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Simple optimal lattice structures for arbitrary loadings

Yiqiang Wang*, Jeroen P. Groen, Ole Sigmund

Department of Mechanical Engineering, Solid Mechanics, Technical University of Denmark, Building 404, 2800 Kongens Lyngby, Denmark

Abstract

This paper identifies four categories of optimal truss lattice structures (TLSs) that together provide ultimate stiffness for arbitrary multi-loading scenarios in the low volume fraction limit. Each category consists of 7 periodic sets of straight bars, forming periodic parallelepiped unit cells. Compared to other optimal TLSs, the identified TLSs most probably have the simplest possible geometries with the least number of bar sets. Macroscopic properties of a TLS are estimated using a superposition model, and an optimization problem is solved to determine the exact geometries of the optimal TLSs. Systematic optimization results, run for thousands of random multi-loading conditions, are compared to (postulated) theoretical bounds for both truss and plate lattice structures. The results clearly demonstrate near-optimality of the identified TLSs (relative difference mostly within machine precision except in few cases up to 0.1%) for any loading scenarios in linear elasticity. At the same time, the optimal anisotropic TLSs always have inferior stiffness to the corresponding optimal plate lattice structures and this inferiority is bounded between 1 (single uniaxial load) and a factor of 3 (optimal isotropy).

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1. Introduction

Microstructures are called optimal if their effective properties attain the theoretical bounds. Specifically in elasticity, optimal microstructures with ultimate stiffness can be recognized by examining Hashin–Shtrikman energy bounds [1,2]. To date, various groups of optimal elastic microstructures have been proposed. For instance, isotropic sphere assemblages [3] and Vigdergauz-type constructions [4–6] can achieve the maximum bulk modulus bound but they cannot attain the maximum shear modulus bound simultaneously. Independently, [7–9] suggested so-called rank-$n$ laminates for attaining the optimal stiffness in both isotropy and anisotropy, where the laminates are made by sequential layerings at $n$ length scales. Moreover in [10,11], the authors proved that at most $n = 3$ and $n = 6$ are needed to construct optimal 2D and 3D rank laminates, respectively. This type of composite was further improved to cover all the theoretically allowable elastic properties by using infinitely rigid and compliant constituents [12]. Another class of optimal microstructures achieving the stiffness bounds was proposed by Sigmund [13], formed by a combination of solid material regions and rank-$n$ laminates ($n = 1$ in 2D and $n = 3$ in 3D). One should note that the optimal stiffness can be attained only by closed-walled plate lattice structures (PLSs). Nowadays, persistent efforts are made to look for other types of optimal microstructures, especially when considering their manufacturability [14,15].

Despite possessing sub-optimal stiffness, truss lattice structures (TLSs) may have superior strength and buckling performance [16]. Also, additive manufacturing technology, like powder-based Selective Laser Melting, may hinder realization of plate-like optimal composites. Therefore, it remains of great importance to look for optimal TLSs with maximum stiffness. Most studies in this area have focused on identification of optimal isotropic TLSs, especially in the low volume fraction limit (i.e. volume fraction going to zero). In that case, the maximum Young’s modulus is analytically calculated [17] and the optimal TLSs can be explicitly represented using single-scale bars. In 2D, triangular-type TLSs have long been recognized as optimal [18]; and in 3D a class of optimal isotropic TLSs was formed by combining two or three elementary TLSs [19–21], where the compound TLSs can be named following the system developed in [22]. If a moderate density is concerned, either rounded corners [23] or multi-scale features are demanded to improve the stiffness. Beyond isotropy, Deshpande et al. [24] proposed an octet-type TLS, whose stiffness and strength scale almost linearly up to moderate densities; and it was later fabricated using advanced 3D printing technique at sub-micrometre length scales [25]. However, there still remain big gaps, first to recognize 3D optimal anisotropic TLSs with ultimate stiffness for any prescribed loading scenarios, and second, to quantitatively study stiffness inferiority of the optimal TLSs compared to the corresponding optimal PLSs.

