ON MODULI SPACES OF RICCI SOLITONS

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Abstract. We study deformations of shrinking Ricci solitons on a compact manifold $M$, generalising the classical theory of deformations of Einstein metrics. Using appropriate notions of twisted slices $S_f$ inside the space of all Riemannian metrics on $M$, we define the infinitesimal solitonic deformations and the local solitonic pre-moduli spaces. We prove the existence of a finite dimensional submanifold of $S_f \times C^\infty(M)$, which contains the pre-moduli space of solitons around a fixed shrinking Ricci soliton as an analytic subset. We define solitonic rigidity and give criteria which imply it.

1. Introduction

In this paper we study Ricci solitons on manifolds setting up the theory of their deformations. We recall that a Ricci soliton on a manifold $M$ is a complete Riemannian metric $g$, with Ricci tensor satisfying the equation $\text{Ric} + \frac{1}{2} \mathcal{L}_X g - cg = 0$ for some complete vector field $X$ and a constant $c$. Solitons are remarkably interesting metrics as they evolve in a particularly easy way under the Ricci flow, namely by diffeomorphisms and homotheties, and they appear as natural generalisations of Einstein metrics, which are the Ricci solitons with $X = 0$. The reader is referred to [4, 5, 11] for a detailed exposition of the theory of Ricci solitons.

It is known that on a compact manifold $M$ every Ricci soliton is of gradient type, i.e., with associated vector field equal to the gradient $X = \nabla f$ of a smooth function $f$. It is then natural to consider the space

$$\mathcal{P} = \left\{ (g, f) \in \mathcal{M} \times C^\infty(M) \mid \int_M e^{-f} \mu_g = (2\pi)^{\frac{n}{2}} \right\},$$

where $\mathcal{M}$ is the space of all Riemannian metrics on the $n$-dimensional manifold $M$, and to define the set of normalised Ricci solitons $\text{Sol}$ as the set of all pairs $(g, f) \in \mathcal{P}$ satisfying the normalised equation

$$\text{Ric} + \text{Hess}(f) - g = 0.$$

We remark that $\text{Sol}$ coincides with the set of all critical points in $\mathcal{P}$ of Perelman’s entropy functional $W(g, f)$ or, equivalently, with the zero set of the corresponding Euler-Lagrangian operator $\mathcal{S} = (S_1, S_2) : \mathcal{P} \rightarrow C^\infty(S^2(T^*M)) \times C^\infty(M)$ (see (2.3)). If we compare all this to the classical

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theory of Einstein metrics (which can be considered as the critical points of the total scalar curvature $H(g)$, in the space $\mathcal{M}_1$ of Riemannian metrics of unitary volume) we have that Perelman’s functional $W(g, f)$ and the equations $S_1(g, f) = S_2(g, f) = 0$ are non-trivial analogues of $H(g)$ and Einstein equation, respectively, that are well suited to our purposes.

The aim of this paper is to study the moduli space $\mathcal{S}ol/\mathcal{D}$ of solitons w.r.t. the natural action of the set $\mathcal{D}$ of all diffeomorphisms of $M$, around a fixed normalised Ricci soliton $(g_o, f_o)$. In order to do this, we first define and prove the existence of modified Ebin slices, called $f$-twisted slices $S_f$, for the $\mathcal{D}$-action on the space $\mathcal{M}$. We then define the solitonic pre-moduli space at $(g_o, f_o)$ as the intersection of $\mathcal{S}ol$ with the set $S_{f_o} \times C^\infty(M)$. Such solitonic pre-moduli space is obviously acted on by the isometry group $\text{Iso}(g_o)$ and the orbit space of this action is a local model for the moduli space $\mathcal{S}ol/\mathcal{D}$ around the class of $(g_o, f_o)$.

The space of infinitesimal solitonic deformations is then defined as the linear subspace of the tangent space $T_{(g_o, f_o)}\mathcal{P}$ given by the kernel of the differential of $S$ at $(g_o, f_o)$. An element $(h, a) \in T_{(g_o, f_o)}\mathcal{P} \simeq C^\infty(S^2(T^*M)) \times C^\infty(M)$ in such a linear subspace is called essential if the tensor field $h$ belongs to the tangent space of the $f_o$-twisted slice $S_{f_o}$ at $g_o$. Our first main result states that the space of all essential infinitesimal solitonic deformations is finite dimensional.

In order to provide a description of the solitonic pre-moduli space, we work in the Hilbert manifold $\mathcal{P}^s \ (s \geq [n/2] + 3)$ given by

$$\mathcal{P}^s = \left\{ (g, f) \in \mathcal{M}^s \times H^s(M) \mid \int_M e^{-f} \mu_g = (2\pi)^{\frac{n}{2}} \right\},$$

where $H^s(M)$ is the Sobolev space of order $s$ and $\mathcal{M}^s$ denotes the space of all Riemannian metrics which are $H^s$-sections of the bundle $S^2(T^*M)$. This setting is convenient because it allows the use of the Implicit Function Theorem and brings to our main result, which consists in a local description of the pre-moduli space as an analytic subset of a finite dimensional analytic submanifold $\mathcal{Z}^s \subset \mathcal{P}^s$ through $(g_o, f_o)$, whose tangent space is given by the set of essential infinitesimal solitonic deformations.

We then introduce the notion of (weak) solitonic rigidity and provide some criteria that imply it, in particular when the fixed metric $g_o$ is Einstein. We analyse the case of compact symmetric spaces of rank one, showing that the complex projective spaces $\mathbb{C}P^n$ are the only ones with non trivial spaces of essential infinitesimal solitonic deformations. Other general results on solitonic rigidity are given, when the curvature is sufficiently positive or the metric $g_o$ is Kähler.

We conclude observing that these results appear to be the natural analogues of the outcomes of the theory of deformations of Einstein metrics, developed by Koiso ([13, 15, 14, 1]), albeit they are based on various non-trivial modifications of Koiso’s settings and techniques.
The structure of the paper is as follows. In §2, we consider the set of normalised solitons inside $P^s$, define and prove the existence of $f$-twisted slices and give the notion of solitonic pre-moduli space. In §3, we define the space of essential infinitesimal solitonic deformations, we show that it is finite dimensional and we prove our main result, which gives a local description of the solitonic pre-moduli space. In §4, we introduce the notion of the solitonic rigidity and study the rigidity of the standard metrics of a compact symmetric spaces of rank one, showing that the space of essential infinitesimal solitonic deformations of $\mathbb{C}P^n$ is non trivial. We also treat the solitonic rigidity of Einstein metrics with certain conditions on the curvature. In §5, we define the weak solitonic rigidity of Einstein metrics and obtain some related results under conditions on the diameter of the manifold or in the Kähler situation.

**Notation.**
Throughout the paper, $M$ is a compact $n$-dimensional manifold, $\mathcal{F}(M) = C^\infty (M, \mathbb{R})$ is the space of $C^\infty$ real functions, $\mathcal{D}$ is the set of all diffeomorphisms from $M$ into itself and $\mathcal{M}$ is the space of all Riemannian metrics on $M$, i.e., the cone of smooth sections of $S^2(T^*M)$ that determine positive definite inner products on each tangent space of $M$.

For a fixed Riemannian metric $g$ on $M$, we denote by $D$ its Levi Civita connection and by $R$ its Riemann curvature tensor, defined by $R_{XY} = D_X[Y, Z] - D_Y[X, Z]$ for any pair of vector fields $X, Y$. The Ricci tensor, the scalar curvature and the volume form of $g$ are denoted by $\text{Ric}$, $s$ and $\mu$, respectively. When it is necessary to specify the dependence on the given metric $g$, a superscript or subscript “$g$” is sometimes added.

