Multipole Extraction: A novel, model independent method

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Abstract. A novel method for extracting multipole amplitudes in the nucleon resonance region from electroproduction data is presented. The method is based on statistical concepts and it relies heavily on Monte Carlo and simulation techniques; it produces precise identification and determination of the contributing multipole amplitudes in the resonance region and for the first time a rigorous determination of the associated experimental uncertainty. The results are demonstrated to be independent of any model bias. The method is applied in the reanalysis of the $Q^2 = 0.127 \text{ GeV}^2 / c^2$ Bates and Mainz $N \to \Delta$ data.

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MOTIVATION

A significant component of hadronic physics research focuses on the understanding of the excitation spectrum of the proton, the only stable hadron [1]. Impressive progress has been achieved in the last decade largely driven by the advances in accelerator and instrumentation technologies. The high quality of data that have emerged and substantial progress in theory have brought a new level of sophistication to the field. Conjectures, such as the deformation of the nucleon [2] have been confirmed (see [3] and references therein) and transformed into an extensive program examining not its existence but rather the mechanisms that generate it. For the first time contact with QCD through lattice [4] calculations and chiral field effective theories [5, 6] has been established. A most fertile ground of research has been the identification of the nucleon resonances, the isolation and interpretation of the contributing multipoles to their excitation spectrum. Progress has been substantial but slow, due to the large width of the resonances, their overlapping nature and the incompleteness of the data base in terms of observables. Extensive work concerns the investigation of the $\Delta^+(1232)$, the first excited and only isolated state of the proton, through the $N \to \Delta$ reaction [3, 7].

FIGURE 1. The currently employed methods of multipole extraction allow a reasonable determination of statistical and systematic uncertainties and an imprecise determination of model error, which often dominates. The error bands shown on the partial cross sections $\sigma_{LT}$ and $\sigma_{oo} \equiv \sigma_{E2}$ depict the estimated model error with or without the assumption of spherical nucleon for the $N \to \Delta$ transition. While the conjecture of "deformation" is confirmed, understanding its origins will require a precise understanding and reduction of the model uncertainties (adopted from Ref [3]).

The currently employed methodology for extracting multipoles has served the field well, yielding important results. Figure 1 taken from reference [3], in which Bates data [8, 9] are compared with theories that account for them, demonstrates that the assumption of sphericity for both the nucleon and the $\Delta^+(1232)$ is incompatible with the data.
The "deformed" band is mapped by the spread of the model predictions that successfully account for the data, while the dark, "spherical", band spans the predictions of the same models in which "sphericity" is imposed [3]. However, investigating the physical origin of deformation will require the comparison of theoretical results which lie within the uncertainty band to the data. A precise, quantitative definition of uncertainties of both the experimentally derived and the theoretically produced values for quantities such as $R_{LT}$ and of the multipoles that contribute to it, is obviously required. The work presented partly addresses this issue by presenting a method for extracting multipole information from experimental nucleon resonance data in a rigorous, precise, and model independent way.

RESONANCE MULTIPOLe EXTRACTION

It has been assumed up to now that to extract multipole information model independently from nucleon resonance data (cross sections and polarization asymmetries), a complete set of experimental observables is required. Current experiments which measure an incomplete set of observables typically rely on model extraction. Multipoles have been presented in the literature extracted by employing one of the following two approaches: a) The Truncated Multipole Expansion (TME) approximation where most or all of the non resonant multipoles are neglected (e.g. see [10, 11]) assuming that at the peak of the resonance only resonant amplitudes contribute significantly and b) The Model Dependent Extraction (MDE) method where certain multipole amplitudes, often the resonant amplitudes, within a phenomenological model description are adjusted to best describe the data (e.g. see [8, 10, 12]). The second method, MDE, is obviously superior to TME, for it assumes that multipoles that cannot be determined from the data are fixed through a model and not simply ignored. However it suffers from the fact that the extracted values are biased by the model and therefore characterized by a hard to evaluate systematic model uncertainty. To ameliorate this deficiency, an Ansatz has been proposed [3] which is often used [12, 13] whereby the same data are analyzed employing different models which describe the data adequately, and attributing the resulting spread in the extracted quantities to model uncertainty. A critical assessment of this approach in the case of the $N \rightarrow \Delta$ transition and the resulting uncertainties is presented in reference [14].

