Physical States in the $W_3$ String

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ABSTRACT

We present a review of some of the recent developments in the study of the $W_3$ string. One of the interesting features of the theory is that the physical spectrum includes states with non-standard ghost structure, such as excitations of the ghost fields, both for discrete-momentum and continuous-momentum states.

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1. Introduction

The ordinary bosonic string is described by the two-dimensional theory of matter conformally coupled to gravity. Quantisation of the two-dimensional metric tensor requires the fixing of gauge symmetries, and the introduction of associated Fadeev-Popov ghosts. The local holomorphic and anti-holomorphic Virasoro symmetries that remain after fixing the conformal gauge are in general broken by anomalies at the quantum level. The anomaly manifests itself as a central extension of the Virasoro algebra, with the ghosts giving a contribution of $-26$ to the central charge. This can be cancelled by choosing a conformal matter system that gives an equal and opposite central charge. The simplest choice is to take 26 free scalars $X^\mu$, with $T = -\frac{1}{2} \partial X_\mu \partial X^\mu$, giving rise to the usual critical 26-dimensional string. Other choices are also possible. In particular, one can realise the Virasoro algebra with 2 scalars $\varphi$ and $X$, one of which, say $\varphi$, has a background charge; $T = -\frac{1}{2} (\partial X)^2 - \frac{1}{2} (\partial \varphi)^2 - Q \partial^2 \varphi$, with $Q^2 = 2$. The resulting theory, commonly called the non-critical string, is still a critical theory in the sense that the matter system has central charge $+26$, as it must in order to cancel the anomaly. The $\varphi$ field in the two-dimensional theory is usually interpreted as a Liouville field. The two-dimensional string theory has received much attention recently. One of the fascinating features of the theory is that its spectrum of physical states is much richer and more subtle than one might at first have supposed.

The physical states in string theory can most elegantly be described by using the BRST formalism. The BRST operator $Q_B$ is given by

$$Q_B = \oint dz c(z) \left( T(z) + \frac{1}{2} T^{gh}(z) \right),$$

where we focus for now just on the holomorphic sector. Here $T(z)$ is the energy-momentum tensor for the matter, with central charge 26, and $T^{gh}$ is the ghost energy-momentum tensor,

$$T^{gh} = -2b \partial c - \partial b c,$$

where $c$ and $b$ are the ghost and antighost fields arising from the gauge fixing for the two-dimensional metric. The BRST operator is nilpotent, $Q_B^2 = 0$, by virtue of the fact that $T$ has central charge 26.

Physical states are defined to be states in the cohomology of $Q_B$. That is to say, a state $|\chi\rangle$ is physical if it satisfies the physical-state condition

$$Q_B |\chi\rangle = 0,$$

and if in addition it is non-trivial, in other words

$$|\chi\rangle \neq Q_B |\psi\rangle$$

(4)
for any state $|\psi\rangle$. In the case of the 26-dimensional string, the physical states have the "standard" form
\[ |\chi\rangle = |\text{phys}\rangle \otimes |\rangle, \tag{5} \]
where $|\text{phys}\rangle$ involves operators built only from the matter fields, and $|\rangle$ is the ghost vacuum
\[ |\rangle \equiv c_1|0\rangle, \tag{6} \]
on obtained from the $SL(2,C)$-invariant vacuum $|0\rangle$ which satisfies
\[ c_n|0\rangle = 0, \quad n \geq 2, \]
\[ b_n|0\rangle = 0, \quad n \geq -1. \tag{7} \]
For states of the standard form (5), it follows from (1) that the physical-state condition (3) becomes
\[ (L_0 - 1)|\text{phys}\rangle = 0, \]
\[ L_n|\text{phys}\rangle = 0, \quad n \geq 1, \tag{8} \]
where $L_n$ are the Laurent modes of $T(z)$. These are the familiar physical-state conditions for the 26-dimensional bosonic string.