This paper identifies four categories of optimal TLSs for attaining maximum stiffness for any anisotropic loading conditions.
in the low volume fraction limit. Each category is built from 7 periodic sets of straight bars, forming periodic parallelepiped unit cells. Compared to other optimal TLSs with more bar sets or non-extending bars, the identified TLSs have most probably the simplest geometries achievable and are hence preferable in practical applications. In the low volume fraction limit, the elastic properties of a TLS are efficiently estimated using a superposition model. An optimization problem is then solved to determine the cell shapes and bar areas for representing the optimal TLSs. Near-optimality of the identified TLSs are verified through thousands of numerical tests with multiple loading cases, including the special case of optimal stiffness with isotropy. Furthermore, we quantitatively investigate the inferior stiffness of the optimal TLSs to their counterpart PLSS and conclude that the stiffness inferiority of any optimal TLSs is bounded between 1 and 3, referring to single uniaxial stress cases and optimal isotropy, respectively.

In the following, we first formulate an optimization problem to obtain optimal TLSs subject to any prescribed loading conditions. Secondly, a postulated energy bound is established for identifying optimal anisotropic TLSs. Thirdly, we identify four categories of optimal TLSs and discuss their geometrical advantages and mechanical performance. Finally, stiffness optimality of the identified TLSs and stiffness inferiority of the optimal TLSs are verified by numerical experiments.

2. Method

2.1. Design model

This study concerns TLSs formed by \( m \) distinct sets of continuous bars, each individual set involving infinite number of parallel equidistant bars, as illustrated in Fig. 1. In the low volume fraction limit, nodal geometries and locations have marginal effects on the macroscopic properties of the TLS [26]. In this regard, the elastic stiffness matrix \( D \) of a TLS can be estimated by directly adding up stiffness matrices of each individual bar set [17,18]

\[
D = \sum_{i=1}^{m} a_i T_i^T D_0 T_i
\]

(1)

where \( T_i = T(p_i) \) is the 3D rotation matrix; \( p_i \) and \( a_i \) indicate the normalized pointing direction and relative cross-sectional area of the \( i \)-th bar set, respectively; and \( D_0 \) is the stiffness matrix for a single uniaxial bar set.

The optimization problem for identifying optimal TLSs with maximum stiffness is formulated by

\[
\begin{align*}
\text{Minimize} & \quad J = \sum_{k=1}^{N_e} w_k \sigma_k^T C(\chi) \sigma_k - \gamma \| a \|
\\
\text{Subject to} & \quad \rho \leq \bar{\rho}
\\
& \quad \chi \leq \chi \leq \bar{\chi}
\end{align*}
\]

(2)

Here, \( J \) is the weighted complementary energy subject to \( N_e \) prescribed stresses \( \sigma_k \), \( w_k \) the weighting factor and \( C = D^{-1} \) the effective compliance tensor. The regularization term \( \gamma \| a \| \) penalizes duplicated bars in the final design with \( \gamma \geq 0 \) being the regularization factor and \( \| - \| \) denoting the \( L^2 \)-norm of the area vector \( a = [a_i] \). The design variable vector \( \chi \) indicates geometric parameters representing the TLSs, constrained by \( [\chi, \bar{\chi}] \) (their exact definitions for specific design problems are given in Section 2.4). A volume fraction constraint is imposed to evaluate optimized energies from different candidates at the same volume fraction level \( \bar{\rho} \). Because \( a_i \) actually reveals the relative volume fraction of each bar set, the total volume fraction \( \rho \) of a TLS can be obtained by summing up each \( a_i \), i.e.

\[
\rho = \sum_{i=1}^{m} a_i
\]

(3)

2.2. Energy bound for anisotropic TLSs

To our best knowledge, no tight theoretical formulation has been set up for identifying optimal anisotropic TLSs. Therefore, a postulated reference bound is used in this study. Analogously to rank-\( n \) laminate cases [10,18], the postulated bound can be established as the optimal energies obtained by optimizations of TLSs involving finite sets of straight bars. For single loading cases, the optimal stiffness is attained by orientating three bar sets in the principal strain (or stress) directions [27]; and for multi-loading cases, the orientations and areas of at least 6 bar sets are freely optimized by running optimization (2). In order to ensure convergence to true optima, we performed optimizations by varying bar numbers from 6 to 10 and chose the minimum value among all the solutions as the reference energy. It is worth noting that the obtained TLSs using this free optimization strategy could typically encounter connectivity issues. Hence, the obtained energy values only make theoretical sense for evaluating the performance of our identified TLSs.

