FUNDAMENTAL GROUPS OF RANDOM CLIQUE COMPLEXES

Abstract. Clique complexes of Erdős-Rényi random graphs with edge probability between $n^{-\frac{2}{3}}$ and $n^{-\frac{1}{2}}$ are shown to be aas not simply connected. This entails showing that a connected two dimensional simplicial complex for which every subcomplex has fewer than three times as many edges as vertices must have the homotopy type of a wedge of circles, two spheres and real projective planes. Note that $n^{-\frac{1}{2}}$ is a threshold for simple connectivity and $n^{-\frac{1}{3}}$ is one for vanishing first $\mathbb{F}_2$ homology.

1. Introduction

If $n$ is a positive integer and $p \in [0, 1]$ is a probability write $K(n, p)$ for the probability measure on 2-dimensional simplicial complexes obtained by taking vertex set $[n] = \{1, \ldots, n\}$ and edges chosen from all $\binom{n}{2}$ possibilities independently each with probability $p$ and all triangles for which all three edges were chosen. This is the 2-skeleton of the clique complex of the Erdős-Rényi random graph. Write aas for asymptotically almost surely where the limit involved is $\lim_{n \to \infty}$.

Theorem 1.1. For any $\epsilon > 0$ and $n^{-\frac{1}{2}} \leq p_n \leq n^{-\epsilon - \frac{1}{2}}$ the group $\pi_1(K(n, p_n))$ is aas hyperbolic and nontrivial.

This is proven largely by following the notation and blueprint in BHK ([1]). The main difference here is

Theorem 1.2. If $X$ is a finite connected two dimensional simplicial complex for which every subcomplex $Y$ has $f_0(Y) - f_1(Y) > \frac{1}{3}$ then $X$ has the homotopy type of a wedge of circles, two spheres and real projective planes and contains a subcomplex with the homotopy type of a wedge of circles and real projective planes for which the inclusion induces an isomorphism of fundamental groups.

Here $f_i(Y)$ is the number of $i$-dimensional faces in $Y$. This is a corollary of theorem 2.1 and replaces BHK Lemma 4.16 in which $f_0(Y) - f_1(Y) > \frac{1}{3}$ is replaced by $f_0(Y) - f_1(Y) > \frac{1}{2}$.

Note that if $p_n \leq n^{-1-\epsilon}$ then $K(n, p_n)$ is aas a disconnected forest and if $n^{-1+\epsilon} \leq p_n \leq n^{-\frac{1}{3}+\epsilon}$ then by Lemma 3.8 $K(n, p_n)$ is aas connected and collapsible to a graph with cycles. If $n^{-\frac{1}{3}+\epsilon} \leq p_n$ then $K(n, p_n)$ is aas simply connected from Kahle’s [3] Theorem 3.4.

2. Definitions

Recall webs from BHK Definition 4.5, $L$ from Definition 4.6 and modify Definition 4.7 to call a web $W$ $k$-admissible if every $Y \subseteq W$ has $(L + k\chi)Y > 0$. Note that BHK studies 2-admissible webs and this note studies 3-admissible ones.

Theorem 2.1. (Related to Lemma 4.16 of BHK) If $W$ is a connected 3-admissible 2-dimensional web with $g(W) \geq 3$ then $|X|$ has the homotopy type of a wedge of circles, two spheres and real projective planes.

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Lemma 2.2. If the theorem fails then there is a counterexample $W$ with $\delta(W) \geq 2$.

Proof: Recall from BHK the Definition 4.14 of $K(W)$ which has $\delta(K(W)) \geq 2$. $K(W)$ is also a counterexample no larger than $W$. □

As in the proof of Lemma 4.22 in BHK write the admissibility sum locally as

$(L + 3\chi)W = \sum_v K_v + \sum_e K_e + \sum_m K_m$ where the sums are over faces of $W$ with empty boundary and dimensions zero, one and two respectively.

Lemma 2.5. (Related to BHK 4.18) If $W$ is normal choose $N_v(W)$ a disconnected link at $v$ of $W$ and consider $f : W' \to W$ the normalization map and note that every subweb $Y' \subseteq W'$ has $LY' = LY$ and $\chi Y' \leq \chi Y$ so that $W'$ is also admissible and $W' < W$ in the above order since $f^2 W' = f^2 W$, $LW' = LW$ and $\chi W' = \chi W - 1$. Thus some component of $W'$ is a smaller counterexample since $|W|$ is the wedge of the components of $|W'|$ and some possibly circles.

