Weibull Distribution and the multiplicity moments in \( pp (p\bar{p}) \) collisions

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(Dated: October 15, 2018)

A higher moment analysis of multiplicity distribution is performed using the Weibull description of particle production in \( pp (p\bar{p}) \) collisions at SPS and LHC energies. The calculated normalized moments and factorial moments of Weibull distribution are compared to the measured data. The calculated Weibull moments are found to be in good agreement with the measured higher moments (up to 5\textsuperscript{th} order) reproducing the observed breaking of KNO scaling in the data. The moments for \( pp \) collisions at \( \sqrt{s} = 13 \text{ TeV} \) are also predicted.

I. INTRODUCTION

The number of charged particles produced in the final state of a collision is one of the basic and simplest observable measured in high energy nucleon-nucleon collisions. Being a global measure to characterize the final states produced, it provides important insight and constraints to the mechanism of particle production \[1\]. Therefore, the study of final state of multi-particle production has generated intense interest in theoretical and experimental high energy physics. Phenomenologically, the multiplicity distributions are expressed in terms of the probability distribution \((P(n))\) of producing \(n\) number of particles in the final state of a collision. If one assumes an independent production of final state particles, the multiplicity distribution is expected to follow a Poisson distribution. Any deviation from the Poissonian shape can reveal more information about the underlying dynamics of particle production and signal about the existence of multi-particle correlations. The shape of multiplicity distribution varies with system size and collision energies and can be incorporated to study the higher moments of the distribution \[2,3\].

In particular, the study of higher-order moments of the distribution and its cumulants constitute a sensitive tool to investigate the existing multi-particle correlations \[4,5\]. Although the multiplicity distribution in full phase space is constrained by global conservation laws, the dynamical fluctuations arising due to random cascading processes in particle production can lead to correlations \[6,7\]. The deviation from independent production can be quantified if the factorial moments of order \(q\), \(F(n)_q = \langle (n_1(n_1-1)...(n_q+1))\rangle \langle n\rangle^q\) are greater (or less) than unity. These factorial moments reveal about the dynamical fluctuations. The study of reduced moments of order \(q\), \(C(n)_q = \langle n^q \rangle / \langle n \rangle^q\) is also sensitive to see the approximate breaking or holding of KNO (Koba, Nielsen and Olesen) scaling \[12\]. The exact KNO scaling would imply the energy independence of the moments. The violation of scaling was first observed in \( p\bar{p} \) collisions in UA5 experiment \[13,14\]. Recent measurements from CMS and ALICE experiment observed strong violation of KNO scaling for higher order moments \((C_4 \text{ and } C_5)\) which became stronger with an increase of energy and width of \(|\eta|\) intervals. The \(C_2\) and \(C_3\) values were more or less constant across all energies and for narrower \(|\eta|\) intervals \[15,16\].

The energy dependence of higher order moments has been used to improve or reject different models (Monte-Carlo and statistical) of particle production. It is well known that multi-particle production processes involve initial hard scatterings of partons followed by parton fragmentation and sequential branching. One cannot apply perturbative QCD to the softer part of particle production and therefore one uses some phenomenological models or studies based on Monte-Carlo generators. It is important to see whether the higher order moments and their variation with beam energy can be understood in terms of some of the statistical models or Monte-Carlo generators. Previous studies have used the negative binomial distribution (NBD) as the most generalized expression for the multiplicity. But deviations from NBD was observed as the beam energy of the collisions increased \[17\]. The choice of Weibull distribution describing the multiplicity distribution for a broad range of collision energies in both leptonic and hadronic systems has been quite successful recently \[20,21\]. Recently, it was shown \[22\] that Weibull model was found to reproduce the genuine correlation measured in \(e^+e^-\) collisions for OPAL data \[23\]. In this work, the higher order reduced moments and factorial moments has been calculated using the Weibull description of multi-
plicity distribution. The results have been compared to the multiplicity moments data measured by different experiments. The idea is to extend the higher moment calculations to \( pp (p\bar{p}) \) collisions for a broad range of energy and to see whether one can observe the breaking of KNO scaling using the Weibull regularity.

