GERBES ON COMPLEX REDUCTIVE LIE GROUPS

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Abstract Let $K$ be a compact Lie group with complexification $G$. We construct by geometric methods a conjugation invariant gerbe on $G$; this then gives by restriction an invariant gerbe on $K$. Our construction works for any choice of level. When $K$ is simple and simply-connected, the level is just an integer as usual. For general $K$, the level is a bilinear form $b$ on a Cartan subalgebra where $b$ satisfies a quantization condition.

The idea of our construction is to first construct a gerbe on the Grothendieck manifold of pairs $(g, B)$ where $g \in G$ and $B$ is a Borel subgroup containing $g$. Then the main work is to descend that gerbe to $G$. There is an interesting torsion phenomenon in that the restriction of the gerbe to a semisimple orbit is not always trivial.

The paper starts with a discussion of gerbe data and of gerbes as geometric objects (sheaves of groupoids); the relation between the two approaches is presented. The Appendix on equivariant gerbes discusses both points of view.

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0. Introduction

Gerbes are higher analogs of bundles. The gerbes we consider here are the so-called DD gerbes (for Dixmier and Douady), and they are analogs of line bundles. General gerbes were introduced by Giraud [Gi] for the purpose of his theory of degree 2 non-commutative sheaf cohomology. The differential geometry of DD gerbes was developed in the book [Br1], where the analogs of connection and curvature were introduced. In particular, the curvature $\Omega$ of a gerbe is a 3-form, called the 3-curvature. This 3-form is quantized in the same way as the curvature of a line bundle is quantized. Gerbes are classified by their 3-curvature up to so-called flat gerbes. The methods of [Br1] are sheaf-theoretic. Gerbes can be obtained from an open covering $(V_\alpha)$ and from line bundles $(\Lambda_{\alpha\beta})$ on overlaps $V_{\alpha\beta}$ together with trivializations of $\Lambda_{\alpha\beta} \otimes \Lambda_{\beta\gamma} \otimes \Lambda_{\gamma\alpha}$ over $V_{\alpha\beta\gamma}$. This point of view was first introduced in [Br1] for the example of $S^3$ covered by two balls, in connection with the magnetic monopole. It was developed systematically by Chatterjee [Ch] and Hitchin [Hi]. The relation with (smooth and holomorphic) Deligne cohomology is discussed in [Br1].

One of the first instances of gerbes occurs on a simple simply-connected compact Lie group $K$. We have the well-known normalized Chern-Simons 3-form $\nu$ on $K$, and it is proved in [Br1] that there is a completely canonical gerbe $\mathcal{C}$ on

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$K$ with 3-curvature $\Omega = 2\pi i\nu$. We gave an explicit construction of this gerbe in [Br1] using the path-fibration $PK \to K$ and the central extension $\tilde{\Omega}K$ of the based loop group $\Omega K$. One can view this gerbe as a geometric realization of a class in smooth Deligne cohomology (or equivalently, the group of differential characters of Cheeger-Simons).

However for a number of reasons it is very interesting to have a geometric construction of the gerbe $\mathcal{C}$ on $K$ which only invokes finite-dimensional geometry. One reason is that one fundamental use of gerbes is to construct line bundles over the free loop space and in particular central extensions of loop groups [Br1] [Br-ML]. This is a good reason to want a description of the canonical gerbe $\mathcal{C}$ which does not use the central extension of $\Omega K$. Another motivation is the theory of group valued moment maps introduced by Alekseev, Malkin and Meinrenken [A-M-M], which is closely related to the canonical gerbe on a compact Lie group. This theory really belongs to finite-dimensional geometry. Indeed Alekseev, Meinrenken and Woodward show how to use group valued moment maps to study hamiltonian actions of loop groups [A-M-W]. There is a description of a gerbe over the complexification $G$ of $K$ which uses the Bruhat decomposition of $G$ into double cosets $BwB$ where $B$ is a Borel subgroup of $G$. Such a description is implicit in the paper [B-D] where certain cohomology classes are constructed in algebraic K-theory. According to [Br2], these classes yield holomorphic gerbes. The ensuing gerbe on $G$ is manifestly equivariant under left translation by $B$. The point of view here is that one gets stronger results by constructing a holomorphic gerbe on $G$; given a gerbe over $G$ one can simply restrict it to $K$ to get a gerbe on $K$.

Here we also use the complexification $G$ and construct a holomorphic gerbe over $G$. Our method is to use an auxiliary manifold $\tilde{G}$, called the Grothendieck manifold, which is the set of pairs $(g, B)$ where $B \subset G$ is a Borel subgroup and $g \in B$. We have a projection map $q : \tilde{G} \to G$ whose fiber over $g$ is the variety of Borel subgroups which contain $g$. Our method is as follows: first we start from some combinatorial data, which is an element $b \in X^*(T) \otimes X^*(T)$, where $T$ is a maximal torus in $G$ and $X^*(T)$ is the character group of $T$. From this combinatorial data we easily construct some gerbe $\tilde{\mathcal{C}}$ over $\tilde{G}$. The idea is that there are natural mappings from $\tilde{G}$ to the complete flag manifold $G/B$ and to $T$. Thus one can pull-back to $\tilde{G}$ characters of $T$ and equivariant line bundles over $G/B$. Then there is a natural cup-product construction which creates a gerbe $\tilde{\mathcal{C}}$ over $\tilde{G}$, which has an explicit description by gerbe data. Then we want to descend $\tilde{\mathcal{C}}$ to a gerbe on $G$. The idea is the following. First over the open set $G^{reg}$ of regular semisimple elements, the mapping $\tilde{G} \to G$ restricts to give a Galois covering whose Galois group is the Weyl group $W$. If we assume that $b$ is $W$-invariant, then we can construct a gerbe over $G^{reg}$ by descent. Now we need to extend this gerbe to all of $G$. This is done in several steps, assuming that $b(\tilde{\alpha}, \alpha)$ is even for all coroots $\alpha$; first we extend $\mathcal{C}$ across some divisors, by reducing the problem to the case of $SL(2, \mathbb{C})$ (§3 and 4). Then (§5) we have to handle codimension 2 subvarieties by cohomological methods. We also obtain a 0-connection on our gerbe $\mathcal{C}$. We don’t have explicit data for the gerbe $\mathcal{C}$, because we have to use abstract processes to descend it and extend it.

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Instead the gerbe is constructed as a sheaf of groupoids. The construction of the next differentiable structure (a so-called 1-connection) is still conjectural.

Our construction is in fact more general in that we consider an arbitrary complex reductive group $G$ (for instance $GL(n, \mathbb{C})$), and we construct holomorphic gerbes on $G$ which are equivariant under some auxiliary group $H$, which is only equal to $G$ up to center. The combinatorial data then becomes a tensor in $X^*(S) \otimes X^*(T)$, where $S$ is a maximal torus in $H$.

It is interesting to restrict the equivariant gerbe $C$ to the conjugation orbit of some $g \in G$. The obstruction to the triviality of the restricted gerbe is a central extension of the centralizer group. This is discussed in §7 for semisimple elements, where an explicit cocycle is given for the extension. Outside of $SL(n, \mathbb{C})$ the central extensions can be non-trivial, and this is an obstacle to constructing equivariant gerbe data for our gerbe in general, as Alekseev, Meinrenken and Woodward were able to do for $SL(n)$.

We have included in the first section an exposition of the DD gerbes and their differential geometry. First we expose the basics of the gerbe data in the sense of Chatterjee [Ch] and Hitchin [Hi]. This has the advantage of being very concrete and explicit. However for this paper we need to use the more general and abstract formalism of [Br1] based on sheaves of groupoids so we discuss that next, including the differential geometry, and we give a brief comparison between the two approaches. We intend to soon write a more detailed account of this in an expository paper.

The equivariant gerbes are discussed in an Appendix, starting from the gerbe data approach and then leading to the sheaves of groupoids.

The second section gives a discussion of the geometry of the manifold $\tilde{G}$ and of its very interesting cohomology. Particularly noteworthy is the fact that we have $H^*(G) = H^*(\tilde{G})^W$ for cohomology with rational coefficients.

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1. Gerbe data versus gerbes

First we discuss gerbe data in the sense of Chatterjee [Ch] and Hitchin [Hi]. Gerbe data on a manifold $M$ consists of the following:

1) an open covering $(V_\alpha)$ of $\tilde{M}$;

2) a family of line bundles $\Lambda_{\alpha\beta} \rightarrow V_{\alpha\beta}$ such that $\Lambda_{\alpha\alpha} = 1$ is the trivial line bundle

3) an isomorphism $u_{\alpha\beta} : \Lambda_{\alpha\beta}^{-1} \rightarrow \Lambda_{\beta\alpha}$ such that, viewed as a section of $\Lambda_{\alpha\beta} \otimes \Lambda_{\beta\alpha}$, $\phi_{\alpha\beta}$ is symmetric in $(\alpha, \beta)$

4) for each $\alpha, \beta, \gamma$, a non-vanishing section $\theta_{\alpha\beta\gamma}$ of the tensor product $\Lambda_{\alpha\beta} \otimes$
\[
\Lambda_{\beta\gamma} \otimes \Lambda_{\gamma\alpha} \text{ over } V_{\alpha\beta\gamma} \text{ satisfying the cocycle condition }
\]
\[
\theta_{\beta\gamma\delta} \otimes \theta_{\alpha\gamma\delta} \otimes \theta_{\alpha\beta\delta} \otimes \theta_{\alpha\beta\gamma} = 1
\]
over \(V_{\alpha\beta\gamma\delta}\). Note this makes sense as the quadruple tensor product is easily seen to be a section of a trivial line bundle. We require that \(u_{\alpha\beta} \otimes u_{\beta\gamma} \otimes u_{\gamma\alpha}\) transforms \(\theta_{\alpha\beta\gamma}\) into \(\theta_{\gamma\beta\alpha}^{-1}\).

It is of course important to know when two gerbe data are equivalent. The notion of equivalence is generated by two types of operations. The first operation is simply restriction to a finer open covering. The second type of operation is a kind of gauge transformation. There are actually two kinds of gauge transformations acting on gerbe data for a fixed open covering \((V_\alpha)\). Firstly, while keeping the line bundles \(\Lambda_{\alpha\beta}\) fixed, we can pick smooth functions \(h_{\alpha\beta} : V_{\alpha\beta} \to \mathbb{C}^*\) and change the line bundle \(\Lambda_{\alpha\beta}\) to \(h_{\alpha\gamma}^{-1}h_{\alpha\beta}\theta_{\alpha\beta\gamma}\). Secondly, we can introduce auxiliary line bundles \(E_\alpha\) over \(V_\alpha\) and change the line bundle \(\Lambda_{\alpha\beta}\) to \(\Lambda_{\alpha\beta}' = E_\alpha \otimes \Lambda_{\alpha\beta} \otimes E_\beta^{-1}\); then the triple tensor product \(\Lambda_{\alpha\beta}' \otimes \Lambda_{\beta\gamma}' \otimes \Lambda_{\gamma\alpha}'\) is canonically isomorphic to \(\Lambda_{\alpha\beta} \otimes \Lambda_{\beta\gamma} \otimes \Lambda_{\gamma\alpha}\), so we take the same \(\theta_{\alpha\beta\gamma}\), now viewed as a section of \(\Lambda_{\alpha\beta}' \otimes \Lambda_{\beta\gamma}' \otimes \Lambda_{\gamma\alpha}'\). Then two gerbe data are equivalent if they are related by a sequence of the various operations we just discussed.

It is useful to single out the criterion for a gerbe data \((\Lambda_{\alpha\beta}, \theta_{\alpha\beta\gamma})\) to be trivial. This means that we can find line bundles \(E_\alpha\) over \(V_\alpha\), and isomorphisms \(E_\alpha^{-1} \otimes E_\beta^{-1} \to \Lambda_{\alpha\beta}\) over \(V_{\alpha\beta}\) such that \(u_{\alpha\beta}\) is the obvious isomorphism and \(\theta_{\alpha\beta\gamma}\) corresponds to the section 1 of the trivial line bundle

\[
[E_\beta^{-1} \otimes E_\gamma] \otimes [E_\alpha^{-1} \otimes E_\gamma]^{-1} \otimes [E_\alpha^{-1} \otimes E_\beta].
\]

Very often, as in [Hi], we will not include \(u_{\alpha\beta}\) explicitly in the gerbe data.

However perhaps it is clearer to view the notion of equivalence of gerbe data in terms of tensor products of gerbe data, which we discuss next.

Given two gerbe data on \(M\), we can first of all refine the open coverings used to define them so that they become the same. Then we are in the situation of an open covering \((V_\alpha)\) and two gerbe data \((\Lambda_{\alpha\beta}, \theta_{\alpha\beta\gamma})\) and \((M_{\alpha\beta}, \rho_{\alpha\beta\gamma})\). Then their tensor product is the data \((\Lambda_{\alpha\beta} \otimes M_{\alpha\beta}, \theta_{\alpha\beta\gamma} \otimes \rho_{\alpha\beta\gamma})\). The inverse of the gerbe data \((\Lambda_{\alpha\beta}, \theta_{\alpha\beta\gamma})\) is simply \((\Lambda_{\alpha\beta}^{-1}, \theta_{\alpha\beta\gamma}^{-1})\).

Then two gerbe data \((\Lambda_{\alpha\beta}, \theta_{\alpha\beta\gamma})\) and \((M_{\alpha\beta}, \rho_{\alpha\beta\gamma})\) over the same open covering are equivalent if the data \((\Lambda_{\alpha\beta} \otimes M_{\alpha\beta}^{-1}, \theta_{\alpha\beta\gamma} \otimes \rho_{\alpha\beta\gamma}^{-1})\) is a trivial gerbe data.

One nice feature of gerbe data is that it is very easy to pull them back by an arbitrary smooth mapping \(f : Y \to M\): one simply introduces the open covering \(f^{-1}V_\alpha\) of \(Y\), together with the pull-back line bundles \(f^*\Lambda_{\alpha\beta}\) and the pull-backs of the \(\theta_{\alpha\beta\gamma}\).

Then there is the notion of a 0-connection. This is given by a family of connections \(D_{\alpha\beta}\) on the \(\Lambda_{\alpha\beta}\), such that \(u_{\alpha\beta}\) is compatible with the connections and \(\theta_{\alpha\beta\gamma}\) is horizontal with respect to the tensor product connection on \(\Lambda_{\alpha\beta} \otimes \Lambda_{\beta\gamma} \otimes \Lambda_{\gamma\alpha}\). Our two kinds of gauge transformations for gerbe data can be extended to the 0-connections. The first type of extended gauge transformation depends on functions \(h_{\alpha\beta} : V_{\alpha\beta} \to \mathbb{C}^*\) as before; besides transforming \(\theta_{\alpha\beta\gamma}\) into \(h_{\beta\gamma}h_{\alpha\gamma}^{-1}h_{\alpha\beta}\theta_{\alpha\beta\gamma}\),
the connection $D_{\alpha\beta}$ is transformed into $D_{\alpha\beta} + d\log(h_{\alpha\beta})$. The second type depends on auxiliary line bundles $E_\alpha$ equipped with connections $\nabla_\alpha$; then when $\Lambda_{\alpha\beta}$ is changed to $\Lambda'_{\alpha\beta} = E_\alpha \otimes \Lambda_{\alpha\beta} \otimes E_\beta^{-1}$, the new line bundle acquires the tensor product connection $\nabla_\alpha + D_{\alpha\beta} - \nabla_\beta$.

Next we have the notion of a 1-connection. This consists of 2-forms $F_\alpha$ on $V_\alpha$ such that

$$F_\beta - F_\alpha = \text{Curv}(D_{\alpha\beta}),$$

where $\text{Curv}(D_{\alpha\beta})$ denotes the curvature of $D_{\alpha\beta}$. Then only the second kind of gauge transformation acts on the 2-forms, transforming $F_\alpha$ into $F_\alpha + \text{Curv}(D_{\alpha\beta})$.

