INFINTESIMALS IN A RECURSIVELY ENUMERABLE PRIME MODEL

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Abstract. Using methods developed by Robinson, we find a complete theory suitable for a first order description of infinitesimal neighborhoods. We use this to construct a specialisation having universal properties and to find a recursively enumerable model in which the algebraic version of Bezout’s theorem is provable by non-standard methods.

1. Specialisations and Valuations

Let $L$ and $K$ be fields with an imbedding $i : L^* \to K^*$. In the case when $L$ and $K$ have the same characteristic, we will consider $L$ as a subfield of $K$, otherwise we will by some abuse of notation refer to the embedded set $i(L^*) \cup \{0\}$ as $L$. Let $P(K) = \bigcup_{n\geq 1} P^n(K)$ and $P(L) = \bigcup_{n\geq 1} P^n(L)$. By a closed algebraic subvariety of $P^n(K)$, we mean a set $W(K)$ where $W$ is defined by homogeneous polynomial equations with coefficients in $K$. We say that $W(K)$ is defined over $L$ if we can take the coefficients to lie in $L$. Let $W^m_n(K)$ denote the $m$'th Cartesian product of $P^n(K)$. By a closed algebraic subvariety of $W^m_n(K)$, we mean a set $W(K)$ defined by multi-homogeneous polynomial equations with coefficients in $K$, similarly we can make sense of the notion of being defined over $L$. Note that if $K$ is not algebraically closed, it is not necessarily true that the projection maps $pr_{k,n} : W^k_n(K) \to W^m_n(K)$ preserve closed algebraic subvarieties.

Definition 1.1. A specialisation is a map $\pi = \bigcup_{n\geq 1} \pi_n : P(K) \to P(L)$, such that each $\pi_n : P^n(K) \to P^n(L)$ has the following property;

Let $W^m_n(K)$ denote the $m$'th Cartesian product of $P^n(K)$. Then, if $V \subset W^m_n(K)$ is a closed algebraic subvariety defined over $L$ and $\bar{a}$ is an $m$-tuple of elements from $W_n(K)$, such that $V(\bar{a})$ holds, then $V(\pi_n(\bar{a}))$ holds as well.

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The following compatibility requirement must also hold between the $\pi_n$:

Fix the following chain of embeddings $i_n$ of $P^n(K)$ and $P^n(L)$ into $P^{n+1}(K)$ and $P^{n+1}(L)$ for $n \geq 1$.

$$i_n : [x_0 : \ldots : x_n] \mapsto [x_0 : \ldots : x_n : 0].$$

Then we require that $\pi_{n+1} \circ i_n = i_{n+1} \circ \pi_n$.

**Definition 1.2.** A Krull valuation $v$ is a map $v : K \to \Gamma \cup \{\infty\}$ where $\Gamma$ is an ordered abelian group with the following properties:

(i). $v(x) = \infty$ iff $x = 0$.
(ii). $v(xy) = v(x) + v(y)$
(iii). $v(x + y) = \min\{v(x), v(y)\}$

Here, we adopt the convention that $\gamma < \infty$ for $\gamma \in \Gamma$ and extend $+$ naturally to $\Gamma \cup \{\infty\}$.

We let $\mathcal{O}_v = \{x \in K : v(x) \geq 0\}$ be the valuation ring of $v$ and $\mathcal{M}_v = \{x \in K : v(x) > 0\}$ the unique maximal ideal. We also require:

(iv). The inclusion $i : L^* \cup 0 \to \mathcal{O}_v^* \cup 0$ maps $L$ isomorphically onto $\mathcal{O}_v^*/\mathcal{M}_v$, the residue field of $v$.

**Definition 1.3.** We say that two Krull valuations $v_1$ and $v_2$ are equivalent, denoted by $v_1 \sim v_2$ if $\mathcal{O}_{v_1} = \mathcal{O}_{v_2}$.

**Lemma 1.4.** $v_1$ and $v_2$ are equivalent iff there exists $\Theta : \Gamma_1 \to \Gamma_2$ such that $\Theta \circ v_1 = v_2$.

In order to see this, define $\Theta(v_1(x)) = v_2(x)$, this is well defined as if $v_1(x) = v_1(x')$, then $v_1(x/x') = 0$, hence $x/x'$ and $x'/x$ belong to $\mathcal{O}_{v_1}$. If $v_1 \sim v_2$, then $x/x'$ and $x'/x$ belong to $\mathcal{O}_{v_2}$ as well, which gives that $v_2(x) = v_2(x')$. One can easily check that $\Theta$ is an isomorphism of ordered abelian groups as required.

Our main result in this section is the following;

**Theorem 1.5.** Let $X := \{\pi : P(K) \to P(L)\}$ be the set of specialisations and $Y := \{v/\sim : v : K \to \Gamma\}$ be the set of equivalence classes of
Krull valuations. Then there exists a natural bijection between $X$ and $Y$. Specifically, there exists maps $\Phi$ and $\Psi$;

$$\Phi : X \rightarrow Y$$

$$\Psi : Y \rightarrow X$$

with $\Psi \circ \Phi = Id_X$ and $\Phi \circ \Psi = Id_Y$.

We first show;

**Theorem 1.6.** There exists $\Psi : Y \rightarrow X$

**Proof.** Let $[v]$ denote a class of Krull valuations on $K$. We define a specialisation map $\pi_{[v]}$ as follows;

Let $(x_0 : x_1 : \ldots : x_n)$ denote an element of $P^n(K)$ written in homogeneous coordinates. For some $\lambda \in K$, the elements $\{\lambda x_0, \ldots, \lambda x_n\}$ will lie in $O_v$ and not all of them will lie in $M_v$. Let $\pi : O_v \rightarrow L$ denote the unique ring morphism such that $\pi \circ i = Id_L$ where $i$ is the inclusion map from $L$ into $O_v$. Then $((\pi(\lambda x_0)) : (\pi(\lambda x_1)) : \ldots : (\pi(\lambda x_n))$ defines an element of $P^n(L)$. As is easily checked, the mapping is independent of the choice of $\lambda$ and depends only on $O_v$, hence we obtain $\pi_{n,[v]} : P^n(K) \rightarrow P^n(L)$. We need to check that each $\pi_{n,[v]}$ satisfies the property required of a specialisation. We will just verify this in the case when $m \leq 2$ for each $n \geq 1$, the other cases are straightforward generalisations;

For $m = 1$, let $V \subset P^n(K)$ be a closed subvariety defined over $L$, then $V$ is defined by a system of homogeneous equations in the variables $\{x_0, \ldots, x_n\}$ with coefficients in $L$. Taking a tuple $\bar{a}$ belonging to $V$, we can assume that the elements $\{a_0, a_1, \ldots, a_n\}$ belong to $O_v$. Now, using the fact that the residue map $\pi$ is a ring homomorphism fixing $L$, the reduced elements $\{\pi(a_0), \pi(a_1), \ldots, \pi(a_n)\}$ also satisfy the same homogeneous equations as required.

