Regularity properties of nonlinear Schrödinger equations in vector-valued spaces
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Abstract
In this paper, regularity properties of Cauchy problem for linear and nonlinear abstract Schrödinger equations in vector-valued function spaces are obtained.

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1. Introduction, definitions

Consider the Cauchy problem for nonlinear abstract Schrödinger (NLAS) equations
\[i\partial_t u + \Delta u + Au + F(u) = 0, \quad x \in \mathbb{R}^n, \quad t \in [0,T],\]
\[u(0, x) = u_0(x),\quad \text{for a.e.} \quad x \in \mathbb{R}^n\]
where \(A\) is a linear and \(F\) is a nonlinear operators in a Banach space \(E\), \(\Delta\) denotes the Laplace operator in \(\mathbb{R}^n\) and \(u = u(t,x)\) is the \(E\)-valued unknown function. If \(F(u) = \lambda |u|^p u\) in (1.1) we get the nonlinear problem
\[i\partial_t u + \Delta u + Au + \lambda |u|^p u = 0, \quad x \in \mathbb{R}^n, \quad t \in [0,T],\]
\[u(0, x) = u_0(x),\quad \text{for a.e.} \quad x \in \mathbb{R}^n,\]
where \(p \in (1, \infty), \lambda\) is a real number,

By rescaling the values of \(u\) it is possible to restrict attention to the cases \(\lambda = 1\) or \(\lambda = -1\) these call as the focusing and defocusing abstract Schrödinger equations, respectively. The problem (1.1) also contain two critical case. These are the mass-critical abstract Schrödinger equation,
\[i\partial_t u + \Delta u + Au + \lambda |u|^4 u = 0, \quad x \in \mathbb{R}^n, \quad t \in [0,T],\]
which is associated with the conservation of mass,
\[M(u(t)) := \int_{\mathbb{R}^n} ||u(t,x)||^2_E dx\]
and the energy-critical abstract Schrödinger equation (in dimensions \(n > 2\)),
\[i\partial_t u + \Delta u + Au + \lambda |u|^{\frac{4}{n-2}} = 0, \quad x \in \mathbb{R}^n, \quad t \in [0,T],\]
which is associated with the conservation of energy,
\[
E (u (t)) := \int_{\mathbb{R}^n} \left[ \frac{1}{2} \| \nabla u (t, x) \|_E^2 + \lambda \left( \frac{1}{2} - \frac{1}{n} \right) \| u (t, x) \|_{2^*}^2 \right] dx.
\]

Let \( \mathbb{N} \) and \( \mathbb{C} \) denote the sets of all natural and complex numbers, respectively. For \( E = \mathbb{C} \) and \( A = 0 \) the problem (1.2) become the classical Cauchy problem for nonlinear Schrödinger (NLS) equations
\[
i \partial_t u + \Delta u + \lambda |u|^{p-1} u = 0, \quad x \in \mathbb{R}^n, \quad t \in [0, T],
\]
\[
u (0, x) = u_0 (x), \quad \text{for a.e. } x \in \mathbb{R}^n.
\]

The existence of solutions and regularity properties of Cauchy problem for NLS equations studied e.g. in [5−7], [9−10], [15−16], [26], [28] and the references therein. In contrast, to the mentioned above results we will study the regularity properties of the abstract Cauchy problem (1.1). Abstract differential equations studied e.g. in [1], [8], [11], [17], [19], [22−25] and [29]. Since the Banach space \( E \) is arbitrary and \( A \) is a possible linear operator, by choosing \( E \) and \( A \) we can obtain numerous classes of Schrödinger type equations and its systems which occur in a wide variety of physical systems. Our main goal is to obtain the existence, uniqueness and estimates of solution to the problem (1.1).

If we choose \( E \) a concrete space, for example \( E = \mathcal{L}^2 (\Omega) \), \( A = \mathcal{L} \), where \( \Omega \) is a domain in \( \mathbb{R}^m \) with sufficiently smooth boundary and \( \mathcal{L} \) is an elliptic operator in \( \mathcal{L}^2 (\Omega) \) in (1.2), then we obtain existence, uniqueness and the regularity properties of the mixed problem for linear Schrödinger equation
\[
i \partial_t u + \Delta u + L u = F (t, x) , \quad t \in [0, T], \quad x \in \mathbb{R}^n, \quad y \in \Omega,
\]
and the following NLS equation
\[
i \partial_t u + \Delta u + L u + \lambda |u|^{p-1} u = 0, \quad t \in [0, T], \quad x \in \mathbb{R}^n, \quad y \in \Omega,
\]
where \( u = u (t, x, y) \).

Moreover, let we choose \( E = \mathcal{L}^2 (0, 1) \) and \( A \) to be differential operator with generalized Wentzell-Robin boundary condition defined by
\[
D (A) = \left\{ u \in W^{2,2} (0, 1), \ B_j u = A u (j) + \sum_{i=0}^1 \alpha_{ij} u^{(i)} (j), \ j = 0, 1 \right\},
\]
\[
A u = a u^{(2)} + b u^{(1)} + cu,
\]
where \( \alpha_{ij} \) are complex numbers, \( a = a (y), b = b (y), c = c (y) \) are complex-valued functions. Then, from the main our theorem, we get the existence, uniqueness and regularity properties of Wentzell-Robin type mixed problem for the for linear Schrödinger equation
\[
i \partial_t u + \Delta u + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} + cu = F (t, x),
\]
\[ B_{j}u = Au(t,x,j) + \sum_{i=0}^{1} \alpha_{ij} u^{(i)}(t,x,j) = 0, \quad j = 0, 1. \tag{1.8} \]

\[ u(0,x,y) = u_0(x,y), \text{ for a.e. } x \in R^n, \quad y \in \Omega. \tag{1.9} \]

and for the following NLS equation,
\[ i\partial_t u + \Delta u + a \frac{\partial^2 u}{\partial y^2} + b \frac{\partial u}{\partial y} + cu + F(u) = 0, \tag{1.10} \]

where
\[ u = u(t,x,y), \quad t \in [0, T], \quad x \in R^n, \quad y \in (0, 1). \]

Note that, the regularity properties of Wentzell-Robin type BVP for elliptic equations were studied e.g. in [13, 14] and the references therein. Moreover, if put \( E = l_2 \) and choose \( A \) as a infinite matrix \([a_{mj}], m,j = 1, 2, ..., \infty\), then from our results we obtain the existence, uniqueness and regularity properties of Cauchy problem for the linear system of Schrödinger equation Consider at first, the Cauchy problem for infinity many system of linear Schrödinger equations

\[ i\partial_t u_m + \Delta u_m + \sum_{j=1}^{N} a_{mj} u_j = F_j(t,x), \quad t \in [0, T], \quad x \in R^n, \tag{1.11} \]

\[ u_m(0,x) = u_{m0}(x), \text{ for a.e. } x \in R^n, \]

and infinity many system of NLS equation

\[ i\partial_t u_m + \Delta u_m + \sum_{j=1}^{N} a_{mj} u_j + F(u_1, u_2, ... u_N) = 0, \quad t \in [0, T], \quad x \in R^n, \tag{1.12} \]

\[ u_m(0,x) = u_{m0}(x), \text{ for a.e. } x \in R^n, \]

where \( a_{mj} \) are complex numbers, \( u_j = u_j(t,x) \).

Let \( E \) be a Banach space. \( L^p(\Omega; E) \) denotes the space of strongly measurable \( E \)-valued functions that are defined on the measurable subset \( \Omega \subset R^n \) with the norm

\[ \|f\|_p = \|f\|_{L^p(\Omega; E)} = \left( \int_{\Omega} \|f(x)\|^p_E \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty. \]

For \( p = 2 \) and \( E = H \), where \( H \) is a Hilbert space, then \( L^p(\Omega; E) \) become \( L^2(\Omega; H) \) – the Hilbert space of \( H \)-valued functions with inner product:

\[ (f,g)_{L^2(\Omega; H)} = \int_{\Omega} (f(x), g(x))_H \, dx, \text{ for any } f, g \in L^2(\Omega; H). \]
Let $L^q_L^r((a,b) \times \Omega; E)$ denotes the space of strongly measurable $E$-valued functions that are defined on the measurable set $(a,b) \times \Omega$ with the norm

$$
\|f\|_{L^q_L^r((a,b) \times \Omega; E)} = \left( \int_a^b \left[ \int \|f(t,x)\|_E^r \, dx \right]^q \, dt \right)^{\frac{1}{r}} , \quad 1 \leq q, r < \infty .
$$

Let $C(\Omega; E)$ denote the space of $E$-valued, bounded strongly continuous functions on $\Omega$ with norm

$$
\|u\|_{C(\Omega; E)} = \sup_{x \in \Omega} \|u(x)\|_E .
$$

$C^m(\Omega; E)$ will denote the spaces of $E$-valued bounded strongly continuous and $m$-times continuously differentiable functions on $\Omega$ with norm

$$
\|u\|_{C^m(\Omega; E)} = \max_{0 \leq |\alpha| \leq m} \sup_{x \in \Omega} \|D^\alpha u(x)\|_E .
$$

Here, $O_R = \{x \in \mathbb{R}^n, \ |x| < R\}$ for $R > 0$.

Let $E_1$ and $E_2$ be two Banach spaces. $B(E_1, E_2)$ will denote the space of all bounded linear operators from $E_1$ to $E_2$. For $E_1 = E_2 = E$ it will be denoted by $B(E)$.

