PHASE UNWINDING, OR INARIANT SUBSPACE DECOMPOSITIONS OF HARDY SPACES

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Abstract. We consider orthogonal decompositions of invariant subspaces of Hardy spaces, these relate to the Blaschke based phase unwinding decompositions [1, 2, 7]. We prove convergence in $L^p$. In particular we build an explicit multiscale wavelet basis. We also obtain an explicit unwinding decomposition for the singular inner function, $\exp 2i\pi/x$.

1. Introduction

Our goal is to extend and clarify convergence properties of the phase unwinding expansions in [7, 1, 2, 10] as well as expansions obtained by the algorithm of adaptive Fourier decomposition [12, 13, 11, 10] where each function in $H^2(T)$, admits its own adapted (unwound) decomposition in an orthonormal system of basis functions consisting of partial products of Blaschke products. We extend the result to $H^p(T)$ for $p \in (1, +\infty)$. We also discuss the relation to various generalizations of the Takenaka Malmquist bases, both for the Torus and the upper half plane. In particular we show that there is a multiscale analysis of $H^2(R)$, and that, at each level, there is a function whose translates make an orthonormal basis. This in the same spirit as studies of hyperbolic wavelets [9, 8, 4]. The main difference is that we use different grids, which allows to get a formalism very close to wavelets. More precisely, let

$$\phi(x) = \frac{\Gamma(x - 1 + i)}{\sqrt{\pi}\Gamma(x - i)}.$$  

Then the functions $\phi(2^n x + j)\Delta(2^n x)$, for $n$ and $j$ in $Z$, where $\Delta$ is a suitable inner function, form an orthonormal basis of $H^2$. Moreover, $\Delta(x) = e^{2i\pi(x+\gamma(x))},$ where $\gamma$ is an entire function, real on the real line, and such that $|\gamma(x)| \leq 0.004 \pi|x|.$
We also give an explicit unwinding of the singular inner function \( \exp(2i\pi x) \):

\[
\exp(2i\pi x) = e^{-2\pi} + (1 - e^{-4\pi}) \sum_{n \geq 0} (-1)^n e^{-2n\pi} B(x)^{n+1},
\]

where \( B \) is a Blaschke product whose zeros are \( \{1/(j+i)\}_{j \in \mathbb{Z}} \).

2. Preliminaries and notation

For \( p \geq 1 \), \( \mathcal{H}^p(\mathbb{T}) \) stands for the space of analytic functions \( f \) on the unit disk \( \mathbb{D} \) such that

\[
\sup_{0 < r < 1} \int_0^{2\pi} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < +\infty.
\]

Such functions have boundary values almost everywhere, and the Hardy space \( \mathcal{H}^p(\mathbb{T}) \) can be identified with the set of \( L^p \) functions on the torus \( \mathbb{T} = \partial \mathbb{D} \) whose Fourier coefficients of negative order vanish.

A subspace of \( \mathcal{H}^p(\mathbb{T}) \) is invariant if it is invariant by multiplication by \( e^{i\theta} \) (or by \( z \), depending whether these functions are considered as functions on \( \mathbb{T} \) or \( \mathbb{D} \)). An inner function is a bounded analytic function on the unit disk whose boundary values have modulus 1 almost everywhere. It is known that the invariant subspaces are of the form \( u \mathcal{H}^p(\mathbb{T}) \) where \( u \) is an inner function. The inner function \( u \) is determined by the invariant subspace up to multiplication by a constant of modulus 1.

Any \( f \in \mathcal{H}^p(\mathbb{T}) \) decomposes as \( gu \), where \( u \) is inner and \( g \) outer. The inner function in its turn can be further decomposed as \( BS \), where \( B \) is a Blaschke product, which accounts for all the zeros, and \( S \) a singular inner function \([6, 5]\).

If \( f \) and \( g \) are two functions on \( \mathbb{T} \) (in \( L^p \) and \( L^{p/(p-1)} \) for some \( p \in [1, +\infty) \)), let

\[
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\overline{g(e^{i\theta})} \, d\theta.
\]

Let \( H \) be the operator of orthogonal projection of \( L^2(\mathbb{T}) \) onto \( \mathcal{H}^2(\mathbb{T}) \). It results from the properties of the Hilbert that this operator extends as a bounded operator from \( L^p(\mathbb{T}) \) to \( \mathcal{H}^p(\mathbb{T}) \) for \( 1 < p < +\infty \).

