The Analytical Solutions for Stochastic Fractional-Space Burgers’ Equation

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We consider here the stochastic fractional-space Burgers’ equation (SFSBs’E) forced by multiplicative noise. Our goal in this paper is to find the analytical solutions for SFSBs’E via the \((G'/G)\)-expansion method. Also, we generalize some previously results where this equation was not studied before with fractional space and multiplicative noise in the Itô sense. Moreover, we utilize graphical representations to discuss the effect of the stochastic term on the stability of the SFSBs’E solutions.

1. Introduction

The research of fractional differential equations (FDEs) has acquired a lot of interest in the last few years. Fractional derivatives are used in many problems in physics [1, 2], finance [3, 4], biochemistry and chemistry [5], hydrology [6], and biology [7]. One of the most significant advantages of fractional derivatives is that they may be thought of as a superset of integer-order derivatives [8]. In the case of heavy tails or infinity fluctuations, the integer-order derivative fails to fulfill scholars. The lack of a larger neighbourhood is another major reason why integer order is no longer a priority. Also, fractional differentiation gives the probability of various forward and backward motion possibilities. As a result, fractional derivatives can achieve what integer-order derivatives cannot. Due to the importance of FDEs, many researchers have recently concentrated their efforts on finding new exact solutions to FPDIEs including [9–17].

On the other hand, stochastic partial differential equations have become increasingly significant in describing phenomena in biology, chemistry, physics, oceanography, atmosphere, fluid mechanics, etc. [18–21].

In recent years, Zou [22], Mohammed [23], Kamrani [24], Táheri et al. [25], Li and Yang [26], Mohammed et al. [27, 28], and Liu and Yan [29] published some work on approximate solutions of FDEs with stochastic term. However, there is little research on the exact solutions of stochastic FDEs (see for instance [30–32]).

Here, we study the next stochastic fractional-space Burgers’ equation (SFSBs’E):

\[
\frac{\partial}{\partial t}w = D_\alpha^{\alpha}w - wD_\beta^\beta w + \rho \omega \partial_\beta^\beta,
\]

where \(D_\alpha^\alpha\) is a conformable fractional derivative of order \(\alpha \in (0, 1]\), \(\beta(t)\) is the Brownian motion process, \(\rho\) is a noise intensity, and \(\omega \partial_\beta^\beta\) is multiplicative noise in the Itô sense. Burgers’ equation is a basic partial differential equation that arises in a wide range of mathematics applications such as traffic flow, gas dynamics, fluid mechanics, and nonlinear acoustics (cf. [33–35]). Harry Bateman [36] initially proposed the equation in 1915. After then, Johannes Martinus Burgers [37] studied it in 1948.

Many authors have studied equation (1), with \(\rho = 0\) and \(\alpha = 1\) (for instance, [38–41] and references therein). On the
other hand, other authors have discussed the approximate solutions for some stochastic versions of Burgers’ equation such as [42–45]. However, Mohammed et al. [46] have attained the analytical solutions of stochastic Burgers’ equation (1), with \( \alpha = 1 \), derived by multiplicative noise in the Stratonovich sense.

Our objectives in this paper are to find the stochastic fractional-space solutions of SFSBs′E (1) and discuss how the stochastic term affects these solutions. To get these solutions, we apply the \((G'/G)\)-expansion method. Moreover, the findings acquired here are generalization of the previous studies, such as those cited in [40, 41]. This is the first publication that investigated the stochastic fractional-space solution of SFSBs′E (1).

Now, we briefly give an outline of this article. In Section 2, we define the conformable fractional derivative (CFD) and state its features. In Section 3, we define the Brownian motion. In Section 4, the wave equation for equation (1) is derived. In Section 5, by using the \((G'/G)\)-expansion method, the exact solutions of equation (1) is attained, while in Section 6, we provide various graphs to display the impact of the noise on the SFSBs′E solutions. In the end, we introduce the conclusions of this article.

2. CFD and Its Features

We discuss in this section some definitions, theorems, and properties of CFD [47].

