Pisot-Fibonacci q-coherent states

To cite this article: Jean Pierre Gazeau and Mariano A del Olmo 2011 J. Phys.: Conf. Ser. 284 012027

View the article online for updates and enhancements.
Pisot–Fibonacci $q$-coherent states

Jean Pierre Gazeau$^1$ and Mariano A del Olmo$^2$

$^1$ Laboratoire APC, Université Paris Diderot Paris 7, 10 rue A. Domon et L. Duquet, 75205 Paris Cedex 13, France.
$^2$ Departamento de Física Teórica, Universidad de Valladolid, E-47005, Valladolid, Spain.

E-mail: gazeau@apc.univ-paris7.fr, olmo@fta.uva.es

Abstract. A family of $q$-coherent states is constructed allowing us to obtain a new quantized version of the harmonic oscillator. These $q$-states are normalized and form an overcomplete set resolving the unity with respect to the appropriate Jackson measure if $0 < q < 1$. We only consider here those values of $q$ such that $q^{-1}$ is a Pisot number. In this case the $q$-deformed integers ($[n]_q$) form Fibonacci-like sequences of integers. We study the main physical characteristics of the corresponding quantum oscillator: localization in the configuration and in the phase spaces, probability distributions and related statistical features and semi-classical phase space trajectories whose periodicity is related with the fact that $q$ is an algebraic number.

1. Introduction
Starting from a separable (complex) Hilbert space, $\mathcal{H}$, with orthonormal basis $\{|e_n\}_{n=0}^{\infty}$, we can construct a family of “generalized” coherent states (CS) belonging to $\mathcal{H}$ by

$$|v_z\rangle := \sum_{n=0}^{\infty} \frac{1}{\sqrt{N(|z|^2)} x_n!} z^n |e_n\rangle, \quad \forall z \in D_{\sqrt{R}} = \{z \in \mathbb{C}, |z| < \sqrt{R}\},$$

(1)

where $R$ is the convergence radius of the “generalized” exponential

$$N(t) := \sum_{n=0}^{+\infty} \frac{t^n}{x_n!},$$

(2)

provided that the (Stieltjes) moment problem has a solution for the sequence of factorials $(x_n!)_{n\in\mathbb{N}}$ defined by $x_n! := x_1 \times x_2 \times \ldots \times x_n$ (and $x_0! := 1$), being $(x_n)_{n\in\mathbb{N}}$ an infinite and strictly increasing sequence of nonnegative real numbers such that $x_0 = 0$. From a physical point of view these $x_n$ could be obtained through some experimental device, for instance, they could constitute the quantum energy spectrum of a given system, but they could be some other kind of observed data.

These vectors $|v_z\rangle$ enjoy properties similar to those obeyed by standard coherent states $[1,2]$:

1. $\|v_z\| = 1$.
2. The map $z \in D_{\sqrt{R}} \mapsto |v_z\rangle$ is weakly continuous.
3. The map $n \in \mathbb{N} \mapsto |\langle e_n | v_z \rangle|^2 = \frac{|z|^{2n}}{N(|z|^2) x_n!}$ is a Poisson-like distribution in $|z|^2$ with average number of events $|z|^2$ [3].
(4) The map \( z \in \mathcal{D}_{\sqrt{\pi}} \mapsto |\langle e_n|v_z \rangle|^2 = \frac{|z|^{2n}}{\mathcal{N}(|z|^2) x_n!} \) is a (gamma-like) probability distribution \([3]\) (with respect to \(|z|^2\)) with \( x_n \) as a shape parameter and with respect to the measure \( \nu(d^2z) \) on the open disk \( \mathcal{D}_{\sqrt{\pi}} \) given by \([4]\)

\[
\nu(d^2z) := \mathcal{N}(|z|^2) w(|z|^2) \frac{d^2z}{\pi}, \quad d^2z = d\text{Re}z \, d\text{Im}z.
\]

(5) The family of vectors (1) resolves the unity, i.e. \( \int_{\mathcal{D}_{\sqrt{\pi}}} \nu(d^2z) |v_z\rangle \langle v_z| = I_d \).

