GEOMETRY OF QUASI-FREE STATES OF CCR ALGEBRAS

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[2000]46L60, 46L51

Abstract. Geometric positions of square roots of quasi-free states of CCR algebras are investigated together with an explicit formula for transition amplitudes among them.

Introduction

In the operator algebraic approach to quantum physics, fundamental are the algebras associated to canonical (anti)commutation relations, which are known to be CCR or CAR algebras respectively. These are interesting from mathematical viewpoints as well and have their own origins inside mathematics in connection with symplectic geometry or Clifford algebras.

These quantum algebras are tied up with real experiments through probabilistic predictions, which can be described as transition probability between quantum states. In terms of quantum algebras, it is a standard recipe to recognize quantum states as normalized positive linear functionals. For commutative algebras, the notion of state in this sense is more or less equivalent to that of probability measure.

Most naive quantum states are given by (normalized) vectors on which quantum algebras operate linearly and the transition probability between such states is postulated to be the squared modulus of their inner product.

The extension of the notion of transition probability to general states are considered by several researchers, among them, let us notice the one introduced by A. Uhlmann (23, 1), which goes back to Kakutani-Bures analysis of measure-theoretical equivalence of product states. Viewing the observation in [16], Uhlmann’s definition of transition probability takes the following form in the setting of W*-algebras: Let $\varphi$ and $\psi$ be normal states on a W*-algebra $M$ with $\varphi^{1/2}$ and $\psi^{1/2}$ the associated representing vectors in the positive cone of the standard Hilbert space $L^2(M)$. Then their (non-commutative) product $\varphi^{1/2}\psi^{1/2}$ is well-defined as a normal linear functional of $M$. If we denote by $|\varphi^{1/2}\psi^{1/2}|$ the positive part of $\varphi^{1/2}\psi^{1/2}$ in the polar decomposition, then the probability is equal to the value at the unit element $1 \in M$ of the positive linear functional $|\varphi^{1/2}\psi^{1/2}|$. Instead of taking positive parts, if simply evaluated the (not necessarily positive) form $\varphi^{1/2}\psi^{1/2}$, then we obtain another candidate for transition probability between states. Clearly this new transition probability (we call it transition amplitude simply because of its appearance) is not greater than the Uhlmann’s one but both of them are reduced to
the physically well-established one when restricted to vector states of full operator algebras.

Returning to CCR or CAR algebras, among states, most studied are so-called quasifree states, which are free in the sense that they are determined in a canonical fashion and parametrized by their covariance operators (or covariance forms). Let us write \( \varphi_S \) to stand for the quasifree state of a covariance operator \( S \).

In the case of CAR algebras, we have the following formula due to H. Araki (2)

\[
\psi_S^1/2|\psi_T^1/2) = (\text{det}(1-(P-Q)^2))^{1/8},
\]

where the dilated projections \( P \) and \( Q \) are defined by

\[
P = \left( \frac{S}{\sqrt{S(1-S)}} \frac{\sqrt{S(1-S)}}{1-S} \right), \quad Q = \left( \frac{T}{\sqrt{T(1-T)}} \frac{\sqrt{T(1-T)}}{1-T} \right).
\]

An analogous expression is possible to write down for CCR algebras as well under the condition that the relevant symplectic form is non-degenerate.

Our main result in this paper is to establish a similar but undilated formula (see Theorem 3.3 and Theorem 6.1 below) in a quite general form, which reveals a peculiar feature of quasifree states: To each covariance form \( S \), we can associate a gaussian measure \( \mu_S \) in such a way that \( \psi_S^1/2|\psi_T^1/2) \) is equal to the Hellinger integral between \( \mu_S \) and \( \mu_T \). In other words, geometric positions of vectors \( \psi_S^1/2 \) are exactly those of completely classical objects \( \mu_S^{1/2} \).

1. CCR Algebras

Let \( V \) be a real vector space and \( \sigma : V \times V \rightarrow \mathbb{R} \) be an alternating form. The couple \( (V,\sigma) \) is called a presymplectic vector space. When \( \sigma \) is non-degenerate, it is called a symplectic vector space.

Given presymplectic vector spaces \( (V,\sigma) \) and \( (V',\sigma') \), a linear map \( \phi : V \rightarrow V' \) is said to be presymplectic if \( \sigma'(\phi x,\phi y) = \sigma(x,y) \) for \( x,y \in V \). When \( (V',\sigma') = (V,\sigma) \) and \( \phi \) is an isomorphism, it is called a presymplectic automorphism of \( (V,\sigma) \).

The group of presymplectic automorphisms of \( (V,\sigma) \) is denoted by \( \text{Aut}(V,\sigma) \). When \( (V,\sigma) \) is symplectic, all these maps are said to be symplectic.

It is often useful to work with the complexification \( V^C \), which is furnished with the real structure \( (x+iy)^* = x-iy \) so that \( V \) is recovered as the real part of \( V^C \).

Given a sesquilinear form \( S \) on \( V^C \), we set \( \overline{S}(z,w) = \overline{S}(z^*,w^*) \). Likewise, given a \( \mathbb{C} \)-linear transformation \( \phi : V^C \rightarrow V^C \), we set \( \overline{\phi x} = (\phi x^*)^* \).

If the presymplectic form \( \sigma \) is bilinearly extended to \( V^C \), then \( h : V^C \times V^C \ni (z,w) \mapsto i\sigma(z^*,w) \) defines a hermitian form satisfying \( \overline{h} = -h \). Conversely, a hermitian form \( h \) on \( V^C \) satisfying \( \overline{h} = -h \) comes from a presymplectic form \( \sigma(x,y) = -ih(x,y) \) \( (x,y \in V) \).

Associated to a presymplectic vector space \( (V,\sigma) \), we introduce the Weyl form of CCR algebra as a unital \(*\)-algebra \( C(V,\sigma) \) universally generated by the symbols \( \{e^{ix};x \in V \} \) subject to the relations

\[
(e^{ix})^* = e^{-ix}, \quad e^{ix} e^{iy} = e^{-i\sigma(x,y)/2} e^{i(x+y)}, \quad x, y \in V,
\]
which are the exponentiated form of the canonical commutation relations. Note that $e^{i0}$ (the zero in the exponential represents the zero vector in $V$) is the unit element in the algebra.

Since the Weyl form of CCR algebra is generated by unitaries $\{e^{ix}\}$, any $*$-representation is automatically bounded. The operator-norm completion with respect to all $*$-representations is then a $C^*$-algebra $C^*(V,\sigma)$, which is referred to as the CCR $C^*$-algebra. From the very definition, there is a one-to-one correspondence between $*$-representations of the Weyl form of CCR algebra on a Hilbert space $\mathcal{H}$ and $*$-representations of $C^*(V,\sigma)$ on $\mathcal{H}$. There is also a one-to-one correspondence between states on $C^*(V,\sigma)$ and states on the Weyl form of CCR algebra.

Owing to the existence of Schrödinger type representations, we know that the family $\{e^{ix}\}_{x \in V}$ is linearly independent in $C^*(V,\sigma)$.

If we are given a presymplectic map $\phi : V \to V'$, it induces a $*$-homomorphism $C^*(V,\sigma) \to C^*(V',\sigma')$ by universality. In particular, $\text{Aut}(V,\sigma)$ is imbedded into $\text{Aut}(C^*(V,\sigma))$.

A $*$-representation $\pi : C^*(V,\sigma) \to \mathcal{L}(\mathcal{H})$ of a CCR C*-algebra on a Hilbert space $H$ is said to be finite-dimensionally continuous or regular if for any $\xi, \eta \in H$ and for any finite-dimensional subspace $W \subset V$, $(\xi|\pi(e^{ix})\eta)$ is a continuous function of $x \in W$.

2. Quasifree States

2.1. Polarizations.

**Definition 2.1.** Let $(V,\sigma)$ be a presymplectic vector space. A positive form $S$ on $V^\mathbb{C}$ is called a **polarization** of $(V,\sigma)$ if it satisfies

$$S(x,y) - \overline{S}(x,y) = i\sigma(x^*,y) \quad \text{for } x,y \in V^\mathbb{C}.$$ 

Let $\text{Pol}(V,\sigma)$ be the set of polarizations of $(V,\sigma)$, which is a convex set with an obvious action of $\text{Aut}(V,\sigma)$.

**Example 2.2.** For $V = \mathbb{R}^2$, possible presymplectic forms are (redundantly) parametrized up to choice of bases by the matrix

$$\begin{pmatrix} 0 & 2\mu \\ -2\mu & 0 \end{pmatrix}, \quad \mu \in \mathbb{R}.$$ 

Then $S \in \text{Pol}(V,\sigma)$ is described by a matrix of the form

$$\begin{pmatrix} z + x & y + i\mu \\ y - i\mu & z - x \end{pmatrix}, \quad x^2 + y^2 + \mu^2 \leq z^2, z \geq 0,$$

whence $\text{Pol}(V,\sigma)$ is identified with the region bounded by a half of a two-sheeted hyperboloid ($\mu \neq 0$) or a cone ($\mu = 0$). Since

$$\text{Aut}(V,\sigma) = \begin{cases} \text{GL}(2,\mathbb{R}) & \text{if } \mu = 0, \\ \text{SL}(2,\mathbb{R}) & \text{otherwise,} \end{cases}$$

orbits in $\text{Pol}(V,\sigma)$ constitute two or three parts according to $\mu \neq 0$ or $\mu = 0$.

**Lemma 2.3** ([4, Lemma 3.3]). Given a polarization $S$ of $(V,\sigma)$, set $(x,y)_S = S(x,y) + S(y^*,x^*)$ for $x,y \in V^\mathbb{C}$. Then $(\cdot, \cdot)_S$ is a positive form on $V^\mathbb{C}$ satisfying (i) $(y^*,x^*)_S = (x,y)_S$ and (ii) $|\sigma(x^*,y)|^2 \leq (x,x)_S (y,y)_S$ for $x,y \in V^\mathbb{C}$. 

Conversely, any positive form fulfilling these conditions comes from a polarization.

**Corollary 2.4.** Let $V_S$ be the real Hilbert space associated to the positive form $S + \overline{S}$, i.e., $V_S$ is the completion of $V/(V \cap \ker(S + \overline{S}))$ with respect to the induced inner product. Then $\sigma$ is continuously extended to an alternating form $\sigma_S$ of $V_S$ so that the natural map $V \to V_S$ is presymplectic.

### 2.2. Quasifree States.

**Lemma 2.5** (Hadamard-Schur product). Let $(a_{jk})$ and $(b_{jk})$ be positive semidefinite matrices of size $n$. Then the matrix with entries of component-wise multiplication $(a_{jk}b_{jk})_{1 \leq j,k \leq n}$ is positive semidefinite.

**Proof.** Express positive matrices as convex combinations of positive matrices of rank one. \hfill $\square$

**Corollary 2.6.** For a positive semidefinite matrix $(a_{jk})$, the matrix $(e^{ia_{jk}})$ is also positive semidefinite.

Given a polarization $S$, the following formula defines a state (called a quasifree state) on the CCR $C^*$-algebra $C^*(V,\sigma)$.

$$
\varphi_S(e^{ix}) = e^{-S(x,x)/2}, \quad x \in V.
$$

Since $\{e^{ix}\}_{x \in V}$ are linearly independent in $C^*(V,\sigma)$, the linear functional $\varphi_S$ is well-defined on a dense subalgebra. To check the positivity of $\varphi_S$, let $\{x_j\}_{1 \leq j \leq n}$ be a finite family of vectors in $V$ and let $z_j \in \mathbb{C}$ $(1 \leq j \leq n)$. Then

$$
\varphi_S\left(\sum_j z_j e^{ix_j} \right)^* \left(\sum_k z_k e^{ix_k}\right) = \sum_{j,k} \overline{z_j} z_k e^{-S(x_j-x_k,x_j-x_k)/2 + i\sigma(x_j,x_k)/2} = \sum_{j,k} e^{-S(x_j,x_j)/2} \overline{z_j} z_k e^{S(x_j,x_k)}
$$

is non-negative by the above corollary.

Given a quasifree state $\varphi_S$, let $\pi_S$ be the associated GNS-representation. Then $\pi_S(e^{ix})$ is strongly continuous in $x \in V$ with respect to the topology induced from the inner product $(\cdot, \cdot)_S$ because both of $S$ and $\sigma$ are continuous relative to $(\cdot, \cdot)_S$.

Given a $C^*$-algebra $A$, let $L^2(A)$ be the standard Hilbert space of the W*-algebra $A^{**}$. We identify each $\varphi \in A_+^*$ with a normal functional of $A^{**}$ and use the notation $\varphi^{1/2}$ to stand for the associated representing vector in the positive cone of $L^2(A^{**})$.

**Proposition 2.7** ([27]). Let $\phi : (V,\sigma) \to (V',\sigma')$ be a presymplectic map with the induced *-homomorphism $\Phi : C^*(V,\sigma) \to C^*(V',\sigma')$. Let $S,T \in \text{Pol}(V',\sigma')$ and assume that $\phi(V)$ is dense in $V'$ with respect to both of $S + \overline{S}$ and $T + \overline{T}$. Then

$$
((\varphi_S \circ \phi)^{1/2}|(\varphi_T \circ \Phi)^{1/2}) = (\varphi_{S \circ \phi}^{1/2}|\varphi_{T \circ \phi}^{1/2}).
$$

If $\phi \in \text{Aut}(V,\sigma)$, then $\Phi \in \text{Aut}(C^*(V,\sigma))$ and

$$
\varphi_S^{1/2} \mapsto \varphi_{S \circ \phi}^{1/2}
$$

for various $S \in \text{Pol}(V,\sigma)$ gives rise to a unitary operator on the subspace of $L^2(C^*(V,\sigma))$ spanned by $\{\varphi_S^{1/2}; S \in \text{Pol}(V,\sigma)\}$.
Let $S$ be a polarization of $(V, \sigma)$ and $\mathcal{S}$ be the positive operator on $V^\mathcal{S}_S$ defined by $S(x, y) = \langle x, S y \rangle_S$, where $S$ is identified with the induced polarization of $(V_S, \sigma_S)$. We say that $S$ is in the boundary unless $\ker \mathcal{S} = \{0\} = \ker (1 - \mathcal{S})$. By the theory of Fock representations and the method of doubling or purification (cf. \S 5 below), we know that $\varphi_S$ is a pure state if and only if $S$ is extremal in $\text{Pol}(V, \sigma)$, i.e., $S$ is a projection.

If $S$ is not in the boundary, we can define a one-parameter group of unitaries \{e^{itH}\} on $V^\mathcal{S}_S$ by $e^{itH} = S^{-it}(1 - S)^{it}$. Note that the infinitesimal generator $H$ (which is a self-adjoint operator) satisfies $H = -\mathcal{H}$. In accordance with the functional calculus notation for positive forms, the operator $e^{itH}$ is also denoted by $S^{-it}\mathcal{S}$. 

**Proposition 2.8 ([\text{18} \S 3])**. Assume that $S$ is not in the boundary. Then

(i) The family \{e^{itH} = S^{-it}\mathcal{S}^{it}\}_{t \in \mathbb{R}} is, when restricted to $V_S$, a one-parameter group of presymplectic transformations of $(V_S, \sigma_S)$ such that $S(e^{itH} x, e^{itH} y) = S(x, y)$ for $x, y \in V^\mathcal{S}_S$.

(ii) The associated automorphism group of $C^*(V_S, \sigma_S)$ satisfies the $\text{KMS}$-condition for the quasi-free state $\varphi_S$:

$$\varphi_S^* e^{ix} \varphi_S^{-it} = \exp(itH x)$$

for $x \in V_S$ and $t \in \mathbb{R}$. (See [\text{24}] for the meaning of notation $\varphi_S^t$.)

