Analysis of the rare semileptonic $B_c \to P(D, D_s)l^+l^-/\nu\bar{\nu}$ decays within QCD sum rules

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Abstract

Considering the gluon condensate corrections, the form factors relevant to the semileptonic rare $B_c \to D, D_s(J^P = 0^-)l^+l^-$ with $l = \tau, \mu, e$ and $B_c \to D, D_s(J^P = 0^-)\nu\bar{\nu}$ transitions are calculated in the framework of the three point QCD sum rules. The heavy quark effective theory limit of the form factors are computed. The branching fraction of these decays are also evaluated and compared with the predictions of the relativistic constituent quark model. Analyzing of such type transitions could give useful information about the strong interactions inside the pseudoscalar $D_s$ meson and its structure.

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1 Introduction

With the chances that in the future a large amount of Bc mesons will be produced at LHC (with the luminosity values of $\mathcal{L} = 10^{34} \text{cm}^{-2}\text{s}^{-1}$ and $\sqrt{s} = 14\text{TeV}$, the number of $B_c^\pm$ mesons is expected to be about $10^8 \sim 10^{10}$ per year [1, 2]), one might explore the rare Bc decays to pseudoscalar ($D, D_s$) and $l^+l^-/\nu\bar{\nu}$. Such types transitions could be useful because of the following reasons: 1) Analyzing of such type transitions could give valuable information about the nature of the pseudoscalar $D_s$ meson and the strong interactions inside it. 2) The form factors of these transitions could be used in the study of the polarization asymmetries, CP and T violations. 3) These will provide a new framework for more precise calculation of the Cabibbo-Kobayashi-Maskawa (CKM) matrix elements $V_{tq}$ ($q = d, s, b$) and leptonic decay constants of $D_{s,d}$ and $B_c$ mesons. 4) These transitions occur at loop level in standard model (SM) via the flavour changing neutral current (FCNC) transitions of $b \to s, d$, which are sensitive to the new physics beyond the SM, so these decays are useful to constrain the parameters beyond the SM. 5) A possible forth generation, SUSY particles [3] and light dark matter [4] might contribute to the loop transitions of $b \to s, d$. The $B_c$, is the only meson containing two heavy quarks with different charge and flavours and it is the lowest bound state of $b$ and $c$ quarks, so its decay modes properties are expected to be different than flavour neutral mesons. Since the excited levels of $\bar{b}c$ lie below the threshold of decay into the pair of heavy $B$ and $D$ mesons, such states decay weakly and they have no annihilation decay modes due to the electromagnetic and strong interactions (for more about the physics of the $B_c$ meson see for example [5]). This paper describes the annihilation of the $B_c$ into the pseudoscalar ($D, D_s$) $l^+l^-/\nu\bar{\nu}$ in the framework of the three point QCD sum rules as a non-perturbative approach based on the fundamental QCD Lagrangian. This transitions are parameterized in terms of some form factors calculation of which plays crucial role in the analyzing of those decay channels. These decays at quark level proceed by the loop $b \to s, d$ in the SM with the intermediate $u, c$ and $t$ quarks and the main contribution comes from the intermediate top quark. These decay modes have also been studied in the relativistic constituent quark model (RCQM) [6]. Some other possible channels such as $B_c \to \ell\nu\gamma, B_c \to \rho^+\gamma, B_c \to K^{*+}\gamma; B_c \to B_u^*l^+l^-, B_c \to B_u^{*\gamma}, B_c \to D_{s,d}^*\gamma, B_c \to D_{s,d}^*l^+l^- and B_c \to X\nu\bar{\nu}$ with $X$ be axial vector particle, $D_{s1}(2460)$, and vector particles, $D^*, D_s^*$ are studied in the light cone or traditional QCD sum rules methods in [7–13], respectively. For a set of exclusive nonleptonic and semileptonic decays of the $B_c$ meson, which have been studied in the relativistic constituent quark model see [14].

The content of paper is as follows: In section 2, we calculate the sum rules for the related form factors considering the gluon correction contributions to the correlation function. The light quark condensate contributions are killed applying the double borel transformations with respect to momentum of the initial and final states. The heavy quark effective theory (HQET) limit of the form factors are presented in section 3. Section 4 depicts our numerical analysis of the form factors and their comparison with the HQET limit of them, results, discussions and comparison of our results with the prediction of the RCQM model.
2 QCD Sum rules for transition form factors of the 
$B_c \rightarrow (D, D_s) l^+ l^- / \nu \bar{\nu}$

At quark level, the processes $B \rightarrow P l^+ l^- / \nu \bar{\nu} (P = D, D_s)$ are described by the loop $b \rightarrow q_1 l^+ l^- / \nu \bar{\nu}$ transitions, $(q_1 = d, q_2 = s)$ in the SM (see Fig.1), and receive contributions from photon and $Z$-penguin and box diagrams for $l^+ l^-$ and only $Z$-penguin and box diagrams for $\nu \bar{\nu}$. These loop transitions occur via the intermediate $u, c, t$ quarks, where dominant contribution comes from intermediate top quark. The effective Hamiltonian responsible for $b \rightarrow q_1 l^+ l^-$ decays is described in terms of the Wilson coefficients, $C_{7}^{eff}$, $C_{9}^{eff}$ and $C_{10}$ as

$$
\mathcal{H}_{eff} = \frac{G_F \alpha}{2 \sqrt{2} \pi} V_{tb} V_{tq_i}^* \left[ C_{7}^{eff} \bar{q}_i \gamma_\mu (1 - \gamma_5) b \bar{\ell} \gamma_\mu \ell + C_{9}^{eff} \bar{q}_i \gamma_\mu (1 - \gamma_5) b \bar{\ell} \gamma_\mu \gamma_5 \ell + \right. \\
- \left. 2C_{10}^{eff} m_b \frac{q^2}{q^2} \bar{q}_i i \sigma_{\mu \nu} q^\nu (1 + \gamma_5) b \bar{\ell} \gamma_\mu \ell \right], 
$$

(1)

where $G_F$ is the Fermi constant, $\alpha$ is the fine structure constant at $Z$ mass scale, and $V_{ij}$ are elements of the CKM matrix. For $\nu \bar{\nu}$ case, only term containing $C_{10}$ is considered. The amplitudes for for $B_c \rightarrow P l^+ l^- / \nu \bar{\nu}$ decays are obtained by sandwiching of Eq. (1) between initial and final meson states:

$$
\mathcal{M} = \frac{G_F \alpha}{2 \sqrt{2} \pi} V_{tb} V_{tq_i}^* \left[ C_{9}^{eff} < P(p') \mid \bar{q}_i \gamma_\mu (1 - \gamma_5) b \mid \bar{B}_c(p) > \bar{\ell} \gamma_\mu \ell \right].
$$
Next, we calculate the matrix elements \( < P(p') | \bar{q}_i \gamma_\mu (1 - \gamma_5)b | B_c(p) > \) and \( f \) constants of the \( P \) is obtained. The following matrix elements are defined in terms of the leptonic decay currents inserting the complete set of intermediate states with the same quantum numbers as the sum rules expressions for our form factors. The phenomenological part can be obtained by and final states to suppress the contribution of the higher states and continuum, we get and applying the double Borel transformations with respect to the momentum of the initial side, 2) quark gluon language which is the QCD or theoretical side. Equating two sides from the general philosophy of the QCD sum rules, we can calculate the above mentioned correlator in two languages: 1) hadron language called the physical or phenomenological side, 2) quark gluon language which is the QCD or theoretical side. Equating two sides and applying the double Borel transformations with respect to the momentum of the initial and final states to suppress the contribution of the higher states and continuum, we get sum rules expressions for our form factors. The phenomenological part can be obtained by inserting the complete set of intermediate states with the same quantum numbers as the currents \( J_P \) and \( J_{B_c} \). As a result of this procedure is obtained. The following matrix elements are defined in terms of the leptonic decay constants of the \( P \) and \( B_c \) mesons as:

\[
\langle 0 | J_P | P(p) \rangle = -i \frac{f_{pm_P}}{m_c + m_{q_i}}, \]

\[
\langle 0 | J_{B_c} | B_c(p) \rangle = -i \frac{f_{m_{B_c}^2}}{m_b + m_c}. \]

Using Eqs. (3), (4) and (7) in Eq. (6), we obtain

\[
\Pi^V(p^2, p'^2, q^2) = -\frac{f_{B_c} m_{B_c}^2}{(m_b + m_c)(m_c + m_{q_i})(m_{B_c}^2 - p'^2)(m_{B_c}^2 - p^2)} \frac{f_{pm_P}}{m_c + m_{q_i}} \left[ f_+ P_\mu + f_- q_\mu \right] + \ldots , \quad (8)
\]
\[
\Pi^T_\mu(p^2, p'^2, q^2) = \frac{f_B m^2_B}{(m_b + m_b)(m_c + m_c)} \frac{f_P m^2_P}{(m_B^c - p^2)(m_B^c - p'^2)} \times \left[ \frac{f_T}{(m_B + m_p)} [q^2\mathcal{P}_\mu - (m_B^c - m_B^c)q_\mu] \right] + \ldots \tag{9}
\]

For extracting the expressions for form factors \( f_+ (q^2) \) and \( f_- (q^2) \), we choose the coefficients of the structures \( \mathcal{P}_\mu \) and \( q_\mu \) from \( \Pi^V_\mu(p^2, p'^2, q^2) \), respectively and the structure \( q_\mu \) from \( \Pi^T_\mu(p^2, p'^2, q^2) \) is considered for the form factor \( f_T(q^2) \). Therefore, the correlation functions are written in terms of the selected structures as:

\[
\Pi^V_\mu(p^2, p'^2, q^2) = \Pi_+ \mathcal{P}_\mu + \Pi_- q_\mu + \ldots , \tag{10}
\]

\[
\Pi^T_\mu(p^2, p'^2, q^2) = \Pi_T q_\mu + \ldots . \tag{11}
\]

On the other side, to calculate the QCD part of correlation function, we evaluate the three–point correlator by the help of the operator product expansion (OPE) in the deep Euclidean region, where \( p^2 \ll (m_b + m_c)^2 \) and \( p'^2 \ll (m_c + m_q)^2 \). For this aim, we write each \( \Pi_i \) function in terms of the perturbative and non-perturbative parts as:

\[
\Pi_i(p^2, p'^2, q^2) = \Pi_i^{\text{per}}(p_1^2, p_2^2, q^2) + \Pi_i^{\text{nonper}}(p_1^2, p_2^2, q^2) , \tag{12}
\]

where \( i \) stands for \( +, - \) and \( T \) and non-perturbative part contains the light quark \( \langle \bar{q}q \rangle \) and gluon \( \langle G^2 \rangle \) condensates. For the perturbative part, the bare loop diagram (Fig. 1 a) is considered, however, diagrams b, c, d in Fig. 1 are correspond to the light quark condensates contributing to the correlator. In principle, the light quark condensate diagrams give contributions to the correlation function, but applying double Borel transformations omits their contributions, hence as first non-perturbative correction, we consider the gluon condensate diagrams (see Fig. 2 a, b, c, d, e, f).

