Quantum integrability in systems with finite number of levels

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We consider the problem of defining quantum integrability in systems with finite number of energy levels starting from commuting matrices. We argue that if the matrices depend on a (real) parameter, one can provide a definition from this feature alone, leading to specific results such as exact solvability, Poissonian energy level statistics and to level crossings.

I. INTRODUCTION

The field of quantum integrable systems, originally a somewhat abstract topic, has received a great deal of attention in recent years with the realization that many typical models are realizable in atomic, mesoscopic and macroscopic many-body systems. In describing molecular systems such as benzene, we deal with Hamiltonians defined on a finite-dimensional Hilbert space. These arise from a lattice of finite size, with spins, fermions or bosons populating the sites. In condensed matter systems too, one often studies finite size systems as a prelude to taking the thermodynamic limit or to obtain finite size corrections. Our concern in this work is with the characterization and definition of quantum integrability in a finite dimensional context, a task that is considerably more delicate than the corresponding classical case.

We may view typical condensed matter problems as models with a finite number of discrete single-particle energy levels, and sometimes (but not always) with a fixed number of particles. The Hamiltonian is then ‘just’ an \(N \times N\) Hermitian matrix, with a suitable \(N\). This description is, by design, far removed from its parentage in the space of many body models. Given such a Hamiltonian matrix, can we say whether it is integrable or not? What is the precise notion of quantum integrability in this case; can we separate \(N \times N\) Hermitian matrices into two clearly distinct classes – integrable and otherwise? If yes, what are the consequences of quantum integrability that can be derived from its definition, i.e. the characteristic properties of such ‘integrable’ matrices?

In this paper, we present a coherent viewpoint that emerges from our recent exploration of such questions. We present a summary of our main results and also list some open questions that remain unanswered. To put this enquiry into context, note that in the thermodynamic limit \(N \to \infty\), impressive alternative approaches are available. For example Baxter’s work on the spectra of infinite dimensional corner transfer matrices\(^3\) culminates in the realization of extra symmetries and structure arising in that limit, formalized by the Yangian approach\(^2\). Another viewpoint is of geometric origin due to Sutherland\(^4\) and emphasizes non-diffractive scattering as a true hallmark of quantum integrability. Much as we admire these powerful viewpoints, our chosen task of understanding finite dimensional matrix systems takes us in quite a different direction.

In classical mechanics a system with \(n\) degrees of freedom is said to be integrable if it has \(n\) functionally independent integrals of motion that Poisson commute\(^4\), i.e. are in involution. An exact solution of equations of motion follows from this definition, and it further can be shown that the motion is confined to invariant tori cut out in the phase space by the conservation laws. Unfortunately, a straightforward import of this elegant notion of integrability into quantum mechanics is problematic.

The difficulties are at least two fold: firstly what is a degree of freedom? In quantum models with a fixed number of particles, this number seems closest to the classical notion of a degree of freedom. However, in addition to the particle number there are other integer parameters to take into account such as the number of sites or the magnitude of spins. For example, in the Hubbard model with \(L\) sites do we count the number of electrons \(n_e\) or the number of holes \(2L - n_e\)? Or does the Heisenberg model for \(L\) spin-\(\frac{1}{2}\) particles have the same number of degrees of freedom as that for \(L\) spin-5 particles?

The second difficulty is in defining what constitutes a nontrivial integral of motion. From an elementary theorem in algebra\(^5\), we know that an arbitrary \(N \times N\) Hermitian matrix commutes with \(N\) other Hermitian matrices that may be chosen as the projection operators along each component in the diagonal representation. We could equivalently express the matrix and all its commuting partners as a power series in some chosen non degenerate matrix. A similar statement can be proved for general commuting Hermitian operators with discrete or continuous spectra\(^5\). If now we are given a special Hamiltonian matrix, and told that it is quantum integrable (in some specific sense that is yet to be defined), possessing \(n\) equally special commuting partner matrices; we might (rather stubbornly) ignore this extra information and compute the \(N\) elementary commuting partners of the given matrix as per the earlier prescription. Don’t we then have too many partners? What, if any, is the distinction between these two sets of commuting partners? From such questions and considerations one seems to be forced to call integrable either all or none Hamiltonians with finite Hilbert spaces.

Our goal here is to attempt a resolution of these difficulties and provide a practically useful, simple and at
the same time rigorous definition of quantum integrability for Hamiltonian matrices. Our interest is of course not just in the definition per se but in a clear delineation and derivation of properties of quantum integrable systems from the definition. Characteristics normally attributed to integrable models include: exact solution for their spectra, Poisson level statistics, crossings of levels of the same symmetry in parameter-dependent Hamiltonians etc. However, they are never derived from one another or within some unified framework, but each property has to be established independently and on a model by model basis.

Consider, for example, the 1d Hubbard model. Shortly after the model was exactly solved via Bethe’s Ansatz,\textsuperscript{[9]} a separate rigorous study\textsuperscript{[10]} of its parameter independent symmetries revealed numerous level crossings violating the Wigner von Neuman non-crossing rule for eigenvalues in generic systems. These findings can be rationalized by the notion that copious level crossings are a hallmark of quantum integrability, and hint at parameter dependent conservation laws in such a model. Much later, one of us\textsuperscript{[11,12]} found an embedding of the 1d Hubbard model into the Yang Baxter scheme, thereby displaying an explicit set of parameter dependent conservation laws. Not surprisingly, a numerical analysis\textsuperscript{[13]} of level spacings finds Poisson statistics that crosses over to GOE when ‘integrability’ is destroyed by adding an off-site Coulomb repulsion. Further work\textsuperscript{[13]} by us on making a causal link between the conservation laws and the level crossings in the 1d Hubbard model has been a fruitful source of insights that we have subsequently explored and report here.

