A NOTE ON THE MODEL (CO)SLICE CATEGORIES

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Abstract. There are various adjunctions between coslice and slice categories. We characterize when these adjunctions are Quillen equivalences. As an application, a triangle equivalence between the stable category of a Frobenius category and the homotopy category of a non-pointed model category is given.

1. Introduction

Given a category $\mathcal{C}$ and a morphism $f : X \to Y$ in $\mathcal{C}$, one can construct adjunctions between coslice categories $(X \downarrow \mathcal{C})$ and $(Y \downarrow \mathcal{C})$, and slice categories $(\mathcal{C} \downarrow X)$ and $(\mathcal{C} \downarrow Y)$. If $\mathcal{C}$ is a closed model category, then all the (co)slice categories inherit the model structure from $\mathcal{C}$ and thus become closed model categories. In this case, the adjunctions between the coslice and slice categories are Quillen adjunctions. If we start from an Quillen adjunction between two closed model categories $\mathcal{C}$ and $\mathcal{D}$, then we also can construct Quillen adjunctions between coslice and slice categories. In this note, we consider when these adjunctions are Quillen equivalences (See, Proposition 3.1).

For a stable pointed model category, its homotopy category is a triangulated category [7, Chapter I, Section 3, Theorem 2, Proposition 5-6]. Using the characterization of the Quillen equivalences between coslice and slice categories, we construct a triangle equivalence between the stable category of a weakly idempotent complete Frobenius model category and the homotopy category of its coslice category (See, Proposition 3.5). As a byproduct, we get a non-pointed model category whose homotopy category is a triangulated category. This shows that the pointed condition of Quilen’s Theorem is not always necessary.

Supported by JSNU12XLR025.
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2. Preliminaries of (co)slice categories and Quillen equivalences

In this section we recall some basic facts about (co)slice categories and Quillen equivalences. Our main references are Quillen [7, Chapter I], Hovey [5, Chapter 1] and Hirschhorn [4, Chapter 8].

2.1. Quillen equivalences. In a closed model category $C$ [7, Definition I.5.1], there are three classes of morphisms, called cofibrations, fibrations and weak equivalences. We will denote them by $Cof(C)$, $Fib(C)$ and $We(C)$, respectively. A morphism which are both a cofibration (respectively fibration) and a weak equivalence is called acyclic cofibration (respectively acyclic fibration). An object $X \in C$ is called cofibrant if $\emptyset \rightarrow A \in Cof(C)$ and fibrant if $X \rightarrow \ast \in Fib(C)$, where $\emptyset$ is the initial object of $C$ and $\ast$ the terminal object of $C$. We use $C_c$ and $C_f$ to denote the classes of cofibrant and fibrant objects, respectively. The object in $C_{cf} := C_c \cap C_f$ is called bifibrant.

Suppose $C$ and $D$ are closed model categories. An adjunction $F : C \rightleftarrows D : G$ is called a Quillen adjunction if $F$ preserves cofibrations and acyclic cofibrations or equivalently $G$ preserves fibrations and acyclic fibrations [5, Definition 1.3.1, Lemma 1.3.4]. Sometimes we will call $F$ a left Quillen functor and $G$ a right Quillen functor.

**Definition 2.1.** [5, Definition 1.3.12] A Quillen adjunction $(F, G; \varphi) : C \rightarrow D$ is called a Quillen equivalence if for all $X \in C_c$ and $Y \in D_f$, a map $f : F(X) \rightarrow Y \in We(D)$ iff $\varphi(f) : X \rightarrow G(Y) \in We(C)$.

If $(F, G)$ is a Quillen equivalence, then the left derived functor $LF$ and the right derived functor $RG$ exist [7, Chapter I, Section 4]. Furthermore, the derived adjunction $(LF, RG) : Ho(C) \rightarrow Ho(D)$ is an equivalent adjunction [7, Chapter I, Theorem 3].

In a model category $C$, we use $p_X : Q(X) \rightarrow X$ to denote the cofibrant approximation of $X$ and $r_X : X \rightarrow R(X)$ fibrant approximations of $X$ respectively [7, Chapter I, Section 1], [11, Section 5]. The following is the most useful criterion for checking when a given Quillen adjunction is a Quillen equivalence.

**Proposition 2.2.** [5, Proposition 1.3.13, Corollary 1.3.16] Suppose $(F, G, \varphi; \eta, \varepsilon) : C \rightarrow D$ is a Quillen adjunction. The following are equivalent:
(1) \((F, G, \varphi)\) is a Quillen equivalence.

