Some Integer Sequences
Based on Derangements

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Abstract
Sequences whose terms are equal to the number of functions with specified properties are considered. Properties are based on the notion of derangements in a more general sense. Several sequences which generalize the standard notion of derangements are thus obtained. These sequences generate a number of integer sequences from the well-known Sloane’s encyclopedia.

Let $A$ be an $m \times n$ rectangular area whose elements are from a set $\Omega$, and let $c_1, \ldots, c_m$ be from $\Omega$. Following the paper [1], we call each column of $A$ which is equal to $[c_1, \ldots, c_m]^T$ an $i$-column of $A$. As usual by $[n]$ will be denoted the set $\{1, 2, \ldots, n\}$, and by $|X|$ the number of elements of a finite set $X$. Mutually disjoint subsets are called blocks. A block with $k$ elements is called $k$-block. We also denote by $n^{(m)}$ the falling factorials, that is, $n^{(m)} = n(n-1) \cdots (n-m+1)$. Stirling numbers of the second kind will be denoted by $S(m, n)$.

We start with the following:

**Theorem 1** Suppose that $X_1, X_2, \ldots, X_k$ are blocks in $[m]$ and $Y_1, Y_2, \ldots, Y_k$ are subsets in $[n]$. Label all functions $f : [m] \to [n]$ by 1, 2, $\ldots$, $n^m$ arbitrary and form a $k \times n^m$ matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if $f_j(X_i) \subseteq Y_i$, and $a_{ij} = 0$ otherwise. The number $D_1$ of $i$-columns of $A$ consisting of 0’s is equal

$$D = \sum_{I \subseteq [k]} (-1)^{|I|} A(I), \quad (1)$$

where

$$A(I) = n^{[|m| \setminus \bigcup_{i \in I} X_i]} \cdot \prod_{i \in I} |Y_i|^{|X_i|}, \quad (2)$$

and $I$ runs over all subsets of $[m]$. 

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Proof. According to Theorem 1.1 in [1], the number $D$ is equal to the right side of (1) if $A(I)$ is the maximal number of columns $j$ of $A$ such that $a_{ij} = 1$ for all $i \in I$. It follows that $A(I)$ is equal to the number of functions $f : [m] \to [n]$ such that $f(X_i) \subseteq Y_i$, $(i \in I)$. This number is clearly equal to the number on the right side of (2).

In a similar way we obtain the following:

**Theorem 2** Suppose that $X_1, X_2, \ldots, X_k$ are blocks in $[m]$ and $Y_1, Y_2, \ldots, Y_k$ are subsets of $[n]$. Label all functions $f : [m] \to [n]$ by $1, 2, \ldots, n^m$ arbitrary, and form a $k \times n^m$ matrix $B = (b_{ij})$ such that $b_{ij} = 1$ if $f_j(X_i) = Y_i$, and $a_{ij} = 0$ otherwise. The number $N$ of $i$-columns of $A$ consisting of 0’s is equal

$$N = \sum_{I \subseteq [k]} (-1)^{|I|} B(I),$$

where

$$B(I) = n^{|[m] \setminus \cup_{i \in I} X_i|} \cdot \prod_{i \in I} |Y_i| \cdot S(|X_i|, |Y_i|),$$

and $I$ runs over all subsets of $[m]$.

Depending on the number of elements of $X_1, \ldots, X_k; Y_1, \ldots, Y_k$ it is possible to obtain a number of different sequences. Consider first the simplest case when each $X_1, \ldots, X_k; Y_1, \ldots, Y_k$ consists of one element. Then

$$A(I) = n^{m-|I|},$$

so that Theorem 1.2 of [1] may be applied. We thus obtain the following consequence of Theorem 1.

**Corollary 1** Given distinct $x_1, \ldots, x_k$ in $[m]$ and arbitrary $y_1, \ldots, y_k$ in $[n]$, then the number $D_{11}(m, n, k)$ of functions $f : [m] \to [n]$ such that

$$f(x_i) \neq y_i, \ (i = 1, 2, \ldots, k),$$

is equal

$$D_{11}(m, n, k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} n^{m-i} \cdot n^{m-k} (n - 1)^k.$$

A number of sequences in [2] is generated by this simple function. Some of them are stated in the following:
We may again apply Theorem 1.2 in [2] to obtain the following:

|   |   |   |   |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 |
| 25 | 26 | 27 | 28 |
| 29 | 30 | 31 | 32 |
| 33 | 34 | 35 | 36 |
| 37 | 38 |   |   |

Suppose that

\[ |X_1| = |X_2| = \ldots = |X_k| = 1, \quad |Y_1| = |Y_2| = \ldots = |Y_k| = 2. \]

Then

\[ A(I) = 2^i n^{m-|I|}. \]

We may again apply Theorem 1.2 in [2] to obtain the following:

**Corollary 2** Given distinct \(x_1, \ldots, x_k\) in \([m]\) and arbitrary 2-sets \(Y_1, \ldots, Y_k\) in \([n]\), then the number \(D_{12}(m, n, k)\) of functions \(f : [m] \rightarrow [n]\) such that

\[ f(x_i) \notin Y_i, \quad (i = 1, 2, \ldots, k), \]

is equal

\[ D_{12}(m, n, k) = \sum_{i=0}^{k} (-2)^i \binom{k}{i} n^{m-i} \left(= n^{m-k}(n - 2)^k \right). \]
This function also generates a number of sequences in \([2]\). The following table contains some of them.

