Abstract—In this paper, we study the stabilization of a class of state interdependent Markov jump linear systems (MJLS) with partial information. First, we formulate a framework for the interdependent multiple MJLS to capture the interactions between various entities in the system, where the modes of the system cannot be observed directly. Instead, a signal which contains information of the modes can be obtained. Then, depending on the scope of the available system state information (global or local), we design centralized and distributed controllers, respectively, that can stochastically stabilize the overall state interdependent MJLS. In addition, the sufficient stabilization conditions for the system under both types of information structure are derived. Finally, we provide a numerical example to illustrate the effectiveness of the designed controller.

Index Terms—State interdependent, Markov jump linear systems, distributed stabilization.

I. INTRODUCTION

Dynamic systems subject to random abrupt changes in their structures and parameters can be modeled by stochastic jump systems. Particularly, when the random jump process is described by a Markovian process with given transition rates, then the system is categorized into the class of Markov jump system (MJS). Extensive research and investigations have been done on the stability analysis and (optimal) control design of Markov jump linear system (MJLS) [1]–[5]. Two common features in the adopted system model in these literatures are: (i) the state transition rate matrix is time-invariant, i.e., the transition rate matrix is constant; (ii) the Markov parameters of the transition matrix can be accessed.

However, in real cases, the transition rate matrix of a system can be related to the system state. For example, the failure probability of a wind turbine is related to its used time, level of wear, stress and stiffness on the blades, etc. [6], [7]. Another instance can be found in the financial stock market where the trend of the market, up or down, is influenced by the investment state of investors. Thus, the general Markov jump system models considered in [1], [8] are not directly applicable to these real applications. Moreover, the modes of the system cannot be accessed, such as robot navigation problems, machine maintenance, and planning under uncertainty [9]–[11]. In such cases, the modes can only be inferred from the emitted distorted signals.

To address these problems, [12] has modeled the system as a state-dependent Markov jump linear system with partial information, in which the transition rate matrix is time-varying due to the evolution of the dynamical system and the controller only has access to the signals providing partial information of the system modes rather than the modes directly. Note that in all above literature, their focused model contains a single Markov jump system.

With the emerging of advanced information and communication technologies (ICTs), the real-world systems are becoming more complex. One main characteristic of these modern control systems is that they are interdependent rather than isolated which forms the networked control systems or system-of-systems [13]–[15]. Hence, the state/condition of one system will have an impact on the operation of other systems that are depending on it. An illustrative example comes from the cascading failures among various entities in the homogeneous and heterogeneous networks [16]–[18].

To capture these interdependent features in the network, the traditional single Markov jump system is not sufficient. Therefore, to better understand the interdependencies between different systems and also design controllers for the complex system, we establish a multiple state interdependent MJS framework in this paper.

With the interdependent MJLS model, specifically, we first derive its stability criterion and design stochastic stabilization controllers when regarding the multiple Markov jump systems as an integrated system. In addition, in order to preserve the distributed nature of various jump systems, we design the distributed stabilization controllers for each individual system. This distributed controller only requires knowledge of its own local system which reduces the complexity of control system implementation.

The main contribution of this paper are summarized as follows.

1) We establish a state interdependent Markov jump system-of-systems model to capture the interactions and couplings in the complex networks.

2) We derive a sufficient stabilization condition and design stochastic stable controllers for the integrated
Markov jump linear system-of-systems.

3) To reduce the complexity of control system, we design distributed stabilization controllers for each individual system which ensure the stability of the overall system-of-systems.

The rest of the paper is organized as follows: Section II describes the control system and introduces a general state interdependent Markov jump system framework. Section III studies a two interdependent system stabilization problem from an integrated system perspective. Distributed stabilization and system controller design are presented in Section IV. Simulation studies are given in Section V. Section VI concludes the paper.

II. GENERAL FRAMEWORK DESCRIPTION

In this section, we give a general description of the state interdependent Markov jump system model, and also present several useful lemmas which are key to the derivations in the following sections.

Consider K systems:

\[
\begin{align*}
\dot{x}_k(t) &= f_k(t, x_k, u_k, w_k; \theta_k), \\
x_k(t_0) &= x_{k0}, \quad k = 1, 2, \ldots, K,
\end{align*}
\]

where \(x_k(t) \in \mathbb{R}^{n_k}\), \(x_{k0}\) is a fixed (known) initial state of the physical plant at starting time \(t_0\), \(u_k(t) \in \mathbb{R}^{n_k}\) is the control input, \(w_k(t) \in \mathbb{R}^{n_k}\) is the disturbance, and all these quantities lie at the physical and control layers of the entire system. \(\theta_k(t)\) is a Markov jump process with right-continuous sample paths, with initial distribution \(\pi_0\), and with rate matrix \(\eta_k = \{\eta^k_{ij}, i, j \in \mathcal{K}\}\) where \(\mathcal{K} = \{1, 2, \ldots, |\mathcal{K}|\}\) is the state space; \(\eta^k_{ij} \in \mathbb{R}_+\) are the transition rates such that for \(i \neq j, \eta^k_{ij} \geq 0\), and \(\eta^k_i = -\sum_{j \neq i} \eta^k_{ij}\) for \(i \in \mathcal{K}\). For convenience, we define \(\mathcal{X} := \{1, 2, \ldots, |\mathcal{K}|\}\).

