Reconstruction in the Horndeski theory within the scope of the Bianchi I cosmology

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In the previous article Eur. Phys. J. Plus 136, 590 (2021) (arXiv: 2110.15396) we have proposed a reconstruction method for the kinetic gravity braiding theory in the framework of the flat Friedman-Robertson-Walker spacetime. Here we develop this method in the Bianchi I spacetime model for a subclass of the Horndeski theory: $G_5 \sim \phi$, $G_4(X) \neq 0$, $G_3(X) \neq 0$. The Hubble parameter $H(t)$ and the canonical kinetic term $X(t)$ are set a priori. The choice of the function $X(t)$ determines the anisotropic properties of the Universe. This makes it possible to provide believable anisotropy. The presented method allows for a realistic model of the Universe to simply reconstruct some scalar field theory. Reconstruction example is given for anisotropic model of a post-inflationary transition to the radiation-dominated phase. The model is investigated for the absence ghosts and Laplacian instabilities.

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I. INTRODUCTION

To explain accelerated expansion of the Universe and other observational facts, modifications of the gravity theory are used. The Horndeski gravity (HG) has interesting properties \cite{1}. The theory is constructed in such a way that the motion equations are of the order of the derivative no higher than the second. In this sense, the HG is the most general variant of the scalar-tensor theory of gravitation. As we will see below, the action density has a cumbersome form and contains four arbitrary functions. However, this provides a wide field for action. Within the framework of the HG, various cosmological and astrophysical problems are solved \cite{2–7}.

Authors \cite{8} proposed such a parametrization of the action density for the HG:

\[ L_H = \sqrt{-g} \left( L_2 + L_3 + L_4 + L_5 \right), \]

\[ L_2 = G_2(\phi, X), \]

\[ L_3 = -G_3(\phi, X) \Box \phi, \]

\[ L_4 = G_4(\phi, X)R + G_4X(\phi, X) \left( (\Box \phi)^2 - (\nabla_\mu \nabla_\nu \phi)^2 \right), \]

\[ L_5 = G_5(\phi, X)G_{\mu\nu} \nabla_\mu \nabla_\nu \phi - \frac{1}{6}G_{5X} \left( (\Box \phi)^3 - 3 \Box \phi (\nabla_\mu \nabla_\nu \phi)^2 + 2 (\nabla_\mu \nabla_\nu \phi)^3 \right), \]

where $g$ is the determinant of metric tensor $g_{\mu\nu}$; $R$ is the Ricci scalar and $G_{\mu\nu}$ is the Einstein tensor; the functions $G_i$ ($i = 2, 3, 4, 5$) depend on the scalar field $\phi$ and the canonical kinetic term, $X = -\frac{1}{2} \nabla_\mu \phi \nabla^\mu \phi$. The following designations are used $G_{iX} \equiv \partial G_i/\partial X$, $(\nabla_\mu \nabla_\nu \phi)^2 \equiv \nabla_\mu \nabla_\nu \phi \nabla^\nu \nabla^\mu \phi$, and $(\nabla_\mu \nabla_\nu \phi)^3 \equiv \nabla_\mu \nabla_\nu \phi \nabla^\nu \nabla^\mu \phi \nabla^\rho \phi$.

We continue the research begun in the work \cite{7}, which considered the kinetic gravity braiding (KGB) theory ($G_2(X) \neq 0$, $G_3(X) \neq 0$, $G_4 = 1/(16\pi)$) in the flat Friedman-Robertson-Walker spacetime (FRW). Here we present a reconstruction method in the Bianchi I spacetime model (BI) and in the FRW spacetime for the subclass of HG:

\[ G_2(X) \neq 0, G_3(X) \neq 0, G_4 = 1/(16\pi), G_5 = \eta \cdot \phi/2, \eta = \text{const.} \]

The functions $G_2$, $G_3$ are reconstructed based on the given the form of the Hubble parameter $H(t)$ and kinetic density $X(t)$. The choice freedom of functions $H$, $X$ allows you to regulate the issues of model stability \cite{3,10}. As shown in
ref. 11 [13], the presence of the function $G_5$ in the action density can lead to non-standard behavior of the anisotropic Universe. During the reconstruction, the function $X$ directly models the anisotropy behavior of the Universe. This moment is important to ensure believable anisotropy. For example, authors [17]. [15] claim that the isotropization of the Universe occurred quite early. At the start of primordial nucleosynthesis, the universe must already be isotropic.

