimer et al. effectively multiply the uniform disk model result by $\sim 1.6$ and match our own extrapolation (based on scaling up the local disk surface density to the given total disk mass in eq. [35]). The extrapolation introduces uncertainty, but the differences accrue from the following factors. First, the $z$ scale height implied by the Lorimer et al. velocity distribution is larger than that derived by us by about a factor of 2. Second, our inclusion of scintillation effects yields a search volume that is about 30% larger than otherwise. Third, Lorimer et al. include four long-period pulsars in their analysis with $P > 295$ ms that we exclude from the MSP sample. Together, these differences in the assumed spatial distributions and MSP samples explain the size of the differences in the estimated total number of MSPs in the Galaxy.

BL synthesized a luminosity function for MSPs after correcting the observed numbers of pulsars for the volume scale factors. Their luminosity function is consistent with a power-law slope of $-2$ (according to our definition of $f_{p,0}$) but with a roll-off below 10 mJy kpc$^{-2}$. Our method is able to constrain the lower cutoff on the luminosity function because we evaluate our results at values for $L_p$ other than those of actually detected pulsars.

Similarly, BL suggest that the period distribution decreases in going from 1 to 10 ms and roughly estimate that there can be no more than $10^{4.3}$ MSPs with periods with $P = 1$ ms. Our results suggest that the number of pulsars between 1 and 1.5 ms is approximately 50% of the number with $P > 1.5$ ms, or about 5000 pulsars.

10. RELATIONSHIP TO LOW-MASS X-RAY BINARIES

10.1. Scale Heights of LMXBs and MSPs

The evolutionary paths that lead to MSPs are poorly understood (for a review, see Bhattacharya 1995). If all MSPs are “spun-up” by mass transfer from a companion star during an LMXB phase, then the birthrate of LMXBs must exceed that of MSPs. Kulkarni & Narayan (1988) estimated that the birthrate of field LMXBs is about 1%–10% of the birthrate of field MSPs for an assumed LMXB lifetime of $10^7$ yr. With a diminished LMXB lifetime ($10^5$ yr), the birthrates are brought into agreement. Our improved estimate of the total number of MSPs in the Galaxy does not alter significantly the rate mismatch nor its strong dependence on the LMXB lifetime. However, our work of the last section does point out that the extrapolation from the local MSP population to that of the Galaxy is uncertain by a factor of $\sim 2$–4 (and an additional factor of $\sim 3$ for beaming). As we will argue below, the kinematic similarity of the LMXB and MSP population manifested in the observed scale heights is reasonably strong evidence of an evolutionary link; given the great uncertainty in LMXB lifetimes, the best evidence for a causal connection between the two populations is not found in the relative number but in the similar spatial distribution. In addition, a comparison of the scale height distribution of MSPs with that of LMXBs can place significant constraints on evolutionary scenarios leading to MSPs.

The Galactic LMXB scale height derived from analysis of a flux-limited sample (Naylor & Podsadlowski 1993) is $0.44$–$1.17$ kpc. Alternatively, based on distance estimates to a subset of LMXBs, van Paradijs & White (1995) infer a scale height of 0.5 kpc and, furthermore, argue that the LMXBs are located predominantly at Galactocentric radii less than 5 kpc. Although the two vertical scales are comparable, the interpretations are quite different. Van Paradijs & White assume that the LMXBs have Population II progenitors and that the scale height is set by large-velocity kicks at birth (of order 400 km s$^{-1}$) and the local disk acceleration, which they argue is $2.5$–$4$ larger in the relevant inner regions of the Galaxy than locally. This analysis ignores the finite birth scale height and the lifetime of the objects. Naylor & Podsadlowski, on the other hand, infer that the LMXBs derive from Population I stars and have a scale height perpendicular to the plane that is roughly like that of the observed thin disk (with a small additional kick), which has a nearly constant value.

