On the existence of extremals of some nonlinear Fredholm operators

Dmitriy V Kostin, Tatiana I Kostina and Leonid V Stenyuhin

1Department of Mathematics, Voronezh State University, 394018, Universitetskaya pl., 1, Voronezh, Russia
2Department of Mechanical Engineering and Aerospace Engineering, Voronezh State Technical University, 394006, 20 let Oktyabrya st., 84, Voronezh, Russia
E-mail: dvk605@mail.ru, tata_sti@rambler.ru, stenyuhin@mail.ru

Abstract. Theorems on the existence and form of solutions of variational problems with circular symmetry, such as the deflection problem of an elastic plate and the capillarity problem, were obtained.

1. Introduction
The article presents the results of studying the solutions of equations with circular symmetry on the examples of Karman’s problems about the equilibrium of a compressed round elastic plate [1, 2], about the structure of the capillarity problem [3].

The variation version of the Lyapunov–Schmidt reduction is a transition from the original model equation to the equivalent problem of analysis of the key function defined on a finite-dimensional space of key variables [4, 5]. Using the key function allows you to apply new analytical and computational technologies in the construction and analysis of plate deflections and periodic solutions, as well as shed light on the nature of the behavior of solutions when the parameters approach critical and resonant values. The bases of the analysis of nonlinear equations with the use of the modified Lyapunov–Schmidt scheme were laid in the work of M A Krasnoselsky, N A Bobylev, E M Mukhamadiev.

The objectives of this publication are 1) to determine the number of solutions and their visualization, 2) to recognize the topological types of solutions (Morse indices of the critical points of the corresponding energy functional) and 3) the behavior of solutions when the parameters approach the critical and resonant values. The article everywhere assumes that the spectral parameter does not exceed the third critical value. The numerical-analytical technique used in this work was previously tested in the problems of deflection of elastic systems, phase transitions in crystals and satellite oscillation in elliptical orbit [4, 6, 7, 8, 9, 10].

The construction of approximations to a non-locally given key function is carried out by means of a numerical procedure of direct descent along the field of anti-gradients of the energy functional narrowed to the orthogonal complement to the linear shell of the main bifurcation modes [11, 12] (in scalar product $L_2$).

In the works of Yu I Saponova and A V Gnezdilova was established (1994), that the key function obtained by the reducing scheme Lyapunov–Schmidt in the problem of branching axisymmetric configurations of a circular elastic plate with its uniform (along the boundary)
compression in the direction of the normals, globally extendable to the space of key variables with finite supercritical increments of the load parameter [13]. The initial versions of the non-local computational and analytical procedure considered in this paper were tested in a series of articles by Yu I Sapronova, A Yu Borzakova and T I Kostina.

The final section shows that the Capillarity Problem is a continuation of the problem of minimal surfaces investigated in [16] and undergoes the unified variational principle of finite-dimensional reductions. Sufficient conditions have been obtained for changing the shape of a symmetric droplet under external influences.

2. The operator form of the non-local version of the Lyapunov–Schmidt variational method

Let \( f : E \to F \) smooth Fredholm zero index mapping of Banach spaces [14]. And let actually (the preimage of \( f^{-1}(K) \) of an arbitrary compact \( K \subset F \) is compact in \( E \)) and potentially.

\[
\langle f(x), h \rangle \equiv \frac{\partial V}{\partial x}(x)h .
\]  

Here \( V \) is a smooth functional on \( E \) (the potential of the map \( f \)), \( \langle \cdot, \cdot \rangle \) are scalar product in some Hilbert space \( H \), in which \( E \) and \( F \) are continuously embedded. \( E \) is assumed to be continuously embedded in \( F \) and \( E \) is tightly in \( H \). If the condition of positivity (monotonicity)

\[
\left\langle \frac{\partial f}{\partial x}(x)h, h \right\rangle > 0 \quad \forall (x, h) \in E \times (E \setminus 0),
\]  

then the equation \( f(x) = 0 \) has at most one solutions. This solution is the minimum point of \( V \) on \( E \)[4]. If the property condition is satisfied, the equation \( f(x) = 0 \) uniquely solvable (due to the Banach Mazur Kacciopoli [14] theorem) and its solution is a global minimum point \( V \) [4].

