From hermitian matrix model
to lattice gauge theory

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Abstract

I consider a lattice model of a gauge field interacting with matrix-valued scalars in $D$ dimensions. The model includes an adjustable parameter $\sigma$, which plays the role of the string tension. In the limit $\sigma = \infty$ the model coincides with Kazakov-Migdal’s “induced QCD”, where Wilson loops obey a zero area law. The limit $\sigma = 0$, where Wilson loops $W(C) = 1$ independently of the size of the loop, corresponds to the Hermitian matrix model. For $D = 2$ and $D = 3$ I show that the model obeys the same combinatorics as the standard LGT and therefore one may expect the area law behavior. In the strong coupling expansion such a behavior is demonstrated.
1 The model.

In what follows I propose a lattice gauge model interpolating between the Hermitian matrix model \[1\] and Kazakov-Migdal’s “induced QCD” \[2\]. The model is very close to standard Wilson LGT. For \( D = 2 \) and \( D = 3 \) I show that the model obeys the same combinatorics as the standard LGT and, therefore, if we generally expect area law in the latter then one may expect it in the former. Within the strong coupling expansion the area law in the model is demonstrated below. The reason for the combinatorial similarity of our model to the standard LGT is the \( Z_N \) symmetry.

Let us start from the Hermitian matrix model \[1\] in \( D \) dimensions

\[
Z = \int \prod_x D\Phi_x \ e^{\text{tr} V(\Phi_x)} \prod_{<xy>} e^{N \text{tr} [\Phi_x \Phi_y]}.
\]

After diagonalizing the hermitian \( N \times N \) matrices \( \Phi_x \) by \( SU(N) \) matrices \( U_x : \Phi_x \to U_x \Phi_x U_x^\dagger \) and defining for each link \(<xy>\) of the \( D \)-dimensional lattice the \( SU(N) \) variables \( U_{xy} = U_x^\dagger U_y \), eq.\((1)\) reads

\[
Z = \int \prod_x d\Phi_x \Delta^2(\Phi_x) \ e^{\text{tr} V(\Phi_x)} \int \prod_{<xy>} DU_{xy} e^{N \text{tr} [\Phi_x U_{xy} \Phi_y U_{xy}^\dagger]} \prod_f \delta(I, U_f). \tag{2}
\]

where \( \Phi_x \)’s are now diagonal matrices, \( U_f \) is the ordered product of \( U_{xy} \)’s along the face (fundamental polygon) \( f \) \[1\] and \( \Delta(\Phi_x) \) is the Vandermonde determinant. The gauge invariant \( \delta \)-function is equivalent to a non-polynomial gauge self-interaction: by definition, the \( \delta \)-function is a sum over irreducible representations \( r \),

\[
\delta(I, U_f) = \sum_r \dim_r \chi_r(U_f), \tag{3}
\]

where \( \chi_r(U) \) and \( \dim_r = \chi_r(I) \) are characters and dimensions of the \( r \)’s. Using the Cauchy formula, eq.\((3)\) can be expressed as

\[
\sum_r \dim_r \chi_r(U_f) = \prod_k^{N} [1 - (U_f)_{kk}]^{-N} = \exp (-N \text{tr} \log (I - U_f)). \tag{4}
\]

1In the case of the square lattice, \( f \) is the fundamental plaquette.
Hence, the $U$-dependent part of action takes the form

$$S = N \sum_{<xy>} \text{tr} [ \Phi_x U_{xy} \Phi_y U_{xy}^\dagger ] + N \sum_f \sum_{q=1}^\infty \frac{1}{q} \text{tr} (U_f)^q. \quad (5)$$

The first term in (5) coincides with the action of the “induced QCD” [2]. The properties of the model [2] have been investigated intensively [2]-[4] and it has been observed that the local $Z_N$ symmetry of the Kazakov-Migdal action results in the so-called “strong confinement”, or “zero area law” (see Ref.[3] for details): the Wilson loop $W(C)$ vanishes unless the spanned area $A(C)$ equals zero:

$$W(C) = 0, \quad A(C) \neq 0. \quad (6)$$

If the (minimal) area law $W(C) \sim \exp(-\sigma A(C))$ is presumed, then the behavior (6) corresponds to an infinite string tension, $\sigma = \infty$.