The presented method allows for a realistic model of the Universe to simply reconstruct some scalar field theory. For example, we will reconstruct the field theory for a post-inflationary transition to the radiation-dominated phase. Authors R. C. Bernardo and I. Vega [16] proposed their own version of reconstruction the KGB theory for the FRW spacetime. In this theory, the scalar charge is zero. To provide a dynamic solution $H(t)$, $X(t)$, another matter with a density $\rho(t)$ was introduced. In our method, the only source is the scalar field $\phi$. The dynamics of functions $H(t)$ and $X(t)$ is provided by the nonzero scalar charge. For the BI model, there is an additional contribution to the dynamics of $H(t)$, $X(t)$ from the nonzero anisotropic charges. R. C. Bernardo et al [17, 18] significantly develop the ideas of reconstruction in connection with observational data. Authors [17] reconstruct the parameters $H(t)$ from cosmic chronometers, supernovae, and baryon acoustic oscillations compiled data sets via the Gaussian process method and use it to draw out HG that are fully anchored on expansion history data. In the work [18], dark energy is studied through the viewpoints of parametric and nonparametric analyses of late-time cosmological data.

II. FIELD EQUATIONS

The metric of a Bianchi-I geometry may be written as

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)dy^2 + a_3^2(t)dz^2.$$  

We consider a homogeneous model: $a_i = a(t)$ and $\phi = \phi(t)$ . For the action density [4] and the Bianchi-I geometry [4] the gravitation equations have the form [13]:

$$G_0^i \left( \mathcal{G} - 2G_{4X} \dot{\phi}^2 - 2G_{4XX} \dot{\phi}^4 + 2G_{5\phi} \dot{\phi}^2 + G_{5XX} \dot{\phi}^4 \right) = G_2 - G_{2X} \dot{\phi}^2 -$$ \n
$$-3G_{3X} H \ddot{\phi}^3 + G_{3\phi} \dot{\phi}^2 + 6G_{4\phi} H \dot{\phi} + 6G_{4XX} \dot{\phi}^3 H - 5G_{5\phi} H_1 H_2 H_3 \dot{\phi}^3 - G_{5XX} H_1 H_2 H_3 \dot{\phi}^5,$$

$$\mathcal{G} G_i^j - (H_j + H_k) \frac{d\mathcal{G}}{dt} = G_2 - \dot{\phi} \frac{dG_3}{dt} + 2 \frac{d}{dt}(G_{4\phi} \dot{\phi})$$

$$- \frac{d}{dt}(G_{5X} \dot{\phi}^3 H_j H_k) - G_{5XX} \dot{\phi}^3 H_j H_k (H_j + H_k), \quad \mathcal{G} \equiv 2G_4 - 2G_{4X} \dot{\phi}^2 + G_{5\phi} \dot{\phi}^2.$$  

A dot means a derivative with respect to $t$; $H_i = \dot{a}_i/a_i$ – the Hubble parameters; $H = \frac{1}{3} \sum_{i=1}^{3} H_i \equiv \ddot{a}/a$ – the average Hubble parameter with $a = (a_1 a_2 a_3)^{1/3}$. The triples of indices $\{i, j, k\}$ take values $\{1, 2, 3\}$, $\{2, 3, 1\}$, or $\{3, 1, 2\}$.

