SOME MARTINGALES ASSOCIATED WITH MULTIVARIATE JACOBI PROCESSES AND AOMOTO’S SELBERG INTEGRAL

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Abstract. We study \( \beta \)-Jacobi diffusion processes on alcoves in \( \mathbb{R}^N \), depending on 3 parameters. Using elementary symmetric functions, we present space-time-harmonic functions and martingales for these processes \((X_t)_{t \geq 0}\) which are independent from one parameter. This leads to a formula for \( \mathbb{E} \left[ \prod_{i=1}^N (y - X_{t,i}) \right] \) in terms of classical Jacobi polynomials. For \( t \to \infty \) this yields a corresponding formula for Jacobi ensembles and thus Aomoto’s Selberg integral.

1. Introduction

For an integer \( N \) and a constant \( \beta \in [0, \infty] \) consider the \( \beta \)-Jacobi (or \( \beta \)-MANOVA) ensembles which are \([0, 1]^N\)-valued random variables \( X \) with Lebesgue densities

\[
f_{\beta, a_1, a_2}(x) := c_{\beta, a_1, a_2} \prod_{i=1}^N x_i^{a_1}(1 - x_i)^{a_2} \cdot \prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta
\]

with parameters \( a_1, a_2 > -1 \) and known normalizations \( c_{\beta, a_1, a_2} > 0 \) which can be expressed via Selberg integrals; see the survey [FW]. For \( \beta = 1, 2, 4 \), and suitable \( a_1, a_2 \), the variables \( X \) appear as spectrum of the classical Jacobi ensembles; see e.g. [E]. Moreover, for general \( \beta, a_1, a_2 \), the variables appear as eigenvalues of the tridiagonal models in [KN], [K]. Furthermore, for the ordered models on the alcoves \( A_0 := \{ x \in \mathbb{R}^N : 0 \leq x_1 \leq \ldots \leq x_N \leq 1 \} \), the probability measures with densities \( N! \cdot f_{\beta, a_1, a_2} \) appear in log gas models as stationary distributions of diffusions \((X_t)_{t \geq 0}\) with \( N \) particles in \([0, 1]\); see [E] [Dem] [RR1]. These diffusions and their stationary distributions are closely related to Heckman-Opdam hypergeometric functions of type \( BC_N \). In particular, the generator of the transition semigroup of \((X_t)_{t \geq 0}\) is the symmetric part of a Dunkl-Cherednik Laplace operator. Moreover, Heckman-Opdam Jacobi polynomials form a basis of eigenfunctions, where these polynomials are orthogonal w.r.t. the density \( N! \cdot f_{\beta, a_1, a_2} \) on \( A_0 \). For the background see the monograph [HS] and [Dem] [RR1] [L] [BO]. We point out that for \( \beta = 1, 2, 4 \) and suitable \( a_1, a_2 \), the diffusion \((X_t)_{t \geq 0}\) and their stationary distributions on \( A_0 \) are projections of Brownian motions and uniform distributions on compact Grassmann manifolds over \( F = \mathbb{R}, \mathbb{C}, \mathbb{H}; \) see [HS], [RR2].

In this paper we use the elementary symmetric polynomials \( e_0, e_1, \ldots, e_N \) in \( N \) variables and construct polynomials \( p_n \) of order \( n = 1, \ldots, N \) via linear combinations such that for suitable exponents \( r_n \geq 0 \), the processes \( (e^{r_n t}p_n(X_t))_{t \geq 0} \) are
martingales where, after some parameter transform, \( p_n, r_n \) depend only on 2 parameters and not on the third one; see Proposition 3.2 for details. We use this result to show that for particular starting points,

\[
\mathbb{E}\left( \prod_{i=1}^{N} (y - X_{t,i}) \right) = \tilde{P}^{(\alpha, \beta)}_N(y) \quad \text{for all} \quad t \geq 0
\]

where \( \tilde{P}^{(\alpha, \beta)}_N \) is a monic Jacobi polynomial on \([0,1]\) where \( \alpha, \beta \) depend on the 2 relevant parameters of the martingale result (here, \( \beta \) is not the \( \beta \) in \((2.1)\)). In the limit \( t \to \infty \), \((1.2)\) leads to a corresponding formula for the expectation for the random variable \( X \) with density in \((1.1)\) and to Aomoto’s Selberg integral \([A]\). Corresponding results for classical Hermite and Laguerre ensembles are given in \([DG, FG]\); for these results for related multivariate Bessel processes we refer to \([KVW]\). Clearly, \((1.2)\) admits an interpretation for characteristic polynomials of classical Jacobi ensembles and the tridiagonal \( \beta \)-Jacobi models in \([KN]\).

The proof of the martingale result relies on the stochastic differential equation for the diffusion \((X_t)_{t \geq 0}\); for the general background here we refer to \([P, RW]\).

This paper is organized as follows. In Section 3 we briefly recapitulate some facts on Heckman-Opdam Jacobi polynomials of type \( BC \), the associated Dunkl-Cherednik Laplace operator, and the transition semigroup of the diffusion \((X_t)_{t \geq 0}\). In Section 3 we then use stochastic analysis to derive our martingales. Section 4 is then devoted to \((1.2)\). In Section 5 we then discuss some connection between the martingale result and Heckman-Opdam Jacobi polynomials.

A comment about notations and normalizations: We start in Section 2 with a brief survey on the compact Heckman-Opdam theory of type \( BC \) with multiplicity parameters \( k_1, k_2, k_3 \geq 0 \). We transfer all results from the trigonometric case to the interval \([-1,1]\), and start with transformed parameters \( \kappa, p, q \) in the SDE approach in Section 3. We there follow \([Dem]\) where we replace his \( \beta \) by \( \kappa \geq 0 \), in order to avoid any confusion with the classical parameters \( \alpha, \beta > -1 \) of the one-dimensional Jacobi polynomials \( P^{(\alpha, \beta)}_N \). The choice of the interval \([-1,1]\) instead of \([0,1]\) as in \((1.1)\) or \([Dem]\) is caused by the fact that the results should fit to the \( P^{(\alpha, \beta)}_N \).

