INTEGRATION DAY THEORY AND THE SYMBOLISM OF QUANTUM MEASUREMENT

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ABSTRACT. Elements of a novel theory of quantum physics are developed, synthesising the role of symbolism in describing quantum measurement and in the topological representation of fractal invariant sets in nonlinear dynamical systems theory. In this synthesis, the universe $U$ is treated as an isolated deterministic dynamical system evolving precisely on a measure-zero fractal invariant subset $I_U$ of its state space. A non-classical approach to the physics of $U$ is developed by treating the geometry of $I_U$ as more primitive than dynamical evolution equations on $I_U$. A specific symbolic representation of $I_U$ is constructed which encodes quaternionic multiplication and from which the statistical properties of complex Hilbert Space vectors are emergent. The Hilbert Space itself arises as the singular limit of Invariant Set Theory as a fractal parameter $N \to \infty$. Although the Hilbert Space of quantum theory is counterfactually complete, the measure-zero set $I_U$ is counterfactually incomplete, no matter how large is $N$. Such incompleteness allows reinterpretations of familiar quantum phenomena, consistent with realism and local causality. The non-computable nature of $I_U$ ensures that these reinterpretations are neither conspiratorial nor retrocausal and, through a homeomorphism with the ring of $2^N$-adic integers, are robust to noise and hence not fine tuned. The non-commutativity of Hilbert Space observables emerges from the symbolic representation of $I_U$ through the generic number-theoretic incommensurateness of $\phi/\pi$ and $\cos \phi$. Invariant Set Theory implies a much stronger synergy between cosmology and quantum physics than exists in contemporary theory, suggesting a novel approach to synthesising gravitational and quantum physics and providing new perspectives on the dark universe and information loss in black holes.

1. Introduction

The title of this paper was partially inspired by the title of Julian Schwinger’s book ‘Quantum mechanics: Symbolism of Atomic Measurement’ [25]. Motivated by the sequential Stern-Gerlach experiment, Schwinger develops a symbolic algebraic approach to the foundations of quantum theory, introducing what he calls ‘the symbol of measurement $|a' b'|$’ and noting:

...with the conceptual analysis of $|a' b'|$ into two stages, one of annihilation and one of creation, as symbolised by $|a' b'| = |a'\rangle \langle b'|$, the fictitious null state, and the symbols $\Psi$ and $\Phi$ can be discarded.

Superficially, Schwinger’s methodology might appear to be mainly of value to those interested in formalising an operational approach to quantum theory (where one is concerned
The present paper is also underpinned by a belief that the symbolism of quantum measurement can provide a route to an understanding of the laws of fundamental physics. However, here we seek to replace quantum theory with something yet more primitive. In this regard we note that the language of symbolism is also an important tool in nonlinear dynamical systems theory. For example, so-called symbolic dynamic representations allow one to characterise topologically the fractal invariant sets $I_{D_f}$ of certain chaotic dynamical systems $D_c$, without needing to know the underlying differential or difference equations which define $D_c$ [15] [9] [10]. In this framework, the invariant set is imagined partitioned into discrete and distinct subsets, which are then labelled using a set of abstract symbols. Dynamical evolution is described by symbolic strings.

This approach has the potential to lead to a deterministic formulation of fundamental physics which is profoundly non-classical. To understand this, recall that in classical physics one starts by defining a coordinate basis for the Euclidean state space of some assumed dynamical system $D$, relative to which the above-mentioned differential or finite-difference equations are expressed. For example, Hamilton’s equations are differential equations customarily written with respect to position/momentum coordinates in state space. Similarly, a fractal invariant set (e.g. [16]) arises from the asymptotic evolution of a generic class of nonlinear differential or difference equations expressed relative to (but ultimately independent of) a chosen coordinate basis in the system’s state space.

However, the symbolic dynamic approach to dynamical evolution suggests an important metaphysical question: If one is only concerned with dynamical evolution on $I_{D_f}$, is it possible to treat the geometry of $I_{D_f}$ as somehow more fundamental (or more primitive) than the corresponding differential or finite-difference equations which define $D_f$? In the case where $I_{D_f}$ is fractal, this question is mathematically non-trivial since geometric properties of $I_{D_f}$ (e.g. whether a given line intersects $I_{D_f}$) cannot be determined algorithmically from $D_f$ - that is to say, the relationship between $D_f$ and $I_{D_f}$ is non-computational [2] [5].

This question can be put in a context where it has resonance with foundational problems in contemporary physics. Suppose the universe $U$ can itself be treated as an isolated deterministic dynamical system evolving (precisely) on a measure-zero fractal invariant subset $I_U$ of $U$’s state space. Consistent with this, let us assume that the most primitive expression of the laws of physics describes the geometry of $I_U$. This ‘Cosmological Invariant Set Postulate’ [17] represents a departure from the way in which both kinematics and dynamics are specified in classical physics, but not a departure from the realism and determinism of classical physics. The question then arises: Could the statistical properties associated with multi-qubit quantum physics be emergent from a suitably constructed geometry for $I_U$? That is to say, can we construct a $\psi$-epistemic theory of quantum physics from the Cosmological Invariant Set Postulate? We attempt to answer the question in the affirmative, and show, moreover, that the non-computability of $I_U$ plays a key role in ensuring that the corresponding physics does not run into (delayed-choice or Bell-conspiracy) paradoxes.
associated with the freedom experimenters have to choose between different experimental set ups.

Locally, \( I_U \) can be expressed as the Cartesian product \( \mathbb{R} \times C \) where \( C \) is a Cantor set. That is to say, \( I_U \) is locally a Cantor Set of state-space trajectories (each trajectory segment representing a cosmological space-time). In Section 2 a symbolic construction of \( \mathbb{R} \times C \) is described which describes the statistical properties of the complex Hilbert Space. In particular, in Section 2.1 we describe a symbolic labelling of trajectory segments on \( I_U \), based on the existence of an intermittent set of instabilities each of which separates nearby trajectories on \( I_U \) into discrete distinct subsets of \( I_U \). In Section 2.2 we describe a symbolic representation of \( C \) which incorporate unit quaternions as permutation/negation operators on strings of symbolic labels. Such quaternions are linked directly to Pauli spin matrices. These quaternions endow the trajectories associated with each fractal iterate of \( I_U \) with a periodic helical structure. It is claimed that these periodic structures provide a realistic geometric representation of unitary evolution. By contrast, the unravelling of the helices during the intermittent periods of instability is claimed to be consistent with the phenomenon of decoherence and hence measurement in conventional quantum theory. In terms of this geometric picture, Schwinger’s \(|a'b'|\) describes the symbolic structure of \( I_U \) between successive fractal iterates.

To justify these claims quantitatively, a precise correspondence is presented in Section 3 between the Cantor set of trajectories on \( I_U \) and the complex Hilbert Space of multi-qubit states. Because \( I_U \) has measure zero in the embedding space, this correspondence is one-to-one, but not onto. That is to say, the symbolic properties of \( I_U \) do not exhibit the closed algebraic representation of Hilbert Space. In Section 4 we relate the surjective nature of this correspondence to the metaphysical notion of counterfactual incompleteness. Such incompleteness allows Invariant Set Theory to generate realistic and locally causal interpretations of some of the iconic phenomena of quantum physics - quantum interference, sequential spin measurements and violation of Bell inequalities - without violating the concept of experimenter free will. Most attention is focussed on the first of these, since, as Feynman famously said, interference illustrates well the essential mystery of quantum physics.

In Section 5 we use the DeBroglie-Bohm deconstruction of the Schrödinger equation to suggest that the Schrödinger equation should not be considered a precise law of physics, but is rather a computational approximation to dynamical evolution and conservation of probability on the non-computational \( I_U \). It is suggested that this approximation provides a gross distortion of reality, a distortion which severely hinders our attempts to find a unified theory of quantum and gravitational physics. In particular, Invariant Set Theory, challenges the general assumption that ‘quantum gravity’ will emerge from the application of quantum field-theoretic ansätze to a suitable gravitational lagrangian. In Section 6 we discuss new perspectives provided by Invariant Set Theory on the synthesis of gravitational and quantum physics. As a result, new solutions are provided to problems at the cutting edge of contemporary physics e.g. associated with information loss in black holes and the nature of dark energy.
2. Constructing $I_U$

The aim of this section is to define the local structure of a fractal subset of some large-dimensional Euclidean state space whose statistical properties describe quantum physics realistically and causally. We will not define the differential or finite-difference equations for how a point in state space evolves; indeed, as discussed in the Introduction, we do not regard such equations as providing a primitive description of the laws of physics. Moreover, since a goal of this study is a demonstration of the natural emergence of both quantum and gravitational physics from principles of state-space geometry, here we attempt to construct a fractal invariant set $I_U$ of the universe $U$ as a whole - the smallest gravitationally isolated dynamical system.

Below we focus on the structure of Cantor sets of trajectories which locally define $I_U$; each trajectory segment in state space defines a space-time (‘a world’). In this respect, a neighbouring trajectory on $I_U$ does not define ‘another world’, but merely defines $U$ (‘our world’) at some earlier or later aeon. In this respect (discussed further in Section 6) we assume a quasi-cyclic cosmology for the evolution of $U$. Importantly, trajectories of $U$ never ‘branch’ or ‘split’ non-deterministically. For these two reasons, the development, whilst superficially similar, is actually quite different from a many-worlds ‘Everettian’ approach to quantum theory.

2.1. Fractal Trajectories. As discussed, a fractal invariant set such as $I_U$ can be locally written as the Cartesian product $\mathbb{R} \times C$ where $\mathbb{R}$ parametrises a time-evolving state-space trajectory and $C$ is some multidimensional Cantor Set. We write

$$C = \bigcap_{k \in \mathbb{N}} C_k$$

where $C_k$ are iterated approximations to $C$. The simplest Cantor set is the ternary set where the $k$th iteration comprises two copies of the $k-1$th iteration, each copy shrunk by a factor of $1/3$. More generally, let $C^{(N)}$ denote a class of ‘binary’ Cantor sets where $C_k^{(N)}$ comprises $2^N$ copies of $C_{k-1}^{(N)}$, each copy reduced by a factor $2^{N+1}$. The fractal similarity dimension of $C^{(N)}$ is equal to $N/(N + 1) \sim 1$ for large $N$. The fractal construction below and corresponding relationship with quantum theory has a singular limit [1] as $N \to \infty$.

