Next-to-leading nonperturbative
calculation in heavy quarkonium

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Abstract

The next-to-leading nonperturbative contributions to heavy quarkonium systems are calculated. The applicability of the Voloshin-Leutwyler approach to heavy quarkonia systems for the physical cases of Bottomonium and Charmonium is investigated. We study whether the background gluon field correlation time can be considered to be infinity or not, by calculating the leading correction to this assumption and checking whether the expansion is under control. A phenomenological analysis of our results is also performed. The results make us feel optimistic about the \( \Upsilon(1S) \) and to a lesser extent about the \( J/\psi \) but do not about higher levels. We also briefly discuss the connection with different models where a finite gluon correlation time is introduced.

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1 Introduction

In the early eighties Voloshin and Leutwyler (VL) developed a theory for quark-antiquark bound state systems in the limit $m_Q \to \infty$ from first principles. In this approach it was considered that the quarkonia system can be mainly understood as a Coulomb type bound state (this is certainly true if the heavy quark mass is large enough). Moreover, the nonperturbative (NP) contributions are corrections and can be included systematically. The leading NP corrections were calculated in refs. [2, 3].

Let us briefly discuss the main features of this approach. First, for calculation convenience, it is usually chosen the modified Schwinger gauge fixing for the background gluon field (see [4])

$$x^iA^i(x, t) = 0, \quad A^0(\vec{0}, t) = 0,$$

while the Coulomb gauge is used for the perturbative gluon field. The Hamiltonian then reads

$$H = H_{Q\bar{Q}} + H_g + H_I,$$

where

$$H_{Q\bar{Q}} = P_s H_s + P_8 H_8,$$

$$H_s = -\frac{\Delta}{m} - \frac{C_F \tilde{\alpha}_s(\mu)}{r}, \quad H_8 = -\frac{\Delta}{m} + \frac{1}{2N_c} \frac{\tilde{\alpha}_s(\mu)}{r},$$

$P_s$ ($P_8$) is the singlet (octet) projector on a quark-antiquark pair, $\tilde{\alpha}_s$ is defined below, $H_g$ acts on the gluonic and light quark degrees of freedom and

$$H_I = -\frac{g}{2} \xi^a \vec{x} \cdot \vec{E}^a(0),$$

where $\xi^a = t_1^a - t_2^a$ ($t_1^a$ ($t_2^a$) is the color SU(3) generator for the quark (antiquark) with $t_{1,2}^a = \frac{\lambda^a}{2}$) and $\vec{E}^a$ is the background chromo-electric field. Giving (1.5) for $H_I$ we are assuming that the multipole expansion holds. That is

$$|\vec{x}| \Lambda_{QCD} \ll 1$$

but we will also assume $H_I$ to be small as far as we will work within standard quantum mechanics perturbation theory (for consistency, we will demand the NP corrections to be
small). Thus, (1.6) becomes

$$\frac{m_{\beta_n}}{n} \gg \Lambda_{QCD},$$  \hspace{1cm} (1.7)

where $\beta_n \equiv C_F \tilde{\alpha}_s/n$. Nevertheless, for obtaining the standard VL results other assumption is needed, that is, to consider the background gluon field correlation time $T_g$ to be approximately infinity

$$\frac{H_g}{m_{\beta_n}^2} << 1 \rightarrow m_{\beta_n}^2 >> \frac{1}{T_g} \sim \Lambda_{QCD}.$$  \hspace{1cm} (1.8)

Now, we have a double expansion $O((\frac{\Lambda_{QCD}}{m_{\beta_n}})^2, (\frac{\Lambda_{QCD}}{m_{\beta_n}^2})^2)$. We remark that (1.8) is more likely to fail than (1.7) (although, certainly, numerical factors can play a role). The main aim of this paper is studying the assumption (1.8) and its corrections which, obviously, are going to be the dominant NP corrections.

During several years it was generally believed that Bottomonium and Charmonium systems were not heavy enough. Thus, the above mention approach could not be applied. Nevertheless, the situation was not so clear for the $\Upsilon(1S)$ and for the $n = 2$ fine and hyperfine splitting. Later on, assumption (1.8) was relaxed and $T_g$ was taken into account [5, 6, 7] (for a recent study see [8]). A detailed study of the $\Upsilon(1S)$ and the $n = 2$ hyperfine splitting was done in the last paper of ref. [7]. Unfortunately, these approaches are model dependent since they have to deal with an arbitrary function. Usually, an exponential decaying ansatz is used.

Recently, there have been some attempts to perform a rigorous QCD determination of the Bottomonium and Charmonium properties [9, 10] using only the VL approach. It is claimed that a consistent theory for the lower Bottomonium levels ($n = 1, 2$) and to a lesser extent for the $J/\psi$ is found. Radiative, NP and eventually relativistic corrections were put together for the first time. Some general features for spin independent observables can be inferred from this study. The leading perturbative and NP corrections are consistently smaller than the Coulomb energy for the $\Upsilon(1S)$ mass. Likewise, the corrections are under control for the $\Upsilon(1S)$ decay width although the result is quite dependent on the scale. For $n = 2$ observables the situation is much more doubtful since the perturbative and NP corrections are almost as important as the leading term. Nevertheless, quite good agreement with the data was obtained. For these observables and the $\Upsilon(1S)$ decay width the NP corrections play a fundamental role in order to agree with experiment.

\footnote{If we did not consider $H_{int}$ to be small, $|\vec{x}|$ would be an arbitrary function of $m_Q$, $\alpha_s$ and $\Lambda_{QCD}$. This more general situation is beyond the scope of this paper.}
It could be argued that further orders in the relativistic, radiative and NP corrections are needed in order to improve these results. Moreover, so far, no model independent check of approximation (1.8) has been done though it is the most likely to fail. This check is clearly very important to clarify in which situation we are, that is, whether (1.8) holds or not and in which observables. In the former case the VL approach should be applied while in the last case $T_g$ has to be taken into account in some way and the models displayed in refs. [5]-[8] may be applied. In order to disentangle this issue, we calculate the next-to-leading spin independent NP contributions to the energy levels and wavefunctions of a heavy $Q\bar{Q}$ pair. We obtain analytical expressions for them, and we check whether the expansion is under control.

Let us briefly comment upon the fine and hyperfine splitting ($n = 1, 2$). For them the agreement with the data (when available) is quite good according to ref. [10] (although errors are large). This is quite surprising since Dosch and collaborators [7] also find agreement with the data using a model where $T_g$ is taken into account. Undoubtedly, this should require a detailed study which goes beyond the scope of this work and we will not perform it here. We expect these observables to depend on shorter distances being the NP contributions smaller. Combinations of these observables can be built such that the NP corrections are under control as it was done in [11] (see also [12] where an omitted contribution was calculated).

We distribute the paper as follows. In sec. 2 we calculate the next-to-leading NP corrections to the energy and decay width. In sec. 3 we perform a phenomenological study of our results. The last section is devoted to the conclusions and the discussion of models where $T_g$ is taken into account. A few formulas are relegated to Appendix A and we discuss the role played by the octet potential in Appendix B.

2 Theoretical results

We display the calculations and results for the next-to-leading NP corrections to the spin independent energy levels and wavefunctions.

2.1 Energy Levels

In this subsection we calculate the next-to-leading NP contributions to the energy levels.
Using the multipole expansion (1.7) and standard Quantum Mechanics perturbation theory the energy correction reads

\[ \delta E_{nl} = \langle 0 | \langle n, l | H_I \frac{1}{E_n - H_8 - H_g} H_I | n, l | 0 \rangle = \frac{g^2}{18} \langle 0 | E_j^a(0) \langle n, l | \vec{r} \frac{1}{E_n - H_8 - H_g} \vec{r} | n, l \rangle E_j^a(0) | 0 \rangle . \]  

(2.1)

\( E_n \) is the Coulomb singlet bound state energy. \( n, l \) are the radial and orbital quantum numbers. As far as we do not study the fine and hyperfine splittings the corrections do not depend on \( j \) (total angular momentum) and \( s \) (spin) so we will not display these indices in the states. \( H_g \sim \Lambda_{QCD} \) and \( H_8 \sim E_n \).

