Logarithmic corrections to higher twist scaling at strong coupling from AdS/CFT

S. Frolov\textsuperscript{a,1}, A. Tirziu\textsuperscript{b,2} and A.A. Tseytlin\textsuperscript{c,3}

\textsuperscript{a} School of Mathematics, Trinity College, Dublin 2, Ireland
\textsuperscript{b} Department of Physics, The Ohio State University, Columbus, OH43210, USA
\textsuperscript{c} Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.

Abstract

We compute 1-loop correction $E_1$ to the energy of folded string in $AdS_5 \times S^5$ (carrying spin $S$ in $AdS_5$ and momentum $J$ in $S^5$) using a “long string” approximation in which $S \gg J \gg 1$. According to the AdS/CFT the function $E_1$ should represent first subleading correction to strong coupling expansion of anomalous dimension of higher twist $SL(2)$ sector operators of the form $\text{Tr} D^S Z^J$. We show that $E_1$ smoothly interpolates between the $\ln S$ regime (previously found in the $J \to 0$ case) and the $\lambda/J^2 \ln^3(S/J)$ regime (which is the leading correction to the thermodynamic limit on the spin chain side). This supports the universality of the $\ln S$ scaling. As in the previous work, we also find “non-analytic” corrections related to non-trivial 1-loop phase in the corresponding Bethe ansatz S-matrix.

\textsuperscript{1}frolovs@maths.tcd.ie
\textsuperscript{2}tirziu@mps.ohio-state.edu
\textsuperscript{3}Also at Lebedev Institute, Moscow. tseytlin@imperial.ac.uk
1 Introduction and Summary

Logarithmic scaling of anomalous dimensions of composite operators with large Lorentz spin $S$ is of major importance in QCD and was recently also at the center of attention in planar $\mathcal{N} = 4$ SYM theory in the context of AdS/CFT. Ref. [1] made a remarkable observation that the logarithmic behaviour $\Delta = S + (k_1 \lambda + k_2 \lambda^2 + ...) \ln S + O(S^0)$ previously known at weak ‘t Hooft coupling should continue also at strong coupling: the classical energy of a folded rotating string in $AdS_5$ which should be dual to a minimal twist operator scales at large $\lambda$ and large $\sqrt{\lambda}$ as $E_0 = S + \sqrt{\lambda} \ln S + O(S^0)$.

Ref. [2] made a next step by computing the leading quantum string 1-loop ($\frac{1}{\sqrt{\lambda}}$) correction $E_1$ to the string energy, confirming that, as at weak coupling, all terms growing faster than logarithm of $S$ cancel out so that one ends up with $E_1 = -\frac{3}{\pi} \ln S + O(S^0)$. This provided strong support to the conjecture that in planar $\mathcal{N} = 4$ SYM theory the coefficient of $\ln S$ (i.e. the “scaling function” or “cusp anomalous dimension” [3, 4, 5]) should be a function of $\lambda$ that smoothly interpolates between the weak and strong coupling regimes

$$\Delta = E = S + f(\lambda) \ln S + O(S^0). \quad (1.1)$$

Arguments in favour of such interpolation (based on Pade approximations) were further advanced in [6].

Ref. [2] also generalized the rotating folded string solution of [7, 1] to the case when the string center of mass is also moving along big circle of $S^5$ so that the string is carrying in addition to the Lorentz spin $S$ an $SO(6)$ spin $J$. Introducing large non-zero $J$ is important in particular since it makes it easier to identify the corresponding dual gauge theory operators as belonging to the $SL(2)$ sector [8, 9] in gauge theory [10]: $\text{tr}(D^SZ^I) + ...$ (here $D$ is l.c. covariant derivative and $Z$ is a complex scalar). While $J = 2$ corresponds to the minimal-twist case, the opposite case of large $J$ and small $S$ is a BMN-type [11] limit. Since $J$ plays the role of the spin chain length on the gauge theory side having both $S$ and $J$ large is important in order to be able to apply the thermodynamic limit approximation and, more generally, to be able to use the asymptotic Bethe ansatz of [12, 13] in the first place.

The classical string energy for this solution turned out to be a complicated function of two arguments $E_0 = \sqrt{\lambda} \mathcal{E}(S, \nu)$, $S \equiv \frac{S}{\sqrt{\lambda}}$, $\nu \equiv \frac{\nu}{\sqrt{\lambda}}$ (given by a solution of two equations involving Jacobi elliptic functions) but it simplifies in various special limits. First, one may consider either “short” ($\sqrt{1 + \nu^2} \gg 2S$) or “long” string. In the “short string” limit [2, 10] with $J \gg S$ one gets the BMN-type scaling $E_0 = J + S \sqrt{1 + \frac{J^2}{\lambda}} + ...$. In the “long string” limit $S \gg \sqrt{\lambda}$, $S \gg J$ and one should further distinguish the two cases [2, 11]

(i) “slow long string” $J \ll \sqrt{\lambda} \ln S \frac{S}{\sqrt{\lambda}}$ in which case $E_0 \approx S + \frac{S}{\sqrt{\lambda}} \ln S + \frac{\pi J^2}{2\sqrt{\lambda} \ln S} + ...$, i.e. one recovers the $\ln S$ scaling of the $J = 0$ case even though $J$ may still be large in

4Here “slow” and “fast” refers to the string center of mass motion.
absolute terms since $S \gg J \sim \sqrt{\lambda}$.