Markov jump systems (MJSs) are interdependent. Specifically, the transition matrix is dependent on the state of the other MJSs. Without loss of generality, we consider the interdependency in a chain structure, i.e., \(\eta^k = F_k(x_k(t_1))\), for \(k = 1, 2, \ldots, K - 1\), and \(\eta^K = F_k(x_1(t))\).

Next, we present two lemmas which are useful in obtaining the results in this paper.

Lemma 1 ([19]). Let \(Y\) be a symmetric matrix and \(H, E\) be given matrices with appropriate dimensions and \(F\) satisfying \(F^TF \leq I\). Then, for any \(\kappa > 0\), we have

\[
HFE + E^TF^TH \leq \kappa HH^T + \frac{1}{\kappa} F^TF.
\]

Lemma 2 (Schur Complement [20]). Given matrices \(\Omega_1, \Omega_2,\) and \(\Omega_3\), where \(\Omega_1\) is symmetric, and \(\Omega_2 = \Omega_2^T > 0\), then,

\[
\Omega_1 - \Omega_2 \Omega_3^{-1} \Omega_2^T > 0 \iff \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_2 & \Omega_3 \end{bmatrix} > 0.
\]

III. STOCHASTIC STABILITY ANALYSIS AND CONTROL OF THE INTEGRATED SYSTEMS

In this section, we analyze the stability of the integrated Markov jump system and derive the state feedback control law that stabilizes the overall system.

First, we give the definition of stochastic stability of a system as follows.

**Definition 1.** The equilibrium point 0 of system (1) is stochastically stable if for arbitrary \(x_k(t_0) \in \mathbb{R}^{n_k}\) and \(\theta_k(t_0) \in \mathcal{K}_k\), \(\forall k \in \mathcal{K}\),

\[
\mathbb{E} \left[ \int_{t_0}^{\infty} |x(t)|^2 dt \right] < \infty,
\]

where \(x(t) := (x_1(t); x_2(t); \ldots; x_K(t))\) is the state vector of the overall system.

In the rest of this paper, we focus on a system which includes two interdependent MJSs (see Fig. 1) for simplicity. Consider two Markov jump linear systems (system 1 and system 2) as follows:

\[
\begin{align*}
\dot{x}_1 &= A_{11}^{\theta_1} x_1 + B_{11}^{\theta_1} u_1 + D_{11}^{\theta_1} w_1, \\
\dot{x}_2 &= A_{22}^{\theta_2} x_2 + B_{22}^{\theta_2} u_2 + D_{22}^{\theta_2} w_2,
\end{align*}
\]

where \(A_{11}^{\theta_1}, B_{11}^{\theta_1}, D_{11}^{\theta_1}, A_{22}^{\theta_2}, B_{22}^{\theta_2}, D_{22}^{\theta_2}\) are system matrices of appropriate dimensions whose entries are continuous functions of time \(t\); \(x_1\) and \(x_2\) are system states; \(u_1\) and \(u_2\) are control inputs; and \(w_1\) and \(w_2\) are system disturbances. Note that the system disturbances \(w_1\) and \(w_2\) satisfy

\[
\int_{t_0}^{\infty} w_1^T w_1 dt < \infty, \quad \text{and} \quad \int_{t_0}^{\infty} w_2^T w_2 dt < \infty.
\]

Based on the state interdependent structure of two Markov jump linear systems, we have

\[
\begin{align*}
\Pr[\theta_1(t + \Delta) = j | \theta_1(t) = i, x_2(t) \in \mathcal{C}_2]\ &= \\
&= \begin{cases}
\lambda_{ij}^{\Delta} + o(\Delta) & \text{if } i \neq j, \\
1 + \lambda_j^{\Delta} + o(\Delta) & \text{otherwise},
\end{cases} (2)
\end{align*}
\]

and

\[
\begin{align*}
\Pr[\theta_2(t + \Delta) = i | \theta_2(t) = i, x_1(t) \in \mathcal{C}_1]\ &= \\
&= \begin{cases}
\mu_i^{\Delta} + o(\Delta) & \text{if } i \neq j, \\
1 + \mu_i^{\Delta} + o(\Delta) & \text{otherwise},
\end{cases} (3)
\end{align*}
\]

where \(\mathcal{C}_1, \mathcal{C}_2\) are nonempty and disjoint sets, and \(\cup_{m=1}^{M} \mathcal{C}_m\) expand the space containing all the possible states of \(x_1(t)\). Similar definitions apply to \(\mathcal{C}_2, \mathcal{C}_3\). For \(x_1(t) \in \mathcal{C}_1\) and \(x_2(t) \in \mathcal{C}_2\), where \(n \in \mathcal{N} = \{1, 2, \ldots, |\mathcal{N}|\}\), the transition rates for the Markov jump process \(\theta_1\) and \(\theta_2\) for the individual system 1 and system 2 are denoted by \(\lambda_{ij}, i,j \in \mathcal{X}_1\) and \(\mu_i, i \in \mathcal{X}_2\), respectively.