A closed densely defined linear operator $A$ is said to be absolute positive in a Banach space $E$ if every real $\lambda, \ |\lambda| > \omega$ is in the resolvent set $\rho(A)$ and for such $\lambda$

$$
\|R(\lambda, A)\|_{B(E)} \leq M (|\lambda| - \omega)^{-1} .
$$

**Remark 1.1.** It is known that if the operator $A$ is absolute positive in a Banach space $E$ and $0 \leq \alpha < 1$ then it is an infinitesimal generator of group of bounded linear operator $U_A(t)$ satisfying

$$
\|U_A(t)\|_{B(E)} \leq M e^{\omega |t|} , \quad t \in (-\infty, \infty) ,
$$

$$
\|A^\alpha U_A(t)\|_{B(E)} \leq M |t|^{-\alpha} , \quad t \in (-\infty, \infty) \quad (1.10)
$$

(see e.g. [19], [§ 1.6], Theorem 6.3).

Let $E$ be a Banach space. $S = S(\mathbb{R}^n; E)$ denotes $E$-valued Schwartz class, i.e. the space of all $E$-valued rapidly decreasing smooth functions on $\mathbb{R}^n$ equipped with its usual topology generated by seminorms. $S(\mathbb{R}^n; \mathbb{C})$ denoted by just $S$.

Let $S'(\mathbb{R}^n; E)$ denote the space of all continuous linear operators, $L: S \to E$, equipped with the bounded convergence topology. Recall $S(\mathbb{R}^n; E)$ is norm dense in $L^p(\mathbb{R}^n; E)$ when $1 < p < \infty$. 

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The Banach space $E$ is called a UMD-space and written as $E \in \text{UMD}$ if only if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is bounded in the space $L^p(R, E)$, $p \in (1, \infty)$ (see e.g. [3], [4]). UMD spaces include $L^p$, $l^p$ spaces, Lorentz spaces $L^{pq}$, $p, q \in (1, \infty)$.

Let $F$ denote the Fourier transformation, $\hat{u} = Fu$ and

$$s \in \mathbb{R}, \; \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n, \; |\xi|^2 = \sum_{k=1}^{n} \xi_k^2,$$

$$\langle \xi \rangle := \left(1 + |\xi|^2\right)^{\frac{1}{2}}.$$

Consider $E$-valued Sobolev space $W^{s,p}(\mathbb{R}^n; E)$ and homogeneous Sobolev spaces $\mathring{W}^{s,p}(\mathbb{R}^n; E)$ defined by respectively,

$$W^{s,p}(\mathbb{R}^n; E) = \{ u : u \in S'(\mathbb{R}^n; E), \; \|u\|_{W^{s,p}(\mathbb{R}^n; E)} = \left\| F^{-1} \left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{u} \right\|_{L^p(\mathbb{R}^n; \mathcal{H})} < \infty \},$$

$$\mathring{W}^{s,p}(\mathbb{R}^n; E) = \{ u : u \in S'(\mathbb{R}^n; E), \; \|u\|_{\mathring{W}^{s,p}(\mathbb{R}^n; E)} = \left\| F^{-1} |\xi|^s \hat{u} \right\|_{L^p(\mathbb{R}^n; \mathcal{H})} < \infty \}.$$

For $\Omega = \mathbb{R}^n \times G$, $p = (p_1, p_2)$, $s \in \mathbb{R}$ and $l \in \mathbb{N}$ we define the $E$-valued anisotropic Sobolev space $W^{s,l,p}(\Omega; E)$ by

$$W^{s,l,p}(\Omega; E) := \{ u \in S'(\Omega; E), \; \|u\|_{W^{s,l,p}(\Omega; E)} = \|u\|_{W^{s,p}(\Omega; E)} + \|u\|_{W^{l,p}(\Omega; E)} \},$$

where

$$\|u\|_{W^{s,p}(\Omega; E)} = \left\| F^{-1} \left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{u} \right\|_{L^p(\Omega; E)} < \infty,$$

$$\|u\|_{W^{l,p}(\Omega; E)} = \|u\|_{L^p(\Omega; E)} + \sum_{|\beta|=l} \|D^\beta u\|_{L^p(\Omega; E)}.$$

The similar way, we define homogeneous anisotropic Sobolev spaces $\mathring{W}^{s,l,p}(\Omega; E)$ as:

$$\mathring{W}^{s,l,p}(\Omega; E) := \{ u \in S'(\Omega; E), \; \|u\|_{\mathring{W}^{s,l,p}(\Omega; E)} = \|u\|_{W^{s,p}(\Omega; E)} + \|u\|_{\mathring{W}^{l,p}(\Omega; E)} \},$$

where

$$\|u\|_{\mathring{W}^{s,p}(\Omega; E)} = \left\| F^{-1} |\xi|^s \hat{u} \right\|_{L^p(\Omega; E)} < \infty.$$
Let $A$ be a linear operator in a Banach space $E$. Consider Sobolev-Lions type homogeneous and in homogeneous abstract spaces, respectively

$$\dot{W}^{s,p}(\mathbb{R}^n; E(A), E) = \dot{W}^{s,p}(\mathbb{R}^n; E) \cap L^p(\mathbb{R}^n; E(A)), $$

$$\|u\|_{\dot{W}^{s,p}(\mathbb{R}^n; E(A), E)} = \|u\|_{\dot{W}^{s,p}(\mathbb{R}^n; E)} + \|u\|_{L^p(\mathbb{R}^n; E(A))} < \infty, $$

$$W^{s,p}(\mathbb{R}^n; E(A), E) = W^{s,p}(\mathbb{R}^n; E) \cap L^p(\mathbb{R}^n; E(A)), $$

$$\|u\|_{W^{s,p}(\mathbb{R}^n; E(A), E)} = \|u\|_{W^{s,p}(\mathbb{R}^n; E)} + \|u\|_{L^p(\mathbb{R}^n; E(A))} < \infty. $$

Sometimes we use one and the same symbol $C$ without distinction in order to denote positive constants which may differ from each other even in a single context. When we want to specify the dependence of such a constant on a parameter, say $\alpha$, we write $C_\alpha$.

**Definition 1.1.** Consider the initial value problem (1.1) for $u_0 \in \dot{W}^{2,s}(\mathbb{R}^n; E)$. This problem is critical when $s = s_c := \frac{n}{2} - \frac{2}{p}$, subcritical when $s > s_c$, and supercritical when $s < s_c$.

**Definition 1.2.** (Solution). A function $[0, T] \times \mathbb{R}^n \to E$ is a (strong) solution to (1.1) if it lies in the class $C^0_t(\dot{W}^{2,s}_x(\mathbb{R}^n; E(A), E))$ and obeys the Duhamel formula

$$u(t) = U_{A+\Delta}(t) u_0 + \int_0^t U_{A+\Delta}(t-s) F(u(s)) \, ds \text{ for all } t \in (0, T),$$

where $U_{A+\Delta}(t)$ is a bounded group in $E$ generated by operator $i(\Delta + A)$.

We write $a \lesssim b$ to indicate that $a \leq Cb$ for some constant $C$, which is permitted to depend on some parameters.

### 3. Dispersive and Strichartz inequalities for linear Schrödinger equation

It can be shown that fundamental solution of the free abstract Schrödinger equation:

$$i \partial_t u + \Delta u + Au = 0, \quad t \in [0, T], \quad x \in \mathbb{R}^n$$

can be expressed as

$$U_{A+\Delta}(t) (x, y) = U_A(t) U_\Delta(t) (x, y),$$

$U_A(t)$ is a group generated by $iA$ and $U_\Delta(t) (x, y) = e^{i\Delta t} (x, y)$ is a fundamental solution of the free Schrödinger equation:

$$i \partial_t u + \Delta u = 0, \quad x \in \mathbb{R}^n, \quad t \in [0, T],$$

i.e.

$$U_\Delta(t) (x, y) = (4\pi it)^{-\frac{n}{2}} e^{i|x-y|^2/4t}, \quad t \neq 0,$$
\[ U_{\Delta}(t) f(x) = (2\pi it)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(x-y)^2/2} f(y) \, dy. \]

**Lemma 3.1.** Let \( A \) be absolute positive in a Banach space \( E \) and \( 0 \leq \alpha < 1 \). Then the following dispersive inequalities hold

\[
\| A^{\alpha} U_{\Delta + A}(t) f \|_{L^p_t(R^n; E)} \lesssim t^{-n\left(\frac{1}{2} - \frac{1}{p}\right) + \alpha} \| f \|_{L^p_t(R^n; E)},
\]

\[
\| A^{\alpha} U_{\Delta + A}(t-s) f \|_{L^\infty_t(R^n; E)} \lesssim |t-s|^{-\left(\frac{n}{2} + \alpha\right)} \| f \|_{L^1_t(R^n; E)}
\]

for \( t \neq 0, \ 2 \leq p \leq \infty, \ \frac{n}{p} + \frac{n}{2} = 1. \)

**Proof.** By using (3.3) and Young’s integral inequality we have

\[
\| U_{\Delta}(t) f \|_{L^p_t(R^n; E)} \lesssim |t|^{-n\left(\frac{1}{2} - \frac{1}{p}\right)} \| f \|_{L^p_t(R^n; E)},
\]

\[
\| U_{\Delta}(t) f \|_{L^\infty_t(R^n; E)} \lesssim |t|^{-\frac{n}{2}} \| f \|_{L^1_t(R^n; E)}.
\]

By (1.10) we get

\[
\| A^{\alpha} U_A(t) \|_{B(E)} \lesssim |t|^{-\alpha}, \ \ t \neq 0.
\]

By using then the properties of \( U_{\Delta + A}(t) = U_{\Delta}(t) U_A(t) \), the estimates (3.7) and (3.6) we obtain (3.4) and (3.5).

**Condition 3.1.** Assume \( n \geq 1, \)

\[
\frac{2}{q} + \frac{n}{r} \leq \frac{n}{2}; \quad 2 \leq q, r \leq \infty \quad \text{and} \quad (n, q, r) \neq (2, 2, \infty).
\]

**Remark 3.1.** If \( \frac{2}{q} + \frac{n}{r} = \frac{n}{2} \), then \((q, r)\) is called sharp admissible, otherwise \((q, r)\) is called nonsharp admissible. Note in particular that when \( n > 2 \) the endpoint \( \left( \frac{2n}{n-2}, \frac{2n}{n-2} \right) \) is called sharp admissible.

