Three Flavor Neutrino Oscillations in Matter: Flavor Diagonal Potentials, the Adiabatic Basis and the CP phase

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We discuss the three neutrino flavor evolution problem with general, flavor-diagonal, matter potentials and a fully parameterized mixing matrix that includes CP violation, and derive expressions for the eigenvalues, mixing angles and phases. We demonstrate that, in the limit that the mu and tau potentials are equal, the eigenvalues and matter mixing angles $\tilde{\theta}_{12}$ and $\tilde{\theta}_{13}$ are independent of the CP phase, although $\tilde{\theta}_{23}$ does have CP dependence. Since we are interested in developing a framework that can be used for $S$ matrix calculations of neutrino flavor transformation, it is useful to work in a basis that contains only off-diagonal entries in the Hamiltonian. We derive the “non-adiabaticity” parameters that appear in the Hamiltonian in this basis. We then introduce the neutrino $S$ matrix, derive its evolution equation and the integral solution. We find that this new Hamiltonian, and therefore the $S$ matrix, in the limit that the mu and tau neutrino potentials are the same, is independent of both $\theta_{23}$ and the CP violating phase. In this limit, any CP violation in the flavor basis can only be introduced via the rotation matrices, and so effects which derive from the CP phase are then straightforward to determine. We show explicitly that the electron neutrino and electron antineutrino survival probability is independent of the CP phase in this limit. Conversely, if the CP phase is nonzero and mu and tau matter potentials are not equal, then the electron neutrino survival probability cannot be independent of the CP phase.

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I. INTRODUCTION

Neutrino flavor transformation in vacuum [1–4] and in matter [5, 6] continues to be the subject of much attention both experimentally and theoretically. On the experimental side we have moved from a situation where little was known about the mixing parameters roughly a decade ago to one where half of the mixing parameters are reasonably well known. The experimental status upon the mixing parameters is that \( \delta m_{12}^2 = 8.0^{+0.4}_{-0.3} \text{ eV}^2, 1.9 	imes 10^{-3} \text{ eV}^2 < |\delta m_{23}^2| < 3.0 \times 10^{-3} \text{ eV}^2, \sin^2 2\theta_{12} = 0.86^{+0.03}_{-0.04}, \sin^2 2\theta_{23} > 0.92, \) and \( \sin^2 2\theta_{13} < 0.19 \) (at 90\%). The sign of \( \delta m_{23}^2 \), whether \( \theta_{23} \) is less than or greater than 45°, and the extent of CP violation or the Majorana phases are not currently known.

On the theoretical side the focus while \( \theta_{12} \) was completely unknown was primarily upon solar neutrinos but after the results of SNO [9] development of neutrino flavor transformation theory has increasingly focused upon supernova neutrinos which display much richer phenomena. Two primary shifts in supernova neutrino flavor transformation theory have occurred. First, simplified and static profiles have increasingly been abandoned, in favor of more realistic, dynamic and turbulent density profiles [10–19]. Secondly, the implications of more complete descriptions of the potentials are now being studied, such as neutrino self interactions [20–51] and the “matter mu-tau” potential [42, 44, 53]. These studies have uncovered new behaviors of the neutrinos, and in the process it has become apparent that the calculational tools employed in the past have limited applicability to these newer situations.

While some progress has been made in joining together these two new developments [54], in this work we are interested primarily in the first. An approximation that is often made in treating the evolution of the neutrino wavefunction through more realistic density profiles is that the mixing of the neutrinos can be treated in a series of two flavor mixing schemes even though there are three neutrino flavors. The computationally desirable reduction from three to two flavors is based upon the observation that typically only two out of the three neutrino states will be mixing with each other at a resonance and that resonances are widely separated. But with the use of more realistic density profiles multiple, closely spaced, resonances can occur and it has been shown [14, 16, 19] that finely grained energy dependent phase effects appear in two flavor mixing. This suggests that the phenomenon also extends to three flavors, although this has not yet been investigated. These effects can only be seen when the phase information of the wavefunctions is retained throughout the entire calculation, so most approximation methods will miss them. The full range of effects of the CP phase [55–57] can also be hard to determine with approximation methods and these effects are purely three-flavor phenomenon. Finally, the small difference between the matter potentials for the \( \mu \) and \( \tau \) flavors may also become important even in cases where the neutrino potentials are small, because this potential difference produces a third matter resonance.

As an alternative to the usual integration of the neutrino wave functions, we consider the \( S \) matrix formulation of neutrino flavor transformation which was shown to be computationally more efficient in the two flavor case [16]. In this paper, we derive a convenient basis which can be used with the \( S \) matrix, which we call the “adiabatic” basis. We shall consider the full, generalized three flavor mixing including all phases and all three matter potentials. In section §II, we derive the eigenvalues and discuss their behavior. In section §III we present expressions for the matter mixing angles and phases and the Hamiltonian in the matter basis. We then derive the three flavor Hamiltonian in the “adiabatic basis” [16] - the basis where the Hamiltonian is completely off-diagonal - in section §IV and from it we discover the expressions for the three-flavor non-adiabaticity parameters. We demonstrate how the \( S \) matrix is found from this Hamiltonian in section §V, its equation of motion and integral solution and discuss its parameterization. We then discuss the two flavor approximation and observe some identities of neutrino propagation that have implications for the CP phase. Finally in section §VI we present our conclusions.

II. THE FLAVOR BASIS

In this section we write down the Hamiltonian which describes neutrino flavor transformation in the flavor basis without approximations and determine its eigenvalues.

The \( 3 \times 3 \) Hamiltonian \( \tilde{H}^{(f)} \) in the flavor basis is composed of two terms: the rotated vacuum Hamiltonian \( UKU^\dagger \) and the matter induced potentials \( V^{(f)} \)'s which are diagonal in the flavor basis if we ignore neutrino collective effects [58]. The full Hamiltonian is

\[
\tilde{H}^{(f)} = UKU^\dagger + V^{(f)}
\] (1)
and $E$ is the neutrino energy, $m_i$ the neutrino masses and $V_a$ the matter potentials that are, possibly, functions of position. We have included here three matter potentials for the neutrinos for generality. We could have removed one of these potentials, since we are free to add to the Hamiltonian an arbitrary multiple of the unit matrix including a term that is a function of position since the only effect of such a term is to introduce a phase. As a consequence of this property only the relative difference between potentials (and eigenvalues) are important for observable quantities, not the absolute values. However, we choose not to impose this feature from the start, and instead allow it to appear automatically as we proceed. For neutrinos, the difference in the potentials $V_\nu$ and $V_\mu$ is the well known $\delta V_{\nu\mu} \approx \sqrt{2} G_F (n_e^- - n_e^+) \ [6]$ where $G_F$ is Fermi’s constant and $n_e$ is the electron number density. For antineutrinos, the potentials change sign. The potential difference between $V_\nu$ and $V_\mu$ arises due to radiative corrections to neutral current scattering and is smaller than $\delta V_{\nu\mu}$ and $\delta V_{\mu\nu}$ by a factor of $\sim 10^{-3}$ in typically encountered matter. Thus, the splitting is usually only important for neutrino propagation through supernova profiles at high density.

In equation (1) $U$ is a unitary matrix that relates the flavor and mass bases. The mixing matrix $U$ has nine elements but the unitary conditions mean that four elements may be expressed in terms of the remaining five after specifying the phase of the determinant. The unitary conditions also place two further restrictions upon the magnitudes of the five independent elements by establishing two relationships between them. Thus, in general, $U$ is parameterized by three magnitudes and six phases. (See e.g. Ref. [59] for discussion of the construction of an SU(3) matrix.) It is traditional to select the independent elements to be those in the first row and last column but other choices are also valid. The three independent magnitudes, which must all be smaller than unity, may be expressed in terms of three mixing angles $\theta_{12}, \theta_{13}$ and $\theta_{23}$. The form for the matrix $U$ that we use for neutrinos is

$$U = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\delta} & 0 \\
0 & 0 & e^{i(\beta+\delta)}
\end{pmatrix}
\begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & s_{13} \\
-s_{12}c_{23}e^{i\epsilon} - c_{12}s_{13}s_{23} & c_{12}c_{23}e^{i\epsilon} - s_{12}s_{13}s_{23} & c_{13}s_{23} \\
(s2_{12}s_{23}e^{i\epsilon} - c_{12}s_{13}s_{23}) & -(s2_{12}s_{23}e^{i\epsilon} - c_{12}s_{13}s_{23}) & c_{13}c_{23}
\end{pmatrix}
\begin{pmatrix}
e^{-i\alpha_1} & 0 & 0 \\
0 & e^{i\alpha_2} & 0 \\
0 & 0 & e^{-i\alpha_3}
\end{pmatrix}
$$

where $c_{12} = \cos \theta_{12}$, $s_{12} = \sin \theta_{12}$ etc. and $\alpha_1$, $\alpha_2$, $\alpha_3$, $\beta$, $\delta$ and $\epsilon$ are the six phases. For antineutrinos, instead of $U$ we take $U^*$ for the unitary transformation matrix. Other than the three $\alpha$’s we have tried to make the expression for $U$ as similar as possible to that found in [7] but note that in this more general parametrization the CP phase is not $\delta$, but is instead $\epsilon$. The phases $\beta$ and $\delta$ may be absorbed by redefinitions of the neutrino fields in the standard model Lagrangian showing that their absolute values are not observable. This degree of freedom can be used to recover the usual convention for CP by setting $\delta=-\epsilon$. Even though the values of these two phases cannot affect observables we will continue to keep them because, for consistency, during the calculation we must keep track of their derivatives. The mixing matrix $U$ in equation (4) can be written as the product of six terms

$$U = B(\beta,\delta) \Theta_{23}(\theta_{23}) E(\epsilon) \Theta_{13}(\theta_{13}) \Theta_{12}(\theta_{12}) A(\alpha_1, \alpha_2, \alpha_3),$$

$$= \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\delta} & 0 \\
0 & 0 & e^{i(\beta+\delta)}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\epsilon} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
c_{13} & 0 & s_{13} \\
-s_{12}c_{13} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
e^{-i\alpha_1} & 0 & 0 \\
0 & e^{i\alpha_2} & 0 \\
0 & 0 & e^{-i\alpha_3}
\end{pmatrix}.$$
With this unitary matrix we find that the elements of $\tilde{H}^{(f)}$ are, in full,

\begin{align}
\tilde{H}_{ee} &= V_e + (k_1 c_{12}^2 + k_2 s_{12}^2) c_{13}^2 + k_3 s_{13}^2, \\
\tilde{H}_{e\mu} &= e^{-\delta} \left[ (k_3 - k_1 c_{12}^2 - k_2 s_{12}^2) c_{13} s_{13} s_{23} - (k_1 - k_2) c_{12} s_{12} c_{13} c_{23} e^{-i\tau} \right], \\
\tilde{H}_{e\tau} &= e^{-i(\beta+\delta)} \left[ (k_3 - k_1 c_{12}^2 - k_2 s_{12}^2) c_{13} s_{13} s_{23} + (k_1 - k_2) c_{12} s_{12} c_{13} c_{23} e^{-i\tau} \right], \\
\tilde{H}_{\mu\mu} &= V_\mu + (k_1 + k_2) c_{23} + k_3 s_{23}^2 + (k_1 c_{12}^2 + k_2 s_{12}^2) (s_{13}^2 s_{23}^2 - c_{13}^2 c_{23}^2) + 2 (k_1 - k_2) c_{12} s_{12} s_{13} s_{23} c_e, \\
\tilde{H}_{\mu\tau} &= e^{-i\beta} \left[ (k_3 - k_1 c_{12}^2 - k_2 s_{12}^2) c_{23} s_{23} - (k_1 - k_2) c_{12} s_{12} s_{13} c_{23} e^{-i\tau} \right], \\
\tilde{H}_{\tau\tau} &= V_\tau + (k_1 + k_2) s_{23}^2 + k_3 s_{13}^2 c_{23}^2 + (k_1 c_{12}^2 + k_2 s_{12}^2) (s_{13}^2 c_{23}^2 - s_{23}^2) - 2 (k_1 - k_2) c_{12} s_{12} s_{13} s_{23} c_e. 
\end{align}

The remaining elements are given by $\tilde{H}_{\mu e} = \tilde{H}_{e\mu}^* = \tilde{H}_{e\tau}^* = \tilde{H}_{e\tau}^*$ and $\tilde{H}_{\tau\mu} = \tilde{H}_{\mu\tau}^*$ but, since $V^{(f)}$ is diagonal, all the off-diagonal elements are equal to the vacuum values i.e. $\tilde{H}_{e\mu} = H_{e\mu}$ etc. Note also that $\alpha_1$, $\alpha_2$ and $\alpha_3$ do not appear in these expressions, although the phases $\delta$, $\beta$ and the CP phase $\epsilon$ do appear. We also point out that $H_{\mu\mu} + \tilde{H}_{\tau\tau}, H_{\mu\mu} - |H_{\mu\tau}|^2$ and $|H_{e\mu}|^2 + |H_{e\tau}|^2$ are all independent of $\epsilon$.

