The monodromy of unit-root $F$-isocrystals with geometric origin

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Abstract

Let $C$ be a smooth curve over a finite field of characteristic $p$ and let $M$ be an overconvergent $F$-isocrystal over $C$. After replacing $C$ with a dense open subset, $M$ obtains a slope filtration. This is a purely $p$-adic phenomenon; there is no counterpart in the theory of lisse $\ell$-adic sheaves. The graded pieces of this slope filtration correspond to lisse $p$-adic sheaves, which we call geometric. Geometric lisse $p$-adic sheaves are mysterious, as there is no $\ell$-adic analogue. In this article, we study the monodromy of geometric lisse $p$-adic sheaves with rank one. More precisely, we prove exponential bounds on their ramification breaks. When the generic slopes of $M$ are integers, we show that the local ramification breaks satisfy a certain type of periodicity. The crux of the proof is the theory of $F$-isocrystals with log-decay. We prove a monodromy theorem for these $F$-isocrystals, as well as a theorem relating the slopes of $M$ to the rate of log-decay of the slope filtration. As a consequence of these methods, we provide a new proof of the Drinfeld–Kedlaya theorem for irreducible $F$-isocrystals on curves.

1. Introduction

1.1 Motivation

Let $C$ be a smooth curve over a finite field $\mathbb{F} = \mathbb{F}_q$ in characteristic $p$. Classically, the study of motives over $C$ has focused on lisse $\ell$-adic étale sheaves on $C$, where $\ell \neq p$. It is natural to ask for a $p$-adic counterpart to the $\ell$-adic theory. However, there are far too many lisse $p$-adic étale sheaves and they tend to be poorly behaved compared with their $\ell$-adic counterparts. For example, if we have a family of ordinary elliptic curves $f : E \to C$, the relative first-degree $p$-adic étale cohomology $R^1_{\text{ét}} f_* \mathbb{Q}_p$ has rank one. In contrast, the relative $\ell$-adic cohomology sheaf $R^1_{\text{ét}} f_* \mathbb{Q}_\ell$ has rank two, as is expected. Instead, the correct $p$-adic coefficient objects are overconvergent $F$-isocrystals, which were first introduced by Berthelot (see [Ber96]).

Overconvergent $F$-isocrystals have a remarkable extra structure that is absent in the $\ell$-adic theory: a slope filtration. Without giving any definitions, consider the overconvergent $F$-isocrystal $M$ that acts as the $p$-adic counterpart to the lisse sheaf $R^1_{\text{ét}} f_* \mathbb{Q}_\ell$. The properties of $M$ follow those of $R^1_{\text{ét}} f_* \mathbb{Q}_\ell$. First, $M$ has rank two. Just as in the $\ell$-adic case, for any $x \in C$ we may consider the fiber $M_x$ and the action of Frobenius on $M_x$. The characteristic polynomial of this action will describe the zeta function of the elliptic curve $E_x$:

$$Z(E_x, s) = \frac{\det(1 - \text{Frob}^* s, M_x)}{(1 - s)(1 - q^\deg(x)s)}.$$

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Here, we see a fundamental difference between the $\ell$-adic and $p$-adic situations. The roots of the numerator of $Z(E_x, s)$ are both $\ell$-adic units. However, because $E_x$ is ordinary, one root is a $p$-adic unit and the other root has $q$-adic valuation one. Even before the modern definition of an $F$-isocrystal was in place, Dwork discovered something miraculous with no $\ell$-adic analogue: these unit roots come from a rank-one subobject $M^{\text{unit}}$ of $M$ existing in a larger category of convergent $F$-isocrystals. It was later demonstrated by Katz in [Kat73] that any ‘unit-root’ $F$-isocrystal corresponds to a $p$-adic étale sheaf. As one may expect, the $p$-adic étale sheaf corresponding to $M^{\text{unit}}$ is $R^1_{et} f_* \mathbb{Q}_p$.

This phenomenon generalizes. Let $N$ be an overconvergent $F$-isocrystal on $C$ and assume that the Newton polygon of $\det(1 - \text{Frob}^s N_x)$ remains constant as we vary $x \in C$. Katz proved in [Kat79] that $N$ obtains an increasing filtration in the larger category of convergent $F$-isocrystals. The graded pieces of this filtration are ‘twists’ of unit-root $F$-isocrystals and thus correspond to lisse $p$-adic étale sheaves on $C$. We say that a lisse $p$-adic étale sheaf is geometric if it arises in this manner. Geometric $p$-adic étale sheaves remain mysterious. When one studies properties of overconvergent $F$-isocrystals, such as their cohomology or Frobenius distributions, the $\ell$-adic theory often serves as a guiding light suggesting what is true and occasionally how it should be proven. However, as there is no $\ell$-adic analogue to the slope filtration, it is less clear how to proceed in developing a coherent theory. It is natural to ask whether all geometric $p$-adic étale sheaves share certain properties or, more ambitiously, is it possible to determine when a $p$-adic étale sheaf is geometric?

In this article, we study the monodromy of geometric $p$-adic étale sheaves of rank one and the ‘growth’ properties of the slope filtration. In the case where the $F$-isocrystal has integral slopes, we prove a monodromy stability result for geometric $p$-adic étale sheaves. This result says that the ramification breaks satisfy a certain type of periodicity. We are naturally led to consider $F$-isocrystals with logarithmic decay and we prove a monodromy theorem for these $F$-isocrystals. We also establish a relationship between the Frobenius slopes and the rate of logarithmic decay of the slope filtration. This allows us to give a new proof of the Drinfeld–Kedlaya theorem.

### 1.2 Monodromy results

1.2.1 Local results. Let $F$ be either $\mathbb{R}((T))$ or a finite extension of $\mathbb{Q}_p$ and let $G_F$ be the absolute Galois group of $F$. We let $L$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$. For $k \geq 0$, we set $O_k^x = \{ x \in O_L^x \mid v_p(1 - x) > k \}$. Consider a continuous character $\rho : G_F \to O_L^x$. We define $s_k(\rho)$ to be the largest upper numbering ramification break of the Galois extension of $F$ corresponding to $\rho^{-1}(O_k^x)$. When $F$ has characteristic zero, a celebrated result of Sen (see [Sen72]) tells us that there exists a positive rational number $c$ such that $ke - c \leq s_k(\rho) \leq ke + c$, where $e$ is the ramification index of $F$ over $\mathbb{Q}_p$. Sen’s theorem fails dismally in equal characteristic, because $s_k(\rho)$ may grow arbitrarily fast with respect to $k$. In this article, we study the growth of $s_k(\rho)$ when $F = \mathbb{R}((T))$ and $\rho$ has geometric origin. We show that $s_k(\rho)$ grows exponentially and under some additional geometric assumptions, we show that $s_k(\rho)$ satisfies a certain periodicity.

**Definition 1.1.** We say that $\rho$ has **finite monodromy** if the image of the inertia subgroup of $G_F$ is finite. For $r > 0$, we say that $\rho$ has **$r$-bounded monodromy** if there exists a positive rational number $c$ such that

$$s_k(\rho) < cp^r,$$

for all $k > 0$ (note that when $\rho$ has finite monodromy there exists $c > 0$ such that $s_k(\rho) < c$ for all $k$, and thus $\rho$ has $r$-bounded monodromy for all $r > 0$). Let $a \in \mathbb{Z}_{\geq 1}$ and let $s = v_p(q^a)$,
where \( v_p \) denotes the \( p \)-adic valuation normalized so that \( v_p(p) = 1 \). We say that \( \rho \) has a-stable monodromy if for every \( k \in [0, s] \), there exists \( m_k \) and \( b_k \) such that

\[
s_{k+n}(\rho) = m_k q^a + b_k
\]

for \( n \gg 0 \). We say that \( \rho \) has stable monodromy if it has a-stable monodromy for some \( a \).

We now restrict ourselves to the case where \( F = \mathbb{R}((T)) \). Let us explain what it means for a character of \( G_F \) to be geometric. Let \( M \) be an overconvergent \( \mathbb{F} \)-isocrystal over \( \text{Spec}(F) \) with coefficients in \( L \) and let \( \iota(M) \) be the corresponding convergent \( \mathbb{F} \)-isocrystal (see §4.2). Then \( \iota(M) \) has a Frobenius slope filtration (see §4.2.1.4):

\[
0 = M_0 \subset M_1 \subset \cdots \subset M_d = \iota(M),
\]

where \( gr_i(M) = M_i/M_{i-1} \) is isoclinic of slope \( \alpha_i \). After enlarging \( L \), we may associate to \( \text{det}(M_i) \) a character \( \rho_i : G_F \to \mathcal{O}_L^\times \). This character is well-defined up to twist by an unramified character. We say that a character of \( G_F \) is geometric if it arises this way.

**Theorem 1.2.** Let \( r_i = 1/(\alpha_{i+1} - \alpha_i) \).

(i) Then \( \rho_i \) has \( r_i \)-bounded monodromy.

(ii) Assume that \( M \) is irreducible and that the slopes of \( M \) are integers. Then \( \rho_i \) has stable monodromy.

