GLOBAL-IN-TIME STRICHARTZ ESTIMATES AND CUBIC SCHRÖDINGER EQUATION ON METRIC CONE

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ABSTRACT. We study the Strichartz estimates for Schrödinger equation on a metric cone \(X\), where the metric cone \(X = C(Y) = (0, \infty) \times Y\) and the cross section \(Y\) is a \((n - 1)\)-dimensional closed Riemannian manifold \((Y, h)\). The equipped metric on \(X\) is given by \(g = dr^2 + r^2 h\), and let \(\Delta_g\) be the Friedrich extension positive Laplacian on \(X\) and \(V = V_0 r^{-2}\) where \(V_0 \in C^\infty(Y)\) is a real function such that the operator \(\Delta_h + V_0 + (n - 2)^2/4\) is a strictly positive operator on \(L^2(Y)\). We establish the full range of the global-in-time Strichartz estimate without loss for the Schrödinger equation associated with the operator \(\mathcal{L}_V = \Delta_g + V_0 r^{-2}\) including the endpoint estimate both in homogeneous and inhomogeneous cases. As an application, we study the well-posed theory and scattering theory for the Schrödinger equation with cubic nonlinearity on this setting.

Key Words: Local smoothing estimate, metric cone, Strichartz estimate, scattering theory

AMS Classification: 58J47, 42B37, 35Q40, 47J35.

1. Introduction and Statement of Main Result

The purpose of this paper is to study the Strichartz estimates for the Schrödinger equations on the setting of metric cone and also to study the well-posed theory and scattering theory for the Schrödinger equation with a nonlinearity.

1.1. The setting model. The setting we considered here is a conic singular space. The space \((X, g)\) is given by \(X = C(Y) = (0, \infty) \times Y\) and \(g = dr^2 + r^2 h\) where \((Y, h)\) is a \((n - 1)\)-dimensional closed Riemannian manifold, i.e. the boundary \(\partial Y = \emptyset\). Let \(\Delta_g\) denote the Friedrichs extension of Laplace-Beltrami from the domain \(C_c^\infty(X^\circ)\), compactly supported smooth functions on the interior of the metric cone. Consider the Schrödinger operator \(\mathcal{L}_V = \Delta_g + V\) where \(V = V_0(y) r^{-2}\) and \(V_0(y)\) is a smooth function on the section \(Y\) such that the operator \(\Delta_h + V_0 + (n - 2)^2/4\) is a strictly positive operator on \(L^2(Y)\) space. This operator has attracted many interests from different studies such as geometry, analysis and physics. In the cases without the potential, this operator was first studied from the perspective of wave diffraction from the cone point; see \([13, 40]\): Cheeger and Taylor \([7, 8]\) studied the diffraction phenomenon of the wave associated with this operator from the functional calculus and the regularity properties of wave propagation \([34]\); the heat kernel and Riesz transform kernel were studied in \([29, 33]\). On the other hand, since the decay of the inverse-square potential is closely related to the fact that the angular momentum as \(r \to \infty\), we are known that inverse square decay of the potential is in some sense critical for the spectral and scattering

\[\text{1The operator } \Delta_h \text{ is the positive Laplacian on } Y \text{ and the number } (n - 2)^2/4 \text{ is the sharp constant for the Hardy inequality which guarantees that the operator } \mathcal{L}_V \text{ is strictly positive.} \]
1.2. The Strichartz estimates. In this paper, we study a spacetime-type estimate of the solution $u : \mathbb{R} \times X \to \mathbb{C}$ to the initial value problem (IVP) for the Schrödinger equation on metric cone $X$,

$$i\partial_t u(t, z) + L_V u(t, z) = 0, \quad u(t, z) |_{t=0} = u_0(z), \quad (t, z) \in \mathbb{R} \times X.$$  

We are particularly interested in the Strichartz estimates since it has been known that the Strichartz estimates are a powerful tool for studying the behaviour of solutions to nonlinear dispersive equations, e.g. Schrödinger equation. More precisely, let $u$ be the solution to (1.1), we aim to establish the inequality in the form of

$$\|u(t, z)\|_{L^q_t L^r_z(I \times X)} \leq C \|u_0\|_{H^s(X)},$$

where $I$ is a subset interval of $\mathbb{R}$ and $H^s$ denotes the $L^2$-Sobolev space over $X$, and $(q, r)$ is an admissible pair, i.e.

$$(q, r) \in \Lambda_0 := \{2 \leq q, r \leq \infty, \quad 2/q + n/r = n/2, \quad (q, r, n) \neq (2, \infty, 2)\}.$$  

In particular, when $I = \mathbb{R}$, we say the Strichartz estimate is global-in-time. If $s = 0$, then the Strichartz estimate has no loss of derivative. For example, in the case of the Euclidean space and $V = 0$, one has global-in-time Strichartz estimates without loss of derivative. In general, it is harder to obtain a global-in-time Strichartz estimate than a local-in-time one. There is a large number of work studying Strichartz inequalities on Euclidean space or manifolds, we can not cite all of the papers here but refer the reader to see [18, 26, 37, 38] and references therein. However, we additionally mention a few of the most relevant references about the Strichartz estimate for the Schrödinger equation on exact cone or with perturbation of inverse-square potentials or the slightly different setting of asymptotically conic manifold. We recall that an asymptotically conic manifold $(M, g)$ is a complete non-compact Riemannian manifold of dimension $n \geq 3$ with one end diffeomorphic to $(0, \infty) \times Y$ and where $Y = \partial \overline{M}$ and $\overline{M}$ is a compactified manifold of $M$. If the metric $g$ in the end $[1, \infty) \times Y$ can be written as

$$g = dr^2 + r^2 \tilde{h}(r, y)$$

$\tilde{h} \in C^\infty(Sym^2(T^*\overline{M}))$ is a smooth family of metrics on $\partial \overline{M}$, then the manifold $(M, g)$ is called an asymptotically conic manifold. Note that the metric cone (considered in this paper) away from the cone tip is isometric to the end of the asymptotically conic manifold. Thus the establishment of the Strichartz estimate on each manifold are closely related. We say $M$ is non-trapping, if every geodesic $z(s)$ in $\overline{M}$ reaches the boundary $\partial \overline{M}$ as $s \to \pm \infty$. On the non-trapping asymptotically conic manifold $M$, the local-in-time Strichartz estimates were established in Hassell, Tao and Wunsch [21,22] and Mizutani [32]. Hassell and the first author [24] improved the results by showing the global-in-time Strichartz inequality and fixing the endpoint estimate. Burq, Guillarmou and Hassell [24] proved a local-in-time Strichartz estimate without loss on the asymptotically conic manifold with a trapped set which is hyperbolic and of sufficiently small fractal dimension. Very recently, Bouclet and Mizutani [11] and the authors [48]...
showed the same result but global-in-time under the same condition of [3] for the asymptotically conic manifold.

On the flat cone \( C(S^1_\rho) \) of dimension two, Ford [12] proved the full range of global-in-time Strichartz estimates. For high dimension, the first author [46] obtained a global-in-time Strichartz estimate from the restriction estimate but with a loss of angular derivatives. On the other hand, as mentioned above, the perturbation of the inverse-square potential is non-trivial since the inverse-square decay of the potential has the same scaling to the Laplacian. In [46], Burq, Planchon, Stalker, and Tahvildar-Zadeh generalized the Euclidean standard Strichartz estimate for Schrödinger and wave to the case in which an additional inverse-square potentials is present as a perturbation. The first main purpose of this paper is to prove the full range of global-in-time Strichartz estimates for Schrödinger equation associated with the operator \( L^V \), which is on the metric cone and with an inverse-square potential.

### 1.3. The main results

Now we state the following main results.

**Theorem 1.1** (Global-in-time Strichartz estimate). Suppose that \((X, g)\) is a metic cone of dimension \( n \geq 3 \). Let \( L^V = \Delta_g + V \) where \( r^2V =: V_0 \in C^\infty(Y) \) such that \( \Delta_h + V_0(y) + (n-2)^2/4 \) is a strictly positive operator on \( L^2(Y) \). Then the homogenous Strichartz estimate

\[
\| e^{itL^V} u_0 \|_{L^q_t L^r_x(R \times X)} \leq C \| u_0 \|_{L^2(X)},
\]

holds for the admissible pair \((q, r)\) \( \in [2, \infty]^2 \) satisfies (1.2); the inhomogeneous inequality

\[
\left\| \int_0^t e^{i(t-s)L^V} F(s) ds \right\|_{L^q_t L^r_x(R \times X)} \leq C \| F \|_{L^q_t L^r_x(R \times X)}
\]

holds for admissible pairs \((q, r)\), \((\tilde{q}, \tilde{r})\) including the endpoint \( q = \tilde{q} = 2 \).

**Remark 1.1.** The result shows the full range of global-in-time Strichartz estimate without loss of derivative including the endpoint estimates both for homogeneous and inhomogeneous cases.

**Remark 1.2.** The physical electrical point-dipole potential \( V = ar^{-2}y_3 \) with the constant \( a > -(n-2)^2/4 \) and \( y_3 \in Y = S^2 \) satisfies the above assumptions. An example of the Schrödinger equation with this potential is from the study of electron capture by polar molecules; see [3, 28].

We sketch the idea and argument for the proof here. We first prove the Strichartz estimate for \( L_0 \), i.e. without potential and then obtain Theorem 1.1 by using a perturbation argument. To prove the first result, we analyze the spectral measure associated with \( L_0 \) and study the dispersive estimate for the Schrödinger propagator. The proof for this combines the methods of [24] in which we developed a micro-localized spectral measure to capturing the decay and oscillation of the Schrödinger propagator, and of [46] in which we employed the Cheeger-Taylor [7]’s method to write the propagator as a linear combination of the Hankel transform of the radial part and eigenfunctions. Having the Strichartz estimate for \( L_0 \) in hand, we perform the perturbation argument to obtain the Strichartz estimate for \( L^V \) through a global-in-time local smoothing estimate. The local smoothing is directly proved by using the formulas with separating
variables expression. The endpoint inhomogeneous Strichartz estimate is proved by an iterated argument and a resolvent estimate.

It is worth making some remarks. The first one is due to the non-trivial perturbation from the metric $h$ which is possible to bring some conjugated points. We have to microlocalize the propagator to separating the conjugated points as did in [24] since the usual dispersive estimate fails due to the conjugated points; see [23]. The second remark comes from the perturbation of the inverse-square type potential. As remarked in [24, Remark 3.7], if $V_0(y)$ takes values in the range $(-(n-2)^2/4, 0)$, then one follows from [16, Corollary 1.5] that the $L^1 \to L^\infty$ norm of the propagator is at least a constant times $t^{-(\nu_0+1)}$ as $t \to \infty$, where $\nu_0^2$ is the smallest eigenvalue of $\Delta_h + V_0 + (n-2)^2/4$. Under the above assumption on the range of $V_0$, we see that $\nu_0 < (n-2)/2$. This implies that the dispersive estimate (1-12) in [24] will no longer be valid as $|t-s| \to \infty$, hence we can not obtain dispersive estimate and hence can not use Keel-Tao’s abstract method to obtain the full set of Strichartz estimate as [24] did. The third remark is from the perturbation argument to obtain the Strichartz estimate for $L_V$. The perturbation argument is the usual Rodnianski-Schlag method [35], for example, the Strichartz estimate established in [5] for negative inverse-square potentials on $\mathbb{R}^n$. But the key thing is a global-in-time local smoothing estimate for $L_V$. The general method to obtain the local smoothing is through establishing the resolvent estimate and Kato’s method. We avoid the resolvent estimate for showing the homogeneous Strichartz estimate although we also prove a resolvent estimate in the proof the endpoint inhomogeneous Strichartz estimate. Since the required global-in-time local smoothing estimate here is a estimate a $L^2$-based weight space, hence the formulas with separating variables expression can be directly used to prove the local smoothing inequality, though the estimates for general $L^p$ will give rise to a loss of angular derivative as shown [16].

Another remark is for the double endpoint inhomogeneous Strichartz estimate. The endpoint inhomogeneous Strichartz estimate is proved by using the argument of [11] and [2] via a resolvent estimate. In [5], Burq et al. used a method of integration by parts to prove a resolvent estimate. The method works well in our situation, hence we also establish a resolvent estimate which implies the weight $r^{-1}$ is $L_V$-supersmooth in the terminology of [25]. As an application, we show an uniform Sobolev inequality for independent interest.

The final remark is from the difference between the models considered in [24] and here. The main difference lies in the cone tip and the singular potential. We note that the metric cone is much simpler than the end of asymptotically conic manifold since the metric $h$ is independent $r$ (while $\tilde{h}$ not), hence we can use a scaling argument. Another natural property from the geometry is that the metric cone $X$ is non-trapping. Indeed, let $r(t)$ be the $r$ coordinate of the geodesic at time $t$ and $\mu$ is the angular momentum. The symbol of the Laplacian with respect to $g = dr^2 + r^2 h$ is $\sigma(\Delta_g) = \tau^2 + r^{-2} h^{-1}_{ij}(y) \mu_i \mu_j$ and along geodesics we have

$$\frac{d^2 r(t)}{dt^2} = 2 \frac{d\tau}{dt} = 4r^{-3} h^{-1}_{ij}(y) \mu_i \mu_j > 0,$$
hence there is no trapped set in $X$. Therefore we can use the results in [24] which is under the non-trapping condition. Compared with [24], the methods here have some analogues and also contain somewhat difference.

1.4. Application to NLS. As an application of the global-in-time Strichartz estimates, we study the cubic Schrödinger equation

$$\begin{cases}
i\partial_t u + \mathcal{L}_V u + \gamma |u|^2 u = 0, & (t, z) \in \mathbb{R} \times X, \\
{u(t, z)|_{t=0} = u_0(z),} & z \in X.
\end{cases}$$

(1.5)

where $\gamma = \pm 1$ which corresponds to the defocusing and focusing case respectively. Here we consider global existence and scattering for the cubic initial value problem. In particular the dimension of the metric cone $n = 2$, we [46] obtained the global solution and scattering result for the mass-critical (1.5) with small $L^2$-norm radial data. Due to the Strichartz estimate in Theorem 1.1 one can follow the arguments of Cazenave-Weissler [10] or Tao [43] to obtain the similar result for the high dimension mass-critical equation without the radial assumption. We here consider the well-posedness problem for (1.5) in energy space and we will meet a new difficulty caused by chain rule associated with our operator $\mathcal{L}_V$. Before stating the second result, we need some notation.

Let $(X, g)$ be the metric manifold above and let $dv = \sqrt{g}dz = r^{n-1}drdh$ be the measure induced by the metric $g$, we define the complex Hilbert space $L^2(X)$ to be given by the inner product

$$(f, h)_{L^2(X)} = \int_X f(z)\overline{h}(z)dv.$$  

We say that $f \in L^p(X)$ for $1 < p < \infty$ if $\int_X |f|^p dv < \infty$. For $1 \leq p < \infty$ we denote the inhomogeneous Sobolev space over $X$ by $H^1_p(X) = (\text{Id} + \mathcal{L}_V)^{-\frac{1}{2}}L^p(X)$ and write $H^1(X) = H^1_2(X)$ with $p = 2$.

In this paper, as mentioned above, we are interested in the global existence and scattering for nonlinear equation (1.5) with $u_0 \in H^1(X)$ when the dimension $n = 3$. The initial value problem falls into a class of energy-subcritical one on the metric cone. Solutions to (1.5) preserve the energy,

$$E(u)(t) = \int_X \left(\frac{1}{2}\sqrt{\mathcal{L}_V}u(t, z)^2 + \frac{\gamma}{4}|u(t, z)|^4\right) dv$$  

(1.6)

along with the mass

$$M(u)(t) = \int_X |u(t, z)|^2 dv.$$  

(1.7)

Our second main result is about the well-posedness and nonlinear scattering of the cubic Schrödinger equation.

