STABILITY AND INDECOMPOSABILITY OF THE REPRESENTATIONS OF QUIVERS OF $A_N$-TYPE

PENGEI HUANG AND ZHI HU

Abstract. In his paper [11], Markus Reineke proposed a conjecture that there exists a stable weight system $\Theta$ for every indecomposable representation of Dynkin type quiver. In this paper, we showed this conjecture is true for quivers of $A_n$-type by combinatorial construction of a special weight system. We also reinterpret this weight system in terms of semi-invariant theory.

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1. Introduction

In his remarkable paper [11] published on Invent. Math. in 2003, Markus Reineke proposed the following conjecture:

**Conjecture 1.1** (Reineke, [11]). *If $Q$ is a quiver of Dynkin type, there exists a weight system $\Theta$ on $Q$ such that the stable representations are precisely the indecomposable ones.*

If this conjecture is true, it will has some valuable applications. For example, it can be used to study the stratification of representation varieties of Dynkin quiver [11], and to study identities between products of quantum dilogarithm series associated with Dynkin quivers [6]$.^1$ In this paper, we will give an elementary proof of Reineke’s conjecture for the quivers of $A_n$-type by combinatorial construction of a special weight system$^2$.

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$^1$ We thank the referee for pointing out this reference to us.

$^2$ From the communications with Prof. Reineke, Prof. Hille and Prof. Juteau, we know that Hille and Juteau have a proof for this conjecture in $A_n$-case, however it is unpublished. Meanwhile, from the report of the referee, we know that in his paper [6], Keller suggests to resolve this conjecture by the method of [4].
On the other hand, Juteau [5] has found some counterexamples for Reineke’s conjecture in the quivers of \( D \)- and \( E \)-type by computer program (unpublished). Therefore, a modified version of original Reineke’s conjecture is proposed as follows\(^3\).

**Conjecture 1.2** (\( = \) Conjecture 3.1). Let \( Q \) be a Dynkin quiver, then the abelian category \( \text{Rep}_k(Q) \) is a maximal stable category.

A natural extension is to consider the Reineke-type conjecture for certain triangulated categories with Bridgeland stability conditions\(^4\).

## 2. Preliminaries

In this section, we collect some basic materials of quiver theory (for more details, see [8]). Throughout the paper, \( k \) is assumed to be a fixed algebraically closed field.

**Definition 2.1.** \( \quad (1) \) A quiver \( Q = (Q_0, Q_1, s, t) \) is a 4-tuple, where
- \( Q_0 \) and \( Q_1 \) are finite sets of vertices and arrows respectively,
- \( s, t : Q_1 \to Q_0 \) map each arrow \( a \in Q_1 \) to its starting vertex \( s(a) \) and terminal vertex \( t(a) \).

(2) Let \( Q = (Q_0, Q_1) \) be a quiver, \( Q' = (Q'_0, Q'_1) \) is called a subquiver if
- \( Q'_0 \subset Q_0 \) and \( Q'_1 \subset Q_1 \),
- \( s(a), t(a) \in Q'_0 \) for all \( a \in Q'_1 \).

In particular, a subquiver \( Q' \) is called a full subquiver if furthermore, \( a \in Q'_1 \) for all \( a \in Q_1 \) satisfying \( s(a) \in Q'_0 \).

(3) A path \( \gamma \) of a quiver \( Q \) is a sequence \( a_1 \cdots a_n (n \geq 1) \) of arrows that satisfies \( s(a_{i+1}) = t(a_i) \) for \( 1 \leq i \leq n - 1 \), and the starting vertex of \( a_1 \) and terminal vertex of \( a_n \) are called starting vertex and terminal vertex of \( \gamma \), respectively.

**Definition 2.2.** \( \quad (1) \) Let \( Q = (Q_0, Q_1) \) be a quiver, a representation of \( Q \) over \( k \) is a pair 
\[ X = \{(X_i)_{i \in Q_0}, (X_a)_{a \in Q_1}\}, \]
where \( (X_i)_{i \in Q_0} \) is a family of finite dimensional \( k \)-vector spaces and \( (X_a)_{a \in Q_1} \) a family of \( k \)-linear maps associated to all arrows, i.e. \( X_a : X_{s(a)} \to X_{t(a)} \).
Define \( d = (d_i)_{i \in Q_0} \in \mathbb{Z}^{[Q_0]} \) for \( d_i = \dim_k X_i \), and call it the dimension vector of \( X \). Let \( X, Y \) be two \( k \)-representations of \( Q \), a morphism \( u : X \to Y \) is a collection of linear maps \( u_i : X_i \to Y_i \) for all \( i \in Q_0 \) such that for each arrow \( a \in Q_1 \), the following diagram commutes:
\[
\begin{array}{ccc}
X_{s(a)} & \xrightarrow{X_a} & X_{t(a)} \\
\downarrow u_{s(a)} & & \downarrow u_{t(a)} \\
Y_{s(a)} & \xrightarrow{Y_a} & Y_{t(a)}
\end{array}
\]

We say the morphism \( u \) is an isomorphism if moreover each \( u_i \) is an isomorphism, and denote it as \( X \cong Y \). We denote by \( \text{Rep}_k(Q) \) the category of representation of \( Q \) over \( k \).

(2) The direct sum \( W = X \oplus Y \) of two representations \( X \) and \( Y \) of \( Q \) is defined by the pair
\[
W = \{(W_i)_{i \in Q_0}, (W_a)_{a \in Q_1}\} = \{(X_i \oplus Y_i)_{i \in Q_0}, (X_a \oplus Y_a)_{a \in Q_1}\},
\]
where each linear map is given by
\[
W_a = X_a \oplus Y_a : X_{s(a)} \oplus Y_{s(a)} \to X_{t(a)} \oplus Y_{t(a)}.
\]

\(^3\)We do not know whether such modified version is true for the Dynkin quivers (even for tame quivers), but so far as we know, Juteau and Hille [5, 3] are attempting to prove it.

\(^4\)Very recently, we note that the authors of [7] prove that for the derived category \( D^b(Q) \) of a Dynkin quiver \( Q \), there exists a Bridgeland stability condition \( \sigma \) such that for an object \( E \in D^b(Q) \) the following are equivalent: i) \( E \) is indecomposable, ii) \( E \) is exceptional, iii) \( E \) is \( \sigma \)-stable.
A representation \( W \) of \( Q \) is said to be *decomposable* if there exist non-zero representations \( X \) and \( Y \) such that \( W \cong X \oplus Y \), otherwise it is said to be *indecomposable*.

(3) Let \( X \) and \( Y \) be two representations of \( Q \). \( X \) is said to be a *subrepresentation* \( Y \) if \( X_i \subseteq Y_i \) for all \( i \in Q_0 \) and \( X_a = Y_a|_{X_{s(a)}} : X_{s(a)} \to X_{t(a)} \) for all \( a \in Q_1 \). A representation is called *simple* if it has no proper non-zero subrepresentations, and called *semisimple* if it is the direct sum of simple representations.

(4) We say a representation \( X \) of \( Q \) is *thin* if \( \dim_k(X_i) \leq 1 \) for all \( i \in Q_0 \), that is, if each linear space \( X_i \) is either 0 or \( k \).