If $W$ is not 2-normal choose $N_v(W)$ with a cut point and consider the zipping map $z : W' \to W$ so that $W'$ has one more vertex and one more edge than $W$. Note that for any $Y' \subseteq W'$ there is $LY' \geq LzY'$ and $\chi Y' \geq \chi z Y'$ so that $W'$ is also admissible and $W' < W$ in the above partial order since $f^2 W' = f^2 W$, $\chi W' = \chi W$, $LW' = LW + 2\mu$. Since $|z|$ is a homotopy equivalence this makes $W'$ a smaller counterexample.

Recall from BHK Lemma 4.11 the Definition of $C(W)$, which is defined for 2-normal webs is again a counterexample and $W \leq CW$, so $W = CW$ and hence $\delta(W) \geq 3$. □

Lemma 2.6. (Related to BHK 4.20) If $W$ is a minimal counterexample then $W$ has no monogons or digons.

Proof: If $F$ is a digon in $W$ with edges $e$ and $f$ having $\mu e \leq \mu f$ consider the collapse of $W'$ to the shorter edge $e$ and the collapse map $\phi : W \to W'$. Note that for every subweb $Y' \subseteq W'$ there is $Y = \phi^{-1}Y'$ and sometimes $Y_e = Y - \{e, F\}$ or $Y_f = Y - \{f, F\}$ are also subwebs. Each of these has $\chi Y'_e = \chi Y'$ and at least one of $LY'_e \leq LY'$ so that $W'$ is also admissible and $W' < W$ in the above order since $f^2 W' = f^2 W - 1$. 
If $F$ is a monogon in $W$ with edge $e$ consider the collapse $\phi : W \rightarrow W'$ of $F$ to a point. Note that for every subweb $Y' \subseteq W'$ there is $Y = \phi^{-1}Y'$ and sometimes $Y_e = Y - \{e, F\}$ is also a subweb. Both of these has $\chi Y_e = \chi Y'$ and at least one of $LY_e(\phi) \leq LY'$ so that $W'$ is also admissible and $W' < W$ in the above order since $f^2W' = f^2W - 1$. \hfill \qed

Lemma 2.7. If $W$ is a minimal counterexample with $v$ a vertex, $e$ and $f$ edges containing $v$, $c$ a circular 1-face, $F$ a 2-face containing $e$ and $f$ and $G$ a 2-face containing $c$ then each of the following variables is a non negative integer:

\[ f^1v = f^1v - 3, \]
\[ f^2e = f^2e - 3, \]
\[ \mu_e = \mu_e - 1, \]
\[ \hat{\chi}F = -\chi F + 1, \]
\[ \hat{\alpha}(v, e, f, F) = \hat{\mu}e + \hat{\mu}f, \]
\[ \hat{m}(v, e, f, F) = \mu\partial F - \hat{\alpha}(v, e, f, F) - 3, \]
\[ \mu\partial(c, G) = \mu\partial G - f^Gc\mu c \text{ and } f^Gc = f^Gc - 1. \]

Here $f^i m$ is the number of $i$-dimensional faces containing the face $m$ and $f^Gc$ is the degree of the map from the boundary of $G$ to $c$. See BHK.

Note that if $v$ is a vertex of $W$ then using

\[ \sum_{\{e \mid v \in e\}} \hat{\mu}e = - \sum_{\{e \mid v \in e\}} \frac{1}{3} \hat{\mu}ef^2e + \sum_{\{e, f, v\mid v \in e, e \in F, e \neq f\}} \frac{1}{3} \hat{\alpha}(v, e, f, F) \]

yields

\[
= 3 \frac{3}{2} f^1v + \sum_{\{e, f, v\mid v \in e, e \in F\}} \frac{3\hat{\chi}F \hat{\alpha}(\mu e + \mu f)}{\mu\partial F} + \sum_{\{e \mid v \in e\}} \mu e - \sum_{\{e, f, v\mid v \in e, e \in F\}} \frac{1}{2} (\mu e + \mu f) \\
= 3 \frac{3}{2} f^1v - \sum_{\{e \mid v \in e\}} \frac{1}{3} \hat{\mu}ef^2e - \frac{3\hat{\chi}F + \hat{m}(v, e, f, F) + \hat{\alpha}(v, e, f, F)}{3 + \hat{m}(v, e, f, F) + \hat{\alpha}(v, e, f, F)} \\
= 3 \frac{3}{2} f^1v - \sum_{\{e \mid v \in e\}} \frac{1}{3} \hat{\mu}ef^2e - \frac{3\hat{\chi} + \hat{m} + \hat{\alpha}[rac{3\hat{\chi}}{3} + \frac{1}{3} \hat{m} + \frac{1}{3} \hat{\alpha}]}{3 + \hat{m} + \hat{\alpha}}.
\]

