A QUENCHED INVARIANCE PRINCIPLE FOR STATIONARY PROCESSES

CHRISTOPHE CUNY AND DALIBOR VOLNÝ

Abstract. In this note, we prove a conditionally centered version of the quenched weak invariance principle under the Hannan condition, for stationary processes. In the course, we obtain a (new) construction of the fact that any stationary process may be seen as a functional of a Markov chain.

1. Introduction and results

Let \((X, A, \mu)\) be a probability space and \(\theta\) be an invertible bimeasurable transformation of \(X\), preserving \(\mu\), and assume that \(\theta\) is ergodic. Let \(F_0\) be a sub-\(\sigma\)-algebra of \(A\) such that \(F_0 \subset \theta^{-1}(F_0)\). Define a filtration \((F_n)_{n \in \mathbb{Z}}\), by \(F_n = \theta^{-n}F_0\) and denote \(F_{-\infty} = \cap_{n \in \mathbb{Z}} F_n\). For every \(n \in \mathbb{Z}\), we denote by \(E_n\) the conditional expectation with respect to \(F_n\) and we define the projection \(P_n := E_n - E_{n-1}\).

Let \(f\) be \(F_0\)-measurable. We want to study the stationary process \((f \circ \theta^n)_{n \in \mathbb{Z}}\).

Let \(\mu(\cdot, \cdot)\) denote a regular conditional probability on \(A\) given \(F_0\), see [1] p. 358-364, and for every \(x \in X\), write \(\mu_x := \mu(x, \cdot)\). Thus, for every \(x \in X\), \(\mu_x\) is a probability measure on \(A\), and for every \(A \in A\), \(\mu(\cdot, A)\) is a version of \(\mu(A|F_0)\).

We say that the process \((f \circ \theta^n)\) satisfies the Hannan Condition if

\[
\sum_{i=0}^{\infty} \|P_i f\|_2 = \sum_{i=0}^{\infty} \|P_i (f \circ \theta^n)\|_2 < \infty.
\]

If \(E_{-\infty}(f) = 0\), the Hannan Condition, introduced by E.J. Hannan in [6], guarantees the CLT and the weak invariance principle (WIP). The condition has been shown to be very useful in applications (cf. [3], also for the WIP). In general, as shown in [4], the Hannan Condition is independent of the so-called Dedecker-Rio and Maxwell-Woodroofe conditions, that are also sufficient for the WIP.

Let us denote \(S_n = S_n(f) = \sum_{i=1}^{n} f \circ \theta^i\). It was shown in [10] that the CLT is not quenched (i.e. \(S_n\) does not satisfy the CLT under \(\mu_x\) for \(\mu\)-almost every \(x \in X\)) but it follows from [11] or [2] that \(\tilde{S}_n = S_n - E_0(S_n)\) satisfies the quenched CLT. Here we prove that \(\tilde{S}_n\) satisfies the quenched WIP as well.

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For every $t \in [0,1]$, write $S_n(t) = S_{[nt]} + (nt - [nt])f \circ \theta^{[nt]+1}$ and $\bar{S}_n(t) = S_n(t) - \mathbb{E}_0(S_n(t))$. Our main result is the following.

**Theorem 1.** Let $f \in L^2(X, \mathcal{A}, \mu)$ satisfy the Hannan condition. Then there exists a martingale $(M_n)_n$ with stationary ergodic increments such that

$$\mathbb{E}_0(\max_{1 \leq n \leq N}(\bar{S}_n - M_n)^2) = o(N) \; \mu\text{-a.s.}$$

In particular, $\sigma^2 := \lim_{n} \mathbb{E}(S_n^2)/n$ exists and for $\mu$-almost every $x \in X$, for every bounded continuous function $\varphi$ on $C([0,1], \| \cdot \|_\infty)$, we have

$$\int_X \varphi(S_n(t)/\sqrt{n})d\mu_x \rightarrow_{n \to +\infty} \mathbb{E}(\varphi(\sigma W_t)),$$

where $(W_t)_{0 \leq t \leq 1}$ stands for a standard brownian motion.