Given a quiver \( Q \), an important aim of representation theory is to classify all representations and all morphisms up to isomorphism. Krull-Schmidt theorem makes this classification problem easier, it states that every representation of a given quiver can be uniquely decomposed into indecomposable representations up to ordering, so we only need to classify the indecomposable representations. A quiver \( Q \) is said to be of finite type if it has finitely many isomorphism classes of indecomposable representations. Gabriel’s classification theorem states that for a connected quiver \( Q \) without oriented cycles the following are equivalent

- \( Q \) is of finite type,
- the underlying graph of \( Q \) is a simply laced Dynkin diagram, namely one of the followings

\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \\
\bullet \quad \bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\end{array}
\]

- the quadratic form

\[
q_Q(\alpha) = \sum_{i \in Q_0} \alpha_i^2 - \sum_{a \in Q_1} \alpha_{s(a)} \alpha_{t(a)},
\]

where \( \alpha \in \mathbb{Z}^{Q_0} \), is positive definite.

Moreover one has the following bijective correspondence:

| isomorphism classes of indecomposable representations \( X \) | \( \Downarrow \) | positives roots of the quadratic form \( q_Q \) | \( \Downarrow \) | noninitial cluster variables \( c_X \) |

For a quiver \( Q \), we express the underlying graph \( \Gamma_Q \) as a binary set \( \Gamma_Q =: \{(Q_0, Q'_1)\} \) of vertices and edges, where \( Q'_1 \) is obtained by taking all arrows in \( Q_1 \) as edges. For any edge \( l \in Q'_1 \), it
corresponds to a unique arrow \( a \in Q_1 \), we denote \( s(l) = s(a) \) and \( t(l) = t(a) \) the starting vertex and terminal vertex of \( l \), respectively.

**Definition 2.3.** The **support** of \( X \) is a subset of \( \Gamma_Q \) consisting of all vertices \( i \) with the assigning linear space \( X_i \neq 0 \) and all edges connecting these vertices, that is:

\[
\text{supp}(X) := \{(i,\tilde{Q}_0,\tilde{Q}_1)|i \in \tilde{Q}_0 \text{ if } X_i \neq 0; l \in \tilde{Q}_1 \text{ if } s(l), t(l) \in \tilde{Q}_0 \} \subset \Gamma_Q.
\]

The **support quiver** \( \text{supp}_X(Q) \) is given by recovering the arrows of all edges in \( \text{supp}(X) \), that is, given by the vertices \( i \) with \( X_i \neq 0 \) and the arrows \( a \) with \( X_a \neq 0 \).

The following facts are very obvious.

**Fact 2.4.** Let \( Q \) be a quiver, then

1. If \( X \) is an indecomposable representation, then its support quiver \( \text{supp}_X(Q) \) is connected.
2. If two representations \( X \) and \( Y \) are isomorphic, then \( \text{supp}_X(Q) = \text{supp}_Y(Q) \).
3. Let \( X \) be a thin representation of \( Q \). Then \( X \) is indecomposable if and only if the support quiver \( \text{supp}_X(Q) \) is connected.

**Definition 2.5 ([2]).** (1) Let \( A \) be a category, and \( w, r \) be two functions on \( A \), called **weight function** and **rank function** respectively, such that \( r(X) \neq 0 \) for any nonzero object in \( A \). A object \( X \in A \) is called \((w, r)\)-stable (respectively, \((w, r)\)-semistable) if for any nonzero subobject \( U \) of \( X \), we have \( \mu(U) < \mu(X) \) (respectively, \( \mu(U) \leq \mu(X) \)), where the slope function \( \mu(X) \) is defined by \( \mu(X) = \frac{w(X)}{r(X)} \).

2. Let \( w, r \) be weight function and rank function on a category \( A \) respectively, all \((w, r)\)-stable (respectively, \((w, r)\)-semistable) objects of \( A \) form a full subcategory of \( A \), called \((w, r)\)-stable (respectively, \((w, r)\)-semistable) subcategory. Two pairs \((w, r)\) and \((w', r')\) is called **stable-equivalent** (respectively, **semistable-equivalent**) if they induce the same stable (respectively, semistable) subcategories.

3. Let \( A \) be an abelian category, if there exist additive weight function \( w \) and rank function \( r \) on \( A \) such that \((w, r)\)-stable subcategory consists of all indecomposable objects, we call \( A \) be a **maximal stable category**.

### 3. Reineke’s Conjecture for Quivers of \( A_n \)-Type

**Conjecture 3.1** (Modified Reineke’s conjecture). Let \( Q \) be a Dynkin quiver, then the abelian category \( \text{Rep}_k(Q) \) is a maximal stable category.

In this section, we confirm the above conjecture for quivers of type \( A_n \). Namely, we will prove the following main theorem.

**Theorem 3.2.** If \( Q \) is a quiver of type \( A_n \), then there exists a weight system \( \Theta = (\theta_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|} \) such that the stable representations with respect to the weight function \( w(X) = \sum_{i \in Q_0} \theta_i \dim X_i \) and rank function \( r(X) = \sum_{i \in Q_0} \dim X_i \) are precisely the indecomposables, namely \( \text{Rep}_k(Q) \) is a maximal stable category.

#### 3.1. Intrinsic Weight System

Let \( Q \) be a quiver of type \( A_n \), we put it horizontally and fix a reference direction from left to right so that assign numbers \( 1, \ldots, n \) to the vertices of \( Q \) along the reference direction, then we can classify all vertices into the following four types according to the directions of arrows attached to them:

- a vertex \( i \in Q_0 \) is called of type I if it is only as the starting vertex of arrows linking to it;
- a vertex \( i \in Q_0 \) is called of type II if it is only as the terminal vertex of arrows linking to it;
• a vertex \( i \in Q_0 \) is called of type III if there is a path with the reference direction such that \( i \) is a vertex but neither a staring nor a terminal one of that path;

• a vertex \( i \in Q_0 \) is called of type IV if there is a path with the direction opposite to the reference direction such that \( i \) is a vertex but neither a staring nor a terminal one of that path.

Now we define a weight system \( \Theta = \{ \theta_i \}_{i \in Q_0} \) according to the type of each vertex as follows:

\[
\theta_i = \begin{cases} 
  l_i + r_i + 2l_ir_i, & \text{if } i \text{ is a vertex of type I;} \\
  -l_i - r_i - 2l_ir_i, & \text{if } i \text{ is a vertex of type II;} \\
  r_i - l_i, & \text{if } i \text{ is a vertex of type III;} \\
  l_i - r_i, & \text{if } i \text{ is a vertex of type IV}
\end{cases}
\] (3.1)

where \( l_i \) and \( r_i \) stands for the number of vertices on the left and right of the vertex \( i \), respectively. Such weight system is called the intrinsic weight system, and the corresponding weight function \( w_\Theta(X) = \sum_{i \in Q_0} \theta_i \dim X_i \) for a representation \( X \) of \( Q \) is called the intrinsic weight function.

