Abstract. The existence of strong solutions and pathwise uniqueness are established for one-dimensional stochastic Volterra equations with locally Hölder continuous diffusion coefficients and sufficiently regular kernels. Moreover, we study the sample path regularity, the integrability and the semimartingale property of solutions to one-dimensional stochastic Volterra equations.

Key words: Hölder regularity, stochastic Volterra equation, pathwise uniqueness, non-Lipschitz coefficient, semimartingale, strong solution, Yamada–Watanabe theorem.

MSC 2020 Classification: 60H20, 45D05.

1. Introduction

Stochastic Volterra equations (SVEs) have been studied in probability theory starting with the works of Berger and Mizel [BM80a, BM80b]. This class of integral equations constitutes a generalization of ordinary stochastic differential equations and serves as well suited mathematical model for numerous random phenomena appearing, e.g., in biology, physics and mathematical finance.

In the present work, we investigate the strong existence and pathwise uniqueness of solutions to one-dimensional stochastic Volterra equations with locally Hölder continuous diffusion coefficients and sufficiently regular kernels. More precisely, we consider SVEs of the form

\begin{equation}
X_t = x_0(t) + \int_0^t K_\mu(s,t)\mu(s,X_s) \, ds + \int_0^t K_\sigma(s,t)\sigma(s,X_s) \, dB_s, \quad t \in [0,T],
\end{equation}

where $x_0$ denotes the initial condition, $(B_t)_{t \in [0,T]}$ is a Brownian motion, the kernels $K_\mu, K_\sigma$ are sufficiently regular functions, the coefficient $\mu$ is locally Lipschitz continuous, and the diffusion coefficient $\sigma$ is locally Hölder continuous.

The motivation to study stochastic Volterra equations with non-Lipschitz coefficients is twofold. On the one hand, it is a natural question to explore to what extent the famous results of Yamada and Watanabe [YW71], ensuring pathwise uniqueness and the existence of strong solutions for ordinary stochastic differential equations, generalizes to stochastic Volterra equations. On the other hand, stochastic Volterra equations with only 1/2-Hölder continuous coefficients recently got a great deal of attention in mathematical finance as so-called rough volatility models, see e.g. [AJEE19b, EER19], which have demonstrated to fit remarkably well historical and implied volatilities of financial markets, see e.g. [BFG16]. Furthermore, SVEs with non-Lipschitz continuous coefficients arise as scaling limits of branching processes in population genetics, see [MS14, AJ21].
The existence of unique strong solutions for stochastic Volterra equations with Lipschitz continuous coefficients is well investigated. Indeed, classical existence and uniqueness results for SVEs with sufficiently regular kernels are due to [BM80a, BM80b, Pro85]. These results have been generalized in various directions such as allowing for anticipating and path-dependent coefficients [PP90, ØZ93, AN97, Kal21], singular kernels [CLP95, CD01] or an infinite dimensional setting [Zha10]. A slight extension beyond Lipschitz continuous coefficients can be found in [Wan08].

The classical approach to prove the existence of strong solutions to ordinary stochastic differential equations with less regular diffusion coefficients is to first show the existence of a weak solution, since this, in combination with pathwise uniqueness, guarantees the existence of a strong solution, see [YW71]. Only recently, the existence of weak solutions for stochastic Volterra equations was derived in the work of Abi Jaber, Cuchiero, Larsson and Pulido [AJCLP21] (see also [MS15, AJLP19, AJ21]), assuming that the kernels in the stochastic Volterra equations are of convolution type, i.e. in our setting \( K_{\mu}(s,t) = K_{\sigma}(s,t) = K(t-s) \) for some function \( K: \mathbb{R} \to \mathbb{R} \). Assuming additionally that the coefficients \( \mu, \sigma \) lead to affine Volterra processes, weak uniqueness was obtained in [MS15, AJEE19a, AJ21, CT20]. However, as we do not impose a convolution structure on the stochastic Volterra equation (1.1), we cannot rely on the known results regarding the existence of weak solutions.

Our first main contribution is to establish the existence of a strong solution to the SVE (1.1) provided the diffusion coefficient \( \sigma \) is locally \( 1/2 + \xi \)-Hölder continuous for \( \xi \in [0, 1/2] \). To that end, we prove the convergence of an Euler type approximation of the SVE (1.1) and do not use the concept of weak solutions. For ordinary stochastic differential equations such an approach was developed by Gyöngy and Rásonyi [GR11], using ideas coming from [YW71]. As a number of results used to deal with ordinary stochastic differential equations are not available in the context of SVEs, the presented proof for the existence of a strong solution to the SVE (1.1) requires various different techniques such as a transformation formula for Volterra processes à la Protter [Pro85] and a Grönwall lemma allowing weakly singular kernels.

Our second main contribution is to establish pathwise uniqueness for the SVE (1.1) provided that the diffusion coefficient \( \sigma \) is locally \( 1/2 + \xi \)-Hölder continuous for \( \xi \in [0, 1/2] \) or even, more generally, satisfies the classical Yamada–Watanabe condition [YW71]. To that end, we generalize the classical approach of Yamada and Watanabe [YW71] to the more general setting of stochastic Volterra equations. The presented proof for pathwise uniqueness is based on similar techniques as the proof of existence and is inspired by the work of Mytnik and Salisbury [MS15]. In [MS15], pathwise uniqueness is proven for one-dimensional stochastic Volterra equations with smooth kernels and without drift (i.e. \( \mu = 0 \)). For SVEs of convolutional type with continuous differentiable kernels admitting a resolvent of the first kind, pathwise uniqueness was shown in [AJEE19a].

Let us remark, while we need to require sufficient regularity on the kernels \( K_{\mu}, K_{\sigma} \) to obtain the existence of a unique strong solution (see Theorem 2.3 and Corollary 2.6), the imposed regularity conditions on the coefficients are essentially the classical regularity conditions of Yamada–Watanabe. Already in case of ordinary stochastic differential equations, it is well-known that these regularity conditions cannot be relaxed in the sense that pathwise uniqueness does not hold in general if, e.g., the diffusion coefficient \( \sigma \) is only Hölder continuous of order strictly less than 1/2.

**Organization of the paper:** Section 2 presents the setting and main result: an existence and uniqueness theorem for stochastic Volterra equations with Hölder continuous diffusion
coefficients. The properties of solutions to SVEs are provided in Section 3. The existence of a strong solution is proven in Section 4 and that pathwise uniqueness holds in Section 5.

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2. Main result and assumptions

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$ be a filtered probability space, which satisfies the usual conditions, $(B_t)_{t \in [0,T]}$ be a standard Brownian motion and $T \in (0, \infty)$. We consider the one-dimensional stochastic Volterra equation (SVE)

\begin{equation}
X_t = x_0(t) + \int_0^t K_\mu(s,t)\mu(s,X_s)\,ds + \int_0^t K_\sigma(s,t)\sigma(s,X_s)\,dB_s, \quad t \in [0,T],
\end{equation}

where $x_0: [0,T] \to \mathbb{R}$ is a continuous function, the coefficients $\mu, \sigma: [0,T] \times \mathbb{R} \to \mathbb{R}$ and the kernels $K_\mu, K_\sigma: \Delta_T \to \mathbb{R}$ are measurable functions, using the standard notation $\Delta_T := \{(s,t) \in [0,T] \times [0,T]: 0 \leq s \leq t \leq T\}$. Furthermore, $\int_0^T K_\mu(s,t)\mu(s,X_s)\,ds$ is defined as a Riemann–Stieltjes integral and $\int_0^T K_\sigma(s,t)\sigma(s,X_s)\,dB_s$ as an Itô integral.

Let $K: \Delta_T \to \mathbb{R}$ be a measurable function. We say $K(\cdot, t)$ is absolutely continuous for every $t \in [0,T]$ if there exists an integrable function $\partial_1 K(u, t) \, du$ for $(s,t) \in \Delta_T$. We say $K(s, \cdot)$ is absolutely continuous for every $s \in [0,T]$ if there exists an integrable function $\partial_2 K: \Delta_T \to \mathbb{R}$ such that $K(s,t) - K(s,0) = \int_0^t \partial_1 K(s,u) \, du$ for $(s,t) \in \Delta_T$. Moreover, for $p \in [1, \infty)$, we denote $K \in L^p(\Delta_T)$ if $\int_0^T \int_0^T |K(s,t)|^p \, ds \, dt < \infty$.

For the kernels $K_\mu, K_\sigma$ and the initial condition $x_0$ we make the following assumptions.

Assumption 2.1. Let $\gamma \in (0, \frac{1}{2}]$, and $K_\mu, K_\sigma: \Delta_T \to \mathbb{R}$ and $x_0: [0,T] \to \mathbb{R}$ be continuous functions such that:

(i) $K_\mu(s, \cdot)$ is absolutely continuous for every $s \in [0,T]$ and $\partial_2 K_\mu$ is bounded on $\Delta_T$.

(ii) $K_\sigma(\cdot, t)$ is absolutely continuous for every $t \in [0,T]$, $K_\sigma(s, \cdot)$ is absolutely continuous for every $s \in [0,T]$ with $\partial_2 K_\sigma \in L^2(\Delta_T)$, and $\partial_2 K_\sigma(\cdot, t)$ is absolutely continuous for every $t \in [0,T]$. Furthermore, there is a constant $C > 0$ such that $|K_\sigma(t, t)| \geq C$ for any $t \in [0,T]$, and there exist $C > 0$, $\alpha \in [0, \frac{1}{2})$ and $\epsilon > 0$ such that

\[\int_0^s |K_\sigma(u, t) - K_\sigma(u, s)|^{2+\epsilon} \, du \leq C|t - s|^{(2+\epsilon)} \quad \text{and} \]
\[|\partial_1 K_\sigma(s, t)| + |\partial_2 K_\sigma(s, s)| + \int_s^t |\partial_2 K_\sigma(s, u)| \, du \leq C(t - s)^{-\alpha}\]

hold for any $(s,t) \in \Delta_T$.

(iii) $x_0$ is $\beta$-Hölder continuous for every $\beta \in (0, \gamma)$.

The regularity properties of the coefficients $\mu$ and $\sigma$ are formulated in the next assumption. We start with assuming global Lipschitz and Hölder continuity of $\mu$ and $\sigma$, respectively. An extension to local regularity conditions are treated in Corollary 2.6 below.

Assumption 2.2. Let $\mu, \sigma: [0,T] \times \mathbb{R} \to \mathbb{R}$ be measurable functions such that:
(i) $\mu$ and $\sigma$ are of linear growth, i.e. there is a constant $C_{\mu,\sigma} > 0$ such that
\[ |\mu(t,x)| + |\sigma(t,x)| \leq C_{\mu,\sigma}(1 + |x|), \]
for all $t \in [0,T]$ and $x \in \mathbb{R}$.

(ii) $\mu$ is Lipschitz continuous and $\sigma$ is Hölder continuous of order $\frac{1}{2} + \xi$ for some $\xi \in [0,\frac{1}{2}]$ in the space variable uniformly in time, i.e. there are constants $C_{\mu,\sigma} > 0$ such that
\[ |\mu(t,x) - \mu(t,y)| \leq C_{\mu}|x - y| \quad \text{and} \quad |\sigma(t,x) - \sigma(t,y)| \leq C_{\sigma}|x - y|^{\frac{1}{2} + \xi} \]
hold for all $t \in [0,T]$ and $x, y \in \mathbb{R}$.

