A CRYSTAL TO RIGGED CONFIGURATION BIJECTION FOR
NONEXCEPTIONAL AFFINE ALGEBRAS

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Abstract. Kerov, Kirillov, and Reshetikhin defined a bijection between high-
est weight vectors in the crystal graph of a tensor power of the vector representa-
tion, and combinatorial objects called rigged configurations, for type $A_n^{(1)}$.
We define an analogous bijection for all nonexceptional affine types, thereby
proving (in this special case) the fermionic formulas conjectured by Hatayama,
Kuniba, Takagi, Tsuboi, Yamada, and the first author.

1. Introduction

The fermionic formula, denoted by $M$, is a certain polynomial expressed as a sum
of products of $q$-binomial coefficients. It originates in the Bethe Ansatz analysis of
solvable lattice models in two dimensional statistical mechanics. The prototypical
example is given by the Kostka polynomial $K_{\lambda\mu}(q) \in \mathbb{Z}_{\geq 0}[q]$, which is indexed by
a pair of partitions $\lambda, \mu$. According to Lascoux and Schützenberger [12],

$$K_{\lambda\mu}(q) = \sum_{\lambda \in \mathcal{T}(\lambda,\mu)} q^{c(T)}.$$ 

Here $\mathcal{T}(\lambda,\mu)$ is the set of semistandard tableaux of shape $\lambda$ and weight $\mu$, and $c(T)$
is the charge of the tableau $T$.

We consider the case that $\mu$ is a single column $(1^L)$. Kirillov and Reshetikhin [8] gave a fermionic formula for the Kostka polynomial:

$$K_{\lambda,(1^L)}(q) = q^{L_2} M(\lambda,(1^L);q^{-1})$$

where

$$M(\lambda,(1^L);q) = \sum_{\{m\}} q^{cc(\{m\})} \prod_{1 \leq a \leq n, i \geq 1} \left[ \frac{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}} \right],$$

$$cc(\{m\}) = \frac{1}{2} \sum_{1 \leq a, b \leq n} C_{ab} \sum_{i,j \geq 1} \min(i,j)m_i^{(a)} m_j^{(b)},$$

$$p_i^{(a)} = L\delta_{i1} - \sum_{1 \leq b \leq n} C_{ab} \sum_{j \geq 1} \min(i,j)m_j^{(b)},$$

$$\left[ \frac{p + m}{m} \right] = (q)_p m! / (q)_m q^m$$ is the $q$-binomial coefficient, $(q)_m = (1 - q)(1 - q^2) \cdots (1 - q^m)$, the sum $\sum_{\{m\}}$ is taken over $\{m_i^{(a)} \in \mathbb{Z}_{\geq 0} \mid 1 \leq a \leq n, i \geq 1\}$, satisfying $p_i^{(a)} \geq 0$ for $1 \leq a \leq n, i \geq 1$ and $\sum_{i \geq 1} i m_i^{(a)} = \lambda_{a+1} + \lambda_{a+2} + \cdots + \lambda_{n+1}$

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for $1 \leq a \leq n$. Here $n$ is an integer not less than the length of $\lambda$ minus 1, and $(C_{ab})_{1 \leq a, b \leq n}$ is the Cartan matrix of $\mathfrak{sl}_{n+1}$.

To prove that the Kostka polynomial is given by the fermionic formula, Kerov, Kirillov and Reshetikhin (KKR) defined a bijection between $T(\lambda, (1^t))$ and combinatorial objects called rigged configurations. Expanding the $q$-binomial coefficients in $M(\lambda, (1^t); q)$, to each term $q^c$ one can associate a rigged configuration having a statistic $c$. Under the bijection, the charge of a tableau agrees with the statistic on the rigged configuration. This bijection was extended to the larger class of Littlewood-Richardson tableaux and corresponding rigged configurations.

The Kostka polynomial is related to the affine Lie algebra of type $\hat{A}_n^{(1)}$, since the corresponding fermionic formula is derived from the integrable model associated to the quantum affine algebra $U_q(\hat{A}_n^{(1)})$. The Kostka polynomial $K_{\lambda \mu}(q)$ gives the graded multiplicity of the $\lambda$-th irreducible $U_q(\hat{A}_n^{(1)})$-module in the restriction of the tensor product of certain finite-dimensional $U_q(\hat{A}_n^{(1)})$-modules that have crystal bases. The situation generalizes to the context of any affine Lie algebra. One can define the analogous tensor product modules and graded multiplicities, and a corresponding fermionic formula $M$. The new combinatorial objects which replace tableaux are called paths. A path is a highest weight element of the aforementioned tensor product crystal base. Paths have a natural statistic called energy. In the case of the Kostka polynomial, paths biject with rigged configurations: one may send the path (which may be viewed as a word) to its Robinson-Schensted recording tableau, which is then sent to a rigged configuration by the KKR bijection. The generating function of paths by energy is called the “one dimensional sum” $X$. The equality $X = M$ was conjectured in full generality.

The purpose of the paper is to construct the analogue of the KKR bijection and thereby prove the $X = M$ conjecture, for all nonexceptional affine Lie algebras, in the case of the simplest crystal bases. For $\hat{A}_n^{(1)}$, this case corresponds to the Kostka polynomial $K_{\lambda (1^t)}(q)$ discussed above.

2. QUANTUM AFFINE ALGEBRAS AND CRYSTALS

2.1. AFFINE ALGEBRAS. We adopt the notation of \cite{Kac}. Let $\mathfrak{g}$ be a Kac-Moody Lie algebra of nonexceptional affine type $X_N^{(r)}$, that is, one of the types $A_n^{(1)} (n \geq 1)$, $B_n^{(1)} (n \geq 3)$, $C_n^{(1)} (n \geq 2)$, $D_n^{(1)} (n \geq 4)$, $A_2^{(2)} (n \geq 1)$, $A_{2n}^{(2)} (n \geq 1)$, $A_{2n-1}^{(2)} (n \geq 2)$, $D_{n+1}^{(2)} (n \geq 2)$. Note that $A_{2n}^{(2)}$ is the same diagram as $A_{2n}^{(2)}$ but with the opposite labeling.

The Dynkin diagram of $\mathfrak{g} = X_N^{(r)}$ is depicted in Table \cite{Kac} (Table Aff 1-3 in \cite{Kac}). Its nodes are labeled by the set $I = \{0, 1, 2, \ldots, n\}$.

Let $\alpha_i, h_i, A_i (i \in I)$ be the simple roots, simple coroots, and fundamental weights of $\mathfrak{g}$. Let $\delta$ and $c$ denote the generator of imaginary roots and the canonical central element, respectively. Recall that $\delta = \sum_{i \in I} a_i \alpha_i$ and $c = \sum_{i \in I} a_i h_i$, where the Kac labels $a_i$ are the unique set of relatively prime positive integers giving the linear dependency of the columns of the Cartan matrix $A$ (that is, $A(a_0, \ldots, a_n)^t = 0)$. 2
Table 1. Dynkin diagrams for $X_N^{(r)}$. The labeling of the nodes (by elements of $I$) is specified under or the right side of the nodes. The numbers $t_i$ (resp. $t_i^r$) defined in (2.4) are attached above the nodes for $r = 1$ (resp. $r > 1$) if and only if $t_i \neq 1$ (resp. $t_i^r \neq 1$).

$A_1^{(1)}$: \[ \begin{array}{c}
0 \\
\end{array} \]

$A_n^{(1)}$: \[ \begin{array}{c}
0 \\
1 \\
2 \\
n-1 \\
n \\
\end{array} \]

$B_n^{(1)}$: \[ \begin{array}{c}
0 \\
1 \\
2 \\
3 \\
n-1 \\
n \\
\end{array} \]

$C_n^{(1)}$: \[ \begin{array}{c}
0 \\
1 \\
2 \\
n-1 \\
n \\
\end{array} \]

$D_n^{(1)}$: \[ \begin{array}{c}
0 \\
1 \\
2 \\
n-1 \\
n \\
\end{array} \]

$A_2^{(2)}$: \[ \begin{array}{c}
0 \\
1 \\
2 \\
\end{array} \]

$A_{2n}^{(2)}$: \[ \begin{array}{c}
0 \\
1 \\
2 \\
n-1 \\
n \\
\end{array} \]

$A_{2n}^{(2)\dagger}$: \[ \begin{array}{c}
0 \\
1 \\
\end{array} \]

$A_{2n-1}^{(2)}$: \[ \begin{array}{c}
0 \\
1 \\
2 \\
n-1 \\
n \\
\end{array} \]

$D_{n+1}^{(2)}$: \[ \begin{array}{c}
0 \\
1 \\
2 \\
n-1 \\
n \\
\end{array} \]
Explicitly,
\[
\delta = \begin{cases} 
\alpha_0 + \cdots + \alpha_n & \text{if } g = A_n^{(1)} \\
\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + \alpha_{n-1} + \alpha_n & \text{if } g = B_n^{(1)} \\
\alpha_0 + 2\alpha_1 + \cdots + \alpha_{n-1} + 2\alpha_n & \text{if } g = C_n^{(4)} \\
\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + \alpha_{n-1} + \alpha_n & \text{if } g = D_n^{(4)} \\
2\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } g = A_{2n}^{(2)} \\
\alpha_0 + 2\alpha_1 + \cdots + 2\alpha_{n-1} + 2\alpha_n & \text{if } g = A_{2n+1}^{(2)} \\
\alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n & \text{if } g = A_{2n-1}^{(2)} \\
\alpha_0 + \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n & \text{if } g = D_{2n+1}^{(2)} 
\end{cases}
\]

The dual Kac label $\tilde{a}_i^\vee$ is the label $a_i$ for the affine Dynkin diagram obtained by “reversing the arrows” of the Dynkin diagram of $g$, or equivalently, the coefficients giving the linear dependency of the rows of the Cartan matrix $A$. Note that $a_0^\vee = 2$ for $g = A_{2n}^{(2)}$ and $a_0^\vee = 1$ otherwise.

Let $(\cdot | \cdot)$ be the normalized invariant form on $P$. It satisfies
\[(\alpha_i | \alpha_j) = \frac{a_i^\vee}{a_i} A_{ij}
\]
for $i, j \in I$. In particular
\[(\alpha_a | \alpha_a) = \frac{2r}{a_0}
\]
if $\alpha_a$ is a long root.

For $i \in I$ let
\[t_i = \max\left(\frac{a_i}{\tilde{a}_i}, \frac{a_i^\vee}{\alpha_i}ight), \quad t_i^\vee = \max\left(\frac{a_i^\vee}{\alpha_i}, a_0\right).
\]
The values $t_i$ are given in Table 1. We shall only use $t_i^\vee$ and $t_i$ for $i \in I^* = I \setminus \{0\}$.

For $a \in I^*$ we have
\[t_i^\vee = 1 \quad \text{if } r = 1, \quad t_a = a_0^\vee \quad \text{if } r > 1.
\]

We consider two finite-dimensional subalgebras of $g$: $\mathfrak{g}$, whose Dynkin diagram is obtained from that of $g$ by removing the 0 vertex, and $\mathfrak{g}_{\tau}$, the subalgebra of $X_N$ fixed by the automorphism $\sigma$ given in [4, Section 8.3].

| $\mathfrak{g}$ | $X_N^{(1)}$ | $A_{2n}^{(2)}$ | $A_{2n}^{(2)}$ | $A_{2n-1}^{(2)}$ | $D_{2n+1}^{(2)}$ |
|----------------|-------------|-----------------|-----------------|-----------------|-----------------|
| $\overline{g}$ | $X_N$       | $C_n$           | $B_n$           | $C_n$           | $B_n$           |
| $\mathfrak{g}_{\tau}$ | $X_N$       | $B_n$           | $B_n$           | $C_n$           | $B_n$           |

Let $\overline{g}$ (resp. $\mathfrak{g}_{\tau}$) have weight lattice $\overline{P}$ (resp. $\overline{P}$), with simple roots and fundamental weights $\alpha_a, \overline{\alpha}_a$ (resp. $\bar{\alpha}_a, \bar{\bar{\alpha}}_a$) for $a \in I^*$. Note that $\mathfrak{g} = \mathfrak{g}_{\tau}$ for $g \neq A_{2n}^{(2)}$. For $g = A_{2n}^{(2)}$, $\overline{g} = C_n$ and $\mathfrak{g}_{\tau} = B_n$.

$\overline{P}$ is endowed with the bilinear form $(\cdot | \cdot)$, normalized by
\[(\bar{\alpha}_a | \bar{\alpha}_a) = 2r/a_0^\vee \quad \text{if } \bar{\alpha}_a \text{ is a long root of } \mathfrak{g}_{\tau}.
\]

For $A_{2n}^{(2)}$, the unique simple root $\bar{\alpha}_1$ of $\mathfrak{g}_{\tau} = B_1$ is considered to be short.
Note that \( \alpha_a, \Lambda_a \) and \( \langle \cdot | \cdot \rangle \) may be identified with \( \hat{\alpha}_a, \hat{\Lambda}_a \) and \( \langle \cdot | \cdot \rangle' \) if \( g \neq A^{(2)}_{2n} \).

Define the \( \mathbb{Z} \)-linear map \( \iota : \hat{P} \to \hat{P} \) by

\[
\iota(\Lambda_a) = \epsilon_a \hat{\Lambda}_a \quad \text{for } a \in I^*,
\]

where \( \epsilon_a \) is defined by

\[
\epsilon_a = \begin{cases} 
2 & \text{if } g = A^{(2)}_{2n} \text{ and } a = n \\
1 & \text{otherwise}.
\end{cases}
\]

In particular \( \iota(\alpha_a) = \epsilon_a \hat{\alpha}_a \) for \( a \in I^* \). We have

\[
\langle \iota(\alpha_a) | \iota(\alpha_b) \rangle' = a_0(\alpha_a | \alpha_b) \quad \text{for all } b \in I^*.
\]

If \( g = A^{(2)}_{2n} \) both sides of (2.8) are equal to 8 if \( b = n \) and 4 otherwise. Especially for \( g = A^{(2)}_2 \) \((n = 1)\), we have \( (\hat{\alpha}_1 | \hat{\alpha}_1)' = 2 \) and \( (\alpha_1 | \alpha_1) = 4 \). In the rest of the paper we shall write \( \langle \cdot | \cdot \rangle \) in place of \( \langle \cdot | \cdot \rangle' \).

2.2. Simple subalgebras. For later use, specific realizations are given for the simple roots and fundamental weights of the simple Lie algebras of types \( B_n, C_n, \) and \( D_n \), which appear as the subalgebras \( g \) and \( g_0 \) of \( g \). In each case the sublattice of \( P \) given by the weights appearing in tensor products of the vector representation, is identified with \( \mathbb{Z}^n \). Let \( \{ \epsilon_i \mid 1 \leq i \leq n \} \) be the standard basis of \( \mathbb{Z}^n \).

**The simple Lie algebra \( B_n \).**

\[
\alpha_a = \epsilon_a - \epsilon_{a+1} \quad \text{for } 1 \leq a < n
\]

\[
\alpha_n = \epsilon_n
\]

\[
\Lambda_a = \epsilon_1 + \cdots + \epsilon_a \quad \text{for } 1 \leq a < n
\]

\[
\Lambda_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_n).
\]

\( \lambda \in \mathbb{Z}^n \) is \( B_n \)-dominant if and only if

\[
\begin{align*}
\lambda_a - \lambda_{a+1} & \geq 0 \\
\lambda_n & \geq 0.
\end{align*}
\]

**The simple Lie algebra \( C_n \).**

\[
\alpha_a = \epsilon_a - \epsilon_{a+1} \quad \text{for } 1 \leq a < n
\]

\[
\alpha_n = 2\epsilon_n
\]

\[
\Lambda_a = \epsilon_1 + \cdots + \epsilon_a \quad \text{for } 1 \leq a \leq n.
\]

\( \lambda \in \mathbb{Z}^n \) is \( C_n \)-dominant if and only if it is \( B_n \)-dominant (2.10).

**The simple Lie algebra \( D_n \).**

\[
\alpha_a = \epsilon_a - \epsilon_{a+1} \quad \text{for } 1 \leq a < n
\]

\[
\alpha_n = \epsilon_{n-1} + \epsilon_n
\]

\[
\Lambda_a = \epsilon_1 + \cdots + \epsilon_a \quad \text{for } 1 \leq a \leq n - 2
\]

\[
\Lambda_{n-1} = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} - \epsilon_n)
\]

\[
\Lambda_n = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_{n-1} + \epsilon_n)
\]
\( \lambda \in \mathbb{Z}^n \) is \( D_n \)-dominant if and only if

\[
\begin{align*}
\lambda_a - \lambda_{a+1} & \geq 0 \quad \text{for } 1 \leq a < n, \\
\lambda_{n-1} + \lambda_n & \geq 0.
\end{align*}
\]

2.3. **Crystals.** Let \( g' \) be the derived subalgebra of \( g \). Denote the corresponding quantized universal enveloping algebras of \( g \supset g' \supset \mathfrak{g} \) by \( U_q(g) \supset U'_q(g) \supset U_q(\mathfrak{g}) \).

