The Abel-Zeilberger Algorithm

William Y. C. Chen\textsuperscript{1}, Qing-Hu Hou\textsuperscript{2}, and Hai-Tao Jin\textsuperscript{3}

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300731, P. R. China
\textsuperscript{1}chen@nankai.edu.cn, \textsuperscript{2}hou@nankai.edu.cn, \textsuperscript{3}jinht1006@mail.nankai.edu.cn

Dedicated to Professor Doron Zeilberger on the occasion of his 60th birthday

Abstract

We use both Abel’s lemma on summation by parts and Zeilberger’s algorithm to find recurrence relations for definite summations. The role of Abel’s lemma can be extended to the case of linear difference operators with polynomial coefficients. This approach can be used to verify and discover identities involving harmonic numbers and derangement numbers. As examples, we use the Abel-Zeilberger algorithm to prove the Paule-Schneider identities, the Apéry-Schmidt-Strehl identity, Calkin’s identity and some identities involving Fibonacci numbers.

Keywords: Abel’s lemma, Zeilberger’s algorithm, holonomic sequence, linear difference equation

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1. Introduction

The main idea of this paper is to employ the classical lemma of Abel and Zeilberger’s algorithm for hypergeometric sums to verify and to discover various identities on summations that are not hypergeometric. Abel’s lemma \cite{Pi} on summation by parts is stated as follows.

\textbf{Lemma 1.1} For two arbitrary sequences \(\{a_k\}\) and \(\{b_k\}\), we have

\[
\sum_{k=m}^{n-1} (a_{k+1} - a_k)b_k = \sum_{k=m}^{n-1} a_{k+1}(b_k - b_{k+1}) + a_n b_n - a_m b_m.
\]

For a sequence \(\{\tau_k\}\), define the forward difference operator \(\Delta\) by

\[
\Delta \tau_k = \tau_{k+1} - \tau_k.
\]

Then Abel’s lemma may be written as

\[
\sum_{k=m}^{n-1} b_k \Delta a_k = - \sum_{k=m}^{n-1} a_{k+1} \Delta b_k + a_n b_n - a_m b_m. \quad (1.1)
\]
Graham, Knuth and Patashnik [11] reformulated Abel’s lemma in terms of finite calculus to evaluate indefinite sums. Recently, Chu [8] utilized Abel’s lemma to prove basic hypergeometric identities including Bailey’s very well poised $\psi_6$-series identity by finding pairs $(a_k, b_k)$. Applying Abel’s lemma to the pairs $(a_k, b_k)$, one obtains contiguous relations for the basic hypergeometric sums. Chen, Chen and Gu [6] presented a systematic approach to finding pairs $(a_k, b_k)$ by using the $q$-Gosper algorithm.

In this paper, we combine Abel’s lemma and Zeilberger’s algorithm to verify and discover identities. Moreover, we use an extended version of Abel’s lemma to deal with sums involving holonomic sequences. Let us illustrate our approach by considering identities involving harmonic numbers. The $k$-th harmonic number $H_k$ is given by

$$H_k = \sum_{j=1}^{k} \frac{1}{j}. \quad (1.2)$$

Note that by definition, $H_k = 0$ whenever $k \leq 0$. Let $f_k$ be a hypergeometric term, i.e., $f_{k+1}/f_k$ is a rational function of $k$. We focus on the summation

$$\sum_{k=m}^{n-1} f_k H_k. \quad (1.3)$$

We can use Gosper’s algorithm [10] to determine whether there exists a hypergeometric term $a_k$ such that $\Delta a_k = f_k$. If such $a_k$ exists, by Abel’s lemma we get

$$\sum_{k=m}^{n-1} f_k H_k = \sum_{k=m}^{n-1} H_k \Delta a_k = - \sum_{k=m}^{n-1} \frac{a_{k+1}}{k+1} + a_n H_n - a_m H_m. \quad (1.4)$$

Hence we can transform a summation involving harmonic numbers into a hypergeometric summation. We call such an approach the Abel-Gosper method.

The same idea applies to the definite summation

$$S(n) = \sum_{k=0}^{n} F(n, k) H_k,$$

where $F(n, k)$ is a proper hypergeometric term such that $F(n, k) = 0$ for $k > n$. In this case, we can apply Zeilberger’s algorithm to find a hypergeometric term $G(n, k)$ and polynomials $p_0(n), \ldots, p_d(n)$ such that

$$\sum_{j=0}^{d} p_j(n) F(n + j, k) = G(n, k + 1) - G(n, k).$$

Taking

$$f_k = \sum_{j=0}^{d} p_j(n) F(n + j, k) \quad \text{and} \quad a_k = G(n, k)$$

in (1.4) and summing over $k$ from 0 to $n + d$, we deduce that

$$\sum_{j=0}^{d} p_j(n) S(n + j) = - \sum_{k=0}^{n+d} \frac{G(n, k + 1)}{k+1} + G(n, n + d + 1) H_{n+d+1}. \quad (1.5)$$
Moreover, we see that the summation on the right hand side of (1.5) is again a hypergeometric sum. This approach is called the Abel-Zeilberger method.