There is a well known homeomorphism between Cantor sets and p-adic integers [27]. For prime $p$, the field of p-adic integers provides an alternative extension of the ordinary arithmetic of the rational numbers to that of the reals [24]. For non-prime $p$, the p-adic numbers have the algebraic structure of a ring, allowing addition and multiplication. As discussed below, the existence of such a homeomorphism is conceptually important as it ensures the structural stability of some of the emergent properties on the invariant set (in the sense that these properties are robust to small perturbations on $I_U$).

Fig 1 shows schematically a fractal set of trajectory segments based on $\mathbb{R} \times C^{(N)}$, at three levels of iteration ($k, k+1$ and $k+2$ for arbitrary $k$) and projected onto a two dimensional
Figure 1. A schematic illustration of state space trajectories of a dynamical system evolving on its fractal invariant set, with time varying instability. See text for details. In a), the probabilities of being attracted to regions $a$ and $b$ are equal, whilst in b) these probabilities are arbitrary, but with the constraint that since each trajectory of $\mathbb{R} \times C^{(N)}_k$ itself comprises $2^N$ trajectories of $\mathbb{R} \times C^{(N)}_{k+1}$, these probabilities must be describable with $N + 1$ bits.

subset of state space. An evolving nonlinear dynamical systems will have temporally varying stability and predictability\footnote{If $\dot{X} = F[X]$, where $F$ is at least quadratic in $X$, then the growth of small perturbations is determined by $\delta \dot{X} = (dF/dX) \delta X$ where $dF/dX$ is at least linear in $X$.} This is manifest in Fig. 1 showing periods of metastability (with no divergence between neighbouring trajectories) punctured by intermittent periods of instability which separate neighbouring trajectories into two distinct classes coloured red and blue according to whether trajectories are attracted to the distinct state-space regions labelled ‘$a$’ and ‘$d$’, or ‘$b$’ and ‘$f$’. A physical criterion for the notion of ‘distinctness’ is defined in Section 6 based on the notion of gravitational interaction energy. With time $t$ parametrising trajectory length, let us suppose the system is metastable during periods $t_1 < t < t_0$ and $t_3 < t < t_2$, and is unstable during periods $t_2 < t < t_1$ and $t_4 < t < t_3$. Between $t_1 < t < t_0$, Fig. 1 illustrates a single trajectory of $\mathbb{R} \times C^{(N)}_k$. Under magnification, this trajectory is found to comprise a bundle of $2^N$ trajectories of $\mathbb{R} \times C^{(N)}_{k+1}$ also coloured red or blue. For $t > t_2$ we focus attention on one of the $k + 1$th iterate trajectories. Under a further magnification between $t_2$ and $t_3$, the $k + 1$th-iterate trajectory is also found to comprise a bundle of $2^N$ trajectories of $\mathbb{R} \times C^{(N)}_{k+2}$. The instability at $t_3$ again separates the
trajectory segments into discrete regions and hence allows each of the $2^N$ $k + 2$nd-iterate trajectories to be coloured red or blue. In Fig 1a, the probability $P(k)$ that a trajectory is coloured red or blue is equal to 1/2. A more general $P(k)$ is illustrated in Fig 1b. However, since each trajectory of $\mathbb{R} \times C_k^{(N)}$ contains $2^N$ trajectories of $\mathbb{R} \times C_{k+1}^{(N)}$, each $P(k)$ must be describable with $N + 1$ bits; for example, with $N = 2$, allowable probabilities are 0.00, 0.01, ..., 0.11, 1.00 in binary. In particular, no matter how large is $N$, the $P(k)$ are never irrational numbers. This is crucial for the discussion in the Sections below.

If we are dealing with a genuinely fractal set of trajectories, the revelation of self-similar trajectory structure under magnification will repeat ad infinitum. This raises the metaphysical question of whether a ‘truly’ fractal construction is needed in describing fundamental physics, where, it might be thought, the notion of infinity should play no explicit role. We address this issue in Section 4 below.

In the Invariant Set Theory developed below, the process of dynamical instability and separation into discrete regions of state space is presumed to mirror the process of decoherence occurring during quantum measurement. That is to say, the symbolic labelling of trajectories in state space is presumed to mirror the symbolism of quantum measurement. On the other hand, the corresponding regions of metastability manifestly do not describe pure-state unitary evolution - not least, there is nothing in the construction in Fig 1 that corresponds to the vital role played by complex numbers in quantum theory. To make progress we need to develop a representation of complex numbers that can be injected into the Fig 1 schematic.

2.2. Complex numbers. To start, consider the bit string

$$S = (a_1 \ a_2 \ a_3 \ldots \ a_{2^N})$$

where $a_i \in \{a, \bar{a}\}$. This bit string will describe the labelling of a bundle of $2^N$ trajectories of $\mathbb{R} \times C_k^{(N)}$ associated with a single trajectory of $\mathbb{R} \times C_k^{(N)}$. Two particular instances of $S$ are

$$S_a = (a \ a \ a \ldots \ a); \quad -S_a = (\bar{a} \ \bar{a} \ \bar{a} \ldots \ \bar{a})$$

More generally, we define a set of bit strings based on sets of quaternionic permutation/negation operators acting on $S_a$. As described below, these operators are represented by $2^N \times 2^N$ matrices where each row and column is full of ‘0’s except for one entry which is either equal either the identity and negation operator, such that

$$1(a) = a; \quad 1(\bar{a}) = \bar{a}; \quad -(a) = \bar{a}; \quad -(\bar{a}) = a$$

With $N = 2$, consider first the set $\{E_j(2)\}$ of $2^2 \times 2^2$ matrices defined by

$$E_1(2) = \begin{pmatrix} i & \bar{i} \end{pmatrix}; \quad E_2(2) = \begin{pmatrix} i & \bar{i} \end{pmatrix}; \quad E_3(2) = \begin{pmatrix} -1 & 1 \end{pmatrix}$$

whose elements are $2^1 \times 2^1$ matrices. Here and below, blank entries denote the zero matrix,

$$i \equiv \begin{pmatrix} 1 \ 0 \\ 0 \ 1 \end{pmatrix}$$
and $1(n)$ denotes the $2^n \times 2^n$ identity matrix. The matrices $E_j(2)$ satisfy the quaternion relationships

$$-1(2) = E_1^2(2) = E_2^2(2) = E_3^2(2) = E_1(2) \times E_2(2) \times E_3(2)$$

(and hence are in correspondence with Pauli spin matrices) and generate the cyclic group $\mathbb{Z}/4\mathbb{Z}$ represented by the set $\{E_j^2(2)\}$ over all integer $\alpha \mod 4$.

Using the notion of self-similarity, the operators $\{E_j(2)\}$ can in turn be used as block matrix elements to define the set of 7 square-root-of-minus-one operators, given by the $2^3 \times 2^3$ matrices

$$\{E_1(3) = \begin{pmatrix} E_1(2) & E_1(2) \\ E_1(2) & -E_1(2) \end{pmatrix}, E_2(3) = \begin{pmatrix} E_2(2) & E_2(2) \\ E_2(2) & -E_2(2) \end{pmatrix}, E_3(3) = \begin{pmatrix} E_3(2) & E_3(2) \\ E_3(2) & -E_3(2) \end{pmatrix},$$

$$E_4(3) = \begin{pmatrix} E_1(2) & -E_1(2) \\ E_1(2) & E_1(2) \end{pmatrix}, E_5(3) = \begin{pmatrix} E_2(2) & -E_2(2) \\ E_2(2) & E_2(2) \end{pmatrix}, E_6(3) = \begin{pmatrix} E_3(2) & -E_3(2) \\ E_3(2) & E_3(2) \end{pmatrix},$$

$$E_7(3) = \begin{pmatrix} -1(2) & 1(2) \\ 1(2) & -1(2) \end{pmatrix}\}$$

These contain the following 3 quaternionic relationships:

$$-1(3) = E_1(3) \times E_4(3) \times E_7(3)$$
$$= E_2(3) \times E_5(3) \times E_7(3)$$
$$= E_3(3) \times E_6(3) \times E_7(3)$$

More generally, the notion of self similarity allows an inductive definition

$$\{E_1(N) = \begin{pmatrix} E_1(N-1) \\ E_1(N-1) \end{pmatrix}, \ldots , E_{2^{N-1}-1}(N) = \begin{pmatrix} E_{2^{N-1}-1}(N-1) \\ E_{2^{N-1}-1}(N-1) \end{pmatrix},$$

$$E_{2N-1}(N) = \begin{pmatrix} E_1(N-1) \\ -E_1(N-1) \end{pmatrix}, \ldots , E_{2^{N-2}}(N) = \begin{pmatrix} E_{2^{N-2}}(N-1) \\ -E_{2^{N-2}}(N-1) \end{pmatrix},$$

$$E_{2^{N-1}}(N) = \begin{pmatrix} -1(N-1) \\ 1(N-1) \end{pmatrix}\}$$

of a set $\{E_j(N)\}$ of ‘square root of minus one’ operators where $1 \leq j \leq 2^N - 1$. This set contains the following $2^{N-1} - 1$ quaternionic relationships:

$$-1(N) = E_1(N) \times E_{2^{N-2}}(N) \times E_{2^{N-1}}(N)$$
$$= E_2(N) \times E_{2^{N-1}}(N) \times E_{2^{N-2}}(N)$$
$$\ldots$$
$$= E_{2^{N-1}-2}(N) \times E_{2^{N-2}}(N) \times E_{2^{N-1}}(N)$$
The operators $E_j(N)$ are designed to act on bit strings $S_\alpha$ which can be considered a $2^N$-element row vector. In particular, from the last entry in (4)

$$E_{2^N-1}(N) \times S_\alpha^T = \begin{pmatrix} a a a \ldots a & \ldots & a d d \ldots d \end{pmatrix}_{2^N-1 \times 2^N-1}$$

