Abstract

We prove that the definitions of the Kato class by the semigroup and by the resolvent of the Lévy process on $\mathbb{R}^d$ coincide if and only if 0 is not regular for $\{0\}$. If 0 is regular for $\{0\}$ then we describe both classes in detail. We also give an analytic reformulation of these results by means of the characteristic (Lévy-Khintchine) exponent of the process. The result applies to the time-dependent (non-autonomous) Kato class. As one of the consequences we obtain a simultaneous time-space smallness condition equivalent to the Kato class condition given by the semigroup.

1 Introduction

The Kato class plays an important role in the theory of stochastic processes and in the theory of pseudo-differential operators that emerge as generators of stochastic processes. The definition of the Kato class may therefore differ according to the underlying probabilistic or analytical problem. In the first case the primary definition of the Kato condition is

$$\lim_{t \to 0^+} \left[ \sup_x \mathbb{E}^x \left( \int_0^t |q(X_s)| ds \right) \right] = 0 .$$

Here $q$ is a Borel function on the state space of the process $X = (X_t)_{t \geq 0}$. As shown in [13, section 3.2] through Khas’minskii Lemma the condition yields sufficient local regularity of the corresponding Schrödinger (Feynman-Kac) semigroup $T_t f(x) = \mathbb{E}^x [\exp(\int_0^t q(X_s) ds) f(X_t)]$. In particular, the existence of a density, strong continuity or strong Feller property are inherited under (1) from properties of the original semigroup $P_t f(x) = \mathbb{E}^x f(X_t)$ (for details and further results see [13, Theorems 3.10–3.12]). The fact that the Schrödinger operator is essentially self-adjoint and has bounded and continuous eigenfunctions is another consequence of (1), see [11], [31] and [17]. Applications of (1) to quadratic forms of Schrödinger operators are also known and we describe them shortly in Remark [12].

The condition (1) can be understood as a particular smallness with respect to time. The alternative definition of the Kato condition is given by the following space smallness,

$$\lim_{r \to 0^+} \left[ \sup_x \mathbb{E}^x \left( \int_0^\infty e^{-\lambda t} 1_{B(x,r)}(X_t) |q(X_t)| dt \right) \right] = 0 ,$$

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for some \( \lambda > 0 \) (equivalently for every \( \lambda > 0 \), see Lemma 3.2).

In this paper we obtain a precise description of the equivalence of (1) and (2) for Lévy processes in \( \mathbb{R}^d \), \( d \in \mathbb{N} \). In order to formulate the result we recall that a point \( x \in \mathbb{R}^d \) is said to be regular for a Borel set \( B \subset \mathbb{R}^d \) if \( \mathbb{P}^x(T_B = 0) = 1 \), where

\[
T_B = \inf\{t > 0 : X_t \in B\}.
\]

**Theorem 1.1.** Let \( X \) be a Lévy process in \( \mathbb{R}^d \). The conditions (1) and (2) are equivalent if and only if 0 is not regular for \( \{0\} \).

To complete the picture we note that if 0 is regular for \( \{0\} \) and \( X \) is not a compound Poisson process, then we fully describe (1) and (2) in Theorem 4.6 and Theorem 4.12. The descriptions of (1) and (2) in the case of the compound Poisson process \( X \) are given in Proposition 3.8.

To easily read the results of Section 4 we refer the reader to Definition 2 and Section 2.2.

A similar result is proposed by Carmona, Masters and Simon in [11, Theorem III.1] under additional assumption on the transition density of the Lévy process. However, the equivalence of (i) and (iii) from [11, Theorem III.1] which is claimed therein does not even hold for the Brownian motion in \( \mathbb{R} \) and, more generally, for a one-dimensional unimodal Lévy process for which \( \{0\} \) is not polar (see Theorem 4.3). For example the function \( q(x) = \sum_{k=1}^{\infty} 2^k 1_{(k,k+2^{-k})}(x) \) satisfies (i) but fails to satisfy (iii) in [11, Theorem III.1] for such processes. The paper [11] was very influential and the resulting confusion reappears in the literature. For instance (1) and (3) of [16, Proposition 4.5] are not equivalent in general.

The special character of the one-dimensional case can also be seen in [24, Remark 3.1]. In [24, Definition 3.1 and 3.2] the authors discuss the Kato class of measures for symmetric Markov processes admitting upper and lower estimates of transition density with additional integrability assumptions, see [24, Theorem 3.2].

Theorem 1.1 allows also for results on the time-dependent Kato class for Lévy processes in \( \mathbb{R}^d \). Such class is used for instance in [34], [35], [7], [9], [5]. See [30] for a wider discussion of the Brownian motion case, c.f. [30, Theorem 2].

**Corollary 1.2.** Let \( X \) be a Lévy process in \( \mathbb{R}^d \). For \( q : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R} \) we have

\[
\lim_{t \to 0^+} \sup_{s,x} \mathbb{E}^x \left( \int_0^t |q(s + u, X_u)| du \right) = 0,
\]

if and only if

\[
\lim_{r \to 0^+} \sup_{s,x} \mathbb{E}^x \left( \int_0^r 1_{B(x,r)}(X_u) |q(s + u, X_u)| du \right) = 0.
\]

See Section 4 for the proof. If one uses Corollary 1.2 for time-independent \( q \), i.e., let \( q : \mathbb{R}^d \to \mathbb{R} \) and put \( q(u,z) = q(z) \), then the quantity in (3) coincides with (1) and we obtain the following reinforcement of (1) to a time-space smallness condition.

**Corollary 1.3.** Let \( X \) be a Lévy process in \( \mathbb{R}^d \). Then (1) holds if and only if

\[
\lim_{r \to 0^+} \sup_x \mathbb{E}^x \left( \int_0^r 1_{B(x,r)}(X_u) |q(X_u)| du \right) = 0.
\]
In view of the equivalence of (1) and (5) for every Lévy process (see Proposition 3.4 for other description of (1) true for Hunt processes) these conditions should be compared with (2) by its alternative form provided by Proposition 3.6 in a generality of a Hunt process, i.e., (2) holds if and only if

$$\lim_{r \to 0^+} \left[ \sup_x \mathbb{E}^x \left( \int_0^t 1_{B(x,r)}(X_u)|q(X_u)| \, du \right) \right] = 0,$$

for some (every) fixed $t > 0$. The closeness or possible differences between (1) and (2) are now more evident for Lévy processes through (5) and (6).

The variety of conditions we point out is due to possible applications where one can choose a version suitable to the knowledge on the process and derive a clear analytic description of the Kato condition (1). See also Theorem 4.14 and Theorem 4.15 for other conditions. For instance, in Example 1 we apply Theorem 1.1 and we make use of (6). On the other hand, by Theorem 1.1 and (2) we obtain that for a large class of subordinators (1) is equivalent to

$$\lim_{r \to 0^+} \sup_{x \in \mathbb{R}} \int_0^r |q(z + x)| \frac{\phi'(z^{-1})}{z^2 \phi^2(z^{-1})} \, dz = 0,$$

where $\phi$ is the Laplace exponent of the subordinator. See Section 5.2 for details. This is usual that from (2) and (6) one learns about acceptable singularities of $q$.

A discussion of analytic counterparts of (1) should begin with the fundamental example of the standard Brownian motion in $\mathbb{R}^d$, $d \in \mathbb{N}$. The famous result of Aizenman and Simon [1, Theorem 4.5] says that in this case (1) is equivalent to

$$\lim_{t \to 0^+} \sup_{x \in \mathbb{R}} \int_0^r |q(z + x)| \ln \frac{t}{|z - x|^2} \, dz = 0,$$

for $d = 2$, and

$$\sup_{x \in \mathbb{R}} \int_{|z - x| < 1} |q(z)| \, dz < \infty,$$

for $d = 1$.

Here we also refer to Simon [31, Proposition A.2.6], Chung and Zhao [13, Theorem 3.6], Demuth and van Casteren [14, Theorem 1.27]. The above remains true if $\ln(t/|z - x|^2)$ is replaced by $\ln(1/|z - x|)$ for $d = 2$ and if $|q(z)|$ is multiplied by $|z - x|$ for $d = 1$. In fact, the expressions in square brackets of (1) and (7) are comparable for $d \geq 3$, while for $d = 2$ and $d = 1$ similar but slightly different results hold (see Bogdan and Szczypkowski [9], Demuth and van Casteren [14, Theorem 1.28]). We emphasise that (7) was used by Kato [19] to prove by analytic methods that the operator $-\Delta + q$ is essentially self-adjoint (see [20] for extensions to second order elliptic operators). The equivalence of (1) with (7) and (8) follow also by Theorem 1.1 (see [36]). The one-dimensional case is covered by Theorem 4.6 of this paper.

A major contribution to the understanding of this subject in a general probabilistic manner is made by Zhao [36]. Zhao considers a Hunt process $X = (\Omega, \mathcal{F}_t, X_t, \vartheta_t, \mathbb{P}^x)$ with state space $(S, \rho)$ and life-time $\zeta$, where $S$ is a locally compact metric space with a metric $\rho$ (see [1]). For a strong sub-additive functional $A_t$ of $X$, $t \geq 0$, he discusses relations between the following...
In presence of three hypotheses on the process $X$

$$h_1(X) \equiv \sup_{t>0} \sup_{r>0} \sup_{x \in S} \mathbb{P}^x (\tau_{B(x,r)} > t) < 1,$$  \hspace{1cm} (H1)

$$h_2(X) \equiv \sup_{r>0} \sup_{t>0} \sup_{x \in S} \mathbb{P}^x (\tau_{B(x,r)} < t) < 1,$$  \hspace{1cm} (H2)

$$h_3(X) \equiv \sup_{u>0} \sup_{r>0} \sup_{x,y \in S} \mathbb{P}^y (T_{B(x,r)} < \zeta) < 1.$$  \hspace{1cm} (H3)

Here for any Borel set $B$ in $S$, $T_B = \inf \{ t > 0 : X_t \in B \}$ is the first hitting time of $B$, $\tau_B = T_{S \setminus B}$ is the first exit time of $B$ (we let $\inf \emptyset = \infty$) and $B(x,r) = \{ y \in S : \rho(x,y) < r \}$, $x \in S$, $r > 0$.

We present the main theorem of Zhao \cite{36} on Figure 1 below; for instance, under (H3), (C3) implies (C1).

![Figure 1: Zhao \cite{36} hypotheses and conditions.](image-url)

In this paper we assume that $A_t$, $t \geq 0$, is the additive functional of the form

$$A_t = \int_0^t |q(X_s)|ds,$$ \hspace{1cm} (9)

and we note that any additive functional is a strong sub-additive functional; see \cite{36} Lemma 1]. Then (C2) coincides with (11) and as such becomes the principal object of our considerations. We explain the origin and the choice of (2) using the concept of $\lambda$-subprocess $X^{\lambda}$, $\lambda > 0$, of the process $X$ (see \cite{4} for definition). We should first notice that (C2) (and (11)) holds for $X$ if and only if it holds for $X^{\lambda}$, $\lambda > 0$ (see Remark \cite{10} and Definition \cite{2}). A similar statement is not true in general for (C1). In fact for the standard Brownian motion in $\mathbb{R}^d$, $d \geq 3$, (C1) and (7) coincide, but for $d = 2$ or $d = 1$ the expectation in (C1) is infinite for constant non-zero $q$, while this never happens for (C2) (and (11)) and bounded $q$. This indicates that (C1) is too strong for a general equivalence result and therefore to characterize (11) we need to rely on the relations in Figure 1 for $X^{\lambda}$, $\lambda > 0$. This results in (2) which is exactly (C1) for $X^{\lambda}$, $\lambda > 0$. We also observe that (2) holds for $X$ if and only if it holds for $X^{\lambda'}$, $\lambda' > 0$ (see Remark \cite{10} and
To ultimately clarify the choice of $X^\lambda$, $\lambda > 0$, we note that $h_1(X^\lambda) = h_1(X)$, $h_2(X^\lambda) = h_2(X)$ and $h_3(X^\lambda) \leq h_3(X)$ (see Lemma 2.8 and 2.9).

We now restrict ourselves to the case of the Lévy process in $\mathbb{R}^d$. Besides being a Hunt process in $\mathbb{R}^d$, $X$ is also translation invariant. We note that (H2) holds for every Lévy process, and (H1) holds if and only if $X$ is not a compound Poisson process (see Remark 9). As already mentioned, the case of the compound Poisson process can be treated directly and we fully describe conditions (1) and (2) in Proposition 3.8. Thus, in the remaining cases, (H3) for $X^\lambda$ becomes decisive for a complete applicability of Figure 1 to $X^\lambda$. By Proposition 2.14 the study of the expression $h_3(X^\lambda)$ reduces to the analysis of the first hitting time of a single point set by the original Lévy process $X$. Namely, we consider the function (see also Lemma 4.2)

$$h^\lambda(x) = \mathbb{E}_0 e^{-\lambda T(x)}, \quad x \in \mathbb{R}^d.$$ (10)

In Corollary 2.15 we obtain that if $X$ is not a compound Poisson process, then (H3) holds for $X^\lambda$ if and only if $\{0\}$ is polar, i.e., $h^\lambda(x) = 0$, $x \in \mathbb{R}^d$ (see Remark 1). We finally summarize the discussion of (H1)-(H3) in Proposition 2.16 which says that all hypotheses (H1), (H2) and (H3) are satisfied for $X^\lambda$ if and only if $X$ is not a compound Poisson process and $\{0\}$ is polar.

Some of the above observations are made by Zhao [36] as motivation for general considerations on (C1), (C2) and (C3). Eventually, he uses them to re-prove the result of Aizenman and Simon [1] for $d \geq 2$. He also verifies hypotheses (H1)-(H3) directly for $X$ (instead of $X^\lambda$) in the case of Lévy processes admitting symmetric density with additional assumption on the behaviour of the density time integrals [36, Lemma 5]. Afterwards he applies this to describe (C) for symmetric $\alpha$-stable processes, $d > \alpha$, and the relativistic process. We generalize [36, Lemma 5] in Theorem 4.15.

Note that Theorem 1.1 goes much beyond the results of [36] indicated by Figure 1 (see Remark 1 and 2). Nevertheless, we crucially combine the outcomes and methods of Zhao [36] with those of Bretagnolle [10], where the function $h^\lambda$ is examined. The study effects in a list of cases which we additionally classify according to non-degeneracy hypothesis (H0). A full layout of our development is presented in Section 2. Theorem 1.1 results as a summary of Proposition 3.8 and 6 theorems of Section 4. We want to stress that the non-symmetric cases or those close to the compound Poisson process (without (H0)) are more delicate and require more precision.