For the case when $m = 2$, we use the Segre embedding which is defined by;

$$Segre : P^n(K) \times P^n(K) \rightarrow P^{n+2}(K)$$
The following diagram is easily checked to commute:

\[
P^n(K) \times P^n(K) \xrightarrow{\text{Segre}} P^{n(n+2)}(K) \\
\downarrow {\pi_n,[v]} \times {\pi_n,[v]} \quad \downarrow {\pi_n(n+2),[v]} \\
P^n(L) \times P^n(L) \xrightarrow{\text{Segre}} P^{n(n+2)}(L)
\]

Therefore, it is sufficient to prove that the property holds for \(\pi_n(n+2),[v] : P^{n(n+2)}(K) \to P^{n(n+2)}(L)\) when \(m = 1\). This is the case covered above.

Finally, we need to check the compatibility requirement for the \(\pi_n,[v]\), this is a trivial calculation.

Denote the specialisation map we have obtained by \(\pi_{[v]}\) and let \(\Psi([v]) = \pi_{[v]}\).

Therefore, it is sufficient to prove that the property holds for \(\pi_{n(n+2),[v]} : P^{n(n+2)}(K) \to P^{n(n+2)}(L)\) when \(m = 1\). This is the case covered above.

Finally, we need to check the compatibility requirement for the \(\pi_{n,[v]}\), this is a trivial calculation.

Denote the specialisation map we have obtained by \(\pi_{[v]}\) and let \(\Psi([v]) = \pi_{[v]}\).

We now show;

**Theorem 1.7.** There exists \(\Phi : X \to Y\)

**Proof.** Suppose that we are given a specialisation \(\pi\). In particular we have a map \(\pi_1 : P^1(K) \to P^1(L)\) satisfying the requirements above. We want to show how to recover a Krull valuation on \(K\).

Let \(\gamma : K \to P^1(K)\) be the map \(\gamma : k \mapsto [k : 1]\), so \(\pi_1 \circ \gamma : K \to P^1(L)\). Let \(U \subset P^1(L)\) be the open subset defined by \(P^1 \setminus [1 : 0]\). Let \(O_K = (\pi_1 \circ \gamma)^{-1}(U)\) and \(M_K = (\pi_1 \circ \gamma)^{-1}([0 : 1])\). We now claim the following:

**Lemma 1.8.** \(O_K\) is a subring of \(K\) with \(\text{Frac}(O_K) = K\) and \(M_K\) is an ideal of \(O_K\).

**Proof.** Suppose that \(\{x, y\} \subset O_K\), then both \(\pi_1([x : 1])\) and \(\pi_1([y : 1])\) are in \(U\). Let \(C \subset P^1(K) \times P^1(K) \times P^1(K)\) be the closed set defined in coordinates \(([u : v], [w : x], [y : z])\) by the equation \(uwz = yvx\). As is easily checked, we have that \(C([x : 1], [y : 1], [xy : 1])\). By the defining property of \(\pi_1\), \(C(\pi_1([x : 1]), \pi_1([y : 1]), \pi_1([xy : 1]))\) also
holds. Therefore, $C([\lambda : 1], [\mu : 1], [\alpha, \beta])$ where $\lambda, \mu, \alpha, \beta$ are in $L$. By definition of $C$, we have $\lambda \mu \beta = \alpha$ which forces $\beta \neq 0$. Hence, $\pi_1([xy : 1]) \in U$ and therefore $xy \in \mathcal{O}_K$. Let $D \subset P^1(K) \times P^1(K) \times P^1(K)$ be defined using the same choice of coordinates by the equation $uxz + wvz = yvx$. Then we have that $D([x : 1], [y : 1], [x + y : 1])$ and therefore $D(\pi_1([x : 1]), \pi_1([y : 1]), \pi_1([x + y : 1]))$. Again, we must have $D((\lambda : 1), [\mu, [1, [\delta, \epsilon]]))$ where $\lambda, \mu, \delta, \epsilon$ are in $L$. This forces $(\lambda + \mu)\epsilon = \delta$ and therefore $\epsilon \neq 0$, so $x + y \in \mathcal{O}_K$. Clearly, $1 \in \mathcal{O}_K$ which shows that $\mathcal{O}_K$ is a subring of $K$ as required. In order to see that $\mathcal{M}_K$ is an ideal of $\mathcal{O}_K$, let $x \in \mathcal{O}_K$ and $y \in \mathcal{M}_K$. We have that $C((\lambda : 1), [0 : 1], [\alpha, \beta])$ where $\pi_1([xy : 1]) = [\alpha, \beta]$. Then $0.1 \beta + 1.0 \epsilon = 1.1 \delta$. Then we obtain $D(0 : 1), [0 : 1], [\delta, \epsilon]$ where $\pi_1([x + y : 1]) = [\delta, \epsilon]$. Then $0.1 \beta + 1.0 \epsilon = 1.1 \delta$, so $\delta = 0$ and $\epsilon = 1$, hence $x + y \in \mathcal{M}_K$ as required. Finally, we show that $Frac(\mathcal{O}_K) = K$. Suppose $x \notin \mathcal{O}_K$, then $\pi_1([x : 1]) = [1 : 0]$. We have that $C([x : 1], [1/x : 1], [1 : 1])$, hence $C([1 : 0], [\alpha, \beta], [1 : 1])$ where $\pi_1([1/x : 1]) = [\alpha, \beta]$. This forces $1.0 \alpha = 0, \beta, 1$, hence $\alpha = 0$ and $\beta = 1$. Therefore $1/x \in \mathcal{O}_K$ as required.

