GLOBAL WELL-POSEDNESS OF THE VELOCITY-VORTICITY-VOIGT MODEL OF THE 3D NAVIER-STOKES EQUATIONS

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Abstract. The velocity-vorticity formulation of the 3D Navier-Stokes equations was recently found to give excellent numerical results for flows with strong rotation. In this work, we propose a new regularization of the 3D Navier-Stokes equations, which we call the 3D velocity-vorticity-Voigt (VVV) model, with a Voigt regularization term added to momentum equation in velocity-vorticity form, but with no regularizing term in the vorticity equation. We prove global well-posedness and regularity of this model under periodic boundary conditions. We prove convergence of the model’s velocity and vorticity to their counterparts in the 3D Navier-Stokes equations as the Voigt modeling parameter tends to zero. We prove that the curl of the model’s velocity converges to the model vorticity (which is solved for directly), as the Voigt modeling parameter tends to zero. Finally, we provide a criterion for finite-time blow-up of the 3D Navier-Stokes equations based on this inviscid regularization.

Keywords: Vorticity-Velocity formulation, Euler-Voigt, Navier-Stokes-Voigt, Global existence, Inviscid-regularization, Turbulence models, Blow-up criteria, Voigt-regularization, Turbulence models, α-models, Mathematics Subject Classification: 35A01, 35B44, 35B65, 35Q30, 35Q35, 76D03, 76D05, 76D17, 76N10

1. Introduction

In recent years, the Voigt-regularization and the velocity-vorticity formulation have seen much study as promising approaches to alleviating some of the analytical and computational difficulty inherent in the 3D Navier-Stokes equations of incompressible fluid flow. However, as one might expect, neither of these approaches overcomes every difficulty in the equations. For instance, the Voigt-regularization has a strong regularizing effect, so much so that it destroys certain fundamental qualities of the equations, such as parabolicity and viscosity-driven energy decay. On the other hand, the velocity-vorticity formulation is merely a reformulation of the equations, and therefore it has no regularizing effect at all, although it is
the basis of many well-behaved numerical algorithms. In this paper, we combine these two approaches, with the intent that the resulting system will retain the best qualities of both systems. Namely, the intent is that the new system will have solutions that are closer to the actual physics of fluids, while still having enough regularization that the equations are better behaved from the standpoints of mathematical analysis, numerical stability, and computational efficiency. In this work, we only address the global well-posedness and convergence properties of the system, but a follow-up work will study the numerical and computational properties of the system.

The incompressible, constant density, 3D Navier-Stokes equations are given by

$$(1.1a) \begin{cases} \frac{\partial \tilde{u}}{\partial t} - \nu \Delta \tilde{u} + (\tilde{u} \cdot \nabla)\tilde{u} + \nabla \tilde{p} = f, \\ \nabla \cdot \tilde{u} = 0, \\ \tilde{u}(\cdot, 0) = \tilde{u}_0, \end{cases}$$

$$\text{(1.1b)} \quad \text{for more details on 3D Navier-Stokes equations. Note that for smooth solutions, the momentum equation (1.1a) can also be written as}$$

$$\frac{\partial \tilde{u}}{\partial t} - \nu \Delta \tilde{u} + (\nabla \times \tilde{u}) \times \tilde{u} + \nabla (\tilde{p} + \frac{1}{2} |\tilde{u}|^2) = f,$$

where $\tilde{u}$ represents the velocity of the fluid, $\tilde{p}$ represents the (density normalized) pressure, and $f$ represents a body force. We now propose the following system, which we refer to as the velocity-vorticity-Voigt (or “VVV”) equations over the three-dimensional periodic box $T^3 = \mathbb{R}^3 / \mathbb{Z}^3 = [0, 1]^3$,

$$\text{(1.3a)} \quad \begin{cases} (I - \alpha^2 \Delta) \frac{\partial \tilde{u}}{\partial t} - \nu \Delta \tilde{u} + w \times u + \nabla p = f, \\ \frac{\partial w}{\partial t} - \nu \Delta w + (u \cdot \nabla)w - (w \cdot \nabla)u = \nabla \times f, \\ \nabla \cdot u = 0, \\ u(\cdot, 0) = u_0 \quad \text{and} \quad w(\cdot, 0) = w_0, \end{cases}$$

where $u = (u_1, u_2, u_3)$ represents an averaged velocity, $w = (w_1, w_2, w_3)$, which plays the role of vorticity but for which we do not assume $w = \nabla \times u$, and $f$ is an external forcing term. Without loss of generality, we assume for our analysis in later sections that the viscosity $\nu = 1$. Note that in the case where $\alpha = 0$, the system formally reduces the velocity-vorticity formulation, while for $\alpha > 0$, if one imposes $w = \nabla \times u$, the system formally reduces to the Navier-Stokes-Voigt equations.

The term $-\alpha^2 \Delta \partial_t u$ in (1.3a) is often referred to as the “Voigt-term”, due to an application of modeling Kelvin-Voigt fluids by A.P. Oskolkov \cite{49, 50} (see also \cite{28}). In the context of the velocity formulation use of the Voigt term was first proposed as a regularization for either the Navier-Stokes (for $\nu > 0$) or Euler (for $\nu = 0$) equations in \cite{4}, for small values of the regularization parameter $\alpha$. This paper also proved global well-posedness of the Voigt-regularized versions of the 3D Euler and 3D Navier-Stokes equations. These equations have been studied analytically and extended in a wide variety of contexts (see, e.g., \cite{3, 6, 4, 36, 45, 44, 42, 41, 40, 39, 38, 37, 36, 35, 34, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3}, and the references therein). Voigt-regularizations of parabolic equations are a special case of pseudoparabolic equations, that is, equations of the form $Mu_t + Nu = f$, where $M$ and $N$ are (possibly non-linear, or even non-local) operators. For more about pseudoparabolic equations, see, e.g., \cite{13, 52, 51, 50, 49, 48, 47, 46, 45, 44, 43, 42, 41, 40, 39, 38, 37, 36, 35, 34, 33, 32, 31, 30, 29, 28, 27, 26, 25, 24, 23, 22, 21, 20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3}. 


Directly computing for the vorticity variable has recently become popular in simulations of the incompressible Navier-Stokes equations, as it can be the primary variable of interest in vortex dominated and rotating flows [63, 42, 43, 21, 23, 64, 24]. Such formulations use the equation of vorticity dynamics,

\[
\frac{\partial \tilde{w}}{\partial t} - \nu \Delta \tilde{w} + (\tilde{u} \cdot \nabla)\tilde{w} - (\tilde{w} \cdot \nabla)\tilde{u} = \nabla \times f,
\]

and close the system with some relation of \(\tilde{u}\) and \(\tilde{w}\), the most common being \(-\Delta \tilde{u} = \nabla \times \tilde{w}\), and boundary conditions such as \(\tilde{w} \mid_{\partial \Omega} = \nabla \times \tilde{u}\). While such a boundary condition is easily and accurately implementable in finite difference methods on uniform grids, it is not generally appropriate for finite element methods on unstructured meshes. For this reason, in the recent works [47, 40, 46, 24], the system is instead closed by coupling to a momentum equation using the \(\tilde{w}\) variable, such as

\[
\tilde{u}_t + \tilde{w} \times \tilde{u} + \nabla \tilde{P} - \Delta \tilde{u} = f,
\]

where \(\tilde{P}\) represents the Bernoulli pressure. This formulation is able to produce efficient and accurate numerical methods in such settings, in particular due to its use of a natural vorticity boundary condition corresponding to no-slip velocity derived in [46].

Because of the numerical successes of such velocity-vorticity systems, it is both natural and important to consider their analysis at the PDE level, as fundamental questions such as well-posedness should be addressed. It is easy to show that determining the global well-posedness of such a system (i.e. \(\alpha = 0\)) would solve the Millennium Prize Problem for the 3D Navier-Stokes equations. We choose in this work to consider the velocity-vorticity system with a Voigt modeling term in (1.3), for several reasons: first, it allows for analysis of the system to be performed; second, the VVV system limiting behavior as \(\alpha \to 0\) can give insight into the behavior of the \(\alpha = 0\) case; third, the discretized Voigt term corresponds to a commonly used numerical stabilization for second (and lower) order methods [51, 16, 13, 2, 33], and thus the VVV system is in this sense the PDE generalization of stabilized numerical discretizations of Navier-Stokes equations in velocity-vorticity form. We note that a steady velocity-vorticity system without regularization terms was analyzed in [45] in the case of no slip velocity boundary conditions, no penetration vorticity boundary conditions, and a natural tangential condition for vorticity that is weakly implemented in a boundary functional involving the pressure; well-posedness of this system was proven, however, it (seemingly) required some new analytic techniques.

We note that we do not add a Voigt modeling term \(-\alpha^2 \Delta \frac{\partial w}{\partial t}\) to the vorticity equation of the system. This could be done, and could make sense as a continuous level generalization of a vorticity equation stabilization. However, from an analysis point of view, it is more challenging to consider the system (1.3) where the Voigt modeling is only applied to the momentum equation, and extension is straightforward for the case when Voigt modeling is also applied to the VVV vorticity equation. Moreover, it may lead to more accurate capturing of the vorticity at the numerical level. The reason for this is that in rooted in a computational study [32] of the magnetohydrodynamic (MHD) equations with Voigt-regularization. Voigt-regularization for MHD was first proposed and studied in [37], with further study in [6, 7, 38]. The MHD system, with Voigt-regularization added only to the momentum equation, is strikingly similar to system (1.3). Indeed, if one were to add the term \((w \cdot \nabla)w\) to the right-hand side of equation (1.3a), the systems would be identical in the \(f = 0\) case. In [32], it was found that in computational tests of the 2D case on a coarse mesh, putting a Voigt term only on (the MHD analogue of) equation (1.3a) resulted in
a better match of level curves of the current density (the analogue of $\nabla \times w$) in fine-mesh simulations than putting a Voigt regularization on both equation, or neither equation. This is the reason for us only applying Voigt-regularization to equation (1.3a).

**Remark 1.1.** We note that the analysis of the 3D VVV system is somewhat distinct that the analysis of the 3D MHD system with Voigt-regularization added only to the momentum equation. This is because the cancellation of the nonlinear terms and that occurs in energy estimates for the MHD-Voigt equations does not occur in the VVV system, and therefore one must deal directly with the analogue of the vortex-stretching term $(w \cdot \nabla)u$. The key is to notice that one may first obtain an energy estimate purely in terms of $u$, and then use this bound to obtain a bound on $w$. Higher-order estimates on $u$ are not $w$-independent, but can be obtained using a bootstrapping technique, going back and forth between the two equations.

**Remark 1.2.** One might ask whether, in the inviscid (i.e., $\nu = 0$) case, results analogous to those in this paper still hold. This is especially in light of global well-posedness results for the so-called Euler-Voigt equations [4, 37], which are formally the inviscid version of the Navier-Stokes-Voigt equations. Two fundamental differences arise. Firstly, the vorticity stretching term $(w \cdot \nabla)u$ can no longer be controlled in the same way, as higher-order derivatives cannot be absorbed into the viscosity. Thus, one must resort to higher-order estimates, but as in case of the 3D Navier-Stokes equations and related $\alpha$-models [13, 26, 10, 9, 25, 8, 11], it is far from clear how to close these estimates. Secondly, in the proof of convergence as $\alpha \to 0$ (Theorem 2.11 below), the estimates depend crucially on the fact that $\int_{0}^{T} \|\nabla u(t)\|_{L^2}^2 \, dt$ is bounded independently of $\alpha \in (0, 1]$, which is a property that one does not have in the Euler-Voigt equations. Thus, it is not clear how to extend the results of this paper to the inviscid version of the VVV system.

The paper is organized as follows. We first provide the necessary preliminaries for our work in subsequent sections in Section 2, then, we define weak and strong solutions to system (1.3), and state our main theorems. In Section 3 we prove the existence and uniqueness of global weak solution for (1.3) by Galerkin approximation following the ideas from [12, 61] (also c.f.[35] with similar approach in full details). In view of the similar behavior of $w$ and the vorticity $\omega = \nabla \times u$, we prove that $w$ indeed tends to $\omega$ in $L^2$ norm as $\alpha \to 0$ in Section 4 as well as the convergence of the velocity in (1.3) to that of the Navier-Stokes equations. We point out that the numerical and computational studies of the VVV system will be the subject of a forthcoming work.

