When Is It Possible to Use Perturbation Technique in Field Theory?

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Abstract

The vector pion form factor is used as an example to analyze this question. Given the experimental radius of the pion, the crucial question is whether perturbative methods could be used for the effective chiral lagrangian to calculate the pion form factor. Our analysis shows that the pion rms radius is far too large (or the related $\rho$ resonant mass is too low) for the perturbation theory to be valid.
1 Introduction

Given an interaction Lagrangian characterized by a dimensionless coupling constant \( g \), one can usually guess whether perturbation theory can be used by comparing \( g^2/4\pi \) with unity. If it is much less than unity, there is a good chance that the perturbation technique could be used. Effective Chiral Lagrangian gives on the other hand a power series expansion in momenta and hence the above rule cannot be applied. It is important to set up a set of rules to test whether perturbation theory could be used.

Chiral Perturbation Theory (ChPT) [1, 2, 3, 4] is a well-defined perturbative procedure for the Effective Chiral Lagrangian allowing one to calculate systematically low energy phenomenon involving soft pions. It is now widely used to analyze low energy pion physics even in the presence of resonance as long as the energy region of interest is sufficiently far from the resonance. In this scheme, the unitarity relation is satisfied perturbatively order by order.

The standard procedure of testing ChPT calculation of the pion form factor [5], which claims to support the perturbative scheme, is shown here to be unsatisfactory. This is so because the calculable terms are extremely small, less than 1.5% of the uncalculable terms at an energy of 0.5 GeV or lower whereas the experimental errors are of the order 10-15%. This obvious fact escapes the attention of many due to the complexity of the calculations.

We show how this situation can be dealt with without asking for a new measurement of the pion form factor with a precision much better than 1.5%. This can be done using dispersion relation which establishes in a model independent way a relation between the real and imaginary part of the amplitude.

From these results, we set up a procedure to test whether it is possible to use the perturbation technique for a given lagrangian.

2 Dispersion Relation, Sum Rules and Unitarity

Because the vector pion form factor \( V(s) \) is an analytic function with a cut from \( 4m^2_\pi \) to \( \infty \), the \( n \)th times subtracted dispersion relation for \( V(s) \) reads:

\[
V(s) = a_0 + a_1 s + ...a_{n-1}s^{n-1} + \frac{s^n}{\pi} \int_{4m^2_\pi}^{\infty} \frac{ImV(z)dz}{z^n(z-s-i\epsilon)}
\]

(1)

where \( n \geq 0 \) and, for our purpose, the series around the origin is considered. Because of the real analytic property of \( V(s) \), it is real below \( 4m^2_\pi \). By taking the real part of this equation, \( ReV(s) \) is related to the principal part of the dispersion integral involving the \( ImV(s) \) apart from the subtraction constants \( a_n \).

The polynomial on the R.H.S. of Eq. (1) will be referred in the following as the subtraction constants and the last term on the R.H.S. as the dispersion integral (DI). The evaluation of DI as a function of \( s \) will be done later. Notice that \( a_n = V^n(0)/n! \) is the coefficient of the Taylor series expansion for \( V(s) \), where \( V^n(0) \) is the nth derivative of
\( V(s) \) evaluated at the origin. The condition for Eq. (1) to be valid was that, on the real positive \( s \) axis, the limit \( s^{-n}V(s) \to 0 \) as \( s \to \infty \). The coefficient \( a_{n+m} \) of the Taylor’s series is given by:

\[
a_{n+m} = \frac{1}{\pi} \int_{m_{\pi}^2}^{\infty} \frac{\text{Im}V(z)dz}{z^{n+m+1}}
\]

where \( m \geq 0 \). The meaning of this equation is clear: under the above stated assumption, not only the coefficient \( a_n \) can be calculated but all other coefficients \( a_{n+m} \) can also be calculated. The larger the value of \( m \), the more sensitive is the value of \( a_{n+m} \) to the low energy values of \( \text{Im}V(s) \). In theoretical work such as in ChPT approach, to be discussed later, the number of subtraction is such that to make the DI converges.