We now make the simple observation that the second term in (5), arising from $\delta$-function (3), breaks local $Z_N$ and leads to “deconfinement”

$$W(C) = 1, \quad A(C) \neq 0, \quad (7)$$

independently of the size of the loop $C$. Indeed, any product of $U_{xy}$’s along a contour $C$ is equal to the product of $U_f$’s, where faces $f$’s fill $C$. Each such $U_f = I$, due to the $\delta$-function and, hence, $\prod_{<xy> \in C} U_{xy} = I$. It is also clear if one recalls that the matrix model (1) is actually independent of the $U$’s.

Formally, this corresponds to zero string tension, $\sigma = 0$.

It is natural to try to generalize this model to one interpolating between the two limiting cases (6) and (7).

Consider the following simple extension:

$$\text{tr} \log (I - U_f) \rightarrow \text{tr} \log (I - \alpha U_f) \quad (8)$$

where $0 \leq \alpha \leq 1$, with $\alpha = e^{-\sigma}$. Thus $0 \leq \sigma \leq \infty$, and $\sigma = \infty$ corresponds to the Kazakov-Migdal “strong confinement” (6), while $\sigma = 0$ corresponds to deconfinement (7).

The action now reads

$$S = N \sum_{<xy>} \text{tr} [ \Phi_x U_{xy} \Phi_y U_{xy}^\dagger ] + N \sum_f \sum_{q=1}^\infty \frac{\alpha^q}{q} \text{tr} (U_f)^q. \quad (9)$$
This implies that instead of (3) for each face $f$, we have a factor
\[ \sum_r \alpha^{\nu_r} \dim_r \chi_r(U_f) \] (10)
where $\nu_r$ is the sum of all components of highest weight of an irreducible representation $r \equiv \{n_1, \ldots, n_N\}$:
\[ \nu_r = \sum_{k=1}^N n_k. \] (11)

2 Loop averages.

If we expect an area law in the standard LGT, then it is natural to expect this property in the model (9). Naively, we have checked this already in the two limits, (3) and (4) corresponding to $\sigma = \infty$ and $\sigma = 0$ respectively. Now I demonstrate for $D = 2$ and $D = 3$ that the model with arbitrary $0 < \sigma < \infty$ obeys the same combinatorics as the standard LGT and that for large $\sigma$ the model obeys an area law.

Remarkably, a crucial role is again played by the invariance of the term $\text{tr} [ \Phi_x U_{xy} \Phi_y U_{xy}^\dagger ]$ with respect to $Z_N$ transformations $\mathbb{Z}^N$
\[ U \rightarrow Z_N U \quad (U \rightarrow UZ_N). \] (12)

2.1 $D = 2$.

To calculate loop averages and partition functions we need to average the product (over all faces) of the quantities (3) with respect to the Kazakov-Migdal action. The invariance of this action and of the Haar measure under transformations (12) immediately gives selection rules for one-link integrals:
– for free links (only such links contribute to the partition function):
\[ \int DU e^{N \text{tr} [ \Phi_x U \Phi_y U^\dagger ]} \chi_{r_1}(UV_1) \chi_{r_2}(U^\dagger V_2) \sim \delta_{\nu_1,\nu_2} \quad (\text{mod } N) \] (13)
– while for links belonging to a Wilson loop:
\[ \int DU e^{N \text{tr} [ \Phi_x U \Phi_y U^\dagger ]} \chi_{r_1}(UV_1) \chi_{r_2}(U^\dagger V_2) \text{ tr } (UV) \sim \delta_{\nu_1+1,\nu_2} \quad (\text{mod } N) \] (14)

In fact, there is also invariance with respect to arbitrary diagonal $SU(N)$-left and -right rotations.
These selection rules provide all the combinatorics and, in particular, allow us to calculate Wilson loops at strong coupling (large $\sigma$). The calculation in this case reduces to counting powers of $\alpha$ and to selecting the lowest order terms in the same way as it can be done in the standard Wilson lattice $SU(N)$ gauge theory. For example, for a simple loop with the topology of a circle the lowest order non-zero term corresponds to the trivial representation for each face outside the contour and to the fundamental representation inside. Then, the result is that the total lowest power is equal to the number of faces $A(C)$ enclosed by loop $C$. Thus, we obtain the area law,

$$W(C) \sim e^{-\sigma A(C)}$$

up to a factor independent of $\sigma$.