Therefore, if we define the bit string

$$S_j(a; \alpha) = E_\alpha^j \times E_{2^N-1} \times S_\alpha^T$$

where $\alpha$ is an integer mod 4 and $1 \leq j \leq 2^N - 2$, then $S_j(a; \alpha)$ contains equal numbers of $a$ and $d$ symbols. For example, with $N = 3$,

$$\{S_1(a; 0), S_1(a; 1), S_1(a; 2), S_1(a; 3)\} = \{aaa, aдаа trava, да trava, trava, да trava, trava, да trava, trava\}$$

$$\{S_2(a; 0), S_2(a; 1), S_2(a; 2), S_2(a; 3)\} = \{aдаа trava, да trava, trava, да trava, trava, да trava, trava\}$$

$$\{S_3(a; 0), S_3(a; 1), S_3(a; 2), S_3(a; 3)\} = \{aдаa trava, да trava, trava, да trava, trava, да trava, trava\}$$

(8)

In Fig 2 we illustrate $\{S_1(a; \alpha)\}$, $\{S_1(a; \alpha)\}$ and $\{S_1(a; \alpha)\}$ for $N = 3$, as $\alpha$ increases in unit steps from the bottom up. The symbolic patterns associated with this construction are illustrated by red lines which join the ‘$a$’ symbols together, and blue lines which join the ‘$d$’ symbols together. By drawing the symbols on a cylinder, the lines are mapped to periodic helices.

We now replace the metastable trajectory segments in Fig 2 with the helical structures defined by the quaternionic operators $E_j$ - see Fig 3. The helices become unravelled and the individual trajectories separated into two distinct basins during and following the periods of instability. Below we associated the periods where trajectories are described by these periodic helices with unitary evolution of the quantum state vector.

In order to relate quantitatively this construction with quantum theory, it must be generalised in two ways. Firstly we need to generalise the parameter $\alpha$ so that it can take fractional as well as integer values. This is possible providing $\alpha$ can be described by at most $N$ bits (c.f. Appendix A). Secondly, although in Fig 3 illustrates a situation where half the trajectory segments are attracted to ‘$a$’, and half the segments are attracted to ‘$d$’, this is easily generalised so that a fraction $\beta = P(k)$ of trajectories are attracted to ‘$a$’ as in Fig 2b. This is achieved by premultiplying $S(a; \alpha)$ with a $2^N \times 2^N$ diagonal matrix whose leading $\beta \ 2^N$ diagonal entries equal $+1$ and whose remaining $(1 - \beta) \ 2^N$ diagonal entries equal $-1$. As a consequence, within the helices, a fraction $\beta$ of trajectories will be coloured red (and a fraction $1 - \beta$ will be coloured blue).

We denote the corresponding generalisation to the bit string $S_j(a; \alpha)$ by the extension $S_j(a; \alpha, \beta)$. It is again crucial to recognise that $0 \leq \beta \leq 1$ must be describable by at most $N + 1$ bits.

3. Complex Hilbert Space

In this section we discuss a mapping between the symbolic bit strings $S_j(a; \alpha, \beta)$ representing trajectory bundles of $\mathbb{R} \times C^{(N)}_{k+1}$ associated with a single trajectory of $\mathbb{R} \times C^{(N)}_k$ for
some arbitrary iterate $k$, and elements of the complex Hilbert Space for finite $M$ qubits at some arbitrary time $t$. Importantly, because the parameters $\alpha$ and $\beta$ must be describable using $O(N)$ bits, and therefore are not real numbers, the mapping is an injection and not a bijection. That is to say, in Invariant Set Theory, certain Hilbert Space states do not correspond to states of physical reality. Such ‘unphysical’ states correspond to certain counterfactual space-times which lie off $I_U$. From a mathematical point of view, the algebraic closure of the Hilbert Space may seem a desirable property. However, we will argue later in the paper that, instead, it is the origin of so-called quantum ‘weirdness’.

3.1. One Qubit. It should first be recalled that the elements of a real Hilbert Space are normalised vectors and, because of Pythagoras’ theorem, a unit vector is a natural way to represent probability assignments, even in elementary classical physics. Consider a two dimensional space, spanned by orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$. If $P_a$ denotes the (frequentist-defined) probability of event $a$ so that $P_\tilde{a} = 1 - P_a$ is the probability of the mutually exclusive event $\tilde{a}$, then by Pythagoras’s theorem the vector

\begin{equation}
\mathbf{v}(P_a) = \sqrt{P_a} \mathbf{i} + \sqrt{P_\tilde{a}} \mathbf{j}
\end{equation}
Figure 3. As Fig 1, but where the quaternionic helical trajectory bundles replace the linear metastable trajectory bundles. The instabilities now unravel and separate these helical structures. In the discussion below, evolution on the periodic helical trajectories will correspond to unitary evolution in quantum theory. Unravelling and separation will correspond to decoherent evolution to measurement eigenstates.

has unit norm for any probability assignment $P_a$. With (9) in mind let us consider the injection

\begin{equation}
S_j(a; \alpha, \beta) \mapsto |\psi_a(\theta; \phi)\rangle = \cos \frac{\theta}{2} |a\rangle + \sin \frac{\theta}{2} e^{i\phi} |\phi\rangle
\end{equation}

where $1 \leq j \leq 2^N - 1$. Using the correspondence developed in Section 2.2

\begin{equation}
\alpha = \frac{2\phi}{\pi}
\end{equation}
Moreover, since $\beta$ denotes the probability that a trajectory has the label $a$, and in quantum theory $\cos^2 \theta/2$ denotes the probability that (under measurement) the state if found in the $|a\rangle$ state, then any correspondence between the two representations requires

$$\beta = \cos^2 \frac{\theta}{2}$$

Since $\alpha$ is describable by $N$ bits, then so must $2\phi/\pi$. Also, since $\beta$ is describable by $N + 1$ bits, then so must $\cos^2 \theta/2$. That is to say, values $(\alpha, \beta)$ correspond to a subset of points $(\theta, \phi)$ on the (Bloch) sphere $S^2$. This subset can be made arbitrarily dense for large enough $N$. However, no matter how dense is this subset, i.e. no matter how large is $N$, the following elementary result in number theory must hold:

**Theorem**. Let $\phi/\pi \in \mathbb{Q}$. Then $\cos \phi \notin \mathbb{Q}$ except when $\cos \phi = 0, \pm 1/2, \pm 1$.

**Proof.** Assume that $2 \cos \phi = a/b$ where $a, b \in \mathbb{Z}, b \neq 0$ have no common factors. Since $2 \cos 2\phi = (2 \cos \phi)^2 - 2$ then

$$2 \cos 2\phi = \frac{a^2 - 2b^2}{b^2}$$

Now $a^2 - 2b^2$ and $b^2$ have no common factors, since if $p$ were a prime number dividing both, then $p | b^2 \implies p | b$ and $p | (a^2 - 2b^2) \implies p | a$, a contradiction. Hence if $b \neq \pm 1$, then the denominators in $2 \cos \phi, 2 \cos 2\phi, 2 \cos 4\phi, 2 \cos 8\phi \ldots$ get bigger without limit. On the other hand, if $\phi/\pi = m/n$ where $m, n \in \mathbb{Z}$ have no common factors, then the sequence $(2 \cos 2^k \phi)_{k \in \mathbb{N}}$ admits at most $n$ values. Hence we have a contradiction. Hence $b = \pm 1$ and $\cos \phi = 0, \pm 1/2, \pm 1$ QED.

**Corollary.** Let $0 < \Delta < \pi/2$ denote some (typically-small) interval on $[0, 2\pi]$. Let $\Delta_n \subset \Delta$ comprising $n > 1$ angles $\phi$, such that $2\phi/\pi$ can be described by (finite) $N$ bits. Then there exists no $\Delta_n$ where the corresponding $\cos \phi$ can be described by $N + 1$ bits.

This result has important implications about the interpretation of familiar quantum phenomena. The physical consequences of this are discussed in the next section.

### 3.2. Two Qubits.

As we discuss below, $M$ bit strings in Invariant Set Theory corresponds to $M$ qubits in quantum theory correspond to. In particular, the injection between 2 bit strings the 2-qubit Hilbert Space state can be written

$$\{a_1, a_2, a_3, \ldots a_{2N}\} \mapsto |\psi_{ab}\rangle$$

where

$$|\psi_{ab}\rangle = \gamma_0^2 |a\rangle |b\rangle + \gamma_1^2 e^{i\chi_1} |a\rangle |\bar{b}\rangle + \gamma_2^2 e^{i\chi_2} |\bar{a}\rangle |b\rangle + \gamma_3^2 e^{i\chi_3} |\bar{a}\rangle |\bar{b}\rangle$$

and $\gamma_0^2 + \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$. 

