On the structure of modules indexed by small categories

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Abstract

Given a small category $C$ and $C$-diagram of vector spaces $M$ over a field $k$ (referred to as a $C$-module), we show that $M$ admits a quasi-tame $C$-module cover $QT(C)(M)$, determined by its local structure $F(M)$ when the local structure is stable (which holds whenever both $C$ and $k$ are finite). $QT(C)(M)$ is a finite direct sum of quasi-blocks determined by the associated graded local structure $F(M)_\ast$ of $M$, and for all $M$ there is a natural isomorphism $F(QT(C)(M))_\ast \cong F(M)_\ast$ of associated graded local structures. When the local structure is stable, it determines an effectively computable non-negative integer invariant $e(M)$ - the excess of $M$ - that quantifies the degree to which the local structure fails to be in general position at each object of $C$. When $M$ admits an inner product there is a surjection of $C$-modules $QT(C)(M) \twoheadrightarrow M$ inducing the above isomorphism of associated graded local structures, which is an isomorphism iff $e(M) = 0$. In the very special case $C$ is the categorical realization of a finite totally ordered set, these methods recover the classical result that any finite one-dimensional persistence module decomposes as a direct sum of interval submodules in a manner unique up to re-ordering.

1 Introduction

It is a consequence of Gabriel’s structure theorem for quiver representations of type $A_n$ [7] that any finite 1-dimensional persistence module over a field $k$ decomposes into a direct sum of indecomposable interval modules; moreover, the decomposition is unique up to reordering. The barcodes (with multiplicity) associated to the original module correspond to these interval submodules, which are indexed by the set of connected subgraphs of the finite directed graph associated to the finite, totally ordered indexing set of the module. Modules that decompose in such a fashion are conventionally referred to as tame.

A central problem in topological data analysis has been to determine what, if anything, holds true for more complex types of modules, for example $n$-dimensional persistence modules [4, 5], given the fact that these more general types of modules are almost never tame.

This paper provides an answer to that question; what follows is a summary of the contents.

In section 2.1, we define the framework in which we will be working. We consider modules indexed by an arbitrary small category $C$ (referred to as $C$-modules in this paper) equipped with i) no additional structure, and ii) an inner product. The structural properties we later establish for such modules in section 2.2 are based on the fundamental notion of a multi-flag of a vector space $V$, defined as a collection of subspaces of $V$ that includes $\{0\}$, $V$, and is closed under intersection. Equally important is the notion of general position for such an array. Using terminology made precise in section 2.3, a $C$-module $M$ determines a functor $\mathcal{F} : C \to (\text{multi-flags}/k)$ which associates to each $x \in \text{obj}(C)$ a multi-flag $\mathcal{F}(M)(x)$ of the vector space $M(x)$, referred to as the local structure of $M$ at $x$. The local structure is naturally the direct limit of a directed system of recursively defined multi-flags $\{\mathcal{F}_n(M), \iota_n\}$, and is stable when this directed system stabilizes at a finite stage (with the limit written as $\mathcal{F}(M)$). This structure can be refined by the existence of an inner product on $M$ (section 2.4). Section 2.5 describes the behavior of

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MSC primary 16-XX; secondary 55-XX
the local structure with respect to sums and tensors of $C$-modules, and for tensor products proves an
algebraic Künneth Theorem for local structures (used later on in section 4).

Assuming a $C$-module $M$ has stable local structure, the associated graded $F_*(M)$ defines a $C$-module
$QT_C(M)$, referred to as the quasi-tame cover of $M$, constructed in section 3.1. Under suitable conditions
on $M$ the quasi-tame cover is $b$-tame (section 3.2), meaning it is a direct sum of blocks (generalizing the
interval decomposition of 1-dimensional persistence modules). For all $C$-modules $M$ there is a canonical
isomorphism of associated gradeds

$$F(QT_C(M))_* \cong F(M)_*$$

(1)

The module $QT_C(M)$ represents the closest approximation to $M$ by a quasi-tame module (one which
decomposes as a finite direct sum of quasi-blocks), and equals $M$ precisely when $M$ itself is quasi-tame.
In section 3.3 we show that when $M$ has stable local structure and admits an inner product (is an
IPC-module), the map of associated gradeds in (1) is induced by a surjection of $C$-modules

$$QT_C(M) \rightarrow M$$

(2)

Moreover, for IPC-modules the following statements are equivalent

- The surjection in (2) is an isomorphism;
- $M$ is quasi-tame;
- the local structure of $M$ is in general position at each object of $C$;
- the excess $e(M)$ of $M$ - a computable non-negative integral invariant of $M$ - is zero.

In section 3.4 we show that when $M$ is a 1-dim. persistence module (i.e., when it is a $C$-module where $C$
is the categorical realization of a finite totally ordered set), there is an isomorphism $QT_C(M) \cong M$ which
recovers the classical decomposition for such modules. In section 3.5 we show that all finite $n$-dimensional
persistence modules have stable local structure.

Finally in section 4 we prove two results for topologically based $C$-modules - those resulting via applying
$H_*(-)$ to a $C$-diagram of spaces. The second of the two is a Künneth Theorem for such modules, derived
via application of the algebraic Künneth Theorem of section 2.5 to this topologically based situation.

We would like to thank Dan Burghelea and Fedor Manin for their helpful comments on earlier drafts of
this work, as well as Bill Dwyer for his contribution to the proof of the cofibrancy replacement result
presented in section 4. We are indebted to Jakob Hansen for valuable feedback and discussions at various
later stages of this work. This paper is part of a larger joint project with Sanjeevi Krishnan.

2 $C$-modules

2.1 Preliminaries

Throughout we work over a fixed field $k$. Let $(\text{vect}/k)$ denote the category of finite-dimensional vector
spaces over $k$, and linear homomorphisms between such. Additionally we denote by $(\text{vect}/k)_*$ the category
of finite-dimensional vector spaces over $k$, and linear homomorphisms enriched over pointed sets with the
base point corresponding the the zero linear map.

**Definition 1.** Given a small category $C$, a $C$-module is a covariant functor $M : C \rightarrow (\text{vect}/k)$. Similarly,
given a small category $C_*$ enriched over pointed sets, a basepointed $C$-module is a basepointed functor
$M : C_* \rightarrow (\text{vect}/k)_*$.

[Note: This basepointed structure above ensures that $\text{Hom}(M(x), M(y))$ always contains the zero linear
map. The only place where this zero map will be needed is in showing that the associated graded of the
local structure for basepointed $C$-modules distributes over finite sums of modules.]
The category \((\mathcal{C}\text{-mod})\) of \(\mathcal{C}\text{-modules}\) then has these functors as objects, with morphisms represented in the obvious way by natural transformations. All functorial constructions on vector spaces extend to the objects of \((\mathcal{C}\text{-mod})\) by objectwise application. In particular, one has the appropriate notions of

- monomorphisms, epimorphisms, short and long-exact sequences;
- kernel and cokernel;
- direct sums, Hom-spaces, tensor products;
- linear combinations of morphisms.

With these constructs \((\mathcal{C}\text{-mod})\) is an abelian category.

### 2.2 Multi-flags and general position

Recall that a flag in a vector space \(V\) consists of a finite sequence of proper inclusions beginning at \(\{0\}\) and ending at \(V\):

\[ W := \{W_i\}_{0 \leq i \leq n} = \{0\} = W_0 \subset W_1 \subset W_2 \subset \cdots \subset W_m = V \]

We will relax this structure in two different ways.

**Definition 2.** A semi-flag in \(V\) is a sequence of not necessarily proper inclusions \(\{0\} = W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_m = V\). More generally, a multi-flag in \(V\) is a collection \(F = \{W_\alpha \subseteq V\}\) of subspaces of \(V\) containing \(\{0\}, V\), partially ordered by inclusion, and closed under intersection. It is not required to be finite.

Given an element \(W \subseteq V\) of a multi-flag \(F\) associated to \(V\), let \(SS_F(W) := \left( \sum_{U \in S_F(W)} U \right) \cup \{0\}\) be the set of elements of \(F\) that are proper subsets of \(W\), and set

\[ F^p := \{(W, SS_F(W))\}_{W \in F} \]

For each \(W \in F\), \(SS_F(W)\) is a subspace of \(W\). The set of pairs of subspaces

\[ F^p := \{(W, SS_F(W))\}_{W \in F} \]

is canonically isomorphic to \(F\) via the identification

\[ F \ni W \leftrightarrow (W, SS_F(W)) \in F^p \]

We need to consider different possible ways a subquotient might be represented by a pair in \(F^p\). To that end, note that if \(W', W \in F\) with \(W' \subseteq W\), then \(SS_F(W') \subseteq SS_F(W)\) and there is an inclusion of pairs \(\iota : (W', SS_F(W')) \hookrightarrow (W, SS_F(W))\). The inclusion \(\iota\) is a q-isomorphism if it induces an isomorphism of quotients

\[ \tau : W' := W/SS_F(W) \cong W'/SS_F(W') = W'_F \]

Finally we say that two elements \((W, SS_F(W)), (W', SS_F(W'))\) of \(F^p\) are q-equivalent if they are connected by a zig-zag sequence of q-isomorphisms. This defines an equivalence relation \(\sim_q\) on \(F^p \cong F\) and the associated graded \(F_*\) of \(F\) is defined as

\[ F_* := F^p/\sim_q \]

An element of \(F_*\) will be typically written as \([W_F]\). Thus \([W_F] = [W'_F]\) iff \((W, SS_F(W)) \sim_q (W', SS_F(W'))\).

We say \([W_F] \in F\) is without multiplicity iff it has only one representative (in other words, if \((W, SS_F(W))\) is only q-equivalent to itself).
Proposition 1. If \([\{0\}] \neq [W_F] \in \mathcal{F}_*\), then \([W_F]\) is without multiplicity.

Proof. If \((W, SS_F(W)) \sim (W', SS_F(W'))\) and \(W \neq W'\), then in the zig-zag sequence of \(q\)-isomorphisms connecting \((W, SS_F(W))\) and \((W', SS_F(W'))\) there must be a \(q\)-isomorphism \((W_1, SS_F(W_1)) \rightarrow (W_2, SS_F(W_2))\) where \(W_1 \rightarrow W_2\) is a proper inclusion. As \(W_1, W_2 \in \mathcal{F}\), this implies \(W_1 \in S_F(W_2)\), and so the induced isomorphism on quotients \(W_1/SS_F(W_1) \xrightarrow{\sim} W_2/SS_F(W_2)\) must be the zero map. Hence
\[
[W_F] = [(W_1)_F] = [W_F'] = [[\{0\}]]
\]

Observation 1. For any multi-flag \(F\) of \(V\), \[
\sum_{[W_F] \in \mathcal{F}} \dim(W_F) \geq \dim(V).
\]

Definition 3. The excess of a multi-flag \(F\) is \(e(F) := \left[\sum_{[W_F] \in \mathcal{F}} \dim(W_F)\right] - \dim(V)\).