The characteristic polynomial for the eigenvalues $k_1, k_2$ and $k_3$ of $\tilde{H}$ can be written as

\begin{equation}
\tilde{k}_i - T \tilde{k}_i^2 + \left( \frac{T^2}{3} + 3Q \right) \tilde{k}_i - \left( \frac{T^3}{2} + \tilde{Q} T + 2 \tilde{R} \right) = 0
\end{equation}

where $\tilde{k}_i$ is any one of the three eigenvalues and $\tilde{T}, \tilde{Q}$ and $\tilde{R}$ are three functions equal to

\begin{align}
\tilde{T} &= \tilde{H}_{ee} + \tilde{H}_{\mu\mu} + \tilde{H}_{\tau\tau}, \\
\tilde{Q} &= -\frac{1}{18} \left[ (\tilde{H}_{ee} - \tilde{H}_{\mu\mu})^2 + (\tilde{H}_{ee} - \tilde{H}_{\tau\tau})^2 + (\tilde{H}_{\mu\mu} - \tilde{H}_{\tau\tau})^2 \right] - \frac{1}{3} \left[ |H_{e\mu}|^2 + |H_{e\tau}|^2 + |H_{\mu\tau}|^2 \right], \\
\tilde{R} &= \frac{1}{54} \left[ 2 \tilde{H}_{ee} - (\tilde{H}_{e\mu} + \tilde{H}_{e\tau}) \right] \left[ 2 \tilde{H}_{\mu\mu} - (\tilde{H}_{e\mu} + \tilde{H}_{e\tau}) \right] \left[ 2 \tilde{H}_{\tau\tau} - (\tilde{H}_{e\mu} + \tilde{H}_{e\tau}) \right] + \frac{1}{2} \left( H_{e\mu} H_{\mu\tau} + H_{e\tau} H_{\mu\tau} H_{\mu\mu} \right) - \frac{1}{6} \left[ |H_{\mu\tau}|^2 \left( 2 \tilde{H}_{ee} - \tilde{H}_{\mu\mu} - \tilde{H}_{\tau\tau} \right) + |H_{e\tau}|^2 \left( 2 \tilde{H}_{ee} - \tilde{H}_{\mu\mu} - \tilde{H}_{\tau\tau} \right) + |H_{e\mu}|^2 \left( 2 \tilde{H}_{ee} - \tilde{H}_{\mu\mu} - \tilde{H}_{\tau\tau} \right) \right]. 
\end{align}

All three functions $\tilde{T}(V_e, V_\mu, V_\tau), \tilde{Q}(V_e, V_\mu, V_\tau)$ and $\tilde{R}(V_e, V_\mu, V_\tau)$ are polynomials in $V_e, V_\mu$, and $V_\tau$ and $\tilde{T}, \tilde{Q}$ and $\tilde{R}$ at a given $V_e, V_\mu$ and $V_\tau$ are independent of the basis. In full these polynomials are

\begin{align}
\tilde{T} &= T + V_e + V_\mu + V_\tau, \\
\tilde{Q} &= Q + 2 q_e V_e + 2 q_\mu V_\mu + 2 q_\tau V_\tau - \frac{1}{18} \left[ \delta V_{e\mu}^2 + \delta V_{e\tau}^2 + \delta V_{\mu\tau}^2 \right], \\
\tilde{R} &= R + r_e V_e + r_\mu V_\mu + r_\tau V_\tau - q_e (V_e^2 + 2 V_\mu V_\tau) - q_\mu (V_\mu^2 + 2 V_e V_\tau) - q_\tau (V_\tau^2 + 2 V_e V_\mu) + \frac{1}{54} (\delta V_{e\mu} + \delta V_{e\tau}) (\delta V_{e\mu} + \delta V_{e\tau}) (\delta V_{e\mu} + \delta V_{e\tau}), 
\end{align}

where $T, Q$ and $R$ are the values of $\tilde{T}, \tilde{Q}$ and $\tilde{R}$ in the vacuum, $\delta V_{e\beta} = V_e - V_\beta, \delta V_{e\beta}^2 = (V_e - V_\beta)^2$ and the six functions $q_e, q_\mu, q_\tau$ and $r_e, r_\mu, r_\tau$ are

\begin{align}
q_e &= \frac{1}{18} (H_{e\mu} + H_{\tau\tau} - 2 H_{ee}), \\
q_\mu &= \frac{1}{18} (H_{ee} + H_{\tau\tau} - 2 H_{e\mu}), \\
q_\tau &= \frac{1}{18} (H_{ee} + H_{e\mu} - 2 H_{\tau\tau}), \\
r_e &= \frac{1}{18} \left[ (H_{ee} - H_{e\mu})^2 + (H_{ee} - H_{e\tau})^2 - 2 (H_{e\mu} - H_{\tau\tau})^2 \right] + 3 \left( |H_{e\mu}|^2 + |H_{e\tau}|^2 - 2 |H_{\mu\tau}|^2 \right), \\
r_\mu &= \frac{1}{18} \left[ (H_{e\mu} - H_{\tau\tau})^2 + (H_{e\mu} - H_{ee})^2 - 2 (H_{\tau\tau} - H_{ee})^2 \right] + 3 \left( |H_{\mu\tau}|^2 + |H_{e\mu}|^2 - 2 |H_{e\tau}|^2 \right), \\
r_\tau &= \frac{1}{18} \left[ (H_{\tau\tau} - H_{ee})^2 + (H_{\tau\tau} - H_{e\mu})^2 - 2 (H_{ee} - H_{e\mu})^2 \right] + 3 \left( |H_{e\tau}|^2 + |H_{\mu\tau}|^2 - 2 |H_{e\mu}|^2 \right). 
\end{align}
In the above expressions, we use $H_{ee}$ etc. to refer to the first element of $H^{(f)} = U K U^\dagger$, i.e. the flavor basis Hamiltonian in vacuum. Since the quantities, $T$, $Q$ and $R$ are independent of the basis and their vacuum values are most easily expressed in the diagonal, mass basis. In this basis their values are

\begin{align}
T &= k_1 + k_2 + k_3, \\
Q &= -\frac{1}{18} \left[ \delta k_{12}^2 + \delta k_{13}^2 + \delta k_{23}^2 \right], \\
R &= \frac{1}{54} \left[ (\delta k_{12} + \delta k_{13}) (\delta k_{21} + \delta k_{23}) (\delta k_{31} + \delta k_{32}) \right],
\end{align}

where $\delta k_{ij} = k_i - k_j$ and $\delta k_{ij}^2 = (k_i - k_j)^2$. The discriminant $\tilde{D} = \tilde{Q}^3 + \tilde{R}^2$ is negative definite consequently all three eigenvalues are real. In terms of $\tilde{T}$, $\tilde{Q}$ and $\tilde{R}$ the eigenvalues $\tilde{k}_1$, $\tilde{k}_2$ and $\tilde{k}_3$ are $[60]:$

\[ \tilde{k}_i = \frac{\tilde{T}}{3} + 2 \sqrt{-\tilde{Q}} \cos \left( \omega \frac{\omega_i}{3} \right), \]

where $\cos \omega = \tilde{R}/\sqrt{-\tilde{Q}^3}$. The three angles $\omega_1$, $\omega_2$ and $\omega_3$ are chosen to ensure that each eigenvalue takes on its appropriate value in vacuum, i.e $\tilde{k}_1$ becomes $k_1$ in the vacuum and similarly $\tilde{k}_2 \rightarrow k_2$ and $\tilde{k}_3 \rightarrow k_3$. As the orderings of the vacuum values depend on the hierarchy, there are two possible choices of values for these angles $\omega_1$, $\omega_2$ and $\omega_3$. In the case of the normal hierarchy (NH) where $k_1 < k_2 < k_3$, the angles must be $\omega_1 = 2\pi$, $\omega_2 = 4\pi$ and $\omega_3 = 0$, while in the case of the inverted hierarchy (IH) where $k_3 < k_1 < k_2$, the angles must be $\omega_1 = 4\pi$, $\omega_2 = 0$ and $\omega_3 = 2\pi$. Antineutrinos have the same vacuum eigenvalues as neutrinos, so these angles are the same for both neutrinos and antineutrinos.

In the limit where $V_\mu$ is $V_\nu = 0$ we find that the characteristic polynomial, equation (7), is independent of the CP phase. It is clear that the vacuum values of $T$, $Q$, and $R$, 11a-11c, are independent of the CP phase, but a careful inspection of equations (9a) - (9c) using previously established identities between the elements of the Hamiltonian shows that the $\epsilon$ dependence drops out here as well. This implies the eigenvalues are also independent of $\epsilon$.

In addition to the hierarchy, the behavior of the matter eigenstates is strongly influenced by whether $\theta_{23} > \pi/4$ or $\theta_{23} < \pi/4$. Thus, when considering all the matter potentials, our present knowledge of the neutrino mixing parameters allows four possibilities for the general evolution of $\tilde{k}_1$, $\tilde{k}_2$ and $\tilde{k}_3$ [42] with $V$. These are shown in figure (1). The solid vertical line in the middle of each panel indicates vacuum with $V_\nu = 0$, so the left portion of each panel corresponds to antineutrinos and the right portion to neutrinos. Note that in the figure, the axes are scaled linearly, not logarithmically. To make the figure we set both $\theta_{12}$ and $\theta_{13}$ to be small and included a non-zero potential for $V_\tau$ proportional to, but smaller in magnitude than, $V_\nu$ and set $V_\mu = 0$. In reality, $\theta_{12} \approx 34$ degrees, but the behavior of the eigenstates is still qualitatively similar to what is shown in the figure. Also $V_\mu$ is not zero, but in order to best illustrate the general behavior of the eigenvalues in a figure, we have used the freedom of adding to the Hamiltonian a multiple of the unit matrix.

When the mixing angles are small each eigenvalue tends to track one of the potentials. At “resonances” they switch which potential they follow therefore ensuring that the ordering of the eigenvalues is invariant. With three potentials there are three resonances commonly referred to as the “L”, “H” and the “$\mu\tau$” resonance. The L resonance, L standing for “low density”, occurs at the smallest positive value of $V_\nu$. From there, $\delta m_{21}^2$ is known to be positive, the two states that mix are always the neutrino states $\tilde{\nu}_2$ and $\tilde{\nu}_3$, for an inverted hierarchy (IH) it is the antineutrinos states $\tilde{\bar{\nu}}_1$ and $\tilde{\bar{\nu}}_3$. The H resonance, H standing for “high density” occurs at the intermediate value of $|V_\nu|$. Which two states mix depends upon the hierarchy: for a normal hierarchy (NH) it is states $\tilde{\nu}_2$ and $\tilde{\nu}_3$, for an inverted hierarchy (IH) it is the antineutrinos states $\tilde{\bar{\nu}}_1$ and $\tilde{\bar{\nu}}_3$. The two states that mix at the $\mu\tau$ resonance, which occurs at the highest value of $|V_\nu|$ depends both upon the hierarchy and whether $\theta_{23} < \pi/4$ or $\theta_{23} > \pi/4$. For a NH and $\theta_{23} < \pi/4$ it is states $\tilde{\nu}_1$ and $\tilde{\nu}_2$ but if $\theta_{23} > \pi/4$ then it is states $\tilde{\nu}_2$ and $\tilde{\nu}_3$. In an IH states $\tilde{\bar{\nu}}_1$ and $\tilde{\bar{\nu}}_2$ mix when $\theta_{23} < \pi/4$ but $\tilde{\bar{\nu}}_1$ and $\tilde{\bar{\nu}}_3$ when $\theta_{23} > \pi/4$. Out of the four possibilities the only one which causes two eigenvalues to mix twice, when considering only a monotonically decreasing density profile, is the scheme of a NH and $\theta_{23} < \pi/4$ because one sees that states $\tilde{\nu}_1$ and $\tilde{\nu}_2$ mix once at the $\mu\tau$ resonance and again at the L resonance. This double mixing of $\tilde{\nu}_1$ and $\tilde{\nu}_2$ raises the possibility that interference effects might appear even in monotonic profiles. We will examine the consequences of this in future work, but now we use the results of this section to compute the Hamiltonian in the matter basis for use with the $S$ matrix.

