General conditions for the generation of long-distance entanglement

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Abstract. We generally investigate necessary conditions for the generation of long-distance entanglement. We consider a quantum system in which a system mediates the indirect interaction between two spins, which we refer to as probe spins. Firstly, we weaken the coupling between each probe spin and the mediator system to the infinitesimal strength in order to generate the long-distance entanglement. We give two necessary conditions for the mediator system to generate the long-distance entanglement. We prove that indirect interaction cannot generate the entanglement if it is ‘classical’. We also give a necessary condition for the effective fields on the probe spins to satisfy. Secondly, we generate the long-distance entanglement by the use of only external fields. We show that external fields on the adjacent spins to the probes are necessary in addition to external fields on the probe spins. Finally, we consider the cases where the coupling strength between each probe spin and the mediator system is finite. In particular, we show two examples where the external fields on the mediator system highly enhance the long-distance entanglement.
1. Introduction

The generation of quantum entanglement [1] is of great importance because entanglement plays an essential role in quantum information processing [2, 3]. Many people have sought efficient generation of entanglement between remote quantum systems [4]. Let us consider that two systems (probe systems) indirectly interact with each other via another quantum system (mediator system); we summarize in table 1 the definition of the original terms in our paper. In this case, entanglement of each probe system with the mediator system would decrease the purity of the probe systems (figure 1). As a result, entanglement between the two probe systems rapidly decays and vanishes as the probe systems get separated distantly. Indeed, we cannot usually obtain entanglement between systems far apart even in the low-temperature limit [5, 6]. For the generation of entanglement between remotely separated probe systems, we therefore need to suppress entanglement between each probe system and the mediator system. It then appears to be a dilemma; we need to weaken the interaction between each probe system and the mediator system, and yet we need to keep the indirect interaction between the probe systems.

A recent study [7], however, reported the successful generation of entanglement between a far-separated spin pair in specific models; such entanglement is referred to as long-distance entanglement. In their models, entanglement with the mediator system is suppressed enough although the two spins still indirectly interact with each other via the mediator system. After the first paper on long-distance entanglement, various systems have turned out to be usable for generating long-distance entanglement [8–17]. In experiments, we will have to prepare a quantum system with which we can control the system parameters. It is expected that we can generate long-distance entanglement by the use of the optical lattice [18, 19]. Because of its usefulness, theoretical and experimental studies have rapidly progressed in recent years [20].

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**Table 1**

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|---------|-------|-------|
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Figure 1. Indirectly interacting systems. The two probe systems respectively interact with the mediator system. They can indirectly interact with each other via the mediator system although the two probe systems do not directly interact with each other. The entanglement between each probe system and the mediator system decreases the purity of the probe systems.

Table 1. Definition of the original terms in the paper.

| Term               | Definition                                                                 |
|--------------------|-----------------------------------------------------------------------------|
| Probe spins        | The spins between which we mainly consider the entanglement                |
| Mediator system    | The quantum system which mediates the indirect interaction between two probe spins |
| Local fields       | The external fields on the probe spins                                     |
|                    | The Hamiltonian is given in (2)                                            |

In order to utilize long-distance entanglement for practical applications, however, there are still many problems to solve. In this paper, we find answers to the following questions on long-distance entanglement:

(i) What are the conditions for a mediator system to generate long-distance entanglement?
(ii) What are the conditions to generate long-distance entanglement by the use of external fields only on the probe systems?
(iii) In what ways can we enhance the capability of the mediator system to generate long-distance entanglement?

These questions are closely related to practical applications.

One of the most popular methods for the generation of long-distance entanglement is to weaken the coupling between each probe system and the mediator system to infinitesimal strength [7–11]. It has been shown for several models that we obtain maximum entanglement between far-separated spins in the limit of weak coupling. In some models, analytical calculations of long-distance entanglement were possible and important quantities such as the first excitation energy were obtained in [8–10]. In [9], a mathematical analysis was also given as to why we can generate strong entanglement in this framework. In [11], experimental realization of long-distance entanglement has been theoretically discussed in the optical system. However, there is no analytical argument on a general condition to generate long-distance entanglement, which is one of the most important problems in discussing the usefulness of long-distance entanglement.
It is also an important question whether we can generate long-distance entanglement with external fields or not. It is a popular problem how much entanglement can be generated by modulating system parameters [5, 21–29]. One of the easiest parameters which we can control freely is external fields \( \{ h_{\xi} \} \) applied on a spin of the Pauli matrices \( \{ \sigma_{\xi} \} \) in the form of the Hamiltonian \( h_{\xi} \sigma^+ + h_{\xi} \sigma^- + h_{\xi} \sigma^z \). Plastina and Apollaro [23] discussed the generation of large entanglement between two distantly separated spins in an \( XX \) spin chain; they showed that local control of the external fields is an effective method of generating large entanglement between the two separated sites in this specific spin chain. In general systems, however, we cannot generate long-distance entanglement if we modulate the external fields only on the two probe spins.

Answering the above questions, we discuss in this paper the generation of long-distance entanglement between two probe spins which indirectly interact with each other through a mediator system. Our main results are summarized as follows:

(i) We first consider the system in figure 2(a) and decrease the coupling strength between the mediator system and each of the probe spins 1 and \( N \). In the weak coupling limit, we give two necessary conditions for the mediator system to generate long-distance entanglement in the form of two sufficient conditions for the non-existence of long-distance entanglement. Firstly, the indirect interaction must not be ‘classical’. Secondly, the effective fields on the probe spins must not be much larger than the indirect interaction between these two spins. We will give mathematical expressions of these conditions below.

(ii) We next consider the generation of long-distance entanglement with external fields. As has been expected, it is not enough to control the external fields only on the two probe spins. In addition to the external fields on the probe spins, we also have to control the external fields on the system adjacent to the probe spins, as depicted in figure 2(b). We give a necessary condition for the external fields to give rise to infinitesimal effective couplings between the mediator system and each of the probe spins 1 and \( N \), thus generating long-distance entanglement between the probe spins owing to the mechanism given in the previous item (i).

(iii) We finally consider the quantum system in figure 2(a) with finite couplings. We can highly enhance long-distance entanglement by modulating the external fields on the mediator system.

We show these results analytically and numerically.

This paper consists of the following sections. In section 2, we review the general framework of the generation of long-distance entanglement. Sections 3–5 describe items (i)–(iii) above, respectively. In section 6, a discussion concludes the paper.

2. General framework for the generation of long-distance entanglement

In the present section, we review the general framework of the generation of long-distance entanglement [7–11]. We consider a quantum system in which two ‘probe’ spins are connected to an arbitrary mediator system (figure 2(a)); that is,

\[
H_{\text{tot}} = H_{\text{int}} + H_{\text{LF}}
\] (1)

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Figure 2. (a) A schematic illustration of the generation of long-distance entanglement in sections 3 and 5. Two spins are connected to the mediator system with the coupling Hamiltonians $H_{1,\text{media}}$ and $H_{N,\text{media}}$. We refer to the spins 1 and $N$ as the probe spins and the entanglement between the probe spins as long-distance entanglement. In order to generate entanglement between the probe spins, we decrease the amplitude of the coupling Hamiltonian. We consider infinitesimally weak couplings in section 3 and finite couplings in section 5. 

(b) A schematic picture of the generation of long-distance entanglement by local fields in section 4. We connect four spins to the mediator system and consider the entanglement generation between the probe spins 1 and $N$. We apply the local fields on the four spins. By increasing the local fields $\vec{h}_2$ and $\vec{h}_{N-1}$, we can decrease the effective coupling between each of the probe spins and the mediator system.

With

$$H_{\text{int}} = H_{\text{media}} + H_{\text{couple}},$$

$$H_{\text{couple}} = \sum_{i=x,y,z} (\sigma_i^1 \otimes H_{1,\text{media}}^i + \sigma_i^N \otimes H_{N,\text{media}}^i),$$

$$H_{\text{LF}} = \sum_{i=x,y,z} (h_i^1 \sigma_i^1 + h_i^N \sigma_i^N),$$

(2)

where $\{\sigma^i\}_{i=x,y,z}$ are the Pauli matrices of $S = 1/2$ spins, while $(H_{1,\text{media}}^i, H_{N,\text{media}}^i)_{i=x,y,z}$ and $H_{\text{media}}$ are arbitrary (but not the identity) $2^{N-2}$-dimensional Hamiltonians which do not include the spins 1 and $N$. We assume that the ground state of $H_{\text{media}}$ is not degenerate. We define $H_{\text{int}}$ as $H_{\text{tot}} - H_{\text{LF}}$, which characterizes the indirect interaction between the spins 1 and $N$. Note that the external fields on the mediator system are also included in the Hamiltonian $H_{\text{int}}$; it means that the indirect interaction between the spins 1 and $N$ depends on the external fields on the
mediator system. We hereafter refer to the spins 1 and N as the probe spins and to the external fields \( \{ h_i \} = \{ h_{1x}, h_{1y}, h_{1z}, h_{Nx}, h_{Ny}, h_{Nz} \} \) as the local fields (table 1). We also refer to the entanglement between the probe spins 1 and N as long-distance entanglement. In fact, long-distance entanglement is usually defined as entanglement between infinitely separated two spins [7]; if the distance between the two probe spins is large but finite, entanglement between such a spin pair is often referred to as quasi-long-distance entanglement [8]. In this paper, however, we refer to all entanglement which is generated by the indirect interaction as long-distance entanglement for simplicity.

Let us consider the problem of enhancing the ground-state entanglement between the spins 1 and N by modulating the coupling Hamiltonian \( H_{\text{couple}} \). The density matrix of the total system in the ground state is given by

\[
\rho_{\text{tot}} = \lim_{\beta \to \infty} \frac{e^{-\beta H_{\text{tot}}}}{Z_{\text{tot}}(\beta)},
\]

where \( \beta \) is the inverse temperature \((k_B T)^{-1}\) with \( k_B \) the Boltzmann constant and \( Z_{\text{tot}}(\beta) = \text{tr}(e^{-\beta H_{\text{tot}}}) \) is the partition function. The density matrix of the probe spins 1 and N is

\[
\rho_{1N} = \text{tr}_{1N} \rho_{\text{tot}},
\]

where \( \text{tr}_{1N} \) denotes the trace operation on the system except the probe spins 1 and N. In order to quantify the entanglement, we adopt here the concurrence [30], which is most commonly used as an entanglement measure. The concurrence \( C(\rho_{1N}) \) is defined as follows:

\[
C(\rho_{1N}) \equiv \max(\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0),
\]

where \( \{ \lambda_n \}_{n=1}^{4} \) are the eigenvalues of the 4 \times 4 matrix

\[
\sqrt{\rho_{1N}(\sigma_1^y \otimes \sigma_N^y) \rho_{1N}^* (\sigma_1^y \otimes \sigma_N^y)}
\]

in the non-ascending order \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \). Note that \( C(\rho_{1N}) > 0 \) is a necessary and sufficient condition for the existence of entanglement.

