Spectral triples and Toeplitz operators

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Abstract

We give examples of spectral triples, in the sense of A. Connes, constructed using the algebra of Toeplitz operators on smoothly bounded strictly pseudoconvex domains in $\mathbb{C}^n$, or the star product for the Berezin–Toeplitz quantization. Our main tool is the theory of generalized Toeplitz operators on the boundary of such domains, due to Boutet de Monvel and Guillemin.

VERSION – 2.1, February 11, 2014

¹Research supported by GA ČR grant no. 201/12/G028.
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1 Introduction

The purpose of this work is to construct spectral triples $(A, \mathcal{H}, D)$ using the algebra $A$ of Toeplitz operators acting on Bergman or Hardy spaces $\mathcal{H}$ on a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$.

Recall that a spectral triple consists, loosely speaking, of an algebra $A$ of operators, acting on a Hilbert space $\mathcal{H}$, and a certain operator $D$ which has bounded commutators with elements from $A$. The notion was introduced by Connes, the model example being that of $A$ the algebra of $C^\infty$ functions on a Riemannian manifold $M$, $\mathcal{H}$ the space of $L^2$-spinors and $D$ the Dirac operator; and a remarkable highlight of the theory is his reconstruction theorem that, in fact, every commutative spectral triple satisfying certain conditions arises in this way (up to isomorphism) [10]. Spectral triples are thus quintessential for the “noncommutative differential geometry” program [9,11] and there exists an extensive literature on the subject, see e.g. the books [19,27], and the references therein.

Our main tool are the so-called generalized Toeplitz operators, or Toeplitz operators with pseudodifferential symbols, on the boundaries of such domains, whose theory was developed by Boutet de Monvel and Guillemin [6]. Upon passing from holomorphic functions on the domain to their boundary values, the generalized Toeplitz operators turn out to include also the ordinary Toeplitz operators on (weighted) Bergman spaces of the domain [6,22] (see also [25] for related constructions), which have long been used in quantization on Kähler manifolds (Berezin and Berezin–Toeplitz quantizations) [3,13,26,30].

The set $\Psi(M)$ of classical pseudodifferential operators on a compact Riemannian manifold is an essential tool to understand the geometry of $M$. For instance, if $A$ is the algebra $C^\infty(M)$ and $D \in \Psi(M)$ is of order one, both acting on $\mathcal{H} = L^2(M)$, then, assuming $D = D^*$, all commutators $[D,a]$ are bounded operators for any $a \in A$ and $(A, \mathcal{H}, D)$ is a spectral triple. If $\Psi^0(M)$ is the algebra of pseudodifferential operators of order less or equal to zero, one can check that $(\Psi^0(M), L^2(M), D)$ is also a spectral triple. An extension to non-compact manifold is possible by imposing for instance that $a(1 + D^2)^{-1}$ is compact for any $a \in A$. The same phenomenon occurs abstractly, since a regular spectral triple $(A, \mathcal{H}, D)$ generates a pseudodifferential calculus $\Psi(A)$ which is the algebra of operators $P$ on $\mathcal{H}$ with asymptotic expansion $P \sim a_0|D|^q + a_{q-1}|D|^{q-1} + \cdots$ where the $a_k$ are in the algebra generated by the $\delta^n(a)$, $n \in \mathbb{N}$ with $\delta(a) = [|D|, a]$. Then $(\Psi^0(A), \mathcal{H}, |D|)$ will be also a spectral triple.

Here we play around similar notions in the framework of Toeplitz operators on an open bounded domain $\Omega \subset \mathbb{C}^n$. By a result of Howe, the algebra $\Psi(\mathbb{R}^n)$ is locally isomorphic to the algebra of Toeplitz operators on the unit ball of $\mathbb{C}^n$ [25,33]. This result has been generalized by Boutet de Monvel and Guillemin [6,8,22]: the generalized Toeplitz operators on a compact manifold possessing so-called Toeplitz structure form an algebra microlocally isomorphic (modulo smoothing operators) to the algebra of pseudodifferential operators in $\mathbb{R}^n$. Indeed, this general framework can be brought back to a microlocal model for generalized Toeplitz operators, via a Fourier integral operator constructed modulo smoothing operators [5], which sets up a bijection with the algebra of pseudodifferential operators on $\mathbb{R}^n$. In a way, the pseudodifferential operators are nothing else but Toeplitz operators in disguise [22].

Of course, $\Omega$ is not compact, but we do not fall in the technicalities related to non unital spectral triples; but the price to pay is the intricate study on the role of the boundary of $\Omega$ which is of interest in complex analysis. This analysis has a long history which we intersect here only at few points: the Heisenberg algebra with its Fock and Bergman space representations and another quantization process than the Weyl one based on this Heisenberg algebra, namely
the so called Berezin–Toeplitz quantization.

We give several spectral triples for the Bergman and Hardy spaces with a natural extension for the Berezin–Toeplitz quantization based on a star product. We compute their spectral dimension, a point related to the appearance of the Dixmier trace. Such appearance was already detected in [7] and studied in [1, 2, 16–18] (see also [23] for the bidisk).

A spectral triple on the algebra \( \{ T_f \mid f \in C(S^1) \} \cap \Psi_0(S^1) \) acting on \( \mathcal{H} = H^2(S^1) \oplus H^2(S^1) \) with an operator \( D \) based on the shift and on the “number operator” has already been proposed in [12].

We expect that our spectral triples encode information about the (CR-)geometry of \( \Omega \) or \( \partial \Omega \) in much the same way as they encode the information about the Riemannian geometry of the manifold in the original model example mentioned above; we plan to treat this question in a subsequent work.

More concretely, we recall in Section 2 the role of the Poisson kernel and the trace map between the Sobolev spaces on \( \partial \Omega \) and Sobolev spaces of harmonic functions on \( \Omega \) which can be associated to a weight \( w \) on \( \Omega \), extending known results on the analysis of Toeplitz operators of Szegö type or Bergman type (see for instance [6, 14]). Section 3 is devoted to the Fock and Bergman representations of the Lie algebra of the Heisenberg group, especially for the model case of \( \Omega \) the unit ball of \( \mathbb{C}^n \) with the standard weights, which are used in the next section on possible Dirac-like operators on \( H^2(\partial \Omega) \) or \( A^2_w \). In Section 5 we consider spectral triples with several candidates for operators \( D \). The first one is defined by an elliptic generalized Toeplitz operator of order one; an example is the inverse of the Toeplitz operator associated with the defining function of \( \Omega \), but since it is positive (so with trivial K-homology class), we double the Hilbert space and construct two classes of unitary operators compatible with the notion of generalized Toeplitz operators. A second idea is to start from the usual Dirac operator on \( \mathbb{R}^n \) and to construct the corresponding operator acting on \( \mathcal{H} \) through isomorphisms which involve a set of representations of the Lie algebra of the Heisenberg group. In the last section, previous results are applied to the star product arising in the Berezin–Toeplitz quantization of \( \Omega \) equipped with a natural Kähler structure [3, 26, 30].

2 Bergman and Hardy spaces, and Toeplitz operators

We gather in this section the preliminary material and known results.

2.1 Notations and definitions

We consider a open strictly pseudoconvex bounded set \( \Omega \subset \mathbb{C}^n \approx \mathbb{R}^{2n} \), with smooth boundary \( \partial \Omega \) for \( n \) a positive integer, so \( \overline{\Omega} = \Omega \cup \partial \Omega \) is compact. Let \( r : \overline{\Omega} \to \mathbb{R}^+ \) be a positively signed defining function of \( \Omega \), i.e.:

\[
r \in C^\infty(\Omega) \text{ with } r|_\Omega > 0, \ r|_{\partial \Omega} = 0 \text{ and } (\partial_n r)|_{\partial \Omega} \neq 0,
\]

where \( \partial_n \) is the normal derivative to the boundary. The condition on the normal derivative means that \( r \) behaves like the distance to the boundary near it.

We decompose the exterior differential \( d = \partial + \bar{\partial} \) in holomorphic and antiholomorphic parts and use \( \partial_i := \partial_{z_i}, \ \bar{\partial}_i := \partial_{\bar{z}_i}. \)

The strict pseudoconvexity guarantees that the restriction \( \eta := \frac{1}{n+1}(\bar{\partial}r - \partial r)|_{\partial \Omega} \) to the boundary of the one-form \( \text{Im}(\bar{\partial}r) \) is a contact form, i.e. \( \nu := \eta \wedge (d\eta)^{n-1} \) is a volume element on \( \partial \Omega \).
Let us also consider the half-line bundle
\[ \Sigma := \{(x', s \eta x') \in T^*(\partial \Omega), s > 0\}, \] (2)
and a positive weight function \( w \) on \( \Omega \) which is decomposed the following way:
\[ w =: r^{m_w} g_w, \text{ where } m_w \in \mathbb{R} \text{ with } m_w > -1 \text{ and } g_w \in C^\infty(\overline{\Omega}) \text{ with } g_w|\partial \Omega > 0. \] (3)

As Hermitian structure, we choose the Euclidean one on \( \mathbb{C}^n \simeq \mathbb{R}^{2n} \), so \( \|dz_j\| = \sqrt{2} \) and \( \|dr\| = \sqrt{2} \|d\eta\| \).

In the following, all pseudodifferential operators are classical and considered in \( OPS_{1,0}^d \), for some \( d \in \mathbb{R} \), using Hörmander notation (see [24, Definition 7.8.1]). For any pseudodifferential operator \( Q \), we will denote by \( \sigma(Q) \) its principal symbol.

**Definition 2.1.** The weighted Bergman space is
\[ A^2_w(\Omega) := L^2_{hol}(\Omega, w) := \{ f \in L^2(\Omega, w \, d\mu), f \text{ is holomorphic on } \Omega \}, \] endowed with the norm derived from the weighted scalar product \( \langle f, g \rangle_w := \int_{\Omega} f \bar{g} \, w \, d\mu \), where \( \mu \) is the Lebesgue measure.

The Bergman space \( A^2(\Omega) \) is the unweighted Bergman space where \( w = r^0 = 1 \).

When \( \Omega = \mathbb{B}^n \) and the defining function \( r \) is radial, i.e. \( r(z) = r(|z|) \), and for a weight \( w = r^{m_w}, m_w \in \mathbb{N} \), we have the following orthonormal basis for \( A^2_w(\mathbb{B}^n) \) (see [21, Corollary 2.5]):
\[ v^w_\alpha(z) = b_\alpha z^\alpha := \left[ \int_{\mathbb{B}^n} z^\alpha \bar{z}^\alpha w(|z|) \, d\mu(z) \right]^{1/2} z^\alpha, \quad \alpha \in \mathbb{N}^n. \] (4)

In particular, an orthonormal basis of the unweighted Bergman space \( A^2(\mathbb{B}^n) \) is given by the family
\[ v_\alpha(z) = b_\alpha z^\alpha := \left( \frac{(n!)}{n ! \alpha ! \mu(\mathbb{B}^n)} \right)^{1/2} z^\alpha, \] (5)
(see [34, Lemma 1.11] for more details) where for a multiindex \( \alpha \in \mathbb{N}^n \), we set \( \alpha ! := \prod_{k=0}^{n} \alpha_k ! \) and \( |\alpha| := \sum_{k=0}^{n} \alpha_k \).

We use the standard definition of Sobolev spaces of order \( s \in \mathbb{R} \) on a subset of \( \mathbb{R}^n \) or its boundary, and on \( \Omega \subset \mathbb{C}^n \) or \( \partial \Omega \), whose construction is given in [28, Chapitre 1], and also in [20, Appendix]. We denote them by \( W^s(\Omega) \) and \( W^s(\partial \Omega) \), respectively. We assume that the norms in these spaces have been chosen so that \( W^0(\Omega) = L^2(\Omega, d\mu) \) with \( \mu \) as above, and \( W^0(\partial \Omega) = L^2(\partial \Omega) \) with respect to some smooth volume element on \( \partial \Omega \), absolutely continuous with respect to the surface measure (for instance, \( \nu = \eta \wedge (d\eta)^{n-1} \)), which we fix from now on.

**Definition 2.2.** For \( s \geq 0 \), the holomorphic (resp. harmonic) Sobolev space on \( \Omega \) of order \( s \) is defined by
\[ W^s_{hol}(\Omega) \text{ (resp. } W^s_{har}(\Omega) \text{) := } \{ f \in W^s(\Omega), f \text{ is holomorphic (resp. harmonic) on } \Omega \}. \]

Thus, \( W^0_{hol}(\Omega) = A^2(\Omega) \).

The set of harmonic function in \( L^2(\Omega, w \, d\mu) \) is denoted \( L^2_{har}(\Omega, w) \).

**Definition 2.3.** The Poisson operator \( K \) is the harmonic extension operator which solves the Dirichlet problem: \( \Delta K u = 0 \text{ on } \Omega, \ K u|_{\partial \Omega} = u \), where \( \Delta = \partial \bar{\partial} \) is the complex Laplacian.
Thus $K$ acts from functions on $\partial \Omega$ into harmonic functions on $\Omega$ and by elliptic regularity theory (see [28]), $K$ extends to a continuous map from $W^s(\partial \Omega)$ onto $W^{s+1/2}_{\text{harm}}(\Omega, w)$, for all $s \in \mathbb{R}$. In particular $K : C^\infty(\partial \Omega) \to C^\infty(\Omega)$. We denote by $K_w$ the operator $K$ considered as acting from $L^2(\partial \Omega)$ into $L^2(\Omega, w \, d\mu)$, and by $K_w^*$ its Hilbert space adjoint. For an arbitrary weight $w$, a simple computation shows $K_w^*$ is related to $K(=K_1)$ through

$$K_w^* u = K^* (wu).$$

In particular, $K_w^* K_w = K^* w K$.

Note that $K_w$ is injective since $0 = K_w u \Rightarrow K_w u|_{\partial \Omega} = 0 \Leftrightarrow u = 0$. The operator $K^*$ acts continuously from $W^s(\Omega)$ into $W^{s+1/2}(\partial \Omega)$, for all $s \in \mathbb{R}$. Thus $K : C^\infty(\partial \Omega) \to C^\infty(\Omega)$.

We consider the operator

$$\Lambda_w := K^* w K.$$  

Actually $\Lambda_w$ is an elliptic and selfadjoint pseudodifferential operator of order $-(m_w + 1)$ on $\partial \Omega$ (hence compact) with principal symbol (see [6])

$$\sigma(\Lambda_w)(x', \xi') = 2^{-1} \Gamma(m_w + 1) g_w(x') \eta_x^m \| \xi' \|^{-(m_w + 1)} \quad x' \in \partial \Omega, \xi' \in T^* \partial \Omega,$$

so, when $m_w \in \mathbb{N}$,

$$\sigma(\Lambda_w)(x', \xi') = 2^{-(m_w + 1)} (\partial_{\eta}^m w)(x') \| \xi' \|^{-(m_w + 1)}, \quad x' \in \partial \Omega, \xi' \in T^* \partial \Omega.$$ 

This is actually a subject of the extensive theory of calculus of boundary pseudodifferential operators due to Boutet de Monvel [4].

In particular, $\Lambda_w$ acts continuously from $W^s(\partial \Omega)$ into $W^{s+1/2}(\partial \Omega)$, for any $s \in \mathbb{R}$. $\Lambda_w$ is an injection since for $u \in \text{Ker}(\Lambda_w)$ and using the injectivity of $K_w$, we have $0 = \langle \Lambda_w u, u \rangle = \| K_w u \|^2$.