In \( [3] \) it is conjectured that there is a family of finite-dimensional irreducible \( U'_q(g) \)-modules \( \{ W_i^{(a)} \mid a \in I^* , i \in \mathbb{Z}_{>0} \} \) which, unlike most finite-dimensional \( U'_q(g) \)-modules, have crystal bases \( B^{a,i} \). This family is conjecturally characterized in several different ways:

1. Its characters form the unique solutions of a system of quadratic relations (the \( Q \)-system) \( [3] \).
2. Every crystal graph of an irreducible integrable finite-dimensional \( U'_q(g) \)-module, is a tensor product of the crystals \( B^{a,i} \).
3. For \( \lambda \in P \) let \( V(\lambda) \) be the extremal weight module defined in \([3], \text{Section 3}\) and \( B(\lambda) \) its crystal base, with unique vector \( u_\lambda \in B(\lambda) \) of weight \( \lambda \). Then the affinization of \( B^{a,i} \) (in the sense of \([3]\)) is isomorphic to the connected component of \( u_\lambda \) in \( B(\lambda) \), for the weight \( \lambda = i\lambda_0 \) (except when \( g = A_{2n}^{(2)} \) and \( a = n \), in which case \( \lambda = 2i\lambda_0 \)).

In light of point (2) above, we consider the category of crystal graphs given by tensor products of the crystals \( B^{a,i} \).

We introduce notation for tensor products of \( B^{a,i} \). Let \( \mu = (L_i^{(a)})_{a \in I^* , i \in \mathbb{Z}_{>0}} \) be a matrix of nonnegative integers, almost all zero. Define

\[
B^{(a)} = \bigotimes_{(a,i) \in I^* \times \mathbb{Z}_{>0}} (B^{a,i})^{L_i^{(a)}}.
\]

In type \( A_n^{(1)} \), this is the tensor product of modules, which, when restricted to \( A_n \), are irreducible modules indexed by rectangular partitions. The set of classically restricted paths (or classical highest weight vectors) in \( B^{(\mu)} \) of weight \( \lambda \in P^\vee = \bigoplus_{i \in I^*} \mathbb{Z}_{\geq 0} \lambda_i \) is by definition

\[
\mathcal{P}(\lambda, \mu) = \{ b \in B^{(\mu)} \mid \text{wt}(b) = \lambda \text{ and } \bar{e}_i b \text{ undefined for all } i \in I^* \}.
\]

Here \( \bar{e}_i \) is given by the crystal graph. For \( b, b' \in B^{a,i} \) we have \( b' = \bar{e}_i (b) \) if there is an arrow \( b' \xrightarrow{i} b \) in the crystal graph; if no such arrow exists then \( \bar{e}_i (b) \) is undefined. Similarly, \( b' = \bar{f}_i (b) \) if there is an arrow \( b \xrightarrow{-i} b' \) in the crystal graph; if no such arrow exists then \( \bar{f}_i (b) \) is undefined. If \( B_1 \) and \( B_2 \) are crystals, then for \( b_1 \otimes b_2 \in B_1 \otimes B_2 \) the action of \( \bar{e}_i \) is defined as

\[
\bar{e}_i (b_1 \otimes b_2) = \begin{cases} 
\bar{e}_i b_1 \otimes b_2 & \text{if } \varepsilon_i (b_1) > \varphi_i (b_2), \\
b_1 \otimes \bar{e}_i b_2 & \text{else},
\end{cases}
\]

where \( \varepsilon_i (b) = \max\{ k \mid \tilde{e}_i^k \text{ is defined} \} \) and \( \varphi_i (b) = \max\{ k \mid \tilde{f}_i^k \text{ is defined} \} \).

**Assumption 2.1.** In this paper we shall restrict our attention to the case \( B^{(\mu)} = B^\otimes L \) where \( B = B^{1,1} \). We shall write \( B(b) = B^\otimes (L-1) \).

The crystal graphs \( B^{1,1} \) are listed in Table \( [3] \).
In each case (other than $A_n^{(1)}$) the elements of $B = B^{1,1}$ consist of \{1, k | 1 \leq k \leq n\} and possibly elements 0 and $\phi$.

**Remark 2.2.** By glancing at Table 3 one may check that the following are equivalent for $b = b_L \otimes b_{L-1} \otimes \cdots \otimes b_1 \in B^\otimes L$ and $\lambda \in \mathcal{P}^+$:

1. $b$ is a classically restricted path of weight $\lambda \in \mathcal{P}^+$.
2. $\lambda - \text{wt}(b_L) \in \mathcal{P}^+$, $b_{L-1} \otimes \cdots \otimes b_1$ is a classically restricted path of weight $\lambda - \text{wt}(b_L)$, and if $b_L = 0 \in B$ then $\lambda_n > 0$ (where $\lambda$ is viewed as an element of $\mathbb{Z}^n$).
The weight function $\text{wt} : B \rightarrow \mathbb{Z}^n$ is given by
\[
\text{wt}(k) = \epsilon_k \quad \text{for } 1 \leq k \leq n \\
\text{wt}(\Phi) = -\epsilon_k \quad \text{for } 1 \leq k \leq n \\
\text{wt}(0) = \text{wt}(\phi) = 0.
\]

The weight function $\text{wt} : B^\otimes L \rightarrow \mathbb{Z}$ is defined by $\text{wt}(b_L \otimes \cdots \otimes b_1) = \sum_{j=1}^L \text{wt}(b_j)$. So if $\lambda = \text{wt}(p)$ where $p \in B^\otimes L$, then $\lambda_k$ is the multiplicity of $k$ in $p$ minus the multiplicity of $\Phi$ in $p$.

2.4. **One-dimensional sums.** The energy function $D : B^{(\mu)} \rightarrow \mathbb{Z}$ gives the grading on $B^{(\mu)}$. In the case $B^{(\mu)} = B^\otimes L$, it takes a simple form. Due to the existence of the universal $R$-matrix and the fact that $W_1^{(1)}$ is irreducible, by [1] there is a unique (up to global additive constant) function $H : B^{1,1} \otimes B^{1,1} \rightarrow \mathbb{Z}$ called the local energy function, such that
\[
(2.16) \quad H(\bar{e}_i(b \otimes b')) = H(b \otimes b') + \begin{cases} 
-1 & \text{if } i = 0 \text{ and } \bar{e}_0(b \otimes b') = b \otimes \bar{e}_0b' \\
1 & \text{if } i = 0 \text{ and } \bar{e}_0(b \otimes b') = \bar{e}_0b \otimes b' \\
0 & \text{otherwise}.
\end{cases}
\]

Let $b^0 \in B^{1,1}$ be the unique element such that $\varphi(b^0) = \Lambda_0$. We normalize $H$ by the condition
\[
(2.17) \quad H(1 \otimes 1) = 0.
\]

Then
\[
(2.18) \quad E(b_L \otimes \cdots \otimes b_1) = L \cdot H(b_1 \otimes b^0) + \sum_{j=1}^{L-1} (L - j) \cdot H(b_{j+1} \otimes b_j),
\]

\[
D(b_L \otimes \cdots \otimes b_1) = E(b_L \otimes \cdots \otimes b_1) - E(1 \otimes \cdots \otimes 1).
\]

Define the one-dimensional sum $X(\lambda, \mu; q) \in \mathbb{Z}[q, q^{-1}]$ by
\[
(2.19) \quad X(\lambda, \mu; q) = \sum_{b \in B(\lambda, \mu)} q^{D(b)}.
\]

Since $B^{(\mu)}$ is completely reducible as a $U_q(\mathfrak{g})$-crystal, one has
\[
\sum_{b \in B^{(\mu)}} e^{\text{wt}(b)}q^{D(b)} = \sum_{\lambda \in \mathcal{P}^+} \chi^\lambda X(\lambda, \mu; q)
\]

where $\chi^\lambda$ is the character of the irreducible $U_q(\mathfrak{g})$-module of highest weight $\lambda$. It can be shown that $X(\lambda, \mu; q) \in \mathbb{Z}_{\geq 0}[q^{-1}]$. For convenience we define
\[
(2.20) \quad \overline{H} = -H, \quad \overline{D} = -D, \quad \overline{X}(\lambda, \mu; q) = X(\lambda, \mu; q^{-1}).
\]

3. **Rigged configurations and the bijection**

3.1. **The fermionic formula.** $\mathfrak{g} \neq A_{2n}^{(1)}$. This subsection reviews definitions of [1, 2]. Let $\mathfrak{g}$ be a Kac-Moody algebra of nonexceptional affine type that is not of the form $A_{2n}^{(1)}$. Fix $\lambda \in \mathcal{P}^+$ and a matrix $\mu = (L_{i}^{(a)})$ of nonnegative integers as in subsection 2.4.
Let \( \nu = (m_i^{(a)}) \) be another such matrix. Say that \( \nu \) is a \( \lambda \)-configuration if

\[
\sum_{a \in I^*} i m_i^{(a)} \alpha_a = t \left( \sum_{a \in I^*} i L_i^{(a)} \tau_a - \lambda \right).
\]

Say that a configuration \( \nu \) is \( \mu \)-admissible if

\[
p_i^{(a)} \geq 0 \quad \text{for all } a \in I^* \text{ and } i \in \mathbb{Z}_{>0},
\]

where

\[
p_i^{(a)} = \sum_{k \in \mathbb{Z}_{>0}} \left( L_k^{(a)} \min(i, k) - \frac{1}{k_a} \sum_{b \in I^*} (\tilde{\alpha}_a | \tilde{\alpha}_b) \min(t_b i, t_a k) m_k^{(b)} \right).
\]

Write \( C(\lambda, \mu) \) for the set of \( \mu \)-admissible \( \lambda \)-configurations. Define

\[
cc(\nu) = \frac{1}{2} \sum_{a, b \in I^*} \sum_{j, k \in \mathbb{Z}_{>0}} (\tilde{\alpha}_a | \tilde{\alpha}_b) \min(t_b j, t_a k) m_j^{(a)} m_k^{(b)}.
\]

The fermionic formula is defined by

\[
\overline{M}(\lambda, \mu; q) = \sum_{\nu \in C(\lambda, \mu)} q^{cc(\nu)} \prod_{a \in I^*} \prod_{i \in \mathbb{Z}_{>0}} \left[ p_i^{(a)} + m_i^{(a)} \right] q^{i_a}.
\]

The \( X = M \) conjecture of [1, 2] states that

\[
\overline{X}(\lambda, \mu; q) = \overline{M}(\lambda, \mu; q).
\]

### 3.2. Rigged configurations, \( g \neq A^{(2)}_{2n} \)

The fermionic formula \( \overline{M}(\lambda, \mu) \) can be interpreted using combinatorial objects called rigged configurations. These objects are a direct combinatorialization of the fermionic formula \( \overline{M}(\lambda, \mu; q) \). Our goal is to prove [1, 2] under Assumption 2.4 by defining a statistic-preserving bijection from rigged configurations to paths. For this purpose it is convenient to use an indexing slightly differing from that used above.

For \( a \in I^* \), define

\[
v_a = \begin{cases}
  2 & \text{if } a = n \text{ and } g = C_n^{(1)} \\
  \frac{1}{2} & \text{if } a = n \text{ and } g = B_n^{(1)} \\
  1 & \text{otherwise}.
\end{cases}
\]

\( v_a \) is half the square length of \( \alpha_a \) for untwisted affine types and is equal to 1 for twisted types.

A quasipartition \( \lambda \) of type \( a \in I^* \) is a finite multiset taken from the set \( v_a \mathbb{Z}_{>0} \). Denote by \( m_i(\lambda) \) the number of times \( i \in v_a \mathbb{Z}_{>0} \) occurs in \( \lambda \). The diagram of such a quasipartition has, for each \( i \in v_a \mathbb{Z}_{>0} \), \( m_i(\lambda) \) rows consisting of \( i \) boxes, where each box has width \( v_a \). Set

\[
\mathcal{H} = \{ (a, i) \mid a \in I^*, i \in v_a \mathbb{Z}_{>0} \}.
\]

Denote by \( (\nu^*, J^*) \) a pair where \( \nu^* = \{ \nu^{(a)} \}_{a \in I^*} \) is a sequence of quasipartitions with \( \nu^{(a)} \) of type \( a \) and \( J^* = \{ J^{(a, i)} \}_{(a, i) \in \mathcal{H}} \) is a double sequence of partitions. For
\((a, i) \in \mathcal{H}, \) define

\[
\begin{align*}
P_i^{(a)}(\nu^*) &= P_i^{(a)} \\
m_i^{(a)}(\nu^*) &= m_i^{(a)} = m_i(\nu) = 0.
\end{align*}
\]

(3.9)

Then a rigged configuration is a pair \((\nu^*, J^*)\) subject to the restriction (3.1) and the requirement that \(J^{(a,i)}\) be a quasipartition contained in a \(m_i^{(a)}(\nu^*) \times P_i^{(a)}(\nu^*)\) rectangle. The set of rigged configurations for fixed \(\lambda\) and \(\mu\) is denoted by \(\text{RC}(\lambda, \mu)\). Then (3.3) is equivalent to

\[
F(\lambda, \mu) = \sum_{(\nu^*, J^*) \in \text{RC}(\lambda, \mu)} q^{cc(\nu^*, J^*)}
\]

where \(cc(\nu^*, J^*) = cc(\nu) + |J^*|\) and \(|J^*| = \sum_{(a, i) \in \mathcal{H}} \mathcal{L}^{(a)}(J^{(a,i)})\) for \(\nu\) corresponding to \(\nu^*\) under (3.4).

3.3. \(A_{2n}^{(2)\dagger}\) rigged configurations. In this subsection let \(g = A_{2n}^{(2)\dagger}\). As this case is not considered in [13], we shall only give the definition in terms of rigged configurations, although it is easy to express the result as a sum of a product of \(q\)-binomials (see [13], Section 7.6]). The important feature is that the riggings of odd-sized parts of \(\nu^{(1)}\), must have the form \(x/2\) where \(x\) is an odd integer. So let \(\mu\) and \(\lambda\) be as in subsection 3.1. Given a matrix \(\nu = (m_i^{(1)})\), let \(P_i^{(1)}(\nu^*)\) and \(m_i^{(1)}(\nu^*)\) be defined as before. Call \(\nu^*\) \(\mu\)-admissible if \(P_i^{(1)}(\nu^*) \geq 0\) for all \(a \in I^*\) and \(i \in \mathbb{Z}_{\geq 0}\), together with the extra condition that

\[
P_i^{(1)}(\nu^*) \geq 1 \quad \text{if} \quad i \text{ is odd and } m_i^{(1)}(\nu^*) > 0.
\]

(3.10)

A rigging \(J^*\) consists of quasipartitions \(J^{(a,i)}\) for \(a \in I^*\) and \(i \in \mathbb{Z}_{\geq 0}\). For \(a \neq n\) or \(i\) even, \(J^{(a,i)}\) is an ordinary partition satisfying the usual properties. For \(a = n\) and \(i\) odd, \(J^{(n,i)}\) is a quasipartition contained in a rectangle with \(P_i^{(1)}(\nu^*)\) columns and \(m_i^{(1)}(\nu^*)\) rows, but it has cells of width \(1/2\) and each part size must be of the form \(x/2\) for \(x\) an odd integer. This defines the set \(\text{RC}(\lambda, \mu)\) for \(g = A_{2n}^{(2)\dagger}\). Then \(F(\lambda, \mu)\) is defined as before where \(|J^*|\) is the sum of the areas of all the quasipartitions \(J^{(a,i)}\). This definition is compatible with the virtual crystal realization which embeds paths (and rigged configurations) of type \(A_{2n}^{(2)\dagger}\) into those of type \(A_{2n-1}^{(1)}\). [3]

3.4. The bijection from RCs to paths. We now describe the general form of the bijection \(\Phi : \text{RC}(\lambda, \mu) \rightarrow \mathcal{P}(\lambda, \mu)\) under Assumption 2.1. Let \(\mu = (\mathcal{L}_i^{(1)})\) be such that \(B^{(\mu)} = B^{\otimes L}\), that is, \(\mathcal{L}_i^{(1)} = L \delta_{i1}\). Let \(\tilde{\mu}\) be such that \(B^{(\tilde{\mu})} = B^{\otimes (L-1)}\).

Let \((\nu^*, J^*) \in \text{RC}(\lambda, \mu)\). We shall define a map \(\text{rk} : \text{RC}(\lambda, \mu) \rightarrow B\) which associates to \((\nu^*, J^*)\) an element of \(B\) called its rank.

Denote by \(\text{RC}_b(\lambda, \mu)\) the elements of \(\text{RC}(\lambda, \mu)\) of rank \(b\). We shall define a bijection \(\delta : \text{RC}_b(\lambda, \mu) \rightarrow \text{RC}(\lambda - \text{wt}(b), \tilde{\mu})\). The disjoint union of these bijections then defines a bijection \(\delta : \text{RC}(\lambda, \mu) \rightarrow \bigcup_{b \in B^1} \text{RC}(\lambda - \text{wt}(b), \tilde{\mu})\).

The bijection \(\Phi\) is defined recursively as follows. For \(b \in B\) let \(\mathcal{P}_b(\lambda, \mu)\) be the set of paths in \(B^{(\mu)} = B^{\otimes L}\) that have \(b\) as leftmost tensor factor. For \(L = 0\) the bijection \(\Phi\) sends the empty rigged configuration (the only element of the set \(\text{RC}(\lambda, \mu)\)) to the empty path (the only element of \(\mathcal{P}(\lambda, \mu)\)). Otherwise assume that
Φ has been defined for \( B^{\otimes (L-1)} \) and define it for \( B^{\otimes L} \) by the commutative diagram

\[
\begin{array}{c}
\text{RC}_b(\lambda, \mu) \xrightarrow{\Phi} \mathcal{P}_b(\lambda, \mu) \\
\downarrow \quad \downarrow \\
\text{RC}(\lambda - \text{wt}(b), \tilde{\mu}) \xrightarrow{\Phi} \mathcal{P}(\lambda - \text{wt}(b), \tilde{\mu})
\end{array}
\]

(3.11)

where the right hand vertical map removes the leftmost tensor factor \( b \). In short,

\[
\Phi(\nu^*, J^*) = \text{rk}(\nu^*, J^*) \otimes \Phi(\delta(\nu^*, J^*')).
\]

(3.12)

Remark 3.1. For \( \Phi \) to be well-defined, by Remark 2.2 it must be shown that if \( b = \text{rk}(\nu^*, J^*) \), then \( \rho = \lambda - \text{wt}(b) \) is dominant, and if \( b = 0 \) then \( \lambda_n > 0 \).

