SOERGEL BIMODULES FOR UNIVERSAL COXETER GROUPS

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ABSTRACT. We produce an explicit recursive formula which computes the idempotent projecting to any indecomposable Soergel bimodule for a universal Coxeter system. This gives the exact set of primes for which the positive characteristic analogue of Soergel’s conjecture holds. Along the way, we introduce the multicolored Temperley-Lieb algebra.

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1. INTRODUCTION

1.1. Kazhdan-Lusztig theory in positive characteristic. In the year 1979, Kazhdan and Lusztig (abbreviated “KL”) introduced their celebrated KL polynomials for any Coxeter system [14]. These polynomials have become a fundamental tool in representation theory, geometry, and combinatorics. However, they are also a fundamental mystery. Despite countless papers exploring the combinatorics of KL polynomials, very little is known outside of specific cases. The only infinite families of Coxeter groups for which we have a complete understanding of KL polynomials are the dihedral groups (a simple exercise) and the universal Coxeter groups (a result of Dyer [5]). Recall that universal Coxeter groups are groups generated by involutions, with no other relations.

As we will discuss shortly, KL polynomials encode multiplicities attached to certain important categories (representation-theoretic, geometric, or otherwise) defined in characteristic zero. For crystallographic Coxeter groups, these categories possess an integral form. Therefore, they can also be defined in characteristic $p$, making the situation more difficult and more interesting. One can encode the new multiplicities in so-called $p$-KL polynomials, which depend strongly on the specific value of $p$, and eventually (for $p$ large) agree with the ordinary KL polynomials. Far less is known about the $p$-KL polynomials (nor is there any algorithm to compute them directly), and the following question is already of extreme interest.
Question 1.1. Given a crystallographic Coxeter group $W$ and a prime $p$, for which $w \in W$ does there exist a $y \leq w$ such that the $p$-KL polynomial $h^p_{y,w} \in \mathbb{Z}[v, v^{-1}]$ disagrees with the ordinary KL polynomial $h_{y,w}$? For example, recent work of Williamson [25] has found an infinite family of such quadruples $(W, p, w, y)$ in type $A$, refuting a well-known conjecture about Lusztig’s character formula. Answering this question for Weyl groups and affine Weyl groups would have significant import for modular representation theory.

Below, we reformulate this question in a language suited to general Coxeter groups. In this paper, we provide an answer for the baby case of universal Coxeter groups. The answer for the other baby case, dihedral groups, can be easily extrapolated from the first author’s work [6]. This is a small step along a very long and difficult road.

1.2. Formulation in terms of Soergel bimodules. In 1992, Soergel [19] introduced an additive category $B = B(W, S, V, k)$ of graded bimodules over a polynomial ring, whose objects have come to be known as Soergel bimodules. This category depends on a Coxeter system $(W, S)$, a field $k$, and a finite dimensional representation $V$ of $W$ over $k$ (see [20] for this general definition). One important special case will be when $k = \mathbb{R}$ or $\mathbb{C}$ and $V = V_{\text{rootic}}$ is the rootic representation of $(W, S)$. When $W$ is crystallographic, its rootic representation can be defined over $\mathbb{Z}$, and thus can be specialized to finite characteristic.

The motivation for introducing $B$ is that, when $W$ is a Weyl group and $V$ its rootic representation in characteristic zero, there is an isomorphism between (a simplified version of) $B$ and additive subcategories of the representation-theoretic and geometric categories which KL polynomials study. More precisely, this simplified version is equivalent to the projective objects in the principal block of category $O$, or the semisimple $N$-equivariant perverse sheaves on the flag variety $G/B$. One advantage of Soergel’s approach is that $B$ can be defined for any Coxeter group, even non-crystallographic groups for which there is no corresponding geometry or representation theory. Another advantage is that $B$ has a simple algebraic definition, allowing one to study KL theory using low-tech methods.

Soergel proved ([19], [20]) that Soergel bimodules are a categorification of the Hecke algebra $H(W)$ of $W$. In other words, there is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras from the split Grothendieck group

$$\eta : [B(W, S, V, k)] \longrightarrow H(W).$$

This is proven whenever $V$ is reflection vector faithful (see [20] for the definition) and $k$ is an infinite field of characteristic $\neq 2$. The indecomposable objects $\{B_x\}_{x \in W}$ in $B$
are classified by elements of $W$, and they descend to some positive basis $\{[B_w]\}$ of the Hecke algebra (i.e. certain coefficients are positive). The Hecke algebra possesses a natural (positive) basis, the KL basis $\{b_x\}_{w \in W}$ (encoded by the KL polynomials). This raises the following question.

**Question 1.2.** Given $(W, S, V, k)$ with $V$ reflection vector faithful, for which $w \in W$ will it be the case that $[B_w] = b_w$?

When $V$ is reflection vector faithful and $k$ has characteristic 0, it was conjectured by Soergel that every $w \in W$ has this property. When $W$ is a Weyl group, this conjecture is equivalent to the famed Kazhdan-Lusztig conjecture, proven by Brylinski-Kashiwara [4] and Beilinson-Bernstein [2] using difficult geometric techniques. Soergel hoped that the algebraic setting of $B$ would allow for a simpler solution. The baby case of the dihedral group was proven by Soergel [19]. The universal Coxeter group case was done by Fiebig in [10], using the tool of moment graphs. It was also shown later by the second author in unpublished work, by constructing idempotents using singular Soergel bimodules [26]. The general case was recently proven by the first author and Williamson [8] for $V$ rootic when $k = \mathbb{R}$; therefore, the KL basis $b_w$ really does encode something about characteristic 0 Soergel bimodules.

Now we address what happens when $k$ has finite characteristic. When $W$ is a Weyl group, $V$ rootic will be reflection vector faithful in characteristic $\neq 2$, and Question 1.2 is equivalent to Question 1.1. However, an infinite Coxeter group does not possess a faithful representation in positive characteristic, so that Question 1.2 is not the right question to ask.

In [9] the first author and Williamson generalize the situation. They define a diagrammatic category $\mathcal{D}$ depending on a realization, which is roughly the data of $(W, S, V, k)$ together with a choice of simple roots and coroots (although $k$ can be any commutative ring). There is an equivalence $\mathcal{D} \cong \mathcal{B}$ of monoidal categories when the latter is “well behaved”, i.e. when (1.1) gives an isomorphism and the indecomposable objects are parametrized by $W$. Under some minimal assumptions, $\mathcal{D}$ is well behaved in this sense even when $\mathcal{B}$ is not (such as when the representation is not reflection vector faithful), justifying the statement that $\mathcal{D}$ is the appropriate replacement for $\mathcal{B}$. We still denote the indecomposable objects of $\mathcal{D}$ by $[B_w]$ for $w \in W$.

