Analytical Solutions in a First-order Cubic Nonlinear System with Periodically Forced Oscillator

Danping Sun1+, Lanzheng Chen1+, Renzhuo Wan1 and Guopeng Zhou2*

1Wuhan Textile University, Wuhan, Hubei, 430200, China
2Hubei University of Science and Technology, Hubei Xiangcheng Intelligent Electromechanical Industry Technology Institute, Xianning, 437100, China

+The first two authors have the same contribution to this work
*Corresponding author’s e-mail: zhgpeng@mail.hbust.com.cn

Abstract. In this paper, the analytical solutions for the period-1 motion of a cubic nonlinear dynamical system with two periodic forced terms are obtained through the generalized harmonic balance method. From the method, the analytical solutions are transformed to the Fourier series expansion in all harmonic terms at the equilibrium position. The stability and bifurcation analysis are investigated through eigenvalues analysis, and the accuracy of the analytical solutions is verified by numerical simulations. The harmonic amplitude distributions are presented to show the characteristic of the different-order terms with different excitation frequencies. From this study, the system parameters that make the system stable are determined. Furthermore, the robustness of the system is guaranteed by selecting the parameters appropriately.

1. Introduction

The precise analytical solutions of periodic-1 motion are of great significance for the system to understand the characteristic of nonlinear system. The generalized balance method is a significant method for solving nonlinear dynamic problems by analytic approximation to periodic solutions. In 1788, Lagrange[1] formulated the gravitational three-body problem as a perturbation of the two-body problem through the idea of averaging method. In 1973, Poincare[2] intensively investigated the perturbation theory for celestial movement. In the same year, Nayfeh[3] presented the multiscale perturbation method to apply such a perturbation method for obtaining approximate solutions in the Duffing oscillators. In 1964, Hayashi[4] applied the averaging method and harmonic balance method to discuss the stability of periodic solutions. Then, perturbation theory, averaging method and harmonic balance theory were used to determine the approximate solutions of nonlinear oscillators. In 1997, Luo and Han[5] analytically presented the stability and bifurcation conditions of periodic motions of the Duffing oscillator through the first order harmonic balance methods. In 2008, Peng et al presented the analytical period-1 solution for the Duffing oscillator by the harmonic balance method, and numerical simulation are carried out to check it though the 4th order Runge-Kutta method[6]. In 2012, Luo and Huang[7] proposed a generalized harmonic balance method to obtain the analytical solutions of period-m motion in nonlinear system. Luo put forward a methodology to explain the balance method(also see, Luo and Huang[8]. In 2016, Zhou[9] obtained that infinite harmonic balance terms must be introduced to approximate chaotic systems. In 2017, Ying[10] studied further analytical solutions for periodic motion in the duffing oscillator. Later, analytical solutions for periodic forced
oscillations was discussed in[11-12]. The dynamic system in engineering almost always contains a variety of nonlinear factors, such as the gap in the mechanical system, the nonlinear control strategy of the control system.

In this paper, a 1-D cubic nonlinear dynamic system will be discussed via the generalized harmonic balance method. The stability and bifurcation are investigated through eigenvalues analysis, and the 4th order Runge-Kutta method are carried out to check the analytical prediction of period-1 motion. The innovations of this paper are as follows.

1) Compared with traditional methods such as perturbation method, average method and so on, the analytical solution of the complex nonlinear dynamic system of third order or higher order is more accurate and the computational complexity is lower.

2) In this paper, the stability and bifurcation of third-order systems are analyzed in detail, which provides a theoretical basis for the structural design and parameter selection of nonlinear systems in practical engineering.

The plan of this paper is as follows. In Section 2, the system model is briefly introduced and the analytical solution and eigenvalue of the system are calculated. In Section 3, the amplitude-frequency characteristics of the system are analyzed. In Section 4, the solutions are verified by numerical simulation. In Section 5, we will give an conclusions for this paper.

