Shellable Drawings and the Cylindrical Crossing Number of $K_n$

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Abstract The Harary–Hill Conjecture states that the number of crossings in any drawing of the complete graph $K_n$ in the plane is at least $Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$. In this paper, we settle the Harary–Hill conjecture for shellable drawings. We say that a drawing $D$ of $K_n$ is $s$-shellable if there exist a subset $S = \{v_1, v_2, \ldots, v_s\}$ of the vertices and a region $R$ of $D$ with the following property: For all $1 \leq i < j \leq s$, if $D_{ij}$ is the drawing obtained from $D$ by removing $v_1, v_2, \ldots, v_{i-1}, v_{j+1}, \ldots, v_s$, then $v_i$ and $v_j$ are on the boundary of the region of $D_{ij}$ that contains $R$. For $s \geq \lceil n/2 \rceil$, we prove that the number of crossings of any $s$-shellable drawing of $K_n$ is at least the long-conjectured value $Z(n)$. Furthermore, we prove that all cylindrical, $x$-bounded, monotone, and 2-page drawings of $K_n$ are $s$-shellable for some $s \geq n/2$ and thus they all have at least $Z(n)$ crossings. The techniques developed provide a unified proof of the Harary–Hill conjecture for these classes of drawings.

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1 Introduction

In the late 1950s, the British artist Anthony Hill got interested in producing drawings of the complete graph $K_n$ with the least possible number of edge crossings. His general technique, explained in a paper he wrote jointly with Harary [7], is best described by drawing $K_n$ on a cylinder as follows. Draw a with $\lceil n/2 \rceil$ vertices regular $\lceil n/2 \rceil$-gon on the rim of the top lid, and a $\lfloor n/2 \rfloor$-gon on the rim of the bottom lid. Then draw the remaining edges joining vertices on the same lid using the straight line joining them across the lid. Finally, for any two vertices on distinct lids, draw the edge joining them along the geodesic that connects them on the side of the cylinder. (See Fig. 1, left, for a planar representation of such a drawing.) It is an elementary exercise to show that such a drawing of $K_n$ has exactly $Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$ crossings.

The Harary–Hill constructions are a particular instance of cylindrical drawings (see formal definition in Sect. 3).

At about the same time as the Harary–Hill paper was published, Blažek and Koman got independently interested in drawing $K_n$ with as few crossings as possible [5]. In their construction (see Fig. 1, right), they start by drawing a cycle as a regular $n$-gon, and then drawing all diagonals with positive slope (as straight line segments) and all other edges outside the cycle. The Blažek–Koman construction also yields drawings of $K_n$ with exactly $Z(n)$ crossings, and it is a particular instance of 2-page drawings (see below for the definition).

To this date, no drawing of $K_n$ with fewer than $Z(n)$ crossings is known. Moreover, all general constructions (for arbitrary values of $n$) known with exactly $Z(n)$ crossings are obtained from insubstantial alterations of either the Harary–Hill or the Blažek–Koman constructions (a few exceptions are known, but only for some small values of $n$). The tantalizingly open Harary–Hill conjecture $\text{cr}(K_n) = Z(n)$ has been confirmed only for $n \leq 12$ [10].

Fig. 1  Left Harary–Hill construction for 10 points. (A cylindrical drawing.) Right Blažek–Koman construction for 8 points (a 2-page drawing)
The main contribution of this paper is the introduction of shellable drawings, a large class of drawings for which (as we shall show) the Harary–Hill conjecture holds. Shellability captures the essential features of 2-page drawings we previously used [1,3] to prove that the 2-page crossing number of $K_n$ is $Z(n)$, and allows us to extend the lower bound to a larger family of drawings, including cylindrical, monotone, and $x$-bounded drawings (see definitions below).

If a drawing $D$ of a graph is regarded as a subset of the plane, then a region of $D$ is a connected component of $\mathbb{R}^2 \setminus D$. (If $D$ is an embedding, then the regions of $D$ are the faces). A drawing $D$ of $K_n$ is $s$-shellable if there exists a subset $S = \{v_1, v_2, \ldots, v_s\}$ of the vertices and a region $R$ of $D$ with the following property. For $1 \leq i < j \leq s$, if $D_{ij}$ denotes the drawing obtained from $D$ by removing $v_1, v_2, \ldots, v_{i-1}, v_{j+1}, v_{j+2}, \ldots, v_s$, then for all $1 \leq i < j \leq s$, the vertices $v_i$ and $v_j$ are on the boundary of the region of $D_{ij}$ that contains $R$. The set $S$ is an $s$-shelling of $D$ witnessed by $R$.

