Anomaly-free scalar perturbations with holonomy corrections in loop quantum cosmology

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Abstract

Holonomy corrections to scalar perturbations are investigated in the loop quantum cosmology framework. Due to the effective approach, modifications of the algebra of constraints generically lead to anomalies. In order to remove those anomalies, counter terms are introduced. We find a way to explicitly fulfill the conditions for anomaly freedom and we give explicit expressions for the counter terms. Surprisingly, the $\bar{\mu}$-scheme naturally arises in this procedure. The gauge-invariant variables are found and equations of motion for the anomaly-free scalar perturbations are derived. Finally, some cosmological consequences are discussed qualitatively.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Loop quantum gravity (LQG) is a tentative non-perturbative and background-independent quantization of general relativity (GR) \cite{1}. Interestingly, it has now been demonstrated that different approaches, based on canonical quantization of GR, on covariant quantization of GR and on formal quantization of geometry lead to the very same LQG theory. Although this is rather convincing, a direct experimental probe is still missing. One can easily argue that cosmology is the most promising approach to search for observational features of LQG or, more specifically, to its symmetry-reduced version, loop quantum cosmology (LQC) \cite{2}.

Much effort has been devoted to the search of possible footprints of LQC in cosmological tensor modes (see \cite{3}). At the theoretical level, the situation is easier in this case as the
algebra of constraints is automatically anomaly free. But, as far as observations are concerned, scalar modes are far more important. They have already been observed in great detail by WMAP [4] and are currently even better observed by the Planck mission. The question of a possible modification of the primordial scalar power spectrum (and of the corresponding TT $C_l$ spectrum) in LQC is therefore essential in this framework.

Gravity is described by a set of constraints. However, for the (effective) theory to be consistent, it is mandatory that the evolution generated by the constraints remains compatible with the constraints themselves. This is always true if their mutual Poisson brackets vanish when evaluated in fields fulfilling the constraints, i.e. if they form a first-class algebra. This means that the evolution and the gauge transformations are associated with vector fields that are tangent to the manifold of null constraints. This obviously holds at the classical level. However, when quantum modifications are added, the anomaly freedom is not anymore automatically ensured. Possible quantum corrections must be restricted to those which close the algebra. This means that, for consistency reasons, the Poisson brackets between any two constraints must be proportional to one constraint of the algebra. This paper is devoted to the search for such an algebra for scalar perturbations.

Our approach will follow the one developed by Bojowald et al in [5]. There are two main quantum corrections expected from LQC: inverse volume terms, basically arising for inverse powers of the densitized triad, which when quantized become an operator with zero in its discrete spectrum thus lacking a direct inverse, and holonomy corrections coming from the fact that loop quantization is based on holonomies, rather than direct connection components. In [5], the authors focused exclusively on inverse volume corrections. Here, we extend with work to the holonomy corrections. Scalar perturbations with holonomy corrections have been studied in [6]. However, the issue of anomaly freedom was not really addressed. Recently, a new possible way of introducing holonomy corrections to the scalar perturbations was proposed in [7]. Although it was interestingly shown that the formulation is anomaly free, the approach is based on the choice of the longitudinal gauge and the extension of the method to the gauge-invariant case is not straightforward. In contrast, the approach developed in our paper does not rely on any particular choice of gauge and the gauge-invariant cosmological perturbations are easily constructed.

The theory of anomaly-free scalar perturbations developed in this paper is obtained on a flat FRW background, such that the line element is given by

$$ds^2 = a^2[-(1 + 2\phi) d\eta^2 + 2\partial_a B d\eta dx^a + ((1 - 2\psi)\delta_{ab} + 2\partial_a \partial_b E) dx^a dx^b],$$

(1)

where $\phi$, $\psi$, $E$ and $B$ are scalar perturbation functions. The matter content is assumed to be a scalar field. This will allow us to investigate the generation of scalar perturbations during the phase of cosmic inflation while taking into account the quantum gravity effects.

Our analysis of the scalar perturbations is performed in the Hamiltonian framework developed in [5, 8]. As was shown there, the background variables are $(\vec{k}, \vec{p}, \vec{\phi}, \vec{\pi})$, while the perturbed variables are $(\delta K^a_i, \delta E^a, \delta \phi, \delta \pi)$. The Poisson bracket for the system can be decomposed as follows:

$$\{\cdot, \cdot\} = \{\cdot, \cdot\}_{\vec{k}, \vec{p}} + \{\cdot, \cdot\}_{\delta K, \delta E} + \{\cdot, \cdot\}_{\delta \phi, \delta \pi},$$

(2)

where

$$\{\cdot, \cdot\}_{\vec{k}, \vec{p}} := \frac{\kappa}{3V_0} \left[ \frac{\partial \vec{\cdot}}{\partial k} \frac{\partial \vec{\cdot}}{\partial \vec{p}} - \frac{\partial \vec{\cdot}}{\partial \vec{p}} \frac{\partial \vec{\cdot}}{\partial k} \right],$$

(3)

$$\{\cdot, \cdot\}_{\delta K, \delta E} := \kappa \int \Sigma d^3 x \left[ \frac{\partial \vec{\cdot}}{\partial \delta K^j_i} \frac{\partial \vec{\cdot}}{\partial \delta E^a} - \frac{\partial \vec{\cdot}}{\partial \delta E^a} \frac{\partial \vec{\cdot}}{\partial \delta K^j_i} \right],$$

(4)
\[ \{\cdot,\cdot\}_\psi,\pi := \frac{1}{V_0} \left[ \frac{\partial}{\partial \psi} \frac{\partial}{\partial \pi} - \frac{\partial}{\partial \pi} \frac{\partial}{\partial \psi} \right], \quad (5) \]

\[ \{\cdot,\cdot\}_\psi,\delta \pi := \int_{\Sigma} d^3x \left[ \frac{\delta}{\delta \psi} \frac{\delta}{\delta \delta \pi} - \frac{\delta}{\delta \delta \pi} \frac{\delta}{\delta \psi} \right]. \quad (6) \]

Here, \( V_0 \) is the volume of the fiducial cell and \( \kappa = 8\pi G \).

The holonomy corrections are introduced by the replacement \( \tilde{k} \to \mathbb{K}[n] \) in the classical Hamiltonian. We follow the notation introduced in [9], where

\[ \mathbb{K}[n] := \begin{cases} \sin(n \tilde{\mu} \gamma \tilde{k}) \overline{n \mu \gamma} & \text{for } n \in \mathbb{Z}/\{0\}, \\ \tilde{k} & \text{for } n = 0, \end{cases} \quad (7) \]

for the correction function. In cases where \( \tilde{k} \) appears quadratically, the integer \( n \) is fixed to 2 (see [9]). In the other cases, the integers remain to be fixed from the requirement of anomaly freedom. The coefficient \( \gamma \) is the Barbero–Immirzi parameter and \( \tilde{\mu} \propto \tilde{p}^\beta \) where \(-1/2 \leq \beta \leq 0\). In what follows, the relation

\[ \tilde{p} \frac{\partial}{\partial \tilde{p}} \mathbb{K}[n] = [\tilde{k} \cos(n \tilde{\mu} \gamma \tilde{k}) - \mathbb{K}[n]] \beta \quad (8) \]

will be useful.

The organization of the paper is as follows. In section 2, the holonomy-corrected gravitational Hamiltonian constraint is defined. We calculate the Poisson bracket of the Hamiltonian constraint with itself and with the gravitational diffeomorphism constraint. In section 3, scalar matter is introduced. The Poisson brackets between the total constraints for the system under consideration are calculated. In section 4, the conditions for anomaly freedom are solved and the expressions for the counter terms are derived. Based on this, in section 5, equations of motion for the scalar perturbations are derived. The system of equations is then investigated in the case of the longitudinal gauge. Finally, gauge-invariant variables are found and the equations for the corresponding Mukhanov variables are derived. In section 6, we summarize our results and draw out some conclusions.

