CONGRUENCES CONCERNING GENERALIZED CENTRAL TRINOMIAL COEFFICIENTS

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Abstract. For any \( n \in \mathbb{N} = \{0, 1, 2, \ldots \} \) and \( b, c \in \mathbb{Z} \), the generalized central trinomial coefficient \( T_n(b, c) \) denotes the coefficient of \( x^n \) in the expansion of \( (x^2 + bx + c)^n \). Let \( p \) be an odd prime. In this paper, we determine the summation \( \sum_{k=0}^{p-1} T_k(b, c)^2/m^k \) modulo \( p^2 \) for integers \( m \) with certain restrictions. As applications, we confirm some conjectural congruences of Sun [Sci. China Math. 57 (2014), 1375–1400].

1. Introduction

For \( n \in \mathbb{N} = \{0, 1, 2, \ldots \} \), the \( n \)th central trinomial coefficient \( T_n \) is the coefficient of \( x^n \) in the expansion of \( (x^2 + x + 1)^n \). By the multinomial theorem, it is easy to see that

\[
T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k}. \tag{1.1}
\]

Central trinomial coefficients have some interesting combinatorial interpretations. For example, \( T_n \) counts the lattice paths from \((0, 0)\) to \((n, 0)\) with steps \( U = (1, 1) \), \( D = (1, -1) \) and \( H = (1, 0) \). For more combinatorial interpretations and formulas of \( T_n \), one may consult [14, A002426].

For \( b, c \in \mathbb{Z} \), the generalized central trinomial coefficients \( T_n(b, c) \) are defined as the coefficients of \( x^n \) in the expansion of \( (x^2 + bx + c)^n \). By the multinomial theorem, we have the following formula similar to :

\[
T_n(b, c) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} b^{n-2k} c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k} b^{n-2k} c^k. \tag{1.2}
\]

Some well-known combinatorial sequences are the special cases of \( T_n(b, c) \). For example,

\[
T_n = T_n(1, 1), \quad \binom{2n}{n} = T_n(2, 1), \quad D_n = T_n(3, 2),
\]

where \( D_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \) is the \( n \)th central Delannoy number (cf. [14 A001850]). Throughout the paper, we set \( d = b^2 - 4c \). The generalized central trinomial coefficients are also related

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to the well-known Legendre polynomials \( P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \left( \frac{x - 1}{2} \right)^k \) via the following identity (see \([10, 20, 21]\)):

\[
T_n(b, c) = (\sqrt{d})^n P_n \left( \frac{b}{\sqrt{d}} \right).
\]

It is known that sums involving central binomial coefficients sometimes have beautiful congruence properties (cf. e.g., \([3, 5–8, 15–18]\)). As mentioned above, the central binomial coefficients \( \binom{2n}{n} = T_n(2, 1) \). Motivated by this, Sun \([20, 21]\) investigated the congruences for sums involving generalized central trinomial coefficients systematically. For any odd prime \( p \), Sun \([20]\) proved some congruences modulo \( p \) for sums of the following forms:

\[
p-1 \sum_{k=0}^{p-1} \binom{2k}{k} T_k(b, c)/m^k \quad \text{and} \quad p-1 \sum_{k=0}^{p-1} \binom{2k}{k} T_{2k}(b, c)/m^k.
\]

Moreover, he also posed some conjectures, one of which states as follows: for any prime \( p > 3 \) we have

\[
\sum_{k=0}^{p-1} \binom{2k}{k} T_k(b, c)/m^k \equiv \left( \frac{p}{3} \right) \frac{3^{p-1} + 3}{4} \pmod{p^2}, \quad (1.3)
\]

where \( \left( \frac{\cdot}{p} \right) \) stands for the Legendre symbol. It is worth mentioning that \((1.3)\) has been confirmed by the second author and Sun \([23]\).

Sun \([21]\) obtained some congruences involving \( T_n(b, c)^2 \). For instance, he proved that for any \( b, c \in \mathbb{Z} \) and prime \( p > 3 \) with \( p \nmid d \) we have

\[
p-1 \sum_{k=0}^{p-1} T_k(b, c)^2 \equiv \left( \frac{16c}{d} \right)^{(p-1)/2} + p \sum_{k=0}^{p-1} \binom{2k}{k} T_{2k}(b, c)/m^k \pmod{p^3}. \quad (1.4)
\]

In particular, taking \( b = 2, c = -1 \) in \((1.4)\) and via some further computation, he deduced that

\[
p-1 \sum_{k=0}^{p-1} T_k(2, -1)^2 \equiv \left( \frac{-2}{p} \right) \pmod{p^2}. \quad (1.5)
\]

Moreover, Sun \([21]\, Conjecture 5.4\) posed the following conjecture.

