Comments on Spontaneous Spin–Statistics Violation by Fermion Condensates

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ABSTRACT

Scalar condensation, the well-established Higgs phenomenon, is the standard paradigm for building up renormalizable gauge-invariant theories of massive gauge bosons. In this short note, we demonstrate the uniqueness of the Higgs vacuum state under the possible presence of fermion condensates in a renormalizable U(1) model. In the same context, we explain why spontaneous spin-statistics violation is technically not admitted in conventional Quantum Field Theory.
Scalar condensation, the so-called Higgs phenomenon [1], remains until now the standard paradigm for building up renormalizable gauge-invariant theories of massive gauge bosons. According to this paradigm, the gauge symmetry is spontaneously broken by the vacuum expectation value (VEV) of a scalar, also known as the Higgs boson. An earlier and alternative approach to the Higgs mechanism was the one that was put forward by Nambu and Jona-Lasinio (NJL) [2]. Based on an analogy with superconductivity, NJL presented a model where the gauge symmetry is broken dynamically by a non-vanishing VEV of a fermion–anti-fermion condensate. In 1979, Dimopoulos and Susskind [3] revived the idea of NJL, within the context of technicolour theories. In these theories, the Standard Model (SM) Higgs boson is not an elementary particle, but rather a bound state of strongly interacting fermion–anti-fermion degrees of freedom. Almost one decade later, in 1990, Bardeen, Hill and Lindner (BHL) [4] reshaped this idea within the SM and suggested that the Higgs boson may be a composite particle made up by a pair of top–anti-top quarks. One serious shortcoming of the NJL-type of models is that they become non-renormalizable and occasionally non-perturbative by the presence of local four-fermion operators [5]. Moreover, all NJL-type models tacitly assume the existence of an unspecified strong dynamics that makes the fermions condense.

Despite the undisputed success of the Higgs mechanism in the model-building of renormalizable theories, one may still ask the following simple questions. Does the ground state resulting from spontaneous symmetry breaking (SSB) always represent the lowest energy state about which perturbation field theory can be successfully formulated? Why only scalars and not fermions or gauge bosons do play a role in the Higgs mechanism for the SM vacuum? More specifically, can fermions receive a non-vanishing VEV?

Our interest in this note is to comment on the above questions within a minimal $U(1)$ renormalizable model. In doing so, we calculate the 1-loop Coleman–Weinberg (CW) effective potential [6] of the $U(1)$ model in the presence of background fermions. The resulting effective potential contains nilpotent terms, for which no physical interpretation does yet exist or can be ascribed to within the conventional framework of Quantum Field Theory (QFT). This difficulty arises due to the intrinsic Grassmann nature of the fermionic fields. If we extend our system of complex numbers to include Lorentz-invariant nilpotent terms, we may then be able to formally trigger SSB of a symmetry due to a non-zero fermionic VEV, but assigning a meaningful value to the latter seems to be ambiguous.

For our illustrations, let us first consider a simple ungauged Higgs model with a complex scalar $\Phi$ and a Weyl fermion $\psi$. The Lagrangian describing the model is given by

$$\mathcal{L} = (\partial_\mu \Phi^\ast (\partial^\mu \Phi) + \bar{\psi} i \sigma^\mu \partial_\mu \psi - \frac{h}{2} \left( \Phi \psi \psi + \Phi \bar{\psi} \bar{\psi} \right) - m^2 \Phi^\ast \Phi - \frac{\lambda}{4} (\Phi^\ast \Phi)^2 ,$$

where $h$ is assumed to be real without loss of generality and $\sigma^\mu = (1_2, -\sigma)$, with $\sigma_{1,2,3}$ being the usual Pauli matrices. Moreover, $\psi_\alpha$ and $\bar{\psi}_{\dot{\alpha}}$ are the 2-component Weyl fermion and its complex dual, respectively, with $\alpha, \dot{\alpha} = 1, 2$, according to the Van der Waerden notation. The Lagrangian (1) has a global $U(1)$ symmetry, since it is invariant under the field transformations: $\Phi \rightarrow e^{i e \Phi}$ and $\psi \rightarrow e^{-i e \Phi}$. For the specific choice of parameters, $m = 0$ and $h^2 = \lambda$, the model becomes identical to the Wess–Zumino (WZ) model [8] with
superpotential $W = \frac{h}{6} \Phi^3$, where $\Phi$ is a chiral superfield, with physical degrees of freedom $\Phi$ and $\psi_{\alpha}$.

