Extended-body effects and rocket-free orbital maneuvering

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Celestial mechanics is most often concerned with the motion of pointlike, or at least spherically-symmetric, masses. However, the orbits of more-complicated extended bodies can differ considerably from their point-particle counterparts. This is especially so if such a body—say a spacecraft—is able to actively modulate its shape. Except where forbidden by symmetries and their associated conservation laws, shape-changing spacecraft can in fact adjust their orbits arbitrarily (though slowly) without the use of reaction mass. We show that such maneuvers can be understood using elementary methods and without constructing any detailed model of a spacecraft’s interior. In many cases, all of a spacecraft’s design can be abstracted to the specification of a single time-dependent eigenvalue of its quadrupole moment. Appropriately varying this eigenvalue allows the energy and eccentricity of an orbit to be increased or decreased and its apses to be rotated arbitrarily. In other contexts, we show that extended-body effects can be used instead to stabilize unstable orbits.

I. INTRODUCTION

How can a spacecraft adjust its orbit? The obvious answer is to use a rocket. But there is another possibility: A spacecraft can generically “grab onto” or “push off of” gradients in a gravitational field, effectively using those gradients as invisible, omnipresent handholds. More precisely, bodies can take advantage of the fact that the gravitational forces and torques which act on them can be manipulated by controlling their internal mass distributions. Although extended-body effects such as these typically perturb gravitational forces only slightly from their point-particle counterparts, the integrated consequences of these perturbations can be considerable; large, fuel-free maneuvers are possible for those who wait.

Such maneuvers have been discussed before, typically by considering systems of tethered masses in which the lengths of those tethers are varied according to prescribed rules [1–5]. Explicit models such as these have the advantage of being conceptually concrete. However, they obscure the underlying physics. It is unclear which results are fundamental and which are artifacts of the particular configuration which was analyzed. Here we explain how rocket-free maneuvers can be understood much more generally, without adopting any particular model. Doing so provides a better understanding of what is allowed by the laws of physics and what is not.

We first explain how extended-body effects may be used to modify orbits around a single spherically-symmetric mass, assuming Newtonian gravity in three dimensions. The symmetry of the system constrains the force and torque to conserve a spacecraft’s total (orbital plus spin) angular momentum, although we specialize to cases in which its orbital and spin components are conserved individually. A particular eigenvalue of a spacecraft’s quadrupole moment then acts as a natural control parameter, and cyclic variations of this eigenvalue can be used to secularly increase or decrease the energy and eccentricity of an orbit. There is in general an apsidal precession which arises in such processes, although we show that cycles can be designed which control the precession and the eccentricity independently.

As another application, we show that extended-body effects may be used not only to change orbits, but also to maintain them; unstable orbits can be stabilized. This is illustrated using a model in which a spacecraft is placed in a nearly-circular orbit around a spherically-symmetric mass in five or more dimensions. Circular orbits in this context are unstable for point particles, but not for appropriately-engineered extended bodies.

II. MOTION OF EXTENDED BODIES

An extended body with mass $m$ in three dimensions is characterized in part by its center-of-mass position $z$ and its spin angular momentum $S$. To review, these satisfy $m\ddot{z} = F$ and $\dot{S} = \tau$, where the net gravitational force,

$$F = \int \rho g d^3 x,$$

and the net gravitational torque about $z$,

$$\tau = \int \rho (x - z) \times g d^3 x,$$

both depend on the gravitational field $g$ and on the body’s mass density $\rho$. The gravitational self-field has no effect on the integrals here, so $g$ may be taken to be the external gravitational field, a field which satisfies $\nabla^2 g = 0$ throughout the body.

If the mass distribution is spherically-symmetric, in the sense that $\rho$ depends on position only via $|x - z(t)|$, the mean-value property of harmonic functions [6] may be used to show that the force and torque reduce to their point-particle values

$$F(t) = mg(z(t), t), \quad \tau(t) = 0.$$

Crucially, forces and torques which act on less symmetric mass distributions can be quite different.
\section{The quadrupole approximation}

In principle, (2.1) and (2.2) may be used to compute corrections to the point-particle force and torque for general extended bodies. Evaluating the relevant integrals might at first appear to require detailed knowledge of \( \rho \), and thus the full specification of a body’s internal structure. This is rarely necessary, however.