**Conjecture 1.1.** Let \( p \) be an odd prime. Then

\[
p-1 \sum_{k=0}^{p-1} T_k(2, 2)^2 \equiv p-1 \sum_{k=0}^{p-1} \binom{2k}{k}^2 \pmod{p^{5+(\frac{1}{p})/2}}. \quad (1.6)
\]

If \( p > 3 \), then

\[
p-1 \sum_{k=0}^{p-1} T_k(4, 1)^2 \equiv p-1 \sum_{k=0}^{p-1} \frac{T_k(4, 1)^2}{36^k} \equiv \left( \frac{-1}{p} \right) \pmod{p^2}. \quad (1.7)
\]
Clearly, Conjecture 1.1 cannot be deduced from (1.4).

Let us introduce some concepts which are necessary to state our main result. Let \( p \) be a prime. Then for any integer \( a \neq 0 \pmod{p} \) we have \( a^{p-1} \equiv 1 \pmod{p} \) by the Fermat’s little theorem, and then \( q_p(a) := (a^{p-1} - 1)/p \in \mathbb{Z}_p \). As usual, we call \( q_p(a) \) a Fermat quotient with base \( a \). For any \( p \)-adic integer \( x \), let \( (x)_p \) denote the least nonnegative residue of \( x \) modulo \( p \). Set

\[
S_{p-1}(x) := \sum_{k=1}^{p-1} \binom{2k}{k} x^k.
\]

Now we state our main theorem which gives different parametric congruences for the summation \( \sum_{k=0}^{p-1} T_k(b,c)^2/m^k \).

**Theorem 1.1.** Let \( p \) be an odd prime and let \( b, c \in \mathbb{Z} \) and \( d = b^2 - 4c \).

(i) If \( p \) does not divide \( d \), then we have

\[
\sum_{k=0}^{p-1} \frac{T_k(b,c)^2}{(-d)^k} \equiv \sum_{k=0}^{(p-1)/2} \binom{2k}{k} \left( \frac{-c}{4d} \right)^k \pmod{p^2}.
\]  

(ii) Let \( m \) be an integer satisfying the equation \((m - d)^2 = 16mc \). If \( p \nmid md(m - d) \), then we have

\[
\sum_{k=0}^{p-1} \frac{T_k(b,c)^2}{m^k} \equiv \left( \frac{-1}{p} \right) + \frac{pd}{d - m} \left( \frac{-1}{p} \right) \left( q_p(d) - q_p(m) + S_{p-1} \left( \frac{m + d}{4m} \right) - S_{p-1} \left( \frac{m + d}{4d} \right) \right) \pmod{p^2}.
\]  

In fact, \( S_{p-1}((m + d)/(4m)) \) and \( S_{p-1}((m + d)/(4d)) \) modulo \( p \) in (1.9) can be determined. If \( p \mid m + d \), then the two sums are both equivalent to 0 modulo \( p \). Now suppose that \( p \nmid m + d \). Sun and Tauraso [16, Theorem 1.2] proved that for any prime \( p \) and integer \( t \) with \( p \nmid t \) we have

\[
\sum_{k=1}^{p-1} \binom{2k}{k} \frac{t^k}{k(-t)^k} \equiv \frac{2t^p - 2V_p(t)}{pt} \pmod{p},
\]  

where the polynomial sequence \( \{V_n(x)\}_{n \in \mathbb{N}} \) is defined as follows:

\[
V_0(x) = 2, \quad V_1(x) = x, \quad \text{and} \quad V_{n+1}(x) = x(V_n(x) + V_{n-1}(x)) \quad (n \in \mathbb{Z}^+).
\]

