Schur-Weyl type duality for quantized $\mathfrak{gl}(1|1)$, the Burau representation of braid groups and invariants of tangled graphs

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Abstract
We show that the Schur-Weyl type duality between $\mathfrak{gl}(1|1)$ and $GL_n$ gives a natural representation-theoretic setting for the relation between reduced and non-reduced Burau representations.

Introduction
The goal of this note is to clarify the relation between reduced Burau representations of braid groups, non-reduced Burau representations, and the representation of the braid group defined by $R$-matrices related to $U_q(\mathfrak{gl}(1|1))$.

A lot is known about the relation between the quantized universal enveloping algebra $U_q(\mathfrak{gl}(1|1))$ of the Lie superalgebra of $\mathfrak{gl}(1|1)$, multivariable Alexander-Conway polynomials on links, and the Burau-Magnus representations of braid groups.

In this paper we show that the Schur-Weyl type duality between $\mathfrak{gl}(1|1)$ and $GL_n$ gives a natural representation-theoretic setting for the relation between reduced and non-reduced Burau representations. We use this simple fact as an excuse to sum up some known (but partly folklore) facts about these representations and the invariants of knots.

In Section 1 we recall the definition of and basic facts about quantized $\mathfrak{gl}(1|1)$. Section 2 describes the duality between $GL_n$ and $U_q(\mathfrak{gl}(1|1))$. In Section 3 we show how the Burau representation naturally reduces on the space of multiplicities. Section 4 relates the Alexander-Conway polynomial to the trace on the multiplicity space.

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1 Quantum $\mathfrak{gl}(1|1)$ and its representations

1.1

Consider the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(1|1)$. Explicitly, this means we consider the super vector space $M$ of all complex $2 \times 2$-matrices with even part $M_0$ spanned by the matrix units $E_{1,1}$ and $E_{2,2}$, and odd part $M_1$ spanned by the matrix units $E_{1,2}$ and $E_{2,1}$, equipped with the Lie superalgebra structure given by the super commutator. The universal enveloping super algebra $U(\mathfrak{g})$ has a quantum version $U_h(\mathfrak{gl}(1|1))$ defined as follows (see e.g. [7]):

Let $\mathbb{C}[[h]]$ denote the ring of formal power series in $h$. The $\mathbb{C}[[h]]$-super algebra $U_h(\mathfrak{gl}(1|1))$ is generated freely as a $\mathbb{C}[[h]]$-algebra by (odd) elements $X,Y$ and (even) elements $G,H$ modulo the defining relations:

\[
\{X,Y\} = e^{hH} - e^{-hH}, \quad X^2 = Y^2 = 0,
\]

\[
[G,X] = X, \quad [G,Y] = -Y
\]

\[
[H,X] = 0, \quad [H,Y] = 0, \quad [H,G] = 0
\]

, (using the common abbreviation $[A,B] := AB - BA$ and $\{A,B\} = AB + BA$.)

$U_h(\mathfrak{gl}(1|1))$ is a Hopf superalgebra with the comultiplication

\[
\Delta X = X \otimes e^{\frac{hH}{2}} + e^{-\frac{hH}{2}} \otimes X, \quad \Delta Y = Y \otimes e^{\frac{hH}{2}} + e^{-\frac{hH}{2}} \otimes Y
\]

\[
\Delta H = H \otimes 1 + 1 \otimes H, \quad \Delta G = G \otimes 1 + 1 \otimes G
\]

The Hopf superalgebra is quasi-triangular with $R$-matrix

\[
R = \exp(h(H \otimes G + G \otimes H))(1 - e^{\frac{hH}{2}}X \otimes e^{-\frac{hH}{2}}Y)
\]

That is this element satisfies the following identities:

\[
\Delta(a)^{op} = R\Delta(a)R^{-1}
\]

and

\[
(\Delta \otimes \text{id})(R) = R_{13}R_{23}, \quad (\text{id} \otimes \Delta)(R) = R_{13}R_{12}
\]

There is an integral form $U_q(\mathfrak{gl}(1|1)) \subset U_h(\mathfrak{gl}(1|1))$, which is generated by $X,Y,G$ and the invertible element $t = e^{\frac{hH}{2}}$ as a $\mathbb{C}[e^h, e^{-h}]$-algebra. As usual, we write $q = e^h$. 

