Abstract. We establish the exponential convergence with respect to the $L^1$-Wasserstein distance and the total variation for the semigroup corresponding to the stochastic differential equation (SDE)
\[ dX_t = dZ_t + b(X_t)\, dt, \]
where $(Z_t)_{t \geq 0}$ is a pure jump Lévy process whose Lévy measure $\nu$ fulfills
\[ \inf_{x \in \mathbb{R}^d, |x| \leq \kappa_0} [\nu \wedge (\delta_x * \nu)](\mathbb{R}^d) > 0 \]
for some constant $\kappa_0 > 0$, and the drift term $b$ satisfies that for any $x, y \in \mathbb{R}^d$,
\[ \langle b(x) - b(y), x - y \rangle \leq \begin{cases} 
\Phi_1(|x - y|)|x - y|, & |x - y| \leq l_0; \\
-K_2|x - y|^2, & |x - y| > l_0 
\end{cases} \]
with some positive constants $K_2, l_0$ and positive measurable function $\Phi_1$. The method is based on the refined basic coupling for Lévy jump processes. As a byproduct, we obtain sufficient conditions for the strong ergodicity of the process $(X_t)_{t \geq 0}$.

Keywords: Refined basic coupling; Lévy jump process; Wasserstein-type distance; strong ergodicity.

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1. Introduction and Main Results

In this paper we study the following $d$-dimensional stochastic differential equation (SDE) with jumps
\[ dX_t = b(X_t)\, dt + dZ_t, \quad X_0 = x \in \mathbb{R}^d, \]
where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable function, and $Z = (Z_t)_{t \geq 0}$ is a pure jump Lévy process on $\mathbb{R}^d$.

Throughout this paper, we suppose that the SDE (1.1) has a non-explosive and pathwise unique strong solution, and $b$ satisfies the assumption $B(\Phi_1(r), \Phi_2(r), l_0)$ that for any $x, y \in \mathbb{R}^d$,
\[ \frac{\langle b(x) - b(y), x - y \rangle}{|x - y|} \leq \Phi_1(|x - y|) - [\Phi_1(|x - y|) + \Phi_2(|x - y|)] \mathbb{1}_{\{|x-y| \geq l_0\}}, \]
where $\Phi_1$ and $\Phi_2$ are two positive measurable functions, and $l_0 \geq 0$ is a constant. For example, when $\Phi_2(r) = K_2 r$ for some positive constant $K_2$, $B(\Phi_1(r), \Phi_2(r), l_0)$
is reduced into $B(\Phi_1(r), K_2r, l_0)$:

$$\langle b(x) - b(y), x - y \rangle \leq \begin{cases} 
\Phi_1(|x - y|)|x - y|, & |x - y| \leq l_0; \\
-K_2|x - y|^2, & |x - y| > l_0.
\end{cases}$$

(1.3)

This holds if the drift term $b$ is dissipative outside some compact set. In particular, when $\Phi_1(r) = K_1r$ for some constant $K_1 \geq 0$, it follows from (1.3) that for any $x \in \mathbb{R}^d$,

$$\langle b(x), x \rangle \leq \langle b(0), x \rangle + K_1|x|^2 \leq C_1(1 + |x|^2),$$

which, along with (1.3), yields that the SDE (1.1) has a non-explosive and pathwise unique strong solution, see [1, Chapter 6, Theorem 6.2.3] (in the standard Lipschitz case) or [15, Chapter 3, Theorem 115]. Note that, since we are sometimes concerned with only measurable drift term $b$, non-Lipschitz condition like $B(K_1, K_2r, l_0)$ will also be adopted in our results below. The reader can refer to [4, 11, 12, 16, 21] and references therein for recent studies on the existence and uniqueness of strong solution to (1.1) with non-regular drift term. In particular, assuming that $Z$ is the truncated symmetric $\alpha$-stable process on $\mathbb{R}^d$ with $\alpha \in (0, 2)$, and $b$ is bounded and $\beta$-Hölder continuous with $\beta > 1 - \alpha/2$, it was proved in [4, Corollary 1.4(i)] that the SDE (1.1) has a unique strong solution for each $x \in \mathbb{R}^d$. Furthermore, in one-dimensional case, it was proved in [16, Remark 1, p. 82] that if $\alpha > 1$, then the SDE (1.1) also enjoys a unique strong solution for each $x \in \mathbb{R}$, even though the drift $b$ is only bounded and measurable.

We denote by $\nu$ the Lévy measure of the pure jump Lévy process $Z$. We assume that there is a constant $\kappa_0 > 0$ such that

$$\inf_{x \in \mathbb{R}^d, |x| \leq \kappa_0} \left[ \nu \wedge (\delta_x * \nu) \right](\mathbb{R}^d) > 0.$$  

(1.4)

Condition (1.4) was first used in [13] to study the coupling property of Lévy processes. It is satisfied by a large class of Lévy measures. For instance, if

$$\nu(dz) \geq 1_{B(z_0, \varepsilon)} \rho_0(z) dz$$

for some $z_0 \in \mathbb{R}^d$ and some $\varepsilon > 0$ such that $\rho_0(z)$ is positive and continuous on $B(z_0, \varepsilon)$, then such Lévy measure $\nu$ fulfills (1.4), see [14, Proposition 1.5] for details.

Let $(P_t)_{t \geq 0}$ be the transition semigroup associated with the process $(X_t)_{t \geq 0}$. In this paper we are interested in the asymptotics of the Wasserstein-type distances (including the $L^1$-Wasserstein distance and the total variation) between probability distributions $\delta_x P_t = P_t(x, \cdot)$ and $\delta_y P_t = P_t(y, \cdot)$ for any $x, y \in \mathbb{R}^d$, when the drift term $b$ is dissipative outside some compact set, i.e. $b$ satisfies $B(\Phi_1(r), K_2r, l_0)$ for some positive measurable function $\Phi_1$, and some constants $K_2 > 0$ and $l_0 \geq 0$.

This kind of problems have already been studied by Eberle [5, 6] in the diffusion case, i.e., the pure jump Lévy process $(Z_t)_{t \geq 0}$ in (1.1) is replaced by a Brownian motion $(B_t)_{t \geq 0}$. He proved that the $L^1$-Wasserstein distance between $\delta_x P_t$ and $\delta_y P_t$ decays exponentially fast. This result was slightly strengthened in [8], where we obtained some convergence result with respect to the $L^p$-Wasserstein distance for any $p \geq 1$. In the general settings of Riemannian manifold and of SDEs with multiplicative noises, F.-Y. Wang [17] obtained the exponential contractivity in the $L^2$-Wasserstein distance under $B(K_1r, K_2r, l_0)$, i.e., (1.3) holds with $\Phi_1(r) = K_1r$ for some $K_1 > 0$; moreover, similar results for the $L^p$-Wasserstein distance for all $p \geq 1$ are proved provided that the diffusion semigroup is ultracontractive. Some developments in the jump case can be found in [20, 9] under $B(K_1r, K_2r, l_0)$. In
particular, the second author [20] obtained exponential convergence rate in the $L^p$-Wasserstein distance for any $p \geq 1$ when the Lévy noise in (1.1) has an $\alpha$-stable component. In the recent paper [9], Majka considered a larger class of Lévy processes without $\alpha$-stable components, and obtained the exponential convergence rates with respect to both the $L^1$-Wasserstein distance and the total variation. See the remark at the end of Subsection 1.1 for more explicit discussions. We mention that in [9] the associated Lévy measure of the Lévy process $Z$ essentially has a rotationally invariant absolutely continuous component.

In order to present our results, we first introduce some notations. Let $\psi$ be a strictly increasing function on $[0, \infty)$ satisfying $\psi(0) = 0$. Given two probability measures $\mu_1$ and $\mu_2$ on $\mathbb{R}^d$, we define the following quantity

$$W_\psi(\mu_1, \mu_2) = \inf_{\Pi \in \mathcal{C}(\mu_1, \mu_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \psi(|x - y|) \, d\Pi(x, y),$$

where $| \cdot |$ is the Euclidean norm and $\mathcal{C}(\mu_1, \mu_2)$ is the collection of measures on $\mathbb{R}^d \times \mathbb{R}^d$ having $\mu_1$ and $\mu_2$ as marginals. When $\psi$ is concave, the above definition gives rise to a Wasserstein distance $W_\psi$ in the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures $\mu$ on $\mathbb{R}^d$ such that $\int \psi(|z|) \, d\mu(z) < \infty$. If $\psi(r) = r$ for all $r \geq 0$, then $W_\psi$ is the standard $L^1$-Wasserstein distance (with respect to the Euclidean norm $| \cdot |$), which will be denoted by $W_\psi(\mu_1, \mu_2)$ throughout this paper. Another well-known example for $W_\psi$ is given by $\psi(r) = \mathbf{1}_{(0, \infty)}(r)$, which leads to the total variation distance $W_\psi(\mu_1, \mu_2) = \frac{1}{2} \| \mu_1 - \mu_2 \|_{\text{Var}}$.

### 1.1. Exponential convergence in Wasserstein-type distances.

Throughout this paper, we denote by

$$J(s) := \inf_{|x|=s} \left[ \nu \land (\delta_x \ast \nu) \right](\mathbb{R}^d), \quad s > 0.$$ 

Condition (1.4) implies that $\inf_{0<s<\kappa_0} J(s) > 0$ for some $\kappa_0 > 0$. The following result is the first main contribution of our paper on exponential convergence in the $L^1$-Wasserstein distance and the total variation for the SDE (1.1). Refer to Theorems 4.2 and 4.4 below for more general statements.

**Theorem 1.1.** The following two assertions hold.

(a) Assume that there are constants $\alpha \in [0, 1)$ and $\theta \in (0, \infty)$ such that

$$\lim_{r \to 0} \inf_{s \in [0, r]} J(s) s^\alpha \left( \log \frac{1}{s} \right)^{-1-\theta} > 0. \tag{1.5}$$

If the drift term $b$ satisfies $B(K_1 r^\beta, K_2 r l_0)$ with some constants $\beta \in [1 - \alpha, 1]$, $K_1, l_0 \geq 0$ and $K_2 > 0$, then there exist constants $\lambda, c > 0$ such that for any $x, y \in \mathbb{R}^d$ and $t > 0$,

$$W_1(\delta_x P_t, \delta_y P_t) \leq ce^{-\lambda t} |x - y|. \tag{1.6}$$

(b) Assume that (1.5) holds with $\alpha = 0$, i.e. there is a constant $\theta \in (0, \infty)$ such that

$$\lim_{r \to 0} \inf_{s \in [0, r]} J(s) \left( \log \frac{1}{s} \right)^{-1-\theta} > 0. \tag{1.7}$$
If the drift term $b$ satisfies $B(K_1, K_2, l_0)$ with some constants $K_1, l_0 \geq 0$ and $K_2 > 0$, then there exist constants $\lambda, c > 0$ such that for any $x, y \in \mathbb{R}^d$ and $t > 0$,

$$
\|\delta_x P_t - \delta_y P_t\|_{\text{var}} \leq ce^{-\lambda t}(1 + |x - y|).
$$

Let us make some comments on Theorem 1.1. First, by Example 1.2 below, the condition (1.5) is satisfied for any (truncated) symmetric $\alpha'$-stable process with $\alpha' \in (\alpha, 2)$. The condition $B(K_1 r^{\beta}, K_2 r, l_0)$ in part (a) holds if the drift coefficient $b$ is $\beta$-Hölder continuous with $\beta \geq 1 - \alpha$. When the Lévy noise $Z$ in the SDE (1.1) is the (truncated) symmetric $\alpha'$-stable process with $\alpha' \in (\alpha, 2)$, the latter is weaker than the assumptions on $b$ in [4, Corollary 1.4(i)], which further implies that the SDE (1.1) has a unique strong solution.

**Example 1.2.** Suppose that

$$
\nu(dz) \geq \mathbf{1}_{\{0 < z \leq 1\}} \frac{c_{d,\alpha}}{|z|^{d+\alpha}} dz
$$

for some $\alpha \in (0, 2)$ and $c_{d,\alpha} > 0$. Then, $J(s) \geq \hat{c}_{d,\alpha} s^{-\alpha}$ for any $s > 0$ small enough.

From Example 1.2 above, we can immediately get exponential rates in the $L^1$-Wasserstein distance and the total variation for the SDE (1.1), when the Lévy noise $Z$ has a (truncated) $\alpha$-stable component for all $\alpha \in (0, 2)$ and the drift term $b$ is dissipative outside some ball. Therefore, Theorem 1.1 covers the main result of [20] (see Theorem 1.2 therein). On the other hand, Example 1.2 indicates that Theorem 1.1 works for Lévy processes whose associated Lévy measure does not necessarily have a rotationally invariant component. Therefore, Theorem 1.1 essentially extends the framework of [9].

The approach of Theorem 1.1 is based on the coupling for Lévy processes, as in [20, 9]. It seems that the couplings used in [20, 9] depend heavily on the rotational symmetry of the Lévy measure, and so they do not work in our general setting, since we do not assume that the Lévy process $Z$ has a symmetric $\alpha$-stable component or the associated Lévy measure of $Z$ has a rotationally invariant absolutely continuous component. Therefore, some new ideas are required for the construction of the coupling. It is worth pointing out that our choice of the test function $\psi(r) \asymp r$ (see Theorem 4.2 below) is quite simple. The choice explicitly reflects the properties of the Lévy measure $\nu$ and the drift $b$, and also yields the explicit expression of $\lambda$ in (1.6), which is optimal in the sense that it is the same as that when $b$ satisfies the uniformly dissipative condition (see Remark 4.3 below).

Second, assuming that the Lévy measure $\nu$ of $Z$ satisfies only (1.4) (not the stronger condition (1.5)), but has a rotationally invariant density function with respect to the Lebesgue measure such that

$$
\int_{\{|z| \geq 1\}} |z| \nu(dz) < \infty,
$$

and the drift term $b$ satisfies $B(K_1 r^{\beta}, K_2 r, l_0)$ with some constants $K_1, l_0 \geq 0$ and $K_2 > 0$, Majka [9] actually proved that there exist constants $\lambda, c > 0$ such that for any $x, y \in \mathbb{R}^d$ and $t > 0$,

$$
W_1(\delta_x P_t, \delta_y P_t) + \|\delta_x P_t - \delta_y P_t\|_{\text{var}} \leq ce^{-\lambda t}(1 + |x - y|);
$$

(1.10)
that is, denoting by \( \psi(r) = r + 1_{(0, \infty)}(r) \),

\[
W_\psi(\delta_x P_t, \delta_y P_t) \leq c e^{-\lambda t} \psi(|x - y|).
\]

It is obvious that (1.10) implies (1.8), but does not imply (1.6).

Applying Theorem 1.1 and using some standard arguments (e.g. see [6, Corollary 2] or [8, Corollary 1.8]), we can also obtain that, under assumptions of Theorem 1.1 and the additional condition (1.9), there exist a unique invariant probability measure \( \mu \), some constants \( c, \lambda > 0 \) and a positive measurable function \( c(x) \) such that

\[
W_1(\delta_x P_t, \mu) \leq c e^{-\lambda t} W_1(\delta_x, \mu), \quad x \in \mathbb{R}^d, t > 0
\]

and

\[
\|\delta_x P_t - \mu\|_{\text{Var}} \leq c(x) e^{-\lambda t}, \quad x \in \mathbb{R}^d, t > 0.
\]

In the literature, (1.12) is called the exponential ergodicity for the process \( (X_t)_{t \geq 0} \).

Note that from (1.10), one can only obtain the exponential ergodicity with respect to \( W_\psi \) with \( \psi(r) = r + 1_{(0, \infty)}(r) \). In particular, one only has

\[
W_1(\delta_x P_t, \mu) \leq c_1 W_1(\delta_x, \mu) + c_2(x), \quad x \in \mathbb{R}^d, t > 0
\]

for some positive constant \( c_1 > 0 \) and some positive measurable function \( c_2(x) \), instead of (1.11). See [9, Corollary 1.6] for more details.

1.2. Strong ergodicity. We are also interested in obtaining the exponential rate for total variation which is stronger than (1.8); that is, we want to prove

\[
\|\delta_x P_t - \delta_y P_t\|_{\text{Var}} \leq c e^{-\lambda t}, \quad x, y \in \mathbb{R}^d, t > 0
\]

for some positive constants \( c \) and \( \lambda \). Note that, compared with (1.8), (1.13) is equivalent to

\[
W_\psi(\delta_x P_t, \delta_y P_t) \leq 2 c e^{-\lambda t} \psi(|x - y|)
\]

with \( \psi(r) = 1_{(0, \infty)}(r) \), which enjoys the same form as that of (1.6).

As shown by the result below, (1.13) can be established by imposing stronger dissipative condition on the drift term \( b \) outside some compact set. See Theorem 4.6 below for more general statement.

**Theorem 1.3.** Assume that the drift term \( b \) satisfies \( \mathcal{B}(K_1, \Phi_2(r), l_0) \) with some constants \( K_1, l_0 \geq 0 \) and some positive measurable function \( \Phi_2 \) such that \( \Phi_2(r) \) is bounded from below for \( r \) large enough, and

\[
\int_{r_0}^{\infty} \frac{1}{\Phi_2(s)} \, ds < \infty \quad \text{for some } r_0 > 0.
\]

If (1.7) holds, then there exist constants \( \lambda, c > 0 \) such that for any \( x, y \in \mathbb{R}^d \) and \( t > 0 \), (1.13) holds true.

