Half-quantized helical hinge currents in axion insulators

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ABSTRACT

Fractional quantization can emerge in noncorrelated systems due to the parity anomaly, while its condensed matter realization is a challenging problem. We propose that in axion insulators (AIs), parity anomaly manifests a unique fractional boundary excitation: the half-quantized helical hinge currents. These helical hinge currents microscopically originate from the lateral Goos-Hänchen (GH) shift of massless side-surface Dirac electrons that are totally reflected from the hinges. Meanwhile, due to the presence of the massive top and bottom surfaces of the AI, the helical current induced by the GH shift is half-quantized. The semiclassical wave packet analysis uncovers that the hinge current has a topological origin and its half-quantization is robust to parameter variations. Lastly, we propose an experimentally feasible six-terminal device to identify the half-quantized hinge channels by measuring the nonreciprocal conductances. Our results advance the realization of the half-quantization and topological magnetoelectric responses in AIs.

Keywords: axion insulator, parity anomaly, half-quantization, Goos-Hänchen effect, topological magnetoelectric effect

INTRODUCTION

Fractional quantization in condensed matter materials is usually accompanied by the emergence of quasi-particles driven by strong correlations. A prominent example is the fractional quantum Hall effect [1–3], which is the fairyland of fractionally charged quasi-particle excitations, and has attracted intense attention in the condensed matter community since its discovery. Interestingly, fractional quantization can also emerge in noncorrelated systems. Such a fractional quantization is triggered by parity anomaly [4–6], which generates a parity-violating current with the half-integer-quantized Hall conductance. However, realizing the parity anomaly in (2+1) dimensions and observing the half-quantized transport signals in condensed matter systems have been challenging problems for over 40 years [7–11]. A critical issue is that the bulk-boundary correspondence principle implies that some kind of half-quantized boundary excitations should exist in the parity anomaly systems. Unfortunately, the physical picture of such excitations, their material realization and the roadmap of their experimental characterization are elusive.

Encouragingly, the axion insulator (AI) [12–15] provides an ideal platform to realize parity anomaly on its top and bottom surfaces [16–18]. As a nontrivial topological phase, the AI manifests a unique topological magnetoelectric (TME) effect [15,19–24] and has stimulated extensive research interests [25–48]. Experimentally, the zero Hall conductance plateau has been observed in magnetic topological insulator (TI) heterostructures [28–30] and the antiferromagnetic TI MnBi₂Te₄ [33]. However, the zero Hall plateau cannot provide smoking-gun evidence of the AI, and, more importantly, parity-anomaly-induced half-quantized signals in AIs have not been observed. Such a dilemma originates from shallow knowledge of the boundary excitations in AIs. To give a definitive answer, a concrete and in-depth understanding of the parity-anomaly-induced boundary excitations in the AI is highly desirable.

In this work, we find that parity anomaly in AIs induces a unique boundary excitation: the half-quantized helical hinge currents. Based on the semiclassical wave packet dynamics, we establish the microscopic picture of these hinge currents. We proposed that the massless Dirac electrons on the side surfaces of the AI undergo a lateral Goos-Hänchen (GH) shift when they are reflected from the massive top or bottom surface (see Fig. 1), reminiscent of the GH shift [49–53] of the totally
reflected light beam. Moreover, due to the breaking of time-reversal \((T)\) symmetry, the GH shift favors a specific direction on the hinge. Consequently, helical GH shift currents accumulate on the hinges of the AI when side-surface electrons bounce back and forth between the gapped top and bottom surfaces (see Fig. 1(a) and (b)), which is distinct from the chiral net currents on the edge of the Chern insulator (CI) (see Fig. 1(c)). Interestingly, we find that the differential GH shift current \(\delta I_{\text{GH}}\) is exactly half-quantized with respect to the differential Fermi energy \(\delta E_F\), i.e. \(\delta I_{\text{GH}} = e\delta E_F/2h\). We demonstrate that the GH shift current originates from the nonvanishing Berry curvature during the scattering process, and its half-quantization is robust to the variation of parameters. Then, we numerically verify the half-quantized helical hinge currents through a three-dimensional (3D) lattice model. Finally, we propose a six-terminal device to identify the half-quantized hinge currents in the AI by measuring the nonreciprocal conductances.