By the help of the double dispersion representation, the bare–loop contribution is written as

\[
\Pi_i^{\text{per}} = -\frac{1}{(2\pi)^2} \int ds' \int ds \frac{\rho_i^{\text{per}}(s, s', Q^2)}{(s - p^2)(s' - p'^2)} + \text{subtraction terms} , \tag{13}
\]

where \( Q^2 = -q^2 \). The spectral densities \( \rho_i^{\text{per}}(s, s', Q^2) \) are calculated by the help of the Gutkovsky rule, i.e., the propagators are replaced by Dirac–delta functions

\[
\frac{1}{p^2 - m^2} \rightarrow -2i\pi\delta(p^2 - m^2) , \tag{14}
\]

expressing that all quarks are real. The integration region in Eq. (13) is obtained by requiring that the argument of three delta vanish, simultaneously. This condition results in the following inequality

\[
-1 \leq \frac{2ss' + (s + s' + Q^2)(m_b^2 - m_c^2 - s) + 2s(m_c^2 - m_q^2)}{\lambda^{1/2}(s, s', -Q^2}\lambda^{1/2}(m_b^2, m_c^2, s)} \leq +1 , \tag{15}
\]
where \( \lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc \). From this inequality, to use in the lower and upper limit of the integration over \( s \) in subtractions, it is easy to express \( s \) in terms of \( s' \) i.e. \( f_{\pm}(s') \) in the \( s = s' \) plane.

Straightforward calculations end up in the following results for the spectral densities:

\[
\rho_+^\pm(s, s', q^2) = I_0 \, N_c \{ \Delta + \Delta' + \\
-2m_c [(+2 + E_1 + E_2)m_c - (1 + E_1 + E_2)m_{q_i}] \\
+2m_0[(1 + E_1 + E_2)m_c - (E_1 + E_2)m_{q_i}] \\
+(E_1 + E_2)u \},
\]

\[
\rho_-^\pm(s, s', q^2) = I_0 \, N_c \{ -\Delta + \Delta' \\
-2m_c [(E_2 - E_1 - 1)m_{q_i} + (E_1 - E_2)m_c] \\
-2m_0[(1 - E_1 + E_2)m_c + (E_1 - E_2)m_{q_i}] \\
+(E_1 - E_2)u \},
\]

\[
\rho_T^\pm(s, s', q^2) = -I_0 \, N_c \{ \Delta (2m_c - m_b - m_{q_i}) + \Delta' (m_b - 2m_c + m_{q_i}) \\
+2[2m_c(E_1 - E_2 - 1) + m_{q_i}(E_2 - E_1)]s \\
-2[m_b(E_1 - E_2) - m_c(E_1 - E_2 + 1)]s' \\
+(E_1 - E_2)(m_b - 2m_c + m_{q_i})u \},
\]

where

\[
I_0(s, s', Q^2) = \frac{1}{4\lambda^{1/2}(s, s', Q^2)},
\]

\[
\lambda(s, s', Q^2) = s^2 + s'^2 + Q^4 + 2sQ^2 + 2s'Q^2 - 2ss',
\]

\[
E_1 = \frac{1}{\lambda(s, s', Q^2)}[2s\Delta - \Delta' u],
\]

\[
E_2 = \frac{1}{\lambda(s, s', Q^2)}[2s\Delta' - \Delta u],
\]

\[
u = s + s' + Q^2,
\]

\[
\Delta = s + m_c^2 - m_b^2,
\]

\[
\Delta' = s' + m_c^2 - m_{q_i}^2,
\]

(16)

and \( N_c = 3 \) is the color factor.

Now as first correction to the non-perturbative part of the correlator, we calculate the gluon condensate contributions (see diagrams in Fig. 2). The calculations proceed the same as \[13\] ( see also \[9, 11, 12, 18\]) and the Fock-Schwinger fixed-point gauge \[15–17\], \( x^\mu G^a_\mu = 0 \), where \( G^a_\mu \) is the gluon field is used. In calculations, the following type of integrals are encountered:

\[
I_0[a, b, c] = \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m_b^2]^a [(p + k)^2 - m_c^2]^b [(p' + k)^2 - m_{q_i}^2]^c}.
\]
Performing integration over loop momentum and applying double Borel transformations with respect to the $p^2$ and $p'^2$, we obtain the Borel transformed form of the integrals as follows:

$$ I_{\mu}[a, b, c] = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{[k^2 - m_b^2]^{a}[(p + k)^2 - m_c^2]^{b}[(p' + k)^2 - m_q^2]^{c}.} $$

Hat in Eq. (18) denotes the double Borel transformed form of integrals. $M_1^2$ and $M_2^2$ are the Borel parameters in the $s$ and $s'$ channels, respectively, and the function $U_0(a, b)$ is
defined as:

$$U_0(a, b) = \int_0^\infty dy(y + M_1^2 + M_2^2)^a y^b \exp \left[-\frac{B_{-1}}{y} - B_0 - B_1 y\right],$$

where

$$B_{-1} = \frac{1}{M_1^2 M_2^2} \left[ m_{q_i}^2 M_1^4 + m_b^2 M_1^4 + M_2^2 M_1^2 (m_b^2 + m_{q_i}^2 + Q^2) \right],$$

$$B_0 = \frac{1}{M_1^2 M_2^2} \left[ (m_{q_i}^2 + m_c^2) M_1^2 + M_2^2 (m_b^2 + m_c^2) \right],$$

$$B_1 = \frac{m_c^2}{M_1^2 M_2^2}. \quad (20)$$

After straightforward but lengthy calculations, we get the following results for the gluon condensate contributions:

$$\Pi_i^{G^2} = i \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{C_i}{6}, \quad (21)$$

where the explicit expressions for $C_i$ are given in appendix–A.

Next step is to apply the Borel transformations with respect to the $p^2$ ($p^2 \to M_1^2$) and $p^2$ ($p^2 \to M_2^2$) on the phenomenological as well as the perturbative parts of the correlation function, continuum subtraction and equate these two representations of the correlator. The following sum rules for the form factors $f_+$, $f_-$ and $f_T$ are derived:

$$f_+ = \frac{(m_b + m_c)(m_c + m_{q_i})}{f_{B_c} m_{B_c} f_{F_p} m_{F_p}} e^{m_{B_c}/M_1^2} e^{m_{F_p}/M_2^2}$$

$$\times \frac{1}{4\pi^2} \left\{ \int_{(m_c + m_{q_i})^2}^{s_0} ds' \int_{f_-(s') \min(s_0, f_+(s'))}^{\min(s_0, f_+(s'))} ds \rho_+^V(s, s', Q^2) e^{-s/M_1^2} e^{-s'/M_2^2} - iM_1^2 M_2^2 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{C_+}{6} \right\}, \quad (22)$$

$$f_- = \frac{(m_b + m_c)(m_c + m_{q_i})}{f_{B_c} m_{B_c} f_{F_p} m_{F_p}} e^{m_{B_c}/M_1^2} e^{m_{F_p}/M_2^2}$$

$$\times \frac{1}{4\pi^2} \left\{ \int_{(m_c + m_{q_i})^2}^{s_0} ds' \int_{f_-(s') \min(s_0, f_+(s'))}^{\min(s_0, f_+(s'))} ds \rho_+^V(s, s', Q^2) e^{-s/M_1^2} e^{-s'/M_2^2} - iM_1^2 M_2^2 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{C_-}{6} \right\}, \quad (23)$$

$$f_T = \frac{(m_b + m_c)(m_c + m_{q_i})}{f_{B_c} m_{B_c} f_{F_p} m_{F_p}(m_{B_c} - m_p)} e^{m_{B_c}/M_1^2} e^{m_{F_p}/M_2^2}$$

$$\times \frac{1}{4\pi^2} \left\{ \int_{(m_c + m_{q_i})^2}^{s_0} ds' \int_{f_-(s') \min(s_0, f_+(s'))}^{\min(s_0, f_+(s'))} ds \rho_T^V(s, s', Q^2) e^{-s/M_1^2} e^{-s'/M_2^2} - iM_1^2 M_2^2 \left\langle \frac{\alpha_s}{\pi} G^2 \right\rangle \frac{C_T}{6} \right\}. \quad (24)$$

where $s_0$ and $s_0'$ are the continuum thresholds and $s = f_\pm(s')$ in the lower and upper limit of the integral over $s$ are obtained from inequality (15). The $\min(s_0, f_+(s'))$ means that for
each value of the \( q^2 \), the smaller one between \( s_0 \) and \( f_+ \) is selected. In above equations, in order to subtract the contributions of the higher states and the continuum the quark-hadron duality assumption is also used

\[
\rho^{\text{higher states}}(s, s') = \rho^{\text{OPE}}(s, s') \theta(s - s_0) \theta(s' - s'_0) . \tag{25}
\]

At the end of this section, we would like to present the differential decay width of \( B_c \to P l^+ l^- / \nu \bar{\nu} \) decays. Using the parametrization of these transitions in terms of form factors and amplitude in Eq.(2), we get

\[
\frac{d\Gamma}{dQ^2} (B_c^\pm \to P^\pm \nu \bar{\nu}) = \frac{G^2 \alpha^2}{24 \pi^5} |V_{tq}|^2 |V_{tb}|^2 \phi_P^{3/2} (1, r_p, s) m_{B_c}^3 |C_{10}|^2 |f_+(Q^2)|^2 , \tag{26}
\]

where, \( \phi_P (1, r_p, s) \) is the usual triangle function

\[
\phi_P (1, r_p, s) = 1 + r_p^2 + s^2 - 2 r_p - 2s - 2 r_p s ,
\]

with \( r_p = \frac{m_p^2}{m_{B_c}^2} \), \( s = - \frac{Q^2}{m_{B_c}^2} \), and

\[
\frac{d\Gamma}{dQ^2} (B_c^\pm \to P^\pm l^+ l^-) = \frac{G^2 |V_{tq}|^2 |V_{tb}|^2 m_{B_c}^2 \alpha^2}{3 \cdot 2^9 \pi^5} v \phi_P^{1/2} (1, r_p, s) \left[ (1 + \frac{2t}{s}) \phi_P (1, r_p, s) \alpha_1 + 12 t \beta_1 \right] , \tag{27}
\]

where \( t = m_l^2 / m_{B_c}^2 \) and the expressions of \( \alpha_1 \) and \( \beta_1 \) and \( v \) are given as:

\[
v = \sqrt{1 + \frac{4m_l^2}{Q^2}} , \quad \alpha_1 = |C^\text{eff}_9 f_+(Q^2) + \frac{2 \hat{m}_b C^\text{eff}_7 f_T(Q^2)}{1 + \sqrt{r_p}}|^2 + |C_{10} f_+(Q^2)|^2 , \quad \beta_1 = |C_{10}|^2 \left[ \left( 1 + r_p - \frac{s}{2} \right) |f_+(Q^2)|^2 + \left( 1 - r_p \right) \text{Re}(f_+(Q^2) f_-^*(Q^2)) + \frac{1}{2} s |f_- (Q^2)|^2 \right] ,
\]

where \( \hat{m}_b = m_b / m_{B_c} \).

3 HQET limit of the form factors

In this section, we present the infinite heavy quark mass limit of the form factors for \( B_c \to (D, D_s) l^+ l^- / \nu \bar{\nu} \) transitions. To this aim, we use the following parametrization (see also [19–23]):

\[
y = \nu \nu' = \frac{m_{B_c}^2 + m_{D_{q_i}}^2 - q^2}{2m_{B_c} m_{D_{q_i}}} \tag{28}
\]

where \( \nu \) and \( \nu' \) are the four-velocities of the initial and final meson states, respectively and \( y = 1 \) are so called zero recoil limit. Now, to obtain the \( y \) dependent expressions of the
form factors we define $m_b \to \infty$, $m_c = \frac{m_b}{\sqrt{z}}$, where $z$ is given by $\sqrt{z} = y + \sqrt{y^2 - 1}$ and we also set the mass of light quarks to zero. In this limit the new Borel parameters $T_1$ and $T_2$ take the form $T_1 = M_1^2/2m_b$ and $T_2 = M_2^2/2m_c$.