2
1.2

Recall that there is up to isomorphism precisely one irreducible \( \mathfrak{gl}(2) \)-module of a fixed dimension \( n \) (for instance the natural representation for \( n = 2 \)). In contrast, the algebra \( U_q(\mathfrak{gl}(1|1)) \) has 2-(complex) parameter family of irreducible representations on \( \mathbb{C}^{1|1} \) for \( z \in \mathbb{C}^* \), \( n \in \mathbb{C} \) denote by \( V_{z,n} \) the irreducible 2-dimensional representation \( V_{z,n} = \mathbb{C}v \oplus \mathbb{C}u \) with \( v \) even and \( u \) odd such that

\[
Xv = u, \quad Yv = 0, \quad Gv = nv, \quad tv = zv, \quad (2)
\]

(from which \( Xu = 0, \, Gu = (n + 1)u, \, Yu = (z^2 - z^{-2})v \) and \( tu = zu \) follows). Obviously, one can also consider the representation \( \Pi V_{z,n} \) with the parity of the elements reversed. The representation \( \Pi V_{z,n} \) can be realized as \( \epsilon \otimes V_{z,n} \) where \( \epsilon \) is an odd one-dimensional representation. These representations and their tensor products will in fact be essentially the only \( \mathfrak{gl}(1|1) \)-representations of interest to us. For more details on the representation theory see e.g. [13, §11].

1.3

Let \( V \) be a finite dimensional representation of \( U_q(\mathfrak{gl}(1|1)) \). It decomposes into a direct sum of weight spaces for \( G \),

\[
V = \bigoplus_{n \in \mathbb{C}} V(n).
\]

Note that we do not assume the weights to be integral. As usual, the elements \( X \) and \( Y \) act from one weight space to another

\[
X : V(n) \to V(n + 1), \quad Y : V(n) \to V(n - 1),
\]

and we have \( X^2 = Y^2 = 0 \). Hence, \( V \) can be viewed as a complex with two differentials acting in opposite directions. The DeRham complex of any Kähler manifold carries an action of \( \mathfrak{gl}(1|1) \), such that the element \( H \) acts as the Laplace operator. Thus, the algebra \( U_h(\mathfrak{gl}(1|1)) \), and for the same reasons \( U(\mathfrak{gl}(1|1)) \), is in a certain sense, an abstraction of the structures of Hodge theory.

These are, in fact, isomorphic as algebras; the difference between them lies in the action of the differential on \( V \otimes W \): the usual, diagonal action for \( U(\mathfrak{gl}(1|1)) \), the comultiplication for \( U_q(\mathfrak{gl}(1|1)) \) gives another action.

Alternatively, any \( \mathfrak{gl}(1|1) \)-representation can be thought of as a matrix factorization with extra structure (primarily, an upgrade of the \( \mathbb{Z}_2 \)-grading to a \( \mathbb{Z} \)-grading). The underlying super vector space remains unchanged, with \( X + Y \) giving the differential, and

\[
(X + Y)^2 = \{X, Y\} = t^2 - t^{-2}
\]

as the potential.
The decomposition of the tensor product

2.1

Let \( \text{Cl}_N \) be the Clifford algebra (over \( \mathbb{C} \)) with 2\( N \) generators:

\[
\text{Cl}_N = \langle a_i, b_i; i = 1, \ldots, N | \{a_i, a_j\} = \{b_i, b_j\} = 0, \{a_i, b_j\} = \delta_{ij} \rangle
\]

The algebra \( \text{Cl}_N \) has an irreducible 2\( N \)-dimensional representation \( U_N \) generated by a cyclic vector \( v \) with \( b_i v = 0 \). We might identify the basis vectors with the set of \( \{0, 1\} \)-sequences of length \( N \), such that \( v = (0, 0, 0, \ldots) \) and \( a_i \) annihilates all basis vectors \( S = (s_1, s_2, \ldots, s_N) \) with \( s_i = 1 \), and otherwise sends \( S \) to \((-1) \sum_{i=1}^{N-1} s_i S'\) where \( S' \) differs from \( S \) exactly in the \( i \text{th} \) entry. If we consider the subspace \( U = \text{span}(a_1, \ldots, a_n) \) of \( \text{Cl}_N \), then there is a natural isomorphism of graded vector spaces:

\[
U_N \longrightarrow \bigwedge U
\]  \( (3) \)

where \( s_{j_1}, s_{j_2}, \ldots, s_{j_k} \) are precisely the 1’s appearing (in this order) in \( S \). The action of \( a_i \) gets turned into \( x \mapsto a_i \wedge x \).