(2) \((\mathbb{L}F, \mathbb{R}G) : \mathcal{H}o(\mathcal{C}) \to \mathcal{H}o(\mathcal{D})\) is an adjoint equivalence of categories.

(3) If \(F(f)\) is a weak equivalence for a map \(f\) in \(\mathcal{C}\), so is \(f\). And the map \(F(Q(G(Y))) \xrightarrow{F(p_G(Y))} FG(Y) \xrightarrow{\varepsilon_Y} Y\) is a weak equivalence for every \(Y \in \mathcal{D}_f\).

(4) If \(G(g)\) is a weak equivalence for a map \(g\) in \(\mathcal{D}_f\), so is \(g\). And the map \(X \xrightarrow{\eta_X} GF(X) \xrightarrow{G(r_F(X))} G(R(F(X)))\) is a weak equivalence for every \(X \in \mathcal{C}_c\).

2.2. The model (co)slice categories.

Definition 2.3. Let \(\mathcal{C}\) be a category. For an object \(X\) in \(\mathcal{C}\), the coslice category \((X \downarrow \mathcal{C})\) is the category in which an object is a map \(X \xrightarrow{f} \mathcal{C}\) in \(\mathcal{C}\), and a map from \(X \xrightarrow{f} \mathcal{C}\) to \(X \xrightarrow{f'} \mathcal{C}'\) is a map \(\alpha : \mathcal{C} \to \mathcal{C}'\) such that \(f' = \alpha \circ f\). The composition of maps is defined by the composition of maps in \(\mathcal{C}\).

Dually, we define the slice category \(\mathcal{C} \downarrow X\) over \(X\).

Now let \(\mathcal{C}\) be a closed model category. If we define a map in \((X \downarrow \mathcal{C})\) and \((\mathcal{C} \downarrow X)\) is weak equivalence, cofibration, or fibration if it is one in \(\mathcal{C}\), then both the coslice category \((X \downarrow \mathcal{C})\) and slice category \((\mathcal{C} \downarrow X)\) become closed model categories [4, Theorem 7.6.5].

From now on, when we talk about model coslice and slice categories, we always mean that their model structures are given as above.

Lemma 2.4. Let \(\mathcal{C}\) be a model category. Then

1. \((X \downarrow \mathcal{C})_c = \{X \xrightarrow{u} \mathcal{C} | u \in \text{Cof}(\mathcal{C})\}\) and \((X \downarrow \mathcal{C})_f = \{X \xrightarrow{u} \mathcal{C} | \mathcal{C} \in \mathcal{C}_f\}\).

2. \((\mathcal{C} \downarrow X)_c = \{C \xrightarrow{u} X | C \in \mathcal{C}_c\}\) and \((\mathcal{C} \downarrow X)_f = \{C \xrightarrow{u} X | u \in \text{Fib}(\mathcal{C})\}\).

Proof. (1). Note that, in \(X \downarrow \mathcal{C}\), the initial object is \(X \xrightarrow{1} X\) and the terminal object is \(X \rightarrow \ast\). So by (1) of Theorem 2.4 and the definitions of cofibrant and fibrant objects, we know that (1) holds. The proof of assertion (2) is dually. \(\square\)
2.3. **Quillen adjunctions between (co)slice categories.** Now let \( \mathcal{C} \) be a bicomplete category and let \( g : X \to Y \) be a map in \( \mathcal{C} \). Then there are adjunctions \( (g_!, g^*) : (X \downarrow \mathcal{C}) \to (Y \downarrow \mathcal{C}) \) and \( (g_*, g^!) : (\mathcal{C} \downarrow X) \to (\mathcal{C} \downarrow Y) \) \[4, Lemma 7.6.6\]. Where \( g_! \) takes the object \( X \to \mathcal{C} \) to its pushout along \( g \) and \( g^* \) takes the object \( Y \to \mathcal{D} \) to its composition with \( g \). Dually, \( g_* \) takes the object \( C \to X \) to its composition with \( g \) and \( g^! \) takes the object \( D \to Y \) to its pullback along \( g \). In particular, if we take \( X = \emptyset \) the initial object of \( \mathcal{C} \), then \( g^* \) is just the forgetful functor from \((Y \downarrow \mathcal{C})\) to \( \mathcal{C} = (\emptyset \downarrow \mathcal{C}) \).