In order to define this mapping explicitly, let us first use (10) to write (15) in the two equivalent forms (each with six degrees of freedom)

\[
|\psi_{ab}\rangle = \cos \frac{\theta_1}{2} |a\rangle |\psi_b(\theta_2; \phi_2)\rangle + \sin \frac{\theta_1}{2} e^{i\phi_1} |b\rangle |\psi_a(\theta_4; \phi_4)\rangle
\]

\[
= |\psi_a(\theta_4; \phi_4)\rangle \cos \frac{\theta_6}{2} |b\rangle + |\psi_a(\theta_5; \phi_5)\rangle \sin \frac{\theta_6}{2} e^{i\phi_6} |b\rangle
\]

where

\[
\gamma_0 = \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} = \cos \frac{\theta_4}{2} \cos \frac{\theta_5}{2}
\]
\[
\gamma_1 = \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} = \sin \frac{\theta_4}{2} \cos \frac{\theta_5}{2}
\]
\[
\gamma_2 = \sin \frac{\theta_1}{2} \cos \frac{\theta_3}{2} = \sin \frac{\theta_4}{2} \cos \frac{\theta_5}{2}
\]
\[
\gamma_3 = \sin \frac{\theta_1}{2} \sin \frac{\theta_3}{2} = \sin \frac{\theta_4}{2} \sin \frac{\theta_5}{2}
\]
\[
\chi_1 = \phi_1 = \phi_5
\]
\[
\chi_2 = \phi_1 = \phi_4
\]
\[
\chi_3 = \phi_1 + \phi_3 = \phi_5 + \phi_6
\]

Relative to the first form in (16), consider the three (1-qubit) injections

\[
S_{j_1}(a; \alpha_1, \beta_1) = \{a_1, a_2, a_3 \ldots a_{2N}\} \rightarrow |\psi_a(\theta_1; \phi_1)\rangle
\]
\[
S_{j_2}(b; \alpha_2, \beta_2) = \{b_1', b_2, b_3' \ldots b_{2N}'\} \rightarrow |\psi_b(\theta_2; \phi_2)\rangle
\]
\[
S_{j_3}(b; \alpha_3, \beta_3) = \{b_1'', b_2', b_3'' \ldots b_{2N}'\} \rightarrow |\psi_b(\theta_3; \phi_3)\rangle
\]

where \(b_1', b_2' \in \{b, b\}\). The injection (14) is then obtained from (18) by setting

\[
b_i = b_i' \quad \text{if} \quad a_i = a
\]
\[
b_i = b_i'' \quad \text{if} \quad a_i = \bar{a}
\]

Consistent with (12) and (11), we have

\[
(20) \quad \alpha_k = 2\phi_k/\pi; \quad \beta_k = \cos^2 \theta_k/2
\]

for \(k \in \{1, 2, 3\}\). For example, with \(N = 3\) let

\[
S_1(a; 1, 1/2) = \{a \neq a \neq a \neq a \neq a\}
\]
\[
S_1(b; 2, 1/2) = \{b b b b b b b b\}
\]
\[
(21) \quad S_1(b; 3, 1/2) = \{b b b b b b b b\}
\]

then, using (19), it is easily shown that

\[
\{a \neq a \neq a \neq a \neq a\} \rightarrow |\psi_{ab}\rangle
\]

It can be noted that there are six degrees of freedom associated with the two bit strings on the in (14), consistent with quantum theory. Also, as the \((M = 2)\) example illustrates,
the bit string \{b_1, b_2, b_3 \ldots b_{2N}\} cannot itself be written in the form \(S_j(b; \alpha, \beta)\), consistent with the fact that the state space of the general 2-qubit state cannot be expressed as the Cartesian product of Bloch Spheres.

For general \(N\), it is easy to show that estimates of probability from the bit strings \{a_i\} and \{b_i\} in (14) are consistent with those from quantum theory. For example, let us estimate the probability that \(a_i = a\) and \(b_i = b\). The probability that \(a_i = a\) is equal to \(\beta_1 = \cos^2 \theta_1/2\) from (20). Now if \(a_i = a\) then by definition \(b_i = b'_i\) from (19). The probability that \(b'_i = b\) is equal to \(\beta_2 = \cos^2 \theta_2/2\) from (20). Hence the probability that \(a_i = a\) and \(b_i = b\) is equal to \(\cos^2 \theta_1/2 \cos^2 \theta_2/2 = \gamma_0^2\) from (17) consistent with quantum theory.

We can consider two special cases of (14). Firstly, suppose \(b'_i = b''_i\). Then the probability that \(a_i = a\) is independent of the probability that \(b_i = b\). In other words, we can write

\[
S_{j_1} \left( a; \alpha_1, \beta_1 \right) \quad S_{j_2} \left( b; \alpha_2, \beta_2 \right) \mapsto |\psi_{ab}\rangle
\]

consistent with the factorisation \(|\psi_{ab}\rangle = |\psi_a\rangle|\psi_b\rangle\) in quantum theory.

In the second special case, let \(\beta_1 = 1/2\), and \(\beta_2 + \beta_3 = 1\). Then the probability that \(a_i = a\) is equal to 1/2. When \(a_i = a\) then by definition \(b_i = b'_i\). The probability that \(b'_i = b\) is equal to \(\beta_2\). Hence the probability that \(a_i = a\) and \(b_i = b\) is equal to \(\beta_2/2\). Similarly, the probability that \(a_i = \alpha\) is equal to 1/2, and when \(a_i = \alpha\) then by definition \(b_i = b''_i\). The probability that \(b''_i = b\) is equal to \(1 - \beta_3 = \beta_2\). Hence, by (20), the probability that either \(a_i = a\) and \(b_i = b\), or \(a_i = \alpha\) and \(b_i = b\) (i.e. the labels ‘agree’) is equal to \(\beta_2/2 + \beta_2/2 = \beta_2 = \cos^2 \theta_2/2\) consistent with the quantum theoretic correlations of measurement outcomes on the Bell state

\[
|\psi_{ab}\rangle = \frac{|a\rangle|b\rangle + |\alpha\rangle|\beta\rangle}{\sqrt{2}}
\]

where \(\theta_2\) denotes the relative orientation of the measurement apparatuses. Because of this, correlations from Invariant Set Theory necessarily violate Bell inequalities. In Section 4.3 we discuss the reason why Invariant Set Theory can be considered both locally causal and realistic despite such violation.

We conclude by noting that equivalent injection

\[
\begin{align*}
S_{j_1} \left( a; \alpha_4, \beta_4 \right) &= \left\{ a'_1, a'_2, a'_3 \ldots a'_{2N} \right\} \mapsto |\psi_a(\theta_4, \phi_4)\rangle \\
S_{j_2} \left( a; \alpha_5, \beta_5 \right) &= \left\{ a''_1, a''_2, a''_3 \ldots a''_{2N} \right\} \mapsto |\psi_a(\theta_5, \phi_5)\rangle \\
S_{j_3} \left( b; \alpha_6, \beta_6 \right) &= \left\{ b_1, b_2, b_3 \ldots a_2N \right\} \mapsto |\psi_b(\theta_6, \phi_6)\rangle
\end{align*}
\]

(25)

can be defined relative to the second form in (16), where \(a'_i, a''_i \in \{a, \alpha\}\). Consistent with (12) and (11),

\[
\phi_k = \pi \alpha_k/2; \quad \cos^2 \theta_k/2 = \beta_k
\]

for \(k \in \{4, 5, 6\}\). It is straightforward to show that (18) and (23) are completely equivalent in terms of statistical properties.
3.3. M Qubits. As discussed in Appendix B, the injections defined in Section 3.2 also provide an inductive definition (define $M$ given $M - 1$) of injections between the set of $M$ bit strings and $M$-qubit Hilbert Space.

4. Reinterpreting Quantum Phenomena

We now consider three of most iconic and hence conceptually puzzling of quantum phenomena - interference, sequential spin measurement and nonlocality - from the realistic $\psi$-epistemic perspective of Invariant Set Theory.

4.1. Interference. Consider a single photon passing through a Mach-Zehnder interferometer (Fig 4a). We first describe the transformation of the Hilbert Space state from input to output using conventional quantum theory. Let the input state be denoted by $|a\rangle$ (‘the particle is input through the ‘a’ channel’). Passing through the first beam splitter, this state is subject to the unitary Hadamard operator

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix};$$

so that

$$|a\rangle \rightarrow \frac{1}{\sqrt{2}} \{ |c\rangle + |d\rangle \} \tag{27}$$

(‘the particle is simultaneously in the upper and lower arms of the interferometer’). A phase shifter is positioned in the lower arm of the interferometer, so that

$$\frac{1}{\sqrt{2}} \{ |c\rangle + |d\rangle \} \rightarrow \frac{1}{\sqrt{2}} \{ |c\rangle + e^{i\phi} |d\rangle \} \tag{28}$$

(‘the particle is simultaneously in a complex superposition of upper and lower arm positions’). Finally, the state is transformed by $U$ again as it leaves the interferometer, so that

$$\frac{1}{\sqrt{2}} \{ |c\rangle + e^{i\phi} |d\rangle \} \rightarrow \frac{1}{2} \{ (1 + e^{i\phi}) |e\rangle + \frac{1}{2} (1 - e^{i\phi}) |f\rangle \} \tag{29}$$

$$= \cos \frac{\phi}{2} |e\rangle + \sin \frac{\phi}{2} |f\rangle$$

where ‘$=$’ denotes equality modulo a global phase. Hence, by the Born rule, the probability of detecting a photon at the detectors $D_e$ and $D_f$ is $\cos^2 \phi/2$ and $\sin^2 \phi/2$ respectively.

Now according to the discussion in Section 3 the Hilbert space state

$$\cos \frac{\phi}{2} |e\rangle + \sin \frac{\phi}{2} |f\rangle \tag{30}$$

corresponds to a bundle of $2^N$ state-space trajectories lying on $I_U$, providing $\cos^2 \theta/2$ is describable by at most $N$ bits. In such a case, a fraction $\cos \phi/2$ of trajectories on $I_U$ describe space-times where a photon passes through the interferometer and is detected by $D_e$, and a fraction $\sin \phi/2$ of trajectories on $I_U$ describe space-times where the photon is detected by $D_f$. 

Figure 4. a) The Mach-Zehnder Interferometer. b) A schematic of an experiment to determine through which channel a photon passes. In Invariant Set Theory, the incompatibility of a) and b) for a given photon is associated with the incommensurability of $\phi/\pi$ and $\cos \phi$, where $\phi$ denotes the phase shift. Such incommensurability corresponds to a type of counterfactual incompleteness on $I_U$.