**Corollary 2.** Let $f \in L^2(X, \mathcal{A}, \mu)$ be such that

$$\sum_{n \geq 1} \frac{\|\mathbb{E}_0(X_n)\|_2}{\sqrt{n}} < \infty$$

Then, (1) holds and the conclusion of Theorem 1 is true with $S_n(t)$ in place of $\bar{S}_n(t)$.

2. PROOF OF THE RESULTS

To avoid technical difficulties (and since it is also convenient for the next section) we assume that $X$ is a Polish space and that $\mathcal{A}$ is the $\sigma$-algebra of its Borel sets. It is known (see for instance Neveu [5, Proposition V.4.3]) that in this case there exists a regular version of the conditional probability given $\mathcal{F}_0$ on $\mathcal{A}$. We then use the notations of the introduction.

Recall that $UP_i = P_{i+1}U$ where $U$ is defined by $Uf = f \circ \theta$. For an adapted function $f \in L^2$ we thus have

$$f = \sum_{i=0}^{\infty} P_{-i}f + \mathbb{E}_{-\infty}(f) = \sum_{i=0}^{\infty} U^{-i}P_0U^i f + \mathbb{E}_{-\infty}(f) = \sum_{i=0}^{\infty} U^{-i}f_i + \mathbb{E}_{-\infty}(f)$$

where $f_i = P_0U^i f$, $i = 0, 1, \ldots$. Therefore, since for every $n \geq 0$, $\mathbb{E}_0(\mathbb{E}_{-\infty}(f)) = \mathbb{E}_{-\infty}(f)$, we have

$$\bar{S}_n(f) = S_n(f) - \mathbb{E}_0(S_n(f)) = \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} U^j f_i.$$

Denote, for $h \in L^1$, $Qh = \mathbb{E}_0(Uh)$. Then $Q$ is a Dunford-Schwartz operator (it is a contraction in all $L^p$, $1 \leq p \leq \infty$). Notice that $Q^n h = \mathbb{E}_0(U^n f)$. The use of the operator $Q$ is crucial in our proof. Its relevance to the problem is made more clear in the next section.

Let us recall several facts from ergodic theory that will be needed in the sequel.
By the Dunford-Schwartz (or Hopf) ergodic theorem (cf. [7, Lemma 6.1]), for every $h \in L^1$, denoting $h^* = \sup_{n \geq 1} (1/n) \sum_{i=0}^{n-1} Q^i(|h|)$, we have
\[(5) \sup_{x > 0} x \mu(h^* > x) \leq \|h\|_1.\]

We will make use of the weak $L^2$-space
\[L^{2,w} := \{ f \in L^1 : \sup_{\lambda > 0} \lambda^2 \mu\{ |f| \geq \lambda \} < \infty \}.\]

Recall that there exists a norm $\| \cdot \|_{2,w}$ on $L^{2,w}$ that makes it a Banach space and which is equivalent to the pseudo-norm $(\sup_{\lambda > 0} \lambda^2 \mu\{ |f| \geq \lambda \})^{1/2}$.

Then it follows from (5), that for every $h \in L^2$,
\[(6) (h^2)^{1/2} \in L^{2,w}.\]

We obtain
\[\text{Lemma 3. Let } f \text{ be as above. We have}
\[(7) (\mathbb{E}_0(\max_{1 \leq n \leq N} \bar{S}_n^2(f)))^{1/2} \leq \sqrt{N} \sum_{i=0}^{\infty} ((f_i^2)^*)^{1/2} < \infty \mu\text{-a.s.}\]

In particular, if $f$ satisfies the Hannan condition, then, by (6)
\[\sup_{N \geq 1} \frac{\mathbb{E}_0(\max_{1 \leq n \leq N} \bar{S}_n^2(f))}{N} < \infty \mu\text{-a.s.}\]

\[\text{Proof. Let } N \geq n \geq 1. \text{ From (4) it follows that}
\[|\bar{S}_n(f)| \leq \sum_{i=0}^{N-1} \max_{1 \leq k \leq N} \left| \sum_{j=1}^{k} U_j f_i \right| .\]

Notice that for every $i \geq 0$, the process $(U_j f_i)_j$ is a sequence of martingale increments. We will use the Doob maximal inequality conditionally, in particular we will use
\[(E_0(\max_{n \leq N} |\bar{S}_n(f_i)|^2))^{1/2} \leq 2 [E_0(\bar{S}_N^2(f_i))]^{1/2}.\]

