The Cartan Model for Equivariant Cohomology

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Abstract

In this article, we will discuss a new operator $d_C$ on $W(g) \otimes \Omega^*(M)$ and to construct a new Cartan model for equivariant cohomology. We use the new Cartan model to construct the corresponding BRST model and Weil model, and discuss the relations between them.

1 Introduction

The standard Cartan model for equivariant cohomology is construction on the algebra $W(g) \otimes \Omega^*(M)$ with operator

$$ d_C \phi^i = 0, \phi^i \in S(g^*), i = 1, \cdots, n; $$

$$ d_C \eta = (1 \otimes d - \sum_{b=1}^n \phi^b \otimes \iota_b) \eta, \eta \in \Omega^*(M), $$

where $\iota_b$ is $\iota_{e_b}$ (see [4],[5],[7],[8]). We can also introduce a new operator on $W(g) \otimes \Omega^*(M)$ by

$$ d_C \phi^i = 0, \phi^i \in S(g^*), i = 1, \cdots, n; $$

$$ d_C \eta = (1 \otimes d - \sum_{b=1}^n \phi^b \otimes (\iota_b + \sqrt{-1} f_b^a \iota_a)) \eta, \eta \in \Omega^*(M) \otimes \mathbb{C}, $$

where $\iota_b$ is $\iota_{e_b}$. In this article we construct the new model for equivariant cohomology which also called Cartan model. The idea comes form the article [3]. We also use the new Cartan model to construct the corresponding BRST model and Weil model.

2 Cartan model

Let $G$ ba a compact Lie group with Lie algebra $g$, $g^*$ be the dual of $g$. We known the Weil algebra is

$$ W(g) = \wedge(g^*) \otimes S(g^*). $$

The contraction $i_X$ and the exterior derivative $d_W$ on $W(g)$ defined as follow:

Choose a basis $e_1, \cdots, e_n$ for $g$ and let $e_1^*, \cdots, e_n^*$ be the dual basis of $g^*$. Let $\theta^1, \cdots, \theta^n$ be the dual basis of $g^*$ generating the exterior algebra $\wedge(g^*)$ and let $\phi^1, \cdots, \phi^n$ be the dual basis of $g^*$ generating the symmetric algebra $S(g^*)$. Let $c^i_{jk}$ be the structure constants of
that is \([e_j, e_k] = \sum_{i=1}^{n} c^i_{jk} e_i\). We kown that \(S(\mathfrak{g}^*)\) is identified with the polynomial ring \(\mathbb{C}[\phi^1, \cdots, \phi^n]\).

Define the contraction \(i_X\) on \(W(\mathfrak{g})\) for any \(X \in \mathfrak{g}\) by
\[
i_e(\theta^s) = \delta^s_r, \quad i_e(\phi^s) = 0
\]
for all \(r, s = 1, \cdots, n\) and extending by linearity and as a derivation.

Define \(d_W\) by
\[
d_W\theta^i = -\frac{1}{2} \sum_{j,k} c^i_{jk} \theta^j \wedge \theta^k + \phi^i
\]
and
\[
d_W\phi^i = - \sum_{j,k} c^i_{jk} \theta^j \phi^k
\]
and extending \(d_W\) to \(W(\mathfrak{g})\) as a derivation.

The Lie derivative on \(W(\mathfrak{g})\) is defined by
\[
L_X = d_W \cdot i_X + i_X \cdot d_W.
\]

**Lemma 1.** \(L_e\theta^i = -\sum_k c^i_{ik} \theta^k\) and \(L_e\phi^j = -\sum_k c^j_{ik} \phi^k\).

**Proof.** Because
\[
L_e\theta^j = (d_W \cdot i_{e_k} + i_{e_k} \cdot d_W) \theta^j = i_{e_k}(\frac{1}{2} \sum_{i,k} c^i_{jk} \theta^j \wedge \theta^k + \phi^j) = - \sum_{k} c^j_{ik} \theta^k
\]
\[
L_e\phi^j = (d_W \cdot i_{e_k} + i_{e_k} \cdot d_W) \phi^j = i_{e_k}(\sum_{i,k} c^j_{ik} \phi^k) = - \sum_{k} c^j_{ik} \phi^k
\]