The new continuum thresholds $\nu_0$, and $\nu'_0$ are defined as:

$$
\nu_0 = \frac{s_0 - m_b^2}{m_b}, \quad \nu'_0 = \frac{s'_0 - m_c^2}{m_c},
$$

and the new integration variables become:

$$
\nu = \frac{s - m_b^2}{m_b}, \quad \nu' = \frac{s' - m_c^2}{m_c}.
$$

The leptonic decay constants are rescaled:

$$
\hat{f}_{B_c} = \sqrt{m_b} f_{B_c}, \quad \hat{f}_{D_{q_i}} = \sqrt{m_c} f_{D_{q_i}}.
$$

The correspond expressions for $\hat{I}_0(a, b, c)$, $\hat{I}_1(a, b, c)$ and $\hat{I}_2(a, b, c)$ in this limit are defined as:

$$
\hat{I}_0(a, b, c)^{HQET} = \frac{(-1)^{a+b+c}}{16\pi^2 \Gamma(a) \Gamma(b) \Gamma(c)} (T_1)^{2-a-b} (T_2)^{2-a-c} U_0^{HQET}(a + b + c - 4, 1 - c - b),
$$

$$
\hat{I}_1(a, b, c)^{HQET} = i \frac{(-1)^{a+b+c+1}}{16\pi^2 \Gamma(a) \Gamma(b) \Gamma(c)} (T_1)^{2-a-b} (T_2)^{3-a-c} U_0^{HQET}(a + b + c - 5, 1 - c - b),
$$

$$
\hat{I}_2(a, b, c)^{HQET} = i \frac{(-1)^{a+b+c+1}}{16\pi^2 \Gamma(a) \Gamma(b) \Gamma(c)} (T_1)^{3-a-b} (T_2)^{2-a-c} U_0^{HQET}(a + b + c - 5, 1 - c - b),
$$

where $T_1$ and $T_2$ are the Borel parameters in the $s$ and $s'$ channel, respectively, and the function $U_0^{HQET}(m, n)$ is defined as

$$
U_0^{HQET}(m, n) = \int_0^\infty (x + T_1 + T_2)^m x^n \exp\{\hat{A} + \hat{B} + \hat{C}\} dx,
$$

with

$$
A = -\frac{m_b^2 T_2^2 + T_1 T_2 (m_b^2 + y)}{T_1 T_2 x},
$$

$$
B = -\frac{T_2 (m_b^2 + m_c^2) + T_1 m_b^2}{T_1 T_2},
$$

$$
C = -\frac{m_b^2}{T_1 T_2}.
$$

In order to the calculations be easy, the following redefinitions for the form factors are applied

$$
\tilde{f}_i = f_i \{m_{B_c} + m_{D_{q_i}}\}
$$
After some calculations, we obtain the $y$-dependent expressions of the form factors as follows:

\[
\hat{f}^{\text{HQET}}_+(y) = \frac{1}{32\pi^2 f_B f_F} e^{\frac{\Lambda}{m_b}} e^{\frac{\Lambda}{m_T}} \left\{ \frac{3(1 + \sqrt{z})^2}{(-1 + \sqrt{z}) F(y, z)} \right. \\
-1 + (1 + 3y)\sqrt{z} - (2 + y + 4y^2)z + (3 + 2y(2 + y))z^{3/2} - 4yz^2 + 2z^{5/2} \\
\left[ \int_0^{\nu_0} d\nu \int_0^{\nu'_0} d\nu' e^{-\frac{\nu}{m_T} e^{-\frac{\nu'}{m_T}}} \theta(2y\nu - \nu^2) \\
+ \lim_{m_b \to \infty} \left( i \frac{5z^2}{24m_b^5} (1 + \sqrt{z}) \left( \frac{\alpha_s}{\pi} G^2 \right) C^{\text{HQET}}_+ \right) \right\} ,
\]

(36)

\[
\hat{f}^{\text{HQET}}_-(y) = \frac{1}{32\pi^2 f_B f_F} e^{\frac{\Lambda}{m_b}} e^{\frac{\Lambda}{m_T}} \left\{ \frac{-3(1 + \sqrt{z})^2}{(-1 + \sqrt{z}) F(y, z)} \right. \\
-1 + (1 + 3y)\sqrt{z} - (2 + 5y)z + (7 + 2y(y - 2))z^{3/2} - 4(y - 1)z^2 + 2z^{5/2} \\
\left[ \int_0^{\nu_0} d\nu \int_0^{\nu'_0} d\nu' e^{-\frac{\nu}{m_T} e^{-\frac{\nu'}{m_T}}} \theta(2y\nu - \nu^2) \\
+ \lim_{m_b \to \infty} \left( i \frac{5z^2}{24m_b^5} (1 + \sqrt{z}) \left( \frac{\alpha_s}{\pi} G^2 \right) C^{\text{HQET}}_- \right) \right\} ,
\]

(37)

\[
\hat{f}^{\text{HQET}}_T(y) = \frac{1}{32\pi^2 f_B f_F} e^{\frac{\Lambda}{m_b}} e^{\frac{\Lambda}{m_T}} \left\{ \frac{-3(1 + \sqrt{z})^2}{(-1 + \sqrt{z}) F(y, z)} \right. \\
-4y^2 z^2 + 2(2 + y)(1 + 4y) z^{3/2} + (3 + 2y(y + 2))z^2 \\
\left[ 3 - (1 + 9y)\sqrt{z} + (4 + y(3 + 8y))z - ((2 + y)(1 + 4y))z^{3/2} + (3 + 2y(y + 2))z^2 \\
-4y^2 z^2 + z^3 \right] \int_0^{\nu_0} d\nu \int_0^{\nu'_0} d\nu' e^{-\frac{\nu}{m_T} e^{-\frac{\nu'}{m_T}}} \theta(2y\nu - \nu^2) \\
+ \lim_{m_b \to \infty} \left( i \frac{5z^2}{48 z m_b^5} (1 + \sqrt{z}) \left( \frac{\alpha_s}{\pi} G^2 \right) C^{\text{HQET}}_T \right) \right\} ,
\]

(38)

where

\[
F(y, z) = z^{3/4} [1 + z + y^2 z + z^2 - 2y\sqrt{z}(1 + z)]^{3/2}.
\]

(39)

In the heavy quark limit expressions of the form factors, $\Lambda = m_{B_q} - m_b$ and $\bar{\Lambda} = m_{D_q} - m_c$, and the explicit expressions of the coefficients $C_i^{\text{HQET}}$ are given in the appendix–B.

## 4 Numerical analysis

This section encompasses our numerical analysis of the form factors $f_+, f_-$ and $f_T$ and their HQET limit, branching fractions, comparison of our results with the prediction of the RCQM and discussion. The sum rules expressions of the form factors depict that the main input parameters entering the expressions are gluon condensate, Wilson coefficients $C_7^{\text{eff}}$, $C_9^{\text{eff}}$ and $C_{10}$, elements of the CKM matrix $V_{tb}$, $V_{ts}$ and $V_{td}$, leptonic decay constants;
$f_{B_C}$, $f_D$ and $f_{D_s}$, Borel parameters $M_1^2$ and $M_2^2$, as well as the continuum thresholds $s_0$ and $s'_0$. In further numerical analysis, we choose the values of the Gluon condensate, leptonic decay constants, CKM matrix elements, Wilson coefficients, quark and meson masses as: $<\frac{\alpha_s}{\pi}G^2> = 0.012 \text{ GeV}^4 [24]$, $C_2^{qff} = -0.313, C_6^{qff} = 4.344, C_{10} = -4.669 [25, 26]$, $V_{tb} = 0.77_{-0.18}^{+0.18}$, $V_{ts} = (40.6 \pm 2.7) \times 10^{-3}$, $V_{td} = (7.4 \pm 0.8) \times 10^{-3} [27]$, $f_{D_s} = 274\pm 13 \pm 7 \text{ MeV} [28]$, $f_D = 222.6 \pm 16.7 \pm 3.4 \text{ MeV} [29]$, $f_{B_s} = 350 \pm 25 \text{ MeV} [30-32]$, $m_c = 1.25 \pm 0.09 \text{ GeV}$, $m_s = 95 \pm 25 \text{ MeV} \ , \ m_b = (4.7 \pm 0.07) \text{ GeV} \ , \ m_d = (3 - 7) \text{ MeV}$, $m_{D_s} = 1.968 \text{ GeV}$, $m_D = 1.869 \text{ GeV}$, $m_{B_C} = 6.258 \text{ GeV} [33]$, $\Lambda = 0.62 \text{ GeV} [34]$ and $\overline{\Lambda} = 0.86 \text{ GeV}[35]$.

The expressions for the form factors contain also four auxiliary parameters: Borel mass squares $M_1^2$ and $M_2^2$ and continuum threshold $s_0$ and $s'_0$. These are not physical quantities, so the physical quantities, form factors, should be independent of them. The parameters $s_0$ and $s'_0$, which are the continuum thresholds of $B_c$ and $P$ mesons, respectively, are determined from the conditions that guarantees the sum rules to have the best stability in the allowed $M_1^2$ and $M_2^2$ region. The values of continuum thresholds calculated from the two-point QCD sum rules are taken to be $s_0 = (45 - 50) \text{ GeV}^2$ and $s'_0 = (6 - 8) \text{ GeV}^2 [7, 24, 36]$. The working regions for $M_1^2$ and $M_2^2$ are determined by requiring that not only contributions of the higher states and continuum are effectively suppressed, but the gluon condensate contributions are small, which guarantees that the contributions of higher dimensional operators are small. Both conditions are satisfied in the regions $10 \text{ GeV}^2 \leq M_1^2 \leq 25 \text{ GeV}^2$ and $4 \text{ GeV}^2 \leq M_2^2 \leq 10 \text{ GeV}^2$.

The dependence of the form factors $f_+, f_-$ and $f_T$ on $M_1^2$ and $M_2^2$ for $B_c \to D_s l^+ l^- /\nu \bar{\nu}$ are shown in Figs. 3, 4 and 5, respectively. The figures 6, 7, and 8 also depict the dependence of the form factors on Borel mass parameters for for $B_c \to D l^+ l^- /\nu \bar{\nu}$. This figures show a good stability of the form factors with respect to the Borel mass parameters in the working regions. Our numerical analysis shows that the contribution of the non-perturbative part (the gluon condensate diagrams) is about $8^0/0$ of the total and the main contribution comes from the perturbative part of the form factors.

The values of the form factors at $q^2 = 0$ are shown in Table 1: The sum rules for the form factors are truncated at about $2 \text{ GeV}^2$ below the perturbative cut, so to extend our results to the full physical region, we look for parametrization of the form factors in such a way that in the region $0 \leq q^2 \leq 19.26 (18.41) \text{ GeV}^2$ for $D(D_s)$, this parametrization coincides with the sum rules prediction. Our numerical calculations shows that the sufficient parametrization of the form factors with respect to $q^2$ is as follows:

$$f_i(q^2) = \frac{f_i(0)}{1 + \alpha q^2 + \beta q^4} \quad (40)$$

| $B_c \to D$ | $B_c \to D_s$ |
|------------------|------------------|
| $f_+(l^+ l^- /\nu \bar{\nu})$ | 0.22 ± 0.045 | 0.16 ± 0.032 |
| $f_-(l^+ l^- /\nu \bar{\nu})$ | −0.29 ± 0.056 | −0.18 ± 0.038 |
| $f_T(l^+ l^-)$ | −0.27 ± 0.054 | −0.19 ± 0.040 |

Table 1: The values of the form factors at $q^2 = 0$.
where \( \hat{q} = q^2/m_{B_c}^2 \). The values of the parameters \( f_i(0) \), \( \alpha \) and \( \beta \) are given in the Tables 2, 3.

| \( f_+(l^+l^−/ν\bar{ν}) \) | \( f_−(l^+l^−/ν\bar{ν}) \) | \( f_T(l^+l^−) \) |
|-----|-----|-----|
| 0.22 | -0.29 | -0.27 |
| -1.10 | -0.63 | -0.72 |
| -2.48 | -4.06 | -3.24 |

Table 2: Parameters appearing in the form factors of the \( B_c \rightarrow Dl^+l^−/ν\bar{ν} \) decay for \( M_1^2 = 15 \text{ GeV}^2, M_2^2 = 8 \text{ GeV}^2 \).

| \( f_+(l^+l^−/ν\bar{ν}) \) | \( f_−(l^+l^−/ν\bar{ν}) \) | \( f_T(l^+l^−) \) |
|-----|-----|-----|
| 0.16 | -0.18 | -0.19 |
| -1.55 | -0.77 | -1.43 |
| -2.80 | -6.71 | -3.06 |

Table 3: Parameters appearing in the form factors of the \( B_c \rightarrow Dsl^+l^−/ν\bar{ν} \) decay for \( M_1^2 = 15 \text{ GeV}^2, M_2^2 = 8 \text{ GeV}^2 \).