For $\mu$-a.e. $x \in X$ and every $i \geq 0$, $(U_j f_i)_j$ remains a sequence of martingale increments under $\mu_x$. Denoting by $\| \cdot \|_{1,\mu_x}$ the norm in $L^2(\mu_x)$, it follows from the Doob maximal inequality that
\[\| \max_{n \leq N} |\bar{S}_n(f)|\|_{2,\mu_x} \leq \sum_{i=0}^{N-1} \| \max_{1 \leq n \leq N} |\bar{S}_n(f_i)|\|_{2,\mu_x} \leq 2 \sum_{i=0}^{N-1} \|\bar{S}_N(f_i)\|_{2,\mu_x}\]
hence
\[
\left( \mathbb{E}_0(\max_{n \leq N} |\bar{S}_n(f)|^2) \right)^{1/2} \leq 2 \sum_{i=0}^{N-1} \left[ \mathbb{E}(\bar{S}_N^2(f_i)) \right]^{1/2} = 2 \sum_{i=0}^{N-1} \left[ \mathbb{E}(\sum_{j=1}^N U^j f_i^2) \right]^{1/2} = \\
2 \sum_{i=0}^{N-1} \left( \sum_{j=1}^N Q^j f_i^2 \right)^{1/2} \leq 2 \sqrt{N} \sum_{i=0}^{\infty} ((f_i^2)^*)^{1/2}.
\]

Now, using (5), (6), we see that \(\sum_{i=0}^{\infty} ((f_i^2)^*)^{1/2}\) is in \(L^{2,w}\), which finishes the proof. \(\square\)

**Proof of Theorem 1.** By Hannan’s condition \(m = \sum_{k \geq 0} P_0(U^k f)\) is well defined and \(M_n = \sum_{k=1}^n U^k m\) is a martingale with stationary and ergodic increments.

Let \(r \geq 1\). We have
\[
f = \sum_{k=0}^r P_0(U^k f) - \sum_{k=1}^r \left( \mathbb{E}_0(U^k f) - \mathbb{E}_0(U^{k-1} f) \right) + \mathbb{E}_0(U^r f).
\]

Hence, denoting \(m^{(r)} = \sum_{k=0}^r P_0(U^k f)\) and \(M^{(r)}_n = \sum_{l=1}^n U^l m^{(r)}\), we obtain
\[
S_n - M_n = M^{(r)}_n - M_n - U^n \left( \sum_{k=1}^r \mathbb{E}_0(U^k f) \right) + \sum_{k=1}^r \mathbb{E}_0(U^k f) + \sum_{l=1}^n U^l \left( \mathbb{E}_0(U^r f) \right)
\]
and
\[
(8) \quad S_n - M_n - \mathbb{E}_0(S_n) = \\
M^{(r)}_n - M_n - \left[ U^n \left( \sum_{k=1}^r \mathbb{E}_0(U^k f) \right) - \mathbb{E}_0(U^n \left( \sum_{k=1}^r \mathbb{E}_0(U^k f) \right) ) \right] + \\
+ \sum_{l=1}^n U^l \left( \mathbb{E}_0(U^r f) \right) - \mathbb{E}_0(\sum_{l=1}^n U^l \left( \mathbb{E}_0(U^r f) \right))
\]

By Doob maximal inequality, denoting \(h^{(r)} := (m - m^{(r)})^2\), we have
\[
(9) \quad \mathbb{E}_0\left( \max_{1 \leq n \leq N} (M^{(r)}_n - M_n)^2 \right) \leq 4 \sum_{1 \leq k \leq N} Q^k h^{(r)} \leq CN(h^{(r)})^*
\]
(recall that \(h^* = \sup_{n \geq 1} (1/n) \sum_{i=0}^{n-1} Q^i(|h|)\)).
Let $K > 0$. Denote $Z^{(r)} = \sum_{k=1}^{r} \mathbb{E}_0(U^k f)$ and $Z^{(r)}_K = Z^{(r)} 1_{|Z^{(r)}|>K}$

$$
\mathbb{E}_0(\max_{1 \leq n \leq N} |U^n(\sum_{k=1}^{r} \mathbb{E}_0(U^k f))|^2) - \mathbb{E}_0(U^n(\sum_{k=1}^{r} \mathbb{E}_0(U^k f))|^2)
\leq 4\mathbb{E}_0(\max_{1 \leq n \leq N} |U^n Z^{(r)}|^2) \leq 4K^2 + 4\mathbb{E}_0(\sum_{n=1}^{N} |U^n Z^{(r)}_K|^2)
$$