For a space-time slab \([0, T] \times R^n\), we define the \( E \)-valued Strichartz norm

\[
\| u \|_{S^0(I; E)} = \sup_{(q,r) \ \text{admissible}} \| u \|_{L^q_t L^r_x(I \times R^n; E)},
\]

where \( S^0 ([0, T]; E) \) is the closure of all \( E \)-valued test functions under this norm and \( N^0 ([0, T]; E) \) denotes the dual of \( S^0 ([0, T]; E) \).

Assume \( H \) is an abstract Hilbert space and \( Q \) is a Hilbert space of function. Suppose for each \( t \in \mathbb{R} \) an operator \( U(t) : Q \to L^2(\Omega; E) \) obeys the following estimates:

\[
\| U(t) f \|_{L^2_t(\Omega; H)} \lesssim \| f \|_Q \quad (3.7)
\]

for all \( t, \ \Omega \subset R^n \) and all \( f \in Q; \)

\[
\| U(s) U^*(t) g \|_{L^\infty_t(\Omega; H)} \lesssim |t-s|^{-\frac{n}{2}} \| g \|_{L^2_t(\Omega; H)} \quad (3.8)
\]
\[ \| U(s) U^*(t) g \|_{L^p_x(\Omega; H)} \lesssim \left( 1 + |t-s|^{-\frac{1}{2}} \right) \| g \|_{L^p_x(\Omega; H)} \]  
for all \( t \neq s \) and all \( g \in L^p_x(\Omega; H) \).

For proving the main theorem of this section, we will use the following bilinear interpolation result (see [2], Section 3.13.5(b)).

**Lemma 3.2.** Assume \( A_0, A_1, B_0, B_1, C_0, C_1 \) are Banach spaces and \( T \) is a bilinear operator bounded from \((A_0 \times B_0, A_0 \times B_1, A_1 \times B_0)\) into \((C_0, C_1, C_1)\), respectively. Then whenever \( 0 < \theta_0, \theta_1 < \theta < 1 \) are such that \( 1 \leq \frac{1}{\theta} + \frac{1}{\theta'} \) and \( \theta = \theta_0 + \theta_1 \), the operator is bounded from

\[ (A_0, A_1)_{\theta, \theta'} \to (B_0, B_1)_{\theta_0, \theta_1} \]

to \((C_0, C_1)_{\theta_r}\).

By following [15, Theorem 1.2] we have:

**Theorem 3.1.** Assume \( U(t) \) obeys (3.8) and (3.9). Let \( U(t) \) generates absolute positive infinitesimal generator operator \( A \) and \( 0 < \alpha < 1 \). Then the following estimates are hold

\[ \| U(t) f \|_{L^q_t L^r_x(H)} \lesssim \| f \|_Q, \]  
\[ \left\| \int U^*(s) F(s) ds \right\|_Q \lesssim \| F \|_{L^q_t L^r_x(E^*)}, \]  
\[ \int_{s<t} \| A^\alpha U(t) U^*(s) F(s) ds \|_{L^q_t L^r_x(H)} \lesssim \| F \|_{L^q_t L^r_x(H)} \]  
for all sharp admissible exponent pairs \((q, r), (\tilde{q}, \tilde{r})\). Furthermore, if the decay hypothesis is strengthened to (3.9), then (3.10), (3.11) and (3.12) hold for all admissible \((q, r), (\tilde{q}, \tilde{r})\).

**Proof. The first step:** Consider the nonendpoint case, i.e. \((q, r) \neq (2, \frac{2}{\alpha-1})\) and will show firstly, the estimates (3.10), (3.11). By duality, (3.10) is equivalent to (3.11). By the TT* method, (3.11) is in turn equivalent to the bilinear form estimate

\[ \left| \int \int \left( (A^\alpha U(s))^* F(s), (A^\alpha U(t))^* G(t) \right) dsdt \right| \lesssim \| F \|_{L^q_t L^r_x(H)} \| G \|_{L^q_t L^r_x(H)}. \]  

(3.13)

By symmetry it suffices to show the to the retarded version of (3.13)

\[ |T(F, G)| \lesssim \| F \|_{L^q_t L^r_x(E^*)} \| G \|_{L^q_t L^r_x(E^*)}, \]  

(3.14)

where \( T(F, G) \) is the bilinear form defined by

\[ T(F, G) = \int \int_{s<t} \left( (A^\alpha U(s))^* F(s), (A^\alpha U(t))^* G(t) \right) dsdt \]
By real interpolation between the bilinear form of (3.7) and due to estimate (1.10) we get
\[
\left| \langle (A^2 U(s))^* F(s), (A^2 U(t))^* G(t) \rangle \right| \lesssim \| F(s) \|_{L^2_x} \| G(t) \|_{L^2_x}.
\]
By using the bilinear form of (3.8) and (1.10) we have
\[
\left| \langle (A^2 U(s))^* F(s), (A^2 U(t))^* G(t) \rangle \right| \lesssim \left| t - s \right|^{-\frac{n}{2}} \| F(s) \|_{L^2_x(\Omega;H)} \| G(t) \|_{L^2_x(\Omega;H)}.
\]
In a similar way, we obtain
\[
\left| \langle (A^2 U(s))^* F(s), (A^2 U(t))^* G(t) \rangle \right| \lesssim \left| t - s \right|^{1 - \beta(r,r)} \| F(s) \|_{L^{q'}(\Omega;H)} \| G(t) \|_{L^{q'}(\Omega;H)},
\]
where \(\beta(r,\tilde{r})\) is given by
\[
\beta(r,\tilde{r}) = \frac{n}{2} - 1 - \frac{n}{2} \left( \frac{1}{r} - \frac{1}{\tilde{r}} \right).
\]
It is clear that \(\beta(r,r) \leq 0\) when \(n > 2\). In the sharp admissible case we have
\[
\frac{1}{q} + \frac{1}{q'} = -\beta(r,r),
\]
and (3.14) follows from (3.16) and the Hardy-Littlewood-Sobolev inequality ([20]) when \(q > q'\).

If we are assuming the truncated decay (3.9), then (3.16) can be improved to
\[
\left| \langle (A^2 U(s))^* F(s), (A^2 U(t))^* G(t) \rangle \right| \lesssim \left( 1 + |t - s| \right)^{1 - \beta(r,r)} \| F(s) \|_{L^{q'}(\Omega;H)} \| G(t) \|_{L^{q'}(\Omega;H)}
\]
and now Young’s inequality gives (3.14) when
\[
-\beta(r,r) + \frac{1}{q} > \frac{1}{q'},
\]
i.e. \((q, r)\) is nonsharp admissible. This concludes the proof of (3.10) and (3.11) for nonendpoint case.

The second step; It remains to prove (3.10) and (3.11) for the endpoint case, i.e. when
\[
(q, r) = \left( 2, 2n \frac{n}{n-2} \right), n > 2.
\]
It suffices to show (3.14). By decomposing $T(F, G)$ dyadically as $\sum_j T_j(F, G)$, where the summation is over the integers $\mathbb{Z}$ and

$$T_j(F, G) = \int_{t-2^{-j-1}<s\leq t-2^{-j}} \langle (A^\dagger U(s))^* F(s), (A^\dagger U(t))^* G(t) \rangle ds dt \quad (3.19)$$

we see that it suffices to prove the estimate

$$\sum_j |T_j(F, G)| \lesssim \|F\|_{L^2_x L^q(t)'} \|G\|_{L^2_x L^q(t)'}.$$  

(3.20)

For this aim, before we will show the following estimate

$$|T_j(F, G)| \lesssim 2^{-j\beta(a,b)} \|F\|_{L^2_x L^q(t)'} \|G\|_{L^2_x L^q(t)'} \quad (3.21)$$

for all $j \in \mathbb{Z}$ and all $(\frac{1}{q}, \frac{1}{p})$ in a neighbourhood of $(\frac{1}{2}, \frac{1}{2})$. For proving (3.21) we will use the real interpolation of $H$-valued Lebesgue space and sequence spaces $l^a_q(H)$ (see e.g. [27] § 1.18.2 and 1.18.6). Indeed, by [27, § 1.18.4.] we have

$$(L^2_x L^p_0(H), L^2_x L^p_1(H))_{\theta, 2} = L^2_x l^{p,2}_x(H) \quad (3.22)$$

whenever $p_0, p_1 \in [1, \infty], p_0 \neq p_1$ and $\frac{1}{p} = \frac{1-q}{p_0} + \frac{q}{p_1}$ and $(l^a_\infty(H), l^b_\infty(H))_{a, 1} = l^s_1(H)$ for $s_0, s_1 \in \mathbb{R}, s_0 \neq s_1$ and

$$\frac{1}{s} = \frac{1 - \theta}{s_0} + \frac{\theta}{s_1},$$

where

$$l^a_q(H) = \left\{ u = \{u_j\}_{j=1}^{\infty}, u_j \in E, \|u\|_{l^a_q(H)} = \left( \sum_{j=1}^{\infty} 2^{j\theta q} \|u_j\|^q_H \right)^{\frac{1}{q}} < \infty \right\}.$$ 

By (3.22) the estimate (3.21) can be rewritten as

$$T : L^2_x l^{s,2}_x(H) \times L^2_x l^{s,2}_x(H) \to l^a_\infty,$$  

(3.23)

where $T = \{T_j\}$ is the vector-valued bilinear operator corresponding to the $T_j$. We apply Lemma 3.2 to (3.23) with $r = 1, p = q = 2$ and arbitrary exponents $a_0, a_1, b_0, b_1$ such that

$$\beta(a_0, b_1) = \beta(a_1, b_0) \neq \beta(a_0, b_0).$$

Using the real interpolation space identities we obtain

$$T : L^2_x l^{a,2}_x(E^*) \times L^2_x l^{b,2}_x(E^*) \to l^a_1,$$

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for all \((a, b)\) in a neighbourhood of \((r, r)\). Applying this to \(a = b = r\) and using the fact that \(L^r (H) \subset L^{r', 2}(H)\) we obtain

\[
T : L^2 \mathcal{L}_x L^{r', 2}(H) \times L^2 \mathcal{L}_x L^{r', 2}(H) \to \mathcal{L}_1(H)
\]

which implies (3.21).

