A comparison of parallel and anti-parallel Werner states

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Abstract

The Werner states \( W(t) = \frac{1-t}{4} I + t |\psi^-\rangle \langle \psi^- | \) for \(-\frac{1}{3} \leq t \leq \frac{1}{3}\) are separable states which have quantum correlations. The geometric measure of these correlations is invariant under \( t \leftrightarrow -t \) and yet a closer scrutiny of these states reveals a profound difference between their quantum correlations as measured by more probing measures. Although these two types of states are prepared by the same type of quantum operations acting on classically correlated states with equal classical correlations, the amount of final quantum correlation is different. We investigate this difference and trace it back to the hidden classical correlation which exists in their preparation process. We also compare these states with regard to their usefulness for entanglement distribution and their robustness against noise.

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1 Introduction

Entanglement and superposition are distinctive quantum mechanical features which can be used to surpass the limitations of classical information processing [1, 2]. The physical and technological impact of these effects are so large that entanglement is considered as a resource, in the same way as energy. Entanglement is classified [3, 4, 5], quantified [6, 7, 8, 9, 10], manipulated [11, 12] and distributed [13, 14, 15]. Even various types of networks of entangled states are being investigated [13, 14, 16, 17]. Like any other resource, it is questioned whether this is the only resource which we can rely on, or there are other cheaper and less fragile resources which can be equally effective in at least certain subclasses of our quantum communication tasks. It is now known that there are indeed separable states which do have some type of quantum correlations [18, 19, 20, 21, 22, 23]. In recent years, the same type of study as mentioned above, has begun to emerge for these kinds of states [24, 25, 26, 27, 28, 29, 30]. For example, questions like: how much correlation exist in a separable state, how such a correlation can be produced, how robust it is against noise, and whether or not this correlation can be distilled are of conceptual and practical relevance. In this paper we want to investigate some of these questions for an important class of states, namely mixed states with maximally mixed marginals or the so-called Bell diagonal states. While we do a rather general study of these states and their properties, we would like to emphasize the interplay of two specific factors,
namely the method of preparation and the amount of quantum correlations in these states. To this end we restrict our attention to a smaller subclass, namely Werner states \([31]\) which are defined as:

\[ W(t) = \frac{1-t}{4}I + t|\psi_{-}\rangle\langle\psi_{-}|, \quad (1) \]

where \(|\psi_{-}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\). The importance of these states stems from the fact any two qubit state can be converted to a Werner state by bi-local unitary operations \([32]\). It is well known that such a state is separable when the parameter \(t\) is restricted to the range \([-\frac{1}{3}, \frac{1}{3}]\).

For concreteness, consider the case where \(t = \frac{1}{3}\) and \(t = -\frac{1}{3}\), where we denote the states \(W(-\frac{1}{3})\) and \(W(\frac{1}{3})\) respectively by \(W^{↑↑}\) and \(W^{↑↓}\). These states are separable and can be decomposed respectively as follows:

\[ W^{↑↑} = \frac{1}{6} \sum_{n=x,y,z} (|n^{↑},n^{↑}\rangle\langle n^{↑},n^{↑}| + |n^{↓},n^{↓}\rangle\langle n^{↓},n^{↓}|) \quad (2) \]

and

\[ W^{↑↓} = \frac{1}{6} \sum_{n=x,y,z} (|n^{↑},n^{↓}\rangle\langle n^{↑},n^{↓}| + |n^{↓},n^{↑}\rangle\langle n^{↓},n^{↑}|) . \quad (3) \]

Therefore for \(t = \frac{1}{3}\), a Werner state is a uniform mixture of parallel spins and for \(t = \frac{1}{3}\), it is a uniform mixture of anti-parallel spins. At a first glance, it seems that there is not much difference between the above two states. However, as we will see, these two states have different amount of quantum correlations, and have different efficiency in performing certain quantum communication tasks. Thinking of them as resources, we may ask the following questions:

- Which of the above two states has higher value of quantum correlation and what is the origin of it?

- Which of the above states is more useful for quantum communication tasks?

- Which one is harder to prepare by local operations and classical communication?

- Which one is more robust under local noise?

We will try to present comprehensive answers to these questions. To this end we study them in the more general setting of one-parameter family of Werner states which we write as follows:

\[ W^{↑↑}(t) = \frac{1+t}{4}I - t|\psi_{-}\rangle\langle\psi_{-}|, \quad 0 \leq t \leq \frac{1}{3}, \quad (4) \]

and

\[ W^{↑↓}(t) = \frac{1-t}{4}I + t|\psi_{-}\rangle\langle\psi_{-}|, \quad 0 \leq t \leq \frac{1}{3}. \quad (5) \]

Analyzing the one parameter family of states in \((4)\) and \((5)\) gives us a comprehensive answer to the questions above. These are special lines in an even larger class, namely states with maximally mixed marginals (see below).

Stated briefly, we find that the parallel Werner states \(W^{↑↑}(t)\) have more quantum correlation and are more efficient for performing certain quantum communication tasks, than the anti-parallel Werner states \(W^{↑↓}(t)\). This is intriguing in view of the fact that both these states are prepared by acting on two equally classically correlated states by identical quantum channels. We relate this difference to a hidden classical correlation which is needed for setting up aligned coordinate systems by the two
Figure 1: (Color online) One can prepare separable states with quantum correlations by the action of bi-correlated unitary maps. For those states whose marginals are maximally mixed, only three types of unitary operators are necessary to produce all such states, as described in the discussion leading to eq. (34).

It turns out that for producing parallel states a more precise alignment is necessary compared with the case when they want to produce anti-parallel states. This extra classical correlation is what goes into the total quantum correlation of the parallel states. Of course one can argue that the state $W_{↑↓}$ can be prepared (at less cost) and then covered to $W_{↑↑}$ by an optimal NOT operation [33]. But as we will show, a successful conversion via the optimal NOT (with good fidelity) also amounts to having more initial classical correlations for preparation of the parallel Werner states.

The structure of this paper is as follows: in section (2) we review some preliminary facts about two-qubit separable states, in section (3) we review quantum correlation and some of its measures which are relevant for our discussion. Equipped with these tools, we perform in section (4) a detailed study of the two classes of states and compare them with regard to the questions above. Finally we end the paper with a discussion.