Definition 4. A multi-flag \(F\) in \(V\) is in general position iff \(e(F) = 0\).

Any semi-flag \(G\) of \(V\) is in general position; this is a direct consequence of the total ordering. Also the multi-flag \(G\) formed by a pair of subspaces \(W_1, W_2 \subset V\) and their common intersection (together with \(\{0\}\) and \(V\)) is always in general position. For the proof of the next lemma, as well as in section 2.4 below, we will need the following definition.

Definition 5. Let \(V = (V, <, >)\) be an inner product (IP) space. If \(W_1 \subseteq W_2 \subset V\), we write \(W_1 \subset W_2\) for the relative orthogonal complement of \(W_1\) viewed as a subspace of \(W_2\) equipped with the induced inner product (so that \(W_2 \cong W_1 \oplus (W_1 \subset W_2)\)).

Note that \((W_1 \subset W_2) = W_1^{\perp} \cap W_2\) when \(W_1 \subseteq W_2\) and \(W_2\) is equipped with the induced inner product.

Lemma 1. If \(G_i, i = 1, 2\) are two semi-flags in the inner product space \(V\) and \(F\) is the smallest multi-flag containing \(G_1\) and \(G_2\) (in other words, it is the multi-flag generated by these two semi-flags), then \(F\) is in general position.

Let \(G_i = \{W_{i,j}\}_{0 \leq j \leq m_i}, i = 1, 2\). Set \(W^{i,k} := W_{1,j} \cap W_{2,k}\). Note that for each \(i\), \(\{W^{i,k}\}_{0 \leq k \leq m_2}\) is a semi-flag in \(W_{1,i}\), with the inclusion maps \(W_{1,i} \rightarrow W_{1,i+1}\) inducing an inclusion of semi-flags \(\{W^{i,k}\}_{0 \leq k \leq m_2} \hookrightarrow \{W^{i+1,k}\}_{0 \leq k \leq m_2}\). By induction on length in the first coordinate we may assume that the multi-flag of \(W := W_{1,m_i-1}\) generated by \(G_1 := \{W_{1,j}\}_{0 \leq j \leq m_i-1}\) and \(G_2 := \{W \cap W_{2,k}\}_{0 \leq k \leq m_2}\) are in general position. To extend general position to the multi-flag on all of \(V\), the induction step allows reduction to considering the case where the first semi-flag has only one middle term:

Claim 1. Given \(W \subseteq V\), viewed as a semi-flag \(G'\) of \(V\) of length 3, and the semi-flag \(G_2 = \{W_{2,j}\}_{0 \leq j \leq m_2}\) as above, the multi-flag of \(V\) generated by \(G'\) and \(G_2\) is in general position.

Proof. Without loss of generality, we may assume \(V\) is equipped with an inner product. The multi-flag \(F\) in question is constructed by intersecting \(W\) with the elements of \(G_2\), producing the semi-flag \(G'_W := W \cap G_2 = \{W \cap W_{2,j}\}_{0 \leq j \leq m_2}\) of \(W\), which in turn includes into the semi-flag \(G_2\) of \(V\). Constructed this way the direct-sum splittings of \(W\) induced by the semi-flag \(W \cap G_2\) and of \(V\) induced by the semi-flag \(G_2\) are compatible, in that if we write \(W_{2,j}\) as \((W \cap W_{2,j}) \oplus (W \cap W_{2,j} \subset W_{2,j})\) for each \(j\), then the orthogonal complement of \(W_{2,k}\) in \(W_{2,k+1}\) is given as the direct sum of the orthogonal complement of \((W \cap W_{2,k})\) in \((W \cap W_{2,k+1})\) and the orthogonal complement of \((W \cap W_{2,k} \subset W_{2,k})\) in \((W \cap W_{2,k+1} \subset W_{2,k+1})\), which yields a direct-sum decomposition of \(V\) indexed by the associated graded elements of \(F\), completing the proof both of the claim and of the lemma.

On the other hand, one can construct simple examples of multi-flags which are not - in fact cannot be - in general position, as the following illustrates.

Example 1. Let \(\mathbb{R} \cong W_1 \subset \mathbb{R}^2\) be three distinct 1-dimensional subspaces of \(\mathbb{R}^2\) intersecting in the origin, and the \(F\) be the multi-flag generated by this data. Then \(F\) is not in general position.
Note: Example 1 also illustrates the important distinction between a configuration of subspaces being of finite type (having finitely many isomorphism classes of configurations), and the stronger property of being in general position.

Lemma 2. General position is preserved when closing a multi-flag under

- subspace sums and intersections;
- relative complements (for a fixed inner product)

Moreover, any submultiflag of a multi-flag in general position is itself in general position.

Proof. These results follow from an equivalent characterization of general position in terms of special bases. We say that \( B \) is an \( \mathcal{F} \)-basis for \( V \) if every non-zero subspace in \( \mathcal{F} \) has a basis which is a subset of \( B \). Similarly we say that \( B \) is an \( \mathcal{F}_s \)-basis if \( B \) can be written as the (disjoint) union over \( W_F \in \mathcal{F}_s \) of bases for \( W_F \). It is straightforward to see these two properties are equivalent - \( B \) is an \( \mathcal{F} \)-basis iff \( B \) is an \( \mathcal{F}_s \)-basis. It is also clear that an \( \mathcal{F}_s \) basis for \( V \) exists iff \( \mathcal{F} \) is in general position.

Thus if \( \mathcal{F} \) is in general position, we may fix a \( \mathcal{F} \)-basis \( B \) for \( V \). Then the closure \( \tilde{\mathcal{F}} \) of \( \mathcal{F} \) under subspace sums and intersections is again in general position because \( B \) is seen to also be a \( \tilde{\mathcal{F}} \)-basis for \( V \). For any inner product on \( V \) such that \( V \) has an orthogonal \( \mathcal{F} \)-basis, the closure of \( \mathcal{F} \) under relative orthogonal complements, sums, and intersections is in general position by the same argument. Finally, if \( B \) is an \( \mathcal{F} \) basis, then for any submultiflag \( \mathcal{F}' \subset \mathcal{F} \), \( B \) is also a \( \mathcal{F}' \) basis.

Corollary 1. A multi-flag \( \mathcal{F} \) on \( V \) is in general position iff there is an isomorphism of vector spaces over \( k \):

\[
V \cong \bigoplus_{W_F \in \mathcal{F}_s} W_F
\]

Proof. By the proof of the previous lemma, \( \mathcal{F} \) is in general position iff there exists an \( \mathcal{F} \)-compatible basis for \( V \). This latter statement is equivalent to the existence of an isomorphism as in (4).

A multi-flag \( \mathcal{F} \) of \( V \) is a poset in a natural way; if \( V_1, V_2 \in \mathcal{F} \), then \( V_1 \leq V_2 \) as elements in \( \mathcal{F} \) iff \( V_1 \subseteq V_2 \) as subspaces of \( V \).

Definition 6. If \( \mathcal{F} \) is a multi-flag of \( V \), \( \mathcal{G} \) a multi-flag of \( W \), a morphism of multi-flags \((L, f) : \mathcal{F} \to \mathcal{G}\) consists of

- a linear map from \( L : V \to W \) and
- a map of posets \( f : \mathcal{F} \to \mathcal{G} \) such that
- for each \( U \in \mathcal{F} \), \( L(U) \subseteq f(U) \).

Definition 7. A morphism \((L, f)\) of multi-flags is closed if for each \( U \in \mathcal{F} \), \( L(U) = f(U) \) (in this case the inclusion of \( f \) is superfluous, and we will often write the morphism simply as \( L \)). \( L \) is inverse-closed if \( L^{-1}(U') \in \mathcal{F} \) for every \( U' \in \mathcal{G} \). It is bi-closed if it is both closed and inverse-closed.

We will denote the category of multi-flags and morphisms between such by \{multi-flags\}.

Given an arbitrary collection of subspaces \( T = \{W_\alpha\} \) of \( V \), the multi-flag generated by \( T \) is the smallest multi-flag containing each element of \( T \). It can be constructed as the closure of \( T \) under the operations i) inclusion of \{0\}; \( V \) and ii) taking intersections.

If \( L : V \to W \) is a linear map of vector spaces and \( \mathcal{F} \) is a multi-flag of \( V \), the multi-flag generated by \( \{L(U) \mid U \in \mathcal{F} \} \cup \{W\} \) is a multi-flag of \( W \) which we denote by \( L(\mathcal{F}) \) (or \( \mathcal{F} \) pushed forward by \( L \)). In the other direction, if \( \mathcal{G} \) is a multi-flag of \( W \), we write \( L^{-1}[\mathcal{G}] \) for the multi-flag \( \{L^{-1}(U) \mid U \in \mathcal{G} \} \cup \{\{0\}\} \) of \( V \) (i.e., \( \mathcal{G} \) pulled back by \( L \)); as intersections are preserved under taking inverse images, this will be a multi-flag once we include - if needed - \{0\}. Obviously \( L \) defines morphisms of multi-flags \( \mathcal{F} \xrightarrow{(L, f)} L(\mathcal{F}), L^{-1}[\mathcal{G}] \xleftarrow{(L, f)} \mathcal{G} \).
Definition 8. The bi-closure of $L$ (with respect to the multi-flags $\mathcal{F}$ and $\mathcal{G}$) is formed inductively as follows:

- Set $\mathcal{F}_0 = \mathcal{F}$, $\mathcal{G}_0 = \mathcal{G}$;
- For $n \geq 0$ let $\mathcal{F}_{n+1}$ be the multi-flag generated by $\mathcal{F}_n$ and $L^{-1}(\mathcal{G}_n)$;
- For $n \geq 1$ let $\mathcal{G}_n$ be the multi-flag generated by $\mathcal{G}_{n-1}$ and $L(\mathcal{F}_n)$;
- Let $\mathcal{F}_\infty = \lim_{\to} \mathcal{F}_n$, $\mathcal{G}_\infty = \lim_{\to} \mathcal{G}_n$.

The original morphism $L$ extends uniquely to a bi-closed morphism of multi-flags $L : \mathcal{F}_\infty \to \mathcal{G}_\infty$. Moreover, it is easily seen that $\mathcal{F}_\infty$ and $\mathcal{G}_\infty$ are the smallest multi-flags containing $\mathcal{F}$ and $\mathcal{G}$ respectively for which the linear transformation $L$ induces a bi-closed morphism $L : \mathcal{F}_\infty \to \mathcal{G}_\infty$.