Sun and Tauraso also showed that \( V_p(t) \equiv t \pmod{p} \). Thus the right-hand side of (1.10) are always \( p \)-adic integers. By (1.10) we obtain

\[
S_{p-1} \left( \frac{m + d}{4m} \right) \equiv \frac{2 \left\langle -\frac{4m}{m+d} \right\rangle_p - 2V_p \left( \left\langle -\frac{4m}{m+d} \right\rangle_p \right)}{p \left\langle -\frac{4m}{m+d} \right\rangle_p} \pmod{p}
\]
and
\[ S_{p-1} \left( \frac{m+d}{4d} \right) \equiv 2 \left\langle -\frac{4d}{m+d} \right\rangle_p - 2V_p \left( \left\langle -\frac{4d}{m+d} \right\rangle_p \right) \pmod{p}. \]

By Theorem 1.1, we can obtain the following results. (The reader may try to find more applications of Theorem 1.1)

**Corollary 1.1.** Let \( p \) be an odd prime. Then modulo \( p^2 \) we have
\[ \sum_{k=0}^{p-1} \frac{T_k(2, 2)^2}{4^k} \equiv \sum_{k=0}^{p-1} \left( \frac{2^k}{8k} \right)^2 \]
\[ \equiv \begin{cases} \left( \frac{2}{p} \right) \frac{(2x - \frac{p}{2x})}{2x} & \text{if } p \equiv 1 \pmod{4} \text{ and } p = x^2 + 4y^2 \text{ (} x, y \in \mathbb{Z} \text{) with } x \equiv 1 \pmod{4}, \\ \left( \frac{-1}{p} \right) \frac{(p+1)/4}{2p} & \text{if } p \equiv 3 \pmod{4}, \end{cases} \] (1.11)
and
\[ \sum_{k=0}^{p-1} \frac{T_k(2, -1)^2}{(-8)^k} \equiv \begin{cases} 2x - \frac{p}{2x} & \text{if } p \equiv 1 \pmod{4} \text{ and } p = x^2 + 4y^2 \text{ (} x, y \in \mathbb{Z} \text{) with } x \equiv 1 \pmod{4}, \\ 0 & \text{if } p \equiv 3 \pmod{4}. \end{cases} \] (1.12)

If \( p > 3 \), then we have
\[ \sum_{k=0}^{p-1} \frac{T_k(4, 1)^2}{4^k} \equiv \sum_{k=0}^{p-1} \frac{T_k(4, 1)^2}{36^k} \equiv \left( \frac{-1}{p} \right) \pmod{p^2}. \] (1.13)

Note that (1.11) and (1.13) confirm Conjecture 1.1 for the modulus \( p^2 \) cases. (1.12) was brought to our attention by Sun and it was originally conjectured by Sun in his notebook where he has proved the modulus \( p \) case. We also note that any prime \( p \equiv 1 \pmod{4} \) can be uniquely represented as \( x^2 + 4y^2 \), where \( x, y \in \mathbb{Z} \) and \( x \equiv 1 \pmod{4} \) (cf. [4, p. 106]). In [21], Sun also showed that for any prime \( p \nmid b - 2c \), then
\[ \sum_{k=0}^{p-1} \frac{T_k(b, c)^2}{(b - 2c)^{2k}} \equiv \left( \frac{-c^2}{p} \right) \pmod{p}. \]

Trivially, the above congruence holds for \( c \equiv 0 \pmod{p} \). If \( c \not\equiv 0 \pmod{p} \), we can obtain the modulus \( p^2 \) extension of the congruence by (1.9).
Our method to prove Theorem 1.1 is quite different from the one used by Sun to prove (1.4). We need the symbolic summation package Sigma developed by Schneider [13] to establish some identities. We shall prove Theorem 1.1 and Corollary 1.1 in the next section.

2. PROOFS OF THEOREM 1.1 AND COROLLARY 1.1

To show the main results, we need the following preliminary results.

We need the following identity obtained by Sun [21, Lemma 4.1] which is deduced from the well-known Clausen identity (cf. [1, p. 116]).

Lemma 2.1. Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. For any $n \in \mathbb{N}$ we have

$$T_n(b, c)^2 = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}^2 c^k d^{n-k}. \quad (2.1)$$

For any $n \in \mathbb{N}$, define the $n$th harmonic $H_n := \sum_{k=1}^{n} 1/k$. Harmonic numbers have many congruence properties. For example, for any odd prime we have $H_p - 1 \equiv 0 \pmod{p}$. (See [4, p. 37].) Note that the above congruence was extended to the modulus $p^2$ case by Wolstenholme [24] for $p > 3$.

We now give an identity involving harmonic numbers.