Useful approximations arise when the gravitational field varies only slightly inside the object of interest, which occurs whenever its size is small compared to the distances to other masses. Assuming that this is the case, \( g \) may be expanded for \( x \) near \( z \). Substituting the first three terms of that expansion into the force integral while adopting the Einstein summation convention for repeated indices then shows that

\[
F_i = mg_i + \frac{1}{2} Q^{jk} \partial_j \partial_k g_i + \ldots, \tag{2.4}
\]

where the field and its derivatives are evaluated here at \( z \) and \( Q^{ij} = Q_i^j \) denotes the body’s quadrupole moment. The form for the quadrupole moment which appears first when deriving this result is a volume integral of \( \rho(x^i - z^i)(x^j - z^j) \). However, recalling that \( g \) must be harmonic, arbitrary multiples of \( \delta^{ij} \) may be added to \( Q^{ij} \) without affecting \( F \). The trace of the quadrupole moment is thus irrelevant, an observation which may be used to redefine the \( Q^{ij} \) appearing in (2.4) as the trace-free symmetric tensor with components

\[
Q^{ij} \equiv \int \rho \left[ (x - z)^i (x - z)^j - \frac{1}{2} \delta^{ij} |x - z|^2 \right] d^3 x. \tag{2.5}
\]

This vanishes when \( \rho \) is spherically symmetric, but not more generally.

Crucially, the quadrupole moment can depend on time. In fact, there are no general laws of physics which constrain its time dependence: A suitably-engineered spacecraft can vary its quadrupole moment essentially at will, strain its time dependence: A suitably-engineered spacecraft can vary its quadrupole moment essentially at will, rapidly rearrange itself into a configuration for which \( Q^{ij} = 0 \). In this sense, it might be viewed as a kind of “oscillating energy.”

The second notion of energy we consider differs by incorporating all of the gravitational potential

\[
U = \int \rho \Phi d^3 x, \tag{2.9}
\]

rather than only its point-particle approximation; let

\[
E \equiv \frac{1}{2} m |\dot{z}|^2 + U. \tag{2.10}
\]

This still ignores any internal contributions to the kinetic energy, as well as, e.g., non-gravitational reservoirs of potential energy.

Expanding \( \Phi \) about \( z \) shows that the two energies here are related via

\[
E = E_{\text{pt}} - \frac{1}{2} Q^{ij} \partial_i g_j \tag{2.11}
\]

in the quadrupole approximation. Moreover,

\[
\dot{E} = \left[ m \partial_t \Phi - \frac{1}{2} Q^{ij} \partial_i (\partial_j g_j) \right] - \frac{1}{2} \dot{Q}^{ij} \partial_i g_j. \tag{2.12}
\]

The first group of terms here vanishes if the gravitational field is static, and in those cases, \( E \) is conserved whenever

\[
\dot{Q}^{ij} \partial_i g_j = 0. \tag{2.13}
\]

If quadrupole moments are changed only stepwise at discrete times in a static field—which is an idealization of many useful control strategies—the conservation of \( E \) can thus be used in between each step. Furthermore, changes in \( E \) from one step to the next are determined by

\[
\delta E = -\frac{1}{2} \delta Q^{ij} \partial_i g_j. \tag{2.14}
\]

This contrasts with the point-particle energy, for which \( \delta E_{\text{pt}} = 0 \) after a rapid configuration change.
III. ORBITS AROUND A SPHERICAL MASS

We now analyze the motion of a shape-changing spacecraft with mass \( m \) and (possibly time-dependent) quadrupole moment \( Q_{ij} \) which is in orbit around a spherically-symmetric body with mass \( M \gg m \) in three dimensions. The gravitational potential at a point \( x \) is then given by

\[
\Phi = -\frac{GM}{|x|}. \tag{3.1}
\]

Defining \( r \equiv |z| \) and \( \dot{z} \equiv z/r \) while using (2.4), the force in the quadrupole approximation is found to be

\[
F_i = mg_i(z) + \frac{3GM}{r^5} \left[ Q_{ij} \dot{z}^j - \frac{5}{2} (Q_{jk} \dot{z}^j \dot{z}^k) \dot{z}_i \right]. \tag{3.2}
\]

Similarly, (2.6) implies that

\[
\tau_i = -\frac{3GM}{r^5} \epsilon_{ijk} \dot{z}^j (Q^{kl} \dot{z}_i). \tag{3.3}
\]

The force and torque thus depend on the quadrupole moment only in the combination \( Q^{ij} \dot{z}_j \).

\[\text{A. Torque-free spacecraft}\]

The spherical symmetry of the gravitational field implies that the total angular momentum about the origin must be conserved: Regardless of extended-body effects, \( \tau = -z \times F \) and \( L_{\text{tot}} = L + S = \) constant, where \( L \equiv mz \times \dot{z} \) is the orbital angular momentum. A rocket-free spacecraft attempting to change \( L \) can therefore do so only at the cost of spinning itself up. However, the degree of spin-up required to significantly change \( L \) would typically be so large as to result in structural failure.