To give a full description of the physical states of the string, we should also consider states of the form
\[ |\chi\rangle = |\text{phys}\rangle \otimes |+\rangle, \tag{9} \]
where $|+\rangle \equiv c_0|\rangle$. The reason for this is that a state (5) actually has zero inner product with itself. To see this, note that $c_0|+\rangle = 0$ since the modes $c_n$ anticommute, and similarly $|\rangle = b_0|+\rangle$ so $b_0|\rangle = 0$. Thus we have $\langle -|\rangle = \langle +|b_0|\rangle$, which therefore vanishes. In fact the fundamental non-zero inner product in the ghost sector is $\langle +|\rangle = \langle -|+\rangle = 1$. Thus we must consider states of both forms (5) and (9) in order to get non-vanishing inner products. If we assign ghost number $G = 0$ to $|\rangle$, so $|0\rangle$ has $G = -1$, then $|+\rangle$ has $G = 1$. In general, a state with ghost number $G$ will have non-zero inner product with some state at ghost number $1 - G$, which we may think of as its "conjugate" state. The inner product on the Hilbert space of states will then be off-diagonal, leading to an indefinite-signature metric. To circumvent this we then truncate out half of the states, by identifying states and their conjugates. For the physical states of the 26-dimensional string, this means identifying states of the forms (5) and (9). One may think of the standard physical states of the 26-dimensional string as lying at ghost number $G = 0$, with their conjugates at ghost number $G = 1$.

In the two-dimensional string, things are rather different. Naively, one might expect that there should be no physical states (i.e. having non-zero norm) at all, apart from the tachyons. This is because all the excited states of the $d$-dimensional string describe gauge fields, with gauge invariances that imply that the physical degrees of freedom are confined to
the \((d-2)\)-dimensional transverse space. Thus if \(d = 2\), one might think that nothing would survive in the excited spectrum. As is now well known, this naive expectation is incorrect. It turns out that the analysis in two dimensions has to be handled more carefully, and whilst it is true for generic values of the on-shell momenta that there are no positive-norm excited states, there are special cases that arise at specific values of the on-shell momenta [1]. These are known as discrete states. The first examples were of the “standard” form (5). Subsequently, it was found that there are more general kinds of discrete state in the two-dimensional string, which do not have the standard ghost structure. Rather than having the form (5), they instead involve excitations of the ghost and antighost operators as well as the matter operators [2]. By considering all possible states that one could build in this way, it is easy to see that at a given level number \(\ell\), the range of ghost numbers \(G\) that could conceivably occur is 
\[
\left[ \frac{1}{2} - \frac{1}{2}\sqrt{8\ell + 1} \right] \leq G \leq \left[ \frac{1}{2} + \frac{1}{2}\sqrt{8\ell + 1} \right],
\]
where \([x]\) denotes the integer part of \(x\). In fact it turns out that the physical states of the two-dimensional string occupy a band within this allowed range, namely from \(G = -1\) to \(G = 2\). (This is wider than the band \(G = 0\) to \(G = 1\) for the continuous states of the 26-dimensional string.) The states themselves lie in the range \(-1 \leq G \leq 1\), and their conjugates lie in the range \(0 \leq G \leq 2\).

Because of the background charge \(Q\) in the \(\varphi\) direction, which can be viewed as an insertion of momentum \(-2iQ\) in the \(\varphi\) direction at infinity, momentum conservation in correlation functions is modified. We may focus just on the \(\varphi\) direction to illustrate the point. The momentum operator \(\alpha_0\) is not hermitean, but instead satisfies \(\alpha_0^\dagger = \alpha_0 - 2iQ\). Thus in the two-point function we have \(\langle p'|\alpha_0|p\rangle = p\langle p'|p\rangle = (p^* + 2iQ)\langle p'|p\rangle\), so \(\langle p'|p\rangle\) vanishes unless \(p = p^* + 2iQ\). This leads to a situation that is analogous to that described above for the ghost vacua, in that one now has an off-diagonal inner product that leads to an indefinite-signature metric. Again the remedy is to halve the dimension of the Hilbert space, by identifying states and their conjugates. If we have a physical state with ghost number \(G\) and momentum \(\vec{p}\), then its conjugate, with which it has a non-zero inner product, has ghost number \(1 - G\) and momentum \(\vec{p}^* + (2iQ, 0)\).

The simplest example of a non-standard discrete physical state is the \(SL(2, C)\) vacuum \(|0\rangle\). This may be written as \(b_{-1}|\rangle - \rangle\), since \(|\rangle - \rangle = c_1|0\rangle\); it is manifestly invariant under \(Q_B\). Thus we see that \(|0\rangle\) is a level \(\ell = 1\) physical state, with ghost number \(G = -1\), and momentum \(\vec{p} = 0\). Its conjugate is the state \(\langle \vec{p}, 0|c_{-1}c_0c_1\rangle\), where \(\vec{p} = (2iQ, 0)\). In fact the \(SL(2, C)\) vacuum \(|0\rangle\) is a physical state even in the 26-dimensional string; it appears to be the only discrete state in that case.

