A comparison of Noether charge and Euclidean methods for Computing the Entropy of Stationary Black Holes

Vivek Iyer and Robert M. Wald

Enrico Fermi Institute and Dept. of Physics

University of Chicago

5640 S. Ellis Ave.

Chicago, IL 60637

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Abstract

The entropy of stationary black holes has recently been calculated by a number of different approaches. Here we compare the Noether charge approach (defined for any diffeomorphism invariant Lagrangian theory) with various Euclidean methods, specifically, (i) the microcanonical ensemble approach of Brown and York, (ii) the closely re-
lated approach of Bañados, Teitelboim, and Zanelli which ultimately expresses black hole entropy in terms of the Hilbert action surface term, (iii) another formula of Bañados, Teitelboim and Zanelli (also used by Susskind and Uglum) which views black hole entropy as conjugate to a conical deficit angle, and (iv) the pair creation approach of Garfinkle, Giddings, and Strominger. All of these approaches have a more restrictive domain of applicability than the Noether charge approach. Specifically, approaches (i) and (ii) appear to be restricted to a class of theories satisfying certain properties listed in section 2; approach (iii) appears to require the Lagrangian density to be linear in the curvature; and approach (iv) requires the existence of suitable instanton solutions. However, we show that within their domains of applicability, all of these approaches yield results in agreement with the Noether charge approach. In the course of our analysis, we generalize the definition of Brown and York’s quasilocal energy to a much more general class of diffeomorphism invariant, Lagrangian theories of gravity. In an appendix, we show that in an arbitrary diffeomorphism invariant theory of gravity, the “volume term” in the “off-shell” Hamiltonian associated with a time evolution vector field $t^a$ always can be expressed as the spatial integral of $t^a C_a$, where $C_a = 0$ are the constraints associated with the diffeomorphism invariance.

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1 Introduction

In two recent papers [1], [2], the first law of black hole mechanics was derived for an arbitrary diffeomorphism invariant Lagrangian theory of gravity. A simple, general expression for the entropy of a black hole was thereby obtained, namely

\[ S = 2\pi \int_{\mathcal{H}} Q[t] \]  \hspace{1cm} (1)

where \( \mathcal{H} \) denotes the bifurcation surface of the horizon, \( t^a \) is the horizon Killing field, normalised to have unit surface gravity, and \( Q \) is the Noether charge \((n-2)\)-form (see section (2) below).

It was shown in [2] that the entropy so defined is also given by

\[ S = -2\pi \int_{\mathcal{H}} E_{abcd}^R \epsilon_{ab} \epsilon_{cd} \]  \hspace{1cm} (2)

where \( \epsilon_{ab} \) is the binormal to \( \mathcal{H} \) and \( E_{abcd}^R \) is the functional derivative of the Lagrangian with respect to the Riemann tensor, \( R_{abcd} \), with the metric and connection held fixed, i.e., \( E_{abcd}^R \) is the equation of motion for \( R_{abcd} \) which would be obtained from the Lagrangian if \( R_{abcd} \) were treated as a field independent of the metric.

The relationship between the Noether charge approach to calculating the entropy of a black hole and a Euclidean approach first given in [3] was already analyzed in [1]. However, recently a variety of other Euclidean approaches to calculating black hole entropy have been given. These approaches appear to bear little, if any, resemblance to the Noether charge approach and, as presented, they usually have had their range of applicability restricted to general rela-
tivity. Although these Euclidean methods agree with the Noether charge method in yielding the Bekenstein-Hawking formula \( S = A/4 \) in the case of general relativity, this fact alone does not go far towards establishing that they are equivalent to (or even directly related to) the Noether charge method.

The purpose of this paper is to examine the relationship between the Noether charge method and (i) the microcanonical ensemble approach of Brown and York [4, 5] for general relativity, (ii) the Hilbert action surface term formula of Bañados, Teitelboim, and Zanelli [6], applicable to general relativity and Lovelock gravity, (iii) the conical deficit angle formula of Bañados, Teitelboim and Zanelli [6] and Susskind and Uglum [7], applicable to general relativity, and (iv) the pair creation approach of Garfinkle, Giddings, and Strominger [9], given for a particular process in general relativity but applicable, in principle, to an arbitrary theory. One of our main goals is to generalize and widen the domain of applicability of these approaches as much as possible so that they can be compared in a meaningful way with the Noether charge approach.

In sections 2 and 3, we will examine the microcanonical ensemble approach of Brown and York [4, 5]. In section 2, we will encounter little difficulty in generalizing their notion of quasilocal energy to an arbitrary, diffeomorphism invariant, Lagrangian theory for which appropriate boundary conditions have been specified for variations of the action. However, in order for the quasilocal energy to have properties suitable for defining a microcanonical action, we need to restrict consideration to theories satisfying certain additional properties specified in section 2. It is not clear to us how restrictive these additional properties are, but we explicitly verify at the end of section 2 that they are satisfied by general relativity. In section 3, we define a microcanonical action and show that for the class of theories satisfying these properties the entropy of a black hole as computed from this microcanonical action always agrees with eq.(1).

The starting point of the Euclidean approach given in [6] is essentially the same as that
of [4, 5], but a formula (valid for general relativity and, more generally, Lovelock gravity) is then presented which expresses black hole entropy as the limit – as a suitable \((n - 1)\)-surface approaches the horizon bifurcation \((n - 2)\)-surface, \(\mathcal{H}\) – of the surface term appearing in the Hilbert action of the theory. In section 4, we give a simple derivation of this formula for the general class of theories satisfying the properties listed in section 2.

In section 5, we briefly examine the conical deficit angle formula of Bañados, Teitelboim and Zanelli [6], which also has been used by Susskind and Uglum [7]. It appears that the range of validity of this formula for black hole entropy is fundamentally limited to theories whose Lagrangians are at most linear in the Riemann curvature. The relationship between this formula and eq.(1) was previously analyzed by Nelson [8].

In section 6 we analyze the approach of Garfinkle, Giddings, and Strominger [9] for calculating black hole entropy by comparing the pair creation rate for black holes (in a process mediated by an instanton) to the corresponding pair creation rate for monopoles. Although the language of this approach is extremely different from that of the Noether charge approach, we shall show that in an arbitrary theory of gravity, this pair creation rate calculation will always yield a formula for black hole entropy which is equivalent to eq.(1).