If \( u \) is an inner function, let \( \chi_u \) be the operator of multiplication by \( u \) (which is an isometry of all the \( L^p \)). Then the operator \( H_u = \chi_u H \chi_u^{-1} \) is the operator of orthogonal projection of \( L^2 \) onto \( u \mathcal{H}^2(\mathbb{T}) \). It results that this operator extends as a bounded operator from \( L^p(\mathbb{T}) \) to \( \mathcal{H}^p(\mathbb{T}) \) for all \( p \in (1, +\infty) \) with a norm independent of \( u \). In other terms, for all \( p > 1 \), there exists \( C_p \) such that, for all \( u \) and all \( f \in L^p(\mathbb{T}) \),

\[
\|H_u f\|_p \leq C_p \|f\|_p.
\]
There is a parallel theory for analytic functions on the upper half plane \( \mathbb{H} = \{ x + iy : y > 0 \} \). The space of analytic functions \( f \) on \( \mathbb{H} \) such that
\[
\sup_{y>0} \|f(\cdot + iy)\|_{L^p(\mathbb{R})} < +\infty
\]
is denoted by \( H^p(\mathbb{R}) \). These functions have boundary values in \( L^p(\mathbb{R}) \) when \( p \geq 1 \). The space \( H^p(\mathbb{R}) \) is identified to the space of \( L^p \) functions whose Fourier transform vanishes on the negative half line \((-\infty, 0)\).

A subspace of \( H^2(\mathbb{R}) \) is said to be invariant if it is stable by multiplication by the functions \( e^{2\pi i \xi x} \) for all \( \xi > 0 \). As previously, the invariant subspaces are of the form \( uH^2 \) where \( u \) is an inner function, i.e., an analytic function on \( \mathbb{H} \) whose boundary values are of modulus 1 almost everywhere.

As previously, the operators of orthogonal projections on invariant subspaces extend, for any \( p \in (1, +\infty) \), as continuous operators on \( H^p(\mathbb{R}) \) with a uniform bound for their norms.

3. Phase unwinding on the torus

In this section, one simply writes \( H^p \) instead of \( H^p(\mathbb{T}) \).

3.1. Phase unwinding. The following construction is a slight generalization of the one described in [1, 2, 7].

One starts with \( f \in H^p \). We choose a projector \( Q_0 \) on some subspace of \( H^p \) and write \( g_0 = Q_0 f \) and \( f = g_0 + u_1 f_1 \), where \( u_1 \) is inner and \( f_1 \) outer. Then choose a projector \( Q_1 \), not necessarily different from \( Q_0 \), and write \( g_1 = Q_1 f_1 \) and \( f_1 = g_1 + u_2 f_2 \), and so on indefinitely unless reaching 0. This leads to the expansion
\[
f = g_0 + u_1 g_1 + u_1 u_2 g_2 + \cdots + g_{n-1} \prod_{1 \leq j < n} u_j + f_n \prod_{1 \leq j \leq n} u_j, \tag{2}
\]
which is orthogonal when \( p = 2 \).

When the process does not stop, it is natural to ask in what sense the series
\[
g_0 + \sum_{n \geq 1} g_n \prod_{0 \leq k \leq n} u_k \tag{3}
\]
represents \( f \). We shall answer this question in the next section.

In previous works, the projectors \( Q_j \) were of the form \( \text{Id} - H_v \), where \( v \) is an inner function. Let us investigate this case.