We now further claim the following;

**Lemma 1.9.** If $\pi_1$ is non-trivial, that is, $\pi_1$ is not a bijection between $P^1(K)$ and $P^1(L)$, then $\mathcal{O}_K$ is a proper subring of $K$.

**Proof.** By the same argument as above we have that $\pi_1 \circ \gamma(1/\mathcal{M}_K) = [1 : 0]$, hence $\mathcal{O}_K = K$, using the previous lemma, we must have that $\mathcal{M}_K = 0$. If $\pi_1$ is non-trivial, we can find $x \in K$ and $y \in K$ distinct such that $\pi_1([x : 1]) = \pi_1([y : 1])$. By the usual arguments, we then have that $\pi_1([x - y : 1]) = [0 : 1]$, so $x - y \in \mathcal{M}_K$ contradicting the fact that $\mathcal{M}_K = \{0\}$.

We can now construct a Krull valuation on $K$ by a standard method. Let $\Gamma = K^*/\mathcal{O}_K^*$ and define $v : K \rightarrow \Gamma$ by $v(x) = x \mod \mathcal{O}_K^*$ and $v(0) = \infty$. Define an ordering on the abelian group $\Gamma$ by declaring $v(x) \leq v(y)$ iff $y/x \in \mathcal{O}_K$. This is well defined as if $v(x) = v(x')$ and $v(y) = v(y')$, then $y'/y, y'/y, x/x'$ and $x'/x$ are all in $\mathcal{O}_K$. We have that $y'/x' = y/x, y'/y, x/x'$ and $y/x = y'/x', y'/y, x/x'$, therefore $y'/x' \in \mathcal{O}_K$ iff $y'/x' \in \mathcal{O}_K$ as required. Transitivity of the ordering follows from the fact that $\mathcal{O}_K$ is a subring of $K$. $\leq$ is a linear ordering as if $x \in K^*$ and $y \in K^*$ then either $x/y$ or $y/x$ lies in $\mathcal{O}_K$. Finally, we clearly have that if $y/x \in \mathcal{O}_K$ then $yz/xz \in \mathcal{O}_K$, hence $v(x) \leq v(y)$ implies
$v(x) + v(z) \leq v(y) + v(z)$. This turns $\Gamma$ into an ordered abelian group. Properties (i) and (ii) of the axioms for a Krull valuation are trivial to check. Suppose property (iii) fails, then we can find $x, y$ with $v(x + y) < v(x)$ and $v(x + y) < v(y)$. Therefore $(x + y)/x \notin \mathcal{O}_K$ and $(x + y)/y \notin \mathcal{O}_K$. As $1 \in \mathcal{O}_K$, we have that $x/y \notin \mathcal{O}_K$ and $y/x \notin \mathcal{O}_K$ which is a contradiction. Finally, we check property (iv). By definition of $\pi_1$, we have that $L^* \subset \mathcal{O}_K^*$, hence $v|L$ is trivial. If $k \in \mathcal{O}_v^*$, we can find $l \in L^*$ such that $\pi_1([k : 1]) = [l : 1]$, then $\pi_1([k - l : 1]) = [0 : 1]$ and $k - l \in \mathcal{M}_K$. It follows that $L$ maps onto $\mathcal{O}_K/\mathcal{M}_K$, and $\mathcal{O}_K/\mathcal{M}_K \cong L$ as required. Denote the valuation we have obtained by $v_\pi$ and set $\Phi(\pi) = [v_\pi]$. This ends the proof of Theorem 1.7.

We now complete the proof of Theorem 1.5;

Proof. $\Phi \circ \Psi = Id_Y$.

Let $[v]$ be a class of Krull valuations on $K$ with corresponding specialisation $\pi_{[v]}$ provided by $\Psi$. Let $\pi_{1,[v]}$ be the restriction to $P^1(K)$. By definition, if $k \in \mathcal{O}_v$ then $\pi_{1,[v]}([k : 1]) = [\pi(k), 1]$ where $\pi$ is the residue map for $v$. If $k \notin \mathcal{O}_v$, then $\pi_{1,[v]}([k : 1]) = [0, 1]$, so we see that $\mathcal{O}_K$ as defined above is exactly $\mathcal{O}_v$. The valuation $v_{\pi_{[v]}}$ constructed from $\pi_{[v]}$ therefore has the same valuation ring $\mathcal{O}_v$, so $v \sim v_{\pi_{[v]}}$ which gives the result.

$\Psi \circ \Phi = Id_X$.

Let $\pi$ be a given specialisation and $[v_\pi]$ the corresponding class of Krull valuations. Let $\pi_1$ be the restriction of $\pi$ to $P^1(K)$ and $\pi_{1,v_\pi}$ the specialisation constructed from $v_\pi$ restricted to $P^1(K)$. We have;

(i). $\pi_{1,v_\pi}([k : 1]) = [0 : 1]$ iff $v_\pi(k) > 0$ iff $k \in \mathcal{M}_{v_\pi}$ iff $k \in \mathcal{M}_K$ as defined above iff $\pi_1([k : 1]) = [0 : 1]$

(ii). $\pi_{1,v_\pi}([k : 1]) = [1 : 0]$ iff $v_\pi(k) < 0$ iff $k \notin \mathcal{O}_{v_\pi}$ iff $k \notin \mathcal{O}_K$ as defined above iff $\pi_1([k : 1]) \notin U$ iff $\pi_1([k : 1]) = [1 : 0]$

(iii). $\pi_{1,v_\pi}([1 : 0]) = \pi_1([1 : 0]) = [1 : 0]$ trivially.

