Weakly interacting Bose gases with generalized uncertainty principle: Effects of quantum gravity

Abdelâali Boudjemâa

Department of Physics, Faculty of Exact Sciences and Informatics, Hassiba Benbouali University of Chlef, P.O. Box 78, 02000 Ouled-Fares, Chlef, Algeria

Received: 4 October 2021 / Accepted: 12 February 2022
© The Author(s), under exclusive licence to Società Italiana di Fisica and Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract We investigate quantum gravity corrections due to the generalized uncertainty principle on three-dimensional weakly interacting Bose gases at both zero and finite temperatures using the time-dependent Hatree–Fock–Bogoliubov theory. We derive useful formulas for the depletion, the anomalous density, and some thermodynamic quantities such as the chemical potential, the ground-state energy, the free energy, and the superfluid density. It is found that the presence of a minimal length leads to modify the fluctuations of the condensate and its thermodynamic properties in the weak and strong quantum gravitational regimes. Unexpectedly, the interplay of quantum gravity effects and quantum fluctuations stemming from interactions may lift both the condensate and the superfluid fractions. We show that quantum gravity minimizes the interaction force between bosons leading to the formation of ultradilute Bose condensates. Our results which can be readily probed in current experiments may offer a new attractive possibility to understand gravity in the framework of quantum mechanics.

1 Introduction

Two crowning intellectual achievements of modern physics that exquisitely explain how nature works: quantum mechanics and theory of general relativity. Attempts to reconcile these incompatible theories usually involve quantizing gravity to formulate a theory of quantum gravity (QG) (see, e.g., [1]). However, no experiment today provides evidence that gravity has a quantum mechanical origin. The most serious obstacle is the incredible energies needed \( \sim 10^{28} \) ev (Planck energy) or equivalently a length scale of the order of the Planck length \( \approx 10^{-35} \) m. Given current technologies, a direct observation of such intriguing QG effects would likely require to build a particle accelerator bigger than our galaxy.

With recent progress in quantum information science, the topic of experimentally testing QG has been gaining renewed interest. Several proposals based on quantum information science such as quantum entanglement between two microspheres [2–6] and non-Gaussianity in matter [7] have been used to witness QG. Ultracold atoms including macroscopic Bose–Einstein condensates (BECs) offer another promising possibility of testing the quantum nature of the gravitational field due to their extraordinary degree of control and sensitivity to ultraweak forces [7–14]. An additional advantage in using ultracold gases is that the electromagnetic interactions are adjustable by means of an external magnetic or optical fields [7,12].

Numerous approaches to QG such as string theory and loop QG, as well as black hole physics, predict a minimum measurable length in nature, below which no other length can be observed. One of the most intriguing aspect linked to the existence of such a minimum length is the modification of the Heisenberg uncertainty principle to a generalized uncertainty principle (GUP) [15–18]. This latter predicts corrections to diverse quantum phenomena [15–32]. Furthermore, based on minimum observable length, implications of QG on the statistical properties of ideal Bose gases have been widely investigated (see, e.g., [33–39] and references therein).

However, to the best of our knowledge, the effects of QG on weakly interacting Bose gases remain rarely examined. The model of weakly interacting Bose gases which is universal is extremely interesting since it simultaneously covers the low-energy edge of the effective theory and the high-energy physics [40]. One remarkable property of such quantum ensembles is their diluteness which enables us to treat problems either completely, or perturbatively or even numerically in reasonable times, opening a feasible route for testing whether or not the gravitational field displays quantum properties.

The aim of this paper is to investigate the effects of QG due to the GUP on weakly interacting homogeneous Bose gases at both zero and finite temperatures. To achieve this goal, we use the time-dependent Hatree–Fock–Bogoliubov (TDHFB) theory [41–43]. Basically, the TDHFB theory is a self-consistent approach describing the dynamics of ultracold Bose gases at any temperatures which involves interactions between the condensate and the thermal cloud. The TDHFB equations offer an elegant starting point to treat many-body dynamics and have been successfully applied to a wide variety of problems [41–56]. They could also provide an
ideal alternative dynamical model involving the cosmological constant for understanding the accelerated expansion of the universe caused by dark matter.

When considering the effects of minimal length, the density of states is modified giving rise to substantially affect the Bogoliubov dispersion relation, the quantum fluctuations, and the thermodynamic properties of the system (see, e.g., [34,36,39]). In this work, we compute correction terms arising from a natural deformation of QG governed by the HFB equations based on a minimal length framework to the condensed depletion, the anomalous density, the chemical potential, the ground-state energy, the free energy, and the superfluid density. One key hurdle routinely encountered while implementing anomalous correlations and ground-state energy is the ultraviolet divergences caused by the short-range contact potential. In this regard, a number of techniques have been used to cure these difficulties such as the renormalization of the coupling constant [57–60] and the dimensional regularization [41,61–63]. Here, we show that the presence of a minimal length scale in the HFB formalism regularizes naturally the coupling constant (i.e., it damps short distance modes) and hence overcomes the well-known ultraviolet divergence problem.

At zero temperature, we find that QG effects may reduce both the quantum fluctuations and the thermodynamic quantities. At finite temperature, corrections due to weak QG effects to the condensate thermal fluctuations, the free energy, and the normal density of the superfluid are obtained analytically. We point out that these amendments are significant only at temperatures well below the transition. On the other hand, effects of strong QG are calculated using numerical simulations. The results reveal that QG which behaves itself as quantum correlations provides an additional attractive term which competes with repulsive mean-field and Lee–Huang–Yang (LHY) energies (resulting from quantum and thermal fluctuations) leading to lower the above observables of the condensate. Crucially, we demonstrate that the presence of gravitational effects may strikingly enhance the condensed and the superfluid densities. The validity criterion of the present HFB theory is accurately established at both zero and finite temperatures. Furthermore, by measuring the corrections in the quantum depletion, we can constrain the deformation parameter and the minimum measurable length.

