Diagonalization of $\text{Sp}(2)$ matrices

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Abstract

The two-by-two $\text{Sp}(2)$ matrix has three parameters with unit determinant. Yet, there are no established procedures for diagonalizing this matrix. It is shown that this matrix can be written as a similarity transformation of the two-by-two Wigner matrix, derivable from Wigner’s little group which dictates the internal space-time symmetries of relativistic particles. The Wigner matrix can be diagonalized for massive and space-like particles, while it takes a triangular form with unit diagonal elements for light-like particles. The most immediate physical application can be made to repeated one-dimensional transfer matrices appearing in many different branches of physics. Another application of current interest could be the dis-entanglement of entangled systems.

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1 Introduction

The two-by-two matrices of the group $Sp(2)$ or its unitary equivalent appear in almost all branches of physics. Indeed, a systematic approach to the study of those matrices is a respectable branch of physics [1, 2].

The $Sp(2)$ matrix is the basic language for linear canonical transformations for phase-space approach to both classical and quantum mechanics [3]. The same is true for classical and quantum optics [3], as well as the current problem of entangled harmonic oscillators [4, 5]. The one-dimensional scattering matrix also takes the form of this $Sp(2)$ matrix. The two-by-two beam transfer matrix is unitarily equivalent to that of the $Sp(2)$ group [6, 7, 8]. The list could be endless. In short, we cannot do modern physics without two-by-two matrices, especially without those of the $Sp(2)$ group.

Do we then know how to diagonalize this simple mathematical expression? We can if they can be diagonalized by a rotation. If not, there are no standard procedures for approaching this problem. In this paper, we address this fundamental issue. Our main interest is whether this matrix can be written as a similarity transformation of a diagonal matrix. If that is the case, we solve the practical problem of calculating repeated applications of the same matrix as we see in finite one-dimensional crystals, including periodic potentials [6, 8], multilayer optics [9, 10], lens optics [11, 12], and laser cavities [13], and many other problems in physics.

We first show that the most general form of the $Sp(2)$ matrix can be written as a similarity transformation of Wigner’s little-group matrix [14]. This Wigner matrix takes three different forms depending on the parameters of the original $Sp(2)$ matrix. However, one of these three matrices is not diagonalizable, but offers the same conveniences as diagonal matrices do. Indeed, the diagonalization of the $Sp(2)$ matrix requires the construction of a two-by-two representation of Wigner’s little groups.

For this purpose, we start with the Bargmann decomposition of the $Sp(2)$ matrix in which it is written as a product of three one-parameter matrices. We also write the matrix as a similarity transformation of Wigner’s little group matrix. In this way, the Wigner parameters can be written as the Bargmann parameters, allowing us to write the $Sp(2)$ matrix as a similarity transformation of the Wigner matrix, namely Wigner’s little-group matrix.

In section 2, we start with the Bargmann representation and transform it into a form convenient for writing it as a similarity transformation of the Wigner’s little-group matrix discussed in section 3. In section 4, the
Bargmann parameters are written in terms of the Wigner parameters. The $Sp(2)$ matrix takes many different forms. They are shown to be unitarily equivalent in section 5. In section 6, we discuss some immediate applications of the diagonalization by Wignerization of the $Sp(2)$ matrix.

2 Bargmann decomposition

The $Sp(2)$ matrix appears in the physics literature in several different forms. This two-by-two unimodular matrix is often written as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

(1)

with $(AD - BC) = 1$. All four elements are real numbers. This matrix is commonly called the $ABCD$ matrix and has three independent parameters.

Furthermore, this unimodular matrix $M$ can also be written as

$$M = R_1(\theta_1)B(2\lambda)R_2(\theta_2),$$

(2)

with

$$B(2\lambda) = \begin{pmatrix} \cosh \lambda & \sinh \lambda \\ \sinh \lambda & \cosh \lambda \end{pmatrix},$$

$$R_i(\theta_i) = \begin{pmatrix} \cos(\theta_i/2) & -\sin(\theta_i/2) \\ \sin(\theta_i/2) & \cos(\theta_i/2) \end{pmatrix}. $$

(3)

This is known as the Bargmann decomposition [15]. We use the notation $B(2\lambda)$ instead of $B(\lambda)$ purely for convenience. The main point is that the original three-parameter matrix is decomposed into three one-parameter matrices.