Definition: Let \( \Lambda : (0, \infty) \rightarrow \mathbb{R} \); then, the CFD of order \( \alpha \) for \( \Lambda \) is defined as

\[
D_{x}^{\alpha} \Lambda(x) = \lim_{h \to 0^-} \frac{\Lambda(x + h x^{1 - \alpha}) - \Lambda(x)}{h}.
\]

(2)

Theorem: Let \( \Lambda, \phi : (0, \infty) \rightarrow \mathbb{R} \) be differentiable and also \( \alpha \) be differentiable functions; then, the next rule holds

\[
D_{x}^{\alpha} (\Lambda\phi)(x) = x^{1 - \alpha} \phi'(x) \Lambda'(\phi(x)).
\]

(3)

Let us state some features of the CFD:

\[
D_{x}^{\alpha}[a\Lambda(x) + b\phi(x)] = aD_{x}^{\alpha}\Lambda(x) + bD_{x}^{\alpha}\phi(x), \quad a, b \in \mathbb{R},
\]

\[
D_{x}^{\alpha}[C] = 0, \quad C \text{ is a constant},
\]

\[
D_{x}^{\alpha}[x^{\gamma}] = \gamma x^{\gamma - 1 - \alpha}, \quad \gamma \in \mathbb{R},
\]

\[
D_{x}^{\alpha}\phi(x) = x^{1 - \alpha} \frac{d\phi}{dx}.
\]

(4)

3. Brownian Motion

The random movement of atoms or molecules in a fluid or gas caused by collisions with other atoms or molecules is known as Brownian motion (BM). Moreover, a macroscopic (visible) representation of a particle that is influenced by a variety of microscopic random effects can be thought of as BM. The significance of defining and describing the BM was that it endorsed modern atomic theory. Now, mathematical models derived by Brownian motion are now applied in chemistry, biology, physics, engineering, economics, and a variety of other fields. The BM \( \beta(t) \) is characterized by the following facts:

1. \( \beta(t) \) is almost surely continuous
2. \( \beta(0) = 0 \)
3. \( \beta(t) \) has independent increments
4. \( \beta(t) - \beta(s) \sim N(0, t - s) \) for \( 0 \leq s < t \)

Here, \( N(0, t - s) \) is the normal distribution with expected 0 and variance \( t - s \).

4. Wave Equation for SFSBs′E

To find the wave equation for SFSBs′E (1), we perform the following wave transformation:

\[
w(x, t) = \chi(t) e^{(\rho \beta(t) - 1/2 t^{\gamma})}, \quad \rho = \frac{1}{\alpha} x^{\alpha} - ct,
\]

(5)

where \( \chi \) is the deterministic function and \( c \) is a constant. We get by differentiating equation (5) with regard to \( x \) and \( t \)

\[
\tau_{x} = -\rho \chi' + \frac{1}{2} \rho^{2} \chi - \frac{1}{2} \rho^{2} \chi e^{(\rho \beta(t) - 1/2 t^{\gamma})},
\]

\[
D_{x}^{\alpha} w = \chi' e^{(\rho \beta(t) - 1/2 t^{\gamma})}, \quad D_{x}^{2\alpha} w = \chi'' e^{(\rho \beta(t) - r^{2} t^{\gamma})},
\]

(6)

where the term \( +1/2 \rho^{2} \chi \) is the Itō correction. Hence, we have by substituting equation (6) into (1)

\[
\chi'' - \rho \chi' e^{(\rho \beta(t) - 1/2 t^{\gamma})} + \chi' = 0.
\]

(7)

Keeping into mind that \( \chi \) is a deterministic function and taking expectation on both sides, we obtain

\[
\chi'' - \rho \chi' e^{(\rho \beta(t) - 1/2 t^{\gamma})} + \chi' = 0.
\]

(8)

We have for any real number \( \rho \) that \( E(e^{\rho \beta(t)}) = e^{\rho^{2}/2t} \) because \( \beta(t) \) is Brownian process; then, equation (8) has the form

\[
\chi'' - \rho \chi' + \chi' = 0.
\]

(9)

5. The Exact Solutions of the SFSBs′E

To find the solutions of equation (9), we apply the \((G'/G)\)-expansion method [48]. After then, we attain the solutions of SFSBs′E (1). To begin, let us define the solution of equation (9) as

\[
\chi = \sum_{i=0}^{N} b_{i} \left( \frac{G'}{G} \right)^{i},
\]

(10)

where \( b_{i} \) for \( i \in \{0, 1, N\} \) are unknown constants that will have to be determined later, and \( G \) solves

\[
G'' + \lambda G' + \nu G = 0,
\]

(11)

where \( \lambda \) and \( \nu \) are unknown constants. By balancing \( \chi' \) with \( \chi'' \) in equation (9), we have
\[ N = 1. \] (12)

From equation (12), we can rewrite equation (10) as
\[ \chi = b_0 + b_1 \frac{G'}{G}. \] (13)