This last property is fundamental for the quantization of \( \mathcal{D}_{\sqrt{\pi}} \) since it allows one to define:

(i) a normalized positive operator-valued measure on \( \mathcal{D}_{\sqrt{\pi}} \) equipped with the measure \( \nu(d^2z) \)

(3) and its \( \sigma \)-algebra \( \mathcal{F} \) of Borel sets

\[
\Delta \in \mathcal{F} \mapsto \int_{\Delta} \nu(d^2z) |v_z\rangle \langle v_z| \in \mathcal{L}(\mathcal{H})^+,
\]

where \( \mathcal{L}(\mathcal{H})^+ \) is the cone of positive bounded operators on \( \mathcal{H} \);

(ii) a Berezin-Klauder-Toeplitz (or “anti-Wick”) quantization (coherent state quantization) of the disk \( \mathcal{D}_{\sqrt{\pi}} \) \([5-7]\), i.e. to the function \( f(z, \bar{z}) \) in \( \mathcal{D}_{\sqrt{\pi}} \) corresponds the operator \( A_f \) in \( \mathcal{H} \) defined by

\[
f \mapsto A_f := \int_{\mathcal{D}_{\sqrt{\pi}}} \nu(d^2z) f(z, \bar{z}) |v_z\rangle \langle v_z| = \sum_{n,n' = 0}^{\infty} (A_f)_{nn'} |e_n\rangle \langle e_{n'}| \tag{4}
\]

with matrix elements

\[
(A_f)_{nn'} = \frac{1}{\sqrt{x_n!x_{n'}!}} \int_{\mathcal{D}_{\sqrt{\pi}}} \frac{d^2z}{\pi} w(|z|^2) f(z, \bar{z}) z^n \bar{z}^{n'}.
\]

For instance, let us consider some interesting functions from the physical point of view:

1) \( f(z, \bar{z}) = z \) and \( f(z, \bar{z}) = \bar{z} \) give

\[
A_z = a, \quad a |e_n\rangle = \sqrt{x_n} |e_{n-1}\rangle, \quad a |e_0\rangle = 0 \quad \text{(lowering operator)},
\]

\[
A_{\bar{z}} = a^\dagger, \quad a^\dagger |e_n\rangle = \sqrt{x_{n+1}} |e_{n+1}\rangle \quad \text{(raising operator)}.
\]

The operators \( a \) and \( a^\dagger \) verify the non-canonical commutation rule

\[
[a, a^\dagger] = x_{N+1} - x_N
\]

where the “deformed” number-operator \( x_N \) is

\[
x_N = a^\dagger a \tag{5}
\]

having spectrum \( \{x_n\}_{n \in \mathbb{N}} \) and eigenvectors \( \{|e_n\rangle\}_{n \in \mathbb{N}} \), i.e. \( x_N |e_n\rangle = x_n |e_n\rangle \).

2) The function \( f(z, \bar{z}) = |z|^2 \) can be considered as the classical Hamiltonian of the harmonic oscillator when we have a particle moving on the line and, hence, the complex plane is the phase space of it, i.e. \( z = \frac{1}{\sqrt{2}} (q + ip) \) and \( |z|^2 = \frac{1}{2} (p^2 + q^2) \). Now we get from (4) that

\[
A_{zz} = A_{\bar{z}\bar{z}} = A_z A_{\bar{z}} = a a^\dagger = x_{N+1} \tag{6}.
\]
Therefore, the spectrum of the quantized version of $|z|^2$ is the sequence $(x_n)_{n\geq 1}$. Note that $A_xA_z = x_N$.

The operators $Q$ and $P$, defined in terms of $a$ and $a^\dagger$

$$Q = \frac{1}{\sqrt{2}}(a + a^\dagger) , \quad P = \frac{1}{i\sqrt{2}}(a - a^\dagger) ,$$

allow us to study the localization properties in $\mathbb{C}$, from the point of view of the sequence $\{x_n\}_{n\in\mathbb{N}}$ by considering the behaviour of the lower symbols $Q(z) := \langle v_z|Q|v_z \rangle$ and $P(z) := \langle v_z|P|v_z \rangle$, and the noncommutativity of the complex plane by $\langle v_z||Q,P||v_z \rangle$. The dispersions verify

$$\langle \Delta_{v_z}Q \rangle^2 = \langle \Delta_{v_z}P \rangle^2 = \frac{1}{2}\langle v_z|(x_{N+1} - x_N)|v_z \rangle = \frac{1}{2}\langle v_z||Q,P||v_z \rangle , \quad \forall \langle v_z \rangle .$$