If $k \in \mathcal{O}_{v_\pi}$, then $\pi_{1,v_\pi}([k : 1]) = [\alpha(k) : 1]$ where $\alpha$ is the residue mapping associated to $v_\pi$. We also have that $\pi_1([k : 1]) \in U$, hence as $\pi_1$ is
a specialisation that \( \pi_1([k : 1]) = [\beta(k) : 1] \) where \( \beta \) is a homomorphism from \( \mathcal{O}_{v_x} \) to \( L \). We thus obtain two homomorphisms \( \alpha, \beta : \mathcal{O}_{v_x} \to L \) such that (by (i)) \( \text{Ker}(\alpha) = \text{Ker}(\beta) = \mathcal{M}_{v_x} \) and with the property that \( \alpha \circ i = \beta \circ i = \text{Id}_L \) where \( i \) is the natural inclusion of \( L \) in \( \mathcal{O}_{v_x} \). We thus obtain the splitting \( \mathcal{O}_{v_x} = L \oplus \text{Ker}(\alpha) = L \oplus \text{Ker}(\beta) = L \oplus M \) with \( \text{Ker}(\alpha) = \text{Ker}(\beta) = M \). Now, using this fact, we can write any element of \( \mathcal{O}_{v_x} \) uniquely in terms of \( L \) and \( M \), hence the corresponding projections \( \alpha \) and \( \beta \) are the same.

We have shown that \( \pi_1 = \pi_{1,v_x} \), it remains to check that \( \pi_n = \pi_{n,v_x} \) for all \( n \geq 1 \). We prove this by induction on \( n \), the case \( n = 1 \) having been established.

By the induction hypothesis and the compatibility requirement between the \( \pi_n \), for \( \{k_0, k_1, \ldots, k_n\} \subset \mathcal{O}_{v_x} \);
\[
\pi_{n+1}([k_0 : k_1 : \ldots : k_n : 0]) = [\pi(k_0) : \pi(k_1) : \ldots : \pi(k_n) : 0] \quad (*)
\]
where \( \pi \) is the residue map on \( \mathcal{O}_{v_x} \).

Let \( C \subset P^{n+1}(K) \) be the closed subvariety defined using coordinates \([x_0 : x_1 : \ldots : x_{n+1}]\) by the equations \( x_0 = x_1 = \ldots = x_{n-1} = 0 \). Then by arguments as above and the fact that \( C \) is preserved by \( \pi_{n+1} \), we can find a Krull valuation \( v' \) on \( K \) with corresponding residue mapping \( \pi' \) such that;
\[
\pi_{n+1}([0 : \ldots : 0 : 1 : k_{n+1}]) = [0 : \ldots : 0 : 1 : \pi'(k_{n+1})] \text{ if } v'(k_{n+1}) \geq 0
\]
\[
= [0 : \ldots : 0 : 1] \text{ otherwise (**)}
\]

Now let \( D \) be the closed subvariety of \( P^{n+1}(K) \) defined by the equations \( x_1 = \ldots = x_n \) and \( x_0 = x_{n+1} \). Again, we have that \( \pi_{n+1} \) preserves \( D \), hence there exists a Krull valuation \( v'' \) on \( K \) with corresponding residue mapping \( \pi'' \) such that;
\[
\pi_{n+1}([k : 1 : \ldots : 1 : k]) = [\pi''(k) : 1 : \ldots : 1 : \pi''(k)] \text{ if } v''(k) \geq 0
\]
\[
= [1 : 0 : \ldots : 0 : 1] \text{ otherwise (***)}
\]

Let \( Sum \) be the closed subvariety of \( P^{n+1}(K) \times P^{n+1}(K) \times P^{n+1}(K) \) defined using coordinates \([x_0 : x_1 : \ldots : x_{n+1}]\), \([y_0 : y_1 : \ldots : y_{n+1}]\) and \([z_0 : z_1 : \ldots : z_{n+1}]\) by the equations \( x_0y_1z_1 + y_0x_nz_1 = z_0x_ny_1 \)
and $x_{n+1}y_1z_1 + y_{n+1}x_nz_1 = z_{n+1}x_ny_1$. Then, for $k \in K$, we have that $Sum([0:0:\ldots:1:k],[k:1:\ldots:0:0],[k:1:\ldots:1:k])$, hence by the properties of a specialisation that $Sum(\pi_{n+1}([0:0:\ldots:1:k]),\pi_{n+1}([k:1:\ldots:0:0]),\pi_{n+1}([k:1:\ldots:1:k])).$

In the generic case when $v_\pi(k), v'(k), v''(k)$ are all non-negative, we obtain $Sum([0:0:\ldots:1:\pi'(k)],[\pi(k):1:\ldots:0:0],[\pi''(k):1:\ldots:1:\pi''(k)])$ which gives the relations $0.1.1 + \pi(k).1.1 = \pi''(k).1.1$ and $\pi'(k).1.1 + 0.1.1 = \pi''(k).1.1$, so $\pi(k) = \pi'(k) = \pi''(k)$.

A simple calculation shows that $v_\pi(k) < 0$ iff $v'(k) < 0$ iff $v''(k) < 0$, hence $\mathcal{O}_{v_\pi} = \mathcal{O}_{v'} = \mathcal{O}_{v''}$. We have now shown the following further compatibility between $\pi_1$ and $\pi_{n+1}$. Namely;

If $\gamma : P^1(K) \rightarrow P^{n+1}(K)$ is given by $\gamma : [x_0,x_1] \mapsto [0:0:\ldots:x_0:x_1]$ then $\pi_{n+1} \circ \gamma = \gamma \circ \pi_1$. (†)

Finally, let $Sum'$ be the closed subvariety of $P^{n+1}(K) \times P^{n+1}(K) \times P^{n+1}(K)$ defined in coordinates $[x_0: \ldots : x_{n+1}],[y_0: \ldots : y_{n+1}],[z_0: \ldots : z_{n+1}]$ by the $(n+1)$ equations $x_jy_1z_1 + y_jx_nz_1 + z_jx_ny_1$ for $j \neq n$. Let $[k_0: \ldots : k_{n+1}]$ be an arbitrary element of $P^{n+1}(K)$. Without loss of generality, we may assume that $\{k_0: \ldots : k_{n+1}\} \subset \mathcal{O}_{v_\pi}$ and that $k_n \in \mathcal{O}_{v_\pi}^*$. Hence, dividing by $k_n$, the element is of the form $[k_0: \ldots : k_{n-1}:1:k_{n+1}]$ with $\{k_0, \ldots , k_{n-1}, k_{n+1}\} \subset \mathcal{O}_{v_\pi}$. We have that $Sum'([0:0:\ldots:0:1:k_{n+1}],[k_0: \ldots : k_{n-1}:1:0],[k_0: \ldots : k_{n-1}:1:k_{n+1}])$, hence by specialisation and (†), $Sum'([0:0:\ldots:0:1:\pi(k_{n+1})],[\pi(k_0): \ldots : \pi(k_{n-1}):1:0],[l_0: \ldots : l_n:l_{n+1}])$ where $\{l_0, \ldots , l_{n+1}\} \subset L$. As is easily checked, the case when $l_n = 0$ leads to a contradiction, hence we can assume that $l_n = 1$ (multiplying by $1/l_n$). Now the equations give that $l_j = \pi(k_j)$ for $j \neq n$. We have therefore shown that $\pi_{n+1} = \pi_{n+1,v_\pi}$ as required.