Meanwhile if we have an adjunction \( (S, U; \varphi, \eta, \varepsilon) : \mathcal{C} \to \mathcal{D} \) between the categories \( \mathcal{C} \) and \( \mathcal{D} \). Then for any object \( X \in \mathcal{C} \) and \( Y \in \mathcal{D} \), \((S, U)\) induces the following two adjunctions:

- \( \overline{S} : (X \downarrow \mathcal{C}) \to (S(X) \downarrow \mathcal{D}) \) with \( u \mapsto S(u) \) and \( f : u \to u' \mapsto S(f) \);
- \( \overline{U} : (S(X) \downarrow \mathcal{D}) \to (X \downarrow \mathcal{C}) \) with \( S(X) \mapsto D \mapsto X \xrightarrow{\eta_Y} U S(X) \xrightarrow{U(u)} U(D) \) and \( f : u \to u' \mapsto U(f) \).

- \( \overline{\mathcal{S}} : (\mathcal{C} \downarrow U(Y)) \to (\mathcal{D} \downarrow Y) \) with \( C \to U(Y) \mapsto S(C) \to SU(Y) \xrightarrow{\varepsilon_Y} Y \) and \( f : u \to u' \mapsto S(f) \);
- \( \overline{U} : (\mathcal{D} \downarrow Y) \to (\mathcal{C} \downarrow U(Y)) \) with \( u \mapsto U(u) \) and \( f : u \to u' \mapsto U(f) \).

With above notations, we have the following proposition, part of which in some special case can be found in \[5, Chapter 1, Section 3\] and \[6, Chapter 16, Section 2\].

**Proposition 2.5.** (1) Let \( \mathcal{C} \) be a model category and \( g : X \to Y \) a map in \( \mathcal{C} \). Then
\[\begin{align*}
(i) & \quad (g_!, g^*) : (X \downarrow \mathcal{C}) \to (Y \downarrow \mathcal{C}) \text{ is a Quillen adjunction.} \\
(ii) & \quad (g_*, g^!) : (\mathcal{C} \downarrow X) \to (\mathcal{C} \downarrow Y) \text{ is a Quillen adjunction.}
\end{align*}\]

(2) Let \( \mathcal{C} \) and \( \mathcal{D} \) be two model categories. If \((S, U) : \mathcal{C} \to \mathcal{D}\) is a Quillen adjunction and \( X \in \mathcal{C}, Y \in \mathcal{D} \) two objects, then
\[\begin{align*}
(i) & \quad (\overline{S}, \overline{U}) : (X \downarrow \mathcal{C}) \to (S(X) \downarrow \mathcal{D}) \text{ is a Quillen adjunction.} \\
(ii) & \quad (\overline{\mathcal{S}}, \overline{U}) : (\mathcal{C} \downarrow U(Y)) \to (\mathcal{D} \downarrow Y) \text{ is a Quillen adjunction.}
\end{align*}\]

**Proof.** By the definition of Quillen adjunction and the pushout, pullback axioms of model categories. \( \square \)

Assume now that \( \mathcal{C} \) and \( \mathcal{D} \) are two closed model categories and denote by \( \mathcal{C}_* = (\ast \downarrow \mathcal{C}) \) and \( \mathcal{D}_* = (\ast \downarrow \mathcal{D}) \) the slice categories of \( \mathcal{C} \) and \( \mathcal{D} \).
induced by the terminal object *. If \((S,U) : C \rightarrow D\) is a Quillen adjunction, Hovey constructs a functor \(U_* : D_* \rightarrow C_*\) by mapping object \(* \rightarrow D\) to \(U(*) = * \rightarrow U(X)*\). He has shown that this functor is a right Quillen functor. Note that if we denote by the map \(S(*) \rightarrow *\) as \(g\) in \(D_*\), then \(U_*\) is the composition the functors \(D_* \rightarrow (S(*) \downarrow D) \rightarrow C_*\). So by Proposition 2.5, this functor has a left adjoint \(S_* = S \circ g\), then we can get Proposition 1.3.5 of [5] directly.

**Corollary 2.6.** [5] Proposition 1.3.5| A Quillen adjunction \((S,U) : C \rightarrow D\) induces a Quillen adjunction \((S_*,U_*) : C_* \rightarrow D_*\).

### 3. Main results

In this section we will characterize when the various Quillen adjunctions as in Proposition 2.5 are Quillen equivalences.

**Proposition 3.1.** (1) If \(C\) is a closed model category and \(g : X \rightarrow Y\) a map in \(C\), then

- \((g_!,g^*) : (X \downarrow C) \rightarrow (Y \downarrow C)\) is a Quillen equivalence if and only if the cobase change of \(g\) along \(u\) for each \(u \in (X \downarrow C)_c\) is a weak equivalence.

- \((g_*,g^!): (C \downarrow X) \rightarrow (C \downarrow Y)\) is a Quillen equivalence if and only if the base change of \(g\) along \(u\) for each \(u \in (C \downarrow Y)_f\) is a weak equivalence.