Finally, in the Appendix, we examine the structure of the “off-shell” Hamiltonian, \(H\), arising in an arbitrary diffeomorphism invariant, Lagrangian theory. We show that \(H\) always can be written as the sum of a surface term plus a volume integral of the form \(\int_{\Sigma} t^a \mathcal{C}_a\) where \(t^a\) denotes the time evolution vector field and \(\mathcal{C}_a = 0\) are the constraints associated with the diffeomorphism invariance.

In the following, we shall follow the notation and conventions of [10]. We will use boldface letters to denote differential forms on spacetime, and shall, in general, suppress their tensor indices.
2 Actions, Hamiltonians, and Quasilocal Energy

In this section we will consider theories arising from a diffeomorphism covariant Lagrangian on a manifold \( M \) with boundary \( \partial M \). We pose the issue of whether an action, \( I_X \), exists for variations satisfying some (arbitrary) specified boundary conditions (denoted “\( X \)”) on \( \partial M \). We will show that this issue is closely related to the issue of whether a Hamiltonian, \( H_X \), exists when the same boundary conditions \( X \) are imposed upon the fields at \( \partial M \). The value of \( H_X \) (when it exists) will then be used to define a notion of quasilocal energy. We will show that for the case of general relativity, this notion of quasilocal energy agrees with that of Brown and York [4], so our analysis may be viewed as a generalization of the definition of Brown and York to a much wider class of theories.

We begin by reviewing the elements of Lagrangian field theory which will be needed for our analysis below. (A much more complete discussion can be found in [2].) We consider theories on an \( n \)-dimensional spacetime \(( M, g_{ab})\) derived from a diffeomorphism covariant Lagrangian \( n \)-form \( L \) with functional dependence

\[
L = L \left( g_{ab}, \nabla a_1 R_{bcde}, \ldots, \nabla (a_1 \ldots a_m) R_{bcde}, \psi, \nabla a_1 \psi, \nabla (a_1 \ldots a_l) \psi \right)
\] (3)

where \( R_{abcd} \) is the curvature of the connection \( \nabla \) compatible with the metric \( g_{ab} \), and \( \psi \) denotes any matter field(s). In the following, we shall we will collectively denote the dynamical fields, \((g_{ab}, \psi)\), by \( \phi \). The variation of the Lagrangian defines the equations of motion form, \( E \), and symplectic potential \((n - 1)\)-form \( \Theta \) (both of which are local in the dynamical fields) via the relation

\[
\delta L = E \delta \phi + d \Theta
\] (4)

where \( \Theta \) is a function of the dynamical fields and their variations,

\[
\Theta = \Theta(\phi, \delta \phi).
\] (5)
As discussed in [2], \( \Theta \) may be chosen to be covariant (that is, have no dependence on any “background”, non-dynamical fields such as a coordinate system), and we will assume here that a covariant choice has been made. The symplectic current, \( \omega \), is defined by taking an antisymmetrized variation of \( \Theta \),

\[
\omega(\phi, \delta_1 \phi, \delta_2 \phi) = \delta_1 \Theta(\phi, \delta_2 \phi) - \delta_2 \Theta(\phi, \delta_1 \phi).
\] (6)

For every diffeomorphism generated by a smooth vector field \( \xi^a \) on the spacetime, there is an associated Noether current \((n-1)\)-form \( J \) defined by

\[
J[\xi] = \Theta(\phi, L_\xi \phi) - \xi \cdot L
\] (7)

where the centered dot denotes contraction of the vector into the first index of the form. When the field equations hold, the Noether current is identically closed (that is \( dJ \equiv 0 \)) for all smooth vector fields \( \xi^a \), and so can be written (see [11]) in terms of a globally defined Noether charge \((n-2)\)-form \( Q \) which is covariant and is locally constructed out of dynamical fields,

\[
J[\xi] = dQ[\xi].
\] (8)

The Noether charge form \( Q \) will play a prominent role in our analysis below.

Now consider the case where the spacetime manifold, \( M \), is compact, with boundary \( \partial M \).

We define the action, \( I \), associated with \( L \) by

\[
I = \int_M L
\] (9)

Then, using eq.(4), we obtain,

\[
\delta I = \int_M \delta L = \int_M E \delta \phi + \int_{\partial M} \Theta(\phi, \delta \phi)
\] (10)

Thus, we see that, in general, the action \( I \) will not be extremized (i.e., will not satisfy \( \delta I = 0 \) for all field variations \( \delta \phi \)) by solutions to the equations of motion. However, suppose that
(for reasons related to quantum physics or otherwise) we seek a modified action, $I_X$, having the property that the solutions to the equations of motion are true extrema of $I_X$ for all field variations $\delta \phi$ satisfying certain specified boundary conditions (which we denote as “$X$”) on $\partial M$.

(An example of such boundary conditions for general relativity would be to hold the induced metric on $\partial M$ fixed.) Clearly, a sufficient (and, presumably, necessary) condition for $I_X$ to exist is for there to exist an $(n-2)$-form $\mu(\phi, \delta \phi)$ and an $(n-1)$-form $B(\phi)$ defined on $\partial M$ such that the pull-back, $\Theta$, of $\Theta$ to $\partial M$ is given by

$$\Theta(\phi, \delta \phi) |_{\partial M} = \delta B(\phi) + d\mu(\phi, \delta \phi) |_{\partial M}$$

for all $\delta \phi$ satisfying the boundary conditions $X$. Namely, if we find such a $B$, we can define

$$I_X = \int_M L - \int_{\partial M} B$$

and immediately obtain the desired relation

$$\delta I_X = \int_M E \delta \phi$$

whenever $\delta \phi$ satisfies conditions $X$ on $\partial M$.