We denote by $H^s(M)$ the Sobolev space of real functions with partial derivatives in $L^2(M)$ up to order $s$. Similarly, if $\pi : E \rightarrow M$ is a vector bundle, $H^s(E)$ is the space of sections of $E$ with square integrable partial derivatives up to order $s$, so that $C^\infty(E) = \bigcup_{s \geq 0} H^s(E)$ is the space of smooth sections of $E$.

Recall that, by Sobolev embedding’s Theorem, $H^s$-differentiability implies $C^{\frac{n}{2}-1}$-differentiability. This allows to consider for any $s > \frac{n}{2}$ the collection $\mathcal{M}^s$ of $C^0$ Riemannian metrics that are in $H^s(S^2(T^*M))$, and the group $\mathcal{D}^{s+1}$ of $H^{s+1}$-diffeomorphisms of $M$, i.e. of $C^1$-diffeomorphisms $\varphi$, with coordinate expressions of $\varphi$ and $\varphi^{-1}$ both in $H^{s+1}$. As it is shown in [9, 10], $\mathcal{M}^s$ and $\mathcal{D}^{s+1}$ are Hilbert manifolds that naturally include $\mathcal{M}$ and $\mathcal{D}$, respectively, and $\mathcal{D}^{s+1}$ is a topological group acting on $\mathcal{M}^s$ via a right action, which extends the standard right action of $D$ on $\mathcal{M}$, namely $\varphi \ast g := \varphi^*(g)$ with $\varphi \in \mathcal{D}^{s+1}$ and $g \in \mathcal{M}^s$.

We finally recall that the tangent spaces $T_g\mathcal{M}^s$ and $T_\varphi\mathcal{D}^{s+1}$ of $\mathcal{M}^s$ and $\mathcal{D}^{s+1}$, respectively, are naturally identifiable with the Hilbert spaces

$$T_g\mathcal{M}^s \simeq H^s(S^2(T^*M)) \ , \quad T_\varphi\mathcal{D}^{s+1} \simeq H^{s+1}(TM)$$

(1.1)
and that the scalar products
\[ < \cdot , \cdot >_g : T_g \mathcal{M}^s \times T_g \mathcal{M}^s \longrightarrow \mathbb{R} , \ < h , k >_g := \int_M g(h , k) \mu_g \] determine a \( D^{s+1} \)-invariant, smooth (weak) Riemannian structure on \( \mathcal{M}^s \).

2. Moduli and pre-moduli spaces of Ricci solitons

2.1. Normalised Ricci solitons and Perelman’s entropy functional.

Let \( M \) be an \( n \)-dimensional compact manifold. We recall that a Riemannian metric \( g \) on \( M \) is called (gradient) Ricci soliton if there exist \( f \in \mathcal{F}(M) \) and a constant \( c \in \mathbb{R} \) so that
\[
\text{Ric} + D_g df = c \cdot g
\] (2.1)
If \( g \) is a Ricci soliton, there exists a unique \( c \) and a unique \( f \), determined up to a constant, satisfying (2.1). The Einstein metrics are Ricci solitons of a special kind, with \( f \) constant. The non-Einstein Ricci solitons are called non-trivial.

It is known that the non-trivial Ricci solitons on compact manifolds are necessarily shrinking, i.e. with \( c > 0 \) (see e.g. [11]), so that any Ricci soliton on \( M \), which is not Ricci flat, can be rescaled to have (2.1) satisfied with the constant \( c = 1 \).

In particular, if we consider the set of pairs
\[
\mathcal{P} := \left\{ (g,f) \in \mathcal{M} \times \mathcal{F}(M) : \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M e^{-f} \mu_g = 1 \right\},
\]
we have that the equivalence classes up to homotheties of the Ricci solitons that are not Ricci flat, are in one-to-one correspondence with the subset of \( \mathcal{P} \) defined by
\[
\text{Sol} := \{ (g,f) \in \mathcal{P} : \text{Ric}_g + D^g df = 0 \}.
\]
We call the elements of this space normalised Ricci solitons.

Consider the map \( W : \mathcal{P} \longrightarrow \mathbb{R} \), given by the restriction to triples \( (g,f,\frac{1}{2}) \in \mathcal{P} \times \{ (\frac{1}{2}) \} \) of Perelman’s entropy functional (21), i.e.
\[
W(g,f) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_M \left( \frac{1}{2} |\nabla f|_g^2 + \frac{1}{2} s_g + f - n \right) e^{-f} \mu_g .
\] (2.2)
The Euler-Lagrange equations for its critical points are (see e.g. [17, 5])
\[
\begin{cases}
\text{Ric}_g + D^g df - g = 0 , \\
\Delta_g f - \frac{1}{2} |\nabla f|_g^2 + \frac{1}{2} s_g + f - n - W(g,f) = 0 ,
\end{cases}
\] (2.3)
so that any such critical point is in \( \text{Sol} \) (it satisfies the first of (2.3)). Conversely, if \( (g,f) \in \text{Sol} \), a well known argument (see e.g. [11]) shows that the function \( \Delta_g f - \frac{1}{2} |\nabla f|_g^2 + \frac{1}{2} s_g + f \) is equal to a constant, which, by integration,
is directly seen to be \( n + W(g, f) \). This means that the following conditions are equivalent:

a) \((g, f) \in \text{Sol}\);

b) \((g, f)\) is a solution of \((2.3)\);

c) \((g, f)\) is a critical point of Perelman’s entropy functional \((2.2)\),

and indicates a useful parallelism between Ricci solitons and Einstein metrics. In the following, we denote by \( S \) the Euler-Lagrange operator

\[
S = (S_1, S_2) : \mathcal{P} \longrightarrow C^\infty(S^2T^*M) \times \mathcal{F}(M),
\]

\[
S(g, f) = \left( \text{Ric}_g + D^g df - g, \Delta_g f - \frac{1}{2} |\nabla f|^2 + \frac{1}{2} s_g + f - n - W(g, f) \right),
\]

so that \( \text{Sol} = S^{-1}(0, 0) \).

2.2. Slices and pre-moduli spaces of Ricci solitons.

Let \( g \in \mathcal{M} \subset \mathcal{M}^s \) be a \( C^\infty \) Riemannian metric and \( \text{Iso}(g) \subset \mathcal{D} \) its isometry group. We recall that Ebin’s Slice Theorem \([9, 10]\) states that, for any \( s \geq \left[ \frac{n}{2} \right] + 3 \), there exist

- a \( \mathcal{D}^{s+1}\)-invariant neighbourhood \( \mathcal{U}^s \subset \mathcal{M}^s \) of \( g \),
- a Hilbert submanifold \( \mathcal{S}^s \subset \mathcal{U}^s \) through \( g \) (called the Ebin slice),
- a neighbourhood \( \mathcal{V}^{s+1} \subset \mathcal{D}^{s+1} / \text{Iso}(g) \) of the coset \( o = \text{Iso}(g) \) and a local cross section \( \chi : \mathcal{V}^{s+1} \longrightarrow \mathcal{D}^{s+1} \),

such that

1) \( \mathcal{S}^s \) is \( \text{Iso}(g) \)-invariant and for \( \varphi \in \mathcal{D}^{s+1} \)

\[
\mathcal{S}^s \cap (\varphi \ast g) \neq \emptyset \iff \varphi \in \text{Iso}(g);
\]

2) the map

\[
L : \mathcal{V}^{s+1} \times \mathcal{S}^s \longrightarrow \mathcal{M}^s, \quad L(u, h) = \chi(u) \ast h
\]

determines a homeomorphism between \( \mathcal{S}^s \times \mathcal{V}^{s+1} \) and \( \mathcal{U}^s \).