**Lemma 2.** The operators \(i_X, d_W, L_X\) on \(W(\mathfrak{g})\) satisfy the following identities:

1. \(d_W^2 = 0\);
2. \(L_X \cdot d_W - d_W \cdot L_X = 0\), for any \(X \in \mathfrak{g}\);
3. \(i_X i_Y + i_Y i_X = 0\), for any \(X, Y \in \mathfrak{g}\);
4. \(L_X i_Y - i_Y L_X = i_{[X,Y]}\), for any \(X, Y \in \mathfrak{g}\);
5. \(L_X L_Y - L_Y L_X = L_{[X,Y]}\), for any \(X, Y \in \mathfrak{g}\);
6. \(d_W i_X + i_X d_W = L_X\), for any \(X \in \mathfrak{g}\).

**Proof.** see [4].

So, there is a complex \((W(\mathfrak{g}), d_W)\), the cohomology of \((W(\mathfrak{g}), d_W)\) is trivial (see [5]), i.e. \(H^*(W(\mathfrak{g})) \cong \mathbb{R}\).

Let \(M\) be a smooth closed manifold with \(G\) acting smoothly on the left. Let \(X^M\) be the vector field generated by the Lie algebra element \(X \in \mathfrak{g}\) given by
\[
(X^M f)(x) = \frac{d}{dt}f(\exp(-tX) \cdot x) \Big|_{t=0}.
\]

Set \(d, \iota_X^M, \mathcal{L}_X^M\) be the exterior derivative, contraction and Lie derivative on \(\Omega^*(M)\). Denote \(\iota_X = \iota_X^M\) and \(\mathcal{L}_X = \mathcal{L}_X^M\) acting on \(\Omega^*(M)\).
Definition 1. The Cartan model is defined by the algebra
\[ S(\mathfrak{g}^*) \otimes \Omega^*(M) \]
and the differential
\[ d_C \phi^i = 0, \phi^i \in S(\mathfrak{g}^*), i = 1, \cdots, n; \]
\[ d_C \eta = (1 \otimes d - \sum_{i=1}^{n} \phi^i \otimes (\iota_i + \sqrt{-1} f^j_i \iota_j))\eta, \eta \in \Omega^*(M) \otimes \mathbb{C}, \]
where \( \iota_i \) is \( \iota_{e_i} \) and \( f^j_i \in \mathbb{R} \). The operator \( d_C \) is called the equivariant exterior derivative.

Its action on forms \( \alpha \in S(\mathfrak{g}^*) \otimes \Omega^*(M) \) is
\[ (d_C \alpha)(X) = (d - \iota_X - \sqrt{-1} \iota_Y)(\alpha(X)) \]
where \( X^M = \mathfrak{c}^i X^i_M \) is the vector field on \( M \) generated by the Lie algebra element \( X = \mathfrak{c}^i e_i \in \mathfrak{g}, Y^M = f^j_i \mathfrak{c}^j X^M_i \) (see [2]). In the article [3] we use the operator \( d + \iota_X + \sqrt{-1} \iota_Y \) to construct an complex \((\Omega^*(M) \otimes \mathbb{C}, d + \iota_X + \sqrt{-1} \iota_Y)\) and cohomology group \( H^*_X + \sqrt{-1} Y(M) \), we can do it in the same way by the operator \( d - \iota_X - \sqrt{-1} \iota_Y \).