Similarly, if $c$ is a circular one dimensional face of $W$ then

\[ K_c = \mu c \left[ 2 - \sum_{\{G, c \in G\}} f^Gc(\hat{f}^Gc + \hat{\chi}G)\mu c + \hat{\mu}\partial(c, G) \right] / f^Gc\mu c + \hat{\mu}\partial(c, G). \]

Finally if $m$ is two dimensional with empty boundary then

\[ K_m = 3\chi(m). \]

Since $(L + 3\chi)W > 0$ there is some face $F$ with empty boundary and $K_F > 0$. If $m$ is 2 dimensional, without boundary and $K_m > 0$ then $\chi(W) > 0$ so $|W|$ is a sphere or projective plane.
Lemma 2.8. If $W$ is a minimal counterexample and $c$ is a circular face then $K_c \leq 0$.

Proof: Assume $K_c > 0$. If $G$ contains $c$ and $\hat{\partial}(c, G) = 0$ then the contribution of $G$ to $\frac{K_c}{\mu_c} = -f^G_c - \hat{\chi}G$ so only twice wrapped disks ($\hat{f} = 1$, $\hat{\chi} = 0$) cross caps ($\hat{f} = 0$, $\hat{\chi} = 1$) and singly wrapped disks ($\hat{f} = 0$, $\hat{\chi} = 0$) can occur if $K_c$ is to be positive. If $\hat{\partial}(c, G) \neq 0$ then $\hat{\mu}(c, G) \geq g(W) \geq 3$ and $\hat{\chi}G \geq 1$ so that $\hat{f}^G_c = 0$. The only faces which do not subtract at least one are the singly wrapped disks but if $W$ is a minimal counterexample and $c$ a circular face there is at most one of these.

This leaves only the case of one doubly wrapped and one singly wrapped disk, which has the homotopy type of a sphere and is therefore not a counterexample. \qed

Lemma 2.9. If $W$ is a minimal counterexample and $v$ is a vertex then $K_v \leq 0$.

Proof: Assume that $v$ is a vertex and $K_v > 0$.

Lemma 2.10. (only long double edges in links) If $W$ is a minimal counterexample, $K_v \geq K_u$ for every $u$ adjacent to $v$ and there are edges $e$ and $f$ and 2-faces $F$ and $G$ with $F$ and $G$ forming a double edge connecting $e$ and $f$ in the link of $v$ then $\hat{\mu}(F) > 2(\hat{\mu} + f)$ (or equivalently $\hat{m}(F) > \hat{\mu} + \hat{\mu}f + 1$).

Note that this implies that every double edge subtracts at least $\frac{4}{9}$ from $K_v$.

Proof: Assume not and consider $j : W'' \rightarrow W$ the deletion of $G$ and $i : W'' \rightarrow W'$ the addition of $G'$ which slides $G$ across $F$. Note that $|W|$ and $|W''|$ are homotopy equivalent and if $G' \in X' \subseteq W'$ then $(L + 3\chi)X' \geq (L + 3\chi)Y$ for either $Y = Y' - G' + G$ or $Y = Y' - G' + G + F$ so that $W'$ is admissible. The former works if $e$ and $f$ are in $Y'$, in which case $\chi Y' = \chi Y$ and $\mu Y' = \mu Y - \hat{\mu}F + 2\hat{\mu} + 2f \geq LX'$. Otherwise the latter works, with four cases depending on the intersection of $Y'$ with $e$ and $f$. If the intersection is empty then $LY' = LY$ and $\chi Y' = \chi Y$. If the intersection is only $v$ then $LY' = LY$ and $\chi Y' = \chi Y + 1$. If the intersection is an edge (wlog $e$) then $LY' = LX + 2\hat{\mu}$ and $\chi Y' = \chi Y$. Also $X' < X$ in the above order since $f^2X' = f^2X$, $\chi X' = \chi X$, $LX' = LX + \hat{\mu}F - 2\mu - 2f \leq LX$ and $K_v > K_u$. \qed

Lemma 2.11. (only long triangles in links) If $W$ is a minimal counterexample, $K_v \geq K_u$ for every $u$ adjacent to $v$ and there are edges $e$, $f$ and $g$ and 2-faces $E$, $F$ and $G$ with $E$, $F$ and $G$ forming the edges $e$, $f$ and $g$ the vertices of a triangle in the link of $v$ then $mF + mE - 2\mu > 2(\mu + \mu f)$ (or equivalently $\hat{m}(E) + \hat{m}(F) > \hat{\mu} + \hat{\mu}f + 2$).

