Remarks on defining the DLCQ of quantum field theory as a light-like limit

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ABSTRACT

The issue of defining discrete light-cone quantization (DLCQ) in field theory as a light-like limit is investigated. This amounts to studying quantum field theory compactified on a space-like circle of vanishing radius in an appropriate kinematical setting. While this limit is unproblematic at the tree-level, it is non-trivial for loop amplitudes. In one-loop amplitudes, when the propagators are written using standard Feynman $\alpha$-parameters we show that, generically, in the limit of vanishing radius, one of the $\alpha$-integrals is replaced by a discrete sum and the (UV renormalized) one-loop amplitude has a finite light-like limit. This is analogous to what happens in string theory. There are however exceptions and the limit may diverge in certain theories or at higher loop order. We give a rather detailed analysis of the problems one might encounter. We show that quantum electrodynamics at one loop has a well-defined light-like limit.

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1. Introduction

Light-cone quantization is based on the idea that one might use the light-cone coordinate $x^+ \sim x^0 + x^1$ as “time” and the corresponding $P_+ \sim P^-$ as hamiltonian. Discrete light cone quantization, or DLCQ for short, in addition takes $x^- \sim x^0 - x^1$ to be compact, i.e. to take values on a circle of radius $R_0$: $x^- \simeq x^- + 2\pi R_0$. One should note that the value of $R_0$ has no invariant meaning since the proper length of this circle is zero. The value of $R_0$ can be changed at will by a Lorentz transformation.

One may view this setup as resulting from a standard quantization with an ordinary time $t \equiv x^0$ and compactified space-like coordinate $x^1$ through an infinite Lorentz boost. More precisely (see section 2) one starts with a space like circle of radius $R = \epsilon R_0$: $x^1 \simeq x^1 + 2\pi \epsilon R_0$ with small but finite $\epsilon$. Through a large Lorentz boost this is mapped to an almost light-like circle. In the $\epsilon \to 0$ limit, the Lorentz transformation becomes infinite and the circle truly light-like. It has been proposed [1] to use this procedure as a definition for the DLCQ: Carry out the quantization of a given theory compactified on a space-like circle of radius $\epsilon R_0$. If the theory is Lorentz invariant and if the $\epsilon \to 0$ limit exists, the latter provides a clear definition of the DLCQ of the same theory.

In particular this procedure should also provide a straightforward way to transpose the standard renormalisation into the DLCQ. This would be an advantage with respect to certain current DLCQ treatments of QCD where renormalisation often looks a bit ad hoc.

Unfortunately, at present, it is not clear for which quantum theories this light-like ($\epsilon \to 0$) limit exists. Recent interest in this question arose in the context of the DLCQ of M-theory [2,3,4]. However, it was pointed out in [1] that already in the simple $\phi^4$-theory the 4-point one-loop amplitude diverges, as $\epsilon \to 0$, if no momentum in the compact dimension is transferred across the loop. These authors [1] advocated that this problem is generic, except in certain susy field theories.

Lateron it was shown that this problem does not occur in type II superstring theory and that there the $\epsilon \to 0$ limit exists, at least at one loop [5]. It was argued that the same should also hold at higher loops and even non-pertubatively, and hence probably also in M-theory [6]. It appeared from these studies that the mechanism did not mainly rely on susy cancellations as might have been expected from [1] but was more stringy in nature.
One might wonder whether the field theory limit \((\alpha' \to 0)\) of the \(\epsilon \to 0\) limit of the string amplitude gives a finite result or whether some divergence appears. Since the resulting field theories are highly supersymmetric and hence rather non-generic, we have preferred to study general quantum field theories directly, trying to follow as much as possible the string computation. It is well known that a string one-loop amplitude can be rewritten as a quantum field theory one-loop amplitude but with an infinite number of particle species running around the loop. The string loop amplitude contains an integral over the moduli \(\nu_r\) of the punctures (localisations of the external propagators on the torus). The field theory analogue of the complex \(\nu_r\) are the real Feynman \(\alpha_r\)-parameters. In the string computation \([5]\) the \(\epsilon \to 0\) limit gives rise to a complex \(\delta\)-function eliminating the integration over one of the moduli \(\nu_r\), replacing it by a finite sum. The resulting amplitude is finite except for precisely those singularities required by unitarity.

We will show here that similarly in quantum field theory the \(\epsilon \to 0\) limit gives rise to a now real \(\delta\)-function. Generically, this \(\delta\)-function eliminates one of the integrations over the Feynman parameters \(\alpha_r\), replacing it by a finite sum, the resulting expression having again only the singularities required by unitarity (after renormalisation).

This is the generic situation. If however no external momentum in the compact direction flows through the loop, then the argument of the \(\delta\)-function can be vanishing, the \(\delta(0)\) signalling a divergent \(\epsilon \to 0\) limit. This is precisely the situation that was encountered in \([1]\) for the \(\phi^4\)-theory. In \(\phi^3\)-theory this may not happen at one loop. Indeed, since the external lines must have non-vanishing momenta in the compact direction,\(^*\) for any theory with cubic vertices only these momenta always flow through the loop. In particular, we show that the light-like limit exists for QED at one loop. We also discuss some subtleties related to renormalisation.

At higher \(L\)-loop order the situation is less clear. Again, the generic situation is straightforward: there is one real \(\delta\)-function for each loop, eliminating \(L\) integrals over \(\alpha_r\)-parameters. This is quite encouraging for the string case of \([5]\) where the higher loop case is technically more involved. But we also encounter non-generic situations (even in theories with cubic vertices only) where a loop subgraph inside a bigger loop has vanishing compact momenta on its external legs (which are just internal lines of the bigger loop). In this case the \(\epsilon \to 0\) limit

\(^*\) As will be explained in section 2, this is necessary in order to correspond to finite energy states in the DLCQ.
diverges. Although this is likely to ruin the existence of the light-like limit beyond one-loop, there might be cancellations that save it. Clearly a more detailed analysis is needed beyond one loop to provide a clear answer, whether and for which theories the DLCQ may be defined as a light-like limit to all orders in perturbation theory.

In section 2, we begin by setting up the relevant kinematic framework. In section 3 we study the scalar $\phi^3$-theory in quite some detail, while in section 4 we extend these results to quantum electrodynamics. Section 5 contains some discussion.

2. Kinematics

We will start by considering a particle of mass $m$ described in a coordinate system $(x^0, x^1, x^i)$ with $i = 2, \ldots d-1$, in a $d$-dimensional space-time with signature $(+\ldots-)$. $x^0$ is time and the spatial coordinate $x^1$ takes values on a circle of radius $R = \epsilon R_0$. All other spatial coordinates $x^i$, called transverse, as well as the time $x^0$ are ordinary non-compact coordinates:

$$x^0 \simeq x^0, \quad x^1 \simeq x^1 + 2\pi \epsilon R_0, \quad x^i \simeq x^i.$$  \hfill (2.1)

The momentum for an on-shell particle is

$$p^1 = -p_1 = \frac{n}{\epsilon R_0}, \quad p^i \text{ arbitrary }, \quad p^0 = [p_1^2 + p_i^2 + m^2]^{1/2} \geq \frac{|n|}{\epsilon R_0}. \hfill (2.2)$$

At this stage $n$ is any integer, positive, negative or zero.