In order to apply the above approach to general holonomic sequences, we extend Abel’s lemma by replacing the operator \( \Delta \) with a linear difference operator \( L \) of the form

\[
L a_k = r_0(k) a_k + r_1(k) a_{k+1} + \cdots + r_d(k) a_{k+d},
\]

where each \( r_i(k) \) is a rational function of \( k \). By combining a variation of Zeilberger’s algorithm and the extended Abel’s lemma, we can find a recurrence relation for a sum of the form

\[
S(n) = \sum_k f(n,k)g(n,k),
\]

where \( f(n,k) \) is hypergeometric and \( g(n,k) \) satisfies two recurrence relations

\[
g(n+k) = r_1(n,k)g(n,k) + \cdots + r_d(n,k)g(n,k+d-1) + u(n,k),
\]

and

\[
g(n+1,k) = s(n,k)g(n,k) + v(n,k),
\]

where all the coefficients \( r_i(n,k) \) and \( s(n,k) \) are rational functions and \( u(n,k) \), \( v(n,k) \) are hypergeometric terms. The algorithm for finding recurrence relations for the sum (1.6) is called the Abel-Zeilberger algorithm.

The paper is organized as follows. In Section 2, we give examples to demonstrate that many indefinite sums involving harmonic numbers can be reduced to hypergeometric sums by the Abel-Gosper method. Section 3 shows how to apply the Abel-Zeilberger method to find recurrence relations of definite sums involving harmonic numbers. For example, the Paule-Schneider identities fall into this framework. In Section 4, by extending Abel’s lemma and using a variation of Zeilberger’s algorithm, we present the Abel-Zeilberger algorithm. The last section provides several examples of the Abel-Zeilberger algorithm including identities involving Fibonacci numbers and derangement numbers, as well some identities on multiple sums.

2. The Abel-Gosper method

In this section, we give several examples to illustrate how to combine Abel’s lemma with Gosper’s algorithm to evaluate indefinite summations. We shall focus on summations involving harmonic numbers.

We begin with a simple example. Consider the sum

\[
S(n) = \sum_{k=1}^{n} H_k.
\]

Expressing \( 1 \) as \( \Delta k \), we obtain

\[
S(n) = \sum_{k=1}^{n} H_k \Delta k = - \sum_{k=1}^{n} (k+1) \Delta H_k + (n+1)H_{n+1} - H_1 = (n+1)H_n - n.
\]

(2.1)
The idea of the above example can be generalized to indefinite sums of products of polynomials and harmonic numbers. The following result is due to Spieß [15]. Here we give a derivation based on Abel’s lemma.

**Theorem 2.1** Let $H_k$ be the $k$-th harmonic number and $u(n)$ be a polynomial of degree $m$ in $n$. Then

$$\sum_{k=0}^{n} u(k)H_k = p(n)H_n - q(n), \quad n = 0, 1, 2, \ldots, \tag{2.2}$$

where $p(n)$ and $q(n)$ are both polynomials of degree $m+1$ in $n$. Moreover, $p(n)$ is divisible by $n + 1$.

**Proof.** It is well-known that there exists a polynomial $f(k)$ of degree $m+1$ in $k$ such that $\Delta f(k) = u(k)$ and the constant term of $f(k)$ is zero. Therefore, we may write $f(k) = kg(k)$, where $g(k)$ is a polynomial of degree $m$. From (1.4) it follows that

$$\sum_{k=0}^{n} u(k)H_k = \sum_{k=0}^{n} \Delta(kg(k))H_k = -\sum_{k=0}^{n} g(k+1) + (n+1)g(n+1)H_{n+1}.$$

Since the sum $\sum_{k=0}^{n} g(k+1)$ is a polynomial of degree $m+1$ in $n$ and

$$(n+1)g(n+1)H_{n+1} = (n+1)g(n+1)H_n + g(n+1),$$

we arrive at (2.2). \hfill \square

Setting $u(n) = 1$ in (2.2), we obtain (2.1). When $u(n) = n^m$ for $m = 1, 2, 3$, we have

$$\sum_{k=1}^{n} kH_k = \frac{n(n+1)}{2}H_n - \frac{(n-1)n}{4}, \tag{2.3}$$

$$\sum_{k=1}^{n} k^2H_k = \frac{n(n+1)(2n+1)}{6}H_n - \frac{(n-1)n(4n+1)}{36}, \tag{2.4}$$

$$\sum_{k=1}^{n} k^3H_k = \frac{n^2(n+1)^2}{4}H_n - \frac{(n-1)n(n+1)(3n-2)}{48}. \tag{2.5}$$

The same idea also applies to the bonus problem 69 proposed by Graham, Knuth and Patashnik [11, Chapter 6].