**Theorem 1.2.** Let $X$ be metric cone of dimension $n = 3$ and $\mathcal{L}_V = \Delta_g + V$ as in Theorem 1.1 and $\gamma = \pm 1$ and suppose that the initial data $u_0 \in H^1(X)$. Then there
exists $T = T(\|u_0\|_{H^1}) > 0$ such that the nonlinear Schrödinger equation (1.5) has a unique solution $u$ satisfying
\begin{equation}
(1.8) \quad u \in C(I; H^1(X)) \cap L^q(I; H^1_r(X)), \quad I = [0, T).
\end{equation}

The solution for the defocusing case, i.e. $\gamma = 1$, can be extended to a global one. Moreover assume $\|u_0\|_{H^1(X)} \leq \epsilon$ for a small constant $\epsilon$, there exists a global solution $u$ and the solution $u$ scatters in sense that there are $u_\pm \in H^1(X)$ such that
\begin{equation}
\lim_{t \to \pm \infty} \|u(t) - e^{itL_V}u_\pm\|_{H^1(X)} = 0.
\end{equation}

This theorem is an analogue of the well known result for nonlinear Schrödinger on Euclidean space and the result about the global well posedness and scattering theory for small data still holds on the metric cone manifold. We know that the key things of the proof are the global-in-time Strichartz estimate and Leibniz chain rule in the establishment of well-posedness and scattering theory for the small initial data problem. As far as we known, there is no result about the chain rule for the operator $L_V$ which is a bit different from the classical one due to the perturbation of the inverse-square potential. For example consider the special operator $L_V$ on Euclidean space and $V = ar^{-2}$, Killip, Miao, Visan and the authors [27] proved the fractional chain rule whose range of the index $p$ is restricted by the value of $a$. In our situation, the related results are relevant to the smallest eigenvalue of the operator $\Delta_h + V_0(y) + (n-2)^2/4$, therefore we have to choose the admissible pairs adapted to the smallest eigenvalue $\nu_0^2$ in the argument of fixed point argument. Since we consider the Cauchy problem in energy space $H^1(X)$, the result on the boundedness of Riesz transform in Hassel-Lin [20] and the dual argument (see [36, 47]) are enough to give a chain rule required in proving our result. In the process, we apply the asymptotical behavior of resolvent in [20] to prove a Hardy inequality associated with the operator $L_V$ which has its own interest. Due to the independent interest of the fraction chain rule associated with $L_V$ (including the boundedness of a generalized Riesz transform operator), we will discuss more in a forthcoming paper [49].

Now we introduce some notation. We use $A \lesssim B$ to denote $A \leq CB$ for some large constant $C$ which may vary from line to line and depend on various parameters, and similarly we use $A \ll B$ to denote $A \leq C^{-1}B$. We employ $A \sim B$ when $A \lesssim B \lesssim A$. If the constant $C$ depends on a special parameter other than the above, we shall denote it explicitly by subscripts. For instance, $C_\epsilon$ should be understood as a positive constant not only depending on $p, q, n$, and $M$, but also on $\epsilon$. Throughout this paper, pairs of conjugate indices are written as $p, p'$, where $\frac{1}{p} + \frac{1}{p'} = 1$ with $1 \leq p \leq \infty$. We denote $a_\pm$ to be any quantity of the form $a \pm \epsilon$ for any small $\epsilon > 0$.

This paper is organized as follows: In Section 2, we use the Hankel transform and Bessel function to give the expression of the solution, and we also establish the $L^p$-product chain rule. Section 3 is devoted to considering the spectral measure associated with the operator $L_0 = \Delta_g$ on the metric cone and hence we prove the Strichartz estimate for Schrödinger without perturbation of the potential. In Section 4, we prove a local-smoothing estimate and then obtain the homogeneous estimates in Theorem 1.1. In Section 5, we prove the endpoint inhomogeneous Strichartz estimate and an
uniform Sobolev inequality. In the final section, we use the Strichartz estimates and $L^p$-product chain rule to show Theorem 1.2.

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2. Preliminary: analysis results associated with $L_V$

In this section, we study the operator $L_V$ over metric cone $X$. By following the separation of variable method [7], we first introduce an orthogonal decomposition of $L^2(Y)$ associated with the eigenfunctions of $\Delta_h + V_0(y) + (n-2)^2/4$ and we next write the Schrödinger propagator as a linear combination of products of the Hankel transform of the radial part and eigenfunctions of $\Delta_h + V_0(y) + (n-2)^2/4$. We finally close this section by studying the Sobolev space and the $L^p$-product chain rule which will serve the existence theory of nonlinear solution.

2.1. Hankel transform and Bessel function. Let $(r, y) \in \mathbb{R}_+ \times Y$ be some polar coordinates, recall the operator

$$L_V = \Delta_g + \frac{V_0(y)}{r^2},$$

on the metric cone $X = (0, \infty)_r \times Y$ where $V_0(y)$ is a real continuous function and the metric $g$ in coordinates $(r, y) \in \mathbb{R}_+ \times Y$ is a metric of the form

$$g = dr^2 + r^2h(y, dy).$$

We remark that the Riemannian metric $h$ on $Y$ is independent of $r$ hence we can use the separation of variable method. In the coordinate $(r, y)$, we write $L_V$

$$L_V = -\partial_r^2 - \frac{n-1}{r} \partial_r + \frac{1}{r^2} (\Delta_h + V_0(y)).$$

where $\Delta_h$ is the Laplace-Beltrami operator on $(Y, h)$. We assume the operator $L_V$ is strictly positive in $L^2(X; dg(z))$, that is, for any nonzero function $f(r, y) \in C_c^\infty((0, \infty) \times Y)$

$$\langle L_V f, f \rangle = \int_0^\infty \int_Y \left( |\partial_r f|^2 + \frac{1}{r^2} (|\nabla_h f|^2 + V_0(y)|f|^2) \right) r^{n-1} dr dh > 0.$$  

From the Hardy inequality with the sharpest constant $(n-2)^2/4$ (e.g. see [6]), the positivity can be guaranteed by assuming that $V_0$ is a smooth function on $Y$ such that

$$\Delta_h + V_0(y) + (n-2)^2/4 > 0$$

is strictly positive on $L^2(Y)$ in sense that for any $f \in L^2(Y) \setminus \{0\}$

$$\langle (\Delta_h + V_0(y) + (n-2)^2/4) f, f \rangle_{L^2(Y)} > 0.$$  

We denote the smallest eigenvalue of \( \Delta_h + V_0(y) + (n-2)^2/4 \) by \( \nu_0^2 \) and second lowest eigenvalue \( \nu_1^2 \), with \( \nu_0, \nu_1 > 0 \). Following [44], we define \( \chi_\infty \) to be the set
\[
\chi_\infty = \left\{ \nu : \nu = \sqrt{(n-2)^2/4 + \lambda}; \; \lambda \text{ is eigenvalue of } \Delta_h + V_0(y) \right\}.
\] (2.5)

For \( \nu \in \chi_\infty \), let \( d(\nu) \) be the multiplicity of \( \lambda_\nu = \nu^2 - \frac{1}{4}(n-2)^2 \) as eigenvalue of \( \tilde{\Delta}_h := \Delta_h + V_0(y) \). Let \( \{ \varphi_{\nu,\ell}(y) \}_{1 \leq \ell \leq d(\nu)} \) be the eigenfunctions of \( \tilde{\Delta}_h \), that is
\[
\tilde{\Delta}_h \varphi_{\nu,\ell} = \lambda_\nu \varphi_{\nu,\ell}, \quad \langle \varphi_{\nu,\ell}, \varphi_{\nu,\ell'} \rangle_{L^2(Y)} = \delta_{\ell,\ell'} = \begin{cases} 1, & \ell = \ell' \\ 0, & \ell \neq \ell' \end{cases}.
\] (2.6)

Let \( \mathcal{H}^\nu = \text{span}\{ \varphi_{\nu,1}, \ldots, \varphi_{\nu,d(\nu)} \} \), we have the orthogonal decomposition of the \( L^2(Y) \) in sense that
\[
L^2(Y) = \bigoplus_{\nu \in \chi_\infty} \mathcal{H}^\nu.
\]

Define the orthogonal projection \( \pi_\nu \) on \( f \in L^2(X) \)
\[
\pi_\nu f = \sum_{\ell=1}^{d(\nu)} \varphi_{\nu,\ell}(y) \int_Y f(r,y) \varphi_{\nu,\ell}(y) \, dh := \sum_{\ell=1}^{d(\nu)} \varphi_{\nu,\ell}(y) a_{\nu,\ell}(r)
\]
where \( dh \) is the measure on \( Y \) under the metric \( h \). For any \( f \in L^2(X) \), we can write \( f \) in the form of separation of variable
\[
f(z) = \sum_{\nu \in \chi_\infty} \pi_\nu f = \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} a_{\nu,\ell}(r) \varphi_{\nu,\ell}(y)
\] (2.7)

and furthermore
\[
\| f(r,y) \|_{L^2(Y)}^2 = \sum_{\nu \in \chi_\infty} \sum_{\ell=1}^{d(\nu)} |a_{\nu,\ell}(r)|^2.
\] (2.8)

Let \( \nu > \frac{-1}{2} \) and \( r > 0 \) and define the Bessel function of order \( \nu \) by its Poisson representation formula
\[
J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu + \frac{1}{2}) \Gamma(1/2)} \int_{-1}^{1} e^{i sr} (1 - s^2)^{(2\nu - 1)/2} ds
\]
which satisfies the equation
\[
r^2 \frac{d^2}{dr^2} (J_\nu(r)) + r \frac{d}{dr} (J_\nu(r)) + (r^2 - \nu^2) J_\nu(r) = 0.
\]
A simple computation gives the rough estimates
\[
|J_\nu(r)| \leq \frac{C r^\nu}{2^\nu \Gamma(\nu + \frac{1}{2}) \Gamma(1/2)} \left( 1 + \frac{1}{\nu + 1/2} \right),
\] (2.9)

where \( C \) is a absolute constant. Let \( f \in L^2(X) \), using the Bessel function, we define the Hankel transform of order \( \nu \) by
\[
(\mathcal{H}_\nu f)(\rho, y) = \int_0^\infty (r \rho)^{-\nu/2} J_\nu(r \rho) f(r, y) r^{n-1} dr.
\] (2.10)
Briefly recalling functional calculus for cones \cite{41}, for well-behaved functions $F$, we have by \cite[(8.45)]{41} 
\begin{equation}
(2.11) \quad F(\mathcal{L}_V)g(r, y) = \sum_{\nu \in \chi} d(\nu) \sum_{\ell = 1}^{d(\nu)} \varphi_{\nu, \ell}(y) \int_{0}^{\infty} F(\rho^2)(\rho \rho)^{-\frac{n-2}{2}} J_{\nu}(r \rho)b_{\nu, \ell}(\rho)\rho^{n-1}d\rho 
\end{equation}
where $b_{\nu, \ell}(\rho) = (\mathcal{H}_{\nu} a_{\nu, \ell})(\rho)$ with $g(r, y) = \sum_{\nu \in \chi} \sum_{\ell = 1}^{d(\nu)} a_{\nu, \ell}(r) \varphi_{\nu, \ell}(y)$. For $u_0 \in L^2(X)$, we write it in the form of separation of variables by (2.7)
\begin{equation}
(2.12) \quad u_0(z) = \sum_{\nu \in \chi} d(\nu) \sum_{\ell = 1}^{d(\nu)} a_{\nu, \ell}(r) \varphi_{\nu, \ell}(y),
\end{equation}
therefore the solution of the Cauchy problem
\begin{equation}
\begin{cases}
\begin{aligned}
i \partial_t u + \mathcal{L}_V u &= 0, \\
u(0, z) &= u_0(z),
\end{aligned}
\end{cases}
\end{equation}
can be written in a form of Hankel transform representation with separation of variable, by using (2.11) with $F(\rho^2) = e^{it\rho^2}$
\begin{equation}
(2.13) \quad u(t, z) = e^{it\mathcal{L}_V} u_0 = v(t, r, y) = \sum_{\nu \in \chi} d(\nu) \sum_{\ell = 1}^{d(\nu)} \varphi_{\nu, \ell}(y) \int_{0}^{\infty} (\rho \rho)^{-\frac{n-2}{2}} J_{\nu}(r \rho)e^{it\rho^2} b_{\nu, \ell}(\rho)\rho^{n-1}d\rho \cdot \mathcal{H}_{\nu}[e^{it\rho^2} b_{\nu, \ell}(\rho)](r).
\end{equation}
where $b_{\nu, \ell}(\rho) = (\mathcal{H}_{\nu} a_{\nu, \ell})(\rho)$. We refer alternatively the reader to \cite{46} for more details about this.

Next, we recall the properties of Bessel function $J_{\nu}(r)$ in \cite{39,45}, and we record here the properties which is needed for our purpose as the following

**Lemma 2.1 (Asymptotics of the Bessel function).** Assume $\nu \gg 1$. Let $J_{\nu}(r)$ be the Bessel function of order $\nu$ defined as above. Then there exist a large constant $C$ and a small constant $c$ independent of $\nu$ and $r$ such that:
\begin{itemize}
    \item when $r \leq \frac{\nu}{2}$
    \begin{equation}
(2.14) \quad |J_{\nu}(r)| \leq Ce^{-c(\nu+r)};
\end{equation}
    \item when $\frac{\nu}{2} \leq r \leq 2\nu$
    \begin{equation}
(2.15) \quad |J_{\nu}(r)| \leq C\nu^{-\frac{1}{2}}(\nu^{-\frac{1}{2}}|r - \nu| + 1)^{-\frac{1}{2}};
\end{equation}
    \item when $r \geq 2\nu$
    \begin{equation}
(2.16) \quad J_{\nu}(r) = r^{-\frac{1}{2}} \sum_{\pm} a_{\pm}(r, \nu)e^{\pm ir} + E(r, \nu),
\end{equation}
\end{itemize}
where $|a_{\pm}(r, \nu)| \leq C$ and $|E(r, \nu)| \leq Cr^{-1}$.

As a direct consequence, we have \cite[Lemma 2.2]{31}
Lemma 2.2. Let $R \gg 1$, then there exists a constant $C$ independent of $\nu$ and $R$ such that

\begin{equation}
\int_{R}^{2R} |J_\nu(r)|^2 dr \leq C.
\end{equation}

2.2. $L^p$-product chain rule. The $L^p$-product rule for fractional derivatives in Euclidean spaces was first proved by Christ and Weinstein [10]. The chain rules for differential operators of non-integer order. For example, if $1 < p < \infty$ and $s > 0$, then

\[ \|(-\Delta_{\mathbb{R}^n})^s (uv)\|_{L^p(\mathbb{R}^n)} \lesssim \|(-\Delta_{\mathbb{R}^n})^s u\|_{L^p(\mathbb{R}^n)} \|v\|_{L^{p_1}(\mathbb{R}^n)} + \|u\|_{L^{p_2}(\mathbb{R}^n)} \|(-\Delta_{\mathbb{R}^n})^s v\|_{L^{p_2}(\mathbb{R}^n)} \]

whenever \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_2} \). For a textbook presentation of these theorems and original references, see [42]. However, there is a bit difference from this if one considers the perturbation of inverse-square potential; we refer the reader to [27]. Since we here consider the solution of the nonlinear equation on $H^1(X)$, the result corresponding to Riesz transform is easy to be obtained due to Hassell-Lin [20] which concluded that: let \( \nabla_g = (\partial_r, r^{-1} \nabla_h) \) the gradient on $X$, there exists a constant $C$ such that

\begin{equation}
\|\nabla_g f\|_{L^p(X)} \leq C \|\sqrt{L_V} f\|_{L^p(X)};
\end{equation}

if and only if $p$ is in the interval

\begin{equation}
R_p := \left( \frac{n}{\min\{1 + \frac{n}{2} + \nu_0, n\}}, \frac{n}{\max\{\frac{n}{2} - \nu_0, 0\}} \right)
\end{equation}

where $\nu_0 > 0$ and $\nu_0^2$ is the smallest eigenvalue of the $\Delta_h + V_0 + (n - 2)^2/4$. We record that the chain rule is enough for our application even though it is not optimal in the range of the index.

Proposition 2.1. Let $L_V$ as above and let $\nu_0 > 0$ such that $\nu_0^2$ is the smallest eigenvalue of the $\Delta_h + V_0 + (n - 2)^2/4$. Then for all $u, v \in C_0^\infty(X^c)$, compactly supported smooth functions on the interior of the metric cone, we have

\begin{equation}
\|\sqrt{L_V} (uv)\|_{L^p(X)} \lesssim \|\sqrt{L_V} u\|_{L^p_1(X)} \|v\|_{L^p_2(X)} + \|u\|_{L^{p_1}(X)} \|\sqrt{L_V} v\|_{L^{p_2}(X)},
\end{equation}

for any exponents satisfying $p, p', p_1, p_2 \in R_p$ defined in (2.19) and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_1} + \frac{1}{p_2} \). (2.20)

Remark 2.1. The result is not optimal due to the restriction of $p' \in R_p$. In particular $n = 3$, we have $R_p = \left( \frac{6}{\min\{3 + 2\nu_0, 6\}}, \frac{6}{\max\{3 - 2\nu_0, 0\}} \right)$.