Replace the relation (2) with a weaker condition

\[
\left\langle \frac{\partial f}{\partial x}(x)h, h \right\rangle > 0 \quad \forall (x, h) \in E \times (\tilde{E} \setminus 0),
\]  

when \( \tilde{E} = E \cap N^\perp, \ N = Lin(e_1, \ldots, e_n), \ N^\perp \) is orthogonal addition to \( N \) in \( H \), \( e_1, \ldots, e_n \) is some orthonormal in \( H \) system of vectors in \( e \). Some orthonormal system in \( H \) vector system in \( E \). Under this condition, it is possible locally (near zero and at small increments of parameters) to determine the key function Lyapunov–Schmidt

\[
W(\xi) := \inf_{x:\langle x,e_j \rangle = \xi_j, \ \forall j} V(x), \quad \xi = (\xi_1, \ldots, \xi_n)^\top,
\]  

"responsible" for the local behavior of the functional \( V \).

The condition of the property \( f \) can also be weakened by replacing it property condition for every \( \xi \) "layered" mapping (condition "layered property")

\[
\tilde{f}_\xi : \tilde{E} \to \tilde{F},
\]

\[
\tilde{F} = F \cap N^\perp,
\]

\[
\tilde{f}_\xi(v) := P_{\tilde{F}}(f(l(\xi) + v)) = f(l(\xi) + v) - \sum_{j=1}^{n}(e_j, f(l(\xi) + v))e_j, \quad l(\xi) = \sum_{j=1}^{n} \xi_j e_j.
\]
When the condition of stratified property and the condition (3) is satisfied, the equation \( \tilde{f}_\xi(v) = 0 \) is uniquely solvable for all \( \xi \). According to the implicit function theorem solution \( v = \Phi(\xi) \) depends smoothly on \( \xi \). Left part (4) can be represented as

\[ W(\xi) \equiv V(l(\xi) + \Phi(\xi)). \]  

(5)

The reflection of \( \Phi : \xi \mapsto l(\xi) + \Phi(\xi) \) is named em relating. It is mutual simply binds the decisions of key equalization to the decisions of initial equalization. For key equation

\[ \theta(\xi) = 0, \quad \xi \in \mathbb{R}^n, \]

in that

\[ \theta(\xi) = (\theta_1(\xi), \ldots, \theta_n(\xi))^T, \quad \theta_j(\xi) = \langle f(l(\xi) + \Phi(\xi)), e_j \rangle, \]

we have [4]:

\[ \theta(\xi) = \text{grad } W(\xi). \]

In the case of an equation that depends smoothly on parameters, all previous conclusions retain their strength while maintaining smooth dependence on parameters.

3. Fredholm functionals with circular symmetry

The analysis of bifurcations of periodic solutions is a special case of the general scheme of analyzing the critical orbits of Fredholm functionals with circular symmetry.

Let \( V : E \to \mathbb{R} \) be Fredholm functional with gradient in the three spaces \{\( E, F, H \)\} and let there are given such a representation \( T \) of the groups \( SO(2) \) in the group \( O(H) \) of orthogonal operators \( H \to H \), that \( T_g(E) \subset E, \ T_g(F) \subset F, \ \forall g \in SO(2) \) and the functional \( V \) is invariant with respect to the action of \( SO(2) \) on \( E \) (possesses circular symmetry):

\[ V(T_g x) = V(x) \quad \forall \ x \in E, \ g \in SO(2). \]

Suppose also that is given mapping \( p : E \to \mathbb{R}^m, \ p(T_g(x)) = T_g(p(x)) \ \forall \ x \in E, \ g \in SO(2) \)  

\( (SO(2) \) action on \( \mathbb{R}^m \) is denoted by the same \( T) \).