The rest of the paper is organized as follows. In Sect. 2, we introduce the fundamental concepts of the TDHFB model with GUP implying the existence of a minimal length. We derive in addition the essential formulas to tackle the problem under investigation. The effects of QG on the quantum and thermal fluctuations, the thermodynamics, and on the superfluidity in BEC are deeply discussed in Sects. 3 and 4, respectively. In Sect. 6, we discuss the experimental relevance of our predictions. Our conclusions are drawn in Sect. 7.

2 TDHFB Theory with GUP

We consider a three-dimensional dilute Bose gas with the atomic mass \( m \) at a temperature \( T \). We make the plausible assumptions that bosons are weakly interaction via a contact potential \( V(x - x') = g \delta(x - x') \), where \( g = (4\pi\hbar^2/m)a \) is the coupling constant with \( a \) being the \( s \)-wave scattering length. The Hamiltonian of such a system reads

\[
\hat{H} = \int dx \hat{\psi}^\dagger(x) \left[ \frac{\hbar^2}{2m} \nabla^2 + \frac{g}{2} \hat{\psi}^\dagger(x) \hat{\psi}(x) \right] \hat{\psi}(x),
\]

where \( \hat{\psi}^\dagger \) and \( \hat{\psi} \) are the boson destruction and creation field operators, respectively, satisfying the usual canonical commutation rules \( \{\hat{\psi}(x), \hat{\psi}(x')\} = \delta(x - x') \). The single particle Hamiltonian is defined by \( h^{sp} = -(\hbar^2/2m) \Delta + V(x) - \mu \), where \( V(x) \) is the external potential and \( \mu \) is the chemical potential.

At finite temperature, we usually perform our analysis in the mean-field framework relying on the TDHFB equations. These latter are based on the time-dependent Balian–Véroni variational principle [64–66] with Gaussian trial time-dependent density operator \( D(t) \). This ansatz is associated with the partition function \( Z \), the one-boson field expectation values \( \langle \hat{\psi}(x, t) \rangle \), \( \langle \hat{\psi}^\dagger(x, t) \rangle \), and the single-particle density matrix \( \rho(x, x', t) \) [42]. Upon inserting these variational parameters into the BV action, one obtains the TDHFB equations [41,67,68]

\[
i \hbar \frac{d\Phi}{dt} = \frac{d\mathcal{E}}{d\Phi}, \quad (2)
\]

\[
i \hbar \frac{d\rho}{dt} = -2 \left[ \rho, \frac{d\mathcal{E}}{d\rho} \right], \quad (3)
\]

where \( \mathcal{E} = \langle \hat{H} \rangle \) is the energy of the system, and the single particle density matrix of a thermal component is defined as:

\[
\rho = \left( \begin{array}{cc} \langle \hat{\psi}\hat{\psi}^\dagger \rangle & -\langle \hat{\psi}^\dagger \hat{\psi} \rangle \\ -\langle \hat{\psi}^\dagger \hat{\psi} \rangle & \langle \hat{\psi}\hat{\psi}^\dagger \rangle \end{array} \right),
\]

where \( \hat{\psi}(x) = \hat{\psi}(x) - \Phi(x) \) is the noncondensed part of the field operator with \( \Phi(x) = \langle \hat{\psi}(x) \rangle \) being the condensate wave-function. Equations (2) and (3) imply that the energy \( \mathcal{E} \) is conserved when the Hamiltonian \( H \) does not depend explicitly on time. They constitute a closed set of equations for a condensate coexisting with a thermal cloud and a pair anomalous density.
An important feature of the TDHF formalism is that it allows unitary evolution of $\rho$. Then, the conservation of the Von Neumann entropy $S = T_r D \ln D$ yields

$$(I - 1)/4 = \rho(\rho + 1),$$

where $I$ is often known as the Heisenberg invariant $[41,67,68]$. It represents the variance of the number of noncondensed particles. For pure state and at zero temperature, one has $I = 1$.

Many QG models predict a minimal uncertainty length. The simplest form of GUP relation which implies the appearance of a nonzero minimal uncertainty is proposed as $[18]$:

$$\Delta X \Delta P \geq \frac{\hbar}{2} \left[1 + \beta(\Delta P)^2\right].$$

where $\beta > 0$ is the deformation parameter which is often expressed in terms of the dimensionless parameter $\beta_0$, and the Planck length $l_p$ as: $\beta = \beta_0 l_p^2 / \hbar^2 = \beta_0 (M_p c)^2$, where $M_p = \sqrt{\hbar c / G}$ is the Planck mass with $G$ being the gravitational constant and $c$ denotes the speed of light in vacuum. Current experiments can set upper bounds on the GUP parameter. For instance, the standard model of high-energy physics implies that $\beta_0 < 10^{34}$ $[9]$. According to the same reference $[9]$, the scanning tunneling microscope delivers the best one $\beta_0 < 10^{21}$ $[9]$. Other upper bounds have been provided by different approaches, namely the Lamb shift and Landau levels $[9]$, optical systems $[30]$, the light deflection and perihelion precession $[69]$, cold atoms $[70]$, and gravitational systems $[71–73]$. However, in the present work we will address QG effects for arbitrary $\beta_0$. Evidently, for $\beta = 0$, the standard Heisenberg uncertainty principle is recovered.