The octet propagator mixes low \( O(\Lambda_{QCD}) \) and high energies \( O(E_n) \). No assumption about (1.8) has still been done in (2.1). Let us assume (1.8) holds, it follows that an OPE can be performed where the parameter expansion is of order

\[ \left( \frac{H_g}{E_n - H_8} \right)^2 \sim \left( \frac{\Lambda_{QCD}}{m_{\beta}^2} \right)^2 \]

(2.2)

and we obtain

\[ \delta E_{nl} = \sum_{r=0}^{\infty} C_r O_r , \]

(2.3)

where

\[ C_r = \langle n, l | \vec{r} \left( \frac{1}{E_n - H_8} \right)^{2r+1} \vec{r} | n, l \rangle \]

(2.4)

and

\[ O_r = \frac{g^2}{18} \langle 0 | E_j^a H_g^{2r} E_j^a | 0 \rangle . \]

(2.5)

Using the equation of motion, the gauge fixing and Lorentz covariance we obtain

\[ O_r = -\frac{g^2}{54} v^{\beta_0} ... v^{\beta_r} v^{\alpha_0} ... v^{\alpha_r} \times \langle 0 | Tr \left( [D_{\beta_1}(0), [...[D_{\beta_r}(0), G_{\beta_\rho (0)}]...] [D_{\alpha_1}(0), [...[D_{\alpha_r}(0), G_{\alpha_\rho (0)}]...] | 0 \rangle \]

(2.6)

where \( v \) is the velocity of the center of masses frame with \( v^2 = 1 \) (in the comoving frame \( v = (1, \vec{0}) \)) and the trace is in the adjoint representation.
Let us give the order of each term ($<r> \sim \frac{n}{m_\beta}$)

$$\delta E_{nl}^{(r)} \equiv C_r O_r \sim m_\beta^2 \left( \frac{\Lambda_{QCD}}{m_\beta} \right)^2 \left( \frac{\Lambda_{QCD}}{m_\beta^2} \right)^{2r+2} \quad (2.7)$$

For $\delta E_{nl}^{(0)}$ we obtain the standard VL result [2, 3]. Nevertheless, we are interested in the next correction

$$\delta E_{nl}^{(1)} = C_1 O_1 \sim m_\beta^2 \frac{A_{QCD} n^2}{m_\beta^{10}} \quad (2.8)$$

where

$$C_1 = \langle n, l | \vec{r} \left( \frac{1}{E_n - H_8} \right)^3 \vec{r} | n, l \rangle . \quad (2.9)$$

After a rather lengthy calculation we obtain

$$C_1 = \frac{-1}{m_\beta^5} H(n, l) , \quad (2.10)$$

where $H(n, l)$ is a dimensionless function which can be obtained using techniques developed in ref. [4]. We refer the reader to Appendix A for the analytical expression which can be written as a rational function. There, we also provide some numbers for the lower levels. It can be easily checked that it has the right leading $n^{10}$ dependence.

For $O_1$, using Lorentz covariance, Bianchi identities and the equation of motion, we obtain

$$O_1 = \frac{1}{108} \left\{ -g^4 \sum_{A,B} \langle 0 | : \bar{q}^A t^a \gamma_\nu q^A q^B t^a \gamma_\nu q^B : | 0 \rangle + \frac{3}{4} \langle G^3 \rangle \right\} , \quad (2.11)$$

where

$$\langle G^3 \rangle \equiv g^3 f_{abc} \langle 0 | : G^\mu_{ab} G^\nu_{bp} G^\rho_{cv} : | 0 \rangle . \quad (2.12)$$

and $A, B$ are SU(3) flavor indices.

We can work (2.11) further using SU(3) flavor symmetry and the factorization hypothesis. Finally, it reads

$$O_1 = \frac{1}{108} \left\{ \frac{26}{3} \pi^2 \alpha_s^2 \langle 0 | \bar{q} q | 0 \rangle^2 + \frac{3}{4} \langle G^3 \rangle \right\} . \quad (2.13)$$

The physical cases will be studied in the next section. Anyway, let us comment that $O_1$ is going to be positive and, therefore, the energy correction to be negative.

In Appendix B the error made if we neglect the octet potential in (2.9) is discussed.
2.2 Decay Width

In this subsection the corrections to the decay width of the $n^3S_1$ levels of quarkonium are calculated. For further details we refer to refs. [1, 2].

The decay width of the $n^3S_1$ levels of quarkonium reads

\[
\Gamma(n^3S_1 \rightarrow e^+e^-) = \pi \left[ \frac{4\alpha_{em} Q}{M(n^3S_1)} \right]^2 \left( 1 - \frac{16\alpha_s(\mu)}{3\pi} \right) \text{Res}_{E=E_{pole}} \langle \vec{x} = 0|G_s(E)|\vec{y} = 0 \rangle, \tag{2.14}
\]

where $Q$ is the quark charge and

\[
G_s(E) = P_s\langle 0|\frac{1}{H-E}|0 \rangle P_s
\]

is the nonrelativistic propagator (Green function) of the system projected on the colorless sector of a quark-antiquark pair and the gluonic vacuum. $E$ is the energy measured from the threshold $2m$. Near the pole $n$ we have the expansion

\[
\langle \vec{x} = 0|G_s(E)|\vec{y} = 0 \rangle = \frac{\rho_n + \delta \rho_n}{E_n + \delta E_{n0} - E} + O((E_n + \delta E_{n0} - E)^0) + O((E_n - E)^0) + O(\delta E^2_{n0}). \tag{2.16}
\]

On the other hand using (1.7) we obtain

\[
\langle \vec{x} = 0|G_s(E)|\vec{y} = 0 \rangle \approx \langle \vec{x} = 0|G_s^{(0)}(E)|\vec{y} = 0 \rangle + \langle \vec{x} = 0|\delta G_s(E)|\vec{y} = 0 \rangle, \tag{2.17}
\]

where

\[
\langle \vec{x} = 0|G_s^{(0)}(E)|\vec{y} = 0 \rangle = \langle \vec{x} = 0|\frac{1}{H_s - E}\vec{y} = 0 \rangle = \frac{\rho_n}{E_n - E} + O((E_n - E)^0) \tag{2.18}
\]

and

\[
\rho_n = \frac{1}{\pi} \left( \frac{m\beta_n}{2} \right)^3
\]

and

\[
\langle \vec{x} = 0|\delta G_s(E)|\vec{y} = 0 \rangle = P_s\langle 0|\langle \vec{x} = 0|\frac{1}{H_s - E}\frac{1}{H_s - H_t}H_s + H_g - E\frac{1}{H_s - E}\vec{y} = 0 \rangle |0 \rangle P_s
\]

\[
= \frac{g^2}{18} \langle 0|E_j^n(0)|\vec{x} = 0|\frac{1}{H_s - E}\frac{1}{H_s - H_t}H_s + H_g - E\frac{1}{H_s - E}\vec{y} = 0 \rangle E_j^n(0)|0 \rangle
\]

\[
= -\frac{\rho_n\delta E_{n0}}{(E_n - E)^2} + \frac{\delta \rho_n}{E_n - E} + O((E_n - E)^0). \tag{2.19}
\]
Proceeding in the same way than in the preceding section we are able to split low from high energies (OPE).

\[ \langle \vec{x} = 0 | \delta G_s(E) | \vec{y} = 0 \rangle = \sum_{r=0}^{\infty} C_r^G O_r , \tag{2.20} \]

where

\[ C_r^G = \langle \vec{x} = 0 | \frac{1}{H_s - E} \vec{r} \left( \frac{1}{H_8 - E} \right)^{2r+1} \vec{r} \frac{1}{H_s - E} | \vec{y} = 0 \rangle = \frac{A(r)}{(E_n - E)^2} + \frac{A(r+1)}{(E_n - E)} + O((E_n - E)^0) \tag{2.21} \]

and \( O_r \) is the same than above. Now, from these expressions we can read off the observables we are interested in, namely

\[ \delta \rho_n \equiv \sum_{r=0}^{\infty} \delta \rho_n^{(r)} = \sum_{r=0}^{\infty} A_{-1}^{(r)} O_r , \]

\[ \delta E_{n0} = -\frac{1}{\rho_n} \sum_{r=0}^{\infty} A_{-2}^{(r)} O_r . \tag{2.23} \]

This provides a new method to obtain the energy corrections for \( l = 0 \) states. We have used them in order to check our results of the previous subsection. \( \delta \rho_n^{(0)} \) and \( \delta E_{n0}^{(0)} \) were already calculated in ref. [2]. We have used their results to check ours for this lower order. We are mainly interested in the next correction \( \delta \rho_n^{(1)} \) which, after some effort, has been calculated exactly. We write it within the dimensionless quantity

\[ \frac{\delta \rho_n^{(1)}}{\rho_n} = \frac{O_1}{m^6 \beta_1^{10}} W(n) , \tag{2.24} \]

where the analytical expression and numerical values for \( W(n) \) are displayed in Appendix A. We can see that it has the expected leading \( n^{12} \) dependence.

