Global Dynamics of the Chaotic Disk Dynamo System
Driven by Noise

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The disk dynamo system, which is capable of chaotic behaviours, is obtained experimentally from two disk dynamos connected together. It models the geomagnetic field and is used to explain the reversals in its polarity. Actually, the parameters of the chaotic systems exhibit random fluctuation to a greater or lesser extent, which can carefully describe the disturbance made by environmental noise. The global dynamics of the chaotic disk dynamo system with random fluctuating parameters are concerned, and some new results are presented. Based on the generalized Lyapunov function, the globally attractive and positive invariant set is given, including a two-dimensional parabolic ultimate boundary and a four-dimensional ellipsoidal ultimate boundary. Furthermore, a set of sufficient conditions is derived for all solutions of the stochastic disk dynamo system being global convergent to the equilibrium point. Finally, numerical simulations are presented for verification.

1. Introduction

The magnetic field has reversed its polarity many times along geological history [1]. To geophysics, their fundamental goal is a coherent understanding of the structure and dynamics of the Earth’s interior. A number of investigators worked hard in order to establish the state of the Earth’s dynamo. Bullard studied a disk dynamo with the intention of discussing possible analogies between them and those of a homogeneous dynamo which is supposed to be the origin of the magnetic field of the Earth and other celestial bodies. Before long, Japanese geophysicist Rikitake [2] found that reversals of electric current generated by a circuit can often occur even in a very simple system such as the one with two disk dynamos. The behaviour of the system is far different from that of the single disk dynamo, which never has a reversal of the electric current. Then, a simple mechanical model used to study the reversals of the Earth’s magnetic field is a two-disc dynamo system idealized by Rikitake. The model consists of two identical single Faraday-disk dynamos of the Bullard type coupled together. For simplicity, we denote the angular velocities of their rotors by $x_3$ and $x_4$ and the currents generated by $x_1$ and $x_2$, respectively. Then, with appropriate normalization of variables, the dynamical equations can be described by the following set of ordinary differential equations [3, 4]:

$$\begin{align*}
\dot{x}_1 &= -\mu_1 x_1 + x_2 x_3, \\
\dot{x}_2 &= -\mu_2 x_2 + x_1 x_4, \\
\dot{x}_3 &= q_1 - \epsilon_1 x_3 - x_1 x_2, \\
\dot{x}_4 &= q_2 - \epsilon_2 x_4 - x_1 x_2.
\end{align*} \tag{1}$$

where $q_1$ and $q_2$ are the torques applied to the rotors and $\mu_1$, $\mu_2$, $\epsilon_1$, and $\epsilon_2$ are the positive constants representing dissipative effects of the disk dynamo system. Rather, from the physical meaning of the equation, the parameters $\mu$ and $\epsilon$
Finally, coupled dynamos (1) can be written in the following differential equations:

\[
\begin{align*}
\frac{dx_1}{dt} &= (-\mu_1 x_1 + x_2 z_1)dt + \sigma_1 x_1 dW(t), \\
\frac{dx_2}{dt} &= (-\mu_2 x_2 + z_1 x_1)dt + \sigma_2 x_2 dW(t), \\
\frac{dx_3}{dt} &= (q_1 - e_1 x_3 - x_1 x_2)dt + (\alpha_1 x_1 + q_{10})dW(t), \\
\frac{dx_4}{dt} &= (q_2 - e_2 x_4 - x_1 x_2)dt + (\alpha_2 x_4 + q_{20})dW(t).
\end{align*}
\]

To illustrate the stochastic effects clearly, we performed simulations for the corresponding stochastic case of Figure 1. The corresponding stochastic case uses the same parameters and initial values. Let \( \mu_1 = 3, \mu_2 = 1, \epsilon_1 = 0.1, \epsilon_2 = 0.2, q_1 = 3, \) and \( q_2 = 1 \) for initial states \((x_1(0), x_2(0), x_3(0), x_4(0)) = (2.2, 2.0, 10.5, 20)\), the numerical simulation shows that the corresponding Lyapunov exponents are \(0.28, 0, -0.10, \) and \(-4.47\). There exists one positive Lyapunov exponent suggest that system (1) has a chaotic attractor. The chaotic attractor’s projections in the coordinate planes \(x_1 - x_2 - x_3\) and \(x_2 - x_3 - x_4\) are shown in Figure 1.