II. WEIBULL DISTRIBUTION AND MOMENTS

The probability density distribution of Weibull for a continuous random variable \( x \) is given by:

\[
f(x; \lambda, k) = \frac{k}{\lambda} \left(\frac{x}{\lambda}\right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}
\]

(1)

where \( k \) is the shape parameter and \( \lambda \) is the scale parameter of the distribution.

The \( n^{th} \) raw moment is given by:

\[
m_n = \lambda^n \Gamma \left(1 + \frac{n}{k}\right)
\]

(2)

The mean of the distribution, i.e. \( m_1 \) is denoted by \( \langle n \rangle \) and is given by,

\[
\langle n \rangle = \lambda \Gamma \left(1 + \frac{1}{k}\right)
\]

(3)

The \( n^{th} \) factorial moment of a variable \( x \) is defined as:

\[
f_n = \langle \frac{x!}{(x-n)!} \rangle = \langle x(x-1)(x-2)\ldots(x-n+1) \rangle
\]

(4)

The \( n^{th} \) reduced raw moment \( C_n \) and factorial moments \( F_n \) are defined as:

\[
C_n = m_n/m_1^n; \quad F_n = f_n/m_1^n
\]

(5)

The first few reduced raw moments for Weibull distribution are given by:

\[
C_2 = \frac{m_2}{m_1^2} = \frac{\Gamma(1 + \frac{2}{k})}{(\Gamma(1 + \frac{1}{k}))^2}
\]

(6)

\[
C_3 = \frac{m_3}{m_1^3} = \frac{\Gamma(1 + \frac{3}{k})}{(\Gamma(1 + \frac{1}{k}))^3}
\]

(7)

\[
C_4 = \frac{m_4}{m_1^4} = \frac{\Gamma(1 + \frac{4}{k})}{(\Gamma(1 + \frac{1}{k}))^4}
\]

(8)

\[
C_5 = \frac{m_5}{m_1^5} = \frac{\Gamma(1 + \frac{5}{k})}{(\Gamma(1 + \frac{1}{k}))^5}
\]

(9)

The first few reduced factorial moments for Weibull distribution are given by:

\[
F_2 = \frac{\langle x(x-1) \rangle}{m_1^2} = \frac{-\lambda \Gamma(1 + \frac{1}{k}) + \lambda^2 \Gamma(1 + \frac{2}{k})}{(\lambda \Gamma(1 + \frac{1}{k}))^2}
\]

(10)

\[
F_3 = \frac{\langle x(x-1)(x-2) \rangle}{m_1^3} = \frac{2\lambda \Gamma(1 + \frac{2}{k}) - 3\lambda^2 \Gamma(1 + \frac{3}{k}) + \lambda^3 \Gamma(1 + \frac{4}{k})}{(\lambda \Gamma(1 + \frac{1}{k}))^3}
\]

(11)

\[
F_4 = \frac{\langle x(x-1)(x-2)(x-3) \rangle}{m_1^4} = \frac{-6\lambda \Gamma(1 + \frac{1}{k}) + 11\lambda^2 \Gamma(1 + \frac{2}{k}) - 6\lambda^3 \Gamma(1 + \frac{3}{k}) + \lambda^4 \Gamma(1 + \frac{4}{k})}{(\lambda \Gamma(1 + \frac{1}{k}))^4}
\]

(12)

\[
F_5 = \frac{\langle x(x-1)(x-2)(x-3)(x-4) \rangle}{m_1^5}
\]

\[
= \frac{24\lambda \Gamma(1 + \frac{1}{k}) - 50\lambda^2 \Gamma(1 + \frac{2}{k}) + 35\lambda^3 \Gamma(1 + \frac{3}{k}) - 10\lambda^4 \Gamma(1 + \frac{4}{k}) + \lambda^5 \Gamma(1 + \frac{5}{k})}{(\lambda \Gamma(1 + \frac{1}{k}))^5}
\]