2 Preliminary facts about two-qubit mixed states

The states (4) and (5) mentioned in the introduction are special subsets in the large class of states, namely those with maximally mixed marginals:

$$\rho = \frac{1}{4}(I + \sum_{i=1}^{3} t_i \sigma_i \otimes \sigma_i).$$

(6)

where $\sigma_i$ ($i = 1, 2, 3$) are Pauli matrices and summation over repeated indices is understood. By local unitary actions, the matrix $t_{ij}$ can be diagonalized:

$$\rho = \frac{1}{4}(I + \sum_{i=1}^{3} t_i \sigma_i \otimes \sigma_i).$$

(7)

Let us denote the space of such states by $\Lambda$. To ensure positivity, the parameters $t_i$ should be confined within a regular tetrahedron, whose vertices are given by the vectors:

$$e_0 := (-1, -1, -1), \quad e_1 := (-1, 1, 1), \quad e_2 := (1, -1, 1), \quad e_3 := (1, 1, -1).$$

Not all the states in this tetrahedron are separable. To be a separable state, the parameters $t_i$ should be restricted to a regular octahedron inscribed in the above tetrahedron. The vertices of this octahedron are given by:

$$v_1^\pm := (\pm 1, 0, 0), \quad v_2^\pm := (0, \pm 1, 0), \quad v_3^\pm := (0, 0, \pm 1).$$
The local unitary action of Alice by a Pauli matrix $\sigma_1 \otimes I$ changes the signs of $t_2$ and $t_3$ while leaving the sign of $t_1$ intact. A similar thing happens with other local Pauli operators. Therefore depending on the sign of $t_1 t_2 t_3$, $A$ is divided into two inequivalent classes of states denoted by $\Lambda^{\uparrow \uparrow}$ and $\Lambda^{\uparrow \downarrow}$. The representative elements of these classes are respectively as follows:

\[ \rho^{\uparrow \uparrow} = \frac{1}{4} (I + t_1 \sigma_1 \otimes \sigma_1 + t_2 \sigma_2 \otimes \sigma_2 + t_3 \sigma_3 \otimes \sigma_3), \quad 0 \leq t_1, t_2, t_3 \leq 1, \quad (8) \]

and

\[ \rho^{\uparrow \downarrow} = \frac{1}{4} (I - t_1 \sigma_1 \otimes \sigma_1 - t_2 \sigma_2 \otimes \sigma_2 - t_3 \sigma_3 \otimes \sigma_3), \quad 0 \leq t_1, t_2, t_3 \leq 1, \quad (9) \]

both subject to the condition $0 \leq t_1 + t_2 + t_3 \leq 1$ (needed for positivity of the matrix). As we will show, the states (8) and (9) are constructed from a mixture of maximally mixed state and parallel ($\uparrow \uparrow$) or anti-parallel ($\uparrow \downarrow$) spins respectively, and this is the reason for the notation that we have used. Note that these two sets are joined to each other along a subset of measure zero, where $t_1 t_2 t_3 = 0$. Since any state of the form (8) can be converted to (8) or (9) by local unitary actions, in order to study the correlation properties of general two-qubit states of the form (8), we need only study the properties of these special classes.

First let us decompose these two states to a convex combination of pure states, this decomposition shows their difference in a transparent way and turns out to be important in our subsequent discussion. To do this, we define the pure states:

\[ P_i^{\pm} = \frac{1}{2} (I \pm \sigma_i), \quad (10) \]

and note that $\sigma_i \otimes \sigma_i$ can be written in two different ways in terms of product of these pure states, namely:

\[ \sigma_i \otimes \sigma_i = 2(P_i^{\uparrow} \otimes P_i^{\uparrow} + P_i^{\downarrow} \otimes P_i^{\downarrow}) - I, \quad (11) \]

or

\[ \sigma_i \otimes \sigma_i = I - 2(P_i^{\uparrow} \otimes P_i^{\downarrow} + P_i^{\downarrow} \otimes P_i^{\uparrow}). \quad (12) \]

In order to write the states (8) and (9) as a convex combination of product states, we use one of the above formulas as appropriate. One finds that

\[ \rho^{\uparrow \uparrow} = \frac{1}{4} \left( (1 - t_1 - t_2 - t_3)I + \sum_{i=1}^{3} 2t_i (P_i^{\uparrow} \otimes P_i^{\uparrow} + P_i^{\downarrow} \otimes P_i^{\downarrow}) \right), \quad (13) \]

and

\[ \rho^{\uparrow \downarrow} = \frac{1}{4} \left( (1 - t_1 - t_2 - t_3)I + \sum_{i=1}^{3} 2t_i (P_i^{\uparrow} \otimes P_i^{\downarrow} + P_i^{\downarrow} \otimes P_i^{\uparrow}) \right). \quad (14) \]

Therefore $\rho^{\uparrow \uparrow}$ is a mixture of maximally mixed state with a convex combination of states of parallel spins along the three axes $x$, $y$ and $z$, while $\rho^{\uparrow \downarrow}$ is a mixture of maximally mixed states and a combination of anti-parallel spin states along the same axes. When one of the parameters $t_i$ say $t_3 = 0$, the two states are locally convertible to each other, i.e. $(I \otimes \sigma_3) \rho^{\uparrow \uparrow} (I \otimes \sigma_3) = \rho^{\uparrow \downarrow}$. This is in fact due to the existence of a universal NOT operator for equatorial states, which is nothing but the $\sigma_3$ operator. This operator can easily reverse the orientation of any spin state in the equatorial plane: i.e. $\sigma_3 : |\phi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + e^{i\phi} |1\rangle) \rightarrow \frac{1}{\sqrt{2}} (|0\rangle - e^{i\phi} |1\rangle) = |\phi^{\uparrow}\rangle$. The same is also true if any other parameters $t_1$ or $t_2$ are zero. However, when all the parameters are different from zero, the two states $\rho^{\uparrow \uparrow}$ and $\rho^{\uparrow \downarrow}$ are not exactly convertible to each other, due to the non-existence of a universal NOT operator.
Remark: Although a universal NOT operation does not exist \[33\], one can come close to it with arbitrary fidelity. In fact in \[33\] it is shown that such an optimal NOT operator can be constructed by first estimating a state from \(N\) copies of a given state \(\sigma\) (with fidelity \(F = \frac{N+1}{N+2}\)) and then preparing the complement state \(\sigma^\perp\). We will further discuss this in subsection (4.4).

We will see that the two states \(\rho^{\uparrow\uparrow}\) and \(\rho^{\uparrow\downarrow}\), although very similar to each other, do not have the same performance in quantum information processing tasks. In fact this is due to the difference in their quantum correlation content. The origin of this difference is subtle and we will argue why this is so, after we have shown how these two states can be prepared from a classically correlated state. Before doing this we need to make a short review of three computable measures of quantum correlations.