Our local determination of the MSP scale height is $0.53$–$0.81$ kpc, comparable with the above LMXB estimates. If MSPs are descendants of the LMXB phase and if the scale height increases with age, then the local MSP population should have a scale height greater than or equal to the LMXB value. If the kicks were as large as suggested by van Paradijs & White, the minimum local scale height of the MSPs would be $1.25$–$2$ kpc, clearly inconsistent with our results. In addition, our upper limit on the diffuse number density of pulsars suggests that the observed MSPs were born in the disk (Population I). A consistent interpretation of the LMXB and MSP data is that both are Population I and that both are derived from a similar evolutionary channel.

10.2. Origin of MSP Space Velocities

In future work, we will discuss how the observed scale height of MSPs and the inferred $z$ velocities ($\sim 50$ km s$^{-1}$) place stringent constraints on the evolution of binary systems that lead to MSP formation. One of the main problems in understanding the LMXB evolution path is that the formation rate ($10^{-6}$ yr$^{-1}$) in the Galaxy is so small that the
pathway is a priori special. A proposed scenario is as follows (Webbink & Kalogera 1994). A binary composed of a massive star \((M_1 = 10-20 \, M_\odot)\) and a light companion \((M_2 < 0.12 M_1)\) with an initial orbital separation less than about 1000 R\(_\odot\) will pass through an epoch of unstable mass transfer and common envelope evolution once the massive star begins to swell. The interaction ejects much of the envelope, drawing the pair to very small distances of separation. If the helium core of the primary is sufficiently massive, it is able to continue to burn and collapse, even after its outer hydrogen envelope has been removed. The resultant neutron star has an orbit far smaller than the size of the original giant primary, an essential requirement if an LMXB phase is to take place. Starting with the presupernova system, we have analyzed how a tight binary is affected by a combination of (1) an asymmetric supernova (SN) kick, (2) the impact of an ejected shell, and (3) the dissipative processes in the eccentric, surviving binary.

Two effects sculpt the properties of the binaries that survive to give LMXBs and/or MSPs. The intrinsic kick given a neutron star by the SN explosion unbinds loosely bound binaries, while the impact of the SN shell on the secondary is responsible for destroying and/or unbinding tight binaries. The surviving binaries occupy relatively narrow ranges in pre-SN orbital separation, primary and secondary masses, and have a limited range of center-of-mass velocities. Most of this analysis is independent of the specific evolutionary pathway leading to the pre-SN progenitor. (We present the details of this analysis in Chernoff & Cordes 1997b.)

11. SELECTION EFFECTS AGAINST FAST BINARIES

Orbital motion causes MSPs in compact binaries to be missed in surveys that assume the pulse period to be constant rather than Doppler-shifted (e.g., Johnston & Kulkarni 1991). All blind surveys for MSPs, including the eight analyzed in this paper, make this assumption. One circumstance in which the results of previous sections may be altered is if the spin period and/or luminosity depend in some way on orbital period. For example, a relation between spin and orbital period might be expected on general evolutionary grounds for spun-up MSPs (Alpar et al. 1982; Ruderman & Shaham 1983). Some observations suggest a weak positive correlation (Lundgren, Zepka, & Cordes 1995), implying that the selection against detecting spin periods less than 1.5 ms may be stronger than we have estimated. Because the measured correlation is weak, we believe that any modification to the distribution of spin periods on this account will be modest. In any case, fast binaries have been missed in MSP surveys, and their ultimate detection can only increase our estimated space densities for MSPs.