Discrete states occur at all higher levels in the two-dimensional string. Especially interesting are the examples at level \(\ell = 2\) and \(G = -1\). They can be described in terms of
operators $x$ and $y$ acting on the $SL(2,C)$ vacuum, with \[2\]

$$
x = \left( bc + \frac{1}{2}Q(\partial \varphi + i\partial X) \right)e^{\frac{1}{2}Q(\varphi - X)},$$

$$
y = \left( bc + \frac{1}{2}Q(\partial \varphi - i\partial X) \right)e^{\frac{1}{2}Q(\varphi + X)},$$

with $Q^2 = 2$. These operators $x$ and $y$ have conformal spin 0, and ghost number $G = 0$. By taking products of the form $x^m y^n$, for $m$ and $n$ integer, one can obtain operators that generate higher-level $G = -1$ states \[2\]. In fact, it can be shown that all higher-level $G = -1$ states are obtained in this way \[2,3,4\].

Finally, to fill out the complete spectrum of discrete states in the two-dimensional string, we note that the operators $a_\varphi \equiv [Q_B, \varphi] = c\partial \varphi - Q\partial c$ and $a_X \equiv [Q_B, X] = c\partial X$, being BRST commutators, map solutions of the physical-state condition (3) into new solutions \[5\]. One might think that these new solutions would be trivial, in the sense that they could be written as $Q_B$ acting on some state. This is, however, not the case, since $\varphi$ and $X$ are not themselves primary fields. The situation is closely analogous to writing the field strength of a Dirac monopole as the exterior derivative of a potential that has a string singularity. Since $a_\varphi$ and $a_X$ each have ghost charge 1, it follows that by starting with a suitable discrete state $|G\rangle$ at ghost number $G$, we can create two new discrete states $a_\varphi |G\rangle$ and $a_X |G\rangle$ at ghost number $G + 1$, and one new discrete state $a_\varphi a_X |G\rangle$ at ghost number $G + 2$. One can only apply each of $a_\varphi$ and $a_X$ once, since a repeated application of the same operator will give a BRST-trivial state. The situation is reminiscent of an $N = 2$ supermultiplet.

The upshot of the above considerations is that one obtains all $G = -1$ discrete states of the two-dimensional string by taking arbitrary integer powers $x^m y^n$ of the “ground-ring” generators $x$ and $y$ acting on the $SL(2,C)$ vacuum. For each such state, one can then fill out a quartet of states, with two at $G = 0$ obtained by acting with $a_\varphi$ or $a_X$, and one at $G = 1$ obtained by acting with both $a_\varphi$ and $a_X$. There will then be a conjugate quartet too, with ghost numbers $0 \leq G \leq 2$, and momentum conjugate to that of the original quartet.

2. The $W_3$ String

The essential ingredient for the construction of the bosonic string was the existence of a nilpotent BRST operator for the Virasoro algebra. This has a Lagrangian interpretation as the existence of an anomaly-free worldsheet theory of conformal matter coupled to two-dimensional gravity. We can now apply similar considerations to the $W_3$ extended conformal algebra. This is generated by two currents; the spin-2 energy-momentum tensor $T(z)$, and
a spin-3 primary current \(W(z)\). The algebra takes the form \[6\]

\[
\begin{align*}
T(z)T(w) & \sim \frac{\partial T(w)}{z-w} + \frac{2T}{(z-w)^2} + \frac{c/2}{(z-w)^4}, \\
T(z)W(w) & \sim \frac{\partial W}{z-w} + \frac{3W}{(z-w)^2}, \\
W(z)W(w) & \sim \frac{16}{22 + 5c} \left( \frac{\partial \Lambda}{z-w} + \frac{2\Lambda}{(z-w)^2} \right) + \frac{c/3}{(z-w)^6} \\
& \quad + \frac{1}{15} \left( \frac{\partial^3 T}{z-w} + \frac{9}{2} \frac{\partial^2 T}{(z-w)^2} + 15 \frac{\partial T}{(z-w)^3} \right) + 30 \frac{T}{(z-w)^4},
\end{align*}
\]

(11)

where \(\Lambda = (TT) - \frac{3}{10} \partial^2 T\), with the normal ordering of the product of operators defined by \((AB)(z) = \frac{1}{2\pi i} \oint dz A(z)B(w)/(z-w)\).