This implies that the space \( \mathcal{M}^s / \mathcal{D}^{s+1} \) of isometry classes in \( \mathcal{M}^s \) has an open neighbourhood \( \mathcal{U}^s = \mathcal{U}^s / \mathcal{D}^{s+1} \) of \([g] = \mathcal{D}^{s+1} \ast g\), which is naturally identifiable with the quotient \( \mathcal{S}^s / \text{Iso}(g) \).

We remark that the Ebin slice \( \mathcal{S}^s \) is in fact the image under the exponential map \( \exp : T_g \mathcal{M}^s \longrightarrow \mathcal{M}^s \) of an open subset of a subspace \( V^s \subset T_g \mathcal{M}^s \), which is complementary to the tangent space \( T_g(D^{s+1} \ast g) \). Such complementary subspace can be shortly described as follows. Consider the operators

\[
\alpha_g : H^{s+1}(TM) \longrightarrow H^s(S^2T^*M), \quad \alpha_g(X) = \mathcal{L}_X g,
\]

\[
\delta_g : H^s(S^2T^*M) \rightarrow H^{s-1}(T^*M) \simeq H^{s-1}(TM), \quad \delta_g h_x := (D_{e_i} h)_x(e_i,)
\]

(2.5)
where $\mathcal{L}_X$ is the Lie derivative along the vector field $X$ and $(e_i)$ is an arbitrary $g_x$-orthonormal basis of $T_xM$, $x \in M$. One can check that $\alpha_g$ has injective symbol and that $-\delta_g$ is the adjoint $\alpha^*_g$ of $\alpha_g$ w.r.t. $(\ref{eq:innerproduct})$, i.e.

$$<\alpha_g(X), k>_g = \int_M g(\mathcal{L}_X g, k) \mu_g = -\int_M \delta_g k(X) \mu_g = <X, \alpha^*_g(k)>_g .$$

From this, one gets the $<,>_g$-orthogonal decomposition $(\ref{eq:orthogonal})$

$$H^s(S^2T^*M) = \text{Im} \alpha_g|_{H^{s+1}(TM)} \oplus \ker \delta_g|_{H^s(S^2T^*M)} .$$

Under the natural identification $T_g\mathcal{M}^s = H^s(S^2T^*M)$, the subspace $T_g(D^{s+1} * g) \subset T_g\mathcal{M}^s$ corresponds to $\text{Im} \alpha_g|_{H^{s+1}(TM)}$ and the subspace $V^s$ can be chosen to be $\ker \delta_g|_{H^s(S^2T^*M)}$, which is then the tangent space $T_g\mathcal{S}^s$ at $g$.

Imitating Ebin’s construction, for any Ricci soliton $(g, f) \in \mathcal{P}$ we now want to construct a special submanifold $\mathcal{S}_f^s \subset \mathcal{M}^s$ with the same property of the Ebin slice, but with a tangent space $T_g\mathcal{S}_f^s \subset T_g\mathcal{M}^s$ characterised in a different way, more convenient for our purposes.

Let $g \in \mathcal{M}^s$ and assume that $f \in F(M)$ is a smooth function, which is $\text{Iso}(g)$-invariant (this surely occurs when $(g, f)$ is a Ricci soliton). The corresponding twisted divergence is the operator

$$\delta_{(g, f)} : H^s(S^2T^*M) \to H^{s-1}(T^*M) , \quad \delta_{(g, f)} h := e^f \cdot \delta_g (e^{-f} \cdot h) = \delta_g h - \tau f h .$$

When $g$ is considered as known, we will shortly write $\delta_f$ instead of $\delta_{(g, f)}$.

Notice that, in analogy with $-\delta$, the operator $-\delta_f$ is the formal adjoint of $\alpha_g$ w.r.t. the “twisted” inner product on $T_g\mathcal{M}^s = H^s(S^2T^*M)$

$$<h, k>_{(g, f)} := \int_M g(h, k)e^{-f} \mu_g .$$

We have that

**Proposition 2.1.** For $s \geq \left\lceil \frac{n}{2} \right\rceil + 3$

$$H^s(S^2T^*M) = \alpha_g(H^{s+1}(TM)) \oplus \ker \delta_{(g, f)}|_{H^s(S^2T^*M)}$$

and there exists a submanifold $\mathcal{S}_f^s \subset \mathcal{M}^s$, which satisfies the same conditions (1) and (2) of $\mathcal{S}^s$, but has tangent space $T_g\mathcal{S}_f^s = \ker \delta_{(g, f)}|_{H^s(S^2T^*M)}$.

**Proof.** We follow the line of arguments of $\cite{10}$, Thm.7.1. Since $\alpha$ has injective symbol and $-\delta_{(g, f)}$ is the formal adjoint of $\alpha_g$ w.r.t. the inner product $<,>_g$, one can check that $\delta_{(g, f)} \circ \alpha_g$ is elliptic and infer that $(\ref{eq:orthogonal})$ holds (see e.g. $\cite{10}$, Cor. 6.9).

Consider now the orbit $\mathcal{O}_s := D^{s+1} * g$ in $\mathcal{M}^s$ and the restricted tangent bundle $T(\mathcal{M}^s)|_{\mathcal{O}_s}$ over $\mathcal{O}_s$. For any $g' = \varphi^*(g) \in \mathcal{O}_s$, consider the subspace of $T_{g'}\mathcal{M}^s$ defined by

$$\nu_{g'} = \varphi^*(\ker \delta_{(g, f)}) \subset H^s(S^2T^*M) = T_{g'}\mathcal{M}^s$$
and set $\nu = \bigcup_{g' \in O^s} \nu_{g'}$. Note that the $\nu_{g'}$’s (and hence $\nu$) are well defined, because $f$ is $\text{Iso}(g)$-invariant.

We claim that $\pi : \nu \to O^s$ is a smooth subbundle of $\pi : TM^s|_{O^s} \to O^s$. In fact, one can consider the smooth family of inner products $\langle , \rangle_{(g', f')}$, determined by $g' = \varphi^*(g) \in O^s$ and $f' = \varphi^*(f)$, and the corresponding family of orthogonal projectors

$$P_{g'} : T_{g'}M^s \longrightarrow T_{g'}O^s = \text{Im} \alpha_{g'}, \quad P_{g'} := \alpha \circ (\delta_{f'} \circ \alpha)^{-1} \circ \delta_{f'},$$

where, for shortness, we denote by $\alpha := \alpha_g$ and $\delta_{f'} := \delta_{(g', f')}$. The map $P_{g'}$ is actually well defined because the composition

$$(\delta_{f'} \circ \alpha)^{-1} \circ \delta_{f'}|_{H^s(S^2T^*M)} : H^s(S^2T^*M) \longrightarrow H^{s+1}(TM)$$

is a single valued map: to see this, one needs to observe that $\delta_{f'}(H^s(S^2T^*M)) = (\delta_{f'} \circ \alpha)(H^{s+1}(TM))$ and that the self-adjoint elliptic operator $\delta_{f'} \circ \alpha : H^{s+1}(TM) \longrightarrow H^{s-1}(TM)$ induces a bijection

$$\delta_{f'} \circ \alpha|_{(\delta_{f'} \circ \alpha)(H^{s+1}(TM))} : (\delta_{f'} \circ \alpha)(H^{s+1}(TM)) \simto (\delta_{f'} \circ \alpha)(H^{s-1}(TM)).$$

Consider now the projection $P : TM^s|_{O^s} \longrightarrow TO^s$, given at any $g' \in O^s$ by the operator $P_{g'}$ and observe that $\nu = \ker P$. Following exactly the same arguments in [10], p. 31–32, one can check that $P$ is a smooth map and hence that $\pi : \nu = \ker P \to O^s$ is a smooth bundle, as claimed.