The inverse operator $\Lambda_w^{-1}$ is well defined on $\text{Ran}(K_w^*)$, thus we have

$$\Lambda_w^{-1} K_w^* K_w = \mathbbm{1}_{L^2(\partial \Omega)} \quad \text{and} \quad K_w \Lambda_w^{-1} K_w^* = \Pi_{\omega, \text{harm}},$$

where $\Pi_{\omega, \text{harm}}$ is the orthogonal projection from $L^2(\Omega, w)$ onto $L^2_{\text{harm}}(\Omega, w)$: the first equality is direct. Applying $K_w$ on both side of it, we deduce that $K_w \Lambda_w^{-1} K_w^*$ is the identity on $\text{Ran}(K_w)$ which is the closure of $W^1_{\text{harm}}(\Omega)$ in $L^2(\Omega, w)$, i.e. $L^2_{\text{harm}}(\Omega, w)$. Moreover, $K_w \Lambda_w^{-1} K_w^*$ vanishes on $\text{Ran}(K_w^*) = \text{Ker}(K_w^*)$, and we get the second equality of [9].

As a bounded operator, $K_w$ has the polar decomposition $K_w = U_w (K_w^* K_w)^{1/2} = U_w \Lambda_w^{1/2}$, where $U_w$ is a unitary from $L^2(\partial \Omega)$ onto $\text{Ran}(K_w) = L^2_{\text{harm}}(\Omega, w)$: $U_w^* U_w$ maps $L^2(\partial \Omega)$ to itself since

$$U_w^* U_w = \Lambda_w^{-1/2} K_w^* K_w \Lambda_w^{-1/2} = \Lambda_w^{-1/2} \Lambda_w \Lambda_w^{1/2} = \mathbbm{1}_{L^2(\partial \Omega)},$$

while $U_w^* U_w = K_w \Lambda_w^{-1} K^*_{\omega} |_{L^2_{\text{harm}}(\Omega, w)} = \mathbbm{1}_{L^2_{\text{harm}}(\Omega, w)}$ by [9].

**Definition 2.4.** The trace operator $\gamma_w : L^2(\Omega, w) \to L^2(\partial \Omega)$ is defined by

$$\gamma_w := \Lambda_w^{-1} K_w^*.$$

In particular, $\gamma_w$ gives

$$K_w \gamma_w |_{L^2_{\text{harm}}(\Omega, w)} = \mathbbm{1}_{L^2_{\text{harm}}(\Omega, w)} \quad \text{and} \quad \gamma_w K_w = \mathbbm{1}_{L^2(\partial \Omega)}.$$
The operator $\gamma_w$ is thus a left inverse of $K_w$, so it takes the boundary value of any function $f$ in $L^2_{\text{harm}}(\Omega, w)$. Again, the index $w$ is just here to recall that the operator is defined on a weighted Hilbert space. The operator $\gamma_w$ extends continuously to $\gamma_w: W^s_{\text{harm}}(\Omega) \to W^{s-1/2}(\partial \Omega)$ for any $s \in \mathbb{R}$.

We now define the following spaces:

**Definition 2.5.** The holomorphic Sobolev space on $\partial \Omega$ of order $s \in \mathbb{R}$ is

$$W^s_{\text{hol}}(\partial \Omega) := \{ u \in W^s(\partial \Omega), Ku \text{ is holomorphic on } \Omega \},$$

and the Hardy space is

$$H^2 := H^2(\partial \Omega) := W^0_{\text{hol}}(\partial \Omega),$$

with the usual norm on $L^2(\partial \Omega)$.

Note that the Hardy space is the closure of $C^\infty_{\text{hol}}(\partial \Omega)$ in $L^2(\partial \Omega)$.

We will use two types of Toeplitz operators using bold letters to refer to operators acting on Hilbert spaces defined over the domain $\Omega$ whereas the regular roman ones concern those over its boundary $\partial \Omega$:

**Definition 2.6.** If $u \in C^\infty(\partial \Omega)$, the Toeplitz operator $T_u: H^2(\partial \Omega) \to H^2(\partial \Omega)$ is

$$T_u := \Pi M_u,$$

where $\Pi: L^2(\partial \Omega) \to H^2(\partial \Omega)$ is the Szegö projection and $M_u$ (or just $u$) is the multiplication operator by $u$.

For $f \in C^\infty(\overline{\Omega})$, the Toeplitz operator $T_f: A^2(\Omega) \to A^2(\Omega)$ is defined as

$$T_f := \Pi_w M_f,$$

where $\Pi_w: L^2(\Omega, w) \to A^2(\Omega)$ is the orthogonal projection onto the space of holomorphic functions in $L^2(\Omega, w)$ and $M_f$ is the multiplication by $f$.

For the Hardy (resp. Bergman) case, we have

$$u \to T_u \text{ is linear, } T_u^* = T_{\overline{u}}, \quad T_1 = \mathbb{1}, \quad \|T_u\| \leq \|u\|_{\infty}, \quad (\text{resp. idem for } T_f).$$

**Remark 2.7.** For any strictly positive function $u$ in $L^\infty(\partial \Omega)$, $T_u$ is a selfadjoint and positive definite operator on $H^2(\partial \Omega)$ since $\langle T_u v, v \rangle = \int_{\partial \Omega} u|v|^2 > 0$ for any $v \neq 0$. In particular, it is an injection, so there exists an unbounded inverse $T_u^{-1}$, which is densely defined on $H^2(\partial \Omega)$. The same is true in the Bergman case for $T_f$, $f \in L^\infty(\Omega)$.

Boutet de Monvel and Guillemin studied in [8] (see also [6]) a more general notion of Toeplitz operators acting on Hardy spaces:

**Definition 2.8.** For a pseudodifferential operator $P$ on $L^2(\partial \Omega)$ of order $m \in \mathbb{R}$, let $T_P$ be the generalized Toeplitz operator (GTO): $W^m_{\text{hol}}(\partial \Omega) \to H^2(\partial \Omega)$ defined by

$$T_P := \Pi P|_{W^m_{\text{hol}}(\partial \Omega)}.$$
Moreover, a GTO also the usual properties π because operators, there exists another pseudodifferential operator. As shown in [8], for any GTO acting on the boundary number s.

The order can be −∞, in which case the symbol is not defined.

As shown in [8], for any GTO T_P there exists a pseudodifferential operator Q such that T_P = T_Q and [Q, Π] = 0. As a consequence, the GTO’s form an algebra: if P, Q are two pseudodifferential operators, there exists another pseudodifferential operator R such that T_P T_Q = T_R. We have also the usual properties

\[ \text{ord}(T_P T_Q) = \text{ord}(T_P) + \text{ord}(T_Q), \]
\[ \sigma(T_P T_Q) = \sigma(T_P) \sigma(T_Q). \]

Moreover, a GTO T_P of order m maps continuously holomorphic Sobolev spaces, namely

\[ T_P : W^{s+m}_{\text{hol}}(\partial \Omega) \to W^s_{\text{hol}}(\partial \Omega), \]

because Π is (or rather extends to) a continuous map from \( W^s(\partial \Omega) \) onto \( W^s_{\text{hol}}(\partial \Omega) \) for any real number s.

A GTO is said to be elliptic if its principal symbol does not vanish. Like classical pseudodifferential operators, an elliptic GTO T_P of order m admits a parametrix T_Q which is a GTO of order −m, verifying

\[ \sigma(T_Q) = \sigma(T_P)^{-1}, \quad T_P T_Q \sim 1 \quad \text{and} \quad T_Q T_P \sim 1. \]

Here and below A ∼ B means that A − B is a smoothing operator (i.e. of order −∞, or equivalently having Schwartz kernel in \( C^\infty(\partial \Omega \times \partial \Omega) \)).

Finally, if T_P is elliptic of order m ≠ 0, positive and selfadjoint as an operator on \( H^2(\partial \Omega) \), with \( \sigma(T_P) > 0 \), then the power \( T_P^s \), \( s \in \mathbb{C} \) (in the sense of the spectral theorem) is a GTO of order ms. In particular, for \( s = -1 \), the inverse of T_P is a GTO of order −m (see [14, Proposition 16] for the details).

Let \( P \) be a differential operator on \( \mathbb{C}^n \) of order \( d \in \mathbb{N} \) of the form

\[ P = \sum_{|\nu| \leq d} a_{\nu}(x) r(x)^j \partial^\nu + \sum_{|\nu'| \leq d} b_{\nu'}(x) r(x)^{j'} \overline{\partial^{\nu'}} \]  

for some \( d \in \mathbb{N}, j, j' \in \mathbb{R}^+, \nu, \nu' \in \mathbb{N}^n \) and some functions \( a_{\nu}, b_{\nu'} \in C^\infty(\overline{\Omega}) \).

We can generalize the definition of \( \Lambda_w \) and construct the operator

\[ \Lambda_w P := K^*wK \quad \text{acting on the boundary} \quad \partial \Omega. \]

By Boutet de Monvel’s theory [4, 20, 29], \( \Lambda_w P \) is again a pseudodifferential operator on the boundary and the following proposition gives a formula for its principal symbol.
Proposition 2.10. The operator $\Lambda_{w} P$ is a pseudodifferential operator on the boundary $\partial \Omega$ and $T_{\Lambda_{w} P}$ is a GTO of order $d - (m_w + 1 + j)$ with principal symbol

$$
\sigma(T_{\Lambda_{w} P})(x', \xi') = \frac{(-1)^{d+1}(m_w + 1 + j)}{2|\xi'|^{m_w + 1 + j}} g_w(x') \|\eta_{x'}\|^{-d + m_w + j} \sum_{|\nu|=d} a_{\nu}(x') \prod_{k=1}^{n} (\partial_k x')^{\nu_k}(x')
$$

(when \[12\] vanishes, $T_{\Lambda_{w} P}$ is in fact of lower order).

Proof. For $k \in \{1, \ldots, n\}$, define the tangential operators $Z_k$ and $\overline{Z}_k$ on $\partial \Omega$ by

$$Z_k := \gamma \partial_k K, \quad \overline{Z}_k := \overline{\gamma} \partial_k K,$$

As $K^* P K = \sum_{\nu} K^* a_{\nu} r^j w^\nu K + \sum_{\nu'} K^* b_{\nu'} r^j w^\nu K = \sum_{\nu} \Lambda_{a_{\nu} r^j w^\nu} + \sum_{\nu'} \Lambda_{b_{\nu'} r^j w^\nu}$ with $Z = \gamma \partial K$ and the same for $\overline{Z}$, we get, using the injectivity of $\Pi$ from [7] that indeed $K^* P K$ is a pseudodifferential operator on $\partial \Omega$ of order $d - (m_w + 1 + j)$ (or less if there are some cancellations in the summation on $\nu$). Since $T_{\Lambda_{w} P} = \Pi \Lambda_{w} P \Pi$, we have

$$\sigma(T_{\Lambda_{w} P})(x', \xi') = \sigma(\Lambda_{w} P)(x', \|\xi'\| \eta_{x'}) = \sigma(\sum_{\nu} \Lambda_{a_{\nu} r^j w^\nu} + \sum_{\nu'} \Lambda_{b_{\nu'} r^j w^\nu})(x', \|\xi'\| \eta_{x'})$$

By a direct computation, $\sigma(Z_k)(x', \xi') = i\langle \xi', Z_k \rangle = -\frac{\|\xi'\|}{\|\eta_{x'}\|} \partial_k x'$ (see also [14, p. 1440]) and from [7], we have

$$\sigma(T_{\Lambda_{w} P})(x', \xi') = \frac{(-1)^{d+1}(m_w + 1 + j)}{2|\xi'|^{m_w + 1 + j}} g_w(x') \|\eta_{x'}\|^{-d + m_w + j} \sum_{|\nu|=d} a_{\nu}(x') \prod_{k=1}^{n} \sigma(Z_k)^{\nu_k}(x', \xi'),$$

so the result follows. □

We remark that equation \[12\] is still valid when $m_w \in \mathbb{C}$ with Re$(m_w) > -1$. Also, when $m_w \in \mathbb{N}$, using $\|\partial x\| = \sqrt{2} \|\eta_{x'}\|$, the right-hand side can be written as

$$\frac{(-1)^{d+1}(m_w + 1 + j)}{2^{d + 2 + j} (m_w + 1 + j)^{-d + j}} g_w(x') \sum_{|\nu|=d} a_{\nu}(x') \prod_{k=1}^{n} (\partial_k x')^{\nu_k}(x').$$

2.2 Links between the spaces on $\Omega$ and $\partial \Omega$

The operator $T_{\Lambda_{w}}$ exists as a positive, elliptic and compact GTO of order $-(m_w + 1)$ on $L^2(\partial \Omega)$ and maps continuously $W^s_{\text{hol}}(\partial \Omega)$ into $W^{s+m_w+1}_{\text{hol}}(\partial \Omega)$, for any $s \in \mathbb{R}$.

Let $u \in \text{Ker}(T_{\Lambda_{w}}) \subset W^s_{\text{hol}}(\partial \Omega)$ for a certain $s \in \mathbb{R}$, then

$$0 = \langle T_{\Lambda_{w}} u, u \rangle_{W^s_{\text{hol}}(\partial \Omega)} = \langle \Pi \Lambda_{w} u, u \rangle_{W^s_{\text{hol}}(\partial \Omega)} = \langle \Lambda_{w} u, \Pi u \rangle_{W^s_{\text{hol}}(\partial \Omega)}.$$ 

Since $\Pi u = u$ we get, using the injectivity of $\Lambda_{w}$

$$0 = \langle \Lambda_{w} u, u \rangle_{W^s_{\text{hol}}(\partial \Omega)} = \|\Lambda_{w}^{1/2} u\|^2 \Rightarrow u = 0.$$ 

Thus, for any $s \in \mathbb{R}$, the inverse operator $T_{\Lambda_{w}}^{-1}$ exists from Ran$(T_{\Lambda_{w}}) = W^{s+m_w+1}_{\text{hol}}(\partial \Omega)$ onto $W^s_{\text{hol}}(\partial \Omega)$.

For completeness, we give the proof of the following result from [15, Theorem 4].
Proposition 2.11. Let $T$ be a positive selfadjoint operator on $H^2(\partial \Omega)$ such that $T \sim T_P$, where $P$ is an elliptic pseudodifferential operator of order $s \in \mathbb{R}$ such that $\sigma(T_P) > 0$. Let $W^T_{hol}(\partial \Omega)$ be the completion of $C^\infty_{hol}(\partial \Omega)$ with respect to the norm

$$\|u\|^2_T := \langle Tu, u \rangle_{H^2}.$$  

Then, we have

$$W^T_{hol}(\partial \Omega) = W^{|\text{ord}(T_P)|/2}(\partial \Omega).$$

Proof. We may assume that $P$ commutes with $\Pi$. The equivalence between $T$ and $T_P$ induces $T^{1/2} \sim T_P^{1/2} \sim T_P^{1/2} = \Pi P^{1/2}|_{H^2} = P^{1/2}|_{H^2}$. Since $P^{1/2}$ is elliptic, the GTO $T_P^{1/2}$ admits a parametrix, so is Fredholm. As a consequence $T^{1/2}$ is a positive, so an injective Fredholm operator, thus an isomorphism from $W^{|\text{ord}(T_P)|/2}(\partial \Omega)$ onto $H^2(\partial \Omega)$. Now if $u \in W^{|\text{ord}(T_P)|/2}(\partial \Omega)$, $T^{1/2}u$ belongs to $H^2(\partial \Omega)$. So we have the finite quantities

$$\|T^{1/2}u\|^2_{H^2} = \langle Tu, u \rangle_{H^2} = \|u\|^2_{W^T_{hol}},$$

which proves the equality $W^T_{hol}(\partial \Omega) = W^{|\text{ord}(T_P)|/2}(\partial \Omega)$.