MODEL HAMILTONIAN AND THE GOOS-HÄNCHEN SHIFT

To begin with, we model the AI by a 3D magnetic TI, of which the top and bottom surfaces are gapped oppositely while the side surfaces remain gapless. Such a model is in accordance with the experimental setups utilized to realize the AI state in magnetic TI heterostructures or the antiferromagnetic TI MnBi\(_2\)Te\(_4\) [28–30,33]. According to the bulk-boundary correspondence theorem, the top/bottom surface and the side surface of the AI can be described by the 2D effective Hamiltonians

\[
\mathcal{H} = \begin{cases} 
\hbar v_F(-i\sigma_x\partial_x - i\sigma_y\partial_y) - U \quad \text{(side)}, \\
\hbar v_F(-i\sigma_x\partial_x - i\sigma_y\partial_y) + m\sigma_z \quad \text{(top/bottom)}. 
\end{cases}
\]

Here, \(v_F\) denotes the Fermi velocity, \(U\) is the gate potential and \(m\) is the mass term induced by the magnetization. In the following discussions, we unfold the top, side and bottom surfaces of the AI (see Fig. 1(a)) into the \(x-y\) plane (see Fig. 1(b)).

We use the probability flux method (see [51,52] and the online supplementary material) to calculate the GH shift on the hinge of the AI. As shown in Fig. 2(a), we place the massless side surface in \(x \leq 0\) and the massive top/bottom surface in \(x > 0\). Firstly, we solve the scattering problem by matching \(\psi_{\text{in}}(r) = \psi_{\text{in}} + r\psi_{\text{re}}(x \leq 0)\) and \(\psi_{\text{eva}}(x > 0)\) at \(x = 0\), where \(\psi_{\text{in}}(r)\) is the interference superposition of the incident plane wave \(\psi_{\text{in}}(r) = e^{ik_x x + i k_y y} \left[ e^{-i\pi/2}, e^{i\pi/2} \right]^T / \sqrt{2}\) and reflected plane wave \(\psi_{\text{re}}(r) = e^{-ik_x x + i k_y y} \left[ e^{-i(\pi - \alpha)/2}, e^{i(\pi - \alpha)/2} \right]^T / \sqrt{2}\) in the massless region. Here \(\psi_{\text{eva}}(r)\) is the evanescent wave in the massive region, \(\alpha = \arctan(k_y/k_x)\) denotes the incident angle (see Fig. 2(a)) and \(r = (x, y)\). Here we only consider the total reflection process, which captures the physics within the magnetization gap. The reflection coefficient \(r = e^{i\delta}\) is given in the online supplementary material. Then, imagine that the incident and reflected waves have finite width, as depicted in Fig. 2(a); the GH shift can be obtained under the probability flux conservation constraint. As labeled in Fig. 2(a), \(J_{\text{in}}, J_{\text{eva}}\) and \(J_{\text{GH}}\) represent the fluxes carried by the interference wave, the evanescent wave and the part proportional to the GH shift. We denote by \(J_d\) the flux through the cross section colored blue with width \(d\). Suppose that the probability density of the incident/reflected beam is normalized as \(\psi_{\text{in}}^\dagger(r)\psi_{\text{in}}(r) = \psi_{\text{re}}^\dagger(r)\psi_{\text{re}}(r) = 1\); then

\[
J_d = v_F d \sin \alpha, \\
J_{\text{GH}} = v_F \Delta_{\text{GH}} \cos \alpha.
\]
half-quantized shift current

To give a microscopic picture of the chiral hinge current induced by the GH shift, we consider the electron in the scattering problem as a point particle, which bounces back and forth between double massive barriers as sketched in Fig. 1(b). Suppose that the width between the opposite hinges is \( L_x \); thus, the average time interval between two consecutive bounces is \( \Delta \tau = 2L_x/v_F \cos \alpha \) for \(-\pi/2 < \alpha < \pi/2\). When it bounces off the hinge, it undergoes a lateral GH shift \( \Delta_{\text{GH}} \). Such a lateral shift induces an anomalous velocity of electrons along the hinge as

\[
v_{\text{GH}} = \frac{\Delta_{\text{GH}}}{\Delta \tau} = \frac{\Delta_{\text{GH}} v_F \cos \alpha}{2 L_x}.
\]

