ON THE HOCHSCHILD COHOMOLOGIES OF ASSOCIATIVE
CONFORMAL ALGEBRAS WITH A FINITE FAITHFUL
REPRESENTATION

P.S. KOLESNIKOV AND R.A. KOZLOV

Abstract. Associative conformal algebras of conformal endomorphisms are of
essential importance for the study of finite representations of conformal Lie al-
gebras (Lie vertex algebras). We describe all semisimple algebras of conformal
endomorphisms which have the trivial second Hochschild cohomology group
with coefficients in every conformal bimodule. As a consequence, we state a
complete solution of the radical splitting problem in the class of associative
conformal algebras with a finite faithful representation.

1. Introduction

An algebraic formalization of the properties of the operator product expansion
(OPE) in 2-dimensional conformal field theory [4] gave rise to a new class of al-
gebraic systems, vertex operator algebras [7, 21]. The singular part of the OPE
describes the commutator of two fields, and the corresponding algebraic structures
are called conformal (Lie) algebras [22] (or vertex Lie algebras [20]).

Namely, suppose $V$ is a vertex operator (super)algebra with a translation oper-
ator $\partial$ and a state-field correspondence $Y$. Then, due to the locality axiom, the
OPE of two fields $Y(a, z)$ and $Y(b, z)$, $a, b \in V$, has a finite singular part:

$$ Y(a, w)Y(b, z) = \sum_{n=0}^{N(a,b)-1} Y(c_n, z) \frac{1}{(w-z)^{n+1}} + \text{(regular part)}. $$

The coefficients of the singular part are completely determined by the (super)com-
mutator of the fields:

$$ [Y(a, w), Y(b, z)] = \sum_{n=0}^{N(a,b)-1} Y(c_n, z) \frac{1}{n!} \frac{\partial^n \delta(w-z)}{\partial z^n}, $$

where $\delta(w-z) = \sum_{s \in \mathbb{Z}} w^s z^{-s-1}$ is the formal delta-function. The correspondence

$$(a, b) \mapsto c_n, \quad n \geq 0,$$

defines an infinite series of bilinear operations ($n$-products) on $V$. Together with
the translation operator $\partial$, these operations turn $V$ into what is called a conformal
Lie (super)algebra.

The most natural analogues of finite-dimensional algebras in the class of confor-
mal algebras are finite ones, i.e., those finitely generated as modules over $H = \mathbb{C}[\partial]$.

An algebraic study of this class of conformal algebras is an interesting mathemat-
ical problem with numerous ties to other areas. Structure theory of finite Lie
conformal algebras was developed in [15], simple and semisimple finite Lie conformal superalgebras were described in [12, 18, 19]. Representations and cohomologies of conformal algebras were studied in [2, 10, 9, 11, 13, 14, 29].

The study of universal structures for conformal algebras was initiated in [33]. The classical theory of finite-dimensional Lie algebras often needs universal constructions like free algebras and universal enveloping algebras. This was a motivation for the development of combinatorial issues in the theory of conformal algebras [5, 6].

One of the most intriguing questions in this field is related with the classical Ado Theorem. The latter states that every finite-dimensional Lie algebra has a faithful finite-dimensional representation. The Ado Theorem is a crucial point for understanding why every Lie algebra integrates globally into a Lie group. A formal approach to Lie theory (see [30]) allows us to hope that the “fundamental triangle” of Lie theory can be established for conformal algebras. To that end, an analogue of the Ado Theorem for conformal algebras will be required. It was shown in [26, 27] that a finite (torsion-free) Lie conformal algebra has a faithful finite representation provided that its semisimple part splits as a subalgebra, i.e., the analogue of the Levi Theorem holds. However, it is known that the Levi Theorem does not hold for finite Lie conformal algebras in general (see, e.g., [2, 14]). In order to get further advance in the study of existence of faithful finite representations we need to explore the structure of associative conformal envelopes of finite Lie algebras. These algebraic structures belong to the class of associative conformal algebras [22] with a finite faithful representation (FFR, for short).

The structure theory of finite associative conformal algebras is very much similar to ordinary associative algebras: simple objects are isomorphic to current conformal algebras \( \text{Cur} M_n(\mathbb{C}) \) over matrix algebras [15], semisimple algebras are direct sums of simple ones, the maximal nilpotent ideal (radical) always exists, and the semisimple part splits as a subalgebra [34]. Namely, for every finite associative conformal algebra \( C \) there exists a maximal nilpotent ideal \( R \) such that \( E/R \) is isomorphic to the current conformal algebra \( \text{Cur} A \) over a semisimple finite-dimensional associative algebra \( A \). Moreover, the following analogue of the Wedderburn Principal Theorem holds: \( C \) is isomorphic to the semi-direct product of \( \text{Cur} A \) and \( R \), \( C \cong \text{Cur} A \rtimes R \). The latter result also follows from the study of Hochschild cohomologies for finite associative conformal algebras [16], where it was shown that \( H^2(\text{Cur} A, M) = 0 \) for every semisimple algebra \( A \), \( \dim A < \infty \), and for every conformal bimodule \( M \) over \( \text{Cur} A \).

Associative conformal algebras with a FFR form a more general class than (torsion-free) finite associative conformal algebras [26]. The conjecture on the structure of simple algebras in this class was posed in [8] and proved in [24]. Semisimple associative conformal algebras with a FFR turn to be direct sums of simple ones, but was shown in [25] that the analogue of the Wedderburn Principal Theorem does not hold in general.

Since the splitting of a semisimple part plays crucial role in the study of Ado-type problems for finite Lie conformal algebras, it is reasonable to investigate the similar problem for associative conformal algebras with a FFR. A natural tool for such investigation is the computation of Hochschild cohomologies for conformal algebras. The latter were proposed in [2], but we prefer using the pseudo-tensor category approach of [1]. In this paper, we explicitly describe all those semisimple associative conformal algebras with a FFR that have trivial second Hochschild cohomology...
group relative to every conformal bimodule. The main technical statement is to show that conformal algebras of type Cend\(_{n,Q}\), where \(Q = \text{diag}\{1, \ldots, 1, x\}\), always have trivial second cohomology group. For \(n = 1\), it was done in [28]. In this paper, we use a different method of proof that does not work for \(n = 1\).

Throughout the paper, \(\kappa\) is an algebraically closed field of characteristic zero, \(\mathbb{Z}_+\) is the set of nonnegative integers.

2. Preliminaries

A conformal algebra [22] is a linear space \(C\) equipped with a linear map \(\partial : C \to C\) and with a countable family of bilinear operations \((\cdot, \cdot)_n : C \otimes C \to C\), \(n \in \mathbb{Z}_+\), satisfying the following axioms:

(C1) for every \(a, b \in C\) there exists \(N \in \mathbb{Z}_+\) such that \((a \langle n \rangle b) = 0\) for all \(n \geq N\);

(C2) \(\partial(a \langle n \rangle b) = -n(a \langle n-1 \rangle b)\);

(C3) \((a \langle n \rangle \partial b) = \partial(a \langle n \rangle b) + n(a \langle n-1 \rangle b)\).

Every conformal algebra \(C\) is a left module over the polynomial algebra \(H = \kappa[\partial]\). The structure of a conformal algebra on an \(H\)-module \(C\) may be expressed by means of a single polynomial-valued map called \(\lambda\)-product:

\[
(\cdot, (\lambda) \cdot) : C \otimes C \to C[\lambda], \quad (a \langle \lambda \rangle b) = \sum_{n \in \mathbb{Z}_+} \lambda^{(n)} (a \langle n \rangle b),
\]

where \(\lambda\) is a formal variable, \(\lambda^{(n)} = \lambda^n/n!\). The axioms (C2) and (C3) turn into the linearity property

\[
(\partial a \langle \lambda \rangle b) = -\lambda(a \langle \lambda \rangle b), \quad (a \langle \lambda \rangle \partial b) = (\partial + \lambda) (a \langle \lambda \rangle b).
\]

For every conformal algebra \(C\) there exists a uniquely defined coefficient algebra \(A(C)\) such that \(C\) is isomorphic to a conformal algebra of formal distributions over \(A(C)\) [23]. Conformal algebra \(C\) is called associative (commutative, Lie, Jordan, etc.) if so is \(A(C)\) [33]. Every identity on \(A(C)\) may be expressed as a family of identities on \(C\). For example, \(A(C)\) is associative if and only if

\[
(a \langle n \rangle (b \langle m \rangle c)) = \sum_{s \geq 0} \binom{n}{s} (a \langle n-s \rangle b) \langle m+s \rangle c)
\]

for all \(a, b, c \in C\), \(n, m \in \mathbb{Z}_+\). This family of identities may be expressed in terms of \(\lambda\)-product (1) as

\[
(a \langle \lambda \rangle (b \langle \mu \rangle c)) = ((a \langle \lambda \rangle b) \langle \lambda+\mu \rangle c), \quad a, b, c \in C,
\]

where \(\lambda\) and \(\mu\) are independent commuting variables [23].