Proposition 2.12. The operator $K_w$ maps bijectively the space $W^T_{hol}(\partial \Omega)$ onto $A^2_w(\Omega)$.

Proof. Let $f \in C^\infty_{hol}(\overline{\Omega}) \subseteq A^2_w(\Omega)$ and $u = \gamma_w f \in C^\infty_{hol}(\overline{\Omega})$. Then

$$\|f\|^2_{A^2_w} = \langle K_wu, K_wu \rangle_{L^2(\Omega)} = \langle \Lambda_wu, u \rangle_{L^2(\Omega)} = \langle \Pi \Lambda_wu, u \rangle_{L^2(\Omega)} = \langle T_{\Lambda_w}u, u \rangle_{L^2(\Omega)}.$$  

Thus $K_w$ is an isometry of $W^T_{hol}(\partial \Omega)$ onto the completion of $C^\infty_{hol}(\overline{\Omega})$ in $A^2_w$.

In a manner completely similar to (9) (see [18] for details), we get

$$\Pi_w = K_w \Pi T_{\Lambda_w}^{-1} \Pi K_w^s.$$  

(14)  

Since $C^\infty_{hol}(\overline{\Omega})$ is dense in $A^2_w$ (just note that $C^\infty(\overline{\Omega})$ is dense in $L^2(\Omega, w)$, while the weighted Bergman projection $\Pi_w$ maps each $W^s_{hol}(\Omega)$, and, hence, $C^\infty(\overline{\Omega})$ into itself), the claim follows.

From the fact that $K_w$ is an isomorphism of $W^s_{hol}(\partial \Omega)$ onto $W^{s+1/2}_{hol}(\Omega)$, we also see that $A^2_w(\Omega) = K_w W^{-(m_w+1)/2}(\partial \Omega) = W^{-(m_w+1)/2}_{hol}(\Omega)$ and $\gamma_w$ is an isomorphism of $A^2_w$ onto $W^{-(m_w+1)/2}(\partial \Omega)$.

As we already said, $T_{\Lambda_w}^{1/2}$ is an isomorphism of $W^s_{hol}(\partial \Omega)$ onto $W^{s+\frac{m_w+1}{2}}(\partial \Omega)$ for all $s \in \mathbb{R}$, with equivalent norms. As a consequence,  

Lemma 2.13. The operator

$$V_w := K_w T_{\Lambda_w}^{-1/2}$$  

is a unitary which maps $H^2(\partial \Omega)$ onto $A^2_w(\Omega)$.

(15)  

Proof. We have $V_w^* V_w = T_{\Lambda_w}^{-1/2} K_w^* K_w T_{\Lambda_w}^{-1/2} = T_{\Lambda_w}^{-1/2} T_{\Lambda_w} T_{\Lambda_w}^{-1/2} = \mathbb{1}_{H^2}$. Similarly, $V_w V_w^* = \mathbb{1}_{A^2_w(\Omega)}$, see (14).  

Now we identify a Toeplitz operator $T_f$ on $A^2_w(\Omega)$ with generalized Toeplitz operators acting on $H^2(\partial \Omega)$ via $\gamma_w$ and $K_w$ or $V_w$ and $V_w^*$.
Proposition 2.14. For $f \in \mathcal{C}^\infty(\Omega)$, we have
\[ \gamma_w T_f K_w = T_{\Lambda_w}^{-1} T_{\Lambda_w f} \quad \text{on } W_{\text{hol}}^{-(m+1)/2}(\partial\Omega), \]
\[ T_f = V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_w f} T_{\Lambda_w}^{-1/2} V_w^* \quad \text{on } A_w^2(\Omega). \tag{16} \]

Proof. For any $u$ and $v$ in $W_{\text{hol}}^{-(m+1)/2}(\partial\Omega)$, we get
\[ \langle T_f K_w u, K_w v \rangle_{A_w^2(\Omega)} = \langle \Pi_w f K_w u, K_w v \rangle_{A_w^2(\Omega)} = \langle f K_w u, K_w v \rangle_{A_w^2(\Omega)} = \langle f K_w u, K_w v \rangle_{\mathcal{H}^2(\partial\Omega)} = \langle L_{\Lambda_w f} u, \Pi v \rangle_{\mathcal{H}^2(\partial\Omega)} = \langle T_{\Lambda_w f} u, v \rangle_{\mathcal{H}^2(\partial\Omega)} = \langle K_w T_{\Lambda_w}^{-1} T_{\Lambda_w f} u, K_w v \rangle_{\mathcal{H}^2(\partial\Omega)}. \]

Thus $T_f K_w = K_w T_{\Lambda_w}^{-1} T_{\Lambda_w f}$ on $W_{\text{hol}}^{-(m+1)/2}(\partial\Omega)$, hence $\gamma_w T_f K_w = T_{\Lambda_w}^{-1} T_{\Lambda_w f}$. Finally, we get $V_w^* T_f V_w = V_w^* (K_w \gamma_w) T_f (K_w \gamma_w) V_w = T_{\Lambda_w}^{-1/2} T_{\Lambda_w f} T_{\Lambda_w}^{-1/2}$. \qed

From the GTO’s theory and the mapping properties of $K_w$ and $\gamma_w$, we see that the right-hand side in (16) extends to a bounded operator on any $W_{\text{hol}}^s(\Omega)$, hence the left-hand side enjoys the same property.

To any differential operator $P$ with coefficients as in (10), we can associate the “Toeplitz operator with symbol $P$”
\[ T_P := \Pi_w P \quad \text{acting on } A_w^2(\Omega). \tag{17} \]

Lemma 2.15. For $P$ as above, we have
\[ \gamma_w T_P K_w = T_{\Lambda_w}^{-1} T_{\Lambda_w P} \quad \text{on } W_{\text{hol}}^{-(m+1)/2}(\partial\Omega), \]
\[ T_P = V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_w P} T_{\Lambda_w}^{-1/2} V_w^* \quad \text{on } A_w^2(\Omega). \tag{18} \]

Moreover, $T_P$ is selfadjoint on $A_w^2(\Omega)$ when $P$ has a selfadjoint extension on $L^2(\Omega, w)$.

Proof. Similar calculation as in the proof of (16) shows (18).

We only need to prove that $(T_{\Lambda_w P})^*$ is selfadjoint, which follows from (11): for $u, v \in H^2(\partial\Omega)$,
\[ \langle (T_{\Lambda_w P})^* u, v \rangle_{H^2(\partial\Omega)} = \langle u, K^* w P K v \rangle_{H^2(\partial\Omega)} = \langle K u, P K v \rangle_{L^2(\Omega, w)} = \langle K w P K u, v \rangle_{L^2(\Omega, w)} = \langle w P K u, v \rangle_{H^2(\partial\Omega)} = \langle T_{\Lambda_w P} u, v \rangle_{H^2(\partial\Omega)}. \]

Remark 2.16. An interesting example of selfadjoint operator $T_P$, where $P$ is not selfadjoint on $L^2(\Omega, w)$ is given by the “weighted normal derivative” operator
\[ P_w := \sum_{j=1}^n \frac{\partial_j(rw)}{w} \partial_j; \]

note that $\partial_j(rw)/w$ is smooth up to the boundary. Using Stokes’ formula, for $f, g$ in $A_w^2(\Omega)$,
\[ \int_{\Omega} d\mu \partial_j(rw f g) = - \int_{\partial\Omega} d\sigma rw f g \frac{\partial_j r}{2|\partial\sigma|^2} = 0 \]

since $rw = 0$ on $\partial\Omega$ (here $d\sigma$ is the ordinary surface measure on $\partial\Omega$). Applying Leibniz rule to the LHS gives
\[ \int_{\Omega} w d\mu \partial_j(rw f g) = - \int_{\Omega} w d\mu (r \partial_j f) g \quad \text{hence } T_{\partial_j(rw)/w} + T_r \partial_j = 0 \]

10
where the Toeplitz operators act on $A^2_w(\Omega)$. Since $T_h^* = T_h^*$ and $T_{h\partial_j} = T_h T_{\partial_j}$, we get

$$T_{P^w} = \sum_{j=1}^n (T_{\partial_j(rw)/w})^* T_{\partial_j} = -\sum_{j=1}^n (T_{rw})^* T_{\partial_j} = -\sum_{j=1}^n T_{\partial_j}^* T_r T_{\partial_j}.$$  

So $T_{P^w}$ is not only selfadjoint but even positive on $A^2_w(\Omega)$.

Now in the context of the unweighted Bergman space, we can even compute the symbol of the operator $\gamma T_{P^w=1}$. Indeed, using Stokes’ formula, we get for $f, g \in A^2(\Omega)$,

$$\langle T_{\partial_j} f, g \rangle_{A^2(\Omega)} = \langle \partial_j f, g \rangle_{A^2(\Omega)} = \int_\Omega d\mu (\partial_j f) \bar{g} = -\int_{\partial\Omega} d\sigma f \bar{g} \frac{\partial \eta}{2|\partial r|} - \int_\Omega d\mu f \partial_j (\bar{g}).$$

It follows that, on $W^{-1/2}_0(\partial \Omega)$, $K^* T_{\partial_j} K = -T_{\partial_j r/2|\partial r|}$ and

$$\gamma T_{\partial_j} K = -\Lambda^{-1} T_{\partial_j r/2|\partial r|},$$

$$T_{\partial_j} = -V T_{\Lambda^{-1/2} T_{\partial_j r/2|\partial r|} T_{\Lambda^{-1/2} V^*}}.$$  

The use of (7), (12) and (18) gives

$$\sigma(\gamma T_{P^w=1} K)(x', \xi') = -2 \|\eta x'\| \|\xi'\|, \quad (x', \xi') \in \Sigma.$$  

In particular $-V^* T_{P^w=1} V$ is an elliptic GTO.

**Remark 2.17.** The hypothesis $P = P^*$ in Lemma 2.15 is quite strong because the equality $T^*_P = T^*_{P^w}$ is not necessarily true: when $w = 1$, one deduces from (19) that

$$T_{\partial_j} = -V T_{\Lambda^{-1/2} T_{\partial_j r/2|\partial r|} T_{\Lambda^{-1/2} V^*}} \neq 0 = T_{-\overline{\partial}_j},$$

while $-\overline{\partial}_j$ is the formal adjoint of $\partial_j$ on the domain of smooth compactly supported functions on $\Omega$. Of course, any selfadjoint extension of a differential operator $P$ needs to take care of the boundary conditions on $\partial \Omega$.

We now establish a result on unitaries $U$ on $A^2_w(\Omega)$ such that $V_w^* U V_w$ is a GTO, result which will be used later on in Section 6.

**Lemma 2.18.** Let $g \in S^0(\mathbb{R})$, i.e.

$$\forall j \in \mathbb{N} \text{ there exists } c_j(g) > 0 \text{ with } |\partial_j^2 g(\xi)| \leq c_j(g)(1 + \|\xi\|)^{-j}, \forall \xi \in \mathbb{R},$$

and let $A$ be an elliptic selfadjoint pseudodifferential operator of order 1 on a compact manifold $M$. Then the operator $\exp(ig(A))$ is a pseudodifferential operator of order 0 on $M$.

**Proof.** The Faà di Bruno’s formula states that for a couple $f, g$ of smooth functions on $\mathbb{R}$,

$$\partial^n (f \circ g)(\xi) = \sum_{(k_1, \ldots, k_n) \in K_n} \frac{n!}{k_1! \cdots k_n!} (\partial^{k_1} f)(g(\xi)) \prod_{j=1}^n (\frac{1}{j!} \partial^j g)^{k_j}(\xi), \quad \forall n \in \mathbb{N},$$
where \( K_n = \{(k_1, \ldots, k_n) \in \mathbb{N}^n, k_1 + 2k_2 + \cdots + nk_n = n\} \).

Thus, if \( f : \xi \in \mathbb{R} \mapsto \exp(i\xi) \) and \( g \in S^0(\mathbb{R}) \), we get
\[
|\partial^n(f \circ g)(\xi)| \leq \sum_{(k_1, \ldots, k_n) \in K_n} \frac{n!}{k_1! \cdots k_n!} \left[ \prod_{j=1}^n (\frac{1}{j!} c_j(g))^{k_j} \right] (1 + ||\xi||)^{-n}.
\]

So there are constants \( c_n(f \circ g) > 0 \) such that
\[
|\partial^n(f \circ g)(\xi)| \leq c_n(f \circ g)(1 + ||\xi||)^{-n},
\]
and we have shown that the function \( \xi \in \mathbb{R} \mapsto \exp(ig(\xi)) \) is in \( S^0(\mathbb{R}) \).

Applying [31, Theorem 1] (or [32, Theorem 1.2]), we conclude that \( \exp(ig(A)) \) is a pseudodifferential operator of order 0 on \( M \).

**Corollary 2.19.** Let \( \varphi \) be any function in \( S^0(\mathbb{R}) \) (for instance, \( \varphi(\xi) := (1 + \xi^2)(1 + \xi^2)^{-1} \)) and choose a pseudodifferential operator \( A \) on \( \partial \Omega \) of order 1 such that \( A \) commutes with \( \Pi \) and \( T_A = (T_{\Lambda w})^{-1/(\text{deg } A - 1)} \). Denote \( V_\varphi := \exp(i\varphi(A)) \).

Then \( TV_\varphi \) is a unitary generalized Toeplitz operator on \( H^2(\partial \Omega) \).

**Proof.** By previous lemma, \( V_\varphi := \exp(i\varphi(A)) \) is a unitary classical pseudodifferential operator on \( M = \partial \Omega \) which commutes with \( \Pi \), thus \( TV_\varphi = \exp[i\varphi(T_{\Lambda w}^{-1/(\text{deg } A - 1)})] \) is a unitary generalized Toeplitz operator.

**Remark 2.20.** In Section 5.2, we will need to find unitary operators on \( A^2_w(\Omega) \) of the form \( V_wTV_w^* \), with \( T \) a GTO, to deduce non-positive Dirac-like operators from positive ones. So we can take \( U := V_wTV_w^*V_w^* \), with \( T \) defined in Corollary 2.19.

Another class of unitary operators on \( A^2_w(\Omega) \) can be obtained as follows. Take any GTO \( T \) which is invertible and not a constant multiple of a positive operator. (For instance, \( T = T_f \) with \( f \) a nonconstant zero-free holomorphic function: the zero-free condition ensures \( T_f = M_f \) is invertible, while \( T_f = cA \) with \( A > 0 \) would mean that multiplication by the nonconstant holomorphic function \( f/c \) is a positive operator, which is quickly seen to lead to contradiction.)

