Transfer matrix spectrum for cyclic representations of the 6-vertex reflection algebra by quantum separation of variables

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Abstract. In this proceeding, we recall the notion of quantum integrable systems on a lattice and then introduce the Sklyanin’s Separation of Variables method. We sum up the main results for the transfer matrix spectral problem for the cyclic representations of the trigonometric 6-vertex reflection algebra associated to the Bazanov-Stroganov Lax operator. These results apply as well to the spectral analysis of the lattice sine-Gordon model with open boundary conditions. The transfer matrix spectrum (both eigenvalues and eigenstates) is completely characterized in terms of the set of solutions to a discrete system of polynomial equations. We state an equivalent characterization as the set of solutions to a Baxter’s like T-Q functional equation, allowing us to rewrite the transfer matrix eigenstates in an algebraic Bethe ansatz form.

1. Introduction
One of the main tasks of statistical mechanics is to understand macroscopic behaviours, like transport properties, specific heat or susceptibility, from microscopic models. The main theoretical tool turns out to be the correlation function, from which the physical quantities can be derived.

Starting with a quantum system described by an Hamiltonian $H$, the research program is clearly established: the first step is to compute the spectrum, i.e. both the eigenvalues and the eigenstates of the Hamiltonian. Then, the computation of form factors (the matrix elements of an operator on a left and a right eigenstate) should be achieved. At zero temperature, the correlation function can be expressed as the mean value of a string of operators on the ground state, and can be decomposed in terms of form factors. In this proceeding, we focus on the first step of such a research program, for models associated to cyclic representations of the 6-vertex reflection algebra [1,2].

The study of quantum models with integrable open boundary conditions has attracted a large research interest, as they are of physical interest to describe both equilibrium and out of equilibrium physics. See [1,2] and the references therein. In this proceeding, we analyze the class of open integrable quantum models associated to cyclic representations of the 6-vertex reflection algebra. Let us recall that in [3], Sklyanin has shown how to construct classes of quantum integrable models with integrable boundaries in the framework of the so-called Quantum Inverse Scattering Method [8,9], constructing families of commuting transfer matrices (conserved charges of the model). In fact, Sklyanin’s construction allows to use solutions of
the Yang-Baxter equation to generate new solutions of the reflection equation [4] once a scalar solution of this last equation is known. Then as a consequence of the reflection equation these new solutions generate commuting transfer matrices [3]. More generally, one can associate to any closed integrable quantum model (characterized by a solution of the Yang-Baxter equation) new open integrable quantum models (characterized by the associated Sklyanin’s solutions of the reflection equation).

The literature of integrable quantum models associated to cyclic representations of the 6-vertex reflection algebra is so far rather sparse, with the exception of some special representations and boundary conditions, like the open XXZ chains at the roots of unity, that can be traced back to these representations under some special constraints, and for which some results are known in the framework of algebraic Bethe ansatz (ABA) [8,9]. In order to study the class of models in the Sklyanin’s construction associated to general scalar solutions of the 6-vertex reflection equation [5,6] and the general Bazhanov-Stroganov cyclic solution of the 6-vertex Yang-Baxter algebra [12], we have to go beyond traditional methods [8,9] which do not apply for these general settings. This is done by developing the Sklyanin Separation of Variables (SoV) method [10] for this class of models, a method that has the advantage to lead (mainly by construction) to the complete characterization of the spectrum and has proven to be applicable for a large variety of integrable quantum models where traditional methods fail.

The aim of this proceeding is to give a short introduction to the works [1,2], where we have generalized this type of results to the 6-vertex cyclic representations of the reflection algebra. In section 2 we recall generalities about the notion of quantum integrability and focus on integrable systems on a lattice. We also introduce the cyclic 6-vertex Yang-Baxter algebra. In section 3, we deal with closed and open chains, introducing representations of the cyclic 6-vertex reflection algebra. The section 4 introduces the main tool, ie Sklyanin’s Separation of Variables, while section 5 is dedicated to the results, giving the spectrum characterization of the considered representations.