2.3. Identified optimal TLSs

The key contribution of our study is to identify four categories of optimal TLSs with simple connected geometries that are able to attain maximum stiffness subject to any loading conditions. These TLSs can be represented by periodic parallelepiped unit cells, and they are built by joining cell vertices to form edge, face and body bars in various combinations. All the bars are continuously extended when periodically repeating the unit cell over space. Note that multiple parallel bars belong to an identical bar set, and thus from a macroscopic view, each TLS is comprised of only 7 periodic sets of bars. The identified TLSs in cubic unit cells are illustrated in Fig. 1(a) (labelled \( L_7 - I \) to \( L_7 - IV \)).

The reasons to use 7 sets of well-connected bars are twofold. On one hand, at least 7 sets of bars are needed to produce optimal isotropic TLSs (using 3 edge and 4 body bar sets) [19,21]; and on the other hand, making use of 7 sets of bars turn out to provide near-optimal stiffness based on extensive numerical experiments. There must exist other optimal TLSs with more bar sets, such as the intuitive TLS with 13 sets of bars shown in Fig. 1(b) (labelled \( L_13 \)). However, the TLSs consisting of 7 bar sets are preferable as they have the simplest geometries with the least number of necessary bar sets and can provide the same maximum stiffness as the \( L_13 \).

Because of their distinct geometries, the four \( L_7 \) TLSs cover different mechanical properties. Here, \( L_7 - I \) and \( L_7 - II \) may behave rigidly in different directions due to their triangular-type rigid frames on various crossed planes, and therefore they may produce optimal rigid TLSs. Conversely, the other two TLSs may be preferred to produce rigidity only in certain directions but may be compliant in other directions. Numerical results (see Section 3.1) show that the suggested four \( L_7 \) TLSs together are able to attain near-optimal stiffness subject to any loading conditions, although there exist many other \( L_7 \) TLSs formed by various combinations of bar sets.

2.4. Numerical implementation

Geometries of the optimal TLSs are determined by optimizing three groups of design variables (see Fig. 2). The first group is referring to the relative areas of the bar sets and has \( m = 7 \) variables for the \( L_7 \) TLSs. The other two groups of variables are used to determine the shapes of the parallelepiped unit cells,
referring to the orientation angles $\theta_j$ and the length ratios $s_k$ of the cell edges. As the orientation of each cell edge is characterized by two angles, the optimization totally demands six $\theta_j$. Moreover, two $s_k$ are used to indicate length ratios of three edges. Orientations of all the other bar sets with the predefined connections can be calculated in terms of $\theta_j$ and $s_k$. Using these variables, the properties of the TLS are estimated by Eq. (1). To this end, the design variable vector in Eq. (2) is written as

$$\chi = [a_i, \theta_j, s_k] \quad (i = 1, 2, \ldots, m; \ j = 1, 2, \ldots, 6; \ k = 1, 2) \quad (4)$$

restricted by $0 \leq a_i \leq \pi$, $-2\pi \leq \theta_j \leq 2\pi$ and $0.1 \leq s_k \leq 3.$

To solve the optimization problems, we first employ a gradient-based interior-point algorithm to get close-to-optimal solutions and then use a global pattern search algorithm to further decrease the optimized energy values. All the optimized TLSs satisfy the optimality conditions, i.e. the sum of the strain energy density in each bar set over all loading cases is equal to a constant. Also, the gradients with respect to the angle and length ratio variables are zeros. Due to symmetries both in the topologies and in the loadings, we provide additional observations regarding optimal strain, stress and strain energy densities on each bar set for the optimal isotropic TLSs in Appendix A.