In case \( N = 1 \), \( U_N \) is 2-dimensional, and the irreducible 2-dimensional representation \( \text{(2)} \) is obtained by pulling back the Clifford algebra action to \( U_q(\mathfrak{gl}(1|1)) \) via the algebra homomorphism \( U_q(\mathfrak{gl}(1|1)) \to \text{Cl}_1 \)

\[
X \mapsto a_1, \quad Y \mapsto (z - z^{-1}) b_1, \quad t \mapsto z, \quad G \mapsto n + a_1 b_1
\]

This formalism can be extended to the \( N \)-fold tensor product (via the comultiplication \( \Delta \)) of these representations:

**Proposition 2.1.** Let \( V(n, z) = V_{z_1, n_1} \otimes \cdots \otimes V_{z_N, n_N} \). Then the mapping

\[
X \mapsto \sum_{i=1}^{N} z_i^{-1} \ldots z_{i+1}^{-1} z_{i+1} \ldots z_N a_i, \quad t \mapsto z_1 \ldots z_N, \\
Y \mapsto \sum_{i=1}^{N} z_i^{-1} \ldots z_{i+1}^{-1}(z_i^2 - z_i^{-2}) z_1 \ldots z_N b_i, \quad G \mapsto \sum_{i=1}^{N} (n_i + a_i b_i).
\]

defines uniquely an algebra homomorphism \( \Phi_{n, z} : U_q(\mathfrak{gl}(1|1)) \to \text{Cl}_N \). Pulling back via this map the representation \( U_N \) of \( \text{Cl}_N \) gives the tensor product representation \( V(n, z) \).

**Proof.** One easily verifies that the map is compatible with the relations of \( U_q(\mathfrak{gl}(1|1)) \). The second statement follows then also by explicit calculations. \( \square \)

2.2

The vector \( v_N = v \otimes \cdots \otimes v \in V(n, z) \) is a lowest weight vector of lowest weight \( \lambda = \sum_{i=1}^{N} n_i \), i.e. \( Y v_N = 0 \) and \( G v_N = \lambda v_N \).

The subspaces \( U = \text{span}(a_1, \ldots, a_N) \) and \( U' = \text{span}(b_1, \ldots, b_N) \) of \( \text{Cl}_N \) can be paired via \( U \otimes U' \to \mathbb{C}, \ a_j \otimes b_i \mapsto \delta_{i,j} \). Abbreviate \( \Phi = \Phi_{n, z} \) and let \( W = (\mathbb{C} \Phi(Y))^\perp \) and \( W' = (\mathbb{C} \Phi(X))^\perp \).
Lemma 2.2. Let \( z := z_1 z_2 \cdots z_N \). Assume \( z^2 - z^{-2} \neq 0 \). Then

1. \( U = C \Phi(X) \oplus W \), and \( U' = C \Phi(Y) \oplus W' \).

2. The subspaces \( W, W' \) generate a subalgebra \( C(X,Y) \) of \( \text{Cl}_N \) isomorphic to \( \text{Cl}_{ \text{dim}(W) } \), which is the super-commutant of the subalgebra generated by \( X \) and \( Y \).

Proof. The inclusion \( U \subseteq C \Phi(X) \oplus W \) holds by definition. For the inverse it is enough to find (for \( 1 \leq i \leq N \)) \( \beta_i \in \mathbb{C} \) such that \( a_i - \beta_i \Phi(X) \in W \).

One easily verifies that \( \beta_i = \frac{z^2 (1 - z^{-4})}{z - z^{-2}} \) does the job. The sum is direct, since an element \( u \) in the intersection is of the form \( u = \sum_{i=1}^{N} \gamma_i \Phi(X) \), hence with our assumption \( \alpha = 0 \) and so \( u = 0 \). The argument for \( U' \) is similar. Part 1 follows.

Now \( C(X,Y) \) is clearly contained in the commutant of \( X \) and \( Y \). Since \( \text{dim}(X,Y) = 4 \), and the action of this subalgebra on \( U_N \) is semi-simple, by \( 2^{N-1} \) copies of the unique 2-dimensional irreducible representation of \( (X,Y) \). Thus, its commutant is of dimension \( 2^{2(N-1)} \). Since \( C(X,Y) \) has this dimension, it must be the entire commutant, obviously isomorphic to the Clifford algebra as claimed.

In order to find the super-commutant not just of \( \Phi(X) \) and \( \Phi(Y) \), but all of \( U_q(\mathfrak{gl}(1|1)) \), we must find the subalgebra which also commutes with \( \Phi(G) \).