**Example 2.2** Find a closed form for

$$\sum_{k=1}^{n} k^2H_{n+k}.$$

The above sum can be rewritten as

$$\sum_{k=n+1}^{2n} (k-n)^2H_k = \sum_{k=1}^{2n} (k-n)^2H_k - \sum_{k=1}^{n} (k-n)^2H_k. \tag{2.6}$$
Expanding the summands and applying formulas (2.1), (2.3), (2.4), we obtain that
\[
\sum_{k=1}^{n} k^2 H_{n+k} = \frac{n(n+1)(2n+1)}{6} (2H_{2n} - H_n) - \frac{n(n+1)(10n-1)}{36}.
\] (2.7)

We may also apply Gosper’s algorithm directly to \((k-n)^2\) and get its indefinite sum
\[
\frac{1}{3} k^3 - \frac{2n+1}{2} k^2 + \frac{6n^2 + 6n + 1}{6} k + C.
\]

Setting \(C = 0\) and using (1.4), we also obtain (2.7).

We remark that Chyzak [9] and Schneider [13, 14] have proved (2.7) by an extension of Zeilberger’s algorithm and Karr’s algorithm, respectively.

**Example 2.3** Evaluate the sum
\[
\sum_{k=0}^{n-1} \frac{1}{4^k} \binom{2k}{k} H_k.
\]

By Gosper’s algorithm, we find that
\[
\Delta \frac{2k}{4^k} \binom{2k}{k} = \frac{1}{4} \binom{2k}{k}.
\]

Therefore,
\[
\sum_{k=0}^{n-1} \frac{1}{4^k} \binom{2k}{k} H_k = -2 \sum_{k=0}^{n-1} \frac{1}{4^{k+1}} \binom{2k+2}{k+1} + \frac{2n}{4^n} \binom{2n}{n} H_n
\]
\[
= 2 - \frac{n+1}{4^n} \binom{2n+2}{n+1} + \frac{2n}{4^n} \binom{2n}{n} H_n.
\]

**Example 2.4** We have
\[
\sum_{k=0}^{n} H_k^2 = (n+1)H_n^2 - (2n+1)H_n + 2n.
\] (2.8)

**Proof.** Setting \(a_k = k\) and \(b_k = H_k^2\) in (1.1), we deduce that
\[
\sum_{k=0}^{n} H_k^2 = -\sum_{k=0}^{n} (k+1) \Delta H_k^2 + (n+1)H_{n+1}^2
\]
\[
= -2 \sum_{k=0}^{n} H_k - \sum_{k=0}^{n} \frac{1}{k+1} + (n+1)H_{n+1}^2
\]
\[
= -2(n+1)H_n + 2n - H_{n+1} + (n+1)H_{n+1}^2 \quad \text{(By (2.1))}
\]
\[
= (n+1)H_n^2 - (2n+1)H_n + 2n.
\]
Similarly, by setting $b_k = H_k^3$, we deduce the following identities, see \[9, 15\],

\[
\sum_{k=0}^{n} H_k^3 = (n + 1)H_n^3 - \frac{3}{2}(2n + 1)H_n^2 + 3(2n + 1)H_n + \frac{1}{2}H_n^{(2)} - 6n, \tag{2.9}
\]

\[
\sum_{k=0}^{n} (2k + 1)H_k^3 = (n + 1)^2H_n^3 - \frac{3}{2}(n + 1)H_n^2 + \frac{3n^2 + 3n + 1}{2}H_n - \frac{3}{4}(n + 1). \tag{2.10}
\]

3. The Abel-Zeilberger method

In this section, we use Abel’s lemma and Zeilberger’s algorithm to find recurrence relations of the definite summation of the form

\[
S(n) = \sum_{k=0}^{n} F(n, k)H_k,
\]

where $F(n, k)$ is a hypergeometric term in two variables $n$ and $k$. We shall give two examples to explain the method. First, we consider an identity due to Chu and De Donno \[7\]

Example 3.1 For $n \geq 0$, we have

\[
\sum_{k=0}^{n} \binom{n}{k}^2 H_k = (2H_n - H_{2n}) \binom{2n}{n}.
\]

Proof. By applying Zeilberger’s algorithm to $\binom{n}{k}^2$, we obtain a skew recurrence relation

\[
(n + 1)\binom{n + 1}{k}^2 - 2(2n + 1)\binom{n}{k}^2 = G(n, k + 1) - G(n, k),
\]

where

\[
G(n, k) = (-3 - 3n + 2k)\binom{n}{k - 1}^2.
\]

Let

\[
S(n) = \sum_{k=0}^{n} \binom{n}{k}^2 H_k.
\]

Substituting $F(n, k) = \binom{n}{k}^2$ and $G(n, k) = (-3 - 3n + 2k)\binom{n}{k - 1}^2$ into (1.5), we find that

\[
(n + 1)S(n + 1) - 2(2n + 1)S(n) = \sum_{k=0}^{n+1} \frac{3n - 2k + 3}{k + 1} \binom{n}{k}^2 = \frac{4n + 1}{n + 1} \binom{2n}{n}, \tag{3.2}
\]

where the second equality can be justified by applying Zeilberger’s algorithm.