Lemma 2.2. For any nonnegative integer $n$ we have

$$\sum_{k=0}^{n} \binom{n}{k} H_k x^k = (1 + x)^n H_n - \sum_{k=1}^{n} \frac{(1 + x)^{n-k}}{k}. \quad (2.2)$$

Proof. We prove this identity by the symbolic summation package Sigma in Mathematica. Denote the left-hand side of (2.2) by $S_n$. Using the command GenerateRecurrence in Sigma we find $S_n$ satisfies the following recurrence relation:

$$-(n+1)(x+1)^2 S_n + (2n+3)(x+1) S_{n+1} - (n+2) S_{n+2} = -x \ (n \in \mathbb{N}).$$

It can be also verified that the right-hand side of (2.2) satisfies the same recurrence relation. Moreover, it is easy to see that the both sides of (2.2) coincide for $n = 0, 1$. Thus the desired result follows. \(\square\)

Remark 2.1. Taking $x = -1$ in (2.2) we obtain the following known identity (cf. [2 (1.45)]):

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k H_k = -H_n.$$

From [2 (2.1) and (2.4)] we know for any $n \in \mathbb{N}$ and $x \neq -1$,

$$\sum_{k=0}^{n} \frac{x^k}{\binom{n}{k}} = (n+1) \sum_{k=1}^{n+1} \frac{x^k + 1}{k(x+1)} \left(\frac{x}{x+1}\right)^{n+1-k}. \quad (2.3)$$

The next lemma is very similar to (2.3).
Lemma 2.3. For any \( n \in \mathbb{Z}^+ \) and \( x \neq -1 \), we have
\[
\sum_{k=0}^{n-1} \frac{x^k}{\binom{2n-1}{k}} = \frac{2n}{x+1} \sum_{k=1}^{2n} \frac{1}{k} \left( \frac{x}{x+1} \right)^{2n-k} + \frac{2n(x-1)}{(x+1)^2} \sum_{k=1}^{n} \frac{x^k}{k(2k)} \left( \frac{x}{x+1} \right)^{2n-2k}.
\] (2.4)

Proof. In fact, by \( \Sigma \) we originally find that
\[
\sum_{k=0}^{n-1} \frac{x^k}{\binom{2n-1}{k}} = \frac{n}{(x+1)^2} \sum_{k=1}^{n} \frac{(4k-1)x + 2k - 1}{k(2k-1)} \left( \frac{x}{x+1} \right)^{2n-2k} + \frac{2n(x-1)}{(x+1)^2} \sum_{k=1}^{n} \frac{x^k}{k(2k)} \left( \frac{x}{x+1} \right)^{2n-2k}.
\] (2.5)

It is easy to verify that the both sides of (2.5) satisfy the following recurrence relation:
\[
(n + 1)x^2 S_n - n(1 + x)^2 S_{n+1} = -\frac{n((4n + 3)x + 2n + 1)}{2n + 1} + \frac{(n^2 + n)(x^{n+1} - x^{n+2})}{(2n + 1)^2}.
\]

Since the both sides of (2.5) coincide for \( n = 1 \). Thus (2.5) holds. Now it suffices to show
\[
\frac{n}{(x+1)^2} \sum_{k=1}^{n} \frac{(4k-1)x + 2k - 1}{k(2k-1)} \left( \frac{x}{x+1} \right)^{2n-2k} = \frac{2n}{x+1} \sum_{k=1}^{2n} \frac{1}{k} \left( \frac{x}{x+1} \right)^{2n-2k}.
\] (2.6)

Clearly,
\[
\frac{n}{(x+1)^2} \sum_{k=1}^{n} \frac{(4k-1)x + 2k - 1}{k(2k-1)} \left( \frac{x}{x+1} \right)^{2n-2k} = \frac{n}{(x+1)^2} \sum_{k=1}^{n} \left( \frac{2x}{2k-1} + \frac{x+1}{k} \right) \left( \frac{x}{x+1} \right)^{2n-2k}
= \frac{2nx^{2n}}{(x+1)^{2n+1}} \left( \sum_{k=1}^{n} \frac{1}{2k-1} \left( \frac{x}{x+1} \right)^{1-2k} + \sum_{k=1}^{n} \frac{1}{2k} \left( \frac{x}{x+1} \right)^{-2k} \right)
= \frac{2n}{x+1} \sum_{k=1}^{2n} \frac{1}{k} \left( \frac{x}{x+1} \right)^{2n-2k}.
\]