A typical example for (1.14) is that \( \Phi_2(s) = K_2 s^{1+\theta} \) for some \( K_2, \theta > 0 \). In this case, the drift term \( b \) satisfies that for any \( x, y \in \mathbb{R}^d \) with \( |x - y| \geq l_0 \),

\[
\langle b(x) - b(y), x - y \rangle \leq -K_2 |x - y|^{2+\theta}.
\]

For instance, \( b(x) = \nabla V(x) \) with \( V(x) = -|x|^{2+\theta} (\theta > 0) \) satisfies the condition above, see [8, Example 1.7] or [20, Example 1.3].

Next we will consider the strong ergodicity (with respect to the total variation) by making use of Theorem 1.3. We emphasize that, to the best of our knowledge,
the proposition below is the first result concerning the strong ergodicity of SDEs with Lévy jumps via the coupling approach. We also note that (1.13), rather than (1.8), is a key point to yield the strong ergodicity.

**Proposition 1.4.** Suppose that the Lévy measure $\nu$ of the process $Z$ fulfills (1.7) and that
\begin{equation}
\int_{\{|z| \geq 1\}} \log(1 + |z|) \nu(dz) < \infty. \tag{1.15}
\end{equation}
If $b$ satisfies $B(K_1 r, \Phi_2(r), l_0)$ with some constants $K_1, l_0 \geq 0$ and some positive measurable function $\Phi_2$ satisfying $\lim \inf_{r \to \infty} \frac{\Phi_2(r)}{r} > 0$ and (1.14), then the process $(X_t)_{t \geq 0}$ is strongly ergodic, i.e. there exist a unique invariant probability measure $\mu$ and some constants $c, \lambda > 0$ such that
\[ \|\delta_x P_t - \mu\|_{\text{Var}} \leq ce^{-\lambda t}, \quad x \in \mathbb{R}^d, \quad t > 0. \]

The remainder of this paper is arranged as follows. In the next section, we will present the refined basic coupling process for Lévy processes, which has its own interest. To reveal the new idea behind this refined basic coupling, we begin with the construction of coupling operator for Lévy processes. Then we consider the corresponding coupling operator for the SDE (1.1). In particular, we directly prove that there exists a system of SDEs, which is associated with this coupling operator and admits a unique strong solution. Based on the coupling process constructed above, general approaches via the coupling idea to exponential convergence rates in Wasserstein distance for the SDE (1.1) are presented in Section 3. Proofs of all the results in Section 1 are given in Section 4. We present in Section 5 two other applications of the refined basic coupling for Lévy processes; namely, the regularity of the semigroup $(P_t)_{t \geq 0}$ associated to the SDE (1.1) under the one-sided Lipschitz condition and the extension of results in this section to SDEs with special multiplicative noises. Some properties related to (1.4) are given in the appendix.

2. **Refined basic coupling for Lévy processes**

In this section we shall first construct a new coupling operator for pure jump Lévy processes, and then find the corresponding SDE for the coupling process. The reason that we choose to begin with the construction of the coupling operator is that it clearly reveals the idea behind the coupling.

2.1. **Coupling operator for Lévy processes.** Recall that a $d$-dimensional pure jump Lévy process $Z = (Z_t)_{t \geq 0}$ is a stochastic process on $\mathbb{R}^d$ with $Z_0 = 0$, stationary and independent increments and càdlàg sample paths. Its finite-dimensional distributions are uniquely characterized by the characteristic exponent or the symbol of characteristic function $\mathbb{E}e^{i\langle \xi, Z_t \rangle} = e^{-i\Phi_Z(\xi)}$ with
\[ \Phi_Z(\xi) = \int \left( 1 - e^{i\langle \xi, z \rangle} + i \langle \xi, z \rangle 1_{B(0,1)}(z) \right) \nu(dz), \]
where $\nu$ is the Lévy measure, i.e. a $\sigma$-finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ such that $\nu(\{0\}) = 0$ and the integral $\int (1 \wedge |z|^2) \nu(dz) < \infty$. Its infinitesimal generator acting on $C^2_b(\mathbb{R}^d)$ is given by
\begin{equation}
L_Z f(x) = \int \left( f(x + z) - f(x) - \langle \nabla f(x), z \rangle 1_{B(0,1)}(z) \right) \nu(dz). \tag{2.1}
\end{equation}
Recall that an operator $\tilde{L}_Z$ acting on $C^2_b(\mathbb{R}^d \times \mathbb{R}^d)$ is called a coupling of $L_Z$, if for any $f, g \in C^2_b(\mathbb{R}^d)$, setting $h(x, y) = f(x) + g(y)$ for all $x, y \in \mathbb{R}^d$, then we have

\begin{equation}
\tilde{L}_Z h(x, y) = L_Z f(x) + L_Z g(y).
\end{equation}

If the coupling operator $\tilde{L}_Z$ generates a Markov process $(Z^1_t, Z^2_t)_{t \geq 0}$ on $\mathbb{R}^d \times \mathbb{R}^d$, then the latter is called a coupling process of $Z$. The coupling time is the first time that the two marginal processes $(Z^1_t)_{t \geq 0}$ and $(Z^2_t)_{t \geq 0}$ meet each other; that is, the stopping time $T = \inf\{t \geq 0 : Z^1_t = Z^2_t\}$. If $T$ is almost surely finite, then the coupling is called successful. After the coupling time, we often let the two marginal processes move together.

We first give the intuitive ideas that lead to the particular construction of our coupling. In the construction of a coupling process for pure jump Lévy process $Z$, we often require the coupling time $T$ to be as small as possible, which provides better convergence speed. To this end, the natural idea is to make the two marginal processes jump to the same point with the biggest possible rate. This is exactly the maximum common part of the jump intensities. In our setting, it takes the form $\mu_{y-x}(dz) := [\nu \wedge (\delta_{y-x} \ast \nu)](dz)$, where $x \neq y$ are the positions of the two marginal processes before the jump.

**Remark 2.1.** We claim that $\mu_x$ is a finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for any $x \neq 0$. Indeed, for any $x, z \in \mathbb{R}^d$ with $x \neq 0$ and $|z| \leq |x|/2$, $|z - x| \geq |x| - |z| \geq |x|/2$, which implies

\[
\int_{\{|z| \leq |x|/2\}} (\delta_x \ast \nu)(dz) = \int_{\{|z| \leq |x|/2\}} \nu(d(z - x)) \leq \int_{\{|u| \geq |x|/2\}} \nu(du).
\]

Consequently,

\[
\mu_x(\mathbb{R}^d) = \int_{\{|z| \leq |x|/2\}} \mu_x(dz) + \int_{\{|z| > |x|/2\}} \mu_x(dz)
\leq \int_{\{|z| \leq |x|/2\}} (\delta_x \ast \nu)(dz) + \int_{\{|z| > |x|/2\}} \nu(dz)
\leq 2 \int_{\{|z| \geq |x|/2\}} \nu(dz) < \infty.
\]

The operator corresponding to the basic coupling can be written as follows: for any $f \in C^2_b(\mathbb{R}^d \times \mathbb{R}^d)$,

\[
\tilde{L}_Z f(x, y) = \int \left( f(x + z, y + z + (x - y)) - f(x, y) - \langle \nabla_x f(x, y), z \rangle 1_{\{|z| \leq 1\}} \\
- \langle \nabla_y f(x, y), x - y + z \rangle 1_{\{|z| + (x - y) \leq 1\}} \right) \mu_{y-x}(dz)
+ \int \left( f(x + z, y) - f(x, y) - \langle \nabla_x f(x, y), z \rangle 1_{\{|z| \leq 1\}} \right) (\nu - \mu_{y-x})(dz)
+ \int \left( f(x, y + z) - f(x, y) - \langle \nabla_y f(x, y), z \rangle 1_{\{|z| \leq 1\}} \right) (\nu - \mu_{x-y})(dz).
\]

Here and in what follows, $\nabla_x h(x, y)$ and $\nabla_y h(x, y)$ are defined as the gradient of $h(x, y)$ with respect to $x, y \in \mathbb{R}^d$, respectively. The last two integrals come from
the marginality (2.2) of the coupling operator and the following crucial identity (see Corollary 6.2):

\begin{equation}
\mu_x(d(z-x)) = (\delta_x \ast \mu_x)(dz) = \left[\delta_x \ast \left(\nu \wedge (\delta_x \ast \nu)\right)\right](dz)
\end{equation}

This coupling can be illustrated as follows:

\begin{equation}
(x, y) \rightarrow \begin{cases}
(x + z, y + z + (x - y)), & \mu_{y-x}(dz); \\
(x + z, y), & (\nu - \mu_{y-x})(dz); \\
(x, y + z), & (\nu - \mu_{x-y})(dz).
\end{cases}
\end{equation}

The first row of this coupling is quite good in applications, since the distance between the two marginals decreases from \(|x - y|\) to \(|(x + z) - (y + z + (x - y))| = 0\). The second row, however, is not so welcome, because the new distance is \(|x - y + z|\), which can be much bigger than the original one when the jump size \(z\) is large. The same problem appears in the last row of the coupling.

Therefore, we have to modify the basic coupling to make it behave better. As a first step, we want to change the second row in (2.5) so that the distance after the jump is comparable with \(|x - y|\). Inspired by the first row, a simple choice is \((x, y) \rightarrow (x + z, y + z + (y - x))\) with rate \(\frac{1}{2}\mu_{x-y}(dz)\), where the distance after the jump is \(2|x - y|\). The price to pay is that we need to modify at the same time the first row in (2.5), so that the two marginal processes cannot jump to the same point with the biggest possible rate, but only half of it. For the last row, we simply let them jump with the same size and their distance remains unchanged. So the coupling (2.5) becomes

\begin{equation}
(x, y) \rightarrow \begin{cases}
(x + z, y + z + (x - y)), & \frac{1}{2}\mu_{y-x}(dz); \\
(x + z, y + z + (y - x)), & \frac{1}{2}\mu_{x-y}(dz); \\
(x + z, y + z), & (\nu - \frac{1}{2}\mu_{y-x} - \frac{1}{2}\mu_{x-y})(dz).
\end{cases}
\end{equation}

Thanks to the identity (2.4) again, we are able to verify the marginality (2.2) for this modified coupling.

The above coupling (2.6) has a drawback too. If the original pure jump Lévy process \(Z\) is of finite range, then the jump intensity \(\mu_{y-x}(dz)\) is identically zero for \(|y-x|\) large enough. Thus the two marginal processes of the coupling (2.6) will never get closer if they are initially far away. Our intuitive idea to overcome this difficulty is that if the distance between the marginal processes is already small, then we let them jump as in (2.6); while if the distance is too large, then it would be more reasonable to reduce it by a small amount after each jump, since the requirement that their distance decreases to zero seems too greedy. Thus, we introduce a parameter \(\kappa > 0\) which serves as the threshold to determine whether the marginal processes jump to the same point or become slightly closer to each other. Let \(\kappa_0\) be the constant in (1.4). For any \(x, y \in \mathbb{R}^d\) and \(\kappa \in (0, \kappa_0]\), define

\begin{equation}
(x - y)_{\kappa} = \left(1 \wedge \frac{\kappa}{|x - y|}\right)(x - y).
\end{equation}
We make the convention that \((x - x)_\kappa = 0\). Then our coupling is given as follows:

\[
(x, y) \mapsto \begin{cases} 
(x + z, y + z + (x - y)_\kappa), & \frac{1}{2}\mu_{(y-x)_\kappa}(dz); \\
(x + z, y + z + (y - x)_\kappa), & \frac{1}{2}\mu_{(x-y)_\kappa}(dz); \\
(x + z, y + z), & \left(\nu - \frac{1}{2}\mu_{(y-x)_\kappa} - \frac{1}{2}\mu_{(x-y)_\kappa}\right)(dz).
\end{cases}
\]

(2.8)

We see that if \(|x - y| \leq \kappa\), then the above coupling is the same as that in (2.6). If \(|x - y| > \kappa\), then according to the first two rows, the distances after the jump are \(|x - y| - \kappa\) and \(|x - y| + \kappa\), respectively. We will call the coupling given by (2.8) the refined basic coupling for pure jump Lévy processes.

We can now write explicitly the coupling operator \(\widetilde{L}_Z\) corresponding to (2.8). Fix \(h \in C^2_0(\mathbb{R}^d)\). For any \(x, y \in \mathbb{R}^d\), we define

\[
\begin{align*}
\widetilde{L}_Z h(x, y) &= \frac{1}{2} \left( h(x + z, y + z + (x - y)_\kappa) - h(x, y) - \langle \nabla_x h(x, y), z \rangle \mathbb{1}_{\{|z| \leq 1\}} \
& \quad - \langle \nabla_y h(x, y), z + (x - y)_\kappa \rangle \mathbb{1}_{\{|z + (x-y)_\kappa| \leq 1\}} \right) \mu_{(y-x)_\kappa}(dz) \\
& \quad + \frac{1}{2} \left( h(x + z, y + z + (y - x)_\kappa) - h(x, y) - \langle \nabla_x h(x, y), z \rangle \mathbb{1}_{\{|z| \leq 1\}} \
& \quad - \langle \nabla_y h(x, y), z + (y - x)_\kappa \rangle \mathbb{1}_{\{|z + (y-x)_\kappa| \leq 1\}} \right) \mu_{(x-y)_\kappa}(dz) \\
& \quad + \sum_{\kappa} \left( h(x + z, y + z) - h(x, y) - \langle \nabla_x h(x, y), z \rangle \mathbb{1}_{\{|z| \leq 1\}} \
& \quad - \langle \nabla_y h(x, y), z \rangle \mathbb{1}_{\{|z| \leq 1\}} \right) \left( \nu - \frac{1}{2}\mu_{(y-x)_\kappa} - \frac{1}{2}\mu_{(x-y)_\kappa}\right)(dz).
\end{align*}
\]

(2.9)

Below, we prove rigorously that \(\widetilde{L}_Z\) is indeed a coupling operator of the operator \(L_Z\) given by (2.1). For this we let \(h(x, y) = g(y)\) for any \(x, y \in \mathbb{R}^d\), where \(g \in C^2_0(\mathbb{R}^d)\). Then, according to (2.9),

\[
\begin{align*}
\widetilde{L}_Z h(x, y) &= \frac{1}{2} \left( g(y + z + (x - y)_\kappa) - g(y) \
& \quad - \langle \nabla g(y), z + (x - y)_\kappa \rangle \mathbb{1}_{\{|z + (x-y)_\kappa| \leq 1\}} \right) \mu_{(y-x)_\kappa}(dz) \\
& \quad + \frac{1}{2} \left( g(y + z + (y - x)_\kappa) - g(y) \
& \quad - \langle \nabla g(y), z + (y - x)_\kappa \rangle \mathbb{1}_{\{|z + (y-x)_\kappa| \leq 1\}} \right) \mu_{(x-y)_\kappa}(dz) \\
& \quad + \sum_{\kappa} \left( g(y + z) - g(y) - \langle \nabla g(y), z \rangle \mathbb{1}_{\{|z| \leq 1\}} \right) \left( \nu - \frac{1}{2}\mu_{(y-x)_\kappa} - \frac{1}{2}\mu_{(x-y)_\kappa}\right)(dz).
\end{align*}
\]

Changing the variables \(z + (x - y)_\kappa \rightarrow u\) and \(z + (y - x)_\kappa \rightarrow u\) respectively leads to

\[
\begin{align*}
\widetilde{L}_Z h(x, y) &= \frac{1}{2} \left( g(y + u) - g(y) - \langle \nabla g(y), u \rangle \mathbb{1}_{\{|u| \leq 1\}} \right) \mu_{(y-x)_\kappa}(d(u - (x - y)_\kappa)) \\
& \quad + \frac{1}{2} \left( g(y + u) - g(y) - \langle \nabla g(y), u \rangle \mathbb{1}_{\{|u| \leq 1\}} \right) \mu_{(x-y)_\kappa}(d(u - (y - x)_\kappa))
\end{align*}
\]
+ \int \left( g(y + z) - g(y) - \langle \nabla g(y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \left( \nu - \frac{1}{2} \mu_{(x-y),\kappa} - \frac{1}{2} \mu_{(y-z),\kappa} \right) (dz).

By (2.4), for any \( x, y \in \mathbb{R}^d \), we arrive at
\[
L_Z h(x, y) = \frac{1}{2} \int \left( g(y + u) - g(y) - \langle \nabla g(y), u \rangle \mathbf{1}_{\{|u| \leq 1\}} \right) \mu_{(x-y),\kappa} (du)
+ \frac{1}{2} \int \left( g(y + u) - g(y) - \langle \nabla g(y), u \rangle \mathbf{1}_{\{|u| \leq 1\}} \right) \mu_{(y-x),\kappa} (du)
+ \int \left( g(y + z) - g(y) - \langle \nabla g(y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \times \left( \nu - \frac{1}{2} \mu_{(x-y),\kappa} - \frac{1}{2} \mu_{(y-z),\kappa} \right) (dz)
= L_Z g(y).
\]

Thanks to (2.10), we can easily conclude that the operator \( \tilde{L}_Z \) defined by (2.9) is a coupling operator of \( L_Z \), i.e. (2.2) holds.