To formulate our results, let us briefly recall the concepts of strong solutions and pathwise uniqueness. For this purpose, let $L^p(\Omega \times [0,T])$ be the space of all real-valued, $p$-integrable functions on $\Omega \times [0,T]$. We call an $(\mathcal{F}_t)_{t \in [0,T]}$-progressively measurable stochastic process $(X_t)_{t \in [0,T]}$ in $L^p(\Omega \times [0,T])$ on the given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$, a (strong) $L^p$-solution of the SVE (2.1) if $\int_0^T ([K_{\mu}(s,t)\mu(s,X_s)] + [K_{\sigma}(s,t)\sigma(s,X_s)^2]) \, ds < \infty$ for all $t \in [0,T]$ and the integral equation (2.1) hold $\mathbb{P}$-almost surely. As usual, a strong $L^1$-solution $(X_t)_{t \in [0,T]}$ of the SVE (2.1) is often just called solution of the SVE (2.1). We say pathwise uniqueness in $L^p(\Omega \times [0,T])$ holds for the SVE (2.1) if $\mathbb{P}(X_t = \tilde{X}_t, \forall t \in [0,T]) = 1$ for two $L^p$-solutions $(X_t)_{t \in [0,T]}$ and $(\tilde{X}_t)_{t \in [0,T]}$ of the SVE (2.1) defined on the same probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$. Moreover, we say there exists a unique strong $L^p$-solution $(X_t)_{t \in [0,T]}$ to the SVE (2.1) if $(X_t)_{t \in [0,T]}$ is a strong $L^p$-solution to the SVE (2.1) and pathwise uniqueness in $L^p(\Omega \times [0,T])$ holds for the SVE (2.1). We say $(X_t)_{t \in [0,T]}$ is $\beta$-Hölder continuous for $\beta \in (0,1]$ if there exists a modification of $(X_t)_{t \in [0,T]}$ with sample paths that are $\mathbb{P}$-almost surely $\beta$-Hölder continuous.

The main results of the present work are summarized in the following theorem.

**Theorem 2.3.** Suppose Assumptions 2.1 and 2.2 and let $p > \max\{\frac{1}{\beta}, 1 + \frac{1}{\epsilon}\}$, where $\gamma \in (0,\frac{1}{2}]$ and $\epsilon > 0$ are given by Assumption 2.2. Then, there exists a unique strong $L^p$-solution $(X_t)_{t \in [0,T]}$ to the stochastic Volterra equation (2.1). Moreover, the solution $(X_t)_{t \in [0,T]}$ is $\beta$-Hölder continuous for every $\beta \in (0,\gamma)$, $\sup_{t \in [0,T]} \mathbb{E}[|X_t|^q] < \infty$ for every $q \in [1,\infty)$ and $(X_t - x_0(t))_{t \in [0,T]}$ is a semimartingale.

**Proof.** The existence of a strong solution $(X_t)_{t \in [0,T]}$ to the stochastic Volterra equation (2.1) is provided by Theorem 4.1 and its pathwise uniqueness by Theorem 5.3. The assertions that $\sup_{t \in [0,T]} \mathbb{E}[|X_t|^q] < \infty$ for every $q \in [1,\infty)$ and of the $\beta$-Hölder continuity as well as the semimartingale property of $(X_t - x_0(t))_{t \in [0,T]}$ follow by Corollary 5.7. □

Note that the regularity assumptions (Assumption 2.2), as required in Theorem 2.3, on the coefficients $\mu, \sigma$ are essentially optimal. Indeed, it is well-known for ordinary stochastic differential equations that pathwise uniqueness does not hold in general if $\mu$ is only Hölder continuous of order strictly less than 1 or $\sigma$ is only Hölder continuous of order strictly less than $1/2$, see for instance [KS01, page 287] and [KS01, Chapter 5, Example 2.15].

**Remark 2.4.** Recall that Yamada and Watanabe derived pathwise uniqueness for ordinary stochastic differential equations under the slightly weaker assumption of $|\sigma(t,x) - \sigma(t,y)| \leq \rho(|x-y|)$ for a function $\rho: [0,\infty) \to [0,\infty)$ with $\int_0^\infty \rho(s)^{-2} \, ds = \infty$ for every $\epsilon > 0$, cf. [YW71, Theorem 1]. While the proof of pathwise uniqueness presented in Section 3 is given under this Yamada–Watanabe condition, in the proof of the existence of a strong solution via an
approximation scheme the Hölder regularity of \( \sigma \) is explicitly used in various estimates, see e.g. (4.9), and a modification of these estimates allowing for the Yamada–Watanabe condition appears not straightforward.

**Remark 2.5.** Assumption (2.1) is satisfied, for instance, if \( K_\mu \) is continuously differentiable, \( K_\sigma \) is twice continuously differentiable with \( K_\sigma(t,t) > 0 \) for \( t \in [0,T] \) and \( x_0 \) is \( \beta \)-Hölder continuous for some \( \beta \in (0,1) \).

While the condition \( |K_\sigma(t,t)| \geq C \) for \( t \in [0,T] \) is crucial for implementing the present method to prove Theorem (2.6), it might appear to be of technical nature. However, assuming \( K_\sigma(t,t) = 0 \) for every \( t \in [0,T] \) and keeping in mind the semimartingale decomposition in Lemma (3.4) any solution of the SVE (2.1) would be a semimartingale of bounded variation without any diffusion part and, thus, some care is needed to not lose the regularization effects of a Brownian motion.

Based on a localization argument, the assumptions of global Lipschitz and Hölder continuity on the coefficients of the SVE (2.1) can be relaxed to local regularity assumptions. In the following, \( C > 0 \) denotes a generic constant that might change from line to line. To emphasize the dependence of the constant \( C \) on parameters \( p,q \) or functions \( f,g \), we write \( C_{p,q,f,g} \).

Moreover, for \( x,y \in \mathbb{R} \) we set \( x \wedge y := \min\{x,y\} \).

**Corollary 2.6.** Suppose Assumptions (2.1), (2.2) (i), and that \( \mu \) is locally Lipschitz continuous and \( \sigma \) is locally Hölder continuous of order \( \frac{1}{2} + \xi \) for some \( \xi \in [0,\frac{1}{2}] \) in the space variable uniformly in time, i.e. for every \( n \in \mathbb{N} \) there are constants \( C_{\mu,n}, C_{\sigma,n} > 0 \) such that

\[
|\mu(t,x) - \mu(t,y)| \leq C_{\mu,n}|x - y| \quad \text{and} \quad |\sigma(t,x) - \sigma(t,y)| \leq C_{\sigma,n}|x - y|^{\frac{1}{2} + \xi}
\]

hold for all \( t \in [0,T] \) and \( x, y \in \mathbb{R} \) with \( |x|, |y| \leq n \). Let \( p > \max\{\frac{1}{\gamma}, 1 + \frac{2}{\gamma}\} \), where \( \gamma \in (0,\frac{1}{2}] \) and \( \epsilon > 0 \) are given by Assumption (2.4). Then, there exists a unique strong \( L^p \)-solution \( (X_1)^{\mu,n}_{t \in [0,T]} \) to the stochastic Volterra equation (2.1). Moreover, the solution \( (X_1)^{\mu,n}_{t \in [0,T]} \) is \( \beta \)-Hölder continuous for every \( \beta \in (0,\gamma) \), \( \sup_{t \in [0,T]} E[|X_1|^q] < \infty \) for every \( q \in [1,\infty) \) and \( (X_t - x_0(t))_{t \in [0,T]} \) is a semimartingale.

**Proof.** By Assumptions (2.1) and (2.2) (i), Lemma (3.3), Corollary (3.5) and Lemma (3.6) imply the integrability, \( \beta \)-Hölder continuity and semimartingale property of the solution. For the well-posedness, we adapt the proofs of Theorems (4.1) and (5.4) and the notation therein.

For the uniqueness, consider two \( L^p \)-solutions \( (X_1^1)_{t \in [0,T]} \) and \( (X_1^2)_{t \in [0,T]} \), and define \( \tilde{X}_t := X_1^1 - X_1^2 \) for \( t \in [0,T] \) and the hitting times \( \tau_k := \inf\{t \in [0,T] : \max\{|X_1^1|, |Y_1^1|\} \geq k \} \wedge T \) for \( k \in \mathbb{N} \) which are stopping times with \( \tau_k \rightarrow T \) a.s. by the same reasoning as for the hitting times defined in (3.3). By bounding \( \phi_n(X_1^1 \mathbbm{1}_{t \leq \tau_k}) \leq \phi_n(X_1^{1 \wedge \tau_k}) \) and applying Itô’s formula to the right-hand-side, we obtain after performing the same steps as in (3.3)-(5.8) and sending \( n \rightarrow \infty \), that

\[
E[|\tilde{X}_t| \mathbbm{1}_{t \leq \tau_k}]
\leq C \int_0^t E[|\tilde{X}_s| \mathbbm{1}_{s \leq \tau_k}] \, ds + \int_0^t E[|\tilde{Y}_s| \mathbbm{1}_{s \leq \tau_k}](\partial_2 K_\sigma(s,s) + \int_s^t |\partial_2 K_\sigma(s,u)| \, du) \, ds,
\]

for \( t \in [0,T] \). Similarly, we get a bound on \( E[|\tilde{Y}_t| \mathbbm{1}_{t \leq \tau_k}] \) analogue to (5.11) and denoting

\[M_k(t) := \sup_{s \in [0,t]} \left( E[|\tilde{X}_s| \mathbbm{1}_{s \leq \tau_k}] + E[|\tilde{Y}_s| \mathbbm{1}_{s \leq \tau_k}] \right)\]

...
we obtain \( M_k(t) = 0 \) for all \( t \in [0, T] \), and sending \( k \to \infty \) yields the uniqueness.

For the existence, we adapt the standard localization argument from the SDE case. We introduce for \( n \in \mathbb{N} \) the localized coefficients

\[
\mu_n(t, x) := \begin{cases} 
\mu(t, x), & \text{if } |x| \leq n, \\
\mu(t, \frac{x}{|x|}), & \text{if } |x| > n,
\end{cases}
\]

and analogously \( \sigma_n \), which fulfill the regularity properties globally, such that corresponding strong solutions exist by Theorem 3.1 that we denote by \( X^n \). Moreover, let \( \kappa_n := \inf \{ t \in [0, T] : |X^n_t| > n \} \land T \) and define \( X(t) := X^n(t) \) for \( \kappa_{n-1} < t \leq \kappa_n(t) \). By the pathwise uniqueness, it holds \( X^n_{\kappa_{n-1}} = X^n_{\kappa_{n-1}} \) for all \( n \in \mathbb{N} \) such that \( X \) is continuously well-defined and we must only show that it cannot explode, i.e. that \( \kappa_n \to T \) a.s. By the Garsia–Rodemich–Rumsey inequality (see [GRR 71, Lemma 1.1]), Markov’s inequality and Lemma 3.1, we obtain for any \( \alpha \in (0, \gamma) \) and \( p > 2 \) chosen such that \( \alpha p > 1 \)

\[
\mathbb{P} \left( \sup_{t \in [0, T]} |X^n_t - X^n_0| > n \right) \leq \mathbb{P} \left( \sup_{t \in [0, T]} \left( C_{\alpha, p, T} \left( \int_0^T \frac{|X_s - X_u|^p}{|s-u|^\alpha p + 1} \, du \, ds \right)^{\frac{1}{p}} \right) > n \right) 
\]

\[
\leq n^{-p} \mathbb{E} \left[ C_{\alpha, p, T} \left( \int_0^T \frac{|X_s - X_u|^p}{|s-u|^\alpha p + 1} \, du \, ds \right) \right] 
\]

which tends to 0 sufficiently fast such that the Borel–Cantelli lemma (see [Kle 14, Theorem 2.7]) implies \( \kappa_n \to T \) a.s. \( \square \)

The rest of the paper is largely devoted to prove Theorem 2.3. However, we will formulate and prove the partial findings under weaker assumptions if possible without additional effort.