A popular method of generating long-distance entanglement in the ground state is to reduce the coupling Hamiltonian \( H_{\text{couple}} \) to zero [7–11]; namely

\[
\| H_{\text{couple}} \| \to 0,
\]

where \( \| \cdots \| \) denotes the matrix norm. It has been shown that long-distance entanglement is generated in the limit (7) as

\[
\lim_{\| H_{\text{couple}} \| \to 0} C(\rho_{1N}) > 0
\]

in some quantum systems [7–11], for example, an XX spin chain.

Let us explain qualitatively why we can generate long-distance entanglement in the limit (7). A key feature for the generation of long-distance entanglement is the monogamy [31], which is a property that multiple pairs cannot share strong entanglement simultaneously. Because of the monogamy, if the components of the mediator system are entangled strongly with each other, large entanglement cannot be generated between each probe spin and the mediator system. In the limit (7) of weak coupling, the components of the mediator system may be entangled strongly with each other, but are scarcely entangled with each probe spin. Then, the probe spins can be entangled strongly only with each other.
Figure 3. Comparison of the entangled ground state and the non-entangled ground state. In the limit of weak coupling, the probe spins 1 and \( N \) cannot entangle strongly with the mediator system because of monogamy. Therefore, we can consider the two cases (a) and (b). In case (a), the energy of the entangled state is less than that of the non-entangled state, while in case (b), the energy of the non-entangled state is less than that of the entangled state.

It then appears to be claimed that the probe spins are entangled without interaction, which is obviously not the case. The fact is that the interaction is weak but not zero, and hence the quantum correlation between the two probe spins can decrease the energy of the total system. Then the ground state may support the entanglement between the probe spins (figure 3). We can describe the indirect interaction in terms of the effective Hamiltonian in the limit (7). We will give the effective Hamiltonian in section 3.2 in order to obtain the general properties of long-distance entanglement.

The next question is how to realize the weak-coupling limit (7) in realistic situations. We answer it by modulating local fields in the system in figure 2(b), extending figure 2(a) slightly. By increasing the local fields on the spins 2 and \( N - 1 \), we can effectively decrease the coupling between the mediator system and each of the probe spins 1 and \( N \). In fact, as will be shown in section 4, we cannot achieve long-distance entanglement by simply letting \( |\vec{h}_2| \) and \( |\vec{h}_{N-1}| \) to infinity but also by adjusting \( \vec{h}_1 \) and \( \vec{h}_N \) accordingly. The merit of using external fields is that we can control them more easily and precisely than coupling interactions. We then consider in section 4 the following Hamiltonian:

\[
H_{\text{tot}} = \sum_{i,j=x,y,z} \left( J_{A}^{i,j} \sigma_i^x \sigma_j^x + J_{A}^{i,j} \sigma_i^y \sigma_j^y + J_{B}^{i,j} \sigma_i^z \sigma_j^z + \tilde{J}_{A}^{i,j} \sigma_i^{x-1} \sigma_j^{x-1} + J_{B}^{i,j} \sigma_i^{x-1} \sigma_j^{x-1} \right) \\
+ \sum_{i=x,y,z} \left( h_i^1 \sigma_i^1 + h_i^2 \sigma_i^2 + h_{N-1}^i \sigma_i^{N-1} + h_N^i \sigma_i^N \right) + H_{\text{media}},
\]

(9)

where the Hamiltonian \( H_{\text{media}} \) is now an arbitrary \( 2^{N-4} \)-dimensional Hamiltonian and we assume that the ground state of \( H_{\text{media}} \) is not degenerate. We modulate only the local fields \( \{h_1^i, h_2^i, h_{N-1}^i, h_N^i\}_{i=x,y,z} \) in order to generate entanglement between spins 1 and \( N \). In section 4, we will clarify the exact correspondence between the decrease of the interaction and the increase of the local fields.

3. General conditions for generation of long-distance entanglement

In the previous section, we presented general statements on the generation of long-distance entanglement. However, they do not tell us whether the mediator system contributes to the
generation of long-distance entanglement or not. We present here two cases in which we cannot generate long-distance entanglement. In the first case, indirect interactions in a particular class cannot generate entanglement at all; we refer to such an interaction as a ‘classical’ interaction. In the second case, the effective fields on the probe spins seriously destroy entanglement. These constitute sufficient conditions for the non-existence of long-distance entanglement, thereby giving two necessary conditions for its existence as their contrapositions. Throughout the present section, we consider the system (1) in figure 2(a).

3.1. Classical interaction

The first necessary condition for the generation of long-distance entanglement is that the indirect interaction between the spins 1 and \( N \) must not be a ‘classical’ interaction. We define that the indirect interaction between the probe spins 1 and \( N \) is ‘classical’ if there exists the following separation of the Hamiltonian \( H_{\text{int}} \) in (1):

\[
H_{\text{int}} = H_A(\sigma_1) + H_B(\sigma_N) \quad \text{with} \quad [H_A(\sigma_1), H_B(\sigma_N)] = 0.
\]

As has been shown in equation (1), the Hamiltonian \( H_{\text{int}} \) is defined as the Hamiltonian except for the external fields on spins 1 and \( N \). Note that \( H_A \) does not contain \( \sigma_N \), nor \( H_B \) contain \( \sigma_1 \). We can prove the following theorem.

**Theorem 1.** If the interaction Hamiltonian \( H_{\text{int}} \) is ‘classical’, we cannot generate entanglement between spins \( \sigma_1 \) and \( \sigma_N \) for any values of the local fields \( \{h^1_i\}_{i=x,y,z} \) and \( \{h^N_i\}_{i=x,y,z} \); namely

\[
C(\rho_{1N}) = 0,
\]

(11)

where the density matrix \( \rho_{1N} \) is defined in equations (3) and (4) and the concurrence \( C(\rho_{1N}) \) is defined in equation (5).

**Comments.** For example, we can separate the following Hamiltonian in the form (10):

\[
H_{\text{int}} = J_1^z \sigma_1^z \sigma_2^z + J_2^z \sigma_2^z \sigma_3^z + J_3^z \sigma_3^z \sigma_4^z,
\]

(12)

where the spin pair (1, 2) interact with each other through the Ising interaction along the \( z \)-axis, while the spin pairs (2, 3) and (3, 4) interact with each other through the Ising interaction along the \( x \)-axis. We can separate this Hamiltonian into \( H_A(\sigma_1) \) and \( H_B(\sigma_4) \) as

\[
H_A(\sigma_1) = J_1^z \sigma_1^z \sigma_2^z + J_2^z \sigma_2^z \sigma_3^z + J_3^z \sigma_3^z \sigma_4^z,
\]

\[
H_B(\sigma_4) = J_3^z \sigma_3^z \sigma_4^z.
\]

(13)

These Hamiltonians satisfy the condition \([H_A(\sigma_1), H_B(\sigma_4)] = 0\), and hence we cannot generate entanglement between probe spins 1 and 4 in this system for any values of the local fields \( \{h^1_i\}_{i=x,y,z} \) and \( \{h^4_i\}_{i=x,y,z} \). Note that spins 1 and 4 are classically correlated with each other. If we replace \( J_3^z \sigma_3^z \sigma_4^z \) in equation (12) by \( J_3^z \sigma_3^z \sigma_4^z \) as

\[
H_{\text{int}} = J_1^z \sigma_1^z \sigma_2^z + J_2^z \sigma_2^z \sigma_3^z + J_3^z \sigma_3^z \sigma_4^z,
\]

(14)

we cannot separate the Hamiltonian \( H_{\text{int}} \) into the forms of \( H_A(\sigma_1) \) and \( H_B(\sigma_4) \) which satisfy \([H_A(\sigma_1), H_B(\sigma_4)] = 0\) anymore, and hence the spins 1 and 4 can entangle with each other.

It is worth noting that by the external fields on the mediator system the interaction Hamiltonian \( H_{\text{int}} \) can be transformed from a ‘classical’ one to a ‘non-classical’ one, that is, entanglement generation becomes possible. For example, if we add the external field \( h_3^3 \sigma_3^z \) on spin 3 in equation (12), we cannot separate the Hamiltonian as in equation (13). We will also
show in section 5 that the external fields on the mediator system can enhance the capability of the interaction Hamiltonian $H_{\text{int}}$.

Eigenstates can have entanglement even if condition (10) is satisfied. For example, the Hamiltonian for the probe spins 1 and 3,

\[
H_{\text{int}} = H_{\text{int}} + h_1^z \sigma_1^z + h_3^z \sigma_3^z, \\
H_{\text{int}} = J_1^z \sigma_1^z \sigma_2^z + J_2^z \sigma_2^z \sigma_3^z 
\]

with $h_1^z = h_3^z = J_1^z = J_2^z$ satisfies condition (10), but it has an eigenstate $(|\uparrow_1 \uparrow_2 \uparrow_3\rangle + |\downarrow_1 \uparrow_2 \downarrow_3\rangle) / \sqrt{2}$, which is highly entangled. Mixing of all the eigenstates with the Boltzmann weight always destroys entanglement between the probe spins.

Finally, under this condition, for appropriate values of the local fields we can generate the quantum discord, which is one of the non-classical correlations. We discuss the quantum discord in the appendix.

**Proof.** We prove the following equality under condition (10):

\[
\text{tr}_{1N}(e^{-\beta H_{\text{int}}}) = \sum_n \rho_1^n \otimes \rho_N^n, 
\]

where $\text{tr}_{1N}$ denotes the trace operation on the system except the probe spins 1 and $N$, and the density matrices $\rho_1^n$ and $\rho_N^n$ are physical states, namely, positive matrices. Then, spins 1 and $N$ are not entangled with each other by definition. By proving equation (16), we can also prove in the limit $\beta \to \infty$ that the density matrix (4) is decomposed into a mixture of the product states.