It is easy to verify that

\[
R(n) = (2H_n - H_{2n}) \binom{2n}{n}
\]
satisfies the same recurrence relation (3.2). Since $S(0) = R(0) = 0$, we get (3.1). This completes the proof.

In [12], Paule and Schneider considered the following summations:

$$T^{(\alpha)}_n = \sum_{k=0}^{n} (1 + \alpha(n - 2k)H_k) \binom{n}{k}^\alpha, \quad \alpha = 1, 2, \ldots$$  \hfill (3.3)

They found closed forms of $T^{(\alpha)}_n$ for $1 \leq \alpha \leq 4$ and derived recurrence relations of $T^{(\alpha)}_n$ for $5 \leq \alpha \leq 9$. As will be seen, we can combine Abel’s lemma and Zeilberger’s algorithm to deal with the summations $T^{(\alpha)}_n$. As an example, let us consider the case $\alpha = 3$.

**Example 3.2** For $n \geq 0$, we have

$$T^{(3)}_n = (-1)^n.$$  

*Proof.* Let

$$F(n, k) = (n - 2k) \binom{n}{k}^3.$$

By Zeilberger’s algorithm, we find that

$$F(n, k) + F(n + 1, k) = G(n, k + 1) - G(n, k),$$

where

$$G(n, k) = (2n - k + 2) \binom{n}{k-1}^3.$$

Let

$$S(n) = T^{(3)}_n = \sum_{k=0}^{n} (1 + 3(n - 2k)H_k) \binom{n}{k}^3.$$

By (1.5), we deduce that

$$S(n) + S(n + 1) = -3 \sum_{k=0}^{n+1} \frac{2n - k + 1}{k + 1} \binom{n}{k}^3 + \sum_{k=0}^{n} \binom{n}{k}^3 + \sum_{k=0}^{n+1} \binom{n+1}{k}^3.$$

By Zeilberger’s algorithm, we find that the right hand side, denoted by $R(n)$, satisfies

$$(n + 1)R(n) + (n + 2)R(n + 1) = 0.$$  

Since $R(0) = 0$, we have $R(n) = 0$ for $n = 0, 1, \ldots$. It is clear that $S(0) = 1$. So we get $S(n) = (-1)^n$. This completes the proof.  

Moreover, as a direct consequence of (1.5), we have the following property.

**Theorem 3.3** Let

$$U^{(\alpha)}_n = \sum_{k=0}^{n} (n - 2k) \binom{n}{k}^\alpha.$$
Assume that the minimal recurrence relation for $U^{(\alpha)}_n$ computed by Zeilberger’s algorithm is
\[ \sum_{i=0}^{d} p_i(n)U^{(\alpha)}_{n+i} = 0. \]

Then the summation
\[ \sum_{i=0}^{d} p_i(n)T^{(\alpha)}_{n+i} \] (3.4)
is a hypergeometric summation.

We find that the sum (3.4) equals zero for $\alpha = 1, 2, \ldots, 9$ and conjecture that it holds for any nonnegative integer $\alpha$. We note that this conjecture implies the conjecture of Schneider and Paule [12], which says that $T^{(\alpha)}_{n+i}$ satisfies the minimal recurrence relation for $U^{(\alpha)}_n$ computed by Zeilberger’s algorithm.

4. The Abel-Zeilberger algorithm

In this section, we give a description of the Abel-Zeilberger algorithm. Notice that in the applications of Abel’s lemma given in previous sections, the main idea lies in the fact that $\Delta H_k$ is a hypergeometric term. In fact, there are other sequences satisfying similar properties that lead us to consider an extension of Abel’s lemma.

Let $\{a_k\}$ be an arbitrary sequence. We consider a linear operator $L$ of the form
\[ La_k = r_0(k)a_k + r_1(k)a_{k+1} + \cdots + r_d(k)a_{k+d}, \]
where each $r_j(k)$ is a rational function of $k$. We associate the operator $L$ with a dual operator $L^*$ defined by
\[ L^*a_k = r_0(k)a_k + r_1(k-1)a_{k-1} + \cdots + r_d(k-d)a_{k-d}. \]

In the above notation, Abel’s lemma can be extended to the following form.