We have argued that the MDE method of extraction has up to now served the field well; however, the refined questions that are now emerging, such as the detailed understanding of the origin of deformation, require the development of a methodology which addresses the following limitations of MDE:

- A preconceived and somewhat arbitrary choice of the multipoles to extracted is required.
- The extracted values are model dependent and the model error is hard to quantify.
- Extraction of multipole values requires the existence of a model that successful describes the given resonance and the estimation of model error requires a multitude of them.
- Multipole amplitudes to which the experimental data are only moderately sensitive are impossible to extract.

A Model Independent Analysis Scheme, "AMIAS", is described here which addresses most of the above mentioned limitations. Although it is presented in this paper for the case of the $N \rightarrow \Delta$ transition, it provides a framework and a methodology of analysis which is applicable to all nucleon resonances. In the sections that follow we shall develop the methodology and will subsequently apply it to the re-analysis of the $Q^2 = 0.127\text{ GeV}^2/c^2$ Bates and Mainz $N \rightarrow \Delta$ data. We will conclude by discussing the advantages of the new method and the possibilities that it offers which they need to be explored and researched.

The AMIAS Method and Postulate

The AMIAS method has general applicability in a variety of physical problems including the analysis of electroproduction data in the nucleon resonance region. The method has a minimal requirement that the parameters to be determined are linked in an explicit way to the measured experimental quantities. There is no requirement that this set of parameters provide an orthogonal basis. Moreover these parameters can be subjected to explicit constraints.

The essence of the method derives from the following postulate:

Postulate: A set of parameters $A_j^1, A_j^2, \ldots, A_j^N = \{A_j\}$ which completely and explicitly describes a physical process can be determined from relevant experimental observables $O^M = \{V^M_k \pm \epsilon^M_k\}$, (value $\pm$ uncertainty), by assuming that any set $\{A_j\}$ constitutes a solution having a probability $P(j, M)$ of representing the "correct" solution which
is equal to

\[ P(j, M) = \text{erf}(\chi^2(j, M)(A^j_V, O^M_k)) \]

where

\[ \chi^2(j, M) = \sum_k \left( \frac{U^j_k - V^M_k}{\varepsilon^M_k} \right)^2 \]

Thus \( P(j, M) \) is a function of the \( \chi^2 \) resulting by the comparison to the experimental data \( O^M = \{ V_k, \varepsilon_k \} \) of the predicted values \( U^j_k \) by the \( \{ A^j_V \} \) solution.

We call an ensemble \( Z^M \) of such solutions \textit{Canonical Ensemble of Solutions}, which has properties that depend only on the experimental data set \( O^M \). Similarly a \textit{Microcanonical Ensemble of Solutions} can be defined as the collection of solutions which are characterized by

\[ \chi^2 \leq \chi^2_{\text{min}} + C \]

where \( C \) is usually taken to be the constant equal to the effective degrees of freedom of the problem.

In the sections that follow we further develop the methodology through the specific problem of extracting multipole values from nucleon resonances electroproduction data.

**Multipole Extraction employing AMIAS**

In applying AMIAS to the problem of multipole extraction in the \( N \to \Delta \) transition the parameters to be extracted, \( \{ A^j_V \} \) in the general formulation, are the multipole amplitudes \( M_{L\pm} \), and if the data allow, the isospin separated amplitudes \( M_{L\pm}^{1/2}, M_{L\pm}^{3/2} \). The experimental observables (data) \( \{ O_i \} \) are typically cross section and polarization asymmetries. Furthermore, the parameters of our problem, the multipole amplitudes, are subjected to the constraint of unitarization, by imposing the Fermi Watson theorem. The linkage of observables to multipoles in the case of nucleon resonances is shown schematically in Figure 2. In the demonstration case we will examine which concerns the Bates-Mainz data, the multipoles used refer to the \( \pi^0p \) charge channel which are connected to the \( (A_1^{1/2}, A_1^{3/2}, A_3^{1/2}) \) isospin-set through the relation:

