ABSTRACT. In this paper we concentrate on the relations between the structure of small Galois groups, arithmetic of fields, Bloch-Kato conjecture, and Galois groups of maximal pro-p-quotients of absolute Galois groups.

1. Introduction

The second author fondly remembers how a number of years ago Paulo Ribenboim helped him to escape to the West and immediately upon his arrival welcomed him with beautiful lectures on the Galois group of the Pythagorean closure of $\mathbb{Q}$. Paulo Ribenboim’s lectures, writings, and research have influenced us strongly, and in particular this paper reflects his influence on the choice of topics and our way of thinking about them. The paper is a selective survey of results on small quotients of absolute Galois groups and their relations with the Bloch-Kato conjecture. It is by no means a comprehensive historical survey. Instead, it focuses only on some selective topics from the work of the authors and their collaborators.

The main idea of our paper, and a key point we want to illustrate, is that already relatively small quotients of absolute Galois groups encode substantial information about them. Absolute Galois groups of fields play a central role in arithmetic, geometry, and
topology. Yet these profinite groups are mysterious and not much is known about the fundamental problem of characterizing absolute Galois groups among all profinite groups. Therefore, it is natural to investigate the maximal pro-$p$-quotients of absolute Galois groups for a fixed prime $p$ which are in general much simpler than the absolute Galois groups. But even these pro-$p$-quotients are quite mysterious and one would like to find more manageable, yet interesting, quotients of maximal pro-$p$-quotients of absolute Galois groups. One extremely interesting family of such quotients are the $W$-groups defined below in Section 6. These are pro-$p$-groups of nilpotent class at most 2 and they have exponent dividing $p^2$. So they are rather simple groups in comparison with the maximal pro-$p$-quotients of absolute Galois groups. Nevertheless, when the primitive $p$-th root of unity is contained in the base field, these groups carry complete information about the entire Galois cohomology with $\mathbb{F}_p$-coefficients of the absolute Galois groups. This is a consequence of the Bloch-Kato conjecture which was proved recently by M. Rost and V. Voevodsky with C. Weibel’s patch. (See [V1] and [V2] for an overview of the proof, and other references [HW, MVW, R1, R2, SJ, V3, W1, W2, W3] for the foundation and completion of the proof, and some further exposition.) Therefore, it is clear that the $W$-groups of fields are good candidates for thorough investigation.

The plan of this survey paper is as follows. We begin with Šafarevič’s early work as a motivation for studying the Galois module structure of $p$-power classes of fields. This is interesting even in the case when we consider just cyclic extensions of degree $p$. We show that the answer in this case already leads to a description of relatively small pro-$p$-quotients of absolute Galois groups called $T$-groups. We then describe some recent results on the Galois module structure of Galois cohomology. After that we give a description of $W$-groups, their relationship with Witt rings of quadratic forms, Galois cohomology, valuations, and the structure of maximal pro-$p$-extensions. Here the Bloch-Kato conjecture plays a very important role, especially in the case $n = 2$ which was established by Merkurjev and Suslin almost 30 years ago. At the conclusion of the paper we touch upon our recent work with D. Benson and J. Swallow in progress whose goal is to provide a refinement of the Bloch-Kato conjecture with group cohomology, combinatorics, and Galois theoretic consequences.

2. Some work of Šafarevič

I.R. Šafarevič initiated the very interesting program of studying Galois groups of maximal $p$-extensions. Let $F$ be a field and $F_{sep}$ the separable closure of $F$. Given a prime number $p$, let $F(p)$ be the maximal $p$-extension over $F$. Thus $F(p)$ is the compositum in $F_{sep}$ of all Galois extensions $K/F$ which have degree a power of $p$. Let $G_F(p) := \text{Gal}(F(p)/F)$ be the Galois group of the maximal $p$-extension. In 1947 Šafarevič showed that if $\mathbb{Q}_p \subseteq F$, $[F : \mathbb{Q}_p] < \infty$, and if $F$ does not contain a primitive $p$th root of unity, then $G_F(p)$ is a free pro-$p$-group on $[F : \mathbb{Q}_p] + 1$ generators. The key part of Šafarevič’s argument was the determination of the number $d(H)$ of minimal generators for all open subgroups $H$ of $G_F(p)$. This number $d(H)$ is equal to $\dim_{\mathbb{F}_p} H/\Phi(H)$, where $\Phi(H) = H^p[H, H]$ is the Frattini subgroup of $H$. By local class field theory it follows that $d(H) = \dim_{\mathbb{F}_p} K^*/K^{*p}$ where $K$ is the fixed field of $H$ and $K^* = K \setminus \{0\}$ is the multiplicative group of $K$. We can calculate $d(H)$ explicitly in terms of invariants of
the extension $K/F$. It turns out that

$$\dim_{\mathbb{F}_p} K^*/K^{*p} - 1 = [K : F](\dim_{\mathbb{F}_p} F^*/F^{*p} - 1).$$