Let \( (v_j)_{j \geq 1} \) be a sequence of inner functions. Let \( f \) in \( H^p \) for some \( p \geq 1 \). Define by recursion three sequences (maybe finite) of functions
\( (f_n)_{n \geq 0}, (g_n)_{n \geq 0}, \) and \( (u_n)_{n \geq 1} \), where, for \( n \geq 1 \), the \( u_n \) are inner, and the \( f_n \) outer:

- \( f_0 = f \),
- to pass from step \( n \) to step \( n + 1 \), consider the projection \( h_n = H_{v_{n+1}} f_n \) of \( f_n \) on \( v_{n+1} \mathcal{H}^p \); if \( h_n = 0 \) then stop, otherwise let \( f_{n+1} \) be the outer part of \( h_n \) and \( u_{n+1} \) its inner part, and set \( g_n = f_n - h_n = f_n - u_{n+1} f_{n+1} \).

Notice that, since \( h_n \in v_n \mathcal{H}^p \), \( u_{n+1}/v_{n+1} \) is an inner function. In particular \( u_{n+1} \) vanishes at the zeros of \( v_{n+1} \) with a not smaller multiplicity.

When \( v_{n+1} \) is a convergent Blaschke products, consider a Malmquist-Takenaka basis \( e_1, e_2, \ldots \) associated with this product (see [17] and the next section). Then, for \( f \in \mathcal{H}^2 \), \( g_n = \sum_j \langle f_n, e_j \rangle e_j \). As the functions \( e_j \) are bounded, these scalar products are well defined if \( f \in \mathcal{H}^p \) and the above expression for \( g_n \) holds as well.

The above construction, when it does not stop, appeals further comments. Any zero of \( v_n \) is also a zero of \( u_n \) (with a multiplicity not smaller). Consider the following decreasing sequence of subspaces of \( \mathcal{H}^p \):

\[
V_n = u_1 u_2 \cdots u_n \mathcal{H}^p
\]

and the space \( V_\infty = \bigcap_{n \geq 0} V_n \).

If a function \( h \) is in \( V_\infty \), the set of its zeros contains all the zeros of the \( v_n \) counted with their multiplicities at least. This means that if \( \sum |1 - z_j| = \infty \) (where the \( z_j \) are the zeros of the \( (v_n)_{n \geq 1} \) repeated according to their multiplicities) \( h = 0 \). In other word, under this hypothesis, \( V_\infty = \{0\} \).

Now, let us describe a few choices of the \( v_j \). If we take \( v_j(z) = z \) for all \( z \), then \( g_n(z) = f_n(0) \). This is a variant of the case studied in [7,11,2]. Indeed in these articles, in the recursion \( f_n - f_n(0) \) is decomposed as gSB, where \( g \) is outer, \( S \) singular inner, and \( B \) a Blaschke product; then one sets \( f_{n+1} = g S \) instead of \( f_{n+1} = g \).

Another possibility is to take \( v_n(z) = \frac{z - a_n}{1 - a_n z} \) with \( |a_n| < 1 \) and \( \sum (1 - |a_n|) = \infty \). Then \( g_n(z) = f_n(a_n)(1 - |a_n|^2)/(1 - a_n z) \).

An important remark is that the sequence \( v_j \) can be defined ‘on the fly’ to be well adapted to the function \( f \) under analysis. For instance, one can choose \( v_{n+1} \) within a collection of Blaschke products to maximize the \( L^2 \)-norm of \( g_n \). This is the greedy algorithm of Qian et al. [14,12,13].
Here is a variant of the above construction. Let $a_j$ be a sequence of numbers of moduli less than 1. Given a $f$ one considers the recursion:

- $f_0 = f$
- for $n \geq 1$, $c_{n-1} = f_{n-1}(z_{n-1})$, $f_{n-1} - c_{n-1} = u_n f_n$, where $u_n$ is inner and $f_n$ outer.