We will also assume that the condition of weak smoothness of the action \( SO(2) \) and the condition for the absence of nonzero fixed points. The space \( N \) of bifurcation modes in this case has an even dimension and is broken up into a direct sum of two-dimensional subspaces

\[ N = N_1 \oplus N_2 \oplus \ldots \oplus N_n. \]

These spaces are invariants with respect to a given action of a circle. And also the restrictions of the action of \( SO(2) \) to each of these subspaces is irreducible. If we identify each subspace of \( N^k \) with the complex plane \( \mathbb{C} \), then the induced action a circle on \( N^k \) is reduced to the standard action of a circle on a complex plane of multiplicity \( p_k \):

\[ \{\varphi, z\} \mapsto e^{ip_k \varphi} z. \]

The local algebraic structure of the key function corresponding to this functional depends on the set of multiplicities \( \{p_k\} \), or, more precisely, on resonances, i.e. by such non-trivial sets of integers \( M = \{m_1, m_2, \ldots m_n\} \), for which relations \( \sum_{k=1}^n m_k p_k = 0 \).

It is clear that the set \( \mathcal{R} \) of all resonances is a subgroup of the integer lattice \( \mathbb{Z}^n \). Arbitrary subgroup basis \( \mathcal{R} \) is called a basic set of resonances. In particular, if \( n = 2 \), then the basis set is the only pair \( \{m_1, m_2\} \).
Note that there is a direct analogy of the resonances considered here with resonances in the theory of the origin of cycles of dynamic systems from a complex focus. As in the theory of dynamical systems, the most difficult to study are cases of resonances with orders of $|m|$ not exceeding four:

$$|m| := \sum_{k=1}^{q} |m_k| \leq 4$$

(strong resonances).

Let, further, $V(x,\lambda)$ — Fredholm functional with gradient, that smoothly depends on a parameter $\lambda \in R^l$, and let such a representation of $T$ the group be given $G := SO(2)$ into the group $O(H)$ of orthogonal operators $H \to H$ together with the representation of the $R$ group $SO(2)$ in the group $O(H)$, that $T_g(E) \subset E$, $T_g(F) \subset F$, $\forall g \in SO(2)$, and the functional $V$ is invariant under the action of $SO(2)$ on $E$ (that is, it has generalized circular symmetry): $V(T_gx, R_g\lambda) = V(x,\lambda) \forall x \in E, \lambda \in R^l$, $g \in SO(2)$. The following assertion holds [4]: let the condition of weak smoothness of the action $SO(2)$ be satisfied and let $SO(2)$-equivariant reducing submersion $p : E \to R_m^m$ be given or a smooth family of $SO(2)$-invariant generalized smooth Fredholm functionals $V(x,\lambda)$, here let $\phi$ be a connecting map. Then the induced action $SO(2)$ on the subvariety $N := \phi(R_m^m)$ automatically smooth, and the corresponding key function inherits a generalized symmetry.

4. The problem of the deflection of a round elastic plate
The forms of elastic equilibria of a circular plate, uniformly compressed along the edge (along the normals), in the Karman model are described by a system of two equations [2]

$$\Delta^2 w + \lambda \Delta w - [w, \phi] = \Delta^2 \phi + \frac{1}{2} [w, w] = 0,$$

where $\Delta$ — Laplace operator, $[w, \phi] = w_{xx} \phi_{yy} + w_{yy} \phi_{xx} - 2w_{xy} \phi_{xy}$, $w$ — deflection function, $\phi$ — stress function, $\lambda$ — load parameter.

The equation (6) is supplemented by the boundary conditions corresponding to the nature of the fixing of the edge of the plate. Consider the case of hard fastening:

$$\phi = \phi_x = \phi_y = w = w_x = w_y = 0 |_{\partial \Omega}. \quad (7)$$

Here $\Omega$ — the domain of the functions $w, \phi$, given as a unit circle and interpreted as the geometric shape of the unloaded plate:

$$\Omega = \{(x, y) \in R^2 : x^2 + y^2 \leq 1\}.$$ 

The equation (6) under boundary conditions (7) is the Euler–Lagrange equation for the extremals of the total energy functional

$$V(w, \phi, \lambda) = \frac{1}{2} \iint_{\Omega} \left( |\Delta w|^2 - |\Delta \phi|^2 - \lambda |\nabla w|^2 - \phi[w, w] \right) \, dx \, dy. \quad (8)$$