Proposition 2. Let $L : \mathcal{F} \to \mathcal{G}$ be a bi-closed morphism of multi-flags, with $L_* : \mathcal{F}_* \to \mathcal{G}_*$ the induced map of associated gradeds. Then $L_*^{-1} : \text{im}(L_*) \to \mathcal{F}_*$ is a well-defined function on $\text{im}(L_*) \setminus \{0\}$.

Proof. Assume given $X \in \mathcal{F}$ with $\{0\} \neq [L(X)]_G \in \mathcal{G}_*$. Note that $L(X) \in \mathcal{G}$ as $L$ is closed, and so $X' := L^{-1}(L(X)) \in \mathcal{F}$ as $L$ is inverse-closed. There is an obvious inclusion of subspaces $X \subset X'$. If this were a proper inclusion, then as both $X, X' \in \mathcal{F}$, $X \in S_{\mathcal{F}}(X')$, implying that the map of quotients

$$X/SS_{\mathcal{F}}(X) \to X'/SS_{\mathcal{F}}(X')$$

is the zero map. However, $L(X) = L(X')$, implying

$$L_*([X_{\mathcal{F}}]) \xrightarrow{L_*(\iota)} L_*([X'_{\mathcal{F}}]) = [L(X)]_G$$

is the identity map. If $0 = L_*(\iota) = \text{Id} : [L(X)]_G \to [L(X)]_G$, then $[L(X)]_G = \{0\}$, contradicting the original assumption. Thus $X = X'$. Suppose $[Y_{\mathcal{F}}] \in \mathcal{F}_*$ with $L_*([Y_{\mathcal{F}}]) = [L(Y)]_G = [L(X)]_G \neq \{0\}$. By Proposition 1 both $[L(X)]_G$ and $[L(Y)]_G$ are without multiplicity, implying $L(X) = L(Y)$. But then $L^{-1}(L(X)) = L^{-1}(L(Y))$; by the above argument this implies $X = Y$. \hfill \Box

Theorem 1. Let $\mathcal{F}$ be a multi-flag in $\mathcal{V}$, $\mathcal{G}$ a multi-flag in $\mathcal{W}$, and $L : \mathcal{F} \to \mathcal{G}$ be a bi-closed morphism of multi-flags induced by a linear transformation $L : \mathcal{V} \to \mathcal{W}$. If $[X_{\mathcal{F}}] \in \mathcal{F}_*$ and $L_*([X_{\mathcal{F}}]) = [L(X)]_G \neq \{0\} \in \mathcal{G}_*$, then $L$ maps $X_{\mathcal{F}} = X/SS_{\mathcal{F}}(X)$ isomorphically to $L(X)/L(SS_{\mathcal{F}}(X)) = L(X)/SS_{\mathcal{G}}(L(X))$.

Proof. As $[L(X)]_G \neq \{0\}$ it must also hold that $[X_{\mathcal{F}}] \neq \{0\}$, and so both must be without multiplicity by Proposition 1. Clearly $L$ maps $X \subseteq V$ surjectively to its image $L(X) \subseteq W$. Because $L$ is closed, $L(SS_{\mathcal{F}}(X)) \subseteq SS_{\mathcal{G}}(L(X))$. As $L$ is also inverse-closed, $X = L^{-1}(L(X))$ by the proof of Proposition 2. Hence $Z \in SS_{\mathcal{G}}(L(X))$ implies $Z' = L^{-1}(Z) \in SS_{\mathcal{F}}(X)$, and as $Z \subseteq \text{im}(L)$, $L(Z') = Z$, yielding the equality $L(SS_{\mathcal{F}}(X)) = SS_{\mathcal{G}}(L(X))$. The fact that $X = L^{-1}(L(X))$ implies $K = \ker(L) \in \mathcal{F}$ is a proper subset of $X$, and thus $K \subseteq SS_{\mathcal{F}}(X)$. Taken together, these facts imply that the surjection $X_{\mathcal{F}} = X/SS_{\mathcal{F}}(X) \xrightarrow{L} L(X)/L(SS_{\mathcal{F}}(X)) = L(X)/SS_{\mathcal{G}}(L(X))$ must be an isomorphism. \hfill \Box

2.3 The local structure of a $\mathcal{C}$-module

Let $M$ be a $\mathcal{C}$-module.

Definition 9. A multi-flag of $M$ or $\mathcal{M}$-multi-flag is a functor $F : \mathcal{C} \to \{\text{multi-flags}\}$ which assigns

- to each $x \in \text{obj}(\mathcal{C})$ a multi-flag $F(x)$ of $M(x)$;
- to each $\phi \in \text{Hom}_C(x, y)$ a morphism of multi-flags $F(x) \to F(y)$.
To any $C$-module $M$ we may associate the multi-flag $F_0(M)$ which assigns to each $x \in \text{obj}(C)$ the multi-flag $\{ \emptyset, M(x) \}$ of $M(x)$. This is referred to as the trivial multi-flag of $M$. A multi-flag on $M$ is closed resp. inverse-closed resp. bi-closed if that property holds for each morphism in the module.

A $C$-module $M$ determines a bi-closed multi-flag on $M$. Precisely, the local structure $F(M)$ of $M$ is defined recursively at each $x \in \text{obj}(C)$ as follows: let $S_1(x)$ denote the set of morphisms of $C$ originating at $x$, and $S_2(x)$ the set of morphisms terminating at $x$, $x \in \text{obj}(C)$ (note that both sets contain $Id_x : x \to x$). Then

\begin{align*}
\text{LS1} & \quad F_0(M)(x) = \text{the multi-flag of } M(x) \text{ generated by } \\
\{ \ker(M(\phi) : M(x) \to M(y)) \}_{\phi \in S_1(x)} \cup \{ \text{im}(M(\psi)) : M(z) \to M(x) \}_{\psi \in S_2(x)};
\end{align*}

\begin{align*}
\text{LS2} & \quad \text{For } n \geq 0, F_{n+1}(M)(x) = \text{the multi-flag of } M(x) \text{ generated by }
\{ \ker(M(\phi) : M(x) \to M(y)) \}_{\phi \in S_1(x)} \cup \{ \text{im}(M(\psi)) : M(z) \to M(x) \}_{\psi \in S_2(x)};
\end{align*}

\begin{align*}
\text{LS3} & \quad F(M)(x) = \lim_{\to} F_n(M)(x).
\end{align*}

More generally, starting with a multi-flag $F$ on $M$, the local structure of $M$ relative to $F$ is arrived at in exactly the same fashion, but starting in LS1 with the multi-flag $F(x)$ at each object $x$. The resulting direct limit is denoted $F^F(M)$ (note: by definition, any multi-flag $F$ on $M$ must contain the trivial multi-flag $F_0(M)$). Thus the local structure of $M$ (without superscript) is the local structure of $M$ relative to the trivial multi-flag on $M$, which we will denote by $F_{-1}(M)(x)$ at each object. In almost all cases we will only be concerned with the local structure relative to the trivial multi-flag on $M$.

**Proposition 3.** For any multi-flag $F$ on $M$, $F^F(M)$ is the smallest bi-closed multi-flag on $M$ containing $F$.

**Proof.** This is an immediate consequence of property (LS2).

**Definition 10.** The local structure of a $C$-module $M$ is the functor $F(M) : C \to \{ \text{multi-flags} \}$, which associates to a vertex $x \in \text{obj}(C)$ the multi-flag $F(M)(x)$, and a morphism $\phi \in \text{Hom}_C(x,y)$ the induced bi-closed morphism of multi-flags $F(\phi) : F(M)(x) \to F(M)(y)$.

A key question arises as to whether the direct limit used in defining $F(M)(x)$ stabilizes at a finite stage. This motivates

**Definition 11.** The local structure on $M$ is locally stable at $x \in \text{obj}(C)$ iff there exists $N = N_x$ such that $F_n(M)(x) \to F_{n+1}(M)(x)$ is the identity map whenever $n \geq N$. It is stable if it is locally stable at each object. It is strongly stable if for all finite multi-flags $F$ on $M$ there exists $N = N(F)$ such that $F^F(M)(x) = F^F_N(M)(x)$ for all $x \in \text{obj}(C)$.

In almost all applications of this definition we will only be concerned with stability, not the related notion of strong stability.

The following result identifies the effect of a morphism in $M$ on the associated graded limit $F(M)_*$.

**Theorem 2.** Let $M$ be a $C$-module with stable local structure. Then for all $x,y,z \in \text{obj}(C)$, $W \in F(M)(x)$, $\phi \in \text{Hom}_C(z,x)$, and $\psi \in \text{Hom}_C(x,y)$

\begin{enumerate}
\item The morphisms of $M$ induce well-defined maps of associated graded sets

$M(\psi)_* : F(M)_*(x) \to F(M)_*(y)$

while their inverses yield multi-functions of associated graded sets

$M(\phi)^{-1}_* : \text{im}(M(\phi)_*) \to F(M)_*(z)$

with $M(\phi)^{-1}_*$ defining a function on $\text{im}(M(\phi)_*) \setminus \{0\}$.
\end{enumerate}
2. \( M(\psi)(W) \in \mathcal{F}(M)(y) \), and either \( M(\psi)(W_{\mathcal{F}(M)(z)}) = \{0\} \), or \( M(\psi) : W_{\mathcal{F}(M)(x)} \xrightarrow{\sim} M(\psi)(W_{\mathcal{F}(M)(x)}) = (M(\psi)(W))_{\mathcal{F}(M)(y)} \);

3. if \( W_{\mathcal{F}} \neq \{0\} \), then either \( W_{\mathcal{F}(M)(x)} \notin \text{im}(M(\phi)_*) \), or there is a unique element \( U \in M(\phi)_*^{-1}(\text{im}(M(\phi)_*)) \subset \mathcal{F}(M)_*(z) \) with \( M(\phi) : U_{\mathcal{F}(M)(z)} \xrightarrow{\sim} W_{\mathcal{F}(M)(x)} \).

**Proof.** Given that \( \mathcal{F}(M) \) is bi-closed with respect to each morphism in \( \mathcal{C} \), these three statements are an immediate consequence of Theorem[1].

We will use the notion of general position, discussed above, to define excess.

**Excess for local structure** The *excess* of a \( \mathcal{C} \)-module \( M \) is

\[
e(M) = \sum_{x \in \text{obj}(\mathcal{C})} e(\mathcal{F}(M)(x))
\]

**Definition 12.** The local structure \( \mathcal{F}(M) \) is in *general position* iff \( e(M) = 0 \).

Note that as \( M(x) \) is finite-dimensional for each \( x \in \text{obj}(\mathcal{C}) \), \( \mathcal{F}(M)(x) \) must be locally stable at \( x \) if it is in object general position (although general position is a much more restrictive property).

### 2.4 Inner product structures

It will be useful to consider two refinements of the category \( (\text{vect}/k) \).