We also require the bijection \( \hat{\Phi} : \text{RC}(\lambda, \mu) \rightarrow \mathcal{P}(\lambda, \mu) \) given by \( \hat{\Phi} = \Phi \circ \text{comp} \) where \( \text{comp} : \text{RC}(\lambda, \mu) \rightarrow \text{RC}(\lambda, \mu) \) with \( \text{comp}(\nu^*, J^*) = (\nu^*, J^*) \) is the function which complements the riggings, meaning that \( J^* \) is obtained from \( J^* \) by complementing all partitions \( J^{(a,i)} \) in the \( m_i^{(a)} \times P_i^{(a)}(\nu^*) \) rectangle.

Theorem 3.2. \( \Phi : \text{RC}(\lambda, \mu) \rightarrow \mathcal{P}(\lambda, \mu) \) is a bijection such that

\[
cc(\nu^*, J^*) = D(\Phi(\nu^*, J^*)) \quad \text{for all} \quad (\nu^*, J^*) \in \text{RC}(\lambda, \mu).
\]

(3.13)

For type \( A_n^{(1)} \), a generalization of this theorem for all \( \mu \) was proven in [10]. For other types Theorem 3.2 is proved in section 3.

4. The bijection for each root system

In this section the maps \( \text{rk} \) and \( \delta \) are defined in a case-by-case manner. For each \( g \), an explicit formula is given for the vacancy numbers \( P^{(a)}(\nu^*) \) (see (3.3)), obtained by writing (3.3) in terms of the function \( Q_i \) (see (3.1)) using the data for the simple Lie algebras given in section 2.2. Then for \( (\nu^*, J^*) \in \text{RC}(\lambda, \mu) \), an algorithm is given which defines \( b = \text{rk}(\nu^*, J^*) \), the new smaller rigged configuration \( (\tilde{\nu}^*, \tilde{J}^*) = \delta(\nu^*, J^*) \) such that \( (\tilde{\nu}^*, \tilde{J}^*) \in \text{RC}(\rho, \tilde{\mu}) \) (where \( \rho = \lambda - \text{wt}(b) \)), and the new vacancy numbers in terms of the old.

For a quasipartition \( \tau \) with boxes of width \( v \) and \( i \in v\mathbb{Z}_{\geq 0} \), define

\[
Q_i(\tau) = \sum_j \min(\tau_j, i),
\]

(4.1)

the area of \( \tau \) in the first \( i \) quasicolumns.

The quasipartition \( J_i^{(a,b)} \) is called singular (with respect to the configuration \( \nu^* \)) if it has a part of size \( P_i^{(a)}(\nu^*) \). If \( A \) is a statement then \( \chi(A) = 1 \) if \( A \) is true and \( \chi(A) = 0 \) if \( A \) is false. We also use the Kronecker delta notation \( \delta_a,b = \chi(a = b) \).

4.1. Bijection algorithm for type \( D_n^{(1)} \).

Vacancy numbers.

\[
P_i^{(a)}(\nu^*) = Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L\delta_{a,1} \quad \text{for} \quad 1 \leq a < n-2
\]

(4.2)

\[
P_i^{(n-2)}(\nu^*) = Q_i(\nu^{(n-3)}) - 2Q_i(\nu^{(n-2)}) + Q_i(\nu^{(n-1)}) + Q_i(\nu^{(n)})
\]

\[
P_i^{(n-1)}(\nu^*) = Q_i(\nu^{(n-2)}) - 2Q_i(\nu^{(n-1)})
\]

\[
P_i^{(n)}(\nu^*) = Q_i(\nu^{(n-2)}) - 2Q_i(\nu^{(n)})
\]
Constraints.

\[
|\mu(a)| = L - \sum_{b=1}^{a} \lambda_b \quad \text{for } 1 \leq a \leq n - 2
\]

(4.3) \[
|\mu(n-1)| = \frac{1}{2}(L - \sum_{b=1}^{n-1} \lambda_b + \lambda_n)
\]

|\mu(n)| = \frac{1}{2}(L - \sum_{b=1}^{n} \lambda_b)

**Algorithm δ.** Set \(\ell(0) = 0\) and repeat the following process for \(a = 1, 2, \ldots, n - 2\) or until stopped. Find the minimal index \(i \geq \ell(a-1)\) such that \(J(a,i)\) is singular. If no such \(i\) exists, set \(b = a\) and stop. Otherwise set \(\ell(a) = i\) and continue with \(a + 1\).

If the process has not stopped at \(a = n - 2\) continue as follows. Find the minimal indices \(i, j \geq \ell(n-2)\) such that \(J(n-1,i)\) and \(J(n,j)\) are singular. If neither \(i\) nor \(j\) exist, set \(b = n - 1\) and stop. If \(i\) exists, but not \(j\), set \(\ell(n-1) = i, b = n\) and stop. If \(j\) exists, but not \(i\), set \(\ell(n) = j, b = \pi\) and stop. If both \(i\) and \(j\) exist, set \(\ell(n-1) = i, \ell(n) = j\) and continue with \(a = n - 2\).

Now continue for \(a = n - 2, n - 3, \ldots, 1\) or until stopped. Find the minimal index \(i \geq \tilde{\ell}(a+1)\) where \(\tilde{\ell}^{(n-1)} = \max(\ell^{(n-1)}, \ell^{(n)})\) such that \(J(n,i)\) is singular (if \(i = \ell(a)\) then there need to be two parts of size \(P_i^{(a)}(\nu^*)\) in \(J(n,i)\)). If no such \(i\) exists, set \(b = a + 1\) and stop. If the process did not stop, set \(b = 1\).

Set all yet undefined \(\ell(a)\) and \(\tilde{\ell}(a)\) to \(\infty\).

**New RC.**

(4.4) 
\[
m_i^{(a)}(\nu^*) = m_i^{(a)}(\nu^*) + \begin{cases} 
1 & \text{if } i = \ell(a) - 1 \\
-1 & \text{if } i = \ell(a) \\
1 & \text{if } i = \tilde{\ell}(a) - 1 \text{ and } 1 \leq a \leq n - 2 \\
-1 & \text{if } i = \tilde{\ell}(a) \text{ and } 1 \leq a \leq n - 2 \\
0 & \text{otherwise}
\end{cases}
\]

The partition \(\tilde{J}^{(a,i)}\) is obtained from \(J^{(a,i)}\) by removing a part of size \(P_i^{(a)}(\nu^*)\) for \(i = \ell(a)\) and \(i = \tilde{\ell}(a)\), adding a part of size \(P_i^{(a)}(\nu^*)\) for \(i = \ell(a) - 1\) and \(i = \tilde{\ell}(a) - 1\), and leaving it unchanged otherwise.

**Change in vacancy numbers.**

(4.5) 
\[
P_i^{(a)}(\nu^*) = P_i^{(a)}(\nu^*) - \chi(\ell(\ell(a-1)) \leq i) + 2\chi(\ell(\ell(a)) \leq i) - \chi(\ell(\ell(a+1)) \leq i) - \chi(\ell(\tilde{\ell}(a)) \leq i) + 2\chi(\ell(\tilde{\ell}(a)) \leq i) - \chi(\ell(\tilde{\ell}(a+1)) \leq i)
\]

for \(1 \leq a < n - 2\)

\[
P_i^{(n-2)}(\nu^*) = P_i^{(n-2)}(\nu^*) - \chi(\ell(\ell(n-3)) \leq i) + 2\chi(\ell(\ell(n-2)) \leq i) - \chi(\ell(\ell(n-1)) \leq i) - \chi(\ell(\tilde{\ell}(n-3)) \leq i) + 2\chi(\ell(\tilde{\ell}(n-2)) \leq i) - \chi(\ell(\tilde{\ell}(n-1)) \leq i)
\]

\[
P_i^{(n-1)}(\nu^*) = P_i^{(n-1)}(\nu^*) - \chi(\ell(\ell(n-2)) \leq i) - \chi(\ell(\tilde{\ell}(n-2)) \leq i) + 2\chi(\ell(\ell(n-1)) \leq i)
\]

\[
P_i^{(n)}(\nu^*) = P_i^{(n)}(\nu^*) - \chi(\ell(\ell(n-2)) \leq i) - \chi(\ell(\tilde{\ell}(n-2)) \leq i) + 2\chi(\ell(\ell(n)) \leq i)
\]

4.2. Bijection algorithm for type \(B_n^{(1)}\).
Vacancy numbers.
\[ P_i^{(a)}(\nu^\bullet) = Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L\delta_{a,1} \text{ for } i \in \mathbb{Z}_{\geq 0} \]
\[ 1 \leq a \leq n - 2 \]
\[ (4.6) \]
\[ P_i^{(n-1)}(\nu^\bullet) = Q_i(\nu^{(n-2)}) - 2Q_i(\nu^{(n-1)}) + 2Q_i(\nu^{(n)}) \text{ for } i \in \mathbb{Z}_{\geq 0} \]
\[ P_i^{(n)}(\nu^\bullet) = 2Q_i(\nu^{(n-1)}) - 4Q_i(\nu^{(n)}) \text{ for } i \in \frac{1}{2}\mathbb{Z}_{\geq 0} \]

Constraints.
\[ |\nu^{(a)}| = L - \sum_{b=1}^{a} \lambda_b \quad \text{for } 1 \leq a \leq n - 1 \]
\[ (4.7) \]
\[ |\nu^{(n)}| = \frac{1}{2}(L - \sum_{b=1}^{n} \lambda_b) \]

Algorithm δ. Call a partition quasi-singular if it is not singular and has a part of size \( P_i^{(a)}(\nu^\bullet) - 1 \).

Set \( \ell^{(0)} = 0 \) and repeat the following process for \( a = 1, 2, \ldots, n - 1 \) or until stopped. Find the minimal index \( i \geq \ell^{(a-1)} \) such that \( J^{(a,i)} \) is singular. If no such \( i \) exists, set \( b = a \) and stop. Otherwise set \( \ell^{(a)} = i \) and continue.

If the process has not yet stopped, continue as follows. For brevity let us denote by (S) and (Q) the following conditions:

(S) \( i \geq \ell^{(n-1)} \) and \( J^{(n,i)} \) is singular.
(Q) \( i = \ell^{(n-1)} - \frac{1}{2} \) and \( J^{(n,i)} \) is singular; or \( i \geq \ell^{(n-1)} \) and \( J^{(n,i)} \) is quasi-singular.

Find the minimal index \( i \geq \ell^{(n-1)} - \frac{1}{2} \) such that (S) or (Q) holds (note that (S) and (Q) are mutually excluding). If no such \( i \) exists, set \( b = n \) and stop. If (S) holds set \( \ell^{(n)} = i \) and find the minimal index \( j > i \) such that (S) holds. If no such \( j \) exists, set \( b = 0 \) and stop. Say that case (Q) holds. Otherwise set \( \ell^{(n)} = j \) and say that case (Q,S) holds.

If the process has not yet stopped continue in the following fashion for \( a = n - 1, n - 2, \ldots, 1 \) or until stopped. Find the minimal index \( i \geq \ell^{(a+1)} \) such that \( J^{(a,i)} \) is singular (if \( \ell^{(a)} = i \) then \( J^{(a,i)} \) actually needs to have two parts of size \( P_i^{(a)}(\nu^\bullet) \)). If no such \( i \) exists, set \( b = a + 1 \) and stop. Otherwise set \( \ell^{(a)} = i \) and continue. If the process did not stop for \( a \geq 1 \) set \( b = \overline{1} \).

Set all undefined \( \ell^{(a)} \) and \( \ell^{(a)} \) for \( 1 \leq a \leq n \) to \( \infty \).

New RC.
\[ m_i^{(a)}(\nu^\bullet) = m_i^{(a)}(\nu^\bullet) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - \nu_a \\ -1 & \text{if } i = \ell^{(a)} \\ 1 & \text{if } i = \overline{\ell^{(a)}} - \nu_a \\ -1 & \text{if } i = \overline{\ell^{(a)}} \\ 0 & \text{otherwise} \end{cases} \]
\[ (4.8) \]

Note that if two or more conditions hold, all of the changes should be performed.

For \( 1 \leq a < n \) the partition \( \overline{J}^{(a,i)} \) is obtained from \( J^{(a,i)} \) by removing a part of size \( P_i^{(a)}(\nu^\bullet) \) for \( i = \ell^{(a)} \) and \( i = \overline{\ell^{(a)}} \), adding a part of size \( P_i^{(a)}(\nu^\bullet) \) for \( i = \ell^{(a)} - 1 \) and \( i = \overline{\ell^{(a)}} - 1 \) and leaving it unchanged otherwise. If case (S) occurred \( \overline{J}^{(n,i)} \) is
If case (Q) holds remove the largest part in \( J^{(a)} \) for \( a = \tilde{a}^{(n)} - 1 \), and leaving it unchanged otherwise.

If case (Q) holds remove the largest part in \( J^{(a)} \) for \( a = \tilde{a}^{(n)} - 1 \), and leaving it unchanged otherwise.

If case (Q) holds remove the largest part in \( J^{(a)} \) for \( a = \tilde{a}^{(n)} - 1 \), and leaving it unchanged otherwise.

If case (Q) holds remove the largest part in \( J^{(a)} \) for \( a = \tilde{a}^{(n)} - 1 \), and leaving it unchanged otherwise.

### Change in vacancy numbers.

\[
P_i^{(a)}(\nu^*) = P_i^{(a)}(\nu^*) + \frac{\chi(\tilde{a}^{(a-1)} \leq i) - \chi(\tilde{a}^{(a+1)} \leq i)}{1} - 2\chi(\tilde{a}^{(a-1)} \leq i) - 2\chi(\tilde{a}^{(a+1)} \leq i)
\]

for \( 1 \leq a \leq n - 1 \) and

\[
P_i^{(n)}(\nu^*) = P_i^{(n)}(\nu^*) + \frac{\chi(\tilde{a}^{(n-1)} \leq i) - \chi(\tilde{a}^{(n+1)} \leq i)}{1} - 2\chi(\tilde{a}^{(n-1)} \leq i) - 2\chi(\tilde{a}^{(n+1)} \leq i)
\]

for \( a = n \).

### 4.3. Bijection algorithm for type \( C_n^{(1)} \).

#### Vacancy numbers.

\[
P_i^{(a)}(\nu^*) = Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L\delta_{a,1} \quad \text{for } i \in \mathbb{Z}_{\geq 0}
\]

(4.10)

\[
P_i^{(n)}(\nu^*) = Q_i(\nu^{(n-1)}) - Q_i(\nu^{(n)}) \quad \text{for } i \in 2\mathbb{Z}_{\geq 0}
\]

#### Constraints.

\[
|\nu^{(a)}| = L - \sum_{b=1}^{a} \lambda_b \quad \text{for } 1 \leq a \leq n
\]

### Algorithm \( \delta \). Set \( \tilde{a}^{(0)} = 0 \) and repeat the following process for \( a = 1, 2, \ldots, n \) or until stopped. Find the minimal index \( a \geq \tilde{a}^{(a-1)} \) such that \( J^{(a,i)} \) is singular. If no such \( i \) exists, set \( b = a \) and stop. Otherwise set \( \tilde{a}^{(a)} = i \) and continue.

If the process has not stopped continue as follows for \( a = n - 1, n - 2, \ldots, 1 \) or until stopped. Set \( \tilde{a}^{(n)} = \tilde{a}^{(n)} \) and reset \( \tilde{a}^{(i)} = \tilde{a}^{(n-1)} - 1 \). If \( \tilde{a}^{(a)} = \tilde{a}^{(a+1)} \) set \( \tilde{a}^{(a)} = \tilde{a}^{(a-1)} \) and reset \( \tilde{a}^{(i)} = \tilde{a}^{(a-1)} - 1 \). Say case (S) holds. Otherwise find the minimal index \( a \geq \tilde{a}^{(a-1)} \) such that \( J^{(a,i)} \) is singular. If no such \( i \) exists, set \( b = a + 1 \). Otherwise set \( \tilde{a}^{(a)} = i \) and continue. If the process does not stop for \( a \geq 1 \) set \( b = \tilde{T} \).

Set all undefined \( \tilde{a}^{(a)} \) and \( \tilde{a}^{(a)} \) for \( 1 \leq a \leq n \) to \( \infty \).
New RC.

\[ m_i^{(a)}(\nu^*) = m_i^{(a)}(\nu^*) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1 \\ -1 & \text{if } i = \ell^{(a)} \\ 1 & \text{if } i = \bar{\ell}^{(a)} - 1 \\ -1 & \text{if } i = \bar{\ell}^{(a)} \\ 0 & \text{otherwise} \end{cases} \tag{4.12} \]

If two or more conditions hold then all changes should be performed.

