On a reverse form of the Brascamp-Lieb inequality

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Abstract

We prove a reverse form of the multidimensional Brascamp-Lieb inequality. Our method also gives a new way to derive the Brascamp-Lieb inequality and is rather convenient for the study of equality cases.

Introduction

We will work on the space $\mathbb{R}^n$ with its usual Euclidean structure. We will denote by $\langle, \rangle$ the canonical scalar product. In [BL], H. J. Brascamp and E. H. Lieb showed that for $m \geq n$, $p_1, \ldots, p_m > 1$ and $a_1, \ldots, a_m \in \mathbb{R}^n$, the norm of the multilinear operator $\Phi$ from $L_{p_1}(\mathbb{R}) \times \cdots \times L_{p_m}(\mathbb{R})$ into $\mathbb{R}$ defined by

$$\Phi(f_1, \ldots, f_m) = \int_{\mathbb{R}^n} \prod_{i=1}^m f_i(\langle x, a_i \rangle) \, dx$$

can be computed as the supremum over centered Gaussian functions $g_1, \ldots, g_m$ of

$$\frac{\Phi(g_1, \ldots, g_m)}{\prod_{i=1}^m \|g_i\|_{p_i}}.$$ 

In other words, $\Phi$ is saturated by Gaussian functions. This theorem is a very convenient tool to derive sharp inequalities. Brascamp and Lieb applied it successfully to prove the optimal version of Young’s convolution inequality (also derived independently and simultaneously by Beckner [Bed]), to rederive Nelson’s hypercontractivity. Their proof is based on a rearrangement inequality of Brascamp, Lieb and Luttinger [BLL] and on the fact that radial functions of a large number of variables behave like Gaussians. However, their method left opened, except in some special cases, the multidimensional problem: let $m \geq n$, $p_1, \ldots, p_m > 1$ and let $n_1, \ldots, n_m$ be integers; for each $i \leq m$ let $B_i$ be a linear mapping from $\mathbb{R}^n$ into $\mathbb{R}^{n_i}$. Is the multilinear operator on $L_{p_1}(\mathbb{R}^{n_1}) \times \cdots \times L_{p_m}(\mathbb{R}^{n_m})$ defined by

$$\Psi(f_1, \ldots, f_m) = \int_{\mathbb{R}^n} \prod_{i=1}^m f_i(B_i x) \, dx$$

saturated by Gaussian functions?

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This question was solved positively by Lieb in his article “Gaussian kernels have only Gaussian maximizers” [Lie]. The point is that Ψ can be viewed as a limit case of multilinear operators with Gaussian kernels.

In [Bar3], the author gave a simple proof, for functions of one real variable, of the Brascamp-Lieb inequality and of a new family of inequalities which can be understood as a reverse form, or as a dual form of the Brascamp-Lieb inequalities. These inequalities can be stated as follows: let \( m \geq n \), \( p_1, \ldots, p_m > 1 \) and \( a_1, \ldots, a_m \in \mathbb{R}^n \), the largest constant \( E \) such that

\[
\int_{\mathbb{R}^n} \sup_{x=\sum c_i \theta_i a_i} \prod_{i=1}^{m} f_i(\theta_i) \, dx \geq E \prod_{i=1}^{m} \| f_i \|_{p_i},
\]

holds for all \( f_1, \ldots, f_m \) is also the largest constant such that the inequality holds for centered Gaussian functions, where \( f^* \) is the outer integral. Again, Gaussian functions play an extremal role. This new inequality was inspired by convexity theory. The strength of the Brascamp-Lieb inequality for volume estimates of convex bodies was noticed by K. Ball (see [Bal1], [Bal2] and [Bal3]), who also remarked in [Bal3] that a reverse inequality would give dual results. For geometric applications of the reverse Brascamp-Lieb inequality see [Bar1] and also section III of the present paper.

In the first section, we prove a fully multidimensional version of the reverse Brascamp-Lieb inequality. Our method also gives a new proof of the multidimensional Brascamp-Lieb inequality. It is very similar to the one dimensional case and uses a theorem of Brenier ([Bre1], [Bre2]) refined by McCann ([McC1], [McC2]) on measure preserving mappings deriving from convex potentials. Notice that this result was applied by McCann in [McC1] to prove the Prékopa-Leindler inequality ([Pre], [Lei]), which is a particular case of the reverse Brascamp-Lieb inequalities.

In section II, we focus on the one-dimensional case to deal in detail with equality cases. This problem was left open for the Brascamp-Lieb inequality because the previous proofs were depending on limit processes. We push further the study of [BL] in the spirit of [Lie] to see when there is a Gaussian maximizer for the Brascamp-Lieb inequality (or a Gaussian minimizer for the reverse form) and whether it is unique.

In section III, we study the particular case of the Brascamp-Lieb inequality which was noticed by K. Ball [Bal] and which is so useful in convexity. We state the corresponding converse inequality. The equality cases are completely solved, which allows us to find new characteristic properties of simplices and parallelotopes. The multidimensional version of the reverse Brascamp-Lieb inequality implies a theorem similar to Brunn-Minkowski inequality for sets that are contained in subspaces.

In section IV, we develop an idea of [BL]: after proving a reverse and sharp form of Young’s inequality, Brascamp and Lieb take limits in some parameters and rederive the Prékopa-Leindler inequality. We rederive Ball’s version of the Brascamp-Lieb inequality and its converse from a generalized form of Young’s inequality and its converse (which we prove by the method that we developed in [Bar2]). The goal of this section is to show the unity of the topics.
1 Proof of the Brascamp-Lieb inequality and its converse

We first introduce some notations. We will denote by $S^+(\mathbb{R}^n)$ the set of $n \times n$ symmetric definite positive matrices. For $A \in S^+(\mathbb{R}^n)$ we will denote by $G_A$ the centered Gaussian function on $\mathbb{R}^n$

$$G_A(x) = \exp(-\langle Ax, x \rangle).$$

We will also denote by $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ the set of linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$, identified with $m \times n$-matrices. If $B \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$, then $B^* \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ will be its Euclidean adjoint. We will work in the set of integrable non-negative functions on $\mathbb{R}^n$, denoted by $L^+_1(\mathbb{R}^n)$.

The fully multidimensional version of the Brascamp-Lieb inequality and its converse is as follows:

**Theorem 1** Let $m \geq n$ be integers. Let $(c_i)_{i=1}^m$ be positive real numbers and $(n_i)_{i=1}^m$ be integers smaller than $n$ such that

$$\sum_{i=1}^m c_i n_i = n.$$

For $i = 1 \ldots n$ let $B_i$ be a linear surjective map from $\mathbb{R}^n$ onto $\mathbb{R}^{n_i}$. Assume that

$$\cap_{i \leq m} \ker B_i = \{0\}.$$

We define two applications $I$ and $J$ on $L_1^+(\mathbb{R}^{n_1}) \times \cdots \times L_1^+(\mathbb{R}^{n_m})$ as follows: if $f_i \in L_1^+(\mathbb{R}^{n_i})$, $i = 1, \ldots, m$ then

$$I((f_i)_{i=1}^m) = \left( \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^m f_i^{c_i}(y_i) : \sum_{i=1}^m c_i B_i^* y_i = x \text{ and } y_i \in \mathbb{R}^{n_i} \right\} \right) dx,$$

and

$$J((f_i)_{i=1}^m) = \int_{\mathbb{R}^n} \prod_{i=1}^m f_i^{c_i}(B_i x) dx.$$

Let $E$ be the largest constant such that for all $(f_i)_{i=1}^m$,

$$I((f_i)_{i=1}^m) \geq E \prod_{i=1}^m \left( \int_{\mathbb{R}^{n_i}} f_i \right)^{c_i},$$

(RBL)

and let $F$ be the smallest one such that for all $(f_i)_{i=1}^m$,

$$J((f_i)_{i=1}^m) \leq F \prod_{i=1}^m \left( \int_{\mathbb{R}^{n_i}} f_i \right)^{c_i}.$$

(BL)

Then $E$ and $F$ can be computed just with centered Gaussian functions, that is

$$E = \inf \left\{ \frac{I((g_i)_{i=1}^m)}{\prod_{i=1}^m \left( \int_{\mathbb{R}^{n_i}} g_i \right)^{c_i}} : g_i \text{ centered Gaussian on } \mathbb{R}^{n_i}, i = 1, \ldots, m \right\},$$

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and
\[ F = \sup \left\{ \frac{J((g_i)_{i=1}^m)}{\prod_{i=1}^m \left( \int_{\mathbb{R}^{n_i}} g_i \right)^{c_i}} ; \text{ } g_i \text{ centered Gaussian on } \mathbb{R}^{n_i}, i = 1, \ldots, m \right\} , \]

Moreover, if we denote by \( D \) the largest real number such that
\[ \det \left( \sum_{i=1}^m c_i B_i^* A_i B_i \right) \geq D \prod_{i=1}^m (\det A_i)^{c_i} , \]
for all \( A_i \in S^+(\mathbb{R}^{n_i}), i = 1, \ldots, m \), then
\[ E = \sqrt{D} \quad \text{and} \quad F = \frac{1}{\sqrt{D}} . \]

**Remark 1:** the hypothesis \( \sum_{i=1}^m c_i n_i = n \) is just a necessary homogeneity condition for \( E \) to be positive and for \( F \) to be finite. The condition on \( \cap \ker B_i \) ensures that \( \sum_{i=1}^m c_i B_i^* A_i B_i \) is an isomorphism. Actually, the conclusion of the theorem remains valid without this condition, but it is obvious because \( D = 0 \).