The paper is organized as follows. In Section 2 we introduce the non-degeneracy hypothesis (H0) for a Lévy process and we use it to classify Lévy processes. This classification provides a detailed plan of our research. In the last part of Section 2 we prove preparatory results on hypotheses (H1)-(H3). In Section 3, for a Hunt process $X$, we define Kato classes $\mathcal{K}(X)$ and $\mathcal{K}(X^\lambda)$ of functions $q$ satisfying (1) and (2), respectively. We prove other general descriptions of both of those classes and we establish their initial relations for Lévy processes. Next, in Section 4 we prove the main description theorems for Lévy processes, separately under and without (H0). Section 4 ends with additional equivalence results involving the class $\mathcal{K}(X^\lambda)$ (see (26)). In Section 5 we present supplementary discussion on isotropic unimodal Lévy processes and subordinators. The paper finishes with examples.
2 Preliminaries

Our main focus in this paper is on a (general) Lévy process $X$ in $\mathbb{R}^d$ (see [28]). The characteristic exponent $\psi$ of $X$ defined by $\mathbb{E}^0 e^{i(x,X_t)} = e^{-t\psi(x)}$ equals

$$\psi(x) = -i \langle x, \gamma \rangle + \langle x, Ax \rangle - \int_{\mathbb{R}^d} \left( e^{i(x,z)} - 1 - i \langle x, z \rangle 1_{|z|<1} \right) \nu(dz), \quad x \in \mathbb{R}^d,$$

where $\gamma \in \mathbb{R}^d$, $A$ is a symmetric non-negative definite matrix and $\nu$ is a Lévy measure, i.e., $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d} (1 \wedge |z|^2) \nu(dz) < \infty$. If $\int_{\mathbb{R}^d} (1 \wedge |z|) \nu(dz) < \infty$, then the above representation simplifies to

$$\psi(x) = -i \langle x, \gamma_0 \rangle + \langle x, Ax \rangle - \int_{\mathbb{R}^d} \left( e^{i(x,z)} - 1 \right) \nu(dz), \quad x \in \mathbb{R}^d,$$

where $\gamma_0 = \gamma - \int_{\mathbb{R}^d} z 1_{|z|<1} \nu(dz)$. Further, if $\gamma_0 = 0$, $A = 0$ and $\nu(\mathbb{R}^d) < \infty$ then $X$ is called a compound Poisson process (see [28] Remark 27.3). We say that $X$ is non-Poisson if $X$ is not a compound Poisson process. A Borel set $B \subseteq \mathbb{R}^d$ is called polar if $\mathbb{P}^x(T_B = \infty) = 1$ for all $x \in \mathbb{R}^d$. A point $x \in \mathbb{R}^d$ is said to be regular for $B$ if $\mathbb{P}^x(T_B = 0) = 1$. Recall that $\mathbb{E}^x F(X) = \mathbb{E}^0 F(X + x)$ for $x \in \mathbb{R}^d$ and Borel functions $F \geq 0$ on paths. In particular $h^\lambda(x) = \mathbb{E}^x e^{-\lambda T(0)}$, and thus the following hold.

**Remark 1.** $\{0\}$ is polar if and only if $h^\lambda(x) = 0$, $x \in \mathbb{R}^d$.

**Remark 2.** 0 is regular for $\{0\}$ if and only if $h^\lambda(0) = 1$.

If necessary we specify which a Lévy process we have in mind by adding a superscript, for instance $h^{Z,\lambda}$ is the function given by (10) that corresponds to the process $Z$.

2.1 Non-degeneracy hypothesis (H0) for Lévy processes

Before we introduce the main non-degeneracy hypothesis on a Lévy process $X$ we recall the basic matrix notation. By $\mathcal{M}_{d \times d}$ we denote the set of all real $d \times d$ dimensional matrices. Consider $\mathbb{R}^d$ as a real linear space with the usual scalar product $\langle \cdot, \cdot \rangle$. For $M \in \mathcal{M}_{d \times d}$ we let $M^*$ to be the transpose of $M$, $N(M) = \{ x \in \mathbb{R}^d : Mx = 0 \}$ to be the kernel of $M$ and $M(\mathbb{R}^d) = \{ Mx : x \in \mathbb{R}^d \}$ the range of $M$. A matrix $M \in \mathcal{M}_{d \times d}$ is symmetric if $M^* = M$, and we call it a projection if it is symmetric and $M^2 = M$. For a subset $V$ by $V^\perp$ we denote the orthogonal complement of $V$ in $\mathbb{R}^d$. For a symmetric matrix $M$ we have $M(\mathbb{R}^d) = N(M)^\perp$. Further, if $M$ is symmetric and $\langle x, Mx \rangle = 0$ holds for every $x$ in a linear subspace $V \subseteq \mathbb{R}^d$, then $M = 0$ on $V$.

**Lemma 2.1.** Let $M, S \in \mathcal{M}_{d \times d}$. If $M$ is symmetric and $S^*MS = 0$ then $M(\mathbb{R}^d) \subseteq S(\mathbb{R}^d)^\perp$.

**Proof.** Since $S^*MS = 0$ implies $M = 0$ on $S(\mathbb{R}^d)$, we have $S(\mathbb{R}^d) \subseteq N(M)$. Finally, $M(\mathbb{R}^d) = N(M)^\perp \subseteq S(\mathbb{R}^d)^\perp$. \hfill $\Box$

**Remark 3.** Let $X$ be a Lévy process in a linear subspace $V$ of $\mathbb{R}^d$ (see [28] Proposition 24.17) and denote $d_0 = \dim(V)$. Then there exists a rotation given by matrix $O \in \mathcal{M}_{d_0 \times d}$ such that $Y = OX$ is a Lévy process in $\mathbb{R}^{d_0}$; the correspondence between $X$ and $Y$ is one-to-one.

**Remark 4.** Let $X$ be a Lévy process in $\mathbb{R}^d$ and let $S$ be a projection on a linear subspace $V$ of $\mathbb{R}^d$. If $\{0\}$ is polar for the process $Y = SX$ then it is polar for $X$. Indeed, if $X_t + x = 0$ then $SX_t + Sx = 0$, thus $\inf\{ t > 0 : X_t + x > 0 \} \geq \inf\{ t > 0 : SX_t + Sx = 0 \}$ and

$$\mathbb{P}^x(T_{\{0\}} < \infty) \leq \mathbb{P}^{SX}(T^{SX}_{\{0\}} < \infty) = 0,$$
where $T_{\{0\}}^{SX}$ is the first hitting time of $\{0\}$ by the process $SX$. The opposite is not true in general.

**Definition 1.** We say that $(H0)$ holds for $X$ if there is no linear subspace $V$ in $\mathbb{R}^d$ such that

$$\dim(V) \leq \min\{1, d - 1\}, \quad A(\mathbb{R}^d) \subseteq V, \quad \nu(\mathbb{R}^d \setminus V) < \infty,$$

and

$$\gamma - \int_{\mathbb{R}^d \setminus V} z 1_{B(0,1)}(z) \nu(dz) \in V. \quad (11)$$

The hypothesis $(H0)$ excludes compound Poisson process and some other processes in $d > 1$. We give a precise description in the following remark.

**Remark 5.** We have

i) $X$ is a compound Poisson process if and only if $(11)$ holds with $V = \{0\}$. Moreover, if $d > 1$ and $X$ is a compound Poisson process then $(11)$ holds with any one-dimensional subspace $V$.

ii) For $d = 1$, $(H0)$ holds if and only if $X$ is not a compound Poisson process.

iii) For $d > 1$, $(H0)$ holds if and only if $X$ is not a compound Poisson process and is not of the form $(12)$, below.

**Proof.** The first part of i) plainly holds. For the proof of the second part we observe that if $X$ is a compound Poisson process, then $\gamma - \int_{\mathbb{R}^d \setminus V} z 1_{B(0,1)}(z) \nu(dz) = 0$ and for any linear subspace $V$ we have $\gamma - \int_{\mathbb{R}^d \setminus V} z 1_{B(0,1)}(z) \nu(dz) = \int_{\mathbb{R}^d \cap V} z 1_{B(0,1)}(z) \nu(dz) \in V$. For $d = 1$ the only proper subspace of $\mathbb{R}$ is $V = \{0\}$, and ii) follows from i). The statement iii) is proved in Proposition 2.2 below.

**Proposition 2.2.** Let $d > 1$ and $X$ be non-Poisson. Then $(11)$ holds if and only if

$$X = Y + Z, \quad (12)$$

and there exist a linear subspace $V$ of $\mathbb{R}^d$, $\dim(V) = 1$, such that

i) $Y$ and $Z$ are independent,

ii) $Y$ is either zero or a compound Poisson process with the Lévy measure vanishing on $V$,

iii) $Z$ is not a compound Poisson process,

iv) $Z$ is supported on $V$.

**Proof.** Since we assume that $X$ is non-Poisson if $(11)$ holds then $\dim(V) = 1$. We let $Y$ to be a compound Poisson process with the Lévy measure $\nu = [\nu]_{\mathbb{R}^d \setminus V}$ and let $Z$ to be a Lévy process with the Lévy triplet $\left( A, \gamma - \int_{\mathbb{R}^d \setminus V} z 1_{B(0,1)}(z) \nu(dz), [\nu]_V \right)$, where $[\nu]_B$ denotes the measure $\nu$ restricted to a set $B$. By definition $\psi = \psi^Y + \psi^Z$, hence $X = Y + Z$ and i), ii) and iii) are satisfied. The property iv) follows from [28 Proposition 24.17]. Conversely, if $X$ is of the form $(12)$ then its Lévy triplet is given by $A = A^Z$, $\gamma = \gamma^Z + \int_{\mathbb{R}^d \setminus V} z 1_{B(0,1)}(z) \nu^Y(dz)$ and $\nu = \nu^Y + \nu^Z$. Then $(11)$ holds since $\nu = \nu^Y$ on $\mathbb{R}^d \setminus V$. 

Remark 6. We stress that for \( d = 1 \) the hypothesis (H0) agrees with the hypothesis (H) from [10], both excluding only compound Poisson process. In particular, for \( d = 1 \) under (H0) we have that \( \{0\} \) is essentially polar if and only if \( \{0\} \) is polar. As known, in \( d > 1 \) \( \{0\} \) is essentially polar (see [3] Theorem 16 and Corollary 17).

Proposition 2.3. Let \( d > 1 \) and assume (H0). Then \( \{0\} \) is polar.

Proof. Let \( V \) be the smallest in dimension linear subspace in \( \mathbb{R}^d \) satisfying

\[
A(\mathbb{R}^d) \subseteq V, \quad \nu(\mathbb{R}^d \setminus V) < \infty, \quad \text{and} \quad \gamma - \int_{\mathbb{R}^d \setminus V} z1_{B(0,1)}(z)\nu(dz) \in V. \tag{13}
\]

Now, let \( T \) be the projection on \( V \) and define \( Y = TX \) the projection of the process \( X \) on \( V \). Observe that by (H0) we have \( \dim(V) \geq 2 \). We claim that there is no one-dimensional subspace \( W \subset V \) such that the projection of \( Y \) on \( W \) is a compound Poisson process. For the proof assume that there is such \( W \) and let \( S \) be the projection on \( W \). Then \( Z = SY = STX = SX \) is a compound Poisson process. By [28 Proposition 11.10] we have the following consequences. First, \( SAS = 0 \) and by Lemma [27] we obtain \( A(\mathbb{R}^d) \subseteq V \cap W^\perp \). Next, \( \nu(\mathbb{R}^d \setminus W^\perp) = \nu S^{-1}(\mathbb{R}^d \setminus \{0\}) < \infty \) and then \( \nu(\mathbb{R}^d \setminus (V \cap W^\perp)) \) < \( \infty \). Further, since \( S_z = 0 \) on \( V \cap W^\perp \) we have

\[
0 = S\gamma - \int_{\mathbb{R}^d} Sz1_{B(0,1)}(z)\nu(dz) = S\gamma - \int_{\mathbb{R}^d \setminus (V \cap W^\perp)} Sz1_{B(0,1)}(z)\nu(dz)
\]

\[
= S\left(\gamma - \int_{\mathbb{R}^d \setminus (V \cap W^\perp)} z1_{B(0,1)}(z)\nu(dz)\right) .
\]

Thus \( \gamma_1 = \gamma - \int_{\mathbb{R}^d \setminus (V \cap W^\perp)} z1_{B(0,1)}(z)\nu(dz) \in W^\perp \). Finally, by \( \mathbb{R}^d \setminus (V \cap W^\perp) = (\mathbb{R}^d \setminus V) \cup (V \setminus W^\perp) \) and by (13),

\[
\gamma_1 = \left(\gamma - \int_{\mathbb{R}^d \setminus V} z1_{B(0,1)}(z)\nu(dz)\right) - \int_{V \setminus W^\perp} z1_{B(0,1)}(z)\nu(dz) \in V ,
\]

which is a contradiction, because then (13) holds with \( V \cap W^\perp \) in place of \( V \) and \( \dim(V \cap W^\perp) < \dim(V) \). Now, by Remark 4 we can treat \( Y \) as a process in \( \mathbb{R}^{d_0} \), \( d_0 = \dim(V) \geq 2 \), and then by [10, Theoreme 4] the set \( \{0\} \) is a polar set for \( Y \) as well as for \( X \) by Remark 4.

2.2 Classification of Lévy processes

We now outline our workflow to analyze every Lévy process \( X \). Though in what follows we assume that \( X \) is not a compound Poisson process, since we deal with this process separately.

We start with \( X \) satisfying (H0). For \( d > 1 \) by Proposition 2.3 and Remark 1 we have \( h^\lambda(x) = 0 \), \( x \in \mathbb{R}^d \), which is satisfactory. According to Remark 6 and [10 Theoreme 3 and 6] one of the following excluding situations holds. For \( d = 1 \) and \( \lambda > 0 \)

(A) \( h^\lambda(x) = 0 \), \( x \in \mathbb{R} \),

(B) \( h^\lambda(0) = \lim\inf_{x \to 0} h^\lambda(x) < \lim\sup_{x \to 0} h^\lambda(x) = 1 \),

(C) \( h^\lambda(0) = \lim_{x \to 0} h^\lambda(x) = 1 \).
This translates equivalently into probabilistic properties of $X$, see [10] Theoreme 6, 8] and Remark [13. We have

(A) $\{0\}$ is polar,
(B) $X$ has finite variation and non-zero drift,
(C) 0 is regular for $\{0\}$.