**Question 1.3.** Given a realization over a complete local ring $k$ where $[\mathcal{D}] \cong \mathbf{H}(W)$, for which $w \in W$ will it be the case that $[B_w] = b_w$?

When $W$ is crystallographic and the realization is rootic in finite characteristic, $\mathcal{D}$ agrees with the category of parity sheaves [13], the appropriate finite characteristic analog of perverse sheaves, whose multiplicities are encoded by the $p$-KL polynomials. Therefore, Question 1.3 is equivalent to Question 1.1 in this case. For other realizations and other Coxeter groups, it is not just the characteristic of $k$ which is important, but specific features of the realization itself can affect the behavior of $\mathcal{D}$.
1.3. **Results.** We give a precise answer to Question 1.3 for (the baby case of) universal Coxeter groups. We provide this answer for an arbitrary realization; Remark 3.13 indicates how to return to the rootic representation in arbitrary characteristic.

In [6], the first author demonstrated that Soergel bimodules for the infinite dihedral group were intimately related to the Temperley-Lieb algebra which arises in sl$_2$ representation theory. The familiar Jones-Wenzl idempotents in the Temperley-Lieb algebra were transformed into idempotent endomorphisms of Soergel bimodules, projecting to the indecomposable summands. This paper takes these ideas to their natural conclusion, producing a relationship between the multicolored Temperley-Lieb 2-category and Soergel bimodules for the corresponding universal Coxeter group.

In §2 we define the multicolored Temperley-Lieb 2-category and explore its representation theory. We define the analogues of Jones-Wenzl idempotents. We provide a recursive formula for these idempotents, allowing one to categorify Dyer’s inductive formula for the KL basis. We also provide an immediate formula for these idempotents in terms of the Jones-Wenzl idempotents in the usual Temperley-Lieb algebra (which unfortunately have no easy closed formula, though see [18]). This second formula implies a criterion for when the idempotent is not defined, which will lead to the answer to Question 1.3. Specifically, $[B_w] = b_w$ so long as certain “colored quantum binomial coefficients” are invertible.

In §3 we define the diagrammatic category $\mathcal{D}$ associated to the most general realization of a universal Coxeter group. Our definition is purely diagrammatic, using the results of [9], and thus we never mention Soergel bimodules. We prove the main theorem: that the multicolored Temperley-Lieb 2-category encodes all the morphisms of minimal degree in $\mathcal{D}$. Therefore, the Jones-Wenzl analogues provide all the indecomposable idempotents in $\mathcal{D}$.

2. **The $n$-colored Temperley-Lieb 2-category**

2.1. **Definitions.** We assume that the reader is familiar with several topics, for which we give some references. Introductory material on the Temperley-Lieb category can be found in [23, 11]. An introduction to (strict) 2-categories and their diagrammatic presentations can be found in [15, section 2]. An introduction to Karoubi envelopes can be found in [1], or on Wikipedia.

Let $S$ be a finite set with size $n$. We associate a color to each element of $S$, blue to $b$ and red to $r$, etcetera.

**Definition 2.1.** The $S$-colored or $n$-colored Temperley-Lieb 2-category $STL$ is the $\mathbb{Z}[\delta]$-linear 2-category with objects $S$, having the following presentation. There is a generating 1-morphism from $b$ to $r$, for each pair of distinct elements $b \neq r \in S$. Therefore, a general 1-morphism can be represented uniquely by the (non-empty) sequence $\underline{x} = s_1 s_2 \ldots s_m$ of colors through which it passes, satisfying $s_i \neq s_{i+1}$ for all $i$. We read 1-morphisms from right to left, so that $\underline{x}$ has source $s_m$ and target $s_1$. We say the 1-morphism has length $\ell(\underline{x}) = m$ (this is not additive under composition; it would be additive if we set $\ell(\underline{x}) = m - 1$ instead). For instance, the identity 1-morphism of any...
object \( s \in S \) has length 1. We represent a composition of 1-morphisms diagrammatically as a sequence of dots on the line, separating regions of different colors.

**Example 2.2.** The 1-morphism \( brgryb \):

The 2-morphisms are generated by colored cups and caps. More precisely, for each \( b \in S \) and for each \( r \in S \setminus b \) there is a cap map \( brb \rightarrow b \) and a cup map \( b \rightarrow brb \), as pictured below.

There are two types of relations, which hold for every possible coloring of regions.

\[
(2.1) \quad \begin{array}{c}
\includegraphics[width=1cm]{cup}
\end{array} = \begin{array}{c}
\includegraphics[width=1cm]{cap}
\end{array} = \begin{array}{c}
\includegraphics[width=1cm]{cupcap}
\end{array}
\]

\[
(2.2) \quad \begin{array}{c}
\includegraphics[width=1cm]{cup2}
\end{array} = -\delta \begin{array}{c}
\includegraphics[width=1cm]{cap2}
\end{array}
\]

This ends the definition.

Let \( CM(m, k) \) denote the set of \((m, k)\)-crossingless matchings in the planar strip (see [23, section 1] for the definition). Given any element of \( CM(m, k) \), one can color the regions by elements of \( S \) so that no two adjacent regions have the same color. The resulting diagram will represent some 2-morphism in \( STL \). Conversely, every 2-morphism in \( STL \) is a \( \mathbb{Z}[\delta] \)-linear combination of such colored crossingless matchings.

**Definition 2.3.** For fixed 1-morphisms \( \underline{x} = s_1 s_2 \ldots s_{m+1} \) and \( \underline{y} = t_1 t_2 \ldots t_{k+1} \), we let \( CM(\underline{x}, \underline{y}) \) denote the subset of \((m, k)\)-crossingless matchings which can be consistently colored to yield a 2-morphism in \( \text{Hom}(\underline{x}, \underline{y}) \).

For example, \( CM(\underline{x}, \underline{y}) = \emptyset \) unless \( s_1 = t_1 \) and \( s_{m+1} = t_{k+1} \).

**Example 2.4.** An element of \( CM(grgrybgbyb, gyrorybrb) \):

The reader can verify that only four elements of \( CM(10, 8) \) actually give rise to an element of \( CM(grgrybgbyb, gyrorybrb) \).

**Lemma 2.5.** The set \( CM(\underline{x}, \underline{y}) \) forms a \( \mathbb{Z}[\delta] \)-basis for \( \text{Hom}(\underline{x}, \underline{y}) \).

**Proof.** This can be proven in exactly the same way that one proves that \( CM(m, k) \) is a \( \mathbb{Z}[\delta] \)-basis for \( \text{Hom}(m, k) \) in the usual Temperley-Lieb category. \( \square \)
Example 2.6. Note that $CM(x, x)$ always contains the identity crossingless matching $1$, but may contain no others. For instance, if $s_i \neq s_{i+2}$ for all $i$, then there can be no cups or caps, and therefore $CM(x, x)$ only contains the identity.