Putting equation (13) into (12) and utilizing equation (11), we obtain
\[
(2b_1 + b_1^2) \left( \frac{G'}{G} \right)^3 + (3\lambda b_1 + \lambda b_1^2 + b_0 b_1 - cb_1) \left( \frac{G'}{G} \right)^2 + (\lambda^2 b_1 + 2b_1 v + \lambda b_1 b_0 + b_1^2v - c\lambda b_1)
\]
\[ + (\nu \lambda b_1 + \nu b_0 b_1 - c\nu b_1) = 0. \] (14)

Equating each coefficient of \([G'/G]^i\) \((i = 3, 2, 1, 0)\) by zero, we get
\[ 2b_1 + b_1^2 = 0, \]
\[ 3\lambda b_1 + \lambda b_1^2 + b_0 b_1 - cb_1 = 0, \]
\[ \lambda^2 b_1 + 2b_1 v + \lambda b_1 b_0 + b_1^2v - c\lambda b_1 = 0, \]
and
\[ \nu \lambda b_1 + \nu b_0 b_1 - c\nu b_1 = 0. \] (16)

Solving this system, we have two cases as follows. The first case is
\[ b_0 = 0, b_1 = -2, \lambda = \frac{1}{c}, v = 0. \] (17)

Then, equation (9) has the solution
\[ \chi(\mu) = b_1 \left( \frac{G'}{G} \right). \] (18)

Now, we have by solving equation (10) with \(v = 0\)
\[ G(\mu) = c_1 + c_2 \exp(-\lambda \mu), \] (19)
where \(c_1\) and \(c_2\) are constants. Putting equation (19) into (18), we have
\[ \chi(\mu) = -2 \left[ \frac{-\lambda c_2 \exp(-\lambda \mu)}{c_1 + c_2 \exp(-\lambda \mu)} \right] = 2c \left[ \frac{c_2 \exp(-\mu)}{c_1 + c_2 \exp(-\mu)} \right]. \] (20)

Consequently, SFSBsE (1) has the analytical solution
\[ w_1(x,t) = 2c \left[ c_2 \exp(-c(1/\alpha x^\alpha - ct)) \right] \left[ c_1 + c_2 \exp(-c(1/\alpha x^\alpha - ct)) \right] \mu \exp(-\rho \beta t). \] (21)

The second case is
\[ b_0 = c, b_1 = -2, \lambda = 0, v = v. \] (22)

Therefore, equation (9) has the solution
\[ \chi(\mu) = c \left[ \frac{G'}{G} \right]. \] (23)

We have by solving equation (10) with \(\lambda = 0\)
\[ G(\mu) = \begin{cases} c_1 \exp(\sqrt{\nu} \mu) + c_2 \exp(-\sqrt{-\nu} \mu), & \text{if } v < 0, \\ c_1 \cos(\sqrt{\nu} \mu) + c_2 \sin(\sqrt{\nu} \mu), & \text{if } v > 0, \\ c_1 + c_2 \mu, & \text{if } v = 0. \end{cases} \] (24)

Substituting equation (24) into (23), we have
\[ \chi(\mu) = c - 2\sqrt{-\nu} \left[ \frac{c_1 \exp(-\sqrt{-\nu} \mu) + c_2 \exp(-\sqrt{-\nu} \mu)}{c_1 \exp(\sqrt{-\nu} \mu) + c_2 \exp(-\sqrt{-\nu} \mu)} \right] \text{if } v < 0, \] (25)

or
\[ \chi(\mu) = c - 2\sqrt{\nu} \left[ \frac{-c_1 \sin(\sqrt{\nu} \mu) + c_2 \cos(\sqrt{\nu} \mu)}{c_1 \cos(\sqrt{\nu} \mu) + c_2 \sin(\sqrt{\nu} \mu)} \right] \text{if } v > 0, \] (26)

or
\[ \chi(\mu) = c - \frac{2c_2}{c_1 + c_2 \mu} \text{if } v = 0. \] (27)

As a result, SFSBsE (1), by using equation (2), has the exact solution
\[
w_2(x,t) = e^\left(\rho \beta (t - 1/2 \rho \beta t)\right) \left\{ c - 2\sqrt{-\nu} \left[ \frac{c_1 \exp(-\sqrt{-\nu}(x^\alpha /\alpha - ct)) - c_2 \exp(-\sqrt{-\nu}(x^\alpha /\alpha - ct))}{c_1 \exp(\sqrt{-\nu}(x^\alpha /\alpha - ct)) + c_2 \exp(-\sqrt{-\nu}(x^\alpha /\alpha - ct))} \right] \right\}, \] (28)