It is easy to verify that the functional (8) is invariant with respect to the action of the group of orthogonal rotations of the plane around the origin. In principle, this allows us to apply the considerations outlined above when studying the branching of extremals of the functional $V$. However, here we restrict ourselves to the consideration of a simpler case connected with the functional reduction of $V$ (by means of contraction) to the subspace of symmetric functions (not dependent on the angular variables).
After the transition to the polar coordinates $x = r \cos(\phi)$, $y = r \sin(\phi)$ and "descent" into axisymmetric space of functions $w = w(r)$, $\phi = \phi(r)$ we obtain the integral of energy in the form $V = 2\pi \hat{V}$,

\[
\hat{V} = \frac{1}{2} \int_0^1 r \left( |\Delta w|^2 - |\Delta \phi|^2 - \lambda \left( \frac{dw}{dr} \right)^2 + \frac{1}{2} \frac{d\phi}{dr} \left( \frac{dw}{dr} \right)^2 \right) dr,
\]

\[
\Delta = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr}.
\]

And after replacing $v = \frac{dw}{dr}$, $\psi = \frac{d\phi}{dr}$ will get

\[
U(w, \psi, \lambda) = \hat{V}(v, \phi, \lambda) = \frac{1}{2} \int_0^1 r \left( A(v)v - A(\psi)\psi - \lambda v^2 + \frac{\psi v^2}{2r} \right) dr,
\]

where

\[
A = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{1}{r^2} I
\]

differential bessel operator

Standard reducing transition to the functional

\[
\mathcal{U}(v, \lambda) = \sup_{\psi} U(v, \psi, \lambda) = \frac{1}{2} \int_0^1 r \left( A(v)v - \lambda v^2 + \frac{1}{4} A^{-1} \left( \frac{v^2}{r} \right) \frac{v^2}{r} \right) dr
\]

get rid of the variable $\psi$.

Stable axisymmetric plate configurations correspond to local minimum points of the functional (9). From the boundary conditions (7) and the conditions of axial symmetry, it follows that $v$ satisfies the boundary conditions

\[
v(0) = v(1) = 0.
\]

Denote by $E$ the space of functions of class $C^2$ on the interval $[0, 1]$ satisfying the boundary conditions (10), and by $\mathcal{H}$ the space of functions with summable weight square on $[0, 1]$ and scalar product $\langle u, v \rangle = \int_0^1 r u v dr$. Obviously, the functional $\mathcal{U}$ is smooth on this space.

**Theorem 1.** Let $e_1, \ldots, e_n$ be such a linearly independent set of functions in $E$ such that for $\lambda \leq l$, where $l$ is some positive number, the quadratic form

\[
\langle (A - \lambda I)v, v \rangle
\]

is positive on $E \cap \text{Lin}(e_1, \ldots, e_n)^\perp$ orthogonal complement in the metric $\langle u, v \rangle$ of space $\mathcal{H}$) Then

1) function

\[
W(\xi, \lambda) = \inf_{v : \langle v, e_j \rangle = \xi_j, j = 1, \ldots, n} \mathcal{U}(v, \lambda), \quad \xi = (\xi_1, \ldots, \xi_n), \quad \lambda \leq l,
\]

is smooth on $R^n$ 2) exists mutually one-to-one correspondence between sets of critical points functional $\mathcal{U}$ and function $W$,