2. Scalar perturbations with holonomy corrections

The holonomy-modified Hamiltonian constraint can be written as

\[ \mathcal{H}_G^2[N] = \frac{1}{2\kappa} \int_{\Sigma} d^3x \left[ \mathcal{N}(\mathcal{H}_G^0 + \mathcal{H}_G^{(2)}) + \delta N \mathcal{H}_G^{(1)} \right], \quad (9) \]

where

\[ \mathcal{H}_G^0 = -6 \sqrt{\tilde{p}} (\mathbb{K}[1])^2, \]

\[ \mathcal{H}_G^{(1)} = -4 \sqrt{\tilde{p}} (\mathbb{K}[1] + \alpha_1) \delta_j^i \delta K^j_i - \frac{1}{\sqrt{\tilde{p}}} (\mathbb{K}[1]^2 + \alpha_2) \delta_j^i \delta E^j_i + \frac{2}{\sqrt{\tilde{p}}} (1 + \alpha_3) \delta_j^i \delta E^j_i, \]

\[ \mathcal{H}_G^{(2)} = \sqrt{\tilde{p}} (1 + \alpha_4) \delta K^j_i \delta K^k_j - \sqrt{\tilde{p}} (1 + \alpha_5) (\delta K^j_i \delta^j_i)^2 \]

\[ - \frac{2}{\sqrt{\tilde{p}}} (\mathbb{K}[1] + \alpha_6) \delta E^j_i \delta K^k_j - \frac{1}{2\tilde{p}^{3/2}} (\mathbb{K}[1]^2 + \alpha_7) \delta E^j_i \delta E^k_j \delta^k_j \]

\[ + \frac{1}{4\tilde{p}^{3/2}} (\mathbb{K}[1]^2 + \alpha_8) \delta E^j_i \delta E^k_j \delta^k_j - \frac{1}{2\tilde{p}^{3/2}} (1 + \alpha_9) \delta \delta \psi \delta \phi \delta \psi \delta \phi. \]

The standard holonomy corrections are parametrized by two integers \( s_1 \) and \( s_2 \). The \( \alpha_i \) are counter terms, which are introduced to remove anomalies. Those factors are defined so that
they vanish in the classical limit ($\bar{\mu} \to 0$). The counter terms could be, in general, functions of all the canonical variables.

In principle, one could also consider other terms that are indeed allowed in the general case, e.g. $M^{abcd}_{\mu} \delta_{\alpha} \delta K^c_{\alpha} \delta K^d_{\alpha}$ multiplied by some new anomaly terms. Such terms are not present in the classical Hamiltonian but may however appear at the quantum level. In this study, we have only considered counter terms depending on functions which may be present at the classical level and which depend on the gravitational background variables only.

In our approach, the diffeomorphism constraint holds the classical form

$$D_G[N^a] = \frac{1}{\kappa} \int d^4x N^a \left[ \bar{\mu} \delta_{\alpha} \left( \delta E^a_{\alpha} \right) - \bar{\mu} \left( \delta_0 \delta K^a_{\alpha} \right) - \bar{\mu} \left( \delta_0 \delta E^a_{\alpha} \right) \right].$$

(10)

In general, the diffeomorphism constraint could also be holonomy corrected. This possibility was studied, e.g., in [6]. Because the underlying LQG maintains diffeomorphism covariance and because the isotropic LQC (about which the scalar perturbation theory is developed) is obtained by solving the diffeomorphisms classically, one can justifiably assume that diffeomorphism constraints and their algebra retain the classical form. Due to this, in this paper, the diffeomorphism constraint is not modified by the holonomies. It is worth stressing that the classicality of the diffeomorphism constraint is also imposed by the requirement of anomaly cancellation. Namely, if one replaces $\bar{k} \to \bar{k}[n]$ in (10), the condition $n = 0$ would anyway be required by the introduction of scalar matter. Indeed, the Poisson bracket $\{H^0_G, D_G^0\}$ leads to an anomaly term proportional to $(\cos(n \bar{\mu} \gamma \bar{k}) - 1)$, which is vanishing only for $n = 0$. In fact, the same condition was obtained for vector modes with holonomy corrections [9].

Let us now calculate the possible Poisson brackets for the constraints $H^0_G[N]$ and $D_G[N^a]$.

2.1. The $\{H^0_G, D_G\}$ bracket

Using the definition of the Poisson bracket (2), we derive

$$\{H^0_G, D_G[N^a]\} = -H^0_G[N^a \delta_0 \delta N] + B \{D_G[N^a]\}$$

$$+ \frac{\sqrt{\bar{\mu}}}{\kappa} \int d^4x N^a \left( \delta_0 \delta N^0 \right) A_1 + \frac{\bar{\kappa} \sqrt{\bar{p}}}{\kappa} \int d^4x \delta^a \delta^0 \left( \delta_0 \delta N^0 \right) A_2$$

$$+ \frac{\bar{\kappa} \sqrt{\bar{p}}}{\kappa} \int d^4x \delta^a \delta^0 \left( \delta_0 \delta N^0 \right) A_3 + \frac{\bar{\kappa}}{2 \kappa \sqrt{\bar{p}}} \int d^4x \delta^a \delta^0 \left( \delta_0 \delta N^0 \right) A_4,$$

(11)

where

$$B = \frac{\bar{\kappa}}{\sqrt{\bar{p}}} \left[ -2\kappa^2 [2] + \bar{k} (1 + \alpha_5) + \kappa [s_2] + \alpha_6 \right],$$

(12)

and

$$A_1 = 2\bar{k} (\kappa [s_1] + \alpha_1) + \alpha_2 - 2\kappa [1]^2,$$

(13)

$$A_2 = \alpha_5 - \alpha_4,$$

(14)

$$A_3 = -\kappa [1]^2 - \bar{p} \frac{\partial}{\partial \bar{p}} \kappa [1]^2 - \frac{1}{2} \alpha_7 + \bar{k} (-2\kappa [2] + \bar{k} (1 + \alpha_5) + 2\kappa [s_2] + 2\alpha_6),$$

(15)

$$A_4 = \alpha_8 - \alpha_7.$$  

(16)

The functions $A_1, \ldots, A_4$ are the first anomalies coming from the effective nature of the Hamiltonian constraint. Later, we will set them to zero so as to fulfill the requirement of anomaly freedom. This will lead to constraints on the form of the counter terms.

Besides the anomalies, the $\{H^0_G, D_G\}$ bracket contains the $-H^0_G[N^a \delta_0 \delta N]$ term, which is expected classically. There is also an additional contribution from the diffeomorphism constraint $B \{D_G[N^a]\}$. This term is absent in the classical theory. This is however consistent as, for $\bar{\mu} \to 0$, the $B$ function tends to zero.
### 2.2. The \( \{H^0_G, H^0_G\} \) bracket