Consider the Cauchy problem for forced Schrodinger equation

\[
i \partial_t u + \Delta u + Au = F, \quad t \in [0, T], \quad x \in \mathbb{R}^n,
\]

(3.24)

\[
u(t_0, x) = u_0(x), \quad t_0 \in [0, T],
\]

where \(A\) is a linear operator in a Hilbert space \(H\).

We are now ready to state the standard Strichartz estimates:

**Theorem 3.2.** Assume the Conditions 3.1 is satisfied and suppose \(A\) is absolute positive in \(H\). Let \(0 \leq s \leq 1\), \(0 \leq \alpha < 1\), \(u_0 \in \mathcal{W}^s(\mathbb{R}^n; H(A^\alpha))\), \(F \in \mathcal{N}^0([0, T]; \mathcal{W}^{s, 2}(\mathbb{R}^n; H))\) and let \(u : [0, T] \times \mathbb{R}^n \to H\) be a solution to (3.24). Then

\[
\frac{2}{q} + \frac{n}{r} = \frac{2}{\tilde{q}} + \frac{n}{\tilde{r}} = \frac{n}{2}.
\]

If \(n = 2\), we also require that \(q, \tilde{q} > 2\). Consider first, the nonendpoint case. The linear operators in (3.26) and (3.27) are adjoints of one another; thus, by the method of \(TT^*\) both will follow once we prove

\[
\frac{\mathcal{L}_s \mathcal{L}_{s'}(H)}{\mathcal{L}_s \mathcal{L}_{t'}(H)} \leq \frac{\mathcal{L}_s \mathcal{L}_{t'}(H)}{\mathcal{L}_s \mathcal{L}_{t'}(H)}.
\]

(3.26)

Apply Theorem 3.1 with \(Q = \mathcal{L}_x^2(\mathbb{R}^n; H) = \mathcal{L}_x^2(H)\). The energy estimate (3.10):

\[
\|U_{\Delta + A}(t) f\|_{\mathcal{L}_x^2(H)} \leq \|f\|_{\mathcal{L}_x^2(H)}
\]
follows from Plancherel’s theorem, the untruncated decay estimate
\[\|U_\Delta (t - s) f\|_{L^\infty_x (H)} \lesssim |t - s|^{-\frac{n}{2}} \|f\|_{L^1_x (H)},\]
from the equality
\[U_{\Delta + A} (t) f = U_\Delta (t) U_A (t) f\]
and explicit representation of the Schrödinger evolution operator
\[U_\Delta (t) f (x) = (2\pi it)^{-\frac{n}{2}} \int \int e^{i (x - y)^2 / 2t} f (y) dy.\]

Due to properties of the operator \(A\), groups \(U_{\Delta + A} (t)\) and by the dispersive estimate (3.4) we have
\[
\|A^\alpha \Phi\|_E \lesssim \int \|A^\alpha U_{\Delta + A} (t - s)\|_{B (H)} \|F (s)\|_H ds \lesssim \int |t - s|^{-n (\frac{1}{2} - \frac{1}{p}) - \alpha} \|F (s)\|_H ds,
\]
where
\[\Phi = \int A^\alpha U_{\Delta + A} (t - s) F (s) ds.
\]

Moreover, from above estimate by the Hardy-Littlewood-Sobolev inequality, we get
\[
\|A^\alpha \Phi\|_{L^q_t L^r_x (H)} \lesssim \left\| \int |t - s|^{-n (\frac{1}{2} - \frac{1}{p}) - \alpha} \|F (s)\|_{L^{r'}_x (H)} ds \right\|_{L^q_t (\mathbb{R})} \lesssim \|F\|_{L^q_1 L^{r'}_x (H)},
\]
where
\[\frac{1}{q_1} = \frac{1}{q} + \frac{1}{p} + \frac{1}{2} - \frac{\alpha}{n}.
\]

The argument just presented also covers (3.27) in the case \(q = \bar{q}, r = \bar{r}\). It allows to consider the estimate in dualized form:
\[
\left| \int \int \langle U_{\Delta + A} (t - s) F (s), G (t) \rangle ds \right| \lesssim \|F\|_{L^q_t L^{r'}_x (H)} \|G\|_{L^q_1 L^{r'}_1 (H)}
\]
when
\[\frac{1}{q_1} = \frac{1}{q} + \frac{1}{p} + \frac{1}{2} - \frac{\nu}{n}.
\]
The case \( \tilde{q} = \infty, \tilde{r} = 2 \) follows from (3.27), i.e.

\[
K \lesssim \left\| \int_{s<t} U_{\Delta+A} (t-s) F (s) \, ds \right\|_{L^q_t L^r_x (H)} \lesssim \left\| G \right\|_{L^1_t L^2_x (H)} \lesssim (3.29)
\]

\[
\left\| F \right\|_{L^q_t L^r_x (H)} \left\| G \right\|_{L^1_t L^2_x (H)} .
\]

where

\[
K = \left| \int_{s<t} \left( U_{\Delta+A} (t-s) F (s) , G (t) \right) ds \right| .
\]

From (3.29) we obtain the estimate (3.28) when \( s = 0 \). The general case is obtained by using the same argument.

Now, consider the endpoint case, i.e. \((q,r) = \left(2, \frac{2n}{n-1} \right)\). It is suffices to show the following estimates

\[
\left\| A^\alpha U_{\Delta+A} (t) u_0 \right\|_{L^q_t L^r_x (H)} \lesssim \left\| A u_0 \right\|_{W^{s,2} (R^n; H)} ; \tag{3.30}
\]

\[
\left\| A^\alpha U_{\Delta+A} (t) u_0 \right\|_{C^0 (L^2_x (H))} \lesssim \left\| A u_0 \right\|_{W^{s,2} (R^n; H)} ; \tag{3.31}
\]

\[
\left\| \int_{s<t} A^\alpha U_{\Delta+A} (t-s) F (s) \, ds \right\|_{L^q_t L^r_x (H)} \lesssim \left\| F \right\|_{L^q_t L^r_x (H)} ; \tag{3.32}
\]

\[
\left\| \int_{s<t} A^\alpha U_{\Delta+A} (t-s) F (s) \, ds \right\|_{C^0 L^2_x (H)} \lesssim \left\| F \right\|_{L^q_t L^r_x (H)} . \tag{3.33}
\]

Indeed, applying Theorem 3.1 for

\[
Q = L^2 (R^n; H) , U (t) = \chi_{[0,T]} U_{\Delta+A} (t)
\]

with the energy estimate

\[
\left\| U (t) \right\|_{L^2 (R^n; H)} \lesssim \left\| f \right\|_{L^2 (R^n; H)}
\]

which follows from Plancherel’s theorem, the untruncated decay estimate (3.8) and by using of Lemma 3.1 we obtain the estimates (3.30) and (3.32). Let us temporarily replace the \( C^0_t L^2_x (H) \) norm in estimates (3.30), (3.32) by the \( L^\infty_t L^2_x (H) \) . Then, all of the above the estimates will follow from Theorem 3.1, once we show that \( U (t) \) obeys the energy estimate (3.7) and the truncated decay estimate (3.9). The estimate (3.7) is obtain immediate from Plancherel’s theorem, and (3.9) follows in a similar way as in [21, p.223-224]. To show that the quantity

\[
GF (t) = \int_{s<t} A^\alpha U_{\Delta+A} (t-s) F (s) \, ds
\]


is continuous in $L^2(R^n;H)$, we use the identity
\[ GF(t + \varepsilon) = U(\varepsilon)GF(t) + G \left( \chi_{[t,t+\varepsilon]}F \right)(t), \]
the continuity of $U(\varepsilon)$ as an operator on $L^2(R^n;H)$, and the fact that
\[ \left\| \chi_{[t,t+\varepsilon]}F \right\|_{L^2_t L^r_x(H)} \to 0 \text{ as } \varepsilon \to 0. \]

From the estimates (3.30) – (3.33) we obtain (3.25) for endpoint case.

4. Strichartz type estimates for solution to nonlinear Schrödinger equation

Consider the initial-value problem
\[ i\partial_t u + \Delta u + Au = F(u), \quad x \in R^n, \quad t \in [0,T], \]
\[ u(0, x) = u_0(x), \text{ for a.e. } x \in R^n \]
for $p \in (1, \infty)$, where $A$ is a linear and $F$ is a nonlinear operator in a Hilbert space $H$, $\lambda$ is a real number, $\Delta$ denotes the Laplace operator in $R^n$ and $u = u(t, x)$ is the $H$-valued unknown function.

**Condition 4.1.** Assume that the function $F : H \to H$ is continuously differentiable and obeys the power type estimates
\[ F(u) = O \left( \|u\|^{1+p} \right), \quad F_u(u) = O \left( \|u\|^p \right), \]
\[ F_u(v) - F_u(w) = O \left( \|v - w\|^\min\{p, 1\} + \|w\|^\max\{0, p-1\} \right) \]
for some $p > 0$, where $F_u(u)$ denotes the derivative of operator function $F$ with respect to $u \in H$.

From (4.2) we obtain
\[ \left\| F(u) - F(v) \right\| \lesssim \|u - v\| (\|u\|^p + \|v\|^p). \]

**Remark 4.1.** The model example of a nonlinearity obeying the conditions above is $F(u) = |u|^p u$, for which the critical homogeneous Sobolev space is $W^{2,s_c}(R^n;H)$ with $s_c := \frac{n}{2} - \frac{2}{p}$.  