We can rewrite the Bargmann decomposition of equation (2) as

$$M = R_1BR_2 = (LR)B(RL^{-1}) = L(RBR)L^{-1},$$

(4)

where

$$R_1 = LR, \quad R_2 = RL^{-1},$$

(5)

and

$$R(\theta) = \sqrt{R_1R_2} = \sqrt{R_2R_1} = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix},$$

$$L(\delta) = \sqrt{R_1R_2^{-1}} = \begin{pmatrix} \cos(\delta/2) & -\sin(\delta/2) \\ \sin(\delta/2) & \cos(\delta/2) \end{pmatrix},$$

(6)
with
\[ \theta = \frac{\theta_1 + \theta_2}{2}, \quad \delta = \frac{\theta_1 - \theta_2}{2}. \]

Then, the matrix \( M \) can be written as a similarity transformation of \((RBR)\) with respect to \( L \), or a rotation of the system by \( L \). Thus, for all practical purposes, it is sufficient to study the core matrix \( RBR \) with two parameters. This is still a form of the Bargmann decomposition.

Now the question is how to diagonalize the core matrix \((RBR)\) and write it as a similarity transformation. This core matrix takes the form
\[
R(\theta) B(2\lambda) R(\theta) = \begin{pmatrix}
(cosh \lambda) \cos \theta & -(cosh \lambda) \sin \theta + sinh \lambda \\
(cosh \lambda) \sin \theta + sinh \lambda & (cosh \lambda) \cos \theta
\end{pmatrix}. \tag{7}
\]
We assume here that both \( \sin \theta \) and \( sinh \lambda \) are positive.

The product of the two off-diagonal components becomes
\[
cosh^2 \lambda [\tanh \lambda + \sin \theta][\tanh \lambda - \sin \theta] = (cosh \lambda)^2 \cos^2 \theta - 1, \tag{8}
\]
Thus, if one of the off-diagonal components vanishes, the diagonal elements become one, and the \( RBR \) matrix becomes
\[
H_+(sinh \lambda) = \begin{pmatrix}
1 & 0 \\
2 sinh \lambda & 1
\end{pmatrix}, \quad \text{or} \quad H_-(sinh \lambda) = \begin{pmatrix}
1 & -2 sinh \lambda \\
0 & 1
\end{pmatrix}, \tag{9}
\]
respectively. This form is known as the Iwasawa decomposition, and is a special case of the Bargmann decomposition. This matrix is not diagonalizable.

Yet, we are interested in diagonalizing the \( RBR \) matrix by following the usual procedure of calculating the eigenvalues of the matrix, which leads to the eigenvalues
\[
E_\pm = (cosh \lambda) \cos \theta \pm \sqrt{(cosh \lambda)^2 \cos^2 \theta - 1}. \tag{10}
\]
with \( E_+ E_- = 1 \). The quantity inside the square root sign is the same as the product of the two off-diagonal elements given in equation (8). Then the diagonal matrix takes the form
\[
D = \begin{pmatrix}
E_+ & 0 \\
0 & E_-
\end{pmatrix}. \tag{11}
\]

If \((cosh \lambda)^2 \cos^2 \theta \) is smaller than one, and the quantity inside the square-root sign is negative, the eigenvalues are
\[
E_\pm = \exp (\pm i\phi/2), \quad \tag{12}
\]
with
\[ \cos(\phi/2) = (\cosh \lambda) \cos \theta. \] (13)

If \((\cosh \lambda)^2 \cos^2 \theta \) is greater than one, the eigenvalues become
\[ E_\pm = \exp(\pm \chi/2), \] (14)

with
\[ \cosh(\chi/2) = (\cosh \lambda) \cos \theta. \] (15)

If \((\cosh \lambda) \cos \theta = 1\), the quantity inside the square-root sign vanishes, the eigenvalues collapse to one, and the \(RBR\) matrix becomes one of the triangular matrices of equation (9).

In view of the triangular matrices of equation (9), the best we can do is to write the \(RBR\) matrix as a similarity transformation of
\[
\left( \begin{array}{cc}
  e^{i\phi/2} & 0 \\
  0 & e^{-i\phi/2}
\end{array} \right), \quad \left( \begin{array}{cc}
  e^{\chi/2} & 0 \\
  0 & e^{-\chi/2}
\end{array} \right),
\] (16)
or one of the triangular matrices of equation (9). We shall call the set of these three matrices the Wigner matrix, use the notation \(W_\pm\) for the set of three matrices with \(H_\pm\), and \(W\) collectively for both.