A more conceptual approach to the theory of conformal algebras, their identities, representations, cohomologies, etc., is provided by the notion of a pseudo-algebra [1]. Indeed, in ordinary algebra all basic definitions may be stated in terms of linear spaces, polylinear maps, and their compositions. For pseudo-algebras, the base field is replaced with a Hopf algebra \(H\), the class of linear spaces is replaced with the class \(\mathcal{M}(H)\) of left \(H\)-modules, and the role of \(n\)-linear maps is played by \(H^{\otimes n}\)-linear maps of the form

\[
\varphi : V_1 \otimes \cdots \otimes V_n \to H^{\otimes n} \otimes_H V, \quad V_i, V \in \mathcal{M}(H),
\]
where $H^\otimes n$ is considered as the outer product of regular right $H$-modules. Compositions of such maps are naturally defined by means of the expansion of $\varphi$ to an $H^\otimes (m_1+\cdots+m_n)$-linear map

$$
(H^\otimes m_1 \otimes_H V_1) \otimes \cdots \otimes (H^\otimes m_n \otimes_H V_n) \to H^\otimes (m_1+\cdots+m_n) \otimes_H V, 
$$

(5)

$m_1,\ldots,m_n \in \mathbb{Z}_+$, given by the following rule:

$$
\varphi(1^\otimes m_1 \otimes_H v_1,\ldots,1^\otimes m_n \otimes_H v_n) = ((\Delta^m_1 \otimes \cdots \otimes \Delta^m_n) \otimes_H \text{id}_V)\varphi(v_1,\ldots,v_n),
$$

where $\Delta^m : H \to H^\otimes m$ is the iterated coproduct on $H$.

A pseudo-algebra is a left $H$-module $C$ equipped with a $H^\otimes 2$-linear map $*: C \otimes C \to H^\otimes 2 \otimes_H C$ called pseudo-product, $*: a \otimes b \mapsto a \ast b$ (similar to the definition of an ordinary algebra as a linear space equipped with a bilinear product).

Conformal algebras are exactly pseudo-algebras over the polynomial Hopf algebra $H = \mathbb{k}[\partial]$ with coproduct $\Delta f(\partial) = f(\partial \otimes 1 + 1 \otimes \partial)$, counit $\varepsilon(f(\partial)) = f(0)$, and antipode $S(f(\partial)) = f(-\partial)$. The relation between pseudo-product and conformal $\lambda$-product is given by

$$
a \ast b = (a (\lambda) b)|_{\lambda=-\partial \otimes 1}.
$$

A conformal algebra $C$ satisfies (3) if and only if

$$
a \ast (b \ast c) = (a \ast b) \ast c, \quad a,b,c \in C,
$$

where $a \ast (b \ast c) \in H^\otimes 3 \otimes_H C$ is the result of the composition $*(\text{id}_C,*)$ on $a \otimes b \otimes c$ (the right-hand side is defined similarly).

**Remark 1.** The class $\mathcal{M}(H)$ together with $H$-polylinear maps and their compositions described above forms a non-symmetric pseudo-tensor category [3]. This language is enough to describe associative algebra features; to include commutativity into consideration (or any other relation involving permutations of variables) one needs a symmetric pseudo-tensor category structure on $\mathcal{M}(H)$ as described in [4]. For example, the anti-commutativity identity in the language of pseudo-product is $a \ast b = -(\sigma_{12} \otimes_H \text{id}_C)(b \ast a)$, where $\sigma_{12}$ acts on $H \otimes H$ as the permutation of tensor factors.

Suppose $C$ is an associative conformal algebra considered as a pseudo-algebra over $H = \mathbb{k}[\partial]$. A conformal bimodule over $C$ is a left $H$-module $M \in \mathcal{M}(H)$ equipped with $H^\otimes 2$-linear maps $l : C \otimes M \to H^\otimes 2 \otimes_H M$ and $r : M \otimes C \to H^\otimes 2 \otimes_H M$ satisfying three associativity identities:

$$
l(*,\text{id}_M) = l(\text{id}_C, l), \quad l(l, \text{id}_C) = r(l, \text{id}_C), \quad r(\text{id}_M, *) = r(r, \text{id}_C).
$$

In terms of ordinary operations, it means that we have two families of $M$-valued $n$-products defined on $C \otimes M$ and $M \otimes C$ satisfying the analogues of (C1)–(C3) and (3). We will also describe these infinite families by their generating functions denoted $(\cdot (\lambda) \cdot)$. Then

$$
l(a,u) = (a (\lambda) u)|_{\lambda=-\partial \otimes 1}, \quad r(u,a) = (u (\lambda) a)|_{\lambda=-\partial \otimes 1},
$$

and the analogues of (2) and (4) hold.

Let us describe the Hochschild cohomology complex $C^*(C,M)$ for an associative conformal algebra $C$ and a conformal bimodule $M$ over $C$ [10]. The space of $n$-cochains $C^n(C,M)$ consists of all $H^\otimes n$-linear maps

$$
\varphi : C^\otimes n \to H^\otimes n \otimes_H M.
$$
The conformal Hochschild differential \( d_n : C^n(C, M) \to C^{n+1}(C, M) \) is defined similarly to the ordinary one, assuming the expansion (5):

\[
(d_n \varphi)(a_1, \ldots, a_{n+1}) = a_1 \ast \varphi(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi(a_1, \ldots, a_i \ast a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} \varphi(a_1, \ldots, a_n) \ast a_{n+1}. \tag{6}
\]

Denote by \( Z^n(C, M) \) and \( B^n(C, M) \) the subspaces of \( n \)-cocycles and \( n \)-coboundaries, respectively. As for ordinary algebras, the quotient space \( H^n(C, M) = Z^n(C, M) / B^n(C, M) \) is called the \( n \)th Hochschild cohomology group of \( C \) with coefficients in \( M \).

The complex \( C^*(C, M) \) may be described by means of \( \lambda \)-products. For every \( \varphi \in C^n(C, M) \) and \( a_1, \ldots, a_n \in C \) we may write

\[
\varphi(a_1, \ldots, a_n) = \sum_{s_1, \ldots, s_{n-1} \in \mathbb{Z}_+} (\varphi_i \otimes \cdots \otimes \varphi_{n-1}) \otimes H u_{s_1, \ldots, s_{n-1}},
\]

where \( u_{s_1, \ldots, s_{n-1}} \in M \) are uniquely defined. Then one may consider a map

\[
\varphi_{\lambda_1, \ldots, \lambda_{n-1}} : C^\otimes n \to M[\lambda_1, \ldots, \lambda_{n-1}]
\]

defined by

\[
\varphi_{\lambda_1, \ldots, \lambda_{n-1}}(a_1, \ldots, a_n) = \sum_{s_1, \ldots, s_{n-1} \in \mathbb{Z}_+} (-1)^{s_1 + \cdots + s_{n-1}} \lambda_1^{s_1} \cdots \lambda_{n-1}^{s_{n-1}} u_{s_1, \ldots, s_{n-1}}.
\]

This map has the following sesquilinearity properties:

\[
\varphi_{\lambda_1, \ldots, \lambda_{n-1}}(a_1, \ldots, \partial a_i, \ldots, a_n) = -\lambda_i \varphi_{\lambda_1, \ldots, \lambda_{n-1}}(a_1, \ldots, a_n), \quad i = 1, \ldots, n-1,
\]

\[
\varphi_{\lambda_1, \ldots, \lambda_{n-1}}(a_1, \ldots, \partial a_n) = (\partial + \lambda_1 + \cdots + \lambda_{n-1}) \varphi_{\lambda_1, \ldots, \lambda_{n-1}}(a_1, \ldots, a_n).
\]

The differential (6) turns into

\[
(d_n \varphi)_{\lambda_1, \ldots, \lambda_{n-1}}(a_1, \ldots, a_{n+1}) = a_1 \varphi_{\lambda_2, \ldots, \lambda_n}(a_2, \ldots, a_{n+1}) + \sum_{i=1}^{n} (-1)^i \varphi_{\lambda_1, \ldots, \lambda_{i+1}, \ldots, \lambda_n}(a_1, \ldots, a_i \ast a_{i+1}, \ldots, a_{n+1}) + (-1)^{n+1} \varphi_{\lambda_1, \ldots, \lambda_{n-1}}(a_1, \ldots, a_n) \ast a_{n+1}.
\]

For example, the space of 2-cocycles \( Z^2(C, M) = \text{Ker} d_2 \subset C^2(C, M) \) consists of all sesquilinear maps \( \varphi : C \otimes C \to M[\lambda] \) such that

\[
a_1(\lambda) \varphi \mu(a_2, a_3) - \varphi \lambda + \mu(a_1(\lambda) a_2, a_3) + \varphi \lambda(a_1, a_2; \mu a_3) - \varphi \lambda(a_1, a_2; (\lambda + \mu) a_3 = 0.
\]

**Remark 2.** It is easy to see that \( C^*(C, M) \) coincides with the complex described in [2], where \( C^n(C, M) \) consists of adjacent classes of sesquilinear maps \( \gamma_{\lambda_1, \ldots, \lambda_n} : C^\otimes n \to M[\lambda_1, \ldots, \lambda_n] \) modulo the multiples of \( (\partial + \lambda_1 + \cdots + \lambda_n) \). The correspondence is given by

\[
\gamma_{\lambda_1, \ldots, \lambda_n} \leftrightarrow \varphi_{\lambda_1, \ldots, \lambda_{n-1}} = \gamma_{\lambda_1, \ldots, \lambda_{n-1}, -\partial - \lambda_1 - \cdots - \lambda_{n-1}}.
\]

Recall that a null extension of an associative conformal algebra \( C \) by means of a \( C \)-bimodule \( M \) is an associative conformal algebra \( E \) in a short exact sequence

\[
0 \to M \to E \to C \to 0,
\]
where \((M,\lambda)M = 0\) in \(E\). Two null extensions \(E_1\) and \(E_2\) are equivalent if there exists an isomorphism \(E_1 \to E_2\) such that the diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & M & \longrightarrow & E_1 & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow \text{id}_M & & \downarrow & & \downarrow \text{id}_C & & \downarrow & & \downarrow \\
0 & \longrightarrow & M & \longrightarrow & E_2 & \longrightarrow & C & \longrightarrow & 0
\end{array}
\]

is commutative.

