On certain inequalities for the prime counting function

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Abstract: We study certain inequalities for the prime counting function $\pi(x)$. Particularly, a new proof and a refinement of an inequality from [1] is provided.

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1 Introduction

Let $\pi(x)$ denote the number of primes $\leq x$, where $x \geq 1$ is an integer. The famous Hardy–Littlewood conjecture states that the inequality

$$\pi(x + y) \leq \pi(x) + \pi(y)$$

is valid for all $x, y \geq 2$. Neither a proof nor a counterexample is known up to now.

There exist many inequalities in the literature, related to (1). For a survey of results, see the recent paper [1] of the author and H. Alzer and M. K. Kwong.

Many earlier results on $\pi(x)$ can be found in Chapter VII of the monograph [4]. For connections with other arithmetic functions, see the recent book [5] (see pp. 159–160).

One of the main results, proved in [1], is the inequality (see Theorem 1 of [1])

$$\pi^2(x + y) \geq \frac{16}{9} \pi(x) \cdot \pi(y), \quad (2)$$

where $x, y \geq 2$ and with equality only for $x = y = 5$. 
For $x = y$ in (2), we get the result that
\[ \pi(2x) \geq \frac{4}{3} \pi(x), \] (3)
with equality only for $x = 5$. This is a converse of the famous Landau inequality
\[ \pi(2x) \leq 2\pi(x), \quad (x \geq 2). \] (4)

Another result of [1] is the following (see left-hand side of Theorem 6 of [1]):
\[ \frac{1}{2} \leq \frac{\pi(x)^{x/(x+y)} \pi(y)^{y/(x+y)}}{\pi(x+y)}. \] (5)

The aim of this paper is to prove that, a converse inequality of (1) holds true, and this gives a new proof, as well as a refinement of (2). Another result will be motivated by relation (5).

2 Main results

The following classical inequality due to Rosser and Schoenfeld [3] will be used:

**Lemma** For all $x \geq 67$ one has
\[ \frac{x}{\log x - \frac{3}{2}} < \pi(x) < \frac{x}{\log x - \frac{3}{2}}. \] (6)

The first main result of this paper gives a converse to inequality (1):

**Theorem 1.** For all $x, y \geq 2$ one has
\[ \pi(x + y) \geq \frac{2}{3} [\pi(x) + \pi(y)], \] (7)
with equality only for $(x, y) = (5, 5); (3, 7); (7, 3)$.

**Proof.** Let $f(x) = \frac{x}{\log x - \frac{3}{2}}$. We shall prove that, this function is strictly concave for $x > e^{\frac{7}{2}}$. Indeed, one has $f'(x) = (\log x - \frac{5}{2})/(\log x - \frac{3}{2})^2$, and after some elementary computations, we get $f''(x).x.(\log x - \frac{3}{2})^3 = -\log x + \frac{7}{2} < 0$ if $\log x > \frac{7}{2}$, i.e., $x > e^{\frac{7}{2}} \approx 33.111 \ldots$.

The concavity of $f(x)$ gives the inequality:
\[ f(x) + f(y) \leq 2f\left(\frac{x+y}{2}\right) \text{ for all } x, y \geq e^{\frac{7}{2}}. \] (8)

By the right-hand side of (6) and (8) we can write:
\[ \pi(x) + \pi(y) < f(x) + f(y) \leq \frac{x + y}{\log\left(\frac{x+y}{2}\right) - \frac{3}{2}}. \] (9)

Now, by the left-hand side of (6) one has $\frac{3}{2} \pi(x + y) > \frac{3}{2} \frac{x + y}{\log(x+y) - \frac{3}{2}}$, so at a first step, in attempt to have (7), we want to prove the inequality:
\[ \frac{x + y}{\log\left(\frac{x+y}{2}\right) - \frac{3}{2}} < \frac{3}{2} \frac{x + y}{\log(x+y) - \frac{1}{2}}. \] (10)
which is equivalent with
\[
\log(x + y) > 3 \log 2 + \frac{9}{5} - \frac{1}{2} = 3.379\ldots,
\]
i.e., \(x + y > e^{3.379\ldots} \approx 29.3\ldots\)

This is clearly true, if \(x, y \geq 67\). Therefore, inequality (7) is proved for all \(x, y \geq 67\).

Now, suppose that \(x \geq y\) and \(y \leq 66\). Then \(\pi(y) \leq 18\), so \(\frac{2}{3} [\pi(x) + \pi(y)] \leq \frac{2}{3} [\pi(x) + 18] = \frac{2}{3} \pi(x) + 12\). We have to prove that \(\frac{2}{3} \pi(x) + 12 \leq \pi(x + y)\), or
\[
2\pi(x) + 36 \leq 3\pi(x + y). 
\]
(11)

As \(3\pi(x) \leq 3\pi(x + y)\), it will be sufficient to consider the inequality \(2\pi(x) + 36 \leq 3\pi(x)\), i.e., \(\pi(x) \geq 36\). This is true, if \(x \geq 151\).

Finally, we have to verify the case:
\[
2 \leq y \leq x \leq 150, y \leq 66.
\]
(12)

This can be verified by a computer (for example, a Maple 13 program). This finishes the proof of Theorem 1.

\[\square\]

**Corollary 1.**
\[
\pi^2(x + y) \geq \frac{4}{9} [\pi(x) + \pi(y)]^2 \geq \frac{16}{9} \pi(x) \pi(y),
\]
(13)
which is a refinement of inequality (2).