One of the one hand, since the Lorenz system [5] was presented, there is a huge volume of the literature devoted to the studies of the Lorenz system and other classical chaotic systems, which are closely related but not topologically equivalent to the Lorenz system, such as Chen system [6], Lü system [7], and Yang system [8]. In a sense defined by Vaněček and Čelikovský [9, 10], the Chen system is a dual system to the Lorenz system and the Lü system and Yang system represent a transition between the Lorenz and the Chen systems. For the Lorenz family system, mathematicians, physicists, and engineers from various fields have studied the characteristics of systems, bifurcations, routes to chaos, essence of chaos, and chaos synchronization. By ignoring mechanical damping dissipation that parameters \(\epsilon_1 = \epsilon_2 = 0\) and setting \(q_1 = q_2 = 1\) and \(\mu_1 = \mu_2 = \mu,\) we can write \(x_3 = x\) and \(x_4 = x - a,\) where \(a\) is a constant of the motion. Finally, coupled dynamos (1) can be written in the following simple form [11]:

\[
\begin{align*}
\dot{x}_1 &= -\mu x_1 + x_2 z, \\
\dot{x}_2 &= -\mu x_2 + x_1 (x - a), \\
\dot{z} &= 1 - x_1 x_2.
\end{align*}
\]

System (2) has a three-dimensional attractor similar to the Lorenz attractor although both systems are obviously not topologically equivalent [11]. The chaotic behavior and other properties, synchronization and control of the disk dynamo system and disk dynamo-like chaotic systems (2), were extensively studied (see, for instance, [11–16] and their references).

On the other hand, Arnold [17] has pointed out that the parameters in the chaotic systems exhibit random fluctuation to a greater or lesser extent due to various environmental noise. Scholars usually estimate them by average values plus some error terms [18]. In general, by the well-known central limit theorem, the error terms follow normal distributions. For the best incorporate (natural) randomness into the mathematical description of the phenomena and to provide a more accurate description of it, we model the stochastic disk dynamo system by replacing the parameters \(\mu_1, \mu_2, \epsilon_1, \epsilon_2, q_1, \) and \(q_2\) by \(\mu_1 \rightarrow \mu_1 + \sigma_1 dW(t), \)

\[
\mu_2 \rightarrow \mu_2 + \sigma_2 dW(t), \quad \epsilon_1 \rightarrow \epsilon_1 + \sigma_3 dW(t), \quad \epsilon_2 \rightarrow \epsilon_2 + \sigma_4 dW(t), \quad q_{100} \rightarrow q_1 + q_{10} dW(t), \quad \text{and} \quad q_{20} \rightarrow q_2 + q_{20} dW(t),
\]

where \(W(t)\) are the mutually independent Brownian motions. Then, one gets the following system of stochastic differential equations:

\[
\begin{align*}
\frac{dx_1}{dt} &= (-\mu_1 x_1 + x_2 z_1)dt + \sigma_1 x_1 dW(t), \\
\frac{dx_2}{dt} &= (-\mu_2 x_2 + z_1 x_1)dt + \sigma_2 x_2 dW(t), \\
\frac{dx_3}{dt} &= (q_1 - e_1 x_3 - x_1 x_2)dt + (\alpha_1 x_1 + q_{10})dW(t), \\
\frac{dx_4}{dt} &= (q_2 - e_2 x_4 - x_1 x_2)dt + (\alpha_2 x_4 + q_{20})dW(t).
\end{align*}
\]
Let $l_{13} = \min \{2(\mu_1 - (1/2)\sigma_1^2), \sigma_1 - (1/2)\sigma_1^2 \}$ and $l_{24} = \min \{2(\mu_2 - (1/2)\sigma_2^2), \sigma_2 - (1/2)\sigma_2^2 \}$. Suppose that the parameters $2\mu_i > \sigma_i^2$, $2\mu_i > \sigma_i^2$, $2\varepsilon_i > \sigma_i^2$, and $2\varepsilon_i > \sigma_i^2$, $\sigma_i \geq 0 (i = 1, 2, 3, 4)$. Then, the set $\Omega$ is the bound for system (3), in the sense that system (3) is the cylindrical bound, where