1. \((\text{WIP}/k)\), the category whose objects are inner product (IP)-spaces \( V = (V, <, >_V) \) and whose morphisms are linear transformations (no compatibility required with respect to the inner product structures on the domain and range);

2. \((\text{PIP}/k)\), the wide partial subcategory of \((\text{WIP}/k)\) whose morphisms

   \[ L : (V, <, >_V) \rightarrow (W, <, >_W) \]

   are partial isometries; that is, \( \tilde{L} : \ker(L)^\perp \rightarrow W \) is an isometric embedding, where \( \ker(L)^\perp \subset V \) denotes the orthogonal complement of \( \ker(L) \subset V \) in \( V \) with respect to the inner product \( <, >_V \), and \( \tilde{L} \) is the restriction of \( L \) to \( \ker(L)^\perp \).

[Note of explanation: A partial subcategory (aka pre-subcategory) \( \mathcal{C} \) of a category \( \mathcal{D} \) refers to i) a collection of objects \( \text{obj}(\mathcal{C}) \subset \text{obj}(\mathcal{D}) \) and morphisms \( \text{hom}(\mathcal{C}) \subset \text{hom}(\mathcal{D}) \) for which the composition of morphisms in \( \mathcal{C} \), which exists in \( \mathcal{D} \), may not lie in \( \mathcal{C} \). A fundamental complication in dealing with partial isometries vs isometries is that they need not be closed under composition - a good discussion of this issue is given in [11]. Partial categories arise in other contexts; for example partial monoids as defined in [12] may be viewed as a partial category with a single object.]

There are obvious transformations

\[
(\text{IP}/k) \xrightarrow{\iota_{\text{IP}}} (\text{WIP}/k) \xrightarrow{\text{proj}_{\text{IP}}} (\text{vect}/k)
\]

where the first map is the inclusion which is the identity on objects, while the second map forgets the inner product on objects and is the identity on transformations between two fixed objects. We will call \( \mathcal{D} \) an *IP-category* if \( \mathcal{D} \) is a subcategory of the partial category \((\text{PIP}/k)\).

**Definition 13.** Given a \( \mathcal{C} \)-module \( M : \mathcal{C} \rightarrow (\text{vect}/k) \) a *weak inner product* on \( M \) is a factorization

\[
M : \mathcal{C} \rightarrow (\text{WIP}/k) \xrightarrow{\text{proj}_{\text{IP}}} (\text{vect}/k)
\]

while an *inner product* on \( M \) is a further factorization through (or lift to) an IP category \( \mathcal{D} \subset (\text{IP}/k) \):

\[
M : \mathcal{C} \rightarrow \mathcal{D} \rightarrow (\text{IP}/k) \xrightarrow{\iota_{\text{IP}}} (\text{WIP}/k) \xrightarrow{\text{proj}_{\text{IP}}} (\text{vect}/k)
\]
A WTPC-module will refer to a C-module $M$ equipped with a weak inner product, while an IPC-module is a C-module that is equipped with an actual inner product, in the above sense. As any vector space admits a (non-unique) inner product, we see that

**Proposition 4.** Any C-module $M$ admits a non-canonical representation as a WTPC-module.

The question as to whether a C-module $M$ can be represented as an IPC-module, however, is a much more delicate issue.

Given a C-module $M$ and a morphism $\phi \in Hom_C(x, y)$, we set $K_M := \ker(\phi : M(x) \to M(y))$. We note that a C-module $M$ is an IPC-module, iff

- for all $x \in \text{obj}(C)$, $M(x)$ comes equipped with an inner product $\langle , \rangle_x$;
- for all $\phi \in Hom_C(x, y)$, the map $\tilde{\phi} : K_M^\perp \to M(y)$ is an isometric embedding, where $\tilde{\phi}$ denotes the restriction of $\phi$ to $K_M^\perp$ = the orthogonal complement of $K_M \subset M(x)$ with respect to the inner product $\langle , \rangle_x$. In other words,

$$\langle \phi(v), \phi(w) \rangle_y = \langle v, w \rangle_x, \quad \forall v, w \in K_M^\perp$$

**Definition 14.** Given a multi-flag $\mathcal{F}$ on an inner product space $(V, \langle , \rangle)$, we will say the inner product is $\mathcal{F}$-compatible - and that $V$ is an FIP-space - if every q-isomorphism $\iota : (W, SS_F(W)) \to (W', SS_F(W'))$ induces, upon restriction to relative orthogonal complements, an isomorphism

$$\tilde{W}_\mathcal{F} := (SS_F(W) \subset W)^\perp \xrightarrow{\iota} (SS_F(W'), W')^\perp = \tilde{W}_\mathcal{F}'$$

An FIPC-module $M$ is an IPC-module for which the inner product $\langle , \rangle_x$ on $M(x)$ is $\mathcal{F}(M)(x)$-compatible for each $x \in \text{obj}(C)$.

For an arbitrary multi-flag on $M$ that is not necessarily biclosed, admitting an FIPC-structure is in general a strictly stronger condition than admitting an IPC-structure. However these notions turn out to be equivalent for the local structure of a C-module $M$.

**Lemma 3.** If $M$ is an IPC-module with stable local structure, then it is an FIPC-module.

**Proof.** Let $W \in \mathcal{F}(M)(x)$. Then $[W] = \{0\}$ iff $(SS_F(W) \subset W)^\perp = \{0\}$, for which the condition in (3) is trivially satisfied. On the other hand, if $[W] \neq \{0\}$, Proposition implies the q-isomorphism $\iota$ in the above definition must be the identity map, again implying $\mathcal{F}$-compatibility for trivial reasons.

For this reason, in what follows we will only need to assume an IPC-structure on $M$, as the $\mathcal{F}$-compatibility of that structure comes along for free. Assume now that $M$ is equipped with such a structure. In this case all morphisms $M(\phi) : M(x) \to M(y)$ are partial isometries which preserve relative orthogonal complements:

$$\tilde{W}_\mathcal{F} = (SS_{\mathcal{F}(M)(x)}(W) \subset W)^\perp \xrightarrow{M(\phi)} (SS_{\mathcal{F}(M)(y)}(M(\phi)(W)) \subset M(\phi)(W))^\perp$$

This map is either 0 or an isometry by the same argument appearing in the proof of Theorem moreover, these relative orthogonal complements remain invariant under q-isomorphisms. A similar analysis applies for inverse images. Consequently, we can lift the arguments given in Theorem from subquotients to orthogonal complements, yielding

**Theorem 3.** Let $M$ be an IPC-module with stable local structure. Then for all $x, y, z \in \text{obj}(C)$, $W \in \mathcal{F}(M)(x)$, $\phi \in Hom_C(z, x)$, and $\psi \in Hom_C(x, y)$

1. The morphism $M(\psi)$ induces a well-defined map of associated graded sets

$$M(\psi)_* : \mathcal{F}(M)_*(x) \to \mathcal{F}(M)_*(y)$$

while the inverse of $M(\phi)_*$ yields a multi-function

$$M(\phi)^{-1}_* : \text{im}(M(\phi)_*) \to \mathcal{F}(M)_*(x)$$

which is a function on $\text{im}(M(\phi)_*) \setminus \{0\}$.
2. \( M(\psi)(W) \in \mathcal{F}(M)(y) \), and either \( M(\psi)(\widetilde{W}_{\mathcal{F}(M)(x)}) = \{0\} \), or \( M(\psi) : \widetilde{W}_{\mathcal{F}(M)(x)} \xrightarrow{\cong} M(\psi)(\widetilde{W}_{\mathcal{F}(M)(x)}) = (M(\psi)(W))_{\mathcal{F}(M)(y)} \);

3. if \( [W_{\mathcal{F}(M)(x)}] \neq \{0\} \), then either \( [W_{\mathcal{F}(M)(x)}] \notin \text{im}(M(\phi)_*) \), or there is a unique element \( U = M(\phi)^{-1}(W) \in \mathcal{F}(M)(z) \) with \( M(\phi)_*([U_{\mathcal{F}(M)(z)}]) = [W_{\mathcal{F}(M)(x)}] \) and \( M(\phi) : \widetilde{U}_{\mathcal{F}(M)(z)} \xrightarrow{\cong} \widetilde{W}_{\mathcal{F}(M)(x)} \).

\[ \text{Proof. Same as above.} \]

\[ \text{2.5 Sums, Tensors, and the Algebraic Künneth Theorem} \]

Let \( \Delta \mathcal{C} \) be the full subcategory of \( \mathcal{C} \times \mathcal{C} \) whose object set is the diagonal of \( \mathcal{C} \times \mathcal{C} \). We define the direct sum of 2 basepointed modules \( M : \mathcal{C}_* \to (\text{vect}/k)_* \) and \( N : \mathcal{C}_* \to (\text{vect}/k)_* \) to be the functor \( M \oplus N : \Delta \mathcal{C}_* \to (\text{vect}/k)_* \) defined componentwise on objects and morphisms. Due to the necessity of introducing zero maps, this definition is reminiscent of the definition of the Whitney Sum for vector bundles. We define the direct sum of multiflags to be the multiflag of pairwise direct sums, i.e. \( \mathcal{F} \oplus \mathcal{G} := \{U \oplus W \mid U \in \mathcal{F}, W \in \mathcal{G}\} \)

While the local structure for basepointed modules does not distribute over direct sums on the level of multiflags, it does at the level of the associated graded. In particular, the associated graded of a stable local structure distributes over direct sums.

\[ \text{Lemma 5. Let } M, N \text{ be basepointed as above. For } i \geq -1, \text{ the following hold} \]

\[ \begin{align*}
1. & \quad \mathcal{F}_i(M \oplus N)(x) \subseteq \mathcal{F}_i(M)(x) \oplus \mathcal{F}_i(N)(x) \\
2. & \quad \mathcal{F}_i(M \oplus N)_*(x) = (\mathcal{F}_i(M)(x) \oplus \mathcal{F}_i(N)(x))_* \\
3. & \quad \mathcal{F}(M \oplus N)_*(x) = (\mathcal{F}(M)(x) \oplus \mathcal{F}(N)(x))_* = \mathcal{F}(M)_*(x) \oplus \mathcal{F}(N)_*(x) \text{ if the local structures of } M \text{ and } N \text{ are stable.}
\end{align*} \]

\[ \text{Proof. 1 follows from the construction of direct sums of modules and of local structure. 2 follows from 1 by applying Lemma 4. 3 is a restatement of 2 when } i = \infty. \]

We now return to the normal unbasepointed setting.