Notice that when $x_n = n$ and $R = \infty$ we recover the canonical quantization of $\mathcal{D}_\infty = \mathbb{C}$ as the classical phase space for the motion of a particle on the line, equipped with the usual Lebesgue measure $\nu(d^2z) = d^2z/\pi$. The quantized version of the classical harmonic oscillator Hamiltonian $H = \frac{1}{2}(p^2 + q^2)$ is the number operator $N + 1$ with spectrum $1, 2, 3, \ldots $

A different statistical interpretation [4] of the phase space can be given by using the measure

$$\nu(d^2z) = w(|z|^2)\mathcal{N}(|z|^2)H(|z|^2 - R)d^2z/\pi ,$$

where $\mathcal{H}(\cdot)$ is the Heaviside function. In this case, classical states are described by the distribution $z \mapsto w(|z - z_0|^2)\mathcal{N}(|z - z_0|^2)H(|z|^2 - R)$ instead of Dirac distributions $z \mapsto \delta_{z_0}(z)$ (i.e. by points $z_0$). Hence, from the “generalized” CS quantization we can consider the associated sequence $(x_n)_{n\in\mathbb{N}}$ as the energy spectrum according to $A_{1/2}(p^2 + q^2) = A_{xz} = a a^\dagger = x_{N+1}$ (6).

On the other hand, in different versions of $q$-harmonic oscillators there appear $q$-integer numbers $\{[n]_q\}$ like

$$\frac{1 - q^{\pm n}}{1 - q}, \quad \frac{q^n - q^{-n}}{q - q^{-1}}, \quad n \in \mathbb{N}, \quad 0 < |q| < 1 ,$$

as forming the energy spectrum, also $q$-coherent states have been constructed for these $q$-oscillators [9–13]. These $q$-integer numbers, $\{[n]_q\}$, are real numbers, deformations of nonnegative integers, that in the limit $q \rightarrow 1$ go to the original number $n$. In general, these sequences of $q$-numbers are non-integer numbers but we are going to search for those values of $q$ such that their associated $q$-numbers are integer for all $n$. These integer sequences $\{[n]_q\}_{n\in\mathbb{N}}$ will be used to construct “generalized” CS and $q$-harmonic oscillators with spectrum made of positive integer numbers $\{[n]_q\}$.

The paper is organized as follows. In Section 2 we obtain integer sequences of $q$-deformed integers, which are Fibonacci-like sequences and are characterized by $0 < |q| < 1$ such that $1/|q|$ is a quadratic Pisot (more fairly Pisot-Vijayaraghavan) unit. Section 3 is devoted to the resolution of the moment problem, i.e. to finding a (discrete) measure such that $[n]_q! = \int_0^\infty t^n w_q(t)dt$. This point is crucial in order to have a family of vectors (1) resolving the unity. In section 4 we present some computations and graphics for different $q$-sequences showing the main physical properties of these $q$-CS. Finally some remarks end the paper.

2. $q$-Pisot–Fibonacci numbers

Let us start from the following $pq$-deformation [14] of natural numbers:

$$\{[n]_{pq}\} := \frac{q^n - p^n}{q - p} , \quad n \in \mathbb{N}, \quad p, q \in \mathbb{R}.$$
Our task is to find particular values of \( p \) and \( q \) such that \( u_n \equiv [n]_q \in \mathbb{N} \). Obviously this is true for \( u_0 = 0 \) and \( u_1 = 1 \). From \( u_2 = q + p \in \mathbb{N} \) and \( u_3 = q^2 + qp + p^2 \in \mathbb{N} \), we see that \( q \) and \( p \) are quadratic integers, i.e. both are roots of the quadratic equation

\[
X^2 - sX + r = 0, \tag{9}
\]

with \( s = u_2 = p + q \in \mathbb{N} \) and \( r = pq = u_2^2 - u_3 \in \mathbb{Z} \). Imposing that \( q \in \mathbb{R}^+ \) and \( p \in \mathbb{R}^+ \) we get that \( 0 \leq s^2 - 4r = 4u_3 - 3u_2^2 = u_3 - 3r \). So, from \( u_3 \in \mathbb{N} \) and \( r \in \mathbb{Z} \) we see that \( q \) and \( p \) are real if \( r < 0 \) and \( u_3 \geq 3r \) if \( r > 0 \). The choice of \( r, s \in \mathbb{Z}^+ \) characterizes completely the sequence \( \{u_n\}_{n \in \mathbb{N}} \) via a three-term recurrence

\[
u_{n+1} - su_n + ru_{n-1} = 0, \quad u_0 = 0, \quad u_1 = 1.
\]