If \( a = n \) or case (S) holds for \( 1 \leq a < n \) the partition \( J^{(a,i)} \) is obtained from \( J^{(a,i)} \) by removing a part of size \( P_i^{(a)}(\nu^*) \) for \( i = \bar{\ell}^{(a)} \), adding a part of size \( P_i^{(a)}(\bar{\nu}^*) \) for \( i = \bar{\ell}^{(a)} - 2 \), and leaving it unchanged otherwise. Otherwise \( J^{(a,i)} \) is obtained from \( J^{(a,i)} \) by removing a part of size \( P_i^{(a)}(\bar{\nu}^*) \) for \( i = \ell^{(a)} \) and \( i = \bar{\ell}^{(a)} \), adding a part of size \( P_i^{(a)}(\bar{\nu}^*) \) for \( i = \ell^{(a)} - 1 \) and \( i = \bar{\ell}^{(a)} - 1 \), and leaving it unchanged otherwise.

**Change in vacancy numbers.**

\[ P_i^{(a)}(\bar{\nu}^*) = P_i^{(a)}(\nu^*) - \chi(\ell^{(a-1)} - \ell^{(a)}) + 2\chi(\ell^{(a)} - \ell^{(a-1)}) - \chi(\ell^{(a+1)} - \ell^{(a)}) \]

\[ -\chi(\ell^{(a-1)} - \ell^{(a)}) + 2\chi(\ell^{(a+1)} - \ell^{(a)}) - \chi(\ell^{(a+1)} - \ell^{(a)}) \]

for \( 1 \leq a \leq n - 1 \) and

\[ P_i^{(n)}(\bar{\nu}^*) = P_i^{(n)}(\nu^*) - \chi(\ell^{(n-1)} - \ell^{(n)}) - \chi(\ell^{(n+1)} - \ell^{(n)}) + \chi(\ell^{(n+1)} - \ell^{(n)}) \]

\[ +\chi(\ell^{(n)} - \ell^{(n)}) + \chi(\ell^{(n+1)} - \ell^{(n)}) \]

4.4. **Bijection algorithm for type \( A_{2n}^{(2)} \).** Recall here that \( \mathfrak{m} = C_n \) and \( \mathfrak{m} = B_n \).

**Vacancy numbers.** The vacancy numbers are the same as for type \( C_n^{(1)} \) with the only exception that now \( i \in \mathbb{Z}_{\geq 0} \) even for \( a = n \).

**Constraints.** The constraints are the same as for type \( C_n^{(1)} \).

**Algorithm \( \delta \).** Set \( \ell^{(0)} = 0 \) and repeat the following process for \( a = 1, 2, \ldots, n \) or until stopped. Find the minimal index \( i \geq \ell^{(a-1)} \) such that \( J^{(a,i)} \) is singular. If no such \( i \) exists, set \( b = a \) and stop. Otherwise set \( \ell^{(a)} = i \) and continue.

If \( \ell^{(n)} = 1 \) set \( b = \phi \) and stop. Otherwise say case (S) holds for \( a = n \) and continue.

If the process has not stopped, set \( \ell^{(n)} = \ell^{(n)} \) and reset \( \ell^{(n)} = \ell^{(n)} - 1 \). Continue as follows for \( a = n - 1, n - 2, \ldots, 1 \) or until stopped. If \( \ell^{(a)} = \ell^{(a+1)} \) set \( \ell^{(a)} = \ell^{(a)} \) and reset \( \ell^{(a)} = \ell^{(a)} - 1 \). Say case (S) holds. Otherwise find the minimal index \( i \geq \ell^{(a+1)} \) such that \( J^{(a,i)} \) is singular. If no such \( i \) exists, set \( b = a + 1 \). Otherwise set \( \ell^{(a)} = i \) and continue. If the process does not stop for \( a \geq 1 \) set \( b = \mathfrak{m} \).

Set all undefined \( \ell^{(a)} \) and \( \ell^{(a)} \) for \( 1 \leq a \leq n \) to \( \infty \).

**New RC.** The configuration changes in the same way as for type \( C_n^{(1)} \).

If case (S) holds for \( 1 \leq a \leq n \) the partition \( J^{(a,i)} \) is obtained from \( J^{(a,i)} \) by removing a part of size \( P_i^{(a)}(\nu^*) \) for \( i = \ell^{(a)} \), adding a part of size \( P_i^{(a)}(\bar{\nu}^*) \) for \( i = \ell^{(a)} - 2 \), and leaving it unchanged otherwise. Otherwise \( J^{(a,i)} \) is obtained from \( J^{(a,i)} \) by removing a part of size \( P_i^{(a)}(\nu^*) \) for \( i = \ell^{(a)} \) and \( i = \bar{\ell}^{(a)} \), adding a part of size \( P_i^{(a)}(\bar{\nu}^*) \) for \( i = \ell^{(a)} - 1 \) and \( i = \bar{\ell}^{(a)} - 1 \), and leaving it unchanged otherwise.
Change in vacancy numbers. The change in the vacancy numbers is the same as for type $C_{n}^{(1)}$ \[1.13\].

4.5. Bijection algorithm for type $A_{2n-1}^{(2)}$.

Vacancy numbers.

Algorithm (4.14)

\[
P_i^{(a)}(\nu^*) = Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L\delta_{a,1} \quad \text{for } 1 \leq a < n - 1
\]

\[
P_i^{(n-1)}(\nu^*) = Q_i(\nu^{(n-2)}) - 2Q_i(\nu^{(n-1)}) + 2Q_i(\nu^{(n)})
\]

\[
P_i^{(n)}(\nu^*) = Q_i(\nu^{(n-1)}) - 2Q_i(\nu^{(n)})
\]

Constraints.

\[
|\nu^{(a)}| = L - \sum_{b=1}^{a} \lambda_b \quad \text{for } 1 \leq a < n
\]

\[
|\nu^{(n)}| = \frac{1}{2}(L - \sum_{b=1}^{n} \lambda_b)
\]

Algorithm $\delta$. Set $\ell(0) = 0$ and repeat the following process for $a = 1, 2, \ldots, n$ or until stopped. Find the minimal index $i \geq \ell^{(a-1)}$ such that $J^{(a,i)}$ is singular. If no such $i$ exists, set $b = a$ and stop. Otherwise set $\ell^{(a)} = i$ and continue.

If the process has not stopped set $\bar{\ell}(n) = \ell(n)$ and continue as follows for $a = n - 1, n - 2, \ldots, 1$ or until stopped. Find the minimal index $i \geq \bar{\ell}(a+1)$ such that $J^{(a,i)}$ is singular (if $i = \ell(a)$ then there need to be two parts of size $P_i^{(a)}(\nu^*)$ in $J^{(a,i)}$). If no such $i$ exists, set $b = a + 1$ and stop. If the process did not stop, set $b = 1$.

Set all yet undefined $\ell^{(a)}$ and $\bar{\ell}(a)$ to $\infty$.

New RC.

(4.16) $m_i^{(a)}(\nu^*) = m_i^{(a)}(\nu^*) + \begin{cases} 1 & \text{if } i = \ell^{(a)} - 1 \\ -1 & \text{if } i = \ell^{(a)} \\ 1 & \text{if } i = \bar{\ell}^{(a)} - 1 \text{ and } 1 \leq a \leq n - 1 \\ -1 & \text{if } i = \bar{\ell}^{(a)} \text{ and } 1 \leq a \leq n - 1 \\ 0 & \text{otherwise.} \end{cases}$

The partition $\bar{J}^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^*)$ for $i = \ell^{(a)}$ when $1 \leq a \leq n$ and $i = \bar{\ell}^{(a)}$ when $1 \leq a < n$, adding a part of size $P_i^{(a)}(\nu^*)$ for $i = \ell^{(a)} - 1$ when $1 \leq a \leq n$ and $i = \bar{\ell}^{(a)} - 1$ when $1 \leq a < n$, and leaving it unchanged otherwise.

Change in vacancy numbers.

(4.17) $P_i^{(a)}(\nu^*) = P_i^{(a)}(\nu^*) - \chi(\ell^{(a-1)} \leq i) + 2\chi(\ell^{(a)} \leq i) - \chi(\ell^{(a+1)} \leq i) - \chi(\bar{\ell}^{(a-1)} \leq i) + 2\chi(\bar{\ell}^{(a)} \leq i) - \chi(\bar{\ell}^{(a+1)} \leq i)$ for $1 \leq a \leq n - 1$ and

$P_i^{(n)}(\nu^*) = P_i^{(n)}(\nu^*) - \chi(\ell^{(n-1)} \leq i) + 2\chi(\ell^{(n)} \leq i) - \chi(\ell^{(n-1)} \leq i)$. 

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4.6. Bijection algorithm for type $P_{n+1}^{(2)}$.

Vacancy numbers.

\begin{align}
P_i^{(a)}(\nu^*) &= Q_i(\nu^{(a-1)}) - 2Q_i(\nu^{(a)}) + Q_i(\nu^{(a+1)}) + L\delta_{a,1} \text{ for } 1 \leq a \leq n-1 \\
\bar{P}_i^{(a)}(\nu^*) &= 2Q_i(\nu^{(n-1)}) - 2Q_i(\nu^{(n)})
\end{align}

Constraints. The constraints are the same as for type $C_n^{(1)}$ [4.11].

Algorithm \( \delta \). Call a partition quasi-singular if it is not singular and has a part of size $P_i^{(a)}(\nu^*) - 1$.

Set \( \ell(0) = 0 \) and repeat the following process for \( a = 1, 2, \ldots, n-1 \) or until stopped. Find the minimal index \( i \geq \ell^{(a-1)} \) such that \( J^{(a,i)} \) is singular. If no such \( i \) exists, set \( b = a \) and stop. Otherwise set \( \ell(a) = i \) and continue.

If the process has not yet stopped, continue as follows. Consider the following conditions:

1. \( J^{(n,i)} \) is singular and \( i > 1 \);
2. \( J^{(n,i)} \) is singular and \( i = 1 \);
3. \( J^{(n,i)} \) is quasi-singular.

Find the minimal index \( i \geq \ell^{(n-1)} \) such that one of the mutually exclusive conditions (S), (P) or (Q) holds. If no such \( i \) exists, set \( b = a \) and stop. If (P) holds set \( \ell(n) = i, b = \phi \) and stop. If (S) holds set \( \ell(n) = i-1, \ell(n) = i \), say case (S) holds for \( a = n \) and continue. If (Q) holds set \( \ell(n) = i \). Find the minimal index \( j > i \) such that (S) holds. If no such \( j \) exists, set \( b = 0 \) and stop. Else set \( \ell(n) = j \), say case (Q,S) holds and continue.

If the process has not stopped continue in the following fashion for \( a = n-1, n-2, \ldots, 1 \) or until stopped. If \( \ell(a) = \ell(a+1) \) set \( \ell(a) = \ell(a) \) and reset \( \ell(a) = \ell(a) - 1 \). Say case (S) holds for \( a \). Otherwise find the minimal index \( i \geq \ell^{(a+1)} \) such that \( J^{(a,i)} \) is singular. If no such \( i \) exists, set \( b = a+1 \) and stop. Otherwise set \( \ell(a) = i \) and continue. If the process did not stop for \( a \geq 1 \) set \( b = T \).

Set all undefined \( \ell(a) \) and \( \ell(a) \) for \( 1 \leq a \leq n \) to \( \infty \).

New RC. The new configuration \( \nu^* \) is given by [4.12].

If case (S) holds for \( 1 \leq a \leq n \) the partition \( J^{(a,i)} \) is obtained from \( J^{(a,i)} \) by removing a part of size \( P_i^{(a)}(\nu^*) \) for \( i = \ell(a) \), adding a part of size \( P_i^{(a)}(\nu^*) \) for \( i = \ell(a) - 2 \), and leaving it unchanged otherwise. If (Q) or (Q,S) holds for \( a = n \), then \( J^{(n,i)} \) is obtained from \( J^{(n,i)} \) by removing a part of size \( P_i^{(n)}(\nu^*) - 1 \) (resp. \( P_i^{(n)}(\nu^*) \)) for \( i = \ell(n) \) (resp. \( i = \ell(n) \)), adding a part of size \( P_i^{(n)}(\nu^*) \) (resp. \( P_i^{(n)}(\nu^*) - 1 \)) for \( i = \ell(n) - 1 \) (resp. \( i = \ell(n) - 1 \)), and leaving it unchanged otherwise. Otherwise \( J^{(a,i)} \) is obtained from \( J^{(a,i)} \) by removing a part of size \( P_i^{(a)}(\nu^*) \) for \( i = \ell(a) \) and \( i = \ell(a) \), adding a part of size \( P_i^{(a)}(\nu^*) \) for \( i = \ell(a) - 1 \) and \( i = \ell(a) - 1 \), and leaving it unchanged otherwise.

Change in vacancy numbers.

\begin{align}
P_i^{(a)}(\nu^*) &= P_i^{(a)}(\nu^*) - \chi(\ell^{(a-1)} \leq i) + 2\chi(\ell(a) \leq i) - \chi(\ell^{(a+1)} \leq i) \\
&\quad - \chi(\ell^{(a-1)} \leq i) + 2\chi(\ell(a) \leq i) - \chi(\ell^{(a+1)} \leq i)
\end{align}
for $1 \leq a \leq n - 1$ and
\[
P_i^{(n)}(\nu^*) = P_i^{(n)}(\nu^*) - 2\chi(\ell(n-1) \leq i) + 2\chi(\ell(n) \leq i) - 2\chi(\bar{\ell}(n-1) \leq i) + 2\chi(\bar{\ell}(n) \leq i).
\]

4.7. Bijection algorithm for type $A_{2n}^{(2)}$.

**Vacancy numbers.** The vacancy numbers are given by the same formula as for type $C_n^{(1)}$ \ref{4.10}, with the only exception that in this case $i \in \mathbb{Z}_{\geq 0}$ for all $a \in I^*$.

**Algorithm $\delta$.** If $a = n$ and $i$ is odd, then $J^{(n,i)}$ is never singular. For $i$ odd, call $J^{(n,i)}$ quasi-singular if it has a part of size $P_i^{(n)}(\nu^*) - 1/2$.

Set $\ell(0) = 0$ and repeat the following process for $a = 1, 2, \ldots, n - 1$ or until stopped. Find the minimal index $i \geq \ell(a-1)$ such that $J^{(a,i)}$ is singular. If no such $i$ exists, set $b = a$ and stop. Otherwise set $\ell(a) = i$ and continue.

If the process has not yet stopped, continue as follows. Consider the conditions

(S) $i$ is even and $J^{(n,i)}$ is singular;
(Q) $i$ is odd and $J^{(n,i)}$ is quasi-singular.

Find the minimal index $i \geq \ell(n-1)$ such that one of the mutually exclusive conditions (S) or (Q) holds. If no such $i$ exists, set $b = n$ and stop. If (S) holds set $\ell(n) = i - 1, \bar{\ell}(n) = i$, say case (S) holds for $a = n$ and continue. If (Q) holds set $\ell(n) = i$. Find the minimal $j > i$ such that (S) holds for $j$. If no such $j$ exists, set $b = 0$ and stop. Else set $\ell(n) = j$, say case (Q,S) holds and continue.

If the process has not stopped continue in the following fashion for $a = n - 1, n - 2, \ldots, 1$ or until stopped. If $\ell(a) = \ell(a+1)$ set $\bar{\ell}(a) = \ell(a)$ and reset $\ell(a) = \bar{\ell}(a) - 1$. Say case (S) holds for $a$. Otherwise find the minimal index $i \geq \bar{\ell}(a+1)$ such that $J^{(a,i)}$ is singular. If no such $i$ exists, set $b = a + 1$ and stop. Otherwise set $\ell(a) = i$ and continue. If the process did not stop for $a \geq 1$ set $b = \mathbb{T}$.

Set all undefined $\ell(a)$ and $\bar{\ell}(a)$ for $1 \leq a \leq n$ to $\infty$.

**New RC.** The new configuration $\nu^*$ is given by \ref{4.13}.

If case (S) holds for $1 \leq a \leq n$ the partition $J^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^*)$ for $i = \ell(a)$, adding a part of size $P_i^{(a)}(\nu^*)$ for $i = \ell(a) - 2$, and leaving it unchanged otherwise.

If (Q) or (Q,S) holds for $a = n$, then $J^{(n,i)}$ is obtained from $J^{(n,i)}$ by removing a part of size $P_i^{(n)}(\nu^*) - 1/2$ for $i = \ell(n)$ (and a part of size $P_i^{(n)}(\nu^*)$ for $i = \bar{\ell}(n) < \infty$), adding a part of size $P_i^{(n)}(\nu^*)$ for $i = \ell(n) - 1$ (and a part of size $P_i^{(n)}(\nu^*) - 1/2$ for $i = \bar{\ell}(n) - 1 < \infty$), and leaving it unchanged otherwise.

Otherwise $J^{(a,i)}$ is obtained from $J^{(a,i)}$ by removing a part of size $P_i^{(a)}(\nu^*)$ for $i = \ell(a)$ and $i = \bar{\ell}(a)$, adding a part of size $P_i^{(a)}(\nu^*)$ for $i = \ell(a) - 1$ and $i = \bar{\ell}(a) - 1$, and leaving it unchanged otherwise.

**Change in vacancy numbers.** The vacancy numbers $P_i^{(a)}(\nu^*)$ change as in \ref{4.13}.
5. Proof of Theorem 3.2

In the following subsections Theorem 3.2 is proved case-by-case for the various root systems. The following notation is used. Let \((\nu^*, J^*) \in \mathcal{P}(\lambda, \mu)\), \(b = \text{rk}(\nu^*, J^*) \in B\), \(\rho = \lambda - \text{wt}(b)\), and \((\tilde{\nu}^*, J^*) = \delta(\nu^*, J^*)\). There are three things that must be verified:

(I) \(\rho\) is dominant and \(b\) can be appended to any path in \(\mathcal{P}(\rho, \tilde{\mu})\) to give an element of \(\mathcal{P}(\lambda, \mu)\).