(10) \quad \leq 4K^2 + 4 \sum_{n=1}^{N} Q^n((Z^{(r)}_K)^2) \leq 4(K^2 + N((Z^{(r)}_K)^2)^*)

To deal with the last term in (8), we apply Lemma 3 to $\mathbb{E}_1(U^r f)$, noticing that in this case $f_i$ is replaced with $P_0(U^i \mathbb{E}_1(U^r f)) = P_0(U^{i+r} f) = f_{i+r}$ when $i \geq 1$ and for $i = 0$, $P_0(\mathbb{E}_1(U^r f)) = 0$). Hence

$$
\mathbb{E}_0(\max_{1 \leq n \leq N} |U^i(\mathbb{E}_1(U^r f))|^2) - \mathbb{E}_0(\sum_{l=0}^{n-1} U^i(\mathbb{E}_1(U^r f))|^2) \leq N \sum_{i \geq r} ((f^2_i)^*)^{1/2}.
$$

(11) \quad \mathbb{E}_0(\max_{1 \leq n \leq N} |\mathbb{S}_n - M_n|^2) \leq C(h^{(r)})^* + ((Z^{(r)}_K)^2)^* + \sum_{i \geq r} ((f^2_i)^*)^{1/2} \quad \mu\text{-a.s.}

Combining (9), (10) and (11), we obtain that for every $K > 0$ and every $r \in \mathbb{N}$,

$$
\limsup_{N \to \infty} \frac{\mathbb{E}_0(\max_{1 \leq n \leq N} |\mathbb{S}_n - M_n|^2)}{N} \leq C(h^{(r)})^* + ((Z^{(r)}_K)^2)^* + \sum_{i \geq r} ((f^2_i)^*)^{1/2} \quad \mu\text{-a.s.}
$$

Now, $\|((Z^{(r)}_K)^2)^*\|_{2,w} \leq C\|Z^{(r)}_K\|_2 \to 0$. Hence there exists a sequence $(K_l)$ going to infinity such that

$$
((Z^{(r)}_K)^2)^* \to 0 \quad \text{a.s.}
$$

Hence

$$
\limsup_{N \to \infty} \frac{\mathbb{E}_0(\max_{1 \leq n \leq N} |\mathbb{S}_n - M_n|^2)}{N} \leq C(h^{(r)})^* + \sum_{i \geq r} ((f^2_i)^*)^{1/2} < \infty \quad \mu\text{-a.s.}
$$

The second term clearly goes to 0 $\mu$-a.s., when $r \to \infty$ (by Lemma 3), and the first one goes to 0 $\mu$-a.s. (along a subsequence) by (6).

**Proof of Corollary 2.** As noticed by Cuny-Peligrad, the condition (3) implies the Hannan condition and the fact that $\mathbb{E}_0(S_n) = o(\sqrt{n}) \mu$-a.s., hence the result. □

3. Markov Chains

In most of the literature, quenched limit theorems for stationary sequences use a Markov Chain setting: the process is represented as a functional $(f(W_n))_n$ of a stationary and homogeneous Markov Chain $(W_n)$; the limit theorem is said “quenched” it it remains true for almost every starting point.
Every (strictly) stationary sequence of random variables admits a Markov Chain representation. This has been observed by Wu and Woodroofe in [14], using an idea from [9]. A remark-survey on equivalent representations of stationary processes can be found in [13]. Here we show that the operator $Q$ introduced above leads to another Markov Chain representation of stationary processes.

Let $(X, \mathcal{A}, \mu)$ be a probability space and $\theta$ be an invertible bi-measurable transformation of $X$ preserving the measure $\mu$.