2. Generalities on quantum integrability

2.1. Quantum integrability

Let a quantum system be described by an Hamiltonian \( H \) in a Hilbert space \( \mathcal{H} \).

**Definition 2.1** (Quantum integrability). This system is quantum integrable if there exists an operator family \( T(\lambda) \) acting on \( \mathcal{H} \), parametrized thanks to \( \lambda \in \mathbb{C} \), such that :

- \( \forall (\lambda, \mu) \in \mathbb{C}^2 \) \( \quad [T(\lambda), T(\mu)] = 0 \)
- \( \forall \lambda \in \mathbb{C} \) \( \quad [T(\lambda), H] = 0 \)
- \( \forall \lambda \in \mathbb{C} \), \( T(\lambda) \) is diagonalizable and with non degenerated spectrum.

With this definition, the set of the eigenstates of \( T(\lambda) \) gives an eigenbasis of the space \( \mathcal{H} \) and thus we can characterize the spectrum of the Hamiltonian of a quantum integrable system by characterizing the spectrum of the family \( T(\lambda) \).

Starting with a given quantum system, it is difficult to directly exhibit such a commuting family \( T(\lambda) \). The standard computation consists in rather constructing a family of commuting operators, thanks to an algebraic structure, and then to give Hamiltonians which commute with this family. These Hamiltonians are constructed from the elements of the family, and depending on the previous algebraic structure, one can reproduce physical models, such as spin chain Hamiltonians for example.

The heart of integrability is thus the above-mentioned algebraic structure, the so-called Yang-Baxter algebra :

Let \( R(x_1, x_2) \in \mathcal{A} \otimes \mathcal{A} \), a matrix solution of the Yang-Baxter equation

\[
R_{12}(\lambda, \mu)R_{13}(\lambda, \nu)R_{23}(\mu, \nu) = R_{23}(\mu, \nu)R_{13}(\lambda, \nu)R_{12}(\lambda, \mu) \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \quad (1)
\]
The Yang-Baxter algebra associated to \( R \) is then
\[
R_{12}(\lambda, \mu) M_{1Q}(\lambda, \xi) M_{2Q}(\mu, \xi) = M_{2Q}(\mu, \xi) M_{1Q}(\lambda, \xi) R_{12}(\lambda, \mu) \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{H} \tag{2}
\]
The elements of the matrix \( M \) are operators on the total Hilbert space \( \mathcal{H} : M_{0Q}(\lambda) \in \text{End}(\mathcal{A} \otimes \mathcal{H}) \).

By taking the trace on the spaces 1 and 2 of the equation (2), it is very easy to show the following proposition :

**Proposition 2.1.** If the \( R \)-matrix \( R_{12} \) is invertible, then a commuting family of operators \( T(\lambda) \) on \( \mathcal{H} \) is given by :
\[
T(\lambda) = tr_0(M_{0Q}(\lambda)) \tag{3}
\]

The question is now how to find a matrix \( M \) satisfying the Yang-Baxter algebra. It might be involved, as the matrix elements are acting on the total Hilbert space \( \mathcal{H} \).

### 2.2. Integrable lattice systems

Let us study systems on a lattice, which are decomposed in \( N \) sites, with degrees of freedom on each site. The Hilbert space takes the form of a tensor product of \( N \) local Hilbert spaces
\[
\mathcal{H} = \bigotimes_{n=1}^{N} \mathcal{H}_n \tag{4}
\]

For example, on each site, the local operators can be taken as canonical operators \( \hat{x} \) and \( \hat{p} \) (satisfying \( [\hat{x}, \hat{p}] = i \)), or Weyl operators \( u_n \) and \( v_n \) (satisfying \( u_n v_m = q^{\delta_{nm}} v_m u_n \), \( q \in \mathbb{C} \)) or also spin operators \( S^\alpha_n \), \( \alpha \in \{1, 2, 3\} \) (satisfying \( [S^\alpha_n, S^\beta_m] = i\varepsilon_{\alpha\beta\gamma} S^\gamma_n \delta_{nm} \)), acting in \( \mathcal{H}_n \).