2. Analytical solutions

Consider a cubic nonlinear dynamic systems

\[ \ddot{x} + (\alpha_1 + Q_1 \cos \Omega t)x + \alpha_2 x^2 + \alpha_3 x^3 = Q_0 \cos \Omega t \]  

(1)

where \( \dot{x} \) is velocity and \( x \) is displacement. \( \alpha_1 \) is a linear coefficients, \( \alpha_2 \) and \( \alpha_3 \) are quadratic and cubic nonlinear coefficients. \( Q_0, Q_1, \cos \Omega t \) and \( \Omega \) are excitation amplitude and frequency.

In Luo[7], the standard form of formula(1) for the coefficient equilibrium analysis is

\[ \dot{x} = g(x, t) \]  

(2)

Where

\[ g(x, t) = -(\alpha_1 + Q_1 \cos \Omega t)x - \alpha_2 x^2 - \alpha_3 x^3 + Q_0 \cos \Omega t \]  

(3)

As in Luo[8], the Fourier expansion formula can be used to describe periodic motion with period \( T = \frac{2\pi}{\Omega} \) in the nonlinear dynamic system. So, the analytical solutions for system(1) can be given by

\[ x^*(t) = a_0(t) + \sum_{k=1}^{N} \left[ b_k(t) \cos(k\Omega t) + c_k(t) \sin(k\Omega t) \right] \]  

(4)

where \( x^*(t) \) is the predicted approximate solution of periodic motions. and \( k \) is the number of harmonic term \( (k = 1, 2, \cdots, N) \). The corresponding first order derivatives is

\[ \dot{x}^* = \dot{a}_0 + \sum_{k=1}^{N} \left[ (\dot{b}_k + k\Omega c_k) \cos(k\theta) + (\dot{c}_k - k\Omega b_k) \sin(k\theta) \right] \]  

(5)

Let

\[ z = (a_0, b, c)^T \]  

(6)

And

\[ b = (b_1, b_2, \cdots, b_N)^T, c = (c_1, c_2, \cdots, c_N)^T \]  

(7)

Substitute of (4) and (5) into (2) and using Fourier coefficient equilibrium, it has
\[ \dot{a}_0 = F_0(a_0, b, c), \dot{b} = -\Omega kc + F_1(a_0, b, c), \dot{c} = \Omega kb + F_2(a_0, b, c) \]
\[ b = [b_1, b_2, \cdots, b_N]^T, c = [c_1, c_2, \cdots, c_N]^T, k = \text{diag}(1, 2, \cdots, N) \]
\[ F_0 = [F_{11}, F_{12}, \cdots, F_{1N}]^T, F_1 = [F_{21}, F_{22}, \cdots, F_{2N}]^T \]

for \( N = 1, 2, \cdots, \infty \).

And
\[ F_0(a_0, b, c) = \frac{1}{T} \int_0^T f(x, t) dt, F_{1k}(a_0, b, c) = \frac{2}{T} \int_0^T f(x, t) \cos(k \Omega t) dt \]
\[ F_{2k}(a_0, b, c) = \frac{2}{T} \int_0^T f(x, t) \sin(k \Omega t) dt \]

With
\[ F_0 = -\alpha_0 a_0 - \frac{1}{2} b Q_1 - \alpha_2[a_0^2 + c_0^2] - \alpha_3[a_0^3 + \frac{3}{2} \sum_{i=1}^N (b_i^2 + c_i^2)] - \frac{1}{4} \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N b_i b_j b_l (\delta^{0}_{i-j+l} + \delta_0 + \delta^0_{i-j-l}) + \frac{3}{4} \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N b_i c_j c_l (\delta^{0}_{i-j+l} + \delta_0 + \delta^0_{i-j-l}) \]

\[ F_{1k} = -\alpha_1 b_k - \alpha_2[2a_0 b_k + G_{11} + G_{12}] - \alpha_3[3a_0^2 b_k + G_{13} + G_{14} + G_{15} + G_{16}] + Q_0 \delta^0_k + G_{17} \]
\[ F_{2k} = -\alpha_1 c_k - \alpha_2[2a_0 c_k + G_{21} + G_{22}] - \alpha_3[3a_0^2 c_k + G_{23} + G_{24} + G_{25} + G_{26}] + G_{27} \]

