Quantum contextuality turns out to be a necessary resource for universal quantum computation and important in the field of quantum information processing. It is therefore of interest both for theoretical considerations and for experimental implementation to find new types and instances of contextual sets and develop methods of their optimal generation. We present an arbitrary exhaustive hypergraph-based generation of the most explored contextual sets—Kochen-Specker (KS) ones—in 4, 6, 8, 16, and 32 dimensions. We consider and analyse twelve KS classes and obtain numerous properties of theirs, which we then compare with the results previously obtained in the literature. We generate several thousand times more types and instances of KS sets than previously known. All KS sets in three of the classes and in the upper part of a fourth are novel. We make use of the MMP hypergraph language, algorithms, and programs to generate KS sets strictly following their definition from the Kochen-Specker theorem. This approach proves to be particularly advantageous over the parity-proof-based ones (which prevail in the literature), since it turns out that only a very few KS sets have a parity proof (in six KS classes < 0.01% and in one of them 0%). MMP hypergraph formalism enables a translation of an exponentially complex task of solving systems of nonlinear equations, describing KS vector orthogonalities, into a statistically linearly complex task of evaluating vertex states of hypergraph edges, thus exponentially speeding up the generation of KS sets and enabling us to generate billions of novel instances of them. The MMP hypergraph notation also enables us to graphically represent KS sets and to visually discern their features.

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I. INTRODUCTION

An assumed property of a classical system is that any of its measurements has values independent of other compatible measurements that might have been carried out on the system previously or counterfactually simultaneously, i.e., that the values are predetermined. The property is called the non-contextuality. This is in contrast to quantum mechanical systems whose measurements might be contextual, i.e., dependent on the context of previous or counterfactually simultaneous measurements. Such property of quantum systems is called (quantum) contextuality.

The so-called Kochen-Specker (KS) sets provide constructive proofs of quantum contextuality and therefore provide straightforward blueprints for their implementation and experimental setups. KS sets are likely to find applications in the field of quantum information, similarly to ones recently found for the Bell setups in implementing entanglements [1, 2]. The assumption is supported by a recent result of A. Cabello [3], according to which local contextuality can be used to reveal quantum nonlocality.

Along that road, it has been most recently “demonstrate[d] that ... contextuality is the source of a quantum computer’s power” [4]. In particular, Howard, Wallman, Veittech, and Emerson [5] “uncover a remarkable connection between the power of quantum computers and ... contextuality” [4] and prove that “contextuality is a necessary resource for universal quantum computation via magic state distillation” [3, p. 354]. (“The way of initializing the quantum bits [by means of] ... superposition ... is called magic” [4].) The scheme of Howard et al. [5] has been extended by Deflosse, Guerin, Bian, and Raussendorf so as to include Wigner function negativity [6].

It has also been recently shown by Raussendorf that “the measurement-based quantum computations which compute a nonlinear Boolean function with a high probability are contextual” [7].

A contextual kind of quantum gates—indispensable ingredients of quantum computational circuits—can be straightforwardly constructed from the scheme which served Waegegg and Aravind to build 4-dim complex KS sets [8].

On the other hand, Pavićić, McKay, Megill, and Fresl have shown that KS sets can serve as a generator of a new kind of lattices within Hilbert lattice representation of the Hilbert space [9, Fig. 8], where a Hilbert lattice is an algebra underlying every Hilbert space. In addition, Megill and Pavićić have shown how new generalized orthoarguesian equations—the only known equations, apart from the orthomodularity equation itself, holding in the algebra of closed subspaces of a Hilbert space—can be generated from KS sets [10].

Another quantum information contextual KS set application is a quantum cryptography protection, as outlined by Cabello, D’Ambrosio, Nagali, and Sciarrino [11]. It has even been shown by Nagata that the KS theorem is a precondition for secure quantum key distribution (QKD) in the sense that in each QKD protocol KS non-contextuality is violated [12].

A series of KS experiments have been carried out during the last ten years. They were implemented for 4-dim systems with photons [13–15], neutrons [16–21] trapped ions [22], and molecular nuclear spins in the solid states [23, 24], for 6-dim systems via six path possibilities for the photon transmission through a diffractive aperture [24, 25], and for 8-dim systems by means of the linear transverse momentum of single photons transmitted by diffractive apertures addressed in spatial light modulators [26].

The aforementioned role of contextual sets in “supply[ing] ‘magic’ for quantum computation” [5] would require numerous instances of contextual sets and here KS sets as the most numerous contextual sets are likely to have an important role in designing appropriate schemes for implementations and applications. Then, in order to test different quantum gates for KS sets we should be able to engineer sufficiently large number of vectors for them, i.e., KS sets of different complexities. For constructing new algebraic structures and equations for the Hilbert space we should also have an arbitrary increasing number of KS sets as explicitly shown in [10]. Finally, it is of theoretical significance to know the structure, features, and sizes of various KS sets. Taken together, it is important to find new classes and new instances of non-redundant non-isomorphic KS sets as well as different coordinatizations for them. It is also of importance to design algorithms and programs with the help of which we can generate an arbitrary number of different KS sets.

In this paper, we describe the discovery of large numbers (billions of them) of critical non-redundant non-isomorphic KS sets in 4-, 6-, 8-, 16-, and 32-dim Hilbert spaces. “Critical” means that they are minimal in the sense that a removal of any n-tuple of mutual orthogonalities, of n vectors from an n-dim Hilbert space, turns a KS set into a non-KS set. In other words, they represent a KS setup that has no redundancy.

We describe the features of KS sets within particular KS classes which emerge as we generate the sets. We also outline patterns of distribution and generation and compare them with the other methods of generation in the literature. For instance, huge blocks of KS sets and even whole classes of KS sets turn out to be completely invisible with the latter methods.

The paper is organised as follows.

In Sec. [11] we provide the reader with a constructive version of the KS theorem, define KS sets as well as the critical KS sets, define the parity proof for KS sets, and present the formalism, algorithms, and programs we make the use of in the paper.
In Secs. II IV VII and IX we deal with KS sets in 4-dim Hilbert space.

In Sec. III we review the oldest KS class, the 24-24 one, which is actually a subclass of the 60-105 class we introduce in Sec. V. In Sec. IV we obtain three orders of magnitude more sets from the class 60-74 then in our previous paper 27 and 15 times more than reported in other literature (the class is also known as the 60-75 and/or 600-cell based KS class): we denoted the 60-74 class as “tentative” in the title of the section because it is particular subclass of the class 300-675 we introduce in Sec. VI. In Sec. VII we elaborate on the 60-105 class defined by means of Pauli operators for two qubits in the complex Hilbert space and obtain ca. 2.5 more types of KS criticals and $3 \times 10^4$ more instances of them than known from the literature. In Sec. VIII we analyze the recently discovered highly complex and extremely interwoven 300-675 class and find an important subclasses which have been completely overlooked in the literature at the higher end of the class. In Sec. VII we generate a completely new class of ca. 250,000 KS criticals from the so-called Witting’s master set, recently found in the literature; none of the criticals has a parity proof and therefore all the obtained sets from class are completely invisible in the standard approach via parity-proof-based algorithms and programs.

In the 6-dimensional Hilbert space, the so-called “seven context” star-like 21-7 KS critical set has recently been discovered and a challenge was issued to find bigger 6-dim KS sets in response to which we in Sec. VIII generate 3.7 $\times 10^6$ 6-dim KS criticals in the novel 236-1216 class; all but 8 of the criticals lack a parity proof; we also show that the vector components of the seven-context-star KS set can be simplified and that the set itself is not contained in the latter class.

In Sec. IX we generate ten times more types of and KS sets themselves, from the Lie algebra E8 based 120-2024 master set, than previously achieved in the literature—due to the very low number of the parity proofs (0.1%); we also construct a real star-like KS critical set and show that it is not contained in the 120-2024 class.

In Sec. X we enter a sparsely charted territory of 16-dim 4-qubit KS sets and generate ca. $2.5 \times 10^6$ more instances and ca. 70 more types of sets than known from the literature from an 80-265 master set.

In Sec. XI we generate ca. $2.5 \times 10^5$ more instances and ca. 153 more types of 32-dim 5-qubit KS criticals (from a 160-661 master set) than known from the literature.

In Sec. XII we revisit the only four known 3-dim KS criticals and show that recently spotted 13-vector set does not prove the Kochen-Specker theorem.

In Sec. XIII we discuss and compare, both mutually and with those in the literature, all the KS sets we generated.

In Appendix A we give all hypergraph strings we refer to in the main body of the paper.

In Appendix B we provide the reader with chosen KS critical sets from most of the types from all classes we considered.

II. KS SETS, MMP HYPERGRAPHS, FORMALISM, ALGORITHMS, AND PROGRAMS

Our aim is to present new results in the realm of contextual setups and KS sets, methods that served us to generate them, and we introduce formalism and representation that enable us to handle them. The input and output data are extremely massive and numerous and they contain all known (from the previous literature, including our own previous papers) setups, sets, figures, hypergraphs, and diagrams as a very special and tiny portion of the ones obtained here. Before we dwell on details of the formalism we will make use of, we briefly introduce contextuality vs. non-contextuality features, quote the KS theorem, and define a KS set.

The notion of non-contextuality of a system, whose observables we measure after its passing through a device, boils down to a statement that measurements of a system corresponds to predetermined values of the observables during the interaction of the system with the device. A stronger statement, which is usually called the KS theorem is that non-contextual theories assume that a predetermined result of a particular measurement of an observable of a system does not depend on measurements simultaneously carried out on other observables of the system, while quantum, contextual theories do not assume any predetermined values for outcomes of measurements, clicks, 0-1s, and might depend on simultaneous measurements.

Theorem II.1. (Kochen-Specker 28 30) In $H^n$, $n \geq 3$, there are sets of $n$-tuples of mutually orthogonal vectors to which it is impossible to assign 1s and 0s in such a way that

1. No two orthogonal vectors are both assigned the value 1;
2. In any group of $n$ mutually orthogonal vectors, not all of the vectors are assigned the value 0.

The sets of such vectors are called KS sets and the vectors themselves are called KS vectors.

Any KS set defined for a quantum system provides a constructive proof of the KS theorem and of the contextuality of quantum mechanics. A collection of related measurements provides an experimental verification of the theorem.
Within quantum mechanics we can formalize KS set properties in the following manner. To every quantum observable of a quantum system there corresponds a linear Hermitian operator in a Hilbert space and to every state of the system associated to the observable there corresponds an eigenvector of the operator in the same space. The result of a measurement of the observable is associated with the eigenvalue of the operator. Any KS set is represented by a collection of \( n \)-tuples of mutually orthogonal (eigen) vectors from an \( n \)-dim Hilbert spaces.

In this paper we consider 3-, 4-, 5-, 6-, 8-, 16-, and 32-dim KS sets. They can be implemented in a laboratory in two different ways. By means of qubits in an \( n \)-dim (where \( n = 2^k \), where \( k \) is a natural number \( \geq 2 \)) Hilbert space \( H^n = H^2 \otimes \cdots \otimes H^2 \) and by means of spin-\( \frac{1}{2} \) systems. The examples of the former way are KS sets in 4-dim \( H^4 \) by means of 2 qubits from the class 60-105 in Sec. V and from the 24-24 class \([31]\) in Sec. III in 8-dim \( H^8 \) by means of 3 qubits, or in 16- and 32-dim spaces via 4 and 5 qubits in Secs. X and XI respectively. The examples of the latter way are 4-dim 60-74 class in Sec. IV, 6-dim star/triangle set and the 236-1216 class in Sec. VIII, and the star/triangle set in Sec. IX. In our hypergraphs approach, the calculational treatment of and the elaboration on all classes are the same, though. Only experimental implementations differ and we will discuss them when needed.

General formalism of \( n \)-dim (\( n \geq 3 \); \( n \in \mathbb{N} \)) KS sets and their implementation via spin-\( \frac{1}{2} \) particles (say via, e.g., generalized Stern-Gerlach devices with a simultaneous usage of magnetic and electric fields by means of which it is possible to generate an arbitrary spin state \([32]\)), covers any possible experimental implementation in contrast to qubit approach which covers only \( n \)-dim= \( 2^k \)-dim cases (\( k \in \mathbb{N} \), \( k \geq 2 \)).