Often, one of the above characteristics is singled out and adopted as a definition.\textsuperscript{[14]} It seems to us that while level statistics and crossings are useful, and even powerful tests of integrability, these are hardly suitable as definitions. These criteria arguably encompass a broader class of systems than those normally thought integrable\textsuperscript{[15]}. It is also difficult to see how, starting from either of them, one could derive other properties, e.g. an exact solution. Universal statistics emerges only in the limit of large matrices, \( N \to \infty \), or for an ensemble of matrices, while here we are looking for a notion that also works for a stand-alone Hamiltonian matrix with fixed \( N \). In addition, there are exceptional points in parameter space of systems usually recognized as integrable, where the statistics is non-Poissonian, e.g. for Gaudin magnets.\textsuperscript{[16]}

Starting from a well-defined notion of integrability that we propose below, we indeed find that there are such deviations in a much broader class of systems and explain their origin. Similarly, we find examples that fail the level crossing test – such as ‘accidental’ degeneracies in non-integrable systems and (rare) instances of integrable systems without crossings.

It turns out that such a simple and yet well-defined notion can in fact be formulated as we recognized in a series of papers\textsuperscript{[14,16,21]}. The main idea is to consider the dependence of commuting operators on a real parameter, which we denote \( u \). This is typically an interaction constant or an applied field, e.g. Coulomb interaction constant in the Hubbard model, magnetic field in Gaudin magnets,\textsuperscript{[22]} pairing strength in the BCS and anisotropy in the XXZ Heisenberg models, etc. Besides usual space-time and internal space symmetries, which are parameter independent, these models have \( u \)-dependent conservation laws (alternatively termed dynamical symmetries or conserved currents) as discussed above. For the BCS and Gaudin models the Hamiltonian and all conserved currents are linear in the parameter\textsuperscript{[22,23]}. In the XXZ and Hubbard models the Hamiltonian and the first dynamical symmetry are linear in \( u \); the rest are polynomials in it of order two and higher\textsuperscript{[11,24,25]}.\textsuperscript{[21]}

\section{II. Definition}

In most of the examples of condensed matter models believed to be quantum integrable, there is a set of commuting operators linear in \( u \) including the Hamiltonian. This seems to be the most common situation in parameter dependent integrable models and leads us\textsuperscript{[16,23]} to consider operators of the form \( H(u) = T + uV \), where \( T \) and \( V \) are \( N \times N \) Hermitian matrices. A key observation is that fixed \( u \)-dependence implies a natural well-defined notion of a nontrivial integral of motion. A typical (e.g. randomly generated) \( H(u) \) commutes only with \((a + bu)1 + cH(u)\), a trivial operator linear in \( u \). Here \( a, b, \) and \( c \) are real numbers and \( 1 \) is the identity operator.

The requirement that there exist a nontrivial commuting partner linear in \( u \) severely constrains the matrix elements of \( H(u) \). In fact, as we will see below, a real symmetric \( H(u) \) that has such a partner is fixed by less than \((N - 1)(N + 8)/2 \) real parameters. In contrast, \( N(N + 1) \) real parameters are necessary to specify the matrix elements of \( T \) and \( V \) for a generic \( H(u) \), indicating that matrices with fixed parameter-dependence split into two non-overlapping sets – those with nontrivial commuting partners and those without.

Moreover, there is a natural classification of integrable families according to the number \( n \) of linearly independent commuting operators. Specifically, we define an integrable family as a vector space of \( n \) linearly independent \( N \times N \) Hermitian matrices

\[ H^k(u) = T^k + uV^k \] such that,

\[ [H^i(u), H^j(u)] = 0 \text{ for all } u \text{ and } i, j = 1, \ldots, n. \] (1)

In addition, we impose an (optional) condition that \( H^i(u) \) have no common \( u \)-independent symmetries – there is no constant matrix \( \Omega \) (\( \neq a1 \)) such that \([\Omega, H^i(u)] = 0 \text{ for all } u \) and \( i \). If there are such symmetries, \( H^i(u) \) are simultaneously block-diagonal and Eq. (1) reduces to that for smaller matrices (blocks) without \( u \)-independent symmetries\textsuperscript{[19]}

Linear independence means that \( \sum_i a_i H^i(u) = (a + bu)1 \) with real \( a_i \) if and only if all \( a_i = 0 \) and \( a = b = 0 \).
Note that, for convenience, we chose to separate multiples of identity from our list of nontrivial commuting matrices \( H^k(u) \). Finally, \( n \) is defined as the maximum number of nontrivial independent commuting matrices in a given family, i.e. any \( H(u) = T + uV \) that commutes with all \( H^i(u) \) can be written as

\[
H(u) = \sum_{i=1}^{n} a_k H^k(u) + (a + bu)1. \tag{2}
\]

Thus \( H^i(u) \) act as basis vectors in the \( n \)-dimensional vector space \( V_n \) of commuting matrices, defined up to a multiple of identity matrix \( 1 \).

We propose the following classification of integrable matrices linear in a parameter into types. The maximum possible number of linearly independent \( N \times N \) commuting Hermitian matrices is \( n = N - 1 \), not counting multiples of the identity matrix. We call this a type 1 or maximally commuting family of matrices. Similarly, families with \( n = N - 2 \) independent commuting \( H^i(u) \) are termed type 2, and a general type \( M \) family is defined through \( n = N - M \). The maximum value of \( M \) is \( M = N - 2 \) when there are two nontrivial members in the family in addition to \( (a + bu)1 \).

We argue that the following properties are direct consequences of the above definition of quantum integrability: (1) an exact solution for the eigenspectra of \( H^i(u) \) in terms of roots of a single algebraic equation, (2) \( H^i(u) \) satisfy Yang-Baxter equation, (3) eigenvalues of \( H^i(u) \) typically (but not always) cross as functions of \( u \); the number of crossings depends on both \( N \) and \( M \) (4) eigenvalues have Poisson statistics in \( N \to \infty \) limit except for some special cases of certain measure zero in the space of all integrable families. Below we prove statements (1) through (3) for type 1 families (in this case crossings are always present) and comment on similar results \cite{24} for other types. Poisson statistics will be demonstrated numerically in a separate paper; here we briefly discuss some of its main findings.