The analytic counterpart by means of characteristic exponent or Lévy triplet is (see [10, Theoreme 3, 7 and 8])

(A) $\int_R \Re \left( \frac{1}{\lambda + \psi(z)} \right) dz = \infty,$
(B) $A = 0$, $\gamma_0 \neq 0$ and $\int_R (|x| \wedge 1) \nu(dx) < \infty,$
(C) $A \neq 0$ or (A) does not hold and $\int_R (|x| \wedge 1) \nu(dx) = \infty.$

Next, we consider the case when (H0) does not hold. Since we assume that $X$ is not a compound Poisson process, by Remark 3 and Proposition 2.2 we have $d > 1$ and $X$ is given by (12). According to Remark 3 we can treat $Z$ from the decomposition as a non-Poisson process in $\mathbb{R}$ and thus by [10] there are three excluding cases for $Z$:

(A') $h_{Z,\lambda}(v) = 0$, $v \in V$,
(B') $h_{Z,\lambda}(0) = \liminf_{v \to 0} h_{Z,\lambda}(v) < \limsup_{v \to 0} h_{Z,\lambda}(v) = 1,$
(C') $h_{Z,\lambda}(0) = \lim_{v \to 0} h_{Z,\lambda}(v) = 1.$

We could similarly reformulate these cases for $Z$ but in proofs of Theorem 4.11 and Theorem 4.12 we just use the following description.

(A') $\int_V \Re \left( \frac{1}{\lambda + \psi(z)} \right) dv = \infty$, ($dv$ denotes the one-dimensional Lebesgue measure on $V$),
(B') $A^Z = 0$, $\gamma_0^Z \neq 0$ and $\int_V (|x| \wedge 1) \nu^Z(dx) < \infty,$
(C') 0 is regular for $\{0\}$.

We next translate (A'), (B') and (C') into $X$ (given by (12)).

**Lemma 2.4.** $\{0\}$ is polar for $X$ if and only if $\{0\}$ is polar for $Z$.

**Proof.** If $\{0\}$ is polar for $Z$ then $\int_V \Re \left( \frac{1}{\lambda + \psi_S(v)} \right) dv = \infty$. By Remark 4 to verify that $\{0\}$ is polar for $X$ it suffices to show that it is polar for $SX = S(Y + Z) = SY + Z$, where $S$ is the projection on $V$. Since $\psi^{SX} = \psi^{SY} + \psi^Z$ and $\psi^{SY}$ is bounded ($SY$ is a compound Poisson process) we have by our assumption

$$\int_V \Re \left( \frac{1}{\lambda + \psi^{SX}(v)} \right) dv = \infty.$$ 

Thus Remark 3 and [10] Theoreme 7, 3] end this part of the proof. If $\{0\}$ is not polar for $Z$, $\mathbb{P}^0(T_x^Z < \infty) > 0$ for some $x \in V$, we have for large $t > 0$

$$\mathbb{P}^0(T_x < \infty) \geq \mathbb{P}^0(Y_t = 0, T_x = T_x^Z < t) = \mathbb{P}^0(Y_t = 0) \mathbb{P}^0(T_x^Z < t) > 0,$$

\[ \square \]
Lemma 2.5. \( \{0\} \) is not polar for \( X \) if and only if \( \limsup_{x \to 0} h^\lambda(x) = 1 \).

Proof. If \( \limsup_{x \to 0} h^\lambda(x) = 1 \) then \( h^\lambda(x) > 0 \) for some \( x \in \mathbb{R}^d \) and \( \mathbb{P}^0(T_{\{x\}} < \infty) > 0 \). Conversely, if \( \{0\} \) is not polar for \( X \) then by Lemma 2.4 it is not polar for \( Z \) and \( \limsup_{v \in V,v \to 0} h_{Z,\lambda}^\lambda(v) = 1 \). This implies \( \limsup_{v \in V,v \to 0} \mathbb{P}^0(T_{\{v\}}^Z < t) = 1 \) for every fixed \( t > 0 \). Thus we have for \( t > 0 \)

\[
h^\lambda(x) \geq \mathbb{E}^0 \left( Y_t = 0, T_{\{x\}}^Z < t; e^{-\lambda T_{\{x\}}} \right) = \mathbb{E}^0 \left( Y_t = 0, T_{\{x\}}^Z < t; e^{-\lambda T_{\{x\}}} \right)
\]

which gives \( \limsup_{x \to 0} h^\lambda(x) \geq \mathbb{P}^0(Y_t = 0)e^{-\lambda t} \). Finally, we let \( t \to 0^+ \).

Lemma 2.6. \( 0 \) is regular for \( \{0\} \) for \( X \) if and only if \( 0 \) is regular for \( \{0\} \) for \( Z \).

Proof. We just observe that \( T_{\{0\}} = 0 \) if and only if \( T_{\{0\}}^Z = 0 \) on the set \( \{Y_s = 0 \text{ for all } s \in [0, \delta] \text{ for some } \delta > 0\} \), which is of measure one with respect to \( \mathbb{P}^0 \) (see Remark 8).

Corollary 2.7. For the process \( X \) of the form (12) we have

(A') \( \{0\} \) is polar,

(B') \( X \) has finite variation and non-zero drift (see Remark 13),

(C') \( 0 \) is regular for \( \{0\} \).

The next observation follows by summarizing the above considerations, especially by Lemma 2.5. It will facilitate the discussion of (H3) in the next section.

Remark 7. For a non-Poisson Lévy process we have \( \limsup_{x \to 0} h^\lambda(x) = 1 \) or \( h^\lambda(x) = 0, \ x \in \mathbb{R}^d \).

2.3 Hypotheses (H1)-(H3)

We start with a general case of a Hunt process \( X \) on \( S \) with life-time \( \zeta \). In the proofs of Lemma 2.8 and 2.9 all objects corresponding to \( X^\lambda \), the \( \lambda \)-subprocess of \( X \), are indicated with a bar, e.g., \( \overline{T}_B = \inf \{t > 0 : X^\lambda_t \in B\} \).

Lemma 2.8. Let \( \lambda > 0 \). We have \( h_1(X^\lambda) = h_1(X) \) and \( h_2(X^\lambda) = h_2(X) \).

Proof. Recall that \( \inf \emptyset = \infty \). For any Borel set \( B \) in \( S \) and \( t > 0 \) we have \( \{\overline{T}_B > t\} = \{\tau_B > t\} \times [0, \infty) \cup \{\tau_B \leq t\} \times [0, \tau_B) \) and \( \{\overline{T}_B < t\} = \{\tau_B < t\} \times (\tau_B, \infty) \). Thus,

\[
\mathbb{P}^\pi(\overline{T}_B > t) = \mathbb{P}^\pi(\tau_B > t) + \mathbb{E}^\pi(\tau_B \leq t; 1 - e^{-\lambda \tau_B}) \leq \mathbb{P}^\pi(\tau_B > t) + 1 - e^{-\lambda t},
\]

and

\[
\mathbb{P}^\pi(\overline{T}_B < t) = \mathbb{E}^\pi(\tau_B < t; e^{-\lambda \tau_B}) = \mathbb{P}^\pi(\tau_B < t) + \mathbb{E}^\pi(\tau_B < t; e^{-\lambda \tau_B} - 1)
\]

\[
\geq \mathbb{P}^\pi(\tau_B < t) + e^{-\lambda t} - 1.
\]

Since we may change sup\(t>0\) with lim sup\(t\to0^+\), \( h_1(X) \leq h_1(X^\lambda) \leq h_1(X) + \lim_{t \to 0^+}(1 - e^{-\lambda t}) \) and since we may replace inf\(t>0\) with lim inf\(t\to0^+\), \( h_2(X) \geq h_2(X^\lambda) \geq h_2(X) + \lim_{t \to 0^+}(e^{-\lambda t} - 1) \). This ends the proof.
Lemma 2.9. Let $\lambda > 0$. We have $h_3(X^\lambda) \leq h_3(X)$, more precisely
\[
h_3(X^\lambda) = \sup_{u>0} \inf_{r>0} \sup_{x,y \in S} \mathbb{E}^y(T_{B(x,r)} < \zeta; e^{-\lambda T_{B(x,r)}}).
\]

Proof. For any Borel set $B$ in $S$ we have $\{ T_B < \zeta \} = \{ T_B < \zeta \} \times (T_B, \infty)$. This results in $\mathbb{P}^\theta(T_B < \zeta) = \mathbb{E}^y(T_B < \zeta; e^{-\lambda T_B})$.

Now, let $S = \mathbb{R}^d$ be the Euclidean space and $\zeta = \infty$. The following lemmas and corollary address the question whether $h_3(X^\lambda) = \sup_{u>0} \inf_{r>0} \sup_{|x-y| \geq u} \mathbb{E}^y e^{-\lambda T_{B(x,r)}} < 1$.

Lemma 2.10. Let $x \in \mathbb{R}^d$ be fixed. Then
\[
\lim_{r \to 0^+} T_{B(x,r)} = T(x) \quad \mathbb{P}^0 \text{ a.s.}
\]

Proof. Fix $x \in \mathbb{R}^d$. Define the stopping times $T_r = T_{B(x,r)}$ and $T = \lim_{r \to 0^+} T_r, \ r > 0$. Obviously, $T_r < T < T(x)$. It suffices to consider $\{ T < \infty \}$ on the set $\{ T < \infty \}$, otherwise both sides of (14) are infinite. Since $T_r$ is non-increasing in $r > 0$ we have by the quasi-left continuity $\lim_{r \to 0^+} X_{T_r} = X_T$ a.s. on $\{ T < \infty \}$. On the other hand, by the right continuity we have $X_{T_r} \in B(x, r)$ and thus $\lim_{r \to 0^+} X_{T_r} = x$ a.s. on $\{ T < \infty \}$. Finally, $X_T = x$ and consequently $T \geq T(x)$ a.s. on $\{ T < \infty \}$.

Lemma 2.11. Let $\tau_n = \tau_{B_n}$. Then $\lim_{n \to \infty} \tau_n = \infty$ $\mathbb{P}^0$ a.s.

Proof. Denote $\tau = \lim_{n \to \infty} \tau_n$. Since $\tau_n$ is non-decreasing, by the quasi-left continuity $X_{\tau_n} \xrightarrow{n \to \infty} X_\tau$ a.s. on $\{ \tau < \infty \}$. On $\{ \tau < \infty \}$ for $n \geq |X_\tau| + 1$ by the right continuity we have $|X_{\tau_n}| \geq |X_\tau| + 1$, which is a contradiction; it shows that a.s $\tau < \infty$ does not occur.

Lemma 2.12. Let $\lambda > 0$. Then
\[
\sup_{u>0} \inf_{r>0} \sup_{|x| \geq u} \mathbb{E}^0 e^{-\lambda T_{B(x,r)}} = \sup_{x \neq 0} \mathbb{E}^0 e^{-\lambda T(x)}.
\]

Proof. Let $f_r(x) = \mathbb{E}^0 e^{-\lambda T_{B(x,r)}}, \ r \geq 0, x \in \mathbb{R}^d$, where $B(x, 0) = \{ x \}$. Notice that $f_r(x) \geq f_0(x)$. Therefore
\[
a = \sup_{u>0} \inf_{r>0} f_r(x) \geq \sup_{u>0} \inf_{r>0} \sup_{|x| \geq u} f_0(x) = \sup_{x \neq 0} f_0(x) = \sup_{x \neq 0} f_0(x) \geq 0.
\]

It suffices to prove the reverse inequality in the case $a \neq 0$, otherwise (15) holds by (16). Thus let $a \in (0, 1]$. Then for $\varepsilon > 0$ there is $u > 0$ such that for all $r > 0$ we have $\sup_{|x| \geq u} f_r(x) > a - \varepsilon$. Hence, there is a sequence $\{ x_n \}$ such that $f_{1/n}(x_n) > a - \varepsilon$ and $|x_n| \geq u$. We will show that $\{ x_n \}$ is bounded. For $r \in (0, 1], m \in \mathbb{N}$ and $|x| \geq m + 2$, we have $T_{B(x,r)} \geq \tau_{B_m}$ thus by Lemma 2.11 and the dominated convergence theorem there is $m_0$ such that
\[
\sup_{|x| \geq m_0 + 2} f_r(x) \leq \mathbb{E}^0 e^{-\lambda \tau_{m_0}} \leq a - \varepsilon.
\]

This proves that $m_0 + 2 \geq |x_n| \geq u > 0$ for every $n$. We let $y \neq 0$ to be the limit point of $\{ x_n \}$. Observe that for every $r > 0$ there is $n$ such that $B(x_n, 1/n) \subseteq B(y, r)$, which implies $T_{B(y,r)} \leq T_{B(x_n,1/n)}$ and $f_r(y) \geq f_{1/n}(x_n) > a - \varepsilon$. Finally, by Lemma 2.10 and the dominated convergence theorem we obtain
\[
\sup_{x \neq 0} \mathbb{E}^0 e^{-\lambda T(x)} \geq \mathbb{E}^0 e^{-\lambda T(y)} = \lim_{r \to 0} \mathbb{E}^0 e^{-\lambda T_{B(y,r)}} = \lim_{r \to 0} f_r(y) \geq a - \varepsilon.
\]

This ends the proof since $\varepsilon > 0$ was arbitrary.
We continue discussing \((\text{H}1)-(\text{H}3)\) for a Lévy process \(X\) in \(\mathbb{R}^d\).

Lemma 2.13. Let \(X\) be non-Poisson. Then \(\mathbb{P}^0(X_t = 0) = 0\) except for countably many \(t > 0\).

Proof. By [28, Theorem 27.4] it suffices to consider compound Poisson process with non-zero drift. Let then \(\nu\) and \(\gamma_0\) be its Lévy measure and drift. According to the decomposition \(\nu = \nu^d + \nu^c\) for discrete and continuous part (see [28, Chapter 5, Section 27]) we write \(X_t = X_t^d + X_t^c + \gamma_0 t\). For \(t > 0\), by [28, Remark 27.3] \(\mathbb{P}^0(X_t^c \in dz)\) is continuous on \(\mathbb{R}^d \setminus \{0\}\), therefore \(\mathbb{P}^0(X_t^c \in C \setminus \{0\}) = 0\) for any countable set \(C \subset \mathbb{R}^d\). By [28, Corollary 27.5 and Proposition 27.6] there is a countable set \(C_{X^d} \subset \mathbb{R}^d\) such that \(\mathbb{P}^0(X_t^d + \gamma_0 t = 0) > 0\) if and only if \((-\gamma_0 t) \in C_{X^d}\). Thus \(\mathbb{P}^0(X_t^d + \gamma_0 t = 0) = 0\) except for countably many \(t > 0\). Finally,

\[
\mathbb{P}^0(X_t^d + X_t^c + \gamma_0 t = 0) = \mathbb{P}^0(X_t^d = 0, X_t^d + \gamma_0 t = 0) + \mathbb{P}^0(X_t^c = -X_t^d + \gamma_0 t), \quad X_t^d + \gamma_0 t \neq 0 \\
\leq \mathbb{P}^0(X_t^d + \gamma_0 t = 0) + \mathbb{P}^0(X_t^c \in -(C_{X^d} + \gamma_0 t) \setminus \{0\}) = 0,
\]

except for countably many \(t > 0\).