\[ \text{Definition 15. The tensor product of two modules } M : \mathcal{C} \to (\text{vect}/k) \text{ and } N : \mathcal{D} \to (\text{vect}/k) \text{ is the functor } M \otimes N : \mathcal{C} \times \mathcal{D} \to (\text{vect}/k) \text{ given componentwise on both objects and morphisms. In the same vein, we define the tensor product of multiflags to be the multiflag of pairwise tensor products, i.e. } \mathcal{F} \otimes \mathcal{G} := \{U \otimes W \mid U \in \mathcal{F}, W \in \mathcal{G}\}. \]

\[ \text{Lemma 6. The associated graded distributes over tensor products, i.e. } (\mathcal{F} \otimes \mathcal{G})_* = (\mathcal{F})_* \times (\mathcal{G})_* \]

\[ \text{Proof. We first note that} \]

\[ \frac{U \otimes W}{SS_{\mathcal{F} \otimes \mathcal{G}}(U \otimes W)} = \frac{U}{SS_{\mathcal{F}}U} \otimes \frac{W}{SS_{\mathcal{G}}W}. \]

\[ \text{As} \]

\[ \frac{U}{SS_{\mathcal{F}}U} \otimes \frac{W}{SS_{\mathcal{G}}W} = \frac{U \otimes W}{(U \otimes SS_{\mathcal{G}}W) + (SS_{\mathcal{F}}U \otimes W)} \]
the desired equality follows from
\[ SS_{F \otimes G}(U \otimes W) = (U \otimes SS_G W) + (SS_F U \otimes W). \]
In a similar vein, these equalities show that \((U, SS_F(U)) \hookrightarrow (U', SS_F(U'))\) and \((W, SS_G(W)) \hookrightarrow (W', SS_G(W'))\) are q-isomorphisms iff \((U \otimes W, SS_{F \otimes G}(U \otimes W)) \hookrightarrow (U' \otimes W', SS_{F \otimes G}(U' \otimes W'))\) is one as well. The result follows.

**Theorem 4.** (K"unneth Theorem for Local Structure) The local structure of \(M \otimes N\) distributes over the tensor product at every finite stage (hence so does the stable local structure) both at the level of multflags and at the level of the associated graded. i.e. for all \(i \geq -1\) and \(i = \infty\),

1. \(F_i(M \otimes N)(x, x') = F_i(M)(x) \otimes F_i(N)(x')\)
2. \(F_i(M \otimes N)_*(x, x') = F_i(M)_*(x) \times F_i(N)_*(x')\)

**Proof.** Proceed by induction. When \(i = -1\), we have:
\[ F_{-1}(M \otimes N)(x, x') = \{0, M(x) \otimes N(x')\} = F_{-1}(M)(x) \otimes F_{-1}(N)(x') \]
Suppose the claim holds for \(i\). Let \(\phi_{ab} \in Hom_C(a, b)\) denote an arbitrary morphism. Then \(F_{i+1}(M)(x) \otimes F_{i+1}(N)(x')\) is generated (via closure under intersections) by the following four sets of images and inverse images:
\[ \{M(\phi_{xx})(U_z) \otimes N(\phi_{x'x'})(W_z')\}\]
\[ \{M(\phi_{zz})(U_y) \otimes N(\phi_{zz}^{-1})(W_y)\}\]
\[ \{M(\phi_{xy})^{-1}(U_y) \otimes N(\phi_{xy}^{-1})(W_y)\}\]
\[ \{M(\phi_{yy})^{-1}(U_y) \otimes N(\phi_{yy}^{-1})(W_y)\}\]
where \(U_a \in F_i(M)(a)\) and \(W_b \in F_i(N)(b)\). By the induction hypothesis, \(F_{i+1}(M \otimes N)(x, x')\) is generated in the same manner by the following two sets:
\[ \{M(\phi_{xx})(U_z) \otimes N(\phi_{x'x'})(W_z')\}\]
\[ \{M(\phi_{xy})^{-1}(U_y) \otimes N(\phi_{xy}^{-1})(W_y)\}\]
These two descriptions clearly imply \(F_{i+1}(M \otimes N)(x, x') \subseteq F_{i+1}(M)(x) \otimes F_{i+1}(N)(x')\). Inclusion in the other direction follows from the equalities
\[ M(\phi_{xx})(U_z) \otimes N(\phi_{x'x'}^{-1})(W_y) = (M(\phi_{xx})(U_z) \otimes N(x')) \cap (M(x) \otimes N(\phi_{x'x'}^{-1})(W_y))\]
\[ M(\phi_{yy})^{-1}(U_y) \otimes N(\phi_{xy}^{-1})(W_y) = (M(\phi_{yy})^{-1}(U_y) \otimes N(x')) \cap (M(x) \otimes N(\phi_{x'x'}^{-1})(W_y))\]
Statement 2. follows from 1. by the previous lemma.

## 3 Coverings

We construct - for any \(C\)-module \(M\) - a quasi-tame module (defined below) which covers it in an appropriate sense. We also establish a sufficient condition for the quasi-tame covering to be b-tame (also defined below).
### 3.1 The quasi-tame covering of a \(C\)-module

Let \(M\) be a \(C\)-module with stable local structure.

**Definition 16.** Fixing an object \(x \in \text{obj}(C)\) and an element \(\{0\} \neq \{W_{\mathcal{F}(M)(x)}\} \in \mathcal{F}(M)_*(x)\) with unique representative \(W_{\mathcal{F}(M)(x)} \in \mathcal{F}(M)(x)\), \(\text{Supp}(W_{\mathcal{F}(M)(x)})\) is the smallest \(C\)-module satisfying

- \(\mathcal{F}(\text{Supp}(W_{\mathcal{F}(M)(x)}))\) is a sub-\(C\)-set of \(\mathcal{F}(M)_*\)
- \(W_{\mathcal{F}(M)(x)} \subseteq \text{Supp}(W_{\mathcal{F}(M)(x)})(x)\)

Precisely, for each object \(y \in \text{obj}(C)\) and element \(U_{\mathcal{F}(M)(y)} \in \mathcal{F}(M)(y)\), \(U_{\mathcal{F}(M)(y)} \in \mathcal{F}(\text{Supp}(W_{\mathcal{F}(M)(x)}))(y)\) iff \(U_{\mathcal{F}(M)(y)}\) is connected to \(W_{\mathcal{F}(M)(x)}\) by a zig-zag sequence of isomorphisms in \(\mathcal{F}(M)_*\) induced by restrictions to subquotients of morphisms in the \(C\)-module \(M\). We refer to a \(C\)-module arising in this fashion (i.e., isomorphic to \(\text{Supp}(W_{\mathcal{F}(M)(x)})\) for some \(\{0\} \neq \{W_{\mathcal{F}(M)(x)}\}\)) as a **quasi-block**.

Suppose \(x \neq y \in \text{obj}(\text{Supp}(W_{\mathcal{F}(M)(x)}))\) and \(U_{\mathcal{F}(M)(y)}\) is the unique representative of the equivalence class \([U_{\mathcal{F}(M)(y)}]\) in \(\mathcal{F}(M)_*(y)\) connected by a zig-zag sequence of isomorphisms to \([W_{\mathcal{F}(M)(x)}]\) \(\in \mathcal{F}(M)_*(x)\). There is then a canonical identification of \(C\)-modules

\[
\text{Supp}(W_{\mathcal{F}(M)(x)}) \cong \text{Supp}(U_{\mathcal{F}(M)(y)})
\]

When two \(C\)-quasi-blocks are identified in this fashion (including the case \(x = y\)), we will refer to them as being **quasi-block-equivalent**, indicated by the symbol \(\sim\).

**Definition 17.** A \(C\)-module is **quasi-tame** if it can be written as a finite direct sum of \(C\)-quasi-blocks. The quasi-tame cover of \(M\) is the \(C\)-module

\[
\mathcal{QTC}(M) := \left( \bigoplus_{x \in \text{obj}(C)} \left( \bigoplus_{\{0\} \neq \{W_{\mathcal{F}(M)(x)}\} \in \mathcal{F}(M)_*} \text{Supp}(W_{\mathcal{F}(M)(x)}) \right) \right) / \sim
\]

**Theorem 5.** There is a natural isomorphism \(P(M)_* : \mathcal{F}(\mathcal{QTC}(M))_* \cong \mathcal{F}(M)_*\) of associated graded local structures given on elements by

\[
\mathcal{F}(\mathcal{QTC}(M))_*(x) \ni \{W_{\mathcal{F}(M)(x)}\} \xrightarrow{P(M)_*} [W_{\mathcal{F}(M)(x)}] \in \mathcal{F}(M)_*(x)
\]

In addition, \(\mathcal{QTC}(M)\) has stable local structure, and is in general position: \(\epsilon(\mathcal{QTC}(M)) = 0\).

**Proof.** The first statement follows directly from the construction of \(\mathcal{QTC}(M)\). Any quasi-block associated to a \(C\)-module with stable local structure will have stable local structure, and has vanishing excess (again, by construction). Therefore \(\mathcal{QTC}(M)\), which is a finite direct sum of quasi-blocks, satisfies the same properties.

### 3.2 Tame coverings

A \(C\)-module \(N\) is a **block** (referred to as a \(C\)-block) if whenever \(N(x) \neq \{0\} \neq N(y)\), there is a zig-zag sequence of isomorphisms in \(N\) connecting \(N(x)\) and \(N(y)\). We say a \(C\)-module is **b-tame** if it is a finite direct sum of blocks. If every \(\text{Supp}(W_{\mathcal{F}(M)(x)})\) in (7) above is a block, the quasi-tame cover is b-tame, and referred to as the **b-tame cover**, written \(\mathcal{T}(M)\).

**Lemma 7.** Suppose \(M\) is a \(C\)-module such that for all \(x \in \text{obj}(C)\), \(\{0\} \neq \{W_{\mathcal{F}(M)(x)}\} \in \mathcal{F}_*(M(x))\) and zig-zag composition of isomorphisms \(\mathcal{F}(M)(x) \ni W_{\mathcal{F}(M)(x)} \overset{U_{\mathcal{F}(M)(x)}}{\rightarrow} U_{\mathcal{F}(M)(x)} \in \mathcal{F}(M)(x)\) (induced by morphisms in \(M\)), \(W_{\mathcal{F}(M)(x)} = U_{\mathcal{F}(M)(x)}\). Then each quasi-block in the quasi-tame cover is a block, and so \(\mathcal{QTC}(M) = \mathcal{T}(M)\).