$$\Omega = \left\{ X \mid E\left[ x_1^2 + x_2^2 \right] \leq \frac{(q_1 + q_{10}\sigma_1)^2}{l_{13}(\varepsilon_1 - (1/2)\sigma_1^2)} + \frac{q_{10}^2}{l_{13}} E\left[ x_2^2 + x_4^2 \right] \right\}$$

$$\leq \frac{(q_2 + q_{20}\sigma_2)^2}{l_{24}(\varepsilon_2 - (1/2)\sigma_2^2)} + \frac{q_{20}^2}{l_{24}}.$$

(4)

**Proof**

**Step 1.** Construct a positive definite and radically unbounded Lyapunov function on $\mathbb{R}^2$:

$$V_{13}(x_1, x_3) = \frac{1}{2}(x_1^2 + x_3^2).$$

(5)

Applying Itô's formula, one has
\[ dV_{13} = \left[ -\mu_1 x_1^2 - \epsilon_1 x_1^3 + q_1 x_3 + \frac{1}{2} \left( \sigma_1^2 x_1^2 + (\sigma_2 x_3 + q_{10})^2 \right) \right] dt \\
+ \left( \sigma_1 x_1^2 + \sigma_3 x_3^2 + q_{10} x_3 \right) dW(t) \\
\leq \left[ -\left( \mu_1 - \frac{1}{2} \sigma_1^2 \right) x_1^2 - \frac{1}{2} \left( \epsilon_1 - \frac{1}{2} \sigma_2^2 \right) x_3^2 + L_{13} \right] dt \\
+ \left( \sigma_1 x_1^2 + 2\sigma_3 x_3^2 + \frac{q_{10}^2}{4\sigma_3} \right) dW(t) \\
\leq (L_{13} V_{13} + L_{13}) dt + \left( L_4 V_{13} + \frac{q_{10}^2}{4\sigma_3} \right) dW(t), \]

where
\[
\begin{align*}
L_{13} &= \min \left\{ 2\left( \mu_1 - \frac{1}{2} \sigma_1^2 \right), \epsilon_1 - \frac{1}{2} \sigma_2^2 \right\}, \\
L_3 &= \max \{ 2\sigma_1, 4\sigma_3 \}; \\
L_{13} &= \frac{(q_1 + q_{10}\sigma_1)^2}{2(\mu_1 - (1/2)\sigma_1^2)} + \frac{1}{2} q_{10}^2.
\end{align*}
\]

Similar to the proof of Theorem 1, we can obtain
\[
E \left[ V_{13} - \frac{L_{13}}{L_{13}} \right] \leq \left[ V_{13}(x_1(t_0), x_3(t_0)) - \frac{L_{13}}{L_{13}} \right] \exp [-L_{13} (t - t_0)].
\]

Therefore, one has \( \lim_{t \to +\infty} E V_{13} \leq (L_{13}/L_{13}) \); that is to say, the following inequality holds as \( t \to +\infty \):
\[
E \left[ x_1^2 + x_3^2 \right] \leq \frac{2L_{13}}{L_{13}}.
\]

Step 2. Construct a positive definite and radically unbounded Lyapunov function on \( \mathbb{R}^2 \):
\[
V_{24}(x_2, x_4) = \frac{1}{2} (x_2^2 + x_4^2).
\]

Similar to the proof of Step 1, we can obtain
\[
dV_{24} \leq (L_{24} V_{24} + L_{24}) dt + I_{24} V_{24} dW(t),
\]

where
\[
L_{24} = \min \left\{ 2\left( \mu_2 - \frac{1}{2} \sigma_2^2 \right), \epsilon_2 - \frac{1}{2} \sigma_4^2 \right\}, \\
I_1 = \max \{ 2\sigma_3, 4\sigma_4 \}; \\
L_{24} = \frac{(q_2 + q_{20}\sigma_2)^2}{2(\epsilon_2 - (1/2)\sigma_2^2)} + \frac{1}{2} q_{20}^2.
\]

Therefore, the following inequality holds as \( t \to +\infty \):
\[
E \left[ x_2^2 + x_4^2 \right] \leq \frac{2L_{24}}{L_{24}}.
\]

Remark 1. Let \( \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0 \); then system (3) is deterministic. Theorem 2 contains the results given in [12] as special cases.