**Definition 4.1.** A function $F : [0,T] \times R^n \to H$ is called a (strong) solution to (4.1) if it lies in the class
\[ \mathcal{C}_t^0 \left( [0,T] ; \tilde{W}^{2,s_c}(R^n;H(A)) \right) \cap \mathcal{L}^{p+2}_{t} L^{\frac{np+2}{2}}_{x} (([0,T] \times R^n;H(A))) \]
and obeys the Duhamel formula

\[ u(t) = U_{\Delta + A}(t) u_0 + \int_0^t U_{\Delta + A}(t-s) F(u)(s) \, ds, \quad \text{for all } t \in [0,T]. \]

We say that \( u \) is a global solution if \( T = \infty \).

Let \( E \) be a Banach space and \( B(x,\delta) \) denotes the ball in \( \mathbb{R}^n \) centred in \( x \) with radius \( \delta \) and \( M \) denote the \( H \)-valued Hardy-Littlewood type maximal operator that is defined as:

\[ Mf(x) = \sup_{\delta > 0} [\mu(B(x,\delta))]^{-1} \int_{B(x,\delta)} \|f(y)\|_E \, dy. \]

For proving the main result of this section we need the following:

By following [20, Ch.1, § 3, Theorem 1], we obtain the following result:

**Proposition 4.1.** Let \( f \in L^p(\mathbb{R}^n;E) \) for \( 1 < p \leq \infty \). Then \( Mf(x) \in L^p(\mathbb{R}^n;E) \) and

\[ \|Mf\|_{L^p(\mathbb{R}^n;E)} \leq M_p \|f\|_{L^p(\mathbb{R}^n;E)}. \]

**Proof.** For \( E = \mathbb{R} \), the result is obtained from [20, §3, Theorem 1]. The \( E \)-valued case can be obtained from the scalar case by applying it to \( \tilde{f}(x) = \|f(x)\|_E \).

A sequence of random variables \( \{r_k\}_{k \geq 0} \) on \( \Omega \) is called a Rademacher sequence (see e.g. [4]) if

\[ \mathbb{P}(\{r_k = 1\}) = \mathbb{P}(\{r_k = -1\}) = \frac{1}{2} \]

for \( k \geq 0 \) and \( \{r_k\}_{k \geq 0} \) are independent. For instance, one can take \( \Omega = (0,1) \) with the Lebesgue measure and

\[ r_k(t) = \text{sign} [\sin(2^{k+1}\pi t)] \text{ for } t \in \Omega. \]

Let \( \eta \in C_0^\infty(\mathbb{R}) \) nonnegative, supported in \( \sigma = \{ \frac{1}{2} < |\xi| < 2 \} \) and satisfying

\[ \sum_{j=-\infty}^{\infty} \eta(2^j \xi) = 1. \]

Let \( l^p(E) \) denotes \( E \)-valued sequence space (see e.g [27, § 1.18.1.]). Define Fourier multiplier operators

\[ Q_j f = F^{-1} \eta(2^{-j} \xi) \hat{f}. \]

From [18, Proposition 3.2] we have the following Littlewood-Paley type result for \( f \in L^p(\mathbb{R}^n;E) \):

\[ \]
Proposition 4.2. Assume $E$ is UMD space, $p \in (1, \infty)$ and $\{r_j\}_{j \geq 0}$ is a Rademacher sequence. Then
\[ \|f\|_{L^p(R^n; E)} \lesssim \left\| \{r_j Q_j f\}_{j \geq 0} \right\|_{L^p(R^n; E)} \lesssim \|f\|_{L^p(R^n; E)}. \]

Consider the vector-valued version of the Fefferman-Stein type maximal inequality for $E-$valued functions:

Proposition 4.3. Let $E$ be a Banach space, $1 < p < \infty$, $1 < q \leq \infty$. Then there exists a constant $C(p, q)$ such that for all $\{f\}_{k \geq 0} \in L^p(R^n; E)$ one has
\[ \left\| \{Mf\}_{k \geq 0} \right\|_{L^p(R^n; E)} \leq C(p, q) \left\| \{f\}_{k \geq 0} \right\|_{L^p(R^n; E)}. \]

Proof. For $q = \infty$ one uses that
\[ \|Mf_k(x)\|_{L^\infty(E)} \leq M \|f_k(x)\|_{L^\infty(E)}, \quad x \in R^n, \quad k \geq 0 \]
and applies the boundedness of $M$ on $L^p(R^n)$ to the function $\hat{f} (x) = \|f_k(x)\|_{L^\infty(E)}$. If $1 < q < 1$ and $E = \mathbb{R}$, the result can be found in [20](Ch.2, § 1, Theorem 1).

The $E-$valued case can be obtained from the scalar case by applying it to
\[ \left\| \{f_k(x)\}_{k \geq 0} \right\|_E \subset L^p(R^n). \]

Then
\[ \|D^\alpha f\|_{L^r(R^n; E)} \lesssim \left\| \sum_{j = -\infty}^{\infty} 2^{j \alpha} Q_j f \right\|_{L^r(R^n; E)} \lesssim \left\| \sum_{j = -\infty}^{\infty} 2^{2j \alpha} \|Q_j f(.)\|_E^2 \right\|_r \]
for all $f \in W^{\alpha, r}(R; E)$ by multiplier theorem in $L^r(R^n; E)$ spaces (see e.g.[12]) and by Proposition 4.2. Moreover, if the right-hand side is finite then $D^\alpha f \in L^r(R^n; E)$ in the sense of $E-$valued distributions. $Q_j$ may be realized as a convolution operator $Q_j f = \psi_j * f$, where $\psi_j \in S(R)$ and
\[ |\psi_j(x)| + 2^{-j} |\partial_x \psi_j(x)| \leq C_N 2^j (1 + 2^j |x|)^{-N} \quad (4.5) \]
for all $N$ uniformly in $j \in \mathbb{Z}$, and
\[ \int \psi_j(x) \, dx = 0. \quad (4.6) \]

By following [7, Proposition 3.1] we obtain.

Lemma 4.1. For any $g \in W^{\alpha, r}(R; E)$,
\[ \left\| \tilde{Q}_j g(y) - \tilde{Q}_j g(x) \right\|_E \leq C \begin{cases} 2^j |x - y| \, M_g(x) \, \text{if } |x - y| \leq C 2^{-j} \\ M_g(x) + M_g(y) \, \text{for all } x, y \end{cases}. \]

Proof. Construct also $\tilde{\eta} \in C_0^\infty(\sigma)$ satisfying $\tilde{\eta} \eta \equiv \eta$. Define
\[ \tilde{Q}_j f = F^{-1} \tilde{\eta}(2^{-j} \xi) \hat{f}. \]
so that the identity operator may be resolved as

\[ I = \sum_{j=-\infty}^{\infty} Q_j = \sum_{j=-\infty}^{\infty} \tilde{Q}_j Q_j, \]

and \( \tilde{Q}_j \) is realized by convolution with a Schwartz function \( \tilde{\psi}_j \) satisfying (4.5) and (4.6).

It is clear that

\[ \| \tilde{Q}_j g (y) - \tilde{Q}_j g (x) \|_E \leq \int |\psi_j (y - z) - \psi_j (x - z)| \| g (z) \|_E \, dz. \]

For any \( x \), we get

\[ \| \tilde{Q}_j g (x) \|_E \leq C M g (x) \]

due to (4.5). If \( |x - y| \leq C 2^{-j} \) then

\[ |\psi_j (y - z) - \psi_j (x - z)| \leq C 2^j |x - y| (1 + 2^j |x - z|)^{-2}, \]

gain by (4.5). By a standard calculation this implies the desired estimate (see, e.g., [20, p. 62-63]).

**Proposition 4.4.** Assume \( E \) is a UMD space and \( F \in C^{(1)} (\mathbb{R}; E) \). Suppose \( \alpha \in (0, 1) \), \( 1 < p, q, r < \infty \) and \( r^{-1} = p^{-1} + q^{-1} \). If \( u \in L^\infty (\mathbb{R}; E) \), \( D^\alpha u \in L^q (\mathbb{R}; E) \) and \( F' (u) \in L^p (\mathbb{R}; E) \), then \( D^\alpha (F(u)) \in L^r (\mathbb{R}; E) \) and

\[ \| D^\alpha (F(u)) \|_{L^r (\mathbb{R}; E)} \lesssim \| F'(u) \|_{L^p (\mathbb{R}; E)} \| D^\alpha u \|_{L^q (\mathbb{R}; E)}. \]

**Proof.** In view of (4.6) we have

\[ O_j F(u) (x) = \int F(u) (y) \psi_j (x - y) \, dy = \int [F(u) (y) - F(u) (x)] \psi_j (x - y) \, dy = \]

\[ \int \left[ \int_0^1 F' (tu (y) + (1 - t) u (x)) \, dt \right] [u (y) - u (x)] \, \psi_j (x - y) \, dy. \]  

By properties of \( E \)-valued Hardy-Littlewood maximal operator we get

\[ \left\| \int_0^1 F' (tu (y) + (1 - t) u (x)) \, dt \right\|_E \leq 2M (F' (u (x))). \]

To estimate (4.7) decompose \( u = \sum_j Q_j u = \sum_j \tilde{Q}_j Q_j u \) to obtain

\[ \| Q_j F(u) (x) \|_E \leq \]

\[ CM (F'(u(x))) \sum_{j=-\infty}^{\infty} \int \| \tilde{Q}_j Q_j u (y) - \tilde{Q}_j Q_j u (x) \| \psi_j (x - y) \, dy. \]

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Break the sum over \( j \) into the cases \( j < m \) and \( j \geq m \). From (4.8) we get that

\[
\sum_{j<m} \int \left| \tilde{Q}_j Q_j u(x) - \tilde{Q}_j Q_j u(x) \right| \left| \psi_j (x-y) \right| dy \leq C \sum_{j<m} \int_{|x-y| \leq 2^{-j}} 2^j |x-y| M(Q_j)(x) 2^m (1+2^m) |x-y|^{-3} dy +
\]

(4.9)

\[
C \sum_{j<m} \int_{|x-y| > 2^{-j}} [M(Q_j)(x) + M(Q_j)(y)] 2^m (1+2^m) |x-y|^{-3} dy \leq
\]

\[
C \sum_{j<m} 2^{j-m} [M(Q_j)(x) + M^2(Q_j)(x)] \leq C \sum_{j<m} 2^{j-m} M^2(Q_j)(x),
\]

where \( M^2 = M \circ M \).