**Proof.** The condition implies that \([W_{\mathcal{F}(M)(x)}]\) cannot be connected by a zig-zag sequence of isomorphisms to an element \([U_{\mathcal{F}(M)(x)}]\) \(\neq [W_{\mathcal{F}(M)(x)}]\), forcing \(\text{Supp}(W_{\mathcal{F}(M)(x)})\) to satisfy the precise conditions needed to be a block. The result follows.
It is currently unknown if the isomorphism $P_\ast(M)$ of associated graded local structures is always induced by a morphism of $\mathcal{C}$-modules $\mathcal{QT}C(M) \rightarrow M$ (although it is if $M$ is equipped with an $\mathcal{T}\mathcal{P}\mathcal{C}$-structure, as discussed below). Nevertheless, the following is clear

**Lemma 8.** If $M$ is a quasi-tame $\mathcal{C}$-module, then $\mathcal{QT}C(M) = M$, with the isomorphism $P_\ast(M)$ induced by this equality. Moreover if $M$ is b-tame, then $\mathcal{QT}C(M) = T\mathcal{C}(M)$.

### 3.3 Coverings of $\mathcal{T}\mathcal{P}\mathcal{C}$-modules

We assume now that $M$ is an $\mathcal{T}\mathcal{P}\mathcal{C}$-module. In this case for each $x \in \text{obj}(\mathcal{C})$ and $(W, SS_{\mathcal{F}(M)(x)}(W)) \in \mathcal{F}(M)(x)^p$, the canonical isomorphism

$$\left( SS_{\mathcal{F}(M)(x)}(W) \subset W \right)^\perp \cong W_{\mathcal{F}(M)(x)}$$

provides an embedding $W_{\mathcal{F}(M)(x)} \hookrightarrow M(x)$ compatible with respect to taking direct and inverse images of morphisms of $M$ mapping from or to $M(x)$. The result, again for each $x \in \text{obj}(\mathcal{C})$ and $[W_{\mathcal{F}(M)(x)}] \in \mathcal{F}(M)_*(x)$, is an extension of $\mathbb{K}$ to an embedding of $\mathcal{C}$-modules

$$\text{Supp}(W_{\mathcal{F}(M)(x)}) \hookrightarrow M$$

**Theorem 6.** For an $\mathcal{T}\mathcal{P}\mathcal{C}$-module $M$, the isomorphism $P_\ast(M)$ of associated graded local structures in Theorem 4 is induced by a surjection of $\mathcal{C}$-modules

$$P_\ast(M) : \mathcal{QT}C(M) \rightarrow M$$

This surjection is an isomorphism (i.e., $M$ is quasi-tame) iff $e(M) = 0$. Moreover, if in addition all of the quasi-block summands of $\mathcal{QT}C(M)$ are blocks, then $M$ is b-tame iff $e(M) = 0$.

**Proof.** Choosing a representative for each $B$-equivalence class and summing these inclusions as in (7) produces a the required surjection $P_\ast(M)$ of $\mathcal{C}$-modules by the above. The total dimension of the kernel $\ker(P_\ast(M))$ is exactly the object excess $e(M)$, implying the second statement. The final statement follows from the results of the previous section.

### 3.4 Decomposition of 1-dimensional persistence modules via local structure

We adopt the following (slightly) restrictive definition.

**Definition 18.** A finite 1-dimensional persistence module over $k$ is a functor $M : \mathbb{N} \cong \mathcal{C} \rightarrow (\text{vect}/k)$ where $\mathbb{N}$ denotes the categorical realization of the totally ordered set $1 < 2 < 3 < \cdots < n$.

We will need some additional terminology.

**Definition 19.** Given $1 \leq i < j \leq n$, the interval $[i,j]$ is the full subcategory of $\mathbb{N}$ on objects $\{k \in \mathbb{N} \mid i \leq k \leq n\}$ (using this notation, $\mathbb{N}$ can alternatively be written as $[1,n]$). If $M$ is an $\mathbb{N}$-module, and $1 \leq i < j \leq n$, $M$ is an $[i,j]$-block if it is a block, and $M(j) \neq \{0\}$ if $i \leq j \leq k$. $M$ is an interval module of type $[i,k]$ if it is an $[i,k]$-block and $\dim_k(M(j)) \leq 1$ for all $1 \leq j \leq n$.

It is a consequence of Gabriel’s Theorem 9 that persistence modules decompose as a direct sum of interval submodules and that the decomposition is unique up to reordering.

In this section we give an alternative proof of this result using local structure.

**Lemma 9.** Any persistence module $M : \mathbb{N} \rightarrow (\text{vect}/k)$ admits an inner product structure, hence can be realized as an $\mathcal{T}\mathcal{N}$-module.
Proof. The first statement is true trivially for \( n = 1 \). For \( 1 \leq i \leq j \leq n \) denote the unique morphism from \( M(i) \) to \( M(j) \) by \( \phi_{i,j} \). By induction, we may assume that we have fixed an inner product structure on the sub-module \( M' := \{ M(2) \xrightarrow{\phi_{1,2}} M(3) \to \cdots \xrightarrow{\phi_{(n-1),n}} M(n) \} \). For \( 1 < j \leq n \) let \( \ker(i,j) := \ker(\phi_{i,j}) \). This yields the semiflag \( \ker(1,2) \subseteq \ker(1,3) \subseteq \cdots \subseteq \ker(1,n) \subseteq M(1) \).

Next, choose a direct sum decomposition
\[
M(1) \cong \ker(1,2) \oplus \ker(1,3)/\ker(1,2) \oplus \ker(1,4)/\ker(1,3) \oplus \ldots \oplus \ker(1,n)/\ker(1, n-1) \oplus M(1)/\ker(1,n)
\]

We observe that \( \ker(1,j)/\ker(1,j-1) \) maps isomorphically to its image in \( M(j-1) \) under \( \phi_{1,(j-1)} \), and then to \( \{0\} \subseteq M(j) \) under \( \phi_{(j-1),j} \). On this summand choose the inner product to be that induced by the one on \( M(j-1) \) via the embedding \( \phi_{1,(j-1)} \); similarly equip \( M(1)/\ker(1,n) \) with the one induced by its embedding via \( \phi_{1,n} \) into \( M(n) \). Choose an arbitrary inner product on \( \ker(1,2) \). This collection of inner products extends to a unique inner product on \( M(1) \) which agrees with the previously defined inner product on each summand, and where the summands themselves are mutually orthogonal. By construction, each \( \phi_{1,j} \) will be a partial isometry, yielding the requisite extension of the \( \mathcal{IP} \)-structure on \( M' \) to \( M \).

Lemma 10. If a finite 1-dim persistence module \( M : \mathfrak{n} \to \text{vect} / k \) is a quasi-block, it is a block. Moreover, if \( M \) is a block, then \( \text{supp}(M) = [k, l] \) for some \( 1 \leq k \leq l \leq n \).

Proof. Associate the symbol \( x_i \) with the morphism \( \phi_{i,i+1}, 1 \leq i \leq n-1 \), and let \( F_{n-1} \) be the free group on \( x_1, \ldots, x_{n-1} \). Via this association, we see that a zig-zag sequence of isomorphisms beginning and ending at \( M(k) \) corresponds in \( F_{n-1} \) to a word \( w \) which in reduced form equals \( e \) (\( e \) denoting the identity element). This implies the self-map \( M(i) \xrightarrow{\cong} M(i) \) must be the identity, and hence the quasi-block is a block. The support of this block must be a connected subcategory of \( \mathfrak{n} \), implying it is an interval subcategory of the form \([k, l]\) as claimed.

For convenience we will abbreviate \( \text{im}(\phi_{i,j}) \) as \( \text{im}(i,j) \). A fact we will exploit extensively in the following discussion is that at each object the semiflag of images and kernels are each nested:
\[
\text{im}(0,k) \subseteq \cdots \subseteq \text{im}(k,k)
\]
\[
\ker(k,k) \subseteq \cdots \subseteq \ker(k,n)
\]

Let
\[
\mathcal{I}(k) := \left\{ \left[ \sum_{i=1}^{m} [\text{im}(i,k) + \ker(k,j_i)] \right] \mid m \in \mathbb{N}, \ i_i \in [0,k], \ j_i \in [k,n+1] \right\}
\]
\[
\mathcal{S}(k) := \left\{ \left[ \sum_{i=1}^{m} [\text{im}(i,k) \cap \ker(k,j_i)] \right] \mid m \in \mathbb{N}, \ i_i \in [0,k], \ j_i \in [k,n+1] \right\}
\]

By construction, \( \mathcal{S}(k) \) is closed and \( \mathcal{I}(k) \) is inverse closed. Since the semiflags of kernels and images are nested, each element of \( \mathcal{I}(k) \) and \( \mathcal{S}(k) \) can be written in multiple ways. We will want a way for eliminating obviously redundant subspaces. For \( i \leq j \in \mathbb{N} \), \( \mathcal{S}[i,j] \) will denote the set of integers \( i \leq k \leq j \) equipped with its standard total ordering. This induces the usual partial ordering on the Cartesian product \( \mathcal{S}[i,j] \times \mathcal{S}[i',j'] \). We recall that an \textit{anti-chain} in a poset is a subset in which each distinct pair of elements are unrelated (incomparable).

We now consider the following subcollections:
\[
\mathcal{I}_r(k) := \left\{ \left[ \sum_{i=1}^{m} [\text{im}(i,k) + \ker(k,j_i)] \right] \mid m \in \mathbb{N}, \ (i_i, j_i) \in [0,k] \times [k,n+1], \ \{(i_i, j_i)\}_{i=1}^{m} \text{ is an antichain} \right\}
\]
\[
S_r(k) := \left\{ \sum_{i=1}^{m} [\text{im}(i, k) \cap \ker(k, j_i)] \mid m \in \mathbb{N}, \ (i, j_i) \in [0, k] \times [k, n + 1],
\right. \\
\left. \{(i, j_i)\}_{i=1}^m \text{ is an antichain} \right\}
\]

**Lemma 11.** Let \( V \) be a vector space and

\[
A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m \subseteq X
\]

\[
X \supseteq B_1 \supseteq B_2 \supseteq \cdots \supseteq B_m
\]

a pair of semi-flags in \( V \), with \( m \geq 2 \). Then we can interchange unions and intersections in the following shifted manner:

\[
\sum_{i=1}^{m} A_i \cap B_i = B_1 \cap \left( \bigcap_{i=1}^{m-1} (A_i + B_{i+1}) \right) \cap A_m
\]

**Proof.** For \( m = 2 \) the equality reads

\[
A_1 \cap B_1 + A_2 \cap B_2 = B_1 \cap (A_1 + B_2) \cap A_2
\]

Both \( A_1 \cap B_1 \) and \( A_2 \cap B_2 \) are contained in \( B_1 \cap (A_1 + B_2) \cap A_2 \), and so their sum is as well. For inclusion in the other direction, we note that an element \( v \) of the 3-fold intersection appearing on the right can be written in three different ways:

\[
v = b_1 = a_1 + b_2 = a_2
\]

for elements \( a_i \in A_i, b_i \in B_i \). Then \((b_1 - b_2) = a_1 \in A_1 \Rightarrow (b_1 - b_2) \in A_1 \cap B_1\) and \(b_2 = (a_2 - a_1) \in A_2 \Rightarrow b_2 \in A_2 \cap B_2\), yielding \( v = (b_1 - b_2) + b_2 \in A_1 \cap B_1 + A_2 \cap B_2\).