With this formalism, one can describe the Heinsenberg spin chains or the discrete sine-Gordon 1D model.

Thanks to the work of Fadeev, Takhtajan and Sklyanin [8,9], we know that given an \( R \)-matrix satisfying (1) we can obtain a solution to the Yang-Baxter algebra (2) by considering the (local) Yang-Baxter algebra :
\[
R_{12}(\lambda, \mu) L_{1n}(\lambda, p_n) L_{2n}(\mu, p_n) = L_{2n}(\mu, p_n) L_{1n}(\lambda, p_n) R_{12}(\lambda, \mu) \text{ in } \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{H}_n \tag{5}
\]
\( L_{0n}(\lambda) \) are matrices called the Lax operators, with matrix elements acting on the local Hilbert space \( \mathcal{H}_n : L_{0n}(\lambda) \in \text{End}(\mathcal{A} \otimes \mathcal{H}_n) \).

The next proposition gives the expression of a solution to the Yang-Baxter algebra using solutions to the local algebra :

**Proposition 2.2** (The monodromy matrix). For a system on a lattice, a solution to the Yang-Baxter algebra associated to \( \mathcal{R} \) can be computed by taking the ordered product of solutions of the local Yang-Baxter algebras on each site :
\[
M_{0Q}(\lambda) = L_{0,qN}(\lambda)L_{0,qN-1}(\lambda)...L_{0,q1}(\lambda) \tag{6}
\]
This matrix is then called the monodromy matrix.

### 2.3. The cyclic 6-vertex Yang-Baxter algebra

As described before, an \( \mathcal{R} \) matrix and a Lax operator are enough to describe an integrable system on a lattice. In this proceeding, we consider

- the standar trigonometric 6-vertex \( \mathcal{R} \) matrix : \( (\mathcal{A} = \mathbb{C}^2) \)

\[
\mathcal{R}(\lambda, \mu) = \begin{pmatrix}
q\lambda/\mu - (q\lambda/\mu)^{-1} & 0 & 0 & 0 \\
0 & \lambda/\mu - (\lambda/\mu)^{-1} & q - q^{-1} & 0 \\
0 & q - q^{-1} & \lambda/\mu - (\lambda/\mu)^{-1} & 0 \\
0 & 0 & 0 & q\lambda/\mu - (q\lambda/\mu)^{-1}
\end{pmatrix} \tag{7}
\]
- the Bazhanov-Stroganov Lax operator [12]:

\[
L_{0n}(\lambda) = \left( \begin{array}{c}
\lambda \alpha_n v_n - \frac{\beta_n}{\lambda} \nu^{-1} \\
u_n^{-1} \left( q^{1/2} \alpha_n v_n + q^{-1/2} \beta_n \nu^{-1} \right)
\end{array} \right)
\]

(8)

The free parameters \((a_n, b_n, c_n, d, \alpha_n, \beta_n, \gamma_n, \delta_n)\) have to satisfy the constraints

\[
a_n c_n = \alpha_n \gamma_n \quad \text{and} \quad b_n d_n = \beta_n \delta_n
\]

(9)

to satisfy the Yang Baxter equation.

The local operators \(u_n\) and \(v_n\) are generators of a Weyl algebra of dimension \(p\):

\[
u_n^p = 1 \quad ; \quad q^p = 1
\]

(10)

We emphasize the fact that the free parameter \(q\) is taken to be a \(p\)th root of unit.

The considered Lax operator is thus the cyclic local generator of the trigonometric 6-vertex Yang-Baxter algebra.