3. Properties of a solution

In this section we establish some properties of solutions to stochastic Volterra equations. We start by the regularity and integrability of \( L \)-solutions, which requires only the linear growth condition of the coefficients and allows for singular kernels in the SVE 2.1.

Lemma 3.1. Suppose Assumption 2.2 (i) and let \( K_\mu, K_\sigma : \Delta_T \to \mathbb{R} \) be measurable functions such that, for some \( \epsilon > 0 \) and \( L > 0 \),

\[
\int_0^t |K_\mu(s, t') - K_\mu(s, t)|^{1+\epsilon} \, ds + \int_t^{t'} |K_\mu(s, t')|^{1+\epsilon} \, ds \leq L|t' - t|^{\gamma(1+\epsilon)},
\]

\[
\int_0^t |K_\sigma(s, t') - K_\sigma(s, t)|^{2+\epsilon} \, ds + \int_t^{t'} |K_\sigma(s, t')|^{2+\epsilon} \, ds \leq L|t' - t|^{\gamma(2+\epsilon)},
\]

for all \( (t, t') \in \Delta_T \), and (3.2) holds. Furthermore, let \( x_0 : [0, T] \to \mathbb{R} \) be \( \beta \)-Hölder continuous for every \( \beta \in (0, \gamma) \) for some \( \gamma \in (0, \frac{1}{2}) \) and let \( (X_t)_{t \in [0, T]} \) be a \( L^p \)-solution of the SVE 2.1 for some \( p > \max\{\frac{1}{\gamma}, 1 + \frac{2}{\gamma}\} \). Then, for any \( \beta \in (0, \gamma) \), there is a constant \( C_{x_0, p, L, T, \mu, \sigma, \epsilon} > 0 \) such that

\[
\mathbb{E}[|X_{t'} - X_t|^p] \leq C_{x_0, p, L, T, \mu, \sigma, \epsilon} |t' - t|^{\beta p},
\]

holds for all \( t, t' \in [0, T] \). Consequently, \( (X_t)_{t \in [0, T]} \) is \( \beta \)-Hölder continuous for any \( \beta \in (0, \gamma - \frac{1}{p}) \).
Proof. Let $p > 2$ be given by the assumption. Since $x_0$ is $\beta$-Hölder continuous, we observe for $t, t' \in [0, T]$ that
\[
E[|X_{t'} - X_t|^p] \leq C_p, x_0 |t' - t|^{\beta p} + C_p, E[|\tilde{X}_{t'} - \tilde{X}_t|^p]
\]
with $\tilde{X}_t := X_t - x_0(t)$.
For $(t, t') \in \Delta_T$ we note that
\[
|\tilde{X}_{t'} - \tilde{X}_t|^p = \left| \int_0^{t'} K_\mu(s, t') \mu(s, X_s) \, ds + \int_0^{t'} K_\sigma(s, t') \sigma(s, X_s) \, dB_s \right|^p - \left| \int_0^{t} K_\mu(s, t) \mu(s, X_s) \, ds - \int_0^{t} K_\sigma(s, t) \sigma(s, X_s) \, dB_s \right|^p.
\]
\[
\leq C_p \left( \left| \int_0^{t} \mu(s, X_s) \left( K_\mu(s, t') - K_\mu(s, t) \right) \, ds \right|^p + \left| \int_0^{t} \mu(s, X_s) K_\mu(s, t') \, ds \right|^p \right.
\]
\[
+ \left. \left| \int_0^{t} \sigma(s, X_s) \left( K_\sigma(s, t') - K_\sigma(s, t) \right) \, dB_s \right|^p + \left| \int_0^{t} \sigma(s, X_s) K_\sigma(s, t') \, dB_s \right|^p \right)
\]
\[
=: C_p (A + B + C + D).
\]
We shall bound the expectation of the terms $A-D$ in the following. For $A$, we use Hölder’s inequality, the linear growth of $\mu$ (Assumption 2.2 (i)), (3.1) and that $X \in L^{\frac{2\pi}{\gamma p}} (\Omega \times [0, T])$ since $\frac{1 + \gamma}{\pi} < p$ to obtain
\[
E[A] \leq E \left[ \left| \int_0^{t} \mu(s, X_s) \right|^{\frac{2\pi}{\gamma p}} \, ds \right] \left( \int_0^{t} \left| K_\mu(s, t') - K_\mu(s, t) \right|^{1 + \gamma} \, ds \right)^{\frac{\gamma p}{\gamma p + 1}}
\]
\[
\leq C_{p, L, \mu, T, \gamma} \left( \int_0^{t} \left| K_\mu(s, t') - K_\mu(s, t) \right|^{1 + \gamma} \, ds \right)^{\frac{\gamma p}{\gamma p + 1}}
\]
\[
\leq C_{x_0, p, L, \mu, \sigma, \gamma} |t' - t|^\gamma p.
\]
Note that the second inequality follows either with Jensen’s inequality, if $\frac{\gamma p}{\gamma p + 1} \leq 1$, or else with Hölder’s inequality and Fubini’s theorem. Applying the analog estimates to $B$ gives
\[
E[B] \leq E \left[ \left| \int_t^{t'} \mu(s, X_s) \right|^{\frac{2\pi}{\gamma p}} \, ds \right] \left( \int_t^{t'} \left| K_\mu(s, t') \right|^{1 + \gamma} \, ds \right)^{\frac{\gamma p}{\gamma p + 1}} \leq C_{x_0, p, L, T, \mu, \sigma, \gamma} |t' - t|^\gamma p.
\]
For term $C$, relying on the Burkholder–Davis–Gundy inequality, Hölder’s inequality, using the linear growth of $\sigma$ (Assumption 2.2 (i)), $X \in L^{\frac{2\pi}{\gamma p}} (\Omega \times [0, T])$ and (3.1), we get
\[
E[C] \leq E \left[ \left( \int_0^{t} \left| \sigma(s, X_s) \right|^{\frac{2\pi}{\gamma p}} \, ds \right)^{\frac{\gamma p}{\gamma p + 1}} \right] \left( \int_0^{t} \left| K_\sigma(s, t') - K_\sigma(s, t) \right|^{2 + \gamma} \, ds \right)^{\frac{\gamma p}{\gamma p + 1}}
\]
\[
\leq C_{p, L, \sigma, T, \gamma} \left( \int_0^{t} \left| K_\sigma(s, t') - K_\sigma(s, t) \right|^{2 + \gamma} \, ds \right)^{\frac{\gamma p}{\gamma p + 1}}
\]
\[
\leq C_{x_0, p, L, \mu, \sigma, \gamma} |t' - t|^\gamma p.
\]
Applying (3.1) and analog estimates to term $D$ reveals
\[
E[D] \leq C_{x_0, p, L, T, \mu, \sigma, \gamma} \left( \int_t^{t'} \left| K_\sigma(s, t') \right|^{2 + \gamma} \, ds \right)^{\frac{\gamma p}{\gamma p + 1}} \leq C_{x_0, p, L, T, \mu, \sigma, \gamma} |t' - t|^\gamma p.
\]
Hence, with the above estimates we arrive at
\[
\mathbb{E}[|X_{t'} - X_t|^p] \leq C_{p,x_0}|t' - t|^\beta p + C_{x_0,p,L,T,\mu,\sigma}|t' - t|^\gamma p \leq C_{x_0,p,L,T,\mu,\sigma,\epsilon}|t' - t|^\beta p,
\]
as \( \beta < \gamma \). Hence, by Kolmogorov–Chentsov’s theorem (see e.g. [Kle14, Theorem 21.6]) and sending \( \beta \to \gamma \), there exists a modification of \((X_t)_{t \in [0,T]}\) which is \( \delta' \)-Hölder continuous for every \( \delta' \in (0, \gamma - 1/p) \).

**Remark 3.2.** Suppose that the kernels \( K_\mu \) and \( K_\sigma \) fulfill Assumption 2.1. In this case it follows from Kolmogorov’s continuity criterion and the estimates in the proof of Lemma 3.1, that, for every progressively measurable stochastic process \( u \in L^p([0,T] \times \Omega) \) for some \( p > \max\{1, 2, 1 + 2/\epsilon\} \), the process \((M_u t)_{t \in [0,T]}\), defined by \( M_{u,t} := \int_0^t K_\mu(s,t)u_s \, ds + \int_0^t K_\sigma(s,t)u_s \, dB_s \), has \( \mathbb{P} \)-a.s. \( \beta \)-Hölder-continuous paths for every \( \beta \in (0, \gamma - 1/p) \).

**Remark 3.3.** Note that the constant \( C_{x_0,p,L,T,\mu,\sigma,\epsilon} \) in Lemma 3.1 depends on \( \mu \) and \( \sigma \) only through the constant appearing in the linear growth condition (Assumption 2.2 (i)).

The integrability of solutions to the SVE (2.1) is the content of the next lemma.

**Lemma 3.4.** Suppose Assumption 2.2 (i) and that \( K_\mu, K_\sigma : \Delta_T \to \mathbb{R} \) are measurable functions such that, for some \( \epsilon > 0 \) and \( L > 0 \),
\[
\int_0^t |K_\mu(s,t)|^{1+\epsilon} \, ds + \int_0^t |K_\sigma(s,t)|^{2+\epsilon} \, ds \leq L, \quad t \in [0,T].
\]
Let \((X_t)_{t \in [0,T]}\) be a \( L^p \)-solution to the SVE (2.1) for some \( p > \max\{2, 1 + 2/\epsilon\} \). Then,
\[
\sup_{t \in [0,T]} \mathbb{E}[|X_t|^q] \leq C_{q,L,T,\mu,\sigma} \left( 1 + \sup_{t \in [0,T]} |x_0(t)|^q \right),
\]
holds for any \( q \geq 1 \), where the constant \( C_{q,L,T,\mu,\sigma} \) depends only on \( q, L, T \) and the growth constants of \( \mu \) and \( \sigma \).

**Proof.** Let us introduce the hitting times
\[
\tau_k := \inf\{t \in [0,T] : |X_t| \geq k\} \wedge T, \quad k \in \mathbb{N}.
\]
Note that \( \tau_k \to T \) a.s. as \( k \to \infty \), since the paths of the solution \( X \) are \( \mathbb{P} \)-a.s. continuous by Lemma 3.1. Since the underlying filtered probability space satisfies the usual conditions, by the Début theorem (see [RY99, Chapter I, (4.15) Theorem]), the hitting times \( (\tau_k)_{k \in \mathbb{N}} \) are stopping times.