In Appendix B we give analytical expressions for \( \delta \rho_n^{(1)} \) and \( \delta E_{n0}^{(1)} \) (\( l = 0 \)) with an arbitrary octet potential, \( \frac{1}{8} \rightarrow \frac{1}{N_c^2 - 1} \). There, we study the error made by neglecting the octet potential \( (N_c \rightarrow \infty) \) in our results.

3 Phenomenological analysis

In this section we confront our results with the data. We expect our corrections to be small when the theory works and the opposite could be a signal that the whole approach
breaks down. We will try to see the effect of the new contributions on the results found by the authors of ref. [9, 10] and we will mainly follow their notation.

Let us briefly discuss the set of parameters we are mainly going to use here. We make $\Lambda_{QCD}$ running between a range of values compatible with those used by the authors of ref. [9]:

$$\Lambda_{n_f=3}^{n_f=3} = 250^{+50}_{-50} \text{ Mev}. \quad (3.1)$$

The mean value approximately corresponds to the value found by Voloshin in ref. [13] using sum rules. The upper bound corresponds to the standard value of $\Lambda_{QCD}$ found in DIS (Deep Inelastic Scattering). The lower bound is closer to the values proposed by sum rules some years ago. For the two gluon condensate we quote the value given in ref. [9]

$$\langle \alpha G^2 \rangle \simeq 0.042 \text{ GeV}^4. \quad (3.2)$$

We will allow $O_1$ to have some errors. It is not our aim to give rigorous errors but to see whether the results are sensitive to sensible variations of $O_1$. For the three gluon condensate we choose the value

$$\langle G^3 \rangle \simeq 0.045 \pm 0.009 \text{ GeV}^6, \quad (3.3)$$

which comes from the dilute instanton gas approximation [14] while the errors have been estimated using the value obtained in ref. [15]. Similar results are obtained using the three-gluon condensate values given in ref. [16] (sum rules) or ref. [17] (lattice).²

In the sum rule approach it is usually found the quantity

$$\kappa \equiv \alpha_s < 0|\bar{q}q|0 >^2 (\mu), \quad (3.4)$$

which appears to be very weak scale dependent once the anomalous dimension is taken into account so it is usually considered to be a constant. In principle this expression could be calculated once the quark condensate and $\alpha_s$ are known at some scale. However, there has been some controversy about the error made within the vacuum saturation hypothesis and that may be the four-quark condensate has been underestimated by around a factor 2 [18]–[20]. We just take the average value of these references given in ref. [21]

$$\kappa = 3.8 \pm 2.0 \times 10^{-4} \text{ GeV}^6. \quad (3.5)$$

²Notice that $< G^3 >$ calculated in euclidean space has an opposite sign compared to that in Minkowski space.
Nevertheless, instead of Eq. (3.4) we actually have
\[ \alpha_s^2 < 0 |q \bar{q}| 0 >^2 (\mu) = \alpha_s(\mu) \kappa \] (3.6)
in \( O_1 \). The only remaining scale dependence is in \( \alpha_s(\mu) \). We stress that \( \alpha_s \) has to be computed at the subtraction point scale, generally, the inverse Bohr radius. This value depends on the physical system one is studying. Therefore, \( O_1 \) is going to be scale dependent, although weakly. So finally, we obtain using (3.3), (3.5) and their errors (slightly larger errors would not change the conclusions)
\[ O_1(\mu) = \left( 3.13^{+0.62}_{-0.63} + \alpha_s(\mu) 7.41^{+3.9}_{-3.9} \right) \times 10^{-4} GeV^6 . \] (3.7)

Let us finally remark that the range of values given in Eq. (3.1) does not include those obtained from LEP and \( \tau \)-decay data [21]-[23]. We quote the value from reference ref. [24]
\[ \alpha_s(M_Z) = 0.118 \pm 0.003 \] (3.8)
which approximately corresponds to
\[ \Lambda_{QCD}^{n_f=3} \simeq 383^{+52}_{-48} . \] (3.9)

There has also been some controversy about the value of the gluon condensate \( < \alpha_s G^2 > \) and several claims of larger values can be found in the literature. See [24] and references therein where the average value
\[ < \alpha_s G^2 > = 0.081 GeV^4 \] (3.10)
was given excluding the SVZ-like value [14]. In fact, for many of these studies the input values of \( \Lambda_{QCD} \) are compatible with those given in Eq. (3.9). This region of parameters was uncovered in ref. [9, 10]. We have also studied these new possibilities although our principal aim will be to compare with the results of ref. [9, 10]. I remark, in order to be consistent, that \( < G^3 > \) should be also changed since [14, 17]
\[ < G^3 > \propto \alpha_s G^2 . \] (3.11)

\(^3\)A complete analysis of this new range of parameters, including the static two loop potential and the relativistic corrections, is under way [25].
In our case (changing accordingly the errors)

\[ <G^3> \rightarrow <G^3> = 0.087^{+0.017}_{-0.018} GeV^6 \]  

(3.12)

and (using the value of \( \kappa \) given in (3.5))

\[ O_1(\mu) \rightarrow O_1(\mu) = \left(6.04^{+1.18}_{-1.25} + \alpha_s(\mu)7.41^{+3.9}_{-3.9}\right) \times 10^{-4} GeV^6. \]  

(3.13)

Now we have two possible sets of parameters. We choose Eqs. (3.1), (3.2) and (3.7) to be our first set of parameters or set I, while Eqs. (3.9), (3.10) and (3.13) are our second set of parameters or set II.

Let us now write the general mass formula for any state

\[ M(n,l) = 2m_b + A_2(n) + A_3(n,l) + \delta E_{nl}^{(0)} + \delta E_{nl}^{(1)}, \]  

(3.14)

where

\[ A_2(n) = -2m_b \frac{C_f^2 \alpha_s^2(\mu)}{8n^2}, \]

\[ A_3(n,l) = -2m_b \frac{C_f^2 \beta_0 \alpha_s^2(\mu) \bar{\alpha}_s(\mu)}{8\pi n^2} \left( \ln \left[ \frac{\mu n}{m_b C_f \bar{\alpha}_s(\mu)} \right] + \psi(n + l + 1) \right), \]

\[ \delta E_{nl}^{(0)} = m_b \frac{\epsilon_{nl} n^6 \pi \langle \alpha_s G^2 \rangle}{(m_b C_f \bar{\alpha}_s(\mu))^4}. \]  

(3.15)

\( \epsilon_{nl} \) were first calculated by Leutwyler in ref. [3]. For the levels we are interested in they read

\[ \epsilon_{10} = \frac{1872}{1275}, \quad \epsilon_{20} = \frac{2102}{1326}, \quad \epsilon_{21} = \frac{9929}{9945}. \]  

(3.16)

\( \alpha_s \) is the two-loop running coupling constant. \( A_3 \), which include radiative corrections, and

\[ \bar{\alpha}_s(\mu) = \left[ 1 + \left( a_1 + \gamma_E \beta_0 / 2 \right) \frac{\alpha_s(\mu)}{\pi} \right] \alpha_s(\mu) \]  

(3.17)

\[ a_1 = \frac{31C_A - 20T_F n_f}{36} \]

have been quoted from ref. [4]. We remark that the relativistic corrections have not been included in this analysis. This is due to the incomplete knowledge of the \( O(m_\alpha^4) \) corrections for the mass (or the equivalent for the decay width) in the \( \overline{MS} \) scheme because the static two loop potential has not been calculated\(^4\). Although naively, these corrections are expected

\(^4\) Recently, we became aware that the static two loop potential had been finally calculated [26]. We expect to introduce those results in the near future [25].
to be small they could blow up if \( \alpha_s, \tilde{\alpha}_s \) are big enough. In order to make quantitative this statement the next-to-next-to-leading perturbative contribution, including radiative and relativistic corrections, should be calculated. Moreover, it has been seen, according to the recent sum rule derivation of the Balmer mass formula [24], that the relativistic corrections tends to compensate the radiative coulombic corrections.