Such sequences of numbers generalize the Fibonacci sequence corresponding to \( s = 1 \) and \( r = -1 \). In this case \( p = \frac{1 + \sqrt{5}}{2} = \tau \) (the golden mean) and \( q = \frac{1 - \sqrt{5}}{2} = -1/\tau \).

We will only consider the cases \( r = \pm 1 \) and \( s > 0 \) (\( s < 0 \) corresponds to a change of sign for both roots):
1. \( r = -1 \), then \( s \geq 1 \) and the roots \((q, p)\) of eq. (9) are such that \(-1 < q < 0 \) and \( 1 < p = -1/q \).
2. \( r = 1 \), then \( s \geq 3 \) and the roots of eq. (9) verify \( 0 < q < 1 \) and \( p = 1/q > 1 \). The case \( s = 2 \) has been excluded since \( p = q = 1 \).

The algebraic integer \( p > 1 \) is a quadratic (the degree of the equation (9)) Pisot-Vijayaraghavan (since \( p > 1 \) and \( |q| < 1 \)) unit (because in eq. (9) \( r = \pm 1 \) [8].

The antisymmetric or fermionic \( q \)-deformation of natural numbers appear in the case 1:

\[
[f][n]_q = q^n - (-1)^n q^{-n} \quad \frac{q + q^{-1}}{q + q^{-1}} = (-1)^n [f][n]^{-1}_{q^{-1}}. \tag{10}
\]

The case 2 corresponds to the symmetric or bosonic \( q \)-deformation of natural numbers that we will use hereafter:

\[
[s][n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} = [s][n]^{-1}_{q^{-1}}. \tag{10}
\]

3. Moment measure for the symmetric deformation of integers

In order to solve the moment problem for the sequence \((x_n \equiv [s][n]_q)_{n \in \mathbb{N}}\) given by (10) it is necessary to find a probability distribution \( t \in [0, R] \mapsto w_q(t) \) such that

\[
x_n! = [s][n]_q! = \int_0^R t^n w_q(t) \, dt.
\]

In this case the associated exponential (2) defines an analytic entire function \( \mathcal{E}_q(z) \) in \( \mathbb{C} \) for any \( q > 0 \) since \( R = \lim_{n \to \infty} x_n = \infty \), \( \forall q > 0 \).

A useful tool is the “auxiliary” exponential

\[
\mathcal{E}_q(t) := \sum_{n=0}^{\infty} q \left( \frac{t^n}{x_n!} \right) = \sum_{n=0}^{\infty} q^n (n+1)! \frac{t^n}{x_n!},
\]

whose radius of convergence is \( \infty \) for \( 0 < q \leq 1 \) and \( 1/(q - q^{-1}) \) for \( q > 1 \). It is connected with the two standard \( q \)-exponentials [15]

\[
\mathcal{E}^q_s := \sum_{n=0}^{\infty} \frac{x^n}{[n]_q^{s}}, \quad \mathcal{E}^q_s := \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{[n]_q^{s}}, \quad [n]_q := \frac{1 - q^n}{1 - q},
\]
by
\[ \mathcal{E}_q(t) = e^{qxt} = E_{q^2}^{\gamma}. \] (11)

From the relation \( e^{-x} E_{q^2}^x = 1 \) we get that \( \mathcal{E}_q(t) \left|_{t=-1} = \mathcal{E}_{q^{-1}}(-t) \). The auxiliary exponential \( \mathcal{E}_q(t) \) is involved in the \( q \)-integral representation of the \( q \)-gamma function \( \Gamma_q(x) \) defined by [15]

\[ \Gamma_q(x) = \frac{(1-q)^{x-1}}{(1-q)^{x-1}}, \quad x > 0, \]

where
\[ (1+a)^\infty_q = \prod_{j=0}^{\infty} (1+aq^j), \quad (1+a)_q^t = \frac{(1+a)^\infty_q}{(1+q^t a)_q^\infty}, \quad t \in \mathbb{C}. \]