We represent KS sets by hypergraphs in the MMP hypergraph notation specified below. In a KS set, the vertices correspond to vertices of an MMP hypergraph. Vertices representing \( n \)-tuples of orthogonal eigenvectors are organized in edges of MMP hypergraphs \([33]\).

**Definition II.1.** MMP hypergraphs are hypergraphs in which

(i) Every vertex belongs to at least one edge;

(ii) Every edge contains at least 3 vertices;

(iii) Edges that intersect each other in \( n - 2 \) vertices contain at least \( n \) vertices.

A KS set with \( n \) vertices and \( m \) edges is denoted as \( n \times m \).

Only minimal KS sets, called critical KS set are relevant for experimental implementations since their supersets just contain additional orthogonalities that do not change the KS property of the smallest critical set.

**Definition II.2.** KS sets that do not properly contain any KS subset, meaning that if any of its edges were removed, they would stop being KS sets, are called critical KS sets.

Some authors make use of a coarser notion of {vertex}-critical KS sets: “A KS set is termed critical iff it cannot be made smaller by deleting the [vertices]” \([34]\). However, this definition lacks operationality in identifying a huge number of critical sets which turn into a non-KS set when an edge of theirs is removed while the number of vertices remains unaltered as allowed by Def. II.2. On the other hand, deleting a vertex means a removal of at least one edge.

We encode MMP hypergraphs by means of alphanumeric and other printable ASCII characters. Each vertex is denoted by one of the following characters: 1 2 \ldots 9 A B \ldots Z a b \ldots z ! " # $ % & ' ( ) * - / : ; < = > ? @ \[ \] ^ _ `{ } | \~ \^ \| \{ \} \sim \[31\]. When all these characters are exhausted we reuse them so as to prefix them by ‘+’, then by ‘++’, and so on. An example is shown in the graphical representation of a hypergraph of KS set 18-9 in the figure in Sec. III where ASCII characters printed next to corresponding vertices from the hypergraph belong the MMP hypergraph string 1234, 4567, 789A, ABCD, DEFG, GHI1, 29BI, 35CE, 68FH. So encoded, MMP hypergraphs are generated by our algorithms and programs or introduced into our programs to be processed. Each edge is represented by a string of characters separated by commas and all of them together form a hypergraph, i.e., a KS set, as a single textual line/string which ends with a full stop. When dealing with such ASCII line encoding of MMP hypergraphs we call them MMP hypergraphs lines or strings when needed. The order of the strings and characters is irrelevant; gaps in characters are allowed and its number is not limited; tens of thousands of them are not a problem for our programs shortd.c, mmpstrip.c, subgraph.c, vectorfind.c, states01.c and others \([8, 27, 31, 36, 37]\).

To visualise the hypergraphs we represent them as figures showing vertices as dots and edges as straight or curved lines each connecting \( n \)-tuples of vertices. We often draw hypergraphs so as to start with the biggest loop they contain. Usually we do not attach characters to vertices in a figure because one can always arbitrary attach them and then use program vectorfind to ascribe vector components to each vertex. In chosen figures in the following sections below we show graphical representations of some of the KS sets that we found in this study in the MMP hypergraph notation.

Our standard and compact definition of MMP hypergraphs enables us to smoothly design algorithms for generation, handling, and analysis of KS sets what together amounts to MMP hypergraph language. In this work, we generate subgraphs of big chosen KS hypergraphs, which we call master sets, by deleting a specified number of edges from such
master sets via our program \texttt{mmpstrip}. Then we filter them on the KS property via our program \texttt{states01} which just verifies whether they violate the conditions of the Def. II.1, i.e., whether they are KS sets. Program \texttt{states01} carries out an exhaustive search according to a backtracking algorithm. This is a much less demanding task than a constructive upward generation we used previously in \cite{32} and although we have to deal with a huge volume of hypergraphs, on computing clusters we can carry out generations successfully, as we show in subsequent sections below.

A collection of all KS subsets of a particular master set $i - j$, with $i$ vertices and $j$ edges, we call an $i - j$ class of KS sets.

We can generate members of an $i - j$ class from a master set $i - j$, on a computing grid, as follows. First, we strip edges from the master set with \texttt{mmpstrip} and then filter them with \texttt{sed, states01, sed, and shortd} to obtain, say, $m$ $k - l$ files, where $l = 1, \ldots, m$ and $k$ depends on $l$ in a rather involved manner depending of how many vertices, if any, were stripped together with stripped edges. Each file might contain millions of KS sets (all with billions of them). This can be compared with less than a dozen of KS sets discovered by several researchers between correspondence between nonlinear systems and MMP hypergraphs enables us to generate KS sets on a large scale more computationally efficient than solving for Hilbert space vectors directly when searching for KS sets. Such a problem of statistically polynomial complexity. In other words, solving for MMP hypergraphs is exponentially 0-1 valuations, required by the definition of a KS set given in Theorem II.1, to vertices of MMP hypergraphs is

The latter system is an unsolvable problem on any supercomputer, even for the smallest KS sets while ascribing odd number of edges, there should also be an odd number of edges with 1s, i.e., we have a contradiction. Parity proofs face several problems, though.

- KS sets with even number of edges cannot have parity proofs per definition;
- Many KS sets with odd number of edges turn out not to have a parity proof, either;
- In some classes of KS sets we obtained, less than 0.1% have a parity proof, and in some others, none at all.

Parity proofs are just special and particular cases of our general MMP hypergraph verification but sometimes they turn out to offer a complementary method of generation of KS sets since-parity-proof based programs are much faster than general MMP hypergraph based ones, when applicable.

### III. TENTATIVE 24-24 CLASS OF 4-DIM KS SETS; GENERATION OF KS SETS VIA STRIPPING OF MASTER SETS

In this section we shall make use of the results about the 24-24 class of 4-dim KS sets we obtained in \cite{31,32,33,34} to introduce the main steps and strategy we shall undertake to obtain the results in the subsequent sections.

In 2002 Pavičić \cite{35} realised that one can establish a correspondence between MMP hypergraphs and systems of nonlinear equations describing mutual orthogonalities of vectors as, for instance, in the following 3-dim example

\begin{align*}
\mathbf{x} \cdot \mathbf{y} &= x_1 y_1 + x_2 y_2 + x_3 y_3 = 0, \\
\mathbf{x} \cdot \mathbf{z} &= x_1 z_1 + x_2 z_2 + x_3 z_3 = 0, \\
\mathbf{y} \cdot \mathbf{z} &= y_1 z_1 + y_2 z_2 + y_3 z_3 = 0.
\end{align*}

(1)

The latter system is an unsolvable problem on any supercomputer, even for the smallest KS sets while ascribing 0-1 valuations, required by the definition of a KS set given in Theorem II.1 to vertices of MMP hypergraphs is a problem of statistically polynomial complexity. In other words, solving for MMP hypergraphs is exponentially more computationally efficient than solving for Hilbert space vectors directly when searching for KS sets. Such a correspondence between nonlinear systems and MMP hypergraphs enables us to generate KS sets on a large scale (billions of them). This can be compared with less than a dozen of KS sets discovered by several researchers between
1967 and the end of the 20th century [29, 42, 47], mostly exploring highly symmetrical geometrical structures defined by mutually orthogonal vectors.

In 2004 Pavičić, McKay, Merlet, and Megill [36, 40] generated non-isomorphic MMP hypergraphs and filtered them by means of a program which was written for our algorithm of assigning 0s and 1s to their vertices and another algorithm for assigning vector components to vertices. The generation and assignments are exponentially complex tasks in general but applied to our KS MMP hypergraphs they turned out to be polynomially complex for the great majority of jobs. We say that they are statistically polynomially complex. Nevertheless, when we reached 24 vertices, the task became forbiddingly CPU-time consuming—we obtained over 300 KS sets with up to 23 vertices on a cluster with on average 100 CPUs running for several months. Among them there were only 5 critical KS sets. So, we started to search for another way of generating KS sets. We arrived at the idea of a faster generation as follows.

Kernaghan [44] and Cabello, Estebaranz and García-Alcaine [46] realised that their 18-9 and 20-11 KS sets were subsets of Peres’ 24-24 set [12] but since they did not make use of graphical representation it took them a while to find their two sets and neither they nor Peres were able to find any more KS subsets the 24-24 set (Peres even wrote a computer program for the purpose [48]).

After we generated the first few hundred KS sets in [36, 40] and started to draw their hypergraphs we visually recognised—see Fig. 1—that they were all subgraphs of the hypergraph we drew for Peres’ 24-24 set. Then Pavičić, Megill, and Merlet designed an algorithm for stripping (peeling) edges off the latter hypergraph and obtained 1232 KS subsets [31] (including all 6 criticals from Fig. 1) within less than 2 minutes on a PC. These 1233 KS sets form a 24-24 class of KS sets and Peres’ 24-24 set is their master set. (We would just like to mention here that we generated and scanned, during 3 CPU-months, all non-isomorphic hypergraphs with 24 vertices and 24 edges and that among all millions of them there is only one KS set—Peres’ 24-24 one.)

![FIG. 1. 4-dim KS sets from the 24-24 KS class; (a)-(f) are all critical KS sets from the class; (a) and (c) were found in [44] and [46], respectively; (b), (d), and (e) were found in [36]; (f) was found in [31]; (g) and (h) are two isomorphic representation of a non-critical Peres’ 24-24 KS set found in [42]—it contains all (a)-(f) as well as all the other 1226 KS sets from the 24-24 class; (a) shows that its vertices cannot satisfy the conditions of Theorem II.1 encircled vertices represent possible 1-assignments.]

This led us to another aspect of generating KS sets. All vectors forming KS sets in the 24-24 class have components from the set \{-1,0,1\} since Peres’ 24-24 set has components from this class. However, we also found KS sets that were not subsets of the 24-24 set (e.g., the 22-11 one, shown in Fig. 3(a) of [31]) and those subsets have the components from a wider set of values (see Table 2 of [31] for the aforementioned 22-11 set). That indicated that there is another class or other classes which contain those sets or both kinds of sets and we started stripping master sets Meanwhile discovered [8, 37, 47, 49, 50].

We designed algorithms and programs which exhaustively generate all KS sets from all stripped subsets of chosen master KS sets that we introduced and described in Sec. III. They are computationally rather demanding and require many CPU months of running on clusters and supercomputers but that is feasible with today’s resources. In the rest of the paper we present various outcomes of such calculations with our algorithms and the features of the critical KS sets we obtained on our clusters.

IV. TENTATIVE 60-74 CLASS OF 4-DIM KS SETS

Waegell and Aravind have derived a 60-75 KS set from a 4-dim regular polytope (600-cell) with 60 pairs of vertices [49]. The vertices correspond to vectors whose components have values from the set \( V = \{0, \pm (\sqrt{3} - 1)/2, \pm 1, \pm (\sqrt{3} + 1)/2, 2\} \) and one can use them to write down the 60-75 set [49, Table 2]. MMP hypergraph of the 60-75 generated in [37] is given in Appendix A.1.

Generation of smaller KS sets from the master sets will be carried out by relying on the MMP hypergraph structure only and the vertices of the obtained set can be ascribed values from \( V \) later on, if needed, via (a) our program **vectorfind** randomly, or (b) via our program **subgraph** so as to trace down vertices which survived stripping of edges. We need to ascribe values from \( V \) to the vertices, e.g., for an experiment (cf [54, Fig. 1]).

In 2011, Megill, Fresl, Waegell, Aravind, and Pavičić [27] presented preliminary and partial results of generating subsets of the 60-75 set by stripping it of its edges and obtaining their features. Here we present in many respects an
almost exhaustive analysis of these subsets.

We start by stripping just one edge at a time of the 60-75 set in 75 different ways so as to obtain seventy five 60-74 sets. To be more explicit, we remove one edge from the 60-75 set to get the 1st 60-74, then we put it back and remove another edge to the 2nd 60-74 and so forth. It turns out that all 75 of the so obtained 60-74 sets are isomorphic to each other and that they all reduce to a single MMP hypergraph string 60-74 given in Appendix A1.