III. THE MATTER BASIS

In this section, while keeping track of all the angles, phases and their derivatives, we write down the Hamiltonian in the matter basis. We then make a few remarks about the large density limit.
FIG. 1: The four possibilities for ordering and evolution of the eigenvalues $\tilde{k}_1$, $\tilde{k}_2$ and $\tilde{k}_3$. The potential $V_\mu$ is set to zero while $V_\tau \propto V_e$ but $|V_e| > |V_\tau|$. The solid vertical line is located at $V_e = 0$. The vacuum mixing angles $\theta_{12}$ and $\theta_{13}$ were set to small values so that the eigenvalues closely followed the potentials. In all four panels the solid line is $\tilde{k}_1$, the short dashed line is $\tilde{k}_2$ and the long dashed line is $\tilde{k}_3$. In the top two panels the hierarchy is normal, and in the bottom pair the hierarchy is inverted (IH). For the left pair of panels the vacuum mixing angle $\theta_{23}$ obeys $\theta_{23} < \pi/4$, for the right pair $\theta_{23} > \pi/4$. The units of either axis are linear but arbitrary.

With the eigenvalues determined we can begin by deducing the angles $\tilde{\theta}_{12}$, $\tilde{\theta}_{13}$, $\tilde{\theta}_{23}$ and phases $\tilde{\beta}$, $\tilde{\delta}$ and $\tilde{\epsilon}$ in the unitary transformation $\tilde{U}$ that relates $\tilde{H}^{(f)}$ to the matrix where $\tilde{k}_1$, $\tilde{k}_2$ and $\tilde{k}_3$ appear on the diagonal i.e

$$
\tilde{U}^\dagger \tilde{H}^{(f)} \tilde{U} = \tilde{K} = \begin{pmatrix}
\tilde{k}_1 & 0 & 0 \\
0 & \tilde{k}_2 & 0 \\
0 & 0 & \tilde{k}_3
\end{pmatrix}.
$$

(13)

After making the change of basis we find that the Schrodinger equation is
\[
\frac{d\psi^{(m)}}{dx} = \left( \hat{K} - i\hat{U}^\dagger \frac{d\hat{U}}{dx} \right) \psi^{(m)}, \tag{14a}
\]
\[
= \hat{H}^{(m)} \psi^{(m)}. \tag{14b}
\]

The term \( \hat{U}^\dagger \frac{d\hat{U}}{dx} \) appears because the eigenvalues are functions of position which requires that \( \hat{U} \) also be a function of position. Obviously if \( \hat{U}^\dagger \frac{d\hat{U}}{dx} = 0 \) then the matter basis Hamiltonian would be diagonal but this can only occur if the density is constant otherwise, in general, \( \hat{H}^{(m)} \) has non-zero off-diagonal elements that come from \( \hat{U}^\dagger \frac{d\hat{U}}{dx}. \) In order to evaluate \( \hat{H}^{(m)} \) we need to compute \( \hat{U}^\dagger \frac{d\hat{U}}{dx}. \)

The matter mixing matrix \( \hat{U} \) is parameterized in exactly the same way as \( U \) in equation (4) i.e. as

\[
\hat{U} = \begin{pmatrix}
1 & 0 & 0 \\
0 & e^{i\beta} & 0 \\
0 & 0 & e^{i(\beta+\delta)}
\end{pmatrix}
\begin{pmatrix}
\hat{c}_{12}
\hat{c}_{13}
-\hat{s}_{12}
\hat{c}_{23}
\hat{s}_{12}
\hat{c}_{23}
\hat{s}_{12}
\hat{s}_{13}
\hat{c}_{13}
\hat{c}_{13}
\hat{c}_{23}
\end{pmatrix}
\begin{pmatrix}
\hat{\varepsilon}
\hat{s}_{13}
\hat{c}_{13}
\hat{s}_{12}
\hat{c}_{23}
\hat{s}_{12}
\hat{s}_{13}
\hat{c}_{13}
\hat{c}_{13}
\hat{c}_{23}
\end{pmatrix} (e^{-i\alpha_1} & 0 & 0 \\
0 & e^{-i\alpha_2} & 0 \\
0 & 0 & e^{-i\alpha_3}) \tag{15}
\]

where \( \hat{c}_{ij} = \cos \hat{\theta}_{ij} \) etc. After some rather lengthy algebra we find that the matter mixing angles \( \hat{\theta}_{12}, \hat{\theta}_{13} \) and \( \hat{\theta}_{23} \), and the matter phases \( \beta, \delta \) and \( \epsilon \) can be expressed in terms of the eigenvalues and the elements of the flavor basis Hamiltonian as

\[
\tan^2 \hat{\theta}_{12} = -\frac{\hat{\delta}_{k_{13}} [ (\hat{H}_{\mu\mu} - \hat{k}_2)(\hat{H}_{\tau\tau} - \hat{k}_2) - |\hat{H}_{\mu\tau}|^2 ]}{\hat{\delta}_{k_{23}} [ (\hat{H}_{\mu\mu} - \hat{k}_3)(\hat{H}_{\tau\tau} - \hat{k}_3) - |\hat{H}_{\mu\tau}|^2 ]}, \tag{16a}
\]
\[
\sin^2 \hat{\theta}_{13} = \frac{(\hat{H}_{\mu\mu} - \hat{k}_3)(\hat{H}_{\tau\tau} - \hat{k}_3) - |\hat{H}_{\mu\tau}|^2}{\hat{\delta}_{k_{13}} \hat{\delta}_{k_{23}}}, \tag{16b}
\]
\[
\tan^2 \hat{\theta}_{23} = \frac{\hat{H}_{\tau\mu} H_{\tau\epsilon} - H_{\mu\epsilon} (\hat{H}_{\tau\tau} - \hat{k}_3)}{\hat{H}_{\tau\mu} H_{\mu\epsilon} - H_{\mu\tau} (\hat{H}_{\mu\mu} - \hat{k}_3)} \left( \frac{H_{\epsilon\tau} H_{\tau\mu} - H_{\epsilon\mu} (\hat{H}_{\tau\tau} - \hat{k}_3)}{H_{\epsilon\mu} H_{\mu\tau} - H_{\epsilon\tau} (\hat{H}_{\mu\mu} - \hat{k}_3)} \right), \tag{16c}
\]
\[
\frac{\hat{\delta}}{2} = \frac{H_{\tau\epsilon} H_{\mu\epsilon} - H_{\mu\tau} (\hat{H}_{\tau\tau} - \hat{k}_3)}{H_{\tau\mu} H_{\epsilon\epsilon} - H_{\epsilon\tau} (\hat{H}_{\mu\mu} - \hat{k}_3)}. \tag{16d}
\]

These equations match those already written down in the literature in the appropriate limits. The last equation, (16f), for \( \epsilon \) is the Naumov [61] and Harrison & Scott [62] identity and comes from the fact that the off-diagonal elements of \( \hat{H} \) given in equations (6b), (6c) and (6e) are not functions of position. The Toshev [63] and Kimura, Takamura & Yokomakura [64] identities are no longer valid when \( \delta V_{\mu\tau} \neq 0 \). Other than a re-arrangement of the ordering of the eigenvalues and some notational changes the expressions for \( \tan^2 \hat{\theta}_{12} \) and \( \sin^2 \hat{\theta}_{13} \) are the same as those one may find in Bellandi et al. [65]; our expression for \( \tan^2 \hat{\theta}_{23} \) differs from Bellandi et al. because we allow the possibility of \( \epsilon \neq 0 \) and do not set \( \beta = 0 \). When performing numerical calculations, the expressions (16a) - (16f) become increasingly difficult to use when the densities become large. In section III A we discuss this limit, how one can derive asymptotic expressions for equations (16a) to (16f) and demonstrate why the numerical difficulties arise.

The most straightforward way to use these equations is to first derive the eigenvalues \( \hat{k}_1, \hat{k}_2, \hat{k}_3 \) at each density and then determine the matter phases and angles. But since these are trigonometric expressions, more than one value of the angles or phases can satisfy the identity. The matter angles \( \hat{\theta}_{12}, \hat{\theta}_{13} \), always remain the quadrant in which they are first defined, which removes any ambiguity in using equations (16a) and (16b). Furthermore, these two angles are independent of \( \epsilon \) if \( V_{\mu} = V_{\tau} \). This can be observed because we have already established that the eigenvalues and the expressions \( H_{\mu\mu} + H_{\tau\tau}, H_{\mu\mu} H_{\tau\tau} - |H_{\mu\tau}|^2 \) are independent of \( \epsilon \). However \( \hat{\theta}_{23} \) can migrate with \( V \) into the second quadrant if its vacuum value is defined to lie in the first quadrant so we do have an ambiguity to resolve when using equation (16c) to find \( \hat{\theta}_{23} \). An examination of equation (16c) shows that \( \hat{\theta}_{23} = \pi/2 \) when

\[
\hat{k}_3 = \hat{H}_{\mu\mu} - \frac{H_{\tau\mu} H_{\mu\epsilon}}{H_{\tau\tau}} = k_{g0}, \quad \hat{k}_3 = \hat{H}_{\mu\mu} - \frac{H_{\epsilon\mu} H_{\mu\tau}}{H_{\epsilon\tau}} = k_{g0}. \tag{17}
\]
In general $k_{90}$ is complex and so at no density can we obtain an eigenvalue $\tilde{k}_3$ that satisfies the condition and, therefore, $\theta_{23}$ never wanders out from the first quadrant. But when $H_{\mu\nu}H_{\mu\tau}H_{\tau\nu}$ is pure real (which requires the CP violating phase $\epsilon$ to be zero or $\pi$) then $k_{90}$ is also pure real and therefore there is a density at which $\theta_{23}$ approaches and then passes $\pi/2$. Thus we see that the evolution of $\theta_{23}$ with the potential depends upon $\epsilon$ in a way that $\theta_{12}$ and $\theta_{13}$ do not. If we have a case where $\epsilon$ is either zero or $\pi$ then by keeping track of the behavior of these denominators, equation (17), we can resolve the ambiguity and determine the correct quadrant for $\theta_{23}$. If we have set $\theta_{23}$ in the vacuum to be less than $\pi/2$ and $k_3 < k_{90}$ then $\theta_{23}$ must become greater than $\pi/2$ for $k_3 > k_{90}$. If on the other hand we have set $\theta_{23}$ in the vacuum to be less than $\pi/2$ but $k_3 > k_{90}$, then $\theta_{23}$ must remain in the first quadrant for all positive definite $V$. Similar arguments can be made for the zeros of the numerator of equation (16c) that become relevant in the inverted hierarchy. One finds that the numerators vanish at the root $k_0$ and, again, $k_0$ is generally complex except when $\epsilon$ is either zero or $\pi$. So, if $\theta_{23}$ in the vacuum is greater than zero and $k_3 > k_0$ then $\theta_{23} < 0$ if $k_3 < k_0$. The CP phase $\epsilon$ can range from zero to $\pi$, but since we can determine not only $\sin \epsilon$ but also $\cos \epsilon$ there is no ambiguity in the CP phase. In principle, there are ambiguities in the phases $\beta$ and $\delta$, but in practice, only their derivatives are used not their absolute values. Finally we note there are no expressions for $\alpha_1$, $\alpha_2$ and $\alpha_3$ so we are free to pick anything for them including functions of $x$. Finally, once the matter mixing angles and phases are found we note that, like $U$, the mixing matrix $\hat{U}$ can be written as the product of six terms

$$
\hat{U} = B(\tilde{\beta}, \tilde{\delta}) \Theta_{23}(\tilde{\theta}_{23}) E(\tilde{\gamma}) \Theta_{13}(\tilde{\theta}_{13}) \Theta_{12}(\tilde{\theta}_{12}) A(\tilde{\alpha}, \tilde{\alpha}_2, \tilde{\alpha}_3),
$$