Now, if $\exp : TM^s \longrightarrow M^s$ is the exponential map relative to the weak Riemannian metric [12], there exists a neighbourhood $V \subset \nu$ of the zero section, such that $\exp|_V : V \longrightarrow U \subset M^s$ is a diffeomorphism onto a $D^{s+1}$-invariant neighbourhood $U$ of $O^s$. By the same arguments of [9], p. 32–33, the submanifold $S_f^\infty = \exp(V \cap \nu|_g)$ satisfies all required conditions. \[\Box\]

We call $S_f^s$ the $f$-twisted slice at $g$ in $M^s$; when $g$ is smooth, we set $S_f^{\infty} := S_f^s \cap M$. As in [9] Thm. 7.4, one can check that also the set $S_f^{\infty}$ satisfies the conditions (1) and (2) of Ebin slices.

For any $s \geq \left[\frac{n}{2}\right] + 3$, we denote by $\mathcal{P}^s$ the Hilbert manifold

$$\mathcal{P}^s := \left\{ (g, f) \in M^s \times H^s(M) : \frac{1}{(2\pi)^{n/2}} \int_M e^{-f} \mu_g = 1 \right\},$$

which naturally extends $\mathcal{P}$ and we obviously extend (2.1) to the operator

$$S^s : \mathcal{P}^s \longrightarrow H^{s-2}(S^2T^*M) \times H^{s-2}(M).$$

We also set $\text{Sol}^s := (S^s)^{-1}(0)$. Since any Ricci soliton of class $C^2$ is necessarily real analytic (8), by standard arguments one gets that $\text{Sol}^s = \text{Sol}$ for every $s \geq \left[\frac{n}{2}\right] + 3$, in complete analogy with the Einstein case (see e.g. [15], Lemma 2.5).
Let \((g, f) \in \text{Sol}\) and recall that \(f\) is \(\text{Iso}(g)\)-invariant, so that one can consider the \(f\)-twisted slice \(S^s_f \subset M^s\) through \(g\) and the subset \(S^\infty_f = S^s_f \cap M\) of \(M\). In analogy with [15] (see also [1], Ch. 12), we introduce the following:

**Definition 2.2.** A solitonic pre-moduli space at \((g, f)\) is the set 
\[
\mathcal{A}_{(g, f)} := \text{Sol}^s \cap (S^s_f \times H^s(M)) = \text{Sol} \cap (S^\infty_f \times F(M))
\]
where \(S^s_f\) is a \(f\)-twisted slice through \(g\) and \(s \geq \left\lceil \frac{n}{2} \right\rceil + 3\).

Note that \(\mathcal{A}_{(g, f)}\) is invariant under the natural action of \(\text{Iso}(g)\) on \(P^s\).

Moreover, from the properties of slices, for any \(s \geq \left\lceil \frac{n}{2} \right\rceil + 3\), the quotient \((\text{Sol} \cap U)^s_s / D^{s+1}_s\) of a sufficiently small neighbourhood \((\text{Sol} \cap U)^s_s\) of \((g, f) \in \text{Sol}^s\) can be naturally identified with the quotient \(\mathcal{A}_{(g, f)} / \text{Iso}(g)\) of the corresponding solitonic pre-moduli space \(\mathcal{A}_{(g, f)}\). This means that the local behavior of the moduli space \(\text{Sol} / D\) is determined by the quotients of the pre-moduli spaces by the actions of (finite dimensional) groups of isometries.

### 3. The solitonic pre-moduli spaces are real analytic sets in finite-dimensional manifolds

Consider a \(C^1\)-curve \((g_t, f_t)\) in \(\text{Sol}\) passing through a fixed Ricci soliton \((g_0, f_0) = (g, f)\). The tangent vector \((h = g|_{t=0}, a = f|_{t=0})\) is necessarily in the kernel of the linearized operator \(dS_{(g, f)}\) of the operator (2.1). If furthermore \((g_t, f_t)\) takes values in a pre-moduli space \(\mathcal{A}_{(g, f)}\), its tangent vector \((h, a)\) satisfies the additional condition \(\delta_{(g, f)} h = 0\). These observations lead to the following definition.

**Definition 3.1.** For a given \((g, f) \in \text{Sol}\), the elements of the subspace 
\[
\ker dS_{(g, f)} \subset T_{(g, f)}P
\]
are called infinitesimal solitonic deformations (shortly, i.s.d.) of \((g, f)\). The i.s.d.’s \((h, a)\) such that \(\delta_{(g, f)} h = 0\) are called essential.

Let us denote by \(Z_{(g, f)}\) the space of the essential i.s.d.’s at \((g, f)\) and set 
\[
Z^s_{(g, f)} = \ker dS^s_{(g, f)} \cap (T_{(g, f)}S^s_f \times H^s(M)) .
\]
We will shortly see that \(Z^s_{(g, f)}\) is actually identifiable with \(Z_{(g, f)}\).

In the next lemma, we give the explicit expression for the linearization \(dS_{(g, f)}\) of the operator \(S = (S_1, S_2)\) at a normalised Ricci soliton \((g, f)\). In the following formulas, \(\Delta_f\) is the twisted Laplacian (also called Bakry-Emery Laplacian or Witten Laplacian), acting on symmetric 2-tensors as 
\[
\Delta_f h = \text{Tr}(D^2 h) - D\nabla f h
\]
and \(\mathcal{R} : \mathcal{C}^\infty(S^2T^* M) \longrightarrow \mathcal{C}^\infty(S^2T^* M)\) is the operator 
\[
\mathcal{R}(h)(X, Y) = \text{Tr}(h(R_X Y, \cdot)) .
\]
Lemma 3.2. For any \((h, a) \in T_{(g,f)}P\)
\[
dS_1|_{(g,f)}(h, a) = -\frac{1}{2}\Delta_fh - \mathcal{R}(h) - \frac{1}{2}Dd(\text{Tr}(h) - 2a) + \frac{1}{2}\mathcal{L}_{(\delta_fh)}g \quad , \quad (3.1)
\]
\[
dS_2|_{(g,f)}(h, a) = \Delta_f(\text{Tr}(h) - 2a) + (\text{Tr}(h) - 2a) - \delta_fh(\delta_fh) \quad . \quad (3.2)
\]

Proof. By the classical formulas for the variations of Ricci tensors and Hessians (see e.g. \([1, 25]\)) we have that
\[
2dS_1|_{(g,f)}(h, a) = -\Delta_h - DD\text{tr}(h) + \mathcal{L}_{(\delta_fh)}g - 2\mathcal{R}(h) + \text{Ric} \circ h + h \circ \text{Ric} - 2h + 2Dd\text{a} + D\nabla_fh - [Dh \cdot \nabla f] \quad , \quad (3.3)
\]
where \([Dh \cdot \nabla f] is the symmetric tensor
\[
[Dh \cdot \nabla f](X, Y) = D_Xh(\nabla f, Y) + D_Yh(\nabla f, X)
\]
and \text{Ric} \circ h denotes the \((2,0)\)-tensor, associated by the metric \(g\) to the composition of \text{Ric} and \(h\), viewed as \((1,1)\)-tensors. From Ricci soliton equation, the definition of \(\delta_fh\) and the fact that \(\mathcal{L}_{(\gamma,h)}g = [Dh \cdot \nabla f]\), equation \((3.1)\) follows. Equation \((3.2)\) follows from the computations in \([17]\), p. 3332, together with the fact that \(dW|_{(g,f)} = 0\) because \((g,f) \in P\) is a soliton.\(\square\)

From Lemma 3.2, we directly get the following

Theorem 3.3. For any \((g,f) \in \text{Sol}\), the space \(Z_{(g,f)}\) is given by
\[
Z_{(g,f)} = \left\{(h, a) \in T_{(g,f)}P \mid \delta_fh = 0, \frac{1}{2}\Delta_fh + \mathcal{R}(h) = 0, a = \frac{\text{Tr}(h)}{2}\right\} \quad (3.4)
\]
and it is finite dimensional.