With $\cos^2 \theta/2$ describable by $N$ bits, let us now ask the question. Could we have detected the photon as it passed through the interferometer? This would be possible according to Invariant Set Theory if the intermediate quantum state $|c\rangle + e^{i\phi}|d\rangle$ also corresponds to a trajectory bundle on $I_U$. However, with the exceptions $\phi = 0, \pi/2, \pi, 3\pi/2$, the number theorem in Section 3.1 prevents any such correspondence. That is to say, in a situation where an experiment such as shown in Fig 4a was performed (i.e. was associated with states
of the universe on $I_U$), then a counterfactual experiment with the same photon, such as shown in Fig 4b, could not have been performed (i.e. could not have been associated with states of the universe on $I_U$).

Conversely, if $2\phi/\pi$ is describable by $N$ bits, so that an experiment such as shown in Fig 4b lies on $I_U$, then a counterfactual experiment with the same photon, such as shown in Fig 4a, could not have been performed and would not be associated with state of the universe on $I_U$.

The isolated exceptions $\phi = 0, \pi/2, \pi, 3\pi/2$ can be ignored for the following reason. Since the phase shifter is a macroscopic object, the phase angle $\phi$ is not a single precise value, but a variable which fluctuates over a small interval $\Delta$, a measure of the finite precision of the phase shifter. In the case where there is correspondence between $\cos \phi/2|e\rangle + \sin \phi/2$ and the bit strings of Invariant set theory, then the only allowable phases $\phi$ in this interval $\Delta$ are those where $\cos^2 \phi/2$ is describable by at most $N$ bits. By the corollary to the number theorem in Section 3.1, there are no intervals $\Delta$ around $\phi = 0, \pi/2, \pi, 3\pi/2$ containing multiple angles $\phi$, where both $\cos^2 \phi/2$ and $2\phi/\pi$ are describable by $N$ bits.

Hence, by applying elementary number theory, Invariant Set Theory provides a realistic explanation of what is known to be true by experiment and what is accounted for in quantum theory by the non-commutativity of Hilbert Space observables (and hence the Uncertainty Principle) - that Figs 4a, b are fundamentally incommensurate.

At this stage one might object that with $N \gg 1$, the difference between an irrational phase angle and one describable by $N$ bits may be so utterly tiny as to be rendered irrelevant by the existence of the tiniest amount of noise in the system. Such a conclusion is not correct. To understand this, it is important to distinguish noise which is random with respect to the full measure of the euclidean space in which $I_U$ is embedded, and noise $\epsilon$, such that if $U \in I_U$, then $U + \epsilon \in I_U$. Since Cantor Sets have the cardinality of the continuum, there are as many small perturbations of the latter type as of the former type, and the discussion above is robust to small perturbations of the latter type. Moreover, by the Cosmological Invariant Set postulate, the latter type of noise can be considered physical, whereas the former type of noise is not. One can put this argument on a more rigorous footing by noting, as discussed in Section 2, that there exists a homeomorphism between points of $C^{(N)}$ and the ring of $2^N$-adic integers. Hence if one interprets the addition symbol in $U + \epsilon$ in the sense of such a ring, then the results above are completely robust, and not at all fine-tuned.

Feynman famously claimed [7] that quantum interference demonstrates that the Laplacian laws for combining probabilities necessarily fail in quantum physics. We can use the discussion above to argue otherwise. Since Invariant Set Theory is $\psi$-epistemic (i.e. rejects the notion of superposition as a fundamental concept), then one can infer that photons indeed travel either through the upper arm or the lower arm of the interferometer. Hence, the probability $p$ of detection at $D_e$ is the sum of the probability $p_1$ of detection when the particle comes through the upper arm, and the probability $p_2$ of detection when the particle comes through the lower arm. In the language of Invariant Set Theory, $p_1$ denotes the fraction of the $2^N$ helical trajectories in $\mathbb{R} \times C^{(N)}_{k+1}$ associated with a single trajectory of
\( \mathbb{R} \times C_k^{(N)} \) that correspond to space times where a photon travels through the upper arm. Similarly, \( p_2 \) is the fraction of the same helical trajectories where a photon travels through the lower arm. To determine experimentally whether \( p = p_1 + p_2 \), we might considering performing an experiment like that in Fig 4b. If we were to interpret the frequentist probabilities arising from this second experiment as determining \( p_1 \) and \( p_2 \), then indeed it would be the case that \( p \neq p_1 + p_2 \). However, according to Invariant Set Theory, an experiment like that in Fig 4b, does not measure \( p_1 \) and \( p_2 \) as defined above for the simple reason that the helical structure associated with the trajectory iterates of Fig 4a (from which \( p_1 \) and \( p_2 \) are defined) simply does not exist for an experiment like Fig 4b - the helices have already begun to unravel at the beam splitter.

One may object to this argument on the basis that the experimenter may not have decided which experiment to perform (Fig 4a or Fig 4b) at the moment \( t_c \) when the photon passes through the first beam splitter - instead enacting some ‘delayed-choice’ experiment. Such an experiment usually elicits the question: How does the photon ‘know’ whether it has entered a configuration like that in Fig 1a and hence must behave like a wave, or has entered a configuration like that in Fig 1b and hence must behave like a particle? In the language of Invariant Set Theory, the question becomes: How do the state space trajectories at \( t_c \) ‘know’ to be helical (thus ensuring the photon has wave-like properties) or to unravel and separate (thus ensuring the photon has particle-like properties)?

This is where the non-computability of \( I_U \) becomes crucial. How would one determine at \( t_c \) whether \( I_U \) had helical structure (implying an experimental configuration like Fig 1a) or had unravelling structure (implying an experimental configuration like Fig 1b)? In order to answer this question, one needs an algorithm for determining whether or not a putative trajectory segment (e.g. a helix segment) lies on \( I_U \) at \( t_c \)? However, for fractal invariant sets, there is no such algorithm \[2\] (Interestingly the problem of determining whether a given line intersects a fractal set is equivalent to the famous undecidable Post Correspondence Problem \[5\]). Now non-computability may appear an overly exotic concept for primitive physical theory. However, one should reflect on an analogy from general relativity theory - the black hole event horizon \[19\]. Like \( I_U \), the event horizon is an entirely causal and realistic entity defined by a global property - the boundary of null rays which escape to (future null) infinity. This means that it is impossible to determine the position of the event horizon from a knowledge of the Riemann tensor in the neighbourhood of the event horizon; its position at some time \( t_c \) in space-time can be influenced by an event potentially in the far future of \( t_c \), e.g. whether or not a massive object falls into the black hole. For this reason, the event horizon is an atemporal concept. So too is \( I_U \). Whether \( I_U \) has helical or unravelling structure at \( t_c \) can be influenced by an event potentially in the far future of \( t_c \) - i.e. whether an experimenter chooses to perform an experiment like that in Fig 1a or Fig 1b. There is nothing ‘retrocausal’ about this state of affairs in the sense that it does not imply that signals are somehow propagating backwards in time. Rather, the analysis makes use of the fact that \( I_U \) is a global state-space concept. We might want to use the word ‘nonlocal’ to describe this property. However, in quantum physics, the word ‘nonlocal’ is generally defined to mean ‘not locally causal’, and that is
certainly not the appropriate word here - $I_U$ is a perfectly causal concept (by the same token, one would not wish to describe the causal event horizon as a ‘nonlocal’ concept). Rather, let us use the expression ‘alocal’.

The mathematical concept of non-computability strictly refers to systems with infinite degrees of freedom, e.g. strict fractals which are self-similar on all scales, no matter how small. It can be argued whether strict infinitesimals are relevant to physical theory. In fact none of the above requires us to adopt such a strict definition here. It could in fact be that $I_U$ is a limit cycle than a strict fractal, with self-similar properties down to some sufficiently small but nevertheless finite limit (what is sometimes called a ‘fat fractal’). In the discussion above, non-computability merely requires that the properties of $I_U$ cannot be determined by an algorithm which can be solved by some sub-system of $U$.

Hence, in Invariant Set Theory it is meaningful to describe the photons as passing through either the upper or lower arm of the interferometer - language that is forbidden in conventional quantum theory. How then can we understand the ‘wave-like’ behaviour in the interferometer for single photon experiments? Recall the basic postulate underpinning Invariant Set Theory: that the rules which describe the geometry of $I_U$ provide a primitive expression of the laws of physics (a more primitive expression than differential or finite-difference evolution equations along state-space trajectories. In the present context, this implies that properties of the universe manifest on an individual $k+1$ th iterate trajectory necessarily reflects the periodic helical geometry of the bundle of neighbouring iterate trajectories.

One can find another pertinent analogy from general relativity. The rate of change of the kinetic energy of a test particle is determined by the space-time geodesic on which the particle moves. This geodesic itself is determined by the geometry of the surrounding space-time. Through the Jacobi equation, this space-time geometry is manifest through the deviation of neighbouring geodesics. That is to say, the inertial properties of a test particle on some fiducial geodesic express the collective properties of the geodesics which neighbour this fiducial geodesic.

Similarly, the energy/momentum characteristics of single photons should also express the collective property of the neighbouring bundle of $k+1$ th trajectory helices - their periodicity in particular. Here one can note that the helices are periodic relative both to the time parameter $t$ which parametrises length along the $k$th trajectory, and to transverse angular directions around the $k$th trajectory. Hence, Invariant Set Theory provides plausible geometric explanations for the de Broglie relationships $E = h\omega$ and $p = h\kappa$, which in turn lead to a novel geometric interpretation of the Schrödinger equation as a Liouville equation for conservation of probability on helical trajectory bundles.

In the case of photons, the collective properties of a large ensemble of neighbouring trajectories of single photons passing through the Mach-Zehnder interferometer of Fig[4] are described by the equations of classical electromagnetism. This suggests that the laws of classical electromagnetism can be derived directly from the quaternionic structure of the state-space trajectories in Invariant Set Theory - we discuss this possibility further in Section[9]. Hence, if the properties of any single $k+1$ th iterate trajectory expresses the collective properties of the bundle of $k+1$ th iterate trajectories associated with a single $k$th
iterate trajectory, a single photon experiment should manifest wave interference as found in classical electromagnetism.