Now perform a Lorentz boost with boost parameter $\beta = \frac{1-\epsilon^2/2}{1+\epsilon^2/2}$. If $\epsilon$ is small, this is a large boost. In the new coordinate system $\tilde{x}^\mu$ it is convenient to define $\tilde{x}^\pm = (\tilde{x}^0 \pm \tilde{x}^1)/\sqrt{2}$. Then simply $\tilde{x}^+ = \frac{1}{\epsilon}(x^0 + x^1)$, $\tilde{x}^- = \frac{1}{\epsilon}(x^1 - x^0)$ with periodicities

$$x^- \simeq x^- + 2\pi R_0, \quad x^+ \simeq x^+ + \epsilon^2 \pi R_0.$$  \hfill (2.3)

The corresponding transformed momenta are

$$\tilde{p}_+ = \frac{1}{\epsilon}(p_0 + p_1) = \frac{p_0}{\epsilon} - \frac{n}{\epsilon^2 R_0}, \quad \tilde{p}_- = \frac{\epsilon}{2}(p_1 - p_0) = -\frac{\epsilon}{2} p_0 - \frac{n}{2 R_0}. \hfill (2.4)$$

Remember that $p_0 \geq \frac{|n|}{\epsilon R_0}$ and hence $\epsilon p_0$ is $O(1)$ while $\frac{p_0}{\epsilon}$ is $O(1/\epsilon^2)$, so that $\tilde{p}_-$ is $O(1)$ and $\tilde{p}_+$ is a priori $O(1/\epsilon^2)$.  

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Equation (2.3) shows that the light-cone coordinate $\tilde{x}^-$ is periodic as we want, but also $\tilde{x}^+$ has a (small) periodicity which is unwanted. This can be eliminated by a further coordinate redefinition (not a Lorentz transformation). Let

$$t = \tilde{x}^+ - \frac{\epsilon^2}{2} \tilde{x}^- , \quad x^- = -\tilde{x}^- \implies t \approx t , \quad x^- \approx x^- - 2\pi R_0 . \quad (2.5)$$

Then $t$ is a non-periodic coordinate and $x^-$ has the desired periodicity. The metric is

$$ds^2 = 2dt dx^- - \epsilon^2(dx^-)^2 - (dx^i)^2 \quad (2.6)$$

so that the circle in the $x^-$ direction is not exactly light-like. It becomes truly light-like in the limit $\epsilon \to 0$. This is to be expected since the original circle (2.1) has invariant length $2\pi \epsilon R_0$ and a light-like circle must have zero invariant length. The $\epsilon \to 0$ limit gives the DLCQ setting. In the coordinates (2.5) the momenta are

$$p_t = \tilde{p}_+ = \frac{p_0}{\epsilon} - \frac{n}{\epsilon^2 R_0} , \quad p_- = -\tilde{p}_- - \frac{\epsilon^2}{2} \tilde{p}_+ = \frac{n}{R_0} . \quad (2.7)$$

Hence $p_- = \frac{n}{R_0}$ as expected in the DLCQ.

It is now easy to see that $n$ must be positive if $m \neq 0$ and non-negative if $m = 0$. Let first $n \neq 0$. Then, expanding the square-root in (2.2) for very small $\epsilon$: $p_0 = p^0 = \frac{|n|}{\epsilon R_0} + \frac{\epsilon R_0}{2|n|}(p_i^2 + m^2) + O(\epsilon^3)$ and hence

$$p_t = \frac{|n| - n}{\epsilon^2 R_0} + \frac{R_0}{2|n|}(p_i^2 + m^2) + O(\epsilon^2) . \quad (2.8)$$

Hence states with $n < 0$ have infinite DLCQ energy $p_t$ as $\epsilon \to 0$, while all state with $n > 0$ have finite $p_t$. Let now $n = 0$. Then $p_0 = (p_i^2 + m^2)^{1/2}$ and $p_t = \frac{p_0}{\epsilon}$. Hence the only state with $n = 0$ and finite DLCQ energy $p_t$ must have $m = p_i = 0$, i.e. is degenerate with the vacuum.

In the following, we will work in the coordinates (2.1), i.e. with a space-like circle of radius $R = \epsilon R_0$. In the $\epsilon \to 0$ limit this is Lorentz equivalent to the DLCQ with radius $R_0$. We have just seen that finite DLCQ energies for on-shell states require the restriction $n > 0$. Hence we will be interested in $N$-point amplitudes in quantum field theory with all external states having strictly positive momenta in the compact direction, $p^1_{(r)} = \frac{n_r}{\epsilon R_0}$ and study their $\epsilon \to 0$ limit.
Possible divergencies can only occur in loop diagrams since the tree amplitudes are entirely expressible in terms of scalar products of on-shell momenta \( p_r \cdot p_s \) and these are always finite as \( \epsilon \to 0 \) (provided \( n_r, n_s > 0 \)):

\[
p_r \cdot p_s = \frac{n_s}{2n_r}((p_r^i)^2 + m_r^2) + \frac{n_r}{2n_s}((p_s^i)^2 + m_s^2) - p_r^i p_s^i + \mathcal{O}(\epsilon^2) . \tag{2.9}
\]

### 3. Scalar quantum field theory : \( \phi^3 \)

The simplest quantum field theories to study are the scalar \( \phi^3 \) or \( \phi^4 \) theories. We already know [1] that the simplest one-loop diagram in \( \phi^4 \), namely the 4-point amplitude, diverges as \( \epsilon \to 0 \) if \( n_1 = n_2, \ n_3 = n_4 \). We will study instead \( \lambda \phi^3 \) theory which does not present this pathology. In the course of our investigation we will also better understand the origin of the problem of \( \phi^4 \). The same problem will occur for all \( \phi^k \) theories with \( k \geq 4 \). However, since many interesting theories, like QED, only have cubic vertices it is useful to study the \( \phi^3 \) theory in some detail.

#### 3.1. The 2-point one-loop amplitude

We will begin with a very detailed computation of the simplest one-loop diagram: the two-point function. This will exhibit the basic mechanism of the \( \epsilon \to 0 \) limit. If we call the external momentum \( P \) and the loop momentum \( k \) then the relevant self-energy diagram is

\[
i\Pi(P) = \frac{1}{2} \frac{\lambda^2 m^4 - d}{(2\pi)^d} \int \frac{d^d k}{(k^2 - m^2)((P - k)^2 - m^2)} \tag{3.1}
\]

where \( \frac{1}{2} \) is the symmetry factor. We work with dimensional regularisation. (The coupling constant \( \lambda \) has dimension of mass as appropriate in \( d = 4 \).) Upon compactifying \( x^1 \) on the circle of radius \( \epsilon R_0 \) we have to replace \( k^1 \to \frac{n}{\epsilon R_0} \) and \( \int d k^1 \to \frac{1}{\epsilon R_0} \sum_n \). We further do a Wick rotation to Euclidean signature and denote \( k_\perp \equiv (k^0, k^3) \). The external momentum is \( P^1 = \frac{N}{\epsilon R_0} \) and \( P_\perp \). Thus

\[
\Pi(P) = \frac{\lambda^2 m^4 - d}{4\pi \epsilon R_0} \sum_n \int \frac{d^{d-1} k_\perp}{(2\pi)^{d-1}} \frac{1}{\left[ \left( \frac{n}{\epsilon R_0} \right)^2 + k_\perp^2 + m^2 \right] \left[ \left( \frac{N-n}{\epsilon R_0} \right)^2 + (P_\perp - k_\perp)^2 + m^2 \right]} . \tag{3.2}
\]