Proof. If \((h, a) \in Z_{(g,f)}\), the function \(u := \text{Tr}(h) - 2a\) satisfies \(\Delta_fu + u = 0\) by \((3.2)\). Since the equation \(\Delta_fu + \lambda u = 0\) admits non-trivial solutions only if \(\lambda > 1\) \((\mathbb{Z})\), \(u = 0\) and, by definitions and Lemma 3.2 \((3.1)\) follows. Being the operator \(\frac{1}{2}\Delta_fh + \mathcal{R}\) on \(C^\infty(S^2T^*M)\) elliptic, \(Z_{(g,f)}\) is finite dimensional.\(\square\)

We remark here that \(Z^o_{(g,f)} = Z_{(g,f)}\). Indeed the same argument as in the proof of Theorem 3.3 shows that \(Z^o_{(g,f)}\) consists of elements \((h, a)\) in \(T_{(g,f)}P^s\) such that \(2a = \text{Tr}(h)\) and \(\frac{1}{2}\Delta_fh + \mathcal{R}(h) = 0\), hence \(h\) and \(a\) are smooth.

The next theorem is an analogue of \([15]\), Thm. 3.1 on Einstein metrics and is a crucial property of solitonic pre-moduli spaces.

Theorem 3.4. Let \((g,f) \in \text{Sol}\) be a normalised Ricci soliton and denote by \(S^*_f\) an \(f\)-twisted slice at \(g\) in \(\mathcal{M}^s\), \(s \geq \left\lfloor \frac{n}{2} \right\rfloor + 3\). Then there exists an open neighbourhood \(\mathcal{U}^*\) of \((g,f)\) in \((S^*_f \times \mathbb{H}^s(M)) \cap P^s\) and a finite dimensional, real analytic submanifold \(Z^* \subset \mathcal{U}^*\) such that:

i) \(T_{(g,f)}Z^* = Z_{(g,f)}\);

ii) the pre-moduli space \(A_{(g,f)} = (S^*_f \cap \mathbb{H}^s(M)) \cap \text{Sol}\) is a real analytic subset of \(Z^*\).
Proof. From definitions, \( A_{(g,f)} = (S^s|_{(S_f^* \times \mathcal{H}^s(M)) \cap \mathcal{P}^s})^{-1}(0,0) \) and if we can show that the image
\[
dS^s|_{(g,f)}(T_{(g,f)}((S_f^* \times \mathcal{H}^s(M)) \cap \mathcal{P}^s))
\]
is a closed subspace of \( H^{s-2}(S^2T^*M) \times H^{s-2}(M) \), the claim follows from the Implicit Function Theorem in Hilbert spaces and the same arguments of the proof of [15], Thm. 3.1. Now, we observe that the tangent space
\[
V^s_{(g,f)} := T_{(g,f)}((S_f^* \times \mathcal{H}^s(M)) \cap \mathcal{P}^s) =
\]
\[
= \left\{ (h, a) \in H^s(S^2T^*M) \times H^s(M) \mid \delta f h = 0, \int_M (\text{Tr}(h) - 2a) e^{-f} \mu_g = 0 \right\}
\]
is identifiable with the vector space
\[
W^s_{(g,f)} := \left\{ (h, u) \in H^s(S^2T^*M) \times H^s(M) \mid \delta f h = 0, \int_M u e^{-f} \mu_g = 0 \right\}.
\]
We also note that
\[
dS^s(V^s_{(g,f)}) = F^s(W^s_{(g,f)})
\]
where \( F^s \) denotes the elliptic differential operator
\[
F^s : H^s(S^2T^*M) \times H^s(M) \rightarrow H^{s-2}(S^2T^*M) \times H^{s-2}(M)
\]
\[
F^s(h, u) = \left( -\frac{1}{2} \Delta_f h - \mathcal{R}(h) - \frac{1}{2} D du, \Delta_f u + u \right).
\]
Hence, the proof reduces to showing that \( F^s(W^s_{(g,f)}) \) is a closed subspace of \( H^{s-2}(S^2T^*M) \times H^{s-2}(M) \).

Consider the linear differential operator
\[
\beta_g : H^s(S^2T^*M) \times H^s(M) \rightarrow H^{s-1}(T^*M), \quad \beta_g(h, u) := \delta f h - \frac{1}{2} du.
\]

Lemma 3.5. For every \( g \in \mathcal{M}^s \) and \( f \in H^s(M) \), with \( s \geq \left[ \frac{n}{2} \right] + 3 \),
\[
\delta f(Ric + Ddf - g) = \frac{1}{2} d(2 \Delta f - |\nabla f|^2 + s + 2f). \tag{3.5}
\]

Proof. The claim is a consequence of the contracted second Bianchi identity
\[
2 \delta \text{Ric} = ds \quad \text{and the equalities}
\]
\[
d(||\nabla f|^2) = 2 \delta(Ddf), \quad d\Delta f = \delta(Ddf) - \imath \nabla f \text{Ric}. \quad \square
\]

As a corollary we have that for every \( (g', u) \in \mathcal{P}^s \)
\[
\beta_g \circ S^s(g', u) = 0. \tag{3.6}
\]
So, taking the differential of (3.6) at \( (g, f) \) and by the fact that \( S(g, f) = 0 \),
\[
\beta_g \circ dS^s|_{(g,f)}(h, a) = 0 \quad \text{for any } (h, a) \in T_{(g,f)}\mathcal{P}^s. \tag{3.7}
\]
Using Lemma 3.2 the equality (3.7) can be rewritten as
\[
\beta_g \circ F^s(h, \text{Tr}(h) - 2a) = -\delta f \left( \frac{1}{2} \mathcal{L}_{(\delta f)^2} g \right) - \frac{1}{2} d\delta f \delta f(h) = G^{s-1}(\delta f h). \tag{3.8}
\]
where we indicate by $G^s$ the elliptic operator

$$G^s : H^s(T^*M) \to H^{s-2}(T^*M), \quad G^s(\omega) = -\frac{1}{2}(\delta f(L_{\omega}g + d\delta f\omega)).$$

Now by (3.8) we have

$$F^s(W_{(g,f)}) \subseteq \ker \beta_g \cap \Im F^s. \quad (3.9)$$

Notice also that $\ker \beta_g \cap \Im F^s$ is a closed subspace, because $F^s$ is elliptic.