$$
(18a)
$$

$$
= \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & e^{\delta} & 0 \\
0 & e^{(\beta+\delta)} & 0 \\
\end{array} \right) \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & \tilde{c}_{23} & \tilde{s}_{23} \\
0 & -\tilde{s}_{23} & \tilde{c}_{23} \\
\end{array} \right) \left( \begin{array}{ccc}
1 & 0 & 0 \\
0 & e^{\epsilon} & 0 \\
0 & 0 & 1 \\
\end{array} \right) \left( \begin{array}{ccc}
\tilde{c}_{13} & 0 & \tilde{s}_{13} \\
0 & 1 & 0 \\
-\tilde{s}_{13} & 0 & \tilde{c}_{13} \\
\end{array} \right) \left( \begin{array}{ccc}
\tilde{c}_{12} & \tilde{s}_{12} & 0 \\
-\tilde{s}_{12} & \tilde{c}_{12} & 0 \\
0 & 0 & 1 \\
\end{array} \right) \left( \begin{array}{ccc}
e^{-i\alpha_1} & 0 & 0 \\
0 & e^{-i\alpha_2} & 0 \\
0 & 0 & e^{-i\alpha_3} \\
\end{array} \right).
$$

$$
(18b)
$$

Next we need to calculate the derivatives of these angles and phases. A surprisingly simple result is that the derivatives of the eigenvalues are given by the expression

$$
\frac{d\tilde{k}_i}{dx} = [\tilde{U}_{ei}]^2 \frac{dV_e}{dx} + |\tilde{U}_{\mu i}|^2 \frac{dV_\mu}{dx} + |\tilde{U}_{\tau i}|^2 \frac{dV_\tau}{dx}.
$$

$$
(19)
$$

By unitarity then $dk_1/dx + dk_2/dx + dk_3/dx = dV_e/dx + dV_\mu/dx + dV_\tau/dx$ which is also an identity that stems from the invariance of the trace of the Hamiltonian. With this result we find

$$
\frac{d\tilde{c}_{12}}{dx} = -\frac{1}{\delta k_{12} \delta k_{23}} \left[ \tilde{c}_{12} \tilde{s}_{12} \tilde{k}_{13} \tilde{\delta}_{k_{23}} \left( (\tilde{c}_{23} - \tilde{s}_{13} \tilde{s}_{23}) \frac{dV_{\mu}}{dx} + (\tilde{s}_{23} - \tilde{s}_{13} \tilde{c}_{23}) \frac{dV_{\tau}}{dx} \right) \right]
$$

$$
(20a)
$$

$$
-\tilde{c}_{12} \tilde{s}_{12} \tilde{k}_{12} \left( \tilde{s}_{23} \frac{dV_{\mu}}{dx} + \tilde{c}_{23} \frac{dV_{\tau}}{dx} \right) - \tilde{s}_{13} \tilde{c}_{23} \tilde{s}_{23} \tilde{c}_{e} \left( \tilde{c}_{12} \tilde{k}_{12} \tilde{s}_{13} \tilde{s}_{23} \tilde{c}_{e} \tilde{d} \frac{dV_{\mu}}{dx} \right),
$$

$$
\frac{d\tilde{c}_{13}}{dx} = -\frac{1}{\delta k_{13} \delta k_{23}} \left[ \tilde{c}_{13} \tilde{s}_{13} \tilde{k}_{13} \tilde{\delta}_{k_{23}} \left( \tilde{s}_{23} \frac{dV_{\mu}}{dx} + \tilde{c}_{23} \frac{dV_{\tau}}{dx} \right) \right]
$$

$$
(20b)
$$

$$
\tilde{c}_{12} \tilde{s}_{12} \tilde{k}_{12} \tilde{\delta}_{k_{23}} \left( \tilde{s}_{23} \frac{dV_{\mu}}{dx} + \tilde{c}_{23} \frac{dV_{\tau}}{dx} \right) + \tilde{c}_{12} \tilde{s}_{12} \tilde{c}_{13} \tilde{s}_{23} \tilde{c}_{e} \tilde{d} \tilde{k}_{12} \frac{dV_{\mu}}{dx} \right),
$$

$$
(20c)
$$

$$
\frac{d\tilde{c}_{23}}{dx} = \frac{1}{\delta k_{12} \delta k_{23}} \left[ \tilde{c}_{23} \tilde{s}_{23} \tilde{k}_{12} \tilde{\delta}_{k_{23}} \left( \tilde{s}_{23} \frac{dV_{\mu}}{dx} + \tilde{c}_{23} \frac{dV_{\tau}}{dx} \right) \right]
$$

$$
(20d)
$$

$$
\frac{d\tilde{c}_{23}}{dx} = \frac{1}{\delta k_{12} \delta k_{23}} \left[ \tilde{s}_{23} \tilde{s}_{13} \tilde{\delta}_{k_{23}} \left( \tilde{c}_{12} \tilde{k}_{12} \tilde{s}_{13} \tilde{s}_{23} \tilde{c}_{e} \tilde{d} \tilde{k}_{23} \frac{dV_{\mu}}{dx} \right) \right],
$$

$$
(20e)
$$

$$
\frac{d\tilde{c}_{23}}{dx} = \frac{1}{\delta k_{12} \delta k_{23}} \left[ \tilde{s}_{23} \tilde{s}_{13} \tilde{\delta}_{k_{23}} \left( \tilde{c}_{12} \tilde{k}_{12} \tilde{s}_{13} \tilde{k}_{12} \tilde{c}_{e} \tilde{d} \tilde{k}_{23} \frac{dV_{\mu}}{dx} \right) \right],
$$

$$
(20f)
$$

Note that $d\tilde{\beta}/dx$, $d\tilde{\delta}/dx$ and $d\tilde{c}/dx$ are all proportional to $\sin \epsilon$ and it is the appearance of this term in equation (20f) that validates our previous statement that $\tilde{c} = 0$ for all $V$ if $\epsilon = 0$. One also sees that when $dV_{\tau} = 0$ there is a very simple relationship between $d\tilde{\beta}/dx$, $d\tilde{\delta}/dx$, $d\tilde{c}/dx$ and $d\tilde{\beta}/dx$ the last of which can be easily solved to give the Toshev identity [63].
So we have finally reached the point where we are able to show that the matrix $\hat{U}^\dagger d\hat{U}/dx$ has the form

$$
\hat{U}^\dagger \frac{d\hat{U}}{dx} = \frac{d\hat{U}_{12}}{dx} \begin{pmatrix}
0 & -e^{-i\delta_{12}} e^{i\delta_{12}} & 0 \\
-e^{-i\delta_{12}} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} + \frac{d\hat{U}_{13}}{dx} \begin{pmatrix}
0 & 0 & c_{12} e^{i\delta_{13}} \\
0 & 0 & -s_{12} e^{-i\delta_{23}} \\
\bar{s}_{12} e^{-i\delta_{12}} & s_{12} e^{-i\delta_{12}} & 0
\end{pmatrix}
$$

$$
+ \frac{d\hat{U}_{23}}{dx} \begin{pmatrix}
-2i c_{12} \bar{s}_{12} \bar{s}_{13} \bar{s}_{2} & s_{13} (\bar{c}_{12} e^{-i\tau} + \bar{s}_{12} e^{i\tau}) e^{i\delta_{12}} & \bar{s}_{12} c_{13} e^{i\delta_{23}} \\
-s_{13} (\bar{c}_{12} e^{-i\tau} + \bar{s}_{12} e^{i\tau}) e^{-i\delta_{12}} & 2i c_{12} \bar{s}_{12} s_{13} \bar{s}_{2} & \bar{s}_{12} c_{13} e^{i\delta_{23}} \\
\bar{s}_{12} c_{13} e^{-i\delta_{12}} & \bar{s}_{12} c_{13} e^{-i\delta_{12}} & 0
\end{pmatrix}
$$

$$
+ \frac{d\hat{U}_{23}}{dx} \begin{pmatrix}
\bar{s}_{13} (\bar{c}_{12} e^{i\tau} + \bar{s}_{12} e^{-i\tau}) e^{i\delta_{12}} & \bar{s}_{12} c_{13} e^{i\delta_{23}} & -s_{12} c_{13} e^{i\delta_{23}} \\
-\bar{s}_{12} c_{13} e^{-i\delta_{12}} & \bar{s}_{12} c_{13} e^{-i\delta_{12}} & -s_{12} c_{13} e^{-i\delta_{23}} \\
\bar{s}_{12} c_{13} e^{-i\delta_{12}} & \bar{s}_{12} c_{13} e^{-i\delta_{12}} & 0
\end{pmatrix}
$$

$$
+ \frac{d\hat{U}_{23}}{dx} \begin{pmatrix}
\bar{s}_{13} (\bar{c}_{12} e^{-i\tau} + \bar{s}_{12} e^{i\tau}) e^{-i\delta_{12}} & \bar{s}_{12} c_{13} e^{-i\delta_{23}} & -s_{12} c_{13} e^{-i\delta_{23}} \\
-\bar{s}_{12} c_{13} e^{i\delta_{12}} & \bar{s}_{12} c_{13} e^{i\delta_{12}} & -s_{12} c_{13} e^{i\delta_{23}} \\
\bar{s}_{12} c_{13} e^{i\delta_{12}} & \bar{s}_{12} c_{13} e^{i\delta_{12}} & 0
\end{pmatrix}
$$

(21)

where the vector $u_{\beta}$ is given by

$$
u_{\beta} = \begin{pmatrix}
\bar{s}_{12} s_{23} e^{i\tau} - \bar{s}_{12} \bar{s}_{13} c_{23}
\end{pmatrix} e^{-i\delta_{12}}, \quad \begin{pmatrix}
\bar{s}_{12} s_{23} e^{i\tau} + \bar{s}_{12} \bar{s}_{13} c_{23}
\end{pmatrix} e^{-i\delta_{12}}, \quad \begin{pmatrix}
\bar{s}_{12} \bar{s}_{23} e^{-i\delta_{12}}
\end{pmatrix}.
$$

(22)

So far $\delta_{1}, \delta_{2}$ and $\delta_{3}$ are unconstrained which permits us to choose anything for them. It is at this point that we make our selection that $\delta_{1}, \delta_{2}$ and $\delta_{3}$ be constants. Since when using the scattering matrix approach, it is convenient to work with a Hamiltonian that has zeros along the diagonal, another, sensible, alternative would be to assign $\delta_{1}, \delta_{2}$ and $\delta_{3}$ so that the diagonal elements of $\hat{U}^\dagger d\hat{U}/dx$ were exactly zero i.e. the contributions from $d\delta_{1}/dx$, $d\delta_{2}/dx$ and $d\delta_{3}/dx$ in the equation above exactly canceled the contributions from $d\delta_{1}/dx$, $d\delta_{2}/dx$, $d\delta_{3}/dx$ and $d\delta_{3}/dx$ that appear along the diagonal. But, as we shall shortly show in sec IV, our desire to cancel the diagonal elements of $\hat{U}^\dagger d\hat{U}/dx$ can be achieved more conveniently by another route hence we opt to set the three phases to be constants.

### A. The High Density Limit

In the limit of high density, the expressions for the matter angles and phases, (16a) to (16f), become difficult to work with. The on-diagonal components of the flavor space Hamiltonian, $H_{ee}$, $H_{\mu\mu}$ and $H_{\tau\tau}$, are dominated by the matter potentials which are much larger than the off-diagonal components. Further, the eigenvalues $k_{1}, k_{2}$, and $k_{3}$ are also dominated by the matter potentials so when using the expressions (16a) to (16f) one is sometimes in the situation where two large numbers must be subtracted to find a small difference. For this reason it is useful to find expansions for the eigenvalues in powers of the potentials. In this section we derive the asymptotic limits of the neutrino and antineutrino eigenvalues when $V_{e} \to \infty$ assuming $V_{e} \gg V_{\mu}$ and $V_{e} \gg V_{\tau}$.