Proposition 2.3 (Schur-Weyl duality). Let still \( z^2 - z^{-2} \neq 0 \). The subalgebra of \( \text{Cl}_W \) commuting with \( \Phi(G) \) is that of Euler degree 0, i.e. that generated by elements of the form \( W \cdot W' \). There is a natural map \( U(\mathfrak{gl}(W)) \rightarrow \text{Cl}_W \subset \text{Cl}_N \) whose image is this subalgebra.

Proof. The first statement is obvious. Recall that \( W \) and \( W' \) generate a Clifford algebra, say with generators \( a'_i, b'_j \). Note that \( W \cdot W' \) forms a Lie subalgebra of \( \text{Cl}_W \) isomorphic to \( \mathfrak{gl}(W) \) (by mapping \( a'_i b'_j \) to the matrix unit \( E_{i,j} \)), hence this extends to an algebra map \( U(\mathfrak{gl}(W)) \rightarrow \text{Cl}_W \subset \text{Cl}_N \) The image of this map is precisely the commutant, because by the PBW theorem for Clifford algebras, the subspace of Euler degree 0 is that of the form \( \bigoplus_n W^n \cdot (W')^n = \bigoplus_n (W \cdot W')^n \).

Under the action of \( \text{Cl}_W \), \( U_N \) decomposes into two copies of \( U_W = \bigwedge W \), one with parity reversed. Thus, \( V(z,n) \) is completely decomposable and, up to grading shift and parity-reversal, the summands are precisely the 2-dimensional simple modules from above. Of course, the highest weight vector \( v_N \) generates a copy of \( V_{z,\lambda} \), so all simple submodules must be of the form \( V_{z,\lambda+k} \) for some \( k \) (possibly with parity reversed). Thus,
Proposition 2.4 (Tensor space decomposition).

The multiplicity space of $V_{z,\lambda+k}$ in $V(n,z)$ is the space of weight $k$ (for $G$) in $UW$. That is

$$V(z,n) \simeq \bigoplus_{k=0}^{N-1} \bigwedge^k W \otimes \Pi^k V_{z,\lambda+k} \quad (5)$$

where $\Pi$ is the shift of parity, $\Pi^2 = id$.

2.3

This decomposition of the tensor product can be made more explicit if we chose a basis $c_i$, $i = 1, \ldots, N-1$ in the subspace $W \subset U$ complementary to $C\Phi(X)$, hence fixing a decomposition $U = C\Phi(X) \oplus \bigoplus_{i=1}^{N-1} Cc_i$. From now on we will just write $X, Y$ instead of $\Phi(X), \Phi(Y)$.

Lemma 2.5. We have the following formulas

$$Xc_{i_1} \ldots c_{i_k} w = (-1)^k c_{i_1} \ldots c_{i_k} Xw$$

$$Yc_{i_1} \ldots c_{i_k} w = (-1)^k c_{i_1} \ldots c_{i_k} Yw + \sum_{j=1}^{k} y_{i_j} (-1)^{j-1} c_{i_1} \ldots \widehat{c_{i_j}} \ldots c_{i_k} w$$

where $w \in U$ and the $y_i$'s are defined by $YC_i + c_i Y = y_i$.

Proof. Obvious.

For a vector $v \in U$ define

$$(v)_{i_1,\ldots,i_k} = c_{i_1} \ldots c_{i_k} v + (-1)^k \frac{1}{z - z^{-1}} \sum_{a=1}^{k} y_a (-1)^{a-1} c_{i_1} \ldots \widehat{c_{i_a}} \ldots c_{i_k} v$$

Proposition 2.6. The space

$$V_{i_1,\ldots,i_k} = C(v_N)_{i_1,\ldots,i_k} \oplus C X(v_N)_{i_1,\ldots,i_k}$$

where $v_N$ is the highest weight vector (see section 2.2), is an irreducible submodule isomorphic to $V_{z, (\sum_{i=1}^{N} n_i) + k}$. This submodule corresponds to the monomial $c_{i_1} \wedge \cdots \wedge c_{i_k}$ in the decomposition (5).