if \(v < 0\), or
\[
w_3(x,t) = e^\left(\rho \beta (t - 1/2 \rho \beta t)\right) \left\{ c - 2\sqrt{\nu} \left[ \frac{-c_1 \sin(\sqrt{\nu}(x^\alpha /\alpha - ct)) + c_2 \cos(\sqrt{\nu}(x^\alpha /\alpha - ct))}{c_1 \cos(\sqrt{\nu}(x^\alpha /\alpha - ct)) + c_2 \sin(\sqrt{\nu}(x^\alpha /\alpha - ct))} \right] \right\}, \] (29)

if \(v > 0\), or
$$w_4(x, t) = e^{ \left( \frac{\rho \beta(t) - 1/2 \rho^2 t}{\alpha} \right)} \left[ c - \frac{2c_2}{c_1 + c_2(x/\alpha - ct)} \right]$$ \text{if } \gamma = 0. \tag{30}

Special cases are as follows:

Case 1: taking $c_1 = c_2 = 1$, equation (28) becomes

$$w(x, t) = \left\{ c - 2\sqrt{-\gamma} \tanh\left( \sqrt{-\gamma} \left( \frac{1}{\alpha} x^2 - ct \right) \right) \right\} e^{\left( \rho \beta(t) - 1/2 \rho^2 t \right)},$$ \text{if } \gamma < 0. \tag{31}

Case 2: taking $c_1 = 1$ and $c_2 = -1$, equation (28) becomes

$$w(x, t) = \left\{ c - 2\sqrt{-\gamma} \coth\left( \sqrt{-\gamma} \left( \frac{1}{\alpha} x^2 - ct \right) \right) \right\} e^{\left( \rho \beta(t) - 1/2 \rho^2 t \right)},$$ \text{if } \gamma < 0. \tag{32}

Case 3: taking $c_1 = 1$ and $c_2 = 0$, equation (29) becomes

$$w(x, t) = \left\{ c + 2\sqrt{-\gamma} \tan\left( \sqrt{-\gamma} \left( \frac{1}{\alpha} x^2 - ct \right) \right) \right\} e^{\left( \rho \beta(t) - 1/2 \rho^2 t \right)},$$ \text{if } \gamma > 0. \tag{33}

Case 4: taking $c_1 = 0$ and $c_2 = 1$, equation (29) becomes

$$w(x, t) = \left\{ c - 2\sqrt{-\gamma} \cot\left( \sqrt{-\gamma} \left( \frac{1}{\alpha} x^2 - ct \right) \right) \right\} e^{\left( \rho \beta(t) - 1/2 \rho^2 t \right)},$$ \text{if } \gamma > 0. \tag{34}

Remark 1. If we choose $\rho = 0$, $\alpha = 1$, and $\gamma = c^2/4$ in equations (22) and (33), then we get the same results as in [40, 41] as follows:

$$w(x, t) = c \left\{ 1 + \tan\left( \frac{c}{2} (x - ct) \right) \right\},$$ \text{if } \gamma > 0. \tag{35}

Remark 2. We can utilize a variety of techniques to find different solutions, such as sine-cosine, tanh-coth, exp($-\varphi$)-expansion, auxiliary equation, first integral, Backlund transformation, variational iteration, generalized Riccati equation, Painlevé expansion, and Lie symmetry methods.
Figure 2: The solution $\omega$ in equation (21) with $\alpha = 0.5$.

Figure 3: The solution $\omega$ in equation (30) with $\alpha = 1$. 
We display here the influence of noise on the solutions of SFSBs’E (1). We introduce a number of graphs for a distinct value of \( \rho \) (noise strength). Let us set the parameters \( \alpha = \frac{1}{4} \) and \( c = 1 \) and use the MATLAB program to simulate the solutions \( w(t, x) \) that are defined in equations (21) and (30) for \( t \in [0, 3] \) and \( x \in [0, 6] \) as follows.

In Figures 1–4, the width of the surface extends when the fractional parameter is reduced. On the other hand, when the noise strength \( \rho = 0 \), the surface is not flat (kink-like, waves), as shown in the first graph in each picture. Moreover, when the noise occurs and its strength starts to increase (\( \rho = 0.5, 1, 2 \)), the surface gets more planar after minor transit behaviors. This means that the solutions of stochastic fractional-space Burgers’ equation (1) become stable as a result of the multiplicative noise effect.

**6. The Influence of Stochastic Term on the SFSBs’E**

We applied the MATLAB tool to create some graphs to address the influence of the noise on the solutions of SFSBs’E (1).

**Data Availability**

The data used to support the findings of this study are included within the article.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors equally contributed to the writing of this paper. All authors read and approved the final manuscript.

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