### 2.3. Central Decomoposition of Quasi-Free States

We assume that $V$ is endowed with a topology of Hilbert space ($V$ is then said to be Hilbertian). A polarization $S$ of $(V, \sigma)$ is said to be admissible if $S + \mathcal{S}$ gives the topology of $V$. Let $V_0 = \ker \sigma$ be the central part of $V$ and $S_0$ be the restriction of $S$ to $V^\mathcal{S}_S$. Let $V_1 = V \ominus S_0$ be the $(S + \mathcal{S})$-orthogonal complement of $V_0$.

We choose an auxiliary linear map of Hilbert-Schmidt class $\Theta : L \to V_0$ with $\ker \Theta = \{0\}$ and having a dense range ($L$ being a real Hilbert space). Then we can realize the state $\varphi_{S_0}$ by a gaussian measure $\nu_{S_0}$ on the topological dual space $\Omega = (\Theta L)^*$, where $\Theta L$ is furnished with the topology induced from the norm $\|\nu_0\|_\Theta = \|\Theta^{-1}\nu_0\|$:

$$\int_{\Omega} e^{i\omega(x_0)} \nu_{S_0}(d\omega) = e^{-S(x_0)/2}, \quad x_0 \in \Theta L.$$ 

and

$$C^*(V_0)_{S_0}^{1/2} = L^2(\Omega, \nu) \quad \text{by} \quad e^{ix_0} \varphi_{S_0}^{1/2} \longleftrightarrow e^{i\omega(x_0)} \sqrt{\nu_{S_0}(d\omega)}.$$ 

**Remark 1.** For an infinite-dimensional $V_0$, $\nu_{S_0}(V_0) = 0$, where $V_0$ is imbedded into the space $\Omega = (\Theta L)^*$ through the inner product $S_0$.

We now introduce a $C^*$-algebra $C^*(V/\omega, \sigma)$ for each $\omega \in \Omega$, which is the quotient of the CCR $C^*$-algebra $C^*(V_1 + \Theta L, \sigma)$ by imposing the condition $e^{ix_0} = e^{i\omega(x_0)}1$ ($x_0 \in \Theta L$). Namely, if we denote the quotient image of $e^{ix}$ by $e^x_\omega$, the $C^*$-algebra $C^*(V/\omega, \sigma)$ is generated by unitaries \{e^x_\omega : x \in V_1 + \Theta L\} subject to the relations

$$e^x_\omega e^y_\omega = e^{-i\sigma(x,y)/2} e^{i(x+y)}_\omega, \quad e^x_\omega = e^{i\omega(x_0)}1$$

for $x, y \in V_1 + \Theta L$ and $x_0 \in \Theta L$. Notice here that $C^*(V/\omega, \sigma)$ depends not only on $\Theta$ but also on $S$ through the choice $V_1 + \Theta L \subset V$. Clearly

$$C^*(V/\omega, \sigma) \ni e^{ix} \mapsto e^{ix_1} \in C^*(V_1, \sigma_1), \quad x_1 \in V_1$$
gives an isomorphism of C*-algebras with \( \sigma_1 = \sigma|_{V_1 \times V_1} \) the induced symplectic form on \( V_1 \), whence we can define a state \( \varphi_{S,\omega} \) of \( C^*(V/\omega, \sigma) \) so that it corresponds to the quasifree state \( \varphi_{S_1} \) of \( C^*(V_1, \sigma_1) \) via the above isomorphism:

\[
\varphi_{S,\omega}(e^{ix}) = e^{-S(x)/2} e^{ix_0} \quad \text{for } x = x_1 + x_0 \in V_1 + \Theta L.
\]

**Proposition 2.9.** Given a polarization \( S \) of the presymplectic vector space \( (V, \sigma) \) with \( \Theta : L \to V_0 \) as above, we have the following unitary isomorphism

\[
C^*(V, \sigma) \varphi_{S_1}^{1/2} C^*(V, \sigma) \cong \int_{\Omega} \varphi_{S_0}(d\omega) C^*(V/\omega, \sigma) \varphi_{S,\omega}^{1/2} C^*(V/\omega, \sigma)
\]

\[
e^{-iy} \varphi_{S_1}^{1/2} \varphi_{S,\omega}^{1/2} \quad \text{with } \varphi_{S,\omega}^{1/2} = e^{-H/2}.
\]

Proof. Step 1. By splitting out the pure state part (corresponding to the spectral range \( \{0, 1\} \)) from \( S \), we may assume that \( S \) is not in the boundary (the isomorphism in question turns out to be the identity on the factor arising from pure state part).

Step 2. If \( y \in V_1 + \Theta L \) is an entirely analytic vector for the self-adjoint operator \( H = \log((1 - S)/S) \),

\[
e^{-iy} \varphi_{S,\omega}^{1/2} = e^{-S(-1/2) S^{1/2} y} \varphi_{S,\omega}^{1/2} \quad \text{with } S^{-1/2} S^{1/2} = e^{-H/2}.
\]

Note here that \( e^{iy} \varphi_{S,\omega}^{1/2} \) is well-defined as an entirely analytic function of \( z \in (V_1 + \Theta L) \) (cf. Lemma 5.9 below).

By the isomorphism \( C^*(V_1, \sigma_1) \ni e^{ix_1} \mapsto e^{ix}, e^{-S(x)/2} \), \( \varphi_{S},\omega \) corresponds to \( \varphi_{S_1} \), whence

\[
\varphi_{S_1}^{1/2} e^{iy} = e^{iS_1^{-1/2} S^{1/2} y} \varphi_{S,\omega}^{1/2}
\]

for \( y_1 \in V \cap V_0 \). Since \( S^{-1/2} S^{1/2} y = y_0 + S_1^{1/2} S^{1/2} y_1 \) for \( y = y_0 + y_1 \) and \( e^{iy_0} \) is in the center, the assertion follows.

Step 3. For \( x \in V_1 + \Theta L \),

\[
(\varphi_{S_1}^{1/2} e^{ix} \varphi_{S,\omega}^{1/2}) = e^{-S(x)/2} e^{-S(x)/2} = \int_{\Omega} \varphi_{S_0}(d\omega) e^{ix_0} e^{-S(x)/2} = \int_{\Omega} \varphi_{S_0}(d\omega) \varphi_{S_0}^{1/2} e^{ix} \varphi_{S_0}^{1/2}.
\]

\[
= \int_{\Omega} \varphi_{S_0}(d\omega) (\varphi_{S,\omega}^{1/2} e^{ix} \varphi_{S,\omega}^{1/2})
\]

Corollary 2.10. Let \( T \) be another admissible polarization of \( (V, \sigma) \) such that \( (V \ominus_T V_0) + \Theta L = (V \ominus_S V_0) + \Theta L \) and \( \nu_T \) is equivalent to \( \nu_S \). Then

\[
(\varphi_{S_1}^{1/2} | \varphi_{T_1}^{1/2}) = \int_{\Omega} \sqrt{\nu_S \nu_T} (d\omega) \left( \varphi_{S_1}^{1/2} | \varphi_{T_1}^{1/2} \right).
\]

3. **Finite-Dimensional Analysis**

Let \( (V, \sigma) \) be a finite-dimensional presymplectic vector space, which is assumed throughout this section unless otherwise stated, and we shall fix a Lebesgue measure on \( V \) once for all.
3.1. Hilbert Algebras. Let \( S(V) \) be the space of rapidly decreasing functions on \( V \). For \( f \in S(V) \), regard the integral

\[
\int_V f(x)e^{ix} \, dx
\]

as defining a virtual element in \( C^*(V,\sigma) \) (more precisely, it belongs to the von Neumann algebra generated by a regular representation), which suggests a \(^*\)-algebra structure in the vector space \( S(V) \):

\[
(f * g)(z) = \int_V f(z')g(z - z')e^{-i\sigma(z',z)/2} \, dz', \quad f^*(x) = \overline{f(-x)}.
\]

It is immediate to check that these operations in fact make \( S(V) \) into a \(^*\)-algebra (denoted by \( \hat{S}(V,\sigma) \)), which turns out to be a Hilbert algebra with respect to the inner product

\[
(f|g) = \int_V \overline{f(x)}g(x) \, dx.
\]

The (associated) trace functional is then defined by

\[
\tau : S(V) \ni f \mapsto f(0) \in \mathbb{C}.
\]

Formally this is equivalent to requiring \( \tau(e^{ix}) = \delta(x) \) for \( x \in V \) (\( \delta(x) \) being the delta function with respect to the preassigned Lebesgue measure). Moreover, the multiplier product by \( e^{iz} \in C^*(V,\sigma) \) leaves \( S(V) \) invariant and we have the multiplier product formula

\[
(e^{iz}f)e^{iy}(z) = f(z-x-y)e^{i(\sigma(x,y)-\sigma(x,z)-\sigma(z,y))/2}
\]

for \( x,y,z \in V \), which follows from the identity

\[
e^{ix} \left( \int_V f(z)e^{iz} \, dz \right) e^{iy} = \int_V f(z-x-y)e^{i(\sigma(x,y)-\sigma(x,z)-\sigma(z,y))/2}e^{iz} \, dz.
\]

In other words, if we denote by \( C^*_{\text{reg}}(V,\sigma) \) the \( C^* \)-closure of \( S(V,\sigma) \), then \( C^*(V,\sigma) \) is identified with a multiplier subalgebra of \( C^*_{\text{reg}}(V,\sigma) \).

Since the quasifree state \( \varphi_S \) gives rise to the functional

\[
\int_V f(x)e^{ix} \, dx \mapsto \int_V f(x)e^{-S(x,x)/2} \, dx,
\]

the associated density operator \( \rho_S \), i.e., \( \rho_S \in S(V,\sigma) \) satisfying \( \tau(\rho_S * f) = \varphi_S(\int dx f(x)e^{ix}) \) for \( f \in S(V) \), is given by the function \( \rho_S(x) = e^{-S(x,x)/2} \) (\( x \in V \)).

To get an expression for the square root of \( \rho_S \) in \( S(V,\sigma) \), we need therefore to seek for a function \( f : V \to \mathbb{C} \) satisfying \( f(-x) = \overline{f(x)} \) and

\[
e^{-S(x,x)/2} = \int_V f(y)f(x-y)e^{i\sigma(x,y)/2} \, dy = (f * f)(x)
\]

for \( x \in V \).

3.2. Non-Degenerate \( \sigma \). From here on \( \sigma \) is assumed to be non-degenerate for the time being. Let \( S \) be the operator representing the polarization \( S \) with respect to the inner product \( \langle \cdot, \cdot \rangle_S \). From the relation \( S + \overline{S} = 1 \), we read off the following spectral property on \( S \): If \( \xi \) is an eigenvector of eigenvalue \( \lambda \), then so is \( \xi^* \) with eigenvalue replaced by \( 1-\lambda \), i.e., \( S\xi^* = (1-\lambda)\xi^* \). To utilize this property, we assume
$0 \leq \lambda < 1/2$, normalize the eigenvector $\xi$ and introduce orthonormal vectors in $V$
by
$$e = \frac{\xi + \xi^*}{\sqrt{2}}, \quad f = \frac{\xi - \xi^*}{\sqrt{2}}, \quad \xi = \frac{e + if}{\sqrt{2}}.$$  
Relative to the basis \{e, f\}, $S$ is represented by the matrix

$$S = \begin{pmatrix} 1/2 & i\mu \\ -i\mu & 1/2 \end{pmatrix} \quad \text{with} \quad S = \begin{pmatrix} 1/2 & -i\mu \\ i\mu & 1/2 \end{pmatrix} \quad \text{and} \quad S = i \begin{pmatrix} 0 & 2\mu \\ -2\mu & 0 \end{pmatrix},$$
where $2\mu \equiv 1 - 2\lambda$.

Consequently the canonical (Liouville) measure is of the form $2\mu ds dt$ with respect to the (partial) coordinates $(s, t) \in \mathbb{R}^2$ representing the vector $se + tf$ in a two-dimensional subspace of $V$, whereas the reference measure is of the form $2mds dt$ with $m > 0$. As the relevant forms are calculated to be

$$S(se + tf, se + tf) = \frac{1}{2}(s^2 + t^2), \quad i\sigma(se + tf, s'e + t'f) = 2i\mu(st' - s't),$$
the equation to determine $f$ takes the expression

$$e^{-(s^2 + t^2)/4} = 2m \int_{\mathbb{R}^2} f(s', t') f(s - s', t - t') e^{i\mu(st' - s't')} ds' dt'$$

with the hermiticity condition given by $f(-s, -t) = \overline{f(s, t)}$. We shall deal with a slightly more general situation: for $f, g \in S(V)$, consider

$$(f \ast g)(se + tf) = 2m \int_{\mathbb{R}^2} f(s', t') g(s - s', t - t') e^{i\mu(st' - s't')} ds' dt'.$$
If we write

$$f(s, t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\xi, \eta) e^{is\xi + it\eta} \, d\xi d\eta$$

with the Fourier transform $\hat{f}$ defined by

$$\hat{f}(\xi, \eta) = \int_{\mathbb{R}^2} f(s, t) e^{-is\xi - it\eta} \, ds dt$$

and similarly for $g$, then

$$(f \ast g)(s, t) = \frac{2m}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\xi, \eta) \hat{g}(\xi - \mu t, \eta + \mu s) e^{is\xi + it\eta} \, d\xi d\eta.$$

For the choice

$$\hat{f}(\xi, \eta) = A e^{-a(\xi^2 + \eta^2)/2}, \quad \hat{g}(\xi, \eta) = B e^{-b(\xi^2 + \eta^2)/2}$$

with $A, B, a, b$ positive reals, explicit computations are worked out by gaussian integrals: The results are

$$f(s, t) = \frac{\mu A}{2\pi a} e^{-\mu(s^2 + t^2)/2a}, \quad g(s, t) = \frac{\mu B}{2\pi b} e^{-\mu(s^2 + t^2)/2b}$$

and

$$(f \ast g)(s, t) = \frac{ABm\mu}{\pi(a + b)} e^{-\mu(s^2 + t^2)/(a + b)},$$

where

$$a \ast b = \frac{a + b}{ab + 1}.$$
Notice here that the function \( h(s, t) = \exp(-\mu(s^2 + t^2)/2c) \) with \( c > 0 \) defines a positive element
\[
h = 2m \int_{\mathbb{R}^2} h(s, t) \pi(e^{is(\sigma + tf)}) \, dsdt
\]
in any regular representation \( \pi \) of \( C^*(V, \sigma) \) if and only if the associated positive functional \( \tau(h) \) is a quasifree state, which means that
\[
\frac{\mu}{c}(s^2 + t^2) = (s \quad t) \begin{pmatrix} z & i\mu \\ -i\mu & z \end{pmatrix} (s \quad t)
\]
with \( z = \mu/c > 0 \) satisfying \( \mu^2 \leq z^2 \), namely \( c \leq 1 \).