From Theorem 1.2, we see that rank-one geometric \( p \)-adic étale sheaves are intricate and fascinating objects. This is in stark contrast to the \( \ell \)-adic situation, where rank-one objects have finite monodromy and are easily understood. The stable monodromy of \( \rho_i \) when \( M \) has integral slopes is particularly surprising. Indeed, Theorem 1.2 shows that the ramification filtration of \( \rho_i \) is completely determined by the first few ramification breaks. This is in contrast to a general \( p \)-adic character of \( \rho \), where there are essentially no restrictions on \( s_n(\rho) \).

**Example 1.3.** Let \( f : X \to \text{Spec}(F) \) be a smooth proper morphism and assume that \( R_{\text{ét}}^i f_* \mathbb{Q}_p \) is a rank-one \( p \)-adic lisse sheaf. The corresponding character \( \rho \) is geometric (see [Ked00]). In particular, Theorem 1.2 applies to many unit-root \( \mathbb{F} \)-isocrystals studied by Dwork and others.

(i) Let \( M \) be the \( \mathbb{F} \)-isocrystal associated to an elliptic curve \( E \) over \( \text{Spec}(\mathbb{R}[[T]]) \), whose generic fiber is ordinary and whose special fiber is supersingular. By Theorem 1.2, the \( p \)-adic Tate module of the generic fiber of \( E \) has stable monodromy. This was previously known by work of Katz–Mazur (see [KM85, Chapter 12.9]). These types of ramification bounds for Abelian varieties play a crucial role in the theory of \( p \)-adic modular forms and canonical subgroups.

(ii) Let \( A \to \text{Spec}(\mathbb{R}[[T]]) \) be a generically ordinary Abelian variety of dimension \( 2g \) with a non-ordinary special fiber. Assume that \( A \) has multiplication by a real field \( K \) of degree \( g \) over \( \mathbb{Q} \). Then the \( \mathbb{F} \)-isocrystal \( M \) associated to \( A \) with coefficients in \( \mathbb{Q}_p \) has rank \( 2g \) and has a linear action by \( \mathbb{Q}_p \otimes K \). If there is only one prime in \( K \) above \( p \), so that \( \mathbb{Q}_p \otimes K \) is a field, we may regard \( M \) as an \( \mathbb{F} \)-isocrystal with coefficients in \( \mathbb{Q}_p \otimes K \) of rank two. The unit-root subcrystal of the generic fiber of \( M \) is a rank-one \( \mathbb{F} \)-isocrystal with coefficients in \( \mathbb{Q}_p \otimes K \). The corresponding Galois representation is surjective by [Rib75] and by Theorem 1.2 it has stable monodromy.

(iii) The rank \( n + 1 \) Kloosterman \( \mathbb{F} \)-isocrystal on \( \mathbb{G}_m \) is irreducible and ordinary at every point with slopes \( \{0, \ldots, n\} \), due to work of Sperber in [Spe80]. The unit-root subcrystal has stable monodromy at 0 and \( \infty \) by Theorem 1.2.
we may deduce an interesting result about genera growth along towers of curves. Let $U$ be a smooth curve over $\mathcal{R}$ and let $C$ be its smooth compactification. Let $\rho: \pi_1(U) \to \mathbb{Z}_p^\dagger$ be a continuous representation. For any $k \geq 1$, we let $g_k(\rho)$ denote the genus of the compact curve corresponding to $\rho^{-1}(1 + p^k\mathbb{Z}_n) \subset \pi_1(U)$.

**Definition 1.4.** Let $a \in \mathbb{Z}_{\geq 1}$ and let $s = v_p(q^a)$. We say that $\rho$ is has $a$-stable genus growth if for every $k \in [0, s]$, there exists a polynomial $b_k(x) \in \mathbb{Q}[x]$ of degree $2s$ such that

$$g_{k+an}(\rho) = b_k(q^n),$$

for $n \gg 0$. We say that $\rho$ has pseudo-stable genus growth if $\rho$ has $a$-stable genus growth for some $a$.

Let $M$ be an overconvergent $\mathbb{F}$-isocrystal on $U$ with coefficients in $\mathbb{Q}_p$. After replacing $U$ with an open dense subset, there exists a slope filtration:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_d = \ell^!(M),$$

where $gr_i(M) = M_i/M_{i-1}$ is isoclinic of slope $\alpha_i$. As in §1.2.1, we may associate a character $\rho_i: \pi_1(U) \to \mathbb{Z}_p^\times$ to det($M_i$) (see §4.2.1.3).

**Theorem 1.5.** Assume $M$ is irreducible and has integral slopes. Then for each $i < d$, the representation $\rho_i$ has pseudo-stable genus growth.

Let $f: X \to U$ be a smooth proper morphism and let $M$ be the overconvergent $\mathbb{F}$-isocrystal $R_{\text{cris}}^i f_* \mathcal{O}_{X,\text{cris}}$ (see Appendix A for an explanation of why this is overconvergent). After shrinking $U$ we may assume that $M$ has a slope filtration. Thus, det($R_{\text{cris}}^i f_* \mathbb{Q}_p$) corresponds to the character $\rho_i$. Assume that $M$ is irreducible and has integral slopes. Theorem 1.5 implies that $\rho_i$ has pseudo-stable genus growth. This proves a weaker version of a conjecture of Wan, which states that there exists a quadratic $b(x) \in \mathbb{Q}[x]$ such that $s_k(\rho) = b(p^k)$ for large $k$ (see [Wan19, Conjecture 5.2]).

### 1.2.2 Global results

By applying the Riemann–Hurwitz formula together with Theorem 1.2, we may deduce an interesting result about genera growth along towers of curves. Let $U$ be a smooth curve over $\mathcal{R}$ and let $C$ be its smooth compactification. Let $\rho: \pi_1(U) \to \mathbb{Z}_p^\dagger$ be a continuous representation. For any $k \geq 1$, we let $g_k(\rho)$ denote the genus of the compact curve corresponding to $\rho^{-1}(1 + p^k\mathbb{Z}_n) \subset \pi_1(U)$.

**Definition 1.4.** Let $a \in \mathbb{Z}_{\geq 1}$ and let $s = v_p(q^a)$. We say that $\rho$ is has $a$-stable genus growth if for every $k \in [0, s]$, there exists a polynomial $b_k(x) \in \mathbb{Q}[x]$ of degree $2s$ such that

$$g_{k+an}(\rho) = b_k(q^n),$$

for $n \gg 0$. We say that $\rho$ has pseudo-stable genus growth if $\rho$ has $a$-stable genus growth for some $a$.

Let $M$ be an overconvergent $\mathbb{F}$-isocrystal on $U$ with coefficients in $\mathbb{Q}_p$. After replacing $U$ with an open dense subset, there exists a slope filtration:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_d = \ell^!(M),$$

where $gr_i(M) = M_i/M_{i-1}$ is isoclinic of slope $\alpha_i$. As in §1.2.1, we may associate a character $\rho_i: \pi_1(U) \to \mathbb{Z}_p^\times$ to det($M_i$) (see §4.2.1.3).

**Theorem 1.5.** Assume $M$ is irreducible and has integral slopes. Then for each $i < d$, the representation $\rho_i$ has pseudo-stable genus growth.

Let $f: X \to U$ be a smooth proper morphism and let $M$ be the overconvergent $\mathbb{F}$-isocrystal $R_{\text{cris}}^i f_* \mathcal{O}_{X,\text{cris}}$ (see Appendix A for an explanation of why this is overconvergent). After shrinking $U$ we may assume that $M$ has a slope filtration. Thus, det($R_{\text{cris}}^i f_* \mathbb{Q}_p$) corresponds to the character $\rho_i$. Assume that $M$ is irreducible and has integral slopes. Theorem 1.5 implies that $\rho_i$ has pseudo-stable genus growth. This proves a weaker version of a conjecture of Wan, which states that there exists a quadratic $b(x) \in \mathbb{Q}[x]$ such that $s_k(\rho) = b(p^k)$ for large $k$ (see [Wan19, Conjecture 5.2]).

### 1.3 Logarithmic decay and slope filtrations

We now give an informal overview of our results on $\mathbb{F}$-isocrystals with logarithmic decay. For simplicity, we restrict ourselves to $\mathbb{Q}_p$-coefficients here. See §9 for more general statements. Let $E = \text{Frac}(W(\mathcal{R}))$, where $W(\mathcal{R})$ is the $p$-typical Witt vectors of $\mathcal{R}$. We define the integral Amice ring

$$\mathcal{O}_\mathcal{E} := \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \mid \text{We have } a_n \in W(\mathcal{R}) \text{ and } \lim_{n \to -\infty} v_p(a_n) = \infty. \right\}.$$

We let $\mathcal{E}$ be the field of fractions of $\mathcal{O}_\mathcal{E}$ and let $\mathcal{E}^\dagger$ be the subring of $\mathcal{E}$ consisting of Laurent series convergent on some annulus $r < |T|_p < 1$. Let $\sigma : \mathcal{O}_\mathcal{E} \to \mathcal{O}_\mathcal{E}$ act on $E$ as the $p$-Frobenius map and send $T \mapsto T^p$. A convergent $\mathbb{F}$-isocrystal over $\text{Spec}(F)$ is a finite-dimensional vector space over $\mathcal{E}$ with an isomorphism $\varphi : M \otimes_{\sigma} \mathcal{E} \to M$ and a compatible differential equation (see §4). An overconvergent $\mathbb{F}$-isocrystal is a finite-dimensional vector space over $\mathcal{E}^\dagger$ with the same extra structure.