Given a 1-connection, we obtain a global closed 3-form $\Omega$ such that $\Omega/V_\alpha = dF_\alpha$. This 3-form is called the 3-curvature.

**Theorem 1.** [Br1] The 3-curvature $\Omega$ is quantized, i.e., the periods of $(2\pi i)^{-1}\Omega$ are integral. Conversely, every quantizable 3-form occurs as the 3-curvature of some gerbe data.

The cohomology class (when $H^3(M, \mathbb{Z})$ has no torsion) is the only obstruction to the triviality of the gerbe data.

Let us consider the question of classifying the equivalence classes of gerbe data over $M$. By picking a fine enough open covering, we may assume that all line bundles $\Lambda_{\alpha\beta}$ are trivial, in which case we have $\theta_{\alpha\beta\gamma} = g_{\alpha\beta\gamma}$ where $g_{\alpha\beta\gamma} : V_{\alpha\beta\gamma} \to \mathbb{C}^*$ is a Čech 2-cocycle with values in the sheaf $\mathbb{C}^*$ of smooth $\mathbb{C}^*$-valued functions. Then we have

**Proposition 1.** The equivalence classes of gerbe data over $M$ are classified by the Čech cohomology group $H^2(M, \mathbb{C}^*) = H^3(M, \mathbb{Z})$.

The point here is that when the $\Lambda_{\alpha\beta}$ are trivial, the only gauge transformation we can still use on gerbe data is $g_{\alpha\beta\gamma} \mapsto g_{\alpha\beta\gamma} h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta}$, which amounts to multiplying the Čech cocycle $(g_{\alpha\beta\gamma})$ by a coboundary.

We now consider gerbes with 0-connections: as $\Lambda_{\alpha\beta}$ is the trivial line bundle, the connection $D_{\alpha\beta}$ is simply a 1-form $A_{\alpha\beta}$. These 1-forms satisfy

$$A_{\alpha\beta} + A_{\beta\gamma} + A_{\gamma\alpha} = d\log(g_{\alpha\beta\gamma}).$$

This means that the pair $(g_{\alpha\beta\gamma}, A_{\alpha\beta})$ is a Čech cocycle with respect to the complex of sheaves $\mathbb{C}^* \xrightarrow{d\log} \mathbb{A}^1_M$, where $\mathbb{A}^1_M$ is the sheaf of 1-forms on $M$. The gauge transformation

$$(g_{\alpha\beta\gamma}, A_{\alpha\beta}) \mapsto (g_{\alpha\beta\gamma} h_{\beta\gamma} h_{\alpha\gamma}^{-1} h_{\alpha\beta}, A_{\alpha\beta} + d\log h_{\alpha\beta})$$

corresponds to a Čech coboundary. Thus gerbe data with 0-connection are parameterized up to equivalence by the Čech hypercohomology group $H^2(M, \mathbb{C}^* \xrightarrow{d\log} \mathbb{A}^1_M)$. This is the smooth Deligne cohomology group $H^3(M, \mathbb{Z}(2))$; see [Br1] and [D-F] for smooth Deligne cohomology groups and their relation to classical field theory.
Similarly given gerbe data with 0-connection and 1-connection, one obtains Čech cocycle \((g_{\alpha\beta\gamma}, A_{\alpha\beta}, F_{\alpha})\) which is unique up to a coboundary. Thus gerbe data equipped with 0-connection and 1-connection are classified by the Čech hypercohomology group \(H^2(M, \Omega^* \log A^1_M \to A^2_M)\). Again this is a smooth Deligne cohomology group.

As concrete and flexible as the formalism of gerbe data is, it does have some limitations. First of all, it may hard to manipulate in situations where it is not possible to work with a unique open covering. For instance, when a group action is present and we look for equivariant gerbes, it will not always be possible to work with invariant open sets (see Appendix). Secondly, it leaves open the question of a geometric interpretation of what lies behind the gerbe data. In other words, if we view gerbes as the higher analog of line bundles, what takes the place of the total space of a gerbe?

Although there is nothing like a manifold intrinsically associated to a gerbe, there are plenty of geometric objects which arise when we think of the gerbe data \((A_{\alpha\beta}, \phi_{\alpha\beta}, \theta_{\alpha\beta\gamma})\) as instructions to solve a geometric construction problem. The problem is to trivialize the gerbe, which means to find line bundles \(E_{\alpha}\) over \(V_{\alpha}\), and isomorphisms \(\psi_{\alpha\beta} : E_{\alpha}^{\infty-1} \otimes E_{\beta} \sim A_{\alpha\beta}\) over \(V_{\alpha\beta}\) such that \(\theta_{\alpha\beta\gamma}\) corresponds to the section 1 of the trivial line bundle \([E_{\alpha}^{\infty-1} \otimes E_{\gamma}] \otimes [E_{\alpha}^{\infty-1} \otimes E_{\gamma}]^{-1} \otimes [E_{\alpha}^{\infty-1} \otimes E_{\beta}].\)

The problem is that if the gerbe data is not trivial, there is no solution, so we are dealing with an insoluble construction problem. However, we can solve the problem at least locally, over some small enough open set \(U\) of \(M\). This means that the line bundles \(E_{\alpha}\) are defined over the intersection \(U \cap V_{\alpha}\) and the isomorphisms \(\phi_{\alpha\beta}\) are defined over \(U \cap V_{\alpha\beta}\).

Then what takes the place of a total space for a gerbe is the collection \(C_U\) of all such data \((E_{\alpha}, \psi_{\alpha\beta})\) as \(U\) ranges over all open sets in \(M\). In this collection, it would be a fatal mistake to identify all isomorphic objects. Indeed, it is essential to keep precise track of how objects of \(C_U\) restrict to smaller open sets and conversely how we can glue together objects of \(C_{U_i}\) into an object of \(C_U\), when \((U_i)\) is an open covering of \(U\). For this purpose, we need to take full account of the inner structure of \(C_U\): it is not just a collection of objects, but we have the notion of isomorphism of objects. Thus each \(C_U\) is a groupoid (i.e., a category where each morphisms is invertible), and we can view the collection of all \(C_U\) as a sheaf of groupoids over \(M\). This means that for \(V \subset U\) we have a restriction functor \(C_U \to C_V\) (usually denoted on objects of \(C_U\) by \(P \mapsto P_{|V}\)), and if \((U_\alpha)\) is an open covering of some open set \(U\), it amounts to the same to give an object of \(C_U\) or to give objects \(P_\alpha\) of \(C_{U_\alpha}\) together with transition isomorphisms

\[
\psi_{\alpha\beta} : [P_\beta]/U_{\alpha\beta} \sim [P_\alpha]/U_{\alpha\beta}
\]

with \(\psi_{\alpha\beta} = \psi_{\beta\alpha}^{-1}\), which satisfy the cocycle condition \(\psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha} = Id\).

We refer to the book [Br1] for a full discussion of sheaves of groupoids, which are essentially the stacks introduced by Grothendieck around 1961. A sheaf of
groupoids is called a Dixmier-Douady gerbe (or DD-gerbe) if it satisfies the following:

1) all objects of $\mathcal{C}_U$ are locally isomorphic;

2) for any $x \in M$, there is an open set $U$ containing $x$ such that $\mathcal{C}_U$ is non-empty;

3) for any object $P$ of $\mathcal{C}_U$, the automorphisms of $P$ are exactly the smooth functions $U \rightarrow \mathbb{C}^*$.

We have seen how gerbe data leads to such DD-gerbes. Hereafter, when we talk about gerbes, we always have in mind DD gerbes. Next we explain briefly how a gerbe leads to gerbe data. Given two objects $Q, R$ of some $\mathcal{C}_U$, the isomorphisms between $Q$ and $R$ are the sections of some $\mathbb{C}^*$-bundle $\text{Isom}(Q, R)$. This follows from axioms 1) and 3). Using axiom 2) we can find an open covering $(V_\alpha)$ and objects $P_\alpha \in \mathcal{C}_{V_\alpha}$. Then over $V_{\alpha\beta}$ we have the line bundle $\Lambda_{\alpha\beta}$ associated to the $\mathbb{C}^*$-bundle $\text{Isom}(P_\beta, P_\alpha)$. Then the composition law for isomorphisms immediately yields the isomorphism $u_{\alpha\beta}$ as well as a non-vanishing section $\theta_{\alpha\beta\gamma}$ of $\Lambda_{\alpha\beta} \otimes \Lambda_{\beta\gamma} \otimes \Lambda_{\gamma\alpha}$. The notion of gerbe gives a geometric explanation for the two types of gauge transformations on gerbe data. The first type, corresponding to functions $h_{\alpha\beta} : V_{\alpha\beta} \rightarrow \mathbb{C}^*$ amounts to keeping the objects $P_\alpha$ and the line bundles $\Lambda_{\alpha\beta}$ fixed but changing the isomorphisms $\phi_{\alpha\beta} : \Lambda_{\alpha\beta} \setminus (0 - \text{section}) \rightarrow \text{Isom}(P_\beta, P_\alpha)$ by multiplication by $h_{\alpha\beta}$. The second type, given by line bundles $E_\alpha$, consists in changing the objects $P_\alpha$ to $P_\alpha \otimes E_\alpha$ by twisting so that $\Lambda_{\alpha\beta} = \text{Isom}(P_\beta, P_\alpha)$ is changed to $\text{Isom}(E_\beta \otimes P_\beta, E_\alpha \otimes P_\alpha) = E_\alpha \otimes \Lambda_{\alpha\beta} \otimes E_{\beta}^{-1}$.

See [Br1, Chap. 4 and 5] for a detailed discussion of gerbes and degree 2 cohomology as well as many examples.

We have seen that gerbes and gerbe data (up to gauge equivalence) are essentially equivalent notions, and one can go back and forth between the two. It is interesting to note that geometric situations where gerbe occur usually lead directly to a gerbe, and only secondarily to gerbe data, after some auxiliary choices are made. This is the case for a principal $G$-bundle $Q \rightarrow M$, when we are given a central extension $\tilde{G}$ of $G$ by $\mathbb{C}^*$. Then the objects of $\mathcal{C}_U$ are the principal $\tilde{G}$-bundles over $U$ which lift $Q|_U$. The gerbe data appear when we pick liftings $\tilde{Q}_\alpha \rightarrow V_\alpha$; then we have $\tilde{Q}_{\beta} = \tilde{Q}_{\alpha} \otimes \Lambda_{\alpha\beta}$ for a unique line bundle $\Lambda_{\alpha\beta}$, and the construction automatically yields the section $\theta_{\alpha\beta\gamma}$. The case where $\tilde{G}$ is the group of invertible linear transformations in a Hilbert space, and $G$ is the projective linear group, is studied in depth in the article of Dixmier and Douady [D-D]. This explains the name: Dixmier-Douady gerbes. The book [Br1] presents another example, where $\tilde{G}$ is a generalized Heisenberg group in the sense of Weil.

The trivial gerbe is still interesting to think about, as it has lots of structure and is the local model for non-trivial gerbes. The objects of $\mathcal{C}_U$ are line bundles over $U$, and isomorphisms are line bundle isomorphisms. Equivalently, we can think of the objects of the trivial gerbe as being $\mathbb{C}^*$-bundles. As soon as a gerbe has an object over some open set, its restriction to that open set becomes trivial. More precisely, we have:
Proposition 2. (1) Given a DD gerbe $C$ over $M$, for each open set $U$ of $M$, and for any given object $P$ of $C_U$, we have an equivalence of categories between $C_U$ and the category of $\mathbb{C}^*$-bundles over $U$, given on objects by $Q \mapsto \text{Isom}(P, Q)$.

(2) If $C_U$ is not empty, its objects are classified up to isomorphism by the group $H^2(U, \mathbb{Z})$.

Thus for any line bundle $L \rightarrow U$ and for any object $P$ of $C$, we have a well-defined twist $P \otimes L$ of $P$ by $L$ such that $\text{Isom}(P, P \otimes L)$ is the $\mathbb{C}^*$-bundle associated to $L$. This can be constructed by explicit gluing as follows. Let $(g_{\alpha\beta})$ be some transition functions for $L$, with respect to some open covering $U_\alpha$ of $U$. Then $g_{\alpha\beta}$ gives an automorphism of $P$ over $U_{\alpha\beta}$, which satisfies the cocycle condition, hence can be used to construct an object of $C_U$.

The notion of equivalence of gerbes is also very natural from the point of view of sheaves of groupoids. Thus if $C$ and $D$ are two DD-gerbes on $M$, an equivalence $\phi : C \rightarrow D$ of gerbes is a family of functors $C_U \rightarrow D_U$ (for $U$ an arbitrary open set in $M$) which is compatible with restriction to smaller open sets. Given such $\phi$, each $\phi_U$ is an equivalence of categories by Proposition 2 (2).

We next discuss the tensor product $C \otimes D$ of two DD gerbes. The tensor product gerbe is such that if $P$ is an object of $C_U$ and $Q$ is an object of $D_U$, then there is a tensor product object $C \otimes D$ in $[C \otimes D]_U$. We want any automorphism $g$ of $C$ or of $D$ to induce an automorphism of $C \otimes D$, and these automorphisms should coincide. However, such tensor products of objects do not give enough objects to have the gluing properties of a sheaf of groupoids, so one needs to add more objects by gluing as follows. Let $(U_\alpha)$ be some open covering of $U$ over which we have objects $P_\alpha \in C_{U_\alpha}$ and $Q_\alpha \in D_{U_\alpha}$. Assume we have isomorphisms $\phi_{\alpha\beta} : [P_\beta]/U_{\alpha\beta} \rightarrow [P_\alpha]/U_{\alpha\beta}$ and $\psi_{\alpha\beta} : [Q_\beta]/U_{\alpha\beta} \rightarrow [Q_\alpha]/U_{\alpha\beta}$. We do not assume that the $\phi_{\alpha\beta}$ and the $\psi_{\alpha\beta}$ satisfy the cocycle condition; instead we require that the defects of the cocycle condition compensate each other, in other words if we have $\phi_{\alpha\beta} \otimes \phi_{\beta\gamma} \otimes \phi_{\gamma\alpha} = c_{\alpha\beta\gamma}$, we require that $\psi_{\alpha\beta} \otimes \psi_{\beta\gamma} \otimes \psi_{\gamma\alpha} = c_{\alpha\beta\gamma}^{-1}$. Then we postulate that the objects $P_\alpha \otimes Q_\alpha$ can be glued together, via the transition isomorphisms $\phi_{\alpha\beta} \otimes \psi_{\alpha\beta}$, to an object of $[C \otimes D]_U$. This gives enough objects in the category $[C \otimes D]_U$ so that it is a DD gerbe.

The inverse $C^{\otimes-1}$ of a gerbe $C$ has objects over $U$ given by the gerbe equivalences from $C$ to the trivial gerbe. Thus there is an equivalence of gerbes from $C^{\otimes-1} \otimes C$ to the trivial gerbe, given by evaluating an equivalence of gerbes on an object of $C$.

The operations of tensor product of DD gerbes and inverse of a gerbe correspond to the group structure on $H^2(M, \mathbb{C}^*)$ (cf. Proposition 1).

It is harder to define the pull-back of a gerbe than the pull-back of gerbe data. For a smooth mapping $f : N \rightarrow M$ and for a gerbe $C$ on $M$, the pull-back gerbe $f^*C$ is characterized by the fact that any object $P \in C_U$ yields an object of $(f^*C)_{f^{-1}(U)}$. The construction of $f^*C$ involves several steps, and we only give an outline. We can first use the procedure for pulling back a sheaf of groups. Thus for $V$ open in $N$, one can define the category $D_V$ to be the direct limit of the categories $C_U$, where $U$ runs over open neighborhoods of $f(V) \subseteq M$. There are two problems
with this construction. First the automorphisms of objects are wrong: they are functions from open subsets of \( M \) to \( \mathbb{C}^* \), rather than from open subsets of \( N \) to \( \mathbb{C}^* \). One remedies this problem by formally enlarging the isomorphisms in \( D \) so that the automorphism group of any \( P \in D_N \) is equal to the group of smooth functions \( V \to \mathbb{C}^* \). Secondly there are not enough objects: one needs to add objects obtained by gluing objects defined on an open covering \( (V_\alpha) \) of \( V \subset N \), together with gluing data using the enlarged notion of isomorphisms. After this second step, one gets a DD-gerbe on \( N \). We refer to [Br1, §5.2] for details of the construction of the pull-back.