**Theorem 1** ([2, 16]). Equivalence classes of null extensions of \(C\) by means of \(M\) are in one-to-one correspondence with \(\mathcal{H}^2(C, M)\).

A cocycle \(\varphi \in Z^2(C, M)\) corresponds to an extension \(E = C \oplus M\) with a new \(\lambda\)-product \((\cdot, \cdot, \cdot)\) given by \((u, v) = 0\) for \(u, v \in M\), \((a, u) = (a, u)\), \((u, a) = (u, a)\) for \(a \in C, u \in M\), and

\[
(a, b) = (a, b) + \varphi(a, b)
\]

for \(a, b \in C\).

**Corollary 1.** Suppose \(C\) is an associative conformal algebra such that \(\mathcal{H}^2(C, M) = 0\) for every \(C\)-bimodule \(M\). Then \(C\) splits in every extension with a nilpotent kernel. Namely, if \(E\) is an associative conformal algebra with a nilpotent ideal \(R\) such that \(E/R \cong C\) then \(E \cong C \times R\).

**Example 1.** Let \(A\) be an ordinary algebra, and let \(H = k[\partial]\). Then the free \(H\)-module \(H \otimes A\) equipped with a \(\lambda\)-product given by

\[
(h(\partial) \otimes a)(\lambda)(g(\partial) \otimes b) = h(\lambda g(\partial + \lambda) \otimes ab), \quad h, g \in H, \ a, b \in A,
\]

is called current conformal algebra \(\text{Cur} A\).

**Theorem 2** ([17]). Let \(A\) be a finite direct sum of matrix algebras over \(k\), \(H = k[\partial]\). Then \(\mathcal{H}^2(\text{Cur} A, M) = 0\) for every conformal bimodule \(M\) over \(\text{Cur} A\).

**Corollary 2** ([34]). Every finite associative conformal algebra is a semi-direct sum of its nilpotent radical and semisimple part.

**Example 2.** Consider the space \(M_n(k[\partial, x])\) equipped with the multiplication action of \(\partial\) and with a \(\lambda\)-product

\[
A(\partial, x)(\lambda)B(\partial, x) = A(-\lambda, x)B(\partial + \lambda, x + \lambda).
\]

This is an associative conformal algebra denoted by \(\text{Cend}_n\).

The conformal algebra \(\text{Cend}_n\) has a finite faithful representation (FFR) on \(M = k[\partial] \otimes k^n \cong k^n[\partial]\) given by

\[
A(\partial, x)(\lambda)v(\partial) = A(-\lambda, \partial)v(\partial + \lambda).
\]

Every associative conformal algebra with a FFR is obviously isomorphic to a conformal subalgebra of \(\text{Cend}_n\) for an appropriate \(n\). Therefore, \(\text{Cend}_n\) plays the same role in the theory of conformal algebras as the matrix algebra \(M_n(k)\) does in the ordinary linear algebra. In [26], it was shown that every finite associative torsion-free conformal algebra has a FFR. However, \(\text{Cend}_n\) is infinite, so associative conformal algebras with a FFR form a more general conformal analogue of the class of finite-dimensional algebras.
Theorem 3 ([17]). One has $H^2(C_{end}, M) = 0$ for every conformal bimodule $M$ over the conformal algebra $C_{end}$.

Conformal subalgebra $M_n(k[\partial]) \subset C_{end}$ is isomorphic to the current conformal algebra $\text{Cur} M_n(k)$ which is known to be simple [14]. Given a matrix $Q = Q(x) \in M_n(k[x])$, the set $C_{end,n,Q} = Q M_n(k[\partial, x])$ is a conformal subalgebra (even a right ideal) of $C_{end}$. If det $Q \neq 0$ then $C_{end,n,Q}$ is simple, and vice versa [22].

Theorem 4 ([24]). Let $C$ be a simple associative conformal algebra with a FFR. Then either $C \simeq \text{Cur} M_n(k)$ or $C \simeq C_{end,n,Q}$, det $Q \neq 0$. Semisimple associative conformal algebra with a FFR is a direct sum of simple ones.

It was shown in [8] that, up to an isomorphism, one may assume $Q(x)$ is in the canonical diagonal form, i.e.,

$$Q(x) = \text{diag}(f_1, \ldots, f_n), \quad f_1 \mid \cdots \mid f_n.$$  

Moreover, if $\deg f_n > 0$ then one may assume $f_n(0) = 0$ since the shift map $x \mapsto x - \alpha, \alpha \in k$, is an automorphism of $C_{end}$. 

Remark 3. Note that $C_{end,n,Q}$ is isomorphic as a conformal algebra to the left ideal $M_n(k[\partial, x])Q(x - \partial)$ of $C_{end}$ [8]. It is easy to check by the definition of $\lambda$-product in Example 2 that the map $\theta : Q(x)A(\partial, x) \mapsto A(\partial, x)Q(x - \partial), A \in M_n(k[\partial, x])$, is an isomorphism.

Every associative conformal algebra $E$ with a FFR has a maximal nilpotent ideal $R$ such that $C = E/R$ also has a FFR. The conformal algebra $C$ obtained in this way is a direct sum of simple associative conformal algebras described by Theorem 1. Theorems 2 and 3 imply $E = C \ltimes R$ if all summands in $C$ are of the form $\text{Cur}_n$ or $C_{end}$. In this paper, we consider all possible cases and explicitly determine those semisimple associative conformal algebras with a FFR that split in every extension with a nilpotent kernel.

3. Extensions of $C_{end,n,Q}, Q = \text{diag}(1, \ldots, 1, x)$

Throughout this section, $C$ denotes the associative conformal algebra $C_{end,n,Q}$ for $Q = \text{diag}(1, \ldots, 1, x)$. The main purpose of this section is to prove that for $n \geq 2$ we have $H^2(C, M) = 0$ for every conformal bimodule $M$ over $C$.

Proposition 1. The conformal algebra $C$ is generated by the set

$$X = \{ e_{ij} | i = 1, \ldots, n - 1, j = 1, \ldots, n \} \cup \{ x_{ij} | i, j = 1, \ldots, n \}$$

relative to the following defining relations:

$$e_{ij} (\lambda) e_{kl} = \delta_{jk} e_{il}, \quad \text{(8)}$$  

$$x_{ij} (\lambda) e_{kl} = \delta_{jk} x_{il}, \quad \text{(9)}$$

$$e_{ij} (\lambda) x_{kl} = \delta_{jk} (x_{il} + \lambda e_{il}), \quad \text{(10)}$$

$$x_{ij} (1) x_{jl} = x_{il}, \quad \text{(11)}$$

$$x_{ij} (\lambda) x_{kl} = 0, \quad j \neq k, \quad \text{(12)}$$

$$x_{ij} (0) x_{jl} = x_{ik} (0) x_{kl}. \quad \text{(13)}$$

Proof. Let $\text{ConfAs}(X, 2)$ be the free associative conformal algebra generated by $X$ with locality bound $N = 2$ [33]. Denote by $\text{ConfAs}(X, 2 \mid S)$ the quotient of $\text{ConfAs}(X, 2)$ modulo the ideal generated by defining relations ([8], [13]). Obviously,
there is a homomorphism \( \psi : \text{ConfAs}(X, 2 \mid S) \rightarrow C \) sending \( e_{ij} \) to the corresponding unit matrix and \( x_{ij} \) to \( xe_{ij} \). To study \( \text{ConfAs}(X, 2 \mid S) \), we apply the Gröbner—Shirshov bases technique for associative conformal algebras \([5, 6]\).