**Example 3.3.** The following quiver of \( A_n \)-type

\[
\begin{array}{ccccccc}
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 & \leftarrow & 5 & \leftarrow & 6 & \rightarrow & 7,
\end{array}
\] (3.2)

has four types of vertices:

\[
\begin{align*}
I : & \quad 1, 6, \\
II : & \quad 4, 7, \\
III : & \quad 2, 3, \\
IV : & \quad 5,
\end{align*}
\]

Then the intrinsic weight system \( \Theta \) is given by \( \theta_1 = 6, \theta_2 = 4, \theta_3 = 2, \theta_4 = -24, \theta_5 = 2, \theta_6 = 16, \theta_7 = -6 \).

**Lemma 3.4.** Let \( Q = \{ Q_0, Q_1 \} \) be a quiver of \( A_n \)-type with the intrinsic weight system \( \Theta = \{ \theta_i \}_{i \in Q_0} \). Then

1. along any path, the weights at the vertices contained in the path decrease.
2. the sum \( \sum_{i \in Q_0} \theta_i \) of weights at all vertices is exactly zero.

**Proof.** (1) We just need to show along each arrow \( a \in Q_1 \), the weight decreases, i.e., \( \theta_{s(a)} > \theta_{t(a)} \) for each arrow \( a \in Q_1 \). We can draw the quiver \( Q \) as follows:

\[
\begin{array}{ccccccc}
\cdots & \bullet & \xrightarrow{a} & \bullet & \xrightarrow{a} & \bullet & \cdots
\end{array}
\]

if we consider the two vertices and arrows closed to \( \begin{array}{ccccccc}
\bullet & \leftarrow & \bullet & \rightarrow & \cdots
\end{array} \). then we have the following four cases:

(i) \( Q : \begin{array}{ccccccc}
\cdots & \bullet & \xrightarrow{a} & \bullet & \xrightarrow{a} & \bullet & \cdots
\end{array} \), in this case, we have

\[
\theta_{s(a)} = r_s(a) - l_s(a), \quad \theta_{t(a)} = -l_{t(a)} - r_{t(a)};
\]

(ii) \( Q : \begin{array}{ccccccc}
\cdots & \bullet & \xrightarrow{a} & \bullet & \xrightarrow{a} & \bullet & \cdots
\end{array} \), in this case, we have

\[
\theta_{s(a)} = l_s(a) + r_s(a) + 2l_s(a) \cdot r_s(a), \quad \theta_{t(a)} = r_{t(a)} - l_{t(a)};
\]

(iii) \( Q : \begin{array}{ccccccc}
\cdots & \bullet & \xrightarrow{a} & \bullet & \xrightarrow{a} & \bullet & \cdots
\end{array} \), in this case, we have

\[
\theta_{s(a)} = r_s(a) - l_s(a), \quad \theta_{t(a)} = r_{t(a)} - l_{t(a)};
\]
Then for each vertex \( i \), in this case, we have
\[
\theta_{s(a)} = l_{s(a)} + r_{s(a)} + 2l_{s(a)} \cdot r_{s(a)}, \quad \theta_{t(a)} = -l_{t(a)} - r_{t(a)} - 2l_{t(a)} \cdot r_{t(a)},
\]
Since \( r_{s(a)} > r_{t(a)} \geq 0 \) and \( 0 \leq l_{s(a)} < l_{t(a)} \), in each case, we have \( \theta_{s(a)} > \theta_{t(a)} \).

(2) For any subquiver \( Q' = \{Q'_0, Q'_1\} \) of \( Q \), we give each vertex \( i \in Q'_0 \) a new weight:
\[
\theta^Q_i = \#\{a \mid s(a) = i, \ a \in Q'_0\} - \#\{a \mid t(a) = i, \ a \in Q'_1\} \in \{\pm2, 0\},
\]
the difference of numbers of arrows in \( Q'_1 \) starting at \( i \) and numbers of arrows in \( Q'_1 \) terminating at \( i \), this construction immediately gives
\[
\sum_{i \in Q'_0} \theta^Q_i = 0.
\]
Then for each vertex \( i \in Q_0 \), its weight \( \theta_i \) is the sum of all such new weights \( \theta^Q_i \) for the connected subquiver \( Q' \) contains \( i \):
\[
\theta_i = \sum_{Q' \subset Q \text{ connected subquiver} \atop i \in Q'_0} \theta^Q_i.
\]

If \( i \) is of type I, then near the vertex \( i \), the quiver locally looks like \( \leftarrow \bullet \rightarrow \), all subquivers contain \( i \) can be divided into three classes:

(i) \( \cdots \leftarrow \bullet \rightarrow \),

(ii) \( \bullet \rightarrow \cdots \),

(iii) \( \cdots \leftarrow \bullet \rightarrow \cdots \).

The first class has \( l_i \) subquivers, for each subquiver \( Q' \), we have \( \theta^Q_i = 1 \); the second class has \( r_i \) subquivers, for each subquiver \( Q' \), we have \( \theta^Q_i = 1 \); the third class has \( l_i r_i \) subquivers, for each subquiver \( Q' \), we have \( \theta^Q_i = 2 \). Consequently,
\[
\sum_{Q' \subset Q \text{ connected subquiver} \atop i \in Q'_0} \theta^Q_i = l_i + r_i + 2l_i r_i = \theta_i.
\]

If \( i \) is of type III, then near the vertex \( i \), the quiver locally looks like \( \rightarrow \bullet \rightarrow \), all subquivers contain \( i \) can be divided into three classes:

(i) \( \rightarrow \bullet \rightarrow \),

(ii) \( \bullet \rightarrow \cdots \),

(iii) \( \cdots \rightarrow \bullet \rightarrow \cdots \).

The first class has \( l_i \) subquivers, for each subquiver \( Q' \), we have \( \theta^Q_i = -1 \); the second class has \( r_i \) subquivers, for each subquiver \( Q' \), we have \( \theta^Q_i = 1 \); the third class has \( l_i r_i \) subquivers, for each subquiver \( Q' \), we have \( \theta^Q_i = 0 \). Hence,
\[
\sum_{Q' \subset Q \text{ connected subquiver} \atop i \in Q'_0} \theta^Q_i = r_i - l_i = \theta_i.
\]
The cases of type II and IV are similar.
Therefore, the sum of all weights is calculated as
\[
\sum_{i \in Q_0} \theta_i = \sum_{i \in Q_0} \left( \sum_{Q' \subset Q \text{ connected subquiver}} \theta_{Q'} \right) = \sum_{Q' \subset Q \text{ connected subquiver}} \left( \sum_{i \in Q'_0} \theta_{Q'} \right) = 0.
\]
We complete the proof. \(\square\)

We denote the following indecomposable thin representation (the orientation in the graph is just an example)
\[
0 \to \cdots \to k \overset{p}{\to} \cdots \overset{1}{\to} k \overset{q}{\to} 0 \to \cdots \to 0,
\]
where \(1 \leq p \leq q \leq n\), by \(I_{p,q}\). Then the indecomposable representations of \(Q\) are classified by \(I_{p,q}\)'s, more precisely, we have

**Proposition 3.5** ([9, 10]). Let \(X\) be a representation of a quiver \(Q\) of type \(A_n\). Then \(X\) is indecomposable if and only if \(X\) is a thin representation whose support quiver is connected, that is, \(X\) is isomorphic to some \(I_{p,q}\).