In particular, for the choice \( a = b = c > 0 \) and \( A = B = C > 0 \),
\[
(f \ast f)(se + tf) = \frac{C^2 m\mu}{2\pi c} e^{-\mu(s^2 + t^2)/2(\sigma + \sigma^*)},
\]
where \( c \ast c = 2c/(c^2 + 1) \leq 1 \) for any \( c > 0 \) as being expected (the square of a hermitian element being positive). If this is identified with \( e^{-(s^2 + t^2)/4} \), we find a solution
\[
f(s, t) = \sqrt{\frac{\mu}{2\pi cm}} e^{-\mu(s^2 + t^2)/2c}
\]
satisfying \( 0 < c \leq 1 \) (recall \( 0 < 2\mu \leq 1 \)) by the choice
\[
c = \frac{2\mu}{1 + \sqrt{1 - 4\mu^2}}, \quad C = \frac{\sqrt{2\pi c}}{\sqrt{m\mu}}.
\]

From the expression
\[
S + \overline{S} + 2\sqrt{SS} = \begin{pmatrix} 1 + \sqrt{1 - 4\mu^2} & 0 \\ 0 & 1 + \sqrt{1 - 4\mu^2} \end{pmatrix},
\]
we see that
\[
\frac{\mu}{2c}(s^2 + t^2) = \frac{1}{4} (s \quad t) \begin{pmatrix} S + \overline{S} + 2\sqrt{SS} \end{pmatrix} (s \quad t).
\]

The last relation immediately gives rise to the general formula for \( \rho_S^{1/2} = \int_V \rho_S^{1/2}(x)e^{ix} \, dx \):
\[
\rho_S^{1/2}(x) = \frac{1}{\sqrt{N_S}} \exp\left(-\frac{1}{4} \left(S + \overline{S} + 2\sqrt{SS}\right)(x, x)\right)
\]
with
\[
N_S = \int_V \exp\left(-\frac{1}{2} \left(S + \overline{S} + 2\sqrt{SS}\right)(x, x)\right) \, dx.
\]
Here the geometric mean \( \sqrt{SS} \) of positive forms is in the sense of Pusz-Woronowicz ([19]).

The expression for the normalization constant is obtained from the identity
\[
\int_V \rho_S^{1/2}(x)\rho_S^{1/2}(x) \, dx = \rho_S(0) = 1.
\]

Compare this with
\[
\rho_S(x) = e^{-S(x, x)/2} = \exp\left(-\frac{1}{4}(S + \overline{S})(x, x)\right),
\]
which suggests the following rule how the exponential part is changed under the replacement \( \rho_S \mapsto \rho_S^{1/2} \): \( S + \overline{S} \mapsto (\sqrt{S} + \sqrt{\overline{S}})^2 \).

Also remark that \((S + \overline{S} + 2\sqrt{SS})(x, x) = (x, x)_{S + \sqrt{SS}}\) with \( S + \sqrt{SS} \in \text{Pol}(V, \sigma) \).

3.3. **Non-Degenerate** \( S + \overline{S} \). From here on, the alternating form \( \sigma \) is relaxed to be presymplectic and consider a polarization \( S \) such that \((,)_S\) is non-degenerate. The kernel of \( \sigma \) is then captured as the eigenspace of \( S \) corresponding to an eigenvalue 1/2. To deal with the degenerate part of \( \sigma \), let \( \eta \in V \) be a normalized eigenvector \((S\eta = \eta/2 \text{ and } (\eta, \eta)_S = 1)\) and assume that the reference measure is of the form \( m_0 dt \) with respect to the coordinate \( t \) representing the vector \( t\eta \) in \( V \).

Then the equation for \( f \) takes the form

\[
e^{-t^2/4} = m_0 \int_{-\infty}^{\infty} f(s) f(t-s) \, ds
\]

with the positive solution given by

\[
\rho_S^{1/2}(t\eta) = f(t) = N_0^{-1/2} e^{-t^2/2} = N_0^{-1/2} e^{-S(t\eta, t\eta)}.
\]

Here the normalization constant is given by

\[
N_0 = m_0 \int_{-\infty}^{\infty} e^{-2S(t\eta, t\eta)} \, dt = m_0 \sqrt{\pi}.
\]

Thus the formula for \( \rho_S^{1/2} \) remains valid even for the degenerate case \( \mu = 0 \) as far as \( S + \overline{S} \) is non-degenerate.

For \( S, T \in \text{Pol}(V, \sigma) \), let \( A \) and \( B \) be positive quadratic forms on \( V \) defined by

\[
A(x) = \frac{S(x, x) + \overline{S}(x, x)}{2} + \sqrt{SS}(x, x),
\]

\[
B(x) = \frac{T(x, x) + \overline{T}(x, x)}{2} + \sqrt{TT}(x, x).
\]

Assume that both of \( A \) and \( B \) are non-degenerate on \( V \). If the multiplier product formula is applied to

\[
\rho_S^{1/2}(x) = N_S^{-1/2} e^{-A(x)/2}, \quad N_S = \int_V e^{-A(x)} \, dx
\]

and to a similar expression for \( \rho_T^{1/2}(x) \), we obtain

\[
\tau \left( \rho_S^{1/2} e^{ix} \rho_T^{1/2} e^{iy} \right) = \frac{1}{\sqrt{N_S N_T}} \int_V e^{-A(z-x)/2 - i\sigma(z,x)/2} e^{-B(z+y)/2 + i\sigma(z,y)/2} \, dz
\]

for \( x, y \in V \). In particular, we have

\[
\left( \rho_S^{1/2} \rho_T^{1/2} \right)^1 = \frac{\int_V e^{-(A(x)+B(x))/2} \, dx}{\sqrt{\int_V e^{-A(x)} \, dx \int_V e^{-B(x)} \, dx}}.
\]

Notice that the right hand side does not depend on the choice of volume form \( dx \) in so far as it is translationally invariant. Also note that the right hand side is of the form \((\xi|\eta)\) with \( \xi, \eta \) unit vectors in \( L^2(V) \) (here \( V \) being assumed to be finite-dimensional).
Thus the relative position of vectors \( \{ \varphi^{1/2}_S \}_{S \in \text{Pol}'(V,\sigma)} \) in \( L^2(C^*(V,\sigma)) \) is same with that of unit vectors \( \{ \xi_S \}_{S \in \text{Pol}'(V,\sigma)} \) in \( L^2(V) \), where

\[
\xi_S = \frac{1}{\sqrt{\int_V e^{-A(x)/2} \, dx}} e^{-A(x)/2} \sqrt{dx}
\]

and \( \text{Pol}'(V,\sigma) \) denotes the set of polarizations \( S \) such that \( (\ , \ )_S \) is non-degenerate.

We now rewrite the transition amplitude formula into a form which can be used without the non-degeneracy restriction on polarizations. Choose an auxiliary inner product \( (\ , \ ) \) in \( V \) so that \( A \) and \( B \) are represented by commuting positive invertible operators \( \hat{A} \) and \( \hat{B} \) respectively \( ((\ , \ )_S + (\ , \ )_T \) can be used for example). The reference measure is then normalized regarding the inner product \( (\ , \ ) \); use the ordinary Lebesgue measure associated to an orthonormal basis of \( (\ , \ ) \). Then

\[
N_S = \int_V e^{-\langle x|\hat{A}x \rangle} \, dx = \frac{\pi^{n/2}}{\sqrt{\det(\hat{A})}}
\]

and we have

\[
(\varphi^{1/2}_S |\varphi^{1/2}_T) = \left( \frac{\det \hat{A} \hat{B}}{\pi^{n/2}} \right)^{1/4} \int_V e^{-\langle x|\hat{A}x \rangle/2 - \langle x|\hat{B}x \rangle/2} \, dx
\]

\[
= \frac{\det^{1/4}(\hat{A} \hat{B})}{\sqrt{\det(\hat{A}/2 + \hat{B}/2)}} = \sqrt{\det \left( \frac{2\sqrt{\hat{A} \hat{B}}}{\hat{A} + \hat{B}} \right)}.
\]

With the help of Pusz-Woronowicz’ functional calculus, the last formula takes a coordinates-free (independent of the choice of auxiliary inner products) expression:

\[
(\varphi^{1/2}_S |\varphi^{1/2}_T) = \sqrt{\det \left( \frac{2\sqrt{AB}}{A + B} \right)}.
\]

See Appendix A for the meaning of determinant in the right hand side.

**Example 3.1.** Let \( \sigma \) be defined on \( V = \mathbb{R}^2 \) by the matrix \( \begin{pmatrix} 0 & 2\mu \\ -2\mu & 0 \end{pmatrix} \) with \( \mu \in \mathbb{R} \) and consider a polarization \( S \) in the boundary of \( \text{Pol}(V,\sigma) \). From the spectral expression discussed above, we see that

\[
\sqrt{SS} = \begin{cases} 
0 & \text{if } \mu \neq 0, \\
S & \text{if } \mu = 0.
\end{cases}
\]

Thus, if \( S \) is associated to a matrix \( \begin{pmatrix} z + x & y + i\mu \\ y - i\mu & z - x \end{pmatrix} \) \((z^2 = x^2 + y^2 + \mu^2)\), then the quadratic form \( A \) corresponds to the matrix \( \begin{pmatrix} z + x & y \\ y & z - x \end{pmatrix} \) for \( \mu \neq 0 \), whereas it is multiplied by the factor 2 for \( \mu = 0 \).

Notice that the matrix \( \begin{pmatrix} z + x & y + i\mu \\ y - i\mu & z - x \end{pmatrix} \) \((z^2 = x^2 + y^2 + \mu^2)\), does not commute with its complex conjugate unless \((x, y, \mu) = (0, 0, 0)\).
Let \( S' \) be another boundary polarization with the associated quadratic form denoted by \( A' \). Then both of \( A \) and \( A' \) are positive definite for \( \mu \neq 0 \) and we have
\[
\langle \varphi^{1/2}_S | \varphi^{1/2}_{S'} \rangle = 2 \frac{(z^2 - x^2 - y^2)^{1/4}(x^2 - \mu^2 - y^2)^{1/4}}{\sqrt{(z + x)^2 - (x + y)^2}}.
\]

3.4. Degenerate Case. Now we shall remove the non-degeneracy condition on polarizations and extend the transition amplitude formula between quasifree states to the general (but finite-dimensional) case. Let \((V, \sigma)\) be a finite-dimensional presymplectic vector space.

Lemma 3.2. Let \((V, \sigma)\) be a general presymplectic vector space. Let \( S, T \in \text{Pol}(V, \sigma) \). If \( \ker(S + T) \neq \ker(T + T) \), then \( \langle \varphi^{1/2}_S | \varphi^{1/2}_T \rangle = 0 \).

Proof. We may assume that there is \( x \in V \) such that \((x, x)_S = 0 \) and \((x, x)_T \neq 0 \). Consider the homomorphism \( \pi : C^*(\mathbb{R}x, 0) \to C^*(V, \sigma) \) induced from the presymplectic map \( \mathbb{R}x \subset V \). Then it is easy to see that \( \varphi_S \circ \pi \) is represented by the Dirac measure concentrated at \( 0 \in \mathbb{R} \) whereas \( \varphi_T \circ \pi \) is described by a gausian measure on \( \mathbb{R} \). Thus by the transition amplitude inequality (27, Lemma 4.1)
\[
0 \leq \langle \varphi^{1/2}_S | \varphi^{1/2}_T \rangle \leq ((\varphi_S \circ \pi)^{1/2}(\varphi_T \circ \pi)^{1/2}) = 0.
\]

As a result, we see that the transition amplitude formula is valid if \( \ker(\cdot, \cdot)_S \neq \ker(\cdot, \cdot)_T \). So assume that these kernels (denoted by \( K \)) coincide.

Let \( V' \) be the quotient space of \( V \) by \( K \) with \( \sigma', S' \) and \( T' \) the induced forms on \( V' \). Let \( \pi : C^*(V, \sigma) \to C^*(V', \sigma') \) be the homomorphism induced from the presymplectic map \( V \to V' \). Then we see that \( \varphi_S = \varphi_S' \circ \pi \), \( \varphi_T = \varphi_T' \circ \pi \) and \( \pi \) satisfies the weak approximation property in [27] with respect to the states \( \varphi_S' \), \( \varphi_T' \) and therefore by [27] Corollary 2.6
\[
\langle \varphi^{1/2}_S | \varphi^{1/2}_T \rangle = \langle \varphi^{1/2}_S' | \varphi^{1/2}_T' \rangle = \sqrt{\det \left( \frac{2 \sqrt{AB'} }{A' + B'} \right)}.
\]

Theorem 3.3. Let \((V, \sigma)\) be a finite-dimensional presymplectic vector space. For polarizations \( S \) and \( T \) of \((V, \sigma)\), we have
\[
\langle \varphi^{1/2}_S | \varphi^{1/2}_T \rangle = \sqrt{\det \left( \frac{2 \sqrt{AB} }{A + B} \right)},
\]
where positive quadratic forms \( A, B \) are defined by
\[
2A = (\sqrt{S} + \sqrt{T})^2, \quad 2B = (\sqrt{T} + \sqrt{T})^2.
\]
Notice that \( 2 \sqrt{AB} \leq A + B \) (geometric mean is majorized by arithmetic mean) and the determinat in the above formula is always well-defined.

3.5. Central Decomposition. We shall decompose regular representations of \( C^*(V, \sigma) \) with the help of Fourier transform on the central subalgebra \( C^*(V_0, 0) \), where \( V_0 = \ker \sigma \).
To this end, choose Lebesgue measures \( dx \) on \( V \) and \( dx_0 \) on \( V_0 \) respectively. A Lebesgue measure \( d\hat{x} \) on the quotient space \( V/V_0 \) is then specified by the relation

\[
\int_V f(x) \, dx = \int_{V/V_0} d\hat{x} \int_{V_0} f(x + x_0) \, dx_0.
\]

If \( \pi \) is a regular representation of \( C^*(V, \sigma) \) satisfying \( \pi(e^{ix_0}) = e^{ix(x_0)}1 \) for \( x_0 \in V_0 \) with \( \omega \in V_0^* \), then we have the integration identity for \( f \in S(V) \)

\[
\int_V f(x) \pi(e^{ix}) \, dx = \int_{V/V_0} d\hat{x} \int_{V_0} f(x + x_0) \pi(e^{i(x + x_0)}) = \int_{V/V_0} d\hat{x} f_\omega(x) \pi(e^{ix}),
\]

where the function \( f_\omega \) on \( V \), which is defined by

\[
f_\omega(x) = \int_{V_0} e^{i\omega(x_0)} f(x + x_0) \, dx_0,
\]

satisfies (i) \( f_\omega(x + x_0) = f_\omega(x) e^{-i\omega(x_0)} \) for \( x_0 \in V_0 \) and (ii) \( f_\omega \) is rapidly decreasing when restricted to a complementary subspace of \( V_0 \).

Denote by \( S(V/\omega) \) the totality of functions satisfying these two conditions. The integration \( \int_{V/V_0} f(x) \pi(e^{ix}) \, d\hat{x} \) is well-defined for \( f \in S(V/\omega) \) and suggests the following \(*\)-algebra structure in \( S(V/\omega) \):

\[
(f * g)(z) = \int_{V/V_0} f(z') g(z - z') e^{-i\sigma(z', z)/2} \, d\hat{z}'
\]

from

\[
\int_{V/V_0} f(x) \pi(e^{ix}) \, d\hat{x} \int_{V/V_0} g(y) \pi(e^{iy}) \, d\hat{y} = \int_{V/V_0} (f * g)(z) \pi(e^{iz}) \, d\hat{z}
\]

and \( f^*(x) = \overline{f(-x)} \) from

\[
\left( \int_{V/V_0} f(x) \pi(e^{ix}) \, d\hat{x} \right)^* = \int_{V/V_0} f^*(x) \pi(e^{ix}) \, d\hat{x}.
\]

The \(*\)-algebra \( S(V/\omega) \) is then a Hilbert algebra with respect to the inner product

\[
(f | g)_{\omega} = \int_{V/V_0} \overline{f(x)} g(x) \, d\hat{x}
\]

with the associated trace functional on \( S(V/\omega) \) given by

\[
\tau_\omega(f) = f(0) \quad \text{for} \quad f \in S(V/\omega).
\]

Note that \( \tau_\omega(f^* \ast g) = (f | g)_{\omega} \) for \( f, g \in S(V/\omega) \). For \( x, y, z \in V \) and \( f \in S(V/\omega) \), the multiplier product formula

\[
(e^{ix} e^{iy})(z) = f(z - x - y) e^{i(\sigma(x, y) - \sigma(z, x) - \sigma(z, y))/2},
\]

remains valid.