Equation (5) immediately leads to the definition of the minimum measurable length

$$\Delta X_{\text{min}} = \hbar \sqrt{3\beta} = \sqrt{3\beta_0 l_p}.$$  

In the case of mirror-symmetric states (i.e., $\langle \hat{P} \rangle = 0$), it is possible to obtain Eq. (5) from the modified commutation relation:

$$[\hat{X}, \hat{P}] = i\hbar \left(1 + \beta P^2 \right).$$

According to Eqs. (5) and (7), the deformed density of states can be given by $[18,20]$

$$D(P) dP = \frac{V}{2\pi^2 \hbar^3} \frac{P^2 dP}{(1 + \beta P^2)^3}.$$  

Equation (8) shows that the GUP may modify the statistics and the thermodynamics of a weakly interacting Bose gas.

From now on, we focus on uniform Bose gases and assume that the dynamics of the thermal cloud and the anomalous density is not important at low temperatures. Effects of QG in quantum systems are implied by the GUP given in Eq. (5). Therefore, the above TDHF equations take the explicit form:

$$i\hbar \frac{d\Phi}{dt} = \left[ E_P + g(n_c + 2\tilde{n} + \tilde{m}) - \mu \right] \Phi,$$

where $E_P = P^2 / 2m$ is the energy of free particle, $n_c = |\Phi|^2$ is the condensed density, $\tilde{n} = |\tilde{\psi} \tilde{\psi}|$ is the noncondensed density, and $\tilde{m} = \langle \tilde{\psi} \tilde{\psi} \rangle$ accounts for the anomalous correlation. The total density is given by $n = n_c + \tilde{n}$. Equation (9) is the generalized Gross-Pitaevskii equation which allows us to study the static and the dynamics of the condensate in terms of the paradigms developed in the GUP framework.

The chemical potential reads

$$\mu = g(n_c + 2\tilde{n} + \tilde{m}) = gn + g(\tilde{n} + \tilde{m}).$$

The term $g(\tilde{n} + \tilde{m})$ represents the quantum corrections to the chemical potential owing to the BEC fluctuations.

The Bogoliubov excitations energy can be obtained by linearizing Eq. (9) employing the standard transformation: $\Phi(P, t) = \sqrt{\pi_c} u_P e^{-i\varepsilon_P t} + v_P e^{i\varepsilon_P t}$, where $\sqrt{\pi_c} = \Phi_0 = \Phi_0^*$ is the equilibrium solution, and $u_P, v_P = (\sqrt{\varepsilon_P/E_P} \pm \sqrt{E_P/E_P})/2$ are the Bogoliubov quasiparticle amplitudes $[74]$. This gives

$$\varepsilon_P = \sqrt{(P^2 / 2m)^2 + c_s^2 P^2},$$

where $c_s = \sqrt{g(n_c + \tilde{m}) / m}$ is the sound velocity. For small momenta $P \to 0$, the Bogoliubov dispersion relation is phonon-like $\varepsilon_P = c_s P$ (quanta of sound waves). For the curl-free superfluids such sound waves play the role of “gravitational waves” $[40]$. In the opposite limit $P \to \infty$, the excitations spectrum (11) reduces to the free particle law: $\varepsilon_P = E_P$.

The effects of the minimal length on the behavior of quantum and thermal fluctuations of BEC can be calculated from Eq. (3) which can be written in momentum space as:

$$\tilde{n} = \frac{1}{4\pi^2 \hbar^3} \int \frac{P^2 dP}{(1 + \beta P^2)^3} \left[ \frac{E_P + m c_s^2}{\varepsilon_P} \sqrt{I_P} - 1 \right].$$
and

\[ \tilde{m} = -\frac{1}{4\pi^2\hbar^3} \int P^2 dP \frac{mc^2}{(1 + \beta P^2)^3} \sqrt{\varepsilon_P}. \]  

(13)

In the absence of the GUP (\(\beta = 0\)), the anomalous density (13) becomes ultraviolet divergent due to the use of the short-range contact potential. To circumvent this problem, we renormalize the coupling constant \(g\) and introduce the Beliaev-type second-order corrections \(g_R(P) = g + g^2 \int_{P<P_c} \frac{dP}{(2\pi\hbar)^3} \frac{1}{2\varepsilon_P} \) [57, 59, 60]. Another strategy to cure such an ultraviolet divergence is the dimensional regularization that is asymptotically accurate for weak interactions [41, 61–63].

The Heisenberg equality (4) serves to provide a useful relation between the noncondensed (12) and anomalous (13) densities [41, 67]:

\[ IP = (2\tilde{n}P + 1)^2 - |2\tilde{m}P|^2 = \coth^2 \left( \frac{\varepsilon_P}{2T} \right), \]

(14)

here we used the identity \(2\tilde{n}P(x) + 1 = \coth(x/2)\) with \(nP = [\exp(\varepsilon_T/T) - 1]^{-1}\) being occupation numbers for the excitations. Equation (14) enables us to determine in a very useful way the critical temperature and the superfluid density of Bose quantum liquids (see below).

3 Fluctuations

Our target now is to calculate corrections owing to the effects of QG governed by the HFB to the condensed depletion and the anomalous density at both zero and finite temperatures. Let us assume the limit \(\tilde{m}/nc \ll 1\), which is valid at low temperature and necessary to ensure the diluteness of the system [41, 62]. Therefore, the sound velocity reduces to \(c_s = \sqrt{gnc/\tilde{m}}\).

3.1 Quantum fluctuations

The noncondensed density can be straightforwardly evaluated from integral (12). One finds

\[ \tilde{n} = \frac{(mc_s)^3}{4\pi^2\hbar^3} f(\beta), \]

(15)

where the deformation function \(f(\beta)\) is given by

\[ f(\beta) = \frac{1}{16(1 - 4m^2c_s^2\beta)^{3/2}(m^2c_s^2\beta)^{3/2}} \left\{ \sqrt{1 - 4m^2c_s^2\beta} \left[ 4m^2c_s^2\beta (2m^2c_s^2\beta (8m^2c_s^2\beta - 1) + 1) - \pi (1 - 4m^2c_s^2\beta)^2 \right] \right\} , \]

(16)

which is unrelated to atom mass \(m\). Importantly, for \(\beta \rightarrow 0\), \(f(\beta) \approx 4/3\) (see Fig. 1 solid line), we reproduce the result of a dilute Bose gas without QG, \(\tilde{n} = (mc_s)^3/(3\pi^2\hbar^3)\). In terms of a small parameter of the theory, the depletion takes the form \(\tilde{n}/n_c = 8\sqrt{n_c a^2}/\pi/3\), where the condensed density \(n_c\) which constitutes our corrections with respect to the Bogoliubov results [74], appears as a key parameter instead of the total density \(n\).