Note that although the external \( N \) is positive, the \( n \) of the loop momentum has no a priori reason to be restricted to positive values only.
Now introduce an \( \alpha \) parameter for each propagator, using
\[
\frac{1}{A_a} = \int_0^\infty d\alpha_a e^{-\alpha_a A_a}
\]
and then change variables \( \alpha_1 + \alpha_2 = \alpha, \frac{\alpha_1}{\alpha} = \gamma, \frac{\alpha_2}{\alpha} = 1 - \gamma \), so that after completing the squares in \( n \) and \( k_{\perp} \) we get
\[
\Pi(P) = \frac{\lambda^2 m^{4-d}}{4\pi \epsilon R_0} \int_0^\infty d\alpha \int_0^1 d\gamma \sum_n \int_{(2\pi)^{d-1}} d^{d-1} k_{\perp} \times
\]
\[
\times \exp \left\{ -\alpha \left[ \left( \frac{n - \gamma N}{\epsilon R_0} \right)^2 + (k_{\perp} - \gamma P_{\perp})^2 + \gamma(1 - \gamma) \left( \frac{N^2}{\epsilon^2 R_0^2} + P_{\perp}^2 \right) + m^2 \right] \right\}.
\]
(3.3)

Now, \( \frac{N^2}{\epsilon^2 R_0^2} + P_{\perp}^2 \) is just \( P^2 \). We assume that \( P^2 \) is finite as \( \epsilon \to 0 \). Of course this is the case if \( P \) is on shell, but we can consider more general cases. The \( k_{\perp} \) integration is trivially done after shifting the integration variables as usual:
\[
\Pi(P) = \frac{\lambda^2 m^{4-d}}{2(2\pi)^d} \int_0^\infty d\alpha \left( \frac{\pi}{\alpha} \right)^{\frac{d-1}{2}} \int_0^1 d\gamma \frac{1}{\epsilon R_0} \sum_n \exp \left\{ -\alpha \left[ \left( \frac{n - \gamma N}{\epsilon R_0} \right)^2 + \gamma(1 - \gamma) P^2 + m^2 \right] \right\}.
\]
(3.4)

Exactly as in [5] the \( \epsilon \)-dependent part combines to give a \( \delta \)-function:
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon R_0} \exp \left\{ -\alpha \left( \frac{n - \gamma N}{\epsilon R_0} \right)^2 \right\} = \left( \frac{\pi}{\alpha} \right)^{1/2} \delta(n - \gamma N)
\]
(3.5)

so that
\[
\Pi(P) = \frac{\lambda^2 m^{4-d}}{2(2\pi)^d} \int_0^\infty d\alpha \left( \frac{\pi}{\alpha} \right)^{\frac{d}{2}} \int_0^1 d\gamma \sum_n \delta(n - \gamma N) \exp \left\{ -\alpha \left[ \gamma(1 - \gamma) P^2 + m^2 \right] \right\}.
\]
(3.6)

The \( \epsilon \to 0 \) limit has produced a \( \delta \)-function which eliminates one of the integrations over the \( \alpha \)-parameters (here the one over \( \gamma \)) and replaces it by a discrete sum. Indeed \( \delta(n - \gamma N) = \)

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* This point is slightly delicate: as long as we have Euclidean signature, \( P^2 \) is a sum of positive terms, and if \( (P^1)^2 = \frac{N^2}{\epsilon^2 R_0^2} \to \infty \) there is no way \( P^2 \) can remain finite if all components are real. But we have to keep in mind that ultimately we are interested in Minkowski signature in which case \( P^2 \) is finite provided \( P^0 = \frac{N}{\epsilon R_0} + O(\epsilon) \) as discussed in section 2.

† Clearly, if \( N = 0 \) one is in trouble. The problematic situation in \( \phi^4 \) corresponds effectively to the \( N = 0 \) case with no external discrete momentum flowing through the loop. Here however, \( N > 0 \) and the problem does not arise.
\( \frac{1}{N} \delta(\gamma - \gamma_n) \) with \( \gamma_n = \frac{n}{N} \). However, only \( n = 0, \ldots, N \) can contribute since only these \( \gamma_n \) are within the interval of the \( \gamma \)-integration. The momenta \( k^1 = \frac{n}{cH_0} \) and \( P^1 - k^1 = \frac{N-n}{cH_0} \) of the loop propagators are thus restricted to non-negative values. There is a slight subtlety here: \( n = 0 \) and \( n = N \) correspond to \( \gamma = 0 \) and \( \gamma = 1 \) which are just on the border of the integration interval. With the convention \( \int_0^0 \delta(x) f(x) dx = \frac{1}{2} f(0) \) and similarly for the upper bound, the values \( n = 0 \) and \( n = N \) should only contribute with an extra factor \( \frac{1}{2} \) to the sum. We denote \( \sum_{n=0}^{N} f(n) = \frac{1}{2} f(0) + \frac{1}{2} f(N) + \sum_{n=1}^{N-1} f(n) \).

Finally, doing the trivial \( \alpha \)-integral and going back to Minkowski signature we get

\[
\Pi(P) = \frac{\lambda^2 m^{4-d}}{2(4\pi)^{d/2}} \Gamma \left( \frac{4-d}{2} \right) \frac{1}{N} \sum_{n=0}^{N} \left[ m^2 - \frac{n}{N} \left( 1 - \frac{n}{N} \right) P^2 \right]^{d-4} .
\] (3.7)

Note that the somewhat ambiguous terms \( n = 0 \) and \( n = N \) are independent of the external momentum \( P \) and thus can be absorbed into the mass renormalisation.

**Renormalisation in \( d = 4 \)**

In \( d = 4 \), \( \Pi(P) \) is logarithmically divergent. Setting \( \tilde{\epsilon} = \frac{4-d}{2} \) we get

\[
\Pi(P) = -\frac{\lambda^2}{2(4\pi)^2} \frac{1}{N} \sum_{n=0}^{N} \log \left[ 1 - \frac{n}{N} \left( 1 - \frac{n}{N} \right) \frac{P^2}{m^2} \right] - \delta m^2
\] (3.8)

with

\[
\delta m^2 = -\frac{\lambda^2}{2(4\pi)^2} (\Gamma(\tilde{\epsilon}) + \log(4\pi)) .
\] (3.9)

The infinite part of \( \Pi(P) \) is \(-\delta m^2\) and is cancelled by a mass counterterm \(+\delta m^2\). We note that the latter does not depend on \( N \), i.e. is independent of the external momentum \( P \) as it should. Also note that \( n = 0 \) and \( n = N \) actually do not contribute to the finite part \( \Pi(P) + \delta m^2 \).