(13)
III. ANALYSIS METHOD AND RESULTS

In this work, we have used the multiplicity distributions of $p\bar{p}$ collisions measured by UA5 experiment [14, 17] at SPS energies (200 GeV, 540 GeV and 900 GeV) and $pp$ collisions measured by ALICE [16] and CMS [15] experiments at LHC energies (0.9 TeV, 2.36 TeV, 7 TeV and 8 TeV). The analysis method is similar to what has been done previously for higher moment analysis of $e^+e^-$ collisions at LEP energies [3, 22]. The values of Weibull parameters i.e $k$ and $\lambda$ were estimated from the measured values of lower order reduced moment, $C_3$ as defined in Eq. 7 and the measured mean multiplicity respectively. The obtained values of $k$ and $\lambda$ were then used to calculate the higher order reduced moments as well as the reduced factorial moments. One can also use $C_2$ to extract the $k$ values but the agreement with the higher order moments was found to be better if $C_3$ was used. This is because the higher moments are very sensitive to $k$. The value of $C_2$ obtained from $k$ parameter estimated from $C_3$ values are in excellent agreement with the measured ones. The analysis was also performed for different pseudorapidity ($\eta$) intervals.

The variation of $k$ and $\lambda$ with collision energy and for different $\eta$ intervals are presented in Fig. 1. The dependence is described by a power law shown by the red dashed curve. One can observe that the variation of $k$ and $\lambda$ is consistent with previous multiplicity analysis [18, 20].

The calculated reduced moments ($C_n$) are shown in different $\eta$ intervals for broad range of energies in Fig. 2. It can be observed from the figure that the obtained Weibull moments agree remarkably well with the measured data within the uncertainties up to the fifth order for different energies and $\eta$ intervals.

Fig. 3 shows the variation of reduced factorial moments ($F_n$) with beam energy for different $\eta$ intervals. We need experimental measurements at LHC energies to see how good the calculations agree. The agreement is very nice for measured values of factorial moments by UA5 experiment. The model values confirm the KNO scaling violation which is seen to increase with beam energy and width of $\eta$ interval.

The values of $k$ and $\lambda$ for $\sqrt{s} = 13$ TeV is obtained from the parameterization shown in Fig. 1. These values are used to predict the raw and factorial moments at 13 TeV for the above mentioned $\eta$ ranges. The values are listed in Table 1.

IV. CONCLUSION

The Weibull model of multiplicity distribution was used to calculate the reduced raw and factorial moments of multiplicity distributions in $pp$ ($p\bar{p}$) collisions and compared with the experimental data. The obtained moments (up to 5th order) are in good agreement (within experimental uncertainties) with the multiplicity moments measured for a broad range of energies. The model predictions have reproduced the violation of KNO scaling as observed for higher moments in measured data. The predictions for reduced factorial moments are also made for available $pp$ collisions at LHC energies. This study further establishes that Weibull distribution seems to be the optimal statistical model to describe multiplicity distribution as well as the higher moments of the same for a broad range of energies in hadron-hadron collisions.

V. ACKNOWLEDGEMENTS

The authors would like to thank Department of Science and Technology (DST), India for supporting the present work.

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FIG. 1. The variation of Weibull parameters $k$ and $\lambda$ with center of mass energy in $pp$ ($\bar{p}p$) collisions. The variation is parameterized by a power law of the form $A + B \times \log(\sqrt{s})^C$, shown by the red dashed curve.

TABLE I. The predicted values for mean and other higher moments and factorial moments in $pp$ collisions at $\sqrt{s} = 13$ TeV.

| $|\eta|$ | $\langle n \rangle$ | $C_2$ | $C_3$ | $C_4$ | $C_5$ | $F_2$ | $F_3$ | $F_4$ | $F_5$ |
|------|-------------|------|------|------|------|------|------|------|------|
| 0.5  | 6.38        | 2.05 | 6.38 | 26.73 | 140.78 | 1.89 | 5.47 | 21.26 | 103.98 |
| 1.0  | 13.03       | 1.95 | 5.65 | 21.62 | 102.68 | 1.87 | 5.21 | 19.13 | 87.22  |
| 1.5  | 19.57       | 1.92 | 5.40 | 20.02 | 91.72  | 1.87 | 5.12 | 18.41 | 81.97  |
| 2.4  | 35.15       | 1.73 | 4.19 | 12.90 | 47.81  | 1.71 | 4.05 | 12.20 | 44.26  |

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FIG. 2. The variation of the reduced raw moments (closed symbols) with the collision energies. The Weibull calculations are compared with experimentally measured normalized raw moments (open symbols).

FIG. 3. The variation of the reduced factorial moments (closed symbols) with the collision energies. The Weibull calculations are compared with experimentally measured normalized factorial moments (open symbols) where data is available.