The equation for the scalar field $\phi(t)$ can be represented as

$$\frac{1}{a^3} \frac{d}{dt}(a^3 \mathcal{J}) = \mathcal{P},$$

with

$$\mathcal{J} = \dot{\phi} \left[ G_{2X} - 2G_{3\phi} + 3H \dot{\phi}(G_{3X} - 2G_{4X} \dot{\phi}) + 

+ G_0^i (-2G_{4X} - 2\dot{\phi}^2 G_{4XX} + 2G_{5\phi} + G_{5XX} \dot{\phi}^2) + H_1 H_2 H_3 (3G_{5X} \dot{\phi} + G_{5XX} \dot{\phi}^3) \right],$$

$$\mathcal{P} = G_{2\phi} - \dot{\phi}^2 (G_{3\phi} + G_{3XX} \dot{\phi}) + RG_{4\phi} + 2G_{4XX} \dot{\phi} (3\dot{\phi} H - \dot{\phi} G_{5\phi}^0) +$$

$$+ G_0^i G_{5XX} \dot{\phi}^2 + G_{5XX} \dot{\phi}^3 H_1 H_2 H_3.$$
For convenience, we consider the parametrization of the metric
\[
ds^2 = -dt^2 + a(t)^2 [e^{2(\beta_+ + \sqrt{3}\beta_-)} dx^2 + e^{2(\beta_+ - \sqrt{3}\beta_-)} dy^2 + e^{-4\beta_+} dz^2].
\] (10)

The functions \(e^{\beta_+ + \sqrt{3}\beta_-}, e^{\beta_+ - \sqrt{3}\beta_-}\) and \(e^{-2\beta_+}\) are the deviation from isotropy, and \(a(t)\) is the isotropic part. The Hubble parameters in the direction of \(x, y\) and \(z\) are given by
\[
H_1 = H + \beta_+, H_2 = H + \beta_+ - \sqrt{3}\beta_-, H_3 = H - 2\beta_+.
\] (11)

We choose the subclass of HG:
\[
G_2 = G_2(X), G_3 = G_3(X), G_4 = \frac{1}{16\pi}, G_5 = \frac{\eta\phi}{2}.
\] (12)

Equations (5), (6) and (7) will change as follows
\[
3(H^2 - \sigma^2) \left( \frac{1}{8\pi} + \frac{3\eta\dot{\phi}^2}{2} \right) = -G_2 + \dot{\phi}^2 G_{2X} + 3G_{3X} H \dot{\phi}^3,
\] (13)
\[
(2\ddot{H} + 3H^2 + 3\sigma^2) \left( \frac{1}{8\pi} + \frac{\eta\dot{\phi}^2}{2} \right) + \eta H \cdot \frac{d(\dot{\phi}^2)}{dt} = -G_2 + G_{3X} \dot{\phi}^2 \ddot{\phi},
\] (14)
\[
\beta_+ \left( \frac{1}{8\pi} + \frac{\eta\dot{\phi}^2}{2} \right) = \frac{C_+}{a^3}, \quad \sigma^2 = \beta_+ + \beta_-,
\] (15)
\[
\dot{\phi} \left[ G_{2X} + 3HG_{3X} \dot{\phi} - 3\eta(H^2 - \sigma^2) \right] = \frac{C_0}{a^3},
\] (16)
where \(C_0, C_+\) and \(C_-\) are the integration constants. Constants \(C_\pm\) correspond to the anisotropic charges; \(C_0\) is the scalar charge. The system (13), (14), (15) and (16) contains three independent equations and five functions \((H(a), X(a), \sigma^2, G_2, G_3)\). There is freedom to choose two functions out of five. For example, we can set the law of the Universe development \(H(a)\) and the canonical kinetic term \(X(a)\). Then we find the functions \(\beta_\pm, G_2(X)\) and \(G_3(X)\). If all charges are simultaneously \(C_0 = C_\pm = 0\), then the system (13), (14), (15), (16) can only have a stationary solution \(H, X = \text{const}, \beta_\pm = 0\). Equality \(\beta_\pm = 0\) means that space-time has become the FRW spacetime. In case \(C_0 \neq 0, C_\pm \neq 0\), we will have a rich variety of cosmological models.