The \( q \)-gamma function obeys
\[ \Gamma_q(1) = 1, \quad \Gamma_q(x+1) = [x]_q \Gamma_q(x), \quad \Gamma_q(x+1) = q^{x-1} [x]_q \Gamma_q(x) \quad \forall x > 0. \]

A first integral representation of \( \Gamma_q(x) \) is
\[ \Gamma_q(x) = \int_0^1 E_{q^t}^{-q} d_q t. \]

The \( q \)-integral [16,17]
\[ \int_0^a f(t) d_q t = (1-q) \sum_{j=0}^{\infty} a q^j f(a q^j) \]

becomes an ordinary integral with a discrete measure
\[ \int_0^a f(t) d_q t = \int_0^\infty \rho_q(t; a) f(t) dt, \quad \rho_q(t; a) = (1-q) \sum_{j=0}^{\infty} a q^j \delta(t - a q^j). \]

Hence
\[ \Gamma_q(x) = \int_0^\infty \rho_q(t; 1/(1-q) \ t^{x-1} E_{q^t}^{-q} dt. \] (12)

Taking nonnegative integer values \( x = n \) for \( \Gamma_q^n(x+1) \) we get that
\[ \Gamma_q^n(x+1) = q^{n(n+1)/2} [s]_q^n! = q^{-n} q^{n(n+1)/2} x_n!. \] (13)

From (13), (12) and (11) we get the solution for the moment problem with a positive measure for the sequence \( \left( q^{n(n+1)/2} x_n! \right)_{n \in \mathbb{N}} \)
\[ q^{n(n+1)/2} x_n! = \int_0^\infty \varpi_q(t) t^n \mathcal{E}_q(-t) dt, \quad \varpi_q(t) = \sum_{j=0}^{\infty} q^{2j} \delta \left( t - \frac{q^{2j}}{q^{-1} - q} \right). \] (14)

In order to solve the moment problem for \( (x_n!)_{n \in \mathbb{N}} \) it is necessary to solve it for \( \left( q^{-n(n+1)/2} \right)_{n \in \mathbb{N}}. \)

From [18,19] the solution to the moment problem for \( \left( q^{-n(n+1)/2} \right)_{n \in \mathbb{N}}, \) with \( 0 < q < 1, \) is given by
\[ q^{-n(n+1)/2} = \int_0^\infty t^n g_q(t) dt, \quad g_q(t) = \frac{1}{\sqrt{2\pi |\ln q|}} \exp \left[ -\frac{(\ln(\sqrt{q}t))^2}{2|\ln q|} \right]. \] (15)
In Fig. 1 we display the function \( g_q(z) \) when \( q = (3 - \sqrt{5})/2 \) which is solution of eq. (9) when \( s = 3 \) and \( r = 1 \).

Finally, the application of a composition formula for moments [18] to the product of (14) and (15) yields the solution to the moment problem for the sequence \( (x_n!) \):

\[
x_n! = \int_0^\infty t^n w_q(t) \, dt, \quad w_q(t) = \int_0^\infty g_q(t/u) \varpi_q(u) \frac{du}{u},
\]

where \( w_q(t) \) is a positive measure density (for a detailed proof see Ref. [20]).

4. Pisot–Fibonacci \( q \)-CS quantization of the complex plane

After the moment problem for the sequence \( (x_n^q = [s][n]^q)_{n \in \mathbb{N}} \) is solved, we are able to construct a family of \( q \)-CS associated to these Pisot–Fibonacci symmetric sequences (section 2) according to formula (1). We now proceed with numerical explorations by choosing the lowest cases \( s = 3, 4 \) and 5 in eq. (9) with \( r = 1 \), that correspond to \( q = (3 - \sqrt{5})/2 \) (= \( 1/\tau^2 \) with \( \tau \) the golden mean), \( q = 2 - \sqrt{3} \) and \( q = (5 - \sqrt{21})/2 \), respectively. The corresponding sequences of \( q \)-numbers \( [s][n]^q \) are generalizations of the Fibonacci sequence: in particular for \( q = (3 - \sqrt{5})/2 \) we obtain Fibonacci numbers occupying the odd place in the Fibonacci series.