Lemma 4.1 For two arbitrary sequences $\{a_k\}$ and $\{b_k\}$, we have
\[ \sum_{k=m}^{n-1} L^*a_k \cdot b_k = \sum_{k=m}^{n-1} a_k \cdot Lb_k - T(n) + T(m), \] (4.1)
where
\[ T(k) = \sum_{i=1}^{d} \sum_{j=1}^{i} r_i(k-j)a_{k-j}b_{k+i-j}. \] (4.2)
Proof. It is easy to verify that

\[
\sum_{k=m}^{n-1} L^*a_k \cdot b_k = \sum_{k=m}^{n-1} \sum_{i=0}^{d} r_i(k-i)a_{k-i}b_k
\]

\[
= \sum_{i=0}^{d} \sum_{k=m-i}^{n-1-i} r_i(k) a_k b_{k+i}
\]

\[
= \sum_{k=m}^{n-1} r_0(k) a_k b_k + \sum_{i=1}^{d} \left[ \sum_{k=m-i}^{n-1-i} r_i(k) a_k b_{k+i} - \sum_{k=n-i}^{n-1} r_i(k) a_k b_{k+i} \right]
\]

\[
= \sum_{k=m}^{n-1} a_k \cdot b_k + \sum_{i=1}^{d} \sum_{k=m-i}^{m-1} r_i(k) a_k b_{k+i} - \sum_{i=1}^{d} \sum_{k=n-i}^{n-1} r_i(k) a_k b_{k+i}.
\]

Let \( f(n, k) \) be a bivariate hypergeometric term and \( g(n, k) \) be a bivariate function. We aim to find a linear recurrence relation for the definite sum

\[ S(n) = \sum_{k=m}^{\ell} f(n, k) g(n, k). \]

Suppose that there exist rational functions \( r_j(n, k) \) such that

\[
Lg(n, k) = \sum_{j=0}^{d} r_j(n, k) g(n, k+j)
\]

is a bivariate hypergeometric term. We shall try to find polynomials \( p_i(n) \), which are independent of \( k \) and not all zero, together with hypergeometric terms \( a(n, k) \) and \( w(n, k) \) such that

\[
\sum_{i=0}^{I} p_i(n) f(n+i, k) g(n+i, k) = g(n, k) L^*a(n, k) + w(n, k).
\]

Summing (4.3) over \( k \) and applying the extended Abel’s lemma, we deduce that

\[
\sum_{i=0}^{I} p_i(n) S(n+i) = \sum_{i=0}^{I} \sum_{k=m}^{\ell} p_i(n) f(n+i, k) g(n+i, k)
\]

\[
= \sum_{k=m}^{\ell} \left( g(n, k) L^*a(n, k) + w(n, k) \right)
\]

\[
= \sum_{k=m}^{\ell} a(n, k) Lg(n, k) + \sum_{k=m}^{\ell} w(n, k) - T(\ell+1) + T(m),
\]

where \( T(k) \) is given by (4.2). Notice that in the last expression, the two summands are hypergeometric. By Zeilberger’s algorithm, the two sums satisfy linear recurrence relations, which lead to a non-homogenous linear recurrence relation for \( S(n) \).
To solve equation (4.3), we impose the condition that $g(n, k)$ satisfies the relation

$$g(n + 1, k) = s(n, k)g(n, k) + v(n, k),$$

where $s(n, k)$ is a rational function and $v(n, k)$ is a hypergeometric term. By induction, it is easy to show that there exist rational functions $s_i(n, k)$ and hypergeometric terms $v_i(n, k)$ such that

$$g(n + i, k) = s_i(n, k)g(n, k) + v_i(n, k). \quad (4.5)$$

Now we can solve the following equation for $p_i(n)$ and $a(n, k)$ by a variation of Zeilberger’s algorithm

$$\sum_{i=0}^l p_i(n)f(n + i, k)s_i(n, k) = L^*a(n, k). \quad (4.6)$$

It can be seen that $a(n, k)$ is similar to $f(n, k)$, that is,

$$R(n, k) = \frac{a(n, k)}{f(n, k)}$$

is a rational function of $n$ and $k$. Hence (4.6) is equivalent to

$$\sum_{i=0}^l p_i(n)\frac{f(n + i, k)}{f(n, k)}s_i(n, k) = \sum_{j=0}^d \frac{f(n, k - j)}{f(n, k)}r_j(n, k - j)R(n, k - j). \quad (4.7)$$

Since $f(n, k)$ is hypergeometric, both $f(n + i, k)/f(n, k)$ and $f(n, k - j)/f(n, k)$ are rational functions. Therefore, (4.7) is a non-homogenous linear recurrence equation on $R(n, k)$ with parameters $p_i(n)$, which can be solved by Abramov’s algorithm [2].

Once we find a solution $(p_0(n), \ldots, p_l(n), a(n, k))$ to (4.6), it is easy to check that

$$(p_0(n), \ldots, p_l(n), a(n, k), w(n, k))$$

is a solution to (4.3), where

$$w(n, k) = \sum_{i=0}^l p_i(n)f(n + i, k)v_i(n, k). \quad (4.8)$$

In summary, the Abel-Zeilberger algorithm can be described as follows.