This completes the proof. \qed

Lemma 2.4. For any \( n \in \mathbb{Z}^+ \) and \( dm(m+d) \neq 0 \), we have
\[
\sum_{l=1}^{n} \binom{n}{l} \binom{n+l}{l} (-1)^l \sum_{k=1}^{l} \frac{1}{k(2k)} \left( -\frac{(m-d)^2}{md} \right)^k
= \frac{d - m}{d + m} (-1)^n \left( \sum_{k=1}^{n} \frac{(-d/m)^k}{k} - \sum_{k=1}^{n} \frac{(-m/d)^k}{k} \right).
\] (2.7)
Proof. (2.7) is also found by Sigma. It is easy to see that both sides of (2.7) satisfy the same recurrence relation:

\[ md(n+1)S_n + (-2d^2 + md - 2m^2 - d^2n + mdn - m^2n)S_{n+1} \\
+ (-2d^2 + 3md - 2m^2 - d^2n + mdn - m^2n)S_{n+2} + md(n+3)S_{n+3} = 0. \]

Moreover, the both sides of (2.7) coincide when \( n = 0, 1, 2 \). This proves the desired lemma. \( \square \)

Remark 2.2. If we replace the \(- (m - d)^2 / (md)\) by arbitrary \( x \in \mathbb{C} \) in the left-hand side of (2.7), then no simple form can be found by Sigma.

Let \( p \) be a prime and \( x \) a \( p \)-adic integer. Define the finite polylogarithms (cf. [7]) as follows:

\[ \mathcal{L}_d(x) = \sum_{k=1}^{p-1} x^k k^d. \]

Let \( Q_p(x) = (x^p + (1-x)^p - 1)/p \). By Fermat’s little theorem, \( Q_p(x) \in \mathbb{Z}_p \). The following lemma concerns the congruence relation between \( \mathcal{L}_d(x) \) and \( Q_p(x) \).

Lemma 2.5 (Mattarei and Tauraso [7]). For any odd prime \( p \) and \( p \)-adic integer \( x \) with \( p \nmid x(1-x) \) we have

\[ \mathcal{L}_1(x) \equiv -Q_p(x) \pmod{p}. \]

Lemma 2.6. Let \( p \) be an odd prime. Then for any \( p \)-adic integer \( x \) we have

\[ \sum_{k=1}^{(p-1)/2} \frac{(1-x)^k}{k} \equiv H_{(p-1)/2} + S_{p-1} \left( \frac{x}{4} \right) \pmod{p}. \] (2.8)

Proof. By the binomial theorem, we have

\[ \sum_{k=1}^{(p-1)/2} \frac{(1-x)^k}{k} = \sum_{k=1}^{(p-1)/2} \frac{1}{k} \sum_{s=0}^{k} \left( \begin{array}{c} k \\ s \end{array} \right) (-x)^s \]

\[ = H_{(p-1)/2} + \sum_{s=1}^{(p-1)/2} (-x)^s \sum_{k=s}^{(p-1)/2} \left( \begin{array}{c} k \\ s \end{array} \right) \]

\[ = H_{(p-1)/2} + \sum_{s=1}^{(p-1)/2} \frac{(-x)^s}{s} \sum_{k=s}^{(p-1)/2} \left( \begin{array}{c} k-1 \\ s-1 \end{array} \right) \]

\[ = H_{(p-1)/2} + \sum_{s=1}^{(p-1)/2} \frac{(-x)^s}{s} \binom{(p-1)/2}{s} \]

\[ \equiv H_{(p-1)/2} + S_{p-1} \left( \frac{x}{4} \right) \pmod{p}, \]