Example 2.7. When $n = 1$ and $S = \{b\}$, the 2-category $STL$ is very boring, having a unique 1-morphism $b$ with $\text{End}(b) = \mathbb{Z}[\delta]$.

Example 2.8. When $n = 2$ and $S = \{r, b\}$, the 1-morphisms are alternating sequences $x = rbrb \ldots$. When $x$ and $y$ both begin with $r$, and have lengths $m + 1$ and $k + 1$ respectively, then $CM(x, y) = CM(m, k)$. In fact, there is an equivalence of categories between the usual Temperley-Lieb category and the full subcategory of $STL$ obtained by considering only 1-morphisms beginning with $r$. In many senses, the two-colored Temperley-Lieb category is more natural than the usual Temperley-Lieb category, because the representation theory of $sl_2$ is naturally $\mathbb{Z}/2\mathbb{Z} \cong \Lambda_{wt}/\Lambda_n$ graded, into even and odd representations. For more on this, see [6] and [7].

Exercise 2.9. Conversely, let $S$ be arbitrary, and suppose that $x$ and $y$ begin with $r$ and have lengths $m + 1$ and $k + 1$ respectively. Then one has an equality $CM(x, y) = CM(m, k)$ if and only if both spaces are empty (i.e. $k + m$ is odd), or $x$ and $y$ both alternate between $r$ and another color $b$.

By flipping diagrams upside-down, one obtains a bijection between $CM(x, y)$ and $CM(y, x)$. This extends to an antiinvolution $\iota$ on $STL$.

2.2. The Karoubi envelope. In this paper, $\mathbb{k}$ will always be a commutative ring, perhaps with extra structure. In this section and the next, $\mathbb{k}$ will be a $\mathbb{Z}[\delta]$-algebra. We now work in the 2-category $STL \otimes_{\mathbb{Z}[\delta]} \mathbb{k}$ obtained by base change, and abusively denote this category $STL$.

Remark 2.10. It is well-known that the usual Temperley-Lieb category is cellular (see [24, section 2] for the definition). In fact, it is an especially nice kind of cellular category known as an object-adapted cellular category, meaning roughly that the cells correspond to some objects in the category, and that the top cell of each of these objects contains only the identity map. The monoidal structure is usually ignored when studying the cellular structure (certainly the theory of monoidal cellular categories has not been thoroughly explored).

Similarly, the 2-category $STL$, when viewed as a 1-category by forgetting the structure of horizontal concatenation, is an (object-adapted) cellular category, using a direct adaptation of the structure on the usual Temperley-Lieb category. The features of the Karoubi envelope of $STL$ that we discuss below are in fact rather general properties of object-adapted cellular categories, but we give complete proofs. In particular, references to “shorter sequences” below should be replaced with references to the cellular partial order. Future work of the first author will contain more discussion of object-adapted cellular categories.
Fix a 1-morphism $x$ of length $m+1$. A key property of the set $CM(m, m)$, which we used implicitly in Example 2.6, is that every diagram except the identity contains a cap on bottom and a cup on top. In particular, the span of the non-identity diagrams in $CM(x, x)$ forms a two-sided ideal $I_{<x} \subset \text{End}(x)$, whose quotient is free of rank 1 over $k$, spanned by the identity.

Suppose that one can decompose $1 \in \text{End}(x)$ into a sum $1 = \sum e_i$ of orthogonal indecomposable idempotents. It is easy to see, by working modulo $I_{<x}$, that there is a unique idempotent $e_0$ with a non-zero coefficient of the identity (in the basis $CM(x, x)$), and this coefficient is 1. The remaining idempotents lie within $I_{<x}$.

Our goal is to prove that within the Karoubi envelope $\text{Kar}(STL)$, the object $x$ has a unique indecomposable summand $V_x$ which is not a summand of $y$ for any shorter sequence; it is the image of $e_0$. In other words, the idempotents within $I_{<x}$ actually factor through shorter sequences $y$.

**Lemma 2.11.** Suppose that $k$ is a complete local ring. Then $\text{Kar}(STL)$ has the Krull-Schmidt property.

**Proof.** This is a general fact for $k$-linear categories with finite dimensional Hom spaces. □

**Proposition 2.12.** Suppose that $k$ is a complete local ring. For each sequence $x$ choose a decomposition $1 = \sum e_i$ into orthogonal indecomposable idempotents, such that $e_0 = 1$ modulo $I_{<x}$. Let $V_x$ denote the image of $e_0$, an object in $\text{Kar}(STL)$. Then the collection of all $V_x$ over all sequences $x$ form a complete list of non-isomorphic indecomposable objects in $\text{Kar}(STL)$, and

$$x \cong V_x \bigoplus \bigoplus_{\ell(y) < \ell(x)} V_y^{\oplus m_y}.$$ 

In particular, by the Krull-Schmidt property, the object $V_x$ is independent of the choice of idempotent decomposition, up to isomorphism.

The sections which follow will give a more intuitive and obvious proof under some additional assumptions, and the novice reader should skip there. We now provide a general proof, which is adapted directly from the proof of the Soergel Categorification Theorem found in [9, section 6.6].

**Proof.** It is not hard to reduce to the following statement: for each $x$ and each indecomposable idempotent $e \in \text{End}(x)$, the corresponding object $V$ in $\text{Kar}(STL)$ is isomorphic to $V_y$ for some $y \leq x$, with equality if and only if $e = 1$ modulo $I_{<x}$.

Any diagram in $CM(x, x)$ factors as $S \circ T$, for some triple $(z, S, T)$ where $\ell(z) \leq \ell(x)$, $T \in C(x, z)$ is a cap diagram, and $S \in C(z, x)$ is a cup diagram (see [23, section 2] for the definition of cap and cup diagrams). Then one can expand $e$ in the diagram basis

$$e = \sum_{z} \sum_{(z, S, T)} a_{S, T} S \circ T$$
with some coefficients $a_{S,T} \in k$. Choose a sequence $y$ of maximal length such that there exists a triple $(y, S, T)$ with $a_{S,T} \neq 0$. Note that $y = x$ precisely when $e = 1$ modulo $I_{<x}$.

We wish to show that there is some triple $(y, X, Y)$ such that

$$Y \circ e \circ X \in k^\times \subset k = \End(y) / I_{<y}.$$ 

Let $I_{<y}$ denote the ideal of all morphisms which factor through any sequence shorter than $y$, or any other sequence of the same length. We now proceed to work in the quotient category $STL/I_{<y}$. The image of $e$ is still a nonzero idempotent, expanded as above except only using triples with $z = y$. Let $m$ denote the maximal ideal of $k$. Suppose that $T \circ e \circ S \in m \subset k = \End(y) / I_{<y}$ for all triples $(y, S, T)$. By expanding $e^3 = e$ one can deduce that each $a_{S,T} \in m$. But this is a contradiction, as $m\End(x)$ is contained in the Jacobson radical of $\End(x)$, and no non-zero idempotent can be contained in the Jacobson radical.