Thus

$$d(H) - 1 = (d(G_F(p)) - 1)[K : F]$$

and therefore the number of generators $d(H)$, for each $H$ open subgroup of $G_F(p)$, grow in the same way as if $G_F(p)$ were a free pro-$p$-group. It was the insight of Safarevič that in fact this property is enough to prove that $G_F(p)$ is a free pro-$p$-group. Safarevič did not use the convenient language of profinite groups as this terminology was not available at that time. Similarly, the language of Galois cohomology appeared much later, and all became wide-spread only after the appearance of Serre’s lecture notes in 1964; see [Ser02] for the latest edition of Serre’s book. In particular, now we can rewrite the above equation as

$$\dim_{\mathbb{F}_p} H^1(H, \mathbb{F}_p) - 1 = (\dim_{\mathbb{F}_p} H^1(G_F(p), \mathbb{F}_p) - 1)[G_F(p) : H].$$

Let $G$ be a pro-$p$-group with $\dim_{\mathbb{F}_p} H^r(G, \mathbb{F}_p) < \infty$ for $1 \leq r \leq n$. Following H. Koch we can set the $n$th partial Euler-Poincaré characteristic $\chi_n(G)$ as

$$\chi_n(G) = \sum_{r=0}^{n} (-1)^r \dim_{\mathbb{F}_p} H^r(G, \mathbb{F}_p).$$

Koch proved that if $W$ is a system of open subgroups of $G$ which form a neighbourhood basis at 1 and $\chi_n(U) = [G : U]\chi_n(G)$ for all $U$ in $W$, then the cohomological dimension $cd(G)$ of $G$ is at most $n$. Because $cd(G) = 1$ if and only if $G$ is a free pro-$p$-group, we see that Koch’s criterion for $cd(G) \leq n$ generalises Safarevič’s criterion for $G_F(p)$ being a free pro-$p$-group.

### 3. The Bloch-Kato conjecture

We briefly recall the Bloch-Kato conjecture for the novice. This conjecture will be used in this article at various places. Let $F$ be a field that has a primitive $p$-th root $\zeta_p$ of unity. Consider the Kummer sequence

$$1 \longrightarrow \mu_p \longrightarrow F^*_\text{sep} \xrightarrow{x \mapsto x^p} F^*_\text{sep} \longrightarrow 1$$

of modules over the absolute Galois group $G_F$, where $\mu_p$ denotes the group of $p$-th roots of unity. The boundary map

$$H^0(G_F, F^*_\text{sep}) = F^* \rightarrow H^1(G_F, \mathbb{F}_p)$$

in the induced long exact sequence in Galois cohomology extends naturally to a map

$$T(F^*) \rightarrow H^*(G_F, \mathbb{F}_p),$$

where $T(F^*)$ is the tensor algebra on $F^*$ and $H^*(G_F, \mathbb{F}_p)$ is the Galois cohomology ring of $G_F$. Bass and Tate verified that the Steinberg relations $(a) \cup (1-a) = 0$ for $a \neq 0, 1$ hold in $H^*(G_F, \mathbb{F}_p)$. Milnor $K$-theory $K_*(F)$ is a graded ring obtained by taking the quotient of $T(F^*)$ by the graded two-sided ideal generated by the elements $a \otimes (1-a)$, $a \in F^* \setminus \{1\}$. Thus we get a map, known as the norm-residue map,

$$\eta: K_*(F)/pK_*(F) \longrightarrow H^*(F, \mathbb{F}_p)$$
from the reduced Milnor $K$-theory to the Galois cohomology of $F$. The Bloch-Kato conjecture claims that the norm-residue map $\eta$ is an isomorphism. The case $p = 2$ was implicitly conjectured by J. Milnor in 1970; see [Mil70]. The Milnor conjecture was eventually proved by Voevodsky [V1, V2]. For this spectacular work Voevodsky was awarded a Fields medal in 2002. His work used some sophisticated machinery such as motivic cohomology operations and the development of $\mathbb{A}^1$ stable homotopy theory. The proof of the Bloch-Kato conjecture is even more subtle. Although Voevodsky announced a proof of the Bloch-Kato conjecture in 2003, not until September 2007 were all of the details for the Rost-Voevodsky proof made available by Voevodsky, Rost and Weibel; see [R1, R2, SJ, V3, W1, W2, W3]. The work on the proof of the Bloch-Kato conjecture and the resulting theorem has already had tremendous impact on contemporary mathematics and is expected to have an even broader impact in the coming years. Note that the Bloch-Kato conjecture gives a presentation of the rather mysterious Galois cohomology $H^*(F, \mathbb{F}_p)$ by generators and relations. In particular, it tells us that $H^*(F, \mathbb{F}_p)$ is generated by one dimensional classes.