**Remark 1.** For \(y \leq x\) there is equality in the first inequality of (13) for \(y = 3, x = 7\) and \(y = 5, x = 5\); while in the second inequality only for \(y = 5, x = 5\).

Indeed, the first inequality follows by (7), while the second one by \((a + b)^2 \geq 4ab\), where \(a = \pi(x), b = \pi(y)\).

Now, by the weighted arithmetic mean—geometric mean inequality one has:
\[
u^\alpha \cdot v^\beta \leq \alpha u + \beta v
\]
for \(u, v, \alpha, \beta > 0; \alpha + \beta = 1\). By letting \(u = \pi(x), \alpha = x/(x + y), v = \pi(y), \beta = y/(x + y)\), by (5) and (14) we get
\[
\pi(x + y) \leq 2\pi(x)^{x/(x+y)} \pi(y)^{y/(x+y)} \leq 2\left[\frac{x}{x + y} \pi(x) + \frac{y}{x + y} \pi(y)\right],
\]
i.e.,
\[
(x + y) \pi(x + y) \leq 2 [x \pi(x) + y \pi(y)].
\]
(15)

In 2001, Panaitopol [2] proved the inequality:
\[
\pi^2(x + y) \leq 2 [\pi^2(x) + \pi^2(y)].
\]
(16)

Motivated by these two inequalities, in what follows, we shall prove:
**Theorem 2.** For all \( x, y \geq 2 \) one has

\[
\pi^2(x + y) \leq \frac{8}{7} \left[ x \pi(x) + y \pi(y) \right],
\]

with equality only for \((x, y) = (3, 4); (4, 3)\).

**Proof.** Let us consider the function \( g(x) = \frac{x^2}{\log x - \frac{1}{2}} \) \((x > 0)\). After elementary computations we can deduce that

\[
\frac{1}{2} g''(x) \left( \log x - \frac{1}{2} \right)^2 = \log^2 x - \frac{3}{2} \log x + 1.
\]  

(18)

Letting \( \log x = t \), clearly \( t^2 - \frac{3}{2} t + 1 > 0 \) (having a negative discriminant), so we get that the function \( g(x) \) is strictly convex.

By the left-hand side of (6) one has

\[
x \pi(x) + y \pi(y) > g(x) + g(y) \geq 2g \left( \frac{x + y}{2} \right) = \left( \frac{x + y}{2} \right)^2 / \left( \log \left( \frac{x + y}{2} \right) - \frac{1}{2} \right),
\]

by the convexity of \( g(x) \).

By the right-hand side of (6), in order to prove (17), we have first to consider the validity of inequality

\[
\frac{8}{7} \cdot \left( \log(x + y) - 2 \right) > \frac{(x + y)^2}{(\log(x + y) - \frac{3}{2})^2}.
\]

(19)

Letting \( \log(x + y) = m \), this becomes after elementary computations:

\[
2m^2 - 13m + 7 \log^2 + 8 > 0.
\]

Solving this quadratic inequality, it follows that it is true for \( m > 2.64 \ldots \), i.e., \( x + y > e^{2.64} = 14.01 \ldots \), which is clearly true for \( x, y \geq 67 \).

Now, let \( x \geq y \) and \( y \leq 66 \). As \( y \pi(y) \geq 2 \), it is sufficient to consider the inequality:

\[
(\pi(x) + 18)^2 \leq \frac{8}{7} [x \pi(x) + 2].
\]

(20)

This can written as

\[
7\pi^2(x) + 252\pi(x) + 2268 \leq 8x\pi(x) + 16. \text{ Now } 8x\pi(x) \geq 12\pi^2(x) \text{ by the elementary inequality}
\]

\[
\frac{\pi(x)}{x} \leq \frac{2}{3} x, \quad (x \geq 2).
\]

(21)

Therefore, we have to consider

\[
5\pi^2(x) - 252\pi(x) - 2252 \geq 0,
\]

which is valid for \( \pi(x) \geq 38 \), i.e., \( x \geq 163 \).
It remains to verify inequality (17) for

\[2 \leq y \leq x \leq 163.\]  \hspace{1cm} (22)

This can be verified by a computer, but we can could reduce the numbers of verifications as follows:

Segal [6] proved in 1962 that inequality (1) holds true for any \(x, y \geq 2\) and \(x + y \leq 101081\). Thus we can write for the values from (22) that

\[7\pi^2(x+y) \leq 7\pi^2(x) + 7\pi^2(y) + 14\pi(x)\pi(y).\]  \hspace{1cm} (23)

Now, if we can prove that \(8x\pi(x) \geq 14\pi^2(x)\), then we would have \(8x\pi(x) + 8y\pi(y) \geq 14\pi^2(x) + 14\pi(y)\) and inequality (23) would follow on base of \(7a^2 + 7b^2 > 14ab\) (i.e., \(7(a-b)^2 > 0\)) for \(a = \pi(x), b = \pi(y)\).

The inequality \(8x\pi(x) \geq 14\pi^2(x)\) is in fact

\[\pi(x) \leq \frac{4}{7}x,\]  \hspace{1cm} (24)

which is similar to (21), and is valid for all \(x \geq 6\).

This is a simple exercise, so (22) can be reduced to

\[2 \leq y \leq x \leq 5.\]  \hspace{1cm} (25)

For these cases, even a verification by hand can be done. This finishes the proof of (17).

Remark 2. The constants 2/3 and 8/7 in Theorems 1 and 2 are the best possible. In a forthcoming paper some other inequalities of a new type will be presented.

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