Given an overconvergent $\mathbb{F}$-isocrystal $M$, the convergent $\mathbb{F}$-isocrystal $M \otimes_{\mathcal{E}^\dagger} \mathcal{E}$ has a slope filtration. In general, the steps $M_i$ of the slope filtration will not be overconvergent. However, it turns out that there are intermediate ‘logarithmic decay’ rings between $\mathcal{E}^\dagger$ and $\mathcal{E}$, over which $M_i$ are defined. This builds on an idea of Dwork–Sperber and utilized by Wan (see [DS91, Wan99]),
where they consider Frobenius structures with logarithmic decay. To define the $r$-log-decay ring, we need to introduce naive partial valuations on $E$. For any $a(T) = \sum a_n T^n \in E$ we define

$$w_k(a(T)) = \min_{v_p(a_n) \leq k} \{n\}.$$  

That is, $w_k(a(T))$ is the $T$-adic valuation of $a(T)$ reduced modulo $p^{k+1}$. We define $E^r$ to be the subring of $E$ consisting of $a(T)$ such that for some $c > 0$, we have $w_k(a(T)) \geq -c p^k$ for $k$ large. Roughly, a convergent $\mathbb{F}$-isocrystal has $r$-log-decay if the Frobenius and differential equation descend to $E^r$ (the actual definition we use is a bit more subtle, see §4.2.0.2). The following theorem states that the rate of logarithmic decay is closely related to differences between consecutive slopes. We further conjecture that if $M$ is irreducible, then the slopes are entirely determined by the rate of log-decay (see Conjecture 4.11).

**Theorem 1.6.** Let $r = 1/(\alpha_{i+1} - \alpha_i)$. Then $M_i$ has $r$-log-decay.

We also study the monodromy of rank-one $\mathbb{F}$-isocrystals with logarithmic decay.

**Proposition 1.7.** Let $N$ be a rank-one convergent $\mathbb{F}$-isocrystal with $r$-log-decay. Let $\rho : G_F \to \mathbb{Z}_p^\times$ be the corresponding character (well-defined up to unramified twist). Then $\rho$ has $r$-bounded monodromy.

The first part of Theorem 1.2 follows from Theorem 1.6 and Proposition 1.7. There is another somewhat surprising consequence of Theorem 1.6 and Proposition 1.7: a proof of the Drinfeld–Kedlaya theorem for irreducible $\mathbb{F}$-isocrystals on curves. This result first appears in (see [DK17] and [Ked22, Appendix A]), though a local version appeared in Kedlaya’s thesis (see [Ked00]). See Remark 9.8 for a comparison of our approach with the work of Kedlaya.

**Corollary 1.8 (Drinfeld–Kedlaya).** Let $M$ be an irreducible $\mathbb{F}$-isocrystal on a smooth curve $U$ and let $\alpha_1 < \cdots < \alpha_d$ be the generic slopes of $M$. Then $|\alpha_{i+1} - \alpha_i| \leq 1$.

**Remark 1.9.** It is possible to prove the Drinfeld–Kedlaya theorem using logarithmic decay without studying representations with infinite monodromy. This is the content of work of the author (see [Kra21]), where the Drinfeld–Kedlaya theorem follows from studying connections with logarithmic decay. It was discovered somewhat accidentally that one could deduce the Drinfeld–Kedlaya theorem by studying ramification filtrations, and we view it as a fortunate consequence of the main results of this paper.

### 1.4 The question of more general ground fields

It would be interesting to see if Theorem 1.2 or Theorem 1.5 still hold if we only require that $\mathfrak{R}$ have characteristic $p$. In this case, defining the ramification filtration is more nuanced (see the work of Abbes–Saito [AS02]). However, in the case of $\mathbb{F}$-isocrystals with finite monodromy, the differential structure determines the ramification invariants through a differential Swan conductor defined by Kedlaya (see [Ked07]). This is due to work of Chiarellotto–Pulita and Xiao (see [CP09, Xia11]). This suggests that monodromy stability may hold for more general $\mathfrak{R}$.

### 1.5 Outline

In §3 we introduce several rings that will be used throughout the article. In §4 we give an overview of $\mathbb{F}$-isocrystals and introduce the notion of logarithmic decay. Next, in §5 we discuss ramification theory for $p$-adic characters and in §6 we prove a monodromy theorem for rank-one $\mathbb{F}$-isocrystals. We prove several results on recursive Frobenius equations in §7 and we study the growth of the slope filtration in §8. Finally, in §9 we combine the results of §§6–8 to prove our main results.
2. Conventions

Let $R$ be any ring. If $R$ has a valuation $v : R \to \mathbb{R}$ and $x \in R$ satisfies $v(x) > 0$, we let $v_p(\cdot)$ denote the normalization of $v$ satisfying $v_p(x) = 1$. Let $\mathcal{O}_R \subset R$ be the subring of elements with $v(x) \geq 0$. For a matrix $A \in M_{n \times m}(R)$, we let $v(A)$ denote the infimum of the valuations of its entries. For $f \in \mathcal{O}_R$, we let $\tilde{f}$ denote the image of $f$ in the residue field. For any field $F$ we let $G_F$ denote the absolute Galois group of $F$. If $L$ is a Galois extension of $F$ we let $G_{L/F}$ denote the Galois group of $L$ over $F$. We assume all characters/representations of $G_F$ are continuous.

The following conventions are used throughout the article. Let $p$ be prime and let $q = p^j$. We always take $L$ to be a finite extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. Let $e$ be the ramification index of $L$ over $\mathbb{Q}_p$ and let $\pi$ be a uniformizing element of $L$. We let $\Theta$ denote the set of embeddings of $L$ into $\mathbb{Q}_p^{\text{alg}}$. Let $\mathfrak{K}$ be a perfect field containing $\mathbb{F}_q$ and let $E$ be $\Frac(\mathcal{O}_L \otimes_{\mathcal{O}(\mathfrak{K})} \mathcal{W}(\mathfrak{K}))$, where $\mathcal{W}(R)$ denotes the ring of $p$-typical Witt vectors of $R$. Then $E$ is totally ramified over $\Frac(\mathcal{W}(\mathfrak{K}))$. Let $v$ denote the $p$-Frobenius endomorphism on $\Frac(\mathcal{W}(\mathfrak{K}))$ and let $\sigma$ denote the endomorphism $1 \otimes v^f$ on $E$.

3. Some rings

3.1 The Amice ring and the bounded Robba ring

Let $F = \mathfrak{K}((T))$ and let $F^{\text{un}} = \mathfrak{K}^{\text{alg}}((T))$. We define the following $E$-algebras:

$$\mathcal{E}_{F,L} := \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \mid \text{We have } a_n \in E, \lim_{n \to -\infty} v_p(a_n) = \infty, \text{ and the } v_p(a_n) \text{ is bounded below.} \right\},$$

$$\mathcal{E}^\dagger_{F,L} := \left\{ \sum_{n=-\infty}^{\infty} a_n T^n \in \mathcal{E}_{F,L} \mid \text{There exists } m > 0 \text{ such that } v_p(a_n) \geq -mn \text{ for } n \leq 0 \right\}.$$

If there is no ambiguity, we omit $F$ and $L$. Note that $\mathcal{E}^\dagger$ and $\mathcal{E}$ are local fields with residue field $F$. The valuation $v_p$ on $L$ extends to the Gauss valuation on each of these fields. We also define

$$\mathcal{E}_{\infty} = \mathcal{E} \otimes_{\mathcal{W}(\mathfrak{K})} \mathcal{W}(\mathfrak{K}^{\text{alg}}), \quad \text{and} \quad \mathcal{E}_{\infty} = \mathcal{E}_{\infty} \cap \mathcal{E}^\dagger_{F^{\text{un}},L},$$

where $\otimes$ denotes the completed tensor product.