**Remark:** The number of modes of two Markov jump systems are equal to \(|\mathcal{X}_1|\) and \(|\mathcal{X}_2|\), respectively. In addition, we define \(\mathcal{F}_1 := \{1, \ldots, |\mathcal{X}_1|\}\) and \(\mathcal{F}_2 := \{1, \ldots, |\mathcal{X}_2|\}\). Define \(\hat{\mathcal{S}} = \mathcal{F}_1 \times \mathcal{F}_2\).

The modes \(\hat{\theta}(t) := (\theta(t), \hat{\theta}(t))\) cannot be observed directly by the agent, instead, a signal \(\hat{\theta}(t) := (\hat{\theta}(t), \hat{\theta}(t))\) in \(\hat{\mathcal{F}} = \hat{\mathcal{F}}_1 \times \hat{\mathcal{F}}_2\) is emitted. \(\hat{\mathcal{F}}_1 = \{1, 2, \ldots, |\mathcal{F}_1|\}\) and \(\hat{\mathcal{F}}_2 = \{1, 2, \ldots, |\mathcal{F}_2|\}\) are the sets in which \(\hat{\theta}(t)\) and \(\hat{\theta}(t)\) belong to, respectively. The observation probability is given by the following

\[
\Pr[ \hat{\theta}(t) = \hat{i} | \theta(t) = i, x(t) \in \mathcal{C}_1 \cap \mathcal{C}_2] = \alpha_{ii}^{m,n}.
\]
Specifically, we assume that $\mathscr{T} = \mathscr{F}$. That is, the number of possible observations is the same as the number of possible states. For each pair of $(m, n)$, we define the following

$$[\beta_{ij}^{m,n}]_{i \in \mathscr{I}, j \in \mathscr{I}} = \left( [\alpha_{ij}^{m,n}]_{i \in \mathscr{I}, j \in \mathscr{I}} \right)^{-1}. $$

That is, $\beta_{ij}^{m,n}$ is the $(i, j)$-th entry of the inverse of the observation formed by $\alpha_{ij}^{m,n}$. Note that the existence of the inverse is guaranteed by the fact that $[\alpha_{ij}^{m,n}]_{i \in \mathscr{I}, j \in \mathscr{I}}$ is a probability matrix.

As $\theta(t)$ cannot be observed directly, the control inputs can only be designed based on $\hat{\theta}(t)$ and $x(t)$. The control inputs are designed to be of the following state-feedback linear form

$$u_k(t) = G_k(\hat{\theta}(t)) x_k(t), \quad k = 1, 2. $$

That is, the control gain is dependent on the observation $\hat{\theta}(t)$. The integrated system can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} A_1^0 + B_1^0 \tilde{G}_1^0 & 0 \\ 0 & A_2^0 + B_2^0 \tilde{G}_2^0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} D_1^0 \\ 0 \\ D_2^0 \end{bmatrix} [w_1 \\ w_2]. \tag{4} $$

Define

$$\bar{A}^\theta = \begin{bmatrix} A_1^0 & 0 \\ 0 & A_2^0 \end{bmatrix}, \quad \bar{B}^\theta = \begin{bmatrix} B_1^0 & 0 \\ 0 & B_2^0 \end{bmatrix}, $$

and

$$\bar{G}^\theta = \begin{bmatrix} G_1^0 & 0 \\ 0 & G_2^0 \end{bmatrix}, \quad \bar{D}^\theta = \begin{bmatrix} D_1^0 \\ 0 \\ D_2^0 \end{bmatrix}. $$

Hence, in a more compact way, we have

$$\dot{x} = A^\theta x + D^\theta w, $$

where

$$A^\theta = \bar{A} + \bar{B} \bar{G}^\theta. $$

For the ease of notation, we let let $i = (i_1, i_2)$, and $j = (j_1, j_2)$. In the rest of the paper, we use $i$ and $j$ to denote general indices of $\theta(t)$ and $\hat{\theta}(t)$, respectively.

Before deriving the stochastic stability criterion of the integrated system, we present the definitions of Dynkin’s formula and infinitesimetal generator.