Let \( \mu_1 = 3, \mu_2 = 1, \epsilon_1 = 0.1, \epsilon_2 = 0.2, q_1 = 3, q_2 = 1, \sigma_1 = \sigma_2 = 0.1, \sigma_3 = \sigma_4 = 0.01, \) and initial values \( (x_0, y_0, z_0, u_0) = (2.2, 2.2, 2.5, 3) \). Calculate \( L_{13} = 0.099950 \) and \( L_{24} = 0.199950 \). We give the following estimate of the ultimate boundary:
\[
\Omega = \left\{ x \mid E \left[ x_1^2 + x_3^2 \right] \leq 901.601426, E \left[ x_2^2 + x_4^2 \right] \leq 25.112567 \right\}.
\]

(14)

The corresponding projections of exponentially attractive sets are shown in Figure 4.

And we also have the following results:
\[
\begin{align*}
| E x_1 | &\leq 30.026679, \\
| E x_2 | &\leq 5.011244, \\
| E x_3 | &\leq 30.026679, \\
| E x_4 | &\leq 5.011244.
\end{align*}
\]

(15)

The numerical solutions, which are stochastic processes, of stochastic dynamo system (3) are obtained by the Euler–Maruyama method. All the stochastic processes’ scopes and the ultimate boundary of the corresponding expectations are listed in Table 1. From Table 1, we are pleased to see that the simulation results and the theoretical results of (14) and (15) are consistent.

3. Globally Exponentially Attractive Set

Theorem 2. Let \( I_0 = \min \{ \mu_1 - (1/2)\sigma_1^2, \mu_2 - (1/2)\sigma_2^2, \epsilon_1 - (1/2)\sigma_3^2, \epsilon_2 - (1/2)\sigma_4^2 \} \), and \( L = (L_1/I_0) \). Suppose that the parameters \( 2\mu_1 > \sigma_1^2, 2\mu_2 > \sigma_2^2, 2\epsilon_1 > \sigma_3^2, \) and \( 2\epsilon_2 > \sigma_4^2 \). Then, for any constant \( \lambda > 0 \), the following estimate holds on system (3):
\[
E [V(X) - L] \leq [V(X_0) - L] \exp [-I_0 (t - t_0)].
\]

(16)

In particular,
\[
\Omega = \{ x \mid EV(X) \leq 2L \} = \{ x \mid E \left[ x_1^2 + \lambda x_2^2 + x_3^2 + \lambda x_4^2 \right] \leq 2L \}
\]

(17)

is a globally exponential attractive set of system (3), where \( V(X) = \frac{1}{2} \left( x_1^2 + \lambda x_2^2 + x_3^2 + \lambda x_4^2 \right) \).

\[
L_1 = \frac{(q_1 + q_{10}\sigma_1)^2}{2(\epsilon_1 - (1/2)\sigma_1^2)} + \frac{\lambda \left( q_2 + q_{20}\sigma_2 \right)^2}{2(\epsilon_2 - (1/2)\sigma_2^2)} + \frac{1}{2} q_{10}^2 + \frac{\lambda}{2} q_{20}^2.
\]

(18)

Proof. Define the Lyapunov on \( \mathbb{R}^4 \), where
\[
V(X) = \frac{1}{2} \left( x_1^2 + \lambda x_2^2 + x_3^2 + \lambda x_4^2 \right).
\]

Applying Itô’s formula to (19), one has
$\sigma_1 = \sigma_2 = 0.1, \sigma_3 = \sigma_4 = 0.01, \text{ and } q_{10} = q_{20} = 0.1, \text{ and initial values } (x_0, y_0, z_0, u_0) = (2.2, 2, 2.5, 3).$