The next bracket is as follows:

\[
\{H^0_G[N], H^0_G[N]\} = (1 + \alpha_3)(1 + \alpha_5)D_G \left[ \frac{\tilde{N}}{\bar{\rho}} \partial^\alpha \delta N_2 - \delta N_1 \right] \\
+ \frac{\tilde{N}}{\kappa} \int_\Sigma d^3x \partial^\alpha \delta N_2 - \delta N_1 \left( \partial_\alpha \delta K^\alpha_4 \right) (1 + \alpha_3) A_5 \\
+ \frac{\tilde{N}}{\kappa \bar{\rho}} \int_\Sigma d^3x \delta N_2 - \delta N_1 \left( \partial^\alpha \delta E^\alpha_n \right) A_6 \\
+ \frac{\tilde{N}}{\kappa} \int_\Sigma d^3x \delta N_2 - \delta N_1 \left( \delta^\alpha_i \delta E^\alpha_i \right) A_7, \\
\]

where

\[
A_5 = \alpha_5 - \alpha_4, \\
A_6 = (1 + \alpha_3)(\kappa[s_1] + \alpha_1) - (1 + \alpha_3)(\kappa[s_2] + \alpha_6) + \kappa[2](1 + \alpha_3) \\
- 2\kappa[2] \frac{\partial}{\partial \bar{\rho}} \frac{\partial\alpha_3}{\partial \bar{\rho}} + \frac{1}{2} \left( \kappa[1]^2 + 2\bar{\rho} \frac{\partial}{\partial \bar{\rho}} \kappa[1]^2 \right) \frac{\partial\alpha_3}{\partial \bar{\rho}} - \bar{k}(1 + \alpha_3)(1 + \alpha_5), \\
A_7 = 4\kappa[2] \frac{\partial}{\partial \bar{\rho}} \left( \kappa[s_1] + \alpha_1 \right) - \left( \kappa[1]^2 + 2\bar{\rho} \frac{\partial}{\partial \bar{\rho}} \kappa[1]^2 \right) \frac{\partial}{\partial \bar{\rho}} \left( \kappa[s_1] + \alpha_1 \right) \\
+ \left( 1 + \frac{3}{2} \alpha_5 - \frac{1}{2} \alpha_4 \right) (\kappa[1]^2 + \alpha_2) - 2(\kappa[s_2] + \alpha_6)(\kappa[s_1] + \alpha_1) \\
+ 2\kappa[2](\kappa[s_1] + \alpha_1), \\
A_8 = \frac{1}{2} (\kappa[s_2] + \alpha_6)(\kappa[1]^2 + \alpha_2) - (\kappa[s_1] + \alpha_1)(\kappa[1]^2 + \alpha_7) \\
+ \frac{3}{2} (\kappa[s_1] + \alpha_1)(\kappa[1]^2 + \alpha_8) - \frac{1}{2} \kappa[2](\kappa[1]^2 + \alpha_2) \\
+ \kappa[2] \frac{\partial}{\partial \bar{\rho}} (\kappa[1]^2 + \alpha_2) - \frac{1}{4} \left( \kappa[1]^2 + 2\bar{\rho} \frac{\partial}{\partial \bar{\rho}} \kappa[1]^2 \right) \frac{\partial}{\partial \bar{\rho}} (\kappa[1]^2 + \alpha_2). 
\]

The \( A_5, \ldots, A_8 \) are the next four anomalies. Moreover, the diffeomorphism constraint is multiplied by the factor \((1 + \alpha_3)(1 + \alpha_5)\).

### 2.3. The \( \{D_G, D_G\} \) bracket

The Poisson bracket between the diffeomorphism constraints is as follows:

\[
\{D_G[N_1], D_G[N_2]\} = 0. \\
\]

### 3. Scalar matter

In this section, we introduce scalar matter. The scalar matter diffeomorphism constraint is

\[
D_M[N_m] = \int_\Sigma \delta N_m \bar{\pi} \left( \partial_\alpha \delta \varphi \right). \\
\]

The scalar matter Hamiltonian can be expressed as

\[
H^0_M[N] = H_M[\tilde{N}] + H_M[\delta N], \\
\]
Here, we have introduced the counter term $\alpha$.

The Poisson bracket between two total diffeomorphism constraints is vanishing:

$$H_M[N] = \int_\Sigma d^3x \bar{\mathcal{N}} \left[ (\mathcal{H}_\pi^{(0)} + \mathcal{H}_\psi^{(0)}) + (\mathcal{H}_\pi^{(2)} + \mathcal{H}_\psi^{(2)}) \right].$$  \hspace{1cm} (25)

$$H_M[\delta N] = \int_\Sigma d^3\delta N \left[ \mathcal{H}_\pi^{(1)} + \mathcal{H}_\psi^{(1)} \right].$$  \hspace{1cm} (26)

The factors in equations (25) and (26) are

$$\mathcal{H}_\pi^{(0)} = \frac{\hat{\pi}^2}{2\bar{p}^{3/2}},$$

$$\mathcal{H}_\psi^{(0)} = \bar{p}^{3/2} V(\bar{\phi}),$$

$$\mathcal{H}_\pi^{(1)} = \frac{\hat{\pi} \delta \pi}{\bar{p}^{3/2}} - \frac{\hat{\pi}^2}{2\bar{p}^{3/2}} \frac{\delta \mathcal{E}_j}{2\bar{p}},$$

$$\mathcal{H}_\psi^{(1)} = \frac{\bar{p}^{3/2}}{2} \left[ V_\psi(\bar{\phi}) \delta \psi + V(\bar{\phi}) \frac{\delta \mathcal{E}_j}{2\bar{p}} \right],$$

$$\mathcal{H}_\pi^{(2)} = \frac{1}{2} \frac{\delta \pi^2}{\bar{p}^{3/2}} - \frac{\hat{\pi} \delta \pi}{\bar{p}^{3/2}} \frac{\delta \mathcal{E}_j}{2\bar{p}} + \frac{1}{2} \frac{\hat{\pi}^2}{\bar{p}^{3/2}} \left[ \frac{(\delta \mathcal{E}_j)^2}{8\bar{p}^2} + \frac{\delta \mathcal{E}_j \delta \mathcal{E}_k \delta \mathcal{E}_l}{4\bar{p}^2} \right],$$

$$\mathcal{H}_\psi^{(2)} = \frac{1}{2} \bar{p}^{3/2} V_\psi(\bar{\phi}) \delta \psi^2 + \bar{p}^{3/2} V(\bar{\phi}) \delta \psi \frac{\delta \mathcal{E}_j}{2\bar{p}}$$

$$+ \bar{p}^{3/2} V(\bar{\phi}) \left[ \frac{(\delta \mathcal{E}_j)^2}{8\bar{p}^2} - \frac{\delta \mathcal{E}_j \delta \mathcal{E}_k \delta \mathcal{E}_l}{4\bar{p}^2} \right].$$  \hspace{1cm} (27)

Here, we have introduced the counter term $\alpha_{10}$ in the factor $\mathcal{H}_\psi^{(2)}$. Thanks to this, the Poisson bracket between two matter Hamiltonians takes the following form:

$$\{ H_M^0[N_1], H_M^0[N_2] \} = (1 + \alpha_{10}) D_M \left[ \frac{\mathcal{N}}{\bar{p}} \bar{\theta}^0 (\delta N_2 - \delta N_1) \right].$$  \hspace{1cm} (30)

As will be explained later, the appearance of the front factor $(1 + \alpha_{10})$ will allow us to close the algebra of total constraints. In principle, other prefactors could have been expected; however, they do not help removing anomalies.

### 3.1. Total constraints

The total Hamiltonian and diffeomorphism constraints are as follows:

$$H_{\text{tot}}[N] = H_M^0[N] + H_M^0[N],$$  \hspace{1cm} (31)

$$D_{\text{tot}}[\delta N] = D_G[\delta N] + D_M[\delta N].$$  \hspace{1cm} (32)

The Poisson bracket between two total diffeomorphism constraints is vanishing:

$$\{ D_{\text{tot}}[N_1^0], D_{\text{tot}}[N_2^0] \} = 0.$$  \hspace{1cm} (33)

The bracket between the total Hamiltonian and diffeomorphism constraints can be decomposed as follows:

$$\{ H_{\text{tot}}[N], D_{\text{tot}}[\delta N] \} = \{ H_M^0[N], D_{\text{tot}}[\delta N] \} + \{ D_G[\delta N], H_M^0[N] \} + \{ H_M^0[N], D_M[\delta N] \}.$$  \hspace{1cm} (34)
The first bracket in sum (34) is given by
\[
\{H^0_c[N], D_{cd}[N^a]\} = -H^0_c[\delta^a \partial_a N].
\] (35)

The second contribution to equation (34) is given by (11), while the last contribution is
vanishing:
\[
\{H^0_c[N], D_{cf}[N^a]\} = 0.
\] (36)