**Definition 2.** Let a random process $(x(t), \theta(t))$ be a Markov process, and its stopping times are denoted by $\zeta_0, \zeta_1, \ldots$ at step 0, 1, ..., respectively. For Lyapunov function $V(x(t), \theta(t))$, the Dynkin’s formula is

$$E[V(x(t), \theta(t))|x(t_0), \theta(t_0)] = \int_0^{\tau} E\left[ \int_{\tau_0}^{\tau_0 + \tau} \mathcal{L}V(x(u), \theta(u))du | x(t \wedge \tau), \theta(t \wedge \tau) \right] \, dt. \tag{5} $$

where $\tau_0 = 0$, $i = 0, 1, \ldots, l^*$, $l^* < \infty$, $\tau_l = \infty$, and $\mathcal{L}V(x(t), \hat{\theta}(t))$ is the infinitesimal generator given by

$$\mathcal{L}V(x(t), \hat{\theta}(t)) = \lim_{\Delta \to \Delta} \frac{1}{\Delta} \left\{ E[V(x(t + \Delta), \hat{\theta}(t + \Delta))|x(t), \theta(t)] - V(x(t), \hat{\theta}(t)) \right\}. $$

Specifically, we choose the Lyapunov function to be the following quadratic form

$$V(x(t), \theta(t)) = x^T(t) \left( \sum_{\theta(t) \in \mathscr{J}} \alpha_{\theta(t)}(\theta(t)) P_{\theta(t)} \right) x(t), $$

where $P_{\theta(t)}$ is a symmetric positive definite matrix. Besides, define

$$Pr[\theta(t + \Delta) = j|\theta(t) = i, x(t) \in \mathscr{E}_1 \times \mathscr{E}_2] = \begin{cases} \gamma_{ij}^m + \alpha(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ij}^m + \alpha(\Delta) & \text{otherwise.} \end{cases} $$

**Lemma 3.** Without loss of generality, suppose $x(t) \in \mathscr{E}_1 \times \mathscr{E}_2$, and $\theta(t) = \theta_0(t) \in \mathscr{J} \times \mathscr{J}$, and $\hat{\theta}(t) = t = (i_1, i_2)$. The infinitesimal generator of $V$ is equal to

$$\mathcal{L}V(x(t), \theta(t)) = x^T(t) \left( P_i A_i^m + A_i^m x_i + \sum_{j \in \mathscr{J} \setminus \theta} \gamma_{ij}^m x_j \right) x(t) + 2x^T(t) P_i D^i w(t), $$

where

$$A_i^m = \sum_{j \in \mathscr{J}} \alpha_{ij}^m A^i_j. $$

**Proof.** See Appendix. A. \(\square\)

The following theorem gives a sufficient condition that ensures the stochastic stability of the overall Markov jump linear systems.

**Theorem 1.** The system can be stochastically stabilized if there exist positive definite matrices $X_i, Y_i$, for all $i \in \mathscr{J}$, $n_i = 1, \ldots, N$, and $m_i = 1, \ldots, M$, satisfying

$$X_i A_i^T + Y_i m x_i x_i + B_i Y_i m x_i + \sum_{j \in \mathscr{J} \setminus \theta} \gamma_{ij}^m x_j \leq 0. \tag{6} $$

By using Schur complement lemma, (6) is equivalent to

$$\begin{bmatrix} \frac{m_i}{m_i} & \frac{1}{m_i} \\ \frac{1}{m_i} & -\mathscr{T} \end{bmatrix} < 0. \tag{7} $$

where $e_i^{m,n} := X_i A_i^{m,n} + Y_i^{m,n} B_i + \sum_{j \in \mathscr{J} \setminus \theta} \gamma_{ij}^m X_j$, and $\mathscr{T} = \{X_i, \ldots, X_1, \ldots, X_n \}$. The control gain is given by $G_i^m = \sum_{j \in \mathscr{J}} \alpha_{ij}^m Y_i^{m,n} X_j^{-1}$. 

Proof. Based on Lemma 1, we obtain, for any $\kappa_i > 0$, $i \in \mathcal{S}$,

$$2x^T(t)P_iD^i w(t) \leq \frac{1}{\kappa_i}x^T(t)P_iD^iD^iT P_ix(t) + \kappa_i w^T(t)w(t).$$

Since $P_i$ is symmetric for all $i \in \mathcal{S}$, we obtain that

$$\mathcal{L}V(x(t), \theta(t)) = x^T(t)(P_i\tilde{A}_i^{m,n} + \tilde{A}_i^{m,n}P_i + \sum_{j \in \mathcal{S}} \gamma_{ij}^{m,n} P_j)x(t) + 2x^T(t)P_iD^i w(t)$$

$$\leq x^T(t)(P_i\tilde{A}_i^{m,n} + \tilde{A}_i^{m,n}P_i + \sum_{j \in \mathcal{S}} \gamma_{ij}^{m,n} P_j)x(t) + \frac{1}{\kappa_i}x^T(t)P_iD^iD^iT P_ix(t) + \kappa_i w^T(t)w(t)$$

$$= x^T(t)(P_i\tilde{A}_i^{m,n} + \tilde{A}_i^{m,n}P_i + \sum_{j \in \mathcal{S}} \gamma_{ij}^{m,n} P_j + \frac{1}{\kappa_i}P_iD^iD^iT P_i)x(t) + \kappa_i w^T(t)w(t)$$

$$= x^T(t)\Psi_i^{m,n}x(t) + \kappa_i w^T(t)w(t), \quad (8)$$

where $\Psi_i^{m,n} := P_i\tilde{A}_i^{m,n} + \tilde{A}_i^{m,n}P_i + \sum_{j \in \mathcal{S}} \gamma_{ij}^{m,n} P_j + \frac{(1/\kappa_i)P_iD^iD^iT P_i}.$

Then,

$$\mathcal{L}V(x(t), \theta(t)) - \kappa_i w^T(t)w(t) \leq x^T(t)\Psi_i^{m,n}x(t) \leq r_\sigma(\Psi_i^{m,n})x^T(t)x(t), \quad (9)$$

where $r_\sigma(\cdot)$ is the spectral radius of a matrix.