2. Preliminaries and Main Results

2.1. Preliminaries. All through this paper $C$ represents some absolute constant varying line by line, and similarly $C_\alpha$ indicates the dependence of the constant on $\alpha$. We denote $\phi_j = \partial \phi / \partial x_j$ and $\phi_t = \partial \phi / \partial t$. Also, we denote the mean-free versions of the usual Lebesgue and Sobolev spaces on $\mathbb{T}$ by $L^p$ for $1 \leq p \leq \infty$ and $H^s = W^{s,2}$ for $s > 0$, respectively; we denote by $C_w(I; X)$ the space of weakly continuous functions from an interval $I$ to a Banach space $X$. Let $\mathcal{F}$ be the set of all trigonometric polynomials over $\mathbb{T}^3$ and define the subset of $\mathcal{F}$ with divergence-free and zero-average trigonometric polynomials

$$\mathcal{V} := \left\{ \phi \in \mathcal{F} : \nabla \cdot \phi = 0, \text{ and } \int_{\mathbb{T}^3} \phi \, dx = 0 \right\}.$$
We follow the standard convention of denoting by $H$ and $V$ the closures of $V$ in $L^2$ and $H^1$, respectively, with inner products

$$(v, \overline{v}) = \sum_{i=1}^{3} \int_{\mathbb{T}^3} v_i \overline{v}_i \, dx$$

and

$$((v, \overline{v})) = \sum_{i,j=1}^{3} \int_{\mathbb{T}^3} \partial_j v_i \partial_i \overline{v}_i \, dx,$$

respectively, associated with the norms $\|v\|_H = (v, v)^{1/2}$ and $\|v\|_V = ((v, v))^{1/2}$. For the sake of convenience, we use $\|v\|_{L^2}$ and $\|v\|_{H^1}$ to denote the above norms in $H$ and $V$, respectively. The latter is a norm due to the Poincaré inequality

$$(2.5) \quad \sqrt{\lambda_1} \|\phi\|_{L^2} \leq \|\nabla \phi\|_{L^2},$$

holding for all $\phi \in V$, where $\lambda_1$ is the first eigenvalue of the Stokes operator $A$ discussed below. Note that we also have the following compact embeddings (see, e.g., [12, 61])

$$V \hookrightarrow H_{\alpha} \hookrightarrow V', \quad V' \hookrightarrow H_{\alpha} \hookrightarrow V,$$

where $V'$ denotes the dual space of $V$.

We denote by $H_{\text{curl}}$ a subspace of $H$ whose elements are in $H$ and their curl (taken in the distributional sense) is in $L^2$, i.e.,

$$H_{\text{curl}} := \{ f \in H \mid \nabla \times f \in L^2 \}, \quad \text{with norm} \quad \|f\|_{H_{\text{curl}}} := (\|f\|_{L^2}^2 + \|\nabla \times f\|_{L^2}^2)^{1/2},$$

and $H_{\text{curl}}^s$ the subspace of $V$ whose elements are in $H^s \cap V$ and their curl is in $H^s$, i.e.,

$$H_{\text{curl}}^s := \{ f \in V \mid \nabla \times f \in H^s \cap V \} \quad \text{with norm} \quad \|f\|_{H_{\text{curl}}^s} := (\|f\|_{H^s}^2 + \|\nabla \times f\|_{H^s}^2)^{1/2}.$$  

For more discussion on the curl-spaces, we refer the readers to [15, 22] and the references therein.

The following interpolation result is frequently used in this paper (see, e.g., [14] for a detailed proof). Assume $1 \leq q, r \leq \infty$, and $0 < \gamma < 1$. For $f \in L^q(\mathbb{T}^n)$, such that $\partial^\alpha v \in L^r(\mathbb{T}^n)$, for $|\alpha| = m$, then

$$(2.6) \quad \|\partial^\alpha v\|_{L^p} \leq C\|\partial^\alpha v\|_{L^r}^{\frac{s}{n}} \|v\|_{L^n}^{1-\gamma}, \quad \text{where} \quad \frac{1}{p} - \frac{s}{n} = \left(\frac{1}{r} - \frac{m}{n}\right) \gamma + \frac{1}{q} (1 - \gamma).$$

The following results are standard in the study of fluid dynamics, in particular for the Navier-Stokes equations and related PDEs, and we refer to reader to [12, 61] for more details. We define the Stokes operator $A := -P_\sigma \Delta$ with domain $\mathcal{D}(A) := D(A)$, where $P_\sigma$ is the Leray-Helmholtz projection. Notice that due to the periodic boundary conditions, it holds that $A = -\Delta P_\sigma$. Moreover, the Stokes operator can be extended as a linear operator from $V$ to $V'$ such that

$$(Av, \overline{v}) = ((v, \overline{v})) \quad \text{for all} \quad v, \overline{v} \in V.$$

It is well-known that $A^{-1} : H \hookrightarrow \mathcal{D}(A)$ is a positive-definite, self-adjoint, compact operator from $H$ into itself, and $H$ possesses an orthonormal basis of eigenfunctions $\{w_k\}_{k=1}^\infty$ of $A^{-1}$, corresponding to a sequence of non-increasing sequence of positive eigenvalues. Therefore, $A$ has non-decreasing eigenvalues $\lambda_k$, i.e., $0 < \lambda_1 \leq \lambda_2, \ldots$ since $\{w_k\}_{k=1}^\infty$ are also eigenfunctions of $A$. Furthermore, for any integer $M > 0$, we define $H_M := \text{span}\{w_1, w_2, \ldots, w_M\}$ and $P_M : H \to H_M$ be the $L^2$-orthogonal projection onto $H_M$. Next, for any $v, \overline{v}, w \in \mathcal{V}$, we introduce the convenient notation for the bilinear term

$$B(v, \overline{v}) := P_\sigma((v \cdot \nabla)\overline{v}),$$
which can be extended to a continuous map \( B : V \times V \to V' \), such that, for smooth functions \( v, \bar{v}, w \in V \),

\[
\langle B(v, \bar{v}), w \rangle = \int_{\Omega} (v \cdot \nabla)\bar{v} \cdot w \, dx.
\]

Moreover \( B \) has certain symmetry properties, and can be extended as a continuous map \( B \) on various spaces, as in the following lemma, which is proved in, e.g., [12, 20].

**Lemma 2.1.** The operator \( B \) can be extended to an operator, still denoted by \( B \), over spaces indicated below, with the following properties.

\[
\begin{align*}
(2.7a) \quad & \langle B(u, v), w \rangle_V = -\langle B(u, w), v \rangle_V, & \forall \, u, v, w \in V, \\
(2.7b) \quad & \langle B(u, v), v \rangle_V = 0, & \forall \, u, v, w \in V, \\
(2.7c) \quad & |\langle B(u, v), w \rangle_V| \leq C|u|^{1/2}_L \|\nabla u\|^{1/2}_L \|\nabla v\|_L \|\nabla w\|_L, & \forall \, u, v, w \in V, \\
(2.7d) \quad & |\langle B(u, v), w \rangle_V| \leq C|\nabla u|_L \|\nabla v\|_L \|w\|^{1/2}_L \|\nabla w\|_L, & \forall \, u, v, w \in V, \\
(2.7e) \quad & |\langle B(u, v), w \rangle_V| \leq C|u|^{1/2}_L \|\nabla v\|^{1/2}_L \|\nabla w\|_L, & \forall \, u \in H, v, w \in D(A), \forall \, v, w \in V, \\
(2.7f) \quad & |\langle B(u, v), w \rangle_V| \leq C|\nabla u|_L \|\nabla v\|^{1/2}_L \|w\|^{1/2}_L \|\nabla w\|_L, & \forall \, u \in V, v \in D(A), \forall \, w \in H, \\
(2.7g) \quad & |\langle B(u, v), w \rangle_V| \leq C|\nabla u|^{1/2}_L \|\nabla v\|_L \|w\|^{1/2}_L \|\nabla w\|_L, & \forall \, u \in D(A), v \in V, \forall \, w \in H, \\
(2.7h) \quad & |\langle B(u, v), w \rangle_V| \leq C|\nabla u|_L \|\nabla v\|_L \|w\|^{1/2}_L \|\nabla w\|_L, & \forall \, u \in H, v \in D(A), \forall \, v, w \in V.
\end{align*}
\]

If we formally apply \( P_\sigma \) to equation (1.3a) and (1.3b), we obtain the following functional formulation of system (1.3)

\[
\begin{align*}
(2.8a) \quad & \frac{d}{dt}(u + \alpha^2 Au) + Au + P_\sigma(w \times u) = P_\sigma f, \\
(2.8b) \quad & \frac{dw}{dt} + Au + B(u, w) - B(w, u) = \nabla \times f,
\end{align*}
\]

Formulation (2.8), taken to hold in the sense of \( L^2(0, T; V') \), can be shown to be equivalent to formulation (1.3). In particular, the pressure gradient can be recovered using a corollary of a deep result of G. de Rham. The corollary states that, for any distribution \( g \), the equality \( g = \nabla p \) holds for some distribution \( p \) if and only if \( \langle g, w \rangle = 0 \) for all \( w \in V \). See [22] for an elementary proof of the corollary.

We recall the Agmon inequalities in 3D (see, e.g., [12] [11]). Namely, for any \( \phi \in D(A) \),

\[
\|\phi\|_{L^\infty} \leq C\|\nabla \phi\|^{1/2}_L \|A\phi\|^{1/2}_L \quad \text{and} \quad \|\phi\|_{L^\infty} \leq C\|\phi\|^{1/2}_L \|A\phi\|^{3/4}_L.
\]

The following Aubin-Lions Compactness Lemma is needed in order to construct solutions for (1.3).

**Lemma 2.2.** Let \( X, Y, \) and \( Z \) be separable, reflexive Banach spaces, where \( X \) is compactly embedded in \( Y \), and \( Y \) is continuously embedded in \( Z \). Let \( T > 0, p \in (1, \infty) \) and let \( \{f_n(t, \cdot)\}_{n=1}^\infty \) be a bounded sequence in \( L^p([0, T]; X) \) such that \( \{|\partial f_n/\partial t\}_{n=1}^\infty \) is bounded in \( L^p([0, T]; Z) \). Then \( \{f_n\}_{n=1}^\infty \) has a strongly convergent subsequence in \( C([0, T]; Y) \).

Typically in the theory of the Navier-Stokes equations, to write \( \frac{1}{2} \frac{d}{dt} \|u\|_2^2 = (u_t, u) \), one needs \( u_t \in L^2(0, T; V') \), \( u \in L^2(0, T; V) \) using the Lions-Magenes Lemma (cf. [11] p. 176) or [24] Corollary 7.3). However, in our context, we have \( u, u_t \in L^2(0, T; V) \). Therefore, the following lemma is useful. The proof
is a straight-forward exercise via mollification in time, and follows closely the proofs of the Lions-Magenes Lemma in the aforementioned references.

**Lemma 2.3.** Let $V$ be a Hilbert space with norm $\| \cdot \|$ and inner product $(\cdot, \cdot)$. Suppose that for some $T > 0$, $u \in L^2(0, T; V)$ and $u_t \in L^2(0, T; V)$. Then the following equality holds in the scalar distribution sense on $(0, T)$.

$$
\frac{1}{2} \frac{d}{dt} \|u\|^2 = ((u_t, u)).
$$

For the sake of completeness, we state the following uniform Grönwall’s inequality, proved in [27] (see also [18] and the references therein), which will be used frequently throughout the paper.

**Lemma 2.4.** Suppose that $Y(t)$ is a locally integrable and absolutely continuous function that satisfies the following:

$$
\frac{dY}{dt} + \alpha(t)Y \leq \beta(t), \quad \text{a.e. on } (0, \infty),
$$

such that

$$
\lim \inf_{t \to \infty} \int_s^{t+\tau} \alpha(s) \, ds \geq \gamma, \quad \text{lim sup}_{t \to \infty} \int_s^{t+\tau} \alpha^{-}(s) \, ds < \infty,
$$

and

$$
\lim_{t \to \infty} \int_s^{t+\tau} \beta^{+}(s) \, ds = 0,
$$

for fixed $\tau > 0$, and $\gamma > 0$, where $\alpha^{-} = \max\{-\alpha, 0\}$ and $\beta^+ = \max\{\beta, 0\}$. Then, $Y(t) \to 0$ at an exponential rate as $t \to \infty$.

We record the following local well-posedness result for strong solutions to equations (1.1) (see, e.g., [60], Theorem 4.2).

**Theorem 2.5.** Let $\overline{u}_0 \in H^s \cap V$, $f \in L^2(0, T; H^{s-1} \cap H)$ for some $s \geq 0$. Then there exists a $T > 0$ and unique solution $(\overline{u}, \overline{p})$ to (1.1) such that $\overline{u} \in C([0, T]; H^s \cap V) \cap L^2(0, T; H^{s+1} \cap V)$.