**Table 2.**

1. \(A_{000027}(n) = D_{12}(1, n, 1)\),  
2. \(A_{005563}(n) = D_{12}(2, n, 1)\),  
3. \(A_{027620}(n) = D_{12}(3, n, 1)\),  
4. \(A_{000244}(n) = D_{12}(n, 3, 1)\),  
5. \(A_{004171}(n) = D_{12}(n, 4, 1)\),  
6. \(A_{005053}(n) = D_{12}(n, 5, 1)\),  
7. \(A_{067411}(n) = D_{12}(n, 6, 1)\),  
8. \(A_{000290}(n) = D_{12}(2, n, 2)\),  
9. \(A_{0002444}(n) = D_{12}(n, 3, 2)\),  
10. \(A_{000578}(n) = D_{12}(3, n, 3)\),  
11. \(A_{081294}(n) = D_{12}(n, 4, 3)\),  
12. \(A_{000583}(n) = D_{12}(4, n, 4)\),

If, in the conditions of Theorem 1, hold

\[|X_1| = \cdots = |X_k| = 2; \quad |Y_1| = \cdots |Y_k| = 1,\]

then

\[A(I) = n^{m-2|I|},\]

so that we have the following:

**Corollary 3** Suppose that \(X_1, \ldots, X_k\) are 2-blocks in \([m]\), and \(y_1, \ldots, y_k\) arbitrary elements in \([n]\), then the number \(D_{21}(m, n, k)\) of functions \(f : [m] \rightarrow [n]\) such that

\[f(X_i) \neq \{y_i\}, \quad (i = 1, 2, \ldots, k)\]

is equal

\[D_{21}(m, n, k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} n^{m-2i} \left(= n^{m-2k} (n^2 - 1)^k \right).\]

We also state some sequences in \([2]\) generated by this function.

**Table 3.**

1. \(A_{005563}(n) = D_{21}(2, n, 1)\),  
2. \(A_{007531}(n) = D_{21}(3, n, 1)\),  
3. \(A_{047982}(n) = D_{21}(4, n, 1)\),  
4. \(A_{005051}(n) = D_{21}(n, 3, 1)\),  
5. \(A_{005010}(n) = D_{21}(n, 2, 2)\),

Take finally the case \(|X_i| = |Y_i| = 2, \quad (i = 1, 2, \ldots, k)\). We have now

\[A(I) = 4^{|I|} \cdot n^{m-2|I|}.\]

We thus obtain the following consequence of Theorem 1.
Corollary 4 Let $X_1, \ldots, X_k$ in $[m]$ be 2-blocks, and $Y_1, \ldots, Y_k$ in $[n]$ be arbitrary 2-sets. Then the number $D_{22}(m, n, k)$ of functions $f : [m] \to [n]$ such that
\[ f(X_i) \not\subset Y_i, \ (i = 1, 2, \ldots, k) \]
is equal
\[ D_{22}(m, n, k) = \sum_{i=0}^{k} (-4)^i \binom{k}{i} n^{m-2i} \left( = n^{m-2k}(n^2 - 4)^k \right). \]

A few sequences in [2], given in the next table, is defined by this function.

| Table 4. |
|-----------|
| 1. $A005030(n) = D_{22}(n, 3, 1)$, 2. $A002001(n) = D_{22}(n, 4, 1)$ |
| 3. $A002063(n) = D_{22}(n, 4, 2)$, |

Take now the case $|X_i| = |Y_i| = 2$, $(i = 1, 2, \ldots, k)$ in the conditions of Theorem 2. We have
\[ B(I) = 2^{|I|} \cdot n^{m-2|I|}. \]

Thus we have the next:

Corollary 5 Let $X_1, \ldots, X_k$ be 2-blocks in $[m]$ and $Y_1, \ldots, Y_k$ arbitrary 2-sets in $[n]$. Then the number $S_{22}(m, n, k)$ of functions $f : [m] \to [n]$ such that
\[ f(X_i) \not= Y_i, \ (i = 1, 2, \ldots, k) \]
is equal
\[ S_{22}(m, n, k) = \sum_{i=0}^{k} (-2)^i \binom{k}{i} n^{m-2i} \left( = n^{m-2k}(n^2 - 2)^k \right). \]

The sequence A005032 in [2] is generated by this function.