\[ A_{\pi^0p} = A_1^{1/2} + \frac{2}{3} A_3^{3/2} \]
as described in [15]. Following the standard practice of the field, we shall also consider the phases of the multipoles as known (from $\pi N$ scattering, see [15, 16, 17, 18]) with extreme precision and therefore fixed. The above choices allow a meaningful comparison with previous analyses and model results; however they are not inherent to the scheme. It is quite easy to consider the phases, as experimental parameters characterized by uncertainties or even to perform a combined analysis of electroproduction and $\pi N$ scattering data.

The AMIAS method produces numerical results, i.e. determines the multipole amplitudes and the associated uncertainties, from the experimental data by examining the properties of the ensemble of solutions, canonical or microcanonical. In implementing the above stated postulate to construct ensembles of solutions, statistical concepts and extensive use of computational techniques are employed. In the sections that follow we employ data and pseudodata concerning the $\mathcal{N} \rightarrow \Delta$ transition. This approach would have been impossible to employ few years earlier; it is the nowadays readily available formidable computational power that has made such a scheme possible.

The Algorithm and the Ensemble of Solutions

The AMIAS postulate is implemented using a Monte Carlo technique. Given a specific set "M" of experimental data $\{O^M_k\}$ the following procedure is taken, employing a formulation that connects multipoles to experimental observables [15] (see Figure 2):

1. Choose a maximum cutoff, $L_{\text{cut}}$, for the multipoles that are to be explored in the analysis. This choice implies that a set $\{A_\nu\}$ of $\nu$ parameters constitutes a solution to our problem.
2. Assume a particular range of "acceptable" values for each of the multipoles. These ranges define the "phase volume" which the Monte Carlo method explores.
3. Within the domain of acceptable values, randomly select a value (with equal probability) for the particular multipole. The randomly chosen values constitute a solution, the $j^{th}$ solution, $\{A_\nu\}_j$, to the problem.
4. Using the data calculate the particular value of the $\chi^2(A_\nu, O_k)_j$ corresponding to the particular solution $\{A_\nu\}_j$.
5. Repeating the above steps (1-4) a large number, typically a few hundred thousand, of solutions are generated which taken collectively they constitute the ensemble of solutions to our problem.

**FIGURE 3.** Typical histogram of solutions that is generated by randomly varying the model parameters $\{A_\nu\}$ (multipoles) within a given phase volume. It can be seen that, as the phase volume increases, more "bad" solutions are generated while the distribution of "good solutions" (area in the dotted ellipse and insert), gets saturated. "$\sigma$" represents a unit of length in the phase volume and "good" or "bad" solutions refer to the $\chi^2$ value of the solution (see discussion in text).
The method is totally insensitive to the choice of the multipole cutoff $L_{cut}$, which should be chosen "sufficiently high" so that the derived results do not change if $L_{cut} + 1$ is used as a maximum cutoff instead. The computational time required increases roughly quadratically to the number of multipoles being explored.

The results presented below are derived by using $L_{cut} = 5$. Similarly the size of the phase volume that the Monte Carlo method is sampling does not affect the solution provided that the volume is "sufficiently large" to include all "good" solutions. By "good solutions" we denote solutions with small or reasonable $\chi^2$ values as compared to the degrees of freedom of the problem. The determination of the phase volume might require successive attempts so it is appropriately bounded or the employment of self adjusting algorithms, as will comment below. The computational time required depends critically on a "reasonable" choice for the phase volume to be explored.