Given this, our analysis can be simplified by restricting to spacecraft in which \( L \) does not change at all. This requires that \( F \) be proportional to \( z \), implying that the direction of the force cannot be controlled. Torque-free extended bodies may nevertheless modulate the magnitude of the force which acts upon them. To understand what is possible in this context, recall that for an elliptical Keplerian orbit with semi-major axis \( a \) and eccentricity \( e \),

\[
L \equiv |L| = m\sqrt{GMa(1-e^2)} \tag{3.4}
\]

A process which conserves \( L \) may thus increase \( a \), but only by increasing \( e \) as well: these parameters cannot be controlled independently. Note in particular that this precludes the ability to move from one circular orbit to another.

Regardless, the design of a shape-changing, torque-free spacecraft can be abstracted to the specification of a single control parameter. To see this, first note that it follows from (3.3) that \( \tau = 0 \) if and only if \( \dot{z}_j \) is an eigenvector of \( Q^{ij} \). We thus require that

\[
Q^{ij} \dot{z}_j = q \dot{z}_i \tag{3.5}
\]

for some eigenvalue \( q \). The sign of \( q \) is unconstrained in general, as its time dependence. It is the relevant control parameter for torque-free spacecraft in spherically-symmetric gravitational fields. In terms of it, (3.2) implies that the force is

\[
F = [1 + \frac{3}{2} (q/mr^2)] mg. \tag{3.6}
\]

As a simple example in which the eigenvector condition (3.5) is satisfied, consider a spacecraft which consists of two equal masses connected by a massless strut with length \( \ell \). If the strut is perpendicular to the orbital plane, (2.5) implies that \( q = -m\ell^2/12 \). If the strut is instead oriented radially, \( q = +m\ell^2/6 \). The sign and magnitude of \( q \) may thus be controlled by varying the length and orientation of the strut—the latter perhaps with the help of a reaction wheel or similar mechanism.

Regardless of specific implementation, we now suppose that \( q \) is piecewise constant. This allows the orbital dynamics to be understood largely from the conservation laws alone. In particular, (2.13) is satisfied when \( \dot{q} = 0 \), implying that \( E \) is conserved whenever \( q \) is constant. Moreover, the conservation of \( L \) allows the angular contribution to the kinetic energy to be rewritten so (2.10) reduces to

\[
E = \frac{1}{2} m\dot{r}^2 + U_{\text{eff}}(r), \tag{3.7}
\]

where

\[
U_{\text{eff}} \equiv \frac{L^2}{2m\dot{r}^2} - \frac{GMm}{r} \left[ 1 + \frac{3}{2} (q/mr^2) \right]. \tag{3.8}
\]

For fixed \( q \), a deformable spacecraft’s radial motion is thus equivalent to the motion of a point particle in only one dimension with effective potential energy \( U_{\text{eff}} \).

If \( q \) switches from one fixed value to another, \( U_{\text{eff}} \) and \( E \) both change. For the latter, (2.14) implies that

\[
\delta E = - \frac{3GM}{2r^3} \delta q. \tag{3.9}
\]

The change in energy thus depends on the distance \( r \) at which a spacecraft rearranges its mass distribution, with larger jumps occurring at smaller radii where the tidal gradient is larger. A cycle in which \( q \) increases when \( r \) is small and returns to its original value when \( r \) is large may be seen to lead to a net decrease in \( E \). Reversing the order here would instead result in a net increase in \( E \). Regardless, if \( q \) is cycled many times in such a manner, the energy of an orbit can be considerably altered without the use of a rocket. This is not unlike the process by which a child uses cyclic motion to pump up a swing.

\[\text{B. Controlling the eccentricity}\]

A bound, torque-free spacecraft in which \( q \) is fixed and small for all time has an orbit which is only slightly perturbed from a Keplerian ellipse. However, appropriately-timed cyclic changes in \( q \) may be used to slowly transition
from one nearly-Keplerian orbit to another. Such a process may be described by introducing a slowly-varying semi-major axis and eccentricity via the Kepler-like relations

\[ a \equiv -\frac{GMm}{2E_{pt}}, \quad e \equiv \sqrt{1 - \left(\frac{L/m}{GMa}\right)^2}. \tag{3.10} \]

The processes considered here conserve \( L \) but not \( E_{pt} \).