The errors are estimated by the variation of the Borel parameters \( M_1^2 \) and \( M_2^2 \), the variation of the continuum thresholds \( s_0 \) and \( s_0' \), the variation of \( b \) and \( c \) quark masses and leptonic decay constants \( f_{B_c} \) and \( f_{D,(D_s)} \). The main uncertainty comes from the thresholds and the decay constants, which is about \( \sim 18\% \) of the central value, while the other uncertainties are small, constituting a few percent.

Now, we compare the extrapolation values for the form factors and their HQET values obtained from Eqs. (36-38) in Tables 4 and 5 for \( B_c \rightarrow Dl^+l^−/ν\bar{ν} \) and \( B_c \rightarrow Dsl^+l^−/ν\bar{ν} \), respectively.

| \( y \) | 1 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 | 1.8 |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( q^2 \) | 19.26 | 16.93 | 14.59 | 12.25 | 9.91 | 7.57 | 5.23 | 2.89 | 0.55 |
| \( f_+(q^2) \) | 2.19 | 1.36 | 0.88 | 0.56 | 0.36 | 0.29 | 0.27 | 0.24 | 0.23 |
| \( f_−(q^2) \) | -3.01 | -1.93 | -1.20 | -0.75 | -0.52 | -0.39 | -0.33 | -0.32 | -0.31 |
| \( f_T(q^2) \) | -2.52 | -1.53 | -1.12 | -0.70 | -0.49 | -0.37 | -0.31 | -0.29 | -0.28 |
| \( f_+^{HQET}(y) \) | ? | 1.35 | 0.50 | 0.29 | 0.20 | 0.15 | 0.12 | 0.10 | 0.08 |
| \( f_−^{HQET}(y) \) | ? | -1.90 | -0.75 | -0.44 | -0.30 | -0.22 | -0.18 | -0.15 | -0.12 |
| \( f_T^{HQET}(y) \) | ? | -1.51 | -0.58 | -0.33 | -0.23 | -0.17 | -0.14 | -0.11 | -0.10 |

Table 4: The comparison of the extrapolation values for the form factors and their HQET limit for \( B_c \rightarrow Dl^+l^− \) at \( M_1^2 = 15 \text{ GeV}^2, M_2^2 = 8 \text{ GeV}^2 \) and corresponding \( T_1 = 1.6 \text{ GeV}, T_2 = 3.2 \text{ GeV} \).

At the \( y = 1 \) called the zero recoil limit, the HQET limit of the form factors are not finite and at this value, we can determine only the ratio of the form factors. For other
values of $y$ and corresponding $q^2$, the behavior of the form factors and their HQET values are the same, i.e., when $y$ increases ($q^2$ decreases) both the form factors and their HQET values decrease. Moreover, at high $q^2$ values, the form factors and their HQET values are close to each other while at low $q^2$, the form factor values are about 2-3 times greater than that of their HQET limit.

At the end of this section we would like to present the values of the branching ratios. Integrating Eqs. (26) and (27) over $q^2$ in the whole physical region and using the total mean life time $\tau \simeq 0.46 \text{ ps}$ of $B_c$ meson [37], the branching ratio of the $B_c \to P(D, D_s)l^+l^-/\nu\bar{\nu}$ decays are obtained as Table 6. This Table also includes a comparison of our results with the prediction of the RCQM. This Table presents a good agreement between two models especially when the errors are taken into account. Any experimentally measurements on the branching fractions of these decays and those comparisons with the results of the phenomenological models like QCD sum rules could give valuable information about the nature of the $D_s$ meson and strong interactions inside it.

In summary, we investigated the $B_c \to P(D, D_s)l^+l^-/\nu\bar{\nu}$ channels and computed the relevant form factors and their HQET limits considering the gluon condensate corrections. We also evaluated the total decay width and the branching fractions of those decays and compared our results with the predictions of the RCQM. Detection of these channels and their comparison with the phenomenological models like QCD sum rules could give useful information about the structure of the $D_s$ meson.

| $y$ | 1 | 1.1 | 1.2 | 1.3 | 1.4 | 1.5 | 1.6 | 1.7 |
|-----|---|-----|-----|-----|-----|-----|-----|-----|
| $q^2$ | 18.41 | 15.94 | 13.48 | 11.02 | 8.55 | 6.09 | 3.63 | 1.16 |
| $f^+_\pm(q^2)$ | 2.17 | 1.12 | 0.79 | 0.53 | 0.31 | 0.22 | 0.18 | 0.17 |
| $f^-_\pm(q^2)$ | -2.50 | -1.53 | -0.79 | -0.43 | -0.29 | -0.24 | -0.23 | -0.22 |
| $f^+_\pm(y)$ | 2.25 | -1.23 | -0.70 | -0.37 | -0.27 | -0.23 | -0.21 | -0.20 |
| $f^-_\pm(y)$ | -1.08 | -0.41 | 0.24 | 0.16 | 0.12 | 0.10 | 0.08 |
| $f^+_{HQET}(y)$ | ? | -1.52 | -0.60 | -0.35 | -0.24 | -0.18 | -0.14 | -0.12 |
| $f^-_{HQET}(y)$ | ? | -1.22 | -0.46 | -0.27 | -0.18 | -0.14 | -0.11 | -0.10 |

Table 5: The comparison of the extrapolation values for the form factors and their HQET limit for $B_c \to D_s l^+l^-$ at $M^2_1 = 15 \text{ GeV}^2$, $M^2_2 = 8 \text{ GeV}^2$ and corresponding $T_1 = 1.6 \text{ GeV}, T_2 = 3.2 \text{ GeV}$.
| Decay                        | Our Results          | RCQM [6]           |
|------------------------------|----------------------|-------------------|
| $B_c \rightarrow D\nu\bar{\nu}$ | $(3.48 \pm 0.71) \times 10^{-8}$ | $3.28 \times 10^{-8}$ |
| $B_c \rightarrow D_s\nu\bar{\nu}$ | $(0.49 \pm 0.12) \times 10^{-6}$ | $0.7 \times 10^{-6}$ |
| $B_c \rightarrow D\ell^+\ell^-$ | $(1.34 \pm 0.25) \times 10^{-8}$ | -                 |
| $B_c \rightarrow D_s\ell^+\ell^-$ | $(1.47 \pm 0.32) \times 10^{-7}$ | -                 |
| $B_c \rightarrow D\mu^+\mu^-$  | $(0.31 \pm 0.06) \times 10^{-8}$ | $0.44 \times 10^{-5}$ |
| $B_c \rightarrow D_s\mu^+\mu^-$ | $(0.61 \pm 0.15) \times 10^{-7}$ | $0.97 \times 10^{-4}$ |
| $B_c \rightarrow D\tau^+\tau^-$ | $(0.13 \pm 0.03) \times 10^{-8}$ | $0.11 \times 10^{-5}$ |
| $B_c \rightarrow D_s\tau^+\tau^-$ | $(0.23 \pm 0.05) \times 10^{-7}$ | $0.22 \times 10^{-7}$ |

Table 6: Values for the branching fractions of the $B_c \rightarrow P(D, D_s)l^+l^-/\nu\bar{\nu}$ decays and their comparison with the predictions of the RCQM [6]

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**Appendix–A**

In this appendix, the explicit expressions of the coefficients of the gluon condensate entering the sum rules of the form factors $f_+, f_-$ and $f_T$ are given.