Equations (13), (14) and (16) have consequences
\[
G_2 = -3 \left( \frac{1}{8\pi} + \eta X \right) (H^2[a(X)] - \sigma^2[a(X)]) + \frac{\varepsilon C_0 \sqrt{2X}}{a^3(X)},
\] (17)
\[
G_{3X} = \frac{2\eta H}{\varepsilon\sqrt{2X}} + \frac{2}{\varepsilon\sqrt{2X}} \left( \frac{1}{8\pi} + \eta X \right) \left( H' + \frac{3a'X\sigma^2}{a(X)H} \right) + \frac{C_0 a'}{Ha^4(X)},
\] (18)
where \(\varepsilon = \pm 1\) defines the sign of the derivative \(\dot{\phi} = \varepsilon\sqrt{2X}\). It is easy to verify that \(\dot{\phi} X = aH/a_X\). As can be seen from the corollary of equation (15)
\[
\dot{\beta}_\pm = \frac{C_\pm}{a_3 (\frac{1}{8\pi} + \eta X(a))},
\] (19)
the anisotropic properties of the Universe are directly determined by the choice of the density \(X(a)\).

**III. REDUCTION TO THE ISOTROPIC CASE**

To begin with, we consider reconstruction in isotropic space. If we set \(\dot{\beta}_\pm\) to zero, then we obtain the isotropic spacetime:
\[
ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right),
\] (20)
then the field equations (13), (14) and (16) take the form

$$3H^2 \left( \frac{1}{8\pi} + \frac{3\eta\phi}{2} \right) = -G_2 + \dot{\phi}^2 G_{2X} + 3G_{3X}H\dot{\phi}^3, \quad (21)$$

$$2\dot{H} + 3H^2 \left( \frac{1}{8\pi} + \frac{\eta\phi^2}{2} \right) + \eta H \cdot \frac{d(\phi^2)}{dt} = -G_2 + G_{3X}\dot{\phi}^2, \quad (22)$$

$$\dot{\phi} \left[ G_{2X} + 3HG_{3X}\dot{\phi} - 3\eta H^2 \right] = \frac{C_0}{a^3}. \quad (23)$$

The restored functions (17), (18) will take the form

$$G_2 = -3 \left( \frac{1}{8\pi} + \eta X \right) H^2[a(X)] + \frac{\varepsilon C_0 \sqrt{2X}}{a^3(X)} , \quad (24)$$

$$G_{3X} = \frac{2\eta H}{\varepsilon \sqrt{2X}} + \frac{2H'}{\varepsilon \sqrt{2X}} \left( \frac{1}{8\pi} + \eta X \right) + \frac{C_0 a_H}{H a^4(X)}. \quad (25)$$

When considering the modified theories of gravity, one has to worry about the presence of ghosts as well as Laplacian instabilities [9, 10]. The scalar perturbations will be stable if [9]:

$$c_S^2 = \frac{3(2w_1^2w_2H-w_3^2w_4+4w_1w_3w_2H-w_1^2w_3^2)}{w_1(4w_1w_3+9w_2^2)} \geq 0, \quad (26)$$

$$Q_S = \frac{w_1(4w_1w_3+9w_2^2)}{3w_2^2} > 0, \quad (27)$$

where

$$w_1 = 2\left( G_4 - 2XG_{4X} - 2X(G_{5X}\dot{\phi}H - G_{5\phi}) \right), \quad (28)$$

$$w_2 = -2G_{3X}\dot{\phi} + 4G_4H - 16X^2G_{4XX}H + 4(\dot{\phi}G_{4\phi} - 4HG_{4X})X + 2G_4\dot{\phi} + 8X^2HG_{5\phi}X + 2H X(6G_{5\phi} - 5G_{5X}\dot{\phi}H) - 4G_{5XX}\dot{\phi}X^2H^2, \quad (29)$$