4. Classical Hilbert 90 and absolute Galois groups

Šafarevič’s approach to $G_F(p)$ made clear that the $p$-th power class group $F^*/F^{*p}$ is a very useful and fundamental object to study. In 1960 Faddeev began to study the Galois module structure of $p$-th power classes of cyclic extensions of local fields, and during the mid 1960s he and Borevič established the Galois module structure of $p$-th power class groups of local fields using basic arithmetic invariants attached to these extensions. (See [Pada, Bor65]).

In the theory of quadratic forms, the exact sequence of square class groups associated with the quadratic extension $K = F(\sqrt{a})$, $a \in F^*/F^{*2}$, $\text{char}(F) \neq 2$ has been playing an important role. This sequence is $1 \rightarrow \{F^{*2}, aF^{*2}\} \rightarrow F^*/F^{*2} \overset{i_{F/K}}{\rightarrow} K^*/K^{*2} \overset{N_{K/F}}{\rightarrow} F^*/F^{*2} \rightarrow B(F)$, where $i_{F/K}$ is the map induced by inclusion map $F \rightarrow K$ and $N_{K/F}: K^*/K^{*2} \rightarrow F^*/F^{*2}$ is the map induced by the norm map $K^* \rightarrow F^*$, and $\epsilon$ is the homomorphism from $F^*/F^{*2}$ to the Brauer group $B(F)$ defined by $bF^{*2} \rightarrow [(\frac{a}{b})]$ in $B(F)$, where $[(\frac{a}{b})]$ is the class of Quaternion algebra

\[
(\frac{a}{b}) = \{f_0 + f_1i + f_2j + f_3ij | f_i \in F, i^2 = j^2 = -1, ij = -ji \}
\]

in the Brauer group of $F$. (See [Lam05 Pg 200, Thm 3.2].) Observe that this sequence completely determines the size of $K^*/K^{*2}$ provided we know the size $N_{K/F}(K^*)/F^{*2}$.

In fact, this sequence determines the structure of $K^*/K^{*2}$ as an $\mathbb{F}_2[\text{Gal}(K/F)]$-module provided we know $N_{K/F}(K^*)/F^{*2}$. Therefore it is desirable to extend the work of Borevič and Fadeev from the case of local fields to general fields. Borevič, Fadeev and Šafarevič used local class field theory to establish their results in the case of local fields. However, in [MS03] it was observed that it is possible to determine the structure of the $\mathbb{F}_2[\text{Gal}(K/F)]$-module $K^*/K^{*p}$ in the case when a primitive $p$-th root of unity is contained in $F$ using just Hilbert 90 in place of local class field theory. In [MS06], the work of [MS03] was extended to all cyclic extensions $K/F$ of degree $p^n$ with no restriction on the base field. The remarkable feature of the final result is that $K^*/K^{*p}$
can be written as a sum of cyclic modules over $\mathbb{F}_p[\text{Gal}(K/F)]$ of dimensions over $\mathbb{F}_p$ all powers of $p$ with the possible single cyclic module exception of dimension $p^m + 1$, $0 \leq m \leq n - 1$.

As an example we formulate here the main result of [MSS06] when the exceptional summand does not occur. For the more complicated case when the exceptional summand does occur, we refer the reader to [MSS06]. Let $F$ be any field. Consider a cyclic Galois extension $K/F$ with Galois group $G = \text{Gal}(K/F) = \langle \sigma \rangle$, a cyclic group of order $p^n$. We set $J = K^*/K^{*p}$. Then $J$ has the obvious $\mathbb{F}_pG$-module structure.

**Theorem 4.1.** Assume that $F$ does not contain any primitive $p$-th root of unity. Then the $\mathbb{F}_pG$-module $J$ decomposes as

$$J \cong Y_0 \oplus Y_1 \oplus \cdots \oplus Y_n,$$

where $Y_i$ is a direct sum of cyclic $\mathbb{F}_pG$-modules of dimension $p^i$.