Proof. The proof is based on the boundness of Riesz transform in [20] and a dual argument in [36]. Let $f = u v$. Since $\sqrt{L_V} C_0^\infty$ is dense in $L^{p'}$ (see [36] Appendix), then

\[ \|\sqrt{L_V} f\|_{L^{p'}} = \sup \left\{ \langle \sqrt{L_V} f, \sqrt{L_V} h \rangle : h \in C_0^\infty(X^c), \|\sqrt{L_V} h\|_{L^{p'}(X)} \leq 1 \right\} \]

Therefore by the definition of the square root of $L_V = \nabla_g \nabla_g + V(z)$, we see

\[ \|\sqrt{L_V} f\|_{L^{p'}} \leq \|L_V f\|_{L^{p'}} \leq \|\nabla_g f\|_{L^p} \|\nabla_g h\|_{L^{p'}} + \|V_0\|_{L^\infty(X)} \|r^{-1} f\|_{L^p} \|r^{-1} h\|_{L^{p'}} . \]
If \( p, p' \in R_p \), (2.18) and the Hardy inequality (2.23) below imply

\[
\| \sqrt{L_V} f \|_{L^p} \leq C \| \sqrt{L_V} h \|_{L^{p'}} \left( \| \nabla g f \|_{L^p} + \| r^{-1} f \|_{L^p} \right).
\]

Note that for any \( X_i \) is the vector field on the manifold \( X \), then

\[
X_i(\sqrt{L_V} f) = X_i(\sqrt{L_V} h) + \| \nabla g \|_{X} L^p(X) + \| u \|_{L^{q_2}(X)} \| \nabla g \|_{X} L^2(X).
\]

For each vector field \( \partial_r, r^{-1} \nabla h \), it works and thus the chain rule works for \( \nabla g \). Hence we prove by the chain rule for \( \nabla g \) and the H"older inequality

\[
\| \sqrt{L_V} (uv) \|_{L^p} \leq C \left( \| \nabla g (uv) \|_{L^p} + \| r^{-1} uv \|_{L^p} \right).
\]

Hence by (2.18) and the Hardy inequality (2.23), we prove (2.20) if \( p_1, q_2 \) belong to (2.19).

\[
\text{□}
\]

2.3. **Hardy inequality.** In this subsection, we prove the hardy inequality associated with \( L_V \) by following the argument in [20] with minor modification. The hardy inequality claims that

**Proposition 2.2** (Hardy inequality for \( L_V \)). Let \( n \geq 3 \) and \( \nu_0 \) be as above. Suppose \( 0 < s < \min \{1 + \nu_0, 2\} \), and \( 1 < p < \infty \). Then

\[
\| r^{-s} f(z) \|_{L^p(X)} \leq C \| L_V^{-\frac{s}{2}} f \|_{L^p(X)}
\]

holds for

\[
n/ \min \{1 + \frac{n}{2} + \nu_0, n\} < p < n/ \max \{s + \frac{n}{2} - 1 - \nu_0, 0\}.
\]

**Remark 2.2.** Note that \( \nu_0 > 0 \), if we choose \( s = 1 \), then (2.20) is exactly same as the interval \( R_p \) defined in (2.19).

**Proof of Proposition 2.2.** The estimate (2.23) is equivalent to

\[
\| r^{-s} L_V^{-\frac{s}{2}} h \|_{L^p(X)} \leq \| h \|_{L^p(X)}
\]

where the operator \( L_V^{-\frac{s}{2}} \) is defined by the Riesz potentials kernel

\[
L_V^{-\frac{s}{2}}(z, z') := \int_0^\infty \lambda^{1-s}(L_V + \lambda^2)^{-1}(z, z')d\lambda.
\]

Therefore we consider the operator \( T = r^{-s} L_V^{-\frac{s}{2}} \) with its kernel

\[
T(z, z') = r^{-s} L_V^{-\frac{s}{2}}(z, z').
\]

Following the method [20], we study the kernel \( T(z, z') \) based on the resolvent kernel \((L_V + \lambda^2)^{-1}(z, z')\).
Step 1: $L^2$-bounded. Note $dv = \sqrt{|g|}dz = r^{n-1}drdh$. In particular $p = 2$, we compute that by 2.27 and 2.28
\[
\left\| r^{-1}f(z) \right\|^2_{L^2(X)} = \int_0^\infty \int_Y \left| \sum_{\nu \in \chi, \ell = 1} d(\nu) a_{\nu, \ell}(r) \varphi_{\nu, \ell}(y) \right|^2 r^{n-1} dr dh \\
= \sum_{\nu \in \chi, \ell = 1} d(\nu) \int_0^\infty \frac{|a_{\nu, \ell}(r)|^2}{r^2} r^{n-1} dr \\
\lesssim \sum_{\nu \in \chi, \ell = 1} \left\| \partial_r a_{\nu, \ell}(r) \right\|^2_{L^2(r^{n-1} dr)} \lesssim \left\| \nabla f \right\|^2_{L^2(X)} \lesssim \left\| L^{1/2}_f \right\|^2_{L^2(X)}.
\]

Therefore $T$ is bounded on $L^2(X)$.

Step 2: $L^p$-bounded on far away diagonal region. Let $\chi [0, \infty) \rightarrow [0, 1]$ be a smooth cutoff function such that $\chi([0,1/2]) = 1$ and $\chi([1, \infty)) = 0$, define the operators
\[
T_1(z, z') = \chi(4r/r')r^{-s} \int_0^\infty \lambda^{1-s}(L_V + \lambda^2)^{-1}(z, z') d\lambda,
\]
\[
T_2(z, z') = \chi(4r'/r)r^{-s} \int_0^\infty \lambda^{1-s}(L_V + \lambda^2)^{-1}(z, z') d\lambda.
\]

Then we decompose $T = T_0 + T_1 + T_2$ where
\[
T_0(z, z') = (1 - \chi(4r/r') - \chi(4r'/r))r^{-s} \int_0^\infty \lambda^{1-s}(L_V + \lambda^2)^{-1}(z, z') d\lambda.
\]

In this subsection, we consider the $L^p$-boundedness of $T_1$ and $T_2$. Since $L_V$ is homogeneous of degree $-2$, then we have by scaling
\[
(L_V + \lambda^2)^{-1}(z, z') = \lambda^{n-2}(L_V + 1)^{-1}(\lambda z, \lambda z').
\]
To this end, we need two results from [20]. One is about the resolvent kernel $(L_V + 1)^{-1}(z, z')$ in [20] Theorem 4.11 and the other is about the boundedness of a kind of operator in [20 Corollary 5.9]. By [20] Theorem 4.11, we have for any $N, M > 0$
\[
|\chi(4r/r')(L_V + 1)^{-1}(z, z')| \lesssim r^{1-\frac{n}{2}+\nu_0} r^{1-\nu_0} \langle r' \rangle^{-N},
\]
and
\[
|\chi(4r'/r)(L_V + 1)^{-1}(z, z')| \lesssim r^{1-\frac{n}{2}-\nu_0} r^{1-\nu_0} \langle r \rangle^{-M}.
\]
Therefore we have that by (2.29) for any $N > 1 - s$ and $s < 2$
\[
|T_1(z, z')| \lesssim r^{-s} \int_0^\infty \lambda^{n-1-s} \chi(4r/r')(L_V + 1)^{-1}(\lambda z, \lambda z') d\lambda \\
\lesssim r^{-s} r^{2-n} (r/r')^{1-\frac{n}{2}+\nu_0} \left( \int_1^{r} \lambda^{1-s} d\lambda + r^{-N} \int_{1/r'}^{\infty} \lambda^{1-s-N} d\lambda \right) \\
\lesssim r^{-n} (r/r')^{1-\frac{n}{2}+\nu_0}.
\]
Similarly we have
\[ |T_2(z, z')| \lesssim d(z, z')^{-2-n} \left( \int_0^{1/r} \lambda^{1-s} d\lambda + r^{-N} \int_{1/r}^{\infty} \lambda^{1-s-N} d\lambda \right) \]
(2.32)
\[ \lesssim d^{-1} r^{2-n} (r'/r)^1 - d^{-1} \nu_0 . \]
Hence by [20, Corollary 5.9], we have that $T_1$ is bounded on $L^p(X)$ for $1 < p < \max\{s + \frac{2}{p} - 1 - \nu_0, 0\}$ and $T_2$ is bounded on $L^p(X)$ for $p > n/\min\{1 + \frac{2}{p} + \nu_0, n\}$.

**Step 3: $L^p$-bounded on diagonal region.** Recall the distance on a metric cone is
\[ d(z, z') = \begin{cases} \sqrt{r^2 + r'^2 - 2rr'\cos(dy(y', y'))}, & dy(y, y') \leq \pi; \\ r + r', & dy(y, y') \geq \pi. \end{cases} \]
In the diagonal region, i.e. the support of $1 - \chi(4r/r') - \chi(4r'/r)$, we have
\[ d(z, z')^{-1} \geq (r + r')^{-1} = r^{-1}(1 + r'/r)^{-1} \geq r^{-1}/9. \]
We claim the following conclusions:
(i) $T_0$ is bounded on $L^2(X)$;
(ii) $|T_0(z, z')| \leq Cd(z, z')^{-n}$;
(iii) $|
abla_z T_0(z, z')| \leq Cd(z, z')^{-(n+1)}$ and $|
abla_{z'} T_0(z, z')| \leq Cd(z, z')^{-(n+1)}$.

Now we verify these conclusions. We have seen that $T, T_1, T_2$ are bounded on $L^2(X)$ from the first two steps if $s < 1 + \nu_0$, hence $T_0 = T - T_1 - T_2$ is also bounded on $L^2(X)$. To verify (ii) and (iii), we recall [20, Lemma 5.4] which implies for any integer $j \geq 0$ and any $N > 0$
\[ |\nabla_j K(z, z')| \lesssim \begin{cases} d(z, z')^{2-j}, & d(z, z') \leq 1; \\ d(z, z')^{-N}, & d(z, z') \geq 1. \end{cases} \]
where $K(z, z') = (1 - \chi(4r/r') - \chi(4r'/r))(\mathcal{L} + 1)^{-1}(z, z')$. Therefore we compute that by using $d(\lambda z, \lambda z') = \lambda d(z, z')$
\[ |K(\lambda z, \lambda z')| \lesssim \begin{cases} \lambda^{-n} d(z, z')^{2-n}, & d(z, z') \leq 1/\lambda; \\ \lambda^{-N} d(z, z')^{-N}, & d(z, z') \geq 1/\lambda. \end{cases} \]
and
\[ |\nabla_{z'} (K(\lambda z, \lambda z'))| \lesssim \begin{cases} \lambda^{-n} d(z, z')^{1-n}, & d(z, z') \leq 1/\lambda; \\ \lambda^{-N+1} d(z, z')^{-N}, & d(z, z') \geq 1/\lambda. \end{cases} \]
Recall $s < 2$ and let $N > n - s$, we note (2.34) to obtain
\[ |T_0(z, z')| = r^{s-n} \int_0^{\infty} \lambda^{n-1-s}|K(\lambda z, \lambda z')|d\lambda \]
\[ \lesssim d(z, z')^{-s} \left( \int_0^{1/d(z, z')} \lambda^{1-s} d\lambda + d(z, z')^{-N} \int_{1/d(z, z')}^{\infty} \lambda^{n-1-s-N} d\lambda \right) \]
\[ \lesssim d(z, z')^{-n} \]
which verifies (ii). Similarly we have
\[ |\nabla z T_0(z, z')| \lesssim r^{-s-1} \int_0^\infty \lambda^{n-1-s} |K(\lambda z, \lambda z')| d\lambda + r^{-s} \int_0^\infty \lambda^{n-1-s} |\nabla z (K(\lambda z, \lambda z'))| d\lambda \]
\[ \lesssim d(z, z')^{-n-1} + d(z, z')^{-s} \left( d(z, z')^{1-n} \int_0^{1/d(z, z')} \lambda^s d\lambda + d(z, z')^{-N} \int_\infty^\infty \lambda^{n-s-N} d\lambda \right) \]
\[ \lesssim d(z, z')^{-n-1}. \]

We also have \(|\nabla z T_0(z, z')| \lesssim d(z, z')^{-n-1}\), thus we prove (iii). As a consequence of Calderón-Zygmund theory, we have \(T_0\) is bounded from \(L^1(X)\) to weak \(L^1(X)\). By using interpolation, we obtain \(T_0\) is bounded on \(L^p(X)\) for all \(1 < p \leq 2\). By dual, we show the following result

**Proposition 2.3.** The operator \(T_0\) is a bounded operator on \(L^p(X)\) for all \(p > 1\).

Collecting all results from the last two steps, we prove \(T\) is bounded on \(L^p(X)\) for all \(p\) satisfies \((2.24)\). Therefore we prove Proposition 2.2.

\[ \square \]

3. **Spectral measure and Strichartz estimate associated with \(L_0\)**

In this section, we study the spectral measure associated with the operator \(H = \sqrt{L_0}\) where \(L_0 = \Delta_g\) is the positive Laplacian on the metric cone. And then we apply the result to prove the Strichartz estimate when \(V = 0\).

3.1. **Spectral measure.** We first brief review the spectral measure. For the following definition, we follow Kato [25].

Consider the given self-adjoint operator \(H : D(H) \subset \mathcal{H} \rightarrow \mathcal{H}\) where \(D(H)\) is the domain of \(H\) and \(\mathcal{H}\) is a Hilbert space. The **spectral family** \(\{E(\lambda)\}\) of \(H\), also known as a resolution of the identity, is a family of projection operators in \(\mathcal{H}\) with the following properties that:

- \(E(\lambda)\) is nondecreasing: \(E(\mu) \leq E(\lambda)\) for \(\mu \leq \lambda\);
- the strong convergences hold: \(\lim_{\lambda \rightarrow -\infty} E(\lambda) = 0\), \(\lim_{\lambda \rightarrow \infty} E(\lambda) = 1\).

Given any \(u, v \in \mathcal{H}\) from the spectral family, one can define a complex function of bounded variation on the real line

\[ \mathbb{R} \ni \lambda \rightarrow (E(\lambda)u, v)_\mathcal{H}. \]

It is known that such this function gives a measure (depending on \(u, v\)) called spectral measure. In our situation, the subspace of absolute continuity \(\mathcal{H}_{ac}\) with respect to the operator \(H\) is equal to \(\mathcal{H} = L^2\), in other word, the operator \(H\) is spectrally absolutely continuous. From [25] Chapter X, Theorem 1.7], the bilinear form

\[ \frac{d}{d\lambda} (E(\lambda)u, v)_{L^2} : L^2 \times L^2 \rightarrow \mathbb{C} \]

is bounded, and then it induces a bounded operator \(A(\lambda) : L^2 \rightarrow L^2\) defined via

\[ (A(\lambda)u, v)_{L^2} = \frac{d}{d\lambda} (E(\lambda)u, v)_{L^2} = \frac{d}{d\lambda} E(\lambda)u, v)_{L^2}, \quad u, v \in L^2. \]
The operator $A(\lambda)$ is called the density of states of the operator $H$. Finally we define the spectral measure associated with $H = \sqrt{\mathcal{L}_0}$

\begin{equation}
    dE_{\sqrt{\mathcal{L}_0}}(\lambda) = A(\lambda)d\lambda
\end{equation}

where $d\lambda$ is the Lebesgue measure. The main purpose of this section is to prove

**Proposition 3.1.** Let $(X, g)$ be metric cone manifold and $\mathcal{L}_0 = \Delta_g$. Then there exists a $\lambda$-dependent operator partition of unity on $L^2(X)$

$$
    \text{Id} = \sum_{j=0}^{N} Q_j(\lambda),
$$

with $N$ independent of $\lambda$, such that for each $0 \leq j \leq N$ we can write

\begin{equation}
    (Q_j(\lambda)dE_{\sqrt{\mathcal{L}_0}}(\lambda)Q_j^*(\lambda))(z, z') = \lambda^{n-1}\left(\sum_{\pm} e^{\pm i\lambda d(z, z')}a_{\pm}(\lambda; z, z') + b(\lambda; z, z')\right),
\end{equation}

and $0 \leq j, j' \leq N$ with either $j = 0$ or $j' = 0$

\begin{equation}
    (Q_j(\lambda)dE_{\sqrt{\mathcal{L}_0}}(\lambda)Q_{j'}^*(\lambda))(z, z') = \lambda^{n-1}c(\lambda, z, z'),
\end{equation}

where $a(\lambda, z, z'), b(\lambda, z, z')$ and $c(\lambda, z, z')$ satisfy the estimates for any $\alpha \geq 0$

\begin{equation}
    |\partial_\lambda^\alpha a_{\pm}(\lambda, z, z')| \leq C_{\alpha, \lambda}^{-\alpha}(1 + \lambda d(z, z'))^{-\frac{n-1}{2}},
\end{equation}

\begin{equation}
    |\partial_\lambda^\alpha b(\lambda, z, z')| \leq C_{\alpha, \lambda}^{-\alpha}(1 + \lambda d(z, z'))^{-K} \text{ for any } K,
\end{equation}

and

\begin{equation}
    |\partial_\lambda^\alpha c(\lambda, z, z')| \leq C_{\alpha, \lambda}^{-\alpha}.
\end{equation}

Here $d(\cdot, \cdot)$ is the distance on $X$.