3. Results and discussions

3.1. Stiffness optimality

Near-optimality of the identified TLSs is verified by optimization for thousands of random loading conditions, where the number of stress cases $N_s$ is varied from 2 to 7. For each $N_s$, 500 loading conditions are randomly generated$^3$ with $w_s = 1/N_s$. For each loading condition, the four identified L7 TLSs are tested using a fixed $\gamma \in [0, 0.01]$, and for each TLS, hundreds of random initial guesses are used to ensure the best solutions. The procedure is terminated if maximum changes in both objective functions and design variables are smaller than $10^{-15}$.

The optimization results are shown in Fig. 3. For a specific loading condition, the performance of the optimized TLS is measured by the relative energy difference $d = (J - J_\gamma)/J$, where $J$ and $J_\gamma$ are the optimized energy of the best-performing TLS among four L7 TLSs and the reference energy by the free design strategy in Section 2.2, respectively. In each group of $N_s$, the worst case among 500 solutions has the difference $d_{\text{max}}$, as stated in Table 1. In addition, the minimum eigenvalue $\epsilon_{\text{min}}$ of $D/\rho$ is used to examine if an optimal TLS behaves rigidity only in specific directions or in all directions. This measure is normalized by the minimum eigenvalue for the optimal isotropic TLS, which is $\epsilon_{\text{min}} = 1/15$ (obtained by Eq. (A.1) in Appendix A), and the normalized eigenvalue is denoted by $\epsilon_{\text{min}} = \epsilon_{\text{min}}/\epsilon_{\text{iso}}$.

From Fig. 3, it is seen that the optimal TLSs can be divided into compliant TLSs (left part) and rigid TLSs (right part). Compliant TLSs can be observed for $N_s \leq 5$, and have close-to-zero $\epsilon_{\text{min}}$. They offer stiffness only in specific directions and perform compliant in other directions. For $N_s = 2$ and 3, the maximum stiffness can be obtained by using any one of the four L7 TLSs. The small but non-zero numerical errors of $10^{-6}$ and $10^{-7}$ arise from the fact that the compliant TLSs are highly sensitive to small misalignments of geometric parameters. For $N_s = 4$, the optimal TLSs offer more complex directional stiffness as required by the increased number of stress cases, hence the optimal performance is more challenging to achieve and a bigger $d_{\text{max}}$ is found. This error comes from the fixed configurations of the identified TLSs and the strong restriction implicitly introduced by periodicity. In that case, L7 − III and L7 − IV dominate the best solutions as they can behave more compliant than the two other TLSs. Nevertheless, the claimed $d_{\text{max}}$ is below 0.1% for all these cases, which is fully acceptable for practical means.

In the right part of Fig. 3, the optimal TLSs show rigidity with moderate $\epsilon_{\text{min}}$, which are found in parts of the results for $N_s = 5$ and in all the results for $N_s = 6$ and 7. Here, L7 − I and L7 − II are preferred to attain maximum stiffness with negligible $d_{\text{max}}$ (machine precision). Particularly, L7 − I itself provides maximum stiffness in almost all the tests for $\epsilon_{\text{min}} > 0.2$. This is mainly attributed to its configuration fully formed by tetrahedron frames. Furthermore, the identified L7 TLSs can together obtain stiffness as close to those using L13 (see Table 1).