And
\[ \delta^0 = \begin{cases} 1, i = k, \text{sgn}(\theta) = \{ & 1, \theta > 0 \\ 0, i \neq k, \text{sgn}(\theta) = \{ & -1, \theta < 0 \end{cases} \]

The functions of \( G_{1q} \) and \( G_{2q} \) (q=1,2,...,7) in Eqs.(10)-(13) are written as follows.
\[ G_{11} = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_i b_j (\delta^{0}_{i-j} + \delta^{0}_{i-j}) \]
\[ G_{12} = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N c_i c_j (\delta^{0}_{i-j} - \delta^{0}_{i-j}) \]
\[ G_{13} = \frac{3}{2} \sum_{i=1}^N \sum_{j=1}^N b_i b_j (\delta^{0}_{i-j} + \delta^{0}_{i-j}) \]
\[ G_{21} = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N b_i c_j (\delta^{0}_{i-j} - \text{sgn}(i-j) \delta^{0}_{i-j}) \]
\[ G_{22} = \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N c_i b_j (\delta^{0}_{i-j} + \text{sgn}(i-j) \delta^{0}_{i-j}) \]
\[ G_{23} = \frac{3}{2} \sum_{i=1}^N \sum_{j=1}^N b_i c_j (\delta^{0}_{i-j} - \text{sgn}(i-j) \delta^{0}_{i-j}) \]

And let
\[ f = [F_0, -k \Omega c + F_1, -k \Omega b + F_2]^T, \]

Then, Eq.(8) becomes
\[ \dot{z} = f(z) \]

The steady-state solutions for period motion in Eq.(1) can be determined by setting \( f(z^*) = 0 \),
\[ F_0(a_0^*, b^*, c^*) = 0, -\Omega kc^* + F_1(a_0^*, b^*, c^*) = 0, \Omega kb^* + F_2(a_0^*, b^*, c^*) = 0 \]

With the Newton-Raphson method, the \( 2N + 1 \) nonlinear equations in Eq.(19) will be solved. Referring to Luo[8], the linearized equations of Eq.(18) at the equilibrium point
\[ z^* = (a_0^*, b^*, c^*)^T \]

is given by \( \Delta z = Df(z^*) \Delta z \)
With

$$Df(z^*) = \frac{\partial f(z)}{\partial z}.$$  

The stability is analyzed via the eigenvalues of Jacobian matrix at the equilibrium point. The corresponding eigenvalues are calculated by

$$|Df(z^*) - \lambda I_{(2N+1)^2(2N+1)}| = 0$$  

(17)

The Jacobian matrix is given by

$$Df = \begin{bmatrix}
\frac{\partial F_0}{\partial a_0} & \frac{\partial F_0}{\partial b} & \frac{\partial F_0}{\partial c} \\
\frac{\partial F_{1k}}{\partial a_0} & \frac{\partial F_{1k}}{\partial b} & \frac{\partial F_{1k}}{\partial c} - \Omega k \\
\frac{\partial F_{2k}}{\partial a_0} & \frac{\partial F_{2k}}{\partial b} + \Omega k & \frac{\partial F_{2k}}{\partial c}
\end{bmatrix}$$  

(18)

According to Luo[8], the eigenvalues of $Df(z^*)$ can be classified as $(n_1, n_2, n_3 | n_4, n_5, n_6)$ where $n_1$ is the total number of positive real eigenvalues, $n_2$ is the total number of negative real eigenvalues, $n_3$ is the total number of zero eigenvalues; $n_4$ is the total pair number of complex eigenvalues with positive real parts, $n_5$ is the total pair number of complex eigenvalues with negative real parts, $n_6$ is the total pair number of complex eigenvalues with zero real parts.

If $Re(\lambda_k) < 0(k = 1, 2, \cdots, 2N + 1)$, the approximate steady-state solution is stable.

If $Re(\lambda_k) > 0(k = 1, 2, \cdots, 2N + 1)$, the truncated approximate steady-state solution is unstable.