It is now obvious to see that the partial Fourier transform

\[
S(V) \ni f \mapsto \int_{V_0^*} \int_{V/V_0} f_\omega \, d\omega \in \int_{V_0^*} S(V/\omega) \, d\omega
\]

gives rise to a decomposition of the relevant Hilbert algebra:

\[
(f \ast g)_{\omega} = f_\omega \ast g_\omega, \quad (f^*)_\omega = (f_\omega)^*, \quad (f | g) = \int_{V_0^*} d\omega \, (f_\omega | g_\omega)_{\omega}
\]
for \( f, g \in S(V) \). Here \( d\omega \) is the measure on \( V_0^* \) in the duality relation with \( dx_0 \):

\[
\int_{V_0^*} d\omega \int_{V_0} dx_0 g(x_0) e^{i\omega(x_0)} = g(0), \quad \int_{V_0} dx_0 \int_{V_0^*} d\omega h(\omega) e^{i\omega(x_0)} = h(0)
\]

for \( g \in S(V_0) \) and \( h \in S(V_0^*) \).

If we denote by \( C^*_\text{reg}(V/\omega, \sigma) \) the C*-closure of \( S(V/\omega) \), then the C*-algebra \( C^*(V/\omega, \sigma) \) introduced in the previous section is a multiplier subalgebra of \( C^*_\text{reg}(V/\omega, \sigma) \).

The following is a consequence of standard Fourier analysis.

**Lemma 3.4.** We have a decomposition of the C*-algebra \( C^*_\text{reg}(V, \sigma) \) into a continuous field \( \{ C^*_\text{reg}(V/\omega, \sigma) \}_{\omega \in V_0^*} \) of C*-algebras so that any regular representation \( \pi \) of \( C^*(V, \sigma) \) is covariantly decomposed into the form

\[
\pi = \int_{V_0^*} \pi_\omega \nu(d\omega),
\]

where \( \nu \) is a measure on \( V_0^* \) and \( \{ \pi_\omega \} \) is a \( \nu \)-measurable field of regular representations of \( \{ C^*(V/\omega, \sigma) \} \).

Given a polarization \( S \) of \( (V, \sigma) \), if one applies the above decomposition to the GNS-representation of the quasifree state \( \varphi_S \) and compare it with Proposition 2.9, an \( L^1 \)-decomposition of \( \varphi_S \) is obtained:

\[
\varphi_S = \int_{V_0^*} \varphi_{S,\omega} \nu_{S_0}(d\omega),
\]

where \( \nu_{S_0} \) is a gaussian measure of covariance form \( S_0 = S|_{V_0^* \times V_0^*} \), \( \varphi_{S,\omega} \) is the state of \( C^*(V/\omega, \sigma) \) introduced in § 2.3 and \( \{ \varphi_{S,\omega} \} \) is a \( \nu_{S_0} \)-measurable family of states of \( \{ C^*(V/\omega, \sigma) \} \).

Now the results on density operators are rewritten in terms of the Hilbert algebra \( S(V/\omega) \): The density function

\[
\rho_{S,\omega}(x) = e^{-i\omega(x_0)} e^{-S(x_1, x_1)/2} \quad \text{for} \quad x = x_0 + x_1 \in V_0 + V_1,
\]

where \( V_1 = \{ x \in V; S(x, V_0) = 0 \} \), satisfies

\[
\tau_\omega(\rho_{S,\omega} \ast f) = \int_{V_0} f(x) \varphi_{S,\omega} (e^{i\omega} \, dx), \quad f \in S(V/\omega)
\]

and its square root is given by

\[
\rho_{S,\omega}^{1/2}(x) = \frac{1}{\sqrt{N_{S_1}}} e^{-A(x_1, x_1)/2} e^{-i\omega(x_0)} \quad \text{with} \quad N_{S_1} = \int_{V_1} e^{-A(x_1, x_1)} \, dx_1.
\]

Given a polarization \( S \) of a presymplectic vector space \( (V, \sigma) \), let \( \dot{S} \) be the polarization of the quotient symplectic vector space \( (V/V_0, \dot{\sigma}) \) defined by

\[
\dot{S}(\xi, \eta) = S(x_1, y_1),
\]

where \( x_1 \) (resp. \( y_1 \)) denotes the projection of \( x \in V^C \) (resp. \( y \in V^C \)) to the \( (S + \overline{S}) \)-orthogonal complement of \( V_0^C \).

The following is immediate from the definition.
Lemma 3.5. For $x \in V^C$, we have
\[
\dot{S}(x) = \inf \{s(x+y_0); y_0 \in V_0^C\},
\]
\[
(\dot{S} + R)(x) = \inf \{(x+y_0, x+y_0) s; y_0 \in V_0^C\}
\]
and
\[
\dot{A}(x) = \inf \{A(x+y_0); y_0 \in V_0^C\}
\]
with $\dot{A} = (\dot{S} + R)/2 + \sqrt{RS}$.

Let $T$ be another polarization of $(V, \sigma)$ and let $F_0$ be the $(T + T^T)$-orthogonal projection to the subspace $V_0^C$.

**Proposition 3.6.** The transition amplitude $(\varphi_S^{1/2}|\varphi_T^{1/2})$ is given by
\[
(\varphi_S^{1/2}|\varphi_T^{1/2}) = (\varphi_S^{1/2}|\varphi_T^{1/2}) e^{ -(\dot{A} + \dot{B})^{-1}(\Delta \omega)/2}.
\]
Here $\Delta : V_0^* \to (V/V_0)^*$ is a linear map defined by
\[
\langle \Delta \omega, \dot{x} \rangle = \omega((E_0 - F_0)x), \ x \in V.
\]

**Proof.** We use the expression
\[
\rho_{S,\omega}^{1/2}(x) = (N_{S,N_T})^{-1/2} e^{-(\dot{A} + \dot{B})^{-1}(\Delta \omega)/2} e^{i\omega((E_0 - F_0)x)}
\]
in the following rewriting:
\[
(\varphi_S^{1/2}|\varphi_T^{1/2}) = \tau_{\omega}(\rho_{S,\omega}^{1/2} \ast \rho_{S,\omega}^{1/2}) = \int_{V/V_0} \rho_{S,\omega}^{1/2}(x) \rho_{T,\omega}^{1/2}(x) d\dot{x}
\]
\[
= \frac{1}{\sqrt{N_{S,N_T}}} \int_{V/V_0} e^{-(\dot{A} + \dot{B})(\dot{x})/2} e^{i(\Delta \omega, \dot{x})} d\dot{x}
\]
\[
= \sqrt{\det \left( \frac{2\sqrt{AB}}{A + B} \right)} e^{-(\dot{A} + \dot{B})^{-1}(\Delta \omega)/2}
\]
\[
= (\varphi_S^{1/2}|\varphi_T^{1/2}) e^{-(\dot{A} + \dot{B})^{-1}(\Delta \omega)/2}.
\]

\[\Box\]

4. Infinite-Dimensional Analysis

From here on, we shall work with an infinite-dimensional presymplectic vector space $(V, \sigma)$.

4.1. Topological Equivalence on Polarizations.

**Lemma 4.1.** Let $S, T \in \text{Pol}(V, \sigma)$ be such that we can find $0 \neq x \in V^C$ and $\epsilon > 0$ satisfying $(x, y)_T \neq 0$ and $(x, y)_S \leq \epsilon(x, y)_T$. Then $(\varphi_S^{1/2}|\varphi_T^{1/2}) \leq 2\epsilon^{1/4}$.

**Proof.** Consider the real subspace $W^C = \mathbb{C}x + \mathbb{C}x^*$ and let $S_W$ and $T_W$ be the restrictions of $S$ and $T$ to $W^C$. Set
\[
2A_W = S_W + S_W^* + 2\sqrt{S_W S_W^*}, \quad 2B_W = T_W + T_W^* + 2\sqrt{T_W T_W^*}.
\]
From the inequality $T_W + T_W^* \geq 2\sqrt{T_W T_W^*}$, $\frac{1}{2}(x, y)_T \leq B_W(x, x) \leq (x, y)_T$ and similarly for $A_W$ and $(\ , \ )_S$. We then have
\[
\frac{1}{2}(x, x)_T \leq A_W(x, x) + B_W(x, x)
\]
and
\[\sqrt{A_W B_W}(x, x) \leq A_W(x, x)^{1/2} B_W(x, x)^{1/2} \leq \epsilon^{1/2}(x, x)^{1/2}(x, x)^{1/2},\]
whence
\[\frac{2\sqrt{A_W B_W}(x, x)}{(A_W + B_W)(x, x)} \leq 4\epsilon^{1/2}.
\]
Let \(C\) be the operator on \(W^C\) defined by
\[2\sqrt{A_W B_W}(\xi, \eta) = (A_W + B_W)(\xi, C\eta).
\]
Then \(0 \leq C \leq 1\) with respect to the inner product \(A_W + B_W\) and the above inequality implies \(\text{Sp}(C) \cap [0, 4\epsilon^{1/2}] \neq \emptyset\) and therefore
\[\det\left(\frac{2\sqrt{A_W B_W}}{A_W + B_W}\right) \leq 4\epsilon^{1/2}.
\]
Now apply the increasing property of transition amplitude ([27, Lemma 4.1]) to get
\[\langle \varphi_S^{1/2} | \varphi_T^{1/2} \rangle \leq \langle \varphi_S^{1/2} | \varphi_{T_W}^{1/2} \rangle \leq 2\epsilon^{1/4}.
\]

\[
\square
\]

\textbf{Corollary 4.2.} Let \(S, T \in \text{Pol}(V, \sigma)\). Then \(\langle \varphi_S^{1/2} | \varphi_T^{1/2} \rangle = 0\) unless \((\ , \ , S)\) and \((\ , \ , T)\) are equivalent.

\section{Hilbert-Schmidt Estimates.}

In what follows, assume that \((\ , \ , S)\) and \((\ , \ , T)\) are equivalent and complete on \(V^C\). Then we can find a bounded operator \(R\) with a bounded inverse which is positive with respect to the inner product \((\ , \ , S)\) (\(R\) being said to be \(S\)-positive) and satisfies
\[(x, y)_T = (Rx, Ry)_S, \quad x, y \in V^C.
\]

In the notation of functional calculus (see Appendix A), \(R = \left(\frac{T - S}{S + S}\right)^{1/2}\).

Since the norm of a Laurent polynomial \(f(R)\) of \(R\) satisfies \(\|f(R)\|_S = \|R^{-1}f(R)R\|_T = \|f(R)\|_T\), we shall omit the reference subscript.

Let \(\delta = 2\log(\|R\| \|R^{-1}\|)\), which is the projective distance between \((\ , \ , S)\) and \((\ , \ , T)\). Let \(A_S\) and \(B_S\) be \(S\)-positive operators representing \(A\) and \(B\) relative to the inner product \((\ , \ , S)\). We also set \(S = (S + S) \setminus S\) and \(T = (T + T) \setminus T\) as before.

In the next lemma, Hilbert-Schmidt norm as well as operator norm is the one based on the inner product \((\ , \ , S)\).

\textbf{Lemma 4.3.}

(i) \[\|A_S - B_T\|_{HS} \leq 2\sqrt{2}(1 + \epsilon^{\delta/2}) \left\|\sqrt{S} - \sqrt{T}\right\|_{HS}.
\]

(ii) \[\|A_S - B_S\|_{HS} \leq 2\|1 - R^2\|_{HS} + 2\sqrt{2}(1 + \epsilon^{\delta/2})\|R^2\| \left\|\sqrt{S} - \sqrt{T}\right\|_{HS}.
\]

(iii) \[\left\|\sqrt{S} - \sqrt{T}\right\|_{HS} \leq \frac{\epsilon^{\delta/4}}{\sqrt{2}} \epsilon^{\delta}(1 + \epsilon^{\delta/2})\|R^{-2}\| \|A_S - B_S\|_{HS}
\]
and \[\|1 - R^2\|_{HS} \leq \epsilon^{\delta/2}\|A_S - B_S\|_{HS} + 2\sqrt{2}(\epsilon^{\delta} + \epsilon^{\delta/2}) \left\|\sqrt{S} - \sqrt{T}\right\|_{HS}.
\]
Proof. Imitate the computations in \cite{5} Lemma 8.4 \hfill \Box

**Corollary 4.4.** The following two conditions are equivalent.

(i) The operator \( \frac{A^2B}{A+B} \) is in the Hilbert-Schmidt class.

(ii) Both of \( \frac{S+T}{S+S} - 1 \) and \( \left( \frac{S}{S+S} \right)^{1/2} - \left( \frac{T}{T+T} \right)^{1/2} \) are in the Hilbert-Schmidt class.

**Proof.** Use the expression

\[
A_S - B_S = \frac{A + B}{S + S} \frac{A + B}{A - B}
\]

and compute as follows (see Appendix A for the first equality):

\[
\|A_S - B_S\|_{HS} = \left\| \left( \frac{S + S}{A + B} \right)^{1/2} \frac{A - B}{S + S} \left( \frac{S + S}{A + B} \right)^{-1/2} \right\|_{(A + B) - HS}
\]

\[
= \left\| \left( \frac{A + B}{S + S} \right)^{1/2} \frac{A - B}{A + B} \left( \frac{A + B}{S + S} \right)^{1/2} \right\|_{(A + B) - HS}
\]

\[
\leq \left\| \frac{A + B}{S + S} \right\| \left\| \frac{A - B}{A + B} \right\|_{(A + B) - HS}
\]

\[
\leq \|1 + R^2\| \left\| \frac{A - B}{A + B} \right\|_{(A + B) - HS}.
\]

Here at the last inequality, we have applied the operator inequality \( \frac{S + T}{S + S} \leq 1 + R^2 \) relative to the inner product \( S + S \), which is a consequence of \( A \leq S + S \) and \( B \leq T + T \). \hfill \Box

4.3. **Orthogonality of Quasifree States.**

**Lemma 4.5.** Given admissible polarizations \( S, T \) on a complete presymplectic vector space \((V, \sigma)\), we can find a family of closed separable subspaces \( \{V_i\}_{i \in I} \) of \( V \) such that \( S(V_i, V_j) = 0 = T(V_i, V_j) \) if \( i \neq j \) and \( V = \bigoplus_{i \in I} V_i \).