We know there exists another pseudodifferential operator \( Q \) such that \( \Pi Q = Q \Pi \) and \( T_P = T_Q \). Hence also \( T_PT_P = T_QT_Q = T_Q = T_Q^2 = T_Q^2 \), implying that \( U := T_P T_{\Pi}^{-1} \) is a unitary generalized Toeplitz operator. From Proposition 2.14, the operator \( U := V_wU^*V_w^* \) is unitary from \( A^2_w(\Omega) \) onto \( A^2_w(\Omega) \). Furthermore, \( U \) is not a multiple of the identity; for, if it were, then so would be \( U \), hence \( T_P = U T_{\Pi} \Pi \) would be a constant multiple of the positive operator \( T_{\Pi}^{1/2} \), contrary to the hypothesis.

### 3 Fock space and Heisenberg algebra

#### 3.1 Fock space and Segal–Bargmann transform

If \((x, y) \in \mathbb{R}^{2n}\), we denote \( xy := \sum_{k=1}^n x_k y_k \). For \((z, z') \in \mathbb{C}^{2n}\), we use the usual scalar product \( (z, z') := \sum_{k=1}^n z_k \bar{z}'_k \). Also, for \((x, z) \in \mathbb{R}^n \times \mathbb{C}^n\), we denote \( xz \) the complex number \( \sum_{k=1}^n x_k z_k \) (so \( x^2 = \sum_{k=1}^n x_k^2 \) and \( z^2 = \sum_{k=1}^n z_k^2 \)), and \( |z|^2 := \sum_{k=1}^n |z_k|^2 \).

We recall now some known results about the Fock space:

**Definition 3.1.** For \( t > 0 \), the Fock space is
\[
\mathcal{F}_t := L^2_{hol}(\mathbb{C}^n, dm_t(z)), \ \text{where} \ dm_t(z) := (\pi t)^{-n} e^{-|z|^2/4t} \, dm(z).
\]
This Fock space is denoted $F$ in [25, Chap. 1.7] when $t = \pi^{-1}$. Here, the measure $dm_t$ has been chosen such that $m_t(1) = 1$. The family of functions $\{u_\alpha\}_{\alpha \in \mathbb{N}^n}$, where
\[
u_\alpha(z) = a_\alpha z^\alpha := (t^{\alpha!} \alpha!)^{-1/2} z^\alpha,
\]forms an orthonormal basis of $F_t$.

**Definition 3.2.** The Segal–Bargmann transform $W_t : L^2(\mathbb{R}^n, dx) \rightarrow F_t$ is
\[
(W_t f)(z) := (\pi t)^{-n/4} \int_{x \in \mathbb{R}^n} e^{(-z^2 + 2\sqrt{2}xz - x^2)/2t} f(x) dx.
\]

**Proposition 3.3.** The Segal–Bargmann transform $W_t$ is a unitary map from $L^2(\mathbb{R}^n)$ to $F_t$. Moreover, for any $f \in F_t$,
\[
(W_t^{-1} f)(x) = (\pi t)^{-5n/4} \int_{z \in \mathbb{C}^n} e^{(-z^2 + 2\sqrt{2}z\bar{z} - x^2)/2t} f(z) e^{-|z|^2/t} d\mu(z).
\]
A proof is given in the Appendix.

The map $W_t$ plays an important role since it links unitarily functions of real variables to holomorphic functions of complex variables.

### 3.2 The Heisenberg group and its Lie algebra

**Definition 3.4.** The Heisenberg group $\mathbb{H}^n$ is the set $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with product:
\[(q, p, s) (q', p', s') = (q + q', p + p', s + s' + \frac{1}{2}(qp' - p'q')),(q, p) \in \mathbb{R}^{2n} \text{ and } s \in \mathbb{R}.
\]The unit element is $(0, 0, 0)$ and $(q, p, s)^{-1} = (-q, -p, -s)$. Let us define for $j \in \{1, \cdots, n\}$ the generators $Q_j, P_j$ and $T$ of $\mathbb{H}^n$ as:
\[
\begin{align*}
\exp(Q_j) &:= (1_j, (0, \cdots, 0), 0), \\
\exp(P_j) &:= ((0, \cdots, 0), 1_j, 0), \\
\exp(T) &:= ((0, \cdots, 0), (0, \cdots, 0), 1),
\end{align*}
\]
where $1_k$ denotes the multiindex being zero everywhere and 1 at the $k^{th}$ position. These generators form a basis of the Lie algebra $\mathfrak{h}^n$ of $\mathbb{H}^n$. The only non null commutation relations are $[Q_j, P_k] = \delta_{j,k} T$. We also define the two elements of $\mathfrak{h}^n$
\[
a_j := \frac{1}{\sqrt{2}} (Q_j + iP_j) \text{ and } a_j^+ := \frac{1}{\sqrt{2}} (Q_j - iP_j),
\]
which verify $[a_j, a_k^+] = -\frac{1}{2} ([Q_j, P_k] + [Q_k, P_j]) = -i\delta_{j,k} T$. We will also use the element $N$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{h}^n)$ of $\mathfrak{h}^n$:
\[
N := \frac{1}{2} \sum_{j=1}^{n} a_j^+ a_j + a_j a_j^+.
\]
3.3 Representations of $\mathfrak{h}^n$

Different representations of $\mathfrak{h}^n$ on the functions spaces we consider here have been studied in [25] and later by [33] with different approaches.

3.3.1 Schrödinger representation

**Definition 3.5.** The Schrödinger representation $\rho_t$ of $\mathfrak{h}^n$ on $L^2(\mathbb{R}^n, dx)$ is defined by

$$\rho_t(Q_j)f(x) := x_j f(x), \quad \rho_t(P_j)f(x) := -it' \partial x_j f(x), \quad \rho_t(T)f(x) := it f(x),$$

where $t'$ is a strictly positive parameter.

With this representation, the canonical commutation relations from the quantum mechanics can be recovered by setting $t' = \hbar$, the Planck constant, $\bar{x}_j = \rho(Q_j)$ and $\hat{p}_j = \rho(P_j)$, respectively the position and momentum operators: $[\bar{x}_j, \hat{p}_j] = -i\hbar \partial x_j + i\hbar \partial x_j = i\hbar \hat{1} = \rho([Q_j, P_j])$.

We have also

$$\rho_t(a_j) = \frac{1}{\sqrt{2}}(x_j + t' \partial x_j), \quad \rho_t(a^+_j) = \frac{1}{\sqrt{2}}(x_j - t' \partial x_j), \quad \rho_t(N) = \frac{i}{\hbar} \sum_{j=1}^n (x_j^2 - t' \partial^2 x_j).$$

3.3.2 Fock representation

From the Schrödinger representation $\rho_t$, we use the unitary map $W_t$ to get a unitary representation of $\mathfrak{h}^n$ on the Fock space $\mathcal{F}_t$ by choosing $t' = t$.

**Definition 3.6.** The Fock representation $\nu_t$ of an element $h \in \mathfrak{h}^n$ on $\mathcal{F}_t$ is

$$\nu_t(h) := W_t \rho_t(h) W_t^{-1}.$$  

**Proposition 3.7.** The explicit actions on the basis vector of $\mathcal{F}_t$ are given by

$$\nu_t(Q_j) u_\alpha = \left( \frac{t}{2} \right)^{1/2} \left( \sqrt{\alpha_j} u_{\alpha - 1_j} + \sqrt{\alpha_j + 1} u_{\alpha + 1_j} \right),$$

$$\nu_t(P_j) u_\alpha = -i \left( \frac{t}{2} \right)^{1/2} \left( \sqrt{\alpha_j} u_{\alpha - 1_j} - \sqrt{\alpha_j + 1} u_{\alpha + 1_j} \right),$$

$$\nu_t(T) u_\alpha = it u_\alpha,$$

$$\nu_t(a_j) u_\alpha = t^{1/2} \sqrt{\alpha_j} u_{\alpha - 1_j}, \quad \nu_t(a^+_j) u_\alpha = t^{1/2} \sqrt{\alpha_j + 1} u_{\alpha + 1_j}, \quad \nu_t(N) u_\alpha = t (|\alpha| + \frac{n}{2}) u_\alpha.$$

**Proof.** Differentiating in Definition 3.2, we get (see Appendix)

$$(\partial z_j W_t f)(z) = (W_t (\frac{z_j}{t} + \frac{\sqrt{2}z_j}{t}) f)(z), \quad \frac{z_j + \sqrt{2}z_j}{t} W_t = W_t \frac{\sqrt{2}z_j}{t}, \quad \text{or} \quad W_t \frac{\sqrt{2}z_j}{t} W_t^{-1} = \frac{t}{2} z_j + \partial z_j.$$

Similarly, integrating by parts in Definition 3.2, we get $(W_t (\partial z_j f)) (z) = (W_t (\frac{z_j}{t} - \frac{\sqrt{2}z_j}{t}) f)(z)$, so

$$W_t \left( \frac{z_j}{t} - \partial z_j \right) = \frac{\sqrt{2}z_j}{t} W_t, \quad \text{or} \quad W_t \left( \frac{z_j}{t} - \partial z_j \right) W_t^{-1} = \frac{\sqrt{2}z_j}{t}.$$

Consequently,

$$W_t z_j W_t^{-1} = \frac{1}{\sqrt{2}} (z_j + t \partial z_j), \quad W_t \partial z_j W_t^{-1} = \frac{1}{\sqrt{2}} (-z_j + t \partial z_j).$$

Using the formulas in §3.3.1, we get the result. \qed
3.3.3 Bergman representation

The Fock and Bergman spaces are linked through a simple change of basis via the unitary operator $\mathcal{Y}_t^w : \mathcal{F}_t \rightarrow A^2_w(\Omega)$:

$$\mathcal{Y}_t^w (u_\alpha) := v_\alpha^w,$$

(22)

where $u_\alpha$ as in [21] and $v_\alpha^w$ any orthonormal basis of $A^2_w(\Omega)$.

We will also need the map

$$U_t^w := \mathcal{Y}_t^w W_t,$$

(23)

which is by construction, a unitary operator from $L^2(\mathbb{R}^n, dx)$ to $A^2_w(\Omega)$.

For the peculiar case $\Omega = \mathbb{B}^n$, $\partial \Omega = S^{2n-1}$ and $w = r^0 = 1$, we can get explicit formulas. Using the basis $[3]$ of $A^2(\mathbb{B}^n)$, we denote the above change of basis by $\mathcal{Y}_t : \mathcal{F}_t \rightarrow A^2(\mathbb{B}^n)$:

$$\mathcal{Y}_t (u_\alpha) := v_\alpha.$$

(24)

Here again, we fix the constant $t'$ from the Schrödinger representation $\rho'_t$ with $t' = t$ to define the Bergman representation.

**Definition 3.8.** The Bergman representation $\tau_t^w$ of an element $h \in \mathfrak{h}^n$ on $A^2_w(\Omega)$ is given by

$$\tau_t^w(h) := \mathcal{Y}_t^w \rho_t(h) \mathcal{Y}_t^{w-1} = U_t^w \rho_t(h) U_t^{w-1}.$$

We denote by $\tau_t$ this representation in the case $\Omega = \mathbb{B}^n$, $\partial \Omega = S^{2n-1}$ and $w = r^0 = 1$.

We get directly the following results, since the Bergman representation differs from the Fock representation by a simple change of basis:

**Proposition 3.9.** The representation $\tau_t^w$ has the following properties:

$$\tau_t^w(Q_j) v_\alpha^w = (t_j^2)^{1/2} (\sqrt{\alpha_j} v_{\alpha-1,j}^w + \sqrt{\alpha_j + 1} v_{\alpha+1,j}^w),$$

$$\tau_t^w(P_j) v_\alpha^w = -i (t_j^2)^{1/2} (\sqrt{\alpha_j} v_{\alpha-1,j}^w - \sqrt{\alpha_j + 1} v_{\alpha+1,j}^w),$$

$$\tau_t^w(T) v_\alpha^w = it v_\alpha^w,$$

$$\tau_t^w(a_j) v_\alpha^w = t^{1/2} \sqrt{\alpha_j} v_{\alpha-1,j}^w, \quad \tau_t^w(a_j^+) v_\alpha^w = t^{1/2} \sqrt{\alpha_j + 1} v_{\alpha+1,j}^w, \quad \tau_t^w(N) v_\alpha^w = t(|\alpha| + \frac{n}{2}) v_\alpha^w.$$

3.4 A diagram as a summary

The figure below shows a diagram with all the representations and maps involved here: the respectively dotted, dashed, simple, thick arrows refer to injections, surjections, isomorphisms, isometries. Double arrows indicate the action of the Lie algebra $\mathfrak{h}^n$ on the Hilbert spaces.

We must highlight the fact that only composition of maps which does not involve dotted or dashed lines (projectors) are commutative. Indeed, for instance $U_w \Pi_w \neq \Pi V_w$.

Note that $K_w$ maps $L^2(\partial \Omega)$ into $L^2_{\text{varm}}(\Omega, w)$ with dense range and that $\gamma \Pi_w K_w = T_{\lambda, w}^{-1} \Pi \Lambda_w$ on $L^2(\partial \Omega)$ (see [15, Proposition 8]), so $\gamma \Pi_w K_w \Pi = \Pi$.

For the case $\Omega = \mathbb{B}^n$, $w = r^0 = 1$, replace $W_{\text{hol}}^{-\langle mw + 1 \rangle/2} (\partial \Omega)$, $\mathcal{Y}_t^w$, $U_t^w$ and $V_w$ of [15] respectively by $W_{\text{hol}}^{-1/2} (\partial \Omega)$, $\mathcal{Y}_t$ as in [24], $U_t$ as in [23] and $V$.  

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4 Pseudodifferential and Toeplitz operators

The relation between pseudodifferential operators and Toeplitz operators has been studied in [25], [8] and [22]. The authors show that these operators enjoy a similar symbolic calculus. Moreover, there exists an isomorphism between pseudodifferential operators of order \( k \) on \( L^2(\mathbb{R}^n) \) and Toeplitz operators of order \( k/2 \) on the Bergman space \( A^2(\mathbb{B}^n) \). This isomorphism is nothing else than the conjugation by the unitary \( U_t \) defined above (see [33, Appendix B] for a detailed proof).

4.1 Toeplitz operators on \( A^2(\mathbb{B}^n) \) as representation of elements in \( \mathcal{U}(\mathfrak{h}^n) \)

We express here the Toeplitz operators \( T_{z^\alpha} \) and \( T_{\partial^\alpha} \) acting on the Bergman space \( A^2(\mathbb{B}^n) \) as representations of elements in the enveloping algebra \( \mathcal{U}(\mathfrak{h}^n) \). We follow [25, Chap. 4.2] for the first result.

**Proposition 4.1.** For \( \alpha \in \mathbb{N}^n \), let \( a^\alpha := \prod_{j=1}^n a_j^{\alpha_j} \), \( (a^+)^\alpha := \prod_{j=1}^n (a_j^+)^{\alpha_j} \) (the order is not important since \( [a_j, a_k] = [a_j^+, a_k^+] = 0 \)), and \( g^{\pm}_\alpha := \prod_{k=1}^n (N - i(\frac{\nu}{2} \pm k)) \) be elements of \( \mathcal{U}(\mathfrak{h}^n) \). The Toeplitz operators \( T_{z^\alpha} \) and \( T_{\partial^\alpha} \) from \( A^2(\mathbb{B}^n) \) to \( A^2(\mathbb{B}^n) \) can be written as

\[
T_{z^\alpha} = \tau_1((a^+)^\alpha \ (\tau(g^+_{\alpha}))^{-1/2}) \quad \text{and} \quad T_{\partial^\alpha} = \tau_1(a^\alpha \ (\tau(g^-_{\alpha}))^{-1/2}).
\]

Notice that the \( t \)-dependence in the respective right hand sides compensate, so these sides do not depend on \( t \).