(II) \((\tilde{\nu}^*, J^*) \in \mathcal{P}(\rho, \tilde{\mu})\) where \(B\tilde{\mu} = B^{\otimes (L-1)}\).

(III) The conditions of Lemma 5.1 are satisfied.

Parts (I) and (II) show that \(\delta\) is well-defined. The proof that \(\delta\) has an inverse, is omitted as it is very similar to the proof of well-definedness. Part (III) suffices to prove that \(\tilde{\Phi}\) preserves statistics.

For \((\nu^*, J^*) \in \mathcal{P}(\lambda, \mu)\), define \(\Delta(\text{cc}(\nu^*, J^*)) = \text{cc}(\nu^*, J^*) - \text{cc}(\delta'(\nu^*, J^*))\) and \(\Delta^2(\text{cc}(\nu^*, J^*)) = \Delta(\text{cc}(\nu^*, J^*)) - \Delta(\text{cc}(\delta'(\nu^*, J^*)))\) where \(\delta' = \text{comp} \circ \delta \circ \text{comp}\).

Lemma 5.1. To prove that (3.13) holds, it suffices to show that it holds for \(L = 1\), and that for \(L \geq 2\) with \(\tilde{\Phi}(\nu^*, J^*) = b_1 \otimes \cdots \otimes b_1\), we have

\[
\Delta(\text{cc}(\nu^*, J^*)) = \frac{t^V}{a_0} \alpha_1^{(1)}(\nu) - \chi(b_L = \phi),
\]

and

\[
\underline{H}(b_L \otimes b_{L-1}) = \frac{t^V}{a_0} (\alpha_1^{(1)}(\nu) - \delta_1^{(1)}(\nu) - \chi(b_L = \phi) + \chi(b_{L-1} = \phi)
\]

where \(\alpha_1^{(1)}(\nu)\) and \(\delta_1^{(1)}(\nu)\) are the lengths of the first columns in \((\nu)\) and \((\nu^*)\) respectively, and \(\delta(\nu^*, J^*) = (\tilde{\nu}^*, J^*)\).

Proof. If \(L = 0\), \(\mathcal{P}(\lambda, \mu)\) and \(\mathcal{P}(\lambda, \mu)\) are both empty unless \(\lambda = 0\), in which case \(\mathcal{P}(\lambda, \mu)\) (resp. \(\mathcal{P}(\lambda, \mu)\)) is the singleton set containing the empty rigged configuration (resp. the empty path). Both of these objects have statistic zero. The case \(L = 1\) is given by hypothesis. For \(L \geq 2\), by the definition (2.18) and (2.20) of \(\mathcal{D}\),

\[
\mathcal{D}(b_L \otimes \cdots \otimes b_1) - \mathcal{D}(b_{L-1} \otimes \cdots \otimes b_1) = \underline{H}(b_1 \otimes b_L) + \sum_{j=1}^{L-1} \underline{H}(b_{j+1} \otimes b_j).
\]

Therefore by induction on \(L\) it suffices to prove that \(\Delta(\text{cc}(\nu^*, J^*))\) is given by the right hand side of (2.13). By induction and again “taking the difference” it suffices to prove that

\[
\Delta^2(\text{cc}(\nu^*, J^*)) = \underline{H}(b_L \otimes b_{L-1}).
\]

But this follows from (2.1) and (5.2). \(\square\)

We also need several preliminary lemmas on the convexity and nonnegativity of the vacancy numbers \(P^{(a)}_i(\nu^*)\).
Lemma 5.2. For large \( i \), we have

\[
P_{i}^{(a)}(\nu^{\bullet}) = \lambda_{a} - \lambda_{a+1} \quad \text{for} \quad 1 \leq a < n
\]

\[
P_{i}^{(n)}(\nu^{\bullet}) = \begin{cases} 
2\lambda_{n} & \text{for } B_{n}^{(1)} \text{, } D_{n+1}^{(2)} , \\
\lambda_{n} & \text{for } C_{n}^{(1)} , A_{2n}^{(2)} , A_{2n+1}^{(2)} , A_{2n-1}^{(2)} , \\
\lambda_{n-1} + \lambda_{n} & \text{for } D_{n}^{(1)} .
\end{cases}
\]

**Proof.** This follows from the formulas for the vacancy numbers (4.2), (4.6), (4.10), (5.6), (5.7), Q, for \( 1 \leq a < n \), the constraints (4.3), (4.7), (4.14), (4.18), and the fact that for large \( i \), \( Q_{i}(\nu^{(a)}) = |\nu^{(a)}| \).

Direct calculations show that

(5.4) **Type** \( D_{n}^{(1)} \)

\[
- P_{i-1}^{(a)}(\nu^{\bullet}) + 2P_{i}^{(a)}(\nu^{\bullet}) - P_{i+1}^{(a)}(\nu^{\bullet}) = \begin{cases} 
m_{i}^{(a-1)}(\nu^{\bullet}) - 2m_{i}^{(a)}(\nu^{\bullet}) + m_{i}^{(a+1)}(\nu^{\bullet}) + L\delta_{a,1} \delta_{i,1} & \text{for } 1 \leq a \leq n - 3 \\
m_{i}^{(n-3)}(\nu^{\bullet}) - 2m_{i}^{(n-2)}(\nu^{\bullet}) + m_{i}^{(n-1)}(\nu^{\bullet}) + m_{i}^{(n)}(\nu^{\bullet}) & \text{for } a = n - 2 \\
m_{i}^{(n-2)}(\nu^{\bullet}) - 2m_{i}^{(n)}(\nu^{\bullet}) & \text{for } a = n - 1, n.
\end{cases}
\]

(5.5) **Type** \( B_{n}^{(1)} \)

\[
- P_{i}^{(a)}_{i-v_{a}}(\nu^{\bullet}) + 2P_{i}^{(a)}(\nu^{\bullet}) - P_{i+v_{a}}^{(a)}(\nu^{\bullet}) = \begin{cases} 
m_{i}^{(a-1)}(\nu^{\bullet}) - 2m_{i}^{(a)}(\nu^{\bullet}) + m_{i}^{(a+1)}(\nu^{\bullet}) + L\delta_{a,1} \delta_{i,1} & \text{for } 1 \leq a \leq n - 2 \\
m_{i}^{(n-2)}(\nu^{\bullet}) - 2m_{i}^{(n-1)}(\nu^{\bullet}) & \text{for } a = n - 1 \\
+2(2m_{i}^{(a)}(\nu^{\bullet}) + m_{i+1}^{(n)}(\nu^{\bullet}) + m_{i-1}^{(n)}(\nu^{\bullet})) & \text{for } a = n.
\end{cases}
\]

(5.6) **Type** \( C_{n}^{(1)} \)

\[
- P_{i}^{(a)}_{i-v_{a}}(\nu^{\bullet}) + 2P_{i}^{(a)}(\nu^{\bullet}) - P_{i+v_{a}}^{(a)}(\nu^{\bullet}) = \begin{cases} 
m_{i}^{(a-1)}(\nu^{\bullet}) - 2m_{i}^{(a)}(\nu^{\bullet}) + m_{i}^{(a+1)}(\nu^{\bullet}) + L\delta_{a,1} \delta_{i,1} & \text{for } 1 \leq a \leq n - 1 \\
m_{i}^{(n-1)}(\nu^{\bullet}) + 2m_{i}^{(n-1)}(\nu^{\bullet}) + m_{i+1}^{(n-1)}(\nu^{\bullet}) - 2m_{i}^{(n)}(\nu^{\bullet}) & \text{for } a = n.
\end{cases}
\]

(5.7) **Types** \( A_{2n}^{(2)} \) and \( A_{2n+1}^{(2)} \)

\[
- P_{i-1}^{(a)}(\nu^{\bullet}) + 2P_{i}^{(a)}(\nu^{\bullet}) - P_{i+1}^{(a)}(\nu^{\bullet}) = \begin{cases} 
m_{i}^{(a-1)}(\nu^{\bullet}) - 2m_{i}^{(a)}(\nu^{\bullet}) + m_{i}^{(a+1)}(\nu^{\bullet}) + L\delta_{a,1} \delta_{i,1} & \text{for } 1 \leq a \leq n - 1 \\
m_{i}^{(n-1)}(\nu^{\bullet}) - m_{i}^{(n)}(\nu^{\bullet}) & \text{for } a = n.
\end{cases}
\]

(5.8) **Type** \( A_{2n-1}^{(2)} \)

\[
- P_{i-1}^{(a)}(\nu^{\bullet}) + 2P_{i}^{(a)}(\nu^{\bullet}) - P_{i+1}^{(a)}(\nu^{\bullet}) = \begin{cases} 
m_{i}^{(a-1)}(\nu^{\bullet}) - 2m_{i}^{(a)}(\nu^{\bullet}) + m_{i}^{(a+1)}(\nu^{\bullet}) + L\delta_{a,1} \delta_{i,1} & \text{for } 1 \leq a < n - 1 \\
m_{i}^{(n-2)}(\nu^{\bullet}) - 2m_{i}^{(n-1)}(\nu^{\bullet}) + 2m_{i}^{(n)}(\nu^{\bullet}) & \text{for } a = n - 1 \\
m_{i}^{(n-1)}(\nu^{\bullet}) - 2m_{i}^{(n)}(\nu^{\bullet}) & \text{for } a = n.
\end{cases}
\]
(5.9) **Type** $D_{n+1}^{(2)}$

\[ -P_{i-1}^{(a)}(v^*) + 2P_i^{(a)}(v^*) - P_{i+1}^{(a)}(v^*) \]

\[ = \left\{ \begin{array}{ll}
m_i^{(a-1)}(v^*) - 2m_i^{(a)}(v^*) + m_i^{(a+1)}(v^*) + L\delta_{a1}\delta_i,1 & \text{for } 1 \leq a \leq n-1 \\
2m_i^{(n-1)}(v^*) - 2m_i^{(n)}(v^*) & \text{for } a = n.
\end{array} \right. \]

In particular these equations imply the convexity condition

(5.10) \[ P_i^{(a)}(v^*) \geq \frac{1}{2}(P_{i-1}^{(a)}(v^*) + P_{i+1}^{(a)}(v^*)) \quad \text{if } m_i^{(a)}(v^*) = 0. \]

**Lemma 5.3.** Let $v^*$ be a configuration in $C(\lambda, \mu)$. The following are equivalent:

1. $P_i^{(a)}(v^*) \geq 0$ for all $i \in v_k\mathbb{Z}_{>0}$, $a \in I^*$;
2. $P_i^{(a)}(v^*) \geq 0$ for all $i \in v_k\mathbb{Z}_{>0}$, $a \in I^*$ such that $m_i^{(a)}(v^*) > 0$.

**Proof.** This follows immediately from Lemma 5.2 and the convexity condition (5.10). (See also [1], Lemma 10).

5.1. **Proof for type** $D_n^{(1)}$.

**Proof of (I) for** $D_n^{(1)}$. Here it suffices to show that $\rho$ satisfies (2.13). Suppose not. If $b = k$ with $1 \leq k \leq n$ then

(a) $\lambda_k = \lambda_{k+1}$ if $1 \leq k \leq n - 2$
(b) $\lambda_{n-1} = |\lambda_n|$ if $k = n - 1$
(c) $\lambda_{n-1} = -\lambda_n$ if $k = n$.

In case (a) we have $P_i^{(k)}(v^*) = 0$ for large $i$ by Lemma 5.2. Let $\ell$ be the largest part in $\nu^{(k)}$. By convexity this implies $P_i^{(k)}(v^*) = 0$ for all $i \geq \ell$. Equation (6.4) in turn yields $m_i^{(k-1)}(v^*) = 0$ for all $i > \ell$ so that $1 \leq \ell^{(k-1)} \leq \ell$. But this is a contradiction since there is a singular string of length $\ell$ in $(\nu^*, J^*)^{(k)}$ since $P_\ell^{(k)}(v^*) = 0$ and $m_\ell^{(k)}(v^*) > 0$ so that we would have $rk(\nu^*, J^*) > k$. In case (b) let us first assume that $\lambda_{n-1} = \lambda_n$. Then for large $i$, $P_i^{(n-1)}(v^*) = 0$ and by convexity $P_i^{(n-1)}(v^*) = 0$ for $i \geq \ell$ where $\ell$ is the largest part in $\nu^{(n-1)}$. By (5.4) we have $m_i^{(n-2)}(v^*) = 0$ for $i > \ell$. Hence $1 \leq \ell^{(n-2)} \leq \ell$ which yields a contradiction since there is a singular string of length $\ell$ in $(\nu^*, J^*)^{(n-1)}$ so that $rk(\nu^*, J^*) \neq n-1$. If $\lambda_{n-1} = -\lambda_n$ the same argument goes through with $n - 1$ replaced by $n$. The case (c) is analogous to the second part of case (b).

Now suppose $b = \ell$ for some $1 \leq k \leq n$. We show again that $\rho$ not dominant will yield a contradiction. If $\rho$ is not dominant one of the following has to be true:

(d) $\lambda_k = \lambda_{k-1}$ if $2 \leq k \leq n - 1$
(e) $\lambda_n = \lambda_{n-1}$ if $k = n$.

Case (e) is analogous to case (b). In case (d) some caution is in order. By lemma 5.2 and convexity (5.10) we have $P_i^{(k-1)}(v^*) = 0$ for $i \geq \ell$ where $\ell$ is the largest part in $\nu^{(k-1)}$. By (5.4) it follows that $m_i^{(k)}(v^*) = 0$ for $i > \ell$. Hence $\ell^{(k-1)} \leq \ell$. Since $P_\ell^{(k-1)}(v^*) = 0$ and $m_\ell^{(k-1)}(v^*) > 0$ there is a singular string of length $\ell$ in $(\nu^*, J^*)^{(k-1)}$. Hence $\ell^{(k-1)} \leq \ell$ unless $\ell^{(k-1)} = \ell$ and $m_\ell^{(k-1)}(v^*) = 1$. We
will show that the latter case cannot occur. Equation (5.4) with \( i = \ell \) implies that \( P_{i-1}^{(k-1)} = 0 \) and \( m_{i}^{(k-2)}(\nu^*) = 0 \) since by assumption \( \ell^{(k-1)} = \ell^{(k)} = \bar{\ell}(k) = \ell \) and hence \( m_{i}^{(k)}(\nu^*) \geq 2 \) (or \( m_{i}^{(n-1)}(\nu^*) \geq 1 \) and \( m_{i}^{(n)}(\nu^*) \geq 1 \) for \( k = n-1 \)). However this implies that \( m_{i-1}^{(n-2)}(\nu^*) = 0 \) since otherwise \( \ell^{(k-1)} \leq \ell - 1 \) and not \( \ell \) since there is a singular string of length \( \ell - 1 \) in \( (\nu^*, J^*)^{(k-1)} \). Now by induction on \( i = \ell - 1, \ell - 2, \ldots, 1 \) it follows from (5.4) at \( a = k-1 \) that \( P_{i}^{(k-1)}(\nu^*) = m_{i}^{(k-2)}(\nu^*) = m_{i}^{(k-1)}(\nu^*) = 0 \). However, this means in particular that \( m_{i}^{(k-2)}(\nu^*) = 0 \) for all \( 1 \leq i \leq \ell \) so that \( \ell^{(k-2)} > \ell \) which contradicts \( \ell^{(k-1)} = \ell \). \( \square \)

Proof of (II) for \( D_{1}^{(1)} \). Denote by \( J_{\text{max}}^{(a,i)}(\nu^*, J^*) \) the biggest part in \( J^{(a,i)}(\nu^*, J^*) \). To prove admissibility of \( (\bar{\nu}^*, \bar{J}^*) \) we need to show for all \( i \geq 1, 1 \leq a \leq n \) that

\[
0 \leq J_{\text{max}}^{(a,i)}(\bar{\nu}^*, \bar{J}^*) \leq P_{i}^{(a)}(\bar{\nu}^*) .
\]

Fix \( a \geq 1 \). Only one string of size \( \ell^{(a)} \) and one string of size \( \bar{\ell}^{(a)} \) change in the transformation \( (\nu^*, J^*)^{(a)} \rightarrow (\bar{\nu}^*, \bar{J}^*)^{(a)} \). Hence

\[
J_{\text{max}}^{(a,i)}(\bar{\nu}^*, \bar{J}^*) = P_{i}^{(a)}(\bar{\nu}^*) \quad \text{for } i = \ell^{(a)} - 1 \quad \text{and} \quad i = \bar{\ell}^{(a)} - 1
\]

\[0 \leq J_{\text{max}}^{(a,i)}(\bar{\nu}^*, \bar{J}^*) \leq J_{\text{max}}^{(a,i)}(\nu^*, J^*) \quad \text{else.}
\]

Hence by (4.3) the inequality (5.11) can only be violated when \( \ell^{(a-1)} \leq i < \ell^{(a)} \) or \( \bar{\ell}^{(a+1)} \leq i < \bar{\ell}^{(a)} \) where \( \bar{\ell}^{(n-1)} = \max(\ell^{(n-1)}, \bar{\ell}^{(n)}) \). By the construction of \( \ell^{(a)} \) and \( \bar{\ell}^{(a)} \) there are no singular strings of length \( i \) in \( (\nu^*, J^*)^{(a)} \) for \( \ell^{(a-1)} \leq i < \ell^{(a)} \) or \( \bar{\ell}^{(a+1)} \leq i < \bar{\ell}^{(a)} \). This means that \( J_{\text{max}}^{(a,i)}(\nu^*, J^*) \leq P_{i}^{(a)}(\nu^*) - 1 \) if \( i \) occurs as a part in \( \nu^{(a)} \), that is \( m_{i}^{(a)}(\nu^*) = 0 \). Hence (5.11) is fulfilled for these \( i \). It remains to prove that \( P_{i}^{(a)}(\bar{\nu}^*) \geq 0 \) for all \( i \) such that \( m_{i}^{(a)}(\nu^*) = 0 \) and \( \ell^{(a-1)} \leq i < \ell^{(a)} \) or \( \bar{\ell}^{(a+1)} \leq i < \bar{\ell}^{(a)} \). Hence by lemma (5.3) it suffices to prove (5.11) for all \( a \) and \( i \) such that \( m_{i}^{(a)}(\nu^*) = 0 \). Therefore the only remaining case for which (5.11) might be violated occurs when

For \( 1 \leq a \leq n - 2 \):

\[
m_{\ell-1}^{(a)}(\nu^*) = 0, \quad P_{\ell-1}^{(a)}(\nu^*) = 0, \quad \ell^{(a-1)} < \ell \quad (\text{resp. } \bar{\ell}^{(a+1)} < \ell)
\]

and \( \ell \) finite where \( \ell = \ell^{(a)} \) (resp. \( \ell = \bar{\ell}^{(a)} \)).