Eq. \( 2 \) can be solved for matrix elements of \( H^i(u) \) at least for some types of integrable families. All type 1 families were constructed in \cite{24}. In \cite{24} all type 2, 3 and some type \( M \) for arbitrary \( M \) were obtained. To facilitate further discussion, let us first cast Eq. \( 1 \) into a different form. Using \( H^k(u) = T^k + uV^k \) and equating to zero terms at all orders of \( u \), we obtain

\[
[T^i, V^j] = [T^j, V^i], \quad [T^i, T^j] = [V^i, V^j] = 0. \tag{3}
\]

It is convenient to choose the basis in the target Hilbert space to be the common eigenbasis of the mutually commuting matrices \( V^i \). The first commutation relation in Eq. \( 3 \) in this basis reads \( T^k_{mn} = T^k_{nm} \equiv A_{km} \), where \( d^i_k \) are the diagonal elements of \( V^i \). This implies that \( T^i \) can be written as

\[
T^i = W^i + [V^i, A] \tag{4}
\]

where \( W^i \) is a diagonal matrix. Note that the anti-hermitian matrix \( A \) is the same for all members of the family and is therefore independent of the basis in \( \mathcal{V}_n \). Now the commutation relations \( [V^i, V^j] = 0 \) and \( [T^i, V^j] = [T^j, V^i] \) are satisfied. The remaining equation, \( [T^i, T^j] = 0 \), takes the form

\[
[[V^i, A], [V^j, A]] = [[V^j, A], W^i] - [[V^i, A], W^j]. \tag{5}
\]

Before we proceed with type 1, let us discuss the number of parameters involved in constructing a generic real symmetric integrable family.

We can generate integrable families of various types by solving Eq. \( 5 \) numerically. \cite{24} The algorithm is as follows. First, we arbitrarily specify some diagonal matrices \( V^i, V^j, W^i \) and \( W^j \) \((4N \) real inputs) and solve for the antisymmetric matrix \( A \). This yields a discrete set of solutions for \( A \) and two commuting matrices \( H^i(u) \) in the family with a given \( A \), we take \( j = k \) in Eq. \( 5 \) (with \( V^i, W^i \) and \( A \) obtained before) and solve this equation for \( V^i \) and \( W^i \).

Some choices of the \( 4N \) inputs in the above procedure produce the same commuting family. In fact, as seen from Eq. \( 2 \), there is \( 2(n+2) \)-parametric freedom (values of \( a_k, a \) and \( b \)) in picking two matrices within the family. This means that we can fix \( 2(n+2) \) out of \( 4N \) parameters by taking linear combinations within the family, i.e. by going to a different basis in the vector space of commuting matrices. In addition, Eqs. \( 5 \) and \( 4 \) are invariant with respect to rescaling \( V^i \to aV^i \) and \( A \to A/\alpha \), which fixes one more parameter in \( V^i \). This leaves \( 4N - 2(n+2) - 1 = 2N + 2M - 5 \) real parameters to specify a generic type \( M \) family.

To further select a particular matrix \( H(u) = T + uV \) within the family, one needs to pick \( n + 2 \) coefficients in Eq. \( 2 \) in addition to \( 2N + M - 5 \) that specify the commuting family, i.e. a general type \( M \) matrix involves \( 4N - n - 3 = 3N + M - 3 \) real parameters. Since \( M \leq N - 2 \) the maximum number is \( 4N - 5 \). This counting has been done in the common eigenbasis of \( V^i \).

Going to an arbitrary basis in the target Hilbert space adds another \( N(N - 1)/2 \) real parameters for a general orthogonal transformation. We see that the total number of parameters is less than \( (N+8)(N-1)/2 \), about half of \( N(N+1) \) for a non-integrable real-symmetric matrix of the form \( B + uC \).

**III. TYPE 1**

The ‘master’ equation \( 5 \) is simplest for type 1. In this case there are \( N \) linearly independent \( H^i(u) \) (including the identity) and consequently \( N \) linearly independent diagonal matrices \( V^i \). By taking linear combinations we can go to a basis in the vector space \( \mathcal{V}_{N-1} \) such that \( V^i_{kk} \equiv d_k = \delta_{ik} \). Eq. \( 5 \) reads \( f^i_{jk} \equiv V^i_{jk} - V^j_{ik} = -A_{ij}A_{ik}/A_{jk} \). These equations are consistent when \( f^i_{jk} + \)
\[ f_{kl} + f_{lj} = 0, \text{ yielding the following four index relation} \]
\[ A_{ij} A_{jk} A_{kl} A_{ii} + A_{ik} A_{kl} A_{lj} A_{jj} + A_{il} A_{lj} A_{jk} A_{ki} = 0. \]  
(6)
The most general solution of this equation is \[ A_{jk} = \frac{\gamma_j \gamma_k^*}{\epsilon_j - \epsilon_k}, \]  
(7)
where real \( \epsilon_i \) and complex \( \gamma_i \) are unrestricted parameters that fix the commuting family.