**Remark 3.1.** The expression of the spectral measure captures both the decay and the oscillatory behaviour of the spectral measure, which is crucial in obtaining sharp dispersive estimates.

**Remark 3.2.** The expression (3.3) with estimate (3.6) are only used in the proof of inhomogeneous Strichartz estimate, not for homogeneous estimate.

We prove this Proposition by considering two separate regimes, near the cone tip and far away from the cone tip. Near the cone tip, we use a variable separating expression for the spectral measure as in Lemma 3.1 below. Far away from the cone tip, we follow the argument of [24, Proposition 1.5].

Now we show an explicit formula for the spectral measure which will be used to treat the regime near the cone tip.

**Lemma 3.1.** Let $\nu_j^2$ be the eigenvalues of the positive operator $\Delta_h + V_0(y) + (n - 2)^2/4$ and let $\varphi_j(y)$ be $L^2$-normalized corresponding eigenfunction, we have the explicit formula for the spectral measure

\begin{equation}
    dE_{\sqrt{\mathcal{L}_0}}(1, z, z') = \frac{\pi}{2} (rr')^{-\frac{n-2}{2}} \sum_j \varphi_j(y)\overline{\varphi_j(y')}J_{\nu_j}(r)J_{\nu_j}(r').
\end{equation}
Remark 3.3. We show here the result with the potential $V$ even though we only need the result with $V = 0$ for considering the spectral measure associated with $\mathcal{L}_0$ near the cone tip in the following argument.

Remark 3.4. This expression provides little asymptotic behavior of the kernel, as both $r, r'$ go to $\infty$, but gives good converges as both $r, r'$ go to $0$.

Proof. The proof is from [16,20] which give an explicit formula for the resolvent $(\mathcal{L}_V - (1 + i0))^{-1}$. We consider the expression of the spectral measure here. Write the operator $\mathcal{L}_V$ on $X$ as

$$\mathcal{L}_V = -\partial_r^2 - \frac{n-1}{r} \partial_r + \frac{\Delta_h + V_0(y)}{r^2}$$

(3.8)

$$= r^{1-\frac{n}{2}} \left( - (r \partial_r)^2 + \Delta_h + V_0(y) + (n-2)^2/4 \right) r^{\frac{n}{2}-1}$$

$$:= r^{1-\frac{n}{2}} P_b r^{\frac{n}{2}-1}$$

where

$$P_b = - (r \partial_r)^2 + \Delta_h + V_0(y) + (n-2)^2/4.$$ 

Let $g_b = r^{-2}g = r^{-2}dr^2 + h$ the conformal metric to $g$. Then we can check that $\mathcal{L}_V$ is formally self-adjoint with respect to $g_b$, and $P_b$ is formally self-adjoint with respect to $g_b$. Indeed, for any $f, \tilde{f} \in \mathcal{C}_0^\infty(X)$, we have by integration by parts

$$\langle \mathcal{L}_V f, \tilde{f} \rangle_{L^2_b(X)} = \int_X (r^{1-\frac{n}{2}} P_b r^{\frac{n}{2}-1} f) \tilde{f} r^{n-1} dr dh = \langle f, \mathcal{L}_V \tilde{f} \rangle_{L^2_b(X)}.$$ 

(3.9)

and

$$\langle P_b f, \tilde{f} \rangle_{L^2_b(X)} = \int_X P_b f \tilde{f} \frac{dr}{r} dh = \langle f, P_b \tilde{f} \rangle_{L^2_b(X)}.$$ 

(3.10)

Our purpose is to obtain the Schwartz kernel of the operator $(\mathcal{L}_V + k^2)^{-1}$ so we first consider the kernel of the operator $\mathcal{L}_V + k^2$. To this end, we regard this operator as acting on half-densities, using the flat connection on half-densities that annihilates the Riemannian half-density $|dg|^{\frac{n}{2}}$. It is natural to introduce this operators $\mathcal{L}_V$ as acting on half-densities $|dg|^{\frac{n}{2}}$ by the formula

$$(\mathcal{L}_V + k^2)(f|dg|^{\frac{n}{2}}) = ((\mathcal{L}_V + k^2)f)|dg|^{\frac{n}{2}}, \quad \text{with } dg = r^{n-1} dr dh.$$ 

A way to understand these formulas is that the term vanishes if the derivatives hit the half-densities. This is because that the derivatives are endowed with the flat connection on the half-density bundle which annihilates the half-density $|dg|^{1/2}$.

However we want to regard the operator $\mathcal{L}_V + k^2$ acting on half-density $|dg_b|^{1/2}$, as did in [16], therefore we obtain from [18,43]

$$r(\mathcal{L}_V + k^2) r(f|dg|^{\frac{n}{2}}) = r((\mathcal{L}_V + k^2) r(f|dr dh|^{\frac{n}{2}}))$$

$$= r^{1-\frac{n}{2}} \left( r^{1+\frac{n}{2}} (\mathcal{L}_V + k^2) r^{1-\frac{n}{2}} \right) (f|dr dh|^{\frac{n}{2}})$$

$$= r^{1-\frac{n}{2}} \left( (P_b + k^2 r^2) f \right) (|dr dh|^{\frac{n}{2}}) = (P_b + k^2 r^2 f) (|dg_b|^{\frac{n}{2}}).$$

We use that the flat connection on half-densities that annihilates the Riemannian half-density $|dg|^{1/2}$ in the third equality. This implies that $r(\mathcal{L}_V + k^2) r$ is equivalent to the
corresponding to the $L^2$-normalized eigenfunction $\varphi_j(y) = \varphi_{\nu_j}(y)$, that is
\[
(\Delta_h + V_0(y) + (n - 2)^2/4)\varphi_j = \nu_j^2 \varphi_j.
\]

Let $\Pi_j = \pi_{\nu_j}$ be projection onto the $\varphi_j$ eigen-space. Then we have the decomposition
\[
P_b + k^2 r^2 = \sum_j \left( -(r\partial_r)^2 + k^2 r^2 + \nu_j^2 \right)\Pi_j.
\]

Therefore
\[
(P_b + k^2 r^2)^{-1} = \sum_j \Pi_j \left( -(r\partial_r)^2 + k^2 r^2 + \nu_j^2 \right)^{-1}.
\]

Let $T_j = -(r\partial_r)^2 + k^2 r^2 + \nu_j^2$ and $T_j^{-1}(r, r')$ be the kernel of the inverse $T_j^{-1}$. When $r \neq r'$, as in [19], the solution space is spanned by the modified Bessel functions
\[
I_{\nu_j}(r) = \frac{2^{-\nu_j} \nu_j}{\sqrt{\pi} \Gamma(\nu_j + \frac{1}{2})} \int_{-1}^{1} (1 - t^2)^{\nu_j - \frac{1}{2}} e^{-rt} dt,
\]
\[
K_{\nu_j}(r) = \frac{\sqrt{\pi} 2^{\nu_j} \nu_j}{\Gamma(\nu_j + \frac{1}{2})} \int_{1}^{\infty} (t^2 - 1)^{\nu_j - \frac{1}{2}} e^{-rt} dt.
\]

Therefore, as in [19], we have
\[
T_j^{-1}(r, r') = \begin{cases}
I_{\nu_j}(kr)K_{\nu_j}(kr') \left| \frac{dr}{r} \frac{d\nu}{r'} \right|^{\frac{1}{2}}, & r < r'; \\
K_{\nu_j}(kr)I_{\nu_j}(kr') \left| \frac{dr}{r} \frac{d\nu}{r'} \right|^{\frac{1}{2}}, & r > r'.
\end{cases}
\]

Hence from the Sturm-Liouville theory, we obtain a explicit formulae separating the $r$ and $y$ variables for the resolvent kernel
\[
(P_b + k^2 r^2)^{-1} = \begin{cases}
\sum_j \varphi_j(y)\overline{\varphi_j}(y')I_{\nu_j}(kr)K_{\nu_j}(kr') \left| \frac{dr}{r} \frac{d\nu}{r'} \right|^{\frac{1}{2}}, & r < r'; \\
\sum_j \varphi_j(y)\overline{\varphi_j}(y')K_{\nu_j}(kr)I_{\nu_j}(kr') \left| \frac{dr}{r} \frac{d\nu}{r'} \right|^{\frac{1}{2}}, & r > r'.
\end{cases}
\]

This formula analytically continues to the imaginary axis, so setting $k = -i$, and using the formulae
\[
I_{\nu}(-iz) = e^{-\nu \pi i/2} J_{\nu}(z), \quad K_{\nu}(-iz) = \frac{\pi i}{2} e^{\nu \pi i/2} H_{\nu}^{(1)}(z),
\]

Let $\Pi$ be projection onto the...
we see that

\[(L_V - (1+i0))^{-1} = \begin{cases} \frac{\pi r r'}{2} \sum_j \varphi_j(y_0) \overline{\varphi_j(y')} J_{\nu_j}(r) H_{\nu_j}^{(1)}(r') \left| \frac{dr dr'}{rr'} dh(y) dh(y') \right|^\frac{1}{2}, & r < r'; \\
\frac{\pi r r'}{2} \sum_j \varphi_j(y_0) \overline{\varphi_j(y')} J_{\nu_j}(r) H_{\nu_j}^{(1)}(r) \left| \frac{dr dr'}{rr'} dh(y) dh(y') \right|^\frac{1}{2}, & r > r', \end{cases}\]

where $J_{\nu}$ and $H_{\nu}^{(1)}$ are standard Bessel and Hankel functions, see e.g. [45]. Since $\text{Im}(i H_{\nu}^{(1)})(r) = J_{\nu}(r)$, let $H$ be the Heaviside function, we have

\[dE_{\sqrt{L_V}}(1, z, z') = \frac{\pi}{2} (rr')^{-\frac{3}{2}} \sum_j \varphi_j(y_0) \overline{\varphi_j(y')} J_{\nu_j}(r) J_{\nu_j}(r') \left| \frac{dr dr'}{rr'} dh(y) dh(y') \right|^\frac{1}{2}.\]

Now we return to half-density $|dg|^{1/2}$ and obtain

\[dE_{\sqrt{L_V}}(1, z, z') = \frac{\pi}{2} (rr')^{-\frac{3}{2}} \sum_j \varphi_j(y_0) \overline{\varphi_j(y')} J_{\nu_j}(r) J_{\nu_j}(r') \left| dg \right|^\frac{1}{2}.\]

\[\square\]

**The proof of Proposition 33.1** Now we prove the main result of this section. The main idea is microlocalizing the spectral measure associated with $L_0$ to capture the decay and oscillation behavior, as described in Proposition 33.1. According the Stone’s formulae, the Schwartz kernel of the spectral measure can be expressed in terms of the difference between the outgoing and incoming resolvents $R(\lambda \pm i0)$, where $R(\sigma) = (L_0 - \sigma^2)^{-1}$. More precisely,

\[dE_{\sqrt{L_0}}(\lambda) = \frac{d}{d\lambda} E_{\sqrt{L_0}}(\lambda) d\lambda = \frac{\lambda}{\pi i} \left( R(\lambda + i0) - R(\lambda - i0) \right) d\lambda, \quad \lambda > 0.\]

Write the resolvent as a kernel

\[R(\sigma)(z, z') = (L_0 - \sigma^2)^{-1}(r, y, r', y')\]

where $z = (r, y)$ and $z' = (r', y')$ are the left and right variables respectively. Since $L_0$ is homogeneous of degree $-2$, then

\[(L_0 - \lambda^2)^{-1}(r, y, r', y') = \lambda^{n-2} (L_0 - 1)^{-1}(\lambda r, y, \lambda r', y'),\]

hence

\[dE_{\sqrt{L_0}}(\lambda; r, y, r', y') = \frac{\lambda^{n-1}}{\pi i} dE_{\sqrt{L_0}}(1; \lambda r, y, \lambda r', y').\]

Recall $X = (0, \infty) \times Y$, then $(z, z') = (r, y, r', y') \in X \times X$. Define $\bar{r} = \lambda r$ and $\bar{r}' = \lambda r'$. Due to the scaling invariant of $X$, $(\bar{r}, \bar{y}, \bar{r}', \bar{y}') \in X \times X$. Hence, from now on, we instead consider the spectral measure $dE_{\sqrt{L_0}}(1; \bar{r}, \bar{y}, \bar{r}', \bar{y}')$ on $X \times X$. 
We employ a similar partition of the identity as in \cite{17,24} defined by specifying the symbols of these operators, which must form a partition of unity on the phase space. Let \( \tilde{x} = 1/\tilde{r} \) and \( \epsilon = 1/R \) with \( R \) being chosen later.

- **The region near the cone tip.** Let \( O_0 \) consist of all points near the cone tip, that is, the points with \( \tilde{x} > \epsilon \), i.e. \( \tilde{r} < R \). Define the operator \( Q_0 = \chi(\tilde{r}) \) where \( \chi \in C_0^\infty(\mathbb{R}^+) \) satisfies \( \chi(s) = 1 \) for \( s \leq R \) and \( \chi(s) = 0 \) for \( s \geq 2R \). Now we prove the properties in Proposition 3.1 associated with \( Q_0 \).

From Lemma 3.1 we have

\[
Q_0 dE \sqrt{E_0} (1, \tilde{z}, \tilde{z}') Q_0^* = \frac{\pi}{2} (\lambda^2 r r')^{-\frac{n-2}{2}} \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda r') \chi(\lambda r) \chi(\lambda r').
\]

Recall that if \( \text{Re} \nu > -1/2 \), then one has the recursion relation, e.g. \cite{15} Page 45, 3.2-(4)]

\[
\frac{d}{dt} (t^{-\nu} J_{\nu}(t)) = -t^{-\nu} J_{\nu+1}(t),
\]

hence we have that

\begin{equation}
\left| \left( \frac{d}{d\lambda} \right)^\alpha \left( (\lambda^2 r r')^{-\frac{n-2}{2}} \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda r') \chi(\lambda r) \chi(\lambda r') \right) \right| \lesssim \lambda^{-\alpha} (\lambda^2 r r')^{-\frac{n-2}{2}} \sum_{\alpha_1+\alpha_2+\alpha_3+\alpha_4 \leq \alpha} \left( \lambda r \right)^{\alpha_1+\alpha_3} (\lambda r')^{\alpha_2+\alpha_4} \times \left| \sum_{j \geq 0} \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j+\alpha_1}(\lambda r) J_{\nu_j+\alpha_2}(\lambda r') \chi^{(\alpha_1)}(\lambda r) \chi^{(\alpha_4)}(\lambda r') \right|.
\end{equation}

On the other hand, recall \cite{2.1}, the Bessel function satisfies

\[
|J_{\nu}(t)| \lesssim \frac{Ct^\nu}{2^\nu \Gamma(\nu + 1/2) \Gamma(1/2)} \left( 1 + \frac{1}{\nu + 1/2} \right).
\]

We remark that we here consider \( V = 0 \), hence the smallest eigenvalue \( \nu_0 > (n-2)/2 \). By Hörmander’s \( L^\infty \)-estimate, e.g. see \cite{19}, we know that \( \| \varphi_j \|_{L^\infty} \lesssim C \nu^{(n-1)/2} \). Hence we obtain that

\begin{equation}
\left| \left( \frac{d}{d\lambda} \right)^\alpha \left( (\lambda^2 r r')^{-\frac{n-2}{2}} \sum_j \varphi_j(y) \overline{\varphi_j(y')} J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda r') \chi(\lambda r) \chi(\lambda r') \right) \right| \lesssim \lambda^{-\alpha} \sum_j \nu_j^{\alpha-1} \frac{(\lambda r)^{\nu_j - \frac{n-2}{2}}}{2^{\nu_j} \Gamma(\nu_j + 1/2)} \frac{(\lambda r')^{\nu_j - \frac{n-2}{2}}}{2^{\nu_j} \Gamma(\nu_j + 1/2)} \chi^{(\alpha_1)}(\lambda r) \chi^{(\alpha_4)}(\lambda r') \lesssim \lambda^{-\alpha} \sum_j \nu_j^{\alpha-1} \frac{R^{2\nu_j -(n-2)}}{2^{2\nu_j} \Gamma(\nu_j + 1/2) \Gamma(\nu_j + 1/2)} \lesssim \lambda^{-\alpha}.
\end{equation}
On the other hand, using notation \( \tilde{z} = (\tilde{r}, y) \), since \( |\tilde{z}| \leq R \) and \( |\tilde{z}'| \leq R \), hence \( d(\tilde{z}, \tilde{z}') \leq C(R) \). Then we have that for any \( K > 0 \)

\[
(3.18) \quad \left( \frac{d}{d\lambda} \right)^{\alpha} \left( \lambda^{2} r \right)^{-\frac{\nu}{2}} \sum_{j} \varphi_{j}(y) \varphi_{j}(y') J_{\nu(r)}(\lambda r) J_{\nu(r')}(\lambda r') \chi(\lambda r) \chi(\lambda r') \right)
\]

\[
\leq C(R) \lambda^{-\alpha} (1 + d(\tilde{z}, \tilde{z}') )^{-K} \leq \tilde{C}(R) \lambda^{-\alpha} (1 + \lambda d(z, z') )^{-K}
\]

which has the property (3.3) of \( b(\lambda, z, z') \).