We shall therefore consider this 60-74 KS set to be a master set for all smaller KS sets we obtain from it. Therefore, we shall call the collection not a 60-75 but a 60-74 class of 4-dim KS sets. The number of sets from the class we generated and analysed by running our programs over a century of CPU time on our clusters are given in Fig. 2. The stripping technique applied to the sets means a removal of one edge at the time and filtering out the KS sets. We obtained no critical sets with 27, 28, 29, 31, or 35 vertices. To be more explicit, we remove one edge from the 60-75 set to get the 1st 60-74, then we put it back and remove another edge to the 2nd 60-74 and so forth. It turns out that all 75 of the so obtained 60-74 sets are isomorphic to each other and that they all reduce to a single MMP hypergraph string 60-74 given in Appendix A1.

In [27] we obtained only about 8,000 KS sets and many were missing. Here we have 1.54 × 10^9 sets and among them all types of sets that were 60-74 critical sets that were obtained by means of much faster parity proofs and which were missing in [27, Table 2]. We also obtained new types of KS sets with both even (mostly) and odd number (23) of vertex-edge sets; before that, 59-32 after 59-32, 55-37 after 55-37, and all the other 150 sets; before that, 59-32 after 59-32, 55-37 after 55-37, and all the other 150 sets; before that, 59-32 after 59-32, 55-37 after 55-37, and all the other 150.

Our aforementioned conjecture that the table in Fig. 2 shows all the types of KS criticals from the 60-74 class is based on the following statistics. The table now shows 1.54 × 10^9 KS criticals. The last new type, 47-30, started to appear after we reached 1.07 × 10^9 sets; before that, 59-32 after 5.5 × 10^8, 55-37 after 3.37 × 10^8, and all the other 150 types were already appearing within 2.15 × 10^8 generated sets. Here we stress that our method of generating sets is as random as a program can possibly be and that therefore the “late” appearance of the aforementioned three types is due only to their very low occurrence among the sets, i.e., to a minuscule probability to appear at all.

This can be well illustrated by looking at the KS criticals with parity proofs. Among all 1.5 × 10^9 criticals only 1.2 × 10^5 have parity proofs and among them some are still missing. In particular, we have 3 × 10^6 60-39 criticals and 3.5 × 10^6 60-41 criticals and none of them has a parity proof although there are at least two (60-39 and 60-41 whose criticals only 1.5 × 10^9 criticals only 1.2 × 10^5 have parity proofs and among them some are still missing. In particular, we have 3 × 10^6 60-39 criticals and 3.5 × 10^6 60-41 criticals and none of them has a parity proof although there are at least two (60-39 and 60-41 whose MMP hypergraph strings are given in Appendix A1) that do have such a proof which we obtained by means of a...
parity-proof program in [50]. The strings are presented with their maximal loops, hexadecagon and heptadecagon (first 16 and 17 edges up to “,”), respectively, to facilitate graphical representation. In Fig. 3 60-41 is drawn (vertex “2” is indicated and and other vertices from the loop follow anti-clockwise) and we can see that there are 22 encircled (in red online) vertices that share four edges and, of course (otherwise we would not have a parity proof), not a single one of which would share three edges. The probability that a randomly generated hypergraph has such a structure is extremely low and this explains why we did not get them even after more than $10^{15}$ runs.

We used a procedure that strips one edge at a time of smaller and smaller sets and simultaneously checks them on KS property, KS criticality, maximal loops, number of iterations, level of classical non-contextuality of each set, etc. A choice of them is represented graphically by means of MMP hypergraphs in Fig. 3

![MMP hypergraphs](image_url)

**FIG. 3.** MMP hypergraphs from the 60-74 class shown with the help of their maximal loops; 26-13 is the smallest set from the class—the arrow points at a “graphical proof” of contextuality (all zeros, while rings (green online) denote “1”); 26-13 through 36-19 all have parity proofs; the first two 30-15 and the last 34-17 have two axes of symmetry; three middle 34-17, one axis; 38-22 are the smallest sets that have even number of edges; 39-23 is the smallest set with an odd number of edges which does not have a parity proof; 54-29 and 54-30 are the two smallest sets with the biggest loops (18-gon); 54-34 is a typical large set; 60-41 belongs to the largest sets of criticals; it does have a parity proof, while other 60-41 criticals do not have it (see text).

The KS criticals 26-13 to 36-19 all have parity proofs and among the sets with up to 38 vertices and odd number of edges there is no one which fails the parity proof. The first sets with odd number of edges without parity proofs are 39-23 sets. One of them is shown in Fig. 3 in which arrows point to vertices that share an odd number of edges and therefore violate the parity proof condition from Def. 1.3. Actually, none of the 39-23 sets satisfy the parity proofs and this is the reason why this type of sets is missing in Table 1 of [50].

None of the sets with even number of edges can have a parity proof per definition. Two of the smallest such sets are 38-22 and 38-22a shown in Fig. 3

Two of the smallest sets with the biggest maximal loops in the class, octadecagon, are 54-29 and 54-30. They show an interesting property of having all vertices contained in the maximal loop like the smallest sets 26-13 and 30-15. Set 54-29 does not have a parity proof because it contains vertices that share three edges. It also has a property that some of its vertices share only one edge which most smaller set do not posses.

As we can see from Fig. 3 the maximal loops range from octagon (26-13) to octadecagon (54-29) in contrast to the sets from the sets from the 24-24 class in Fig. 1. On the other hand, the majority of sets from the 24-24 class have edges which intersect each other at more than one vertex, while in the vast 60-74 class there is not a single such set. It follows that not only the two classes are disjoint but that is also unlikely that they would belong to a wider class which would contain them both.

However, there is a class which contains the 24-24 class—the 60-105 one, which we present in the next section.
V. 60-105 CLASS OF 4-DIM KS SETS DEFINED BY HILBERT SPACE OPERATORS AND PROPERLY CONTAINING 24-24 CLASS

When we envisage an application of KS sets in the field of quantum computation and communication, a qubit implementation comes forward as most interesting. And while the real vectors of the KS sets from the 24-24 class do enable a qubit representation, as recent experiments have shown, it is not clear whether the vector components of the real vectors defining the 60-74 class offer us a qubit representation. Recall that the dimension of the Hilbert space of a quantum system and the spin of this system satisfy \( \dim \mathcal{H}_s = 2s + 1 \). So, a 4-dim KS set can be realised either via an \( s = 3/2 \) particle, say by means of a Stern-Gerlach device, or via two qubits: \( \dim(\mathcal{H}^2 \otimes \mathcal{H}^2) = 2^2 = 4 \)

In order to achieve a qubit representation in the complex 4-dim Hilbert space, by means of complex vectors, Aravind and Waegell [5] made use of Pauli operators (e.g., \( \sigma_x^{(1)} \otimes \sigma_y^{(2)} \)), where the superscripts refer to one of two qubits. In a 4-dim Hilbert space they form 9 mutual tensor products and 6 tensor products with the unit vectors. Altogether, these 4-dim operators form 15 commuting triplets each of which has four eigenvectors (tetrads) in common. There are 60 different eigenvectors that form the resulting 105 tetrads as given in Tables 1 and 2 of [5]. Their components take values from the set \( \{0, \pm 1, \pm i\} \). A few lines of the former Table are given in Table I below.

| Pauli product triple | 4 eigenvectors of each product from the triple |
|----------------------|-----------------------------------------------|
| \( \sigma_x^{(1)} \otimes I^{(2)} \) | \( |000\rangle \), \( |010\rangle \), \( |001\rangle \), \( |011\rangle \) |
| \( \sigma_z^{(1)} \otimes I^{(2)} \) | \( |100\rangle \), \( |101\rangle \), \( |110\rangle \), \( |111\rangle \) |
| \( \sigma_y^{(1)} \otimes I^{(2)} \) | \( |010\rangle \), \( |01i\rangle \), \( |0i0\rangle \), \( |i00\rangle \) |
| \( \sigma_x^{(1)} \otimes \sigma_y^{(2)} \) | \( |001\rangle \), \( |01i\rangle \), \( |100\rangle \), \( |10i\rangle \) |
| \( \sigma_z^{(1)} \otimes \sigma_y^{(2)} \) | \( |010\rangle \), \( |01i\rangle \), \( |0i0\rangle \), \( |i00\rangle \) |

TABLE I. A sample from a complete list of 15 Pauli operator products and their eigenvectors given in Ref. [5].

The latter table represents a 60-105 master set. Its MMP hypergraph string is given in Appendix A.2. By removing one of 105 edges from the string at a time, each time a different one, we obtain 105 sets. They all turn out to belong to two non-isomorphic non-critical KS sets in contrast to the 60-75 set which reduces to the unique 60-74 one. By applying the same technique as in Sec. IV we generate critical KS sets listed in the table in Fig. 4. They make the 60-105 class of KS sets. Although the generated critical KS sets from the 60-105 class are more than two orders of magnitude less numerous than the ones from the 60-74 class, the statistics indicate that the majority of types has been generated. MMP hypergraph of the master set 60-105 properly contains all MMP hypergraphs from the 24-24 class [5] and also the ones we obtained by means of our down-up generation in [31, 36] but which did not belong to the 24-24 class as well as new ones, which do not belong to either of those two kinds, shown in Fig. 5. That is why we called 24-24 class tentative in the title of Sec. III.

Still, with respect to vector representation, the 24-24 class is not uniquely determined by the coordinatization of the 60-105 master set. The vector components of the 60-105 set are complex (taking values from the set \( \{0, \pm 1, \pm i\} \)) and Peres’ 24-24 master set can take over them directly as shown in Appendix A.2.

But, as we mentioned above, for the master set 24-24 and therefore all of its subsets there exist real coordinatizations, e.g., the one originally found by Peres, and that is what Waegell and Aravind meant when they said that ‘60-105 system contain[ed] (in ten different ways) 24-24 systems of rays and bases used by Peres and others’ [5].

The fact that the 24-24 class can have both real and complex coordinatization depend on particular structure of its sets. In contrast, the systems 21-11 shown in Fig. 4 do not possess real coordinatizations, apparently due to the \( \delta \)-feature of their structure—see below.

Another example of different coordinatizations within the 60-105 class is Pavičić, Merlet, McKay, and Megill’s 20-11a [36], shown in Fig. 1. Its 60-105 coordinatizations might be complex as given in Appendix A.2 as well as real. If we compared the components with those of the 24-24 set, we would see that 20-11a might be generated (stripped) directly from the 24-24. The 20-11a also possesses real coordinatizations, though, one of which is given in [36].

On the other hand, Cabello, Estebaranz, and García-Alcaine’s 18-9 [46] and Kernaghan’s 20-11b [44] (both shown in Fig. 1) have real coordinatizations with components from \( \{0, \pm 1\} \) in 60-105 as given in Appendix A.2. By comparing their components we can see that they are not generated directly from the presented 24-24 set with the given complex coordinatization (it does not have enough real components) but from some other subsets of 60-105.
Critical 4-dim KS sets from the 60-105 class with 9 to 40 edges (columns) and 18 to 60 vertices (rows)  

| ⊗ |  9 | 11 | 13 | 15 | 17 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | ⊗  |
|---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|---|
| 18| 18  | 20  | 21  | 22  | 23  | 24  | 25  | 26  | 27  | 28  | 29  | 30  | 31  | 32  | 33  |
| 20| 20  | 21  | 22  | 23  | 24  | 25  | 26  | 27  | 28  | 29  | 30  | 31  | 32  | 33  | 34  |
| 21| 21  | 22  | 23  | 24  | 25  | 26  | 27  | 28  | 29  | 30  | 31  | 32  | 33  | 34  | 35  |
| 22| 22  | 23  | 24  | 25  | 26  | 27  | 28  | 29  | 30  | 31  | 32  | 33  | 34  | 35  | 36  |
| 23| 23  | 24  | 25  | 26  | 27  | 28  | 29  | 30  | 31  | 32  | 33  | 34  | 35  | 36  | 37  |
| 24| 24  | 25  | 26  | 27  | 28  | 29  | 30  | 31  | 32  | 33  | 34  | 35  | 36  | 37  | 38  |
| 25| 25  | 26  | 27  | 28  | 29  | 30  | 31  | 32  | 33  | 34  | 35  | 36  | 37  | 38  | 39  |
| 26| 26  | 27  | 28  | 29  | 30  | 31  | 32  | 33  | 34  | 35  | 36  | 37  | 38  | 39  | 40  |
| 27| 27  | 28  | 29  | 30  | 31  | 32  | 33  | 34  | 35  | 36  | 37  | 38  | 39  | 40  | 41  |
| 28| 28  | 29  | 30  | 31  | 32  | 33  | 34  | 35  | 36  | 37  | 38  | 39  | 40  | 41  | 42  |
| 29| 29  | 30  | 31  | 32  | 33  | 34  | 35  | 36  | 37  | 38  | 39  | 40  | 41  | 42  | 43  |

FIG. 4. List of 7,720,539 non-isomorphic KS critical sets from the 60-105 class we obtained on our cluster. We conjecture that all possible types of vertex-edge sets are given here. We obtained no critical sets with 10, 12, or 14 vertices.