Next, using Eq. (6) and \( V_{ij} = \delta_{ij} \delta_{kk} \), we determine \( W^i \) from Eq. (5), which is linear in \( W^i \), and \( T^i \) from Eq. (4). The most general member of a type 1 family, \( H(u) = \sum_{i=1}^{N} d_i H^i(u) \) with arbitrary real \( d_i \), is

\[
[H(u)]_{mn} = \gamma_m \gamma_n^* \frac{(d_m - d_n)}{(\epsilon_m - \epsilon_n)}, \quad m \neq n, \\
[H(u)]_{mm} = u d_m - \sum_{k \neq m} |\gamma_k|^2 \left( \frac{d_m - d_k}{\epsilon_m - \epsilon_k} \right).
\]
(8)

Note that \( d_m \) are eigenvalues of \( V \) by design. In particular, for basic operators \( H^i(u) \) we have \( d_k = \delta_{ik} \), i.e. their nonzero matrix elements are

\[
[H^i(u)]_{ij} = \gamma_i \gamma_j^* \frac{\delta_{ij}}{\epsilon_i - \epsilon_j}, \quad j \neq i, \\
[H^i(u)]_{jj} = u \delta_{ij} - \sum_{k \neq j} |\gamma_k|^2 \left( \frac{\delta_{ij} - \delta_{ik}}{\epsilon_j - \epsilon_k} \right).
\]
(9)

A similar construction is possible for real symmetric integrable families of arbitrary type \( M \). It involves \( M + 1 \) new parameters in addition to \( \gamma_i, \epsilon_i, \) and \( d_i \), which play a role similar to that in Eq. (8). However, this construction produces all real symmetric families only for \( M = 1, 2, 3 \), while for \( M > 3 \) it yields only a subset of such families.

A correspondence between real symmetric type \( M \) families and compact Riemann surfaces of genus \( g \geq M - 1 \) was conjectured in [21]. The construction of ansatz type \( M \) produces families that correspond to hyperelliptic Riemann surfaces of genus \( g = M - 1 \). All Riemann surfaces of genus 0, 1, and 2 are hyperelliptic which explains the completeness of the construction for \( M = 1, 2, 3 \).

IV. EXACT SOLUTION

The exact spectra of type 1 matrices were obtained in [20]. The components of an eigenvector (column) \( \vec{v}_m(u) \) of \( H(u) \) given by (8) are

\[
[\vec{v}_m(u)]_j = \frac{\gamma_j}{\lambda_m - \epsilon_j}, \quad (10)
\]
with respective eigenvalue

\[
E_m(u) = \sum_{k=1}^{N} \frac{d_k |\gamma_k|^2}{\lambda_m - \epsilon_k}, \quad (11)
\]
where the \( \lambda_i, i = 1, \ldots, N \) are determined from a single algebraic equation

\[
f(\lambda_m) = \sum_{j=1}^{N} \frac{|\gamma_j|^2}{\lambda_m - \epsilon_j} = u. \quad (12)
\]
These equations can be verified directly by evaluating \( \sum_j [H(u)]_{ij} [\vec{v}_m(u)]_j \). Ansatz type \( M \) (see the last paragraph in Sect. (11)) families have an exact solution in terms of a single equation similar to Eq. (12).

All \( N \) roots \( \lambda_i \) of Eq. (12) are real. Indeed, \( f(\lambda) \to +\infty \) as \( \lambda \to \epsilon_{k}^+ \) and \( f(\lambda) \to -\infty \) as \( \lambda \to \epsilon_{k+1}^- \), where \( \epsilon_k \) are ordered, \( \epsilon_1 < \epsilon_2 < \cdots < \epsilon_N \). It follows that \( f(\lambda) = u \) has a real solution between \( \epsilon_k \) and \( \epsilon_{k+1} \) for any \( k \), i.e. \( \epsilon_m < \lambda_m < \epsilon_{m+1} \). The last root \( \lambda_N > \epsilon_N \) for \( u > 0 \) and \( \lambda_N < \epsilon_{N+1} \equiv \epsilon_1 \) for \( u < 0 \), where we identify indices that differ by a multiple of \( N \).

V. LEVEL CROSSINGS

First, we show that any type 1 matrix \( H(u) = T + u V \) has at least one level crossing [20]. To this end, let us analyze the evolution of eigenvalues \( E_m(u) \) with \( u \). We observe that \( \lambda_m \to \epsilon_m \) as \( u \to +\infty \). In this limit the main contribution to Eqs. (11) and (12) comes from the \( j = m \) term, yielding \( E_m \to |u|d_m \). Similarly, we obtain \( x_m \to \epsilon_{m+1} \) and \( E_m \to -|u|d_{m+1} \) for \( u \to -\infty \). It is not surprising that eigenvalues of \( H(u) \) tend to \( \pm |u|d_k \), which are eigenvalues of \( V \) and \( u V \) dominates \( H(u) \) for large \( u \). What is important however is that we know to which particular \( u d_k \) a given \( E_m(u) \) tends in both limits.

Symbolically, we can write \( k \to k - 1 \) (mod \( N \)) meaning the eigenvalue goes from \( -|u|d_k \) on the left \( (u \to -\infty) \) to \( |u|d_{k-1} \) on the right \( (u \to +\infty) \).

The presence of level crossings can now be proved by contradiction. Suppose there are no crossings. Since eigenvalues are continuous functions of \( u \), this implies that their ordering must be the same at all \( u \). The top level must connect the largest eigenvalue at \( u \to -\infty \) to the largest eigenvalue at \( u \to +\infty \), the bottom level goes from the lowest eigenvalue at \( u \to -\infty \) to that at \( u \to +\infty \), etc. Let \( d_k \) be ordered as \( d_1 < d_2 < \cdots < d_m \). Then, the largest (lowest) eigenvalue at \( u \to -\infty \) is \( -|u|d_1 \) and the largest (lowest) eigenvalue at \( u \to +\infty \) is \( |u|d_m \) (i.e. we have \( i = m \) and \( m \to i \) for the top and bottom levels, respectively. On the other hand, according to the \( k \to k - 1 \) rule established above this implies \( m = i - 1 \) (mod \( N \)) and at the same time \( i = m - 1 \) (mod \( N \)). We obtain \( 0 = 2 \) (mod \( N \)), which does not hold for any \( N \geq 3 \), i.e. the above assumption that levels do not cross cannot be true. Thus, at least one level crossing is inevitable.