2. **Heckman-Opdam Jacobi polynomials of type \( BC \)**

We first recapitulate some general facts on Heckman-Opdam theory from \([HS]\).

**General Heckman-Opdam theory 2.1.** Let \((\mathfrak{a}, \{\cdot, \cdot\})\) be a Euclidean space of dimension \( N \). Let \( R \) be a crystallographic, possibly not reduced root system in \( \mathfrak{a} \) with associated reflection group \( W \). Fix a positive subsystem \( R_+ \) of \( R \) and a \( W \)-invariant multiplicity function \( k : R \to [0, \infty] \). The Cherednik operators associated with \( R_+ \) and \( k \) are

\[
D_\xi(k)f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} \frac{k(\alpha)}{1 - e^{\langle \alpha, x \rangle}} (f(x) - f(\sigma_\alpha(x))) - \langle \rho(k), \xi \rangle f(x)
\]

for \( \xi \in \mathbb{R}^n \) with the half-sum \( \rho(k) := \frac{1}{2} \sum_{\alpha \in R_+} k(\alpha) \alpha \) of positive roots.

The \( D_\xi(k) \) \((\xi \in \mathfrak{a})\) commute, and for each \( \lambda \in \mathfrak{a}_C \) there exists a unique analytic function \( G(\lambda, k; \cdot, \cdot) \) on a common \( W \)-invariant tubular neighborhood of \( \mathfrak{a} \) in the complexification \( \mathfrak{a}_C \), the so called Opdam-Cherednik kernel, satisfying

\[
D_\xi(k)G(\lambda, k; \cdot, \cdot) = \langle \lambda, \xi \rangle G(\lambda, k; \cdot) \quad \forall \xi \in \mathfrak{a}; \quad G(\lambda, k; 0) = 1.
\]
The hypergeometric function associated with $R$ is defined by

$$F(\lambda, k; z) = \frac{1}{|W|} \sum_{w \in W} G(\lambda, k; w^{-1}z).$$

For the Heckman-Opdam polynomials, we write $\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}$ for $\alpha \in R$ and use the weight lattice and the set of dominant weights associated with $R$ and $R_+,$

$$P = \{ \lambda \in \mathfrak{a} : (\lambda, \alpha^\vee) \in \mathbb{Z} \ \forall \alpha \in R \}, \quad P_+ = \{ \lambda \in P : (\lambda, \alpha^\vee) \geq 0 \ \forall \alpha \in R_+ \} \supset R_+$$

where $P_+$ carries the usual dominance order. Let $T := \text{span}_\mathbb{C} \{ e^{i\lambda}, \lambda \in P \}$ be the vector space of trigonometric polynomials associated with $R.$ The orbit sums

$$M_\lambda = \sum_{\mu \in W\lambda} e^{i\mu}, \quad \lambda \in P_+$$

form a basis of the subspace $T^W$ of $W$-invariant polynomials in $T.$ For $Q^\vee := \text{span}_\mathbb{Z} \{ \alpha^\vee, \alpha \in R \},$ consider the torus $\mathbb{T} = a/2\pi Q^\vee$ with the weight function

$$\delta_k(t) := \prod_{\alpha \in R_+} \left| \sin \left( \frac{(\alpha, t)}{2} \right) \right|^{2k}. \quad (2.3)$$

The Heckman-Opdam polynomials associated with $R_+$ and $k$ are given by

$$P_\lambda(k; z) := M_\lambda(z) + \sum_{\nu < \lambda} c_{\lambda, \nu}(k) M_\nu(z) \quad (\lambda \in P_+, z \in \mathfrak{a}c)$$

where the $c_{\lambda, \nu}(k) \in \mathbb{R}$ are determined by the condition that $P_\lambda(k; )$ is orthogonal to $M_\nu$ in $L^2(\mathbb{T}, \delta_k)$ for $\nu \in P_+$ with $\nu < \lambda.$ It is known that $\{ P_\lambda(k; ), \lambda \in P_+ \}$ is an orthonormal basis of the space $L^2(\mathbb{T}, \delta_k)^W$ of all $W$-invariant functions in $L^2(\mathbb{T}, \delta_k).$ By [HS], the normalized polynomials

$$R_\lambda(k, z) := P_\lambda(k; z)/P_\lambda(k; 0)$$

satisfy

$$R_\lambda(k, z) = F(\lambda + \rho(k), k; iz). \quad (2.4)$$

We next introduce the Heckman-Opdam Laplacian

$$\Delta_k := \sum_{j=1}^N D_{\xi_j}(k)^2 - \|\rho(k)\|_2^2$$

with an orthonormal basis $\xi_1, \ldots, \xi_N$ of $\mathfrak{a}.$ The operator $\Delta_k$ does not depend on the basis and, by [Sch1, Sch2], has for a $W$-invariant function $f$ the form

$$\Delta_k f(x) = \Delta f(x) + \sum_{\alpha \in R_+} k(\alpha) \coth \left( \frac{(\alpha, x)}{2} \right) \cdot \partial_\alpha f(x).$$

If we take the factor $i$ in (2.4) into account as in [RR1], we now consider the operator

$$\tilde{\Delta}_k f(t) := \Delta f(t) + \sum_{\alpha \in R_+} k(\alpha) \cot \left( \frac{(\alpha, t)}{2} \right) \cdot \partial_\alpha f(t). \quad (2.5)$$