Since \(\mathbb{P}^0(\tau_{\{0\}} > t) \leq \mathbb{P}^0(X_t = 0), t > 0\), Lemma 2.13 reinforces remarks following [36, Lemma 3].

Remark 8. A Lévy process \(X\) is non-sticky (\(\mathbb{P}^0(\tau_{\{0\}} > 0) = 0\)) if and only if \(X\) is non-Poisson.

Remark 9. Using Remark 8 it is shown in [36, Lemma 2 and Lemma 3] that:

\(\text{(H1)}\) holds for every non-Poisson Lévy process \(X\) with \(h_1(X) = 0\).

\(\text{(H2)}\) holds for every Lévy process \(X\) with \(h_2(X) = 0\).

Clearly \(\text{(H1)}\) does not hold for any compound Poisson process.

Proposition 2.14. Let \(X\) be a Lévy process in \(\mathbb{R}^d\) and \(\lambda > 0\). For \(h^\lambda(x)\) defined by \((10)\) we have

\[
h_3(X^\lambda) = \sup_{x \neq 0} h^\lambda(x).
\]

Proof. By Lemma 2.9 \(B(x, r/2) \subseteq B(x, r) \subseteq \overline{B}(x, r)\) and Lemma 2.12

\[
h_3(X^\lambda) = \sup_{u > 0} \inf_{r > 0} \sup_{|x-y| \geq u} \mathbb{E}^y(T_{B(x,r)} < \infty; e^{-\lambda T_{\overline{B}(x,r)}}) = \sup_{u > 0} \inf_{r > 0} \sup_{|x-y| \geq u} \mathbb{E}^0(e^{-\lambda T_{\overline{B}(x,y,r)}}) \\
= \sup_{x \neq 0} \mathbb{P}^0 e^{-\lambda T(x)}.
\]

As a consequence of Proposition 2.14, Remark 7 and Remark 1 we obtain the following improvement of [36, Lemma 4].

Corollary 2.15. Let \(X\) be non-Poisson and \(\lambda > 0\). Then \(\text{(H3)}\) holds for \(X^\lambda\) if and only if \(\{0\}\) is polar for \(X\). If this is the case, then we have \(h_3(X^\lambda) = 0\).

We end the above discussion with the summary statement.

Proposition 2.16. Let \(X\) be a Lévy process in \(\mathbb{R}^d\) and \(\lambda > 0\). All hypotheses \(\text{(H1)}, \text{(H2)}\) and \(\text{(H3)}\) are satisfied for \(X^\lambda\) if and only if \(X\) is not a compound Poisson process and \(\{0\}\) is polar.
3 Kato class

Let $X$ be a Hunt process in $\mathbb{R}^d$. For $t \geq 0$, $\lambda \geq 0$ we define the transition kernel $P_t(x, dz)$ and $\lambda$-potential kernel $G^\lambda(x, dz)$ by

$$P_t(x, B) = \mathbb{P}^x(X_t \in B), \quad G^\lambda(x, B) = \int_0^\infty e^{-\lambda t} P_t(x, B) dt.$$

The corresponding transition operator $P_t$ and $\lambda$-potential operator $G^\lambda$ are given by

$$P_t f(x) = \int_{\mathbb{R}^d} f(z) P_t(x, dz), \quad G^\lambda f(x) = \int_{\mathbb{R}^d} f(z) G^\lambda(x, dz) = \int_0^\infty e^{-\lambda t} P_t f(x) dt,$$

whenever the integrals exist. Moreover, we use the following (truncated) measures

$$G^\lambda_t(x, B) = \int_0^t e^{-\lambda s} P_s(x, B) ds, \quad 0 < t < \infty, \lambda \geq 0.$$

We unify the notation by putting $G^\lambda_\infty = G^\lambda$.

**Definition 2.** Let $q : \mathbb{R}^d \to \mathbb{R}$. We say that $q \in \mathbb{K}(X)$ if (1) holds, i.e.,

$$\lim_{t \to 0^+} \left[ t \sup_{x \in \mathbb{R}^d} \int_0^t P_s |q|(x) ds \right] = 0. \quad (17)$$

We say that $q \in \mathcal{K}(X)$ if for some $\lambda > 0$ (all $\lambda > 0$) (2) holds, i.e.,

$$\lim_{r \to 0^+} \left[ r \sup_{x \in \mathbb{R}^d} \int_{B(x, r)} |q(z)| G^\lambda(x, dz) \right] = 0. \quad (18)$$

If the process $X$ is understood from the context we will write in short $\mathbb{K}$, $\mathcal{K}$ for $\mathbb{K}(X)$, $\mathcal{K}(X)$, respectively.

In the next two lemmas we show that the definition of $\mathcal{K}$ is consistent. The first lemma is an apparent reinforcement of (2) and (18).

**Lemma 3.1.** For every $\lambda \geq 0$, $0 < t \leq \infty$,

$$\left[ t \sup_{x,y \in \mathbb{R}^d} \int_{B(x,r)} |q(z)| G^\lambda_t(y, dz) \right] \leq \left[ r \sup_{x \in \mathbb{R}^d} \int_{B(x,2r)} |q(z)| G^\lambda_t(x, dz) \right], \quad r > 0.$$

**Proof.** Let $T = T_{B(x,r)}$. The strong Markov property leads to

$$\mathbb{E}^y \left( \int_0^\infty e^{-\lambda s} 1_{[0,t]}(s) 1_{B(x,r)}(X_s) |q(X_s)| ds \right) = \mathbb{E}^y \left( T < \infty; \int_0^\infty e^{-\lambda s} 1_{[0,t]}(s) 1_{B(x,r)}(X_s) |q(X_s)| ds \right)$$

$$\leq \mathbb{E}^y \left( T < \infty; e^{-\lambda T} \int_0^\infty e^{-\lambda u} 1_{[0,t]}(u) 1_{B(x,r)}(X_u \theta_T) |q(X_u \theta_T)| du \right)$$

$$= \mathbb{E}^y \left( T < \infty; e^{-\lambda T} \mathbb{E}^{X_T} \left( \int_0^\infty e^{-\lambda u} 1_{[0,t]}(u) 1_{B(x,r)}(X_u) |q(X_u)| du \right) \right).$$
where $\theta$ denotes the usual shift operator. By the right continuity $X_T \in \overline{B}(x, r)$ and $B(x, r) \subseteq B(X_T, 2r)$ on $\{ T < \infty \}$. Thus eventually
\[
\int_{B(x, r)} |q(z)| G^\lambda_t(y, dz) \leq \mathbb{E}^y \left( T < \infty ; e^{-\lambda T} E_{X_T} \left( \int_0^\infty e^{-\lambda u} 1_{(0,t]}(u) 1_{B(X_T, 2r)}(X_u) |q(X_u)| \, du \right) \right) \leq \sup_{x \in \mathbb{R}^d} \mathbb{E}^x \left[ \int_0^\infty e^{-\lambda u} 1_{(0,t]}(u) 1_{B(x, 2r)}(X_u) |q(X_u)| \, du \right] = \sup_{x \in \mathbb{R}^d} \int_{B(x, 2r)} |q(z)| G^\lambda_0(x, dz).
\]

\section*{Lemma 3.2.} If (2) or (18) holds for some $\lambda_0 > 0$, then it holds for every $\lambda > 0$.

\section*{Proof.} Clearly, by the resolvent formula (see [41 Chapter 1, (8.10)]) it suffices to consider the measure $A \mapsto \int 1_A(z) G^{\lambda_0} G^\lambda(x, dz) = \int \int 1_A(z) G^{\lambda_0}(y, dz) G^\lambda(x, dy)$. We have
\[
\int_{B(x, r)} |q(z)| G^{\lambda_0} G^\lambda(x, dz) = \int_{\mathbb{R}^d} \left( \int_{B(x, r)} |q(z)| G^{\lambda_0}(y, dz) \right) G^\lambda(x, dy) \leq \lambda^{-1} \left[ \sup_{x, y \in \mathbb{R}^d} \int_{B(x, r)} |q(z)| G^{\lambda_0}(y, dz) \right].
\]
This ends the proof due to Lemma 3.1.

\section*{Remark 10.} Let $\lambda > 0$. Then $\mathbb{K}(X) = \mathbb{K}(X^\lambda)$ and $\mathcal{K}(X) = \mathcal{K}(X^\lambda)$. The first equality follows by the comparability of $\mathbb{E}^x |q(X_t^\lambda)| = e^{-\lambda u} \mathbb{E}^x |q(X_u)|$ and $\mathbb{E}^x |q(X_u)|$ for $0 \leq u \leq t$. The other arises as a straightforward consequence of the definition of $\mathcal{K}(X)$.

Now, we give alternative characterisations of $\mathbb{K}(X)$ and $\mathcal{K}(X)$.

\section*{Lemma 3.3.} We have
\[
(1 - e^{-1}) \sup_x \left[ G^{1/t} |q|(x) \right] \leq \sup_x \left[ \int_0^t P_s |q|(x) ds \right] \leq e \sup_x \left[ G^{1/t} |q|(x) \right].
\]

\section*{Proof.} Actually, the upper bound holds pointwise as follows,
\[
\int_0^t P_s |q|(x) ds \leq e \int_0^t e^{-s/t} P_s |q|(x) ds \leq e G^{1/t} |q|(x).
\]
We prove the lower bound,
\[
G^{\lambda} |q|(x) \leq \sum_{k=0}^\infty e^{-k} \int_{k/\lambda}^{(k+1)/\lambda} P_{k/\lambda} P_{t-k/\lambda} |q|(x) dt = \sum_{k=0}^\infty e^{-k} P_{k/\lambda} \left( \int_0^{1/\lambda} P_t |q|(\cdot) dt \right)(x) \leq (1 - e^{-1})^{-1} \sup_{z \in \mathbb{R}^d} \int_0^{1/\lambda} P_t |q|(z) dt.
\]

Recall from a general theory that for resolvent operators $R^\lambda$, $\lambda > 0$, of a strongly continuous contraction semigroup on a Banach space we have $\lim_{\lambda \to \infty} \lambda R^\lambda \phi = \phi$. Thus $\lim_{\lambda \to \infty} R^\lambda \phi = 0$ in the norm for every element $\phi$ of the Banach space. For a Markov process the counterparts of the resolvent operators are the $\lambda$-potential operators. By Lemma 3.3 below we express (17) throughout the behaviour of $\lambda$-potential operators at infinity ($\lambda \to \infty$).
Lemma 3.5. For a fixed $\lambda > 0$, Remark 11. Proposition 3.4 extends the equivalence of (i) and (ii) of [11, Theorem III.1] from a subclass of Lévy processes to any Hunt process. Similar result is proved in [23, Lemma 3.1] where authors discuss the Kato class of measures for Markov processes possessing transition densities which satisfy the Nash type estimate (see also [24] for symmetric case). Later in Lemma 3.7 we also show that the assumption of uniform local integrability of $V$ ([11, Theorem III.1]) is a necessary condition for $V \in K(X)$ for any Lévy process $X$ in $\mathbb{R}^d$.

We explain briefly the importance of Proposition 3.4.

Remark 12. In the case of the Brownian motion, as mentioned in [25], by Stein’s interpolation theorem the inequality $\sup_{x \in \mathbb{R}^d} (G^\lambda |q|(x)) \leq \gamma \implies \sup_{x \in \mathbb{R}^d} \|q|^{1/2} \|_2^2 \leq \gamma \| \nabla \phi \|_2^2 + \lambda \| \phi \|_2^2$, $\phi \in C_0^\infty (\mathbb{R}^d)$ (a partial reverse result is proved in [11, Theorem 4.9]). For a counterpart of such implication for other processes see remarks preceding [16, Theorem 4.10]. The latter inequality with $\gamma < 1$ allows to define a self-adjoint Schrödinger operator in the sense of quadratic forms, cf. [26, Theorem 3.17], the analogue of Kato-Rellich theorem, for a precise and general formulation.

We can use Lemma 3.4 to get a better insight into the result of Lemma 3.3.

Lemma 3.5. For $0 < t < \infty$ we have $G_0^t(x,dz) \leq e G^{1/t}(x,dz)$ and

$$(1 - e^{-1}) \sup_{x \in \mathbb{R}^d} \left[ \int_{B(x,r)} |q(z)| G^{1/t}(x,dz) \right] \leq \sup_{x \in \mathbb{R}^d} \left[ \int_{B(x,2r)} |q(z)| G_0^t(x,dz) \right], \quad r > 0.$$ 

Proof. For a fixed $y \in \mathbb{R}^d$ by Lemma 3.3 with $\tilde{q}(z) = q(z) 1_{B(y,r)}(z)$ we have

$$(1 - e^{-1}) \int_{B(y,r)} |q(z)| G^{1/t}(y,dz) = (1 - e^{-1}) G^{1/t} \tilde{q}(y)$$

$$\leq \sup_{x \in \mathbb{R}^d} \int_0^t \sup_{x \in \mathbb{R}^d} \int_{B(y,r)} |\tilde{q}(z)| G_0^t(x,dz) = \sup_{x \in \mathbb{R}^d} \int_{B(y,r)} |q(z)| G_0^t(x,dz).$$

Thus, by Lemma 3.1 we obtain

$$(1 - e^{-1}) \sup_{y \in \mathbb{R}^d} \int_{B(y,r)} |q(z)| G^{1/t}(y,dz) \leq \sup_{x \in \mathbb{R}^d} \int_{B(x,2r)} |q(z)| G_0^t(x,dz).$$

As a consequence in Proposition 3.6 we obtain a description of $K(X)$ by $G_0^t(x,dz)$. This result is about cutting at time $0 < t < \infty$ rather than taking $\lambda = 0$, since the measures $G_0^t(x,dz)$ and $G_0^t(x,dz)$ are comparable if $t$ is finite. This truncation in time is useful when the distribution $P^x(X_s \in dz)$ is well estimated for $s \in (0,t]$ near $x \in \mathbb{R}^d$. See [18, 12] Theorems 2.4 and 3.1 for such estimates. In view of [24, (A2.3), Lemma 4.1 and 4.3] the following result can also be regarded as an extension or counterpart of [24, Theorem 3.1].

Proposition 3.6. $q \in K(X)$ if and only if

$$\lim_{r \to 0^+} \left[ \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} |q(z)| G_0^t(x,dz) \right] = 0.$$ 

for some (equivalently for all) $0 < t < \infty$. 