**Proof.** This follows from a standard maximality argument relying on Zorn’s lemma (cf. \cite{5} Lemma 6.9)). \hfill \Box

In view of this lemma, we may restrict ourselves to a separable \( V \). Choose an increasing sequence of finite-dimensional vector space \( \{V_n\}_{n \geq 1} \) such that \( \dim V_n = n \) and \( V = \bigcup_{n \geq 1} V_n \). Let \( S_n = S|_{V_n^C} \) and \( T_n = T|_{V_n^C} \) be the restrictions with \( S_n = (S_n + S_n) \backslash S_n \) and \( T_n = (T_n + T_n) \backslash T_n \) be the associated linear operators on \( V_n^C \). Let \( E_n \) (resp. \( F_n \)) be the \((S + S)-\)orthogonal (resp. \((T + T)\)-orthogonal) projection to the subspace \( V_n^C \). Then, for \( x, y \in V^C \), we have

\[
E_n S E_n = S_n E_n, \quad F_n T F_n = T_n F_n
\]

and hence

\[
S^{1/2} = \lim_{n \to \infty} S_n^{1/2} E_n \quad \text{and} \quad T^{1/2} = \lim_{n \to \infty} T_n^{1/2} F_n
\]

in the strong operator topology.
Choose an \((S + \overline{S})\)-orthonormal basis \(\{\xi_n\}_{n \geq 1}\) of \(V^C\) so that \(V_n^C = C\xi_1 + \cdots + C\xi_n\). Then, for \(1 \leq m \leq n\),

\[
\sum_{j=1}^{m} \|(S_{n}^{1/2}E_{n} - T_{n}^{1/2}F_{n})\xi_{j}\|_{S}^2 = \sum_{j=1}^{m} \|(S_{n}^{1/2} - T_{n}^{1/2})\xi_{j}\|_{S}^2 \leq \sum_{j=1}^{n} \|(S_{n}^{1/2} - T_{n}^{1/2})\xi_{j}\|_{S}^2.
\]

Taking \(n \to \infty\) and then \(m \to \infty\), we have

\[
\|S_{1}^{1/2} - T_{1}^{1/2}\|_{HS}^2 \leq \liminf_{n \to \infty} \|S_{n}^{1/2} - T_{n}^{1/2}\|_{HS}^2.
\]

Next, for \(x, y \in V^C\), \(E_nR^2E_n = E_nR_n^2E_n = R_n^2E_n\) with \(R_n = (T_n + \overline{T}_n)/(S_n + \overline{S}_n)\) implies

\[
R = \lim_{n \to \infty} (E_nR_n^2E_n)^{1/2} = \lim_{n \to \infty} (R_n^2E_n)^{1/2} = \lim_{n \to \infty} R_nE_n
\]

in the strong operator topology. As a result, we have

\[
\|R - 1\|_{HS} \leq \liminf_{n \to \infty} \|R_n - 1\|_{HS}.
\]

**Lemma 4.6.** Unless both of \(S_{1}^{1/2} - T_{1}^{1/2}\) and \(R - 1\) are in the Hilbert-Schmidt class, we have \((\varphi_{S}^{1/2}|\varphi_{T}^{1/2}) = 0\).

**Proof.** Let \(2A_n = (S_{n}^{1/2} + \overline{S}_{n}^{1/2})^2\), \(2B_n = (T_{n}^{1/2} + \overline{T}_{n}^{1/2})^2\) and set \(C_n = \frac{A_n - B_n}{A_n + B_n}\). Then \(-1 \leq C_n \leq 1\) and

\[
\det \left( \frac{2\sqrt{A_nB_n}}{A_n + B_n} \right) = \sqrt{\det(1 - C_n^2)}.
\]

From the estimate discussed above, the assumption implies

\[
\liminf_{n \to \infty} \|S_{n}^{1/2} - T_{n}^{1/2}\|_{HS} = +\infty \quad \text{or} \quad \liminf_{n \to \infty} \|R_n - 1\|_{HS} = +\infty,
\]

which, in turn, gives

\[
\liminf_{n \to \infty} \left\| \frac{A_n}{S_n + \overline{S}_n} - \frac{B_n}{S_n + \overline{S}_n} \right\|_{HS} = +\infty
\]

by the inequalities in Lemma 4.3. Since \(\|1_n + R_n^2\| = 1 + \|R_n^2\| \leq 1 + \|R^2\|\), the inequality

\[
\left\| \frac{A_n - B_n}{S_n + \overline{S}_n} \right\|_{HS} \leq \|1_n + R_n^2\| \left\| \frac{A_n - B_n}{A_n + B_n} \right\|_{(A_n + B_n) - HS}
\]

\[
\leq (1 + \|R\|^2) \left\| \frac{A_n - B_n}{A_n + B_n} \right\|_{(A_n + B_n) - HS}
\]

implies

\[
\liminf_{n \to \infty} \text{tr} \left( \frac{A_n - B_n}{A_n + B_n} \right)^2 = +\infty,
\]
whence the equality $(\varphi_T^{1/2} | \varphi_T^{1/2}) = \lim_{n \to \infty} (\varphi_n^{1/2} | \varphi_n^{1/2})$ ([24, Theorem 4.3]) is used to have

$$(\varphi_T^{1/2} | \varphi_T^{1/2}) = \lim_{n \to \infty} (\varphi_n^{1/2} | \varphi_n^{1/2}) = \lim_{n \to \infty} \det \left( \frac{2 \sqrt{A_n B_n}}{A_n + B_n} \right)$$

$$= \lim_{n \to \infty} \sqrt{\det(1 - C_n)} = \lim_{n \to \infty} \exp \left( \frac{1}{2} \text{tr} \log(1 - C_n^2) \right)$$

$$\leq \limsup_{n \to \infty} \exp \left( -\frac{1}{2} \text{tr}(C_n^2) \right) = \exp \left( -\frac{1}{2} \liminf_{n \to \infty} \text{tr}(C_n^2) \right) = 0.$$

$\square$

5. Quadrature on Polarizations

5.1. Quadrature on Polarizations. Start with a presymplectic vector space $(V, \sigma)$ such that $V^\mathbb{C}$ is a Hilbert space relative to an inner product of the form $S + \overline{S}$ with $S$ a polarization of $(V, \sigma)$ (recall that a polarization possessing this property is said to be admissible). Note that any Hilbert space inner product on $V^\mathbb{C}$ is unique up to metrical equivalence and represents $\sigma$ by a bounded operator. We shall now review the construction of phase-space doubling (see [4, 24] for example).

Consider a presymplectic vector space of the form $(V \oplus V, \sigma \oplus -\sigma)$. Given an admissible polarization $S$, define a positive form $P$ on $V^\mathbb{C} \oplus V^\mathbb{C}$ by

$$P(x \oplus y, x' \oplus y') = \left( \begin{array}{cc} x^* & y^* \\ \sqrt{SS} & \overline{S} \end{array} \right) \left( \begin{array}{c} x' \\ y \end{array} \right)$$

$$= S(x, x') + \sqrt{SS}(x, y') + \sqrt{SS}(y, x') + \overline{S}(y, y'),$$

which is a polarization of $(V \oplus V, \sigma \oplus -\sigma)$.

To look into the topology induced from $P + \overline{P}$, we consider the ‘rotation’ of $(V \oplus V, \sigma \oplus -\sigma)$ by an angle $\pi/4$. Let $R_{\pi/4} : V \oplus V \to V \oplus V$ be defined by

$$R_{\pi/4}(x \oplus y) = \frac{x - y}{\sqrt{2}} + \frac{x + y}{\sqrt{2}} \otimes \frac{x + y}{\sqrt{2}}.$$  

Then the identities

$$P_{\pi/4} = R_{\pi/4}^{-1} P R_{\pi/4} = \frac{1}{2} \left( \begin{array}{cc} S^{1/2} + \overline{S}^{1/2} & \overline{S} - S \\ \overline{S} - S & (S^{1/2} - \overline{S}^{1/2})^2 \end{array} \right)$$

and

$$(\sigma \oplus -\sigma)_{\pi/4} = R_{\pi/4}^{-1} \left( \begin{array}{cc} \sigma & 0 \\ 0 & -\sigma \end{array} \right) R_{\pi/4} = \left( \begin{array}{cc} 0 & -\sigma \\ -\sigma & 0 \end{array} \right)$$

show that $R_{\pi/4}$ is a presymplectic isomorphism of $(V \oplus V, (\sigma \oplus -\sigma)_{\pi/4})$ onto $(V \oplus V, \sigma \oplus -\sigma)$ and $P(R_{\pi/4} \xi, R_{\pi/4} \eta) = P_{\pi/4}(\xi, \eta)$ for $\xi, \eta \in V^\mathbb{C} \oplus V^\mathbb{C}$.

The kernel of $P + \overline{P}$ corresponds to that of

$$P_{\pi/4} + \overline{P}_{\pi/4} = \left( \begin{array}{cc} (S^{1/2} + \overline{S}^{1/2})^2 & 0 \\ 0 & (S^{1/2} - \overline{S}^{1/2})^2 \end{array} \right)$$

by the presymplectic isomorphism $R_{\pi/4}$, which is equal to $0 \oplus (\ker \sigma)^\mathbb{C}$ (cf. $(S^{1/2} - \overline{S}^{1/2})^2 = (S^{1/2} + \overline{S}^{1/2})^2 - 2(S - \overline{S})^2$). Thus

$$\ker(\ , \ )_P = \{ x \oplus -x; x \in (\ker \sigma)^\mathbb{C} \}.$$

and $P$ induces a non-degenerate polarization on the quotient presymplectic vector space $(V \oplus V)/\{x \oplus -x; x \in \ker \sigma\}$.

Clearly the quotient by the kernel of $P_{\pi/4} + P_{\pi/4}$ is equal to $V \oplus \hat{V}$ with $\hat{V} = V/\ker \sigma$ and $R_{\pi/4}$ induces a presymplectic isomorphism $V \oplus \hat{V} \rightarrow (V \oplus V)/\{x \oplus -x; x \in \ker \sigma\}$.

Since $S + \frac{\pi}{4} \leq (S^{1/2} + \frac{\pi}{4})^2 \leq 2(S + \frac{\pi}{4})$, the topology induced from $P_{\pi/4} + P_{\pi/4}$ is hilbertian when restricted to $V^C \oplus 0$, while the topology on $0 \oplus V^C$ is associated to the positive form $(S^{1/2} - \frac{\pi}{4})^2$, which is generally different from that of $V^C$ even for a non-degenerate $\sigma$.

**Lemma 5.1.** Given an admissible polarization $T$ of a hilbertian presymplectic vector space $(V, \sigma)$, let $W_T$ be the Hilbert space associated to the positive form $(T^{1/2} - \overline{T}^{1/2})^2$ ($W_T$ being therefore a completion of the quotient space $\hat{V} = V/\ker \sigma$). Then we have an isometry $U_T : W_T^C \rightarrow V^C$ ($W_T$ and $V^C$ being furnished with inner products $(T^{1/2} - \overline{T}^{1/2})^2$ and $(T^{1/2} + \overline{T}^{1/2})^2$ respectively) such that

$$U_T \hat{x} = \frac{T^{1/2} - \overline{T}^{1/2}}{T^{1/2} + \overline{T}^{1/2}} x$$

for $\hat{x} \in V^C$ with $x \in V^C$.

Furthermore, the composition $\hat{U}_T : W_T^C \rightarrow V^C \rightarrow \hat{V}^C$ is unitary if $\hat{V}^C$ is furnished with the inner product $(\hat{T}^{1/2} + \overline{\hat{T}}^{1/2})^2$. Here the positive form $\hat{T}$ on $\hat{V}$ is defined by $\hat{T}(x, y) = T(x, y)$ with representatives $x$ and $y$ taken from the $(T + \overline{T})$-orthogonal complement of $V_0^C$. Note that $\hat{T}(\hat{x}, \hat{x}) = \inf\{T(x, x); \hat{x} = x + V_0^C\}$.

**Proof.** By passing to the quotient $\hat{V}$, we may assume that $\ker \sigma = \{0\}$.

Let $\overline{T} = (T + \overline{T})\setminus T$ be a $(T + \overline{T})$-positive ratio operator, which satisfies the inequality $1 \leq \sqrt{T} + \sqrt{1 - T} \leq \sqrt{2}$ and hence $\sqrt{T} + \sqrt{1 - T}$ has a bounded inverse. The identity

$$(T^{1/2} + \overline{T}^{1/2})^2 \left(\frac{\sqrt{T} - \sqrt{1 - T}}{\sqrt{T} + \sqrt{1 - T}}, \frac{\sqrt{T} - \sqrt{1 - T}}{\sqrt{T} + \sqrt{1 - T}}\right) x,
\left(\frac{\sqrt{T} - \sqrt{1 - T}}{\sqrt{T} + \sqrt{1 - T}}, \frac{\sqrt{T} + \sqrt{1 - T}}{\sqrt{T} + \sqrt{1 - T}}\right) y
\right) = (x, (\sqrt{T} - \sqrt{1 - T})^2 y) = (T^{1/2} - \overline{T}^{1/2})^2(x, y)$$

then shows that we have an isometry $U : W^C \rightarrow V^C$ such that

$$U x = \frac{T^{1/2} - \overline{T}^{1/2}}{T^{1/2} + \overline{T}^{1/2}} x \quad \text{for} \quad x \in V^C \subset W^C. \quad \square$$

**Lemma 5.2.** Let $S$ and $T$ be admissible polarizations of a hilbertian presymplectic vector space $(V, \sigma)$. Then positive forms $(S^{1/2} - \frac{\pi}{4})^2$, $(T^{1/2} - \overline{T}^{1/2})^2$ on $V^C$ are equivalent and therefore $W_S = W_T$. The common hilbertian space is denoted by $W$. 


Since both of the initial and final expressions are continuous in $y \in W^C$, we have
\[(T^{1/2} - \overline{T}^{1/2})^2(x, y) = (T^{1/2} - T^{1/2})^2(T^{1/2} - \overline{T}^{1/2}x', y')\]
for $x \in V^C$ and $y \in W^C$.

Taking the relation $S - \overline{S} = T - \overline{T}$ into account, we have the following for $y \in V^C \subset W^C$ and $x = Ux'$ with $x' \in V^C$:
\[
(S^{1/2} - \overline{S}^{1/2})^2(x, y) = (S - \overline{S})(x, \frac{S - \overline{S}}{(S^{1/2} + \overline{S}^{1/2})^2}y)
\]
\[= (T - \overline{T})(x, \frac{T - \overline{T}}{(S^{1/2} + \overline{S}^{1/2})^2}y)
\]
\[= (T - \overline{T})(x, \frac{(T^{1/2} + \overline{T}^{1/2})^2}{(S^{1/2} + \overline{S}^{1/2})^2}T - \overline{T}y)
\]
\[= (T^{1/2} - \overline{T}^{1/2})^2(x', \frac{(T^{1/2} + \overline{T}^{1/2})^2}{(S^{1/2} + \overline{S}^{1/2})^2}Uy)
\]
\[= (T^{1/2} - \overline{T}^{1/2})^2(x, Ux', \frac{(T^{1/2} + \overline{T}^{1/2})^2}{(S^{1/2} + \overline{S}^{1/2})^2}Uy)
\]
\[= (T^{1/2} - \overline{T}^{1/2})^2(x, Ux', \frac{S - \overline{S}}{(S^{1/2} + \overline{S}^{1/2})^2}Uy).
\]

Since $\frac{T^{1/2} - \overline{T}^{1/2}}{T^{1/2} + \overline{T}^{1/2}}V^C$ as well as $V^C$ is dense in $W^C$, we obtain the equivalence of positive forms $(S^{1/2} - \overline{S}^{1/2})^2, (T^{1/2} - \overline{T}^{1/2})^2$ on $W^C$ and the equality
\[
\frac{(S^{1/2} - \overline{S}^{1/2})^2}{(T^{1/2} - \overline{T}^{1/2})^2} = U^* \frac{(T^{1/2} + \overline{T}^{1/2})^2}{(S^{1/2} + \overline{S}^{1/2})^2}U
\]
holds at the same time. \qed
**Corollary 5.3** ([4] Lemma 6.1). The topology on \((V \oplus V) / \{x \oplus -x; x \in \ker \sigma\}\) induced from \(P + \overline{P}\) does not depend on the choice of an admissible polarization \(S\). Let \(\hat{V}\) be its hilbertian completion and \(\hat{V}_{\pi/4}\) be the rotated space of \(\hat{V}\). Then we have \(\hat{V}_{\pi/4} = V \oplus W\).