Let us briefly comment upon some general features of the numerical study. Large values of \( \alpha_s, \tilde{\alpha}_s \) appear in the problem. Moreover, for larger values of \( \Lambda_{QCD} \) the value of \( \alpha_s \) also increases. Therefore, for the second set of parameters, the results are going to be more doubtful. This is especially so for \( n = 2 \) Bottomonium states and Charmonium. Nevertheless, without the knowledge of the next-to-next-to-leading perturbative contribution, we do not have a real check in order to know when the \( \alpha_s \) perturbative expansion breaks down. Therefore, we refrain from displaying numbers for \( n = 2 \) Bottomonium states and Charmonium with the second set of parameters here (although we do not rule out the possibility the VL approach to work for them). This is relegated to future work where the static two loop potential and the relativistic corrections will be taken into account [25].

### 3.1 \( \Upsilon(1S) \) mass

We start our study with the \( \Upsilon(1S) \) mass where the formalism is expected to apply better. We proceed as follows. First of all, we fix \( m_b \) and the inverse Bohr radius \( a_{bb,1}^{-1} \) from the self-consistency equation

\[
a_{bb,1}^{-1} = \frac{m_b C_f \tilde{\alpha}_s(a_{bb,1}^{-1})}{2}\tag{3.18}
\]

and the \( \Upsilon(1S) \) mass set at the inverse Bohr radius scale \((\mu \rightarrow a_{bb,1}^{-1})\). We allow for the both above mentioned sets of parameters (without using errors for \( O_1 \)) and give the relative weight of each contribution in table I. It can be seen that \( \Lambda_{QCD} \) turns out to be the main source of error. We also get that perturbation theory works nicely here. Similar results than in ref. [8] are found being not to much affected by the new contribution.

For the first set of parameters, plugging the errors given in Eq. (3.7) into \( \delta E^{(1)}_{10} \) and making the ratio with \( \delta E^{(0)}_{10} \), we find

\[
0.19 < \left| \frac{\delta E^{(1)}_{10}}{\delta E^{(0)}_{10}} \right| < 0.38, \quad 0.14 < \left| \frac{\delta E^{(1)}_{10}}{\delta E^{(0)}_{10}} \right| < 0.27, \quad 0.10 < \left| \frac{\delta E^{(1)}_{10}}{\delta E^{(0)}_{10}} \right| < 0.20, \tag{3.19}
\]

for \( \Lambda_{QCD}^{n_f=3} = 200, 250, 300 \text{ MeV} \) respectively. Likewise, for the second set of parameters, we
Table 1: We display $A_2(1,0)$, $A_3(1,0)$, $\delta E_{10}^{(0)}$ and $\delta E_{10}^{(1)}$ for the $\Upsilon(1S)$. The last two columns give our results for $m_b$ and $a_{bb,1}^{-1}$. All the quantities are displayed in units of MeV. We have used $\Lambda_{n_f=3}^{QCD} = 330^{+52}_{-48}$ MeV for the second set of parameters.

| $\Lambda_{n_f=3}^{QCD}$ | $A_2$ | $A_3$ | $\delta E_{10}^{(0)}$ | $\delta E_{10}^{(1)}$ | $m_b$ | $a_{bb,1}^{-1}$ |
|--------------------------|-------|-------|---------------------|---------------------|--------|-----------------|
| $250^{+50}_{-50}$       | $-376^{+64}_{-62}$ | $61^{+13}_{-12}$ | $18^{+5}_{-7}$     | $-4^{+2}_{-3}$     | $4881^{+26}_{-27}$ | $1355^{+114}_{-121}$ |
| $383^{+52}_{-48}$       | $-535^{+74}_{-68}$ | $93^{+16}_{-15}$ | $17^{+4}_{-5}$     | $-1^{+0}_{-1}$     | $4943^{+26}_{-28}$ | $1626^{+115}_{-111}$ |

obtain

$$0.08 < \left| \frac{\delta E_{10}^{(1)}}{\delta E_{10}^{(0)}} \right| < 0.14, \quad 0.06 < \left| \frac{\delta E_{10}^{(1)}}{\delta E_{10}^{(0)}} \right| < 0.11, \quad 0.05 < \left| \frac{\delta E_{10}^{(1)}}{\delta E_{10}^{(0)}} \right| < 0.09,$$

for $\Lambda_{n_f=4}^{QCD} = 282, 330, 382$ MeV respectively. We see $\delta E_{10}^{(1)}$ is always reasonably smaller than $\delta E_{10}^{(0)}$ and we can safely conclude we are inside the VL regime for this observable.

Let us comment on some features of the results. They are very stable under variations of $\mu$ because the $\mu$ scale dependence of $A_2$ and $A_3$ cancel each other. This can be seen in more detail in Figure I where we plot the $\Upsilon(1S)$ mass as a function of the scale. Moreover, the scale of minimum sensitivity and the inverse Bohr radius scale are quite close. Indeed, the difference between the $M_{\Upsilon(1S)}$ values for those two scales is completely negligible. Other very important point is that not only the NP corrections are small but also under control. That is, higher order NP corrections can be calculated giving smaller contributions. The values we obtain for $m_b$ are compatible with the lattice calculation [27] although somewhat larger than those obtained in sum rules [28] if the comparison is done with the two-loop perturbative pole mass. Nevertheless, the agreement is quite good if the comparison is done with the three-loop pole mass.

Let us finally point out one prediction which follows directly from our results, $\bar{\Lambda}$, the nonperturbative parameter relating the mass of the heavy meson ($B, D$) to $m_Q (m_b, m_c)$ in the HQET [29]. We obtain

$$\bar{\Lambda} = 433^{+27}_{-27} MeV (\Lambda_{n_f=3}^{QCD} = 250^{+50}_{-50} MeV); \quad \bar{\Lambda} = 370^{+31}_{-28} MeV (\Lambda_{n_f=4}^{QCD} = 330^{+52}_{-48} MeV),$$

for the first and the second set of parameters respectively. Similar comments than for $m_b$ apply here when comparing with the literature.
Figure 1: Plot of $M_{T(1S)}$ (GeV) versus $\mu$ (GeV). The NP corrections have a little effect on the draw. We have used the first set of parameters with $\Lambda_{QCD}^{n_f=3} = 250 \text{MeV}$ and $m_b = 4881 \text{MeV}$. The same features hold for the other values of $\Lambda_{QCD}^{n_f=3}$.