Allowed values of the total number of crossings \( n_x \) in a type 1 matrix can be determined by analyzing the representation of an arbitrary type 1 matrix \( \tilde{H}(u) \) in terms of powers of any other nontrivial \( \tilde{H}(u) \) that belongs to the
same commuting family. One finds\cite{19} that this expansion is necessarily of the form
\[
\tilde{H}(u) = \sum_{m=0}^{N-1} \frac{Q^m(u)}{P^H(u)} [H(u)]^m, \tag{13}
\]
where \(P^H(u)\) is a polynomial in \(u\) of degree \((N-1)(N-2)/2\) with real coefficients that depend on matrix elements of \(H(u)\) only. \(Q^m(u)\) are polynomials in \(u\) of order \(m-1\) lower than \(P^H(u)\). This expansion breaks down at a given value of \(u = u_\infty\) only when \(H(u)\) has a crossing at \(u = u_\infty\) and \(P^H(u_\infty) = 0\). Thus, crossings of \(H(u)\) occur at the roots of \(P^H(u)\) (see \cite{19} for a detailed proof).

We conclude that the maximum number of crossings in a type 1 matrix is
\[
N_{\infty}^{\text{max}} = \frac{(N-1)(N-2)}{2}. \tag{14}
\]
This upper bound is realized e.g. for matrices \(3\) such that \(d_1 > d_2 > \cdots > d_N\), see Fig. 1. \(n_\infty\) has a definite parity, that of \(n_\infty^{\text{max}}\), because the coefficients of \(P^H(u)\) are real and its complex roots therefore come in conjugate pairs. For example, \(4 \times 4\) type 1 matrices have either 1 or 3 crossings, \(n_\infty = 2, 4, 6\) for \(5 \times 5\) etc.

Crossings are also ubiquitous in type \(M > 1\) integrable families. It is possible however to deliberately engineer \(4 \times 4\) type 2 matrices without crossing\cite{19}. Other than that, we do not have rigorous results for higher types. Empirically, one finds\cite{19} that the expansion \(13\) still holds, but the degree of \(P^H(u)\) and therefore the maximum number of crossings is reduced to \((N-1)(N-2)/2 - g\), where \(g \geq M - 1\) is the genus of the corresponding Riemann surface, typically \(g = M - 1\) (see the end of Sect. \(\Pi\)).

Interestingly, one can also construct \(4 \times 4\) real symmetric matrices \(A + uB\) that have crossings but no \(u\)-independent symmetry and no commuting partners linear in \(u\) (this is impossible for \(N = 3\), see \cite{19} for details). This indicates that either ‘accidental’ degeneracies are possible or one can meaningfully introduce nonlinear in \(u\) conserved currents in such cases.

\section{VI. Yang-Baxter Equation for the Matrix Model}
We next show how the type 1 matrices can be fit into the Yang Baxter formulation of integrable systems, by displaying matrix objects \(S\), Eq. \(17\), that play the role of scattering amplitudes. Thus we consider a purely matrix model in \(N\) dimensions with states \(|i\rangle\), \(1 \leq i \leq N\), the projection operators \(\pi_{ij}\) and the identity matrix \(1\):
\[
\pi_{ij} = |i\rangle \langle j| \quad 1 = \sum_i \pi_{ii}
\]
With \(i \neq j\), let us define a ‘dressed’ permutation operator
\[
\Pi_{ij} = \frac{\gamma_i^2}{2} 1 + \gamma_i \gamma_j \pi_{ij} - |\gamma_i|\pi_{jj} - |\gamma_j|^2 \pi_{ii}.
\]
If we set \(\gamma_i \rightarrow \gamma_j\), \(\Pi\) reduces to the permutation operator
\[
\Pi_{ij} \rightarrow |\gamma_j|^2 P_{ij}, \tag{15}
\]
where \(P_{ij}\) acts as:
\[
P_{ij}|k\rangle = \delta_{ik}|j\rangle + \delta_{jk}|i\rangle + (1 - \delta_{ik})(1 - \delta_{jk})|k\rangle. \tag{16}
\]
Let us introduce \(g\) as a coupling parameter and also the composite parameter \(x = (\varepsilon, \gamma)\) so that the scattering operator \(S_{ij}\) is defined as:
\[
S_{ij}(x_{j} | x_{i}) = S_{ij} = \frac{(\varepsilon_j - \varepsilon_i) 1 + 2g \Pi_{ij}}{(\varepsilon_j - \varepsilon_i) + g (|\gamma_i|^2 + |\gamma_j|^2)} \tag{17}
\]
The action of a particular scattering matrix \(S_{nm}\) on the relevant states is given by
\[
S_{nm} |k\rangle = |k\rangle, \quad (k \neq n, m)
\]
\[
S_{nm} |n\rangle = t_{nm}^+ |n\rangle + r_{nm} |m\rangle
\]
\[
S_{nm} |m\rangle = t_{nm}^- |m\rangle + r_{nm} |n\rangle \tag{18}
\]
with the reflection \((r)\) and transmission \((t)\) amplitudes defined by
\[
r_{nm} = \frac{2g\gamma_n \gamma_m^*}{\varepsilon_m - \varepsilon_n + g(|\gamma_m|^2 + |\gamma_n|^2)},
\]
\[
t_{nm}^+ = \frac{\varepsilon_m - \varepsilon_n + g(|\gamma_m|^2 - |\gamma_n|^2)}{\varepsilon_m - \varepsilon_n + g(|\gamma_m|^2 + |\gamma_n|^2)},
\]
\[
t_{nm}^- = \frac{\varepsilon_m - \varepsilon_n + g(|\gamma_m|^2 - |\gamma_n|^2)}{\varepsilon_m - \varepsilon_n + g(|\gamma_m|^2 + |\gamma_n|^2)}. \tag{19}
\]
We state the two operator relations that are needed, and readily verified, for any three indices 1, 2, 3:
\[
\Pi_{12} \Pi_{32} \Pi_{31} = \Pi_{31} \Pi_{32} \Pi_{12}, \quad \Pi_{12}, (\Pi_{31} + \Pi_{32})] = 0. \tag{20}
\]
With \(C_{ij}\) as constants we can now verify the condition
\[
(C_{12} 1 + g \Pi_{12}) (C_{32} 1 + g \Pi_{32}) (C_{31} 1 + g \Pi_{31}) = (C_{31} 1 + g \Pi_{31}) (C_{32} 1 + g \Pi_{32}) (C_{12} 1 + g \Pi_{12}), \tag{22}
\]
provided \(C\)'s satisfy the triangle law:
\[
C_{32} = C_{31} + C_{12}. \tag{23}
\]
Dividing by a suitable constant and consulting Eq. (18) we therefore verify the Yang Baxter equation for the \(S\)
\[
S_{ik} S_{jk} S_{ij} = S_{ij} S_{jk} S_{ik}, \tag{24}
\]
and the initializing condition
\[
\lim_{x_j \to x_i} S_{ij} (x_j | x_i) \to P_{ij}. \tag{25}
\]
We consider the \(N\) sites (indices) and add an auxiliary index \(\alpha\) (that is outside the original space of \(N\) states) so that \(\pi_{i\alpha}, \pi_{\alpha i}, \pi_{\alpha \alpha}\) are added to the list of operators. Now define a (monodromy) matrix
\[
T_\alpha = B_\alpha (x_\alpha) S_{N\alpha} S_{N-1 \alpha} \cdots S_{1\alpha}. \tag{26}
\]
This is a function of all the indicated variables:
\[
T_\alpha = T_\alpha (x_\alpha \{x_1, x_2, \ldots, x_N\}), \tag{27}
\]
and also a boundary field term \(B_\alpha (x_\alpha)\). For the boundary term (twist) to give commuting operators, this term must be chosen to satisfy the condition\[22\]
\[
[S_{\alpha \beta}, B_\alpha B_\beta] = 0, \tag{28}
\]
whereby we choose
\[
B_\alpha (x_\alpha) = 1 + \frac{g}{u} \pi_{\alpha \alpha}, \tag{29}
\]
with the freedom of an arbitrary parameter \(u\).