We now give a brief account of survey sensitivity to binary orbital period. The observation time \(T \lesssim 1\) minute for most of the Arecibo surveys, so that orbital effects are negligible for \(P_{\text{orb}} \gtrsim 1.6 P^{-3/4}\) (\(P\) in milliseconds) for white dwarf (WD) companions with \(M_2 = 0.3 \, M_\odot\). However, surveys 4, 7, and 8 with \(T \sim 3\) minutes are insensitive to orbital motion only for \(P_{\text{orb}} \gtrsim 8.5 P^{-3/4}\). Weighted by volumes searched, the surveys with longer \(T\) contribute strongly to an overall selection against MSP binaries with short periods. Indeed, J0751 + 1807 with \(P_{\text{orb}} = 6.5\)\(\) was discovered in survey 4 in a single harmonic, the higher harmonics having been attenuated by orbital motion (Lundgren et al. 1995). That yet faster binaries with fairly massive WD companions exist is certain because objects like J0751 + 1807 experience orbital decay due to gravitational radiation on less than a Hubble time. Indeed, if MSP-WD binaries achieve \(P_{\text{orb}} \lesssim 8^{\circ}\) solely because of such inspiral, then it may be shown that the orbital period distribution \(dn/dP_{\text{orb}} \propto P_{\text{orb}}^{-5/3}\). Overall, our conclusions are unaffected for MSPs in binaries with orbital periods \(\gtrsim 6\) hr and with companion masses \(\lesssim 0.3 \, M_\odot\).

Proper consideration of orbital effects—and estimation of the MSP orbital period distribution—requires an analysis of search volumes as a function of orbital period and companion mass, as well as spin period and luminosity. Such a study is beyond the scope of the present paper.

Future surveys made with greater sensitivities than heretofore available, using the upgraded Arecibo Observatory and the new GBT, will be able to probe to greater distances while also circumventing orbital suppression of Fourier harmonics. Furthermore, algorithms that correct for orbital motion are becoming much more feasible with the prospect of computers with teraflops capability.

12. OPTIMAL SEARCHES FOR MILLISECOND PULSARS

The population distributions we have derived may be used to optimize new searches for MSPs. Search sensitivities (Appendix A) depend on sky background, dispersion, and scattering, as well as on the flux densities and periods of the MSPs. Consequently, the optimal search is frequency- and telescope-dependent. Here we illustrate the contributions from different effects by showing \(\delta V_s\), the search volume (the volume searched in a beam area, averaged over \(P\) and \(L_p\); eq. [8]) and \(\delta V_d\), the detection volume (the volume searched weighted by the dimensionless density of MSPs; eq. [18]). To calculate each, we use the best-fit distributions for \(P\) and \(L_p\) for the exponential disk model of § 5 and Table 3. By definition, \(\delta V_s \leq \delta V_d\) surveys that search much more deeply than the scale height of the MSP population yield \(\delta V_s \leq \delta V_d\). For concreteness, we consider a survey conducted by telescopes like the under-construction GBT and a hypothetical analog in the southern hemisphere, in order that we may consider a full-sky survey. We assume receiver and survey parameters such that the minimum detectable flux density is about 2 mJy when looking at high Galactic latitudes and long periods.

Figure 11 shows \(\delta V_s\) and \(\delta V_d\) per square degree for a search at 430 MHz. Search volumes (top portion of figure) increase more or less monotonically with Galactic latitude but level off for \(b > 30^\circ\). The volume is smallest toward the Galactic center, where the sky background is high and dispersion and scattering effects are large. By contrast, the detection volume (bottom part of figure) is maximum for \(b \sim 20^\circ\) and \(l \gtrsim 50^\circ\) and corresponds to directions that allow the largest \(I_{90}^{1807}\) through directions that allow the largest \(I_{90}^{1807}\) through directions that allow the largest \(I_{90}^{1807}\) through directions that allow the largest \(I_{90}^{1807}\) through directions that allow the largest \(I_{90}^{1807}\). The latitude constraint ensures that the search depth does not exceed the MSP scale height. The longitude restriction follows from the variation of the search volume in the plane. The detailed shapes of the contours are dependent on the survey frequency and duration (per direction) but suggest that future surveys that concentrate on low latitudes will maximize the number of new discoveries. However, deep high-latitude surveys will better constrain the Z of the disk population as well as place tighter constraints on, or make detections of, any bona fide diffuse or halo population pulsars. For the hypothetical survey depicted in Figure 11, a
total detection volume $\sim 13.3$ kpc$^3$ is sampled, corresponding to the discovery of $585 \pm 330$ disk MSPs. This number is for a uniform disk component; any Galactocentric radial dependence, likely to increase the number of MSPs toward the inner Galaxy, will only increase the number of detected MSPs.