Note that this implies that every triangle in the link of $v$ subtracts at least $\frac{3}{7}$ from $K_v$ and every square with diagonal subtracts at least $\frac{6}{7}$.

Proof: Assume not and consider $j : X'' \rightarrow X$ the deletion of $G$ and $i : X'' \rightarrow X'$ the addition of $G'$ which slides $G$ across $E$ and $F$. Note that $|X|$ and $|X'|$ are homotopy equivalent and if $G' \in X' \subseteq X'$ then $(L + 3\chi)X' \geq (L + 3\chi)Y$ for $Y = Y' - G' + G$ or $Y = Y' - G' + G + F + E$. Also $X' < X$ in the above order since $f^2X' = f^2X$, $\chi X' = \chi X$, $LX' \geq LX$ and $K_v > K_u$. \qed

A case analysis now eliminates any minimal counterexample, proving Lemma 2.9. \qed
This completes the proof of Theorem 2.1. □

Proof of Theorem 1.2: The first part follows from Theorem 2.1. Since Theorem 2.1 also holds for subcomplexes the argument in the proof of Theorem 4.1 in BHK completes the proof. □

3. FUNDAMENTAL GROUPS

The fundamental group restriction is much like in BHK.

Definition 3.1. If $X$ is a 2-dimensional connected simplicial complex then

$$e_1^0 X = \min_{Y \subseteq X} \frac{f_0^0 Y}{f_1 Y}$$

if also $X$ contains the vertices $\{1, \ldots, w\}$ then

$$e_1^0 X_w = \min_{\{1, \ldots, w\} \subseteq Y \subseteq X} \frac{f_0^0 Y - w}{f_1 Y}.$$  

This is similar to $e$ in BHK, but involves the ratio of vertices to edges rather than to 2-faces.

Lemma 3.2. If $X$ is a 2-dimensional connected simplicial complex with $e_1^0 X > \frac{1}{3}$ then

$$f_1^1 X \leq \frac{3\chi X - 3w + LX}{3e_1^0 X_w - 1}.$$  

Proof: See the proof of BHK Lemma 5.1. □

Lemma 3.3. For every $\epsilon > \frac{1}{3}$ there is $\beta$ so that every connected 2-complex with $e_1^0 (X) > \epsilon$, $L(X) \leq 0$ and $\chi(X) \leq 1$ and any contractible loop $\gamma : C_r \to X$ satisfies $A(\gamma) < \beta L(\gamma)$.

Proof: See the proof of BHK 5.2. In this case the bound on $f^1$ from Lemma 3.2 replaces that on $f^2$ to yield only finitely many complexes to check. □

Lemma 3.4. For every $\epsilon > \frac{1}{3}$ there is some $\beta$ so that every minimal filling $(C_r \to D \to X)$ with $e_1^0 X \geq \epsilon$ and $\chi Z \leq 1$ for every connected $Z \subseteq X$ has

$$f^2 D < \beta (r + f^2 (D - D_{\leq 0})).$$  

Proof: See the proof of BHK 5.9 and the definition before 5.5 in BHK. □

Lemma 3.5. If $\epsilon > \frac{1}{3}$ and $(C_r \to D \to X)$ is a minimal filling with $e_1^0 X \geq \epsilon$ then

$$f^2 (D - D_{\leq 0}) \leq \frac{8r}{9e - 3}.$$  

Proof: See the proof of BHK 5.10 and use $LX^\pi_{ij} = 2f^1 X^\pi_{ij} - 3f^2 X^\pi_{ij} \geq 1$ so that $(3e - 1)f^1 X^\pi_{ij} \geq \frac{2}{3}(3e - 1)f^2 X^\pi_{ij}$. □

Lemma 3.6. For every $\epsilon > \frac{1}{3}$ there is $\lambda$ such that for every $X$ with $e_1^0 X \geq \epsilon$, every contractible loop $\gamma : C \to X$ satisfies

$$A_\gamma \leq \lambda L_\gamma.$$
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Proof: See the proof of BHK 3.7.