2.2. Coupling process for Lévy processes. The aim of this subsection is to find the SDE associated with the coupling operator \( \tilde{L}_Z \) defined above. This will help us with constructing the coupling process by solving the SDE.

For a pure jump Lévy process \( Z \), by the Lévy–Itô decomposition, there exists a Poisson random measure \( N(ds, dz) \) associated with \( Z \) in such a way that
\[
Z_t = \int_0^t \int_{\{|z| > 1\}} z N(ds, dz) + \int_0^t \int_{\{|z| \leq 1\}} z \tilde{N}(ds, dz),
\]
where
\[
\tilde{N}(ds, dz) = N(ds, dz) - ds \nu(dz)
\]
is the compensated Poisson measure. Recall that there exist a sequence of random variables \( \{\tau_j\}_{j \geq 1} \) in \( \mathbb{R}_+ \) encoding the jump times and a sequence of random variables \( \{\xi_j\}_{j \geq 1} \) in \( \mathbb{R}^d \) encoding the jump sizes such that
\[
N\left( ([0, t], A) \right)(\omega) = \sum_{j=1}^{\infty} \delta_{(\tau_j(\omega), \xi_j(\omega))} \left( ([0, t] \times A) \right), \quad \omega \in \Omega, A \in \mathcal{B}(\mathbb{R}^d).
\]
To construct a coupling process, let us follow the idea in [9, Section 2.2] and begin with extending the Poisson random measure \( N \) on \( \mathbb{R}_+ \times \mathbb{R}^d \) to a Poisson random measure on \( \mathbb{R}_+ \times \mathbb{R}^d \times [0, 1] \), by replacing the \( d \)-dimensional random variables \( \xi_j \) determining the jump sizes of \( (Z_t)_{t \geq 0} \) with the \((d + 1)\)-dimensional random variables \( (\xi_j, \eta_j) \), where each \( \eta_j \) is a uniformly distributed random variable on \([0, 1]\). Thus, we have
\[
N\left( ([0, t], A \times [0, 1]) \right)(\omega) = \sum_{j=1}^{\infty} \delta_{(\tau_j(\omega), \xi_j(\omega), \eta_j(\omega))} \left( ([0, t] \times A \times [0, 1]) \right), \quad \omega \in \Omega, A \in \mathcal{B}(\mathbb{R}^d).
\]
To save notations, we still denote the extended Poisson random measure by \( N \), and write
\[
Z_t = \int_0^t \int_{\{|z| > 1\} \times [0, 1]} z N(ds, dz, du) + \int_0^t \int_{\{|z| \leq 1\} \times [0, 1]} z \tilde{N}(ds, dz, du).
\]
For simplicity, we set
\[ \tilde{N}(ds, dz, du) = 1_{\{|z|>1\} \times [0,1]} N(ds, dz, du) + 1_{\{|z| \leq 1\} \times [0,1]} \tilde{N}(ds, dz, du) \]
and hence
\[ Z_t = \int_0^t \int_{\mathbb{R}^d \times [0,1]} z \tilde{N}(ds, dz, du). \]
or equivalently,
\[ dZ_t = \int_{\mathbb{R}^d \times [0,1]} z \tilde{N}(dt, dz, du). \tag{2.11} \]

We want to find the SDE for the process \( Z^* := (Z_t^*)_{t \geq 0} \) so that \( (Z_t, Z_t^*)_{t \geq 0} \) is a Markov process on \( \mathbb{R}^{2d} \), and has the coupling operator \( \tilde{L}_Z \) constructed in (2.9) as its generator.

With the above notations and taking into account the construction (2.8) of the coupling operator \( \tilde{L}_Z \), if a jump occurs at time \( t \), then the process \( Z \) moves from the point \( Z_t^- \) to \( Z_t^- + z \), and we draw a random number \( u \in [0,1] \) to determine whether the process \( Z^* \) should jump from the point \( Z_t^- \) to the points \( Z_t^- + z + (Z_t^* - Z_t^-)_\kappa \), \( Z_t^- + z + (Z_t^- - Z_t^-)_\kappa \) and \( Z_t^- + z \), respectively. To this end, we define the control function \( \rho \) as follows: for any \( x, z \in \mathbb{R}^d \),
\[ \rho(x, z) = \frac{\nu \wedge (\delta_x * \nu)(dz)}{\nu(dz)} \in [0,1]. \]

By convention, \( \rho(0, z) \equiv 1 \) for all \( z \in \mathbb{R}^d \). For simplification of notations, we write \( U_t = Z_t - Z_t^* \) and consider the following SDE:
\[ dZ_t^* = \int_{\mathbb{R}^d \times [0,1]} \left[(z + (U_t^-)_\kappa) 1_{\{u \leq \frac{1}{2}\rho((-U_t^-)_\kappa, z)\}} + \left( z + (-U_t^-)_\kappa \right) 1_{\{\frac{1}{2}\rho((-U_t^-)_\kappa, z) < u \leq \frac{1}{2}[\rho((-U_t^-)_\kappa, z) + \rho(U_t^-)_\kappa, z)]\}} + z 1_{\{\frac{1}{2}[\rho((-U_t^-)_\kappa, z) + \rho(U_t^-)_\kappa, z)] < u \leq 1\}} \right] \tilde{N}(dt, dz, du) \]
\[ - \int_{\mathbb{R}^d \times [0,1]} \left[(z + (U_t^-)_\kappa) \left(1_{\{|z + (U_t^-)_\kappa| \leq 1\}} - 1_{\{|z| \leq 1\}} \right) 1_{\{u \leq \frac{1}{2}\rho((-U_t^-)_\kappa, z)\}} + (z + (-U_t^-)) \left(1_{\{|z + (U_t^-)_\kappa| \leq 1\}} - 1_{\{|z| \leq 1\}} \right) \times 1_{\{\frac{1}{2}[\rho((-U_t^-)_\kappa, z) + \rho(U_t^-)_\kappa, z)] < u \leq \frac{1}{2}[\rho((-U_t^-)_\kappa, z) + \rho(U_t^-)_\kappa, z)]\}} \right] \nu(dz) du dt. \tag{2.12} \]

Here, the first integral with respect to the Poisson random measure corresponds to three jumps in (2.8), while the second integral is needed to ensure that \( (Z_t, Z_t^*)_{t \geq 0} \) has the generator \( \tilde{L}_Z \), see the proof of Proposition 2.3 below.

The equation (2.12) looks a little complicated, thus we have to simplify it before moving forward. Recall that for \( x, y \in \mathbb{R}^d \) and \( \kappa \in (0, \kappa_0] \), \( (x - y)_\kappa \) is given by (2.7).
By collecting the terms involving $z$, we can rewrite the above equation as

\[
d Z^*_t = \int_{\mathbb{R}^d \times [0,1]} z \tilde{N}(dt, dz, du) \\
+ \int_{\mathbb{R}^d \times [0,1]} \left( (U_{t-})_n \mathbb{1}_{\{u \leq \frac{1}{2} \rho((-U_{t-})_n, z)\}} \right) \tilde{N}(dt, dz, du) \\
+ \left(\frac{1}{2} \int_{\mathbb{R}^d} \left( z + (U_{t-})_n \right) \left( \mathbb{1}_{\{|z+(U_{t-})_n| \leq 1\}} - \mathbb{1}_{\{|z| \leq 1\}} \right) \rho((-U_{t-})_n, z) \nu(dz) \right) dt
\]

(2.13)

Furthermore, observe that if $U_{t-} = Z_{t-} - Z^*_t = 0$, then $d Z^*_t = d Z_t$; if $U_{t-} \neq 0$, then, by the fact that $\mu_x = \nu \wedge (\delta_z * \nu)$ is a finite measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ for any $x \neq 0$,

\[
\int_{\mathbb{R}^d \times [0,1]} \mathbb{1}_{\{u \leq \frac{1}{2} \rho((-U_{t-})_n, z)\}} \nu(dz) \, du = \frac{1}{2} \mu((-U_{t-})_n)(\mathbb{R}^d) < \infty
\]

(2.14) and

\[
\int_{\mathbb{R}^d \times [0,1]} \mathbb{1}_{\{z \leq \frac{1}{2} \rho((-U_{t-})_n, z)\}} \nu(dz) \, du = \frac{1}{2} \mu((U_{t-})_n)(\mathbb{R}^d) < \infty
\]

(2.15)

Furthermore,

\[
\int_{\mathbb{R}^d \times [0,1]} \mathbb{1}_{\{u \leq \frac{1}{2} \rho((-U_{t-})_n, z)\}} \tilde{N}(dt, dz, du)
\]

and

\[
\int_{\mathbb{R}^d \times [0,1]} \mathbb{1}_{\{z \leq \frac{1}{2} \rho((-U_{t-})_n, z)\}} \tilde{N}(dt, dz, du)
\]

are well defined.

We denote by $J_i$ ($1 \leq i \leq 3$) the three terms on the right hand side of (2.13). On the one hand, using (2.4) and changing variable $z + (U_{t-})_n \to z$ lead to

\[
\frac{1}{2} \int_{\mathbb{R}^d} \left( z + (U_{t-})_n \right) \left( \mathbb{1}_{\{|z+(U_{t-})_n| \leq 1\}} - \mathbb{1}_{\{|z| \leq 1\}} \right) \rho((-U_{t-})_n, z) \nu(dz)
= \frac{1}{2} \int_{\mathbb{R}^d} z \left( \mathbb{1}_{\{|z| \leq 1\}} - \mathbb{1}_{\{|z+(U_{t-})_n| \leq 1\}} \right) \rho((U_{t-})_n, z) \nu(dz).
\]

Thus,

\[
J_3 = \frac{1}{2} (U_{t-})_n \int_{\mathbb{R}^d} \left( \mathbb{1}_{\{|z+(U_{t-})_n| \leq 1\}} - \mathbb{1}_{\{|z| \leq 1\}} \right) \rho((U_{t-})_n, z) \nu(dz) \, dt.
\]

On the other hand, the subtracted term in the martingale part of $J_2$ is

\[
\int_{\{|z| \leq 1\} \times [0,1]} \left( (U_{t-})_n \mathbb{1}_{\{u \leq \frac{1}{2} \rho((-U_{t-})_n, z)\}} \right) \tilde{N}(dt, dz, du) \\
+ \left(\frac{1}{2} \int_{\mathbb{R}^d} \left( z + (U_{t-})_n \right) \left( \mathbb{1}_{\{|z+(U_{t-})_n| \leq 1\}} - \mathbb{1}_{\{|z| \leq 1\}} \right) \rho((-U_{t-})_n, z) \nu(dz) \right) dt
\]

\[
= \frac{1}{2} (U_{t-})_n \int_{\{|z| \leq 1\}} \rho((-U_{t-})_n, z) \nu(dz) - \int_{\{|z| \leq 1\}} \rho((U_{t-})_n, z) \nu(dz) \, dt
\]
the solution coupling process for the Lévy process
\[2.3.\]
postpone it in the next subsection.

\[dZ_t = \int_{\mathbb{R}^d \times [0,1]} z \tilde{N}(dt, dz, du) + (U_t)_\kappa \int_{\mathbb{R}^d \times [0,1]} \left[ \mathbb{1}_{\{u \geq \frac{1}{2} \rho((-U_t)_\kappa, z)\}} - \mathbb{1}_{\{\frac{1}{2} \rho((-U_t)_\kappa, z) < u \leq \frac{1}{2} \rho((-U_t)_\kappa, z) + \rho(U_t)_\kappa, z)\}} \right] N(dt, dz, du).\]

We denote by
\[V_t(z, u) = (U_t)_\kappa \left[ \mathbb{1}_{\{u \geq \frac{1}{2} \rho((-U_t)_\kappa, z)\}} - \mathbb{1}_{\{\frac{1}{2} \rho((-U_t)_\kappa, z) < u \leq \frac{1}{2} \rho((-U_t)_\kappa, z) + \rho(U_t)_\kappa, z)\}} \right]\]
and
\[dL_t^* = \int_{\mathbb{R}^d \times [0,1]} V_t(z, u) N(dt, dz, du).\]

Then (2.12) reduces to
\[dZ^*_t = dZ_t + dL_t^*.\]

By Remark 2.5 below, the process \((Z_t, Z_t^*)_t \geq 0\) constructed above is a Markov coupling process for the Lévy process \(Z_t\), and its infinitesimal generator is \(\tilde{L}_Z\) defined in (2.9). Since the proof is similar to that of the coupling for the SDE (1.1), we postpone it in the next subsection.

2.3. Coupling for the SDE (1.1). In this part we study the coupling process of the solution \((X_t)_t \geq 0\) to the SDE (1.1). The infinitesimal generator of \((X_t)_t \geq 0\) is

\[L_X f(x) = \int \left( \left( f(x + z) - f(x) - \langle \nabla f(x), z \rangle \mathbb{1}_{\{|z| \leq 1\}} \right) \right) \nu(dz) + \langle b(x), \nabla f(x) \rangle = L_Z f(x) + \langle b(x), \nabla f(x) \rangle.\]

Given the coupling operator \(\tilde{L}_Z\) in (2.9) for the pure jump Lévy process \(Z_t\), it is natural to define \(L_X\) as follows: for any \(h \in C_0^2(\mathbb{R}^d \times \mathbb{R}^d)\),

\[\tilde{L}_X h(x, y) = \tilde{L}_Z h(x, y) + \langle b(x), \nabla_x h(x, y) \rangle + \langle b(y), \nabla_y h(x, y) \rangle.\]

Since \(\tilde{L}_Z\) is a coupling operator of \(L_Z\), it is easy to see that \(\tilde{L}_X\) is a coupling operator of \(L_X\) too.

Next we present the coupling equation corresponding to \(\tilde{L}_X\). Recall that the process \((X_t)_t \geq 0\) is generated by the SDE
\[dX_t = b(X_t) dt + dZ_t, \quad X_0 = x.\]

Therefore, taking into account the equation (2.16), we denote by \(U_t = X_t - Y_t\) and
\[V_t(z, u) = (U_t)_\kappa \left[ \mathbb{1}_{\{u \geq \frac{1}{2} \rho((-U_t)_\kappa, z)\}} - \mathbb{1}_{\{\frac{1}{2} \rho((-U_t)_\kappa, z) < u \leq \frac{1}{2} \rho((-U_t)_\kappa, z) + \rho(U_t)_\kappa, z)\}} \right]\]
for \(z \in \mathbb{R}^d\) and \(u \in [0, 1]\). Then the marginal process \((Y_t)_t \geq 0\) of the coupling process \((X_t, Y_t)_t \geq 0\) should fulfill the equation
\[dY_t = b(Y_t) dt + dZ^*_t.\]
with \(dZ_t^* = dZ_t + dL_t^*\), where
\[
(2.19) \quad dL_t^* = \int_{R^d \times [0,1]} V_t(z, u) N(dt, dz, du).
\]

Fix any \(x, y \in R^d\) with \(x \neq y\). We consider the system of equations:
\[
(2.20) \quad \begin{cases} 
  dX_t = b(X_t) dt + dZ_t, & X_0 = x; \\
  dY_t = b(Y_t) dt + dZ_t + dL_t^*, & Y_0 = y.
\end{cases}
\]

**Proposition 2.2.** The system of equations (2.20) has a unique strong solution.

**Proof.** In the setting of our paper, we always assume that the equation (1.1) (i.e., the first equation in (2.20)) has a non-explosive and pathwise unique strong solution \((X_t)_{t \geq 0}\). We show that the sample paths of \((Y_t)_{t \geq 0}\) can be obtained by repeatedly modifying those of the solution of the following equation:
\[
(2.21) \quad d\tilde{Y}_t = b(\tilde{Y}_t) dt + dZ_t, \quad \tilde{Y}_0 = y.
\]

Denote by \(Y_t^{(1)}\) the solution to (2.21). Take a uniformly distributed random variable \(\zeta_1\) on \([0,1]\), and define the stopping times \(T_1 = \inf \{t > 0 : X_t = Y_t^{(1)}\}\) and
\[
\sigma_1 = \inf \left\{ t > 0 : \zeta_1 \leq \frac{1}{2} \left[ \rho((Y_t^{(1)} - X_t)_+, \Delta Z_t) + \rho((X_t - Y_t^{(1)})_+, \Delta Z_t) \right] \right\}.
\]

We consider two cases:

(i) On the event \(\{T_1 \leq \sigma_1\}\), we set \(Y_t = Y_t^{(1)}\) for all \(t < T_1\); moreover, by the pathwise uniqueness of the equation (1.1), we can define \(Y_t = X_t\) for \(t \geq T_1\).