FIG. 5. MMP hypergraphs of KS critical sets from the 60-105 class with up to 24 vertices that are not isomorphic to the ones shown in Fig. 1. They share edges of the form \( \alpha \), \( \beta \), and \( \gamma \) which characterise 24-24 sets but not, e.g., the one of the form \( \delta \), which is specific to the 60-105 sets. Maximal loops of the criticals shown here range from hexagons to octagons.

Of course, here we should pose a question on whether there is an even wider class which properly contains the criticals from the 60-105 class and this is an open question. Since the biggest such criticals contain only 60 vertices such a bigger class might exist (the master set from Sec. VI has 300 vertices). However, a wider class which would properly contain both 60-74 and 60-105 might not exist, since these two classes have too disparate properties. First, not a single critical KS set from the 60-105 class is isomorphic to any of the \( 1.5 \times 10^9 \) critical KS sets from the 60-74 class. Second, there is an important structural difference together with all similarities.

The similarities are of the \( \alpha \), \( \beta \), and \( \gamma \) kind shown at 24-13b and 24-13c in Fig. 5. \( \alpha \) is an edge whose vertices each share a single edge from the maximal loop; \( \beta \) consists of 2 such vertices, 1 which shares two loop edges and a third edge and 1 which shares only that third edge; \( \gamma \) is the third edge from the previous \( \beta \) definition.

A definite difference with and a dominant feature of 65-105 sets is the \( \delta \)-feature (see Fig. 5). It refers to two neighbouring edges from the maximal loop exclusively sharing two vertices, i.e., intersecting each other at two vertices.
which do not share any third edge. The $\delta$-feature characterises most of the criticals shown in Figs. 5 and 7. It might correspond to a rank-2 projector and be related to the fact that in a KS test one need not distinguish which of the two vertices that share two edges was assigned a 1. The role of projectors of a higher rank in a description of KS sets has been explored by Waegell and Aravind in details in [8].

The portion of sets from the 60-105 class with an odd number of edges which possess the parity proofs and the overall number of sets from the class with the parity proofs is much higher than in the 60-74 class. Of $7.5 \times 10^6$ 60-105 criticals, we obtained, $5.72 \times 10^6$ have parity proofs, i.e. 76.3%. The latter number includes 6 criticals from the former 24-24 class which all have parity proofs. There are 132 types of KS criticals with an odd number of edges of which 45 were previously reported by Waegell and Aravind [8] and additional 12 by Pavičić [51] and 111 with an even number of edges of which 22 were previously found by Pavičić [51].

A general feature of all classes is that smaller sets have only odd number of edges and that they all have parity proofs. On the other hand, among large sets with odd number of edges there are only a very few ones with the parity proofs. As we saw in Sec. [LV] we did not obtain a single such 60-39 or 60-41 set in the 60-74 class although they exist (and are given above) and of 21 60-39 sets in the 60-105 class no one has a parity proof and, to our knowledge, it is not known whether such a set exists. One of 11 smallest sets without a parity proof is the 26-15 shown in Fig. 6.

Encircled vertices (in red online) do not satisfy the parity proof condition; they do not share an even number of edges.

The smallest sets with even number of edges are 29-16. In Fig. 6 a sample of them is shown with vertices which share only one edge drawn as rings (red online). As the number of vertices and edges increase there are fewer and fewer such vertices which are dominant among 3-dim KS criticals (see Sec. [XI]). Yet, there is one of them in the 60-40 set.

Our generation of KS sets via stripping of master sets was so far completely random. As we already stressed this does require a considerable amount of CPU time. What slows down the generation is not the stripping itself, which is extremely fast, but filtering on the KS property and criticality. Algorithms which would be focused on particular arrangement of vertices and edges might prove more efficient and even serve us to obtain KS sets without previous stripping from any master set. Possible arrangements of such a kind are the ones which would have all vertices contained within a single loop as 26-13 to 46-23 or nearly so as 50-25 and 54-27 in Fig. 7.

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VI. 300-675 CLASS OF 4-DIM KS SETS CONTAINING 60-74 CLASS

In 2011 Waegell, Aravind, Megill, and Pavičić made use of 600-cell convex regular 4-polytope to obtain a 60-75 master KS set and a huge number of KS criticals which we call the 60-74 class [52]. Three years later, in 2014, Waegell and Aravind considered its dual 120-cell and obtained a 300-675 master set and from it a number of different KS sets via parity proofs [52]. In particular, using parity proof algorithms and programs, they found the following 102 types of critical KS sets from their 300-675 master sets: 38-19, 42-21, 44-46-23, 48-50-25, 50-54-27, 52-58-29, 54-62-31, 56-66-33, 58-70-35, 60-74-37, 53-78-39, and 65-82-41. We show MMP hypergraphs for some of them (38-19, 42-21, 48-25) in Fig. 8.

![Fig. 8. KS criticals from the 300-675 class together with one non-KS set (see text); KS criticals from the higher vertex-edge group (211 to 283 vertices and 127 to 188 edges) are represented by circles since the vertices and edges are too numerous to be discernible in a figure—they are all listed in Fig. 9 through; black circles represent maximal loops with 47 (Schläfli symbol (47)) and 57 edges ([57]).](image-url)

Among the smallest KS criticals we generated from the 300-675 master set are one 26-13 (shown in Fig. 8), two (non-isomorphic) 30-15, one 32-17, one 33-17, four 34-17, two 38-19 (one of them, 38-19b, is shown in Fig. 8), one 43-24, and one 44-26. Apart from the last two, all of them have parity proofs. These KS criticals with parity proofs are subgraphs of the master set 60-74 and therefore belong to the 60-74 class.

Waegell and Aravind actually show in [52], by the very construction of the 300-675 master set, that the 60-75 master set is properly contained in it, i.e., that the hypergraph of the 60-75 master sets is contained in the hypergraph of the 300-675 master set.

Next, in Fig. 8 we present three hypergraph MMP representations of the KS criticals obtained in [52]: 38-19 (max loop: 10-gon), 42-21 (11-gon), and 48-25 (12-gon). Their MMP hypergraphs are given in Appendix A3. Our program subgraph shows that these KS criticals are not subgraphs of the master set 60-74 (or 60-75) and that, therefore, the class 60-74 does not contain them.

Here we use the opportunity to show yet another advantage of the hypergraph approach to KS sets. Waegell and Aravind made a misprint somewhere in their Table 7 [52] which should have defined their 48-25 set but an automated translation gives a hypergraph denoted “non-KS 42-21” in Fig. 8. To find the misprint in their list of vertices and edges one should invest a considerable amount of time and most likely they themselves as well. However, in our representation it is immediately visually apparent that in a parity proof the vertex can neither share three edges, denoted by "\*", nor just one, denoted by "?". Therefore we can easily amend the misprint by disconnecting the brown edge from the "\*"-vertex and extending it to the "?"-vertex, so as to obtain the 48-25 KS critical set shown as the next set in the figure; its ASCII MMP representation is given above. It provably does not belong to the 60-74 class.

Further advantages are obvious from the generation of a cluster of unprecedentedly big KS criticals indicated in the last two figures in Fig. 8 and listed in all details in Fig. 9. The generation of KS criticals in the 300-675 class is an extremely demanding task due to the intricacy of the master set 300-675 itself which stems from the high number of vertices. If we strip too many loops in the first step with mmpstrip we shall find ourselves in the non-KS desert, i.e., the probability of finding a KS set will be too small. If we strip only, say, 500 loops, from the master set, verification of whether a single obtained MMP hypergraph is a KS set and if it is to reduce it to a critical KS set will take between one and three CPU-months (3 GHz). At the first glance it might look as if Waegell and Aravind also stumbled upon this problem of intricately interwoven edges and orthogonalities: “we have not found any [set] with more than 41 bases, but we cannot be sure about the upper limit because our searches have been limited to only the reduced sets in Table 4” [52, p. 1093, bottom].

However, they actually could not have found them because with their parity-proof-based programs they could not have seen them at all. More precisely, 14% of all criticals in the 300-675 class have a parity proof but the probability KS sets having them is not uniformly distributed throughout the class. All KS sets with parity proofs are in the bottom part of the class. KS sets from the top part of the class do not have parity proofs—none of the 221-127 to 283-188 generated KS criticals has a parity proof. Hence, they are invisible for parity-proof-based algorithms and programs and since search algorithms in the literature rely almost exclusively on parity proofs we give an MMP representation of the 221-127 critical KS set (47-gon) in Appendix A3.
Higher criticals from the class 300-675 we obtained and presented in Fig. 9 are far less numerous than KS criticals from any other class we presented in this paper. This is, however, not due to a small number of sets in the class—their overall number is according to our tests staggeringly huge; this is due to the fact that their generation is computationally extremely demanding and time consuming. Therefore we generated the sets in stages and subjected them to several levels of filtering. We first randomly stripped 400 to 550 edges from the master set 300-675 by mmpstrip thus obtaining 150 groups of sets with 275 down to 125 edges. Then we filtered these sets for the KS property and randomly reduced them to criticals by means of states01. This procedure takes up to three CPU months for each single critical. We generated higher criticals from the KS noncritical sets in the range from 190 to 275 edges. E.g., the 211-127 KS critical we obtained from a set with 190 edges. For sets with less than 190 edges we observed a sudden drop to criticals with up to 40 edges.
VII. 148-265 CLASS OF 4-DIM KS SETS

In January 2017 Waegell and Aravind [53] showed that that the Penrose dodecahedron, Zimba and Penrose used to construct their 40-40 non-critical KS set [30], can be extracted from the Witting polytope in $\mathbb{C}^4$. Actually Waegell and Aravind consider a 148-265 KS master sets and their subsets, 40-40 being one of them. Since this is a work in progress, we shall not go into details but will only list the types of KS criticals the master set 148-265 can be reduced to and give two of their hypergraphs, in Fig. 10, so as to round up our presentation of generation of KS criticals from all known 4-dim KS master sets and their classes.

![Hypergraph of the 148-265 KS set](image)

FIG. 10. KS criticals from the 4-dim 148-265 class; none of them has a parity proof; 40-40 circle (red online) indicates the Penrose 40-40 non-critical KS set; inset (a) shows one of smallest KS criticals generated from Penrose’s 40-40 set; inset (b) shows one of the smallest KS criticals not contained in the 40-40 set.

Waegell and Aravind in [53] make use of a rather involved coordinatization but they also indicate that a simpler one, in which vector components take the values from the set $\{0, \pm 1, \pm \omega, \pm \omega^2\}$, where $\omega = e^{2\pi i/3} = (-1 + i\sqrt{3})/2$, can be used [53, Eq. (6)]. We explicitly verified that, in the master set 148-265, vectors can indeed be ascribed a valuation from this set which means that all sets from the 148-265 class can easily be given a random valuation with the help of our program $\text{vectorfind}$ by simply introducing the 7 values given above as its options. Two examples of such a valuation are 40-23 and 49-27 KS criticals. 40-23 MMP hypergraph, shown in Fig. 10(a) is one of 56 40-23 subgraphs of Penrose’s 40-40 KS hypergraph. 49-27, shown in Fig. 10(b), is the smallest critical from the 148-265 class which is not contained in its 40-40 set. MMP hypergraph strings of 40-23 and 49-27 are given in Appendix A 4.

In contrast to the smallest sets from the other 4dim KS classes the above smallest hypergraphs do not show geometrical symmetries and that is caused by the geometrical features of the Witting polytope which in turn cause that the vertices share both even and odd number of edges, i.e., that they do not have parity proofs. Actually, none of 250140 KS criticals in the 148-265 class we obtained has a parity proof, so, they are completely invisible for the parity-proof-based algorithms and programs.

The maximal loops of the criticals are up to 36-gons big and therefore smaller than the ones of all the higher KS criticals from the 300-675 class but bigger than ones of all the other KS criticals from any other class.