there is a one-to-one correspondence between the sets of critical points of the functional $\mathcal{U}$ and the function $W$, which preserves coranks and Morse indices of critical points.
Let
\[ E_0 = E \cap \text{Lin}(e_1, \ldots, e_n)^\perp, \quad F_0 = C([0,1]) \cap \text{Lin}(e_1, \ldots, e_n)^\perp. \]
The extremal \( v = \Phi(\xi) \) of the functional \( \bar{U}_{|E_\xi} \) is given by the equation
\[ g(v, \lambda) = 0, \quad v \in E_\xi, \quad (12) \]
here \( E_\xi = \{ v \in E : \langle v, e_j \rangle = \xi_j, \quad j = 1, \ldots, n \} \), \( g : E_\xi \to F_0 \),
\[ v \mapsto f(v, \lambda) - \sum_{j=1}^n \langle f(v, \lambda), \hat{e}_j \rangle \hat{e}_j, \]
\[ f(v, \lambda) := A(v) - \lambda v + \frac{v^2}{2r} A^{-1} \left( \frac{v^2}{r} \right) - q, \]
\( \hat{e}_1, \ldots, \hat{e}_n \) — any orthonormal basis in \( \text{Lin}(e_1, \ldots, e_n) \). The mapping \( g \) is locally diffeomorphically invertible. The latter follows from the Fredholm property of \( g \) (index 0), positivity (11) and the following statement.

**Theorem 2.** For the functional \( S(v) = \langle A^{-1} \left( \frac{v^2}{r} \right), \frac{v^2}{r} \rangle \), the relation (convexity condition)
\[ \frac{d^2}{dv^2} S(v, \lambda)(h, h) \geq 0, \quad \forall v, h \in E, \lambda < 1 \] is satisfied.

**Proof.** The validity of this statement is carried out by direct verification. First we need to note that
\[ \frac{d^2}{dv^2} S(v, \lambda)(h, h) = 2 \left( \langle A^{-1} \left( \frac{v^2}{r} \right), \frac{h^2}{r} \rangle + 2 \langle A^{-1} \left( \frac{vh}{r} \right), \frac{vh}{r} \rangle \right). \]
The second term here is non-negative due to the positive definiteness of the operator \( A \). The positivity of the first term follows from the following assertion. This completes the proof.

**Theorem 3.** The operator \( A^{-1} : F \to E \) is positive \( v(r) \geq 0 \forall r, \quad A(v)(r) \geq 0 \forall r \).

**Proof.** Suppose the opposite
\[ \min A(v)(r) \geq 0, \quad \min v(r) < 0. \]
Let the minimum \( v \) be reached at \( c \in (0,1) \). Then \( v_r(c) = 0, \quad v(c) < 0 \) and \( A(v)(c) = -v_{rr}(c) + \frac{v(c)}{r^2} \geq 0 \). Therefore \( v_{rr}(c) < 0 \). The latter contradicts the minimal value \( v(c) \). This completes the proof.

These statements imply the convexity and coercivity of the energy functional (9) on the orthogonal complement to \( \text{Lin}(e_1, \ldots, e_n) \) in the space of energy \( H^1 \) (see below proof of the theorem) and therefore takes place uniqueness and smooth dependence of the solution \( v = \varphi(\xi, \lambda) \) equations (12) of \( \lambda \) if a solution exists.

**Theorem 4.** For each value of the \( \lambda \) parameter for which the condition (11), the equation (12) is solvable.

**Proof.** Let \( H^1 \) be the Hilbert space of functions, received by closure in norm \( \| v \|_1 := \langle A(v), v \rangle \) of the space of smooth functions satisfying the condition (10). Let \( \langle u, v \rangle_1 := \langle A(u), v \rangle \) — scalar product in \( H^1 \). Then for \( \mathcal{U} \) we have the representation
\[ \mathcal{U}(v, \lambda, q) = \frac{1}{2} \left( \| v \|_1 - \lambda \langle A^{-1}(v), v \rangle_1 + S(v) \right) + \langle A^{-1}(v), \tilde{q} \rangle_1, \quad (14) \]
here
\[ S(v) = \frac{1}{4} \left\| A^{-1} \left( \frac{v^2}{r} \right) \right\|_1^2. \]
Let \( H_1^\xi \) — closure in \( H^1 \) affine subspace \( E_\xi \). Coercivity \( U \) in \( H_1^\xi \) follows from the positivity of the form (11). Convexity and coercivity of \( U \) guarantee extremal existence \( \Phi(\xi, \lambda, q) \) in \( H_1^\xi \). As usual (for such problems), this extremal is actually in \( E_\xi \). "Swap" smoothness of the extremal \( v(r) \) can produce due to the ratio:

\[
v + \lambda A^{-1}(v) + A^{-1}\left(\frac{v}{2r} A\left(\frac{e^2}{r}\right)\right) = \sum_{j=1}^{n} (\mu_j + q \alpha_j) \tilde{e}_j, \quad \tilde{e}_j = A^{-1}(e_j),
\]

which is the Euler–Lagrange equation for extremals (conditional), the considered functional in \( H_1^\xi \). This completes the proof.

