Reconstructing one- and two-bifurcation diagrams of all components in the Rössler equations only from time-series data sets

Yoshitaka Itoh\textsuperscript{1a)} and Masaharu Adachi\textsuperscript{2}

\textsuperscript{1} Department of Electrical and Electronic Engineering, Hokkaido University of Science
7-Jo 15-4-1 Maeda, Teine, Sapporo, Hokkaido 006-8585, Japan

\textsuperscript{2} Department of Electrical and Electronic Engineering, Tokyo Denki University
5 Senju Asahi-cho, Adachi-ku, Tokyo 120-8551, Japan

\textsuperscript{a)} ito-yo@hus.ac.jp

Received December 31, 2020; Revised March 24, 2021; Published July 1, 2021

Abstract: We reconstruct bifurcation diagrams of all components in the Rössler equations only from time-series data sets, thereby estimating the attractors when the parameter values are changed. In this study, we show that the bifurcation diagrams of all components can be reconstructed from time-series data of all components. In addition, we estimate the Lyapunov spectrum of the reconstructed bifurcation diagrams. We expect that the reconstruction requires a shorter length of training data when using time-series data sets of all components compared with one component. Accordingly, in numerical experiments, we reconstruct the bifurcation diagrams using training data whose length is shorter than when a bifurcation diagram is reconstructed using training data of one component.

Key Words: chaos, reconstruction of bifurcation diagrams, time-series prediction, extreme learning machine

1. Introduction
In 1994, Tokunaga et al. proposed a method for reconstructing bifurcation diagrams (BDs) [1] that several research groups then studied [2–10]. BD reconstruction allows attractors to be estimated when the bifurcation parameter values of a target system are changed. Therefore, this method is suited to finding optimal parameters from several data sets. In a previous example, we demonstrated reconstruction of the BDs of induction motor drives [11]. To apply BD reconstruction to real-world data, we have shown that the BD reconstruction is robust against dynamical and observational noise because time-series data are always affected by noise [12, 13]. In the aforementioned studies, the time-series data sets used for BD reconstruction were generated from electronic circuits. In these studies, the target systems are logistic and sine maps and the Rössler equations, and we seek to confirm that the method of BD reconstruction is robust against noise in both discrete and continuous systems. We also use BD reconstruction to analyze the bifurcation structure of an unknown system. In previous
work, we tracked bifurcation curves in parameter space as estimated by BD reconstruction [14]. This tracking method is based on a method proposed in 1981 by Kawakami [15].

Although several research groups have studied BD reconstruction, BDs of several components have not yet been reconstructed. Previous studies assumed that time-series data sets of only one component are obtained from a target system, but here we consider the case in which time-series data sets of several components are obtained. In this paper, we reconstruct BDs of all components of the Rössler equations only from time-series data sets to show that BDs of several components can be reconstructed from time-series data of several components. We therefore assume that time-series data sets of all components of a system are obtained for reconstructing the associated BDs. In the BD reconstruction, time-series predictors are trained to model the obtained time-series data sets of all components, and the method allows the attractors of all components to be estimated when the bifurcation parameter values are changed. We reconstructed the BDs for various systems and numerical conditions using time-series data sets of a component in [8–14]. In this study, we show that the reconstruction requires a shorter length of training data when using time-series data sets of all components than when using the time series of only one component, compared by the previous studies. Therefore, this reconstruction method is suitable when time-series data sets of various kinds and short length are obtained.

The rest of this paper is organized as follows. In Section 2, we explain how to reconstruct BDs of all components using an extreme learning machine (ELM) only from time-series data sets. In Section 3, we explain how to estimate the Lyapunov spectrum of the reconstructed BDs. In Section 4, we present results of our numerical experiments. Finally, we draw conclusions in Section 5.

2. Reconstructing bifurcation diagrams of all components using an extreme learning machine

In this section, we explain how to reconstruct BDs of all components using an ELM only from time-series data sets. First we explain the ELM as a time-series predictor, and then we explain the algorithm for BD reconstruction using the ELM [8].

2.1 Extreme learning machine

The ELM proposed by Huang et al. in 2006 [16] is a three-layer neural network. The training targets of the synaptic weights in the ELM are the synaptic weights of only the output neurons. On the other hand, the synaptic weights and the biases of the hidden neurons are generated as random numbers and are not trained. Given its extremely high training speed and good generalization, we use the ELM as a time-series predictor. In previous work, we showed that the ELM is useful for reconstructing BDs [8–13].