In the presence of constant background scalar and fermion fields, $\Phi$ and $\psi$, the 1-loop effective potential can be calculated diagrammatically. Specifically, the effective potential terms that involve only the background field $\Phi$ can be computed following the standard CW approach [6], by resumming an infinite series of $\Phi$ insertions. For the fermion fields, however, one must notice that the series of external fermion insertions terminates very quickly at order $\psi^2 \bar{\psi}^2$, because of the Grassmann nature of the background fermions, where $\psi^2 = \psi_{\alpha} \psi_{\alpha}$ and $\bar{\psi}^2 = \bar{\psi}_{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}$. Hence, we only need to calculate the Feynman graphs shown in Fig. 1 in the presence of a non-vanishing scalar field $\Phi$. Putting everything together, the complete 1-loop effective potential of the model may be expressed as a sum of three terms:

$$V_{\text{eff}} = V^{(0)} + V^{(1)}_{\Phi} + V^{(1)}_{\psi}$$

where $V^{(0)}$ is the tree-level piece [1].

$$V^{(0)} = \frac{h}{2} \left( \Phi \psi^2 + \Phi^* \bar{\psi}^2 \right) + m^2 \Phi^* \Phi + \frac{\lambda}{4} (\Phi^* \Phi)^2$$

$V^{(1)}_{\Phi}$ is the standard CW effective potential result [6], calculated in dimensional reduction with minimal subtraction (DR) [9],

$$V^{(1)}_{\Phi} = \frac{1}{32 \pi^2} \left\{ \left( m^2 + \lambda |\Phi|^2 \right)^2 \left[ \ln \left( \frac{m^2 + \lambda |\Phi|^2}{\mu^2} \right) - \frac{3}{2} \right] 
- h^4 |\Phi|^4 \ln \left( \frac{h^2 |\Phi|^2}{\mu^2} \right) - \frac{3}{2} \right\}.$$  (4)

Finally, $V^{(1)}_{\psi}$ is a new $\psi$-dependent effective potential term, which is calculated from the diagrams shown in Fig. 1 and is given by

$$V^{(1)}_{\psi} = -\frac{h^4}{64 \pi^2} \psi^2 \bar{\psi}^2 \left[ I(m^2 + \lambda |\Phi|^2, h^2 |\Phi|^2) + h^2 |\Phi|^2 L(m^2 + \lambda |\Phi|^2, h^2 |\Phi|^2) \right],$$

where the loop functions $I(a, b)$ and $L(a, b)$ are given by

$$I(a, b) = \int \frac{d^4k}{i\pi^2} \frac{k^2}{(k^2 - a)^2 (k^2 - b)^2} = -\frac{a + b}{(a - b)^2} - \frac{2ab}{(a - b)^3} \ln \left( \frac{b}{a} \right),$$

$$L(a, b) = \int \frac{d^4k}{i\pi^2} \frac{(k^2)^2}{(k^2 - a)^4 (k^2 - b)^2} = -\frac{11}{6(a - b)} + \frac{5a}{(a - b)^3} - \frac{3a^2 + b^2}{(a - b)^4}
+ \frac{b^3 + 3ab^2}{(a - b)^5} \ln \left( \frac{a}{b} \right).$$

1Unless specified otherwise, we use thereafter the same notation for the fields $\Phi$, $\psi_{\alpha}$ and $\bar{\psi}_{\dot{\alpha}}$ to indicate that they are classical fields obtained as quantum averages over the path-integral under the action of external sources, i.e. $\Phi$ instead of $\langle \Phi \rangle_J$, $\psi_{\alpha}$ instead of $\langle \psi_{\alpha} \rangle_J$ etc.
In the limit $a \to b$, the loop functions simplify to: $I(a, a) = -1/(3a)$ and $L(a, a) = -1/(5a^2)$. Note that the complete 1-loop effective potential $V_{\text{eff}}$ is invariant under the global $U(1)$ symmetry. Because of the $U(1)$ symmetry, there is no 1-loop contribution to the trilinear couplings $\Phi \psi^2$ and $\Phi^* \bar{\psi}^2$. Moreover, it is easy to establish that in the WZ limit of the theory, with $m^2 = 0$ and $h^2 = \lambda$, $V_{\Phi}^{(1)}$ vanishes identically, whereas $V_{\psi}^{(1)}$ is not zero. The non-vanishing of $V_{\psi}^{(1)}$ is consistent with the non-renormalization theorem of the superpotential in exact $\mathcal{N} = 1$ supersymmetric theories, since a term proportional to $\psi^2 \bar{\psi}^2$ can arise from the non-renormalizable operator $(\hat{\Phi}^2 \hat{\Phi}^2)$ [10] as derived by the 1-loop effective Kähler potential [11].