- **The region away from the cone tip.** Now we consider the rest points which are away from the cone tip. Recalling the definition of the asymptotically conic manifold in the introduction, the metric \( g \), outside the compact set \( \{(r, y) \in X : \frac{1}{2} < r \} \), is a scattering metric. Therefore the method in [24] works well for our situation here, although we only consider the spectral at a fixed energy \( \lambda = 1 \) by using the scaling invariant. Indeed, in this region, the operator \( L \) is a scattering operator introduced by Melrose [30]. Without confusion, we use the notation \( r \) and \( x \) instead of \( \tilde{r} \) and \( \tilde{x} \) for adapting to Melrose’s notation. Write the the operator \( \Delta_g \) on \( X \) as

\[
\Delta_g = -(x^2 \partial_x)^2 + (n - 1) x \partial_r + \frac{\Delta Y}{r^2}
\]

With \( x = 1/r \), it gives

\[
\Delta_g = -(x^2 \partial_x)^2 + (n - 1) x \partial_r + x^2 \Delta Y
\]

which is an elliptic scattering differential operator near \( x = 0 \). Using the scattering symbol calculus defined in [30], we have the principal symbol of \( \Delta_g \) is \( |\mu|_h^2 + \nu^2 \) where \( (\nu, \mu) \) as rescaled cotangent variables. In other word, if \( (\xi, \eta) \) is the dual cotangent variables to \( (x, y) \), then \( \nu = x^2 \xi, \mu = x \eta \). Next we define the operator \( Q_j \) whose micro-support is relied to the characteristic set.

Define \( O_1 \) to consist of points away the cone tip, say \( \tilde{x} < 2 \epsilon \), but away from the characteristic variety, that is, satisfying \( |\mu|_h^2 + \nu^2 < 1/2 \) or \( |\mu|_h^2 + \nu^2 > 3/2 \). Finally divide the set \( \{ \tilde{x} < 2 \epsilon, |\mu|_h^2 + \nu^2 \in [1/4, 2] \} \) into a finite number of sets \( O_2, \ldots, O_N \) such that, for each set \( O_j \) is equal to \( \{ \tilde{x} < 2 \epsilon, |\mu|_h^2 + \nu^2 \in [1/4, 2]; |\nu - \nu_j| \leq \delta \} \), where \( \delta \) is taken sufficiently small such that we can separate the conjugated points and \( \nu_j \) is the finite points in this set, as we did in [24].

Therefore we have an open cover \( O_0 \cup \cdots \cup O_N \) of phase space. We then construct a partition of unity subordinate to the above open cover, and take these as the principal symbols of pseudo-differential operators \( Q_j \) \( (j \geq 1) \) in the scattering pseudo-differential operator class \( \Psi_0^k \) described in [24] and are micro-locally supported in \( O_j \), \( (j \geq 1) \). As mentioned above, the setting is a special case (i.e. \( h \) independent of \( r \)) of [24] in this region. Hence by using [24, Proposition 1.5] at fixed energy \( \lambda = 1 \), we have for \( j \geq 1 \)

\[
Q_j dE_{\sqrt{L_0}}(1; \tilde{r}, y, \tilde{r}', y') Q_j^* = \sum_{\pm} e^{\pm id(\tilde{r}, y, \tilde{r}', y')} a_{\pm}(1; \tilde{r}, y, \tilde{r}', y') + b(1; \tilde{r}, y, \tilde{r}', y').
\]

From the proof of Proposition 1.5 in [24, section 4] and recall \( \tilde{r} = \lambda r, \tilde{r}' = \lambda r' \), we see that

\[
a_{\pm}(1, \tilde{r}, y, \tilde{r}', y') = a_{\pm}(\lambda; r, y, r', y), \quad b(1, \tilde{r}, y, \tilde{r}', y') = b(\lambda; r, y, r', y).
\]
On the other hand, the conic metric distant function is homogeneous such that
\[ d(\lambda r, y, \lambda r', y) = \lambda d(r, y, r', y), \]
which can be obtained from the explicit expression \( d(z, z') = r + r' \) when \( d_Y(y, y') \geq \pi \) and when \( d_Y(y, y') \leq \pi \)
\[ d(z, z') = d(r, y, r', y') = \sqrt{r^2 + r'^2 - 2rr' \cos d_Y(y, y')} \]
Therefore we have (3.2) when \( j \geq 1 \). From [24, Proposition 1.5], we obtain the analogue property (3.3) and (3.5).

Finally we verify (3.3) and (3.6). Recall that \( Q_0 \) is constructed to micro-localizely supported in \( \{|\lambda r| \leq 1\} \) and \( Q_j, (j \geq 1) \) is constructed to micro-localizely supported in \( \{|\lambda r| \gg 1\} \). Indeed we similarly estimate as proving (3.3)
\[
\left| \left( \frac{d}{d\lambda} \right)^\alpha \left( (\lambda^2 r^2)' - \frac{n-2}{2} \sum_j \varphi_j(y) \varphi_j(y') J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda r') \chi(\lambda r)(1 - \chi(\lambda r')) \right) \right| \\
\quad \lesssim \lambda^{-\alpha} \left( \lambda^2 r^2 \right)' - \frac{n-2}{2} \sum_{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \leq \alpha} (\lambda r)^{\alpha_1 + \alpha_3} (\lambda r')^{\alpha_2 + \alpha_4} \\
\quad \times \left| \sum_j \varphi_j(y) \varphi_j(y') J_{\nu_j + \alpha_1} J_{\nu_j + \alpha_2} (\lambda r) (\lambda r')^{(\alpha_3)(\lambda r)(1 - \chi(\lambda r'))} \right| 
\]
If \( \alpha_4 \geq 1 \), then it follows from the same argument
\[
\left| \left( \frac{d}{d\lambda} \right)^\alpha \left( (\lambda^2 r^2)' - \frac{n-2}{2} \sum_j \varphi_j(y) \varphi_j(y') J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda r') \chi(\lambda r)(1 - \chi(\lambda r')) \right) \right| \\
\quad \lesssim \lambda^{-\alpha} \sum_j \nu_j^{\alpha - (n-2)/2} \frac{(\lambda r)^\nu_j - \frac{n-2}{2}}{2^\nu_j \Gamma(\nu_j + \frac{1}{2}) \Gamma(1/2)} \chi^{(\alpha_3)}(\lambda r) \frac{(\lambda r')^{(\nu_j - \frac{n-2}{2}}}{2^\nu_j \Gamma(\nu_j + \frac{1}{2}) \Gamma(1/2)} \chi^{(\alpha_4)}(\lambda r') \\
\quad \lesssim \lambda^{-\alpha} \sum_j \nu_j^{-1} \frac{R^{\nu_j - (n-2)/2}}{2^2 \nu_j \Gamma(\nu_j + \frac{1}{2})} \lesssim \lambda^{-\alpha}. 
\]
From Lemma 2.1, we have the fact that \( |J_{\nu}(r)| \lesssim C r^{-1/3}, r \gg 1 \) with constant \( C \) independent of \( \nu \). When \( \alpha_4 = 0 \), then we obtain
\[
\left| \left( \frac{d}{d\lambda} \right)^\alpha \left( (\lambda^2 r^2)' - \frac{n-2}{2} \sum_j \varphi_j(y) \varphi_j(y') J_{\nu_j}(\lambda r) J_{\nu_j}(\lambda r') \chi(\lambda r)(1 - \chi(\lambda r')) \right) \right| \\
\quad \lesssim \lambda^{-\alpha} \sum_j \nu_j^{\alpha - (n-2)/2} \frac{(\lambda r)^\nu_j - \frac{n-2}{2}}{2^\nu_j \Gamma(\nu_j + \frac{1}{2}) \Gamma(1/2)} \chi^{(\alpha_3)}(\lambda r) (\lambda r')^{-\frac{n-2}{2} - \frac{1}{2}} (1 - \chi(\lambda r')) \\
\quad \lesssim \lambda^{-\alpha} \sum_j \nu_j^{-1} \frac{R^{\nu_j - (n-2)/2}}{2^2 \nu_j \Gamma(\nu_j + \frac{1}{2})} \lesssim \lambda^{-\alpha}. 
\]
This proves the property \( 3.6 \) of \( c(\lambda, z, z') \). Collecting all of these, we prove Proposition 3.1.
3.2. **Strichartz estimate for the case** \( V = 0 \). Our next main result in this section is to prove the Strichartz estimate for the Schrödinger associated with \( \mathcal{L}_0 \).

**Proposition 3.2.** Let \((q, r), (\tilde{q}, \tilde{r})\) be admissible pairs satisfying \((1, 2)\), the following Strichartz estimates hold: the homogeneous inequality
\[
\|e^{it\mathcal{L}_0}u_0\|_{L^q(\mathbb{R}; L^r; \mathcal{X}(X))} \leq C\|u_0\|_{L^2(X)}
\]
and inhomogeneous Strichartz estimate
\[
\|\int_0^te^{i(t-s)\mathcal{L}_0}F(s)ds\|_{L^q(\mathbb{R}; L^r; \mathcal{X}(X))} \leq C\|F\|_{L^{\tilde{q}}(\mathbb{R}; L^{\tilde{r}}; \mathcal{X}(X))}.
\]

**Remark 3.5.** We obtain the full set of global-in-time Strichartz estimate both in homogeneous and inhomogeneous inequalities.

**Proof.** Having the Proposition \(3.1\) in hand, we can follow the argument in \([24]\). For convenience, we sketch the proof here. We first microlocalize (in phase space) Schrödinger propagators \(U_j(t)\) associated with \(Q_j\) by
\[
U_j(t) = \int_0^\infty e^{it\lambda^2} Q_j(\lambda) dE\sqrt{\tau}(\lambda), \quad 0 \leq j \leq N,
\]
where \(Q_j(\lambda)\) is a partition of the identity operator in Proposition \(3.1\). Then the operator \(U_j(t)U_j(s)^*\) is given
\[
U_j(t)U_j(s)^* = \int e^{i(t-s)\lambda^2} Q_j(\lambda) dE\sqrt{\tau}(\lambda)Q_j(\lambda)^*.
\]
Mimicking the argument \([24\) Proposition 5.1], we prove that there exists a constant \(C\) such that
\[
\|U_j(t)\|_{L^2 \to L^2} \leq C.
\]
On the other hand, using the stationary phase argument as did in \([24\) Proposition 6.1], we prove that the uniform boundedness of the dispersive estimate for \(U_j(t)U_j(s)^*\) with norm \(O(|t-s|^{-\frac{\theta}{2}})\). Then by Keel-Tao’s formalism \([26\), we have for each \(j\)
\[
\|U_j u_0\|_{L^q(\mathbb{R}; L^r; \mathcal{X}(X))} \leq C\|u_0\|_{L^2(X)}.
\]
As remarked in \([26\), we sharp the inequality to a Lorentz space norm \(L^{r,2}(X)\). Summing over \(j\), the homogeneous Strichartz estimate \((3.19)\) for \(e^{it\mathcal{L}_0}\) finally is proved. The non-endpoint inhomogeneous Strichartz estimate follows from the homogeneous estimates and the Christ-Kiselev lemma. For the endpoint inhomogeneous estimate, we required additional argument \([24\) section 8] to obtain dispersive estimate on \(U_j(t)U_{j'}(s)^*\) for \(j' \neq j\) and the Keel-Tao’s argument showed the desirable endpoint inhomogeneous Strichartz estimate. We sketch the proof in the following.

Recall that \(Q_j\) with \(j \geq 1\) are micro-localized away from the cone tip, in which as mentioned before the problem are same. By using \([24\) Lemma 8.2] (see also \([15\) Lemmas 5.3 and 5.4]), we can divide \((j, j')\), \(1 \leq j, j' \leq N\) into three classes
\[
\{1, \ldots, N\}^2 = J_{near} \cup J_{not-out} \cup J_{not-\text{inc}},
\]
so that
- if \((j, j') \in J_{near}\), then \(Q_j(\lambda) dE\sqrt{\tau}(\lambda)Q_{j'}(\lambda)^*\) satisfies the conclusions of Proposition \(3.1\).
We use integration by parts and let 

Then we have

\textbf{Proposition 3.3.} Let \( U_j(t) \) be defined in (3.21). Then there exists a constant \( C \) independent of \( t, z, z' \) for all \( j, j' \geq 0 \) such that the following dispersive estimates on \( U_j(t)U_{j'}(s) \) hold:

- If \( (j, j') \in J_{\text{near}} \) or \( (j, j') = (0, j'), (j, 0) \), then for all \( t \neq s \) we have
  \begin{equation}
  \| U_j(t)U_{j'}(s) \|_{L^1 \to L^\infty} \leq C|t - s|^{-n/2},
  \end{equation}

- If \( (j, j') \) such that \( Q_j \) is not outgoing related to \( Q_{j'} \) and \( t < s \), then
  \begin{equation}
  \| U_j(t)U_{j'}(s) \|_{L^1 \to L^\infty} \leq C|t - s|^{-n/2},
  \end{equation}

- Similarly, if \( (j, j') \) such that \( Q_j \) is not incoming related to \( Q_{j'} \), and \( s < t \), then
  \begin{equation}
  \| U_j(t)U_{j'}(s) \|_{L^1 \to L^\infty} \leq C|t - s|^{-n/2}.
  \end{equation}

\textbf{Proof.} The results are proved [24, Lemma 8.6] except for the cases \( (j, j') = (0, j') \) or \( (j, 0) \). If \( (j, j') = (0, j') \) or \( (j, 0) \), by Proposition 3.1 i.e. (3.3), we can use integrate by parts to show (3.23). Indeed, we only need consider the case \( (j, j') = (0, j') \). For \( j = 0 \), then we can use (3.3) with estimate (3.6) to obtain

\begin{equation}
\left| \left( \frac{d}{d\lambda} \right)^N \left( Q_0(\lambda)dE_{\sqrt{\lambda \delta}}(\lambda)Q_{j'}^*(\lambda) \right)(z, z') \right| \leq C_N \lambda^{n-1-N} \quad \forall N \in \mathbb{N}.
\end{equation}

Let \( \phi \in C_c^\infty([\frac{1}{2}, 2]) \) such that \( \sum_{k \in \mathbb{Z}} \phi(2^{-k} \lambda) = 1 \); we denote \( \phi_0(\lambda) = \sum_{k \leq -1} \phi(2^{-k} \lambda) \) and let \( \delta \) be a small constant to be chosen later. Then

\begin{equation}
\left| \int_0^\infty e^{it\lambda^2} \left( Q_0(\lambda)dE_{\sqrt{\lambda \delta}}(\lambda)Q_{j'}^*(\lambda) \right)(z, z') \phi_0(\frac{\lambda}{\delta})d\lambda \right| \leq C \int_0^\delta \lambda^{n-1}d\lambda \leq C\delta^n.
\end{equation}

We use integration by parts \( N \) times and use (3.26) to show

\begin{equation}
\left| \int_0^\infty e^{it\lambda^2} \sum_{k \geq 0} \phi(\frac{\lambda}{2^k \delta}) \left( Q_0(\lambda)dE_{\sqrt{\lambda \delta}}(\lambda)Q_{j'}^*(\lambda) \right)(z, z')d\lambda \right|
\leq \sum_{k \geq 0} \left| \int_0^\infty \left( \frac{1}{2\lambda t} \frac{\partial}{\partial \lambda} \right)^N \left( e^{it\lambda^2} \phi(\frac{\lambda}{2^k \delta}) \right) \left( Q_0(\lambda)dE_{\sqrt{\lambda \delta}}(\lambda)Q_{j'}^*(\lambda) \right)(z, z')d\lambda \right|
\leq C_N |t|^{-N} \sum_{k \geq 0} \int_{n/2 - 10}^{2k + 10} \lambda^{n-1-2N}d\lambda \leq C_N |t|^{-N} \delta^{2N}.
\end{equation}

Choosing \( \delta = |t|^{-\frac{1}{4}} \), we have thus proved

\begin{equation}
\left| \int_0^\infty e^{it\lambda^2} \left( Q_0(\lambda)dE_{\sqrt{\lambda \delta}}(\lambda)Q_{j'}^*(\lambda) \right)(z, z')d\lambda \right| \leq C_N |t|^{-\frac{3}{2}}.
\end{equation}
Therefore we have finished the proof. □

Hence using the above dispersive estimate (3.23)–(3.24) and the Keel-Tao’s argument [26], we show that there exists a constant $C$ such that for each pair $(j, j') \in J_{\text{near}}$ or $(j, j') = (0, j'), (j, 0)$ or $(j, j') \in J_{\text{non-inc}}$

\begin{equation}
\left(3.28\right) \int_{s<t} \langle U_j(t)U_j^*(s)F(s), G(t) \rangle_{L^2} ds dt \leq C \|F\|_{L_t^2 L_x^\frac{n}{n+2}} \|G\|_{L_t^2 L_x^\frac{n}{n+2}},
\end{equation}

and $(j, j') \in J_{\text{non-out}}$

\begin{equation}
\left(3.29\right) \int_{s>t} \langle U_j(t)U_j^*(s)F(s), G(t) \rangle_{L^2} ds dt \leq C \|F\|_{L_t^2 L_x^\frac{n}{n+2}} \|G\|_{L_t^2 L_x^\frac{n}{n+2}}.
\end{equation}

On the other hand, we have proved that for all $0 \leq j \leq N$,

\[ \|U_j(t)u_0\|_{L^2_x} \leq \|u_0\|_{L^2}, \]

hence it follows by duality that for all $0 \leq j', j' \leq N$,

\begin{equation}
\left(3.30\right) \int_{\mathbb{R}^2} \langle U_j(t)U_j^*(s)F(s), G(t) \rangle_{L^2} ds dt \leq C \|F\|_{L_t^2 L_x^\frac{n}{n+2}} \|G\|_{L_t^2 L_x^\frac{n}{n+2}}.
\end{equation}

Subtracting (3.29) from (3.30) shows that (3.28) holds for each pair $(j, j') \in J_{\text{non-out}}$. Therefore we show that (3.28) holds for every pair $(j, j') \in \{0, 1, \cdots, N\}^2$. Then, by summing over all $j$ and $j'$, we obtain

\begin{equation}
\left(3.31\right) \int_{s<t} \langle U(t)U^*(s)F(s), G(t) \rangle_{L^2} ds dt \leq C \|F\|_{L_t^2 L_x^\frac{n}{n+2}} \|G\|_{L_t^2 L_x^\frac{n}{n+2}}.
\end{equation}

which is equivalent to (3.20). □

4. local-smoothing and Strichartz estimates

In this section, we prove a local-smoothing estimate and then obtain the Strichartz estimate by using the Rodnianski-Schlag method [35]. We remark here that we directly prove the local smoothing by a method in [31] avoiding the resolvent estimate of $\mathcal{L}_V$.