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We use Proposition 3.6 in Example 1 below.

**Lemma 3.7.** Let $X$ be a Lévy process in $\mathbb{R}^d$. Assume that there are $t > 0$ and $0 \leq M < \infty$ such that for all $x \in \mathbb{R}^d$,

$$
\int_0^t P_s|q|(x) \, ds \leq M.
$$

Then there is a constant $0 \leq M' < \infty$ independent of $q$ such that

$$
\sup_x \int_{B(x,1)} |q(z)| \, dz \leq M'.
$$

**Proof.** Let $\varphi \in C_0(\mathbb{R}^d)$ be such that $\varphi \geq 0$, $\varphi = 1$ on $B(0,1)$ and $\int_{\mathbb{R}^d} \varphi(x) \, dx = N < \infty$. For $x_0 \in \mathbb{R}^d$ we have, for $h \leq t$,

$$
MN \geq \int_0^h \int_{\mathbb{R}^d} P_u|q|(x)\varphi(x_0-x) \, dxd\nu = \int_0^h \int_{\mathbb{R}^d} E^0|q(X_u+x)|\varphi(x_0-x) \, dxd\nu
$$

$$
= \int_0^h \int_{\mathbb{R}^d} E^0 \left[ \int_{\mathbb{R}^d} |q(X_u+x)|\varphi(x_0-x) \, dx \right] \, du = \int_0^h \int_{\mathbb{R}^d} E^0 \left[ \int_{\mathbb{R}^d} |q(z)|\varphi(X_u+x_0-z) \, dz \right] \, du
$$

$$
= \int_0^h \int_{\mathbb{R}^d} |q(z)|P_u\varphi(x_0-z) \, dzu \geq \int_0^h \int_{B(x_0,1)} |q(z)|P_u\varphi(x_0-z) \, dzu
$$

$$
\geq (\varepsilon/2) \int_{B(x_0,1)} |q(z)| \, dz,
$$

where $0 < \varepsilon \leq h$ is such that $\|P_u\varphi - \varphi\|_\infty \leq 1/2$ for $u \leq \varepsilon$ (see [28, Theorem 31.5]). \qed

We write $q \in (L^1_{loc})_{uni}(\mathbb{R}^d)$ if (19) holds. At the end of this section we collect basic properties of $K(X)$ and $M(X)$ for a Lévy process $X$ in $\mathbb{R}^d$.

**Proposition 3.8.** We have

1. $K \subseteq K \subseteq (L^1_{loc})_{uni}(\mathbb{R}^d)$ for every Lévy process,
2. $B(\mathbb{R}^d) \subseteq K$ for every Lévy process,
3. $B(\mathbb{R}^d) \subseteq K$ for every non-Poisson Lévy process,
4. $K = \{0\}$ and $K = B(\mathbb{R}^d)$ for every compound Poisson process.

**Proof.** The inclusion $K \subseteq (L^1_{loc})_{uni}(\mathbb{R}^d)$ follows by Lemma 3.7. To complete the proof of 1 we let $q \in K(X)$, which reads as [C1] for $X^\lambda$, $\lambda > 0$, and $A_t = \int_0^t |q(X^\lambda)| \, ds$. Since by Remark 9 and Lemma 2.8 $h_2(X^\lambda) = 0$ the hypothesis [H2] holds for $X^\lambda$ and we can apply the result of Zhao [36] presented on Figure 1. Thus, [C2] holds for $X^\lambda$. This means $q \in K(X^\lambda)$, and $q \in K(X)$ follows by Remark 10. Plainly, 2 holds. Now, let $X$ be non-Poisson. By Lemma 2.13 we get $P_t(\{0\}) = 0$ for almost all $t > 0$ and consequently $G^\lambda(\{0\}) = 0$. Further, since $G^\lambda(dx)$ is a finite measure, for $q \in B(\mathbb{R}^d)$ we have

$$
\lim_{r \to \infty} \sup_{x \in \mathbb{R}^d} \int_{B_r} |q(x+z)|G^\lambda(dz) \leq \lim_{r \to 0^+} G^\lambda(B_r) \sup_{x \in \mathbb{R}^d} |q(x)| = G(\{0\}) \sup_{x \in \mathbb{R}^d} |q(x)| = 0,
$$

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and it holds. Finally, if \( X \) is a compound Poisson process, then \( G^\lambda(\{0\}) \geq (\lambda + \nu(\mathbb{R}^d))^{-1} > 0 \) and for every \( r > 0 \)

\[
\sup_{x \in \mathbb{R}^d} \int_{B_r} |q(x + z)|G^\lambda(dz) \geq \sup_{x \in \mathbb{R}^d} |q(x)|(\lambda + \nu(\mathbb{R}^d))^{-1}.
\]

Hence \( q \in \mathcal{K} \) if and only if \( q \equiv 0 \). Moreover,

\[
\sup_{x \in \mathbb{R}^d} \int_0^t P_s|q|(x)ds \geq \sup_{x \in \mathbb{R}^d} |q(x)| \int_0^t e^{-\nu(\mathbb{R}^d)s}ds,
\]

which proves \( \Box \)

4 Main Theorems

In this section we consider a Lévy process \( X \) in \( \mathbb{R}^d \) and we pursue according to the cases of Section 2.2. Before we start we give a short proof of Corollary 1.2 directly from Theorem 1.1.

**Proof of Corollary 1.2.** Consider a Lévy process \( Y \) in \( \mathbb{R}^{d+1} = \mathbb{R} \times \mathbb{R}^d \) defined by \( Y_t = (t, X_t), t \geq 0 \), where \( X \) is an arbitrary Lévy process in \( \mathbb{R}^d, d \geq 1 \). Observe that for \( (s, x) \in \mathbb{R}^{d+1} \) we have \( \mathbb{P}(s, x)(Y_u \in B) = \mathbb{E}^s[1_B(s + u, X_u)], u \geq 0 \). Since for \( Y \) \( 0 \) is not regular for \( \{0\} \) Theorem 1.1 applies to \( Y \). Finally, we use (2) taking into account that \( 1_{B_{d+1}(x,r)}(Y_t) = 1_{B_{d+1}(x,r)}(t, X_t) \), where \( B_{d+1}(x, r) \) denotes a ball in \( \mathbb{R}^{d+1} \), can be replaced with \( 1_{[0, r]}(t)1_{B(x,r)}(X_t) \) and that \( e^{-\lambda t} \) is comparable with one for \( t \in [0, r] \). \( \Box \)

4.1 Under (H0)

In this subsection we consider a Lévy process \( X \) satisfying (H0).

**Theorem 4.1.** For \( d > 1 \) or \( d = 1 \) under (A) we have \( \mathcal{K}(X) = \mathbb{K}(X) \).

**Proof.** By Proposition 3.8 we concentrate on \( \mathbb{K}(X) \subseteq \mathcal{K}(X) \). Let \( q \in \mathbb{K}(X) = \mathbb{K}(X^\lambda), \lambda > 0 \). This reads as (C2) for \( X^\lambda \). Since \( X \) is non-Poisson, by Remark 9 and Lemma 2.8 the hypothesis (H1) holds for \( X^\lambda \). To obtain (C1) for \( X^\lambda \), that is to prove \( q \in \mathcal{K}(X) \), it remains to verify (H3) for \( X^\lambda \). In view of Corollary 2.15 it suffices to justify that \( \{0\} \) is a polar set. For \( d > 1 \) this is assured by Proposition 2.3. For \( d = 1 \) it is our assumption.

From now on in this subsection we discuss the case of \( d = 1 \) and we analyze \( \mathbb{K}(X^\lambda) \) rather than \( \mathbb{K}(X) \) (see Remark 10). To this end we use \( G^\lambda_t(dz), \lambda > 0, 0 < t < \infty, \) and observe that

\[
\int_{\mathbb{R}} |q(x + z)|G^\lambda_t(dz) = \int_0^t e^{-\lambda s}P_s|q|(x)ds.
\]

For convenience we recall from [10, Theoreme 7, 1, 5, 6 and 8] the following facts.

**Lemma 4.2.** Let \( d = 1 \) and \( \int_{\mathbb{R}} \Re \left( \frac{1}{\lambda + \psi(z)} \right) dz < \infty, \lambda > 0 \). Then \( G^\lambda(dz) \) has a bounded density \( G^\lambda(z) = k^\lambda h^\lambda(z), z \in \mathbb{R} \), with respect to the Lebesgue measure which is continuous on \( \mathbb{R} \setminus \{0\} \). Further, \( G^\lambda(z) \) is continuous at 0 if and only if \( \{0\} \) is regular for itself (i.e. \( h^\lambda(0) = 1 \)), and then \( 0 < h^\lambda(z) \leq 1 \) for \( z \in \mathbb{R} \).

We investigate the properties of \( G^\lambda_t(dz) \).
Lemma 4.3. Let $d = 1$ and $\int_{\mathbb{R}} \text{Re} \left( \frac{1}{1 + e(z)} \right) dz < \infty$, $\lambda > 0$. Then $G^\lambda_t(z)$ has a bounded density $G^\lambda_t(z)$ with respect to the Lebesgue measure which is lower semi-continuous on $\mathbb{R} \setminus \{0\}$.

Proof. According to Lemma 4.2 we define $F^\lambda(z) = G^\lambda(z)$ on $\mathbb{R} \setminus \{0\}$ and $F^\lambda(0) = \lim sup_{z \to 0} F^\lambda(z)$. Then $F^\lambda(z)$ is a density of $G^\lambda(z)$. Since $G^\lambda_t(B) \leq G^\lambda(B)$ and $G^\lambda_t(B) = G^\lambda(B) - e^{-\lambda t} \int_{\mathbb{R}} G^\lambda(B - z)P_t(dz)$, $G^\lambda_t(dx)$ is absolutely continuous and its density $G^\lambda_t(x)$ can be chosen to satisfy

$$G^\lambda_t(x) = F^\lambda(x) - e^{-\lambda t} \int_{\mathbb{R}} F^\lambda(x - z)P_t(dz).$$

To prove the semi-continuity we observe that for $x_0 \in \mathbb{R} \setminus \{0\}$,

$$G^\lambda_t(x_0) = F^\lambda(x_0) - e^{-\lambda t} \left( \int_{\mathbb{R} \setminus \{x_0\}} F^\lambda(x - z)P_t(dz) + F^\lambda(x - x_0)P_t(\{x_0\}) \right),$$

and by the bounded convergence theorem

$$\liminf_{x \to x_0} G^\lambda_t(x) = F^\lambda(x_0) - e^{-\lambda t} \left( \lim_{x \to x_0} \int_{\mathbb{R} \setminus \{x_0\}} F^\lambda(x - z)P_t(dz) + \limsup_{x \to x_0} F^\lambda(x - x_0)P_t(\{x_0\}) \right) = G^\lambda_t(x_0).$$

Remark 13. A Lévy process $X$ has a Lévy triplet such that $A = 0$, $\gamma_0 \in \mathbb{R}^d$, $\int_{\mathbb{R}^d} (|x| \wedge 1)\nu(dx) < \infty$ if and only if $X$ has finite variation on finite time intervals (see [28, Theorem 21.9]). Then by [32, Theorem 1] $\mathbb{P}^0(\lim_{s \to 0^+} s^{-1}X_s = \gamma_0) = 1$ (see also [28, Theorem 43.20]).

Theorem 4.4. For $d = 1$ under (B) we have

$$\mathcal{K}(X) = \mathbb{K}(X) = \left\{ q : \limsup_{r \to 0^+} \int_{B(x,r)} |q(z)|dz = 0 \right\}.$$

Proof. Without loss of generality we may and do assume that $\gamma_0 > 0$. Due to Proposition 3.3 and Lemma 4.2 (boundedness of the function $G^\lambda$) it remains to prove $\mathbb{K}(X) \subseteq \left\{ q : \limsup_{r \to 0^+} \int_{B(x,r)} |q(z)|dz = 0 \right\}$. By Remark 13 we get $\mathbb{P}^0(\lim_{s \to 0^+} s^{-1}X_s = \gamma_0) = 1$. Hence, there is $\varepsilon > 0$ such that $\mathbb{P}^0(|X_s - \gamma_0s| < \gamma_0s) \geq 1/2$ for $s \leq \varepsilon$. This implies that for $t \leq \varepsilon$,

$$G^\lambda_t(0, 2\gamma_0t) = \int_0^t e^{-\lambda s}\mathbb{P}^0(X_s \in (0, 2\gamma_0t))ds \geq \int_0^t e^{-\lambda s}\mathbb{P}^0(|X_s - \gamma_0s| < \gamma_0s)ds \geq \frac{1 - e^{-\lambda t}}{2\gamma_0}.$$

Hence, $\sup_{t \in [0, 2\gamma_0t]} G^\lambda_t(z) \geq \frac{1 - e^{-\lambda t}}{4\gamma_0} \geq \frac{1 - e^{-\lambda t}}{4\gamma_0}$. Since $G^\lambda_t(z)$ is lower semi-continuous on $\mathbb{R} \setminus \{0\}$ there exist $0 < a_t < b_t \leq \varepsilon$ such that $G^\lambda_t(z) \geq \frac{1 - e^{-\lambda t}}{8\gamma_0}$ for $z \in (a_t, b_t)$. Now, let $q \in \mathbb{K}(X)$. We obtain for $t \leq \varepsilon$,

$$\int_{\mathbb{R}} |q(x + z)|G^\lambda_t(dz) \geq \frac{1 - e^{-\lambda t}}{8\lambda \varepsilon \gamma_0} \int_{a_t}^{b_t} |q(x + z)|dz.$$

Thus,

$$0 = \limsup_{t \to 0^+} \int_{a_t}^{b_t} |q(x + z)|dz \geq \limsup_{r \to 0^+} \int_{B(x,r)} |q(z)|dz.$$

□

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Lemma 4.5. Let \( 0 < M_{G,\lambda} < \infty \) such that
\[
G^\lambda(x) \leq M_{G,\lambda} G^\lambda(y), \quad x, y \in \mathbb{R}, \quad |x - y| \leq 1. \tag{22}
\]
Further, \( G^\lambda_t(x) \) given by (21) is continuous on \( \mathbb{R} \) and
\[
G^\lambda_t(x) \leq G^\lambda(x)(\lambda t + ||P_tf - f||_\infty), \quad f(x) = h^\lambda(-x) \in C_0(\mathbb{R}).
\]