The map $Y \circ e$ induces a map $V \rightarrow y$, and $e \circ X$ induces a map in the other direction. By composing these further with the chosen idempotent $e_0$ inside $y$, we obtain maps $e_0 \circ Y \circ e : V \rightarrow V_y$ and $e \circ X \circ e_0 : V_y \rightarrow V$. Composing these maps we get an endomorphism of $V_y$ which projects to an invertible map in $\End(V_y) / I_{<y} = k$, so that it must be invertible in the local ring $\End(V_y)$. Therefore, $V_y$ occurs as a summand of $V$, and since $V$ is indecomposable, we have $V \cong V_y$. □

2.3. Orthogonality. In the rest of this chapter, we discuss the case when $e_0$ has an alternate description as the unique idempotent perpendicular to $I_{<x}$. In particular, $e_0$ is canonically defined, and $V_x$ is well-defined up to unique isomorphism. In this case, the recursive formula of the following section will make the fact that all other idempotents factor through shorter expressions immediately obvious.

Let $T = T(x) \subset \End(x)$ be the right perpendicular space to $I_{<x}$. In other words, $T$ is the $k$-module consisting of all $f \in \End(x)$ such that $cf = 0$ for any cap $c$.

![Diagram](image)

(The lack of color in some regions is supposed to represent the irrelevance of those colors.) Similarly, let $B = B(x) \subset \End(x)$ be the left perpendicular space, the $k$-module consisting of all $f \in \End(x)$ such that $fc = 0$ for any cup $c$. Clearly $\iota(B) = T$.

Claim 2.13. Suppose that $B(x)$ contains an element $f$ for which the coefficient of the identity is invertible in $k$. Then $B = T$ and both are spanned by $f$. Every element of $B$ is fixed by $\iota$. Moreover, $B$ contains a unique idempotent $JW(x)$, which is determined within $B$ by the
fact that the coefficient of the identity is 1. The idempotent $JW(x)$ is indecomposable and central, and it is the unique indecomposable idempotent not contained in $I_{<x}$.

Proof. Let us write $f = \lambda \mathbb{1} + f'$, where $f' \in I_{<x}$ and $\lambda \in \mathbb{k}$ is invertible. For any $g \in T$ one has $f'g = 0$, since any non-identity diagram has a cap on bottom. Therefore $fg = \lambda g$. By the same token, if $g = \mu \mathbb{1} + I_{<x}$ then $fg = \mu f$. In particular, $g = \mu \lambda^{-1} f$. This proves that every element of $T$ is in the $\mathbb{k}$-span of $f$. By the same token, every element of $B$ is in the $\mathbb{k}$-span of $\iota(f)$. Thus $\iota(f) \in T$ is a multiple of $f$, and the coefficient of the identity is also $\lambda$, so that $\iota(f) = f$. Therefore $B = T = \mathbb{k} \cdot f$, and every element is $\iota$-fixed. Moreover, letting $JW(x) = \lambda^{-1} f$, the above argument proves that $JW(x)^2 = JW(x)$.

If $g \in \text{End}(x)$ is any element, and $g = \mu \mathbb{1} + I_{<x}$, then $JW(x)g = g JW(x) = \mu JW(x)$, so that $JW(x)$ is central. In particular, the ideal of $JW(x)$ is one-dimensional, from which it follows that the idempotent is indecomposable. Moreover, $JW(x)g = 0$ if and only if $g \in I_{<x}$, so that every other indecomposable idempotent is contained in $I_{<x}$. □

This idempotent $JW(x)$, which we call the top idempotent, is akin to the Jones-Wenzl projectors defined for usual Temperley-Lieb algebras. We draw $JW(x)$ as a box labeled by $x$, as in this example.

\[
\text{rgybrbg}
\]

It is possible that $B$ does not contain any element with invertible coefficient of the identity, in which case we say that the top idempotent does not exist. In this case, the special idempotent $e_0$ discussed above can be more complicated, and will not be orthogonal to $I_{<x}$.

2.4. A recursive formula for top idempotents. We use quantum number notation to indicate certain elements in $\mathbb{k}$. Let $[2] \in \mathbb{k}$ be the image of $\delta$, and let $[1] = 1$ and $[0] = 0$. One defines the quantum number $[m] \in \mathbb{k}$ for $m \in \mathbb{Z}$ by the recursive formula

\[
[2][m] = [m + 1] + [m - 1].
\]

Given a sequence of colors $x$, a subsequence $y \subset x$ will always indicate a consecutive subsequence. A subsequence is alternating if it alternates between two colors in $S$; it is maximal alternating if it can not be extended to a longer alternating subsequence. An initial subsequence is a sequence consisting of the first $k$ colors in $x$, and a final subsequence is a sequence consisting of the last $k$ colors in $x$, for any $1 \leq k \leq \ell(x)$. The tail of $x$ is the maximal alternating final subsequence. For example, the tail of $\text{rgybrbg}$ is the final 5 colors $\text{brbrb}$.

In this section we provide a recursive formula for top idempotents under the assumption that certain quantum numbers are invertible in $\mathbb{k}$. This imitates a formula from [22], giving the Jones-Wenzl projectors in the usual Temperley-Lieb category.
Proposition 2.14. When $\ell(x) \leq 2$, the idempotent $JW(x)$ exists, and is equal to the identity map. Now suppose that $x = \ldots rb$ and that $JW(y)$ exists for all initial subsequences of $x$. Extending $x$ by a color $g \neq r, b$ one has

\begin{equation}
(2.4)
\end{equation}

Extending $x$ by $r$, when $[k]$ is invertible one has

\begin{equation}
(2.5)
\end{equation}

The sequence $\underline{z} = \ldots r$ is the initial subsequence of $x$ which is only missing the final $b$. The number $k$ appearing in (2.5) is the length of the tail of $x$. Moreover, the map

\begin{equation}
(2.6)
\end{equation}

is an idempotent in $\text{End}(x^r)$ orthogonal to $JW(x^r)$. If $[k]$ is not invertible then $JW(x^r)$ does not exist.

Note that the idempotent $JW(\underline{z})$ which appears in (2.6) can be replaced by the identity map of $\underline{z}$. After all, any non-identity term will have a cup on top, which will annihilate $JW(x)$. We included the idempotent $JW(\underline{z})$ because it implies that the idempotent (2.6) factors through $V_{\underline{z}}$ inside $\underline{z}$.