The core of this paper is the following statement, whose proof is given in Sect. 2.

**Theorem 1** Let $D$ be an $s$-shellable drawing of $K_n$, for some $s \geq \lfloor n/2 \rfloor$. Then $D$ has at least $Z(n)$ crossings.

We use this to settle the Harary–Hill conjecture for several classes of drawings:

- In a 2-page book drawing (or simply 2-page drawing), the vertices are placed on a line (the spine of the book), and each edge (except for its endvertices) lies entirely on an open halfplane spanned by the spine (one of the 2 pages of the book). (See Fig. 2, right.)

- Following Schaefer [12], in a cylindrical drawing of a graph, there are two concentric circles that host all the vertices, and no edge is allowed to intersect these circles, other than at its endvertices. (Schaefer defines cylindrical drawings only for bipartite graphs, but his definition obviously applies to arbitrary graphs). (See Fig. 2, left.)

We remark that Hill’s drawings can be naturally regarded as cylindrical drawings. Indeed, even though in Hill’s drawings the edges joining consecutive rim vertices are
placed on the rims, such drawings are easily adapted to this definition, since those edges can be drawn arbitrarily close to a rim.

- A drawing is monotone if each vertical line intersects each edge at most once. (See Fig. 3, left.)
- A drawing is $x$-bounded if by labelling the vertices $v_1, v_2, \ldots, v_n$ in increasing order of their $x$-coordinates, for all $1 \leq i < j \leq n$ the edge $v_iv_j$ is contained in the strip bounded by the vertical line that contains $v_i$ and the vertical line that contains $v_j$. (See Fig. 3, right.)

In Sect. 3, we find a condition on drawings of $K_n$ that guarantees that they are $s$-shellable for some $s \geq \lfloor n/2 \rfloor$. Then we show that if $D$ is a crossing minimal 2-page, cylindrical, monotone, or $x$-bounded drawing, then $D$ satisfies this condition, thus settling (in view of Theorem 1) the Harary–Hill conjecture for all these families of drawings. Section 4 contains some concluding remarks.

2 $k$-Edges in Shellable Drawings and Proof of Theorem 1

We recall that in a good drawing of a graph, no two edges share more than one point and no edge crosses itself. It is easy to show that every crossing minimal drawing of a graph is good.

We generalized the geometrical concept of a $k$-edge to arbitrary (topological) good drawings of $K_n$ [1,3], as follows. Let $D$ be a good drawing of $K_n$, $pq$ a directed edge of $D$, and $r$ a vertex of $D$ distinct from $p$ and $q$. Then $pqr$ denotes the oriented closed curve defined by concatenating the edges $pq$, $qr$, and $rp$. An oriented, simple, and closed curve in the plane is oriented counterclockwise (respectively clockwise) if the bounded region it encloses is on the left (respectively right) hand side of the curve. Further, $r$ is on the left (respectively right) side of $pq$ if $pqr$ is oriented counterclockwise (respectively clockwise). We say that the edge $pq$ is a $k$-edge of $D$ if it has exactly $k$ points of $D$ on one side (left or right), and thus $n - 2 - k$ points on the other side. Hence, as in the geometric setting, a $k$-edge is also an $(n - 2 - k)$-edge. The direction
of the edge $pq$ is no longer relevant and every edge of $D$ is a $k$-edge for some unique $k$ such that $0 \leq k \leq \lfloor n/2 \rfloor - 1$. Finally, we denote by $E_k(D)$ the number of $k$-edges of the drawing $D$.

Following our previous work [1,3], if $D$ is a good drawing of $K_n$, then for each $0 \leq k \leq \lfloor n/2 \rfloor - 1$ we define the set of $\leq k$-edges of $D$ as all $j$-edges in $D$ for $j = 0, \ldots, k$. The number of $\leq k$-edges of $D$ is denoted by

$$E_{\leq k}(D) := \sum_{j=0}^{k} E_j(D).$$

Similarly, we denote the number of $\leq k$-edges of $D$ by

$$E_{\leq k}(D) := \sum_{j=0}^{k} E_{\leq j}(D) = \sum_{j=0}^{k} \sum_{i=0}^{j} E_i(D) = \sum_{i=0}^{k} (k+1-i) E_i(D). \quad (1)$$