5. Construction of a key function of two variables

Let \( e_k \) is Bessel function \([1, 2, 15]\):

\[
e_k = \mu_k B_0(\alpha_k \theta), \quad B_0(\theta) \text{ zero order Bessel function of the first kind, } \{\alpha_k\} \text{ — sequence of zeros of this function, numbered in ascending order of magnitudes, } \mu_k \text{ — normalization constant (relative to the scalar weight product used above). Functions } \{e_k\} \text{ — eigenfunctions of the operator from the main linear part of the Karman equation (taking into account the boundary conditions) (7).}
\]

Suppose that the value of the \( \lambda \) parameter does not exceed third critical value. Let \( N = Lin(\xi_1, \xi_2) \), \( R \) — orthogonal addition to \( N \) in energy space. Below, the \( \lambda, q \) parameters will be dropped (take into account “default”).

Based on the previously formulated sentences, it can be argued that the key function exists and is smooth:

\[
W(\xi) = \inf_{v \in N^\perp} V(u + v), \quad u = \xi_1 e_1 + \xi_2 e_2.
\]

This function can be approximately represented as:

\[
W(\xi) \equiv V(u + \Phi(\xi)), \quad u := \xi_1 e_1 + \xi_2 e_2, \quad \Phi(\xi) \in R.
\]

Its construction will be carried out by the method of the shortest descent (along the gradient), according to which the construction of approximations to the key function is carried out in the form:

\[
W_k = V(a_k + v_k), \quad a_0 = u, \quad v_k = s \nabla k, \quad \nabla k = -\text{grad}V(a_k).
\]

As a result of applying these procedures, we obtain an algorithm for constructing a key function with any predetermined accuracy.

After obtaining the approximate key function can be done approximate construction of the discriminant set (\textit{caustics}) — sets of parameter values \((\lambda, q)\), for which degenerate critical points exist. Caustics are determined by the system of equations:

\[
\begin{aligned}
\text{grad } W_j(\xi_1, \xi_2) &= 0, \\
\text{Hessian } W_j(\xi_1, \xi_2) &= 0,
\end{aligned}
\]

The figures show images of approximately calculated key function and plate deflections (corresponding to critical points of the reduced energy functional).

6. On The Structure Of The Capillary Problem

The problem of symmetrically lying a small drop on the surface of a fixed volume \( V \) with a contact angle of \( 0 \leq \gamma \leq \pi \) is determined by minimizing the energy functionals:

1) surface tension energy (area functional);
2) the potential energy of gravity;
3) the energy of volume bonds, determining the constancy of the volume.

The total energy functional has the form:

\[
E(u) = \int_{\Omega} \sqrt{EG - F^2} \, dx + \int_{\Omega} \frac{\gamma \rho u}{\sigma} \, dx + \lambda \int_{\Omega} u \, dx,
\]

(17)
Figure 1. An example of computer-generated images of perturbed key function level lines.

**Figure 2.** Graphic image of plate deflections.

$E, G, F$ – coefficients of the first quadratic surface shape, $\sigma$ – surface tension, $\Upsilon$ – potential energy per unit mass, $\rho$ – density, $\lambda$ – Lagrange multiplier, responsible for volume.

In the upper part of the free surface, the height function $u(x, y)$ above the plane is a solution to the equation

$$\text{div } T u = B u,$$

where $B = \frac{2\rho a^2}{\sigma}$ – Bond number characterizing size, $a$ – radius. For the bottom of the free surface the sign $\text{div } T u$ is reversed. Boundary condition tasks with constant contact angle $\gamma$

$$\pi T u = \cos \gamma,$$

$\pi$ – unit normal vector.