Lemma 3.
\[ d_C^2 = - \sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f^j_i \mathcal{L}_j) \]

Proof. By the lemma 2. we have
\[ d_C^2 = (1 \otimes d - \sum_{i=1}^{n} \phi^i \otimes (\iota_i + \sqrt{-1} f^j_i \iota_j))(1 \otimes d - \sum_{i=1}^{n} \phi^i \otimes (\iota_i + \sqrt{-1} f^j_i \iota_j)) \]
\[ = - \sum_{i=1}^{n} \phi^i \otimes [d(\iota_i + \sqrt{-1} f^j_i \iota_j)) + (\iota_i + \sqrt{-1} f^j_i \iota_j))d] \]
\[ = - \sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f^j_i \mathcal{L}_j) \]

Let \((S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\mathcal{G}}\) be the subalgebra of \( S(\mathfrak{g}^*) \otimes \Omega^*(M) \) which satisfied
\[ (\sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f^j_i \mathcal{L}_j))\alpha = 0, \forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\mathcal{G}} \]
So we get the complex \((S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\mathcal{G}}, d_C)\). The equivariantly closed form is \( \forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\mathcal{G}} \) with \( d_C \alpha = 0 \), the equivariantly exact form is \( \forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\mathcal{G}} \) there is \( \beta \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\mathcal{G}} \) with \( \alpha = d_C \beta \).

As in [8] we can define the equivariant connection
\[ \nabla_{\mathfrak{g}} = 1 \otimes \nabla - \sum_{i=1}^{n} \phi^i \otimes (\iota_i + \sqrt{-1} f^j_i \iota_j) \]
and the equivariant curvature of the connection
\[ F_{\mathfrak{g}} = (\nabla_{\mathfrak{g}})^2 + \sum_{i=1}^{n} \phi^i \otimes (\mathcal{L}_i + \sqrt{-1} f^j_i \mathcal{L}_j) \]
3 BRST model

This section is inspired by [5]. First, we will to construct the BRST differential algebra. The algebra is

\[ B = W(g) \otimes \Omega^*(M). \]

The BRST operator is

\[ \delta = d_W \otimes 1 + 1 \otimes d + \sum_{i=1}^{n} \theta^i \otimes (L_i + \sqrt{-1} f^{i}_{j} l_j) - \sum_{a=1}^{n} \phi^a \otimes (\tau_a + \sqrt{-1} f^{a}_{k} t_k) + \frac{1}{2} \sum_{j,k} c^{j}_{kj} \theta^j \theta^k \otimes (\tau_j + \sqrt{-1} f^{j}_{i} l_j) \]

\[ - \sum_{j<k} \theta^j \theta^k \otimes ((L_j + \sqrt{-1} f^{j}_{k} L_k)(\tau_k + \sqrt{-1} f^{k}_{i} l) - (\tau_j + \sqrt{-1} f^{k}_{j} l_j)(L_k + \sqrt{-1} f^{k}_{i} L_i)) \]

where \( L_i \) is \( L_{e_i} \) and \( \tau_a \) is \( \tau_{e_a} \).

Lemma 4. On the algebra \( W(g) \otimes \Omega^*(M) \), we have \( \delta^2 = 0 \).

Proof. By computation, we have

\[ \delta = \exp(\sum_{i=1}^{n} \theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{i} l_i))(d_W \otimes 1 + 1 \otimes d) \exp(-\sum_{i=1}^{n} \theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{i} l_i)) \]

where \( \tau_a \) is \( \tau_{e_a} \). So we have

\[ \delta^2 = \exp(\sum_{i=1}^{n} \theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{i} l_i))(d_W \otimes 1 + 1 \otimes d) \exp(-\sum_{i=1}^{n} \theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{i} l_i)) \cdot \]

\[ \exp(\sum_{i=1}^{n} \theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{i} l_i))(d_W \otimes 1 + 1 \otimes d) \exp(-\sum_{i=1}^{n} \theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{i} l_i)) \]

\[ = \exp(\sum_{i=1}^{n} \theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{i} l_i))(d_W \otimes 1 + 1 \otimes d)^2 \exp(-\sum_{i=1}^{n} \theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{i} l_i)) \]

\[ = 0 \]

So we get the BRST differential algebra \( (W(g) \otimes \Omega^*(M), \delta) \).