Exercise 2.15. When (2.5) holds, show that the coefficient of

\begin{equation}
(2.7)
\end{equation}

in $JW(x^r)$ is precisely $\frac{[k-1]}{[k]}$. Hint: replace each $JW(x)$ with a linear combination of crossingless matchings in the RHS of (2.5), and observe that only a single term could possibly contribute to this coefficient.
Proof. When $\ell(x) \leq 2$, $CM(x,x)$ only contains the identity map, and $I_{\leq 2} = 0$. It is clear that $B = T = \text{End}(x)$ and $JW(x) = 1_x$. We assume henceforth that $\ell(x) \geq 2$, and that $JW(y)$ exists for all initial subsequences $y$ of $x$. If $k$ is the length of the tail of $\gamma$, then our inductive hypothesis implies that $[l]$ is invertible for all $l < k$.

Suppose that $g \neq r, b$. Any non-identity diagram in $CM(xg, xg)$ must begin with a cup, and it is inconsistent with the coloring for this cup to involve the final strand. Therefore any non-identity diagram will kill the RHS of (2.4), because a cup enters $JW(x)$. The RHS of (2.4) is clearly in $B(xg)$, and the coefficient of the identity is equal to 1 since this is true also in $JW(x)$, so that the RHS is equal to $JW(xg)$.

Now we extend $x$ by $r$. We claim that

$$ (2.8) \quad = -\frac{[k]}{[k-1]} $$

To show this we use induction, assuming that $JW(x)$ was defined using either (2.4) or (2.5) to extend $z$ by $r$. When $k = 2$, $JW(x)$ is defined using (2.4), and (2.8) is clear since the value of a circle is $-2$. When $k > 2$, $JW(x)$ is defined using (2.5), and the size of the tail of $z$ is $k - 1$. Writing $JW(x)$ as a linear combination of crossingless matchings, the only ones with nonzero contribution to (2.8) are the identity and the diagram in (2.7). The identity contributes $-2$ times $JW(z)$, and the diagram in (2.7) contributes $\frac{[k-2]}{[k-1]}$ times $JW(z)$. Adding these, one obtains $\frac{[k-2]}{[k-1]} \frac{[k]}{[k-1]} = \frac{[k]}{[k-1]}$ times $JW(z)$, as desired.

Suppose that $[k]$ is invertible. The RHS of (2.5) is obviously killed by any cup other than a cup on the final subsequence $rrb$. This final cup also kills the RHS, by (2.8). The coefficient of the identity in the RHS is only affected by the first term, and is therefore equal to 1. Thus the RHS of (2.5) is by definition equal to $JW(xr)$. The statement about the orthogonal idempotent is also clear.

Now suppose that $[k]$ is not invertible. Multiplying the RHS of (2.5) by $[k]$, one obtains a map in $B$ which is well-defined. The coefficient of (2.7) is now $[k-1]$, which is invertible, but the coefficient of the identity is $[k]$, which is not invertible. If $JW(xr)$ exists then any element of $B$ is $JW(xr)$ multiplied by the coefficient of the identity; if the coefficient of the identity is non-invertible, then every coefficient is non-invertible. This is a contradiction, so that $JW(xr)$ can not exist. □

Remark 2.16. One could also prove this recursive formula using the usual recursive formula from [22] for Jones-Wenzl projectors, combined with Proposition 2.20 below. However, we felt this proof was still useful and motivational.
Corollary 2.17. Suppose that all quantum numbers are invertible. Let $V_x$ denote the image of $JW(x)$, an indecomposable object of the Karoubi envelope $\text{Kar}(S\mathcal{TL})$. If $x$ ends in $rb$, and $\bar{z}$ is the initial sequence missing only the final $b$, then one has

\begin{align}
V_x V_{bg} &\cong V_{bg}, \\
V_x V_{br} &\cong V_{br} \oplus V_{z}.
\end{align}

(2.9) \hspace{1cm} (2.10)

Proof. This is implied by (2.4) and (2.5), which give a decomposition of the identity of $V_x V_{bg}$ and $V_x V_{br}$ respectively into orthogonal idempotents which factor through the appropriate objects. \square

2.5. A descriptive formula for top idempotents. The recursive formula of Proposition 2.14 does not completely answer the question of when the map $JW(x)$ exists. After all, it is possible for $JW(x)$ to exist even when $JW(y)$ does not exist for an initial subsequence $y \subset x$. As an example, consider the case when $x$ is an alternating sequence, so that the question reduces to the usual Temperley-Lieb algebra and its Jones-Wenzl projectors.

Claim 2.18. Let $x = rbrb \ldots$ be an alternating sequence of length $k + 1$. Then $JW(x)$ exists if and only if the quantum binomial coefficients $\begin{bmatrix} k \\ m \end{bmatrix}$ are invertible in $k$, for all $0 \leq m \leq k$.

Example 2.19. $JW(rbrb)$ exists when $[3]$ is invertible. This can happen even when $[2]$ is not invertible (e.g., when $[2] = 0$ one has $[3] = -1$), in which case $JW(rbr)$ does not exist.

We have not been able to find this claim in the literature on Temperley-Lieb algebras. It was explained to the first author by Ben Webster, using a representation-theoretic argument [21].

Now we use Claim 2.18 to give an exact condition for whether $JW(x)$ exists.

Proposition 2.20. Suppose that $\begin{bmatrix} k \\ m \end{bmatrix}$ is invertible whenever $0 \leq m \leq k$ and $k + 1$ is the length of a maximal alternating subsequence of $x$. Then one has

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Now suppose that some $\begin{bmatrix} k \\ m \end{bmatrix}$ is not invertible, where $k+1$ is the length of a maximal alternating subsequence $y$. By the algebraic proof of Claim 2.18, there is some element $f$ of $B(y)$ with a noninvertible coefficient of the identity, but with an invertible coefficient of some non-identity diagram $D$. Taking the horizontal concatenation of $f$ with $JW(\underline{z})$ for the other maximal alternating subsequences $\underline{z}$ of $\underline{x}$, one obtains an element of $B(\underline{x})$ with a noninvertible coefficient of the identity, but with an invertible coefficient of $1 \otimes D \otimes 1$. This contradicts the existence of $JW(\underline{x})$, using the same argument as in the end of the proof of Proposition 2.14.

\hfill \Box

2.6. Generalizations. We no longer assume that $\mathbb{k}$ is a $\mathbb{Z}[\delta]$-algebra. Let $A = (a_{s,t})$ be an $S \times S$ matrix with values in $\mathbb{k}$. We will only be interested in the values of $a_{s,t}$ for $s \neq t$. By convention, we set $a_{s,s} = 2$, so that $A$ is a Cartan matrix (in the sense of the next chapter).