Our plan is to write \(RBR\) as a similarity transformation of the Wigner matrix:
\[ RBR = S^{-1}WS. \] (17)

As we shall see in Sec. 3, the \(W\) matrices are also derivable from Wigner’s little group which dictate the internal space-time symmetries of elementary particles [14]

Then the question is which among the two Wigner matrices is to be chosen. In order to address this question, let us note the Lorentz group has two branches. The \(Sp(2)\) group is generated by
\[
J_2 = \frac{1}{2} \left( \begin{array}{cc}
  0 & -i \\
  i & 0
\end{array} \right), \quad K_3 = \frac{1}{2} \left( \begin{array}{cc}
  i & 0 \\
  0 & -i
\end{array} \right), \quad K_1 = \frac{1}{2} \left( \begin{array}{cc}
  0 & i \\
  i & 0
\end{array} \right),
\] (18)
forming a closed set of commutation relations:
\[ [J_2, K_3] = iK_1, \quad [J_2, K_1] = -iK_2, \quad [J_2, K_3] = -iJ_2. \] (19)
The rotation generator \(J_2\) is antisymmetric, and \(K_3\) and \(K_1\) are symmetric. Furthermore, this set of commutation relations is invariant under the sign
change of the $K_i$ matrices while it is not when $J_2$ changes its sign. The rotation matrices generated by $J_2$ matrix are anti-symmetric, while the squeeze matrices generated by $K_i$ are symmetric.

Let us go back to the $RBR$ matrix of equation (7). Since the $B(2\lambda)$ matrix is symmetric and is generated by $K_1$. We should therefore consider the case where $B(2\lambda)$ is replaced by $B(-2\lambda)$. The resulting $RBR$ matrix is

$$R(\theta)B(-2\lambda)R(\theta) = \begin{pmatrix} (\cosh \lambda) \cos \theta & -(\cosh \lambda) \sin \theta - \sinh \lambda \\ (\cosh \lambda) \sin \theta - \sinh \lambda & (\cosh \lambda) \cos \theta \end{pmatrix}. \tag{20}$$

For convenience, we shall use the notations $(RBR)_+$ and $(RBR)_-$ for equations (7) and (20) respectively:

$$(RBR)_+ = R(\theta)B(2\lambda)R(\theta),$$

$$(RBR)_- = R(\theta)B(-2\lambda)R(\theta). \tag{21}$$

We shall still use the notation $RBR$ collectively for $(RBR)_+$ and $(RBR)_-$. We cannot obtain $(RBR)_-$ from $(RBR)_+$ by simply changing the sign of the $\lambda$ parameter. It is a parity operation.

If both $\sin \theta$ and $\sinh \lambda$ are positive, the upper-right element of $(RBR)_+$ can vanish, while the lower-left element can become zero for $(RBR)_-$. They are therefore consistent with $H_+(\sinh \lambda)$ and $H_-(\sinh \lambda)$ respectively.

Let us see why the $W$ matrix should be called the Wigner matrix in section 3.

3 Wigner’s little groups

Since the group $Sp(2)$ is locally isomorphic to the Lorentz group applicable to two space dimensions and one time variable, we can import useful matrix identities from the kinematics of the Lorentz group, particularly from Wigner’s little group [14].

Wigner’s little group is the maximal subgroup of the Lorentz group whose transformations leave the given four-momentum of a particle invariant [14]. Let us first consider a particle moving along the negative $z$ direction with the four-momentum

$$P = (E, 0, 0, -p), \tag{22}$$

with $E = \sqrt{p^2 + m^2}$. We use here the four-vector convention $(t, x, y, z)$, and the $c = \hbar = 1$ unit system.
It is possible to bring this four-momentum to 
\[(m, 0, 0, 0),\] (23)
by applying the transformation matrix 
\[S(\eta) = \begin{pmatrix}
cosh \eta & 0 & 0 & \sinh \eta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
sinh \eta & 0 & 0 & \cosh \eta \\
\end{pmatrix},\] (24)
width 
\[E = m(\cosh \eta), \quad p = m(\sinh \eta).\] (25)

The four-momentum of equation (23) is invariant under the rotation 
\[R(\phi) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \phi & 0 & \sin \phi \\
0 & 0 & 1 & 0 \\
0 & -\sin \phi & 0 & \cos \phi \\
\end{pmatrix},\] (26)

After this rotation, we can bring back the four momentum of equation (23) to the starting momentum of equation (22) by applying the inverse of the boost matrix \(S(\eta)\). Thus the transformation \(S(-\eta)R(\phi)S(\eta)\) will leave the four-momentum of equation (22) invariant.