Assume the order on \( \text{ConfAs}(X, 2) \) is induced as in \([5]\) by the following order on \( X \): \( e_{ij} < x_{kl}, e_{ij} < e_{kl} \) or \( x_{ij} < x_{kl} \) if and only if \( (ij) < (kl) \) lexicographically. The set of \( S \)-reduced words consists of

\[
\partial^s e_{ij}, \quad i = 1, \ldots, n - 1, \quad j = 1, \ldots, n,
\]

\[
\partial^t x_{kl}, \quad k, l = 1, \ldots, n,
\]

\[
\partial^s (x_{kl} (0) x_{11} (0) \cdots (0) x_{11} (0) x_{1l}), \quad k, l = 1, \ldots, n,
\]

\[
s \geq 0, \quad t \geq 1.
\]

The images of these words under \( \psi \) are linearly independent in \( C \), so \( S \) is a Gröbner—Shirshov basis of \( C \) with respect to the generators \( X \).

Let us now reduce the set of generators.

**Corollary 3.** The conformal algebra \( C \) is generated by \( X' = \{ e_{ij}, e_{1n}, x_{1n} \mid i, j = 1, \ldots, n - 1 \} \) relative to the following defining relations:

\[
e_{ij} (\lambda), \quad e_{kl} = \delta_{jk} e_{il}, \quad i, j, k, l = 1, \ldots, n - 1,
\]

\[
e_{1n} (\lambda), \quad e_{ij} = 0, \quad i, j = 1, \ldots, n - 1,
\]

\[
e_{11} (\lambda), \quad e_{1n} = e_{1n},
\]

\[
e_{ij} (\lambda), \quad x_{1n} = 0, \quad i, j = 1, \ldots, n - 1,
\]

\[
x_{1n} (\lambda), \quad e_{11} = x_{1n},
\]

\[
e_{1n} (1), \quad e_{11} = e_{1n}, \quad e_{1n} (m), x_{1n} = 0, \quad m > 1,
\]

**Proof.** Since \( C \) is simple, it is enough to note that if \([14] - [19]\) hold then the elements of \( X' \) together with

\[
e_{in} = e_{i1} (0) e_{1n}, \quad i = 2, \ldots, n - 1,
\]

\[
x_{nj} = x_{1n} (0) e_{1j}, \quad j = 2, \ldots, n,
\]

\[
x_{ij} = e_{in} (0) x_{nj}, \quad i = 1, \ldots, n - 1, \quad j = 1, \ldots, n
\]

satisfy all relations \([5] - [13]\). For example, let us check \([11]\). First, \( x_{1n} (m) e_{1l} = (x_{1n} (0) e_{11}) (m) e_{1l} = 0 \) for \( m > 0 \) by \([15]\) and \([14]\). Next, it follows from conformal associativity that

\[
x_{ij} (m) x_{jl} = (e_{i1} (0) e_{1n} (0) x_{1n} (0) e_{1j} (0) e_{j1} (m)) (e_{j1} (0) e_{1n} (0) x_{1n} (0) e_{1l} (m)) e_{1n} (0) x_{1n} (0) e_{1l} (m)
\]

\[
= (e_{i1} (0) e_{1n} (0) x_{1n} (0) e_{1j} (0) e_{j1} (m)) (e_{1n} (0) x_{1n} (0) e_{1l} (m)) e_{1n} (0) x_{1n} (0) e_{1l} (m)
\]

\[
= (e_{i1} (0) e_{1n} (0) x_{1n} (0) e_{1j} (0) e_{j1} (m)) (e_{1n} (0) x_{1n} (0) e_{1l} (m))
\]

\[
\sum_{s \in \mathbb{Z}_+} \binom{m}{s} (e_{1n} (s) x_{1n} (0) e_{1l} (m-s)) e_{1l} (m)
\]

\[
= (e_{i1} (0) e_{1n} (0) x_{1n} (0) e_{1j} (0) e_{j1} (m)) (e_{1n} (0) x_{1n} (0) e_{1l} (m)) = \begin{cases} x_{il}, & m = 1, \\ 0, & m > 1. \end{cases}
\]

Other relations \([8] - [13]\) can be checked in a similar way. \( \square \)
Let \( M \) be an arbitrary conformal bimodule over \( C \).

**Lemma 1.** For every 2-cocycle \( \varphi \in Z^2(C,M) \) there exists \( \varphi' \in Z^2(C,M) \) such that \( \varphi - \varphi' \in B^2(C,M) \) and

\[
\varphi'_\lambda(u_{ij},u_{kl}) = 0, \quad 1 \leq i,j,k,l \leq n-1, \quad u,v \in \{e,x\}.
\]  

(20)

**Proof.** Note that the subalgebra \( C_0 \subset C \) generated by \( u_{ij}, 1 \leq i,j \leq n-1, \ u \in \{e,x\} \), is isomorphic to \( \text{Cend}_{n-1} \) (upper left block of size \( n-1 \) in \( \text{Cend}_{n,Q} \)). The restriction of \( \varphi \) on \( C_0 \) belongs to \( Z^2(C_0,M) \). By Theorem 3 \( H^2(C_0,M) = 0 \), so there exists \( \tau \in C^1(C_0,M) \) such that \( (d_1\tau)_\lambda(u,v) = \varphi_\lambda(u,v) \) for all \( u,v \in C_0 \).

Let us choose an arbitrary extension of \( \tau \) to an \( H \)-linear map \( C \rightarrow M \) and note that \( \varphi' = \varphi - d_1\tau \) is the desired cocycle. \( \square \)

In the subsequent computations, we will use the following notation. For \( a,b \in C \) and \( \varphi \in C^2(C,M) \), denote

\[
\{ a \left( \lambda \right) b \} = \langle a \left( \lambda \right) b \rangle = \sum_{n,s \in \mathbb{Z}_+} \frac{(-\lambda)^n}{n!} \frac{(-\partial)^s}{s!} \langle a \left( n+s \right) b \rangle,
\]

and, similarly,

\[
\varphi_\lambda \{ a,b \} = \varphi_{-\lambda} \langle a,b \rangle = \sum_{n,s \in \mathbb{Z}_+} \frac{(-\lambda)^n}{n!} \frac{(-\partial)^s}{s!} \varphi_{n+s}(a,b).
\]

It is well-known (see [23]) that the following relation holds on an associative conformal algebra:

\[
a \left( \lambda \right) \{ b \left( \mu \right) c \} = \{ a \left( \lambda \right) b \} \left( \mu \right) c.
\]

(21)

Therefore, similar relations hold for conformal bimodule multiplications.

For a 2-cocycle \( \varphi \in Z^2(C,M) \), Theorem 1 and relation (21) imply

\[
\varphi_\lambda(a,\{ b \left( \mu \right) c \}) + a \left( \lambda \right) \varphi_\mu \{ b,c \} = \varphi_\mu(a_{\lambda}b,c) + \{ \varphi_\lambda(a,b) \left( \mu \right) c \}.
\]

(22)

Since \( \lambda \)-product is sesquilinear, we also have

\[
(\varphi_\mu \{ a,b \} \left( \lambda \right) c) = (\varphi_{-\lambda}^{-\mu} (a,b) \left( \lambda \right) c)
\]

for all \( a,b,c \in C \).

**Lemma 2.** For every 2-cocycle \( \varphi \in Z^2(C,M) \) there exists \( \varphi' \in Z^2(C,M) \) such that \( \varphi - \varphi' \in B^2(C,M) \), (20) holds, and

\[
\varphi'_\lambda(e_{1n},e_{ij}) = 0, \quad i,j = 1,\ldots,n-1,
\]

(23)

\[
\varphi'_\lambda(e_{11},e_{1n}) = 0.
\]

(24)

**Proof.** Without loss of generality we may assume that \( \varphi \) satisfies (20). Denote \( e = e_{11} + \cdots + e_{n-1n-1} \in C_0 \subset C \), where \( C_0 \) stands for the same subalgebra as in the proof of Lemma 1. Define \( \tau \in C^1(C,M) \) in such a way that \( \tau(e_{1n}) = \varphi_0(e_{11},e_{1n}) - \{ e_{11} \left( 0 \right) \varphi_0(e_{1n},e) \} \) and \( \tau(u) = 0 \) for other generators \( u \) of \( C \) as of \( H \)-module. Then, in particular, \( \tau(C_0) = 0, \) so \( (d_1\tau)_\lambda(C_0,C_0) = 0 \). Let us compute

\[
(d_1\tau)_\lambda(e_{1n},e_{ij}) = \varphi_\lambda(e_{11},e_{1n}) \left( \lambda \right) e_{ij} - e_{11} \left( \lambda \right) (\varphi_0(e_{1n},e) \left( 0 \right) e_{ij})
\]

\[
= \varphi_\lambda(e_{11},e_{1n} \left( 0 \right) e_{ij}) + e_{11} \left( \lambda \right) \varphi_0(e_{1n},e_{ij}) - \varphi_\lambda(e_{11} \left( \lambda \right) e_{1n},e_{ij})
\]

\[- e_{11} \left( \lambda \right) (\varphi_0(e_{1n},e) \left( 0 \right) e_{ij}) + e_{1n} \left( 0 \right) \varphi_0(e, e_{ij}) - \varphi_0(e_{1n} \left( 0 \right) e, e_{ij})
\]

\[= -\varphi_\lambda(e_{1n},e_{ij})
\]
for $i, j = 1, \ldots, n - 1$, and
\[
(d_1 \tau)_\lambda(e_{11}, e_{11}) = -\tau(e_{11}) + e_{11} (\lambda) \tau(e_{11})
\]
\[
= -\varphi_0(e_{11}, e_{11}) + \{e_{11} (0) \varphi_0(e_{11}, e)\} + e_{11} (\lambda) \varphi_0(e_{11}, e_{11})
\]
\[
- \{(e_{11} (\lambda) e_{11}) (0) \varphi_0(e_{11}, e)\} = -\varphi_0(e_{11}, e_{11}) + \varphi_0(e_{11} (\lambda) e_{11}, e_{11})
\]
\[
+ \{\varphi_\lambda(e_{11}, e_{11}) (0) e_{11}\} - \varphi_\lambda(e_{11}, (e_{11} (0) e_{11})\} = -\varphi_\lambda(e_{11}, e_{11}).
\]