**Definition 2.21.** Let $STL(A)$ be the 2-category defined as in Definition 2.1 except that instead of (2.2) one has

\begin{equation}
\begin{array}{c}
\begin{array}{c}
\text{circle} \\
\end{array}
= a_{b,r}.
\end{array}
\end{equation}

In other words, a circle still evaluates to a scalar, but which scalar depends on the color both inside and outside. When $\mathbb{k}$ is a $\mathbb{Z}[\delta]$-algebra, the special matrix with $a_{s,t} = -\delta$ for all $s \neq t$ will recover the original 2-category $STL$.

For any two fixed colors $s \neq t \in S$, there is a notion of two-colored quantum numbers $[m]_{s,t}$. These are defined by recursive formulae, starting with $[0]_{s,t} = 0$ and $[1]_{s,t} = 1$. Then one has

\begin{equation}
[2]_{s,t} = -a_{s,t},
[2]_{t,s} = -a_{t,s},
\end{equation}

\begin{equation}
[2]_{s,t}[m]_{t,s} = [m - 1]_{s,t} + [m + 1]_{s,t},
[2]_{t,s}[m]_{s,t} = [m - 1]_{t,s} + [m + 1]_{t,s}.
\end{equation}

It is not difficult to see that $[m]_{s,t} = [m]_{t,s}$ when $m$ is odd, and that $[m]_{s,t}a_{t,s} = [m]_{t,s}a_{s,t}$ when $m$ is even. There are many analogies between these two-colored quantum numbers and usual quantum numbers; see the appendix of [6] for more details. In particular, two-colored quantum binomial coefficients exist.
The results of Claim 2.13 still hold. Proposition 2.14 and its corollaries will still hold, after replacing quantum numbers with the suitable two-colored quantum numbers. More precisely, one should replace (2.5) with

\[ (2.15) \]

\[ x^r = x + \frac{[k-1]_r}{[k]_{r,b}} x^s x^t. \]

Corollary 2.17 holds once one replaces the first sentence with “Suppose all two-colored quantum numbers are invertible, for all pairs \( s \neq t \in S \).” The analogue of Proposition 2.20 is:

**Corollary 2.22.** Suppose that \([k]_{m,s,t}\) is invertible whenever \( 0 \leq m \leq k \) and there exists a maximal alternating subsequence \( \ldots s t s \subset x \) of length \( k + 1 \). Then \( JW(x) \) exists and (2.11) holds. If one of these two-colored quantum binomial coefficients is not invertible, then \( JW(x) \) does not exist.

### 3. Universal Soergel Bimodules

#### 3.1. Universal Coxeter groups and Hecke algebras

The universal Coxeter group associated to a finite set \( S \) is a group \( W \) with a Coxeter presentation having generators \( S \) and relations \( s^2 = 1 \) for each \( s \in S \). An expression is a sequence \( x = s_1 s_2 \ldots s_d \) of elements of \( S \); it has length \( d \) and is reduced if \( s_i \neq s_{i+1} \) for any \( i \). Removing the underline, we let \( x \) denote the corresponding product of generators in \( W \). Each element of \( W \) has a unique reduced expression.

The Hecke algebra \( H \) of \( W \) is the \( \mathbb{Z}[v, v^{-1}] \)-algebra having a presentation with generators \( H_s \) for \( s \in S \), and relations

\[ (3.1) \]

\[ (H_s + v)(H_s - v^{-1}) = 0 \]

for each \( s \in S \). It has a Kazhdan-Lusztig or KL basis \( \{ b_w \}_{w \in W} \) as a \( \mathbb{Z}[v, v^{-1}] \)-module. One has \( b_1 = 1 \) and \( b_s = H_s + v \) for each \( s \in S \). Dyer [5, Lemma 6.1] has shown that the KL basis is given by the following recursive formula.

**Proposition 3.1.** (The Dyer formula) If \( x = s_1 s_2 \ldots r b \) is a reduced expression and \( g \neq r, b \in S \) then

\[ (3.2) \]

\[ b_x b_g = b_x g. \]

On the other hand,

\[ (3.3) \]

\[ b_x b_r = b_x r + b_x b. \]
Note that the reduced expression for $xb$ is the initial subsequence $z$ of $x$ which is missing only the final $b$.

Whenever $x$ is a reduced expression but $xs$ is not, one can show that

\[(3.4) \quad b_x b_s = (v + v^{-1})b_x.\]

Between this equation and the Dyer formula, one can compute the product of any $b_w$ for $w \in W$ with any $b_s$ for $s \in S$.

3.2. Realizations.

**Definition 3.2.** A realization of a universal Coxeter group $W$ over $\mathbb{k}$ is a free, finite rank $\mathbb{k}$-module $h$ together with its dual $h^*$, a choice of simple roots $\{\alpha_s\}_{s \in S} \subset h^*$ and a choice of simple coroots $\{\alpha_s^\vee\}_{s \in S} \subset h$, satisfying $\langle \alpha_s, \alpha_s^\vee \rangle = 2 \in \mathbb{k}$.

The Cartan matrix attached to a realization is the $S \times S$ matrix $A = (a_{s,t})$ with values in $\mathbb{k}$, given by $a_{s,t} = \langle \alpha_t, \alpha_s^\vee \rangle$. We do not assume that the simple roots span $h^*$ or that the simple coroots span $h$, so that $A$ need not determine the realization.

We assume in this paper that our realization satisfies Demazure surjectivity, which is the condition that $\langle \alpha_s, \cdot \rangle$ and $\langle \cdot, \alpha_s^\vee \rangle$ are both surjective maps to $\mathbb{k}$.

Given any realization, there is an action of $W$ on $h$ defined on generators by the formula $s(v) = v - \langle \alpha_s, v \rangle \alpha_s^\vee$. The contragredient action on $h^*$ is given by $s(f) = f - \langle f, \alpha_s^\vee \rangle \alpha_s$. Let $R$ be the coordinate ring of $h$, or in other words, the $\mathbb{k}$-linear polynomial ring whose linear terms are $h^*$. We give $R$ a grading, so that $\deg(h^*) = 2$. The commutative ring $R$ has a natural homogeneous action of $W$.

There is a Demazure map $\partial_s : R \rightarrow R^s$ whose image is the set of $s$-invariant polynomials. On linear polynomials $f \in h^* \subset R$, one has $\partial_s(f) = \langle f, \alpha_s^\vee \rangle \in \mathbb{k}$.

3.3. Diagrammatics for Soergel bimodules. Instead of defining and developing the theory of Soergel bimodules, we prefer to follow the diagrammatic approach developed in [9]. Our object of study will be a certain monoidal category with graded Hom spaces, given diagrammatically.