Proof. Let \( F^\lambda \) be defined as in the proof of Lemma 1.5. By Lemma 1.2, the functions \( G^\lambda \) and \( F^\lambda \) are equal and continuous on \( \mathbb{R} \). Further, Lemma 2.11 implies that the function \( h^\lambda(x) = G^\lambda(x)/k^\lambda = \mathbb{E}^0e^{-\lambda T(x)} \) is in \( C_0(\mathbb{R}) \). Since \( h^\lambda(x + y) \geq h^\lambda(x)h^\lambda(y) \), \( x, y \in \mathbb{R} \) (see remarks after [10, Lemma 2]), we get
\[
\frac{G^\lambda(x - z)}{G^\lambda(x)} = \frac{h^\lambda(x - z)}{h^\lambda(x)} \geq h^\lambda(-z).
\]
By positivity and continuity of \( h^\lambda \) we obtain (22) with \( M_{G,\lambda} = \sup_{|z| \leq 1} 1/|h^\lambda(z)| < \infty \). Eventually, by (21),
\[
G^\lambda_t(x) = G^\lambda(x) \left( 1 - e^{-\lambda t} + e^{-\lambda t} \int_{\mathbb{R}} \left( 1 - \frac{G^\lambda(x - z)}{G^\lambda(x)} \right) P_t(dz) \right)
\leq G^\lambda(x) \left( \lambda t + \int_{\mathbb{R}} (h^\lambda(0) - h^\lambda(-z)) P_t(dz) \right).
\]

\[\square\]

Theorem 4.6. For \( d = 1 \) under (C) we have \( \mathcal{K}(X) \subset \mathcal{K}(X) \),
\[
\mathcal{K}(X) = \left\{ q : \limsup_{r \to 0^+} \sup_{x \in \mathbb{R}} \int_{B(x,r)} |q(z)| \, dz = 0 \right\},
\]
and
\[
\mathcal{K}(X) = (L^1_{loc})_{uini}(\mathbb{R}) = \left\{ q : \sup_{x \in \mathbb{R}} \int_{B(x,1)} |q(z)| \, dz < \infty \right\}.
\]

Proof. For \( \mathcal{K}(X) \) we just observe that \( G^\lambda(z) \) is bounded and \( G^\lambda(z) \geq \varepsilon > 0 \) if \( |z| \leq 1 \). Now, we describe \( \mathcal{K}(X) \). The condition \( q \in (L^1_{loc})_{uini}(\mathbb{R}) \) is necessary by Lemma 3.7. We show that it is sufficient. Let \( \lambda > 0 \) and denote \( c_t = \lambda t + ||P_tf - f||_\infty \), where \( f(x) = h^\lambda(-x) = \mathbb{E}e^{-\lambda T(-x)} \).

By Lemma 4.5
\[
\int_{\mathbb{R}} |q(x + z)|G^\lambda_t(dz) \leq c_t \int_{\mathbb{R}} |q(x + z)|G^\lambda(x)dz = c_t \sum_{k=-\infty}^{\infty} \int_{k-1/2}^{k+1/2} |q(x + z)|G^\lambda(x)dz
\leq c_t M_{G,\lambda} \sum_{k=-\infty}^{\infty} G^\lambda(k) \int_{k-1/2}^{k+1/2} |q(x + z)|dz \leq c_t M_{G,\lambda} \sup_{x \in \mathbb{R}} \int_{B(x,1)} |q(z)|dz \sum_{k=-\infty}^{\infty} G^\lambda(k)
\leq c_t (M_{G,\lambda})^2 \lambda^{-1} \sup_{x \in \mathbb{R}} \int_{B(x,1)} |q(z)|dz. \tag{23}
\]
Since \( f \in C_0(\mathbb{R}) \) we get \( c_t \to 0 \) as \( t \to 0^+ \).

\[\square\]
4.2 Without (H0)

In this subsection we assume that (H0) does not hold. In view of Proposition 3.8 we assume that $X$ is non-Poisson. Remark 5 and Proposition 2.2 imply then that $d > 1$ and $X$ is given by (12). Thus the transition kernel of $X$ equals

$$P_t(dx) = P_t^Z * \sum_{n=0}^{\infty} e^{-t\nu^Y(\mathbb{R}^d)} \frac{n^n (\nu^Y)^n}{n!} (dx).$$

The characteristic exponent $\psi$ of $X$ can be written as $\psi = \psi^Y + \psi^Z$. We note that $\psi^Z(z) = \psi^Z(v)$ for $z = v + w \in \mathbb{R}^d$, $v \in V$, $w \in V^\perp$.

We use results of Section 4.1 and analyze the cases (A'), (B') and (C').

Theorem 4.7. Under (A') we have $K(X) = K(X)$.

Proof. Following the proof of Theorem 4.1 it remains to show that $\{0\}$ is polar for the process $X$. This is assured by Lemma 2.7. We proceed to the remaining cases. In particular we investigate $\lambda$-potentials and truncated resolvents of $X$ and $Z$. For $t \geq 0$, $\lambda > 0$, $n \in \mathbb{N}$ we define

$$G^{\lambda}(dx) = \int_0^\infty e^{-s \lambda} P_s(dx) ds, \quad G^{Z,\lambda,n}(dv) = \int_0^\infty s^n e^{-s \lambda} P_s^Z(dv) ds,$$

$$G^X_t(dx) = \int_0^t e^{-s \lambda} P_s(dx) ds, \quad G^Z_t(dv) = \int_0^t e^{-s \lambda} P_s^Z(dv) ds,$$

and we write $G^{Z,\lambda}(dv)$ for $G^{Z,\lambda,0}(dv)$. The measures $G^{Z,\lambda}$, $G^{Z,\lambda}_t$, $G^{Z,\lambda,n}$ are concentrated on $V$.

We reformulate Lemma 4.3 and Lemma 4.5 in view of Remark 3.

Lemma 4.8. Let $\int_V \text{Re}\left(\frac{1}{\lambda + \psi^Z(u)}\right) dv < \infty$, $\lambda > 0$. Then $G^{Z,\lambda}_t(dv)$ has a bounded density $G^{Z,\lambda}_t(v)$ with respect to the Lebesgue measure on $V$ which is lower semi-continuous on $V \setminus \{0\}$. If $0$ is regular for $\{0\}$ for $Z$ then there is $0 < M_{G^{Z,\lambda}} < \infty$ such that

$$G^{Z,\lambda}(v) \leq M_{G^{Z,\lambda}} G^{Z,\lambda}(v'), \quad v, v' \in V, \quad |v - v'| \leq 1,$$

$G^{Z,\lambda}_t(v)$ is continuous on $V$ and

$$G^{Z,\lambda}_t(v) \leq G^{Z,\lambda}(v)(\lambda t + ||P_t^Z f - f||_\infty), \quad f(v) \in C_0(V).$$

Observe that

$$G^{\lambda}(dx) = \sum_{n=0}^{\infty} \frac{1}{n!} G^{Z,\lambda + \nu^Y(\mathbb{R}^d),n} * (\nu^Y)^n(dx). \quad (24)$$

Lemma 4.9. Let $\int_V \text{Re}\left(\frac{1}{\lambda + \psi^Z(u)}\right) dv < \infty$, $\lambda > 0$. Then $G^{Z,\lambda,n}(dv)$ has a density $G^{Z,\lambda,n}(v)$ with respect to the Lebesgue measure on $V$, and

$$G^{Z,\lambda,n}(v) \leq \frac{n!}{\lambda^n} \int_V \text{Re}\left(\frac{1}{\lambda + \psi^Z(u)}\right) du. \quad (25)$$
Proof. By Remark 3 we assume that \( V = \mathbb{R} \) and we observe that the Fourier transform of \( G^{Z,\lambda,n} \) equals
\[
\int_0^\infty t^n e^{-\lambda t} e^{-t\psi Z(\xi)} dt = \frac{n!}{(\lambda + \psi Z(\xi))^{n+1}}, \quad \xi \in \mathbb{R}.
\]
Since \( \text{Re}(1/z) = \text{Re}(\bar{z})/|z|^2 \) and \( \text{Re}[\psi] \geq 0 \) we obtain
\[
\frac{1}{\lambda + \psi Z(\xi)^{n+1}} \leq \lambda^{-n+1} e^{-\lambda t} e^{-t\psi Z(\xi)} dt = \lambda^{-n} \text{Re} \left( \frac{1}{\lambda + \psi Z(\xi)} \right).
\]
This implies that the Fourier transform is integrable and (25) follows by the inversion formula. \( \square \)

Lemma 4.10. Let \( \int V \text{Re} \left( \frac{1}{\lambda + \psi Z(0)} \right) dv < \infty, \lambda > 0. \) Then
\[
\sup_{x \in \mathbb{R}^d} \left( \int_{B(0,r)} q(x+z) G^\lambda(dz) \right) \leq \sup_{x \in \mathbb{R}^d} \left( \int_{B(0,r) \cap V} |q(x+v)| dv \right) C \left[ 1 + \nu^V(\mathbb{R}^d)/\lambda \right],
\]
where \( dv \) is the one-dimensional Lebesgue measure on \( V \) and \( C = \int V \text{Re} \left( 1/[\lambda + \nu^V(\mathbb{R}^d) + \psi Z(u)] \right) du. \)

Proof. By (24) and (25) we have
\[
\int_{B(0,r)} |q(x+z)| G^\lambda(dz) = \sum_{n=0}^\infty \frac{1}{n!} \left( \int_V 1_{B(0,r)}(v+w) |q(x+v+w)| G^{Z,\lambda+n\nu^V(\mathbb{R}^d),n}(dv) \right) \nu^Y)^n(dw)
\]
\[
\leq \sup_{x \in \mathbb{R}^d} \left( \int_{B(0,r)} |q(x+v)| dv \right) C \sum_{n=0}^\infty \left( \frac{\nu^Y(\mathbb{R}^d)}{\lambda + \nu^Y(\mathbb{R}^d)} \right)^n,
\]
and
\[
\sup_{x \in \mathbb{R}^d} \left( \int_{B(0,r)} |q(x+v+w)| dv \right) = \sup_{x \in \mathbb{R}^d} \left( \int_{B(-w,r) \cap V} |q(x+v)| dv \right)
\]
\[
= \sup_{x \in \mathbb{R}^d} \left( \int_{B(-w,r) \cap V} |q(x+v)| dv \right) = \sup_{x \in \mathbb{R}^d} \left( \int_{B(0,r) \cap V} |q(x+v)| dv \right),
\]
where the last equality follows by the translation invariance of the Lebesgue measure on \( V. \) This ends the proof. \( \square \)

Theorem 4.11. Under (B') we have
\[
\mathcal{K}(X) = \mathbb{K}(X) = \left\{ q : \lim_{r \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{B(0,r) \cap V} |q(x+v)| dv = 0 \right\},
\]
where \( dv \) is the one-dimensional Lebesgue measure on \( V. \)

Proof. Lemma 4.10 gives \( \left\{ q : \lim_{r \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{B(0,r) \cap V} |q(x+v)| dv = 0 \right\} \subseteq \mathcal{K}(X). \) By Proposition 3.8 it suffices to show \( \mathbb{K}(X) \subseteq \left\{ q : \lim_{r \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{B(0,r) \cap V} |q(x+v)| dv = 0 \right\}. \) Since for \( t > 0 \) and \( x \in \mathbb{R}^d \) we have
\[
\int_0^t P_s |q|(x) ds \geq \int_0^t \int_{\mathbb{R}^d} |q(x+z)| e^{-s\nu^Y(\mathbb{R}^d)} P_s Z(dz) ds = \int_{\mathbb{R}^d \cap V} |q(x+v)| G_t^{Z,\nu^Y(\mathbb{R}^d)}(dv),
\]
the inclusion follows by adapting the proof of Theorem 4.4 to the one-dimensional process \( Z \) with the support of Lemma 4.8 and Remark 13. \( \square \)
Theorem 4.12. Under (C') we have \( \mathcal{K}(X) \subsetneq \mathbb{K}(X) \),

\[
\mathcal{K}(X) = \left\{ q : \lim_{r \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{B(0,r) \cap V} |q(x+v)| \, dv = 0 \right\},
\]

and

\[
\mathbb{K}(X) = \left\{ q : \sup_{x \in \mathbb{R}^d} \int_{B(0,1) \cap V} |q(x+v)| \, dv < \infty \right\},
\]

where \( dv \) is the one-dimensional Lebesgue measure on \( V \).

Proof. The condition postulated for the description of \( \mathcal{K}(X) \) is sufficient by Lemma 4.10. Next, by Remark 3 and Lemma 4.2 the \( \lambda \)-potential kernel of \( Z \), that is \( G^{Z,\lambda}(dv) := G^{Z,\lambda,0}(dv) \), has a density \( G^{Z,\lambda}(v) \) with respect to the Lebesgue measure on \( V \), such that \( G^{Z,\lambda}(v) \geq \varepsilon > 0 \) if \( v \in B(0,1) \cap V \) (\( \varepsilon \) may depend on \( \lambda \)). Thus,

\[
\int_{B(0,r)} |q(x+z)| G^{\lambda}(dz) \geq \int_{B(0,r) \cap V} |q(x+v)| G^{Z,\lambda+\nu^Y(R^d)}(dv) \geq \varepsilon \int_{B(0,r) \cap V} |q(x+v)| \, dv,
\]

which proves the necessity. Further, the necessity of the condition proposed to describe \( \mathbb{K}(X) \) follows by Remark 3, Lemma 3.7 and

\[
\int_{0}^{t} P_s|q|(x) \, ds \geq \int_{0}^{t} \int_{\mathbb{R}^d \cap V} |q(x+v)| e^{-s\nu^Y(R^d)} P_s^Z(dv) \, ds \geq e^{-t\nu^Y(R^d)} \int_{0}^{t} \int_{\mathbb{R}^d \cap V} |q(x+v)| P_s^Z(dv) \, ds.
\]

For the sufficiency we partially follow the proof of Theorem 4.6. Note that \( \int_{0}^{t} s^n e^{-\lambda s} P_s^Z(dv) \, ds \leq t^n G_t^{Z,\lambda}(dv) \) which gives

\[
G_t^\lambda(dx) \leq \sum_{n=0}^{\infty} \frac{t^n}{n!} G_t^{Z,\lambda+\nu^Y(R^d)} * (\nu^Y)^n(dx).
\]

Thus by Lemma 4.8 and adaptation of (23) we have with \( c_t = (\lambda + \nu^Y(R^d))t + ||P_t^Z f - f||_{\infty} \),

\[
\int_{\mathbb{R}^d} |q(x+z)||G_t^\lambda(dz) \leq \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\mathbb{R}^d} \left( \int_{V} |q(x+v+w)||G_t^{Z,\lambda+\nu^Y(R^d)}(dv) \right) (\nu^Y)^n(dw)
\]

\[
\leq \left( c_t \left( M_{G^{Z,\lambda+\nu^Y(R^d)}} \right) ^2 (\lambda + \nu^Y(R^d))^{-1} \sup_{x \in \mathbb{R}^d} \int_{B(0,1) \cap V} |q(x+v)| \, dv \right) \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_{\mathbb{R}^d} (\nu^Y)^n(dw),
\]

which ends the proof.

\begin{flushright}
\( \Box \)
\end{flushright}

4.3 Zero-potential kernel

In the previous sections and subsections (see 158 and 201) we have already used measures \( G_t^\lambda, \lambda \geq 0, 0 < t \leq \infty \) (with notation \( G_{\infty}^\lambda = G^\lambda \)). Below we present additional sufficient assumptions on a Lévy process \( X \) under which the measure with \( \lambda = 0 \) and \( t = \infty \) can be used to describe the class \( \mathbb{K}(X) \).