As a special but important subcase, optimal isotropic TLSs (see the red solid triangle in Fig. 3) are obtained by applying a specific multi-loading condition (see Eq. (A.3) or (A.4) in Appendix A). Besides being able to reproduce the stiffest TLSs proposed in [19], the optimization also yields many other optimal isotropic TLSs. Two examples are presented in Appendix A (see Fig. A.1), which are formed by a number of rigid triangular frames and have the maximum Young’s modulus matching the theoretical value. To our knowledge, these two TLSs have not appeared in the literature before.
Table 1

| $N_\sigma$ | Compliant TLSs | Rigid TLSs |
| --- | --- | --- |
| 2 | $d_\text{max}^{\text{TLS1}}$ | $7.59 \times 10^{-14}$ | $2.34 \times 10^{-14}$ | $2.54 \times 10^{-14}$ |
| 3 | $3.99 \times 10^{-7}$ | $2.34 \times 10^{-14}$ | $5.49 \times 10^{-14}$ | $5.04 \times 10^{-14}$ |
| 4 | $1.09 \times 10^{-3}$ | $5.49 \times 10^{-14}$ | $3.45 \times 10^{-14}$ | $2.54 \times 10^{-14}$ |
| 5 | $6.71 \times 10^{-4}$ | $2.34 \times 10^{-14}$ | $5.04 \times 10^{-14}$ | $2.54 \times 10^{-14}$ |
| 6 | $7.59 \times 10^{-14}$ | $2.34 \times 10^{-14}$ | $5.04 \times 10^{-14}$ | $2.54 \times 10^{-14}$ |
| 7 | $2.54 \times 10^{-14}$ | $5.04 \times 10^{-14}$ | $2.54 \times 10^{-14}$ | $2.54 \times 10^{-14}$ |

Fig. 3. Relative difference of the best-performing L7 TLSs for 6 × 500 testing problems. Different colours distinguish various L7 TLSs; different symbols distinguish different load numbers $N_\sigma$; and different scale factors and ranges are separately used for the optimized results of compliant TLSs (left part) and rigid TLSs (right part).

3.2. Stiffness inferiority

The stiffness inferiority of optimal TLSs compared to known optimal PLSs can be evaluated by studying their energy ratios $R = J^{\text{TLS}}/J^{\text{PLS}}$. Here, $J^{\text{TLS}}$ is obtained by using rank-6 laminates (in the low volume fraction limit) [1] and $J^{\text{TLS}}$ corresponds to the postulated energy from the free optimization strategy. The results for single stress cases are also taken into account. In addition, we use an index $M = |e|/\|e\| - 1$ to measure the degree of anisotropy of each optimal TLS, $|e|$ and $\|e\|$ for $L_1$- and $L_2$-norms of eigenvalue vector $e$, indicating averaged and total stiffness, respectively [28]. It can be checked that $M$ is bounded on the interval $[0, 1]$, where $M = 1$ corresponds to an optimal isotropic TLS and $M = 0$ stands for a single bar member. Note that $M$ is invariant to orientations of TLSs. Also note that $M$ can have various values for a specific loading condition since different TLSs might yield the same energy. For the comparisons, the base material has assumed Poisson’s ratio of $\nu_0 = 1/3$, but conclusions do not change for other Poisson’s ratios.

The obtained $M - R$ chart is plotted in Fig. 4. The results clearly verify that the optimal TLSs always have lower stiffness than the optimal PLSs. More importantly, one can observe that the stiffness inferiority of any optimal anisotropic TLS is well bounded. Therein, minimum $R = 1$ appears for a single uniaxial stress, where the two microstructures have a single bar and plate set in the loading direction, indicating no stiffness reduction by using TLSs; and maximum $R = 3$ (function of $\nu_0$ as seen in Appendix A) is reached if all the plates in the PLSs are fully loaded, e.g. for a single loading case with three equal principal stresses, where the TLSs will have the maximum stiffness inferiority. The values of $1 < R < 3$ (see coloured dots) means that the TLSs can have intermediate inferior stiffness, if there exist plates in the PLSs not fully loaded. Especially, for single stress cases, $R$ can be (analytically) evaluated by using three orthonormal bar and plate sets (see grey regions in Fig. 4 and discussions in Supplementary Material).

Fig. 4. Energy ratios between optimal TLSs and PLSs, grey regions for single stress cases with 3 bar and plate sets and coloured dots for various groups of stress case numbers.

Note that the optimal isotropic TLSs have the maximum inferior stiffness, with only $1/3$ Young’s modulus of the optimal isotropic rank-6 laminates, which matches the theoretical estimation in [17] (also see Appendix A). Finally, increasing the number of random stress cases will make optimal TLSs behave more isotropically.