**Proof.** Since \( z \mapsto z^\alpha \) is holomorphic, \( T_{z^\alpha} = M_{z^\alpha} \) and we have also \( T_{\partial^\alpha} = \Pi_z \partial^\alpha = \partial^\alpha \) on \( A^2(\mathbb{B}^n) \). Applied on the basis, they yield

\[
T_{z^\alpha} v_\beta = z^\alpha v_\beta = \frac{b_\alpha}{b_{\beta+a}} v_{\beta+a} = \left( \frac{(\beta+a)!}{\beta!} \frac{(|\beta|+n)!}{(|\beta|+|\alpha|+n)!} \right)^{1/2} v_{\beta+a}, \quad \forall \alpha, \beta,
\]

\[
T_{\partial^\alpha} v_\beta = \partial^\alpha v_\beta = \frac{\beta}{(\beta-\alpha)} \frac{b_\alpha}{b_{\beta-a}} v_{\beta-a} = \left( \frac{\beta!}{(\beta-\alpha)!} \frac{(|\beta|+n)!}{(|\beta|-|\alpha|+n)!} \right)^{1/2} v_{\beta-a}, \quad \beta \geq \alpha,
\]

(25)
Moreover, the representations of the elements \( g^\pm_\alpha \) act on \( A^2(\mathbb{B}^n) \) as

\[
\tau_\lambda (g^\pm_\alpha) v_\beta = \prod_{k=1}^{[\alpha]} (\tau_\lambda (N) - i(n \pm k) \tau_\lambda (T)) v_\beta = t^{[\alpha]} \prod_{k=1}^{[\alpha]} ([\beta] + n \pm k) v_\beta = t^{[\alpha]} |(\beta \pm |\alpha| + n)! / (\beta + n)! | v_\beta.
\]

Thus the operators \( \tau_\lambda (g^\pm_\alpha) \) are invertible on \( A^2(\mathbb{B}^n) \) and we get the claimed formulae. \( \square \)

**Lemma 4.2.** The operator

\[
R := \sum_{j=1}^n T_{z_j} T_{\partial_j} = \sum_{j=1}^n T_{z_j \partial_j} \quad \text{on } A^2(\mathbb{B}^n)
\]

is positive and

\[
\tau_\lambda (P_j) = -i (\frac{\lambda}{2})^{1/2} [T_{\partial_j} (R + n)^{-1/2} - (T_{\partial_j} (R + n)^{-1/2})^*]
\]

(showing again that \( \tau_\lambda (P_j) \) is selfadjoint).

The operator \( \gamma \tau_\lambda (P_j) K \) is a GTO of order \( \frac{1}{2} \) with principal symbol

\[
\sigma(\gamma \tau_\lambda (P_j) K)(\xi', \xi') = 2^{3/4} (\frac{\lambda}{2})^{1/2} (\frac{\|\xi\|}{R(\xi')})^2 \frac{\xi'}{\|\xi\|^2}, \quad (\xi', \xi') \in \Sigma.
\]

**Proof.** By \([28]\), \( R v_\beta = |\beta| v_\beta \) for any multiindex \( \beta \), so \( R \geq 0 \). Since

\[
T_{\partial_j} (R + n)^{-1/2} v_\alpha = \sqrt{\alpha_j} v_{\alpha - 1_j}, \quad (R + n)^{1/2} T_{z_j} v_\alpha = (\alpha_j + 1)^{1/2} v_{\alpha + 1_j}
\]

we get

\[
(T_{z_j})^* v_\alpha = (\frac{\alpha_j}{|\alpha| + n})^{1/2} v_{\alpha - 1_j}.
\]

(These relations could also be deduced from \( (T_{z_j} F)^* = t T_{\partial_j} F \), where \( T_{\partial_j} F \) refers to the Toeplitz operator on the Fock space, and for instance, \( T_{\partial_j} = \sqrt{\gamma} T_{\partial_j} \gamma^{-1} (R + n)^{1/2} \).

By Proposition \([39]\) this yields

\[
\tau_\lambda (P_j) v_\alpha = -i (\frac{\lambda}{2})^{1/2} \left[ \frac{1}{|\alpha| + n} T_{\partial_j} v_\alpha - \sqrt{|\alpha| + n + 1} T_{z_j} v_\alpha \right],
\]

and \( \tau_\lambda (P_j) = -i (\frac{\lambda}{2})^{1/2} [T_{\partial_j} (R + n)^{-1/2} - T_{z_j} (R + n + 1)^{1/2}] \). Thus, \([28]\) implies \([26]\).

As in the proof of Remark \([2.16]\) we get for \( f, g \in A^2(\Omega) \) (omitting the sum over \( j \)):

\[
\langle (R + n) f, g \rangle = \int_{\Omega} d\mu \partial_j (z_j f) \overline{g} = \int_{\Omega} d\mu \partial_j (z_j f) \overline{g} = - \int_{\partial \Omega} d\sigma f \overline{g} \frac{z_j \partial_j r}{2 |\sigma|}. \quad (29)
\]
Moreover, since $\gamma(17)$, with images of the usual Dirac operator as Bergman and Hardy spaces. The natural candidates $A$ of Toeplitz operators acting on Hilbert spaces $H(\Omega)$.

Using (26) with $\gamma_t$, we get $\sigma(\gamma(P_t)K) = -i(\frac{1}{2})^{1/2}[\sigma(A) - \sigma(A')] = 2(\frac{1}{2})^{1/2}\text{Im}(\sigma(A))$

Finally, $\sigma(\gamma(P_t)K)(x', \xi') = -2(\frac{1}{2})^{1/2}\bigg(\sigma(A)^{-1/2}\sigma(X)^{-1/2}\bigg(2\|\partial r\|^{-1}\text{Im}(\partial_j r)\bigg)\bigg)(x', \xi')$

Moreover, $\text{Im}(\partial_j r)(x') = (\eta_{x'})_j = \|\xi\|^{-1/2}\xi_j$, which yields the result.

4.2 Dirac-like operators on $A_w(\Omega)$ and $H^2(\partial\Omega)$

Our goal is to construct spectral triples $(\mathcal{A}, \mathcal{H}, D)$ (see Definition 5.1 below) using the algebra $\mathcal{A}$ of Toeplitz operators acting on Hilbert spaces $\mathcal{H}$ of functions on bounded domains such as Bergman and Hardy spaces. The natural candidates $D_0\Omega$ and $D_\Omega$ defined below are the images of the usual Dirac operator $D := -i\sum_{j=1}^n\gamma_j\partial_{x_j}$ on $\mathbb{R}^n$ through the maps involved in the diagram of Section 3.4. Here the $\Gamma_j$ are the usual selfadjoint gamma matrices which represent the $n$-dimensional Clifford algebra $\mathbb{C}_n \cong \mathbb{C}^{2^{[n/2]}}$. The resulting operators act on $H^2(\partial\Omega) \otimes \mathcal{C}_n$ or $A_w^2(\Omega) \otimes \mathcal{C}_n$, respectively, and are even a generalized Toeplitz operator of order 1/2, according to previous result.

Since the representation of $P_j \in \mathfrak{h}^n$ on $L^2(\mathbb{R}^n)$ is $\rho_t(P_j) = -it\partial_j$, we define:

$$D_0\Omega := (\frac{1}{2})^{-1/2} \sum_{j=1}^n \Gamma_j V_w^* \gamma_j^w(P_j) V_w,$$

$$D_{\Omega} := V_w D_0\Omega V_w^* = (\frac{1}{2})^{-1/2} \sum_{j=1}^n \Gamma_j U_t^w \rho_t(P_j) U_t^{w*},$$

acting respectively on $H^2(\partial\Omega) \otimes \mathcal{C}_n$ and $A_w^2(\Omega) \otimes \mathcal{C}_n$. They are selfadjoint and again, do not depend on $t$.

Another way to construct operators $D$ on $A_w^2(\Omega)$ is to consider an operator $T_P$ of the form $[17]$, with $P$ a selfadjoint differential operator on $L^2(\Omega, w)$ as in Proposition 2.10.
5 Spectral triples

Definition 5.1. A (unital) spectral triple is defined by the data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ with
- an involutive unital algebra $\mathcal{A}$,
- a faithful representation $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$,
- a selfadjoint operator $\mathcal{D}$ acting on $\mathcal{H}$ with compact resolvent such that for any $a \in \mathcal{A}$, the extended operator of $[\mathcal{D}, \pi(a)]$ is bounded.

The spectral dimension of the triple is $d := \inf\{d' > 0 \text{ such that } \text{Tr} |\mathcal{D}|^{-d'} < \infty\}$.

The spectral triple is called regular if the spaces $\mathcal{A}$ and $[\mathcal{D}, \mathcal{A}]$ are contained in the domain of $\delta^k$, for all $k \in \mathbb{N}$, where $\delta(a) := |\mathcal{D}|, a \in \mathcal{A}$.

5.1 Spectral triples for Hardy space on $\Omega$

Proposition 5.2. For a bounded domain $\Omega$ as in Section 2.1, let $\mathcal{A}_H$ be the algebra of all GTO’s of order $0$, with the identity representation $\pi$ on $\mathcal{H} := H^2(\partial \Omega)$, and $\mathcal{D}$ be a selfadjoint elliptic generalized Toeplitz operator of order 1 on $\mathcal{H}$.

Then $(\mathcal{A}_H, \mathcal{H}, \mathcal{D})$ is a regular spectral triple of dimension $n = \text{dim}_C \Omega$.

Proof. Clearly $\mathcal{A}_H$ is an algebra with unit $T_1 = 1$, and involution $T_P^* = T_P^*$, where $P^*$ is the adjoint of $P$ in $L^2(\partial \Omega)$, and trivially $\pi$ is faithful. Since $\mathcal{D}$ is elliptic of order 1, it has a parametrix of order $-1$, hence compact, so $\mathcal{D}$ has compact resolvent. Moreover, for any $T_P \in \mathcal{A}_H$, the commutator $[\mathcal{D}, T_P]$ is bounded since, as commutator of GTO’s, $\text{ord}([\mathcal{D}, T_P]) \leq \text{ord}([\mathcal{D}, T_P]) - 1 \leq 1 + 0 - 1 = 0$. So $(\mathcal{A}_H, \mathcal{H}, \mathcal{D})$ is a spectral triple.

Since $[\mathcal{D}]$ is of order 1 (see for instance [14, Proposition 16]), one can check recursively that for all $k \in \mathbb{N}$ and $T_P \in \mathcal{A}_H$, $\delta^k(T_P) = [[\mathcal{D}, T_k], T_k]$, where $T_k$ is a GTO of order 0, so the commutator is bounded. The same is true for elements of the form $T_P = [\mathcal{D}, T_Q], T_Q \in \mathcal{A}_H$, so the regularity follows.

For the dimension computation, we follow [18, Theorem 3]. We sort the points $\lambda_j$ of the spectrum of $[\mathcal{D}] := (\mathcal{D}^* \mathcal{D})^{1/2}$ counting multiplicities as $0 < \lambda_1 \leq \lambda_2 \leq \cdots$. Let $M(\lambda)$ be the number of $\lambda_j$’s less than $\lambda$. We can apply [8, Theorem 13.1] to $[\mathcal{D}]$ which is of order 1:

$$M(\lambda) = \lambda \rightarrow \infty \frac{\text{vol}(|\Sigma_D|)}{(2\pi)^n} \lambda^n + \mathcal{O}(\lambda^{-1}),$$

where, using $\mathcal{D}$, $\Sigma_D := \{(x, \xi) \in \Sigma : \sigma(\mathcal{D})(x, \xi) \leq 1\}$. So, if $c = (2\pi)^{-n} \text{vol}(\Sigma_D)$, we get for large $\lambda$

$$\lambda^n = \frac{M(\lambda)}{c} + \mathcal{O}(\lambda^{-1}) = \frac{M(\lambda)}{c} + \mathcal{O}(\lambda^{-1}M(\lambda)).$$

Since $M(\lambda)^{-1/n} \sim \mathcal{O}(\lambda^{-1})$, we have $\lambda^n = \frac{M(\lambda)}{c} + \mathcal{O}(M(\lambda)^{1-1/n}) = \frac{M(\lambda)}{c}[1 + \mathcal{O}(M(\lambda)^{-1/n})]$ as $\lambda \rightarrow \infty$, so given $d \in \mathbb{R}$,

$$\lambda^{-d} = \frac{e^{d/n}[1+\mathcal{O}(M(\lambda)^{-1/n})]}{M(\lambda)^{d/n}} = \frac{e^{d/n}}{M(\lambda)^{d/n}} + \mathcal{O}(\frac{1}{M(\lambda)^{(d+1)/n}}).$$

Thus

$$\text{Tr} |\mathcal{D}|^{-d} = \sum_{j=1}^{\infty} \lambda_j^{-d} = \int_{\lambda_1}^{\infty} \lambda^{-d} dM(\lambda) = \int_{\lambda_1}^{\infty} \left( \frac{e^{d/n}}{M(\lambda)^{d/n}} + \mathcal{O}(\frac{1}{M(\lambda)^{(d+1)/n}}) \right) dM(\lambda)$$

$$= \int_{1}^{\infty} \left( \frac{e^{d/n}}{M^{d/n}} + \mathcal{O}(\frac{1}{M^{(d+1)/n}}) \right) dM$$
is finite if and only if \( d > n \).

\[ \square \]

**Remark 5.3.** If we assume in above proposition that \( D \) is of order \( a < 1 \), then the commutators with \( T_P \) will then be GTOs of order \( a - 1 \), hence not only bounded but even compact.

### 5.2 Spectral triples for Bergman space on \( \Omega \)

In the Bergman case, we have a similar result as Proposition 5.2.

**Proposition 5.4.** For a bounded domain \( \Omega \) as in Section 2.7 let \( \mathcal{A}_B \) be the algebra generated by the Toeplitz operators \( T_f \), with \( f \in C^\infty(\bar{\Omega}) \), with the identity representation \( \pi \) on \( \mathcal{H} := \mathcal{A}^2_w(\Omega) \), and \( \mathcal{D} := V_w TV^*_w \), where \( T \) is a selfadjoint elliptic GTO of order 1 and \( V_w \) as in (15).

Then \( (\mathcal{A}_B, \mathcal{H}, \mathcal{D}) \) is a regular spectral triple of dimension \( n = \dim_{\mathbb{C}} \Omega \).

**Proof.** As in the Hardy case, clearly \( \mathcal{A}_B \) is a unital involutive algebra with a faithful representation on \( \mathcal{H} \). Since \( T \) has a parametrix of order \( -s \), for \( s \in \mathbb{R} \), the regularity and dimension computation are shown by using the same arguments as in Proposition 5.2. \( \square \)

Let \( T^{-\infty} \) denote the ideal in \( \mathcal{A}_H \) of GTO’s of order \( -\infty \), i.e. of smoothing generalized Toeplitz operators with Schwartz kernel in \( C^\infty(\partial \Omega \times \partial \Omega) \).