**Figure 4:** The projection of exponentially attractive set of the stochastic dynamo system with $\mu_1 = 3, \mu_2 = 1, \epsilon_1 = 0.1, \epsilon_2 = 0.2, q_1 = 3, q_2 = 1,$ $\sigma_1 = \sigma_2 = 0.1, \sigma_3 = \sigma_4 = 0.01, \text{ and } q_{10} = q_{20} = 0.1, \text{ and initial values } (x_0, y_0, z_0, u_0) = (2.2, 2, 2.5, 3).$

**Table 1:** Ultimate boundary for stochastic dynamo system.

| Scopes of stochastic processes | Simulated results of expectation | Theoretical estimates of expectation |
|-------------------------------|----------------------------------|-------------------------------------|
| $-7.692934 \leq x_1 \leq 7.736921$ | $E_{x_1} = 0.104873$ | $|E_{x_1}| \leq 30.026679$ |
| $-1.484651 \leq x_2 \leq 2.785851$ | $E_{x_2} = 0.033916$ | $|E_{x_2}| \leq 5.011244$ |
| $1.160790 \leq x_3 \leq 22.686710$ | $E_{x_3} = 19.921792$ | $|E_{x_3}| \leq 30.026679$ |
| $-1.372995 \leq x_4 \leq 3.000000$ | $E_{x_4} = 0.248853$ | $|E_{x_4}| \leq 5.011244$ |
| $2.652947 \leq r_1 \leq 526.915239$ | $E_{r_1} = 410.415404$ | $0 \leq E_{r_1} \leq 901.601426$ |
| $0.000000 \leq r_2 \leq 13.000000$ | $E_{r_2} = 0.525692$ | $0 \leq E_{r_2} \leq 25.112567$ |

$r_1 = x_1^2 + x_2^2; r_2 = x_3^2 + x_4^2.$

$\begin{align*}
\text{dV}(X) &= \left[ x_1(-\mu_1 x_1 + x_2 x_3) + \lambda x_2(-\mu_2 x_2 + x_1 x_4) \\
&+ x_3(q_1 - \epsilon_1 x_1 - x_2) + \lambda x_4(q_2 - \epsilon_2 x_4 - x_1 x_2) \\
&+ \frac{1}{2}[\sigma_1 x_1^2 + \lambda \sigma_2 x_2^2 + (\sigma_3 x_3 + q_{10})^2] \\
&+ \lambda (\sigma_4 x_4 + q_{20})^2 \right] dt \\
&+ \left[ -\left( \mu_1 - \frac{1}{2}\sigma_1^2 \right) x_1^2 - \lambda \left( \mu_2 - \frac{1}{2}\sigma_2^2 \right) x_2^2 - \left( \epsilon_1 - \frac{1}{2}\sigma_3^2 \right) x_3^2 \\
&- \lambda \left( \epsilon_2 - \frac{1}{2}\sigma_4^2 \right) x_4^2 + (q_1 + q_{10}) x_1 \\
&+ \lambda (q_2 + q_{20}) x_2 + \frac{1}{2}(q_{10}^2 + \lambda q_{20}^2) \right] dt \\
&+ \left[ \sigma_1 x_1^2 + \lambda \sigma_2 x_2^2 + \sigma_3 x_3^2 + q_{10} x_3 + \lambda \sigma_4 x_4^2 \\
&+ \lambda q_{20} x_4 \right] dW(t) \\
&= \left[ \frac{1}{2}(-\mu_1 + \frac{1}{2}\sigma_1^2) x_1^2 + \frac{1}{2}(-\mu_2 + \frac{1}{2}\sigma_2^2) x_2^2 \\
&- \frac{1}{2}\left( \epsilon_1 - \frac{1}{2}\sigma_3^2 \right) x_3^2 - \frac{1}{2}\left( \epsilon_2 - \frac{1}{2}\sigma_4^2 \right) x_4^2 + F(X) \right] dt \\
&+ \left[ 2\sigma_1 x_1^2 + 2\lambda \sigma_2 x_2^2 + 2\sigma_3 x_3^2 + 2\lambda \sigma_4 x_4^2 + G(X) \right] dW(t),
\end{align*}$