13. DISCOVERING FAST PULSARS

Our fitting indicates that available survey data already place useful constraints on the minimum spin period in the MSP population (cf. §5 and Table 3). The reason such constraints may be placed is found in Figure 3, which shows that, for periods less than 1 ms, a nonzero (though small) volume has been searched. Here we estimate how much additional volume must be searched in order to expect to find pulsars with $P_1 < P < P_{\text{fast}}$, where $P_{\text{fast}}$ is the maximum period of interest.

Using equation (7), we derive an upper bound on the volume that must be searched (evaluated at the minimum MSP period, $P_1$), in order that we find pulsars with periods faster than $P_{\text{fast}}$. Since surveys at low periods do not see to large $D_{\text{max}}$, we assume that they do not see as far as the $z$ scale height $\sim 0.5$ kpc. Performing the integrals, we may write

$$\langle N_p \rangle \approx n_d(1 - P_1/P_{\text{fast}})[\Omega_b L_p^{3/2} S_m^{3/2}(P)]$$

(38)

Defining the term in brackets as $V_d(P_1)$ and requiring $\langle N_p \rangle \sim 1$ to obtain a likely detection, we find an upper bound on the required search volume to be

$$V_d(P_1) = \frac{1}{n_d(1 - P_1/P_{\text{fast}})}$$

(39)

Evaluating equation (39) for $P_{\text{fast}} = 1.5$ ms and $n_d \sim 44$ kpc$^{-3}$, we find that, as a function of the minimum period, $V_d(P_1)$ ranges from $\sim 1/25$ kpc$^3$ for $P_1 = 0.65$ ms (our 99% lower bound on $P_1$) to $\sim 1$ kpc$^3$ for $P_1 = 1.4$ ms. Comparing with Figure 3 shows that, for the luminosity assumed for that figure ($L_p = 16$ mJy kpc$^{-2}$), the Parkes survey (No. 7) yields search volumes at these $P_1$ that are comparable to those needed to yield a detection of a pulsar faster than 1.5 ms. However, the assumed $L_p$ for the figure is larger than average, and the Parkes survey observes to depths that, for some directions, exceed the $z$ scale height. Consequently, it is not surprising that a pulsar faster than PSR B1937+21 ($P = 1.56$ ms) has not been found. Nonetheless, future surveys should be able to probe this region of period space and either find fast pulsars or determine better the period cutoff to the MSP population. Our results indicate that deeper surveys at low Galactic latitudes (e.g., $b \lesssim 10^\circ$) will yield the search volume needed to accomplish these goals.

14. DISCUSSION

Through a likelihood analysis, we have constrained the period and pseudoluminosity distributions to be steep power laws with slopes $\sim -2$. The distributions imply that the population of MSPs increases rapidly to smaller periods and smaller pseudoluminosities. We infer a minimum period $P_{\text{min}} > 0.65$ ms at 99% confidence and a minimum luminosity cutoff $L_{p1} = 1.1^{+0.4}_{-0.3}$ mJy kpc$^{-2}$. The column density of MSPs in the local vicinity of the solar system is $N_{H_1} \sim 50^{+50}_{-20}$ kpc$^{-2}$. The limits on a diffuse halo-like component are $\lesssim 1\%$ of the midplane density. All these results are essentially identical for each of the models we have analyzed. Estimates of the total number of MSPs in the Galaxy are uncertain. Extrapolating on a per mass basis from the local disk surface density to a total disk mass, $M_{\text{disk}}$, we find $N_{\text{MSP}} \approx 3.0^{+1.8}_{-1.2} \times 10^4 (M_{\text{disk}}/4 \times 10^{10} M_\odot)$ for cutoff period 1.56 ms and cutoff luminosity 1.1 mJy kpc$^{-2}$ (without correction for beaming).