### 3.2. Nested invariant subspaces.

**Theorem 1.** Let $(V_n)_{n \geq 0}$, with $V_0 = H^2$, be a decreasing sequence of invariant subspaces. Set $V_\infty = \cap V_n$ and let $P_n$ stand for the operator associated with the inner function defining $V_n$. Then, for all $p \in (1, +\infty)$ and $f \in H^p$, one has

$$\lim_{n \to +\infty} \|P_n f - P_\infty f\|_p = 0.$$ 

**Proof.** Fix $p \in (1, +\infty)$ ($p \neq 2$) and $p_0 \in (1, +\infty)$ such that $p$ lies in the open interval delimited by 2 and $p_0$. Let $g \in H^2 \cap H^{p_0}$. By Hölder inequality, there exists $\alpha \in (0, 1)$, depending only on $p$ and $p_0$, such that $\|P_n g - P_\infty g\|_p \leq \|P_n g - P_\infty g\|^\alpha_2 \|P_n g - P_\infty g\|_p^{1-\alpha}$. It results from (1) that

$$\|P_n g - P_\infty g\|_p \leq (2C_p)^{1-\alpha} \|P_n g - P_\infty g\|_2^\alpha \|g\|_p^{1-\alpha}$$

and

$$\lim_{n \to +\infty} \|P_n g - P_\infty g\|_p = 0.$$ 

Now, if $f \in H^p$, for all $g \in H^2 \cap H^{p_0}$, one has

$$\|P_n f - P_\infty f\|_p \leq \|P_n g - P_\infty g\|_p + \|P_n (f - g) - P_\infty (f - g)\|_p,$$

therefore (due to (1))

$$\limsup_{n \to +\infty} \|P_n f - P_\infty f\|_p \leq 2C_p \inf_{g \in H^2 \cap H^{p_0}} \|f - g\|_p = 0.$$ 

**Corollary 2.** Let $Q_n = P_n - P_{n+1}$. Then, for all $p \in (1, +\infty)$ and $f \in H^p$, the series

$$\sum_{n \geq 0} Q_n f$$

converges to $f - P_\infty f$ in $L^p$.

In particular, this proves that the series (3) converges to $f$ in $H^p$ provided that $1 < p < +\infty$.

This corollary also contains the theorem by Szabó [16] and by Qian et al. [11] on the $H^p$-convergence of Malmquist-Takenaka series.
3.2.1. Malmquist-Takenaka bases. For the reader’s convenience we give an account of Malmquist-Takenaka bases.

**Lemma 3.** Let $a$ be a complex number of modulus less than 1. Then $(z - a)H^2$ has codimension 1 in $H^2$ and \( \frac{\sqrt{1 - |a|^2}}{1 - az} \) is a unit vector in the orthogonal complement of $(z - a)H^2$ in $H^2$.

**Proof.** One has

\[
\left\langle (z - a)f(z), \frac{\sqrt{1 - |a|^2}}{1 - \bar{a}z} \right\rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{i\theta} - a)f(e^{i\theta}) \frac{\sqrt{1 - |a|^2}}{1 - ae^{-i\theta}} \, d\theta
\]

\[
= \frac{\sqrt{1 - |a|^2}}{2\pi} \int_{-\pi}^{\pi} e^{i\theta} f(e^{i\theta}) \, d\theta = 0.
\]

Also, if $f$ is orthogonal to $(1 - \bar{a}z)^{-1}$ one has

\[
0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{i\theta})}{1 - ae^{-i\theta}} \, d\theta = \frac{1}{2i\pi} \oint f(z) \frac{dz}{z - a} = f(a),
\]

so $f \in (z - a)H^2$.

Now \((a_n)_{n>0}\) is a sequence of complex numbers of moduli less than 1 such that

\[
\sum_{n \geq 1} (1 - |a_j|^2) = +\infty. \quad (4)
\]

For $n \geq 0$, let

\[
B_n(z) = \prod_{0 \leq j < n} \frac{z - a_j}{1 - \bar{a}_j z} \quad \text{and} \quad \phi_n(z) = B_n(z) \frac{\sqrt{1 - |a_n|^2}}{1 - \bar{a}_n z}.
\]

It results from Lemma 3 that the functions $\phi_n$ form an orthonormal basis of $H^2$. Indeed, the spaces $B_nH^2$, for $n \geq 0$, form a nested sequence of invariant subspaces, and $\phi_n$ is a basis of the unidimensional space $B_nH^2 \ominus B_{n+1}H^2$. The bases so obtained are the Malmquist-Takenaka bases. Theorem 4 implies that, if $1 < p < +\infty$ and $f \in H^p$, the series \( \sum_{n \geq 0} \langle f, \phi_n \rangle \phi_n \) converges to $f$ in $H^p$.