Lemma 5. Fixing the index \( i \) and \( k \)

\[ (\theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{j} l_j))(\theta^k \otimes (\tau_k + \sqrt{-1} f^{k}_{j} l_k)) = (\theta^k \otimes (\tau_k + \sqrt{-1} f^{k}_{i} l_i))(\theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{i} l_i)) \]

Proof. If \( i = k \), we have

\[ (\theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{i} l_i))(\theta^k \otimes (\tau_k + \sqrt{-1} f^{k}_{i} l_k)) = 0 = (\theta^k \otimes (\tau_k + \sqrt{-1} f^{k}_{i} l_k))(\theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{i} l_i)) \]

If \( i \neq k \), then because

\[ (\theta^i \otimes \tau_i)(\theta^k \otimes \tau_k) = -\theta^i \wedge \theta^k \otimes \tau_i \tau_k = -\theta^k \wedge \theta^i \otimes \tau_k \tau_i = (\theta^k \otimes \tau_k)(\theta^i \otimes \tau_i) \]

\[ (\theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{j} l_j))(\theta^k \otimes \tau_k) = -\theta^i \wedge \theta^k \otimes (\tau_i + \sqrt{-1} f^{i}_{j} l_j) = -\theta^k \wedge \theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{j} l_j) = (\theta^k \otimes \tau_k)(\theta^i \otimes (\tau_i + \sqrt{-1} f^{i}_{j} l_j)) \]

So we get the result. \( \square \)
Let \( \psi : W(\mathfrak{g}) \otimes \Omega^*(M) \to W(\mathfrak{g}) \otimes \Omega^*(M) \) be the map
\[
\psi = \prod_i (1 - \theta^i \otimes (\iota_i + \sqrt{-1}f^j_i t_j)).
\]

By computation
\[
(1 - \theta^1 \otimes (\iota_1 + \sqrt{-1}f^j_1 t_j))(1 - \theta^2 \otimes (\iota_2 + \sqrt{-1}f^j_2 t_j)) \cdots (1 - \theta^n \otimes (\iota_n + \sqrt{-1}f^j_n t_j))
\]
we have
\[
\psi = \exp(-\sum_{i=1}^n \theta^i \otimes (\iota_i + \sqrt{-1}f^j_i t_j)).
\]

In the section 5. we will discuss the map \( \psi \).

### 4 Weil model

The exterior derivative operator on \( W(\mathfrak{g}) \otimes \Omega^*(M) \) is defined by
\[
D \doteq dW \otimes 1 + 1 \otimes d,
\]
the contraction operator is defined by
\[
\tilde{i}_X \doteq i_X \otimes 1 + 1 \otimes \iota_X
\]
and Lie derivative be defined by
\[
\tilde{L}_X \doteq L_X \otimes 1 + 1 \otimes \mathcal{L}_X
\]

**Lemma 6.** The operators \( \tilde{i}_X, D, \tilde{L}_X \) on \( W(\mathfrak{g}) \otimes \Omega^*(M) \) satisfy the following identities:

1. \( D^2 = 0 \);
2. \( \tilde{L}_X \cdot D - D \cdot \tilde{L}_X = 0 \), for any \( X \in \mathfrak{g} \);
3. \( \tilde{i}_X \tilde{i}_Y + \tilde{i}_Y \tilde{i}_X = 0 \), for any \( X, Y \in \mathfrak{g} \);
4. \( \tilde{L}_X \tilde{i}_Y - \tilde{i}_Y \tilde{L}_X = \tilde{i}_{[X,Y]} \), for any \( X, Y \in \mathfrak{g} \);
5. \( \tilde{L}_X \tilde{L}_Y - \tilde{L}_Y \tilde{L}_X = \tilde{L}_{[X,Y]} \), for any \( X, Y \in \mathfrak{g} \);
6. \( \tilde{L}_X = D \cdot \tilde{i}_X + \tilde{i}_X \cdot D \), for any \( X \in \mathfrak{g} \).