(i) “fast long string” $S \gg J \gg \sqrt{\lambda} \ln \frac{S}{\sqrt{\lambda}}$, in which case one finds a familiar “fast spinning string” \cite{14} scaling $E_0 \approx S + J[1 + \frac{\lambda}{2\pi^2} h_1(\frac{S}{J}) + \frac{\lambda^2}{8\pi^4} h_2(\frac{S}{J}) + \ldots]$ with $h_1 = \frac{1}{2\pi^2} \ln^2 \frac{S}{J}$, $h_2 \sim \ln^4 \frac{S}{J}$, etc. As was shown in \cite{10} (see also \cite{17}), the leading $h_1$ term here is reproduced as the corresponding term in the 1-loop anomalous dimension on the gauge theory side, implying its non-renormalizability as one goes from weak to strong coupling in the above large charge limit.

More recently, ref. \cite{15} made an interesting observation that the expressions for the string energy in \cite{2, 10} imply that the $\ln^k$-terms appearing in the “fast long string” case can be resummed in a closed form: in the case when $J \sim \sqrt{\lambda} \ln \frac{S}{\sqrt{\lambda}}$ the energy can be written as

$$E_0 = S + J \sqrt{1 + \frac{\lambda}{2\pi^2} J^2 \ln^2 \frac{S}{J}} + \ldots = S + J \sqrt{1 + x^2} + \ldots,$$

which is valid when

$$S \gg J \quad \text{and} \quad x \equiv \frac{\sqrt{\lambda}}{\pi J} \ln \frac{S}{J} = \text{fixed}.$$  \hfill (1.3)

Remarkably, this formula captures both the “slow long string” limit\footnote{This resummation is formally true for small $x$ but the result can then be extended to large $x$ as well, see sect. 2. As we show in section 2, a natural quantity to be kept fixed in the “slow long string” limit is $\frac{\sqrt{\lambda}}{\pi J} \ln \frac{S}{\sqrt{\lambda}}$. For $S \gg J \gg \sqrt{\lambda}$ the difference between this variable and $x$ is negligible. To have a smooth limit $J \to 0$ one should, however, replace $\ln \frac{S}{J}$ by $\ln S$ in eqs.\footnote{12} and \footnote{13}. Let us mention that a discussion of the logarithmic scaling at the classical string side appeared also in \cite{16}.} and the “fast long string” limit smoothly interpolating between them:

$$E_0(x \gg 1) = S + \frac{\sqrt{\lambda}}{\pi} \ln \frac{S}{J} + \frac{\pi J^2}{2\sqrt{\lambda} \ln \frac{S}{J}} + \ldots,$$

$$E_0(x \ll 1) = S + J + \frac{\lambda}{2\pi^2 J} \ln^2 \frac{S}{J} - \frac{\lambda^2}{8\pi^4 J^3} \ln^4 \frac{S}{J} + \frac{\lambda^3}{16\pi^6 J^5} \ln^6 \frac{S}{J} + \ldots.$$  \hfill (1.4)

In this paper we will extend the computation \cite{2} of string 1-loop correction $E_1 = E_1(\frac{S}{\sqrt{\lambda}}, \frac{J}{\sqrt{\lambda}})$ to the energy of folded spinning string to the case of $J \neq 0$ in the parameter space region \footnote{12} and thus find a closed expression for the 1-loop string counterpart of the classical expression \footnote{13}. The result can be written as

$$E_1 = J \sqrt{\frac{\lambda}{1 + x^2}} F(x) + O(\kappa^0), \quad \kappa \equiv \frac{J}{\sqrt{\lambda}} \sqrt{1 + x^2} \gg 1,$$

$$F(x) = \frac{1}{1 + x^2} \left[ x \sqrt{1 + x^2} - x^2 - 2(1 + x^2) \ln(1 + x^2) - (1 + 2x^2) \ln[\sqrt{1 + 2x^2}(x + \sqrt{1 + x^2})] \right].$$  \hfill (1.7)
with $x$ defined in (1.3). The counterparts of the “slow” and “fast” expansions (1.4) and (1.5) are now

$$E_1(x \gg 1) = -\frac{3 \ln 2}{\pi} \ln \frac{S}{J} + \frac{2\pi^2 J^2 \ln \ln \frac{S}{J}}{\lambda} + \frac{\pi^3 J^4 \ln \ln \frac{S}{J}}{\lambda^2} + \frac{\pi^2 J^2 (\ln \ln \frac{S}{J})^2}{\lambda^3} + \ldots , \quad (1.8)$$

$$E_1(x \ll 1) = -\frac{4\lambda}{3\pi^3 J^2} \ln^3 \frac{S}{J} + \frac{4\lambda^2}{5\pi^5 J^4} \ln^5 \frac{S}{J} + \frac{\lambda^{5/2}}{3\pi^6 J^5} \ln^6 \frac{S}{J} + \ldots . \quad (1.9)$$

Figure 1 shows the plot of the 1-loop function $\sqrt{1 + x^2 F(x)}$: it illustrates that this function smoothly interpolates between the linear $-x$ one at large $x$ and $-x^3$ one at small $x$.

Eq. (1.8) provides further support for the universality of the $\ln S$ coefficient in the quantum string energy, i.e. in the strong-coupling expansion: we reproduce the same $-\frac{3 \ln 2}{\pi} \ln S$ term as was found earlier at $J = 0$ as a regular limit of the large $J$ expression.