\section{Three computable measures of quantum correlations}

For a bipartite system, quantum correlation can be defined as the difference between the total and classical correlations \[18\] \[19\], maximized over all possible local measurements on one of the two subsystems. Naturally this requires a formidable optimization which can be carried out only for a restricted class of states \[34\]. As substitutes, other computable measures have been proposed in the literature. Some are based on the distance between a given state and the closest classically correlated state \[35\], others are based on fidelity with such a state \[36\], or on non-commutativity of reduced density matrices of one of the parties \[37\]. Finally the most recent one is based on the local quantum uncertainty for observables of one part, the uncertainty being related to the correlations in a bi-partite state. In this article we consider only two types of continuous measures, namely geometric measure denoted by \(D_A(\rho)\) and the local quantum uncertainty, denoted by \(LQU_A(\rho)\) \[23\]. (Note that all our measures are defined with respect to the party \(A\). The measure in principle is not symmetric.) For our discussion it is also important to pay attention to a discrete measure of correlation, called rank \[22\] which shows how much useful a state is for a specific quantum processing task \[28\].

\subsection{Geometric measure of quantum correlation}

The geometric measure is defined as
\[ D_A(\rho) := 2 \min_{\sigma \in \Omega} ||\rho - \sigma||^2, \]
in which \(||A||^2 = tr(AA^\dagger)\) is the square of the Hilbert-Schmidt norm and \(\Omega\) is the set of classically-correlated states, i.e. \(\sigma^{AB} = \sum_i p_i |i\rangle^A \otimes \rho_i^B\) with \(p_i\) being a probability distribution and \(\{|i\rangle\}\) is an orthogonal basis for the Hilbert space of \(A\) \[38\]. For a general two-qubit density matrix
\[ \rho = \frac{1}{4} (I + x_i \sigma_i \otimes I + y_i I \otimes \sigma_i + t_{ij} \sigma_i \otimes \sigma_j), \]
(16) gives
\[ D_A(\rho) = \frac{1}{2} (||\vec{X}\||^2 + ||T||^2 - k_{\max}), \]
(17) where \(k_{\max}\) is the largest eigenvalue of the operator \(K = \vec{X}\vec{X}^\dagger + TT^\dagger\) \[35\].

For the states which we are considering in this article, an even more explicit expression can be found:
\[ D_A(\rho^{\uparrow\uparrow}) = D_A(\rho^{\uparrow\downarrow}) = \frac{1}{2} (t_{<e} + t_{<2}), \]
(18)
where $t_{\ge}$ and $t_{<c}$ are the smallest parameters in the set \{\(t_1, t_2, t_3\)\}. Finally we should note that although this measure is easily computable, it has the undesirable property of increasing under local reversible operations of part $B$ \cite{19}. Recently an alternative quantum correlation measure $(LQU)_A$ has been proposed \cite{23} which circumvents this shortcoming while having all the necessary requirements of a measure of quantum correlation \cite{19}.

### 3.2 Rank

It has been shown in \cite{22} that a continuous measure cannot specify completely the quantum correlation inherent in a separable state. In fact there is also an important discrete measure, called rank, which should be taken into account. The rank is nothing but the number of orthogonal operators which is needed in the expansion of a density matrix. In particular for the states \(\mathbf{9}\) and \(\mathbf{9}\).

\[
\text{Rank}(\rho) = 1 + \text{the number of non-zero } t_i's.
\]

It is known that starting from a classically correlated state, like $\rho_{cc} = \frac{1}{2}(|00\rangle\langle00| + |11\rangle\langle11|)$, rank cannot be increased by local operations of one party alone \cite{22}. Hence such local operations can produce only rank-2 quantum correlated states like $\rho_{qc} = \frac{1}{2}(|00\rangle\langle00| + |+1\rangle\langle+1|)$ \cite{36}. Although such rank-2 states are locally producible, it is not easy to decide whether or not all rank-2 states can be produced locally. On the other hand it is certain that no rank-3 or rank-4 state can be produced by local operations of one of the parties \cite{22}. Therefore these states are not locally producible.

### 3.3 Local Quantum Uncertainty

The definition of Local Quantum Uncertainty (LQU) \cite{23} is based on the observation that the existence of correlation with a far away party $B$, prevents exact determination of even a single observable in a state possessed by a party $A$. It is defined as \cite{23}

\[
U_A(\rho) = \min_{K_A} \mathcal{I}(\rho, K_A \otimes I_B),
\]

(19)

in which $K_A$ is an observable on part $A$ and

\[
\mathcal{I}(\rho, K) = -\frac{1}{2} \text{Tr}\{[\rho^{\frac{1}{2}}, K]^2\},
\]

(20)

is called the skew information \cite{40, 41}. For a $2 \times d$ dimensional system, \cite{19} can be cast into a closed form and is given by \cite{23}

\[
U_A(\rho) = 1 - \lambda_{max}\{W\},
\]

(21)

where $\lambda_{max}$ denotes the largest eigenvalue, and $W$ is a symmetric matrix with elements

\[
W_{ij} = tr\{\rho^{\frac{1}{2}}(\sigma_i \otimes I)\rho^{\frac{1}{2}}(\sigma_j \otimes I)\},
\]

(22)

with $i, j = 1, 2, 3$.

For the state \(\mathbf{9}\), the square root of the density matrix can be calculated in closed form. Lengthy but straightforward calculation gives the local quantum uncertainty in terms of the eigenvalues of the matrix $\rho^{\frac{1}{2}}$, which are

\[
\lambda_0 = \frac{1}{4}(1 - t_1 - t_2 - t_3),
\]

\[
\lambda_1 = \frac{1}{4}(1 - t_1 + t_2 + t_3),
\]
\[ \lambda_2 = \frac{1}{4} (1 + t_1 - t_2 + t_3), \]
\[ \lambda_3 = \frac{1}{4} (1 + t_1 + t_2 - t_3). \]  
(23)

The final result is
\[ LQU(\rho^{\uparrow \uparrow}) = 1 - \max_i \{ w_i \}, \]  
(24)
in which
\[
\begin{align*}
  w_1 &= \frac{2}{\sqrt{\lambda_0 \lambda_1 + \lambda_{i+1} \lambda_{i+2}}} \\
  &= \frac{1}{2} \left( \sqrt{(1 - t_1)^2 - (t_{i+1} + t_{i+2})^2} + \sqrt{(1 + t_1)^2 - (t_{i+1} - t_{i+2})^2} \right), \\
  w_2 &= \frac{1}{2} \left( \sqrt{1 - (t_1 + t_2)^2} + \sqrt{1 - (t_1 - t_2)^2} \right), \\
  w_3 &= \frac{1}{2} \left( \sqrt{1 - (t_1 + t_2)^2} + \sqrt{1 - (t_1 - t_2)^2} \right),
\end{align*}
\]  
(25)

and the summations in the subscripts are done in mod 3. Correspondingly for \( \rho^{\uparrow \downarrow} \), we use the same formula as in (25), with all \( t_i \) replaced with \(-t_i\). The important point now is that this measure is not symmetric under the change \( t \leftrightarrow -t \).