**Remark 2:** Notice that the reverse Brascamp-Lieb inequality for \( m = 2, n_1 = n_2 = n, B_1 = B_2 = B_1^* = B_2^* = I_n \) and \( c_1 = \alpha = 1 - c_2 \), where \( I_n \) is the identity map on \( \mathbb{R}^n \) and \( 0 < \alpha < 1 \), is the inequality of Prékopa-Leindler. Indeed the constant \( D \) is
\[ D = \inf_{A_1, A_2 \in S^+(\mathbb{R}^n)} \frac{\det(\alpha A_1 + (1-\alpha) A_2)}{\det A_1^\alpha \det A_2^{1-\alpha}} = 1 \]
by the arithmetic-geometric inequality. So (RBL) becomes, for all \( f, g \in L_1^+(\mathbb{R}^n) \),
\[ \int_{\mathbb{R}^n}^{\ast} \sup_{x = \alpha u + (1-\alpha) v} f^{\ast}(u) g^{1-\alpha}(v) d^n x \geq \left( \int_{\mathbb{R}^n} f \right)^{\alpha} \left( \int_{\mathbb{R}^n} g \right)^{1-\alpha} . \]

It is well-known that this inequality implies the celebrated Brunn-Minkowski theorem: for \( A, B \) compact non-void subsets of \( \mathbb{R}^n \),
\[ \text{Vol}^\frac{1}{n}(A + B) \geq \text{Vol}^\frac{1}{n}(A) + \text{Vol}^\frac{1}{n}(B) . \]

The proof of Theorem 1 is divided into lemmas. We deal first with the study of the behaviour of \( I \) and \( J \) with respect to centered Gaussian functions. We set
\[ E_g = \inf \left\{ \frac{I((g_i)_{i=1}^m)}{\prod_{i=1}^m \left( \int_{\mathbb{R}^{n_i}} g_i \right)^{c_i}} ; \text{ } g_i \text{ centered Gaussian on } \mathbb{R}^{n_i}, i = 1, \ldots, m \right\} , \]
and
\[ F_g = \sup \left\{ \frac{J((g_i)_{i=1}^m)}{\prod_{i=1}^m \left( \int_{\mathbb{R}^{n_i}} g_i \right)^{c_i}} ; \text{ } g_i \text{ centered Gaussian on } \mathbb{R}^{n_i}, i = 1, \ldots, m \right\} , \]
our aim is to prove that \( E = E_g = \sqrt{D} \) and \( F = F_g = D^{-1/2} \). We begin by a classical computation, done in [BL]; its only uses the fact that if \( M \in S^+(\mathbb{R}^k) \) then
\[ \int_{\mathbb{R}^k} \exp(-\langle x, Mx \rangle) dx = \sqrt{\frac{\pi^k}{\det M}} . \]
Lemma 1 With the notations of Theorem 1, we have
\[ F_g = \frac{1}{\sqrt{D}}. \]

Our next lemma links \(E_g\) and \(F_g\) by means of duality between quadratic forms.

Lemma 2 With the previous notations, we have
\[ E_g \cdot F_g = 1, \]
and \(E_g = 0\) if and only if \(F_g = +\infty\).

Proof: For \(i = 1, \ldots, m\), let \(A_i \in S^{+}(\mathbb{R}^{n_i})\) and let \(Q\) be the quadratic form on \(\mathbb{R}^n\) defined by
\[ Q(y) = \langle \sum_{i=1}^{m} c_i B_i^* A_i B_i y, y \rangle. \]

Let \(Q^*\) be the dual quadratic form of \(Q\), we recall that it is defined on \(\mathbb{R}^n\) by
\[ Q^*(x) = \sup \left\{ |\langle x, y \rangle|^2 ; Q(y) \leq 1 \right\}. \]

We also introduce the application \(R\) on \(\mathbb{R}^n\) such that for all \(x \in \mathbb{R}^n\),
\[ R(x) = \inf \left\{ \sum_{i=1}^{m} c_i \langle A_i^{-1} x_i, x_i \rangle ; x = \sum_{i=1}^{m} c_i B_i^* x_i \text{ and for all } i, x_i \in \mathbb{R}^{n_i} \right\}. \]

We show now that \(R = Q^*\). Indeed, assume that \(x = \sum_{i=1}^{m} c_i B_i^* x_i\) with \(x_i \in \mathbb{R}^{n_i}\) for \(i = 1, \ldots, m\), then
\[ |\langle x, y \rangle|^2 = |\langle \sum_{i=1}^{m} c_i B_i^* x_i, y \rangle|^2 = |\sum_{i=1}^{m} \langle \sqrt{c_i} A_i^{-1/2} x_i, \sqrt{c_i} A_i^{1/2} B_i y \rangle|^2. \]

By the Cauchy-Schwartz inequality, applied to the quadratic form \(\phi\) on \(\mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_m}\) defined by \(\phi(x_1, \ldots, x_m) = \sum_{i=1}^{m} \langle X_i, X_i \rangle\), one gets:
\[ |\langle x, y \rangle|^2 \leq \left( \sum_{i=1}^{m} |\sqrt{c_i} A_i^{-1/2} x_i|^2 \right) \left( \sum_{i=1}^{m} |\sqrt{c_i} A_i^{1/2} B_i y|^2 \right) = \left( \sum_{i=1}^{m} c_i \langle x_i, A_i^{-1} x_i \rangle \right) \left( \sum_{i=1}^{m} c_i B_i^* A_i B_i y, y \right). \]

In fact, one easily checks that there is equality in the previous argument if one takes
\[ y = \left( \sum_{i=1}^{m} c_i B_i^* A_i B_i \right)^{-1} x \]
and
\[ x_i = A_i B_i y \quad i = 1, \ldots, m, \]
therefore $R = Q^*$. We apply this result to our integrals of Gaussian functions. Straightforward computations give that

$$J(G_{A_1}, \ldots, G_{A_m}) \prod_{i=1}^{m} (\int G_{A_i})^{c_i} = \sqrt{\prod_{i=1}^{m} (\det A_i)^{c_i} \det Q},$$

and

$$I(G_{A_1^{-1}}, \ldots, G_{A_m^{-1}}) \prod_{i=1}^{m} (\int G_{A_i}^{-1})^{c_i} = \sqrt{\prod_{i=1}^{m} (\det A_i)^{-c_i} \det R}.$$

Using the result $R = Q^*$ and the classical duality relation $\det Q \cdot \det Q^* = 1$, one has

$$J(G_{A_1}, \ldots, G_{A_m}) \prod_{i=1}^{m} (\int G_{A_i})^{c_i} \cdot I(G_{A_1^{-1}}, \ldots, G_{A_m^{-1}}) \prod_{i=1}^{m} (\int G_{A_i}^{-1})^{c_i} = 1,$$

therefore $E_g = F_g^{-1}$. \hfill \Box

Remark: We emphasize the equivalence for $A_i \in S^+(\mathbb{R}^n)$, $i = 1, \ldots, m$, of the assertions

- $\det \left( \sum_{i=1}^{m} c_i B_i A_i B_i \right) = D \prod_{i=1}^{m} (\det A_i)^{c_i}$.
- The $m$-tuple of centered Gaussians $(G_{A_1}, \ldots, G_{A_m})$ is a maximizer for (BL).
- The $m$-tuple of centered Gaussians $(G_{A_1^{-1}}, \ldots, G_{A_m^{-1}})$ is a minimizer for (RBL).

We state now the fundamental result which, combined with the two previous lemma, will suffice to establish Theorem 1. Since the theorem is already established if $D = 0$, we assume from now on that $D$ is positive.