On the other hand, given $(k,v) \in \ker \beta_g \cap \Im F^s$, we may write $(k,v) = F^s(h,u)$ for some $(h,u) \in H^s(S^2T^*M) \times H^s(M)$. Using (2.6) we can find some $X \in H^{s+1}(TM)$ such that $h = \alpha(X) + h_1$ with $\alpha(X) := L_X g$ and $\delta f h_1 = 0$. Moreover we can decompose $u = u_o + u_1$, with $u_o \in \mathbb{R}$ and $u_1$ satisfying $\int_M u_1 e^{-f} \mu_g = 0$. Since $(k,v) \in \ker \beta_g$, we have that $G^s(\delta f k) = 0$ and hence $G^s(\delta f (\alpha(X))) = 0$, i.e. $X \in V := \ker (G^s \circ \delta f \circ \alpha) \subset H^{s+1}(TM)$, which is a finite dimensional space being $G^s$ elliptic. Substituting, we get that

$$(k,v) = F^s(h,u) = F^s(h_1,u_1) + F^s(\alpha(X),u_o),$$

which means that

$$\ker \beta_g \cap \Im F^s \subseteq F^s(W_{(g,f)}) + F^s(\alpha(V) \times \mathbb{R}), \quad (3.10)$$

where $F^s(\alpha(V) \times \mathbb{R})$ is a finite dimensional space. From (3.9) and (3.10), it follows that $F^s(W_{(g,f)})$ has finite codimension in the closed subspace $\ker \beta_g \cap \Im F^s$. Since $F^s(W_{(g,f)})$ is the image of a bounded linear operator, a standard argument shows that $F^s(W_{(g,f)})$ is closed (see e.g. [20], p. 119). □

### 4. Solitonic rigidity

In the following, we constantly assume $s \geq \left\lceil \frac{n}{2} \right\rceil + 3$.

**Definition 4.1.** A normalised Ricci soliton $(g,f)$ is said to be *solitonic rigid* or, shortly, *sol-rigid* (resp. *solitonic rigid in $\mathcal{P}^s$*) if there exists a neighbourhood $\mathcal{U} \subset \mathcal{P}$ (resp. $\mathcal{U} \subset \mathcal{P}^s$) of $(g,f)$ such that $\mathcal{U} \cap \text{Sol}$ consists only of the $\mathcal{D}$-orbit (resp. $\mathcal{D}^{s+1}$-orbit) of $(g,f)$.

By a classical result of Palais ([19]; see also [13], p. 53) if two normalised Ricci solitons are in the same $\mathcal{D}^{s+1}$-orbit, they both lie in the same $\mathcal{D}$-orbit. So, if $(g,f)$ is solitonic rigid in $\mathcal{P}^s$, it is automatically sol-rigid.

**Example 4.2.** The constant curvature metrics of the standard sphere $S^n$ and of the real projective space $\mathbb{R}P^n$ are sol-rigid. In fact, by a result of Böhm and Wilking ([22]), the shrinking Ricci solitons with 2-positive curvature have constant sectional curvature, so that any Ricci soliton on $S^n$ or $\mathbb{R}P^n$, which is close to the standard metric $g_o$, is surely isometric to $g_o$.

The next proposition is a direct consequence of the notion of pre-moduli spaces and Theorem 3.4 and gives a useful tool for proving sol-rigidity.
Proposition 4.3. Let \((g, f)\) be a normalised Ricci soliton. If the space \(Z_{(g, f)}\) of essential i.s.d.’s is trivial, the soliton \((g, f)\) is sol-rigid.

Recall now that an Einstein metric \(g\) on \(M\) with Einstein constant \(c = 1\) corresponds to a normalised Ricci soliton \((g, f) \in \text{Sol}\) with \(f\) constant and equal to \(f = \log \left( \frac{\text{vol}(M, g)}{(2\pi)^{n/2}} \right)\). So, the set \(\mathcal{E}\) of all (normalised) Einstein metrics can be identified with the subset of \(\mathcal{P}\) given by

\[
\mathcal{E} := \left\{ \left( g, -\log \left( \frac{\text{vol}(M, g)}{(2\pi)^{n/2}} \right) \right) \in \mathcal{P} \mid \text{Ric}_g - g = 0 \right\} = (4.1)
\]

For any \((g, f) \in \mathcal{E}\), we call space of essential infinitesimal Einstein deformations of \((g, f)\) the subspace of \(T_{(g, f)} \mathcal{P}\)

\[
E_{(g, f)} := Z_{(g, f)} \cap \left\{ (h, a) \in T_{(g, f)} \mathcal{P} \mid da = 0 \right\} .
\]

Note that it can be naturally identified with the spaces of essential EID considered by Koiso (15, Def. 1.4).

From Theorem 3.3, we have that if \((h, a) \in E_{(g, f)}\), then \(\text{Tr}(h)\) is constant and

\[
\text{Tr}(\Delta h + 2\mathcal{R}(h)) = \Delta \text{Tr}(h) + 2 \text{Tr}(h) = 2 \text{Tr}(h) = 0 .
\]

This implies

\[
E_{(g, f)} = \left\{ (h, 0) \in T_{(g, f)} \mathcal{P} \mid \Delta h + 2\mathcal{R}(h) = 0 \text{ and } a = \text{Tr}(h) = 0 \right\} \cong (4.4)
\]

\[
\cong \left\{ h \in C^\infty(S^2 T^* M) \mid \delta_g h = 0, \text{Tr}(h) = 0, \Delta h + 2\mathcal{R}(h) = 0 \right\},
\]

recovering the classical results on deformations of Einstein metrics (see e.g. 2 Lemma 7.1, 15 Lemma 1.5 or 1, Ch. 12).

The following proposition has useful applications

Proposition 4.4. Let \((g, f) \in \mathcal{E}\). If \(2 \not\in \text{Spec}(-\Delta, \mathcal{F}(M))\), then

\[
Z_{(g, f)} = E_{(g, f)} .
\]

Proof. From (4.3) and the hypotheses, for any \((h, a) \in Z_{(g, f)}\), we have \(\text{Tr}(h) = 0\). Since \(E_{(g, f)} = \{(h, a) \in Z_{(g, f)} \mid \text{Tr}(h) = 0\}\), the claim follows.

Next theorem is a remarkable example of how Proposition 4.4 can be used.

Theorem 4.5. Let \(M = G/H\) be a compact rank one symmetric space and \(g_o\) its standard Einstein metric, corresponding to the normalised Ricci soliton \((g_o, f_o = \log \left( \frac{\text{vol}(M, g_o)}{(2\pi)^{n/2}} \right))\).

(a) If \(M \neq \mathbb{C}P^n\), \(n \geq 2\), then \((g_o, f_o)\) is sol-rigid.
b) If \( M = \mathbb{CP}^n = SU_{n+1}/S(U_1 \times U_n) \), \( n \geq 2 \), the space of essential infinitesimal Einstein deformations \( E_{(g_o,f_o)} \) is trivial, while the space of essential infinitesimal solitonic deformations \( Z_{(g_o,f_o)} \) is \( SU_{n+1} \)-equivariantly isomorphic to \( \mathfrak{su}_{n+1} \).

Proof. (a) The sol-rigidity of \( M = S^n \) and \( M = \mathbb{RP}^n \) have been already discussed in Example 4.12. For the cases \( M = \mathbb{HP}^n \) and \( M = \text{Ca} P^2 \), by the results in [13], we know that \( E_{(g_o,f_o)} = 0 \). We claim that \( 2 \notin \text{Spec}(\Delta, \mathcal{F}(M)) \) and this immediately implies (a) by Proposition 4.4 and Theorem 3.4. This claim can be checked using the results in [6], where the spectrum of the Laplacian is computed for every compact rank one symmetric space. How-ever, in that paper, the Laplacian is computed w.r.t. the invariant metric \( \bar{g} \) induced by the Cartan-Killing form of the Lie algebra of isometries. Since the Ricci tensor of such a metric is equal to \( \frac{1}{2} g_o \), the needed check corresponds to verify that \( 2 \notin \text{Spec}(\Delta, \mathcal{C}^\infty(N)) \) for \( N = \mathbb{HP}^n \) or \( \text{Ca} P^2 \). From [6], we have that

\[
\text{Spec}(\Delta, \mathcal{C}^\infty(\mathbb{HP}^n)) = \left\{ \frac{k^2 + k(2n + 1)}{2(n + 2)} \mid k \in \mathbb{N} \right\},
\]

\[
\text{Spec}(\Delta, \mathcal{C}^\infty(\text{Ca} P^2)) = \left\{ \frac{k^2 + 11k}{18} \mid k \in \mathbb{N} \right\}
\]

and the claim follows.