The elastic unitarity relation for the pion form factor is \( \text{Im}V(s) = V(s)e^{-i\delta(s)}\sin\delta(s) \) where \( \delta(s) \) is the elastic P-wave pion phase shifts. Below the inelastic threshold of \( 16m_{\pi}^2 \) where \( m_{\pi} \) is the pion mass, \( V(s) \) must have the phase of \( \delta(s) \) \[6\]. It is an experimental fact that below 1.3 GeV the inelastic effect is very small, hence, to a good approximation, the phase of \( V(s) \) is \( \delta \) below this energy scale.

\[\text{Im}V(z) = |V(z)| \sin\delta(z)\]

and

\[\text{Re}V(z) = |V(z)| \cos\delta(z)\]

where \( \delta \) is the strong elastic P-wave \( \pi\pi \) phase shifts. Because the real and imaginary parts are related by dispersion relation, it is important to know accurately \( \text{Im}V(z) \) over a large energy region. Below 1.3 GeV, \( \text{Im}V(z) \) can be determined accurately because the modulus of the vector form factor \[4, 8\] and the corresponding P-wave \( \pi\pi \) phase shifts are well measured \[4, 10, 11\] except at very low energy.

It is possible to estimate the high energy contribution of the dispersion integral by fitting the asymptotic behavior of the form factor by the expression, \( V(s) = -(0.25/s)\ln(-s/s_{\rho}) \) where \( s_{\rho} \) is the \( \rho \) mass squared.

Using Eq. (3) and Eq. (4), \( \text{Im}V(z) \) and \( \text{Re}V(s) \) are determined directly from experimental data and are shown, respectively, in Fig.1 and Fig.2.

### 3 Analysis of the Experimental Data and Test of Dispersion Relation

In the following, for definiteness, one assumes \( s^{-1}V(s) \to 0 \) as \( s \to \infty \) on the cut, i.e. \( V(s) \) does not grow as fast as a linear function of \( s \). This assumption is a very mild one because theoretical models assume that the form factor vanishes at infinite energy as \( s^{-1} \). In this case, one can write a once subtracted dispersion relation for \( V(s) \), i.e. one sets \( a_0 = 1 \) and \( n = 1 \) in Eq. (1).

From this assumption on the asymptotic behavior of the form factor, the derivatives of the form factor at \( s = 0 \) are given by Eq. (2) with \( n=1 \) and \( m=0 \). In particular one has:
where the standard definition $V(s) = 1 + \frac{1}{s} < r^2_V > s + cs^2 + ds^3 + ...$ is used. Eq. (3) is a sum rule relating the pion rms radius to the magnitude of the time like pion form factor and the P-wave $\pi\pi$ phase shift measurements. Using these data, the derivatives of the form factor are evaluated at the origin:

$$< r^2_V > = 0.45 \pm 0.015 \text{fm}^2; c = 3.90 \pm 0.20 \text{GeV}^{-4}; d = 9.70 \pm 0.70 \text{GeV}^{-6}$$

where the upper limit of the integration is taken to be $1.7 \text{GeV}^2$. By fitting $ImV(s)$ by the above mentioned asymptotic expression, the contribution beyond this upper limit is completely negligible.

The only experimental data on the derivatives of the form factor at zero momentum transfer is the rms radius of the pion, $r^2_V = 0.439 \pm 0.008 \text{fm}^2$ [12]. This value is very much in agreement with that determined from the sum rules. In fact the sum rule for the rms radius gets overwhelmingly contribution from the $\rho$ resonance as can be seen from Fig.1. The success of the calculation of the r.m.s. radius is a first indication that causality is respected and also that the extrapolation procedures to low energy for the P-wave $\pi\pi$ phase shifts and for the modulus of the form factor are legitimate.

Dispersion relation for the pion form factor is now shown to be well verified by the data over a wide energy region. Using $ImV(z)$ as given by Eq. (3) together with the once subtracted dispersion relation, one can calculate the real part of the form factor $ReV(s)$ in the time-like region and also $V(s)$ in the space like region. Because the space-like behavior of the form factor is not sensitive to the calculation schemes, it will not be considered here. The result of this calculation is given in Fig.2. As it can be seen, dispersion relation results are well satisfied by the data.

4 Inadequacy of ChPT

The i-loop ChPT result can be put into the following form, similar to Eq. (1):

$$V_{\text{pert}}^{(i)}(s) = 1 + a_1 s + a_2 s^2 + ... + a_i s^i + D^{\text{pert}(i)}(s)$$

where $i + 1$ subtraction constants are needed to make the last integral on the RHS of this equation converges and

$$D^{\text{pert}(i)}(s) = \frac{s^{1+i}}{\pi} \int_{4m_h^2}^{\infty} \frac{ImV^{\text{pert}(i)}(z)dz}{z^{1+i}(z - s - i\epsilon)}$$

with $ImV^{\text{pert}(i)}(z)$ calculated by the $ith$ loop perturbation scheme.