Fig. 1 Deformation functions \(f(\beta), h(\beta)\) and \(S(\beta)\) which govern the dependence of the condensate depletion, the anomalous fraction and the ground-state energy corrections vs. the deformation parameter \(\beta\) in units of \((mc_s)^2\)
Thanks to GUP the anomalous density (13) does not suffer from ultraviolet divergences. Consequently, integral (13) yields
\[ \tilde{m} = -\frac{(mc_s)^3}{4\pi^2 h^3} h(\beta), \]
where the QG correction function \( h(\beta) \) is given by
\[ h(\beta) = -\frac{1}{4(1 - 4m^2c_s^2\beta)^{5/2}\sqrt{m^2c_s^2\beta}} \left[ (10 - 16m^2c_s^2\beta) \right. \\
\times \sqrt{m^2c_s^2\beta(1 - 4m^2c_s^2\beta)} - 3 \arccos \left( \frac{2\sqrt{m^2c_s^2\beta}}{h(\beta)} \right) \left. \right]. \]

Again for \( \beta \to 0, h(\beta) \approx 3\pi/(8\sqrt{m^2c_s^2\beta}) - 4 \), thus, the anomalous density (17) reduces to \( \tilde{m} = (mc_s)^3/(\pi^2 h^3) \left[ 1 - 3/(32\pi\sqrt{m^2c_s^2\beta}) \right] \), where the leading term, \( (mc_s)^3/(\pi^2 h^3) = 8n_c\sqrt{n_c}\alpha^3/\pi \), is the standard anomalous density for Bose gases [41,58,62]. Remarkably, \( \tilde{m} \) diverges as \( \sim 1/\sqrt{\beta} \) for small \( \beta \) (see Fig. 1 dashed line).

Figure 1 shows that both functions \( f \) and \( h \) are decreasing with \( \beta \) for any value of \( (mc_s)^2 \) indicating that QG effects lead to strongly reduce the condensed depletion and the anomalous density and thus, enhance the condensed fraction \( n_c/n \). For instance, for \( \beta = 1/(mc_s)^2 \) or equivalently \( \beta_0 = (M\rho c/mc_s)^2 \), the depletion reduces to \( \tilde{n}/n_c \approx 0.2\sqrt{n_c}\alpha^3 \) whatever the value of \( \alpha \). The reason of such a decrease in \( \tilde{n} \) and \( \tilde{m} \) is most probably attributed to QG which acts as an extra force blocking interactions between bosons. We observe also from the same figure that the anomalous density is larger than the noncondensed density in the whole range of \( \beta \).

In weakly interacting Bose gases, the fluctuations must be small. Therefore, the validity of the present theory at \( T = 0 \) requires the condition:
\[ \sqrt{n_c}\alpha^3 f(\beta) \ll 1, \]
which differs by the factor \( f(\beta) \) from the universal small parameter of the theory. The condition (19) tells us that the presence of QG conducts to the emergence of an ultradilute BEC since \( f(\beta) \) is very small.

### 3.2 Thermal fluctuations

Let us now extend our results for the case of uniform BEC at finite temperature under the GUP. Evaluation of integrals (12) and (13) depends on the energy-momentum relation which unfortunately exhibits a divergent behavior at non-zero temperatures for the usual Bogoliubov excitation energy. For \( \beta \ll 1 \), we expand the correction factor \( (1 + \beta P^2)^{-3} \) up to third order as:
\[ (1 + \beta P^2)^{-3} = 1 - 3\beta P^2 + 6(\beta P^2)^2 - 10(\beta P^2)^3 + \cdots \]
(see e.g. [34,36,37]). This permits us to earn small QG corrections to the condensate fluctuations and to its thermodynamics. However, when \( \beta \) approaching unity, the system reaches the regime in which effects of QG become important. In such a case we should use numerical simulations in order to further analyze the influence of the GUP on the thermal properties of BEC.

At low temperatures \( T \ll mc_s^2 \), the main contribution to integrals (12) and (13) comes from the phonon region. Using the identity
\[ \int_0^\infty x^a\ln x/(e^x - 1) = \Gamma(a + 1)\zeta(a + 1), \]
where \( \Gamma(x) \) is the gamma function and \( \zeta(x) \) is the Riemann zeta function, one obtains the temperature-dependence of the anomalous density up to third order in \( \beta \):
\[ \tilde{n}_T = |\tilde{n}_T| = \frac{mT^2}{12\hbar^2c_s} \mathcal{F}(\beta, T), \]
(20)
where
\[ \mathcal{F}(\beta, T) = 1 - \frac{6\pi^2\beta}{5} \left( \frac{T}{c_s} \right)^2 + \frac{128\pi^4\beta^2}{21} \left( \frac{T}{c_s} \right)^4 \\
- 16\pi^6\beta^3 \left( \frac{T}{c_s} \right)^6 + \cdots. \]

Unlike the zero-temperature case, the function \( \mathcal{F}(\beta, T) \) depends on boson mass and varies as \( m^{2a}(\alpha a)^{-a} \), where \( \alpha > 0 \). It is obvious that for \( \beta = 0 \), Eq. (20) coincides with the standard prediction for weakly-interacting BEC without considering QG which is \( \propto T^2 \). This differs from the result of the ideal gas where \( \tilde{n}_T \) behaves like \( T^{3/2} \). Equation (20) shows that at low temperatures \( \tilde{n}_T \) and \( \tilde{m}_T \) are of the same order of magnitude but with opposite signs.