Our analysis assumes specific forms for the period and luminosity distributions, namely, power-law functions, that undoubtedly influence the specific values for numbers of pulsars in a given period range and also on the minimum-period cutoff. We have not tested other mathematical forms for these distributions, so the true cutoff for the period distribution may be different than we have derived. Nonetheless, because the period distribution increases monotonically with decreasing period, our quoted minimum period is larger than it would be for a function that plateaus or decreases with decreasing period below 1.56 ms. We consider the most important implication of our derived minimum period to be that MSPs faster than those already found may indeed be present in the Galaxy: the
surveys done heretofore cannot rule out their existence. In addition, modeling of the spin-up process using full general relativity (Cook et al. 1994a, 1994b) implies that gravitational instabilities do not prohibit the formation of very fast MSPs. Of course, the ultimate proof of existence for MSPs with $P < 1.56$ ms lies in future surveys that can explore large volumes of the Galaxy at these small periods. Such surveys will be feasible with new spectrometers that can sample more frequency channels at faster rates and with postprocessing that can contend with the motion of fast pulsars in binaries.

Another implication of our results on the period distribution is that, if MSPs exist because of the accretion-driven spin-up of neutron stars, then accretion must ensue for sufficiently long times that periods shorter than 1.56 ms can be achieved. From the work of Cook et al. (1994a, 1994b), such accretion appears possible without requiring typical ages for LMXBs that are so long as to resurrect the discrepancy between birth rates for MSPs and LMXBs.

The observed scale height of MSPs implies that they are a low-velocity population among neutron stars, having an rms speed that is about a factor of 5 smaller than that of young pulsars with much stronger magnetic fields. We attribute a part of the total inferred dispersion to a kick unique to the evolution of MSP systems ($\sim 40$ km s$^{-1}$), and the rest to the effect of diffusive processes that increase the dispersion of old objects. A number of kinematic signatures that should be evident in larger MSP samples (transverse motions, asymmetric drift, shape of velocity ellipsoid) are described.

The disparity in velocity between the low-field MSPs and the high-field pulsars might be taken as evidence that the two empirical classes of neutron stars are born through substantially different processes. If MSPs are produced largely through accretion-induced collapse of a white dwarf, and if that process yields only a small kick to the resultant NS compared with Type II supernova, then the observed dispersions of MSPs and high-field pulsars may find a natural explanation. In any case, the similarity in scale height of MSPs and LMXBs shows that the formation of the NS in both objects is accomplished without substantial center-of-mass impulses and supports the notion of an evolutionary connection. On the other hand, if binary survival after the Type II supernova is the most significant bottleneck in the production of LMXBs and their MSP descendants, it is possible that the processes that dictate survival of the binary system are also responsible for allowing only a limited range of center-of-mass velocities. Correlations between spin and orbital periods and space velocity, such as those suggested by Bailes et al. (1994), depend critically on the details of mass transfer and on the number of evolutionary paths that lead to MSP formation. Elsewhere, we will present our detailed analysis of the effects that sculpt binary survival.

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APPENDIX A

SEARCH SENSITIVITIES

The pulsar searches we consider involve the removal of dispersion delays between the outputs of a multichannel receiver using trial values for the dispersion measure. The resultant time series is then Fourier analyzed. Let us suppose that an $N_{\text{FFT}}$-length fast Fourier transform (FFT) is calculated from the time series for each trial dispersion measure. With a sample time $\Delta t$ and pulse period $P$, harmonics appear in frequency bins

$$ k_l = \frac{l \Delta t N_{\text{FFT}}}{P}, \quad l = 0, 1, \ldots, $$

(A1)