Consider the first term of the energy functional (17)

$$S(u) = \int_{\Omega} \sqrt{EG - F^2} \, dx \, dy.$$
Let $u_0$ be the surface of a lying drop, $\Gamma_{t,\tau}$ be a curvilinear rectangle on $u_0$. We define surfaces close to $u_0$ in the coordinates of the normal bundle to $u_0$. Get the equation
\[
\frac{\delta S}{\delta u}(u_0 + \eta \bar{n}) = 0,
\]
or
\[
\frac{\delta S}{\delta \eta}(\eta) = 0, \quad \eta \bigg|_{\Gamma_{t,\tau}} = 0
\]
from which is the normal coordinate $\eta = \eta(x,y)$.

**Theorem 5.** Functional (20) close to $u_0$ surfaces $S(\eta)$ and its Euler operator $\frac{\delta S}{\delta \eta}(\eta)$ has the following structure
\[
S(u) = \int_\Omega \sqrt{D} \, dx \, dy,
\]
\[
\frac{\delta S}{\delta u}(\eta) = E_0^2 D^{-\frac{3}{2}} (A \eta_{xx} - 2B \eta_{xy} + C \eta_{yy} + G).
\]
Here
\[
E_0 = u_{0x}^2, \quad D = \sum_{p=2}^{4} d_{ijk} \eta_{x}^i \eta_{y}^j \eta_{x}^k + E_0^2,
\]
\[
A = \sum_{p=1}^{6} a_{ijk} \eta_{x}^i \eta_{y}^j \eta_{x}^k + 1, \quad B = \sum_{p=1}^{6} b_{ijk} \eta_{x}^i \eta_{y}^j \eta_{x}^k, \quad C = \sum_{p=1}^{6} c_{ijk} \eta_{y}^i \eta_{y}^j \eta_{x}^k + 1,
\]
\[
G = \sum_{p=2}^{7} g_{ijk} \eta_{x}^i \eta_{y}^j \eta_{x}^k + gn,
\]
where $i,j,k$ – non-negative integers, $p = i + j + k$. Everything odds
\[
a_{ijk}, b_{ijk}, c_{ijk}, d_{ijk}, g_{ijk}, g
\]
are analytic functions and are found by formulas similar to the following
\[
g = (\bar{n}, \bar{n}_{xx} + \bar{n}_{yy}) + \frac{4}{E_0} \left[ (\bar{n}, u_{xx})^2 + (\bar{n}, u_{yy})^2 \right].
\]

The linear part of the operator $A \eta_{xx} - 2B \eta_{xy} + C \eta_{yy} + G$ is
\[
\Delta \eta + g \eta,
\]
$\Delta$, where $\Delta$ – laplacian.

For all values $(t, \tau) \in \mathbb{R}^2_+$ the problem (21) has a trivial solution $\eta = 0$, corresponding to the surface $u_0$. If $u_0$ – extremal functional (20), then $u_0$ is called the minimal surface. In [16], for minimal surfaces, the existence of the values of the parameters $(t, \tau)$ for rectangular and circular contours on the surface for which there exist nontrivial solutions of problem (21) is established.

Replacing the variables $x = tx'$, $y = \tau y'$, of transition to areas of unit size, leads to the appearance of the parameters $t$ and $\tau$ in the structure of the operator of problem (21), which we denote $P(\eta, t, \tau)$.

**Theorem 6.** The operator $P(\eta, t, \tau)$ acts in the spaces
\[
P : W^m_2(\Omega) \times \mathbb{R}^2_+ \to W^{m-2}_2(\Omega), \quad m \geq 2
\]
and is \( C^\infty \) smooth on set of variables. Frechet derivative

\[
P_t(0, t, \tau) h = A(t, \tau) h
\]

for all \((t, \tau) \in \mathbb{R}^2_+\) is a Fredholm operator of zero index. Its linearization \( A(t, \tau) \) is self-adjoint on the space \( W^2_2(\Omega) \) with respect to the scalar product in \( L^2(\Omega) \).