**Input:** a hypergeometric term $f(n, k)$ and a term $g(n, k)$ satisfying two recurrence relations

$$g(n, k + d) = r_1(n, k)g(n, k) + \cdots + r_d(n, k)g(n, k + d - 1) + u(n, k), \quad (4.9)$$

and

$$g(n + 1, k) = s(n, k)g(n, k) + v(n, k), \quad (4.10)$$

where $r_i(n, k)$ and $s(n, k)$ are rational functions, and $u(n, k)$ and $v(n, k)$ are hypergeometric terms.
Output: polynomials \( p_i(n) \) that are independent of \( k \), two hypergeometric terms \( t_1(n, k) \), \( t_2(n, k) \) and a term \( T(k) \) satisfying

\[
\sum_{i=0}^{I} p_i(n)S(n + i) = \sum_{k=m}^{\ell} t_1(n, k) + \sum_{k=m}^{\ell} t_2(n, k) - T(\ell + 1) + T(m),
\]

where

\[
S(n) = \sum_{k=m}^{\ell} f(n, k)g(n, k).
\]

The algorithm consists of the following steps.

Initially, we set \( I = 0 \).

Step 1. For \( 0 \leq i \leq I \), compute the rational functions \( s_i(n, k) \) and the hypergeometric terms \( v_i(n, k) \) defined by \((4.5)\) by using the recurrence relations

\[
s_{i+1}(n, k) = s(n + i, k)s_i(n, k),
\]

\[
v_{i+1}(n, k) = s(n + i, k)v_i(n, k) + v(n + i, k),
\]

with the initial values \( s_0(n, k) = 1 \) and \( v_0(n, k) = 0 \).

Step 2. Let

\[
Lg(n, k) = -r_1(n, k)g(n, k) - \cdots - r_d(n, k)g(n, k + d - 1) + g(n, k + d). \tag{4.11}
\]

According to \((4.7)\), construct an equation on \( p_i(n) \) and \( R(n, k) = a(n, k)/f(n, k) \). That is, compute polynomials \( P_j(n, k), 0 \leq j \leq d \) and \( Q_i(n, k), 0 \leq i \leq I \) such that

\[
\sum_{j=0}^{d} P_j(n, k)R(n, k - j) = \sum_{i=0}^{I} p_i(n)Q_i(n, k). \tag{4.12}
\]

Step 3. Solve equation \((4.12)\) for \( R(n, k) \) and \( p_i(n), 0 \leq i \leq I \) by using Abramov’s algorithm. If all the polynomials \( p_i(n) \) are zeros, then we increase \( I \) by one and repeat steps 1–3.

Step 4. Compute \( w(n, k) \) based on \((4.8)\). For \( L \) given by \((4.11)\), compute \( T(k) \) according to \((4.2)\). Finally, set

\[
t_1(n, k) = a(n, k)u(n, k) \quad \text{and} \quad t_2(n, k) = w(n, k).
\]

Then \((p_i(n), t_1(n, k), t_2(n, k), T(k))\) is the desired output.
5. Examples

In this section, we provide several examples to compute summations by using the Abel-Zeilberger algorithm.

We first consider the case when \(g(n, k)\) is independent of \(n\) in the Abel-Zeilberger algorithm as described in the previous section, i.e., \(s(n, k) = 1\) and \(v(n, k) = 0\) in (4.10). In this case, we have \(s_i(n, k) = 1\) and \(v_i(n, k) = 0\) so that \(t_2(n, k) = 0\).

Let \(F_n\) be the \(n\)-th Fibonacci number which is defined by the recurrence relation

\[
F_{n+2} = F_{n+1} + F_n,
\]

for \(n \geq 0\), with initial values \(F_0 = 0\), \(F_1 = 1\). Employing the Abel-Zeilberger algorithm, we can prove the following identities on the product of binomial coefficients and Fibonacci numbers, see [4, 17].

Example 5.1 We have

\[
\begin{align*}
\sum_{k=0}^{n} \binom{n}{k} F_k &= F_{2n}, \quad (5.1) \\
\sum_{k=0}^{n} (-1)^k \binom{n}{k} F_k &= -F_n, \quad (5.2) \\
\sum_{k=0}^{n} \binom{n}{k} F_{3k} &= 2^n F_{2n}, \quad (5.3) \\
\sum_{k=0}^{n} \binom{n}{k} F_{4k} &= 3^n F_{2n}. \quad (5.4)
\end{align*}
\]

Proof. For equation (5.1), taking \(f(n, k) = \binom{n}{k}\) and \(g(n, k) = F_k\) as the input of the Abel-Zeilberger algorithm, we obtain

\[
p_0(n) = 1, \quad p_1(n) = -3, \quad p_2(n) = 1, \quad t_1(n, k) = t_2(n, k) = 0,
\]

and

\[
T(k) = \left((2k - n - 3)F_k + (n + 2 - k)F_{k+1}\right) \frac{n!}{(n + 2 - k)!(k - 1)!}.
\]

We see that \(T(n + 3) = T(0) = 0\). Therefore, the summation

\[
S(n) = \sum_{k=0}^{n} \binom{n}{k} F_k
\]

satisfies the recurrence relation

\[
S(n) - 3S(n + 1) + S(n + 2) = 0.
\]

Since \(F_{2n}\) satisfies the same recurrence relation with initial values \(F_0 = 0\) and \(F_2 = 1\), we deduce that \(S(n) = F_{2n}\).
The other three identities can be proved in the same fashion. The detailed arguments are omitted. This completes the proof.