We may shift any finite constant between the finite part of \( \Pi(P) \) and \( \delta m^2 \). If we impose the standard renormalisation condition \( \Pi_R(p^2 = m^2) = 0 \) then it is easy to see that

\[
\Pi_R(P) = -\frac{\lambda^2}{2(4\pi)^2} \frac{1}{N} \sum_{n=1}^{N-1} \left\{ \log \left[ 1 - \frac{n}{N} \left( 1 - \frac{n}{N} \right) \frac{P^2}{m^2} \right] - \log \left[ 1 - \frac{n}{N} \left( 1 - \frac{n}{N} \right) \right] \right\} .
\] (3.10)

The terms \( n = 0 \) and \( n = N \) have disappeared from the sum. This means in particular that after renormalisation the discrete components of the propagators in the loop are strictly
positive, just as is the case for external on-shell states. This is an important feature of the DLCQ loop diagrams. However it appears below that this seems not to hold for higher-point one-loop diagrams in $\phi^3$. Note that this implies $\Pi_R(P) = 0$ for $N = 1$.

From eq. (3.10) it is clear that the two-point function has branch cuts starting at $P^2 = \frac{N^2}{n(N-n)} m^2$ for $n = 1, \ldots [\frac{N}{2}]$. This is easily seen to correspond to the threshold of production of two on-shell particles with $p_{(1)} = \frac{n}{\epsilon R_0}$ and $p_{(2)} = \frac{N-n}{\epsilon R_0}$.

So far, all is satisfactory: the $\epsilon \to 0$ limit exists, the mass counterterm does not depend on $N$, the internal propagators have strictly positive $n$, and the renormalised two-point function has the appropriate unitarity cuts.

**Renormalisation in $d = 6$**

$d = 6$ is the critical dimension of $\phi^3$ beyond which it is not renormalisable, so it is interesting to look at this case as well. Let now $\tilde{\epsilon} = \frac{6-d}{2}$. Then $\Pi(P)$ diverges quadratically and (3.8) is replaced by

$$
\Pi(P) = \frac{\lambda^2}{2(4\pi)^3} \frac{1}{N} \sum_{n=1}^{N-1} \left[ 1 - \frac{n}{N} \left( 1 - \frac{n}{N} \right) \frac{P^2}{m^2} \right] \log \left[ 1 - \frac{n}{N} \left( 1 - \frac{n}{N} \right) \frac{P^2}{m^2} \right] - \delta m^2 - (Z-1)P^2
$$

with

$$
\delta m^2 = \frac{\lambda^2}{2(4\pi)^3} (\Gamma(\tilde{\epsilon}) + 1 + \log(4\pi))
$$

$$
(Z-1) = \left( \frac{\lambda^2/m^2}{12(4\pi)^3} (\Gamma(\tilde{\epsilon}) + 1 + \log(4\pi)) \right) \left( 1 - \frac{1}{N^2} \right)
$$

Again, $\delta m^2$ does not depend on $N$, but $(Z-1)$ does. Although this dependence disappears at large $N$ as it should, it is likely that this $N$-dependence of the wave-function renormalisation constant $Z$ signals some inconsistency. It is not clear to us whether and how this could be resolved.
3.2. $\mathcal{N}$-point one-loop amplitudes

The basic feature that has emerged from the above computation was the appearance of a $\delta$-fct whose argument involves a certain combination of $\alpha$-parameters and external discrete momentum quantum numbers $n_r$, allowing to trivially perform one of the $\alpha$-integrations. It is clear that the same will happen for any one-loop diagram with cubic vertices only. The purpose of this subsection is to actually perform the calculation for an arbitrary number $\mathcal{N}$ of external legs in the $\phi^3$-theory and check that there are no hidden difficulties. We will give much less details than in the previous subsection. We will be interested in 4 dimensions so that all one-loop $\mathcal{N}$-point functions with $\mathcal{N} \geq 3$ are UV convergent. Otherwise, for $d = 6$, one could use dimensional regularisation for the 3-point function, all others being finite.

The main technical issue for the $\mathcal{N}$-point function is to find the most convenient change of variables for the $\alpha$-parameters so that the $\delta$-function can be used efficiently to eliminate one integration and the resulting integrals are over a simple domain. The correct change of variables is inspired from the string computation [5].

Let the external momenta be $p_r$, $r = 1, \ldots, \mathcal{N}$, all taken to be incoming. Momentum conservation then is $\sum_{r=1}^\mathcal{N} p_r = 0$. Obviously, not all $p_0^r = \frac{n_r}{\sqrt{\mu_0}}$ then are positive, a negative $n_r$ just means that we are actually dealing with an outgoing particle rather than an incoming. However, all $n_r$ are non-vanishing. The momentum of the $r^{\text{th}}$ propagator in the loop then is

$$k_r = k - p_1 - \ldots - p_{r-1} = k + p_r + \ldots + p_\mathcal{N}. \quad (3.13)$$

Using $\alpha$-parameters the product of the (Euclidean) propagators is

$$I_\mathcal{N}(k; p_r) = \int_0^\infty \ldots \int_0^\infty \prod_{r=1}^\mathcal{N} d\alpha_r \exp \left\{ - \sum_{r=1}^\mathcal{N} \alpha_r (k_r^2 + m^2) \right\}. \quad (3.14)$$

We change variables to

$$\beta_i = \sum_{r=1}^i \alpha_r \quad (3.15)$$

and introduce the notation $\beta_{ij} \equiv \beta_i - \beta_j$ which equals $\sum_{r=j+1}^i \alpha_r$ if $i > j$, and $\beta_{ij} = -\beta_{ji}$. 
The Jacobian obviously is 1. One has

\[ N \sum_{r=1}^{N} \alpha_r k_r^2 = \left( \sum_{r=1}^{N} \alpha_r \right) k^2 - 2k \cdot \left( \sum_{r=1}^{N} \alpha_r \sum_{i=1}^{r-1} p_i \right) + \sum_{r=1}^{N} \alpha_r \left( \sum_{i=1}^{r-1} p_i \right)^2. \]  

(3.16)

Now we have the following identities:

\[ N \sum_{r=1}^{N} \alpha_r \sum_{i=1}^{r-1} p_i = -\sum_{r=1}^{N} \alpha_r \sum_{i=r}^{N} p_i = -\sum_{i \geq r} \alpha_r p_i = -\sum_{i=1}^{N} \beta_i p_i, \]

\[ N \sum_{r=1}^{N} \alpha_r \left( \sum_{i=1}^{r-1} p_i \right)^2 = -\sum_{r=1}^{N} \alpha_r \sum_{i=1}^{r-1} p_i \cdot \sum_{j=r}^{N} p_j = -\sum_{i \leq r \leq j} \alpha_r p_i \cdot p_j = -\sum_{i < j} \beta_{ji} p_i \cdot p_j, \]

(3.17)

\[ \left( \sum_{i} \beta_i p_i \right)^2 = -\frac{1}{2} \sum_{i,j} \beta_{ji}^2 p_i \cdot p_j = -\sum_{i < j} \beta_{ji}^2 p_i \cdot p_j \]

where we have used momentum conservation. We can then rewrite eq. (3.16) as

\[ N \sum_{r=1}^{N} \alpha_r k_r^2 = \beta_N \left( k + \frac{1}{\beta_N} \sum_{i} \beta_i p_i \right)^2 + \sum_{i < j} \left( \frac{\beta_{ji}^2}{\beta_N} - \beta_{ji} \right) p_i \cdot p_j. \]  

(3.18)