The BRST operator has the form

\[
Q_B = \oint dz \left( c(T + \frac{1}{2}T_{gh}) + \gamma(W + \frac{1}{2}W_{gh}) \right),
\]

(12)

and is nilpotent provided that the matter currents \(T\) and \(W\) satisfy (11) with central charge \(c = 100\), and the ghost currents are chosen to be

\[
\begin{align*}
T_{gh} & = -2b \partial c - b \partial c - 3\beta \partial \gamma - 2\partial \beta \gamma, \\
W_{gh} & = -\partial \beta c - 3\beta \partial c - \frac{8}{261} \left[ \partial(b \gamma T) + b \partial \gamma T \right] \\
& \quad + \frac{25}{6 \cdot 261} \left( 2\gamma \partial^3 b + 9\partial \gamma \partial^2 b + 15\partial^2 \gamma \partial b + 10\partial^3 \gamma b \right),
\end{align*}
\]

(13)

(14)

where the ghost-antighost pairs \((c,b)\) and \((\gamma,\beta)\) correspond respectively to the \(T\) and \(W\) generators [7]. Thus we need a matter realisation of \(W_3\) with central charge 100. Such realisations can be given in terms of \(n \geq 2\) scalar fields, as follows [8]:

\[
\begin{align*}
T & = -\frac{1}{2}(\partial \varphi_1)^2 - Q \partial^2 \varphi_1 + \tilde{T}, \\
W & = -\frac{2i}{\sqrt{261}} \left( \frac{1}{3}(\partial \varphi)^3 + Q \partial \varphi_1 \partial^2 \varphi_1 + \frac{1}{3}Q^2 \partial^3 \varphi_1 + 2\partial \varphi_1 \tilde{T} + Q \partial \tilde{T} \right),
\end{align*}
\]

(15)

where \(Q^2 = \frac{49}{8}\) and \(\tilde{T}\) is an energy-momentum tensor with central charge \(\frac{51}{2}\) that commutes with \(\varphi_1\). Since \(\tilde{T}\) has a fractional central charge, it cannot be realised simply by taking free scalar fields. We can use \(d\) scalar fields \(X^\mu\), with a background-charge vector \(a_\mu\):

\[
\tilde{T} = -\frac{1}{2} \partial X_\mu \partial X^\mu - a_\mu \partial^2 X^\mu,
\]

(16)

with \(a_\mu\) chosen so that \(\frac{51}{2} = d + 12a_\mu a^\mu\) [9,10].
As in the case of the ordinary string, we can again look for physical states that satisfy the condition (3) along with the condition (4) of non-triviality. Let us again first consider states of the “standard” form analogous to (5):

$$|\chi\rangle = |\text{phys}\rangle \otimes |\cdot\cdot\cdot\rangle,$$

where the ghost vacuum is now given by

$$|\cdot\cdot\cdot\rangle = c_1 \gamma_1 \gamma_2 |0\rangle.$$

We assign ghost number $G = 0$ to $|\cdot\cdot\cdot\rangle$, so for the $W_3$ string the $SL(2, C)$ vacuum has $G = -3$. It satisfies (7), along with

$$\gamma_n |0\rangle = 0, \quad n \geq 3,$$
$$\beta_n |0\rangle = 0, \quad n \geq -2. \quad (19)$$

For states of the form (17), the condition of BRST invariance becomes

$$\begin{align*}
(L_0 - 4) |\text{phys}\rangle &= 0, \\
W_0 |\text{phys}\rangle &= 0, \\
L_n |\text{phys}\rangle &= W_n |\text{phys}\rangle = 0, \quad n \geq 1.
\end{align*} \quad (20)$$

The consequences of these physical-state conditions have been studied in some detail in various papers [10,11,12,13]. The main features that emerge are the following. The excited states can be divided into two kinds, namely those for which there are no excitations in the $\varphi_1$ direction, and those where $\varphi_1$ is excited too. For the first kind, we may write $|\text{phys}\rangle$ as

$$|\text{phys}\rangle = e^{i\beta \varphi_1(0)} |\text{phys}\rangle_{\text{eff}}.$$