We can also rotate the entire system around the z axis without changing the four-momenta of equation (22) and equation (23), and the matrix \(S(\eta)\). Thus, we can obtain the three-dimensional rotation group by taking into account this degree of freedom along with the rotation matrix of equation (26) \[16\]. The group represented by this Lorentz-boosted rotation matrix is known as Wigner’s \(O(3)\)-like little group for massive particles.

If the four-momentum is space-like, it can be brought to \((0, 0, 0, p)\). This four-momentum is invariant under the boost 
\[X(\chi) = \begin{pmatrix}
cosh \chi & \sinh \chi & 0 & 0 \\
\sinh \chi & \cosh \chi & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},\] (27)
along the \(x\) direction. The little group is thus represented by \(S^{-1}X(\chi)S\), along with rotations around the \(z\) axis. Wigner’s little group for space-like four-momentum is a Lorentz-boosted \(O(2, 1)\) group.
If the particle is massless, there are no Lorentz frames in which the particle is at rest, but its four-momentum can be brought to \((k, 0, 0, k)\). This four-vector is invariant under rotations around the \(z\) axis. In addition, it is invariant under

\[
N(\gamma) = \begin{pmatrix}
1 + \gamma^2/2 & -\gamma & 0 & -\gamma^2/2 \\
-\gamma & 1 & 0 & \gamma \\
0 & 0 & 1 & 0 \\
\gamma^2/2 & \gamma & 0 & 1 - \gamma^2/2
\end{pmatrix}.
\] (28)

In his 1939 paper, Wigner showed that this \(N\) operator together with the above-mentioned rotation around the \(z\) axis form a three-parameter group which is locally isomorphic to the two-dimensional Euclidean group. Its rotational degree of freedom corresponds to the helicity of the particle while the two-translational degrees collapse into one gauge degree of freedom \[17\].

This summarizes what Wigner did in his 1939 paper and later papers on massless particles \[14, 17\]. The transformation matrix which leaves the four-momentum invariant is

\[
S(-\eta)WS(\eta),
\] (29)

where \(W\) takes the form of \(R(\phi), X(\chi),\) or \(N(\gamma)\).

It is well-known that the six-parameter Lorentz group can also be represented by two-by-two unimodular matrices. This group of two-by-two matrices is called \(SL(2,c)\), and \(Sp(2)\) is one of its subgroups.

Wigner’s approach is not the only method to obtain the momentum-preserving transformations. If we rotate the four-vector of equation (22) around the \(y\) axis using the rotation matrix

\[
R(\theta) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \cos \theta & 0 & \sin \theta \\
0 & 0 & 1 & 0 \\
0 & -\sin \theta & 0 & \cos \theta
\end{pmatrix},
\] (30)

the resulting four-momentum is

\[
(E, -p \sin \theta, 0, -p \cos \theta).
\] (31)

The above four-by-four matrix corresponds to the two-by-two matrix \(R(\theta)\) of equation (3).
It is possible to boost this four-vector along the positive $x$ direction using the boost matrix corresponding to $B(2\lambda)$ of equation (3). Then the four-momentum becomes

$$(E, p \sin \theta, 0, -p \cos \theta).$$

(32)

The four-by-four boost matrix is

$$B(2\lambda) = \begin{pmatrix}
\cosh(2\lambda) & \sinh(2\lambda) & 0 & 0 \\
\sinh(2\lambda) & \cosh(2\lambda) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.$$

(33)

We can then rotate this four-momentum to the original form of equation (22), using again the rotation matrix $R(\theta)$ of equation (30). This method was discussed in detail in Ref. [18], but only for a massive particle whose four-momentum takes the form of equation (22).

It should be noted that, in order that the four-vector of equation (22) return to the original form after the rotation-boost-rotation process, the boost parameter $\lambda$ cannot be arbitrary. It should be determined from the rotation angle $\theta$ and the four-momentum of equation (22).

On the other hand, if the parameters $\theta$ and $\lambda$ are chosen first, they determine the form of the four-vector. It could be time-like (non-zero massive), space-like (for imaginary mass), or light-like (zero mass), covering all three possible little groups [14].