Therefore, $\varphi' = \varphi + d_1 \tau$ satisfies (20), (23), and (24). □

**Lemma 3.** For every 2-cocycle $\varphi \in Z^2(C, M)$ there exists $\varphi' \in Z^2(C, M)$ such that $\varphi - \varphi' \in B^2(C, M)$, (20), (23), (24) hold, and
\[
\varphi'_\lambda(x_{n1}, e_{11}) = 0,
\]
\[
\varphi'_\lambda(e_{ij}, e_{n1}) = 0, \quad i, j = 1, \ldots, n - 1.
\]

**Proof.** Define a 1-cochain $\tau \in C^1(C, M)$ in such a way that
\[
\tau(x_{n1}) = \varphi_0(x_{n1}, e_{11}) - \varphi_0(e, x_{n1}) (0) e_{11}
\]
and $\tau(u) = 0$ for other generators of $C$ as of $H$-module. Then $(d_1 \tau)_\lambda(C_0, C_0) = 0$ and $(d_1 \tau)_\lambda(e_{1n}, C_0) = (d_1 \tau)_\lambda(C_0, e_{1n}) = 0$. Moreover,
\[
(d_1 \tau)_\lambda(x_{n1}, e_{11}) = \tau(x_{n1}) (\lambda) e_{11} - \tau(x_{n1}) = \varphi_0(x_{n1}, e_{11}) (\lambda) e_{11}
\]
\[
- \varphi_0(e, x_{n1}) (0) e_{11} (\lambda) e_{11} - \varphi_0(x_{n1}, e_{11}) + \varphi_0(e, x_{n1}) (0) e_{11} = \varphi_0(x_{n1}, e_{11} (\lambda) e_{11})
\]
\[
+ x_{n1} (0) \varphi_\lambda(e_{11}, e_{11}) - \varphi_\lambda(x_{n1} (0) e_{11}, e_{11}) - \varphi_0(x_{n1}, e_{11}) = -\varphi_\lambda(x_{n1}, e_{11})
\]
and
\[
(d_1 \tau)_\lambda(e_{ij}, x_{n1}) = e_{ij} (\lambda) \tau(x_{n1})
\]
\[
= e_{ij} (\lambda) \varphi_0(x_{n1}, e_{11}) - \{e_{ij} (\lambda) \varphi_0(e, x_{n1})\} (\lambda) e_{11} = \varphi_\lambda(e_{ij} (\lambda) x_{n1}, e_{11})
\]
\[
+ \varphi_\lambda(e_{ij}, x_{n1}) (\lambda) e_{11} - \varphi_\lambda(e_{ij}, x_{n1} (0) e_{11}) - (\varphi_\lambda(e_{ij}, (\lambda) e, x_{n1})
\]
\[
+ \varphi_\lambda(e_{ij}, e) (\lambda) x_{n1} - \varphi_\lambda(e_{ij}, e (0) x_{n1})) (\lambda) e_{11} = -\varphi_\lambda(e_{ij}, x_{n1}).
\]

Therefore, $\varphi' = \varphi + d_1 \tau$ is the desired cocycle. □

**Lemma 4.** For every 2-cocycle $\varphi \in Z^2(C, M)$ there exists $\varphi' \in Z^2(C, M)$ such that $\varphi - \varphi' \in B^2(C, M)$, (20), (23), (24), (25), (26) hold, and
\[
\varphi'_\lambda(e_{1n}, x_{n1}) = \varphi'_0(e_{1n}, x_{n1}).
\]

**Proof.** Given a 2-cocycle $\varphi \in Z^2(C, M)$, denote by $S$ the set of all $\varphi' \in Z^2(C, M)$ such that $\varphi - \varphi' \in B^2(C, M)$ and $\varphi'$ satisfies (20), (23), (24), (25), (26). Lemmas [1], [2], [3] imply $S \neq \emptyset$. Without loss of generality, we may assume $\varphi \in S$ and $m = \deg_\lambda \varphi_\lambda(e_{1n}, x_{n1})$ is minimal among all $\deg_\lambda \varphi_\lambda'(e_{1n}, x_{n1})$, $\varphi' \in S$. If $m = 0$ then there is nothing to prove. If $m > 0$ then define $\tau(x_{n1}) = \frac{1}{m} x_{n1} (0) \varphi_1(e_{1n}, x_{n1})$ and $\tau(u) = 0$ for other $H$-linear generators $u$ of $C$. Let us compute
\[
\varphi_\mu(e_{1n}, x_{n1}) (\lambda) e_{11} = \varphi_\mu(e_{1n}, x_{n1} (\lambda - \mu) e_{11})
\]
\[
+ e_{1n} (\mu) \varphi_{\lambda - \mu}(x_{n1}, e_{11}) - \varphi_\lambda(e_{1n} (\mu) x_{n1}, e_{11}) = \varphi_\mu(e_{1n}, x_{n1})
\]
due to the choice of \( S \). Hence,

\[
(d_1 \tau)_\lambda(x_{n1}, e_{11}) = \tau(x_{n1}) (\lambda) e_{11} - \tau(x_{n1}) = \frac{1}{m} x_{n1} (0) (\varphi_1(e_{1n}, x_{n1}) (\lambda) e_{11} - \varphi_1(e_{1n}, x_{n1})) = 0,
\]

\[
(d_1 \tau)_\lambda(e_{ij}, x_{n1}) = e_{ij} (\lambda) \tau(x_{n1}) = e_{ij} (\lambda) \frac{1}{m} x_{n1} (0) \varphi_1(e_{1n}, x_{n1}) = \frac{1}{m} (e_{ij} (\lambda) x_{n1}) (\lambda) \varphi_1(e_{1n}, x_{n1}) = 0.
\]

Therefore, the conditions of Lemmas 2 and 3 hold for \( d_1 \tau \), so the cocycle \( \varphi - d_1 \tau \) belongs to the set \( S \). Moreover,

\[
(d_1 \tau)_\lambda(e_{1n}, x_{n1}) = e_{1n} (\lambda) \tau(x_{n1}) = \frac{1}{m} e_{1n} (\lambda) (x_{n1} (\lambda) \varphi_1(e_{1n}, x_{n1})) = \frac{1}{m} (x_{11} + \lambda e_{11}) (\lambda) \varphi_1(e_{1n}, x_{n1}) = \frac{1}{m} (x (0) e_{11} (\lambda) \varphi_1(e_{1n}, x_{n1}) + \lambda e_{11} (\lambda) \varphi_1(e_{1n}, x_{n1}))
\]

has the same principal term (with respect to \( \lambda \)) as \( \varphi_\lambda(e_{1n}, x_{n1}) \). Indeed, let us evaluate

\[
e_{11} (\lambda) \varphi_\mu(e_{1n}, x_{n1}) = \varphi_\lambda + \mu(e_{11} (\lambda) e_{1n}, x_{n1}) + \varphi_\lambda(e_{11}, e_{1n} (\mu) x_{n1}) = \varphi_\lambda + \mu(e_{1n}, x_{n1}) \in M[\lambda, \mu].
\]

The coefficient at \( \mu \) of the latter polynomial is equal \( e_{11} (\lambda) \varphi_1(e_{1n}, x_{n1}) \). If \( \varphi_\lambda(e_{1n}, x_{n1}) = \lambda^{(m)} u_m + \lambda^{(m-1)} u_{m-1} + \ldots \) then

\[
e_{11} (\lambda) \varphi_1(e_{1n}, x_{n1}) = \lambda^{(m-1)} u_m + \lambda^{(m-2)} u_{m-1} + \ldots
\]

Finally,

\[
(d_1 \tau)_\lambda(e_{1n}, x_{n1}) = \frac{1}{m} \lambda^{(m-1)} x (0) u_m + \frac{1}{m} \lambda^{(m-2)} x (0) u_{m-1} + \ldots
\]

and for \( \varphi' = \varphi - d_1 \tau \in S \) we have

\[
\deg \varphi'_\lambda(e_{1n}, x_{n1}) < m = \deg \varphi_\lambda(e_{1n}, x_{n1})
\]

in contradiction to the choice of \( \varphi \).

\[ \square \]

**Theorem 5.** Let \( C = \text{Cend}_n Q \), \( Q = \text{diag}(1, \ldots, 1, x) \), \( n > 1 \), and let \( M \) be an arbitrary conformal bimodule over \( C \). Then \( \mathcal{H}^2(C, M) = 0 \).