$$w_3 = 3X(G_{2X} + 2XG_{2XX}) + 6X(3X\dot{\phi}H_{3XX} - G_{3XX}X - G_{3\phi} + 6H\dot{\phi}G_{3X}) + 18H(4X^3G_{4XX} - HG_4 - 5X\dot{\phi}G_{4\phi} - G_{4\phi}H + 7HG_{4X}X + 16HX^2G_{4XX} - 2X^2\dot{\phi}G_{4\phi}X) + 6H^2X(2H\dot{\phi}G_{5XX}X^2 - 6X^2G_{5\phi}XX + 13XH\dot{\phi}G_{5XX} - 27G_{5\phi}X + 15H\dot{\phi}G_{5XX} - 18G_{5\phi}), \quad (30)$$

$$w_4 = 2G_4 - 2XG_{5\phi} - 2XG_{5XX}\dot{\phi}. \quad (31)$$

The speed square $c_S^2 > 0$ excludes the Laplacian instabilities. The condition (27) guarantees the absence of ghosts. Similarly, for tensor perturbations we have [9]:

$$Q_T = \frac{w_1}{4} > 0, \quad c_T^2 = \frac{w_4}{w_1} \geq 0. \quad (32)$$

Taking into account (24), (25) and $G_5 = \eta\phi/2$, we rewrite functions $Q_T$, $c_T^2$, $c_S^2$ and $Q_S$ in the form

$$Q_T = \frac{1}{4} \left( \frac{1}{8\pi} + \eta X(a) \right), \quad c_T^2 = \frac{1 - 8\pi\eta \cdot X(a)}{1 + 8\pi\eta \cdot X(a)},$$

$$Q_S = \frac{3 \left( \frac{1}{8\pi} + \eta X \right) }{w_2^2} \left[ S \left( \frac{1}{\pi} + 16\eta X \right) \left( \frac{X^{5/2}H_a'}{a^3(X_a)^2} - \frac{X^{3/2}}{2\alpha^3X_a} + \frac{4S^2X^3}{a^5(X_a)^2H^2} \right) \right], \quad (34)$$
\[ c_S^2 = \frac{1}{Q_S} \left[ -\omega_4 + 2H \left( \frac{\omega^2 a}{w_2} \right)' \right] , \quad S = \varepsilon C_0 \sqrt{2} . \] (35)

where \( w_1 = 1/(8\pi) + \eta X \), \( w_4 = 1/(8\pi) - \eta X \) and
\[ w_2 = 2 \left( \frac{1}{8\pi} + \eta X \right) \left( H - \frac{2H'X}{X_a'} \right) - \frac{2SX^{3/2}}{a^4X_aH} . \] (36)

The Hubble parameter \( H(a) \) is set from cosmological considerations. The choice of the function \( X(a) \) makes it possible to satisfy conditions (26), (27) and (32) in the region of applicability of the model \( H(a) \).

### A. Post-inflationary transition to the radiation-dominated phase

Now we will show reconstruction example of the gravitating scalar field theory for a post-inflationary transition to the radiation-dominated phase. Further, the theory will be tested for the absence of pathologies.

We define the Hubble parameter \( H(a) \) as follows
\[ H^2 = \frac{8\pi \rho}{3} , \quad \rho = \frac{2\rho_d}{a^4 + 1} , \quad \rho_d = \text{const} . \] (37)

The effective energy density \( \rho \) corresponds to the equation of state
\[ p = \frac{\rho}{3} - \mu \rho^2 , \quad \mu > 0 \] (38)
describes the transition from the inflationary era to the radiation era in the the early Universe [19]. The linear term \( \rho/3 \) describes the radiation. The nonlinear part \(-\mu \rho^2\) may be due to Bose-Einstein condensates with self-interaction.