In the following we study some physical properties of these Pisot–Fibonacci \( q \)-CS for different values of \( q \) and we compare with the standard \((q = 1)\) CS.

The explicit expression of these Pisot–Fibonacci \( q \)-coherent states (1) is

\[
|v_z^q\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{\mathcal{N}_q(|z|^2)}} \frac{z^n}{\sqrt{[s][n]^q!}} |e_n\rangle.
\]

It is easy to see that the Pisot–Fibonacci \( q \)-CS \( |v_z^q\rangle \) goes to the standard CS \( |v_z\rangle \) in the limit \( q \to 1 \) [5, 21, 22]

\[
|v_z\rangle = \sum_{n=0}^{\infty} \frac{1}{\exp(|z|^2)} \frac{z^n}{\sqrt{n!}} |e_n\rangle,
\]

since \( [s][n]^q \to n \) and \( \mathcal{N}_q(|z|^2) \to \exp(|z|^2) \) when \( q \) goes to 1. A graphical way to see this limit is to consider the ratio-function \( d_q(n) = [s][n]^q!/n! \) and display it versus \( n \) for different values of \( q \) [13] (see Fig. 2a and also Fig. 2b).
Figure 2. Representation of $d_q(n)$ and $N_q(t)$ with $t \geq 0$ for $q = 1$ (red), $(3 - \sqrt{5})/2$ (blue), $2 - \sqrt{3}$ (green), $(5 - \sqrt{21})/2$ (purple).

Figure 3. Representations of Poisson-like distributions $\rho_q(n, |z|)$ versus $n$ for different values of the parameters $|z|$ and $q$.

Let $|e_n\rangle$ be the state of $n$ bosons. The probability of finding $n$ bosons in the $q$-CS $|v_z\rangle_q$ is given by the $q$-Poisson distribution

$$\rho_q(n, |z|) = \frac{|z|^{2n}}{\mathcal{N}(|z|^2) x_n^n}.$$ 

Its limit when $q \to 1$ is the standard Poisson distribution. The $q$-Poisson distributions are left displaced with respect to the standard Poisson distribution, hence $\rho_q(n, |z|)$ is a sub-Poissonian distribution (see Fig. 3a and Fig. 3b).

Expression (8) shows us that these generalized-CS are intelligent states for the operators $Q$ and $P$ (7). In particular, for our Pisot–Fibonacci $q$-CS the variances of $Q$ and $P$ (8) are

$$(\Delta_{v_z} Q)^2 = (\Delta_{v_z} P)^2 = \frac{1}{2} \left[ \frac{1}{\mathcal{N}(|z|^2)} + (s - 1)|z|^2 - |z|^4 \left\langle \frac{1}{x_{n+2}} \right\rangle \right] \geq 1/2, \quad \forall z \in \mathbb{C}. \quad (16)$$

According to expression (16) only for $z = 0$, i.e. for the vacuum state $|v_0\rangle = |e_0\rangle$, $(\Delta_{v_0} Q)^2 = (\Delta_{v_0} P)^2 = 1/2$. So, the states $|v_z\rangle_q$ are not squeezed coherent states for any $q$ and $z$. 
Another point of interest is the behavior of the trajectories in phase space. So, we consider the time-evolution of $a = A_z$, i.e. $A_z(t) = e^{-iHt} A_z e^{iHt}$, which is given by the mean value in terms of the $q$-coherent states $|v_z\rangle_q$

$$\tilde{z}(t) = q \langle v_z | e^{-iHt} A_z e^{iHt} | v_z \rangle_q = \frac{z}{N_x(z^2)} \sum_{n=0}^{+\infty} \frac{|z|^{2n}}{x_n!} \exp\left(-i (x_{n+2} - x_{n+1}) t \right).$$

In Fig. 4 we plot $\Im \tilde{z}(t)$ versus $\Re \tilde{z}(t)$ for different values of $q$ and $0 \leq t \leq 8\pi$. The phase space trajectories are periodic with period $2\pi$ because the $q$-numbers involved in these CS are integers.