Now, we relax the condition that $M$ be compact but restrict attention to the case where $\partial M$ is a timelike hypersurface. We assume, in addition, that we have foliated $M$ by achronal, spacelike hypersurfaces, $\Sigma_t$ labeled by parameter $t$, which intersect $\partial M$ orthogonally in compact $(n-2)$-surfaces, denoted as $C_t$. (Note that the requirement that $\Sigma_t$ be orthogonal to $\partial M$ will impose some restrictions on the allowed variations of the spacetime metric.) We also assume that we have chosen a time evolution vector field, $t^a$, which satisfies $t^a \nabla_a t = 1$ on $M$ and is tangent to $\partial M$. We shall say that a Hamiltonian conjugate to $t^a$ exists for the boundary conditions $X$ if we can find a functional, $H_X$, of the fields and their derivatives on $\Sigma_t$ such that for all solutions $\phi$ of the field equations and for all field variations $\delta \phi$ compatible with the
conditions $X$ on $\partial M$, we have

$$\Omega(\phi, \delta \phi, \mathcal{L}_t \phi) = \delta H_X$$

(14)

where

$$\Omega(\phi, \delta_1 \phi, \delta_2 \phi) = \int_{\Sigma_t} \omega(\phi, \delta_1 \phi, \delta_2 \phi).$$

(15)

with $\omega$ defined by eq.(3). Note, however, that $H_X$ will truly be a Hamiltonian in the usual sense only when a phase space has been defined which incorporates the chosen boundary conditions $X$, so that the $X$-conditions hold automatically for all variations, $\delta \phi$, in the phase space. (Even when this is done, there is, of course, no guarantee that the resulting Hamilton’s equations on phase space will have a well posed initial value formulation, or even admit any solutions at all.) Finally, note that there would be essentially no change in the above discussion if we were to replace the conditions, $X$, on a finite boundary $\partial M$ with suitable asymptotic conditions at “spatial infinity” for a manifold $M$ without boundary.

A key point to note here is that there is an intimate relation between the existence of $I_X$ and $H_X$. As shown in [2], we have

$$\Omega(\phi, \delta \phi, \mathcal{L}_t \phi) = \int_{C_t} \delta Q[t] - t \cdot \Theta(\phi, \delta \phi)$$

(16)

whenever the equations of motion hold for $\phi$ and the linearized equations of motion hold for $\delta \phi$.

Now, suppose that the forms $\mu$ and $B$ can be found so that eq.(11) holds, thereby guaranteeing the existence of an action $I_X$. Suppose, in addition, that $\mu$ satisfies the additional condition that for all $t$,

$$\overline{\mu} |_{C_t} = 0$$

(17)

Then $H_X$ exists and is given by

$$H_X = \int_{C_t} Q[t] - t \cdot B$$

(18)
since, when pulled back to $C_t$, we have

$$ t \cdot \delta B = t \cdot \Theta - t \cdot d\mu $$

$$ = t \cdot \Theta - \mathcal{L}_t \mu + d(t \cdot \mu) $$

$$ = t \cdot \Theta + d(t \cdot \mu) $$

where eq. (17) was used in the last step and we omitted writing bars over all of the differential forms. Eq. (14) then follows immediately from eqs. (16) and (19). Note that $H_X$ has the surface integral form (18) only when the equations of motion hold for $\phi$. In the appendix we analyze the “off shell” structure of $H_X$.

Suppose, now, that $\mu$ and $B$ can be found so that eqs. (11) and (17) hold. Equation (18) then motivates the following definitions: We define the quasilocal energy (associated with the boundary conditions $X$) of the cut $C_t$ by

$$ E_t \equiv \int_{C_t} Q[u] - u \cdot B. $$

where $u^a$ is the unit normal to $\Sigma_t$. Similarly we define the quasilocal momentum conjugate to a vector field $N^a$ tangent to the cut $C_t$ by

$$ J_t \equiv -\int_{C_t} (Q[N^a] - N \cdot B). $$

As we shall see below, these definitions generalize the those given by Brown and York [4] for general relativity (see also [12]). Note also that in the asymptotically flat case, the definitions (20) and (21) (for $N^a$ chosen to be an asymptotic rotation) correspond to the definitions of total energy and angular momentum given in [2].

Although the formula (20) provides a local expression for an energy-like quantity defined on a 2-surface $C_t$, it should be noted that this expression would suffer from the following deficiencies if one were to interpret it as defining a notion of the energy contained in the region bounded by
$\mathcal{E}_t$: (1) The definition of $\mathcal{E}_t$ depends upon the choice of boundary conditions $X$. (2) The choice of $B$ clearly is ambiguous up to addition of terms which are constant under field variations which keep $X$ fixed. (3) The Noether charge $Q$ has the ambiguities discussed in [2]. (4) $\mathcal{E}_t$ does not depend on the choice of $C_t$ in a suitably continuous manner; specifically, $\mathcal{E}$ can take very different values on two surfaces which are very close to each other in spacetime, but one of which is much “wigglier” than the other. Note, however, that these difficulties (1)-(4) do not occur (or, at least, are greatly alleviated) when defining the total energy or momentum of an asymptotically flat spacetime.

Until this point, we have considered an essentially arbitrary Lagrangian theory, with arbitrary boundary conditions $X$ such that $\mu$ and $B$ can be found so that eqs.(11) and (17) hold. We now shall restrict consideration to theories where the following 3 additional conditions hold. We shall explicitly verify below that general relativity satisfies these conditions, but we have not investigated the precise range of theories for which these conditions (or suitable generalisations of them) are valid, nor have we even investigated the extent to which these conditions are independent of each other. For simplicity, we restrict attention here to the case where matter fields are absent, so that the only dynamical field is the spacetime metric $g_{ab}$.

1. We assume that the boundary conditions, $X$, correspond to the fixing of the induced metric $\gamma_{ab}$ on $\partial M$. Now, for a fixed choice of slicing, $\Sigma_t$, and time evolution vector field $t^a$, we can express $\gamma_{ab}$ in terms of the induced 2-metric $\sigma_{ab}$ on the cross-sections $C_t$ together with the lapse, $N$, and shift $N^a$, defined via

$$t^a = Nu^a + N^a.$$  \hspace{1cm} (22)

Since $\bar{\Theta} = \delta B + \delta \mu$ on $\partial M$ when the boundary conditions $X$ are
satisfied (see eq. (11)), it follows that for arbitrary metric variations, \( \Theta - \delta B - d\mu \) at any point \( p \in \partial M \) can depend upon \( \delta g_{ab} \) only via \( \delta N, \delta N^a, \delta \sigma_{ab} \) and their derivatives tangential to \( \partial M \). The freedom available in the choice of \( \mu \) permits us to eliminate any dependence upon the tangential derivatives, i.e., we may always choose \( \mu \) so that on \( \partial M \)

\[
\Theta - \delta B = \alpha \delta N + \beta_a \delta N^a + \lambda^{ab} \delta \sigma_{ab} + d\mu
\]  

(23)

We now assume that a choice of \( \mu \) can be made which is simultaneously compatible with both eqs. (17) and (23).