We begin by considering the two quantities $\bar{Q}$ and $\bar{R}$. The original expressions for these quantities may be rewritten as

$$
\bar{Q} = \frac{-V_{e}^2}{9} \left[ 1 - \frac{18 q_{e} + V_{\mu} + V_{\tau}}{V_{e}} \right] + \frac{(V_{\mu}^2 - V_{\mu} V_{e} + V_{e}^2 - 9 (Q + 2 q_{e} V_{\mu} + 2 q_{e} V_{e}))}{V_{e}^2},
$$

(23)

$$
\bar{R} = \frac{V_{e}^2}{27} \left[ 1 - \frac{3 (18 q_{e} + V_{\mu} + V_{\tau})}{2 V_{e}} \right] + \frac{3 (V_{\mu}^2 - 4 V_{\mu} V_{e} + V_{e}^2 - 18 [r_{e} - 2 q_{e} V_{e} - 2 q_{e} V_{\mu}])}{2 V_{e}^2} + \frac{(2 V_{e}^2 - 3 V_{e}^2 V_{e} - 3 V_{e} V_{e}^2 + 2 V_{e}^2 + 54 [R + r V_{e} + r V_{e} - 2 q_{e} V_{e} V_{e} - 2 q_{e} V_{\mu}])}{2 V_{e}^2},
$$

(25)

$$
\bar{R} = \frac{V_{e}^2}{27} \left[ 1 + \frac{3 a q_{e}}{2 V_{e}} \right] + \frac{c R V_{e}^2}{V_{e}^5},
$$

(26)
For antineutrinos

\[
\hat{Q} = \frac{-V_e^2}{9} \left[ 1 + \frac{(18 q_e - V_\mu - V_\tau)}{V_e} + \frac{(V_\mu^2 - V_\mu V_\tau + V_\tau^2 - 9 [Q - 2 q_\mu V_\mu - 2 q_\tau V_\tau])}{V_e^2} \right],
\]

(27)

\[
\hat{R} = \frac{V_e^3}{27} \left[ 1 + \frac{3 (18 q_e - V_\mu - V_\tau)}{2 V_e} - \frac{3 (V_\mu^2 - 4 V_\mu V_\tau + V_\tau^2 - 18 |r_e + 2 q_\mu V_\mu + 2 q_\tau V_\tau|)}{2 V_e^2} \right.

\[
\left. + \frac{(2V_\mu^3 - 3V_\mu^2 V_\tau - 3V_\mu V_\tau^2 + 2V_\tau^3 - 54 [R - r_\mu V_\mu - r_\tau V_\tau - 2q_e V_\mu V_\tau - q_\mu V_\mu^2 - q_\tau V_\tau^2])}{2 V_e^2} \right],
\]

(29)

Using these expressions we find that \( \cos \omega \to 1 \), i.e. \( \omega \to 0 \) in the \( V_e \to \infty \) limit. If we expand \( \cos \omega \) in terms of powers of \( \omega \) then

\[
1 - \frac{\omega^2}{2} + \frac{\omega^4}{24} + \cdots = \frac{1 + a_R/V_e + b_R/V_e^2 + c_R/V_e^3}{1 + a_Q/V_e + b_Q/V_e^2}.
\]

(31)

Solving for \( \omega \) and expanding out the right-hand side of the resulting expression up to order \( 1/V_e^2 \) we are eventually led to

\[
\omega = \sqrt{\frac{3a_Q^2 + 12b_Q - 8b_R}{2V_e}} - \frac{[5a_Q^3 + 12a_Q (b_Q - b_R) + 8c_R]}{4V_e^2 \sqrt{3a_Q^2 + 12b_Q - 8b_R}},
\]

(32a)

\[
\equiv \frac{a_\omega}{V_e} + \frac{b_\omega}{V_e^2},
\]

(32b)

which defines the quantities \( a_\omega \) and \( b_\omega \). When we substitute in the definitions of \( a_Q, b_Q \) etc. we find that

\[
a_\omega = \frac{3\sqrt{3}}{2} \sqrt{4 |H_{\mu\tau}|^2 + \left( \hat{H}_{\mu\mu} - \hat{H}_{\tau\tau} \right)^2},
\]

(33a)

\[
b_\omega = \frac{3\sqrt{3}}{4} \left[ \hat{H}_{\mu\mu} + \hat{H}_{\tau\tau} - 2 H_{ee} \right] \sqrt{4 |H_{\mu\tau}|^2 + \left( \hat{H}_{\mu\mu} - \hat{H}_{\tau\tau} \right)^2}

\[
- \frac{3\sqrt{3}}{2} \left( 2 |H_{e\mu} H_{\mu\tau} H_{e\tau} + H_{e\tau} H_{\mu\mu} H_{\mu\tau}| + (|H_{e\mu}|^2 - |H_{e\tau}|^2) (\hat{H}_{\mu\mu} - \hat{H}_{\tau\tau}) \right)
\]

\[
\sqrt{4 |H_{\mu\tau}|^2 + \left( \hat{H}_{\mu\mu} - \hat{H}_{\tau\tau} \right)^2}.
\]

(33b)

For antineutrinos we find instead that \( \cos \bar{\omega} \to -1 \) which indicates the angle \( \bar{\omega} \) approaches \( \pi \) as \( V_e \to \infty \). In this case we instead expand \( \cos \bar{\omega} \) in terms of \( \bar{\omega} - \pi \) and find the asymptotic limit is

\[
\bar{\omega} = \pi - \frac{a_\omega}{V_e} - \frac{b_\omega}{V_e^2}.
\]

(34)

where \( \bar{a}_\omega \) and \( \bar{b}_\omega \) are given by exactly the same expressions as \( a_\omega \) and \( b_\omega \), although the elements of the Hamiltonian are, in this case, taken from the Hamiltonian appropriate for antineutrinos. With the asymptotic expression for \( \omega \) and \( \bar{\omega} \) determined we can then proceed to the eigenvalues. For neutrinos these are given by

\[
\tilde{k}_i = \frac{(T + V_e + V_\mu + V_\tau)}{3} + \frac{2V_e}{3} \cos \left( \frac{\omega + \omega_i}{3} \right) \sqrt{1 + a_Q/V_e + b_Q/V_e^2}.
\]

(35)

Using the expression for the expansion of \( \omega \) then, up to \( 1/V_e \), we eventually find
\[
\tilde{k}_i = \frac{4 V_e}{3} \cos \left( \frac{\omega_i - \pi}{6} \right) \cos \left( \frac{\omega_i + \pi}{6} \right) + \frac{T + V_{\mu} + V_{\tau}}{3} + \frac{a_Q}{3} \cos \left( \frac{\omega_i}{3} \right) - \frac{2 a_\omega}{9} \sin \left( \frac{\omega_i}{3} \right)
\]

\[
- \frac{1}{9 V_e} \left[ \left( 3 \frac{a_Q^2}{4} - 3 b_Q + \frac{a_\omega^2}{3} \right) \cos \left( \frac{\omega_i}{3} \right) \right. + \left. (a_Q a_\omega + 2 b_\omega) \sin \left( \frac{\omega_i}{3} \right) \right],
\]

\[
\equiv \frac{4 V_e}{3} \cos \left( \frac{\omega_i - \pi}{6} \right) \cos \left( \frac{\omega_i + \pi}{6} \right) + C_i + \frac{a_i}{V_e}.
\]

(36a)

Notice that if \( \omega_i = 2\pi \) or \( \omega_i = 4\pi \) then the term linear in \( V_e \) vanishes: it only survives when \( \omega_i = 0 \). Substituting in for \( a_\omega \) etc. we find

\[
C_i = \frac{H_{ee}}{3} \left[ 1 + 2 \cos \left( \frac{\omega_i}{3} \right) \right] + \left( \frac{H_{\mu\mu} + H_{\tau\tau}}{3} \right) \left[ 1 - \cos \left( \frac{\omega_i}{3} \right) \right] - \frac{\sqrt{4 |H_{\mu\tau}|^2 + (H_{\mu\mu} - H_{\tau\tau})^2}}{\sqrt{3}} \sin \left( \frac{\omega_i}{3} \right),
\]

(37a)

\[
a_i = \sqrt{3} \left[ \frac{2 (H_{\mu\mu} H_{\tau\tau} + H_{\mu\tau} H_{\tau\mu}) + (|H_{\mu\mu}|^2 - |H_{\tau\tau}|^2) (H_{\mu\mu} - H_{\tau\tau})}{|H_{\mu\mu}|^2 + (H_{\mu\mu} - H_{\tau\tau})^2} \right] \sin \left( \frac{\omega_i}{3} \right)
\]

\[
+ \frac{1}{9 V_e} \left[ \left( 3 \frac{a_Q^2}{4} - 3 b_Q + \frac{a_\omega^2}{3} \right) \cos \left( \frac{\omega_i}{3} \right) \right. + \left. (a_Q a_\omega + 2 b_\omega) \sin \left( \frac{\omega_i}{3} \right) \right],
\]

(37b)

Note that when \( \omega_i = 0 \) then \( C_i = H_{ee} \) so that the first two terms of this eigenvalue \( k_i \) are \( k_i = V_e + H_{ee} = \tilde{H}_{ee} \).

The expression for the eigenvalues in the case of antineutrinos is instead

\[
\tilde{k}_i = \frac{(T - V_e - V_{\mu} - V_{\tau})}{3} + \frac{2 V_e}{3} \cos \left( \frac{\omega_i + \pi}{3} \right) \sqrt{1 + \frac{\tilde{a}_Q}{V_e} + \frac{\tilde{b}_Q}{V_e^2}}.
\]

(38)

When we substitute in the asymptotic expansion for \( \tilde{\omega} \) we find

\[
\tilde{k}_i = \frac{-4 V_e}{3} \cos \left( \frac{\tilde{\omega}_i - \pi}{6} \right) \cos \left( \frac{\tilde{\omega}_i + 3 \pi}{6} \right) + \frac{T - V_e - V_{\tau}}{3} + \frac{\tilde{a}_Q}{3} \cos \left( \frac{\tilde{\omega}_i + \pi}{3} \right) + \frac{2 \tilde{a}_\omega}{9} \sin \left( \frac{\tilde{\omega}_i + \pi}{3} \right)
\]

\[
- \frac{1}{9 V_e} \left[ \left( 3 \frac{\tilde{a}_Q^2}{4} - 3 \tilde{b}_Q + \frac{\tilde{a}_\omega^2}{3} \right) \cos \left( \frac{\tilde{\omega}_i + \pi}{3} \right) \right. + \left. (\tilde{a}_Q \tilde{a}_\omega + 2 \tilde{b}_\omega) \sin \left( \frac{\tilde{\omega}_i + \pi}{3} \right) \right],
\]

(39a)

\[
\equiv \frac{-4 V_e}{3} \cos \left( \frac{\tilde{\omega}_i - \pi}{6} \right) \cos \left( \frac{\tilde{\omega}_i + 3 \pi}{6} \right) + \tilde{C}_i + \frac{\tilde{a}_i}{V_e}.
\]

(39b)

Here the linear term in \( V_e \) vanishes if \( \omega_i \neq 2\pi \). In this case the expression \( \tilde{C}_i \) is equal to

\[
\tilde{C}_i = \frac{\tilde{H}_{ee}}{3} \left[ 1 - 2 \cos \left( \frac{\tilde{\omega}_i + \pi}{3} \right) \right] + \left( \frac{\tilde{H}_{\mu\mu} + \tilde{H}_{\tau\tau}}{3} \right) \left[ 1 + \cos \left( \frac{\tilde{\omega}_i + \pi}{3} \right) \right]
\]

\[
+ \frac{\sqrt{4 |\tilde{H}_{\mu\tau}|^2 + (\tilde{H}_{\mu\mu} - \tilde{H}_{\tau\tau})^2}}{\sqrt{3}} \sin \left( \frac{\tilde{\omega}_i + \pi}{3} \right),
\]

(40a)

\[
\tilde{a}_i = \sqrt{3} \left[ \frac{2 (\tilde{H}_{\mu\mu} \tilde{H}_{\tau\tau} + \tilde{H}_{\mu\tau} \tilde{H}_{\tau\mu}) + (|\tilde{H}_{\mu\mu}|^2 - |\tilde{H}_{\tau\tau}|^2) (\tilde{H}_{\mu\mu} - \tilde{H}_{\tau\tau})}{|\tilde{H}_{\mu\mu}|^2 + (\tilde{H}_{\mu\mu} - \tilde{H}_{\tau\tau})^2} \right] \sin \left( \frac{\tilde{\omega}_i + \pi}{3} \right)
\]

\[
+ \frac{1}{9 V_e} \left[ \left( 3 \frac{\tilde{a}_Q^2}{4} - 3 \tilde{b}_Q + \frac{\tilde{a}_\omega^2}{3} \right) \cos \left( \frac{\tilde{\omega}_i + \pi}{3} \right) \right. + \left. (\tilde{a}_Q \tilde{a}_\omega + 2 \tilde{b}_\omega) \sin \left( \frac{\tilde{\omega}_i + \pi}{3} \right) \right].
\]

(40b)