2.2. Main results of the paper. We first define the weak and strong solutions to system (1.3), respectively.

**Definition 2.6.** Let $T > 0$ be arbitrary. Suppose $u_0 \in V$, $w_0 \in H$, and $f \in L^2(0, T; H)$. We call the pair $(u, w)$ a weak solution on the time interval $[0, T]$ to system (1.3), if $u \in C(0, T; V)$, $u_t \in L^2(0, T; V)$, $w \in C_w(0, T; H) \cap L^2(0, T; V)$, $w_t \in L^2(0, T; H^{-1})$, and moreover, $(u, w)$ satisfies system (1.3) in the weak sense, i.e.,

$$
\begin{align*}
\alpha^2((u_t, \psi)) + (u_t, \psi) + ((u, \psi)) + \langle w \times u, \psi \rangle = (f, \psi), \\
\langle w_t, \psi \rangle + ((w, \psi)) - \langle B(u, \psi), w \rangle - \langle \overline{B}(w, u), \psi \rangle = -(f, \nabla \times \psi),
\end{align*}
$$

holds for any $\psi \in L^2(0, T; V)$.

Note that by taking $\psi = v\phi$ for $v \in V$ and $\phi \in C^1_c((0, T))$, it follows that formulation (2.8) is equivalent to formulation (2.10), interpreted as an operator equation holding in an appropriate distributional sense.

**Definition 2.7.** Let $T > 0$ be an arbitrarily given time. Suppose $u_0 \in V$, $w_0 \in V$, and $f \in L^2(0, T; H_{\text{curl}})$. We call the pair $(u, w)$ a strong solution on the time interval $[0, T]$ to system (1.3), if it is a weak solution as in Definition 2.6 and satisfies additionally $w \in C([0, T]; V) \cap L^2(0, T; D(A))$, and $w_t \in L^2(0, T; H)$. 
The following theorem provides the global existence and uniqueness of weak solution to system (1.3).

**Theorem 2.8.** Suppose $u_0 \in V$, $w_0 \in H$, and $f \in L^2(0,T;H)$. Then, the velocity-vorticity-Voigt system (1.3) possesses a unique global weak solution $(u, w)$ in the sense of Definition 2.2 that satisfies $\nabla \cdot w = 0$. Moreover, the following energy equality holds.

$$
\alpha^2 \|\nabla u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(s) - \nabla u(t)\|_{L^2}^2 \, ds = \alpha^2 \|\nabla u_0\|_{L^2}^2 + \|u_0\|_{L^2}^2 + 2 \int_0^t (u(s), f(s)) \, ds
$$

The next theorem is about the global existence and uniqueness of strong solution to system (1.3), as well as the higher-order regularity of the solution.

**Theorem 2.9.** For the initial data $u_0 \in V$, $w_0 \in V$, and $f \in L^2(0,T;H\text{curl})$, there exists a unique strong solution $(u, w)$ in the sense of Definition 2.2. Moreover, if we further assume that the initial data $w_0 \in H^s \cap V$, $w_0 \in H^s \cap V$, and $f \in L^2(0,T;H^{s-1})$ for $s \geq 2$, $s \in \mathbb{N}$, then, the solution $u \in C_w(0,T;H^s \cap V)$ and $w \in C_w(0,T;H^s \cap V) \cap L^2(0,T;H^{s+1} \cap V)$.

The following theorem relates the quantity $w$ in (1.3b) to the vorticity $\omega = \nabla \times u$.

**Theorem 2.10.** Denote by $\omega := \nabla \times u$ the vorticity of the flow and let $u_0 \in H^1 \cap V$, $f \in H^2$. Then, we have

$$
\|\omega(t) - w(t)\|_{L^2}^2 + \alpha^2 \|\nabla \omega(t) - \nabla w(t)\|_{L^2}^2 + \int_0^t \|\nabla \omega(s) - \nabla w(s)\|_{L^2}^2 \, ds \leq C_0 \alpha^2 + \frac{\tilde{K} \alpha^2}{C} (e^{Ct} - 1),
$$

where $C_0$ depends on the initial data and $\tilde{K}$ is explained in the proof. If we further assume $w_0 = \nabla \times u_0$, then,

$$
\|\omega(t) - w(t)\|_{L^2}^2 + \alpha^2 \|\nabla \omega(t) - \nabla w(t)\|_{L^2}^2 + \int_0^t \|\nabla \omega(s) - \nabla w(s)\|_{L^2}^2 \, ds \leq K \alpha^2 (e^{Ct} - 1),
$$

for a.e. $t > 0$, i.e., $\|w - \omega\|_{L^2(0,T;L^2)} \sim O(\alpha)$ and $\|w - \omega\|_{L^2(0,T,V)} \sim O(\alpha)$. In particular, we have $\|w - \omega\|_{L^2(0,T;L^2)} \to 0$ and $\|w - \omega\|_{L^2(0,T,V)} \to 0$ as $\alpha \to 0$.

The next theorem describes the relation between the velocity-vorticity-Voigt equations (1.3) and the 3D Navier-Stokes equations (1.1).

**Theorem 2.11.** Denote by $\tilde{\omega} := \nabla \times \tilde{u}$ the vorticity of $\tilde{u}$ in (1.1a) and let $u_0$, $f$, and $T > 0$ be the same as in Theorem 2.10 and set $w_0 = \nabla \times u_0$ and $\tilde{u}_0 = u_0$. Then, for any $\alpha \in (0,1]$,

$$
\|\omega(t) - \tilde{\omega}(t)\|_{L^2}^2 + \|u(t) - \tilde{u}(t)\|_{L^2}^2 + \alpha^2 \|\nabla u(t) - \nabla \tilde{u}(t)\|_{L^2}^2 + \int_0^t \|\nabla u(s) - \nabla \tilde{u}(s)\|_{L^2}^2 \, ds \leq C \alpha^2
$$

for a.e. $t > 0$ in the interval of existence of the solution to (1.1a), say, up to $T > 0$ and the constant $C$ depends on $\|\tilde{u}\|_{H^3}$, $\|u\|_{H^3}$, as well as $\|f\|_{H^{2+1}_\text{curl}}$. In particular, we have $\|\omega - \tilde{\omega}\|_{L^2(0,T;H^3)} \to 0$, $\|u - \tilde{u}\|_{L^2(0,T;H^3)} \to 0$, and $\|u - \tilde{u}\|_{L^2(0,T;V)} \to 0$ as $\alpha \to 0$.

**Remark 2.12.** We point out that the global well-posedness of (1.3) still holds if we remove the divergence-free condition on the initial data $w_0$, i.e., we only assume $w_0 \in L^2$ for the weak solution, and $w_0 \in H^1$ for the strong solution. Consequently, we obtain the weak solutions $w \in C_w(0,T;L^2) \cap L^2(0,T;H^1)$ and the strong solution $w \in C_w(0,T;H^1) \cap L^2(0,T;H^2)$ by modifying the proofs in Section 2 and Section 3 accordingly. However, Theorem 2.10 is no longer valid. Also, the assumptions that $\tilde{u}_0 = u_0$ and $w_0 =
\[ \nabla \times u_0 \text{ can be removed at the cost of obtaining "convergence up to and error" as } \alpha \to 0. \text{ For the sake of clarity, we do not include these details.} \]

The above result yields the following blow-up criterion, which looks identical to the blow-up criterion for the 3D Euler-Voigt and 3D Navier-Stokes equations [37, 38] (see also [31]).

**Corollary 2.13.** Assume the hypotheses and the notation of Theorem 2.11. Suppose that there is a \( T > 0 \) and an \( \epsilon > 0 \) such that

\[
\sup_{t \in [0, T]} \limsup_{\alpha \to 0^+} \alpha \|\nabla u\|_{L^2} \geq \epsilon > 0.
\]

Then solutions to the 3D Navier-Stokes equations with initial data \( u_0 \) develop a singularity on the time interval \([0, T]\), in the sense that there does not exist a strong solution \( \tilde{u} \in L^2(0, T; V \cap H^2) \cap C([0, T], V) \).

### 3. Proof of Theorem 2.8

In this section, we provide the construction of weak solution to system (1.3) via Galerkin approximation. We also show that the obtained global weak solution is unique.

#### 3.1. Proof of global existence of weak solutions.

Consider the following finite-dimensional Galerkin ODE system for (1.3):

\[
\begin{align*}
(3.13a) \quad & \frac{d}{dt}(u_M + \alpha^2 A u_M) + \nu A u_M + P_M P_M(w_M \times u_M) = f_M, \\
(3.13b) \quad & \frac{d w_M}{dt} + A w_M + P_M B(u_M, w_M) - P_M B(w_M, u_M) = \nabla \times f_M,
\end{align*}
\]  

with initial data \( u_M(0) = P_M u_0, \ w_M(0) = P_M w_0 \) and forcing \( f_M = P_M f \).

Notice that all the terms in (3.13a) and (3.13b) except the time-derivatives are at most quadratic, and thus, they are locally Lipschitz continuous. Therefore, by the Picard-Lindelöf Theorem, we know that there exists a unique solution up to some time \( T_M > 0 \).

Next we take inner-products with the above two equations by \( u_M \) and \( w_M \), respectively, integrate by parts, and obtain

\[
\begin{align*}
(3.14a) \quad & \frac{1}{2} \frac{d}{dt} \left( \|u_M\|_{L^2}^2 + \alpha^2 \|\nabla u_M\|_{L^2}^2 \right) + \|\nabla u_M\|_{L^2}^2 = \int_{\mathbb{T}^3} u_M \cdot f_M \, dx, \\
(3.14b) \quad & \frac{1}{2} \frac{d}{dt} \|w_M\|_{L^2}^2 + \|\nabla w_M\|_{L^2}^2 = \int_{\mathbb{T}^3} (w_M \cdot \nabla) u_M \cdot w_M \, dx + \int_{\mathbb{T}^3} (\nabla \times f_M) \cdot w_M \, dx,
\end{align*}
\]

where we used \( \nabla \cdot u_M = 0 \) in (3.14a). Then, by applying Cauchy-Schwarz inequality to the right side of (3.14a), we obtain

\[
(3.15) \quad \frac{d}{dt} \left( \|u_M\|_{L^2}^2 + \alpha^2 \|\nabla u_M\|_{L^2}^2 \right) + 2\|\nabla u_M\|_{L^2}^2 \leq 2\|f_M\|_{L^2} \|u_M\|_{L^2} \leq \|f_M\|_{L^2}^2 + \|\nabla u_M\|_{L^2}^2.
\]

Thus, for any \( T \in (0, T_M) \), integrating in time, it holds that for a.e. \( t \in (0, T) \),

\[
\|u_M(t)\|_{L^2}^2 + \alpha^2 \|\nabla u_M(t)\|_{L^2}^2 \leq C \|f_M\|_{L^2(0,T;V)}^2 + \|u_M(0)\|_{L^2(0,T;H)}^2 + \alpha^2 \|\nabla u_M(0)\|_{L^2}^2 + \alpha^2 \|\nabla u_0\|_{L^2}^2 =: K_T.
\]

Since the right-hand side is finite, \( u_M \) can be extended beyond \( T_M \), so that the above inequality holds for arbitrary \( T > 0 \) and a.e. \( t \in (0, T) \). In particular, the interval of existence is independent of \( M \).
Moreover, \( \{u_M\}_{M=1}^{\infty} \) is uniformly bounded in \( L^\infty(0, T; V) \). Using the Banach-Alaoglu Theorem, and extracting a subsequence if necessary (which we relabel if necessary and still denote by \( u_M \)), we obtain a \( u \in L^\infty(0, T; V) \) such that \( u_M \) converges to \( u \) in the weak-* sense of \( L^\infty(0, T; V) \).

Next, we estimate the right side of (3.14b) and obtain
\[
\frac{1}{2} \frac{d}{dt} \|w_M\|^2_{L^2} + \|\nabla w_M\|^2_{L^2} \leq C \|\nabla u_M\|_{L^2} \|w_M\|_{L^2}^{1/2} \|\nabla w_M\|_{L^2}^{3/2} + \frac{1}{2} \|\nabla \times f_M\|^2_{L^2} + \frac{1}{2} \|w_M\|^2_{L^2}.
\]
(3.16)

where we used Lemma 2.1. Then, after rearranging and using Grönwall’s inequality, it follows that
\[
\|w_M(t)\|^2_{L^2} \leq K_T := e^{TC\kappa_T} \|w_M(0)\|^2_{L^2} + \int_0^T e^{C\kappa_T(t-s)} \|\nabla \times f(s)\|^2_{L^2} \, ds < \infty,
\]
from which we conclude that \( w_M \) can be extended beyond \( T_M \) up to any \( t < T \) and is uniformly bounded in \( L^\infty(0, T; L^2) \). Using this fact and integrating (3.16) on \([0, T]\), we also find that \( w_M \) is uniformly bounded in \( L^2(0, T; H^1) \).

By similar arguments as above for \( u_M \), we extract a weak-* convergent subsequence, still denoted by \( w_M \), with limit \( w \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \).