First, under condition (10), we can decompose the density matrix as follows:

\[
e^{-\beta H_{\text{int}}} = e^{-\beta H_A(\sigma_I)} e^{-\beta H_B(\sigma_N)}. 
\]

We can express $e^{-\beta H_A(\sigma_I)}$ and $e^{-\beta H_B(\sigma_N)}$ as

\[
e^{-\beta H_A(\sigma_I)} = \sum_{\mu=0,x,y,z} \sigma_1^\mu \otimes \rho_{\text{media}}^{1\mu} \otimes I_N, \\
e^{-\beta H_B(\sigma_N)} = \sum_{v=0,x,y,z} I_1 \otimes \rho_{\text{media}}^{Nv} \otimes \sigma_N^v, 
\]

where $I_1$ and $I_N$ are the identity matrices in the spaces of the spins 1 and $N$, respectively, and we define $\sigma_{I}^0 = I_1$ and $\sigma_{N}^0 = I_N$. We also define that the matrices $\rho_{\text{media}}^{1\mu}$ and $\rho_{\text{media}}^{Nv}$ are Hermitian operators in the mediator space. Because $H_A(\sigma_I)$ and $H_B(\sigma_N)$ are assumed to commute with each other, the matrices $e^{-\beta H_A(\sigma_I)}$ and $e^{-\beta H_B(\sigma_N)}$ also commute with each other. Therefore, we obtain the following equation:

\[
\text{tr}^{1N}[\sigma_1^\mu \otimes \sigma_N^v e^{-\beta H_A(\sigma_I)} e^{-\beta H_B(\sigma_N)}] = \text{tr}^{1N}[\sigma_1^\mu \otimes \sigma_N^v e^{-\beta H_B(\sigma_N)} e^{-\beta H_A(\sigma_I)}], 
\]

where $\text{tr}^{1N}$ denotes the trace operation only on the spins 1 and $N$. From this equation we obtain

\[
\rho_{\text{media}}^{1\mu} \rho_{\text{media}}^{Nv} = \rho_{\text{media}}^{Nv} \rho_{\text{media}}^{1\mu} \]

for $\mu, v = 0, x, y, z$. Therefore, the matrices $\rho_{\text{media}}^{1\mu}$ and $\rho_{\text{media}}^{Nv}$ have simultaneous eigenstates. Then, we can express $\rho_{\text{media}}^{1\mu}$ and $\rho_{\text{media}}^{Nv}$ as

\[
\rho_{\text{media}}^{1\mu} = \sum_{n=1}^{2^{N-2}} \lambda_n^{1\mu} |n, \mu_1, v_N\rangle \langle n, \mu_1, v_N| 
\]

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and

$$\rho_{\text{media}}^{N\nu} = \sum_{n=1}^{2^{N-2}} \tau_v^n |n, \mu_1, \nu_N\rangle \langle n, \mu_1, \nu_N|,$$

(22)

where \{\{|n, \mu_1, \nu_N\}\} are \(2^{N-2}\) pieces of the simultaneous eigenstates of \(\rho_{\text{media}}^{1\mu}\) and \(\rho_{\text{media}}^{N\nu}\). As a result, we obtain

$$e^{-\beta H_A(\sigma_1)} e^{-\beta H_B(\sigma_N)}$$

$$= \left(\sum_{n,\mu} \lambda_n^\mu \sigma_1^\mu \otimes |n, \mu_1, \nu_N\rangle \langle n, \mu_1, \nu_N| \otimes I_N\right) \left(\sum_{n',v} \tau_v^{n'} I_1 \otimes |n', \mu_1', \nu_N\rangle \langle n', \mu_1', \nu_N| \otimes \sigma_N^{v'}\right)$$

$$= \sum_{n,\mu,v} \lambda_n^\mu \tau_v^n \sigma_1^\mu \otimes |n, \mu_1, \nu_N\rangle \langle n, \mu_1, \nu_N| \otimes \sigma_N^{v'},$$

(23)

where the indices \(\mu'\) and \(v'\) in the first line can be arbitrarily chosen \((v', \mu' = 0, x, y, z)\), and hence we choose \(\mu'\) and \(v'\) in accordance with \(\mu\) and \(v\). By tracing out the mediator space, we have

$$\text{tr}_{1N} e^{-\beta H_A(\sigma_1)} e^{-\beta H_B(\sigma_N)} = \sum_{n,\mu,v} \lambda_n^\mu \tau_v^n \sigma_1^\mu \otimes \sigma_N^{v'}$$

$$= \sum_n \left(\sum_{\mu} \lambda_n^\mu \sigma_1^\mu\right) \otimes \left(\sum_v \tau_v^n \sigma_N^{v'}\right).$$

(24)

At this moment, we cannot say that \(\sum_{\mu} \lambda_n^\mu \sigma_1^\mu\) and \(\sum_v \tau_v^n \sigma_N^{v'}\) are necessarily physical states, namely, positive matrices. In the following, we prove that equation (24) can be reduced to the mixture of the product states as in the form (16).

For the purpose, we should pay attention to the degeneracies of the matrices \(\rho_{\text{media}}^{1\mu}\) and \(\rho_{\text{media}}^{N\nu}\). In fact, if there are no degeneracies in the eigenspaces of all these matrices for \(\mu, v = 0, x, y, z\), we can easily prove that each of \(\sum_{\mu} \lambda_n^\mu \sigma_1^\mu\) and \(\sum_v \tau_v^n \sigma_N^{v'}\) (\(n = 1, 2, \ldots, 2^{N-2}\)) in equation (24) is a positive matrix. Since the matrices \(\rho_{\text{media}}^{1\mu}\) and \(\rho_{\text{media}}^{N\nu}\) commute with each other as well as \(\rho_{\text{media}}^{1\mu}\) and \(\rho_{\text{media}}^{N\nu}\) do, the matrices \(\rho_{\text{media}}^{1\mu}\) and \(\rho_{\text{media}}^{N\nu}\) should also have simultaneous eigenstates if there are no degeneracies. If there are absolutely no degeneracies in all eigenspaces of \(\rho_{\text{media}}^{1\mu}\) and \(\rho_{\text{media}}^{N\nu}\) \((\mu, v = 0, x, y, z)\), we have an orthonormal set of \(2^{N-2}\) pieces of states \(|n\rangle\), each of which is the simultaneous eigenstate \(|n, \mu_1, \nu_N\rangle\) for all of \(\mu, v = 0, x, y, z\). Then, we have from equation (18)

$$\sum_{\mu=0,x,y,z} \lambda_n^\mu \sigma_1^\mu = \text{tr}_{1N} (e^{-\beta H_A(\sigma_1)} |n\rangle \langle n|),$$

(25)

$$\sum_{v=0,x,y,z} \tau_v^n \sigma_N^{v'} = \text{tr}_{1N} (e^{-\beta H_B(\sigma_N)} |n\rangle \langle n|),$$

(26)

for \(n = 1, 2, \ldots, 2^{N-2}\). This means that each of \(\sum_{\mu} \lambda_n^\mu \sigma_1^\mu\) and \(\sum_v \tau_v^n \sigma_N^{v'}\) (\(n = 1, 2, \ldots, 2^{N-2}\)) is a positive matrix, and hence equation (24) indeed takes the form (16).

If there are degeneracies in some of the eigenspaces of the matrices \(\rho_{\text{media}}^{1\mu}\) and \(\rho_{\text{media}}^{N\nu}\) there is a possibility that we cannot choose a common state \(|n\rangle\) that represents the simultaneous eigenstate \(|n, \mu_1, \nu_N\rangle\) for \(\mu, v = 0, x, y, z\) (figure 4). Let us then inspect the degeneracies in more detail.
Suppose that the $n_0$th and the $n_1$th eigenvalues of $\rho_{\text{media}}^{N\nu}$ are not degenerate ($\tau_v^{n_0} \neq \tau_v^{n_1}$); then the eigenstates $|n_0, \mu_1, \nu_N\rangle$ and $|n_1, \mu_1', \nu_N\rangle$ are orthogonal to each other. Suppose that all the eigenstates of $\rho_{\text{media}}^{N\nu}$ are degenerate at the $(n_1 + 1)$th and the $(n_1 + 2)$th levels, namely, $\tau_v^{n_1+1} = \tau_v^{n_1+2}$ for $v = 0, x, y, z$. Then, any superposition of the states $|n_1 + 1, \mu_1, \nu_N\rangle$ and $|n_1 + 2, \mu_1, \nu_N\rangle$ can be the $(n_1 + 1)$th eigenstates of $\rho_{\text{media}}^{N\nu}$ and hence it is possible that $\rho_{\text{media}}^{N\nu}$ and $\rho_{\text{media}}^{N\nu'}$ do not have simultaneous eigenstates in the space $\{|n_1 + 1, \mu_1, \nu_N\rangle, |n_1 + 2, \mu_1, \nu_N\rangle\}$. We then form a block which is composed of the states $|n_1 + 1, \mu_1, \nu_N\rangle$ and $|n_1 + 2, \mu_1, \nu_N\rangle$.

We then form a block which is composed of the states $|n_1 + 1, \mu_1, \nu_N\rangle$ and $|n_1 + 2, \mu_1, \nu_N\rangle$.

We can thereby break down the whole eigenspace into blocks. We form a block of eigenspace in which all the matrices $\rho_{\text{media}}^{N\nu}$ (with the dimensionality $D_\nu$) have degeneracies in the corresponding eigenspaces and/or all the matrices $\rho_{\text{media}}^{N\nu}$ (with the dimensionality $D_\nu$) have degeneracies in the corresponding eigenspaces.
This shows that equation (24) can be summarized into the form of equation (16), where the summation on the right-hand side of equation (16) is taken over the blocks $\tilde{n}$. Thus, theorem 1 is proved.

The contraposition of theorem 1 gives us a necessary condition to generate entanglement between the probe spins by interaction (2). We can see from the proof that we can extend this theorem to be applicable to any bipartite systems which indirectly interact with each other at arbitrary temperatures. Let us consider the interaction between the bipartite system with $\mathcal{L}$ levels; namely

$$H_{\text{int}} = \sum_{i=1}^{\mathcal{L}^2-1} (\tilde{\sigma}^i_1 \otimes H^i_{1,\text{media}} + \tilde{\sigma}^i_N \otimes H^i_{N,\text{media}}) + H_{\text{media}}, \quad (28)$$

where $\{\tilde{\sigma}^i_1, \tilde{\sigma}^i_N\}_{i=1}^{\mathcal{L}^2-1}$ are the bases of the bipartite systems and we define $\{\tilde{\sigma}^0_1, \tilde{\sigma}^0_N\}$ as the identity matrices. In equation (18), we had $\mu, \nu = 0, x, y, z$ for the two $S = 1/2$ spins, while we have $\mu, \nu = 0, 1, 2, \ldots, \mathcal{L}^2 - 1$ for this bipartite system. The extension of the proof is straightforward.