Theorem 1.5 is now proved.

2. A Model Theoretic Language of Specialisations

We now introduce a model theoretic language which will enable us to describe specialisations in the context of algebraic geometry. In this section, we will assume that $K$ and its residue field have the same characteristic. We will use a many sorted structure $\{\bigcup S_n : n \in \mathcal{N}\}$. Each sort will be the domain of $P^n(K)$ for an algebraically closed field.
We fix an algebraically closed constant field $L$ which we assume to be countable and let $K$ be some non-trivial extension of $L$, having the same characteristic. In order to describe algebraic geometry, we introduce sets of predicates $\{V^m_n\}$ on the Cartesian powers $S^m_n$ to describe closed algebraic subvarieties of $P^n(K)$ defined over $L$. In particular, we have constants to denote the individual elements of $P^n(L)$ on each sort $S_n$. We introduce function symbols $i_n : S_n \to S_{n+1}$ to describe the imbeddings $P^n(K) \to P^{n+1}(K)$ defined above. Finally, we will have symbols $\{\pi_n : n \in \mathbb{N}\}$ to describe the specialisation map $\pi = \cup_{n \geq 1} \pi_n$. Strictly speaking, as $P^n(L)$ is not definable, each $\pi_n$ will be a union over $l \in P^n(L)$ of unary predicates defined as $\{x \in P^n(K) : \pi_n(x) = l\}$. We denote the language $\langle \{V^m_n\}, i_n, \pi_n \rangle$ by $L_{\text{spec}}$ and the theory of the structure $< P(K), P(L), \pi >$ in this language by $T_{\text{spec}}$. We denote the theory of the structure $< P(K), P(L) >$ in the language $L_{\text{spec}} \setminus \{\pi_n\}$ by $T_{\text{alg}}$. Note that the structure $< K, 0, 1, +, >$ is interpretable in the structure $< P(K), P(L) >$ in the language $L_{\text{spec}} \setminus \{\pi_n\}$ (\*). This follows by noting that the points $[1 : 0], [0 : 1]$ and $[1 : 1]$ are named as elements in the sort $S_1$ and the operations of $+,$ define algebraic subvarieties in the sorts $S_3^1$. The structure $< L, 0, 1, +, >$ is not interpretable but any model of $T_{\text{alg}}$ will contain an isomorphic copy of $P(L)$ as a substructure. It follows that the models of $T_{\text{alg}}$ are exactly of the form $< P(K), P(L) >$ for some algebraically closed field $K$ properly extending $L$ (use the fact that the axiomatisation of $Th(< K, 0, 1, +, >)$ can be interpreted in $T_{\text{alg}}$ and the field structure can be related to the predicates $\{V^m_n\}$ using the imbeddings $i_n$). We now claim the following;

**Theorem 2.1.** The theory $T_{\text{spec}}$ is axiomatised by $T_{\text{axioms}} = T_{\text{alg}} \cup \Sigma$ where $\Sigma$ is the set of sentences given by;

(i). The mappings $\{\pi_n\}$ preserve the predicates $\{V^m_n\}$.

(ii). The compatibility requirement $\pi_{n+1} \circ i_n = i_{n+1} \circ \pi_n$ holds.

(see definition 1.1). In particular, $T_{\text{axioms}}$ is complete. Moreover, $T_{\text{axioms}}$ is model complete.

The proof of this theorem will be based on Theorem 1.5 and the following result by Robinson, given in [6];

**Theorem 2.2.** Let $K$ be an algebraically closed field with a non trivial Krull valuation $v$ and residue field $l$. Then $T_K$ is model complete in the language $L_{\text{val}}$ and admits quantifier elimination in the language
Moreover, the completions of $K$ are determined by the pair $(\text{char}(l), \text{char}(K))$, that is $T_K \cup \Sigma$ is complete where $\Sigma$ is the possibly infinite set of sentences specifying the characteristic of $K$ and $l$.

Here, by the language $\mathcal{L}_{\text{rob}}$ we mean the language of algebraically closed fields together with a binary predicate $\text{Div}(x, y)$ denoting $v(x) \leq v(y)$. By the language $\mathcal{L}_{\text{val}}$, we mean a 2-sorted language for the value group and the field, with the usual language for the field sort and the language of ordered groups on the group sort. $T_K$ is the theory which asserts that $K$ is an algebraically closed field, the value group $\Gamma$ is linearly ordered and abelian, the valuation is non-trivial. For our purposes, we will require a slightly refined version of this result. Namely, we will fix a set of constants for an algebraically closed field $L$ which we can assume to be countable, add to $T_K$ the atomic diagram of $L$, relativized to the field sort, the requirement that $v|L$ is trivial and $\pi$, the residue mapping, maps $L$ injectively and homomorphically into the residue field. (Note, the condition that $L$ maps onto the residue field is not definable and that the homomorphism requirement ensures that the residue field $l$ and $K$ have equal characteristic, hence the characteristic of $K$ is already determined by the characteristic of $L$.) We will denote the corresponding theory by $T_{K,L}$ and the expanded languages by $\mathcal{L}_{\text{rob}}$ and $\mathcal{L}_{\text{val}}$ again. It is no more difficult to prove that $T_{K,L}$ is model complete, Robinson’s original proof in [6] requires the solution of certain valuation equations in the model $K$ given that these equations have a solutions in an extension $K'$, it makes no difference if some of the elements from $K$ are named. In order to show that $T_{K,L}$ is complete, it is sufficient to exhibit a prime model of the theory;

Case 1. $\text{Char}(K, L) = (p, p)$, with $p \neq 0$. Take $L(\epsilon)^{\text{alg}}$ where $\epsilon$ is transcendental over $L$, define the valuation on $L$ to be zero and extend it to $L(\epsilon)$ non-trivially using say $v_{\text{ord}, \epsilon}$, the order valuation in $\epsilon$. Take any extension to $L(\epsilon)^{\text{alg}}$.