We suppose that \( E_{pt} < 0 \) and consider a cycle which increases this energy, thus increasing \( a \) and \( e \). It is somewhat easier to instead consider changes in \( E \), and recalling (2.11), this is related to \( E_{pt} \) via

\[ E = E_{pt} - \frac{3GMq}{2a^3}. \tag{3.11} \]

Supposing that a spacecraft’s design allows

\[ q \in [-q_-, q_+] \tag{3.12} \]

for some constants \( q_\pm \geq 0 \), it follows from (3.9) that net increases in \( E \) may be obtained with the following strategy:

1. Upon reaching pericenter, set \( q = -q_- \).
2. Upon reaching apocenter, set \( q = +q_+ \).

This cycle is illustrated in Fig. 1, where it may be seen that repeated executions result in a spacecraft which “climbs up” the two effective potential energy curves associated with the two values of \( q \). If \( \Delta \) is used to denote the change in some quantity from one pericenter to the next, (3.11) implies that \( \Delta E = \Delta E_{pt} \) through first order in \( q \). Using the same approximation, (3.9) and (3.10) imply that

\[ \frac{\Delta a}{a} = \frac{\Delta E}{|E|} = 6e \left( \frac{q_+ + q_-}{ma^2} \right) \frac{3 + e^2}{(1 - e^2)^3} \tag{3.13} \]

and

\[ \Delta e = 3 \left( \frac{q_+ + q_-}{ma^2} \right) \frac{3 + e^2}{(1 - e^2)^2}. \tag{3.14} \]

As \( (q_+ + q_-)/ma^2 \) is of order \( (\ell/a)^2 \ll 1 \), where \( \ell \) again denotes the spacecraft’s characteristic size, \( \Delta a/a \) and \( \Delta e \) are both small. However, as expected from the stronger tidal gradients encountered along orbits with smaller pericenters, the effectiveness of this method at fixed \( a \) is enhanced at high eccentricities.

If \( q \) is cycled \( N \) times in the given manner, the net eccentricity change may be found by repeatedly applying (3.14). More generally, that equation can be viewed as a recurrence relation for \( e = e(N) \). This relation is difficult to solve as-is, although its solutions may be approximated by those of the differential equation obtained by replacing \( \Delta e \) with \( de/dN \) and allowing \( N \) to take non-integer values. Using the conservation of \( L \) and defining

\[ N_{\text{char}}^{-1} = \frac{3\sqrt{3}}{(1 - e_0^2)^2} \left( \frac{q_+ + q_-}{ma_0^2} \right), \tag{3.15} \]

\[ \Delta E \]

\[ N_{\text{esc}} = \left[ \frac{\pi}{6} - \tan^{-1} \left( \frac{e_0}{\sqrt{3}} \right) \right] \left[ N_{\text{char}} \right]. \tag{3.17} \]

Before an escape could occur, however, the spacecraft would likely crash into the central mass. Regardless, the approximation (3.16) is plotted in Fig. 2 together with the eccentricity obtained by numerically solving the full equations of motion.

The process just described increases a spacecraft’s eccentricity and semi-major axis. However, \( a \) and \( e \) may be decreased instead, by modifying steps 1 and 2 above so that \( q \) is set equal to \( q_+ \) at pericenter and \(-q_-\) at apocenter. Such a strategy might be used to, e.g., circularize an initially-eccentric orbit. Angular momentum conservation implies that the radius of the final circular orbit would then be \( r_{\text{circ}} = a_0(1 - e_0^2) \), and noting that (3.16) remains valid in this context with the replacement \( N_{\text{char}} \rightarrow -N_{\text{char}} \), the number of cycles required to eliminate all eccentricity is \( N_{\text{circ}} = N_{\text{char}} \tan^{-1}(e_0/\sqrt{3}) \).
If $q$ varies according to the eccentricity-raising cycle discussed in Sect. III B, direct integration shows that

$$\Delta \psi = \frac{9\pi}{2(1-e^2)^2} \left( \frac{q_+ - q_-}{ma^2} \right). \quad (3.22)$$

This is determined by $q_+ - q_-$ while (3.14) implies that $\Delta e$ is determined by $q_+ + q_-$. Appropriate choices for $q_+$ and $q_-$ may therefore be used to control the eccentricity and the precession rate independently. Nevertheless, switching $q$ at apocenter and pericenter does not maximize the precession rate.