The Poisson bracket between the two total Hamiltonian constraints can be decomposed in the
following way:
\[
\{H_{tot}[N_1], H_{tot}[N_2]\} = \{H^0_c[N_1], H^0_c[N_2]\} + \{H_c[N_1], H_c[N_2]\}
+ \{H^0_c[N_1], H_{cf}[N_2]\} -(N_1 \leftrightarrow N_2).
\] (37)

The contribution from the last brackets can be expressed as
\[
\{H^0_c[N_1], H_{cf}[N_2]\} = \frac{1}{2} \int \Sigma d^3x \tilde{N} (\delta N_2 - \delta N_1) \left(\frac{\pi^2}{2p^3} - V(\bar{\varphi})\right) \partial_a \partial^a E_j A_9
+ 3 \int \Sigma d^3x \tilde{N} (\delta N_2 - \delta N_1) \left(\frac{\pi \delta \pi}{p^3} - \bar{p} V_c(\bar{\varphi}) \delta \varphi\right) A_{10}
+ \int \Sigma d^3x \tilde{N} (\delta N_2 - \delta N_1) (\delta_j \delta E^j) \left(\frac{\pi^2}{2p^3} - V(\bar{\varphi})\right) \bar{p} A_{11}
+ \frac{1}{2} \int \Sigma d^3x \tilde{N} (\delta N_2 - \delta N_1) (\delta_j \delta E^j) \left(\frac{\pi^2}{2p^3}\right) A_{12}
+ \frac{1}{2} \int \Sigma d^3x \tilde{N} (\delta N_2 - \delta N_1) (\delta_j \delta E^j) V(\bar{\varphi}) A_{13},
\] (38)

where
\[
A_9 = \frac{\partial \alpha_1}{\partial \bar{k}},
\] (39)
\[
A_{10} = K[2] - K[s_1] - \alpha_1,
\] (40)
\[
A_{11} = -\frac{\partial}{\partial \bar{k}} (K[s_1] + \alpha_1) + \frac{3}{2} (1 + \alpha_5) - \frac{1}{2} (1 + \alpha_6),
\] (41)
\[
A_{12} = -\frac{1}{2} \frac{\partial}{\partial \bar{k}} (K[1]^2 + \alpha_2) + 5 (K[s_1] + \alpha_1) - 5K[2] + K[s_2] + \alpha_6,
\] (42)
\[
A_{13} = \frac{1}{2} \frac{\partial}{\partial \bar{k}} (K[1]^2 + \alpha_2) + K[s_1] + \alpha_1 - K[2] - K[s_2] - \alpha_6.
\] (43)

The functions \(A_9, \ldots, A_{13}\) are the last five anomalies.

4. Anomaly freedom

The requirement of anomaly freedom is equivalent to the conditions \(A_i = 0\) for \(i = 1, \ldots, 13\).

Let us start form the condition \(A_9 = 0\). Since \(\alpha_3\) cannot be a constant, this condition
implies \(\alpha_3 = 0\). The condition \(A_{10} = 0\) gives \(\alpha_1 = K[2] - K[s_1]\). Using this, the condition
\(A_1 = 0\) can be written as \(\alpha_2 = 2K[1]^2 - 2kK[2]\). The conditions \(A_2 = 0\) and \(A_3 = 0\) are
equivalent and lead to \(\alpha_4 = \alpha_5\). Based on this, the requirement \(A_{11} = 0\) leads to
\[
1 + \alpha_4 = \frac{\partial K[2]}{\bar{k}} = \cos(2\mu \gamma \bar{k}) =: \Omega.
\] (44)
For the sake of simplicity, we have defined here the $/Omega_1$-function. With the use of this, the condition $A_6 = 0$ leads to
\[
\alpha_6 = \kappa[2](2 + \alpha_9) - \kappa[x_2] - \tilde{k}\Omega .
\] (45)
So equation (42) simplifies to
\[
A_{12} = \alpha_9 \kappa[2] .
\] (46)
Therefore, requiring $A_{12} = 0$ is equivalent to the condition $\alpha_9 = 0$. Furthermore, $A_4 = 0$ gives $\alpha_7 = \alpha_9$. The expression for $\alpha_7$ can be derived from the condition $A_3 = 0$. Namely, using equation (45), one obtains
\[
\alpha_7 = 2(2\beta - 1)\kappa[1]^2 + 4(1 - \beta)\tilde{k}\kappa[2] - 2\tilde{k}^2\Omega .
\] (47)
The condition $A_{13} = 0$ is fulfilled by using the expressions derived for $\alpha_1$, $\alpha_2$ and $\alpha_6$. The last two anomalies (20) and (21) can be simplified to
\[
A_7 = 2(1 + 2\beta)(\Omega \kappa[1]^2 - \kappa[2]^2) ,
\] (48)
\[
A_8 = \tilde{k}(1 + 2\beta)(\tilde{k}\kappa[2]^2 - \Omega \kappa[1]^2) .
\] (49)
The anomaly-freedom conditions for those last terms, $A_7 = 0$ and $A_8 = 0$, are fulfilled if and only if $\beta = -1/2$.

It is also worth noting that the function $B$ given by equation (12) is equal to zero when the expression obtained for $\alpha_6$ is used. There is finally no contribution from the diffeomorphism constraint in the $[H_G^0, D_G]$ bracket.

Using the anomaly-freedom conditions given above, the bracket between the total Hamiltonian constraints simplifies to
\[
[H_{tot}[N_1], H_{tot}[N_2]] = \Omega D_{tot} \left[ \frac{\tilde{N}}{\tilde{P}} \partial^a (\delta N_2 - \delta N_1) \right] + (\alpha_{10} - \alpha_4)D_M \left[ \frac{\tilde{N}}{\tilde{P}} \partial^a (\delta N_2 - \delta N_1) \right] .
\] (50)
The closure of the algebra of total constraints implies the last condition $\alpha_{10} = \alpha_4 = \Omega - 1$.

To summarize, the counter terms allowing the algebra to be anomaly free are uniquely determined, and are given by
\[
\alpha_1 = \kappa[2] - \kappa[x_1] .
\] (51)
\[
\alpha_2 = 2\kappa[1]^2 - 2\tilde{k}\kappa[2] .
\] (52)
\[
\alpha_3 = 0 .
\] (53)
\[
\alpha_4 = \Omega - 1 .
\] (54)
\[
\alpha_5 = \Omega - 1 .
\] (55)
\[
\alpha_6 = 2\kappa[2] - \kappa[x_2] - \tilde{k}\Omega .
\] (56)
\[
\alpha_7 = -4\kappa[1]^2 + 6\tilde{k}\kappa[2] - 2\tilde{k}^2\Omega .
\] (57)
\[
\alpha_8 = -4\kappa[1]^2 + 6\tilde{k}\kappa[2] - 2\tilde{k}^2\Omega .
\] (58)
\[
\alpha_9 = 0 .
\] (59)
\[
\alpha_{10} = \Omega - 1 .
\] (60)
It is straightforward to check that the counter terms $\alpha_1, \ldots, \alpha_{10}$ are vanishing in the classical limit ($\bar{\mu} \rightarrow 0$), as expected.

Those counter terms are defined up to the two integers $s_1$ and $s_2$, which appear in (51) and (56). However, in Hamiltonian (9), the factor $\alpha_1$ appears with $\mathcal{K}[s_1]$ and the factor $\alpha_6$ appears with $\mathcal{K}[s_2]$. Namely, we have $\mathcal{K}[s_1] + \alpha_1 = \mathcal{K}[2]$ and $\mathcal{K}[s_2] + \alpha_6 = 2\mathcal{K}[2] - \bar{k}\Omega$. Therefore, the final Hamiltonian will not depend on the parameters $s_1$ and $s_2$. No ambiguity remains to be fixed.