Checking this universality was one of our motivations in view of recent remarkable results [19, 18, 20, 21] (see also [22, 23]). This universality of the $\ln S$ coefficient at $S \gg J \gg 1$ for lowest dimension states in the $SL(2)$ sector was argued for at weak coupling in [15] (at one loop) and also more generally in [24]. Following [15] and [24], the work of [18] was based on the assumption that the coefficient of the $\ln S$ term is the same when computed at small $J$ and at large $J$ – large enough to be able to apply the asymptotic Bethe ansatz approach.

The importance of eqs. (1.2), (1.7) is that they allow us to connect the $\ln S$ terms to $J^{-k} \ln^n \frac{S}{J}$ terms which are “visible” in the thermodynamic limit in the weakly coupled gauge theory spin chain. Indeed, like the $\frac{\lambda}{\pi} \ln^3 \frac{S}{J}$ term in (1.5) which is reproduced [10] in the 1-loop $SL(2)$ spin chain, the leading $\frac{\lambda}{\pi} \ln^3 \frac{S}{J}$ term in (1.9) agrees precisely with the leading correction to the thermodynamic limit of 1-loop gauge theory chain found in [15] (see below). Thus the coefficient of the $\ln S$ term is encoded as a limit of the full string 1-loop correction determined from the string Bethe ansatz [26, 27, 28] by the BDS [12] part as well as by the 1-loop phase of [27, 28].

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6The results of [20, 21] that confirm the agreement between the strong-coupling expansion of the
To connect (1.5), (1.9) to weakly coupled gauge theory let us recall that as was emphasized in [15], in the 1-loop gauge theory spin chain the logarithmic scaling \( \lambda \ln S \) appears universally in the thermodynamic limit \( J \to \infty \) provided \( \ln S/J \gg J \): it takes over the leading-order “fast string” scaling \( \frac{1}{J} \ln^2 \frac{S}{J} \) as one increases the value of the “gauge-theory” parameter \[ \xi \equiv \frac{\pi}{\sqrt{\lambda}} x = \frac{\ln S}{J}. \] (1.10)

Motivated by the analysis of the 1-loop gauge spin chain and also by the strong-coupling string result (1.2), ref. [15] made the following proposal for the all-order behaviour of the minimal gauge-theory anomalous dimensions in the region \( S \gg J \gg 1 \) with fixed \( \xi \):

\[ E = S + J + J \sum_{n=1}^{\infty} c_n(\xi)(\lambda \xi^2)^n, \] (1.11)

where

\[ c_n(\xi \ll 1) = c_{n0} + c_{n1} \xi + c_{n2} \xi^2 + \ldots, \] (1.12)

\[ c_n(\xi \gg 1) = \frac{a_n}{\xi^{2n-1}} + \ldots. \] (1.13)

The terms in the “fast string” [2, 14] or BMN-type scaling (1.12) with \( \xi \ll 1 \) which multiply \( c_{n1}, c_{n2}, \ldots \) scale as \( \frac{1}{J}, \frac{1}{J^2}, \ldots \) and thus represent corrections to the thermodynamic \( J \to \infty \) limit. The large \( \xi \) scaling behaviour of \( c_n \) in (1.13) translates into the familiar perturbative \( \ln S \) scaling

\[ E = S + J + (a_1 \lambda + a_2 \lambda^2 + \ldots) \ln S + \ldots. \] (1.14)

The coefficients in (1.12) were assumed in [15] to have no additional dependence on \( \lambda \).

However, it is known [27] that in the strong coupling limit as described by string theory, where \( E \) is expressed in terms of \( \lambda \gg 1 \) and the semiclassical string parameter \( x \) in (1.10) (in which one can subsequently expand assuming \( x < 1 \)) the “fast string” scaling (i.e. (1.11) written in terms of \( x \)) breaks down starting with \( x^6 \) (or “3-loop”) term. As we shall explain below, eq. (1.9) provides a direct indication of that, in complete analogy with the results of [27, 37, 28] for circular string solutions. The strong-coupling corrections to the dressing phase [28] translate [22, 18] into the weak-coupling correction which should contribute starting with 4-loop term in the weak-coupling expansion in (1.11).

This suggests that (1.11) viewed as an exact expression for the energy interpolating between weak and strong coupling regions should be modified following [27, 30, 18]: integral equation of [24, 18] and the 1-loop result \(- \frac{3 \ln 2}{\pi} \ln S\) is a check of the strong-to-weak coupling continuation of the S-matrix phase in [18]; the starting point of [18] was the strong-coupling expansion of the phase [22] which already encodes this \( \ln S \) result.

Footnote 7: At the transition point \( \xi \sim 1 \) the semiclassical expansion based on the thermodynamic limit of the Bethe equations breaks down [15] due to the collision of the two cuts at the origin [10].
most of the coefficients $c_n$ in (1.11) except for the few leading “protected” ones should
develop additional dependence on $\lambda$, allowing, in particular, for the “3-loop” coefficient
$c_{30}$ to have different limiting values at $\lambda \to 0$ and $\lambda \to \infty$. Explicitly, one may expect
(see also below)

$$E = S + J \left[ 1 + \frac{\lambda}{J^2} \ln^2 \frac{S}{J} \left( c_{10} + c_{11} \ln \frac{S}{J} + c_{12} \frac{\ln^2 S}{J^2} + \ldots \right) \right. $$

$$+ \frac{\lambda^2}{J^4} \ln^4 \frac{S}{J} \left( c_{20} + c_{21} \ln \frac{S}{J} + c_{22}(\lambda) \frac{\ln^2 S}{J^2} + \ldots \right) \quad (1.15)$$

$$+ \frac{\lambda^3}{J^6} \ln^6 \frac{S}{J} \left( c_{30}(\lambda) + c_{31}(\lambda) \frac{\ln S}{J} + c_{32}(\lambda) \frac{\ln^2 S}{J^2} + \ldots + \ldots \right) \right] ,$$