First, let \( q > 2 \) be big enough such that \( q' := \frac{q}{q-1} \leq 1 + \epsilon \) and \( \tilde{q} := \frac{q}{q-2} \leq 1 + \epsilon/2 \). Using Hölder’s inequality, the Burkholder–Davis–Gundy inequality, and the linear growth condition
Lemma 3.6. The statement follows by applying Lemma 3.4 and Lemma 3.1 with

\[ C \lesssim \int_0^t K_\rho(s,t) \sigma(s,X_s) \, dB_s \]
solution to the SVE (2.1) such that \( \mathbb{E}[|X_t|^2] \leq C \) for all \( t \in [0,T] \) and some constant \( C \). Then, \( (X_t - x_0(t))_{t \in [0,T]} \) is a semimartingale with decomposition \( X_t - x_0(t) = M_t + A_t \) where

\[
M_t := \int_0^t K_\sigma(s,s)\sigma(s, X_s) \, dB_s \quad \text{and} \quad A_t := \int_0^t K_\mu(s,s)\mu(s, X_s) \, ds + \int_0^t \left( \int_s^t \partial_2 K_\mu(u,s)\mu(u, X_u) \, du + \int_s^t \partial_2 K_\sigma(u,s)\sigma(u, X_u) \, dB_u \right) \, ds
\]

for \( t \in [0,T] \).

\textbf{Proof.} Setting

\[
Y_t := \int_0^t \sigma(s, X_s) \, dB_s \quad \text{and} \quad Z_t := \int_0^t \mu(s, X_s) \, ds, \quad \text{for} \ t \in [0,T],
\]

and using the absolute continuity of \( K_\mu, K_\sigma \), we get

\[
X_t = \int_0^t K_\mu(s,s) \, dZ_s + \int_0^t \left( \int_s^t \partial_2 K_\mu(s,u) \, du \right) \, dZ_s + \int_0^t \left( \int_s^t \partial_2 K_\sigma(s,u) \, du \right) \, dY_s + \int_0^t K_\sigma(s,s) \, dY_s.
\]

Since

\[
\mathbb{E} \left[ \int_{\Delta_T} |\partial_2 K_\mu(s,u)\mu(s, X_s)| \, ds \, du \right] + \mathbb{E} \left[ \int_{\Delta_T} (\partial_2 K_\sigma(s,u)\sigma(s, X_s))^2 \, ds \, du \right] < \infty
\]

due to \( \mathbb{E}[|X_t|^2] \leq C \) for all \( t \in [0,T] \), \( \partial_2 K_\mu \in L^1(\Delta_T) \) and \( \partial_2 K_\sigma \in L^2(\Delta_T) \), we can apply the classical and the stochastic Fubini theorem (see e.g. [Ver12 Theorem 2.2]) to get

\[
X_t = \int_0^t K_\mu(s,s) \, dZ_s + \int_0^t \left( \int_0^s \partial_2 K_\mu(s,u) \, du \right) \, dZ_s + \int_0^t \left( \int_0^s \partial_2 K_\sigma(s,u) \, du \right) \, dY_s + \int_0^t K_\sigma(s,s) \, dY_s,
\]

which completes the proof. \( \square \)

Applying the previous lemmas to the setting of Theorem 2.3 leads to the following corollary.

\textbf{Corollary 3.7.} Suppose Assumptions 2.1 and 2.2. Let \( (X_t)_{t \in [0,T]} \) be a \( L^p \)-solution to the SVE (2.1) for some \( p > \max\{\frac{1}{2}, 1 + \frac{2}{p} \} \). Then, \( (X_t)_{t \in [0,T]} \) satisfies \( \sup_{t \in [0,T]} \mathbb{E}[|X_t|^q] < \infty \) for every \( q \in [1, \infty] \), \( (X_t)_{t \in [0,T]} \) is \( \beta \)-Hölder continuous for every \( \beta \in (0, \gamma) \) for \( \gamma \in (0, 1/2) \) given in Assumption 2.1 and \( (X_t - x_0(t))_{t \in [0,T]} \) is a semimartingale.

\textbf{Proof.} Note that the existence and boundedness of \( \partial_2 K_\mu \) from Assumption 2.1(i) imply that

\[
\int_0^s |K_\mu(u,t) - K_\mu(u,s)|^{1+\epsilon} \, du = \int_0^s \left( \int_s^t \partial_2 K_\mu(u,r) \, dr \right)^{1+\epsilon} \, du
\]

holds for some \( C > 0 \) and any \( (s,t) \in \Delta_T \), using \( \epsilon > 0 \) and \( \gamma \in (0, 1/2) \) from Assumption 2.1. Furthermore, the continuity of \( K_\mu \) and \( K_\sigma \) ensures that condition (3.2) holds and, thus,
Lemma 4.2. Suppose Assumptions 2.1 and 2.2, and let $p > \max\{\frac{1}{\gamma}, 1 + \frac{2}{\gamma}\}$. Then, there exists a strong $L^p$-solution $(X_t)_{t \in [0, T]}$ to the SVE (2.1).

The construction of a strong solution relies on an Euler type approximation. To set up the approximation, we use the sequence $(\pi_m)_{m \in \mathbb{N}}$ of partitions defined by

$$\pi_m := \{t^m_0, \ldots, t^m_{2^m}\} \quad \text{with} \quad t^m_i := \frac{iT}{2^m} \quad \text{for} \quad i = 0, \ldots, 2^m - 1,$$

and introduce, for every $m \in \mathbb{N}$, the function $\kappa_m : [0, T] \to [0, T]$ by

$$\kappa_m(T) := T \quad \text{and} \quad \kappa_m(t) := t^m_i \quad \text{for} \quad t^m_i \leq t < t^m_{i+1}, \quad \text{for} \quad i = 0, 1, \ldots, 2^m - 1.$$

For every $m \in \mathbb{N}$, we iteratively define the process $(X^m(t))_{t \in [0, T]}$ by $X^m(0) := x_0(0)$ and for $t \in (t^m_i, t^m_{i+1}]$ by

$$X^m(t) := x_0(t) + \int_0^{t^m_i} K_{\mu}(s, t)\mu(s, X^m(\kappa_m(s))) \, ds + \int_{t^m_i}^t K_{\mu}(s, t)\mu(s, X^m(t^m_i)) \, ds + \int_0^{t^m_i} K_{\sigma}(s, t)\sigma(s, X^m(\kappa_m(s))) \, dB_s + \int_{t^m_i}^t K_{\sigma}(s, t)\sigma(s, X^m(t^m_i)) \, dB_s,$$

for $i = 0, \ldots, 2^m - 1$.

Note that we neither discretize the kernels $K_{\mu}, K_{\sigma}$ nor the time-component in the coefficients $\mu, \sigma$. While these additional discretizations might be desirable to derive an implementable numerical scheme, for our purpose of proving the existence of a strong solution, it is sufficient to avoid this additional approximation.

Lemma 4.3. Suppose Assumptions 2.1 and 2.2. $X^m \in L^q(\Omega \times [0, T])$ for every $m \in \mathbb{N}$ and any $q \in [1, \infty)$. In particular, $X^m \in L^p(\Omega \times [0, T])$ for every $m \in \mathbb{N}$ and $p > \max\{\frac{1}{\gamma}, 1 + \frac{2}{\gamma}\}$.

Proof. For $m \in \mathbb{N}$ and $q \in (2, \infty)$ we define

$$g_m(t) := \mathbb{E}[|X^m(t)|^q] \quad \text{for} \quad t \in [0, T].$$

To prove that $X^m \in L^q(\Omega \times [0, T])$, it is sufficient to show that the function $g_m$ is bounded on $[0, T]$ since

$$\mathbb{E}\left[\int_0^T |X^m(t)|^q \, dt\right] = \int_0^T g_m(t) \, dt \leq T \sup_{t \in [0, T]} g_m(t).$$

For $t = 0$ we have $\mathbb{E}[|X^m(0)|^q] = |x_0(0)|^q < \infty$ and, thus, $g_m$ is bounded on $[0, t^m_0]$. For $t \in (t^m_i, t^m_{i+1}]$ with $i = 1, \ldots, 2^m - 1$, using similar estimates as in (3.3), we iteratively get

$$\sup_{t \in [0, T]} \mathbb{E}[|X^m(t)|^q] < \infty \quad \text{for} \quad q \in [1, \infty).$$

Moreover, since Assumption 2.1 implies (3.1), Corollary 3.5 states the claimed $\beta$-Hölder continuity. The semimartingale property follows by Lemma 3.6. \qed
that
\[
\mathbb{E}[|X^m(t)|^q] 
\leq C \left( |x_0(t)|^q + \int_0^t \mathbb{E}[|\mu(s, X^m(\kappa_m(s)))|^q] \, ds + \int_{t_i}^t \mathbb{E}[|\sigma(s, X^m(t_i^m))|^q] \, ds \right)
\]
\[
+ \int_0^t \mathbb{E}[|\sigma(s, X^m(\kappa(s)))|^q] \, ds + \int_{t_i}^t \mathbb{E}[|\sigma(s, X^m(t_i^m))|^q] \, ds
\]
\[
\leq C \left( 1 + \int_0^t \mathbb{E}[|X^m(\kappa(s))|^q] \, ds + \int_{t_i}^t \mathbb{E}[|X^m(t_i^m)|^q] \, ds \right) < \infty.
\]
Therefore, sup\(_{t \in [0,T]} g_m(t) < \infty. \, \square

It can be quickly seen that the integrability and regularity results from Section 3 also hold for the process \((X^m(t))_{t \in [0,T]}\).

**Proposition 4.3.** Suppose Assumptions 2.1 and 2.2. Let \(\gamma \in [0,1/2]\) be as given in Assumption 2.7. Then, for any \(m \in \mathbb{N}\), there is a constant \(C > 0\) such that
\[
\sup_{t \in [0,T]} \mathbb{E}[|X^m(t)|^q] \leq C \left( 1 + \sup_{t \in [0,T]} |x_0(t)|^q \right).
\]
holds for any \(q \geq 1\). Moreover, for any \(\beta \in (0, \gamma)\), there is a constant \(C > 0\) such that
\[
\mathbb{E}[|X^m(t') - X^m(t)|^q] \leq C |t' - t|^\beta q
\]
holds for all \(t', t \in [0,T]\). Consequently, \((X^m(t))_{t \in [0,T]}\) is \(\beta\)-Hölder continuous for any \(\beta \in (0, \gamma)\).

**Proof.** The \(L^q\)-bound of \((X^m(t))_{t \in [0,T]}\) follows by similar arguments as used in the proof of Lemma 3.4.

