Heat Superconductivity and Electrical Activity of Superfluid Systems

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It has been shown that a heat flux induced by the temperature gradient $V T$ in He-II placed in a magnetic field $H$ generates an electric field $E = H \times V T$. The effect occurs in superfluid dielectric systems because of their heat superconductivity. The magnitude of the field significantly depends on the shape of a sample with helium and on the direction of the magnetic field with respect to the sample. It has been shown that the effect exists at both static and time dependent temperature gradient (at the propagation of the second sound).

The role of polarized surface layers is particularly large in thin superfluid films. A third sound wave propagates in such a film in the presence of time-periodic temperature difference at the edge of the film. This wave is responsible for oscillations of the thickness of the film, which result in oscillations of the wall-induced dipole moment and in the appearance of an oscillating electric field in the environment. This effect was predicted in [16, 17].

In this work, we focus on the thermal polarization mechanism that is possible in superfluid dielectric systems in the presence of a magnetic field. It is known that He-II has two unique properties: superfluidity and heat superconductivity (this term was proposed in [18]). The thermal conductivity in the superfluid state is huge because the heat transfer in He-II is due to the motion of the normal component. In this case, the mass flux carried by the normal component is compensated by the flux carried by the superfluid component. It is very important that the condition of absence of mass flux should be satisfied only on average; i.e., the mass flux averaged over the area of the system should vanish. Since the motion of any insulator in a magnetic field polarizes it, the local mass flux generated by the temperature gradient in the presence of the magnetic field induces a local dipole moment in superfluid helium and the average electric field can appear beyond the system. It can be said that the discussed effect is an analog of the Nernst–Ettingshausen effect.

However, it appeared that the electric potential arising in superfluid systems is very sensitive to the geometry of a system and to the orientation of the magnetic field. In particular, the effect is absent in a circular...
capillary. In the case of an elliptic capillary with the semiaxes $a \gg b$, the electric field strengths in the cases of the magnetic field directed along the major and the minor semiaxis differ by more than three orders of magnitude. The aim of this work is to reveal the character of the dependence of the electric field on the magnitude and direction of the magnetic field and on the observation point.

We begin with the following expression for the electric displacement field $\mathbf{D}$ in the insulator that has the permittivity $\varepsilon$ and permeability $\mu$ and moves at the velocity $\mathbf{v}$, which was obtained by Minkowski (see, e.g., Eq. (76,10) in [19]):

$$\mathbf{D} = \varepsilon \mathbf{E} + \frac{\mu - 1}{c}(\mathbf{v} \times \mathbf{H}).$$  \hspace{1cm} (1)

Here, $\mathbf{E}$ and $\mathbf{H}$ are the static electric and magnetic field strengths, respectively. Expression (1) has an accuracy of $\nu/c$.

For $^4\text{He}$, $\mu = 1$ with a high accuracy. The relation between the permittivity $\varepsilon$ and the polarizability $\alpha$ of the $^4\text{He}$ atom and the density of atoms $n$ can be obtained from the Clausius–Mossotti relation

$$\frac{\varepsilon - 1}{\varepsilon + 2} = \frac{4\pi n\alpha}{3}.$$  \hspace{1cm} (2)

This relation follows from the fact that the atom is polarized by the local electric field rather than by the average field $\mathbf{E}$. For $^4\text{He}$, the density is $n = 2 \times 10^{22}$ cm$^{-3}$, the polarizability is $\alpha = 2 \times 10^{-25}$ cm$^3$, and the product $n\alpha$ is much smaller than unity. Consequently, it is sufficient to retain only the first term $4\pi n\alpha$ in the expansion of $\varphi = 1$ in powers of $n\alpha$.