Rescaling the \( \beta_i \) as

\[ \beta_N = \alpha, \quad \frac{\beta_i}{\beta_N} = \gamma_i, \quad i = 1, \ldots, N - 1 \]  

(3.19)

(with \( \gamma_N \equiv 1 \) and \( \gamma_{ij} \equiv \gamma_i - \gamma_j \)) the product (3.14) of the loop propagators becomes

\[ I_N(k, p_r) = \int_{0}^{\infty} \alpha^{N-1} \int_{0}^{1} d\gamma_{N-1} \int_{0}^{\gamma_{N-1}} d\gamma_{N-2} \ldots \int_{0}^{\gamma_2} d\gamma_1 \times \]

\[ \times \exp \left\{ -\alpha \left[ \left( k + \sum_{i} \gamma_i p_i \right)^2 + \sum_{i < j} \left( \gamma_{ji}^2 - \gamma_{ji} \right) p_i \cdot p_j + m^2 \right] \right\}. \]  

(3.20)

The domain of integraton of the \( \gamma_i \) is \( 0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{N-1} \leq 1 \). If the integrand were symmetric under permutation of any two \( \gamma_i \) one could instead integrate over the full hypercube \( 0 \leq \gamma_1, \gamma_2, \ldots, \gamma_{N-1} \leq 1 \) (and divide by \( N! \)). But permuting \( \gamma_i \) and \( \gamma_j \) does two things: it
permutes $p_i$ and $p_j$ and it changes the sign of the $\gamma_{ji}$. The latter is easily fixed by writing $|\gamma_{ji}|$ instead. The former is more interesting. We should really compute the $\mathcal{N}$-point one-loop Green’s function. This is obtained from a single diagram by adding all permutations of the external lines, i.e. by adding all permutations of the $p_i$. To avoid overcounting, one has to keep one external line (say $p_N$) fixed, and moreover divide by 2 to avoid counting a diagram and its mirror image twice. This Bose symmetrisation thus precisely amounts to extending the integration range of the $\gamma_i$, $i = 1, \ldots \mathcal{N} - 1$ to the full hypercube. Thus the correct Green’s function is (not writing any coupling constants explicitly):

$$G_{\mathcal{N}}(p_r) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} \sum_{\text{permutations of } p_1, \ldots, p_{\mathcal{N}-1}} I_{\mathcal{N}}(k, p_r)$$

$$= \frac{1}{2} \int_0^\infty d\alpha \alpha^{\mathcal{N}-1} \prod_{r=1}^{\mathcal{N}-1} d\gamma_r \int \frac{d^d k}{(2\pi)^d} \times$$

$$\times \exp \left\{ -\alpha \left[ (k + \sum_i \gamma_i p_i)^2 + \frac{1}{2} \sum_{i,j} \left( \gamma_{ji}^2 - |\gamma_{ji}| \right) p_i \cdot p_j + m^2 \right] \right\}.$$  (3.21)

The sequel is straightforward. For a compact $x^1$ coordinate with radius $R = \epsilon R_0$ one again replaces $\int dk^1 \to \frac{1}{\epsilon R_0} \sum n$ and $k^1 = \frac{n}{\epsilon R_0}$, while $p_i^1 = \frac{n_i}{\epsilon R_0}$. The integral over the $d-1$ other components of $k$ is trivial, while the $\epsilon$ dependent part gives

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon R_0} \exp \left\{ -\alpha \left( \frac{n + \sum_i \gamma_i n_i}{\epsilon R_0} \right)^2 \right\} = \left( \frac{\pi}{\alpha} \right)^{1/2} \delta \left( n + \sum_i \gamma_i n_i \right).$$  (3.22)

Thus, after performing the $\alpha$-integration, we end up with

$$G_{\mathcal{N}}(p_r) = \frac{1}{2(4\pi)^{d/2}} \Gamma \left( \mathcal{N} - \frac{d}{2} \right) \prod_{r=1}^{\mathcal{N}-1} d\gamma_r \sum n \delta \left( n + \sum_i \gamma_i n_i \right) \times$$

$$\times \left[ m^2 + \frac{1}{2} \sum_{i,j} \left( \gamma_{ji}^2 - |\gamma_{ji}| \right) p_i \cdot p_j \right]^{\frac{d}{2} - \mathcal{N}}.$$  (3.23)

It is straightforward to check that for $\mathcal{N} = 2$ this coincides with eq. (3.6) (after doing the $\alpha$-integral, and up to the coupling constant we have dropped).

* As before we assume that all $p_i \cdot p_j$ have finite limits as $\epsilon \to 0$. This is in particular the case if all $p_i$ are on shell, cf eq. (2.9).
At this point we can use the δ-function to trivially do one of the γᵣ-integrals. In terms of the initial αᵣ-parameters this would have been most complicated. We choose to eliminate the γₙ₋₁ integration. To do so, we introduce the notation {x}ᵢ, meaning the fractional part of x, i.e. {x}ᵢ = x + ˘n for some ˘n ∈ ℤ and {x}ᵢ ∈ [0, 1). Then γₙ₋₁ only takes the following |nₙ₋₁| discrete values

$$\gamma^l_{n-1} = \frac{1}{|n_{n-1}|} \left( \left\{ -\text{sign} n_{n-1} \sum_{i=1}^{N-2} \gamma_i n_i \right\}_f + l \right), \quad l = 0, \ldots |n_{n-1}| - 1.$$  \hspace{1cm} (3.24)

For N = 2 this simply gives γ¹ᵢ = l |n₁| as before. Thus

$$G_N(p_r) = \frac{1}{2(4\pi)^{d/2}} \Gamma \left( N - \frac{d}{2} \right) \prod_{r=1}^{N-2} d\gamma_r \frac{1}{|n_{n-1}|} \sum_{l=0}^{|n_{n-1}|-1} \left[ m^2 + \frac{1}{2} \sum_{i,j} \left( \gamma^2_{ji} - |\gamma_{ji}| \right) p_i \cdot p_j \right]^{-\frac{d}{2}-N} \bigg|_{\gamma_N=1, \gamma_{n-1}=\gamma_{n-1}}.$$  \hspace{1cm} (3.25)

Although this final expression looks asymmetric between the Nᵗʰ, the (N − 1)ᵗʰ and the other external legs, it is clear from the derivation that it is completely symmetric. This will also be obvious for the examples considered below. It is a perfectly well-defined expression and we see that the ε → 0 limit exists for the N-point amplitude under consideration. Of course, Gₑ has those cuts required by unitarity. We have already seen this for N = 2.