For different choices of the free parameters, one can describe the discretized sine-Gordon model, the chiral-Potts model, or the XXZ spin \(s\) chain at \(p\)th-root of unity, with \(2s + 1 = p\).

3. On the integrable boundary conditions

3.1. Closed chains

Thanks to propositions 2.1 and 2.2, the transfer matrix

\[
T(\lambda) = tr_0 (M_0Q(\lambda))
\]

(11)

defines a one parameter family of commuting operators. However, when constructing the local Hamiltonians from this transfer matrix, the chain is closed. (cf figure 1.)

\[\text{Figure 1. A closed spin chain. The last site N and the first one are neighbours.}\]

Considering a scalar solution \(P\) of the Yang-Baxter algebra (2), one can describe twisted integrable boundary conditions, by using the commuting family:

\[
T(\lambda) = tr_0 (P_0.M_0Q(\lambda))
\]

(12)

Different matrices \(P\) lead to the anti-periodic chain or more generally to twisted chains.

3.2. More general integrable boundary conditions

In order to describe more general integrable boundary conditions [3,4], we follow the general procedure introduced by Sklyanin, associating to any solution \(M_0Q(\lambda) \in \text{End}(C^2 \otimes H)\) of the 6-vertex Yang-Baxter equation, a solution \(U_{-0Q}(\lambda) \in \text{End}(C^2 \otimes H)\) of the 6-vertex reflection equation:

\[
R_{12}(\lambda/\mu)U_{-1Q}(\lambda)R_{12}(\lambda/\mu)U_{-2Q}(\mu) = U_{-2Q}(\mu)R_{12}(\lambda/\mu)U_{-1Q}(\lambda)R_{12}(\lambda/\mu)
\]

(13)
The solution, called open monodromy matrix, takes the form:

\[ U_{-0Q}(\lambda) = M_{0Q}(\lambda) K_{-0}(\lambda) M_{0Q}^{-1}(\lambda) \]  

(14)

where \( K_{-}(\lambda) \) is a scalar solution of the reflection algebra.

Moreover, we can consider the following dual 6-vertex reflection equation:

\[
\mathcal{R}_{12}(\frac{\mu}{\lambda}) U_{+1Q}(\lambda) \mathcal{R}_{12}(\frac{1}{\lambda \mu q}) U_{+2Q}(\mu) = U_{+1Q}(\lambda) \mathcal{R}_{12}(\frac{1}{\lambda \mu q}) U_{+2Q}(\mu) \mathcal{R}_{12}(\frac{\mu}{\lambda}) \]  

(15)

**Proposition 3.1** (Boundary transfer matrix). If \( K_{+0}(\lambda) \) is a scalar solution of the dual 6-vertex reflection equation, a one parameter family of commuting transfer matrices is given by

\[ T(\lambda) = tr_0 (K_{+0}(\lambda) U_{-0Q}(\lambda)) \]  

(16)

\( T(\lambda) \) is then called the boundary transfer matrix.

Thanks to this boundary transfer matrix, one can describe integrable quantum Hamiltonians with general integrable boundary conditions, linked with the \( K_{+} \) matrices. For example, in the case of a XXZ spin chain, diagonal boundary matrices describe z-oriented boundary magnetic fields.

**Figure 2.** An open chain: the boundary sites are exposed to external actions. The study of such systems is more complicated than in closed systems (the algebra is quadratic in the \( R \) matrix), but is worth as it can lead to the description of non equilibrium physics.