4. Summary

We have identified four optimal TLS categories with simple geometries attaining maximum stiffness for arbitrary loading scenarios. Their near-optimal stiffness (below 0.1% of bounds) are verified through thousands of tests with various number of stress cases. In addition, we conclude that the optimal TLSs always have inferior stiffness within the range of 1 and 3 compared to PLSs. At this stage, one can readily use these optimal TLSs in stiffness-preferred multi-scale design problems [29].
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Appendix A. Optimal isotropy

Analytical properties

In the low volume fraction, Christensen [17] analytically studied the elastic properties of optimal isotropic microstructures. For optimal TLSs, it has

\[
\frac{E^*}{\rho} = \frac{1}{6}, \quad \nu^* = \frac{1}{4} \tag{A.1}
\]

with \( E^* \) and \( \nu^* \) the relative Young’s modulus and effective Poisson’s ratio of the optimal TLSs, respectively.

For optimal PLSs, it has

\[
\frac{E^*}{\rho} = \frac{2 (7 - 5 \nu_0)}{3 (1 - \nu_0) (9 + 5 \nu_0)}, \quad \nu^* = \frac{1 + 5 \nu_0}{9 + 5 \nu_0} \tag{A.2}
\]

with \( \nu_0 \) denoting the Poisson’s ratio of the base material. The stiffness and compliance matrices can be calculated accordingly.

Loading conditions

Among many others, two loading conditions are used to obtain isotropic TLSs. The first condition applies 6 uniaxial stress cases, whose loading directions (denoted by \( \mathbf{n} \)) match the normal directions of 6 ranks in the optimal isotropic rank-6 laminates [11], stated by

\[
\mathbf{n}_1 = [0, 0, 1], \quad \mathbf{n}_k = [\sin 2 \beta \cos 2k \alpha, \sin 2 \beta \sin 2k \alpha, \cos 2 \beta] \quad (k = 2, \ldots , 6) \tag{A.3}
\]

with \( \alpha = \pi / 5 \) and \( \cos 2 \beta = 1/\sqrt{5} \). The weighting factors are set by \( w_k = 1/6 \) for \( k = 1, 2, \ldots , 6 \).

The second condition involves 7 uniaxial stress cases, whose loading directions are

\[
\mathbf{n}_i = \begin{bmatrix} 1 & 0 & 0 & q & -q & q \\ 0 & 1 & 0 & q & -q & q \\ 0 & 0 & 1 & q & -q & q \end{bmatrix}^T, \quad w_i = \frac{8}{9}, \quad (i = 1, 2, 3; \ j = 4, 5, 6, 7) \tag{A.4}
\]

with \( q = \sqrt{3} / 3 \).

The components of each stress case can then be calculated by rotating a uniaxial stress vector \( \mathbf{\sigma}_q = [1, 0, 0, 0, 0, 0]^T \) to the above directions through \( \mathbf{\sigma}_i = \mathbf{R}_i \mathbf{\sigma}_q \mathbf{R}_i^T \), with \( \mathbf{R}_i = \mathbf{R}(\mathbf{n}_i) \) being the rotation matrix.

Optimal isotropic TLSs

Two new optimal isotropic TLSs are presented in Fig. A.1, together with the corresponding geometric parameters. Both the obtained TLSs have only two different bar areas. Although running optimization (2) can yield other optimal isotropic TLSs, they may have complex geometries with more than two different bar areas. Specifically note that the optimal isotropic TLSs have unit axial strain for all the involved bar sets, i.e., \( \epsilon_i = \sum_{k=1}^{N_k} w_k \mathbf{p}_k^T \left( \mathbf{C} \right) \mathbf{p}_k = 1 \). Furthermore, because of the same constituent material, the bar sets also have a constant strain stress, and therefore, they have an equal strain energy density, hence meeting the optimality conditions.

Appendix B. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.eml.2019.03.004.

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