where we have used the Chu identity (cf. [2 (1.51)]) and the fact \( \binom{(p-1)/2}{s} \equiv \binom{2s}{s}/(-4)^s \pmod{p} \). \( \square \)
Now we are in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** Suppose that $p \mid md(m - d)$. Note that
\[
\binom{2k}{k} \equiv 0 \pmod{p}
\]
for $k$ among $(p + 1)/2, \ldots, p - 1$. Therefore, in view of Lemma 2.1 we have
\[
\sum_{k=0}^{p-1} \frac{T_k(b, c)^2}{m^k} = \sum_{k=0}^{p-1} \frac{d^k}{m^k} \sum_{l=0}^{k} \binom{k+l}{2l} \frac{(2l)^2 c^l}{d^l}
\]
\[
= \sum_{l=0}^{(p-1)/2} \frac{(2l)^2 c^l}{d^l} \sum_{k=l}^{p-1} \binom{k+l}{2l} \frac{d^k}{m^k}
\]
\[
= \sum_{l=0}^{(p-1)/2} \frac{(2l)^2 c^l}{d^l} \sum_{k=p-2l}^{p-1} \binom{k+2l}{2l} \frac{d^k}{m^k}
\]
\[
= \Sigma_1 + \Sigma_2 \pmod{p^2},
\]
where
\[
\Sigma_1 := \sum_{l=0}^{(p-1)/2} \frac{(2l)^2 c^l}{m^l} \sum_{k=p-2l}^{p-1} \binom{k+2l}{2l} \frac{d^k}{m^k}
\]
and
\[
\Sigma_2 := \sum_{l=0}^{(p-1)/2} \frac{(2l)^2 c^l}{m^l} \sum_{k=p-2l}^{p-l-1} \binom{k+2l}{2l} \frac{d^k}{m^k}.
\]

We first evaluate $\Sigma_1$ modulo $p^2$. Clearly, for any $k \in \{0, 1, \ldots, p - 2l - 1\}$ we have
\[
\binom{p-2l-1}{k} = \prod_{j=1}^{k} \frac{p-2l-j}{j} \equiv \binom{-2l-1}{k}(1 - p(H_{2l+k} - H_{2l})) \pmod{p^2}.
\]
Hence we have
\[
\Sigma_1 = \sum_{l=0}^{(p-1)/2} \frac{(2l)^2 c^l}{m^l} \sum_{k=0}^{p-2l-1} \binom{-2l-1}{k} \left( -\frac{d}{m} \right)^k
\]
\[
= \sum_{l=0}^{(p-1)/2} \frac{(2l)^2 c^l}{m^l} \sum_{k=0}^{p-2l-1} \binom{p-2l-1}{k} \left( -\frac{d}{m} \right)^k (1 + p(H_{2l+k} - H_{2l}))
\]
\[
= \sum_{l=0}^{(p-1)/2} \frac{(2l)^2 c^l}{m^l} \left( \frac{m-d}{m} \right)^{p-2l-1} (1 - pH_{2l})
\]
\[
+p \sum_{l=0}^{(p-1)/2} \frac{(2l)^2 c^l}{m^l} \sum_{k=0}^{p-2l-1} \binom{p-2l-1}{k} \left( -\frac{d}{m} \right)^k H_{2l+k} \pmod{p^2}. \tag{2.9}
\]
Note that for any $0 \leq k \leq p - 1$,
\[
H_{p-1-k} = \sum_{j=1}^{p-1-k} \frac{1}{j} = \sum_{j=k+1}^{p-1} \frac{1}{p-j} \equiv H_k - H_{p-1} \equiv H_k \pmod{p}.
\]
Thus
\[
\sum_{k=0}^{p-2l-1} \binom{p-2l-1}{k} \left( \frac{d}{m} \right)^k H_{2l+k} = \sum_{k=0}^{p-2l-1} \binom{p-2l-1}{k} \left( \frac{d}{m} \right)^{p-2l-1-k} H_{p-1-k}
\]
\[
\equiv \left( -\frac{m}{d} \right)^{2l} \sum_{k=0}^{p-2l-1} \binom{p-2l-1}{k} \left( \frac{m}{d} \right)^k H_k
\]
\[
\equiv \left( \frac{m}{m-d} \right)^{2l} H_{2l} + \frac{d}{d-m} \left( \frac{m}{m-d} \right)^{2l} \sum_{k=2l+1}^{p-1} \frac{1}{k} \left( \frac{d-m}{d} \right)^k \pmod{p},
\]
where we have used Lemma 2.2 with $x = -m/d$. Substituting this into (2.9) we arrive at
\[
\Sigma_1 \equiv \left( \frac{m}{m-d} \right)^{p-1} \frac{(p-1)^{2l}}{(m-d)^2} \sum_{l=0}^{(p-1)/2} \binom{2l}{l} \left( \frac{cm}{m-d} \right)^l
\]
\[
+ \frac{pd}{d-m} \sum_{l=0}^{(p-1)/2} \binom{2l}{l} \left( \frac{cm}{m-d} \right)^l \sum_{k=2l+1}^{p-1} \frac{1}{k} \left( \frac{d-m}{d} \right)^k \pmod{p^2}. \tag{2.10}
\]
Now we turn to consider $\Sigma_2$ modulo $p^2$. Note that for $l \in \{1, 2, \ldots, (p-1)/2\}$ we have
\[
\binom{p+k}{2l} = \frac{(p+k) \cdots (p+1)p(p-1) \cdots (p+k-2l+1)}{(2l)!}
\]
\[
\equiv (-1)^{k+1} \frac{p}{2l} \frac{k!(2l-1-k)!}{(2l-1)!} = \frac{p}{2l} \binom{-1}{k+1} \frac{1}{\binom{2l-1}{k}} \pmod{p^2}.
\]
So using Lemma 2.3 with $x = -d/m$ we have
\[
\Sigma_2 = \sum_{l=1}^{(p-1)/2} \binom{2l}{l} \frac{cm}{m^2} \sum_{k=p-2l}^{p-1} \frac{k+2l}{2l} \frac{d^k}{m^k} = \sum_{l=1}^{(p-1)/2} \binom{2l}{l} \frac{cm}{l^2} \sum_{k=0}^{l-1} \frac{(p+k)(d)}{2l} \frac{1}{\binom{2l-1}{k}}
\]
\[
\equiv -\frac{pd}{2m} \sum_{l=1}^{(p-1)/2} \frac{(cm)}{l} \frac{1}{\binom{2l-1}{k}} \sum_{k=0}^{l-1} \left( -\frac{d}{m} \right)^k \frac{1}{\binom{2l-1}{k}}
\]
\[
\frac{pd}{d-m} \sum_{l=1}^{(p-1)/2} \left( \begin{array}{c} 2l \\ l \end{array} \right)^2 \left( \frac{cm}{(m-d)^2} \right)^l \left( \frac{d-m}{d} \right)^k \\
+ \frac{pd(m+d)}{(d-m)^2} \sum_{l=1}^{(p-1)/2} \left( \begin{array}{c} 2l \\ l \end{array} \right)^2 \left( \frac{cm}{(m-d)^2} \right)^l \sum_{k=1}^{l} \frac{1}{k \binom{2k}{k}} \left( \frac{-(m-d)^2}{md} \right)^k 
\equiv \text{mod } p^2. \quad (2.11)
\]