Since the electric displacement field is $\mathbf{D} = \varepsilon \mathbf{E} + 4\pi \mathbf{P}$ and the mass flux is $\mathbf{j} = Mn\mathbf{v}$ ($M$ is the mass of the helium atom), it follows from Eq. (1) in the absence of the electric field that

$$\mathbf{P} = \frac{\alpha}{Mc} [\mathbf{j} \times \mathbf{H}].$$  \hspace{1cm} (3)

In the general case, in superfluid helium, $\mathbf{j} = \rho_n \mathbf{v}_n + \rho_s \mathbf{v}_s$, where $\mathbf{v}_n$ and $\mathbf{v}_s$ are the velocity and density of the normal (superfluid) component, respectively. Expression (3) is valid for any method of initiation of the motion of He-II. Below, we assume that motion is induced by the temperature gradient.

We consider a steady-state heat flux in He-II filling a capillary with a small temperature difference $\Delta T$ between the ends of the capillary. We analyze the laminar motion where the velocities $\mathbf{v}_n$ and $\mathbf{v}_s$ are independent of the longitudinal coordinate (along the capillary). In this case, the equations of two-fluid hydrodynamics have the form

$$\eta_n \nabla^2 \mathbf{v}_n = \frac{\rho_n}{\rho} \nabla P + \rho_s S \nabla T,$$  \hspace{1cm} (4)

$$\nabla P = \rho S \nabla T.$$  \hspace{1cm} (5)

Here, $\eta_n$ is the coefficient of viscosity of the normal component, $\rho = \rho_n + \rho_s$ is the total mass density, $S$ is the specific entropy, and $\nabla P$ and $\nabla T$ are the pressure and temperature gradient, respectively. Equation (5), called the London equation, follows from the mechanical equilibrium arising at the flow of a superfluid liquid between the ends of the capillary. Equations (4) and (5) give the equation

$$\eta_n \nabla^2 \mathbf{v}_n = \nabla P,$$  \hspace{1cm} (6)

which is equivalent to the Poiseuille equation in classical hydrodynamics.

The velocity of the superfluid component cannot depend on the transverse coordinate because $\nabla \times \mathbf{v}_s = 0$. This velocity can be found from the condition of the absence of the total mass flux, i.e., from the condition $\rho_n \langle \mathbf{v}_n \rangle + \rho_s \mathbf{v}_s = 0$. Here, $\langle \mathbf{v}_n \rangle$ is the velocity of the normal component averaged over the cross section of the capillary. Taking into account this condition, we obtain from Eq. (3)

$$\mathbf{P} = \frac{\alpha \rho_n}{Mc} [\langle \mathbf{v}_n - (\langle \mathbf{v}_n \rangle) \rangle \times \mathbf{H}].$$  \hspace{1cm} (7)

It is seen that the polarization in the presence of the magnetic field is locally nonzero, whereas the total dipole moment vanishes together with the total mass flux. The electric potential outside of the system is given by the expression

$$\varphi(r_0) = \int \frac{\mathbf{P} \cdot (r_0 - r) d^3r}{|r_0 - r|},$$  \hspace{1cm} (8)

where $r_0 = (\eta, \theta_0, z_0)$ is the position vector of the observation point. Consequently, vanishing of the total dipole moment generally does not lead to the absence of the electric field outside of the system.

The solution of Eq. (6) depends on the geometry of the problem. It seems reasonable to use the capillary in the form of a circular cylindrical tube. However, the calculation shows that the potential outside the capillary is identically zero in this case. This result is due to the high symmetry of such a system. The inclusion of nonlinear terms in $n\alpha$ in the expansion of the expression for $\varphi$ does not change the symmetry of the problem. For this reason, an axisymmetric flow in the cylindrical capillary does not induce an electric field outside the capillary even with the inclusion of the nonlinear dependence of $\varphi(n)$. The potential $\varphi(r_0)$ is nonzero for a capillary with, e.g., an elliptic cross section.