It is known [1,3] that if $D$ is a good drawing, then $D$ has exactly

$$2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq k}(D) - \frac{1}{2} \binom{n}{2} \left\lfloor \frac{n-2}{2} \right\rfloor - \frac{1}{2} (1 + (-1)^n) E_{\leq \lfloor n/2 \rfloor - 2}(D) \quad (2)$$
crossings. Thus we now concentrate on bounding $E_{\leq k}(D)$. We need a few more definitions. If $D_y$ is the drawing of $K_{n-1}$ obtained from $D$ by deleting a vertex $y$, then an edge non-incident to $y$ is $(D, D_y)$-invariant if for some $0 \leq k \leq \lfloor (n-3)/2 \rfloor$ it is a $k$-edge in both $D$ and $D_y$. We let $E_{\leq k}(D, D_y)$ denote the number of $(D, D_y)$-invariant $\leq k$-edges.

### 2.1 $k$-Edges Incident to a Boundary Point

The unbounded region of a drawing $D$ is its unique region with noncompact closure. We refer to the topological boundary of the unbounded region of $D$ simply as the boundary of $D$.

Let $D$ be a good drawing of $K_n$ and assume that $x$ is a vertex on the boundary of $D$. We define a linear order for the vertices adjacent to $x$ using the rotation of vertex $x$ (the cyclic order of the edges incident to $x$): Let $\Omega$ be a small enough topological disk containing $x$. Because $x$ is on the boundary of $D$, there are exactly two vertices, say $y$ and $z$, such that $xy \cap \Omega$ and $xz \cap \Omega$ are on the boundary of $D$. Suppose without loss of generality that the triangle $xyz$ is oriented counter-clockwise. Then we can label the vertices adjacent to $x$ by $x_1, x_2, \ldots, x_{n-1}$ so that $x_1 = y$, $x_{n-1} = z$. We refer to this as the order induced by $x$ in $D$.

**Proposition 2** Let $n \geq 1$ and consider a good drawing $D$ of the complete graph $K_n$. Let $x$ be a vertex on the boundary of $D$, and let $x_1, x_2, \ldots, x_{n-1}$ be the order induced by $x$ in $D$. Then $xx_i$ and $xx_{n-i}$ are two distinct $(i-1)$-edges of $D$ for $1 \leq i \leq \lfloor (n-1)/2 \rfloor$. 

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Fig. 4 The order induced by \( x \)

**Proof** Consider a disk \( \Omega \) as above. Then any point \( p \) in \( \Omega \) and outside the triangle \( xyz \) is in the unbounded region of \( D \). (See Fig. 4.) This means that \( p \) cannot be in the interior of any triangle of \( D \). In particular, if \( j < i \), then the triangle \( xx_jx_i \) is oriented counter-clockwise as otherwise its interior would contain \( p \). This means that \( x_j \) is to the right of \( xx_i \) if \( j < i \), and to the left if \( j > i \). Thus there are exactly \( i - 1 \) vertices to the right of \( xx_i \) and \( n - 1 - i \) to the left. This means that \( xx_i \) is a \( \min(i - 1, n - 1 - i) \)-edge of \( D \), implying the result.

**Proposition 3** Let \( 0 \leq i - 1 \leq k \leq \lfloor (n - 3)/2 \rfloor \), \( D \) a good drawing of the complete graph \( K_n \), and \( x \) and \( y \) vertices of \( D \). Let \( U \) be a subset of \( i - 1 \) vertices of \( D \) not including \( x \) and \( y \). Assume that \( x \) is on the boundary of the drawing \( D(U) \) obtained from \( D \) by removing \( U \). Then there exist at least \( k - i + 2 \) edges incident to \( x \) and non-incident to vertices in \( U \) that are \( (D, D_y) \)-invariant \( \leq k \)-edges.

**Proof** Consider the order \( x_1, x_2, \ldots, x_{n-1} \) induced by \( x \) in \( D(U) \). As before, \( x_\ell \) is to the right of \( xx_j \) if \( \ell < j \), and to the left if \( \ell > j \). Thus there are exactly \( j - 1 \) vertices in \( D(U) \) to the right of \( xx_j \) and \( n - i - j \) to the left. Including \( U \), this means that there are at most \( i - 1 + j - 1 = i + j - 2 \) vertices to the right of \( xx_j \) in \( D \) and at most \( i - 1 + n - i - j = n - j - 1 \) to the left.