The ELM used in this work is trained to output the time-series data of all components at time $t + 1$ when given the time-series data of all components at time $t$. Therefore, we set the numbers of input and output neurons to the number of components in the target system. The ELM is represented as

\[
\mathbf{h}(t) = g\left(\mathbf{W}^T \mathbf{x}(t) + \mathbf{b}\right),
\]

\[
\mathbf{x}(t + 1) = \mathbf{B}^T \mathbf{h}(t),
\]

where $\mathbf{x}(t) \in \mathbb{R}^Z$ and $\mathbf{x}(t + 1) \in \mathbb{R}^Z$ are the state variables of all components in the target system at times $t$ and $t + 1$, respectively, $\mathbf{h}(t) \in \mathbb{R}^Y$ is the output vector of the hidden neurons, $g(\cdot)$ is a nonlinear function, $\mathbf{W} \in \mathbb{R}^{Z \times Y}$ and $\mathbf{b} \in \mathbb{R}^Y$ are the matrix of synaptic weights and the vector of biases, respectively, for the hidden neurons, and $\mathbf{B} \in \mathbb{R}^{Y \times Z}$ is the synaptic weight matrix for the output neurons. Here, $Y$ is the number of hidden neurons and $Z$ is the number of components in the target system. For the nonlinear function, we use the sigmoid function

\[
g(\chi) = \frac{\epsilon_1}{1 + e^{\epsilon_3 \chi}} - \epsilon_2,
\]

where $\epsilon_1$ and $\epsilon_2$ are parameters used to adjust the range of the sigmoid function and $\epsilon_3$ is a parameter determined by trial and error.
The matrix of synaptic weights $W$ and the vector of biases $b$ are randomly generated in the range of $[-1, 1]$, and then fixed. The matrix of synaptic weights for the output neurons is trained by

$$B = H^\dagger D,$$

where $H^\dagger$ is the pseudo-inverse of the output matrix $H = [h(1) h(2) \cdots h(L)]^T$ for the hidden neurons, and $D \in \mathbb{R}^{L \times Z}$ is the matrix of the desired output. Here, $L$ is the length of the training data sets.

### 2.2 Reconstruction of bifurcation diagrams

In this section, we explain the algorithm for BD reconstruction using the ELM [8]. The BD reconstruction begins by training the ELM to model the obtained time-series data sets of all components. Next, principal component analysis is applied to the trained synaptic weights in the ELM. Finally, the BDs are reconstructed from the results of the principal component analysis.

First, we train the synaptic weights of the output neurons in the ELM to model the target system using time-series data sets of all components as in Section 2.1. The synaptic weights and biases of the hidden neurons are randomly generated before the modeling and are fixed in the BD reconstruction.

The nonlinear function $P(\cdot, \cdot)$ of the ELM is described by

$$x(t + 1) = P\left(\beta^{(n)}, x(t)\right), \quad (1 \leq n \leq N),$$

where $N$ is the number of time-series data sets obtained for the BD reconstruction, and $\beta^{(n)} \in \mathbb{R}^{YZ}$ is the trained synaptic weight vector for the $n$th time-series data set that is reshaped from the synaptic weight matrix $B^{(n)} = [\beta_{1}^{(n)} \beta_{2}^{(n)} \cdots \beta_{Z}^{(n)}]$ as follows:

$$\beta^{(n)} = \begin{bmatrix} \beta_{1}^{(n)} \\ \beta_{2}^{(n)} \\ \vdots \\ \beta_{Z}^{(n)} \end{bmatrix}.$$  (6)

Next, eigenvalues and eigenvectors are obtained from the results of applying principal component analysis to $[\Delta \beta^{(1)} \Delta \beta^{(2)} \cdots \Delta \beta^{(N)}]$, where $\Delta \beta^{(n)}$ is a deviation vector of $\beta^{(n)}$ as follows:

$$\Delta \beta^{(n)} = \beta^{(n)} - \bar{\beta}.$$  (7)