2. Consider a slice \( \Sigma_t \) and a vector field \( v^a \) on \( M \) which vanishes on \( \Sigma_t \), so that the diffeomorphisms generated by \( v^a \) leave each point of \( \Sigma_t \) invariant. The infinitesimal change, \( \mathcal{L}_v g_{ab} \), induced by \( v^a \) on \( g_{ab} \) will, in general, be nonzero on \( \Sigma_t \), since the “time components” of the metric can change. However, we assume that these induced variations are “dynamically trivial” in the sense that they are degeneracy directions of \( \Omega \), i.e., we assume that

\[
\Omega(g_{ab}, \mathcal{L}_v g_{ab}, \delta g_{ab}) = 0
\]  

(24)

for all \( v^a \) which vanish on \( \Sigma_t \) and all metric variations \( \delta g_{ab} \).

3. The integrand appearing on the right side of eq. (18) is linear in \( t^a \) and its derivatives. The freedom involved in the choice of \( Q \) ensures that we may assume that this integrand depends only upon \( t^a \) and its first antisymmetrized derivative. We now assume that (using integration by parts if necessary) we can eliminate the dependence of
the integral on the derivatives of $t^a$, so that we may write eq. (18) in the form

$$H_X[t^a] = \int_{C_t} t^a e_a$$ \hspace{1cm} (25)$$

where $e_a$ is independent of $t^a$ and any other background structure not invariant under diffeomorphisms which leave each point of $\Sigma_t$ fixed in a following sense: For all metric variations of the form $\delta g_{ab} = \mathcal{L}_v g_{ab}$ with $v^a = 0$ on $\Sigma_t$, we have $\delta e_a = \mathcal{L}_v e_a$.

Using the above assumptions, we may relate the quasilocal energy and momentum to the coefficients $\alpha$ and $\beta_a$ appearing in eq. (23). Let $\hat{\delta}$ denote the variation induced in any quantity by the variation $\mathcal{L}_v g_{ab}$ of the spacetime metric, where $v^a$ vanishes on $\Sigma_t$. Then, by eqs. (16) and (24), we have

$$0 = \Omega(g_{ab}, \mathcal{L}_v g_{ab}, \mathcal{L}_t g_{ab}) = \int_{C_t} \delta Q[t] - t \cdot \Theta(g_{ab}, \mathcal{L}_v g_{ab})$$

$$= \hat{\delta} H_X[t^a] - \int_{C_t} (t \cdot \Theta - t \cdot \hat{\delta} B)$$ \hspace{1cm} (26)$$

On the other hand, using property (3) and $\hat{\delta} t^a = 0$, we have

$$\hat{\delta} H_X[t^a] = \int_{C_t} t^a \delta e_a$$

$$= \int_{C_t} t^a \mathcal{L}_v e_a$$

$$= \int_{C_t} (\mathcal{L}_v (t^a e_a) - e_a \mathcal{L}_v t^a)$$

$$= - \int_{C_t} e_a \mathcal{L}_v t^a$$ \hspace{1cm} (27)$$

Finally, using eq. (23), we obtain from (26)\hspace{1cm}

$$0 = - \int_{C_t} e_a \mathcal{L}_v t^a + t \cdot \alpha \delta N + t \cdot \beta_a \delta N^a$$ \hspace{1cm} (28)$$
for all $v^a$ which vanish on $\Sigma_t$. Now, using eq.\((22)\) and the fact that $\mathcal{L}_v N = \mathcal{L}_v N^a = 0$, we have
\[
\mathcal{L}_v t^a = N \mathcal{L}_v u^a = N \delta u^a
\] (29)

On the other hand, we have
\[
0 = \delta t^a = (\delta N) u^a + N \delta u^a + \delta N^a.
\] (30)
and, consequently
\[
\mathcal{L}_v t^a = - (\delta N u^a + \delta N^a).
\] (31)
Thus, (28) implies
\[
t \cdot \alpha = u^a e_a,
\]
\[
t \cdot \beta_a = \sigma^b_a e_b.
\] (32)
Thus, we see that, under our above assumptions, the quasilocal energy and momentum densities are simply the coefficients of $\delta N$ and $\delta N^a$ appearing in the pullback to $C_t$ of $t \cdot (\Theta - \delta B)$, and we have
\[
H[X] = \int_{C_t} N t \cdot \alpha + N^a t \cdot \beta_a.
\] (33)
This formula corresponds to the definition of the quasilocal energy and momentum densities given by Brown and York \[4\].

We conclude this section by verifying explicitly that all of the above conditions hold for general relativity. We start with the expression for the pullback of $\Theta$ to $\partial M$ given in \[14\]:
\[
\tilde{\Theta}_{abc}\lvert_{\partial M} = - \frac{1}{16\pi} (K_{mn} - \gamma_{mn} K) \delta\gamma_{mn} \epsilon_{abc} - \delta\left(\frac{1}{8\pi} K \epsilon_{abc}\right) + \frac{1}{16\pi} d(n^m \delta n^n \epsilon_{abmn})
\] (34)
where $n^a$ is the unit "outward pointing" normal to $\partial M$, $\epsilon_{abc} \equiv n^d \epsilon_{dabc}$ is the induced volume element on $\partial M$, $K_{ab}$ denotes the extrinsic curvature of $\partial M$ and $K = K^a_a$. It follows immediately that for the boundary conditions, $X$, of condition (1), the choices
\[
B_{abc} = - \frac{1}{8\pi} \epsilon_{abc} (K + S_0)
\] (35)
and
\[ \mu_{ab} = \frac{1}{16 \pi} n^m \delta n^n \epsilon_{abmn} \] (36)
satisfy eq. (11), where \( S_0 \) is any quantity only depending on the intrinsic geometry of \( \partial M \).
Furthermore, using the fact that both \( n^m \) and \( \delta n^n \) are tangent to \( \Sigma_t \), we see that eq. (17) holds.
Since no derivatives of \( \delta \gamma_{ab} \) appear in the first term in eq. (34), it also is manifest that eq. (23) holds, so condition (1) is indeed satisfied.

Using the form for \( \Theta \) given in (34) but now pulled back to a spacelike slice \( \Sigma_t \), one finds
\[ \Omega(g_{ab}, \delta g_{ab}, \mathcal{L}_v g_{ab}) = \int_{\Sigma_t} \delta \pi_{ab} \mathcal{L}_v h_{ab} - \mathcal{L}_v \pi_{ab} \delta h_{ab} = 0 \] (37)
for all \( v^a \) which vanish on \( \Sigma_t \), where \( h_{ab} \) is the induced spatial metric on \( \Sigma_t \) and \( \pi_{ab} \) is its canonically conjugate momentum. Thus, condition (2) is satisfied.