The 0 and 1-connections take a different meaning in the context of gerbes viewed as sheaves of groupoids. A 0-connection becomes what we call a connective structure in [Br 1]: this associates to any object \( P \in \mathcal{C}_U \) a sheaf \( Co(P) \) on which the complex-valued 1-forms \( A^1 \) operate “locally simply-transitively”. In more details, for any open set \( U \), the set \( A^1(U) \) of 1-forms on \( U \) operates on the sections of \( Co(P) \) over \( U \). For any \( x \in M \), there exists \( U \ni x \) such that for any \( x \in V \subseteq U \), \( Co(P)(V) \) is isomorphic to \( A^1(V) \), where \( A^1(V) \) acts on itself by translations. One says that the sheaf \( Co(P) \) is a torsor under the sheaf \( A^1_M \) of 1-forms. The trivial example is the trivial 0-connection on the trivial gerbe. Then an object over \( U \) is a line bundle \( L \to U \), and \( Co(L)(U) \) is defined to be the set of connections on the line bundle \( L \).

For instance, if \( C \) is the trivial gerbe, then \( P \) is a line bundle and we can take \( Co(P) \) to be the set of connections on the line bundle. If we consider the gerbe attached to a principal \( G \)-bundle \( Q \to M \) and a central extension \( \tilde{G} \) of \( G \), then we get a connective structure once we fix a connection on \( Q \); then for an object \( \tilde{Q} \to U \) of \( \mathcal{C}_U \), the sheaf \( Co(P) \) is the sheaf of connections on \( \tilde{Q} \) which lift the connection on \( Q \).

Then a 1-connection becomes what is dubbed curving in [Br1]. This associates to a section \( \nabla \) of the sheaf \( Co(P) \) some 2-form \( K(\nabla) \) (called the curvature of \( \nabla \)), in such a way that
\[
K(\nabla + \alpha) = K(\nabla) + d\alpha
\]
for any 1-form \( \alpha \). Then the 3-curvature becomes the 3-form \( \Omega \) such that \( \Omega = dK(\nabla) \).

The notion of curving lead to the interesting notion of flat object of a gerbe. A flat object of \( C \) over \( U \) is an object \( P \in \mathcal{C}_U \) equipped with a section \( \nabla \) of the sheaf \( Co(P) \) such that \( K(\nabla) = 0 \). Such a flat object can exist only if the 3-curvature \( \Omega \) vanishes. If this is the case, flat objects exist locally. From the point of view of gerbe data, gerbe data \( (\Lambda_{\alpha\beta}, \phi_{\alpha\beta}, \theta_{\alpha\beta\gamma}) \) are flat if the line bundles \( \Lambda_{\alpha\beta} \) are flat and the \( \phi_{\alpha\beta} \) and \( \theta_{\alpha\beta\gamma} \) are horizontal. Flat gerbes are classified by the cohomology group \( H^2(M, \mathbb{C}^*) \). Indeed if we pick an open covering \( (V_\alpha) \) and a flat object \( (P_\alpha, \nabla_\alpha) \) over each \( V_\alpha \), as well as isomorphisms \( \psi_{\alpha\beta} : P_\beta \to P_\alpha \) which carry \( \nabla_\beta \) to \( \nabla_\alpha \), then \( \theta_{\alpha\beta\gamma} = \psi_{\alpha\beta}\psi_{\beta\gamma}\psi_{\gamma\alpha}^{-1} \) is an automorphism of \( P_\gamma \) which maps \( \nabla_\gamma \in Co(P_\gamma) \) to itself; this implies that \( \theta_{\alpha\beta\gamma} \) is (locally) constant, hence yields a Čech cohomology class in \( H^2(M, \mathbb{C}^*) \). If we fix the 3-curvature \( \Omega \) of a DD-gerbe, then the DD-gerbes with 0 and 1-connection whose 3-curvature is \( \Omega \) differ from one another by tensoring by a flat gerbe: hence they are parameterized by \( H^2(M, \mathbb{C}^*) \).
Curvings also occur in the geometric interpretation of the group-valued
moment maps of $[\text{A-M-M}]$ in terms of gerbes. This goes roughly as follows: we
suppose given on the compact simple Lie group $K$ a gerbe $\mathcal{C}$ together with 0 and
1-connections, whose 3-curvature is $2\pi i$ times the Chern-Simons form. Given a
smooth $K$-action on the manifold $M$ and an invariant 2-form $\omega$ on $M$, a
group-valued moment map is a $K$-equivariant mapping $\mu : M \to K$ (where $K$ acts on
itself by conjugation), together with an object $P$ of the pull-back gerbe $\mu^*\mathcal{C}$ and a
section $\nabla \in \text{Co}(P)$ such that $K(\nabla) = \omega$. There is an extra condition in $[\text{A-M-M},$
Def. 2.2], which has to do with the object $P$ and $\nabla$ being $K$-equivariant.

In this paper we will mostly deal with holomorphic gerbes over a complex
manifold $M$. This means, in terms of gerbe data, that the line bundles $\Lambda_{\alpha\beta}$, the
isomorphisms $\phi_{\alpha\beta}$ and the sections $\theta_{\alpha\beta\gamma}$ are holomorphic. In terms of gerbes, a
holomorphic gerbe is a sheaf of groupoids $\mathcal{C}$ such that the group of automorphisms
of an object $P$ of $\mathcal{C}_U$ is the group of holomorphic functions $U \to \mathbb{C}^*$. A holomor-
phic gerbe data yields a standard smooth data by simply forgetting that the data
$(\lambda_{\alpha\beta}, \phi_{\alpha\beta}, \theta_{\alpha\beta\gamma})$ are. From the point of view of sheaves of groupoids, a holomorphic
gerbe yields a smooth gerbe by first extending the isomorphisms between objects
to accept smooth functions to $\mathbb{C}^*$ as automorphisms, and then adding new objects
obtained by gluing with the help of the new automorphisms. Similarly we have
the notion of a holomorphic 0-connection, meaning that the connections $D_{\alpha\beta}$ on
the $\Lambda_{\alpha\beta}$ are holomorphic. On the side of sheaves of groupoids, one needs to attach
to each object $P$ of any $\mathcal{C}_U$ a torsor $\text{Co}_{\text{hol}}(P)$ under the sheaf $\Omega^1_M$ of holomorphic
1-forms. The one recovers the torsor under the sheaf $\text{A}^1_M$ of smooth 1-forms by
enlarging the torsor $\text{Co}_{\text{hol}}(P)$ accordingly. For 1-connections to be holomorphic,
one requires the corresponding 2-forms to be holomorphic.

The classification now involves Čech cocycles $g_{\alpha\beta\gamma}$ which are holomorphic. In
other words, they are Čech 2-cocycles with values in the sheaf $\mathcal{O}^*_M$ of invertible
holomorphic functions. Thus equivalence classes of holomorphic gerbes are classified
by the Čech cohomology group $H^2(M, \mathcal{O}^*)$. Similarly, holomorphic gerbes
with holomorphic 0-connection are classified by the Čech hypercohomology group
$H^2(M, \mathcal{O}^* \to \Omega^1_M)$, and holomorphic gerbes with holomorphic 0 and 1-connections
by the Čech hypercohomology group $H^2(M, \mathcal{O}^* \to \Omega^1_M \to \Omega^2_M)$. These are (holo-
morphic) Deligne cohomology groups; see $[\text{Br1}]$ $[\text{Br2}]$ for details.

2. The Grothendieck manifold

Let $G$ be a connected reductive algebraic complex Lie group. Let $X$ be the the
variety of Borel subgroups of $G$. Recall that the $G$-orbits in $X \times X$ are canonically
parameterized by a finite group $W$ called the Weyl group of $G$. Denote by $Y_w$
the orbit corresponding to $w \in W$. This a version of the Bruhat decomposition.
Thus given two Borel subgroups $B, B'$ of $G$, we say that $(B, B')$ are in position $w$
(notation $B \overset{w}{\rightarrow} B'$) if $(B, B') \in Y_w$. It follows that $W$ acts naturally on the set $X^T$
of Borel subgroups containing a given Cartan subgroup $T$. Given $B \supset T$ and given
$w \in W$, there exists a unique $B' \supset T$ such that $B \overset{w}{\rightarrow} B'$. We let $w \cdot B = B'$. This
action is simply-transitive. Now it easily follows that given \( T \subset B \) we can identify the geometric Weyl group \( W \) with the concrete Weyl group \( W_T = N(T)/T \). Indeed both groups act on \( X^T \) and the two actions commute and are simply-transitive. Thus the choice of a point \( B \in X^T \) yields an isomorphism between the two groups.

Now let \( \tilde{G} \subset G \times X \) be the closed subset consisting of pairs \((B,g)\) where \( g \in B \). Then the projection \( \tilde{G} \to X \) is a locally trivial fiber bundle with fiber over \( B \in X \) equal to \( B \). Hence \( \tilde{G} \) is an algebraic bundle of groups over \( X \) and is a smooth algebraic variety. We call \( \tilde{G} \) the Grothendieck manifold; it was introduced by Grothendieck for the purpose of resolving simultaneously the singularities of all closures of conjugation orbits in \( G \). On the other hand, the projection \( q : \tilde{G} \to G \) is a proper algebraic mapping. Let \( G^{reg} \subset G \) be the open set of regular semisimple elements.

**Lemma 1.** The restriction of \( q \) to \( G^{reg} \) is a finite Galois cover with Galois group \( W \).

**Proof.** There is a natural action of \( W \) on \( q^{-1}(G^{reg}) \): given \((g,B) \in q^{-1}(G^{reg})\), there is a unique Cartan subgroup \( T \) containing \( g \). The fiber of \( \tilde{G} \) over \( g \) thus identifies with \( X^T \), which admits a natural action of \( W \) as we discussed above. It is clear that this action is simply-transitive on the fibers of \( q \), and makes the restriction of \( q \) to \( G^{reg} \) into a Galois covering.

The action of \( W \) on the open set \( q^{-1}(G^{reg}) \) of \( \tilde{G} \) does not extend to a global action on \( \tilde{G} \). However there is a natural action of \( W \) on the cohomology of \( \tilde{G} \). This can be seen as follows. We have a projection map \( p : \tilde{G} \to X \times T \), which is a homotopy equivalence. Now there is a well-known action of \( W \) on \( X = G/B \), which depends on the choice of maximal compact subgroup \( K \) of \( G \). Indeed, introducing the maximal torus \( T_c = T \cap K \) of \( K \), we have \( X \simeq K/T_c \) and \( W_T \) is the Weyl group of \( T_c \), which acts on \( K/T_c \) by right multiplication.

Then we have the diagonal action of \( W \) on the product \( X \times T \). Thus we get an action of \( W \) on \( H^\ast(\tilde{G}) = H^\ast(X \times T) \). The restriction of \( p \) to \( q^{-1}(G^{reg}) \) is \( W \)-equivariant.

The submanifold \((K/T_c) \times T_c \) of \( \tilde{G} \) has half the real dimension of \( \tilde{G} \) and is a \( W \)-equivariant deformation retract of \( \tilde{G} \). We will call it the core of \( \tilde{G} \).

In the following Proposition, cohomology groups are taken with coefficients in a field of characteristic 0.

**Proposition 3.** The pull-back map induces an isomorphism of \( H^\ast(G) \) with \( H^\ast(\tilde{G})^W \).

**Proof.** There are several ways of proving this result. The most concrete is to first observe that since \( q \) is a proper map which is generically finite, the pull-back map \( H^\ast(G) \to H^\ast(\tilde{G}) \) is injective. Clearly its image is contained in the \( W \)-invariant subspace. Now it is easy to compute \( H^\ast(\tilde{G}) \) together with its \( W \)-action. For this
we use the core $X \times T_c$ of $\tilde{G}$. We have $H^*(X \times T_c) = H^*(K/T_c) \otimes H^*(T_c)$. We have natural actions of $W$ on $H^*(K/T_c)$ and on $H^*(T_c)$. The $W$-module $H^*(K/T_c)$ is isomorphic to the regular representation. Hence the $W$-invariant subspace of $H^*(X) \otimes H^*(T_c)$ identifies with $H^*(T_c)$. Since $H^*(G)$ and $H^*(T_c)$ have the same dimension, the statement follows.

**Corollary 2.** The pull-back map induces an isomorphism of the equivariant cohomology ring $H^*_G(G)$ with $H^*_G(\tilde{G})^W$.

**Proof.** This follows immediately from Proposition 1, using the spectral sequence from ordinary to equivariant cohomology.

We can compute $H^*_G(\tilde{G})^W$ as follows. We have

$$H^*_G(\tilde{G}) = H^*_K(\tilde{G}) = H^*_K(X) \otimes H^*(T_c) = R(T_c) \otimes H^*(T_c) = \mathbb{Q}[X^*(T_c)] \otimes \wedge^* X^*(T_c),$$

where $X^*(T_c) = Hom(T_c, S^1)$ is the character group of $T_c$ and $R(T_c)$ is the representation ring of $T_c$. We can identify $X^*(T_c)$ with the algebraic character group $X^*(T)$ of the algebraic torus $T$ and $R(T_c)$ with the algebraic representation ring $R(T)$. The tensor product algebra $\mathbb{Q}[X^*(T_c)] \otimes \wedge^* X^*(T_c)$ identifies with the ring $\Omega^*(R(T) \otimes \mathbb{Q})$ of Grothendieck differentials of the algebra $\mathbb{Q}[X^*(T)] = R(T) \otimes \mathbb{Q}$. The following result is proved in [B-Z].

**Lemma 2.** We have:

$$\Omega^*(R(T) \otimes \mathbb{Q})^W = \Omega^*(R(T)^W \otimes \mathbb{Q}).$$

Thus we obtain

**Proposition 4.** We have a natural isomorphism of graded algebras:

$$H^*_G(G) = \Omega^*(R(T)^W \otimes \mathbb{Q}).$$

The nice thing about the core $(K/T_c) \times T_c$ of $\tilde{G}$ is that its equivariant cohomology can easily be represented by explicit equivariant differential forms. The cohomology of $X$ is generated by Chern classes of line bundles. Then any character $\chi \in T$ extends uniquely to a character of $B$ hence defines a $G$-homogeneous holomorphic line bundle $L(\chi)$ over $G/B$. We can get a hermitian structure on $L(\chi)$ by picking a maximal compact subgroup $K$ of $G$. Then for any $\chi \in X(T)$ the line bundle $L(\chi)$ acquires a hermitian structure from the facts that $X \simeq K/T_c$ and the restriction of $\chi$ to $S \subset T_{sc}$ takes values in the circle group $T \subset \mathbb{C}^*$. 


Since the line bundle $L(\chi)$ over $X$ is homomorphic and hermitian, it has a canonical connection. Let $R_\chi$ be the curvature of this connection. This is a purely imaginary $K$-invariant closed 2-form, and its value at the base point can be computed as follows. The tangent space to $K/T_c$ is the quotient space $\mathfrak{k}/\mathfrak{t}_c$; for $\xi \in \mathfrak{k}$ we denote by $\bar{\xi}$ the corresponding tangent vector. Then for elements $\xi, \eta$ of $\mathfrak{k}$ we have:

$$R_\chi(\bar{\xi}, \bar{\eta}) = d\chi([\xi, \eta]).$$

The 2-form $\omega_\chi = \frac{1}{2\pi\sqrt{-1}} R_\chi$ has integer periods, so its cohomology class belongs to $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{C})$. We note that the curvature $R$ of the principal bundle $K \to K/T_c$ is the $\mathfrak{t}$-valued 2-form $R(\bar{\xi}, \bar{\eta}) = p_t \chi([\xi, \eta])$ where $p_t : k \to \mathfrak{t}$ is the projection; then we have $R_\chi = \langle d\chi, R \rangle$.