Lemma 3.7. If $X$ has an embedded cycle $\gamma : C_6 \to X$ having $\gamma(2i) = i$ for every $i \in \{1, 2, 3\}$ and $e_1^0 X > \frac{1}{3}$ then $\gamma$ is not contractible in $X$.

Proof: (See the proof of BHK 3.13) If $\gamma = (1, a, 2, b, 3, c)$ is a contractible cycle in $X$ then by the second part of theorem 2.1 there is $Z \subseteq X$ such that every connected $Z' \subseteq Z$ has $\chi Z' \leq 1$ and $\gamma$ is contractible in $Z$. Let $(C_6 \to D \to Z)$ be a minimal filling of $\gamma$ in $Z$. By BHK Lemma 5.4, $\pi$ is a 1-immersion so that no images of interior edges contribute positively to $L(\gamma)$ and

$$L(\text{Im}(\pi)) \leq LD \leq 6.$$ 

By Lemma 3.1 there is

$$f^1(\text{Im}(\pi)) \leq \frac{3\chi(\text{Im}(\pi)) - 3 \cdot 3 + L(\text{Im}(\pi))}{3e_1^0(\text{Im}(\pi))_3 - 1} \leq \frac{3 - 9 + 6}{\cdots} = 0.$$ 

This is a contradiction and $\gamma$ is not contractible in $X$.

Definition 3.8. (See BHK Definition 3.9) A 2-dimensional simplicial complex $X$ is $(e_1^0, m, r)$-sparse if every 2-dimensional simplicial subcomplex $Z \subseteq X$ containing the vertices $\{1, \ldots, r\}$ with $f^0 Z \leq m$ satisfies $e_1^0 Z_r < \epsilon$. It is $(e_1^0, m, r)$-full if every such complex $Z$ occurs as a subcomplex of $X$.

Lemma 3.9. If $m$ and $r$ are positive integers, $\epsilon > 0$ and every $p_n \leq n^{-\epsilon}$ then $K(n, p_n)$ is aas $(e_1^0, m, r)$-sparse, while if every $p_n \geq n^{-\epsilon}$ then $K(n, p_n)$ is aas $(e_1^0, m, r)$-full.

Proof: See the proof of BHK 3.10 for the sparsity. Full follows from an easy second moment argument.

Lemma 3.10. For every $\epsilon > \frac{1}{3}$ there are $m$ and $\rho$ such that every contractible loop $\gamma : C_r \to X$ in an $(e_1^0, m, 0)$-sparse complex $X$ satisfies $A(\gamma) < \rho L(\gamma)$.

Proof: See the proof of the first part of Lemma 3.12 in BHK and use Lemma 3.5 in place of BHK Lemma 3.7.

Proof of Theorem 1.1: Since $\epsilon < \frac{1}{3}$ Lemma 3.9 with $r = 3$ implies that $K(n, p_n)$ has aas a cycle $\gamma : C_6 \to X$ with $\gamma(2i) = i$ for $i \in \{1, 2, 3\}$. By Lemma 3.6 $\gamma$ is aas not contractible in $X$.

I am assured by Matthew Kahle that the arguments his paper [4] give an aas spectral gap larger than $\frac{1}{2}$ for appropriate Laplacians at all vertex links of $K(n, p_n)$ if $p_n \geq n^{-\frac{1}{2} + \epsilon}$ and that this together with a Garland type argument of Žuk imply

Theorem 3.11. If $\epsilon > 0$ and $n^{-\frac{1}{2} + \epsilon} \leq p_n \leq n^{-\frac{1}{2} - \epsilon}$ then $\pi_1(K(n, p_n))$ aas has Kazhdan’s property T.
4. Questions

Write $K_4(n, p)$ for the measure on cell complexes given by adding a two cell to all possible cycles of length 3 and of length 4 in the Erdős-Rényi random graph $K(n, p)$ and $\pi_1(K_4(n, p))$ for the associated measure on groups.

**Question:** For which $\epsilon$ is $\pi_1(K_4(n, n^{-\epsilon}))$ a.s. trivial?

For this question 4-admissible webs appear to replace the 3-admissible ones arising in the clique complexes, but the local reduction methods used here do not seem to work as easily.

**References**

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