Finally we show that condition (3) is satisfied in general relativity. We have [2]
\[ Q[t]_{ab} = -\frac{1}{16 \pi} \epsilon_{abcd} \nabla^c t^d \] (38)
and hence
\[ \int_C t \quad Q[t] - t \cdot B = \int_C -\frac{1}{16 \pi} \epsilon_{abcd} \nabla^c t^d + \frac{1}{8 \pi} t^c \epsilon_{cab} (K + S_0) \]
\[ = \int_C \epsilon_{ab} \left( \frac{1}{16 \pi} (u_c n_d - u_d n_c) \nabla^c t^d + t^a u_a \frac{1}{8 \pi} (K + S_0) \right) \]
\[ = \int_C \epsilon_{ab} \frac{1}{8 \pi} \left( u_c n_d \nabla^c t^d + t^a u_a (K + S_0) \right) \]
\[ - \int_C \epsilon_{ab} \left( \frac{1}{16 \pi} (u^c n^d + u^d n^c) \nabla_c t_d \right). \] (39)
The last term can be seen to vanish as follows:
\[ (u^c n^d + u^d n^c) \nabla_c t_d = u^c n^d \mathcal{L}_t g_{cd} \]
\[ = u^c \mathcal{L}_t n_c - u_c \mathcal{L}_t n^c \]
\[ = u^c \left( t^a 2 \nabla_{[a} n_{c]} - \nabla_c (t^a n_a) \right) - u_c \mathcal{L}_t n^c \] (40)
However, the last term on the right side vanishes on $\partial M$ since $n^c$ is tangent to each $\Sigma_t$ and $t^a$ generates diffeomorphisms which map the $\Sigma_t$'s into themselves, so $\mathcal{L}_t n^c$ also is tangent to each $\Sigma_t$. The middle term vanishes on $\partial M$ since $t^a n_a = 0$. Finally, using the hypersurface orthogonality of $n_c$ on $\partial M$, we see that the first term also vanishes. Thus, we obtain

$$\int_{C_t} Q^c[t] - t \cdot B = \int_{C_t} \epsilon_{ab} \frac{1}{8\pi} \left( u_c n_d \nabla^c t^d + t^d u_d (K + S_0) \right)$$

$$= \int_{C_t} \epsilon_{ab} \frac{1}{8\pi} \left( -u^c t^d \nabla_c n_d + t^d u_d (K + S_0) \right)$$

$$= \int_{C_t} \epsilon_{ab} \frac{1}{8\pi} \left( -u^c t^d K_{cd} + t^d u_d (K + S_0) \right) \quad (41)$$

which is seen to satisfy condition (3) with

$$\epsilon_a = \frac{1}{8\pi} (u^c K_{ca} + u_a (K + S_0)) \quad (42)$$

We have therefore verified that general relativity satisfies the conditions (1)-(3) posed above.

3 The Microcanonical Action and Black Hole Entropy

Consider a diffeomorphism invariant theory of a metric $g_{ab}$ derived from an action, $I_X$, for boundary conditions, $X$, which satisfies the conditions stated in the previous section. In particular, in such a theory, the quasilocal energy and momentum densities are defined on each cut $C_t$. Following Brown and York [5], we say that an action $I_m$ is a microcanonical action for the theory if, for arbitrary metric variations about an arbitrary metric (subject only to the restriction that the hypersurfaces $\Sigma_t$ are orthogonal to $\partial M$), we have

$$\delta I_m = \int_M E^{ab} \delta g_{ab} - \int_{\partial M} dt \wedge (N \delta (u^a e_a) + N^a \delta (\sigma^c e_c)) - \lambda^{ab} \delta \sigma_{ab}$$

$$= \int_M E^{ab} \delta g_{ab} - \int_{\partial M} N \delta \alpha + N^a \delta \beta_a - \lambda^{ab} \delta \sigma_{ab} \quad (43)$$

where $E^{ab} = 0$ are the equations of motion for $g_{ab}$ and eq.(32) was used in the second line. Here, when comparing signs in our formulas with those of Brown and York, it should be noted...
that the relationship between our choice of orientations \((n-1)\epsilon\) of \(\partial M\) and \((n-2)\epsilon\) of \(C_t\) is given by \((n-1)\epsilon = -dt \wedge (n-2)\epsilon\). (This arises because we choose \((n-1)\epsilon_{a_1...a_{n-1}} = n^b\epsilon_{ba_1...a_{n-1}}\) and \((n-2)\epsilon_{a_1...a_{n-2}} = n^b\epsilon_{cba_1...a_{n-1}}\); the orientation of \(\Sigma_t\) is chosen to be \(u^b\epsilon_{ba_1...a_{n-1}}\).

We now show that for theories satisfying the conditions of the previous section, \(I_m\) is given by

\[
I_m = \int_M L - \int_{\partial M} dt \wedge Q[t] \tag{44}
\]
i.e., to get \(I_m\) we replace \(B\) in eq.(12) by \(dt \wedge Q[t]\). Namely, taking the variation of eq.(44), we obtain

\[
\delta I_m = \int_M E^{ab} \delta g_{ab} - \int_{\partial M} dt \wedge (\delta Q[t] - t \cdot \Theta) \tag{45}
\]

However, we have

\[
\int_{C_t} \delta Q[t] - t \cdot \Theta = \int_{C_t} \delta (Q[t] - t \cdot B) - t \cdot (\Theta - \delta B)
\]
\[
= \int_{C_t} \delta HX[t] - t \cdot (\Theta - \delta B)
\]
\[
= \int_{C_t} t \cdot (δN\alpha + N^aδβ_a - Λ^aδσ_{ab})
\]
where eqs. (18), (19), (23), and (33) were used. Thus, we obtain

\[
\delta I_m = \int_M E^{ab} \delta g_{ab} - \int_{\partial M} dt \wedge t \cdot (Nδ\alpha + N^aδβ_a - Λ^aδσ_{ab}) \tag{47}
\]

which agrees precisely with eq. (43).