3.2 $n = 2$ states

We now discuss $n = 2$ states and their masses $M(2,0)$ and $M(2,1)$. We will only use the first set of parameters in this subsection. We expect the theory to work worse or even to fail completely here. In fact, there are several signals that for $n = 2$ the theory does not really work. For instance, we can now predict the physical masses since we have got $m_b$ and, in order to set the scale, we can use the self-consistency equation for $n = 2$, that is

$$a_{bh,2}^{-1} = \frac{m_b C_f \tilde{\alpha}_s (a_{bh,2})}{4}.$$  \hfill (3.22)

We display the results in table II and III. We find out that $|\delta E_{nl}^{(0)}| > |A_2|$ and also $|\delta E_{nl}^{(1)}| > |\delta E_{nl}^{(0)}|$, excepting the case $n = 2, l = 1$ for $\Lambda_{QCD}^{n_f=3} = 300 \text{MeV}$, so the theory is not really trustworthy. Agreement with the data is also very bad. We have also studied the dependence

\footnote{Nevertheless the reliability of perturbation theory is marginal since $|\delta E_{21}^{(1)}| \lesssim |\delta E_{21}^{(0)}|$ and $|\delta E_{21}^{(0)}| \gtrsim \frac{2}{3}|A_2|$. Moreover, $\alpha_s = 0.403$ and $\tilde{\alpha}_s = 0.628$ are quite big making the $\alpha_s$ expansion rather doubtful. Therefore, knowledge of perturbative and NP next order contributions would be welcome in order to discern whether the $\alpha_s$ and/or the NP expansion work or not.}
\[ \Lambda_{QCD}^{n_f=3} \]

|       | \( A_2 \)       | \( A_3 \)       | \( \delta E_{20}^{(0)} \) | \( \delta E_{20}^{(1)} \) | \( M(2, 0) \) | \( a_{bb, 2}^{-1} \) |
|-------|-----------------|-----------------|--------------------------|--------------------------|----------------|------------------|
| 250\( ^{+50}_{-50} \) | \(-178\( ^{-37}_{+35} \) | \(-29\( ^{+8}_{-7} \) | \(339\( ^{-109}_{+190} \) | \(-439\( ^{+228}_{-586} \) | \(9454\( ^{+128}_{-408} \) | \(932\( ^{+095}_{-100} \) |

Table 2: We display \( A_2(2, 0), A_3(2, 0), \delta E_{20}^{(0)} \) and \( \delta E_{20}^{(1)} \). The last two columns give our results for \( M(2, 0) \) and \( a_{bb, 2}^{-1} \). All the quantities are displayed in MeV. The experimental value is \( M(2, 0) = 10023 \) MeV.

|       | \( A_2 \)       | \( A_3 \)       | \( \delta E_{21}^{(0)} \) | \( \delta E_{21}^{(1)} \) | \( M(2, 1) \) |
|-------|-----------------|-----------------|--------------------------|--------------------------|----------------|
| 250\( ^{+50}_{-50} \) | \(-178\( ^{-37}_{+35} \) | \(-71\( ^{-19}_{+17} \) | \(213\( ^{-68}_{+120} \) | \(-247\( ^{+128}_{-329} \) | \(9479\( ^{+057}_{-211} \) |

Table 3: We display \( A_2(2, 1), A_3(2, 1), \delta E_{21}^{(0)} \) and \( \delta E_{21}^{(1)} \). The last column gives our results for \( M(2, 1) \). All the quantities are displayed in MeV. The experimental value, averaged over angular momentum, is \( M(2, 1) = 9900 \) MeV.

on the scale. We find that the scale we reach the minimum and the scale we obtain doing the self consistency equation (with \( n = 2 \)) are not so near now, especially for the \( n = 2, l = 1 \) state. Therefore, we rather try with mass shifts like

\[
\Delta M(2, 0) = M(2, 0) - M(1, 0), \quad \Delta M(2, 1) = M(2, 1) - M(1, 0),
\]

as it was indeed done in ref. [9]. Let us briefly comment upon this work. The results found there were already quite delicate because the corrections were around to be as large as the leading term. However, this fact was compensated with the good agreement with the data obtained.

\[
\Delta M(2, 0) = 479 \text{MeV (expt. 563 MeV)}, \quad \Delta M(2, 1) = 417 \text{MeV (expt. 450 MeV)}.
\]  

(3.24)

From this point of view we should say that the new contribution completely destroys this agreement as we can see if we simply plug our result with their input set of parameters \((m_b = 4906 \text{MeV}, \mu = 986 \text{MeV} \text{ for } \Delta M(2, 0) \text{ and } \mu = 1062 \text{MeV for } \Delta M(2, 1))\) and with the errors given by (3.7).

\[
\Delta M(2, 0) = -182^{+231}_{-232} \text{MeV}, \quad \Delta M(2, 1) = -211^{+276}_{-218} \text{MeV}.
\]

(3.25)

The new contribution even changes the sign of \( \Delta M \). This result is due to \( |\delta E^{(1)}| > |\delta E^{(0)}| \).

It can also be seen that the dependence on the scale is very strong. Let us briefly explain
what is going on here. Let us first neglect \( \delta E^{(1)} \). Then, for consistency of the theory, one demands

\[
|A_2| > |A_3| \quad |A_2| > |\delta E^{(0)}|.
\] (3.26)

The first constraint does not allow to lower the scale while the second one does not allow to increase it, hence, only a very small window is permitted. (3.24) was found for a \( \mu \) scale such as radiative and NP contributions were equal in absolute value and even in this optimum situation (3.26) was not well satisfied \(|A_2| \sim |A_3| \sim |\delta E^{(0)}|\). Now, if we introduce the new contribution a new consistency constraint \(|\delta E^{(0)}| > |\delta E^{(1)}|\) arises which is also not well satisfied or, even more frequently, it is not satisfied at all.

We do not give more numbers here as far as we conclude the theory is not really trustworthy for \( n = 2 \) states. Indeed, all these results make us feel that probably both approximations (1.7) and (1.8) fail. In fact, that can be seen by the large mass gap between the \( n = 1 \) and \( n = 2 \) states which can not be obtained by only perturbing about the Coulomb spectrum. Therefore, \( n = 2 \) states seem to behave completely different than \( n = 1 \) states. For the former the NP contributions are large and can not be treated as corrections so other approaches should be attempted. For example, a hopeful approach has been developed by Dosch and Simonov [30] where they seem to be able to connect the VL regime with the regime where potential models work.

### 3.3 Charmonium

We now briefly discuss the Charmonium case. We will only use the first set of parameters. For this particle the Bohr scale is rather small and consequently \( \alpha_s \) quite large (indeed larger than for \( n = 2 \) Bottomonium states) so the \( \alpha_s \) perturbative expansion should be taken with care. Let us anyway say a few words about it. First let us find the value of our contribution with the set of parameters given in ref. \[m_c = 1570\text{MeV} \text{ and } \mu = 1\text{GeV}\). We obtain (using the errors given in (3.7))

\[
\delta E^{(1)}_{10} = -154^{+54}_{-51}\text{MeV}
\] (3.27)

\(|\delta E^{(1)}_{10}| \lesssim |\delta E^{(0)}_{10}| \text{ and } |A_2| = |\delta E^{(0)}_{10}|\) (this was the condition used to fix the scale in ref. [31]) for those parameters so the results are a little bit doubtful. Let us now use our standard procedure by fitting the scale with the self-consistency Bohr radius equation (notice that we use \( \eta_c \) rather than \( J/\psi \) in our procedure). We display the results in table IV. Everything
Table 4: We display $A_2(1,0)$, $A_3(1,0)$, $\delta E_{10}^{(0)}$ and $\delta E_{10}^{(1)}$ for the $\eta_c$. The last two columns give our results for $m_c$ and $a_{cc,1}^{-1}$. All the quantities are displayed in MeV.

It seems to behave quite properly, especially the NP contributions, but for those scales $\alpha_s$ is quite big so it would be desirable to obtain the next-to-next-to-leading order correction $A_4$ in order to discern whether the $\alpha_s$ expansion expansion can be applied or not. We notice here that $\delta E_{10}^{(1)}$ and $\delta E_{10}^{(0)}$ behave quite well now, say, the inequalities $|A_2| > |\delta E_{10}^{(0)}| > |\delta E_{10}^{(1)}|$ are well satisfied. The reason being that $n=1$ and the blowing up of the NP corrections with $n$ does not arise.

Just as in the $m_b$ case the values we obtain for $m_c$ are somewhat larger than those obtained in sum rules [28] if the comparison is done with the two-loop perturbative pole mass. Nevertheless, the agreement is quite good if the comparison is done with the three-loop pole mass.