It is now important to derive the vacuum stability conditions related to this model. Requiring that the effective potential be convex for large field values along the $\Phi$ and $\psi$ directions implies that $V_{\Phi}^{(0)} + V_{\psi}^{(1)} \geq 0$ and the coefficient of $V_{\psi}^{(1)}$ is positive. At the tree-level, we have the following extremal or tadpole conditions:

$$T_H \equiv \left\langle \frac{\delta V^{(0)}}{\delta H} \right\rangle = \frac{h}{2\sqrt{2}} \left( \psi^2 + \bar{\psi}^2 \right) + \frac{v}{\sqrt{2}} \left( 2m^2 + \lambda v^2 \right) = 0, \quad (8)$$

$$T_G = \left\langle \frac{\delta V_{\text{eff}}}{\delta G} \right\rangle = \frac{ih}{2\sqrt{2}} \left( \psi^2 - \bar{\psi}^2 \right), \quad (9)$$

$$T_{\psi}^\alpha = \left\langle \frac{\delta V^{(0)}}{\delta \psi^\alpha} \right\rangle = hv_{\psi^\alpha} = 0, \quad (10)$$

where $v$ is the VEV of the complex Higgs scalar $\Phi$, which is linearly decomposed as $\Phi = v + \frac{1}{\sqrt{2}}(H + iG)$. If $m^2 > 0$, the minimum occurs at $v = 0$, whereas for $m^2 < 0$, the true minimum is for the non-zero VEV $v^2 = -2m^2/\lambda$, for which the $U(1)$ symmetry gets spontaneously broken. In both cases, the Weyl fermions $\psi$ and $\bar{\psi}$ have vanishing VEVs. Evidently, a non-zero $\psi^\alpha$ would have immediately signalled spontaneous violation of spin–statistics and of the Lorentz symmetry. Whether this is still possible will be elaborated in more detail below at the 1-loop quantum level.

To simplify matters, let us consider the WZ limit of the model, i.e. $\lambda = h^2$, softly broken by the 2-dimensional operator $m^2 \Phi^* \Phi$. Moreover, we assume that $|m^2| \ll \lambda v^2$. 

Figure 1: Diagrams contributing to the 1-loop effective potential term $V_{\psi}^{(1)}$. All internal lines are evaluated in the background of a non-vanishing complex field $\Phi$. 

\[ \begin{array}{cc}
\psi & \bar{\psi} \\
\bar{\psi} & \psi \\
\end{array} \hspace{1cm} \begin{array}{cc}
\psi & \bar{\psi} \\
\bar{\psi} & \psi \\
\end{array} \hspace{1cm} \begin{array}{cc}
\Phi^* & \Phi \\
\Phi & \Phi^* \\
\end{array} \]
An assumption that needs be checked \textit{a posteriori}. To leading order in an expansion of $|m^2|/(\lambda v^2)$, the effective potential (2) takes on the simple form:

\[
V_{\text{eff}} = \frac{h}{2} \left( \Phi \psi^2 + \Phi^* \bar{\psi}^2 \right) + m^2 |\Phi|^2 + \frac{h^2}{4} |\Phi|^4 + \frac{h^2}{120 \pi^2} \frac{\psi^2 \bar{\psi}^2}{|\Phi|^2},
\]

(11)

where we ignored $V_{\Phi}^{(1)}$ next to the tree-level contribution. Notice that the coefficient of the potential term $\psi^2 \bar{\psi}^2$ is positive, thus satisfying the convexity condition mentioned above.

As done before in (8) and (10), we calculate the tadpole conditions from the approximate 1-loop effective potential (11). These are given by

\[
T_H = \left\langle \frac{\partial V_{\text{eff}}}{\partial H} \right\rangle = \frac{1}{\sqrt{2}} \left( \frac{h}{2} \psi^2 + \frac{h}{2} \bar{\psi}^2 + 2 m^2 v + h^2 v^3 - \frac{h^2}{60 \pi^2} \frac{\psi^2 \bar{\psi}^2}{v^3} \right) = 0,
\]

(12)

\[
T_G = \left\langle \frac{\partial V_{\text{eff}}}{\partial G} \right\rangle = \frac{ih}{2\sqrt{2}} \left( \psi^2 - \bar{\psi}^2 \right),
\]