We can now write down the eigenvalues in the high density limit. For neutrinos and the normal hierarchy \( \omega_1 = 2\pi \),
\( \omega_2 = 4\pi \) and \( \omega_3 = 0 \) so in the infinite density limit

\[
\tilde{k}_1 \rightarrow \tilde{H}_{\mu\mu} + \tilde{H}_{\tau\tau} - \sqrt{\left|H_{\mu\tau}\right|^2 + \frac{\left(\tilde{H}_{\mu\mu} - \tilde{H}_{\tau\tau}\right)^2}{4}} + \frac{a_1}{V_e} + O(1/V_e),
\]

(41a)

\[
\tilde{k}_2 \rightarrow \tilde{H}_{\mu\mu} + \tilde{H}_{\tau\tau} + \sqrt{\left|H_{\mu\tau}\right|^2 + \frac{\left(\tilde{H}_{\mu\mu} - \tilde{H}_{\tau\tau}\right)^2}{4}} + \frac{a_2}{V_e} + O(1/V_e),
\]

(41b)

\[
\tilde{k}_3 \rightarrow \tilde{H}_{ee} + \frac{a_3}{V_e} + O(1/V_e^2).
\]

(41c)

For antineutrinos in the normal hierarchy

\[
\tilde{k}_1 \rightarrow \tilde{H}_{ee} + \tilde{a}_1 \frac{1}{V_e} + O(1/V_e^2),
\]

(42a)

\[
\tilde{k}_2 \rightarrow \tilde{H}_{ee} + \tilde{a}_2 \frac{1}{V_e} + O(1/V_e^2),
\]

(42b)

\[
\tilde{k}_3 \rightarrow \tilde{H}_{ee} + \tilde{a}_3 \frac{1}{V_e} + O(1/V_e^2).
\]

(42c)

For neutrinos and an inverted hierarchy \( \omega_1 = 4\pi, \omega_2 = 0 \) and \( \omega_3 = 2\pi \)

\[
\tilde{k}_1 \rightarrow \tilde{H}_{ee} + \tilde{a}_1 \frac{1}{V_e} + O(1/V_e^2),
\]

(43a)

\[
\tilde{k}_2 \rightarrow \tilde{H}_{ee} + \tilde{a}_2 \frac{1}{V_e} + O(1/V_e^2),
\]

(43b)

\[
\tilde{k}_3 \rightarrow \tilde{H}_{ee} + \tilde{a}_3 \frac{1}{V_e} + O(1/V_e^2).
\]

(43c)

And for antineutrinos in the inverted hierarchy

\[
\tilde{k}_1 \rightarrow \tilde{H}_{ee} + \tilde{a}_1 \frac{1}{V_e} + O(1/V_e^2),
\]

(44a)

\[
\tilde{k}_2 \rightarrow \tilde{H}_{ee} + \tilde{a}_2 \frac{1}{V_e} + O(1/V_e^2),
\]

(44b)

\[
\tilde{k}_3 \rightarrow \tilde{H}_{ee} + \tilde{a}_3 \frac{1}{V_e} + O(1/V_e^2).
\]

(44c)

The expressions we have derived for \( \tilde{k}_1, \tilde{k}_2 \) and \( \tilde{k}_3 \) in the asymptotic limit can be employed together with equations (16a) to (16f) to more easily determine the correct values for the mixing angles and phases in situations where the matter potential is large.

**IV. THE ADIABATIC BASIS**

Returning to the problem of neutrino propagation we now introduce a new basis \( \psi^{(a)} \), which we call the adiabatic basis, with \( \psi^{(m)} = W(x)\psi^{(a)} \).

After making this change of basis we find the Schrödinger equation has become

\[
\frac{d \psi^{(a)}}{dx} = \left( W^\dagger \tilde{K}W - iW^\dagger \frac{dW}{dx} - iW^\dagger \tilde{U}^\dagger \frac{d\tilde{U}}{dx}W \right) \psi^{(a)},
\]

(45a)

\[
\equiv \tilde{H}^{(a)} \psi^{(a)}.
\]

(45b)
and we choose $W$ so that it removes the diagonal of $\hat{H}^{(a)}$. The matrix $W$ is simply

$$
W = \left(\begin{array}{ccc}
\exp[-2i\pi\phi_1] & 0 & 0 \\
0 & \exp[-2i\pi\phi_2] & 0 \\
0 & 0 & \exp[-2i\pi\phi_3]
\end{array}\right),
$$

(46)

where $\phi_1$, $\phi_2$ and $\phi_3$ are defined to be

$$
\frac{d\phi_1}{dx} = \frac{1}{2\pi} \left( \tilde{k}_1 - \frac{\tilde{s}_{12}\tilde{s}_{13}\tilde{c}_{23}\tilde{s}_{e}}{\tilde{c}_{12}} \frac{\delta\tilde{k}_{23}}{\delta k_{12}} \frac{d\delta V_{\mu\tau}}{dx} \right),
$$

(47a)

$$
\frac{d\phi_2}{dx} = \frac{1}{2\pi} \left( \tilde{k}_2 - \frac{\tilde{c}_{12}\tilde{s}_{13}\tilde{c}_{23}\tilde{s}_{e}}{\tilde{s}_{12}} \frac{\delta\tilde{k}_{13}}{\delta k_{12}} \frac{d\delta V_{\mu\tau}}{dx} \right),
$$

(47b)

$$
\frac{d\phi_3}{dx} = \frac{1}{2\pi} \left( \tilde{k}_3 - \frac{\tilde{c}_{12}\tilde{s}_{13}\tilde{c}_{23}\tilde{s}_{e}}{\tilde{s}_{13}} \frac{\delta\tilde{k}_{23}}{\delta k_{13}} \frac{d\delta V_{\mu\tau}}{dx} \right).
$$

(47c)

If we had used $\tilde{\alpha}_1$, $\tilde{\alpha}_2$ and $\tilde{\alpha}_3$ to cancel off the diagonal elements of $\tilde{U}^\dagger \frac{d\tilde{U}}{dx}$ then equations (47a) to (47c) would only contain the first terms in the above expression, i.e. only the contribution proportional to $\tilde{k}_1$, $\tilde{k}_2$ and $\tilde{k}_3$. The name “adiabatic” comes from the consideration of the two-flavor oscillation problem where the phase analogous to the $\phi_i$’s is the adiabatic phase of the matter states. For three flavors since there are multiple phases which also contain additional terms proportional to $d\delta V_{\mu\tau}/dx$ the situation is more complex. However, this basis is useful for providing insight into the phase effects, e.g. Ref [16], and the effects of the CP phase $\epsilon$ as described in Sec. V B. Further, it is an ideal basis in which to perform neutrino flavor transformation calculations using the $S$ matrix prescription. After removing the diagonal elements of $\hat{H}^{(a)}$ we can write out $\hat{H}^{(a)}$ as

$$
\hat{H}^{(a)} = \left(\begin{array}{cccc}
0 & \delta_{k_{12}} \Gamma_{12} e^{2i\pi\phi_{12}} & \delta_{k_{13}} \Gamma_{13} e^{2i\pi\phi_{13}} & 0 \\
\delta_{k_{12}} \Gamma_{12} e^{2i\pi\phi_{12}} & \frac{\delta_{k_{12}}}{2\pi} & 0 & \delta_{k_{13}} \Gamma_{13} e^{2i\pi\phi_{13}} \\
n & \frac{\delta_{k_{13}}}{2\pi} \Gamma_{13} e^{2i\pi\phi_{13}} & \frac{\delta_{k_{13}}}{2\pi} \Gamma_{23} e^{2i\pi\phi_{23}} & 0 \\
0 & \delta_{k_{13}} \Gamma_{13} e^{2i\pi\phi_{13}} & \delta_{k_{13}} \Gamma_{23} e^{2i\pi\phi_{23}} & 0
\end{array}\right),
$$

(48)

where, as usual, $\delta_{k_{ij}} = \phi_i - \phi_j$. This equation defines three functions $\Gamma_{12}, \Gamma_{13}$ and $\Gamma_{23}$ which are the non-adiabaticity parameters for 3 flavor neutrino oscillations. By matching the expressions we find

$$
\Gamma_{12} = -\frac{2\pi e^{i\delta_{k_{12}}}}{\delta k_{12}} \left( \frac{d\delta_{k_{12}}}{dx} + \tilde{s}_{13} \left( \tilde{c}_{12}\tilde{e}^{ie} + \tilde{s}_{12}\tilde{e}^{-ie} \right) \frac{d\delta_{k_{23}}}{dx} \right)
$$

$$
\times \left( \tilde{c}_{12}\tilde{s}_{23}\tilde{e}^{-ie} - \tilde{c}_{12}\tilde{s}_{13}\tilde{c}_{23} \left( \tilde{c}_{12}\tilde{s}_{23}\tilde{e}^{ie} + \tilde{s}_{12}\tilde{s}_{13}\tilde{c}_{23} \right) \frac{\delta_{k_{12}}}{dx} - i\tilde{c}_{12}\tilde{s}_{12}\tilde{c}_{13} \frac{d\delta}{dx} + i\tilde{c}_{12}\tilde{s}_{12}\tilde{d}_{12} \frac{d\delta}{dx} \right),
$$

(49a)

$$
\Gamma_{13} = -\frac{2\pi e^{i\delta_{k_{13}}}}{\delta k_{13}} \left( \tilde{c}_{12} \frac{d\delta_{k_{13}}}{dx} - \tilde{s}_{12}\tilde{c}_{13}\tilde{e}^{-ie} \frac{d\delta_{k_{23}}}{dx} + i \left( \tilde{s}_{12}\tilde{c}_{23}\tilde{e}^{-ie} - \tilde{c}_{12}\tilde{s}_{13}\tilde{c}_{23} \right) \tilde{c}_{13}\tilde{c}_{23} \frac{d\delta}{dx} - i\tilde{c}_{12}\tilde{s}_{13}\tilde{d}_{13} \frac{d\delta}{dx} \right),
$$

(49b)

$$
\Gamma_{23} = -\frac{2\pi e^{i\delta_{k_{23}}}}{\delta k_{23}} \left( \tilde{s}_{12} \frac{d\delta_{k_{23}}}{dx} + \tilde{c}_{12}\tilde{c}_{13}\tilde{e}^{-ie} \frac{d\delta_{k_{23}}}{dx} - i \left( \tilde{c}_{12}\tilde{s}_{23}\tilde{e}^{-ie} + \tilde{s}_{12}\tilde{s}_{13}\tilde{c}_{23} \right) \tilde{c}_{13}\tilde{c}_{23} \frac{d\delta}{dx} - i\tilde{s}_{12}\tilde{c}_{13}\tilde{d}_{13} \frac{d\delta}{dx} \right). 
$$

(49c)

Note how the derivatives of the two phases $\tilde{\beta}$ and $\tilde{\delta}$ appear in these expressions. When we substitute in the expressions for the derivatives we find

$$
\Gamma_{12} = \frac{2\pi e^{i\delta_{k_{12}}}}{\delta k_{12}^2} \left[ \tilde{c}_{12}\tilde{s}_{12}\tilde{c}_{13} \left( \tilde{s}_{23} \frac{d\delta V_{\mu\mu}}{dx} + \tilde{c}_{23} \frac{d\delta V_{\mu\tau}}{dx} \right) - \left( \tilde{c}_{12}\tilde{s}_{12} \left( \tilde{c}_{23}^2 - \tilde{s}_{23}^2 \right) + \tilde{s}_{12}\tilde{c}_{13}\tilde{s}_{23} \left( \tilde{c}_{12}^2\tilde{e}^{-ie} - \tilde{s}_{12}^2\tilde{e}^{-ie} \right) \right) \frac{d\delta V_{\mu\tau}}{dx} \right],
$$

(50a)

$$
\Gamma_{13} = \frac{2\pi e^{i\delta_{k_{13}}}}{\delta k_{13}^2} \left[ \tilde{c}_{12}\tilde{c}_{13}\tilde{s}_{13} \left( \tilde{s}_{23} \frac{d\delta V_{\mu\mu}}{dx} + \tilde{c}_{23} \frac{d\delta V_{\mu\tau}}{dx} \right) - \tilde{s}_{12}\tilde{c}_{13}\tilde{c}_{23}\tilde{s}_{23} \frac{d\delta V_{\mu\tau}}{dx} \right],
$$

(50b)

$$
\Gamma_{23} = \frac{2\pi e^{i\delta_{k_{23}}}}{\delta k_{23}^2} \left[ \tilde{s}_{12}\tilde{c}_{13}\tilde{s}_{13} \left( \tilde{s}_{23} \frac{d\delta V_{\mu\mu}}{dx} + \tilde{c}_{23} \frac{d\delta V_{\mu\tau}}{dx} \right) + \tilde{c}_{12}\tilde{c}_{13}\tilde{s}_{23} \frac{d\delta V_{\mu\tau}}{dx} \right].
$$

(50c)
If we restrict ourselves to $\delta V_{\mu\tau} = 0$ then we see that $\Gamma_{12} \propto \tilde{c}_{12}\tilde{s}_{12}/\delta k^2_{12} dV_c/dx$ which is exactly the two flavor non-adiabaticity parameter as described in [16]. The other similar terms, $\Gamma_{13} \propto \tilde{c}_{13}\tilde{s}_{13}/\delta k^2_{13} dV_c/dx$ and $\Gamma_{23} \propto \tilde{c}_{13}\tilde{s}_{13}/\delta k^2_{13} dV_c/dx$ also take on similar meanings. This Hamiltonian is constructed in such a way that, similar to the two flavor case, the focus is on the “non-adiabatic” pieces of the solution, i.e. the places near the resonances where the matter eigenstates are likely to swap. There are some corrections for the case of three flavors that come from the terms $\tilde{c}_{13}^*, \tilde{c}_{12}$ and $\tilde{s}_{12}$ and the three, arbitrary, complex exponentials in $\Gamma_{12}, \Gamma_{13}$ and $\Gamma_{23}$ respectively. It can be seen that it is the behavior of $\tilde{\theta}_{13}$ that selects between mixing of states $\tilde{\nu}_1$ and $\tilde{\nu}_3$ or between $\tilde{\nu}_2$ and $\tilde{\nu}_3$.