Similarly to 60-74 and 300-675 and unlike 24-24 and 60-105 classes, no two edges share more than one vertex.

Program $\text{subgraph}$ verified that the master set 148-265 is not a subgraph of the master set 300-675. Program $\text{shortd}$ verifies that the classes 148-265 and 300-675 are completely disjoint.

VIII. 21-7 6-DIM KS SET AND 236-1216 CLASS OF 6-DIM KS SETS

Lisoněk, Badziąg, Portillo, and Cabello [24] recently found a 6-dim 21-7 KS set which they drew in the form of a seven pointed star, a regular heptagram with Schläfli symbol $\{7/3\}$, as shown in Fig. 11 (It was experimen-
tally implemented in [23].) They chose vector component values from the set \( \{0,1,\omega,\omega^2\} \), as we did in Sec. 7, however, since \( \omega \) is a cube root of 1 and therefore \( \omega \times \omega^2 = 1 \), and 1 is already present as a component, for their set \( \omega^2 \) is not needed. To see this, we start with the MMP hypergraph representation of the seven star set: 123456, 6789AB, BCD3EF, F5G8HI, IAJD2K, KE4G7L, LH9JC1. We assign 1 to any point and then proceed along the edges. Our program vectorfind can then assign the components to vertices: \( 1=(0,0,0,0,0,1), 2=(0,0,0,1,0,1), \ldots, 6=(1,0,0,0,0,0), 7=(0,1,0,1,1,0), 8=(0,0,1,1,0,0) \), \( A=(0,0,1,0,1,1), B=(0,1,0,0,1,0), C=(1,0,1,0,0,0), D=(0,1,1,0,0,1), E=(0,0,0,0,1,1), F=(1,0,0,1,0,0), G=(1,0,0,1,0,1), H=(0,1,0,1,0,0), I=(0,0,1,0,0,1), J=(1,1,0,0,0,0), K=(1,0,0,0,0,1), L=(0,0,0,1,1,0) \). Recall that dot products (orthogonality of vectors) involve the complex conjugates, e.g., \( K \cdot L^s = \omega^* + \omega + 0 + 0 + 1 + 0 + 0 = (−1 − i\sqrt{3})/2 + (−1 + i\sqrt{3})/2 + 1 = 0 \).

The heptagram is isomorphic to a triangle (\( \Delta \)) hypergraph shown in Fig. 11 below the star (\( \star \)). The advantage of the triangular representation is that it can describe both odd and even dimensional sets while the star like representation is limited to the even dimensional sets. 4-dim triangle (5 pointed star), which does not admit a 0-1 state, would be a 10-5 KS set if it had a vectorial representation in the complex Hilbert space, but apparently it does not have it. The triangle is up to 1,000 times faster; the 5-dim triangle (15-6 set) is isomorphic to a 3-dim triangle; the 7-dim triangle (28-8) is KS hypergraphs (they do admit 0-1 states), but the 8-dim one (36-9) is. There are also, hypergraphs which do not admit non-contextual 0-1 states, e.g., those smaller than the 18-9 [36], or the above 10-5 one, for which we actually do not know whether they have a vectorial representation from some other sets. A direct solving of nonlinear equations which would answer this question is rather demanding.

Neither the 5-dim triangle (15-6 set) nor the 7-dim triangle (28-8) are KS hypergraphs (they do admit 0-1 states), but the 8-dim one (36-9) is. The latter KS set is also not a subgraph of the 120-2024 class (see Sec. IX).

The authors of [24] have made a big step in trying to find a bigger 6-dim set but did not find any.

Waegell and Aravind appreciated the approach as the first one “in a dimension that is not of the form \( 2^N \) [52], meaning that the 6-dim space cannot “host” qubits (recall that two qubits reside in the \( 2^2 \)-dim, i.e., 4-dim Hilbert space, three qubits in the \( 2^3 \)-dim, i.e., 8-dim space, etc.). Subsequently Aravind and Waegell [54] designed a 6-dim 236-1216 master KS set but since it did not allow parity proofs they could not generate smaller sets with their parity proof programs. So, they sent the master set to us and we generated \( 3.7 \times 10^6 \) KS criticals in this paper. We say that they make the 236-1216 class. Their statistics is shown in Fig. 12. The vector components take values from the set \( \{0, \pm 1/2, \pm 1/\sqrt{3}, \pm 1/\sqrt{2}, 1\} \). The class does not contain the 21-7 KS set, though (verified with subgraph).

Aravind [34] has arrived at the 236-1216 master KS set by considering hypercubes which led him to a hexeract (6-cube, 6-dim cube) with Schläfli symbol \( \{4,3,3,3,3\} \) or \( \{4,3^4\} \). The master set written in the MMP notation occupies more than 3 pages so we do not print it here.

The approach of Aravind and Waegell is very geometrical and unorthodox and by no means straightforward, so, it is outside of the scope of the present paper. It will be presented in detail in a separate publication. The master set in the MMP notation is given in our repository.

The features of the 236-1216 class are:

- Its KS sets cannot be implemented via qubits but can via spin-\( \frac{5}{2} \) quantum systems;
- Its smallest KS sets have an even number of edges and small sets with odd and even number of edges are evenly distributed, unlike in any other class;
- Although the number of vertices of the master set is comparable with the 4-dim 300-675 and the number of edges is twice as high, the criticals are computationally much easier to generate; a generation of a single KS critical is up to 1,000 times faster;
- Types of sets with a definite number of edges and different number of vertices are more numerous than in other classes (columns in Fig. 12 are higher than in other tables); a dynamic algorithm compensated for a lower occurrence of smaller KS sets;

![FIG. 11. 6-dim KS critical sets: \( \star/\Delta \) 21-7 and KS from the 236-1216 KS class; \( \star \) 21-7 is from [23]; \( \Delta \) 21-7 is isomorphic to \( \star \); others are critical KS sets from the 236-1216 class; 53-21 has a parity proof.](image-url)
FIG. 12. List of 3,714,503 non-isomorphic 6-dim K5 critical sets from the 236-1216 class; 16 to 87 edges (columns); 34 to 177 vertices (rows); 169-78 to 87-177 sets are show in the inset.
— Edges connect vertices in much more irregular way than in other classes as the figures in Fig. 11 show. We were not able to find a single symmetric hypergraph;
— Statistics from Fig. 12 shows gaps in the KS sets with high number of edges indicating that a more extensive generation would generate many more sets possibly with higher number of edges and vertices.

There is another peculiarity we should mention.

As already stressed above, in the literature, most of the KS proof have been found via parity proofs. However, in the 236-1216 class among $3.7 \times 10^6$ KS critical sets we generated we found only 8 KS critical sets with a parity proof. Their edges are in the interval from 21 to 39. We shall present and discuss them in detail in a subsequent publication and here we only show one of them (53-21) in Fig. 11.

IX. 120-2024 CLASS OF 8-DIM KS SETS AND $\star/\triangle$ 36-9 8-DIM KS SET

We start with a brief description of generation of 8-dim KS sets which can be realised with either 3 qubits ($2^3 = 8$) or spin-$\frac{7}{2}$ systems. In 1995 Kernaghan and Peres produced a 36-11 KS critical set and a 40-25 non-critical one (experimentally implemented in [26]) from which several smaller ones including 36-11 can be obtained [48]; in 2006 Ruuge and van Oystaeyen gave a scheme for constructing 8-dim KS proofs but did not themselves construct any [55]; in 2012 Ruuge claimed to have given an example of a 36-vertex 8-dim KS set [34] but we were not able to identify its octads of orthogonal vertices in [34] (nor to contact him), so, we could not verify whether it is isomorphic to 36-11 from [13] as claimed in [34] and finally, also in 2012, Planat discussed 8-dim KS sets that can be obtained from the Kernaghan-Peres’ 40-25 KS set [45]. In 2015 Waegell and Aravind obtained a KS master set with 120 vertices and 2025 edges and, from it, many smaller 8-dim KS sets, including non-critical Kernaghan-Peres’ 40-25 one [57] (see also [58]). In the present paper we generate $6.9 \times 10^6$ non-isomorphic KS criticals, listed in the table in Fig. 13 from that Waegell-Aravind’s 120-2025 master set. We also produce a new star/triangle ($\star/\triangle$) 36-9 KS set which is not a subgraph of the 120-2025 master set.

To obtain KS sets, in Refs. [55, 57], the authors made use of the Lie algebra E8. Waegell and Aravind reduced it to a collection of 120 vectors (rays, vertices) and 2025 bases (octads, edges) [57] to obtain their 120-2025 KS master set. We verified that by peeling off one edge at the time we obtain 2025 varieties of the 120-2024 KS sets which are all isomorphic to each other and therefore reduce to a single 120-2024 KS master set from which we generate the 120-2024 KS class, i.e., smaller KS criticals. Critical KS sets from the 120-2024 class are given in the table in Fig. 13.

The coordinatization (vector components) in [57] is taken over from D. Richter and is based on tetrads formed by expressions $r_m e^{i n \pi /30}$ (values of constants $r_m$ and $n$ are given in [57]) so that their real and imaginary parts form octads. Using this coordinatization Waegell and Aravind generate sets of bases (edges) which define their KS sets. We, however, do not need the coordinatization to obtain KS sets. We start with the master set 120-2024 and simply strip off edges. Then we filter the smaller sets via states01 to obtain critical KS sets. We can always add vector components later on, if needed.

The distribution of sets from the 120-2024 KS class is different from the above 6-dim class as well as from the three of the 4-dim ones and somewhat similar to the 300-675 4-dim class with respect to the following feature. The critical sets are split so as to be clustered in two groups of subsets with respect to the number of vertices and edges: first one, sparsely spread, over 9 to about 40 edges and 34 to about 100 vertices and the second one, densely populated, over about 41 to 58 edges and about 100 to 120 vertices as shown in the table in Fig. 13. The split structure of the 120-2024 class resembles the similarly split structure of the 4-dim class. We conjecture that there is only one or at most a few KS noncritical sets with about 100 vertices and 40 edges which most the smaller critical sets are subsets of.

Similarly to the 4-dim classes (with the exception of the 60-105 one) and the 6-dim class, the number of critical sets which exhibit a parity proof is very small with respect to the total number of critical sets, but on the other hand, parity proof algorithms used by Waegell and Aravind [57] are very efficient in generating the sets so that the two approaches (via the MMP algorithms for bare hypergraphs and the parity-proof-based ones for vectors corresponding to vertices of hypergraphs) turn out to be complementary. In particular, Waegell and Aravind [57] obtained the following sets which still did not appear in the course of our computer generation so far: 36-11 (Kernaghan-Peres), 38,39-13, 40,41,44,45-15, 48-17, 60-15,17,19,21,23,27, and 85-25 (we do show these sets in the table in Fig. 11 as $\bigcirc$); both Waegell and Aravind [57] and we in the present paper obtained 34-9, 36-9, 37-11, and 95-35; Waegell and Aravind [57] have not obtained all the other sets we obtained in the table in Fig. 13 and most of them they actually cannot obtain due to the features of the parity based algorithm they make use of but, still, the parity proof based programs confirm themselves as a powerful complimentary method of providing us with KS critical sets since our general MMP hypergraph algorithms are CPU-time demanding.

In Fig. 11 we show five chosen KS criticals from the 120-2024 class. KS criticals 34-9 are the smallest in the
FIG. 13. List of 6,925,540 non-isomorphic 8-dim KS critical sets from the 120-2042 class we obtained on our cluster and of those, denoted as $\otimes$, obtained by Waegell and Aravind [57] and still not by us.

class. KS 36-9 is particularly interesting because it can be viewed as an 8-dim version of 18-9 from Fig. 1(a) with graphically analogous edges where each vertex from the 18-9 is represented by a pair of vertices in the 36-9. KS 44-11 is one of the critical KS sets with the biggest maximal loop (heptagon) among the sets with 11 edges (second smallest number of edges). KS 52-16 has the smallest even number of edges. One of 14 KS 120-58 has the biggest maximal loop—tetradecagon (14-gon); it is not shown in the figure.

In Sec. [V] we have seen that the 24-24 class is contained in the 60-105 class and in Sec. [VI] that the 60-74 class is contained in the 300-675 class. On the other hand in Sec. [VIII] we have shown that the $\alpha/\Delta$ KS set is not contained in the much bigger 236-1216 class of the 6-dim KS classes. Here we verified that 8-dim $\alpha/\Delta$ 36-9 KS critical set, shown in Fig. [I] is not contained in also much bigger 120-2024 class of 8-dim KS sets.