Proposition 4.1 (local-smoothing). Let $u$ be the solution of (1.1) and $z = (r, y) \in X$, then there exists a constant $C$ independent of $u_0$ such that

\begin{equation}
\left(4.1\right) \|r^{-\beta} \partial^\alpha \mathcal{L}_V^\frac{s}{2} u(t, z)\|_{L_t^2(R; L_x^2(X))} \leq C \|u_0\|_{H^{2\alpha+s+\beta-1}(X)}
\end{equation}

where $\alpha, s \in \mathbb{R}$ and $1/2 < \beta < 1 + \nu_0$ with $\nu_0$ being the positive square root of the smallest eigenvalue of $\Delta_h + V_0(y) + (n - 2)^2/4$.

Proof. Recall $dv = r^{n-1} dr dh(y)$. By the Plancherel theorem with respect to time $t$, it suffices to estimate

\[ \int_{\mathbb{R}} \int_{X} |\partial_t^\alpha \mathcal{L}_V^\frac{s}{2} u(t, z)|^2 \frac{dt dv}{r^{2\alpha}} = \int_{\mathbb{R}} \int_{X} |r^\alpha \mathcal{L}_V^\frac{s}{2} \hat{u}(z, \tau)|^2 \frac{d\tau dv}{r^{2\alpha}}. \]

Recall

\[ u_0(z) = \sum_{\nu \in \chi_{\infty}} \sum_{\ell=1}^{d(\nu)} a_{\nu, \ell}(r) \varphi_{\nu, \ell}(y), \quad b_{\nu, \ell}(\rho) = (\mathcal{H}_\nu a_{\nu, \ell})(\rho), \]
we have by (2.11) with $F(p^2) = \rho^ne^{it\rho^2}$
\[
\mathcal{L}_V^\frac{\pi}{2}u(t) = \int_\mathbb{R} e^{-it\tau} \mathcal{L}_V^\frac{\pi}{2}u(z)dt
\]
\[= \sum_{\nu \in \mathbb{N}_\infty} \sum_{\ell = 1}^{d(\nu)} \varphi_{\nu,\ell}(y) \int_\mathbb{R} \int_0^\infty (\rho - \frac{\nu - 2}{2} J_\nu(\rho) e^{it(\rho^2 - \tau)} b_{\nu,\ell}(\rho) \rho^{n-1} d\rho dt.
\]
Put this into above formula, we need to estimate
\[
\int_\mathcal{X} \int_0^\infty \sum_{\nu \in \mathbb{N}_\infty} \sum_{\ell = 1}^{d(\nu)} \varphi_{\nu,\ell}(y) \int_\mathbb{R} \int_0^\infty (\rho - \frac{\nu - 2}{2} J_\nu(\rho) e^{it(\rho^2 - \tau)} b_{\nu,\ell}(\rho) \rho^{n-1} d\rho dt| \frac{d\tau dv}{\rho^{2\beta}}
\]
\[= \int_\mathcal{X} \int_0^\infty \sum_{\nu \in \mathbb{N}_\infty} \sum_{\ell = 1}^{d(\nu)} \varphi_{\nu,\ell}(y) \int_0^\infty (\rho - \frac{\nu - 2}{2} J_\nu(\rho) \delta(\tau - \rho^2) b_{\nu,\ell}(\rho) \rho^{n-1} d\rho | \frac{d\tau dv}{\rho^{2\beta}}.
\]
Using the delta function definition and changing the role of $\rho$ and $\tau$, we are reduced to estimate the integral
\[
\int_\mathcal{X} \int_0^\infty \sum_{\nu \in \mathbb{N}_\infty} \sum_{\ell = 1}^{d(\nu)} \varphi_{\nu,\ell}(y)(\sqrt{\rho}) \frac{\nu - 2}{2} J_\nu(\sqrt{\rho}) b_{\nu,\ell}(\sqrt{\rho}) \rho^{n-1} d\rho dt| \frac{d\rho dv}{\rho^{2\beta}}
\]
\[= \int_\mathcal{X} \int_0^\infty \sum_{\nu \in \mathbb{N}_\infty} \sum_{\ell = 1}^{d(\nu)} \varphi_{\nu,\ell}(y)(\rho) \frac{\nu - 2}{2} J_\nu(\rho) b_{\nu,\ell}(\rho) \rho^{2\alpha + s} \rho^{n-2} | \frac{d\rho dv}{\rho^{2\beta}}.
\]
By the orthogonality, since
\[
\int_\mathcal{Y} \sum_{\nu \in \mathbb{N}_\infty} \sum_{\ell = 1}^{d(\nu)} \varphi_{\nu,\ell}(y) J_\nu(\rho) b_{\nu,\ell}(\rho) | dy = \sum_{\nu \in \mathbb{N}_\infty} \sum_{\ell = 1}^{d(\nu)} | J_\nu(\rho) b_{\nu,\ell}(\rho) |^2
\]
we see that the above equals
\[
\sum_{\nu \in \mathbb{N}_\infty} \sum_{\ell = 1}^{d(\nu)} \int_0^\infty \int_0^\infty |(\rho - \frac{\nu - 2}{2} J_\nu(\rho) b_{\nu,\ell}(\rho) \rho^{2\alpha + s} \rho^{n-2} |^2 d\rho dv | \rho^{n-1-2\beta} dr.
\]
To estimate it, we make a dyadic decomposition into the integral. Let $\chi$ be a smoothing function supported in $[1,2]$, we see that the above is less than
\[I_3\]
\[
\sum_{\nu \in \mathbb{N}_\infty} \sum_{\ell = 1}^{d(\nu)} \sum_{M \in 2^\mathbb{Z}} \int_0^\infty \int_0^\infty |(\rho - \frac{\nu - 2}{2} J_\nu(\rho) b_{\nu,\ell}(\rho) \rho^{2\alpha + s} \rho^{n-2} |^2 d\rho dv | \rho^{n-1-2\beta} dr.
\]
\[\lesssim \sum_{\nu \in \mathbb{N}_\infty} \sum_{\ell = 1}^{d(\nu)} \sum_{M \in 2^\mathbb{Z}} \sum_{R \in 2^\mathbb{Z}} M^{n+4\alpha+2\beta-2} \int_0^\infty |(\rho - \frac{\nu - 2}{2} J_\nu(\rho) b_{\nu,\ell}(M \rho) \chi(\rho) |^2 dp dr.
\]
Define
\[G_{\nu,\ell}(R, M) = \int_0^\infty |(\rho - \frac{\nu - 2}{2} J_\nu(\rho) b_{\nu,\ell}(M \rho) \chi(\rho) |^2 dp dr.
\]
Proposition 4.2. We have the following inequality
\[
G_{\nu, \ell}(R, M) \lesssim \begin{cases} 
R^{2\nu - n + 3}M^{-n}\|b_{\nu, \ell}(\rho)\chi(\frac{\rho}{M})\rho^{\frac{n-1}{2}}\|_{L^2}^2, & R \lesssim 1; \\
R^{-(n-2)}M^{-n}\|b_{\nu, \ell}(\rho)\chi(\frac{\rho}{M})\rho^{\frac{n-1}{2}}\|_{L^2}^2, & R \gg 1.
\end{cases}
\]

Proof. To prove (4.5), we break it into two cases.

- **Case 1:** $R \ll 1$. Since $\rho \sim 1$, thus $r \rho \lesssim 1$. By the property of the Bessel function (2.9), we obtain
\[
G_{\nu, \ell}(R, M) \lesssim \int_0^2 \int_0^\infty \frac{(r\rho)^\nu(r\rho)^{-\frac{n-2}{2}}b_{\nu, \ell}(M\rho)\chi(\rho)}{2^\nu \Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})} \rho^\nu d\rho dr 
\lesssim R^{2\nu - n + 3}M^{-n}\|b_{\nu, \ell}(\rho)\chi(\frac{\rho}{M})\rho^{\frac{n-1}{2}}\|_{L^2}^2.
\]

- **Case 2:** $R \gg 1$. Since $\rho \sim 1$, thus $r \rho \gg 1$. We estimate by Lemma 2.2 on Bessel function (4.7)
\[
G_{\nu, \ell}(R, M) \lesssim R^{-(n-2)}\int_0^\infty \|b_{\nu, \ell}(M\rho)\chi(\rho)\|^2 dr d\rho \lesssim R^{-(n-2)}M^{-n}\|b_{\nu, \ell}(\rho)\chi(\frac{\rho}{M})\rho^{\frac{n-1}{2}}\|_{L^2}^2.
\]

Thus we prove (4.5). \qed

Now we turn to estimate
\[
\sum_{\nu \in \mathbb{C}} \sum_{\ell = 1 \atop \ell \in \mathbb{Z}} M^{2\nu} \int_0^\infty \int_0^\infty \frac{(r\rho)^{-\frac{n-2}{2}}J_\nu(\rho)\rho^{2\nu - 2}r \rho \rho^{-2} \chi(\frac{\rho}{M})^2 \rho \rho r \rho - 1 - 2\beta} d\rho dr 
\lesssim \sum_{\nu \in \mathbb{C}} \sum_{\ell = 1 \atop \ell \in \mathbb{Z}} \sum_{R \in \mathbb{Z}} \sum_{M \in \mathbb{Z}} M^{n+4\alpha + 2s + 2\beta - 2} R^{n - 1 - 2\beta} G_{\nu, \ell}(R, M)
\lesssim \sum_{\nu \in \mathbb{C}} \sum_{\ell = 1 \atop \ell \in \mathbb{Z}} \sum_{R \in \mathbb{Z}} M^{n+4\alpha + 2s + 2\beta - 2} R^{n - 1 - 2\beta} R^{2\nu - n + 3} M^{-n}
\begin{align*}
&\quad + \sum_{R \in \mathbb{Z}, R \gg 1} M^{n+4\alpha + 2s + 2\beta - 2} R^{n - 1 - 2\beta} R^{-(n-2)} M^{-n} \|b_{\nu, \ell}(\rho)\chi(\frac{\rho}{M})\rho^{\frac{n-1}{2}}\|_{L^2}^2 \\
&\quad \lesssim \sum_{\nu \in \mathbb{C}} \sum_{\ell = 1 \atop \ell \in \mathbb{Z}} \sum_{R \in \mathbb{Z}} M^{4\alpha + 2s + 2\beta - 2} R^{2\nu - n + 3} M^{-n}
\begin{align*}
&\quad + \sum_{R \in \mathbb{Z}} M^{4\alpha + 2s + 2\beta - 2} R^{2\nu - n + 3} M^{-n} \|b_{\nu, \ell}(\rho)\chi(\frac{\rho}{M})\rho^{\frac{n-1}{2}}\|_{L^2}^2.
\end{align*}
\]

Note that if $1/2 < \beta < 1 + \nu$ the summations in $R$ converges and further converges to $\|u_0\|_{L^2}^2$. Hence we prove (4.3). \qed

Note $\nu_0 > 0$, we can choose $\alpha = s = 0$ and $\beta = 1$ to obtain
Corollary 4.1. Let $u$ be the solution of (1.1), then there exists a constant $C$ independent of $u_0$ such that

\[
\|r^{-1}u(t,z)\|_{L^2_t(\mathbb{R};L^2(x))} \leq C\|u_0\|_{L^2(x)}.
\]

This corollary is enough for our purpose of this paper, we remark the following additional results due to the independent interest of local smoothing. We find that the above argument only gives the result with a small $\epsilon > 0$ loss of regularity

\[
\|r^{-(1/2+\epsilon)}L^\frac{1}{2}_t u(t,z)\|_{L^2_t(\mathbb{R};L^2(x))} \leq C\|u_0\|_{H^s(X)}.
\]

However if we replace the weight $r^{-\beta}$ by $\beta(r)$ where $\beta$ is a smooth function compact supported, we obtain

Corollary 4.2. Let $u$ be the solution of (1.1) and let $\beta \in \mathcal{C}^\infty([0,1])$, then there exists a constant $C$ independent of $u_0$ such that

\[
\|\beta(r)L^\frac{1}{2}_t u(t,z)\|_{L^2_t(\mathbb{R};L^2(x))} \leq C\|u_0\|_{L^2(x)}.
\]

Proof. We use the above argument with $\alpha = 0$ and $s = 1/2$ and replace the weight $r^{-\beta}$ by $\beta(r)$, we will only need to sum in $R \lesssim M$. We modify the argument to obtain

\[
\sum_{\nu \in \chi^\infty} \sum_{\ell = 1}^{d(\nu)} \sum_{M \in 2^\mathbb{Z}} \int_0^\infty \int_0^\infty \left| (r\rho)^{-\frac{n-2}{2}} J_\nu(r\rho)b_{\nu,\ell}(\rho)\rho^{1/2}\rho^{n-2} \chi(\frac{\rho}{M}) \right|^2 \rho d\rho r^{n-1} \beta^2(r) dr
\]

\[
\lesssim \sum_{\nu \in \chi^\infty} \sum_{\ell = 1}^{d(\nu)} \sum_{M \in 2^\mathbb{Z}} \sum_{R \in 2^\mathbb{Z}, R \leq \min\{1,M\}} M^{n-1} R^{n-1} R^{2\nu-n+3} M^{-n}
\]

\[
+ \sum_{R \in 2^\mathbb{Z}, 1 < R \leq M} M^{n-1} R^{n-1} R^{-(n-2)} M^{-n} \|b_{\nu,\ell}(\rho)\chi(\frac{\rho}{M})\rho^{\frac{n-1}{2}}\|^2_{L^2}
\]

\[
\lesssim \sum_{\nu \in \chi^\infty} \sum_{\ell = 1}^{d(\nu)} \sum_{M \in 2^\mathbb{Z}} \|b_{\nu,\ell}(\rho)\chi(\frac{\rho}{M})\rho^{\frac{n-1}{2}}\|^2_{L^2} = \|u_0\|_{L^2(x)}^2.
\]

Now we prove Theorem 1.1. By Duhamel formula and Proposition 3.2, we have for $r \geq 2$

\[
\|e^{it\Delta}u_0\|_{L^2_t(\mathbb{R};L^{2^*}_x)} \lesssim \|e^{it\Delta}u_0\|_{L^2_t(\mathbb{R};L^{2^*}_x)}
\]

\[
\lesssim \|e^{it\Delta}u_0\|_{L^2_t(\mathbb{R};L^{2^*}_x)} + \int_0^t e^{i(t-s)\Delta} V(z)e^{is\Delta}u_0 ds \|_{L^2_t(\mathbb{R};L^{2^*}_x)}
\]

\[
\lesssim \|u_0\|_{L^2_x} + \|V(z)e^{is\Delta}u_0\|_{L^2_t(\mathbb{R};L^{2^*}_x)} + \|rV(z)\|_{L^{n,\infty}} \|r^{-1} e^{is\Delta}u_0\|_{L^2_x}
\]

Note that $\|rV(z)\|_{L^{n,\infty}} \leq C\|V_0\|_{L^{n}(\mathbb{Y})}$. By Corollary 4.1, we have the global-in-time local smoothing

\[
\|r^{-1} e^{it\Delta}u_0\|_{L^2_t(\mathbb{R};L^2(x))} \leq C\|u_0\|_{L^2}.
\]
Hence we prove the homogeneous Strichartz estimate \((1.3)\) for all admissible pair \((q, r)\).
By duality, the estimate is equivalent to
\[
\left\| \int_{\mathbb{R}} e^{i(t-s)\mathcal{L}_V} F(s) ds \right\|_{L^q_t L^r_x} \lesssim \|F\|_{L^q_t L^r_x'},
\]
where both \((q, r)\) and \((\tilde{q}, \tilde{r})\) satisfy \((1.2)\). By the Christ-Kiselev lemma \([9]\), we obtain for \(q > \tilde{q}'\)
\[
(4.12) \quad \left\| \int_{s<t} e^{i(t-s)\mathcal{L}_V} F(s) ds \right\|_{L^q_t L^r_x} \lesssim \|F\|_{L^q_t L^r_x'}.
\]
Notice that \(\tilde{q}' \leq 2 \leq q\), therefore we have proved all inhomogeneous Strichartz estimates except the endpoint \((q, r) = (\tilde{q}, \tilde{r}) = (2, \frac{2n}{n-2})\).