We choose the notation $W$ for the set of the four-by-four matrices $R(\phi)$ of equation (26), $X(\chi)$ of equation (27), and $N(\gamma)$ of equation (28), and call it the “Wigner matrix.” Then, in terms of $W$, the matrix $RBR$ can be written as a similarity transformation

$$R(\theta)B(2\lambda)R(\theta) = S(-\eta)WS(\eta).$$

(34)

We are then interested in whether the same relation can be derived for the two-by-two matrices of the $Sp(2)$ group.

Since the group $Sp(2)$ is locally isomorphic to $O(2, 1)$, we expect to be able to write this similarity transformation in terms of the corresponding two-by-two matrices. Indeed, there are two-by-two $Sp(2)$ matrices corresponding to the four-by-four matrices for $R(\phi), X(\chi), N(\gamma)$ given equation (26), equation (27), and equation (28) respectively. They are

$$R(\phi) = \begin{pmatrix}
\cos(\phi/2) & -\sin(\phi/2) \\
\sin(\phi/2) & \cos(\phi/2)
\end{pmatrix},$$

and

$$X(\chi) = \begin{pmatrix}
\cos(\chi/2) & 0 \\
0 & \cos(\chi/2)
\end{pmatrix},$$

$$N(\gamma) = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.$$
\[
X_\pm(\chi) = \begin{pmatrix} \cosh(\chi/2) & \pm \sinh(\chi/2) \\ \pm \sinh(\chi/2) & \cosh(\chi/2) \end{pmatrix},
\]
\[
N_+(\gamma) = \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}, \quad \text{or} \quad N_-(\gamma) = \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}.
\] (35)

The two-by-two counterpart of the boost matrix of equation (24) is
\[
S(\eta) = \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}.
\] (36)

Thus the two-by-two representation of Wigner’s little group should also be written as \( S(\mp \eta) W S(\pm \eta) \). However, we have to confront the question of which \( N(\gamma) \) matrix from equation (35) is to be chosen. If we go back to the \((RBR)_\pm\) matrices of equations (7) and (20), the upper-right element can vanish for \((RBR)_+\), while \((RBR)_-\) can have a vanishing lower-left element.

We shall also see that the transition from \( X_+(\chi) \) to \( X_-(-\chi) \) cannot be achieved through an analytic continuation from positive \( \chi \) to negative \( \chi \) in equation (35). We shall assume that \( \lambda \) and \( \chi \) have the same sign.

Indeed, the two-by-two Wigner matrix has two branches. Thus \( N_+ \) and \( X_+ \) should be chosen for \((RBR)_+\). We shall call this set of Wigner matrices \( W_+ \), and write
\[
(RBR)_+ = S(-\eta) W_+ S(\eta).
\] (37)

What then happens to \( N_- \) and \( X_- \)?

In order to answer this question, let us start with the four-momentum
\[
P = (E, 0, 0, p),
\] (38)

instead of \((E, 0, 0, -p)\) of equation (22). We can boost this four-momentum to \( E = (m, 0, 0, 0) \), using the inverse of \( S(\eta) \) given in equation (24). This four-vector is invariant under rotations. We can then boost back to the starting form of equation (38).

As for the Bargmann decomposition, we can rotate the four-vector of equation (38) to
\[
P = (E, p \sin \theta, 0, p \cos \theta),
\] (39)

by applying the rotation matrix \( R(\theta) \). We then apply the boost matrix \( B(-2\lambda) \) to this four vector to get
\[
P = (E, -p \sin \theta, 0, p \cos \theta).
\] (40)
The rotation matrix $R(\theta)$ will bring this vector back to the original form of equation (38). If we complete this kinematics, the resulting matrix is $(RBR)_-$ of equation (20). This version of the Lorentz kinematics was thoroughly discussed in reference [18].

The resulting $RBR$ matrix is that of equation (20), which we write as $(RBR)_-$. The lower-left element of this matrix can vanish. The corresponding Wigner matrix should include $N_-$ instead of $N_+$, and

$$ (RBR)_- = S(\eta)W_-S(-\eta). \tag{41} $$

If we combine equation (37) and equation (41), we can write

$$ (RBR)_\pm = S(\mp\eta)W_\pm S(\pm\eta). \tag{42} $$

### 4 The $\text{Sp}(2)$ matrix as a similarity transformation of the Wigner matrix

Our main concern is to see whether the $M$ matrix or its core $RBR$ of equation (7) or equation (20) can be written as a similarity transformation of the diagonal $D$ matrix of equation (11). We have already shown that this is not always possible when the matrix is triangular. On the other hand the triangular matrix has one parameter, and is as simple as the diagonal matrix.