The physical-state conditions (20) imply that

$$(\beta - iQ)(\beta - \frac{6}{7}iQ)(\beta - \frac{8}{7}iQ) = 0, \quad (22)$$

together with the effective physical-state conditions

$$\begin{align*}
(\tilde{L}_0 - \tilde{a}) |\text{phys}\rangle_{\text{eff}} &= 0, \\
\tilde{L}_n |\text{phys}\rangle_{\text{eff}} &= 0, \quad n \geq 1.
\end{align*} \quad (23)$$

where $|\text{phys}\rangle_{\text{eff}}$ involves only the $X^\mu$ fields and not $\varphi_1$. The value of the effective intercept $\tilde{a}$ is 1 when $\beta = \frac{6}{7}iQ$ or $\frac{8}{7}iQ$, and it equals $\frac{15}{16}$ when $\beta = iQ$. Thus these states of the $W_3$ string are described by two effective Virasoro-string spectra, for an effective energy-momentum tensor $\tilde{T}$ with central charge $c = \frac{51}{2}$ and intercepts $\tilde{a} = 1$ and $\tilde{a} = \frac{15}{16}$ [10,11,12,13]. The first
of these gives a mass spectrum similar to an ordinary string, with a massless vector at level 1, whilst the second gives a spectrum of purely massive states. Explicitly, we have

\[-p'^\mu(p_\mu - 2ia_\mu) = 2\ell - 2, \quad \tilde{a} = 1,\]
\[-p'^\mu(p_\mu - 2ia_\mu) = 2\ell - \frac{15}{8}, \quad \tilde{a} = \frac{15}{16},\]  

(24)

where \(\ell\) is the level number.

A crucial property of the states described above is that the values of allowed \(\varphi_1\) momentum, given by (22), occur in conjugate pairs. As discussed in the previous section, when a background charge is present the momentum-conservation law is modified. Thus for a two-point function to be non-zero, it must be that the \(\varphi_1\) momenta of the two states must satisfy \(\beta = \beta'^{*} + 2iQ\). This is indeed possible for the states (21), for which \(\beta = iQ\) is self-conjugate, whilst \(\beta = \frac{5}{4}iQ\) and \(\beta = \frac{8}{7}iQ\) are conjugates of each other. Of course we must also introduce the appropriate notion of conjugation with respect to ghosts as well, just as we did for the ordinary string. Here, we do this by defining

\[|++\rangle = c_0\gamma_0|--\rangle,\]

(25)

and noting that we therefore have also that \(|--\rangle = b_0\beta_0|--\rangle, c_0\gamma_0|--\rangle = 0\) and \(b_0\beta_0|--\rangle = 0\). Thus it follows that \(\langle --|--\rangle = \langle ++|--\rangle = 0\) and \(\langle ++|--\rangle = \langle --|--\rangle = 1\). The discussion proceeds in parallel to that for the ordinary string, and we see that here we must introduce the states built over the \(|++\rangle\) ghost vacuum as conjugates of the states (17). Again the Hilbert space has off-diagonal inner product, leading to an indefinite-signature metric, which we remedy by identifying states and conjugates so as to project onto the positive-definite subspace.

Now let us consider states that still have the “standard” structure (17), but where now there are excitations in the \(\varphi_1\) direction as well as the \(X^\mu\) directions. It is hard to give a full analysis in this case, but some partial results have been obtained at low-lying levels. In all cases that have been examined, it turns out that the \(\varphi_1\) momentum \(\beta\) is again frozen to specific values, but these values no longer occur in conjugate pairs [10,11,12,13]. For example, at level 1 one finds solutions to the physical-state conditions (20) with \(\beta = \frac{10}{7}iQ\) and \(\beta = \frac{11}{7}iQ\). Thus it is not possible to form any non-zero inner products between such states, and so they are evidently of zero norm. One would expect therefore to be able to write them as BRST-trivial states \(Q_B|\psi\rangle\), and indeed this can be done. This pattern seems to persist at all higher levels; details are discussed in [13].

This completes the discussion of “standard” states of the form (17) in the \(W_3\) string. Until recently, it was believed that these would constitute the full set of physical states in the theory, at least in the case of generic multi-scalar realisations when the number \(d\) of scalars \(X^\mu\) exceeds 1. We now know, however, that the story is a more complicated one. For the ordinary string, it was only in the case of a two-dimensional spacetime that it was necessary
to entertain the possibility of having “non-standard” physical states with ghost as well as matter excitations. For the $W_3$ string, however, it turns out that for any dimension for the spacetime, i.e. for any number $d \geq 1$ of $X^\mu$ fields, states with non-standard ghost structure will play a rôle.* Thus we consider general states built from creation operators for the $b$, $c$, $\beta$ and $\gamma$ ghosts as well as the matter fields, acting on tachyonic states of the form $|\bar{p}, -\rangle$. Since we have assigned ghost number $G = 0$ to the tachyonic state $|\bar{p}, -\rangle \equiv e^{i\beta \varphi_1 + ip_\mu X^\mu}|\bar{p}, -\rangle$, then it is easy to see by considering all possible excited states that at a given level number $\ell$ their ghost numbers can lie in the range $[1 - \sqrt{4\ell + 1}] \leq G \leq \left[1 + \sqrt{4\ell + 1}\right]$ [15]. As for the case of the ordinary string, it looks as if the physical-state condition (3), together with the condition of non-triviality (4), singles out states in a restricted band of ghost numbers within the limits given above. The evidence accumulated so far suggests that this band for non-trivial physical states may lie in the range $-3 \leq G \leq 5$ (i.e. $-3 \leq G \leq 1$ for the states themselves, and $1 \leq G \leq 5$ for their conjugates).