V. THE SCATTERING MATRIX

The Schrodinger equation for the evolution of the neutrino wavefunction is

$$\frac{d\psi}{dx} = -\iota \hat{H}(x) \psi(x). \quad (51)$$

When we integrate equation (51) we obtain

$$\psi(X) = \psi(X_0) - \iota \int_{X_0}^{X} dx_1 \hat{H}(x_1) \psi(x_1). \quad (52)$$

We choose the initial point to be at $X_0$. Repeated substitution of this result into itself yields

$$\psi(X) = \psi(X_0) - \iota \int_{X_0}^{X} dx_1 \hat{H}_1 \psi(x_1) + (-\iota)^2 \int_{X_0}^{X} dx_1 \hat{H}_1 \int_{X_0}^{x_1} dx_2 \hat{H}_2 \psi(x_2) + \ldots, \quad (53a)$$

$$= \left\{1 - \iota \int_{X_0}^{X} dx_1 \hat{H}_1 + (-\iota)^2 \int_{X_0}^{X} dx_1 \hat{H}_1 \int_{X_0}^{x_1} dx_2 \hat{H}_2 + \ldots\right\} \psi(X_0) \quad (53b)$$

where the subscripts on the $\hat{H}$’s mean $\hat{H}_i = \hat{H}(x_i)$. This equation defines the scattering matrix $S(X, X_0)$ since

$$\psi(X) = S(X, X_0) \psi(X_0). \quad (54)$$

The upper limits on the integrals appearing in equation (53b) indicate the space ordering but we can change all the upper limits to $X$ by using identities such as

$$\int_{X_0}^{X} dx_1 \hat{H}_1 \int_{X_0}^{x_1} dx_2 \hat{H}_2 = \frac{1}{2!} \int_{X_0}^{X} dx_1 \int_{X_0}^{x_1} dx_2 \left\{\hat{H}_1 \hat{H}_2 \Theta(x_1 - x_2) + \hat{H}_2 \hat{H}_1 \Theta(x_2 - x_1)\right\} \quad (55)$$

where $\Theta(x_1 - x_2)$ is the Heaviside step function. Using this result and similar identities for the higher order multiple integrals, allows us to write $S$ as

$$S(X, X_0) = 1 + (-\iota) \int_{X_0}^{X} dx_1 \hat{H}_1 + \frac{(-\iota)^2}{2!} \int_{X_0}^{X} dx_1 \int_{X_0}^{x_1} dx_2 \Theta(\hat{H}_1 \hat{H}_2) \quad (56)$$

$$+ \frac{(-\iota)^3}{3!} \int_{X_0}^{X} dx_1 \int_{X_0}^{x_1} dx_2 \int_{X_0}^{x_2} dx_3 \Theta(\hat{H}_1 \hat{H}_2 \hat{H}_3) + \ldots,$$

where $\Theta$ is the space/time-ordering operator. Now that it is defined we can simply insert equation (54) into the Schrodinger equation and find that $S$ also obeys the differential equation

$$\iota \frac{dS}{dx} = \hat{H} S \quad (57)$$

which describes 9 coupled equations for the elements of $S$. From this equation we can also derive that the phase of the determinant $|S| = e^{\iota \Phi}$ is simply

$$\Phi(X, X_0) = -\int_{X_0}^{X} \text{Tr}(H) dx. \quad (58)$$
So we find that in the adiabatic basis \( \Phi^{(a)}(X, X_0) = 0 \) and does not vary because the Hamiltonian is traceless. Since the initial condition is that \( S^{(a)}(X_0, X_0) = 1 \) we see that in the adiabatic basis \( S \) is a member of \( SU(3) \). For all bases \( S \) has the property that it obeys the product rule

\[
S(X, X_0) = S(X, X_*)S(X_*, X_0).
\]

and the probability that a neutrino with initial state \( |\tilde{\nu}_i(X_0)\rangle \) is later detected as state \( |\tilde{\nu}_i(X)\rangle \) is

\[
P(|\tilde{\nu}_i) = |S_{ij}(X, X_0)|^2.
\]

\( S \) is, in general, a member of \( U(3) \) and this restriction means that any element of \( S(X, X_0) \) satisfies the relationship

\[
S_{ij}(X, X_0) = e^{\Phi(X, X_0)} C^{*}_{ij}(X, X_0)
\]

where \( C^{*}_{ij}(X, X_0) \) is the cofactor of the element. This identity allows us to remove four of the elements of \( S \) if we know the determinant \( |S| \) and there remain two unitary conditions upon the magnitudes of the remaining, independent, elements. Thus in the end we see that \( S \), like \( U \), is parameterized by nine real numbers; three magnitudes and six phases though the phase of the determinant may be stationary if the Hamiltonian in that basis is traceless.

A. The two flavor approximation

We have, so far, described everything in terms of three flavors but a quick scan through the literature by the reader will reveal that many studies have used only two. In this section we show how the three flavor \( S \) matrix formalism using the adiabatic basis can be separated in pieces which contain only two flavors.

The motivation for the reduction in number of flavors one must consider comes from observation of the structure of the adiabatic Hamiltonian, equation (48). If we make the assumption that only one \( \Gamma_{ij} \) is significant at any given location, never two (or all three) simultaneously, then neutrino mixing occurs only between states \( |\tilde{\nu}_i\rangle \) and \( |\tilde{\nu}_j\rangle \) and the third state, \( |\tilde{\nu}_k\rangle \), is decoupled. It is not immediately obvious that only one \( \Gamma_{ij} \) is significant at any given location: each \( \Gamma_{ij} \) is proportional to the same derivatives of the potential \( \partial \bar{\nu} \alpha / \partial \alpha \) and the only difference is the matter mixing angle prefactors and the difference between the eigenvalues \( \delta \kappa_{ij} \) in the denominators. Nevertheless, this is often the case for most density profiles.

We can express the full Hamiltonian \( \bar{H}^{(a)} \) as the sum of three terms

\[
\bar{H}^{(a)} = \bar{H}_{(12)} + \bar{H}_{(13)} + \bar{H}_{(23)}
\]

\[
= \frac{i}{2 \pi} \delta \kappa_{12} \begin{pmatrix} 0 & \Gamma_{12} e^{2 \pi \delta \phi_{12}} & 0 \\ -\Gamma_{12} e^{-2 \pi \delta \phi_{12}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{i}{2 \pi} \delta \kappa_{13} \begin{pmatrix} 0 & 0 & \Gamma_{13} e^{2 \pi \delta \phi_{13}} \\ 0 & 0 & 0 \\ -\Gamma_{13} e^{-2 \pi \delta \phi_{13}} & 0 & 0 \end{pmatrix}
\]

\[
+ \frac{i}{2 \pi} \delta \kappa_{23} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \Gamma_{23} e^{2 \pi \delta \phi_{23}} \\ 0 & \Gamma_{23} e^{-2 \pi \delta \phi_{23}} & 0 \end{pmatrix}
\]

and then introduce the three \( S \)-matrices \( S_{(12)} \), \( S_{(13)} \) and \( S_{(23)} \) we obtain from the substitution of each \( H_{(ij)} \) into equation (56). Due to the structure of each \( H_{(ij)} \) the three \( S_{(ij)} \) have the form

\[
S_{(12)} = \begin{pmatrix} \zeta_{(12)} & \eta_{(12)} & 0 \\ -\eta_{(12)} & \zeta_{(12)} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
S_{(13)} = \begin{pmatrix} \zeta_{(13)} & 0 & \eta_{(13)} \\ 0 & 1 & 0 \\ -\eta_{(13)} & 0 & \zeta_{(13)} \end{pmatrix}
\]

\[
S_{(23)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \zeta_{(23)} & \eta_{(23)} \\ 0 & -\eta_{(23)} & \zeta_{(23)} \end{pmatrix}
\]

where the \( \zeta_{(ij)} \)'s and \( \eta_{(ij)} \)'s are Cayley Klein parameters. Now it is simply a case of adapting this approximation to the situation at hand. In figure (1) we see that the L resonance always involves mixing between states \( |\tilde{\nu}_1\rangle \) and \( |\tilde{\nu}_2\rangle \)
so if the density profile under consideration possesses only an L resonance then the S matrix describing the evolution of the neutrinos through the profile will have the structure of $S_{(12)}$. The H resonance mixes states $|\tilde{\nu}_2\rangle$ and $|\tilde{\nu}_3\rangle$ for a normal hierarchy so for a profile containing just an H resonance the S matrix will have the structure of $S_{(23)}$. By applying the same reasoning for all the different possibilities we can assign the appropriate $S_{(ij)}$ for any resonance shown in the figure. For a density profile possessing multiple resonances we can apply the group product rule for the evolution operator, equation (59), to divide the profile into sub-domains such that, within each, there is just one resonance. Since the structure of the S matrix for each sub-domain is given by the reasoning above the S matrix for the entire profile is then the time-ordered product of the appropriate $S_{(ij)}$’s.

Once we have decided which pair of states are mixing we can then either solve the reduced problem

$$
\frac{dS^{(2)}}{dx} = \tilde{H}^{(2)}(x)S^{(2)}(x)
$$

where $S^{(2)}$ is a 2x2 matrix and

$$
\tilde{H}^{(2)} = \frac{i}{2\pi} \left( \begin{array}{cc}
0 & \Gamma_{ij} e^{2i\delta_{\phi_{ij}}} \\
-\Gamma_{ij} e^{-2i\delta_{\phi_{ij}}} & 0
\end{array} \right)
$$

using the expression for $\Gamma_{ij}$ given in equations (50a)-(50c), use a straight two flavor calculation a la Kneller and McLaughlin [16], or utilize some other alternative or approximate method. However one determines $S^{(2)}$, once it has been found one simply constructs the appropriate three flavor S matrix (or matrices in the case profiles with L, H and/or $\mu\tau$ resonances) as in, for example, Kneller, McLaughlin & Brockman [19]. The disadvantage of applying a series of two flavor approximations is that some phase information can be lost. Nevertheless, many features of a flavor transformed neutrino signal can often be determined in this way.

### B. Identities of the Scattering Matrix

The problem of neutrino propagation through supernovae has received considerable attention the in the past few years. The signal from the next Galactic supernova has the potential to reveal a great deal of information about both the supernova and the mixing parameters for the neutrinos. For example, if the angle $\theta_{13}$ is not too small then dynamic MSW effects may be observed and some authors have also considered the possibility of observing effects from a non-zero CP phase. It is upon this possibility of observing the non-zero CP phase that we now focus our attention.