Choose $X_i = P_i^{-1}$. By pre- and post- multiplying $P_i\tilde{A}_i^{m,n} + \tilde{A}_i^{m,n}P_i + \sum_{j \in \mathcal{S}} \gamma_{ij}^{m,n} P_j + \frac{(1/\kappa_i)P_iD^iD^iT P_i}$ with $X_i$ and setting $\gamma_{i,j}^{m,n} = G_i \alpha_{i,j}^{m,n}X_i$, we observe that if (6) holds, then $\Psi_i^{m,n} < 0$. Here, $\gamma_{i,j}^{m,n} = \sum_{i \in \mathcal{S}} \alpha_{i,j}^{m,n} G_i$.

By using Dynkin’s formula (5), and for any $x(t_0) \in \mathcal{E}_1^{m_0} \times \mathcal{E}_2^{m_0}$, and let $\{(m_0, n_0), (m_1, n_1), (m_2, n_2), \ldots\}$ be the successive clusters of states visited. We obtain

$$\mathbb{E}[V(x(t), \theta(t))|x(t_0), \theta(t_0)] - V(x(t_0), \theta(t_0))$$

$$= \mathbb{E}\left[\int_{t_0}^{t_1} \mathcal{L}V(x(u), \theta(u))du|x(t_0), \theta(t_0)\right] + \mathbb{E}\left[\int_{t_1}^{t_2} \mathcal{L}V(x(u), \theta(u))du|x(t_0), \theta(t_0)\right]$$

$$+ \ldots + \mathbb{E}\left[\int_{t_{r-1}}^{t_r} \mathcal{L}V(x(u), \theta(u))du|x(t \wedge \tau_r), \theta(t \wedge \tau_r)\right]$$

Therefore, by (8) and (9),

$$\mathbb{E}[V(x(t), \theta(t))|x(t_0), \theta(t_0)] - V(x(t_0), \theta(t_0)) \leq \max\{r_\sigma(\Psi_i^{m,n})\}\mathbb{E}\left[\int_{t_0}^{t} x^T(u)x(u)du\right]$$

$$+ \kappa_i \int_{t_0}^{t} w^T(u)w(u)du.$$

Hence,

$$-\max\{r_\sigma(\Psi_i^{m,n})\}\mathbb{E}\left[\int_{t_0}^{t} x^T(u)x(u)du\right]$$

$$\leq -\max\{r_\sigma(\Psi_i^{m,n})\}\mathbb{E}\left[\int_{t_0}^{t} x^T(u)x(u)du\right]$$

$$+ \mathbb{E}[V(x(t), \theta(t))|x(t_0), \theta(t_0)]$$

$$\leq V(x(t_0), \theta(t_0)) + \kappa_i \int_{t_0}^{t} w^T(u)w(u)du,$$

which leads to

$$\mathbb{E}\left[\int_{t_0}^{t} x^T(u)x(u)du\right] \leq V(x(t_0), \theta(t_0)) + \kappa_i \int_{t_0}^{t} w^T(u)w(u)du,$$  \quad (10)

Letting $t \to \infty$ implies that $\mathbb{E}\left[\int_{t_0}^{\infty} x^T(u)x(u)du\right]$ is bounded by the right hand side of (10). Therefore, the system is stochastically stable if (6) holds. \quad \Box

In the case of full information, we immediately have the following proposition.

**Proposition 1.** The system can be stochastically stabilized if there exist positive definite matrices $X_i, Y_i$, satisfying

$$X_i\tilde{A}_i^{T} + Y_i^{T}B_i^{T} + \tilde{A}_iX_i + B_iY_i + \gamma_{i,j}^{m,n}X_i$$

$$+ X_i \left( \sum_{j \in \mathcal{S}/j} \gamma_{ij}^{m,n} X_j \right)^{-1} X_i + \frac{1}{\kappa_i}P_iD^iD^iT P_i < 0, \quad (11)$$

By using Schur complement lemma, (11) is equivalent to

$$\begin{bmatrix} \phi_i^{m,n} & \gamma_i^{m,n} \\ \gamma_i^{m,n} & \phi_i^{m,n} \end{bmatrix} < 0, \quad (12)$$

where

$$\phi_i^{m,n} := \left( \begin{array}{c} X_i \tilde{A}_i^{T} + Y_i^{T}B_i^{T} + \tilde{A}_iX_i + B_iY_i + \gamma_{i,j}^{m,n}X_i \\ \gamma_{i,j}^{m,n}X_i + (1/\kappa_i)P_iD^iD^iT P_i \end{array} \right),$$

and $\phi_i^{m,n}$ is a symmetric matrix.

