Quantum information science as an approach to complex quantum systems

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What makes quantum information science a science? These notes explore the idea that quantum information science may offer a powerful approach to the study of complex quantum systems. We discuss how to quantify complexity in quantum systems, and argue that there are two qualitatively different types of complex quantum system. We also explore ways of understanding complex quantum dynamics by quantifying the strength of a quantum dynamical operation as a physical resource. This is the text for a talk at the “Sixth International Conference on Quantum Communication, Measurement and Computing”, held at MIT, July 2002. Viewgraphs for the talk may be found at [http://www.qinfo.org/talks/](http://www.qinfo.org/talks/).

I. INTRODUCTION

Good afternoon, ladies and gentlemen. My name is Michael Nielsen and I’m from the University of Queensland. My talk today is about using quantum information science as an approach to the study of complex quantum systems. The work I will describe has involved many collaborators at the University of Queensland and MIT, but I would especially like to emphasize the key role played by Tobias Osborne in this research.

Let me begin by asking what it is that makes quantum information science a science? Friends outside the field sometimes comment that it seems to be largely engineering, with little science. A standard response to this question from physicists has been that in the course of building devices like quantum computers, we'll discover lots of interesting and fundamental physics.

This is undoubtedly true, and is an excellent reason for doing quantum information science. But it seems a little like the argument sometimes used to justify going to the moon, namely, that it resulted in valuable spin-off technologies in fields such as computation and aeronautical engineering. Of course, this misses a large part of the point, since going to the moon has an intrinsic worth, a point obvious to those with even the slightest romance in their soul.

What is the intrinsic worth of quantum information science? In my talk today I’ll argue that quantum information science may be regarded as an approach to the study of complex quantum systems. Related ideas have been advocated by many people. Aharonov [1], Nielsen [2] and Preskill [3] argued that there may be interesting connections between the quantitative theory of entanglement and the properties of many-body quantum systems, and there is now a burgeoning literature exploring these connections [10].

More explicitly, the concluding paragraph of DiVincenzo’s [4] paper on the physical requirements for quantum computation suggests that quantum information science may offer valuable insights into complex quantum systems. This theme was explored in more detail by Osborne and Nielsen [5, 6, 7, 8], and the present talk is an outgrowth of this work, with some illustrative examples drawn from [9, 10].

II. COMPLEX QUANTUM SYSTEMS

What is a complex system? Complexity is an elusive concept: it is difficult to define, but we know it when we see it. In response to this difficulty one is naturally led to ask whether it is possible to come up with quantitative measures of complexity.

In the 1980s Bennett proposed just such a measure, which he called the logical depth of a system [1][2]. The basic idea is that a system should be called complex, or logically deep, if that system can be generated by a few simple rules, but those rules require a long time to run. So, for example, a human body is complex in that it is specified by a relatively small amount of information encoded in DNA, but it takes a great deal of processing to get from that DNA to the human body.

Another example is a regular pattern on a checkerboard, which is not complex because it can be quickly generated by a simple rule. More subtle is the case of a random pattern on a checkerboard. That is not complex either, because there is no simple rule generating the pattern. Indeed, the simplest rule generating the pattern is simply the program which contains (and prints) a complete listing of the states of all the elements of the checkerboard, and this program runs very quickly.

Now let me give you an example of something complex, that is, with high logical depth. Suppose I take a point $x$ in the plane, for example, $x = (0,0)$. I now bounce the point around the plane by applying one of the following four rules at random, over and over again [3].

$$x \rightarrow \begin{cases} 0 & 0 \\ 0 & 0.16 \end{cases} x$$

with probability 0.01

(2.1)
There are some technicalities hidden in this definition, but they operate very slowly. If it's quantum then it seems likely that this system is not logically deep, and thus is not complex, because we can quickly generate the list using a short quantum program, namely, Shor's algorithm [15, 16]. If the definition of logical depth uses a classical computer, and there really is, as we suspect, no fast classical factoring algorithm, then the list of factors has high logical depth, since there are simple computer programs capable of generating such a list, but they operate very slowly.

Thus, according to this discussion there are two distinct notions of logical depth: classical logical depth, and quantum logical depth. We can summarize the situation by dividing the class of possible data sets into three distinct types. First, there are data sets which have both high classical logical depth and high quantum logical depth. We can summarize the situation by dividing the class of possible data sets into three distinct types. First, there are data sets which have both

25195908475657893494027183240048398571429282
126204032027771378360436620207059555626401
85258807844069182906412495150821892985519491
76184502808489120072844992687392807287776735
97141834727026189637501497182469116507761337
9859095700973304597488084284017974291006424
5869181719518746121515172654632282216869987
54918242243363725908541186546204357679842338
71847744792073993246584823824281981638150
1067481045160037730602501169676256133844143
603833904419526343219011465754445417842402
0924616515723350778707748198117257724769629263
8635637328991215483143816789988504045364023
5273819513786365439121201039712282212072035
7.