3.2. Effective fields

We next show that the effective fields on the probe spins have serious effects on the generation of long-distance entanglement. In order to make the discussion clear, we first give the formal expression of the effective Hamiltonian of spins 1 and $N$.

We investigate the ground state of the total Hamiltonian $H_{\text{tot}}$ of the system in figure 2(a). For this purpose, we break down the Hamiltonian (1) as follows:

$$H_{\text{tot}} = H_0 + H_1, \quad H_0 \equiv H_{\text{media}}, \quad H_1 \equiv H_{\text{couple}} + H_{\text{LF}}. \quad (29)$$

We define the ground state of $H_0$ as $|\psi^\text{media}_0\rangle$ with the eigenvalue $E^\text{media}_0$. Because we assumed that the ground state of the mediator system is not degenerate, we also define the first excitation energy of $H_{\text{media}}$ as $\delta E^\text{media}_1$ ($> 0$).

We then assume that

$$\| H_1 \| \ll \delta E^\text{media}_1 \quad (30)$$

and consider the term $H_1$ as perturbation. The unperturbed ground state is given by

$$P_0 \equiv I_{1N} \otimes |\psi^\text{media}_0\rangle \langle \psi^\text{media}_0|, \quad (31)$$

where $I_{1N}$ is the identity matrix of the spins 1 and $N$.

In order to define the effective Hamiltonian, we consider a Green’s function

$$\frac{1}{E - P_0 H_{\text{tot}} P_0} = P_0 \frac{1}{E - H_{\text{tot}}} P_0 \quad (32)$$

and define the effective Hamiltonian as

$$\frac{1}{E - P_0 H_{\text{tot}} P_0} \equiv \frac{1}{E - H_{\text{eff}}}. \quad (33)$$
We calculate the effective Hamiltonian by expanding equation (32) with respect to $H_1$ to have

$$\frac{1}{E - H_{\text{tot}}} P_0 = \frac{1}{E - H_0 - H_1} P_0 = P_0 \frac{1}{E - H_0} P_0 + P_0 \frac{1}{E - H_0} H_1 \frac{1}{E - H_0} P_0 + P_0 \frac{1}{E - H_0} H_1 \frac{1}{E - H_0} P_0 + O (\|H_1\|^2).$$

(34)

We can rewrite equation (34) as

$$\frac{1}{E - H_{\text{tot}}} P_0 = \frac{1}{E - P_0 H_0 P_0 - P_0 H_1 P_0 - P_0 H_1 Q_0 \frac{1}{E - H_0} Q_0 H_1 P_0} + O (\|H_1\|^3),$$

(35)

where $Q_0 \equiv I_{\text{tot}} - P_0$ with $I_{\text{tot}}$ the identity matrix of the total system. We can indeed confirm equation (35) by expanding it. The pole of the Green’s function (35) is given by $E = E_0^{\text{media}} + O (\|H_1\|)$, and hence we substitute $E = E_0^{\text{media}}$ into the term $P_0 H_1 Q_0 \frac{1}{E - H_0} Q_0 H_1 P_0$. We thereby obtain the effective Hamiltonian around the ground state $E_0^{\text{media}}$ as

$$H_{\text{eff}}^{\text{media}} = P_0 H_0 P_0 + P_0 H_1 P_0 + P_0 H_1 Q_0 \frac{1}{E_0^{\text{media}} - H_0} Q_0 H_1 P_0.$$  

(36)

Let us calculate the effective Hamiltonian (36) using (29). Because the Hamiltonian $H_{\text{LF}}$ and the projection operators $P_0$ and $Q_0$ commute with each other, we have

$$P_0 H_1 P_0 = H_{\text{LF}} \otimes \langle \psi_0^{\text{media}} | \langle \psi_0^{\text{media}} | + P_0 H_{\text{couple}} P_0$$

(37)

and

$$P_0 H_1 Q_0 \frac{1}{E_0^{\text{media}} - H_0} Q_0 H_1 P_0 = P_0 H_{\text{couple}} Q_0 \frac{1}{E_0^{\text{media}} - H_0} Q_0 H_{\text{couple}} P_0.$$  

(38)

where we utilized the equation $P_0 Q_0 = P_0 (I_{\text{tot}} - P_0) = 0$. Then, we obtain the effective Hamiltonian for the spins 1 and $N$, which we refer to as $H_{1N}^{\text{eff}}$, up to the constant $E_0^{\text{media}}$ in the form

$$H_{1N}^{\text{eff}} \equiv \langle \psi_0^{\text{media}} | H_{\text{eff}}^{\text{media}} | \psi_0^{\text{media}} \rangle$$

$$= H_{\text{LF}} + \langle \psi_0^{\text{media}} | H_{\text{couple}} | \psi_0^{\text{media}} \rangle + \langle \psi_0^{\text{media}} | H_{\text{couple}} Q_0 \frac{1}{E_0^{\text{media}} - H_0} Q_0 H_{\text{couple}} | \psi_0^{\text{media}} \rangle.$$  

(39)

Note that the effective Hamiltonian can be divided into the part that includes only $H_{\text{LF}}$ and the part that includes only $H_{\text{couple}}$. Expression (39) is essentially the same as the one derived in [9].

In order to relate the effective Hamiltonian (39) to the argument in section 2, we prove the following equation:

$$\text{tr}_{1N} \frac{1}{Z_{\text{tot}}} e^{-\beta H_{\text{tot}}} = \frac{1}{Z_{\text{tot}}} e^{-\beta H_{1N}^{\text{eff}}} + O (\|H_1\|^2)$$

(40)

in the low-temperature limit $\beta \to \infty$, where $\text{tr}_{1N}$ denotes the trace operation on the system except the probe spins 1 and $N$. Equation (40) means that the effective Hamiltonian defined in equation (39) gives the density matrix up to second order. The proof is given as follows. First, equation (33) is Laplace transformed to the following equation:

$$P_0 e^{-\beta H_{\text{tot}}} P_0 = e^{-\beta H_{\text{eff}}}.$$  

(41)

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Because of inequality (30), perturbation theory yields

\[
\frac{1}{Z_{\text{tot}}} e^{-\beta H_{\text{tot}}} = \frac{1}{Z_{\text{tot}}} (P_0 + Q_0) e^{-\beta H_{\text{tot}}} (P_0 + Q_0)
\]

\[
= \frac{1}{Z_{\text{tot}}} P_0 e^{-\beta H_{\text{tot}}} P_0 + O(\|H_1\|) (P_0 A_1 Q_0 + Q_0 A_1^\dagger P_0) + O(\|H_1\|^2),
\]

(42)

in the low-temperature limit \(\beta \to \infty\), where we define \(A_1\) as an \(O(1)\) operator of the total system. We can obtain equation (40) from equations (41) and (42).

Let us characterize each term of the effective Hamiltonian \(H_{1N}^{\text{eff}}\) in equation (39). The first term in equation (39) is the local fields on the probe spins, whereas the second term gives the effective fields of the order of \(\|H_1\|\); the second term does not generate the interaction term because \(H_{\text{couple}}\) given in equation (2) does not include terms such as \(\sigma_i^x \otimes \sigma_j^y\) \((i, j = x, y, z)\). The third term gives the effective interaction between the probe spins as well as the effective fields of order \(\|H_1\|^2\). The effective Hamiltonian \(H_{1N}^{\text{eff}}\) is thereby summarized as follows:

\[
H_{1N}^{\text{eff}} = H_{\text{LF}} + \sum_{i=x,y,z} (h_i^{\text{eff}} \sigma_i^x + h_i^{\text{eff}} \sigma_i^y) + \sum_{i,j=x,y,z} J_{ij}^{\text{eff}} \sigma_i^x \otimes \sigma_j^z,
\]

(43)

where

\[
\sum_{i=x,y,z} (h_i^{\text{eff}} \sigma_i^x + h_i^{\text{eff}} \sigma_i^y) = \langle \psi_0^{\text{media}} | H_{\text{couple}} | \psi_0^{\text{media}} \rangle + O(\|H_1\|^2),
\]

\[
\sum_{i,j=x,y,z} J_{ij}^{\text{eff}} \sigma_i^x \otimes \sigma_j^z = O(\|H_1\|^2).
\]

(44)

We show that a necessary condition for the mediator system to generate the long-distance entanglement is that the fields of the order of \(\|H_1\|\) vanish, namely

\[
\langle \psi_0^{\text{media}} | H_{\text{couple}} | \psi_0^{\text{media}} \rangle + H_{\text{LF}} = 0.
\]

(45)

If the above condition is not satisfied, the effective fields of order \(O(\|H_1\|)\) totally destroy long-distance entanglement in the limit of (7). When we explicitly know the form of the effective Hamiltonian, we can cancel the effective fields by choosing the local fields on the spins 1 and \(N\) as

\[
H_{\text{LF}} = \sum_{i=x,y,z} \left( h_i^1 \sigma_i^x + h_i^N \sigma_i^y \right) = -\langle \psi_0^{\text{media}} | H_{\text{couple}} | \psi_0^{\text{media}} \rangle.
\]

(46)

Note that condition (45) does not ensure maximum long-distance entanglement, namely \(C(\rho_{1N}) = 1\); effective fields of the order of \(\|H_1\|^2\) can still exist even if condition (45) is satisfied, which are of the same order as the indirect interaction in the limit of (7).

Let us consider the following special case:

\[
H_{\text{couple}} = \sum_{i,j=x,y,z} \left( J_{ij}^A \sigma_i^x \otimes \sigma_j^z + J_{ij}^B \sigma_i^y \otimes \sigma_j^z \right)
\]

(47)

with the condition

\[
\langle \psi_0^{\text{media}} | H_{\text{couple}} | \psi_0^{\text{media}} \rangle = 0.
\]

(48)
In this case, we do not need $H_{\text{LF}}$ to satisfy equation (45), namely we can put $H_{\text{LF}} = 0$. Condition (48) is reduced to

$$\sum_{j=x,y,z} \langle \psi_0^{\text{media}} | J_A^{i,j} \sigma_2^j | \psi_0^{\text{media}} \rangle = 0 \quad \text{and} \quad \sum_{j=x,y,z} \langle \psi_0^{\text{media}} | J_B^{i,j} \sigma_N^{j-1} | \psi_0^{\text{media}} \rangle = 0$$

(49)

for $i = x, y, z$. This indeed occurs in any systems with time-reversal symmetry; quantum spin chains without external fields and other odd-body interactions typically satisfy equation (49), even if the Dzyaloshinskii–Moriya interaction [32, 33] is included in the spin–spin interaction.