Here, we are interested in the most general scalar (boundary matrices) solutions of the trigonometric 6-vertex reflection equation and its dual [5, 6]

\[
K_{-0}(\lambda) = \begin{pmatrix} a_{-}(\lambda) & b_{-}(\lambda) \\ c_{-}(\lambda) & d_{-}(\lambda) \end{pmatrix}_{[0]} = \frac{1}{\xi_{-} - 1/\xi_{-}} \begin{pmatrix} \lambda \xi_{-}^{1/2} - q^{1/2} / \lambda_{-} & \kappa_{-} e^{-\tau_{-}} \left( \frac{\lambda^{2}}{q} - q^{2} \right) \\ \kappa_{-} e^{\tau_{-}} \left( \frac{\lambda^{2}}{q} - q^{2} \right) / \xi_{-}^{-1/2} - q^{1/2} / \lambda_{-} \end{pmatrix}_{[0]} \]  

(17)

\[
K_{+0}(\lambda) = \begin{pmatrix} a_{+}(\lambda) & b_{+}(\lambda) \\ c_{+}(\lambda) & d_{+}(\lambda) \end{pmatrix}_{[0]} = \frac{1}{\xi_{+} - 1/\xi_{+}} \begin{pmatrix} \lambda q^{1/2} / \xi_{+} - q^{1/2} / \xi_{+} & \kappa_{+} e^{\tau_{+}} \left( q^{2} - \lambda_{+}^{2} / q \right) \\ \kappa_{+} e^{-\tau_{+}} \left( q^{2} - \lambda_{+}^{2} / q \right) / \xi_{+}^{1/2} - q^{1/2} / \lambda_{+} \end{pmatrix}_{[0]} \]  

(18)

Let us use the following stansars notation for the monodromy matrices:

\[
M_{0Q}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[0]} \quad \text{and} \quad U_{-0Q}(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_{[0]} \]  

(19)

Using Sklyanin procedure, equation (14) and the quantum determinant, the generators of the open monodromy matrix are quadratic in the generators of the closed monodromy matrix. For example,

\[
B(\lambda) = -a_{-}(\lambda) A(\lambda) B(\frac{1}{\lambda}) + b_{-}(\lambda) A(\lambda) A(\frac{1}{\lambda}) - c_{-}(\lambda) B(\lambda) B(\frac{1}{\lambda}) + d_{-}(\lambda) B(\lambda) A(\frac{1}{\lambda}) \]

Then, the boundary transfer matrix has the expression

\[ T(\lambda) = a_{+}(\lambda) A(\lambda) + d_{+}(\lambda) D(\lambda) + b_{+}(\lambda) C(\lambda) + c_{+}(\lambda) B(\lambda) \]  

(20)
The characterization of the spectrum (eigenvalues and eigenstates) of this class of transfer matrices is the main subject of our works \cite{1,2} and of this proceeding. In order to give this characterization, \textit{ie} to diagonalize this $T(\lambda)$ operator, let us introduce the main tool : the quantum separation of variables.

4. The quantum separation of variables

This method, introduced by Sklyanin, is an alternative to the Bethe ansatz.

\textbf{Definition 4.1 (The separate basis).} Let $O(\lambda)$ be a co-diagonalizable one parameter family of operators on $\mathcal{H}$. If

- the spectrum of the $O(\lambda)$ operators is non degenerated
- and the spectral problem for the transfer matrix $T(\lambda)$, in the eigenbasis of $O(\lambda)$, separates into several one variable independant problems

then the eigenbasis of $O(\lambda)$ constitutes a separate basis.

In the case of the Yang-Baxter algebra, Sklyanin’s idea is to use the operator $B(\lambda)$ of the monodromy matrix to construct a separate basis \cite{10}. This idea is in fact generalizable to the reflection algebra case for a certain class of models \cite{7,11}. For example, $B(\lambda)$ can be used for models with a lower triangular boundary matrix $K_+$. Let indeed $\langle \xi_1, ..., \xi_n, ..., \xi_M |$ be the eigenbasis of the $B(\lambda)$, parametrized such that :

$$\langle \xi_1, ..., \xi_n, ..., \xi_M | B(\lambda) = b_0 \prod_{k=1}^{M} \left( \frac{\lambda}{\xi_k} - \frac{\xi_k}{\lambda} \right) \left( \lambda \xi_k - \frac{1}{\lambda \xi_k} \right) \langle \xi_1, ..., \xi_n, ..., \xi_M |$$ (21)