(13)

\[
T_{\psi} = \left\langle \frac{\partial V_{\text{eff}}}{\partial \psi^\alpha} \right\rangle = \psi^\alpha \left( hv + \frac{h^2}{60 \pi^2 v^2} \right) = 0.
\]

(14)

Our interest is to see whether an additional solution beyond the standard one, with $\psi^\alpha = 0$ and $v^2 = -2m^2/\lambda$, does exist. In doing so, we should first notice that the tadpole conditions consist not only of terms that are defined over real numbers, but also of terms that involve fermions and are nilpotent, e.g. $\psi_\alpha \psi_\beta \psi_\gamma = 0$. Equating separately nilpotent and non-nilpotent terms to zero, we find that the only admissible solution is the standard one.

The above result seems to be unavoidable for a theory, in which all kinematic parameters, such as masses and couplings, are real or complex numbers. There is no way that one could attribute a nilpotent or Grassmann behaviour to a classical fermionic field, in terms of real kinematic parameters only. Consequently, the Higgs vacuum state $v$ is uniquely determined within the standard framework of QFT.

The fact that classical nilpotent fields appear in the effective action or potential could one motivate to consider more exotic realizations of QFT, in which kinematic parameters contain a Lorentz-invariant nilpotent piece. For instance, one may consider mixed complex numbers of the form: $p(\theta) = a + b \theta^\alpha \theta_\alpha$, where $a, b \in C$ and $\theta_\alpha$ is a 2-component complex Grassmann number. These mixed complex numbers satisfy all axioms of the field defined over the binary operations of addition and multiplication, unless they are singular, i.e. $a = 0$, for $b \neq 0$ [12]. In such an unconventional framework, one could write down Lagrangians that can, in principle, trigger SSB of spin-statistics and Lorentz symmetry. One minimal model may be formulated by means of the non-renormalizable Lagrangian

\[
\mathcal{L} = (\partial^\mu \Phi^*)(\partial^\mu \Phi) + \bar{\psi} i\sigma^\mu \partial_\mu \psi + \left( t + w \theta^2 \right) \Psi + \left( t^* + w^* \bar{\theta}^2 \right) \Psi^\dagger - m^2 \Psi^\dagger \Psi,
\]

(15)

where $t, w$ are complex of dimension of mass$^3$ and

\[
\Psi = \Phi + \frac{\bar{\psi}^2}{\Lambda^2}.
\]

(16)
Working out the extremization conditions, one readily finds that \( \langle \Phi \rangle = v = t/m^2 \) and
\[
\langle \psi_\alpha \rangle = w \frac{\Lambda^2}{m^2} \theta_\alpha \quad (17)
\]
Nevertheless, the difficulty of interpreting the Grassmann constants \( \theta_\alpha \) physically still remains. At this point, one could only speculate [13] that constant background fermions, identified with \( \theta_\alpha \), are present in the vacuum converting bosons into fermions and vice versa. The hope is that one will somehow be able to trade the Grassmann constants, like \( \theta_\alpha \), with 2-component complex vectors at the probabilistic level, so as to obtain a meaningful result for physical observables. However, there is no obvious mechanism of how such a mapping between Grassmann and ordinary numbers could take place. Here, one might be tempted to define a kind of scalar product between Grassmann-valued transition amplitudes \( T_1 \) and \( T_2 \), e.g.
\[
(T_1, T_2) \equiv \int d^2 \theta d^2 \theta' d^2 \bar{\theta} d^2 \bar{\theta}' e^{-\theta \theta' - \bar{\theta} \bar{\theta}'} T_1^\dagger(\theta', \bar{\theta}') T_2(\theta, \bar{\theta}) \quad (18)
\]
Although the result obtained with the aid of (18) lies in \( \mathbb{C} \), it is still not clear how unitarity by means of the optical theorem can be enforced in this context.

In summary, condensates of single fermions are technically not admitted within the framework of standard QFT. Possible attempts to provide physical interpretation to fermion condensates face serious problems of unitarity and analyticity. The Higgs vacuum state is therefore uniquely determined and calculable (for comparison, the corresponding vacuum state of non-Abelian gauge theories is more involved [14]). It would be difficult for the LHC and other possible future colliders to obtain a fairly good reconstruction of the Higgs potential [15], independently of any theoretical prejudice. Nonetheless, it is not only of high interest to us to simply discover the SM Higgs boson at the LHC, but also think of ways of how to probe the nature of the ground state of the theory about which the Higgs phenomenon is realized.

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