including a “DC” term ($l = 0$) and the fundamental ($l = 1$). Let the intrinsic pulse shape be $\tilde{s}(t)$, $0 \leq t \leq P$ so that a (short) discrete Fourier transform (DFT) of this shape over a single pulse period is $\tilde{s}(l)$, $l = 1, \ldots, M$. This function would determine the envelope of harmonic amplitudes in the long FFT were it not for additional contributions that derive from the postdetection averaging time (or “time constant”), from dispersion smearing across individual frequency channels, and from pulse broadening due to interstellar scattering (for distant sources). In many surveys, postdetection smoothing is simply an RC filter whose time domain response is a one-sided exponential function. Interstellar scattering produces nearly the same kind of time response, while the dispersion time function is dictated by the shapes of receiver filters, usually approximately Gaussian in form. Letting the M-point DFTs of the time constant, dispersion, and scattering functions be $\tilde{s}_{\text{tc}}$, $\tilde{s}_{\text{d}}$ and $\tilde{s}_{\text{s}}$, respectively, we may write the effective envelope function of harmonics as

$$ \tilde{s}_{\text{eff}}(l) = \tilde{s}(l) \tilde{s}_{\text{tc}}(l) \tilde{s}_{\text{d}}(l) \tilde{s}_{\text{s}}(l). $$

(A2)

It is useful to define the ratio of the $l$th harmonic to the DC value as

$$ R_l \equiv \left| \frac{\tilde{s}_{\text{eff}}(l)}{\tilde{s}_{\text{eff}}(0)} \right|. $$

(A3)

Survey FFTs are analyzed by constructing partial sums of harmonics (of the FFT magnitude or squared magnitude) for different trial periods. These sums are typically of $N_{\ell}$ = 1, 2, 3, 4, 8, and 16 harmonics, although there are variations on this.
Let us suppose that a threshold $\eta_p$ is chosen that represents the number of standard deviations in the FFT’s magnitude. This is typically $\eta_p \sim 6-9$ in order to minimize false alarms when testing large numbers of spectral values (typically multiples of 10$^9$) in a survey.

The minimum detectable flux density for a sum of harmonics $1, \ldots, N_h$ is

$$S_{\text{min}, N_h} = \frac{\eta_p T_{\text{sys}}}{G \sqrt{N_{\text{pol}} \Delta v \Delta t_{\text{FFT}}}} \left( \frac{\sqrt{N_k}}{\sum_{i=1}^{N_h} R_i} \right), \quad (A4)$$

where $T_{\text{sys}}$ is the system temperature (kelvins), $G$ is the telescope gain (K Jy $^{-1}$), $N_{\text{pol}} = 2$ is the number of independent polarization channels included, $\Delta v$ is the total bandwidth, and $\Delta t$ is the sample interval. For searches that analyze $|\text{FFT}|^2$ rather than $|\text{FFT}|$, $R_i \rightarrow R_i^2$. The actual minimum flux density depends on the number of harmonics that contribute significantly, which in turn depends on the duty cycle of the pulse. Because extrinsic effects (viz., dispersion and scattering) broaden the pulse, the optimal $N_h$ and corresponding $S_{\text{min}}$ are strongly dependent on the observation frequency, distance, direction, and pulse period. The direction dependence is manifested in the dispersion measure to which a pulsar of given direction, and pulse period may be detected. Consequently, $S_{\text{min}}$ is the minimum over all considered in the analysis and may be written with dependences

$$S_{\text{min}} = S_{\text{min}}(l, b, P, DM, \nu, \Delta v, N_{\text{ch}}, T_{\text{sys}}, G, \ldots) \quad (A5)$$

In practice, surveys usually test only a subset of all possible harmonic sums. We take this into account when computing $S_{\text{min}}$ for each survey.

Note that our expression for the minimum flux density differs from that often quoted in the literature (e.g., Camilo et al. 1996), which replaces the factor in large parentheses in equation (A4) with a factor $\left[w/(P - w)\right]^{1/2}$, where $w$ is the pulse width. The divergence of this factor as $w \rightarrow P$ is equivalent to assuming $R_i = 0$, which overestimates the true $S_{\text{min}}$ because, even when pulse smearing exceeds a pulse period, the variable flux remaining at the fundamental frequency can still be detectable for a luminous pulsar. Our expression takes this possibility into account, which corresponds to $0 < R_i < 1$.