By construction, the $P_\lambda$ ($\lambda \in P_+$) are eigenfunctions of $\Delta_k$ with with eigenvalues $-\langle \lambda, \lambda + 2\rho(k) \rangle \leq 0.$ This is used in [RR1] to construct the transition densities of the diffusions with the generators $L_k.$
The compact BC-case 2.2. We now turn to the nonreduced root system

\[ R = BC_N = \{ \pm e_i, \pm 2e_i, \pm (e_i \pm e_j) ; \ 1 \leq i < j \leq N \} \subset \mathbb{R}^N \]

with weight lattice \( P = \mathbb{Z}^n \) and torus \( T = (\mathbb{R}/2\pi\mathbb{Z})^N \). The multiplicity on \( R \) are written as \( k = (k_1, k_2, k_3) \) with \( k_1, k_2, k_3 \) as the values on the roots \( e_i, 2e_i, e_i \pm e_j \). Now fix a positive subsystem \( R_+ \) and consider the associated normalized Heckman-Opdam Jacobi polynomials \( R_\lambda = R_\lambda^{BC} (\lambda \in \mathbb{Z}_+^N) \) as e.g. in [BO, L, RR1]. (2.3) and some calculation show that the polynomials \( \tilde{R}_\lambda \) with

\[ \tilde{R}_\lambda(\cos t) := R_\lambda(\cos t; k) \quad (\lambda \in \mathbb{Z}_+^N) \]

form an orthogonal basis of \( L^2(A_N, \mu_k) \) on the alcove

\[ A_N := \{ x \in \mathbb{R}^N | -1 \leq x_1 \leq ... \leq x_N \leq 1 \} \]

with the weight function

\[ w_k(x) := \prod_{i=1}^{N} (1 - x_i)^{k_1+i+k_2-1/2} (1 + x_i)^{k_2-1/2}, \prod_{i<j} |x_i - x_j|^{2k_3}. \] (2.6)

The operator \( \tilde{\Delta}_k \) from (2.5) is given by

\[ \tilde{\Delta}_k f(t) := \Delta f(t) + \sum_{i=1}^{N} \left( k_1 \cot \left( \frac{t_i}{2} \right) + 2k_2 \cot(t_i) \right. \]

\[ + k_3 \sum_{j:j \neq i} \left( \cot \left( \frac{t_i - t_j}{2} \right) + \cot \left( \frac{t_i + t_j}{2} \right) \right) \partial_i f(t). \] (2.7)

The substitutions \( x_i = \cos t_i \) and elementary calculations lead to the operator

\[ L_k f(x) := \sum_{i=1}^{N} (1-x_i^2) f_{x_i,x_i}(x) + \sum_{i=1}^{N} \left( -k_i - (1+k_1+2k_2)x_i + 2k_3 \sum_{j:j \neq i} \frac{1-x_i^2}{x_i - x_j} \right) f_{x_i}(x). \] (2.8)

In summary, the Heckman-Opdam Jacobi polynomials \( \tilde{R}_\lambda \) are eigenfunctions of \( L_k \) with eigenvalues \( -\langle \lambda, \lambda + 2\rho(k) \rangle \) where \( \rho(k) \) has the coordinates

\[ \rho(k)_i = (k_1 + 2k_2 + 2k_3(N - i))/2 \quad (i = 1, \ldots, N). \] (2.9)

By [RR1], \( L_k \) is the generator of a Feller semigroup with transition operators whose smooth densities admit series expansions involving the \( \tilde{R}_\lambda \). Moreover, by standard stochastic calculus, the associated Feller processes \( (X_t)_{t \geq 0} \) with coordinates \( X_{t,i} \) should be solutions of the SDEs

\[ dX_{t,i} = \sqrt{2(1 - X_{t,i}^2)} dB_{t,i} + \left( -k_1 - (1+k_1+2k_2)X_{t,i} + 2k_3 \sum_{j:j \neq i} \frac{1-X_{t,i}^2}{X_{t,i} - X_{t,j}} \right) dt. \] (2.10)

for \( i = 1, \ldots, N \) and \( N \)-dimensional Brownian motion \( (B_t)_{t \geq 0} \). In fact, it is shown in Theorem 2.1 of [Dem] that for any starting point \( x \) in the interior of \( A_N \) and all \( k_1, k_2, k_3 > 0 \), the SDE (2.10) has a unique strong solution \( (X_t)_{t \geq 0} \) where the paths are reflected when they meet the boundary \( \partial A_N \) of \( A_N \). We study these Jacobi processes in the next section.
Example 2.3. For $N = 1$, the $\tilde{R}_{\lambda}$ are one-dimensional Jacobi polynomials

$$P_n^{(\alpha, \beta)}(x) := \binom{n + \alpha}{n} \binom{n + \beta + 1}{n + 1} \binom{n + \alpha + 1 + 1}{n + 1} \binom{n + \beta}{n + 1} \binom{n + \alpha + 1}{n + 1} (x - 1)^k$$

for $\alpha, \beta > -1$, where the $P_n^{(\alpha, \beta)}$ are orthogonal w.r.t. the weights $(1 - x)^{\alpha}(1 + x)^{\beta}$ on $[-1, 1]$; see Ch. 4 of [Sz]. With these notations we see from (2.6) that

$$\binom{n + \alpha}{n} \tilde{R}_n^{BC}(k; .) = P_n^{(\alpha, \beta)} \quad \text{with} \quad \alpha = k_1 + k_2 - \frac{1}{2}, \beta = k_2 - \frac{1}{2}.$$ 

Moreover, (2.8) corresponds with the classical differential equation for the $P_n^{(\alpha, \beta)}$.

3. SOME MARTINGALES RELATED TO JACOBI PROCESSES

In this section we study the $\beta$-Jacobi processes $(X_t)_{t \geq 0}$ on $A_N$ which satisfy (2.10). We follow [Dem] and introduce new parameters $p, q, \kappa > 0$ instead $k_1, k_2, k_3$ where we replace the parameter $\beta$ in [Dem] by $\kappa$ in order to avoid problems with the classical Jacobi polynomials below. For $\kappa > 0$ and $p, q > N - 1 + 1/\kappa$, we now define the Jacobi process $(X_t)_{t \geq 0}$ as the unique strong solution of the SDEs

$$dX_{t,i} = \sqrt{2(1 - X_{t,i}^2)} dB_{t,i} + \kappa \left( (p - q) - (p + q)X_{t,i} + \sum_{j: j \neq i} \frac{(1 + X_{t,i})(1 - X_{t,j}) + (1 + X_{t,j})(1 - X_{t,i})}{X_{t,i} - X_{t,j}} \right) dt$$