Figure 2a depicts that as \( \beta \) and \( T \) rise, the function \( \mathcal{F}(\beta, T) \) decreases which means that the effects of QG are crucial specifically at low temperatures \( T \ll T_c \). This behavior holds true also in ideal Bose gases [39]. For instance, in the case of \( ^{133}\text{Cs} \) BEC with parameters: \( a = 450a_0 \) (\( a_0 \) is the Bohr radius) [75], \( n = 10^{20} \text{m}^{-3} \), the function \( \mathcal{F}(\beta, T) \) lowers by \( \sim 13% \) from \( \beta = 0.006/(mc_s)^2 \) to \( \beta = 0.06/(mc_s)^2 \) at temperature \( T \approx 0.27T_c^0 \), whereas it decreases by \( \sim 44% \) at \( T \approx 0.33T_c^0 \) for the same values of \( \beta \). This confirms that the condensate under the GUP still survives even in the limit of high temperature. Here we use the fact that
worth stressing that the Tc value of the criterion (21) reduces to that obtained for BEC without considering the GUP [76].

\[ T / mc^2 = (T / mc^2_0)/(\sqrt{n}\pi / (3/2))^2 \]

where \( mc^2_0 = 2\pi\hbar^2 n / (\zeta(3/2))^{2/3} / m \) is the critical temperature of an ideal Bose gas. It is worth stressing that \( T_c \) strongly depends on the density and on the interaction strength.

The occurrence of the extra factor \([\mathcal{F}(\beta, T)(T / mc^2_0)]\) is due to the interplay of QG and the thermal fluctuations. For \( \mathcal{F}(\beta, T) = 1 \), the criterion (21) reduces to that obtained for BEC without considering the GUP [76].

At \( T \gg mc^2 \) where the main contribution to integrals (12) and (13) comes from the single particle excitations, there is copious evidence that \( \tilde{n}_T \) becomes identical to the noncondensed density of an ideal Bose gas. This implies that corrections to all thermodynamic quantities are closer to the values obtained for an ideal Bose gas. However, the anomalous density being proportional to the condensed density, tend to zero together and hence, their contributions become negligibly small [41,58,62].

4 Thermodynamic quantities

In this section, we analyze the influence of the GUP on the thermodynamic properties of weakly interacting Bose gases.

The chemical potential can be easily obtained from Eq. (10),

\[ \mu = gn + g(mc)^3 / 4\pi^2 h^3 \right [ f(\beta) - h(\beta) \right ] . \]

Using the asymptotic behavior of the functions \( f \) and \( h \) for \( \beta \to 0 \), the equation of state (22) becomes \( \mu = gn + gn_c (32/3) \sqrt{n\pi / (128m^2 c^2 \beta)} \). The quantum corrections term \( gn_c (32/3) \sqrt{n\pi / (128m^2 c^2 \beta)} \) was first derived by the LHY [77].

Again, the chemical potential decays as \( \sim 1/\sqrt{\beta} \) for small \( \beta \).

The ground-state energy of ultracold Bose gases in the presence of the GUP is defined as:

\[ E / V = gn^2 / 2 + \frac{1}{4\pi^2 h^3} \int \frac{P^2 dP}{(1 + P^2)^3} \left [ E_P - E_P - mc^2_0 \right ] . \]

Unlike the ordinary BEC, the ground-state energy (23) is safe from the ultraviolet divergence due to the presence of a minimal length. After a straightforward calculation, we find

\[ E / V = gn^2 + \frac{m^4 c^6_0}{4\pi^2 h^3} S(\beta) . \]

where the energy deformation function \( S(\beta) \) is given by

\[ S(\beta) = -\frac{1}{32(1 - 4m^2 c^2_0 \beta)^{3/2} m^2 c^2_0 \beta^{5/2}} \left \{ 1 - 4m^2 c^2_0 \beta \left [ 4m^2 c^2_0 \beta (8m^2 c^2_0 \beta - 3) + \pi (3 - 2m^2 c^2_0 \beta (4m^2 c^2_0 \beta + 5)) \right ] + (32m^2 c^2_0 \beta - 6) \arccos \left (2\sqrt{m^2 c^2_0 \beta} \right ) \right \} . \]
For $\beta \to 0$, $S(\beta) \approx -3\pi/(16\sqrt{m^2c_s^2}\beta) + 32/15$, hence Eq. (24) simplifies to $E/V = gn^2/2[1 + (128gn^2/15)\sqrt{na^3/\pi}(1 - 45\pi/512\sqrt{mc_s^2}\beta)]$. We can see from Fig. 1 (dotted line) that the function $S(\beta)$ is increasing with $\beta$ and remains negative signaling that the effects of QG tend to lower the ground-state energy. This emphasizes that the subleading term in Eq. (24) which arises from the LHY quantum fluctuations and QG corrections is much smaller than the leading contribution to the energy which originates from interactions.

In the frame of our formalism, the free energy can be written as:

$$F = E + \frac{T}{2\pi^2\hbar^3} \int \frac{P^2\,dP}{(1 + \beta P^2)^3} \ln \left( \frac{2}{\sqrt{TP + 1}} \right),$$

where $E$ is the ground-state energy given in Eq. (24). Integrating the subleading term in Eq. (26) for temperatures satisfying the inequality $T \ll mc_s^2$, we obtain up to third order in $\beta$:

$$F = E + \frac{\pi^2T^4}{90\hbar^3c_s^3}\mathcal{H}(\beta, T),$$

where

$$\mathcal{H}(\beta, T) = -1 + \frac{24\pi^4\beta}{7} \left( \frac{T}{c_s} \right)^2 - \frac{144\pi^6\beta^2}{7} \left( \frac{T}{c_s} \right)^4 + \frac{6400\pi^8\beta^3}{33} \left( \frac{T}{c_s} \right)^6 - \cdots.$$  

Expression (27) clearly shows that for $\beta = 0$, one recovers the famous $T^4$-law for the free energy, in contrast with the $T^{5/2}$ behavior found for the non-interacting Bose gas [59].

