A NOTE ON THE VAN DER WAERDEN COMPLEX

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Abstract. Ehrenborg, Govindaiah, Park, and Readdy recently introduced the van der Waerden complex, a pure simplicial complex whose facets correspond to arithmetic progressions. Using techniques from combinatorial commutative algebra, we classify when these pure simplicial complexes are vertex decomposable or not Cohen-Macaulay. As a corollary, we classify the van der Waerden complexes that are shellable.

1. Introduction

Let $V = \{x_1, \ldots, x_n\}$ and suppose that $0 < k < n$. The van der Waerden complex of dimension $k$ on $n$ vertices, denoted $\text{vdW}(n, k)$, is the pure simplicial complex on $V$ whose facet set is given by

$$\text{vdW}(n, k) = \langle \{x_i, x_{i+d}, x_{i+2d}, \ldots, x_{i+kd}\} \mid d \in \mathbb{Z} \text{ with } 1 \leq i < i + kd \leq n \rangle.$$ 

In other words, the facets of $\text{vdW}(n, k)$ correspond to all arithmetic progressions of length $k + 1$ whose largest element is less than or equal to $n$. The complexes $\text{vdW}(n, k)$ were introduced by Ehrenborg, Govindaiah, Park, and Readdy as part of a recent program to study the topology of complexes that arise within number theory. In particular, the work of [2] focused on the homotopy type of $\text{vdW}(n, k)$.

The van der Waerden complex is a pure simplicial complex. It is known that pure simplicial complexes may have additional combinatorial and topological properties, e.g., vertex decomposable, shellable, and Cohen-Macaulay. Specifically, we have the following chain of implications (definitions are postponed until the next section):

vertex decomposable $\Rightarrow$ shellable $\Rightarrow$ Cohen-Macaulay $\Rightarrow$ pure.

In general, these implications are all strict. It is natural to ask when $\text{vdW}(n, k)$ has these additional properties in terms of $n$ and $k$. We answer this question in this note; precisely:

Theorem 1.1. Let $0 < k < n$ be integers. Then

(i) $\text{vdW}(n, k)$ is vertex decomposable if and only if
   - $n \leq 6$, or
   - $n > 6$ and $k = 1$, or
   - $n > 6$ and $\frac{n}{2} \leq k < n$.

(ii) $\text{vdW}(n, k)$ is pure but not Cohen-Macaulay if and only if $n > 6$ and $2 \leq k < \frac{n}{2}$.

As a corollary, we can recover a result of [5] first proved using different techniques.

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Corollary 1.2. Let $0 < k < n$ be integers. Then $\text{vdW}(n, k)$ is shellable if and only if

- $n \leq 6$, or
- $n > 6$ and $k = 1$, or
- $n > 6$ and $\frac{n}{2} \leq k < n$.

Proof. If $k$ and $n$ satisfy the above conditions, then $\text{vdW}(n, k)$ is vertex decomposable by Theorem 1.1 and consequently, shellable. Otherwise $\text{vdW}(n, k)$ is not Cohen-Macaulay by Theorem 1.1, so it cannot be shellable. \qed

Our paper is structured as follows. We first recall the relevant background in Section 2. In Section 3 we prove Theorem 1.1 using some tools from combinatorial commutative algebra. In particular, to show that $\text{vdW}(n, k)$ is not Cohen-Macaulay, we will show that the Stanley-Reisner ideal of the Alexander dual of $\text{vdW}(n, k)$ has nonlinear first syzygies.

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2. Background

In this section we recall the relevant combinatorial and algebraic background.

Let $V = \{x_1, \ldots, x_n\}$ be a vertex set. A simplicial complex on $V$ is a subset $\Delta \subseteq 2^V$ such that (a) if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$, and (b) $\{x_i\} \in \Delta$ for all $i \in \{1, \ldots, n\}$. Elements of $\Delta$ are called faces, and maximal faces under inclusion are called facets. If $F_1, \ldots, F_s$ is a complete list of facets of $\Delta$, we usually write $\Delta = \langle F_1, \ldots, F_s \rangle$. The dimension of a face $F$, denoted $\dim(F)$, is $\dim(F) = |F| - 1$. The dimension of $\Delta$, denoted $\dim \Delta$, is $\dim \Delta = \max \{\dim(F) \mid F$ a facet of $\Delta\}$. A simplicial complex is pure if all its facets have the same dimension.