Moreover, the multiplicity of cyclic summands of dimension $p^i$ in $Y_i$ is completely determined by the filtration of $F^*$ by norm groups

$$N_{K/F}(K^*) \subset N_{K_{i-1}/F}(K^*_{i-1}) \subset \cdots \subset F^*$$

where, for each $i = 0, 1, \ldots, n$, $K_i$ is the unique subfield of $K$ which has dimension $p^i$ over $F$. Indeed if

$$Y_i = \oplus C_{p^i}$$

where $C_{p^i}$ is the cyclic $\mathbb{F}_pG$-module of dimension $p^i$, then the cardinality of $I$ is just

$$\dim_{\mathbb{F}_p} N_{K_i/F}(K^*_i)/N_{K_{i+1}/F}(K^*_i).$$

Further set $[K^*_i] = K_i/K^{*p}$ and set $H_i = \text{Gal}(K/K_i)$. Then we have Galois descent in the sense that $[K^*_i]^* = J^{H_i}$ – the fixed elements of $J$ under the action of $H_i$.

The results in the case when $F$ contains a primitive $p$-th root of unity are similar, but technically more challenging due to the occurrence of the exceptional module mentioned earlier.

These results are remarkable because of the absence of summands of dimensions not equal to a power of $p$. (Except in the case of exceptional summands which have dimension $p^m + 1$.) These results severely restrict possible small quotients of absolute Galois groups. We shall illustrate this by describing a result from a recent paper [BLMS].

We call a pro-$p$-group $R$ elementary abelian if it has the form $R = \prod I C_p$, where $C_p$ is a cyclic group of order $p$, and $I$ is some possibly infinite index set. We say that a pro-$p$-group $G$ is a $T$-group if $G$ contains a maximal closed subgroup $N$, $N \neq G$, of exponent dividing $p$. Then $N$ is a normal subgroup of $G$ and the factor group $G/N$ acts naturally on $N$ via conjugation. Furthermore, the subgroup $N$ is uniquely determined by $G$ provided that $G$ is neither an elementary abelian group of order greater than $p$ nor the direct product of an elementary abelian group and a non-abelian group of order $p^3$ of exponent $p$ if $p > 2$, and the dihedral group of order 8 if $p = 2$. Given any profinite group $A$ with a closed normal subgroup $B$ of index $p$, the factor group $A/\langle B^p \rangle$ is a $T$-group. Now suppose that $E/F$ is a cyclic field extension of degree $p$. We define the $T$-group of $E/F$ to be $T_{E/F} := G_F/G_E^p[G_E, G_E]$, where $G_F$ and $G_E$ are absolute Galois groups of $F$ and $E$ respectively. (For the benefit of topologists, in order to avoid a possible confusion, we remark that the name “$T$-group” is not motivated by Kazhdan’s property (T), and in fact we do not know of any connection between $T$-groups and
groups with property (T) in Kazhdan’s sense.) We shall now classify those $T$-groups which are realisable as $T_{E/F}$ for fields containing a primitive $p$-th root of unity.

In order to illustrate the restrictions on those $T$-groups which are realisable as $T_{E/F}$ for fields which contain a primitive $p$-th root of unity we shall introduce a simple set of invariants which determine $T$-groups up to isomorphism. We shall then see that for $p > 2$ we obtain restrictions on possible invariants of $T$-groups which are $T_{E/F}$ for suitable $E/F$. On the other hand there are no restrictions in the case $p = 2$. The proof of these statements can be found in [BLMS].

For a pro-$p$-group $A$, denote $Z(A)$ its center and $Z(A)[p]$ the elements of $Z(A)$ of order dividing $p$. Let $A(n)$ be the $n$-th group in the central series of $A$. Thus $A(1) = A$, and $A(n+1) = [A(n), A]$. Here we always consider closed subgroups of $A$. Hence $A(n+1)$ is the closed subgroup of $A$ generated by commutators $[x, y]$, $x \in A(n)$, $y \in A$. For a $T$-group $A$ we define:

$$t_1 = \dim_{\mathbb{F}_p} H^1(\frac{Z(A)[p]}{T(A) \cap A(2)}, \mathbb{F}_p)$$

$$t_i = \dim_{\mathbb{F}_p} H^1(\frac{Z(A)[p] \cap A(i)}{T(A) \cap A(i+1)}, \mathbb{F}_p) \quad 2 \leq i \leq p$$

$$\mu = \max\{i : 1 \leq i \leq p, A^p \subset A(i)\}.$$

These invariants are convenient for describing the $\mathbb{F}_p[A/N]$-module $N$ associated with our $T$-group $A$; see [BLMS, Section 1]. We have

**Proposition 4.2.** For arbitrary cardinalities $t_i$, $i = 1, 2, \cdots, p$, and $\mu$ with $1 \leq \mu \leq p$, the following are equivalent:

1. The $t_i$ and $\mu$ are invariants of some $T$-group.
2. (a) If $\mu < p$, then $t_\mu \geq 1$, and
   (b) If $\mu = p$ and $t_i = 0$ for all $2 \leq i \leq p$, then $t_1 \geq 1$.