Preliminary investigations of the new kinds of physical states for the $W_3$ string, with “non-standard” ghost structure, were carried out in [15]. The main thrust of this work was a study of physical discrete states in the two-scalar $W_3$ string (i.e. the case when there is just one $X^\mu$ coordinate in addition to the field $\varphi_1$). These could be viewed as the direct analogues of the discrete states of the two-scalar ordinary string. We shall describe them in more detail later. However, it also emerged from this work that there are new physical states with non-standard ghost structure in the multi-scalar $W_3$ string too; i.e. when there is an arbitrary number $d \geq 1$ of additional coordinates $X^\mu$. This is a rather surprising result, which indicates that the $W_3$ string theory is very different, and in some sense richer, than the ordinary string. It is not only for discrete states, but also for true physical states with continuous on-shell momenta in the multi-scalar $W_3$ string, that states with non-standard ghost structure appear. We shall illustrate this with an example that was found in [15], namely states at level $\ell = 1$ and $G = -1$.

From the general formula given above, one sees that the allowed range of ghost numbers at level $\ell = 1$ is $-1 \leq G \leq 2$. Let us consider the most negative ghost number, namely $G = -1$. Since $Q_B$ has ghost number $G = 1$, it follows that any $G = -1$ state at $\ell = 1$ that is annihilated by $Q_B$ must be BRST non-trivial, since no state $|\psi\rangle$ at $G = -2$ can exist to allow the existence of a $G = -1$ trivial state $Q_B|\psi\rangle$. Just two structures are possible for states at $G = -1$, corresponding to the operator

$$ \left( c \gamma + \alpha \gamma \partial \gamma \right) e^{i\beta \varphi_1 + ip_\mu X^\mu}, $$

where $\alpha$ is a constant. Requiring that this be annihilated by the BRST operator leads to the following results. There are two discrete states, with momenta

$$ (i\beta, p_\mu) = (-\frac{6}{7}Q, 0), \quad (-\frac{8}{7}Q, 0), $$

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* An example of a discrete state with non-standard ghost structure in the special case of the two-scalar $W_3$ string was first exhibited in [14].
and two continuous-momentum states, with momenta

$$(i\beta, p_\mu) = \left( -\frac{3}{7}Q, p_\mu \right), \quad \left( -\frac{4}{7}Q, p_\mu \right),$$  \hspace{1cm} (28)

where for each state the “spacetime” momentum $p_\mu$ satisfies

$$-p^\mu (p_\mu - 2ia_\mu) = -1.$$ \hspace{1cm} (29)

The constant $\alpha$ in (26) is equal to $i/\sqrt{522}$ for the continuous state with $\beta = \frac{4}{7}iQ$ and the discrete state with $\beta = \frac{6}{7}iQ$. In the other two cases, $\alpha = -i/\sqrt{522}$.

These level $\ell = 1$ states at $G = -1$ provide the first examples of physical states with non-standard ghost structures in the multi-scalar $W_3$ string. Just as in the case of the two-dimensional ordinary string, we may now build “multiplets” of further states, at higher ghost numbers, by acting with the operators $a_\varphi \equiv [Q_B, \varphi_1]$ and $a_{X\mu} \equiv [Q_B, X^\mu]$. Their form is more complicated than for the ordinary string; details may be found in [15]. By this means we may obtain from the $G = -1$ level-1 states discussed above further $\ell = 1$ states at higher ghost numbers including $G = 0$. The $G = -1$ states (26) have excitations exclusively in the ghost operators. The “multiplet” states that we build by acting with $a_\varphi$ and $a_{X\mu}$ will have matter excitations as well as ghost excitations. These will include excitations in the $\varphi_1$ direction as well as $X^\mu$. Thus the previous conclusion that all states with excitations in the $\varphi_1$ direction have zero norm is seen to be true only for states with the “standard” ghost structure (17). The states that we have just been describing, which were constructed in [15], all of course have non-zero norm, since they are BRST non-trivial.