Now consider the point \( y \), which is equal to \( x_w \) for some \( 1 \leq w \leq n - i \). If \( w > k + 2 - i \), then for \( 1 \leq j \leq k + 2 - i \) the edge \( xx_j \) has at most \( i + j - 2 \leq i + (k + 2 - i) - 2 = k \) points to its right and \( y \) on its left (because \( w > k + 2 - i \geq j \)). If \( w \leq k + 2 - i \), then for \( n - k - 1 \leq j \leq n - i \) the edge \( xx_j \) has at most \( n - j - 1 \leq n - (n - k - 1) - 1 = k \) points to its left and \( y \) on its right (because \( k \leq (n - 3)/2 < (n - 3 + i)/2 \) and thus \( w \leq k + 2 - i < n - k - 1 \leq j \)). In either case, the \( k + 2 - i \) edges \( xx_j \) are \( (D, D_y) \)-invariant \( \leq k \)-edges.

2.2 Bounding the Number of \( \leq k \)-Edges in Shellable Drawings of \( K_n \)

We now bound the number of \( \leq k \)-edges of \( s \)-shellable drawings of \( K_n \) for a certain interval of \( k \) determined by \( s \).

**Proposition 4** Let \( D \) be an \( s \)-shellable good drawing of the complete graph \( K_n \), in which the region \( R \) that witnesses the \( s \)-shellability of \( D \) is its unbounded region. Then \( E_{\leq k}(D) \geq 3\left(\frac{k + 3}{3}\right) \) for all \( 0 \leq k \leq \min(s - 2, \lfloor (n - 3)/2 \rfloor) \).

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Proof The statement is vacuous for \( n \leq 2 \). We assume \( n \geq 3 \). Let \( V \) be the set of vertices of \( D \) and \( S = \{v_1, v_2, \ldots, v_s\} \) an \( s \)-shelling of \( D \) witnessed by the unbounded region \( R \). Fix \( k \) with \( 0 \leq k \leq \min(s - 2, [(n - 3)/2]) \). We prove that

\[
E_{\leq i}(D_{1,s-k+i}) \geq 3\left(\frac{i+3}{3}\right)
\]

for \( 0 \leq i \leq k \) by induction on \( i \). For \( i = 0 \), because \( S \) is an \( s \)-shelling of \( D \), and the unbounded region witnesses this \( s \)-shellability, it follows that \( v_1 \) and \( v_{s-k} \) are on the boundary of \( D_{1,s-k} \). By Proposition 2 each of these two vertices (they are different because \( k \leq s - 2 \)) is incident to two 0-edges and they can share at most one 0-edge. That is, \( E_{\leq 0}(D_{1,s-k}) \geq 3 \). We now compare the following two identities obtained from (1). For \( 1 \leq r \leq s \) and \( 0 \leq k' \leq [(n - s + r)/2] \),

\[
E_{\leq k'}(D_{1,r}) = \sum_{j=0}^{k'} (k' + 1 - j) E_j(D_{1,r})
\]

and

\[
E_{\leq k'-1}(D_{1,r-1}) = \sum_{j=0}^{k'-1} (k' - j) E_j(D_{1,r-1}).
\]

As shown in our previous work [2], for a \( j \leq k' \) a \( j \)-edge incident to \( v_r \) contributes \( k' + 1 - j \) to (4) and nothing to (5), a \( (D_{1,r}, D_{1,r-1}) \)-invariant edge contributes 1 more to (4) than to (5), and all other edges contribute the same to (4) and (5). Therefore,

\[
E_{\leq k'}(D_{1,r}) = E_{\leq k'-1}(D_{1,r-1}) + \sum_{\ell=0}^{k'} (k' + 1 - \ell) e_\ell(v_r) + E_{\leq k'}(D_{1,r}, D_{1,r-1}),
\]

where \( e_\ell(r) \) is the number of \( \ell \)-edges incident to \( v_r \) in \( D_{1,r} \). Now, choose \( i \) such that \( 1 \leq i \leq k \) and assume that

\[
E_{\leq i-1}(D_{1,s-k+i-1}) \geq 3\left(\frac{i+2}{3}\right).
\]

By (6) for \( k' = i \) and \( r = s - k + i \), we have that

\[
E_{\leq i}(D_{1,s-k+i}) = E_{\leq i-1}(D_{1,s-k+i-1}) + \sum_{\ell=0}^{i} (i + 1 - \ell) e_\ell(v_{s-k+i})
\]

\[+ E_{\leq i}(D_{1,s-k+i}, D_{1,s-k+i-1}).
\]