Also, by taking the divergence of (3.14b) and denoting \( v_M := \nabla \cdot w_M \), we obtain
\[
\frac{d}{dt} \|v_M\|^2_{L^2} + A v_M + P_M B(u_M, v_M) = 0.
\]
(3.17)

Multiplying the above equation by \( v_M \) and using (2.7c) and (2.9), we obtain
\[
\frac{1}{2} \frac{d}{dt} \|v_M\|^2_{L^2} + \lambda_1 \|v_M\|^2_{L^2} \leq 0.
\]

Then, Grönwall’s inequality the implies for a.e. \( t \in (0, T) \),
\[
\|v_M(t)\|^2_{L^2} \leq \|v_M(0)\|^2_{L^2} e^{-\lambda_1 t}.
\]

If the initial data \( w_0 \) is in \( H \), we have \( v_M(0) = 0 \), which implies \( \nabla \cdot w_M = v_M = 0 \) in \( L^2([0, T]; L^2) \). Since \( L^2([0, T]; H) \) is closed in \( L^2([0, T]; L^2) \), this also implies \( \nabla \cdot w = 0 \), so long as \( \nabla \cdot w_0 = 0 \). Namely, we have \( w \in L^\infty(0, T; H) \cap L^2(0, T; V) \). Note that (3.17) also implies
\[
\|v_M(T)\|^2_{L^2} + 2 \int_0^T \|\nabla v_M\|^2_{L^2} \, dt \leq \|v_M(0)\|^2_{L^2},
\]
from which we obtain that \( v_M \) is uniformly bounded in \( L^2(0, T; H^1) \).

We consider the pair \( (u, w) \) as our candidate solution. Next, we obtain bounds on \( du_M / dt \) in \( L^2(0, T; V) \) and bounds on \( dw_M / dt \) in \( L^2(0, T; V') \), uniformly with respect to \( M \). Note that
\[
(I + \alpha^2 A) \frac{du_M}{dt} = -Au_M - P_M (u_M \times w_M) + f_M,
\]
(3.18a)
\[
\frac{dw_M}{dt} = -Au_M - P_M (u_M \times w_M) + P_M B(w_M, u_M) + \nabla \times f_M.
\]
(3.18b)

Note in (3.18a) that \( -Au_M \) is uniformly bounded in \( L^2(0, T; V') \) due to the fact that \( u_M \) is also uniformly bounded in \( L^2(0, T; V) \). On the other hand, by Lemma 2.1 we have
\[
\left| \int_{\mathbb{R}^3} P_M (u_M \times w_M) \cdot \phi \, dx \right| \leq \int_0^T \left( u_M \times w_M \right) \cdot P_M \phi \, dx \leq C \|u_M\|^{1/2}_{L^2} \|\nabla u_M\|^{1/2}_{L^2} \|w_M\|_{L^2} \|P_M \phi\|_{H^1} \leq C \|u_M\|^{1/2}_{L^2} \|\nabla u_M\|^{1/2}_{L^2} \|w_M\|_{L^2} \|\phi\|_{H^1},
\]
for all test functions $\phi \in V$. Thus, $P_M(u_M \times w_M)$ is also uniformly bounded in $L^2(0,T;V')$. It is easily seen that, uniformly in $M$, $f_M$ and $\nabla \times f_M$ are bounded in $L^2(0,T;V')$, as well as $A u_M, A w_M$. Therefore,

$$(I + \alpha^2 A) \frac{du_M}{dt} \text{ is uniformly bounded in } L^2(0,T;V').$$

By inverting the Helmholtz operator $(I + \alpha^2 A)$ with respect to zero-mean, periodic boundary conditions, we obtain

$$\frac{du_M}{dt} \text{ is uniformly bounded in } L^2(0,T;V).$$

Next, notice in (3.18b) that $\Delta w_M$ is bounded in $L^2(0,T;H^{-1}) \subset L^2(0,T;V')$, and for the nonlinear terms, we integrate by parts and use Lemma 2.2 in order to get

$$\int_{T^3} P_M((u_M \cdot \nabla)w_M) \cdot \phi \, dx = \int_{T^3} (u_M \cdot \nabla)w_M \cdot P_M \phi \, dx \leq \|u_M\|_{L^2}^{1/2} \|
abla u_M\|_{L^2}^{1/2} \|w_M\|_{H;1} \|\phi\|_{H;1}$$

for all test functions $\phi \in V$. Therefore, $-P_M B(u_M, w_M)$ is uniformly bounded in $L^2(0,T;V')$. Similar estimates show that $P_M B(w_M, u_M)$ is also bounded in $L^2(0,T;V')$, uniformly in $M$. Thus, we get

$$\frac{dw_M}{dt} \text{ is uniformly bounded in } L^2(0,T;V').$$

Also, by the bounds we obtained above and Lemma 2.2, there is a subsequence, still labeled as $(u_M, w_M)$ that satisfies

(3.19a) $u_M \to u \text{ strongly in } L^2(0,T;V)$ and $w_M \to w \text{ strongly in } L^2(0,T;H)$,

(3.19b) $w_M \to w \text{ weakly in } L^2(0,T;V)$,

(3.19c) $u_M \to u \text{ weak-}^* \text{ in } L^\infty([0,T];V)$ and $w_M \to w \text{ weak-}^* \text{ in } L^\infty([0,T];H)$,

for all $T > 0$. Hence, for $0 < \bar{T} < T$, by taking inner products of (3.13a) and (3.13b) with test function $\psi \in C^1_c([0,\bar{T});V)$, integrating in time over $[0,\bar{T}]$, and integrating by parts, we obtain

(3.20a) $\begin{cases} -\int_0^{\bar{T}} (u_M, \psi_t) \, dt + (u_M(\cdot,\bar{T}), \psi(\cdot,\bar{T})) - (u_M(\cdot,0), \psi(\cdot,0)) \\ -\alpha^2 \int_0^{\bar{T}} ((u_M, \psi_t)) \, dt - \alpha^2((u_M(\cdot,\bar{T}), \psi(\cdot,\bar{T}))) + \alpha^2((u_M(\cdot,0), \psi(\cdot,0))) \\ + \int_0^{\bar{T}} ((u_M, \psi)) \, dt + \int_0^{\bar{T}} \langle (w_M \times u_M), P_M \psi \rangle \, dt = \int_0^{\bar{T}} (f_M, \psi) \, dt, \\ -\int_0^{\bar{T}} (w_M, \psi_t) \, dt + (w_M(\cdot,\bar{T}), \psi(\cdot,\bar{T})) - (w_M(\cdot,0), \psi(\cdot,0)) + \int_0^{\bar{T}} ((u_M, \psi)) \, dt \\ + \int_0^{\bar{T}} \langle B(u_M, w_M), P_M \psi \rangle \, dt - \int_0^{\bar{T}} \langle B(w_M, u_M), P_M \psi \rangle \, dt = \int_0^{\bar{T}} (f_M, \nabla \times \psi) \, dt. \end{cases}$

Using the standard arguments from the theory of the Navier-Stokes equations (see, e.g., [12, 61]), we have that each of the integrals in (3.20a) and (3.20b) converges to the time integral of the corresponding term in (3.20). For the sake of completeness, we provide the details below. First, convergence of integrals of the linear terms follows from (3.19a) as well as $f \in H$. The choice of $\psi$ and (3.19a) imply the convergence of the boundary terms at $t = 0$ and $t = \bar{T}$ in both (3.20a) and (3.20b). Regarding the nonlinear term in
(3.20a), we have
\[
\left| \int_0^T \langle (w_M \times u_M), P_M \psi \rangle \ dt - \int_0^T \langle (w \times u), \psi \rangle \ dt \right| \\
\leq \int_0^T \left| \langle (w_M \times (u_M - u)), P_M \psi \rangle \right| \ dt + \int_0^T \left| \langle (w_M - w) \times u, P_M \psi \rangle \right| \ dt \\
+ \int_0^T \left| \langle w \times u, P_M \psi - \psi \rangle \right| \ dt \\
\leq \|w_M\|_{L^2(H^1)} \|u - u_M\|_{L^2(H^1)} \|\psi\|_{L^\infty(H^1)} \|w_M\|^{1/2}_{L^2(L^2)} \|w_M\|^{1/2}_{L^2(H^1)} \\
+ \|w_M - w\|_{L^2(H^1)} \|\psi\|_{L^\infty(H^1)} \|w_M\|^{1/2}_{L^2(L^2)} \|w_M\|^{1/2}_{L^2(H^1)} \\
+ \|w\|_{L^2(H^1)} \|\psi\|_{L^\infty(H^1)} \|w_M - w\|^{1/2}_{L^2(L^2)} \|w_M - w\|^{1/2}_{L^2(H^1)} \\
\to 0 \text{ as } M \to \infty,
\]
which converges to 0 in view of (3.19a) and the uniform bounds on $w_M$ and $u$. As for the first nonlinear term in (3.20a), we use (3.19a) and estimate as
\[
\left| \int_0^T \langle B(u_M, P_M \psi), w_M \rangle \ dt - \int_0^T \langle B(u, \psi), w \rangle \ dt \right| \\
\leq \int_0^T \left| \langle B(u_M, P_M \psi - \psi), w_M \rangle \right| \ dt + \int_0^T \left| \langle B(u_M - u, \psi), w_M \rangle \right| \ dt \\
+ \int_0^T \left| \langle B(u_M, \psi), w_M - w \rangle \right| \ dt \\
\leq \|w_M\|_{L^2(H^1)} \|P_M \psi - \psi\|_{L^\infty(H^1)} \|w_M\|^{1/2}_{L^2(L^2)} \|w_M\|^{1/2}_{L^2(H^1)} \\
+ \|u_M - u\|_{L^2(H^1)} \|\psi\|_{L^\infty(H^1)} \|w_M\|^{1/2}_{L^2(L^2)} \|w_M\|^{1/2}_{L^2(H^1)} \\
+ \|u\|_{L^2(H^1)} \|\psi\|_{L^\infty(H^1)} \|w_M - w\|^{1/2}_{L^2(L^2)} \|w_M - w\|^{1/2}_{L^2(H^1)} \\
\to 0 \text{ as } M \to \infty,
\]
where we used (3.19a), the uniform boundedness of $w_M$ in $H$ and integrated by parts in the last integral. Finally, convergence of the second nonlinear term in (3.20b) is obtained as
\[
\left| \int_0^T \langle B(w_M, u_M), P_M \psi \rangle \ dt - \int_0^T \langle B(w, u), \psi \rangle \ dt \right| \\
\leq \int_0^T \left| \langle B(w_M, u_M), P_M \psi - \psi \rangle \right| \ dt + \int_0^T \left| \langle B(w_M, \psi), u_M - u \rangle \right| \ dt \\
+ \int_0^T \left| \langle B(w_M - w, u), \psi \rangle \right| \ dt \\
\leq \|w_M\|^{1/2}_{L^2(L^2)} \|w_M\|^{1/2}_{L^2(H^1)} \|u_M\|_{L^2(H^2)} \|P_M \psi - \psi\|_{L^\infty(L^2)} \\
+ \|w_M\|^{1/2}_{L^2(L^2)} \|w_M\|^{1/2}_{L^2(H^1)} \|\psi\|_{L^\infty(H^1)} \|u_M - u\|_{L^2(H^1)} \\
+ \|w_M - w\|_{L^2(L^2)} \|u\|_{L^2(H^2)} \|\psi\|_{L^\infty(L^2)} \|\psi\|_{L^\infty(H^1)} \\
\to 0 \text{ as } M \to \infty,
\]
due to (3.19b) and the uniform boundedness of $w_M$. Similar arguments also apply to equation (3.17) for $v_M := \nabla \cdot w_M$. Namely, each term in

\[
- \int_0^\bar{T} (v_M, \psi_t) dt + (v_M(\cdot, \bar{T}), \psi(\cdot, \bar{T})) - (v_M(\cdot, 0), \psi(\cdot, 0)) \\
+ \int_0^\bar{T} ((v_M, \psi)) dt + \int_0^\bar{T} \langle B(u_M, v_M), P_M \psi \rangle dt = 0
\]

converges to the time integral of the following weak formulation of $v$ in view of (3.17),

\[
\langle v_t, \psi \rangle + ((v, \psi)) + \langle B(u, v), \psi \rangle = 0.
\]