Here, $\bar{\beta} \in \mathbb{R}^{YZ}$ is the mean vector of the vectors $\beta^{(n)}, (n = 1, \ldots, N)$ of trained synaptic weights. We can estimate the number of bifurcation parameters from the $c$th cumulative contribution ratio as follows:

$$CCR_c = \sum_{i=1}^{c} \frac{\lambda_i}{\sum_{j=1}^{C} \lambda_j} \times 100\%,$$  (8)

where $\lambda_j$ is the $j$th eigenvalue and $C$ is the number of trained synaptic weights (i.e., $C = YZ$). The number of bifurcation parameters $A$ is estimated to be $c$, which satisfies $CCR_c > 90\%$. Using the eigenvectors with an estimated parameter vector $\gamma \in \mathbb{R}^A$, the new synaptic weight vector is obtained by

$$\tilde{\beta} = U\gamma + \bar{\beta},$$  (9)

where $U \in \mathbb{R}^{C \times A}$ is the matrix of eigenvectors $[u_1 u_2 \cdots u_A]$ of the significant principal components. Here, $u_i \in \mathbb{R}^C$ is the $i$th eigenvector.

Finally, the BDs are reconstructed using the new synaptic weight vector. The nonlinear function of the ELM for BD reconstruction is described by

$$x(t + 1) = P\left(\tilde{\beta}, x(t)\right).$$  (10)

To reconstruct the BDs, time-series data are repeatedly generated using the nonlinear function with new synaptic weights $\tilde{\beta}$ while changing $\gamma$. Here, the $n$th estimated parameter vector $\gamma_n$ corresponded to $\beta^{(n)}$ can be calculated as follows:
\[ \gamma_n = U^{-1} \Delta \beta^{(n)}. \]  

Therefore, we expect that the BD reconstructed in the range of \([\gamma_i, \gamma_{i+1}]\) is corresponded to the original BD in the range of \([\rho_i, \rho_{i+1}]\) where \(\rho_n \in \mathbb{R}^A\) is the parameter vector used to generate \(n\)th time-series data sets. We would like to emphasize that the BD can be reconstructed in extrapolation range \([8]\).

3. Estimating Lyapunov spectrum of reconstructed bifurcation diagrams

In previous work, we proposed a method for estimating the Lyapunov spectrum of reconstructed BDs \([8]\) based on \([17–19]\). The Lyapunov spectrum is obtained from the results of applying QR decomposition to the Jacobian matrix of the nonlinear function \(P(\cdot, \cdot)\) in Eq. (10) as follows:

\[ JP(\tilde{\beta}, x(t))Q(t) = Q(t+1)R(t+1), \quad (t = 0, \ldots, T), \]  

where \(Q(t)\) is an orthogonal matrix, \(R\) is an upper triangular matrix, and \(JP(\cdot, \cdot)\) is the Jacobian matrix of the nonlinear function \(P(\cdot, \cdot)\). Here, \(Q(0)\) is set to be the identity matrix. We describe a supplement for the Jacobian matrix of the nonlinear function in the Appendix A.

The Lyapunov exponents are obtained by

\[ \mu_i = \lim_{\phi \to \infty} \frac{1}{\phi} \sum_{t=1}^{\phi} \log r_{ii}(t), \quad (i = 1, \ldots, Z), \]  

where \(r_{ii}(t)\) is the \(i\)th diagonal component of \(R(t)\) and \(\phi\) is the number of trials.

4. Numerical experiments

In this section, we present the results of BDs reconstruction with original BDs. In these numerical experiments, the target system was the Rössler equations \([20]\). Here, we show the results of BD reconstruction, compare the length of the training data with those used in previous studies, quantitatively compare the Lyapunov experiments on the original and reconstructed BDs, and show the results of reconstructing a two-dimensional BD only from time-series data sets of all components. First, we show the experimental conditions for and the results of reconstructing one-dimensional BDs. Then, we show the results of reconstructing a two-dimensional BD.