Let $F \subset \mathcal{A}$ be a $\sigma$-algebra such that $F \subset \theta^{-1}F$. Denote $E(\cdot|F)$ the conditional expectation with respect to $F$ and define an operator $Q$ on $L^\infty(X, \mathcal{F}, \mu)$ by

\begin{equation}
Qh = E(h \circ \theta|\mathcal{F}).
\end{equation}

Then $Q$ is a positive contraction satisfying $Q1 = 1$ and it is the dual of a positive contraction $T$ of $L^1(X, \mathcal{F}, \mu)$, namely $Tg = (E(g|\mathcal{F})) \circ \theta^{-1}$. By Neveu [8, Proposition V.4.3], if $X$ is a Polish space and $\mathcal{A}$ the $\sigma$-algebra of its Borel sets, there exists a transition probability $Q(x, dy)$ on $X \times \mathcal{F}$ such that for every $h \in L^\infty(X, \mathcal{F}, \mu)$,

\begin{equation}
Qh = \int_X h(y)Q(\cdot, dy).
\end{equation}

Clearly, the transition probability $Q$ preserves the measure $\mu$, hence the canonical Markov chain induced by $Q$, with initial distribution $\mu$, may be extended to $\mathbb{Z}$.

Now define a sequence of random variables $(W_n)_{n \in \mathbb{Z}}$ defined from $(X, \mathcal{A})$ to $(X, \mathcal{F})$ by $W_n(x) = \theta^n(x)$. Then we have

**Proposition 4.** Let $(X, \mathcal{A})$ be a Polish space with its Borel $\sigma$-algebra. Let $\mu$ be a probability on $\mathcal{A}$ and $\theta$ be an invertible bi-measurable transformation of $X$ preserving the measure $\mu$. Let $\mathcal{F} \subset \mathcal{A}$ be a $\sigma$-algebra such that $\mathcal{F} \subset \theta^{-1}\mathcal{F}$. Then $(W_n)_{n \in \mathbb{Z}}$ is a Markov chain with state space $(X, \mathcal{F})$, transition probability $Q$ (given by (12)) and stationary distribution $\mu$. In particular, for every $f \in L^2(X, \mathcal{F}, \mu)$, the process $(f \circ \theta^n)$ is a functional of a stationary Markov chain.

**Proof.**

It suffices to show that for every $n \geq 1$, and any $\varphi_0, \ldots, \varphi_n$ bounded measurable functions from $\mathbb{R}$ to $\mathbb{R}$, we have

\begin{equation}
\int_X \varphi_0(W_0) \ldots \varphi_n(W_n) d\mu = \int_X \varphi_0(W_0) \ldots \varphi_{n-1}(W_{n-1})Q\varphi_n(W_{n-1})d\mu,
\end{equation}

the result for general blocks with possibly negative indices follows by stationarity. By definition of $(W_n)$, (13) holds for $n = 1$.

Assume that (13) holds for a given $n \geq 1$. Let $\varphi_0, \ldots, \varphi_{n+1}$ be bounded $\mathcal{F}$-measurable functions from $X$ to $\mathbb{R}$. Using the definition of $Q$, (13) for our given
n, and stationarity, we obtain
\[
\int_X \varphi_0(W_0) \ldots \varphi_n(W_{n+1}) d\mu = \int_X \varphi_0 \circ \theta \ldots \varphi_{n+1} \circ \theta^{n+1} d\mu
\]
\[
= \int_X \varphi_0(\theta^{-n}) \ldots \varphi_{n-1} \circ \theta^{-1} \varphi_n \varphi_{n+1} \circ \theta d\mu = \int_X \varphi_0(\theta^{-n}) \ldots \varphi_n Q \varphi_{n+1} d\mu
\]
\[
= \int_X \varphi_0 \ldots \varphi_n \circ \theta^n Q \varphi_n \circ \theta^n d\mu = \int_X \varphi_0(W_0) \ldots \varphi_n(W_n) Q \varphi_n(W_n) d\mu
\]
which proves our result by induction. ☐

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Current address: Laboratoire MAS, Ecole Centrale de Paris, Grande Voie des Vignes, 92295 Chatenay-Malabry cedex FRANCE
E-mail address: christophe.cuny@ecp.fr

Current address: Département de Mathématiques, Université de Rouen, 76801 Saint-Etienne du Rouvray, FRANCE
E-mail address: dalibor.volny@univ-rouen.fr