We propose to write this expression as a similarity transformation of the Wigner matrix $W$ given in equation (35). Let us combine $RBR$ matrices of equation (7) and equation (20) into one expression:

$$ R(\theta)B(\pm 2\lambda)R(\theta) = \begin{pmatrix} (\cosh \lambda) \cos \theta & -(\cosh \lambda) \sin \theta \pm \sinh \lambda \\ (\cosh \lambda) \sin \theta \pm \sinh \lambda & (\cosh \lambda) \cos \theta \end{pmatrix}. \tag{43} $$

If the diagonal elements are smaller than one, then

$$ (\cosh \lambda) \sin \theta > \sinh \lambda, \tag{44} $$

and the off-diagonal elements of equation (43) have opposite signs, the $RBR$ matrix can be written as

$$ R(\theta)B(\pm 2\lambda)R(\theta) = S(\mp \eta)R(\phi)S(\pm \eta). \tag{45} $$
The right-hand side takes the form
\[
\begin{pmatrix}
\cos(\phi/2) & -e^{\mp\eta} \sin(\phi/2) \\
e^{\pm\eta} \sin(\phi/2) & \cos(\phi/2)
\end{pmatrix}.
\tag{46}
\]

By comparing this expression with equation (43), we can write \( \phi \) and \( \eta \) in terms of \( \lambda \) and \( \theta \), and the result is
\[
\cos(\phi/2) = \cosh \lambda \cos \theta,
\]
\[
e^{2\eta} = \frac{\cosh \lambda \sin \theta + \sinh \lambda}{\cosh \lambda \sin \theta - \sinh \lambda}.
\tag{47}
\]

If the diagonal elements of equation (43) are greater than one, then
\[
(cosh \lambda) \sin \theta < \sinh \lambda, \tag{48}
\]
and the off-diagonal elements have the same sign. We should then use \( X(\chi) \) as the Wigner matrix, and write \( RBR \) as
\[
R(\theta)B(\pm 2\lambda)R(\theta) = S(\pm \eta)X_{\pm}(\chi)S(\mp \eta). \tag{49}
\]

The right-hand side of this expression can be written as
\[
\begin{pmatrix}
\cosh(\chi/2) & \pm e^{\mp\eta} \sinh(\chi/2) \\
\pm e^{\pm\eta} \sinh(\chi/2) & \cosh(\chi/2)
\end{pmatrix}.
\tag{50}
\]

If we compare this expression with the \((RBR)_\pm\) matrix of equation (43),
\[
\cosh(\chi/2) = \cosh \lambda \cos \theta,
\]
\[
e^{2\eta} = \frac{\cosh \lambda \sin \theta + \sinh \lambda}{\sinh \lambda - \cosh \lambda \sin \theta}, \tag{51}
\]

The upper-right or the lower-left component of equation (43) can go through zero starting either from a small negative number or from a small positive number.

For the first case, if \((\cosh \lambda \sin \theta - \sinh \lambda)\) goes through zero from a small negative to a small positive number, for the \((RBR)_+\) branch one should start from \(S(-\eta)R(\phi)S(\eta)\). Then
\[
H_+(2 \sinh \lambda) = S(-\eta)R(\phi)S(\eta), \tag{52}
\]
hence in this negative region we have

\[ 2 \sinh \lambda = e^{\eta} \sin(\phi/2). \tag{53} \]

For the \((RBR)_-\) branch one should start from \(S(\eta)X_- (\chi)S(-\eta)\), then

\[ H_- (-2 \sinh \lambda) = S(\eta)X_- (\chi)S(-\eta). \tag{54} \]

So in this negative region we have

\[ 2 \sinh \lambda = e^{\eta} \sinh(\chi/2). \tag{55} \]

When \(\eta\) goes to infinity, the upper-right component of \(S(-\eta)R(\phi)S(\eta)\) section of equation \((46)\), namely \(- \exp^{-\eta} \sin(\phi/2)\) goes to zero. Similarly, \(\pm \exp^{-\eta} \sinh(\chi/2)\) vanishes, which is the lower-left component of \(S(\eta)X_- (\chi)S(-\eta)\) section of equation equation \((50)\). In those cases \(\phi\) and \(\chi\) should be very small so that the diagonal elements collapse to one. This limiting process is known as group contraction, and is applicable to various problems in physics \([12, 17]\).