$$C_+ = -5 I_1(3, 2, 2)m_c^6 - 5 I_2(3, 2, 2)m_c^4 - 5 I_0(3, 2, 2)m_c^6 + 5 I_2(3, 2, 2)m_c^5 m_b + 5 I_1(3, 2, 2)m_c^5 m_b + 5 I_0(3, 2, 2)m_c^5 m_b + 5 I_2(3, 2, 2)m_c^4 m_b^2 - 5 I_1(3, 2, 2)m_c^2 m_b^3 - 5 I_0(3, 2, 2)m_c^3 m_b^3 - 5 I_2(3, 2, 2)m_c m_b^4 + 15 I_1^{[0,1]}(3, 2, 2)m_c^4 - 15 I_1(2, 2, 2)m_c^4 + 15 I_2^{[0,1]}(3, 2, 2)m_c^4 - 5 I_2(3, 1, 2)m_c^4 - 5 I_1(3, 1, 2)m_c^4 - 15 I_2(4, 1, 1)m_c^4 - 5 I_1(3, 2, 1)m_c^4 - 15 I_0(2, 2, 2)m_c^4 + 15 I_0^{[0,1]}(3, 2, 2)m_c^4 - 15 I_1(4, 1, 1)m_c^4 - 15 I_2(2, 2, 2)m_c^4 + 5 I_1(3, 1, 2)m_c^3 m_b - 10 I_0^{[0,1]}(3, 2, 2)m_c^3 m_b + 10 I_1(2, 2, 2)m_c^3 m_b + 10 I_0(3, 2, 1)m_c^3 m_b^2 - 10 I_2^{[0,1]}(3, 2, 2)m_c^3 m_b^2 + 5 I_0(3, 1, 2)m_c^3 m_b^2 - 10 I_2(2, 3, 1)m_c^3 m_b^2 + 15 I_0(4, 1, 1)m_c^3 m_b^2 + 10 I_1(2, 2, 2)m_c^3 m_b^2 - 10 I_0(2, 3, 1)m_c^3 m_b^2 + 10 I_2^{[0,1]}(3, 2, 2)m_c^3 m_b^2 + 5 I_2(3, 1, 2)m_c^3 m_b^2 + 10 I_0(2, 2, 2)m_c^3 m_b + 5 I_2(3, 2, 1)m_c^3 m_b$$
\[+5I_0^{[0,1]}(3, 2, 2)m_c^2m_b^2 + 30I_0(1, 4, 1)m_c^2m_b^2 - 5I_0(2, 2, 2)m_c^2m_b^2 - 10I_1(3, 2, 1)m_c^2m_b^2 + 30I_2(1, 4, 1)m_c^2m_b^2 - 10I_2(3, 2, 1)m_c^2m_b^2 + 30I_1(1, 4, 1)m_c^2m_b^2 - 5I_0^{[0,1]}(3, 2, 2)m_c^2m_b^2 - 30I_0(1, 4, 1)m_c^2m_b^2 + 10I_2(2, 3, 1)m_c^2m_b^3 + 10I_1(2, 3, 1)m_c^2m_b^3 - 5I_1^{[0,1]}(3, 2, 2)m_c^2m_b^3 - 30I_1(1, 4, 1)m_c^2m_b^3 + 10I_0(3, 2, 1)m_c^2m_b^4 + 15I_1^{[0,1]}(3, 2, 1)m_c^2m_b^2 - 5I_0(3, 1, 1)m_c^2m_b^2 - 5I_0(2, 2, 1)m_c^2m_b^2 - 5I_0(2, 1, 2)m_c^2m_b^2 + 30I_1^{[0,1]}(2, 2, 2)m_c^2m_b^2 + 15I_0^{[0,1]}(4, 1, 1)m_c^2m_b^2 + 15I_2^{[0,1]}(3, 2, 1)m_c^2m_b^2 + 20I_0^{[0,1]}(3, 2, 1)m_c^2m_b^2 - 15I_2^{[0,2]}(3, 2, 2)m_c^2m_b^2 + 20I_0^{[0,1]}(3, 1, 2)m_c^2m_b^2 + 15I_2^{[0,1]}(4, 1, 1)m_c^2m_b^2 - 15I_1^{[0,2]}(3, 2, 2)m_c^2m_b^2 + 15I_2^{[0,1]}(3, 1, 2)m_c^2m_b^2 - 10I_1(1, 2, 2)m_c^2m_b - 10I_0(1, 2, 2)m_c^2m_b + 30I_0^{[0,1]}(2, 2, 2)m_c^2m_b + 15I_1^{[0,1]}(3, 1, 2)m_c^2m_b - 15I_0^{[0,2]}(3, 2, 2)m_c^2m_b - 25I_0(2, 2, 1)m_c^2m_b - 5I_2(2, 1, 2)m_c^2m_b - 10I_0^{[0,1]}(3, 2, 1)m_c^2m_b - 20I_1(2, 2, 1)m_c^2m_b - 40I_0(1, 3, 1)m_c^2m_b - 10I_0^{[0,1]}(2, 2, 2)m_c^2m_b - 10I_1(1, 2, 2)m_c^2m_b - 40I_1(1, 3, 1)m_c^2m_b - 5I_1(2, 1, 2)m_c^2m_b - 5I_2^{[0,1]}(3, 2, 1)m_c^2m_b - 10I_0^{[0,1]}(2, 2, 2)m_c^2m_b - 15I_1^{[0,1]}(3, 1, 2)m_c^2m_b - 5I_0^{[0,1]}(2, 2, 1)m_c^2m_b - 5I_1^2(3, 2, 2)m_c^2m_b + 5I_0^{[0,1]}(3, 2, 1)m_c^2m_b - 5I_2^{[0,2]}(3, 2, 2)m_c^2m_b - 5I_0(1, 2, 2)m_c^2m_b - 15I_0^{[0,1]}(3, 1, 2)m_c^2m_b - 10I_0(2, 2, 1)m_c^2m_b + 5I_1^{[0,2]}(3, 2, 2)m_c^2m_b - 15I_0^{[0,1]}(3, 1, 2)m_c^2m_b - 10I_0^{[0,1]}(2, 2, 2)m_c^2m_b - 5I_0^{[0,1]}(3, 1, 2)m_c^2m_b - 10I_0(1, 3, 1)m_c^2m_b - 5I_0^{[0,1]}(2, 2, 2)m_c^2m_b - 5I_0(1, 2, 2)m_c^2m_b - 5I_0(3, 1, 1)m_c^2m_b - 5I_0^{[0,1]}(1, 4, 1)m_c^2m_b - 5I_0(1, 2, 2)m_c^2m_b - 5I_1^{[0,1]}(3, 2, 2)m_c^2m_b - 30I_0^{[0,1]}(1, 4, 1)m_c^2m_b - 5I_0^{[0,1]}(2, 2, 2)m_c^2m_b - 30I_0^{[0,1]}(1, 4, 1)m_c^2m_b - 5I_0(2, 1, 2)m_c^2m_b + 15I_0^{[0,1]}(2, 2, 2)m_c^2m_b + 5I_1(2, 2, 1)m_c^2m_b - 10I_1^{[0,1]}(3, 1, 2)m_c^2m_b - 10I_0(2, 1, 2)m_c^2m_b + 10I_0^{[0,1]}(2, 2, 2)m_c^2m_b - 5I_1(2, 2, 1)m_c^2m_b - 10I_2^{[0,2]}(3, 2, 2)m_c^2m_b - 5I_0^{[0,1]}(3, 2, 2)m_c^2m_b - 5I_0^{[0,1]}(2, 2, 2)m_c^2m_b - 30I_0^{[0,1]}(1, 4, 1)m_c^2m_b - 5I_1^{[0,1]}(3, 2, 1)m_c^2m_b + 10I_2^{[0,1]}(2, 2, 2)m_c^2m_b - 30I_1^{[0,1]}(1, 4, 1)m_c^2m_b - 5I_0(2, 1, 2)m_c^2m_b + 15I_0^{[0,1]}(2, 2, 2)m_c^2m_b + 5I_1(2, 2, 1)m_c^2m_b - 10I_2^{[0,2]}(3, 2, 2)m_c^2m_b - 15I_0^{[0,2]}(2, 2, 2) - 10I_0^{[0,2]}(3, 2, 2) + 5I_0(1, 1, 2) + 5I_1^{[0,1]}(2, 2, 1) + 5I_2^{[0,1]}(2, 2, 1) - 5I_1^{[0,1]}(2, 2, 2) - 5I_0^{[0,1]}(2, 2, 2) - 15I_2^{[0,2]}(2, 2, 2) + 10I_1(1, 2, 1) + 10I_1(2, 1, 1) + 10I_2(2, 1, 1) + 10I_0^{[0,1]}(3, 1, 1) - 10I_2^{[0,2]}(3, 1, 2) + 10I_0^{[0,1]}(1, 2, 2)\]
\[ C_\omega = 5 I_2(3,2,2)m_c^6 - 5 I_1(3,2,2)m_c^6 - 5 I_0(3,2,2)m_c^5m_b \\
+ 5 I_1(3,2,2)m_c^6m_b - 5 I_2(3,2,2)m_c^5m_b + 5 I_1(3,2,2)m_c^4m_b^2 \\
- 5 I_1(3,2,2)m_c^4m_b^2 - 5 I_1(3,2,2)m_c^3m_b^3 + 5 I_0(3,2,2)m_c^3m_b^3 \\
+ 5 I_2(3,2,2)m_c^3m_b^3 - 15 I_2^{[0,1]}(3,2,2)m_c^4 - 5 I_1(3,1,2)m_c^4 \\
+ 15 I_2(2,2,2)m_c^4 - 15 I_1(4,1,1)m_c^4 + 5 I_2(3,2,1)m_c^4 \\
+ 5 I_2(3,1,2)m_c^4 + 15 I_1^{[0,1]}(3,2,2)m_c^4 - 5 I_1(3,2,1)m_c^4 \\
- 15 I_1(2,2,2)m_c^4 + 15 I_2(4,1,1)m_c^4 + 5 I_1(3,1,2)m_c^3m_b \\
+ 10 I_2^{[0,1]}(3,2,2)m_c^3m_b + 10 I_0^{[0,1]}(3,2,2)m_c^3m_b - 5 I_2(3,1,2)m_c^3m_b \\
- 10 I_1(2,3,1)m_c^3m_b - 15 I_0(4,1,1)m_c^3m_b + 10 I_2(2,3,1)m_c^3m_b \\
+ 15 I_1(4,1,1)m_c^3m_b - 15 I_2(4,1,1)m_c^3m_b - 10 I_0(2,3,1)m_c^3m_b \\
+ 5 I_0(3,2,1)m_c^3m_b - 5 I_0(3,1,2)m_c^3m_b - 10 I_2(2,2,2)m_c^3m_b \\
- 10 I_0(2,2,2)m_c^3m_b - 10 I_1^{[0,1]}(3,2,2)m_c^3m_b + 5 I_1(3,2,1)m_c^3m_b \\
- 5 I_2(3,2,1)m_c^3m_b + 10 I_1(2,2,2)m_c^3m_b - 5 I_0^{[0,1]}(3,2,2)m_c^2m_b^2 \\
+ 5 I_0(2,2,2)m_c^2m_b^2 + 30 I_1(1,4,1)m_c^2m_b^2 + 10 I_2(3,2,1)m_c^2m_b^2 \\
- 10 I_1(3,2,1)m_c^2m_b^2 - 30 I_2(1,4,1)m_c^2m_b^2 + 5 I_0^{[0,1]}(3,2,2)m_c^3m_b^3 \\
+ 30 I_0(1,4,1)m_c^3m_b^3 - 30 I_1(1,4,1)m_c^3m_b^3 + 10 I_1(2,3,1)m_c^3m_b^3 \\
+ 10 I_1(3,2,1)m_c^3m_b^3 - 5 I_1^{[0,1]}(3,2,2)m_c^3m_b^3 + 30 I_2(1,4,1)m_c^3m_b^3 \\
- 10 I_2(2,3,1)m_c^3m_b^3 - 10 I_2(3,2,1)m_c^3m_b^3 + 5 I_2^{[0,1]}(3,2,2)m_c^3m_b^3 \\
+ 5 I_0(3,2,1)m_b^4 - 15 I_0(1,4,1)m_b^4 + 15 I_1^{[0,1]}(4,1,1)m_b^4 \\
- 10 I_0(2,2,1)m_b^4 + 10 I_0^{[0,1]}(3,2,2)m_b^4 + 10 I_0(2,1,2)m_b^4 \\
+ 30 I_1^{[0,1]}(2,2,2)m_b^4 + 10 I_2(1,2,2)m_b^4 - 10 I_1(1,2,2)m_b^4 \\
- 30 I_2^{[0,1]}(2,2,2)m_b^4 - 15 I_2^{[0,1]}(3,1,2)m_c^2m_b^2 + 15 I_2^{[0,1]}(4,1,1)m_c^2m_b^2 \\
+ 15 I_1^{[0,1]}(3,1,2)m_c^2m_b^2 - 15 I_2^{[0,1]}(3,2,1)m_c^2m_b^2 + 15 I_1^{[0,1]}(3,2,1)m_c^2m_b^2 \\
- 10 I_0^{[0,1]}(3,1,2)m_c^2m_b^2 - 15 I_1^{[0,1]}(3,2,2)m_c^2m_b^2 + 15 I_2^{[0,1]}(3,2,2)m_c^2m_b^2 \\
+ 40 I_2(1,3,1)m_c^4m_b - 5 I_0^{[0,2]}(3,2,2)m_c^2m_b - 5 I_1^{[0,1]}(3,2,1)m_c^2m_b \\
+ 15 I_0^{[0,1]}(3,1,2)m_c^2m_b - 10 I_1(1,2,2)m_c^2m_b + 15 I_2^{[0,1]}(3,1,2)m_c^2m_b \\
- 15 I_1^{[0,1]}(3,1,2)m_c^2m_b - 10 I_1^{[0,1]}(2,2,2)m_c^2m_b + 5 I_2^{[0,1]}(3,2,1)m_c^2m_b \\
+ 20 I_0(1,3,1)m_c^2m_b + 5 I_1^{[0,2]}(3,2,2)m_c^2m_b + 5 I_2(2,1,2)m_c^2m_b \\
+ 20 I_0(2,2,1)m_c^4m_b - 20 I_1(2,2,1)m_c^4m_b + 10 I_2(1,2,2)m_c^4m_b \\
+ 10 I_0^{[0,1]}(2,2,2)m_c^2m_b - 10 I_2^{[0,1]}(2,3,1)m_c^2m_b + 15 I_0^{[0,1]}(3,2,1)m_c^2m_b \\
- 5 I_1(2,1,2)m_c^2m_b + 10 I_2^{[0,1]}(2,2,2)m_c^2m_b + 10 I_1^{[0,1]}(2,3,1)m_c^2m_b \\
+ 10 I_0(1,2,2)m_c^2m_b + 5 I_0(2,1,2)m_c^2m_b - 5 I_2^{[0,2]}(3,2,2)m_c^2m_b \\
+ 20 I_2(2,2,1)m_c^4m_b - 40 I_1(1,3,1)m_c^4m_b + 10 I_0^{[0,1]}(2,3,1)m_c^4m_b \\
- 15 I_0(1,3,1)m_b^2 - 30 I_1^{[0,1]}(1,4,1)m_b^2 + 5 I_0(2,1,2)m_b^2
where

\[ \hat{I}_{[a, b, c]}^i \left( M_1^2 \right) \left( M_2^2 \right)^j \left( \frac{d^i}{d(M_1^2)} \frac{d^j}{d(M_2^2)} \right) \left[ (M_1^2)^i (M_2^2)^j \hat{I}_n(a, b, c) \right] . \]