The functions $c_{nk}(\lambda)$ should have regular expansion at weak coupling, e.g.,\footnote{Note that there is an ambiguity in how one splits corrections between different terms, e.g., $\lambda^2 c_{22}(\lambda)$ and $\lambda^3 c_{30}(\lambda)$ which both multiply $J^{-3} \ln^3 \frac{S}{J}$ in the brackets in (1.15).}

$$c_{30}(\lambda \ll 1) = c_{30,0} + c_{30,1}\lambda + O(\lambda^2) , \quad \text{etc.} \quad (1.16)$$

The weak-coupling coefficient $c_{30,0}$ should be different from the strong-coupling one\footnote{The result for $c_{11}$ is given at the end of sect. 3.2 in [18] (here $s = \frac{1}{2}$ since the string states we consider are dual to scalar operators). We thank A. Belitsky for pointing this out to us.}

$$c_{30,0} = \frac{1}{16\pi^2} \quad \text{in (1.5)} \quad \text{(manifesting “3-loop disagreement” [35, 25]) while the first violation of the “semiclasical” scaling $c_{30,1} \neq 0$ at weak coupling should thus appear at four loops.}

The weak-coupling expansion of the dressing phase proposed in [18] suggests that a
direct gauge-theory computation of this coefficient should give $c_{30,1} \sim \zeta(3)$.\footnote{The coefficients $c_{20}$ and $c_{21}$ were not yet computed on the gauge theory side. It would be nice to check the above values directly by extending the methods of [15] to 2-loop order.}

Let us now compare (1.11), (1.15) with the string theory predictions $E_0 + E_1$ in
(1.4), (1.5) and (1.8), (1.9) for the leading strong-coupling corrections. The leading term
in (1.9) has the same structure as the $c_{11}$ term in (1.15). Remarkably, like $c_{10} = \frac{1}{2\pi^2}$ that
matches [10] the coefficient of the leading $J^{-1} \ln^2 \frac{S}{J}$ correction in (1.5), its coefficient
in (1.9) is exactly the same as the 1-loop gauge theory coefficient

$$c_{11} = -\frac{4}{3\pi^3} \quad (1.17)$$

computed as the leading correction to the thermodynamic limit in the $SL(2)$ Heisenberg
chain in [15].\footnote{\textcolor{red}{The result for $c_{11}$ is given at the end of sect. 3.2 in [15] (here $s = \frac{1}{2}$ since the string states we consider are dual to scalar operators). We thank A. Belitsky for pointing this out to us.}

Looking at higher orders, the $c_{20}$ term in (1.15) has the same form and should also
have the same coefficient as the $J^{-3} \ln^4 \frac{S}{J}$ term in the classical string energy (1.5), i.e.
$c_{20} = -\frac{1}{8\pi^2}$. We also observe that the absence of the $J^{-3} \ln^4 \frac{S}{J}$ term in the string 1-loop correction (1.9) is in full agreement with (1.15): the subleading 2-loop $c_{21}$ term there scales as $J^{-4} \ln^3 \frac{S}{J}$ and should have the same coefficient as the corresponding term in
(1.9), i.e. $c_{21} = \frac{4}{5\pi^2}$.\footnote{The coefficients $c_{20}$ and $c_{21}$ were not yet computed on the gauge theory side. It would be nice to check the above values directly by extending the methods of [15] to 2-loop order.}
The non-renormalization of the few leading 1- and 2-loop coefficients in (1.15) in going from weak to strong coupling is of course expected on the basis of consistency with previous results for similar circular string solutions. The 1-loop and 2-loop leading and the first subleading \[\lambda^3, \lambda^{5/2}\] corrections to the thermodynamic limit should match precisely between gauge and string theory: the effect of the non-trivial phase in the Bethe ansatz should first become visible in leading thermodynamic limit and strong-coupling expansion only starting with terms of \(\lambda^3\) order in “semiclassical” expansion like (1.15) \[\text{11}\]

Indeed, combining the \(J^{-5} \ln^6 \frac{S}{J}\) terms in (1.5) and (1.9) and comparing them to (1.15) we conclude that as in \[\text{27}\] the presence of “non-analytic” \(\lambda^{5/2}\) term in string 1-loop correction \[\text{27, 37}\] implies the renormalization of the “3-loop” coefficient \(c_{30}\) in (1.15):

\[
c_{30}(\lambda \gg 1) = 1 + \frac{16}{3\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right). \tag{1.18}\]

As was found in \[\text{27}\] on the examples of circular solutions in \(SL(2)\) \[\text{43}\] and \(SU(2)\) \[\text{42}\] sectors \[\text{12}\] the non-analytic \(\lambda^{5/2}\) correction in the string 1-loop energy should universally account for the difference between the string and gauge predictions for the coefficient of the \(\lambda^3\) term, corresponding to the function

\[
c_2(\lambda) = 1 - \frac{16}{3\sqrt{\lambda}} + O\left(\frac{1}{\lambda}\right) \tag{1.19}\]

in the dressing phase in the Bethe ansatz which should interpolate between 1 \[\text{26}\] at strong coupling and 0 \[\text{12}\] at weak coupling. For example, in the case of \(J_1 = J_2\) circular solution in the \(SU(2)\) sector one finds from the classical energy \(E_0 = \sqrt{J^2 + \lambda m^2}\) and the 1-loop correction in \[\text{27}\] (see also appendix C in \[\text{30}\]) that the string prediction is \(E_0 + E_1 = \ldots + \frac{\lambda^3 \ln^6}{J^2}(1 - \frac{16}{3\sqrt{\lambda}}) + \ldots\) This is consistent with the above expression for the phase function (1.19) since it is known that in this case the BDS ansatz gives zero contribution at order \(\lambda^3\) \[\text{25}\] (for the discussion of the case of circular solution with \(J_1 \neq J_2\) see appendix A in \[\text{30}\]). Similarly, our present string result (1.18) will be consistent with the universal phase (1.19) provided the weak-coupling gauge-theory coefficient \(c_{30,0}\) in (1.16) is twice its strong-coupling limit, i.e. the classical string value