For \(t \in (t_i^m, t_i^{m+1}]\) and fixed \(m \in \mathbb{N}\) and \(q \geq 2\), we get
\[
\mathbb{E}[|X^m(t)|^q] \leq C \left( |x_0(t)|^q + \int_0^t \mathbb{E}[|X^m(\kappa_m(s))|^q] \, ds + \int_{t_i}^t \mathbb{E}[|X^m(t_i^m)|^q] \, ds \right),
\]
where we used Hölder’s inequality, Burkholder–Davis–Gundy’s inequality, and the linear growth condition (Assumption 2.2 (i)). Hence, we arrive at
\[
\sup_{u \in [0,t]} \mathbb{E}[|X^m(u)|^q] \leq C \left( \sup_{u \in [0,T]} |x_0(u)|^q + \int_0^t \sup_{u \in [0,s]} \mathbb{E}[|X^m(u)|^q] \, ds \right).
\]
Since \( t \mapsto \sup_{u \in [0,t]} \mathbb{E}[|X^m(u)|^q] \) is bounded by the proof of Lemma 4.2, we can apply Grönwall’s lemma (see e.g. [Kle14, Lemma 26.9]) to get

\[
\sup_{t \in [0,T]} \mathbb{E}[|X^m(t)|^q] \leq C \left(1 + \sup_{t \in [0,T]} |x_0(t)|^q\right), \quad t \in [0,T],
\]

which reveals the assertion.

The regularity statement follows by adapting the proof of Lemma 3.1. Indeed, the regularity assumption on the kernels (Assumption 2.1) yields that condition (3.1) is fulfilled. Thus, performing similar estimations as in the proof of Lemma 3.1 and using the just established \( L^q \)-bound of \( X^m \), we obtain

\[
\mathbb{E}[|X^m(t') - X^m(t)|^q] \leq C|t' - t|^\beta q,
\]

for \( \beta \in (0, \gamma) \). Hence, by Kolmogorov–Chentsov’s theorem (see e.g. [Kle14, Theorem 21.6]), there exists a modification of \( (X^m(t))_{t \in [0,T]} \) which is \( \delta \)'-Hölder continuous for \( \delta \in (0, \beta - 1/q) \).

Sending \( \beta \to \gamma \) and \( q \to \infty \) leads to the claimed Hölder regularity.

Due to Proposition 4.3, for every \( m \in \mathbb{N} \) the process \( (X^m(t))_{t \in [0,T]} \) has a continuous modification. Hence, keeping the definition of \( (X^m(t))_{t \in [0,T]} \) in mind, we see that \( (X^m(t))_{t \in [0,T]} \) fulfills the integral equation

(4.1) \( X^m(t) = x_0(t) + \int_0^t K_\mu(s,t)\mu(s,X^m(\kappa_m(s))) \, ds + \int_0^t K_\sigma(s,t)\sigma(s,X^m(\kappa_m(s))) \, dB_s, \)

for \( t \in [0,T] \). Moreover, using the just derived regularity estimates of \( (X^m(t))_{t \in [0,T]} \), we obtain the following bound.

**Corollary 4.4.** Suppose Assumptions 4.1 and 4.2. Then, for any \( q, \delta \in (0, \infty) \), there is a constant \( C > 0 \) such that

\[
\mathbb{E} \left[ \left( \int_0^T |X^m(s) - X^m(\kappa_m(s))|^\delta \, ds \right)^q \right] \leq C 2^{-\delta q m^5},
\]

holds for all \( \beta \in (0, \gamma) \) and \( m \in \mathbb{N} \).

**Proof.** Let \( \delta > 0 \) be fixed. First, assume \( q \geq 1 \) is sufficiently large such that \( q\delta > 2 \). For \( \beta \in (0, \gamma) \) and \( m \in \mathbb{N} \), we use Hölder’s inequality, Fubini’s theorem and Proposition 4.3 to get

(4.2)

\[
\mathbb{E} \left[ \left( \int_0^T |X^m(s) - X^m(\kappa_m(s))|^\delta \, ds \right)^q \right] \leq C \mathbb{E} \left[ \int_0^T |X^m(s) - X^m(\kappa_m(s))|^{\delta q} \, ds \right]
\]

\[
= C \int_0^T \mathbb{E} \left[ |X^m(s) - X^m(\kappa_m(s))|^{\delta q} \right] \, ds
\]

\[
\leq C \int_0^T |s - \kappa_m(s)|^{\delta q \beta} \, ds
\]

\[
\leq C 2^{-\delta q m^5}.
\]
For $0 < q \leq \frac{2}{3}$, we choose $q' > q$ is sufficiently large such that $q'\delta > 2$. Applying Jensen’s inequality and (4.2), we obtain
\[
\mathbb{E} \left( \left( \int_0^T |X^m(s) - X^m(\kappa_m(s))|^{\delta} \, ds \right)^q \right)^{\frac{1}{q}} \leq C 2^{-\delta q\beta m^5}.
\]
\[\square\]

Lemma 4.5. Suppose Assumptions [2.1] and [2.2]. Then, there is a sequence $(C_m)_{m \in \mathbb{N}}$ of constants such that
\[\mathbb{E}[|X^{m+1}(t) - X^m(t)|] \leq C_m\]
holds for every $t \in [0, T]$, and $\sum_{m=1}^{\infty} C_m^{1/4} < \infty$.

Proof. Following Gyöngy–Rásonyi [GR11] and Yamada–Watanabe [YW71], we approximate the function $\phi(x) := |x|$ by smooth functions $\phi_{\delta \epsilon}(x)$ for $\delta > 1$ and $\epsilon > 0$. To that end, note that
\[
\int_{\frac{\delta}{\epsilon}}^{\epsilon} \frac{1}{x} \, dx = \ln(\delta),
\]
and, thus, there is a continuous, non-negative function $\psi_{\delta \epsilon} : \mathbb{R} \to \mathbb{R}$, that is zero outside the interval $[\frac{\delta}{\epsilon}, \epsilon]$, $\int_0^\infty \psi_{\delta \epsilon}(x) \, dx = 1$ and satisfies
\[\psi_{\delta \epsilon}(x) \leq \frac{2}{x \ln(\delta)}\]

We define
\[\phi_{\delta \epsilon}(x) := \int_0^{|x|} \int_0^y \psi_{\delta \epsilon}(z) \, dz \, dy \quad \text{for} \quad x \in \mathbb{R},\]
such that the inequalities
\[|x| \leq \phi_{\delta \epsilon}(x) + \epsilon, \quad 0 \leq |\phi_{\delta \epsilon}'(x)| \leq 1 \quad \text{and} \quad \phi_{\delta \epsilon}''(x) = \psi_{\delta \epsilon}(|x|) \leq \frac{2}{|x| \ln(\delta)} \mathbf{1}_{[\frac{\delta}{\epsilon}, \epsilon]}(|x|)\]
hold for all $x \in \mathbb{R}$, where $\mathbf{1}_{[\frac{\delta}{\epsilon}, \epsilon]}$ denotes the indicator function of the interval $[\frac{\delta}{\epsilon}, \epsilon]$.

To apply Itô’s formula to $\phi_{\delta \epsilon}(\tilde{X}_t^m)$, where
\[\tilde{X}_t^m := X^{m+1}(t) - X^m(t), \quad t \in [0, T],\]
we need to find the semimartingale decomposition of $(\tilde{X}_t^m)_{t \in [0, T]}$. For this purpose, we introduce the local martingale
\[\tilde{Y}_t^m := Y_t^{m+1} - Y_t^m \quad \text{with} \quad Y_t^m := \int_0^t \sigma(s, X^m(\kappa_m(s))) \, dB_s\]
and the process of finite variation
\[\tilde{Z}_t^m := \int_0^t \mu(s, X^{m+1}(\kappa_{m+1}(s))) \, ds - \int_0^t \mu(s, X^m(\kappa_m(s))) \, ds, \quad \text{for} \ t \in [0, T].\]
Since $\partial_2 K_\mu \in L^1(\Delta_T), \partial_2 K_\sigma \in L^2(\Delta_T)$ (see Assumption [2.1]) and the integrability property of $(X^m(t))_{t \in [0, T]}$ as presented in Proposition [4.3], we obtain, as in the proof of Lemma [3.6], the
Hence, using (4.3) and applying Itô’s formula for fixed \( \tilde{\omega} \), we set

\[
X_t^m = \int_0^t K_\mu(s, t) \, d\tilde{Z}_s^m + \int_0^t K_\sigma(s, t) \, d\tilde{Y}_s^m
\]

\[
= \int_0^t K_\mu(s, s) \, d\tilde{Z}_s^m + \int_0^t \left( \int_0^s \partial_2 K_\mu(u, s) \, d\tilde{Z}_u^m \right) \, ds
\]

\[
+ \int_0^t \tilde{H}_s^m \, ds + \int_0^t K_\sigma(s, s) \, d\tilde{Y}_s^m,
\]

where \( \tilde{H}_t^m := H_t^{m+1} - H_t^m \) with \( H_t^m := \int_0^t \partial_2 K_\sigma(s, t) \, d\tilde{Y}_s^m \). Note that the quadratic variation of \( (X_t^m)_{t \in [0, T]} \) is given by

\[
\langle X_t^m \rangle_t = \left( \int_0^t K_\sigma(s, s) \left( \sigma(s, X^{m+1}(\kappa_{m+1}(s))) - \sigma(s, X^m(\kappa_m(s))) \right) \, dB_s \right)_t
\]

\[
= \int_0^t K_\sigma(s, s) \left( \sigma(s, X^{m+1}(\kappa_{m+1}(s))) - \sigma(s, X^m(\kappa_m(s))) \right)^2 \, ds, \quad t \in [0, T].
\]

Hence, using (4.3) and applying Itô’s formula for fixed \( \epsilon > 0 \) and \( \delta > 1 \) yields

\[
|\tilde{X}_t^m| \leq \epsilon + \phi_\delta(\tilde{X}_t^m)
\]

\[
= \epsilon + \int_0^t \phi_\delta'(\tilde{X}_s^m) \, d\tilde{X}_s^m + \frac{1}{2} \int_0^t \phi_\delta''(\tilde{X}_s^m) \, d(\tilde{X}_s^m)_s
\]

\[
= \epsilon + \int_0^t \phi_\delta'(\tilde{X}_s^m) K_\mu(s, s) \, d\tilde{Z}_s^m + \int_0^t \phi_\delta'(\tilde{X}_s^m) \left( \int_0^s \partial_2 K_\mu(u, s) \, d\tilde{Z}_u^m \right) \, ds
\]

\[
+ \int_0^t \phi_\delta'(\tilde{X}_s^m) \tilde{H}_s^m \, ds + \int_0^t \phi_\delta'(\tilde{X}_s^m) K_\sigma(s, s) \, d\tilde{Y}_s^m
\]

\[
+ \frac{1}{2} \int_0^t \phi_\delta''(\tilde{X}_s^m) K_\sigma(s, s)^2 \left( \sigma(s, X^{m+1}(\kappa_{m+1}(s))) - \sigma(s, X^m(\kappa_m(s))) \right)^2 \, ds
\]

\[\text{(4.4)}\]

\[
=: \epsilon + I_{1,t}^\delta + I_{2,t}^\delta + I_{3,t}^\delta + I_{4,t}^\delta + I_{5,t}^\delta
\]

for \( t \in [0, T] \).