**Proposition 5.5.** The map \( \psi : a \in \mathcal{A}_B \mapsto V^*_w a V_w \in \mathcal{A}_H \) is a \( ^* \)-isomorphism of \( \mathcal{A}_B \) onto a subalgebra \( \psi(\mathcal{A}_B) \subset \mathcal{A}_H \). Moreover, \( \mathcal{A}_H = \psi(\mathcal{A}_B) + T^{-\infty} \).

**Proof.** Let us first show the inclusion: thanks to (16), \( \psi(T_f) = V^*_w T_f V_w = T_{\Lambda_w}^{-1/2} T_{\Lambda_w f} T_{\Lambda_w}^{-1/2} \in \mathcal{A}_H \). Since \( \mathcal{A}_B \) is generated by the \( T_f \), the map defines an isomorphism from \( \mathcal{A}_B \) into \( \psi(\mathcal{A}_B) \) which preserves the adjoint.

For the second part, let \( T \in \mathcal{A}_H \), denote by \( -s \leq 0 \) the order of \( T \) and by \( u_0(x')\|\xi'\|^{-s} \), \( u_0 \in C^\infty(\partial \Omega) \), its principal symbol. If

\[ f_0(x) := \frac{\Gamma(m_{w+1})}{\Gamma(m_{w+s+1})} K(\|\eta\|^s u_0)(x), \]

then \( f_0 \in C^\infty(\bar{\Omega}) \) and by Proposition 2.14 (7) and (12), the operator \( \psi(T_{r^s f_0}) \) is a GTO also of order \( -s \) and with the same principal symbol as \( T \); thus \( T_1 := T - \psi(T_{r^s f_0}) \) is a GTO of order \( -s - 1 \). Applying the same reasoning to \( T_1 \) in the place of \( T \) yields \( f_1 \in C^\infty(\bar{\Omega}) \) such that \( \psi(T_{r^{s+1} f_1}) \) has the same order and principal symbol as \( T_1 \), hence \( T_2 := T - \psi(T_{r^s f_0 + r^{s+1} f_1}) \) is a GTO of order \( -s - 2 \). Continue in this way to construct \( f_2, f_3, \ldots \), and then let \( f \in C^\infty(\bar{\Omega}) \) be a function which has the same boundary jet as the formal sum \( \sum_{j=0}^{\infty} r^j f_j \); that is, such that

\[ f - \sum_{j=0}^{k} r^j f_j = O(r^{k+1}) \]

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vansishes to order \( k + 1 \) at the boundary, for any \( k = 0, 1, 2, \ldots \). (Such function can be obtained in a completely standard manner along the lines of the classical Borel’s theorem.) Set \( g := r^s f \).

Then by Proposition \ref{prop:boundedness} and \cite{7}, again, for any \( k \in \mathbb{N} \), the difference

\[
R := T - \psi(T g) = T - \psi(\sum_{j=0}^{k} x^{s+j} f_j) - \psi(T r^s \sum_{j=0}^{k} x^{s+j} f_j) = T_{k+1} - \psi(T \sigma(r^{h+1} s))
\]

is a GTO of order (at most) \(-s - k - 1\). Since \( k \) is arbitrary, \( R \) is a GTO of order \(-\infty\), i.e. \( R \in \mathcal{T}^{-\infty} \), and the proof is complete.

For a function \( f \) in \( C^\infty(\overline{\Omega}) \) vanishing at order \( j \in \mathbb{N} \) on the boundary, the order of \( \psi(T f) \) is \(-j\), since, from \cite{8} and \cite{12}, the expression for its the principal symbol is

\[
\sigma(\psi(T f))(x', \xi') = 2^{-j} \|\xi'\|^{-j} \partial^j_n f(x').
\]

Hence all functions \( f \) that vanish to infinite order on the boundary (so they are not analytic on the boundary) are such that \( \psi(T f) \in \mathcal{T}^{-\infty} \).

**Question 5.6.** Is the inclusion \( \psi(\mathcal{A}_B) \subset \mathcal{A}_H \) strict? (Or, is \( \mathcal{T}^{-\infty} \) not contained in \( \psi(\mathcal{A}_B) \)?)

We now give two examples of operator \( \mathcal{D} \) for these Bergman triples:

Using Proposition \ref{prop:boundedness}, we may take \( \mathcal{D} = \mathcal{T}_p \), with any differential operator \( \mathcal{P} \) of order 1 on \( \overline{\Omega} \) such that \( \mathcal{T}_p = \mathcal{T}_p \) and \( \mathcal{T}_1 \) is nonzero on \( \Sigma \). So the first example is given in Remark \ref{rem:boundedness}.

As a second example, we deduce from Remark \ref{rem:boundedness} that \( (\mathcal{T}_r)^{-1} \) exists on \( \text{Ran}(\mathcal{T}_r) \) which is dense in \( \mathcal{A}_w^2(\Omega) \). So we can take \( \mathcal{D} = (\mathcal{T}_r)^{-1} \) as an example of Dirac operator on \( \mathcal{A}_w^2(\Omega) \) and construct the spectral triple \( (\mathcal{A}, \mathcal{H}, \mathcal{D}) \) with the same \( \mathcal{A} \) and \( \mathcal{H} \) as in of Proposition \ref{prop:boundedness}. However, the positivity of \( \mathcal{D} = (\mathcal{T}_r)^{-1} \) induces a trivial K-homology class for the spectral triple. We now get around this triviality:

**Proposition 5.7.** Let \( (\mathcal{A}, \mathcal{H}, \mathcal{D}) \) be the spectral triple of Proposition \ref{prop:boundedness} with \( \mathcal{D} = (\mathcal{T}_r)^{-1} \).

Define \( \tilde{\mathcal{A}} \) as the algebra of all \( \tilde{\mathcal{T}}_f \)’s acting diagonally on \( \tilde{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H} \) and let \( \tilde{\mathcal{D}} \) be the operator

\[
\tilde{\mathcal{D}} := \begin{pmatrix} 0 & \mathcal{U}(\mathcal{T}_r)^{-1} \mathcal{U}^* \\ (\mathcal{T}_r)^{-1} \mathcal{U}^* & 0 \end{pmatrix}
\]

where \( \mathcal{U} \) is a unitary operator on \( \mathcal{A}_w^2(\Omega) \). If \( \mathcal{U} \) is such that

\[
\mathcal{V}_w^* \mathcal{U} \mathcal{V}_w \text{ is a unitary GTO}, \tag{33}
\]

then \( (\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{\mathcal{D}}) \) is a regular spectral triple.

The triples \( (\mathcal{A}, \mathcal{H}, \mathcal{D}) \) and \( (\tilde{\mathcal{A}}, \tilde{\mathcal{H}}, \tilde{\mathcal{D}}) \) have the same dimension.

**Proof.** We first check the boundedness of \( [\tilde{\mathcal{D}}, \tilde{\mathcal{T}}_f] \). For any \( \tilde{\mathcal{T}}_f \in \tilde{\mathcal{A}} \), we have \( [\tilde{\mathcal{D}}, \tilde{\mathcal{T}}_f] = \begin{pmatrix} 0 & \mathcal{D}_1 \\ \mathcal{D}_2 & 0 \end{pmatrix} \)

\[
\mathcal{D}_1 := [\mathcal{U}(\mathcal{T}_r)^{-1}, \mathcal{T}_f] \text{ and } \mathcal{D}_2 := [\mathcal{T}_r^{-1} \mathcal{U}^*, \mathcal{T}_f].
\]

From Proposition \ref{prop:boundedness}, we have the relations
\[ T_f = V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2} V_w^* \], and \( (T_r)^{-1} = V_w T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2} V_w \). We get

\[
D_1 = U T_r^{-1} T_f - T_f U T_r^{-1} = U \left( V_w T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2} V_w^* \right) \left( V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2} V_w^* \right) - \left( V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2} V_w^* \right) U \left( V_w T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2} V_w \right)
\]

\[
= U \left( V_w T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2} \right) \left( V_w T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2} V_w^* \right) - V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_{wf}} T_{\Lambda_w}^{-1/2} \left( V_w U V_w^* \right) T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1/2} T_{\Lambda_w}^{1/2} V_w^* \]

\[
= V_w \left( V_w U V_w^* V_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1/2} T_{\Lambda_{wr}}^{-1/2} \right) V_w^* \]

From the hypothesis, \( V_w U V_w^* \) is a bounded GTO, \( T_{\Lambda_{wr}}^{-1} T_{\Lambda_w} \) is a GTO of order 1 and \( T_{\Lambda_{wf}}^{-1} T_{\Lambda_{wf}} \) is a GTO of order less or equal to 0, so the commutator is a GTO of order less or equal to 0, thus it is a bounded operator on \( H^2_w(\Omega) \). Similar arguments show that

\[
D_2 = V_w \left( T_{\Lambda_w}^{1/2} T_{\Lambda_{wr}}^{-1} T_{\Lambda_w}^{1/2} \right) \left( V_w U V_w^* \right) T_{\Lambda_w}^{-1/2} T_{\Lambda_{wr}}^{-1/2} V_w^* \]

is also bounded on \( A^2_w(\Omega) \), which makes \( [\tilde{D}, \tilde{T}_f] \) bounded on the direct sum \( \tilde{H} \).

We remark that the expression of \( D_1 \) and \( D_2 \) differ from (32) by the term \( V_w U V_w^* \) which is a GTO of order 0. So the regularity of the spectral triple is proven as in Proposition 5.4.

Finally \( \tilde{D} \) has compact resolvent since \( \tilde{D}^{-1} = \begin{pmatrix} 0 & U \Lambda^* \cr T_r & 0 \end{pmatrix} \) is compact because the operators \( U T_r \) and \( T_r \Lambda^* \) are compact.

Since \( \tilde{D}^2 = \begin{pmatrix} U T_r \Lambda^* & 0 \cr 0 & T_r \end{pmatrix} \), we deduce that the unitary \( U \) does not influence the calculation of eigenvalues. \( \square \)

**Remark 5.8.** The two classes of unitaries \( U \) defined in Remark 2.20 satisfy (33), so provide examples of spectral triples \((\tilde{A}, \tilde{H}, \tilde{D})\) on (the sum of two copies of) the Bergman space with non-positive \( \tilde{D} \) when \( D = (T_r)^{-1} \).

For the case of the unit ball with a radial weight, the Proposition 5.4 can be made much more explicit. Indeed, if \( f \) is a radial function in \( C^\infty(\mathbb{B}^n) \) and the weight \( w \) is as in (3), the family \( \{v_\alpha\}_{\alpha \in \mathbb{N}^n} \) defined in (1) diagonalizes \( T_f : A^2_w(\mathbb{B}^n) \to A^2_w(\mathbb{B}^n) \) and the eigenvalues only depend on \( |\alpha| \). Namely,

\[
\langle T_f v_\alpha, v_\beta \rangle_{A^2_w} = \int_0^1 \frac{\delta_{\alpha, \beta}}{t^{2n+2|\alpha|-1} w(t)} \int_0^1 t^{2n+2|\alpha|-1} f(t) w(t) dt \]

as is easily seen by passing to the polar coordinates.

Thus, assume that \( w = r^{m_w}, m_w \in \mathbb{N} \) where the function \( r \) of the form \( (1) \) depends only on the variable \( |x| \), for \( x \) in \( \mathbb{B}^n \). For convenience, we temporarily denote here \( \partial := \partial_\alpha \). Then, since the first \( m_w - 1 \) derivatives of \( w \) vanish on \( \partial \mathbb{B}^n = S^{2n-1} \), and \( \partial^k(w r) \) is non-zero only for \( k > m_w \), we have for \( \alpha \in \mathbb{N}^n \):

\[
\int_0^1 w(t) t^{2n+2|\alpha|-1} dt = \frac{(-1)^{m_w}}{\prod_{k=0}^{m_w} (2n+2|\alpha|+k)} \int_0^1 \frac{\partial^{m_w+1} w(t)}{t^{2n+2|\alpha|+m_w+1}} dt,
\]

\[
\int_0^1 w(t) r(t) t^{2n+2|\alpha|-1} dt = \frac{(-1)^{m_w+1} r(w r)}{\prod_{k=0}^{m_w+1} (2n+2|\alpha|+k)} \int_0^1 \frac{\partial^{m_w+2} (w r)(t)}{t^{2n+2|\alpha|+m_w+2}} dt.
\]
Since \( \partial^{m_w}w(1) \neq 0 \) by hypothesis, we deduce, applying the Leibniz formula for \( \partial^{m_w+1}(w f)(1) \),

\[
\int_0^1 w(t) t^{2n+2|\alpha|-1} dt \sim (-1)^m w \frac{\partial^{m_w} w(1)}{|\alpha|} \prod_{k=0}^m (2n+2|\alpha|+k),
\]

\[
\int_0^1 w(t) r(t) t^{2n+2|\alpha|-1} dt \sim (-1)^{m_w+1} \frac{\partial^{m_w} w(1)}{|\alpha|} \prod_{k=0}^{m_w+1} (2n+2|\alpha|+k)
\]

We choose now \( D = T \), and from the previous result, \( D \) diagonal in the basis \( \{ v_\alpha \}_{\alpha \in \mathbb{N}^n} \) and we obtain the asymptotic behavior of the eigenvalues:

\[
\langle D v_\alpha, v_\alpha \rangle_{A^2_n} \sim (-1)^{|\alpha|} \frac{1}{2|\alpha|} \frac{\partial r(1)}{|\alpha|} \sim -\frac{1}{2|\alpha|} \partial r(1).
\]

Since \( \mathrm{Tr}|D|^{-d'} = \sum_{k=0}^\infty \binom{n-1+k}{n-1} \frac{\partial r(1)}{(2k)!} \) and \( \sum_{k=0}^\infty \frac{k}{(n-1)!} \), we have \( \mathrm{Tr}|D|^{-d'} < \infty \) if and only if \( \frac{1}{(n-1)!} \sum_{k=0}^\infty \frac{k^{d'}}{(n-1)!} < \infty \), so for each \( d' > d = n \).

To give an example, we choose the function \( r : z \in \Omega \mapsto 1 - |z|^2 \), and the weight \( w = r^0 = 1 \). A direct calculation shows that \( \{ v_\alpha \}_{\alpha \in \mathbb{N}^n} \) defined in \( \text{[5]} \) diagonalizes the operator \( T_{1-|z|^2} \), acting on \( A^2(\mathbb{B}^n) \) with eigenvalues \( \lambda_\alpha := \frac{1}{n+|\alpha|+1} \) and multiplicity \( \binom{n-1+|\alpha|}{n-1} \). Indeed:

\[
T_{1-|z|^2} v_\alpha = v_\alpha - \sum_{j=1}^n \sum_{\beta \in \mathbb{N}^n} (z_j \bar{z}_j v_\alpha, v_\beta) v_\beta = v_\alpha - \sum_{j=1}^n \sum_{\beta \in \mathbb{N}^n} (b_{\alpha j} z^{\alpha+1}_j, b_{\beta j} z^{\beta+1}_j) v_\beta
\]

\[
= (1 - \sum_{j=1}^n b^2_{\alpha j} |z^{\alpha+1}_j|^2) v_\alpha = (1 - \sum_{j=1}^n b^2_{\alpha j} \frac{1}{b_{\alpha j}^{n+1}}) v_\alpha.
\]

Since \( b_{\alpha j}^2 = \frac{(n+|\alpha|+1)!}{n! |\alpha|! (n+|\alpha|+1) |\alpha|!} \), we get \( \lambda_\alpha = 1 - \sum_{j=1}^n \frac{(n+|\alpha|)! n! |\alpha|! (n+|\alpha|+1)!}{(n+|\alpha|+1)!} = 1 - \frac{n+|\alpha|}{n+|\alpha|+1} = \frac{1}{n+|\alpha|+1} \), and the multiplicity follows because \( \lambda_\alpha \) depends only on \( |\alpha| \) and we see directly the appearance of the dimension \( n \).