Assume \( m > 2 \). By induction on \( m \) we have

\[
\sum_{i=1}^{m} A_i \cap B_i = \left( \sum_{i=1}^{m-1} A_i \cap B_i \right) + A_m \cap B_m = B_1 \cap \left( \bigcap_{i=1}^{m-2} (A_i + B_{i+1}) \right) \cap A_m - 1 + A_m \cap B_m
\]

Set \( W_1 := B_1 \cap \left( \bigcap_{i=1}^{m-2} (A_i + B_{i+1}) \right) \cap A_{m-1}, W_2 := A_m \cap B_m \), and \( W_3 := B_1 \cap \left( \bigcap_{i=1}^{m-1} (A_i + B_{i+1}) \right) \cap A_m \). There are evident inclusions \( W_1 \subseteq W_3, W_2 \subseteq W_3 \), implying \( W_1 + W_2 \subseteq W_3 \). For the other direction, we see (as above) that an element \( v \in W_3 \) can be expressed in \( (m + 1) \) different ways

\[
W_3 \ni v = b_1 = a_1 + b_2 = a_2 + a_3 = \cdots = a_{m-1} + b_m = a_m, \quad a_i \in A_i, b_j \in B_j
\]

Let \( v_1 = b_1 - b_m \). From \( \text{(11)} \) we have

\[
v_1 = (b_1 - b_m) = a_1 + (b_2 - b_m) = a_2 + (b_3 - b_m) = \cdots = a_{m-1} \Rightarrow v_1 \in B_1 \cap \left( \bigcap_{i=1}^{m-2} (A_i + B_{i+1}) \right) \cap A_{m-1}
\]

and

\[
b_m = a_m - a_{m-1} \in A_m \Rightarrow b_m \in A_m \cap B_m
\]

yielding \( v = v_1 + b_m \in W_1 + W_2 \).

**Lemma 12.** \( S(k) = S_r(k) = I_r(k) = I(k) \)

**Proof.** The middle equality follows by applying the previous lemma. One inclusion of the two outer equalities is obvious. The other follows by reducing non-antichains to antichains.

**Proposition 5.** For all \( k \in [1, n] \), the local structure of \( M \) stabilizes after 1 stage, and is \( I(k) = S(k) \).

**Proof.** The middle equality follows by applying the previous lemma. One inclusion of the two outer equalities is obvious. The other follows by reducing non-antichains to antichains.

**Hence** \( c(M) = 0 \).
Proof. Since $\mathcal{F}_0(M)(k) = \{\im(i,k) \cap \ker(k,j) \mid 0 \leq i \leq j \leq n+1\}$, the first equality holds by applying LS1. The second equality holds since $\mathcal{I}(k) = S(k)$ is biclosed. By Lemma 10 the multilag generated by the semiflags of images and of kernels is in general position, and therefore so is its sum-intersection closure (Lemma 11). The multilag $\mathcal{I}(k)$ is a subflag of the sum-intersection closure, hence is in general position for each $k$. We conclude that $e(M) = 0$. □

While 0 and $n+1$ are not objects in $n$, we unify our notation by taking $M(0) = M(n+1) = \{0\}$ and $\phi_{0,1} : \{0\} \rightarrow V_1$ and $\phi_{n,n+1} : V_n \rightarrow \{0\}$ to be zero maps. Note that $\ker(k,k) = 0$ and $\ker(k,n+1) = M(k)$. Let $\im(i,k) = \im(\phi_{i,k})$. Then $\im(k,k) = M(k)$ and $\im(0,k) = \{0\}$.

For $k \in [i,j]$, define the subquotient

$$\tilde{A}_k = \frac{N_k}{D_k} = \frac{\ker(k,j+1) \cap \im(i,k)}{\ker(k,j) \cap \im(i,k) + \im(i-1,k) \cap \ker(k,j+1)}$$

Lemma 13. For $i \leq l < j$, $\tilde{\phi}_{l,l+1} : \tilde{A}_l \rightarrow \tilde{A}_{l+1}$ is an isomorphism, and $\tilde{\phi}_{j,j+1}(\tilde{A}_j) = 0$

Proof. If $i \leq l < j$, $\tilde{\phi}_{l,l+1}$ maps $N_l$ surjectively to $N_{l+1}$, implying $\tilde{\phi}_{l,l+1}$ is surjective. In addition, $\ker(\phi_{l,l+1}\vert_{N_k}) = \ker(k,j+1) \cap \im(i,k) \subset D_k$, so $\tilde{\phi}_{l,l+1}$ is injective. Finally $\phi_{j,j+1}$ maps $N_j$ to zero, implying the second statement. □

As $M$ is equipped with an inner product, $A_k := (D_k \subset N_k)^\perp \cong \tilde{A}_k$, and so

$$M[i,j] := 0 \rightarrow \cdots \rightarrow 0 \rightarrow A_i \overset{\phi_{i,i+1}}{\sim} \cdots \overset{\phi_{j-1,j}}{\sim} A_j \rightarrow 0 \rightarrow \cdots \rightarrow 0 \quad (12)$$

is a submodule of $M$ and also a block supported on $[i,j]$.

Theorem 7. Let $M : n \rightarrow (\text{vect}/k)$ be a finite persistence module. Then $M$ decomposes uniquely (up to reordering) as a direct sum of blocks:

$$M = \bigoplus_{[i,j] \subseteq [1,n]} M[i,j]$$

Proof. It is clear that the support of any block must be an interval submodule of $[1,n]$, and moreover that any interval submodule can occur as the support of a block in $M$. By Proposition 7, $M$ is an IPC-module with no object-excess; it follows by the results of the previous section and Lemma 10 that $M$ is the direct sum of its blocks. It is straightforward to show that for each $1 \leq i < j \leq n$ the $[i,j]$-block appearing in the associated graded of $\mathcal{F}(M) = \mathcal{F}_1(M)$ is precisely $M[i,j]$, as defined in (12). The decomposition is unique up to reordering, as it is completely determined by the local structure on $M$. □

It is additionally worth noting that this decomposition is basis-free, arising directly from the local structure. To get the traditional decomposition of $M$ as a direct sum of interval modules, one needs to further express each $M[i,j]$ as a direct sum of interval modules of type $[i,j]$. This is a simple exercise which however does require a choice of basis (although the number of terms in the direct sum does not).

3.5 Modules with stable local structure

One-dimensional persistence modules generalize naturally to higher dimensions.

Definition 20. A finite $n$-dimensional persistence category $\mathcal{C}$ is a finite category $\mathcal{C}$ for which there exists an isomorphism of categories

$$m_1 \times m_2 \times \ldots \times m_n \cong \mathcal{C}$$

for some set of positive integers $m_1, m_2, \ldots, m_n$. An $n$-dimensional persistence category $\mathcal{C}$ has multi-dimension $(m_1, m_2, \ldots, m_n)$ if $\mathcal{C}$ is isomorphic to $m_1 \times m_2 \times \cdots \times m_n$; note that this $n$-tuple is a well-defined invariant of the isomorphism class of $\mathcal{C}$ up to reordering. An $n$-dimensional persistence module is a $\mathcal{C}$-module for some finite $n$-dimensional persistence category $\mathcal{C}$.

The proof of the next theorem illustrates the usefulness of strong stability.

Theorem 8. Finite $n$-dimensional persistence modules have strongly stable local structure for all $n \geq 0$. 
Proof. We may assume, without loss of generality, that \( M \) is a \( C \)-module for an \( n \)-dimensional persistence category \( C \) with multi-dimension \( (m_1, m_2, \ldots, m_n) \) where the dimensions \( m_i \) have been arranged in non-increasing order. We assume the objects of \( C \) have been labeled with multi-indices \( (i_1, i_2, \ldots, i_n), 1 \leq i_j \leq m_j \), so that a morphism exists in \( C \) from \( (i_1, i_2, \ldots, i_n) \) to \( (j_1, j_2, \ldots, j_n) \) if \( i_k \leq j_k, 1 \leq k \leq n \) (as \( C \) is a poset category, if such a morphism exists it is unique). We will reference the objects of \( C \) by their multi-indices. The proof is by induction on dimension; the base case \( n = 0 \) is trivially true as there is nothing to prove.

Assume then that \( n \geq 1 \). For \( 1 \leq i \leq j \leq m_n \), let \( C[i, j] \) denote the full subcategory of \( C \) on objects \( (k_1, k_2, \ldots, k_n) \) with \( i \leq k_n \leq j \), and let \( M[i, j] \) denote the restriction of \( M \) to \( C[i, j] \). Let \( F_1 \) resp. \( F_2 \) denote the local structures on \( M[1, m_n - 1] \) and \( M[m_n] \) respectively; by induction on the cardinality of \( m_n \) we may assume these local structures are stable with stabilization in \( 1 \)-dimensions. We will reference the objects of \( C \) and on their multi-indices. The proof is by induction on dimension; the base case \( n = 0 \) is trivially true as there is nothing to prove.

**Theorem 9.** Assume the base field \( F \) is finite. Then for all finite categories \( C \) and \( C \)-modules \( M \), \( M \) has stable local structure.

4 **Topologically based \( C \)-modules**

A \( C \)-module \( M \) is said to be *topologically based* if \( M = H_*(F;k) \) (for either a particular value of \( * \) in the ungraded case, or more generally viewed as a graded element of \( \text{vect}/k \)) where \( F : C \to D \) is a functor from \( C \) to a category \( D \), equaling either

- **f-s-sets** - the category of simplicial sets with finite skeleta and morphisms of simplicial sets, or
- **f-s-com** - the category of finite simplicial complexes and morphisms of simplicial complexes.
In what follows we will restrict ourselves to the category $\textbf{f-s-sets}$, as it is slightly easier to work in (although all results carry over to $\textbf{f-s-complexes}$). We show that any topologically-based module indexed on a poset category $\mathcal{C}$ admits a presentation by $\mathcal{IPC}$-modules (hence a presentation by $\mathcal{FTPc}$-modules by our results above). We also prove a general Künneth Theorem for topologically based $\mathcal{C}$-modules without restrictions on $\mathcal{C}$ (answering a question posed by G. Carlsson).

### 4.1 An $\mathcal{IPC}$-presentation

For this subsection we assume $\mathcal{C}$ to be a connected, finite poset-category, so that all $\mathcal{C}$-modules are finite poset-modules.