$$= \sqrt{2(1 - X_{t,i}^2)} dB_{t,i} + \kappa \left( (p - q) - (p + q)X_{t,i} + 2 \sum_{j: j \neq i} \frac{1 - X_{t,i}X_{t,j}}{X_{t,i} - X_{t,j}} \right) dt$$

$$= \sqrt{2(1 - X_{t,i}^2)} dB_{t,i} + \kappa \left( (p - q) + (2(N - 1) - (p + q))X_{t,i} + 2 \sum_{j: j \neq i} \frac{1 - X_{t,i}^2}{X_{t,i} - X_{t,j}} \right) dt$$

for $i = 1, \ldots, N$ with an $N$-dimensional Brownian motion $(B_t)_{t \geq 0}$ where the paths of $(X_t)_{t \geq 0}$ are reflected on $\partial A_N$ and where we start in some point in the interior of $A_N$; see Theorem 2.1 of [Dem]. Clearly the SDEs (3.1) and (2.10) are equal for

$$\kappa = k_3, \quad q = N - 1 + \frac{1 + 2k_1 + 2k_2}{2k_3}, \quad p = N - 1 + \frac{1 + 2k_2}{2k_3}.$$ 

It is known by [Dem, Dou] that for $\kappa \geq 1$ and $p, q \geq N - 1 + 2/\kappa$, the process does not meet $\partial C_N^2$ almost surely.

Besides the original processes for $\kappa > 0$ we also consider the transformed processes $(\tilde{X}_t := X_{t/\kappa})_{t \geq 0}$. We use the obvious formulas

$$\int_0^t Z_{s/\kappa} \, ds = \kappa \int_0^{t/\kappa} Z_s \, ds \quad \text{and} \quad \int_0^t Z_{s/\kappa} \, dB_s = \sqrt{\kappa} \int_0^{t/\kappa} Z_s \, dB_s$$
with Brownian motions \((B_t)_{t \geq 0}\), \((\tilde{B}_t)_{t \geq 0}\) starting in 0 related by \(\tilde{B}_t = \sqrt{\kappa} \cdot B_{t/\kappa}\). We then obtain the renormalized SDEs
\[
dX_{t,i} = \frac{\sqrt{2}}{\sqrt{\kappa}} \sqrt{1 - \tilde{X}_t^2} \ d\tilde{B}_{t,i} + \left( (p-q) - (p+q)\tilde{X}_{t,i} + 2 \sum_{j \neq i} \frac{1 - \tilde{X}_{t,i} \tilde{X}_{t,j}}{\tilde{X}_{t,i} - \tilde{X}_{t,j}} \right) dt \tag{3.3}
\]
for \(i = 1, \ldots, N\). The generator of the diffusion semigroup associated with \((\tilde{X}_t)_{t \geq 0}\) is the operator \(L_k := \frac{1}{\kappa} L_k\).

We now derive some results for symmetric polynomials of \((\tilde{X}_t)_{t \geq 0}\) and \((\tilde{X}_t)_{t \geq 0}\). For this we recapitulate that the elementary symmetric polynomials \(e_n^m\) in \(m\) variables for \(n = 0, \ldots, m\) are characterized by
\[
\prod_{j=1}^m (z - x_j) = \sum_{j=0}^m (-1)^{m-j} e_{m-j}^n(x) z^j \quad (z \in \mathbb{C}, \ x = (x_1, \ldots, x_m)). \tag{3.4}
\]
In particular, \(e_0^m = 1\), \(e_1^m(x) = \sum_{j=1}^m x_j, \ldots, e_m^m(x) = \prod_{j=1}^m x_j\).

We need a further notation: For a non-empty set \(S \subset \{1, \ldots, N\}\), let \(\tilde{X}^S\) be the \(\mathbb{R}^{|S|}\)-valued variable with the coordinates \(\tilde{X}_{t,i}\) for \(i \in S\) in the natural ordering on \(S \subset \{1, \ldots, N\}\). We need the following technical observation:

**Lemma 3.1.** For all \(r \in \mathbb{R}\), \(n = 0, 1, \ldots, N\), \(\kappa \geq 1\) and \(p, q \geq N - 1 + 2/\kappa\),
\[
d(e^{rt} \cdot e_n^N(\tilde{X}_t)) = \frac{\sqrt{2} \cdot e^{rt}}{\sqrt{\kappa}} \sum_{j=1}^N \sqrt{1 - \tilde{X}^2_{t,j} \cdot e_{n-1}^{N-1}(\tilde{X}_t^{\{1, \ldots, N\}\setminus\{j\}})} \ d\tilde{B}_{t,j} + e^{rt} \left( (r-n(p+q-n+1))e_n^N(\tilde{X}_t) + (p-q)(N-n+1)e_n^N(\tilde{X}_t) \right.
\]
\[
- (N-n+2)(N-n+1)e_{n-2}^N(\tilde{X}_t) \right) dt
\]
where, for \(n = 0, 1\), we assume that \(e_{-2} \equiv e_{-1} \equiv 0\).