The total GH shift current induced by \( v_{\text{GH}} \) is obtained by counting the contributions of all filled electrons as

\[
I_{\text{GH}} = \sum_{\text{filled}} \frac{e v_{\text{GH}}}{L_y} = \frac{e}{h} \int_{-\infty}^{E_F} dE \int_{-K}^{K} \frac{dk_x}{2\pi} \Delta_{\text{GH}}.
\]

where \( L_y \) is the circumference of the side surface and \( K = (E + U)/hv_F \) is the Fermi wave vector at energy \( E \). Details of the calculations can be found in the online supplementary material. By employing the stationary phase method (see [50,54–56] and the online supplementary material), we find that the GH shift can be written as \( \Delta_{\text{GH}} = -\partial \phi_F/\partial k_x \). Substituting this result into (6), we obtain the differential shift current \( \delta I_{\text{GH}} = \delta E_F \phi_F(-\pi/2) - \phi_F(\pi/2) \) e/2 \pi h.

\[
\delta I_{\text{GH}} = \delta E_F \phi_F(-\pi/2) - \phi_F(\pi/2) \frac{e}{2\pi h}.
\]
After the reflection, the total contribution from the adiabatic current gives rise to the GH shift. The local Hamiltonian are varying under the constraint $\hbar H$ as a Zeeman type charge transport problem in the $y$ direction. The nontrivial Berry curvature $\Omega_{k_y}$ induces an anomalous velocity $v(k_y)$ in the $y$ direction. After the reflection, the total contribution from the adiabatic current gives rise to the GH shift.

In Fig. 2(c), we plot $\phi_\tau$ as a function of $\alpha$ for different $m$. It is clear that $\phi_\tau(-\pi/2) - \phi_\tau(\pi/2) = \pi$ and robust to the variation of $m$ (see the online supplementary material). Therefore, $\delta I_{GH}$ is exactly half-quantized with respect to $\delta E_F$ as

$$\delta I_{GH} = \frac{e}{2\hbar} \delta E_F. \tag{7}$$

Moreover, $\delta I_{GH}$ can be decomposed according to its contributions from the evanescent wave and the interference wave, i.e., $\delta I_{GH} = \delta I_{GH,eva} + \delta I_{GH,int}$. We emphasize that $\delta I_{GH,eva}$ shows a power-law ($x^{-1/2}$) decay from the hinge, while $\delta I_{GH,eva}$ decays exponentially from the hinge ($e^{-x/2}$); see the online supplementary material. Therefore, $\delta I_{GH}$ is different from the current carried by the topologically protected edge or hinge state, which decays exponentially on both sides of the hinge. The inset of Fig. 2(c) plots $\delta I_{GH,eva}$ and $\delta I_{GH,int}$ versus $m$. The differential shift current $\delta I_{GH}$ is mainly contributed from $\delta I_{GH,eva}$ when $m$ is small. As $m$ increases, the contribution from $\delta I_{GH,int}$ becomes dominant.

**TOPOLOGICAL ORIGIN OF THE HALF QUANTIZATION**

We provide a topological viewpoint of the half-quantized GH shift current in the frame of adiabatic charge transport theory [57–60]. Here, we use Hamiltonian $H(r) = \hbar v_F(-i\sigma_x \partial_x - i\sigma_y \partial_y) + m(x)\sigma_z$ to describe the scattering process, where $m(x)$ is a smooth function connecting the gapless and the gapped regions with $m(x) \rightarrow m$ for $x \gg 0$ and $m(x) \rightarrow 0$ for $x \ll 0$ (see Fig. 3(a)). During the reflection process, the energy $E$ and momentum $k_y$ are conserved. Therefore, the relation $\hbar^2 v_F^2 k_y^2 + \hbar^2 v_F^2 k_y^2 + m^2 = E^2$ holds. We take $t = -\hbar v_F k_x$ as the virtual time and the scattering process is now described by the time-dependent 1D Hamiltonian $H(k_y, t) = -i\sigma_x - \hbar v_F k_x \sigma_y + m(t)\sigma_z$, which describes the Zeeman coupling of a Pauli spinor to a time-dependent magnetic field $B(t) = [-t, \hbar v_F k_y, m(t)]$ (see Fig. 3(b)). The study of the GH shift is reduced to the study of an adiabatic charge transport problem in the $y$ direction, with an internal adiabatic spin procession under the magnetic field $B(t)$ [61].