We first show that any $\mathcal{C}$-diagram in $\textbf{f-s-sets}$ can be cofibrantly replaced, up to weak homotopical transformation. Precisely,

**Theorem 10.** If $F: \mathcal{C} \to \textbf{f-s-sets}$, then there is a $\mathcal{C}$-diagram $\tilde{F}: \mathcal{C} \to \textbf{f-s-sets}$ and a natural transformation $\eta: \tilde{F} \cong F$ which is a weak equivalence at each object, where $\tilde{F}(\phi_{xy})$ is a closed cofibration (inclusion of simplicial sets) for all morphisms $\phi_{xy}$.

**Proof.** The simplicial mapping cylinder construction $Cyl(-)$ applied to any morphism in $\textbf{f-s-sets}$ verifies the statement of the theorem in the simplest case $\mathcal{C}$ consists of two objects and one non-identity morphism. Suppose $\mathcal{C}$ has $n$ objects; we fix a total ordering on $\text{obj}(\mathcal{C})$ that refines the partial ordering: $\{x_1 \prec x_2 \prec \cdots \prec x_n\}$ where if $\phi_{x_i x_j}$ is a morphism in $\mathcal{C}$ then $i \leq j$ (but not necessarily conversely). Let $\mathcal{C}(m)$ denote the full subcategory of $\mathcal{C}$ on objects $x_1, \ldots, x_m$, with $F_m = F|_{\mathcal{C}(m)}$. By induction, we may assume the statement of the theorem for $F_m: \mathcal{C}(m) \to \textbf{f-s-sets}$, with cofibrant lift denoted by $\tilde{F}_m$; with $\eta_m: \tilde{F}_m \cong F_m$.

Now let $\mathcal{D}(m)$ denote the slice category $\mathcal{C}/x_{m+1}$; as “$\prec$” is a refinement of the poset ordering “$<$”, the image of the forgetful functor $P_m: \mathcal{D}(m) \to \mathcal{C}; (y \to x_{m+1}) \mapsto y$ lies in $\mathcal{C}(m)$. And as $\mathcal{C}$ is a poset category, the collection of morphisms $\{\phi_{yx_{m+1}}\}$ uniquely determine a map

$$f_m: \colim_{\mathcal{D}(m)} \tilde{F}_m \circ P_m \xrightarrow{\eta_m} \colim_{\mathcal{D}(m)} F_m \circ P_m \to F(x_{m+1})$$

Define $\tilde{F}_{m+1}: \mathcal{C}(m+1) \to \textbf{f-s-sets}$ by

- $\tilde{F}_{m+1}|_{\mathcal{C}(m)} = \tilde{F}_m$;
- $\tilde{F}_{m+1}(x_{m+1}) = Cyl(f_m)$;
- If $\phi_{xx_{m+1}}$ is a morphism from $x \in \text{obj}(\mathcal{C}(m))$ to $x_{m+1}$, then

$$\tilde{F}_{m+1}(\phi_{xx_{m+1}}): \tilde{F}_m(x) = \tilde{F}_{m+1}(x) \to \tilde{F}_{m+1}(x_{m+1})$$

is given as the composition

$$\tilde{F}_m(x) = \tilde{F}_m \circ P_m(x \xrightarrow{\phi_{xx_{m+1}}} x_{m+1}) \hookrightarrow \colim_{\mathcal{D}(m)} \tilde{F}_m \circ P_m \hookrightarrow Cyl(f_m) = \tilde{F}_{m+1}(x_{m+1})$$

where the first inclusion into the colimit over $\mathcal{D}(m)$ is induced by the inclusion of the object

$$(x \xrightarrow{\phi_{xx_{m+1}}} x_{m+1}) \hookrightarrow \text{obj}(\mathcal{D}(m)).$$

As all morphisms in $\mathcal{D}(m)$ map to simplicial inclusions under $\tilde{F}_m \circ P_m$ the resulting map of $F_m(x)$ into the colimit will also be a simplicial inclusion. Finally, the natural transformation $\eta_m: \tilde{F}_m \to F_m$ is extended to $\eta_{m+1}$ on $\tilde{F}_{m+1}$ by defining $\eta_{m+1}(x_{m+1}) : \tilde{F}_{m+1}(x_{m+1}) \to F_{m+1}(x_{m+1})$ as the natural collapsing map $Cyl(f_m) \to F(x_{m+1})$, which has the effect of making the diagram

---

1The proof following is a minor elaboration of an argument communicated to us by Bill Dwyer [6].
then yields the isomorphisms of (14) and (15). We note that as equality in (13) follows from the traditional Künneth Theorem. Applying Theorem 4 to the RHS of (13)

\[ F_{m+1}(x) \xrightarrow{\phi_{xy}} F_{m+1}(y) \]

\[ \eta_{m+1}(x) \xrightarrow{} \eta_{m+1}(y) \]

\[ F_{m+1}(x) \xrightarrow{F_{m+1}(\phi_{xy})} F_{m+1}(y) \]

commute for morphisms \( \phi_{xy} \in \text{Hom}(C_{m+1}) \). This completes the induction step, and the proof. \( \square \)

**Corollary 2.** Any topologically based \( C \)-module \( M \) admits a presentation by \( C \)-modules \( N_1 \rightarrow N_2 \rightarrow M \) where \( N_i \) is an \( IPC \)-module and \( N_1 \rightarrow N_2 \) is an isometric inclusion of \( IPC \)-modules.

**Proof.** By the previous result and the homotopy invariance of homology, we may assume \( M = H_n(F) \) where \( F : C \rightarrow \text{i-f-s-sets} \), the subcategory of \( \text{f-s-sets} \) on the same set of objects, but where all morphisms are simplicial set injections. In this case, for each object \( x \), \( C_n(F(x)) \) admits a canonical inner product determined by the natural basis of \( n \)-simplices \( F(x)_n \), and each morphism \( \phi_{xy} \) induces an injection of basis sets \( F(x)_n \rightarrow F(y)_n \), resulting in an isometric inclusion \( C_n(F(x)) \rightarrow C_n(F(y)) \).

In this way the functor \( C_n(F) := C_n(F;k) : C \rightarrow (\text{vect}/k) \) inherits a natural \( IPC \)-module structure. If \( Q \) is an \( IPC \)-module where all of the morphisms are isometric injections, then any \( C \)-submodule \( Q' \subset Q \), equipped with the same inner product, is an \( IPC \)-submodule of \( Q \). Now \( C_n(F) \) contains the \( C \)-submodules \( Z_n(F) \) (\( n \)-cycles) and \( B_n(F) \) (\( n \)-boundaries); equipped with the induced inner product the inclusion \( B_n(F) \hookrightarrow Z_n(F) \) is an isometric inclusion of \( IPC \)-modules, for which \( M \) is the cokernel \( C \)-module.

[Note: The results for this subsection have been stated for \( \text{f-s-sets} \); similar results can be shown for \( \text{f-s-complexes} \) after fixing a systematic way for representing the mapping cylinder of a map of simplicial complexes as a simplicial complex; this typically involves barycentrically subdividing.]

### 4.2 A Künneth Theorem for topologically based \( C \)-modules

The following theorem represents the natural generalization of the traditional Künneth Theorem for singular homology with field coefficients.

**Theorem 11.** Let \( F_i : C_i \rightarrow \text{f-s-sets} \), \( 1 \leq i \leq n \), with \( M_i := H_*(F_i;k) \) denoting the corresponding functor from \( C_i \) to the category (\( \text{gr-vect}/k \)) of finite-dimensional graded vector spaces over \( k \). Similarly, define

\[ F := \text{diag}(F_1 \times F_2 \times \ldots F_n) : C_1 \times C_2 \times \ldots C_n \rightarrow \text{f-s-sets} \]

\[ M := H_*(F;k) \]

Then there is a natural isomorphism

\[ M \cong M_1 \otimes M_2 \otimes \cdots \otimes M_n \]  \hspace{1cm} (13)

of topological \( C_1 \times C_2 \times \ldots C_n \)-modules which induces isomorphisms

\[ \mathcal{F}(M) \cong \mathcal{F}(M_1) \otimes \mathcal{F}(M_2) \otimes \ldots \mathcal{F}(M_n) \]  \hspace{1cm} (14)

\[ \mathcal{F}(M)_* \cong \mathcal{F}(M_1)_* \otimes \mathcal{F}(M_2)_* \otimes \ldots \mathcal{F}(M_n)_* \]  \hspace{1cm} (15)

**Proof.** The proof is short, as the main technical component has already been established above. The equality in (13) follows from the traditional Künneth Theorem. Applying Theorem 4 to the RHS of (13) then yields the isomorphisms of (14) and (15). We note that as \( M \) is graded, the tensor products in all three equations are to be taken in the graded sense. \( \square \)
Corollary 3. If $M_i, 1 \leq i \leq n$ (as in the previous theorem) are tame for $1 \leq i \leq n$, then so is $M_1 \otimes M_2 \otimes \ldots M_n$.

This applies in particular to the case that each $M_i$ is a finite 1-dimensional persistence module.

Funding and Competing Interests

No funding was received to assist with the preparation of this manuscript. The authors have no competing interests to declare that are relevant to the content of this article.

Data Availability Statement

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

References

[1] M. Auslander, I. Reiten, S. Smalø, *Representation Theory of Artin Algebras*, Camb. Studies in Adv. Math. **36** (1995) Camb. Univ. Press

[2] Michael Brion, *Representations of Quivers*, Geometric methods in representation theory - Lecture Notes, Summer School, Inst. Fourier (2008).

[3] G. Carlsson, V. deSilva, *Zig-zag persistence*, Found. Comput. Math. **10** (2010), 367 – 405.

[4] G. Carlsson, J. Skryzalin, *Numeric invariants from multidimensional persistence*, Jour. App. and Comp. Top. **1** (2017) pp 89 – 119.

[5] G. Carlsson, A. Zomorodian, *The theory of multidimensional persistence*, Disc. Comput. Geom. **42** (2009), 71 – 93.

[6] W. Dwyer, *Private communication*.

[7] P. Gabriel, *Unzerlegbare Darstellungen I*, Manuscr. Math. **6** (1972), 71 – 103.

[8] P. Gabriel, A. Roiter, *Representations of finite-dimensional algebras*, Springer-Verlag (1997).

[9] S. Krishnan, C. Ogle, *Invertibility in Category Representations*, https://arxiv.org/abs/2010.11276 (2020).

[10] E. Miller, *Data structures for real multiparameter persistence modules*, [arXiv:1709.08155v1](https://arxiv.org/abs/1709.08155) (2017).

[11] F. Fernandez-Polo, Antonio Peralta, *Partial Isometries: A Survey*, Adv. Oper. Theory **3** (2018) 75 – 116.

[12] Graeme Segal, *Configuration spaces and iterated loop-spaces*, Invent. Math. **21** (1973), 213 – 221.