Each one of the solutions in the ensemble $\{A_\nu\}$ is tagged by a $\langle \chi^2 \rangle$ value. It is obvious that the overwhelming majority of the solutions generated in this fashion are characterized by very large $\chi^2$ values. This can be seen in the histogram of $\chi^2$ values which result for a given choice of the phase volume shown in Figure 3. The behavior of the distribution is altered by changing the "size" of the phase volume. To visualize the effect, we have defined a scale "$\sigma$" within which a given multipole is allowed to vary. We subsequently have increased the phase volume explored by increasing the range within which multipole is allowed to vary by multiples of this basic scale. As the volume is increased, the distribution is changed, basically by producing more solutions characterized by terrible $\chi^2$ values, as expected. However, as is also evident in the insert, a saturation of the "good values" is achieved; we simply are not able to identify new regions of good solutions by exploring wider range of values for the multipoles.

The behavior of a given model parameter $A_\nu$ (multipole) in the ensemble of solutions can be visualized by projecting the solutions on a two dimensional surface where the value of the parameter is projected in the ordinate and its corresponding $\chi^2$ value on the abscissa. This is done in the left column of Figure 4. The top row concerns the $L_{1+}$ multipole, for which the data are known to be particularly sensitive, while the bottom row that of the $L_{3+}$ multipole to which the particular data set exhibits little or no sensitivity. For a specific value of the multipole the ensemble contains solutions characterized by a wide range of $\chi^2$ values depending of the choice of the other multipoles whose values are not visualized. For a given value of the multipole it can be seen that a solution can be identified which
is characterized by a minimum value of $\chi^2$. For the $L_{1+}$ multipole, but not for the $L_{3+}$, the observed minima show strong dependence on the value of the multipole, thus manifesting its sensitivity to the data. This sensitivity becomes explicit and quantifiable by applying successive cuts on the $\chi^2$ values and constructing out of the selected population of solutions histograms, shown in the middle column of the figure. It can be seen that the distribution of solutions is uniform (flat) if the entire population is examined, manifesting the randomness of the choice (no model bias). However, as we progressively limit the solutions accepted by restricting the acceptable $\chi^2$ values, a distribution of values with a well defined maximum and width emerges, but only for the ($L_{1+}$) multipole, to which the data are sensitive; for the non sensitive ($L_{3+}$), no distribution with a finite width can be found. Finally, in the righthand panel of the top row of Figure 4, the dependence on the values of the maxima and of the widths of the resulting peaks is explored as a function of $\chi^2_{\text{cut}}$. A stable saturation value for the peak position emerges but not for its width which grows as a function of $\chi^2_{\text{cut}}$.

Certain features emerge, which are found to hold for all multipoles, not only $L_{1+}$ and $L_{3+}$: The peak of the distribution is highly insensitive to the $\chi^2$ cut while the width grows as a function of the $\chi^2$ cut. The method does not treat differently "sensitive" from "non sensitive" multipoles; they naturally emerge as such. The widths of the histograms, ranging from very narrow to very wide or infinite widths, naturally select and order the various multipoles according to their sensitivity to the data set.

**Correlations**

A central issue of the problem of extracting multipole amplitudes which is properly treated in the AMIAS method, is the handling of correlations. Both the TME and the MDE methods, by freezing all the "insensitive" multipoles, exclude any possibility of determining them. In addition the derived solution has in it embedded the correlations to those multipoles; the result is that this source of model error is reflected both by "shifting" the extracted values for the dominant amplitudes and by underestimating the model uncertainty.

![FIGURE 5.](image)

The AMIAS method, addresses these shortcomings and it provides for an easy visualization of the existence of such correlations. All possible correlations are accounted by allowing all multipoles to randomly vary, and to yield solutions with all allowed values of the "insensitive" multipoles. The visualization of at least the dominant correlations is accomplished in a two-dimensional scatter plot in which the ensemble of solutions is projected on the plane defined by their values and color coded according to the $\chi^2$ value. In the left panel of Figure 5, the strong correlation between the $L_{0+}$ and $L_{1+}$ multipoles is obvious. The right panel does not indicate any correlation between the $E_{2+}$ and $L_{2+}$ multipoles.