Moreover, the anomaly cancellation requires $\beta = -\frac{1}{2}$, which fixes the functional form of the $\bar{\mu}$ factor. The fact that anomaly freedom requires $\beta = -\frac{1}{2}$ is a quite surprising result. The exact value of $\beta$ is highly debated in LQC. The only a priori obvious statement is that $\beta \in [-1/2, 0]$. The choice $\beta = -1/2$ is called the $\bar{\mu}$-scheme (new quantization scheme) and is preferred by some authors for physical reasons [10]. Our result seems to show that the $\bar{\mu}$-scheme is embedded in the structure of the theory and this gives a new motivation for this particular choice of quantization scheme. The quantity $\bar{\mu}^2 \bar{p}$ can be interpreted as the physical area of an elementary loop along which the holonomy is calculated. Because, in the $\bar{\mu}$-scheme, $\bar{\mu}^2 \propto \bar{p}^{-1}$, the physical area of the loop remains constant. This elementary area is usually set to be the area gap $\Delta$ derived in LQG. Therefore, in the $\bar{\mu}$-scheme,

$$\bar{\mu} = \sqrt{\frac{\Delta}{\bar{p}}}.$$  \hspace{1cm} (62)

### 4.1. Algebra of constraints

Taking into account the previous conditions of anomaly freedom, the non-vanishing Poisson brackets for the gravity sector are as follows:

$$\{H^{\text{G}}_1[N], D^{\text{G}}[N^a]\} = -H^{\text{G}}_1[\delta N^a \partial_a \delta N],$$  \hspace{1cm} (63)

$$\{H^{\text{G}}_1[N_1], H^{\text{G}}_2[N_2]\} = \Omega D^{\text{G}} \left[ \frac{\bar{N}}{\bar{p}} \bar{g}^a (\delta N_2 - \delta N_1) \right].$$  \hspace{1cm} (64)

This clearly shows that the gravity sector is anomaly free. The remaining non-vanishing brackets are as follows:

$$\{H^{\text{tot}}[N], D^{\text{tot}}[N^a]\} = -H^{\text{tot}}[\delta N^a \partial_a \delta N],$$  \hspace{1cm} (65)

$$\{H^{\text{tot}}[N_1], H^{\text{tot}}[N_2]\} = \Omega D^{\text{tot}} \left[ \frac{\bar{N}}{\bar{p}} \bar{g}^a (\delta N_2 - \delta N_1) \right].$$  \hspace{1cm} (66)

The algebra of total constraints therefore takes the following form:

$$\{D^{\text{tot}}[N^a], D^{\text{tot}}[N^b]\} = 0,$$  \hspace{1cm} (67)

$$\{H^{\text{tot}}[N], D^{\text{tot}}[N^a]\} = -H^{\text{tot}}[\delta N^a \partial_a \delta N],$$  \hspace{1cm} (68)

$$\{H^{\text{tot}}[N_1], H^{\text{tot}}[N_2]\} = D^{\text{tot}} \left[ \frac{\bar{N}}{\bar{p}} \bar{g}^a (\delta N_2 - \delta N_1) \right].$$  \hspace{1cm} (69)

Although the algebra is closed, there are however modifications with respect to the classical case, due to presence of the factor $\Omega$ in equation (69). Therefore, not only the dynamics, as a result of the modification of the Hamiltonian constraint, is modified but the very structure of the
spacetime itself is also deformed. This is embedded in the form of the algebra of constraints. The hypersurface deformation algebra generated by (69) is pictorially represented in figure 1. As $\Omega \in [-1, 1]$, the shift vector

$$N^a = \frac{\bar{N}}{p} \cos(2\bar{\mu}y\bar{k}) \partial^a(\delta N_2 - \delta N_1)$$

appearing in (69) can change sign in time.

In order to see when this might happen, let us express the parameter $\Omega$ as

$$\Omega = \cos(2\bar{\mu}y\bar{k}) = 1 - \frac{\rho}{\rho_c},$$

where $\rho$ is the energy density of the matter field and

$$\rho_c = \frac{3}{\kappa \gamma \Delta \bar{\mu}}.$$

In the low-energy limit, $\rho \to 0$, the classical case ($\Omega \to 1$) is correctly recovered. However, while approaching the high-energy domain, the situation drastically changes. Namely, for $\rho = \rho_c/2$, the shift vector (70) becomes null. At this point, the maximum value of the Hubble parameter is also reached. The maximum allowed energy density is $\rho = \rho_c$ and corresponds to the bounce. Then the shift vector (70) fully reverses with respect to the low-energy limit. One can interpret this peculiar behavior as a geometry change. Namely, when the universe is in its quantum stage ($\rho > \rho_c/2$), the effective algebra of constraints shows that the space is Euclidian. At the particular value $\rho = \rho_c/2$, the geometry switches to the Minkowski one [11]. This will become even clearer when analyzing the Mukhanov equation in section 5. The consequences of this have not yet been fully understood, but it is interesting to note that this model naturally exhibits properties related to the Hartle–Hawking no-boundary proposal [12].

5. Equations of motion

Once the anomaly-free theory of scalar perturbations with holonomy corrections is constructed, the equations of motion for the canonical variables can be derived. This can be achieved through the Hamilton equation

$$\dot{f} = \{f, H[N, N^a]\},$$

where the Hamiltonian $H[N, N^a]$ is the sum of all constraints

$$H[N, N^a] = H^G_0[N] + H_M[N] + D_G[N^a] + D_M[N^a].$$
5.1. Background equations

Based on the Hamilton equation (73), the equations for the canonical background variables are as follows [13]:

\[
\dot{k} = -\frac{\tilde{N}}{2\sqrt{\bar{p}}} \bar{k}[1]^2 - \tilde{N}\sqrt{\bar{p}} \frac{\partial}{\partial \bar{p}} \bar{k}[1]^2 + \frac{\kappa}{2} \sqrt{\bar{p}} N \left[ -\frac{\bar{\pi}^2}{2\bar{p}^3} + V(\bar{\phi}) \right], \tag{75}
\]

\[
\dot{\bar{p}} = 2\tilde{N}\sqrt{\bar{p}} \bar{k}[2], \tag{76}
\]

\[
\dot{\bar{\phi}} = \tilde{N} \frac{\bar{\pi}}{p^{3/2}}, \tag{77}
\]

\[
\dot{\bar{\pi}} = -\tilde{N} \frac{\bar{\pi}}{p^{3/2}} V(\bar{\phi}). \tag{78}
\]

In the following, we choose the time to be conformal by setting \( \tilde{N} = \sqrt{\bar{p}} \). The ‘\( \cdot \)’ then means differentiation with respect to conformal time \( \eta \).

Equations (77) and (78) can be now combined into the Klein–Gordon equation

\[
\ddot{\bar{\phi}} + 2\bar{k}[2]\dot{\bar{\phi}} + \bar{p} V(\bar{\phi}) = 0. \tag{79}
\]

Equation (76), together with the background part of the Hamiltonian constraint

\[
\frac{1}{V_0} \frac{\partial}{\partial \bar{N}} H = \frac{1}{2\kappa} \left[ -6\sqrt{\bar{p}} \bar{k}[1]^2 \right] + \bar{p}^{3/2} \left[ \frac{\bar{\pi}^2}{2\bar{p}^3} + V(\bar{\phi}) \right] = 0, \tag{80}
\]

leads to the modified Friedmann equation

\[
\mathcal{H}^2 = \frac{\bar{p}^2}{3} \rho \left( 1 - \frac{\rho}{\rho_c} \right). \tag{81}
\]

Another useful expression is

\[
3\bar{k}[1]^2 = \frac{\bar{\pi}^2}{2\bar{p}^2} + \bar{p} V(\bar{\phi}). \tag{82}
\]

Here \( \mathcal{H} \) stands for the conformal Hubble factor

\[
\mathcal{H} := \frac{\dot{p}}{2\bar{p}} = \bar{k}[2]. \tag{83}
\]

The energy density and pressure of the scalar field are given by

\[
\rho = \frac{\bar{\pi}^2}{2\bar{p}^3} + V(\bar{\phi}), \tag{84}
\]

\[
P = \frac{\bar{\pi}^2}{2\bar{p}^3} - V(\bar{\phi}). \tag{85}
\]