Likewise, we get

\[
\sum_{j \geq m} \int \left| \tilde{Q}_j Q_j u(y) - \tilde{Q}_j Q_j u(x) \right| \left| \psi_j (x-y) \right| dy \leq C \sum_{j<m} 2^{j-m} M^2(Q_j)(x).
\]

Putting (4.9) and (4.10) into (4.8), we have

\[
\left( \sum_{m=-\infty}^{\infty} 2^{m\alpha} \|Q_m u(x)\|_E^2 \right)^{\frac{1}{2}} \leq CM(F'u(x)) \times
\]

(4.11)

\[
\left\{ \sum_{m} 2^{m\alpha} \left[ \sum_{j<m} 2^{j-m} M^2(Q_j)(x) + \sum_{j \geq m} M^2(Q_j)(x) \right] \right\}^{\frac{1}{2}} \leq CM(F'u(x)) \sum_{k=-\infty}^{\infty} 2^{-\varepsilon k} \left( \sum_{j=-\infty}^{\infty} 2^{j\alpha} \|Q_j u(x)\|_E^2 \right)^{\frac{1}{2}}
\]

by substituting \( m = j - k \) after applying Minkowski’s inequality, where

\[
\varepsilon = 2 \min(\alpha, 1-\alpha) > 0.
\]

Finally, from (4.11) by using Proposition 4.3 we obtain

\[
\|D^\alpha (F(u))\|_{L^r(\mathbb{R}_x;E)} \leq C \left( \sum_{m=-\infty}^{\infty} 2^{m\alpha} \|Q_m u(x)\|_E^2 \right)^{\frac{1}{2}} \leq
\]

\[
C \left( M(F'u(x)) \left( \sum_{j=-\infty}^{\infty} 2^{2j\alpha} \|M^2 Q_j u(x)\|_E^2 \right)^{\frac{1}{2}} \right) \leq
\]

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defined by

\[ u_0 = 1 \text{ is absolute positive in a Banach space } \mathcal{A}. \]

The Strichartz estimates (3.25), we will show that the solution map \( \Phi(u) \) defined by

\[ \Phi(u)(t) := U_A(t)u_0 + \int_0^t U_A(t-s)F(u(s))ds \]

is a contraction on the set \( B_1 \cap B_2 \) under the metric given by

\[ d(u,v) = \|u-v\|_{L_t^p L_x^r([0,T] \times \mathbb{R}^n; H)}. \]

\[ C \| M(F^t(u)(x)) \|_{L^p(\mathbb{R}; E)} \left\| \left( \sum_{j=-\infty}^{\infty} 2^{j\alpha} \|Q_j u(x)\|^2_E \right)^{\frac{1}{2}} \right\| \leq C \|F^t(u)\|_{L^p(\mathbb{R}; E)} \|D^\alpha u\|_{L^p(\mathbb{R}; E)}. \]

**Theorem 4.1.** Assume the Conditions 3.1., 4.1 are satisfied and suppose \( A \) is absolute positive in a Banach space \( \mathcal{A} \). Let \( 0 \leq s \leq 1, 0 \leq \alpha < 1, u_0 \in W^{s,2} (\mathbb{R}^n; H(A^\alpha)) \) and \( n \geq 1 \). Then there exists \( \eta_0 = \eta_0(n) > 0 \) such that if \( 0 < \eta \leq \eta_0 \) such that

\[ \|\nabla^\alpha U_{\Delta+A}(t) A^\alpha u_0\|_{L_t^p L_x^r([0,T] \times \mathbb{R}^n; H)} \leq \eta, \quad (4.5) \]

then here exists a unique solution \( u \) to (4.1) on \([0,T] \times \mathbb{R}^n\). Moreover, the following estimates hold

\[ \|\nabla^\alpha U_{\Delta+A} A^\alpha u\|_{L_t^{p+2} L_x^r([0,T] \times \mathbb{R}^n; H)} \leq 2\eta, \quad (4.6) \]

\[ \|\nabla^\alpha u\|_{S^0([0,T] \times \mathbb{R}^n; H)} + \|A^\alpha u\|_{C^0([0,T]; W^{s,2}(\mathbb{R}^n; H))} \lesssim \|A^\alpha \nabla^\alpha u_0\|_{L_t^2(\mathbb{R}^n; H)} + \eta^{1+p}, \quad (4.7) \]

\[ \|A^\alpha u\|_{S^0([0,T] \times \mathbb{R}^n; H)} \lesssim \|A^\alpha u_0\|_{L_t^2(\mathbb{R}^n; H)}, \quad (4.8) \]

where

\[ r = r(p,n) = \frac{2n(p+2)}{2(n-2) + np}. \]

**Proof.** We apply the standard fixed point argument. More precisely, using the Strichartz estimates (3.25), we will show that the solution map \( u \rightarrow \Phi(u) \), defined by

\[ \Phi(u)(t) := U_A(t,u_0) + \int_0^t U_A(t-s) F(u(s))ds \]

for all \( t \in [0,T] \), is a contraction on the set \( B_1 \cap B_2 \) under the metric given by

\[ d(u,v) = \|u-v\|_{L_t^p L_x^r([0,T] \times \mathbb{R}^n; H)} \]

where

\[ B_1 = \left\{ u \in W^{\infty,s_{\alpha},2} = L_t^\infty W_x^{s_{\alpha},2}([0,T] \times \mathbb{R}^n; H(A^{\alpha})) : \|u\|_{W^{\infty,s_{\alpha},2}} \leq 2 \|A^\alpha u_0\|_{W^{s_{\alpha},2}(\mathbb{R}^n; H)} + C(n)(2\eta)^{1+p} \right\}, \]

\[ B_2 = \left\{ u \in W^{p+2,s_{\alpha},r} = L_t^{p+2} W_x^{s_{\alpha},r}([0,T] \times \mathbb{R}^n; H) : \|A^\alpha \nabla^\alpha u\|_{L_t^{p+2} L_x^r([0,T] \times \mathbb{R}^n; H)} \leq 2\eta, \right\}. \]

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here $C(n)$ denotes the constant from the Strichartz inequality in (3.25).

Note that both $B_1$ and $B_2$ are closed in this metric. Using the Strichartz estimate (3.25), Proposition 4.4 and Sobolev embedding in $H$—valued fractional Sobolev spaces [23], we get that for $u \in B_1 \cap B_2$,

$$\|A^\alpha u\|_{L_t^{p+2}L_x^r([0,T] \times \mathbb{R}^n; H)} \leq 2C(n) \|A^\alpha u_0\|_{L_x^2(\mathbb{R}^n; H)} \right\},$$

we estimate (3.25). In view of Definition 4.1, uniqueness follows from uniqueness in the contraction mapping theorem.

Similarly,

$$\| \Phi(u) \|_{L_t^{p+2}L_x^r([0,T] \times \mathbb{R}^n; H)} \leq C(n) \|A^\alpha u_0\|_{L_x^2(\mathbb{R}^n; H)} + C(n) \|u\|_{L_t^{p+2}L_x^r(H)} \leq$$

$$\|A^\alpha u_0\|_{W^{s,2}_x(\mathbb{R}^n; H)} + C(n) \left( 2\eta + 2C(n) \|A^\alpha u_0\|_{L_x^2(\mathbb{R}^n; H)} \right) \|\nabla^{s\epsilon} u\|_{L_t^{p+2}L_x^r(H)} \leq$$

$$\|A^\alpha u_0\|_{W^{s,2}_x(\mathbb{R}^n; H)} + C(n) \left( 2\eta + 2C(n) \|A^\alpha u_0\|_{L_x^2(\mathbb{R}^n; H)} \right) (2\eta)^p,$$

where

$$L_t^qL_x^r(H) = L_t^qL_x^r([0,T] \times \mathbb{R}^n; H), \ r_1 = r_1(p,n) = \frac{2n(p+2)}{2(n+2)+np}.$$

Similarly,

$$\| \Phi(u) \|_{L_t^{p+2}L_x^r(\mathbb{R}^n; H)} \leq C(n) \|A^\alpha u_0\|_{L_x^2(\mathbb{R}^n; H)} + C(n) \|u\|_{L_t^{p+2}L_x^r(H)} \leq$$

$$\|A^\alpha u_0\|_{W^{s,2}_x(\mathbb{R}^n; H)} + 2C^2(n) \|A^\alpha u_0\|_{L_x^2(\mathbb{R}^n; H)} (2\eta)^p.$$

Arguing as above and invoking (4.5), we obtain

$$\|\nabla^{s\epsilon} \Phi(u)\|_{L_t^{p+2}L_x^r} \leq \eta + C(n) \|\nabla^{s\epsilon} \Phi(u)\|_{L_t^{p+2}/(p+1)L_x^r} \leq$$

$$\eta + C(n)(2\eta)^{1+p}.$$

Thus, choosing $\eta_0 = \eta_0(n)$ sufficiently small, we see that for $0 < \eta \leq \eta_0$ the function $\Phi$ maps the set $B_1 \cap B_2$ to itself. To see that it is a contraction, we repeat the computations above and use (4.4) to obtain

$$\|F(u) - F(v)\|_{L_t^{p+2}L_x^r} \leq C(n) \|F(u) - F(v)\|_{L_t^{p+2}/(p+1)L_x^r} \leq$$

$$C(n)(2\eta)^p \|u - v\|_{L_t^{p+2}L_x^r}.$$

Thus, choosing $\eta_0 = \eta_0(n)$ small enough, we can guarantee that is a contraction on the set $B_1 \cap B_2$. By the contraction mapping theorem, it follows that has a fixed point in $B_1 \cap B_2$. Since $\Phi$ maps into $C^{0}_{t}W^{s_{\epsilon},2}_{x}([0,T] \times \mathbb{R}^n; H)$ we derive that the fixed point of $\Phi$ is indeed a solution to (4.1).