The function $\mathcal{H}(\beta, T)$ is displayed in Fig. 3a. We see that for small $\beta$, it increases monotonically with $T/mc_s^2$ and remains negative. As $\beta$ increases $\mathcal{H}(\beta, T)$ changes its character from positive at low temperatures to negative at relatively higher $T$. In this case, the system is viewed as being dominated by QG effects reflecting the formation of an unstable BEC.

In Fig. 3b, we represent the numerical simulation of the free energy for large $\beta$ in units of $(mc_s)^2$. We observe that corrections due to QG remain negative and tiny except at very low temperatures $T \lesssim 0.08T_c$. Here the negative term is smaller than the positive one signaling that the system is in its stable state.

5 Superfluidity

Fluctuations due to QG may lead also to modify the superfluid density. In our formalism, it is defined as [44]

$$n_s = 1 - n_n = 1 - \frac{2Q}{3T},$$

where $n_n = 2Q/3T$ accounts for the normal density of the Bose-condensed liquid, and $Q$ is the dissipated heat defined in an equilibrium system through the average of the total kinetic energy per particle. This yields

$$Q = \frac{1}{2\pi^2\hbar^3} \int \frac{P^2\,dP}{(1 + \beta P^2)^3} \left( \frac{E_P}{4} - 1 \right).$$
Deformation function $S(\beta, T)$ which governs the dependence of normal component of the superfluid density as a function of the temperature for several values of the deformation parameter $\beta$ in units of $(mc_s)^2$. Solid line: $\beta = 0.002/(mc_s)^2$. Dashed line: $\beta = 0.008/(mc_s)^2$. Dotted line: $\beta = 0.02/(mc_s)^2$. Dotted-dashed line: $\beta = 0.06/(mc_s)^2$.

**b** Numerical simulation of the normal density of the superfluid from Eq. (29). Solid line: $\beta = 0.5/(mc_s)^2$. Dashed line: $\beta = 0.8/(mc_s)^2$. Dotted line: $\beta = 1/(mc_s)^2$

In the absence of QG, the dissipated heat (29) becomes identical to that of Ref. [62]. Such two-fluid hydrodynamics developed by Landau and Khalatnikov [78,79] incorporates the motion of both the superfluid background (gravitational field) and excitations (matter). This is equivalent of the Einstein equations which encompass both gravity and matter [40].

Again at $T \ll mc_s^{-2}$, a straightforward calculation up to third order in $\beta$ gives for the superfluid density:

$$n_s = 1 - \frac{2\pi^2 T^4}{45mh^3c_s^5} S(\beta, T),$$

where

$$S(\beta, T) = 1 - \frac{60\pi^4 \beta}{7} \left( \frac{T}{c_s} \right)^2 + 96\pi^6 \beta^2 \left( \frac{T}{c_s} \right)^4 - \frac{12800\pi^8 \beta^3}{11} \left( \frac{T}{c_s} \right)^6 + \cdots.$$

For $\beta = 0$, the superfluid density returns to the seminal Landau’s formula $n_s = 1 - 2\pi^2 T^4/(45mh^3c_s^5)$ [79].

Figure 4a shows that for fixed $(mc_s)^2$, the function $S$ is decreasing with increasing $\beta$ which may lead to strongly reduce the normal density of the superfluid. Therefore, QG may enhance the superfluid density even for small $\beta$. Another important remark is that at $T = 0$, the whole liquid is superfluid, i.e., $n_s = n$.

Corrections to the normal density due to QG for large $\beta$ are computed numerically. We can see from Fig. 4b that as $\beta$ gets larger (approaches to unity), $n_s$ diminishes. The normal density remains minor even at higher temperatures in contrast with the thermal cloud of the condensate.

Notice that at $T \gg mc_s^{-2}$, the normal density agrees with the noncondensed density of an ideal Bose gas at least for small $\beta$.

### 6 Experimental test of quantum gravity

In this section, we discuss the possible experimental tests of our theoretical predictions that include QG effects. The measurement of quantum depletion of an interacting homogeneous BEC has been recently reported in [80]. In such an experiment, the quantum depletion is of the order of 1%, while by largely increasing the scattering length using Feshbach resonances, $n$ as increases as about 10% [80].

Table 1 shows typical values of $\beta_0$ and $\Delta X_{\text{min}}$ obtained from Eq. (6) for $^{39}\text{K}$ BEC [80] and $^{133}\text{Cs}$ BEC [75]. We see that for a sufficiently dilute Bose gas, our model predicts much more large QG parameter $\beta_0 \sim 10^{35} - 10^{57}$ and a minimum measurable

| $n$       | $\alpha$       | $\tilde{n}/n$ | $\beta_0$     | $\Delta X_{\text{min}}$ |
|-----------|----------------|---------------|---------------|--------------------------|
| $^{39}\text{K}$ | $3.5 \times 10^{17}$ m$^{-3}$ | 200 $a_0$     | 0.001         | $4.2 \times 10^{57}$     | $\sim 1.8 \mu$m           |
|           | $3.5 \times 10^{17}$ m$^{-3}$ | 3000 $a_0$    | 0.1           | $7.5 \times 10^{56}$     | $\sim 0.7 \mu$m           |
| $^{133}\text{Cs}$ | $1.2 \times 10^{18}$ m$^{-3}$ | 450 $a_0$     | 0.01          | $1.3 \times 10^{57}$     | $\sim 1 \mu$m             |
|           | $1.2 \times 10^{18}$ m$^{-3}$ | 2000 $a_0$    | 0.1           | $3.3 \times 10^{56}$     | $\sim 0.5 \mu$m           |

Here the values of $\alpha$ can be adjusted via Feshbach resonances [80,81].
length $\Delta X_{\text{min}}$ around 1 $\mu$m. For large depletion, our results for the QG parameter ($\beta_0 \sim 10^{56}$) are comparable to those obtained recently for optical systems [30], whereas our findings for a minimum length are higher by 100 of order of magnitude than those predicted for graphene $\Delta X_{\text{min}} \sim 2.3$ nm [82].