The Alexander dual of $\Delta$, denoted $\Delta^\vee$, is the simplicial complex whose facets are complements of the minimal non-faces of $\Delta$. That is, $\Delta^\vee = \{V \setminus F \mid F \notin \Delta\}$.

To any simplicial complex $\Delta$, the Stanley-Reisner ideal of $\Delta$ is a monomial ideal $I_\Delta$ in the polynomial ring $R = k[x_1, \ldots, x_n]$ where

$$I_\Delta = \langle x_{i_1} x_{i_2} \cdots x_{i_t} \mid \{x_{i_1}, \ldots, x_{i_t}\} \notin \Delta \rangle.$$

The following result allows us to directly write out the minimal generators of the Stanley-Reisner ideal of the Alexander dual of $\Delta$ from the facets of $\Delta$.

Lemma 2.1 ([4, Corollary 1.5.5]). Let $\Delta = \langle F_1, F_2, \ldots, F_s \rangle$. Then

$$I_{\Delta^\vee} = \langle m_{F_1}, \ldots, m_{F_s} \rangle \text{ where } m_{F_i} = \prod_{x \notin F_i} x.$$

We recall three families of pure simplicial complexes. The first family was introduced by Provan and Billera [6]; a pure simplicial complex $\Delta$ on $V$ is vertex decomposable if

(i) $\Delta = \emptyset$, or $\Delta = \{\{x_1, \ldots, x_n\}\}$, i.e., a simplex; or
(ii) there exists a vertex $x \in V$ such that the link of $x$, i.e.,

$$\text{lk}_\Delta(x) = \{H \in \Delta \mid H \cap \{x\} = \emptyset \text{ and } H \cup \{x\} \in \Delta\},$$

is a pure simplicial complex on $V \setminus \{x\}$.
and the deletion of $x$, i.e., $\text{del}_\Delta(x) = \{H \in \Delta \mid H \cap \{x\} = \emptyset\}$, are both vertex decomposable simplicial complexes.

The second family is the family of shellable simplicial complexes. A pure complex $\Delta$ is shellable if the facets of $\Delta$ can be ordered, say $F_1, \ldots, F_s$, such that for all $1 \leq i < j \leq s$, there exists some $x \in F_j \setminus F_i$ and some $\ell \in \{1, \ldots, j - 1\}$ with $F_j \setminus F_\ell = \{x\}$.

Finally, a pure simplicial complex $\Delta$ is Cohen-Macaulay if the facets of $\Delta$ can be ordered, say $F_1, \ldots, F_s$, such that for all $1 \leq i < j \leq s$, there exists some $x \in F_j \setminus F_i$ and some $\ell \in \{1, \ldots, j - 1\}$ with $F_j \setminus F_\ell = \{x\}$.

These complexes to prove Theorem 1.1. We begin with

**Theorem 2.2.** Let $\Delta$ be a pure simplicial complex.

(i) If $\Delta$ is vertex decomposable, then $\Delta$ is shellable.

(ii) If $\Delta$ is shellable, then $\Delta$ is Cohen-Macaulay.

(iii) If $\dim \Delta = 1$ and $\Delta$ is connected, then $\Delta$ is vertex decomposable.

**Proof.** (i) is [6, Corollary 2.9]; (ii) is [7, Theorem 5.3.18]; and (iii) is [6, Theorem 3.1.2].

We now state some of the basic results that we require, with references to their proofs.