**Appendix–B**

In this appendix, the explicit expressions of the coefficients of the gluon condensate entering the HQET limit of the form factors $\hat{f}^{\text{HQET}}_+, \hat{f}^{\text{HQET}}_-$, and $\hat{f}^{\text{HQET}}_\perp$ are given. Note that only in this appendix, by $\hat{I}_n(a, b, c)$ we mean $\hat{I}_n(a, b, c)^{\text{HQET}}$ which are defined in Eq. (32).

\[
C^{\text{HQET}}_+ = 2 \left( \frac{\hat{I}_{[0, 1]}^{[0, 1]}(3, 2, 2) m_b^5}{\sqrt{z}} \right) + \left( \frac{\hat{I}_{[0, 1]}^{[0, 1]}(3, 2, 2) m_b^4}{\sqrt{z}} \right) - 32 \left( \frac{\hat{I}_2(2, 1, 1) m_b^4}{\sqrt{z}} \right)
\]
\begin{equation}
-4 \frac{i_0(2,2,1) m_b^4}{\sqrt{z}} - 8 \frac{i_0(2, 1, 1) m_b^3}{\sqrt{z}} + 16 \frac{i_2(3, 1, 2) m_b^3}{\sqrt{z}} \\
-8 \frac{i_{2,1}^{[0,1]}(3, 1, 1) m_b^2}{\sqrt{z}} + 4 \frac{i_0(2, 1, 2) m_b^4}{\sqrt{z}} + 12 \frac{i_2(3, 1, 2) m_b^4}{\sqrt{z}} \\
-16 \frac{i_0(1, 1, 2) m_b^4}{\sqrt{z}} - 12 \frac{i_{0,1}^{[1]}(2, 2, 2) m_b^4}{\sqrt{z}} - 8 \frac{i_2(3, 1, 2) m_b^4}{\sqrt{z}} \\
+8 \frac{i_2(2, 1, 2) m_b^5}{\sqrt{z}} + 8 \frac{i_0(2, 1, 2) m_b^4}{\sqrt{z}} - 16 \frac{i_2(2, 1, 2) m_b^4}{\sqrt{z}} \\
+6 \frac{i_0(1, 1, 2) m_b^3}{\sqrt{z}} - 16 \frac{i_0(1, 2, 1) m_b^3}{\sqrt{z}} - 64 \frac{i_1(1, 1, 2) m_b^5}{\sqrt{z}} \\
+4 \frac{i_2(3, 2, 2) m_b^4}{\sqrt{z}} - 4 \frac{i_2(3, 2, 1) m_b^5}{\sqrt{z}} - 8 \frac{i_2(2, 2, 1) m_b^5}{\sqrt{z}} \\
-16 \frac{i_{0,1}^{[1]}(2, 2, 2) m_b^5}{\sqrt{z}} + 16 \frac{i_{0,1}^{[2]}(3, 1, 2) m_b^3}{\sqrt{z}} - 2 \frac{i_0(2, 2, 2) m_b^3}{\sqrt{z}} \\
+24 \frac{i_{0,1}^{[2]}(2, 2, 2) m_b^3}{\sqrt{z}} - 6 \frac{i_2(4, 1, 1) m_b^3}{\sqrt{z}} + 4 \frac{i_0(3, 2, 1) m_b^3}{\sqrt{z}} \\
-64 \frac{i_2(2, 2, 1) m_b^5}{\sqrt{z}} - 16 \frac{i_0(1, 2, 1) m_b^3}{\sqrt{z}} - 64 \frac{i_1(1, 1, 2) m_b^5}{\sqrt{z}} \\
+2 \frac{i_0(3, 1, 1) m_b^3}{\sqrt{z}} - 8 \frac{i_1(2, 2, 1) m_b^5}{\sqrt{z}} - 3/2 \frac{i_0(4, 1, 1) m_b^3}{\sqrt{z}} \\
+16 \frac{i_2(3, 2, 2) m_b^4}{\sqrt{z}} + 2 \frac{i_0(2, 2, 2) m_b^5}{\sqrt{z}} + 1/2 \frac{i_0(3, 2, 2) m_b^5}{\sqrt{z}} \\
-24 \frac{i_0(1, 3, 1) m_b^5}{\sqrt{z}} + \frac{i_2(3, 2, 2) m_b^6}{\sqrt{z}} + 16 \frac{i_0(1, 2, 2) m_b^5}{\sqrt{z}} \\
-8 \frac{i_0(3, 2, 1) m_b^3}{\sqrt{z}} - 3 \frac{i_{0,1}^{[1]}(4, 1, 1) m_b^2}{\sqrt{z}} + 2 \frac{i_1(3, 2, 2) m_b^5}{\sqrt{z}} \\
-4 \frac{i_1(3, 2, 1) m_b^5}{\sqrt{z}} - 2 \frac{i_2(3, 1, 2) m_b^5}{\sqrt{z}} - 16 \frac{i_2(2, 2, 1) m_b^4}{\sqrt{z}} \\
+8 \frac{i_1(2, 1, 2) m_b^5}{\sqrt{z}} + 4 \frac{i_2(3, 2, 1) m_b^5}{\sqrt{z}} - 4 \frac{i_{0,1}^{[2]}(3, 2, 2) m_b^4}{\sqrt{z}} \\
+12 \frac{i_{0,1}^{[1]}(3, 1, 2) m_b^2}{\sqrt{z}} - 32 \frac{i_1(2, 1, 1) m_b^4}{\sqrt{z}} + 16 \frac{i_{0,1}^{[2]}(2, 2, 2) m_b^5}{\sqrt{z}} \\
-12 \frac{i_{0,1}^{[2]}(3, 1, 2) m_b^4}{\sqrt{z}} + 32 \frac{i_2(2, 2, 1) m_b^5}{\sqrt{z}} - \frac{i_0(3, 1, 2) m_b^4}{\sqrt{z}} \\
+20 \frac{i_0(2, 2, 1) m_b^4}{\sqrt{z}} - 8 \frac{i_1(3, 2, 1) m_b^4}{\sqrt{z}} + 8 \frac{i_{0,1}^{[1]}(2, 2, 2) m_b^4}{\sqrt{z}} \\
-3 \frac{i_2(4, 1, 1) m_b^4}{\sqrt{z}} - \frac{i_{0,1}^{[1]}(3, 2, 2) m_b^4}{\sqrt{z}} + 48 \frac{i_{0,1}^{[2]}(2, 2, 2) m_b^4}{\sqrt{z}}
\end{equation}
\[-8 \frac{z^3}{3} \hat{I}_2(2,3,1) m_b^6 - 16 \frac{z^3}{3} \hat{I}_0(1,2,1) m_b^4 - 16 \frac{z^3}{3} \hat{I}_1^{[0,1]}(2,2,2) m_b^5 \]

\[+4 \frac{z^2}{2} \hat{I}_0(2,1,2) m_b^4 - 64 \frac{z^3}{3} \hat{I}_2(1,2,1) m_b^4 - 32 \frac{z^3}{3} \hat{I}_0^{[0,1]}(1,2,2) m_b^4 \]

\[+12 \frac{z^3}{3} \hat{I}_0(1,4,1) m_b^6 + 4 \frac{z^3}{3} \hat{I}_2(1,2,2) m_b^6 + 3 \frac{z^3}{3} \hat{I}_2(4,1,1) m_b^4 \]

\[+16 \frac{z^{3/2}}{2} \hat{I}_1^{[0,1]}(2,2,2) m_b^5 - 3 \frac{z^{3/2}}{2} \hat{I}_1(4,1,1) m_b^4 + 4 \frac{z^{3/2}}{2} \hat{I}_1(3,2,1) m_b^5 \]

\[-12 \frac{z^{3/2}}{2} \hat{I}_1^{[0,1]}(3,1,2) m_b^4 - 16 \frac{z^{3/2}}{2} \hat{I}_1^{[0,1]}(2,2,1) m_b^4 - 4 \frac{z^{3/2}}{2} \hat{I}_0(2,2,2) m_b^5 \]

\[+6 \frac{z^{3/2}}{2} \hat{I}_1^{[0,1]}(3,1,2) m_b^4 + 48 \frac{z^{3/2}}{2} \hat{I}_0^{[0,1]}(1,4,1) m_b^5 + 16 \frac{z^{3/2}}{2} \hat{I}_1^{[0,2]}(3,2,1) m_b^3 \]

\[-64 \frac{z^{3/2}}{2} \hat{I}_1(1,2,1) m_b^5 - 8 \frac{z^{3/2}}{2} \hat{I}_0^{[0,1]}(2,3,1) m_b^4 - 4 \frac{z^{3/2}}{2} \hat{I}_1^{[0,2]}(3,2,2) m_b^4 \]

\[+4 \frac{z^{3/2}}{2} \hat{I}_0^{[0,1]}(3,2,1) m_b^4 + 3/2 \frac{z^{3/2}}{2} \hat{I}_0(4,1,1) m_b^3 + 2 \frac{z^{3/2}}{2} \hat{I}_2(3,1,2) m_b^5 \]

\[+12 \frac{z^{3/2}}{2} \hat{I}_1^{[0,1]}(2,2,2) m_b^4 - 4 \frac{z^{3/2}}{2} \hat{I}_1^{[0,1]}(2,2,2) m_b^5 - 64 \frac{z^{3/2}}{2} \hat{I}_1^{[0,1]}(1,2,2) m_b^5 \]

\[-64 \frac{z^{3/2}}{2} \hat{I}_1^{[0,1]}(1,2,2) m_b^5 - 2 \frac{z^{3/2}}{2} \hat{I}_0(3,2,1) m_b^4 + 4 \frac{z^{3/2}}{2} \hat{I}_0(2,2,1) m_b^4 \]

\[-2 \frac{z^{3/2}}{2} \hat{I}_2(3,2,1) m_b^5 + 16 \frac{z^{3/2}}{2} \hat{I}_0(1,2,2) m_b^5 + 32 \frac{z^{3/2}}{2} \hat{I}_1(1,2,2) m_b^6 \]

\[-8 \frac{z^{3/2}}{2} \hat{I}_1(2,3,1) m_b^6 + 32 \frac{z^{3/2}}{2} \hat{I}_2(1,2,2) m_b^6 - 6 \frac{z^{3/2}}{2} \hat{I}_1(4,1,1) m_b^3 \]

\[+128 \frac{z^{3/2}}{2} \hat{I}_2(1,3,1) m_b^6 + 32 \frac{z^{3/2}}{2} \hat{I}_1(1,2,2) m_b^5 + 4 \frac{z^{3/2}}{2} \hat{I}_0^{[0,1]}(3,2,2) m_b^5 \]

\[-24 \frac{z^{3/2}}{2} \hat{I}_0^{[0,1]}(2,2,2) m_b^4 - 8 \frac{z^{3/2}}{2} \hat{I}_0^{[0,1]}(3,2,1) m_b^3 + 64 \frac{z^{3/2}}{2} \hat{I}_0(1,3,1) m_b^5 \]

\[-2 \frac{z^{3/2}}{2} \hat{I}_1(3,1,2) m_b^5 - 16 \frac{z^{3/2}}{2} \hat{I}_2(2,2,2) m_b^6 - 8 \frac{z^{3/2}}{2} \hat{I}_2(2,2,2) m_b^6 \]