It is important to calculate accurately the minimum detectable flux density because it determines the Galactic volume searched. This volume is small but not zero for very short periods $\lesssim 1.5$ ms.

APPENDIX B

INTERSTELLAR SCINTILLATIONS

Interstellar scintillations are intensity variations in both time and frequency caused by multipath propagation through ionized gas. At 400 MHz, both diffractive (DISS) and refractive (RISS) interstellar scintillations contribute to the flux variations of pulsars. Here we restrict the discussion to DISS, which will dominate RISS at 400 MHz and is especially important because its probability density is skewed whereas that of RISS is symmetric.

DISS causes the pulsar flux density $S$ to vary as $S = gS$, where $g$ is the DISS gain that has a one-sided exponential distribution when DISS is saturated and not quenched by time-bandwidth averaging (Rickett 1990; Cordes & Lazio 1991). Except for the very nearest pulsars ($D < 100$ pc), DISS is saturated at 400 MHz. The characteristic time and frequency scales of DISS diminish with increasing distance. Use of finite bandwidth $B$ and data-span length $T$ will average over distinct scintillation maxima, increasing the number of degrees of freedom from 2 (for unquenched DISS) to $2n_{\text{iss}}$, where

$$n_{\text{iss}} \sim \left(1 + 0.2 \frac{B}{\Delta v_g}\right) \left(1 + 0.2 \frac{T}{\Delta t_g}\right), \quad (B1)$$

$\Delta v_g$ is the characteristic bandwidth of DISS, and $\Delta t_g$ is the characteristic timescale (Cordes 1986). The characteristic bandwidth and timescale have been measured for many pulsars and were used as input to the TC model for pulsar distances. For our purposes, we use the TC model’s estimation of the scattering measure, along with the distance and frequency, to estimate the scintillation parameters.

The pdf of $g$ is

$$f_g(g, n_{\text{iss}}) = \frac{(g n_{\text{iss}})^{n_{\text{iss}}} e^{-g n_{\text{iss}}}}{\Gamma(n_{\text{iss}})} U(g), \quad (B2)$$

with $U(g)$ the Heaviside function and $\Gamma$ the gamma function. As $n_{\text{iss}} \rightarrow \infty$ (i.e., pulsars at large distances or observed at low frequencies), $f_g$ tends toward a delta function, $\delta(g - 1)$.

We include scintillations in our analysis by defining a scintillated pseudoluminosity, $L'_p = g L_p$. For a luminosity function $f_{L_p}(L_p)$, the corresponding scintillated luminosity function is

$$f_{L'_p}(L'_p) = \int dg^{-1} f_g(g, n_{\text{iss}}) f_{L_p}(L_p/g). \quad (B3)$$

The distinctive effect of DISS is that, if the intrinsic luminosity function has cutoffs at low and high luminosities, the scintillated luminosity function will not. In fact, the scintillated luminosity function will extend to zero luminosity because the
most probable scintillation gain (for $n_{iss} = 1$) is zero. Luminosities larger than the upper cutoff will be seen because of the long exponential tail of $f_g$ (again for $n_{iss} = 1$). Figure 12 shows examples of scintillated luminosity functions for several values of $n_{iss}$. As $n_{iss} \to \infty$, the scintillated luminosity function tends toward the original, unscintillated luminosity function.

In surveys where DISS is saturated and unquenched and single trials are made on each sky position, the volume surveyed is effectively increased by a factor $\langle \rho^{5/2} \rangle = \Gamma(5/2) \sim 1.33$. Multiple trials can increase or decrease this volume factor, depending on how the results of the various trials are combined.

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FIG. 12.—Luminosity functions with and without the effects of scintillations included. Heavy solid line: intrinsic luminosity function having a power-law slope of $-2$. Solid line: the scintillated luminosity function when interstellar scintillations are saturated and unquenched by time-bandwidth averaging. Dotted lines: scintillated luminosity functions with various degrees of averaging, indicated by $n_{iss}$ (cf. eq. [B3]).
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