Lemma 3 For $i = 1, \ldots, m$, let $f_i$ and $h_i$ belong to $L^+_1(\mathbb{R}^{n_i})$ and satisfy $\int_{\mathbb{R}^{n_i}} f_i = \int_{\mathbb{R}^{n_i}} h_i$. Then

$$I(f_1, \ldots, f_m) \geq D \cdot J(h_1, \ldots, h_m).$$

In [Bar3], the author proved this result for functions of one real variable, using measure-preserving mappings; given $f$ and $h$, two non-negative functions on $\mathbb{R}$ with integral one, there exists a non-decreasing mapping $u$ such that for all $x \in \mathbb{R}$:

$$\int_{-\infty}^{u(x)} f = \int_{-\infty}^{x} h.$$

In other words, $u$ maps the probability measure of density $h$ onto the probability measure of density $f$. Our proof in the general case (i.e. for functions of several variables) is also based on measure-preserving mappings. But, in dimension larger than one, there is a large choice of such mappings between two sufficiently regular probability measures. For our purpose, the Brenier mapping (see [Bre1], [Bre2]) fits perfectly; it has the additional convenient property of deriving from a convex potential. Brenier proved its existence and uniqueness under certain integrability assumptions on the moments of the measures, which where later removed by McCann [McC1], [McC2]. Let us state the result that we need.
Theorem 2. Let $f_1, f_2$ be non-negative measurable functions on $\mathbb{R}^n$ with integral one. There exists a convex function $\phi$ on $\mathbb{R}^n$ such that the map $u = \nabla \phi$ has the following property: for every non-negative borelian function $b$ on $\mathbb{R}^n$,
\[
\int_{\mathbb{R}^n} b(u(x))f_2(x) \, dx = \int_{\mathbb{R}^n} b(x)f_1(x) \, dx.
\]
The function $\phi$ given by this theorem represents a generalized solution of the Monge-Ampère equation
\[
\det(\nabla^2 \phi(x))f_2(\nabla \phi(x)) = f_1(x).
\]
In fact, the gradient of $\phi$ is unique $f_1 \, dx$-almost everywhere. Since it will be convenient to work with strong solutions, we recall here a corollary of a theorem of Caffarelli [Caf], who has developed a regularity theory for these convex solutions.

Theorem 3. For $i = 1, 2$, let $\Omega_i$ be bounded domains of $\mathbb{R}^n$ and let $f_i$ be non-negative functions, supported on $\Omega_i$. Assume that $f_i$ and $1/f_i$ are bounded on $\Omega_i$ and that $\Omega_2$ is convex. If $f_i$, $i = 1, 2$ are Lipschitz then the Brenier mapping $\phi$ is twice continuously differentiable.

Let $\mathcal{C}_L(\mathbb{R}^n)$ be the set of functions $f \in L^+_1(\mathbb{R}^n)$ which are the restriction to some opened Euclidean ball of a positive Lipschitz function on $\mathbb{R}^n$.

Let us remark that it suffices to establish (BL) and (RBL) for functions in $\mathcal{C}_L(\mathbb{R}^n)$. We use strongly the monotonicity of the applications $I$ and $J$. By the regularity of measure, for any $f \in L^+_1(\mathbb{R}^n)$ and any $\varepsilon > 0$, there exists a function $s$, which is a positive combination of characteristic functions of compact sets, such that
\[
f \geq s \text{ and } \int f - \int s \leq \varepsilon,
\]
so it suffices to prove (RBL) for such functions. As $s$ is clearly the pointwise limit of some decreasing sequence of Lipschitz functions, it suffices to work on Lipschitz functions. Moreover, we can assume these functions to be positive (by adding some Gaussian $G/N$, where $N$ tends to infinity). Eventually, by truncation, it is enough to work with functions in $\mathcal{C}_L(\mathbb{R}^n)$.

The same kind of argument is valid for (BL). Moreover, since (BL) is equivalent to the boundedness of a multilinear operator which is, with respect to each function, a linear kernel operator (because the $B_i$’s are surjective), it clearly suffices to show (BL) for a dense subset of $L_1$.

Proof of Lemma 3. By homogeneity we can assume that $\int f_i = \int h_i = 1$ for all $i$. The previous remark allows us to work with functions $f_i, h_i$ belonging to $\mathcal{C}_L(\mathbb{R}^n)$, so that we can use Caffarelli’s regularity result and Brenier theorem. We denote by $\Omega_{h_i}$ the domain where $h_i$ is positive. We get, for $i = 1, \ldots, m$, differentiable mappings $T_i$ deriving from convex potentials and such that, for all $x \in \Omega_{h_i}$,
\[
det(dT_i(x)) \cdot f_i(T_i(x)) = h_i(x).
\]
Since $T_i$ derives from convex potential, its differential is symmetric semi-definite positive and because of the previous equation and of the non-vanishing property of $h_i$, we know that for all $x \in \Omega_{h_i}$, $dT_i(x) \in \mathcal{S}^+(\mathbb{R}^n)$. 

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We define a function $\Theta$ from $\cap_{i=1}^m B_i^{-1}(\Omega_{h_i}) \subset \mathbb{R}^n$ into $\mathbb{R}^n$ by

$$\Theta(y) = \sum_{i=1}^m c_i B_i^*(T_i(B_iy)).$$

Its differential is symmetric semi-definite positive

$$d\Theta(y) = \sum_{i=1}^m c_i B_i^*dT_i(B_iy)B_i,$$

and it is actually definite positive because:

$$\det \left( \sum_{i=1}^m c_i B_i^*dT_i(B_iy)B_i \right) \geq D \prod_{i=1}^m (\det dT_i(B_iy))^{c_i} > 0$$

In particular for all $v \neq 0$ of $\mathbb{R}^n$,

$$\langle d\Theta(y) \cdot v, v \rangle > 0$$

so $\Theta$ is injective. Denoting $S = \cap_{i=1}^m B_i^{-1}(\Omega_{h_i})$, we can write

$$\int_{\mathbb{R}^n} \prod_{i=1}^m h_{c_i}^i(B_iy) \, dy = \int_{S} \prod_{i=1}^m h_{c_i}^i(B_iy) \, dy$$

$$= \int_{S} \prod_{i=1}^m (f_i(T_i(B_iy)) \det dT_i(B_iy))^{c_i} \, dy$$

$$\leq \frac{1}{D} \int_{S} \prod_{i=1}^m f_i(T_i(B_iy))^{c_i} \det \left( \sum_{i=1}^m c_i B_i^*dT_i(B_iy)B_i \right) \, dy$$

$$\leq \frac{1}{D} \int_{S} \sup_{\Theta(y)=\sum_{i=1}^m c_i B_i^*x_i} \left( \prod_{i=1}^m f_i(x_i) \right)^{c_i} \det(d\Theta(y)) \, dy$$

$$\leq \frac{1}{D} \int_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^m c_i B_i^*x_i} \left( \prod_{i=1}^m f_i(x_i) \right)^{c_i} \, dx$$

which concludes the proof. \qed

2 Equality cases

In this section, we restrict to functions of one real variable. With the notations of Theorem 1 there are vectors $v_1, \ldots, v_m$, in $\mathbb{R}^n$ such that $\text{span}((v_i)_{i=1}^m) = \mathbb{R}^n$ and for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$,

$$B_i(x) = \langle x, v_i \rangle$$

We are going to study the best constant in inequalities (BL) and (RBL) and to characterize equality cases. We call maximizers the non-zero functions that give equality in (BL) and minimizers those that provide equality in (RBL).
2.1 The geometric structure of the problem

We introduce some notations. For a subset $K$ of $\{1, \ldots, m\}$, we denote by $E_K$ the linear span in $\mathbb{R}^n$ of the vectors $(v_k)_{k \in K}$. We will call **adapted partition** a partition $S$ of $\{1, \ldots, m\}$ such that:

$$\mathbb{R}^n = \bigoplus_{K \in S} E_K.$$  

These partitions are useful because this splitting of the space $\mathbb{R}^n$ implies a splitting of the Brascamp-Lieb inequality and of its converse, so that one just needs to work separately on each piece. We shall show first that there exists a best adapted partition.

**Proposition 1** Let $\bowtie$ be the relation on $N_m = \{1, \ldots, m\}$ defined by as follows: $i \bowtie j$ if and only if there exists a subset $K$ of $N_m$ of cardinality $n - 1$ such that both $(v_i, (v_k)_{k \in K})$ and $(v_j, (v_k)_{k \in K})$ are basis of $\mathbb{R}^n$. Let $\sim$ be the transitive completion of $\bowtie$ ($i \sim j$ means that there exists a path between $i$ and $j$ in which two consecutive elements are in relation for $\bowtie$).

Then $\sim$ is an equivalence relation and the subdivision $C$ of $N_m$ into equivalence classes for $\sim$ is the most accurate adapted partition.

**Proof:** We establish first that $C$ is more accurate than any adapted partition $S$. Let $I, J \in S$, $I \neq J$ and let $i \in I$, $j \in J$. It suffices to show that $i \bowtie j$ is impossible.

Assume precisely that $i \bowtie j$, there exists $K \subset N_m$ such that

$$e_i = (v_i, (v_k)_{k \in K}) \text{ and } e_j = (v_j, (v_k)_{k \in K})$$

form basis of $\mathbb{R}^n$. As $S$ is adapted, we have $\mathbb{R}^n = \bigoplus_{H \in S} E_H$, each of them being spanned by certain $v_i$’s. So, every basis of $\mathbb{R}^n$ which is formed of some of the $v_i$’s must contain $\dim(E_H)$ elements in $E_H$. But our basis $e_i$ and $e_j$ do not have the same number of vectors in $E_I$ because $v_i \in E_I$ and $v_j \in E_J$. Thus we have a contradiction.