Let us conclude by discussing experimenter free will. In the discussion above, we described the experimenter as choosing a particular experimental configuration (Fig 4a or 4b). The fact that Invariant Set Theory is a deterministic theory raises the metaphysical question as to whether the experimenter really is free to choose which experiments to perform - indeed more than this, whether the experimenter is merely an algorithmically driven automaton. We would argue that the experimenter does have free will in any practical sense. Firstly, recalling the ‘compatibilist’ philosophies of Hobbes, Hume, Mill and others [14], experimenters can be said to have free will if there is nothing constraining them from performing the experiments they want to perform. As discussed above, since for large \( N \), allowable choices of \( \phi \) are effectively dense on the interval \( [0, 2\pi] \), Invariant Set Theory does not lead to any practical restriction on available choices. However, we can go further than this. Deciding what experiment an experimenter ‘wants’ to perform will be based on cognitive processing in their brains. Consistent with the fact that the processing capacity of the brain requires only a few 10s of Watts of power (orders of magnitude less than contemporary bit-reproducible supercomputers) there is considerable evidence that signal propagation along the extremely slender \( 0.1 \mu \) axons of human neurons is subject to thermal noise whose ultimate origin is quantum decoherence (i.e. helical unravelling) [18]. Hence, cognitive processing is sensitive to processes which, by Invariant Set Theory, are deterministic but not computably so. In conclusion, the experimenter can perform the experiments they wish to perform without any effective constraint is is not a mere automaton. In these two senses, experimenters can be said to have free will, the counterfactual incompleteness of Invariant Set Theory notwithstanding. This discussion is relevant to an analysis of Bell’s Theorem below.

4.2. The Sequential Stern-Gerlach Experiment. Schwinger’s symbolic approach to quantum theory [25] was guided by the iconic sequential Stern Gerlach (SG) experiment (Fig 5a). Consider a beam of spin-1/2 particles moving in the x direction and entering the first beam splitter (SG1) whose magnetic field is oriented in the z direction. The spin-up beam is input into the second beam splitter (SG2) whose magnetic field is oriented at an angle \( \theta_2 \) relative to the z direction. A fraction \( \cos^2 \theta_2/2 \) are input into the third beam splitter (SG3), oriented at an angle \( \theta_3 \) relative to SG2. A fraction \( \cos^2 \theta_3/2 \) are measured spin ‘up’ by detectors in the output beams of SG3.

From the perspective of Invariant Set Theory, the sequential SG experiment (and hence properties of Schwinger’s symbol \( |a', b'| \); see Introduction) can be understood by considering the self-similar (fractal) structure of \( I_U \) as shown in Fig 3. Let us start by considering a single trajectory of \( \mathbb{R} \times C_k^{(N)} \), approximating a state of the universe where a particle leaves the spin-up output port of SG1. Under magnification, this comprises \( 2^N \) trajectories of the higher-order iterate \( \mathbb{R} \times C_k^{(N)} \). These \( k + 1 \) iterates are unravelled by SG2, with fraction \( \cos^2 \theta_2/2 \) approximating a state of the universe where a particle leaves the spin-up output port of SG2. We now consider one of these unravelled trajectories. Under magnification,
it comprises $2^N$ trajectories of the higher-order iterate $\mathbb{R} \times C_{N+1}^{(N)}$. These $k+2$th iterates are unravelled by SG3, with fraction $\cos^2 \theta_3/2$ approximating a state of the universe where a particle leaves the spin-up output port of SG3. In this respect, the iterate label $k$ is a descriptor of the passage of time on $I_U$.

Invariant Set Theory provides a realistic explanation for the non-commutativity of spin operators in quantum theory. As discussed above, both $\cos \theta_2$ and $\cos \theta_3$ must be describable with $N$ bits. These $N$-bit restrictions are crucial if we ask the following counterfactual question: Given an experiment in the order SG1-SG2-SG3 was performed (i.e. SG1-SG2-SG3 occurs in a space-time on $I_U$), could the order have been inverted to perform the experiment SG1-SG3-SG2 (Fig 5)? If this were possible then one could infer the existence of simultaneous values of spin relative to the two orientations $\theta_2$ and $\theta_2 + \theta_3$ (the orientation of SG3 relative to the z axis is $\theta_2 + \theta_3$). If simultaneous values of spin were possible in Invariant Set Theory, then it would be necessary that $\cos(\theta_2 + \theta_3)$ is describable by $N$ bits when $\{\theta_2\}$ and $\{\theta_3\}$ both are. However, since $\cos(\theta_2 + \theta_3) = \cos \theta_2 \cos \theta_3 - \sin \theta_2 \sin \theta_3$, then in general this is not possible (typically, if $\cos \theta$ can be described by $N$ bits, $\sin \theta = \sqrt{\cos^2 \theta - 1}$ is not). Hence, if the sequential Stern-Gerlach experiment SG1-SG2-SG3 lies on $I_U$, then the counterfactual experiment SG1-SG3-SG2 does not lie on $I_U$: it is not a physically realisable experiment. Conversely, if SG1-SG3-SG2 lies on $I_U$, then SG1-SG2-SG3 does not.

As in Section 4.1 this result is not destroyed by small perturbations, providing such perturbations keep the state of the universe on $I_U$. As before, this is achieved mathematically by adding small perturbations within the ring of $2^N$ adic integers. Also, as in Section 4.1 the non-computability of $I_U$ also plays a crucial role here. Suppose, SG3 was in fact replaced with an inverted copy of SG2 (Fig 5). Then in this case, the trajectories of $\mathbb{R} \times C_{k+1}^{(N)}$ would not start to unravel on exit from SG2 but would retain their helical periodic structure between SG2 and -SG2. However, an experimenter may not decide to perform this experiment until after the time when a particle has left SG2. One can then ask how the trajectories would ‘know’ whether to unravel or not. See Section 4.1 for a discussion on the role of non-computability on this ‘delayed-choice’ experiment.

4.3. Bell’s Theorem. The Bell inequality

$$|\text{Corr}_\rho(\hat{a}, \hat{b}) - \text{Corr}_\rho(\hat{a}, \hat{c})| \leq 1 + \text{Corr}_\rho(\hat{b}, \hat{c})$$

is violated in quantum theory. It is also violated in Invariant Set Theory (since, as shown in Section 3, Invariant Set Theory can replicate the statistical correlations of entangled qubit physics). Since Invariant Set Theory is both realistic and locally causal, this must imply a partial violation of the measurement independence condition [11][12] - sometimes called the ‘free-will’ condition. As discussed below, this is achieved without conspiratorial or retrocausal correlations, or, indeed without violating the notion of experimenter free will in any practical sense.

In order to establish that (31) is violated experimentally (e.g. in a EPR-Bohm apparatus), we need to perform three separate sub-experiments: one where the measuring devices are oriented with respect to $\hat{a}$ and $\hat{b}$ respectively, one where the devices are oriented with
Let the directions \( \hat{a}, \hat{b} \) and \( \hat{c} \) be represented by three points \( a, b \) and \( c \) respectively on the unit sphere. In Invariant Set Theory, it is necessary that the cosine of the three angles - \( \theta_{ab} \) between \( \hat{a} \) and \( \hat{b} \), \( \theta_{ac} \) between \( \hat{a} \) and \( \hat{c} \), and \( \theta_{bc} \) between \( \hat{b} \) and \( \hat{c} \) - are all describable by \( N \) bits. The cosine rule for spherical triangles gives

\[
\cos \theta_{ab} = \cos \theta_{ac} \cos \theta_{bc} + \sin \theta_{ac} \sin \theta_{bc} \cos \phi
\]

where \( \phi \) is the (typically small) angle subtended by the two sides of \( \triangle_{abc} \) at \( c \) and \( \phi/\pi \) is describable by \( N \) bits (see Fig 3).

As shown in [19], and consistent with the analysis in Sections 4.1 and 4.2 ([32] combines the two), it is impossible for \( \cos \theta_{ab}, \cos \theta_{ac} \) and \( \cos \theta_{bc} \) to all be describable by \( N \) bits, no matter how large is \( N \). Hence, according to Invariant Set Theory, in the three sub-experiments, what is actually measured are correlations relative to the following three pairs of orientations - \( \hat{a} \) and \( \hat{b} \), \( \hat{a} \) and \( \hat{c} \), and \( \hat{b} \) and \( \hat{c}' \) - such that \( \cos \theta_{ab}, \cos \theta_{ac} \) and \( \cos \theta_{bc} \) are all describable by \( N \) bits. Here we again make use of the inherently finite-precision nature of the experiments to ensure the angle \( \theta_{cc'} \) between \( \hat{c} \) and \( \hat{c}' \) is undetectable experimentally. (The corresponding analysis of the CHSH experiment is given in [19].) By contrast, in attempting to derive mathematically whether a putative locally causal hidden-variable theory is constrained by the Bell inequalities, then it must be assumed that \( \hat{c} = \hat{c}' \) precisely,
and not just approximately. Because all three cosines cannot be described by $N$ bits, the derivation therefore fails.

Again, the argument is not fragile in the sense that it could be destroyed by the additional of small perturbations, providing these small perturbations are faithful to the Cosmological Invariant Set Postulate and map points on $I_U$ to $I_U$. As discussed in Section 4.1, there exists a homeomorphism between points of the Cantor set $C^{(N)}$ of trajectories and the ring of $2^N$-adic integers - itself an algebraically-closed completion of the rational integers. Hence the discussion above is robust to the addition of noise, where addition is meant in the sense of this ring algebra.