**IV. DISTRIBUTED STABILIZATION OF THE INTERDEPENDENT MARKOV JUMP SYSTEMS**

In Section III we have studied the stability of the integrated interdependent Markov jump system which requires to know global system’s state information. However, due to the distributed structure and different types of the jump systems, obtaining the overall system information is not possible/convenient in most cases. Thus, to enable the distributed control of the interdependent Markov jump systems, we aim to investigate the criterion that leads to the stochastic stability of each individual system in this section.

Specifically, in this section, we assume that the transition probabilities satisfy the following

$$\text{Pr}[\theta_i(t + \Delta)|\theta_i(t), x(t)] = \text{Pr}[\theta_i(t + \Delta)|\theta_i(t), x_2(t)],$$

$$\text{Pr}[\theta_2(t + \Delta)|\theta_2(t), x(t)] = \text{Pr}[\theta_2(t + \Delta)|\theta_2(t), x_1(t)].$$
Besides, the observation probabilities are assumed to have the following properties

\[ \Pr[\hat{\theta}_1(t)|\theta_1(t),x(t)] = \Pr[\hat{\theta}_2(t)|\theta_2(t),x_1(t)] = \alpha_{1,i_1}^m, \]
\[ \Pr[\hat{\theta}_2(t)|\theta_2(t),x(t)] = \Pr[\hat{\theta}_2(t)|\theta_2(t),x_2(t)] = \alpha_{2,i_2}^m. \]

Similarly, define

\[ [\beta^m_{i_1,i_1}]_{i_1,i_1}\in \mathcal{J}_1 = \left( [\alpha^m_{i_1,i_1}]_{i_1,i_1}\in \mathcal{J}_1 \right)^{-1}, \]
\[ [\beta^m_{i_2,i_2}]_{i_2,i_2}\in \mathcal{J}_2 = \left( [\alpha^m_{i_2,i_2}]_{i_2,i_2}\in \mathcal{J}_2 \right)^{-1}. \]

Also, define

\[ \hat{A}_{i_1}^m = \sum_{i_1\in \mathcal{J}_1} \alpha_{i_1,i_1}^m \hat{A}_i^{i_1}, \quad \hat{A}_{i_2}^m = \sum_{i_2\in \mathcal{J}_2} \alpha_{i_2,i_2}^m \hat{A}_i^{i_2}, \]
where

\[ \hat{A}_i^{i_1} = A_i^{i_1} + B_i^{i_1}G_i^{i_1}, \quad \hat{A}_i^{i_2} = A_i^{i_2} + B_i^{i_2}G_i^{i_2}. \]

Before we proceed to the main result of this section, we give the following corollary, which presents how the individual stabilization and stabilizing control of each system leads to a stable integrated system.

**Corollary 1.** The stochastic stability of both subsystems ensures a stochastically stable system. In addition, for \( x_1 \in \mathbb{R}^{n_1}, m \in \{1,2,\ldots,M\} \), and \( x_2 \in \mathbb{R}^{n_2}, n \in \{1,2,\ldots,N\} \), the stabilizing control \( G_{1,i}^m \) and \( G_{2,i}^m \), for all \( i_1 \in \mathcal{J}_1, i_2 \in \mathcal{J}_2, i_1 \in \mathcal{J}_1, \) and \( i_2 \in \mathcal{J}_2, \) of individual system 1 and system 2 leads to a stable integrated interdependent MJLS.

**Proof.** Recall that the individual stabilizing controller of one subsystem is designed by considering all the possible states of the other system. In addition, the two subsystems satisfy

\[ \mathbb{E} \int_0^\infty |x_1(t)|^2 dt < \infty, \quad \text{and} \quad \mathbb{E} \int_0^\infty |x_2(t)|^2 dt < \infty. \]

Our goal is to show

\[ \mathbb{E} \int_0^\infty |x(t)|^2 dt < \infty, \]
where \( x := [x_1, x_2]^T \). First, let the Lyapunov function be

\[ V(x(t), \theta(t)) = x^T(t) \begin{bmatrix} P_{1,i} & 0 \\ 0 & P_{2,i} \end{bmatrix} x(t), \]
\[ = x_1^T(t)P_{1,i}x_1(t) + x_2^T(t)P_{2,i}x_2(t), \]
where \( P_{1,i} \in \mathbb{R}^{n_1 \times n_1} \) and \( P_{2,i} \in \mathbb{R}^{n_2 \times n_2} \) are real, symmetric and positive definite matrices. Then, the infinitesimal generator of \( V \) is equal to

\[ \mathcal{L}V(x(t), \theta(t)) = x_1^T(t) \begin{bmatrix} P_{1,i}\hat{A}_{1,i}^m & \lambda_{1,i}\hat{A}_{1,i}^m \end{bmatrix}, \]
\[ + x_2^T(t) \begin{bmatrix} \lambda_{2,i}\hat{A}_{2,i}^m & P_{2,i}\hat{A}_{2,i}^m \end{bmatrix}, \]

where

\[ \hat{A}_{1,i}^m = \sum_{i_1\in \mathcal{J}_1} \alpha_{i_1,i_1}^m A_{1,i_1}^{i_1}, \quad \hat{A}_{2,i}^m = \sum_{i_2\in \mathcal{J}_2} \alpha_{i_2,i_2}^m A_{2,i_2}^{i_2}. \]

The last equality is a direct application of Lemma 3.