In the derivation (39) of the effective Hamiltonian, we assumed that the effective Hamiltonian can be obtained by the second-order perturbation. However, there are cases in which we cannot obtain the effective interaction by the second-order perturbation. For example, in the cases where the indirect interaction is ‘classical’, we cannot obtain the effective interaction by the second-order perturbation; if the effective Hamiltonian for a classical interaction were given as in equation (39), we could always obtain the maximum entanglement by choosing the local fields $\{h_1^i, h_N^i\}_{i=x,y,z}$ properly [28], but this is contradictory to theorem 1. If the second-order perturbation vanishes, the effective interaction may depend on the local fields $\{h_1^i, h_N^i\}_{i=x,y,z}$ and takes a more complicated form in higher-order perturbations.

4. The generation of long-distance entanglement by local fields

In the present section, we discuss the generation of long-distance entanglement by the use of only local fields. Throughout this section, we consider system (9) in figure 2(b).

4.1. Effective Hamiltonian

We can realize a condition equivalent to (7) in the system (9) by increasing the amplitudes of the local fields $\{h_2^i\}_{i=x,y,z}$ and $\{h_{N-1}^i\}_{i=x,y,z}$ with the local fields $\{h_1^i\}_{i=x,y,z}$ and $\{h_N^i\}_{i=x,y,z}$ canceling the resulting effective fields. In the limits $|\vec{h}_2| \to \infty$ and $|\vec{h}_{N-1}| \to \infty$, the total Hamiltonian (9) can be transformed into the following effective Hamiltonian after tracing out the spins 2 and $N - 1$:

$$H_{\text{tot}}^{\text{eff}} = \sum_{i,j=x,y,z} \left( J_A^{i,j,\text{eff}} \sigma_i^1 \sigma_j^3 + J_B^{i,j,\text{eff}} \sigma_{N-2}^j \sigma_N^i \right) + \sum_{i=x,y,z} \left( (h_i^1 + h_i^{1,\text{eff}}) \sigma_i^1 + (h_N^i + h_N^{i,\text{eff}}) \sigma_N^i \right) + H_{\text{media}} + H_{\text{media}}^{\text{eff}}.$$  

(50)

The coupling parameters $\{J_A^{i,j,\text{eff}}\}_{i=x,y,z}$ and $\{J_B^{i,j,\text{eff}}\}_{i=x,y,z}$ in equation (50) approach zero as $|\vec{h}_2|^{-1}$ and $|\vec{h}_{N-1}|^{-1}$. The amplitudes of the effective fields $\{h_i^{1,\text{eff}}\}_{i=x,y,z}$ and $\{h_N^{i,\text{eff}}\}_{i=x,y,z}$ are also of the order of $|\vec{h}_2|^{-1}$ and $|\vec{h}_{N-1}|^{-1}$, respectively, which is the same as the coupling parameters $\{J_A^{i,j}\}_{i=x,y,z}$ and $\{J_B^{i,j}\}_{i=x,y,z}$. In order to achieve condition (45), we apply the weak local fields $\vec{h}_1 = -\vec{h}_1^{\text{eff}}$ and $\vec{h}_N = -\vec{h}_N^{\text{eff}}$ on spins 1 and $N$ to cancel the effective fields, so that

$$\sum_{i=x,y,z} (h_i^1 + h_i^{1,\text{eff}}) \sigma_i^1 = 0,$$

$$\sum_{i=x,y,z} (h_N^i + h_N^{i,\text{eff}}) \sigma_N^i = 0.$$  

(51)
In the following, for simplicity, we let the coupling parameters \( \{ J_{A}^{i,j}, \tilde{J}_{A}^{i,j}, \tilde{J}_{B}^{i,j}, J_{B}^{i,j} \}_{i,j=x,y,z} \) be the XYZ interaction and \( \{ h_{2}^{i}, h_{N-1}^{i} \}_{i=x,y,z} \) applied only in the z direction:

\[
H_{\text{tot}} = \sum_{i=x,y,z} \left( J_{A}^{i} \sigma_{i}^{1} \sigma_{2}^{i} + J_{A}^{i} \sigma_{2}^{i} \sigma_{3}^{i} + \tilde{J}_{A}^{i} \sigma_{N-2}^{i} \sigma_{N-1}^{i} + J_{B}^{i} \sigma_{N-1}^{i} \sigma_{N}^{i} \right) \\
+ \sum_{i=x,y,z} \left( h_{2}^{i} \sigma_{1}^{i} + h_{N}^{i} \sigma_{N}^{i} \right) + h_{2}^{\text{z}} \sigma_{z}^{2} + h_{N-1}^{\text{z}} \sigma_{z}^{N-1} + H_{\text{media}}. 
\]  

(52)

In section 4.2, we show the decrease of the effective coupling interaction mathematically and discuss how precisely we can achieve condition (7) by increasing \( |\vec{h}_{2}| \) and \( |\vec{h}_{N-1}| \).

4.2. The equivalence between the increase of the local fields and the decrease of the interaction

We show here the equivalence between the increase of the local fields and the decrease of the interaction. In order to study the mathematical structure generally, we consider the quantum system shown in figure 5 instead of the system in figure 2(b); spins 1 and 3 indirectly interact with each other through spin 2, while spin 3 is coupled to the environmental system. We define the environment as an arbitrary \( n_{\text{env}} \)-dimensional quantum system; that is, the dimensionality of the total system is \( 8n_{\text{env}} \). The total Hamiltonian is given as follows:

\[
H_{\text{tot}} = h_{2}^{\text{z}} \sigma_{z}^{2} + H_{J,J'} + H_{3,\text{env}}, \\
H_{J,J'} = \sum_{i=x,y,z} \left( J_{A}^{i} \sigma_{i}^{1} \sigma_{2}^{i} + J_{A}^{i} \sigma_{2}^{i} \sigma_{3}^{i} + J_{B}^{i} \sigma_{N-2}^{i} \sigma_{N-1}^{i} \right), 
\]  

(53)

where \( H_{3,\text{env}} \) is an arbitrary \( 2n_{\text{env}} \)-dimensional Hamiltonian of spin 3 and the environment. The effective Hamiltonian \( H_{1,3,\text{env}}^{\text{eff}} \) after tracing out spin 2 takes a simple form in the limit \( |h_{2}^{\text{z}}| \to \infty \).

We apply formula (36) to the Hamiltonian (53) to obtain the form of \( H_{1,3,\text{env}}^{\text{eff}} \).

**Theorem 2.** The effective Hamiltonian \( H_{1,3,\text{env}}^{\text{eff}} \) for the Hamiltonian (53) is given as follows in the limit of \( |h_{2}^{\text{z}}| \to \infty \):

\[
H_{1,3,\text{env}}^{\text{eff}} = \sum_{i=x,y,z} J_{A}^{i} \sigma_{i}^{1} \sigma_{3}^{i} + H_{3,\text{env}} + h_{1}^{\text{z}} \sigma_{1}^{z} + h_{3}^{\text{z}} \sigma_{3}^{z}, 
\]  

(54)
where

\[
\begin{align*}
J^x_{\text{eff}} &= - \frac{J^x J^x}{h_2^2}, & J^y_{\text{eff}} &= - \frac{J^y J^y}{h_2^2}, & J^z_{\text{eff}} &= 0, \\
J^z_{\text{eff}} &= - J^z - \frac{J^x J^y}{h_2^2}, & J^z_{\text{eff}} &= - J^z - \frac{J^x J^y}{h_2^2}.
\end{align*}
\]

In addition, the Hamiltonian \( H_{1,3,\text{env}}^{\text{eff}} \) satisfies

\[
\text{tr}^2 \frac{e^{-\beta H_{\text{tot}}}}{Z_{\text{tot}}} = \frac{e^{-\beta H_{1,3,\text{env}}^{\text{eff}}}}{Z_{\text{tot}}} + O(|h_2^z|^{-2})
\]

in the limit of \( \beta \to \infty \), where \( \text{tr}^2 \) denotes the trace operation only on spin 2.

**Comment.** The influence from spin 2 is effectively expressed as the fields on spins 1 and 3 after the trace-out of spin 2. The interaction with spin 2 also mediates the indirect interaction between spins 1 and 3. As the external fields \( |h_2^z| \) increase, the state of spin 2 is approximately fixed to \( |\uparrow_2\rangle \), and hence, the indirect interaction via spin 2 is weakened. The indirect interactions along the \( x \)- and \( y \)-axis decay as \( |h_2^x|^{-1} \) in the limit \( |h_2^z| \to \infty \), while the indirect interaction in the \( z \)-axis decays as \( |h_2^z|^{-i} \) with \( i \geq 2 \). In other words, the indirect interaction in the direction of the local field on spin 2 decays more rapidly.

**Proof.** In order to prove the present theorem, we follow the same calculations as in section 3.2. We break down the Hamiltonian (53) as follows:

\[
H_{\text{tot}} = H_0 + H_1, \quad H_0 = h_2^z \sigma_z^2, \quad H_1 = H_{j,j'} + H_{3,\text{env}}.
\]

We define the ground state of \( H_0 \) as \( |\uparrow_2\rangle \) with the eigenvalue \( -h_2^z \).