We separately bound each term of the right-hand side of (8). The first term is bounded in (7). For the second term, Proposition 2 (for \( x = v_{s-k+i} \) is on the boundary of
\[ D_{1,s-k+i} \] implies that \( e_\ell(v_{s-k+i}) = 2 \) and thus

\[
\sum_{\ell=0}^{i} (i + 1 - \ell) e_\ell(v_{s-k+i}) = \sum_{\ell=0}^{i} (i + 1 - \ell) 2 = 2 \binom{i + 2}{2}.
\]

Finally, we show that

\[
E_{\leq i}(D_{1,s-k+i}, D_{1,s-k+i-1}) \geq \sum_{\ell=1}^{i+1} (i - \ell + 2) = \binom{i + 2}{2}.
\]

We use Proposition 3 for the drawing \( D_{1,s-k+i}, x = v_\ell, y = v_{s-k+i} \), and \( U = \{v_1, v_2, \ldots, v_{\ell-1}\} \). Note that \( k \leq s - 2 \) implies \( 1 \leq \ell \leq i + 1 < s - k + i \) and thus \( v_\ell \) and \( v_{s-k+i} \) are different and do not belong to \( \{v_1, v_2, \ldots, v_{\ell-1}\} \). Moreover, \( v_\ell \) and \( v_{s-k+i} \) are on the boundary of \( D(U) = D_{i,s-k+i} \) because \( S \) is an \( s \)-shelling of \( D \).

Also, \( D_{1,s-k+i} \) has \( n - s + (s - k + i) = n - k + i \) vertices and thus we must check that

\[ 0 \leq \ell - 1 \leq i \leq (n-k+i-3)/2. \]

The first two inequalities hold because \( 1 \leq \ell \leq i + 1 \). The last inequality follows from \( k \leq \min(s-2, [(n-3)/2]) \leq [(n-3)/2]/2 \), which implies \( k + i \leq 2k \leq n - 3 \). Therefore, Proposition 3 implies that for \( 1 \leq \ell \leq i + 1 \) there are at least \( i - \ell + 2 \) edges incident to \( v_\ell \) and non-incident to \( v_1, v_2, \ldots, v_{\ell-1} \) (so all these edges are different) that are \((D_{1,s-k+i}, D_{1,s-k+i-1})\)-invariant \( \leq i \)-edges. \( \square \)

2.3 Proof of Theorem 1

Let \( D \) be an \( s \)-shellable drawing of \( K_n \), for some \( s \geq \lfloor n/2 \rfloor \). By using a suitable inversion, if needed, we transform \( D \) into a drawing \( D' \), with the same number of crossings as \( D \), such that the region that witnesses the \( s \)-shellability of \( D' \) is the unbounded region. Since \( \min(s - 2, [(n - 3)/2]) \geq [(n/2) - 2 \), it follows from Proposition 4 that \( E_{\leq k}(D') \geq 3^{(k+3)/3} \) for all \( 0 \leq k \leq [(n/2) - 2 \). Since \( D' \) is a good drawing, then by (2) \( D' \) has exactly

\[
2 \sum_{k=0}^{\lfloor n/2 \rfloor - 2} E_{\leq k}(D') - \frac{1}{2} \binom{n}{2} \left[ \frac{n-2}{2} \right] - \frac{1}{2} (1 + (-1)^n) E_{\leq \lfloor n/2 \rfloor - 2}(D')
\]

crossings. Using this fact, a straightforward calculation [1,3] shows that if \( D' \) is a drawing of \( K_n \) that satisfies \( E_{\leq k}(D') \geq 3^{(k+3)/3} \) for all \( 0 \leq k \leq [(n - 3)/2] \), then \( D' \) has at least \( Z(n) \) crossings. \( \square \)

3 Verifying the Harary–Hill Conjecture for 2-Page, Cylindrical, Monotone, and \( x \)-Bounded Drawings

The workhorse of this section is a property of a drawing that guarantees its shellability:
Lemma 5 Let $D$ be a drawing of $K_n$. Suppose that $C = v_1v_2\ldots v_s$ is a cycle that satisfies the following: (i) the edge $v_sv_1$ has no crossings; and (ii) for $k = 1, \ldots, s-1$ all crossings in the edge $v_kv_{k+1}$ involve edges $v_i v_j$ with $i < k$ and $j > k + 1$. Then $D$ is $s$-shellable.