Case 2. $\text{Char}(K, L) = (0, 0)$, define a similar valuation on $L(\epsilon)^{\text{alg}}$.

We now show the following lemma;

\textbf{Lemma 2.1.} \textit{Amalgamation of Specialisations}
Let \((P(K_1), P(L), \pi_1)\) and \((P(K_2), P(L), \pi_2)\) be models of \(T_{\text{axioms}}\), then there exists a further model \((P(K_3), P(L), \pi_3)\) such that:

\[(P(K_1), P(L), \pi_1) \preceq (P(K_3), P(L), \pi_3)\]

and

\[(P(K_2), P(L), \pi_2) \preceq (P(K_3), P(L), \pi_3)\]

Proof. By Theorem 1.5, we can find Krull valuations \(v_1\) and \(v_2\) on \(K_1\) and \(K_2\) such that \(\pi_1 = \pi_{v_1}\) and \(\pi_2 = \pi_{v_2}\). Using the refined version of Robinson’s completeness result, we can jointly embed \((K_1, v_1)\) and \((K_2, v_2)\) over \(L\) into \((K_3, v_3)\) \((*)\). Let \(L'\) be the residue field of \(v_3\), then as \(K_3\) is algebraically closed, so is \(L'\) and extends the residue field \(L\) of \(v_1\) and \(v_2\). By standard results, we can construct a Krull valuation \(v\) on \(L'\) with residue field \(L\), for example use the construction given in [2]. Using Theorem 1.5 again, we can construct specialisations \((P(K_3), P'(L'), \pi_{v_3})\) and \((P'(L'), P(L), \pi_v)\), the composition gives a specialisation \((P(K_3), P(L), \pi_3)\). It remains to see that in fact \(\pi_3\) extends the specialisations \(\pi_1\) and \(\pi_2\). This follows from the fact that if \(k \in K_1\) or \(k \in K_2\) and there exists \(l \in L\) such \(v_1(k - l) > 0\) or \(v_2(k - l) > 0\) then this relation is preserved in the embedding \((*)\). Hence the specialisation \(\pi_{v_3}\) already extends the specialisations \(\pi_1\) and \(\pi_2\) of \(P(K_1)\) and \(P(K_2)\) into \(P(L)\). As the specialisation \(\pi_v\) fixes \(L\), this proves the lemma.

\[\square\]

**Lemma 2.3. Transfer of Formulas**

Let \((P(K), P(L), \pi)\) be a specialisation with corresponding \((K, v)\), then there exists a map:

\[\sigma : P(K) \to K^{eq}\]

\[\sigma : \mathcal{L}_{\text{spec-formulae}} \to \mathcal{L}_{\text{val-formulae}}\]

such that for any \(\phi(x_1, \ldots, x_n)\) which is a \(\mathcal{L}_{\text{spec-formula}}\) and \((k_1, \ldots, k_n) \subset P(K)\):

\[(P(K), P(L), \pi) \models \phi(k_1, \ldots, k_n) \iff (K, v) \models \sigma(\phi)(\sigma(k_1), \ldots, \sigma(k_n))\]

\[\dagger\]
Moreover, the definition of the map is uniform in $K$.

Proof. The map $\sigma$ is defined on the sorts $P^n(K)$ by sending $[k_0, \ldots, k_n]$ to $(k_0, \ldots, k_n)/\sim_n$ where $\sim_n$ is the equivalence relation defined on $K^{n+1}$ from multiplication by $K^*$. Similarly, $\sigma$ maps a variable from the sort $S^n$ to the corresponding variable from the sort in $K_{eq}$ defined by $\sim_n$. A closed algebraic subvariety in $\{V^m_n\}$ is defined by a multi-homogeneous equation in the variables $\{(x_{01}, \ldots, x_{n1}), \ldots, (x_{0m}, \ldots, x_{nm})\}$. Let $C^m_n$ be the algebraic variety in $K^{m(n+1)}_{eq}$ defined by this equation. Then the corresponding formula in $K_{eq}$ is given by:

$$(y_1, \ldots, y_m) \in (\sim_n)^m[\exists x_1 \ldots x_m(C^m_n(x_1, \ldots, x_m) \land \bigwedge_{i=1}^m x_i/\sim_n = y_i)]$$

For the inclusion maps $i_n$, let us identify each $i_n$ with its graph, then clearly we can define $\sigma$ to map the formula $i(x) = y$ to a corresponding formula relating the sorts $\sim_n$ and $\sim_{n+1}$ in $K_{eq}$.

Note that if $l \in P^n(L)$ is a constant, then $\sigma(l) = (l_0, \ldots, l_n)/\sim_n$ where each $l_i$ is a constant from the atomic diagram of $L$.

Finally, let $\pi_n : P^n(K) \to P^n(L)$ be a specialisation. Again, let us assume that we can identify $\pi_n$ with its graph. We then have that:

$$\pi_n([x_0 : \ldots : x_n]) = [l_0 : \ldots : l_n]$$

iff

$$\exists z \exists z_0 \ldots \exists z_n((\bigwedge_{i=0}^n x_i z = l_i + z_i) \land (\bigwedge_{i=0}^n v(z_i) > 0)).$$

It is now clear how to define $\sigma(\pi_n)$ as a union of formulas in the sort defined by $\sim_n$.