When $N$ is large enough, the predicted solution is infinitely close to the exact solution.

3. Frequency-amplitude characteristics

The accurate steady-state solutions of the nonlinear oscillator can be obtained through the infinite harmonic terms. As the number of harmonics increases, the analytical prediction of periodic motions will become precise. The corresponding solution in Eq.(ref{eq4}) can be written via Fourier series theory

$$x^*(t) = a_0(t) + \sum_{k=1}^{N} A_k \cos(k\Omega t + \phi_k)$$  

(19)

The harmonic amplitude and phase can be obtained by

$$A_k = \sqrt{b_k^2 + c_k^2}, \phi_k = \arctan\left(\frac{c_k}{b_k}\right), k = 1, 2, ..., N$$  

(20)

In the following, a set of parameters as

$$\alpha_1 = 1.0, \alpha_2 = 1.0, \alpha_3 = -1.0, Q_0 = 1.0, Q_1 = 0.5$$
Fig. 1 Harmonic amplitudes and phases of periodic solutions based on ten harmonic terms (HB10). (a) constant term $a_0$, (b) first harmonic amplitude $A_1$, (c) second harmonic amplitude $A_2$, (d) tenth amplitude $A_{10}$.

To analyze stability of the dynamical system and precision of analytical solutions, analytical approximate solutions for period-1 motion are based on 2 harmonic balance terms (HB2) and 10 harmonic balance terms (HB10). We will discuss the stability problem with HB10 in this section. In Fig. 1, the frequency-amplitude characteristics are presented for $\Omega \in (0,9.6)$, Stable motion and unstable motion are represented by solid lines and dashed lines respectively. SN represents saddle-node bifurcation. In Fig. 1(a), there are one stable branch and two unstable branches. For $a_0 > 0$, the branch of the period-1 motion are unstable and there is a trend to slow growth with one zoomed window. For $a_0 < 0$, the stable and unstable periodic motions are symmetric about $a_0 = -0.2306$ and the saddle-node bifurcation of periodic motions is at $\Omega = 2.5272$. In the bottom branch, the periodic motion is unstable. One unstable period-1 motion exists in $(0,\infty)$ and the other unstable and stable periodic motions are in $\Omega \in (2.5272,\infty)$. Amplitude of harmonic oscillations $A_1$ is presented in Fig. 1(b) The range of harmonic amplitude is $A_1 \in (0.0186,0.4279)$ for such a frequency range. When saddle-node bifurcation occurs, the quantity level is $A_1 = 0.4279$. Harmonic amplitude $A_2$ is presented in Fig. 1(c) The range of harmonic amplitude is $A_2 \in (2.5230e^{-4},0.0383)$. For $\Omega > 2.5272$, the corresponding quantity level of harmonic amplitude is in $A_2 = 2.523e^{-4}$. For the purpose to solve
the approximate solutions accurately, the highest order amplitude is \( A_0 \) which is computed to keep the accuracy below \( A_0 < 4.1905e^{-8} \) in Fig.1(d). Based on eigenvalue analysis, the corresponding stability classification is tabulated in Table1.

Table1. Stability classification of periodic motions based on HB10

| Type of eigenvalues   | Excitation frequency | Stability |
|-----------------------|----------------------|-----------|
| (3,0,0|18,0,0)            | (0,0.52)            | Unstable  |
| (1,0,0|20,0,0)            | (0.52,9.6),(2.52,9.6) | Unstable  |
| (0,1,0|20,0,0)            | (2.52,9.6)          | Stable    |

It is easy to see that the approximate steady-state solution is unstable if \( \Omega \in (0.00,0.52) \cup (0.52,9.60) \cup (2.52,9.60) \), it is stable if \( \Omega \in (0.00,0.52) \).