We prove now that the partition $C$ is adapted to our geometric setting. Let $I$ be an equivalence class for $\sim$ and let $E_I$ be the corresponding space. Since the vectors $(v_i)_{i=1}^m$ span all $\mathbb{R}^n$, we find a permutation of indices such that $b = (v_1, \ldots, v_n)$ is a basis of $\mathbb{R}^n$ and $(v_1, \ldots, v_r)$ is a basis of $E_I$ for a certain $r \leq n$. Let us denote by $F$ the span of $v_{r+1}, \ldots, v_n$.

Let $i \in N_m$; the vector $v_i$ can be decomposed in the basis $b$:

$$v_i = \sum_{i=1}^n \alpha_i v_i.$$  

For any $j \leq n$, we notice

$$\det_{b} (v_1, \ldots, v_{j-1}, v_i, v_{j+1}, \ldots, v_n) = \alpha_j,$$

hence $\alpha_j \neq 0$ implies that $v_i$ and $v_j$ belong to neighbour basis, that is $i \sim j$. So, if $i \in I$, as $i$ can be in relation for $\bowtie$ only with elements of $I$ we have $\alpha_{r+1}, \ldots, \alpha_n = 0$. Thus $i \in I$ implies $v_i \in E_I$. By a similar argument, if $i \notin I$, $\alpha_1, \ldots, \alpha_r = 0$ and $v_i$ belongs to $F$. We have proved that $\mathbb{R}^n = \text{span}\{v_i, i \in I\} \bigoplus \text{span}\{v_i, i \notin I\}$, this is the first step of the decomposition. The result follows by induction, noticing that the relation $\sim$ can be restricted to $F$.  

As a conclusion let us notice that is suffices to study the case when the relation $\bowtie$ has only one equivalence class. In this case we say that $(\mathbb{R}^n, (v_i)_{i=1}^m)$ is irreducible.
2.2 The Gaussian case

Let \( v_1, \ldots, v_m \) be the vectors of \( \mathbb{R}^n \) defined at the beginning of this section. For \( I \subset \{1, \ldots, m\} \) of cardinal \( |I| = n \), we denote
\[
d_I = \det((v_i)_{i \in I})^2.
\]
For each \( m \)-tuple \( c = (c_i)_{i=1}^m \) of positive real, we study the constant \( D_c \) defined by
\[
D_c = \inf \left\{ \frac{\det(\sum_{i=1}^m \lambda_i v_i \otimes v_i)}{\prod_{i=1}^m \lambda_i^{c_i}} ; \lambda_i > 0, \ i = 1 \ldots m \right\}.
\]
We wish to know when it is positive and when it is achieved. We will sometimes call minimizers the \( m \)-tuples \( (\lambda_i)_{i=1}^m \) for which \( D_c \) is achieved.

The computation of the previous determinant is made possible by the Cauchy-Binet formula which we recall:

**Proposition 2** Let \( m \geq n \) be integers; let \( A \) be a \( n \times m \) matrix and let \( B \) be a \( m \times n \) matrix. For \( I \subset N_m \) of cardinality \( n \) we denote by \( A_I \) the square matrix obtained from \( A \) by keeping only the columns with indices in \( I \); we denote by \( B_I \) the square matrix obtained from \( B \) by keeping the rows with indices in \( I \). Then we have the formula
\[
\det(AB) = \sum_{|I|=n} \det(A_I) \det(B_I)
\]
where the sum is over the subsets of cardinality \( n \) of \( N_m \).

The relevance of this formula is clear from

**Corollary 1** Let \( m \geq n \) and let \( (v_1, \ldots, v_m) \) be vectors in \( \mathbb{R}^n \), then
\[
\det \left( \sum_{i=1}^m \lambda_i v_i \otimes v_i \right) = \sum_{|I|=n} \lambda_I (\det((v_i)_{i \in I}))^2,
\]
where for \( I \subset N_m \), we have set \( \lambda_I = \prod_{i \in I} \lambda_i \).

The condition for \( D_c \) to be non-zero has a rather nice geometric expression which requires some notations. For \( I \subset \{1, \ldots, m\} \), we denote by \( 1_I \) the vector of \( \mathbb{R}^m \) of coordinates \((1_I)_i = \delta_{i \in I}\) (it is the characteristic function of \( I \)). We denote by \( c \) the vector \((c_i)_{i=1}^m \). One has the following result:

**Proposition 3** The infimum \( D_c \) is positive if and only if the vector \( c \) belongs to the convex hull of the characteristic vectors \( 1_I \) of the subsets \( I \) of cardinal \( n \) such that the vectors \((v_i)_{i \in I}\) form a basis of \( \mathbb{R}^n \).

**Proof:** We shall show first that the condition is sufficient. Assume that we have a family of non-negative real numbers \((t_I)_{|I|=n}\) indexed by the subsets of cardinal \( n \) of \( \{1, \ldots, m\} \), such that
\[
t_I = 0 \text{ whenever } d_I = 0,
\]
\[
c_i = \sum_{|I|=n, i \in I} t_I, \text{ for all } i
\]
Let \( \lambda_i, i = 1, \ldots, m \) be positive. By the Cauchy-Binet formula, we have:

\[
\det\left(\sum_{i=1}^{m} \lambda_i v_i \otimes v_i\right) = \sum_{|I|=n} \prod_{t_I \neq 0} \left( \sum_{t_I = 0} \frac{\lambda_i d_I}{t_I} \right) + \sum \lambda_i d_I.
\]

The second term is non-negative. We apply the arithmetic mean-geometric mean inequality with coefficients \( t_I \) (their sum is indeed one), and for each \( i \) we gather the factors with \( \lambda_i \). Each \( \lambda_i \) will appear with an exponent equal to

\[
\sum_{t_I \neq 0} t_I,
\]

this is \( c_i \) by hypothesis. Thus we have

\[
\det\left(\sum_{i=1}^{m} \lambda_i v_i \otimes v_i\right) \geq \prod_{t_I \neq 0} \left( \sum_{t_I = 0} \frac{d_I}{t_I} \right) \prod_{i=1}^{m} \lambda_i^{c_i}.
\]

Since \( t_I \neq 0 \) implies \( d_I \neq 0 \), the constant \( D_c \) is positive.

Let us prove now that the condition is necessary. Assume that the function

\[
\Delta(\lambda_1, \ldots, \lambda_m) = \frac{\det\left(\sum_{i=1}^{m} \lambda_i v_i \otimes v_i\right)}{\prod_{i=1}^{m} \lambda_i^{c_i}}
\]

is bounded below by a positive \( D_c \) when \( \lambda_i > 0, i = 1 \ldots m \). If we take \( \lambda_i = N^{-x_i} \), where the \((x_i)\) are arbitrary, and \( N \) tends to the infinity, then \( \Delta(\lambda_1, \ldots, \lambda_m) \) is equivalent to a positive constant times \( N \) to the exponent:

\[
\sum_{i=1}^{m} x_i c_i + \max\{-\sum_{i \in I} x_i; |I| = n \text{ and } d_I \neq 0\}.
\]

As by hypothesis, \( \Delta \) cannot tend to 0, the exponent must be non-negative. Thus for all \((x_i)_{i=1}^{m} \in \mathbb{R}^m\), one has:

\[
\sum_{i=1}^{m} x_i c_i \geq \min\left\{\sum_{i \in I} x_i; |I| = n \text{ and } d_I \neq 0\right\}.
\]

Equivalently, for all \( x \in \mathbb{R}^m \),

\[
\min_{|I|=n, d_I \neq 0} \langle x, 1_I \rangle \leq \langle x, c \rangle,
\]

which can be reformulated in terms of convex cones as \( \cap_{d_I \neq 0} C_{1_I} \subset C_c \), where, for \( y \in \mathbb{R}^m \), \( C_y = \{ x \in \mathbb{R}^m; \langle x, y \rangle \geq 0 \} \). By duality of convex cones, this implies that the vector \( c \) belongs to the convex cone generated by the vectors \( 1_I \) such that \( d_I \neq 0 \). Thus there exist non-negative real numbers \((t_I)_{I,d_I \neq 0}\) such that for all \( i \leq m \),

\[
c_i = \sum_{|I|=n \text{ and } i \in I} t_I.
\]
If we make the sum on $i$ of the previous relations, we get $\sum_{d_I \neq 0} t_I = (\sum_{i=1}^m c_i)/n$. But the Hypothesis $D_c > 0$ implies that the numerator and the denominator of $\Delta$ must be of the same homogeneity degree in the variables, so $\sum_{i=1}^m c_i = n$ and we have derived that $c$ belongs to the convex hull of the $1_I$ such that $d_I \neq 0$.

**Remark:** Let $K = \{ x \in [0,1]^m \colon \sum_{i=1}^m x_i = n \}$, it is the convex hull of the vectors $(1_I)|_{|I|=n}$. By the previous result, $D_c$ is non-zero only if $c$ is in $K$. If the vectors $(v_i)$ are in generic position, $D_c \neq 0$ if and only if $c \in K$. But as the $1_I$ are clearly the only extremal points of $K$, any geometrical degeneracy (i.e. any $d_I$ equal to zero) will imply a reduction of the domain where $c$ must be.