The BRST-non-trivial physical states that we have been describing are just the first examples of a complex pattern of states. Work on analysing this in more detail is in progress [16].

3. Discrete states in the two-scalar $W_3$ string

The physical states that we have been considering so far have been for multi-scalar $W_3$ strings. In view of the experience with ordinary string theory, where the two-scalar string has its own interest and surprises, it is natural to wonder what might happen in the analogous situation for the $W_3$ string. As discussed in [15], it is not totally clear what the analogue of the two-scalar string should be in the $W_3$ case. Since the physical-state conditions always lead to the “freezing” of the momentum in the $\varphi_1$ direction, one might think that a total of three scalars, namely $\varphi_1$ plus two more $X^\mu$ coordinates, would be the natural analogue. However, the physical states of the $W_3$ string have fewer gauge symmetries than those of the ordinary string. For example, we saw in the last section that there are excited states described by effective Virasoro strings as in (23), with effective intercept $\tilde{a} = \frac{15}{16}$. One can see from the fact that the level-1 vector is massive that it will have non-zero norm for arbitrary
on-shell momenta in the three-scalar $W_3$ string. Thus there will be continuous-momentum excited states as well as discrete states in a three-scalar $W_3$ string. In order to have a theory with only discrete-momentum states, it is necessary to consider the two-scalar $W_3$ string. Thus we have coordinates $\phi_1$ and $X = \phi_2$, with the background-charge vector $a^\mu$ in (16) becoming just the one-component quantity $a$, such that $a^2 = \frac{49}{24}$.

In section 1, we saw that the $SL(2,C)$ vacuum in the two-scalar ordinary string is a discrete physical state at level $\ell = 1$ and ghost number $G = -1$. In the two-scalar $W_3$ string it is also clear that $|0\rangle$ is a physical state, in that it satisfies $Q_B|0\rangle = 0$ and it cannot be written as $Q_B|\psi\rangle$ for any $|\psi\rangle$. In this case, it is a level $\ell = 4$ state with ghost number $G = -3$, since it can be written as

$$|0\rangle = \beta_{-2}\beta_{-1}b_{-1}|--\rangle.$$  

In the two-scalar string, the $x$ and $y$ operators (10) that give rise to $G = -1$ states at level $\ell = 2$ played a crucial rôle, since they could be used to construct all higher-level $G = -1$ states by acting with $x^m y^n$ on the $SL(2,C)$-invariant vacuum. In [15], a search was undertaken for analogous operators in the two-scalar $W_3$ string. It was found that two such operators exist that correspond to $G = -3$ states at level $\ell = 6$. They have the form

$$x = R_x e^{\frac{2}{7}Q\phi_1},$$
$$y = R_y e^{(\frac{1}{7}Q\phi_1 + \frac{3}{7}a\phi_2)},$$

where the prefactors $R_x$ and $R_y$ involve the 30 different possible excitation structures for this level and ghost number; $R \sim g_1 b \partial\gamma \partial\gamma + \cdots + g_{30} b\gamma(\partial\phi_2)^2$. Demanding BRST invariance fixes the momenta to be those given in (31), and determines all but two of the constant coefficients $g_i$ in each case. Only one of these two parameters represents a BRST-non-trivial state; the other is a reflection of the fact that there exists a unique state at the minimum allowed ghost number $G = -4$ at level 6, and so $Q_B$ of this state gives a trivial BRST-invariant state at $G = -3$.

Thus we find the two non-trivial operators $x$ and $y$ at level 6 [15]. Like the analogous operators in the two-scalar string, they have spin 0 and ghost number $G = 0$. One can thus build new such operators by taking products of powers of $x$ and $y$. In the string case, one can take arbitrary integer powers; this is because the $(z - w)$ pole that one gets from normal ordering $x$ with $x$, $y$ with $y$, or $x$ with $y$ is always of integer degree. In the $W_3$ case, however, this is no longer true. One can see from (31), and the definitions of the background charges ($Q^2 = \frac{49}{8}$, $a^2 = \frac{49}{24}$), that the normal ordering of the exponentials will give factors $(z - w)^{-1/2}$ for $xx$ and $yy$, and a factor of $(z - w)^{-1/4}$ for $xy$. Thus only certain powers of these operators will be well defined. It turns out that the full set of allowed powers is given by [15]

$$x^{4p}y^{4q}\{1, x, y, xy^2, x^2y, x^2y^2\},$$

\[32\]
where \( p \) and \( q \) are arbitrary non-negative integers. The situation here is similar to that for the ordinary string in one dimension rather than two, where again only certain powers of the single \( x \)-type operator that exists in that case can arise [15].