We shall now consider injective functions from $[m]$ to $[n]$, $(m \leq n)$. We start with the following:

Theorem 3 Let $X_1, X_2, \ldots, X_k$ be blocks in $[m]$ and $Y_1, Y_2, \ldots, Y_k$ blocks in $[n]$ such that
\[ |X_i| = |Y_i|, \ (i = 1, 2, \ldots, k). \]
If a \( k \times n \) matrix \( A \) is defined such that \( a_{ij} = 1 \) if \( f_j(X_i) = Y_i \) and \( a_{ij} = 0 \) otherwise, then the number \( I(m, n, k) \) of \( i \)-columns of \( A \) consisting of 0’s is equal

\[
I_k(m, n) = \sum_{I \subseteq [k]} (-1)^{|I|} (n - |\cup_{i \in I} X_i|)(m - |\cup_{i \in I} X_i|) \cdot \prod_{i \in I} |X_i|!.
\]

**Proof.** In this case we have

\[
A(I) = (n - |\cup_{i \in I} X_i|)(m - |\cup_{i \in I} X_i|) \cdot \prod_{i \in I} |X_i|!,
\]

so that theorem follows from Theorem 1.1. in [1].

We shall also state some particular cases of this theorem. Suppose first that

\[ |X_i| = |Y_i| = 1, \quad (i = 1, \ldots, k). \]

The number \( A(I) \) in this case is equal

\[ (n - |I|)(m - |I|). \]

We thus obtain the following:

**Corollary 6** For disjoint \( x_1, \ldots, x_k \) in \([m]\) and disjoint \( y_1, \ldots, y_k \) in \([m]\), the number \( I_1(m, n, k) \) of injections \( f : [m] \to [n] \) such that

\[ f(x_i) \neq y_i, \quad (i = 1, 2, \ldots, k) \]

is equal

\[
I_1(m, n, k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (n - i)(m - i).
\]

**Note 1** Since obviously holds \( D(n) = I(n, n, n) \), where \( D(n) \) is the number of derangements of \( n \) elements, this function is an extension of derangements.

There are a number of sequences in [2] that are generated by this function. We state some of them in the next table.

**Table 5.**
1. \( A000290(n) = I(2, n, 1) \), 2. \( A045991(n) = I(3, n, 1) \)
3. \( A114436(n) = I(3, n, 1) \), 4. \( A047929(n) = I(4, n, 1) \)
5. \( A001563(n) = I(n, n, 1) \), 6. \( A001564(n) = I(n, n, 2) \)
7. \( A001565(n) = I(n, n, 3) \), 8. \( A002061(n) = I(2, n, 2) \)
9. \( A027444(n) = I(3, n, 2) \), 10. \( A058895(n) = I(4, n, 2) \)
11. \( A027444(n) = I(3, n, 2) \), 12. \( A074143(n) = I(n-1, n, 1) \)
13. \( A001563(n) = I(n-1, n, 1) \), 14. \( A001564(n) = I(n-1, n, 2) \)
15. \( A001565(n) = I(n-1, n, 3) \), 16. \( A109074(n) = I(n-1, n, 1) \)
17. \( A001563(n) = I(n-1, n, 4) \), 18. \( A001564(n) = I(n-1, n, 2) \)
19. \( A001565(n) = I(n-1, n, 3) \), 20. \( A001566(n) = I(n-1, n, 4) \)
21. \( A023043(n) = I(n-1, n, 6) \), 22. \( A023044(n) = I(n-1, n, 7) \)
23. \( A023045(n) = I(n-1, n, 8) \), 24. \( A023046(n) = I(n-1, n, 9) \)
25. \( A023047(n) = I(n-1, n, 10) \), 26. \( A001563(n) = I(n-2, n, 1) \)
27. \( A001564(n) = I(n-2, n, 2) \), 28. \( A061079(n) = I(n, 2n, 1) \).

As a special case we also have the following generalization of derangements.

**Corollary 7** If \( X_1, X_2, \ldots, X_n \) is a partition of \([kn]\) such that
\[
|X_i| = k, \ (i = 1, 2, \ldots, n),
\]
then the number \( D(n,k) \) of permutations \( f \) of \([kn]\) such that \( f(X_i) \neq X_i, \ (i = 1, 2, \ldots, n) \) is equal
\[
D(n,k) = \sum_{i=0}^{n} (-1)^i (k!)^i (nk - ik)!.
\]

For \( k = 1 \) we obtain the standard formula for derangements.

**Note 2** From the preceding formula the following sequences in [2] are derived:
A128805, A127888, A116221, A116220, A116219.

**References**

[1] Milan Janjić, Counting on rectangular areas, [arXiv:0704.0851v1](arXiv:0704.0851v1)
[2] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*
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A001477, A002378, A045991, A085537, A085538, A085539, A000079, A008776, A002001, A005054, A052934, A055272, A055274, A055275, A052268, A055276, A000290, A011379, A035287, A099762, A000079, A003946, A002063, A055842, A055846, A055270, A055847, A055995, A055996, A056002, A056116, A076728, A000578, A005051, A056120, A000583, A101362, A118265, A000027, A005563, A027620, A000244, A004171, A005053, A067411, A000290, A002444, A000578, A081294, A000583, A005563, A007531, A047982, A005051, A005010, A005032, A005030, A002001, A002063, A005032, A000290, A045991, A114436, A047929, A001563, A001564, A001565, A002061, A027444, A058895, A027444, A074143, A001563, A094304, A109074, A094258, A001564, A001565, A001688, A001689, A023043, A023044, A023045, A023046, A023407, A001563, A001564, A061079, A128805, A127888, A116221, A116220, A116219.