For the purpose of further considerations, we also derive the relation

\[
\kappa \left( \frac{\bar{\pi}^2}{2\bar{p}^3} \right) = \tilde{k}\bar{k}[2] - \dot{k}, \tag{86}
\]

which comes from equation (75) combined with (80).
5.2. Equations for the perturbed variables

The equations for the perturbed parts of the canonical variables are as follows:

\[
\delta E^a_i = -\tilde{N} \left[ \sqrt{\bar{p}} \Omega \bar{K}_i^j \delta_i^a \delta_j^a - \sqrt{\bar{p}} \Omega (\delta K_i^j \delta_i^a) \delta_j^a - \frac{1}{\sqrt{\bar{p}}} (2\bar{K}[2] - \bar{\kappa} \Omega) \delta E^a_i \right] \\
+ \delta N (2\bar{K}[2] \sqrt{\bar{p}} \delta_i^a) - \bar{p} (\partial_a \delta N^a - (\partial_i \delta N^i) \delta_i^a),
\]

\[
\delta K_a^i = \tilde{N} \left[ - \frac{1}{\sqrt{\bar{p}}} (2\bar{K}[2] - \bar{\kappa} \Omega) \delta K_a^i \\
- \frac{1}{2 \bar{p}^{3/2}} (-3\bar{K}[1] \bar{I}^2 + 6\bar{K}[2] - 2\bar{\kappa} \Omega \delta E^b_j \delta_i^b \delta_j^a \\
+ \frac{1}{4 \bar{p}^2} (-3\bar{K}[1] \bar{I}^2 + 6\bar{K}[2] - 2\bar{\kappa} \Omega \delta E^b_j \delta_i^b) \delta_a^b \delta_i^b + \frac{\delta_i^a}{2 \bar{p}^2} \partial_a \delta E^i_a \\
+ \delta_i^a (\partial_a \delta N^a) + \kappa \delta N \frac{\bar{p}}{\bar{I}} \left[ - \frac{\bar{\kappa}^2}{2 \bar{p}^3} + V(\bar{\psi}) \right] \delta_i^a \\
+ \kappa \tilde{N} \left[ - \frac{\bar{\kappa} \delta \bar{\psi}}{2 \bar{p}^{3/2}} + \frac{\sqrt{\bar{p}}}{2} \delta \bar{\psi} \frac{\delta \bar{\psi}}{\partial \bar{\psi}} \delta_i^a + \left( \frac{\bar{\kappa}^2}{2 \bar{p}^{3/2}} + \bar{p}^{3/2} V(\bar{\psi}) \right) \frac{\delta_i^a \delta E^i_a}{4 \bar{p}^2} \right]
\]

\[
\delta \phi = \delta N \left( \frac{\bar{\kappa}}{\bar{p}^{3/2}} \right) + \tilde{N} \left( \frac{\bar{\kappa} \delta \phi}{\bar{p}^{3/2}} - \frac{\bar{\kappa} \delta \phi \delta E^i_a}{2 \bar{p}} \right),
\]

\[
\delta \bar{\psi} = -\delta N \left( \bar{p}^{3/2} V_\phi(\bar{\psi}) \right) + \tilde{N} (\partial_a \delta N^a)
\]

\[
= -\tilde{N} \left[ -\sqrt{\bar{p}} \Omega \delta^{ab} \partial_a \delta \phi + \bar{p}^{3/2} V_\phi(\phi) \delta \phi + \bar{p}^{3/2} V_\phi(\bar{\psi}) \frac{\delta_i^a \delta E^i_a}{2 \bar{p}} \right].
\]

5.3. Longitudinal gauge

As an example of application, we will now derive the equations in the longitudinal gauge. In this case, the \( E \) and \( B \) perturbations are set to zero. The line element (1) therefore simplifies to

\[
dx^2 = a^2 [-(1 + 2\phi) \, d\eta^2 + (1 - 2\psi) \delta_{ab} \, dx^a \, dx^b],
\]

where \( \phi \) and \( \psi \) are two remaining perturbation functions and \( a \) is the scale factor. From the metric above, one can derive the lapse function, the shift vector and the spatial metric:

\[
N = a \sqrt{1 + 2\phi},
\]

\[
N^a = 0,
\]

\[
q_{ab} = a^2 (1 - 2\psi) \delta_{ab}.
\]

The lapse function can be expanded for the background and perturbation part as \( N = \tilde{N} + \delta N \), where

\[
\tilde{N} = \sqrt{\bar{p}} = a,
\]
\[ \delta N = \dot{N}\phi. \] (96)

Using equation (94), the perturbation of the densitized triad is expressed as
\[ \delta E_i^a = -2\dot{\bar{p}}\delta_i^a. \] (97)

The time derivative of this expression will also be useful and can be written as
\[ \dot{\delta} E_i^a = -2\ddot{\bar{p}}(2\bar{K}[2]\psi + \dot{\psi})\delta_i^a. \] (98)

Let us now find the expression for the perturbation of the extrinsic curvature \( \delta K'_a \) in terms of the metric perturbations \( \phi \) and \( \psi \). For this purpose, one can apply expression (97) to the left-hand side of (87). The resulting equation can be solved for \( \delta K'_a \), leading to
\[ \delta K'_a = -\delta^1_\Omega \left( \dot{\psi} + \ddot{\psi}\Omega + \bar{K}[2]\phi \right). \] (99)

The time derivative of this variable is given by
\[ \ddot{\delta} K'_a = \delta^1_\Omega \left[ -\dddot{\psi} - \ddot{\ddot{\psi}}\Omega + \dot{\psi} \left( \ddot{\bar{K}}\Omega - \ddot{\bar{K}} \right) + \phi\bar{K}[2]\dddot{\bar{K}} \Omega - \phi\bar{K}[2] - \bar{K}[2]\phi \right]. \] (100)

Applying (100) to the left-hand side of (88), the equation containing the diagonal part as well as the off-diagonal contribution is easily obtained. The off-diagonal part leads to
\[ \dot{\delta} \psi = 0. \] (101)

This translates into \( \psi = \phi \). In what follows, we will therefore consider \( \phi \) only. The diagonal part of the discussed equation can be expressed as
\[ \ddot{\phi} + \phi \left[ 3\bar{K}[2] - \frac{\dddot{\bar{K}}}{\Omega} + \frac{\dddot{\bar{K}}}{\bar{K}[2]} \right] + \phi \left[ \ddot{\bar{K}}[2] + 2\bar{K}[2] \dddot{\bar{K}} - \bar{K}[2] \dddot{\bar{K}} \Omega \right] = 4\pi G\Omega \left[ \dddot{\phi} - \dddot{\phi}\Omega \phi V_{\phi}(\phi) \right]. \] (102)

One can now use the diffeomorphism constraint
\[ \kappa \frac{\delta H[N, N^a]}{\delta (\delta N^a)} = \bar{p}\delta \delta_i^a \left( \delta_i^a \delta K'_a \right) - \bar{p} \left( \partial_\phi \delta K'_a \right) - \bar{K} \left( \partial_\phi \delta E_i^a \right) + \kappa \bar{\pi} \left( \partial_\phi \delta \phi \right) = 0. \] (103)

With the expressions for \( \delta K'_a \) and \( \delta E_i^a \), it can be derived that
\[ \dot{\phi} + \phi \left[ 3\bar{K}[2] - \frac{\dddot{\bar{K}}}{\Omega} + \frac{\dddot{\bar{K}}}{\bar{K}[2]} \right] + \phi \left[ \ddot{\bar{K}}[2] + 2\bar{K}[2] \dddot{\bar{K}} - \bar{K}[2] \dddot{\bar{K}} \Omega \right] = 4\pi G\Omega \left[ \dddot{\phi} - \dddot{\phi}\Omega \phi V_{\phi}(\phi) \right]. \] (104)