The 2-form $\omega_\chi$ can be extended to a closed $K$-equivariant 2-form on $X$, namely $(\omega_\chi, \mu_\chi)$ where $\mu_\chi : X = K/T_c \to \mathfrak{k}^*$ is the moment map

$$\mu_\chi(kT_c/T_c) = \frac{1}{2\pi\sqrt{-1}} Ad^*(k)d\chi.$$ 

Here $d\chi \in \mathfrak{t}_c^*$ is extended to an element of $\mathfrak{k}^*$ which vanishes on the orthogonal of $\mathfrak{t}_c$.

So $(\omega_\chi, \mu_\chi)$ yields an equivariant differential form, whose class in $H^2_K(X) = H^2_G(X)$ is the opposite of the equivariant first Chern class $c^1_K(L(\chi))$.

Now any character $u$ of $T$ gives rise to the equivariant 1-form $\frac{1}{2\pi\sqrt{-1}} du$ over $T$, which represents the cohomology class of $u$ in $H^1(T, \mathbb{Z}) = X(T)$.

Then, denoting by $A^*_K(X \times T) = [A^*(X \times T) \otimes S(\mathfrak{t})^*]^K$ the complex of $K$-equivariant differential forms, we have an algebra map

$$c : \mathbb{Z}[X(T)] \otimes \wedge^* X(T) \to A^*_K(X \times T)$$

such that

$$c(\chi \otimes 1) = p_1^*(\omega_\chi, \mu_\chi) , \quad c(1 \otimes \chi) = \frac{1}{2\pi\sqrt{-1}} p_2^* d\chi,$$

where $p_1 : X \times T \to T$ and $p_2 : X \times T \to T$ are the two projections.

It is interesting to compare the canonical equivariant 3-forms on $G$ and on $\tilde{G}$ attached to a $W$-invariant bilinear form $b$ on the cocharacter group $X_*(T) = X^*(T)^*$. Note that $X_*(T)$ is the group of algebraic homomorphisms $\lambda : \mathbb{C}^* \to T$. Then $b$ extends uniquely to an invariant complex-valued bilinear form $b$ on $\mathfrak{g}$.

On one hand, there is the well-known $G$-equivariant 3-form

$$(\nu, \alpha).$$
To describe it, introduce the left (resp right) invariant Maurer-Cartan forms $\theta$ (resp. $\bar{\theta}$) over $G$. Then $\nu$ is the Chern-Simons 3-form

$$\nu = \frac{1}{12} b(\theta, [\theta, \theta])$$

and $\alpha$ is the $g$-valued 1-form such that

$$\alpha = \frac{1}{2} (\theta + \bar{\theta}).$$

We will consider the restriction of $(\nu, \alpha)$ to a $K$-equivariant differential form on $G$. This $K$-equivariant differential form is closed.

On the other hand, we can view $b$ as an element of $X^*(T) \otimes X^*(T) \subset Z[X^*(T)] \otimes \wedge^* X^*(T)$ and then we have the $K$-equivariant differential form $c(b)$ on $X \times T$.

By pulling back both equivariant 3-forms to $\tilde{X}$ we can compare them:

**Proposition 5.** We have the equality of $K$-equivariant differential forms on $\tilde{X}$:

$$q^*(\nu, \alpha) - \frac{1}{2} p^* c(b) = d_K(p^* \beta)$$

where $\beta$ is the invariant 2-form on $X \times T$ defined as follows. $\beta(v, w)$ vanishes if $v$ or $w$ is tangent to the factor $T$. For $\xi, \eta \in k$ we have

$$\beta_1, t(\bar{\xi}, \bar{\eta}) = \frac{1}{2} \left( b(\xi, Ad^t(\eta)) - b(\eta, Ad^t(\xi)) \right).$$

This follows from Theorem 6.2 in [G-H-J-W] or Proposition 3.1 in [A-M-M].

We note a variant of these constructions. To get the full set of homogeneous line bundles, it is important to consider line bundles over $X$ which are equivariant under more general groups. Thus let $f : H \to G$ be an algebraic homomorphism of complex algebraic Lie groups. We will assume that $\text{Ker}(f)$ is central in $H$ and that $G = Z(G) \cdot \text{Im}(f)$, where $Z(G)$ is the center of $G$. Then we have $X \simeq H/f^{-1}(B)$. Thus any character $\chi$ of $f^{-1}(B)$ defines an equivariant line bundle $L(\chi)$ over $X$. Now $f^{-1}(B)$ has the same group of characters as its subgroup $S = f^{-1}(T)$. If we pick a maximal compact subgroup $L'$ of $f^{-1}(K)$, the product $L = L' \cdot Z(H)$ will also be maximal compact in $H$. As the restriction of $\chi$ to $S \cap L$ will take values in $T$, it follows that $L(\chi)$ has a hermitian structure. Hence we can extend the construction of the 2-forms $R_\chi$ and $\omega_\chi$ to this case. We thus obtain an algebra map

$$S^* (X^*(S)) \otimes \wedge^* X(T) \to A^*_L(X \times T).$$

A very interesting case is the following. Introduce the derived subgroup $G'$ of $G$ and its simply-connected covering $G_{sc}$, which is a complex semisimple algebraic group. We then have $X \simeq G_{sc}/B_{sc}$, where $B_{sc} = f^{-1}(B)$. Then $S$ be a maximal torus of $G_{sc}$ contained in $B_{sc}$. Then any character $\chi \in S$ extends uniquely to a
character of $B_{sc}$ hence defines a $G_{sc}$-homogeneous holomorphic line bundle $L(\chi)$ over $G_{sc}/B_{sc}$. The $G_{sc}$-equivariant cohomology of $\tilde{X}$ and of $X \times T$ is then equal to

$$S^*(X^*(T_{sc})) \otimes \wedge^* X(T).$$

This algebra maps to the algebra of $K_{sc}$-equivariant differential forms on $X \times T$.

3. Construction of gerbes over $\hat{G}$

As in section 1, let $f : H \to G$ be an algebraic homomorphism of complex algebraic Lie groups which is an isomorphism modulo centers. We will give a general construction for $H$-equivariant holomorphic gerbes on $\tilde{X}$. The basic data we will use is an element $b \in X^*(S) \otimes X^*(T)$, which we often view as a bilinear form

$$b : X_*(S) \otimes X_*(T) \to \mathbb{Z}.$$ 

We will associate to $b$ the corresponding linear map $\underline{b} : X_*(T) \to X^*(S)$. Thus for any $\lambda \in X_*(T)$ we have the $H$-equivariant line bundle $L(\underline{b}(\lambda))$ on $X$.

We will first give two constructions of gerbe data on $\tilde{X}$. Given a line bundle $L$ over a manifold $X$ and a smooth function $f : X \to \mathbb{C}^*$, there is a gerbe attached to $L$ and $f$, which was discussed and exploited in [B-M2]. This gerbe is given by a cup-product construction. We will discuss two methods to construct gerbe data corresponding to $L$ and $f$. In the first method, we pick an open covering $(V_\alpha)$ of $X$ over which we have a branch $\log f_\alpha$ of a logarithm of $f$. Then we have $\log_{\beta} f - \log_{\alpha} f = 2\pi i m_{\alpha\beta}$ for $m_{\alpha\beta} \in \mathbb{Z}$. Then we set $\Lambda_{\alpha\beta} = L^{\otimes m_{\alpha\beta}}$ with the obvious choice of $\phi_{\alpha\beta}$ (see §1). The trivialization $\theta_{\alpha\beta\gamma}$ is obvious from the cocycle property $m_{\beta\gamma} + m_{\alpha\gamma} + m_{\alpha\beta} = 0$.

The second method uses the Deligne line bundle $(g, f)$ attached to two invertible functions $f, g$ [De]. To describe $(g, f)$, it is enough to work in the universal case where $X = \mathbb{C}^* \times \mathbb{C}^*$, and $g = x, f = y$. Then we start with the trivial bundle over the universal covering space $\mathbb{C}^* \times \mathbb{C}$ of $\mathbb{C}^* \times \mathbb{C}^*$ with covering map $\pi := Id \times exp : \mathbb{C}^* \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*$. The group of deck transformations is $\mathbb{Z}$, where the generator $T$ acts on $(x, w)$ by $(x, w) \mapsto (x, w + 2\pi i)$. We let $T$ act on the trivial line bundle over $\mathbb{C}^* \times \mathbb{C}$ by multiplication by the invertible function $x^{-1}$. In other words, a section of $(f, g)$ over some open set $U$ is a function $h : \pi^{-1}(U) \to \mathbb{C}$ such that $h(x, w + 2\pi i) = x^{-1} h(x, w)$. The Deligne line bundle is equipped with a connection $\nabla$ which is characterized by the equation $\nabla(h) = dh + (2\pi i)^{-1} w x^{-1} h$. The curvature is

$$(2\pi i)^{-1} \frac{dx}{x} \wedge \frac{dy}{y}.$$ 

Note that by construction, a (local) logarithm $\log f$ determines a non-vanishing section of $(g, f)$, which we will denote by $(g, \log f)$. We have the rule $(g, \log f + 2\pi im) = x^m (g, \log f)$. We refer to [De] for more information on the Deligne line bundle, which Deligne invented in relation with algebraic K-theory and regulator.
maps. For our purposes (as in [Br1] [B-M1]), we view it as a very nice way to quantize the 2-torus or its complexification.

Returning to our data \((X, L, f)\), we give our second construction of gerbe data. We pick an open covering \((W_\alpha)\) over which \(L\) has a non-vanishing section \(s_\alpha\). Then we have \(s_\beta = s_\alpha g_{\alpha\beta}\) for the transition cocycle \(g_{\alpha\beta}\). We then define the line bundle \(\Lambda_{\alpha\beta} = (g_{\alpha\beta}, f)\) with obvious choices of \(\theta_{\alpha\beta\gamma}\). Each \(\Lambda_{\alpha\beta}\) is equipped with a connection, for which \(\theta_{\alpha\beta\gamma}\) is horizontal. Thus we have a 0-connection on the gerbe data. The equivalence of this gerbe with the previous one is easy to prove: it amounts to showing that the gerbe data given by the line bundles \((g_{\alpha\beta}, f) \otimes L \otimes s_{m_{\beta\alpha}}\) is trivial. This can be done by using the trivializing sections \((g_{\alpha\beta}, \log_\alpha f) \otimes s_{m_{\beta\alpha}}\).

We will denote by \((L, f)\) the gerbe we have constructed.

Now in our situation we have a tensor \(b\) in \(X^*(S) \otimes X(T)\). If we write \(b = \sum b_\chi \otimes \zeta_j\), then for each \(j\) we have the \(H\)-equivariant line bundle \(L(\chi_j)\) over \(X\) and the invertible function \(\zeta_j\) on \(T\), hence we have the tensor product gerbe \(\otimes_j (L(\chi_j), \zeta_j)\) on the product \(X \times T\). Furthermore this gerbe is \(H\)-equivariant (cf. the appendix for this notion). We can take its pull-back under the map \(p: \tilde{G} \to X \times T\) to get an \(H\)-equivariant gerbe with 0-connection on \(\tilde{G}\). We will denote this gerbe by \(\tilde{C} = \tilde{C}_b\).

We need however to make sure that this construction does not depend on the expression of \(b\) as a sum of tensors. This is true up to natural isomorphisms. For instance, using the second construction, the main point is that for a complex manifold \(Y\) and for an element \(\gamma \in Hol(Y)^* \otimes Hol(Y)^*\), there is a canonical Deligne line bundle \((\gamma)\) which is isomorphic to \(\otimes_j (g_j, f_j)\) for any expression of \(\gamma\) as \(\sum_j g_j \otimes f_j\).

**Proposition 6.** If the bilinear form \(b\) is \(W\)-invariant, then the restriction of the gerbe \(C_b\) to \(\tilde{G}^{\text{reg}}\) is \(W\)-equivariant.

**Proof.** For this purpose, it is best to use the description of \(\tilde{G}^{\text{reg}}\) as \(G \times T^{\text{reg}} = (G/T) \times T^{\text{reg}}\). Then the \(W\)-action on this contracted product can be described as a diagonal action. It is clear that with respect to the action of \(W\) on \(G/T\), we have a canonical isomorphism \(w^* L(\chi) \to L(w^{-1}\chi)\). On the other hand, the \(W\)-action on \(T\) transforms the characters of \(T\) according to the action of \(W\) on \(X(T)\). The result then follows since the construction of \(\tilde{C}\) does not depend on the expression of \(b\) as a sum of tensors. \(\square\)

Our gerbe \(C_b\) is such that the sum \(b_1 + b_2\) of two tensors leads to the tensor product gerbe \(C_{b_1} \otimes C_{b_2}\). The gerbe \(C_b\) has a simple behavior under the inverse map \(\iota: \tilde{G} \to \tilde{G}\) given by \(\iota(g, B) = (g^{-1}, B)\):

**Lemma 3.** The pull-back \(\iota^* C_b\) is equivalent to \(C_b^{-1} = C_{-b}\).
Proof. We have the commutative square

\[
\begin{array}{ccc}
G & \xrightarrow{\iota} & G \\
\downarrow & & \downarrow \\
T & \xrightarrow{\text{inv}} & T
\end{array}
\]

where \(\text{inv} : T \to T\) is the inverse map. On the other hand, \(\iota\) is compatible with the projection \(\tilde{G} \to X\). Then using the construction of \(C_b\) as \(\otimes_j L(\chi_j, \zeta_j)\) we see that \(\iota^* C_b = \otimes_j L(\chi_j, \text{inv}^* \zeta_j) = \otimes_j L(\chi_j, -\zeta_j) = C_b \otimes^{-1}_b\).

We conclude with two reasonably concrete description of our gerbe \(\tilde{C}\). The exponential map \(\exp : t \to T\) is a covering whose Galois group is \(X_*(T)\); for \(\lambda \in X_*(T)\) we denote by \(t_\lambda\) the corresponding deck transformation \(t \to t\) which is translation by \(2\pi i [d\lambda(1)]\). Denote by \(\kappa : \tilde{G} \to T\) and by \(p_2 : \tilde{G} \to X = G/B\) the projection maps. Let \(U \subset \tilde{G}\) be an open set and let \(\tilde{U} = U \times_T t \to U\) be the pull-back covering of \(U\). For \(\lambda \in X_*(T)\) we still have the translation automorphism \(t_\lambda\) of \(\tilde{U}\). Then an object of \(\tilde{C}(U)\) is a holomorphic line bundle \(L\) over \(\tilde{U}\) together with isomorphisms

\[
\beta_\lambda : t_\lambda^* L \to L \otimes L(-b(\lambda))
\]

for \(\lambda \in X_*(T)\), which satisfy the transitivity condition

\[
\beta_{\lambda_1 + \lambda_2} = (t_{\lambda_1}^* \beta_{\lambda_2}) \beta_{\lambda_1}.
\]

Here we still denoted by \(L(-b(\lambda))\) the pull-back to \(\tilde{U}\) of the corresponding line bundle over \(X\).

The second description is derived from the first, but it directly gives gerbe data for an open covering of \(\tilde{G}\). We cover \(T\) by open sets \(V_\alpha\) over which we have a section \(\log_\alpha : V_\alpha \to t\) of the exponential map \(\exp : t \to T\). Over \(V_{\alpha\beta}\) we have \(\log_\beta - \log_\alpha = 2\pi i \lambda_{\alpha\beta}\) for some \(\lambda_{\alpha\beta} \in X_*(T)\). Then we cover \(\tilde{G}\) by the open sets \(U_\alpha = \kappa^{-1}(V_\alpha)\). Over \(U_{\alpha\beta}\) we consider the line bundle \(\Lambda_{\alpha\beta} = p_2^* L(b(\lambda_{\alpha\beta}))\). With obvious choices for \(\theta_{\alpha\beta}\) this gives gerbe data.