**Lemma 3.6.** Let \(Q\) be a quiver of \(A_n\)-type, and \(X\) is the indecomposable representation of type \(I_{1,n}\), then \(X\) is stable with respect to the intrinsic weight function and rank function.

**Proof.** To show the stability, we only need to prove the intrinsic weight function \(w_{\Theta}(X')\) on any proper subrepresentation \(X'\) of \(I_{1,n}\) is negative. Obviously, the support quiver \(Q' = \text{supp}_{X'}(Q)\) is a proper full subquiver of \(Q\). We first assume \(Q'\) is connected, then \(Q\) must look like as follows:

\[
Q : \cdots \to Q' \leftarrow \cdots,
\]
and denoting the two vertices of the boundary of \(Q'\) by \(s(Q')\) and \(t(Q')\), one draws \(Q\) as follows:

\[
Q : \cdots \to s(Q') \cdots \leftarrow t(Q') \leftarrow \cdots.
\]
Let \(l_{Q'}\) and \(r_{Q'}\) denote the number of vertices on the left of the whole \(Q'\) and the number of vertices on the right of the whole \(Q'\), respectively.

To compute the weight function \(w_{\Theta}(X')\), we first separate \(Q'\) from \(Q\), and view \(Q'\) as an independent quiver. Then \(Q'\) carries a weight system \(\Theta'\), called the independent weight system, given by the manner described previously so that the sum denoted by \(\theta_{\text{ind}}(Q')\) of the weights belong to the independent weight system is zero. By Lemma 3.4, the actual weight function \(w_{\Theta}(X')\) is the sum of \(\theta_{\text{ind}}(Q')\) and \(\theta_{\text{add}}(Q')\), where \(\theta_{\text{add}}(Q')\) is the sum of the added new weights at the vertices in \(Q'_0\) caused by the connected subquivers containing not only vertices in \(Q'_0\) but also in \(Q_0\setminus Q'_0\). Such connected quivers are divided into three cases:

(i) the considered connected subquiver (inside the box) contains vertices in \(Q'\) and some vertices only on the right of \(Q'\), like the following:

\[
\cdots \to s(Q') \quad \cdots \leftarrow t(Q') \leftarrow \cdots,
\]

(ii) the considered connected subquiver (inside the box) contains vertices in \(Q'\) and some vertices only on the left of \(Q'\), like the following:

\[
\cdots \cdot s(Q') \quad \cdots \leftarrow t(Q') \leftarrow \cdots,
\]

(iii) the considered connected subquiver (inside the box) contains the whole \(Q'\) and some vertices both on the right and on the left of \(Q'\), like the following:

\[
\cdots \to s(Q') \quad \cdots \leftarrow t(Q') \leftarrow \cdots.
\]
The first case includes \( r_{Q'} \cdot |Q'_0| \) choices. A key observation is that each choice contributes a term \(-1\) to the sum of weights at the vertices of \( Q' \). Indeed, let \( \hat{Q} \) be a such connected subquiver which produce new weight for the vertices in \( \hat{Q}_0 \) as in the proof of Lemma 3.4

\[
\theta^\hat{Q} = \#\{a | s(a) = i, \ a \in \hat{Q}_1\} - \#\{a | t(a) = i, \ a \in \hat{Q}_1\}.
\]

Then once we compute the sum \( \sum_{i\in \hat{Q}_0 \cap Q_0} \theta^\hat{Q} \), the inner arrows of \( \hat{Q} \cap Q' \) do no work, only the arrow closest attaching to \( \hat{Q} \cap Q' \) has effect by providing one term \(-1\) in the sum. Similarly, the second case admits \( l_{Q'} \cdot |Q'_0| \) choices, and each case contributes a term \(-1\) to the sum; the third case contains \( l_{Q'} \cdot r_{Q'} \) choices, and each case contributes a term \(-2\) to the sum. Finally we reach

\[
\theta_{\text{add}}(Q') = -r_{Q'} \cdot |Q'_0| - l_{Q'} \cdot |Q'_0| - 2r_{Q'} \cdot l_{Q'}.
\]

hence the weight function \( w_\Theta(X') \) is given by

\[
\theta(Q') = \theta_{\text{ind}}(Q') + \theta_{\text{add}}(Q') = -r_{Q'} \cdot |Q'_0| - l_{Q'} \cdot |Q'_0| - 2r_{Q'} \cdot l_{Q'} < 0.
\]

If \( Q' \) is not connected, we denote its connected components as \( Q^1, Q^2, \cdots, Q^s \) which correspond to the direct summand \( X^i \) of the representation \( X' \), then

\[
w_\Theta(X') = \sum_{i=1}^{s} w_\Theta(X^i).
\]

For each summand we have shown it is negative. \( \square \)

3.2. Proof of the Main Theorem. We complete our proof of the main theorem by the following lemma.

Lemma 3.7. Let \( Q \) be a quiver of \( A_n \)-type, then every indecomposable representation is stable with respect to the intrinsic weight function and rank function.

Proof. Let \( X \) be an indecomposable representation of \( Q \) and \( X' \subset X \) be any proper subrepresentation. Now \( X \) must be of type \( I_{p,q} \) with support quiver \( Q^X := \text{supp}_X(Q) \) connected, and the support quiver \( Q^{X'} := \text{supp}_{X'}(Q) \) of \( X' \) is a proper full subquiver of \( Q^X \). Therefore our aim is to prove the following inequality holds for any proper full subquiver \( Q^{X'} \) of \( Q^X \):

\[
\frac{w_\Theta(X')}{|Q^{X'}_0|} < \frac{w_\Theta(X)}{|Q^X_0|}.
\] (3.3)

Let \( Q^{X'} \) has \( s \) connected components \( Q^1, \cdots, Q^s \), clearly each \( Q^i \) is a proper full subquiver of \( Q^X \). To calculate the total weights \( w_\Theta(X) \) and \( w_\Theta(X') \), similar as the proof of Lemma 3.6, we first separate \( Q^X \) from the whole quiver \( Q \) to get the sums \( \theta_{\text{ind}}(Q^X) \) (= 0) and \( \theta_{\text{add}}(Q^{X'}) \) (< 0) coming from the independent weight system on \( Q^X \). Secondly, we calculate the sums \( \theta_{\text{add}}(Q^X) \) and \( \theta_{\text{add}}(Q^{X'}) \) when the rest parts of \( Q \) are considered.

According to the relation of \( Q^i \) and \( Q^{X'} \), we can divide our consideration into three different big cases, and when we take the rest part of \( Q \) into account, each big case can be divided into four different small cases.