### 5.3 Example of spectral triples on the unit ball of \( \mathbb{C}^n \) without weight

We consider in this section the model case \( \Omega = \mathbb{B}^n \) with \( w = 1 \).

**Proposition 5.9.** Let \( A = \{ T_u, u \in C^\infty(S^{2n-1}) \} \) be the algebra of Toeplitz operators on the Hardy space \( H^2(S^{2n-1}) \) and \( D_{S^{2n-1}} \) the operator from \( \text{[30]} \). Then \( (A, \mathcal{H}, D_{S^{2n-1}}) \) is a regular spectral triple of dimension \( 2n \).

Let \( A = \{ T_f, f \in C^\infty(\mathbb{B}^n) \} \) be the algebra of Toeplitz operators on the Bergman space \( \mathcal{H} = A^2(\mathbb{B}^n) \) and \( D_{\mathbb{B}^n} \) the operator from \( \text{[31]} \). Then \( (A, \mathcal{H}, D_{\mathbb{B}^n}) \) is a spectral triple of dimension \( 2n \).

**Proof.** The requirement of compact resolvent is fulfilled automatically, since it is fulfilled for the standard Dirac operator on \( \mathbb{R}^n \), from which \( D_{S^{2n-1}} \) and \( D_{\mathbb{B}^n} \) were obtained by transferring via various \( * \)-isomorphisms, which also shows they are selfadjoint. From Lemma \( \text{4.2} \), \( D_{S^{2n-1}} = (l/2)^{1/2} \sum_j \Gamma_j T_A^{1/2} \gamma_j(T_j) K T_j^{-1/2} \), is a GTO of order \( 1/2 \) and we use the same argument as in the proof of Proposition \( \text{5.2} \). The result for the Bergman case follows from the identity \( D_{\mathbb{B}^n} = V D_{S^{2n-1}} V^* \) and using a similar reasoning as in the proof of Proposition \( \text{5.4} \). \( \square \)
5.4 Dixmier traces

In all the examples of spectral triples above, one can also give a formula for the Dixmier traces
\[ \text{Tr}_\omega(a[D]^{-d}) \] where \( a \in \mathcal{A} \) and \( d \) is the spectral dimension.

First, recall that by [18, Theorem 3], if \( T_P \) is a GTO on \( \partial \Omega \) of order \( -n \), then \( T_P \) is in the
Dixmier-class, is measurable, and
\[ \text{Tr}_\omega(T_P) = \frac{1}{(2\pi)^n} \int_{\partial \Omega} \sigma(T_P)(x', \eta_{x'}) \nu_{x'} \quad (34) \]

This formula is independent of the choice of the defining function \( r \) (see [18, Remark 4]).

In the context of the Hardy space spectral triple from Section 5.1, we thus have for any
\( u \in C^\infty(\partial \Omega) \) and \( D \) as in Proposition 5.2 we thus have for any
\[ \text{Tr}_\omega(T_u |D|^{-n}) = \frac{1}{(2\pi)^n} \int_{\partial \Omega} u(x') |\sigma(D)(x', \eta_{x'})|^{-n} \nu_{x'} \quad (35) \]

and similarly for \( T_u \) replaced by any GTO \( T_Q \) of order 0: the \( u(x') \) in the integrand is then
replaced by \( \sigma(T_Q)(x', \eta_{x'}) \).

For the Bergman case, the Dirac operator in Proposition 5.4 is of the form \( D = V_w TV_w^* \),
where \( T \) is a selfadjoint elliptic GTO of order 1. So we have for any \( f \in C^\infty(\Omega) \)
\[ \text{Tr}_\omega(T_f |D|^{-n}) = \text{Tr}_\omega(V_w T_{\Lambda_w}^{-1/2} T_{\Lambda_w} \gamma_{\tau} V_w T_w^* V_w |T|^{-n} V_{w}^*) \]
\[ = \text{Tr}_\omega(T_{\Lambda_w}^{-1/2} T_{\Lambda_w} \gamma_{\tau} T_{\Lambda_w}^{-1/2} |T|^{-n}) \]

which is treated as above.

For the triple \( (\mathcal{A}, \mathcal{H}, \mathcal{D}) \) from Proposition 5.7, the Dixmier traces get multiplied by 2 due to
the appearance of \( 2 \times 2 \) block matrices (see the last paragraph of the proof of the proposition).

Similarly, a factor \( n \) appears in the computation of Dixmier traces in the context of Section 5.3 since
the Dirac operators \( D_{S^{2n}-1} \) and \( D_{S^n} \) involved in Proposition 5.9 contain gamma
matrices. From the identity \( V^* \tau(P_j) V = T_{\Lambda_j}^{1/2} (\gamma \tau(P_j) K) T_{\Lambda_j}^{-1/2} \) and Lemma 4.2, we remark
that \( D_{S^{2n}-1} = \frac{t}{2k}^{-1/2} \sum_j \Gamma_j T_{Q_j} \), where the \( T_{Q_j} \) are GTOs of order 1/2 whose symbols are
known. Hence \( \text{Tr}_\omega(|D_{S^{2n-1}}|^{-2n}) = \text{Tr}_\omega(|D_{S^n}|^{-2n}) \) is finite. We use (35) again to compute
\( \text{Tr}_\omega(T_u |D_{S^{2n-1}}|^{-2n}) \) and the Bergman case follows as above.

Remark 5.10. Note that pseudodifferential operators of order \( k \) on \( \mathbb{R}^n \) are transformed in
GTOs of order \( k/2 \) on the boundary of \( \Omega \) (in the beginning of Section 4), which might seem
to be at odds with the fact that Dixmier-trace operators correspond to both pseudodifferential
operators and GTOs of order \( -n \), respectively on a compact real manifold of dimension \( n \) and
on the boundary of a complex domain of dimension \( n \). The point is that on a compact manifold
of dimension \( n \) this is true, but on \( \mathbb{R}^n \) this fails: for instance the operator \( (\mathbb{1} - \Delta)^{-n/2} \) on \( \mathbb{R}^n \) is not even compact, much less in Dixmier-class. What one needs is first of all not to use
Hörmander but Shubin (also known as Grossman–Louveras–Stein) classes of pseudodifferential
operators (i.e. with prescribed decay of symbols not only as $\xi$ goes to infinity, but as $(x, \xi)$ goes to infinity), and secondly, the order needed for the Dixmier-class is then not $-n$ but $-2n$ (see for instance [1, Theorem 4.1] where the result is stated for pseudodifferential operators on $\mathbb{R}^n$ of Weyl type, but it is the same for the Kohn–Nirenberg type). Since $(-2n)/2 = -n$, this yields precisely the correct order for GTOs, and the contradiction disappears.

Actually, this can be recast the following way: for a non unital spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, the axiom “$\mathcal{D}$ has a compact resolvent” is replaced by “$\pi(a)(\mathbf{1} + D^2)^{-1}$ is a compact operator for any $a \in \mathcal{A}$.” For instance, in a triple on $\mathbb{R}^n$ like $(\text{Functions}(\mathbb{R}^n), \mathcal{H} = L^2(\mathbb{R}^n) \otimes \mathcal{C}_\alpha, \mathcal{D} = -i \sum_j \Gamma_j \partial_{x_j})$, where Functions$(\mathbb{R}^n)$ is a subalgebra of $C^\infty(\mathbb{R}^n)$, one can choose for $\mathcal{A}$ the Schwartz space on $\mathbb{R}^n$ to secure this property. While $(\mathbf{1} + D^2)^{-n/2} = ((\mathbf{1} - \Delta) \otimes \mathbb{1}_{\mathcal{C}_\alpha})^{-n/2}$ is not Dixmier-traceable, $\pi(f)(\mathbf{1} + D^2)^{-n/2}$ is, so the dimension $n$ appears twice: one in the power of $|\mathcal{D}|$ and the other through the algebra $\mathcal{A}$ (via the $n$ variables of $f$).

6 Berezin–Toeplitz star products

One may, in a sense, glue the spectral triples from §5.2 with different weights $w$ into a single “composed” spectral triple, much as Toeplitz operators on weighted Bergman spaces are “glued” together in the Berezin–Toeplitz quantization [3] [30]; this actually yields a spectral triple directly related to the standard Berezin–Toeplitz star product on $\Omega$. Let us give the details.

Assume that log $1/r$ is strictly plurisubharmonic on $\Omega$ (defining functions $r$ with this property exist in abundance due to the strict pseudoconvexity of $\Omega$), so that $g_{\overline{j}k}(z) := \partial_{\overline{j}} \partial_k \log \frac{1}{r(z)}$ defines a Kähler metric on $\Omega$; and let $g := r^{n+1} \det[g_{\overline{j}k}]$.

By elementary matrix manipulations, one can check that $g \in C^\infty(\overline{\Omega})$ and (thanks to strict pseudoconvexity) $g > 0$ on $\partial\Omega$. (In fact $g$ coincides with the Monge–Ampere determinant $g = -\det \left[ \frac{r}{\partial r} \frac{\partial r}{\partial \overline{r}} \right]$.)

Consider the weighted Bergman spaces $A^2_\alpha(\Omega, w_m)$ with $w_m := r^m g$, $m \in \mathbb{N}$, which we will now denote by $A_m^2$ for brevity. Let

$$H := \bigoplus_{m=0}^\infty A_m^2$$

be their orthogonal direct sum, and let $\pi_m$ stand for the orthogonal projection in $H$ onto the $m$-th summand. Denote by $N$ the “number operator” $N := \bigoplus_m (m + n + 1) \pi_m$.

For $f \in C^\infty(\overline{\Omega})$, we then have the orthogonal sums

$$T_f^{\oplus} := \bigoplus_m (T_f \text{ on } A_m^2)$$

of the Toeplitz operators $T_f$ from Section 2, acting on $H$. Clearly each $T_f^{\oplus}$ is again bounded with $\|T_f^{\oplus}\| \leq \|f\|_{C^\infty}, \ (T_f^{\oplus})^* = T_f^{\oplus}$, and $[T_f^{\oplus}, \pi_m] = 0$ for all $m$.

Let $B$ denote the subset of all bounded linear operators $M$ on $H$ for which $[M, \pi_m] = 0$ for all $m$ and which possess an asymptotic expansion

$$M \approx \sum_{j=0}^\infty N^{-j} T_f^{\oplus}_{f_j} \quad (36)$$

with some $f_j \in C^\infty(\overline{\Omega})$ (depending on $M$). Here “$\approx$” means that

$$\left\| \pi_m (M - \sum_{j=0}^{k-1} N^{-j} T_f^{\oplus}_{f_j}) \pi_m \right\| = O(m^{-k}) \quad \text{as } m \to +\infty \text{ for any } k = 0, 1, 2, \ldots.$$
It is the main result of the Berezin–Toeplitz quantization on $\Omega$ that finite products of $T_f^\oplus$ belong to $\mathcal{B}$. More specifically, one has \[ T_f^\oplus T_g^\oplus \approx \sum_{j=0}^{\infty} N^{-j} T_{C_j(f,g)}^\oplus \] where \[ \sum_{j=0}^{\infty} h^j C_j(f,g) =: f \ast g \] defines a star product on $(\Omega, g_{jk})$. Symbolically, we can write \[ T_f^\oplus \ast T_g^\oplus = T_f^\oplus T_g^\oplus. \]

Another result is, incidentally, that $\| \pi_m T_f^\oplus \pi_m \| \to \| f \|_\infty \text{ as } m \to +\infty,$ (37) implying, in particular, that for a given $M \in \mathcal{B}$ the sequence $\{ f_m \}$ in (36) is determined uniquely.

There is a neat representation for this whole situation, as follows. (See e.g. [13], p. 235, for details in this setting; the idea however goes back to Forelli and Rudin.) Consider the “unit disc bundle” over $\Omega$: \[ \tilde{\Omega} := \{(z, t) \in \Omega \times \mathbb{C} : |t|^2 < r(z)\}. \] The fact that $r$ is a defining function for $\Omega$ implies that $\tilde{\Omega}$ is smoothly bounded, and the facts that $\Omega$ is strictly pseudoconvex and $\log 1/r$ is strictly plurisubharmonic imply that $\tilde{\Omega}$ is strictly pseudoconvex. Thus we have the Hardy space \[ \tilde{H} := H^2(\tilde{\Omega}) \] on the GTOs $\tilde{T}_p$ there, whose symbols $P$ are now pseudodifferential operators on $\partial \tilde{\Omega}$. A function in $\tilde{H}$ has the Taylor expansion in the fiber variable \[ f(z, t) = \sum_{m=0}^{\infty} f_m(z) t^m. \]

Denote by $\tilde{H}_m$, $m \in \mathbb{N}$, the subspace in $\tilde{H}$ of functions of the form $f_m(z) t^m$ (i.e. for which all the Taylor coefficients vanish except for the $m$-th); alternatively, $\tilde{H}_m$ is the subspace of functions in $\tilde{H}$ satisfying \[ f(z, e^{i\theta} t) = e^{m\theta} f(z, t), \quad \forall \theta \in \mathbb{R}. \] Then the correspondence \[ f_m(z) t^m \longmapsto f_m(z) \] is an isometric (up to a constant factor) isomorphism of $\tilde{H}_m$ onto $A^2_{m-n-1}$. Thus (note that $A^2_{m-n-1} = \{0\}$ for $m \leq n$) \[ \tilde{H} = \bigoplus_{m=0}^{\infty} \tilde{H}_{m+n+1} \cong \bigoplus_{m=0}^{\infty} A^2_m = \mathbb{H}. \] Furthermore, viewing a function $f \in C^\infty(\Omega)$ also as the function $f(z, t) := f(z)$ on $\partial \tilde{\Omega}$ (i.e. identifying $f$ with its pullback via the projection map), one has, under the above isomorphism, \[ \tilde{T}_f \cong \bigoplus_m (T_f \text{ on } A^2_m) = T_f^\oplus. \]
Finally, let \( \tilde{K} \) be the Poisson operator for \( \tilde{\Omega} \), and set as before \( \tilde{\Lambda} := \tilde{K}^* \tilde{K} \). Thus \( \tilde{\Lambda} \) is a pseudodifferential operator on \( \partial \tilde{\Omega} \) of order \(-1\), and a positive selfadjoint compact operator on \( \tilde{H} \). Since the fiber rotations \( (z,t) \mapsto (z,e^{i\theta}t), \theta \in \mathbb{R} \), preserve holomorphy and harmonicity of functions, both \( \tilde{K}, \tilde{\Lambda} \) and the Szegö projection \( \tilde{S} : L^2(\partial \tilde{\Omega}) \rightarrow \tilde{H} \) must commute with them. The GTO \( \tilde{T}_\Lambda \) on \( \tilde{H} \) therefore likewise commutes with these rotations, and hence commutes also with the projections in \( \tilde{H} \) onto \( H_m \), i.e. is diagonalized by the decomposition \( \tilde{H} = \bigoplus_m H_m \).