This last description is also useful to give a 1-connection on the gerbe restricted to the core \((K/T_c) \times T_c\) of \(\tilde{G}\). Over the intersection of the core with \(U_\alpha\) we have the 2-form \(F_\alpha = b(p_2^\alpha \log_\alpha, p_1^\alpha R)\). The 3-curvature is the 3-form described in §2 as the equivariantly closed 3-form \(c(B)\). We conjecture that if we add \(2\beta\) to this 1-connection, the resulting 1-connection extends holomorphically to \(\tilde{G}\). This should be expected, as the corresponding 3-form is equal to \(2\pi i \nu\) by Proposition 5, and this extends to a holomorphic 3-form over \(G\).

4. Descent to \(G\): the case of \(SL(2)\).

We have constructed a gerbe \(\tilde{C}\) with 0-connection on \(\tilde{G}\), and we now want to descend it to \(G\). The first step is to construct the gerbe \(\tilde{C}\) over the open set \(G^{reg}\). There we can use Lemma 1 which gives us the Galois covering \(\tilde{G}^{reg} \to G^{reg}\) with
Galois group $W$. Since the gerbe $\tilde{C}$ on $\tilde{G}^\text{reg}$ is $W$-equivariant, it automatically descends to a gerbe on $G^\text{reg}$. This is much easier to describe for gerbes than for gerbe data. An object of the gerbe $\tilde{C}$ over an open set $U \subset G^\text{reg}$ will be an object $P$ of $\tilde{C}$ over $q^{-1}(U)$ which is $W$-equivariant, that is equipped with isomorphisms $\eta_w : w^* P \to P$ such that $\eta_{w_1 w_2} = w_1(\eta_{w_2})\eta_{w_1}$ (see the Appendix for a discussion of equivariant gerbes and equivariant objects). Then we have:

**Lemma 4.** This construction describes a gerbe $C$ on $G^\text{reg}$.

Next we want to extend this gerbe to $G$. First we examine the case where $G$ is a complex connected semisimple algebraic group of dimension 3, that is $G = SL(2, \mathbb{C})$ or $G = PGL(2, \mathbb{C})$. Let $Z(G)$ denote the center of $G$, which is a finite group of order 2 or 1. We have the following nested open subsets of $G$:

$$G^\text{reg} \subset V \subset G,$$

where $V = G \setminus Z(G)$. Over $V$, the mapping $\tilde{V} := q^{-1}(V) \to V$ is a ramified double covering with Galois group $W = \mathbb{Z}/2 = \{1, \tau\}$. So we are led to the question of descending the $W$-equivariant gerbe $\tilde{C}$ on $\tilde{V}$ to a gerbe on $V$. The complement $Y$ of $G^\text{reg}$ in $V$ is a smooth hypersurface: for $SL(2)$ it has 2 components $Y_1, Y_2$ corresponding to matrices all of whose eigenvalues are 1 resp. $-1$, and for $PGL(2)$ it is connected.

Now consider a possibly ramified double covering $p : \tilde{V} \to V$ with involution $\tau$, and a $\mathbb{Z}/2$-equivariant holomorphic gerbe $\tilde{C}$ over $\tilde{V}$. First of all, if the covering is not ramified, we construct a holomorphic gerbe $\tilde{C}$ over $V$ whose objects over $U \subset V$ are the $\mathbb{Z}/2$-equivariant objects of $\tilde{C}_{p^{-1}(U)}$ (see Appendix for equivariant objects); the morphisms are $\mathbb{Z}/2$-equivariant isomorphisms. If $\tilde{C}$ has an equivariant connective structure $P \mapsto \text{Co}(P)$, then for any equivariant object $P$ of $\tilde{C}_{p^{-1}(U)}$ the $\Omega^1_{p^{-1}(U)}$-torsor $\text{Co}(P)$ is $\mathbb{Z}/2$-equivariant, hence it descends to an $\Omega^1$-torsor over $U$. Thus $\tilde{C}$ acquires a connective structure.

Now consider the case where the covering $p : \tilde{V} \to V$ is ramified along a smooth hypersurface $Y$. In that case there is an obstruction to descending the equivariant gerbe $\tilde{C}$: this consists of an element of $\mathbb{Z}/2$ attached to each component $Y_j$ of $Y$. The description of this integer mod 2 is purely local along $Y_j$; thus we may assume the gerbe is trivial, so is the gerbe whose objects are line bundles. Then the action of $\tau$ on $\tilde{C}$ must given by $\tau(L) = L \otimes \Lambda$ for some line bundle $\Lambda$ on the complement of $Y_j$ (cf. Proposition 2). The constraint that $\tau^2$ should be (isomorphic to) the identity means that there is an isomorphism $\phi : \Lambda \otimes 1 \to \Lambda$. Now the obstruction arises when we look for a local section $s$ of $\Lambda$ around some point of $Y_j$ such that $\phi(s \otimes \tau^*(s)) = 1$. Indeed, first take any holomorphic section $\sigma$ of $\Lambda$ and consider the order $d$ of $f := \phi(\sigma \otimes \tau^*(\sigma))$ along $Y_j$. When we multiply $\sigma$ by a meromorphic function $g$ (with possible pole along $Y_j$), we change this order into $d + 2l$, where $l$ is the order of $g$ along $Y_j$. Thus the residue modulo 2 of $d$ is an intrinsic invariant, and is our obstruction. It can clearly be measured by restricting the whole geometric situation (including the gerbe) to some small $\tau$-invariant disc.
which meets \( Y_j \) transversally at the origin.

If the obstruction vanishes, then we have a holomorphic gerbe over \( V \), whose objects over \( U \) are again the \( \mathbb{Z}/2 \)-equivariant objects of \( \tilde{\mathcal{C}} \) over \( p^{-1}(U) \). Then an equivariant connective structure on \( \tilde{\mathcal{C}} \) will induce one on \( C \) just as in the unramified case.

When \( G = SL(2, \mathbb{C}) \), there are 2 divisors \( Y_1, Y_2 \) to consider. The open set \( G^{\text{reg}} \) is isomorphic to \( (G/T) \times T^{\text{reg}} \), where \( T = \mathbb{C}^* \) and \( T^{\text{reg}} = \mathbb{C}^* \setminus \{ \pm 1 \} \). The variable in \( T \) will be denoted by \( z \). \( W \) acts on \( T \) by \( z \mapsto z^{-1} \). The character group \( X^*(T) \) is generated by the identity character \( \chi \). \( b \) is determined by the integer \( m = b(\chi \otimes \chi) \). The line bundle \( \Lambda \) occurs because to write down the gerbe data, we need to fix a local branch of the logarithm of \( z \), and this will not in general be \( W \)-equivariant. Near \( z = 1 \) we can make a \( W \)-invariant choice of \( \log(z) \), such that \( |\log(z)| < \pi \), so the line bundle \( \Lambda \) is trivial. However near \( z = -1 \) (corresponding to the divisor \( Y_2 \)), if we choose the branch so that \( \log(-1) = \pi i \), then we have \( \log(z^{-1}) = -\log(z) + 2\pi i \). It follows that the line bundle \( \Lambda \) over a neighborhood of \( Y_2 \) in \( G^{\text{reg}} \) is the pull-back of the line bundle \( L(m\chi) \) over \( G/T \). We must then analyze the element of \( \mathbb{Z}/2 \) attached to this line bundle.

The surface \( G/T \) is an affine quadric, and as such it has two rulings which are exchanged by \( W \). We call the lines of these rulings lines of the first resp. second kind. The line bundle \( \Lambda \) is constant along the lines of the first kind, which are the fibers of the projection \( G/T \to G/B \). Along the lines of the second kind, the line bundle corresponds to the divisor \( m[p] \) for a point \( p \). Its transform \( \tau^*\Lambda \) is constant along the lines of the second kind, and its restriction to lines of the first kind is attached to some point. Now as we approach the divisor \( Y_2 \) along a transverse disc, we stay in a small neighborhood of some line of the second kind (this is because the projection to \( G/B \) of our point in \( G/T \) has a limit). It follows that the pull-back of \( \Lambda \) to this disc has a zero of order \( m \) at the origin, while \( \tau^*\Lambda \) pulls back to an equivariantly trivial bundle. Thus we conclude

\textbf{Lemma 5.} For \( G = SL(2, \mathbb{C}) \), the gerbe \( \tilde{\mathcal{C}} \) can descended from \( \tilde{V} \) to \( V = G \setminus \{ \pm 1 \} \) iff \( m = b(\chi \otimes \chi) \) is even.

The case of \( PGL(2, \mathbb{C}) \) is different as there is only one divisor and the corresponding obstruction vanishes automatically. This can be viewed in the following way: the integer \( m \) is even because \( \chi^2 \) is a character of \( T \).

Next we need to extend our gerbe from \( V \) to \( G \). We use the following Hartogs type theorem from [Br2].

\textbf{Lemma 6.} Let \( X \) be a complex manifold, and let \( Z \) be a closed complex subvariety of codimension \( \geq 3 \). Then the restriction map from holomorphic gerbes with 0-connection on \( X \) to those on \( X \setminus Z \) is an equivalence of categories.

So we obtain a gerbe \( \mathcal{C} \) on \( G \) equipped with a 0-connection.
5. Descent to $G$: the general case.

We will first construct an extension of the gerbe $\mathcal{C}$ over $G^{\text{reg}}$ to $G \setminus Z$, where $Z$ is a Zariski closed subset of codimension $\geq 2$.

The complement $Y$ of $G^{\text{reg}}$ in $G$ is a divisor, whose components $Y_\alpha$ are indexed by the roots $\alpha$ up to the $W$-action; so if the semisimple part of $G$ is simple, there is one component in the simply-laced case and two otherwise. A general point of $Y_\alpha$ is $G$-conjugate to an element $g$ with Jordan decomposition $g = su$, where

- $s$ is an element of $T$ such that $\exp(\pm \alpha)(s) = 1$ but no other root is trivial on $s$;
- $u$ is a general unipotent element of the centralizer $Z_G(s)$.

Note that $Z_G(s)$ is (up to a finite group) the product of a torus of dimension $r - 1$ with a 3-dimensional simple group $R$.

The $W$-covering $\tilde{G}^{\text{reg}} \to G^{\text{reg}}$ has an ordinary ramification of order 2 along each component $Y_\alpha$; the corresponding ramification subgroup is the $\mathbb{Z}/2$-subgroup generated by $s_\alpha$.

Now we have the $W$-equivariant gerbe $\tilde{\mathcal{C}}$ and we wish to descend it to an open subset of $G$ which at least meets each component $Y_\alpha$. We studied in §3 the obstruction to doing this: it is an element of $\mathbb{Z}/2$ attached to each component $Y_\alpha$. Now pick a general point $g = su$ of $Y_\alpha$ as above, so we have a closed embedding $R \hookrightarrow G$ where $j(h) = sh$, and the trace of the divisor $Y_\alpha$ on $R$ is a component $V$ of the complement of $R^{\text{reg}}$ in $R$. We can lift $j$ to an algebraic mapping $\tilde{j} : \tilde{R} \to \tilde{G}$ of Grothendieck manifolds as follows. Fix a Borel subgroup $B$ of $G$ containing $s$ and let $P_\alpha$ be the minimal parabolic subgroup containing $b$ corresponding to the root $\alpha$. Given a Borel subgroup $C$ in $R$, there is a unique Borel subgroup $B'$ of $G$ such that $C \subset B' \subset P_\alpha$. Then we set $\tilde{j}(h, C) = (sh, B')$. Now we wish to describe the pull-back of the gerbe $\tilde{\mathcal{C}}$ under this mapping $\tilde{j}$. For this purpose, we define a maximal torus $T_R$ of $T$ as the intersection of $T$ with $R$. Then we define the algebraic group $L = R \times_G H$ which maps to $R$, and we define a maximal torus $S_R$ of $L$ to fit in the cartesian diagram

\[
\begin{array}{ccc}
L = R \times_G H & \rightarrow & H \\
S_R & \rightarrow & S \\
& & \rightarrow & T
\end{array}
\]

Then the algebraic group homomorphism $L \to H$ satisfies our assumptions and we can view the restriction $j^*\tilde{\mathcal{C}}$ of $\tilde{\mathcal{C}}$ to $R$ as an $H_R$-equivariant gerbe.

We can then consider the restriction map $X^*(S) \otimes X^*(T) \to X^*(S_R) \otimes X^*(T_R)$. Denote by $b_R$ the image of $b \in X^*(S) \otimes X^*(T)$ under this map.

**Lemma 6.** The $L$-equivariant gerbe $j^*\tilde{\mathcal{C}}$ over $R$ is the gerbe associated to the element $b_R$ of $X^*(S_R) \otimes X^*(T_R)$.

It follows then that for each component $Y_\alpha$ of $Y$, the obstruction in $\mathbb{Z}/2$ can also be calculated in terms as the obstruction to extending the gerbe on $R^{\text{reg}}$ along
the divisor \( j^*Y_\alpha \). We know from \( \S 3 \) that this obstruction vanishes if \( b_R(\tilde{\alpha}, \tilde{\alpha}) \) is even. Now this is this equal to \( b(\tilde{\alpha}, \tilde{\alpha}) \). Hence if we make the assumption

\[
b(\tilde{\alpha}, \tilde{\alpha}) \text{ is even for any root } \alpha \quad (EV)
\]

then we can descend \( \mathcal{C} \) to a gerbe on a Zariski open set \( U \supset G^{reg} \) which meets each component of \( Y \), so that its complement \( Z \) has codimension \( \geq 2 \). Also there will be a holomorphic connective structure on \( \mathcal{C} \).

The arguments which will lead to Theorem 2 are quite technical, as they make heavy use of hypercohomology of a complex of sheaves with supports in a closed subset, so many readers may wish to skip ahead to the statement of Theorem 2.

We denote by \( V \subset G \) the set of elements \( g \) of \( G \) which are not regular, or in other words \( q^{-1}(g) \) is not finite. It is well-known that \( V \) has codimension 3 in \( G \). Then the obstruction to extending our gerbe with 0-connection from \( G \setminus Z \) to \( G \setminus V \) is an element of the hypercohomology group \( H^3_{Z \setminus V}(G \setminus V, \mathcal{O}^* \to \Omega^1) \) with supports in \( Z \setminus V \). Furthermore the non-uniqueness of the extension is controlled by the hypercohomology group \( H^2_{Z \setminus V}(G \setminus V, \mathcal{O}^* \to \Omega^1) \). We will only say here a few things about hypercohomology with supports, referring the reader to the book [K-S] for details. For any complex of sheaves \( F^* \) over a space \( X \), and for \( Y \) a closed subset of \( X \), the hypercohomology groups \( H^p_Y(X, F^*) \) with supports in \( Y \) sit in an exact sequence

\[
\cdots \to H^{p-1}(X, F^*) \to H^p_Y(X, F^*) \to H^p(X, F^*) \to H^p(X \setminus Y, F^*) \to \cdots
\]

They can be computed by the Čech method using open coverings \( (V_\alpha) \) of \( X \setminus Y \) such that \( U_\alpha \subseteq V_\alpha \) and all cohomology groups \( H^q(V_\alpha_1 \cdots \alpha_t, F^p) \) and \( H^q(U_\alpha_1 \cdots \alpha_t, F^p) \) vanish for \( q > 0 \). Then we can construct the Čech double complexes \( C^*(V, F^*) \) and \( C^*(U, F^*) \) and we have a natural restriction mapping from the first double complex to the second, which allows to construct a triple complex, whose total cohomology is the hypercohomology with supports. For the complex of sheaves \( \mathcal{O}^* \to \Omega^1 \), we may pick the \( V_\alpha \) and \( U_\alpha \) to be small open discs.