For a lower triangular $K_+$, the boundary transfer matrix reads

$$T(\lambda) = a_+(\lambda) A(\lambda) + d_+(\lambda) D(\lambda) + c_+(\lambda) B(\lambda)$$ (22)

and using now (21) and the reflection algebra commutation relations, one gets

$$\langle \xi_1, ..., \xi_n, ..., \xi_M | T(\xi_n) = a(\xi_n) \langle \xi_1, ..., q \xi_n, ..., \xi_M | + d(\xi_n) \langle \xi_1, ..., \xi_n/q, ..., \xi_M |$$ (23)

where $a(\lambda)$ and $d(\lambda)$ are known functions. The essential point to derive (23) is that in the zeros of $B(\lambda)$, the actions of $A(\lambda)$ and $D(\lambda)$ are just shifts on the spectral parameter.

Now, let $|t\rangle$ be an eigenvector of the boundary transfer matrix with eigenvalue $t(\lambda)$

$$T(\lambda) |t\rangle = t(\lambda) |t\rangle$$ (24)

and let us denote by $\Psi_t$ the wave functions associated to $t(\lambda)$

$$\Psi_t(\xi_1, ..., \xi_M) = \langle \xi_1, ..., \xi_M | t\rangle$$ (25)

Projecting the equation (23) on an eigenvector $|t\rangle$, one gets the Baxter equation for the wave functions :

$$t(\xi_n) \Psi_t(\xi_1, ..., \xi_M) = a(\xi_n) \Psi_t(\xi_1, ..., q \xi_n, ..., \xi_M) + d(\xi_n) \Psi_t(\xi_1, ..., \xi_n/q, ..., \xi_M)$$ (26)

The Baxter equation clearly shows that the spectral problem for the transfer matrix $T(\lambda)$, in the eigenbasis of $B(\lambda)$, separates in 1 variable independant problems. The variables $(\xi_n)_{1 \leq n \leq M}$ are called separate variables.
Moreover, let \( Q_t(\xi_n) \) be the solution of the one variable problem (26). We have the following representation in separate variables

\[
\Psi_t(\xi_1, \ldots, \xi_M) = \prod_{k=1}^{M} Q_t(\xi_k)
\]  

(27)

In the next section, I explain how to implement this method to solve the spectral problem of the 6-vertex cyclic representations of the reflection algebra. For brevity, in this proceeding we only state the main propositions of [1] and [2].

5. Characterization of the transfer matrix spectrum

Here we present the complete characterization of the transfer matrix spectrum using the separation of variables. We first exhibit the separate basis, and then give two different characterizations

- in terms of the solution to a system of polynomial equations
- in terms of the solution to a functional Baxter type equation

5.1. The quantum separation of variables for the considered systems

5.1.1. For a lower triangular \( K_+ \)

Following the idea exposed in section 4, we first focus on a triangular \( K_+ \) boundary matrix [1]. The proposition 5.1 allows us to indeed use the quantum separation of variables technique:

Proposition 5.1 (Existence of separate basis). For general bulk parameters, the \( B(\lambda) \) operators are co-diagonalizable and with simple spectrum.

Moreover, the following proposition gives the form of the SoV basis:

Proposition 5.2 ((Left) separate basis). For almost every parameters, the states

\[
\langle \hat{\mathbf{h}} \rangle = \langle h_1, \ldots, h_N \rangle = \mathcal{N} \prod_{n=1}^{N} \prod_{i=1}^{p-1} A(\xi_{n}^{(i)}) \quad ; \quad h_i \in [0, p-1]
\]  

(28)

define a \( B(\lambda) \)-eigenbasis of \( \mathcal{H}^* \) : \( \langle \hat{\mathbf{h}} \rangle | B(\lambda) = B_{\mathbf{h}}^{\lambda} (\lambda) \langle \hat{\mathbf{h}} \rangle \).