To find a strategy which does optimize this rate, note that the sign of the integrand in (3.21) is given by the sign of $q \cos \nu$. If it is again assumed that $q$ can vary only in the interval (3.12), the maximum amount of prograde precession is thus obtained by setting $q = q_+$ when $\cos \nu > 0$ and $q = -q_-$ otherwise. Doing so results in

$$\Delta \psi = \frac{9\pi}{(1-e^2)^2} \left[ \left( \frac{q_+ + q_-}{ma^2} \right) \left( \frac{1 - 2e^2/3}{\pi e} \right) + \frac{1}{2} \left( \frac{q_+ - q_-}{ma^2} \right) \right]. \quad (3.23)$$

If a large amount of retrograde precession is desired instead, $q$ may be set equal to $-q_-$. When $\cos \nu > 0$ and $q_-$ otherwise. This results in a precession given by (3.23) but with the roles of $q_+$ and $-q_-$ swapped. Regardless, one interesting feature of these strategies is that at fixed $a$, their effectiveness is enhanced at both small and large eccentricities. While large-eccentricity enhancements also arise in, e.g., (3.14), we find low-eccentricity enhancement only for the precession rate. It is perhaps unsurprising that an orbit which is only slightly eccentric should be more easily reoriented.

IV. STABILIZING UNSTABLE ORBITS

As another application of extended-body effects, a shape-changing spacecraft may be able to stabilize itself in an otherwise-unstable orbit. This is not of course applicable to the three-dimensional 2-body problem considered in Sect. III, where there are no unstable orbits (for spherically-symmetric satellites). It is more relevant to the three-dimensional 3-body problem, where one might, e.g., attempt to maintain position around an ordinarily-unstable Lagrange point, or to counteract the Moon-induced eccentricity increase [3, 7]—and subsequent crash into the Earth—of a satellite which is in low polar orbit around the Earth. In general relativity, however, even the 2-body case becomes interesting: The ordinarily-unstable circular orbits very close to a Schwarzschild black hole might be stabilized using extended-body effects.

Here we do not wish to enter into the complexities of general relativity or the Newtonian 3-body problem, but instead discuss stabilization in a model problem which
Now, $E$ has one value, say $E_-$ whenever $r < r_{\text{circ}}$. It has another value, $E_+$, when $r \geq r_{\text{circ}}$, and (2.14) implies that these constants are related by

$$E_+ = E_- - \frac{d(d-2)km}{2r_{\text{circ}}^{d-2}} \left( \frac{q_+ + q_-}{mr_{\text{circ}}^2} \right).$$

(4.3)

Combining the effective potential energies in both regimes, it follows that $E_- = \frac{1}{2} mr^2 + u_{\text{eff}}(r)$ for all $r$, where

$$u_{\text{eff}}(r) = U_{\text{eff}}(r)|_{q=0} + \frac{d(d-2)k}{2r^d} \times \begin{cases} q_- , & r < r_{\text{circ}} , \\ (q_+ + q_-)(r/r_{\text{circ}})^d - q_+ , & r \geq r_{\text{circ}}. \end{cases}$$

(4.4)

The radial motion of a spacecraft may be understood directly from $u_{\text{eff}}$. This is plotted in Fig. 3, which shows that extended-body effects “carve out” a small well on top of the concave-down curve expected from the point-particle problem, at least if $q_-$ and $q_+$ are both nonzero. Orbits in this well are manifestly stable.

V. DISCUSSION

A wide variety of orbital maneuvers may be performed by deformable spacecraft. Roughly speaking, anything which is not forbidden is allowed, and maneuvers are forbidden only by the conservation laws associated with the symmetries of the gravitational field. Unlike previous discussions of rocket-free maneuvers in Newtonian gravity [1–5], no specific model for a deformable spacecraft has been adopted here. Rather, we have shown that the internal dynamics can be abstracted to the specification of a single time-dependent component of a body’s quadrupole moment. This simplifies and generalizes the analysis while also demonstrating that, e.g., eccentricity and orbital precession can be controlled independently.

This difference in approach becomes crucial in general relativity, where it is difficult to construct explicit, physically-plausible models which are also tractable. Indeed, it can be difficult even to verify that certain models fall freely and do not violate energy conditions. Difficulties of this kind have led [8, 9] to certain claims [10] for qualitatively non-Newtonian effects in general relativity, claims which were later refuted by a more general analysis related to the one considered here [11].

Nevertheless, certain aspects of the relativistic problem are indeed different. For example, the momentum of a relativistic extended body is not necessarily proportional to its velocity, and this misalignment—associated with a body’s “hidden momentum” [12, 13]—can be controlled using internal deformations [14]. Relativistic extended-body effects may thus be used to directly modulate not only a body’s acceleration, but also its velocity. This is true for both gravitational and electromagnetic interactions.
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