Let’s optimistically imagine, then, that it’s ten years from now and somebody wants to prove that they have a functioning quantum computer in their lab, but don’t want to reveal all the details of how they built it. One good way of proving this to the world would be to publish a paper containing the prime factors of a large group of big integers — perhaps the prime factors of all numbers between $10^{500}$ and $10^{500} + 100000$.

Now, would this list of factors constitute a complex system or not? The answer is that it depends on whether the computer in the definition of logical depth is quantum or classical. If it’s quantum then it seems likely that this system is not logically deep, and thus is not complex, because we can quickly generate the list using a short quantum program, namely, Shor’s algorithm [15, 16].

Bennett formalized these intuitions by defining the logical depth of a data set to be the running time of a near-optimal computer program generating that data set. There are some technicalities hidden in this definition, like the precise meaning of “near-optimal”, that I’m going to gloss over today. Nonetheless, I would like to give you an intuitive feel for what the definition means.

First of all, by “near-optimal” we mean that the program is nearly the shortest possible. The motivating idea for this requirement is an analogy between computer programs generating data sets and scientific hypotheses. Scientists tend to prefer simple explanations over more complex; this is Occam’s razor. If we think of computer programs as explanations for data sets, then we would tend to prefer short computer programs — simple “explanations” — over longer programs.

With this definition, simple repeated structures and random patterns, like the checkerboards I described earlier, have low logical depth. However, systems like the fern have high logical depth because they have very simple explanations that take a long time to run.

Up until now I’ve said nothing about quantum mechanics. However, it turns out that there is an interesting quantum twist to logical depth. As we know, factoring integers is a hard problem, hard enough that RSA systems offers lots of money to people able to factor integers like this colossus [18]:

$$\begin{array}{c}
25195908475657893494027183240048398571429282 \\
126204032027771378360436620207059555626401 \\
85258807844069182906412495150821892985519491 \\
76184502808489120072844992687392807287776735 \\
97141834727026189637501497182469116507761337 \\
9859095700973304597488084284017974291006424 \\
5869181719518746121515172654632282216869987 \\
54918242243363725908541186546204357679842338 \\
71847744792073993246584823824281981638150 \\
1067481045160037730602501169676256133844143 \\
603833904419526343219011465754445417842402 \\
0924616515723350778707748198117257724769629263 \\
8635637328991215483143816789988504045364023 \\
5273819513786365439121201039712282212072035 \\
7.
\end{array}$$ (2.5)
logical depth. I don’t yet have a really good example of a system which I expect to have this property, but consider some systems likely candidates, for example, the output of a quantum cellular automata that’s been running for a long time. Note that systems with high quantum logical depth but low classical logical depth seem unlikely, because a simple, fast classical computer generating a data set can be simulated by a simple, fast quantum computer.

We thus conclude that there are two different types of complex system, one which we might call semi-classically complex, being systems with high classical logical depth, but low quantum logical depth. The other we might call quantum complex, having both high quantum and classical logical depth.

III. QUANTUM DYNAMICS AS A PHYSICAL RESOURCE

I’ve talked a fair bit about quantifying complexity, and the information-theoretic viewpoint has led us to the idea that there are at least two, if not more, qualitatively different types of complex system. I’d like now to talk about what we can learn about specific complex quantum systems from quantum information science.

As you are all aware, over the past few years a great deal of effort has been devoted to developing a quantitative theory of quantum entanglement [3]. In my opinion, one of the major areas in which this theory will be applied is to obtain insights into the static properties of complex quantum systems. We’re already starting to see this with applications of the theory of entanglement to condensed matter systems [10]. Indeed, when I originally planned this talk I was going to spend about a third of it discussing how the theory of entanglement provides insight into an important numerical tool for studying many-body quantum systems, the so-called density matrix renormalization group [3].

However, in the meantime my group has obtained what I think are some more interesting results that can be used to illustrate how quantum information science provides insight into complex quantum systems, so I will talk about those instead.