(ii) On the event \(\{T_1 > \sigma_1\}\), we define \(Y_t = Y_t^{(1)}\) for all \(t < \sigma_1\) and
\[
Y_{\sigma_1} = Y_{\sigma_1}^{(1)} + \Delta Z_{\sigma_1} + \begin{cases} 
  (X_{\sigma_1 -} - Y_{\sigma_1 -}^{(1)})_+, & \text{if } \zeta_1 \leq \frac{1}{2} \rho((Y_{\sigma_1 -}^{(1)} - X_{\sigma_1 -})_+, \Delta Z_{\sigma_1}); \\
  (Y_{\sigma_1 -}^{(1)} - X_{\sigma_1 -})_+, & \text{if } \zeta_1 > \frac{1}{2} \rho((Y_{\sigma_1 -}^{(1)} - X_{\sigma_1 -})_+, \Delta Z_{\sigma_1}).
\end{cases}
\]

Next, we restrict on the event \(\{T_1 > \sigma_1\}\) and consider the SDE (2.21) with \(t > \sigma_1\) and \(\tilde{Y}_{\sigma_1} = Y_{\sigma_1}\). Denote its solution by \(Y_t^{(2)}\). Similarly, we take another uniformly distributed random variable \(\zeta_2\) on \([0,1]\), and define \(T_2 = \inf \{t > \sigma_1 : X_t = Y_t^{(2)}\}\) and
\[
\sigma_2 = \inf \left\{ t > \sigma_1 : \zeta_2 \leq \frac{1}{2} \left[ \rho((Y_t^{(2)} - X_t)_+, \Delta Z_t) + \rho((X_t - Y_t^{(2)})_+, \Delta Z_t) \right] \right\}.
\]

In the same way, we can define the process \(Y_t\) for \(t < \sigma_2\). We repeat this procedure and note that, thanks to (2.14) and (2.15), only finite many modifications have to be made in any finite interval of time. Finally, we obtain the sample paths \((Y_t)_{t \geq 0}\). \(\square\)

Furthermore, the following conclusion indicates that the process \((X_t, Y_t)_{t \geq 0}\) is indeed the coupling process of \((X_t)_{t \geq 0}\).

**Proposition 2.3.** The infinitesimal generator of the process \((X_t, Y_t)_{t \geq 0}\) is \(\tilde{L}_X\) defined in (2.18).

**Proof.** According to the discussions in the previous subsection, the driven noise \((Z_t^*)_{t \geq 0}\) defined by \(Z_t^* = Z_t + L_t^*\) in the second equation of (2.20) also enjoys the expression (2.12) with \(U_t = X_t - Y_t\) replacing \(U_t = Z_t - Z_t^*\). Then, the desired assertion can be proved by making use of the equations (2.12) and (2.20) and applying
the Itô formula. Indeed, denote by $\tilde{L}_X$ the generator corresponding to $(X_t, Y_t)_{t \geq 0}$. For $h \in C^2_0(\mathbb{R}^d \times \mathbb{R}^d)$, by (2.12) and (2.20), we have

$$\tilde{L}_X h(x, y) = \frac{1}{2} \int_{\mathbb{R}^d} \left( h(x + z, y + z + (x - y)_n) - h(x, y) - \langle \nabla_x h(x, y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) dz$$

$$- \langle \nabla_y h(x, y), z + (x - y)_n \rangle \mathbf{1}_{\{|z| \leq 1\}} \mu_{(y-x)_n}(dz)$$

$$+ \frac{1}{2} \int_{\mathbb{R}^d} \left( h(x + z, y + z + (y - x)_n) - h(x, y) - \langle \nabla_x h(x, y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) dz$$

$$- \langle \nabla_y h(x, y), z + (y - x)_n \rangle \mathbf{1}_{\{|z| \leq 1\}} \mu_{(x-y)_n}(dz)$$

$$+ \int_{\mathbb{R}^d} \left( h(x + z, y + z) - h(x, y) - \langle \nabla_x h(x, y), z \rangle \mathbf{1}_{\{|z| \leq 1\}} \right) \left( \nu(dz) - \frac{1}{2} \mu_{(y-x)_n}(dz) - \frac{1}{2} \mu_{(x-y)_n}(dz) \right)$$

$$- \frac{1}{2} \int_{\mathbb{R}^d} \langle \nabla_y h(x, y), z + (y - x)_n \rangle \left( \mathbf{1}_{\{|z| \leq 1\}} \mu_{(x-y)_n}(dz) - \mathbf{1}_{\{|z| \leq 1\}} \mu_{(x-y)_n}(dz) \right)$$

$$- \frac{1}{2} \int_{\mathbb{R}^d} \langle \nabla_y h(x, y), z + (y - x)_n \rangle \left( \mathbf{1}_{\{|z| \leq 1\}} \mu_{(x-y)_n}(dz) - \mathbf{1}_{\{|z| \leq 1\}} \mu_{(x-y)_n}(dz) \right)$$

$$+ \langle b(x), \nabla_x h(x, y) \rangle + \langle b(y), \nabla_y h(x, y) \rangle,$$

where the first three integrals come from the integral in (2.12) with respect to the Poisson random measure $\tilde{N}(dt, dz, du)$, while the next two terms follow from the second integral in (2.12). Simplifying the above identity, we can easily see that

$$\tilde{L}_X h(x, y) = \tilde{L}_X h(x, y),$$

therefore the proof is complete. \qed

According to the above discussions, $\tilde{L}_X$ is a coupling operator of $L_X$ in (2.17), thus we deduce

**Corollary 2.4.** The process $(Y_t)_{t \geq 0}$ has the same finite dimensional distributions with $(X_t)_{t \geq 0}$.

Summarizing all the conclusions above, the coupling operator $\tilde{L}_X$ generates a non-explosive coupling process $(X_t, Y_t)_{t \geq 0}$ of the process $(X_t)_{t \geq 0}$, and $X_t = Y_t$ for any $t \geq T$, where $T = \inf\{t \geq 0 : X_t = Y_t\}$ is the coupling time of the process $(X_t, Y_t)_{t \geq 0}$.

**Remark 2.5.** Since the drift term $b$ can be chosen to be $b(x) = 0$ for all $x \in \mathbb{R}^d$ in the proofs of Propositions 2.2 and 2.3, one can claim that the process $(Z_t, Z^*_t)_{t \geq 0}$ constructed in Subsection 2.2 is a Markov coupling process for the Lévy process $Z$, and its infinitesimal generator is $\tilde{L}_Z$ defined in (2.9). In particular, the process $(Z^*_t)_{t \geq 0}$ defined by (2.13) is also a Lévy process on $\mathbb{R}^d$ with Lévy measure $\nu$.

### 3. Exponential convergence in Wasserstein-type distances via coupling

By making full use of the coupling operator and the coupling process constructed in Section 2.3, we will provide in this part a general result for exponential convergence in Wasserstein distances including the total variation.
3.1. Preliminary calculations. Let \( \tilde{L}_X \) be the coupling operator given in (2.18). We will compute the expression of \( \tilde{L}_X f(|x - y|) \) for any \( f \in C^1_b([0, \infty)) \) with \( f \geq 0 \).

Let \((X_t,Y_t)\) be the coupling process corresponding to the operator \( \tilde{L}_X \) constructed in Subsection 2.3. Recall that for any \( t \geq 0, \kappa \in (0,\kappa_0) \) and \( z,u \in \mathbb{R}^d \),

\[
V_t(z,u) = (U_t)\kappa \left[ \mathbb{1}_{\{u \leq \kappa \rho((U_t)_n,z)\}} - \mathbb{1}_{\{\kappa \rho((U_t)_n,z) < u \leq \frac{1}{2} \rho((U_t)_n,z) + \rho((U_t)_n,z)\}} \right].
\]

In particular,

\[
V_0(z,u) = (x-y)\kappa \left[ \mathbb{1}_{\{u \leq \frac{1}{2} \rho((y-x)_n,z)\}} - \mathbb{1}_{\{\frac{1}{2} \rho((y-x)_n,z) < u \leq \frac{1}{2} \rho((y-x)_n,z) + \rho((y-x)_n,z)\}} \right].
\]

It follows from the system (2.20) that

\[
dU_t = (b(X_t) - b(Y_t)) dt - \int_{\mathbb{R}^d \times [0,1]} V_t(z,u) N(dt,dz,du).
\]

Take \( f \in C^1_b([0, \infty)) \) with \( f \geq 0 \). By the Itô formula,

\[
f(|U_t|) = f(|x - y|) + \int_0^t \frac{f'(|U_s|)}{|U_s|} (U_s, b(X_s) - b(Y_s)) ds + \int_0^t \int_{\mathbb{R}^d \times [0,1]} \left[ f(|U_{s-} - V_{s-}(z,u)|) - f(|U_{s-}|) \right] N(ds,dz,du).
\]

Therefore,

\[
\tilde{L}_X f(|x - y|) = \frac{f'(|x - y|)}{|x - y|} (b(x) - b(y), x - y)
\]

\[
+ \int_{\mathbb{R}^d \times [0,1]} \left[ f(|(x - y) - V_0(z,u)|) - f(|x - y|) \right] \nu(dz,du).
\]

By the definition of \( V_0 \), the second term on the right hand side is equal to

\[
\frac{1}{2} \int_{\mathbb{R}^d} \left[ \left( f(|(x - y) - (x - y)_\kappa|) - f(|x - y|) \right) \rho((y-x)_n,z) \right.
\]

\[
+ \left. \left( f(|(x - y) + (x - y)_\kappa|) - f(|x - y|) \right) \rho((y-x)_n,z) \right] \nu(dz)
\]

\[
= \frac{1}{2} \left[ \left( f(|(x - y) - (x - y)_\kappa|) - f(|x - y|) \right) \mu_{(y-x)_n}(\mathbb{R}^d)
\]

\[
+ \left. \left( f(|(x - y) + (x - y)_\kappa|) - f(|x - y|) \right) \mu_{(y-x)_n}(\mathbb{R}^d) \right]
\]

Thanks to the fact (also see Corollary 6.2) that

\[
\mu_{(y-x)_n}(\mathbb{R}^d) = \mu_{(-x-y)_n}(\mathbb{R}^d) = \mu_{(y-x)_n}(\mathbb{R}^d),
\]

we can finally conclude that, for any \( x, y \in \mathbb{R}^d \) with \( x \neq y \),

\[
\tilde{L}_X f(|x - y|) = \frac{1}{2} \mu_{(x-y)_n}(\mathbb{R}^d) \left[ f\left(|x - y| + \kappa \wedge |x - y|\right) + f\left(|x - y| - \kappa \wedge |x - y|\right) \right.
\]

\[
- 2 f(|x - y|) \left. + \frac{f'(|x - y|)}{|x - y|} (b(x) - b(y), x - y) \right] - 2 f(|x - y|) \left. + \frac{f'(|x - y|)}{|x - y|} (b(x) - b(y), x - y) \right]
\]

Note that, by (3.1), \( \tilde{L}_X f(|x - y|) \) is pointwise well defined for any \( f \in C^1([0, \infty)) \).
3.2. General result. The following theorem provides us a general result for exponential convergence in Wasserstein-type distance via the coupling method. Recall the definition of $J(s)$ in Section 1.

**Theorem 3.1.** Assume that the drift term $b$ satisfies $\mathcal{B}(\Phi_1, \Phi_2, l_0)$, i.e. (1.2), and that (1.4) holds for the Lévy measure $\nu$ with some $\kappa_0 > 0$. For any $n \geq 1$, let $\psi_n \in C^1([0, \infty))$ be increasing on $[0, \infty)$, satisfying $\psi_n(0) = 0$ and

$$
\psi_n(r+s) + \psi_n(r-s) - 2\psi_n(r) \leq 0 \quad \text{for all } r \geq 1/n, 0 < s \leq r \wedge \kappa_0.
$$

Suppose that there are $\lambda > 0$ and $\kappa \in (0, \kappa_0]$ such that for $n \geq l_0^{-1} \vee l_0$ large enough, $\psi_n$ satisfies the condition $C(\lambda, \kappa, n)$ on $[1/n, n]$ as follows:

(i) for $r \in [1/n, l_0)$,

$$
\frac{1}{2} J(\kappa \wedge r) \left[ \psi_n(r + r \wedge \kappa) + \psi_n(r - r \wedge \kappa) - 2\psi_n(r) \right] + \Phi_1(r) \psi_n'(r) \leq -\lambda \psi_n(r);
$$

(ii) for $r \in [l_0, n]$,

$$
-\Phi_2(r) \psi_n'(r) \leq -\lambda \psi_n(r).
$$

Then for any $t > 0$ and $x, y \in \mathbb{R}^d$,

$$
W_{\psi_n}(\delta_x P_t, \delta_y P_t) \leq \psi_n(|x - y|) e^{-\lambda t},
$$

where $\psi_\infty = \lim \inf_{n \to \infty} \psi_n$.

**Proof.** Step 1. Let $\bar{L} = \bar{L}_X$ be the coupling operator given in (2.18). We first prove that for $n \geq l_0^{-1} \vee l_0$ large enough and for all $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| \leq n$,

$$
\bar{L}\psi_n(|x - y|) \leq -\lambda \psi_n(|x - y|).
$$

For this, we consider the following two cases.

(a) $1/n \leq |x - y| < l_0$. The definition of $J(s)$ leads to

$$
\mu_{(x-y)_*}(\mathbb{R}^d) = [\nu \wedge (\delta(1_{\wedge |x-y|}) \ast \nu)](\mathbb{R}^d) \geq J(|x - y| \wedge \kappa).
$$

Thus by (3.1), (3.2) and (1.2),

$$
\bar{L}\psi_n(|x - y|) \leq \frac{1}{2} J(|x - y| \wedge \kappa) \left[ \psi_n(|x - y| + |x - y| \wedge \kappa)
\right.

$$

$$
+ \psi_n(|x - y| - |x - y| \wedge \kappa) - 2\psi_n(|x - y|)
\left. + \Phi_1(|x - y|) \psi_n'(|x - y|) \right]
\leq -\lambda \psi_n(|x - y|),
$$

where we used the condition (i) in the last inequality.

(b) $l_0 \leq |x - y| \leq n$. In view of (3.1), it is obvious from the conditions (3.2) and (1.2) that

$$
\bar{L}\psi_n(|x - y|) \leq -\Phi_2(|x - y|) \psi_n'(|x - y|) \leq -\lambda \psi_n(|x - y|),
$$

where the last inequality follows from (ii).

Then (3.4) is proved by summarizing these arguments.

**Step 2.** Based on (3.4), the proof of the desired assertion (3.3) is similar to that of [8, Theorem 1.3] or [20, Theorem 1.2] by some slight modifications. For the sake of completeness, we present the details here. Let $(X_t, Y_t)_{t \geq 0}$ be the coupling process.
constructed in Section 2.3. It suffices to verify that for $x, y \in \mathbb{R}^d$ with $|x - y| > 0$ and any $t > 0$,$$
abla^{(x,y)} \psi_\infty(|X_t - Y_t|) \leq \psi_\infty(|x - y|) e^{-\lambda t},$$
where $\nabla^{(x,y)}$ is the expectation of $(X_t, Y_t)_{t \geq 0}$ starting from $(x, y)$.

For any $t > 0$ set $r_t = |U_t| = |X_t - Y_t|$, and for $n \geq 1$ define the stopping time
$$T_n = \inf\{t > 0 : r_t \notin [1/n, n]\}.$$Since the coupling process $(X_t, Y_t)_{t \geq 0}$ is non-explosive, we have $T_n \uparrow T$ a.s. as $n \to \infty$, where $T$ is the coupling time of the process $(X_t, Y_t)_{t \geq 0}$.

For any $x, y \in \mathbb{R}^d$ with $|x - y| > 0$, we take $n \geq l_0^1 \vee l_0$ large enough such that $1/n < |x - y| < n$. For $m \geq n$, let $\psi_m$ be the function and $\lambda$ be the constant given in the statement. Then,
$$\nabla^{(x,y)}[e^{\lambda(T\wedge T_m)} \psi_m(|X_{t\wedge T_m} - Y_{t\wedge T_m}|)]$$
$$= \psi_m(|x - y|) + \nabla^{(x,y)}\left(\int_0^{T\wedge T_m} e^{\lambda s} [\lambda \psi_m(|X_s - Y_s|) + \nabla \psi_m(|X_s - Y_s|)] ds\right)$$
$$\leq \psi_m(|x - y|),$$
where the inequality above follows from (3.4). Hence,
$$\nabla^{(x,y)}[e^{\lambda(T\wedge T_m)} \psi_m(r_{t\wedge T_m})] \leq \psi_m(r_0).$$Thus by Fatou's lemma, first letting $m \to \infty$ and then $n \to \infty$ in the above inequality gives us
$$\nabla^{(x,y)}(e^{\lambda(T\wedge T)} \psi_\infty(r_{t\wedge T})) \leq \psi_\infty(r_0).$$

Thanks to our convention that $Y_t = X_t$ for $t \geq T$, we have $r_t = 0$ and so $\psi_\infty(r_t) = 0$ for all $t \geq T$, which implies
$$\nabla^{(x,y)}(e^{\lambda(T\wedge T)} \psi_\infty(r_{t\wedge T})) = e^{\lambda T} \nabla^{(x,y)}(\psi_\infty(r_{t\wedge T}) \mathbf{1}_{\{T > t\}}) = e^{\lambda T} \nabla^{(x,y)} \psi_\infty(r_t).$$Therefore, the desired assertion follows from all the discussions above. \hfill \Box

4. Proofs

4.1. Proofs of results related to Wasserstein-type distances. The following result is crucial for constructing test functions $\psi_n$ in Theorem 3.1.