**Proof.** Suppose \( \varphi \in \mathcal{Z}^2(C, M) \). By Lemmas \[1\] \[2\], there exists \( \varphi' \in \mathcal{Z}^2(C, M) \) such that \( \varphi - \varphi' \in \mathcal{B}^2(C, M) \) and \( \[20\], \[23\], \[24\], \[25\], \[26\], \[27\] \) hold.

Consider an extension \( E \) defined by \( \varphi' \) as in Theorem \[1\]

\[
0 \to M \to E \to C \to 0.
\]

Then \( E = C + M \), and the pre-images of the elements of \( X' \) from Corollary \[3\] satisfy defining relations \( \[14\] - \[19\] \). Therefore, there exists a homomorphism \( \rho : C \to E \) which maps \( a \in X' \) to \( a + 0 \in C + M = E \). The map \( \rho \) is injective since it has a left inverse \( E \to C \) in the exact sequence above. Hence, the subalgebra of \( E \)
generated by pre-images of \( X' \) is isomorphic to \( C \), the extension \( E \) is split, and \( \varphi' \in \mathcal{B}^2(C, M) \). The latter implies \( \varphi \in \mathcal{B}^2(C, M) \). \( \square \)

**Remark 4.** For \( n = 1 \), the proof stated above does not work since there is no idempotent \( e_{11} \in C \). However, it was proved in [28] that \( \mathcal{H}^2(\text{Cend}_1, M) = 0 \) for every conformal bimodule \( M \) over \( \text{Cend}_1 \). As a corollary, Theorem 4 holds also for \( n = 1 \).

4. **Non-split extensions with finite faithful representation**

In this section, we state a series of examples of non-split null extensions of semisimple associative conformal algebras with a FFR.

**Example 3.** Let \( C = \mathbb{Z}[\partial, x] \oplus \mathbb{Z}[\partial, y] \simeq \text{Cend}_1 \oplus \text{Cend}_2 \). Consider \( M = \mathbb{Z}[\partial, z] \) as a conformal bimodule over \( C \) relative to

\[
xf(\partial, x) (\lambda) z^2 g(\partial, z) = z^2 (z + \lambda) f(-\lambda, z) g(\partial + \lambda, z + \lambda),
\]

\[
z^2 g(\partial, z) (\lambda) yf(\partial, y) = z^2 (z + \lambda) g(-\lambda, z) f(\partial + \lambda, z + \lambda),
\]

\[
yf(\partial, y) (\lambda) z^2 g(\partial, z) = 0,
\]

\[
z^2 g(\partial, z) (\lambda) xf(\partial, x) = 0.
\]

Define a 2-cochain \( \varphi \in C^2(C, M) \) in the following way:

\[
\varphi_\lambda(xf(\partial, x), yg(\partial, y)) = z^2 f(-\lambda, z) g(\partial + \lambda, z + \lambda),
\]

\[
\varphi_\lambda(yf(\partial, x), xg(\partial, x)) = 0, \quad \varphi_\lambda(xf(\partial, x), xg(\partial, x)) = 0,
\]

\[
\varphi_\lambda(yf(\partial, y), yg(\partial, y)) = 0.
\]

It is straightforward to check that \( \varphi \) is a 2-cocycle. Indeed, one may either check the relation (4), or simply note that the set \( E \) of all matrices of the form

\[
\begin{pmatrix}
xf(\partial, x) & \frac{1}{2}xf(\partial, x) + \frac{1}{2}(x - \partial)g(\partial, x) + x(x - \partial)h(\partial, x) \\
0 & (x - \partial)g(\partial, x),
\end{pmatrix}
\]

(28)

\( f, g, h \in \mathbb{Z}[\partial, x] \), is a conformal subalgebra of \( \text{Cend}_2 \) isomorphic to the extension of \( C \) by \( M \) relative to \( \varphi \), where the isomorphism is given by

\[
xf(\partial, x) \mapsto \begin{pmatrix} xf(\partial, x) & \frac{1}{2}xf(\partial, x) \\ 0 & 0 \end{pmatrix}, \quad yg(\partial, y) \mapsto \begin{pmatrix} 0 & 0 \\ \frac{1}{2}(x - \partial)g(\partial, x) & (x - \partial)g(\partial, x) \end{pmatrix},
\]

\[
z^2 h(\partial, z) \mapsto \begin{pmatrix} 0 & x(x - \partial)h(\partial, x) \\ 0 & 0 \end{pmatrix}.
\]

Let us show \( \varphi \not\in \mathcal{B}^2(C, M) \). Assume the converse, i.e., there exists \( \psi \in C^1(C, M) \) such that \( d_1 \psi = \varphi \). Suppose

\[
\psi(x) = z^2 f(\partial, z), \quad \psi(y) = z^2 g(\partial, z).
\]

Then

\[
z^2 = \varphi_\lambda(x, y) = z^2 (z + \lambda) f(-\lambda, z) + z^2 (z + \lambda) g(\partial + \lambda, z + \lambda),
\]

i.e., \( z + \lambda \) divides 1 in \( \mathbb{Z}[\partial, z, \lambda] \). The contradiction just obtained proves

\[
\mathcal{H}^2(\text{Cend}_1, \text{Cend}_2, M) \neq 0.
\]

The example above clarifies the main idea of the following statement.
Proposition 2. Suppose \( C \simeq \text{Cend}_{Q,n} \oplus \text{Cend}_{Q',m} \), where \( Q = \text{diag}(1, \ldots, 1, x) \), \( Q' = \text{diag}(1, \ldots, 1, x) \). Then there exists a conformal \( C \)-bimodule \( M \) such that \( H^2(C, M) \neq 0 \).

Proof. Assume \( n \leq m \). Consider the \( H \)-module \( M = M_{n,m}(k[\partial, x]) \) equipped with the following \( C \)-bimodule structure:

\[
(Q(x)A(\partial, x) + Q'(x)B(\partial, x)) (\lambda) \chi(\partial, x) = A(-\lambda, x)Q(x + \lambda)X(\partial + \lambda, x + \lambda),
\]

\[
X(\partial, x) (\lambda) (Q(x)A(\partial, x) + Q'(x)B(\partial, x)) = X(-\lambda, x)Q'(x + \lambda)B(\partial + \lambda, x + \lambda),
\]

\( X \in M \). Straightforward computation shows that this is indeed a conformal bimodule over \( C \).

Let us define linear maps

\[
\vdash: M_n(k[\partial, x]) \to M_{n,m}(k[\partial, x]), \quad \dashv: M_m(k[\partial, x]) \to M_{n,m}(k[\partial, x])
\]

as

\[
A^\vdash = (A \quad 0), \quad A \in M_n(k[\partial, x])
\]

(add \( m - n \) zero columns) and

\[
B^\vdash = \begin{bmatrix}
    b_{11} & \cdots & b_{1m} \\
    \vdots & \ddots & \vdots \\
    b_{n1} & \cdots & b_{nm}
\end{bmatrix}, \quad B \in M_m(k[\partial, x])
\]

(remove \( m - n \) rows in the bottom). It is clear that

\[
A^\vdash B = AB^\vdash, \quad A_1A_2^\vdash = (A_1A_2)^\vdash, \quad B_1^\vdash B_2 = (B_1B_2)^\vdash.
\]

Consider the following 2-cochain \( \varphi \in \mathcal{C}^2(C, M) \):

\[
\varphi_{\lambda}(QA_1 + Q'B_1, QA_2 + Q'B_2) = A_1(-\lambda, x)B_2^\vdash(\partial + \lambda, x + \lambda). \tag{29}
\]