where

$\begin{align*}
F(X) &= -\frac{1}{2}(\mu_1 - \frac{1}{2}\sigma_1^2) x_1^2 - \lambda \left( \mu_2 - \frac{1}{2}\sigma_2^2 \right) x_2^2 \\
&= -\frac{1}{2} \left( \epsilon_1 - \frac{1}{2}\sigma_3^2 \right) x_3^2 - \frac{1}{2}\left( \epsilon_2 - \frac{1}{2}\sigma_4^2 \right) x_4^2 \\
&+ \lambda q_{10} x_1 x_4 + \lambda q_{20} x_2 x_4 \\
&+ (q_1 + q_{10}) x_1 + \lambda (q_2 + q_{20}) x_2 \\
&+ \frac{1}{2}(q_{10}^2 + \lambda q_{20}^2),
\end{align*}$

$\begin{align*}
G(X) &= -\sigma_1 x_1^2 - \lambda \sigma_2 x_2^2 - \sigma_3 x_3^2 - q_{10} x_3 - \lambda \sigma_4 x_4^2 + \lambda q_{20} x_4.
\end{align*}$

Then,

$\begin{align*}
F(X) &\leq \sup_{X \in \mathbb{R}^4} F(X) = \frac{(q_1 + q_{10}\sigma_3)^2}{2(\epsilon_1 - (1/2)\sigma_1^2)} + \frac{\lambda (q_2 + q_{20}\sigma_4)^2}{2(\epsilon_2 - (1/2)\sigma_2^2)} \\
&+ \frac{1}{2} q_{10}^2 + \frac{1}{2} \lambda q_{20}^2 = L_1,
\end{align*}$

$\begin{align*}
G(X) &\leq \sup_{X \in \mathbb{R}^4} G(X) = \frac{q_{10}^2 + \lambda q_{20}^2}{4\sigma_3} = L_2.
\end{align*}$

From (22) and (23), we can obtain

$\begin{align*}
dV(X) &\leq \left[-I_0 V(X) + L_1\right] dt + \left[I_1 V(X) + L_2\right] dW(t),
\end{align*}$

where
\[ l_0 = \min \left\{ \frac{1}{2} \sigma_1^2, \frac{1}{2} \sigma_2^2, \frac{1}{2} \sigma_3^2, \frac{1}{2} \sigma_4^2 \right\}, \]
\[ l_1 = 4 \max \{ \sigma_1, \sigma_2, \sigma_3, \sigma_4 \}. \tag{25} \]

From (24) and the calculating the expectation, one obtains
\[ EV(X) \leq V(X_0) + \int_{t_0}^{t} [-l_0 EV(X) + L_1] \, ds. \tag{26} \]

From above inequality, one can obtain
\[ EV(X) \leq V(X_0) \exp \left( -l_0 (t - t_0) \right) + L_1 \int_{t_0}^{t} \exp \left( -l_0 (s - t_0) \right) \, ds \]
\[ = V(X_0) \exp \left( -l_0 (t - t_0) \right) + \frac{L_1}{l_0} \left[ 1 - \exp \left( -l_0 (t - t_0) \right) \right]. \tag{27} \]

Let \( L = (L_1/l_0) \). When \( EV(X_0) - L > 0, EV(X_0) - L > 0 \), the following estimate holds:
\[ E[V(X) - L] \leq [V(X_0) - L] \exp \left( -l_0 (t - t_0) \right). \tag{28} \]

Thus,
\[ \lim_{t \to \infty} EV(X) \leq L. \tag{29} \]

That is,
\[ \Omega = \{ X \mid EV(X) \leq 2L \} = \{ X \mid E \left[ x_1^3 + \lambda x_2^2 + x_3^2 + \lambda x_4^2 \right] \leq 2L \}. \tag{30} \]

**Theorem 3.** Let \( l_0 = \min \left\{ \frac{1}{2} \sigma_1^2, \frac{1}{2} \sigma_2^2, \frac{1}{2} \sigma_3^2 \right\}, \) \( e_1 = (1/2) \sigma_1^2, \) \( e_2 = (1/2) \sigma_2^2, \) \( \sigma = (1/2) \sigma_3^2, \) and \( L = (L_1/l_0) \). Suppose that the parameters \( 2 \mu_1 > \sigma_1^2, \) \( 2 \mu_2 > \sigma_2^2, \) \( 2 e_1 > \sigma_3^2, \) and \( 2 e_2 > \sigma_4^2. \) Then, for any constant \( \lambda > 0 \) and \( \eta \in \mathbb{R} \), the following estimate holds on system (3):
\[ E[V(X) - L] \leq \left[ V(X_0) - L \right] \exp \left( -l_0 (t - t_0) \right). \tag{31} \]