\[+24 \frac{z^{3/2}}{2} \hat{I}_0(1,4,1) m_b^6 + 48 \frac{z^{3/2}}{2} \hat{I}_1(3,2,2) m_b^6 + 48 \frac{z^{3/2}}{2} \hat{I}_1(1,4,1) m_b^7 \]

\[-24 \frac{z^{3/2}}{2} \hat{I}_2(3,2,2) m_b^6 + 48 \frac{z^{3/2}}{2} \hat{I}_2(1,4,1) m_b^7 + 96 \frac{z^{3/2}}{2} \hat{I}_2(1,4,1) m_b^6 \]

\[+2 \frac{z^{3/2}}{2} \hat{I}_0^{[0,1]}(3,2,2) m_b^4 - 48 \frac{z^{3/2}}{2} \hat{I}_0^{[0,1]}(3,2,2) m_b^5 + 16 \frac{z^{3/2}}{2} \hat{I}_0(2,2,2) m_b^5 \]

\[+2 \frac{z^{3/2}}{2} \hat{I}_1(3,2,1) m_b^5 - 48 \frac{z^{3/2}}{2} \hat{I}_1^{[0,1]}(2,2,2) m_b^5 - 12 \frac{z^{3/2}}{2} \hat{I}_1^{[0,1]}(3,2,1) m_b^4 \]
\[\begin{align*}
+2 \frac{\hat{I}_1(3,1,2) m_b^5}{z^2} - \frac{\hat{I}_2(3,2,2) m_b^6}{z^2} + 4 \frac{\hat{I}_0(2,3,1) m_b^5}{z^2} \\
+2 \frac{\hat{I}_2(3,2,1) m_b^5}{z^2} - \frac{1}{2} \frac{\hat{I}_0(3,2,2) m_b^5}{z^2} + 48 \frac{\hat{I}_1(1,4,1) m_b^7}{z^2} \\
-3 \frac{\hat{I}_0^{[0,1]}(3,2,2) m_b^4}{z^2} + 3 \frac{\hat{I}_1(4,1,1) m_b^4}{z^2} - 48 \frac{\hat{I}_2(1,4,1) m_b^7}{z^2} \\
+12 \frac{\hat{I}_1^{[0,2]}(3,2,2) m_b^4}{z^2} - 6 \frac{\hat{I}_2^{[0,1]}(3,2,2) m_b^5}{z^2} + 12 \frac{\hat{I}_2(2,2,2) m_b^6}{z^2} \\
-24 \frac{\hat{I}_0(1,4,1) m_b^6}{z^2} - 8 \frac{\hat{I}_1(2,2,2) m_b^6}{z^2} + 8 \frac{\hat{I}_2(2,3,1) m_b^6}{z^2} \\
+96 \frac{\hat{I}_0^{[0,1]}(1,4,1) m_b^6}{z^2} + 4 \frac{\hat{I}_1^{[0,1]}(3,2,2) m_b^5}{z^2} - 16 \frac{\hat{I}_1^{[0,1]}(2,3,1) m_b^5}{z^2} \\
- \frac{\hat{I}_1(3,2,2) m_b^6}{z^2} + 128 \frac{\hat{I}_1(1,3,1) m_b^6}{z^2} + 32 \frac{\hat{I}_1(1,2,2) m_b^6}{z^2} \\
+2 \frac{\hat{I}_0(3,2,1) m_b^5}{z^{5/2}} - 48 \frac{\hat{I}_1(1,4,1) m_b^7}{z^{5/2}} + 12 \frac{\hat{I}_1(2,2,2) m_b^6}{z^{5/2}} \\
+1/2 \frac{\hat{I}_0(3,2,2) m_b^5}{z^{5/2}} - 6 \frac{\hat{I}_1^{[0,1]}(3,2,2) m_b^5}{z^{5/2}} + \frac{\hat{I}_2(3,2,2) m_b^6}{z^{5/2}} \\
- \frac{\hat{I}_1(3,2,2) m_b^6}{z^{5/2}} + 8 \frac{\hat{I}_1(2,3,1) m_b^6}{z^{5/2}} + \frac{\hat{I}_1(3,2,2) m_b^6}{z^3} \\
+4 \frac{\hat{I}_0(2,1,2) m_b^6}{z^2} + 2 \frac{\hat{I}_0(3,1,1) m_b^3}{z} + \frac{\hat{I}_0(3,2,1) m_b^4}{z}
\end{align*}\]

\[C_{\text{-HQET}} = -8 \frac{\hat{I}_2(2,1,2) m_b^5}{\sqrt{z}} + 32 \frac{\hat{I}_2(2,1,1) m_b^4}{\sqrt{z}} - 4 \frac{\hat{I}_0(2,1,2) m_b^6}{\sqrt{z}}
\]

\[+4 \frac{\hat{I}_0^{[0,1]}(2,2,2) m_b^4}{\sqrt{z}} - 12 \frac{\hat{I}_2^{[0,1]}(3,1,2) m_b^4}{\sqrt{z}} + 64 \frac{\hat{I}_2(1,1,2) m_b^5}{\sqrt{z}}
\]

\[-16 \frac{\hat{I}_2^{[0,2]}(3,1,2) m_b^3}{\sqrt{z}} - 6 \frac{\hat{I}_0^{[0,1]}(3,1,2) m_b^3}{\sqrt{z}} + 8 \frac{\hat{I}_0^{[0,1]}(2,1,2) m_b^3}{\sqrt{z}}
\]

\[+8 \frac{\hat{I}_2^{[0,1]}(3,2,1) m_b^4}{\sqrt{z}} + 16 \frac{\hat{I}_2^{[0,1]}(2,2,2) m_b^5}{\sqrt{z}} + 8 \frac{\hat{I}_0(2,2,1) m_b^4}{\sqrt{z}}
\]

\[+16 \frac{\hat{I}_2^{[0,1]}(2,1,2) m_b^4}{\sqrt{z}} - 2 \frac{\hat{I}_0^{[0,1]}(3,2,2) m_b^5}{\sqrt{z}} + 4 \frac{\hat{I}_2(3,2,1) m_b^5}{\sqrt{z}}
\]

\[+4 \frac{\hat{I}_2^{[0,2]}(3,2,2) m_b^4}{\sqrt{z}} + 8 \frac{\hat{I}_2(2,2,1) m_b^5}{\sqrt{z}} - \frac{\hat{I}_0^{[0,1]}(3,2,2) m_b^4}{\sqrt{z}}
\]

\[+4 \frac{\hat{I}_2^{[0,1]}(3,2,1) m_b^4}{z} + 4 \frac{\hat{I}_2^{[0,2]}(3,2,2) m_b^4}{z} - \frac{\hat{I}_0(1,2,2) m_b^5}{z}
\]

\[+16 \frac{\hat{I}_1^{[0,1]}(2,1,2) m_b^4}{z} - \frac{\hat{I}_2(3,2,2) m_b^6}{z} + 16 \frac{\hat{I}_1^{[0,2]}(3,1,2) m_b^3}{z}
\]

\[-8 \frac{\hat{I}_1^{[0,1]}(3,2,1) m_b^4}{z} + 3 \frac{\hat{I}_2(4,1,1) m_b^4}{z} - 32 \frac{\hat{I}_2(2,2,1) m_b^5}{z}
\]
\[ +12 \frac{\hat{I}_0^{[0,1]} (3, 1, 2) m_b^4}{z} + 16 \frac{\hat{I}_2^{[0,1]} (2, 2, 1) m_b^4}{z} - 4 \frac{\hat{I}_1 (3, 2, 1) m_b^5}{z} \]

\[-2 \frac{\hat{I}_0 (2, 2, 2) m_b^5}{z} + \frac{\hat{I}_0 (3, 1, 2) m_b^4}{z} - 16 \frac{\hat{I}_1^{[0,1]} (2, 2, 2) m_b^5}{z} \]

\[-4 \frac{\hat{I}_2 (3, 2, 1) m_b^5}{z} + 4 \frac{\hat{I}_0^{[0,1]} (3, 1, 2) m_b^3}{z} + 2 \frac{\hat{I}_0^{[0,2]} (3, 2, 2) m_b^3}{z} \]

\[-64 \frac{\hat{I}_1 (1, 1, 2) m_b^5}{z} - 16 \frac{\hat{I}_2^{[0,1]} (2, 2, 2) m_b^5}{z} - 6 \frac{\hat{I}_0^{[0,1]} (2, 2, 2) m_b^3}{z} \]

\[+4 \frac{\hat{I}_1^{[0,2]} (3, 2, 2) m_b^4}{z} - 8 \frac{\hat{I}_0^{[0,1]} (2, 2, 2) m_b^3}{z} - 16 \frac{\hat{I}_2^{[0,2]} (3, 2, 1) m_b^3}{z} \]

\[+12 \frac{\hat{I}_0 (1, 4, 1) m_b^6}{z} - 8 \frac{\hat{I}_0 (2, 1, 2) m_b^4}{z} + \frac{\hat{I}_0^{[0,1]} (3, 2, 2) m_b^4}{z} \]

\[+\frac{3}{2} \frac{\hat{I}_0 (4, 1, 1) m_b^3}{z} + 8 \frac{\hat{I}_1 (2, 1, 2) m_b^5}{z} + 8 \frac{\hat{I}_2 (2, 3, 1) m_b^6}{z} \]

\[-\frac{1}{2} \frac{\hat{I}_0 (3, 2, 2) m_b^5}{z} - 48 \frac{\hat{I}_2^{[0,2]} (2, 2, 2) m_b^4}{z} - 8 \frac{\hat{I}_0^{[0,1]} (2, 2, 2) m_b^4}{z} \]

\[+\frac{4}{2} \frac{\hat{I}_2 (3, 1, 2) m_b^5}{z} + 64 \frac{\hat{I}_2^{[0,1]} (1, 2, 2) m_b^5}{z} - 32 \frac{\hat{I}_1 (2, 1, 1) m_b^4}{z} \]

\[+\frac{2}{2} \frac{\hat{I}_2^{[0,1]} (3, 2, 2) m_b^5}{z} + 24 \frac{\hat{I}_0 (1, 3, 1) m_b^5}{z} - 8 \frac{\hat{I}_1 (2, 2, 1) m_b^5}{z} \]

\[+\frac{12}{2} \frac{(C_{0,1}) (3, 1, 2) m_b^4}{z} + 64 \frac{\hat{I}_2 (1, 2, 1) m_b^5}{z} + 48 \frac{\hat{I}_2^{[0,1]} (2, 2, 2) m_b^5}{z^{3/2}} \]

\[+\frac{32}{2} \frac{\hat{I}_1 (2, 2, 1) m_b^5}{z^{3/2}} - 8 \frac{\hat{I}_1 (2, 3, 1) m_b^6}{z^{3/2}} + 16 \frac{\hat{I}_1^{[0,2]} (3, 2, 1) m_b^3}{z^{3/2}} \]

\[-96 \frac{\hat{I}_1^{[0,1]} (1, 4, 1) m_b^6}{z^{3/2}} - 32 \frac{\hat{I}_2 (1, 2, 2) m_b^5}{z^{3/2}} - 64 \frac{\hat{I}_1^{[0,1]} (1, 2, 2) m_b^5}{z^{3/2}} \]

\[+4 \frac{\hat{I}_1^{[0,1]} (3, 2, 1) m_b^4}{z^{3/2}} + 16 \frac{\hat{I}_2^{[0,1]} (2, 3, 1) m_b^5}{z^{3/2}} + 32 \frac{\hat{I}_1 (1, 2, 2) m_b^6}{z^{3/2}} \]