It is worthwhile to remark here that for $D = 2$ the model becomes exactly solvable in the limit of infinite mass scalars. Indeed, following [2], consider a quadratic potential of the scalar field:

$$\text{tr} V(\Phi) = N m_0^2 \text{tr} (\Phi_x^2)$$

After gaussian integration over $\Phi$’s we have the Kazakov-Migdal action of “induced QCD”, which is proportional to negative powers of $m_0^2$. Hence, in the limit $m_0^2 = \infty$, the gauge field is completely decoupled from the scalars and instead of (13) we have an orthogonality condition for characters

$$\int DU \chi_{r_1}(UV_1)\chi_{r_2}(U^\dagger V_2) = \delta_{r_1, r_2} \frac{\chi_{r_1}(V_1V_2)}{\text{dim} r_1}$$

and instead of (14) we have the Wigner coefficient

$$\int DU \chi_{r_1}(U)\chi_{r_2}(U^\dagger) \text{tr} (U) = D_{r_1, r_2}$$

which makes the model to be exactly solvable. The subsequent calculations follow the approach suggested in [3]. (Full combinatorial details, as well as expressions for Wilson loops and partition function on the most general Riemann surface are explicitly given in [3].) In particular, the result for a simple Wilson loop (15) becomes exact for an arbitrary $\sigma$: $W(C) = \exp(-\sigma A(C))$. 

4
2.2 $D = 3$.

In this case, the symmetry (12) leads to the following selection rules:
- for free links:

$$\int DU \, e^{N \text{tr} [\Phi_x U \Phi_y U^\dagger]} \chi_{r_1}(UV_1) \chi_{r_2}(U^\dagger V_2) \chi_{r_3}(UV_3) \chi_{r_4}(U^\dagger V_4) \sim$$

$$\sim \delta_{\nu_1+\nu_3,\nu_2+\nu_4} \pmod{N} \quad (19)$$

- and for links belonging to a Wilson loop:

$$\int DU \, e^{N \text{tr} [\Phi_x U \Phi_y U^\dagger]} \chi_{r_1}(UV_1) \chi_{r_2}(U^\dagger V_2) \chi_{r_3}(UV_3) \chi_{r_4}(U^\dagger V_4) \text{tr}(UV) \sim$$

$$\sim \delta_{\nu_1+\nu_3+1,\nu_2+\nu_4} \pmod{N} \quad (20)$$

The strategy of further calculation of the Wilson loop in the strong coupling expansion is straightforward and closely follows the $D = 2$ case, even though the combinatorial details are somewhat more complicated. As a result, we again obtain the area-like behavior,

$$W(C) \sim \sum_{A(C): \partial A = C} \lambda_A \, e^{-\sigma A(C)}. \quad (21)$$

where $\lambda_A$’s are coefficients growing slower than $\exp(\sigma A(C))$\(^3\). The remark similar to $D = 2$ case can be done: in the $m_0^2 = \infty$ limit the expressions (19) and (20) become exactly the same as in the pure Wilson LGT.

3 Conclusions.

We have a lattice model of a gauge field interacting with matrix-valued scalars in $D$ dimensions. The model includes an adjustable parameter $\sigma$, which plays role of the string tension. In the limit $\sigma = \infty$ the model coincides with Kazakov-Migdal’s “induced QCD”, where Wilson loops obey a zero area law. The limit $\sigma = 0$, where Wilson loops $W(C) = 1$ independently of the size of the loop, corresponds to the Hermitian matrix model. For $D = 2$

\(^3\)I do not discuss here the problems related to the competition of coefficients $\lambda_A$’s with $e^{-\sigma A(C)}$ in the vicinity of RG-stable point, etc. My only purpose here is to demonstrate the usual LGT behavior for the model (1).
and $D = 3$ we have shown that the model obeys the same combinatorics as the standard LGT and, therefore, one may expect the area law behavior. We have demonstrated this property at the strong coupling. In the massive limit, $m_0^2 = \infty$, where the gauge field is completely decoupled from the scalars the model coincides (up to inessential details) with the pure Wilson LGT.