In particular,
\[ \Omega = \{ X \mid EV(X) \leq 2L \} = \{ X \mid E \left[ x_1^3 + \lambda x_2^2 + x_3^2 + \lambda x_4^2 \right] \leq 2L \} \tag{32} \]

is a globally exponential attractive set of system (3), where
\[ V(X) = \frac{1}{2} \left[ x_1^2 + \lambda x_2^2 + (x_3 + \lambda \eta)^2 + \lambda (x_4 - \eta)^2 \right], \]
\[ L_{\lambda \eta} = \frac{(q_1 + q_{10} \sigma_3 - (1/2) \lambda \eta \sigma_3^2)}{2(e_1 - (1/2) \sigma_3^2)} + \frac{1}{2} \lambda^2 \eta^2 \left( e_1 - \frac{1}{2} \sigma_3^2 \right) \]
\[ + \lambda \eta q_1 + \frac{1}{2} q_{10} \]
\[ + \frac{\left[ (q_2 + q_{20} \sigma_3 - (1/2) \lambda \eta \sigma_3^2) \right]^2}{2(e_2 - (1/2) \sigma_3^2)} + \frac{1}{2} \lambda^2 \left( e_2 - \frac{1}{2} \sigma_4^2 \right) \]
\[ - \lambda \eta q_2 + \frac{1}{2} q_{20}. \tag{33} \]

**Proof.** The proof is the same as that for Theorem 2; we omit it here.

**Remark 2.** Let \( \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = q_{10} = q_{20} = 0; \) then, system (3) is deterministic. Theorem 3 contain the results given in [12] as special cases.

In Theorem 3, let \( \mu_1 = 3, \) \( \mu_2 = 1, \) \( e_1 = 0.1, \) \( e_2 = 0.2, \) \( q_1 = 3, \) \( q_2 = 1, \) \( \sigma_1 = \sigma_2 = 0.1, \) \( \sigma_3 = \sigma_4 = 0.01, \) \( q_{10} = q_{20} = 0.1, \) and initial values \( (x_0, y_0, z_0, u_0) = (2.2, 2, 10.5, 20). \) We give the following estimate of the ultimate boundary:
\[ \Omega = \{ X \mid E \left[ x_1^2 + x_2^2 + (x_3 + 1)^2 + (x_4 - 1)^2 \right] \leq 1034.644495 \}. \tag{34} \]

This is the globally exponentially attractive set and positive invariant set of the stochastic disk dynamo system.

Then, we have the following results of the ultimate boundary about \( x_1 - x_2 - x_3, \) \( x_1 - x_2 - x_4, \) \( x_1 - x_3 - x_4, \) and \( x_2 - x_3 - x_4, \) which are the exponentially attractive sets of the stochastic disk dynamo system:
\[ E \left[ x_1^2 + x_2^2 + (x_3 + 1)^2 + (x_4 - 1)^2 \right] \leq (32.165890)^2, \]
\[ E \left[ x_1^2 + x_2^2 + (x_4 - 1)^2 \right] \leq (32.165890)^2, \]
\[ E \left[ x_1^2 + (x_3 + 1)^2 + (x_4 - 1)^2 \right] \leq (32.165890)^2. \tag{35} \]

The numerical solutions, which are stochastic processes, of stochastic dynamo system (3) are obtained by the Euler–Maruyama method. The simulated time series about \( x_1^2 + x_2^2 + (x_3 + 1)^2 + (x_4 - 1)^2 \) is displayed in Figure 5. In addition, the stochastic processes’ scopes is
\[ 9.887650 \leq x_1^2 + x_2^2 + (x_3 + 1)^2 + (x_4 - 1)^2 \leq 580.748156, \tag{36} \]
and the corresponding expectation is
\[ E \left[ x_1^2 + x_2^2 + (x_3 + 1)^2 + (x_4 - 1)^2 \right] = 457.015926. \tag{37} \]

It is nice to see that the simulation results and the theoretical results of (34) are consistent.