Specifically, the linear terms converge due to the fact that $v_M$ is bounded in $L^2(0, T; H^1)$ from (3.17), while the convergence of the nonlinear term follows similar to those in (3.20a), i.e., after integration by parts, we have

\[
\left| \int_0^\bar{T} \langle B(u_M, P_M \psi), v_M \rangle dt - \int_0^\bar{T} \langle B(u, \psi), v \rangle dt \right| \\
\leq \int_0^\bar{T} |\langle B(u_M, P_M \psi - \psi), v_M \rangle| dt + \int_0^\bar{T} |\langle B(u_M - u, \psi), v_M \rangle| dt \\
+ \int_0^\bar{T} \langle B(u_M, \psi), v_M - v \rangle dt \\
\leq \|u_M\|_{L^2(H^1)} \|P_M \psi - \psi\|_{L^\infty(H^1)} \|v_M\|_{L^2(L^2)}^{1/2} \|v_M\|_{L^2(H^1)}^{1/2} \\
+ \|u_M - u\|_{L^2(H^1)} \|\psi\|_{L^\infty(H^1)} \|v_M\|_{L^2(L^2)}^{1/2} \|v_M\|_{L^2(H^1)}^{1/2} \\
+ \|u_M\|_{L^2(H^1)} \|\psi\|_{L^\infty(H^1)} \|v_M - v\|_{L^2(L^2)}^{1/2} \|v_M - v\|_{L^2(H^1)}^{1/2} \\
\to 0 \quad \text{as} \quad M \to \infty,
\]

where we also used $v_M \to v$ in $L^2(0, T; L^2)$ as $M \to \infty$, as well as (3.19a). Note that by the fact $v_M = 0$ in $L^2([0, \bar{T}]; L^2)$, we have $v(t) = \nabla \cdot w(t) = 0$ for a.e. $t < T$. Now, all the above convergence is valid if we take $\psi = v \phi(t)$ where $v \in C^\infty$ and $\phi \in C^1(0, \bar{T})$. In particular, the convergence is valid for all $\phi \in D([0, \bar{T}])$, thus, (2.10) holds in the sense of distributions, which in turn implies that (2.8) is valid as an equation of operators. Namely, in view of the embedding $V \hookrightarrow H \hookrightarrow V'$, we conclude that equations (1.3) hold in the weak sense by Lemma 2.2 while the pressure term $p$ is recovered by the approach mentioned in Section 2. Finally, by integrating in time over $[\bar{t}, \bar{T}]$ for $0 \leq \bar{t} < \bar{T}$ and sending $\bar{T} \to \bar{t}$ one can use similar convergence arguments as above (c.f. [12, 61]) and show that $(u, w)$ is in fact weakly continuous with respect to time in $V \times H$, so that the initial condition is satisfied in the weak sense. Since $T > 0$ is arbitrary, the existence in Theorem 2.8 is thus proven.

3.2. Proof of uniqueness. Suppose there exist two pairs of solution $(u, w, p)$ and $(\bar{u}, \bar{w}, \bar{p})$ to system (1.3), with the same initial data $u_0 \in V$, $w_0 \in H$ and the same forcing $f$ on their common time interval of existence $(0, T)$. By subtracting the equations of the two pairs of solution and denoting $\bar{u} = u - \bar{u}$,
\[ \tilde{w} = w - \bar{w}, \text{ and } \tilde{p} = p - \bar{p}, \text{ we obtain} \]
\[
\left\{ \begin{array}{l}
(I - \alpha^2 \Delta) \frac{\partial \tilde{u}}{\partial t} - \Delta \tilde{u} + \tilde{w} \times u + \tilde{w} \times \bar{u} + \nabla \tilde{p} = 0, \\
\frac{\partial \tilde{w}}{\partial t} - \Delta \tilde{w} + (u \cdot \nabla) \tilde{w} + (\bar{u} \cdot \nabla) \tilde{w} - (w \cdot \nabla) \bar{u} - (\bar{w} \cdot \nabla) \tilde{u} = 0, \\
\nabla \cdot \tilde{u} = 0, \\
\tilde{u}(:,0) = 0 \quad \text{and} \quad \tilde{w}(:,0) = 0.
\end{array} \right. \tag{3.21}
\]

Multiplying the two equations by \( \tilde{u} \) and \( \tilde{w} \), respectively, integrating by parts over \( \mathbb{T}^3 \), and adding, we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \| \tilde{u} \|^2_{L^2} + \alpha^2 \| \nabla \tilde{u} \|^2_{L^2} + \| \tilde{w} \|^2_{L^2} \right) + \| \nabla \tilde{u} \|^2_{L^2} + \| \nabla \tilde{w} \|^2_{L^2} \]
\[
= - \int_{\mathbb{T}^3} (\tilde{w} \times u) \cdot \tilde{u} \, dx - \int_{\mathbb{T}^3} (\tilde{u} \cdot \nabla) \tilde{w} \cdot \tilde{u} \, dx - \int_{\mathbb{T}^3} (w \cdot \nabla) \bar{u} \cdot \tilde{u} \, dx + \int_{\mathbb{T}^3} (\bar{w} \cdot \nabla) \bar{u} \cdot \tilde{u} \, dx,
\]
where we used \( \nabla \cdot u = \nabla \cdot \bar{u} = \nabla \cdot \bar{u} = 0 \). Next, we estimate the four terms on the right side of (3.22).

By Lemma 2.11 Hölder’s and Young’s inequalities, the first integral is bounded by
\[
\int_{\mathbb{T}^3} |u| \| \tilde{u} \| \| \tilde{w} \| \, dx \leq C \| u \|_{H^1} \| \tilde{u} \|_{L^2} \| \tilde{w} \|_{L^2} \leq C \| u \|^2_{H^1} \| \tilde{u} \|_{L^2} \| \tilde{w} \|_{L^2} + \frac{1}{8} \| \nabla \tilde{w} \|^2_{L^2}
\]
\[
\leq C_\alpha \| u \|^2_{L^2} \left( \| \tilde{u} \|^2_{L^2} + \alpha^2 \| \nabla \tilde{u} \|^2_{L^2} \right) + \frac{1}{8} \| \nabla \tilde{w} \|^2_{L^2}.
\]

By similar estimates, the second integral is bounded by
\[
\int_{\mathbb{T}^3} |\tilde{w}| \| \tilde{u} \| \| \tilde{w} \| \, dx \leq C \| \tilde{w} \|_{H^1} \| \tilde{u} \|_{L^2} \| \tilde{w} \|_{L^2} \leq C_\alpha \| \tilde{w} \|^2_{H^1} \left( \alpha^2 \| \nabla \tilde{u} \|^2_{L^2} + \frac{1}{8} \| \nabla \tilde{w} \|^2_{L^2} \right),
\]
where we integrated by parts and used \( \nabla \cdot \tilde{u} = 0 \). By Lemma 2.11 the third integral is bounded by
\[
\int_{\mathbb{T}^3} |u| \| \nabla \tilde{u} \| \| \tilde{w} \| \, dx \leq C \| u \|_{H^{1/2}} \| \nabla \tilde{u} \|_{L^2} \| \tilde{w} \|_{L^2} \leq C \| u \|^4_{H^{1/2}} \| \nabla \tilde{u} \|^4_{L^2} \| \tilde{w} \|^2_{L^2} + \frac{1}{8} \| \nabla \tilde{w} \|^2_{L^2}
\]
\[
\leq C_\alpha \| u \|^2_{L^2} \left( \alpha^2 \| \nabla \tilde{u} \|^2_{L^2} \right) + C_\alpha \| \tilde{w} \|^2_{L^2} + \frac{1}{8} \| \nabla \tilde{w} \|^2_{L^2},
\]
where we applied Hölder’s and Young’s inequalities. As for the last integral, we obtain the upper bound analogously as
\[
\int_{\mathbb{T}^3} |\nabla \tilde{u} | \| \tilde{w} \| \, dx \leq C \| \tilde{u} \|_{H^1} \| \tilde{w} \|^2_{L^2} \| \nabla \tilde{w} \|^3_{L^2} \leq C \| \tilde{u} \|^4_{H^1} \| \tilde{w} \|^2_{L^2} + \frac{1}{8} \| \nabla \tilde{w} \|^2_{L^2}.
\]

Summing up all the above estimates, we obtain
\[
\frac{d}{dt} \left( \| \tilde{u} \|^2_{L^2} + \alpha^2 \| \nabla \tilde{u} \|^2_{L^2} + \| \tilde{w} \|^2_{L^2} \right) + \| \nabla \tilde{u} \|^2_{L^2} + \| \nabla \tilde{w} \|^2_{L^2} \leq M_\alpha \left( \| \tilde{u} \|^2_{L^2} + \alpha^2 \| \nabla \tilde{u} \|^2_{L^2} + \| \tilde{w} \|^2_{L^2} \right).
\]

Here \( M_\alpha := M_1 + M_2(\| \tilde{u} \|^4_{L^2} + \| \tilde{w} \|^4_{L^2}) \), is such that \( M_1 \) depends on \( \| \tilde{u} \|^4_{H^1} \) and \( \| u \|^2_{L^2} \), which are bounded, while \( M_2 \) is an absolute constant. Therefore, by Grönwall’s inequality and \( w, \tilde{w} \in L^2(0, T; V) \), we conclude that
\[
\| \tilde{u}(t) \|^2_{L^2} + \| \tilde{w}(t) \|^2_{L^2} = 0,
\]
since \( \tilde{u}_0 = \tilde{w}_0 = 0 \). Namely, we have \( u(t) = \tilde{u}(t) \) and \( w(t) = \tilde{w}(t) \). Finally, setting \( \psi = u \in L^2(0, T; V) \) in Definition 2.8 using Lemma 2.11, and integrating in time, we obtain (2.11). The proof of Theorem 2.8 is thus complete.
4. Proof of Theorem 2.9

In this section, we show that system (1.3) has a unique global strong solution and provide the a priori estimates for the higher order regularity of such solution \((u, w)\) to (1.3), with \(u_0, w_0 \in V\). In view of Definition 2.4, it suffices to prove the uniform boundedness of \(w_M\) in \(V\).

4.1. Proof of global existence of strong solutions. To begin, we multiply \((3.13a)\) by \(Aw_M\), respectively, integrate by parts over \(T^3\) and obtain

\[
\frac{1}{2} \frac{d}{dt} \| \nabla w_M \|_{L^2}^2 + \| Aw_M \|_{L^2}^2 = \int_{T^3} (u_M \cdot \nabla) w_M \cdot Aw_M \, dx - \int_{T^3} (w_M \cdot \nabla) u_M \cdot Aw_M \, dx \\
- \int_{T^3} (\nabla \times f_M) \cdot Aw_M \, dx.
\]

(4.23)

Then, we estimate the three terms on the right side of (4.23). For the first term, we integrate by parts and apply Lemma 2.1 as

\[
\int_{T^3} (u_M \cdot \nabla) w_M \cdot Aw_M \, dx \leq C \| \nabla u_M \|_{L^2} \| \nabla w_M \|_{L^2}^{1/2} \| Aw_M \|_{L^2}^{3/2} \leq C \| \nabla u_M \|_{L^2}^{1/2} \| \nabla w_M \|_{L^2}^{1/2} + \frac{1}{8} \| Aw_M \|_{L^2}^2.
\]

By (2.9), the second term is bounded by

\[
\int_{T^3} |w_M| \| \nabla u_M \| |Aw_M| \, dx \leq C \| w_M \|_{L^\infty} \| \nabla u_M \|_{L^2} \| Aw_M \|_{L^2} \leq C \| \nabla u_M \|_{L^2} \| \nabla w_M \|_{L^2}^{1/2} \| Aw_M \|_{L^2}^{3/2} \leq C \| \nabla u_M \|_{L^2}^{1/2} \| \nabla w_M \|_{L^2}^{1/2} + \frac{1}{8} \| Aw_M \|_{L^2}^2.
\]

By Hölder’s inequality, the last term is bounded by

\[
\int_{T^3} |\nabla \times f_M| \| Aw_M \| \, dx \leq \| \nabla \times f_M \|_{L^2} \| Aw_M \|_{L^2} \leq C \| \nabla \times f \|_{L^2}^2 + \frac{1}{8} \| Aw_M \|_{L^2}^2.
\]

Combining all the above estimates and denoting

\[
X_M(t) = \| \nabla w_M(t) \|_{L^2}^2 \quad \text{and} \quad Y_M(t) = \| Aw_M(t) \|_{L^2}^2,
\]

for \(0 \leq t \leq T\), we get

\[
\frac{d}{dt} X_M(t) + Y_M(t) \leq C X_M(t) + C \| f \|_{H_{\text{curl}}}^2.
\]

Thus, the uniform bound of \(u\) in \(V\) and Grönwall’s inequality imply that \(w_M\) is uniformly bounded in \(L^\infty(0, T; V)\). Integrating in time over \([0, T]\), we also have that \(w_M\) is uniformly bounded in \(L^2(0, T; D(A))\). As for the time derivative of \(w_M\) in \((3.18b)\), we use the fact that \(w_M\) are bounded in \(L^2(0, T; D(A))\) and all the nonlinear terms are bounded in \(L^2(0, T; L^2)\), and conclude that

\[
\frac{dw_M}{dt} \quad \text{is uniformly bounded in} \quad L^2(0, T; H).
\]

A standard simple argument (see, e.g., [61]) then shows that \(dw/dt \in L^2(0, T; H)\), thanks to the convergence properties of \(w_M\) proven above. Therefore, we obtain the global existence of strong solutions.