We have an exact sequence of complexes of sheaves

\[
0 \to \mathcal{O} \to \mathcal{O} \to \Omega^1 \to \mathcal{O}^* \to \Omega^1 \to 0
\]

Thus for cohomology with supports we have the exact sequence

\[
0 \to C \to H^3_{Z \setminus V}(G \setminus V, \mathcal{O}^* \to \Omega^1) \to H^4_{Z \setminus V}(G \setminus V, \mathcal{O}^* \to \Omega^1)
\]

where \( C \) is a complex vector space, namely a hypercohomology group with coefficients in \( \mathcal{O} \to \Omega^1 \). Now the group \( H^4_{Z \setminus V}(G \setminus V, \mathcal{O}^* \to \Omega^1) \) is the free abelian group generated by the cohomology classes of those components \( Z_j \) of \( Z \) which have codimension 2 in \( G \). Taking the inverse images \( \tilde{V}, \tilde{Z} \) of \( V \) and \( Z \) in \( \tilde{G} \), we have a similar exact sequence

\[
0 \to \tilde{C} \to H^3_{\tilde{Z} \setminus \tilde{V}}(\tilde{G} \setminus \tilde{V}, \mathcal{O}^* \to \Omega^1) \to H^4_{\tilde{Z} \setminus \tilde{V}}(\tilde{G} \setminus \tilde{V}, \mathcal{O}^* \to \Omega^1)
\]
Now the map $C \to \tilde{C}$ is injective because the mapping $\tilde{G} \setminus \tilde{V} \to G \setminus V$ is finite and $C$ is a vector space. The map $H^1_{Z \setminus V}(G \setminus V, \mathbb{Z}) \to H^1_{Z \setminus V}(\tilde{G} \setminus \tilde{V}, \mathbb{Z})$ is injective because the inverse image of a component $Z_j$ of $Z$ is a union of codimension 2 components of $\tilde{Z}$. It follows that the pull-back map $H^3_{Z \setminus V}(G \setminus V, \mathcal{O}^* \to \Omega^1) \to H^3_{Z \setminus V}(\tilde{G} \setminus \tilde{V}, \mathcal{O}^* \to \Omega^1)$ is injective. Thus the obstruction to extending our gerbe $C$ from $G \setminus Z$ to $G \setminus V$ vanishes, because the gerbe $\tilde{C}$ over $\tilde{G} \setminus \tilde{Z}$ extends to $\tilde{G} \setminus \tilde{V}$.

One shows similarly that the map $H^2_{Z \setminus V}(G \setminus V, \mathcal{O}^* \to \Omega^1) \to H^2_{Z \setminus V}(\tilde{G} \setminus \tilde{V}, \mathcal{O}^* \to \Omega^1)$ is injective (this is actually easier, as these cohomology groups are complex vector spaces). Thus the data of the gerbe $\tilde{C}$ over $\tilde{G} \setminus \tilde{V}$ leaves no amount of freedom for the extension of $C$ to $G \setminus V$.

Then using Lemma 6 we obtain

**Theorem 2.** Let $H$ be a connected reductive algebraic complex group, and let $f : H \to G$ be an algebraic homomorphism where $H$ is also algebraic reductive, $\text{Ker}(f)$ is central, and $G = Z(G) \cdot \text{Im}(f)$. For any $W$-invariant tensor $\phi \in X^*(S) \otimes X^*(T)$ such that the bilinear form $b : X_*(S) \otimes X_*(T) \to \mathbb{Z}$ satisfies the condition (EV), the corresponding $H \times W$-equivariant holomorphic gerbe $\tilde{C}$ with 0-connection over $\tilde{G}$ can be descended in an (essentially) unique way to an $H$-equivariant gerbe $C$ over $G$.

**Corollary 1.** If $\text{inv} : G \to G$ denotes the inverse map, then the pull-back gerbe $\text{inv}^* C$ is equivalent to $C^\otimes -1$.

**Proof.** This follows easily from Lemma 3 and the fact that $C$ is obtained from $\tilde{C}$ by quotienting by $W$ and then extending from an open set.

We denote by $C \boxtimes C$ (external tensor product) the gerbe over $G^2$ obtained by tensoring the gerbes $p_1^* C$ and $p_2^* C$. The following result is analogous to Proposition 3.2 in [A-M-M].

**Corollary 2.** Assume $H = G$ so that $C$ is $G$-equivariant. Let $D : G^2 \to G^2$ be the double map $D(a, b) = (ab, a^{-1}b^{-1})$. Then the pull-back of $C \otimes C$ under $D$ is a trivial gerbe.

**Proof.** The equivariance of $C$ implies that for $d_j : G^2 \to G$ the face maps of the Appendix, the tensor product gerbe $d_0^* C \boxtimes d_1^* C^\otimes -1$ is trivial. Now using Lemma 3 this means that the gerbe $d_0^* C \boxtimes d_1^* \text{inv}^* C$ is trivial. If we introduce the mapping $\delta : G^2 \to G^2$ such that $\delta(a, b) = (d_0(a, b), d_1(a, b)^{-1}) = (aba^{-1}, b^{-1})$, we see that the pull-back gerbe $\delta^*(C \boxtimes C)$ is trivial. Now we can write $D = \delta \phi$, where $\phi(a, b) = (ab^{-1}, ba)$, hence $D^*(C \otimes C)$ is trivial too.
Here is a description of $\mathcal{C}$: for an open subset $U$ of $G$, an object of $\mathcal{C}_U$ is an object $P$ of $\tilde{\mathcal{C}}_{f^{-1}(U)}$ together with the structure of a $W$-equivariant object on the restriction $Q$ of $P$ to $\tilde{\mathcal{C}}_{f^{-1}(U)\cap \mathcal{G}^reg}$. This means (see Appendix) that for any $w \in W$ we are given an isomorphism $\eta_w : Q \rightarrow w^*(Q)$ such that

1. the $\eta_w$ satisfy the cocycle condition $\eta_{w_1 w_2} = w_2^* (\eta_1) \eta_2$
2. each $\eta_w$ has no poles along components of $f^{-1}(U) \cap (G \setminus \tilde{\mathcal{C}}^reg)$. This means that if $w^k = 1$, then $(w^{k-1})^* (\eta_w)(w^{k-2})^* (\eta_w) \cdots \eta_w$ is an automorphism of $Q$ which extends holomorphically to an automorphism of $P$ over $U$.

Isomorphisms are isomorphisms of objects of $\tilde{\mathcal{C}}_{f^{-1}(U)}$ which are compatible with the extra data $(\eta_w)$. From this description of the gerbe $\tilde{\mathcal{C}}$ one can show easily that it is $H$-equivariant. The 0-connection on $\mathcal{C}$ can then be described in terms of an $\Omega^1$-torsor $Co(P, \eta_w)$ attached to the data $(P, \eta_w)$: the sections of this sheaf are the holomorphic sections $\nabla$ of $Co(P)$ over $U$ which are $W$-invariant in the sense that $\eta_w$ maps $\nabla$ to $w^* \nabla$. It would of course be very nice to find some explicit gerbe data for the gerbe $\mathcal{C}$.

6. Discussion of the combinatorial data.

The bilinear forms $b : X_*(S) \otimes X_*(T)$ satisfying the conditions of $W$-invariance and (EV) were introduced independently by Toledo in [To] and by the author and Deligne in [B-D].

For $G$ simply-connected and simple, and for $H = G$, the allowable $b \in X^*(T) \otimes X^*(T)$ are the integer multiples of the basic $b_0$ for which $b_0(\alpha, \alpha) = 2$ for a long root $\alpha$ (so that $\check{\alpha}$ is a short coroot). This bilinear form $b_0$ is introduced in [P-S] for the purpose of constructing central extension of loop groups.

For $G = SL(n, \mathbb{C})$, $T$ the group of diagonal matrices, $X^*(T)$ is the quotient of the free group on the diagonal entries $(t_1, \ldots, t_n)$ by the relation $t_1 + \cdots + t_n = 0$. The roots are the $\alpha_{ij} = t_i - t_j$ for $i \neq j$. Let $(e_1, \ldots, e_n)$ be the basis dual to $(t_1, \ldots, t_n)$. The dual group $X_*(T)$ is the subgroup of $\mathbb{Z}e_1 \cdots \mathbb{Z}e_n$ comprised of the linear combinations $\sum n_i e_i$ such that $\sum n_i = 0$. The coroots are the $\check{\alpha}_{ij} = e_i - e_j$. The basic element $b_0$ is $b_0 = \sum^n_{i=1} t_i^2$.

Now take $G_{ad} = PGL(n, \mathbb{C})$ to be the adjoint group of $SL(n, \mathbb{C})$, so that $T$ is replaced by the quotient group $T_{ad} = (\mathbb{C}^*)^n / \mathbb{C}^*_{\text{diag}}$, where $\mathbb{C}^*$ is embedded diagonally in $(\mathbb{C}^*)^n$. Then $X^*(T_{ad}) = Q$ is the coroot lattice generated by the $\check{\alpha}_{ij}$. If we take $H = G_{ad}$ then the allowable $b \in X^*(T_{ad}) \otimes X^*(T_{ad})$ are the integer multiples of $\sum_{i < j} \check{\alpha}_{ij}^2 = n \cdot b_0$. The same situation occurs if we take $H = G$ mapping to $G_{ad}$ in the obvious way. Now let us consider $H = G = GL(n, \mathbb{C})$, and take $T$ to be the group all diagonal matrices, so that $X^*(T) = \mathbb{Z}t_1 \oplus \cdots \oplus \mathbb{Z}t_n$. If we take $H = G$, then the allowable $b$ are of the form $b = l \sum^n_{i=1} t_i^2 + m(t_1 + \cdots + t_n)^2$, for $l, m \in \mathbb{Z}$. If now $H = SL(n, \mathbb{C})$, so that $X^*(S)$ is the quotient of $X^*(T)$ by the relation $t_1 + \cdots + t_n = 0$, then we are left with the integer multiples of $t_1^2 + \cdots + t_n^2$.

If we take $G$ to be of type $B_n$, then for the simply-connected group $G_{sc} = Spin(2n+1, \mathbb{C})$ the character group $X^*(T_{sc})$ is the group $1/2 \sum m_i t_i$ where $m_i \in \mathbb{Z}$ and $m_1 \equiv \cdots \equiv m_n \mod 2$. The simple coroots are $(e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n)$. 23
For \( H = G = \text{Spin}(2n+1, \mathbb{C}) \), the allowable \( b \) are the integer multiples of \( b_0 = \sum_{i=1}^{n} t_i^2 \). For the adjoint group \( G_{ad} = SO(2n+1, \mathbb{C}) \) and for \( H = G_{ad} \) or \( H = G_{sc} \), the allowable \( b \) are integer multiples of \( 2b_0 \). This corresponds to the well-known phenomenon that only the tensor square of the fundamental gerbe over \( \text{Spin}(2n+1) \) can be descended to \( SO(2n+1) \).

If we take \( G \) to be of type \( D_n \), then for \( G_{sc} = \text{Spin}(2n, \mathbb{C}) \) the character group \( X^*(T_{sc}) \) is the group of \( \frac{1}{2} \sum_{i=1}^{n} m_i t_i \), where \( m_1 \equiv m_2 \cdots \equiv m_n \) mod 2. The coroots are the \( e_i - e_j \). Thus for \( H = G = G_{sc} \), the allowable \( b \) are the integer multiples of \( b_0 = \sum_{i=1}^{n} t_i^2 \). Now take \( G = \text{SO}(2n, \mathbb{C}) \) so that \( X^*(T) = \sum_{i=1}^{n} \mathbb{Z} t_i \); then for \( H = G_{sc} \) or \( H = \text{Spin}(2n, \mathbb{C}) \) we find the allowable \( b \) are again the integer multiples of \( 2b_0 \). Now if we take \( G = G_{ad} = \text{SO}(2n, \mathbb{C})/\{\pm 1\} \), then \( X^*(T_{ad}) \) is the group of \( \sum m_i t_i \) such that \( m_1 + \cdots + m_n \) is even. Then for any choice of \( H \) (equal to \( G_{sc} \) divided by a central subgroup), we find that only integer multiples of \( 2b_0 \) is allowable.

7. Restricting the gerbe to conjugation orbits.

**Lemma 7.** For any regular semisimple element \( g \) of \( G \), the restriction of the \( H \)-equivariant gerbe \( C \) to the \( H \)-orbit \( O_g = H \cdot g \subset G \) is \( H \)-equivariantly trivial.

**Proof.** First of all we can lift \( O \) to an \( H \)-orbit \( \tilde{O} \) in \( \hat{G} \) which maps isomorphically to \( O \), namely the orbit of \((g, B)\) for any Borel subgroup of \( G \) containing \( g \). Then it is enough to show that the restriction of the gerbe \( \tilde{C} \) to \( \tilde{O} \) is \( H \)-equivariantly trivial. Using the description of the gerbe \( \tilde{C} \) given in §3, we see that any logarithm \( \xi \) of \( g \) gives such an equivariant trivialization. □

In general, the restriction of \( C \) to an \( H \)-orbit will not be equivariantly trivial. As shown in the Appendix, the obstruction to the triviality is a central extension of the centralizer group. We will describe this central extension in case \( g \) is semisimple. First we recall some well-known facts about central extensions of reductive complex groups. Let \( L \) be a reductive connected algebraic group over \( \mathbb{C} \). We want to describe in combinatorial terms the central extensions \( 1 \to \mathbb{C}^* \to \tilde{L} \to L \to 1 \) of complex algebraic groups. Let \( S \) be a maximal torus in \( L \). The inverse image \( \tilde{S} \) is a maximal torus in \( \tilde{L} \), and we have an extension of free abelian groups

\[
0 \to X^*(S) \to X^*(\tilde{S}) \to X^*(\mathbb{C}^*) = \mathbb{Z} \to 0.
\]

Hence the extension (EXT) is equivariant under the Weyl group of \( S \) in \( L \), which we will simply denote by \( W_L \).

**Lemma 8.** The central extension \( \tilde{L} \) of \( L \) is entirely determined by the extension (EXT) of \( W_L \)-modules.
Now the extension (EXT) gives rise to a degree 1-cocycle group cocycle \( w \mapsto c_w : W_L \to X^*(S) \) as follows: pick an element \( \chi \in X^*(\mathcal{S}) \) which restricts to \( 1 \in X^*(\mathbb{C}^*) \). Then \( c_w = w\chi - \chi \in X^*(S) \) is a 1-cocycle, and its cohomology class vanishes iff the extension (EXT) has a \( W_L \)-equivariant splitting.

Let us apply this to \( f : H \to G \) as in §2 and some semisimple element \( g \in G \). Denote by \( L \) the connected component of the centralizer of \( g \) in \( H \). Then pick a maximal torus \( T \) of \( G \) containing \( g \) and as usual let \( S = f^{-1}(T) \subset H \), which is a maximal torus of \( L \). The Weyl group \( W_L \) of \( S \) in \( L \) is the subgroup of \( W = W_H \) which centralizes \( g \). Pick some element \( \xi \) of \( t \) such that \( \exp(2\pi i \xi) = g \). Then for \( w \in W_L \), the difference \( w\xi - \xi \) has exponential 1, so \( d_w = w\xi - \xi \) belongs to \( X_*(T) \). This gives a 1-cocycle of \( W_{S,L} \) with values in \( X_*(T) \).