4. Numerical Illustration

In this section, numerical illustrations of analytical solutions compared to numerical solutions will be given. The circle and solid lines respectively represent the Analytical solutions and numerical solutions. The initial conditions for numerical solutions are calculated by summation of 2 harmonic terms and 10 harmonic terms, and depicted by solid red circle. The comparison trajectories of analytical and numerical solutions under the same initial conditions are given in Figs.2(a)(c)(e). In Figs.2(a)(b), \( a_0 = A_0 = 1.777 \), the trajectory of solid lines and circle shows that the approximate solutions is not good for overlapping the numerical steady-state solutions. 10 harmonic terms are discussed in order to reduce the approximate error in Fig.2(c)(d), \( a_0 = A_0 = 0.0128 \). The main harmonic amplitudes are \( A_1 \approx 0.1508 \), \( A_2 \approx 3.102e^{-3} \), \( A_3 \approx 8.880e^{-5} \), \( A_4 \approx 3.231e^{-6} \), \( A_5 \approx 1.143e^{-7} \). From Figs.2(a)(c), the approximate error of Fig.2(c) are less than that of Fig.2(a) with the same parameters. From Fig.2(d)(f), it is easy to obtain that, the amplitudes of the harmonic terms are decreased exponentially with the increasing of harmonic terms.
5. Conclusions
In this paper, analytical solutions of period-1 motion in 1-D nonlinear dynamical system are discussed via the generalized harmonic method. There are two periodical excitations in this system. The frequency-amplitude characteristics was discussed with 10 harmonic balance terms. We study the stability of this system through the eigenvalue analysis at the equilibrium point. Compared with analytical approximate and numerical solutions different harmonic terms, the results show that they are in a good agreement. The structural parameters that make the system stable are determined by studying the stability interval of the system. Moreover, the system stability problem can be solved when designing structural parameters.

Acknowledgments
This work is partially supported by Outstanding Youth Science and Technology Innovation Team Program of Education Department of Hubei Province (T201817), Major Project of Scientific and Technological Innovation in Hubei(2018ABA076,2019AAA057), 2019 Science and Technology Plan Project of Hubei Province (the second batch) (2019BEC206)

References
[1] Lagrange J L. (1788) Mécanique Analytique , edition Albert Blanchard.
[2] Poincaré H.(1899) Les méthodes nouvelles de la mécanique céleste. Gauthier-Villars et fils.
[3] Nayfeh A H.(1973) Perturbation Methods Wiley Interscience. New York-London-Sydney.
[4] Hayashi, C.(1989) Nonlinear Oscillators in Physical Systems. Journal of Vibration and Acoustics.
[5] Luo A C J, Han R P S.(1997) A quantitative stability and bifurcation analyses of the generalized duffing oscillator with strong nonlinearity. Journal of the Franklin Institute, 334(3): 447-459.

[6] Peng Z K, Lang Z Q, Billings S A, et al.(2008) Comparisons between harmonic balance and nonlinear output frequency response function in nonlinear system analysis. Journal of Sound and Vibration, 311(1-2): 56-73.

[7] Luo A C J, Huang J.(2012) Approximate solutions of periodic motions in nonlinear systems via a generalized harmonic balance. Journal of Vibration and Control, 18(11):1661-1674.

[8] Luo A C J.(2012) Continuous dynamical systems. L & H Scientific Pub. and Higher Education Press Limited.

[9] Luo A C J, Yu B.(2013) Analytical solutions for stable and unstable period-1 motions in a periodically forced oscillator with quadratic nonlinearity. Journal of Vibration and Acoustics, 135(3).

[10] Zhou G, Luo A C J, Zhou N, et al.(2016) Analytical Solutions of a First-Order Quadratic Nonlinear System With Parametrical and Periodical Excitation. ASME 2016 International Mechanical Engineering Congress and Exposition. American Society of Mechanical Engineers Digital Collection.

[11] Ying J, Jiao y. et al.(2017) Further analytical solutions for periodic motion in the duffing oscillator. International Journal of Dynamics and Control, 947-964.

[12] Chen X, Zhou G.(2018) An approximate solution for period-1 motions in a periodically forced oscillator with quadratic nonlinearity. 2018 Chinese Control And Decision Conference (CCDC). IEEE : 5372-5377.