4.2. Proof of higher regularity. In this subsection, we first provide the \(H^2\) and \(H^3\) a priori estimates for \(u\) and \(w\), then we obtain the \(H^s\) bounds on \(u\) and \(w\) for \(s \geq 4\).
4.2.1. $H^2$ bounds. First we provide the $H^2$ a priori estimates for $u$ with initial data $u_0 \in D(A)$. We work formally for the sake of clarity, we point out that the following estimates can be justified at the Galerkin level following similar arguments to those in Section 3.1 and Section 4.1. We begin by multiplying (1.3a) by $-\Delta u$, integrating by parts over $\mathbb{T}^3$, and obtain

$$
\frac{1}{2} \frac{d}{dt} \left( \| \nabla u \|_{L^2}^2 + \alpha^2 \| \Delta u \|_{L^2}^2 \right) + \| \Delta u \|_{L^2}^2 = \int_{\mathbb{T}^3} w \times u \cdot \Delta u \, dx + \int_{\mathbb{T}^3} f \cdot \Delta u \, dx,
$$

where we used $\nabla \cdot u = 0$. By Lemma 2.1, the first term on the right side of (4.24) is bounded by

$$
\int_{\mathbb{T}^3} |w| \| \Delta u \| \, dx \leq C \| w \|_{L^2}^{1/2} \| \nabla w \|_{L^2}^{1/2} \| \nabla u \|_{L^2} \| \Delta u \|_{L^2} \leq \| w \|_{L^2} \| \nabla w \|_{L^2} \| \nabla u \|_{L^2}^2 + \frac{1}{4} \| \Delta u \|_{L^2}^2,
$$

where we used Young’s inequality. By Hölder’s inequality, we bound the second term by

$$
\int_{\mathbb{T}^3} |f| \| \Delta u \| \, dx \leq \| f \|_{L^2} \| \Delta u \|_{L^2} \leq C \| f \|_{L^2}^2 + \frac{1}{4} \| \Delta u \|_{L^2}^2.
$$

Combining all the above estimates and denoting

$$
\bar{X}(t) = \| \nabla u(t) \|_{L^2}^2 + \alpha^2 \| \Delta u(t) \|_{L^2}^2 \quad \text{and} \quad \bar{Y}(t) = \| \Delta u(t) \|_{L^2}^2,
$$

for $0 \leq t \leq T$, we arrive at

$$
\frac{d}{dt} \bar{X}(t) + \bar{Y}(t) \leq K_1 \bar{X}(t) + C \| f \|_{L^2}^2,
$$

where the constant $K_1$ depends on the $H^1$ norms of $u$ and $w$. Thus, Grönwall’s inequality implies that $u \in L^\infty(0, T; D(A))$.

Next, we show the $H^2$ boundedness of $w$. We multiply the $w$ equation in system (1.3) by $\Delta^2 w$, integrate by parts over $\mathbb{T}^3$, and obtain

$$
\frac{1}{2} \frac{d}{dt} \left( \| \Delta w \|_{L^2}^2 \right) + \| \nabla \Delta w \|_{L^2}^2 = - \int_{\mathbb{T}^3} (u \cdot \nabla)w \cdot \Delta^2 w \, dx + \int_{\mathbb{T}^3} (w \cdot \nabla)u \cdot \Delta^2 w \, dx
$$

$$
+ \int_{\mathbb{T}^3} (\nabla \times f) \cdot \Delta^2 w \, dx.
$$

We then estimate the three terms on the right side of (4.25). After integration by parts, we use Lemma 2.1 and Hölder’s inequality in order to bound the first integral by

$$
\int_{\mathbb{T}^3} |\nabla w| |\nabla u| |\nabla \Delta w| \, dx + \int_{\mathbb{T}^3} |u| |\nabla \nabla u| |\nabla \Delta w| \, dx
$$

$$
\leq C \| \Delta u \|_{L^2} \| \nabla w \|_{L^2}^{1/2} \| \Delta w \|_{L^2}^{1/2} \| \nabla \Delta w \|_{L^2} + C \| \nabla u \|_{L^2}^{1/2} \| \Delta u \|_{L^2}^{1/2} \| \nabla \nabla w \|_{L^2} \| \nabla \Delta w \|_{L^2}
$$

$$
\leq \frac{C}{\sqrt{\lambda_1}} \| \Delta u \|_{L^2}^2 \| \Delta w \|_{L^2}^2 + C \| \nabla u \|_{L^2} \| \Delta u \|_{L^2} \| \Delta w \|_{L^2}^2 + \frac{1}{8} \| \nabla \Delta w \|_{L^2}^2,
$$

where we used (2.3) and Young’s inequality. Similarly, by Lemma 2.1 the second term is bounded by

$$
\int_{\mathbb{T}^3} |\nabla w| |\nabla u| |\nabla \Delta w| \, dx + \int_{\mathbb{T}^3} |w| |\nabla \nabla u| |\nabla \Delta w| \, dx
$$

$$
\leq C \| \Delta u \|_{L^2} \| \nabla w \|_{L^2}^{1/2} \| \Delta w \|_{L^2}^{1/2} \| \nabla \Delta w \|_{L^2} + C \| \nabla \nabla u \|_{L^2} \| \Delta u \|_{L^2}^{1/2} \| \Delta w \|_{L^2}^{1/2} \| \nabla \Delta w \|_{L^2}
$$

$$
\leq \frac{C}{\sqrt{\lambda_1}} \| \Delta u \|_{L^2}^2 \| \Delta w \|_{L^2}^2 + \frac{1}{8} \| \nabla \Delta w \|_{L^2}^2,
$$

where we also used (2.5) and (2.9). The last integral is bounded by

$$
\int_{\mathbb{T}^3} |\nabla (\nabla \times f)| |\nabla \Delta w| \, dx \leq \| \nabla (\nabla \times f) \|_{L^2} \| \nabla \Delta w \|_{L^2} \leq C \| f \|_{L^2}^2 + \frac{1}{8} \| \nabla \Delta w \|_{L^2}^2,
$$

for $0 \leq t \leq T$. .
where we integrated by parts and used Hölder’s inequality. Summing up all the above estimates and denoting
\[ \tilde{X}(t) = \| \Delta u(t) \|_{L^2}^2 \quad \text{and} \quad \tilde{Y}(t) = \| \nabla \Delta u(t) \|_{L^2}^2, \]
for \( 0 \leq t \leq T \), we obtain
\[
\frac{d}{dt} \tilde{X}(t) + \tilde{Y}(t) \leq K_2 \tilde{X}(t) + C \| f \|_{H_{curl}^1}^2,
\]
where the first constant \( K_2 \) depends on \( \| u \|_{H^2} \) and \( \lambda_1 \). Thus, Grönwall’s inequality and the \( H^2 \) bound of \( u \) imply that \( w \) is also bounded in \( H^2 \).

4.2.2. \( H^3 \) bounds. We start by testing \( 1.3a \) with \( \Delta^2 u \), integrating by parts, and obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \| \Delta u \|_{L^2}^2 + \alpha^2 \| \nabla \Delta u \|_{L^2}^2 \right) + \| \nabla \Delta u \|_{L^2}^2 = \int_{T^3} w \times u \cdot \Delta^2 u \, dx + \int_{T^3} f \cdot \Delta^2 u \, dx.
\]
After integration by parts, the first term on the right side of \( 4.26 \) is bounded by
\[
\int_{T^3} |\nabla w| |u| |\nabla \Delta u| \, dx + \int_{T^3} |w| |\nabla u| |\nabla \Delta u| \, dx \leq C \| u \|_{H^2}^{1/2} \| u \|_{H_x^2}^{1/2} \| \nabla w \|_{L^2} \| \nabla \Delta u \|_{L^2}
\]
\[
\quad + C \| u \|_{H^4} \| \nabla w \|_{L^2} \| \nabla \Delta u \|_{L^2} \leq C \| u \|_{H^2} \| \nabla w \|_{L^2} \| \nabla \Delta u \|_{L^2}^2 + C \| w \|_{H^2} \| \nabla u \|_{L^2}^2
\]
\[
\quad + \frac{1}{4} \| \nabla \Delta u \|_{L^2}^2
\]
where we applied \( 2.9 \). By Hölder’s inequality and integration by parts, we bound the second term by
\[
\int_{T^3} |\nabla f| |\nabla \Delta u| \, dx \leq \| \nabla f \|_{L^2} \| \nabla \Delta u \|_{L^2} \leq C \| f \|_{H^1}^2 + \frac{1}{4} \| \nabla \Delta u \|_{L^2}^2.
\]
Combining the above estimates, we have
\[
\frac{d}{dt} \left( \| \Delta u \|_{L^2}^2 + \alpha^2 \| \nabla \Delta u \|_{L^2}^2 \right) \leq K_3 + C \| f \|_{H^1}^2,
\]
where the constant \( K_3 \) depends on the \( H^2 \) norms of \( u \) and \( w \). Therefore, Grönwall’s inequality implies that \( u \in L^\infty(0, T; H^2 \cap V) \).