Motivated by path integral methods for computing entropy, Brown and York [5] proposed a prescription for obtaining the entropy of a black hole in general relativity. When generalized to the class of theories we consider here, their prescription can be reformulated as follows:

Consider a Lorentzian black hole solution \((M, g_{ab})\) with bifurcation surface \(\mathcal{H}\), so that
spacetime manifold $M$ has topology $\mathcal{R}^2 \times \mathcal{H}$. Normalize the Killing field $t^a$ which vanishes on $\mathcal{H}$ so that it has unit surface gravity. Choose a slicing, $\Sigma_t$, of the exterior region of the black hole labeled by Killing parameter $t$, such that each $\Sigma_t$ smoothly intersects $\mathcal{H}$. Define a “Euclidean manifold” $M'$ by writing $T = i t$, taking $T$ to be real, and then periodically identifying $T$ with period $2\pi$ to avoid a conical singularity at $\mathcal{H}$. We choose the boundary of $M'$ to be an orbit of the Killing field, so that $\partial M'$ has topology $S^1 \times \mathcal{H}$. Define a (in general, complex) “Euclidean metric” $g^E$ on $M'$ by analytic continuation of $g$. Let $I'_m$ denote the “Euclidean microcanonical action”, defined by

$$I'_m = -i \left( \int_{M'} L + \int_{\partial M'} dt \wedge Q[t] \right)$$

(48)

where $L$ and $Q$ are analytically continued from the Lorentzian spacetime. (The imaginary factor makes $I_m$ real since the “$dt$” implicitly appearing in $L$ is imaginary on $M'$.) Then, the prescription of Brown and York corresponds to the formula

$$S = -I'_m$$

(49)

We now show that – in its domain of applicability – eq. (49) is equivalent to the Noether charge prescription, eq.(1). Writing $T^a = -i t^a$ (so that $T^a \nabla_a T = 1$), we have

$$I'_m = -i \int_{M'} L + i \int_{\partial M'} dt \wedge Q[t]$$

$$= -i \int_{M'} dT \wedge T \cdot L + \int_{\partial M'} dT \wedge Q[t]$$

$$= -2\pi i \int_{\Sigma_0} T \cdot L - 2\pi \int_{C_0} Q[t]$$

$$= -2\pi \left( \int_{\Sigma_0} t \cdot L + \int_{C_0} Q[t] \right)$$

(50)

where the apparent sign change in the surface term in the third line is due to our orientation conventions (see the first paragraph of this section), and it should be noted that the slice $\Sigma_0$
and cut $C_0$ are common to both $M$ and $M'$. However, we have

$$\int \Sigma_0 t \cdot \mathbf{L} = \int \Sigma_0 (\Theta(\phi, \mathcal{L}_t \phi) - J[t])$$

$$= - \int \Sigma_0 dQ[t]$$

$$= - \int_{C_0} Q[t] + \int_{\mathcal{H}} Q[t]$$

(51)

Thus, we obtain

$$S = 2\pi \left( - \int_{C_0} Q[t] + \int_{\mathcal{H}} Q[t] + \int_{C_0} Q[t] \right)$$

$$= 2\pi \int_{\mathcal{H}} Q[t]$$

(52)

in agreement with eq. (52).

4 Black Hole Entropy as a Hilbert Action Boundary Term

Bañados, Teitelboim and Zanelli [6] have given another approach for computing the entropy of black holes applicable to general relativity and, more generally, Lovelock gravity. The starting point of this approach is essentially the same as the microcanonical action approach discussed in the previous section. However, these authors then quote, without detailed derivation, the following formula for the entropy of a stationary, Euclidean black hole in Lovelock gravity

$$S = - \lim_{\epsilon \to 0} \int_{\partial D_\epsilon \times \mathcal{H}} \mathbf{B}$$

(53)

Here $\mathbf{B}$ is the Hilbert action boundary form, defined by eq.(11) for variations which keep the induced metric fixed on the boundary, and $D_\epsilon$ is a two-dimensional disk of radius $\epsilon$ orthogonal to the bifurcation surface $\mathcal{H}$ (on which the stationary Killing field vanishes).

Our aim here is to explain why (53) – which looks very different from either (49) or (1) – actually produces answers which coincide with these other calculations. To see this, we consider
a theory which satisfies the three assumptions of section (2). Let $\Sigma$ be a smooth hypersurface (transverse to the Killing field $t^a$) in the Euclidean space which passes through the bifurcation surface, and let $\mathcal{H}_\epsilon$ be a smooth, one-parameter family of surfaces in $\Sigma$ which approach the bifurcation surface $\mathcal{H}$ as $\epsilon \to 0$. Then, by assumption (3) of section (2), $e_a$ will be well defined on each $\mathcal{H}_\epsilon$ and will smoothly approach its value on the bifurcation surface as $\epsilon \to 0$. However, it then follows immediately from eq.(25) that on the bifurcation surface (where $t^a$ vanishes) we have $H_X = 0$. Thus, we find that

$$\lim_{\epsilon \to 0} H_X(\epsilon) = 0$$

(54)

From the original definition of $H_X$, eq.(18), we thus obtain

$$\lim_{\epsilon \to 0} \int_{\mathcal{H}_\epsilon} t \cdot B = \lim_{\epsilon \to 0} \int_{\mathcal{H}_\epsilon} t \cdot B$$

(55)

where the boundary surface used to define $B$ at each $\epsilon$ is taken to be the orbit of $H$ under the action of $t^a$, i.e., it is simply $\partial D_\epsilon \times \mathcal{H}$. By eq.(1) above, the left side is simply $S/2\pi$. Therefore we obtain

$$S = 2\pi \lim_{\epsilon \to 0} \int_{\mathcal{H}_\epsilon} t \cdot B$$

$$= -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon \times \mathcal{H}} dt \wedge t \cdot B$$

$$= -\lim_{\epsilon \to 0} \int_{\partial D_\epsilon \times \mathcal{H}} B$$

(56)

where the sign change in the second line results from our orientation conventions as explained at the end of the first paragraph of section 3. This establishes the equivalence of eq. (53) and (1).

The explicit form of eq.(53) for Lovelock gravity given in [6] appears to be closely related to eq.(2) above. (That those two formulae must be equivalent follows from eq. (54) together with the equivalence of eqs.(1) and (2) proven in [2].) If the Lagrangian does not contain derivatives
of the Riemann tensor, then $\Theta$ can always be chosen to have the form \[2\]

$$\Theta = 2\epsilon_{a_1...a_{n-1}}E_R^{abcd}\nabla_c\delta g_{bd} + S_{a_1...a_{n-1}}\delta g_{bd}$$

(57)

where $E_R^{abcd}$ was defined below eq.(2). Hence if a Hilbert action surface term, $B$, exists, one would expect there to be a direct relationship between it and $E_R^{abcd}$. Thus it may be possible to give a simple direct proof of the equivalence of eqs.(53) and (2), although we have not succeeded in doing so.