3.2 Logarithmic decay rings

Let $k \in (1/e)\mathbb{Z}$. We define the partial valuation $w_k : \mathcal{E}_{F^{\text{un}},L} \to \mathbb{Z} \cup \infty$ as follows: for $x = \sum a_n T^n$ we have

$$w_k(x) = \min_{v_p(a_n) \leq k} \{ n \}.$$

Informally, $w_k(x)$ is the smallest power of $T$ occurring in $x$ reduced modulo $\pi^{ke+1}$. These partial valuations satisfy the following inequalities:

$$w_k(x + y) \geq \min(w_k(x), w_k(y)),$$

$$w_k(xy) \geq \min_{i+j \leq k} (w_i(x) + w_j(y)). \quad (1)$$

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In either inequality, there is an equality if the minimum is attained exactly once. For \( r > 0 \) we define

\[
\mathcal{E}_r^\tau = \left\{ x \in \mathcal{E}_\infty \ \left| \ \text{there exists } c > 0 \text{ such that } w_k(x) \geq -cr^k \text{ for } k \gg 0 \right. \right\},
\]

\[
\mathcal{E}_r = \mathcal{E}_\infty^r \cap \mathcal{E}.
\]

By (1) both \( \mathcal{E}_\infty^r \) and \( \mathcal{E}_r \) are rings. Let \( s = 1/r \) and let \( P \subset \mathbb{R}^2 \) denote the lower convex hull of the points

\[(0,0), (s,p), (2s,p^2), \ldots .\]

Then \( P \) is the graph of a continuous piece-wise linear function \( f_r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \). For \( c > 0 \) we define

\[
\mathcal{O}_{\mathcal{E}_\infty}^r = \left\{ x \in \mathcal{O}_{\mathcal{E}_\infty} \ \left| \ w_k(x) \geq -cf_r(k), \text{ for } k \gg 0 \right. \right\},
\]

\[
\mathcal{O}_{\mathcal{E}_r}^c = \mathcal{O}_{\mathcal{E}_\infty}^r \cap \mathcal{E}.
\]

As \( f_r \) is super-additive (i.e. \( f_r(x+y) \geq f_r(x) + f_r(y) \) for all \( x, y \geq 0 \)) we know that \( \mathcal{O}_{\mathcal{E}_r}^c \) (respectively, \( \mathcal{O}_{\mathcal{E}_\infty}^r \)) is a \( p \)-adically closed subring of \( \mathcal{O}_{\mathcal{E}} \) (respectively, \( \mathcal{O}_{\mathcal{E}_\infty}^r \)).

**Lemma 3.1.** Let \( x \in \mathcal{O}_{\mathcal{E}_r}^c \) with \( w_0(x) = 0 \). Then \( x^{-1} \in \mathcal{O}_{\mathcal{E}_\infty}^r \).

**Proof.** This follows from (1) and the super-additivity of \( f^r \). \( \square \)

**Proposition 3.2.** The rings \( \mathcal{E}_r^\tau \) and \( \mathcal{E}_r \) are fields.

**Proof.** Let \( x \in \mathcal{E}_r^\tau \). After multiplying by a power of \( p \) and a power of \( T \) we may assume that \( x \in \mathcal{O}_{\mathcal{E}_r} \) and \( w_0(x) = 0 \). Then for \( c \) sufficiently large, we have \( x \in \mathcal{O}_{\mathcal{E}}^r \). The proposition follows from Lemma 3.1. \( \square \)

### 3.3 Auxiliary spaces of power series

We now introduce subrings of \( \mathcal{E}_\infty \) that are used throughout this article. First, extend \( \sigma \) (respectively, \( \nu \)) to an endomorphism of \( \mathcal{E}_\infty, \mathcal{E}_\infty^\sigma, \) and \( \mathcal{E}_\infty^\nu \) (respectively, \( \mathcal{E}_{F,\mathbb{Q}_p}, \mathcal{E}_{F,\mathbb{Q}_p}^\sigma, \) and \( \mathcal{E}_{F,\mathbb{Q}_p}^\nu \)) sending \( T \mapsto T^q \) (respectively, \( T \mapsto T^n \)). We define

\[
\mathcal{E}_{\infty}^{(p^n)} = \left\{ \sum a_nT^n \in \mathcal{E}_\infty \ \left| \ a_n = 0 \text{ for all } p^n \mid n \right. \right\}.
\]

Note that \( \mathcal{E}_{\infty}^{(q)} \) is a \( \sigma(\mathcal{E}_{\infty}) \)-module. Next, for any \( m > 0 \) we define the \( p \)-adically closed ring

\[
\mathcal{O}_{\mathcal{E}_\infty}^{1,m} = \left\{ x \in \mathcal{O}_{\mathcal{E}_\infty} \ \left| \ w_k(x) \geq mk \text{ for all } k \geq 0 \right. \right\}.
\]

Finally, we define the ring

\[
\mathcal{E}_{\infty}^{\dagger^r} = \left\{ \sum a_nT^n \in \mathcal{E}_{\infty}^{\dagger} \ \left| \ a_n = 0 \text{ for all } n > 0 \right. \right\}.
\]

### 4. F-isocrystals and their slope filtrations

#### 4.1 (\( \sigma, \nabla \))-modules

For this subsection we let \( R \) be \( \mathcal{E}_\star, \mathcal{E}_\star^\tau, \) or \( \mathcal{E}_\star^\dagger, \) where \( \star \) is either \( \infty \) or nothing.

**Definition 4.1.** A \( \sigma \)-module over \( R \) is a finite-dimensional vector space \( M \) over \( R \) equipped with a \( \sigma \)-semilinear endomorphism \( \varphi : M \to M \) whose linearization is an isomorphism. That is, we have \( \varphi(am) = \sigma(a)\varphi(m) \) for \( a \in R \) and \( \sigma \varphi : R \otimes_\sigma M \to M \) is an isomorphism. Given a basis
These categories are denoted by \( \text{Isoc}^{\dagger} \) and \( \text{Isoc}^{\ddagger} \). Let \( F \) be the category whose objects are pairs \((A,f)\) over \( E \). A \( \nabla \)-module over \( R \) is a vector space \( M \) over \( R \) equipped with a connection \( \nabla \). That is, \( M \) comes with an \( E \)-linear map \( \nabla : M \to M \otimes_R \Omega_R \) satisfying the Leibnitz rule.

Definition 4.2. Let \( \Omega_R \) be the module of differentials of \( R \) over \( E \). We define the map \( \delta_T : R \to \Omega_R \) to be the map \( \delta_T = (da(T)/dT) \). A \( \nabla \)-module over \( R \) is a vector space \( M \) over \( R \) equipped with a connection \( \nabla \). That is, \( M \) comes with an \( E \)-linear map \( \nabla : M \to M \otimes_R \Omega_R \) satisfying the Leibnitz rule.

Definition 4.3. By abuse of notation, we let \( \sigma : \Omega_R \to \Omega_R \) be the map defined by \( \sigma(f(T)) = \sigma(f(T))d\sigma(T) \). A \((\sigma, \nabla)\)-module is a \( \sigma \)-module \( M \) with a connection \( \nabla \) such that

\[
\begin{align*}
M \xrightarrow{\nabla} M \otimes \Omega_R \\
\downarrow \phi \downarrow \phi \otimes \sigma \\
M \xrightarrow{\nabla} M \otimes \Omega_R,
\end{align*}
\]

is a commutative diagram. We denote the category of \((\sigma, \nabla)\)-modules over \( R \) by \( M\Phi_{R,\sigma} \).

Remark 4.4. Let \( \sigma_0 : R \to R \) be another lift of the \( q \)-Frobenius morphism that extends \( \sigma|_E \). We may define the category \( M\Phi_{R,\sigma_0} \) in an analogous way. One may show that the categories \( M\Phi_{R,\sigma} \) and \( M\Phi_{R,\sigma_0} \) are equivalent (see [Tsu98b, Proposition 3.4.2] for \( R = E, E^\dagger \) and the case where \( R = E^\sigma \) is similar). However, for the purposes of this article it is enough to only consider \( \sigma \).

4.2 F-isocrystals

Let \( X \) be \( \text{Spec}(\mathbb{R}) \), \( \text{Spec}(F) \), or a smooth geometrically connected variety over \( \text{Spec}(\mathbb{R}) \). We let \( F - \text{Isoc}^{\dagger}(X) \) denote the category of overconvergent \( F \)-isocrystals on \( X \) and we let \( F - \text{Isoc}(X) \) denote the category of convergent \( F \)-isocrystals on \( X \) (see [Ked22, \S 2] for precise definitions). These categories are \( \mathbb{Q}_q \)-linear. We define \( F - \text{Isoc}(X) \otimes L \) (respectively, \( F - \text{Isoc}^{\dagger}(X) \otimes L \)) to be the category whose objects are pairs \((M, f)\), where \( M \) is an object of \( F - \text{Isoc}(X) \) (respectively, \( F - \text{Isoc}^{\dagger}(X) \)) and \( f : L \to \text{End}(M) \) is a \( \mathbb{Q}_q \)-linear map. For a finite extension \( L' \) of \( L \), there is a functor \( F - \text{Isoc}(X) \otimes L \to F - \text{Isoc}(X) \otimes L' \) (respectively, \( F - \text{Isoc}^{\dagger}(X) \otimes L \to F - \text{Isoc}^{\dagger}(X) \otimes L' \)). We have the following equivalences of categories:

\[
\begin{align*}
M\Phi_{F,L,\sigma} \cong F - \text{Isoc}(\text{Spec}(F)) \otimes L, \\
M\Phi_{F,L,\sigma}^{\dagger} \cong F - \text{Isoc}^{\dagger}(\text{Spec}(F)) \otimes L.
\end{align*}
\]

There is a functor \( i^\dagger : F - \text{Isoc}(X) \otimes L \to F - \text{Isoc}(X) \otimes L \). In terms of \((\sigma, \nabla)\)-modules, this functor sends a \((\sigma, \nabla)\)-module \( M \) over \( E_{F,L} \) to \( M \otimes \delta^\dagger_{F,L} E_{F,L} \). This functor is known to be fully faithful (see [Ked04]).