It is not difficult to see that the Abel-Zeilberger algorithm could be used to verify identities involving general \( C \)-finite sequences. For instance, suppose that \( \{G_k\}_{k \geq 0} \) satisfies a recurrence relation

\[
G_{k+2} = bG_{k+1} + cG_k, \quad k \geq 0,
\]

where \( b \) and \( c \) are constants. Then the Abel-Zeilberger algorithm generates the recurrence relation

\[
(b + 1 - c)S(n) - (b + 2)S(n + 1) + S(n + 2) = 0
\]

for the summation

\[
S(n) = \sum_{k=0}^{n} \binom{n}{k} G_k.
\]

The next example involves the \( n \)-th derangement number \( D_n \) as given by

\[
D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.
\]

Using the method of MacMahon’s partition analysis, Andrews and Paule \[3\] the following identity on \( D_n \). We shall give a derivation by applying the Abel-Zeilberger algorithm.

**Example 5.2**

\[
\sum_{j \geq 0} \binom{k}{j} \frac{(k + n - j)!}{(k + N - j)!} D_{k+N-j} = (-1)^n \sum_{j \geq 0} (-1)^j \binom{n}{j} (j + k)!, \quad (5.5)
\]

for \( n \geq N \geq n - k \).

**Proof.** Substituting \( k + N - j \) for \( j \), the left hand side of \( (5.5) \) can be rewritten as

\[
S(N, n) = \sum_{j=N}^{k+N} \binom{k}{j-N} \frac{(n+j-N)!}{j!} D_j.
\]

Because of the recurrence relation

\[
D_n = nD_{n-1} + (-1)^n
\]

for \( D_n \), we may put

\[
f(N, j) = \binom{k}{j-N} \frac{(n+j-N)!}{j!} \quad \text{and} \quad g(N, j) = D_j
\]

as the input of the Abel-Zeilberger algorithm. Then we obtain

\[
S(N, n) - S(N+1, n) = \sum_{j \geq N} (-1)^j \binom{k}{j-N} \frac{(n+j-N)!}{(j+1)!}.
\]
Denote the right hand side by $G(k)$. By Zeilberger’s algorithm, we find that for $k \geq 0,
(1 + N - n + k)G(k) - (2 + k + N)G(k + 1) = 0,
which implies that $G(k) = 0$ for $k \geq n - N \geq 0$. Thus we get $S(N, n) = S(N + 1, n)$. In particular,
\[ S(N, n) = S(n, n) = \sum_{j \geq n} \binom{k}{j-n} D_j. \]
Applying the Abel-Zeilberger algorithm to
\[ f(n, j) = \binom{k}{j-n} \quad \text{and} \quad g(n, j) = D_j, \]
we find
\[ (n + 1)S(n, n) + (n + k + 1)S(n + 1, n + 1) - S(n + 2, n + 2) = \sum_{j \geq n} (-1)^j \binom{k + 1}{j-n} = 0. \]
By Zeilberger’s algorithm, we see that the right hand side of (5.5) satisfies the same recurrence relation. Finally, from the identity
\[ \sum_{j=0}^{k} \binom{k}{j} D_j = k!, \]
we deduce that $S(0, 0) = k!$ and $S(1, 1) = (k + 1)! - k!$, which coincides with the initial values of the right hand side of (5.5). Thus (5.5) holds for $n \geq N \geq n - k$.

The following identity was found by Schmidt and has been proved in several ways, see [16, 18].

**Example 5.3 (The Apéry-Schmidt-Strehl Identity)** For $n \geq 0$, we have
\[ \sum_k \sum_j \binom{n}{k} \binom{n+k}{k} \binom{k}{j}^3 = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2. \] (5.6)

**Proof.** Let
\[ f(n, k) = \binom{n}{k} \binom{n+k}{k}, \quad g(n, k) = \sum_{j=0}^{k} \binom{k}{j}^3, \]
and
\[ S(n) = \sum_{k=0}^{n} f(n, k) g(n, k). \]
By Zeilberger’s algorithm, $g(n, k)$ is annihilated by the operator
\[ -8(k + 1)^2 - (7k^2 + 21k + 16)K + (k + 2)^2 K^2. \]
where $K$ denotes the shift operator on $k$. Now applying the Abel-Zeilberger algorithm to $f(n,k)$ and $g(n,k)$, we obtain

$$S(n) - \frac{(3 + 2n)(39 + 51n + 17n^2)}{(n + 1)^3} S(n + 1) + \frac{(n + 2)^3}{(n + 1)^3} S(n + 2) = 0,$$

Meanwhile, using Zeilberger’s algorithm, we find that the Apéry numbers

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

satisfy the same recurrence relation. Finally, by comparing the initial values, we arrive at (5.6).