**Example 2.3.** We show that both $\text{vdW}(5, 2)$ and $\text{vdW}(6, 2)$ are vertex decomposable. Not only do these examples illuminate our definitions, we require these special arguments for these complexes to prove Theorem 1.1. We begin with

$$\Delta = \text{vdW}(5, 2) = \langle \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \{x_1, x_3, x_5\} \rangle.$$

We form the deletion and link of $x_5$:

$$\text{del}_\Delta(x_5) = \langle \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\} \rangle \text{ and } \text{lk}_\Delta(x_5) = \langle \{x_3, x_4\}, \{x_1, x_3\} \rangle.$$

Now $\text{lk}_\Delta(x_5)$ is vertex decomposable by Theorem 2.2 (iii). Let $\Gamma = \text{del}_\Delta(x_5)$ and form the link and deletion with respect to $x_4$:

$$\text{del}_\Gamma(x_4) = \langle \{x_1, x_2, x_3\} \rangle \text{ and } \text{lk}_\Gamma(x_5) = \langle \{x_2, x_3\} \rangle.$$

Both of these complexes are simplicies, so $\text{del}_\Delta(x_5)$ is vertex decomposable, and consequently, so is $\text{vdW}(5, 2)$

The proof for the complex

$$\Delta = \text{vdW}(6, 2) = \langle \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \{x_4, x_5, x_6\}, \{x_1, x_3, x_5\}, \{x_2, x_4, x_6\} \rangle$$

is similar. We form the deletion and link of $x_6$. In particular,

$$\text{del}_\Delta(x_6) = \langle \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \{x_1, x_3, x_5\}, \{x_2, x_4, x_6\} \rangle = \text{vdW}(5, 2), \text{ and }$$

$$\text{lk}_\Delta(x_6) = \langle \{x_4, x_5\}, \{x_2, x_4\} \rangle.$$

1One normally defines a simplicial complex $\Delta$ to be Cohen-Macaulay either in terms of the depth and dimension of $R/I_\Delta$, or in terms of the reduced simplicial homology of $\Delta$. Our definition uses the characterization of Cohen-Macaulay simplicial complexes due to Eagon and Reiner [1].
We just showed that $\operatorname{vdW}(5, 2) = \operatorname{del}_\Delta(x_6)$ is vertex decomposable, and $\operatorname{lk}_\Delta(x_6)$ is vertex decomposable by Theorem 2.2 (iii). So, $\operatorname{vdW}(6, 2)$ is vertex decomposable.

We complete this section with some results about the first syzygy module of a monomial ideal. Let $I$ be a monomial ideal of $R = k[x_1, \ldots, x_n]$ whose unique set of minimal generators are $G(I) = \{m_1, \ldots, m_s\}$. Let $d_i = \deg(m_i)$ for $i = 1, \ldots, s$, and let $e_{m_i}$ denote the basis element of the shifted $R$-module $R(-d_i)$. We can then construct the following degree zero $R$-module homomorphism

$$\varphi : M = R(-d_1) \oplus R(-d_2) \oplus \cdots \oplus R(-d_s) \longrightarrow I$$

where $e_{m_i} \mapsto m_i$ for $i = 1, \ldots, s$. The first syzygy module of $I$ is then

$$\operatorname{Syz}_R^1(I) = \{(F_1, \ldots, F_s) \in M \mid \varphi(F_1, \ldots, F_s) = F_1m_1 + \cdots + F_sm_s = 0\},$$

i.e., $\operatorname{Syz}_R^1(I) = \ker(\varphi)$. The module $\operatorname{Syz}_R^1(I)$ is a finitely generated $R$-module; in fact:

**Theorem 2.4** ([3 Corollary 4.13]). Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a monomial ideal with minimal generators $G(I) = \{m_1, \ldots, m_s\}$. Then

$$\operatorname{Syz}_R^1(I) = \langle \sigma_{i,j}e_{m_i} - \sigma_{i,j}e_{m_j} \mid 1 \leq i < j \leq s \rangle \text{ where } \sigma_{i,j} = \frac{m_i}{\gcd(m_i, m_j)}.$$

The set of generators in the above result may not be a minimal set of generators. However, some subset of these generators is a minimal set of generators. The first syzygy module is generated by linear first syzygies if there is some subset $T \subseteq \{\sigma_{i,j}e_{m_i} - \sigma_{i,j}e_{m_j} \mid 1 \leq i < j \leq s\}$ that generates $\operatorname{Syz}_R^1(I)$, and for all $\sigma_{i,j}e_{m_i} - \sigma_{i,j}e_{m_j} \in T$, $\deg \sigma_{i,j} = \deg \sigma_{j,i} = 1$.