Then, by defining \( \Psi_{i_1,i_2}^{m,n} := p_{1,i}^{I_1} \hat{A}_{1,i_1}^{i_1} + \hat{A}_{1,i_1}^{i_1} p_{1,i_1} + \mu_{1,i_1,i_1}^{m,n} p_{1,i_1} + \sum_{j_1=1}^{\mathbb{J}_1} \lambda_{1,i_1,j_1}^{m,n} p_{1,j_1} + \sum_{j_2=1}^{\mathbb{J}_2} \mu_{1,i_2,j_2}^{m,n} p_{1,j_2}, \)
and using the properties \( \Psi_{i_1,i_2}^{m,n} < 0 \) and \( \Phi_{i_2}^{m,n} < 0 \) in Theorem [13] we further obtain

\[ \mathbb{E}\left[ V(x(t), \theta(t))\right] - V(x(0), \theta(0)) \]
\[ \leq \max_{n\in \mathbb{N}, m\in \mathbb{M}} \left\{ r_{\sigma}(\Psi_{i_1,i_2}^{m,n}) \right\} \mathbb{E}\left[ \int_0^t x_1^T(u)x_1(u)du \right] \]
\[ + \max_{n\in \mathbb{N}, m\in \mathbb{M}} \left\{ r_{\sigma}(\Phi_{i_2}^{m,n}) \right\} \mathbb{E}\left[ \int_0^t x_2^T(u)x_2(u)du \right]. \]

Therefore,

\[ - \max_{n\in \mathbb{N}, m\in \mathbb{M}} \left\{ r_{\sigma}(\Psi_{i_1,i_2}^{m,n}) \right\} \mathbb{E}\left[ \int_0^t x_1^T(u)x_1(u)du \right] \]
\[ - \max_{n\in \mathbb{N}, m\in \mathbb{M}} \left\{ r_{\sigma}(\Phi_{i_2}^{m,n}) \right\} \mathbb{E}\left[ \int_0^t x_2^T(u)x_2(u)du \right] \]
\[ \leq V(x(0), \theta(0)) - \mathbb{E}\left[ V(x(t), \theta(t))\right] - V(x(0), \theta(0)) \]
\[ \leq \max_{n\in \mathbb{N}, m\in \mathbb{M}} \left\{ r_{\sigma}(\Psi_{i_1,i_2}^{m,n}) \right\} \mathbb{E}\left[ \int_0^t x_1^T(u)x_1(u)du \right] \]
\[ + \max_{n\in \mathbb{N}, m\in \mathbb{M}} \left\{ r_{\sigma}(\Phi_{i_2}^{m,n}) \right\} \mathbb{E}\left[ \int_0^t x_2^T(u)x_2(u)du \right] \]
\[ \leq \max_{n\in \mathbb{N}, m\in \mathbb{M}} \left\{ r_{\sigma}(\Psi_{i_1,i_2}^{m,n}) \right\} \mathbb{E}\left[ \int_0^t x_1^T(u)x_1(u)du \right] \]
\[ + \max_{n\in \mathbb{N}, m\in \mathbb{M}} \left\{ r_{\sigma}(\Phi_{i_2}^{m,n}) \right\} \mathbb{E}\left[ \int_0^t x_2^T(u)x_2(u)du \right] \]
\[ \cdot \mathbb{E}\left[ \int_0^t x_1^T(u)x_1(u)du \right]. \]

which yields

\[ V(x(t), \theta(t)) \geq - \max_{n\in \mathbb{N}, m\in \mathbb{M}} \left\{ r_{\sigma}(\Psi_{i_1,i_2}^{m,n}) \right\} \mathbb{E}\left[ \int_0^t x_1^T(u)x_1(u)du \right] \]
\[ - \max_{n\in \mathbb{N}, m\in \mathbb{M}} \left\{ r_{\sigma}(\Phi_{i_2}^{m,n}) \right\} \mathbb{E}\left[ \int_0^t x_2^T(u)x_2(u)du \right] \]
\[ = r_{\sigma}(\Psi_{i_1,i_2}^{m,n}) \mathbb{E}\left[ \int_0^t x_1^T(u)x_1(u)du \right] \]
\[ + r_{\sigma}(\Phi_{i_2}^{m,n}) \mathbb{E}\left[ \int_0^t x_2^T(u)x_2(u)du \right] \]
\[ := r_{\sigma}(\Psi_{i_1,i_2}^{m,n}) \mathbb{E}\left[ \int_0^t x_1^T(u)x_1(u)du \right] \]
\[ + r_{\sigma}(\Phi_{i_2}^{m,n}) \mathbb{E}\left[ \int_0^t x_2^T(u)x_2(u)du \right] \]

Thus, we obtain that

\[ \mathbb{E}\left[ \int_0^t x_1^T(u)x_1(u)du \right] \leq \frac{V(x(t), \theta(t))}{r_{\sigma}(\Psi_{i_1,i_2}^{m,n}) + r_{\sigma}(\Phi_{i_2}^{m,n})}. \]

This completes the proof.