Moreover, $T$-groups are uniquely determined up to isomorphism by these invariants.

**Theorem 4.3.** For $p$ an odd prime, the following are equivalent.

1. $A$ is a $T$-group with invariants $t_i$ and $\mu$ satisfying
   (a) $\mu \in \{1, 2\}$
   (b) $t_2 = \mu - 1$, and
   (c) $t_i = 0$ for $3 \leq i \leq p$.

2. $A \cong T_{E/F}$ for some cyclic extension $E/F$ of degree $p$ such that $F$ contains a primitive $p$-th root of unity.

Now suppose $p = 2$. Then each $T$-group is isomorphic to $T_{E/F}$ for some cyclic extension of degree 2.

It is interesting to note that these strong restrictions on possible $T$-groups occuring as $T_{E/F}$ are consequences of the classical Hilbert 90 theorem which can now be viewed as the Bloch-Kato conjecture in degree 1, as it just involves basic Kummer theory which depends on Hilbert 90. For the realisation of given $T$-groups with invariants described in our theorem one uses constructions of cyclic extensions $E/F$ with prescribed groups $F^*/F^{*p}$ and $N_{E/F}(E^*)/F^{*p}$ developed in [MS03], which in turn uses results in [Efr1] realising certain semi-direct products and free pro-$p$-products as absolute Galois groups. This theorem provides restrictions on possible relations in $G_F(p)$; see [BLMS].
5. Higher Galois cohomology and the Bloch-Kato conjecture.

In [MS05] it was shown that the classical theorem Hilbert 90 is the key for determining the \( \mathbb{F}_p[\text{Gal}(E/F)] \)-module structure of \( E^*/E^{*p} \) in the case of cyclic extensions of degree \( p^n \). But \( E^*/E^{*p} \) is also \( K_1(E)/pK_1(E) \). On the other hand Merkurjev-Suslin established in [MS82] an analogue of the Hilbert 90 theorem for Milnor \( K \)-theory in degree 2. Further it turned out that the analogue of Hilbert 90 for higher Milnor \( K \)-theory is essentially equivalent to the Bloch-Kato conjecture. This follows from the work of Merkurjev, Suslin, Rost and Voevodsky. Therefore it was a natural idea to extend results on Galois module structure of \( p \)-power classes to Galois module structure of \( K_n(E)/pK_n(E) \) for any positive integer \( n \). This was achieved in the case when \( E/F \) is a cyclic extension of prime degree \( p \) and the primitive \( p \)-th root of unity is in \( E \) in [LMS07]. (An extension of this work for the case of cyclic extension of degree \( p^n \) is work in progress.) Some of the main results are explained in this section on the language of Galois cohomology as we freely use the Bloch-Kato conjecture.

Let \( G \) be the Galois group of \( E/F \). Write \( E = F(\sqrt[p]{a}), a \in F^\times \) and denote \( (a),(\xi_p) \in H^2(F,\mathbb{F}_p) \) to be the cup product of \((a),(\xi_p) \in H^1(F,\mathbb{F}_p) \). For each \( n \in \mathbb{N} \) set also:

\[
\Upsilon_1: \dim_{\mathbb{F}_p}(\text{ann}_{H^n-1(F,\mathbb{F}_p)}(a),(\xi_p)/\text{ann}(a))
\]

and

\[
\Upsilon_2: \dim_{\mathbb{F}_p}H^{n-1}(F,\mathbb{F}_p)/\text{ann}_{H^n-1(F,\mathbb{F}_p)}(a),(\xi_p).
\]

Here \( H^i(F,\mathbb{F}_p) = H^i(G_F,\mathbb{F}_p) \) is the \( i \)-th Galois cohomology and \( G_F \) is the absolute Galois group of \( F \). Here \( \text{ann} \) is an abbreviation for the annihilator. Thus for example \( \text{ann}_{H^n-1(F,\mathbb{F}_p)}(a),(\xi_p) \) is the kernel of the cup product

\[
(a),(\xi_p) : H^{n-1}(F,\mathbb{F}_p) \rightarrow H^{n+1}(F,\mathbb{F}_p).
\]

Set \( U \) to be the absolute Galois group of \( E \) and consider \( H^n(U,\mathbb{F}_p) = H^n(E,\mathbb{F}_p) \) as an \( \mathbb{F}_p[G] \)-module. In [LMS07] we prove the following theorem. Our corestriction map here denotes the map \( H^n(E,\mathbb{F}_p) \rightarrow H^n(F,\mathbb{F}_p) \) and \( \text{res} \) means the restriction map \( \text{res}: H^n(F,\mathbb{F}_p) \rightarrow H^n(E,\mathbb{F}_p) \).