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is frozen at the model values $[12, 13]$. The right panel explores the correlation between the $E_{2+}$ and $L_{2+}$ multipoles which visually is shown to be absent.

**Probability Distributions**

The results that emerge by examining the behavior of the ensemble of solutions provide an understanding of the method, but they do not make use of the AMIAS postulate. Implementing the postulate allows the extraction of values and uncertainties from the data. The following procedure is followed: For each solution of the ensemble $\{A_\nu\}_j$ we take the corresponding value of the multipole of interest $A_\nu^j$ and we assign to it a probability, as our postulate requires, equal to $P(j, k) = \text{erf}[\chi^2(A_\nu^j, O_k)]$. We then form a histogram of the multipole values of the ensemble weighted by the probability $P(j, k)$. The resulting distributions, shown in the right hand panel of the Figure 6 are characterized by a peak position and a width; the histogram for $A_\nu = L_{1+}$ has a well defined maximum and a well defined width while the corresponding histogram for $A_\nu = L_{3+}$ is featureless manifesting its insensitivity to the data. The resulting distribution is the relative probability distribution for this particular parameter (multipole) to represent the physical value. We take its mean to be the derived (extracted) from the data value; this value is characterized by an uncertainty which is completely determined by the probability distribution. A numerical value for the uncertainty can be derived from the distribution for a given confidence level (C.L.). For instance, a "1σ" (68% C.L.) value is given by the $\pm A_\nu$ values which bound $\pm 34\%$ of the integral.

**FIGURE 6.** The two-dimensional projections of the ensemble of solution on a $A_\nu \chi^2$ plane, same as those of Figure 4 are shown in the middle panels. They are mapped into probability distributions by implementing the AMIAS postulate (right hand panels), or to flat distributions (left hand panels) if an equal probability is assumed manifesting the model independence of the method.

In the subsequent section we will demonstrate through the use of pseudodata, for which the generating multipoles are precisely known, that the method can reproduce them with a well understood uncertainty.
Validation with Pseudodata

AMIAS produces values and uncertainties for multipoles for which the data exhibit sensitivity, in a manifestly model independent manner. The values and uncertainties, have been demonstrated to have a precise meaning through the analysis of pseudodata which were generated with predetermined statistical accuracy. We have used as generator the MAID code, with our particular set of input multipole parameters. In the cases presented below, we have frozen the $\alpha_N^{1/2}$ and we have varied the $\alpha_A^{1/2}$ amplitudes. Few demonstrative cases of the pseudodata validation are presented below.

The demonstration case presented here is based on the kinematics of the $Q^2 = 0.127\ (\text{GeV/c})^2$ Bates and Mainz $N \to \Delta$ data. The data set is published, is well understood and it is well described by MAID if the resonant amplitudes are adjusted [12]. Several sets of pseudodata were generated, at identical kinematics to those of the experimental values, in order to test the ability of the AMIAS code to reproduce a) the generator multipole values and b) the input uncertainties imposed on the generator multipoles.

TABLE 1. The multipole values extracted, in units of $10^{-3}/m_\pi$, from the three pseudodata sets are compared to the generator (modified MAID) values. They are shown to be entirely compatible with increasing accuracy as required.

| Multipole | Generator | Set A   | Set B   | Set C   |
|-----------|-----------|---------|---------|---------|
| $M_{1+}$  | 27.248    | 27.23 ± 0.13 | 27.229 ± 0.013 | 27.249 ± 0.001 |
| $L_{0+}$  | 3.500     | 3.70 ± 0.23  | 3.515 ± 0.022  | 3.502 ± 0.002  |
| $L_{1+}$  | 1.048     | 1.03 ± 0.08  | 1.047 ± 0.008  | 1.048 ± 0.001  |
| $E_{1+}$  | 1.481     | 1.49 ± 0.18  | 1.489 ± 0.017  | 1.482 ± 0.002  |
| $E_{0+}$  | 4.225     | 3.68 ± 1.02  | 4.278 ± 0.135  | 4.239 ± 0.013  |
| $M_{1-}$  | 4.119     | 4.47 ± 1.31  | 4.161 ± 0.126  | 4.124 ± 0.013  |
| $L_{1-}$  | 1.205     | 1.05 ± 0.43  | 1.170 ± 0.080  | 1.203 ± 0.008  |
| $E_{2-}$  | 1.024     | 1.07 ± 0.45  | 1.053 ± 0.061  | 1.027 ± 0.006  |
| $L_{2-}$  | 0.007     | 0.02 ± 0.01  | 0.008 ± 0.001  | 0.008 ± 0.001  |
| $E_{2-}$  | 0.006     | 0.01 ± 0.01  | 0.009 ± 0.001  | 0.007 ± 0.001  |