First, we prove equation (54). We can calculate the effective Hamiltonian as in equation (36):

\[
H^{\text{eff}} = P_0 H_0 P_0 + P_0 H_1 P_0 + P_0 H_1 Q_0 \frac{1}{-h_2^2 - h_2^z} Q_0 H_1 P_0.
\]

where \( P_0 = |\uparrow_2\rangle \langle \uparrow_2| \otimes I_{1,3,\text{env}} \) with \( I_{1,3,\text{env}} \) the identity matrix of spins 1, 3 and the environment, and \( Q_0 = I_{\text{tot}} - P_0 \). Because \( H_{3,\text{env}} \) commutes with \( P_0 \), we have

\[
H^{\text{eff}} = |\uparrow_2\rangle \langle \uparrow_2| \otimes \left[ -h_2^z + H_{3,\text{env}} \right] + |\uparrow_2\rangle \langle \uparrow_2| \left( H_{j,j'} + H_{j,j'} Q_0 \frac{1}{-h_2^2 - h_2^z} Q_0 H_{j,j'} \right) |\uparrow_2\rangle.
\]

The effective Hamiltonian \( H_{1,3,\text{env}}^{\text{eff}} \) of spins 1, 3 and the environment is thereby given by

\[
H_{1,3,\text{env}}^{\text{eff}} = |\uparrow_2\rangle H^{\text{eff}} |\uparrow_2\rangle.
\]

We can calculate the term \( \langle \uparrow_2| \left( H_{j,j'} + H_{j,j'} Q_0 \frac{1}{-h_2^2 - h_2^z} Q_0 H_{j,j'} \right) |\uparrow_2\rangle \) as follows:

\[
\langle \uparrow_2| H_{j,j'} |\uparrow_2\rangle + \langle \uparrow_2| H_{j,j'} Q_0 \frac{1}{-h_2^2 - h_2^z} Q_0 H_{j,j'} |\uparrow_2\rangle
\]

\[
= (-J^z \sigma_1^z - J^z \sigma_3^z) + \left( -\frac{J^x J^x}{h_2^2} \sigma_1^z - \frac{J^x J^y}{h_2^2} \sigma_3^z - \frac{J^x J^x}{h_2^2} \sigma_1^x \sigma_3^x - \frac{J^x J^y}{h_2^2} \sigma_1^y \sigma_3^y \right).
\]

From the above calculation, we arrive at equation (54) with equation (55) up to the constant component \(-h_2^z\).
Next, we prove equation (56). Assuming that $h_2^z \gg \|H_2^z\|$, we obtain
\[
e^{-\beta H_{\text{tot}}} \frac{1}{Z_{\text{tot}}} = P_0 e^{-\beta H_{\text{tot}}} \frac{1}{Z_{\text{tot}}} P_0 + O(\|h_2^z\|^{-1})(P_0 A_1 Q_0 + Q_0 A_1^T P_0) + O(\|h_2^z\|^{-2}) \tag{61}
\]
in the low-temperature limit as in equation (42), where we define $A_1$ as an $O(1)$ operator of the total system. We then obtain equation (56) from equations (41) and (61). This completes the proof of theorem 2. \qed

Applying theorem 2 to the Hamiltonian (9), we obtain the effective Hamiltonian $H_{\text{tot}}^\text{eff}$ of the form (50). For the Hamiltonian (52), more specifically, the effective Hamiltonian $H_{\text{tot}}^\text{eff}$ is given by
\[
H_{\text{tot}}^\text{eff} = H_{\text{LF}}^\text{eff} + H_{\text{couple}}^\text{eff} + H_{\text{media}}^\text{eff} + H_{\text{media}}^\text{eff}, \tag{62}
\]
where
\[
H_{\text{LF}}^\text{eff} = -\left(J_A^z + \frac{J_A^x J_A^y}{h_2^z}\right)\sigma_1^z - \left(J_B^z + \frac{J_B^x J_B^y}{h_2^z}\right)\sigma_N^z,
\]
\[
H_{\text{couple}}^\text{eff} = -\left(\frac{J_A^x J_A^y}{h_2^z}\right)\sigma_3^x - \frac{J_B^x J_B^y}{h_2^z}\sigma_3^y - \left(\frac{J_B^x J_B^y}{h_2^z}\right)\sigma_{N-2}^x - \left(\frac{J_B^x J_B^y}{h_2^z}\right)\sigma_{N-2}^y,
\]
\[
H_{\text{media}}^\text{eff} = -\left(\frac{J_A^x J_A^y}{h_2^z}\right)\sigma_3^x - \left(\frac{J_B^x J_B^y}{h_2^z}\right)\sigma_{N-2}^y.
\]
We thus achieve $\|H_{\text{couple}}^\text{eff}\| \to 0$ in the limits $|h_2^z| \to \infty$ and $|h_{N-1}^z| \to \infty$. We can choose the local fields $h_1^z$ and $h_N^z$ as
\[
h_1^z = J_A^z + \frac{J_A^x J_A^y}{h_2^z}, \quad h_N^z = J_B^z + \frac{J_B^x J_B^y}{h_{N-1}^z},
\]
so that they may cancel the effective fields $H_{\text{LF}}^\text{eff}$.

### 4.3. Numerical demonstration

In this subsection, we numerically demonstrate the generation of the long-distance entanglement by the use of the local fields for an $XY$ spin chain. We consider the total Hamiltonian
\[
H_{\text{tot}} = (1 + \gamma)(\sigma_1^x \sigma_2^x + \sigma_2^x \sigma_3^x + \sigma_N^x \sigma_{N-1}^x + \sigma_{N-1}^x \sigma_N^x) + (1 - \gamma)(\sigma_1^y \sigma_2^y + \sigma_2^y \sigma_3^y + \sigma_N^y \sigma_{N-1}^y + \sigma_{N-1}^y \sigma_N^y) + \frac{1 - \gamma^2}{h_0}(\sigma_1^z + \sigma_N^z) + h_0(\sigma_2^z + \sigma_{N-1}^z) + H_{\text{media}},
\]
\[
H_{\text{media}} = \sum_{l=3}^{N-3} \left[(1 + \gamma)\sigma_l^x \sigma_{l+1}^x + (1 - \gamma)\sigma_l^y \sigma_{l+1}^y\right] + \frac{1 - \gamma^2}{h_0}(\sigma_3^z + \sigma_{N-2}^z), \tag{65}
\]
where we choose the local fields on spins 2 and $N - 1$ as $h_2^z = h_{N-1}^z = h_0$. We introduced the third term in $H_{\text{tot}}$ according to equation (55) in order to cancel the effective fields on spins 1 and $N$ which are generated after tracing out spins 2 and $N - 1$. We also introduced the second term.
Figure 6. We plot (a) the long-distance entanglement \( C(\rho_{1N}) \) and (b) the first excitation energy \( \delta E_1 \) against the field \( h_0 \) in the three cases \( \gamma = 0 \) (solid line), \( \gamma = 0.03 \) (broken line) and \( \gamma = 0.05 \) (chained line). The fitting line (thin solid line) to the data points shows that the first excitation energy decreases as \( h_0^{-2} \).

in \( H_{\text{media}} \) to cancel the effective fields on the media spins 3 and \( N-2 \) for simplicity. We obtain the effective Hamiltonian \( H_{\text{tot}}^{\text{eff}} \) of the form (62) as

\[
H_{\text{tot}}^{\text{eff}} = -\frac{(1 + \gamma)^2}{h_0} (\sigma_3^x \sigma_3^x + \sigma_{N-2}^x \sigma_N^x) - \frac{(1 - \gamma)^2}{h_0} (\sigma_3^y \sigma_3^y + \sigma_{N-2}^y \sigma_N^y)
+ \sum_{l=3}^{N-3} \left[ (1 + \gamma) \sigma_l^x \sigma_{l+1}^x + (1 - \gamma) \sigma_l^y \sigma_{l+1}^y \right]
\] (66)

in the limit of \( h_0 \to \infty \). The ground state of the \( XY \) spin chain without fields satisfies condition (48) because the Hamiltonian has time-reversal symmetry. Note that the indirect interaction through the \( XY \) spin chain is not a ‘classical’ one. We can therefore expect that long-distance entanglement exists in this system in the limit of \( h_0 \to \infty \).

Let us show the numerical calculation of long-distance entanglement for the system (65) with \( N = 100 \). In figure 6(a), we show the entanglement between spins 1 and \( N \) in the three cases \( \gamma = 0, 0.03 \) and 0.05. We can see that the entanglement \( C(\rho_{1N}) \) monotonically increases with \( h_0 \). As the parameter \( \gamma \) increases, however, we need a greater value of \( h_0 \) to generate a large long-distance entanglement. This is because the first excitation energy of the mediator system decreases rapidly as the parameter \( \gamma \) increases; as is shown in equation (30), we need to attenuate the coupling strength adequately so that it may be much less than the first energy gap of the mediator system. If condition (30) does not hold, the second term on the right-hand side of equation (40) may not be ignored and may decrease the purity of the total system, which causes the destruction of the long-distance entanglement. We show in figure 6(b) the first excitation energy \( \delta E_1 \) of the total system, which decreases as \( h_0^{-2} \) in the limit of \( h_0 \to \infty \); this dependence comes from the fact that the indirect interaction obtained from equation (39) is of the order of \( h_0^{-2} \). This decrease of the excitation energy would make long-distance entanglement fragile to thermal fluctuation.

The value of the long-distance entanglement is almost the same if we consider the total Hamiltonian (66) with \( N = 98 \) from the beginning. We compare the values in table 2 for \( h_0 = 100 \). The difference of \( C(\rho_{1N}) \) is of the order of \( h_0^{-2} \) as is expected from equation (56).
Table 2. Comparison of the entanglement for $h_0 = 100$. The effective Hamiltonian of the total Hamiltonian (65) is equal to the Hamiltonian (66) by the second-order perturbation. The difference between them is of the order of $h_0^{-2} = 10^{-4}$.

| $\gamma$ | Hamiltonian (65) | Hamiltonian (66) |
|---------|----------------|----------------|
| 0       | 0.9832         | 0.9834         |
| 0.03    | 0.6743         | 0.6745         |
| 0.05    | 0.1631         | 0.1632         |

5. Finite coupling

In the previous sections, we considered the weak coupling limit (7) in order to generate long-distance entanglement. By making the coupling strength zero, however, the first excitation energy of the total system vanishes and long-distance entanglement becomes extremely fragile against thermal fluctuation. In order to avoid the situation, we have to achieve long-distance entanglement by as strong a coupling as possible. In the present section, we consider the Hamiltonian (1) in figure 2(a) with a finite coupling Hamiltonian:

$$\|H_{\text{couple}}\| = \text{const} \neq 0.$$  (67)

We focus on the following two points:

1. We look for a mediator system suitable for the generation of long-distance entanglement.
2. We do not necessarily need condition (45) for the generation of long-distance entanglement when the coupling Hamiltonian is finite.

Firstly, in section 5.1, we introduce two examples of the $XY$ spin chains with the external fields in the $z$ direction in order to demonstrate that the external fields on the mediator system can enhance long-distance entanglement with the coupling strength fixed. With a finite coupling Hamiltonian, the preparation of a suitable mediator system is crucial for the generation of long-distance entanglement. For example, we showed in section 4.3 that long-distance entanglement strongly depends on the parameter $\gamma$ if we keep the coupling strength finite. We instead consider here the possibilities that we can enhance the capability of the mediator system with external fields on the mediator system.