We know that $D_c$ is positive if and only if $c$ can be written as a convex combination of certain vectors. The next proposition states that $D_c$ is achieved if and only if there exists a convex combination with some additional property.

**Proposition 4** The constant $D_c$ is achieved if and only if there exist positive numbers $(t_I)|_{|I|=n}$ and $(\lambda_i)_{i=1}^m$ such that

$$c = \sum_{|I|=n} t_I 1_I$$

and for all $I$

$$t_I = d_I \prod_{i \in I} \lambda_i.$$ 

Notice that $d_I = 0$ implies $t_I = 0$, so the result is coherent with the previous one.

**Proof:** The if part comes from a precise study of the proof of proposition 3: the inequality it gives is an equality for the $m$-tuple $(\lambda_i)$ because for all $I$, $\lambda_I d_I / t_I = 1$ so the arithmetic-geometric inequality is an equality; moreover the term $\sum I, t_I = 0 \lambda_I d_I$ is zero. The only if part is obvious by differentiation. □

We are going to rewrite our problem in the setting of Fenchel duality for convex functions in order to use the following result (see [Roc] p264):

**Proposition 5** Let $\phi$ be a l.s.c. convex function on $\mathbb{R}^m$ and let $\phi^*$ be its Fenchel conjugate, defined for $x \in \mathbb{R}^m$ by

$$\phi^*(x) = \sup_{y \in \mathbb{R}^m} \langle x, y \rangle - \phi(y).$$

Then $\phi^*(x)$, which is a supremum, is achieved if and only if $\phi^*$ is subdifferentiable at the point $x$. In particular, it is achieved when $x$ belongs to the relative interior of $\text{dom}(\phi^*) = \{ y \in \mathbb{R}^m \colon \phi^*(y) < +\infty \}$.

Let us define the function $\phi$ on $\mathbb{R}^m$ by

$$\phi(x_1, \ldots, x_m) = \log \det \left( \sum_{i=1}^m e^{t_i} v_i \otimes v_i \right),$$

the next proposition links our problem on $D$ with the study of the Fenchel conjugate of $\phi$.

**Proposition 6** 1. The function $\phi$ is convex.
2. The constant $D_c$ is equal to $\exp(-\phi^*(c))$.

3. $D_c$ is positive if and only if $c \in \text{dom}(\phi^*)$.

4. $D_c$ is achieved if and only if $\phi^*(c)$ is.

5. $\text{dom}(\phi^*)$ is equal to $K = \text{conv}\{1_I; d_I \neq 0\}$.

6. The constant $D_c$ is achieved when $c$ belongs to the relative interior of $K$.

**Proof:** The convexity of $\phi$ is a consequence of the Cauchy-Schwartz inequality: let $s, t \in \mathbb{R}^m$,

$$
\phi\left(\frac{t + s}{2}\right) = \log \left(\sum_{|I|=n} \left\{ d_I \exp\left(\sum_{i \in I} t_i\right)\right\}^{\frac{1}{2}} \left\{ d_I \exp\left(\sum_{i \in I} s_i\right)\right\}^{\frac{1}{2}}\right)
\leq \log \left(\left\{ \sum_{i \in I} d_I \exp\left(\sum_{i \in I} t_i\right)\right\}^{\frac{1}{2}} \left\{ \sum_{i \in I} d_I \exp\left(\sum_{i \in I} s_i\right)\right\}^{\frac{1}{2}}\right)
= \frac{\phi(t) + \phi(s)}{2}.
$$

The other assertions are also very simple. \hfill \square

The last statement of the previous proposition allows us to recover a result already stated in [BL].

**Corollary 2** If for all $I \subset N_m$ of cardinality $n$, $d_I = \det((v_i))_{i \in I}$ is not zero, then for all $c = (c_i)_{i=1}^m$ such that:

$$
\sum_{i=1}^m c_i = n \quad \text{and} \quad 0 < c_i < 1 \text{ for all } i,
$$

then the constant $D_c$ is achieved for a certain $(\lambda_i)_{i=1}^m$.

The following result shows that the reciprocal statement is almost true.

**Proposition 7** If $(\mathbb{R}^n, (v_i)_{i=1}^m)$ is irreducible and if $c_1 = 1$, then $D_c$ is achieved only when $m = n = 1$.

We come to unicity results: if $D$ is achieved, there is a unique minimizer, up to scalar multiplication.

**Proposition 8** Assume that $(\mathbb{R}^n, (v_i)_{i=1}^m)$ has the irreducibility property. If $(\lambda_i)_{i=1}^m$ and $(\mu_i)_{i=1}^m$ are two minimizers, then there exists $r \in \mathbb{R}$ such that for all $i$, $\lambda_i = r\mu_i$.

**Proof:** Let $t = ((t_i)_{i=1}^m)$ and $s = ((s_i)_{i=1}^m)$ such that for all $i$, one has

$$
\lambda_i = e^{t_i} \quad \text{and} \quad \mu_i = e^{s_i}.
$$

Let $\psi$ be the function on $\mathbb{R}^m$ defined for all $((x_i)_{i=1}^m)$ by

$$
\psi((x_i)) = \phi((x_i)) - \sum_{i=1}^m c_i x_i.
$$
Then \( \psi \) reaches its minimum at the points \( t, s \) and also at \((t + s)/2\) because it is convex. So we have
\[
\frac{\phi(t) + \phi(s)}{2} = \phi \left( \frac{t + s}{2} \right),
\]
and there must be equality in the Cauchy-Schwartz inequality in the proof of proposition \( \square \). Hence, there exists \( a \in \mathbb{R} \) such that for all \( I, |I| = n, \)
\[
d_I \exp \left( \sum_{i \in I} t_i \right) = a \cdot d_I \exp \left( \sum_{i \in I} s_i \right).
\]
In particular, if \( d_I \neq 0 \), one has
\[
\prod_{i \in I} \left( \frac{\lambda_i}{\mu_i} \right) = a.
\]
Let \( i, j \in N_m \) such that \( i \triangleright j \); by definition, there exists \( K \subset N_m \) of cardinality \( n - 1 \), such that \( d_{(i) \cup K} \) and \( d_{(j) \cup K} \) are both non-zero. So, we have
\[
\prod_{l \in (i) \cup K} \left( \frac{\lambda_l}{\mu_l} \right) = \prod_{l \in (j) \cup K} \left( \frac{\lambda_l}{\mu_l} \right),
\]
and after simplification
\[
\frac{\lambda_i}{\mu_i} = \frac{\lambda_j}{\mu_j}.
\]
By the irreducibility property (see Proposition \( \square \)), this implies \( \frac{\lambda_1}{\mu_1} = \cdots = \frac{\lambda_m}{\mu_m} \).

\section{2.3 The general case}

We have studied existence and uniqueness of centered Gaussian maximizers for (BL) and minimizers for (RBL), we turn to the general study. As explained before, we may assume that \((\mathbb{R}^n, (v_i)_{i=1}^m)\) is irreducible. The behaviour of extremal functions is very different for \( n = 1 \) and for \( n \geq 2 \).

\subsection{2.3.1 The case \( n = 1 \)}

If \( n = 1 \), then \( n_i = 1 \) for all \( i \leq m \), the condition on \((c_i)_{i=1}^m\) is just \( \sum_{i=1}^m c_i = 1 \), and the \( v_i \)'s are just real numbers. The inequality (BL) is nothing else than Hölder’s inequality for the functions \( x \mapsto f_i(v_i x) \), whereas (RBL) is the Prékopa-Leindler inequality for \( x \mapsto f_i(x/v_i) \).

The equality cases can be settled from our proof; we will not do it because they are well-known: if \( \sum_{i=1}^m c_i = 1 \), and \( f_i \in L_1^+(\mathbb{R}) \), \( i = 1, \ldots, m \) are non identically zero, then
\[
\int_{\mathbb{R}} \prod_{i=1}^m f_i^{c_i}(x) \, dx = \prod_{i=1}^m \left( \int_{\mathbb{R}} f_i \right)^{c_i}
\]
holds if and only if
\[
\frac{f_1}{\int_{\mathbb{R}} f_1} = \cdots = \frac{f_m}{\int_{\mathbb{R}} f_m}.
\]
Under the same assumptions,
\[
\int \sup_{\sum c_i x_i = x} \prod_{i=1}^{m} f_i^{c_i}(x_i) \, dx = \prod_{i=1}^{m} \left( \int f_i \right)^{c_i}
\]
holds if and only if there exists \( (y_i)_{i=1}^{m} \in \mathbb{R}^m \) such that
\[
\frac{f_1(\cdot - y_1)}{\int_{\mathbb{R}} f_1} = \cdots = \frac{f_m(\cdot - y_m)}{\int_{\mathbb{R}} f_m}
\]
is a log-concave function.