To prove that (29) defines a 2-cocycle, one may check (7) for \( a_i = QA_i + Q'B_i \), \( i = 1, 2, 3 \). Indeed,

\[
a_1 (\lambda) \varphi_{\mu}(a_2, a_3) = (QA_1 + Q'B_1) (\lambda) \varphi_{\mu}(QA_2 + Q'B_2, QA_3 + Q'B_3)
\]

\[
= QA_1 (\lambda) A_2(-\mu, x)B_3^\vdash(\partial + \mu, x + \mu)
\]

\[
= A_1(-\lambda, x)Q(x + \lambda)A_2(-\mu, x + \lambda)B_3^\vdash(\partial + \mu, x + \lambda + \mu)
\]

\[
\varphi_{\lambda+\mu}(a_1 (\lambda) a_2, a_3) = \varphi_{\lambda+\mu}(QA_1(-\lambda, x)Q(x + \lambda)A_2(\partial + \lambda, x + \lambda)
\]

\[
+ Q'B_1(-\lambda, x)Q'(x + \lambda)B_2(\partial + \lambda, x + \lambda), QA_3 + Q'B_3)
\]

\[
= A_1(-\lambda, x)Q(x + \lambda)A_2(-\lambda + \mu, x + \lambda)B_3^\vdash(\partial + \lambda + \mu, x + \lambda + \mu),
\]

\[
\varphi_{\lambda}(a_1, a_2 (\mu) a_3) = \varphi_{\lambda}(QA_1 + Q'B_1, QA_2(-\mu, x)Q(x + \mu)A_3(\partial + \mu, x + \mu)
\]

\[
+ Q'B_2(-\mu, x)Q'(x + \mu)B_3(\partial + \mu, x + \mu))
\]

\[
= A_1(-\lambda, x)B_2^\vdash(-\mu, x + \lambda)Q'(x + \lambda + \mu)B_3(\partial + \lambda + \mu, x + \lambda + \mu),
\]

\[
\varphi_{\lambda}(a_1, a_2) (\lambda+\mu) a_3 = A_1(-\lambda, x)B_2^\vdash(\partial + \lambda, x + \lambda) (\lambda+\mu) QA_3 + Q'B_3
\]

\[
= A_1(-\lambda, x)B_2^\vdash(-\lambda + \mu, x + \lambda)Q'(x + \lambda + \mu)B_3(\partial + \lambda + \mu, x + \lambda + \mu),
\]
so

\[ a_1(\lambda) \varphi_\mu(a_2, a_3) = \varphi_{\lambda+\mu}(a_1(\lambda), a_2, a_3), \quad \varphi_\lambda(a_1, a_2(\mu) a_3) = \varphi_\lambda(a_1, a_2(\lambda+\mu) a_3). \]

Assume \( \varphi = d_1 \psi \in B^2(C, M) \) for some \( \psi \in C^1(C, M) \). Suppose

\[ \psi(Q) = X_1(\partial, x), \quad \psi(Q'e_{nm}) = X_2(\partial, x). \]

Then

\[ e_{nm} = I_n e_{nm}^* = \varphi_\lambda(Q, Q') = X_1(\lambda) Q' - \psi(Q(\lambda) Q') + Q(\lambda) X_2 \]

\[ = X_1(-\lambda, x)Q'(x + \lambda) - Q(x + \lambda)X_2(\partial + \lambda, x + \lambda), \]

but in the right-hand side of the last expression a multiple of \( x + \lambda \) occurs in \( n \)th row and \( m \)th column. Therefore, \( \mathcal{H}^2(C, M) \neq 0 \). \( \square \)

**Remark 5.** The non-split extension

\[ 0 \to M \to E \to C \to 0 \]

constructed with \( C, M \), and \( \varphi \) from Proposition 2 has a FFR.

Indeed, one may present \( E \) as a subalgebra of \( \text{Cend}_{n+m} \) of all matrices of the form

\[
\begin{pmatrix}
Q(x)A & \frac{1}{2}Q(x)A^\gamma + \frac{1}{2}BQ(x - \partial) + Q(x)XQ(x - \partial) \\
0 & BQ(x - \partial)
\end{pmatrix},
\]

where \( A \in M_n(k[\partial, x]), B \in M_m(k[\partial, x]), X \in M_{n,m}(k[\partial, x]) \).

**Proposition 3.** Let \( C = \text{Cend}_{n,q} \), \( n > 1 \), and \( Q = \text{diag}(f_1, f_2, \ldots, f_n) \), where \( f_i \in k[x] \) and \( f_1 | f_2 | \cdots | f_n, \det Q \neq 0 \). If there exist \( 1 \leq i < j \leq n \) such that \( \deg f_j \geq \deg f_i > 0 \) then \( \mathcal{H}^2(C) = \mathcal{H}^2(C, C) \neq 0 \).

**Proof.** Let us choose a pair \( i < j \) such that \( 0 < \deg f_i \leq \deg f_j \) and consider the 2-cochain given by

\[ \varphi_\lambda(Q(x)A, Q(x)B) = Q(x)A(-\lambda, x)e_{ij}B(\partial + \lambda, x + \lambda), \quad (30) \]

where \( e_{ij} \) stands for the corresponding unit matrix.

One may easily check that (7) holds, so \( \varphi \in \mathcal{Z}^2(C) \). Let us show \( \varphi \notin \mathcal{B}^2(C) \).

Indeed, assume there exists \( \psi \in C^1(C) \) such that \( \varphi = d_1 \psi \). Suppose

\[ \psi(f_i e_{ii}) = Q(x)A(\partial, x), \quad \psi(f_j e_{jj}) = Q(x)B(\partial, x) \]

for some \( A, B \in M_n(k[\partial, x]) \). Let us compute

\[ \varphi_\lambda(f_i e_{ii}, f_j e_{jj}) = Q e_{ii}e_{ij}e_{jj} = f_i e_{ij} \]

and compare the result with

\[ (d_1 \psi)_\lambda(f_i e_{ii}, f_j e_{jj}) = Q(x)A(\partial, x)(\lambda)f_i e_{jj} + f_i e_{ii}(\lambda)Q(x)B(\partial, x). \]

The latter is equal to

\[ f_j(x + \lambda)Q(x)A(-\lambda, x)e_{jj} + e_{ii}f_i(x)Q(x + \lambda)B(\partial + \lambda, x + \lambda), \]

so in \( i \)th row and \( j \)th column we get an equation

\[ f_i(x) = f_j(x + \lambda)f_i(x)a_{ij}(-\lambda, x) + f_i(x)f_i(x + \lambda)b_{ij}(\partial + \lambda, x + \lambda), \]

which implies a contradiction since \( f_i | f_j \). \( \square \)

**Remark 6.** The non-split extension \( 0 \to C \to E \to C \to 0 \) constructed with the cocycle from the proof of Proposition 3 is a conformal algebra with a FFR.
Indeed, it is easy to see that
\[
E \simeq \left\{ \begin{pmatrix} Q(x)A & e_{ij}A + Q(x)B \\ 0 & Q(x)A \end{pmatrix} \mid A, B \in M_n(k[\partial, x]) \right\} \subseteq \text{Cend}_{2n}.
\]

**Proposition 4.** Let \( C = \text{Cend}_1, f = f(x) \in k[x], \) \( \deg f > 1, n \geq 1. \) Then there exists conformal bimodule \( M \) such that \( \mathcal{H}^2(C, M) \neq 0. \)

**Proof.** Linear shift of variable allows us to assume \( f(x) = x^N + \alpha x^{N-2} + \ldots, \) i.e., there is no \( x^{N-1}, N = \deg f. \)

Consider \( M = \text{Cend}_1 \) as a conformal bimodule over \( C \) relative to the following operations \((\cdot, [\lambda] \cdot)\):

\[
a a \lambda h = a (\lambda) fh, \quad h a \lambda f = h (\lambda) fa,
\]

\( a, h \in \text{Cend}_1, \) where \((\cdot, (\lambda) \cdot)\) is the standard operation on \( \text{Cend}_1 \) (see Example 2).

This is indeed a bimodule since the right action of \( C \) on \( M \) coincides with the regular module structure, the left one is twisted by the isomorphism \( \theta \) from Remark 3

\((fa \lambda h) = (\theta(fa)) (\lambda) h).\)

Define a sesquilinear map \( \varphi : C \otimes C \to M[\lambda] \) given by

\[
\varphi_\lambda(fa, fb) = a (\lambda) b.
\]

It is straightforward to check that \( \varphi \in \mathcal{Z}^2(C, M). \) Alternatively, one may note that \( M \) is isomorphic to the radical of a conformal algebra \( E \) with a FFR,

\[
E = \left\{ \begin{pmatrix} f(x)a & f(x)a + f(x)hf(x - \partial) \\ 0 & af(x - \partial) \end{pmatrix} \mid a, h \in k[\partial, x] \right\} \subseteq \text{Cend}_2.
\]

Namely, \( h \in M \) corresponds to the matrix \( f(x)hf(x - \partial)e_{12}. \)

Let us show that \( \varphi \notin \mathcal{B}^2(C, M). \) Assume the converse: \( \varphi = d_1 \psi \) for some \( \psi \in \mathcal{C}^1(C, M). \) Define \( \tilde{\psi} : \text{Cend}_1 \to \text{Cend}_1 \) by the rule \( \tilde{\psi}(a) = \psi(fa), a \in \text{Cend}_1. \)

Then

\[
\varphi_\lambda(fa, fb) = \tilde{\psi}(a) \lambda fb - \tilde{\psi}(a (\lambda) fb) + fa \lambda \tilde{\psi}(b)
\]

and (31), (32) imply

\[
\tilde{\psi}(a (n) fb) = \tilde{\psi}(a (n) fb + a (n) f \tilde{\psi}(b) - a (n) b, \quad n \in \mathbb{Z}_+.
\]

In particular, for \( a = b = 1 \) we have \( \tilde{\psi}(1) \langle N \rangle f + 1 \langle N \rangle f \tilde{\psi}(1) = \tilde{\psi}(1 \langle N \rangle f) = N! \tilde{\psi}(1). \) Let us write \( \tilde{\psi}(1) \in \text{Cend}_1 \) as

\[
\tilde{\psi}(1) = \sum_{k \geq 0} a_k(x)(x - \partial)^k, \quad a_k(x) \in k[x].
\]