Thus, the Hermitian matrix model and the “induced QCD” can be considered as the limiting cases of one lattice gauge model.

In order to better understand how the incorporation of the $\alpha$ term changes the matrix model, it is fruitful to look at the $D = 1$ case (the matrix chain). In this case the evolution from $\alpha = 1$ to $\alpha = 0$ is very similar to evolution from the closed to the open matrix chain, which in turn can be described by the action

$$S = N \sum_{k=1}^{M-1} \text{tr} [\Phi_k \Phi_{k+1}] + \alpha N \text{tr} [\Phi_M \Phi_1]$$

where $M$ is the number of sites of the chain, $\Phi$’s are non-diagonalized. However, while in (22) $\alpha = 0$ corresponds to the chain being split, it is not the case in our model. The similarity between our model and the model (22) is the absence of the non-singlet $SU(N)$ states in the $\alpha = 0$ limit (the “splitting” point). Obviously, the “splitting” point, $\alpha = 0$, is a singular one. In the language of LGT, this point corresponds to restoration of local $Z_N$, resulting in the “strong confinement”.

The unbroken local $Z_N$ symmetry makes it possible to apply the Itzykson-Zuber formula in the Kazakov-Migdal model [2] to integrate out the $U$’s reducing the problem to a master field, etc. As a price for solvability, the model [2] obeys an undesirable “strong confinement”. In our model the gauge self-interaction, $\text{tr} \log (I - U_f)$, breaks the local $Z_N$, but now we cannot integrate out the $U$’s exactly.
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References

[1] V.Kazakov, Phys.Lett. 150, 282 (1985);
    F.David, Nucl.Phys. B257, 45 (1985);
    V.Kazakov, I.Kostov and A.Migdal, Phys.Lett. 157, 295 (1985).

[2] V.A.Kazakov and A.A.Migdal, Induced QCD at large N, Paris / Princeton
    preprint LPTENS-92/15 / PUPT-1322, (June, 1992).

[3] I.I.Kogan, G.W.Semenoff and N.Weiss, Induced QCD and hidden local Z_N
    symmetry, UBC preprint UBCTP-92-022 (June, 1992);
    I.I.Kogan, A.Morozov, G.W.Semenoff and N.Weiss, Area law and continuum
    limit in “induced QCD”, UBC preprint UBCTP-92-022 (June, 1992).

[4] A.A.Migdal, Exact solution of induced lattice gauge theory at large N, Prince-
    ton preprint PUPT-1323 (June, 1992);
    A.A.Migdal, 1/N expansion and particle spectrum in induced QCD, Princeton
    preprint PUPT-1332 (July, 1992);
    A.Gocksch and Y.Shen, The phase diagram of the N = 2 Kazakov-Migdal
    model, BNL preprint (July, 1992);
    D.Gross, Some remarks about induced QCD, Princeton preprint PUPT-1335
    (August, 1992);
    M.Caselle, A.D.'Adda and S.Panzeri, Exact solution of D=1 Kazakov-Migdal
    induced gauge theory, Turin preprint DFTT 38/92 (July, 1992);
    S.B.Khokhlachev and Yu.M.Makeenko, The problem of large-N phase transi-
    tion in Kazakov-Migdal model of induced QCD , ITEP-YM-5-92, (July, 1992);
    A.A.Migdal, Phase transitions in induced QCD, Paris preprint LPTENS-92/22,
    (August, 1992);
    Yu.Makeenko, Large-N Reduction, Master Field and Loop Equations in
    Kazakov-Migdal Model, Moscow preprint ITEP-YM-6-92, (August, 1992);
I.I.Kogan, A.Morozov, G.W.Semenoff and N.Weiss, *Continuum limits of “induced QCD”: lessons of the Gaussian model at $D = 1$ and beyond*, UBCTP 92-27, ITEP-M7/92 (August, 1992).

[5] A.Migdal, *ZhETF* **69** (1975) 810 (*Sov.Phys.JETP*. **42** 413).

[6] B.Rusakov, *Mod.Phys.Lett.* **A5** (1990) 693.