### 3.4 $\Upsilon(1S)$ decay width

Let us conclude with the $\Upsilon(1S)$ decay width.

$$
\Gamma(\Upsilon(1S)) = \Gamma^{(0)} (1 + \delta_r) \left( 1 + \delta_{WF} + \frac{\delta \rho_1^{(0)}}{2 \rho_1} + \frac{\delta \rho_1^{(1)}}{2 \rho_1} \right)^2,
$$

where

$$
\Gamma^{(0)} = 2 \left[ \frac{Q_b \alpha_{em}}{M_{\Upsilon(1S)}} \right]^2 (m_b C_f \tilde{\alpha}_s(\mu))^3, \quad \delta_{WF} = 3 \beta_0 \left( \ln \left( \frac{\mu}{m_b C_f \tilde{\alpha}_s(\mu)} \right) - \gamma_E \right) \frac{\alpha_s(\mu)}{4 \pi},
$$

$$
\delta_r = -\frac{4 C_f \alpha_s(\mu)}{\pi}, \quad \delta \rho_1^{(0)} = \frac{2 \rho_1}{2 \rho_1} = \left( \frac{270459}{217600} + \frac{1838781}{5780000} \right) \pi < \alpha_s G^2 > m_b \tilde{\alpha}_s^6(\mu).
$$

We have quoted (3.28) from ref. [4] adding our result. Let us first discuss the set I of physical inputs. We have drawn $\Gamma$ with and without $\delta \rho_1^{(1)}$ in Figure 2. For small values of $\mu$ the results are not reliable because the perturbative corrections become too large. It can also be seen the strong dependence on the scale. The results on Figure 2 should be compared...
Figure 2: Plot of the $\Upsilon(1S)$ decay width (KeV) versus $\mu$ (GeV). We have used the first set of parameters with $\Lambda_{QCD}^{n_f=3} = 250\text{MeV}$ and $m_b = 4881\text{MeV}$. The continuous line draws the decay width with $O_1 = 0\text{GeV}^6$, the dot-dashed line with the mean value displayed in (3.7), the dashed and dotted line, respectively with the lower and upper value displayed in (3.7). The same features hold for the other values of $\Lambda_{QCD}^{n_f=3}$. We also plot the experimental value $\Gamma(\Upsilon(1S)) = 1.34\text{KeV}$.

with the experimental value $\Gamma(\Upsilon(1S)) = 1.34\text{KeV}$ and with $\Gamma(\Upsilon(1S)) = 1.12\text{KeV}$, the result found in ref. [10] using a value of $\mu \simeq 2.33\text{GeV}$ ($n_f = 4$) such that the perturbative and NP corrections cancel each other. We now include our result for such a scale. We find (using the errors given in (3.7))

$$\Gamma(\Upsilon(1S)) = 0.36^{+0.20}_{-0.15}\text{KeV}. \quad (3.30)$$

We see that it destroys agreement with the data. What is going on here is that for such a scale $|\frac{\delta \rho_1^{(1)}}{2\rho_1}| < 1$ and $|\delta \rho_1^{(1)}| > |\delta \rho_1^{(0)}|$ so the result is not reliable. Indeed, one can not get agreement with the data for any scale as it can be seen in Figure 2. If one lowers the scale in order to fulfill $|\delta \rho_1^{(0)}| > |\delta \rho_1^{(1)}|$ the perturbative contributions grow in size with negative sign and make agreement with the data worse.
These results are somehow surprising since we would expect to obtain the same features as those obtained in the $M_{T(1S)}$ case. One point is that here we are faced with a two scale problem, the annihilation scale $O(m_b)$ and the wavefunction scale $O(a_{bb,1}^{-1})$. For this kind of observables factorization holds, that is, we can split the observable in two pieces the behavior of which is dictated by long and short distances respectively. Effective theories are especially useful in these situations. In our case the suitable effective theory is NRQCD [31] which has proved to be extremely successful in separating low from high energies.

Using NRQCD it can be seen that the right scale for $\delta_r$ is $\mu \sim m_b$. The NP contributions behave properly for their natural scales $\mu \sim a_{bb,1}^{-1}$, that is, the inequalities $|\rho_1| > |\delta \rho_1^{(0)}| > |\delta \rho_1^{(1)}|$ are well satisfied, especially the first one. The typical scale of $\Gamma^{(0)}$ and $\delta_{WF}$ is also $\mu \sim a_{bb,1}^{-1}$. The problem is that $\delta_{WF}$ is large and strongly dependent on the scale. For instance, we obtain (with $\Lambda_{QCD}^{n_f=3} = 250 MeV$, $m_b = 4881 MeV$ and $a_{bb,1}^{-1} = 1355 MeV$) $\Gamma(\Upsilon(1S)) = 0.11KeV$ for $\Gamma^{(0)}(a_{bb,1}) = 2.90 KeV$,

$$\delta_{WF}(a_{bb,1}^{-1}) = -0.81, \quad \delta_r(m_b) = -0.31, \quad \frac{\delta \rho_1^{(0)}}{2 \rho_1}(a_{bb,1}^{-1}) = 0.07, \quad \frac{\delta \rho_1^{(1)}}{2 \rho_1}(a_{bb,1}^{-1}) = -0.02,$$

while for $\delta_{WF}(2a_{bb,1}^{-1}) = -0.08$ and the remaining set of corrections kept fixed we obtain $\Gamma(\Upsilon(1S)) = 1.84 KeV$. The experimental value lies in between for the value $\delta_{WF}(2.21 GeV) = -0.23$. Although $\delta_{WF}$ is rather large it is not senseless to suppose that higher orders could improve the result changing significantly the value of $\delta_{WF}$.

For the second set of parameters the same general features are observed. The NP contributions are under control ($|\rho_1| > |\delta \rho_1^{(0)}| > |\delta \rho_1^{(1)}|$) but the $\alpha_s$ perturbative expansion behavior is even worse. For instance, we obtain (with $\Lambda_{QCD}^{n_f=4} = 330 MeV$, $m_b = 4943 MeV$ and $a_{bb,1}^{-1} = 1626 MeV$) $\Gamma(\Upsilon(1S)) = 0.09 KeV$ for $\Gamma^{(0)}(a_{bb,1}^{-1}) = 5.02 KeV$,

$$\delta_{WF}(a_{bb,1}^{-1}) = -0.87, \quad \delta_r(m_b) = -0.36, \quad \frac{\delta \rho_1^{(0)}}{2 \rho_1}(a_{bb,1}^{-1}) = 0.05, \quad \frac{\delta \rho_1^{(1)}}{2 \rho_1}(a_{bb,1}^{-1}) = -0.01.$$

Summarizing we may conclude that the $\Gamma(\Upsilon(1S))$ belongs to the VL regime since for scales $\mu \sim a_{bb,1}^{-1}$ the NP contributions are under control ($|\rho_1| > |\delta \rho_1^{(0)}| > |\delta \rho_1^{(1)}|$). However, no precise determination of $\Gamma(\Upsilon(1S))$ can be given due to the strong dependence on the scale and the large perturbative corrections involved. The solution could be to calculate the two loop $\bar{Q} - Q$ potential and introduce the next order perturbative correction. This may significantly improve the value of $\delta_{WF}$.

---

6The $M_{T(1S)}$ does not have this problem because it is an asymptotic feature of the physical state and only the typical bound state scale, $a_{bb,1}^{-1}$, can appear in the dynamics.
From the whole set of results we can conclude that the $\Upsilon(1S)$ belongs to the VL regime. Therefore, $T_g$ is not needed and the approach developed in ref. [7] or potential models should not be applied here.