**Proof.** see [4]. \( \square \)

Set
\[
\tilde{i}_{X+\sqrt{-1}Y} \doteq i_X \otimes 1 + 1 \otimes (\iota_X + \sqrt{-1}\iota_Y)
\]
be the contraction operator on \( W(\mathfrak{g}) \otimes \Omega^*(M) \) induced by the contraction of \( X + \sqrt{-1}Y \).

Set
\[
\tilde{L}_{X+\sqrt{-1}Y} \doteq L_X \otimes 1 + 1 \otimes (\mathcal{L}_X + \sqrt{-1}\mathcal{L}_Y)
\]
be the Lie derivative on \( W(\mathfrak{g}) \otimes \Omega^*(M) \) about \( X + \sqrt{-1}Y \).
Lemma 7.  
\[ \tilde{L}_{X+\sqrt{-1}Y} = D \cdot i_{X+\sqrt{-1}Y} + \tilde{i}_{X+\sqrt{-1}Y} \cdot D \]
for any \( X, Y \in g \).

Proof.
\[
D \cdot i_{X+\sqrt{-1}Y} + \tilde{i}_{X+\sqrt{-1}Y} \cdot D = (d_W \otimes 1 + 1 \otimes d) \cdot i_{X+\sqrt{-1}Y} + \tilde{i}_{X+\sqrt{-1}Y} \cdot (d_W \otimes 1 + 1 \otimes d)
\]
\[
= d_W i_X \otimes 1 + i_X d_W \otimes 1 + 1 \otimes d(i_X + \sqrt{-1}i_Y) + 1 \otimes (i_X + \sqrt{-1}i_Y)d
\]
\[
= L_X \otimes 1 + 1 \otimes (L_X + \sqrt{-1}L_Y)
\]
\[
= \tilde{L}_{X+\sqrt{-1}Y}
\]
\[ \square \]

Definition 2. An element \( \eta \in W(g) \otimes \Omega^*(M) \) is basic if it satisfies \( \tilde{i}_{X+\sqrt{-1}Y} \eta = 0 \), \( \tilde{L}_{X+\sqrt{-1}Y} \eta = 0 \) for any \( X, Y \in g \). Set \( (W(g) \otimes \Omega^*(M))_{bas} \) be the set of basic elements.

Lemma 8. The operator \( D \) preserves \( (W(g) \otimes \Omega^*(M))_{bas} \).

Proof. Set \( \eta \in (W(g) \otimes \Omega^*(M))_{bas} \), then \( \tilde{i}_{X+\sqrt{-1}Y} \eta = 0 \) and \( \tilde{L}_{X+\sqrt{-1}Y} \eta = 0 \) for any \( X, Y \in g \). So by Lemma 7., we have
\[
(\tilde{i}_{X+\sqrt{-1}Y} \cdot D)\eta = \tilde{i}_{X+\sqrt{-1}Y}(D\eta) = \tilde{L}_{X+\sqrt{-1}Y} \eta - D(\tilde{i}_{X+\sqrt{-1}Y} \eta) = 0
\]
for any \( X, Y \in g \).

And
\[
\tilde{L}_{X+\sqrt{-1}Y}(D\eta) = D(\tilde{i}_{X+\sqrt{-1}Y} \cdot D)\eta + \tilde{i}_{X+\sqrt{-1}Y}(D^2)\eta = 0
\]
for any \( X, Y \in g \).

Then we get
\[
D\eta \in (W(g) \otimes \Omega^*(M))_{bas},
\]
\[ \square \]

Now we can construct the cohomology group as following:

By the complex \((W(g) \otimes \Omega^*(M))_{bas}, D\), we can define the cohomology group as follow,
\[
H^*_G(M) \doteq \frac{\text{Ker}D|_{(W(g) \otimes \Omega^*(M))_{bas}}}{\text{Im}D|_{(W(g) \otimes \Omega^*(M))_{bas}}}
\]

Definition 3. The cohomology group \( H^*_G(M) \) is called the equivariant cohomology groups of \( M \). The equivariant cohomology construct by this way is called Weil model.

5 The main results

In this section we explain the precise relation between the Weil model and the Cartan model for equivariant cohomology defined earlier.