Then

\[
N! \sum_{k \geq 0} a_k(x)(x - \partial)^k = \sum_{k \geq 0} a_k(x)(x - \partial)^k \langle N \rangle f(x) + 1 \langle N \rangle f(x) \sum_{k \geq 0} a_k(x)(x - \partial)^k
\]

\[
= \sum_{k \geq 0} a_k(x) \frac{d^N}{dx^N}(x^k f(x)) + \sum_{k \geq 0} \frac{d^N}{dx^N}(f(x)a_k(x))(x - \partial)^k.
\]
Compare coefficients at \((x - \partial)^k, k \geq 0\), in the left- and right-hand sides of (35) to get
\[
N!a_0(x) = \sum_{k \geq 0} a_k(x) \frac{d^N}{dx^N}(x^k f(x)) + \frac{d^N}{dx^N}(a_0(x)f(x)),
\]
(35)

\[
N!a_k(x) = \frac{d^N}{dx^N}(f(x)a_k(x)), \quad k \geq 1.
\]
(36)

Relation (36) implies \(\deg a_k \leq 0\) for \(k \geq 1\) (it is enough to compare principal terms of the polynomials). Then it follows from (35) that
\[
0 = \frac{d^N}{dx^N} \left( \sum_{k \geq 1} a_k x^k f(x) + a_0(x)f(x) \right),
\]
i.e.,
\[
a_0(x) = - \sum_{k \geq 1} a_k x^k.
\]
Hence,
\[
\tilde{\psi}(1) = \sum_{k \geq 1} a_k ((x - \partial)^k - x^k), \quad a_k \in k.
\]

It remains to show
\[
\tilde{\psi}(h) = h \tilde{\psi}(1)
\]
(37)
for all \(h \in \text{Cend}_1\). It is enough to consider \(h = h(x) \in k[x]\) and proceed by induction on \(\deg h\). Indeed, if (37) holds for some \(h \in k[x]\) then, by (36) for \(a = h, b = 1, n = N - 1\), we have
\[
N!\tilde{\psi}(xh)
= h(x) \sum_{k \geq 1} a_k((x - \partial)^k - x^k) \tau(x^{-1}) f(x) + h \tau(x^{-1}) f(x) \sum_{k \geq 1} a_k((x - \partial)^k - x^k)
= h(x) \sum_{k \geq 1} a_k \left( \frac{d^{N-1}}{dx^{N-1}}(x^k f) - N!x^{k+1} + N!x(x - \partial)^k - \frac{d^{N-1}}{dx^{N-1}}(x^k f) \right)
= N!x(h(x))\tilde{\psi}(1).
\]
(38)

Here we used the initial assumption on the polynomial \(f\) to get \(f^{(N-1)} = N!x\).

Relation (37) means \(\tilde{\psi} \in \mathcal{B}^1(C, M)\), so \(d_1\tilde{\psi} = 0 \neq \varphi\).

**Corollary 4.** Let \(C = \text{Cend}_{n,Q}^n, n \geq 1, Q = \text{diag}(f_1, f_2, \ldots, f_n), \det Q \neq 0\). If \(\deg f_n > 1\) then there exists a bimodule \(M\) over \(C\) such that \(\mathcal{N}^2(C, M) \neq 0\).

**Proof.** A non-split extension \(0 \to M \to E \to C \to 0\) may be constructed as
\[
E = \left\{ \left( \begin{array}{c} Q(x)A \\ 0 \end{array} \right) \begin{array}{c} Q(x)A + Q(x)XQ(x - \partial) \\ AQ(x - \partial) \end{array} \right| A, X \in \text{Cend}_n \right\} \subseteq \text{Cend}_{2n}.
\]

Indeed, \(C\) contains a subalgebra \(C' = e_{nn}C_{ee}\) isomorphic to \(\text{Cend}_{1,f_n}\), \(M\) contains a \(C'\)-submodule \(M' = e_{nn}Me_{ee}\). The latter is isomorphic to \(\text{Cend}_1\) considered as a \(\text{Cend}_{1,f_n}\)-bimodule relative to the operations (31) for \(f = f_n\), and the induced cocycle coincides with (32). If \(E\) was a split extension then so is \(E' = (e_{nn} + e_{2n2n})E(e_{nn} + e_{2n2n}),\) but \(E'\) does not split by Proposition 4.

\(\square\)
5. Conclusion

Let us analyze the general case in order to choose which semisimple associative conformal algebras with a FFR have trivial second Hochschild cohomology group relative to every conformal bimodule.

Let $C$ be a conformal algebra. Recall that a conformal identity (or unit) is an element $e \in C$ such that $e(1) e = e$ and $e(0) a = a$ for all $a \in C$ [31]. Unital associative conformal algebras and pseudo-algebras are well studied, they are closely related with differential associative algebras [32]. From the cohomological point of view they also have nice properties.

**Proposition 5.** Suppose

$$0 \to M \to E \to C \to 0$$

is an exact sequence of associative conformal algebras, $M$ is nilpotent, and $C = C_0 \oplus C_1$, where $\mathcal{H}^2(C_1, M) = 0$, $\mathcal{H}^2(C_0, M) = 0$, and $C_1$ is a unital conformal algebra. Then $\mathcal{H}^2(C, M) = 0$ and, therefore, $E \simeq C \rtimes M$.

**Proof.** If $\bar{e}$ is a conformal identity of $C_1$ then there exists its pre-image $e \in E$ which is a conformal idempotent: $e(0) e = e$, $e(1) e = 0$ for $n > 0$. Hence, we may apply conformal Pierce decomposition procedure to $C$. In particular, the subalgebra $E_0 = \{a - e(0) a - \{a(0) e\} + e(0) \{a(0) e\} \mid a \in E\}$ contains a pre-image of $C_0 \subset C$. Therefore, $E_0$ contains a subalgebra isomorphic to $C_0$. On the other hand, the subalgebra $E_1 = \{e(0) \{a(0) e\} \mid a \in E\}$ contains a subalgebra isomorphic to $C_1$. Finally, $E_0(\lambda) E_1 = E_1(\lambda) E_0 = 0$ since $e(\lambda) E_0 = E_0(\lambda) e = 0$ for all $m \geq \mathbb{Z}_+$, so $E$ contains a subalgebra isomorphic to $C$. \[\square\]

**Corollary 5.** Let $C$ be an associative conformal algebra with a FFR, and let $R$ be the maximal nilpotent ideal $R$, $C/R = \bigoplus_i C_i$, $C_i$ are simple conformal algebras. Suppose all $C_i$ are isomorphic to $\text{Cur}_n$, $\text{Cend}_n$, and no more than one of them is of the form $\text{Cend}_{Q,n}$, $Q = \text{diag}(1, \ldots, 1, x)$. Then $C \simeq C/R \rtimes R$.

Let us summarize Theorem 5, Corollary 3, Example 3, Propositions 2, 4, 5, Corollary 4 and the results of [17, 28] to state the ultimate description of those semisimple associative conformal algebras with a FFR that split in every extension with a nilpotent kernel.

By [24], every semisimple conformal algebra $C$ with a FFR is a direct sum of simple ones, $C = \bigoplus_i C_i$, where either $C_i \simeq \text{Cur}_n$ or $C_i \simeq \text{Cend}_{n,Q}$, $Q = \text{diag}(f_1, \ldots, f_n)$, $\det Q \neq 0$, $f_1 | \cdots | f_n$.

**Theorem 6.** An associative conformal algebra $C$ with a FFR splits in every extension with a nilpotent kernel if and only if $C$ is a direct sum of conformal algebras $C_i$ isomorphic to $\text{Cur}_n$, $\text{Cend}_n$, and no more than one $\text{Cend}_{n,Q}$, $Q = \text{diag}(1, \ldots, 1, x)$.

It is worth mentioning that if $C$ does not satisfy the condition of Theorem 5 then there exists a non-split extension $E$ in an exact sequence $0 \to M \to E \to C \to 0$ which is an conformal algebra with a FFR. Therefore, Theorem 6 may be considered as an analogue of the Wedderburn Principal Theorem for the class of associative conformal algebras with a FFR.

The list of those simple associative conformal algebras with a FFR that split in every null extension seems to be related with irreducible representations of finite Lie conformal superalgebras. Particular observations on simple superalgebras lead to the following...
Conjecture. Let $L$ be a finite Lie conformal superalgebra, and let $M$ be a finite irreducible conformal $L$-module. Then the associative conformal subalgebra of $\text{Cend} M$ generated by the image of $L$ is isomorphic to either $\text{Cur}_n$, or $\text{Cend}_n$, or $\text{Cend}_n, Q$ for $Q = \text{diag}(1, \ldots, 1, x)$, where $n$ is the rank of $M$ over $H = k[\partial]$.

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Sobolev Institute of Mathematics
E-mail address: pavelisk77@gmail.com

Novosibirsk State University
E-mail address: KozlovRA.NSU@yandex.ru