4.2.0.1 Pullbacks. Let \( f : Y \to X \) be a smooth morphism. There are pullback functors:

\[
\begin{align*}
f^* : F - \text{Isoc}(X) \otimes L &\to F - \text{Isoc}(Y) \otimes L, \\
f^* : F - \text{Isoc}(X) \otimes L &\to F - \text{Isoc}^{\dagger}(Y) \otimes L.
\end{align*}
\]

Consider the morphism \( \eta : \text{Spec}(F^{\text{un}}) \to \text{Spec}(F) \). Then \( \eta^* \) sends a \((\sigma, \nabla)\)-module \( M \) over \( E_{F,L} \) (respectively, \( \delta_{F,L}^\dagger \)) to \( M \otimes \delta_{F,L}^{\dagger} E_{F^{\text{un}},L} \) (respectively, \( M 
\otimes \delta_{F,L}^{\dagger} E_{F^{\text{un}},L} \)). In particular, the functor \( \eta^* \) factors through the ‘tensor by \( E_{\infty} \)’ (respectively, ‘tensor by \( \delta_{\infty}^{\dagger} \)’) functors.

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4.2.0.2 The log-decay condition. Let $K = \mathbb{R}((u))$ be a finite separable extension of $F$. An object $M$ of $\mathbf{F} - \text{Isoc}(\text{Spec}(K)) \otimes L$ may be realized as a $(\sigma_u, \nabla)$-module over $\mathcal{E}_{K,L}$, where $\sigma_u$ sends $u$ to $u^q$.

Definition 4.5. Let $r > 0$ and let $M$ be an object of $\mathbf{F} - \text{Isoc}(\text{Spec}(K)) \otimes L$. We say that $M$ has $r$-log-decay for $u$ if there exists a $(\sigma_u, \nabla)$-module $M'$ over $\mathcal{E}_{K,L}$ such that $M' \otimes \mathcal{E}_{K,L} \cong M$. We say that an object $M$ of $\mathbf{F} - \text{Isoc}(\text{Spec}(F)) \otimes L$ has $r$-log-decay if for every finite separable morphism $f : \text{Spec}(\mathbb{R}((u))) \rightarrow \text{Spec}(F)$, the pullback $f^* M$ has $r$-log-decay for $u$. We say that $M$ has strict $r$-log-decay if $M$ has $r$-log-decay and does not have $s$-log-decay for any $s < r$.

Remark 4.6. Our definition of $r$-log-decay is slightly ad hoc. One can prove that if an object of $\mathbf{F} - \text{Isoc}(\text{Spec}(F)) \otimes L$ has $r$-log-decay for $T$, then it has $r$-log-decay. This intrinsic approach to $r$-log-decay will appear in future work of the author, but is not necessary for this article.

4.2.1 Unit-root $\mathbf{F}$-isocrystals and $p$-adic representations.

Definition 4.7. Assume $k$ is algebraically closed. We say an object $M$ of $\mathbf{F} - \text{Isoc}(\text{Spec}(k)) \otimes L$ is étale or unit-root if all of its slopes are zero when viewed as a Dieudonné module (see, e.g., [Kat79, §1.3]). More generally, we say that an object $M$ of $\mathbf{F} - \text{Isoc}(X) \otimes L$ (respectively, $\mathbf{F} - \text{Isoc}^\dagger(X) \otimes L$) is unit-root if for every geometric point $x$ in $X$, the pullback $x^* M$ is unit-root. We denote the category of unit-root objects by $\mathbf{F} - \text{Isoc}(X)^{\text{et}} \otimes L$ (respectively, $\mathbf{F} - \text{Isoc}(X)^{1,\text{et}} \otimes L$).

Theorem 4.8 (Katz, see [Kat73] or [Cre87], Tsuzuki, see [Tsu98a]). There is an equivalence of categories

$$\mathbf{F} - \text{Isoc}^{\text{et}}(X) \otimes L \leftrightarrow \left\{ \text{continuous finite-dimensional representations } \rho : \pi_1^{\text{et}}(X) \rightarrow \text{GL}_n(L) \right\}.$$  

If we restrict ourselves to unit-root $\mathbf{F}$-isocrystals over $\text{Spec}(F)$ that are overconvergent, we obtain

$$\mathbf{F} - \text{Isoc}^{1,\text{et}}(\text{Spec}(F)) \otimes L \leftrightarrow \{ \text{continuous representation } \rho : G_F \rightarrow \text{GL}_n(L), |\rho(I_F)| < \infty \},$$  

where $I_F$ denotes the inertia subgroup of $G_F$.

4.2.1.1 Embeddings of $L$ into $\mathbb{Q}_p^{\text{alg}}$. Let $(M, f)$ be an object of $\mathbf{F} - \text{Isoc}(X) \otimes L$. For any $g \in \Theta$, we obtain an object $(M, f \circ g^{-1})$ of $\mathbf{F} - \text{Isoc}(X) \otimes g(L)$. In particular, if $L$ is a Galois extension of $\mathbb{Q}_p$ there is an action of $G_L/\mathbb{Q}_p$ on $(M, f)$. What does this mean in terms of Galois representations? Assume that $(M, f)$ is unit-root and corresponds to the Galois representation $\rho : \pi_1^{\text{et}}(X) \rightarrow \mathcal{O}_L^\times$. Then $(M, f \circ g^{-1})$ corresponds to the composition $g \circ \rho$, which we denote by $\rho^g$.

4.2.1.2 Pullbacks. Let $f : \text{Spec}(Y) \rightarrow \text{Spec}(X)$ be a finite étale morphism. Let $M$ be an object of $\mathbf{F} - \text{Isoc}(X)^{\text{et}} \otimes L$ corresponding to the representation $\rho$ of $\pi_1^{\text{et}}(X)$. Then the pullback $f^* M$ corresponds to the pullback of $\rho$ along the map $\pi_1^{\text{et}}(Y) \rightarrow \pi_1^{\text{et}}(X)$.

4.2.1.3 Galois representations associated to isoclinic $\mathbf{F}$-isocrystals.

Definition 4.9. Let $\omega \in \mathbb{Q}_p^{\text{alg}}$. After enlarging $L$ we may assume that $\omega \in L$. Let $L(\omega)$ denote the object of $\mathbf{F} - \text{Isoc}(\mathbb{R}) \otimes L$ whose Frobenius structure is multiplication by $\omega^{-1}$. By abuse of notation, we regard $L(\omega)$ as an object of $\mathbf{F} - \text{Isoc}(X) \otimes L$. We say that an object $M$ of $\mathbf{F} - \text{Isoc}(X) \otimes L$ is isoclinic of slope $\alpha$ if there exists $\omega \in \mathbb{Q}_p^{\text{alg}}$ with $v_q(\omega) = \alpha$ such that $M \otimes L(\omega)$ is unit-root.
For any \( k \geq 5 \).

5.1 Ramification and \( \mathfrak{p} \)-adic Lie filtrations

Consider a multiplicative character \( \rho \) of \( \pi_1^{\text{ét}}(X) \) corresponding to \( M_i \). We have \( M_1 \cong M_2 \otimes L(\omega_1/\omega_2) \). The \( \mathbb{F} \)-isocrystal \( L(\omega_1/\omega_2) \) is unit-root, and thus corresponds to a \( \mathbb{p} \)-adic character \( \chi \) of \( \pi_1^{\text{ét}}(X) \). Note that \( L(\omega_1/\omega_2) \) descends along the structure map \( X \to \text{Spec}(\mathbb{R}) \). This means that \( \chi \) descends to a character of \( G_\mathbb{R} \). Thus, we may associate to \( M \) a \( \mathbb{p} \)-adic representation of \( \pi_1^{\text{ét}}(X) \) that is well-defined up to twist by a character of \( G_\mathbb{R} \).

4.2.1.4 Slope filtrations and log-decay.

Let \( M \) be an object of \( \mathbb{F} - \text{Isoc}(X) \otimes L \). After replacing \( X \) with a dense open subset, there is a unique slope filtration

\[ 0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_d = i^\dagger(M), \]

where each graded piece \( gr_i(M) = M_i/M_{i-1} \) is isoclinic of slope \( \alpha_i \) and \( \alpha_1 < \alpha_2 < \cdots < \alpha_d \) (see [Kat79, Theorem 2.4.2] or [Ked22, §4]).

**Definition 4.10.** The Newton polygon \( NP(M) \) of \( M \) is the lower convex hull of the points \((\rho(M), \sum_{i=0}^{d} \rho(gr_i(M))/\alpha_i)\) in the \( xy \)-plane, where \( i \) ranges from 0 to \( d \).

This slope filtration is functorial in \( X \). If \( \alpha_1 = 0 \), then \( M_1 \) is a unit-root convergent \( \mathbb{F} \)-isocrystal. In this case, we denote \( \mathfrak{m} = \{ \mathfrak{m} \in \mathfrak{O}_\mathbb{F} \mid \mathfrak{m}(1-x) > k \} \), and \( \mathfrak{O}_\mathbb{F}^+ = \{ \mathfrak{m} \in \mathfrak{O}_\mathbb{F}^+ \mid \mathfrak{m}(x) > k \} \).