4 Comparison of the states

We are now equipped with necessary tools to investigate in detail the difference between the states in \( \Lambda^{\uparrow \uparrow} \) and \( \Lambda^{\uparrow \downarrow} \). To this end we propose a simple way for preparing these states by local operation and classical communication and we show that all separable states of all ranks can be prepared by correlated unitary actions of Alice and Bob on their qubits and the number of unitary operations does not exceed three in any case. Then we show that the difference in the quantum correlation content of the states \( \Lambda^{\uparrow \uparrow} \) and \( \Lambda^{\uparrow \downarrow} \) is due to the different resources used to prepare them. We also present an entanglement distribution protocol in which parallel mixtures act better than anti-parallel ones. Hence it seems that the higher correlation that is used in the preparation procedure will be accumulated in the state for future uses. We finally compare the states according to their robustness against noise.

4.1 Quantum correlation

It is obvious from (18) that the geometric discord, being an even function of the \( t_i \)'s, cannot distinguish the two states \( \rho^{\uparrow \uparrow} \) and \( \rho^{\uparrow \downarrow} \). However local quantum uncertainty (LQU) is different for the two states. Note that this difference shows itself only for rank four states. For lower rank states, LQU is the same for both types of states. To see this explicitly, let us fix one of the \( t_i \)'s say \( t_3 = 0 \). Then we find from (25) that
\[ w_1 = \frac{1}{2} (\sqrt{(1 - t_1)^2 - t_2^2} + \sqrt{(1 + t_1)^2 - t_2^2}), \]
\[ w_2 = \frac{1}{2} (\sqrt{1 - (t_1 + t_2)^2} + \sqrt{1 - (t_1 - t_2)^2}), \]
\[ w_3 = \frac{1}{2} (\sqrt{1 - (t_1 + t_2)^2} + \sqrt{1 - (t_1 - t_2)^2}), \]  
(26)

which in view of (24), clearly shows the symmetry \( LQU(t_1, t_2) = LQU(-t_1, -t_2) \). This is to be expected owing to the fact that when \( t_3 = 0 \), a local transformation \( (\sigma_3 \otimes I) \), turns \( \rho^{\uparrow \downarrow} \) in (8) into \( \rho^{\downarrow \downarrow} \) in (9).

For the general case to see the difference of LQUs quantitatively, let us use (25) and compare \( w_i \) for \( \rho^{\uparrow \uparrow} \) and \( \rho^{\downarrow \downarrow} \). A simple calculation shows that
\[ w_i^2(\rho^{\uparrow \uparrow}) - w_i^2(\rho^{\downarrow \downarrow}) = \frac{1}{2} (\sqrt{a - 8t_1t_2t_3} - \sqrt{a + 8t_1t_2t_3}), \quad \forall \ i, \]  
(27)
Figure 2: (Color online) The quantum correlation of the parallel and anti-parallel Werner states as compared by Local Quantum Uncertainty (LQU). The solid line shows LQU for $W^{↑↑}(t)$ and the dashed line shows it for $W^{↑↓}(t)$.

where

$$a = (1 - t_1^2 - t_2^2 - t_3^2)^2 - 4(t_1^2 t_2^2 + t_1^2 t_3^2 + t_2^2 t_3^2).$$ (28)

This obviously confirms our earlier result that when rank is less than 4 (i.e. $t_1t_2t_3 = 0$), the quantum correlations for $\rho^{↑↑}$ and $\rho^{↑↓}$ are equal and further shows that when $t_1t_2t_3 > 0$, then for all $i$, $w_i(\rho^{↑↑}) < w_i(\rho^{↑↓})$ and hence

$$LQU(\rho^{↑↑}) > LQU(\rho^{↑↓}).$$

It is seen that the farther the states are from lower rank states (measured by the parameter $t_1t_2t_3$), the higher is the difference.

As a simple but important special case, we look into the Werner states (1), by putting $t_1 = t_2 = t_3 = -t$. In this case, one finds from (25) that

$$w_1 = w_2 = w_3 = \frac{1}{2}(\sqrt{(1+3t)(1-t)} + 1-t),$$ (29)

from which we can find the LQU of both the states $W^{↑↑}(t)$ and $W^{↑↓}(t)$, depicted in figure (2), where it is clearly seen that the LQU of $W^{↑↑}(t)$ is higher than that of $W^{↑↓}(t)$ and the difference becomes maximum when $t = 1/3$ in which case the two states become uniform mixtures of parallel and anti-parallel spin states in the three directions $x, y$ and $z$.

4.2 Methods of preparation

To see why the states $\rho^{↑↑}$ and $\rho^{↑↓}$ have different quantum correlation as measured by their (LQU) despite their equal rank and geometric discord, we investigate the important question of their preparation. First we show that one can start from a simple classically correlated state

$$\rho^{↑↑}_{cc} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|),$$ (30)

and produce all the states in the class $\Lambda^{↑↑}$ by bi-local unitary actions. It has been shown [42] how one can experimentally prepare such classically correlated states in the valence electrons of two $^{40}Ca+$
ions in a linear Paul trap, where a qubit is encoded in an $S^{1/2}$ ground and a $D^{5/2}$ metastable state. Once this state is prepared, Alice and Bob act on their initial state \( \rho \) by the correlated unitary channel
\[
\mathcal{E}(\rho) = \sum_{i=0}^{3} p_i (U_i \otimes V_i) \rho (U_i^\dagger \otimes V_i^\dagger),
\]
where \( \{p_i\} \) is a probability distribution, \( U_i \) and \( V_i \) are unitary operators on single qubits and \( U_0 = V_0 = I \). We explicitly show that by a specific choice of \( U_i \) and \( V_i \) one can produce all the states of \( \Lambda^{↑↑} \) only by appropriate choice of \( p_i \). The same type of production is possible for the states in \( \Lambda^{↑↓} \) if we start from the following classically correlated state,
\[
\rho_{cc}^{↑↓} = \frac{1}{2}(|01\rangle \langle 01| + |10\rangle \langle 10|).
\]
To see this, let us define two types of Hadamard operators
\[
H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad K = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.
\]
The operator \( H \) turns the basis states \( |0\rangle \) and \( |1\rangle \) into \( |+\rangle \) and \( |--\rangle \) respectively. Similarly the operator \( K \) turns these states into \( |y+\rangle \) and \( |y-\rangle \). The operator \( \sigma_z \) leaves the computational states intact.

**Temporary change of notation:** For brevity, in the few lines below, we use the notations \( \rho^{↑↑} \equiv \rho^+ \) and \( \rho^{↑↓} \equiv \rho^- \) and later we resort to our earlier notation.

From (30) and (32), we can verify the following equations:
\[
(H \otimes H)\rho_{cc}^{↑↓}(H^\dagger \otimes H^\dagger) = \frac{1}{4}(I \pm \sigma_1 \otimes \sigma_1),
\]
\[
(K \otimes K)\rho_{cc}^{↑↓}(K^\dagger \otimes K^\dagger) = \frac{1}{4}(I \pm \sigma_2 \otimes \sigma_2),
\]
\[
\rho_{cc}^{↑↓} + (\sigma_1 \otimes I)\rho_{cc}^{↑↓}(\sigma_1 \otimes I) = \frac{I}{2}.
\]
Inserting these into (8), and (9) we find
\[
\rho^+ = \frac{1}{2} - t_1 - t_2 + t_3\rho_{cc}^+ + \frac{1}{2} - t_1 - t_2 - t_3(\sigma_1 \otimes I)\rho_{cc}^+(\sigma_1 \otimes I) + t_1 H \otimes H\rho_{cc}^{↑↓}H^\dagger \otimes H^\dagger + t_2 K \otimes K\rho_{cc}^{↑↓}K^\dagger \otimes K^\dagger =: \mathcal{E}(\rho^{cc}_{cc}).
\]
This relation defines the bi-local channel \( \mathcal{E} \). Hence the same channel which produces \( \rho^{↑↑} \) from \( \rho^{↑↓}_{cc} \), also produces \( \rho^{↑↓} \) from \( \rho^{↑↑}_{cc} \) (see figures (1) and (3)).

Therefore we have shown that Alice and Bob should only use three types of unitary gates (rotations) to produce any state in the classes \( \Lambda^{↑↑} \) or \( \Lambda^{↑↓} \) starting from appropriately chosen classically correlated states (30) or (32).

### 4.3 Origin of correlation difference

In this section we want to see on physical and operational grounds, why the correlations of the two states \( \rho^{↑↑} \) and \( \rho^{↑↓} \) (and specially the two types of Werner states) are different. To see more clearly
the relevance of this question, note that the difference of the two initial classically correlated states \((30)\) and \((32)\), used for production of states in \(\Lambda^{\uparrow\uparrow}\) and \(\Lambda^{\uparrow\downarrow}\), is just a simple local unitary rotation \(|0\rangle \leftrightarrow |1\rangle\). Such a local action does not produce any quantum correlation. Nevertheless when this simple unitary action is followed by the channel \(E\) (defined in \((34)\)), it leads to two states \(\rho^{\uparrow\uparrow}\) and \(\rho^{\uparrow\downarrow}\) with manifestly different quantum correlations (see Fig. (3)). The question is why this simple local action on the initial state, produces different quantum correlation at the end? What is the source of this excess quantum correlation, despite our equal action on the two states \(\rho^{\uparrow\uparrow}_{cc}\) and \(\rho^{\uparrow\downarrow}_{cc}\)? Is there any kind of hidden classical correlation in the initial state or in the channel which is converted into the final quantum correlation in the resulting states and makes their correlation different?

In the following we will argue that there is indeed a hidden classical correlation in the channel which causes this discrepancy. In fact, the difference of the states in \(\Lambda^{\uparrow\uparrow}\) and \(\Lambda^{\uparrow\downarrow}\) should be traced back to the requirement of setting up a standard and agreed-upon frame of coordinate axes between Alice and Bob. We will show that to enact the channel \(E\) on the initial state \((30)\) and produce the state \(\rho^{\uparrow\uparrow}\) with a given fidelity, they need to align their coordinate axes with more precision compared with the case when they want to produce the state \(\rho^{\uparrow\downarrow}\) from \((32)\). This more precise alignment costs them sending back and forth a larger number of bits before they start the process. It is this extra communication of bits which goes into the final higher value of quantum correlation for the states \(\rho^{\uparrow\uparrow}\). Note that in the present work, our method of comparison is not entirely quantitative in this respect, that is, while we prove the above statements quantitatively, we do not exactly relate the extra quantum correlation obtained to the extra classical correlation which is necessary for aligning the coordinate systems. Our argument runs as follows.

First we should note that the two parties agree on the \(z\) axis, since it is assumed that they both agree on the form of classically correlated states which has been given to them. What they need to do is to agree on the coordinate axes \(x\) and \(y\) to enact on this state by their local rotation operators:

\[
H = R_y(\frac{\pi}{2}), \quad K = R_x(-\frac{\pi}{2}), \quad \sigma_1 = -iR_x(\pi).
\]

Note that the operator \(\sigma_1\) is applied only by Alice. This is nothing but a \(\pi\) rotation around any axis in the \(x - y\) plane (which is perpendicular to the \(z\) axis and hence is known to both Alice and Bob). The
Figure 4: (color online) To turn the classically correlated states $\rho_{cc}$ to quantum correlated states, Alice and Bob need to correlate their unitary actions on their qubits. This requires precisely aligned coordinate axes between them. For producing $\rho_{\uparrow\uparrow}^{\pm}$ they need more precise alignment. This extra correlation in setting up the axes and enacting of correlated channels goes into the final quantum correlation of the $\rho_{\uparrow\uparrow}^{\pm}$ state.

difficulty arises when they want to enact in a correlated way the unitary operations $R_y(\frac{\pi}{2})$ or $R_x(-\frac{\pi}{2})$ for which they have to agree on a fixed axis in the $x-y$ plane (For example using the method proposed in [43]). Once this axis is chosen the other axis is automatically chosen to be perpendicular to this one and lying in the $x-y$ plane. Hence instead of $x$ and $y$ axes, Bob has considered $x'$ and $y'$ (see Fig. 4). The correlated channel which now Alice and Bob enact on the states $\rho_{cc}^{\pm\uparrow\uparrow}$ is denoted by $\mathcal{E}_\theta$ rather than $\mathcal{E}$, where

\[
\mathcal{E}_\theta(\rho_{cc}^{\uparrow\uparrow}) = \frac{1}{2}(I \otimes R_z(\theta))\mathcal{E}(\rho_{cc}^{\uparrow\uparrow})(I \otimes R_z^\dagger(\theta)),
\]

in which $H' = R_y'(\frac{\pi}{2})$ and $K' = R_x'(-\frac{\pi}{2})$. The state which is prepared in this way differs from what they wanted to prepare. In fact we see that

\[
\mathcal{E}_\theta(\rho_{cc}^{\uparrow\uparrow}) = \mathcal{E}(\rho_{cc}^{\uparrow\uparrow})(I \otimes R_z(\theta)),
\]

where $\theta$ is the angle of $x$ axis of Bob with respect to that of Alice. Let us see how much this error in aligning the $x$ axis affects the final state. We measure this by the fidelity of the resulting state and the desired state. In the appendix it is shown how the fidelity can be calculated. The result is as follows:

\[
F_\theta^{\uparrow\uparrow} := F(\mathcal{E}(\rho_{cc}^{\uparrow\uparrow}), \mathcal{E}_\theta(\rho_{cc}^{\uparrow\uparrow})) = \frac{1}{2} \sqrt{(1 + t_3)^2 - (t_1 - t_2)^2 \sin^2 \theta + \frac{1}{2}} \sqrt{(1 - t_3)^2 - (t_1 + t_2)^2 \sin^2 \theta}.
\]
By changing $t_i$ to $-t_i$ we obtain

$$ F_{\theta}^{\uparrow \downarrow} := F(E(\rho_{cc}^{\uparrow \downarrow}), E_{\theta}(\rho_{cc}^{\uparrow \downarrow})) = \frac{1}{2} \sqrt{(1 - t_3)^2 - (t_1 - t_2)^2 \sin^2 \theta} + \frac{1}{2} \sqrt{(1 + t_3)^2 - (t_1 + t_2)^2 \sin^2 \theta}. $$

In order to compare the fidelities, we simplify $(F_{\theta}^{\uparrow \uparrow})^2 - (F_{\theta}^{\uparrow \downarrow})^2$ and after some rearrangements we find that:

$$ (F_{\theta}^{\uparrow \uparrow})^2 - (F_{\theta}^{\uparrow \downarrow})^2 = \frac{1}{2} \left( \sqrt{a_{\theta} - 8t_1t_2t_3 \sin^2 \theta} - \sqrt{a_{\theta} + 8t_1t_2t_3 \sin^2 \theta} \right), \quad (38) $$

where $a(\theta)$ has the same form as in (28) except that $t_1$ and $t_2$ should be replaced with $t_1 \sin \theta$ and $t_2 \sin \theta$ respectively. This clearly shows that $(F_{\theta}^{\uparrow \uparrow})^2 < (F_{\theta}^{\uparrow \downarrow})^2$ as long as $t_1t_2t_3 > 0$ and they are equal only when $t_1t_2t_3 = 0$. Again we see that the difference in fidelity is larger the more distant the states are from lower-rank states, as measured by the parameter $t_1t_2t_3$.

Thus for rank-4 states, it is always harder to prepare the state $\rho^{\uparrow \uparrow}$ than the state $\rho^{\uparrow \downarrow}$, in the sense that Alice and Bob need to precisely agree on their corresponding coordinate axes, otherwise they end up with a state which has a lower fidelity with the required state. When $t_1t_2t_3 = 0$, (i.e. when the rank is less than 4), the two fidelities are equal and at the same time the corresponding quantum correlations are also equal as they should be, since in this case the two states are convertible to each other via local unitary actions.

### 4.4 An alternative method of preparation, using optimal NOT operation

Up to now we have emphasized the absence of a universal NOT operator in our arguments (see subsection (4.1) and figure (3)). It is well known that although a universal NOT operator does not exist, there is an optimal NOT operation which can approximate it to any desired degree of accuracy [33, 44, 45]. Therefore it is in order to re-evaluate our arguments in the light of this finding. This is what we do in this subsection.

In [33] it is shown that, although a universal NOT channel violates quantum mechanics and hence cannot exist, one can achieve an optimal NOT which out of $N$ copies of a single pure qubit state, produces a qubit state which can be as orthogonal as we wish to the original state. The method relies on optimal state estimation of the original state and then re-preparation of the orthogonal state. The fidelity of the produced state with the actual orthogonal state is given by $F = \frac{N+1}{N+2}$ [33]. The process can be described by the simple quantum channel [33]

$$ \sigma^{\otimes N} \rightarrow \Phi(\sigma^{\otimes N}) = \frac{N}{N+2} \sigma^\perp + \frac{1}{N+2} I, \quad (39) $$

where $\sigma^\perp$ is the state which is orthogonal to $\sigma$.

Equipped with this new operation, one can now imagine an alternative method for production of $\rho^{\uparrow \uparrow}$ states which at first sight may use less resources than the one mentioned above. For concreteness we restrict ourselves to the production of the state $W^{\uparrow \uparrow} \equiv W^{\uparrow \downarrow}(4)$. In this alternative method, one acts on the $\rho_{cc}^{\uparrow \downarrow}$ from the beginning and prepares $W^{\uparrow \downarrow}$ without the need for much precise alignment between the axes, and then only at the end uses the optimal NOT operation (by Bob for example) to turn $W^{\uparrow \downarrow}$ to a state as close as possible to $W^{\uparrow \uparrow}$. This is shown in figure (5) along with the original method mentioned in subsection (4.2). The price that one should pay is to use more copies of the initial states in order to achieve a given fidelity as per equation (39). In other words, in this new
Figure 5: (Color online) Starting from a classically correlated state of parallel spins, there are two ways to produce the state $W^{↑↑}$. In method A, the channel $\mathcal{E}_\theta$ directly acts on the state $\rho_{cc}^{↑↑}$, and in method B, first a NOT operation on one of the spins turns the state to $\rho_{cc}^{↑↓}$ on which the same channel $\mathcal{E}_\theta$ acts. Finally the optimal NOT produces a state which is to be as close as possible to $W^{↑↑}$. Figure (6) compares the fidelities of the two methods.

method one compromises the precision in $\theta$ (the precision in alignment) for the number of pairs of states to begin with. To make a comparison between the two methods, we assume that the axes have been aligned with a precision $\theta$ and then compare the fidelities of the two methods as follows.

Method A) In this method, the channel $\mathcal{E}_\theta$ is applied to the classical state $\rho_{cc}^{↑↑}$ and we calculate the fidelity of the resulting state with an ideal $W^{↑↑}$ state, which turns out from (38) to be:

$$F_A \equiv F(\mathcal{E}_\theta(\rho_{cc}^{↑↑}), W^{↑↑}) = \frac{1}{2} \left[ 1 + t + \sqrt{(1-t)^2 - 4t^2 \sin^2 \theta} \right].$$

Method B) In this method, the channel $\mathcal{E}_\theta$ is applied to each of the $N$ copies of the classical state $\rho_{cc}^{↑↓}$ and then the optimal NOT channel $(I_N \otimes \Phi)$ is applied to the resulting state to produce a state which is as close as possible to $W^{↑↑}$. The fidelity of the resulting state is:

$$F_B \equiv F((I_N \otimes \Phi)(\mathcal{E}_\theta(\rho_{cc}^{↑↓}))^{\otimes N}, W^{↑↑})$$
$$= \frac{1}{2} \sqrt{2(1+t)(1+st) + \sqrt{(1-t)(1-st) + 4st^2 \cos 2\theta + \sqrt{[(1-t)^2 - 4t^2][(1-st)^2 - 4s^2t^2]}},$$

where $s = \frac{N}{N+2}$. The two fidelities are plotted in figure (6). The results shown in this figure, confirm our earlier result that some sort of hidden classical correlation in the channel is responsible for the higher quantum correlation in the final state $W^{↑↑}$. This hidden classical correlation shows itself in method A as the need for more precise alignment of axes, and in method B as using a larger number of classically correlated states to begin with in order to achieve the final fidelity with the desired states. In both cases Alice and Bob need a larger amount of classical correlation for producing the state $W^{↑↑}$.

Having established the difference of quantum correlations between parallel and anti-parallel Werner states, we can now ask whether the higher amount of quantum correlation makes the parallel Werner states more suitable for quantum communication tasks. This is indeed the case as we show in the following example. Clearly this is only an example and does not exhausts all the communication tasks.
Figure 6: (Color online) Fidelities of preparing the state $W^{↑↑}$ by method A and B. The solid (blue) line corresponds to method A, where other curves show fidelities for different number of initial copies ($N = 1, 5, 10$ and $100$ from bottom to top), used for the optimal NOT gate. Two features are evident: First, for any value of $\theta$ (precision in the alignments), method A gives a higher fidelity than method B, no matter how many copies are used. The two methods give the same fidelity only when the number of copies in method B is infinite. Second, to achieve a given fidelity only when the number of copies in method B is infinite. Second, to achieve a given fidelity only when the number of copies in method B is infinite. Second, to achieve a given fidelity only when the number of copies in method B is infinite. Second, to achieve a given fidelity only when the number of copies in method B is infinite. Second, to achieve a given fidelity only when the number of copies in method B is infinite. Second, to achieve a given fidelity only when the number of copies in method B is infinite. Second, to achieve a given fidelity only when the number of copies in method B is infinite. Second, to achieve a given fidelity only when the number of copies in method B is infinite.
4.5 Effectiveness in quantum information processing

As we mentioned in the introduction, separable states which have some degree of quantum correlation, can be useful for quantum communication tasks. For example, while maximally entangled states are used for teleportation, separable states of rank-4 can be used for sending the information required for reconstruction of an arbitrary state by a remote party. For rank-3 states, the method is used to reconstruct only pure states [28]. Other examples include Remote State Preparations (RSP) [29, 30] where the fidelity is related to the geometric discord inherent in the shared state between the two parties [46]. Therefore one cannot compare the effectiveness of the parallel and anti-parallel Werner states by such tasks. To unravel a difference we resort to another task, namely distribution of entanglement by using separable states.

In the protocols of entanglement distribution [47, 48, 49, 50], a separable state $\rho_{AB}$ is shared between two parties, Alice and Bob. Alice adds a qubit $C$, performs a local operation and sends it to Bob who after a local operation again, will change the original separable state into a mixed state with a definite and non-zero entanglement. The original separable state $\rho_{AB}$ is of a special kind and the mediated qubit $C$ remains separable with the states of $A$, $B$ and $AB$ throughout the process [47]. We now want to see which one of the states $W^{+\uparrow}(t)$ or $W^{+\downarrow}(t)$ are more effective for the above task. Given the above results, one expects that $W^{+\uparrow}(t)$ may lead to more quantum entanglement than $W^{+\downarrow}(t)$ in such a protocol. As we will see, this is indeed the case.

Suppose that the state $W^{+\uparrow}(t)$ is shared between Alice and Bob. Alice prepares the ancilla $C$ in the initial state $\ket{+}$ in her possession and uses her particle $A$ as a controller to apply a $Z$ gate on the ancillary particle $C$, then she sends $C$ to Bob who performs another $CZ$ gate on the particles $B$ and $C$. The initial state of the particles $ABC$ can be written as

$$\rho_{ABC}^0 = W_{AB}^{+\uparrow}(t) \otimes \ket{+}\bra{+}_C,$$

and the final state after the operations of Alice and Bob is

$$\rho_{ABC} = \frac{1 + t}{2} \rho_{AB}^+ \otimes \ket{+}\bra{+}_C + \frac{1 - t}{2} \rho_{AB}^-(t) \otimes \ket{-}\bra{-}_C. \quad (42)$$

If Bob measures $C$ in the $x$ basis, he will find $+1$ with probability $p_+ = \frac{1+t}{2}$ and the state of $AB$ collapses to the separable state $\rho_{AB}^+ = \frac{1}{2} (\ket{00}\bra{00} + \ket{11}\bra{11})$ which is a classically correlated state. He may also find $-1$ with $p_- = \frac{1-t}{2}$ where the state of $AB$ will change to an entangled state $\rho_{AB}^-:

$$\rho_{AB}^-(t) = \frac{1}{4} \left( I \otimes I + \frac{2t}{1-t} \sigma_1 \otimes \sigma_1 + \frac{2t}{1-t} \sigma_2 \otimes \sigma_2 - \sigma_3 \otimes \sigma_3 \right). \quad (43)$$

One can easily change $t$ to $-t$ and get the final state of the process when the shared separable state is $W^{+\downarrow}(t)$. The entanglement of the final states can be compared by calculating their concurrences [7, 8]. The result is:

$$C^{+\uparrow}(t) := C(\rho_{AB}^+(t)) = \frac{2t}{1-t}, \quad 0 \leq t \leq \frac{1}{3}; \quad (44)$$

$$C^{+\downarrow}(t) := C(\rho_{AB}^-(t)) = \frac{2t}{1+t}, \quad 0 \leq t \leq \frac{1}{3}. \quad (45)$$

The result is plotted in figure [7], from which it is evident that the state, $W^{+\uparrow}(t)$ does perform better than the state $W^{+\downarrow}(t)$ in this process.

Finally we turn to the problem of robustness of the two states against noise. We know that entanglement is a fragile quantum property, i. e. when an entangled system is exposed to noise, the
entanglement starts leaking. This observation leads us to the intuition that the states which are more quantum correlated, are more fragile when exposed to a noise. As we will see in the next section, this is indeed the case for the states $W^{↑↑}(t)$ and $W^{↑↓}(t)$.

### 4.6 Robustness against noise

For simplicity and for definiteness we again consider only $W^{↑↑}(t)$. (We could have considered general states of the form (8) and (9) for this purpose, but the essential feature is also revealed in this special isotropic case). A natural type of noise which retains the isotropy of these states is the depolarizing noise $\rho \rightarrow \Phi_p(\rho) = (1 - p)\rho + p \frac{I}{2}$ acting on one of the qubits. It is readily found that such a noise, when acting on these states has the simple effect of changing $t$ to $t(1 - p)$ in both cases. The new states are thus given by $W^{↑↑}_p(t) := W^{↑↑}_p((1 - p)t)$ and $W^{↑↓}_p(t) := W^{↑↓}_p((1 - p)t)$. We can now ask two different questions, namely:

- What are the quantum correlations of the noisy states $W^{↑↑}_p$ and $W^{↑↓}_p$, as measured by LQU, and
- How much the original states have been affected by noise, as measured by their fidelities with the noisy states, i.e. $F(W^{↑↑}(t), W^{↑↑}_p(t))$ and $F(W^{↑↓}(t), W^{↑↓}_p(t))$.

The answer to the first question is readily found by using equation (24) for LQU of isotropic states and one finds

$$LQU(W^{↑↑}_p(t) = 1 - \frac{1}{2}\left(\sqrt{1 - 3t(1 - p))(1 + t(1 - p))} + 1 + t(1 - p)\right).$$ (46)

For the other state $W^{↑↓}_p(t)$, it is enough to change $t$ to $-t$ everywhere in the above formula. Figure (8) shows the plot of quantum correlations (LQU) for these two states for the case $t = \frac{1}{3}$ versus $p$. It is
clearly seen that the state $W^{↑↑}$ when affected by noise keeps its higher value of quantum correlation for all values of $p$ compared with $W^{↑↓}$. We note in passing that the same feature is also observed if both parties are subject to depolarizing channels with parameters $p$ and $p'$, since in this case it can be easily shown that the parameter $t$ changes to $t(1-p)(1-p')$ and the previous argument is again valid in this case.

To answer the second question we need a closed formula for the fidelity of two general $W$ states, $W(t)$ and $W(t')$. Regarding the fact that $W(t)$ and $W(t')$ commute and their eigenvalues can easily be obtained (in the form of $\frac{1-t}{3}$ and $\frac{1+t}{3}$, where the numbers in parentheses indicate degeneracies), we have:

$$F(W(t), W(t')) = \frac{1}{4} \left( 3\sqrt{(1-t)(1-t')} + \sqrt{(1+3t)(1+3t')} \right).$$

(47)

Using this formula we can find closed forms for the fidelities $F(W^{↑↑}(t), W^{↑↑}_p(t))$ and $F(W^{↑↓}(t), W^{↑↓}_p(t))$. For definiteness we consider the case where $t = 1/3$. The result is shown in figure 8. It is seen that the state $W^{↑↑}$ is less robust than the state $W^{↑↓}$. This is perhaps expected in view of its higher correlation. In other words, this result implies that a state which has a higher quantum correlation is more fragile under local noise.

Summing up the results of previous subsections, we have shown that the parallel Werner states are different from anti-parallel Werner states in at least four respects, namely they contain more quantum correlation, and are more effective in quantum information processing tasks. Yet they are harder to prepare, and more fragile against noise, which is the price one should pay for using them as a more powerful resource.
5 Discussion

We have made a detailed study of correlation properties of all separable states with maximally mixed marginal. These states are divided into two separate classes jointed at a set of measure zero. In particular we have focused on two important representatives of these two classes, namely the Werner states \( W(t) = \frac{1}{2}I + t|\psi^-\rangle\langle\psi^-| \) and have shown that depending on the sign of \( t \), these states, which we denote as \( W^{\uparrow\uparrow}(t) \) and \( W^{\uparrow\downarrow}(t) \) have quite different correlation properties. We relate this difference to a hidden classical correlation which is needed for preparation of these states, i.e. although both states are prepared by acting on equally classically correlated states by the same quantum channels, the quantum correlation in \( W^{\uparrow\uparrow}(t) \) is higher than \( W^{\uparrow\downarrow}(t) \), due to this hidden classical correlation. This higher correlation shows itself in several aspects of these states. \( W^{\uparrow\uparrow}(t) \) is harder to produce, once produced, is more fragile against noise, but is more efficient in quantum communication protocols based on separable states. We show this in detail for one entanglement distribution protocol. Throughout the paper we have emphasized the essential difference of these two states in that they cannot be converted to each other exactly by a universal NOT operator. In fact for lower rank states which are convertible to each other by NOT operators, such difference in correlation vanishes. Finally we show that one can use an alternative method of production of parallel Werner states from anti-parallel classically correlated states, simply by using an optimal NOT operation at the end. In view of the less precise alignment of the coordinate axes, this may seem a cheaper way of production of such states. However in order to achieve a good fidelity with optimal NOT operation, one needs multiple copies of anti-parallel classically correlated states at the beginning. We interpret this as yet another reason for an extra hidden classical correlation needed for production of parallel Werner states.

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7 Appendix

In this appendix we detail the steps leading to equations (38). The aim is to calculate the fidelity of two \( 4 \times 4 \) matrices (34) and (37).

Consider the state \( \rho^{\uparrow\uparrow} \). This state has a block structure in the form

\[
\rho^{\uparrow\uparrow} = A \oplus B = \begin{pmatrix}
\frac{1}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{1}{4}
\end{pmatrix},
\]

where \( A \) and \( B \) respectively denote the outer and the inner \( 2 \times 2 \) blocks. We also write the local rotation operator \( (I \otimes R_z(\theta)) \) in the same block structure form as:

\[
I \otimes R_z(\theta) = \begin{pmatrix}
e^{i\theta} & 0 \\
0 & e^{-i\theta}
\end{pmatrix} \oplus \begin{pmatrix}
e^{i\theta} & 0 \\
0 & e^{-i\theta}
\end{pmatrix} = R_z(\theta) \oplus R^\dagger_z(\theta).
\]

This block structure will then give

\[
\rho^{\uparrow\uparrow}\frac{\theta}{2} = A^\frac{\theta}{2} \oplus B^\frac{\theta}{2} \quad \rho^{\uparrow\uparrow}(\theta) = R_z(\theta)AR^\dagger_z(\theta) \oplus R^\dagger_z(\theta)BR_z(\theta)
\]
and greatly facilitates calculation of fidelity. We write

\[ \sqrt{\rho^{\uparrow\uparrow}_1 \rho^{\uparrow\uparrow}_2} = \sqrt{A^\dagger R_z(\theta) A R_z^\dagger(\theta) A^\dagger} \oplus \sqrt{B^\dagger R_z^\dagger(\theta) B R_z(\theta) B^\dagger} \]  \tag{51}

where all matrices are now $2 \times 2$ matrices. We now use the following identity which is valid for any $2 \times 2$ matrix $M$:

\[ \text{tr} \sqrt{M} = \sqrt{\text{tr}(M) + 2 \sqrt{\det(M)}}. \]  \tag{52}

This identity is easily verified by diagonalizing the matrix. Using this identity, we find

\[ F(W, W(\theta)) = \sqrt{\text{tr}(AR_z(\theta) A R_z^\dagger(\theta)) + 2 \det(A)} + \sqrt{\text{tr}(B R_z^\dagger(\theta) B R_z(\theta)) + 2 \det(B)}. \]  \tag{53}

Inserting $A$ and $B$ form (48) into this formula we arrive at (38).

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