2.3.2 The case \( n \geq 2 \)

We prove that if there is a centered Gaussian extremizer, then up to dilatation and scalar multiplication, it is the only extremizer.

**Theorem 4** Let \( n \geq 2 \) and let \( (\mathbb{R}^n, (v_i)_{i=1}^{m}) \) be irreducible. Let \( (c_i)_{i=1}^{m} \) and \( (\lambda_i)_{i=1}^{m} \) be positive numbers such that \( D_c \) is achieved for \( (\lambda_i)_{i=1}^{m} \):

\[
\det \left( \sum_{i=1}^{m} \lambda_i v_i \otimes v_i \right) = D_c \prod_{i=1}^{m} \lambda_i^{c_i}.
\]

Then \( (h_i)_{i=1}^{m} \) is a maximizer for (BL) if and only if there exist \( a > 0 \), \( (\alpha_i)_{i=1}^{m} \) positive and \( y \in \mathbb{R}^n \) such that for all \( i \) and all \( t \in \mathbb{R} \),

\[
h_i(t) = \alpha_i \exp(-\lambda_i (at - \langle y, v_i \rangle)^2).
\]  

(1)

The \( m \)-tuple \( (h_i)_{i=1}^{m} \) is a minimizer for (RBL) if and only if there exist \( b > 0 \), \( (\beta_i)_{i=1}^{m} \) positive and \( (t_i)_{i=1}^{m} \) real such that for all \( i \) and all \( t \in \mathbb{R} \),

\[
h_i(t) = \beta_i \exp(-(bt - t_i)^2/\lambda_i).
\]

**Proof:** By Lemma \([1]\) and by the proof of Lemma \([2]\), we know that \( (G_{\lambda_i})_{i=1}^{m} \) is a maximizer for (BL) and \( (G_{\lambda_i^{-1}})_{i=1}^{m} \) a minimizer for (RBL), so by simple changes of variables in \( \mathbb{R}^n \), one can check that the previous functions are extremizers.

Let \( (h_i)_{i=1}^{m} \) be a maximizer for (BL) and \( (f_i)_{i=1}^{m} \) is a minimizer for (RBL). We may assume that \( f_i, h_i \) are positive and continuous for all \( i \). Indeed by the following lemma (which was communicated to me by K. Ball) we know that \( (h_i * G_{\lambda_i})_{i=1}^{m} \) is a positive and continuous maximizer for (BL). If we know that it is Gaussian, then so is \( (h_i)_{i=1}^{m} \) by the properties of the Fourier transform. The same argument is relevant for (RBL).

**Lemma 4** If \( (f_i)_{i=1}^{m} \) and \( (g_i)_{i=1}^{m} \) are maximizers for (BL), then so is \( (f_i * g_i)_{i=1}^{m} \).

If \( (f_i)_{i=1}^{m} \) and \( (g_i)_{i=1}^{m} \) are minimizers for (RBL), then so is \( (f_i * g_i)_{i=1}^{m} \).

A proof of the first part of this lemma is written in \([2]\), the proof of the second part is very similar. Notice that this lemma is valid for the multidimensional version of the inequalities.

We show now that if \( (h_i)_{i=1}^{m} \) is a positive continuous maximizer for (BL), then it has to be of the form \([1]\); the proof for (RBL) is analogous and a bit simpler. We study precisely the proof.
of Lemma 3 applied with \((h_i)_{i=1}^m\) being the maximizer we study and \((f_i)_{i=1}^m\) being the particular minimizer for (RBL) that we know by hypothesis, namely

\[
f_i(t) = \exp \left( -\frac{x^2}{\lambda_i} \right).
\]

Since our functions are positive, the change of variables \(T_i\)'s are increasing differentiable bijections of \(\mathbb{R}\), such that for all \(t \in \mathbb{R}\),

\[
T'_i(t) \cdot f_i(T_i(t)) = h_i(t).
\]

There must be equality in every step of the proof. In particular, for all \(y \in \mathbb{R}^n\), one has

\[
\det \left( \sum_{i=1}^m T'_i(\langle y, v_i \rangle) v_i \otimes v_i \right) = D_c \prod_{i=1}^m (T'_i(\langle y, v_i \rangle))^{c_i}.
\]

By irreducibility and proposition 3, one gets for all \(y \in \mathbb{R}^n\),

\[
\frac{T'_i(\langle y, v_i \rangle)}{\lambda_i} = \ldots = \frac{T_m'(\langle y, v_i \rangle)}{\lambda_m}.
\]

Since \(n \geq 2\), for all \(i \leq m\) there exists \(j \leq m\) such that \(v_i\) and \(v_j\) are not colinear; so there exits \(z \in \mathbb{R}\) such that \(\langle z, v_i \rangle = 1\) and \(\langle z, v_i \rangle = 0\). The previous relation for \(y = tz\) says that for all \(t \in \mathbb{R}\)

\[
\frac{T'_i(t)}{\lambda_i} = \frac{T'_j(0)}{\lambda_j}.
\]

Consequently, there exist \(a > 0\) and \((s_i)_{i=1}^m\) real such that for all \(i\) and for all \(t \in \mathbb{R}\),

\[
T'_i(t) = a\lambda_i t + s_i
\]

and by the change of variable formula between \(h_i\) and \(f_i\) we get

\[
h_i(t) = T'_i(t) \exp \left( -\frac{T^2_i(t)}{\lambda_i} \right)
= \mu_i \exp(-\lambda_i(at - t_i)^2).
\]

for some positive \((\mu_i)\) and some real \((t_i)\).

It remains to find which translates of a centered Gaussian maximizer are still maximizers. Let \((g_i)_{i=1}^m\) be a maximizer,

\[
g_i(t) = \exp(-\lambda_i t^2),
\]

and let \(x = (x_i)_{i=1}^m \in \mathbb{R}^m\) and for \(i \leq m\), \(h_i(t) = g_i(t - x_i)\). Let us consider \(\mathbb{R}^m\) with the Euclidean metric given by

\[
N^2(w) = \sum_{i=1}^m c_i \lambda_i w_i^2,
\]

and the subspace

\[
K = \{ \langle y, v_i \rangle_{i=1}^m : y \in \mathbb{R}^n \}.
\]

Let \(s\) be the orthogonal projection of \(x\) onto \(A\). Then there exists \(z \in \mathbb{R}^n\) satisfying \(s_i = \langle z, v_i \rangle\) for all \(i\); moreover by the Pythagorean Theorem

\[
N((\langle y, v_i \rangle - x_i)_{i=1}^m) \geq N((\langle y, v_i \rangle - \langle z, v_i \rangle)_{i=1}^m)
\]

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with equality only if \( x \) belongs to \( A \), that is \( x = s \). Thus \( J((h_i)_{i=1}^m) \leq J((g_i)_{i=1}^m) \), with equality only if \( x_i = \langle z, v_i \rangle \) for all \( i \).

**Remark:** There are, for \( n \geq 2 \), some remaining questions. If there is no centered Gaussian maximizer, is there any maximizer at all? The answer seems to be no: if \((f_i)_{i=1}^m\) is a maximizer then by the Brascamp-Lieb-Luttinger inequality [BLL] so is \((f_i^*)_{i=1}^m\), where \( f^* \) is the symmetric rearrangement. As K. Ball remarked, for all \( k \) integer

\[
\left( \sqrt{k} f_i^* \cdots f_i^*(\sqrt{k} \cdot) \right)_{i=1}^m \text{, \( k \) times}
\]

is also a maximizer; moreover, but under some integrability assumptions, it tends towards a centered Gaussian \( m \)-tuple by the Central Limit Theorem.

Notice that our method gives the answer when there are positive continuous maximizers for (BL) and positive continuous minimizers for the corresponding (RBL). The study the equality case of Lemma 3 shows that the constant \( D \) must be achieved, so there is a centered Gaussian maximizer.

Some arguments based on the equality case in the Minkowski inequality (see [Lie]) might help to solve the question.

### 3 Applications to convex geometry

#### 3.1 Dimension one

K. Ball noticed that an additional geometrical hypothesis on the vectors \((v_i)\), which is frequent in convexity, allows an easy computation of the optimal constants in the Brascamp-Lieb inequality. For completeness, we begin by the proof of his observation.

**Proposition 9** Let \( m \geq n \), let \( v_1, \ldots, v_m \) be vectors in \( \mathbb{R}^n \) such that \( \sum_{i=1}^m v_i \otimes v_i = I_n \), where \( I_n \) stands for the identity map; then for every \( m \)-tuple \((\lambda_i)_{i=1}^m\) of positive numbers

\[
\det \left( \sum_{i=1}^m \lambda_i v_i \otimes v_i \right) \geq \prod_{i=1}^m |\lambda_i v_i|^2.
\]

There is equality when \( \lambda_1 = \cdots = \lambda_m \).