As an explicit example one may take the triangle diagram, N = 3 in d = 4. Recall that in this case γ₃ = 1. Let γ₁ ≡ γ, so that γ₂ = γ¹₂ = (\{-γ₁ \text{sign} n₂ ≪ l\} / |n₂|) and

$$G_{N=3}(p_i) = \frac{1}{32\pi^2} \int_0^1 d\gamma \sum_{l=0}^{|n₂|} \left[ m^2 + p^2_1 \gamma (1 - \gamma) + p^2_2 \gamma_2 (1 - \gamma_2) + p_1 \cdot p_2 (\gamma + \gamma_2 - |\gamma - \gamma_2| - 2\gamma_2) \right]^{-1}. \hspace{1cm} (3.26)$$

For fixed l, as γ varies from 0 to 1, γ¹₂ will be discontinuous whenever γ₁n₁ is an integer, so that to actually evaluate the integral, the interval [0, 1] has to be split into pieces. If we take e.g. n₁ = 2, n₂ = n₃ = −1 only l = 0 is present and γ₂ = \{2γ\}_f so that we have to split
the interval into $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. It is easy to see that both pieces actually give the same contribution and, after Wick rotating back to Minkowski signature, we get

$$G_{N=3}^{n_1=2, n_2=n_3=-1}(p_i) = \frac{1}{32\pi^2} \int_0^1 dx \left[ m^2 - \left( \frac{p_2^2}{2} - p_2 \cdot p_3 \right) x + \left( \frac{p_2^2}{4} - p_2 \cdot p_3 \right) x^2 \right]^{-1}$$

(3.27)

where we used $p_2^2 + p_1 \cdot p_2 = -p_2 \cdot p_3$. This expression is manifestly symmetric under exchange of $p_2$ and $p_3$ as it should since $n_2 = n_3$. The integral is elementary and can easily be done. However, we already see from (3.27) that a branch cut will start whenever the square bracket in the integrand is zero at the boundary of the integration interval. At $x = 0$ it is just $m^2 \neq 0$, but vanishing at $x = 1$ gives $p_1^2 = 4m^2$, giving the branch cut as expected. Indeed, an incoming $n_1 = 2$ state can split into two states with $n = 1 > 0$ each. Note also that there are no cuts in $p_2^2$ or $p_3^2$ and this is in agreement with the fact that $|n_2| = |n_3| = 1$ cannot split into two states both with $n > 0$.

It is interesting to trace back what are the discrete components of the loop momenta that contribute. One of these is $n$ as entering the argument of $\delta(n + \sum_i \gamma_i n_i)$. When $\gamma_1 \leq \gamma_2 \leq \ldots$ this $n$ corresponds to the propagator between the first and the $\mathcal{N}^{th}$ external line, but after Bose symmetrisation (any configuration of the $\gamma_i \leq 1$) this can correspond to any of the internal propagators. The $\delta$-function implies $n = -\sum_i \gamma_i n_i = -n_{\mathcal{N}} - n_{\mathcal{N}-1} \gamma_{\mathcal{N}-1} - \sum_{i=1}^{N-2} \gamma_i n_i$. For the above example with $\mathcal{N} = 3$ and $n_1 = 2, n_2 = n_3 = -1$ this gives $n = 1 + \gamma_2 - 2\gamma = 1 + \{2\gamma\} f - 2\gamma = 1 - [2\gamma]$, hence $n = 1$ for $\gamma \in [0, \frac{1}{2})$ and $n = 0$ for $\gamma \in [\frac{1}{2}, 1)$. We see that, contrary to what happened for the renormalised two-point function, internal propagators do occur with vanishing discrete momentum. In this example, it is clear from momentum conservation around the loop that one cannot have $n_1 = 2, n_2 = n_3 = -1$ with all discrete loop momenta strictly positive (for a given “time slicing”). Another example is $\mathcal{N} = 3$ with $n_1 = 4, n_2 = n_3 = -2$ and $\gamma_2 = (\{4\gamma\} f + l)/2$, $l = 0, 1$ and hence $n = 2 + l - [4\gamma]$ which takes the values $-1, 0, 1, 2, 3$. The only way to evade the conclusion that internal propagators may have vanishing (or negative) discrete momentum would be to show that these diagrams actually vanish. But the above explicit example (3.27) tells us that this is not so.

\* One gets $\frac{1}{32\pi} \ln(a + b)^{-1/2} \log \left\{ \left[ 1 + \left( 1 + \frac{a}{b} \right)^{1/2} \right] / \left[ 1 - \left( 1 + \frac{a}{b} \right)^{1/2} \right] \right\}$ with $a = m^2(p_2^2 - 4m^2)$ and $b = \frac{p_2^2}{2} - p_2 \cdot p_3 - 2m^2 = \frac{1}{4}[(p_1^2 - 4m^2) + ((p_3 - p_2)^2 - 4m^2)]$. 

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3.3. \( \phi^3 \) Beyond One Loop

Beyond one loop the situation is less clear. Naively, we proceed in exactly the same way, and expect that the mechanism that produced the \( \delta \)-function at one loop also operates here giving one \( \delta \)-function for each loop, so that \( L \) \( \alpha \)-parameters will be discretised. However, looking more carefully, we see that one important ingredient at one loop was that the external momenta are such that the \( p_r \cdot p_s \) have finite limits as \( \epsilon \to 0 \). Recall for example eq. (3.3) for \( \Pi(p) \) where we used that \( \frac{N^2}{c R_0^2} + P_\perp^2 = P^2 \) is finite. If this self-energy diagram appears as an insertion in some internal line with momentum \( k \) of another loop, then we have \( \Pi(k) \), i.e. \( P \to k, \ N \to n \) and there is no reason why \( \frac{n^2}{c R_0^2} + k_\perp^2 \) should be finite as \( \epsilon \to 0 \). This does not mean that we cannot take the \( \epsilon \to 0 \) limit but it might be different from what one naively expects. Let us proceed anyway.

Consider some \( L \)-loop diagram with \( N \) external lines. Then there are \( I = N + 3L - 3 \) internal propagators. As before they are rewritten using \( I \) \( \alpha \)-parameters. The exponent then is some quadratic form in the \( L \) loop momenta \( k_i, \ i = 1, \ldots L \), and in the external momenta \( p_r, \ r = 1, \ldots N \), with coefficients that are functions of the \( \alpha_s \). Just as one completes the square for \( k \) at one loop, one successively completes the squares for the \( L k_i \), starting with \( k_1 \), then for \( k_2 \), etc. For the non-compact directions one then shifts the integration variables \( k_i^1 \) to \( \tilde{k}_i^1 \) and does the integral trivially. For the compact direction \( x^1 \) one has \( k_1^1 = \frac{l}{c R_0} \) (analogue of \( k_1 = \frac{n}{c R_0} \) at one loop) and \( \int \prod_{i=1}^L dk_i^1 \to \frac{1}{(c R_0)^L} \sum_{l_1, \ldots, l_L} \), as well as

\[
\lim_{\epsilon \to 0} \frac{1}{(c R_0)^L} \exp \left\{ - \sum_{j=1}^L \left( \frac{f_j(l_i, \alpha_s)}{c R_0} \right)^2 \right\} = \frac{\pi^{\frac{L}{2}}}{L!} \prod_{j=1}^L \delta \left( f_j(l_i, \alpha_s) \right). \tag{3.28}
\]

In practice one would like to find the most convenient change of variables \( \alpha_s \to \gamma_s \) analogous to (3.15), (3.19). As a matter of principle, however, this is not necessary. All we need is a subset of \( L \) \( \alpha \)'s (label them \( \alpha_1, \ldots \alpha_L \)) such that

\[
\det \left( \frac{\partial f_j(l_i, \alpha_s)}{\partial \alpha_i} \right)_{i, j = 1, \ldots, L} \bigg|_{f_j = 0} \neq 0 \tag{3.29}
\]

so that we can solve the constraints imposed by the \( \delta \)-functions in terms of the \( \alpha_1, \ldots \alpha_L \). Rather than trying to analyse this condition in general we consider a specific two-loop example.