5 Black Hole Entropy as a Quantity Conjugate to Conical Deficit Angle

Bañados, Teitelboim and Zanelli [6] and Susskind and Uglum [7] have proposed another approach to obtaining the entropy of a static black hole. The starting point of this approach is the fact that in ordinary quantum field theory, the partition function, $Z$, at inverse temperature, $\beta$, is given by a path integral over Euclidean configurations with period $\beta$ of $e^{-I}$, where $I$ is the Euclidean action. Thus, if we apply this formula to a static black hole solution, then in the “zero loop” approximation, we simply have $Z = e^{-I}$, where $I$ is the usual “Hilbert action” of the Euclidean black hole given by

$$I_m = -i \int_{M'} L + i \int_{\partial M'} B$$

(58)

(see eq.(48) above.) Thus, under these assumptions, the Helmholtz free energy, $F$, of a static black hole would be given by

$$F = -\log Z/\beta = I/\beta.$$ 

(59)

The entropy of the black hole should then be given by

$$S = \beta \frac{\partial F}{\partial \beta} = \beta \frac{\partial I}{\partial \beta} - I$$

(60)
where, in ordinary thermodynamics, the partial derivatives would be taken at fixed values of the state parameters (other than energy). A possible interpretation of the partial derivative in our case would be to restrict consideration to variations in which the geometry is changed only by varying the periodicity of the Euclidean time coordinate. Since the contributions to \( I \) from both the ordinary volume term and the surface term at infinity are linearly proportional to the periodicity, \( \beta \), the variations of these terms will not contribute to \( S \) in eq. (60). However, when \( \beta \) is varied away from \( \beta = 2\pi \), a conical singularity is created at the bifurcation surface of the black hole. This conical singularity can be viewed as corresponding to a \( \delta \)-function in the curvature, which is proportional to \( 2\pi - \beta \) rather than \( \beta \). Hence, this \( \delta \)-function contribution to the volume term in \( I \) will yield a nonzero contribution to \( S \) in eq. (60).

For the case of general relativity, the Lagrangian density is linear in the curvature, so there is no difficulty in defining the contribution of the \( \delta \)-function to the volume term in \( I \). Equation (60) then yields the standard result \( S = A/4 \). However, if the Lagrangian density is a nonlinear function of the curvature, it is far from clear that any well defined regularization scheme can be given to define \( I \) when a conical singularity is present – except in the limit in which the curvature of the Euclidean black hole vanishes \([7]\), where the nonlinear curvature terms of the Lagrangian can be neglected in any case. Thus, it would appear that the conical deficit approach to calculating the entropy of a black hole is fundamentally limited to theories where the Lagrangian density is at most linear in the curvature.

The relationship between the conical deficit and Noether charge approaches to the calculation of black hole entropy has been analyzed by Nelson \([8]\). Making use of the fact that the variation of the geometry resulting from a change of \( \beta \) can be induced by the action of a (singular) diffeomorphism, Nelson argues that in the cases where the conical deficit approach can be defined, it should yield the same result as the Noether charge method.
6 Black Hole Entropy as Adduced From Pair Creation Rates

The final Euclidean method for calculating black hole entropy which we shall analyze involves the calculation of pair creation rates of black holes by instanton methods \[9\]. The idea here is to compare the rate for pair creation of black holes with a corresponding rate for the pair creation for objects which have the same exterior field as the black hole but do not have a horizon. The enhancement factor of the black hole rate should measure the number of “internal states” of the black hole, and, thus, determine its entropy. The particular calculation done in \[9\] involved the comparison of the pair creation rates of Reissner-Nordstrom black holes and monopoles in a magnetic field, but we now shall show that this approach always yields results in agreement with the Noether charge approach (see also \[15\]).

The calculation of black hole entropy by the pair creation method proceeds as follows: One first finds a Euclidean instanton solution corresponding to the black hole pair creation process, and asserts that the pair creation rate, $\Gamma$, is proportional to $e^{-I}$, where $I$ is the Euclidean action of the instanton. One then finds an instanton solution describing the monopole (or other object) pair creation process, and asserts that the pair creation rate, $\tilde{\Gamma}$, for these objects is proportional to $e^{-\tilde{I}}$, where $\tilde{I}$ is the Euclidean action of this instanton. The entropy, $S$, of the black hole is then given by

$$S = \ln \frac{\Gamma}{\tilde{\Gamma}} = \tilde{I} - I$$  \hspace{1cm} (61)

As usual, the instanton action will have a volume term and a surface term from infinity. Since, by assumption, the instanton solutions agree near infinity, the surface term contributions from $I$ and $\tilde{I}$ will cancel in eq.(61), so we need only consider the volume term, which is of the form

$$I_V = -i \int L$$  \hspace{1cm} (62)
where $L$ is an analytic continuation of the Lorentzian Lagrangian $n$-form to Riemannian metrics (see eq.(48) above.) This volume term for both $I$ and $	ilde{I}$ can be evaluated as follows. Let $\Sigma$ be a hypersurface which intersects each (circular) orbit of the timelike Killing field once and only once. In the black hole case, $\Sigma$ will terminate at the bifurcation surface of the black hole, so $\partial \Sigma$ will consist of the bifurcation surface together with a two-surface at infinity. In the monopole (or other object) case $\partial \Sigma$ will be comprised by only the two-surface at infinity. In either case, we have by a calculation which parallels eqs.(50) and (51) above,

$$I_V = -i \int_{M'} L = -i \int_{M'} dT \wedge T \cdot L$$

$$= -2\pi i \int_{\Sigma} T \cdot L$$

$$= -2\pi \int_{\Sigma} t \cdot L$$

$$= -2\pi \int_{\Sigma} [\Theta - J]$$

$$= 2\pi \int_{\Sigma} dQ$$

$$= 2\pi \int_{\partial \Sigma} Q$$

(63)

where we used the fact that the instanton is stationary (so that $\Theta(\phi, L_t \phi) = 0$) and is a solution (so that $J = dQ$). Thus, for the black hole instanton, we obtain

$$I_V = 2\pi \left( \int_{\infty} Q - \int_{\mathcal{H}} Q \right)$$

(64)

whereas for the monopole (or other object) we have

$$\tilde{I}_V = 2\pi \int_{\infty} \dot{Q}$$

(65)