Let \(\hat{\sigma}\) be the presymplectic form on \(\hat{V}\) induced from \(\sigma \oplus -\sigma\). We regard \(P\) as defining an admissible polarization of the presymplectic vector space \((\hat{V}, \hat{\sigma})\) and call it the **quadrature** of \(S\).

**Lemma 5.4** ([4] Lemma 5.8]). The spectrum of the ratio operator \((P + \overline{P}) \setminus P\) is a subset of \(\{0, 1/2, 1\}\) with its spectral subspaces given by closures of
\[
\{[-\sqrt{1 - Sx} \oplus \sqrt{S}x]; x \in V^C\}, \{[x \oplus x]; x \in V^C_0\}, \{[\sqrt{S}x \oplus -\sqrt{1 - Sx}]; x \in V^C\}
\]
respectively, where \([x \oplus y]\) denotes the quotient of \(x \oplus y \in V^C \oplus V^C\) with respect to the subspace \(\{z \oplus -z; z \in (\ker \sigma)^C\}\).

**Lemma 5.5** ([3] \S 3]). Let \(y \in V \cap \ker(S(1 - S)(2S - 1))\) be entirely analytic for \(\{e^{itH}\}_{t \in \mathbb{R}}\) (cf. Proposition 2.8). Then
\[
e^{it[0 \oplus y]} \varphi^{1/2}_p = e^{it[H/2 y \oplus 0]} \varphi^{1/2}_p.
\]
Here \((e^{H/2}y)^* = e^{-H/2}y\) and the right hand side is, by definition,
\[
\sum_{n=0}^{\infty} \frac{1}{n!} [e^{H/2}y \oplus 0]^n \varphi^{1/2}_p,
\]
which is norm-convergent in the Hilbert space \(C^*(\hat{V}, \hat{\sigma}) \varphi^{1/2}_p\).

**Proposition 5.6.** Let \(S\) be an admissible polarization. Then we have the unitary map
\[
\overline{C^*(V, \sigma)\varphi^{1/2}_S C^*(V, \sigma)} \to C^*(\hat{V}, \hat{\sigma}) \varphi^{1/2}_p
\]
defined by
\[
e^{ix} \varphi^{1/2}_S e^{iy} \mapsto e^{i[x \oplus y]} \varphi^{1/2}_p, \quad x, y \in V.
\]

**Proof.** Since the decomposition
\[
V^C = (\ker \bar{S} + \ker(1 - \bar{S})) \oplus \ker(2 \bar{S} - 1) \oplus V^C_\bar{S}
\]
with \(V^C_\bar{S}\) the orthogonal complement of \(\ker(S(1 - S)(2S - 1))\) gives rise to tensor product factorizations of \(C^*(V, \sigma)\) and \(\varphi_S\), we are reduced to checking each case separately.

For \(V^C_\bar{S}\) the associated representation is a Fock representation and the assertion follows from
\[
C^*(V_F, \sigma_F) \varphi^{1/2}_{S_F} \to C^*(\bar{V}_F, \bar{\sigma}_F) \varphi^{1/2}_p \varphi^{1/2}_{S_F} \to C^*(V_F, \sigma_F).
\]

Since the part \(\ker(2S - 1)\) produces a commutative algebra, we have
\[
e^{ix} \varphi^{1/2}_S e^{iy} = e^{i(x + y)} \varphi^{1/2}_S,
\]
whereas \([x \oplus y] = [x + y \oplus 0]\) for \(x, y \in \ker(2S - 1)\) shows that the correspondence in question is isometric.
Finally assume that \( \ker(S(1-S)(2S-1)) = \{0\} \). Then, by the previous lemma and the modular relation, the isometricity follows from

\[
(\varphi_p^{1/2} e^{i[x \otimes y]} \varphi_p^{1/2}) = (\varphi_p^{1/2} e^{i[x \otimes 0]} e^{i[0 \otimes y]} \varphi_p^{1/2}) = (\varphi_p^{1/2} e^{i[x \otimes 0]} e^{i[y \otimes 0]} \varphi_p^{1/2}) \\
= (\varphi_S^{1/2} e^{-te^{i[a \otimes b]}y} \varphi_S^{1/2}) = (\varphi_S^{1/2} e^{i[a \otimes b]} e^{-i[a \otimes b]y}).
\]

\[\square\]

**Remark 2.** An antiunitary involution \( J \) (the modular conjugation of \( \varphi_S \)) is defined on the Hilbert space \( C^*(\hat{V}, \sigma) \varphi_p^{1/2} \) by

\[
J(e^{i[x \otimes y]} \varphi_p^{1/2}) = e^{-i[y \otimes x]} \varphi_p^{1/2}, \quad x, y \in V.
\]

### 5.2. Quadrade Polarizations

**Definition 5.7.** An admissible polarization \( P \) of a complete presymplectic vector space is said to be **quadrade** if the spectrum of \( \frac{P}{P+P} \) is included in \( \{0, \frac{1}{2}, 1\} \).

Let \((V, \sigma)\) be a hilbertian presymplectic vector space and set \( V_0 = \ker \sigma \). Let \( P \) and \( Q \) be admissible quadratic polarizations of \((V, \sigma)\) with \( P_0 \) and \( Q_0 \) the restriction of \( P \) and \( Q \) to the subspace \( V_0^C \) respectively.

Let \( E_0 \) be the orthogonal projection to the subspace \( V_0^C \) with respect to the inner product \( P + \overline{P} \). According to the decomposition \( V_0^C = V_0^C + (1 - E_0) V_0^C \), operators \((P + \overline{P}) \setminus P\) and \((Q + \overline{Q}) \setminus Q\) are represented in the following block form

\[
\begin{pmatrix}
P \\
Q \\
\overline{P} \\
\overline{Q}
\end{pmatrix}
= \begin{pmatrix}
1/2 & 0 & E \\
0 & 1/2 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

From the identity \((Q + \overline{Q}) \setminus Q + (Q + \overline{Q}) \setminus \overline{Q} = 1\), we have

\[
F + \overline{F} = 1 - E_0 = E + \overline{E}, \quad f + \overline{f} = 0.
\]

Let \( F_0 \) and \( F_1 \) be the spectral projections of \((Q + \overline{Q}) \setminus Q\) associated to eigenvalues \(1/2\) and \(1\) respectively. Then

\[
F_0 = \begin{pmatrix}
1 & 0 \\
0 & 2f F - F
\end{pmatrix}, \quad F_1 = \begin{pmatrix}
0 & 2f F \\
0 & F
\end{pmatrix}
\]

and consequently

\[
F^2 = F \quad (\iff \ F_1^2 = F_1), \quad F \overline{F} = 0 = \overline{F} F \quad (\iff \ F_1 \overline{F_1} = 0 = \overline{F_1} F_1).
\]

Note here that \( F \) and \( \overline{F} \) are *not* necessarily orthogonal relative to the inner product \( P + \overline{P} \).

**Lemma 5.8.** (Lemma 5.2). We have the following.

(i) \( (E - F)^2 = E \overline{F} F + F \overline{E} F = F E F + E F \overline{F} \) is negative with respect to the inner product \( P + \overline{P} \).

(ii) \( E - \overline{F} = -(E - F) \).

(iii) \( [E, (E - F)^2] = 0 = [F, (E - F)^2] \).

By a computation similar to the proof of this lemma, we also have

**Lemma 5.9.** When \( f = 0 \), \( (E - F)^2 \) is negative with respect to \( Q + \overline{Q} \) as well and, if we denote the kernel projection of \( (E - F)^2 \) within \((1 - E_0) V_0^C \) by \( c (c E_0 = E_0 c = 0) \), then \( c = c \) and

\[
P(cx, (1 - c)y) = 0 = Q(cx, (1 - c)y), \quad P(cx, cy) = Q(cx, cy).
\]
for \( x, y \in V^C \).

5.3. **Hilbert-Schmidt Approximations.** From here on, \((P + \overline{P}) \setminus P - (Q + \overline{Q}) \setminus Q\) and \((P + \overline{P}) \setminus (Q + \overline{Q}) - 1\) are assumed to be in the Hilbert-Schmidt class.

**Lemma 5.10.** We can find an increasing sequence of finite-dimensional subspaces \( \{V_n\}_{n \geq 1} \) with \( \cup_{n \geq 1} V_n \) dense in \( V \) and a sequence of admissible quadratic polarizations \( \{Q_n\}_{n \geq 1} \) such that

(i) if we denote by \( W_n = V \ominus V_n \) the \((P + \overline{P})\)-orthogonal complement of \( V_n \), then
\[
Q_n|_{V_n^C \times V_n^C} = Q|_{V_n^C \times V_n^C} \quad \text{and} \quad Q_n|_{W_n^C \times W_n^C} = P|_{W_n^C \times W_n^C}.
\]

(ii) \( \|Q_n - Q\|_{HS} \to 0 \) as \( n \to \infty \).

**Proof.** Given a bounded operator \( \gamma \) on \( V^C \) such that \( \overline{\gamma} = \gamma \) and \( \gamma = E_0 \gamma (1 - E_0) \),

\[
\Gamma = e^\gamma = \begin{pmatrix} E_0 & \gamma \\ 0 & 1 - E_0 \end{pmatrix}
\]

defines a presymplectic transformation of \((V, \sigma)\) and the composition \(Q\Gamma\) gives an admissible polarization.

If we set \( \gamma = -2f(1 - 2F) = E_0 - F_0 \) in the expression

\[
\frac{Q\Gamma}{Q\Gamma + Q^T} = \begin{pmatrix} 1/2 & f + \gamma(1 - 2F)/2 \\ 0 & F \end{pmatrix},
\]

then we see that \( Q' = Q\Gamma \) satisfies

\[
\frac{Q'}{Q' + Q^T} = \begin{pmatrix} 1/2 & 0 \\ 0 & F \end{pmatrix}.
\]

Choose an increasing sequence \( \{h_n\}_{n \geq 1} \) of projections so that (i) \( h_n \) is orthogonal relative to \( P + \overline{P} \) and \( Q' + \overline{Q}' \), (ii) \( h_n \) commutes with \( E \) and \( F \), (iii) \( h_n \) is of finite rank, (iv) \( h_n = h_n \) and (v) \( \lim_{n \to \infty} h_n = 1 - E_0 \). (Recall that \((E - F)^2\) is in the Hilbert-Schmidt class and \( P = Q_c \) on the kernel of \((E - F)^2\), see Lemma 5.9)

Then \( fh_nV^C \) is an increasing sequence of finite-dimensional \(*\)-invariant subspaces of \( V_0^C \) and we can find an increasing sequence \( \{g_n\}_{n \geq 1} \) of \((P + \overline{P})\)-orthogonal projections such that (i) \( g_n \) is of finite rank, (ii) \( \overline{g_n} = g_n \), (iii) \( \lim_{n \to \infty} g_n = E_0 \) and (iv) \( g_n f h_n = f h_n \).

Let \( e_n = g_n + h_n \). Then \( e_n \) is a \((P + \overline{P})\)-orthogonal projection such that (i) \( e_n \) is of finite rank, (ii) \( \overline{e_n} = e_n \) and (iii) \( \lim_{n \to \infty} e_n = 1 \).

Define an admissible polarization of \((\hat{V}, \hat{\sigma})\) by

\[
Q'_n(x, y) = Q'(e_n x, e_n y) + P((1 - e_n)x, (1 - e_n)y).
\]

In view of the fact that \( h_n \) is \((Q' + \overline{Q}')\)-orthogonal, we see \( Q'_n(x, y) = 0 \) for \( x \in E_0 V^C \) and \( y \in (1 - E_0)V^C \).

Noticing that \( e^{g_n(F_0 - E_0)h_n} \) is a presymplectic transformation due to \( g_n(F_0 - E_0)h_n = g_n(F_0 - E_0)h_n \), we finally introduce an admissible polarization by \( Q_n = Q'_n e^{g_n(F_0 - E_0)h_n} \).

From \( (1 - E_0)e_n = h_n \) and \([F, e_n] = 0\), we see that \( (F_0 - E_0)e_n = e_n(F_0 - E_0)e_n \) and hence

\[
e^{F_0 - E_0}e_n = e^{g_n(F_0 - E_0)h_n} e_n = e_n e^{g_n(F_0 - E_0)h_n} e_n = e_n e^{F_0 - E_0} e_n,
\]
which is used to identify restrictions of \( Q_n \): the results are

\[
Q_n(e_n x, e_n y) = Q(e_n x, e_n y),
\]

\[
Q_n(e_n x, (1 - e_n) y) = 0,
\]

\[
Q_n((1 - e_n) x, (1 - e_n) y) = P((1 - e_n) x, (1 - e_n) y).
\]

In this way, we have checked

\[
\frac{Q_n}{P + \overline{P}} = e_n \frac{Q}{P + \overline{P}} e_n + (1 - e_n) \frac{P}{P + \overline{P}} (1 - e_n)
\]

and therefore

\[
\frac{Q_n}{P + \overline{P}} = \frac{Q - P}{P + \overline{P}} + \frac{Q - P}{P + \overline{P}} (1 - e_n) + (1 - e_n) \frac{P - Q}{P + \overline{P}} (1 - e_n)
\]

converges to 0 in the Hilbert-Schmidt norm because

\[
\frac{Q - P}{P + \overline{P}} = \left( \frac{Q + \overline{Q}}{P + \overline{P}} - 1 \right) \frac{Q}{Q + \overline{Q}} + \left( \frac{Q}{Q + \overline{Q}} - \frac{P}{P + \overline{P}} \right)
\]

is in the Hilbert-Schmidt class.

Choose an auxiliary Hilbert-Schmidt operator \( \Theta : L \rightarrow V_0 \) so that \( \Theta L \) contains

\[
(E_0 - F_0)V^C \cup \bigcup_{n \geq 1} (V_n \cap V_0).
\]

This is possible because \( E_0 - F_0 \) is in the Hilbert-Schmidt class and \( \dim V_n < +\infty \).

Take \( \omega \in (\Theta L)^* \) and consider quasifree states on \( C^*(V/\omega, \sigma) \) as in §2.3.

**Lemma 5.11.** We have

\[
\frac{Q_n}{Q_n + \overline{Q_n}} = \begin{pmatrix} 1/2 & f h_n \\ 0 & F_n \end{pmatrix}
\]

with \( F_n = h_n F + (1 - h_n) E \) and the equality \( \varphi Q_n \omega = \varphi Q_n \omega \) holds on the C*-subalgebra \( C^*(V_n/\omega_n, \sigma_n) \) (\( \omega_n = \omega|_{V_n \cap V_0} \)).

**Proof.** Since \( e_n \) commutes with \( (Q' + \overline{Q'}) \setminus Q' \) and \( (P + \overline{P}) \setminus P \), we have

\[
Q'_n(x, y) = (e_n x, \frac{Q'}{Q' + \overline{Q'}} e_n y) Q' + ((1 - e_n) x, \frac{P}{P + \overline{P}} (1 - e_n) y)_P
\]

\[
= (e_n x, e_n \frac{Q'}{Q' + \overline{Q'}} e_n y) Q' + ((1 - e_n) x, (1 - e_n) \frac{P}{P + \overline{P}} (1 - e_n) y)_P,
\]

which is compared with the expression for \( (Q'_n + \overline{Q'_n})(x, y) \) to get

\[
\frac{Q'_n}{Q'_n + \overline{Q'_n}} = \frac{Q'}{Q' + \overline{Q'}} e_n + \frac{P}{P + \overline{P}} (1 - e_n) = \begin{pmatrix} 1/2 & 0 \\ 0 & F_n \end{pmatrix}
\]

and therefore

\[
\frac{Q_n}{Q_n + \overline{Q_n}} = e^{-g_n(F_0 - E_0) h_n} \frac{Q'_n}{Q'_n + \overline{Q'_n}} e^{g_n(F_0 - E_0) h_n} = \begin{pmatrix} 1/2 & f \frac{F}{F} h_n \\ 0 & F_n \end{pmatrix}.
\]

Thus, for \( x \in h_n V \subset V \), we have \( F_0 n x = F_0 x \). Since

\[
F_0 e_n = \begin{pmatrix} g_n & 2 f (F - F) h_n \\ 0 & 0 \end{pmatrix} = e_n F_0 e_n,
\]
it follows that
\[ \varphi_{Q_n, \omega}(e^{ix}) = e^{i\omega(F_{0,n}x)} e^{-Q_n((1 - F_{0,n})x)/2} = e^{i\omega(F_{0}x)} e^{-Q_n((1 - F_{0})x)/2} \]
\[ = e^{i\omega(F_{0}x)} e^{-Q((1 - F_{0})x)/2} = \varphi_{Q, \omega}(e^{ix}) \]
for \( x \in \mathbb{h}_n V \).