11The presence of higher-order corrections in the strong-coupling expansion of the dressing phase \[\text{22}\] corresponding to 2- and higher loop quantum string corrections and translating into the presence of further subleading terms in the strong-coupling expansion of the coefficient \(c_{30}\) in (1.15) (see (1.18) below) appears to imply that, in contrast to \(c_{20}\) and \(c_{21}\), the “1/J^2\” 2-loop coefficient \(c_{22}\) should not be protected when going from weak to strong coupling. Same should be true also for the “1/J^4\” 1-loop coefficient \(c_{14}\) in (1.15), etc. Similar remark should apply to the expansion of the energy of other semiclassical solutions (note that in the near-BMN case \[\text{35}\] the order at which weak-strong coupling non-renormalization should fail is shifted by one power of \(1/J^2\), see sect. 6 in \[\text{30}\]). It would be important to study in detail how the proposal for the dressing phase in \[\text{22, 18}\] extending \[\text{27, 28}\] to all orders modifies the “semiclassical” expansion originally conjectured in \[\text{14, 41}\]. We are grateful to N. Beisert for a discussion of this issue.

12 Same result is found also for the circular solution \[\text{41}\] in the \(SU(3)\) sector \[\text{38}\].
in (1.5), i.e.

\[ c_{30,0} = \frac{1}{8\pi^6} . \]  

(1.20)

Again, it would be interesting to confirm this prediction by a direct computation on the gauge theory side, thus checking again the universal origin of the 1-loop phase \[27, 28\] that corrects the string Bethe ansatz S-matrix \[26\].

We conclude that the structure of our 1-loop string result (1.6) appears to be perfectly consistent with all so far known facts about the strong coupling expansion of the Bethe ansatz.

2 1-loop correction to folded \((S, J)\) string energy in the “long string” approximation

Let us now describe the details of the computation leading to (1.6). What follows will be heavily based on the results of \[2\].

Let us first recall the form of the folded string solution found in \[2\] (which generalizes the one in \[7, 1\])

\[ t = \kappa \tau, \quad \rho = \rho(\sigma), \quad \phi = \omega \tau, \quad \varphi = \nu \tau . \]  

(2.1)

Here \(t\) is global time, \(\rho\) is the radial coordinate in AdS\(_5\) and \(\phi\) and \(\varphi\) are angles in AdS\(_5\) and S\(_5\) respectively. \(\rho\) is determined from

\[
\rho'' = (\kappa^2 - \omega^2) \sinh \rho \cosh \rho, \quad \rho'^2 = \kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho - \nu^2 .
\]  

(2.2)

The periodicity condition on \(\rho(\sigma) = \rho(\sigma + 2\pi)\) is satisfied by considering a folded string configuration. For \(0 \leq \sigma < \pi/2\) the function \(\rho'\) increases from 0 to its maximal value \(\rho_0\), while for \(\pi/2 < \sigma < \pi\), \(\rho\) decreases from \(\rho_0\) to 0. It is useful to introduce the parameter \(\eta\) as

\[ \coth^2 \rho_0 = \frac{\omega^2 - \nu^2}{\kappa^2 - \nu^2} = 1 + \eta, \quad \eta > 0 \]  

(2.3)

The periodicity condition implies

\[ \sqrt{\kappa^2 - \nu^2} = \frac{1}{\sqrt{\eta}} \, _2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \frac{-1}{\eta}\right) \]  

(2.4)

The non-zero charges are the energy \(E\) and two angular momenta \(S\) and \(J\)

\[ E = \sqrt{\lambda} \kappa \int_0^{2\pi} \frac{d\sigma}{2\pi} \cosh^2 \rho = \sqrt{\lambda} \mathcal{E}, \quad S = \sqrt{\lambda} \omega \int_0^{2\pi} \frac{d\sigma}{2\pi} \sinh^2 \rho = \sqrt{\lambda} S, \quad J = \sqrt{\lambda} \nu \]  

(2.5)

where \(\mathcal{E} = \kappa + \frac{\omega}{\sqrt{\kappa^2 - \nu^2}} S\) and

\[ \mathcal{E} = \frac{\kappa}{\sqrt{\kappa^2 - \nu^2}} \frac{1}{\sqrt{\eta}} \, _2F_1\left(-\frac{1}{2}, \frac{1}{2}; 1; \frac{-1}{\eta}\right), \quad S = \frac{\omega}{\sqrt{\kappa^2 - \nu^2} \sqrt{\eta}} \frac{1}{\sqrt{\eta}} \, _2F_1\left(\frac{1}{2}, \frac{3}{2}; 2; \frac{-1}{\eta}\right) \]  

(2.6)
The above hypergeometric functions can be expressed in terms of standard elliptic integrals (see [10]).