Proof. We have

$$X(v_N)_{i_1,\ldots,i_k} = (Xv_N)_{i_1,\ldots,i_k}$$

and since $Yv_N = 0$, we have

$$Y(v_N)_{i_1,\ldots,i_k} = 0, \quad YX(v_N)_{i_1,\ldots,i_k} = (z^2 - z - 2)(v_N)_{i_1,\ldots,i_k}$$

The statement follows directly from the action of $t$ and $G$ and [2].
3 The relation to the Burau representation

3.1
The action of the universal $R$-matrix $\Pi$ in the tensor product representation $V_{2_1,n_1} \otimes V_{2_2,n_2}$ can easily be computed explicitly. Namely, in terms of the weight basis (by abuse of language we use the basis $\{v, Xv\}$ for either module) this right action looks as follows:

\[
R(v \otimes v) = z_1^{2n_2} z_2^{2n_1} v \otimes v \\
R(v \otimes Xv) = z_1^{2n_2+2} z_2^{2n_1} v \otimes Xv - z_2^{-1} z_1 (z_2^2 - z_2^{-2}) z_1^{2n_2+2} z_2^{2n_1} Xv \otimes v \\
R(Xv \otimes v) = z_1^{2n_2} z_2^{2n_1+2} Xv \otimes v \\
R(Xv \otimes Xv) = z_1^{2n_2+2} z_2^{2n_1+2} Xv \otimes Xv
\]

In the tensor product basis $v \otimes v, v \otimes Xv, Xv \otimes v, Xv \otimes Xv$ it produces the $4 \times 4$ matrix $R^{(z_1, z_2)} = z_1^{-2n_2} z_2^{-2n_1} (R)$,

\[
R^{(z_1, z_2)} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & z_1^2 & -(z_2^{-2} - z_2^2) z_1^3 z_2^{-1} & 0 \\
0 & 0 & z_2^2 & z_1 z_2 \\
0 & 0 & 0 & z_1^{-2} z_2^{-2}
\end{bmatrix}
\]  

with

\[
R^{(z_1, z_2)}^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & z_1^{-2} & (z_2^{-2} - z_2^2) z_1^{-3} z_2 & 0 \\
0 & 0 & z_2^{-2} & z_1^{-1} z_2 \\
0 & 0 & 0 & z_1^{-2} z_2^{-2}
\end{bmatrix}
\]

3.2
Consider the groupoid of braids whose strands are labeled by elements in $\mathbb{C} \times \mathbb{C}^*$. Each $N$-braid with colors $(z_i, n_i), 1 \leq i \leq N$ on its $N$ strands defines a morphism from the tuple $(z, n)$ to the permuted tuple $(\sigma z, \sigma n)$ given by the braid. Assigning to a tuple $(z, n)$ the representation $V(z, n) = V_{2_1,n_1} \otimes \cdots \otimes V_{2_N,n_N}$ and to the single (positive) braid $\beta$, with strands colored by $a := (z_i, n_i)$ and $b := (z_{i+1}, n_{i+1})$ the mapping

\[
\pi(\beta_i)(a, b) : V(z, n) \to V(s_i z, s_i n)
\]

\[
\pi(\beta_i)(a, b) = -z_1^{-3} z_2^{-1} P_{i,i+1} \circ \left( 1 \otimes \cdots \otimes 1 \otimes R^{(z_1, z_2)} \otimes 1 \otimes \cdots \otimes 1 \right)
\]

defines a representation $\pi$ of the colored braid groupoid. Here $R^{(z_1, z_2)}$ is as above, hence up to a multiple, the universal $R$-matrix $\Pi$ acting on $V_{n_{\sigma_1}, z_{\sigma_1}} \otimes V_{n_{\sigma_2}, z_{\sigma_2}}$ and $P_{i,i+1}$ is the flip map of simply swapping the two tensor factors as $x \otimes y \mapsto (-1)^{im} y \otimes x$. To verify the claim note that the braid relations amount to the relations

\[
\pi(\beta_i)(a, b) \circ \pi(\beta_{i+1})(a, c) \circ \pi(\beta_i)(b, c) = \pi(\beta_{i+1})(b, c) \circ \pi(\beta_i)(a, c) \circ \pi(\beta_{i+1})(a, b),
\]

\[
\pi(\beta_i)(a, b) \circ \pi(\beta_j)(c, d) = \pi(\beta_j)(c, d) \circ \pi(\beta_i)(a, b).
\]
for $j \neq i - 1, i, i + 1$ and $a, b, c, d$ arbitrary colors. These relations can easily be checked by direct calculations. In particular, the subgroup $\mathbb{E}_n$ of the braid group that preserves $(z, n)$ acts on $V(z, n)$. Because the operators $\pi(\beta_i)$ commute with the action of $U_q(\mathfrak{gl}(1|1))$, the action is determined by the action on multiplicity spaces.

The first interesting multiplicity space is $W$ considered as a subspace of $U$:

**Proposition 3.1.** In the case where $z_1 = \cdots = z_n$, this braid group representation on $U$ is isomorphic to the Burau representation. Similarly, the action on $W$ gives rise to the reduced Burau representation in case $z_1 = \cdots = z_n$.