**Lemma 5.12.** Let \( E_0 \) (resp. \( F_0 \)) be the orthogonal projection to the subspace \( V_0^\mathbb{C} \) with respect to \( P + \overline{P} \) (resp. \( Q + \overline{Q} \)) and let \( D\omega : V/V_0 \rightarrow \mathbb{R} \) be defined by \( \langle D\omega, \dot{x} \rangle = \omega((E_0 - F_0)x) \). Set \( G = \dot{P} + \overline{P} + \dot{Q} + \overline{Q} \) with \( G^{-1} \) the inverse form on the dual vector space \((V^\mathbb{C}/V_0^\mathbb{C})^*\). Then we have

\[ \langle \varphi_{P, \omega}^{1/2} | \varphi_{Q, \omega}^{1/2} \rangle = \left( \frac{(\varphi_{P}^{1/2} | \varphi_{Q}^{1/2})}{e^{-G^{-1}(D\omega)}} \right)^{-1/2} \]

and

\[ \langle \varphi_{P}^{1/2} | \varphi_{Q}^{1/2} \rangle = \text{det} \left( \frac{(\dot{Q} + \overline{Q})}{P + \overline{P}} \right)^{1/2} \frac{(\dot{P} + \overline{P})}{Q + \overline{Q}} \right)^{-1/2} \]

**Proof.** We use the notation \( G_n \) to stand for the positive form \( \dot{P} + \overline{P} + \dot{Q}_n + \overline{Q}_n \) on \((V/V_0)^\mathbb{C}\). Let \( F_{0,n} \) be the spectral projection of \((Q_n + \overline{Q}_n)\) of an eigenvalue \( 1/2 \) and set \( \langle D_n \omega, \dot{x} \rangle = \omega((E_0 - F_{0,n})x) \) for \( x \in (V/V_0)^\mathbb{C} \). (Note that \((E_0 - F_{0,n})x \in V_n \cap V_0\).)

Since the subalgebras \( C_n^* = C^*(V_n/\omega_n, \sigma_n) \) of \( C^*(V/\omega, \sigma) \) \( (\omega_n = \omega|_{V_n \cap V_0}) \) meet the approximation condition in [27, Theorem 4.3], we see that

\[ \langle \varphi_{P, \omega}^{1/2} | \varphi_{Q, \omega}^{1/2} \rangle = \lim_{n \rightarrow \infty} \langle \varphi_{P, \omega} | C^*(V_n/\omega_n, \sigma_n) \rangle^{1/2} \langle \varphi_{Q, \omega} | C^*(V_n/\omega_n, \sigma_n) \rangle^{1/2} \]

which, in turn, is equal to

\[ \lim_{n \rightarrow \infty} \langle \varphi_{P, \omega} | C^*(V_n/\omega_n, \sigma_n) \rangle^{1/2} \langle \varphi_{Q, \omega} | C^*(V_n/\omega_n, \sigma_n) \rangle^{1/2} \]

by the above lemma.

As \( P \) and \( Q_n \) split according to the decomposition \( V = V_n \oplus W_n \), we obtain

\[ \langle \varphi_{P, \omega}^{1/2} | \varphi_{Q_n, \omega}^{1/2} \rangle = \langle \varphi_{P, \omega} | C^*(V_n/\omega_n, \sigma_n) \rangle^{1/2} \langle \varphi_{Q_n, \omega} | C^*(V_n/\omega_n, \sigma_n) \rangle^{1/2} \]

\[ = \langle \varphi_{P}^{1/2} | \varphi_{Q_n}^{1/2} \rangle \exp(-G_n^{-1}(D_n \omega)) \]

and

\[ \langle \varphi_{P}^{1/2} | \varphi_{Q_n}^{1/2} \rangle = \text{det} \left( \frac{C_{n}^{1/2} + C_{n}^{-1/2}}{2} \right)^{-1/2} \]

with \( C_n = \frac{\dot{Q}_n + \overline{Q}_n}{\dot{P} + \overline{P}} \).

Likewise, by extracting quotient parts, we have

\[ \langle \varphi_{P}^{1/2} | \varphi_{Q_n}^{1/2} \rangle = \lim_{n \rightarrow \infty} \langle \varphi_{P}^{1/2} | \varphi_{Q_n}^{1/2} \rangle \]

Since \( \dot{Q}_n \rightarrow \dot{Q} \) in the Hilbert-Schmidt topology as a part of the convergence \( Q_n \rightarrow Q \), we see \( C_n \rightarrow C = (\dot{P} + \overline{P}) \backslash (\dot{Q} + \overline{Q}) \) in the Hilbert-Schmidt topology; the determinant formula for \( \langle \varphi_{P}^{1/2} | \varphi_{Q}^{1/2} \rangle \) is proved.
Since $G_n \to G$ in the Hilbert-Schmidt topology and $G$ is invertible with $G^{-1}$ bounded, $G_n^{-1} \to G^{-1}$ in the Hilbert-Schmidt topology as well. From the expression of $(Q_n + \overline{Q}_n)/Q_n$ in the previous lemma, we see
\[
E_0 - F_{0,n} = \begin{pmatrix}
0 & -2g_n f h_n(T_n - F_n) \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
0 & 2f(F - T)h_n \\
0 & 0
\end{pmatrix}
\]
Since the domain of $\omega$ is chosen so that it includes $(E_0 - F_0)V$, the boundedness of $D\omega : x \mapsto \langle \omega, (E_0 - F_0)x \rangle$ is equivalent to $\omega(E_0 - F_0) \in (V/V_0)^\ast$ with $D\omega = \omega(E_0 - F_0)((V/V_0)^\ast$ being furnished with a hilbertian topology as the topological dual of the hilbertian space $V/V_0$) and hence $D_n\omega \to D\omega$ in $(V/V_0)^\ast$. Consequently we have
\[
\lim_{n \to \infty} G_n^{-1}(D_n\omega) = G^{-1}(D\omega)
\]
if $D\omega$ is bounded.

Contrarily assume that $D\omega$ is not bounded. Since $G_n^{-1} \to G^{-1}$ in the norm topology, we can find $\epsilon > 0$ so that $G_n^{-1} \geq \epsilon(\hat{P} + \overline{P})$ for $n \geq 1$. Then
\[
\liminf_{n \to \infty} G_n^{-1}(\omega(E_0 - F_0)h_n) \geq \lim_{n \to \infty} \epsilon \langle \omega(E_0 - F_0)h_n, h_n \rangle = +\infty.
\]

6. Transition Amplitude Formula

Our main goal here is a formula for the transition amplitude between square roots of quasifree states.

Recall that two positive forms $A$ and $B$ on a complex vector space $K$ is said to be equivalent if we can find a positive number $M > 0$ such that $A(x, x) \leq MB(x, x)$ and $B(x, x) \leq MA(x, x)$ for any $x \in K$. Equivalent positive forms $A$, $B$ are said to be **HS-equivalent** if $A - B$ is in the Hilbert-Schmidt class, where $A$ and $B$ are operators representing $A$ and $B$ on the completion of $K/\ker A = K/\ker B$ relative to a positive form equivalent to both of $A$ and $B$. Note here that the condition is independent of the choice of a reference inner product.

Two polarizations $S$ and $T$ of a presymplectic vector space are said to be **equivalent** if positive forms $(S^{1/2} + \overline{S}^{1/2})^2$ and $(T^{1/2} + \overline{T}^{1/2})^2$ are HS-equivalent.

**Remark 3** ([5, Proposition 6.6]). The above equivalence on polarizations is equivalent to requiring that (i) $S + \overline{S}$ and $T + \overline{T}$ are equivalent as positive forms and (ii) $S^{1/2} - \overline{T}^{1/2}$ is in the Hilbert-Schmidt class, where $S$ and $T$ are operators representing $S$ and $T$ on the completion-after-quotient of $V/\mathbb{C}$ relative to a positive form equivalent to $S + \overline{S}$.

**Theorem 6.1.** Let $S$, $T$ be polarizations of a presymplectic vector space $(V, \sigma)$ with the associated quasifree states denoted by $\varphi_S$, $\varphi_T$ and define positive forms by $2A = (S^{1/2} + \overline{S}^{1/2})^2$, $2B = (T^{1/2} + \overline{T}^{1/2})^2$. Then we have
\[
(\varphi_{S^{1/2}} | \varphi_{T^{1/2}}) = (\varphi_{A^{1/2}} | \varphi_{B^{1/2}}).
\]
Here the right hand side concerns states on the trivial presymplectic vector space $(V, 0)$, i.e., the gaussian states with covariance forms given by $A/2$ and $B/2$.

**Remark 4.** The correspondance $S \mapsto (S^{1/2} + \overline{S}^{1/2})^2$ is one-to-one.
From the results obtained so far (Corollary 4.2, Corollary 4.4 and Lemma 4.6), we know that both of the transition amplitudes in question are zero if \( S \) and \( T \) are not equivalent as polarizations of \((V, \sigma)\). So the equivalence of \( S \) and \( T \) is assumed in the remaining of this section. By Proposition 2.8, we may further assume that \( V \) is non-degenerate and complete relative to the inner products \( S + \bar{S} \) and \( T + \bar{T} \).

Recall that the rotated quadrature \( P_{\pi/4} \) of \( S \) is of the form

\[
P_{\pi/4} = \frac{1}{2} \begin{pmatrix} (S^{1/2} + \bar{S}^{1/2})^2 & \overline{S} - S \\ \overline{S} - S & (S^{1/2} - \bar{S}^{1/2})^2 \end{pmatrix},
\]

where \((S^{1/2} - \bar{S}^{1/2})^2\) is regarded as a positive form on \((V/V_0)^\mathbb{C}\) \((V_0 = \ker \sigma)\). To avoid confusion, we write \( \tilde{V} = V/V_0 \) and let \( \tilde{S} \) be the induced polarization on \( \tilde{V} \). Then the quotient form of \((S^{1/2} - \bar{S}^{1/2})^2\) on \( V^\mathbb{C} \) admits a continuous extension \((\tilde{S}^{1/2} - \bar{\tilde{S}}^{1/2})^2\) to the hilbertian completion \( W^\mathbb{C} \) of \( \tilde{V}^\mathbb{C} \), which is the precise meaning of the \((2,2)\)-component in the above matrix expression of \( P_{\pi/4} \). Recall also that

\[
P_{\pi/4} + \overline{P_{\pi/4}} = \text{diag}\left((S^{1/2} + \bar{S}^{1/2})^2, (\tilde{S}^{1/2} - \bar{\tilde{S}}^{1/2})^2\right).
\]

Now the rotated version of \((P + \overline{P}) \setminus P\) is given by

\[
\frac{P_{\pi/4}}{P_{\pi/4} + \overline{P_{\pi/4}}} = \left( \frac{P_{\pi/4} + \overline{P_{\pi/4}}}{(1)} \right)^{-1} \left( \frac{P_{\pi/4}}{(1)} \right) = \left( \frac{(\tilde{S}^{1/2} + \bar{\tilde{S}}^{1/2})^2}{0} \begin{pmatrix} 0 & (\tilde{S}^{1/2} - \bar{\tilde{S}}^{1/2})^2 \\ (\tilde{S}^{1/2} - \bar{\tilde{S}}^{1/2})^2 & (\tilde{S}^{1/2} + \bar{\tilde{S}}^{1/2})^2 \end{pmatrix} \right) \times \frac{1}{2} \begin{pmatrix} (S^{1/2} + \bar{S}^{1/2})^2 & \overline{S} - S \\ \overline{S} - S & (S^{1/2} - \bar{S}^{1/2})^2 \end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix} 1V & \bar{U}_S \\ \frac{S^{1/2} + \bar{S}^{1/2}}{S^{1/2} - \bar{S}^{1/2}} & 1W \end{pmatrix}.
\]

Here \((\cdot | \cdot)\) denotes the inner product \((\cdot, \cdot)_{S} \oplus (\cdot, \cdot)_{\bar{S}}\) on \(V^\mathbb{C} \oplus W^\mathbb{C}\) and the off-diagonal operators in the last line are considered to be

\(-\bar{U}_S = \frac{S^{1/2} - \bar{S}^{1/2}}{S^{1/2} + \bar{S}^{1/2}} : W^\mathbb{C} \to V^\mathbb{C}, \quad -U^*_S = \frac{S^{1/2} + \bar{S}^{1/2}}{S^{1/2} - \bar{S}^{1/2}} : V^\mathbb{C} \to W^\mathbb{C}\).

(Strictly speaking, these involve unbounded operators as well as an ill-defined inner product \((\cdot, \cdot)_{\bar{S}}\) on \( W^\mathbb{C} \) and therefore they should be considered as computations on spectral subspaces of \( S \).)