In order to bound \( E[|\tilde{X}_t^m|] \), we shall estimate the five terms appearing in (4.4) separately. We set

\[
U_t^m := |X^m(t) - X^m(\kappa_m(t))|, \quad t \in [0, T].
\]

For \( I_{1,t}^\delta \), we use the boundedness of \( K_\mu \) (Assumption 2.1), the Lipschitz continuity of \( \mu \) (Assumption 2.2 (ii)) and the bound \( \|\phi_\delta''\|_\infty \leq 1 \) to estimate

\[
E[I_{1,t}^\delta] = E \left[ \int_0^t \phi_\delta'(\tilde{X}_s^m) K_\mu(s, s) \left( \mu(s, X^{m+1}(\kappa_{m+1}(s))) - \mu(s, X^m(\kappa_m(s))) \right) \, ds \right]
\]

\[
\leq C E \left[ \int_0^t \left( |\tilde{X}_s^m| + U_s^m + U_s^{m+1} \right) \, ds \right].
\]

Since, by Corollary 4.2

\[
E \left[ \int_0^t (U_s^m + U_s^{m+1}) \, ds \right] \leq C 2^{-\beta m^5}
\]
for any $\beta \in (0, \gamma)$, we get

\begin{equation}
E[I_{1,t}^4] \leq C \left( 2^{-\beta m^5} + \int_0^t E[|\tilde{X}_s^m|] \, ds \right).
\end{equation}

For $I_{2,t}^4$, using the boundedness of $\partial_2 K_\mu(u,s)$ on $\Delta_T$ (Assumption 2.1), the Lipschitz continuity of $\mu$ (Assumption 2.2 (ii)) and the bound $||\phi_{\beta_t}||_\infty \leq 1$, we obtain

\[
E[I_{2,t}^4] = E\left[ \int_0^t \phi'_{\beta_t}(\tilde{X}_s^m) \left( \int_0^s \partial_2 K_\mu(u,s) \left( \mu(u,X^{m+1}(\kappa_{m+1}(u))) - \mu(u,X^m(\kappa_m(u))) \right) \, du \right) \, ds \right] \leq CE \left[ \int_0^t (|\tilde{X}_s^m| + U_m^s + U_{m+1}^s) \, ds \right].
\]

Hence, as for $I_{1,t}^4$, we arrive at

\begin{equation}
E[I_{2,t}^4] \leq C \left( 2^{-\beta m^5} + \int_0^t E[|\tilde{X}_s^m|] \, ds \right).
\end{equation}

For $I_{3,t}^4$, we have

\[
E[I_{3,t}^4] = E\left[ \int_0^t \phi'_{\beta_t}(\tilde{X}_s^m)\tilde{H}_s^m \, ds \right].
\]

Noting that an application of the integration by parts formula for semimartingales (cf. [RW00, Theorem (VI), 38.3]) gives

\[
\tilde{H}_s^m = \int_0^s \partial_2 K_\sigma(u,s) \, d\tilde{Y}_m^u = \partial_2 K_\sigma(s,s)\tilde{Y}_s^m - \int_0^s \tilde{Y}_u^s \partial_{21} K_\sigma(u,s) \, du,
\]

we use $||\phi'_{\beta_t}||_\infty \leq 1$ and the stochastic Fubini theorem to get

\begin{equation}
E[I_{3,t}^4] \leq \int_0^t E[|\tilde{H}_s^m|] \, ds \leq \int_0^t |\partial_2 K_\sigma(s,s)|E[|\tilde{Y}_s^m|] \, ds + \int_0^t \int_0^s |\partial_{21} K_\sigma(u,s)|E[|\tilde{Y}_u^s|] \, du \, ds \leq \int_0^t E[|\tilde{Y}_s^m|]\left( |\partial_2 K_\sigma(s,s)| + \int_s^t |\partial_{21} K_\sigma(s,u)| \, du \right) \, ds.
\end{equation}

For $I_{4,t}^4$, we get

\[
E[I_{4,t}^4] = E\left[ \int_0^t \phi'_{\beta_t}(\tilde{X}_s^m)K_\sigma(s,s) \left( \sigma(s,X^{m+1}(\kappa_{m+1}(s))) - \sigma(s,X^m(\kappa_m(s))) \right) \, dB_s \right] = 0,
\]

since $I_{4,t}^4$ is a martingale by [Pro92, p.73, Corollary 3], since $E[|I_{4,t}^4|] < \infty$ for all $t \in [0,T]$ due to the boundedness of $K_\sigma$ (Assumption 2.1), the growth bound on $\sigma$ and Proposition 4.3.
For $I_{5,t}^{\delta}$, using the boundedness of $K_{\sigma}$ (Assumption 2.1), the Hölder continuity of $\sigma$ (Assumption 2.2 (ii)) and the inequality (4.3), we get that

$$E[I_{5,t}^{\delta}] = E\left[\frac{1}{2}\int_0^t \phi'_\delta(\tilde{X}^m_s)K_{\sigma}(s, s)^2\left(\sigma(s, X^{m+1}(\kappa_{m+1}(s))) - \sigma(s, X^m(\kappa_m(s)))\right)^2 ds\right]$$

$$\leq CE\left[\int_0^t \phi'_\delta(\tilde{X}^m_s)(|\tilde{X}^m(s)| + U^m_s + U^{m+1}_s)^{1+2\delta} ds\right]$$

$$\leq CE\left[\int_0^t 1_{[\delta, \epsilon]}(|\tilde{X}^m(s)|)(|\tilde{X}^m(s)| + U^m_s + U^{m+1}_s)^{1+2\delta} ds\right]$$

(4.9)

Moreover, by Corollary 4.3, we derive that

$$E\left[\int_0^t (U^m_s + U^{m+1}_s)^{1+2\delta} ds\right] \leq C2^{-\frac{1+2\delta}{2}\beta m^5}$$

for any $\beta \in (0, \gamma)$ and, hence, we conclude

(4.10)

$$E[I_{5,t}^{\delta}] \leq C\left(\frac{e^{2\kappa}}{\ln(\delta)} + \frac{\delta}{\epsilon \ln(\delta)}\right).$$

Combining (4.4) with the five estimates (4.5), (4.6), (4.7), (4.8) and (4.10), we end up with

$$E[\tilde{X}^m_t] \leq C\left(2^{-\beta m^5} + \frac{e^{2\kappa}}{\ln(\delta)} + \frac{\delta}{\epsilon \ln(\delta)}\right) + \int_0^t E[\tilde{X}^m_s] ds$$

$$+ \int_0^t E[|u^m_s|] \left(|\partial_2 K_{\sigma}(s, s)| + \int_s^t |\partial_{21} K_{\sigma}(s, u)| du\right) ds.$$ 

Therefore, choosing $\delta := 2^{\rho m^5}$ for $\rho \in (0, ((1 + 2\delta)\beta)/2]$ and $\epsilon := 2^{-\frac{(1+2\delta)\beta}{2}\mu^5}$, we get

$$E[|X^m_t|] \leq C_m + \int_0^t E[|\tilde{X}^m_s|] ds$$

(4.11)

$$+ \int_0^t E[|u^m_s|] \left(|\partial_2 K_{\sigma}(s, s)| + \int_s^t |\partial_{21} K_{\sigma}(s, u)| du\right) ds,$$

with

(4.12)

$$C_m := 2^{-\beta m^5} + m^{-5}2^{-\frac{1+2\delta}{2}\beta m^5} + m^{-5}2^{-\frac{(1+2\delta)\beta}{2}\mu^5}.$$

To apply a Grönwall lemma, we set

$$M_m(t) := \sup_{s \in [0, t]} \left(E[|\tilde{X}^m_s|] + E[|u^m_s|]\right), \quad t \in [0, T],$$

and derive in the following an inequality of the form $M_m(t) \leq C_m + \int_0^t f(t - s)M_m(s) ds$ for a suitable function $f$.
To get a bound for $\mathbb{E}[|\tilde{Y}_t^m|]$, we first apply the integration by part formula to obtain

$$\tilde{X}_t^m = \int_0^t K_\mu(s, t) \left( \mu(s, X^{m+1}_s(\kappa_{m+1}(s))) - \mu(s, X^m_s(\kappa_m(s))) \right) ds + \int_0^t K_\sigma(s, t) d\tilde{Y}_s^m$$

$$= \int_0^t K_\mu(s, t) \left( \mu(s, X^{m+1}_s(\kappa_{m+1}(s))) - \mu(s, X^m_s(\kappa_m(s))) \right) ds$$

$$+ K_\sigma(t, t) \tilde{Y}_t^m - \int_0^t \partial_t K_\sigma(s, t) \tilde{Y}_s^m ds,$$

where we used that $K_\sigma(:, t)$ is absolutely continuous for every $t \in [0, T]$. Since $K_\sigma(t, t) > C$ for some constant $C > 0$, $K_\mu$ is bounded (both by Assumption 2.2) and $\mu$ is Lipschitz continuous (Assumption 2.2), we get

$$\mathbb{E}[|\tilde{Y}_t^m|] \leq C \mathbb{E}[|\tilde{X}_t^m|] + \int_0^t \mathbb{E}[|K_\mu(s, t)|] \left( \mu(s, X^{m+1}_s(\kappa_{m+1}(s))) - \mu(s, X^m_s(\kappa_m(s))) \right) ds$$

$$\leq C \left( \mathbb{E}[|\tilde{X}_t^m|] + \int_0^t \mathbb{E}[|\tilde{X}_s^m|] ds + \mathbb{E} \left[ \int_0^t (U_s^m + U_s^{m+1}) ds \right] \right)$$

$$\leq C \left( 2^{-\beta m^5} + \mathbb{E}[|\tilde{X}_t^m|] + \int_0^t \mathbb{E}[|\tilde{X}_s^m|] ds + \int_0^t |\partial_t K_\sigma(s, t)| \mathbb{E}[|\tilde{Y}_s^m|] ds \right),$$

where we used Corollary 4.4 for the last estimate. Hence, by (4.11) we obtain

$$\mathbb{E}[|\tilde{Y}_t^m|] \leq C \left( C_m + \int_0^t \mathbb{E}[|\tilde{X}_s^m|] ds \right)$$

$$+ \int_0^t \mathbb{E}[|\tilde{Y}_s^m|] \left( |\partial_t K_\sigma(s, t)| + |\partial_2 K_\sigma(s, s)| + \int_s^t |\partial_{21} K_\sigma(s, u)| du \right) ds.$$

(4.13)

By the bound on the partial derivatives of $K_\sigma$ made in Assumption 2.1 (4.11) and (4.13) can be further estimated to

$$\mathbb{E}[|\tilde{X}_t^m|] \leq C \left( C_m + \int_0^t \mathbb{E}[|\tilde{X}_s^m|] ds + \int_0^t (t - s)^{-\alpha} \mathbb{E}[|\tilde{Y}_s^m|] ds \right),$$

$$\mathbb{E}[|\tilde{Y}_t^m|] \leq C \left( C_m + \int_0^t \mathbb{E}[|\tilde{X}_s^m|] ds + \int_0^t (t - s)^{-\alpha} \mathbb{E}[|\tilde{Y}_s^m|] ds \right),$$

for $\alpha \in [0, \frac{1}{2})$ as given in Assumption 2.1. Hence, we arrive at

$$M_m(t) \leq \sup_{s \in [0, t]} \mathbb{E}[|\tilde{X}_t^m|] + \sup_{s \in [0, t]} \mathbb{E}[|\tilde{Y}_t^m|]$$

$$\leq C \left( C_m + \int_0^t (1 + (t - s)^{-\alpha}) M_m(s) ds \right).$$

Note that Proposition 4.3 secures the integrability of $M_m$. An application of the Grönwall’s lemma for weak singularities (see e.g. [Kru14] Lemma A.2) reveals that $M_m(t) \leq CC_m$. The claimed summability of the sequence $(C_m)_{m \in \mathbb{N}}$ follows immediately by (4.12). □
Remark 4.6. The approximation $\phi_\delta$ of the absolute value, as used in the proof of Theorem 4.1, was introduced by Gyöngy and Rásonyi [GR11]. It is a modification of the approximation originally used by Yamada and Watanabe [YW71] and appears to be more involved. While the original approximation of Yamada and Watanabe is sufficient to prove pathwise uniqueness, as we will also see in Section 4 to prove the existence of a solution the approximation $\phi_\delta$ seems necessary. Indeed, one needs $\epsilon \to 0$ to ensure that $\phi_\delta \to |\cdot|$ but the second parameter $\delta$ is essential to obtain the convergence of the Euler type approximation $(X^m)_{m\in\mathbb{N}}$ in the case $\xi = 0$ (i.e., $\sigma$ is 1/2-Hölder continuous), as one can see from (4.11) and (4.12).