5. Endpoint inhomogeneous Strichartz estimate

In this section, we focus on the inhomogeneous Strichartz estimate at the endpoint \((q, r) = (\tilde{q}, \tilde{r}) = (2, \frac{2n}{n-2})\). We prove the endpoint Strichartz estimate by using the argument of \([11]\) and \([2]\), but we need to prove a resolvent estimate by following the method of \([5]\) in which they proved a resolvent estimate for the Euclidean Laplacian with the perturbation of an inverse-square potential.

5.1. A perturbation and iterated reduction. Recall \(\mathcal{L}_V = \Delta_g + V\) and \(\mathcal{L}_0 = \Delta_g\), define the operators
\[
(5.1) \quad \mathcal{N}_0 F(t) = \int_0^t e^{i(t-s)\mathcal{L}_0} F(s) ds, \quad \mathcal{N} F(t) = \int_0^t e^{i(t-s)\mathcal{L}_V} F(s) ds.
\]
Set \(u(t) = e^{i(t-s)\mathcal{L}_V} F(s)\), then we can write
\[
u(t) = e^{i(t-s)\mathcal{L}_0} F(s) - i \int_s^t e^{i(t-\tau)\mathcal{L}_0} \left( V e^{-i(\tau-s)\mathcal{L}_V} F(\tau) \right) d\tau.
\]
Integrating in \(s \in [0, t]\), we have by Fubnii’s formula
\[
\mathcal{N} F(t) = \int_0^t e^{i(t-s)\mathcal{L}_V} F(s) ds
\]
\[
= \int_0^t e^{i(t-s)\mathcal{L}_0} F(s) ds - i \int_0^t \int_s^t e^{i(\tau-\tau')\mathcal{L}_0} \left( V e^{-i(\tau-s)\mathcal{L}_V} F(\tau) \right) d\tau ds
\]
\[
= \mathcal{N}_0 F(t) - i \int_0^t \int_0^\tau e^{i(\tau-\tau')\mathcal{L}_0} \left( V \int_0^\tau e^{-i(\tau-s)\mathcal{L}_V} F(s) ds \right) d\tau
\]
\[
= \mathcal{N}_0 F(t) - i \mathcal{N}_0 \left( V \mathcal{N} F \right)(t).
\]
Therefore
\[
(5.2) \quad \mathcal{N} F(t) = \mathcal{N}_0 F(t) - i \mathcal{N}_0 \left( V \mathcal{N} F \right)(t).
\]
On the other hand, we have by similar argument
\[
\mathcal{N}_0 F(t) = \mathcal{N} F(t) + i \mathcal{N} \left( \mathcal{N}_0 F \right)(t),
\]
hence
\[ \mathcal{N}F(t) = \mathcal{N}_0 F - i \mathcal{N} (V(\mathcal{N}_0 F)) (t). \]
Plugging it into (5.2), we obtain
\[ \mathcal{N}F = \mathcal{N}_0 F - i \mathcal{N}_0 (V(\mathcal{N}_0 F)) - \mathcal{N}_0 (V(\mathcal{N}(\mathcal{N}_0 F))), \]
that is
\[ (5.3) \quad \mathcal{N}F = \mathcal{N}_0 F - i \langle \mathcal{N}_0 V \mathcal{N}_0 \rangle F - \mathcal{N}_0 (VNV) \mathcal{N}_0 F. \]
From now on, let \((q, r) = (\tilde{q}, \tilde{r}) = (2, \frac{2n}{n-2})\). Hence we need to estimate
\[ \| \mathcal{N}F \|_{L^q(\mathbb{R}; L^{r,2})} \]
\[ \lesssim \| \mathcal{N}_0 F \|_{L^q(\mathbb{R}; L^{r,2})} + \| \langle \mathcal{N}_0 V \mathcal{N}_0 \rangle F \|_{L^q(\mathbb{R}; L^{r,2})} + \| \mathcal{N}_0 (VNV) \mathcal{N}_0 F \|_{L^q(\mathbb{R}; L^{r,2})}. \]
By the inhomogeneous Strichartz estimate in Proposition 3.2, we have
\[ (5.4) \quad \| \mathcal{N}_0 F \|_{L^q(\mathbb{R}; L^{r,2})} \lesssim \| F \|_{L^{q'}(\mathbb{R}; L^{r',2})}. \]
Since \( V = r^{-2} V_0(y) \) and \( V_0 \in C^\infty (Y) \), then one has \( V \in L^{\frac{n}{2}, \infty} \). Thus we obtain from the Strichartz estimate in Proposition 3.2 again
\[ (5.5) \quad \| \langle \mathcal{N}_0 V \mathcal{N}_0 \rangle F \|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}, 2})} \lesssim \| \mathcal{N}_0 F \|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}, 2})} \lesssim \| F \|_{L^{q'}(\mathbb{R}; L^{r',2})}. \]
Using the property of \( V \) again, then one has \( rV \in L^{n, \infty} \) and \( r^2 V \in L^\infty \). Similarly as above, we prove
\[ (5.6) \quad \| \mathcal{N}_0 (VNV) \mathcal{N}_0 F \|_{L^q(\mathbb{R}; L^{r,2})} \lesssim \| \langle VNV \rangle \mathcal{N}_0 F \|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}, 2})} \lesssim \| r^{-1} \mathcal{N}_0 F \|_{L^2(\mathbb{R}; L^{\frac{2n}{n-2}, 2})} \lesssim \| F \|_{L^{q'}(\mathbb{R}; L^{r',2})}. \]
Here we use the following lemma about the local smooth estimate

**Lemma 5.1.** Let \( \mathcal{L}_V \) be as above, then we have
\[ (5.7) \quad \| r^{-1} \int_0^t e^{i(t-s) \mathcal{L}_V} r^{-1} F ds \|_{L^2(\mathbb{R}; L^2)} \lesssim C \| F \|_{L^2(\mathbb{R}; L^2)}. \]

**Proof.** This is the consequence of D’Ancona’s result [11, Theorem 2.3] and the resolvent estimate (5.8) below. \( \square \)

5.2. A resolvent estimate. In this subsection, we show a resolvent estimate by following the method of [3].

**Proposition 5.1.** Let \( \mathcal{L}_V \) be as above. We have the resolvent estimate
\[ (5.8) \quad \sup_{\sigma \notin \mathbb{R}^+} \| r^{-1} (\mathcal{L}_V - \sigma)^{-1} r^{-1} f \|_{L^2(X)} \lesssim C \| f \|_{L^2(X)}. \]
Proof. The proof closely follows from [5]. Let $z = \sqrt{-\sigma}$ with the branch such that \( \text{Re} z = \tau > 0 \), then for given \( f \in L^2(X) \) and \( \sigma \in \mathbb{C} \setminus \mathbb{R}^+ \), we consider the Helmholtz equation
\[
(5.9) \quad \mathcal{L}_V u + z^2 u = f.
\]
By density argument, we can take \( f \in \mathcal{C}_0^\infty(X) \). Then \( u \) is a classical solution of \((5.9)\) and define \( v(r, y) : (0, \infty) \times Y \to \mathbb{C} \) by
\[
v(r, y) = r^{\frac{n+1}{2}} e^{rz} u(r, y).
\]
Then we see that
\[
\partial_r v = r^{\frac{n-1}{2}} e^{rz} \left( \frac{n-1}{2r} u + zu + \partial_r u \right)
- \partial_r^2 v = r^{\frac{n-1}{2}} e^{rz} \left( -\partial_r^2 u - 2 \left( \frac{n-1}{2r} + z \right) \partial_r u - \left( \frac{(n-1)(n-3)}{4r^2} + \frac{(n-1)z}{r} + z^2 \right) u \right)

z \partial_r v = r^{\frac{n+1}{2}} e^{rz} \left( z \partial_r u + \left( \frac{(n-1)z}{2r} + z^2 \right) u \right)
\]
therefore \( v \) satisfies
\[
(5.10) \quad -\partial_r^2 v + 2z \partial_r v + \left( \frac{(n-1)(n-3)}{4} - \Delta_h + V_0(y) \right) \frac{v}{r^2}
\]
\[
\quad = r^{\frac{n-1}{2}} e^{rz} \left( -\partial_r^2 u - \frac{n-1}{r} \partial_r u + \left( -\Delta_h + V_0(y) + z^2 \right) u \right)
\]
\[
\quad = r^{\frac{n-1}{2}} e^{rz} f.
\]
For fixed \( M > m > 0 \), let \( \phi = \phi_{m,M}(r) \) be a smooth cut-off function such that \( 0 \leq \phi \leq 1 \) with being zero outside \([0, M+1]\) and equaling to 1 on \([m, M]\). By multiplying \((5.10)\) by \( re^{-2\tau r} \phi(r) \partial_r \bar{v} \) and taking the real part, we show that
\[
(5.11) \quad -\frac{1}{2} re^{-2\tau r} \phi(r) \partial_r |\partial_r v|^2 + 2\tau r e^{-2\tau r} \phi(r) |\partial_r v|^2
+ \frac{1}{2} r e^{-2\tau r} \phi(r) \left( \frac{(n-1)(n-3)}{4} + V_0(y) \right) \partial_r |v|^2
+ \frac{1}{r} e^{-2\tau r} \phi(r) \text{Re}(-\Delta_h v \partial_r \bar{v}) = \tau^{\frac{n-1}{2}} \phi(r) \text{Re} \left( e^{r(z-2\tau)} \partial_r \bar{v} f \right).
\]
Integrating the above formula on \( X = (0, \infty) \times Y \) but with volume \( drdh \) and performing the integration by parts, we have
\[
\frac{1}{2} \int_0^\infty \int_Y \partial_r \left( r e^{-2\tau r} \phi(r) \right) |\partial_r v|^2 drdh
+ 2\tau \int_0^\infty \int_Y r e^{-2\tau r} \phi(r) |\partial_r v|^2 drdh
- \frac{1}{2} \int_0^\infty \int_Y \partial_r \left( r^{-1} e^{-2\tau r} \phi(r) \right) \left( \frac{(n-1)(n-3)}{4} + V_0(y) \right) |v|^2 drdh
- \frac{1}{2} \int_0^\infty \int_Y \partial_r \left( r^{-1} e^{-2\tau r} \phi(r) \right) |\nabla_h v|^2 drdh = \int_0^\infty \int_Y r^{\frac{n-1}{2}} \phi(r) \text{Re} \left( e^{r(z-2\tau)} \partial_r \bar{v} f \right) drdh.
\]
Furthermore we obtain

\[
\frac{1}{2} \int_0^\infty \int_Y e^{-2\tau r} \phi(r)(1 - 2r\tau) |\partial_r v|^2 drdh + \frac{1}{2} \int_0^\infty \int_Y e^{-2\tau r} \phi(r) |\partial_r v|^2 drdh + 2r \int_0^\infty \int_Y e^{-2\tau r} \phi(r) |\partial_r v|^2 drdh \\
+ \frac{1}{2} \int_0^\infty \int_Y e^{-2\tau r} \phi(r) r^{-2} (1 + 2r\tau) \left( \frac{(n-2)^2}{4} - \frac{1}{4} + V_0(y) \right) |v|^2 drdh
\]

\[
= \int_0^\infty \int_Y r^{n+1}(r) \Re \left(e^{i(z-2\tau)|\partial_r \bar{v}|} \right) drdh.
\]

Therefore we show

\[
\frac{1}{2} \int_0^\infty \int_Y e^{-2\tau r} \phi(r)(1 + 2r\tau) \left( |\partial_r v|^2 - \frac{|v|^2}{4r^2} \right) drdh \\
+ \frac{1}{2} \int_0^\infty \int_Y e^{-2\tau r} \phi(r) r^{-2} (1 + 2r\tau) \left( \frac{(n-2)^2}{4} + V_0(y) \right) |v|^2 + |\nabla_h v|^2 drdh \\
+ \frac{1}{2} \int_0^\infty \int_Y re^{-2\tau r} \phi'(r) \left( |\partial_r v|^2 + \frac{1}{4r^2} |v|^2 - \frac{1}{r^2} (|\nabla_h v|^2 + (V_0(y) + \frac{(n-2)^2}{4})|v|^2) \right) drdh
\]

\[
= \int_0^\infty \int_Y r^{n+1}(r) \Re \left(e^{r(z-2\tau)|\partial_r \bar{v}|} \right) drdh.
\]

On the other hand, since \(-\Delta_h + V_0(y) + (n-2)^2/4\) is positive on \(Y\) with the smallest eigenvalue \(\nu_0^2 > 0\), that is,

\[
\int_Y \left( \left( \frac{(n-2)^2}{4} + V_0(y) \right) |v|^2 + |\nabla_h v|^2 \right) dh \geq \nu_0^2 \int_Y |v(r,y)|^2 dh \geq 0.
\]

Hence we show for \(\forall \epsilon > 0\)

\[
\frac{1}{2} \int_0^\infty \int_Y e^{-2\tau r} \phi(r)(1 + 2r\tau) \left( |\partial_r v|^2 + (\nu_0^2 - \frac{1}{4}) |v|^2 \right) drdh \\
+ \frac{1}{2} \int_0^\infty \int_Y re^{-2\tau r} \phi'(r) \left( |\partial_r v|^2 + \frac{1}{4r^2} |v|^2 - \frac{1}{r^2} (|\nabla_h v|^2 + (V_0(y) + \frac{(n-2)^2}{4})|v|^2) \right) drdh
\]

\[
\leq \int_0^\infty \int_Y r^{n+1}(r) \Re \left(e^{r(z-2\tau)|\partial_r \bar{v}|} \right) drdh \leq \frac{1}{4\epsilon^2} \|rf\|_{L^2}^2 + \epsilon^2 \int_0^\infty \int_Y \phi(r)e^{-2\tau r} |\partial_r v|^2 drdh
\]

**Lemma 5.2.** We have following estimate for \(m \to 0, M \to \infty\)

\[
\int_0^\infty \int_Y re^{-2\tau r} \phi'(r) \left( |\partial_r v|^2 + \frac{1}{4r^2} |v|^2 - \frac{1}{r^2} (|\nabla_h v|^2 + (V_0(y) + \frac{(n-2)^2}{4})|v|^2) \right) drdh \to 0.
\]
\[ \text{Proof.} \text{ Note the fact that the compact support of } \phi'(r) \text{ belongs to } [0, m] \cup [M, M + 1]. \text{ One has } 0 \leq \phi' \leq C/m \text{ on } [0, m] \text{ and } -C \leq \phi' \leq 0 \text{ on } [M, M + 1]. \text{ Thus by } (5.13) \text{ it suffices to show the negative terms} \\
\[
\int_M^{M+1} \int_Y r e^{-2r} \left( |\partial_r v|^2 + \frac{1}{4r^2} |v|^2 \right) dr dh \to 0, \quad \text{as } M \to \infty;
\]
and
\[
\int_0^m \int_Y r e^{-2r} \frac{1}{r^2} \left( |\nabla_h v|^2 + (V_0(y) + \frac{(n-2)^2}{4})|v|^2 \right) dr dh \to 0, \quad \text{as } m \to 0;
\]
which can be proved by using the same argument of proving [5, (2.6), (2.7)]. \]