In case (38), the deceleration parameter (DP) has the form
\[ q(\rho) = \frac{d}{dt} \left( \frac{1}{H} \right) - 1 = 1 - \frac{3\mu \rho}{2} = 1 - \frac{\rho}{\rho_d} . \] (39)

The DP changes sign at point \( \rho_d \):
\[ q(\rho_d) = 0 , \quad \rho_d = \frac{2}{3\mu} . \] (40)

The Universe is accelerating when \( \rho > \rho_d \) and decelerating when \( \rho \leq \rho_d \). The point \( a = a_d = 1 \) corresponds to the end of inflation. The density \( \rho \) is a bounded function:
\[ 2\rho_d \leftarrow \rho \to 0 , \quad \text{as } 1 \gg a \gg 1 . \] (41)

The Planck density \( \rho_p \) is approximately equal to \( \rho_p \approx 2\rho_d \). With this in mind, we write
\[ \rho = \frac{\rho_p}{a^4 + 1} . \] (42)

In the early time \( (a \ll 1) \), the model has the quasi-de Sitter behavior:
\[ H^2 \to \frac{8\pi \rho_p}{3} = \text{const} . \] (43)

At late time \( (a \gg 1) \) the Universe enters in the radiation era:
\[ H^2 \propto \frac{1}{a^4} . \] (44)

We define the input function \( X(a) \):
\[ X = \frac{m}{8\pi\eta a^6} , \quad m = \text{const} . \] (45)
Looking ahead, we note that the choice of $X(a)$ will determine the behavior of the anisotropy in the Bianchi I model. Assumption 45 allows us to obtain from (24), (25) the functions $G_2$ and $G_3 X$:

$$G_2 = S \left( \frac{8\pi \eta}{m} \right)^{1/2} X - \frac{\rho_p \cdot X^{2/3}(1 + 8\pi \eta X)}{(8\pi \eta / m)^{2/3} + X^{2/3}},$$

$$G_3 X = \frac{1}{4\varepsilon} \left\{ \frac{3}{\pi \rho_p} \cdot X^{-5/6} \left( \left( \frac{8\pi \eta}{m} \right)^{2/3} + X^{2/3} \right)^{1/2} \times \right.$$ \begin{align*}
&\left\{ \frac{5}{6} \left( \frac{8\pi \eta}{m} \right)^{1/2} + 16\pi \rho_p \frac{\eta X^{2/3}}{3} \left( \frac{8\pi \eta}{m} \right)^{2/3} + X^{2/3} \right\} + \left( \frac{8\pi \eta}{m} \right)^{2/3} + X^{2/3} \right\}. \end{align*}

$$\times \left\{ -\frac{S}{6} \left( \frac{8\pi \eta}{m} \right)^{1/2} + \frac{16\pi \rho_p}{3} \frac{\eta X^{2/3}}{3} + \left( \frac{8\pi \eta}{m} \right)^{2/3} + X^{2/3} \left( \frac{1 + 8\pi \eta X}{24\pi X^{1/3} \left( \left( \frac{8\pi \eta}{m} \right)^{2/3} + X^{2/3} \right)} \right) \right\}. \quad (47)$$

During the radiation era ($a \gg 1$) we have a limit:

$$G_2 \simeq -\rho_p \left( \frac{m}{8\pi \eta} \right)^{2/3} \cdot X^{2/3},$$

$$G_3 \simeq -\varepsilon \sqrt{\frac{\rho_p}{3\pi}} \left( \frac{m}{8\pi \eta} \right)^{1/3} \cdot X^{-1/3} + const. \quad (48)$$

The quasi-de Sitter behavior ($a \ll 1$):

$$G_2 \simeq \left[ -8\pi \eta \rho_p + S \left( \frac{8\pi \eta}{m} \right)^{1/2} \right] \cdot X, $$

$$G_3 \simeq \frac{\varepsilon}{2} \sqrt{\frac{3}{\pi \rho_p}} \left[ \frac{16\pi \eta \rho_p}{3} - \frac{S}{6} \left( \frac{8\pi \eta}{m} \right)^{1/2} \right] \cdot X^{1/2} + const. \quad (49)$$