**Theorem 5.1.** If \( p > 2 \) and \( n \in \mathbb{N} \) then

\[
H^n(E,\mathbb{F}_p) \cong X_1 \oplus X_2 \oplus Y \oplus Z,
\]

where

1. \( X_1 \) is a trivial \( \mathbb{F}_p[G] \)-module of dimension \( \Upsilon_1 \), and

   \[
   X_1 \cap \text{res} H^n(F,\mathbb{F}_p) = \{0\}.
   \]

2. \( X_2 \) is a direct sum of \( \Upsilon_2 \) cyclic \( \mathbb{F}_p[G] \)-modules of dimension 2.

3. \( Y \) is a free \( \mathbb{F}_p[G] \)-module of rank

   \[
   \dim_{\mathbb{F}_p} \text{Im}(\text{cor}: H^n(E,\mathbb{F}_p) \rightarrow H^n(F,\mathbb{F}_p))/(a).H^{n-1}(F,\mathbb{F}_p).
   \]

4. \( Z \) is a trivial \( \mathbb{F}_p[G] \)-module of dimension

   \[
   z = \dim_{\mathbb{F}_p} H^n(F,\mathbb{F}_p)/((\xi_p)H^{n-1}(F,\mathbb{F}_p) + \text{cor} H^n(E,\mathbb{F}_p)) \text{ and } Z \subset \text{res}(H^n(F,\mathbb{F}_p)).
   \]
Theorem 3. Suppose that \( \xi_p \in F \), and let \( n \in \mathbb{N} \). The following are equivalent:

1. \( \text{cd}(G) \leq n \).
2. \( \chi_n(N) = p\chi_n(G) \) for all open subgroups \( N \) of \( G \) of index \( p \).
3. \( \chi_n(U) = p\chi_n(U) \) for all open subgroups \( U \) of \( G \) and all open subgroups \( V \) of \( U \) of index \( p \).

6. Galois theoretic connections

In this section we will explain the role played by certain Galois groups called W-groups in arithmetic. To set the stage, we begin with our notation.

Let \( F \) be a field of characteristic not equal to 2. We shall introduce several subextensions of \( F_{\text{sep}} \):

- \( F^{(2)} \) is the compositum of all quadratic extensions of \( F \).
• $F^{(3)} = \text{compositum of all quadratic extensions of } F^{(2)}$, which are Galois over $F$.
• $F_q = \text{compositum of all Galois extensions } K/F \text{ such that } [K:F] = 2^n$, for some positive integer $n$.

All of these subextensions are Galois and they fit in a tower

$$F \subset F^{(2)} \subset F^{(3)} \subset F_q \subset F_{sep}.$$ 

We denote their Galois groups (over $F$) as

$$G_F \rightarrow G_q \rightarrow G_F^{[3]} (= G_F) \rightarrow G_F^{[2]} (= E) \rightarrow 1.$$ 

Observe that $G_F^{[2]}$ is just $\prod_{i \in I} C_2$, where $I$ is the dimension of $F^*/F^{*2}$ over $F_2$. $G_F$ is the absolute Galois group of $F$. Although the quotients $G_q$ are much simpler than $G_F$ we are far from understanding their structure in general. $F^{[3]}$ and its Galois group over $F$ are considerably much simpler and yet they already contain substantial arithmetic information of the absolute Galois group. The groups $G_F^{[3]} (= G_F)$ are called W-groups.

To illustrate this point, consider $WF$ the Witt ring of quadratic forms; see [Lam05] for the definition. Then we have the following theorem.

**Theorem 6.1.** [MS96] Let $F$ and $L$ be two fields of characteristic not 2. Then $WF \cong WL$ (as rings) implies that $G_F \cong G_L$ as pro-2-groups. Further if we assume additionally in the case when each element of $F$ is a sum of two squares that $\sqrt{-1} \in F$ if and only if $\sqrt{-1} \in L$, then $G_F \cong G_L$ implies $WF \cong WL$.