The condition we want to analyze now is \( q \in \mathcal{K}^0(X) \) defined by

\[
\lim_{r \to 0^+} \sup_{x \in \mathbb{R}^d} \int_{B(0,r)} |q(z+x)||G^0(dz)| = 0.
\]
Since $G^\lambda(dz) \leq G^0(dz)$, (26) implies $q \in \mathcal{K}(X)$ and thus $\mathcal{K}^0(X) \subseteq \mathcal{K}(X) \subseteq \mathbb{K}(X)$ by Proposition 3.3. Our aim is to obtain the equivalence, that is the implication from $q \in \mathbb{K}(X)$ to (26), and this can only to be the subcase of $\mathcal{K}(X) = \mathbb{K}(X)$. Our basic assumptions will be that $X$ is transient and $\{0\}$ is polar (in Theorem 1.15 polarity will follow implicitly by other assumptions). The former one is necessary, otherwise $G^0(dz)$ is a locally unbounded measure (see [28, Theorem 35.4]) and non-zero constant functions do not belong to $\mathcal{K}^0(X)$, but are always included in $\mathbb{K}(X)$. The polarity of $\{0\}$ assures $\mathcal{K}(X) = \mathbb{K}(X)$. Moreover, if $\{0\}$ is not polar, then the class $\mathbb{K}(X)$ is described explicitly by our theorems. As shown in section 2.2, the polarity of $\{0\}$ is to some extent encoded in the characteristic function $\psi$. It is even more so for the transience property of $X$ (see [28, Remark 37.7]). Finally, we note that $q \in \mathcal{K}^0(X)$ is equivalent to (C1) and $q \in \mathbb{K}(X)$ to (C2) (and with $A_t$ given by (9)). Thus according to Figure 1, we focus on showing (H3) for $X$ (see also Remark 9).

Remark 14. If $X$ is transient, then we have

$$\lim_{r \to 0^+} \mathbb{P}^0(T_{\overline{B}(x,r)} < \infty) = \mathbb{P}^0(T_x < \infty), \quad x \in \mathbb{R}^d.$$  \hspace{1cm} (27)

Such statement is not true in general, but here it follows by $\mathbb{P}^0(T_{\overline{B}(x,r)} < \infty) = \mathbb{P}^0(T_{\overline{B}(x,r)} < \infty, T_{\{x\}} < \infty) + \mathbb{P}^0(T_{\overline{B}(x,r)} < \infty, T_{\{x\}} = \infty)$, Lemma 2.10 and $\lim_{t \to \infty} |X_t| = \infty \mathbb{P}^0$ a.s.

We say that a measure $G^0(dz)$ tends to zero at infinity if $\lim_{|x| \to \infty} \int_{\mathbb{R}^d} f(z + x)G^0(dz) = 0$ for any $f \in C_c(\mathbb{R}^d)$.

Remark 15. Observe that under a certain assumption on the group of the Lévy process [28, Definition 24.21] $G^0(dz)$ tends to zero for every transient $X$ if $d \geq 2$, see [28, Exercise 39.14]. As indicated in [28, Exercise 39.14] the case $d = 1$ is more complicated, see also Remark 18.

Lemma 4.13. Let $X$ be transient. If $G^0(dz)$ tends to zero at infinity then

$$h_3(X) = \sup_{x \neq 0} \mathbb{P}^0(T_{\{x\}} < \infty).$$

Proof. The statement follows by the same proof as for Proposition 2.14 but with $\lambda = 0$ and a version of Lemma 2.12 for $\lambda = 0$. To prove the latter one we also repeat its proof with functions $f_r$ extended to $\lambda = 0$, i.e., $f_r(x) = \mathbb{P}^0(T_{\overline{B}(x,r)} < \infty)$ up to a moment when $a > 0$ and a sequence $\{x_n\}$ such that $f_{1/n}(x_n) > a - \varepsilon$ are chosen. The rest of the proof easily applies with (27) in place of Lemma 2.10 as soon as we can show that $\{x_n\}$ is bounded. To this end assume that the sequence is unbounded. Since $f_r(x) = \mathbb{P}^0(T_{\overline{B}(x+y,r)} < \infty)$, $r > 0$, $y \in \mathbb{R}^d$, for $r \in (0,1]$ and $|x - x_n| < 1$ we have

$$a - \varepsilon < f_r(x_n) = \mathbb{P}^{-x}(T_{\overline{B}(x_n-x,r)} < \infty) \leq \mathbb{P}^{-x}(T_{\overline{B}(0,2)} < \infty) = f_2(x),$$  \hspace{1cm} (28)

Next, by [28, Theorem 42.8 and Definition 41.6] for $g \in C_c(\mathbb{R}^d)$ such that $1_{B(0,1)} \leq g$ we get

$$\int_{\mathbb{R}^d} g(x_n-x)f_2(x)\,dx = \int_{\mathbb{R}^d} \left[ \int_{\mathbb{R}^d} g(-v + w + x_n)G(dv) \right] m_{\overline{B}(0,2)}(dw) \xrightarrow{n \to \infty} 0,$$

since $m_{\overline{B}(0,2)}(dw)$ is finite and supported on $\overline{B}(0,2)$ and $G(dv)$ tends to zero at infinity. This contradicts (28) and ends the proof.

Theorem 4.14. Let $X$ be transient, $\{0\}$ be polar and $G^0(dz)$ tend to zero at infinity. Then $q \in \mathbb{K}(X)$ if and only if (26) holds, i.e., $\mathcal{K}^0(X) = \mathcal{K}(X) = \mathbb{K}(X)$. 

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In the next result we improve [36, Lemma 5] and we cover some cases when $G^0(dz)$ may not tend to zero at infinity.

**Theorem 4.15.** Let $X$ be transient and let $G^0(dz)$ have a density $G^0(z)$ with respect to the Lebesgue measure which is unbounded and bounded on $|z| \geq r$ for every $r > 0$. Then $\mathcal{K}^0(X) = \mathcal{K}(X) = \mathbb{K}(X)$.

**Proof.** We note that the polarity of $\{0\}$ follows by our assumptions (see [28, Theorem 41.15 and 43.3]). By [28, Proposition 42.13 and Definition 42.9] for $r > 0$ we have

$$P^x(T_{B(0,r)} < \infty) = \int_{B(0,r)} G^0(y - x) m_{B(0,r)}(dy), \quad x \in \mathbb{R}^d.$$ 

Next, for $u > 0$, $|x| \geq u$ and $0 < r < u/2$ we obtain,

$$P^x(T_{B(0,r)} < \infty) \leq \left[ \sup_{|y| \geq u/2} G^0(y) \right] C(B(0, r)),$$

where $C(\cdot)$ stands for capacity. By [28, Proposition 42.10 and (42.20)] and Remark [14] we have $\lim_{r \to 0^+} C(B(0, r)) = C(\{0\})$ (see also [27, Proposition 8.4]). This gives

$$h_3(X) = \sup_{u > 0} \inf_{r > 0} \sup_{|x| \geq u} P^x(T_{B(0,r)} < \infty) \leq \sup_{u > 0} \left[ \sup_{|y| \geq u/2} G(y) \right] \inf_{0 < r < u/2} C(B(0, r))$$

$$= \sup_{u > 0} \left[ \sup_{|y| \geq u/2} G(y) \right] C(\{0\}).$$

Finally, since $\{0\}$ is polar, by [28, Theorem 42.19] we have $C(\{0\}) = 0$ and so ($H_3$) holds with $h_3(X) = 0$. \hfill \Box

## 5 Further discussion and applications

In this section we give additional results for isotropic unimodal Lévy processes concerning (the implication) $\mathcal{K}(X) \subseteq \mathbb{K}(X)$, we apply general results to a subclass of subordinators and finally we present a number of examples.

We recall from [6] the definition of weak scaling. Let $\theta \in [0, \infty)$ and $\phi$ be a non-negative non-zero function on $(0, \infty)$. We say that $\phi$ satisfies the weak lower scaling condition (at infinity) if there are numbers $\alpha \in \mathbb{R}$ and $c \in (0, 1]$, such that

$$\phi(\eta \theta) \geq c \eta^\alpha \phi(\theta) \quad \text{for} \quad \eta \geq 1, \quad \theta > \theta.$$

In short we say that $\phi$ satisfies $\text{WLSC}(\alpha, \theta, c)$ and write $\phi \in \text{WLSC}(\alpha, \theta, c)$. Similarly, we consider $\bar{\theta} \in [0, \infty)$. The weak upper scaling condition holds if there are numbers $\bar{\alpha} \in \mathbb{R}$ and $\overline{c} \in [1, \infty)$ such that

$$\phi(\eta \theta) \leq \overline{c} \eta^{\bar{\alpha}} \phi(\theta) \quad \text{for} \quad \eta \geq 1, \quad \theta > \bar{\theta}.$$

In short, $\phi \in \text{WUSC}(\bar{\alpha}, \bar{\theta}, \overline{c})$. 

\[24\]
5.1 Isotropic unimodal Lévy processes

A measure on \( \mathbb{R}^d \) is called isotropic unimodal, in short, unimodal, if it is absolutely continuous on \( \mathbb{R}^d \setminus \{0\} \) with a radial non-increasing density function (such measures may have an atom at the origin). A Lévy process \( X \) is called (isotropic) unimodal if all of its one-dimensional distributions \( P_t(dx) \) are unimodal. Unimodal Lévy processes are characterized in [33] by isotropic unimodal Lévy measures \( \nu(dx) = \nu(|x|)dx \). The distribution (transition probability) of \( X_t \) has a radial non-increasing density \( p(t, x) \) on \( \mathbb{R}^d \setminus \{0\} \), and atom at the origin, with mass \( \exp\{-t\nu(\mathbb{R}^d)\} \) (no atom if \( \psi \) is unbounded).

For a continuous non-decreasing function \( \psi : [0, \infty) \to [0, \infty) \), such that \( \psi(0) = 0 \), we let \( \phi(\infty) = \lim_{s \to \infty} \phi(s) \) and we define the generalized left inverse \( \phi^- : [0, \infty) \to [0, \infty) \),

\[
\phi^-(u) = \inf\{s \geq 0 : \phi(s) = u\} = \inf\{s \geq 0 : \phi(s) > u\}, \quad 0 \leq u < \infty,
\]

with the convention that \( \inf \emptyset = \infty \). The function is increasing and càglâd where finite. Notice that \( \phi(\phi^-(u)) = u \) for \( u \in [0, \phi(\infty)] \) and \( \phi^-(\phi(s)) \leq s \) for \( s \in [0, \infty) \). Moreover, by the continuity of \( \phi \) we have \( \phi^- (\phi(s) + \varepsilon) > s \) for \( \varepsilon > 0 \) and \( s \in [0, \infty) \). We also define \( f^*(u) = \sup_{|x| < u} |f(x)| \) for \( f : \mathbb{R}^d \to \mathbb{R} \).

In view of general results for Schrödinger perturbations [8, Theorem 3] and the so-called 3G type inequalities [7, (40) and Corollary 11] it is desirable to have the following results which extend [14, Theorem 1.28] and [9, Proposition 4.3] (see also [8, Remark 2]).

**Proposition 5.1.** Let \( X \) be unimodal. For \( 0 < t_0 \leq \infty \), \( r > 0 \) and \( 0 < t < t_0 \),

\[
\sup_{x \in \mathbb{R}^d} \int_0^t P_s|q|(x)ds \leq \left(1 + \frac{t}{|B(0,1/2)|r^d G^0_{t_0}(r)}\right) \left[ \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} |q(z)|G^0_{t_0}(z-x)dz \right],
\]

where \( G^0_{t_0}(z) = \int_0^{t_0} p(s, z)ds \), \( z \in \mathbb{R}^d \) and \( G^0_{t_0}(r) = G^0_{t_0}(x), \ |x| = r \).

**Proof.** We use [9, Lemma 4.2] with \( k(x) = \int_0^t p(s, x)ds \) and \( K(x) = G^0_{t_0}(x) \).

In what follows we assume that \( d \geq 3 \) and since \( X \) is (isotropic) unimodal with an unbounded Lévy-Khintchine exponent thus the assumptions of Theorem 4.15 are satisfied by [28, Theorem 37.8] and radial monotonicity of \( G^0 \). Hence \( \mathcal{K}_0(X) = \mathcal{K}(X) = \mathbb{K}(X) \). Under additional assumptions we strengthen this relation.

**Remark 16.** Below we use the result of [15, Theorem 3] which says that if \( X \) is unimodal and \( d \geq 3 \) we always have \( G^0(x) \leq C/(|x|^d \psi^*(|x|^{-1})) \), \( x \in \mathbb{R}^d \), for some \( C > 0 \). If additionally \( \psi \in \text{WLSC}(\alpha, \bar{\alpha}, \underline{c}) \), \( \alpha > 0 \), then \( c/(|x|^d \psi^*(|x|^{-1})) \leq G^0(x) \) for \( |x| \) small enough and some \( c > 0 \).