The instantaneous eigenstates of $H(k_y, t)$ are $\left| u_{\pm}(k_y, t) \right\rangle = \left[ E \pm m, -t + i\hbar v_F k_y \right\rangle / \sqrt{2E(E \pm m)}$, where ‘±’ denotes the spin-up and spin-down components of the spinor. Following the analysis in [60], up to first order in the rate of change in the Hamiltonian, the wave function is given by

$$\left| u_{\pm}(k_y, t) \right\rangle = -i\hbar \sum_{n' \neq n} \frac{\left| u_n(k_y, t) \right\rangle \left\langle u_{n'}(k_y, t) \right| \partial u_n(k_y, t)}{E_n - E_{n'}}. \tag{8}$$

where $n(n') = \pm$ represents the spin-up or spin-down components. The average velocity for a given...
\(v_n(k_y) = \partial E_n(k_y) / \hbar \partial k_y\)

\[-i \left[ \frac{\partial u_n}{\partial k_y} \right] = \frac{\partial u_n}{\partial t} \frac{\partial u_n}{\partial k_y} \]

\[\Delta \text{GH}(k_y) = \int_{-\tau(k_y)}^{\tau(k_y)} v(k_y) \, dt\]

\[= \int_{-\tau}^{\tau} \left[ \frac{\partial E}{\partial k_y} - \Omega_{k_y} \right] \, dt\]

where \(\Omega_{k_y} = -2\text{Im}(\partial u_n / \partial k_y \cdot \partial u_n / \partial t)\) is the Berry curvature in \(k_y\)-space. We only consider the conduction band with \(n = +\) where the scattering process happens; thus, from now on we omit the band index \(n\). The GH shift (see Fig. 3(c)) in the \(y\) direction contributed from a given \(k_y\) is

\[\Delta \text{GH}(k_y) = \int_{-\tau}^{\tau} \left[ \partial E / \partial k_y - \Omega_{k_y} \right] \, dt\]

with \(T(k_y) = \sqrt{E^2 - \hbar^2 v_k^2 k_y^2}\). Combining this with (6), one obtains

\[\delta I_{\text{GH}} / \delta E_F = -e / \hbar \int_{-E_{\text{F}} \hbar v}^{E_{\text{F}} / \hbar v} \frac{d k_y}{2 \pi} \Omega_{k_y} \, dt\]

where \(A(C)\) is the Berry phase along boundary \(C\) of the integration manifold. The integration \(\partial E(k_y) / \hbar \partial k_y\) vanishes because the band structure is symmetric with respect to \(k_y\).

Intuitively, \(\Gamma(C) = \pm \pi\) due to the gapless Dirac electrons on the side surfaces of the AI. To pin down the sign of \(\Gamma(C)\), and hence the direction of the differential GH shift current, we perform the integral in polar coordinates. Define \(k = \sqrt{r^2 + \hbar^2 v_k^2 k_y^2}\), \(t = k \cos \theta\) and \(\hbar v_k k_y = k \sin \theta\). The Berry curvature under the polar coordinate system becomes \(\Omega_{k,0} = \left[ /\partial \partial k \right] / \partial \theta - \left[ /\partial \partial k \right] / \partial \theta \partial u / \partial k]/k\)

\[= -\text{sgn}(m) / 2E \sqrt{E^2 - k^2} \]

\[= \left[ E + m, -ik e^{-i\theta} \right] / \sqrt{2E} \left( E + m \right)\]. The Berry phase

\[\Gamma(C) = \int_0^{2\pi} d \theta \int_0^E k \, dk \frac{-\text{sgn}(m)}{2E \sqrt{E^2 - k^2}}\]

\[= -\text{sgn}(m) \pi\]

(12)

Therefore, we conclude that the half-quantized chiral GH shift current \(\delta I_{\text{GH}} / \delta E_F = \text{sgn}(m) e / 2 \hbar\) is protected by the \(\pi\) Berry phase of the massless Dirac electron while its direction is determined by the mass \(m\) of the massive barrier.

**VISUALIZING THE HALF-QUANTIZED HINGE CURRENT DISTRIBUTION**

The half-quantized hinge current can be numerically visualized based on the 3D magnetic TI Hamiltonian. The model Hamiltonian is given by \(H = H_0 + H_M\), where \(H_0 = \sum_{i=x,y,z} \Delta_k \tau_i \otimes \sigma_i + (M_0 - Bk^2) \tau_z \otimes \sigma_i\) is the nonmagnetic part and \(H_M = M(r) \tau_0 \otimes \sigma_z\) represents the magnetization term \([62,63]\). Here, \(\sigma_i\) and \(\tau_i\) are the Pauli matrices acting on the spin and orbital spaces, respectively. The current density \([16]\) in the \(y\) direction across the \(x-z\) plane with \(r = (x, z)\) is

\[J_y(E, r) = -e / \hbar \int_{-\pi}^{\pi} \text{ImTr} \left[ \partial H(k_y) / \partial k_y \right] G_{k_y}^* (E, r, r') \, dk_y\]

(13)

where \(G_{k_y}^* (E, r, r')\) is the retarded Green’s function for the momentum-sliced Hamiltonian \(H(k_y)\).