**Proof.** We are in the situation where the process does not meet the boundary. Itô's formula and the SDE (3.3) show that
\[
d(e^{rt} \cdot e_n^N(\tilde{X}_t)) = r \cdot e^{rt} \cdot e_n^N(\tilde{X}_t) \ dt + e^{rt} \sum_{j=1}^N e_{n-1}^{N-1}(\tilde{X}_t^{\{1, \ldots, N\}\setminus\{j\}}) \ d\tilde{X}_{t,j}.
\]
Therefore, by the second line of the SDE (3.3), and with
\[
dM_t := \frac{\sqrt{2} \cdot e^{rt}}{\sqrt{\kappa}} \sum_{j=1}^N \sqrt{1 - \tilde{X}_t^2} \cdot e_{n-1}^{N-1}(\tilde{X}_t^{\{1, \ldots, N\}\setminus\{j\}}) \ d\tilde{B}_{t,j},
\]
\[
d(e^{rt} \cdot e_n^N(\tilde{X}_t)) = r \cdot e^{rt} e_n^N(\tilde{X}_t) \ dt + dM_t \tag{3.5}
\]
\[
+ e^{rt} \left( (p-q) \sum_{j=1}^N e_{n-1}^{N-1}(\tilde{X}_t^{\{1, \ldots, N\}\setminus\{j\}}) - (p+q) \sum_{j=1}^N e_{n-1}^{N-1}(\tilde{X}_t^{\{1, \ldots, N\}\setminus\{j\}}) \right) \cdot \tilde{X}_{t,j}
\]
\[
+ 2 \sum_{i,j; i \neq j} \frac{1 - \tilde{X}_{t,j} \tilde{X}_{t,i}}{\tilde{X}_{t,i} - \tilde{X}_{t,j}} e_{n-1}^{N-1}(\tilde{X}_t^{\{1, \ldots, N\}\setminus\{j\}}) \ dt.
\]
Simple combinatorial computations for \(i \neq j\) (cf. (2.10), (2.11) in [VW]) yield
\[
\sum_{j=1}^{N} e_{n-1}^{N-1}(\tilde{X}_{t}^{1,...,N}(\{j\})) = (N - n + 1) \cdot e_{n-1}^{N}(\tilde{X}_{t}),\tag{3.6}
\]
\[
\sum_{j=1}^{N} e_{n-1}^{N-1}(\tilde{X}_{t}^{1,...,N}(\{j\})) \tilde{X}_{t,j} = n \cdot e_{n}^{N}(\tilde{X}_{t})
\]
as well as
\[
e_{n-1}^{N-1}(\tilde{X}_{t}^{1,...,N}(\{j\}) - e_{n-1}^{N-1}(\tilde{X}_{t}^{1,...,N}(\{i\})) = (\tilde{X}_{t,i} - \tilde{X}_{t,j})e_{n-2}^{N-2}(\tilde{X}_{t}^{1,...,N}(\{i,j\}))\tag{3.7}
\]
and
\[
\sum_{i,j=1,...,N; i \neq j} e_{n-2}^{N-2}(\tilde{X}_{t}^{1,...,N}(\{i,j\})) = (N - n + 2)(N - n + 1)e_{n-2}^{N}(	ilde{X}_{t}).\tag{3.8}
\]

\[\text{(3.6)-(3.8)}\text{ now imply}
\]
\[
d(e^{t \cdot} \cdot e_{n}^{N}(	ilde{X}_{t})) = dM_{t} + e^{t \cdot} \left( r \cdot e_{n}^{N}(	ilde{X}_{t}) + (p - q)(N - n + 1)e_{n-1}^{N}(	ilde{X}_{t})
\]
\[- n(p + q)e_{n}^{N}(	ilde{X}_{t}) - (N - n + 2)(N - n + 1)e_{n-2}^{N}(	ilde{X}_{t}) + n(n - 1)e_{n}^{N}(	ilde{X}_{t}) \right) dt.
\]

This leads to the lemma for \(n \geq 2\). An inspection of the proof shows that all formulas are also valid for \(n = 0, 1\) with the convention of the lemma. \(\square\)

Lemma \(\text{3.1}\) leads to the following martingales w.r.t. the canonical filtration of the Brownian motion \((\tilde{B}_{t})_{t\geq0}\).

**Proposition 3.2.** Let \(n \in \{1, ..., N\}, \kappa > 0\) and \(p, q > N - 1 + 1/\kappa\). Put
\[
r_{n} := n(p + q - n + 1).
\]
Then there exist coefficients \(c_{n,l} \in \mathbb{R}\) for \(l = 0, ..., n - 1\) such that for all starting points \(x_{0}\) in the interior of \(C_{n,l}^{A}\) and the Jacobi process \((\tilde{X}_{t})_{t\geq0}\) with parameters \(\kappa, p, q\), the process
\[
\left( e^{r_{n} \cdot} \cdot \left( e_{n}^{N}(	ilde{X}_{t}) + \sum_{i=1}^{n} c_{n,l} \cdot e_{n-1}^{N}(\tilde{X}_{t}) \right) \right)_{t\geq0}\tag{3.9}
\]
is a martingale. The \(c_{n,l}\) and \(r_{n}\) do not depend on \(\kappa\).

**Proof.** We first assume that \(\kappa \geq 1\) and \(p, q \geq N - 1 + 2/\kappa\) as in Lemma \(\text{3.1}\). Lemma \(\text{3.1}\) here yields that for \(l = 0, ..., n - 1\), the processes \((e^{r_{n} \cdot} \cdot e_{n-1}^{N}(\tilde{X}_{t}))_{t\geq0}\) are linear combinations of the processes \(I_{n-l-j} := (\int_{0}^{t} e^{r_{n} \cdot} \cdot e_{n-l-j}^{N}(\tilde{X}_{s}) ds)_{t\geq0}\) for \(j = 0, 1, 2\) up to the addition of some integrals w.r.t. \((\tilde{B}_{t})_{t\geq0}\). As the integrands of these Brownian integrals are bounded, these Brownian integrals are obviously martingales.

Let us now consider all linear combinations of the processes \(I_{n-l-j}\) for \(l \geq 0\) and \(j = 0, 1, 2\). The definition of \(r_{n}\) and Lemma \(\text{3.1}\) ensure that the summand \(I_{n}\) does not appear. Moreover, if we put
\[
c_{n,1} := \frac{(p - q)(N - n + 1)}{r_{n} - r_{n-1}} = \frac{(p - q)(N - n + 1)}{p + q - 2n + 2},\tag{3.10}
\]
we see that the summand $I_{n-1}$ also does not appear. If we now define
\[
c_{n,l} := \frac{(p-q)(N-n+l)c_{n,l-1} - (N-n+l)(N-n+l+1)c_{n,l-2}}{r_n - r_{n-l}}
\]

(3.11)
ge for $l = 2, \ldots, n$ with $c_{n,0} = 1$ in an recursive way, we obtain from Lemma 3.1
that the summands $I_{n-1}$ also do not appear. In summary, we conclude that the
proposition holds for $\kappa \geq 1$ and $p > q \geq N - 1 + 2/\kappa$.