We consider the elliptic capillary with a small temperature difference $\Delta T$ between its ends. Let $a$ and $b$ be the semiaxes along the $x$ and $y$ axes, respectively. The expression for the velocity of the normal component can be found in [20]. The substitution of this
expression into Eq. (7) for the magnetic field $H$ directed along the $\hat{y}$ axis gives the polarization

$$ P(x, y) = -P_0 \frac{a^2}{a^2 + b^2} \left( \frac{1}{2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} \right) \hat{x}. $$

(9)

Here,

$$ P_0 = \frac{\alpha \rho_s H \rho S \Delta T}{2 \eta_n L} b^2, $$

(10)

where $L$ is the length of the capillary.

The electric potential outside of the liquid can be obtained by substituting Eq. (9) into Eq. (8). However, integration cannot be performed at arbitrary coordinates $x$ and $y$ of the observation point. An analytical expression can be obtained at $\gamma = 1$. This expression is very lengthy even in this case. We present the results only for two particular cases. The potential on the surface of the capillary at $x_0 = a$ is given by the formula

$$ \varphi = \frac{2 \pi b P_0}{3} \frac{1 - \gamma}{(1 + \gamma^2)(1 + \gamma)^2}, $$

(11)

where $\gamma = b/a$. At $\gamma = 1$, i.e., at $a = b$, when the ellipse becomes a circle, Eq. (11) gives zero in agreement with the above statement.

At $\gamma \ll 1$ (in this case, the system models a slit and the magnetic field is directed across the slit),

$$ \varphi(x_0, 0) = \frac{2 \pi b P_0}{3} \left[ x_0^2 \left( x_0 - \sqrt{x_0^2 - 1} \right) - \frac{1}{3} \right]. $$

(12)

Here, $x_0$ is measured in units of $a$.

At $\gamma \gg 1$, the system again models a slit, but the magnetic field is directed along the slit. In this case, the potential is several orders of magnitude smaller than that at $\gamma \approx 1$ and $\gamma \ll 1$.

At arbitrary $x_0$ and $y_0$ values, the numerical integration makes it possible to determine the potential $\varphi$ at the motion from the points $(x_0, y_0 = 0)$ to the points $(x_0 = 0, y_0)$. A monotonic decrease in the potential $\varphi(x_0, y_0)$ could be expected in this case. However, this is not the case. Figures 1 and 2 shows the dependences of the potential $\varphi$ on the surface of the capillary on the polar angle $\theta = \arctan(y_0/x_0)$ for $\gamma \sim 1$ and $\gamma \ll 1$.

Analysis shows that the case $\gamma \ll 1$ (magnetic field is applied across the slit) is the most favorable for obtaining a high potential $\varphi$. The maximum potential is reached at small angles $\theta$. In particular, the potential at $\theta = 0$ and $x_0 = 1$ is

$$ \varphi = \frac{\pi b^3 \alpha \rho_s H \rho S \Delta T}{3 \eta_n L}. $$

(13)

The temperature dependence of the potential $\varphi$ is determined by the temperature dependences of the normal density $\rho_n$, viscosity $\eta_n$, and specific entropy $S$. The temperature dependence of the total density $\rho$ and polarizability $\alpha$ can be neglected. It seems that the potential $\varphi$ can be significantly changed by varying the semiaxis $b$. However, such a change is possible only in certain limits. The reason is that the increase in $b$ is accompanied by the conversion of laminar motion of the liquid to turbulent motion.