(2) Assume that \(C\) and \(D\) are closed model categories and \((S,U) : C \rightarrow D\) a Quillen adjunction. Then given any objects \(X \in C\) and \(Y \in D\),

- \((i)\) if \(X \in C_c\), the adjunction \((S,U) : (X \downarrow C) \rightarrow (S(X) \downarrow D)\) is a Quillen equivalence.

- \((ii)\) if \(Y \in D_f\), the adjunction \((\tilde{S},\tilde{U}) : (C \downarrow U(Y)) \rightarrow (D \downarrow Y)\) is a Quillen equivalence.

**Proof.** For the proof \((i)\) of (1), recall that \((X \downarrow C)_c = \{ X \rightarrow C \mid u \in Coʃ(C)\}\) and \((Y \downarrow C)_f = \{ Y \rightarrow C \mid C \in C_f\}\). By the construction of \(F\), the unit \(\eta_u : u \rightarrow g^*F(u)\) is the cobase change of \(g\) along \(u\) for any \(u \in (X \downarrow C).\) Since \(g^*(f) = f,\) by Proposition 2.2, \((g_!,g^*)\) is a Quillen equivalence if and only if the composite

\[
u \eta_u \rightarrow g^*F(u) \rightarrow g^*(R(\eta_u))
\]
is a weak equivalence for \( u \in (X \downarrow C)_c \). Note that \( g^*(r_{n(u)}) = r_{g(u)} \) is a weak equivalence. So by the 2-out-of-3 axiom of weak equivalences, this is equivalent to the cobase change of \( g \) along \( u \) is a weak equivalence.

The others can be proved similarly. □

**Corollary 3.2.** [5 Proposition 1.3.7] Suppose \((S, U) : C \to D\) is a Quillen equivalence, and suppose in addition that the terminal object \(* \in C_c\) and that \( S \) preserves the terminal object. Then \((S_*, U_*) : C_* \to D_*\) constructed as in Corollary 2.7 is a Quillen equivalence.

**Proof.** In this case, \((S_*, U_*) = (\overline{S}, \overline{U})\) is a Quillen equivalence by Proposition 3.1. □

Recall that a model category \( C \) is called left proper if every cobase change of a weak equivalence along a cofibration is weak equivalence. Dually, \( C \) is called right proper if every base change of a weak equivalence along a fibration is a weak equivalence. By Proposition 3.1, we can redescribe the left or right properness of a model category by Quillen equivalences:

**Corollary 3.3.**

1. A model category \( C \) is left proper if and only if \((g, g^*)\) is a Quillen equivalence for every weak equivalence \( g : X \to Y \).

2. A model category \( C \) is right proper if and only if \((g_*, g^! )\) is a Quillen equivalence for every weak equivalence \( g : X \to Y \).

**Remark 3.4.** In general, if \( g \) is not a weak equivalence, even \( C \) is proper, \((g, g^*)\) is not necessary a Quillen equivalence. For example, take \( C = \text{mod} \mathbb{k}[x]/(x^2) \) where \( \mathbb{k} \) is a field. Then \( C \) is a proper model category in which weak equivalences are stable equivalences and every object is bifibrant. Take \( g = 0 \to k \), then \( \eta = (\frac{1}{\partial}) \) is the unit of the adjunction \((g, g^*)\). Since in this case every object is fibrant, the maps in (4) of Proposition 2.2 is just \( \eta_C \) for any \( C \in C \). But \( \eta_k : k \to k \oplus k \) is no way to be a weak equivalence. So the Quillen adjunction \((g, g^*) : C \to (k \downarrow C)\) is not a Quillen equivalence.

\(^1\)This should be the right version of Proposition 16.2.4 of [6], there the authors claim that \( C \) is left proper or right proper iff \((g, g^*)\) or \((g_*, g^! )\) is Quillen equivalence for a given map \( g \). For a counter example, see Remark 3.4
If $\mathcal{F}$ is a weakly idempotent complete Frobenius category, then $\mathcal{F}$ has a canonical model structure in which the cofibrations are the monomorphisms, the fibrations are the epimorphisms and the weak equivalences are the stable equivalences [2, Theorem 3.3]. Let $A$ be any nonzero projective-injective object in $\mathcal{F}$. Take $g = 0 : 0 \to A$, then $0_h(0 \to C) = C \xrightarrow{(0)} A \oplus C$ is always a weak equivalence for any $C \in \mathcal{F}$. By Proposition 3.1, we have a Quillen equivalence $(0_!, 0^*) : (0 \downarrow \mathcal{F}) = \mathcal{F} \to (A \downarrow \mathcal{F})$. So the derived functors of $0_!$ and $0^*$ give an equivalence of homotopy categories between $\mathcal{Ho}(\mathcal{F})$ and $\mathcal{Ho}(A \downarrow \mathcal{F})$. Note that in this case, the homotopy category $\mathcal{Ho}(\mathcal{F})$ is just the stable category $\mathcal{F}$ [3, Chapter I, Section 2.2] which is a triangulated category by Theorem 2.6 of [3]. If we can show that the homotopy category $\mathcal{Ho}(A \downarrow \mathcal{F})$ is a triangulated category, then the derived adjunction $(L0_!, R0^*) : \mathcal{F} \to \mathcal{Ho}(A \downarrow \mathcal{F})$ will be a triangle equivalence. Since Quillen equivalences are automatically triangle equivalences if the corresponding homotopy categories are triangulated categories [7, Chapter I, Theorem 3].