**Proof:** By the Cauchy-Binet formula, we have

\[
1 = \det I_n = \det \left( \sum_{i=1}^m v_i \otimes v_i \right) = \sum_{|I|=n} d_I.
\]

Hence we can use the arithmetic-geometric inequality with coefficients \( d_I \):

\[
\det \left( \sum_{i=1}^m \lambda_i v_i \otimes v_i \right) = \sum_{|I|=n} \lambda_I d_I \geq \prod_{|I|=n} \lambda_I^{d_I}.
\]
Each $\lambda_i$ appears with the total exponent $\sum_{I, i \in I} d_I$. But by Corollary 1 applied to the $m$-tuple $(v_1, \ldots, v_{i-1}, 0, v_{i+1}, \ldots, v_m)$ we get:

$$
\sum_{I, i \in I} d_I = \sum_I d_I - \sum_{I, i \notin I} d_I
= 1 - \det(v_1 \otimes v_1 + \cdots + v_{i-1} \otimes v_{i-1} + v_{i+1} \otimes v_{i+1} + \cdots + v_m \otimes v_m)
= 1 - \det(I_n - v_i \otimes v_i) = |v_i|^2.
$$

Ball’s version of the Brascamp-Lieb inequality and the corresponding reverse version are as follows:

**Theorem 5** Let $m \geq n$, let $(u_i)_{i=1}^m$ be unit vectors in $\mathbb{R}^n$ and let $(c_i)_{i=1}^m$ be positive real numbers such that

$$
\sum_{i=1}^m c_i u_i \otimes u_i = I_n.
$$

Then for all $f_i \in L_1^+(\mathbb{R})$, $i = 1, \ldots, m$ one has

$$
\int_{\mathbb{R}^n} \prod_{i=1}^m f_i^{c_i}(\langle x, u_i \rangle) \, dx \leq \prod_{i=1}^m \left( \int f_i \right)^{c_i},
$$

and

$$
\int_{\mathbb{R}^n} \sup_{x = \sum c_i \theta_i u_i} \prod_{i=1}^m f_i^{c_i}(\theta_i) \, dx \geq \prod_{i=1}^m \left( \int f_i \right)^{c_i}.
$$

The equality case is completely settled: the space $\mathbb{R}^n$ is an orthogonal sum of irreducible subspaces. On the irreducible subspaces of dimension one there is equality for (BL) if and only if the functions are equal up to scalar multiplication and, for (RBL), if and only if all the functions are equal up to multiplication and translation to a common log-concave function. On irreducible subspaces of dimension more than or equal to 2, there is equality if and only if the functions are (up to scalar multiplications, up to translations and only coherent translations in the direct form) equal to a common centered Gaussian function. We state a useful corollary which allows to solve the equality case in the geometric applications due to K. Ball.

**Corollary 3** Let $m \geq n$, let $(u_i)_{i=1}^m$ be $m$ different unit vectors in $\mathbb{R}^n$ and let $(c_i)_{i=1}^m$ be positive real numbers such that

$$
\sum_{i=1}^m c_i u_i \otimes u_i = I_n.
$$

If $(f_i)_{i=1}^m$ are non-identically-zero functions in $L_1^+(\mathbb{R})$ such that none of them is a Gaussian and

$$
\int_{\mathbb{R}^n} \prod_{i=1}^m f_i^{c_i}(\langle x, u_i \rangle) \, dx = \prod_{i=1}^m \left( \int f_i \right)^{c_i},
$$

then $(u_i)_{i=1}^m$ is an orthonormal basis of $\mathbb{R}^n$. 

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We give an application: for $K \subset \mathbb{R}^n$ a convex body, we denote by $\nu r(K)$ the ratio of the volume of $K$ by the volume of the maximal volume ellipsoid contained in $K$ (called the John’s ellipsoid, see [Joh]). In [Bal1] and [Bal3], K. Ball proved by means of the Brascamp-Lieb inequality that simplices have maximal volume ratio and that among symmetric bodies, parallelotopes have. By the previous corollary, we can answer the question of equality cases. We denote by $Q_n$ the unit cube and by $\Delta_n$ the regular simplex.

**Proposition 10** Let $K \subset \mathbb{R}^n$ a convex body.

- If $K$ is symmetric and $\nu r(K) = \nu r(Q_n)$ then $K$ is a parallelotope.
- If $\nu r(K) = \nu r(\Delta_n)$ then $K$ is a simplex.

### 3.2 Larger dimensions

We obtain a multidimensional generalization of Ball’s version of the Brascamp-Lieb inequality and its converse. The estimate for Gaussians is a generalization of proposition 9.

**Theorem 6** Let $m, n$ be integers. For $i = 1, \ldots, m$ let $E_i$ be a subspace of $\mathbb{R}^n$ of dimension $n_i$ and let $P_i$ be the orthogonal projection onto $E_i$ (on each $E_i$ there is a Lebesgue measure compatible with the induced Euclidean structure). Assume that there exist positive numbers $(c_i)^m_{i=1}$ satisfying

$$\sum_{i=1}^m c_i P_i = I_n,$$

Then if for $i = 1, \ldots, m$, $f_i$ is a non-negative integrable function on $E_i$, one has

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i^{c_i}(P_i x) \, d^n x \leq \prod_{i=1}^m \left( \int_{E_i} f_i \right)^{c_i},$$

and

$$\int_{\mathbb{R}^n} \sup_{x=\sum_{i=1}^m c_i x_i, x_i \in E_i} \prod_{i=1}^m f_i^{c_i}(x_i) \, d^n x \geq \prod_{i=1}^m \left( \int_{E_i} f_i \right)^{c_i}.$$

**Remark:** When the $f_i$ are taken to be characteristic functions of sets, the reverse inequality provides a Brunn-Minkowski type result for convex bodies which do not have full dimension: if $K_i \subset E_i$ then

$$\text{Vol}_n \left( \sum_{i=1}^m c_i K_i \right) \geq \prod_{i=1}^m \left( \text{Vol}_{E_i}(K_i) \right)^{c_i}. $$

When the sets $K_i$ are segments, we recover a lower estimate for the volume of zonoids, already proved by Ball [Bal2] by a geometrical inductive method.
4 Link with Young’s convolution inequality

In [Bar2], the author gave a proof of Young’s inequality and its converse which is also based on measure-preserving mappings. Following an idea of [BL], we illustrate how a generalization of Young’s inequality and its converse contains Ball’s version of (BL) and the corresponding form of (RBL). For $t > 1$, we define the conjugate number $t'$ by $1/t + 1/t' = 1$.

**Theorem 7** Let $m \geq n$ be integers, let $V$ be an orthogonal $m \times m$ matrix and denote by $(v_i)_{i=1}^m$ its rows. Let $M$ be the $(m-n) \times m$ submatrix of $V$ formed of the last $m-n$ columns. Let $r$ and $(p_i)_{i=1}^m$ be larger than 1 and such that $\sum_{i=1}^m 1/p_i = n + (m-n)/r$; let $D = D_{r,p_i}$ be the largest constant such that for all positive $(\lambda_i)_{i=1}^m$, one has

$$\det (M \cdot \text{diag}(\lambda_i)) \cdot M \geq D \prod_{i=1}^m \lambda_i^{r'/p_i}.$$ 

Then for every continuous positive integrable functions on $\mathbb{R}$, $(f_i)_{i=1}^m$ and $(F_i)_{i=1}^m$ such that for all $i$, $\int f_i = \int F_i$, one has

$$\left[ \int \left[ \int \prod_{i=1}^m f_i^{1/p_i}((x,v_i)) \, dx_1 \cdots dx_n \right]^r \, dx_{n+1} \cdots dx_m \right]^{1/r} \leq D^{-1/r'} \int \left[ \int \prod_{i=1}^m F_i^{r/p_i}(\langle X,v_i \rangle) \, dX_m \cdots dX_{n+1} \right]^{1/r} \, dX_n \cdots dX_1.$$ 

**Proof:** For all $i$, there exists a positive differentiable increasing map $T_i$ satisfying, for all $s \in \mathbb{R}$

$$\int_{-\infty}^{T_i(s)} f_i = \int_{-\infty}^s F_i$$

and by differentiation:

$$T_i'(s) f_i(T_i(s)) = F_i(s).$$

(2)

We consider the change of variable $\Theta$ in $\mathbb{R}^m$ given by $V^{-1}(T_1 \otimes \cdots \otimes T_m)V$. More precisely, $x = \Theta(X)$ means that for all $i$,

$$\langle v_i, x \rangle = T_i(\langle v_i, X \rangle).$$

The application $\Theta$ is clearly bijective, its differential at a point $X \in \mathbb{R}^n$ is

$$d\Theta(X) = V^{-1} \text{diag}(T_i'(\langle v_i, X \rangle))V = V \text{diag}(T_i'(\langle v_i, X \rangle))V,$$

so its jacobian is simply

$$\prod_{i=1}^m T_i'(\langle v_i, X \rangle).$$

We want an upper estimate of the integral

$$I = \left[ \int \left[ \int \prod_{i=1}^m f_i^{1/p_i}((x,v_i)) \, dx_1 \cdots dx_n \right]^r \, dx_{n+1} \cdots dx_m \right]^{1/r},$$