Below we will be interested in the particular “long string” limit $\rho_0 \to \infty$, i.e. $\eta \ll 1$, in which the computation can be drastically simplified [2], assuming one is interested in the leading large $\kappa$ correction to the string energy. In this limit

$$
\kappa^2 \approx \nu^2 + \frac{1}{\pi^2} \ln^2 \eta, \quad \omega^2 \approx \nu^2 + \frac{1}{\pi^2} (1 + \eta) \ln^2 \eta, \quad S \approx -\frac{2\omega}{\eta \ln \eta},
$$

(2.7)
i.e. $\kappa \approx \omega$ and $S \gg 1$. To describe both the “slow long string” and the “fast long string” limits we consider a special scaling of $\nu = J^{\sqrt{\lambda}}$ such that

$$
u \equiv \kappa
$$

(2.8)
is kept fixed in the long string limit $\eta \to 0$. By using (2.7) we see that this scaling implies that

$$
\kappa \approx -\frac{\ln \eta}{\pi \sqrt{1 - u^2}}, \quad S \approx -\frac{2\kappa}{\eta \ln \eta} \approx \frac{2}{\pi} \frac{1}{\sqrt{1 - u^2} \eta},
$$

(2.9)
and, therefore, the parameter $\kappa$ is expressed through the spin $S$ and the parameter $u$ as follows

$$
\kappa \approx \frac{\ln(\frac{\pi}{2}S) + \frac{1}{2} \ln(1 - u^2)}{\pi \sqrt{1 - u^2}}.
$$

(2.10)
Taking into account that $u = \frac{\nu}{\kappa}$, we find that $u$ as a function of $\nu$ and $S$ is given by the solution of the following equation

$$
u \approx \frac{\pi \nu \sqrt{1 - u^2}}{\ln(\frac{\pi}{2}S) + \frac{1}{2} \ln(1 - u^2)}.
$$

(2.11)
In the large $S$ limit, and $u < 1$, and not approaching 1, the solution of this equation takes the form

$$
u \approx \frac{\pi \nu}{\ln S} \frac{1}{\sqrt{1 + \left(\frac{\pi \nu}{\ln S}\right)^2}},
$$

(2.12)
where we replaced $\ln(\frac{\pi}{2}S)$ by $\ln S$. It is worth noting that we do not assume here that $u$ is small. This formula is valid for large $S$, and any finite $u < 1$, e.g., for $u = 1/2$. Since $u$ is kept fixed in the large $S$ limit, eq.(2.12) also implies that the ratio $\nu/\ln S \sim J/\ln S$ is fixed too. Taking into account that $\kappa = \frac{\nu}{\kappa}$, and that in the long string limit $E - S \approx \sqrt{\lambda} \kappa$, we get the following expression for the classical energy

$$
E_0 - S \approx \frac{\sqrt{\lambda}}{\pi} \ln S \sqrt{1 + \left(\frac{\pi \nu}{\ln S}\right)^2} = \sqrt{J^2 + \frac{\lambda}{\pi^2} \ln^2 \frac{S}{\sqrt{\lambda}}},
$$

(2.13)
Expanding this in powers of \( \frac{\pi \nu}{\ln S} \), we recover the expression derived in [2].

To consider the “fast long string” case with \( S \gg \nu \gg \ln \frac{S}{\nu} \) corresponding to the limit \( u \to 1 \), we introduce the variable \( y \) such that
\[
1 - u^2 = \frac{y^2}{\pi^2 \nu^2} \tag{2.14}
\]
and take the large \( \nu \) limit assuming \( y \ll \nu \). Then the equation (2.11) for \( u \) takes the form
\[
\sqrt{1 - \frac{y^2}{\pi^2 \nu^2}} \approx \frac{y}{\ln \left( \frac{S}{\nu}y \right)} \tag{2.15}
\]
and for \( S \gg \nu \) we get
\[
y \approx \frac{\ln \frac{S}{\nu}}{\sqrt{1 + \ln^2 \frac{S}{\nu}}} \tag{2.16}
\]
This leads to the following expression for the energy of the fast long string
\[
E_0 - S \approx \sqrt{\lambda \nu} \sqrt{1 + \frac{\ln^2 \frac{S}{\nu}}{\pi^2 \nu^2}} = \sqrt{J^2 + \frac{\lambda \ln^2 S}{J}} \tag{2.17}
\]
first derived in [15] and already quoted above in (1.2).

To compute the 1-loop string correction to the classical energy we shall start with the bosonic fluctuation action in conformal gauge as found in [2]: \( I = I_1 + I_2 \), where
\[
I_1 = -\frac{1}{4\pi} \int d^2 \sigma \left[ -\partial_a \bar{t} \partial^a \bar{t} - \mu_t^2 \bar{t}^2 + \partial_a \bar{\phi} \partial^a \bar{\phi} + \mu_\phi^2 \bar{\phi}^2 + \partial_a \bar{\rho} \partial^a \bar{\rho} + \mu_\rho^2 \bar{\rho}^2 + 4\bar{\rho}(\kappa \sinh \rho \cdot \partial_0 \bar{t} - \omega \cosh \rho \cdot \partial_0 \bar{\phi}) \right],
\tag{2.18}
\]
\[
I_2 = -\frac{1}{4\pi} \int d^2 \sigma \left[ \partial_a \beta_i \partial^a \beta_i + m_i^2 \beta_i^2 + \partial_a \bar{\varphi} \partial^a \bar{\varphi} + \partial_a \psi_s \partial^a \psi_s + \nu^2 \psi_s^2 \right],
\tag{2.19}
\]
and
\[
\mu_t^2 = 2\rho^2 - \kappa^2 + \nu^2, \quad \mu_\phi^2 = 2\rho^2 - \omega^2 + \nu^2, \quad \mu_\rho^2 = 2\rho^2 - \kappa^2 - \omega^2 + 2\nu^2, \quad m_i^2 = 2\rho^2 + \nu^2
\]
Here \( \beta_i \) \( (i = 1, 2) \) are fluctuations of two angles of \( AdS_5 \) transverse to \( AdS_3 \) in which the string is moving; \( \psi_s \) \( (s = 1, 2, 3, 4) \) are fluctuations of four \( S^5 \) directions transverse to the center of mass motion.