### 4. The upper half plane

In this section, one simply writes $H^p$ instead of $H^p(\mathbb{R})$. 

4.1. Malmquist-Takenaka bases. Among the inner functions \( u \) there are the Blaschke products: let \((a_j)_{1 \leq j} \) be a sequence (finite or not) of complex numbers with positive imaginary parts and such that

\[
\sum_{j \geq 0} \frac{\Im a_j}{1 + |a_j|^2} < +\infty.
\]  

The corresponding Blaschke product is

\[
B(x) = \prod_{j \geq 0} \frac{|1 + a_j^2|}{1 + a_j^2} \frac{x - a_j}{x - \overline{a}_j},
\]

where, 0/0, which appears if \( a_j = i \), should be understood as 1. The factors \( \frac{|1 + a_j^2|}{1 + a_j^2} \) insure the convergence of this product when there are infinitely many zeroes. But, in some situations, it is more convenient to use other convergence factors as we shall see below.

Whatever the series (5) be convergent or not, one defines (for \( n \geq 0 \)) the functions

\[
\phi_n(x) = \frac{1}{\sqrt{\pi}} \left( \prod_{0 \leq j < n} \frac{x - a_j}{x - \overline{a}_j} \right) \frac{1}{x - a_n}.
\]

Then these functions form a orthonormal system in \( H^2 \). If the series (5) diverges, it is a basis of \( H^2 \), otherwise it is a basis of the orthogonal complement of \( B H^2 \) in \( H^2 \).

For \( 1 < p < +\infty \), and \( f \in H^p \), the series \( \sum_{n \geq 0} (f, \phi_n) \phi_n \) converges in \( H^p \) (towards \( f \) if the series (5) diverges). The proof is the same as previously.

4.2. A multiscale decomposition. The infinite products

\[
G_n(x) = \prod_{j \leq n} \frac{j - i}{j + i} \frac{x - j - i}{x - j + i} \quad \text{and} \quad G(x) = \prod_{j \in \mathbb{Z}} \frac{j - i}{j + i} \frac{x - j - i}{x - j + i}
\]

converge. As \( \frac{j - i}{j + i} \times \frac{-j - i}{-j + i} = 1 \), one has

\[
G(x) = -\lim_{n \to +\infty} \prod_{|j| \leq n} \frac{x - j - i}{x - j + i},
\]

which shows that \( G \) is periodic of period 1. It appears that these products can be expressed in terms of known functions.

Lemma 4. We have

\[
G_n(x) = \frac{\Gamma(-i - n)}{\Gamma(i - n)} \frac{\Gamma(x - n + i)}{\Gamma(x - n - i)} \quad \text{and} \quad G(x) = \frac{\sin \pi(i - x)}{\sin \pi(i + x)}.
\]
Proof. The well known formula \( \Gamma(z) = \lim_{n \to +\infty} \frac{n!n^z}{z(z + 1) \cdots (z + n)} \) yields the expression of \( G_0 \).

On the other hand,

\[
\frac{1}{G_0(-x)} = \prod_{j \leq 0} \frac{j + i}{j - i} \frac{x - j + i}{x - j - i} = \prod_{j \geq 0} \frac{-j + i}{-j - i} \frac{-x + j + i}{-x + j - i},
\]

Therefore

\[
G(x) = -\frac{(x + i)G_0(x)}{(x - i)G_0(-x)} = -\frac{(x + i)\Gamma(x + i)\Gamma(-x - i)}{(x - i)\Gamma(x - i)\Gamma(-x + i)} = -\frac{\Gamma(x + i)\Gamma(1 - (x + i))}{\Gamma(x - i)\Gamma(1 - (x - i))} = -\frac{\sin \pi(x - i)}{\sin \pi(x + i)}.
\]

4.3. An orthonormal system. Consider the function

\[
\phi(x) = \frac{\Gamma(x - 1 + i)}{\sqrt{\pi} \Gamma(x - i)}.
\]