Proof Let $R$ be a region of $D$ containing the edge $v_sv_1$ on its boundary. Let $1 \leq i < j \leq s$ and define $D_{ij}$ as before. Let $R'$ be the region of $D_{ij}$ that contains $R$. Since the vertices $v_1, v_2, \ldots, v_{i-1}, v_{j+1}, v_{j+2}, \ldots, v_s$, and consequently any edge incident to one of these vertices, are removed to obtain $D_{ij}$, then $v_1$ and $v_s$ are in the interior of $R'$. Moreover, it follows from the crossing properties of the edges of $C$ that the edges $v_1v_2, v_2v_3, \ldots, v_{i-1}v_i, v_jv_{j+1}, v_{j+1}v_{j+2}, \ldots, v_{s-1}v_s$ are not intersected by any edge of $D_{ij}$. Hence the paths $v_i, v_{i-1}, \ldots, v_1$ and $v_j, v_{j+1}, \ldots, v_s$ are completely contained in $R'$ and thus $v_i$ and $v_j$ are on the boundary of $R$. Therefore, $\{v_1, v_2, \ldots, v_s\}$ is an $s$-shelling of $D$ witnessed by $R$. $\square$

We need the full strength of Lemma 5 to show that monotone and $x$-bounded drawings satisfy the Harary–Hill conjecture. However, it seems worth stating the following weaker form, which is all we need to show that the Harary–Hill conjecture holds for 2-page and cylindrical drawings:

Corollary 6 If a drawing $D$ of $K_n$ has a crossing-free cycle $C$ of size $s$ then $D$ is $s$-shellable.

We are finally ready to verify the Harary–Hill conjecture for several classes of drawings.

Theorem 7 Every cylindrical drawing of $K_n$ has at least $Z(n)$ crossings.

Proof Let $D$ be a crossing-minimal cylindrical drawing of $K_n$. Out of the two concentric cycles that contain all the vertices, let $\rho$ be one that contains at least $n/2$ vertices. Let $v_1, v_2, \ldots, v_s$ be the vertices on $\rho$, in counterclockwise order. Since no two edges cross each other more than once (this follows since $D$ is crossing-minimal) and no edge crosses $\rho$, it follows that the cycle $v_1v_2\ldots v_sv_1$ is uncrossed in $D$. Since $s \geq n/2$, the result follows by Theorem 1 and Corollary 6. $\square$

A 2-page drawing is a particular kind of a cylindrical drawing, namely, a degenerate one with all vertices on one of the concentric circles. Thus Theorem 7 immediately implies our previous result [1,3] for 2-page drawings.

Corollary 8 Every 2-page drawing of $K_n$ has at least $Z(n)$ crossings.

It is straightforward to check that any $x$-bounded drawing $D$ of $K_n$ satisfies the conditions of Lemma 5. Thus the Harary–Hill conjecture holds for $x$-bounded drawings (and hence for monotone drawings, as every monotone drawing is obviously $x$-bounded)

Theorem 9 Every $x$-bounded (and so, in particular, any monotone) drawing of $K_n$ has at least $Z(n)$ crossings. $\square$

It is worth nothing that the statement for $x$-bounded drawings can also be derived from the result for monotone drawings (previously proved by the authors [2] and by Balko et al. [4]), since $x$-bounded drawings can be easily reduced to monotone drawings [6].
4 Concluding Remarks

Cylindrical drawings of $K_n$ were previously investigated by Richter and Thomassen [11]. In that paper, they determined the number of crossings in a cylindrical drawing of $K_m,m$ with one chromatic class on the inner circle and the other chromatic class on the outer circle. From their result it follows that a cylindrical drawing of $K_{2m}$ in which the edges joining vertices on the same circle are not drawn on the annulus (bounded by the two circles) has at least $Z(2m)$ crossings.

As we observed in Sect. 1, the 2-page and the cylindrical constructions (possibly with some insubstantial alterations) are the only known drawings of $K_n$ with $Z(n)$ crossings for arbitrary values of $n$. In his interesting entry at mathoverflow.net, Kynčl [8] asks about the existence of alternative constructions, and observes that there is a plethora of drawings with $Z(n) + O(n^3)$ crossings (noting that Moon [9] showed that a random spherical drawing of $K_n$ has the expected crossing number

$$(1/64)n(n - 1)(n - 2)(n - 3) = Z(n) + O(n^3)).$$

Balko et al. [4] noted that there are cylindrical drawings $D$ that do not satisfy the bound $E_{\leq k}(D) \geq 3(k+3)^3$. However, as shown in this paper, for every such drawing there exists a second drawing $D'$ obtained from $D$ by an appropriate inversion (and thus with the same number of crossings) that satisfies $E_{\leq k}(D') \geq 3(k+3)^3$.

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