Solving for the spectrum of fluctuations in general is a difficult task since \( \rho \) is a non-trivial (elliptic) function of \( \sigma \). Fortunately, in the long string limit the fluctuation
action can be brought to more tractable form. To this end we may first perform a field redefinition
\[ \chi = \tilde{\phi} \cosh \rho - \frac{\kappa}{\omega} \tilde{t} \sinh \rho, \quad \zeta = -\tilde{\phi} \sinh \rho + \frac{\kappa}{\omega} \tilde{t} \cosh \rho \] (2.20)
which simplifies the cross-term on the second line in (2.18) when \( \kappa \approx \omega \) as is true in the long-string limit (2.7) when \( \eta \to 0 \) with \( u = \frac{\zeta}{\kappa} \) fixed. It was shown in [2] (see section 6.2 there) that in this limit \( \rho' \) is approximately constant and is equal to
\[ \rho' \approx \pm \sqrt{\kappa^2 - \nu^2}, \quad \rho'' \approx 0. \] (2.21)
except at the turning points \( \sigma = \frac{\pi}{2}, \frac{3\pi}{2} \) where \( \rho' = 0 \). As was argued in [2] (and as in a similar situation in [39]), contribution of these isolated points may be ignored in the computation of the spectrum to leading order in large \( \kappa \). Then the action (2.18) expressed in terms of the rotated coordinates \( \chi \) and \( \zeta \) has constant coefficients
\[ I_1 \approx -\frac{1}{4\pi} \int d^2 \sigma \left[ \dot{\chi}^2 + \chi'^2 + \dot{\zeta}^2 - \zeta'^2 + 4\sqrt{\kappa^2 - \nu^2} \chi' \zeta - 4\kappa \tilde{\phi} \tilde{\phi}' - \nu^2 + \tilde{\rho}^2 \right] \] (2.22)
and thus the spectrum of characteristic frequencies is readily computable. We find that one combination of the \( AdS_3 \) modes \( \chi, \zeta, \tilde{\rho} \) is massless, and thus, like the massless mode \( \tilde{\phi} \) in (2.19), it does not produce nontrivial contribution to string energy. The remaining two modes have the following frequencies
\[ \Omega_{\pm n} = \sqrt{n^2 + 2\kappa^2 \pm 2\sqrt{\kappa^4 + n^2 \nu^2}}, \quad n = 0, \pm 1, \pm 2, \ldots. \] (2.23)
In addition to these two \( AdS_3 \) modes there are also 2 transverse \( AdS_5 \) bosonic modes with mass \( m_{\beta}^2 \approx 2\kappa^2 - \nu^2 \) and 4 transverse \( S^5 \) bosonic frequencies with mass \( \nu^2 \).

As was shown in [2], after the \( \kappa \)-symmetry gauge fixing (and before making any approximations) the quadratic fermionic part of the \( AdS_5 \times S^5 \) superstring action reduces to the standard action for \( 4 + 4 \) 2\( d \) Majorana fermions with \( \rho(\sigma) \)-dependent masses \( m_F = \pm \sqrt{\rho^2 + \nu^2} \). In the “long string” approximation the square of these masses becomes approximately constant and equal to \( m_F^2 \approx \kappa^2 \), so that the fermionic frequencies are
\[ \Omega_{F n} \approx \sqrt{n^2 + \kappa^2}. \] (2.24)
Therefore, the 1-loop correction to the string energy can be written as:
\[ E_1 \approx \frac{1}{2\kappa} \sum_{n=-\infty}^{\infty} K_n = \frac{1}{\kappa} \left( \sum_{n=1}^{\infty} K_n + \frac{1}{2} K_0 \right), \] (2.25)

\[ ^{13}\text{The contribution of these two massless decoupled modes is cancelled against the conformal gauge ghost contribution [2].} \]

\[ ^{14}\text{The zero-mode contribution } K_0 \text{ in a similar equation (6,6) in [2] was omitted since it gave only subleading contribution to the energy. Same will apply here.} \]
where
\[ K_n = \Omega_{+n} + \Omega_{-n} + 2\sqrt{n^2 + 2\kappa^2 - \nu^2} + 4\sqrt{n^2 + \nu^2} - 8\sqrt{n^2 + \kappa^2} . \] (2.26)

It is easy to check that this sum is convergent in the UV. For \( \nu = 0 \) this sum reduces to the expression of [2] found in the \( J = 0 \) case in the static gauge. This confirms that the computations in the static gauge and the conformal gauge agree as they should.