**Corollary 5.2.** Let \( d \geq 3 \), \( X \) be unimodal with \( \psi \in \text{WLSC}(\alpha, \bar{\alpha}, \underline{c}) \), \( \alpha > 0 \). There exist constants \( C = C(d, \alpha, \underline{c}) \) and \( b = (d, \alpha, \underline{c}) \) such that for any \( 0 < t < 1/\psi^*(\underline{\theta}/b) \) and \( q : \mathbb{R}^d \to \mathbb{R} \),

\[
\sup_{x \in \mathbb{R}^d} \int_0^t P_s|q|(x)ds \leq C \sup_{x \in \mathbb{R}^d} \int_{B(x,1/\psi^*(1/t))} |q(z)|G^0(z-x)dz.
\]

**Proof.** We let \( t_0 = \infty \) in Proposition 5.1. For \( 0 < t < \infty \) we take \( r = 1/(\psi^*)^{-1}(1/t) > 0 \). Since \( \psi^*(r^{-1}) = 1/t \) by [15, Theorem 3] \( r^dG^0(r) \geq c/\psi^*(r^{-1}) = ct \) if \( 1/(\psi^*)^{-1}(1/t) \leq b/\underline{\theta} \) for some constant \( c > 0 \). The last holds if \( t < 1/\psi^*(\underline{\theta}/b) \).
Lemma 5.3. Let \( d \geq 3 \), \( X \) be unimodal and \( \psi \in \mathcal{W} \mathcal{L} \mathcal{S} \mathcal{C}(\alpha, \theta, C) \cap \mathcal{W} \mathcal{U} \mathcal{S} \mathcal{C}(\varpi, \theta, C) \), \( \alpha, \varpi \in (0, 2) \). Then there exist constants \( c = c(d, \alpha, \varpi, C, C, C) \) and \( a = (d, \alpha, \varpi, C, C, C) \) such that for any \( 0 < t < 1/\psi^*(\theta/a) \) and \( q : \mathbb{R}^d \rightarrow \mathbb{R} \),

\[
\sup_{x \in \mathbb{R}^d} \int_0^t P_s|q|(x)ds \geq c \sup_{x \in \mathbb{R}^d} \int_{B(x,1/(\psi^*)-(1/t))} |q(z)|G^0(z-x)dz.
\]

Proof. Let \( x \in \mathbb{R}^d \) be such that \( |x| < 1/(\psi^*)-(1/t) \), which gives \( 1/\psi^*(|x|^{-1}) \leq t \). Further, since \( t < 1/\psi^*(\theta/a) \) implies \( 1/(\psi^*)-(1/t) < a/\theta \) we get \( |x| < a/\theta \) and also \( s \psi^*(\theta/a) < 1 \) if \( s < 1/\psi^*(|x|^{-1}) \). Then \([6]\) Theorem 21 and Lemma 17 \((r_0 = a)\) yield

\[
\int_0^t p(s, x)ds \geq \int_0^{1/\psi^*(|x|^{-1})} p(s, x)ds \geq c^{*} \int_0^{1/\psi^*(|x|^{-1})} s\psi^*(|x|^{-1})ds = \frac{c^{*}}{2|x|^d\psi^*(|x|^{-1})}.
\]

Finally, we apply \([15]\) Theorem 3] to obtain

\[
\int_0^t p(s, x)ds \geq cG^0(x), \quad \text{for} \quad |x| < 1/(\psi^*)-(1/t) .
\]

\[
\square
\]

5.2 Subordinators

Let \( X \) be a subordinator (without killing) with the Laplace exponent \( \phi \). Then \( \phi \) is a Bernstein function (in short BF) with zero killing term. The two important subclasses of BF are special Bernstein functions (SBF) and complete Bernstein functions (CBF). We refer the reader to \([29]\) for definitions and an overview. Since the cases when \( \phi \) is bounded (equivalently \( X \) is a compound Poisson process) or when \( X \) has a non-zero drift \( \gamma_0 \), are completely described by Theorem 3.8 and Theorem 4.4, we assume that

(S1) \( \phi \) is unbounded \((X \text{ is non-Poisson})\) and \( \gamma_0 = 0 \).

Note that under (S1) by Remark 5 the hypothesis \((H0)\) holds and because \( \gamma_0 = 0 \) the assumptions of Theorem 4.1 are satisfied (see section 2.2 or \([10]\) Theoreme 7 and 8)). We conclude this within a remark.

Remark 17. If \( X \) satisfies \( \text{(S1)} \) then \( \{0\} \) is polar and \( \mathcal{K}(X) = \mathbb{K}(X) \).

We impose further assumptions on the Laplace exponent to study \( G^\lambda(dz), \lambda \geq 0 \), and describe its behaviour near the origin:

(S2) \( a + \phi \in \text{SBF} \) for some \( a \geq 0 \) (see \([29]\) Remark 11.21]),

(S3) \( \frac{\partial \phi}{\partial^2} \in \text{WUSC}(-\beta, \bar{\beta}, C), \beta > 0 \).

We shall mention that (S2) is always satisfied if \( \phi \in \text{CBF} \). Indeed, recall that if \( \phi \in \text{CBF} \) then \( a + \phi \in \text{CBF}, a \geq 0, \) and \( \text{CBF} \subset \text{SBF} \).

Remark 18. Recall that \( X \) is a subordinator without killing, i.e., \( \phi \in \text{BF} \) with zero killing term. Note that \( U(dz) = G^a(dz) \) is a potential kernel of (possibly killed) subordinator \( S = X^a \), see \([29]\) (5.2)], which is just the \( a \)-subprocess of \( X \). The Laplace exponent of \( S \) equals \( a + \phi \), thus by \([29]\) Theorem 11.3, formulas (11.9) and Corollary 11.8] we obtain the following
(a) under (S2), the measure \( G^a(dz) \) is absolutely continuous with respect to the Lebesgue measure if and only if \( \nu(0,\infty) = \infty \) (\( X \) is non-Poisson) or \( \gamma_0 > 0 \),

(b) under (S1) and (S2), the density \( G^a(z) \) of \( G^a(dz) \) satisfies: \( G^a(z) = 0 \) on \((-\infty, 0]\), \( G^a(z) \) is finite, positive and non-increasing on \((0, \infty)\), and \( \lim_{z \to 0^+} G^a(z) = \infty \),

(c) under (S2) with \( a = 0 \), \( G^0(dz) \) tends to zero if and only if \( \int_1^\infty xv(dx) = \infty \).

We already know by Remark [17] that \( G^a, a > 0 \), describe \( \mathbb{K}(X) \) by [18]. Now we extend this observation to \( a = 0 \).

**Proposition 5.4.** Assume (S1) and (S2) with \( a = 0 \). Then \( \mathcal{K}^0(X) = \mathcal{K}(X) = \mathbb{K}(X) \), that is \( q \in \mathbb{K}(X) \) if and only if

\[
\lim_{r \to 0^+} \left[ \sup_{z \in \mathbb{R}} \int_0^r |q(z + x)| G^0(z)dz \right] = 0.
\]

**Proof.** Obviously \( X \) is transient and by Remark [18] the result of Theorem [4.15] applies. \( \square \)

**Lemma 5.5.** Assume (S1), (S2) and (S3) and let \( a \geq 0 \) be chosen according to (S2). Then the density \( G^a(z) \) of \( G^a(dz) \) satisfies

\[
G^a(z) \approx \frac{\phi'(z^{-1})}{z^2 \phi^2(z^{-1})}, \quad 0 < z \leq 1.
\]

**Proof.** The Laplace transform of \( G^a(z) \) is given by \( \Phi = 1/|a + \phi| \). Note that

\[
\Phi' = \frac{\phi'}{\phi^2} \left[ \frac{\phi}{a + \phi} \right]^2 \approx \frac{\phi'}{\phi^2} \quad \text{on} \quad [1, \infty).
\]

Thus by [6] Remark 3 \( \Phi' \in \text{WUSC}(\beta, \overline{\theta} \vee 1, \overline{\gamma})/c \), \( c = [\phi(1)/|a + \phi(1)|]^2 \). Next, [6] Lemma 5 and a version of Lemma 13 from [6] imply \( G^a(z) \approx z^{-2} \Phi'(z^{-1}) \approx z^{-2} \phi'(z^{-1})/\phi^2(z^{-1}) \) as \( z \to 0^+ \) (see also [21] Proposition 3.4). The result extends to \( z \in (0, 1] \) by the regularity of both sides of the estimate. \( \square \)

A combination of the above lemma, Remark [17] and Proposition 5.4 implies the following result.

**Proposition 5.6.** Let \( X \) be a subordinator satisfying (S1), (S2) and (S3). Then \( q \in \mathbb{K}(X) \) if and only if

\[
\lim_{r \to 0^+} \sup_{x \in \mathbb{R}} \int_0^r |q(z + x)| \frac{\phi'(z^{-1})}{z^2 \phi^2(z^{-1})} dz = 0.
\]

### 5.3 Examples

We refer the reader to [1], [11], [36] and [24] for basic examples of the Brownian motion, the relativistic process, symmetric \( \alpha \)-stable processes and relativistic \( \alpha \)-stable processes. We can now proceed towards our examples.

**Example 1.** Denote \( A_1 = \{2^n : n \in \mathbb{Z}\} \) and

\[
f(s) = 1_{[0,1]}(s) s^{-\alpha} + e^m 1_{(1,\infty)}(s) e^{-ms^\beta} s^{-\delta}, \quad s > 0,
\]
where \( m > 0, \beta \in (0, 1], \delta > 0 \) and \( \alpha \in (0, 2) \). Define a Lévy measure in \( \mathbb{R} \) as

\[
\nu(dz) = \sum_{y \in A_1} f(|y|) \left( \delta_y(dz) + \delta_{-y}(dz) \right).
\]  

(29)

Let \( X \) be a Lévy process with \( A = 0, \gamma = 0 \) and (an infinite symmetric) \( \nu \) given by [29]. Then \( X \) is a recurrent process, \( \psi(z) \) is a real valued function comparable with \( |z|^2 \wedge |z|^{\alpha} \) (see [18 Example 4] and [28 Corollary 37.6]). Further, if \( \alpha \in (1, 2) \) there is a recurrent process, \( X \) given by (29). Then for some \( c > 0 \)

\[
\int_0^1 p(s, x) \, ds \geq c \, H(|x|), \quad |x| \leq c_2/2.
\]

where

\[
H(r) = \begin{cases} r^{\alpha-1}, & 0 < \alpha < 1, \\ \ln(r^{-1}), & \alpha = 1. \end{cases}
\]

Moreover, by [18 Example 4] there is \( c_3 > 0 \) so that \( p(t, x) \leq c_3 \, t^{-1/\alpha} (1 \wedge t \, |x|^{-\alpha}) \) on \( |x| \leq 1 \), \( t \in (0, 1) \). Thus if \( \alpha \in (1/2, 1] \) then there exists a constant \( c > 0 \) such that

\[
\int_0^1 p(s, x) \, ds \leq c \, H(|x|), \quad |x| \leq 1/2.
\]

Finally, by Proposition 3.6 for \( \alpha \in (1/2, 1] \) we have \( q \in \mathcal{K}(X) = \mathbb{K}(X) \) if and only if

\[
\lim_{r \to 0^+} \int_{B(x,r)} |q(z)| H(|z - x|) \, dz = 0.
\]

We note that this considerations superficially resemble the results of [24] (see especially [24 Definition 3.2]). We explain why [24] does not apply in the present setting. Let \( f(t, x) \) be a function that is non-increasing on \( x \in (0, 1] \) for every fixed \( t \in (0, 1] \). If \( p(t, x) \leq f(t, x) \) by the lower bound for \( p \) and monotonicity of \( f \) we have \( f(t, x) \geq c_4 \, t^{-1/\alpha} (1 \wedge 2^{\alpha k}), x \in (2^{-k-1}, 2^{-k}] \).

Then for \( n(t) = (1/\alpha) \log_2(1/t) \) we obtain

\[
\int_0^1 f(t, x) \, dx \geq c_4 \, t^{-1/\alpha} \sum_{k=0}^{n(t)} 2^{(\alpha-1)k-1} \xrightarrow{t \to 0^+} \infty, \quad \text{if} \quad \alpha \in (0, 1],
\]

Finally, if the upper bound assumption [24 (A2.3)] holds, i.e., \( p(t, x) \leq t^{-1/\beta} \Phi_2(t^{-1/\beta}|x|) = f(t, x) \) for some \( \beta > 0 \), we have

\[
\int_0^{t^{-1/\beta}} \Phi_2(z) \, dz = \int_0^1 f(t, x) \, dx \xrightarrow{t \to 0^+} \infty, \quad \text{if} \quad \alpha \in (0, 1],
\]

which contradicts with the integrability assumption in [24 (A2.3)].

In fact, we have \( p(s, x) \leq c_3 \, t^{-1/\alpha} \Phi_2(t^{-1/\alpha}|x|) \) for \( |x| \leq 1, t \in (0, 1] \) with \( \Phi_2(r) = 1 \wedge r^{-\alpha} \), which is a precise estimate for \( x \in A_1 \) and \( |x| \leq 1 \), and the integrability condition for \( \Phi_2 \) holds only if \( \alpha \in (1, 2) \).
Example 2. Let $\psi(x, y) = |x|^2 + iy$ that is $X_t = (B_t, t)$, where $B_t$ is the standard Brownian motion in $\mathbb{R}^d$ (see [2] 10.4 and Example 13.30). We note that in this case the transition kernel is not absolutely continuous but the potential kernel is. By Theorem 4.15 $q \in K(X)$, i.e.,

$$\lim_{t \to 0^+} \sup_{x \in \mathbb{R}^d, y \in \mathbb{R}} \int_0^t \int \mathbb{R}^d |q(z + x, s + y) - q(z + x, s + y)| s^{-d/2} e^{-|z|^2/(4s)} dz ds = 0,$$

holds if and only if

$$\lim_{r \to 0^+} \sup_{x \in \mathbb{R}^d, y \in \mathbb{R}} \int_0^r \int_{B(0, r)} |q(z + x, s + y) - q(z + x, s + y)| s^{-d/2} e^{-|z|^2/(4s)} dz ds = 0.$$

Now we discuss in detail subordinators. Since functions $\phi$ presented below are unbounded CBF with zero drift term, see [29] Chapter 16: No 2 and 59, Proposition 7.1, they satisfy (S1) and (S2). The assumption (S3) can be easily checked. The first example covers the case of $\alpha$-stable subordinator, $\alpha \in (0, 1)$, and the inverse Gaussian subordinator.

Example 3. Let $\phi(u) = \delta[(u + m)^{\alpha} - m^{\alpha}]$, $\delta > 0$, $m \geq 0$, $\alpha \in (0, 1)$. Then $q \in K(X)$ if and only if

$$\lim_{r \to 0^+} \sup_{x \in \mathbb{R}^d} \int_x^{x+r} |q(z)|(z - x)^{\alpha-1} dz = 0.$$

Example 4. Let $\phi(u) = \ln(1 + u^{\alpha})$, where $\alpha \in (0, 1)$. Then $q \in K(X)$ if and only if

$$\lim_{r \to 0^+} \sup_{x \in \mathbb{R}^d} \int_x^{x+r} \frac{|q(z)|}{(z - x) \ln^2(z - x)} dz = 0.$$

Example 5. Let $\phi(u) = \frac{u}{\ln(1 + u^{\alpha})}$, where $\alpha \in (0, 1)$. Then $q \in K(X)$ if and only if

$$\lim_{r \to 0^+} \sup_{x \in \mathbb{R}^d} \int_x^{x+r} |q(z)| \ln(z - x) dz = 0.$$

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