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which is finite (we may suppose that all our functions are dominated by some Gaussian function). Hence there exists a positive function $h \in L^r(\mathbb{R}^{m-n})$ such that $\|h\|_r = 1$ and

$$I = \int \prod_{i=1}^{m} f_i^{1/p_i}(\langle x, v_i \rangle) h(x_{n+1}, \ldots, x_m) \, dx_1 \cdots dx_m.$$

By the change of variables $x = \Theta(X)$ and by relations (2), we get

$$I = \int \prod_{i=1}^{m} F_i^{1/p_i}(\langle x, v_i \rangle) h(A(x)) \prod_{i=1}^{m} T'_i(\langle v_i, X \rangle) \, dX_1 \cdots dX_m$$

$$= \int \left[ \int \prod_{i=1}^{m} F_i^{1/p_i}(\langle X, v_i \rangle) h(A(x)) \prod_{i=1}^{m} (T'_i(\langle v_i, X \rangle))^{1/p'_i} \, dX_m \cdots dX_{n+1} \right] dX_n \cdots dX_1,$$

where $A$ is defined by

$$A(x) = \left( \sum_{i=1}^{m} M_{n+1,i} T_i(\langle v_i, X \rangle), \ldots, \sum_{i=1}^{m} M_{m,i} T_i(\langle v_i, X \rangle) \right).$$

For fixed $X_1, \ldots, X_n$, we consider $A$ as a function of $X_{n+1}, \ldots, X_m$. Its differential is

$$M \cdot \text{diag}(T'_i(\langle v_i, X \rangle)) \cdot M,$$

so we know, by hypothesis, a lower estimate for its Jacobian

$$\det dA(X) \geq D \prod_{i=1}^{m} \left( T'_i(\langle v_i, X \rangle) \right)^{r'/p'_i}.$$

By Hölder’s inequality with parameters $(r, r')$ applied to the inner integral of the previous expression for $I$ and by the lower estimate for the Jacobian of $A$, we get

$$I \leq D^{-1/r'} \int \left[ \int \prod_{i=1}^{m} F_i^{r/p_i}(\langle X, v_i \rangle) \, dX_m \cdots dX_{n+1} \right]^{1/r}$$

$$\left[ \int h^{r'}(A(X)) \det(dA(X)) \, dX_m \cdots dX_{n+1} \right]^{1/r'} \, dX_n \cdots dX_1.$$

Since for fixed $X_1, \ldots, X_n$, $A$ is injective (indeed its differential is symmetric definite positive),

$$\int h^{r'}(A(X)) \det(dA(X)) \, dX_m \cdots dX_{n+1} \leq \int h^{r'} = 1,$$

so

$$I \leq D^{-1/r'} \left[ \int \prod_{i=1}^{m} F_i^{r/p_i}(\langle X, v_i \rangle) \, dX_m \cdots dX_{n+1} \right]^{1/r} \, dX_n \cdots dX_1.$$

$\square$
This theorem contains both (BL) and (RBL) in the form stated in Theorem 5. We have positive \((c_i)_{i=1}^m\) and unit vectors \((u_i)_{i=1}^m\) in \(\mathbb{R}^n\), linked by
\[
\sum_{i=1}^m c_i u_i \otimes u_i = I_n.
\]

We begin by a very simple fact:

**Lemma 5** Let \(m \geq n\) and let \((c_i)_{i=1}^m\) be positive \((u_i)_{i=1}^m\) be unit vectors in \(\mathbb{R}^n\) with the relation:
\[
\sum_{i=1}^m c_i u_i \otimes u_i = I_n,
\]
then there exists an orthonormal basis \((v_i)_{i=1}^m\) of \(\mathbb{R}^m\) such that for all \(i\),
\[
P(v_i) = \sqrt{c_i} u_i,
\]
where \(P\) stands for the projection from \(\mathbb{R}^m\) onto \(\mathbb{R}^n\) which keeps only the first \(n\) coordinates of a vector.

We apply Theorem 7 with the vectors \((v_i)_{i=1}^m\) given by the lemma and for special values of the parameters: namely for \(R \gg 1\) and very close to one, we chose
\[
p_i = \frac{1}{R c_i}
\]
and
\[
r = \frac{m-n}{\sum_{i=1}^m 1/p_i - n} = \frac{m-n}{n(R-1)}.
\]
Then we take the limits when \(R\) tends to 1 in the inequality provided by the theorem.

Let us describe the asymptotic behaviour of all the related quantities when \(R \to 1\). It is clear that
\[
\frac{1}{p_i} \to c_i, \quad \frac{1}{p_i'} \to 1 - c_i, \quad r \to \infty \quad \text{and} \quad r' \to 1.
\]

We define \(m\) vectors \((w_i)_{i=1}^m\) in \(\mathbb{R}^{m-n}\) as follows: the coordinates of \(w_i\) are the last coordinates of \(v_i\). With these notations, \(D_{r,p_i}\) is the largest constant such that for all positive \((\lambda_i)_{i=1}^m\),
\[
\det \left( \sum_{i=1}^m \lambda_i w_i \otimes w_i \right) \geq D_{r,p_i} \prod_{i=1}^m \lambda_i^{r'/p_i'}.
\]

Since we have the orthogonal decomposition \(v_i = \sqrt{c_i} u_i + w_i\), we know that \(|w_i|^2 = 1 - c_i\). Moreover, \((w_i)_{i=1}^m\) being the orthogonal projection of \(v_i\) and orthonormal basis, the following relation holds
\[
\sum_{i=1}^m w_i \otimes w_i = I_{m-n},
\]
so we get from Lemma 9, for all positive $(\lambda_i)_{i=1}^m$,
\[
\det \left( \sum_{i=1}^m \lambda_i w_i \otimes w_i \right) \geq \prod_{i=1}^m \lambda_i^{1-c_i}.
\]
As $r'/p_i \to 1 - c_i$ when $R$ tends to one it follows that $D_{r,p_i} \to 1$.

We study now the quantities involving the functions. The main point is that when a function decreases fast enough, its $r$-norm tends to its essential supremum when $r$ tends to infinity. We introduce some more notation: for $(x_1, \ldots, x_m)$ we set $y = (x_1, \ldots, x_n)$ and $z = (x_{n+1}, \ldots, x_m)$. When $R \to 1$, $r \to \infty$, and
\[
\left[ \int \left[ \int \prod_{i=1}^m f_i^{1/p_i}(\langle x, v_i \rangle) \, dx_1 \cdots dx_n \right]^r \, dx_{n+1} \cdots dx_m \right]^{1/r}
\]
tends to
\[
\sup_{z \in \mathbb{R}^{m-n}} \int \prod_{i=1}^m f_i^{c_i}(\sqrt{c_i} \langle y, u_i \rangle + \langle z, w_i \rangle) \, dy
\]
which is larger than
\[
\int \prod_{i=1}^m f_i^{c_i}(\langle y, u_i \rangle) \, dy.
\]
On the other hand, the quantity
\[
\left[ \int \left[ \int \left( \prod_{i=1}^m F_i^{1/p_i}(\langle x, v_i \rangle) \right)^r \, dx_m \cdots dx_{n+1} \right]^{1/r} \, dx_n \cdots dx_1
\]
tends to
\[
\int \sup_z \prod_{i=1}^m F_i^{c_i}(\langle x, v_i \rangle) \, dx_n \cdots dx_1,
\]
where the supremum is on the vectors $z = (x_{n+1}, \ldots, x_m) \in \mathbb{R}^{m-n}$. Noticing that the numbers $\langle x, v_i \rangle$ are just the coordinates of $x = y + z$ in the orthonormal basis $(v_i)_{i=1}^m$, and since the existence of a $z \in \mathbb{R}^{m-n}$ such that $y + z = \sum_{i=1}^n \alpha_i v_i$ is equivalent to $y = \sum_{i=1}^n \sqrt{c_i} \alpha_i u_i$ (by taking orthogonal projection onto the first $n$ coordinates), the previous integral is just
\[
\int \sup_{y \in \mathbb{R}^n} \prod_{i=1}^n F_i^{c_i}(\alpha_i) \, dy,
\]
\[
= \int \sup_{y \in \mathbb{R}^n} \prod_{i=1}^n F_i^{c_i}(\sqrt{c_i} \theta_i) \, dy.
\]
Hence the limiting case of Theorem 7 states that for all $f_i, F_i$, one has
\[
\frac{\prod_{i=1}^m \int f_i^{c_i}(\langle y, u_i \rangle) \, dy}{\prod_{i=1}^m \left( \int f_i^{c_i} \right)} \leq \int \sup_{y \in \mathbb{R}^n} \prod_{i=1}^m F_i^{c_i}(\theta_i) \, dy
\]
\[
\leq \int \prod_{i=1}^m \left( \int f_i^{c_i} \right)
\]
\[\]
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When the $F_i$’s are identical centered Gaussians this is Ball’s version of (BL), and when the $f_i$’s are identical centered Gaussians it is the corresponding version of (RBL).

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