4.1 Experimental conditions

The Rössler equations are

\[ \frac{d\xi}{dt} = -\eta - \nu, \]  

\[ \frac{d\eta}{dt} = \xi + p_1 \eta, \]  

\[ \frac{d\nu}{dt} = p_2 \xi + \nu(\xi - p_3), \]  

where \(p_1, p_2,\) and \(p_3\) are parameters. For the one-dimensional BD reconstruction, we used only \(p_3\) as the bifurcation parameter, with the other two fixed as \(p_1 = 0.33\) and \(p_2 = 0.3\). For the numerical experiments, the time-series data sets were generated using a fourth-order Runge–Kutta method in which the time increment and sampling time step were 0.01 and 5, respectively; that is, we used the time-series data sets of all components sampled at steps of 0.05. For the training data, we generated five time-series data sets with the bifurcation parameter \(p_3^{(n)}\) as follows:

\[ p_3^{(n)} = -0.5 \cos(2\pi(n-1)/8) + 3.7, \quad (n = 1, \ldots, 5). \]  

We chose the parameter sets that include at least one parameter value that exhibits chaotic time-series because we expect that the highly precise predictor can be obtained by training chaotic time-series data set that has more information than periodic time-series data set.

394
The length of each time series was 1000, which was shorter than the length of 5000 of the one-component training data used for BD reconstruction of the Rössler equations in [8].

The numbers of input, hidden, and output neurons in the ELM were 3, 15, and 3, respectively. The parameters $\epsilon_1$, $\epsilon_2$, and $\epsilon_3$ of the sigmoid function were set to 16, 8, and 0.05, respectively.

4.2 Reconstructing one-dimensional bifurcation diagrams

First, we estimate the number of bifurcation parameters by using the cumulative contribution ratio shown in Eq. (8). Because the first cumulative contribution ratio is 99.1%, we estimate that there is only one bifurcation parameter.

Figures 1 and 2 show the original and reconstructed one-dimensional BDs of all components in the Rössler equations. Figures 1(a), (b), and (c) show the original BDs of the $\xi$, $\eta$, and $\nu$ components, respectively, and Figs. 2(a), (b), and (c) show the corresponding reconstructed BDs. Although the bifurcation parameter values of the reconstructed BDs differ from those of the original BDs, we adjusted the horizontal axes by the difference between the original and estimated bifurcation parameters. Here, the interpolation region in the reconstructed BDs is $[-74.4, 77.7]$, corresponding to bifurcation parameter values of $[3.2, 4.2]$ in the original BDs; the other areas are extrapolation areas. Comparing the original and reconstructed BDs shows that they correspond overall, but the numbers of periodic windows in the reconstructed BDs differ from those in the original BDs. We attribute this to the bifurcation parameter regions of the reconstructed BDs sometimes being expanded in some intervals and contracted in others.

4.3 Estimating Lyapunov spectrum

Figures 3(a) and (b) show the Lyapunov spectra of the original and reconstructed BDs, respectively. The solid, dashed, and dotted lines in Figs. 3(a) and (b) correspond to the first-, second-, and third-largest Lyapunov exponents, respectively. Comparing the Lyapunov spectra of the original and reconstructed BDs shows that they agree closely. Therefore, we see that the reconstructed BDs correspond quantitatively to the original ones.

4.4 Reconstructing two-dimensional bifurcation diagram

Here, we show the results of reconstructing a two-dimensional BD of the Rössler equations. The numerical conditions are the same as those for the one-dimensional BDs except for the bifurcation parameters used to generate the training data sets. For the training data in this reconstruction, we
generated six time-series data sets with $p_2 = 0.3$ and the bifurcation parameters $p_1^{(n)}$ and $p_3^{(n)}$ as follows:

$$
p_1^{(n)} = 0.03 \sin \left( \frac{2\pi(n-1)}{6} \right) + 0.33, \quad (n = 1, \ldots, 6),
$$

$$
p_3^{(n)} = -0.5 \cos \left( \frac{2\pi(n-1)}{6} \right) + 3.7, \quad (n = 1, \ldots, 6).
$$

Here, the length of each time-series data was 1000, compared with 5000 previously [7]. As with the one-dimensional BD reconstruction, the length of training data is shorter than that for reconstructing the two-dimensional BD of the Rössler equations in [7].

Again, we estimate the number of bifurcation parameters using the cumulative contribution ratio shown in Eq. (8), and now we estimate that there are two bifurcation parameters because the second cumulative contribution ratio is 99.2%.

Figure 4 shows the original and reconstructed two-dimensional BDs, in which the colors indicate the periodicity of each attractor; here, yellow indicates attractors with period 20 or greater, including chaos. We see that the reconstructed two-dimensional BD corresponds qualitatively to the original one. Therefore, we have clarified that a two-dimensional BD can be reconstructed from shorter time-series data sets than those used previously.