The detection threshold for quantum depletion is of the order $10^{-3}$ for weak interactions and about $10^{-2}$ for relatively strong interactions [80]. Therefore, if one is to deal with actual laboratory measurements of $\beta$ including QG effects, the deformation parameter must be bounded as $\beta_0 < 10^{57}$. It is important to mention here that the depletion corrections can be modified merely by changing the $s$-wave scattering length $a$, and hence improve the bound on the QG parameter.

7 Conclusions

We studied QG effects due to the GUP on the properties of weakly interacting homogeneous Bose gases at both zero and finite temperatures using the self-consistent TDHFB theory. The developed approach which is a combined theory of the HFB formalism and the GUP may be useful to test QG in real ultracold dilute Bose gases. We showed that corrections due to the presence of a minimal length provide an extra term modifying the condensate depletion, the anomalous density, the chemical potential, the ground-state energy, the free energy, and the normal density of the superfluid. Our results pointed out also that the condensate depletion is tunable by changing the QG parameter. When the GUP is not considered ($\beta = 0$), the obtained results become consistent with those existing in the literature. The competition between quantum fluctuations induced by interactions and QG effects may also shift the critical temperature allowing the formation of BEC and Bose superfluids even at higher temperatures. Finally, we showed that the experimental observation of such a system requires much more pronounced QG parameter. We expect that our findings will provide deeper insights into the universal properties of the dilute Bose gas under the GUP.

We believe that this work opens new prospects to the unexplored frontier of microscopic theory which will supply a real opportunity for a future tabletop test for unification of quantum theory and general relativity with a BEC. In our future work, we aim to investigate QG effects in dipolar BEC using our HFB model. Ultracold atoms with dipole–dipole interactions [60] which have the same form as the quantum gravitational interactions would be interesting to distinguish the QG signal from electromagnetic force [7] and hence enable studies for testing QG.

Data Availability Statement The data generated and/or analyzed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