The construction of $\operatorname{Syz}_R^1(I)$ is the first step in the construction of the minimal free resolution of $I$. In particular, we have the following fact.

**Theorem 2.5.** If $I$ is a monomial ideal with a linear resolution, then $\operatorname{Syz}_R^1(I)$ is generated by linear first syzygies.

3. Proof of the main theorem

We prove Theorem 2.5 in this section. To do so, we require the following two lemmas about the facets of $\operatorname{vdW}(n, k)$. Given a facet $F = \{x_i, x_{i+d}, x_{i+2d}, \ldots, x_{i+kd}\} \in \operatorname{vdW}(n, k)$, we call $d$ the increment of $F$. Note that every facet has an associated increment.

**Lemma 3.1.** Suppose $n \geq 7$. Let $F \in \operatorname{vdW}(n, 2)$ be any facet such that its increment is the largest possible odd integer $d$. If $G \in \operatorname{vdW}(n, 2)$ is any other facet with increment $d' \neq d$, then $|F \cap G| \leq 1$.

**Proof.** Because $n \geq 7$, the complex $\operatorname{vdW}(n, 2)$ contains the facet $\{1, 4, 7\}$. Thus the largest odd increment $d$ satisfies $d \geq 3$. Let $F = \{x_a, x_{a+d}, x_{a+2d}\}$ be any facet whose increment is $d$ and let $G = \{x_b, x_{b+d'}, x_{b+2d'}\}$ be any other facet whose increment is $d' \neq d$.

It is immediate that $F \neq G$, so $|F \cap G| \leq 2$. So suppose $|F \cap G| = 2$. Since $a < a + d < a + 2d$ and $b < b + d' < b + 2d'$, we have the following possible cases:
Cases (a), (d), (e), (g) and (i) all imply \( d = d' \), so we can eliminate those cases. For cases (b) and (h), we would have \( d = 2d' \), which implies that the odd integer \( d \) is even, so this case cannot happen. Finally, for cases (c) and (f), we would have \( 2d = d' \). But \( d \geq 3 \) is the largest odd increment, so the largest increment of \( vdW(n, 2) \) is either \( d \) or \( d + 1 \). But \( d' = 2d > d + 1 \), so this is not a valid increment, and consequently, this case cannot happen.

Therefore, it must be the case that \( |F \cap G| \leq 1 \). \( \square \)

We now prove a similar lemma, but now we do not require the increment to be odd.

**Lemma 3.2.** Suppose \( n \geq 7 \) and \( 2 < k < \frac{n}{2} \). Let \( F \in vdW(n, k) \) be any facet whose increment \( d \) is the largest possible. If \( G \in vdW(n, k) \) is any other facet with increment \( d' \neq d \), then \( |F \cap G| \leq k - 1 \).

**Proof.** Since \( k < \frac{n}{2} \), we have \( \{x_1, x_3, \ldots, x_{1+2k}\} \in vdW(n, k) \). If \( F \in vdW(n, k) \) has the largest possible increment \( d \), we must therefore have \( d \geq 2 \).

Let \( F = \{x_a, x_{a+d}, \ldots, x_{a+kd}\} \) be a facet with increment \( d \), and suppose that the facet \( G = \{x_b, x_{b+d'}, \ldots, x_{b+kd'}\} \) has increment \( d' \neq d \). Since the facets are distinct, we must have \( |F \cap G| \leq k \).