These other results arise from the observation that static quantum entanglement is only a small part of the story: it would also be interesting to obtain a better understanding of the quantum dynamics of complex systems. To do this, we’ve focused on the question of whether or not it might be possible to quantify the strength of a quantum dynamical operation for quantum information processing.

The motivation behind this idea is the observation, made last year, that quantum dynamics are a fungible physical resource. By fungible, I mean that it is possible to convert different types of dynamical operation, one to the other, just as it is possible to convert one type of entangled state to another.

Let me be a little more precise about what I mean. Suppose a system contains n qudits, that is d-dimensional quantum systems, where d can be any integer. We further suppose that the Hamiltonian for the system contains only two-body terms, and so can be represented by a graph with the vertices representing qudits, and an edge between vertices representing the presence of an interaction between those qudits. We’ll further suppose that the graph is connected, so that different qudits aren’t completely cut off from one another.

What is interesting is that by alternating evolution due to such a Hamiltonian with single-qudit gates, it turns out that we can efficiently simulate any quantum computation, at least in principle [11]. For our purposes, the interesting conclusion to draw from this discussion is that these Hamiltonians form a fungible physical resource, since it is possible to use any one to simulate any other.

Now, this is a very interesting theoretical result, however one might ask whether the result is practically useful for quantum computation. Certainly the schemes proposed last year were not practically useful, since they required extremely frequent and rapid local control to do the simplest of operations, such as a controlled-NOT. Even an extremely optimistic example [17] required something like $10^4$ operations to do a single controlled-NOT.

Recently, however, the situation has changed. Imagine one is given a single entangling two-qubit unitary operation, $U$, and is asked to perform a controlled-NOT using just $U$ and local unitaries. Last year, J.-L. and R. Brylinski [8] were able to show that this is always possible. Indeed, they were even able to generalize the result to the qudit case. However, they needed to use quite a few results from algebraic geometry and the theory of Lie algebras to prove their results, and it is not clear to me whether their proof can be used to give an efficient constructive method for doing the controlled-NOT.

However, just a couple of weeks ago, my group put a paper on the archive [1] that provides a simple and constructive algorithm for doing a controlled-NOT using these resources. Perhaps more importantly, the algorithm even turns out to be near-optimal, in the sense that it uses what is very nearly the minimal number of uses of $U$ to simulate a controlled-NOT. An online implementation of this algorithm, due to Chris Dawson and Alexei Gilchrist, may be found at http://www.physics.uq.edu.au/gqc/.

This algorithm shows that not only are quantum dynamical operations fungible in principle, but they may be fungible in practice as well.

Indeed, I believe that this algorithm is one step along the way in an interesting evolution in the viewpoint of a would-be quantum computer designer. In the early days of quantum computing, the question for such a designer was always “How can I quantum compute, given the interactions in my system?” What I hope our result and other similar results will do is change that question so that it becomes simply “What is the interaction in my system?”, with the method for doing quantum computa-
tion in an optimal way simply pulled up out of a database once the interaction is known.

Knowing that quantum dynamics are a fungible physical resource, and thus qualitatively equivalent to one another, it makes sense to ask whether we can quantify the strength of a particular dynamical operation. The picture to have in mind is an interaction, \( U \), being applied to \( n \) qubits. We then attempt to quantify the strength by some appropriate function, \( K(U) \). The letter \( K \), by the way, comes from the Greek God of strength and power, Krakos. (Note that the following ideas about quantifying dynamic strength are based on a paper by Nielsen, Dawson, Dodd, Gilchrist, Mortimer, Osborne, Bremner, Harrow, and Hines [10], and much more discussion can be found there.)

Let me give you a couple of examples of strength measures. These are just two examples chosen on an ad hoc basis from the much larger collection of strength measures considered in [10]. For simplicity I will state both measures just for two qubits, although extensions to more than two qubits are easily defined. The first measure of strength, the entangling power of a unitary operation, is just the maximal change in entanglement that the unitary can cause, maximized over all possible pure input states,

\[
K_{\Delta}(U) \equiv \max_{\psi} |E(U\psi) - E(\psi)|, \quad (3.1)
\]

where \( E(\cdot) \) is some appropriate measure of entanglement. The second measure of strength is very different in nature. Imagine we have a metric \( D \) on the space of all unitary operations. Then this metric induces a natural strength measure, \( K_D \), as follows:

\[
K_D(U) \equiv \min_{A,B} D(U, A \otimes B). \quad (3.2)
\]

That is, the strength is just the minimal distance between \( U \) and the set of local unitaries. (Note that similar approaches to the definition of entanglement measures have been explored in [14, 20].)