Now given \( b \in X^*(S) \otimes X^*(T) \) which is \( W \)-invariant and satisfies (EV), we have constructed an \( H \)-equivariant holomorphic gerbe over \( G \). Recall that \( b : X_*(T) \to X^*(S) \) is the corresponding linear map. Then according to the Appendix, for any \( g \) we have a central extension of the centralizer group \( Z_g(H) \). We can now identify this central extension:

**Proposition 6.** For \( g \in G \) semisimple, the central extension of the centralizer \( L \) of \( g \in H \) is described by the cohomology class of the group 1-cocycle

\[
w \in W_L \mapsto b(d_w) \in X^*(S).
\]

**Proof.** It follows from the Appendix that the central extension \( \tilde{L} \) of \( L \) has restriction to \( S \) given by the multiplicative \( \mathbb{C}^* \)-bundle \( \otimes_{i=1}^r (\chi_i, \zeta_i(g)) \) if \( b \in X^*(S) \otimes X^*(T) \) is equal to \( \sum \chi_i \otimes \zeta_i \). Here \( \zeta_i(g) \) is just a complex number and \( \chi_i \) is a holomorphic function \( S \to \mathbb{C}^* \). The expression \( (\chi_i, \zeta_i(g)) \) denotes a Deligne line bundle. A multiplicative trivialization of the Deligne line bundle above is obtained using the logarithm \( d\zeta_i(\xi) \) of \( \zeta_i(g) \). This gives a trivialization \( s = \otimes_i (\chi_i, d\zeta_i(\xi)) \) of the central extension of \( S \), which however is not \( W_L \)-equivariant. For \( w \in W_L \), we can measure the defect of equivariance as a group homomorphism \( c_w : S \to \mathbb{C}^* \) given by \( c_w(h) = ws(w^{-1}h) - s(h) \). Now we have

\[
w \cdot s = \otimes_i (w\chi_i, d\zeta_i(\xi)) = \otimes_i (\chi_i, dw^{-1}\zeta_i(\xi)) = \otimes_i (\chi_i, d\zeta_i(w\xi))
\]

using the \( W \)-invariance of \( b \). It follows that

\[
c_w(h) = \prod_i \chi_i(h)^{d\zeta_i}(dw) = [b(d_w)](h).
\]

It appears likely that this central extension is trivial for \( SL(n, \mathbb{C}) \) or \( GL(n, \mathbb{C}) \) but often non-trivial for other groups. In any case, its cohomology class has finite order (bounded by the order of \( W_L \)), so if we replace the gerbe by some tensor
power the obstruction will vanish. This fits with the result in [G-H-J-W] and [A-M-M] that the class in \(H^3_c(K, \mathbb{R})\) restricts trivially to any orbit.

We give an example where the central extension is non-trivial.

Pick for instance \(H = G = \text{Spin}(7, \mathbb{C})\) and \(g = \exp(2\pi i \xi)\) where \(\xi = \frac{e_1 - e_2}{2}\). The centralizer \(L\) of \(g\) is infinitesimally isomorphic to \(SL(2, \mathbb{C})^3\). Then the subgroup \(W_L\) of the Weyl group is the group \((\mathbb{Z}/2)^3\) generated by the change of sign \((-1, -1, 1)\), the change of sign \((1, 1, -1)\) and by the permutation of 1 and 2. The 1-cocycle \(d_w\) of \(W_L\) with values in \(X_*(T)\) is \(d_w = w\xi - \xi\). We now take \(b = b_0 = t_1^2 + t_2^2 + t_3^2\); then \(b : X_*(T) \to X^*(T)\) is the inclusion which maps \(e_i\) to \(t_i\). Now we claim that \(b(d_w)\) is not the coboundary of some \(u \in X^*(T)\). Since there is no vector in \(X^*(T)\) fixed by the group \(W_L\), the only possible choice of \(u\) would be \(u = b(\xi) = \frac{t_1 - t_2}{2}\). However this is a vector in \(X^*(T) \otimes \mathbb{Q}\) which does not belong to \(X^*(T)\). Thus \(b(d_w)\) is not a coboundary and the corresponding central extension \(\tilde{L}\) of \(L\) is not trivial. Clearly by construction it has order 2, hence is induced by a central extension of \(L\) by \(\mathbb{Z}/2\), i.e., a connected double cover \(\tilde{L}\) of \(L\).

We can describe \(L\) more precisely as follows: \(W_L\) is generated by the reflections corresponding to the orthogonal roots \(t_1 - t_2, t_1 + t_2, t_3\). For the maximal torus \(S\) in \((SL(2, \mathbb{C}))^3, X_*(S)\) is spanned by the corresponding coroots \(e_1 - e_2, e_1 + e_2, 2e_3\). This subgroup is on index 2 in \(X_*(T)\), and the quotient subgroup is spanned by the half-sum \(1/2((e_1 - e_2) + (e_1 + e_2) + 2e_3) = e_1 + e_3\). It follows that \(L\) is isomorphic to the quotient of \((SL(2, \mathbb{C}))^3\) by the group \(\mathbb{Z}/2\) embedded diagonally in the center \((\mathbb{Z}/2)^3\), and thus \(L\) is the universal cover \((SL(2, \mathbb{C}))^3\).

**Appendix. Equivariant gerbes**

1) **Equivariant line bundles**

We will review well-known material on equivariant line bundles in a framework which will adapt to gerbes. Let \(G\) be a Lie group which acts smoothly on the smooth manifold \(M\). Let \(m : G \times M \to M\) denote the action and let \(p_2 : G \times M \to M\) be the second projection. First of all if we view a line bundle as given by transition cocycles \(g_{\alpha\beta}\) with respect to an open covering \(V_\alpha\), then the problem we encounter if that we can’t in general assume that each \(V_\alpha\) is \(G\)-invariant. In order to write down the equivariance data for the line bundle in Čech form, what we need to do is cover \(G \times M\) by the open sets \(Z_{\alpha\beta} = m^{-1}(G \times V_\alpha) \cap p^{-1}(G \times V_\beta)\). Then to make our line bundle \(G\)-equivariant we need to introduce a function \(\phi_{\alpha\beta} : Z_{\alpha\beta} \to \mathbb{C}^*\). The meaning of \(\phi_{\alpha\beta}\) is as follows: if \(s_\alpha\) is the chosen non-vanishing section of \(L\) over \(V_\alpha\), then if \(L\) is \(G\)-equivariant we have the section \((g, x) \mapsto [g^*s_\alpha](x)\) over \(m^{-1}(G \times V_\alpha)\), and \(\phi_{\alpha\beta}\) is this section divided by \(s_\beta(x)\). Then we see that \(\phi_{\alpha\beta}\) must satisfy the following requirement:

\[
\phi_{\gamma\beta} = [m^*g_{\gamma\alpha}]\phi_{\alpha\beta}, \quad \phi_{\alpha\delta} = [p_2^*[g_{\delta\beta}]^{-1}]\phi_{\alpha\beta}.
\]

over the relevant open sets in \(G \times M\).

Next a connection on \(L\) amounts to 1-forms \(A_\alpha\) over \(V_\alpha\) such that \(A_\beta - A_\alpha = d \log(g_{\alpha\beta})\). The connection is \(G\)-equivariant iff we have the equality of 1-forms on
$G \times M$ in the direction of $M$:

$$m^*A_\alpha - p_2^*A_\beta = d\log \phi_{\alpha\beta}.$$  

Of course, rather than using such data, it may be more convenient to use the geometric language of line bundles and their pull-backs. For this purpose, we introduce the simplicial manifold $G^\bullet \times M$, which is the family of manifolds $G^n \times M$, equipped with the following face maps $d_i : G^n \times M \to G^{n-1} \times M$:

$$d_0(g_1, \ldots, g_n, x) = (g_2, \ldots, g_n, g_1x)$$
$$d_i(g_1, \ldots, g_n, x) = (g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_n, x) \text{ for } 1 \leq i \leq n-1$$
$$d_n(g_1, \ldots, g_n, x) = (g_1, \ldots, g_{n-1}, x)$$

The significance of the simplicial manifold $G^\bullet \times M$ is that it is a geometric model for the Borel space $EG \times^G M$. Recall that the cohomology of the Borel space is the (Borel) equivariant cohomology $H^*_G(M)$.

Note that $d_0 : G \times M$ is equal to $m$, and $d_1 : G \times M \to M$ is equal to $p_2$.

Then an equivariant line bundle over $M$ is a line bundle $L$ over $M$, equipped with a non-vanishing section $\sigma$ of the line bundle $d_0^* L \otimes d_1^* L^{\otimes -1}$, which satisfies the cocycle condition $d_0^* \sigma \otimes d_1^* \sigma^{\otimes -1} \otimes d_2^* \sigma = 1$. This makes sense as $d_0^* \sigma \otimes d_1^* \sigma^{\otimes -1} \otimes d_2^* \sigma$ is a section of the trivial line bundle over $G^2 \times M$ (due to the relations among the iterated face maps $G^2 \times M \to M$).

When $G$ is discrete, $\sigma$ amounts to a family of isomorphisms $\sigma_g : L \to g^* L$, and the cocycle condition becomes simply $\sigma_{g_1g_2} = g_2^* \sigma_{g_1} \sigma_{g_2}$. For a Lie group $G$, $\sigma$ still amounts to the data of such $\sigma_g$, which must vary smoothly as a function of $g \in G$.

From such $\sigma$ we obtain the previous function $\phi_{\alpha\beta} = \frac{[d_0^* s_\alpha \otimes p_2^* s_\beta^{\otimes -1}]}{\sigma}$ - a ratio of two sections of $d_0^* L \otimes d_1^* L^{\otimes -1}$ over $Z_{\alpha\beta}$.

Then the condition that a connection $\nabla$ is $G$-invariant amounts to the condition that the covariant derivative of $\sigma$ vanishes in the $M$-direction.

Recall that line bundles over $M$ are classified by the cohomology group $H^2(M, \mathbb{Z})$. Similarly we have

**Proposition A-1.** For $G$ a compact Lie group, the $G$-equivariant line bundles over $M$ are classified by the (Borel) equivariant cohomology group $H^2_G(M, \mathbb{Z})$.

**Proof.** The data $(g_{\alpha\beta}, \phi_{\alpha\beta})$ can viewed as a Čech cocycle for the simplicial manifold $G^n \times M$, when we use the covering $(V_\alpha)$ of $M$, the open covering $Z_{\alpha\beta}$ of $G \times M$ and the open covering $(d_0^* V_\alpha) \cap (d_1^* V_\beta) \cap (d_2^* V_\gamma)$ of $G^2 \times M$. Thus it yields a class in the degree 1 hypercohomology $H^1(G^\bullet \times M, \mathbb{C}^*)$. Now the exponential exact sequence gives an exact sequence

$$H^1(G^\bullet \times M, \mathbb{C}) \to H^1(G^\bullet \times M, \mathbb{C}^*) \to H^2(G^\bullet \times M, \mathbb{Z}) \to H^2(G^\bullet \times M, \mathbb{C})$$

Now all the hypercohomology groups $H^p(G^\bullet \times M, \mathbb{C})$ can be computed in terms of the global sections of the sheaves $\mathbb{C}$ over $G^k \times M$, since $\mathbb{C}$ is a fine sheaf. Then
the complex in question becomes the complex of smooth cochains of \( G \) with values in the \( G \)-module \( C^\infty(M) \). Thus the hypercohomology groups \( H^p(G^\bullet \times M, \mathbb{C}) \) are exactly the differentiable cohomology groups \( H^p(G, C^\infty(M)) \), which are 0 for \( p > 0 \) since \( G \) is compact. Thus \( H^1(G^\bullet \times M, \mathbb{C}^* ) \) identifies with \( H^2 (G^\bullet \times M, \mathbb{Z}) \), and the latter group is just the Čech version of \( H^2_G(M, \mathbb{Z}) \).

It is interesting to see more concretely how the data \(( g_{\alpha\beta}, \phi_{\alpha\beta}, A_\alpha ) \) for an equivariant line bundle lead to an equivariantly closed differential form \(( F, \mu )\). First of all \( F \) is the curvature so that \( F_{/V_\alpha} = dA_\alpha \). Next we consider the 1-form \( \omega_{\alpha\beta} \) on \( Z_{\alpha\beta} \subseteq G \times M \) defined as

\[
\omega_{\alpha\beta} = d_0^* A_\alpha - d_1^* A_\beta - d \log \phi_{\alpha\beta}.
\]

This 1-form vanishes in the \( M \)-direction, so is entirely in the \( G \)-direction. Also \( \omega_{\alpha\beta} \) is \( G \)-invariant if \( G \) acts on \( G \) by left multiplication. Further over \( Z_{\alpha\beta} \cap Z_{\gamma\beta} \) we have

\[
\omega_{\gamma\beta} - \omega_{\alpha\beta} = m^* (A_\gamma - A_\alpha) - d \log \left( \frac{\phi_{\gamma\beta}}{\phi_{\alpha\beta}} \right) = d \log (m^* g_{\gamma\alpha}) - d \log \left( \frac{\phi_{\gamma\beta}}{\phi_{\alpha\beta}} \right) = 0
\]

using (A-1) and similarly \( \omega_{\alpha\beta} \) coincides with \( \omega_{\alpha\delta} \) over \( Z_{\alpha\beta} \cap Z_{\alpha\delta} \). Hence the \( \omega_{\alpha\beta} \) glue together to give a global 1-form \( \omega \) on \( G \times M \) which is \( G \)-invariant and lives in the \( G \)-direction. We can write \( \omega = \langle p^* (g^{-1}dg, \mu) \rangle \), where \( g^{-1}dg \) is the Maurer-Cartan 1-form on \( G \) and \( \mu : M \to \mathfrak{g}^* \) is a smooth function. This function is a moment map for the \( G \)-action. We can evaluate it as follows: for \( \xi \in \mathfrak{g} \), denote by \( \xi \) the corresponding vector field on \( M \), and by \( (\xi, 0) \) the vector field on \( G \times M \) which lives in the \( G \)-direction and is the left-invariant vector field defined by \( \xi \). The derivative \( [(\xi, 0)] \cdot \log \phi_{\alpha\beta} \) is equal to \( \frac{\xi \cdot s_\alpha}{s_\alpha} \), where \( \xi \cdot s_\alpha \) denotes the derivative at \( t = 0 \) of \( \exp (t\xi) \cdot s_\alpha \). Then we find:

\[
\langle \mu (x), \xi \rangle = \langle \omega, [(\xi, 0)] \rangle = \frac{\nabla \xi s_\alpha}{s_\alpha} - \frac{\xi \cdot s_\alpha}{s_\alpha}
\]

which is a standard description of the moment map as measuring the difference between two infinitesimal actions of \( \mathfrak{g} \) on sections of \( L \): the one given by the connection evaluated along the \( G \)-orbits and the one given by the \( G \)-action on sections of the equivariant line bundle \( L \) [B-V]. One checks easily that \( d \langle \mu, \xi \rangle = \langle \xi, F \rangle \) as required, so that \( (F, \mu) \) is an equivariantly closed 2-form. This is the equivariant Chern class as constructed by Berline and Vergne [B-V].

2) Equivariant gerbes

We will discuss equivariant gerbes in a similar spirit as we discussed equivariant line bundles. Let \(( V_\alpha, A_{\alpha\beta}, \theta_{\alpha\beta\gamma} )\) be some gerbe data over \( M \) (as mentioned before, we will suppress from the notation the isomorphism between \( \Lambda_{\alpha\beta}^{-1} \) and \( \Lambda_{\beta\alpha} \)). Then to make the gerbe data \( G \)-equivariant we need to pick a line bundle \( E_{\alpha\beta} \) over \( Z_{\alpha\beta} \subseteq G \times M \) together with isomorphisms...
\[ \phi_{\gamma/\alpha,\beta} : d_1^{*} \Lambda_{\gamma\alpha} \otimes E_{\alpha\beta} \to E_{\gamma\beta} \]

and

\[ \phi_{\alpha,\delta/\beta} : d_0^{*} \Lambda_{\delta/\beta} \otimes E_{\alpha\beta} \to E_{\alpha,\delta} \]

which satisfy the compatibility conditions

\[ \phi_{\epsilon/\alpha,\beta} = d_1^{*} \theta_{\epsilon/\gamma,\beta} \phi_{\epsilon/\gamma,\alpha,\beta} \]

\[ \phi_{\alpha,\epsilon/\beta} = d_0^{*} \theta_{\epsilon/\delta,\beta} [\phi_{\alpha,\epsilon/\delta} \phi_{\alpha,\delta/\beta}] \]

and the obvious commutation relation between the two types of isomorphisms.