Next we will show that the homotopy category $\mathcal{Ho}(A \downarrow \mathcal{F})$ is a triangulated category. And then we give the promised example as advertised in Introduction since the coslice category $(A \downarrow \mathcal{F})$ is not pointed by noting that its initial object is $A \xrightarrow{1_A} A$ and its terminal object is $A \to 0$.

**Proposition 3.5.** With the above notations, we have:

1. The homotopy category $(A \downarrow \mathcal{F})$ is a triangulated category.
2. $(L0_!, R0^*) : \mathcal{F} \to \mathcal{Ho}(A \downarrow \mathcal{F})$ is a triangle equivalence.

**Proof.** (1). Since $\mathcal{F}_{cf} = \mathcal{F}$ and $A$ is injective, we know that $(A \downarrow \mathcal{F})_{cf} = (A \downarrow \mathcal{F})$ and cofibrant objects in $\mathcal{Ho}(A \downarrow \mathcal{F})$ are split monomorphisms in $\mathcal{F}$ with domain $A$. So for any object $u \in (A \downarrow \mathcal{F})_{cf}$, we may write $u$ as $A \xrightarrow{(1 \, 0)_u} A \oplus C$. The morphisms from $u = A \xrightarrow{(1 \, 0)_u} A \oplus C$ to $v = A \xrightarrow{(1 \, 0)_v} A \oplus D$ are of the form $(1 \, r \, 0)$. But they are left homotopic to the morphisms of the form $(1 \, 0 \, s)$ in $(A \downarrow \mathcal{F})_{cf}$. Thus by the construction of $\mathcal{Ho}(A \downarrow \mathcal{F})$ [7, Chapter I, Theorem 1], we may assume that all the morphisms in $(A \downarrow \mathcal{F})_{cf}$ are of the form $(1 \, 0 \, 0)$.

Now $1_A$ is a zero object of the category $(A \downarrow \mathcal{F})_{cf}$ of the cofibrant objects. To prove the statement, it is enough to show that $\mathcal{Ho}((A \downarrow \mathcal{F})_{cf})$ is a triangulated category [7, Chapter I, Theorem 1, Theorem...
Note that \( A \xrightarrow{h} A \oplus C \oplus I(C) \) is a very good cylinder object of \( u = A \xrightarrow{h} A \oplus C \), where \( I(C) \) is an injective preenvelope of \( C \). By the construction of the suspension functor of the homotopy category \( \mathcal{H}_0(A \downarrow F) \)\(^7\), the proof of Theorem 2 in Chapter I], we may define \( \Sigma(u) = A \xrightarrow{h} A \oplus \Sigma^F(C) \). Where \( \Sigma^F \) is the suspension functor of the stable category \( \mathcal{F} \) which is an automorphism of \( \mathcal{F} \) \(^3\) Chapter I, Proposition 2.2]. Then it can be verified directly that the suspension functor \( \Sigma \) defined as above on the homotopy category \( \mathcal{H}_0((A \downarrow F)_{\mathcal{C}}) \) is an auto-equivalence and thus \( \mathcal{H}_0((A \downarrow F)) \) is a triangulated categories by Proposition 5-6 in Section I.3 of \(^7\).

(2). Since Quillen equivalences are automatically triangle equivalences if the corresponding homotopy categories are triangulated categories \(^7\) Chapter I, Theorem 3].

Remark 3.6. Dually we can construct a slice category \( (F \downarrow A) \) for a nonzero projective-injective object \( A \), and there is a triangle equivalence \( (L_0, R_0^!): \mathcal{F} \to \mathcal{H}_0(\mathcal{F} \downarrow A) \).

Acknowledgements: I would like to thank Xiao-Wu Chen, Guodong Zhou and Xiaojing Zhang for their helpful discussions and encouragements.

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