What good are these measures of strength, even assuming we could obtain useful computational formulas for them? Let me answer that question by talking about an interesting connection between measures of dynamic strength and computational complexity. Computational complexity theory is concerned with the problem of determining how many elementary quantum gates are needed to perform a particular unitary \( U \). In other words, it’s about quantifying the complexity, not of data sets, as we talked about earlier, but rather the complexity of particular dynamical operations.

It turns out that there’s an interesting relationship between strength measures and computational complexity. Imagine we had a strength measure with the following three properties.

The first property, which I will call chaining, just says that the strength of a product of two unitary operations, \( U \) and \( V \), must be less than the sum of their combined strengths. That is,

\[
K(UV) \leq K(U) + K(V). \quad (3.3)
\]

The intuition behind the chaining property is that the ability to do \( U \) and \( V \) separately should be at least as powerful as the ability to do \( UV \). This property is always satisfied by \( K_{\Delta} \), and while it may not always be true for the metric-based strength measures, it turns out to be true for a large class of them.

The second property, stability, just says that if we imagine adding an extra qubit to our system and doing nothing to it, that should not change the strength of the operation. That is,

\[
K(U \otimes I) = K(U). \quad (3.4)
\]

For example, if \( U \) is a two-qubit unitary operator, then this type of stability implies that the three-qubit strength of \( U \otimes I \) is the same as the two-qubit strength of \( U \). Provided an appropriate measure of \( n \)-party entanglement is chosen, \( K_{\Delta} \) can be shown to satisfy this property. The metric-based measures do not always satisfy stability, but they do in some instances.

The third and final property, locality, just states that a strength measure should be zero for products of local unitary operations. That is,

\[
K(A \otimes B \otimes \ldots) = 0. \quad (3.5)
\]

This is true of both \( K_{\Delta} \) and the metric-based measures.

Well, imagine that we have such a strength measure, and we want to perform a particular unitary operation, \( U \), using a circuit containing just controlled-NOT and single-qubit unitary gates. Imagine the circuit contains \( M \) controlled-NOTs. Then, applying our three properties, we see that the strength of \( U \) can be no more than the sum of the strengths of all the individual controlled-NOTs, plus the strengths of the local unitaries, which are all zero. This gives us an upper bound for the strength of \( U \), namely \( M \) times the strength of the controlled-NOT,

\[
K(U) \leq MK(\text{cnot}). \quad (3.6)
\]

This, in turn, tells us that the number of controlled-NOTs needed in the circuit was at least the strength of \( U \), divided by the strength of the controlled-NOT,

\[
M \geq \frac{K(U)}{K(\text{cnot})}. \quad (3.7)
\]

Because of the stability property the strength of the controlled-NOT is a constant, so we see that if the strength of \( U \) scales superpolynomially, then so must the number of gates needed to do \( U \).

The reason this is an interesting line of thought is because nobody has ever succeeded in proving nontrivial lower bounds on the complexity of problems like the travelling-salesman problem. Thus one reason for being interested in measures of dynamic strength is that by
developing good measures of dynamic strength it might become possible to prove some interesting lower bounds on computational complexity.

There are many other interesting questions and potential applications of this idea of quantifying the dynamic strength of a quantum operation, far too many for me to even describe the questions here, much less the answers. An attempt at fleshing out the theory of dynamic strength may be found in [10]. The key point, however, is that by introducing measures of strength for quantum dynamical operations we may be able to obtain insight into the enormously complicated space of dynamical processes.

Let me conclude by looking again at the big picture. My belief is that the major scientific task of quantum information science is to develop tools that will lead to insight into the properties of complex quantum systems. In quantum mechanics we’re like chess players who’ve just learnt the rules of the game, and are still trying to figure out all the emergent properties those rules imply [21]. We’re doing so by developing overarching theories, like the theory of entanglement and of dynamic strength, which let us understand ever more complex phenomena. I expect that as these theories are further developed they will enable us to better understand complex systems, not only in information processing, but also in other areas of many-body physics.

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[37] See, for example [5, 23, 24, 25, 26] for a selection of recent work and further references.
[38] RSA Systems offers US $200,000 for the following number, at the time of this writing.
[39] See, for example, the recent review articles [27, 28, 29, 30] for an introduction and further references.
[40] See, for example [5, 23, 24, 25, 26] for a selection of recent work and further references.
[41] Many results in this vein have been obtained over the past 18 months or so. Further information on these results may be found in [7, 14, 22, 33, 41, 42, 43], and references therein.