### 4. Stochastic Stability

The purpose of this section is to seek condition for the asymptotic behavior of system (3).

**Theorem 4.** When perturbed parameters \( \sigma_1 = \sigma_2 = \sigma_4 = q_{10} = q_{20} = 0; \) suppose that the parameters \( 2 \mu_1 > \sigma_1^2, \) \( 2 \mu_2 > \sigma_2^2. \) If \( (q_1/e_1) + (q_2/e_2) < \sqrt{(2 \mu_1 - \sigma_1^2)(2 \mu_2 - \sigma_2^2)}, \) the equilibrium position \( (0, 0, (q_1/e_1), (q_2/e_2)) \) of system (3) is stochastically asymptotically stable.

**Proof.** Let \( V(X) = (1/2)(x_1^2 + x_2^2 + (x_3 - (q_1/e_1))^2 + (x_4 - (q_2/e_2))^2). \) Then,
When the parameters $2\mu_1 > \sigma_1^2$, $2\mu_2 > \sigma_2^2$, and $(q_1/\epsilon_1) + (q_2/\epsilon_2) < \sqrt{(2\mu_1 - \sigma_1^2)(2\mu_2 - \sigma_2^2)}$, the matrix $Q$ is positive-definite. Thus, $LV$ is negative-definite. Then, from Theorem 4.2.3 of [35], the equilibrium position $(0,0,(q_1/\epsilon_1),(q_2/\epsilon_2))$ of system (3) is stochastically asymptotically stable.
Remark 3. When the perturbed parameters
\( \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = q_{10} = q_{20} = 0 \), system (3) is deterministic. Suppose \( \mu_1 = \mu_2, \ e_1 = e_2, \) and \( q_1 = q_2; \) then, condition
\( (q_1/e_1) + (q_2/e_2) < \sqrt{(2\mu_1 - \sigma_1^2)(2\mu_2 - \sigma_2^2)} \)
reduced to \( q/e < \mu \). That is to say, the deterministic disk dynamo system is stable when \( q/e < \mu \). That conclusion is coincident with the result of the literature [4].

Remark 4. Comparing the conditions of Theorem 3 and Theorem 4, the results show that the asymptotically stability of the stochastic disk dynamo system occurs when
\( (q_1/e_1) + (q_2/e_2) < \sqrt{(2\mu_1 - \sigma_1^2)(2\mu_2 - \sigma_2^2)} \), which means the stochastic disk dynamo system will not show chaotic behavior.

Let \( u_1 = 2.5, \ u_2 = 2.6, \ e_1 = 0.5, \ e_2 = 1, \ q_1 = 1.5, \ q_2 = 2, \)
\( \sigma_1 = 1, \ \sigma_2 = 1, \) and \( \sigma_3 = \sigma_4 = q_{10} = q_{20} = 0 \), and initial values \( (x_0, y_0, z_0, u_0) = (2.2, 2, 10.5, 20) \). In Figure 6, the number results show that the trivial solution of system (3) is stochastically asymptotically stable.

5. Conclusions

The coupled dynamo system is a nonlinear dynamical system which is capable of chaotic behaviours. It models the geomagnetic field and is used to explain the reversals in its polarity. Actually, the parameters of the chaotic systems exhibit random fluctuation to a greater or lesser extent, which can carefully describe the disturbance made by environmental noise. The global dynamics of the chaotic disk dynamo system with random fluctuating parameters are concerned, and some new results are presented. Based on the generalized Lyapunov function, the globally attractive and positive invariant set is given, including a two-dimensional parabolic ultimate boundary and a four-dimensional ellipsoidal ultimate boundary. Furthermore, a set of sufficient conditions is derived for all solutions of the stochastic disk dynamo system being global convergent to the equilibrium point. The stochastic disk dynamo system will not show chaotic behavior when the system is stable. Finally, numerical simulations are presented for verification.

Data Availability

All data generated or analyzed during this study are included in this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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