This completes the definition of $\sigma$, it is clear that the definition is uniform in $K$ and a straightforward induction on the length of a formula from $\mathcal{L}_{spec}$ shows that it has the required property ($\dagger$).

\[\square\]

Theorem 2.1 is now a fairly straightforward consequence of the above lemmas. We first show model completeness. Suppose that we have models of $T_{axioms}$;
By theorem 1.5, we can find Krull valuations $v_1$ and $v_2$ such that $(K_1, v_1) \leq (K_2, v_2)$ and $(K_1, v_1), (K_2, v_2) \models T_{K,L}$. By the refined model completeness result after Theorem 2.2, we have $(K_1, v_1) \prec (K_2, v_2)$, hence using Lemma 2.3, we must have that;

$$(P(K_1), P(L), \pi_1) \prec (P(K_2), P(L), \pi_2)$$

as required. Completeness now follows directly from Lemma 2.1 and model completeness. Alternatively, one can exhibit a prime model of the theory, this is clearly possible by taking the specialisations corresponding to the prime models of $T_{K,L}$ above.

3. A First Order Definition of Intersection Multiplicity and Bezout’s Theorem

We now formulate a non-standard definition of intersection multiplicity in the language $L_{\text{spec}}$. We will do this only in the case of projective curves inside $P^2(L)$, the reader may wish to try formulating a corresponding definition in higher dimensions.

Let $C_1$ and $C_2$ be projective curves of degree $d$ and degree $e$ in $P^2(K)$ defined over $L$. The parameter spaces for such curves are affine spaces of dimension $(d+1)(d+2)/2$ and $(e+1)(e+2)/2$ respectively. We can give them a projective realisation by noting that if $(l)$ is a non-zero vector defining a curve of degree $d$, then multiplying it by a constant $\mu$ defines the same curve. Let $P^{d(d+3)/2}(K)$ and $P^{e(e+3)/2}(K)$ define these spaces which we will denote by $P_d$ and $P_e$ for ease of notation. Let $\text{Curve}_d$ and $\text{Curve}_e$ be the closed projective subvarieties of $P_d \times P^2(K)$ and $P_e \times P^2(K)$, defined over the prime subfield of $L$, such that, for $l \in P_d$, the fibre $\text{Curve}_d(l)$ defines the corresponding projective curve of degree $d$ in $P^2(K)$. For $l$ in $P^n(L)$, we denote its infinitesimal neighborhood $V_l$ to be the inverse image under the specialisation $\pi_n$.

Now suppose that $C_1$ and $C_2$ (which may not be reduced or irreducible), of degrees $d$ and $e$ respectively, are defined by parameters $l_1$ and $l_2$ and intersect at an isolated point $l$ in $P^2(L)$. Then we define;

$$\text{Mult}(C_1, C_2, l) \geq n$$
iff

\[ \exists x_1, x_2 \in V_1, V_2, \exists y \neq y_n \in V(\{y_1, \ldots, y_n\} \subset Curve_d(x_1) \cap Curve_e(x_2)) \]

Then define \( \text{Mult}(C_1, C_2, l) = n \) iff

\[ \text{Mult}(C_1, C_2, l) \geq n \text{ and } \neg \text{Mult}(C_1, C_2, l) \geq n + 1. \]

Clearly, the statement that \( \text{Mult}(C_1, C_2, l) = n \) naturally defines a sentence in the language \( \mathcal{L}_{\text{spec}} \). One consequence of the completeness result given above is that the statement "The curves \( C_1 \) and \( C_2 \) intersect with multiplicity \( n \) at \( l \)" depends only on the theory \( T_{\text{axioms}} \) and is independent of the particular structure \( (P(K), P(L), \pi) \). In the paper \[3\], we showed that this non-standard definition of multiplicity is equivalent to the algebraic definition of multiplicity when computed in the structure \( (P(K_{\text{univ}}), P(L), \pi_{\text{univ}}) \) (see the next section). It therefore follows that the non-standard definition of multiplicity is equivalent to the algebraic definition even when computed in a prime model of \( T_{\text{axioms}} \) which I will denote by \( (P(K_{\text{prime}}), P(L), \pi_{\text{prime}}) \).

We now turn to the statement of Bezout’s theorem. In algebraic language, this says that if projective algebraic curves \( C_1 \) and \( C_2 \) of degree \( d \) and degree \( e \) in \( P^2(L) \) intersect at finitely many points \( \{l_1, \ldots, l_n\} \), then:

\[ \sum_{i=1}^{n} I(C_1, C_2, l_i) = de \]

where \( I(C_1, C_2, l_i) \) is the algebraic intersection multiplicity. The non-standard version of this result can be formulated in the language \( \mathcal{L}_{\text{spec}} \) by the sentence:

\[ \text{Bezout}(C_1, C_2) \equiv \exists m_1, \ldots, m_n, m_1 + \ldots + m_n = de (\wedge_{i=1}^{n} \text{Mult}(C_1, C_2, l_i) = m_i) \]

Again, in the paper \[3\], we proved the algebraic version of Bezout’s theorem by non-standard methods in the structure \( (P(K_{\text{univ}}), P(L), \pi_{\text{univ}}) \). It follows that the sentences \( \text{Bezout}(C_1, C_2) \) are all proved by the theory \( T_{\text{axioms}} \) and therefore hold in the structure \( (P(K_{\text{prime}}), P(L), \pi_{\text{prime}}) \) as well. This demonstrates the fact that we can prove an algebraic statement of Bezout’s theorem using only infinitesimals from a straightforward extension of \( L \), namely \( L(\epsilon)^{\text{alg}} \), in particular in a structure such
that the infinitesimal neighborhoods $V_i$ are all recursively enumerable. This seems to provide some answer to a general objection concerning the use of infinitesimals, originating in [1]. It may also provide an effective alternative method to compute intersection multiplicities generally in algebraic geometry.