Case I: all \( Q^i \) are in the interior of \( Q^X \), illustrated as follows:

\[
\begin{array}{cccccc}
\cdots & \rightarrow & Q^1 & \rightarrow & \cdots & \rightarrow \end{array}
\]

\[
\begin{array}{cccccc}
\cdots & \rightarrow & Q^2 & \rightarrow & \cdots & \rightarrow \end{array}
\]

\[
\begin{array}{cccccc}
\cdots & \rightarrow & Q^s & \rightarrow & \cdots & \rightarrow \end{array}
\]
STABILITY AND INDECOMPOSABILITY OF THE REPRESENTATIONS OF QUIVERS OF $A_N$-TYPE

Case II: there is a full subquiver of $Q^X$ (without loss of generality, assumed to be $Q^s$) that shares one boundary vertex with $Q^X$, illustrated as follows:

\[ \cdots \rightarrow Q^1 \leftarrow \cdots \rightarrow Q^2 \leftarrow \cdots \rightarrow Q^s \leftarrow \cdots \rightarrow Q^* \]

Case III: there are two full subquivers of $Q^X$ (assumed to be $Q^1$ and $Q^s$) each of which has one boundary vertex coincides with that of $Q^X$, illustrated as follows:

\[ Q^1 \leftarrow \cdots \rightarrow Q^2 \leftarrow \cdots \rightarrow Q^s \leftarrow \cdots \rightarrow Q^* \]

We first consider the Case I, when we added the rest of $Q$, there are following four different cases:

(a) the two arrows near $Q^X$ both point into $Q^X$: \[ \cdots \rightarrow \cdots \rightarrow Q^1 \leftarrow \cdots \rightarrow Q^2 \leftarrow \cdots \rightarrow Q^s \leftarrow \cdots \rightarrow Q^* \]

(b) the left arrow near $Q^X$ is out from $Q^X$ and the right one points into $Q^X$: \[ \cdots \leftarrow \cdots \rightarrow Q^1 \leftarrow \cdots \rightarrow Q^2 \leftarrow \cdots \rightarrow Q^s \leftarrow \cdots \rightarrow Q^* \]

(c) the two arrows near $Q^X$ are both out from $Q^X$: \[ \cdots \leftarrow \cdots \rightarrow Q^1 \leftarrow \cdots \rightarrow Q^2 \leftarrow \cdots \rightarrow Q^s \leftarrow \cdots \rightarrow Q^* \]

(d) the left arrow near $Q^X$ points into $Q^X$ and the right one is out from $Q^X$: \[ \cdots \rightarrow \cdots \rightarrow Q^1 \leftarrow \cdots \rightarrow Q^2 \leftarrow \cdots \rightarrow Q^s \leftarrow \cdots \rightarrow Q^* \]

No matter in what case, the slope of $X'$ is the same, to show $\mu(X') < \mu(X)$, we just need to consider the case (a), since the value of $\mu(X)$ is the minimum among these cases. Now we have

\[
\mu(X') = \frac{w_\Theta(X')}{|Q^X_0|} = \frac{\sum_{i=1}^s w_\Theta(X^i)}{\sum_{i=1}^s |Q^i_0|} = \frac{\sum_{i=1}^s \theta_{\text{ind}}(Q^i) + \sum_{i=1}^s \theta_{\text{add}}(Q^i)}{\sum_{i=1}^s |Q^i_0|},
\]

\[
\mu(X) = \frac{w_\Theta(X)}{|Q^X_0|} = \frac{\theta_{\text{ind}}(Q^X) + \theta_{\text{add}}(Q^X)}{|Q^X_0|},
\]

where

\[
\theta_{\text{add}}(Q^X) = -|Q^X_0|(l_{Q^X} + r_{Q^X}) - 2l_{Q^X} \cdot r_{Q^X},
\]

\[
\theta_{\text{add}}(Q^i) = -|Q^i_0|(l_{Q^i} + r_{Q^i}) - 2l_{Q^i} \cdot r_{Q^i} - 2l_{Q^i} \cdot r_{Q^X} + 2l_{Q^X} \cdot r_{Q^i}
\]

\[
= -|Q^i_0|(l_{Q^i} + r_{Q^i}) - 2l_{Q^i} \cdot (r_{Q^i} - r_{Q^X}) - 2l_{Q^i} \cdot r_{Q^X}, \quad 1 \leq i \leq s,
\]

thus

\[
\sum_{i=1}^s \theta_{\text{add}}(Q^i) = -\sum_{i=1}^s |Q^i_0| (l_{Q^i} + r_{Q^i}) - 2l_{Q^i} \cdot \sum_{i=1}^s (r_{Q^i} - r_{Q^X}) - 2 \sum_{i=1}^s l_{Q^i} \cdot r_{Q^X}.
\]
Then the inequality (3.3) reads
\[
\frac{\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i) - (\sum_{i=1}^{s} |Q^i_0|)(l_{Q^X} + r_{Q^X}) - 2l_{Q^X} \cdot \sum_{i=1}^{s} (r_{Q^i} - r_{Q^X}) - 2 \sum_{i=1}^{s} l_{Q^i} \cdot r_{Q^X}}{\sum_{i=1}^{s} |Q^i_0|} < \frac{-|Q^X_0|(l_{Q^X} + r_{Q^X}) - 2l_{Q^X} \cdot r_{Q^X}}{|Q^X_0|},
\]
or reads
\[
\frac{\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i)}{\sum_{i=1}^{s} |Q^i_0|} < \frac{2l_{Q^X} \cdot \sum_{i=1}^{s} (r_{Q^i} - r_{Q^X})}{\sum_{i=1}^{s} |Q^i_0|} + 2 \sum_{i=1}^{s} l_{Q^i} \cdot r_{Q^X} \left(\frac{1}{\sum_{i=1}^{s} |Q^i_0|} - \frac{1}{|Q^X_0|}\right). \tag{3.4}
\]

Since \(\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i) < 0\), \(l_{Q^X} \geq 0\), \(r_{Q^X} \geq 0\), \(r_{Q^i} > r_{Q^X}\) and \(\sum_{i=1}^{s} |Q^i_0| < |Q^X_0|\), the equality (3.4) holds.

- For the Case II, as the analysis process in Case I, if, we also have the following four different cases when the rest of \(Q\) is added:

\[
\begin{array}{l}
(a) \quad \cdots \rightarrow Q^1 \leftarrow \cdots \rightarrow Q^2 \leftarrow \cdots \rightarrow Q^s \leftarrow \cdots,
(b) \quad \cdots \leftarrow Q^1 \leftarrow \cdots \rightarrow Q^2 \leftarrow \cdots \rightarrow Q^s \leftarrow \cdots,
(c) \quad \cdots \leftarrow Q^1 \leftarrow \cdots \rightarrow Q^2 \leftarrow \cdots \rightarrow Q^s \rightarrow \cdots,
(d) \quad \cdots \rightarrow Q^1 \leftarrow \cdots \rightarrow Q^2 \leftarrow \cdots \rightarrow Q^s \rightarrow \cdots .
\end{array}
\]

Note that in (a) and (b), \(\mu(X')\) is the same, however, \(\mu(X)\) is smaller in (a), so we just need to show \(\mu(X') < \mu(X)\) in (a). In (c) and (d), \(\mu(X')\) is the same, however, \(\mu(X)\) is smaller in (d), so we just need to show \(\mu(X') < \mu(X)\) in (d).