The following theorem provides sufficient conditions for the with controllers designed in the distributed fashion.

**Theorem 2.** The integrated Markov jump linear system can be stochastically stabilized if there exist positive definite
matrices $X_i^{1} > 0$, $X_i^{2} > 0$, $Y_{i_1}^{m,n} > 0$, $Y_{i_2}^{m,n} > 0$, for all $i_1 \in \mathcal{I}_1$, $i_2 \in \mathcal{I}_2$, $n \in \mathcal{N}$, $m \in \mathcal{M}$, satisfying

$$X_i^{1}A_{i_1}^{1} + Y_{i_1}^{m,n}B_{1}^{i_1} + A_{i_1}^{1}X_i^{1} + B_{1}^{i_1}Y_{i_1}^{m,n} + \lambda_{i_1}^{n}X_i^{1} = 0,$$

$$X_i^{2}A_{i_2}^{2} + Y_{i_2}^{m,n}B_{2}^{i_2} + A_{i_2}^{2}X_i^{2} + B_{2}^{i_2}Y_{i_2}^{m,n} + \mu_{i_2}^{m}X_i^{2} = 0,$$

for all $i$ and $m$ which is equivalent to

$$[\begin{array}{c|c}
\varepsilon_{1,i_1}^{m,n} & \Lambda_{i_1}^{n} \\
\hline
\varepsilon_{1,i_2}^{m,n} & \Lambda_{i_2}^{n} \\
\end{array}] < 0, \quad \text{and} \quad [\begin{array}{c|c}
\varepsilon_{2,i_1}^{m,n} & \Lambda_{i_1}^{n} \\
\hline
\varepsilon_{2,i_2}^{m,n} & \Lambda_{i_2}^{n} \\
\end{array}] < 0,$$

where

$$\varepsilon_{1,i_1}^{m,n} := X_i^{1}A_{i_1}^{1} + Y_{i_1}^{m,n}B_{1}^{i_1} + A_{i_1}^{1}X_i^{1} + B_{1}^{i_1}Y_{i_1}^{m,n} + \lambda_{i_1}^{n}X_i^{1},$$

$$\varepsilon_{2,i_2}^{m,n} := X_i^{2}A_{i_2}^{2} + Y_{i_2}^{m,n}B_{2}^{i_2} + A_{i_2}^{2}X_i^{2} + B_{2}^{i_2}Y_{i_2}^{m,n} + \mu_{i_2}^{m}X_i^{2}.$$
Fig. 2: Stabilized state trajectory of System 1.

Fig. 3: Stabilized state trajectory of System 2.

Fig. 4: The sampled Markov chain of System 1.

Fig. 5: The sampled Markov chain of System 2.

Fig.2 and 3 show the state trajectories of the interdependent systems with the initial conditions $x_1(0) = [-6 5]^T$, and $x_2(0) = [2 -5.8]^T$. Fig.4 and 5 show the sampled Markov chains of the underlying parameters $\theta_1(t)$ and $\theta_2(t)$, and their observations $\hat{\theta}_1(t)$ and $\hat{\theta}_2(t)$, respectively.

VI. CONCLUSION

In this paper, we have studied the state interdependent multiple Markov jump linear systems. We have designed distributed stabilizing controllers for each MJLS with partial information which only require the system state information and observations. In addition, these designed controllers are able to stabilize the integrated Markov jump system. The distributed feature of these controllers reduce the information exchange and communication costs between different Markov jump systems.

APPENDIX I

PROOF OF LEMMA 3

Note that

$$x(t + \Delta) = (I + \Delta \cdot A^\theta(t) \cdot \hat{A}(t))x(t) + \Delta \cdot D^\theta(t) w(t). \tag{17}$$

By definition, the infinitesimal generator of $V$ is equal to

$$\mathcal{L}V(x(t), \theta(t)) = \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \mathbb{E} \left[ V(x(t + \Delta), \theta(t + \Delta)) | x(t), \theta(t) \right] - V(x(t), \theta(t)) \right\}$$

$$= \lim_{\Delta \to 0} \frac{1}{\Delta} \left\{ \sum_{i \in \bar{S}} \alpha_i^{m,n} x^T(t + \Delta) \left( P_i + \Delta \sum_{j \in \bar{S}} \gamma_i^{m,n} P_j \right) \cdot x(t + \Delta) - x^T(t) P_i x(t) \right\}$$

$$= x^T(t) \left( \sum_{i \in \bar{S}} \alpha_i \left( P_i \tilde{A}_i^{m,n} + \tilde{A}_i^{m,n} P_i + \sum_{j \in \bar{S}} \gamma_i^{m,n} P_j \right) x(t) \right)$$

$$+ 2x^T(t) P_i D^j w(t).$$

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