The MMP representations of the star and triangle forms (they are mutually isomorphic) of the 36-9 critical are given in Appendix [A]. In Fig. [I] the first three edges correspond to the edges of the triangle as indicated by its vertices $1, 8, F$ and then the inner vertices are denoted in alphabetical order from left to right from the bottom horizontal ones (indicated by $M$ to $Q$) to the single $a$ at the top.

In contrast to 6-dim 21-7 set from Fig. [I] this 8-dim 36-9 can have real vector components from $\{-1, 0, 1\}$. Coordinatizations for the triangle and for the star are given in Appendix [A].
Interestingly, our program \texttt{vectorfind} finds the triangle coordinatization sooner than the one for the star.

The 8-dim star/triangle set is not smaller than the smallest sets from the 120-2024 KS class as the 6-dim one is with respect to the smallest sets from the 6-dim 236-1216 class; the 34-9 sets shown in Fig. 14 are smaller. The 120-2024 class contains at least seven 36-9 criticals but their structure is very different from the star/triangle 36-9 (Cf. 36-9 in the middle of Fig. 14).

Via our program \texttt{subgraph} we prove that the star/triangle 36-9 or any other 36-9 isomorphic to it cannot be obtained by stripping edges and vertices from the master set 120-2024 down to sets with 36 vertices and 9 edges, i.e., that it cannot be a subgraph of the master set and that it therefore does not belong to the 120-2024 class.

Of all sets from the 120-2024 class we generated so far, only ca. 0.1\% have parity proofs, notably 609 of 6,925,540. The star/triangle 36-9 does have a parity proof, though.

X. 80-265 CLASS OF 16-DIM KS SETS

In 2012 Harvey and Chryssanthacopoulos constructed an 80-265 KS master set in the 8-dim real Hilbert space with vector components from the set \{-1, 0, 1\} [59]. They considered it for four qubits (2^4 = 16) although— theoretically—it can also serve as a KS set for spin-\(\frac{15}{2}\) systems. The set has far too many redundant edges, so, Planat promptly designed a procedure to obtain smaller KS sets and he claimed to have obtained three sets with the initial number of vertices: 80-21, 80-22, and 80-23 [56], however, as we show below, his 80-21 and 80-22 are not KS sets and 80-23 is not critical. In this paper we generate \(4.1 \times 10^6\) non-isomorphic critical KS sets from the 80-265 master set. We say that KS critical sets that can be generated by stripping the 80-265 master set form the 80-265 class of KS critical sets. The ones we obtained so far are shown in the table in Fig. 15.

| Critical 16-dim KS sets from the 80-265 class; 11 to 23 edges (columns) and 72 to 80 vertices (rows) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 72 | 1873 | 1 | 14865 | 12 | 278 | | | | | | | | | | | | | |
| 73 | 95559 | 3618 | 6583 | 221 | 53 | 2 | 73 |
| 74 | 221640 | 20968 | 60132 | 6589 | 2098 | 73 | 1 | 74 |
| 75 | 237116 | 32967 | 230599 | 60783 | 33983 | 3899 | 216 | 75 |
| 76 | 123769 | 18717 | 394131 | 214168 | 228704 | 66601 | 9206 | 224 | 76 |
| 77 | 2033 | 273552 | 248801 | 588721 | 407809 | 117876 | 8322 | 118 | 77 |
| 78 | 1 | 193 | 89266 | 90023 | 490986 | 742553 | 446076 | 70408 | 2864 | 29 | 78 |
| 79 | 9 | 14360 | 11250 | 80457 | 106160 | 87482 | 21627 | 1592 | 20 | 79 |
| 80 | 1448 | 2392 | 33573 | 101333 | 169197† | 88464† | 13543 | 500 | 7 | 80 |

The original 80-265 master file printed in [59] took over 11 pages. Its MMP representation is much shorter. Still, it takes over one page. So, we shall consider some smaller examples, but, first, we shall check the sets Planat obtained in [56].

His set 80-21 given by 21 lines of Eq. (17) in [56] has an MMP rendering with 21 edges as given in Appendix A6. But this set is not a KS set. For instance, according to our program \texttt{states01} we can assign “1” to \(G, H, Y, o, r\) and \(u\), so as to exhaust all 21 edges, i.e., when we delete the edges that contain them, then none is left, meaning that
each contains one “1” and therefore the set is non-contextual. Cf. Fig. 1(a) where, e.g., we can assign “1” to none of vertices 789A and for which there is always an edge to which one cannot assign “1” at all vertices contained in it.

Then it is claimed that this 80-21 set together with the 1st line of Eq. (18) from [56], in MMP notation: 2ACEZbnjS(∗::?@, form an 80-22 KS set. However, this 80-22 is not a KS set, either.

All lines from Eqs. (17) and (18), the last line reading notuwty!#)−::<>? in MMP notation, form an 80-23 non-critical KS set. By deleting the first line, Zbhjprsv$:::*:=0, we get a non-critical 80-22 KS set. If we also deleted the eighth line (HIKLMPRQTVUW), we would get a non-critical 80-21 KS set. These sets contain one 80-20 critical set and two non-isomorphic 80-19 criticals—all shown in Appendix A6. Their maximal loops are pentagons. We obtained them from the aforementioned 80-23,22,21 via our program states01.

The goal of [56] was to find small KS sets, but the table in Fig. 15 shows that its non-critical KS set 80-23 is bigger than all 2.5 million KS criticals we generated from the master 80-265 set and listed in the table in Fig. 15. This shows that algorithms for automated exhaustive generation of MMP hypergraphs, although probabilistic until full exhaustion is reached, are indispensable sources for obtaining new KS sets. Still, the 80-20 and 80-19s we obtained with the help of our program states01 are not isomorphic to any of the 80-20s and 80-19s we listed in the table in Fig. 15. This is because the probability of generating any specific KS set via our programs mmpstrip and states01 is very low due to the their probabilistic algorithms. Within established probabilities for obtaining MMP hypergraphs with wanted number of edges and vertices they are generated completely at random.

The 16-dim KS criticals listed in the table in Fig. 15 have maximal loops in the range from a square to a heptagon as illustrated in Fig. 15. The vector components corresponding to vertices from the set (-1,0,1) for the master set listed in [59] can be traced down to any chosen MMP hypergraph from the table in Fig. 15 via any of our programs mmpstrip, states01, mmpshuffle, etc., or, equivalently, program vectorfind can generate the components directly for a given hypergraph.

In contrast to all previous classes of KS sets apart from 4-dim 60-105, 16-dim 80-625 class has a significant number of parity proofs, notably, 28%. There are ca. 64% criticals with an odd number of edges but only 44% of them have parity proofs.

Also in contrast to all previous classes, except the 6-dim 236-1216 class, the KS criticals of the 16-dim 80-265 class do not exhibit symmetries. They have rather intricate and dense structure. In particular, all vertices share at least two edges and some pairs of vertices share eight edges. Also, in contrast to KS sets from the classes in smaller dimensions (not counting the tentative 24-24 class for which we in Sec. V proved to be contained in the 60-105 class), there are no maximal loops bigger than heptagons. There are ca. 10% of squares, 86% of pentagons, 4.6% of hexagons, and 1%/0% of heptagons.

Why is 77-13 missing, while 76-13 has 123,769 non-isomorphic instances and 78-13 is present, is an open question. 16-dim star/triangle set does not admit 0-1 states and is critical and therefore would be a critical KS set if one found a coordinatization for it. We have not found any so far. It has 16 +1 =17 edges and (16+1)/2 =136 vertices, which are 1.7 times the highest number of vertices of the critical sets from the 80-265 class. Its structure is dissimilar to any obtained set from the 80-265 class so it is very unlikely that it might belong to it, however, for the time being, the program subgraph which would give us a definitive answer to this question it is still running.

XI. 160-661 CLASS OF 32-DIM KS SETS

Recently Planat and Saniga, extending Aravind’s and DiVincenzo-Peres’ generalisations of the Bell-Kochen-Specker theorem [60, 61], constructed a 32-dim KS master set with 160 vertices/vectors and 661 edges with a real
coordinatization from the set \{-1, 0, 1\} \[62\]. This is a very big set which corresponds to states of five qubits, so, they did not present it in their paper. But, M. Planat kindly sent us the set in their notation and we translated it to an MMP encoded hypergraph. Planat and Saniga only published a smaller 160-21 KS set they obtained from the master set. MMP hypergraph string of that set is given in Appendix A7 (Edge 14 in \[62\] which reads 08 should read 108.)

However, this KS set is not critical and it contains at least two smaller critical KS sets, 160-19 and 152-19 ones. There is no point in giving their MMP representations here because we obtained thousands of smaller KS criticals from the 160-661 master set as shown in the table in Fig. 17. The non-isomorphic KS criticals with the smallest number of edges (11) all have 144 vertices and we show one of them in Fig. 18.

| Critical 32-dim KS sets from the 160-661 class; 11 to 29 edges (columns) and 135 to 160 vertices (rows) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 135 | 1 | 135 |
| 144 | 31 | 4 |
| 145 | 21 | 2 |
| 146 | 100 | 15 | 59 | 11 |
| 147 | 253 | 54 | 400 | 219 | 124 | 10 |
| 148 | 301 | 61 | 1337 | 1548 | 1780 | 553 | 59 | 1 |
| 149 | 226 | 26 | 798 | 523 | 675 | 212 | 24 | 1 |
| 150 | 217 | 16 | 1895 | 2605 | 7129 | 5507 | 1379 | 53 | 1 |
| 151 | 127 | 3 | 976 | 513 | 1350 | 776 | 286 | 13 | 2 | 1 |
| 152 | 65 | 5 | 1647 | 1733 | 9609 | 13324 | 8862† | 1578 | 60 |
| 153 | 852 | 380 | 2629 | 3016 | 2547 | 683 | 80 | 1 | 1 |
| 154 | 1 | 739 | 440 | 5974 | 11667 | 16824 | 7802 | 1278 | 67 | 1 |
| 155 | 283 | 93 | 2015 | 2998 | 5901 | 3743 | 945 | 91 | 1 |
| 156 | 152 | 66 | 2476 | 5171 | 14443 | 13922 | 6035 | 1027 | 66 | 4 | 1 |
| 157 | 34 | 8 | 700 | 976 | 4478 | 6140 | 4048 | 974 | 117 | 5 | 1 | 1 |
| 158 | 10 | 6 | 451 | 906 | 4866 | 9188 | 9879 | 4395 | 942 | 82 | 2 | 2 |
| 159 | 3 | 3 | 49 | 43 | 475 | 1285 | 2556 | 2269 | 1055 | 228 | 17 | 3 | 1 | 2 |
| 160 | 1 | 57 | 51 | 861† | 2405 | 4272 | 4133 | 2071 | 562 | 74 | 5 | 2 | 2 | 2 | 160 |

FIG. 17. List of 254,318 non-isomorphic 32-dim KS critical sets from the 160-661 class and 52-19 and 60-19 criticals we derived from Planat and Saniga’s \[62\] non-critical 160-21; the latter criticals are included in \[6862 \text{52-19s and 861 \text{60-19s, respectively, and indicated by } † \text{ in the table.}]

| Critical 32-dim KS sets from the 160-661 class; 11 to 29 edges (columns) and 135 to 160 vertices (rows) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 11 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 152 | 65 | 5 | 1647 | 1733 | 9609 | 13324 | 8862† | 1578 | 60 |
| 153 | 852 | 380 | 2629 | 3016 | 2547 | 683 | 80 | 1 | 1 |
| 154 | 1 | 739 | 440 | 5974 | 11667 | 16824 | 7802 | 1278 | 67 | 1 |
| 155 | 283 | 93 | 2015 | 2998 | 5901 | 3743 | 945 | 91 | 1 |
| 156 | 152 | 66 | 2476 | 5171 | 14443 | 13922 | 6035 | 1027 | 66 | 4 | 1 |
| 157 | 34 | 8 | 700 | 976 | 4478 | 6140 | 4048 | 974 | 117 | 5 | 1 | 1 |
| 158 | 10 | 6 | 451 | 906 | 4866 | 9188 | 9879 | 4395 | 942 | 82 | 2 | 2 |
| 159 | 3 | 3 | 49 | 43 | 475 | 1285 | 2556 | 2269 | 1055 | 228 | 17 | 3 | 1 | 2 |
| 160 | 1 | 57 | 51 | 861† | 2405 | 4272 | 4133 | 2071 | 562 | 74 | 5 | 2 | 2 | 2 | 160 |

FIG. 18. 144-11; one of 31 smallest 32-dim critical KS sets.
It is interesting that a single KS set with 21 edges (the number of edges of the KS set from [62]) has 9 vertices less than any other set we found (135); indicated by \( \times \) in the table in Fig. 17. This might stem from some geometrical structure of the set or its smaller subset. We do not show the 135-21 hypergraph because it has almost twice as many edges as 144-11 and its figure would be much more difficult to read. Also 135-21 maximal loop is a square and its hypergraph has 47 vertices outside of the loop as opposed to 24 such vertices of the 144-11 hypergraph shown in Fig. 13.