For the model under consideration, we rewrite functions $Q_T$, $c_T^2$, $Q_S$ and $c_S^2$ in the form:

$$Q_T = \frac{a^6 + m}{32\pi a^6}, \quad c_T^2 = \frac{a^6 - m}{a^6 + m},$$

$$Q_S = \frac{3B(a^6 + m)(1 + a^4)^2[2(a^6 + m)(a^4 + 3) + B(a^4 + 1)^2]}{8\pi a^6[2(a^6 + m)(a^4 + 3) + B(a^4 + 1)^2]^2}, \quad (51)$$

$$c_S^2 = \frac{Q_S}{a} \left[ -\omega_4 + 2H \left( \frac{\omega_4 a}{w_2} \right)_a \right],$$

where

$$\omega_1 = \frac{a^6 + m}{8\pi a^6}, \quad \omega_4 = \frac{a^6 - m}{8\pi a^6},$$

$$\omega_2 = \frac{A[2(a^6 + m)(a^4 + 3) + B(a^4 + 1)^2]}{24\pi a^6(1 + a^4)^{3/2}}, \quad A = \sqrt{\frac{8\pi \rho_p}{3}}, \quad B = \frac{S}{A^2} \left( \frac{8\pi m}{\eta} \right)^{1/2}. \quad (54)$$

Conditions 32 for tensor perturbations are satisfied when $a^6 > |m|$. In the case $m > 0$, the inequality $Q_T > 0$ holds for any $a > 0$. However, $c_T^2$ has a negative limit, $c_T^2 \to -1$, $a \to 0$. The function $Q_S$, $c_S^2$ have the following approximations:

$$Q_S \propto \frac{3Bm(12m + B)}{8\pi (6m + B)^2 a^6}, \quad a \to 0; \quad Q_S \propto \frac{3B}{8\pi a^2}, \quad a \to +\infty; \quad (55)$$

$$c_S^2 \to \frac{(B - 24m)(6m + B)}{3B(12m + B)}, \quad a \to 0; \quad c_S^2 \propto \frac{8a^2}{3B}, \quad a \to +\infty. \quad (56)$$

The profiles of functions $c_S^2$, $Q_S$ are shown in Fig. 1 and Fig. 2 Conditions 26, 27 for the scalar perturbations are satisfied.
IV. BIANCHI I MODEL

Let us consider an anisotropic generalization of the model (37), (45). The assumption (45) for the function $X(a)$ gives a corollary to equation (19):

$$\dot{\beta}_\pm = \frac{8\pi C_\pm}{a^6 + m} \cdot \frac{a^3}{a^9 + m}, \quad \sigma^2 = \frac{(8\pi)^2 \sigma_0^2 a^6}{(a^6 + m)^2},$$

(57)

where $\sigma_0^2 \equiv C_-^2 + C_+^2$. In the general theory of relativity, spatial anisotropies produce in the Einstein equations terms proportional to $1/a^6$, which become dominant when one goes backwards in time. Here, the shear scalar $\sigma^2$ is a
The shear scalar tend to zero for \( a \to 0 \) and their contribution to the total energy balance is \( \propto a^6 \) instead of \( \propto 1/a^6 \). This is the non-standard behavior at small \( a \). The shear scalar is suppressed at late time as well by the factor \( 1/a^6 \). Taking into account anisotropy, we generalize the assumption (37):

\[
H^2 = \sigma^2 + \frac{8\pi \rho}{3},
\]

where

\[
\rho = \frac{\rho_p}{a^4 + 1}.
\]

The mean anisotropy parameter \( A \) is defined as

\[
A = \frac{1}{3} \sum_{i=1}^{3} \left( H_i - H \right)^2 = \frac{2\sigma^2}{H^2} = 
\]

\[
= 2 \left[ 1 + \frac{F(a^6 + m)^2}{a^n(a^4 + 1)} \right]^{-1},
\]

where

\[
F \equiv \frac{\rho_p}{24\pi \sigma_0^2}.
\]