In the adiabatic and matter basis we have shown that the Hamiltonian is independent of the CP phase $\epsilon$ if $V_\mu = V_e = 0$. This occurs because the eigenvalues and $\hat{\theta}_{12}$ and $\hat{\theta}_{13}$ are all independent of $\epsilon$ in this limit so that the non-adiabaticity parameters $\Gamma_{12}$, $\Gamma_{13}$ and $\Gamma_{23}$ are also independent of $\epsilon$. If the Hamiltonian is independent of $\epsilon$ then the S matrix must also be independent of $\epsilon$ and, therefore, we must have $S^{(a)}(X, X_0, \epsilon) = S^{(a)}(X, X_0, 0)$ and $S^{(m)}(X, X_0, \epsilon) = S^{(m)}(X, X_0, 0)$. As a result, all the survival and crossing probabilities for the matter or adiabatic neutrino states, $P(|\tilde{\nu}_j\rangle \rightarrow |\tilde{\nu}_i\rangle) = |S^{(m,a)}_{ij}|^2$, are also independent of $\epsilon$. Any dependence upon $\epsilon$ for the survival/crossing probabilities of states in other bases can only enter explicitly in the unitary transformation to those bases.

For example, let us consider the transformation to the flavor basis. The flavor basis survival/crossing probabilities are found from the $S^{(f)}$ matrix related to $S^{(m)}(X, X_0, \epsilon)$ and $S^{(a)}(X, X_0, \epsilon)$ by

$$
S^{(f)}(X, X_0, \epsilon) = \tilde{U}(X, \epsilon) S^{(m)}(X, X_0, \epsilon) \tilde{U}^\dagger(X, X_0, \epsilon) = \tilde{U}(X, \epsilon) W(X, \epsilon) S^{(a)}(X, X_0, \epsilon) W^\dagger(X, \epsilon) \tilde{U}^\dagger(X, \epsilon).
$$

In above expression we can set $W^\dagger(X, \epsilon) = 1$ if all the phases in equation (46) are set to zero at $X_0$. However when splitting the S matrix into pieces, as in equation (59), then one should retain this phase information, so $W^\dagger = 1$ only for the first, rightmost, S matrix.

Since $S^{(m)}(X, X_0, \epsilon) = S^{(m)}(X, X_0, 0)$ the following identity must be obeyed:

$$
\tilde{U}^\dagger(X, \epsilon) S^{(f)}(X, X_0, \epsilon) \tilde{U}(X, \epsilon) = \tilde{U}^\dagger(X, 0) S^{(f)}(X, X_0, 0) \tilde{U}(X, \epsilon) = 0.
$$

Now let us consider how the CP phase enters into $\tilde{U}$. We saw from equation (18b) that $\tilde{U}$ could be written as

$$
\tilde{U} = B(\tilde{\beta}, \tilde{\delta}) \Theta_{23}(\tilde{\theta}_{23}) E(\tilde{\epsilon}) \Theta_{13}(\tilde{\theta}_{13}) \Theta_{12}(\tilde{\theta}_{12}) A(\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\alpha}_3) \text{ but, when } V_\mu = V_e = 0, \text{ we have also seen that both } \tilde{\theta}_{12} \text{ and } \tilde{\theta}_{13} \text{ are independent of } \epsilon \text{ and, furthermore, we have also made the decision that } \tilde{\alpha}_1, \tilde{\alpha}_2 \text{ and } \tilde{\alpha}_3 \text{ are constants. This means } \Theta_{13}(x, \epsilon) = \Theta_{13}(x, 0), \Theta_{12}(x, \epsilon) = \Theta_{12}(x, 0) \text{ and } A(x, \epsilon) = A(x, 0) \text{ so when we insert the decomposition of } \tilde{U} \text{ into equation (67) we find that what survives can be written as }
$$

$$
B^\dagger(X, \epsilon) S^{(f)}(X, X_0, \epsilon) B(X, \epsilon) = \Theta_{23}(X, \epsilon) E(X, \epsilon) \Theta_{13}^{(f)}(X, 0) B^\dagger(X, 0) S^{(f)}(X, X_0, 0) B(X, 0) \Theta_{23}(X, 0) E^\dagger(X, \epsilon) \Theta_{13}^{(f)}(X, \epsilon).
$$
The two combinations \( \Theta_{23}(X, \epsilon) E(X, \epsilon) \Theta_{23}^\dagger(X, 0) \) and \( \Theta_{23}(X, 0) E^\dagger(X, \epsilon) \Theta_{23}^\dagger(X, 0, \epsilon) \) are

\[
\Theta_{23}(X, \epsilon) E(X, \epsilon) \Theta_{23}^\dagger(X, 0) = \\
\begin{pmatrix}
1 & 0 \\
0 & \hat{s}_{23}(X, \epsilon) \hat{s}_{23}(X, 0) + \hat{c}_{23}(X, \epsilon) \hat{c}_{23}(X, 0) e^{i \epsilon(X)} \\
0 & \hat{c}_{23}(X, \epsilon) \hat{s}_{23}(X, 0) - \hat{s}_{23}(X, \epsilon) \hat{c}_{23}(X, 0) e^{i \epsilon(X)}
\end{pmatrix}
\]

\( \Theta_{23}(X, 0) E^\dagger(X, \epsilon) \Theta_{23}^\dagger(X, 0, \epsilon) = \\
\begin{pmatrix}
1 & 0 \\
0 & \hat{s}_{23}(X, 0, \epsilon) \hat{s}_{23}(X, 0) + \hat{c}_{23}(X, 0, \epsilon) \hat{c}_{23}(X, 0) e^{-i \epsilon(X)} \\
0 & \hat{c}_{23}(X, 0, \epsilon) \hat{s}_{23}(X, 0) - \hat{s}_{23}(X, 0, \epsilon) \hat{c}_{23}(X, 0) e^{-i \epsilon(X)}
\end{pmatrix}
\]

(69)

(70)

respectively. After inserting these matrices into equation (68) we find nine relationships between the elements of \( S^{(f)}(X, X, \epsilon) \) and the elements of \( S^{(f)}(X, X, 0, 0) \). The flavour basis survival/crossing probabilities \( P_{\alpha \beta} = |S_{\alpha \beta}^{(f)}| \) may be found for both \( S^{(f)}(X, X, \epsilon) \) and \( S^{(f)}(X, X, 0, 0) \) and using the relationships between the elements of the two matrices we find that we can derive four, non-trivial identities for the probabilities:

- \( P_{ee}(X, X, 0, 0) = P_{ee}(X, X, 0, 0) \),
- \( P_{\tau \mu}(X, X, 0, \epsilon) + P_{\tau \tau}(X, X, 0, 0) = P_{\tau \mu}(X, X, 0, 0) + P_{\tau \tau}(X, X, 0, 0) \),
- \( P_{\mu \tau}(X, X, 0, \epsilon) + P_{\mu \mu}(X, X, 0, 0) + P_{\mu \tau}(X, X, 0, 0) = P_{\mu \mu}(X, X, 0, 0) + P_{\mu \tau}(X, X, 0, 0) + P_{\tau \mu}(X, X, 0, 0) + P_{\tau \tau}(X, X, 0, 0) \).

The first two identities were also found by Balantekin, Gava and Volpe [56].

However when computing the fluxes at Earth one requires a slightly different set of probabilities: the probability that an initial flavor state emerges as a given matter/mass state. These probabilities are found from the matrix \( S^{(mf)} \) given by

\[
S^{(mf)}(X, X, \epsilon) = S^{(m)}(X, X, 0, \epsilon) U^\dagger(X, 0, \epsilon) = W(X, \epsilon) S^{(a)}(X, X, 0, \epsilon) W^\dagger(X, 0, \epsilon) \dot{U}(X, 0, \epsilon).
\]

(71)

Again the invariance of \( S^{(m)}(X, X, 0, \epsilon) \) with regard to \( \epsilon \) when \( V_\mu = V_\tau = 0 \) means that we have the following identity:

\[
S^{(mf)}(X, X, 0, \epsilon) \dot{U}(X, 0, \epsilon) = S^{(mf)}(X, X, 0, 0) \dot{U}(X, 0, 0).
\]

(72)

which ultimately leads to

\[
S^{(mf)}(X, X, 0, \epsilon) B(X, 0, \epsilon) = S^{(mf)}(X, X, 0, 0) B(X, 0, 0) \Theta_{23}(X, 0) E^\dagger(X, \epsilon) \Theta_{23}^\dagger(X, 0, \epsilon).
\]

(73)

This is very similar to equation (68). Again we have nine relationships between \( S^{(mf)}(X, X, 0, \epsilon) \) and \( S^{(mf)}(X, X, 0, 0) \) which lead to the following, non-trivial identities for the probabilities \( P_{\alpha \beta} = |S_{\alpha \beta}^{(mf)}|^2 \):

- \( P_{ee}(X, X, 0, 0) = P_{ee}(X, X, 0, 0) \),
- \( P_{2e}(X, X, 0, \epsilon) = P_{2e}(X, X, 0, 0) \),
- \( P_{3e}(X, X, 0, \epsilon) = P_{3e}(X, X, 0, 0) \),
- \( P_{1\mu}(X, X, \epsilon) + P_{1\tau}(X, X, 0, \epsilon) = P_{1\mu}(X, X, 0, 0) + P_{1\tau}(X, X, 0, 0) \),
- \( P_{2\mu}(X, X, \epsilon) + P_{2\tau}(X, X, 0, \epsilon) = P_{2\mu}(X, X, 0, 0) + P_{2\tau}(X, X, 0, 0) \),
- \( P_{3\mu}(X, X, \epsilon) + P_{3\tau}(X, X, 0, \epsilon) = P_{3\mu}(X, X, 0, 0) + P_{3\tau}(X, X, 0, 0) \).
At the detector on Earth the supernova neutrinos are measured as flavor eigenstates thus, for example, the electron neutrino signal is the appropriate linear combination of survival probabilities multiplied but the original fluxes. One can see from these identities above that the CP phase effects will not show up in observed supernova neutrino signal if the $\nu_\mu, \nu_\tau$ fluxes emitted from the supernova neutrino sphere are equal \cite{55,56}. However, as pointed out in \cite{56} if the $\nu_\mu, \nu_\tau$ fluxes are not equal when they are emitted, then there is no guarantee that the fluxes as observed in the detector will be independent of the CP phase.

In the case where when $\epsilon \neq 0$ and $\delta V_{\mu \tau} \neq 0$ these identities no longer apply because the eigenvalues and the mixing angles now become functions of $\epsilon$. One might imagine that some sort of cancellation occurs such that the three non-adiabaticity parameters are independent of $\epsilon$. However an examination of the adiabatic basis shows that a complete cancellation cannot occur. For all non-adiabaticity parameters, equations (50a)-(50c), there emerges an imaginary component proportional to $\sin \epsilon \delta V_{\mu \tau} / \mu$. This term cannot be canceled by concomitant changes in the eigenvalues and/or mixing angles. For all three $\Gamma_{ij}$ we have a situation where $\Gamma_{ij}(\epsilon) \neq \Gamma_{ij}(0)$ thus $\tilde{H}^{(a)}(\epsilon) \neq \tilde{H}^{(a)}(0)$, $S^{(a)}(\epsilon) \neq S^{(a)}(0)$ and, finally, $P_{ij}(\epsilon) \neq P_{ij}(0)$ for any basis.

VI. SUMMARY AND CONCLUSIONS

We have considered the problem of a generalized, 3-flavor, neutrino mixing that includes matter potentials for both $\mu$ and $\tau$ flavors and CP violation. We presented expressions for the eigenvalues and matter mixing angles and pointed out that mixing phases that were zeroed in the vacuum are not necessarily zero in matter. We found that in the limit that the mu and tau potentials are equal, the eigenvalues and matter mixing angles $\tilde{\theta}_{12}$ and $\tilde{\theta}_{13}$ are independent of the CP phase, although $\tilde{\theta}_{23}$ does have CP dependence.

We introduced the 3-flavor adiabatic basis. In this basis the Hamiltonian is completely off-diagonal and the behavior of the neutrinos is largely determined by the 3-flavor adiabaticity parameters. This is a useful basis in which to calculate neutrino flavor transformation; further it gives a straightforward picture of the effects of the CP phase. In the limit that the mu and tau neutrino potentials are the same, the Hamiltonian in the adiabatic basis is independent of both $\theta_{23}$ and the CP violating phase, so CP phase effects appear only in rotations into and out of the flavor basis.

We discussed the $S$ matrix, as well as two flavor $S$-matrix approximation. Using the three flavor $S$-matrix, we found several non-trivial identities related to the observability of the CP phase. In a future study \cite{66} we shall discuss how one can formulate efficient algorithms for the calculation of $S$ and present some calculations of three flavor oscillations made with them.

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