Lemma 4.1. Let $g \in C([0, 2l_0]) \cap C^3((0, 2l_0])$ be satisfying
\begin{equation}
(4.1) \quad g'(r) \geq 0, \quad g''(r) \leq 0 \quad \text{and} \quad g'''(r) \geq 0 \quad \text{for any} \quad r \in (0, 2l_0].
\end{equation}

Then for all $c_1, c_2 > 0$ the function
\begin{equation}
(4.2) \quad \psi(r) := \psi_{c_1, c_2}(r) = \begin{cases} c_1 r + \int_0^r e^{-c_2 s} ds, & r \in [0, 2l_0), \\ \psi(2l_0) + \psi'(2l_0)(r - 2l_0), & r \in (2l_0, \infty) \end{cases}
\end{equation}
satisfies
\begin{enumerate}
\item $\psi \in C^1([0, \infty))$ and $c_1 r \leq \psi(r) \leq (c_1 + 1)r$ on $[0, 2l_0]$;
\item $\psi' > 0$, $\psi'' \leq 0$, $\psi''' \geq 0$ and $\psi^{(4)} \leq 0$ on $(0, 2l_0]$;
\item for any $0 \leq \delta \leq r$, $\psi(r + \delta) + \psi(r - \delta) - 2\psi(r) \leq 0$;
\end{enumerate}
(4) for any $0 \leq \delta \leq r \leq l_0$,
$$\psi(r + \delta) + \psi(r - \delta) - 2\psi(r) \leq \psi''(r)\delta^2.$$  

Proof. (1) is trivial. The property (2) follows from (4.1) and the definition of $\psi$ by direct calculations. The assertion (3) is trivial if $\delta = 0$, thus we assume $\delta > 0$ in the sequel. By the mean value formula, there exist constants $\xi_1 \in (r, r + \delta)$ and $\xi_2 \in (r - \delta, r)$ such that
$$\psi(r + \delta) - \psi(r) = \psi'(\xi_1)\delta$$
and
$$\psi(r - \delta) - \psi(r) = -\psi'(\xi_2)\delta.$$  

Therefore,
$$\psi(r + \delta) + \psi(r - \delta) - 2\psi(r) = (\psi'(\xi_1) - \psi'(\xi_2))\delta \leq 0,$$  

since $\psi'$ is decreasing due to the definition of $\psi$.

To prove (4), we will still assume $\delta > 0$. Similar to the proof of (3), by the Taylor formula, there exist constants $\xi_1 \in (r, r + \delta)$ and $\xi_2 \in (r - \delta, r)$ such that
$$\psi(r + \delta) = \psi(r) + \psi'(r)\delta + \frac{1}{2}\psi''(r)\delta^2 + \frac{1}{6}\psi'''(\xi_1)\delta^3,$$
$$\psi(r - \delta) = \psi(r) - \psi'(r)\delta + \frac{1}{2}\psi''(r)\delta^2 - \frac{1}{6}\psi'''(\xi_2)\delta^3.$$  

Therefore,
$$\psi(r + \delta) + \psi(r - \delta) - 2\psi(r) = \psi''(r)\delta^2 + \frac{\delta^3}{6}[\psi'''(\xi_1) - \psi'''(\xi_2)] \leq \psi''(r)\delta^2$$  

since $\psi'''$ is decreasing due to (2). \qed

In the next theorem we establish the exponential contraction in $L^1$-Wasserstein distance which is more general than Theorem 1.1.

Theorem 4.2. Assume that

(a) (1.4) holds for the Lévy measure $\nu$ with some $\kappa_0 > 0$;
(b) the drift $b$ satisfies $B(\Phi_1(r), K_2r, l_0)$ for some constants $K_2 > 0$, $l_0 \geq 0$, and a nonnegative concave function $\Phi_1 \in C([0, 2l_0]) \cap C^2((0, 2l_0])$ such that $\Phi_1(0) = 0$ and $\Phi_1''$ is nondecreasing;
(c) there is a nondecreasing and concave function $\sigma \in C([0, 2l_0]) \cap C^2((0, 2l_0])$ such that for some $\kappa \in (0, \kappa_0]$, one has

(4.3)  
$$\sigma(r) \leq \frac{1}{2r}J(\kappa \land r)(\kappa \land r)^2, \quad r \in (0, 2l_0];$$

and the integrals $g_1(r) = \int_0^r \frac{\Phi_1(s)}{\sigma(s)}ds$ and $g_2(r) = \int_0^r \frac{\Phi_1(s)}{\sigma^2(s)}ds$ are well defined for all $r \in [0, 2l_0]$.

Set $c_2 = (2K_2) \land g_1(2l_0)^{-1}$ and $c_1 = e^{-c_29(2l_0)}$, where the function $g$ is defined by
$$g(r) = g_1(r) + \frac{2}{c_2}g_2(r), \quad r \in (0, 2l_0].$$  

Then for any $x, y \in \mathbb{R}^d$ and $t > 0$,
$$W_1(\delta_x P_t, \delta_y P_t) \leq Ce^{-\lambda t}|x - y|,$$
Using the classical synchronous coupling, one can prove that for any \( x, y \in \mathbb{R}^d \) and \( t > 0 \),
\[
W_1(\delta_x P_t, \delta_y P_t) \leq e^{-K_2t|x-y|}.
\]
In this case, the constants \( C \) and \( \lambda \) given by (4.4) are also equal to 1 and \( K_2 \), respectively.

(2) Assume that \( \Phi_1(s) = K_1s \) for any \( s \in [0, 2l_0] \) with some constant \( K_1 \geq 0 \). Then
\[
g(r) = \int_0^r \frac{ds}{\sigma(s)} + 2 \int_0^r \frac{K_1s}{s\sigma(s)} ds = \frac{c_2 + 2K_1}{c_2} g_1(r), \quad r \in (0, 2l_0],
\]
and so
\[
c_2g(2l_0) = (2K_1 + c_2)g_1(2l_0) = [2K_1 + (2K_2) \wedge g_1(2l_0)^{-1}]g_1(2l_0) \leq 2K_1g_1(2l_0) + 1.
\]
Therefore, for any fixed \( a_0 > 0 \), we have
\[
\lambda \geq \begin{cases} 
(1 + e^{1+2a_0})^{-1} [(2K_2) \wedge g_1(2l_0)^{-1}], & K_1g_1(2l_0) \leq a_0; \\
(2e)^{-1} [(2K_2) \wedge g_1(2l_0)^{-1}] \exp (-2K_1g_1(2l_0)), & K_1g_1(2l_0) \geq a_0.
\end{cases}
\]
If moreover \( Z \) is the (truncated) symmetric \( \alpha \)-stable process with \( \alpha \in (0, 2) \), then, from the proof of Example 1.2 below, we can take \( \sigma(r) = a_1r^{1-\alpha} \) and so \( g_1(r) = a_2r^\alpha \) for some \( a_1, a_2 > 0 \). Thus, taking into account the related discussions in [6, Section 2.3] for diffusions, we find that the lower bounds above for \( \lambda \) are of optimal orders with respect to \( l_0, K_1 \) and \( K_2 \) when \( \alpha \to 2 \) (i.e. \( Z \) is replaced by the standard Brownian motion).

(3) Suppose that (1.4) holds with some \( \kappa_0 > 0 \) and
\[
\lim_{\kappa \to 0} J(\kappa)\kappa^2 = 0,
\]
which are true for (truncated) \( \alpha \)-stable processes with \( \alpha \in (0, 2) \), cf. the proof of Example 1.2. We claim that, if \( l_0 > 0 \), then the constant \( \lambda \) defined in (4.4) tends to 0 as \( \kappa \to 0 \). Indeed, for \( \kappa < r \leq 2l_0 \), one has
\[
\sigma(r) \leq \frac{1}{2r} J(\kappa \wedge r)(\kappa \wedge r)^2 = \frac{1}{2r} J(\kappa)\kappa^2,
\]
hence, as \( \kappa \to 0 \),
\[
g_1(2l_0) = \int_0^{2l_0} \frac{dr}{\sigma(r)} \geq \int_\kappa^{2l_0} \frac{2r}{J(\kappa)\kappa^2} dr = \frac{4l_0^2 - \kappa^2}{J(\kappa)\kappa^2} \to \infty
\]
and so
\[
\lambda \leq (2K_2) \wedge g_1(2l_0)^{-1} \to 0.
\]
Proof of Theorem 4.2. We split the proof into two steps.

Step 1. We first show that the function $g$ defined in the theorem satisfies (4.1). For $r \in (0, 2\tau_0]$, it is clear that

$$g'(r) = \frac{1}{\sigma(r)} \left[ 1 + \frac{2\Phi_1(r)}{c_2 r} \right] \geq 0.$$ 

Next, since $\Phi_1$ is concave and $\Phi_1(0) = 0$, we have $\Phi_1(r) = \int_0^r \Phi_1(s) \, ds \geq \Phi_1'(r) r$. This together with $\sigma' \geq 0$ implies

$$g''(r) = -\frac{\sigma'(r)}{\sigma(r)^2} \left[ 1 + \frac{2\Phi_1(r)}{c_2 r} \right] + \frac{2}{c_2 \sigma(r)} \cdot \frac{\Phi_1'(r) r - \Phi_1(r)}{r^2} \leq 0.$$

Finally,

$$g'''(r) = \frac{2\sigma'(r) - \sigma(r) \sigma''(r)}{\sigma(r)^3} \left[ 1 + \frac{2\Phi_1(r)}{c_2 r} \right] - \frac{4\sigma'(r)}{c_2 \sigma(r)^2} \cdot \frac{\Phi_1'(r) r - \Phi_1(r)}{r^2} + \frac{2}{c_2 \sigma(r)} \cdot \frac{2\Phi_1(r) - 2\Phi_1'(r) r + \Phi_1''(r) r^2}{r^3}.$$ 

As $\sigma''(r) \leq 0$, the first term on the right hand side is nonnegative. The same is true for the second term since $\sigma'(r) \geq 0$ and $\Phi_1'(r) r - \Phi_1(r) \leq 0$. For the last term, we have by Taylor’s formula that there is a constant $\xi \in (0, r)$ such that

$$\Phi_1(0) = \Phi_1(r) - \Phi_1'(r)r + \frac{1}{2} \Phi_1''(\xi)r^2 \leq \Phi_1(r) - \Phi_1'(r)r + \frac{1}{2} \Phi_1''(r)r^2,$$

where the last inequality is due to the fact that $\Phi_1''$ is nondecreasing. Note that $\Phi_1(0) = 0$, we conclude that the third term is also nonnegative. Therefore $g'''(r) \geq 0$.

Step 2. Let $\psi$ be defined as in Lemma 4.1 with $c_1, c_2$ and $g$ given in the theorem. We prove that $\psi$ satisfies $C(\lambda, \kappa, \infty)$ for some $\lambda > 0$ and $\kappa \in (0, \kappa_0]$ (see Theorem 3.1 for its meaning). Note that, by (3) in Lemma 4.1, $\psi$ verifies (3.2) for all $r \geq s \geq 0$. Under the condition $B(\Phi_1(r), K_{2r}, l_0)$, (4) in Lemma 4.1 and (4.3) yield that for all $r \in (0, l_0]$,

$$\Theta(r) := \frac{1}{2} J(\kappa \wedge r) \left[ \psi(r + \kappa \wedge r) + \psi(r - \kappa \wedge r) - 2\psi(r) \right] + \Phi_1(r) \psi'(r)$$

$$\leq \frac{1}{2} J(\kappa \wedge r)(\kappa \wedge r)^2 \psi''(r) + \Phi_1(r) \psi'(r)$$

$$\leq \sigma(r) r \psi''(r) + \Phi_1(r) \psi'(r).$$

By (4.2), we have $\psi'(r) = c_1 + e^{-c_2 g(r)}$ and $\psi''(r) = -c_2 g'(r) e^{-c_2 g(r)}$. Hence, by the definition of $g$, we get that

$$\Theta(r) \leq \sigma(r) r \left[ -c_2 g'(r) e^{-c_2 g(r)} - \Phi_1(r) [c_1 + e^{-c_2 g(r)}] \right]$$

$$\leq -c_2 r e^{-c_2 g(r)} \left[ 1 + \frac{2\Phi_1(r)}{c_2 r} \right] + 2\Phi_1(r) e^{-c_2 g(r)}$$

$$= -c_2 r e^{-c_2 g(r)} \leq -c_1 c_2 \psi(r) \leq -\frac{c_1 c_2}{c_1 + 1} \psi(r),$$

where the last inequality follows from (1) in Lemma 4.1.
Next, if \( r \in (l_0, 2l_0] \), by \( \mathbf{B}(\Phi_1(r), K_2r, l_0) \) and (1) in Lemma 4.1 again,

\[
-K_2r \psi'(r) = -K_2r \left[ c_1 + e^{-c_2g(r)} \right] \leq - \frac{K_2 [c_1 + e^{-c_2g(2l_0)}]}{c_1 + 1} \psi(r) \\
= - \frac{2K_2c_1}{c_1 + 1} \psi(r) \leq - \frac{c_1c_2}{c_1 + 1} \psi(r).
\]

(4.6)

Note that the function

\[
r \mapsto \frac{\psi'(2l_0) r}{\psi(r)} = \frac{2c_1 r}{2c_1 r + \int_0^{2l_0} e^{-c_2g(s)} ds - 2c_1 l_0}
\]

is increasing on \((2l_0, \infty)\), since \( \int_0^{2l_0} e^{-c_2g(s)} ds \geq 2l_0 e^{-c_2g(2l_0)} = 2c_1 l_0 \). Thus for \( r > 2l_0 \), we use again \( \mathbf{B}(\Phi_1(r), K_2r, l_0) \) to obtain

\[
-K_2r \psi'(r) = -K_2\psi'(2l_0) r \leq -K_2 \frac{2l_0 \psi'(2l_0)}{\psi(2l_0)} \psi(r) \\
\leq -2K_2 \frac{2c_1 l_0}{2l_0 (c_1 + 1)} \psi(r) \leq - \frac{c_1c_2}{c_1 + 1} \psi(r).
\]

(4.7)

We conclude from all the estimates above that \( \mathbf{C}(\lambda, \kappa, \infty) \) holds with the positive constant \( \lambda \) given by (4.4). Therefore, we can apply Theorem 3.1 to get that for any \( t > 0 \) and \( x, y \in \mathbb{R}^d \),

\[
W_{\psi}(\delta_x P_t, \delta_y P_t) \leq \psi(|x - y|) e^{-\lambda t}.
\]

Since \( \psi \) is concave on \([0, \infty)\), it is clear that \( (c_1 + 1)r \geq \psi(r) \geq \psi'(2l_0)r = 2c_1 r \) for all \( r \geq 0 \). Hence the desired result holds with \( C = (c_1 + 1)/(2c_1) \). \( \square \)

Similar to Theorem 4.2, we have the following statement about the exponential rates for total variation.

**Theorem 4.4.** Assume that the drift \( b \) satisfies \( \mathbf{B}(K_1, K_2r, l_0) \) for some \( K_1, l_0 \geq 0 \) and \( K_2 > 0 \), and that (1.4) holds for the Lévy measure \( \nu \) with some \( \kappa_0 > 0 \). Moreover, suppose that there is a nondecreasing and concave function \( \sigma \in C([0, 2l_0]) \cap C^2((0, 2l_0]) \) such that for some \( \kappa \in (0, \kappa_0 \wedge l_0] \), one has

\[
\sigma(r) \leq \frac{1}{2r} J(\kappa \wedge r)(\kappa \wedge r)^2, \quad r \in (0, 2l_0];
\]

and the function \( g(r) = \int_0^r \frac{ds}{\sigma(s)} \) is well defined for all \( r \in [0, 2l_0] \). Then there exist constants \( \lambda, c > 0 \) such that for any \( x, y \in \mathbb{R}^d \) and \( t > 0 \),

\[
\|\delta_x P_t - \delta_y P_t\|_{\text{var}} \leq ce^{-\lambda t}(1 + |x - y|).
\]

**Proof.** Step 1. Let \( \psi \) be the function defined by (4.2). For any \( n \geq 1 \), define \( \psi_n \in C^2([0, \infty)) \) such that \( \psi_n \) is strictly increasing and

\[
\psi_n(r) \begin{cases} = \psi(r), & 0 \leq r \leq 1/(n + 1); \\
\leq a + \psi(r), & 1/(n + 1) < r \leq 1/n; \\
= a + \psi(r), & 1/n \leq r < \infty,
\end{cases}
\]

where \( a > 0 \) and the constants \( c_1, c_2 \) in the definition of \( \psi \) are determined later. For any \( n \geq 1 \) and every \( r \in [1/n, \infty) \), we have \( \psi_n(r) = a + \psi(r) \) and \( \psi_n'(r) = \psi'(r) \). Therefore, for any \( \kappa \in (0, \kappa_0] \),

\[
\psi_n(r - \kappa \wedge r) = \psi_n(r - \kappa \wedge r) 1_{\{r > \kappa\}} \leq [a + \psi(r - \kappa \wedge r)] 1_{\{r > \kappa\}}.
\]

(4.8)
This along with (3) in Lemma 4.1 implies that $\psi_n$ fulfills (3.2).