We find the complexity of KS criticals from the 32-dim 160-661 class similar to the one of the 16-dim 80-265 class. Not a single vertex shares only one edge, and some pairs of edges share 16 vertices. The distribution of distances between maximal bases was considered in [62] for a single non-critical 160-21 set. In our approach such a distribution does not play any role for either obtaining thousands of KS criticals or proving that they really are KS sets.

The vector components of vertices from the set \{-1,0,1\} for the master set they are listed in [62]. As for all the sets from the previous classes given above we can either trace them or generate them for any given hypergraph.

Of all \( 2.5 \times 10^5 \) KS criticals we obtained, only 10.7\% have a parity proof.

In contrast to all KS sets from the classes in smaller dimensions, there are no maximal loops bigger than hexagons. There are 11.9\% of squares, 87.4\% of pentagons, and 0.7\% of hexagons.

The star/triangle KS set, although not admitting 0-1 states and being critical, is far too complicated to be considered here. It has \((32+1)/2=528\) vertices and \(32+1=33\) edges which makes it far bigger than any of the critical sets from the present class. However, if one found a coordinatization for it would be the biggest critical KS set of all known ones.

XII. 3-DIM KS SETS

The successful generations of all the above presented KS sets in up to 32 dimensions were enabled by newly found big master sets and they were in turn derived from various polytopes (like, e.g., 120-cell and 600-cell), or Lie algebras, or some involved individual constructions which made use geometric symmetries of even-dimensional spaces. Even without the big master sets a direct generation of smaller KS sets is possible via our MMP algorithms [63] because in four and higher dimensional space those KS sets are pretty small. Disparately, for the 3-dim space, so far, no one has come forward with a master set and of the few known 3-dim KS sets no one is small and all of them are critical and cannot be lessened.

Since it would be very important to find more 3-dim KS sets to gain a better insight into the structure of contextual KS sets and enable new breakthroughs in their generation and application algorithms and programs, in this section we give MMP representations and KS hypergraphs of the four known (the only known ones) 3-dim KS sets and one that was claimed to be of such kind (the Yu-Oh 13-set), but is not, as we show below.

The full specification of all vertices (their vector components) is, as shown by Larsson [64] and Pavičić, Merlet, McKay, and Megill [65], indispensable “for an experimental realisation, which involves procedures equivalent to basis rotations” [66, p. 332, end of the 1st par.]. E.g., spin-1 particle flying through a sequence of generalized Stern-Gerlach devices whose filters/paths correspond to 3 orthogonal eigenprojections of the spin observable [32] and we would not have a correct measurement statistics if we ignored some of the vertices present in particular edges.

As shown in Fig. 19, Bub’s [45], Conway and Kochen’s [48], Peres’ [42], original Kochen and Specker’s [29] KS sets and Yu and Oh’s non-KS set [67], have 49, 51, 57, 192, and 25 vertices, respectively (and 36, 37, 40, 118, and 16 edges, respectively). In Fig. 19 the vertices that share only one edge are denoted by fully greyed dots and grey ASCII characters. If we ignored them in an implementation, we would be left with 33, 31, 33, 117, and 13 vertices, but then the measurements would give us incorrect data as we explained above. Surprisingly, in all presentations of their KS sets the aforementioned authors simply dropped the (grey) vertices that shared only one edge in an attempt to present their KS sets as being smaller and therefore more attractive for possible implementations.

Yet, all that vertices/vertices have definite vector components in the coordinatization they made use of. Thus, it is just the visual presentation of these KS sets in the original papers and subsequent reviews in numerous articles and books of these sets that are misleading, not the actual structures of them (which are perfectly correct).

In 2012, Yu and Oh published a paper [67] in which they introduced a set with 13 vertices which they call a 13-ray set—13-vertex set in our notation. The set is displayed in Fig. 19 where the 13 vertices are shown as red dots (in online version; black dots in printed version). Yu and Oh dropped the vertices that share only one edge, shown as grey dots in the figure (12 of them), following the aforesaid manner. In our figure we see that after restoring the dropped vertices it is possible to assign 0s and 1s to vertices from all edges. So, the Kochen-Specker theorem [111] tells us that Yu-Oh’s 13-vertex set is not a KS set. Actually, Yu and Oh themselves cite the Bell-Kochen-Specker theorem in the same wording as in Theorem [111] and admit that their set does not satisfy the conditions of the theorem [67, p. 3, top]. That can be formulated as the following lemma.

Lemma XII.1. Yu-Oh’s 13-vertex set is not a KS set.
Proof. It is possible to assign 0s and 1s to vertices in such a way that no two orthogonal directions are both assigned 1 and no three mutually orthogonal directions are all assigned 0 as shown by circled 1s in Fig. 19.

Yu and Oh admit the validity of Lemma XIII.4 as follows: “The KS value assignments to the 13-ray set are possible; i.e., no logical contradiction can be extracted by considering conditions 1 and 2 of Theorem XIII.1 only.” Yet, they entitled their paper “State-Independent Proof of Kochen-Specker Theorem with 13 Rays” and on p. 3 they claim to have nevertheless “proved the original KS theorem” [67]. How come, when the Lemma XIII.4 proves the contrary? Well, it seems to be a question of misapplied terminology. In the paper they proceed to define a new kind of contextuality through their inequalities (2), (3), and (4) applied to their 13-non-KS-set, and then they mistakenly claim that their proof of such a newly defined contextuality amounts to the proof of the Kochen-Specker theorem. Only a KS set can be a proof of the KS theorem since a violation of conditions 1 and 2 of the theorem is tantamount to a definition of a KS set and therefore no non-KS can prove the theorem since it does not violate them [68]. This does not mean that the contextuality Yu and Oh proved for their 13-set is wrong. This only means that via such a contextuality for their 13-set one cannot prove the Kochen-Specker theorem simply because the set is not a KS set.

Hence, we are left with the four big KS sets as the only known 3-dim KS sets. (Gould and Aravind have proven that the so-called Penrose’s 3-dim KS set is isomorphic to Peres’ one [69].) It is well-known that all of them are critical and our program states01 confirms that. So, we cannot use them to generate smaller KS sets. (By the way, Yu-Oh’s 13 vertex set is a subset of Peres’ critical set and its criticality is yet another avenue of proving that Yu-Oh’s set cannot be a KS set and therefore that it cannot prove the KS theorem [68].) But when we look at their MMP hypergraphs we notice that the number of grey dots, i.e., the number of vertices that share only one edge, increases with the total number of vertices within a KS set, in contrast to the opposite trend of KS sets from the 4-dim 60-105 class; Cf. red circles in 29-16, 30-16, 31-18, and 60-40 in Fig. 5. In particular, there are 16, 20, 24, and 75 such vertices (grey dots) in Bub, Conway-Kochen, Peres, and Kochen-Specker’s hypergraphs in Fig. 5, respectively.

Therefore, we conjecture that a more complex non-critical KS sets, interwoven similarly to higher dimensional KS sets, with comparatively low number of vertices, c.a. 50, might be found on clusters and supercomputers and used to generate smaller 3-dim KS criticals. This is a work in progress.

XIII. DISCUSSION

In the past ten years the exploration and generation of contextual sets, in particular, Kochen-Specker (KS) sets received a lot of attention (see Sec. I) both for their possible applications and implementations and for their further theoretical usage and development in quantum mechanics and quantum information. The approaches to generation of KS sets diversified and many partial results were achieved, recently. Therefore, we have focused our efforts on the unification of results, features, structure, and mutual relations of different KS sets as well as on the development of technique and method of their arbitrary exhaustive generation and handling.

In pursuing this goal, we have concentrated neither on immediate experimental implementation (small sets), nor on the standard parity proofs based algorithms. Instead, we have made use of the general MMP hypergraph language by means of which we generated a large number of new types of contextual KS critical sets and numerous non-isomorphic instances within each of them, which mostly cannot be generated by other known algorithms. The approach also gave us the results we were not originally concerned with, such as an abundance of novel small KS sets, and in addition provided us with an explanation why the other approaches, like parity proof ones, failed to spot them—it turns out that only a very few KS sets have a parity proof (in some classes under 1% of sets and in some
none at all) what makes them completely invisible for the parity proof based algorithms and programs, predominant in the literature.

Instead of parity proofs of only some KS sets with a particular structure, the MMP hypergraph algorithms and methods enable direct numerical proofs of the KS theorem for any chosen KS set via literal verifying of KS theorem conditions: program states01 gives a maximal number of 1s for a chosen KS set and after deleting all edges that contain these 1s at least one edge should remain. This is all automated but the user can easily check the output MMP hypergraph strings by hand. When the MMP hypergraphs are drawn as figures, the proofs also become “visual” as indicated in Fig. 4(a) and Fig. 5(6-13) by dashed red ellipses.

We developed the hypergraph approach to KS sets introduced in Sec. II from the lattice theory of Hilbert spaces and the way of assigning of 0-1 states to vertices of hypergraphs we took over from the methods of dealing with discrete states defined on those lattices. In particular, we redesigned our programs for analysing the Hilbert lattice features and turned them into the programs we used to build up an MMP-hypergraph-based language (specified in Sec. II) for generating, analysing, filtering, and modifying KS sets. We made use of this language to obtain numerous novel KS sets and classes and their features in 4-dim (in Secs. IV, V, VI, and VII), 6-dim (in Sec. VIII), 8-dim (in Sec. IX), 16-dim (in Sec. X), and 32-dim (in Sec. XI) Hilbert spaces. We also reviewed the 3-dim KS sets Sec. XII.

In the table in Fig. 20 we list the most important properties of generated critical KS sets we obtained and compare them with ones obtained previously.

![Table](image)

FIG. 20. List of properties of generated critical KS sets and their comparison with the previously obtained sets; “na” stands for “not applicable,” e.g., for the biggest known classes; “2?” refers to the claim in [52] that the operators defining the 300-675 might be redefined to allow a representation via 2 qubits; “*” in 0.14 for 300-675 indicates unevenly distributed parity proofs: all lower criticals have it while all higher ones (211-127 to 283-188) lack it; “Edges share ≥ 2 v” = edges share 2 or more vertices, i.e., intersect each other two or more times; “New types” “0” for, e.g., 24-24 means that there are no new types of 24-24 KS criticals in this paper—they are only elaborated on and discussed here; “18.3” (for 60-74) means: 28 (new types obtained in this paper for the first time) / 153 (total number of types) = 18.3%; “100”(for 236-1216) gives a maximal number of 1s for a chosen KS set and after deleting all edges that contain these 1s at least one edge should remain; this is all automated but the user can easily check the output MMP hypergraph strings by hand. When the MMP hypergraphs are drawn as figures, the proofs also become “visual” as indicated in Fig. 4(a) and Fig. 5(6-13) by dashed red ellipses.

The large variety of KS sets we obtain provides a much greater choice for KS experiments and insight into their properties and the properties of gates that handle them as well as the properties of quantum sets in general. While smaller KS sets are currently preferred for a feasibility of experimental implementations, in the future other sets that are intrinsically different (i.e. non-isomorphic) may become desirable for more sophisticated experiments, verification of the KS theorem with different setups, etc., especially because bigger sets do not require higher efficiency of measurements but only a higher number of measurements. Since the sets we found are critical, there will be no redundancy in any experimental setup making use of them.