Let us give an example of the existence of special solutions in the capillarity problem. In [3] it was shown that for a symmetrically lying drop, \( 0 < \gamma < \frac{\pi}{2} \) and a small Bond number \( B \), equation (18) can be reduced to view

\[
\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = Bu - 2 \sin \gamma_0.
\]

The solution is sought on the circle \( \Omega \), \( u = 0 \) on \( \partial \Omega \) and a drop constant volume \( V_0 \). If a drop is exposed external potential \( \varphi \), then

\[
V_0 = \frac{\pi(2 + \cos \gamma_0)(1 - \cos \gamma_0)^2}{3 \sin^3 \gamma_0},
\]

\[
\cos \gamma_0 = \cos \gamma + \frac{\varphi_0}{\sigma}.
\]

The potential \( \varphi \) can be, for example, the temperature of a drop substance or the pressure acting from inside the drop, or the temperature and pressure simultaneously.

The action of the potential leads to a change in the shape of the drop, in particular, to a change in the angle of contact with the plane \( \Pi \), (26).

In this case, \( \gamma_0 \) is the only solution to equation (25). Further impact of the potential \( \varphi \) will lead to a change in the Bond number \( B \) and to a rearrangement (bifurcation) of the drop itself.

To describe the further states of a drop, we put in the equation (24) \( B = 0 \).

**Theorem 7.** Under the above assumptions and \( B = 0 \), there is an exact solution of equation (24)

\[
u_0 = \frac{-\cos \gamma_0 + \sqrt{1 - r^2 \sin^2 \gamma_0}}{\sin \gamma_0}.
\]

Solution (27) is verified by direct verification.

**Theorem 8.** If condition

\[
B = \cos \gamma_0.
\]

is satisfied, then there is an exact analytical solution to problem (24) of type

\[
u = \frac{-\cos \gamma_0 + \sqrt{1 - (r - r_0)^2 \sin^2 \gamma_0}}{\sin \gamma_0},
\]

\( r_0 \) - center radius of the ring \( |r - r_0| \leq 1 \), with the boundary condition (19).

**References**

1. Volmir A S 1956 *Flexible Plates and Covers* (Moscow: Gostekhizdat)
2. Vorovich I I 1989 *Mathematical Problems of the Nonlinear Theory Flat Covers* (Moscow: Nauka)
3. Finn R 1986 *Equilibrium Capillary Surfaces* (Springer)
4. Darinsky B M, Saponov Yu I, Tsaryov S L 2004 *Modern Mathematics. Fundamental Directions* 12 3–140
5. Krasnoselsky M A, Bobyliov N A, Mukhamadiyev E M 1978 *Reports of the USSR Academy of Sciences* 240 (3) 530–533
6. Kostin D V 2016 *Sbornik: Mathematics* 12 1709–1729
[7] Gneushev I A, Kovaleva M I, Sapronov Yu I 2018 Pumps. Turbines. Systems 1(26) 75–82
[8] Darinskii B M, Kolesnikova I V, Sapronov Yu I 2009 Proceedings of Voronezh State University. Series: Physics. Mathematics 1 101–107
[9] Derunova E V, Sapronov Yu I 2015 Russian Mathematics 59(8) 9–18
[10] Kostin T I, Sapronov Yu I 2018 Proceedings of Voronezh State University. Series: Physics. Mathematics 1 99–114
[11] Korotkih A S 2017 Proceedings of Voronezh State University. Series: Physics. Mathematics 1 115–127
[12] Krasnosel’skii M A, Krein S G 1952 Sbornik: Mathematics 31(73) 315–334
[13] Gnezdilov A V 2000 Functional Analysis and Its Applications 34(1) 67–69
[14] Borisovich Yu G, Zvyagin V G, Sapronov Yu I 1977 Russian Mathematical Surveys 32(4) 1–54
[15] Smirnov V I 1974 The course of higher mathematics. Volume 2 (Moscow: Nauka)
[16] Stenyuhin L V, 2012 Russian Mathematics 56(11) 45–51