Thus we see that $G_F$ essentially controls the Witt ring $WF$ and in fact, $G_F$ can be viewed as a Galois theoretic analogue of $WF$. In particular, $G_F$ detects orderings of fields. (Recall that $P$ is an ordering of $F$ if $P$ is an additively closed subgroup of index 2 in $F^*$.) More precisely, we have:

**Theorem 6.2.** [MS90] There is a 1-1 correspondence between the orderings of a field $F$ and cosets $\{\sigma \Phi(G_F) | \sigma \in G_F \sigma \Phi(G_F) \text{ and } \sigma^2 = 1\}$. Here $\Phi(G_F)$ is the Frattini subgroup of $G_F$, which is just the closed subgroup of $G_F$ generated by all squares in $G_F$. The correspondence is as follows:

$$\sigma \Phi(G_F) \rightarrow P_\sigma = \{f \in F^* | \sigma(\sqrt{f}) = \sqrt{f}\}.$$ 

This theorem was generalised considerably for detecting additive properties of multiplicative subgroups of $F^*$ in [MMS04]. In this paper (see [MMS04 Section 8]) it was shown that $G_F$ can be used also for detecting valuations on $F$. The work on extending these ideas is in progress; see [CEM].

Also in [AKM99 Corollary 3.9] it is shown that $G_F \cong G_L$ if and only if $k_*(A) \cong k_*(L)$. Here $k_*(A)$ denotes the Milnor $K$-theory (mod 2) of a field $A$. In particular, in [AKM99 Theorem 3.14] it is shown that if $R$ is the subring of $H^*(G_F, \mathbb{F}_2)$ generated by one dimensional classes, then $R$ is isomorphic to the Galois cohomology $H^*(G_F, \mathbb{F}_2)$ of $F$. Thus we see that $G_F$ also controls Galois cohomology and in fact $H^*(G_F, \mathbb{F}_2)$ contains some further substantial information about $F$ which $H^*(G_F, \mathbb{F}_2)$ does not contain. These results can be extended to the case when $p > 2$ and $F^{(2)}$ contains a primitive $p$-th root of unity; see [CEM, BCA, SLMS]. In summary, $G_F$ is a very interesting object. On the one hand $G_F$ is much simpler than $G_F$ or $G_q$, yet it contains substantial information about the arithmetic of $F$. In fact, consider the case when $p > 2$ and $F$ contains a primitive
p-th root of unity. Then let $G = G_F$ be the absolute Galois group of $F$. The descending $p$-central series of $G$ is defined inductively by $G^{(1)} = G$, and $G^{(i+1)} = (G^{(i)})^p[G^{(i)}, G]$, for $i \geq 1$. Thus $G^{(i+1)}$ is the closed subgroup of $G$ generated by all powers $h^p$ and all commutators $[h, g] = h^{-1}g^{-1}hg$, where $h \in G^{(i)}$ and $g \in G$. Then the fixed fields $F^{(i)}$ of $G^{(i)}$ are precisely analogue of fields

$$F = F^{(1)} \subset F^{(2)} \subset F^{(3)} \subset \cdots \subset F^{(i)} \subset \cdots$$

introduced above in the case $p = 2$ and $i = 1, 2$ and 3.

The special case of the main theorem in [EM] then states:

**Theorem 6.3.** For $p > 2$ and for $G = G_F$ as above, $G^{(3)}$ is the intersection of all normal subgroups $N$ of $G$ such that $G/N$ is isomorphic to one of $\{1\}, C_p^2$, and $M_{p^3}$ (the modular group of order $p^3$ which is the unique non-abelian group of order $p^3$ and exponent $p^2$).

The analogous result in the case $p = 2$ was discovered by Villegas [Vi] in a different formalism. In [MS96, Corollary 2.18] this result was reformulated and reproved using the descending 2-central sequence of $G_F$. Namely, then $G^{(3)} = G^{(3)}_2$ is the intersection of all open normal subgroups $N$ of $G$ such that $G/N$ is isomorphic to $\{1\}, C_2, C_4$, or to the dihedral group of order 8. The main ingredients in the proofs of the above results is the Bloch-Kato conjecture in degree 2 which was proved in [MS82]. The case $p = 2$ was the first break-through in the case of general fields made by Merkurjev who used some $K$-theoretic calculations due to Suslin. For this particular case there is now an elementary proof available due to Merkurjev; see [EM]. If $p > 2$, in the cohomology group $H^2(C_{p^2}, \mathbb{F}_p)$ we have elements not expressible as sums of products of elements in $H^1(C_{p^2}, \mathbb{F}_p)$. To handle these elements, in [EM] there is a detailed consideration of the Bockstein homomorphism

$$B_G: H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p).$$

In fact, in [EM] not the full strength of Merkurjev-Suslin theorem was used. The essential tool was the injectivity of the map

$$k_2 \to H^2(F, \mathbb{F}_p).$$

In [CEM], the surjectivity of this map is used to obtain restrictions on presentation of groups $G_F(p)$ via generators and relations.