This parameter describes the isotropization process of the Universe. The function \( A \) has a limit: \( A \to 0 \) as \( 0 \leftarrow a \to \infty \). Therefore, the anisotropy effect is totally negligible at early and late times. The behavior of the parameter \( A \) is shown in Fig. 3. As we can see, the selected function \( X(a) \) provides isotropization in later times. Since the anisotropy is suppressed at early and late times, the behavior of the model (59) is similar to that of the model (37) at these times:

\[
H^2 \approx \frac{8\pi \rho}{3} \to \frac{8\pi \rho_p}{3} = \text{const}, \ a \ll 1.
\]

\[
H^2 \approx \frac{8\pi \rho}{3} \propto \frac{1}{a^4}, \ a \gg 1.
\]

There is a difference for models in intermediate times. The quasi-de Sitter stage and the radiation era are separated by the anisotropic phase. The profile Fig. 4 compares the behavior of the Hubble parameter for models (37) and (59). The scale factors \( a \) of the two models are shown in Fig 5.

Assumption (45) allows us to obtain from (17), (18) the functions \( G_2 \) and \( G_{3X} \):

\[
G_2 = S \left( \frac{8\pi \eta}{m} \right)^{1/2} X - \rho_p \cdot \frac{X^{2/3}(1 + 8\pi \eta X)}{(8\pi \eta m)^{2/3} + X^{2/3}},
\]

\[
G_{3X} = \frac{1}{\varepsilon \sqrt{2X}} \left( \frac{(8\pi \eta)^3 \sigma_0^2 \eta X}{m(1 + 8\pi \eta X)^2} + \frac{8\pi \rho_p}{3} \cdot \frac{X^{2/3}}{(8\pi \eta m)^{2/3} + X^{2/3}} \right)^{-1/2} \times
\]

\[
\times \left\{ \frac{S}{6} \left( \frac{8\pi \eta}{m} \right)^{1/2} + \frac{16\pi \rho_p}{3} \left[ \frac{\eta X^{2/3}}{(8\pi \eta m)^{2/3} + X^{2/3}} + \frac{8\pi \eta}{m} \right]^{2/3} \right\}.
\]

Here, approximations (48), (49) are also valid. Stability conditions (26), (27), (32) have been derived for the homogeneous and isotropic backgrounds. They can be used also in the Bianchi I model case at early and late times, since the anisotropies are then damped.
FIG. 3: Profile of the parameter $A$ for $m = 0.01, F = 1$.

FIG. 4: Profiles of the parameter $\sqrt{\frac{4}{8\pi \rho_p}} \cdot H$ for isotropic model (red) and anisotropic model (blue).

V. CONCLUSION

We have proposed a way to reconstruct the HG theory on the Hubble evolution and on the canonical kinetic term $X$ in the Bianchi I spacetime model. The experience of many researchers makes it possible to set a scenario for the evolution of the Universe a priori. The freedom of choice $X(a)$ is used to regulate the stability of the model and the anisotropic properties of the Universe.

We have reconstructed the scalar theory, which gives the model of post-inflationary transition to the radiation-dominated phase in the Bianchi I spacetime. As a basis for the reconstruction, we took a fluid with the nonlinear
equation of state. At the Universe stages with suppressed anisotropy, this equation has the form $p = \rho/3 - \mu \rho^2$. The term $-\mu \rho^2$ may be due to Bose-Einstein condensates with repulsive ($\mu < 0$) or attractive ($\mu > 0$) self-interaction. The kinetic dependence $X(a) \sim a^{-6}$ gives the following properties for the model. It is important that in the process of expansion, the Universe is isotropized. Interestingly, the anisotropy is also suppressed in the early times of the Universe. At intermediate times, there is a finite burst of anisotropy with further relaxation. The isotropic analogue of the model does not contain ghosts and Laplacian instabilities for scalar perturbations. For the tensor perturbations, on the interval $(0, a_1)$, the square of the sound speed $c_T^2$ is negative. For the anisotropic model, these judgments are valid at stages of the Universe with suppressed anisotropy.

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