**Definition 3.3.** A Soergel graph is a certain kind of graph embedded in the planar strip $\mathbb{R} \times [0,1]$. The edges in this graph are colored by $s \in S$. The only vertices allowed are trivalent vertices connecting three edges of the same color, and univalent vertices (called dots). We also allow edges which meet no vertices, forming a circle. This graph is allowed to have a boundary on the walls of the strip (i.e. edges may terminate at $\mathbb{R} \times \{0\}$ or $\mathbb{R} \times \{1\}$, though these termination points are not counted as vertices). The edge labels that meet the boundary give two sequences of colors, the bottom boundary and the top boundary. Finally, we may place a homogeneous polynomial in $R$ inside each region of the graph. We consider these graphs up to isotopy, though this isotopy must preserve $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{1\}$. A Soergel graph has a degree, which accumulates $+1$ for every dot, $-1$ for every trivalent vertex, and the degree of each polynomial.
In particular, the connected components of a Soergel graph have a single color. For numerous examples, look ahead.

**Definition 3.4.** Let $\mathcal{D}$ be the monoidal category defined as follows. The objects are monoidally generated by $s \in S$, so that a general object is an expression $x$ (not necessarily reduced, possibly empty). Given two expressions $x$ and $y$, the morphism space $\text{Hom}(x, y)$ will be the $k$-module spanned by Soergel graphs with bottom boundary $x$ and top boundary $y$, modulo the local relations below. This morphism space is graded by the degree of the Soergel graphs, and all the relations below are homogeneous.

The **Needle relation**:

(3.5) \[ \begin{array}{c} \circ \end{array} = 0 \]

The **Frobenius relations**:

(3.6a) \[ \begin{array}{c} \times \end{array} = \begin{array}{c} \times \end{array} \]

(3.6b) \[ \begin{array}{c} \circ \end{array} = \begin{array}{c} \circ \end{array} \begin{array}{c} \circ \end{array} = \begin{array}{c} \circ \end{array} \]

The **Barbell relation**:

(3.7) \[ \begin{array}{c} \bullet \end{array} = \begin{array}{c} \alpha_b \end{array} \]

The **Polynomial forcing relation**:

(3.8) \[ f \begin{array}{c} \circ \end{array} = \begin{array}{c} \partial_b(f) \end{array} + \begin{array}{c} b(f) \end{array} \]

This ends the definition.

We may also consider Soergel graphs on the planar disk; these have a single boundary sequence $x$, which is to be considered only up to rotation. A Soergel graph on the disk does not represent a morphism in $\mathcal{D}$. However, as the relations above are local, one can apply them to any disk within the planar strip, so disk diagrams are useful for local calculations.

Let $\text{Kar}(\mathcal{D})$ denote the Karoubi envelope of the graded, additive closure of $\mathcal{D}$. The following theorem is proven in [9], whose analogue for Soergel bimodules is known as the **Soergel Categorification Theorem**.

**Theorem 3.5.** The indecomposable objects in $\text{Kar}(\mathcal{D})$, up to isomorphism and grading shift, can be labeled by $w \in W$. The indecomposable object $B_w$ is the unique summand inside $w$ for a reduced expression of $w$ which is not a direct summand of $y$ for any shorter expression. There is an isomorphism of $\mathbb{Z}[v, v^{-1}]$-algebras

\[ H \rightarrow [\text{Kar}(\mathcal{D})] \]
from the Hecke algebra to the Grothendieck ring of $\text{Kar}(D)$, which sends $b_s$ to the symbol of the generating object $s$, which is equal to $B_s$.

Soergel conjectured that, when $k$ has characteristic zero and the representation $\mathfrak{h}$ is “reflection-faithful,” this isomorphism sends $b_w$ to $[B_w]$. Our goal in the rest of this paper is to give a criterion for when $b_w \mapsto [B_w]$, which will happen when we can categorify the Dyer formula.

3.4. Maximally connected graphs and minimal degrees. This section is an adaptation of [6, section 5.3.3], which only treated the case of two-colors. However, the arguments are almost identical.

Using the relations in Definition 3.4, it is not hard to show that every graph is in the span of a graph containing only simple trees with polynomials. In other words, each connected component of the graph is a tree with non-empty boundary. Any two trees with the same boundary are equal by (3.6a) and (3.6b). Moreover, this tree contains a dot precisely when the boundary is a single point, in which case the tree has no trivalent vertices; we call such a tree a boundary dot. Moreover, one can assume that there is a single polynomial, and it occurs in a region of one’s choosing (say, the leftmost region). Proofs are essentially given in [6].

Definition 3.6. A Soergel graph containing only simple trees is maximally connected if it has no polynomials, and satisfies the following condition for each $s \in S$. Consider the subgraph $\Gamma$ consisting of all the edges colored by $S \setminus s$. Then $\Gamma$ splits the planar strip into regions, and each region may contain at most one connected component colored $s$.

It is easy to see that maximally connected Soergel graphs with a given boundary exist, and that for any graph which is not maximally connected, one can produce a maximally connected graph of smaller degree by “fusing” two edges (see [6]) or removing a polynomial. However, not all maximally connected graphs have the same degree, as the following examples show.

Given a $S$-colored crossingless matching on the planar disk (with at least two colors), one can obtain a maximally connected Soergel graph by taking a deformation retract of each colored region. A quick inductive argument shows that any such Soergel graph has degree $+2$. The choice of deformation retract is irrelevant, because any two trees with the same boundary are equal.
The regions in the resulting graph (RHS) correspond to the strands in the original crossingless matching (LHS). Therefore, each region is bounded by exactly two colors, and meets the boundary exactly twice. It is easy to recover the colored crossingless matching from the resulting graph: simply deformation retract each region into a strand, and use the colors of the graph to color the regions between strands.

**Proposition 3.7.** Let $x = s_1 s_2 \ldots s_d$ be a sequence representing the boundary of a Soergel graph on the planar disk (so that we only consider $x$ up to rotation). Then any maximally connected Soergel graph on the disk with boundary $x$ has degree $\geq 2 - m$, where $m$ is the number of repetitions in $x$ (i.e. the number of $1 \leq i \leq d$ such that $s_i = s_{i+1}$, where we set $s_{d+1} = s_1$). If $x$ has no repetitions and the graph has degree 2, then it arose as the deformation retract of some colored crossingless matching.

**Proof.** It is easy to reduce to the case where $x$ has no repetitions.

The maximally connected graph splits the disk into regions. Since there are no cycles, each region must meet the boundary of the disk at least once, say between $s_i$ and $s_{i+1}$. Suppose that a region $X$ meets the boundary of the disk $k$ times, so that there are distinct indices $i_1, i_2, \ldots, i_k$ such that $X$ meets the boundary between $s_{i_j}$ and $s_{i_{j+1}}$. By following the walls of $X$ we see that the colors $s_{i_{j+1}}$ and $s_{i_{j+1}}$ are equal, and the maximally connected condition implies that the $k$ colors $s_{i_j}$ are all distinct. Therefore, the number of bordering colors of a region is equal to the number of times that region meets the boundary. This number is always at least 2, since there are no repetitions in $x$.