Consider a multiplicative character \( \rho : G_F \to \mathfrak{O}_\mathbb{F}^+ \) (respectively, an additive character \( \psi : G_F \to \mathfrak{O}_\mathbb{F}^+ \)). We let \( s_k(\rho) \) (respectively, \( s_k(\psi) \)) denote the largest upper numbering ramification break of the extension corresponding to \( \rho^{-1}(\mathfrak{O}_\mathbb{F}^+) \subset G_F \) (respectively, \( \psi^{-1}(\mathfrak{O}_\mathbb{F}^+) \subset G_F \)).

**Lemma 5.1.** Let \( \psi : G_F \to \mathfrak{O}_\mathbb{F}^+ \) be an additive character with \( \psi(G_F) \subset \mathfrak{O}_\mathbb{F}^+ \). Then \( s_k(\psi) = s_k(\exp(\psi)) \) for all \( k \geq 1 \). Similarly, let \( \rho : G_F \to \mathfrak{O}_\mathbb{F}^+ \) be a multiplicative character with \( \rho(G_F) \subset \mathfrak{O}_\mathbb{F}^+ \). Then \( s_k(\rho) = s_k(\log(\rho)) \) for all \( k \geq 1 \).

The image \( \psi(G_F) \subset \mathfrak{O}_\mathbb{F}^+ \) is a free \( \mathbb{Z}_p \)-module whose rank \( d \) is at most \( [L : \mathbb{Q}_p] \). We may break up \( \psi \) as \( \psi = \sum \psi_i \), such that the image of \( \psi_i \) is a rank-one \( \mathbb{Z}_p \)-module. Note that

\[ s_k(\psi) = \max\{s_k(\psi_i)\}. \tag{2} \]

This follows from the ‘quotient property’ of the upper ramification numbers (see, e.g., [Ser79, Proposition 14]). Finally, we mention a natural restriction on the growth of \( s_k(\psi) \) and \( s_k(\rho) \).
Lemma 5.2. Let \( \rho \) (respectively, \( \psi \)) be a multiplicative (respectively, additive) character. For all \( k \geq 0 \) we have \( s_k+1(\rho) \geq ps_k(\rho) \) and \( s_k+1(\psi) \geq ps_k(\psi) \).

Proof. Let \( F_\infty/F \) be the Abelian \( p \)-adic Lie extension corresponding to \( \psi \). By local class field theory, \( F_\infty/F \) corresponds to an open subgroup \( H \) of \( \mathcal{O}_F^\times \) such that \( \mathcal{O}_F^\times/H \cong G_{F_\infty/F} \). The image of the subgroup \( U^s = 1 + T^s\mathcal{O}_F \) in \( \mathcal{O}_F^\times/H \) corresponds to the subgroup \( G_{F_\infty/F}^{s+1} \). For any group \( A \) we let \( A^p \) denote \( \{ a^p \mid a \in A \} \). As we are in characteristic \( p \), we have \( (U^s)^{x_p} \subset U^{ps} \) for any \( s \). On the Galois side of the correspondence this means \( (G_{F_\infty/F})^{x_p} \subset G_{F_\infty/F}^{ps} \). However, because \( G_{F_\infty/F} \) is isomorphic to \( \mathbb{Z}_d \) for some \( d \), we know that \( (G_{F_\infty/F})^{x_p} = G_{F_\infty/F}^{ps} \). The multiplicative case is identical.

Corollary 5.3. Let \( r < 1 \). If \( \rho \) has \( r \)-bounded monodromy, then \( \rho \) has finite monodromy.

Definition 5.4. Let \( \rho \) be a multiplicative character. We say \( \rho \) is harshly ramified if for \( k \gg 0 \), we have \( s_{k+1}(\rho) > ps_k(\rho) \).

5.2 Ramification and base change

Proposition 5.5. Let \( \rho : G_F \to \mathcal{O}_L^\times \) be a multiplicative character. Let \( K \) be a finite Galois extension of \( F \). There exists \( c \geq 0 \) such that for \( k \) sufficiently large we have

\[
s_k(\rho|_{G_K}) = |G_{K/F}^0|s_k(\rho) - c.
\]

Furthermore, \( c = 0 \) if and only if \( K \) is tamely ramified over \( F \).

Proof. Recall the inverse Hasse–Herbrand function

\[
\psi_{K/F}(s) = \int_0^s [G_{K/F}^0 : G_{K/F}^r] dx.
\]

Let \( F_\infty \) (respectively, \( K_\infty \)) be the fixed field of \( \rho^{-1} \{ 1 \} \) (respectively, \( \rho^{-1}_{G_K} \{ 1 \} \)). We claim that for \( s \) sufficiently large, the map \( G_{K_\infty/K} \to G_{F_\infty/F} \) restricts to an isomorphism

\[
G_{K_\infty/K}^{\psi_{K/F}(s)} \cong G_{F_\infty/F}^{s}.
\]

To see this, let \( K_1 \) be a finite Galois extension of \( F \) containing \( K \). Using [Ser79, §4, Proposition 15], we see that

\[
G_{K_1/F}^s \cap G_{K_1/K} = (G_{K_1/F})_{\psi_{K_1/F}(s)} \cap G_{K_1/K} = (G_{K_1/F})_{\psi_{K_1/K}}(s) = (G_{K_1/F})_{\psi_{K/F}}(s).
\]

By taking a limit along the finite subextensions of \( K_\infty \) over \( F \) we obtain

\[
G_{K_\infty/F}^s \cap G_{K_\infty/K} = G_{K_\infty/K}^{\psi_{K/F}(s)}.
\]

As \( K/F \) is a finite extension we have \( G_{K_\infty/F}^s \subset G_{K_\infty/K} \) for large \( s \). In addition, for \( s \) large the restriction map \( G_{K_\infty/F}^s \to G_{F_\infty/F}^s \) is an isomorphism. Combining this with (5) proves (4). From (4) we see that \( s_k(\rho|_{G_K}) = \psi_{K/F}(s_k(\rho)) \) for \( k \) sufficiently large. The proposition follows from (3).

Corollary 5.6. Assume \( \rho(I_F) \) is infinite. If \( K \) is wildly ramified over \( F \), then \( \rho|_{G_K} \) is harshly ramified.
The monodromy of unit-root $F$-isocrystals with geometric origin

Proof. This follows from Lemma 5.2 and Proposition 5.5. □

Corollary 5.7. Adopt the notation from Proposition 5.5. Then $\rho$ has a-stable monodromy (respectively, $r$-bounded monodromy) if and only if $\rho|_{G_K}$ has a-stable monodromy (respectively, $r$-bounded monodromy).

6. Monodromy of rank-one $F$-isocrystals

6.1 Frobenius structures of $p$-adic characters

6.1.1 Rings of periods. Let $\mathcal{E} = L \otimes_{W(F_{\text{alg}})} W(F_{\text{alg}})$. There is an embedding $\iota : \mathcal{E} \hookrightarrow \mathcal{E}$ that sends $T$ to the Teichmuller lift $[T]$. Recall that $\sigma : \mathcal{E} \rightarrow \mathcal{E}$ (respectively, $\nu : \mathcal{E} \rightarrow \mathcal{E}$) is the endomorphism that sends $T$ to $T^q$ (respectively, $T$ to $T^p$) and restricts to endomorphism $\sigma$ (respectively, $\nu$) of $L$ as defined in § 2. If $K$ is a finite separable extension of $F$, there exists a unique unramified extension $\mathcal{E}^K$ of $\mathcal{E}$ contained in $\mathcal{E}$ whose residue field is $K$ (see [Mat95, Theorem 2.2]). Define $\mathcal{E}^\text{un} = \bigcup_{[K:F]<\infty} \mathcal{E}^K$, and let $\mathcal{E}^\text{un}$ be the $p$-adic completion of $\mathcal{E}^\text{un}$. Note that $G_F$ acts continuously on $\mathcal{E}^\text{un}$.

6.1.2 Multiplicative characters. Let $\rho : G_F \rightarrow \mathcal{O}_L^\times$ be a multiplicative character and let $V = e_0 L$ be a one-dimensional vector space over $L$ on which $G_F$ acts through $\rho$. The corresponding object $M$ of $\text{F - Isoc}^{\text{et}}(\text{Spec}(F)) \otimes L$ is $(\mathcal{E}^\text{un} \otimes_L V)^{G_F}$, where the Frobenius acts by $\sigma \otimes \text{id}$. Thus, $M$ consists of elements $x_0 \otimes e_0$ such that $x_0/x_0^q = \rho(\gamma)$. The Frobenius structure of $M$ is given by $m = x_0^q/x_0$. Note that $m$ is well-defined up to multiplication by elements of the form $c^q/c$, where $c \in \mathcal{E}^\times$.