(b) When \( M = \mathbb{CP}^n = SU_{n+1}/S(U_1 \times U_n) \), from [13], we have that \( E_{(g_o,f_o)} = 0 \). It remains to determine \( Z_{(g_o,f_o)} \) and this can be done using standard arguments of Representation Theory and some other results in [13], as follows. Note that the same arguments determine \( Z_{(g_o,f_o)} \) for other Hermitian symmetric spaces.

Let \( G = SU_{n+1}, K = S(U_1 \times U_n) \) and consider the Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \) of \( \mathfrak{g} = \mathfrak{su}_{n+1} \), in which \( \mathfrak{m} \) is naturally identified with the tangent space \( T_{eK} M \) at the origin \( eK \in G/K \). Denote by \( \mathcal{B} \) the Cartan-Killing form of \( \mathfrak{g} \) and by \( C \) the Casimir element (w.r.t. \( \mathcal{B} \)) of the universal enveloping algebra \( U(\mathfrak{g}) \) of \( \mathfrak{g} \). Recall that the \( G \)-invariant Riemannian metric \( \bar{g} \) with \( \bar{g}_{eK} = -B_{|\mathfrak{m} \times \mathfrak{m}} \) is the multiple of the Fubini-Study metric \( g_o \) that satisfies \( \text{Ric} = \frac{1}{2} \bar{g} \).

Since any \( G \)-homogeneous vector bundle \( \pi : E \longrightarrow M \), with fiber \( W = E_{|eK} \), can be naturally identified with the bundle \( \bar{\pi} : G \times K W \longrightarrow G/K \), we have

\[
\mathcal{C}^\infty(S^2 T^* M) \simeq \mathcal{C}^\infty(G \times K S^2(\mathfrak{m}^*)) \simeq \mathcal{C}^\infty(G, S^2(\mathfrak{m}^*))_K,
\]

where

\[
\mathcal{C}^\infty(G, S^2(\mathfrak{m}^*))_K := \{ s : G \longrightarrow S^2(\mathfrak{m}^*) \mid s(xy) = y^{-1} s(x), \ x \in G, \ y \in K \}.
\]

In particular, \( Z_{(g_o,f_o)} \simeq \ker T \), where \( T \) is the operator \( T : \mathcal{C}^\infty(G, S^2(\mathfrak{m}^*))_K \longrightarrow \mathcal{C}^\infty(G, S^2(\mathfrak{m}^*))_K \), corresponding to \( \Delta + 2 \mathcal{R} : \mathcal{C}^\infty(S^2 T^* M) \longrightarrow \mathcal{C}^\infty(S^2 T^* M) \).
Lemma 4.6. The finite-dimensional $G$-module $\ker T$ is such that
\[
(\ker T)^C \cong 2 \cdot \mathfrak{g}^C.
\] (4.5)

Proof. By [13], Prop. 5.3, for any $p \geq 1$, the action of the Casimir element $C \in U(\mathfrak{g})$ on the $G$-module $C^\infty(\mathfrak{g}, \bigotimes^p(\mathfrak{m}^*))_K$ is identifiable with the differential operator $C = -\Delta - 2\mathcal{R} + \frac{2}{p} \text{Id}$. This means that $C|_{\ker T} = \text{Id}_{\ker T}$ and, in particular, that $\ker T$ has no trivial summand. On the other hand, it is known that, for any compact simple Lie algebra $\mathfrak{k}$, the Casimir operator acts as the identity on an irreducible complex $\mathfrak{k}$-module $V$ if and only if $V \cong \mathfrak{k}^C$ (see e.g. [14], p.654). Hence, $(\ker T)^C \cong m \cdot \mathfrak{g}^C$ for some integer $m$.

By Frobenius reciprocity, $m = \dim \text{Hom}_K(\mathfrak{g}^C, S^2(\mathfrak{m}^*)^C)$ and, by standard arguments of Representation Theory, we have $m = 2$. □

Consider now the Hodge-Laplacian eigenspace $F = \{f \in \mathcal{F}(M) | -\Delta f = f\}$ and recall that, by standard facts on compact Kähler-Einstein manifolds (see e.g. [1], Ch. 2), the map $\iota : \mathcal{F}(M) \to \mathcal{X}(M)$, $\iota(f) = J(\nabla f)$, gives a $G$-equivariant isomorphism between $F$ and the Lie algebra $\mathfrak{g} = \mathfrak{su}_{n+1}$ of Killing vector fields of $(M = \mathbb{C}P^n, g_o)$. Consider the maps $\psi_1, \psi_2 : F \to C^\infty(S^2T^*M)$ defined by
\[
\psi_1(f) := f g_o, \quad \psi_2(f) := Ddf
\]
(see also [14], p.659). We want to show that $\ker T = \text{Im} \psi_1 \oplus \text{Im} \psi_2$ and that
\[
Z_{(g_o, f_o)} = \ker T \cap \ker \delta = \left\{ h = Ddf + \frac{1}{2} f g_o \mid f \in F \right\} \cong F = \mathfrak{su}_{n+1},
\]
from which (b) will immediately follow. We first note that $\text{Im} \psi_1 \subset \ker T$ by definitions, while $\text{Im} \psi_2 \subset \ker T$ because of the following argument. Given a local orthonormal frame field $\{e_i\}_{i=1,\ldots,n}$ with $D_{e_i}e_j|_{x_o} = 0$ at a fixed point $x_o \in M$, we have at $x_o$
\[
\Delta(Ddf)_{ij} = D_i D_j D_l D_j f = D_i[D_j D_l D_j f + R_{lijp} D_p f] =
\]
\[
= D_i D_l D_j D_j f + R_{lijp} D_p D_j f + R_{lijp} D_l D_j D_p f + R_{lijp} D_l D_p D_j f + R_{lijp} D_l D_p D_j f =
\]
\[
= D_l[D_j D_i D_j f + R_{lijp} D_p f] + 2R_{lijp} D_l D_p f + \frac{1}{2} D_l D_j f =
\]
\[
= 2R_{lijp} D_l D_p f = -2\mathcal{R}(Ddf)_{ij},
\]
where $R_{ijkl} = g(R_{e_i e_j e_k e_l})$.

Secondly, if $h \in \text{Im} \psi_1 \cap \text{Im} \psi_2$ (i.e. $h = Ddf_1 = f_2 g_o$ for some $f_1, f_2 \in F$), since
\[
(\delta h)_j = D_i D_i D_j f_1 = D_j \Delta f_1 + R_{lijp} D_p f_1 = -D_j f_1 + \frac{1}{2} D_j f_1 = -\frac{1}{2} D_j f_1,
\]
we have that
\[
df_1 = 2\delta f = 2\delta(f_2 g_o) = 2df_2 \quad \text{and} \quad df_1 = d(\Delta f_1) = d(\text{Tr} h) = 2n df_2.
\]
If $n \geq 2$, it follows that $df_1 = df_2 = 0$, so that $f_1 = f_2 = 0$ and $\text{Im} \psi_1 \cap \text{Im} \psi_2 = \{0\}$. From Lemma 4.6 and (4.6), we see that $\ker T = \text{Im} \psi_1 \oplus \text{Im} \psi_2$. 
and that all elements of $\ker T \cap \ker \delta$ are of the form $h = D f + \frac{1}{2} f g_o$ for some $f \in F$. □

**Remark 4.7.** Theorem 3.4 (b) shows that in a neighborhood of the standard metric $g_o$ of $M = \mathbb{C}P^n$, the moduli space $\text{Sol}/\mathcal{D}$ has at most dimension $n = \dim \mathfrak{su}_{n+1}/\mathfrak{SU}_{n+1}$. A detailed study of the solitonic pre-moduli space at the Fubini-Study metric $g_o$ will be the content of some of our future investigations.