**Lemma 6.2.** Let \( P \) and \( Q \) be the quadratures of equivalent polarizations \( S \) and \( T \). Then operators

\[
\frac{P}{P + \overline{P}} - \frac{Q}{Q + \overline{Q}} = \frac{Q + \overline{Q}}{P + \overline{P}} - 1
\]

are in the Hilbert-Schmidt class as well.
Proof. First \( P + \overline{T} \) and \( Q + \overline{Q} \) are HS-equivalent because \((S^{1/2} - \overline{S}^{1/2})^2 - (T^{1/2} - \overline{T}^{1/2})^2 \) is in the Hilbert-Schmidt class as a quotient of \( 2(A - B) \). Then

\[
\frac{P}{P + \overline{T}} - \frac{Q}{Q + \overline{Q}} \sim \frac{P - Q}{P + \overline{T}} = \frac{1}{2} \frac{(P + \overline{T}) - (Q + \overline{Q})}{P + \overline{T}}
\]

is in the Hilbert-Schmidt class. □

Let \( e_0 \) be the \((S + \overline{S})\)-orthogonal projection to the subspace \( V_0^C \subset V^C \), which is the kernel projection of \( U_S^* \), i.e., \( 1 - e_0 = U_S U_S^* \), and set

\[
E_0 = \begin{pmatrix} e_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_1 = \frac{1}{2} \begin{pmatrix} 1V - e_0 & -U_S \\ -U^*_S & 1W \end{pmatrix}.
\]

The spectral decomposition of \((P_{\pi/4} + \overline{P_{\pi/4}})\backslash P_{\pi/4} \) is then given by

\[
\frac{P_{\pi/4}}{P_{\pi/4} + \overline{P_{\pi/4}}} = \frac{1}{2} E_0 + E_1.
\]

The same procedure is applied to \( Q \) and \( T \) to get the spectral decomposition \((Q_{\pi/4} + \overline{Q_{\pi/4}})\backslash Q_{\pi/4} \) is then given by

\[
F = \begin{pmatrix} (1V - e_0)(1V - f_0)(1V - e_0) & -(1V - e_0)U_T \\ -U^*_T(1V - e_0) & 1W \end{pmatrix}.
\]

Lemma 6.3. We have

\[
(\varphi^{1/2}_P | \varphi^{1/2}_Q) = (\varphi^{1/2}_{A/2} | \varphi^{1/2}_{B/2})^2.
\]

Proof. Instead of passing to the quotient, we assume that \( V_0 = \{0\} \) to avoid many dots. Then in the rotated form (the suffix \( \pi/4 \) being omitted also)

\[
\begin{align*}
P + \overline{T} &= \begin{pmatrix} (S^{1/2} + \overline{S}^{1/2})^2 & 0 \\ 0 & (S^{1/2} - \overline{S}^{1/2})^2 \end{pmatrix}, \\
Q + \overline{Q} &= \begin{pmatrix} (T^{1/2} + \overline{T}^{1/2})^2 & 0 \\ 0 & (T^{1/2} - \overline{T}^{1/2})^2 \end{pmatrix};
\end{align*}
\]

Thus

\[
\det \left( \frac{(Q+\overline{Q})^{1/2}}{P+\overline{T}} + \frac{(P+\overline{T})^{1/2}}{Q+\overline{Q}} \right)
\]

is equal to

\[
\det \left( \frac{(A\backslash B)^{1/2} + (B\backslash A)^{1/2}}{2} \right) \det \left( C^{1/2} + C^{-1/2} \right)
\]

with \( C = \frac{(S^{1/2} - \overline{S}^{1/2})^2}{(T^{1/2} - \overline{T}^{1/2})^2} \)

and the problem is reduced to showing that

\[
\det \left( \frac{(A\backslash B)^{1/2} + (B\backslash A)^{1/2}}{2} \right) = \det \left( C^{1/2} + C^{-1/2} \right).
\]
Proof. Recall that the rotated is in the Hilbert-Schmidt class by Lemma 6.2. In Lemma 5.11 is bounded if and only if so is the functional $\Delta$ which should be compared with $\langle U \rangle$. Now we use the relation $U_T^*(A \setminus B)U_T = C$ to have $$U_T^* \left( \frac{B}{A} + \frac{A}{B} + 2 \right)U_T = C + C^{-1} + 2$$ and we are done by the similarity invariance of determinants. \hfill \Box

Choose a Hilbert-Schmidt map $\Theta: L \rightarrow V_0 = \ker \sigma$ so that it has a dense range and satisfies the condition in §5.3 (ker $\sigma$ being identified with $V_0$ by the canonical isomorphism). Recall that

$$\frac{1}{4}(E_0 - F_0) = \frac{P}{P + \varepsilon} - \left( \frac{P}{P + \varepsilon} \right)^2 - \frac{Q}{Q + \varepsilon} + \left( \frac{Q}{Q + \varepsilon} \right)^2$$

is in the Hilbert-Schmidt class by Lemma 6.2.

Lemma 6.4. For a linear functional $\omega: \Theta L \rightarrow \mathbb{R}$, the functional $D\omega: V \oplus W \rightarrow \mathbb{R}$ in Lemma 5.11 is bounded if and only if so is the functional $\Delta\omega: \hat{V} \rightarrow \mathbb{R}$ in Proposition 3.8, and we have

$$G^{-1}(D\omega) = (\hat{A} + \hat{B})^{-1}(\Delta\omega).$$

Proof. Recall that the rotated $G_{\pi/4}$ is a positive form on $V^C \oplus W^C$ defined by

$$G_{\pi/4}(\hat{a} \oplus b, \hat{x} \oplus y) = 2(\hat{A} + \hat{B})(\hat{a}, \hat{x}) + (S^{1/2} - S^{1/2})^2(b, y) + (T^{1/2} - T^{1/2})^2(b, y),$$

where $\hat{a}, \hat{x} \in V^C$ and $b, y \in W$. Since an isometric isomorphism $v_0: V_0 \rightarrow \ker(\sigma \oplus -\sigma)_{\pi/4}$ is given by $v_0(x_0) = x_0/\sqrt{2} \oplus 0 \in V \oplus W$ and $(E_0 - F_0)(\hat{x} \oplus y) = (e_0 - f_0)x \oplus 0$, $D\omega$ is equal to the composition

$$\begin{align*}
V \oplus W &\longrightarrow V_0 \oplus W \\
\hat{x} \oplus y &\longrightarrow (e_0 - f_0)x \oplus 0 \\
&\longrightarrow \sqrt{2}(e_0 - f_0)x \\
&\longrightarrow \sqrt{2}\langle \omega, (e_0 - f_0)x \rangle
\end{align*}$$

which should be compared with $\langle \Delta\omega, \hat{x} \rangle = \langle \omega, (e_0 - f_0)x \rangle$.

To evaluate these functionals by inverse forms, we introduce the representing vectors $\hat{a} \in \hat{V}$ and $\hat{a} + b \in \hat{V} \oplus W$ by the relation

$$\langle \Delta\omega, \hat{x} \rangle = (\hat{A} + \hat{B})(\hat{a}, \hat{x}), (D\omega, \hat{x} \oplus y) = G_{\pi/4}(\hat{a} \oplus b, \hat{x} \oplus y).$$

The equality $\sqrt{2}\langle \Delta\omega, \hat{x} \rangle = (D\omega, \hat{x} \oplus y)$ then forces us to have $\sqrt{2}a = \hat{a}$, $b = 0$ and therefore

$$G^{-1}(D\omega) = G(\hat{a} \oplus 0) = 2(\hat{A} + \hat{B})(\hat{a}) = (\hat{A} + \hat{B})(\hat{a}) = (\hat{A} + \hat{B})^{-1}(\Delta\omega).$$

\hfill \Box

Proof of Theorem: Combining lemmas 5.10, 6.3, 6.4 and [27] Proposition 5.2], we have

$$\langle \phi_{A/2}^{1/2}, \phi_{B/2}^{1/2} \rangle = \langle \phi_{A/2}^{1/2}, \phi_{B/2}^{1/2} \rangle e^{-G^{-1}(D\omega)/2} = \langle \phi_{A/2}^{1/2}, \phi_{B/2}^{1/2} \rangle e^{-(A+B)^{-1}(\Delta\omega)}.$$
Since $A$ and $B$ are HS-equivalent, the spectral decomposition of $\frac{B}{A} - 1$ enables us to apply Proposition 3.11 to get
\[
(\varphi_{A/2}^{1/2} | \varphi_{B/2}^{1/2}) e^{-(\hat{A} + \hat{B})^{-1}(\Delta \omega)} = (\varphi_{A/2,\omega}^{1/2} | \varphi_{B/2,\omega}^{1/2}).
\]
Finally we use Corollary 2.11 twice in the following to get the formula:
\[
(\varphi_S^{1/2} | \varphi_T^{1/2}) = \int_\Omega \sqrt{\nu_S} \nu_T (d\omega) (\varphi_S^{1/2} | \varphi_T^{1/2}) = \int_\Omega \sqrt{\nu_S} \nu_T (d\omega) (\varphi_{A,\omega}^{1/2} | \varphi_{B,\omega}^{1/2})^{1/2} = \int_\Omega \sqrt{\nu_S} \nu_T (d\omega) (\varphi_{A,\omega}^{1/2} | \varphi_{B,\omega}^{1/2})^{1/2} = (\varphi_{A/2}^{1/2} | \varphi_{B/2}^{1/2}).
\]

**Appendix A. Domination and Ratio Operators**

Let $K$ be a complex vector space and $A$, $B$ be sesquilinear forms on $K$. We set $A^*(x, y) = A(y, x)$ for $x, y \in K$. Assume that $B$ is positive, i.e., $B(x, x) \geq 0$ for any $x \in K$. A form $A$ is **dominated** by $B$ if there is $\lambda > 0$ such that
\[
|A(x, y)| \leq \lambda \sqrt{B(x, x)B(y, y), \quad \forall x, y \in K.}
\]
Let $K_B$ be the Hilbert space associated to a positive $B$ with the inner product denoted by $(\ , )_B$: $K_B$ is the completion of the quotient space $K/\ker B$. Then we can find a bounded operator $\hat{A} : K_B \to K_B$ so that $A(x, y) = ([x]_B | [y]_B)_B$. The operator $\hat{A}$ is referred to as a ratio operator and denoted by
\[
B \setminus A = \frac{A}{B} = A / B.
\]
We have $(B \setminus A)^* = B \setminus A^*$, where the adjoint operation is taken relative to the inner product $(\ | )_B$ in $K_B$.

Assume that $A$ is positive as well. Then the condition of domination is equivalent to $A(x, x) \leq \lambda B(x, x)$ for all $x \in K$ (denoted by $A \leq \lambda B$) by Schwarz' inequality. Two positive forms $A$, $B$ are said to be **equivalent** if each of $A$ and $B$ is dominated by the other. Note that, if this is the case, $K_A = K_B$ and the ratio operator $A \setminus B$ is bounded with the bounded inverse $B \setminus A$. Given a bounded linear operator $O$ on $K_A = K_B$, let $O^A$ (resp. $O^B$) be the adjoint of $O$ with respect to the inner product $(\ | )_A$ (resp. $(\ | )_B$). Then we have
\[
O^B = \frac{A}{B} O^A B \frac{A}{B}.
\]
Particularly $O = A \setminus B$ is self-adjoint relative to $(\ | )_B$ and
\[
\left\| \frac{B}{A} \right\|_A = \sup \left\{ \frac{\|x\|_B}{\|x\|_A} : 0 \neq x \in K_A = K_B \right\} = \left\| \frac{A}{B} \right\|_B^{-1}.
\]

**Remark 5.** The operator $A \setminus B$ is positive relative to $(\ | )_B$ because $(A \setminus B)^{1/2}$ is $B$-hermitian.

Let $R = (A \setminus B)^{1/2}$ and regard it as a unitary map $(K_B, (\ | )_B) \to (K_A, (\ | )_A)$. Then, for a bounded operator $O$ on $K_A = K_B$, the commutativity of the diagram
\[
\begin{array}{ccc}
K_B & \xrightarrow{R} & K_A \\
\downarrow & & \downarrow R O R^{-1} \\
K_B & \rightarrow & K_A
\end{array}
\]
shows that \( \|O\|_B = \|RO\|_A^{-1} \| \) and
\[
\sum_j \|O_{\eta_j}\|_B^2 = \sum_j \|RO\|_A^{-1} R_{\eta_j}\|_A^2 = \sum_j \|RO\|_A^{-1} \xi_j\|_A^2
\]
with \( \{\xi_j\} \) and \( \{\eta_j\} \) orthonormal bases for \((|)A\) and \((|)B\) respectively.

**Lemma A.1.** Let \( C \) be a sesquilinear form. Then
\[
(AB)(B\mid C) = A\mid C.
\]

**Proof.** Just compute as
\[
(x\mid BA) = (x\mid C) = (x\mid C) = (x\mid C) = (x\mid C).
\]

**Remark 6.** Notice that \( \frac{C}{B} \neq \frac{C}{A} \) generally: The backslash notation is therefore safer in applying cancellation.

Given equivalent positive forms \( A, B \), we introduce the projective distance \( \delta(A, B) \) between them by
\[
\delta(A, B) = \inf\{\log(\lambda\mu); A \leq \lambda B, B \leq \mu A\}.
\]
The projective distance is in fact a distance on the set of rays and we have \( \delta(A, B) \geq 0 \) and \( \delta(A, B) = 0 \) if and only if \( A \) and \( B \) are proportional.

**Definition A.2.** Let \( Q \) and \( Q' \) be positive quadratic forms on a finite-dimensional real vector space and assume that \( Q' \) is dominated by \( Q \). Then the relative determinant is defined by
\[
\det\left(\frac{Q'}{Q}\right) = \frac{\det_{1 \leq j, k \leq n}(Q'(v_j, v_k))}{\det_{1 \leq j, k \leq n}(Q(v_j, v_k))}.
\]
Here \( \{v_j\} \subset V \) is any representative family of basis in \( V' \).

**Appendix B. Gaussian Measures**

We shall review here relevant results on gaussian measures, which can be found in standard textbooks such as [20, 12].

Let \( \mathbb{R}^\infty \) be the set of sequences of real numbers and \( \mathcal{B}(\mathbb{R}^\infty) \) be the product Borel structure. Given a sequence \( \alpha = (\alpha_n)_{n \geq 1} \) of positive numbers, let \( \nu_\alpha \) be the infinite product measure of gaussian measures of variance \( \alpha_j (j \geq 1) \): If we denote by \( X_j : \mathbb{R}^\infty \to \mathbb{R} \) the projection random variable to \( j \)-th component, then
\[
\int_{\mathbb{R}^\infty} \nu_\alpha(dx) e^{t_j \sum_{j=1}^{n} t_j X_j(x)} = e^{-\sum_{j=1}^{n} \alpha_j t_j^2 / 2}.
\]

**Proposition B.1.** Let \( \alpha, \beta \in \mathbb{R}^\infty_+ \) and set
\[
\mathbb{R}_{\beta}^\infty = \{x = (x_j) \in \mathbb{R}^\infty; \sum_{j=1}^{\infty} \beta_j x_j^2 < +\infty\}.
\]

Then
\[
\nu_\alpha(\mathbb{R}_{\beta}^\infty) = \begin{cases} 1 & \text{if } \sum_j \alpha_j \beta_j < +\infty, \\ 0 & \text{otherwise}. \end{cases}
\]

In other words, \( \sum_j \beta_j x_j^2 < +\infty \) for \( \nu_\alpha \)-a.e. \( x \in \mathbb{R}^\infty \) if \( \sum_j \alpha_j \beta_j < +\infty \), and \( \sum_j \beta_j x_j^2 = +\infty \) for \( \nu_\alpha \)-a.e. \( x \in \mathbb{R}^\infty \) if \( \sum_j \alpha_j \beta_j = +\infty \).
Given an admissible positive form \(S\) on a separable real Hilbert space \(V\), we can realize the Gaussian random process indexed by \((V,S)\) in terms of the topological dual \((\Theta V)^*\) of \(\Theta V\). Here \(\Theta : V \to \Theta V\) is any invertible operator in the Hilbert-Schmidt class and \(V\) is a Hilbert space so that \(V \ni x \mapsto \Theta x \in \Theta V\) is a topological isomorphism. Note also that the topological dual \((\Theta V)^*\) is identified with \(V^*\Theta^{-1}\): any \(f \in (\Theta V)^*\) is of the form \(f(\Theta x) = g(x) (x \in V)\) with \(g \in V^*\).

Let \(B\) be the Boolean algebra of Borel sets in \(V^*\Theta^{-1}\).

**Proposition B.2.** We can find a probability measure \(\nu_S\) on \(((\Theta V)^*, B)\) such that
\[
\int_{(\Theta V)^*} e^{\langle x, \omega \rangle} \nu_S(d\omega) = e^{-S(x)/2}
\]
and
\[
\int_{(\Theta V)^*} \langle x, \omega \rangle \langle y, \omega \rangle \nu_S(d\omega) = S(x,y)
\]
for \(x, y \in \Theta V\).

Furthermore, the correspondence \(\Theta V \ni x \mapsto \langle x, \cdot \rangle \in L^2((\Theta V)^*, \nu_S)\) is extended to a Gaussian random process \(V \to L^2((\Theta V)^*, \nu_S)\) indexed by \((V,S)\). In other words, \(x \mapsto \langle x, \cdot \rangle\) is continuous relative to the subtopology of \(V^*\): if \(\lim x_n = x\) in \(V\) with \(x_n \in \Theta V\) and \(x \in V\), then \(\{\langle x_n, \cdot \rangle\}\) is convergent in \(L^2(\Omega, \nu_S)\).

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