For the case (a), we have
\[
\begin{align*}
\theta_{\text{add}}(Q^X) &= -|Q^X_0|(l_{Q^X} + r_{Q^X}) - 2l_{Q^X} \cdot r_{Q^X}, \\
\theta_{\text{add}}(Q^i) &= -|Q^i_0|(l_{Q^X} + r_{Q^X}) - 2l_{Q^X} \cdot (r_{Q^i} - r_{Q^X}) - 2l_{Q^i} \cdot r_{Q^X}, \quad 1 \leq i \leq s - 1, \\
\theta_{\text{add}}(Q^s) &= -|Q^s_0|(l_{Q^X} + r_{Q^X}) - 2l_{Q^s} \cdot r_{Q^X} - 2l_{Q^X} \cdot r_{Q^s} + 2l_{Q^X} \cdot r_{Q^X} \\
&= -|Q^s_0|(l_{Q^X} + r_{Q^X}) - 2l_{Q^s} \cdot r_{Q^X},
\end{align*}
\]
thus the inequality (3.3) reads
\[
\frac{\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i) - (\sum_{i=1}^{s} |Q^i_0|)(l_{Q^X} + r_{Q^X}) - 2l_{Q^X} \cdot \sum_{i=1}^{s} (r_{Q^i} - r_{Q^X}) - 2 \sum_{i=1}^{s} l_{Q^i} \cdot r_{Q^X}}{\sum_{i=1}^{s} |Q^i_0|} < \frac{-|Q^X_0|(l_{Q^X} + r_{Q^X}) - 2l_{Q^X} \cdot r_{Q^X}}{|Q^X_0|},
\]
or reads
\[
\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i) \cdot \frac{2l_{Q^X} \cdot \sum_{i=1}^{s} (r_{Q^i} - r_{Q^X}) + 2 \sum_{i=1}^{s} l_{Q^i} \cdot r_{Q^X}}{|Q_0^X|} < \frac{2l_{Q^X} \cdot r_{Q^X}}{|Q_0^X|}.
\] (3.5)

This inequality is the same as (3.4), thus holds true.

For the case \((d)\), we have
\[
\theta_{\text{add}}(Q^X) = |Q_0^X| (r_{Q^X} - l_{Q^X}),
\]
\[
\theta_{\text{add}}(Q^i) = -|Q_0^i| (l_{Q^X} + r_{Q^X}) - 2l_{Q^X} \cdot (r_{Q^i} - r_{Q^X}) - 2l_{Q^i} \cdot r_{Q^X}, \quad 1 \leq i \leq s - 1,
\]
\[
\theta_{\text{add}}(Q^s) = -|Q_0^s| (l_{Q^X} - r_{Q^X})
\]
\[-|Q_0^s| (l_{Q^X} + r_{Q^X}) + 2|Q_0^s| \cdot r_{Q^X},
\]
thus the inequality (3.3) reads
\[
\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i) - (\sum_{i=1}^{s} |Q_0^i|) (l_{Q^X} + r_{Q^X}) - 2l_{Q^X} \cdot \sum_{i=1}^{s} (r_{Q^i} - r_{Q^X}) - 2 \sum_{i=1}^{s} l_{Q^i} \cdot r_{Q^X} + 2|Q_0^s| \cdot r_{Q^X}
\]
\[
< \frac{|Q_0^X| (r_{Q^X} - l_{Q^X})}{|Q_0^X|},
\]
or reads
\[
\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i) \cdot \frac{2|Q_0^s| \cdot r_{Q^X}}{|Q_0^X|} < 2r_{Q^X} - \frac{2l_{Q^X} \cdot \sum_{i=1}^{s} (r_{Q^i} - r_{Q^X}) + 2 \sum_{i=1}^{s} l_{Q^i} \cdot r_{Q^X}}{|Q_0^X|},
\] (3.6)

which is true due to again \(\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i) < 0\), \(l_{Q^X} \geq 0\), \(r_{Q^X} \geq 0\), \(r_{Q^i} > r_{Q^X}\) and \(\sum_{i=1}^{s} |Q_0^i| < |Q_0^X|\).

- Last we consider the Case III. The subcases (a) and (d) are similar to the cases I-(a) and II-(d) respectively.

For the case \((b)\)
\[
\begin{array}{c}
\cdots \leftarrow [Q^1] \leftarrow \cdots \rightarrow [Q^2] \leftarrow \cdots \cdots \rightarrow [Q^s] \leftarrow \cdots ,
\end{array}
\]
we have
\[
\theta_{\text{add}}(Q^X) = |Q_0^X| (l_{Q^X} - r_{Q^X}),
\]
\[
\theta_{\text{add}}(Q^i) = |Q_0^i| (l_{Q^X} - r_{Q^X}) = -|Q_0^i| (l_{Q^X} + r_{Q^X}) + 2|Q_0^i| \cdot l_{Q^X},
\]
\[
\theta_{\text{add}}(Q^i) = -|Q_0^i| (l_{Q^X} + r_{Q^X}) - 2l_{Q^X} \cdot (r_{Q^i} - r_{Q^X}) - 2l_{Q^i} \cdot r_{Q^X}, \quad 2 \leq i \leq s - 1,
\]
\[
\theta_{\text{add}}(Q^s) = -|Q_0^s| (l_{Q^X} + r_{Q^X}) - 2l_{Q^X} \cdot r_{Q^s} - 2l_{Q^s} \cdot r_{Q^X} + 2|Q_0^s| \cdot r_{Q^X}
\]
\[-|Q_0^s| (l_{Q^X} + r_{Q^X}) - 2l_{Q^s} \cdot r_{Q^X},
\]
thus the inequality (3.3) reads
\[
\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i) - \left( \sum_{i=1}^{s} |Q^i_0| \right) (l_{Q^i} + r_{Q^i}) - 2l_{Q^i} \cdot \sum_{i=2}^{s} (r_{Q^i} - r_{Q^i}) - 2 \sum_{i=2}^{s} l_{Q^i} \cdot r_{Q^i} + 2 |Q^i_0| \cdot l_{Q^i}
\]
\[
\frac{\sum_{i=1}^{s} |Q^i_0|}{|Q^X_0|},
\]
or reads
\[
\frac{\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i)}{\sum_{i=1}^{s} |Q^i_0|} < 2l_{Q^X} - \frac{2|Q^1_0|}{\sum_{i=1}^{s} |Q^i_0|} \cdot l_{Q^X} + \frac{2l_{Q^X} \cdot \sum_{i=2}^{s} (r_{Q^i} - r_{Q^i}) + 2 \sum_{i=2}^{s} l_{Q^i} \cdot r_{Q^i}}{\sum_{i=1}^{s} |Q^i_0|},
\]
which holds.