Similarly to these equations, the corresponding experimental vector form factor $V^{\text{exp}(i)}(s)$ and $D^{\text{exp}(i)}(s)$ can be constructed using the same subtraction constants as in Eq. (3) but with the imaginary part replaced by $ImV^{\text{exp}(i)}(s)$, calculated using Eq. (3).
The one-loop ChPT calculation requires 2 subtraction constants. The first one is given by the Ward Identity, the second one is proportional to the r.m.s. radius of the pion.

In Fig. 1, the imaginary part of the one-loop ChPT calculation for the vector pion form factor is compared with the result of the imaginary part obtained from the experimental data. It is seen that they differ very much from each other. One expects therefore that the corresponding real parts calculated by dispersion relation should be quite different from each other.

In Fig. 2, the full real part of the one loop amplitude is compared with that obtained from experiment. At very low energy one cannot distinguish the perturbative result from the experimental one due to the dominance of the subtraction constants. At an energy around 0.56 GeV there is a definite difference between the perturbative result and the experimental data. This difference becomes much clearer in Fig. 3 where only the real part of the perturbative DI, \( ReD_I^{pert(1)}(s) \), is compared with the corresponding experimental quantity, \( ReD_I^{exp(1)}(s) \). It is seen that even at 0.5 GeV the discrepancy is clear. Supporters of ChPT would argue that ChPT would not be expected to work at this energy. One would have to go to a lower energy where the data became very inaccurate.

This argument is false as can be seen by comparing the ratio \( ReD_I^{pert(1)}/ReD_I^{exp(1)} \). It is seen in Fig. 4 that everywhere below 0.6 GeV this ratio differs from unity by a factor of 6-7 due to the presence of non perturbative effects.

Similarly to the one-loop calculation, the two-loop results are plotted in Fig. (1) - Fig. (4) \[5\]. Although the two-loop result is better than the one-loop calculation, because more parameters are introduced, calculating higher loop effects will not explain the data because in ChPT both the form factor and scattering amplitude which enter in the imaginary part calculation are dominated by a polynomial behavior.

It is seen that perturbation theory is inadequate for the vector pion form factor even at very low momentum transfer. This fact is due to the very large value of the pion r.m.s. radius or a very low value of the \( \rho \) mass (see below).

5 Consequences of Unitarized Models

Two unitarized models which are relevant are as follows. The first model is obtained by introducing a zero in the calculated form factor to get an agreement with the experimental r.m.s. radius. The pion form factor is now multiplied by \( 1 + \alpha s/s_\rho \) where \( s_\rho \) is the \( \rho \) mass squared \[13\].

The experimental data can be fitted with a \( \rho \) mass equal to 0.773 GeV and \( \alpha = 0.14 \). These results are in excellent agreement with the data \[8, 12\].

The second model, which is more complete at the expense of introducing more parameters, is based on the two-loop ChPT calculation with unitarity taken into account. It has the singularity associated with the two loop graphs. Using the same inverse amplitude method as was done with the one-loop amplitude, but generalizing this method to two-loop calculation, Hannah has recently obtained a remarkable fit to the pion form factor in the time-like and space-like regions. His result is equivalent to the (0,2) Padé
approximant method as applied to the two-loop ChPT calculation \[16\]. Both models contain ghosts which can be shown to be unimportant \[17\].

As can be seen from Figs. 1, 2 and 3 the imaginary and real parts of these two models are very much in agreement with the data. A small deviation of $ImV(s)$ above $0.9GeV$ is due to a small deviation of the phases of $V(s)$ in these two models from the data of the P-wave $\pi\pi$ phase shifts.

6 Criteria for the Validity of Perturbation Theory

Let us examine in details the one-loop ChPT calculation of the vector pion form factor $V(s)$ \[1\]:

$$V_{\text{pert.}}(s) = 1 + \frac{s}{s_R} + \frac{1}{96\pi^2 f_\pi^2} ((s - 4m_\pi^2)H_{\pi\pi}(s) + \frac{2s}{3})$$

(9)

where $f_\pi = 0.93GeV$ and the r.m.s. radius of the vector form factor is related to $s_R$ by the definition $V'(0) = \frac{1}{6}r_V^2 = 1/s_R$. The function $H_{\pi\pi}(s)$ is given by:

$$H_{\pi\pi}(s) = (2 - 2\sqrt{\frac{s - 4m_\pi^2}{s}} \ln \frac{\sqrt{s} + \sqrt{s - 4m_\pi^2}}{2m_\pi}) + i\pi\sqrt{\frac{s - 4m_\pi^2}{s}}$$

(10)

for $s > 4m_\pi^2$; for other values of $s$, $H_{\pi\pi}(s)$ can be obtained by analytic continuation.