Below we prove that by proper choices of $c_1$, $c_2$ and $a > 0$, for $n \geq t_0^{-1} \vee l_0$ large enough, $\psi_n$ satisfies $C(\lambda, \kappa, n)$ with some constants $\lambda > 0$ and $\kappa \in (0, \kappa_0]$ (indeed for all $r \in [1/n, \infty)$). Once this is done, then, by Theorem 3.1 and the fact that

$$\lim_{n \to \infty} \psi_n = a\mathbb{1}_{(0,\infty)} + \psi,$$

we have for any $x, y \in \mathbb{R}^d$ with $x \neq y$,

$$W_{a\mathbb{1}_{(0,\infty)} + \psi}(\delta_x P_t, \delta_y P_t) \leq e^{-\lambda t}(a + \psi(|x - y|)).$$

This implies that

$$\|\delta_x P_t - \delta_y P_t\|_{\text{Var}} \leq 2a^{-1}W_{a\mathbb{1}_{(0,\infty)} + \psi}(\delta_x P_t, \delta_y P_t) \leq 2e^{-\lambda t}\left(1 + \frac{1}{a}\psi(|x - y|)\right),$$

which proves the desired assertion.

Step 2. In the proof below we also aim to give an explicit expression for the exponential rate $\lambda$ in the theorem. First, by (1.4), for any $0 < \kappa \leq \kappa_0$,

$$J_\kappa := \inf_{0 < s \leq \kappa} J(s) > 0.$$

Note that the drift term $b$ satisfies $B(K_1, K_2, l_0)$ for some $K_1, l_0 \geq 0$ and $K_2 > 0$. According to (4.8), for $r \in (1/n, l_0)$, we have

$$\Theta_n(r) := \frac{1}{2}J(\kappa \wedge r)\left[\psi_n(r + r \wedge \kappa) + \psi_n(r - r \wedge \kappa) - 2\psi_n(r)\right] + K_1\psi'_n(r)$$

(4.9)

$$\leq \frac{1}{2}J(\kappa \wedge r)\left[\psi(r + r \wedge \kappa) + \psi(r - r \wedge \kappa) - 2\psi(r)\right] + K_1\psi'(r)$$

$$- \frac{a}{2}J(\kappa \wedge r)\mathbb{1}_{\{r \leq \kappa \wedge l_0\}}.$$

In the following, let $\kappa \in (0, \kappa_0 \wedge l_0]$ be the constant in assumptions of the theorem. By (4.9) and (4) in Lemma 4.1, we find that for all $r \in (\kappa, l_0]$, $\kappa$, $l_0$,

$$\Theta_n(r) \leq \frac{1}{2}J(\kappa)\kappa^2\psi''(r) + K_1\psi'(r)$$

$$\leq \frac{1}{2}J(\kappa)\kappa^2\psi''(r) + \frac{K_1}{\kappa}r\psi'(r).$$

Taking $c_1 = e^{-c_2g(2l_0)}$ and $c_2 = 2K_1/\kappa + [(2K_2) \wedge g(2l_0)^{-1}]$, and following the argument of (4.5), we obtain that for all $r \in (\kappa, l_0]$,

(4.10) $\Theta_n(r) \leq -\frac{c_1}{c_1 + 1}\left[(2K_2) \wedge g(2l_0)^{-1}\right]\psi(r).$

On the other hand, we can deduce from (4.9) and (3) in Lemma 4.1 that for all $r \in [1/n, \kappa]$,

$$\Theta_n(r) \leq K_1(c_1 + e^{-c_2g(r)}) - \frac{a}{2}J_\kappa \leq K_1(c_1 + 1) - \frac{a}{2}J_\kappa.$$

Then, choosing

$$a = \frac{2}{J_\kappa}\left(K_1(c_1 + 1) + \frac{c_1}{c_1 + 1}\left[(2K_2) \wedge g(2l_0)^{-1}\right]\psi(\kappa)\right),$$

we find that for all $r \in [1/n, \kappa]$,

(4.11) $\Theta_n(r) \leq -\frac{c_1}{c_1 + 1}\left[(2K_2) \wedge g(2l_0)^{-1}\right]\psi(\kappa).$
Furthermore, using $\mathcal{B}(K_1, K_2r, l_0)$ and following the arguments of (4.6) and (4.7), it is easy to see that for all $r \geq l_0$,

$$-K_2 r \psi'_n(r) = -K_2 r \psi'(r) \leq -\frac{2K_2c_1}{c_1 + 1} \psi(r).$$

Combining all the estimates above, we can see that $\psi_n$ satisfies $C(\lambda, \kappa, n)$ with

$$\lambda = \frac{c_1}{c_1 + 1} \left[ (2K_2) \land g(2l_0)^{-1} \right] \inf_{r > 0} \frac{\psi(r \lor \kappa)}{a + \psi(r)}$$

(4.12)

$$= \frac{c_1}{c_1 + 1} \left[ (2K_2) \land g(2l_0)^{-1} \right] \left( 1 + \frac{a}{\psi(\kappa)} \right)^{-1} > 0.$$ 

Then, the proof is complete. \hfill \Box

Remark 4.5. Suppose that

$$\lim_{\kappa \to 0} \inf_{0 < s \leq \kappa} J(s) = \infty,$$

which holds true under (1.5). If the drift term $b$ satisfies $\mathcal{B}(K_1, K_2r, l_0)$ with $K_1 = 0$, then for any $x, y \in \mathbb{R}^d$,

$$(b(x) - b(y), x - y) \leq 0.$$ 

In this case, the exponential rate $\lambda$ given by (4.12) is reduced into

$$\lambda = \frac{(2K_2) \land g(2l_0)^{-1}}{1 + \exp \left\{ g(2l_0) \left[ (2K_2) \land g(2l_0)^{-1} \right] \right\}} \times \left( 1 + \frac{2}{J_\kappa} \cdot \frac{(2K_2) \land g(2l_0)^{-1}}{1 + \exp \left\{ g(2l_0) \left[ (2K_2) \land g(2l_0)^{-1} \right] \right\}} \right)^{-1}.$$

Note that, as $\kappa \to 0$,

$$\frac{2}{J_\kappa} \cdot \frac{(2K_2) \land g(2l_0)^{-1}}{1 + \exp \left\{ g(2l_0) \left[ (2K_2) \land g(2l_0)^{-1} \right] \right\}} \leq \frac{4K_2}{J_\kappa} \to 0,$$

thanks to $\lim_{\kappa \to 0} J_\kappa = \infty$. Therefore, the quantity in the big round brackets tends to $1$ as $\kappa \to 0$, which implies that the exponential rate with respect to the total variation can be arbitrarily close to the one with respect to the $L^1$-Wasserstein distance (by choosing $\kappa$ small enough), provided that the condition $\mathcal{B}(K_1, K_2r, l_0)$ holds with $K_1 = 0$.

We can now present the

Proof of Theorem 1.1. Our strategy is to deduce the assertions (a) and (b) from Theorems 4.2 and 4.4, respectively. It is obvious that $\Phi_1(r) = K_1 r^\beta$ with $\beta \in (0, 1]$ satisfies (b) in Theorem 4.2. Hence, it suffices to show that, under the condition (1.5), there exists a function $\sigma \in C([0, 2l_0]) \cap C^2((0, 2l_0])$ satisfying the conditions in Theorems 4.2 and 4.4.

Let $b_0 = 2l_0 e^{(1+\theta)/(1-\alpha)}$. It is easy to see that the function $s \mapsto s^{1-\alpha} (\log \frac{b_0}{s})^{-1+\theta}$ is concave and increasing on the interval $[0, 2l_0]$. Under the condition (1.5), there exist constants $\kappa \in (0, \kappa_0 \land l_0 \land 1]$ and $b_1 > 0$ such that for all $s \in (0, \kappa]$, it holds

$$J(s) \geq b_1 s^{-\alpha} \left( \log \frac{1}{s} \right)^{1+\theta}.$$
By taking a smaller \( b_1 \) we also have
\[
J(s) \geq b_1 s^{-\alpha} \left( \log \frac{b_0}{s} \right)^{1+\theta}, \quad s \in (0, \kappa),
\]
which is equivalent to
\[
\frac{1}{2} s J(s) \geq \frac{b_1}{2} s^{1-\alpha} \left( \log \frac{b_0}{s} \right)^{1+\theta}, \quad s \in (0, \kappa).
\]
For \( s \in (\kappa, 2l_0] \), we have
\[
\frac{1}{2} s J(k) \kappa^2 \geq \frac{1}{4l_0} J(k) \kappa^2 > 0.
\]
From the above two inequalities, we deduce that there is a small enough constant \( b_2 \in (0, b_1/2) \) such that
\[
b_2 s^{1-\alpha} \left( \log \frac{b_0}{s} \right)^{1+\theta} \leq \frac{1}{2s} J(k \wedge s)(k \wedge s)^2 \quad \text{for all } s \in (0, 2l_0].
\]
That is, \((4.3)\) holds with \( \sigma(s) = b_2 s^{1-\alpha} \left( \log \frac{b_0}{s} \right)^{1+\theta} \). It is clear that the integrals \( \int_0^r \frac{1}{\sigma(s)} ds \) and \( \int_0^r K \frac{1}{\sigma(s)} ds \) are well defined since \( \alpha \geq 0 \) and \( \alpha + \beta \geq 1 \). Therefore the function \( \sigma \) satisfies all the requirements in Theorem 4.2. Note that \( \int_0^\infty \sigma(s) ds \) still makes sense when \( \alpha = 0 \), thus it fulfills also the conditions in Theorem 4.4. \( \square \)

To conclude this subsection, we present the proof of Example 1.2.

**Proof of Example 1.2.** Denote by \( q(z) = \mathbb{1}_{\{0 < z_1 \leq 1\}} \frac{c_{d, \alpha}}{|z|^d} \) for any \( z \in \mathbb{R}^d \). Then
\[
q(z) \wedge q(x + z) = \left( \mathbb{1}_{\{0 < z_1 \leq 1\}} \frac{c_{d, \alpha}}{|z|^{d+\alpha}} \right) \wedge \left( \mathbb{1}_{\{0 < x_1 + z_1 \leq 1\}} \frac{c_{d, \alpha}}{|x + z|^{d+\alpha}} \right).
\]
We assume \(|x| \leq 1/4\). If \( x_1 \geq 0 \), then
\[
q(z) \wedge q(x + z) \geq \mathbb{1}_{\{0 < z_1 \leq 1\}} \frac{c_{d, \alpha}}{(|z| + |z|)^{d+\alpha}} \geq \mathbb{1}_{\{0 < z_1 \leq 1\}} \mathbb{1}_{\{|z| \leq 1\}} \frac{c_{d, \alpha}}{(2|z|)^{d+\alpha}} \geq \mathbb{1}_{\{z_1 > 0\}} \mathbb{1}_{\{|z| \leq 1\}} \frac{c_{d, \alpha}}{(2|z|)^{d+\alpha}}.
\]
Therefore, denoting by \( S_{d-1}^d = \{ \theta \in \mathbb{R}^d : |\theta| = 1 \text{ and } \theta_1 > 0 \} \) the half sphere and \( \sigma(d\theta) \) the spherical measure, we have
\[
\int_{\mathbb{R}^d} q(z) \wedge q(x + z) dz \geq \frac{c_{d, \alpha}}{2^{d+\alpha}} \int_{\{z_1 > 0\} \cap \{|z| \leq 1\}} \frac{1}{|z|^{d+\alpha}} dz
\]
\[
= \frac{c_{d, \alpha}}{2^{d+\alpha}} \int_{\{|z| \leq 1\}} \frac{1}{|z|^{d-1}} dr \int_{S_{d-1}^d} \frac{\sigma(d\theta)}{|\theta|^{d+\alpha}}
\]
\[
= \frac{c_{d, \alpha} \omega_d}{2^{d+1+\alpha}} \frac{1}{|x|^\alpha} + \frac{1}{(1 - |x|)^\alpha},
\]
where \( \omega_d = \sigma(S_{d-1}^d) \) is the area of the sphere. Since \(|x| \leq 1/4\), it is clear that
\[
(4.13) \quad \int_{\mathbb{R}^d} q(z) \wedge q(x + z) dz \geq \frac{c_{d, \alpha} \omega_d}{2^{d+1+\alpha}} \frac{1}{|x|^\alpha}.
\]
If $x_1 < 0$, then
\[
q(z) \wedge q(x + z) \geq \mathbf{1}_{\{x_1 < z_1 \leq 1\}} \frac{c_{d,\alpha}}{(|x + z| + |x|)^{d+\alpha}}
\]
\[
\geq \mathbf{1}_{\{x_1 < z_1 \leq 1\} \cap \{(|z| \leq |x + z|) \leq 1\}} \frac{c_{d,\alpha}}{(2|x + z|)^{d+\alpha}}
\]
\[
\geq \mathbf{1}_{\{x_1 + x_1 > 0\} \cap \{(|z| \leq |x + z|) \leq 1\}} \frac{c_{d,\alpha}}{(2|x + z|)^{d+\alpha}}.
\]

Hence, similar to the argument for the case that $x_1 \geq 0$, we have
\[
\int_{\mathbb{R}^d} q(z) \wedge q(x + z) \, dz \geq \frac{c_{d,\alpha}}{2^{d+\alpha}} \int_{\{z_1 + x_1 > 0\} \cap \{(|z| \leq |x + z|) \leq 1\}} \frac{1}{|x + z|^{d+\alpha}} \, dz
\]
\[
= \frac{c_{d,\alpha}}{2^{d+\alpha}} \int_{\{z_1 > 0\} \cap \{(|z| \leq |x + z|) \leq 1\}} \frac{1}{|z|^{d+\alpha}} \, dz
\]
\[
= \frac{c_{d,\alpha} \omega_d}{2^{d+1+\alpha}} \left( 1 - \frac{1}{3^\alpha} \right) \frac{1}{|x|^{\alpha}}.
\]

Combining this with (4.13), we get that for all $0 < s \leq 1/4$,
\[
J(s) \geq \inf_{|x|=s} \int_{\mathbb{R}^d} q(z) \wedge q(x + z) \, dz \geq \frac{c_{d,\alpha} \omega_d}{2^{d+1+\alpha}} \left( 1 - \frac{1}{3^\alpha} \right) s^{-\alpha},
\]
which finishes the proof.  \[\square\]

### 4.2. Proofs of results related to strong ergodicity

Similar to Theorem 1.1(b), Theorem 1.3 is a consequence of the following result.

**Theorem 4.6.** Assume that the drift $b$ satisfies $B(K_1, \Phi_2(r), l_0)$ for some $K_1, l_0 \geq 0$ and some positive measurable function $\Phi_2$ such that $\Phi_2(r)$ is bounded below for $r$ large enough and satisfies (1.14), and that (1.4) holds for the Lévy measure $\nu$ with some $\kappa_0 > 0$. Moreover, suppose that there is a nondecreasing and concave function $\sigma \in C([0, 2l_0]) \cap C^2((0, 2l_0])$ such that for some $\kappa \in (0, \kappa_0 \wedge l_0]$, one has
\[
\sigma(r) \leq \frac{1}{2r} J(\kappa \wedge r)(\kappa \wedge r)^2, \quad r \in (0, 2l_0];
\]
and the function $g(r) = \int_0^r \frac{ds}{\sigma(s)}$ is well defined for all $r \in [0, 2l_0]$. Then there exist constants $\lambda, c > 0$ such that for any $x, y \in \mathbb{R}^d$ and $t > 0$,
\[
\|\delta_x P_t - \delta_y P_t\|_{\text{Var}} \leq ce^{-\lambda t}.
\]

**Proof.** Without loss of generality, we can and do assume that $l_0 \geq 1$ is large enough such that $\inf_{r \geq l_0} \Phi_2(r) > 0$ and $\Phi_2$ is increasing on $[l_0, \infty)$; otherwise, we can use $\Phi_2^*(r) := \inf_{s \geq r} \Phi_2(s)$ instead of $\Phi_2(r)$. Define
\[
\psi(r) = \begin{cases} c_1 r + \int_0^r e^{-r_2 g(s)} \, ds, & r \in [0, 2l_0]; \\
\psi(2l_0) + \psi'(2l_0) \, \Phi_2(2l_0) \int_{2l_0}^r \frac{1}{\Phi_2(s)} \, ds, & r \in (2l_0, \infty), \end{cases}
\]
where $c_1, c_2 > 0$ are determined later. It is easy to see that $\psi \in C_b^1([0, \infty))$ is concave, due to (2) in Lemma 4.1 and the increasing property of $\Phi_2$ on $[l_0, \infty)$. For any $n \geq 1$, define $\psi_n \in C^1([0, \infty))$ such that $\psi_n$ is strictly increasing and
\[
\psi_n(r) = \begin{cases} \psi(r) & 0 \leq r \leq 1/(n + 1); \\
\leq a + \psi(r), & 1/(n + 1) < r \leq 1/n; \\
= a + \psi(r) & 1/n \leq r < \infty, \end{cases}
\]
where \( a > 0 \) is determined below. We still have (4.8), hence the function \( \psi_n \) satisfies (3.2) for all \( n \geq 1 \).