For the case (c)
\[
\cdots \leftarrow Q^1 \leftarrow \cdots \rightarrow Q^2 \leftarrow \cdots \cdots \cdots \rightarrow Q^s \rightarrow \cdots ,
\]
we have
\[
\theta_{\text{add}}(Q^X) = |Q^X_0| (l_{Q^X} + r_{Q^X}) + 2l_{Q^X} \cdot r_{Q^X},
\]
\[
\theta_{\text{add}}(Q^i) = |Q^i_0| (l_{Q^i} - r_{Q^i}) = -|Q^i_0| (l_{Q^i} + r_{Q^i}) + 2 |Q^i_0| \cdot l_{Q^i},
\]
\[
\theta_{\text{add}}(Q^i) = -|Q^i_0| (l_{Q^i} + r_{Q^i}) - 2l_{Q^i} \cdot (r_{Q^i} - r_{Q^i}) - 2l_{Q^i} \cdot r_{Q^i}, \quad 2 \leq i \leq s - 1,
\]
\[
\theta_{\text{add}}(Q^s) = -|Q^s_0| (l_{Q^s} - r_{Q^s}) = -|Q^s_0| (l_{Q^s} + r_{Q^s}) - 2 |Q^s_0| \cdot r_{Q^s},
\]
thus the inequality (3.3) reads
\[
\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i) - \left( \sum_{i=1}^{s} |Q^i_0| \right) (l_{Q^i} + r_{Q^i}) - 2l_{Q^i} \cdot \sum_{i=2}^{s-1} (r_{Q^i} - r_{Q^i}) - 2 \sum_{i=2}^{s-1} l_{Q^i} \cdot r_{Q^i} + 2 |Q^1_0| \cdot r_{Q^X} - 2 |Q^s_0| \cdot r_{Q^X}
\]
\[
\frac{\sum_{i=1}^{s} |Q^i_0|}{|Q^X_0|},
\]
or reads
\[
\frac{\sum_{i=1}^{s} \theta_{\text{ind}}(Q^i)}{\sum_{i=1}^{s} |Q^i_0|} < 2l_{Q^X} + r_{Q^X} \frac{2|Q^1_0|}{\sum_{i=1}^{s} |Q^i_0|} \cdot r_{Q^X}
\]
\[
+ \frac{2l_{Q^X} \cdot \sum_{i=2}^{s-1} (r_{Q^i} - r_{Q^i}) + 2 \sum_{i=2}^{s-1} l_{Q^i} \cdot r_{Q^i} + 2 |Q^1_0| \cdot r_{Q^X}}{\sum_{i=1}^{s} |Q^i_0|} + \frac{2l_{Q^X} \cdot r_{Q^X}}{|Q^X_0|},
\]
which is satisfied.

So far, we complete the proof.
Definition 3.8. Let \( \text{Ind}(Q) \) be the (finite) set of the isomorphism classes indecomposable representations over \( k \) of a Dynkin quiver \( Q \). For a nonempty subset \( U \subseteq \text{Ind}(Q) \), one introduces a subset \( S_{Z}(Q, U) \) of \( \mathbb{Z}^n \) with \( n = |Q_0| \) as
\[
S_{Z}(Q, U) = \left\{ \Theta = (\theta_1, \cdots, \theta_n) \in \mathbb{Z}^n : \text{each element of } U \text{ is stable with respect to the corresponding weight function } w_\Theta \text{ and rank function } r \right\}.
\]
Obviously, \( S_{Z}(Q, U) \subset S_{Z}(Q, V) \) if \( V \subseteq U \). \( S_{Z}(Q, U) \) is determined by finitely many linear inequalities \( f_1(\Theta) > 0, \cdots, f_m(\Theta) > 0 \), then we define a subset \( C_{Z}(Q, U) \) of \( \mathbb{R}^n \) as
\[
C_{Z}(Q, U) = \{ \Theta = (\theta_1, \cdots, \theta_n) \in \mathbb{R}^n : f_1(\Theta) > 0, \cdots, f_m(\Theta) > 0 \},
\]
and define a convex polyhedral cone \( C(Q, U) \) of \( \mathbb{R}^n \) as the closure
\[
C(Q, U) = \overline{C_{Z}(Q, U)}.
\]
The faces of maximal dimension in a cone \( C(Q, U) \) are called the walls in \( \mathbb{R}^n \).

Corollary 3.9. ([11]) Let \( Q \) be a quiver of \( A_n \)-type, then the cardinality of \( S_{Z}(Q, \text{Ind}(Q)) \) is infinite. Moreover, for any element in \( S_{Z}(Q, \text{Ind}(Q)) \), with respect to the corresponding weight function and rank function

\begin{enumerate}
\item any semistable representation is polystable;
\item the Hader-Narasimhan strata are precisely \( GL(Q, d) \)-orbits in \( \text{Rep}(Q, d) \).
\end{enumerate}

3.3. Revisit Intrinsic Weight System via Semi-Invariant Theory.

Proposition 3.10. For each indecomposable representation \( I_{p,q} \) of the quiver \( Q \) of \( A_n \)-type, one defines weight systems \( \Theta(I_{p,q}) \) and \( \Theta'(I_{p,q}) \) as follows
\[
\Theta(I_{p,q})_i = \begin{cases}
1, & p < i < q \text{ and } i \text{ is a vertex of type I}, \\
-1, & p < i < q \text{ and } i \text{ is a vertex of type II}, \\
0, & p < i < q \text{ and } i \text{ is a vertex of type III or IV}, \\
1, & i = p \text{ and } i \text{ is a vertex of type I or III; } i = p = q, \\
0, & i = p < q \text{ and } i \text{ is a vertex of type II or IV}, \\
0, & i = p - 1 \text{ and } i \text{ is a vertex of type I or III,} \\
-1, & i = p - 1 \text{ and } i \text{ is a vertex of type II or IV,} \\
0, & i = q + 1 \text{ and } i \text{ is a vertex of type I or IV,} \\
-1, & i = q + 1 \text{ and } i \text{ is a vertex of type II or III,} \\
0, & i < p - 1; i > q + 1,
\end{cases}
\]
\[
\Theta'(I_{p,q})_i = \begin{cases}
1, & p < i < q \text{ and } i \text{ is a vertex of type I,} \\
-1, & p < i < q \text{ and } i \text{ is a vertex of type II,} \\
0, & p < i < q \text{ and } i \text{ is a vertex of type III or IV,} \\
-1, & i = p \text{ and } i \text{ is a vertex of type II or IV; } i = p = q, \\
0, & i = p < q \text{ and } i \text{ is a vertex of type III,} \\
1, & i = p - 1 \text{ and } i \text{ is a vertex of type I or III,} \\
0, & i = p - 1 \text{ and } i \text{ is a vertex of type II or IV,} \\
1, & i = q + 1 \text{ and } i \text{ is a vertex of type I or IV,} \\
0, & i = q + 1 \text{ and } i \text{ is a vertex of type II or III,} \\
0, & i < p - 1; i > q + 1,
\end{cases}
\]
then the intrinsic weight system \( \Theta \) can be written as
\[
\Theta = \sum_{I_{p,q}} c(I_{p,q}) \Theta(I_{p,q}) \quad \text{(or } \Theta = \sum_{I_{p,q}} c(I_{p,q}) \Theta'(I_{p,q}))
\]
where the sum runs through all indecomposable representations of \( Q \), and the coefficients \( c(I_{p,q}) \)'s are non-negative integers, moreover the sum can be taken over the indecomposable representations.
For a representation $X$ of a general quiver $Q$ with the dimension vector $d$, every weight system $W = (W_i) \in \mathbb{Z}^{[Q]}$ on $Q$ defines a character $\chi_W$ of reductive algebraic group $GL(Q,d) = \prod_{i \in Q_0} GL(d_i)$ acting on $X$ as a homomorphism