The author of the pioneering work [21] showed that, when the heat flux $W = \rho_s ST |v_n - v_j|$ reaches the critical value $W_{c1}$, the laminar motion is restored through the appearance of quantized vortices and vortex rings forming a vortex tangle in the superfluid liquid (T-1 state). Further studies showed that there is the second critical heat flux $W_{c2}$ above which the vortex density increases significantly (T-2 state). The transition from the T-1 state to the T-2 state is associated with the transition of the normal component to the turbulent state (see, e.g., [22, 23]). The result significantly depends on the shape of the channel. Two transitions occur in elliptic channels with semiaxes $a = b$. Only one transition to the T-2 state occurs at $a \gg b$ [24].
Rings with the radius of about the size of the capillary \( b \) in the T-1 state are critical. For the appearance of such rings, the difference \( |v_n - v_\ell| \) should be about \( (h/Mb) \ln (b/\xi) \), where \( \xi \) is the radius of the vortex core [25]. The transition to the T-2 turbulent state should occur when the normal component reaches the critical velocity \( R_n \eta_n/\rho_n b \) (\( R_n \) is the Reynolds number). The existence of critical velocities determines the maximum allowable size \( b \) at which the laminar motion of the liquid holds. This condition implies a constraint on the electric potential given by Eq. (13).

We present constraints on the electric potential in the laminar fluid flow regime. At \( \gamma = 1 \), this constraint is due to the condition \( |v_n - v_\ell| \) \( (h/Mb) \ln (b/\xi) \):

\[
\phi < \frac{4\pi \hbar \alpha H}{3M \alpha c} \rho_n \ln \left( \frac{b}{\xi} \right) = \phi_{cl}. \tag{14}
\]

At \( \gamma \ll 1 \), a constraint is imposed on the velocity of the normal component \( |v_n| < R_n \eta_n/\rho_n b \). Therefore,

\[
\phi < \frac{4\pi \hbar \alpha H}{3M \alpha c} \rho_n \eta_n \equiv \phi_{c2}. \tag{15}
\]

In the temperature range where \( \rho_n \sim \rho_n \sim \rho \), we find \( \phi_{c2} > \phi_{cl} \). In particular, at \( H = 10 \) T, \( \alpha = 2 \times 10^{-25} \) cm\(^3\), \( R_n = 2 \times 10^3 \), \( \eta_n = 2 \) \( \mu \)P, and \( \rho = 10^{-11} \) g/cm\(^3\), we have \( \phi_{cl} = 2 \times 10^{-3} \) V and \( \phi_{c2} = 5 \times 10^{-7} \) V. Thus, to obtain the maximum potential \( \phi \), it is necessary to use a slit in the magnetic field directed across it rather than the circular capillary. We determine the transverse dimension of the slit \( b_c \) corresponding to the maximum potential \( \phi_{c2} \). For the capillary with the length \( L = 1 \) cm and the temperature difference \( \Delta T = 10^{-3} \) K between its ends, from Eq. (13), we find \( b_c = 10^{-3} \) cm in the indicated temperature range.

We considered above the static temperature difference between the ends of the capillary. However, the effect is also possible in the case where such difference periodically depends on time, i.e., in the case of propagation of the second sound in He-II.

Let He-II fill the circular cylinder with the radius \( R \) and the second sound propagate along the \( z \) axis of the cylinder. It is known that, disregarding the thermal expansion, oscillations of the pressure and density occur in the first sound waves, whereas only the temperature and entropy oscillate in waves of the second sound. When the thermal expansion is taken into account, coupling between oscillations of the second and first sounds arises and, as a result, oscillations of the pressure occur at the propagation of the second sound. These oscillations result in a nonzero mass flux \( j_z \), which is due to the oscillating component of the temperature \( T' \). When \( R \gg \lambda \), where \( \lambda \) is the wavelength of the second sound, the dependence of \( j_z \) on the radial coordinate \( r \) can be neglected, and the flux \( j_z \) is related to \( T' \) as (see, e.g., [26])

\[
j_z = -\frac{\beta \rho_n u_2}{u_1 - u_2} T', \tag{16}
\]