Next we need a non-vanishing section \( \psi_{\alpha\beta\gamma} \) of the line bundle \( Q_{\alpha\beta\gamma} = d_0^{*} E_{\beta\gamma} \otimes d_1^{*} \Lambda_{\alpha\beta} \otimes d_2^{*} E_{\alpha\beta} \) over \( d_0^{*} Z_{\beta\gamma} \cap d_2^{*} Z_{\alpha\beta} \subset G^2 \times M \). This section should satisfy three conditions: first of all, \( \psi_{\alpha\beta\gamma} \) should correspond to \( \psi_{\delta\gamma\beta} \) under the tensor product of the isomorphisms \( d_1^{*} \phi_{\gamma/\alpha,\beta} \phi_{\gamma/\alpha,\beta} \) and \( d_2^{*} \phi_{\delta/\alpha,\beta} \phi_{\delta/\alpha,\beta} \); there are two similar conditions involving changing the second and third indices in \( Q_{\alpha\beta\gamma} \). Secondly, we require the cocycle condition

\[ d_0^{*} \psi_{\beta\gamma\delta} \otimes d_0^{*} \psi_{\alpha\gamma\delta} \otimes d_2^{*} \psi_{\alpha\beta\gamma} \otimes d_3^{*} \psi_{\alpha\beta\gamma} = 1. \]

This makes sense as the left hand side is a section of the trivial line bundle over an open set of \( G^3 \times M \).

We can now state

**Proposition A-2.** If \( G \) is a compact Lie group, the equivariant DD-gerbes over \( M \) are classified by the equivariant cohomology group \( H^3_G(M, \mathbb{Z}) \).

**Proof.** The proof is similar to that of Proposition A-1 so we will be brief. One first assumes that all line bundles \( \Lambda_{\alpha\beta} \) and \( E_{\alpha\beta} \) are trivial. Then one can interpret the data \( (\theta_{\alpha\beta\gamma}, \phi_{\gamma/\alpha,\beta}, \phi_{\alpha,\delta/\beta}, \psi_{\alpha\beta\gamma}) \) as yielding as giving a Čech 2-cocycle with values in the simplicial sheaf \( \mathbb{C}_* \) on the simplicial manifold \( G^* \times M \). Then one uses the exponential exact sequence to compare \( H^2(G^* \times M, \mathbb{C}_*) \) with \( H^3(G^* \times M, \mathbb{Z}) \).

There is one case where these unwieldy constructions are simpler: assume that all open sets \( V_\alpha \) are \( G \)-stable, so that \( d_0^{-1}(V_\alpha) = d_0^{-1}(V_\alpha) \) and that the \( \Lambda_{\alpha\beta} \) are equivariant line bundles over \( V_{\alpha\beta} \) and that each \( \theta_{\alpha\beta\gamma} \) is \( G \)-invariant. Then we can pick \( E_{\alpha\beta} = d_0^{*} \Lambda_{\alpha\beta} \); then we can take \( \phi_{\alpha,\delta/\beta} \) to be induced by \( d_0^{*} \theta_{\beta\alpha\delta} \), and since \( d_1^{*} \Lambda_{\alpha\beta} \) is isomorphic to \( d_0^{*} \Lambda_{\alpha\beta} \) (by the equivariance of \( \Lambda_{\alpha\beta} \)), we can take \( \phi_{\gamma/\alpha,\beta} \) to be the isomorphism induced by \( d_1^{*} \theta_{\beta\gamma\alpha} \). Then \( \psi_{\alpha\beta\gamma} \) is induced by \( \theta_{\alpha\beta\gamma} \) and all conditions are satisfied for an equivariant gerbe.

The data also simplify considerably in the case where \( M \) is a point. Then there is no need for the covering \( (V_\alpha) \) and the line bundle \( \Lambda_{\alpha\beta} \), so the construction boils down to a line bundle \( E \) over \( G \) and a non-vanishing section \( \psi \) of the line bundle
The fiber of this line bundle at \((g_1, g_2)\) is equal to \(E_{g_2} \otimes E_{g_1} \otimes -1 \otimes E_{g_1} \). Thus \(\psi\) amounts to a multiplicative structure on the line bundle \(E \to G\), namely an isomorphism \(E_{g_1} \otimes E_{g_2} \to E_{g_1} \otimes -1 \otimes E_{g_1} \). This means also that the total space \(\hat{G}\) of the corresponding \(\mathbb{C}^*\)-bundle acquires a product structure \(\hat{G} \times \hat{G} \to \hat{G}\) which lifts the product on \(G\). The cocycle condition for \(\psi\) just says that this product law is associative. Then \(\hat{G}\) becomes a Lie group, which is a central extension of \(G\) by \(\mathbb{C}^*\). In fact we have

**Proposition A-3.** The equivalence classes of \(G\)-equivariant gerbes over a homogeneous space \(G/H\) correspond precisely to the central extensions

\[ 1 \to \mathbb{C}^* \to \hat{H} \to H \to 1. \]

We have proved this in the case where \(G = H\). We next claim that by restriction to the base point of \(G/H\), we get a bijective correspondence between \(G\)-equivariant gerbes over \(G/H\) and \(H\)-equivariant gerbes over a point. What we need to do is to give an analog for gerbes of the construction of a homogeneous vector bundle over a homogeneous space. So we start with an \(H\)-homogeneous gerbe \(C\) over a point, and we pull it back to a gerbe over \(G\) which is left \(G\)-equivariant and right \(H\)-equivariant. Then we have the following lemma:

**Lemma A-1.** If the Lie group \(H\) acts freely on a manifold \(X\), then pullback gives a bijective correspondence between equivalence of gerbes on \(X/H\) and equivalence classes of \(H\)-equivariant gerbes over \(X\).

Thus we can descend our gerbe to a gerbe on \(G/H\), which is still \(G\)-equivariant as the left action of \(G\) on \(G\) commutes with the right \(H\)-action. This “homogeneous gerbe construction” is inverse to restriction to the base point.

This construction can be generalized to the case where the gerbe data is trivial, so that the line bundles \(\Lambda_{\alpha\beta}\) are trivial and \(\theta_{\alpha\beta\gamma} = 1\). Then the isomorphisms \(\phi_{\gamma\alpha,\beta}\) give, for fixed \(\beta\), gluing isomorphisms between the line bundles \(E_{\alpha\beta}\) and \(E_{\gamma\beta}\) over the overlap \(Z_{\alpha\beta} \cap Z_{\gamma\beta}\) of their domains of definition. Then we can use these gluing data to produce a line bundle \(E_{\beta}\) over \(d_1^*V_{\beta}\). The isomorphisms \(\phi_{\alpha,\delta/\beta}\) then glue together, as \(\alpha\) varies, to yield a global isomorphism between \(E_{\beta}\) and \(E_{\delta}\) (this uses the commutation relations between the two types of isomorphisms). So finally we have a global line bundle \(E\) over \(G \times M\). Then the \(\psi_{\alpha\beta\gamma}\) glue together to give a global non-vanishing section \(\psi\) of \(d_1^*E \otimes d_1^*E^{\otimes -1} \otimes d_2^*E\) over \(G^2 \times M\). To interpret this, recall that \(A = G \times M\) has the structure of a differentiable groupoid, where \((g, x)\) is viewed as the arrow labeled by \(g\) which goes from \(x\) to \(g \cdot x\). Then \(\psi\) gives the total space \(\hat{A}\) of the \(\mathbb{C}^*\)-bundle associated to \(E\) a partial composition law which lifts that on the groupoid \(A\). The cocycle condition for \(\psi\) means that this composition law is associative, so that \(\hat{A}\) becomes a groupoid which is a central extension of \(A\) by \(\mathbb{C}^*\) (we refer to [W-X] for central extensions of groupoids and their applications to geometric quantization).
Next we examine the notion of equivariant 0-connection on an equivariant gerbe. Let then \((D_{\alpha\beta})\) be a 0-connection. To make it \(G\)-equivariant, we need to pick a connection \(\nabla_{\alpha\beta}\) in the direction of \(M\) on each \(E_{\alpha\beta}\); this means that the covariant derivative of a section of \(E_{\alpha\beta}\) on an open set of \(G \times M\) is a 1-form in the \(M\)-direction. Such a connection is also called a relative connection (relative to the projection \(G \times M \to M\)). We require that the isomorphisms \(\phi_{\gamma/\alpha,\beta}\) and \(\phi_{\alpha,\delta/\beta}\) are compatible with the relative connections. Then \(\psi_{\alpha\beta\gamma}\) should be a horizontal section of \(Q_{\alpha\beta\gamma}\), but only in the direction of \(M\).

Given a 1-connection \((F_{\alpha})\), it is equivariant iff it satisfies the constraint

\[
Curv(\nabla_{\alpha\beta}) = d_{\alpha}^{*}F_{\alpha} - d_{\beta}^{*}F_{\beta} \text{ in the direction of } M \text{ (over } Z_{\alpha\beta})
\]

Then the 3-curvature \(\Omega\) satisfies \(d_{\alpha}^{*}\Omega = d_{\beta}^{*}\Omega\), so it is \(G\)-invariant. It is most natural at this point to write down the equivariantly closed 3-form which is the equivariant Chern character of the equivariant gerbe. As in the case of line bundles, we introduce the 2-form \(d_{\alpha}^{*}F_{\alpha} - d_{\beta}^{*}F_{\beta} = Curv(\nabla_{\alpha\beta})\) on \(Z_{\alpha\beta}\) which has zero component in \(A^{0}(G) \hat{\otimes} A^{2}(M)\). We look at the component \(B_{\alpha\beta}\) of this 2-form onto the factor \(A^{1}(G) \hat{\otimes} A^{1}(M)\) of \(A^{2}(G \times M)\). It is easy to see that the \(B_{\alpha\beta}\) glue together to give a global 2-form \(B\) on \(G \times M\). As \(B\) is \(G\)-invariant for the left action of \(G\) on itself, it can be written down as \(B = \langle g^{-1}dg; \otimes E\rangle\), where \(E\) is a \(g\)-valued 1-form on \(M\) and \(B\) is obtained using the evaluation map \(\langle \ , \ \rangle: g^{*} \otimes g \to \mathbb{R}\). Then \((\Omega, E)\) is an equivariantly closed 3-form on \(M\).

Now we interpret the notion of equivariant gerbe data in terms of DD-gerbes. So let \(\mathcal{C}\) be a DD-gerbe over \(M\), viewed as in §1 as a sheaf of groupoids satisfying axioms 1)-3). Then to make \(\mathcal{C}\) \(G\)-equivariant we need two extra pieces of data, First we need an equivalence \(\phi: d_{1}^{*}\mathcal{C} \to d_{0}^{*}\mathcal{C}\) of gerbes over \(G \times M\). Equivalently, \(\phi\) amounts to a global object \(R\) of the gerbe \(d_{0}^{*}\mathcal{C} \otimes d_{1}^{*}\mathcal{C}^{\otimes -1}\) over \(G \times M\). Second we need an isomorphism

\[
\psi: d_{0}^{*}R \otimes d_{1}^{*}R^{\otimes -1} \otimes d_{2}^{*}R \to 1
\]

of objects of the trivial gerbe over \(G^{2} \times M\). This isomorphism must satisfy the cocycle condition

\[
d_{0}^{*}\psi \otimes d_{1}^{*}\psi^{\otimes -1} \otimes d_{2}^{*}\psi \otimes d_{3}^{*}\psi^{\otimes -1} = 1.
\]

For \(G\) discrete, \(\phi\) amounts to gerbe equivalences \(\phi_{g}: \mathcal{C} \to g^{*}\mathcal{C}\) and \(\psi\) amounts to natural transformations

\[
\psi_{g_{1}, g_{2}}: \phi_{g_{1}, g_{2}} \to (g_{2}^{*}\phi_{g_{1}})\phi_{g_{2}}
\]

between equivalence of gerbes which must satisfy a cocycle condition (that condition may be visualized as a commutative tetrahedron). We can write \(\phi_{g}(P)\) as \(g_{*}P\) for an object \(P\) of \(\mathcal{C}\) over some open set. Then \(\psi_{g_{1}, g_{2}}\) is an isomorphism between \((g_{1}g_{2})_{*}P\) and \((g_{1})_{*}[(g_{2})_{*}P]\).

We briefly adumbrate how these data lead to the equivariant gerbe data discussed previously. First take an open covering \((V_{\alpha})\) of \(M\) and objects \(P_{\alpha} \in \mathcal{C}_{V_{\alpha}}\). Then as in §1, we have the line bundle \(\Lambda_{\alpha\beta} \to V_{\alpha\beta}\) associated to the \(\mathbb{C}^{*}\)-bundle

\[
31
\]
$\text{Isom}(P_\beta, P_\alpha)$. Over $Z_{\alpha\beta}$ we have the objects $\phi(d_1^* P_\beta)$ and $d_0^* P_\alpha$ of the pull-back gerbe $d_0^* C$. Then we define the line bundle $E_{\alpha\beta}$ to be the line bundle associated to the $\mathbb{C}^*$-bundle $\text{Isom}(d_0^* P_\beta, \phi(d_1^* P_\alpha))$. The isomorphisms $\phi_{\gamma/\alpha, \beta}$ and $\phi_{\alpha, \beta/\gamma}$ are given by composition of isomorphisms of objects in the gerbe $d_0^* C$. Thus by composition of isomorphisms we obtain the trivialization $\psi_{\alpha\beta\gamma}$ of $Q_{\alpha\beta\gamma}$, which by its construction satisfies a cocycle condition.

Another advantage of the notion of equivariant gerbe $C$ is that we can define an equivariant object of $C$. This means that $P$ is an object of $C_U$, equipped with an isomorphism $\eta : \phi(d_1^* P) \rightarrow d_0^* P$ of objects of $[d_0^* C]_{G \times M}$ which satisfies the associativity condition $d_0^* \eta \otimes d_1^* \eta \otimes d_2^* \eta = 1$. For $G$ discrete, $\eta$ amounts to a family of isomorphisms $\eta_g : \phi_g(P) \rightarrow g^* P$ which satisfy the cocycle condition. This is formally the same description as for equivariant line bundles.

In case $M$ is a point, the obstruction to finding an equivariant object of $C$ is a central extension of $G$ by $\mathbb{C}^*$. Indeed, picking any $P$ and $\eta$ as above, the automorphism $d_0^* \eta \otimes d_1^* \eta \otimes d_2^* \eta$ of $pr_1^* P \in pr_1^* C_G$ is a function $G^2 \rightarrow \mathbb{C}^*$, which is a 2-cocycle. Thus we recover the central extension we previously described using equivariant gerbe data.

Now given a 0-connection on the gerbe $C$, viewed as in §1 as a sheaf $Co(P)$ attached to each object of each $C_U$, to make the 0-connection equivariant we need to extend the equivalence $\phi : d_1^* C \rightarrow d_0^* C$ of gerbes over $G \times M$ to an equivalence of gerbes with 0-connection. This means that for each object $P$ of each $(d_1^* C)_U$ we give an isomorphism of sheaves $\phi_* : Co_{rel}(P) \rightarrow Co_{rel}(\phi_* P)$, where $Co_{rel}(P)$ is the sheaf obtained by dividing $Co(P)$ by the action of the 1-forms on $G \times M$ which are in the direction of $G$. Thus $Co_{rel}(P)$ is a torsor under the sheaf $\Omega^1_{M \times G \rightarrow G}$ of relative 1-forms with respect to the projection $G \times M \rightarrow G$. In case $G$ is discrete, we have $\Omega^1_{M \times G \rightarrow G} = \Omega^1_{M \times G}$, and a section of this sheaf is a family $\omega_g$ of 1-forms on (open sets of) $M$, indexed by $g \in G$. Let us see what $\phi_*$ looks like when $P = d_1^* Q$ where $Q$ is some object of $C$ over some open set. Then we put $\phi_g(Q) = g_* (Q)$ as explained earlier, so that $\phi_*$ amounts to a family of isomorphisms of torsors $Co(Q) \rightarrow Co(g_* Q)$ which satisfy a transitivity condition. If now $Q$ is an equivariant object of $C$, then this family of isomorphisms makes $Co(Q)$ into an equivariant $\Omega^1_M$-torsor.

Then a curving $\nabla \in Co(P) \mapsto K(\nabla)$ is $G$-equivariant if and only if and only if the relative 2-forms $K(\nabla)$ and $K(\phi_* \nabla)$ coincide, for any object $P$ of $[d_1^* C]_U$ and for any $\nabla \in Co(P)$.

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