\[\dagger\] The definition of the \( f_j^2 \) will have to take into account terms like the above-mentioned \( n^2 \).
Take the one-loop two-point function $\Pi(P)$ computed above with internal propagators having momenta $k$ and $P - k$. In one of the loop propagators, say the one with momentum $k$, insert again this same $\Pi(k)$. The “external” momenta of the inserted sub-loop then simply are $k$ and $-k$. The discrete component of $k$ is $k^1 = \frac{n}{\epsilon R_0}$. Suppose the problem mentioned above of diverging $k^2 = \frac{n^2}{\epsilon R_0} + k^2_\perp$ is somehow resolved. The condition (3.29) then in particular means that the $\delta$-function appearing in $\Pi(k)$ can be used to eliminate the $\gamma$-integration appearing in $\Pi(k)$. We have seen that this is the case if $k^1 = \frac{n}{\epsilon R_0} \neq 0$ (now $n$ plays the role of $N$ above). In the previous section, we have shown that in the renormalised $\Pi_R(P)$ only $n = 1, \ldots N - 1$ contribute (recall that $P^1 = \frac{N}{\epsilon R_0}$). Thus $n \neq 0$, and the full (renormalised) two-loop diagram (probably) has a well-defined finite limit as $\epsilon \to 0$. However, this example does not represent the generic situation: consider inserting a one-loop two-point function $\Pi(k)$ into any of the internal propagators of the triangle diagram. As we have discussed at the end of the previous section, in this case it appears the possibility that one of these internal propagators has vanishing discrete momentum so that the inserted $\Pi(k)$ is taken at $k^1 = 0$ and thus its $\epsilon \to 0$ limit diverges.

4. QED

The purpose of this section is to show that the same mechanism that worked for the scalar $\phi^3$ theory also works in a realistic theory containing fermions and having gauge invariance.

We will explicitly study the behaviour of two UV divergent one-loop diagrams in QED in four dimensions, namely the vacuum polarisation and the electron self-energy. Then we will comment on the other one-loop and on higher-loop diagrams. As before, we work with dimensional regularisation performed on the Wick rotated Euclidean integrals.

4.1. Vacuum polarisation

The vacuum polarisation diagram $\omega_{\mu\nu}(p)$ has a superficial degree of divergence 2, but it is well-known that $\omega_{\mu\nu}(p) = i(p_\mu p_\nu - g_{\mu\nu}p^2)\omega(p)$ with $\omega(p)$ only logarithmically divergent. In the present case, the discrete component is $p_1$ and we must treat $\omega_{11}, \omega_{1i}$ and $\omega_{ij}$ ($i, j = 0, 2, 3$)
separately. We begin with the Wick rotated expression

\[
\omega_{\mu\nu}(p) = ie^2 m^{4-d} \int \frac{d^{d-1}k}{(2\pi)^{d-1}2\pi\epsilon R_0} \sum_n \frac{k_{\mu}(k-p)_\nu + k_{\nu}(k-p)_\mu - \delta_{\mu\nu}(k \cdot (k-p) + m^2)}{(k^2 + m^2)((k-p)^2 + m^2)}
\]

(4.1)

with \(m\) being the mass of the charged fermion (electron). We denote \(p^1 = \frac{N}{\epsilon R_0}\) as before and \(k^1 = \frac{n}{\epsilon R_0}\). Introducing \(\alpha\)-parameters and going through the same kind of algebra as for the self-energy diagram in the \(\phi^3\) theory, we get for \(\omega_{1i}\) in the \(\epsilon \to 0\) limit

\[
\omega_{1i}(p) = -ie^2 m^{4-d} \int_0^\infty d\alpha \frac{\alpha^{1-d}}{(4\pi)^{d/2}} \int_0^1 d\gamma \sum_n \left( \frac{n}{\epsilon R_0} (1 - \gamma) p_i + \frac{N - n}{\epsilon R_0} \gamma p_i \right) \times
\]

\[
\times \delta(n - \gamma N) \exp \left\{ -\alpha \left[ m^2 + \gamma(1 - \gamma)p^2 \right] \right\}.
\]

(4.2)

Using \(\delta(n - \gamma N)\) to replace \(n\) by \(\gamma N\) in the parenthesis we can extract a factor \(\frac{N}{\epsilon R_0} p_i = p_1 p_i\) so that indeed \(\omega_{1i} = ip_1 p_i \omega(p)\) with

\[
\omega(p) = -\frac{2e^2 m^{4-d}}{(4\pi)^{d/2}} \int_0^\infty d\alpha \frac{\alpha^{1-d}}{(4\pi)^{d/2}} \int_0^1 d\gamma \sum_n \gamma (1 - \gamma) \delta(n - \gamma N) \exp \left\{ -\alpha \left[ m^2 + \gamma(1 - \gamma)p^2 \right] \right\}
\]

\[
= \frac{e^2}{8\pi^2} \frac{1}{N} \sum_{n=1}^{N-1} \frac{n}{N} \left( 1 - \frac{n}{N} \right) \log \left[ 1 + \frac{n}{N} \left( 1 - \frac{n}{N} \right) \frac{p^2}{m^2} \right] + (Z_3 - 1)
\]

\[
\equiv \omega_R(p) + (Z_3 - 1)
\]

(4.3)

with

\[
(Z_3 - 1) = -\frac{e^2}{8\pi^2} \frac{1}{N} \sum_{n=1}^{N-1} \frac{n}{N} \left( 1 - \frac{n}{N} \right) \left[ \Gamma(\bar{\epsilon}) + \log 4\pi \right]
\]

\[
= -\frac{e^2}{48\pi^2} \left( 1 - \frac{1}{N^2} \right) \left[ \Gamma(\bar{\epsilon}) + \log 4\pi \right].
\]

(4.4)

We note two things. Again, the terms \(n = 0\) and \(n = N\) do not contribute to \(\omega(p)\) or \(Z_3 - 1\). Again also, the counterterm \(Z_3 - 1\) explicitly depends on \(N\). As in the \(\phi^3\) theory, the first fact is encouraging, while the second is puzzling. Let’s compare with the standard (Euclidean) result in the non-compact case. There

\[
\omega_{\text{non-compact}}(p) = \frac{e^2}{8\pi^2} \int_0^1 d\gamma \gamma (1 - \gamma) \log \left[ 1 + \gamma(1 - \gamma) \frac{p^2}{m^2} \right] + (Z_3 - 1).
\]

(4.5)
So all that has happened is $\gamma \to \frac{n}{N}$ and $\int_0^1 d\gamma \to \frac{1}{N} \sum_{n=1}^{N-1}$.