Next, we multiply \( 1.3b \) by \( \partial^\beta w \) after applying the operator \( \partial^\beta \), where \( \beta \) is a multi-index with \( |\beta| = 3 \), and obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \| \partial^\beta w \|_{L^2}^2 \right) + \| \nabla \partial^\beta w \|_{L^2}^2 = - \sum_{0 \leq \gamma \leq \beta} \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) \int_{T^3} \left( \partial^\gamma u \cdot \nabla \right) \partial^{\beta - \gamma} w \cdot \partial^\beta w \, dx
\]
\[
\quad + \sum_{0 \leq \gamma \leq \beta} \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) \int_{T^3} \left( \partial^\gamma \nabla \cdot f \right) \partial^{\beta - \gamma} u \cdot \partial^\beta w \, dx
\]
\[
\quad + \int_{T^3} \partial^\beta \left( \nabla \times f \right) \cdot \partial^\beta w \, dx,
\]
where we used \( \nabla \cdot u = 0 \) and \( \gamma \) is also a multi-index and \( \gamma \leq \beta \) indicates that \( |\gamma| \leq |\beta| \) and \( \gamma_i \leq \beta_i \) for \( i = 1, 2, 3 \). Then, we estimate the first term on the right side of \( 4.27 \) in the following three cases. For \( \gamma \leq \beta \) and \( |\gamma| = 1 \), it is bounded by
\[
\sum_{|\gamma| = 1} \left( \begin{array}{c} \beta \\ \gamma \end{array} \right) \int_{T^3} \left| \partial^\gamma u \right| \left| \nabla \partial^{\beta - \gamma} w \right| \left| \partial^\beta w \right| \, dx \leq C \| \partial^\gamma u \|_{L^\infty} \| \nabla \partial^{\beta - \gamma} w \|_{L^2} \| \partial^\beta w \|_{L^2}
\]
\[
\leq C \| u \|_{H^3} \| w \|_{H^{3/2}}^2.
\]
where we used (2.9). For $\gamma \leq \beta$ and $|\gamma| = 2$, it is bounded by
\[
\sum_{|\gamma|=2} \left( \frac{\beta}{\gamma} \right) \int_{T^3} |\partial^{\gamma} u| |\nabla \partial^{3-\gamma} w| |\partial^\beta w| \, dx \leq C\|\partial^\gamma u\|_{L^2} \|\nabla \partial^{3-\gamma} w\|_{L^6} \|\partial^\beta w\|_{L^2} \leq C\|u\|_{H^2}^{1/2} \|u\|_{H^3} \|w\|_{H^1}^2.
\]
For $\gamma \leq \beta$ and $|\gamma| = 3$, it is bounded from above by
\[
\sum_{|\gamma|=3} \left( \frac{\beta}{\gamma} \right) \int_{T^3} |\partial^{\gamma} u| |\nabla \partial^{3-\gamma} w| |\partial^\beta w| \, dx \leq C\|\partial^\gamma u\|_{L^2} \|\nabla \partial^{3-\gamma} w\|_{L^6} \|\partial^\beta w\|_{L^6} \leq C\|u\|_{H^3} \|w\|_{H^1}^{1/2} \|w\|_{H^2} \|\nabla \partial^\beta w\|_{L^2} \leq C\|u\|_{H^3}^2 \|w\|_{H^1} \|w\|_{H^2} + \frac{1}{8}\|\nabla \partial^3 w\|_{L^2}^2.
\]
The estimates for the second term on the right side of (4.27) follow similarly in the following four cases. For $\gamma \leq \beta$ and $|\gamma| = 0$, we integrate by parts and bound it by
\[
\sum_{|\gamma|=0} \left( \frac{\beta}{\gamma} \right) \int_{T^3} |w| |\partial^{\gamma-\gamma} u| |\nabla \partial^{\beta} w| \, dx \leq C\|w\|_{L^\infty} \|\partial^{\gamma-\gamma} u\|_{L^2} \|\nabla \partial^\beta w\|_{L^2} \leq C\|w\|_{H^2}^2 \|u\|_{H^3}^2 + \frac{1}{8}\|\nabla \partial^\beta w\|_{L^2}^2,
\]
where we used $\nabla \cdot w = 0$. For $\gamma \leq \beta$ and $|\gamma| = 1$, it is bounded by
\[
\sum_{|\gamma|=1} \left( \frac{\beta}{\gamma} \right) \int_{T^3} |\partial^\gamma w| |\nabla \partial^{3-\gamma} u| |\partial^\beta w| \, dx \leq C\|\partial^\gamma w\|_{L^\infty} \|\nabla \partial^{3-\gamma} u\|_{L^2} \|\partial^\beta w\|_{L^2} \leq C\|u\|_{H^3} \|w\|_{H^1}^2,
\]
where we applied (2.44). For $\gamma \leq \beta$ and $|\gamma| = 2$, it is estimated in the same way for the first integral with $|\gamma| = 2$, i.e., we have
\[
\sum_{|\gamma|=2} \left( \frac{\beta}{\gamma} \right) \int_{T^3} |\partial^\gamma w| |\nabla \partial^{3-\gamma} u| |\partial^\beta w| \, dx \leq C\|u\|_{H^2} \|u\|_{H^3} \|w\|_{H^1}^2.
\]
For $\gamma \leq \beta$ and $|\gamma| = 3$, it is bounded by
\[
\sum_{|\gamma|=3} \left( \frac{\beta}{\gamma} \right) \int_{T^3} |\partial^\gamma w| |\nabla \partial^{3-\gamma} u| |\partial^\beta w| \, dx \leq C\|\nabla \partial^{3-\gamma} u\|_{L^\infty} \|\partial^\gamma w\|_{L^2} \|\partial^\beta w\|_{L^2} \leq C\|u\|_{H^3} \|w\|_{H^1}^2.
\]
As for the last term on the right side of (4.27), we integrate by parts and estimate as
\[
\int_{T^3} \partial^\beta (\nabla \times f) \cdot \partial^\beta w \, dx \leq \sum_{\gamma=2} \left( \frac{\beta}{\gamma} \right) \int_{T^3} |\partial^\gamma (\nabla \times f)| |\partial^\beta w| \, dx \leq C\|\nabla \times f\|_{H^2} \|\nabla \partial^\beta w\|_{L^2} \leq C\|f\|_{H^2_{\text{curl}}}^2 + \frac{1}{8}\|\nabla \partial^\beta w\|_{L^2}^2.
\]
Summing up all the above estimates, we have
\[
\frac{d}{dt}\|\partial^\beta w\|_{L^2}^2 + \|\nabla \partial^\beta w\|_{L^2}^2 \leq K_4 \|\partial^\beta u\|_{L^2}^2 + K_5,
\]
where the constants $K_4$ and $K_5$ depend on the $H^3$ norm of $u$, $H^2$ norm of $w$, while $K_5$ also depends on the $H^2_{\text{curl}}$ norm of $f$. Therefore, Grönwall’s inequality implies that $w \in L^\infty(0,T;H^3 \cap V) \cap L^2(0,T;H^4 \cap V)$. Therefore, by repeating similar arguments as above inductively, we get $H^s$ uniform bound on $u$ and $w$ for all integers $s \geq 4$. Proof of Theorem 2.9 is thus complete.
5. Proof of Convergence Results

In this section, we prove our convergence results Theorem 2.10 and Theorem 2.11. Notice that from the proof of Theorem 2.8 we have $\nabla \cdot w(t) = 0$ for a.e. $t > 0$ as long as $\nabla \cdot w_0 = 0$.

5.1. Convergence of $\nabla \times u$ to $w$ in $L^2$. Proof of Theorem 2.10

We start by applying the curl operator “$\nabla \times$” to (1.3a) and after denoting by $\omega$ the vorticity $\nabla \times u$, we obtain

\[
(I - \alpha^2 \Delta) \omega_1 - \nu \Delta \omega + (u \cdot \nabla) w - (\nabla \cdot w) u - (w \cdot \nabla) u = \nabla \times f,
\]

where we used $\nabla \cdot u = 0$ and the identity

\[
\nabla \times (F \times G) = ((\nabla \cdot G) + G \cdot \nabla) F - ((\nabla \cdot F) + F \cdot \nabla) G
\]

for arbitrary smooth vector fields $F$ and $G$ in $\mathbb{R}^3$. Denoting by $\xi$ the difference $\omega - w$ and subtracting the $w$ equation of system (1.3) from (5.28) lead to

\[
(5.29) \quad \xi_t - \alpha^2 \Delta \omega_t - \Delta \xi - (\nabla \cdot w) u = 0.
\]

Then, we use Theorem 2.8 and rewrite (5.29) as

\[
\xi_t - \alpha^2 \Delta \omega_t - \Delta \xi = \alpha^2 \Delta w + (\nabla \cdot w) u,
\]

to which we multiply $\xi$ and integrate by parts over $T^3$, and obtain

\[
(5.30) \quad \frac{1}{2} \frac{d}{dt} \left( \|\omega\|^2_{L^2} + \alpha^2 \|\nabla \xi\|^2_{L^2} \right) + \|\nabla \xi\|^2_{L^2} = \alpha^2 \int_{T^3} \Delta w_t \cdot \xi \, dx + \int_{T^3} \nabla \cdot w u \cdot \xi \, dx.
\]

Note that the second term on the right side of (5.30) vanishes due to Theorem 2.8 since $\nabla \cdot w(0) = 0$. In order to estimate the first term on the right side of (5.30), we integrate by parts and use the equation of $w$ and obtain

\[
\alpha^2 \int_{T^3} \Delta w_t \cdot \xi \, dx = \alpha^2 \int_{T^3} w_t \cdot \Delta \xi \, dx = \alpha^2 \int_{T^3} \Delta w \cdot \Delta \xi \, dx - \alpha^2 \int_{T^3} (u \cdot \nabla) w \cdot \Delta \xi \, dx + \int_{T^3} (w \cdot \nabla) u \cdot \Delta \xi \, dx + \alpha^2 \int_{T^3} \nabla \times f \cdot \Delta \xi \, dx
\]

(5.31)

Then, we estimates the four integrals on the right side of (5.31). After integration by parts, we bound the first integral by

\[
\alpha^2 \int_{T^3} |\nabla \Delta w||\nabla \xi| \, dx \leq C \alpha^2 \|\nabla \Delta w\|_{L^2} \|\nabla \xi\|_{L^2} \leq C \alpha^2 \|w\|^2_{H^3} + \alpha^2 \|\nabla \xi\|^2_{L^2}.
\]

Using Lemma 2.1 the second integral is bounded by

\[
\alpha^2 \int_{T^3} |\nabla u| |\nabla w| |\nabla \xi| \, dx + \alpha^2 \int_{T^3} |u| |\Delta w| |\nabla \xi| \, dx
\leq C \alpha^2 \|u\|_{L^2} \|w\|_{H^2} \|\nabla \xi\|_{L^2} + C \alpha^2 \|u\|_{L^2} \|\Delta w\|_{L^2} \|\nabla \xi\|_{L^2}
\leq C \alpha^2 \|u\|^2_{H^2} \|w\|^2_{H^2} + \alpha^2 \|\nabla \xi\|^2_{L^2}.
\]

Estimates for the third integral is similar and we have

\[
\alpha^2 \int_{T^3} (w \cdot \nabla) u \Delta \xi \, dx \leq C \alpha^2 \|\Delta u\|_{L^2} \|w\|_{H^2} \|\nabla \xi\|_{L^2} + C \alpha^2 \|\Delta u\|_{L^2} \|w\|_{H^2} \|\nabla \xi\|_{L^2}
\leq C \alpha^2 \|u\|^2_{H^2} \|w\|^2_{H^2} + \alpha^2 \|\nabla \xi\|^2_{L^2}.
\]
As for the last integral in (5.31), we integrate by parts and use Hölder’s inequality, and bound it by
\[
\alpha^2 \int_{T^3} |\Delta f| |\nabla \xi| \, dx \leq C \alpha^2 \| \Delta f \|_{L^2} \| \nabla \xi \|_{L^2} \leq C \alpha^2 \| f \|_{H^2} + \alpha^2 \| \nabla \xi \|_{L^2}^2.
\]
Summing up all the above estimates, we obtain
\[
\frac{d}{dt} \left( \| \xi \|_{L^2}^2 + \alpha^2 \| \nabla \xi \|_{L^2}^2 \right) \leq C \left( \| \xi \|_{L^2}^2 + \alpha^2 \| \nabla \xi \|_{L^2}^2 \right) + K \alpha^2
\]
where the constant $K$ depends on the $H^3$ norms of $u$ and $w$, as well as $H^2$ norm of $f$. By Lemma 2.4 we have
\[
\| \xi(t) \|_{L^2}^2 + \alpha^2 \| \nabla \xi(t) \|_{L^2}^2 + \int_0^t \| \nabla \xi(s) \|_{L^2}^2 \, ds \leq K \alpha^2 (e^{Ct} - 1),
\]
where we used $\xi_0 = u_0 - \omega_0 = 0$. Therefore, $\| \xi(t) \|_{L^2}^2 + \xi(T) \leq C \alpha^2 e^{C \alpha T} \to 0$ as $\alpha \to 0$. The proof of Theorem 2.10 is thus complete.

5.2. Convergence of $\varrho$ to $\tilde{\varrho}$ and $u$ to $\tilde{u}$ in $L^2$. Proof of Theorem 2.11. Assume the hypotheses and the notation of Theorem 2.11. Since $u_0 \in H^4 \cap V$, by Theorem 2.3 there exists a time $T > 0$ and a unique strong solution $(\tilde{u}, \tilde{p})$ to (1.1) satisfying $\tilde{u} \in C([0, T]; H^4 \cap V) \cap L^2(0, T; H^5 \cap V)$.

In view of Theorem 2.10 it suffices to show that $\| w - \tilde{\varrho} \|_{L^2} + \| u - \tilde{u} \|_{L^2} \sim O(\alpha)$. We start by applying the curl operator “$\nabla \times$” to (1.1) and obtain
\[
\frac{\partial \tilde{\varrho}}{\partial t} - \Delta \tilde{\varrho} + (\tilde{u} \cdot \nabla)\tilde{\varrho} - (\tilde{\varrho} \cdot \nabla)\tilde{u} = \nabla \times f.
\]
Then, by taking the difference of (1.3b) and (5.33) and denoting by $\theta := w - \tilde{\varrho}$, we have
\[
\frac{\partial \theta}{\partial t} - \Delta \theta + (u \cdot \nabla)\theta + (\zeta \cdot \nabla)\tilde{\varrho} - (\theta \cdot \nabla)u - (\tilde{\varrho} \cdot \nabla)\zeta = 0,
\]
Denoting $\zeta := u - \tilde{u}$, we rewrite the above as
\[
\frac{\partial \theta}{\partial t} - \Delta \theta + (u \cdot \nabla)\theta + (\zeta \cdot \nabla)\tilde{\varrho} - (\theta \cdot \nabla)u - (\tilde{\varrho} \cdot \nabla)\zeta = 0,
\]
with $\theta(\cdot, 0) = \theta_0 = 0$.