The next proposition is another consequence of Proposition 4.4 and shows that Einstein metrics with sufficiently large positive curvatures are sol-rigid.

**Proposition 4.8.** Let $(M,g)$ be an $n$-dimensional Einstein metric with $\text{Ric} = g$. If the sectional curvature $K$ is $\delta$-pinched (i. e. $\delta \cdot K_{\text{max}} \leq K \leq K_{\text{max}}$ for some $\delta \in (0,1]$) such that

$$K_{\text{min}} \geq \frac{1}{n} \quad \text{and} \quad \delta > \frac{n-2}{3n},$$

(4.7) then $(M,g)$ is sol-rigid.

**Proof.** By [22], the first condition implies that $2 \notin \text{Spec}(\Delta, \mathcal{F}(M))$, unless $(M,g)$ is isometric to a sphere. The condition on $\delta$ implies that $(M,g)$ has no non trivial infinitesimal Einstein deformations (see [1], Cor. 12.72, p. 357). By Proposition 4.4, the claim follows. □

5. **Other rigidity properties of Ricci solitons**

**Definition 5.1.** An Einstein metric $g$ is said to be weakly solitonic rigid if there is a neighbourhood $\mathcal{U}$ of $g$ in $\mathcal{M}$ such that every Ricci soliton in $\mathcal{U}$ is Einstein.

The following proposition is a consequence of a deep recent result in [12].

**Proposition 5.2.** Let $g$ be an Einstein metric with Einstein constant $c > 0$ and the diameter $d$. If

$$d \cdot \sqrt{c} < 2(\sqrt{2} - 1)\pi ,$$

(5.1) then $g$ is weakly solitonic rigid.

**Proof.** In [12] it is proved that a shrinking Ricci soliton $(g,f)$, satisfying $\text{Ric} + Df - cg = 0$ for some $c > 0$, is Einstein whenever its diameter $d$ satisfies (5.1). If $D : \mathcal{M} \rightarrow \mathbb{R}$ is the continuous map

$$D(g') = d(g') \cdot \left( \frac{1}{n \cdot \text{Vol}(g')} \int_M s_{g'} \mu_{g'} \right)^{1/2},$$

where $d(g')$ and $\text{Vol}(g')$ are the diameter and the volume of $g'$, respectively, then $\mathcal{U} = D^{-1}((\infty, 2(\sqrt{2} - 1)\pi))$ works in the definition of weak rigidity. □
Consider now the case of a compact Kähler manifold, that is a compact complex manifold \((M,J)\) admitting a \(J\)-Hermitian metric \(g\) with closed Kähler form \(\omega = g(\cdot, J\cdot)\). It is well known that, up to biholomorphisms, there is at most one Kähler Ricci soliton \(g_0\) on \((M,J)\) \((\text{[24]})\). On the other hand, a recent result by Li \((\text{[18]})\) shows that, for a given Kähler Ricci soliton \(g_0\) on \((M,J)\) with \(G = \text{Iso}^0(M,g_0)\), if there exists a smooth family \(\{J_t\}\) of \(G\)-invariant (non biholomorphic) complex structures on \(M\) with \(J_0 = J\), then there is also a family of \(G\)-invariant Ricci solitons \(g_t\) on \(M\), which are Kähler w.r.t. the corresponding complex structures \(J_t\).

The next proposition is a generalization of a result of Koiso \((\text{[15]}, \text{Thm. 10.5})\) and deals with weak solitonic rigidity of Kähler-Einstein metrics.

**Proposition 5.3.** Let \((M,J,g)\) be a \(2n\)-dimensional compact Kähler-Einstein manifold with \(\text{Ric} = g\). Assume that \(2\) is not an eigenvalue of the Hodge Laplacian \(-\Delta\) acting on \(F(M)\) and on the space of forms of type \((1,1)\).

If \(H^2(M,\Theta) = 0\), where \(\Theta\) is the sheaf of germs of holomorphic vector fields, then every Ricci soliton \(g'\) on \(M\), which is sufficiently close to \(g\), is Kähler-Einstein (w.r.t. a suitable complex structure \(J')\).

**Proof.** By \([15]\), Thm. 10.5, if \((M,J,g)\) is a compact Fano Kähler-Einstein manifold, any Einstein metric \(g'\), which is sufficiently close to \(g\), is Kähler with respect to some complex structure \(J'\), provided the following conditions are satisfied:

a) the complex structure \(J\) belongs to a non-singular complete family of complex structures,

b) there are no non-trivial holomorphic vector fields on \(M\) and

c) any essential infinitesimal Einstein deformation, which is \(J\)-Hermitian, is necessarily trivial.

If this is the case, denoting by \((g,f = \text{const.}) \in P^s, s \geq \lfloor n/2 \rfloor + 3\), the pair in \(P^s\) corresponding to \(g\), the proof of \([15]\), Thm. 10.5, shows that the set of Einstein metric belonging to a sufficiently small Ebin’s slice \(S_{(g,f)} \subset M^s\) fills a real analytic submanifold \(E^s \subset S_{(g,f)}\) with \(E_{(g,f)} = T_{(g,f)}E^s\).

We now observe that \(H^2(M,\Theta) = 0\) implies (a), while the assumption \(2 \not\in \text{Spec}(-\Delta,\mathcal{F}(M))\) implies (b) and the equality \(Z_{(g,f)} = E_{(g,f)}\), which are consequences of the Lichnerowicz Theorem \([1], \text{p. 90})\) and Proposition 4.4, respectively. Moreover, if \(h\) is an essential infinitesimal Einstein deformation (hence, \(\delta h = 0\) and \(\Delta h + 2R(h) = 0\)), the \((1,1)\)-form \(\psi := h \circ J\) is coclosed and is such that \(\Delta_H \psi = 2\psi\), where \(\Delta_H\) is the Hodge Laplacian of \((M,J,g)\) (see e.g. \([1], \text{p. 362}\)). Hence, if \(2 \not\in \text{Spec}(-\Delta,\Omega^{1,1}(M))\), also (c) is satisfied and the quoted result by Koiso applies.

Moreover, by Theorem 3.4, the real analytic submanifold \(E^s\) is contained in \(Z^s\) and, since \(T_{(g,f)}Z^s = Z_{(g,f)} = E_{(g,f)} = T_{(g,f)}E^s\), it follows that \(E^s \cap U = Z^s \cap U\) for any sufficiently small neighbourhood \(U\) of \((g,f)\). This implies the claim. \(\square\)
Remark 5.4. The blow-up of the complex projective plane at $\nu$ generic points, with $\nu \geq 5$, is a compact Kähler Einstein manifold ([23]) with no non trivial holomorphic vector fields and $H^2(M, \Theta) = 0$ (see e.g. [16]). We do not know whether it also satisfies the above condition on the spectra of the Laplacian.

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