We now use Dynkin’s formula (see e.g. Section III.10 of [RW]) which implies
that the symmetric functions
\[
f_{N,n} : \mathbb{A}_N \times [0, \infty] \rightarrow \mathbb{R}, \quad (x,t) \mapsto e^{r_nt} \cdot \left( e^N_n(x) + \sum_{l=1}^n c_{n,l} \cdot e^N_{n-l}(x) \right)
\]
are space-time-harmonic w.r.t. the generators of the renormalized Jacobi processes
$(\tilde{X}_t)_{t \geq 0}$ for $\kappa \geq 1$ and $p > q \geq N - 1 + 2/\kappa$, i.e.,
\[
\left( \frac{\partial}{\partial t} + \tilde{L}_k \right) f_{N,n} \equiv 0
\]
(3.12)
for the parameters related via (3.2). As the left hand side of (3.12) is analytic
in $p,q,\kappa$, analytic continuation shows that $f_{N,n}$ is space-time-harmonic also for all
$\kappa > 0$ and $p > q > N - 1 + 1/\kappa$ with the corresponding coefficients $c_{n,l} = c_{n,l}(p,q,N)$
via (3.10) and (3.11). Dynkin’s formula now yields the proposition in general. □

The independence of $r_n$ and $c_{n,l}$ from $\kappa$ in Proposition 3.2 is not surprising by the
space-time-harmonicity argument. In fact, $\kappa$ appears in the differential operator in
(3.12) only as a constant factor in the classical Laplace operator $\Delta$. As $\Delta e^N_j \equiv 0$
for all $j$, the independence of $\kappa$ is obvious.

**Remark 3.3.** The recurrence formulas (3.10) and (3.11) for the $c_{n,l}$ can be simplified
slightly; we however do not have a closed formula for $c_{n,l}$ except for $p = q$.

In the case $p = q$ we have $c_{n,l} = 0$ for $l$ odd, and
\[
c_{n,2l} = \frac{(-1)^l(N-n+2l)}{l! \cdot 2^l \cdot (p+q-2n+3)(p+q-2n+5) \cdots (p+q-2n+2l+1)}
\]
(3.13)
for $l = 1, \ldots, \lfloor n/2 \rfloor$. This follows easily from (3.10) and (3.11).

We return to Lemma 3.1 and Proposition 3.2. An inspection of the proofs shows that
both results are also valid for $\kappa = \infty$ in which case the SDE (3.3) is an ODE,
and the process $(\tilde{X}_t)_{t \geq 0}$ is deterministic whenever so is the initial condition for
$t = 0$. There are several limit theorems (laws of large numbers, CLTs) for the limit transition $\kappa \rightarrow \infty$; see [HV]. In particular, Proposition 3.2 for $\kappa \in [0, \infty]$ leads to:

**Corollary 3.4.** For any starting point $x_0$ in the interior of $\mathbb{A}_N$, let $(\tilde{X}_t)_{t \geq 0}$ be the
associated normalized Jacobi process with $\kappa \in [0, \infty]$ and $p,q > N - 1 + 1/\kappa$. Then
there are constants $a_{n,l} \in \mathbb{R}$ for $0 \leq l \leq n \leq N$ such that
\[
E(e^N_n(\tilde{X}_t)) = \sum_{l=0}^n a_{n,l} e^{-r_l t}
\]
with $r_0 = 0$ where the coefficients $a_{n,l}$ and the exponents $r_l$ do not depend on $\kappa$. 

Proof. Proposition 3.2 shows that
\[ E(e_n^N(\tilde{X}_t)) = e^{-rt} \left( e_n^N(x_0) + \sum_{i=0}^{n-1} c_{n,i} e_i^N(x_0) \right) - \sum_{i=0}^{n-1} c_{n,i} E(e_i^N(\tilde{X}_t)). \]
As \( a_{n,i} := e_n^N(x_0) + \sum_{i=0}^{n-1} c_{n,i} e_i^N(x_0) \) is independent of \( \kappa \), the corollary follows by induction. \( \square \)

4. Results for a special starting point

We now choose special starting points. For this we use the ordered zeros of special Jacobi polynomials \( P_N^{(\alpha,\beta)} \) as introduced in Section 2. We need the following characterization of the ordered zeros \( z_1 \leq \ldots \leq z_N \) of \( P_N^{(\alpha,\beta)} \) due to Stieltjes, which is presented in [Sz] as Theorem 6.7.1:

**Lemma 4.1.** Let \((x_1, \ldots, x_N) \in A_N\). Then \((x_1, \ldots, x_N) = (z_1, \ldots, z_N) =: z\) if and only if for all \( j = 1, \ldots, N\),
\[ \sum_{i=1, i \neq j}^{N} \frac{1}{x_j - x_i} + \frac{\alpha + 1}{2} \frac{1}{x_j - 1} + \frac{\beta + 1}{2} \frac{1}{x_j + 1} = 0. \] (4.1)

We now return to the normalized Jacobi processes \((\tilde{X}_t)_{t \geq 0}\) with parameters \( p, q, \kappa \). We write the drift parts in the SDEs (3.3) as
\[ (p-q) - (p+q)\tilde{X}_{t,i} + 2 \sum_{j: j \neq i}^{N} \frac{1 - \tilde{X}_{t,i} \tilde{X}_{t,j}}{\tilde{X}_{t,i} - \tilde{X}_{t,j}} \]
\[ = (p-q) + 2(N-1) - (p+q)\tilde{X}_{t,i} + 2(1 - \tilde{X}_{t,i}^2) \sum_{j: j \neq i}^{N} \frac{1}{X_{t,i} - X_{t,j}} \]
\[ = 2(1 - \tilde{X}_{t,i}^2) \cdot \left( \frac{p-(N-1)}{2} X_{t,i} + 1 + \frac{q-(N-1)}{2} \frac{1}{X_{t,i} - 1} + \sum_{j: j \neq i}^{N} \frac{1}{X_{t,i} - X_{t,j}} \right) \] (4.2)
for \( i = 1, \ldots, N\). We now compare (4.2) with (4.1) and obtain:

**Corollary 4.2.** Let \( p, q > N - 1 \), and put
\[ \alpha := q - N > -1, \quad \beta := p - N > -1. \] (4.3)
Then, for \( \kappa = \infty \), the SDE (3.3) is an ODE which has the vector \( z \in A_N \) of the preceding lemma as unique constant solution.