4. Constructing a Universal Specialisation

In the papers [2] and [3], we used the existence of a specialisation $(P(K_{univ}), P(L), \pi_{univ})$ having the following ”universal” property;

If $L \subset L_m$ is an algebraically closed extension of $L$ with transcendence degree $m$, and $(P(L_m), P(L), \pi_m)$ is a specialisation, then there exists an $L$-embedding $\alpha_L : L_m \rightarrow K_{univ}$ with the property that $\pi_{univ} \circ \alpha_L = \pi_m$. (*)

Unfortunately, the construction of $K_{univ}$ was flawed. We correct this difficulty here;

Model theoretically, using theorem 2.1, it is easy to show the existence of such a structure. Namely, let $(P(K_{univ}), P(L), \pi_{univ})$ be a $2^\omega$ saturated model of the theory $T_{axioms}$. Then, if $L \subset L_m$ is an algebraically closed extension of $L$ of transcendence degree $m \leq n$, clearly $\bigcup_{n \geq 1} \text{Card}(S^n(\text{Th}(\mathcal{M}))) \leq 2^\omega$, where $\mathcal{M} = (P(L_m), P(L), \pi_m)$. This follows as $L$ was assumed to be countable. Hence, by elementary model theory, there exists an $L$-embedding $\alpha_L$ with the required properties. For the non-model theorist, we give a more algebraic construction, replacing the use of types by an explicit amalgamation of the possible valuations;

Proof. Suppose, inductively, we have already constructed a specialisation $(P(K_n), P(L), \pi_n)$ which has the property (*) for all extensions $L \subset L_m$ with $L_m$ algebraically closed of transcendence degree $m \leq n$. We will construct $K_{n+1}$ having this property for $m \leq n + 1$. By theorem 1.5, we can find a Krull valuation $v_n$ on $K_n$ corresponding to the specialisation $\pi_n$. Let $t$ be a new transcendental element. The extensions of $v_n$ to $K_n(t)$ are completely classifiable. In fact, we have the following result in [3] (Theorem 3.9), we refer the reader to the paper for the definition of each family of valuations;
The extensions of \( v_n \) are of the form:

(i). \( v_{n,a,\gamma} \) where \( a \in K_n \) and \( \gamma \) is an element of some ordered group extension of \( v(K) \).

(ii). \( v_{n,A} \) where \( A \) is a pseudo Cauchy sequence in \((K_n, v_n)\) of transcendental type.

Let \( I \) be a fixed enumeration of these valuations. Inductively, we assume that \( \text{Card}(K_n) \leq 2^\omega \) in which case the dimension of \( v(K_n) \) as a vector space over \( \mathbb{Q} \) has dimension at most \( 2^\omega \) as well. Clearly then the number of non-isomorphic (over \( K_n \)) valuations from (ii) is at most \( 2^\omega \) and the same holds for the valuations obtained from (i) by noting that the number of order types of \( \gamma \) is at most \( 2^\omega \) (it is easily checked that 2 new elements of the value group, \( \gamma_1 \) and \( \gamma_2 \), having the same order type, define isomorphic valuations in the case of (i)). Hence, we can assume that \( I \) is well ordered and apply the method of transfinite induction to construct a series of specialisations \((P(K_n,i), P(L), \pi_n,i)\) as follows;

For \( i = 0 \), set \((P(K_n,0), P(L), \pi_{n,0}) = (P(K_n), P(L), \pi_n)\)

Given \( i \in I \) with \( i \) not a limit ordinal, let \( v_{i+1} \) be the next valuation in the enumeration. Let \((K_n, t, v_{i+1})\) be the completion of \((K_n(t), v_{i+1})\) and let \( \overline{v_{i+1}} \) also denote the unique extension of this valuation to the algebraic closure \( K_n \{t\}^{alg} \). This defines a Krull valuation and hence a specialisation \((P(K_n, t, v_{i+1}), P(L'), \pi_n,i+1)\) where \( L' \) is the algebraic closure of the residue field of \( v_{i+1} \), having transcendence degree at most 1 over \( L \). Using arguments as above, we can construct a specialisation \((P(L'), P(L), \pi)\). Composing these specialisations, we obtain a specialisation \((P(K_n, t, v_{i+1}), P(L), \pi_n,i+1)\). (One can omit this step by enumerating in \( I \) only those valuations which preserve the residue field \( L \)) Now, using Lemma 2.1 and Theorem 2.1, amalgamate the specialisations \((P(K_n \{t\}^{alg}), P(L), \pi_{n,i+1})\) and \((P(K_n,i), P(L), \pi_n,i)\) to form a specialisation;

\((P(K_n,i), P(L), \pi_n,i) \prec (P(K_n,i+1), P(L), \pi_{n,i+1})\).

For \( i \) a limit ordinal, we set;

\((P(K_n,i), P(L), \pi_n,i) = \bigcup_{j<i}(P(K_n,j), P(L), \pi_{n,j})\)
By the usual union of chains arguments we have that;

\((P(K_{n,j}), P(L), \pi_{n,j}) \prec (P(K_{n,i}), P(L), \pi_{n,i})\) for \(j < i\).

Repeating this process, we obtain a structure \((P(K_{n+1}), P(L), \pi_{n+1})\) such that;

\((P(K_n), P(L), \pi_n) \prec (P(K_{n+1}), P(L), \pi_{n+1})\).

It remains to check that this structure has the universal property \((\ast)\) for \(m = n + 1\). Let \(L_{n+1}\) be an algebraically closed extension of \(L\) with transcendence degree \(n + 1\) and specialisation \((P(L_{n+1}), P(L), \pi)\). Let \(v\) be the corresponding valuation and its restriction to \(L \subset L_n \subset L_{n+1}\), a subfield of transcendence degree \(n\). The corresponding specialisation \((P(L_n), P(L), \pi)\) already factors through \((P(K_n), P(L), \pi_{n})\) \((\dagger)\) and the valuation \(v\) appears as \(v_i\) in the enumeration \(I\) when restricted to \(L_n(t)\). By a standard result in valuation theory, see \([5]\), there exists an \(L_n(t)\)-embedding \(\tau : L_n(t)^{alg} \rightarrow L_n\{t\}^{alg}\) such that \(v = v_i \circ \tau\) \((\dagger\dagger)\) (see notation above). Combining \((\dagger)\) and \((\dagger\dagger)\), we obtain an embedding \(\alpha : (P(L_{n+1}), P(L)) \rightarrow (P(K_{n,i}), P(L))\) such that \(\pi = \pi_{n,i} \circ \alpha\). This proves the result. It is now clear that the structure

\[(P(K_{univ}), P(L), \pi_{univ}) = \bigcup_{i>0} (P(K_i), P(L), \pi_i)\]

has the required universal property, is a model of \(T_{axioms}\) and;

\[(P(K_i), P(L), \pi_i) \prec (P(K_{univ}), P(L), \pi_{univ})\) for \(i > 0\).

\[\square\]

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