It is easily checked that

\[
\phi(x - n) = \frac{\Gamma(i - n)}{\Gamma(-i - n)} \frac{G_n(x)}{\sqrt{\pi} (x - (n + 1) + i)}.
\]

Set \( \phi_n(x) = \phi(x - n) \). For fixed \( m \), the functions \( \phi_n/G_m \), for \( n \geq m \), form a Malmquist-Takenaka basis of \( (G/G_m)H^2 \). In other terms, the functions \( \phi_n \), for \( n \geq m \), form an orthonormal basis of \( G_mH^2 \ominus GH^2 \). This means that the functions \( \phi(x - n) \) (for \( n \in \mathbb{Z} \)) form a Malmquist-Takenaka basis of the orthogonal complement of \( GH^2 \) in \( H^2 \).

Then the same proof as the one of Theorem 4 yields the following results.

**Lemma 5.** Let \( f \in H^p \) for some \( p > 1 \). Then both series

\[
\sum_{j < 0} \langle f, \phi_j \rangle \phi_j \quad \text{and} \quad \sum_{j \geq 0} \langle f, \phi_j \rangle \phi_j
\]

are convergent in \( H^p \) and

\[
f = \sum_{n \in \mathbb{Z}} \langle f, \phi_n \rangle \phi_n + P_p f,
\]

where \( P_p \) stands for the extension to \( L^p \) of the orthogonal projector on \( GH^2 \).
4.3.1. Multiscale decomposition. As $|1 - G(x)| \leq C \min\{1, |x|\}$ the product

$$\Delta(x) = \prod_{j < 0} G(2^j x)$$

is convergent and $\lim_{n \to -\infty} \Delta(2^n x) = 1$ uniformly on compact sets.

Consider the following subspaces of $H^2$:

$$V_n = \Delta(2^n x) H^2.$$  

This is a decreasing sequence. The space $V_{+\infty} = \bigcap_{n \in \mathbb{Z}} V_n$ is equal to $\{0\}$ since a non-zero function in this space would have too many zeros, and the space $V_{-\infty} = \text{closure of } \bigcup_{n \in \mathbb{Z}} V_n$ is equal to $H^2$ since $\Delta(2^n x)$ converges to 1 uniformly on compact sets when $n$ goes to $-\infty$.

For all $n$ and $j$, let

$$\phi_{n,j}(x) = 2^{n/2} \phi(2^n x - j) \Delta(2^n x).$$

Then, for all $n$, $(\phi_{n,j})_{j \in \mathbb{Z}}$ is an orthonormal basis of $V_n \ominus V_{n+1}$. At last $(\phi_{n,j})_{n,j \in \mathbb{Z}}$ is an orthonormal basis of $H^2$. The following theorem results from the preceding discussion.

**Theorem 6.** Let $\prec$ stand for the lexicographic order on $\mathbb{Z} \times \mathbb{Z}$. Then, if $f \in H^p$ for some $p > 1$, one has

$$\lim_{(n,j) \to \infty} \sum_{(m,k) \prec (n,j)} \langle f, \phi_{m,k} \rangle \phi_{m,k} = f.$$  

Let us give another expression of $\Delta$. Write $\Delta = e^{i\psi}$. A simple calculation yields

$$G(x) = \exp 2i \left( \pi x + \sum_{n \geq 1} \frac{e^{-2\pi n}}{n} \sin \pi n x \right).$$

Then

$$\psi(x) = 2\pi x + 2 \sum_{n \geq 1} \frac{e^{-2\pi n}}{n} \sum_{k \geq 1} \sin 2\pi 2^{-k} n x.$$  

So, if we set

$$\Xi(t) = \sum_{k \geq 1} \sin 2^{-k} t$$

we get

$$\psi(x) = 2\pi x + 2 \sum_{n \geq 1} \frac{e^{-2\pi n}}{n} \Xi(2\pi n x).$$
It is worth noticing that \( \psi \), as a function on \( \mathbb{R} \), is increasing and that 
\[
|\psi(x) - 2\pi x| < 2\pi|x|e^{-2\pi}/(1 - e^{-2\pi}) < 0.004\pi|x|.
\]
This means that \( \Delta \) is “not far” from being periodic of period 1, and that (8) is reminiscent of the usual formula for wavelets.