In this general case, the reference state \( \langle \Omega \rangle \) is given by a recursion formula. The factor \( \mathcal{N} \) is a normalization.

In order to make completely explicit the SoV basis, we impose here the following constraint on the parameters of the representation at any quantum site

\[
\forall n \in [1, N] \quad ; \quad a_{n}^{b} + b_{n}^{b} = 0
\]  

(29)

With the extra condition (29), we have an explicit expression for the reference state:

\[
\langle \Omega \rangle = \bigotimes_{n=1}^{N} \langle j_n - 1, n \rangle \text{ where } j_n \text{ is given by } b_n = -q^{2j_n - 1} a_n.
\]  

(30)

The vectors \( \langle k, n \rangle \) are a basis for the operators \( u_n \) and \( v_n \):

\[
v_n[k, n] = q^k |k, n\rangle \text{ and } u_n[k, n] = |k + 1, n\rangle \quad \forall (n, k) \in \{1, \ldots, N\} \times [0, p - 1].
\]  

(31)

The eigenvalues have the following expression:

\[
B_{\mathbf{h}}^{\lambda}(\lambda) = \kappa_{-}e_{-}^{\tau_{-}} \left( \frac{\lambda^2}{q} - \frac{q}{\lambda^2} \right) \Lambda_{\mathbf{h}}^{\lambda}(\lambda) \Lambda_{\mathbf{h}}^{-1}(1) \text{ with } \Lambda_{\mathbf{h}}^{\lambda}(\lambda) = (-1)^{N} \prod_{n=1}^{N} (\alpha_n \beta_n)^{1/2} \left( \frac{\lambda}{\xi_{n}^{(h_n)}} - \frac{\xi_{n}^{(h_n)}}{\lambda} \right)
\]  

(32)
5.1.2. For the most general $K_+$

Now we consider the question of the existence of a separate basis for the most general case, with $K_+$ the most general scalar solution of the trigonometric 6-vertex reflection equation. In this proceeding, we just highlight the main steps of [2].

- First, we adapted a gauge transformation originally introduced for spin 1/2 generators by Baxter [13]. This introduces three free gauge parameters, refered later as $\beta$, and we thus constructed gauged generators for a gauged Yang-Baxter algebra and then for a gauged reflection algebra.
- The key point with this transformation is that the gauged generators of the gauged reflection algebra, $A(\lambda|\beta)$, $B(\lambda|\beta)$, $C(\lambda|\beta)$ and $D(\lambda|\beta)$, satisfy similar commutation relations to the ungauged case.
- Then, the gauged boundary matrix $K_+(\lambda|\beta)$ is put lower triangular, specifying an appropriate choice of $\beta$. This way, referring to the ungauged case, the operator $B(\lambda|\beta)$ is a natural candidate to give a separate basis.
- And indeed, we have the following:

**Proposition 5.3.** A separate basis for the general case is the pseudo-eigenbasis of the gauged operator $B(\lambda|\beta)$:

$$
\langle \beta, h_1, ..., h_N | = \frac{1}{N^\beta} \langle \Omega_\beta | \prod_{n=1}^N \prod_{k_n=1}^{h_n} A_-(\xi_n^{(k_n)}) | \beta q^2 \rangle
$$

$$
\langle \beta, h_1, ..., h_N | B_-(\lambda|\beta) = B_h(\lambda|\beta) \langle \beta/q^2, h_1, ..., h_N |
$$

The SoV basis has the same form as in the ungauged case, and as before, $\langle \Omega_\beta|$ is in general known by recursion, but can be explicited under some constraints.

Using this separate basis, we can give the complete characterization of the transfer matrix spectrum, without any constraints neither on the bulk parameters, nor on the boundary matrices.