Let \( L := \bigoplus_m L_m \) be the operator corresponding to \( \tilde{T}_\Lambda \) under the isomorphism \( \tilde{H} \cong H = \bigoplus_m A_m^2 \).

**Proposition 6.1.** Let \( A \) be the algebra (no closures taken) generated by \( T_f^\oplus, f \in C^\infty(\tilde{\Omega}) \) acting (via identity representation) on \( H := H \) and \( D := L^{-1} \).

Then \( (A, H, D) \) is a spectral triple of dimension \( n + 1 \).

**Proof.** Using the above isomorphisms, we can actually switch from \( H \) to the space \( \tilde{H} \), from \( A \) to the algebra generated by \( T_f, f \in C^\infty(\tilde{\Omega}) \) (identified via pullback with functions on \( \partial \tilde{\Omega} \)), and to \( D \) equal to \( \tilde{\Lambda}^{-1} \). Everything then follows in exactly the same way as in Section 5.1 noting that \( \dim \tilde{\Omega} = n + 1 \) (in fact, it is even the special case of the result from that section for functions \( f \) on \( \partial \tilde{\Omega} \) that are pullbacks of functions on \( \Omega \)). \( \square \)

Using (36), we now alternatively define \( A \) in the last result as a certain subalgebra of formal power series generated by \( f \in C^\infty(\tilde{\Omega}) \) with the product \( \star \), taking for \( \pi \) the representation \( f \mapsto T_f^\oplus \). More specifically, let \( \kappa \) be the linear map from \( B \) into the ring of formal power series

\[
A := C^\infty(\Omega)[[h]]
\]

(equipped with the usual involution \( (\sum_m h^m f_m(z))^* := \sum_m h^m \overline{f_m(z)} \)) given by

\[
\kappa : M \mapsto \sum_{m=0}^\infty h^m f_m(z)
\]

for \( M \) as in (36). As noted previously, \( \kappa \) is well defined owing to (37) (although it is not injective), and, extending as usual \( \star \) from functions to all of \( A \) by \( \mathbb{C}[[h]] \)-linearity,

\[
\kappa(MN) = \kappa(M) \star \kappa(N), \quad \kappa(M^*) = \kappa(M)^*,
\]

i.e. \( \kappa : (B, \circ) \rightarrow (A, \star) \) is a \( * \)-algebra homomorphism. Then we have the following:

**Theorem 6.2.** Let \( A \) be the subalgebra over \( \mathbb{C}[h] \) (no closures taken) of \( (A, \star) \) generated by \( \kappa(T_f^\oplus), f \in C^\infty(\Omega) \) endowed with the representation \( \pi \) on \( H := H \) be determined by

\[
\pi(h^m f) := N^{-m} T_f^\oplus, \quad f \in C^\infty(\Omega), \quad m = 0, 1, 2, \ldots ,
\]

(38)

which is well-defined from \( A \) into \( B \), and \( D := \bigoplus_m L_m^1 \) on \( H \).

Then \( (A, H, D) \) is a spectral triple of dimension \( n + 1 \).

**Proof.** In view of the preceding result, the only thing we need to check is that \( \pi \) is well-defined and faithful. The former is immediate from (38) and the fact that \( \kappa : (B, \circ) \rightarrow (A, \star) \) is a \( * \)-algebra homomorphism. For the faithfulness, note that \( \kappa \circ \pi = \text{id} \) on \( A \); thus \( \pi(a) = 0 \) implies \( a = \kappa(\pi(a)) = 0 \). \( \square \)

Again, proceeding as in Proposition 5.7, one can adjoin to the last construction an appropriate unitary GTOs on \( \partial \tilde{\Omega} \) to obtain also non-positive operators \( \tilde{D} \) (cf. Remark 5.8).
Appendix

Proof of Proposition 3.3

Proof. (see also [35]) The integral in $W_t$ converges since the function $x \mapsto e^{\left(-x^2 + \frac{2}{2}zxz^2\right)/2t}$ is square integrable or each $z \in \mathbb{C}^n$. Now, take a closed curve $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ in $\mathbb{C}^n$ and compute:

$$\oint_{z \in \Gamma} (W_t f)(z) \, dz = (\pi t)^{-n/4} \int_{x \in \mathbb{R}^n} e^{\left(-x^2 + \frac{2}{2}zxz^2\right)/2t} f(x) \, dx \, dz.$$  

By Fubini’s theorem, we can invert the integrals and get

$$\oint_{z \in \Gamma} (W_t f)(z) \, dz = (\pi t)^{-n/4} \int_{x \in \mathbb{R}^n} e^{-x^2/2t} f(x) \left[ \int_{z \in \mathbb{R}^n} e^{\left(-x^2 + \frac{2}{2}zxz^2\right)/2t} \, dz \right] \, dx.$$  

For a fixed $x \in \mathbb{R}^n$, the analyticity of each $z_k \mapsto e^{\left(-z_k^2 + \frac{2}{2}x_kz_k\right)/2t}$, $k = 1 \ldots n$, induces the vanishing of the integral over $\Gamma$, and so $\oint_{z \in \Gamma} (W_t f)(z) \, dz = 0$. According to the Morera’s theorem, the map $z \mapsto (W_t f)(z)$ is holomorphic.

The fact that the Segal–Bargmann transform is a unitary will follow by showing that $W_t$ maps the orthonormal basis of $L^2(\mathbb{R}^n)$ given by the Hermite functions, to the orthonormal basis of the Fock space we have introduced. To complete the proof, we define the operator $A_j$ on $L^2(\mathbb{R}^n)$, with domain the usual Sobolev space $H^1(\mathbb{R}^n)$, and its Hilbert adjoint $A_j^*$:

$$A_j := \frac{1}{\sqrt{2}}(x_j + t \partial_{x_j}) \quad \text{and} \quad A_j^* := \frac{1}{\sqrt{2}}(x_j - t \partial_{x_j}),$$

and first show the following result:

$$W_t A_j^* = z_j W_t, \quad \text{for} \quad j = 0, \ldots, n, \quad \text{(A.1)}$$

which follows using integration by parts: for $f \in H^1(\mathbb{R}^n)$,

$$(W_t(A_j^* f))(z) = (\pi t)^{-n/4} \int_{x \in \mathbb{R}^n} e^{\left(-x^2 + \frac{2}{2}zxz^2\right)/2t} \frac{1}{\sqrt{2}}(x_j - t \partial_{x_j}) f(x) \, dx$$

$$= (\pi t)^{-n/4} \int_{x \in \mathbb{R}^n} e^{\left(-x^2 + \frac{2}{2}zxz^2\right)/2t} \frac{1}{\sqrt{2}}(x_j + t \frac{1}{2t} (2 \sqrt{2} z_j - 2 x_j)) f(x) \, dx$$

$$= (\pi t)^{-n/4} \int_{x \in \mathbb{R}^n} e^{\left(-x^2 + \frac{2}{2}zxz^2\right)/2t} z_j f(x) \, dx = z_j (W_t f)(z).$$

Defining now the operator $A := \prod_{j=1}^n A_j$ (the order is not important since the $A_j$ commute), the solution of the equation $(Ah_0)(x) = 0$ is given by $h_0(x) = (\pi t)^{n/4} e^{-x^2/2t}$, where the constant has been chosen to normalize $h_0$ on $L^2(\mathbb{R}^n, dx)$. Moreover $(W_t h_0)(z) = (\pi t)^{-n/4} \int_{x \in \mathbb{R}^n} e^{\left(-x^2 + \frac{2}{2}zxz^2\right)/2t} (\pi t)^{-n/4} e^{-x^2/2t} \, dx = 1$. Intertwining \text{[A.1]} with $W_t h_0 = 1$, we get for $\alpha \in \mathbb{N}^n$, $W_t((A^*)^\alpha h_0)(z) = z^\alpha$, where $(A^*)^\alpha = \prod_{j=1}^n (A_j^*)^\alpha_j$. The functions

$$h_\alpha(x) := (t^{\left|\alpha\right|} \alpha!)^{-1/2} ((A^*)^\alpha h_0)(x), \quad x \in \mathbb{R}^n \quad \text{(A.2)}$$

are the Hermite functions and they form an orthonormal basis of $L^2(\mathbb{R}^n, dx)$. So $W_t$ maps unitarily the space $L^2(\mathbb{R}^n)$ onto $\mathcal{F}_t$ since $W_t : h_\alpha \mapsto (t^{\left|\alpha\right|} \alpha!)^{-1/2} z^\alpha = u_\alpha$.
Since $W_t$ is unitary, its inverse $W_t^{-1}$ is equal to its Hilbert adjoint and for $g \in L^2(\mathbb{R}^n, dx)$,
\[
\langle W_t g, f \rangle_{\mathcal{F}_t} = (\pi t)^{-n/4} \int_{x \in \mathbb{C}^n} (\pi t)^{-n/4} \int_{x \in \mathbb{R}^n} e^{-(z^2 + 2\sqrt{2} z \cdot x - x^2)/2t} g(x) f(z) e^{-|z|^2/t} dx d\mu(z)
\]
\[
= \int_{x \in \mathbb{R}^n} g(x) \left( (\pi t)^{-5n/4} \int_{z \in \mathbb{C}^n} e^{-(z^2 + 2\sqrt{2} z \cdot z - z^2)/2t} f(z) e^{-|z|^2/t} d\mu(z) \right) dx
\]
\[
= \langle g, W_t^* f \rangle_{L^2(\mathbb{R}^n, dx)}.
\]

References

[1] H. Bommier-Hato, M. Engliš and E-H. Youssfi, *Dixmier trace and the Fock space*, Bulletin des Sciences Mathématiques, 2013.

[2] H. Bommier-Hato, M. Engliš and E-H. Youssfi, *Dixmier classes on generalized Segal–Bargmann–Fock spaces*, J. Funct. Anal., 266: 2096–2124, 2014.

[3] M. Bordemann, E. Meinrenken, and M. Schlichenmaier, *Toeplitz quantization of Kähler manifolds and gl(n) → ∞ limits*, Comm. Math. Phys., 165: 281–296, 1994.

[4] L. Boutet de Monvel, *Boundary problems for pseudo-differential operators*, Acta Math., 126: 11–51, 1971.

[5] L. Boutet de Monvel, *Hypoelliptic operators with double characteristics and related pseudo-differential operators*, Comm. Pure Appl. Math., 27: 585–639, 1974.

[6] L. Boutet de Monvel, *On the index of Toeplitz operators of several complex variables*, Inventiones math., 50: 249–272, 1979.

[7] L. Boutet de Monvel, *Vanishing of the logarithmic trace of generalized Szegö projection*, Algebraic Analysis of Differential Equations from Microlocal Analysis to Exponential Asymptotics: Festschrift in Honor of Takahiro Kawai (T. Aoki, H. Majima, Y. Takei, and N. Tose, eds.), Springer, 2008.

[8] L. Boutet de Monvel and V. Guillemin, *The spectral theory of Toeplitz operators*, Princeton University Press, Princeton, N.J, 1981.

[9] A. Connes, *Noncommutative geometry*, Academic Press, 1994.

[10] A. Connes, *On the spectral characterization of manifolds*, J. Noncommut. Geom., 7: 1–82, 2013.

[11] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, Colloquium Publications, Vol. 55, American Mathematical Society, 2008.

[12] A. Connes and H. Moscovici, *Transgression and the Chern character of finite-dimensional K-cycles*, Commun. Math. Phys., 155: 103–122, 1993.

[13] M. Engliš, *Weighted Bergman kernels and quantization*, Comm. Math. Phys., 227: 211–241, 2002.
[14] M. Engliš, *Toeplitz operators and weighted Bergman kernels*, J. Funct. Analysis, 255(6): 1419–1457, 2008.

[15] M. Engliš, *Weighted Bergman kernels for logarithmic weights*, Pure Appl. Math. Quarterly (Kohn special issue), 6: 781–804, 2010.

[16] M. Engliš, K. Guo and G. Zhang, *Toeplitz and Hankel operators and the Dixmier traces on the unit ball of \( \mathbb{C}^n \)*, Proc. Amer. Math. Soc., 137: 3669–3678, 2009.

[17] M. Engliš and R. Rochberg, *The Dixmier trace of Hankel operators on the Bergman space*. J. Funct. Analysis, 257:1445–1479, 2009.

[18] M. Engliš and G. Zhang, *Hankel operators and the Dixmier trace on strictly pseudoconvex domains*, Documenta Mathematica, 15: 601–622, 2010.

[19] J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa. *Elements of Noncommutative Geometry*, Birkhäuser Advanced Texts, Birkhäuser, Boston, 2001.

[20] G. Grubb, *Functional Calculus of Pseudodifferential Boundary Problems*, Birkhäuser, Boston, 1996.

[21] S. Grudsky, A.Karapetyants, and N. Vasilevski, *Toeplitz operators on the unit ball in \( \mathbb{C}^n \) with radial symbols*, J. of Oper. Theory, 49(2): 325–346, 2003.

[22] V. Guillemin, *Toeplitz operators in n-dimensions*, Integral Equations and Operator Theory, 7: 145–205, 1984.

[23] K. Guo, K. Wang and G. Zhang, *Trace formulas and \( p \)-essentially normal properties of quotient modules on the bidisk*, J. Oper. Theory, 67: 511–535, 2012.

[24] L. Hörmander, *The Analysis of Linear Partial Differential Operators I: Distribution Theory and Fourier Analysis*, Springer-Verlag.

[25] R. Howe, *Quantum mechanics and partial differential equations*, J. Funct. Anal., 38(2): 188–254, 1980.

[26] A. Karabegov, M. Schlichenmaier, *Identification of Berezin-Toeplitz deformation quantization*, J. reine angew. Math. 540 (2001), 49–76.

[27] M. Khalkhali, *Basic noncommutative geometry*, EMS, Zürich, 2009.

[28] J. Lions and E. Magenes, *Problèmes aux limites non homogènes (I)*. Dunod Paris, 1968.

[29] S. Rempel, B.-W. Schulze, *Index Theory of Elliptic Boundary Problems*, Akademie-Verlag, Berlin, 1982.

[30] M. Schlichenmaier, *Deformation quantization of compact Kähler manifolds by Berezin-Toeplitz quantization*, Proceedings of the Conference Moshe Flato 1999 (G. Dito and D. Sternheimer, editors), Kluwer 2000, pp. 289–306, arXiv:math.QA/9910137.

[31] R. Strichartz. *A functional calculus for elliptic pseudo-differential operators*, American J. of Mathematics, 94: 711–722, 1972.
[32] M. Taylor, *Fourier integral operators and harmonic analysis on compact manifold (second version)*. Proc. Sympos. Pure Math., 35(Part 2), 1979.

[33] M. Taylor, *Noncommutative microlocal analysis. Part 1 (revised edition)*, Memoirs Amer. Math. Soc. 52, no. 313, 1984.

[34] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Springer, New York, 2005.

[35] K. Zhu, *Analysis on Fock spaces*, Springer, New York, 2012.