4 Discussion and Conclusions

Let us briefly comment upon models where $T_g$ is fully taken into account [7,8]. There, it is obtained

$$\delta E_{nl} = \pi <\alpha_s G^2 > \frac{1}{18} \frac{\langle n, l l | r_i | H_8 - E_n + 1/T_g | n, l \rangle}{r_i}.$$  

(4.1)

In order to compare it with our results we should take the limit $T_g \to \infty$ in (4.1). That is, $1/T_g << m\beta_n^2$. We see that it has the right asymptotic limit, but it fails to describe properly the preasymptotic corrections. For instance, instead of $\delta E^{(1)} \sim \Lambda_{QCD}^6$, (4.1) already has $\Lambda_{QCD}^5$ corrections. This is not so strange if we recall the exponential approximation for the gluon correlator (Euclidean space) which has been used to obtain (4.1)

$$< G(t)G(0) > \sim e^{-t/T_g}$$

(4.2)

is only expected to provide a good approximation for long distances which is indeed the opposite limit that we are studying. That is, we may only expect (4.1) to provide a good description of the leading NP corrections when (1.7) is satisfied and

$$\frac{1}{T_g} >> m\beta_n^2.$$

Indeed, lattice calculations [33] suggest that the right behavior at long distances is (4.2). Therefore, we conclude that (4.1) can not be applied when the approximation (1.8) is satisfied which is the situation we are interested in.

Let us now summarize our conclusions.

We have calculated the next-to-leading NP corrections to the energy and wavefunctions for heavy quarkonium systems with general quantum numbers $n$ and $l$. We have obtained general analytical expressions. We have also been able to obtain analytical expressions for an arbitrary octet potential. We have also exactly calculated the error done if we had neglected the octet potential ($N_c \to \infty$) in our results. We have seen these observables to depend on the three-gluon condensate.

In this situation new NP contributions seem to appear [32].
We have reanalyzed the applicability of the VL approach for the lower levels of Bottomonium and Charmonium. We have studied two possible sets of parameters, especially the one compatible with the values used in [9, 10]. We have carefully studied the multipole expansion (1.7) and the approximation (1.8), especially the last one. As we have remarked throughout the paper, this last point is essential in order to distinguish whether a finite $T_g$ is needed. We also believe our work clarifies whether the $\alpha_s$ perturbative expansion can be applied.

It has been proven that $M_{\Upsilon(1S)}$ lies inside the VL regime since the NP corrections are under control and very small. It follows that neither (4.1) nor potentials models should be applied here. Therefore, this observable could provide one of the best model independent determination of the $m_b$ mass where the principal source of error will come from $\Lambda_{QCD}$ and higher orders in the perturbative expansion which are calculable and do not introduce any new parameter. Nevertheless, these corrections could be large. Other very striking consequence is that $M_{\Upsilon(1S)}$ should not be used fixing parameters in the potentials models. This would obviously change the value of these parameters. It would be rather interesting to see how important these changes are. Finally, we have also seen the weak $\mu$ scale dependence of the result.

For the $n = 1$ Charmonium level, using the first set of parameters, the theory seems to behave properly and the VL approach to work but the largeness of $\alpha_s$ makes the results less reliable.

For the first set of parameters we have also shown that the VL approach is not really applicable for $n = 2$ states and probably both assumptions the multipole expansion (1.7) and the approximation (1.8) fail for those states and obviously for higher levels. Therefore, in order to control the NP contributions, we should look for other approaches, perhaps in the spirit of ref. [30], where the NP contributions are not considered to be corrections.

The $\Upsilon(1S)$ decay width is very interesting. We have seen that calculating naively we do not obtain agreement with the data. The reason is that the new NP contribution blows up before we can reach the scale used in ref. [10]. It has been noted that the $\Upsilon(1S)$ decay width is a two-scale problem and, in the spirit of NRQCD, different scales should be used for the different contributions. We have argued about whether agreement with the data can be obtained. Nevertheless, errors are large owing to the large value of $\delta_{WF}$ for scales $\mu \sim a_{bb,1}^{-1}$. The solution could be to calculate the two loop $\bar{Q} - Q$ potential which could fix the value of $\delta_{WF}$. This may significantly improve the agreement with the experiment. We have also seen that the multipole expansion (1.7) and the approximation (1.8) applies for scales $\mu \sim a_{bb,1}^{-1}$.
We have not performed the calculation for the fine and hyperfine splitting but we believe that they will behave in the same way than the $\Upsilon(1S)$ decay width, being strong scale dependent.

Improvements of our results would come from diminishing errors in the parameters. It would also be very interesting to have the two loop $Q - \bar{Q}$ potential in order to obtain the next-to-next-to-leading perturbative contribution to masses and decay widths. It would improve the $m_b$ results and the $\Upsilon(1S)$ decay width. As we said, for the latter it could be essential in order to set the right value of $\delta_{WF}$. For the $\eta_c$ and $n = 2$ Bottomonium states the knowledge of the next-to-next-to-leading perturbative contribution could eventually discern whether the $\alpha_s$ expansion works or not. Moreover, a complete analysis with the second set of parameters remains to be done for these states (the conclusions obtained above could change for this set of parameters). This analysis will be only reliable once the above perturbative contributions are calculated.

When available, our results may eventually fix parameters in NRQCD and HQET. For both of them we can fix $m_b$ and $m_c$ in a model independent way. Once the masses are known we can obtain the HQET parameter $\bar{\Lambda}$ as it has already been done above. We also expect to be able to give QCD rigorous predictions of NRQCD matrix elements related with the $\Upsilon(1S)$ decay width. Moreover, this theoretical framework is able to deal with octet matrix elements in a natural way so further predictions may, in principle, be given.

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A Formulas of sec. 2

In this section we display analytical expressions and numerical values for the formulas obtained in sect. 2.

$$H(n,l) = \frac{1024 n^8}{81 (9 n^2 - 64) (81 n^2 - 64)^3 (81 n^2 - 256)^3}$$
\[ \times \left( 13295844358881280 - 68153404341354496 n^2 + 145600287615221760 n^4 
- 168309372752363520 n^6 + 114216987240880128 n^8 - 46158344158975776 n^{10} 
+ 10789755579716526 n^{12} - 1327743092409993 n^{14} + 65241222927111 n^{16} + I \left( 32604849090592768 - 128931229385883648 n^2 + 198763892651851776 n^4 
- 154057684466810880 n^6 + 65418421737648384 n^8 - 15363511613472780 n^{10} 
+ 1857043938050595 n^{12} - 88965303991515 n^{14} \right) + l^2 \left( 40459485281517586 - 155542385042915328 n^2 + 231200956108666528 n^4 
- 17178916119588864 n^6 + 70180300981986048 n^8 - 15974396240527980 n^{10} 
+ 1886699039381100 n^{12} - 88965303991515 n^{14} \right) + l^3 \left( 15799724393103360 - 53517313156055040 n^2 + 65200816249896960 n^4 
- 35606723591467008 n^6 + 9549421221866688 n^8 - 1223181401792805 n^{10} 
+ 59310202661010 n^{12} \right) + l^4 \left( 8125992224686080 - 27496161183006720 n^2 + 33417136857415680 n^4 
- 1816261251051104 n^6 + 483886743911744 n^8 - 615121070102415 n^{10} 
+ 29655101330505 n^{12} \right) + l^5 \left( 271356033761280 - 885005525975040 n^2 + 980074478960640 n^4 
- 431100857733120 n^6 + 76988199574080 n^8 - 4236443047215 n^{10} \right) + l^6 \left( 90452011253760 - 295001841991680 n^2 + 326691492986880 n^4 
- 143700285911040 n^6 + 25662733191360 n^8 - 1412147682405 n^{10} \right) \right), \]

\[
H(1, 0) = \frac{141912051712}{844421875}, \quad H(2, 0) = \frac{484859657191424}{20400039729}, \quad (A.2)
\]

\[
H(2, 1) = \frac{10215095135870976}{765014898375}, \quad H(3, 0) = \frac{1299369918513056329728}{79496721732575},
\]

\[
H(3, 1) = \frac{6471153009519628976971776}{529050683130286625}, \quad H(3, 2) = \frac{2239521974640025214976}{397483608662875}.
\]

\[
W(n) = \frac{512 n^{10}}{81 \left( 9 n^2 - 64 \right)^2 \left( 81 n^2 - 1024 \right) \left( 81 n^2 - 64 \right)^4 \left( 81 n^2 - 256 \right)^4} \times \left( -447359221655207230383849472 + 3163532546828444207717285888 n^2 \right)
\]
In this appendix we briefly discuss the role played by the octet coulomb potential and the error done neglecting it.