Secondly, in section 5.2, we discuss the case in which a random Hamiltonian, which may generate the effective fields (44), is added to the system. There, we tune the coupling strength in order to achieve maximum entanglement and see that the coupling strength should be neither too strong nor too weak. In the limit of weak coupling (7) of this system, long-distance entanglement always vanishes in any quantum system which does not satisfy condition (45). If the coupling has a finite value, however, entanglement may exist in the system without condition (45). As has been analyzed in section 3.2, the effective fields on the probe spins are of first order of the coupling amplitude $O(\|H_{\text{couple}}\|)$. On the other hand, the effective interaction is of the order of $\|H_{\text{couple}}\|^2$, and hence the ratio of the effective fields to the effective interaction is of the order of $\|H_{\text{couple}}\|^{-1}$. We therefore expect that by increasing the coupling strength we can relatively reduce the effective fields and enhance the long-distance entanglement.
Figure 7. The distribution of (a) long-distance entanglement and (b) first excitation energy. We determined each of the external fields \( \{ h_l \}_{l=2}^{99} \) stochastically out of the uniform distribution \([-1.5, 1.5]\), to obtain \( 10^7 \) samples. In (a), the value for the bin from 0 to 0.002 is 0.134, which is out of the range of the plot. The average of the entanglement is 0.162. Without random fields, the value of the long-distance entanglement would be \( C = 1.69 \times 10^{-4} \) and the first excitation energy would be \( \delta E_1 = 1.19 \times 10^{-8} \).

5.1. Enhancement of the long-distance entanglement with the external fields

In the present section, we consider the possibilities that we can enhance the capability of the mediator system to generate the long-distance entanglement by the use of external fields on the mediator system. We consider \( XY \) spin chains with external fields in the \( z \) direction in the following two cases: random fields and uniform fields. Firstly, it has been shown that in the \( XX \) spin chain random fields can enhance entanglement between short-range spin pairs [27]. We thereby expect that a randomness may also enhance long-distance entanglement. Secondly, it is known that the quantum phase transition occurs in \( XY \) spin chains with external fields. Then, we expect that the long-distance entanglement is enhanced around the critical point because of strong quantum fluctuation. We indeed show that long-distance entanglement is highly enhanced in these two cases.

We first give the Hamiltonian

\[
H_{\text{tot}} = H_{\text{media}} + 0.02(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \sigma_{N-1}^x \sigma_N^x + \sigma_{N-1}^y \sigma_N^y),
\]

where

\[
H_{\text{media}} = \sum_{l=2}^{N-2} \left[ \alpha (1 + \gamma) \sigma_l^x \sigma_{l+1}^x + (1 - \gamma) \sigma_l^y \sigma_{l+1}^y \right] + \sum_{l=2}^{N-1} h_l^z \sigma_l^z
\]

and \( \alpha \) is an integer which has the values \( \pm 1 \). The property of the mediator system qualitatively changes depending on the sign \( \alpha \) as is shown in the following. In the present system, the indirect interaction between the spins 1 and \( N \) depends on the external fields \( \{ h_l^z \}_{l=2}^{N-1} \). We then expect the possibilities that \( \{ h_l^z \}_{l=2}^{N-1} \) enhance the generation of long-distance entanglement.

Firstly, we apply the random fields \( \{ h_l^z \}_{l=2}^{N-1} \) distributed uniformly in the range \(-1.5 \leq h_l^z \leq 1.5 \) (\( l = 1, 2, \ldots, N-1 \)). We let \( \alpha = +1, \gamma = 0.1 \) and \( N = 100 \). For \( h_l^z = 0 \) (\( l = 2, \ldots, 99 \)), the long-distance entanglement \( C(\rho_{1N}) \) and the first excitation energy \( \delta E_1 \) are equal to \( 1.69 \times 10^{-4} \) and \( 1.19 \times 10^{-8} \), respectively. In figure 7, we show the distributions of \( C(\rho_{1N}) \) and \( \log_{10} \delta E_1 \) in the case of the random fields. The average of \( C(\rho_{1N}) \) over \( 10^7 \) samples is 0.162; we can see...
Figure 8. The entanglement against $h_0$ (a) for $\gamma = 0.3$ (solid line), $\gamma = 0.5$ (broken line) and $\gamma = 0.7$ (chain line) with $N = 100$ and (b) for $N = 50$ (chain line), $N = 100$ (broken line) and $N = 200$ (solid line) with $\gamma = 0.5$. We apply the uniform fields $h_0$ on the XY spin chain (68). The long-distance entanglement can be enhanced around the point $h_0 = 2\gamma$. As the spin number $N$ increases, the peak approaches the critical point $h_0 = 2\gamma$ and becomes sharp.

that the random fields highly improve the entanglement and the first excitation energy. There are even samples for which the entanglement $C(\rho_{1N})$ is more than 0.95; the largest value of the entanglement among the $10^7$ samples is 0.982. This suggests that we can significantly enhance long-distance entanglement by choosing the external fields optimally.

Secondly, we apply the uniform fields in the case $\alpha = -1$: $h_z^l = h_0$ (70) for $l = 2, \ldots, N - 1$. In this case of $\alpha = -1$, a quantum phase transition occurs at the point $h_0 = 2\gamma$ in the limit of $N \to \infty$ [34]. In figure 8(a), we show the plots of the long-distance entanglement against the parameter $h_0$ for $\gamma = 0.3, 0.5, 0.7$ and $N = 100$. We can see that the entanglement is highly enhanced near the critical point $h_0 = 2\gamma$. In figure 8(b), we show for $\gamma = 0.5$ that the peak becomes sharp and close to the point $h_0 = 2\gamma$ as the spin number $N$ increases from 50 to 200. For $\alpha = 1$, we cannot achieve such enhancement by the external fields even at the critical point.

5.2. Small spin chains with the random Hamiltonian

Next, we consider a quantum system that does not satisfy condition (45). In order to discuss the effect of breaking condition (45), we consider the following spin chain with $N = 8$:

\[ H = \sum_{i=2}^{6} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + J (\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \sigma_7^x \sigma_8^x + \sigma_7^y \sigma_8^y) + \sum_{l=1}^{7} \sum_{i,j=0,1,2,3} \delta J_{l}^{ij} \sigma_i^x \sigma_{i+1}^x, \]  

where $\sigma^0$ denotes the identity operator. We add the random coupling $\{\delta J_{l}^{ij}\}_{i,j=0,1,2,3}$ ($l = 1, 2, \ldots, 7$) to the XX spin chains; we assume that the randomness is uniform in the range $-0.05 \leq \delta J_{l}^{ij} \leq 0.05$ for $i, j = 0, 1, 2, 3$. In this system, condition (45) is not satisfied, and hence the entanglement $C(\rho_{1N})$ is equal to zero in the limit $J \to 0$.

In figures 9(a)–(c), we show the distribution of the entanglement $C(\rho_{1N})$ for $J = 0.05, 0.25$ and 0.5. In table 3, we show the averages of the entanglement in these three cases. We also show
Figure 9. The distribution of the entanglement $C(\rho_{1N})$ (a) for $J = 0.05$, (b) for $J = 0.25$ and (c) for $J = 0.5$ for the Hamiltonian (71). We determined each random coupling $\{\delta J_{i,j}^{l}\}_{i,j=0,x,y,z}$ ($l = 1, 2, \ldots, 7$) stochastically out of the uniform distribution $[-0.05, 0.05]$. For panels (a)–(c), we used 250,000 samples for each case. In panel (d), we plot the average of $C(\rho_{1N})$ for each value of $J$ with 2500 samples for each point to calculate the average of the entanglement.

Table 3. The average values of the entanglement without and with the random Hamiltonian in equation (71).

| $J$     | 0.05 | 0.25 | 0.5  |
|---------|------|------|------|
| Entanglement without random Hamiltonian | 0.973 | 0.551 | 0.102 |
| Entanglement with random Hamiltonian   | 0.03  | 0.27  | 0.07  |

the values of the entanglement without the random Hamiltonian, namely $J_{i,j}^{l} = 0$ for $i, j = 0, x, y, z$ and $l = 1, 2, \ldots, 7$. We can see from table 3 that the destruction of the entanglement by the random Hamiltonian becomes smaller as the coupling parameter $J$ increases. However, too strong a coupling parameter does not give the optimum value of the entanglement, nor does too weak a coupling parameter. If the coupling parameter $J$ is small, the destruction of entanglement by the effective fields is serious. On the other hand, if the coupling parameter $J$ is large, entanglement between the probe spins and the mediator system becomes non-negligible, the purity of the probe spins decreases and hence entanglement is destroyed. The mechanism of entanglement destruction is thus different physically in these two cases. These two effects
compete in giving the optimum value of the coupling parameter. As shown in figure 9(d), we can achieve the maximum average entanglement with the coupling parameter $J \simeq 0.25$.

6. Summary and conclusion

We have analytically and numerically studied the generation of long-distance entanglement by the use of weak coupling (7). We gave the two necessary conditions for the mediator system to generate long-distance entanglement. The first one is that the indirect interaction must not be 'classical' as has been defined in equation (10). The second one is that the effective fields of first order have to vanish as in equation (45). The first condition, in particular, is applicable to any bipartite systems indirectly interacting with each other at any temperature. The second condition can be satisfied artificially by applying the local fields so as to cancel the effective fields. In this sense, we may overcome the constraint of the second condition if we explicitly know the effective fields on the probe spins. We have shown that these two conditions can be satisfied in various systems; this means that many quantum systems have potential for the generation of long-distance entanglement.

Next, we discussed the generation of long-distance entanglement by the use of only external fields. As shown in figure 2(b), we have to control the local fields on the probe spins and the part of the system adjacent to the probe spins. We have to apply strong fields on the adjacent spins and weak fields on the probe spins. Then, we achieved the condition mathematically equivalent to equation (7). The strong fields contribute to attenuation of the coupling between the probe spins and the mediator system, while weak fields are necessary to cancel the effective fields on the probe spins. Because we utilize the fields with finite amplitudes, we can achieve condition (7) by second-order approximation as shown in equation (56), but the degree of accuracy rapidly improves as the amplitude of the strong fields increases. Our result also makes it possible to control the interaction parameter by local fields, and hence it may also be applicable to the accurate control of the interaction.