Because Invariant Set Theory is realistic and locally causal, it must partially violate the measurement independence condition ([11] [12]). In particular, we necessarily have

$$\rho(\lambda|\hat{b}, \hat{c}) \neq \rho(\lambda|\hat{b}, \hat{c}')$$

In the case where $\cos \theta_{ab}$ and $\cos \theta_{ac}$ are describable by $N$ bits, then in Invariant Set Theory, the left hand side of (33) is equal to zero, but the right hand side is not. Such violation is often seen as implying implausible ‘conspiratorial’ correlations between the putative hidden-variables $\lambda$ of particles being measured and potentially earlier determinants of instrumental settings. However, the non-computable nature of $I_U$, as noted when discussing ‘delayed choice’ experiments in Sections 4.1 and 4.2 provides a non-conspiratorial explanation. Consider for example a situation where instrumental settings are determined by the frequency $\nu$ of photons emitted at time $t_0$ by quasars billions of years before the particles whose spins are to be measured were themselves produced [8]. The non-computational nature of $I_U$ implies that there is no algorithm which will determine at $t_0$ whether a putative state $U_0$ of the universe in which $\nu$ takes some value $\nu_0$, lies on $I_U$ or not. As with ‘delayed choice’ (see Sections 4.1 and 4.2) - and similar to the example of the event horizon - whether or not $U_0 \in I_U$ depends on the later values $\hat{b}$ and $\hat{c}$ (see also [19]), without retrocausality or conspiracy. Finally, as discussed in Section 4.1, experimenters can be described as free agents - Invariant Set Theory poses practical constraint on free will.

In conclusion, Invariant Set Theory can accommodate the violation of the Bell inequality without abandoning realism, local causality and without being ‘superdeterministic’ in any pejorative sense. As such, Invariant Set Theory suggests it would be wrong to describe the EPR-Bohm experiment as demonstrating that quantum physics is nonlocal (in the sense of ‘not locally causal’). Clearly Bohm-EPR statistics demonstrate the non-classical nature of quantum physics; however, as discussed above, the realistic causal Cosmological Invariant Set Postulate is itself non-classical. Following the discussion on quantum interference, we conclude that the EPR-Bohm experiment demonstrates that physics is alocal and non-classical, but nevertheless does not require us to abandon realism or local causality.

5. The Schrödinger Equation

As discussed above, the complex Hilbert Space can be considered as the singular limit of bit-string representations of bundles of trajectories of $I_U$ as $N \to \infty$. The limit is singular because the mapping [10] is an injection (and not a bijection) no matter how large is
Dynamical evolution \( D_U \) on \( I_U \) does not correspond precisely to Schrödinger dynamics, no matter how large is \( N \). Nevertheless, since the Schrödinger equation is such an extraordinarily precise description of nature, then there must be some sense in which \( D_U \) is described by Schrödinger dynamics to a good computational level of approximation.

From the perspective of some fiducial trajectory of \( \mathbb{R} \times C_k^{(N)} \), there are two separate issues to be considered: conservation of probability for the ensemble of \( 2^N \) trajectories of \( \mathbb{R} \times C_k^{(N)} \) which the fiducial trajectory comprises and \( k \)th iterate deterministic dynamics on the fiducial trajectory itself. It is interesting to note that the De Broglie-Bohm decomposition of the Schrödinger equation comprises exactly these two forms: a Liouville equation for conservation of probability and a Hamilton-Jacobi equation describing dynamical evolution. The latter contains the so-called quantum potential \( Q \), without which the dynamics would be classical.

Given the relationship, developed above, between \( I_U \) and complex Hilbert Space vectors, it is plausible that \( Q \) provides a continuum approximation in configuration space for the fractal geometry of \( I_U \) in state space. A simple analogy would be evolution on the Lorenz attractor \( I_L \), where motion in the projected \( X-Y \) space can be represented to good approximation by replacing the fractal \( I_L \) by a continuum (double) potential function \( Q_L(X,Y) \). However, with such an approximation, the fine-scale fractal structure of \( I_L \) has been completely obliterated. However, because \( Q \) has support on the whole of configuration space, it manifests none of the counterfactual incompleteness which is a defining characteristic of \( I_U \). As discussed above, it is this counterfactual incompleteness that allows Invariant Set Theory to violate the Bell inequalities without violating realism or local causality. By contrast, Bohmian theory is (and must be by the continuum nature of \( Q \)) explicitly nonlocal.

Invariant Set Theory presents a dichotomy. The non-computational nature of \( I_U \) makes it seemingly impossible to use for calculations. As such, approximating its effect through the continuum De Broglie-Bohm quantum potential appears a sensible strategy. As discussed, the corresponding combination of Liouville and deterministic Hamilton-Jacobi dynamics yields the conventional Schrödinger equation. The reward for such a continuum approximation is an accurate computational tool. The penalty, however, is a grossly distorted picture of such reality, not least as manifest by the phenomena discussed in the last Section. That is to say, we conclude that the Schrödinger equation (even in relativistic form) can only be an approximation to the physics of reality.

Consider the following analogy: the inviscid Euler equations are a good approximation to the Navier-Stokes equations for fluid flow in the limit of high Reynolds number. Many high Reynolds-number computations are much simpler using the Euler equations than the full Navier-Stokes equations. However, the Euler equations arise as the singular limit of the Navier-Stokes equations as viscosity goes to zero, and hence the Euler equations necessarily describe a distorted picture of reality. This distortion is revealed by the fact that aeroplanes fly - an impossibility if the world was truly governed by the Euler equations.

It is suggested below that the distortion of reality provided by the Schrödinger equation is put into even sharper focus when we try to synthesise quantum and gravitational physics.
6. INVARIANT SET THEORY AND THE UNIFICATION OF QUANTUM AND GRAVITATIONAL PHYSICS

Penrose’s words [20]:

Despite impressive progress . . . towards the intended goal of a satisfactory quantum theory of gravity, there remain fundamental problems whose solutions do not appear to be yet in sight. . . . [I]t has been argued that Einstein’s equations should perhaps be replaced by something more compatible with conventional quantum theory. There is also the alternative possibility, which has occasionally been aired, that some of the basic principles of quantum mechanics may need to be called into question.

remain as relevant today as when they were written. Using Invariant Set Theory as a guide, we speculate here on this ‘alternative possibility’.

As discussed in Section 4.1, the wave-like character of photons arises from the periodic helical structure of the state-space trajectories on $I_U$. This suggests that instead of thinking of electromagnetism in terms of a field on a fixed background space time, one should instead think of it as in terms of a representation of the collective properties of helical geometry in state space. Consistent with this, the unit quaternions on which these helical structures are based, are themselves closely linked with the spinors $\phi^{AB}$ which provide the most basic representations

$$\nabla_{A'A'}\phi^{AB} = 0$$

of the Maxwell equations [23]. Indeed, if we take electrodynamics as a prototypical gauge invariant theory, we can also speculate that the Standard Model itself be given some geometric representation in terms of the periodic helical geometry of trajectory segments in state space.

How does the phenomenon of gravity fit into this picture? Throughout this paper, we have emphasised two complementary aspects of trajectory structure in state space - metastable helical/unitary evolution and the unstable/decoherent evolution. Penrose, Diósi and others have speculated that gravity would lead to decoherence [4] [21]. On this basis, we speculated here that the phenomenon we call ‘gravity’ is manifest through the unstable/decoherent ‘clumping’ of trajectories into discrete regions of state space. To make this more precise, let us define two trajectory segments on $I_U$ as ‘gravitationally indistinct’ if

$$\int_{t_0}^{t} E_G(M_1, M_2)dt < O(h)$$

where $E_G$ denotes the gravitational interaction energy associated with the space times associated with these two trajectories (usually defined as the energy needed to move the mass distribution in space-time $M_1$ to the mass distribution in $M_2$ against the gravitational field in space-time $M_2$). We assume that this condition is satisfied for two trajectory components belonging to a helical bundle of trajectories (i.e. they are gravitationally indistinct) but is eventually violated by the instability process which unravels the helices.
It can be noted that gravitational interaction energy is a problematic concept in a general relativistic context because the Principle of Equivalence prevents a pointwise identification of distinct space times. In this sense, inequality (35) is a criterion which uses $\hbar$ to define situations where space-times are sufficiently indistinct that a pointwise identification is possible, i.e. where the Principle of Equivalence breaks down.

If this picture is correct, then it suggests that the synthesis of gravitational and quantum physics will not be achieved by replacing Einstein’s equations with something ‘more compatible with quantum theory’. Rather it will be achieved by embracing the geometric concepts of in general relativity and applying them to state space, as well as space-time. Consistent with suggestions by Freeman Dyson [6], Invariant Set Theory predicts that there is no such thing as a graviton.

Invariant Set Theory suggests a much stronger synergy between cosmology and quantum physics than is implied by contemporary physical theory. For example, the whole concept of an invariant set assumes that the universe evolves quasi-periodically between aeons (eg [22], [24]).

In classical dynamical systems theory, fractal invariant sets arise from forced dissipative systems: by the Liouville theorem, initial state-space volumes shrink to zero on the invariant set. What physically is the basis of trajectory convergence onto $I_U$ in fundamental physics? We argue here that it is associated with dynamics in the vicinity of space-time singularities. Not least, Penrose [22] has argued for a convergence of state-space trajectories in regions of state space associated with black holes (and hence space-time singularities more generally). If the late universe is dominated by black holes, then one can expect such trajectory convergence to be substantial in the late universe.

Without some additional forcing, the invariant set would be a (completely static) fixed point in state space in the presence of trajectory convergence. In the case of $I_U$, it would seem plausible to associate this forcing with the existence of a positive cosmological constant. If such a speculation is correct, then it implies that the universe would not be subject to the laws of quantum physics in the absence of a positive cosmological constant. More than this, Invariant Set Theory can readily explain why vacuum fluctuations of quantum fields do not themselves contribute to this positive cosmological constant. By hypothesis, these vacuum fluctuations arise from the helical structure of the invariant set trajectories. By (35) these fluctuations are gravitationally indistinct and therefore not associated with (and hence not coupled to) the phenomenon of gravity.