Let $V_0 = \mathbb{Q}_p e_0$, so that $V = V_0 \otimes_{\mathbb{Q}_p} L$. Assume that $\rho$ factors through a map $\rho_0 : G_F \rightarrow \text{GL}(V_0)$ and let $M_0$ be the corresponding object of $\text{F - Isoc}^{\text{et}}(\text{Spec}(F))$. There is an isomorphism $(\mathcal{E}^\text{un} \otimes_{\mathbb{Q}_p} V_0)^{G_F} \rightarrow (\mathcal{E}^\text{un} \otimes_L V)^{G_F}$ sending $x_0 \otimes_{\mathbb{Q}_p} e_0$ to $x_0 \otimes_L e_0$. Then $m_0 = x_0^q/x_0$ is a Frobenius structure of $M_0$ and we have $m = \prod_{i=0}^{f-1} m_0^{\nu^i}$.

6.1.3 Additive characters. Let $\psi : G_F \rightarrow p\mathcal{O}_L^+$ be an additive character. For some $y_0 \in \mathcal{E}^\text{un}$ and $a \in \mathcal{E}$ we have

\[ y_0 - y_0^\gamma = \psi(\gamma), \]

\[ y_0^\gamma - y_0 = a. \]

We refer to $a$ as the Frobenius structure of $\psi$. It is well-defined up to addition by elements $c^q - c$ for $c \in \mathcal{E}$. If we set $\rho = \exp(\psi)$, then we may take $y_0 = \log(x_0)$ and $a = \log(m)$ (here $x_0$ and $m$ are as in § 6.1.2). If $\psi$ factors through $\psi_0 : G_F \rightarrow \mathbb{Z}_p^+$, we obtain

\[ y_0^\gamma - y_0 = \log(m_0), \]

\[ \log(m) = \sum_{i=0}^{f-1} \log(m_0)^{\nu^i}. \]
6.1.4 Maximal Frobenius structures. Let \( x \in \mathcal{O}_{E_{\text{fun}, L}} \). For \( k \geq 0 \), define the \( k \)th weighted partial valuation as follows:

\[
c_k(x) = \min \{ p^i w_{k-i}(x) \} \cup \{0\}.
\]

These weighted partial valuations satisfy the following properties.

**Lemma 6.1.** Let \( x, y \in \mathcal{O}_{E_{\text{fun}, L}} \). The following hold.

(i) For \( k \geq 0 \) we have \( c_{k+1}(px) = c_k(x) \), \( c_k(x^v) = pc_k(x) \), and \( c_k(x^\sigma) = qc_k(x) \).

(ii) We have \( c_k(x + y) \geq \min(c_k(x), c_k(y)) \). If the minimum is attained exactly once there is equality.

(iii) We have \( c_k(x) + c_k(y) \geq c_k(x \cdot y) \).

(iv) If \( v_p(x - 1) \geq 1 \) (respectively, \( v_p(x) \geq 1 \)), then \( c_k(x) = c_k(\log(x)) \) (respectively, \( c_k(x) = c_k(\exp(x)) \)) for all \( k \geq 1 \).

**Proof.** Statements (i)–(iii) follow from the definition and the statement about exponentials will follow from the statement about logs. Write \( x = 1 + py \), with \( y \in \mathcal{O}_{E_{\text{fun}, L}} \). It is enough to prove \( c_k(p^ny^n/n) > c_k(py) \) for all \( n \geq 2 \). Let \( m = v_p(n) \). Then by statements (i) and (iii), we see

\[
c_k\left(\frac{p^ny^n}{n}\right) \geq \frac{n}{p^{n-1-m}} c_k(py) > c_k(py).
\]

**Definition 6.2.** Let \( x \in \mathcal{O}_{E_{\text{fun}, L}} \). We say that \( x \) is maximal if the following holds: for all \( k \geq 0 \), we have \( q|c_{k+1}(x) \) if and only if \( c_{k+1}(x) = pc_k(x) \).

The term maximal is justified by the following proposition.

**Proposition 6.3.** The following hold.

(i) Let \( x \in \mathcal{O}_{E_{\text{fun}, L}} \) be maximal. If \( y \in \mathcal{O}_{E_{\text{fun}, L}} \), then \( c_k(x) \geq c_k(x + y^\sigma - y) \) for all \( k \geq 0 \).

(ii) Let \( x \in 1 + p\mathcal{O}_{E_{\text{fun}, L}} \) be maximal. If \( y \in 1 + p\mathcal{O}_{E_{\text{fun}, L}} \), then \( c_k(x) \geq c_k(x(y^\sigma/y)) \) for all \( k \geq 0 \).

**Proof.** By Lemma 6.1 it is enough to prove part (i). Let \( z = x + y^\sigma - y \) and let \( k \geq 0 \) be the smallest value with \( c_k(z) > c_k(x) \). As \( x \) is maximal, we have \( c_k(x) < 0 \). By Lemma 6.1 we see that \( c_k(y^\sigma) = c_k(x) \). Thus, \( q|c_k(x) \), so \( c_k(x) = pc_{k-1}(x) \geq pc_{k-1}(z) \). This implies \( c_k(z) > pc_{k-1}(z) \), which is impossible.

**Definition 6.4.** Let \( \rho : G_F \to \mathcal{O}_L^\times \) be a multiplicative character. A maximal Frobenius structure of \( \rho \) is a Frobenius structure \( m \) that is maximal. A \( \Theta \)-maximal Frobenius structure of \( \rho \) is a set \( \{m_\theta\}_{\theta \in \Theta} \), where \( m_\theta \) is a maximal Frobenius structure of \( \rho^\theta \). We make analogous definitions for additive characters.

**Corollary 6.5.** Let \( \rho : G_F \to 1 + p\mathcal{O}_L \) (respectively, \( \psi : G_F \to \mathcal{O}_L^+ \)) be a multiplicative (respectively, additive) character. Let \( \alpha_1, \alpha_2 \) be a maximal Frobenius structure of \( \rho \) (respectively, \( \psi \)). Then \( c_k(\alpha_1) = c_k(\alpha_2) \) for all \( k \geq 0 \).

It will be helpful to distinguish when a Frobenius structure is maximal. This motivates the following.

**Lemma 6.6.** If \( x \in \mathcal{O}_{E_{\text{fun}}} \), then \( x \) is maximal.
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Remark 6.7. Let $\rho : G_F \to 1 + p\mathcal{O}_L$ be a multiplicative character. One may show that $\rho$ has a Frobenius structure contained in $\mathcal{O}_{\mathcal{E}_{\infty}^p}$ (start with a Frobenius structure $\alpha \equiv 1 \mod p$ and successively find a Frobenius structure that looks like it is in $\mathcal{O}_{\mathcal{E}_{\infty}^p}$ modulo powers of $\pi$). Thus, by Lemma 6.6 we know that $\rho$ has a maximal Frobenius structure.

6.2 The additive situation
For this subsection we assume $\mathfrak{R}$ is a finite field.

Proposition 6.8. Let $\psi : G_F \to \mathcal{O}_L^+$ be an additive character and let $\{a_g\}_{g \in \Theta}$ be a $\Theta$-maximal Frobenius structure. Then, for $k \geq 0$, we have

\[ \frac{q}{p} s_k(\psi) = \max_{g \in \Theta} \{-c_k(a_g)\}. \]  

We deduce this proposition from a theorem due to Kosters and Wan as follows.

Theorem 6.9 (Kosters–Wan, see [KW18, Proposition 3.3 or §4.1]). Let $\psi : G_F \to p\mathbb{Z}_p^+$ be an additive character that surjects onto $p\mathbb{Z}_p^+$. Then there exists a maximal Frobenius $a \in \mathcal{O}_{\mathcal{E}_F, \mathfrak{O}_p}$ of $\psi$ and $-c_k(a) = s_k(\psi)$ for all $k \geq 0$.

Proof. Let $F_{\infty}/F$ be the fixed field of $\ker(\psi)$. Then $F_{\infty}/F$ is a $\mathbb{Z}_p$-extension, and thus by Artin–Schreier–Witt theory corresponds to an equivalence class of $W(F)/(1 - \nu)W(F)$ (here $\nu$ denotes the standard $p$-power Frobenius endomorphism on the ring of $p$-typical Witt vectors). By [KW18, Proposition 3.1], for any $b \in \mathfrak{R}$ with $\text{Tr}_{\mathfrak{R}/F}(b) \neq 0$, there exists a unique representative $a_0$ in $z$ of the form

\[ a_0 = r[b] + \sum_{i \geq 1, (i, p) = 1} r_i T^{-i}, \]

where $r \in \mathbb{Z}_p$ and $r_i \in W(\mathfrak{R})$ (here we regard $\mathcal{E}$ as a subring of $W(F)$ via the map $\iota$ defined at the beginning of §6.1). Note that by Lemma 6.6, the element $a_0$ is maximal. As in §6.1.3, from $a_0$ we obtain a surjective character $\psi_0 : \text{Gal}(F_{\infty}/F) \to \mathbb{Z}_p$ (take $y_0$ satisfying $y_0^p - y_0 = a_0$ and then set $\psi_0(g) = y_0^g - y_0$). Thus, for some $b \in \mathbb{Z}_p$, we have $\psi = b\psi_0$, so that $a = ba_0$ is a maximal Frobenius structure of $\psi$. A proposition due to Kosters–Wan [KW18, Proposition 3.3] states that $s_n(\psi_0)$ is equal to $-c_n(a_0)$ (note that [KW18] state their result in terms of conductors, but this easily translates into a statement about higher ramification groups). From here we deduce that $s_n(\psi) = -c_n(a)$. \hfill $\Box$

Corollary 6.10. Let $\psi : G_F \to \mathcal{O}_L^+$ be an additive character such that $\psi(G_F) \subset b\mathbb{Z}_p^+$ for some $b \in \mathcal{O}_L^+$. Let $a$ be a maximal Frobenius structure of $\psi$. Then $(q/p) s_k(\psi) = -c_k(a)$ for all $k \geq 0$.