Frequently the octet potential is neglected in the octet propagator when the gluon correlation time is taken into account \[6, 8\] (see \[7\] where the octet potential is taken into account being solved the differential equations numerically). Therefore, we find pretty interesting to consider the case for a general octet potential and take the limit \(N_c \to \infty\) in our results.

Thus, we recalculate \(H\) and \(W\) with \(\frac{1}{8} = \frac{1}{N_c^2 - 1} \equiv r\) for a general \(r\). That is,

\[
V_s = \frac{1}{2N_c} \frac{\alpha_s}{r} \to \frac{1}{N_c^2 - 1} \frac{C_F \alpha_s}{r} = -rV_s,
\]

\[
H(n, l) \to H_{oct}(n, l, r),
\]

\[
W(n) \to W_{oct}(n, r).
\]

(B.1)

We obtain for \(r = 0\) \(N_c \to \infty\)

\[
H_{oct}(1, 0, 0) = 198 \sim 1.18H(1, 0), \quad H_{oct}(2, 0, 0) = 312576 \sim 1.32H(2, 0),
\]

\[
H_{oct}(2, 1, 0) = 183040 \sim 1.37H(2, 1), \quad H_{oct}(3, 0, 0) = 22609206 \sim 1.38H(3, 0),
\]

\[
H_{oct}(3, 1, 0) = 17058600 \sim 1.39H(3, 1), \quad H_{oct}(3, 2, 0) = 7846956 \sim 1.39H(3, 2),
\]
\[ W_{oct}(1,0) = -\frac{20548}{5} \sim 1.19W(1), \quad W_{oct}(2,0) = -\frac{13338112}{5} \sim 1.34W(2), \]
\[ W_{oct}(3,0) = -\frac{2113375198}{5} \sim 1.39W(3). \]  

(B.2)

We find they are always larger than the values obtained with \( N_c = 3 \). We also see that the error is large though under control. Finally, we have also been able to obtain analytical expressions for \( W \) and \( H \) (with \( l = 0 \) for a general \( r \)). They read

\[
W_{oct}(n, r) = 288n^{10} \left( \frac{3(1 - n^2)}{2(1 + r)^3} \right) + \frac{17(1 - n^2)}{3(1 + r)^3} + \frac{(898 - 873n^2 - 25n^4)}{72(1 + r)^2},
\]
\[
+ \frac{(17976 - 17741n^2 - 235n^4)}{864(1 + r)} + \frac{n(120 - 274n + 225n^2 - 85n^3 + 15n^4 - n^5)}{17280(-4 + n + nr)}
\]
\[
+ \frac{(-3 + n)^2(-40 + 94n - 77n^2 + 26n^3 - 3n^4)}{1440(-3 + n + nr)^2}
\]
\[
+ \frac{(2520 - 9186n + 13531n^2 - 10240n^3 + 4160n^4 - 854n^5 + 69n^6)}{4320(-3 + n + nr)}
\]
\[
+ \frac{(-2 + n^4)(-3 + 4n - n^2)}{24(-2 + n + nr)^4} + \frac{(2 - n)^3(108 - 75n - 56n^2 + 23n^3)}{432(-2 + n + nr)^3}
\]
\[
+ \frac{(-2 + n)^2(-360 - 4743n + 8026n^2 - 3317n^3 + 394n^4)}{4320(-2 + n + nr)^2}
\]
\[
+ \frac{(-10080 + 99624n - 142806n^2 + 42615n^3 + 22325n^4 - 13479n^5 + 1801n^6)}{17280(-2 + n + nr)}
\]
\[
+ \frac{2(2 - n)(-1 + n)^4}{3(-1 + n + nr)^4} + \frac{(-1 + n)^3n(126 - 53n - 5n^2)}{27(-1 + n + nr)^3}
\]
\[
+ \frac{(-1 + n)^2n(8070 - 1103n - 1612n^2 + 73n^3)}{864(-1 + n + nr)^2}
\]
\[
+ \frac{n(-121380 + 87068n + 87765n^2 - 60855n^3 + 9615n^4 - 2213n^5)}{8640(-1 + n + nr)}
\]
\[
- \frac{2n(1 + n)^4(2 + n)}{3(1 + n + nr)^4} + \frac{n(1 + n)^3(-126 - 53n + 5n^2)}{27(1 + n + nr)^3}
\]
\[
+ \frac{n(1 + n)^2(-8070 - 1103n + 1612n^2 + 73n^3)}{864(1 + n + nr)^2}
\]
\[
+ \frac{n(-121380 - 87068n + 87765n^2 + 60855n^3 + 9615n^4 + 2213n^5)}{8640(1 + n + nr)}
\]
\[ \begin{align*}
- \frac{(2 + n)^4 (3 + 4 n^2 + n^3)}{24 (2 + n + n r)^4} + \frac{(2 + n)^3 (-108 - 75 n + 56 n^2 + 23 n^3)}{432 (2 + n + n r)^3} \\
+ \frac{(2 + n)^2 (-360 + 4743 n + 8026 n^2 + 3317 n^3 + 394 n^4)}{4320 (2 + n + n r)^2} \\
+ \frac{(10080 + 99624 n + 142806 n^2 + 42615 n^3 - 22325 n^4 - 13479 n^5 - 1801 n^6)}{17280 (2 + n + n r)} \\
- \frac{(3 + n)^2 (40 + 94 n + 77 n^2 + 26 n^3 + 3 n^4)}{1440 (3 + n + n r)^2} \\
- \frac{(2520 + 9186 n + 13531 n^2 + 10240 n^3 + 4160 n^4 + 854 n^5 + 69 n^6)}{4320 (3 + n + n r)} \\
+ \frac{n (120 + 274 n + 225 n^2 + 85 n^3 + 15 n^4 + n^5)}{17280 (4 + n + n r)} \Bigg) \quad \text{(B.3)}
\end{align*} \]

\[ \begin{align*}
H_{oct}(n, 0, r) &= n^8 \left( \frac{72 (-1 + n^2)}{(1 + r)^3} + \frac{228 (-1 + n^2)}{(1 + r)^2} + \frac{2 (-214 + 229 n^2 - 15 n^4)}{1 + r} \right) \\
+ \frac{(-1 + n)^2 (-24 + 26 n - 9 n^2 + n^3)}{2 (-3 + n + n r)} + \frac{2 (-2 + n)^3 (3 - 4 n + n^2)}{(-2 + n + n r)^3} \\
+ \frac{(-2 + n)^2 (3 + 20 n - 31 n^2 + 8 n^3)}{(-2 + n + n r)^2} + \frac{(12 - 112 n + 117 n^2 + 18 n^3 - 45 n^4 + 10 n^5)}{(-2 + n + n r)} \\
+ \frac{32 (-2 + n) (-1 + n)^3 n}{(-1 + n + n r)^3} + \frac{8 (-1 + n)^2 n (-22 + n + 5 n^2)}{(-1 + n + n r)^2} \\
+ \frac{n (578 - 303 n - 539 n^2 + 255 n^3 + 9 n^4)}{2 (-1 + n + n r)} + \frac{32 n (1 + n)^3 (2 + n)}{(1 + n + n r)^3} \\
+ \frac{8 n (1 + n)^2 (22 + n - 5 n^2)}{(1 + n + n r)^2} + \frac{n (578 + 303 n - 539 n^2 - 255 n^3 + 9 n^4)}{2 (1 + n + n r)} \\
+ \frac{2 (2 + n)^3 (3 + 4 n + n^2)}{(2 + n + n r)^3} + \frac{(2 + n)^2 (3 - 20 n - 31 n^2 - 8 n^3)}{(2 + n + n r)^2} \\
+ \frac{(-12 - 112 n - 117 n^2 + 18 n^3 + 45 n^4 + 10 n^5)}{2 (2 + n + n r)} + \frac{(1 + n)^2 (24 + 26 n + 9 n^2 + n^3)}{2 (3 + n + n r)} \Bigg) \quad \text{(B.4)}
\end{align*} \]

(B.3) and (B.4) do not have a unique limit when \( n \to \text{integer}, r \to 0 \). The proper order is
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