Penrose [22] argues that trajectory convergence is a state-space manifestation of information loss in black holes. However, this is a contentious issue since it would imply a breakdown of quantum unitarity. Is information ‘really’ lost in black holes? Invariant Set Theory provides a new perspective on this question. The concept of information is closely linked to entropy and conventionally requires one to consider (a coarse-graining of) state-space volumes. One can equate information loss to volume shrinkage associated with trajectory convergence. However, whilst hypothetical trajectories neighbouring $I_U$ will converge onto $I_U$, such trajectories do not correspond to states of physical reality - they can be associated with hypothetical counterfactual worlds not belonging to $I_U$. By contrast, the volume of $I_U$ is strictly zero, and hence volumes cannot shrink on $I_U$. Hence, information,
suitably defined to account for the zero-volume nature of $I_U$, can be conserved on $I_U$. This resolution of the black hole information paradox is actually no different to the resolution of the other quantum paradoxes discussed in this paper. For example, the trajectories that describe counterfactual experiments for the Mach Zehnder experiment that would, if physically real, contradict Laplacian addition of probability, lie off $I_U$. Hence Laplacian addition of probability is not ‘really’ violated. Similarly, trajectories that describe the counterfactual experiments necessary to derive the Bell inequality (31) from Invariant Set Theory, also lie off $I_U$ and do not correspond to states of physical reality. Hence, local causality is not ‘really’ violated in quantum physics. Finally, trajectories that describe the counterfactual experiments necessary to infer black-hole information loss also lie off $I_U$ and do not correspond to states of physical reality. Hence, conservation of information is not ‘really’ violated in quantum physics. In this sense, Invariant Set Theory provides a novel approach to unifying and solving a range of conceptual problems in contemporary physics.

This raises another key conceptual issue regarding the alocal character of the invariant set. The existence of state-space trajectory convergence onto $I_U$ associated with those parts of $U$’s state space containing black holes, will necessarily imply a measure-zero $I_U$ even in those parts of $U$’s state space which do not contain black holes. In other words, the fact that dynamical evolution is locally Hamiltonian for laboratory physics, does not itself rule out the notion that these Hamiltonian dynamics evolve on a measure-zero fractal invariant set. Put yet another way, the structure of state space in the neighbourhood of space-time singularities may be central to explaining why the laboratory physical is quantum mechanical. It is often said that quantum mechanics is needed to understand the true nature of space-time singularities. Here we speculate that the converse is even more true!

7. Conclusions

Elements of a novel geometric theory of quantum physics have been developed, based on the non-classical assumption that the universe $U$ can be considered a deterministic dynamical system evolving precisely on a measure-zero fractal invariant geometry $I_U$ in $U$’s state space. Based on a set of permutation/negation operators with the multiplicative properties of quaternions, fractal bundles of trajectories of $U$ on $I_U$ are labelled symbolically and shown to map injectively into the complex Hilbert Space (of finite $M$qubits). That the mapping is not a bijection, characterised by the number-theoretic incommensurability of $\phi$ and $\cos \phi$, reflects the fact that $I_U$ is not counterfactually complete.

A key parameter in the construction of $I_U$ is the number $N$ which defines the fractal dimension of $I_U$. Invariant Set Theory requires $N$ to be large but finite. The complex Hilbert Space of quantum theory can be considered as emergent from Invariant Set Theory in the limit $N \to \infty$. However, it is important to note that this is a singular limit [1]. This notion of a singular limit should not be considered pathological. Indeed, Berry [1] notes that singular limits are commonplace in science, and provide insight into how a more general theory (e.g. the Navier-Stokes theory of viscous fluids) can reduce to a less general
theory (e.g. the Euler theory of inviscid fluids), and therefore how higher-level phenomena can emerge from lower-level ones.

Invariant Set Theory can be considered a \( \psi \)-epistemic theory, but with the caveat that the wavefunction \( \psi \) and the associated Schrödinger equation (relativistic or non-relativistic) must be considered a continuum approximation to laws of dynamical evolution and conservation of probability on the non-computable fractal set \( I_U \). It has been argued in this paper that the continuum nature of the Schrödinger equation is the principal reason, not only why conventional quantum theory is irreconcilable with realism and local causality, but also why it is so difficult to synthesise with general relativity. By contrast, the heterogeneous structure of \( I_U \) suggests a novel approach to unify the physics of the Standard Model with the physics of gravity, an approach that will be further developed in forthcoming papers.

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Appendix A. Fractional Roots of $i$

Here we develop the formalism of Section 2.2 which allows a description of bit strings $S(a; \alpha)$ where $\alpha$ are fractional numbers. Effectively this requires us to describe fractional powers of the quaternion operators $E_j$. This development evolves around the fact that for any matrix $A$,

$$
\begin{pmatrix}
A & A \\
A & A
\end{pmatrix} = 
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
$$

With this in mind, define the operator $\bar{E}_j(M)$, where $1 \leq M < N - 1$, $1 \leq j \leq 2M - 1$, as the $2^N \times 2^N$ block diagonal matrix comprising $2^{N-M}$ copies of $E_j(M)$. For example, if $M = N - 2$, then

$$
\bar{E}_j(M) = \begin{pmatrix}
E_j(M) & E_j(M) \\
E_j(M) & E_j(M)
\end{pmatrix}
$$

allowing us to define the square-root quaternion operator

$$
\bar{E}_j^{1/2}(M) = \begin{pmatrix}
1 & E_j(M) \\
E_j(M) & 1
\end{pmatrix} \Rightarrow \bar{E}_j^{1/2}(M) \times \bar{E}_j^{1/2}(M) = \bar{E}_j(M)
$$

and, generalising (7), the $2^N$-long bit string

$$
S(a; \alpha) = \bar{E}_j^\alpha(M) \times E_{2^{N-1}} \times S_{\alpha}^T,
$$

where $0 \leq \alpha < 4$ is a rational number describable by 3 bits (i.e. $\alpha \in \{00.0, 00.1, \ldots, 11.0, 11.1\}$). For each such $\alpha$, $S(a; \alpha)$ contains equal number of $a$ and $\bar{a}$ symbols.

Repeated application of the identity (36) allows one to define, more generally,

$$
S(a; \alpha) = \bar{E}_j^\alpha(M) \times E_{2^{N-1}} \times S_{\alpha}^T
$$

where $\alpha$ is a rational number mod 4 describable by $N - M + 1$ bits. Since $M \geq 1$, $\alpha$ is therefore a rational number fully describable by at most $N$ bits. This restriction is a crucial aspect of Invariant Set Theory.
A general $M$ qubit state can be built from a general $M-1$ qubit state using the following inductive formula:

$$|\psi_{a,b...d}(\theta_1, \ldots \theta_{2M-1}; \phi_1, \ldots \phi_{2M-1})\rangle =$$

$$\cos \frac{\theta_1}{2} |a\rangle \times |\psi_{b,c...d}(\theta_2, \ldots \theta_{2M-1}; \phi_2 \ldots \phi_{2M-1})\rangle$$

$$+ \sin \frac{\theta_1}{2} e^{i\phi_1} |d\rangle \times |\psi_{b,c...d}(\theta_{2M-1+1}, \ldots \theta_{2M-1}; \phi_{2M-1+1} \ldots \phi_{2M-1})\rangle$$

(40)

The correspondence with bit strings is similarly defined inductively. Let

$$|\psi_{b,c...d}(\theta_2, \ldots \theta_{2M-1}; \phi_2 \ldots \phi_{2M-1})\rangle$$

(41)

$$\sim S_j(b; \alpha_2, \beta_2) = \{b'_1, b'_2, b'_3, \ldots b'_{2N}\}$$

(42)

$$S_j(c; \alpha_3, \beta_3) = \{c'_1, c'_2, c'_3, \ldots c'_{2N}\}$$

$$\ldots$$

$$S_j(d; \alpha_{2M-1}, \beta_{2M-1}) = \{d'_1, d'_2, d'_3, \ldots d'_{2N}\}$$

and

$$|\psi_{b,c...d}(\theta_{2M-1+1}, \ldots \theta_{2M-1}; \phi_{2M-1+1} \ldots \phi_{2M-1})\rangle$$

(43)

$$\sim$$

$$S_j(b; \alpha_{2M-1+1}, \beta_{2M-1+1}) = \{b''_1, b''_2, b''_3, \ldots b''_{2N}\}$$

(44)

$$S_j(c; \alpha_{2M-1+2}, \beta_{2M-1+2}) = \{c''_1, c''_2, c''_3, \ldots c''_{2N}\}$$

$$\ldots$$

$$S_j(d; \alpha_{2M-1}, \beta_{2M-1}) = \{d''_1, d''_2, d''_3, \ldots d''_{2N}\}$$

define a pair of $M-1$ qubit correspondences, then with

$$\cos \frac{\theta_1}{2} |a\rangle + \sin \frac{\theta_1}{2} e^{i\phi_1} |d\rangle \sim \{a_1, a_2, a_3 \ldots a_{2N}\}$$

(45)

the $M$ qubit correspondence can be written as

$$|\psi_{a,b...d}(\theta_1, \ldots \theta_{2M-1}; \phi_1 \ldots \phi_{2M-1})\rangle \sim \{a_1, a_2, a_3 \ldots a_{2N}\}$$

$$\{b_1, b_2, b_3, \ldots b_{2N}\}$$

$$\{c_1, c_2, c_3, \ldots c_{2N}\}$$

$$\ldots$$

$$\{d_1, d_2, d_3, \ldots d_{2N}\}$$

(46)
where

\[ b_i = b'_i \quad c_i = c'_i \quad \ldots \quad d_i = d'_i \quad \text{if} \quad a_i = a \]

\[ b_i = b''_i \quad c_i = c''_i \quad \ldots \quad d_i = d''_i \quad \text{if} \quad a_i = \not{a} \]

(47)

This reduces to \[ (14) \] when \( M = 2 \).

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