Let \( \kappa \in (0, \kappa_0 \wedge l_0] \) be the constant in the statement of the theorem. On the one hand, take \( c_1 = e^{-c_2g(2l_0)} \), \( c_2 = 2(K_2 + K_1/\kappa) \) and

\[
a = \frac{2}{J_\kappa} \left( K_1(c_1 + 1) + \frac{2K_2c_1}{c_1 + 1}\psi(\kappa) \right),
\]

where \( J_\kappa := \inf_{0<s<\kappa} J(s) > 0 \), thanks to (1.4). Using \( B(K_1, \Phi_2(r), l_0) \) and following the arguments of (4.10) and (4.11), we can get that for all \( r \in [1/n, l_0] \),

\[
\Theta_n(r) \leq -\frac{2K_2c_1}{c_1 + 1}\psi(r \vee \kappa) \leq -\frac{2K_2c_1}{c_1 + 1}\psi(\kappa).
\]

On the other hand, by \( B(K_1, \Phi_2(r), l_0) \) again, if \( r \in (l_0, 2l_0] \),

\[
-\Phi_2(r)\psi_n(r) = -\Phi_2(r)\psi(r) = -\Phi_2(r)(c_1 + e^{-c_2g(r)}) \leq -2c_1\Phi_2(l_0);
\]

while for \( r > 2l_0 \),

\[
-\Phi_2(r)\psi_n(r) = -\Phi_2(r)\psi(r) = -\psi(2l_0)\Phi_2(2l_0) = -2c_1\Phi_2(2l_0),
\]

where the last two equalities follow from the definition of \( \psi \). Combining all conclusions above with the fact that \( \psi_n \) is uniformly bounded with respect to \( n \), \( \psi_n \) satisfies \( C(\lambda, \kappa, n) \) with some constant \( \lambda > 0 \) for all \( n \geq 1 \) large enough.

Therefore, by Theorem 3.1, for any \( x, y \in \mathbb{R}^d \),

\[
\|\delta_x P_t - \delta_y P_t\|_{\text{var}} \leq 2a^{-1}W_1(0, x) + \psi(\delta_x P_t, \delta_y P_t) \\
\leq 2e^{-\lambda t} \left( 1 + \frac{1}{a} \psi(\|x - y\|) \right) \\
\leq ce^{-\lambda t}.
\]

By now we have proved the desired assertion. \( \square \)

At the end of this section, we give the

**Proof of Proposition 1.4.** Under \( B(K_1r, \Phi_2(r), l_0) \), it holds that for any \( x \in \mathbb{R}^d \) with \( |x| \) large enough,

\[
\frac{\langle b(x), x \rangle}{|x|} \leq -\Phi_2(|x|) + \frac{\langle b(0), x \rangle}{|x|} \leq -\frac{1}{2} \Phi_2(|x|),
\]

where in the last inequality we have used the fact that \( \liminf_{r \to \infty} \frac{\Phi_2(r)}{r} = \infty \). Let \( f \in C^2(\mathbb{R}^d) \) such that \( f(x) = \log(1 + |x|) \) for all \( |x| \geq 1 \). Then, by (1.15), we can easily establish the following Foster–Lyapunov type condition:

\[
L_X f(x) \leq -c_1 \frac{\Phi_2(|x|)}{1 + |x|} + c_2, \quad x \in \mathbb{R}^d,
\]

where \( L_X \) is the generator of the process \((X_t)_{t \geq 0}\) given by (2.17), and \( c_1, c_2 \) are two positive constants. On the other hand, since \( b \) satisfies \( B(K_1r, \Phi_2(r), l_0) \) and \( \liminf_{r \to \infty} \frac{\Phi_2(r)}{r} = \infty, b \) satisfies \( B(K_1r, K_2r, l_0') \) for some constants \( K_2, l_0' > 0 \), and so Theorem 1.1 holds, also thanks to the fact that the associated Lévy measure \( \nu \) satisfies (1.7). Then, there exist constants \( \lambda, c > 0 \) such that for any \( x, y \in \mathbb{R}^d \) and \( t > 0 \),

\[
W_1(\delta_x P_t, \delta_y P_t) \leq ce^{-\lambda t}|x - y|.
\]
This implies that (e.g. see [2, Theorem 5.10])
\[ \|P_t f\|_{\text{Lip}} \leq ce^{-\lambda t}\|f\|_{\text{Lip}} \]
holds for any \( t > 0 \) and any Lipschitz continuous function \( f \), where \( \|f\|_{\text{Lip}} \) denotes the Lipschitz semi-norm with respect to the Euclidean norm \(|\cdot|\). By the standard approximation, we know that the semigroup \((P_t)_{t \geq 0}\) is Feller, i.e. for every \( t > 0 \), \( P_t \) maps \( C_b(\mathbb{R}^d) \) into \( C_b(\mathbb{R}^d) \). (Indeed, by (1.7) and Corollary 5.3(4) below, the semigroup \((P_t)_{t \geq 0}\) is strongly Feller, i.e. for every \( t > 0 \), \( P_t \) maps \( B_b(\mathbb{R}^d) \) into \( C_b(\mathbb{R}^d) \), where \( B_b(\mathbb{R}^d) \) denotes the class of bounded measurable functions on \( \mathbb{R}^d \).)

This along with (4.14), \( \liminf_{r \to \infty} \frac{\Phi_r(x)}{r^\alpha} = \infty \) and [10, Theorems 4.5] yields that the process \((X_t)_{t \geq 0}\) has an invariant probability measure.

Furthermore, under the assumptions Theorem 1.3 holds. Then, we can deduce from (1.13) that the process \((X_t)_{t \geq 0}\) has at most one invariant probability measure, so by the above arguments, it admits a unique one. Indeed, let \( \mu_1 \) and \( \mu_2 \) be invariant probability measures of the process \((X_t)_{t \geq 0}\). Then,
\[ \|\mu_1 - \mu_2\|_{\text{Var}} = \int \int \|\delta_x P_t - \delta_y P_t\|_{\text{Var}} \mu_1(dy) \mu_2(dx) \leq ce^{-\lambda t}. \]

Letting \( t \to \infty \), we find that \( \mu_1 = \mu_2 \). Denote by \( \mu \) the unique invariant probability measure. Therefore, by (1.13), we have
\[ \|\delta_x P_t - \mu\|_{\text{Var}} \leq \int \|\delta_x P_t - \delta_y P_t\|_{\text{Var}} \mu(dy) \leq ce^{-\lambda t}. \]
The proof is complete. \( \square \)

5. FURTHER APPLICATIONS OF THE REFINED BASIC COUPLING

5.1. Spatial regularity of semigroups. As another application of the refined basic coupling for Lévy processes, we shall study in this subsection the regularity of the semigroup \((P_t)_{t \geq 0}\) for SDEs with Lévy noises, a topic which has attracted lots of interests in recent years. For instance, the Bismut–Elworthy–Li’s derivative formula and gradient estimates for SDEs driven by (multiplicative) Lévy noise have been established in [22, 18]. Note that, when the Lévy noise is reduced to a symmetric \( \alpha \)-stable process, the statement of Corollary 5.3 below is weaker than those in [22, 18]; however, it works for more general Lévy noises. Besides, the drift term \( b \) in our setting only satisfies the one-sided Lipschitz condition; while in [22, 18] it is required to be in \( C_b^1(\mathbb{R}^d) \), which is essentially due to the fact that the Malliavin calculus was used there.

Throughout this part, we assume that (1.4) holds for the Lévy measure \( \nu \) with some \( \kappa_0 > 0 \), and the drift term \( b \) satisfies the following one-sided Lipschitz condition, i.e. there is a constant \( K_1 > 0 \) such that for any \( x, y \in \mathbb{R}^d \),
\[ \langle b(x) - b(y), x - y \rangle \leq K_1|x - y|^2. \]

**Theorem 5.1.** Assume that (1.4) holds and \( b \) satisfies the one-sided Lipschitz condition. For some fixed \( \varepsilon_0 \in (0, \kappa_0] \), let \( \phi \in C^1([0, \varepsilon_0]) \) be such that \( \phi(0) = 0, \phi' \geq 0 \), and for all \( 0 < \varepsilon \leq \varepsilon_0 \)
\[
A_\varepsilon(\phi) := \inf_{0 < r \leq \varepsilon} \left\{ \frac{1}{2} J(r)(2\phi(r) - \phi(2r)) - K_1\phi'(r)r \right\} > 0.
\]
Then, for any $f \in B_b(\mathbb{R}^d)$ and $t > 0$, 
\begin{equation}
(5.1) \quad \sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{\phi(|x - y|)} \leq 2\|f\|_{\infty} \inf_{\epsilon \in (0,\epsilon_0]} \left[ \frac{1}{\phi(\epsilon)} + \frac{1}{tA_\epsilon(\phi)} \right].
\end{equation}

**Proof.** Let $\tilde{L} = \tilde{L}_X$ be the coupling operator given in (2.18). For any $x, y \in \mathbb{R}^d$ with $0 < |x - y| \leq \epsilon \leq \epsilon_0$, by applying (3.1) with $\kappa = \kappa_0$ and noticing that $\epsilon_0 \leq \kappa_0$, we have
\begin{equation}
(5.2) \quad \frac{1}{2} \mu_{x-y}(\mathbb{R}^d)[\phi(2|x-y|) - 2\phi(|x-y|)] + K_1\phi'(|x-y|)|x-y| \leq -A_\epsilon(\phi) < 0.
\end{equation}

Below we follow the same argument in the proof of [7, Theorem 1.2]. We still use the coupling process $(X_t, Y_t)_{t \geq 0}$ constructed in Section 2.3, and denote by $\tilde{P}^{(x,y)}$ and $\tilde{E}^{(x,y)}$ the distribution and the expectation of $(X_t, Y_t)_{t \geq 0}$ starting from $(x, y)$, respectively. For any $n \geq 1$ and $\epsilon \in (0, \epsilon_0]$, we set
\begin{align*}
S_\epsilon & := \inf\{t \geq 0 : |X_t - Y_t| > \epsilon\}, \\
T_n & := \inf\{t \geq 0 : |X_t - Y_t| \leq 1/n\}, \\
T_n,\epsilon & := T_n \wedge S_\epsilon.
\end{align*}
Furthermore, we still use the coupling time defined by
\begin{equation*}
T := \inf\{t \geq 0 : X_t = Y_t\}.
\end{equation*}
Note that $T_n \uparrow T$ as $n \uparrow \infty$. For any $x, y \in \mathbb{R}^d$ with $0 < |x - y| < \epsilon \leq \epsilon_0$, we take $n$ large enough such that $|x - y| > 1/n$. Then, by (5.2),
\begin{align*}
0 & \leq \tilde{E}^{(x,y)}\phi(|X_{t \wedge T_n,\epsilon} - Y_{t \wedge T_n,\epsilon}|) \\
& = \phi(|x - y|) + \tilde{E}^{(x,y)}\left( \int_0^{t \wedge T_n,\epsilon} \tilde{L}\phi(|X_u - Y_u|) \, du \right) \\
& \leq \phi(|x - y|) - A_\epsilon(\phi)\tilde{E}^{(x,y)}(t \wedge T_n,\epsilon).
\end{align*}
Therefore
\begin{equation*}
\tilde{E}^{(x,y)}(t \wedge T_n,\epsilon) \leq \frac{\phi(|x - y|)}{A_\epsilon(\phi)}.
\end{equation*}
Letting $t \to \infty$ and then $n \to \infty$, we arrive at
\begin{equation}
(5.3) \quad \tilde{E}^{(x,y)}(T \wedge S_\epsilon) \leq \frac{\phi(|x - y|)}{A_\epsilon(\phi)}.
\end{equation}
On the other hand, again by (5.2), for any $x, y \in \mathbb{R}^d$ with $1/n \leq |x - y| < \epsilon \leq \epsilon_0$, 
\begin{align*}
& \tilde{E}^{(x,y)}\phi(|X_{t \wedge T_n,\epsilon} - Y_{t \wedge T_n,\epsilon}|) \\
& = \phi(|x - y|) + \tilde{E}^{(x,y)}\left( \int_0^{t \wedge T_n,\epsilon} \tilde{L}\phi(|X_u - Y_u|) \, du \right) \\
& \leq \phi(|x - y|),
\end{align*}
which yields that
\begin{equation*}
\phi(\epsilon)\tilde{P}^{(x,y)}(S_\epsilon < T_n \wedge t) \leq \phi(|x - y|).
\end{equation*}
Letting $t \to \infty$ and then $n \to \infty$ leads to
\begin{equation}
\tilde{P}^{(x,y)}(T > S_\varepsilon) \leq \frac{\phi(|x - y|)}{\phi(\varepsilon)}.
\end{equation}

Therefore, for any $x, y \in \mathbb{R}^d$ with $0 < |x - y| < \varepsilon \leq \varepsilon_0$, by (5.3) and (5.4),
\[\tilde{P}^{(x,y)}(T > t) \leq \tilde{P}^{(x,y)}(T \wedge S_\varepsilon > t) + \tilde{P}^{(x,y)}(T > S_\varepsilon)\]
\[\leq \frac{\tilde{P}^{(x,y)}(T \wedge S_\varepsilon)}{t} + \frac{\phi(|x - y|)}{\phi(\varepsilon)}\]
\[\leq \phi(|x - y|) \left[ \frac{1}{\phi(\varepsilon)} + \frac{1}{tA_\varepsilon(\phi)} \right].\]

Hence, for any $f \in B_b(\mathbb{R}^d)$, $t > 0$ and any $x, y \in \mathbb{R}^d$ with $0 < |x - y| < \varepsilon \leq \varepsilon_0$,
\[|P_t f(x) - P_t f(y)| = |E^{x}(f(x)) - E^{y}(f(y))|\]
\[= |\tilde{E}^{(x,y)}(f(x) - f(y))|\]
\[= |\tilde{E}^{(x,y)}(f(x) - f(y))1_{T > t}|\]
\[\leq 2\|f\|_\infty \tilde{P}^{(x,y)}(T > t)\]
\[\leq 2\|f\|_\infty \phi(|x - y|) \left[ \frac{1}{\phi(\varepsilon)} + \frac{1}{tA_\varepsilon(\phi)} \right].\]

As a result,
\[\sup_{|x - y| \leq \varepsilon} \frac{|P_t f(x) - P_t f(y)|}{\phi(|x - y|)} \leq 2\|f\|_\infty \left[ \frac{1}{\phi(\varepsilon)} + \frac{1}{tA_\varepsilon(\phi)} \right].\]

This along with the fact that
\[\sup_{|x - y| \geq \varepsilon} \frac{|P_t f(x) - P_t f(y)|}{\phi(|x - y|)} \leq \frac{2\|f\|_\infty}{\phi(\varepsilon)}\]

further gives us that for all $\varepsilon \in (0, \varepsilon_0]$,
\[\sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{\phi(|x - y|)} \leq 2\|f\|_\infty \left[ \frac{1}{\phi(\varepsilon)} + \frac{1}{tA_\varepsilon(\phi)} \right].\]

The desired assertion follows from the inequality above by taking infimum with respect to $\varepsilon \in (0, \varepsilon_0]$ in the right hand side. \hfill \Box

As a consequence of Theorem 5.1, we have the following result.

**Proposition 5.2.** Assume that (1.4) holds and $b$ satisfies the one-sided Lipschitz condition. If there exist a constant $\varepsilon_0 \in (0, \kappa_0]$ and a function $\phi \in C^3([0, \varepsilon_0])$ such that $\phi(0) = 0$, $\phi' \geq 0$, $\phi'' \leq 0$ and $\phi''' \geq 0$, and that
\begin{equation}
\lim_{\varepsilon \to 0} \sup_{0 < r \leq \varepsilon} J(r)r^2\phi''(2r) < 0,
\end{equation}
then there are constants $C > 0$ and $\varepsilon'_0 \in (0, \varepsilon_0]$ such that for any $f \in B_b(\mathbb{R}^d)$ and $t > 0$,
\begin{equation}
\sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{\phi(|x - y|)} \leq C\|f\|_\infty \inf_{\varepsilon \in (0, \varepsilon'_0)} \left[ \frac{1}{\phi(\varepsilon)} + \frac{1}{tB_\varepsilon(\phi)} \right],
\end{equation}

with
\[
B_\varepsilon(\phi) := - \sup_{0 < r \leq \varepsilon} J(r) r^2 \phi''(2r).
\]

**Proof.** Since \( \phi''' \geq 0 \), we have
\[
2\phi(r) - \phi(2r) = - \int_0^r \int_s^{r+s} \phi''(u) \, du \, ds \geq -\phi''(2r)r^2.
\]
On the other hand, by \( \phi'' \leq 0 \) and the fact that \( \phi(0) = 0 \),
\[
\phi'(r) \leq \int_0^r \phi'(s) \, ds = \phi(r).
\]
Therefore,
\[
\frac{1}{2} J(r)(2\phi(r) - \phi(2r)) - K_1 \phi'(r)r \geq -\frac{1}{2} J(r) r^2 \phi''(r) - K_1 \phi(r)
\]
\[
\geq \frac{1}{2} B_\varepsilon(\phi) - K_1 \phi(r).
\]
According to (5.5), we know that there is a constant \( \varepsilon'_0 \in (0, \varepsilon_0] \) such that for all \( \varepsilon \in (0, \varepsilon'_0] \),
\[
A_\varepsilon(\phi) \geq \frac{1}{4} B_\varepsilon(\phi) > 0.
\]
Then, the desired assertion (5.6) follows immediately from Theorem 5.1. \( \square \)

Furthermore, we have the following more explicit regularity properties of the semigroup \((P_t)_{t \geq 0}\).