The next equation comes from the perturbed part of the Hamiltonian constraint:
\[ \frac{\delta H[N, N^a]}{\delta (\delta N)} = \frac{1}{2\kappa} \left[ -4\sqrt{\bar{p}} \left[ 2\bar{K}[2] \delta_i^a \delta K'_a - \frac{1}{\sqrt{\bar{p}}} \left( 3\bar{K}[1]^2 - 2\bar{K}\bar{K}[2] \right) \delta_i^a \delta E_j^a + \frac{\bar{p}}{\sqrt{\bar{p}}} \delta_i^a \delta \phi \right] \right] \] (105)

Using the expressions for \( \delta K'_a \) and \( \delta E_i^a \), this can be rewritten as
\[ \Omega \nabla^2 \phi - 3\bar{K}[2] \phi - \left[ \bar{K}[2] + 2\bar{K}[2]^2 \right] \phi = 4\pi G\Omega \left[ \dddot{\phi} - \dddot{\phi}\Omega \phi V_{\phi}(\phi) \right]. \] (106)

The last equality comes from (89) and (90):
\[ \delta \psi + 2\bar{K}[2] \delta \phi - \Omega \nabla^2 \delta \phi + \bar{pV}_{\phi\phi}(\phi) \delta \phi + 2\bar{pV}_{\phi}(\bar{\phi}) \phi - 4\dddot{\phi} \phi = 0. \] (107)

Equations (102), (104) and (106) can be now combined into
\[ \dddot{\phi} + 2 \left( \bar{\mathcal{H}} - \left( \frac{\dddot{\phi}}{\dddot{\phi}} + \epsilon \right) \right) \phi + 2 \left( \bar{\mathcal{H}} - \bar{\mathcal{H}} \left( \frac{\dddot{\phi}}{\dddot{\phi}} + \epsilon \right) \right) \phi - c^2 \nabla^2 \phi = 0. \] (108)
with the quantum correction
\[ \epsilon = \frac{1}{2} \frac{\Omega}{\Omega} = 3K[2] \left( \frac{\rho + P}{\rho_c - 2\rho} \right), \] (109)

and the squared velocity
\[ c_s^2 = \Omega. \] (110)

The squared velocity of the perturbation field \( \phi \) is equal to \( \Omega \). Because \(-1 \leq \Omega \leq 1\), the speed of perturbations is never super-luminal. However, for \( \Omega < 0 \), perturbations become unstable \( (c_s^2 < 0) \). This corresponds to the energy density regime \( \rho > \rho_c^2 \), where the phase of super-inflation is expected.

At the point \( \rho = \rho_c^2 \), the velocity of the perturbation field \( \phi \) is vanishing. Therefore, perturbations do not propagate anymore when approaching \( \rho = \rho_c^2 \), where the Hubble factor reaches its maximal value. Moreover, at this point, the quantum correction \( \epsilon \to \infty \). Because of this, equation (108) diverges and cannot be used to determine the propagation of the perturbations. However, as shown in the next section, the equation for the gauge-invariant Mukhanov variable does not exhibit such a pathology.

It is interesting to note that the equations of motion derived in this subsection are the same as those found in [7]. This is quite surprising, because they were derived following independent paths. In this approach, we have introduced the most general ‘sine’ form for the holonomy corrections to the Hamiltonian, parametrized by some unknown integers. Then, by adding counter terms, anomalies in the algebra of constraints were removed. It has been argued that one could obviously also add other functions agreeing with the classical limit. On the other hand, the method proposed in [7] is based on the diagonal form of the metric in the longitudinal gauge. This enables one to introduce holonomy corrections in almost the same way as in the case of a homogeneous model but with an argument which depends on the spatial position also. It was then shown that a system defined in this way stays on-shell, that is, is free of anomalies. Nevertheless, it is possible to show that starting from the Hamiltonian constraint given in [7] and performing a Taylor expansion around \( \overline{K}_i^a \) and \( \overline{E}_a^i \), one obtains exactly the same Hamiltonian constraint (9) with our values for the counter terms (51)–(60). The non-trivial equivalence of both approaches may suggest uniqueness in defining a theory of scalar perturbations with holonomy corrections in an anomaly-free manner.

5.4. Gauge-invariant variables and Mukhanov equation

Considering the scalar perturbations, there is only one physical degree of freedom. As was shown in [14], this physical variable combines both the perturbation of the metric and the perturbation of matter. The classical expression on this gauge-invariant quantity is
\[ v = a(\eta) \left( \delta \phi^{\text{GI}} + \frac{\dot{\psi}}{H} \right), \] (111)

and its equation of motion is given by
\[ \ddot{v} - \nabla^2 v - \frac{\dot{z}}{z} v = 0, \] (112)

where
\[ z = a(\eta) \frac{\dot{\psi}}{H}. \] (113)

In the canonical formalism with scalar perturbations, the gauge transformation of a variable \( X \) under a small coordinate transformation
\[ x^\mu \to x^\mu + \xi^\mu, \quad \xi^\mu = (\xi_0, \partial^a \xi) \] (114)
is given by (see [8] for details)
$$\delta_{[e',e]}X = [X, H^{(2)}[\bar{N}\xi^0] + D^{(2)}[\bar{\sigma}^a\xi]],$$
(115)
and it is straightforward to see that, classically,
$$\delta_{[e',e]}v = 0.$$  
(116)
This means that \(v\) is diffeomorphism invariant and can be taken as an observable.

Taking into account the holonomy corrections introduced in this paper, the \(\Omega\) function will modify the gauge transformations of the time derivative of a variable \(X\), so that
$$\delta_{[e',e]}\dot{X} = (\delta_{[e',e]}X) = \Omega \cdot \delta_{[0,e']}X.$$  
(117)
Using this relation and gauge transformations of the metric perturbations
$$\delta_{[e',e]}\psi = -\bar{k}[2]\xi^0,$$
(118)
$$\delta_{[e',e]}\phi = \dot{\xi}^0 + \bar{k}[2]\xi^0,$$
(119)
$$\delta_{[e',e]}E = \xi,$$
(120)
$$\delta_{[e',e]}B = \dot{\xi},$$  
(121)
one can define the gauge-invariant variables (Bardeen potentials) as
$$\Phi = \phi + \frac{1}{\Omega} (B - \dot{E}) + \left(\frac{\bar{k}[2]}{\Omega} - \frac{\Omega}{\bar{k}[2]}\right) (B - \dot{E}),$$
(122)
$$\Psi = \psi - \frac{\bar{k}[2]}{\Omega} (B - \dot{E}),$$
(123)
$$\delta\phi^{\text{GI}} = \delta\phi + \frac{\dot{\phi}}{\Omega} (B - \dot{E}).$$  
(124)
The normalization of these variables was set such that, in the longitudinal gauge \((B = \dot{E} = 0)\), we have \(\Phi = \phi, \Psi = \psi\) and \(\delta\phi^{\text{GI}} = \delta\phi\). It is possible to define the analogous of the Mukhanov variable (111):
$$v := \sqrt{\bar{p}} \left( \delta\phi^{\text{GI}} + \frac{\dot{\phi}}{\bar{k}[2]} \Psi \right).$$
(125)
Writing the equations for \(\Psi\) and \(\delta\phi^{\text{GI}}\), which are
$$\ddot{\Psi} + 2 \left[ H - \left(\frac{\dot{\phi}}{\phi} + \epsilon\right)\right] \Psi + 2 \left[ \dot{H} - H \left(\frac{\dot{\phi}}{\phi} + \epsilon\right)\right] \Psi - c_s^2 \nabla^2 \Psi = 0$$
(126)
and
$$\delta\phi^{\text{GI}} + 2\bar{k}[2]\dot{\delta}\phi^{\text{GI}} - \Omega\nabla^2 \delta\phi^{\text{GI}} + \bar{p}V_{,\phi}(\bar{\phi})\delta\phi^{\text{GI}} + 2\bar{p}V_{,\phi}(\bar{\phi})\Psi - 4\dot{\phi}^{\text{GI}} \Psi = 0,$$
(127)
one obtains the equation for variable (125):
$$\ddot{v} - \Omega \nabla^2 v - \frac{\dot{z}}{z} v = 0,$$
(128)
$$\dot{z} = \sqrt{\bar{p}} \frac{\dot{\phi}}{\bar{k}[2]},$$
(129)
which corresponds to the Mukhanov equation for our model. As we see, the difference between the classical and the holonomy-corrected case is the factor \(\Omega\) in front of the Laplacian. This quantum contribution leads to a variation of the propagation velocity of the perturbation \(v\).
This is similar to the case of the perturbation $\phi$ considered in the previous subsection. The main difference is that there is no divergence for $\rho = \rho_c/2$ and the evolution of perturbations can be investigated in the regime of high energy densities. It is once again worth noting that for $\rho > \rho_c/2$, $\Omega$ becomes negative and equation (128) changes from a hyperbolic form to an elliptic one. This basically means that the time part becomes indistinguishable from the spatial one. This can be interpreted as a transition from a Minkowskian geometry to an Euclidean geometry, as mentioned earlier.