If each region meets the boundary exactly twice, then we can deformation retract the regions into strands to obtain a colored crossingless matching, as above. It is easy to see that this yields a bijection between $S$-colored crossingless matchings, and maximally connected graphs where each region meets the boundary exactly twice.

If a region meets the boundary $k$ times, one can use this region to cut the overall graph into $k$ subgraphs, each of which is either a boundary dot, or a maximally connected graph with one repetition. This is illustrated in the picture below, for a region $X$ meeting the boundary 4 times. Using induction therefore, each subgraph has degree at least $+1$. If any region meets the boundary $\geq 3$ times, it follows that the overall degree of the original graph is $\geq 3$. \qed
Proposition 3.7 allows one to place a lower bound on the degree of the Hom space between two objects in $D$.

**Corollary 3.8.** Let $x = s_1 \ldots s_d$ and $y = t_1 \ldots t_k$ be two nonempty reduced expressions. If $s_1 = t_1$ and $s_d = t_k$ then every nonzero morphism in $\text{Hom}_D(x, y)$ has degree $\geq 0$. Otherwise, every nonzero morphism has degree $\geq 1$. Similarly, nonzero morphisms in $\text{Hom}_D(x, \emptyset)$ and $\text{Hom}_D(\emptyset, y)$ have degree $\geq 1$, while every nonzero morphism in $\text{Hom}_D(\emptyset, \emptyset)$ has degree $\geq 0$.

**Proof.** Let $y^{\text{op}}$ denote the sequence $y$ in reverse. Viewing $x(y^{\text{op}})$ as a long sequence on the circle, there is one repetition if $s_1 = t_1$, and one repetition if $s_d = t_k$. By Proposition 3.7, the minimal degree of any map is at least $2$ minus the number of repetitions. Similarly, if $y$ is empty, then $x$ has at most one repetition, if $s_1 = s_d$. □

Given an $S$-colored crossingless matching on the planar strip, one can obtain a maximally connected Soergel graph of degree $0$ by deformation retract, as in the example below.

**Corollary 3.9.** Every degree zero map between nonempty reduced expressions in $D$ arises from an $S$-colored crossingless matching on the planar strip.

The lower bound in Corollary 3.8 can also be obtained from Soergel’s Hom formula. The advantage of this approach is an explicit description of the morphisms in the lowest degree. This description can also be obtained, with some work, from the second author’s light leaves basis for Hom spaces.

3.5. The main theorem.

**Proposition 3.10.** Let $A$ be the Cartan matrix of the realization. There is a non-monoidal functor from $STL(A)$ (viewed as a 1-category) to $D$, which sends the 1-morphism $x$ in $STL(A)$ to the object $x$ in $D$, and sends a 2-morphism in $STL(A)$ corresponding to an $S$-colored crossingless matching to the corresponding degree 0 deformation retract. This functor is essentially surjective (in the Karoubi envelope) and fully faithful onto maps of degree 0.
Proof. First we must show that this deformation retract map preserves the algebra structure. It is easy to check (2.1). Relation (2.12) follows from $\partial b(\alpha_r) = a_{b,r}$, a deduction we leave as an exercise to the reader (a similar computation was performed in [6]). Thus the functor is well-defined. Corollary 3.9 implies that it is full onto degree 0. Soergel’s Hom formula implies that the dimensions of Hom spaces agree, and thus it is also faithful. By the classification of Theorem 3.5, each indecomposable object in $\text{Kar}(\mathcal{D})$ appears as a summand of an object in the image of the functor, so the functor is essentially surjective. □

This non-monoidal functor can be upgraded to a genuine 2-functor to the 2-category of singular Soergel bimodules. For definitions and a proof in the two-color case, see [6].

By Proposition 3.10, every idempotent in $\text{End}_\mathcal{D}(x)$ for a nontrivial reduced expression comes from an idempotent in $\text{End}_{S_{T(L)}}(x)$. In particular, the theory of top idempotents implies that, when $JW(x)$ exists, it must project to the indecomposable object $B_x \in \text{Kar}(\mathcal{D})$. Our main theorem is now immediately implied by Corollary 2.17.

Theorem 3.11. Suppose that all two-colored quantum numbers are invertible in $\mathbb{k}$. Then for a reduced expression $x = \ldots rb$ and a simple reflection $s \in S$ one has the following isomorphism in $\text{Kar}(\mathcal{D})$, which categorifies the Dyer formula. As a consequence, the map $H \to \text{[Kar(\mathcal{D})]}$ sends $b_w \mapsto [B_w]$.

\begin{equation}
B_x B_s \cong \begin{cases} 
B_x(1) \oplus B_x(-1) & \text{if } s = b, \\
B_{xs} & \text{if } s \neq r, b, \\
B_{xr} \oplus B_{xb} & \text{if } s = r.
\end{cases}
\end{equation}

(3.9)

Proof. The case when $s = b$ follows from the case when $x = s$, and was proven in [9]. The other cases follow from the recursion formula for top idempotents. □

Moreover, when certain two-colored quantum binomial coefficients vanish, it is immediately clear for which $w$ the statement that $b_w \mapsto [B_w]$ will fail, as in Corollary 2.22.

Remark 3.12. The vanishing of two-colored quantum numbers determines which Coxeter quotient of $W$ acts faithfully on $h$, as was discussed in [6, appendix]. In particular, if the universal Coxeter group $W$ acts faithfully on $h$, then all two-colored quantum numbers are nonzero. Therefore, $b_w \mapsto [B_w]$ is always satisfied for a faithful realization over a field $\mathbb{k}$.

Remark 3.13. The typical crystallographic setting for universal Coxeter groups is the realization where $\mathbb{k} = \mathbb{Z}$ and $a_{s,t} = -2$ for all $s \neq t$. In this case, the two-colored quantum number $[m]_{s,t}$ is equal to the integer $m$. Specializing to a field of finite characteristic, it is clear which binomial coefficients vanish.

Remark 3.14. It is not difficult, using Soergel’s Hom Formula (see [2]), to extend these results to those reduced expressions $x$ in arbitrary Coxeter groups $W$ for which every
element $y \leq x$ in $W$ has a unique reduced expression. For general Coxeter groups, morphisms are not spanned by univalent-trivalent Soergel graphs as above, requiring a more complicated definition of Soergel graphs. However, the Hom formula implies that univalent-trivalent graphs are sufficient for these kinds of expressions.

Remark 3.15. In unpublished work, the second author proved Soergel’s conjecture for universal Coxeter groups when $k = \mathbb{R}$, while considering the wider study of “large” Coxeter groups. The proof involved singular Soergel bimodules, and produced indecomposable bimodules using a formula analogous to (2.11). For more details, contact the second author.

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