The validation of the ability of the method to reproduce the central values is straightforward. Three sets of pseudodata were generated, characterized by the increasing statistical accuracy: i) "Set A" with statistical accuracy similar to that of the experimental values ii) "Set B" with statistical accuracy ten times better than that of the experimental values and iii) "Set C" with statistical accuracy hundred times better than that of the experimental values. These data were analyzed and the multipoles were extracted which are tabulated and compared with the generator values in Table 1. We have tabulated only extracted values which are derived with uncertainties better than 100% for "Set A". It can be seen that the multipole values are extracted accurately, in complete agreement with the generator values within the stated statistical accuracy. Also, as required, the quoted uncertainties are reduced (tenfold and hundredfold), proportionally to the statistical accuracy of the pseudodata sets.

FIGURE 7. The derived values and uncertainties for the $M_{1+}$ multipole as a function of the $\chi^2$ value of the pseudodata that were generated. The shaded area around the nominal value depicts the $1\sigma$ variation which was imposed on this multipole as a "model uncertainty".
To test the ability of the method to extract uncertainties which have precise statistical interpretation is generally more difficult. The scaling behavior exhibited by the three sets of pseudodata, discussed above, is a necessary but not sufficient condition. The definitive validation was achieved by introducing an arbitrary uncertainty, a “model uncertainty” to the nominal generator values.

The introduction of a model uncertainty allows the generation of pseudodata for which the uncertainties are defined by the model uncertainty and not the statistical precision of the data set (corresponding to the limit of infinite statistics). Pseudodata with increasing $\chi^2$ values, depending on the magnitude of the model error, were analyzed using the AMIAS method. AMIAS is demonstrated to reproduce the central value of a given multipole of the generator and to assign to it an uncertainty which is statistically compatible to the predetermined “model error”. This is shown for the case of the $M_{1+}$ multipole in Figure 7.

The detailed investigation of the method using pseudodata, which is far more extensive than is documented in this article, demonstrated that AMIAS, unlike TME and MDE provides for the first time a methodology and a tool for identifying and precisely extracting multipole amplitudes in a model independent way. In addition the quoted uncertainties have a precise, well understood meaning. We proceed in the following section to apply the method to reanalyze the previously published $Q^2 = 0.127\,\text{GeV}^2/\text{c}^2$ data.

**EXAMPLE: RE-ANALYSIS OF THE $Q^2 = 0.127\,\text{GeV}^2/\text{c}^2$ DATA**

To demonstrate the capabilities of the method we re-analyzed the $H(e,e^\prime)p\pi^0$ measurements performed at $Q^2 = 0.127\,\text{GeV}^2/\text{c}^2$ and $W = 1232$ MeV. This set consists of Bates and Mainz data; a detailed description and analysis of the data can be found in [19]. The data set consists of 31 data points, cross section results for the $\sigma_{TT}$, $\sigma_{LT}$, $\sigma_0$, $\sigma_{E2}$ and the polarized beam cross section $\sigma_{LT'}$. Since the MAID model [16] provides for this $Q^2$ a good description of the data set, the MAID-2003 multipoles are used as a starting point for the AMIAS method. As commented earlier, the starting point can be arbitrary, however a good starting point provides easy convergence and considerable savings in computer time. A $L_{cut} = 5$ value was chosen, so that a sufficiently large number of background amplitudes are included in the computational exploration. Results are derived by uniformly varying the real and imaginary part of the input amplitudes of the model in the $\pi^0$ charge-channel after a reiterative selection of the phase volume to be explored. The unitary box width $w_0$ assigned to each of the input amplitudes was set to the $\pm 10\%$ of their central (MAID) value. Amplitudes in general were allowed to vary normally in a range of $20 \times w_0$; this range was reduced for the sensitive $E_{1+}$ and $L_{1+}$ amplitudes and particularly for the $M_{1+}$ multipole.