Next, by subtracting (1.2) from (1.3a), we obtain the following system for $\zeta := u - \tilde{u}$.
\[
\begin{aligned}
(1 - \alpha^2 \Delta) \frac{\partial \zeta}{\partial t} - \alpha^2 \Delta \frac{\partial \tilde{u}}{\partial t} - \Delta \zeta + w \times u - \tilde{u} \times \tilde{u} + \nabla \Pi = 0, \\
\nabla \cdot \zeta = 0, \\
\zeta(\cdot, 0) = \zeta_0 = 0,
\end{aligned}
\]
where $\Pi = p - \tilde{p} - \frac{1}{2} |\tilde{u}|^2$. We multiply (5.34) by $\theta$ and (5.35a) by $\zeta$, respectively, integrate by parts over $T^3$, and add, to obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \| \theta \|_{L^2}^2 + \| \zeta \|_{L^2}^2 + \alpha^2 \| \nabla \zeta \|_{L^2}^2 \right) + \| \nabla \theta \|_{L^2}^2 + \| \nabla \zeta \|_{L^2}^2 \leq \int_{T^3} (\zeta \cdot \nabla)\tilde{\varrho} \cdot \theta \, dx + \int_{T^3} (\theta \cdot \nabla)u \cdot \theta \, dx + \int_{T^3} (\tilde{\varrho} \cdot \nabla)\zeta \cdot \theta \, dx \\
+ \alpha^2 \int_{T^3} \Delta \frac{\partial \tilde{u}}{\partial t} \cdot \zeta \, dx + \int_{T^3} (\theta \times \tilde{u}) \cdot \zeta \, dx,
\]
(5.36)
where we used $w \times u - \tilde{w} \times \tilde{u} = \theta \times \tilde{u} + w \times \zeta$, $(w \times \zeta) \cdot \zeta = 0$, (2.7f), and (5.36c). Next, we estimate the five integrals on the right side of (5.36). Using (2.7c), the first integral is bounded by

$$\|\nabla \tilde{w}\|_{L^2} \|\zeta\|_{L^2}^{1/2} \|\nabla \zeta\|_{L^2}^{1/2} \|\nabla \theta\|_{L^2} \leq C \|\zeta\|_{L^2}^{2} + \frac{1}{4} \|\nabla \theta\|_{L^2}^{2} + \frac{1}{6} \|\nabla \zeta\|_{L^2}^{2},$$

where $\|\nabla \tilde{w}\|_{L^\infty(0,T;H)} \leq C$. The second integral can also be estimated using (2.7c):

$$\int_{T^3} (\theta \cdot \nabla)u \cdot \theta \, dx \leq \|\nabla \theta\|_{L^2} \|u\|_{L^2}^{1/2} \|\nabla u\|_{L^2}^{1/2} \leq C \|\nabla u\|_{L^2} \|\theta\|_{L^2}^{2}.$$ We note that, as in Remark 4.2, that $\|\nabla u\|_{L^2}$ might not be bounded independently of $\alpha$, but $\int_0^{T} \|\nabla u(t)\|_{L^2} \, dt \leq T^{1/2} \|u\|_{L^2(0,T;V)}$ is bounded independently of $\alpha \in (0, 1)$. The third integral can be estimated using (2.7d):

$$\int_{T^3} (\tilde{u} \cdot \nabla) \zeta \cdot \theta \, dx \leq \|\nabla \tilde{w}\|_{L^2} \|\theta\|_{L^2} \|\nabla \theta\|_{L^2} \|\nabla \zeta\|_{L^2} \leq C \|\theta\|_{L^2}^{2} + \frac{1}{4} \|\nabla \theta\|_{L^2}^{2} + \frac{1}{6} \|\nabla \zeta\|_{L^2}^{2}. $$

By substituting $\partial \tilde{u} / \partial t$ from (1.13) and integration by parts, we bound the fourth term on the right side of (5.36) by

$$\alpha^2 \int_{T^3} |\nabla \tilde{u} \cdot \nabla \zeta| \, dx + \alpha^2 \int_{T^3} |\nabla (\tilde{w} \times \tilde{u})| \cdot \nabla \zeta \, dx + \alpha^2 \int_{T^3} |\nabla f| \cdot \nabla \zeta \, dx$$

$$\leq C \alpha^2 \|\tilde{u}\|_{H^3}^2 + C \alpha^2 \|\tilde{u}\|_{H^1} \|\tilde{u}\|_{H^3} + C \alpha^2 \|\tilde{u}\|_{H^1} \|\tilde{u}\|_{H^3}^2 + C \alpha^2 \|f\|_{H^1}^2 + \frac{1}{6} \|\nabla \zeta\|_{L^2}^2,$$ where we used the hypothesis of the theorem that $\alpha \in (0, 1)$. As for the last integral, we use (2.30) to bound it by

$$\|\tilde{u}\|_{H^3} \|\theta\|_{L^2} \|\zeta\|_{L^2} \leq C \|\theta\|_{L^2}^2 + C \|\zeta\|_{L^2}^2,$$

where $\|\tilde{u}\|_{H^3} \leq C$. Combining all the above estimates and using (5.35c), we obtain

$$\frac{d}{dt} \left( \|\theta(t)\|_{L^2}^2 + \|\zeta(t)\|_{L^2}^2 + \alpha^2 \|\nabla \zeta(t)\|_{L^2}^2 \right) + \|\nabla \theta\|_{L^2}^2 + \|\nabla \zeta\|_{L^2}^2 \leq C \alpha^2 + \tilde{C} \|\nabla u\|_{L^2} \left( \|\theta(t)\|_{L^2}^2 + \|\zeta(t)\|_{L^2}^2 + \alpha^2 \|\nabla \zeta(t)\|_{L^2}^2 \right),$$

where the constants $C$ and $\tilde{C}$ are uniform-in-time bounds for $\|\tilde{u}\|_{H^3}$, as well as $\tilde{K}$. Using Grönwall’s inequality, the integrability of $\|\nabla u(t)\|_{L^2}$, and the fact that $\|u\|_{L^2(0,T;V)}$ is bounded independently of $\alpha \in (0, 1)$, we obtain that $\|\theta(t)\|_{L^2}^2 + \|\zeta(t)\|_{L^2}^2 + \int_0^t \|\nabla \zeta(s)\|_{L^2}^2 \, ds + \int_0^t \|\nabla \theta(s)\|_{L^2}^2 \, ds \leq C \alpha^2 T$. Thus, the proof of Theorem 2.11 is now complete.

5.3. **Proof of Corollary 2.13.** Assume the hypotheses. From Theorem 2.11 $\|u - \tilde{u}\|_{L^\infty(0,T;H)} \to 0$ and $\|u - \tilde{u}\|_{L^2(0,T;V)}$ as $\alpha \to 0$. Thus, passing to the limit as $\alpha \to 0$ in the energy equality (2.11) yields

$$\limsup_{\alpha \to 0} \alpha^2 \|\nabla u(t)\|_{L^2}^2 + \|\tilde{u}(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla \tilde{u}(s)\|_{L^2}^2 \, ds = \|u_0\|_{L^2}^2 + 2 \int_0^t (\tilde{u}(s), f(s)) \, ds$$

However, if $\tilde{u}$ is a strong solution to the Navier-Stokes equation on $[0,T]$, the energy identity

$$\|\tilde{u}(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla \tilde{u}(s)\|_{L^2}^2 \, ds = \|u_0\|_{L^2}^2 + 2 \int_0^t (\tilde{u}(s), f(s)) \, ds,$$

holds (see, e.g., [12, 20]). Thus, (5.37) together with (5.38) contradict (2.12).

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References

[1] S. Agmon. *Lectures on elliptic boundary value problems*. AMS Chelsea Publishing, Providence, RI, 2010. Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr., Revised edition of the 1965 original.

[2] M. Anitescu, W. Layton, and F. Pahlevani. Implicit for local effects and explicit for nonlocal effects is unconditionally stable. *ETNA*, 18:174–187, 2004.

[3] M. Böhm. On Navier–Stokes and Kelvin–Voigt equations in three dimensions in interpolation spaces. *Math. Nachr.*, 155:151–165, 1992.

[4] Y. Cao, E. Lunasin, and E. S. Titi. Global well-posedness of the three-dimensional viscous and inviscid simplified Bardina turbulence models. *Commun. Math. Sci.*, 4(4):823–848, 2006.

[5] R. W. Carroll and R. E. Showalter. *Singular and Degenerate Cauchy Problems*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1976. Mathematics in Science and Engineering, Vol. 127.

[6] D. Catania. Global existence for a regularized magnetohydrodynamic–α model. *Ann. Univ. Ferrara*, 56:1–20, 2010. 10.1007/s11565-009-0069-1.

[7] D. Catania and P. Secchi. Global existence for two regularized MHD models in three space-dimension. *Quad. Sem. Mat. Univ. Brescia*, (37), 2009.

[8] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, and S. Wynne. Camassa-Holm equations as a closure model for turbulent channel and pipe flow. *Phys. Rev. Lett.*, 81(24):5338–5341, 1998.

[9] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, and S. Wynne. The Camassa-Holm equations and turbulence. *Phys. D*, 133(1-4):49–65, 1999. Predictability: quantifying uncertainty in models of complex phenomena (Los Alamos, NM, 1998).

[10] S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, and S. Wynne. A connection between the Camassa-Holm equations and turbulent flows in channels and pipes. *Phys. Fluids*, 11(8):2343–2353, 1999. The International Conference on Turbulence (Los Alamos, NM, 1998).

[11] A. Cheskidov, D. D. Holm, E. Olson, and E. S. Titi. On a Leray–α model of turbulence. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 461(2055):629–649, 2005.

[12] P. Constantin and C. Foias. *Navier-Stokes Equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[13] L. Davis and F. Pahlevani. Semi-implicit schemes for transient Navier-Stokes equations and eddy viscosity models. *Numer. Methods Part. Diff. Equn.*, 25(1):212–231, 2009.

[14] G. Di Molfetta, G. Krstulovic, and M. Brachet. Self-truncation and scaling in Euler-Voigt- and related fluid models. *Phys. Rev. E*, 92:033020, Jul 2015.

[15] E. DiBenedetto and R. E. Showalter. Implicit degenerate evolution equations and applications. *SIAM J. Math. Anal.*, 12(5):731–751, 1981.

[16] W. E and J.-G. Liu. Simple finite element method in vorticity formulation for incompressible flows. *Math. Comp.*, 70:579–593, 1997.

[17] M. A. Ebrahimi, M. Holst, and E. Lunasin. The Navier–Stokes-Voight model for image inpainting. *IMA J. App. Math.*, pages 1–26, 2012. doi:10.1093/imamat/hxr069.

[18] A. Farhat, E. Lunasin, and E. S. Titi. Continuous data assimilation for a 2D Bénard convection system through horizontal velocity measurements alone. *Journal of Nonlinear Science*, pages 1–23, 2017.

[19] C. Foias, D. D. Holm, and E. S. Titi. The three-dimensional viscous Camassa-Holm equations, and their relation to the Navier–Stokes equations and turbulence theory. *J. Dynam. Differential Equations*, 14(1):1–35, 2002.

[20] C. Foias, O. Manley, R. Rosa, and R. Temam. *Navier-Stokes Equations and Turbulence*, volume 83 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2001.

[21] T. Gatski. Review of incompressible fluid flow computations using the vorticity-velocity formulation. *Appl. Numer. Math.*, 7:227–239, 1991.

[22] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations: Theory and Algorithms*. Springer-Verlag, 1986.

[23] G. Guevremont, W. G. Habashi, and M. M. Hafez. Finite element solution of the Navier-Stokes equations by a velocity-vorticity method. *Int. J. Numer. Methods Fluids*, 10:461–475, 1990.

[24] T. Heister, M. A. Olshanskii, and L. G. Rebholz. Unconditional long-time stability of velocity-vorticity method for 2D Navier-Stokes equations. *Numerische Mathematik*, 135:143–167, 2017.

[25] D. Holm and E. Titi. Computational models of turbulence: The LANS-α model and the role of global analysis. *SIAM News*, 38(7), September 2005. Feature Article.

[26] A. A. Ilyin, E. M. Lunasin, and E. S. Titi. A modified-Leray-α subgrid scale model of turbulence. *Nonlinearity*, 19(4):879–897, 2006.

[27] D. A. Jones and E. S. Titi. Determining finite volume elements for the 2D Navier-Stokes equations. *Phys. D*, 60(1-4):165–174, 1992. Experimental mathematics: computational issues in nonlinear science (Los Alamos, NM, 1991).

[28] V. K. Kalantarov. Attractors for some nonlinear problems of mathematical physics. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 152(Kraev. Zadachi Mat. Fiz. i Smezhnye Vopr. Teor. Funktsii18):50–54, 182, 1986.
[58] R. E. Showalter. Nonlinear degenerate evolution equations and partial differential equations of mixed type. *SIAM J. Math. Anal.*, 6:25–42, 1975.

[59] R. E. Showalter. The Sobolev equation. II. *Applicable Anal.*, 5(2):81–99, 1975.

[60] R. Temam. *Navier-Stokes Equations and Nonlinear Functional Analysis*, volume 66 of *CBMS-NSF Regional Conference Series in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 1995.

[61] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*. AMS Chelsea Publishing, Providence, RI, 2001. Theory and numerical analysis, Reprint of the 1984 edition.

[62] X. M. Wang. A remark on the characterization of the gradient of a distribution. *Appl. Anal.*, 51(1-4):35–40, 1993.

[63] K. L. Wong and A. J. Baker. A 3D incompressible Navier-Stokes velocity-vorticity weak form finite element algorithm. *Int. J. Numer. Meth. Fluids*, 38:99–123, 2002.

[64] X. H. Wu, J. Z. Wu, and J. M. Wu. Effective vorticity-velocity formulations for the three-dimensional incompressible viscous flows. *J. Comput. Phys.*, 122:68–82, 1995.