Finally, we want to stress that our generation of KS sets in 16-dim and 32-dim spaces allowing for their implementation by means of 4 and 5 qubits, respectively, is—as, actually, all generations in this paper—vector/vertex-based and therefore complementary to a recent operator-based generation of KS sets for 4, 5, and 6 qubits (explicitly, and more of them, in principle) by Waegell and Aravind [74]. Building a correspondence between the two approaches (via...
The eigenvectors of their operators) is a work in progress so that we, as of yet, cannot say to which extent our results overlap. We can only say that we have not obtained KS criticals with 9 edges (bases, in their terminology) for 4 and 5 qubits, as they have. Our minimal number of edges for the corresponding two classes is 11, as shown in the tables in Figs. 15 and 20. We did not include 16-dim and 32-dim KS criticals with 9 edges from [70] into our tables because we were not able to find out whether they, respectively, belong to the 80-265 and 160-661 classes or not.

The generation of KS critical sets we presented in this work is, with our algorithms and programs, straightforward but demanding and CPU-time consuming. The jobs require cluster and grid calculations and even with them it might take months to obtain a required or desired particular set which might be needed in elaboration, confirmation, or checking particular assumptions about construction, unification, geometry, computation, or implementation of contextual sets. We provide samples of the KS criticals in the MMP hypergraph notation in the Supplemental Material. They are just a tiny fraction of TBs of data we generated but the reader can obtain any KS sets from us upon “appropriate” request (e.g., the 60-74 file has 231 GB). Also, the reader her/himself may generate any KS set by making use of our programs that are freely available at our repository http://goo.gl/xbx8U2.

See Appendix B for MMP hypergraph strings of chosen KS critical sets from most of the types from all classes we considered in the paper.

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Appendix A: MMP Hypergraph Strings Referred to in the Article

### 1. Section IV

Master set 60-75 1234, 5678, 9ABC, DEFG, HIJK, LMNO, PQRS, TUUV, XYZa, bcde, fghi, jklm, maSK, IzRJ, KYqi, jXPH, iwO, hdVN, gcUM, fBT, npq, rstu, vwx, yuqG, xTP, woE, vnd, peY, qzeB, oxAa, nba9, uROC, TQNB, sPM9, rSLA, wUHC, xVIA, yWJ9, vTKB, jgEB, lGA, kh9F, mFDC, pLJ8, qMI7, nNH6, oOK5, vhX8, ygY5, xfZ6, wia7, ukc6, rjd7, b15, sme8, uQq8, TRF7, vPD5, WSE6, tXW4, rZU3, sYT2, uAv1, oFq1, pIP3, ngR4, qhS2, xjO2, wKL4, vIL1, ymN3, eHf1, cJd2, bIE3, dKG4.

Master set 60-74 1234, 5678, 9ABC, DEFG, HIJK, LMNO, PQRS, TUUV, XYZa, WOCG, vNFb, UMEA, TLD9, aSK8, ZRJ7, YQI6, XPH5, bcde, fghi, jklm, nieh, ld3, kgc2, jfB1, ndIB, oeJA, pCk9, qhBC, refB, seD5, tbD6, ucG7, rY9, uhXa, tiaB, szGC, nkT8, plv6, qmU7, ojW5, piRE, ogSD, qhQF, npFG, smNK, ujLI, rMH, tkOJ, sTQ1, uV54, tUp3, vWR2, oYN3, nZM4, qAL2, pXO1, epl, vePL, wCQM, xBRN, ydSO, xkYE, yjZF, wnvJ, xLaG, xvul, yiTH, vhlK, x954, vb71, yA62, wC83.

60-39 4123, 3hsS, SQRP, PcvoO, OLMN, NiX5, 5786, 6teF, FDEG, GVgp, qnpo, oABa, ABC9, 9xkK, KHJH, JUD4., bcde, ujhZ, h8TV, 2MmN, 7IRq, EyY1, WLRb, QCKD, lIXt, Uxj7, buE5, n8QH, 1KcT, arkK, hhME, 5MWW, tlgQ, yHiA, 7DSC, 0B8Y, Vb12, 2ZDv, ArOU.

60-41 2341, 1gQt, 56EF, FJBS, SydM, MLON, NiX5, 5mWw, whCI, 17Rq, qnpo, oAbA, arkK, Kf99, 9EPI, 8Tv, vZD2., 5678, 9ABC, DEFG, HIJK, PQRs, bcde, VgpG, hs3S, 2MmN, 7YH1, WLRb, JUD4, QCKD, Uxj7, buE5, n8QH, 1KcT, 9ZDq, 7Dsc, CtuM, 0B8Y, Vb12, iTFq, BcnX.

### 2. Section V

Master set 60-105 1234, 5678, 9ABC, DEFG, HIJK, LMNO, PQRS, TUUV, XYZa, bcde, fghi, jklm, maSK, IzRJ, KYqi, jXPH, iwO, hdVN, gcUM, fBT, npq, rstu, vwx, yuqG, xTP, woE, vnd, peY, qzeB, oxAa, nba9, uROC, TQNB, sPM9, rSLA, wUHC, xVIA, yWJ9, vTKB, jgEB, lGA, kh9F, mFDC, pLJ8, qMI7, nNH6, oOK5, vhX8, ygY5, xfZ6, wia7, ukc6, rjd7, b15, sme8, uQq8, TRF7, vPD5, WSE6, tXW4, rZU3, sYT2, uAv1, oFq1, pIP3, ngR4, qhS2, xjO2, wKL4, vIL1, ymN3, eHf1, cJd2, bIE3, dKG4.

Master set 60-105 as a subgraph of the master set 60-15 with original complex vector components: 1234, 5678, 9ABC, DEFG, HIJK, LMNO, PQRS, TUUV, XYZa, bcde, fghi, jklm, maSK, IzRJ, KYqi, jXPH, iwO, hdVN, gcUM, fBT, npq, rstu, vwx, yuqG, xTP, woE, vnd, vePL, xBRN, ydSO, xkYE, yjZF, wnvJ, xLaG, xvul, yiTH, vhlK, x954, vb71, yA62, wC83.

Master set 24-24 as a subgraph of the master set 60-15 with original complex vector components: 1234, 5678, 9ABC, DEFG, HIJK, LMNO, PQRS, TUUV, XYZa, bcde, fghi, jklm, maSK, IzRJ, KYqi, jXPH, iwO, hdVN, gcUM, fBT, npq, rstu, vwx, yuqG, xTP, woE, vnd, vePL, xBRN, ydSO, xkYE, yjZF, wnvJ, xLaG, xvul, yiTH, vhlK, x954, vb71, yA62, wC83.
3. Section VIII

40-23 3214, 49AB, BalvL, LMNE, ECO8, S567, 71JX, KR5Q, QOFH, HFG3, TVUS, WXP5, YZSD, abXG, cZN2, deR1, eBjH, djWCA, cUIF, cbOC, daYI, WSLF, eZVP. 1 = \{(1,0), \omega^2 \} , 2 = \{(1,0), \omega \} , 3 = \{(1,0), \omega^2 \} , 4 = \{(1,0), \omega \} , 5 = \{(1,0), \omega \} , 6 = \{(1,0), \omega^2 \} , 7 = \{(1,0), \omega \} , 8 = \{(1,0), \omega^2 \} , 9 = \{(1,0), \omega \} , 10 = \{(1,0), \omega \} , 11 = \{(1,0), \omega^2 \} , 12 = \{(1,0), \omega \} , 13 = \{(1,0), \omega^2 \} , 14 = \{(1,0), \omega \} , 15 = \{(1,0), \omega^2 \} , 16 = \{(1,0), \omega \} , 17 = \{(1,0), \omega^2 \} , 18 = \{(1,0), \omega \} , 19 = \{(1,0), \omega^2 \} , 20 = \{(1,0), \omega \} , 21 = \{(1,0), \omega^2 \} , 22 = \{(1,0), \omega \} , 23 = \{(1,0), \omega^2 \} .

49-27 3241, 1675, 5XWW, wLCL, nNj, jiC, caZb, b1JX, DghO, OQPQ, NSRQ, 98A4, 3B4D, 3EGF, 4HJ, 4KLM, TUUS, 5YFZ, aUMJ, dOHE, dCZ, dBh, geXX, 6kfh, 6UC1, 1kFL, T8Br. 1 = \{(0,0), \omega \} , 2 = \{(0,0), 1 \} , 3 = \{(0,0), \omega \} , 4 = \{(0,0), 1 \} , 5 = \{(0,0), \omega \} , 6 = \{(0,0), \omega \} , 7 = \{(0,0), 1 \} , 8 = \{(0,0), \omega \} , 9 = \{(0,0), 1 \} , 10 = \{(0,0), \omega \} , 11 = \{(0,0), 1 \} , 12 = \{(0,0), \omega \} , 13 = \{(0,0), 1 \} , 14 = \{(0,0), \omega \} , 15 = \{(0,0), 1 \} , 16 = \{(0,0), \omega \} , 17 = \{(0,0), 1 \} , 18 = \{(0,0), \omega \} , 19 = \{(0,0), 1 \} , 20 = \{(0,0), \omega \} , 21 = \{(0,0), 1 \} , 22 = \{(0,0), \omega \} , 23 = \{(0,0), 1 \} , 24 = \{(0,0), \omega \} , 25 = \{(0,0), 1 \} , 26 = \{(0,0), \omega \} , 27 = \{(0,0), 1 \} .

5. Section IX

8dim 12345678, 89ABCDF, FGH14KL, L7MNBOQ, QERSI3TU, UK6VWAX, XDPSHYZ2, ZTJ59MYA, aWQCRYG1.

8dim 12345678, 89ABCDF, FGH14KL, L2MRVYaE, KM3NSWZD, JRN4OUTC, IV5D0PUB, HYWTP6QA, GszXUq79.
Appendix B: Samples of KS Criticals in MMP Hypergraph Notation

Whenever possible, an attempt has been made to show the sets not already drawn above. Smaller sets presented in the MMP hypergraph figure-to-draw format: the edges before "-- " build the maximal loop; "-" following a character/vertex means that the vertex belongs to an edge from the loop. Thus, the reader can easily turn each of the following smaller MMP hypergraph strings into an MMP hypergraph figure reversing the procedure we explained in the body of the paper on how to turn any MMP hypergraph figure into an MMP hypergraph string. Bigger sets are presented just as they were generated (without maximal loop denotations)—only a space is inserted after each comma for easier formatting; these spaces should be taken out before strings into an MMP hypergraph figure processing.

1. 3-dim

---

Maximum loops of all KS sets from the 24-24 class are hexagons.

Master set 24-24

3241, 1576, 691L, IHJD, DEGF, FB3K, 2*89*A, 5*8B*C, E*KL*M, H*K*N*Q, 7*A*J*M, 4*CG*O, 2*5*EH*, 4*B*J*L*, 7*9*G*O*N, 1*8*DK*M, 3*C*IM*, 6*AF*O, 1*2*5*B, 1*I*J*M, F*G*O*N*Q, 6*7*9*A, 3*4*B*C, D*E*H*K.

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2. 4-dim 24-24

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3. 4-dim 60-74

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4. 4-dim 60-74

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Appendix A of the paper.

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Master set is given in Appendix A of the paper.

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
4. 4-dim 60-105

Master set is given in Appendix A of the paper.

5. 4-dim 300-675

The master set is given in the repository. Particular smaller sets that also belong to the 60-74 class are in [52] Sec V. The first three sets below are obtained by Waegel and Arawind [52], just translated into MMP hypergraphs. They do not translate into MMP hypergraphs.
The master set is given in the repository.

72-12 123456789ABCDEFG, HIJKLMNOPQRSTU VWXYZabcdefg, hijkjabcd, lmnopqrs, tuvwxysz, a7bcd8ef9, u9g6h2382154, +123456789, \%$&'()*+-`,./0123456789

10. 18-60-265

The master set is given in the repository.

72-12 123456789ABCDEFG, HIJKLMNOPQRSTU VWXYZabcdefg, hijkjabcd, lmnopqrs, tuvwxysz, a7bcd8ef9, u9g6h2382154, +123456789, \%$&'()*+-`,./0123456789

10. 32-dim 160-661

The master set is given in the repository.

135-21 123456789ABCDEFG, HIJKLMNOPQRSTU VWXYZabcdefg, hijkjabcd, lmnopqrs, tuvwxysz, a7bcd8ef9, u9g6h2382154, +123456789, \%$&'()*+-`,./0123456789

10. 32-dim 160-661
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