$$\chi_W : GL(Q,d) \to \mathbb{C}^{	imes}, g = (g_i : g_i \in GL(d_i)) \mapsto \prod_{i \in Q_0} \det(g_i)^{W_i},$$

conversely, every character of $GL(d)$ must look like the above form. Let

$$\text{Rep}(Q,d) = \bigoplus_{a \in Q_1} \text{Hom}(k^{d_{\mathbf{A}}(a)}, k^{d_{\mathbf{B}}(a)})$$

be the affine variety of representations of $Q$ with dimension vector $d$, a polynomial function $f$ in $k[\text{Rep}(Q,d)]$ is called a $W$-semi-invariant if $g \cdot f = \chi_W(g)f$ for any $g \in GL(X)$. Denote by $SI_W(Q,d)$ the vector space of $W$-semi-invariants, then the direct sum

$$SI(Q,d) = \bigoplus_{W \in \mathbb{Z}^{[Q]_0}} SI_W(Q,d)$$

carries a ring structure, hence called the ring of semi-invariant, moreover $SI(Q,d) = k[\text{Rep}(Q,d)]^{SL(Q,d)}$ for $SL(Q,d) = \prod_{i \in Q_0} SL(d_i)$ is the ring of polynomials in $k[\text{Rep}(Q,d)]$ which is stable under the action of $SL(d)$. Let $X, Y$ be two representations of a quiver $Q$ of $A_n$-type with dimension vectors $d_X, d_Y$ respectively, the Euler inner product is given by

$$\langle d_X, d_Y \rangle = \dim_k \text{Hom}_Q(X,Y) - \dim_k \text{Ext}_Q(X,Y)$$

$$= \sum_{i \in Q_0} (d_X)_i(d_Y)_i - \sum_{a \in Q_1} (d_X)_{a(1)}(d_Y)_{a(2)}$$

$$= \sum_{i,i+1 \in Q_0} ((d_X)_i(d_Y)_{i+1} - (d_X)_{i+1}(d_Y)_i),$$

where $i+1$ stands for the next vertex of $i$ along the reference direction, and

$$(\widehat{d_X})_i = \begin{cases} 
(d_X)_i, & \text{if } i \text{ is a vertex of type I or III}, \\
(d_X)_{i+1}, & \text{if } i \text{ is a vertex of type II or IV};
\end{cases}$$

$$(\widehat{d_Y})_i = \begin{cases} 
(d_Y)_i, & \text{if } i \text{ is a vertex of type I or III}, \\
(d_Y)_{i+1}, & \text{if } i \text{ is a vertex of type II or IV}.
\end{cases}$$

Define a map $f^Y_X : \bigoplus_{i \in Q_0} \text{Hom}(X_i, Y_i) \to \bigoplus_{a \in Q_1} \text{Hom}(X_{a(1)}, Y_{a(2)})$ by

$$(f_i)_{i \in Q_0} \mapsto (f_{a(1)}X_a - Y_{a(2)})_{a \in Q_1}.$$

If $\langle d_X, d_Y \rangle = 0$, the matrix of $f^Y_X$ is a square matrix, then one can define a semi-invariant $c(X,Y) = \det f^Y_X$ of the action $GL(Q,d_X) \times GL(Q,d_Y)$ on $\text{Rep}(Q,d_X) \times \text{Rep}(Q,d_Y)$. For a fixed representation $X$ (or $Y$), the restriction $c(X,Y)$ to $\{X\} \times \text{Rep}(Q,d_Y)$ (or $\text{Rep}(Q,d_X) \times \{Y\}$) defines a semi-invariant $c_X(Y)$ (or $c^Y(X)$) in $SI(Q,d_Y)$ with respect to the weight system $W_X = \{W_X_i\}_{i \in Q_0},$
Derksen and Weyman’s remarkable theorem [1] asserts that the semi-invariants of type \(c_X(Y)\) (or \(c^Y(X)\)) span all the weight spaces in the rings \(\text{SI}(Q,d)\) (or \(\text{SI}(Q,d_X)\)). One can easily check

\[ W_{I_p,q} = \Theta(I_{p,q}), \quad W'_{I_p,q} = \Theta'(I_{p,q}). \]

on the other hand, \(I_{1,n}\) is stable with respect to the intrinsic weight function, thus for each subrepresentations \(R \subset I_{p,q}\) we have \(w(R) < 0\), then by King’s result, there exists an \(m > 0\) and \(f \in \text{SI}(Q, \vec{m})_{m\Theta}\) such that \(f(I_{1,n}) \neq 0\), where \(\vec{m} = (1, \cdots, 1)\) denotes for the dimension vector of \(I_{1,n}\). Derksen-Weyman theorem implies that the set \(\Sigma(Q,d) = \{W : \text{SI}(Q,d)_W \neq 0\}\) is saturated, therefore \(\text{SI}(Q, \vec{m})_{\Theta} \neq 0\). The ring \(\text{SI}(Q, \vec{m})\) is generated by all \(c_{I_{p,q}}’\)s with \(\langle d_{I_{p,q}}, \vec{m} \rangle = 0\) (or \(c_{I_{p,q}}’\)s with \(\langle \vec{m}, d_{I_{p,q}} \rangle = 0\)). All these facts together lead to the final conclusion. \(\square\)

**Remark 3.11.** We write \(\Theta = \Theta^+ - \Theta^-\), where \(\Theta^+ = \{\Theta^+_i\}\) with \(\Theta^+_i = \max\{\theta_i, 0\}\) and \(\Theta^- = \{\Theta^-_i\}\) with \(\Theta^-_i = \max\{-\theta_i, 0\}\). For a dimension vector \(d\), if \(\sum_{i \in Q_0} d_i \Theta_i \neq 0\), then there is only trivial \(\Theta\)-semi-invariant. Therefore, we assume \(\sum_{i \in Q_0} d_i \Theta_i = 0\), i.e., \(\sum_{i \in Q_0} d_i \Theta^+_i = \sum_{i \in Q_0} d_i \Theta^-_i = l\), then for a representation \(X \in \text{Rep}(Q,d)\), one can define an \(l \times l\) matrix

\[ A : \bigoplus_{i \in Q_0} X_i^{\Theta^+_i} \to \bigoplus_{i \in Q_0} X_i^{\Theta^-_i}, \]

where each block \(A_{ij} \in \text{Hom}(X_i, X_j)\) has a form

\[ A_{ij} = \begin{cases} X(p_{i,j}), & \text{if there exists a path } p_{i,j} \text{ from } i \text{ to } j, \\ 0, & \text{otherwise,} \end{cases} \]

with \(X(p_{i,j})\) denoting the composition of the morphisms \(V_a\) for the arrows \(a\)’s consisting of the path \(p_{i,j}\). Then \(\det A\) is a semi-invariant in \(\text{SI}(Q,d)_\Theta\), and such semi-invariants generate the space \(\text{SI}(Q,d)_\Theta\).

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School of Mathematics, University of Science and Technology of China, Hefei 230026, China

E-mail address: pfhwang@mail.ustc.edu.cn

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

E-mail address: halfask@mail.ustc.edu.cn