The transfer matrix is obtained by tracing over the auxiliary index \(\alpha\)
\[
T (x_\alpha \{x_1, x_2, \ldots, x_N\}) = \sum_{\alpha} \langle \alpha | T_\alpha | \alpha \rangle. \tag{30}
\]
Using Baxter’s classic proof\[23\] for commutation of transfer matrices, slightly generalized to the case of twisted boundary conditions\[22;23\] we conclude that
\[
[T (x_\alpha), T (x_\beta)] = 0, \tag{31}
\]
for arbitrary \(x_\alpha\) and \(x_\beta\) while holding \(\{x_j\}\) \(g\), \(u\) fixed. We note that Eq. (31) is also valid as \(x_\alpha \to x_i\) and hence conclude \([T_i, T_j] = 0, \) where
\[
T_j = \lim_{x_\alpha \to x_j} T (x_\alpha \{x_1, x_2, \ldots, x_N\}) = S_{j-1,j} (x_j, x_{j-1}) \cdots S_{1,j} (x_j, x_1),
\]
\[
B_j (x_j) S_{N,j} (x_j, x_N) \cdots S_{j+1,j} (x_j, x_{j+1}). \tag{32}
\]
An expansion in powers of the interaction strength \(g\) produces the currents:
\[
T_j = 1 + \frac{g}{u} H^j (u) + O (g^2), \tag{33}
\]
with
\[
H^j (u) = \pi_{ii} + u \sum \gamma_i \gamma^*_i (\pi_{ij} + \pi_{ji}) - \gamma_i^2 \pi_{jj} - \gamma_j^2 \pi_{ii} \equiv \epsilon_i - \epsilon_j. \tag{34}
\]
Considering terms of order \(O (g)\) in \([T_i, T_j]\), we conclude
\[
[H^j (u), H^i (u)] = 0. \tag{35}
\]
Note that \(H^i (u)\) are the exactly the basis operators of Eq. (9) written in terms of projection operators\[20\].

VII. LINKS TO VARIOUS MODELS

First, note that one can choose an arbitrary Hermitian matrix \(V\) and still have \(H (u) = T + uV\) to be a member of any given type 1 family. Indeed, Eq. (8) is written in the eigenbasis of \(V\). Both this basis and the eigenvalues of \(V\) – parameters \(d_k\) – can be chosen arbitrarily. By symmetry one can instead choose \(T\) at will, though this is not apparent from Eq. (8). But as soon as e.g. \(V\) is fixed, \(T\) is severely constrained – one is left with only 2\(N\) parameters \(\gamma_i, \epsilon_i\) to specify its matrix elements. This ability to choose either \(V\) or \(T\) arbitrarily means in particular that any \(u\)-independent Hamiltonian can be ‘embedded’ into a type 1 family in many different ways. For example, one can choose \(V\) to be the isotropic Heisenberg (XXX) model, or the Haldane-Shastry model, and find \(T\) so that \(T + uV\) belongs to a given type 1 family.