For \( a = n - 1, n \):

\[
m_{\ell-1}^{(a)}(\nu^*) = 0, \quad P_{\ell-1}^{(a)}(\nu^*) = 0, \quad \ell^{(n-2)} < \ell
\]

and \( \ell \) finite where \( \ell = \ell^{(a)} \).

We show that these conditions cannot be met simultaneously. Let \( p < \ell \) be maximal such that \( m_{p}^{(a)}(\nu^*) = 0 \); if no such \( p \) exists set \( p = 0 \). By (5.10) \( P_{p-1}^{(a)}(\nu^*) = 0 \) is only possible if \( P_{i}^{(a)}(\nu^*) = 0 \) for all \( p \leq i \leq \ell \). By (5.4) we find that \( m_{i}^{(a-1)}(\nu^*) = 0 \) (resp. \( m_{i}^{(a+1)}(\nu^*) = 0 \)) for \( p < i < \ell \). Since \( \ell^{(a-1)} < \ell \) (resp. \( \bar{\ell}^{(a+1)} < \ell \)) this implies that \( \ell^{(a-1)} \leq p \) (resp. \( \bar{\ell}^{(a+1)} \leq p \)). If \( p = 0 \) this contradicts the condition \( \ell^{(a-1)} \geq 1 \) (resp. \( \bar{\ell}^{(a+1)} \geq 1 \)). Hence assume that \( p > 0 \). Since \( P_{p}^{(a)}(\nu^*) = 0 \) and \( m_{p}^{(a)}(\nu^*) > 0 \) there is a singular string of length \( p \) in \( (\nu^*, J^*)^{(a)} \) and therefore
\[ \ell(a) = p \text{ (resp. } \bar{\ell}(a) = p) \]. However, this contradicts \( p < \ell \). This concludes the proof that \((\nu^*, J^*)\) is well-defined. 

**Proof of (III) for** \( D_n \). Here \( b^2 = 1 \), \( \overline{\mathcal{P}}(b \otimes b') = 0 \) if \( b \leq b' \), \( \overline{\mathcal{P}}(b \otimes b') = 1 \) if \( b \otimes b' = n \otimes \overline{n}, \overline{n} \otimes n \) or \( b > b' \) where \( b \neq \overline{T}, b' \neq 1 \), and \( H(T \otimes 1) = 2 \).

If \( L = 1 \) then the path is 1, the rigged configuration is empty, and both sides of (3.13) are zero.

Here (3.1) and (3.2) are given by

\begin{align*}
\Delta(cc(\nu^*, J^*)) &= \alpha_1^{(1)} \\
\overline{\mathcal{P}}(b_L \otimes b_{L-1}) &= \chi(\ell(1) = 1) + \chi(\bar{\ell}(1) = 1)
\end{align*}

where \( \ell(i) \) and \( \bar{\ell}(i) \) are determined by the algorithm \( \delta \).

Let \( \tilde{\ell}(a) \) and \( \tilde{\bar{\ell}}(a) \) be the length of the selected strings defined by the algorithm \( \delta \) on \((\nu^*, J^*) = \delta(\nu^*, J^*)\). To check (5.13) note that if \( \ell(1) = 1 \) it follows from (4.5) that \( \ell(a) > \ell(a+1) \) for \( 1 \leq a \leq n - 2 \). Hence if \( b_L \leq n - 1 \) then \( b_{L-1} < b_L \) and both sides of (5.13) yield 1. If \( b_L = n \) then \( b_{L-1} \leq n - 1 \) or \( b_{L-1} = \overline{n} \) and both sides of (5.13) are 1. Similarly, if \( b_L = \overline{n} \) then \( b_{L-1} \leq n \) and both sides if (5.13) are 1.

Finally, if \( b_L \geq n - 1 \) then \( \tilde{\ell}(a) \geq \tilde{\bar{\ell}}(a-1) \) and \( b_{L-1} < b_L \). If \( b_L < \overline{T} \) then both sides of (5.13) are 1. If \( b_L = \overline{T} \) and \( \ell(1) = 1 \) then \( \tilde{\ell}(1) = \infty \) and hence \( b_{L-1} = 1 \). In this case both sides of (5.13) are 2. If \( b_L = \overline{T} \) and \( \ell(1) > 1 \) then there is a singular string in \((\nu^*, J^*)^{(1)} \) so that \( b_{L-1} > 1 \). In this case both sides of (5.13) are 1. If \( \ell(1) > 1 \) then \( \tilde{\ell}(a) < \ell(a) \) for \( 1 \leq a \leq n - 2 \) and the cases can be checked in a similar fashion as before.

To prove (5.12), by (3.4) and (4.4) we have

\[
cc(\nu^*) = \frac{1}{2} \sum_{i,j \geq 1} \sum_{a,b=1}^n \min(i, j)(\alpha_a | \alpha_b) \\
\times \left( m_i^{(a)} - \delta_i, \tilde{\ell}(a) + \delta_i, \bar{\ell}(a) - 1 - \chi(a \leq n - 2)(\delta_i, \tilde{\ell}(a) - \delta_i, \bar{\ell}(a) - 1) \right) \\
\times \left( m_j^{(b)} - \delta_j, \tilde{\ell}(b) + \delta_j, \bar{\ell}(b) - 1 - \chi(b \leq n - 2)(\delta_j, \tilde{\ell}(b) - \delta_j, \bar{\ell}(b) - 1) \right).
\]

Applying the data for \( D_n \) and using (4.3), a tedious but straightforward calculation yields

\[
\Delta cc(\nu^*) = \sum_{a=1}^n \sum_{i \geq 1} \left( P_i^{(a)}(\nu^*) - P_i^{(a)}(\bar{\nu}^*) \right) \\
\times \left( m_i^{(a)} - \delta_i, \ell(a) - \chi(a \leq n - 2)\delta_i, \ell(a) \right) + \sum_{i \geq 1} m_i^{(1)}.
\]

For \( \Delta J^* \) we obtain from the algorithm \( \delta' \)

\[
\Delta J^* = \sum_{a=1}^n \sum_{i \geq 1} \left( P_i^{(a)}(\nu^*) - P_i^{(a)}(\bar{\nu}^*) \right) \left( m_i^{(a)} - \delta_i, \ell(a) - \chi(a \leq n - 2)\delta_i, \ell(a) \right).
\]

Hence altogether, using \( \sum_{i \geq 1} m_i^{(1)} = \alpha_1^{(1)} \), we obtain (5.12). 

\[ \square \]
5.2. Proof for type $B_n^{(1)}$.

Proof of (I) for $B_n^{(1)}$. Let us assume that either $\rho$ is not dominant, or that $b = 0$ (so that $\rho = \lambda$) and $\lambda_n = 0$. For $b = k$ with $1 \leq k < n$ the proof that this cannot happen is the same as for type $D_n^{(1)}$. Now assume that $b = n$ and $\lambda_n = 0$. Then $P_i^{(n)}(\nu^*) = 0$ for $i \geq \ell$ where $\ell$ is the largest part in $\nu^{(n)}$ by Lemma 5.2 and (5.10). By (5.3) with $a = n$ we find that $m_i^{(n-1)}(\nu^*) = 0$ for $i > \ell$, so that $\ell^{(n-1)} \leq \ell$. But there is a singular string of length $\ell$ in $(\nu^*, J^{(n)})$ which contradicts $\ell^{(n)} = \infty$. Next assume that $b = 0$ and $\lambda_n = 0$. By the same arguments as in the previous case $\ell^{(n-1)} \leq \ell$. But there is a singular string of length $\ell$ in $(\nu^*, J^{(n)})$ since $m_i^{(n)}(\nu^*) > 0$ and $P_i^{(n)}(\nu^*) = 0$. Since (Q) must hold for $b = 0$, there must be a singular string at $\ell^{(n-1)} - \frac{1}{n}$ or a quasisingular string at $\ell^{(n-1)} - 1 < \ell$. But then (S) holds for $\ell$ which contradicts $b = 0$. The case $b = k$ with $1 \leq k \leq n$ is the same as for type $D_n^{(1)}$.

Proof of (II) for $B_n^{(1)}$. Denote by $J_{\text{max}}^{(a,i)}(\nu^*, J^*)$ the biggest part in $J^{(a,i)}$. To prove admissibility of $(\nu^*, J^*)$ we need to show for all $i \geq 1, 1 \leq a \leq n$ that

(5.14) \[ 0 \leq J_{\text{max}}^{(a,i)}(\nu^*, J^*) \leq P_i^{(n)}(\nu^*). \]

Up to small alterations, the proof of (5.14) for $1 \leq a < n$ is the same as for type $D_n^{(1)}$. Let us assume that $a = n$. Only one string of size $\ell^{(n)}$ and one string of size $\bar{\ell}^{(n)}$ change in the transformation $(\nu^*, J^*)^{(1)} \rightarrow (\tilde{\nu}^*, \tilde{J}^*)^{(1)}$. Hence for the different cases:

(S) \[ J_{\text{max}}^{(n,i)}(\nu^*, \tilde{J}^*) = P_i^{(n)}(\nu^*) \text{ for } i = \bar{\ell}^{(n)} - 1 \]

\[ 0 \leq J_{\text{max}}^{(n,i)}(\nu^*, \tilde{J}^*) \leq J_{\text{max}}^{(n,i)}(\nu^*, J^*) \text{ else} \]

(Q) \[ J_{\text{max}}^{(n,i)}(\nu^*, \tilde{J}^*) = P_i^{(n)}(\nu^*) \text{ for } i = \ell^{(n)} - 1/2 \]

\[ 0 \leq J_{\text{max}}^{(n,i)}(\nu^*, \tilde{J}^*) \leq J_{\text{max}}^{(n,i)}(\nu^*, J^*) \text{ else} \]

(Q, S) \[ J_{\text{max}}^{(n,i)}(\nu^*, \tilde{J}^*) = P_i^{(n)}(\nu^*) \text{ for } i = \ell^{(n)} - 1/2 \]

\[ J_{\text{max}}^{(n,i)}(\nu^*, \tilde{J}^*) = P_i^{(n)}(\nu^*) \text{ for } i = \bar{\ell}^{(n)} - 1/2, \ell^{(n)} = \bar{\ell}^{(n-1)} \]

\[ J_{\text{max}}^{(n,i)}(\nu^*, \tilde{J}^*) = P_i^{(n)}(\nu^*) - 1 \text{ for } i = \ell^{(n)} - 1/2, \ell^{(n)} < \bar{\ell}^{(n-1)} \]

\[ 0 \leq J_{\text{max}}^{(n,i)}(\nu^*, \tilde{J}^*) \leq J_{\text{max}}^{(n,i)}(\nu^*, J^*) \text{ else} \]

Let us first assume that (S) holds:

By the definition of $\ell^{(n)}$ and $\bar{\ell}^{(n)}$ there is no singular string at $\ell^{(n-1)} - \frac{1}{2}$ and no singular or quasisingular string of length $\ell^{(n-1)} \leq i < \bar{\ell}^{(n)} = \ell$. Hence, if $m_i^{(n)}(\nu^*) > 0$, we have $J_{i-1}^{(n)}(\nu^*, J^*) \leq P_i^{(n)}(\nu^*) - 2$ for $\ell^{(n-1)} \leq i \leq \ell$ and $J_{\text{max}}^{(n,i)}(\nu^*, J^*) \leq P_i^{(n)}(\nu^*) - 1$ for $i = \ell^{(n-1)} - \frac{1}{2}$. Hence (5.14) holds if $m_i^{(n)}(\nu^*) > 0$. By lemma 5.3 (5.14) can only be violated if

$m_i^{(n)}(\nu^*) = 0, P_i^{(n)}(\nu^*) = 0$ or $1, \ell^{(n-1)} \leq \ell - 1, \ell$ finite.

The case $P_i^{(n)}(\nu^*) = 0$ is the same as before. Hence assume that $P_i^{(n)}(\nu^*) = 1$. If $m_{i-1}^{(n)}(\nu^*) = 0$, then by (5.5) and (5.11) $P_i^{(n)}(\nu^*) = 1$ for $p \leq i \leq \ell$ where $p < \ell$ is maximal such that $m_p^{(n)}(\nu^*) > 0$. By (5.3) we also have $m_i^{(n-1)}(\nu^*) = 0$ for
\( p < i < \ell \) so that \( \ell^{(n-1)} \leq p \). But since \( P^i_p(n^*) = 1 \) and \( m^i_p(n^*) > 0 \) there is a (quasi) singular string of length \( p \) in \((n^*, J^*)^{(n)}\) which contradicts \( p < \ell \). If \( m^i_{\ell - \frac{1}{2}}(n^*) > 0 \), then \( \ell^{(n)}(n^*) \geq 2 \) since otherwise there would be a (quasi) singular string of length \( \ell - \frac{1}{2} \) in \((n^*, J^*)^{(n)}\). By convexity (5.10) and (5.3) this implies \( P^i_{\ell - \frac{1}{2}}(n^*) = 0 \) and \( m^i_{\ell - \frac{1}{2}}(n^*) > 0 \). Since \( \ell^{(n-1)} \leq \ell - 1 \), (Q) would hold for \( \ell - \frac{3}{2} \) which contradicts our assumptions.

One more problem might occur when \( \bar{\ell}^{(n-1)} = \bar{\ell}^{(n)} = \ell \). If \( \bar{\ell}^{(n)}(n^*) = 1 \), then the path is 1, the rigged configuration is empty, and both sides of (5.13) are zero. Hence by convexity (5.10) it follows that \( P^i_{\ell - \frac{1}{2}}(n^*) = 0 \) for \( p < i < \ell \) where \( p < \ell \) is maximal such that \( m^i_p(n^*) > 0 \). Equation (5.3) implies that \( m^{i-1}_p(n^*) = 0 \) for \( p < i < \ell \) so that \( \ell^{(n-1)} < p \). But there is a (quasi) singular string of length \( p \) in \((n^*, J^*)^{(n)}\) which contradicts \( p < \ell \).

Finally assume that (Q,S) holds:

For \( i < \ell \) the same arguments hold as for case (Q). Since by definition there are no singular strings of length \( \ell^{(n-1)} < i < \ell^{(n)} = \ell \) in \((n^*, J^*)^{(n)}\), case (Q) holds for \( i = \ell^{(n)} \) and \( m^{(n)}_{\ell^{(n)}}(n^*) > 0 \), the only problem occurs when

\[
\text{if } p < \ell \text{ is maximal such that } m^i_p(n^*) > 0, \text{ then by (5.5) and (5.10) } P^i_{\ell - \frac{1}{2}}(n^*) = 0 \text{ for } p < i < \ell. \quad (\text{case } (Q))
\]

But then there is a singular string of length \( p \) in \((n^*, J^*)^{(n)}\) which contradicts \( \ell^{(n)} = \ell \).

Proof of (III) for \( B_{n}^{(1)} \). Here \( b^2 = \mathbb{T} \). Note that \( \overline{\tau}(b \otimes b') = 0 \) if \( b \leq b' \) and \( b \otimes b' \neq 0 \), \( \overline{\tau}(b \otimes b') = 2 \) if \( b \otimes b' = \mathbb{T} \otimes 1 \), and \( H(b \otimes b') = 1 \) otherwise.

If \( L = 1 \) then the path is 1, the rigged configuration is empty, and both sides of (5.13) are zero. Here (5.1) and (5.2) are given by

\[
\Delta(\text{cc}(n^*, J^*)) = \alpha_1^{(1)}
\]

(5.15)

\[
\overline{\tau}(b_L \otimes b_{L-1}) = \chi(\ell^{(1)} = 1) + \chi(\bar{\ell}^{(1)} = 1).
\]

(5.16)

where \( \ell^{(i)} \) and \( \bar{\ell}^{(i)} \) are determined by the algorithm \( \delta \).