We now combine this with Corollary 3.2 and obtain:

**Corollary 4.3.** Let \( \kappa \in [0, \infty] \), \( p, q > N - 1 \), and take the vector \( z \in A_N \) of the preceding lemma as starting point. Let \((\tilde{X}_t)_{t \geq 0}\) be the associated normalized Jacobi process. Then, for all \( n = 0, 1, \ldots, N \) and \( t \geq 0 \),
\[ E(e_n^N(\tilde{X}_t)) = e_n^N(z). \]
In particular, this expectation does not depend on \( t \) and \( \kappa \).

Proof. This is clear for \( \kappa = \infty \) by Corollary 4.2. As \( E(e_n^N(\tilde{X}_t)) \) is independent of \( \kappa \in [0, \infty] \) by Corollary 3.4, the result is clear. \( \square \)

We immediately obtain:
Corollary 4.4. Let \( \kappa \in [0, \infty[ , p, q > N - 1, \) and \( z \in A_N \) as above. Then for the associated Jacobi process \( (X_t)_{t \geq 0} \) starting in \( z \), and all \( n = 0, 1, \ldots, N \) and \( t \geq 0 \),
\[
\mathbb{E}(e_n^N(X_t)) = e_n^N(z).
\]

As an application, we get the following result:

Theorem 4.5. Let \( \kappa \in [0, \infty[ , p, q > N - 1, \) and \( \alpha, \beta \) as well as \( z \in A_N \) as above. Let \( (X_t)_{t \geq 0} \) be the associated Jacobi process starting in \( z \). Then, for all \( t \geq 0 \),
\[
\mathbb{E}(\prod_{i=1}^{N}(y - X_{t,i})) = \frac{1}{l_N^{(\alpha,\beta)}} \cdot P_N^{(\alpha,\beta)}(y) \quad \text{for } y \in \mathbb{R} \tag{4.4}
\]
with the leading coefficient \( l_N^{(\alpha,\beta)} \) of \( P_N^{(\alpha,\beta)} \). Moreover, for \( n = 0, 1, \ldots, N \),
\[
\mathbb{E}(e_n^N(X_t)) = e_n^N(z) = \frac{2^N}{(2N+\alpha+\beta)_N^2} \sum_{l=0}^{N} (-1)^{N-l} \binom{N}{l} \binom{l}{N-n} \frac{(N+\alpha+\beta+1)(\alpha+l+1)_{N-l}}{N!(\alpha+1)_n} \tag{4.5}
\]

Proof. Corollary 4.4 shows that
\[
\mathbb{E}(\prod_{i=1}^{N}(y - X_{t,i})) = \frac{N}{n=0} (-1)^n \mathbb{E}(e_n^N(X_t)) \cdot y^{N-n} = \sum_{n=0}^{N} (-1)^n e_n^N(z) \cdot y^{N-n} \tag{4.6}
\]
This proves the first statement. We now use \( l_N^{(\alpha,\beta)} = 2^{-N} (2N+\alpha+\beta) \) (see (4.21.6) of [Sz]) and compare the coefficients in (4.6) and (2.11). This in combination with the binomial formulas easily leads to the second statement. \( \blacksquare \)

Remark 4.6. For \( p = q, \) i.e., \( \alpha = \beta, \) Eq. (4.5) can be written in a simpler way by using the \( 2F1 \)-representation (4.7.30) of [Sz] of the Jacobi polynomials in this case.
In fact, a straightforward computation here implies the following:
If \( N = 2R \) is even, then \( \mathbb{E}(e_n^N(X_t)) = e_n^N(z) = 0 \) for \( k \) odd, and, for \( n = 0, \ldots, R, \)
\[
\mathbb{E}(e_n^N(X_t)) = e_{2n}^N(z) = (-1)^n \frac{R! \cdot n!}{(R-n)!} \cdot \frac{(2R+\alpha+1/2-n)_n}{(1/2+R-n)_n}.
\]
Moreover, for \( N = 2R + 1 \) odd, we have \( \mathbb{E}(e_n^N(X_t)) = e_n^N(z) = 0 \) for \( k \) even, and, for \( n = 0, \ldots, R-1, \)
\[
\mathbb{E}(e_{2n+1}^N(X_t)) = e_{2n+1}^N(z) = (-1)^n \frac{R! \cdot n!}{(R-n)!} \cdot \frac{(2R+\alpha+3/2-n)_n}{(3/2+R-n)_n}.
\]

We now apply Theorem 4.5 for \( t \to \infty \) in order to get a corresponding result for \( \beta \)-Jacobi ensembles:

Corollary 4.7. Let \( k_1, k_2, k_3 \in \mathbb{R} \) with \( k_3 > 0, \) \( k_2 > -1/2, \) and \( k_1 + k_2 > -1/2. \) Let \( X \) be an \( A_N \)-valued random variable with Lebesgue density
\[
c_{k_1, k_2, k_3} \prod_{i=1}^{N} (1-x_1)^{k_1+k_2-1/2} (1-x_i)^{k_2-1/2} \cdot \prod_{1 \leq i < j \leq N} |x_i - x_j|^{2k_3} \tag{4.7}
\]
Then
\[ E \prod_{i=1}^{N}(y - X_i) = \frac{1}{l_N^{(\alpha, \beta)}} \cdot P_N^{(\alpha, \beta)}(y) \quad \text{for} \quad y \in \mathbb{R} \] (4.8)
with the Jacobi polynomial \( P_N^{(\alpha, \beta)} \) with leading coefficient \( l_N^{(\alpha, \beta)} \) and with
\[ \alpha = \frac{1 + 2k_1 + 2k_2}{2k_3} - 1 > -1, \quad \beta = \frac{1 + 2k_2}{2k_3} - 1 > -1. \]