5. Conclusions
In this paper we have introduced a $q$-dependent family of coherent states and we have explored some properties of these coherent states as well as the quantum harmonic oscillator obtained through the corresponding coherent state quantization. We have restricted our study to the case in which $q^{-1}$ is a quadratic unit Pisot number, since then the spectrum of the quantum Hamiltonian is made of the $q$-deformed integers $[n]_q = (q^n - q^{-n})/(q - q^{-1})$ which are still integers and form sequences of the Fibonacci type. We have put into evidence interesting quantum features issued from these particular algebraic cases, concerning particularly the localization in the configuration space and in the phase space, probability distributions and related statistical features, time-evolution and semi-classical phase space trajectories. The periodicity of the latter nicely reflects the algebraic Pisot nature of the deformation parameter $q$. By contrast we present the semi-classical phase space trajectories for irrational values of $q$ (Figs. 5). Obviously, the trajectories are not periodic.

Acknowledgments
This work was partially supported by the Ministerio de Educación y Ciencia of Spain (Project FIS2009-03959) and by the Junta de Castilla y León.

6. References
[1] Klauder J R and Skagerstam B S 1985 Coherent States - Applications in Physics and Mathematical Physics (Singapore: World Scientific)
Figure 5. Plots of $\Im \hat{z}(t)$ versus $\Re \hat{z}(t)$ for irrational values of $q$ and $0 \leq t \leq 8\pi$.

[2] Gazeau J P 2009 *Coherent States In Quantum Physics* (Berlin: Wiley-VCH)
[3] Ali S T, Balkova L, Curado E M F, Gazeau J P, Rego-Monteiro M A, Rodrigues L M C S and Sekimoto K, 2009 J. Math. Phys. 50 043517- 1-28;
Curado E M F, Gazeau J P and Rodrigues L M C S 2010 Phys. Scr. 82 038108-1-9
[4] Ali S T, Gazeau J P and Heller B 2008 J. Phys. A: Math. Theor. 41 365302
[5] Klauder J R 1963 J. Math. Phys. 4, 1055 and 1058;
1995 Ann. of Phys. 237 147
[6] Berezin F A 1975 Commun. Math.Phys. 40 153
[7] Chakraborty B, Gazeau J P and Youssef A 2008 Coherent state quantization of angle, time, and more irregular functions and distributions (*Preprint* arXiv: 0805.1847v1)
[8] Bertin M J, Decomps-Guilloux A, Grandet-Hugot M, Pathiaux-Delefosse M, Schreiber J P 1992 *Pisot and Salem Numbers* (Birkhäuser);
Boyd D W 1978 *Math. Comp.* 32 1244
[9] Arik M and Coon D D 1976 J. Math. Phys. 17 524
[10] Jannussis A, Brodimas G, Sourlas D and Zisis V 1981 Lett. Nuovo Cimento 30 123
[11] Biedenharn L C 1989 J. Phys. A: Math. Gen. 22 L873
[12] Macfarlane A J 1989 J. Phys. A: Math. Gen. 22 4581
[13] Quesne C 2002 J. Phys. A: Math. Gen. 35 9213
[14] Chakrabarti R and Jagannathan R 1991 J. Phys. A: Math. Gen. 24 L711
[15] De Sole A and Kac V 2005 Rend. Mat. Acc. Lincei 9 11 (*Preprint* arXiv: math.QA/0302032)
[16] Thomae J 1869 J. reine angew. Math. 70 258
[17] Jackson F H 1910 *Quart. J. Pure and Appli. Math.* 41 193
[18] Bergeron H 2010 Nonlinear coherent states and their generalization: a resolvent-like definition (*Preprint* unpublished)
[19] Gazeau J P, Baldiotti M C and Gittman D M 2009 Phys. Lett. A 373 1916
[20] del Olmo M A and Gazeau J P 2010 Pisot $q$-Coherent states quantization of the harmonic oscillator (*Preprint*)
[21] Glauber R J 1963 Phys. Rev. 130 2529;
1963 Phys. Rev. 131 2766
[22] Sudarshan E C G 1963 Phys. Rev. Lett. 10 277.