With these preparations at hand we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Step 1: The sequence $(X^m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega \times [0,T])$ for $p$ given in the statement of Theorem 4.1.

By Fubini’s theorem and Lemma 4.3 there exists a sequence $(C_m)_{m\in\mathbb{N}}$ such that

$$
\mathbb{E}\left[ \int_0^T |X^{m+1}(s) - X^m(s)|^p \, ds \right] \leq C \sup_{s \in [0,T]} \mathbb{E}\left[ |X^{m+1}(s) - X^m(s)| \right] \leq C_m
$$

for $m \in \mathbb{N}$. Hence, using Hölder’s inequality and the moment bound for $(X^m(t))_{t \in [0,T]}$ from Proposition 4.3, we get

$$
\mathbb{E}\left[ \int_0^T |X^{m+1}(t) - X^m(t)|^p \, dt \right]
\leq \mathbb{E}\left[ \int_0^T |X^{m+1}(t) - X^m(t)|^{2p-1} \, dt \right]^{\frac{1}{2p-1}} \mathbb{E}\left[ \int_0^T |X^{m+1}(t) - X^m(t)| \, dt \right]^{\frac{1}{2}}
\leq 2^{p-1} \left( 1 + \sup_{t \in [0,T]} |x_0(t)|^{2p-1} \right)^{\frac{1}{2p-1}} C_m.
$$

Due to the summability property of $(C_m)_{m\in\mathbb{N}}$, the sequence $(X^m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega \times [0,T])$. Hence, there exists a process $X = (X_t)_{t \in [0,T]} \in L^p(\Omega \times [0,T])$, such that

$$
\lim_{m \to \infty} \mathbb{E}\left[ \int_0^T |X^m(s) - X_s|^p \, ds \right] = 0.
$$

Step 2: $(X_t)_{t \in [0,T]}$ yields a strong solution to the SVE (2.1).

By construction, the processes $(X^m(t))_{t \in [0,T]}$ are $(\mathcal{F}_t)_{t \in [0,T]}$-progressively measurable on the given probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$. Since (4.14) also shows the $L^p([0,t] \times \Omega)$-convergence of $(X^m)_{s \in [0,t]}$ to $(X_s)_{s \in [0,t]}$ for every $t \in [0,T]$, the completeness of the $L^p$ spaces [see e.g. [Kre13] Theorem 7.3] yields $B([0,t]) \otimes \mathcal{F}_t$-measurability of $(s, \omega) \mapsto X_s(\omega)$, $(s, \omega) \in [0,t] \times \Omega$ for every $t \in [0,T]$. Hence, the process $(X_t)_{t \in [0,T]}$ is also $(\mathcal{F}_t)_{t \in [0,T]}$-progressively measurable on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$. Moreover, by the growth conditions on $\mu$ and $\sigma$ (see Assumption 2.2 (ii)) and the integrability properties of $K_\mu$ and $K_\sigma$, we get that

$$
\int_0^t (|K_\mu(s,t)\mu(s,X_s)| + |K_\sigma(s,t)\sigma(s,X_s)|^2) \, ds < \infty \quad \text{for all } t \in [0,T].
$$

It remains to show that the process $(X_t)_{t \in [0,T]}$ fulfills the SVE (2.1). To that end, we show that the two integrals in (4.11) preserve the $L^p(\Omega \times [0,T])$-convergence. For the Riemann–Stieltjes integral, we use the boundedness of $K_\mu$, the Lipschitz continuity of $\mu$, Hölder’s
inequality and Fubini’s theorem to obtain
\[
\mathbb{E} \left[ \int_0^T \left| \int_0^t K_\mu(s,t) (\mu(s,X^m_\sigma(s))) - \mu(s,X_s) \right|^p ds \right] dt \\
\leq C \int_0^T \int_0^T \mathbb{E} \left[ |X^m_\sigma(s) - X_s|^p \right] ds dt \\
\leq C \left( \mathbb{E} \left[ \int_0^T |X^m_\sigma(s) - X^m(s)|^p ds \right] + \mathbb{E} \left[ \int_0^T |X^m(s) - X_s|^p ds \right] \right) \to 0,
\]
as \( m \to \infty \) by Corollary 4.4 and (4.4). For the stochastic integral, we use Fubini’s theorem, Burkholder–Davis–Gundy’s inequality, Hölder’s inequality, the boundedness of \( K_\sigma \), and the Hölder regularity of \( \sigma \) to get that
\[
\mathbb{E} \left[ \int_0^T \left| \int_0^t K_\sigma(s,t) (\sigma(s,X^m_\sigma(s))) - \sigma(s,X_s) \right|^p dB_s \right] dt \\
= \int_0^T \mathbb{E} \left[ \left| \int_0^t K_\sigma(s,t) (\sigma(s,X^m_\sigma(s))) - \sigma(s,X_s) \right|^p dB_s \right] dt \\
\leq \int_0^T \mathbb{E} \left[ \int_0^t K_\sigma(s,t)^2 (\sigma(s,X^m_\sigma(s))) - \sigma(s,X_s)^2 \right] ds \frac{t}{2} dt \\
\leq C \left( \mathbb{E} \left[ \int_0^T \left| X^m_\sigma(s) - X_s \right|^{2 + p \xi} ds \right] + \mathbb{E} \left[ \int_0^T |X^m(s) - X_s|^{2 + p \xi} ds \right] \right).
\]
Thus, by Corollary 4.4 and the convergence \( X^m \to X \) in \( L^{\frac{2}{1+\xi}}(\Omega \times [0,T]) \) as \( m \to \infty \), for \( \xi \in (0,\frac{1}{2}) \), which is implied by the one in \( L^p(\Omega \times [0,T]) \), we see that the stochastic integral does preserve the \( L^p(\Omega \times [0,T]) \)-convergence. Thus, we have proven that the limiting process \( (X_t)_{t \in [0,T]} \) fulfills the SVE (2.1) for almost all \( (t,\omega) \in [0,T] \times \Omega \). By Remark 3.2 \( (X_t)_{t \in [0,T]} \) has an \( \mathbb{P} \)-a.s. continuous version, which fulfills the SVE (2.1) for all \( t \in [0,T] \) for almost all \( \omega \in \Omega \), and hence, is a strong solution of (2.1). \( \blacksquare \)

5. Pathwise uniqueness

In this section we establish the pathwise uniqueness for the stochastic Volterra equation (2.1) under Assumptions 2.1 2.2 (i), and under slightly weaker regularity assumptions on \( \mu \) and \( \sigma \) than Assumption 2.2 (ii), namely an Osgood-type condition on \( \mu \) and the Yamada–Watanabe condition on \( \sigma \), as formulated in the next assumption.

**Assumption 5.1.** Let \( \mu, \sigma : [0,T] \times \mathbb{R} \to \mathbb{R} \) be measurable functions such that:

(i) there is some continuous, non-decreasing and concave function \( \kappa : [0,\infty) \to [0,\infty) \) with \( \kappa(0) = 0 \) and \( \kappa(x) > 0 \) for \( x > 0 \), such that, with the notation \( \tilde{\kappa}(x) := \kappa(x) + |x| \),
\[
\int_0^\epsilon \frac{dx}{(\tilde{\kappa}(\sqrt{x}))^q} = \infty,
\]
holds for all \( \epsilon > 0 \) and \( q \in (\frac{1}{1-\alpha}, \frac{1}{1-\alpha} + \epsilon) \) for some \( \epsilon > 0 \), where \( \alpha \in [0,\frac{1}{2}) \) is given by Assumption 2.2 (ii), and
\[
|\mu(t,x) - \mu(t,y)| \leq \kappa(|x - y|),
\]
for all \( t \in [0, T], \ x, y \in \mathbb{R} \),

(ii) there is some continuous strictly increasing function \( \rho: [0, \infty) \to [0, \infty) \) with \( \rho(0) = 0 \) and \( \rho(x) > 0 \) for \( x > 0 \); such that

\[
\int_0^\epsilon \frac{dx}{\rho(x)^2} = \infty,
\]

holds for all \( \epsilon > 0 \), and

\[
|\sigma(t,x) - \sigma(t,y)| \leq \rho(|x - y|),
\]

for all \( t \in [0, T], \ x, y \in \mathbb{R} \).

**Remark 5.2.** Choosing \( \kappa(x) = C_\mu |x| \) and \( \rho(x) = C_\sigma |x|^{\frac{1}{2} + \epsilon} \) shows that Assumption \([2.2]\) (ii) implies Assumption \([5.1]\). We note that if \( \mu \) is assumed to be Lipschitz continuous and \( \sigma \) to fulfill the Yamada–Watanabe condition, it is sufficient to use a fractional Grönwall lemma like the one in [Kru14 Lemma A.2] instead of the fractional Bihari inequality in \([5.12]\). Moreover, if one considers \( K_\sigma = 1 \), the Osgood-type condition in Assumption \([5.1]\) (i) can be replaced by the classical Osgood condition for SDEs (see e.g. [KS91 Chapter 5, Remark 2.16]) since one can then use the classical instead of the fractional Bihari inequality and the application of integration by parts to the stochastic integral is not required.

The main result of this section reads as follows.

**Theorem 5.3.** Suppose Assumptions \([2.1]\) \([2.2]\) (i) and \([5.1]\). Then, pathwise uniqueness holds for the stochastic Volterra equation \([2.1]\).

**Proof.** Since the proof relies partly on similar techniques as the proof of Lemma \([4.5]\) we try to give a condensed presentation and refer to the analogue calculation in Section \([4]\).