Suppose that \( |F \cap G| = k \). Since \( |G| = k \geq 3 \), there must be \( x_{b+id'}, x_{b+(i+1)d'} \in G \), i.e., two consecutive terms of the arithmetic progression in \( G \) such that

\[
a + \ell d = b + id' \quad \text{and} \quad a + jd = b + (i+1)d' \quad \text{for some } \ell < j.
\]

But these two equations imply that \( (j - \ell)d = d' \), i.e., \( d' \geq d \), contradicting the fact that \( d \) is the largest increment. So \( |F \cap G| \leq k - 1 \). \( \square \)

We now prove Theorem [1.1]

**Proof.** (of Theorem [1.1]) We break the proof into cases depending on \( 0 < k < n \).

**Case 1:** \( k = 1 \) and \( 1 < n \). In this case \( vdW(n, 1) \) is vertex decomposable by Theorem [2.2] (iii) because

\[
vdW(n, 1) = \langle \{x_i, x_j\} \mid 1 \leq i < j \leq n \rangle,
\]

is a connected one-dimensional simplicial complex.

**Case 2:** \( \frac{n}{2} \leq k < n \). If \( 1 = k < 2 \), then \( vdW(2, 1) \) is vertex decomposable by the previous case. We now proceed by induction on \( n \). If \( k = n - 1 \), then \( vdW(n, n - 1) = \langle \{x_1, x_2, x_3, \ldots, x_n\} \rangle \) is a simplex, and hence, vertex decomposable.

So suppose that \( \frac{n}{2} \leq k < n - 1 \). Every facet of \( vdW(n, k) \) must have increment \( d = 1 \) since \( \frac{n}{2} \leq k \). So

\[
\Delta = vdW(n, k) = \{x_1, x_2, \ldots, x_{k+1}\}, \{x_2, x_3, \ldots, x_{k+2}\}, \ldots, \{x_{n-k}, \ldots, x_n\}.
\]
We form the link and deletion of \( x_n \):

\[
\text{del}_\Delta(x_n) = \text{vdW}(n - 1, k) \quad \text{and} \quad \text{lk}_\Delta(x_n) = \langle \{x_{n-k}, \ldots, x_{n-1}\} \rangle.
\]

Since \( \frac{n-1}{2} < k < n-1 \), by induction \( \text{vdW}(n - 1, k) \) is vertex decomposable. Because \( \text{lk}_\Delta(x_n) \) is a simplex, we can now conclude that \( \text{vdW}(n, k) \) is vertex decomposable if \( \frac{n}{2} \leq k < n \).

**Case 3:** \( 0 < k < n \leq 6 \). The only \( n \) and \( k \) in this case not covered by Case 1 or 2 is \((n, k) = (5, 2)\) or \((6, 2)\). We now use Example 2.3 to complete this case.

**Case 4:** \( n > 6 \) and \( 2 \leq k < \frac{n}{2} \). Let \( I = I_{\text{vdW}(n,k)^v} \) be the Stanley-Reisner ideal of the Alexander dual of \( \text{vdW}(n, k) \). We will show that \( \text{Syz}^1_R(I) \) cannot be generated by linear first syzygies. It will then follow by Theorem 2.5 that \( I \) does not have a linear minimal free resolution, and consequently, \( \text{vdW}(n, k) \) is a simplicial complex that is pure but not Cohen-Macaulay.

If \( \text{vdW}(n, k) = \langle F_1, \ldots, F_s \rangle \), then by Lemma 2.1

\[
I = \left\langle m_{F_i} = \prod_{x \notin F_i} x \mid i = 1, \ldots, s \right\rangle.
\]

Since the complex is pure, this ideal is generated by \( s \) monomials all of degree \( n - k - 1 \).

We first consider the case that \( 3 \leq k < \frac{n}{2} \). Let \( F \) be any facet with the largest increment \( d \). Since \( n > 6 \), we know that \( d \geq 3 \). Now take another facet \( G \) with increment \( d' \neq d \).