Finally, it is also possible to define the perturbation of curvature $R$ such that
\begin{equation}
R = \frac{v}{z}.
\end{equation}
Based on this, one can now calculate the power spectrum of scalar perturbations. This opens new possible ways to study quantum gravity effects in the very early universe. Promising applications of the derived equations will be investigated elsewhere.

6. Summary and conclusions

In this paper, we have investigated the theory of scalar perturbations with holonomy corrections. Such corrections are expected because of quantum gravity effects predicted by LQG. They basically come from the regularization of the curvature of the connection at the Planck scale. Because of this, the holonomy corrections become dominant in the high-curvature regime. The introduction of ‘generic-type’ holonomy corrections leads to an anomalous algebra of constraints. The conditions of anomaly freedom impose some restrictions on the form of the holonomy corrections. However, we have shown that the holonomy corrections, in the standard form, cannot fully satisfy the conditions of anomaly freedom. In order to solve this difficulty, additional counter terms were introduced. Such counter terms tend to zero in the classical limit, but play the role of regularizers of anomalies in the quantum (high-curvature) regime. The method of counter terms was earlier successfully applied to cosmological perturbations with inverse-triad corrections [5].

We have shown that, thanks to the counter terms, the theory of cosmological perturbations with holonomy corrections can be formulated in an anomaly-free way. The anomaly freedom was shown to be fulfilled not only for the gravity sector but also when taking into account scalar matter. The requirements of anomaly freedom were used to determine the form of the counter terms. Furthermore, conditions for obtaining an anomaly-free algebra of constraints were shown to be fulfilled only for a particular choice of the $\bar{\mu}$ function, namely for the $\bar{\mu}$-scheme (new quantization scheme). This quantization scheme was shown earlier to be favored because of the consistency of the background dynamics [10]. Our result supports these earlier claims.

In our formulation, the diffeomorphism constraint holds its classical form, in agreement with the LQG expectations. The obtained anomaly-free gravitational Hamiltonian contains seven holonomy modifications. It was also necessary to introduce one counter term into the matter Hamiltonian in order to ensure the closure of the algebra of total constraints. There is no ambiguity in defining the holonomy corrections after imposing the anomaly-free conditions. The only remaining free parameter of the theory is the area gap $\Delta$ used in defining the $\bar{\mu}$ function. This quantity can however be possibly fixed with the spectrum of the area operator in LQG. Based on the equations derived in this paper, it will also be possible to put observational constraints on the value of $\Delta$ and, hence, on the critical energy density $\rho_c$.

Based on the studied anomaly-free formulation, equations of motion were derived. As an example of application, we studied the equations in the longitudinal gauge. We have also found the gauge-invariant variables, which are holonomy-corrected versions of the
Bardeen potentials. Using this, we have derived the equation for the Mukhanov variable. This equation can be directly used to compute the power spectrum of scalar perturbations with quantum gravitational holonomy corrections. Similar considerations were studied in the case of inverse-triad corrections [15]. In that case, observational consequences have been derived and compared with CMB data [16, 17].

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References

[1] Rovelli C 2004 Quantum Gravity (Cambridge: Cambridge University Press)
[2] Thiemann T 2007 Modern Canonical Quantum General Relativity (Cambridge: Cambridge University Press)
[3] Ashtekar A and Lewandowski J 2004 Class. Quantum Grav. 21 R53
[4] Perez A 2004 arXiv:gr-qc/0409061v3
[5] Dona P and Speziale S 2010 arXiv:1007.4020v1
[6] Rovelli C 2011 arXiv:1102.3660v5
[7] Bojowald M 2005 Living Rev. Rel. 8 11
[8] Ashtekar A and Singh P 2011 Class. Quantum Grav. 28 213001
[9] Bojowald M and Hossain G M 2008 Phys. Rev. D 77 023508
[10] Mulryne D and Nunes N 2006 Phys. Rev. D 74 083507
[11] Mielczarek J and Szydlowski M 2007 Phys. Lett. B 657 20
[12] Copeland E J, Mulryne D J, Nunes N J and Shaeri M 2008 Phys. Rev. D 77 023510
[13] Mielczarek J 2008 J. Cosmol. Astropart. Phys. JCAP11(2008)011
[14] Mielczarek J 2009 Phys. Rev. D 79 123520
[15] Copeland E J, Mulryne D J, Nunes N J and Shaeri M 2009 Phys. Rev. D 79 023508
[16] Grain J and Barrau A 2009 Phys. Rev. D 79 084015
[17] Mielczarek J, Cailleteau T, Grain J and Barrau A 2010 Phys. Rev. D 81 104049
[18] Calcagni G and Hossain G M 2009 Adv. Sci. Lett. 2 184
[19] Larson D et al 2011 Astrophys. J. Suppl. 192 16
[20] Bojowald M, Hossain G M, Kagan M and Shankaranarayanan S 2008 Phys. Rev. D 78 063547 (arXiv:0806.3929 [gr-qc])
[21] Wu J P and Ling Y 2010 J. Cosmol. Astropart. Phys. JCAP05(2010)026 (arXiv:1001.1227 [hep-th])
[22] Wilson-Ewing E 2012 Class. Quantum Grav. 29 085005 (arXiv:1108.6265 [gr-qc])
[23] Bojowald M, Hossain G M, Kagan M and Shankaranarayanan S 2009 Phys. Rev. D 79 043505
[24] Bojowald M, Hossain G M, Kagan M and Shankaranarayanan S 2010 Phys. Rev. D 82 109903 (erratum) (arXiv:0811.1572 [gr-qc])
[25] Mielczarek J, Cailleteau T, Barrau A and Grain J 2012 Class. Quantum Grav. 29 085009 (arXiv:1106.3744)
[26] Corichi A and Singh P 2008 Phys. Rev. D 78 024034
[27] Nelson W and Sakellariadou M 2007 Phys. Rev. D 76 104003 (arXiv:0707.0588 [gr-qc])
[28] Bojowald, M private discussion
[29] Hartle J B and Hawking S W 1983 Phys. Rev. D 28 2960
[30] Ashtekar A, Pawlowski T and Singh P 2006 Phys. Rev. D 74 084003 (arXiv:gr-qc/0607039)
[31] Mukhanov V F, Feldman H A and Brandenberger R H 1992 Phys. Rep. 215 203–333
[32] Bojowald M and Calcagni G 2011 J. Cosmol. Astropart. Phys. JCAP03(2011)032 (arXiv:1011.2779 [gr-qc])
[33] Bojowald M, Calcagni G and Tsujikawa S 2011 Phys. Rev. Lett. 107 211302 (arXiv:1101.5391 [astro-ph.CO])
[34] Bojowald M, Calcagni G and Tsujikawa S 2011 J. Cosmol. Astropart. Phys. JCAP11(2011)046 (arXiv:1107.1540 [gr-qc])