Type 1 integrable families are closely related\[19\] to Gaudin magnet\[22\] \(h_i = B \hat{s}_i^z + \sum_{j \neq i} \hat{s}_i \hat{s}_j (\epsilon_i - \epsilon_j)^{-1}\), where \(\hat{s}_j\) are quantum spins of arbitrary length \(s_j\). In the sector with (conserved) \(\hat{S}^z = \sum \hat{s}_j^z\) equal to its maximum (minimum) possible value less (plus) one, \(\hat{h}_i\) are \(N\) commuting \(N \times N\) matrices which form a type 1 family with \(u = B\) and \(\gamma_j^2 = s_j\). The BCS model is obtained\[23\] as \(\sum \epsilon_i \hat{h}_j\) for \(\gamma_j^2 = s_j = 1/2\) and a replacement \(u = B \to 1/g\), where \(g\) is the dimensionful BCS coupling constant.

Some blocks of the 1D Hubbard model characterized by a complete set of \(u\)-independent symmetry quantum numbers are type 1 matrices, though most blocks are type \(M > 21\). A similar typology can be developed for e.g. the 1D XXZ Hamiltonian and other sectors of Gaudin and BCS models using the method of\[24\] for determining the type of parameter-dependent matrices. Interestingly, this implies that at least in some blocks there is an exact solution in terms of a single algebraic equation – Eq. (12) or a similar equation for higher types\[23\] – a vast simplification as compared to Bethe’s Ansatz.

One can also construct fermionic (bosonic) Hamiltonians\[20\] out of type 1 matrices as
\[
\hat{H} = \sum_{mn} [H (u)]_{mn} a^d_m a_n, \tag{35}
\]
where $a_n$ are the usual fermionic (bosonic) destruction operators. \([\hat{H}_1, \hat{H}_2] = 0\) as long as the corresponding matrices $H_1(u)$ and $H_2(u)$ commute, i.e. belong to the same family.

**VIII. LEVEL STATISTICS**

We have performed an extensive numerical study of level statistics of type 1 and higher type matrices\(^{10}\) for various choices of parameters. Almost in all cases the statistics is Poissonian for $N \gg 1$ with high accuracy, even when we deliberately attempt to adjust the parameters to get a different statistics. Let us briefly describe the main results e.g. for the level-spacing distribution.

It is convenient to redefine the parameter $u \to 1/x$ and replace $T + uV \to V + xT$. To get a proper large $N$ limit one has to make sure that the scaling of parameters $d_k, \varepsilon_k, \gamma_k$ in Eq. (36) and $x$ with $N$ is such that the eigenvalues of $V$ and $xT$ scale in the same way for large $N$. As discussed above, the matrix $V$ is arbitrary, so at $x = 0$ one can have any admissible level statistics. Consider, for example, three representative cases: (a) $V$ is a random real symmetric matrix with independent identically distributed matrix elements $V_{jk}$ for $j \leq k$, (b) eigenvalues $d_k$ of $V$ are independent uniformly distributed random numbers, and (c) $d_k$ display level attraction, $P(s) = a \exp(-b s^{1+\omega})$ with $-1 < \omega < 0$. The level-spacing distribution $P(s)$ for $V$ is Wigner-Dyson $P(s) = 2a s^{-\omega/2}$ in (a) and Poissonian $P(s) = e^{-s}$ in (b). As soon as $V$ is chosen, $T$ is no longer arbitrary and we find that it has Poissonian $P(s)$ in all three cases for all choices of parameters $\gamma_k$ and $\varepsilon_k$ we considered as long as $\varepsilon_k$ and $d_k$ are uncorrelated.

Specifically, motivated in part by the BCS and Gaudin examples discussed in the previous section we took: (1) $\varepsilon_k$ that are also eigenvalues of a random matrix and $\gamma_k = \text{const}$ independent of $k$, (2) same as (1) but with random uncorrelated $\gamma_k$ and (3) independent uniformly distributed $\varepsilon_k$ and $\gamma_k = \text{const}$. For all these choices the level-spacing distribution for $T$ is very well approximated by Poissonian $P(s) = e^{-s}$, where $s$ is the level-spacing in units of the mean level-spacing. In case (a) above the level statistics of $H(x) = V + xT$ at $x = 0$ is Wigner-Dyson, but we find that it crosses over to Poisson at $|x| \approx 1/N$ and remains Poisson for larger $|x|$. Case (c) is analogous to (a) – a crossover to Poisson behavior at $|x| \approx 1/N$. In case (b) the statistics is Poissonian for all $x$. Similar behavior is found in spectral rigidity. We conclude that one can arrange for any statistics at a given value of the parameter $x = x_0$, but this becomes an isolated point in $N \to \infty$ limit, while for $x \neq x_0$ integrability as defined in Sect. 14 enforces Poisson statistics.

The only exception to Poisson statistics other than at an isolated value of $x$ we were able to identify is when parameters $d_k$ and $\varepsilon_k$ are correlated, so that $d_k = f(\varepsilon_k)$, where $f(\varepsilon)$ is a smooth function of $\varepsilon$ in $N \to \infty$ limit\(^{14}\) and $\gamma_k = \text{const}$. This is the case in e.g. the BCS model where $d_k = \varepsilon_k$ (see above). In such cases the statistics is distinctly non-Poissonian and, moreover, in case (a) above, for example, $P(s)$ crosses over at $x = O(N^0) \equiv O(1)$ from the Wigner-Dyson $P(s) = 2a s^{-\omega/2}$ to a more repulsive distribution $P(s) \propto s^4$ for small $s$. The repulsion is softened by randomizing $\gamma_k$. More importantly, the statistics quickly becomes Poissonian when the correlation between $d_k$ and $\varepsilon_k$ is destroyed, $d_k = f(\varepsilon_k)(1 + \eta_k)$, where $\eta_k$ are random. We find Poisson distribution already for $\eta_k = O(1/N)$ at $x = O(1)$, see also\(^{14}\) for a similar study of Gaudin model. $d_k = f(\varepsilon_k)$ define exceptional ‘surfaces’ of certain measure zero in parameter space. This seems analogous to the harmonic oscillator exceptions to the Poisson distribution in classical integrable systems.\(^{13}\) There too one finds increased level repulsion for oscillators instead of Poisson $P(s)$.