**Proof.** Choose the basis $b_i = v \otimes v \otimes \cdots \otimes v \otimes Xv \otimes v \otimes \cdots \otimes v$, where $X$ is applied to the $i$-th factor. Then $\pi(\beta_i)$ defined in (8) acts on this basis as follows:

$$\pi(\beta_i) b_j = b_j \quad \text{for} \quad j \neq i, i + 1, \quad \text{and on} \quad b_i \text{ and } b_{i+1} \text{ it acts as the matrix }$$

$$- z_i^{-3} z_{i+1}^{-1} \begin{pmatrix} 0 & z_i^2 & z_{i+1} (z_i^2 - (z_i^2 - z_{i+1}^2) z_i z_{i+1}) \\ z_i^2 & - (z_i^2 - z_{i+1}^2) z_i z_{i+1} \end{pmatrix}$$

(9)

Change the basis as $b_i = A_i b'_i$ where $A_i a_i = -z_i z_{i+1}$, then $\pi(\beta_i)(b'_j) = b'_j$ for $j \neq i, i + 1$ and $\pi(\beta_i)$ acts on $b'_i, b'_{i+1}$ by

$$\begin{pmatrix} 0 & z_i^{-4} \\ 1 & (1 - z_i^{-4}) \end{pmatrix}$$

(10)

In case $z_i = z_j$ for any $i, j$, we set we $t := z_i^{-4}$ and obtain that $\pi(\beta_i)$ acts on $b_j$ an identity when $j \neq i, i + 1$ and on $b_i$ and $b_{i+1}$ by the matrix:

$$\begin{pmatrix} 0 & t^{-1} \\ t & 1 - t^{-1} \end{pmatrix}$$

But this is exactly the Burau representation, see for example [11, p.118 Example 3]. The invariant subspace is $\mathbb{C} X v$. The reduced Burau representation acts in the quotient space $W = U / \mathbb{C} X v$.

In general, we obtain a colored version of the Magnus representation of $\mathbb{E}_n$ obtained from an action on the free group on $N$ generators (see [2] p.102 ff for the non-colored version and for colored version see [3, Section 4]) and thus, Gassner representation of the pure braid group. In other words we proved the following.

**Theorem 3.2.** The mapping $\beta_i \mapsto \pi(\beta_i)(a_i, a_{i+1})$ gives the Magnus-Gassner representation of the pure braid group.

### 4 Multivariable Alexander-Conway polynomial

In this section we will use Theorem 2.4 to obtain the Alexander-Conway polynomial of a knot in terms of $R$-matrices for quantum $\mathfrak{gl}(1|1)$. These results are very closely related to the results in [6] and [8].
4.1

To construct invariants of links and tangled graphs let us start with the explicit decomposition of the two-folded tensor product. We abbreviate

$$
\gamma := (z_1^2 z_2^2 - z_1^{-2} z_2^{-2})^{-1},
$$

(assuming from now on this inverse exists). The following linear maps explicitly describe the decomposition of the tensor product of two generic irreducible two-dimensional representations:

$$
\varphi : V_{z_1, z_2, n + m} \oplus V_{z_1, z_2, n + m + 1} \to V_{z_1, n} \otimes V_{z_2, m}
$$

and

$$
\psi : V_{z_1, n} \otimes V_{z_2, m} \to V_{z_1, z_2, n + m} \oplus V_{z_1, z_2, n + m + 1}
$$

We denote by $w_1, X w_1$ (resp. $w_2, X w_2$) the standard basis in $V_{z_1, z_2, n + m + 1}$ and in $V_{z_1, z_2, n + m}$, and by $v_1, X v_1$ (resp. $v_2, X v_2$) the standard basis in $V_{z_1, n}$ and in $V_{z_2, m}$, respectively. Then the maps are defined as follows:

$$
\begin{align*}
\varphi(w_1) &= v_1 \otimes v_2, \\
\varphi(X w_1) &= z_2 X v_1 \otimes v_2 + (-1)^n z_1^{-1} v_1 \otimes X v_2, \\
\varphi(w_2) &= (-1)^{n+1} z_1^{-1} (z_2^2 - z_2^{-2}) \gamma X v_1 \otimes v_2 + z_2 (z_1^2 - z_1^{-2}) \gamma v_1 \otimes X v_2, \\
\varphi(X w_2) &= X v_1 \otimes X v_2 
\end{align*}
$$

and

$$
\begin{align*}
\psi(v_1 \otimes v_2) &= w_1, \\
\psi(X v_1 \otimes v_2) &= z_2 (z_1^2 - z_1^{-2}) \gamma X w_1 + (-1)^n z_1^{-1} z_2^{-1} w_2, \\
\psi(v_1 \otimes X v_2) &= (-1)^n z_1^{-1} (z_2^2 - z_2^{-2}) \gamma X w_1 + z_2 w_2, \\
\psi(X v_1 \otimes X v_2) &= X w_2
\end{align*}
$$