Since the instanton solutions agree at infinity, we have $Q = \tilde{Q}$ there. Thus, eq.(61) yields

$$S = 2\pi \int_{\mathcal{H}} Q$$

(66)

in agreement with the Noether charge method, as we desired to show.
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Appendix: The Structure of the “Off-Shell” Hamiltonian

In this Appendix, we derive the general form of the Hamiltonian in a diffeomorphism invariant theory of gravity. Consider a theory derived from a Lagrangian $L$ on a manifold $M$, as described above eq. (4). Suppose, in addition, that for the chosen boundary conditions (which could be either suitable asymptotic flatness conditions or conditions at a finite boundary, as considered in section 2), there exist forms $B$ and $\mu$ satisfying eqs. (11) and (17). Then the Hamiltonian, $H$, conjugate to a time translation vector field $t^a$ is determined by 

$$\delta H = \Omega(\phi, \delta \phi, \mathcal{L}_t \phi) = \int_{\Sigma_t} \delta J[t] - \int_{C_t} t \cdot \delta B$$

(67)

where $\phi$ is a solution to the equations of motion, but the variation $\delta$ is to an arbitrary nearby configuration (in contrast to the situation considered in eq.(16) above, where $\delta \phi$ was required to satisfy the linearised equations of motion). By inspection, we have

$$H = \int_{\Sigma_t} J[t] - \int_{C_t} t \cdot B$$

$$= \int_{\Sigma_t} J[t] - dQ[t] + \int_{C_t} Q[t] - t \cdot B$$

(68)

where $Q$ is determined (up to the ambiguities analyzed in [2]) by the relation $dQ = J$ when $\phi$ satisfies the equations of motion and, for the present, its definition is extended in an arbitrary, local, covariant manner to $\phi$ which do not satisfy the equations of motion. Clearly (68) reduces to our previous expression (18) “on shell”, i.e., when $\delta \phi$ satisfies the linearized equations of motion about $\phi$. We now shall show that $Q$ always can be defined “off shell” so that the
integrand of the volume integral in eq. (68) takes the form

\[ J[t] - dQ[t] = t^a C_a \]  \hspace{1cm} (69)

where \( C_a \) is locally constructed out of the dynamical fields in a covariant manner and \( C_a = 0 \) when the equations of motion are satisfied. Indeed, it follows from the analysis of [16] that we may view \( C_a = 0 \) as being the constraint equations of the theory which are associated with its diffeomorphism invariance. Thus, the general form of the Hamiltonian in a theory arising from a diffeomorphism covariant lagrangian is

\[ H = \int_{\Sigma_t} t^a C_a + \int_{C_t} Q[t] - t \cdot B \]  \hspace{1cm} (70)

To prove (69), we note that the freedom in extending the definition of \( Q \) off-shell clearly allows us to make the replacement \( Q \rightarrow (Q + \tau) \), where \( \tau \) is any covariant \((n-2)\)-form locally constructed from the dynamical fields such that \( \tau = 0 \) whenever the equations of motion, \( E = 0 \), are satisfied. To prove that this freedom is sufficient to ensure that eq. (69) holds, we recall [2] that \( J[t] - dQ[t] \) is a linear differential operator on the vector field \( t^a \), so we can write it as a sum

\[ (J - dQ)_{a_1...a_{n-1}} = \sum_{i=0}^{m} (i) A_a{}^{b_1...b_i}{}_{a_1...a_{n-1}} \nabla (b_1...b_i) t^a \]  \hspace{1cm} (71)

where the coefficients \( (i) A_a{}^{b_1...b_i}{}_{a_1...a_{n-1}} \) are locally and covariantly constructed from the dynamical fields and have the symmetries \( (i) A_a{}^{b_1...b_i}{}_{a_1...a_{n-1}} = (i) A_a{}^{(b_1...b_i)}{}_{a_1...a_{n-1}} = (i) A_a{}^{b_1...b_i}{}_{[a_1...a_{n-1}]} \).

In fact, the analysis of [2] shows that we may always choose \( J \) and \( Q \) so that \( m = 2 \), but this fact does not simplify the proof, so we shall leave eq. (71) in the form of a general sum. Since \( J = dQ \) when the field equations hold, we have

\[ (i) A_a{}^{b_1...b_i}{}_{a_1...a_{n-1}} = 0 \quad \text{when} \quad E = 0. \]  \hspace{1cm} (72)

We now parallel the proof of lemma 1 of [11] to show that if eq. (71) holds with \( m \geq 1 \), an \((n-2)\)-form \( \tau \) always can be chosen so that \( J - dQ - d\tau \) is of the form of the right side of
eq. (71), but with the sum terminating at \((m - 1)\). By induction, it then will follow immediately that \(Q\) can be chosen so that eq. (72) holds.

To proceed, we recall that (see, e.g., [16])

\[
d(J[t] - dQ) = dJ = -\mathbf{E}L_{\phi}
\]  

(73)

Now the right side of this equation involves only one derivative of \( t^a \). Consequently, substituting on the left side from eq. (71), and assuming \( m \geq 1 \) we obtain

\[
0 = (i) A_{a}^{(b_{1}...b_{i}c_{1}...c_{n-1}\delta_{d}^{c})} \nabla_{(c} \nabla_{b_{1}...\nabla_{b_{m})t^{a}}}
+ (\text{terms with fewer symmetrised derivatives of } t^a)
\]  

(74)

Since this equation holds for all \( t^a \), it follows that

\[
(i) A_{a}^{(b_{1}...b_{i}c_{1}...c_{n-1}\delta_{d}^{c})} = 0.
\]  

(75)

By inspection, it then follows [11] that

\[
\tau_{a_{2}...a_{n-1}} = \frac{m}{m+1} (i) A_{a}^{c_{b_{2}...b_{i}c_{a_{2}...a_{n-1}}} \nabla_{b_{2}...\nabla_{b_{m}}t^{a}}}
\]  

(76)

satisfies the desired requirement that \( J - dQ - d\tau \) has at most \( m - 1 \) symmetrised derivatives of \( t^a \). Furthermore, eq. (72) implies that \( \tau = 0 \) whenever the equations of motion hold, so the substitution \( Q \to (Q + \tau) \) does not affect the definition of \( Q \) “on shell”.

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