On the other hand when \((\cosh \lambda \sin \theta - \sinh \lambda)\) goes through zero from a small positive number to a small negative number, on the \((RBR)_+\) branch one should start from \(S(-\eta)X_+ (\chi)S(\eta)\), thus

\[ H_+(2 \sinh \lambda) = S(-\eta)X_+ (\chi)S(\eta). \tag{56} \]

In this positive region we have the relation between the parameters as in equation \((55)\). On the \((RBR)_-\) branch we should start from \(S(\eta)R(\phi)S(-\eta)\) thus

\[ H_- (-2 \sinh \lambda) = S(\eta)R(\phi)S(-\eta). \tag{57} \]

So in this positive region we have the relation between the parameters as in equation \((53)\).

When \((\cosh \lambda \sin \theta - \sinh \lambda) = 0\) the diagonal elements are equal to one, one of the off-diagonal elements vanishes. The similarity transformation should be written as

\[ H_\pm (\sinh \lambda) = S(\mp \eta)N_\pm (\gamma)S(\pm \eta). \tag{58} \]

Thus

\[ 2 \sinh \lambda = \gamma e^{\eta}. \tag{59} \]
Since there is only one variable for the Bargmann decomposition collapses into the Iwasawa decomposition, the variables $\gamma$ and $\eta$ become combined into one variable.

Let us now return to the original question of diagonalizing the $ABCD$ matrix or its Bargmann decomposition given in equation (2). We now have $RBR$ matrix as a similarity transformation of the $W$ matrix. The transformation matrix $S$ is given above. Thus, the $M$ matrix can be written as a similarity transformation given in equation (4). The transformation matrix is $G = LS^{-1}$, where the $L$ matrix is defined in equation (6). The $G$ matrix now takes the form

$$G_{\pm} = LS^{\mp} = \begin{pmatrix} e^{\pm \eta/2} \cos(\delta/2) & -e^{\pm \eta/2} \sin(\delta/2) \\ e^{\pm \eta/2} \sin(\delta/2) & e^{\pm \eta/2} \cos(\delta/2) \end{pmatrix}.$$ (60)

We thus conclude that the $Sp(2)$ matrix cannot always be diagonalized, but it can be written as a similarity transformation of the Wigner matrix. The transformation matrix is given above.

5 Further Mathematical Details

The $Sp(2)$ matrices are used in many different branches of physics. They take different forms in the literature, but they are unitarily equivalent. The most convenient way to organize those different expressions is to recognize that the group $Sp(2)$ is a subgroup of the six-parameter $SL(2,c)$ group, which is locally isomorphic to the Lorentz group applicable to three space and one time dimensions. The three-dimensional rotation group is one of its subgroups. In addition, it has three $O(2, 1)$ Lorentz groups applicable to two space and one time dimensions.

The form given in equation (2) corresponds to the $O(2, 1)$ group applicable to $z, x$ and $t$, where rotations around the $y$ axis are allowed. On the other hand, the expression

$$\begin{pmatrix} e^{i\theta_1/2} & 0 \\ 0 & e^{-i\theta_1/2} \end{pmatrix} \begin{pmatrix} \cosh \lambda & \sinh \lambda \\ \sinh \lambda & \cosh \lambda \end{pmatrix} \begin{pmatrix} e^{i\theta_2/2} & 0 \\ 0 & e^{-i\theta_2/2} \end{pmatrix}$$ (61)

is quite common in the literature, especially in layer optics. This is also the form used by Bargmann in his original paper [15]. This expression represents the $O(2, 1)$ subgroup applicable to $(x, y, t)$ while rotations around the $z$ axis.
are allowed. Indeed, the decomposition of equation (2) can be obtained from equation (61) by a conjugate transformation with the matrix \[ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \] (62)

It is also possible to decompose the \( Sp(2) \) matrix in terms of the three matrices of the form
\[
\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}.
\] (63)

These expressions are useful in para-axial lens optics. They are discussed in detail in Ref. [11].