One easily verifies that they are inverse to each other:

$$
\psi \circ \varphi = \text{id}_{V \otimes V}, \quad \varphi \circ \psi = \text{id}_{V \otimes V}
$$

Let $P_0, P_1 \in \text{End}(V_{z_1, z_2, n + m} \oplus V_{z_1, z_2, n + m + 1})$ be the natural orthogonal projections to the first and the second summand respectively.

For any $A \in \text{End}(M)$ with $M$ an arbitrary super space, define the super trace to be $\text{str}(A)$ to be the trace of $A$ restricted to the even part of $M$ minus the trace of $A$ restricted to the odd part of $M$. For instance, if $M = V$, then $\text{str}(A) = A_{v, v} - A_{v, X v}$ where $v, X v = u$ is the weight basis in $V$.

We have the following identities for the super traces:

$$
\begin{align*}
\text{str}_2(\phi P_0 \psi) &= z_2^2 (z_1^2 - z_1^{-2}) \gamma \text{id}_{V_{z_1, n}}, \\
\text{str}_2(\phi P_1 \psi) &= -z_2^2 (z_1^2 - z_1^{-2}) \gamma \text{id}_{V_{z_1, n}}, \\
\text{str}_1(\phi P_0 \psi) &= z_1^{-2} (z_2^2 - z_2^{-2}) \gamma \text{id}_{V_{z_2, n}}, \\
\text{str}_1(\phi P_1 \psi) &= -z_1^{-2} (z_2^2 - z_2^{-2}) \gamma \text{id}_{V_{z_2, n}},
\end{align*}
$$

(11)
Here $\text{str}_{1,2}$ are partial super traces:

$$\text{str}_2(a \otimes b) = a \text{str}(b), \quad \text{str}_1(a \otimes b) = \text{str}(a)b$$

The matrix $PR(z; z)$ has the spectral decomposition:

$$PR(z; z) = z^2 \phi P_0 \psi - z^{-2} \phi P_1 \psi.$$

We also have

$$\text{str}_2(PR(z; z)) = z^2 \text{id}, \quad \text{str}_2((PR(z; z))^{-1}) = z^2 \text{id},$$

and it is easy to check that these identities agree with the spectral decomposition and super trace identities above.

### 4.2

Let $\pi$ be the representation from above and $\beta$ a braid. The partial trace $\text{tr}_{23...N}(\pi(\beta))$ is the evaluation of a central element in $U_h(\mathfrak{gl}(1|1))$ in the irreducible representation $V_{z_1, n_1}$. This is a general fact about the construction of link invariants from quasitriangular Hopf algebras [10], [14, Definition 2.1]. Therefore this partial trace is proportional to the identity. We will write

$$\text{str}_{23...N}(\beta) = \langle \text{str}_{23...N}(\pi(\beta)) \rangle I_1$$

where $I_1$ is the identity operator in $V(z_1, n_1)$.

**Theorem 4.1.** Abbreviating $z = z_1 \ldots z_N$, the following holds:

$$\langle \text{str}_{23...N}(\pi(\beta)) \rangle = \frac{z^2 - z^{-2}}{z^2 - z^{-2}} \sum_{m=0}^{N-1} \text{tr}_{\wedge^m W}(\pi_{W}(\beta))$$

**Proof.** This theorem follows immediately from Proposition 4.1. The decomposition of the tensor product $V(z, n)$ defines linear maps $f_m : \wedge^m W \rightarrow \wedge^m W$ for each element $f \in \text{End}(V(z, n))$. Using the explicit formulae for the decomposition of two irreducible 2-dimensional representations from the previous section and the formulae for partial traces $\text{str}_a(\phi_P \psi)$ from the previous subsection we arrive to the identity:

$$\langle \text{str}_{23...N}(f) \rangle = \sum_{m=0}^{N-1} \text{tr}_{\wedge^m W}(f_m) \frac{z^2_1 - z^{-2}}{z^2 - z^{-2}} z^2$$

Let $\hat{\beta}$ be the link which is the closure of the braid $\beta$. We number its connected components by $1 \leq i \leq k$ and denote by $w_i(\beta)$ the winding number of the $i$-th component of $\hat{\beta}$. Then the following holds:
Theorem 4.2. The function
\[ \tau(\beta) = \langle \text{tr}_{23...N}(\beta) \rangle z^2 \sum_{i=1}^{k} w_i(\beta) \]  
\[ (13) \]
is an invariant of the link \( \hat{\beta} \).