Moreover, for \( n = 0, 1, \ldots, N, \)
\[ E(e_n^N(X)) = \] (4.9)
\[ = \frac{2^N}{(2N + \alpha + \beta)^N} \cdot \sum_{l=N-n}^{N} (-1)^{N-l} \binom{N}{l} \left( \frac{l}{N} \right) \binom{N + \alpha + \beta + 1}{l} \frac{(\alpha + l + 1)_{N-l}}{N!}. \]

**Proof.** By the parameter transform (3.2), we have \( p, q > N - 1. \) Moreover, (3.2) and (4.3) lead to the formula for \( \alpha, \beta \) in the corollary. We now consider the associated Jacobi processes \((X_t)_{t \geq 0}\) as in Theorem 4.5. It follows from [RR1] (see in particular Proposition 3.4 there) that the \( X_t \) tend for \( t \to \infty \) to \( X \) in distribution. Therefore, as \( A_N \) is compact, the corollary follows from Theorem 4.5. \( \square \)

**Example 4.8.** Let \( N = 1. \) Here \( k_3 \) is irrelevant, and we have from (2.11) that
\[ \frac{1}{l_N^{(\alpha, \beta)}} \cdot P_N^{(\alpha, \beta)}(y) = y + \frac{\alpha - \beta}{\alpha + \beta + 2} = y + \frac{k_1}{k_1 + 2k_2 + 1}. \]
Eq. (4.8) can be checked here by via classical beta-integrals.

Corollary 4.7 is equivalent to Aomoto’s Selberg integral [A] which involves additional elementary symmetric polynomials in classical Selberg integrals. These formulas admit even further generalizations like Kadell’s Selberg integral [Ka] where Jack polynomials \( C^{(1/k_3)}_{\lambda} \) instead of elementary symmetric polynomials are used in Selberg integrals.

**Remark 4.9.** If we put \( k_1 = 0 \) and use the tranform \( x_i \mapsto x_i/\sqrt{k_3} \) \((i = 1, \ldots, N), \) then Corollary 4.7 leads for \( k_2 \to \infty \) to a corresponding result for \( \beta \)-Hermite ensembles; see e.g. [FG].

Moreover, if we use the tranform \( x_i \mapsto \frac{\alpha}{2}(x_i + 1) \) \((i = 1, \ldots, N) \) with \( \alpha \) as in Corollary 4.7, then Corollary 4.7 leads for \( k_1 \to \infty \) to a corresponding result for \( \beta \)-Laguerre ensembles; see e.g. [FG].

Corresponding limits are also possible on the level of the diffusions above, and one obtains the results in [KVV] for multivariate Bessel processes associated with the root systems A and B. For these Bessel processes we refer to [CDGRVY] and references there.

5. **An algebraic explanation of some of the preceding results**

We finally discuss Proposition 3.2 from an algebraic point of view. For this we consider the Heckman-Opdam Jacobi polynomials \( \tilde{R}_{\lambda} \) from Section 2.2 for partitions \( \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{Z}_+ \) with \( \lambda_1 \geq \ldots \geq \lambda_N. \) It follows from (2.9) that \( \tilde{R}_{\lambda} \) is an eigenfunction of \( L_k \) with eigenvalue
\[ r(\lambda) := -\langle \lambda, \lambda + 2\rho(k) \rangle = -\sum_{i=1}^{N} \lambda_i (\lambda_i + k_1 + 2k_2 + 2k_3(N - i)). \] (5.1)
Moreover, the $\tilde{R}_\lambda$ form an orthogonal basis of $L^2(A_N, w_k)$.

On the other hand, Proposition 3.2 and its proof with the comments about space-time harmonic functions imply that for $n = 1, \ldots, N$ the polynomials

$$q_n(x) := e_n(x) + \sum_{i=1}^n c_{n,i} \cdot e_{n-i}(x)$$

(5.2)

with the coefficients $c_{n,i}$ from (3.9) are eigenfunctions of $\tilde{L}_k$ and thus of $L_k$. In particular, $q_n$ is an eigenfunction of $L_k$ with eigenvalue

$$-k_3n(p + q - n + 1) = -n(1 + k_1 + 2k_2 + (2N - n - 1)k_3).$$

This is equal to $r(\lambda(n))$ in (5.1) for the partition $\lambda(n) := (1, \ldots, 1, 0, \ldots, 0)$ where 1 appears $n$-times. Therefore, the following result is quite natural.

**Lemma 5.1.** Let $k_1, k_2, k_3 \in \mathbb{R}$ with $k_3 \geq 0$, $k_2 > -1/2$, and $k_1 + k_2 > -1/2$. Then for each $n = 1, \ldots, N$, the polynomials $\tilde{R}_\lambda(n)$ and $q_n$ are equal up to a multiplicative constant. In particular, $\tilde{R}_\lambda(n)$ is independent of $k_3$.

**Proof.** Let $n = 1, \ldots, N$. Then the statement is clear when the eigenvalue $r(\lambda(n))$ has multiplicity 1.

Assume now that $r(\lambda(n))$ has multiplicity $\geq 2$, i.e., there exists a partition $\lambda \neq \lambda(n)$ with

$$\langle \lambda, \lambda + 2\rho(k) \rangle = \langle \lambda(n), \lambda(n) + 2\rho(k) \rangle.$$ 

(5.3)

As $\rho(k)_1 \geq \ldots \geq \rho(k)_N$, a simple monotonicity argument shows that then $\lambda$ satisfies $\sum_{i=1}^N \lambda_i \neq n$. On the other hand, if (5.3) holds for some $\lambda \neq \lambda(n)$ with $\sum_{i=1}^N \lambda_i \neq n$ and some $k_1, k_2, k_3$, then (5.3) fails to hold for any slightly modified parameter $k_1$ by (5.1). Therefore, $\tilde{R}_\lambda(n)$ and $q_n$ are equal up to a multiplicative constant for these modified $k_1$. An obvious continuity argument now shows that this equality also holds for the original $k_1$. This completes the proof. \qed

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