We still need to check that $\omega_{11}(p) = -ip_\perp \omega(p)$ and $\omega_{ij}(p) = i(p_i p_j - \delta_{ij} p^2) \omega(p)$. Here we encounter an interesting subtlety. In both cases the numerator of (4.1) contains a term $k_1(k - p)_1 = \frac{n(n - N)}{\epsilon^2 R_0^2}$ so that the $\epsilon$-dependent terms are of the form

$$
\frac{1}{\epsilon R_0} \left[ a \frac{n(n - N)}{\epsilon^2 R_0^2} + b \right] \exp \left\{ -\alpha \left( \frac{n - \gamma N}{\epsilon R_0} \right)^2 \right\}.
$$

(4.6)

In this case one has to develop in $\epsilon$ to next to leading order:

$$
\frac{1}{\epsilon R_0} \exp \left\{ -\alpha \left( \frac{n - \gamma N}{\epsilon R_0} \right)^2 \right\} \sim \left( \frac{\pi}{\alpha} \right)^{\frac{1}{2}} \left[ \delta(n - \gamma N) + \frac{\epsilon^2 R_0^2}{4\alpha} \delta''(n - \gamma N) + O(\epsilon^4) \right]
$$

(4.7)

as can easily be seen by multiplying by a test function $f(n)$ and integrating $\int d\gamma$ (or equivalently by using a test function $f(\gamma)$ and integrating $\int d\gamma$). As a result, as $\epsilon \to 0$, expression (4.6) gives

$$
\left( \frac{\pi}{\alpha} \right)^{\frac{1}{2}} \left[ a \frac{n(n - N)}{\epsilon^2 R_0^2} + b + \frac{a}{2\alpha} \right] \delta(n - \gamma N).
$$

(4.8)

The extra contribution $\sim \frac{a}{2\alpha}$ is crucial. The rest of the computation is straightforward algebra, and we indeed find that $\omega_{\mu\nu}(p) = i(p_\mu p_\nu - \delta_{\mu\nu} p^2) \omega(p)$ for all $\mu, \nu$.

### 4.2. Fermion self-energy

The computation is straightforward and the result is again identical to the standard result of the non-compact case but with the replacement $\gamma \to \frac{n}{N}$, $\int_0^1 d\gamma \to \sum_{n=0}^{N-1}$. This time the terms $n = 0$ and $n = N$ contribute to the regularised self-energy and both terms must be included with a factor $\frac{1}{2}$ into the sum. Then

$$
\Sigma(p) = \frac{e^2 m^{4-d}}{(4\pi)^{d/2}} \left( \frac{4 - d}{2} \right) \frac{1}{N} \sum_{n=0}^{N-1} \left[ d m - (d - 2) \left( 1 - \frac{n}{N} \right) \phi \right] \times
$$

$$
\times \left[ \frac{n}{N} \left( 1 - \frac{n}{N} \right) p^2 + \frac{n}{N} m^2 + \left( 1 - \frac{n}{N} \right) \mu^2 \right]^{\frac{d-4}{2}}
$$

(4.9)
where $\mu^2$ is an IR regulator (photon mass) needed to keep the $n = 0$ term finite. We have

$$\Sigma(p) = -\frac{e^2}{(4\pi)^2} \frac{1}{N} \sum_{n=0}^{N'} \left[ 4m - 2 \left( 1 - \frac{n}{N} \right) \beta \right] \log \left[ \frac{n}{N} \left( 1 - \frac{n}{N} \right) \frac{p^2}{m^2} + \frac{n}{N} + \left( 1 - \frac{n}{N} \right) \frac{\mu^2}{m^2} \right] + \delta m + (Z_2 - 1)(\beta - m)$$

with

$$\delta m = \frac{e^2}{(4\pi)^2} \left( 3\Gamma(\bar{\epsilon}) + 3 \log 4\pi - 1 \right) m$$

$$Z_2 - 1 = -\frac{e^2}{(4\pi)^2} \left( \Gamma(\bar{\epsilon}) + \log 4\pi - 1 \right).$$

We note that $\delta m$ and $(Z_2 - 1)$ are independent of $N$. We also note that the terms $n = 0$ or $n = N$ in $\Sigma(p)$ are $\sim m$ or $\sim \beta$. They can be eliminated from the renormalised self-energy since we may still change $\Sigma_R$ by finite counterterms. In particular, if we impose the standard renormalisation condition $\Sigma_R(\beta = m) = 0$ and $\partial \Sigma_R / \partial \beta(\beta = m) = 0$ we find

$$\Sigma_R(p) = -\frac{e^2}{(4\pi)^2} \frac{1}{N} \sum_{n=1}^{N-1} \left\{ \left[ 4m - 2 \left( 1 - \frac{n}{N} \right) \beta \right] \log \left[ \frac{N}{n} + \frac{N - 1}{n - 1} \frac{p^2}{m^2} \right] + 4 \left( \frac{N}{n} - \frac{n}{N} \right) (\beta - m) \right\}$$

with no contributions from $n = 0$ or $n = N$! Note that we have set $\mu = 0$ because it was only needed for the $n = 0$ term which is no longer present.

### 4.3. Other one-loop and higher loop diagrams

While the other one-loop diagrams (vertex function, etc) are technically more involved, in particular due to the $\gamma$-matrix algebra, there are no new difficulties to be encountered* and

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* A minor complication arises when one wants to extend the integration domain of the $\gamma_i$ from $0 \leq \gamma_1 \leq \gamma_2 \leq \ldots \leq \gamma_{N-1} \leq 1$ to the full hypercube $0 \leq \gamma_i \leq 1$. In the scalar $\phi^3$ theory this was achieved by Bose symmetrisation. Here, in QED one has to separately Bose symmetrise external photon lines, and separately antisymmetrise incoming and outgoing fermion lines. As a result, we do not get the same type of compact expression with the $\gamma_i$ all integrated over the hypercube. This is easy to see for the vertex function. Nevertheless, one can still use the $\delta$-function to eliminate one of the $\gamma_i$ which is replaced by a finite sum, although the lower and upper bound of the sum now depend on the other integration variable(s). In the case of the vertex function one can very explicitly evaluate the diagram in the $\epsilon \to 0$ limit, which is indeed finite, and study its properties.
we can safely conclude that all one-loop amplitudes in QED have a finite light-like limit. As in $\phi^3$-theory, the situation beyond one loop is less clear.

5. Conclusions

We have studied the issue of defining discrete light-cone quantization in field theory as a light-like limit. While this limit is unproblematic at the tree-level, it is non-trivial for loop amplitudes. We have seen that in theories with cubic vertices only, and in particular in QED, all one-loop amplitudes have a finite and well-defined light-like limit (provided the external states have non-vanishing discrete momenta which is the appropriate kinematic setting corresponding to the DLCQ). Moreover, we have seen that these amplitudes have exactly the cuts required by unitarity. When they are written using standard Feynman $\alpha$-parameters, we showed that the only thing that happens in the light-like limit is the replacement of one of the $\alpha$-integrals by a discrete sum. This is analogous to what happens in string theory. These one-loop amplitudes are actually often easier to compute explicitly than the standard ones since they involve one less integral.

There are however some puzzling features: we have seen in the examples of the scalar self-energy (in $d = 6$) and the QED vacuum polarisation diagrams that certain counterterms depend explicitly on $N$, i.e. on the discrete component of momentum of the external particle. It is not clear to us whether and how this might make sense. Also, while for the self-energy diagrams all internal discrete momenta are strictly positive, this is not always the case for the higher-point functions (vertex function, etc).

Related to this last point, we have argued that beyond one loop we are probably going to encounter problems and the light-like limit is likely to diverge, although a more detailed analysis is needed to settle the question. It should be noted that the troublesome situation occurring at two and more loops is very specifically field theoretic and does not affect the conjectured existence of a light-like limit of string theory to all orders in perturbation theory [6].

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