Theorem 1. \( \psi \) is an isomorphism of differential algebra, i.e., the diagram
\[
\begin{array}{ccc}
W(g) \otimes \Omega^*(M) & \xrightarrow{\psi} & W(g) \otimes \Omega^*(M) \\
\delta & & \downarrow D \\
W(g) \otimes \Omega^*(M) & \xrightarrow{\psi} & W(g) \otimes \Omega^*(M)
\end{array}
\]
commutes.
Proof. By computation in lemma 4., we have
\[ \delta = \psi \cdot D \cdot \psi^{-1} \]

\[ \square \]

**Theorem 2.** We have the following commutative diagram:

\[ \begin{array}{ccc}
(W(\mathfrak{g}) \otimes \Omega^*(M), \delta) & \xrightarrow{\psi} & (W(\mathfrak{g}) \otimes \Omega^*(M), D) \\
\uparrow{id} & & \uparrow{id} \\
(S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\tilde{G}} & \xrightarrow{\psi} & (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}
\end{array} \]

**Proof.** For \( \forall \alpha \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\tilde{G}} \), by
\[ \prod_a (1 - \theta^a \otimes (t_a + \sqrt{-1}f^b_{aj}t_b)) \cdot (i_k \otimes 1) = (i_k \otimes 1 + \sqrt{-1}f^j_k \cdot 1) \cdot \prod_a (1 - \theta^a \otimes (t_a + \sqrt{-1}f^b_{aj}t_b)) \]
we have
\[ (i_k \otimes 1 + \sqrt{-1}f^j_k \cdot 1)(\psi(\alpha)) = 0. \]
Because
\[ [\delta, i_k \otimes 1] = L_k \otimes 1 + 1 \otimes (L_k + \sqrt{-1}f^j_k \cdot L_j) \]
and
\[ \prod_a (1 - \theta^a \otimes (t_a + \sqrt{-1}f^b_{aj}t_b)) \cdot (L_k \otimes 1 + 1 \otimes (L_k + \sqrt{-1}f^j_k \cdot L_j)) \]
\[ = (L_k \otimes 1 + 1 \otimes (L_k + \sqrt{-1}f^j_k \cdot L_j)) \cdot \prod_a (1 - \theta^a \otimes (t_a + \sqrt{-1}f^b_{aj}t_b)) \]
so we have
\[ (L_k \otimes 1 + 1 \otimes (L_k + \sqrt{-1}f^j_k \cdot L_j))(\psi(\alpha)) = 0 \]
Then we get \( \psi(\alpha) \in (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \). So we get the commutative diagram. \[ \square \]

The theorem 2. tell us the relation about BRST model and Cartan model.

**Theorem 3.**
\[ (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\tilde{G}} \xrightarrow{\psi} (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas} \]
is a isomorphism.

**Proof.** For \( \forall \eta \in (W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}, \psi^{-1}\eta = \prod_a (1 + \theta^a \otimes (t_a + \sqrt{-1}f^b_{aj}t_b))\eta. \) By
\[ \prod_a (1 + \theta^a \otimes (t_a + \sqrt{-1}f^b_{aj}t_b))|_{(W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}} = \prod_a (1 - \theta^a \otimes 1)|_{(W(\mathfrak{g}) \otimes \Omega^*(M))_{bas}} \]
and
\[ Im(1 - \theta^a \otimes 1) = Ker(i_a \otimes 1) \]
So
\[ \psi^{-1}\eta \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))_{bas}. \]
Then
\[ (\sum_{i=1}^n \phi^i \otimes (L_i + \sqrt{-1}f^j_i \cdot L_j))\psi^{-1}\eta = 0 \]
i.e., \( \psi^{-1}\eta \in (S(\mathfrak{g}^*) \otimes \Omega^*(M))^{\tilde{G}} \). And by the proof in theorem 2. we get that \( \psi \) is a isomorphism. \[ \square \]

The theorem 3. tell us the relation about Cartan model and Weil model.
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