We know that

\[
\frac{m_G^c}{\gcd(m_{F^c}, m_G^c)} e_{m_{F^c}} - \frac{m_G^c}{\gcd(m_{F^c}, m_G^c)} e_{m_G^c}
\]

is a (possibly non-minimal) generator of \( \text{Syz}^1_R(I) \) by Theorem 2.4. Moreover, this generator is not a linear first syzygy because Lemma 3.2 tells us that \( |F \cap G| \leq k - 1 \), which implies that

\[
\deg \left( \frac{m_G^c}{\gcd(m_{F^c}, m_G^c)} \right) \geq 2 \quad \text{and} \quad \deg \left( \frac{m_{F^c}}{\gcd(m_{F^c}, m_G^c)} \right) \geq 2.
\]

To see why, \( m_{F^c} \) and \( m_G^c \) are squarefree monomials, so

\[
\deg(\gcd(m_{F^c}, m_G^c)) = |F^c \cap G^c| = |(F^c \cup G)^c| = n - |F \cup G|
\]

\[
= n - |F| - |G| + |F \cap G|
\]

\[
\leq n - (k + 1) - (k + 1) + (k - 1) = n - k - 3.
\]

Since \( \deg(m_{F^c}) = \deg(m_G^c) = n - k - 1 \), the result follows.

Now suppose that \( \text{Syz}^1_R(I) \) is generated by linear first syzygies. So, in particular there are facets \( H_1, \ldots, H_t \in \{F_1, \ldots, F_s\} \), not necessarily distinct, so that we can write

\[
(3.1) \quad \frac{m_G^c}{\gcd(m_{F^c}, m_G^c)} e_{m_{F^c}} - \frac{m_G^c}{\gcd(m_{F^c}, m_G^c)} e_{m_G^c} = \sum_{i=1}^{t} A_i \left( \frac{m_{H_{i+1}^c}}{\gcd(m_{H_i^c}, m_{H_{i+1}^c})} e_{m_{H_i^c}} - \frac{m_{H_{i+1}^c}}{\gcd(m_{H_i^c}, m_{H_{i+1}^c})} e_{m_{H_{i+1}^c}} \right),
\]

where each \( \frac{m_{H_{i+1}^c}}{\gcd(m_{H_i^c}, m_{H_{i+1}^c})} e_{m_{H_i^c}} - \frac{m_{H_{i+1}^c}}{\gcd(m_{H_i^c}, m_{H_{i+1}^c})} e_{m_{H_{i+1}^c}} \) is a linear first syzygy.
Note that if the facet $H$ has increment $d$, the largest possible increment, and 
\[
\frac{m_{H^c}}{\gcd(m_{H^c}, m_{K^c})} e_{m_{K^c}} - \frac{m_{K^c}}{\gcd(m_{H^c}, m_{K^c})} e_{m_{H^c}}
\]
is any linear first syzygy involving $H$, then $K$ must also have increment $d$. Indeed, if the increment of $K$ is $d' \neq d$, then we could again use Lemma 3.2 to show that 
\[
\deg\left(\frac{m_{H^c}}{\gcd(m_{H^c}, m_{K^c})}\right) \geq 2 \text{ and } \deg\left(\frac{m_{K^c}}{\gcd(m_{H^c}, m_{K^c})}\right) \geq 2,
\]
contradicting the fact we have a linear first syzygy.

Because $e_{m_{F^c}}$ appears on both sides of (3.1), at least one of the $H_i$s must be $F$. In the light of discussion in the previous paragraph, we are forced to have 
\[
\frac{m_{G^c}}{\gcd(m_{F^c}, m_{G^c})} e_{m_{F^c}} = \sum A_{H,K} \left(\frac{m_{H^c}}{\gcd(m_{H^c}, m_{K^c})} e_{m_{K^c}} - \frac{m_{K^c}}{\gcd(m_{H^c}, m_{K^c})} e_{m_{H^c}}\right),
\]
where all the $H$ and $K$ have increment $d$. That is, all the linear first syzygies involving a facet with increment $d$ must appear together. But this means that 
\[
0 = \varphi\left(\frac{m_{G^c}}{\gcd(m_{F^c}, m_{G^c})} e_{m_{F^c}}\right) = \frac{m_{G^c}}{\gcd(m_{F^c}, m_{G^c})} m_{F^c} \neq 0,
\]
which is false. Here, $\varphi$ is the $R$-module homomorphism used to define $\text{Syz}_R^1(I)$.

The proof for $k = 2$ is similar. The only difference is that $F$ is picked to be any facet with the largest odd increment, and we use Lemma 3.1 instead of Lemma 3.2 \hfill \Box

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