**FIGURE 8.** Probability distributions for the norms of some of the sensitive amplitudes of the analyzed Bates/Mainz data set. The distributions allow the determination of the central value and corresponding uncertainty for each of the multipoles.
The resulting (un-normalized) probability distributions for some of the sensitive amplitudes are shown in Figure 8. The probability distributions, as in the case of the pseudodata, allow the determination of both the value and the uncertainty of each of the sensitive multipoles. The extracted multipole values and (1σ confidence) uncertainties, of the Bates/Mainz data fit with the AMIAS method are presented in the Table 2.

### Table 2

| Multipole | Extracted Value | Relative Error | MAID-2003 | Sato & Lee | DMT |
|-----------|-----------------|----------------|-----------|------------|-----|
| $M_{1+}$  | 27.24 ± 0.20    | 0.73 %         | 27.464    | 27.661     | 27.489 |
| $L_{1+}$  | 0.82 ± 0.20     | 17.7 %         | 1.000     | 0.672      | 0.986 |
| $L_{0+}$  | 2.23 ± 0.41     | 18.4 %         | 2.345     | 1.008      | 1.994 |
| $E_{0+}$  | 3.44 ± 0.70     | 20.3 %         | 2.873     | 2.213      | 3.206 |
| $E_{1+}$  | 1.16 ± 0.32     | 24.1 %         | 1.294     | 1.288      | 1.401 |

The quoted uncertainties for the extracted multipoles include all experimental errors (statistical + systematic) and obviously contain no model error. The relative uncertainties, which can be considered as a measure of the sensitivity for each of the extracted multipoles, are also tabulated. The values of the MAID-2003, Sato & Lee [21] and DMT [22, 23] model are also given in the same table for comparison. It is important to highlight the fact that the AMIAS method yields information on five multipoles, while the MDE method only for the three resonant amplitudes [14, 19].

![Figure 9](image-url)  
**Figure 9.** The experimentally allowed region for the partial cross sections for $Q^2 = 0.127$ GeV$^2$/c$^2$ and $W = 1232$ MeV, as determined by the analysis of the Bates/Mainz data by AMIAS: the orange (shaded) bands depict the allowed region with 1σ confidence. The solid curves show the model predictions while the dashed curves show the "spherical" solutions of the same models.

In Figure 9 the allowed region for partial cross sections for $Q^2 = 0.127$ GeV$^2$/c$^2$ and $W = 1232$ MeV is shown in the shaded area (bands). These bands are defined through the result of the AMIAS solution for the Bates/Mainz...
data set. The shaded band shows the envelope that accommodates all possible solutions that are compatible with the experimental data with $1 \sigma$ (68%) confidence level. These uncertainty bands are model independent.

The error bands of Figure 8 are solely defined by the experimental data, with the minimal assumptions of the AMIAS method, spelled out earlier. In contrast, the bands in Figure 1 are model defined, representing the spread of the model solutions to the same data that “acceptable” models offer at a given time. It can be observed that the experimentally allowed region is broader and covers a slightly different region that that covered by the model bands of Figure 1.

The predictions of the DMT, MAID and SL models are also shown in Figure 9 (solid curves) along with the corresponding results for their "spherical" solutions (dashed curves, same color coding). It can be seen that the model solutions fall within the experimentally allowed region. As expected, only the partial cross sections $\sigma_{LT}$ and $\sigma_{E2}$ discriminate, but with a high level of confidence, between the deformed and spherical solutions.