**Corollary 5.3.** Assume that (1.4) holds for some \( \kappa_0 > 0 \) and \( b \) satisfies the one-sided Lipschitz condition.

1. If for some \( \theta > 0 \),
\[
\lim_{\varepsilon \to 0} \inf_{0 < r \leq \varepsilon} J(r) r \left( \log \frac{1}{r} \right)^{-(1+\theta)} > 0,
\]
then there exist constants \( C > 0 \) and \( \varepsilon'_0 \in (0, \kappa_0] \) such that for all \( f \in B_b(\mathbb{R}^d) \) and \( t > 0 \),
\[
\sup_{\substack{x \neq y \\ \log |x-y|}} \frac{|P_tf(x) - P_tf(y)|}{|x-y|} \leq C \|f\|_{\infty} \inf_{\varepsilon \in (0, \varepsilon'_0]} \left[ \frac{1}{\varepsilon} + \frac{1}{t \inf_{0 < r \leq \varepsilon} J(r) r \left( \log \frac{1}{r} \right)^{-(1+\theta)}} \right].
\]

2. If for some \( \theta > 0 \),
\[
\lim_{\varepsilon \to 0} \inf_{0 < r \leq \varepsilon} J(r) r \left( \log \frac{1}{r} \right)^{\theta-1} > 0,
\]
then there exist constants \( C > 0 \) and \( \varepsilon'_0 \in (0, \kappa_0] \) such that for all \( f \in B_b(\mathbb{R}^d) \) and \( t > 0 \),
\[
\sup_{\substack{x \neq y \\ \log |x-y|}} \frac{|P_tf(x) - P_tf(y)|}{|x-y| \log |x-y|} \leq C \|f\|_{\infty} \inf_{\varepsilon \in (0, \varepsilon'_0]} \left[ \frac{1}{\varepsilon |\log \varepsilon|^\theta} + \frac{1}{t \inf_{0 < r \leq \varepsilon} J(r) r \left( \log \frac{1}{r} \right)^{\theta-1}} \right].
\]
(3) If for some $\theta \in (0, 1)$
\[
\lim_{\varepsilon \to 0} \inf_{0 < r \leq \varepsilon} J(r)r^\theta > 0,
\]
then there exist constants $C > 0$ and $\varepsilon' \in (0, \kappa_0]$ such that for all $f \in B_b(\mathbb{R}^d)$ and $t > 0$,
\[
\sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{|x - y|^\theta} \leq C \|f\| \inf_{\varepsilon \in (0, \varepsilon']} \left[ \frac{1}{\varepsilon^\theta} + \frac{1}{t \inf_{0 < r \leq \varepsilon} J(r)r^\theta} \right].
\]

(4) If for some $\theta > 0$,
\[
\lim_{\varepsilon \to 0} \inf_{0 < r \leq \varepsilon} J(r) \left( \log \frac{1}{r} \right)^{-(1+\theta)} > 0,
\]
then there exist constants $C > 0$ and $\varepsilon' \in (0, \kappa_0]$ such that for all $f \in B_b(\mathbb{R}^d)$ and $t > 0$,
\[
\sup_{x \neq y} \frac{|P_t f(x) - P_t f(y)|}{|x - y|^\theta} \leq C \|f\| \inf_{\varepsilon \in (0, \varepsilon')} \left[ \frac{1}{\log \varepsilon} + \frac{1}{t \inf_{0 < r \leq \varepsilon} J(r) \left( \log \frac{1}{r} \right)^{-(1+\theta)}} \right].
\]

Proof. The assertions follow from Proposition 5.2 by taking $\phi(r) = r(1 - \log^{-\theta}(1/r))$, $\phi(r) = r \log^\theta(1/r)$, $\phi(r) = r^\theta$ and $\phi(r) = \log^{-\theta}(1/r)$ for $r > 0$ small enough, respectively. \hfill \Box

5.2. SDEs with special multiplicative noises. This subsection is concerned with the extension of our results above to the following $d$-dimensional stochastic differential equation with jumps

\[
X_t = x + \int_0^t b(X_s) \, ds + Z_t^{(1)} + \int_0^t \sigma(X_{s-}) \, dZ_s^{(2)}, \quad x \in \mathbb{R}^d, t \geq 0,
\]

where $Z^{(1)} = (Z^{(1)}_t)_{t \geq 0}$ and $Z^{(2)} = (Z^{(2)}_t)_{t \geq 0}$ are two independent pure jump Lévy processes on $\mathbb{R}^d$, and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are two measurable functions.

For simplicity, we only explain in the following the main differences between (5.7) and (1.1) about the associated coupling process and coupling operator, and present some key estimates involved in the expression analogous to (3.1) for the coupling operator. With those at hand, the corresponding extension can be done similarly.

Assume that the SDE (5.7) has a non-explosive and pathwise unique strong solution, which is denoted by $(X_t)_{t \geq 0}$. For any fixed $x, y \in \mathbb{R}^d$ with $x \neq y$, we consider the system of equations:

\[
\begin{align*}
&dX_t = b(X_t) \, dt + dZ_t^{(1)} + \sigma(X_{t-}) \, dZ_t^{(2)}, \quad X_0 = x; \\
&dY_t = b(Y_t) \, dt + dZ_t^{(1)} + dZ_t^{(2)} + \sigma(Y_{t-}) \, dZ_t^{(2)}, \quad Y_0 = y,
\end{align*}
\]

where $L^*_t$ is determined by (2.19) with $Z^{(1)}_t$ instead of $Z_t$. Similar to the arguments in Proposition 2.2, we can prove that the system of equations (5.8) has a unique strong solution, and the solution denoted by $(X_t, Y_t)_{t \geq 0}$ is a coupling of the process $(X_t)_{t \geq 0}$. 
Let $\nu^{(i)}$ be the Lévy measure corresponding to the Lévy process $Z^{(i)}$ for $i = 1, 2$. The generator of the process $(X_t)_{t \geq 0}$ acting on $C^2_b(\mathbb{R}^d)$ is given by

$$Lf(x) = (\nabla f(x), b(x)) + \int \left( f(x+z) - f(x) - \langle \nabla f(x), z \rangle 1_{|z| \leq 1} \right) \nu^{(1)}(dz) + \int \left( f(x+\sigma(x)z) - f(x) - \langle \nabla f(x), \sigma(x)z \rangle 1_{|z| \leq 1} \right) \nu^{(2)}(dz).$$

Therefore, the operator associated with the coupling process $(X_t, Y_t)_{t \geq 0}$ (i.e., the coupling operator of the above operator $L$) is given by

$$\tilde{L}h(x, y) = \langle b(x), \nabla_x h(x, y) \rangle + \langle b(y), \nabla_y h(x, y) \rangle + \tilde{L}_1 h(x, y) + \tilde{L}_2 h(x, y), \quad h \in C^2_b(\mathbb{R}^d \times \mathbb{R}^d),$$

where $\tilde{L}_1 h(x, y) = \tilde{L}_Z h(x, y)$ with $\nu^{(1)}$ replacing $\nu$ in the definition (2.9), and

$$\tilde{L}_2 h(x, y) = \int \left( h(x+\sigma(x)z, y+\sigma(y)z) - h(x, y) - \langle \nabla_x h(x, y), \sigma(x)z \rangle 1_{|z| \leq 1} \right) \nu^{(2)}(dz).$$

Next, we assume that $\nu^{(1)}$ satisfies (1.4) for some constant $\kappa_0 > 0$, and that

$$\int_{\{|z| > 1\}} |z| \nu^{(2)}(dz) < \infty.$$  

According to the arguments in Subsection 3.1, for any $f \in C^1_b([0, \infty))$ with $f \geq 0$, and for any $\kappa \in (0, \kappa_0],$

$$\tilde{L}f(|x-y|) = \frac{1}{2} \left[ \nu^{(1)} \wedge \left( \delta_{\{1 \wedge \frac{|x-y|}{\kappa} \leq 1\}} \nu^{(1)} \right) \right](\mathbb{R}^d) \times \left[ f(|x-y| + \kappa \wedge |x-y|) + f(|x-y| - \kappa \wedge |x-y|) - 2f(|x-y|) \right] + \int \left( f(|(x-y) + (\sigma(x) - \sigma(y))z|) - f(|x-y|) \right.

$$\left. - \frac{f'(|x-y|)}{|x-y|} \langle x-y, (\sigma(x) - \sigma(y))z \rangle 1_{|z| \leq 1} \right) \nu^{(2)}(dz) + \frac{f'(|x-y|)}{|x-y|} \langle b(x) - b(y), x-y \rangle$$

$$=: I_1 + I_2 + I_3.$$  

Note that the sum $I_1 + I_3$ has been treated carefully in the previous sections. For $I_2$, we consider $f \in C^2([0, \infty))$ with $f' \geq 0$ and $f'' \leq 0$. As in Theorems 4.2 and 4.4, this class of functions is sufficient to study the exponential contractivity with respect to the $L^1$-Wasserstein distance and the total variation. Then

$$I_2 = \int \left( f(|(x-y) + (\sigma(x) - \sigma(y))z|) - f(|x-y|) \right.

$$\left. - \frac{f'(|x-y|)}{|x-y|} \langle x-y, (\sigma(x) - \sigma(y))z \rangle 1_{|z| \leq 1} \right) \nu^{(2)}(dz) \leq f'(|x-y|) |\sigma(x) - \sigma(y)| \int_{\{|z| > 1\}} |z| \nu^{(2)}(dz)$$
+ f′(|x − y|) \int_{|z| \leq 1} \left| (x − y) + (\sigma(x) − \sigma(y))z − |x − y| − \frac{1}{|x − y|} (x − y, (\sigma(x) − \sigma(y))z) \right| \nu^{(2)}(dz),

where in the last inequality we have used the fact that

\[ f(a) − f(b) \leq f'(b)(a − b), \quad a, b > 0. \]

Next, the inequality

\[ |x + y| − |x| − \frac{1}{|x|}(x, y) \leq \frac{1}{2|x|}|y|^2, \quad x, y \in \mathbb{R}^d \]

further implies that

\[ I_2 \leq f′(|x − y|) \left( |\sigma(x) − \sigma(y)| \int_{|z| > 1} |z| \nu^{(2)}(dz) + \frac{|\sigma(x) − \sigma(y)|^2}{2|x − y|} \int_{|z| \leq 1} |z|^2 \nu^{(2)}(dz) \right). \]

Combining all the estimates above, we can easily obtain the explicit estimate of \( \tilde{L}f(|x − y|) \) for any \( f \in C^2((0, \infty)) \) with \( f' \geq 0 \) and \( f'' \leq 0 \). This along with the arguments in Sections 3 and 4 can yield the exponential contractivity with respect to the \( L^1 \)-Wasserstein distance and the total variation for the associated semigroup. We omit the details to save pages.

6. APPENDIX: PROPERTIES OF \( \nu \wedge (\delta_x \ast \nu) \)

Recall that for any two finite measures \( \mu_1 \) and \( \mu_2 \) on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \),

\[ \mu_1 \wedge \mu_2 := \mu_1 − (\mu_1 − \mu_2)^+, \]

where \( (\mu_1 − \mu_2)^\pm \) refers to the Jordan–Hahn decomposition of the signed measure \( \mu_1 − \mu_2 \). In particular, \( \mu_1 \wedge \mu_2 = \mu_2 \wedge \mu_1 \) and for any \( A \in \mathcal{B}(\mathbb{R}^d) \),

\[ (\mu_1 − \mu_2)^+(A) = \text{sup}\{\mu_1(B) − \mu_2(B) : B \subset A, B \in \mathcal{B}(\mathbb{R}^d)\}. \]

We first claim that

**Lemma 6.1.** Let \( \mu_1 \) and \( \mu_2 \) be two bounded measures on \( (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \). For any \( x \in \mathbb{R}^d \), it holds that

\[ \delta_x \ast (\mu_1 \wedge \mu_2) = (\delta_x \ast \mu_1) \wedge (\delta_x \ast \mu_2). \]

**Proof.** We have

\[ \delta_x \ast (\mu_1 \wedge \mu_2) = \delta_x \ast \mu_1 − \delta_x \ast (\mu_1 − \mu_2)^+ \]

and

\[ (\delta_x \ast \mu_1) \wedge (\delta_x \ast \mu_2) = \delta_x \ast \mu_1 − (\delta_x \ast (\mu_1 − \mu_2)^+). \]

Note that for any \( A \in \mathcal{B}(\mathbb{R}^d) \),

\[ (\delta_x \ast (\mu_1 − \delta_x \ast \mu_2)^+)(A) = \text{sup}\{(\mu_1 − \mu_2)(B) : B \subset A, B \in \mathcal{B}(\mathbb{R}^d)\} \]

\[ = \text{sup}\{(\mu_1 − \mu_2)(B) : B \subset A, B \in \mathcal{B}(\mathbb{R}^d)\} \]

\[ = \text{sup}\{(\mu_1 − \mu_2)(B) : B \subset A − x, B \in \mathcal{B}(\mathbb{R}^d)\} \]

\[ = (\mu_1 − \mu_2)^+(A − x) \]

\[ = (\delta_x \ast (\mu_1 − \mu_2)^+)(A). \]
Combining all the conclusions above, we prove the desired assertion. □

Let ν be a Lévy measure on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\) for some Lévy process. For any \(n \geq 1\), define \(\nu_n(dz) = 1_{\{|z| \geq 1/n\}}\nu(dz)\) which is a finite measure. By the above arguments, the measures \(\{\nu_n \wedge (\delta_x \ast \nu_n)\}_{n \geq 1}\) are well defined and the sequence is increasing with respect to \(n\). For any \(x \in \mathbb{R}^d\) with \(x \neq 0\), similar to (2.3), we have

\[
[\nu_n \wedge (\delta_x \ast \nu_n)](\mathbb{R}^d) \leq 2\nu(\{z : |z| \geq |x|/2\}) < \infty \quad \text{for all } n \geq 1.
\]

Hence the limit

\[
\nu \wedge (\delta_x \ast \nu) = \lim_{n \to \infty} \nu_n \wedge (\delta_x \ast \nu_n)
\]

exists. In particular, \(\nu \wedge (\delta_x \ast \nu)\) is a finite measure on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\), and \(\nu \wedge (\delta_x \ast \nu) \leq \nu\) and \(\nu \wedge (\delta_x \ast \nu) \leq \delta_x \ast \nu\). As a consequence of Lemma 6.1, we have the following statement.

**Corollary 6.2.** Let \(\nu\) be a Lévy measure on \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))\). Then, for any \(x \in \mathbb{R}^d\),

\[
\delta_x \ast (\nu \wedge (\delta_{-x} \ast \nu)) = \nu \wedge (\delta_x \ast \nu)
\]

and so

\[
\nu \wedge (\delta_{-x} \ast \nu)(\mathbb{R}^d) = \nu \wedge (\delta_x \ast \nu)(\mathbb{R}^d).
\]

**Proof.** Applying Lemma 6.1 with \(\mu_1 = \nu_n\) and \(\mu_2 = \delta_x \ast \nu_n\) and then letting \(n \to \infty\), we can get the first assertion. Since

\[
\nu \wedge (\delta_{-x} \ast \nu)(\mathbb{R}^d) = \delta_x \ast (\nu \wedge (\delta_{-x} \ast \nu))(\mathbb{R}^d) = \nu \wedge (\delta_x \ast \nu)(\mathbb{R}^d),
\]

the second one immediately follows from the first one. □

**Example 6.3.** Let \(\nu(dz) = g(z) \, dz\) for some nonnegative measurable function \(g\). Then for any \(x \in \mathbb{R}^d\),

\[
\nu \wedge (\delta_x \ast \nu)(dz) = (g(z) \wedge g(z-x)) \, dz.
\]

**Proof.** For any \(n \geq 1\), \(\nu_n(dz) = 1_{\{|z| \geq 1/n\}}g(z) \, dz\) and

\[
\delta_x \ast \nu_n(dz) = 1_{\{|z-x| \geq 1/n\}}g(z-x) \, dz.
\]

Then,

\[
\nu_n \wedge (\delta_x \ast \nu_n)(dz) = (1_{\{|z| \geq 1/n\}}g(z)) \wedge (1_{\{|z-x| \geq 1/n\}}g(z-x)) \, dz
\]

\[
= 1_{\{|z| \geq 1/n, |z-x| \geq 1/n\}}(g(z) \wedge g(z-x)) \, dz.
\]

Letting \(n \to \infty\), we prove the desired assertion. □

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