Let us go back to the \( ABCD \) matrix of equation (1). In terms of the Bargmann parameters, they can be written as
\[
A = -\sinh \lambda \sin \delta + \cosh \lambda \cos \theta \\
B = \sinh \lambda \cos \delta - \cosh \lambda \sin \theta \\
C = \sinh \lambda \cos \delta + \cosh \lambda \sin \theta \\
D = \sinh \lambda \sin \delta + \cosh \lambda \cos \theta.
\] (64)

Conversely, the Bargmann parameters are
\[
\tan \theta = \frac{C - B}{A + D}, \quad \tan \delta = \frac{D - A}{B + C},
\] (65)

and
\[
\cosh(2\lambda) = \frac{1}{2C^2} \left\{ (A^2 + C^2)(C^2 + D^2) - 2AD + 1 \right\}.
\] (66)

6 Physical applications

The immediate application of this similarity transformation is in one dimensional crystals requiring repeated applications of the \( Sp(2) \) matrix, such as multilayer optics [10, 9], finite periodic potentials [6, 8] and thus surface problems, and laser cavities [13]. For those problems, we are confronted with the burden of calculating \( M^N \). Now, we have
\[
M^N = \left( GWG^{-1} \right)^N = GW^NG^{-1},
\] (67)
where it becomes trivial to calculate $W^N$ for all its three cases.

These days, the diagonalization becomes the central issue when we deal with “entanglement” problems in physics, especially when the system cannot be diagonalized through a rotation. The system of two entangled harmonic oscillators is a case in point [4, 5]. Let us start with the ground-state wave function

$$
\psi_0(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{2}(x_1^2 + x_2^2) \right\}.
$$

(68)

Here, the variables $x_1$ and $x_2$ are separable. When they are coupled, the wave function takes the form

$$
\psi_\eta(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{4} \left[ e^{-\eta}(x_1 + x_2)^2 + e^{\eta}(x_1 - x_2)^2 \right] \right\}.
$$

(69)

This wave function can be expanded as [5, 20],

$$
\psi_\eta(x_1, x_2) = \frac{1}{\cosh(\eta/2)} \sum_k \left( \tanh \frac{1}{\eta} \right)^k \phi_k(x_1) \phi_k(x_2),
$$

(70)

where $\phi_k(x)$ is the normalized harmonic oscillator wave function for the $k-th$ excited state.

This expansion serves as the mathematical basis for squeezed states of light in quantum optics [3], as an illustrative example of Feynman’s rest of the universe [21, 22], and more recently as an entangled oscillator state [4, 5]. In order to obtain equation (69) from equation (68), we first have to rotate the coordinate system with the matrix

$$
\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.
$$

(71)

Then we have to squeeze the new coordinates using the matrix

$$
\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix},
$$

(72)

which is the same as the matrix given in equation (36). Indeed, this system was coupled by a squeeze $S(\eta)$ followed by a rotation by 45° degrees.

In addition, this paper examines the question of transition from $R(\phi)$ to $X(\chi)$ via $N(\gamma)$ given equation (35). This is known as the stability problem in cavity physics [13] and focal condition in lens optics [12]. While its
mathematical aspect was studied in detail in multilayer optics [9], its physical implication is yet to be studied. There could be many other transitional problems like this in physics.

Let us note again that the group $Sp(2)$ is locally isomorphic to the Lorentz group applicable to two space dimensions and one time variable. The transition from $R(\phi)$ to $N(\gamma)$ is known as a group contraction, which is similar to the transition of the internal space-time symmetries of massive particles to that of massless particles in the limit $v \to c$ [17]. It is interesting to note that this process can be achieved analytically in terms of the Bargmann parameters. This requires further investigation.

Concluding Remarks

Two-by-two matrices appear in almost all branches of physics. For matrices, the basic question is whether they can be brought to a diagonal form. We have addressed this basic question in this paper. We started with the most general form of the $Sp(2)$ matrix and its Bargmann decomposition. We then showed that it can also be written as a similarity transformation of the Wigner matrix, which can be either diagonalized or can be written as a two-by-two matrix with unit diagonal elements and one vanishing off-diagonal element.

Also in this paper, we have found that the $Sp(2)$ matrix can be written both in the Bargmann form of writing it as a product of three-one parameter matrices and also in the Wigner form as a similarity transformation of the Wigner matrix. The fact that $Sp(2)$ matrix can be written in these two different forms may lead to further interesting results.

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