To conclude this paper, we consider the following summations,

$$S_{n}^{(\alpha)} = \sum_{k=0}^{n} \left( \sum_{j=0}^{k} \binom{n}{j} \right)^{\alpha}.$$

For $\alpha = 1, 2$ and $3$, closed forms for $S_{n}^{(\alpha)}$ have been derived by Andrews and Paule [3] by using the method of MacMahon’s partition analysis. It is easy to see that these formulas can be derived by using the Abel-Zeilberger algorithm.

**Example 5.4** We have

$$S_{n}^{(1)} = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{j} = n2^{n-1} + 2^n,$$  (5.7)

and

$$S_{n}^{(2)} = \sum_{k=0}^{n} \left( \sum_{j=0}^{k} \binom{n}{j} \right)^2 = \left( \frac{n}{2} + 1 \right) 2^{2n} - \frac{n}{2} \binom{2n}{n}.$$  (5.8)

**Proof.** Let

$$f(n,k) = 1, \quad g(n,k) = \sum_{j=0}^{k} \binom{n}{j}.$$

It is clear that

$$g(n, k + 1) = g(n, k) + \binom{n}{k + 1} \quad \text{and} \quad g(n + 1, k) = 2g(n, k) - \binom{n}{k}.$$  (5.9)

Applying the Abel-Zeilberger algorithm to $f(n,k)$ and $g(n,k)$, we find that

$$S_{n}^{(1)} = \sum_{k=0}^{n} (C - k) \binom{n}{k+1} - T(n+1) + T(0),$$

where $C$ and $T(k)$ are constants to be determined.
where \( T(k) = (C + 1 - k)g(n, k) \) and \( C \) is a constant. Setting \( C = -1 \), we get \( T(0) = 0 \). Thus

\[
S_n^{(1)} = - \sum_{k=0}^{n} (k + 1) \binom{n}{k+1} + (n + 1)g(n, n + 1)
\]

\[
= -n \sum_{k=0}^{n} \binom{n-1}{k} + (n + 1) \sum_{j=0}^{n+1} \binom{n}{j}
\]

\[
= -n2^{n-1} + (n + 1)2^n
\]

\[
= n2^{n-1} + 2^n.
\]

Now we consider the evaluation of \( S_n^{(2)} \). Let \( f(n, k), g(n, k) \) be given as above, and let \( h(n, k) = g(n, k)^2 \). From (5.9), we see that

\[
h(n, k + 1) = h(n, k) + u(n, k),
\]

\[
h(n + 1, k) = 4h(n, k) + v(n, k),
\]

where

\[
u(n, k) = 2 \binom{n}{k+1} g(n, k) + \left( \binom{n}{k+1} \right)^2,
\]

\[
v(n, k) = -4 \binom{n}{k} g(n, k) + \left( \binom{n}{k} \right)^2.
\]

It should be mentioned that there is actually no need to impose the condition for \( u(n, k) \) and \( v(n, k) \) to be hypergeometric in the Abel-Zeilberger algorithm. Therefore, we can still apply the Abel-Zeilberger algorithm to \( f(n, k) \) and \( h(n, k) \) to deduce that

\[
S_n^{(2)} = \sum_{k=0}^{n} (C - k) \left( 2 \binom{n}{k+1} g(n, k) + \left( \binom{n}{k+1} \right)^2 \right) + (n - C)h(n, n + 1) + (C + 1)h(n, 0),
\]

where \( C \) is a constant. Setting \( C = n/2 - 1 \) and applying the Abel-Zeilberger algorithm, we find that

\[
\sum_{k=0}^{n} \frac{n/2 - 1 - k}{k+1} \binom{n}{k+1} g(n, k) = \sum_{k=0}^{n} \frac{-n + k + 1}{2} \binom{n}{k+1}^2 - \frac{n}{2}g(n, 0).
\]

Hence

\[
S_n^{(2)} = \sum_{k=0}^{n} \frac{-n + k + 1}{2} \binom{n}{k+1}^2 - n + \sum_{k=0}^{n} \frac{n/2 - 1 - k}{k+1} \binom{n}{k+1}^2 + \left( \frac{n}{2} + 1 \right) 2^{2n} + \frac{n}{2}
\]

\[
= -\frac{n}{2} \sum_{k=0}^{n} \binom{n}{k+1}^2 + \left( \frac{n}{2} + 1 \right) 2^{2n} - \frac{n}{2}
\]

\[
= -\frac{n}{2} \left( \binom{2n}{n} - 1 \right) + \left( \frac{n}{2} + 1 \right) 2^{2n} - \frac{n}{2}
\]

\[
= \left( \frac{n}{2} + 1 \right) 2^{2n} - \frac{n}{2} \binom{2n}{n}.
\]

\[\square\]
Using the same argument, we can deduce Calkin’s identity [5]

\[ S_n^{(3)} = n2^{3n-1} + 2^{3n} - 3n2^{n-2} \binom{2n}{n}. \] (5.10)

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