If \( m = -d \), then combining (2.10) and (2.11) we have
\[
\sum_{k=0}^{p-1} \frac{T_k(b,c)^2}{m^k} \equiv \Sigma_1 + \Sigma_2 \\
\equiv 2^{p-1} \sum_{l=0}^{(p-1)/2} \left( \begin{array}{c} 2l \\ l \end{array} \right)^2 \left( -\frac{c}{4d} \right)^l + \frac{1}{2} p \sum_{l=0}^{(p-1)/2} \left( \begin{array}{c} 2l \\ l \end{array} \right)^2 \left( -\frac{c}{4d} \right)^l L_1(2) \\
\equiv \sum_{l=0}^{(p-1)/2} \left( \begin{array}{c} 2l \\ l \end{array} \right)^2 \left( -\frac{c}{4d} \right)^l \text{ (mod } p^2),
\]

where in the last step follows from Lemma 2.5. This proves (1.8).

Below we assume that \( 16mc = (m-d)^2 \). Now from (2.10) and (2.11) we have
\[
\Sigma_1 + \Sigma_2 \equiv \sum_{l=0}^{(p-1)/2} \frac{\left( \begin{array}{c} 2l \\ l \end{array} \right)^2}{16^l} \left( \frac{m-d}{m} \right)^{p-1} + \frac{pd}{d-m} \left( \frac{d-m}{d} \right) + \frac{pd(m+d)}{16mc} \sum_{l=1}^{(p-1)/2} \frac{\left( \begin{array}{c} 2l \\ l \end{array} \right)^2}{16^l} \sum_{k=1}^{l} \frac{1}{k \binom{2k}{k}} \left( \frac{-(m-d)^2}{md} \right)^k \text{ (mod } p^2). \quad (2.12)
\]

It is known from [17 Lemma 3.1] that for any \( 0 \leq l \leq (p-1)/2 \) we have
\[
\left( \begin{array}{c} (p-1)/2 \\ l \end{array} \right) \left( \begin{array}{c} (p-1)/2 + l \\ l \end{array} \right) (-1)^k \equiv \frac{\left( \begin{array}{c} 2l \\ l \end{array} \right)^2}{16^l} \text{ (mod } p^2).
\]