\[+2 \frac{\hat{I}_2 (3, 2, 1) m_b^5}{z^{3/2}} - 32 \frac{\hat{I}_0 (1, 3, 1) m_b^5}{z^{3/2}} + \frac{\hat{I}_2 (3, 2, 2) m_b^6}{z^{3/2}} \]

\[+\frac{4}{2} \frac{\hat{I}_1 (3, 2, 1) m_b^5}{z^{3/2}} - 2 \frac{\hat{I}_2 (3, 1, 2) m_b^5}{z^{3/2}} - 4 \frac{\hat{I}_2^{[0,1]} (3, 2, 2) m_b^5}{z^{3/2}} \]

\[-4 \frac{\hat{I}_0^{[0,2]} (3, 2, 2) m_b^4}{z^{3/2}} - 2 \frac{\hat{I}_0^{[0,1]} (3, 2, 2) m_b^4}{z^{3/2}} + 12 \frac{\hat{I}_2^{[0,1]} (3, 2, 1) m_b^4}{z^{3/2}} \]

\[-4 \frac{\hat{I}_0^{[0,1]} (3, 2, 1) m_b^3}{z^{3/2}} - 3 \frac{\hat{I}_2 (4, 1, 1) m_b^4}{z^{3/2}} + 16 \frac{\hat{I}_1^{[0,1]} (2, 2, 2) m_b^5}{z^{3/2}} \]

\[-16 \frac{\hat{I}_1^{[0,1]} (2, 2, 1) m_b^4}{z^{3/2}} + 8 \frac{\hat{I}_0 (2, 2, 1) m_b^4}{z^{3/2}} - 12 \frac{\hat{I}_2^{[0,2]} (3, 2, 2) m_b^4}{z^{3/2}} \]
\begin{align*}
+8 \frac{\hat{I}_2(2, 2, 2) m_b^6}{z^{3/2}} & - 128 \frac{\hat{I}_2(1, 3, 1) m_b^6}{z^{3/2}} - 3 \frac{\hat{I}_1(4, 1, 1) m_b^4}{z^{3/2}} \\
-8 \frac{\hat{I}_0^{[0,1]}(2, 3, 1) m_b^4}{z^{3/2}} & + 6 \frac{\hat{I}_1^{[0,1]}(4, 1, 1) m_b^3}{z^{3/2}} - 12 \frac{\hat{I}_1^{[0,1]}(3, 1, 2) m_b^4}{z^{3/2}} \\
+4 \frac{\hat{I}_1(3, 1, 2) m_b^5}{z^{3/2}} & - 2 \frac{\hat{I}_0(4, 1, 1) m_b^6}{z^{3/2}} + 48 \frac{\hat{I}_1^{[0,2]}(2, 2, 2) m_b^4}{z^{3/2}} \\
-64 \frac{\hat{I}_1(1, 2, 1) m_b^5}{z^{3/2}} & - \hat{I}_0(3, 2, 1) m_b^4 - 24 \frac{\hat{I}_0(1, 4, 1) m_b^6}{z^{3/2}} \\
-48 \frac{\hat{I}_1(1, 4, 1) m_b^7}{z^{3/2}} & + \hat{I}_0(3, 2, 2) m_b^6 - 12 \frac{\hat{I}_1^{[0,1]}(3, 2, 1) m_b^4}{z^{3/2}} \\
+4 \frac{\hat{I}_1^{[0,1]}(3, 2, 2) m_b^5}{z^{3/2}} & - 16 \frac{\hat{I}_1^{[0,1]}(2, 3, 1) m_b^5}{z^2} + 4 \frac{\hat{I}_0(2, 3, 1) m_b^5}{z^2} \\
+3 \frac{\hat{I}_1(4, 1, 1) m_b^4}{z^{3/2}} & + 128 \frac{\hat{I}_1(1, 3, 1) m_b^6}{z^2} - 2 \frac{\hat{I}_2(3, 2, 1) m_b^5}{z^2} \\
-8 \frac{\hat{I}_1(2, 2, 2) m_b^6}{z^2} & + 96 \frac{\hat{I}_1^{[0,1]}(1, 4, 1) m_b^6}{z^2} - 8 \frac{\hat{I}_2(3, 1, 1) m_b^6}{z^2} \\
+32 \frac{\hat{I}_1(1, 2, 2) m_b^6}{z^2} & - \frac{\hat{I}_1(3, 2, 2) m_b^6}{z^2} + 2 \frac{\hat{I}_1(3, 1, 2) m_b^5}{z^2} \\
+12 \frac{\hat{I}_1^{[0,2]}(3, 2, 2) m_b^4}{z^2} & + 1/2 \frac{\hat{I}_0(3, 2, 2) m_b^5}{z^2} - 2 \frac{\hat{I}_1(3, 2, 1) m_b^5}{z^2} \\
+6 \frac{\hat{I}_2^{[0,1]}(3, 2, 2) m_b^5}{z^2} & - 48 \frac{\hat{I}_1^{[0,1]}(2, 2, 2) m_b^5}{z^2} + \frac{\hat{I}_2(3, 2, 2) m_b^6}{z^2} \\
-12 \frac{\hat{I}_2(2, 2, 2) m_b^6}{z^2} & + 48 \frac{\hat{I}_2(1, 4, 1) m_b^7}{z^2} + 48 \frac{\hat{I}_1(1, 4, 1) m_b^7}{z^2} \\
-12 \frac{\hat{I}_2(3, 2, 2) m_b^6}{z^{5/2}} & - 48 \frac{\hat{I}_1(1, 4, 1) m_b^7}{z^{5/2}} - \frac{\hat{I}_1(3, 2, 2) m_b^6}{z^{5/2}} \\
-6 \frac{\hat{I}_1^{[0,1]}(3, 2, 2) m_b^5}{z^{5/2}} & + 8 \frac{\hat{I}_1(2, 3, 1) m_b^6}{z^{5/2}} + 12 \frac{\hat{I}_0(2, 2, 2) m_b^6}{z^{5/2}} \\
+2 \frac{\hat{I}_1(3, 2, 1) m_b^5}{z^{5/2}} & + \frac{\hat{I}_1(3, 2, 2) m_b^6}{z^3} - 2 \frac{\hat{I}_0(3, 1, 1) m_b^3}{z^3} \\
- \hat{I}_0(3, 2, 1) m_b^4 & - 4 \hat{I}_0(2, 1, 2) m_b^4 \\

C_T^{HQT} &= 32 \frac{\hat{I}_0(1, 2, 2) m_b^5}{\sqrt{z}} - 4 \frac{\hat{I}_0^{[0,1]}(3, 1, 2) m_b^3}{\sqrt{z}} - 4 \frac{\hat{I}_0^{[0,1]}(3, 2, 1) m_b^3}{\sqrt{z}} \\
+4 \frac{\hat{I}_0^{[0,2]}(3, 2, 2) m_b^3}{\sqrt{z}} & + 8 \frac{\hat{I}_0(2, 3, 1) m_b^5}{\sqrt{z}} - \hat{I}_0(3, 2, 2) m_b^5 \\
+8 \frac{\hat{I}_0(2, 2, 1) m_b^4}{\sqrt{z}} & - 32 \frac{\hat{I}_0^{[0,1]}(2, 1, 2) m_b^3}{\sqrt{z}} + 3 \frac{\hat{I}_0(4, 1, 1) m_b^3}{\sqrt{z}} \\
+8 \frac{\hat{I}_0^{[0,2]}(3, 1, 2) m_b^2}{\sqrt{z}} & - 2 \frac{\hat{I}_0(3, 1, 2) m_b^4}{\sqrt{z}} - 16 \frac{\hat{I}_0^{[0,1]}(2, 2, 2) m_b^4}{\sqrt{z}} \\

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\end{align*}
\[-16 \frac{\hat{I}_0(2,1,1) m_b}{\sqrt{z}} + 64 \frac{\hat{I}_0(1,1,2) m_b}{\sqrt{z}} + 8 \frac{\hat{I}_0(2,1,2) m_b}{\sqrt{z}} \]
\[-16 \frac{\hat{I}_0^{[0,1]}(2,3,1) m_b}{z} - \frac{8 \hat{I}_0^{[0,2]}(3,2,1) m_b}{z} + 2 \frac{\hat{I}_0(3,1,2) m_b}{z} \]
\[+6 \frac{\hat{I}_0(3,2,1) m_b}{z} - \frac{8 \hat{I}_0^{[0,1]}(3,1,2) m_b}{z} - \frac{48 \hat{I}_0(1,4,1) m_b}{z} \]
\[-4 \frac{\hat{I}_0^{[0,1]}(3,2,2) m_b}{z} + 16 \frac{\hat{I}_0(2,1,2) m_b}{z} + 16 \frac{\hat{I}_0^{[0,1]}(3,1,1) m_b}{z} \]
\[+4 \frac{\hat{I}_0^{[0,1]}(3,2,1) m_b}{z} + 8 \frac{\hat{I}_0(3,1,1) m_b}{z} - \frac{32 \hat{I}_0(2,1,1) m_b}{z} \]
\[-192 \frac{\hat{I}_0(1,2,1) m_b}{z} - \frac{8 \hat{I}_0(2,2,1) m_b}{z} + \frac{8 \hat{I}_0(2,2,2) m_b}{z} \]
\[-160 \frac{\hat{I}_0(1,3,1) m_b}{z} + \frac{2 \hat{I}_0(3,1,2) m_b}{z^{3/2}} - \frac{2 \hat{I}_0(3,2,1) m_b}{z^{3/2}} \]
\[+8 \frac{\hat{I}_0(2,3,1) m_b}{z^{3/2}} + \frac{\hat{I}_0(3,2,2) m_b}{z^{3/2}} - \frac{8 \hat{I}_0(3,1,1) m_b}{z^{3/2}} \]
\[+8 \frac{\hat{I}_0^{[0,1]}(3,2,1) m_b}{z^{3/2}} - \frac{2 \hat{I}_0(3,2,1) m_b}{z^2} + 4 \frac{\hat{I}_0(3,2,1) m_b}{z} \]
\[+8 \frac{\hat{I}_0^{[0,1]}(3,2,2) m_b}{z^2} + 8 \frac{\hat{I}_0(3,1,1) m_b}{z^2} - 12 \frac{\hat{I}_0^{[0,1]}(3,1,2) m_b}{z^2} \]
\[+4 \frac{\hat{I}_0(2,2,2) m_b}{z^2} + 16 \frac{\hat{I}_0(2,1,2) m_b}{z^2} \]

where

\[\hat{I}_n^{[i,j]}(a, b, c) = \frac{(2 m_b)^{i+j}}{(\sqrt{z})^j} \int \left( T_1^2 \right)^i \left( T_2^2 \right)^j \frac{d^i}{d (T_1^2)^i} \frac{d^j}{d (T_2^2)^j} \left[ \left( T_1^2 \right)^i \left( T_2^2 \right)^j \hat{I}_n(a, b, c) \right] .\]
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Figure 3: The dependence of the form factor $f_+$ on Borel parameters $M_1^2$ and $M_2^2$ for $B_c \to D_s l^+ l^- / \nu \bar{\nu}$.
Figure 4: The same as Fig. 3, but for $f_\_$. 
Figure 5: The dependence of the form factor $f_T$ on Borel parameters $M_1^2$ and $M_2^2$ for $B_c \rightarrow D_s l^+ l^-$. 
Figure 6: The dependence of the form factor $f_+$ on Borel parameters $M_1^2$ and $M_2^2$ for $B_c \to Dl^+l^-/\nu\bar{\nu}$.
Figure 7: The same as Fig. 6, but for $f_-$. 
Figure 8: The dependence of the form factor $f_T$ on Borel parameters $M_1^2$ and $M_2^2$ for $B_c \to Dl^+l^-$. 