Finally, we showed the cases where the coupling strength is non-zero. We first discussed the two cases in which the external fields on the mediator system can enhance the generation of long-distance entanglement; we introduced the $XY$ spin chain with uniform fields and random fields with the Hamiltonian (68). Long-distance entanglement is enhanced by random fields on average; there are a few cases in which the entanglement is enhanced to be nearly equal to unity. This means that we can improve the capability of the mediator system to a great extent. Long-distance entanglement is also highly enhanced near the point $h_0 = 2\gamma$, which corresponds to the point of the quantum phase transition in the limit $N \to \infty$. Second, we discuss a case in which the effective fields of first order remain; namely, the $XX$ spin chain with random interactions. In this system, entanglement always vanishes in the limit of weak coupling because condition (45) is not satisfied. Too strong a coupling causes the decrease of the purity because of entanglement with the mediator system, while too weak a coupling increases the relative amplitude of the effective fields to the effective interaction. The optimal coupling strength is determined so as to make both effects minimum. If we choose the coupling strength properly, long-distance entanglement remains to some extent.

In conclusion, we have generally researched necessary conditions for the generation of entanglement. Our results show that the conditions for generation are not so strict in the ground states. For practical application, however, we cannot consider the low-temperature limit because it is not realized experimentally, and hence the amplitude of the first excitation
energy is important. We should keep a certain coupling strength in order to increase the first excitation energy as much as possible; in the limit of weak coupling, the first excitation energy always vanishes. For this purpose, it is essential to find or construct a mediator system best suited to the generation of long-distance entanglement. We present the possibilities that the capability of the mediator system can be improved by external manipulation. However, we do not understand the principle of how to improve the capability of the mediator system. Then, the analysis of the capability of the mediator system will be the next problem for the practical application of long-distance entanglement. There are several trials [10, 13] aimed at finding efficient mediator systems for entanglement generation.

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Appendix. The relation between theorem 1 and quantum discord

In theorem 1, we show a necessary condition for entanglement to be generated via a mediator system. In this section, we answer the following question: if condition (10) is satisfied, can quantum discord still exist? The answer to this question is yes and we show an example in the following.

First, we review the definition of quantum discord [35]. Let us consider a two-spin system with the density matrix \( \rho_{1N} \). The quantum discord \( Q(\rho_{1N}) \) between the spins 1 and \( N \) is defined as follows:

\[
Q(\rho_{1N}) \equiv I(\rho_{1N}) - J(\rho_{1N}),
\]

(A.1)

where \( I(\rho) \) is the quantum mutual information defined by

\[
I(\rho) \equiv S(\rho_{1}) + S(\rho_{N}) - S(\rho_{1N})
\]

(A.2)

with \( S(\rho) \) the von Neumann entropy \( S(\rho) \equiv \text{tr}(\rho \ln \rho) \). On the other hand, \( J(\rho) \) is the optimized classical mutual information, which is the maximum information obtained from the measurement of spins 1 or \( N \), and is defined by

\[
J(\rho) \equiv S(\rho_{N}) - \min_{\Pi_{j}} \sum_{j} p_{j} S(\rho_{N|\Pi_{j}}),
\]

(A.3)

where \( S(\rho_{N}) \) is the initial von Neumann entropy of spin \( N \) and \( \sum_{j} p_{j} S(\rho_{N|\Pi_{j}}) \) is the average of the von Neumann entropy after the measurement of spin 1 in the basis of \( \Pi_{j} \). If the quantum discord (A.1) has a non-zero value, the correlation between these two spins may not be explained by classical theory.

Let us consider the Hamiltonian

\[
H_{tot} = H_{\text{int}} + H_{\text{LF}},
\]

(A.4)

where

\[
H_{\text{int}} = \sigma_{1}^{x} \sigma_{2}^{x} + \sigma_{2}^{x} \sigma_{3}^{x},
\]

\[
H_{\text{LF}} = \sum_{i=x,y,z} \left( h_{i}^{x} \sigma_{1}^{i} + h_{i}^{y} \sigma_{2}^{i} \right).
\]

(A.5)
The Hamiltonian (A.5) satisfies condition (10) as
\[
[H_A(\sigma_1), H_B(\sigma_3)] = 0, \quad H_A = \sigma_1^x \sigma_2^x, \quad H_B = \sigma_2^x \sigma_3^x. \tag{A.6}
\]
Therefore, entanglement can never exist between spins 1 and 3 in the ground state of \(H_{\text{tot}}\) for any values of the local fields \(\{h_1^1, h_3^1\}_{x,y,z}\). Indeed, the density matrix \(\rho_{13}\) is given by
\[
\rho_{13} = \lim_{\beta \to \infty} \frac{\text{tr}_{13}(e^{-\beta H_{\text{tot}}})}{Z_{\text{tot}}(\beta)} = \begin{pmatrix}
0.7286 & 0 & 0 & 0.1250 \\
0 & 0.1250 & 0.1250 & 0 \\
0 & 0.1250 & 0.1250 & 0 \\
0.1250 & 0 & 0 & 0.02145
\end{pmatrix}, \tag{A.7}
\]
for \(h_1^1 = h_3^1 = 1\) and \(h_1^y = h_3^y = h_1^z = h_3^z = 0\). This system has no entanglement.

However, it has non-zero quantum discord. We utilize the criterion in [36] to prove this. First, we separate the density matrix into the following four blocks:
\[
\rho^{11} = \begin{pmatrix}
0.7286 & 0 \\
0 & 0.1250
\end{pmatrix}, \quad \rho^{12} = \begin{pmatrix}
0 & 0.1250 \\
0.1250 & 0
\end{pmatrix},
\]
\[
\rho^{21} = \begin{pmatrix}
0 & 0.1250 \\
0.1250 & 0
\end{pmatrix}, \quad \rho^{22} = \begin{pmatrix}
0.1250 & 0 \\
0 & 0.02145
\end{pmatrix}. \tag{A.8}
\]
A necessary and sufficient condition for zero discord is given by the following two statements:
\[
[\rho^{ij}, (\rho^{ij})^\dagger] = 0 \quad \text{for } i, j = 1, 2 \tag{A.9}
\]
and
\[
[\rho^{ij}, \rho^{i'j'}] = 0 \quad \text{for } i, j, i', j' = 1, 2. \tag{A.10}
\]
The density matrix (A.7) satisfies the first condition (A.9) because it is a real matrix. However, the second (A.10) condition is not satisfied. Indeed,
\[
\rho^{11} \rho^{12} = \begin{pmatrix}
0 & 0.019107 \\
0.019107 & 0
\end{pmatrix}, \quad \rho^{12} \rho^{11} = \begin{pmatrix}
0 & 0.01562 \\
0.01562 & 0
\end{pmatrix}, \tag{A.11}
\]
and we have \(\rho^{11} \rho^{12} \neq \rho^{12} \rho^{11}\). Therefore, there exists quantum discord between the spins 1 and 3. This shows that the condition in theorem 1 is applicable only to the existence of entanglement and not quantum discord. So far, we are not sure whether there exists a condition for the indirect interaction to generate quantum discord.

References

[1] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Rev. Mod. Phys. 81 865
[2] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[3] Bennett C H and DiVincenzo D P 2000 Nature 404 247
[4] Amico L, Fazio R, Osterloh A and Vedral V 2008 Rev. Mod. Phys. 80 517
[5] Arnesen M C, Bose S and Vedral V 2001 Phys. Rev. Lett. 87 017901
[6] Osborne T J and Nielsen M A 2002 Phys. Rev. A 66 032110
[7] Campos Venuti L, Degli Esposti Boschi C and Roncaglia M 2006 Phys. Rev. Lett. 96 247206
[8] Campos Venuti L, Giampaolo S M, Illuminati F and Zanardi P 2007 Phys. Rev. A 76 052328
[9] Ferreira A and Lopes dos Santos J M 2008 Phys. Rev. A 77 034301
[10] Ferreira A and Lopes dos Santos J M 2010 Phys. Rev. A 82 022320

New Journal of Physics 14 (2012) 123032 (http://www.njp.org/)
[11] Giampaolo S M and Illuminati F 2010 New J. Phys. 12 025019
[12] Giampaolo S M and Illuminati F 2009 Phys. Rev. A 80 050301
[13] Gualdi G, Giampaolo S M and Illuminati F 2011 Phys. Rev. Lett. 106 050501
[14] Reslen J and Bose S 2009 Phys. Rev. A 80 012330
[15] Sangchul O, Mark F and Xuedong H 2010 Phys. Rev. B 82 140403
[16] Wichterich H and Bose S 2009 Phys. Rev. A 79 060302
[17] Campos Venuti L, Degli Esposti Boschi C and Roncaglia M 2007 Phys. Rev. Lett. 99 060401
[18] Bloch I 2005 Nature Phys. 1 23
[19] Bloch I 2008 Nature 453 1016
[20] Bloch I, Dalibard J and Zwerger W 2008 Rev. Mod. Phys. 80 885
[21] Gunlycke D, Kendon V M, Vedral V and Bose S 2001 Phys. Rev. A 64 042302
[22] Wang X, Bayat A, Schirmer S G and Bose S 2010 Phys. Rev. A 81 032312
[23] Plastina F and Apollaro T J G 2007 Phys. Rev. Lett. 99 177210
[24] Apollaro T J G, Cuccoli A, Fubini A, Plastina F and Verrucchi P 2008 Phys. Rev. A 77 062314
[25] Giuliano D, Sindona A, Falcone G, Plastina F and Amico L 2010 New J. Phys. 12 025022
[26] Apollaro T J G, Cuccoli A, Di Franco C, Paternostro M, Plastina F and Verrucchi P 2010 New J. Phys. 12 083046
[27] Fujinaga M and Hatano N 2007 J. Phys. Soc. Japan 76 094001
[28] Kuwahara T and Hatano N 2011 Phys. Rev. A 83 062311
[29] Kuwahara T 2012 arXiv:1204.2337 [quant-ph]
[30] Wootters W K 1998 Phys. Rev. Lett. 80 2245
[31] Coffman V, Kundu J and Wootters W K 2000 Phys. Rev. A 61 052306
[32] Dzyaloshinskii I 1958 J. Phys. Chem. Solids 4 241
[33] Moriya T 1960 Phys. Rev. 117 635
[34] Tong P and Zhong M 2001 Physica B 304 91
[35] Ollivier H and Zurek W H 2001 Phys. Rev. Lett. 88 017901
[36] Huang J H, Wang L and Zhu S Y 2011 New J. Phys. 13 063045

New Journal of Physics 14 (2012) 123032 (http://www.njp.org/)