In view of Definition 4.1, uniqueness follows from uniqueness in the contraction mapping theorem.
5. The existence and uniqueness for the system of Schrödinger equation

Consider at first, the Cauchy problem for linear system of Schrödinger equations

\[ i\partial_t u_m + \Delta u_m + \sum_{j=1}^{N} a_{mj} u_j = F_j (t, x), \quad t \in [0, T], \quad x \in \mathbb{R}^n, \quad (5.1) \]

\[ u_m (0, x) = u_{m0} (x), \quad \text{for a.e. } x \in \mathbb{R}^n, \]

where \( u = (u_1, u_2, ..., u_N), \ u_j = u_j (t, x), \ a_{mj} \) are complex numbers. Let \( l_2 = l_2 (N) \) and \( l_2^* = l_2^* (N) \) (see [27, § 1.18]). Let \( A \) be the operator in \( l_2 (N) \) defined by

\[ D (A) = \left\{ u = \{u_j\} \in l_2, \quad \|Au\|_{l_2(N)} = \left( \sum_{m,j=1}^{N} |a_{mj} u_j|^2 \right)^{\frac{1}{2}} < \infty \right\}, \]

\[ A = \{a_{mj}\}, \quad a_{mj} = a_{jm}, \quad s > 0, \quad m, j = 1, 2, ..., N, \quad N \in \mathbb{N}. \]

From Theorem 3.2 we obtain the following result

**Theorem 5.1.** Assume the Conditions 3.1 are hold. Let \( 0 \leq s \leq 1, \ 0 \leq \alpha < 1, \ u_0 \in W^{s,2} (\mathbb{R}^n; D (A^\alpha)), \ F \in L^0 \left( [0, T]; W^{s,2} (\mathbb{R}^n; l_2) \right) \) and \( n \geq 1 \). Let \( u : [0, T] \times \mathbb{R}^n \rightarrow l_2 (N) \) be a solution to (5.1). Then

\[ \left\| \nabla^s u \right\|_{S^0([0, T];l_2)} + \left\| \nabla^s A^\alpha u \right\|_{L^0([0, T];L^2(\mathbb{R}^n; l_2))} \lesssim \]

\[ \left\| \nabla^s A^\alpha u_0 \right\|_{L^2(\mathbb{R}^n; l_2)} + \left\| \nabla^s F \right\|_{L^0([0, T];l_2)}. \]

**Proof.** It is easy to see that \( A \) is a symmetric operator in \( l_2 \) and other conditions of Theorem 3.2 are satisfied. Hence, from Theorem 4.2 we obtain the conclusion.

Consider now, the Cauchy problem (1.10). We obtain from Theorem 4.1 the following result

**Theorem 5.2.** Assume the Conditions 3.1 and 4.1 are hold. Let \( 0 \leq s \leq 1, \ 0 \leq \alpha < 1, \ u_0 \in W^{s,2} (\mathbb{R}^n; D (A^\alpha)) \) and \( n \geq 1 \). Then there exists \( \eta_0 = \eta_0 (n) > 0 \) such that if \( 0 < \eta \leq \eta_0 \) such that

\[ \left\| \nabla^s U_{\Delta + A} (t) A^\alpha u_0 \right\|_{L^{s+2}_{t}L^{2}_{x}(\mathbb{R}^n; l_2)} \lesssim \eta, \]

then here exists a unique solution \( u \) to (1.10) on \([0, T] \times \mathbb{R}^n\). Moreover, the following estimates hold

\[ \left\| \nabla^s U_{\Delta + A} A^\alpha u \right\|_{L^{s+2}_{t}L^{2}_{x}(\mathbb{R}^n; l_2)} \lesssim 2 \eta, \]

\[ \left\| \nabla^s u \right\|_{S^0([0, T];l_2)} + \left\| A^\alpha u \right\|_{L^0([0, T];W^{s,2} (\mathbb{R}^n; l_2))} \lesssim \]

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\[ \|A^\alpha |\nabla|^s u_0\|_{L^2_2(R^n)} + \eta^{1+p}, \]
\[ \|A^\alpha u\|_{S^0([0,T] \times R^n)} < \|A^\alpha u_0\|_{L^2_2(R^n)} , \]

where
\[ \sigma = \sigma(p,n) = \frac{2n(p+2)}{2(n-2)+np}. \]

**Proof.** It is easy to see that \( A \) is a symmetric operator in \( L^2_2 \) and other conditions of Theorem 4.1 are satisfied. Hence, from Theorem 4.1 we obtain the conclusion.

6. The existence and uniqueness of solution to anisotropic Schrödinger equation

Let \( \Omega = R^n \times G, G \subset R^d, d \geq 2 \) is a bounded domain with \( (d-1) \)-dimensional boundary \( \partial G \). Consider at first, the mixed problem for Schrödinger equation

\[ i\partial_t u + \Delta_x u + \sum_{|\alpha| \leq 2m} a_\alpha(y) D_y^\alpha u = F(t,x) , \]
\[ x \in R^n, y \in G, t \in [0,T], p \geq 0. \]

\[ B_j u = \sum_{|\beta| \leq m_j} b_{j\beta}(y) D_y^\beta u = 0, x \in R^n, y \in \partial G, j = 1,2,...,m, \]

\[ u(0,x,y) = u_0(x,y) \text{ for } x \in R^n, y \in G \]

where \( u = u(t,x,y) \) is a solution, \( a_\alpha, b_{j\beta} \) are the complex valued functions, \( \lambda = \pm 1, \alpha = (\alpha_1, \alpha_2, ..., \alpha_n), \beta = (\beta_1, \beta_2, ..., \beta_n), \mu_i < 2m \) and

\[ D_x^k = \frac{\partial^k}{\partial x^k}, D_j = -i \frac{\partial}{\partial y_j}, D_y = (D_1,..., D_n), y = (y_1,..., y_n). \]

Let
\[ \xi' = (\xi_1, \xi_2, ..., \xi_{n-1}) \in R^{n-1}, \alpha' = (\alpha_1, \alpha_2, ..., \alpha_{n-1}) \in Z^n, \]
\[ A(y_0, \xi', D_y) = \sum_{|\alpha'| + j \leq 2m} a_{\alpha'}(y_0) \xi_1^{\alpha_1} \xi_2^{\alpha_2} ... \xi_{n-1}^{\alpha_{n-1}} D_y^j \text{ for } y_0 \in \tilde{G} \]
\[ B_j(y_0, \xi', D_y) = \sum_{|\beta'| + j \leq m_j} b_{j\beta'}(y_0) \xi_1^{\beta_1} \xi_2^{\beta_2} ... \xi_{n-1}^{\beta_{n-1}} D_y^j \text{ for } y_0 \in \partial G. \]

For \( \Omega = R^n \times G, p = (p_1, p_2), s \in R \text{ and } l \in N \) let \( \tilde{W}^{s,l,p}(\Omega) = \tilde{W}^{s,l,p}(\Omega, \mathbb{C}) . \)

From Theorem 3.2 we obtain the following result
Theorem 6.1. Assume the following conditions be satisfied:

(1) \( G \in C^2, a_\alpha \in C(G) \) for each \(|\alpha| = 2m \) and \( a_\alpha \in L_\infty(G) \) for each \(|\alpha| < 2m; \)

(2) \( b_{j\beta} \in C^{2m-m_j}(\partial G) \) for each \( j, \beta \) and \( m_j < 2m, \sum \nolimits_{j=1}^m b_{j\beta}(y')\sigma_j \neq 0 \), for \(|\beta| = m_j, y' \in \partial G, \) where \( \sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in R^n \) is a normal to \( \partial G; \)

(3) for \( y \in G, \xi \in R^n, \mu \in S(\varphi_0) \) for \( 0 \leq \varphi_0 < \pi, |\xi| + |\mu| \neq 0 \) let \( \mu + \sum \nolimits_{|\alpha|=2m} a_\alpha(y) \xi^\alpha \neq 0; \)

(4) for each \( y_0 \in \partial G \) local BVP in local coordinates corresponding to \( y_0; \)

\[
\mu + A(y_0, \xi', D_y) \vartheta(y) = 0,
\]

\[
B_j(y_0, \xi', D_y) \vartheta(0) = h_j, \ j = 1, 2, \ldots, m
\]

has a unique solution \( \vartheta \in C_0(\mathbb{R}_+) \) for all \( h = (h_1, h_2, \ldots, h_m) \in \mathbb{C}^n \) and for \( \xi' \in R^{n-1}; \)

(5) Assume the Conditions 3.1 are hold. Let \( 0 \leq s \leq 1, 0 \leq \alpha < 1, \)

\( u_0 \in \dot{W}^{s,2}(R^n; D(A^\alpha)), F \in N^0 \left([0,T] ; \dot{W}^{s,2}(R^n; L^2(G)) \right) \) and \( n \geq 1. \) Let \( u : [0,T] \times R^n \rightarrow L^2(G) \) be a solution to (6.1) – (6.3). Then

\[
\|
\nabla|^s u\|_{S^0([0,T];L^2(G))} + \|
\nabla|^s A^\alpha u\|_{C^0([0,T];L^2(R^n;L^2(G)))} \lesssim
\]

\[
\|
\nabla|^s A^\alpha u_0\|_{L^2(R^n;L^2(G))} + \|
\nabla|^s F\|_{N^0([0,T];L^2(G))}.
\]

Proof. Let us consider the operator \( A \) in \( H = L^2(G) \) that are defined by

\[
D(A) = \{ u \in W^{2m,2}(G), B_j u = 0, \ j = 1, 2, \ldots, m \}, \ A u = \sum \nolimits_{|\alpha| \leq 2m} a_\alpha(y) D^\alpha u(y).
\]

Then the problem (6.1) – (6.3) can be rewritten as the problem (4.1), where \( u(x) = u(x, t), f(x) = f(x, t), x \in R^n \) are the functions with values in \( H = L^2(G) \). By virtue of [8, Theorem 8.2], operator \( A + \mu \) is absolute positive in \( L^2(G) \) for sufficiently large \( \mu > 0 \). Moreover, in view of (1)-(5) all conditons of Theorem 3.2 are hold. Then Theorem 3.2 implies the assertion.