A model independent isolation of resonant quadrupole strength ($M_{1+}^{3/2}, E_{1+}^{3/2}, L_{1+}^{3/2}$) and an extraction of the EMR and CMR values is allowed by AMIAS, if the data can support such a separation. Polarization and isospin sensitive observables are required. Such data are emerging from the new CLAS measurements, where both $(e,e'\pi^+)$ and $(e,e'\pi^-)$ have been measured [24].

**TABLE 3.** Comparison of the extracted multipoles from the full and a reduced data set, in which the $\sigma_{LT}$ cross sections have been excluded. The relative change in the derived values is also indicated. The multipoles are in units of $10^{-3}/m_{\pi}$

| Multipole | Full Data Set | Reduced Data Set | Change |
|-----------|--------------|------------------|--------|
| $M_{1+}$  | $27.24 \pm 0.20$ | $27.15 \pm 0.24$ | $+0.3\%$ |
| $L_{1+}$  | $0.82 \pm 0.20$ | $0.90 \pm 0.30$ | $-9.8\%$ |
| $L_{0+}$  | $2.23 \pm 0.41$ | $2.63 \pm 0.60$ | $-18.0\%$ |
| $E_{0+}$  | $3.44 \pm 0.70$ | $3.74 \pm 0.76$ | $-8.7\%$ |
| $E_{1+}$  | $1.16 \pm 0.24$ | $1.19 \pm 0.28$ | $-2.6\%$ |

The ability of the AMIAS method to determine correctly and selectively the multipole sensitivities contained in a given data set, was demonstrated by analyzing a subset of the Bates/Mainz data set in which polarization observables, $(\sigma_{LT})$, were excluded. It is known that the $\sigma_{LT}$ cross section depends on the background amplitudes and particularly on the $L_{0+}$. A new multipole solution was generated with AMIAS based on the reduced data set. The derived results are listed in Table 3, and compared to the values extracted from the full data. The expected sensitivities are borne out as can be seen in the last column in the table. It is clear from the derived values, that the change in the resonant amplitudes is minimal and lies within the range of their extracted uncertainty. As expected the sensitive background amplitudes, and especially the $L_{0+}$, indicate a significant change in their initial values.

**SUMMARY, CONCLUSIONS AND FUTURE WORK**

We have developed and presented aspects of a scheme for the analysis of experimental data (AMIAS) which is very general and it allows the extraction of maximal information from incomplete data sets. The AMIAS method is shown to be suitable for the analysis of nucleon resonance electroproduction data in general although it was discussed only for the case of the $N \rightarrow \Delta$ transition data. The new method offers significant advantages over the currently employed methods and is demonstrated to:

- Be model independent.
- Extract maximum information for all available multipoles, without any bias; it is capable of ranking them in order of significance.
- Account for the correlations amongst the contributing multipoles, and to provide an easy visualization of them.
- Yield uncertainties which have a precise meaning, in terms of confidence levels.
- Be numerically robust, regardless of the data base.

The re analysis of the published and well understood $Q^2 = 0.127 GeV^2/c^2$ Bates and Mainz $N \rightarrow \Delta$ data was presented as a demonstration case. Information on five multipoles is derived, instead of the three that were extracted
using the standard methodology. The derived values for $M_{1+}$, $E_{1+}$ and $L_{1+}$ are compatible with the published experimental values, but they are determined with higher precision.

A number of developments are under way. A $W$ dependent analysis is being currently developed. In parallel, the coding and the algorithms are being improved in order to improve their efficiency. A self adopting algorithm, which searches the starting point and optimally bounds the phase space volume, is also under investigation. A (re) analysis of the high quality data sets at a number of $Q^2$ values that is now emerging is planned.

The model independent, precise determination of the multipole amplitudes and the quantification of the associated experimental uncertainty provides a tool for a meaningful comparison of data with competing interpretation schemes and models. In the case of the issue of hadron deformation such a precise and quantitative comparison is need in order to understand the mechanisms that generate it.

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