5. Conclusion

We reconstructed the one- and two-dimensional BDs of all components in the Rössler equations only from time-series data sets. In addition, we estimated the Lyapunov spectrum of the reconstructed BDs and compared it quantitatively with the Lyapunov spectra the original BDs. Overall, we demonstrated that the BDs of all components can be reconstructed. We expected that the reconstruction would require a shorter length of training data of all components than that of one component. In this
study, we reconstructed the BDs of the Rössler equations using a shorter length of training data of all components than when using training data of one component in [7] and [8]. The lengths of training data in this study were sufficient with one-fifth in both numerical experiments of the one- and two-dimensional BDs reconstruction, compared by the previous studies.

In future work, we intend to reconstruct BDs of all components in other systems. Also, because we cannot always obtain time-series data sets of all components in the target system when we apply the present method to real-world systems, we intend to reconstruct the BDs of several, but not all, components. In addition, we will try to trace unstable sets by analyzing Eq. (10) while changing the bifurcation parameters. Moreover, we will attempt to investigate oscillations when the unknown oscillators are coupled by using Eq. (10) as parameter adjustable oscillators.

Acknowledgments

Part of this work was carried out under the Cooperative Research Project Program of the Research Institute of Electrical Communication, Tohoku University.

Appendix

A. Jacobian matrix of nonlinear function for estimating Lyapunov exponents

As a supplement of Sec. 4.3, we show the Jacobian matrix of the nonlinear function $P(\cdot, \cdot)$ for estimating Lyapunov exponents [8]. The Jacobian matrix is as follows:

$$JP\left(\tilde{\beta}, x(t)\right) = \begin{bmatrix}
\frac{\partial P(\tilde{\beta}_1, x(t))}{\partial x_1(t)} & \cdots & \frac{\partial P(\tilde{\beta}_1, x(t))}{\partial x_Z(t)} \\
\vdots & \ddots & \vdots \\
\frac{\partial P(\tilde{\beta}_Z, x(t))}{\partial x_1(t)} & \cdots & \frac{\partial P(\tilde{\beta}_Z, x(t))}{\partial x_Z(t)}
\end{bmatrix}$$  \hspace{1cm} (A-1)

where $\tilde{\beta}_i \in \mathbb{R}^Y, (i = 1, \ldots, Z)$ is the synaptic weight vector of $i$th output neuron in the new synaptic weights $\check{\beta}$ of Eq. (9) and $x_j(t) \in \mathbb{R}, (j = 1, \ldots, Z)$ is the $j$th state value of state variables $x(t)$. An element $\frac{\partial P(\tilde{\beta}_i, x(t))}{\partial x_j(t)}$ of the matrix $JP(\cdot, \cdot)$ is obtained from Eqs. (1) and (2) as follows:

$$\frac{\partial P(\tilde{\beta}_i, x(t))}{\partial x_j(t)} = \frac{\partial}{\partial x_j(t)} \left( \sum_{k=1}^{Y} \tilde{\beta}_{ik} h_k(t) \right) = \frac{\partial}{\partial x_j(t)} \left( \sum_{k=1}^{Y} \tilde{\beta}_{ik} g \left( \sum_{l=1}^{Z} w_{kl} x_l(t) + b_l \right) \right)$$

$$= \sum_{k=1}^{Y} \tilde{\beta}_{ik} \left( \epsilon_1 \epsilon_3 - \epsilon_3 (h_k(t) + \epsilon_2) \right) \frac{h_k(t) + \epsilon_2}{\epsilon_1} w_{kj}$$  \hspace{1cm} (A-2)

where $\tilde{\beta}_{ik} \in \mathbb{R}$ and $h_k(t) \in \mathbb{R}$ are the $k$th element of $\tilde{\beta}_i$ and $h(t)$, and $w_{kl} \in \mathbb{R}$ and $b_l \in \mathbb{R}$ are the $l$th element of $w_k$ and $b$.

References

[1] R. Tokunaga, S. Kajiwara, and T. Matsumoto, “Reconstructing bifurcation diagrams only from time-waveforms,” *Physica D*, vol. 79, pp. 348–360, 1994.