We first evaluate a semimagnetic TI with a gapped top surface in the \(y-z\) plane, which can be viewed as ‘half’ of an AI (see Fig. 4(a)) \([39,40,64]\). As shown in Fig. 4(b), the current flux \(I_y(x) = \int_0^{2\pi} dx \int_0^\pi dz J_y(x, z)\) oscillates around 0.5. The \(\hat{x}\) averaged \(\langle J_y(\hat{x}) \rangle = \int_0^\pi \hat{x} / \pi \, dx \int_0^\pi dz J_y(x, z)\) further shows the half-quantization of the chiral hinge current, which coincides with the GH shift analysis. For an AI where the top and bottom surfaces are gapped with opposite magnetization (see Fig. 4(c) and (d)), a pair of counterpropagating hinge currents emerges on the hinges. The moving averaged current flux \(\langle J_y(\hat{x}) \rangle_{MA} = \int_0^{1/2} dz \int_{-\pi/\tau}^{\pi/\tau} dx J_y(x, z)\) further demonstrates that the hinge currents are nearly half-quantized and helical.

**EXPERIMENTAL CHARACTERIZATION OF THE HALF-QUANTIZED HELICAL HINGE CURRENTS**

The half-quantized hinge current \(\delta I_{\text{GH}}\) can be experimentally detected by a six-terminal device, as illustrated in Fig. 5(a). Leads 1 and 3 (2 and 4) are contacted near the top (bottom) surface of the AI, while leads 5 and 6 are contacted to the ends of the sample. According to the Landauer-Büttiker formula, the transmission coefficient between terminals \(i\) and \(j\) is \(T_{ij} = \text{Tr} \left[ \Gamma_i G \Gamma_j G^0 \right] \) and the related differential conductance is \(G_{ij} = e^2 / h \cdot T_{ij}\) \([65-68]\). The measurement method of \(G_{ij}\) is given in the online supplementary material. Here \(\Gamma_i\) is the linewidth function of...
The moving averaged current flux \( \langle J_x(x) \rangle \) of the cross section in the \( x-z \) plane. Figure 4. Current density distribution for semimagnetic TI and AI. (a),(c) Schematics of semimagnetic TI and AI (infinite in the \( y \) direction). The thickness in the \( z \) direction is \( L_z = 8 \). The pink areas in (a) and (c) show the region \( 0 \leq z \leq L_z/2 \) and the dotted box in (c) shows the cross section in the \( x-z \) plane. (b) The current flux through region \( 0 \leq x \leq \bar{x} \), \( 0 \leq z \leq L_z/2 \) and \( \bar{x} \) averaged \( \langle J_x(x) \rangle \) for semimagnetic TI. (d) The upper panel shows the distribution of \( J(x, z) \) in the \( x-z \) plane. The lower panel shows the moving averaged current flux \( \langle J_x(x) \rangle \) through the window \( \{x - 7 \leq x \leq x + 7, 0 \leq z \leq L_z/2 \} \). Both \( J_x \) and \( I_x \) are in units of \( e/h \).

As shown in Fig. 5(c), in the AI state \( G_N^{(N)} = e^2/2h \) and \( G_N^{(i)} = -e^2/2h \) are half-quantized with opposite signs and \( G_N^{(0)} = 0 \). This implies that there exist two counterpropagating half-quantized hinge channels. Moreover, the spatial distribution of the local current \( J_{i \rightarrow i}(E) \) from site \( i \) to \( j \) further reveals the helical signature of the transport hinge currents in the AI, as shown in Fig. 5(b). Since the nonreciprocal conductance \( G_N^{ij} \) counts the asymmetric part of \( J_{i \rightarrow i}(E) \), we can deduce that the half-quantized conducting channel originates from the half-quantized chiral GH shift current \( \delta I_{GH} \) on the hinge of the AI. The results are well consistent with those determined from the GH shift (see Fig. 2) as well as the current density distribution (see Fig. 4). Therefore, these transport signals strongly confirm the reliability of the microscopic picture of half-quantized hinge currents proposed in Fig. 1. Since the half-quantized helical hinge currents serve as a fingerprint of the AI, our proposals also promote the experimental identification of AIs.