Proof. We have to verify the invariance with respect to Markov moves. The invariance with respect to the first Markov move means \( \tau(\sigma \beta \sigma^{-1}) = \tau(\beta) \). But this identity follows immediately from the conjugation invariance of the ordinary trace and Theorem 4.1. The second Markov move means that \( \tau(\beta s_{n-1}^{\pm 1}) = \tau(\beta) \) where \( \beta \) is a braid which has no factors \( s_{n-1}^{\pm 1} \). But this identity follows immediately from the property (12) of \( R \)-matrices.

As it was shown in [8] the invariant (13) is the Alexander-Conway polynomial \( \Delta_{z_1,...,z_N} \):
\[ \langle \text{tr}_{23...N}(\pi(\beta)) \rangle = z_1^2 - z_{1}^{-2}z_2^2 z_{2}^{-2} \Delta_{z_1,...,z_N}(\hat{\beta}) \]

Remark 4.3. For any \( U_q(\mathfrak{gl}(1|1))\)-linear map \( f \in \text{End}(V(z_1,n_1) \otimes V(z_2,n_2)) \) we have the following identity:
\[ z_1^{-2} \text{str}_2(f) = z_2^2 \text{str}_1(f) \]
This follows immediately from (11). This property is a projective version of the ambidextrous [15] property of \( U_q(\mathfrak{gl}(1|1))\)-modules. Using this formula the Alexander-Conway polynomial in terms of \( U_q(\mathfrak{gl}(1|1)) \) can be written as a state sum and as a state sum, it can be extended to invariants of framed graphs (see [6] Section 3).

References

[1] J. Birman, Braids, links, and mapping class groups, Annals of Mathematics Studies, No. 82. Princeton University Press, 1974

[2] J. Birman, D. D. Long, J. Moody, Finite-dimensional representations of Artin's braid group, In: The Mathematical Legacy of Wilhelm Magnus: Groups, Geometry and Special Functions, Contemporary Math. 169, Amer. Math. Soc., 1994, pp. 123–132.

[3] F. Constantinescu, F. Toppan, On the linearized Artin braid representation. J. Knot Theory Ramifications 2 (1993), no. 4, 399–412.

[4] N. Geer, B. Patureau-Mirand, An invariant supertrace for the category of representations of Lie superalgebras Pacific J. Math, Vol. 238 (2008), No. 2, 331-348.

[5] N. Geer, B. Patureau-Mirand, V. Turaev, Modified quantum dimensions and re-normalized link invariants, Compos. Math. 145 (2009), no. 1, 196–212.
[6] L. H. Kauffman, L. H., H. Saleur, *Free fermions and the Alexander-Conway polynomial*, Comm. Math. Phys. 141 (1991), no. 2, 293–327.

[7] P. P. Kulish, *Quantum Lie superalgebras and supergroups*, Problems of Modern Quantum Field Theory (Alushta, 1989) Springer, 1989, pp. 1421.

[8] J. Murakami, *A state model for multi-variable Alexander polynomial*, Pacific J. Math. Volume 157, Number 1 (1993), 109-135.

[9] N. Reshetikhin, *Quantum Supergroups*, Proceedings of the NATO advanced research workshop, Quantum Field Theory, Statistical Mechanics, Quantum Groups, and Topology. (Coral Gables, FL, 1991), 264-282.

[10] N. Reshetikhin, V. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. Volume 127, Number 1 (1990), 1-26.

[11] M. Rosso, *Alexander polynomial and Koszul resolution*, Algebra Monpellier Announcements, 1999.

[12] V. Turaev, *Quantum invariants of knots and 3-manifolds*. de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, (1994).

[13] O. Viro, *Quantum relatives of the Alexander polynomial*, Algebra i Analiz 18:3 (2006) 63-157 (in Russian), St. Petersburg Math. J. 18 (2007), no. 3, 391–457 (in English).

[14] N. Geer, B. Patureau-Mirand, *An invariant supertrace for the category of representations of Lie superalgebras* Pacific J. Math, Vol. 238 (2008), No. 2, 331-348.

[15] N. Geer, B. Patureau-Mirand, V. Turaev, *Modified quantum dimensions and re-normalized link invariants*, Compos. Math. 145 (2009), no. 1, 196–212.