Consider now, the mixed problem for nonlinear Schrodinger equation

\[
i \partial_t u + \Delta_x u + \sum \nolimits_{|\alpha| \leq 2m} a_\alpha(y) D^\alpha u + \lambda |u|^p u = 0,
\]

\[
x \in R^n, y \in G, t \in [0,T], p \geq 0,
\]

\[
B_j u = \sum \nolimits_{|\beta| \leq m_j} b_{j\beta}(y) D^\beta_y u = 0, \ x \in R^n, y \in \partial G, j = 1, 2, \ldots, m,
\]

(6.4)
The following estimates hold

\[ u(0, x, y) = u_0(x, y) \] for \( x \in \mathbb{R}^n, \ y \in G \)

(6.6)

**Theorem 6.2.** Assume the following conditions be satisfied:

1. \( G \in C^2, \ a_\alpha \in C_0(\bar{G}) \) for each \(|\alpha| = 2m\) and \( a_\alpha \in L_\infty(G) \) for each \(|\alpha| < 2m\);
2. \( b_{j\beta} \in C^{2m-m_j}(\partial G) \) for each \( j, \beta \) and \( m_j < 2m, \sum_{j=1}^{m} b_{j\beta}(y') \sigma_j \neq 0 \), for \(|\beta| = m_j, \ y' \in \partial G\), where \( \sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in \mathbb{R}^n \) is a normal to \( \partial G \);
3. (3) for \( y \in \bar{G}, \ x \in \mathbb{R}^n, \ \mu \in S(\varphi_0) \) for \( 0 \leq \varphi_0 < \pi, \ |\xi| + |\mu| \neq 0 \) let \( \mu + \sum_{|\alpha| = 2m} a_\alpha(y) \xi^\alpha \neq 0 \);
4. (4) for each \( y_0 \in \partial G \) local BVP in local coordinates corresponding to \( y_0 \):
   \[ \mu + A(y_0, \xi', D_y) \vartheta(y) = 0, \]
   \[ B_j(y_0, \xi', D_y) \vartheta(0) = h_j, \ j = 1, 2, ..., m \]
   has a unique solution \( \vartheta \in C_0(\mathbb{R}_+) \) for all \( h = (h_1, h_2, ..., h_n) \in \mathbb{C}^n \) and for \( \xi' \in \mathbb{R}^{n-1} \);
5. (5) Assume the Condition 3.1 are hold. Let \( 0 \leq s \leq 1, \ 0 \leq \alpha < 1, \ u_0 \in \dot{W}^{s, 2m}(\mathbb{R}^n \times G) \) and \( n \geq 1 \).

Then there exists \( \eta_0 = \eta_0(n) > 0 \) such that if \( 0 < \eta \leq \eta_0 \) such that

\[ \|\nabla^\alpha U_{A+\Lambda}(t) A^\alpha u_0\|_{L^p_t L^2_x([0, T] \times \mathbb{R}^n \times G)} \leq \eta, \]

then there exists a unique solution \( u \) to (6.4) – (6.6) on \([0, T] \times \mathbb{R}^n\). Moreover, the following estimates hold

\[ \|\nabla^\alpha U_{A+\Lambda} A^\alpha u\|_{L^p_t L^2_x([0, T] \times \mathbb{R}^n \times G)} \leq 2\eta, \]
\[ \|\nabla^\alpha u\|_{S^0 L^2([0, T] \times \mathbb{R}^n \times G)} + \|A^\alpha u\|_{C^0([0, T]; \dot{W}^{s, 2}(\mathbb{R}^n \times G))} \lesssim \|A^\alpha \nabla^\alpha u_0\|_{L^2_{t,x}(\mathbb{R}^n \times G)} + \eta^{1+p}, \]
\[ \|A^\alpha u\|_{S^0 L^2([0, T] \times \mathbb{R}^n \times G)} \lesssim \|A^\alpha u_0\|_{L^2_{t,x}(\mathbb{R}^n \times G)}. \]

where

\[ \sigma = \sigma(p, n) = \frac{2n (p + 2)}{2(n - 2) + np}. \]

**Proof.** The problem (6.4) – (6.6) can be rewritten as the problem (1.1), where \( u(x) = u(x, .), \ f(x) = f(x, .), \ x \in \mathbb{R}^n \) are the functions with values in \( H = L^2(G) \). By virtue of [8, Theorem 8.2], operator \( A + \mu \) is absolute positive in \( L^2(G) \) for sufficiently large \( \mu > 0 \). Moreover, in view of (1)-(5) all conditions of Theorem 4.1 are hold. Then Theorem 4.1 implies the assertion.
7. The Wentzell-Robin type mixed problem for Schrödinger equations

Consider at first, the linear problem \((1.7) - (1.9)\). From Theorem 3.2 we obtain the following result

**Theorem 7.1.** Suppose the following conditions are satisfied:
1. \(a\) is positive, \(b\) is a real-valued functions on \((0, 1)\). Moreover, \(a(\cdot) \in C(0, 1)\) and
   \[
   \exp \left( -\int_{\frac{x}{n}} b(t) a^{-1}(t) \, dt \right) \in L_1(0, 1);
   \]
2. Assume the Conditions 3.1 and 4.1 are hold. Let \(0 \leq s \leq 1, 0 \leq \alpha < 1, F \in N^0([0, T]; W^{s,2}(R^n; L^2(0, 1)))\) \(u_0 \in \bar{W}^{s,2,2}(R^n \times (0, 1)), \) and \(n \geq 1.\)

Let \(u : [0, T] \times R^n \rightarrow L^2(G)\) be a solution to \((1.7) - (1.9).\) Then
\[
\begin{align*}
\||\nabla|^s u||_{S^0([0, T]; L^2(0, 1))} + ||\nabla|^s A^\alpha u||_{C^0([0, T]; L^2(R^n; L^2(0, 1)))} \lesssim, \\
\end{align*}
\]
\[
\begin{align*}
\||\nabla|^s A^\alpha u_0||_{L^2(R^n; L^2(0, 1))} + ||\nabla|^s F||_{N^0([0, T]; L^2(0, 1))}.
\end{align*}
\]

**Proof.** Let \(H = L^2(0, 1)\) and \(A\) is a operator defined by \((4.1).\) Then the problem \((1.7) - (1.9)\) can be rewritten as the problem \((1.2).\) By virtue of \([13, 14]\) the operator \(A\) generates analytic semigroup in \(L^2(0, 1).\) Hence, by virtue of \((1)-(5)\) all conditions of Theorem 3.2 are satisfied. Then Theorem 3.2 implies the assertion.

Consider now, the problem \((1.7) - (1.9).\) In this section, from Theorem 4.1 we obtain the following result:

**Theorem 7.2.** Suppose the following conditions are satisfied:
1. \(a\) is positive, \(b\) is a real-valued functions on \((0, 1)\). Moreover, \(a(\cdot) \in C(0, 1)\) and
   \[
   \exp \left( -\int_{\frac{x}{n}} b(t) a^{-1}(t) \, dt \right) \in L_1(0, 1);
   \]
2. Assume the Conditions 3.1 and 4.1 are hold. Let \(0 \leq s \leq 1, 0 \leq \alpha < 1, u_0 \in \bar{W}^{s,2,2}(R^n \times (0, 1))\) and \(n \geq 1.\)

Then there exists \(\eta_0 = \eta_0(n) > 0\) such that if \(0 < \eta \leq \eta_0\) such that
\[
\begin{align*}
\||\nabla|^s U_{\Delta + A}(t) A^\alpha u_0||_{L^{p+2}(L^2(0, T) \times R^n \times (0, 1)))} \leq \eta,
\end{align*}
\]
then there exists a unique solution \(u\) to \((1.8) - (1.10)\) on \([0, T] \times R^n.\) Moreover, the following estimates hold
\[
\begin{align*}
\||\nabla|^s U_{\Delta + A}A^\alpha u||_{L^{p+2}(L^2(0, T) \times R^n \times (0, 1)))} \leq 2\eta, \\
\||\nabla|^s u||_{S^0(L^2(0, T) \times R^n \times (0, 1)))} + ||A^\alpha u||_{C^0([0, T]; W^{s,2}(R^n \times 0, 1))} \lesssim
\end{align*}
\]
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\[ \| A^{\alpha} |\nabla|^s u_0 \|_{L^2_y(R^n \times (0,1))} + \eta^{1+p}, \]
\[ \| A^{\alpha} u \|_{S^0 L^2_y([0,T] \times R^n \times (0,1))} \lesssim \| A^{\alpha} u_0 \|_{L^2_y(R^n \times (0,1))}. \]

where
\[ \sigma = \sigma (p, n) = \frac{2n (p + 2)}{2(n - 2) + np}. \]

**Proof.** Let \( H = L^2 (0,1) \) and \( A \) is an operator defined by (1.4). Then the problem (1.8) – (1.10) can be rewritten as the problem (4.1). By virtue of [13, 14] the operator \( A \) generates analytic semigroup in \( L^2 (0,1) \). Hence, by virtue of (1)-(5) all conditions of Theorem 4.1 are satisfied. Then Theorem 4.1 implies the assertion.

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