Here we are interested in the value of the sum (2.25) in the scaling limit (2.8) when
\[ \kappa = \frac{\nu}{u} \gg 1 , \quad u = \text{fixed} . \] (2.27)

As in [2] (see also [30, 44]) the leading large \( \kappa \) asymptotics of the sum in (2.25) can be found by replacing it by an integral
\[ E_1 \approx \kappa \int_0^\infty dp \left[ \sqrt{p^2 + 2 + 2\sqrt{1 + u^2 p^2}} + \sqrt{p^2 + 2 - 2\sqrt{1 + u^2 p^2}} + 2\sqrt{p^2 + 2 - u^2 + 4\sqrt{p^2 + u^2 - 8\sqrt{p^2 + 1}}} \right] + O(\kappa^0) . \] (2.28)

This integral happens to be essentially the same as in the case of the 1-loop correction to the energy of the circular \( J_1 = J_2 \) string solution in \( SU(2) \) sector [14, 40] considered in [27] and in Appendix C of [30]. The precise relation between the parameters is as follows: \( \kappa \) in [30] is \( \nu \) here, and the winding number \( m \) in [30] is replaced by the imaginary value \( i\sqrt{\kappa^2 - \nu^2} \) here (recall that the circular solution of [14] is unstable unless \( m^2 \) is formally less than 1 or negative; the folded string solution considered here is stable). Finally, the rescaled angular momentum \( J = \sqrt{\kappa^2 - m^2} \) in [30] corresponds then \( \kappa \) here.

The reason why these two 1-loop expressions are related in this curious way is as follows. In the long string limit of the folded string solution eq. (2.21) implies \( \rho \approx m\sigma, \ m \equiv \pm \sqrt{\kappa^2 - \nu^2}, \) i.e. linear in \( \sigma \). But then the above \( AdS_3 \times S^1 \) \((ds^2 = dp^2 - \cosh^2 \rho dt^2 + \sinh^2 \rho d\phi^2 + d\phi^2)\) configuration with \( \omega \approx \kappa \), i.e. \( t = \kappa \tau, \phi \approx \kappa \tau, \rho \approx m\sigma, \varphi = \nu \tau \) is related by a formal analytic continuation as in [10] to the \( J_1 = J_2 \) circular string solution in \( R \times S^3 \) \((ds^2 = -dt'^2 + d\theta^2 + \cos^2 \theta d\varphi_1^2 + \sin^2 \theta d\varphi_2^2)\) taken in its original (unrotated, cf. [40]) form given in [14]: \( t' = \kappa' \tau, \theta = m'\sigma, \varphi_1 = \varphi_2 = w' \tau, \ w' = \sqrt{\kappa'^2 - m'^2} \) [13].

The evaluation of the integral (2.28) is thus done in the same way as in [30] – by introducing an UV cutoff, doing individual integrals and then taking the cutoff to infinity. Using the identity
\[ \sqrt{p^2 + 2 + 2\sqrt{1 + u^2 p^2}} + \sqrt{p^2 + 2 - 2\sqrt{1 + u^2 p^2}} = \sqrt{4u^2 + (p + \sqrt{p^2 + 4 - 4u^2})^2} \]
\[ 15\text{Under the continuation } t \to \varphi'_1, \rho \to i\theta, \phi \to \varphi'_2, \varphi \to t' \text{ and one is to change the overall sign of the action.} \]
and changing the variable in the corresponding integral \((p \to z = p + \sqrt{p^2 + 4 - 4u^2})\) we end up with

\[
E_1 \approx -\frac{\nu}{u} \left[ 1 - u^2 - \sqrt{1-u^2} + (2 - u^2) \ln[\sqrt{2-u^2(1 + \sqrt{1-u^2})}] + 2u^2 \ln u \right] + O(\kappa^0) \quad (2.29)
\]

Written in terms of \(x\) in (1.3) and \(\nu = J\sqrt{\lambda} (\kappa = 2u, u = \frac{1}{\sqrt{1+x^2}})\) this is the same expression as was given earlier in (1.6), (1.7).

### 3 Concluding remarks

In this paper we computed the one-loop string correction to the energy of folded \((S,J)\) string in a special “long string” limit, confirming the universality of the \(\ln S\) coefficient.

It would be interesting to reproduce this one-loop correction by starting with the quantum string Bethe ansatz. It would give an additional nontrivial check of the one-loop correction [28] to the AFS phase [26]. The new scaling in \(\nu = \frac{J}{\sqrt{\lambda}}\) in the large \(S = S\sqrt{\lambda}\) limit (preserving the leading \(\ln S\) behavior of the string energy but leading to a nontrivial dependence on the parameter \(u\) in (2.11)) which we considered in section 2 may serve as a regulator of the complicated singular integral equations describing the two-spin folded string case in the BA approach.

As we mentioned in section 1, there are few perturbative gauge-theory computations of the leading coefficients in (1.15) that remain to be done in order to check explicitly the correspondence between our results and the expected “dressed” Bethe ansatz picture. It would be interesting also to understand the meaning of the \(\ln \ln S\) terms appearing in the subleading terms in strong-coupling result (1.8) on the gauge theory side.

On the string-theory side, it would be very important to find the 2-loop string-theory counterpart of the 1-loop correction discussed here. This computation may be feasible in the “long-string” limit where the string fluctuation Lagrangian is expected to simplify. This would lead to many new checks of the proposed dressing phase [22, 18].

Among other possible string-theory generalizations, let us mention the computation of 1-loop correction to the energy of long rotating strings with \(n\) spikes [30] (with small or large \(n > 2\)) and also the computation of 1-loop energy of strings with two spins \((S_1,S_2)\) in \(AdS_5\) [45]. This may shed further light on the universality of the \(\log S\) scaling and the properties of the quantum string Bethe ansatz.
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