Let \((X^1_i)_{i \in [0,T]}\) and \((X^2_i)_{i \in [0,T]}\) be solutions to the SVE \([2.1]\). Analogously to Section \([4]\) we define \(Y^i_t := \int_0^t \sigma(s, X^i_s) \, dB_s \) and \(H^i_t := \int_0^t \partial_2 K_\sigma(s, t) \, dY^i_s \), for \( i = 1, 2 \), as well as \( \tilde{Y}_t := Y^1_t - Y^2_t \), \( \tilde{X}_t := X^1_t - X^2_t \), \( \tilde{H}_t := H^1_t - H^2_t \), and \( Z_t := \int_0^t (\mu(s, X^1_s) - \mu(s, X^2_s)) \, ds \), for \( t \in [0, T] \). By Lemma \([3.6]\) we obtain the semimartingale decomposition

\[
\tilde{X}_t = \int_0^t K_\mu(s, s) (\mu(s, X^1_s) - \mu(s, X^2_s)) \, ds + \int_0^t \int_0^s \partial_2 K_\mu(u, s) \, d\tilde{Z}_u \, ds
\]

\[
+ \int_0^t \tilde{H}_s \, ds + \int_0^t K_\sigma(s, s) \, d\tilde{Y}_s, \quad t \in [0, T].
\]

To construct an approximation of the absolute value by smooth functions allowing us to apply Itô’s formula, we use the classical approximation of Yamada–Watanabe [YW71] for simplicity, cf. Remark \([4.6]\). Based on the strictly increasing function \( \rho \) from Assumption \([5.1]\) (ii), we define a sequence \((\phi_n)_{n \in \mathbb{N}}\) of functions mapping from \( \mathbb{R} \) to \( \mathbb{R} \) that approximates the absolute value in the following way: Let \((a_n)_{n \in \mathbb{N}}\) be a strictly decreasing sequence with \( a_0 = 1 \) such that \( a_n \to 0 \) as \( n \to \infty \) and

\[
\int_{a_n}^{a_{n-1}} \frac{1}{\rho(x)^2} \, dx = n.
\]

Furthermore, we define a sequence of mollifiers: let \((\psi_n)_{n \in \mathbb{N}} \in C^\infty_0(\mathbb{R})\) be smooth functions with compact support such that \( \text{supp}(\psi_n) \subset (a_n, a_{n-1}) \), and with the properties

\[
0 \leq \psi_n(x) \leq \frac{2}{n \rho(x)^2}, \quad \forall x \in \mathbb{R}, \quad \text{and} \quad \int_{a_n}^{a_{n-1}} \psi_n(x) \, dx = 1.
\]
We set
\[ \phi_n(x) := \int_0^{[x]} \left( \int_0^y \psi_n(z) \, dz \right) \, dy, \quad x \in \mathbb{R}. \]
By (5.2) and the compact support of \( \psi_n \), it follows that \( \phi_n(\cdot) \to |\cdot| \) uniformly as \( n \to \infty \). Since every \( \psi_n \) and, thus, every \( \phi_n \) is zero in a neighborhood around zero, the functions \( \phi_n \) are smooth with
\[ \|\phi_n\|_\infty \leq 1, \quad \phi_n(x) = \text{sgn}(x) \int_0^{[x]} \psi_n(y) \, dy, \quad \text{and} \quad \phi_n''(x) = \psi_n(|x|) \quad \text{for} \quad x \in \mathbb{R}. \]
Since the quadratic variation of the semimartingale \( (\tilde{X}_t)_{t \in [0,T]} \) is given by
\[ (\tilde{X}_t) = \int_0^t K_\sigma(s, s)^2(\sigma(s, X^1_s) - \sigma(s, X^2_s))^2 \, ds, \quad t \in [0, T], \]
we get, by applying Itô’s formula and using the semimartingale decomposition (5.1), that
\[ \phi_n(\tilde{X}_t) = \int_0^t \phi_n'(\tilde{X}_s) \, d\tilde{X}_s + \frac{1}{2} \int_0^t \phi_n''(\tilde{X}_s) \, d(\tilde{X})_s \]
\[ = \int_0^t \phi_n'(\tilde{X}_s) K_\sigma(s, s)(\mu(s, X^1_s) - \mu(s, X^2_s)) \, ds + \int_0^t \phi_n'(\tilde{X}_s) \left( \int_0^s \partial_2 K_\mu(u, s) \, d\tilde{Z}_u \right) \, ds \]
\[ + \int_0^t \phi_n''(\tilde{X}_s) H_s \, ds + \int_0^t \phi_n'(\tilde{X}_s) K_\sigma(s, s) \, d\tilde{Y}_s \]
\[ + \frac{1}{2} \int_0^t \phi_n'''(\tilde{X}_s) K_\sigma(s, s)^2(\sigma(s, X^1_s) - \sigma(s, X^2_s))^2 \, ds \]
(5.3) \[ := I^n_{1,t} + I^n_{2,t} + I^n_{3,t} + I^n_{4,t} + I^n_{5,t} \]
for \( t \in [0, T] \).

For \( I^n_{1,t} \), we use Assumption 5.1 (i), the boundedness of \( K_\mu \) (Assumption 2.1), the bound \( \|\phi_n\|_\infty \leq 1 \) and Jensen’s inequality to estimate
\[ \mathbb{E}[I^n_{1,t}] \leq C \int_0^t \mathbb{E}[\kappa(|\tilde{X}_s|)] \, ds \leq C \int_0^t \kappa(\mathbb{E}[|\tilde{X}_s|]) \, ds. \]
For \( I^n_{2,t} \), we additionally use the boundedness of \( \partial_2 K_\mu(u, s) \) on \( \Delta_T \) to obtain
\[ \mathbb{E}[I^n_{2,t}] \leq C \int_0^t \kappa(\mathbb{E}[|\tilde{X}_s|]) \, ds. \]
For \( I^n_{3,t} \), similarly to (4.7), we use the integration by parts formula to estimate
\[ \mathbb{E}[I^n_{3,t}] \leq \int_0^t \mathbb{E}[|\tilde{H}_s|] \, ds \]
\[ \leq \int_0^t \|\partial_2 K_\sigma(s, s)\| \mathbb{E}[|\tilde{Y}_s|] \, ds + \int_0^t \int_0^s \|\partial_{21} K_\sigma(u, s)\| \mathbb{E}[|\tilde{Y}_u|] \, du \, ds \]
\[ \leq \int_0^t \mathbb{E}[|\tilde{Y}_s|] \left( \|\partial_2 K_\sigma(s, s)\| + \int_0^s \|\partial_{21} K_\sigma(s, u)\| \, du \right) \, ds. \]
(5.6)
For $I^n_{4,t}$, since $I^n_{4,t}$ is a martingale by [Pro92, p.73, Corollary 3] due to the boundedness of $K_\sigma$, the growth bound on $\sigma$ and Lemma 5.4 we get

$$
(5.7) \quad \mathbb{E}[I^n_{4,t}] = \mathbb{E} \left[ \int_0^t \phi'_n(\tilde{X}_s)K_\sigma(s,s)(\sigma(s,s,X^1_s) - \sigma(s,s,X^2_s))\,d\mathbb{B}_s \right] = 0,
$$

For $I^n_{5,t}$, we get by using the boundedness of $K_\sigma$ (Assumption 2.1), the regularity of $\sigma$ from Assumption 5.1 (ii), and the inequality (5.2) that

$$
\mathbb{E}[I^n_{5,t}] \leq C \mathbb{E} \left[ \int_0^t \phi''_n(\tilde{X}_s)\sigma(|\tilde{X}_s|)^2 \,ds \right]
\leq C \mathbb{E} \left[ \int_0^t \frac{2}{n\rho(|X_s|)^2} \sigma(|\tilde{X}_s|)^2 \,ds \right]
\leq C \frac{1}{n},
$$

(5.8)

for some $C > 0$.

Finally, sending $n \to \infty$ and combining the five previous estimates (5.4), (5.5), (5.6), (5.7) and (5.8) with (5.3) implies

$$
(5.9) \quad \mathbb{E}[|\tilde{X}_t|] \leq C \int_0^t \kappa(\mathbb{E}[|\tilde{X}_s|]) \,ds + \int_0^t \mathbb{E}[|\tilde{Y}_s|] \left( |\partial_2 K_\sigma(s,s) - |\sigma(s,s)| \right) d\mathbb{Y}_s.
$$

To apply a Grönwall lemma, we set

$$
M(t) := \sup_{s \in [0,t]} \left( \mathbb{E}[|\tilde{X}_s|] + \mathbb{E}[|\tilde{Y}_s|] \right), \quad t \in [0,T],
$$

and derive in the following an inequality of the form $M(t) \leq \int_0^t f(t-s)\tilde{\kappa}(M(s)) \,ds$ for suitable functions $f$ and $\tilde{\kappa}$. To find a bound for $\mathbb{E}[|\tilde{Y}_t|]$, we apply the integration by part formula to obtain

$$
\tilde{X}_t = \int_0^t K_\mu(s,t)(\mu(s,X^1_s) - \mu(s,X^2_s)) \,ds + \int_0^t K_\sigma(s,t) \,d\tilde{Y}_s
= \int_0^t K_\mu(s,t)(\mu(s,X^1_s) - \mu(s,X^2_s)) \,ds + K_\sigma(t,t)\tilde{Y}_t - \int_0^t \partial_1 K_\sigma(s,t)\tilde{Y}_s \,ds
$$

(5.10)

keeping in mind that that $K_\sigma(.,t)$ is absolutely continuous for every $t \in [0,T]$. Due to $|K_\sigma(t,t)| > C$ for some constant $C > 0$, we can rearrange (5.10) and use (5.9) to get

$$
\mathbb{E}[|\tilde{Y}_t|] \leq C \left( \int_0^t \mathbb{E}[|\mu(s,X^1_s) - \mu(s,X^2_s)|] \,ds + \mathbb{E}[|\tilde{X}_t|] + \int_0^t |\partial_1 K_\sigma(s,t)| \mathbb{E}[|\tilde{Y}_s|] \,ds \right)
\leq C \left( \int_0^t \left( \mathbb{E}[|\tilde{X}_s|] + \kappa(\mathbb{E}[|\tilde{X}_s|]) \right) \,ds + \int_0^t \mathbb{E}[|\tilde{Y}_s|] \left( |\partial_1 K_\sigma(s,t)| + |\partial_2 K_\sigma(s,s)| + \int_s^t |\partial_21 K_\sigma(s,u)| \,du \right) \,ds \right).
$$

(5.11)
Using Assumption 2.1 to bound the partial derivative terms in (5.9) and (5.11), we end up with
\[
M(t) \leq \sup_{s \in [0,t]} E[|\tilde{X}_s|] + \sup_{s \in [0,t]} E[|\tilde{Y}_s|] \\
\leq C \left( \int_0^t \left( \sup_{u \in [0,s]} E[|\tilde{X}_u|] + \kappa \left( \sup_{u \in [0,s]} E[|\tilde{X}_u|] \right) \right) ds + \int_0^t (t-s)^{-\alpha} \sup_{u \in [0,s]} E[|\tilde{Y}_u|] ds \right)
\]
(5.12) \[ \leq C \int_0^t (t-s)^{-\alpha} \tilde{\kappa}(M(s)) ds,
\]
where \( \tilde{\kappa}(x) := \kappa(x) + |x| \). An application of the fractional Bihari inequality, [OHNO21 Theorem 2.3], with sending \( q \to \frac{1}{1-\alpha} \) like in [OHNO21 proof of Theorem 3.1, Step 1] with the condition on \( \tilde{\kappa} \) in Assumption 5.1 (i) that \( M(t) = 0 \) holds. Hence, \( \tilde{X}_t = 0 \) almost surely, and, thus, by the continuity of the solutions, the processes \( (X^1_t)_{t \in [0,T]} \) and \( (X^2_t)_{t \in [0,T]} \) are indistinguishable. \[ \square \]

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