Let $1 \to R \to S \to G \to 1$, where $G = G_F(p)$ with $F$ as above, $S$ a free pro-$p$-group with minimal number of generators (see [K] Chapter 4), and $R$ is the subgroup of $S$ of relations in $G$. Then we have:

**Theorem 6.4.** (CEM) for any $p$, [MS96] for $p = 2$. Let $S \supset S^{(2)} \supset S^{(3)} \supset \cdots$ be the $p$-descending series of $S$. Then we have

$$R^p[R, S] = R \bigcap S^{(3)}.$$

Observe that for any minimal presentation of any pro-$p$-group $G$ as above, we have $R^p[R, S] \subset R \bigcap S^{(3)}$, as $R \subset S^{(2)}$. The equality in the case when $G = G_F(p)$ is the consequence of the surjectivity of the norm residue map $k_2(F) \to H^2(G_F(p), \mathbb{F}_p)$, which follows from the Merkurjev-Suslin’s theorem.
From the above theorem, one can deduce that if $G$ is any pro-$p$-group such that $R \subset S^{(3)}$, and $G = G_F(p)$, then $G$ is a free pro-$p$-group.

**Example** Let $G$ be a pro-$p$-group on $n$ generators $a_1, a_2, \cdots, a_n$ for $n \geq 2$ subject to relations $[[a_i, a_j], a_r] = 1$ for all $1 \leq i < j \leq n$ and $1 \leq r \leq n$. Then $G$ is not $G_F(p)$ for any field $F$ containing a primitive $p$th root of unity. [CEM] contains further restrictions on possible groups $G_F(p)$ by exploring further properties of its quotients $G_F^{[3]} = G_F(p)/G_F(p)^{(3)}$ and close connections between properties of $G_F^{[3]}$ and the existence of non-trivial valuations on $F$.

We now outline a joint project with Benson and Swallow in which our goal is to obtain a refinement of the Bloch-Kato conjecture. Associated to the field $F$, we have a natural tower of subfields $F^{(n)}$ of the separable closure $F_{\text{sep}}$ defined as follows: $F^{(1)} = F$, $F^{(2)}$ is the compositum of all cyclic extensions of degree $p$ over $F$, and for $n \geq 3$, $F^{(n)}$ is the compositum of all cyclic extensions of degree $p$ over $F^{(n-1)}$ which are Galois over $F$. We call this tower the *filtration tower* associated to $F$. We define Galois groups (in agreement with the notation we used above)

$$G_F^{[n]} := \text{Gal}(F^{(n)}/F), \quad \text{and} \quad G_F^{(n)} := \text{Gal}(F_{\text{sep}}/F^{(n)}),$$

which fit in a sequence $1 \to G_F^{(n)} \to G_F \to G_F^{[n]} \to 1$, where $G_F$ is the absolute Galois group $\text{Gal}(F_{\text{sep}}/F)$. In [CEM] (see also [BCMS]), we have shown that the decomposable part of $H^*(G_F^{[3]}, \mathbb{F}_p)$ is isomorphic to $H^*(F, \mathbb{F}_p)$ under the inflation map. The important question therefore is to determine how an indecomposable class in $H^*(G_F^{[3]}, \mathbb{F}_p)$ decomposes under the various inflation maps along the filtration tower. By the Bloch-Kato conjecture, we know that it decomposes completely into one-dimensional classes when it goes all the way up to the separable closure. But what happens in between? A precise knowledge of this gives a refinement of the Bloch-Kato conjecture. We have shown (using the Bloch-Kato conjecture in degree 2!) that every indecomposable class in $H^2(G_F^{[n]}, \mathbb{F}_p)$ decomposes into one-dimensional classes when it goes to the next level $H^2(G_F^{(n+1)}, \mathbb{F}_p)$ under the inflation map. Thus we have obtained a second cohomology refinement of the Bloch-Kato conjecture. The goal of our joint project with Benson and Swallow is to understand this refinement of the Bloch-Kato conjecture for higher cohomology. This is work in progress.

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Department of Mathematics, Illinois State University, Normal, IL 61790, USA
E-mail address: schebol@ilstu.edu

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7, Canada
E-mail address: minac@uwo.ca