where \( \beta \) is the thermal expansion coefficient and \( u_1 \) and \( u_2 \) are the speeds of the first and second sounds, respectively. If the external magnetic field is directed across the cylinder (along the \( y \) axis), the electric polarization \( P_y = -\alpha H j_z/Mc \) appears in helium according to Eq. (3). The substitution of this polarization into Eq. (8) gives the electric potential outside of the cylinder. Details of the calculation of the corresponding integral can be found in [17]. Taking into account that \( u_1 \gg u_2 \) at \( T > 0.8 \) K and assuming that the height of the cylinder \( L \) is much larger than the wavelength of the second sound \( \lambda \), we obtain the electric potential at the point \((r_0, \theta_0, 0)\) in the form

\[
\phi(r_0, \theta_0, t) = 4\pi \beta \rho_n \frac{\alpha HR}{Mc} I_1(kR) I_1(kr_0) \times T' \cos \theta_0 \cos (u_2kt). \tag{17}
\]

Here, \( k \) is the wavenumber, \( T' \) is the temperature amplitude in the second sound wave, and \( I_1 \) and \( K_1 \) are the first-order modified Bessel functions of the first and second kinds, respectively.

When deriving Eq. (17), we assume that \( R \gg \lambda \) and neglect the dependence of the flux \( j_z \) on the radial coordinate \( r \). However, this assumption is invalid if the radius of the cylinder is about the wavelength of the second sound. In this case, to correctly solve the problem of propagation of the second sound, one should use the complete system of hydrodynamic equations for the superfluid liquid with boundary conditions. To solve this system, it is convenient to use an approach developed in [27]. Under the conditions \( u_1 \gg u_2 \) and \( \lambda \gg R \gg \lambda_n \), where \( \lambda_n = \sqrt{2\pi \pi c/\sqrt{\nu \rho_n}} \) is the wavelength of the viscous wave ( \( \nu \) is the frequency of the second sound), the mass flux is given by the expression

\[
j_z = -\left( \beta \rho_n u_2 + \frac{\rho \alpha S J_0(k_n R)}{u_2 J_1(k_n R)} \right) T', \tag{18}
\]

where \( J_0 \) is the zeroth order Bessel function of the first kind and \( k_n \) is the wavenumber of the viscous wave. Using Eqs. (3) and (18), we obtain from Eq. (8) the electric potential at the point with the coordinates \((r_0, \theta_0, 0)\) in the form

\[
\phi(x_0, \theta_0, t) = 4\pi \frac{\alpha HR}{Mc} K_1(\lambda_n) \beta \rho_n u_2 I_1(kR) \times \rho \frac{\lambda_n}{u_2} \cos \left( \frac{\pi}{8} \right) T' \cos \theta_0 \cos (u_2kt). \tag{19}
\]
This potential, as well as the potential in Eq. (17), is proportional to the cosine of the polar angle of the observation point and vanishes at \( \theta_0 = \pi/2 \). In other words, the potential is identically zero on the plane parallel to the magnetic field \( H \) and passing through the axis of the cylinder.

We note that the expressions for the potential \( \varphi \) at the propagation of the second sound, as in the static case, are obtained under the assumption of the laminar regime. Unfortunately, we are not aware of any analytical criteria of the transition from the laminar regime to the turbulent one in the case of the propagation of the second sound. Cylindrical samples used in experiments usually have the radius in the range of cm (see, e.g., [24]). The amplitude of the temperature corresponding to the laminar region does not exceed several millikelvin. At the parameters \( T = 1.5 \) K, \( H = 10 \) T, \( v = 400 \) Hz, \( T_\alpha = 10^{-3} \) K, and \( R = 0.5 \) cm, the potential on the surface of the cylinder at the point with the coordinates \( r_0 = R \) and \( \theta_0 = 0 \) is \( \varphi = 4 \times 10^{-8} \) V.

To conclude, we note that electric fields predicted in this work in experiments with the second sound should be distinguished from the potential difference between the ends of the sample that was reported in [1]. Potentials (17) and (19) have a characteristic angular dependence. Furthermore, we predict electric fields outside of the sample, whereas the potential difference inside the sample was measured in [1].

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