A more precise bound of the form 
\[
|\psi(x) - 2\pi x| \leq c \min\{|x|, \log(1+|x|)\}
\]
can be obtained: one has \( |\Xi(x)| \leq n + 2 \) when \( n \geq 0 \) and \( 2^n \leq |x| \leq 2^{n+1} \).

5. Explicit phase unwindings

The unwinding procedure can be performed for \( \mathcal{H}^2(\mathbb{R}) \) functions in the same way as in Section 3. This time, we take 
\[
h_j(x) = \frac{1}{\sqrt{\pi}} (z_j - \bar{z}_j),
\]
with \( \Re z_j > 0 \).

Let \( f(x) = e^{2i\pi x} \). The functions \( f - f(i) \) and \( G \) have the same zeros. So we can write 
\[
f(x) - f(i) = G(x)h(x):
\]
\[
h(x) = e^{-2\pi \left(e^{2i\pi(x-i)} - 1\right)} \sin \pi(i + x) \over \sin \pi(i - x) = -2i e^{i\pi(x-i+2i)} \sin \pi(i + x) = 1 - e^{2i\pi(x+i)} = 1 - e^{-2\pi} f(x).
\]

It results the following identity
\[
f = e^{-2\pi} + \frac{G}{1 + e^{-2\pi}G}.
\]
(9)

So we get the following representation of \( e^{2i\pi x} \),
\[
e^{2i\pi x} = e^{-2\pi} + (1 - e^{-4\pi}) \sum_{n \geq 0} (-1)^n e^{-2n\pi} G(x)^{n+1},
\]
(10)

which gives an explicit unwinding series for \( \frac{e^{2i\pi x}}{\sqrt{\pi}(x+i)} \):
\[
\frac{e^{2i\pi x}}{\sqrt{\pi}(x+i)} = \frac{e^{-2\pi}}{\sqrt{\pi}(x+i)} + (1 - e^{-4\pi}) \sum_{n \geq 0} (-1)^n e^{-2n\pi} \frac{G(x)^{n+1}}{\sqrt{\pi}(x+i)}.
\]
(11)

Also, by replacing \( x \) by \( \frac{\pi}{2\pi} \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \) in Formula (10) we get an unwinding series of the singular inner function \( \exp \frac{1 + e^{i\theta}}{1 - e^{i\theta}} \) on the torus.
In the same way, more general formulas can be obtained. Indeed, for \( \alpha > 0 \) let us define the following Blaschke product

\[
G_\alpha(x) = \prod_{n \in \mathbb{Z}} \frac{x - n - i\alpha}{x - n + i\alpha}.
\]

We then have \( G_\alpha(x) = \frac{\sin \pi (i\alpha - x)}{\sin \pi (i\alpha + x)} \) and

\[
e^{2i\pi x} = e^{-2\pi \alpha} + G_\alpha(x) (1 - e^{-2\pi \alpha} e^{2i\pi x}).
\]

This last formula leads to a variety of unwinding series: let \( (\alpha_n)_{n > 0} \) be a sequence of positive numbers, set \( a_n = e^{-2i\alpha_n} \); then

\[
e^{2i\pi x} = a_1 + (1 - a_1 a_2) G_{a_1}(x) - a_1 (1 - a_2 a_3) G_{a_1}(x) G_{a_2}(x) + a_1 a_2 (1 - a_3 a_4) G_{a_1}(x) G_{a_2}(x) G_{a_3}(x) - a_1 a_2 a_3 (1 - a_4 a_5) G_{a_1}(x) G_{a_2}(x) G_{a_3}(x) G_{a_4}(x) + \cdots
\]

Also, one can use identity (9) to analyse the singular inner function on the real line defined by the Dirac mass at 0, namely \( e^{-2i\pi/x} \):

\[
e^{-2i\pi/x} = \frac{e^{-2\pi} + B(x)}{1 + e^{-2\pi} B(x)},
\]

where \( B \) is the Blaschke product \( G(1/x) \) whose zeros are the points \(-1/(n + i), \ n \in \mathbb{Z}\).

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