Let \( \hat{\ell}^{(a)} \) and \( \hat{\bar{\ell}}^{(a)} \) be the length of the selected strings defined by the algorithm \( \delta \) on \((n^*, J^*) = \delta(n^*, J^*)\). To check (5.10) note that if \( \hat{\ell}^{(1)} = 1 \) it follows that \( \hat{\ell}^{(a)} > \hat{\ell}^{(a+1)} \) for \( 1 \leq a \leq n - 1 \). Hence if \( b_L < n \) then \( b_{L-1} < b_L \) and both sides of (5.16) yield 1. If \( b_L = 0 \) then \( b_{L-1} < b_L \) by (4.9) and both sides of (5.13) are 1. If \( \pi \leq b_L < \mathbb{T} \), then \( b_{L-1} < b_L \) by (4.9) and both sides of (5.16) are 1.
and \( \bar{\ell}^{(1)} = 1 \), then \( b_{L-1} = 1 \) by (5.13). Hence both sides of (5.16) yield 2. Finally, if 
\[ b_{L} = 1 \] and \( \bar{\ell}^{(1)} > 1 \), then there exists a singular string of length \( \bar{\ell}^{(1)} - 1 \) in \( (\nu^{*}, \bar{J}^{*}) \) so that \( b_{L-1} \neq 1 \). Hence both sides of (5.16) are 1. If \( \ell^{(1)} > 1 \) then \( \bar{\ell}^{(a)} < \ell^{(a)} \) for \( 1 \leq a \leq n - 2 \) and the cases can be checked in a similar fashion as before.

To prove (5.15), by (3.4) and (4.8) we have
\[
\begin{align*}
cc(\nu^{*}) &= \frac{1}{2} \sum_{i,j} \sum_{a,b=1}^{n} \min(t_{b,i}, t_{a,j})(\alpha_{a}\alpha_{b}) \\
&\times \left( m_{i}^{(a)} - \frac{\delta_{i,\ell^{(a)}} + \delta_{i,\ell^{(a)}-1}}{\ell^{(a)}} - \frac{\delta_{i,\bar{\ell}^{(a)}} + \delta_{i,\bar{\ell}^{(a)}-1}}{\bar{\ell}^{(a)}} \right) \\
&\times \left( m_{j}^{(b)} - \frac{\delta_{j,\ell^{(b)}} + \delta_{j,\ell^{(b)}-1}}{\ell^{(b)}} - \frac{\delta_{j,\bar{\ell}^{(b)}} + \delta_{j,\bar{\ell}^{(b)}-1}}{\bar{\ell}^{(b)}} \right).
\end{align*}
\]

Applying the data for \( B_{n} \) and using (4.3), a tedious but straightforward calculation yields
\[
\Delta cc(\nu^{*}) = \sum_{a=1}^{n} \sum_{i,j} \left( P_{i,j}^{(a)}(\nu^{*}) - P_{i,j}^{(a)}(\nu^{*}) \right) \left( m_{i}^{(a)} - \delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}} \right) + \sum_{i,j} m_{i}^{(1)}
- \chi(\ell^{(n)} = \ell^{(n-1)} - 1) + \chi(\bar{\ell}^{(n)} = \ell^{(n-1)}) + \chi(\ell^{(n)} = \infty) \chi(\bar{\ell}^{(n)} < \infty).
\]

For \( \Delta |J^{*}| \) we obtain from the algorithm \( \delta' \)
\[
\Delta |J^{*}| = \sum_{a=1}^{n} \sum_{i,j} \left( P_{i,j}^{(a)}(\nu^{*}) - P_{i,j}^{(a)}(\nu^{*}) \right) \left( m_{i}^{(a)} - \delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}} \right)
+ \chi(\ell^{(n)} = \ell^{(n-1)} - 1) - \chi(\bar{\ell}^{(n)} = \ell^{(n-1)}) - \chi(\ell^{(n)} = \infty) \chi(\bar{\ell}^{(n)} < \infty),
\]
where the last three terms come from the fact that for \( n \)-th rigged partition singular strings can be transformed into quasisingular strings and vice versa. Hence altogether, using \( \sum_{i,j} m_{i}^{(1)} = \alpha_{1}^{(1)} \), we obtain (5.15).

5.3. Proof for type \( C_{n}^{(1)} \).

Proof of (I) for \( C_{n}^{(1)} \). If \( b = k \) with \( 1 \leq k < n \) the proof that \( \rho \) is dominant is analogous to type \( D_{n}^{(1)} \). For \( b = n \) a problem occurs if \( \lambda_{n} = 0 \). In this case \( P_{i}^{(n)}(\nu^{*}) = 0 \) for \( i \geq \ell \) where \( \ell \) is the largest part in \( \nu^{(n)} \) by Lemma 5.2 and (5.10). By (5.11) this implies \( m_{i}^{(n-1)}(\nu^{*}) = 0 \) for \( i > \ell \). Hence \( \ell^{(n-1)} \leq \ell \). But there is a singular string of length \( \ell \) in \( (\nu^{*}, J^{*})^{(n)} \) which contradicts \( \ell^{(n)} = \infty \). If \( k = \pi \) a problem occurs if \( \lambda_{n} = \lambda_{n-1} \). In this case \( P_{i}^{(n-1)}(\nu^{*}) = 0 \) for \( i \geq \ell \) where \( \ell \) is the largest part in \( \nu^{(n-1)} \). By (5.11), \( m_{i}^{(n)}(\nu^{*}) = 0 \) for \( i > \ell \). Hence \( \ell^{(n)} \leq \ell \). But there is a singular string of size \( \ell \) in \( (\nu^{*}, J^{*})^{(n-1)} \) (this also works if \( \ell^{(n-1)} = \ell^{(n)} = \ell \)) which contradicts \( \ell^{(n)} = \infty \). \( \square \)
Proof of (II) for $C_{n}^{(1)}$. We show that $(\bar{\nu}^{*}, \bar{J}^{*}) \in \text{RC}(\rho, \bar{\mu})$. We use the same notation and set-up as in type $D_{n}^{(1)}$. Then

\[
J_{\text{max}}^{(a,i)}(\bar{\nu}^{*}, \bar{J}^{*}) = P_{i}^{(a)}(\bar{\nu}^{*})
\]

for $i = \ell^{(a)} - 1$ and $i = \bar{\ell}^{(a)} - 1$ or $i = \bar{\ell}^{(a)} - 2$ if $\ell^{(a)} = \bar{\ell}^{(a+1)}$

\[0 \leq J_{\text{max}}^{(a,i)}(\bar{\nu}^{*}, \bar{J}^{*}) \leq J_{\text{max}}^{(a,i)}(\nu^{*}, J^{*})\]

else.

The proof that $0 \leq J_{\text{max}}^{(a,i)}(\bar{\nu}^{*}, \bar{J}^{*}) \leq P_{\ell}^{(a)}(\bar{\nu}^{*})$ for $1 \leq a < n$ is the same as usual if $\ell^{(a)} \neq \bar{\ell}^{(a+1)}$. If $\ell^{(a)} = \bar{\ell}^{(a+1)}$, by (4.13) the only problem occurs if

\[
m_{n-1}^{(a)}(\nu^{*}) = 0, \quad P_{\ell-2}^{(a)}(\nu^{*}) = 0, \quad \ell^{(a-1)} < \ell - 1, \quad \ell = \bar{\ell}^{(a)} = \ell^{(a)} + 1 \text{ finite.}
\]

We show that these conditions cannot be met simultaneously. Let $p < \ell - 1$ be maximal such that $m_{p}^{(a)}(\nu^{*}) > 0$; if no such $p$ exists set $p = 0$. By (5.14), $P_{p-2}^{(a)}(\nu^{*}) = 0$ is only possible if $P_{i}^{(a)}(\nu^{*}) = 0$ for $p \leq i < \ell - 1$. By (5.6) this requires $m_{i}^{(a-1)}(\nu^{*}) = 0$ for $p < i < \ell - 1$ so that $\ell^{(a-1)} < \ell - 1$ implies $\ell^{(a-1)} \leq p$. But there is a singular string of length $p$ in $(\nu^{*}, J^{*})^{(a)}$ which contradicts $\ell^{(a)} = \ell - 1 > p$.

Finally for $a = n$ the only problem occurs if

\[
m_{n-1}^{(n)}(\nu^{*}) = 0, \quad P_{\ell-2}^{(n)}(\nu^{*}) = 0, \quad \ell^{(n-1)} < \ell - 1, \quad \ell = \bar{\ell}^{(n)} \text{ finite.}
\]

By convexity (5.10), $P_{i}^{(n)}(\nu^{*}) = 0$ for $p \leq i < \ell$ where $p < \ell$ is largest such that $m_{p}^{(n)}(\nu^{*}) > 0$. Then by (5.6), we also have $m_{i}^{(n-1)}(\nu^{*}) = 0$ for $p < i < \ell$ so that $\ell^{(n-1)} < \ell - 1$ implies $\ell^{(n-1)} \leq p$. But there is a singular string of length $p$ in $(\nu^{*}, J^{*})^{(n)}$ which contradicts $\ell^{(n)} = \ell > p$.

Proof of (III) for $C_{n}^{(1)}$. Here $b^{\circ} = \bar{T}$, $\overline{\Pi}(b \otimes b') = 0$ if $b \leq b'$ and $H(b \otimes b') = 1$ otherwise.

If $L = 1$ then the path is 1, the rigged configuration is empty, and both sides of (5.13) are zero.

Here (5.1) and (5.2) are given by

\[
\Delta(cc(\nu^{*}, J^{*})) = \alpha_{2}^{(1)}
\]

(5.17)

\[
\overline{\Pi}(b_{L} \otimes b_{L-1}) = \chi(\ell^{(1)} = 1)
\]

(5.18)

where $\ell^{(i)}$ is determined by the algorithm $\delta$. Note that there is no contribution from $\ell^{(1)}$ in (5.18) since $\ell^{(1)} > 1$.

Let $\tilde{a}^{(a)}$ and $\tilde{\ell}^{(a)}$ be the length of the selected strings defined by the algorithm $\delta$ on $(\bar{\nu}^{*}, \bar{J}^{*}) = \delta(\nu^{*}, J^{*})$. Note that if $\ell^{(1)} = 1$ then (4.13) implies that $b_{L-1} < b_{L}$ so that both sides of (5.13) are 1. If $\ell^{(1)} > 1$ then $\tilde{a}^{(a)} < \ell^{(a)}$ for $1 \leq a < n$ so that $b_{L-1} \geq b_{L}$ and both sides of (5.13) are 0. For $b_{L} = \bar{T}$, $\tilde{\ell}^{(n)} < \bar{\ell}^{(n)}$ unless $\ell^{(n)} = 2$. But note that in this case $\ell^{(n)} = 1$ and hence $\ell^{(1)} = 1$ which contradicts our assumptions. If $\tilde{\ell}^{(n)} < \bar{\ell}^{(n)}$ then also $\tilde{a}^{(a)} < \ell^{(a)}$ which implies that $b_{L-1} \geq b_{L}$. Hence both sides of (5.18) are 0.
To prove (5.17), by (3.4) and (4.12) we have

\[ cc(\nu^*) = \frac{1}{2} \sum_{i,j \geq 1} \sum_{a,b=1}^n \min(t_{bi}, t_{aj})(\alpha_a | \alpha_b) \]

\[ \times \left( m_i^{(a)} - \chi(a < n)(\delta_{i,\ell^{(a)}} - \delta_{i,\ell^{(a)}-1}) - \delta_{i,\ell^{(a)}} + \delta_{i,\ell^{(a)}-1} \right) \]

\[ \times \left( m_j^{(b)} - \chi(b < n)(\delta_{j,\ell^{(b)}} - \delta_{j,\ell^{(b)}-1}) - \delta_{j,\ell^{(b)}} + \delta_{j,\ell^{(b)}-1} \right) . \]

Applying the data for \( C_n \) and using (4.13), a tedious but straightforward calculation yields

\[ \Delta cc(\nu^*) = \sum_{a=1}^n \sum_{i \geq 1} \left( P_{v,i}^{(a)}(\nu^*) - P_{v,i}^{(a)}(\hat{\nu}^*) \right) \left( m_i^{(a)} - \chi(a < n)(\delta_{i,\ell^{(a)}} - \delta_{v,i,\ell^{(a)}}) \right) \]

\[ + \sum_{i \geq 1} m_i^{(1)} . \]

For \( \Delta |J^*| \) we obtain from the algorithm \( \delta' \)

\[ \Delta |J^*| = \sum_{a=1}^n \sum_{i \geq 1} \left( P_{v,i}^{(a)}(\nu^*) - P_{v,i}^{(a)}(\hat{\nu}^*) \right) \left( m_i^{(a)} - \chi(a < n)(\delta_{i,\ell^{(a)}} - \delta_{v,i,\ell^{(a)}}) \right) . \]

Hence altogether, using \( \sum_{i \geq 1} m_i^{(1)} = \alpha_1^{(1)} \), we obtain (5.17).

5.4. Proof for type \( A_{2n}^{(2)} \). The proofs of (I) and (II) are analogous to the previous cases. In particular, the proof of (II) is very similar to that for type \( C_n^{(1)} \).

Proof of (III) for \( A_{2n}^{(2)} \). Here \( b^* = \phi, H(b \otimes b') = 0 \) if \( b \leq b' \), \( H(b \otimes b') = 2 \) if \( b > b' \) or \( b \otimes b' = \phi \otimes \phi \) and \( H(b \otimes b') = 0 \) otherwise.

If \( L = 1 \) then the path is 1 or \( \phi \). In the former case, the rigged configuration is empty, and both sides of (3.13) are zero. In the other case both sides of (3.13) are 1.

Here (5.1) and (5.2) are given by

(5.19) \[ \Delta (cc(\nu^*, J^*)) = 2\alpha_1^{(1)} - \chi(\ell^{(n)} = 1) \]

(5.20) \[ \mathcal{P}(b_L \otimes b_{L-1}) = 2\chi(\ell^{(1)} = 1) - \chi(\ell^{(n)} = 1) + \chi(\tilde{\ell}^{(n)} = 1) \]

where \( \ell^{(1)} \) is determined by the algorithm \( \delta \). Note that there is no contribution from \( \tilde{\ell}^{(1)} \) in (5.20) since \( \tilde{\ell}^{(1)} > 1 \).

Equation (5.19) can be checked in a similar fashion as to the other cases.

To prove (5.19), applying the data for \( B_n \) and using (4.13), a tedious but straightforward calculation yields

\[ \Delta cc(\nu^*) = 2 \sum_{a=1}^n \sum_{i \geq 1} \left( P_i^{(a)}(\nu^*) - P_i^{(a)}(\hat{\nu}^*) \right) \left( m_i^{(a)} - \chi(a < n)(\delta_{i,\ell^{(a)}} - \delta_{i,\ell^{(a)}-1}) \right) \]

\[ - \chi(\ell^{(n)} = 1) + 2 \sum_{i \geq 1} m_i^{(1)} . \]
For $|J^*|$ we obtain from the algorithm $\delta'$

$$
\Delta|J^*| = 2 \sum_{a=1}^{n} \sum_{i \geq 1} \left( P_i^{(a)}(\nu^*) - P_i^{(a)}(\tilde{\nu}^*) \right) \left( m_i^{(a)} - \chi(a < n)\delta_{i,\ell(a)} - \delta_{i,\ell(a)} \right).
$$

Hence altogether, using $\sum_{i \geq 1} m_i^{(1)} = \alpha_1^{(1)}$, we obtain \((5.19)\). \(\blacksquare\)

### 5.5. Proof for type $A_2^{(2)}_{2n-1}$

**Proof of (I) for $A_2^{(2)}_{2n-1}$**. The proof that $\rho$ is dominant for $b = k$ with $1 \leq k \leq n$ is analogous to the other types. For $b = k$ with $1 \leq k \leq n$, $\rho$ is not dominant if $\lambda_k = \lambda_{k-1}$. In this case $P_i^{(k-1)}(\nu^*) = 0$ for $i \geq \ell$ where $\ell$ is the largest part of $\nu^{(k-1)}$ by Lemma 5.2 and \((5.10)\). By \((5.8)\), $m_i^{(k)}(\nu^*) = 0$ for $i > \ell$ so that $\ell(k) \leq \ell$.

But since $P_i^{(k-1)}(\nu^*) = 0$ and $m_i^{(k-1)}(\nu^*) > 0$, there is a singular string of length $\ell$ in $(\nu^*, J^*)^{(k-1)}$. Hence $\ell(k-1) \leq \ell$ (which contradicts $\ell(k-1) = \infty$ since $\delta = \text{rk}(\nu^*, J^*)$) unless $\ell(k-1) = \ell(k) = \ell$ and $m_i^{(k-1)}(\nu^*) = 1$. Since $m_i^{(k)}(\nu^*) \geq 2$ for $1 \leq k < n$ and $m_i^{(n)}(\nu^*) \geq 1$, \((5.8)\) for $a = k - 1$ and $i = \ell$ implies that $m_i^{(k-2)}(\nu^*) = 0$ and $P_i^{(k-1)}(\nu^*) = 0$. Hence $\ell(k-2) < \ell$ and $m_i^{(k-1)}(\nu^*) = 0$ since otherwise $\ell(k-1) \leq \ell - 1$.

By induction on $i = \ell - 1, \ell - 2, \ldots, 1$ \((5.8)\) for $a = k - 1$ implies that $m_i^{(k-2)}(\nu^*) = 0$ and $P_i^{(k-1)}(\nu^*) = 0$ which in turn requires $m_{i-1}^{(k-1)}(\nu^*) = 0$ since else $\ell(k-1) \leq i - 1$.

But then $m_i^{(k-2)}(\nu^*) = 0$ for all $1 \leq i \leq \ell$ so that $\ell(k-2) > \ell$ which contradicts our assumptions. \(\blacksquare\)

**Proof of (II) for $A_2^{(2)}_{2n-1}$**. To prove that $(\tilde{\nu}^*, J^*)$ is admissible, one finds similarly to the proof of type $D_n^{(1)}$ that the only problem occurs if

$$
m_i^{(a)} = 0, \quad P_{i-1}^{(a)}(\nu^*) = 0, \quad \ell(a-1) < \ell \quad \text{(resp. } \tilde{\ell}(a+1) < \ell \text{ for } 1 \leq a < n), \quad \ell \text{ finite}, \quad \ell = \ell(a) \quad \text{(resp. } \ell = \tilde{\ell}(a) \text{ for } 1 \leq a < n).$$

Analogous to the case $D_n^{(1)}$ it can be shown that these conditions cannot hold simultaneously. \(\blacksquare\)

**Proof of (III) for $A_2^{(2)}_{2n-1}$**. Here $b^\delta = T, \quad \overline{H}(T \otimes 1) = 2, \quad \overline{H}(b \otimes b') = 0$ if $b \leq b'$ and $H(b \otimes b') = 1$ otherwise.