The unitarised version of Eq. (9), obtained by the inverse amplitude, the Padé approximant or the N/D methods, is given by \[13, 14\]:

$$V(s) = \frac{1}{1 - s/s_R - \frac{1}{96\pi^2 f_\pi^2} ((s - 4m_\pi^2)H_{\pi\pi}(s) + 2s/3)}$$

(11)

It is obvious that Eq. (11) has the Breit-Wigner resonance character while that from Eq. (9) does not, although their amplitude and first derivative are identical at $s = 0$. Furthermore, if the parameter $s_R$ was fixed by the the r.m.s. radius, the $\rho$ mass squared, $s_\rho$, would come out to be slightly low compared measured $\rho$ mass. Neglecting this last problem which is unimportant here, the Taylor’s series expansion around $s = 0$ reveals that Eq. (11) gives rise to a coefficient of the $s^2$ term as $(1/s_R)^2 \approx 4.0GeV^{-4}$ which is much larger than that coming from the third term of Eq. (9), $1/(960\pi^2 m_\pi^2 f_\pi^2) \approx 0.63GeV^{-4}$. This is the signal of the failure of the perturbation method.

In other words, ChPT should work if the r.m.s. radius is much smaller than its experimental value or $s_R \gg \sqrt{960\pi f_\pi m_\pi} = 1.3GeV$. This last condition means that the physical $\rho$ mass is far too small for the perturbation theory to be valid.

We can generalize our criteria for the validity of the perturbation method to the nth loop result. For this purpose we have to add one more step in our calculation in order to make it free from criticisms: Eq. (11) could in principle contain unwanted poles due to the unitarisation procedure. This can be eliminated by subtracting out the ghost pole which can be conveniently done by writing down a dispersion relation for $V(s)$ with the same
subtraction constants as used in the perturbative series Eq. (9), but with the imaginary part given by Eq. (11) [17].

Provided that the ghost removing procedure did not change very much the unitarised amplitude, one could then compare the perturbative calculation with the modified unitarized result. If their difference was negligible, one could be sure that the perturbative scheme could then be used.

7 Further Remarks and Conclusion

Because the calculable quantities of the vector pion form factor in the ChPT scheme are too small (well within experimental errors) compared with the uncalculable ones, unless some unitarisation is made, it may be better to give up this perturbative scheme. One would then gain in the transparency of the physics.

Although we have not made here a detailed study of the processes $\pi, \eta \rightarrow \gamma e^+ e^-$ the loop contributions can be shown to be completely negligible compared with the subtracted terms and are therefore not relevant.

In conclusion, higher loop perturbative calculations do not solve the unitarity problem. The perturbative scheme has to be supplemented by the well-known unitarisation schemes such as the inverse amplitude, N/D and Padé approximant methods [13, 14, 16, 17, 18, 19].
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Figure Captions

Fig. 1: The imaginary part of the vector pion form factor $\text{Im} V$, given by Eq. (3), as a function of energy in the GeV unit. The solid curve is the experimental results with experimental errors; the long-dashed curve is the two-loop ChPT calculation, the medium long-dashed curve is the one-loop ChPT calculation, the short-dashed curve is from the modified unitarized one-loop ChPT calculation fitted to the $\rho$ mass and the experimental r.m.s. radius, and the dotted curve is the unitarized two-loop calculation of Hannah [16].

Fig. 2: The real parts of the pion form factor $\text{Re} V$, given by Eq. (4) as a function of energy. The curves are as in Fig. 1. The real part of the form factor calculated by the once subtracted dispersion relation using the experimental imaginary part is also given by the solid line.

Fig. 3: The real parts of the dispersion integral $\text{Re} DI$ as a function of energy. The curves are as in Fig. 1.

Fig. 4: The ratio of the one-loop ChPT to the corresponding experimental quantity, $\text{Re} DI^{\text{pert}(1)}/\text{Re} DI^{\text{exp}(1)}$, defined by Eq. (8), as a function of energy, is given by the solid line; the corresponding ratio for the two-loop result is given by the dashed line. The ratio of the unitarized models to the experimental results is unity (not shown). The experimental errors are estimated to be less than 10%.
Figure 1:

Figure 2:
Figure 3:

Figure 4: