Two Degree-of-freedom $\mu$ Synthesis Control for Turbofan Engine with Slow Actuator Dynamics and Uncertainties

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Abstract. To address the robust control problem of turbofan engine with slow actuator dynamics and uncertainties, we present a two degree-of-freedom (TDOF) $\mu$ synthesis control scheme. The actuators with slow dynamics and parameter uncertainties are considered in the control design. To handle the problem of rapid change of reference command, feed forward control (one freedom) is adopted, and feedback control (another freedom) is introduced to ensure good servo tracking and disturbance attenuation. The performance and the control output weighting functions are properly devised to guarantee servo tracking and disturbance attenuation performance of the devised controller. The D-K iterative algorithm is adopted to calculate the TDOF $\mu$ synthesis controller. A turbofan engine with slow actuator dynamics and parameter uncertainties is adopted for simulation. The effectiveness and superiority of the devised TDOF $\mu$ synthesis controller are verified by a series of simulation analyses. The simulation analyses show that the TDOF $\mu$ synthesis control scheme developed in this paper can handle the robust control problem of turbofan engines, despite the negative influence of the slow actuator dynamics and uncertainties. The comparison shows that the TDOF $\mu$ synthesis controller provides better performance than the PI controller and the LMI optimization gain scheduled controller do.

Nomenclature

$A_8$ Throat area of nozzle, $m^2$
$H$ height, km
$Ma$ Mach number
$N_1$ Low spool speed, r/min
$N_2$ High spool speed, r/min
$W_f$ Fuel flow, kg/s
LMI Linear Matrix Inequality

1. Introduction

With years of development, modern turbofan engines have adopted advanced full authority digital electronics system and multivariable control technology to achieve high control performance. The control system plays a vital role in operating the engine in a stable, safe, and effective manner throughout its operational envelope during the lifetime[1, 2]. In practice, the actuator of turbofan engine generally adopts the hydraulic mechanical structure. The hydraulic mechanical actuator works in the harsh working environment of fuel source medium pollution and high and low temperature for a long time. Because of the motion wear, aging, and performance degradation of the components, the
dynamic characteristics of the actuator become worse and the dynamic uncertainty of the actuator is caused. The slow dynamics of the actuator, especially when it suffers from performance degradation, have negative influence on the control performance of turbofan engines. Therefore, it is necessary to take into account the slow actuator dynamics and uncertainties in the control system design of turbofan engine. However, most of the literatures did not consider actuator dynamics or just consider a fast actuator dynamic in the controller design. Morteza et al.[3] did not consider actuator dynamics in their study for model predictive control of turbofan engine. Yang et al.[4] investigated limit protection control of turbofan engine without considering actuator dynamics. Miao et al.[5] studied the full flight envelope transient control of turbofan engines based on LMI optimization with considering a fast actuator dynamics. Liu et al.[6] investigated the multi-variable adaptive control method for turbofan engines with dynamic and input uncertainties considering a fast actuator dynamics. To address the robust control problem of turbofan engine with slow actuator dynamics and uncertainty, this paper presents a two degree-of-freedom (TDOF) $\mu$ synthesis control scheme.

In 1982, the structure singular value (SSV) was introduced by Doyle to handle the conservative problem of robust control design, which gradually formed the $\mu$ synthesis theory[7-9]. Now, $\mu$ synthesis theory has become one of the most significant methods in robust control design, it has been widely applied in a wide range of engineering area such as distillation column system[10], aircraft engine[11], altitude ground test facilities[12-16], etc. Lundstrom et al.[10] presented a two DOF $\mu$ synthesis control method for a distillation column system. Amato et al.[11] studied the robust control problem of a small commercial aircraft engine by designing a $\mu$ controller under a single point to achieve good robust performance over a large flight envelope. Zhu et al.[12-14] presented the application of $\mu$ synthesis control for synchronous control of pressure and temperature of flight environment simulation system (FESV) over a specific working envelope range. Zhu et al.[16] addressed the robust control problem of FESV by proposing a two freedom LPV $\mu$ synthesis control method. However, the $\mu$ synthesis has not been used in turbofan engine with slow actuator dynamics and uncertainties. We will study the application of $\mu$ synthesis theory in turbofan engine to achieve robust performance.

We organise this paper as follows: Firstly, an augmented linear system with slow actuator dynamics and parameter uncertainties is provided. Secondly, a brief introduction of $\mu$ synthesis theory and the $D-K$ iterative algorithm is given. Then, the two DOF $\mu$ synthesis control scheme is proposed. Finally, the proposed two DOF $\mu$ synthesis method is applied in a turbofan engine with slow actuator dynamics and parameter uncertainties, and the effectiveness and superiority of the devised TDOF $\mu$ synthesis controller is verified through simulation analysis.

2. Problem Formulation

Consider a nonlinear system as

$$x(t) = f(x,u)$$
$$y(t) = g(x,u)$$

where $x(t) \in \mathbb{R}^n$ denotes the system state, $u(t) \in \mathbb{R}^m$ denotes the control input with slow actuators dynamics, and $y(t) \in \mathbb{R}^m$ denotes the measurement output; $f(x,u)$ denotes an n-dimensional nonlinear differentiable function which features the system dynamics, and $g(x,u)$ denotes an m-dimensional nonlinear function which generates the system measurement outputs. Given a desired signal $r(t) \in D_r \subseteq \mathbb{R}^m$, we aim to devise a $\mu$ synthesis controller that achieves robust performance of the system with slow actuators dynamics and uncertainties.

We assume that, for any $r(t) \in D_r$, there exists a unique pair $(x_\epsilon, u_\epsilon)$ that satisfies the equations:

$$0 = f(x_\epsilon, u_\epsilon)$$
$$r(t) = g(x_\epsilon, u_\epsilon)$$

(2)
where \( x_e \) denotes the desired equilibrium point, \( u_e \) denotes the equilibrium control signal that is needed to keep the steady-state at \( x_e \).

The nonlinear system (1) is linearized on a equilibrium point \((x_e, u_e, y_e)\), and we obtain the linearized system as

\[
\begin{align*}
\delta \dot{x} &= A \delta x + B \delta u \\
\delta y &= C \delta x + D \delta u
\end{align*}
\]

where

\[
\begin{align*}
\delta x &= x - x_e, \delta u = u - u_e, \delta y = y - y_e \\
A_{[i,j]} &= \frac{\partial f}{\partial x_j}(x_e, u_e), B_{[i,j]} = \frac{\partial f}{\partial u_j}(x_e, u_e) \\
C_{[i,j]} &= \frac{\partial g}{\partial x_j}(x_e, u_e), D_{[i,j]} = \frac{\partial g}{\partial u_j}(x_e, u_e)
\end{align*}
\]

To ensure the devised TDOF \( \mu \) synthesis controller achieves robust performance, the uncertainty of the actuators is needed to be considered in the control design, and it is considered as a first-order function with parameter uncertainties as

\[
\delta u = \begin{bmatrix}
K_{a1} & \cdots & 0 \\
T_{a1}s+1 & \vdots & \ddots & \vdots \\
0 & \cdots & K_{am} & T_{am}s+1
\end{bmatrix} \delta v, K_{ai} \in [K_{ai}, \bar{K}_{ai}], T_{ai} \in [T_{ai}, \bar{T}_{ai}]
\] (4)

where, \( \delta v = v - v_e \) denotes the actuators input increment, \( v_e \) denotes the actuator input of the steady-state; \( K_{ai} \) denotes the \( i \)-th actuator gain coefficient, \( K_{ai} \) denotes the infimum of \( K_{ai} \), \( \bar{K}_{ai} \) denotes the supremum of the \( K_{ai} \); \( T_{ai} \) denotes the \( i \)-th actuator time coefficient, \( T_{ai} \) denotes infimum of \( T_{ai} \), \( \bar{T}_{ai} \) denotes the supremum of the \( T_{ai} \).

By applying the Laplace transformation in Eq.(4), we have

\[
\delta \hat{u} = \begin{bmatrix}
\frac{1}{T_{a1}} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \frac{1}{T_{am}}
\end{bmatrix} \delta u + \begin{bmatrix}
K_{a1} & \cdots & 0 \\
T_{a1} & \vdots & \ddots & \vdots \\
0 & \cdots & K_{am} & T_{am}
\end{bmatrix} \delta \hat{v}
\] (5)

We introduce two metrics as

\[
M_a = \begin{bmatrix}
\frac{1}{T_{a1}} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \frac{1}{T_{am}}
\end{bmatrix}, N_a = \begin{bmatrix}
K_{a1} & \cdots & 0 \\
T_{a1} & \vdots & \ddots & \vdots \\
0 & \cdots & K_{am} & T_{am}
\end{bmatrix}
\] (6)

Then, the linear system (3) and the actuators model (5) are augmented as

\[
\begin{align*}
\delta \ddot{x} &= \tilde{A} \delta \ddot{x} + \tilde{B} \delta \hat{v} \\
\delta \dot{y} &= \tilde{C} \delta \ddot{x}
\end{align*}
\] (7)
where

\[
\delta \mathbf{x} = \begin{bmatrix} \delta x_1 \\ \delta u \end{bmatrix}, \quad A = \begin{bmatrix} A & B \\ 0_{m \times n} & M \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0_{m \times n} \\ N_u \end{bmatrix}, \quad \mathbf{C} = [C \quad D]
\]

Now, the derivation of the TDOF \( \mu \) synthesis controller design can be based on the augmented system (7).

3. \( \mu \) Synthesis Theory Description

A brief introduction of \( \mu \) synthesis theory is provided in this section. The \( \mu \) synthesis theory is one of the most important robust control design techniques; it is especially good at handling structure uncertainty problem. Generally, the uncertainty of a system can be classified as parametric uncertainty and unmodeled dynamics. In \( \mu \) synthesis theory, the standard \( M-\Delta \) structure is used for robust stability analysis, which is illustrated in Figure 1. In Figure 1, \( K(s) \) denotes the controller, \( P(s) \) denotes the open-loop connection which includes the nominal model of the system, the uncertainty weighting functions, and the weighting functions, it can be partitioned as

\[
P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) & P_{13}(s) \\ P_{21}(s) & P_{22}(s) & P_{23}(s) \\ P_{31}(s) & P_{32}(s) & P_{33}(s) \end{bmatrix}
\]

![Figure 1. Standard \( \mu \) synthesis \( M-\Delta \) structure.](image)

\( M(s) \) denotes the lower Linear Fractional Transformation (LFT) of \( P(s) \) and \( K(s) \), it is obtained from the following equation

\[
M(s) = \begin{bmatrix} M_{11}(s) & M_{12}(s) \\ M_{21}(s) & M_{22}(s) \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} + \begin{bmatrix} P_{13}(s) \\ P_{23}(s) \end{bmatrix} K(s) \left[ I - P_{33}(s) K(s) \right]^{-1} \begin{bmatrix} P_{31}(s) \\ P_{32}(s) \end{bmatrix}
\]

\[\Delta\] denotes the set of all possible uncertainties in the system. Without loss of generality, \( \Delta \) is assumed square. Furthermore, \( w \) is exogenous input that usually includes command signals, noises, disturbances, etc.; \( z \) is error output typically consisting of tracking errors, regulator output, etc.; \( u_K \) denotes the output of \( K(s) \); \( y_K \) denotes the input of \( K(s) \); \( u_\Delta \) denotes the output of \( \Delta \); \( y_\Delta \) denotes the input of \( \Delta \).

In order to define the set of \( \Delta \), an uncertainty structure \( S \) is introduced as follows

\[
S = \left\{ \alpha_1, \ldots, \alpha_{a_1}, \alpha_{n_1+1}, \ldots, \alpha_{n_1+n_2}, \ldots, \alpha_{n_1+n_2+1}, \ldots, \alpha_{N_S} \right\}, \quad \alpha, n_r, n_c, n_C \in \mathbb{R}^+, N_S = n_r + n_c + n_C
\]

Then, a matrix set based on the structure \( S \) can be defined as

\[
\Delta_S = \left\{ \Delta = \text{diag}\{\delta_1 I_{a_1}, \ldots, \delta_{n_r} I_{n_r}, \ldots, \delta_{n_1} I_{a_{n_1}}, \ldots, \delta_{n_1+n_2} I_{a_{n_1+n_2}}, \Delta_c^1, \ldots, \Delta_c^c \} : \delta_i^r \in \mathbb{R}, \delta_i^c \in \mathbb{R}, \Delta_i^C \in \mathbb{R}^{a_{r_i} \times a_{r_i}}, \Delta_c^C \in \mathbb{R}^{a_{c_i} \times a_{c_i}}, 1 \leq i \leq c \right\}
\]

**Definition 1.**[13] The uncertainty set is defined as
\[
\Phi(\Delta_s) = \{ \Delta(s) \in \Re H_{\infty} : \Delta(j\sigma) \in \Delta_s, \forall \sigma \in \Re, \|\Delta\|_{\infty} < \gamma^{-1} \}
\]

where \( \|\Delta\|_{\infty} = \sup_{\sigma} |\bar{\sigma}(j\sigma)| \) denotes the \( \infty \)-norm of \( \Delta(s) \).

**Definition 2.** [7] For a matrix \( M \in \mathbb{C}^{n \times n} \), the SSV of \( M \) with respect to \( \Delta \) is defined as

\[
\mu_{\Delta}(M) = \begin{cases} 
\min_{\Delta}(\bar{\sigma}(\Delta) : \det(I - M(\Delta)) = 0) & \text{if } \det(I - M(\Delta)) \neq 0 \text{ for } \Delta \in \Delta_s \\
0 & \text{otherwise}
\end{cases}
\]

where \( \bar{\sigma}(\Delta) \) denotes the maximum singular value of \( \Delta \). When \( M \) denotes an interconnected transfer matrix as in Figure 1, the SSV of \( M \) with respect to \( \Delta \) is defined by

\[
\mu_{\Delta}(M(s)) = \sup_{\sigma} \mu_{\Delta}(M(j\sigma))
\]

**Theorem 1.** [7, 17] The controller \( K(s) \) in Figure 1 stabilizes the system for \( \forall \Delta(s) \in \Phi(\Delta_s) \), if and only if

\[
\sup_{\sigma} \mu_{\Delta}(M_{ii}(j\sigma)) < \gamma
\]

\[\text{Figure 2. Standard } \mu \text{ synthesis } M-\Delta \text{ structure with } \Delta_F.\]

The \( \mu \) synthesis method not only ensure the robust stability, but also consider the robust performance of the system. To guarantee the closed-loop system has robust performance, a fictitious uncertainty block \( \Delta_F \) is introduced in the control design, as shown in Figure 2. \( \Delta_F \) is the performance uncertainty block, satisfying \( \|\Delta_F\|_{\infty} < \gamma^{-1} \). Thus, the uncertainty structure for robust stability and performance test can be extended as

\[
\Delta_p = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_F \end{bmatrix} : \Delta \in \Delta_s, \Delta_F \in \mathbb{C}^{n \times n_{\infty}}
\]

**Definition 3.** [13] The uncertainty set is redefined as

\[
\Phi(\Delta_p) = \{ \Delta(s) \in \Re H_{\infty} : \Delta(s), \Delta_F(s) \in \Re H_{\infty}, \Delta(j\sigma) \in \Delta_s, \Delta(j\sigma) \in \mathbb{C}^{n \times n_{\infty}}, \forall \sigma \in \Re, \|\Delta\|_{\infty}, \|\Delta_F\|_{\infty} < \gamma^{-1} \}
\]

**Theorem 2.** [16, 17] The closed-loop system illustrated in Figure 2 is internally stable, well posed, and has robust performance for \( \forall \Delta_p(s) \in \Phi(\Delta_p) \), if and only if
\[
\sup_{a \in \mathbb{R}} \mu_{\lambda_r}[M(j\omega)] < \gamma
\]  

Thus, the goal of \( \mu \) synthesis design is to obtain an optimal controller through minimizing the peak value of \( \mu_{\lambda_r}[M(s)] \) over all the stabilizing controllers, which is described as

\[
\min_{K \text{-stabilizing}} \max_{a \in \mathbb{R}} \mu_{\lambda_r}[M(j\omega)]
\]

Currently, no analytic method is available to synthesize Eq.(19), but the \( D-K \) iteration algorithm that combines \( \mu \) analysis and \( \mu \) synthesis produces good results. Thus, the \( D-K \) iteration algorithm is adopted in this paper to synthesize Eq.(19).

**Step 1.** Begin with an initial guess for matrix \( D \), generally set \( D = I \).

**Step 2.** Fix \( D \) and solve the subsequent \( H_\infty \)-optimization problem for \( K \), we obtain

\[
K = \arg \inf_{K(j\omega)} \| DMD^{-1}(j\omega) \|_\infty
\]

**Step 3.** Fix \( K \) and solve the subsequent convex optimization problem for \( D \) at every frequency over a chosen frequency range

\[
D(j\omega) = \arg \sup_{a \in \mathbb{R}} \inf_{D \in D_r} \{ \sigma[DMD^{-1}(j\omega)] \}
\]

**Step 4.** Check the following convergence criteria

\[
\mu_{\lambda_r}[M(j\omega)] < \gamma
\]

If the above convergence criteria is satisfied, the \( K \) obtain in step 2 is the expected \( \mu \) synthesis controller, and end the iteration; or enter step 5.

**Step 5.** Curve fit \( D(j\omega) \) to obtain a stable and minimum phase \( D(s) \); back to step 2 and repeat.

4. Two Degree-of-freedom \( \mu \) Synthesis

![Figure 3. Schematic structure of the TDOF \( \mu \) synthesis.](image)

To guarantee the devised controller of the nonlinear system (1) achieves robust performance with slow actuator dynamics and uncertainties, a TDOF \( \mu \) synthesis design scheme that combines feed forward and feedback control law is proposed. Figure 3 is the schematic structure of the TDOF \( \mu \) synthesis controller design. In Figure 3, the feed forward controller \( K_r \), representing one DOF, is introduced to handle the problem of reference command rapidly change; the output feedback controller \( K_s \), representing another DOF, is adopted to achieve servo tracking and disturbance attenuation. Moreover, the performance weighting function \( W_p \) and the control output weighting function \( W_c \) are devised to ensure robustness and control energy limit. Where, \( e_p \) denotes the performance weighted output, and \( e_c \) denotes the control output weighted output.

Utilizing the scheme illustrated in Figure 3, a TDOF \( \mu \) synthesis controller of a selected equilibrium state is calculated by the dksyn function in MATLAB[18]. The devised TDOF \( \mu \) synthesis controller is depicted as...
\[
\begin{align*}
\delta \dot{x}_c &= A_c \delta x_c + B_c \delta u_c \\
\delta v &= C_c \delta x_c + D_c \delta u_c 
\end{align*}
\] (23)

where, \(\delta x_c\) and \(\delta u_c\) = \([\delta r; \delta y]\) denote the state and the input of the TDOF \(\mu\) synthesis controller.

5. Turbofan Engine Example

The proposed TDOF \(\mu\) synthesis method is applied to a turbofan engine with slow actuator dynamics and uncertainties. The nonlinear model of this turbofan engine is built by using MATLAB and Toolbox for the Modeling and Analysis of Thermodynamic System (T-MATS)[19]. All the characteristic maps and parameters are obtained from Gas turbine Simulation Program (GSP)[20]. This turbofan engine model, which has also been used in other studies[5, 6], is the basis of the following control devise and simulation analyses.

![Figure 4. Simulation platform of turbofan engine.](image)

Firstly, we take the full thrust state of the turbofan engine as an example to devise the TDOF \(\mu\) synthesis controller. Secondly, a simulation platform of the turbofan engine is adopted to verify the effectiveness of the devised \(\mu\) synthesis controller, which is shown in Figure 4. Thirdly, the devised TDOF \(\mu\) synthesis controller controls the turbofan engine from the full thrust state to idle then back to the full thrust state at the standard day sea level condition with slow actuator dynamics to verify the servo tracking performance. Then, the robust performance of the devised TDOF \(\mu\) synthesis controller is validated by the supremum value of the uncertainty parameters. Furthermore, to further verify the robust performance of the devised TDOF \(\mu\) synthesis controller, the same engine process is conducted at flight condition \(H = 10\) km and \(Ma = 1\). Finally, the advantage of the TDOF \(\mu\) synthesis controller is verified through comparing with a PI controller and a LMI optimization gain scheduled controller.

5.1. Two DOF \(\mu\) Synthesis Controller Design

Under the standard day sea level condition, the full thrust state \((N_1 = 10065\) r/min, \(N_2 = 12832\) r/min) of the turbofan engine is taken as an example to devise the TDOF \(\mu\) synthesis controller. Note that all the variables are normalized by their design values: \(x_1(t) = N_1(t)/N_{1d}\) denotes the normalized low spool speed, and the design value \(N_{1d} = 10065\) r/min; \(x_2(t) = N_2(t)/N_{2d}\) denotes the normalized high spool speed, and the design value \(N_{2d} = 12832\) r/min; \(u_1(t) = W_f(t)/W_{fd}\) denotes the normalized main fuel flow, and the design value \(W_{fd} = 1.144\) kg/s; \(u_2(t) = A_8(t)/A_{8d}\) is the normalized nozzle throat area, and the design value \(A_{8d} = 0.2981\) m\(^2\) is. Then the linearized model is
\[
\begin{bmatrix}
\delta x_1 \\
\delta x_2 \\
\delta u_1 \\
\delta u_2
\end{bmatrix} =
\begin{bmatrix}
-6.5499 & 5.5686 \\
0.1298 & -3.7204 \\
0.4829 & 1.9816 \\
0.6643 & -0.01428
\end{bmatrix}
\begin{bmatrix}
\delta x_1 \\
\delta x_2 \\
\delta u_1 \\
\delta u_2
\end{bmatrix} +
\begin{bmatrix}
0.1298 \\
3.7204 \\
0.6643 \\
0.01428
\end{bmatrix}
\begin{bmatrix}
\delta x_1 \\
\delta x_2 \\
\delta u_1 \\
\delta u_2
\end{bmatrix}
\tag{24}
\]

Because of the influence of degradation, the actuators of the turbofan engine suffer slow dynamics and parameter uncertainties. The actuators are considered as two first-order functions with parameter uncertainties as

\[
\delta u =
\begin{bmatrix}
\frac{K_{a1}}{T_{a1}s+1} & 0 \\
0 & \frac{K_{a2}}{T_{a2}s+1}
\end{bmatrix}
\delta v
\tag{25}
\]

where, both the nominal values of \( K_{a1} \) and \( K_{a2} \) are equal to 1, both the nominal values of \( T_{a1} \) and \( T_{a2} \) are equal to 0.5. We assume that both the actual gain coefficient \( K_{a1} \) and \( K_{a2} \) have 15% uncertainties, and there are 25% uncertainties in the time constant \( T_{a1} \) and \( T_{a2} \).

Then, the augmented system (7) is obtained. To guarantee the closed-loop system with good servo tracking and disturbance attenuation performance, \( W_P \) is devised by using the frequency division weighting principle as

\[
W_p =
\begin{bmatrix}
\frac{s+8.5}{s+0.00085} & 0 \\
0 & \frac{s+3.6}{s+0.00036}
\end{bmatrix}
\tag{26}
\]

Figure 5 is the magnitude response of \( W_P \).

To guarantee that the TDOF \( \mu \) synthesis controller with a proper output, \( W_C \) is devised by the principle of free limit in the low frequency, gradually increase the limit in the medium frequency, and maximum limit in the high frequency as follows

\[
W_c =
\begin{bmatrix}
\frac{0.5}{s+1} & 0 \\
0 & \frac{0.5}{s+1}
\end{bmatrix}
\tag{27}
\]

Figure 6 is the magnitude response of \( W_C \).
Utilizing the control scheme illustrated in Figure 3 and the weighting functions in (26) and (27), the TDOF $\mu$ synthesis controller is calculated by the dksyn function in MATLAB. As a result, a $8^{th}$ order TDOF $\mu$ synthesis controller is obtained as

$$
A = \begin{bmatrix}
-149.7282 & -0.9277 & 0.4829 & 1.9816 & -7.1433 \times 10^{-8} & -2.5924 \times 10^{-9} & 0 & 0 \\
-6.3664 & -83.9935 & 0.6643 & -0.01428 & 3.0602 \times 10^{-9} & -3.3458 \times 10^{-8} & 0 & 0 \\
-1.1199 \times 10^4 & -5.3657 \times 10^3 & -1.2083 & -0.04320 & 2.9816 & 7.1433 & 0 & 0 \\
-5.3654 \times 10^4 & 1.0987 \times 10^3 & -0.01534 & -1.2196 & 0.1670 & -0.2065 & -0.004026 & 1.6651 \\
-0.002592 & 6.3989 \times 10^4 & 1.6375 \times 10^{11} & -4.8542 \times 10^{10} & -3.8095 \times 10^{-4} & 3.0737 \times 10^{-12} & 0 & 0 \\
0.001177 & -0.002495 & -3.9625 \times 10^{-10} & 3.0102 \times 10^{-9} & 4.826 \times 10^{-12} & -3.5193 \times 10^{-14} & 0 & 0 \\
-3.2266 & -12.4705 & -5.5769 & -2.4865 & 4.9620 & 15.5249 & -9.9327 & -0.7500 \\
-11.6518 & -2.7297 & -1.7656 & -12.4493 & 19.2202 & -23.7735 & -0.4635 & -8.3050
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
-1.5034 \times 10^{-6} & -6.4412 \times 10^{-8} & -6.4027 \times 10^{-3} & -290.5026 \\
-6.4405 \times 10^{-8} & -8.3131 \times 10^{-7} & -290.5026 & -3.5897 \times 10^{0} \\
-1.1090 \times 10^{-5} & -5.2789 \times 10^{-5} & -5.0077 \times 10^{-4} & -2.3730 \times 10^{5} \\
-5.3458 \times 10^{-5} & 1.1035 \times 10^{-5} & -2.3993 \times 10^{-5} & 4.9131 \times 10^{-6} \\
178.8747 & 7.6370 \times 10^{-11} & 178.7587 & 0.02861 \\
9.4341 \times 10^{-11} & 89.4373 & 0.05264 & 89.3258 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
C = \begin{bmatrix}
-0.001127 & -0.004357 & -0.001948 & -8.6881 \times 10^{-4} & 0.001734 & 0.005425 & 0.06641 & -2.6204 \times 10^{-4} \\
-0.002036 & -4.7688 \times 10^{-4} & -3.0845 \times 10^{-4} & -0.002175 & 0.003358 & -0.004153 & -8.0974 \times 10^{-5} & 0.03349
\end{bmatrix}
$$

$$
D = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

5.2. Simulation Results

We conduct several simulation analyses in the simulation platform to verify the effectiveness and advantages of the devised TDOF $\mu$ synthesis controller. Firstly, with slow actuator dynamics, the devised TDOF $\mu$ synthesis controller controls the turbofan engine from the full thrust state to idle then back to the full thrust state at the sea level condition to verify the servo tracking performance. Secondly, to verify the robustness of the devised TDOF $\mu$ synthesis controller, the same engine process is conducted with the supremum value of the uncertainty parameters. Then, the robust performance of the devised TDOF $\mu$ synthesis controller is further validated at high-altitude condition...
Finally, a PI controller and a LMI optimization gain scheduled controller are used to verify the superiority of the TDOF $\mu$ synthesis controller.

5.2.1. Servo Tracking Performance Verification at Sea Level Condition. Using the simulation platform of the turbofan engine in Figure 4, the servo tracking performance of the devised TDOF $\mu$ synthesis controller is verified at sea level condition through simulation analysis.

Figure 7 is the simulation results of $N_1$. Figure 7 (a) illustrates the histories of $N_1$, and the black line denotes the command signal of $N_1$, the dotted line denotes the response of $N_1$ of the turbofan engine controlled by the devised TDOF $\mu$ synthesis controller. Figure 7 (b) shows the histories of the rotation speed difference $\Delta N_1$ between the response of $N_1$ and the command signal of $N_1$. From Figure 7 (a), we can see the low spool speed can track the command signal even though the turbofan engine is subject to slow actuator dynamics, which means the devised TDOF $\mu$ synthesis controller compensates the influence of the slow actuator dynamics. From Figure 7 (b), we can see that the dynamic error is less than 4%, the steady-state error is less than 0.01%, and the maximum overshoot is less than 0.1%, which demonstrates that the devised TDOF $\mu$ synthesis controller achieves good servo performance on low spool speed control at sea level condition.
The simulation results of $N_2$ are illustrated in Figure 8. Figure 8 (a) is the histories of $N_1$, and the black line is the command signal of $N_2$, the dotted line is the response of $N_2$ of the turbofan engine controlled by the devised TDOF $\mu$ synthesis controller. Figure 8 (b) is the histories of the rotation speed difference $\Delta N_2$ between the response of $N_2$ and the command signal of $N_2$. From Figure 8(a), it can be seen that the high spool speed also can track the command signal with slow actuator dynamics of the turbofan engine. From Figure 8 (b), we can see that the dynamic error is less than 3.5%, the steady-state error is less than 0.01%, and there is no overshoot, which demonstrates that the devised TDOF $\mu$ synthesis controller also achieves good servo performance on high spool speed control at sea level condition.

Figure 9 is the fuel flow during the engine regulation process at sea level condition. In Figure 9, the black line denotes the fuel flow command to the actuator and the dotted line denotes the real fuel flow to the engine. As we can see in Figure 9, the slow actuator dynamics has negative influence on the fuel flow command during the transient process. Figure 10 is the regulating process of throat area at sea level condition. In Figure 10, the black line is the throat area command to the actuator and the red line is the real throat area. The same negative influence of the slow actuator dynamics can be seen in the throat area command.
5.2.2. Robust Performance Verification with the Supremum Values of Uncertainty Parameters. The robust performance of the devised TDOF $\mu$ synthesis controller is verified with the supremum values of the uncertainty parameters. We set the actuators gain coefficient $K_{a1} = K_{a2} = 1.15$ and time constant $T_{a1} = T_{a2} = 0.625$ to conduct the simulation. Figure 11 and 12 are the simulation results of the robustness verification. From Figure 11 and 12, we can see that the control effects of low and high spool speed are consistent with those in Figure 7 and 8, which demonstrates that the devised TDOF $\mu$ synthesis controller achieves good robust performance both on low and high spool speed control of the turbofan engine.
5.2.3. Robust Performance Verification at $H = 10$ km and $Ma = 1$. To further verify the robustness of the devised TDOF $\mu$ synthesis controller, the same controller, simulation platform, and engine process (full thrust state $\rightarrow$ idle $\rightarrow$ full thrust state) at high-altitude condition ($H = 10$ km and $Ma = 1$) is used to conduct the simulation. Figure 13 and 14 illustrate the simulation results of the robust performance verification at the high-altitude condition. From Figure 13 and Figure 14, it can be seen that the
control effects of low and high spool speed are consistent with those in Figure 7 and 8, which demonstrates that the devised TDOF μ synthesis controller has good robust performance both on low and high spool speed control of the turbofan engine at high-altitude condition.

Figure 13. Simulation results of $N_1$ at the high-altitude condition.
Figure 14. Simulation results of $N_2$ at the high-altitude condition.

5.2.4. Comparison Verification. To verify the superiority of the devised TDOF $\mu$ synthesis controller, the control results of the devised TDOF $\mu$ synthesis controller are compared with those of a PI controller and a LMI optimization gain scheduled controller[5, 21-23].

Figure 15. Comparison results of $N_1$.

Figure 15 is the comparison results of $N_1$. Figure 15 (a) is the histories of $N_1$. In Figure 15 (a), the black line is the command signal of $N_1$, the black dotted line is the response of $N_1$ controlled by the devised TDOF $\mu$ synthesis controller, the red dotted line is the response of $N_1$ controlled by the LMI
optimization gain scheduled controller, and the blue dotted line is the response of $N_1$ controlled by the PI controller. Figure 15 (b) is the histories of the rotation speed difference $\Delta N_1$ between the response of $N_1$ and the command signal of $N_1$. In Figure 15 (b), the black dotted line is the $\Delta N_1$ of the devised TDOF $\mu$ synthesis controller, the red dotted line is the $\Delta N_1$ of the LMI optimization gain scheduled controller, and the blue dotted line is the $\Delta N_1$ of the PI controller. From Figure 15, we can see that the devised TDOF $\mu$ synthesis controller provides better control performance on low spool speed control than both the PI controller and the LMI optimization gain scheduled controller do: the devised TDOF $\mu$ synthesis controller has smaller overshoot and shorter settling time compared with the other two controllers.

Figure 16 is the comparison results of $N_2$. Figure 16 (a) is the histories of $N_2$. In Figure 16 (a), the black line is the command signal of $N_2$, the black dotted line is the response of $N_2$ controlled by the designed $\mu$ synthesis controller, the red dotted line is the response of $N_2$ controlled by the LMI optimization gain scheduled controller, and the blue dotted line is the response of $N_2$ controlled by the PI controller. Figure 16 (b) is the histories of the rotation speed difference $\Delta N_2$ between the response of $N_2$ and the command signal of $N_2$. In Figure 16 (b), the black dotted line is the $\Delta N_2$ of the devised TDOF $\mu$ synthesis controller, the red dotted line is the $\Delta N_2$ of the LMI optimization gain scheduled controller, and the blue dotted line is the $\Delta N_2$ of the PI controller. From Figure 16, we can see that the devised TDOF $\mu$ synthesis controller also provides better control performance on high spool speed control than both the PI controller and the LMI optimization gain scheduled controller do.

Based on the above analyses, the devised TDOF $\mu$ synthesis controller achieves good robust performance of the turbofan engine with slow actuator dynamics and parameter uncertainties at both
ground and high altitude condition, and provides better performance than the PI controller and the LMI optimization gain scheduled controller do.

6. Conclusion
This paper presented a TDOF $\mu$ synthesis control scheme to solve the robust control problem of turbofan engines with slow actuator dynamics and parameter uncertainties. The effectiveness of the devised TDOF $\mu$ synthesis controller was verified through a series of turbofan engine regulation missions. The simulation analyses showed that the devised TDOF $\mu$ synthesis controller achieved a good robust performance at both ground and high-altitude conditions. For low spool speed control, the dynamic error is smaller than 4%, the steady-state error is less than 0.01%, and the maximum overshoot is less than 0.1% at both ground and high-altitude conditions. For high spool speed control, the dynamic error is less than 3.5%, the steady-state error is less than 0.01%, and there is no overshoot at both ground and high-altitude conditions. The robustness of the devised TDOF $\mu$ synthesis controller was verified with the supremum values of the uncertainty parameters and the results showed that the control effect was consistent with the simulation results in the nominal condition. Additionally, through comparing with the PI controller and the LMI optimization gain scheduled controller, the devised TDOF $\mu$ synthesis controller provided better control performance than the other two controllers did. The presented TDOF $\mu$ synthesis control scheme provides a solution to achieve robust control of turbofan engines with slow actuator dynamics and parameter uncertainties.

Acknowledgments
This research is supported by the Academic Excellence Foundation of BUAA for PhD Students, China Scholarship Council (CSC), and National Science and Technology Major Project, China (2017-V-0015-0067).

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