DISCUSSION

The half-quantized helical hinge currents can be understood as a consequence of the quantized TME response in the AI \cite{15}. As sketched in Fig. 5(d), a shift of the Fermi energy on the surfaces of the AI induces an interface electric field \( \delta \mathbf{E}_S = -\nabla \delta \mathbf{E}_F/\epsilon \). According to Maxwell’s equations with the axion term, \( \delta j = e^2/2\pi \hbar \cdot \nabla \theta \times \delta \mathbf{E}_S \) \cite{15,70} with the spatially varying axion angle \( \theta \). The half-quantized hinge current \( \delta I = \pm e \delta \mathbf{E}_S/2h \) is obtained by integrating \( \delta j \) over the corner.

In conclusion, we found that the half-quantized helical hinge currents exist in AIs, and established their microscopic picture based on GH shift currents. The half-quantization of the GH shift current has a topological origin and is robust to the variation of parameters. We numerically demonstrated that the half-quantized hinge channel is reflected by the half-quantized nonreciprocal conductances. Our studies deepen the perception of the boundary excitations of the AI and shed light on the detection of AIs through transport experiments.

METHODS

Derivation of the cross-section current density

In calculating the cross-section current density, we take the system to be infinite in the \( y \) direction. The cross section in the \( x-z \) plane is finite in the \( x \) and \( y \) directions for the AI and the CI, and semi-infinite

lead \( i \) and \( G^{(a)} \) is the retarded (advanced) Green’s function of the AI sample.

To demonstrate the existence of the helical hinge channels, we calculate the nonreciprocal conductance \( G_{N}^{ij} = G_{ij} - G_{ji} \) between lead \( i \) and lead \( j \).
in the $x$ direction for the semimagnetic TI. Since the Hamiltonian is infinite in the $y$ direction for the three cases, $k_y$ is a good quantum number and the total Hamiltonian $H$ can be decomposed into summations of the momentum-sliced Hamiltonians as

$$H = \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} H(k_y, r),$$

where $H(k_y, r)$ is the momentum-sliced Hamiltonian with momentum $k_y$ and $r = (x, z)$. We define Green’s function $G^r_{k_y}(E) = \sum_n |\psi_{k_y,n}(r)|^2/(E - E_{k_y,n} + i\delta)$, where $|\psi_{n}(r)|$ is the $n$th eigenstate of $H(k_y, r)$. We write $G^r_{k_y}(E)$ in real-space form as $G^r_{k_y}(E, r, r') = \langle r | G_{k_y}(E) | r' \rangle$.

The velocity operator in the $y$ direction for a given $k_y$ is $v_y(k_y, r) = \partial H(k_y, r)/\hbar \partial k_y$. The local current density in the $x$-$z$ plane can be expressed as

$$j_y(E, r) = \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} j_y(k_y, E, r)$$

with the local current density for a given $k_y$ given by

$$j_y(k_y, E, r) = -\frac{e}{\hbar} \text{Im} \text{Tr} \left[ \frac{\partial H(k_y)}{\partial k_y} G^r_{k_y}(E, r, r) \right].$$

Equation (16) can be derived as follows:

$$j_y(k_y, E, r) = e v_y(k_y, r) \rho(E, r)
= \sum_n e v_n(k_y, r) \delta(E - E_{k_y,n})
= -\frac{e}{\hbar} \text{Im} \text{Tr} \left[ \frac{\partial H(k_y)}{\partial k_y} G^r_{k_y}(E, r, r) \right].$$

Here $\rho(E, r)$ is the local density of states at energy $E$ and position $r$, which can be expanded as $\rho(E, r) = \sum_n \delta(E - E_{k_y,n})$.

**SUPPLEMENTARY DATA**

Supplementary data are available at NSR online.

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**AUTHOR CONTRIBUTIONS**

X.-C.X. conceived the idea from a discussion with H.L.; X.-C.X. supervised the research with H.J.; C.-Z.C. and M.G. performed the analytical and numerical calculations. All authors co-wrote the manuscript.

**Conflict of interest statement.** None declared.

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