Some of these numerical observations can be understood using perturbation theory. Energies to the first order in $x$ are given by the second equation in (36), where we set $|\gamma_j| = 1/N$ to achieve proper scaling for large $N$ as discussed above\(^{13}\). We have

$$E_m(x) \approx d_m - \frac{x}{N} \sum_{j \neq m} \left( \frac{d_m - d_j}{\varepsilon_m - \varepsilon_j} \right). \quad (36)$$

The first term comes from $V$, which we take to have Wigner-Dyson $P(s)$, the second – from $T$, which is determined by the integrability condition\(^{13}\) and whose level statistics we do not control. Let us estimate $x$ at which the two terms in Eq. (36) become comparable. Without loss of generality we can take $d_k = O(N^0) = O(1)$ and we must also take $\varepsilon_k = O(1)$ so that $T$ and $xV$ scale in the same way for large $N$. Suppose $\varepsilon_k$ are ordered as $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_N$. When $d_k$ and $\varepsilon_k$ are uncorrelated $d_m - d_j$ is $O(1)$ when $j$ is close to $m$, i.e. when $(\varepsilon_m - \varepsilon_j) = O(1/N)$. The second term in Eq. (36) is then $x c_m \ln N$, where $c_m = O(1)$ is a random number only weakly correlated with $d_m$.

If we now order $d_m$, $c_m$ in general will not be ordered, i.e. if $d_{m+1} > d_m$ is the closest level to $d_m$ and therefore $(d_{m+1} - d_m) = O(1/N)$, the corresponding difference $(c_{m+1} - c_m) = O(1)$. The contributions to level spacings from the two terms in Eq. (36) become comparable for $x = x_c \approx 1/(N \ln N)$. It makes sense that the second term introduces a trend towards Poisson distribution because it is a (nonlinear) superposition of $\varepsilon_k$ and $d_k$ – eigenvalues of two uncorrelated random matrices. Thus, we expect a crossover from Wigner-Dyson to Poisson distribution at $x = x_c$.

This argument breaks down when $d_k = f(\varepsilon_k)$, since in this case $(d_m - d_j) = O(1/N)$ when $(\varepsilon_m - \varepsilon_j) = O(1/N)$. The two terms in Eq. (36) become comparable only at $x = O(1)$ in agreement with the numerics for this case. Moreover, the second term no longer trends towards Poisson statistics. Relaxing the correlation between $d_k$ and $\varepsilon_k$ with $d_k = f(\varepsilon_k)(1 + \eta_k)$ and going through the same argument, one expects a crossover to Poisson statistics at $x = O(1)$ for $\eta_k = O(1/N)$. 


IX. DISCUSSION

A distinct feature of the notion of quantum integrability proposed in this paper is the parameter \((u)\) dependence. This is in contrast to the classical notion that does not require any such dependence. We however find it necessary to be able to quantize the classical definition in a meaningful way. In this sense our definition is more demanding than its classical counterpart.

On the other hand, there is another distinction from the classical notion that makes our definition seem less stringent – even a single nontrivial integral of motion linear in \(u\) is sufficient to declare the model integrable. This was originally motivated by the absence of a well-defined analog of the number of degrees of freedom in quantum mechanics. There is evidence however that the presence of a single such integral is actually much more consequential in the quantum case. For example, for any \(H(u) = T + uV\) with a single nontrivial commuting partner \(I(u) = K + uW\) one also finds numerous (about \(N\) linearly independent) integrals of motion quadratic in \(u\) that commute with both \(H(u)\) and \(I(u)\). We note that a generic matrix of the form \(A + uB\) has no quadratic integrals other than its own square and a multiple of identity. One of the implications of this is that, for example, currents of higher order in the parameter in the 1D Hubbard and XXZ models might follow from the linear ones discussed here. Secondly, we can study the fermionic representation of type 1 matrices, Eq. (35). Here \(n = L = N\), the sector \(n_e = 1\) is isomorphic to type 1 matrix family \(1\), while other sectors are much larger matrices of high types \(M = (\binom{N}{n} - N + 1)\) i.e. they are apparently distinct and much more complicated integrable matrix families. With the hindsight of Eq. (35) it is evident that they all are different manifestations of the same type 1 model and are in some sense equivalent, but can this be formulated generally so that all such similar matrix integrable models are naturally grouped together and recognized to be related to each other as different representations of the same underlying structure? Further work is needed to obtain clarity on these questions.

Other open questions include: a general construction of type \(M > 3\) integrable families, analytical results for crossings in types \(M > 1\) and for level statistics in all types, the relationship between the exact solution for types \(M > 1\) through a single algebraic equation and Bethe’s Ansatz.

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\[1\] R. J. Baxter, Physica A 106, 18 (1981).

\[2\] V. G. Drinfel’d, Sov. Math. Dokl. 32, 254 (1985), B.
For example, the Anderson localization of electrons in a random potential is often viewed as a transition from Poisson (localized) to GOE (extended) statistics. In this context it is not obvious if or how the concept of quantum many-body problems plays a role.

To obtain $f(\xi)$ in this limit, we scale $\epsilon_k$ and $d_k$ with $N$ so that they fill finite intervals when $N \to \infty$.

Equivalently, one can redefine $x \to x/N$ as in the BCS model for which $x = g = \lambda \delta$, where $\lambda$ is the dimensionless coupling and $\delta \propto 1/N$ is the mean level spacing in $\epsilon_k$. Such examples are available analytically (type $N-2$ matrices) or can be generated numerically by solving Eq. (3), see[21] for details.

Individual sectors do not have additional sector-specific $u$-independent integrals. This is the case for the fermion[20] and Gaudin models, but not for the Hubbard model where there are numerous such sector-specific integrals[21].