As discussed in [15], there are indications that further spin 0, \( G = 0 \) operators should exist at higher levels, which cannot be built from powers of \( x \) and \( y \). It seems likely that four such operators at level \( \ell = 8 \), with momenta \((i\beta, ip) = (\frac{4}{7}Q, -\frac{2}{7}a), (\frac{3}{7}Q, -\frac{12}{7}a), (\frac{1}{7}Q, a)\) and \((-\frac{4}{7}Q, \frac{12}{7}a)\), should be sufficient, together with \( x \) and \( y \) above, to generate all spin-0, \( G = 0 \) physical operators in the two-scalar \( W_3 \) string. Full details of this will appear elsewhere [17].

4. Conclusions

We have seen that in the \( W_3 \) string not only are there physical states with the “standard” ghost structure (17), but also there are further physical states involving ghost excitations as well as matter excitations. The simplest examples, at level \( \ell = 1 \), were found in [15], and are given in (26)-(29). It is indeed fortunate that such physical states occur, since otherwise, in the standard sector (17), only states of the form (21) with no excitations in the \( \varphi_1 \) direction arise. Since it is the \( \varphi_1 \) coordinate that is the truly special and distinctive field of the \( W_3 \) realisations (it is the only one that does not enter exclusively via its energy-momentum tensor), it would otherwise have been the case that the \( W_3 \) string spectrum amounted to little more than a set of two slightly unusual Virasoro string spectra. Instead, it turns out that the truly new aspects of the \( W_3 \) string spectrum occur in the non-standard ghost sectors.

Perhaps the most intriguing aspect of the \( W_3 \) spectrum is that the physical states with non-standard ghost structure are important in the multi-scalar case as well as the two-scalar case. Thus by contrast with the ordinary string, the non-standard states play a rôle for the continuous-momentum physical states of a multi-scalar \( W_3 \) string as well as for the discrete states of a two-scalar \( W_3 \) string. The reason for this may be related to the smaller number of gauge symmetries of the states of the \( W_3 \) string. The ordinary string has gauge symmetries that restrict the degrees of freedom of the continuous states in the \( d \)-dimensional theory to the \( (d-2) \)-dimensional transverse space. Thus, aside from the discrete states, there exists a physical light-cone gauge. (In fact the breakdown of the light-cone gauge occurs precisely at the discrete momenta for which the denominator \( P^+ \) becomes zero.) The gauge symmetries of the \( W_3 \) string are fewer, as indicated for example by the presence of a massive vector in its spectrum, and so there is no notion of a physical light-cone gauge.

An interesting way of understanding the necessity of the new physical states with non-standard ghost structure in the \( W_3 \) string has been discussed recently in [18]. In this paper, the authors calculate the scattering amplitude for four tachyons in the \( \tilde{\alpha} = \frac{15}{16} \) branch of the “standard” states given by (17) and (21). This involves the introduction of the conformal-spin \( \frac{1}{16} \) primary field \( \sigma \) of the Ising model, in order to make up the deficit of the spin \( \frac{15}{16} \) of
the tachyon operators $e^{ip \cdot X}$ to give spin-1 vertex operators. In [18], the resulting scattering amplitude is then expanded in $s$-channel poles, revealing the existence of resonances with masses that are consistent with neither of the mass formulae in (24), which are associated with states of “standard” ghost structure. In fact, the calculation shows that there must be extra states in the $W_3$ string with masses of the form (28), together with higher-level excitations. Thus one sees by looking at the scattering of “standard” states that there must also be states of non-standard ghost structure in the physical spectrum.

Finally, we remark that recently a new kind of BRST operator for $W_3$ has been constructed, starting from two independent and mutually-commuting copies of the $W_3$ algebra [19]. This may be relevant to a description of non-critical $W_3$ strings. A field theory giving rise to this BRST operator has been constructed [20]. Many issues analogous to those discussed in this paper could be investigated for this case.

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Note Added

After this paper was completed a paper considering related issues for the new BRST operator of [19] appeared [21].
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