[2] I. Tokuda, R. Tokunaga, and T. Matsumoto, “Detecting switch dynamics in chaotic time-waveform using a parametrized family of nonlinear predictors,” *Physica D*, vol. 135, pp. 63–78, 2000.

[3] E. Bagarinoa, K. Pakdaman, T. Nomura, and S. Sato, “Reconstructing bifurcation diagrams from noisy time series using nonlinear autoregressive models,” *Physical Review E*, vol. 60, no. 1, 1073, 1999.

[4] E. Bagarinoa, K. Pakdaman, T. Nomura, and S. Sato, “Time series-based bifurcation diagram reconstruction,” *Physica D*, vol. 130, pp. 211–231, 1999.
[5] E. Bagarinoa, K. Pakdaman, T. Nomura, and S. Sato, “Reconstructing bifurcation diagrams of dynamical systems using measured time series,” *Methods of information in medicine*, vol. 39, pp. 146–149, 2000.

[6] G. Langer and U. Parlitz, “Modeling parameter dependence from time series,” *Physical Review E*, vol. 70, 056217, 2004.

[7] Y. Tada and M. Adachi, “Reconstruction of bifurcation diagrams using extreme learning machines,” *2013 IEEE International Conference on Systems, Man, and Cybernetics*, pp. 1127–1131, 2013.

[8] Y. Itoh, Y. Tada, and M. Adachi, “Reconstructing bifurcation diagrams with Lyapunov exponents from only time-series data using an extreme learning machine,” *NOLTA*, vol. 8, no. 1, pp. 2–14, 2017.

[9] Y. Itoh and M. Adachi, “A quantitative method for evaluating reconstructed one-dimensional bifurcation diagrams,” *Journal of Computers*, vol. 13, no. 3, pp. 271–278, 2018.

[10] Y. Itoh and M. Adachi, “Bifurcation diagrams in estimated parameter space using a pruned extreme learning machine,” *Physical Review E*, vol. 98, pp. 013301-1–12, 2018.

[11] Y. Itoh and M. Adachi, “Reconstructing bifurcation diagrams of induction motor drives using an extreme learning machine,” *Proceedings of ELM2017*, ed. J. Cao et al., pp. 58–69, Springer, 2019.

[12] Y. Itoh and M. Adachi, “Reconstruction of bifurcation diagrams using time-series data generated by Electronic circuits of the Rössler equations,” *2017 International Symposium on Nonlinear Theory and its Applications*, pp. 439–442, Cancun, Mexico, December 2017.

[13] Y. Itoh, S. Uenohara, M. Adachi, T. Morie, and K. Aihara, “Reconstructing bifurcation diagrams only from time-series data generated by electronic circuits in discrete-time dynamical systems,” *chaos*, vol. 30, pp. 013128-1–11, 2020.

[14] Y. Itoh and M. Adachi, “Tracking bifurcation curves in the Henon map from only time-series datasets,” *NOLTA*, vol. 10, no. 2, pp. 268–278, 2019.

[15] H. Kawakami, “Bifurcation of periodic responses in forced dynamic nonlinear circuits: Computation of bifurcation values of the system parameters,” *IEEE Transactions on Circuits and Systems*, vol. 31, no. 3, pp. 248–260, 1984.

[16] G.B. Huang, Q.Y. Zhu, and C.K. Siew, “Extreme learning machine: Theory and applications,” *Neurocomputing*, vol. 70, pp. 489–501, 2006.

[17] I. Shimada and T. Nagashima, “A numerical approach to ergodic problem of dissipative dynamical systems,” *Progress of theoretical physics*, vol. 61, no. 6, pp. 1605–1616, 1979.

[18] M. Sano and Y. Sawada, “Measurement of the lyapunov spectrum from chaotic time series,” *Physical review letters*, vol. 55, 1985.

[19] M. Adachi and M. Kotani, “Identification of chaotic dynamical systems with back-propagation neural networks,” *IEICE TRANSACTIONS on Fundamentals of Electronics, Communications and Computer Sciences*, vol. E77-A, no. 1, pp. 324–334, 1994.

[20] O.E. Rössler, “Continuous Chaos,” *Annals of the New York Academy of Sciences*, vol. 31, pp. 376–392, 1979.