Therefore by Lemmas 2.3 and 2.6 we arrive at
\[
\sum_{l=1}^{(p-1)/2} \frac{\left( \begin{array}{c} 2l \\ l \end{array} \right)^2}{16^l} \sum_{k=1}^{l} \frac{1}{k \binom{2k}{k}} \left( -\frac{16c}{d} \right)^k \\
\equiv \sum_{l=1}^{(p-1)/2} \left( \begin{array}{c} (p-1)/2 \\ l \end{array} \right) \left( \begin{array}{c} (p-1)/2 + l \\ l \end{array} \right) (-1)^k \sum_{k=1}^{l} \frac{1}{k \binom{2k}{k}} \left( -\frac{(m-d)^2}{md} \right)^k \\
\equiv \frac{d-m}{d+m} \left( \frac{1}{p} \right) \left( \sum_{k=1}^{(p-1)/2} \frac{(-d/m)^k}{k} - \sum_{k=1}^{(p-1)/2} \frac{(-m/d)^k}{k} \right)
\]
\[
\equiv \frac{d - m}{d + m} \left( \frac{-1}{p} \right) \left( S_{p-1} \left( \frac{m + d}{4m} \right) - S_{p-1} \left( \frac{m + d}{4d} \right) \right) \pmod{p}.
\] (2.13)

Mortenson [9] proved that for any odd prime \( p \) we have
\[
\sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \equiv \left( \frac{-1}{p} \right) \pmod{p^2}
\] (2.14)
as conjectured by Rodriguez-Villegas [12]. Substituting (2.13) and (2.14) into (2.12) and using Lemma 2.5, we immediately obtain (1.9).

The proof of Theorem 1.1 is now complete. \qed

Proof of Corollary 1.1. We first consider (1.11) and (1.12). By (1.8), we have
\[
\sum_{k=0}^{p-1} \frac{T_k(2, 2)^2}{4^k} \equiv \sum_{k=0}^{p-1} \frac{(2k)^2}{8^k} \pmod{p^2}
\]
and
\[
\sum_{k=0}^{p-1} \frac{T_k(2, 2)^2}{(-8)^k} \equiv \sum_{k=0}^{p-1} \frac{(2k)^2}{32^k} \pmod{p^2}.
\]
Sun [18] conjectured that if \( p \equiv 1 \pmod{4} \) and \( p = x^2 + 4y^2 \) with \( x \equiv 1 \pmod{4} \), then
\[
\left( \frac{2}{p} \right) \sum_{k=0}^{p-1} \frac{(2k)^2}{8^k} \equiv \sum_{k=0}^{p-1} \frac{(2k)^2}{32^k} \equiv 2x - \frac{p}{2x}.
\]
Later, Z.-H. Sun [15] confirmed this conjecture and showed that
\[
\sum_{k=0}^{p-1} \frac{(2k)^2}{32^k} \equiv 0 \pmod{p^2}
\]
for \( p \equiv 3 \pmod{4} \). Sun [19] proved that if \( p \equiv 3 \pmod{4} \), then
\[
\sum_{k=0}^{p-1} \frac{(2k)^2}{8^k} \equiv \frac{(-1)^{(p+1)/4}2p}{(p+1)/4} \pmod{p^2}.
\]
In view of the above, we obtained (1.11) and (1.12) as desired.

Now we consider (1.13). Taking \( b = 4, c = 1 \) we have \( d = b^2 - 4c = 12 \). Solving the equation
\[(m - 12)^2 = 16m\]
we obtain \( m = 4, 36 \). Therefore, (1.13) is a special case of (1.9). By (1.9) we deduce that
\[
\sum_{k=0}^{p-1} \frac{T_k(4, 1)^2}{4^k} \equiv \left( \frac{-1}{p} \right) + \frac{3p}{2} \left( \frac{-1}{p} \right) \left( q_p(12) - q_p(4) + S_{p-1}(1) - S_{p-1} \left( \frac{1}{3} \right) \right) \pmod{p^2}.
\]
For any prime \( p > 3 \), Pan and Sun [11] proved that \( S_{p-1}(1) \equiv 0 \pmod{p} \); Sun and Tauraso [16] proved that \( S_{p-1}(1/3) \equiv q_p(3) \pmod{p} \). Combining the above, we obtain (1.13) at once. \qed
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