5.2. A first characterization of the spectrum

**Proposition 5.4** (Eigenvalues characterization I). Let $t(\lambda)$ be an eigenvalue of the transfer matrix. Then $t(\lambda)$ is an even polynomial of degree $2N+2$ in $\lambda$, invariant under $\lambda \rightarrow \frac{1}{\lambda}$, given by an interpolation formula with $N$ coefficients $t(\xi_n)$ which have to satisfy the system:

$$
det (D_t(\xi_n)) = 0 \quad \forall n \in [1, N]
$$

$D_t(\lambda)$ is the $p \times p$ matrix, quasi tri-diagonal, which encodes the Baxter equation (26). The equation (35) can be understood as the compatibility condition, ie the existence of a non trivial solution to the Baxter equation.

**Proposition 5.5** (Eigenvectors characterization I). In the previously introduced separate basis, the eigenvectors $|t\rangle$ have the following form:

$$
\langle \beta, h_1, ..., h_N | t \rangle = \prod_{n=1}^N q_{t,n,h_n}
$$

The coefficients $q_{t,n,h_n}$ are the unique solution, up to normalisation, of the system:

$$
D_t(\xi_n) \begin{pmatrix} q_{t,n,0} \\ \vdots \\ q_{t,n,p-1} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
$$
5.3. A second characterization of the spectrum

The previous discrete characterization of the spectrum can be reformulated in terms of a Baxter type TQ-functional equation.

Proposition 5.6 (Eigenvalues characterization II : inhomogeneous Baxter equation). Let \( t(\lambda) \) be an entire function. Then \( t(\lambda) \) is an eigenvalue if and only if there exists an even polynomial \( Q(\lambda) \), of degree \( N_Q = 2(p - 1)N \) in \( \lambda \), invariant under \( \lambda \rightarrow \frac{1}{\lambda} \), which satisfies the functional equation :

\[
t(\lambda)Q(\lambda) = \tilde{a}(\lambda)Q\left(\frac{\lambda}{q}\right) + \tilde{a}\left(\frac{1}{\lambda}\right)Q(q\lambda) + G(\lambda)
\]

The functions \( \tilde{a}(\lambda) \) and \( G(\lambda) \) are known functions of the parameters. The function \( G(\lambda) \) is here to ensure the same asymptotic behaviour for both sides of the equation. It determines the inhomogeneous term in the Baxter equation.

Remark 1. The polynomial \( Q \) interpolates the coefficients \( q_{t,n,h_n} \) previously introduced in \((37)\) :

\[
Q(\xi_n^{(h_n)}) = q_{t,n,h_n}.
\]

This type of formulation for the spectrum allows us to give an algebraic Bethe ansatz like formulation for the eigenstates, as stated in the next proposition.

Proposition 5.7 (Eigenvectors characterization II). We can construct a vector \( \langle \omega \rangle \), whose components are 1 in the separate basis, such that :

\[
\langle t | = \langle \omega | \prod_{b=1}^{N_Q} B(\lambda_b) \quad (39)
\]

where the \( \lambda_b \) are linked to the roots of \( Q(\lambda) \).

This last characterization makes a link between the Bethe ansatz methods and the quantum separation of variables method.

Moreover, we give here a characterization of the eigenvalues thanks to an inhomogeneous Baxter functional equation. There is quite a sparse literature on this topic, compared to the rich one about the homogeneous Baxter equations, and it is noticeable that with only 1 constraint on the parameters the equation becomes homogeneous.

6. Conclusions

In this proceeding I have reported about the works \([1,2]\) on the transfer matrix spectrum of the class of cyclic representations of the 6-vertex reflection algebra in the case of completely general boundary matrices and for general bulk parameters. Our result gives the complete characterization of the spectrum (eigenvalues and eigenstates) of this class of models both in terms of a discrete system of Baxter’s like second order difference equations and through a single inhomogeneous TQ-functional equation within a class of polynomial Q-functions.

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