Therefore by taking \( m \to 0, M \to \infty, \) we have
\[
\frac{1}{2} \int_0^\infty \int_Y e^{-2r} (1 + 2r) \left( (1 - 2r^2) |\partial_r v|^2 + (\nu_0 - \frac{1}{4}) |v|^2 \right) dr dh \leq \frac{1}{4r^2} \frac{\|rf\|_{L^2}}{}.
\]

**Lemma 5.3** (Weighted Hardy’s inequality). \( \text{Let } w \in C^2(\mathbb{R}^+; \mathbb{R}) \text{ satisfy} \\
\[
w(r) \geq 0, w'(r) \leq 0, \quad r(w'(r)^2 + 2w(r)w''(r)) \geq 2w(r)w'(r), \forall r \geq 0.
\]
\( \text{Let } g : \mathbb{R}^+ \to \mathbb{C} \text{ be such that } g(0) = 0. \text{ Then} \\
\[
\int_0^\infty w^2 \frac{|g(r)|^2}{r^2} dr \leq 4 \int_0^\infty w^2 |g'(r)|^2 dr.
\]

**Proof.** This Lemma is the same to [5, Lemma 2.2]. \]

Let \( w(r) = e^{-\frac{r^2}{2}}(1 + 2r^2)^{-1/2} \) then
\[
w'(r) = -2r^2 e^{-\frac{r^2}{2}}(1 + 2r^2)^{-1/2} \leq 0; \\
w''(r) = 2r^2 e^{-\frac{r^2}{2}}(1 + 2r^2)^{-1/2}(-1 + r^2 + r(1 + 2r)^{-1}).
\]

Hence we have
\[
\frac{1}{4} (w')^2 + \frac{1}{2} w w'' - \frac{ww'}{2r} = r^2 e^{-\frac{r^2}{2}}(1 + 2r^2)^{-1}(2 + 3r) \geq 0.
\]

Let
\[
g(r) = \left( \int_Y |v(r, y)|^2 \frac{dh}{dh} \right)^{1/2},
\]
now we verify \( g(0) = 0. \) Indeed we can argue this as [5] did. Note that \( v(r, y) = r^{n-1} e^{r^2 u(r, y)}, \) then
\[
\int_0^1 \frac{|g(r)|^2}{r^2} dr = \int_0^1 \int_Y \frac{e^{2r^2 u(r, y)^2}}{r^2} dr dh \leq C(\tau) \|u/r\|_{L^2(X)} \leq C \|u\|_{H^1(X)}.
\]
And
\[\int_0^1 |g'(r)|^2 dr = \int_0^1 \left| \partial_r \left( \left( \int \left| v(r, y) \right|^2 dh \right)^{1/2} \right) \right|^2 dr \leq \int_0^1 \int \left| \partial_r v(r, y) \right|^2 dh \, dr\]
\[\leq C \left( \int_0^1 \int \left| r^{-1}u(r, y) \right|^2 dh r^{n-1} dr + \tau^2 \int_0^1 \int \left| u(r, y) \right|^2 dh r^{n-1} dr \right) + \int_0^1 \int \left| \partial_r u(r, y) \right|^2 dh r^{n-1} dr \]
\[\leq C(\tau) \|u\|_{H^1(X)}^2 \]

Then we conclude that \( g \in C^{1/2}(0,1) \), thus \( g(0) = 0 \). Applying Lemma 5.3 to \( w(r) = e^{-rr}(1 + 2r\tau)^{1/2} \) and \( g(r) = \left( \int Y |v(r, y)|^2 dh \right)^{1/2} \), we obtain
\[\int_0^\infty \int_Y e^{-2r\tau}(1 + 2r\tau) \frac{|v|^2}{4r^2} dr dh \leq \int_0^\infty \int_Y e^{-2r\tau}(1 + 2r\tau) \left| \partial_r v \right|^2 dr dh.\]

Using (5.19), we obtain from (5.16)
\[\frac{1}{2} \left( \nu^2 - \frac{\epsilon^2}{2} \right) \int_0^\infty \int_Y e^{-2r\tau}(1 + 2r\tau) \frac{|v|^2}{r^2} dr dh \leq \frac{1}{4c^2} \| rf \|_{L^2}.\]

Recall \( v(r, y) = r^{-\frac{n+1}{2}} e^{\epsilon z} u(r, y) \) and choose \( \epsilon \) small enough, we have
\[\|r^{-1}u\|_{L^2(X)} \leq C \| rf \|_{L^2(X)}\]

which implies 5.3.

5.3. An application of the endpoint inhomogeneous Strichartz estimate. As a direct consequence of
\[\left\| \int_{s<t} e^{i(t-s)\mathcal{L}_V} F(s) ds \right\|_{L^2_t(\mathbb{R}; L^\frac{2n}{n+2}(X))} \lesssim \| F \|_{L^2_t(\mathbb{R}; L^\frac{2n}{n+2}(X))},\]
we have the following uniform Sobolev estimate

\[\text{Proposition 5.2. Let } \mathcal{L}_V \text{ be as above. Then the Sobolev inequality holds for}\]
\[\sup_{\sigma \in \mathbb{R}^+} \left\| (\mathcal{L}_V - \sigma)^{-\frac{1}{2}} f \right\|_{L^\frac{2n}{n+2}(X)} \leq C \| f \|_{L^\frac{2n}{n+2}(X)}.\]

In fact, the inhomogeneous Strichartz estimate implies the uniform Sobolev inequality which was pointed out by Thomas Duyckaerts and Colin Guillarmou; we also refer to [24, Remark 8.8].

\[\text{Proof. Choose } w \in C^\infty_0(X) \text{ and } \chi(t) \text{ equal to 1 on } [-T, T] \text{ and zero for } |t| \geq T + 1, \text{ and let } u(t, z) = \chi(t)e^{i\sigma t}w(z). \text{ Then}\]
\[(i\partial_t + \mathcal{L}_V)u = F(t, z), \quad F(t, z) := \chi(t)e^{i\alpha t}(\mathcal{L}_V - \sigma)w(z) + i\chi'(t)e^{i\alpha t}w(z),\]

hence we have
\[u(t, z) = \int_{-\infty}^t e^{i(t-s)\mathcal{L}_V} F(s) ds.\]

Applying the endpoint inhomogeneous Strichartz estimate (5.21), we show
\[\| u \|_{L^2_t(\mathbb{R}; L^\frac{2n}{n+2}(X))} \lesssim \| F \|_{L^2_t(\mathbb{R}; L^\frac{2n}{n+2}(X))}.\]
Finally taking the limit $T \to \infty$ we see that

$$\|u\|_{L^2_t(\mathbb{R}; L^{2n/(n-2)}(X))} = \sqrt{2T} \|u\|_{L^{2n/(n-2)}(X)} + O(1),$$

$$\|F\|_{L^2_t(\mathbb{R}; L^{2n/(n-2)}(X))} = \sqrt{2T} \|(\mathcal{L}_V - \sigma)u\|_{L^{2n/(n-2)}(X)} + O(1).$$

which implies the uniform Sobolev estimate (5.21).

6. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. The key points are the Strichartz estimate in Theorem 1.1 and the Leibniz chain rule in Proposition 2.1.

6.1. Well-posedness theory.

Proposition 6.1 (Local well-posedness theory). Let $n = 3$. Assume that $u_0 \in H^1(X)$. Then there exists $T = T(\|u_0\|_{H^1}) > 0$ such that the equation (1.5) has a unique solution $u$ with

$$u \in C(I; H^1(X)) \cap L^q_t(I; H^1_r(X)), \quad I = [0, T),$$

where the pair $(q, r)$ is an admissible pair as in (1.2).

Proof. We follow the standard Banach fixed point argument to prove this result. To this end, we consider the map

$$\Phi(u(t)) = e^{it\mathcal{L}_V}u_0 - i\gamma \int_0^t e^{i(t-s)\mathcal{L}_V}(\|u\|^2 u(s))ds$$

on the complete metric space $B_T$

$$B_T := \{ u \in Y(I) \triangleq C_t(I; H^1) \cap L^q_0(I; H^{10}_0) : \|u\|_{Y(I)} \leq 2CC_1\|u_0\|_{H^1}\}$$

with the metric $d(u, v) = \|u - v\|_{L^q_0 L^r_0(X \times X)}$ and $(q_0, r_0)$ satisfies (1.2) and

$$\begin{cases} 
(q_0, r_0) = \left(\left(\frac{30}{97}, \frac{20}{97}\right), \quad \text{if } \nu_0 > \frac{2}{3}, \right) \\
\left(\frac{2}{(\nu_0)^2}, (\frac{3}{2\nu_0})^{1 - \frac{1}{2}}, \right) \quad \text{if } \nu_0 \leq \frac{2}{3}
\end{cases}$$

where $\nu_0$ is the positive square root of the smallest eigenvalue of $\Delta + V_0(y) + (n-2)^2/4$.

We need to prove that the operator $\Phi$ defined by (6.2) is well-defined on $B_T$ and is a contraction map under the metric $d$ for $I$.

Let $u \in B_T$. By Sobolev embedding and equivalence of Sobolev spaces,

$$\|u\|_{L^q_0 L^r_0} \leq C\|u\|_{L^q_0 H^{10}_0} \leq C\|u\|_{L^q_0 H^{-1}_0} \leq \tilde{C}\|u_0\|_{H^1}, \quad \theta = \frac{3}{r_0} - \frac{1}{2}.$$

Then, we have by Strichartz estimate and Proposition 2.1

$$\|\Phi(u)\|_{Y(I)} \leq C\|u_0\|_{H^1} + C\|L^{1/2}_V(\|u\|^2 u)\|_{L^2_0 L^{5/2}_r(\mathbb{R}^3 \times \mathbb{R}^3)}$$

$$\leq C\|u_0\|_{H^1} + CC_1\|I\|^{\frac{1}{2} - \frac{2}{\theta}} \|L^{1/2}_V u\|_{L^2_0 L^2_0} \|u\|^2_{L^0_0 L^0_0}.$$

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Note that \( \|u\|_{Y(I)} \leq 2CC_1 \|u_0\|_{H^1} \) if \( u \in B_T \), we see that for \( u \in B_T \),
\[
\|\Phi(u)\|_{Y(I)} \leq C \|u_0\|_{H^1} + \tilde{C} |I|^{\frac{2}{q_0}} (2CC_1 \|u_0\|_{H^1})^3.
\]

Taking \( |I| \) sufficiently small such that
\[
\tilde{C} |I|^{\frac{2}{q_0}} (2CC_1 \|u_0\|_{H^1})^3 \leq \frac{1}{2} \|u_0\|_{H^1},
\]
we have \( \Phi(u) \in B_T \) for \( u \in B_T \). On the other hand, by the same argument as before, we have for \( u, v \in B_T \),
\[
d(\Phi(u), \Phi(v)) \leq C |I|^{\frac{1}{2} - \frac{2}{q_0}} (\|u\|_{Y(I)}^2 + \|v\|_{Y(I)}^2) d(u, v).
\]
Thus we derive by taking \( |I| \) small enough
\[
d(\Phi(u), \Phi(v)) \leq \frac{1}{2} d(u, v).
\]

The standard fixed point argument and applying again the Strichartz estimate gives a unique solution \( u \) of (1.5) on \( I \times X \) which satisfies the bound (6.1).

By using Proposition 6.1, mass and energy conservations, we conclude the proof of global well-posed result of Theorem 1.2 in defocusing case \( \gamma = 1 \).

6.2. Scattering theory. The scattering result of Theorem 1.2 follows from the following Proposition.

**Proposition 6.2** (Small data implying scattering). Let \( n = 3 \). Assume \( \|u_0\|_{H^1(X)} \leq \epsilon \) for a small constant \( \epsilon \). Then, there exists a global solution \( u \) to (1.5). Moreover, the solution \( u \) scatters in sense that there are \( u_\pm \in H^1(X) \) such that
\[
\lim_{t \to \pm \infty} \|u(t) - e^{it\mathcal{L}V} u_\pm\|_{H^1(X)} = 0.
\]

**Proof.** First, we use the fixed point argument to show the global existence. To do this, we consider the map again
\[
\Phi(u(t)) = e^{it\mathcal{L}V} u_0 - i \gamma \int_0^t e^{i(t-s)\mathcal{L}V} (|u|^2 u(s)) ds
\]
on the complete metric space \( B \)
\[
B := \{ u \in S^1(\mathbb{R}) : \|u\|_{S^1(\mathbb{R})} \leq 2C \epsilon \}
\]
with the metric \( d(u, v) = \|u - v\|_{L^q_t L^r_x(I \times X)} \) \( (q_0, r_0) \) is as in (6.3), and define
\[
\|u\|_{S^1(\mathbb{R})} := \sup_{(q, r) \in \Lambda_0 ; q \geq 2} \|u\|_{L^q_t L^r_x(H^1_x(X))}.
\]
Using Strichartz estimate, Proposition 2.4, Hölder’s inequality and Sobolev embedding, we get for \( u \in B \)
\[
\|u\|_{S^1(\mathbb{R})} \leq C\|u_0\|_{H^1(X)} + C\|u\|^2\|u\|_{L^2_t(\mathbb{R}, H^1_{6/5}(X))} \\
\leq C\|u_0\|_{H^1(X)} + C\|u\|_{L^{q_0}_t(\mathbb{R}, H^1_{r_0})}\|u\|^2_{L^2_t(\mathbb{R}, L^2_x)} \\
\leq C\epsilon + C\|u\|_{L^{q_0}_t(\mathbb{R}, H^1_{r_0})}\|u\|_{L^\infty_t(\mathbb{R}, L^p_x)}\|u\|_{L^2_t(\mathbb{R}, L^q_x)} \\
\leq C\epsilon + C\|u\|_{S^1(\mathbb{R})}^3,
\]
where \((q_0, r_0)\) is as in (6.3) and
\[
\frac{1}{2} = \frac{1}{q_0} + \frac{2}{q}, \quad \frac{5}{6} = \frac{1}{r_0} + \frac{2}{r}, \quad \frac{1}{6} = \frac{1}{r_1} + \frac{1}{r}.
\]
By continuous argument, we obtain
\[
\|u\|_{S^1(\mathbb{R})} \leq 2C\epsilon.
\]
Hence, we have \( \Phi(u) \in B \) for \( u \in B \). On the other hand, by the same argument as before, we have for \( u, v \in B \),
\[
d(\Phi(u), \Phi(v)) \leq C\|u - v\|^2\|u\|_{L^2(\mathbb{R}, L^6/5_x)} \\
\leq C\|u - v\|_{L^{q_0}_t(\mathbb{R}, L^{r_0}_x)}(\|u\|_{L^1_t(\mathbb{R}, L^p_x)}^2 + \|v\|_{L^1_t(\mathbb{R}, L^p_x)}^2) \\
\leq C\|(u, v)\|_{S(\mathbb{R})}^2 d(u, v) \\
\leq C\epsilon^2 d(u, v) \leq \frac{1}{2} d(u, v),
\]
providing that \( \epsilon \) is sufficient small.
Therefore, the standard fixed point argument gives a unique global solution \( u \) of (1.3).

Next, we turn to prove the scattering part. By time reversal symmetry, it suffices to prove this for positive times. For \( t > 0 \), we will show that \( v(t) := e^{-it\mathcal{L}_V}u(t) \) converges in \( H^1_x \) as \( t \to +\infty \), and denote \( u_+ \) to be the limit. In fact, we obtain by Duhamel’s formula
\[
(6.6) \quad v(t) = u_0 - i\gamma \int_0^t e^{-i\tau\mathcal{L}_V}(|u|^2u)(\tau)d\tau.
\]
Hence, for \( 0 < t_1 < t_2 \), we have
\[
v(t_2) - v(t_1) = -i\gamma \int_{t_1}^{t_2} e^{-i\tau\mathcal{L}_V}(|u|^2u)(\tau)d\tau.
\]
Arguing as before, we deduce that
\[
\|v(t_2) - v(t_1)\|_{H^1(X)} = \left\| \int_{t_1}^{t_2} e^{-i\tau\mathcal{L}_V}(|u|^{p-1}u)(\tau)d\tau \right\|_{H^1(X)} \\
\leq \left\| |u|^2u \right\|_{L^2_t H^1_{6/5}([t_1, t_2] \times X)} \\
\leq \|u\|_{L^{q_0}_t H^1_{r_0}([t_1, t_2] \times X)} \|u\|_{L^q_t(\mathbb{R}, L^p_x)}^2 \\
\to 0 \quad \text{as} \quad t_1, t_2 \to +\infty.
\]
As \( t \) tends to \(+\infty\), the limitation of (6.6) is well defined. In particular, we find the asymptotic state

\[
u_+ = \nu_0 - i\gamma \int_0^\infty e^{-i\tau L_V(|\nu|^2\nu)(\tau)}d\tau.
\]

Therefore, we conclude the proof of Proposition 6.2. \( \square \)

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