If $L = 1$ then the path is 1, the rigged configuration is empty, and both sides of \((8.13)\) are zero.

Here \((5.1)\) and \((5.2)\) are given by

\[(5.21)\]
$$
\Delta(cc(\nu^*, J^*)) = \alpha_1^{(1)}
$$

\[(5.22)\]
$$
\overline{H}(b_L \otimes b_{L-1}) = \chi(\ell(1) = 1) + \chi(\tilde{\ell}(1) = 1)
$$

where $\ell(1)$ is determined by the algorithm $\delta$.

The proof that \((5.22)\) holds is very similar to the previous cases.
To prove (5.21) we apply the data for $C_n$ to (3.4). Using (4.10) and (4.17) a tedious but straightforward calculation yields

$$\Delta cc(\nu^*) = \sum_{a=1}^{n} \sum_{i \geq 1} t^\nu_i \left( P^{(a)}_i(\tilde{\nu}^*) - P^{(a)}_i(\nu^*) \right) \left( m_i^{(a)} - \delta^\nu_i, \ell(a) - \chi(a < n) \delta^\nu_i, \bar{\ell}(a) \right) + \sum_{i \geq 1} m_i^{(1)}.$$  

For $\Delta |J^*|$ we obtain from the algorithm $\delta'$

$$\Delta |J^*| = \sum_{a=1}^{n} \sum_{i \geq 1} t^\nu_i \left( P^{(a)}_i(\nu^*) - P^{(a)}_i(\tilde{\nu}^*) \right) \left( m_i^{(a)} - \delta^\nu_i, \ell(a) - \chi(a < n) \delta^\nu_i, \bar{\ell}(a) \right).$$

Hence altogether, using $\sum_{i \geq 1} m_i^{(1)} = \alpha^{(1)}_1$, we obtain (5.21).  

5.6. Proof for type $D^{(2)}_{n+1}$.

Proof of (I) for $D^{(2)}_{n+1}$. The proof proceeds as before except in the cases $b = n$ and $b = 0$ (there is nothing to prove for $b = \phi$). Suppose $b = n$ and $\nu$ is not dominant. Since $\lambda$ is dominant, $\lambda_n = 0$. Then it can be deduced that $P^{(n)}_i(\nu^*) = 0$ for $i \geq \ell$ where $\ell$ is the largest part in $\nu^{(n)}$ by Lemma 5.9 and (6.10) and the admissibility of $\nu^*$. By (6.9) it follows that $m_i^{(n-1)}(\nu^*) = 0$ for $i > \ell$ so that $\ell^{(n-1)} = \ell$. But there is a singular string of length $\ell$ in $(\nu^*, J^{(n)}_\nu)$ which contradicts $\ell^{(n)} = \infty$. To prove that $b = 0$ cannot occur if $\nu$, we find as for the case $k = n$ that $\ell^{(n-1)} = \ell$ and that there is a singular string of length $\ell$ in $(\nu^*, J^{(n)}_\nu)$ since $P^{(n)}_\ell(\nu^*) = 0$. For the $b = 0$ case (Q) must hold so that there must be a quasisingular string of length $\ell^{(n)} < \ell$ in $(\nu^*, J^{(n)}_\nu)$. But observe that there is a singular string of length $\ell > \ell^{(n)}$ in $(\nu^*, J^{(n)}_\nu)$ which contradicts $\ell^{(n)} = \infty$. 

Proof of (II) for $D^{(2)}_{n+1}$. Next we need to show that $(\tilde{\nu}^*, J^{(n)}_\nu) \in RC(\rho, \tilde{\mu})$. The case that $(\tilde{\nu}^*, J^{(n)}_\nu)$ is admissible for $1 \leq a < n$ works as usual. Consider $a = n$. First note that there is no problem in case (Q,S) setting the new string of length $\bar{\ell}^{(n)} = 1$ to be quasisingular since the string of length $\bar{\ell}^{(n)} = 1$ is not singular by definition so that $P^{(n)}_{\bar{\ell}^{(n)}-1}(\nu^*) > 0$ and also $P^{(n)}_{\bar{\ell}^{(n)}-1}(\tilde{\nu}^*) > 0$ by (4.11). The only problem occurs if

$$m^{(n)}_{\ell-1}(\nu^*) = P^{(n)}_{\ell-1}(\nu^*) = 0 \quad or \quad \ell^{(n-1)} < \ell, \quad \bar{\ell}^{(n)} = \ell^{(n)} \quad finite.$$  

Note that $P^{(n)}_{\ell-1}(\nu^*)$ is always even so that $P^{(n)}_{\ell-1} = 1$ is impossible. The proof that these conditions cannot hold simultaneously works as usual.  

Proof of (III) for $D^{(2)}_{n+1}$. Here $b^k = \phi$ and $H(\phi \otimes \phi) = 2$. $\bar{H}(b \otimes \phi) = \bar{H}(\phi \otimes b) = 1$ if $b \neq \phi$, $\bar{H}(b \otimes b')$ if $b, b' \neq \phi$, $b \leq b'$ and $b = b'$ if $b = 0$, and $\bar{H}(b \otimes b') = 2$ if $b > b'$ or $b = b' = 0$.

If $L = 1$ then the path is either 1 or $\phi$. In the former case the rigged configuration is empty, and both sides of (3.13) are zero. In the latter case it is also not hard to check that both sides of (3.13) are 1.
Here (5.1) and (5.2) are given by
\begin{align}
\Delta(cc(\nu^*, J^*)) &= 2\alpha_1^{(1)} - \chi(\ell(n) = 1) \\
\overline{H}(bt \otimes b_L-1) &= 2\chi(\ell(1) = 1) - \chi(\ell(n) = 1) + \chi(\ell(n) = 1)
\end{align}
where $\ell^{(i)}$ and $\overline{\ell}^{(i)}$ are determined by the algorithm $\delta$. To obtain (5.24) we used the fact that by definition $\overline{\ell}(1) > 1$. Here $\bar{\ell}(n)$ is defined by the algorithm $\delta$ on $(\bar{\nu}^*, \bar{J}^*) = \delta'(\nu^*, J^*)$.

It can be checked directly that (5.24) holds. For example, if $\ell^{(1)} = 1$ it follows that $\bar{\ell}(n) \geq \bar{\ell}(a+1)$ for $1 \leq a < n$. Hence if $b_L \leq n$ then $b_L-1 < b_L$ and both sides of (5.24) yield 2. If $b_L = \phi$ then both sides of (5.24) are 1 if $b_{L-1} \neq \phi$ and 2 if $b_{L-1} = \phi$. If $b_L = 0$ then both sides of (5.24) are 2 if $b_{L-1} \leq 0$. Note that $b_{L-1} = \phi$ or $b_{L-1} \geq \pi$ is not possible. Finally, if $b_L \geq \pi$ then $\overline{\ell}(a) = \bar{\ell}(a)$ and $b_{L-1} < b_L$. Note that $b_{L-1} = \phi$ is not possible in this case since $\ell(1) > 1$ which implies $\ell(n) > 1$. Both sides of (5.24) yield 2 in this case. If $\ell(1) > 1$ then $\ell(n) < \ell(a)$ for $1 \leq a \leq n$ and the cases can be checked in a similar fashion as before.

To prove (5.23), from (4.4) and (4.12) we obtain
\[ cc(\bar{\nu}^*) = \frac{1}{2} \sum_{a,b} \sum_{i,j \geq 1} \min(i,j)(\alpha_a|\alpha_b) \times (m_i^{(a)} - \delta_{i,\ell(a)} - \delta_{i,\bar{\ell}(a)} + \delta_{i,\bar{\ell}(a)-1}) \times (m_j^{(b)} - \delta_{j,\ell(a)} - \delta_{j,\bar{\ell}(a)} + \delta_{j,\bar{\ell}(a)-1}). \]

Expanding out and using (4.19) a tedious but straightforward calculation yields
\[ \Delta cc(\nu^*) = \sum_{a=1}^{n} \sum_{i \geq 1} \ell_i^{(a)} \left( P_i^{(a)}(\bar{\nu}^*) - P_i^{(a)}(\nu^*) \right) \left( m_i^{(a)} - \delta_{i,\ell(a)} - \delta_{i,\bar{\ell}(a)} \right) \]
\[ + 2 \sum_{i \geq 1} m_i^{(1)} - \chi(\bar{\ell}(n) = \infty) \chi(\ell(n) < \infty). \]

For $\Delta |J^*|$ we obtain from the algorithm $\delta'$
\[ \Delta |J^*| = \sum_{a=1}^{n} \sum_{i \geq 1} \ell_i^{(a)} \left( P_i^{(a)}(\nu^*) - P_i^{(a)}(\bar{\nu}^*) \right) \left( m_i^{(a)} - \delta_{i,\ell(a)} - \delta_{i,\bar{\ell}(a)} \right) \]
\[ + \chi(\bar{\ell}(n) = \infty) \chi(1 < \ell(n) < \infty) \]
where the last term comes from the fact that in case (Q) a quasisingular string is changed into a singular string. Hence altogether, using $\sum_{i \geq 1} m_i^{(1)} = \alpha_1^{(1)}$ and the fact that $\bar{\ell}(n) = \infty$ if $\ell(n) = 1$ by the algorithm $\delta$, we obtain (5.23).

5.7. Proof for type $A^{(2)}_{2n}$. 

Proof of (I) for $A^{(2)}_{2n}$. The only case that proceeds differently than before is $b = 0$. Suppose $\lambda_n = 0$. Let $\ell$ be the longest part of $\nu(n)$. As in the proof of the $D^{(2)}_{n+1}$ case, $P_i^{(a)}(\nu^*) = 0$ where $\bar{\ell} \geq \ell(n-1)$. If $\ell$ is odd then this is a contradictionary of the admissibility of $\nu^*$; see (3.10). If $\ell$ is even then $J^{(n,\ell)}$ is singular and $\ell(n) < \ell$ (as $\ell(n)$ is odd and $\ell$ is the longest part), contradicting $b = 0$. \qed
Proof of (II) for $A_{2n}^{(2)}$. The admissibility of $(\nu^*, J^*)$ for $1 \leq a < n$ is as before. Let $a = n$. We first observe that in all cases,

$$
(5.25) \quad \ell^{(1)} \leq \ell^{(2)} \leq \cdots \leq \ell^{(n)} \leq \bar{\ell}^{(n-1)} \leq \cdots \leq \bar{\ell}^{(1)},
$$

with $\ell^{(n)}$ odd and $\bar{\ell}^{(n)}$ even (when they are finite). We also note that by (4.13),

$$
(5.26) \quad P_{\iota}^{(n)}(\nu^*) \geq P_{\iota}^{(n)}(\nu^*) - 1
$$

with equality if and only if $\ell^{(n-1)} \leq i < \ell^{(n)}$.

Let us verify (3.10) for $(\nu^*, J^*)$. Let $i$ be odd such that $m_i^{(n)}(\nu^*) > 0$. Suppose first that $m_i^{(n)}(\nu^*) > 0$. By (3.10) for $(\nu^*, J^*)$, $P_{\iota}^{(n)}(\nu^*) > 0$. By (5.24) we may assume that $P_{\iota}^{(n)}(\nu^*) = 1$ and $\ell^{(n-1)} \leq i < \ell^{(n)}$. But then $J_{\iota}$ was quasisingular, which is a contradiction to the definition of $\delta$. So suppose $m_i^{(n)}(\nu^*) = 0$. Since $m_i^{(n)}(\nu^*) > 0$ we are in case $(Q, S)$ with $i = \bar{\ell}^{(n)} - 1$. In case $(Q, S)$ $\ell^{(n)} < \bar{\ell}^{(n)}$, so $\ell^{(n)} \leq i < \ell^{(n)}$. Now $i \neq \ell^{(n)}$ since $m_i^{(n)}(\nu^*) = 0$. So $\ell^{(n)} < i < \ell^{(n)}$ with $\ell^{(n)}$ and $i$ odd. By (4.13) $P_{\iota}^{(n)}(\nu^*) = P_{\iota}^{(n)}(\nu^*)$. There is only a problem if $P_{\iota}^{(n)}(\nu^*) = 0$. By (5.10) it follows that $P_{\iota}^{(n)}(\nu^*) = P_{\iota}^{(n)}(\nu^*) = 0$. Since $i - 1$ is even, if $m_i^{(n)}(\nu^*) > 0$ then $J_{\iota}^{(n-1)}$ would have been singular with $\ell^{(n)} < i - 1 < \ell^{(n)}$, contradicting the choice of $\ell^{(n)}$. So $m_i^{(n)}(\nu^*) = 0$. Applying (5.10) again, $P_{\iota}^{(n)}(\nu^*) = 0$. Since $(\nu^*, J^*)$ was admissible and $i - 2$ is odd, by (3.11) it follows that $m_{i-2}^{(n)}(\nu^*) = 0$. Continuing in this manner, a contradiction is reached since $P_{\iota}^{(n)}(\nu^*) > 0$.

Now suppose $i$ is even. It must be checked that $P_{\iota}^{(n)}(\nu^*) \geq 0$. The only problem is if $P_{\iota}^{(n)}(\nu^*) = 0$ and $\ell^{(n-1)} \leq i < \ell^{(n)}$. If $m_i^{(n)}(\nu^*) > 0$ then $\delta$ would have chosen the singular partition $J_{\iota}^{(n-1)}$. So $m_i^{(n)}(\nu^*) = 0$. By (5.10) it follows that $P_{\iota}^{(n)}(\nu) = 0$. Arguing as above but with the index increasing from $i$, a contradiction is reached since $P_{\iota}^{(n)}(\nu^*) > 0$.

Proof of (III) for $A_{2n}^{(2)}$. One has $b^2 = 1$, $\overline{\Pi}(b_2^* \otimes b_1^*) = 0$ if $b_2^* \leq b_1^*$ (except for $\overline{\Pi}(0 \otimes 0) = 1$), and $\overline{\Pi}(b_2^* \otimes b_1^*) = 1$ for $b_2^* > b_1^*$.

If $L = 1$ then the path is 1, the rigged configuration is empty, and both sides of (3.13) are zero.

Here (5.1) and (5.4) are given by

$$
(5.27) \quad \Delta(\text{cc}(\nu^*, J^*)) = \alpha_i^{(1)}
$$

$$
(5.28) \quad \overline{\Pi}(bL \otimes b_{L-1}) = \chi(\ell^{(1)}) = 1
$$

where $\ell^{(i)}$ and $\bar{\ell}^{(i)}$ are determined by the algorithm $\delta$. The term $\chi(\bar{\ell}^{(1)}) = 1$ disappears since the definition of the algorithm forces $\bar{\ell}^{(1)} \geq 2$. The proof of (5.27) is very similar to that in the $D_{n+1}^{(2)}$ case.

Straightforward computations yield

$$
\Delta \text{cc}(\nu^*) = \sum_{a,b,i} \left( \alpha_a | \alpha_b \right) \chi(i \geq \ell^{(b)}) + \chi(i \geq \bar{\ell}^{(b)}) \left( m_i^{(a)}(\nu^*) - \delta_{i,\ell^{(a)}} - \delta_{i,\bar{\ell}^{(a)}} \right) + \chi(\ell^{(1)} < \infty) + \chi(\bar{\ell}^{(1)} < \infty) - \frac{1}{2} \chi(\ell^{(n)} < \infty) + \frac{1}{2} \chi(\bar{\ell}^{(n)} < \infty)
$$

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and
\[
\sum_{a,i}(P_i^{(a)}(\nu^*) - P_i^{(a)}(\nu^*))((m_i^{(a)}(\nu^*) - \delta_{i,\ell(a)} - \delta_{i,\bar{\ell}(a)})
= \sum_{a,b,i}(\alpha_a|\alpha_b)(\chi(i \geq \ell(b)) + \chi(i \geq \bar{\ell}(b)))(m_i^{(a)}(\nu^*) - \delta_{i,\ell(a)} - \delta_{i,\bar{\ell}(a)})
- \sum_{i} m_i^{(a)}(\nu^*) + \chi(\ell(1) < \infty) + \chi(\bar{\ell}(1) < \infty).
\]
Together these yield
\[
\Delta cc(\nu^*) = \sum_{a,i}(P_i^{(a)}(\nu^*) - P_i^{(a)}(\nu^*))((m_i^{(a)}(\nu^*) - \delta_{i,\ell(a)} - \delta_{i,\bar{\ell}(a)})
+ \alpha_1^{(1)} - \frac{1}{2} \chi(\ell(n) < \infty) + \frac{1}{2} \chi(\bar{\ell}(n) < \infty).
\]
One can also show that
\[
\Delta |J^*| = \sum_{a,i}(P_i^{(a)}(\nu^*) - P_i^{(a)}(\nu^*))((m_i^{(a)}(\nu^*) - \delta_{i,\ell(a)} - \delta_{i,\bar{\ell}(a)})
+ \frac{1}{2} \chi(\ell(n) < \infty) - \frac{1}{2} \chi(\bar{\ell}(n) < \infty).
\]
This proves (5.27). \qed

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