References

1. R. Penrose, On the gravitization of quantum mechanics I: quantum state reduction. Found. Phys. 44, 557 (2014)
2. D. Kafri, J.M. Taylor, G.J. Milburn, A classical channel model for gravitational decoherence. New J. Phys. 16, 065020 (2014)
3. S. Bose, A. Mazumdar, G.W. Morley, H. Ulbricht, M. Torós, M. Paternostro, A. Geraci, P. Barker, M. Kim, G. Milburn, Phys. Rev. Lett. 119, 240401 (2017)
4. C. Marletto, V. Vedral, Phys. Rev. Lett. 119, 240402 (2017)
5. T. Krisnanda, M. Zappardo, M. Paternostro, T. Paterek, Phys. Rev. Lett. 119, 120402 (2017)
6. C. Marletto, V. Vedral, Phys. Rev. D 98, 046001 (2018)
7. R. Howl, V. Vedral, D. Naik, M. Christodoulou, C. Rovelli, A. Iyer, Phys. Rev. X Quantum 2, 010325 (2021)
8. K. Shiraishi, Prog. Theor. Phys. 77, 975 (1987)
9. S. Das, E.C. Vagenas, Phys. Rev. Lett. 101, 221301 (2008)
10. F. Brissese, M. Grether, M. de Llano, Europhys. Lett. 98, 6 (2012)
11. F. Brissese, Phys. Lett. B 718, 214 (2012)
12. J. Hansson, S. Francois, Int. J. Mod. Phys. D 26, 1743003 (2017)
13. M. Jaffe, P. Haslinger, V. Xu, P. Hamilton, A. Upadhye, B. Elder, J. Khoury, H. Müller, Nat. Phys. 13, 938 (2017)
14. S.A. Haine, New J. Phys. 23, 033020 (2021)
15. M. Maggiore, Phys. Lett. B 304, 65 (1993)
16. M. Maggiore, Phys. Lett. B 319, 83 (1993)
17. M. Maggiore, Phys. Rev. D 49, 5182 (1994)
18. A. Kempf, G. Mangano, R.B. Mann, Phys. Rev. D 52, 1108 (1995)
19. F. Scardigli, Phys. Lett. B 452, 39 (1999)
20. L.N. Chang, D. Minic, N. Okamura, T. Takeuchi, Phys. Rev. D 65, 125028 (2002)
21. A.F. Ali, S. Das, E.C. Vagenas, Phys. Rev. D 84, 044013 (2011)
22. M. Sprenger, P. Nicoli, M. Bleicher, Class. Quantum Grav. 28, 235019 (2011)
23. I. Pikovski, M.R. Vanner, M. Aspelmeyer, M. Kim, C. Brucker, Nat. Phys. 8, 393 (2012)
24. V. Hussain, S. Seahra, S. Webster, Phys. Rev. D 88, 024014 (2013)
25. P. Pedram, Phys. Rev. D 91, 065051 (2015)
26. Z. Feng, H.L. Li, X.T. Zu, S.Z. Yang, Eur. Phys. J. C 76, 1 (2016)
27. H. Shababi, W.S. Chung, Phys. Lett. B 770, 445 (2017)
28. G. Gecim, Y. Sucu, Phys. Lett. B 773, 391 (2017)
29. F. Scardigli, G. Lambiase, E.C. Vagenas, Phys. Lett. B 767, 242 (2017)
30. M.C. Braidotti, Z.H. Musslimani, C. Conti, Phys. D 338, 34 (2017)
31. P. Bosso, S. Das, I. Pikovski, M.R. Vanner, Phys. Rev. A 96, 023849 (2017)
32. R. Casadio, F. cardigini, Phys. Lett. B 807, 135558 (2020)
33. T. Fityo, Phys. Lett. A 372, 5872 (2008)
34. B. Vakili, M.A. Gorji, J. Stat. Mech. P10013 (2012)
35. E. Castellanos, C. Laemmerzahl, Phys. Rev. A 96, 023849 (2017)
36. R. Casadio, F. Cardiglini, Phys. Lett. B 807, 135558 (2020)
37. T. Fityo, Phys. Lett. A 372, 5872 (2008)
38. S. Dey, V. Hassan, Int. J. Theor. Phys. 58, 3138 (2019)
39. S. Dai, M. Fridman, Phys. Rev. D 104, 026014 (2021)
40. M. Novello, M. Visser, G. Volovik (eds.), *Artificial Black Holes* (World Scientific, 2002)
41. A. Boudjemâa, *Degenerate Bose Gas at Finite Temperatures* (Lambert Academic Publishing, Saarbrücken, 2017)
42. A. Boudjemâa, M. Benarous, Eur. Phys. J. D 59, 427 (2010)
43. A. Boudjemâa, M. Benarous, Phys. Rev. A 84, 043633 (2011)
44. A. Boudjemâa, Phys. Rev. A 86, 033608 (2012)
45. A. Boudjemâa, Phys. Rev. A 88, 023619 (2013)
46. A. Boudjemâa, Phys. Rev. A 90, 013628 (2014)
47. A. Boudjemâa, Phys. Rev. A 91, 063633 (2015)
48. A. Boudjemâa, Commun. Nonlinear Sci. Numer. Simul. 33, 85 (2016)
49. A. Boudjemâa, Commun. Nonlinear Sci. Numer. Simul. 48, 376 (2017)
50. A. Boudjemâa, Phys. Rev. A 94, 053629 (2016)
51. A. Boudjemâa, N. Gueblï, J. Phys. A: Math. Theor. 50, 425004 (2017)
52. A. Boudjemâa, Phys. Rev. A 98, 033612 (2018)
53. A. Boudjemâa, Phys. Rev. A 97, 033627 (2018)
54. A. Boudjemâa, N. Gueblï, Phys. Rev. A 102, 023302 (2020)
55. N. Gueblï, A. Boudjemâa, Phys. Rev. A 104, 023310 (2021)
56. A. Boudjemâa, Sci. Rep. 11, 21765 (2021)
57. S.T. Beliaev, Sov. Phys. JETP 7, 289 (1958)
58. A. Griffin, H. Shi, Phys. Rep. 304, 1 (1998)
59. C.J. Pethick, H. Smith, *Bose–Einstein Condensation in Dilute Gases*, 2nd edn. (Cambridge University Press, 2008)
60. A. Boudjemâa, J. Phys. B: At. Mol. Opt. Phys. 48, 035302 (2015)
61. J.O. Andersen, Theory of the weakly interacting Bose gas. Rev. Mod. Phys. 76, 599 (2004)
62. V. Yukalov, Phys. Part. Nucl. 42, 460 (2011)
63. A. Boudjemâa, J. Phys. A: Math. Theor. 49, 285005 (2016)
64. R. Balian, M. Vénéroni, Ann. Phys. (NY) 187, 29 (1988)
65. R. Balian, M. Vénéroni, Ann. Phys. (NY) 195, 324 (1989)
66. R. Balian, M. Vénéroni, Ann. Phys. (NY) 362, 838 (2015)
67. C. Martin, Phys. Rev. D 52, 721 (1995)
68. M. Benarous, H. Flocard, Ann. Phys. 273, 242 (1999)
69. F. Scardigli, R. Casadio, Eur. Phys. J. C 75, 425 (2015)
70. D. Gao, M. Zhan, Phys. Rev. A 94, 013607 (2016)
71. Z.W. Feng, S.Z. Yang b, H.L. Li, X.T. Zu, Phys. Lett. B 768, 81 (2017)
72. J.C.S. Neves, Eur. Phys. J. C 80, 343 (2020)
73. A. Das, S. Das, N.R. Mansour, E.C. Vagenas, Phys. Lett. B 819, 136429 (2021)
74. N.N. Bogolubov, J. Phys. (Moscow) 11, 23 (1947)
75. A.D. Lange, K. Pilch, A. Prantner, F. Ferlaino, B. Engeser, H.-C. Nägerl, R. Grimm, C. Chin, Phys. Rev. A 79, 013622 (2009)
76. P.O. Fedichev, G.V. Shlyapnikov, Phys. Rev. A 58, 3146 (1998)
77. T.D. Lee, K. Huang, C.N. Yang, Phys. Rev. 106, 1135 (1957)
78. I.M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965)
79. E.M. Lifshitz, L.P. Pitaevskii, *Statistical Physics, Part 2* (Pergamon Press, Oxford, 1980)
80. R. Lopes, C. Eigen, N. Navon, D. Clément, R.P. Smith, Z. Hadzibabic, Phys. Rev. Lett. 119, 190404 (2017)
81. C. Chin, R. Grimm, P. Julienne, E. Tiesinga, Rev. Mod. Phys. 82, 1225 (2010)
82. L. Menculini, O. Panella, P. Roy, Phys. Rev. D 87, 065017 (2013)