Proof. Let $\psi_0 : G_F \to \mathbb{Z}_p^+$ be the character defined by $b^{-1}\psi$ and let $a_0$ be a maximal Frobenius of $\psi_0$. By Lemma 6.1 we see that $b \sum_{i = 0}^{f-1} a_0^{i}$ is a Frobenius structure of $\psi$. The result follows from Corollary 6.5 and Theorem 6.9. \hfill $\Box$

Definition 6.11. Let $M \subset \mathcal{O}_L$ be a $\mathbb{Z}_p$-module and let $B$ be a basis of $M$. For $i = 0, \ldots, e - 1$, define $B_i = \{b \in B \mid v_p(b) \equiv i/e \mod \mathbb{Z}\}$. In particular, we may write $B_i = \{\pi^i p^{r,i} u_{i,1}, \ldots, \pi^i p^{r,i} u_{i,d_i}\}$. We say $B$ is nice if for each $i$, the reductions of $u_{i,1}, \ldots, u_{i,d_i}$ modulo $\pi$ are linearly independent over $F_p$.

Lemma 6.12. Every $\mathbb{Z}_p$-module $M \subset \mathcal{O}_L$ has a nice basis.
Proof of Proposition 6.8. Let \( B \) be a subset \( B \subset B \subset \mathbb{A} \) that is continuous with \( \sum_{b \in B} b \bmod \pi \neq 0 \). Let \( g \) be an element of \( G_{L/\mathbb{Q}_p} \) that reduces to the \( p \)-th power map on \( \mathcal{O}_{L}/\pi \mathcal{O}_{L} \) and let \( c = g(\pi)/\pi^i \). Then

\[
\sum_{b \in B} x_b b^i \equiv c \pi^i p^n \sum_{b \in B} \frac{d}{\pi} b^j \bmod \pi^{c^r+1}.
\]

□

Corollary 6.14. Consider a subset \( \{y_b\}_{b \in B} \subset \mathcal{O}_{E,F,\mathbb{Q}_p} \). There exists \( g \in \Theta \) such that

\[
\max_{b \in B} \{-c_k(y_b)\} = -c_k \left( \sum_{b \in B} y_b b^j \right).
\]

Proof. Let \( n = \min_{b \in B} v_k(y_b) \) and let \( x_b \in W(\mathbb{A}) \) denote the coefficient of \( T^j \) in \( y_b \). Note that \( \min_{b \in B} v_p(x_b) \leq k \). By Lemma 6.13, there exists \( g \in \Theta \) with \( v_p(\sum_{b \in B} x_b b^j) \leq k \). It follows that the \( p \)-adic valuation of the coefficient of \( T^n \) in \( \sum_{b \in B} y_b b^j \) is less than \( k \).

Proof of Proposition 6.8. Let \( B \) be a nice basis of \( \psi(G_F) \). Decompose \( \psi \) as \( \sum_{b \in B} b \psi_b \), where the image of \( \psi_b \) is \( \mathbb{Z}_p \). For each \( b \in B \), let \( y_b \in \mathcal{O}_{E,F,\mathbb{Q}_p} \) be a maximal Frobenius of \( \psi_b \). Then \( a_g = \sum y_b b^j \) is a maximal Frobenius of \( \psi^g \) for each \( g \in \Theta \). By (2) and Corollaries 6.10 and 6.14 we have

\[
\frac{q}{p} s_k(\psi) = \max_{b \in B} \left\{ \frac{q}{p} s_k(b \psi_b) \right\} = \max_{b \in B} \{-c_k(y_b)\} = \max_{g \in \Theta} \{-c_k(a_g)\}.
\]

□

6.3 The multiplicative situation

Again, we assume \( \mathbb{K} \) is a finite field for this subsection.

Proposition 6.15. Let \( \rho : G_F \to 1 + p\mathcal{O}_L \) and let \( \{m_g\}_{g \in \Theta} \) be a \( \Theta \)-maximal Frobenius structure of \( \rho|_{F^w} \). Then for \( k \geq 0 \), we have

\[
\frac{q}{p} s_k(\rho) = \max_{g \in \Theta} \{-c_k(m_g)\}.
\]

Proof. By Lemma 6.1 we know that \( \{\log(m_g)\}_{g \in \Theta} \) is a \( \Theta \)-maximal Frobenius structure for \( \log \varphi \) and that \( c_k(m_g) = c_k(\log(m_g)) \) for all \( k \geq 0 \). The proposition follows from Lemma 5.1 and Proposition 6.8. □
Corollary 6.16. Adopt the notation of Proposition 6.15 and assume $\rho$ is harshly ramified. For $k$ large, we have

$$\frac{q}{p} s_k(\rho) = \max_{g \in \Theta} \{-w_k(m_g)\}.$$  

Corollary 6.17. Let $\rho$ be a multiplicative character of $G_F$ and let $M$ be the corresponding unit-root $\mathbb{F}$-isocrystal. Let $r > 0$ and assume that $M^g$ has $r$-log-decay for each $g \in \Theta$. If $r \geq 1$, then $\rho$ has $r$-bounded monodromy. If $r < 1$, then $M$ is overconvergent and $\rho$ has finite monodromy.

Proof. After replacing $F$ with a finite extension we may assume $\rho(G_F) \subset 1 + p \mathcal{O}_L$. We may also assume that $\rho$ is either harshly ramified or unramified by Corollary 5.6. Let $\{m_g\}_{g \in \Theta}$ be a $\Theta$-maximal Frobenius structure. There exists $c > 0$ such that $\{m_g\}_{g \in \Theta} \subset \mathcal{O}_c^{r,c}$. By Proposition 6.15, we know $\rho$ has $r$-bounded monodromy. The statement about $r < 1$ follows from Corollary 5.3. □

7. Recursive Frobenius equations

For this section, we define $E_a$ to be $\text{Frac}(\mathcal{O}_L \otimes_{\mathbb{F}} W(\mathbb{F}_{q^a}))$ and $\mathcal{E}_a$ (respectively, $\mathcal{E}_a^\dagger$) to be $\mathcal{E}_{\mathbb{F}_{q^a}((T)),L}$ (respectively, $\mathcal{E}_{\mathbb{F}_{q^a}((T)),L}^\dagger$). Note that $\mathcal{E}_\infty$ is the closure of $\bigcup_{a=1}^\infty \mathcal{E}_a$ and $\bigcup_{a=1}^\infty \mathcal{E}_a^\dagger$ is dense in $\mathcal{E}_\infty$.

7.1 Basic definitions

Let $A \in M_{d \times d}(\mathcal{O}_{\mathcal{E}_\infty})$ with $v_q(A) > 0$ and let $C \in M_{d \times 1}(\mathcal{O}_{\mathcal{E}_\infty})$. We define $R_a(A;C)$ to be the unique $d \times 1$ matrix satisfying the recursive Frobenius equation:

$$x = Ax^a + C. \quad (7)$$

We define $R(A;C)$ to be $R_1(A;C)$. These solutions have the following explicit formula:

$$R_a(A;C) = \sum_{i=0}^{\infty} A^{(1-a)/\sigma}(1-\sigma) C^{\sigma^i}. \quad (8)$$

More generally, for $A_1, \ldots, A_m, B_1, \ldots, B_{m-1} \in M_{d \times d}(\mathcal{O}_{\mathcal{E}_\infty})$ with $v_q(A_i) > 0$, we give the recursive definition:

$$R_a(A_1, \ldots, A_m; B_1, \ldots, B_{m-1}, C) = R_a(A_1, \ldots, A_{m-1}; B_1, \ldots, B_{m-1}, C \cdot R_a(A_m; C)^\sigma).$$

If the $A_i$ are all equal to $A$, we define

$$R_a(A; B_1, \ldots, B_{m-1}, C) = R_a(A, \ldots, A; B_1, \ldots, B_{m-1}, C).$$

When $d = 1$ we drop the pretext of dealing with matrices and view everything as elements of $\mathcal{O}_{\mathcal{E}_\infty}$.

Lemma 7.1. We have

$$R_a(A, B + C) = R_a(A, B) + R_a(A, C), \quad (9)$$

$$R_a(A + B, C) = \sum_{m=0}^{\infty} R_a(A; B, \ldots, B, C)^m. \quad (10)$$

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Proof. The first equation is immediate. Define $S_m = R_n(A; B, \ldots, B, C)$ and note that
\[
S_m = \begin{cases} 
    AS_m^a + BS_m^a & m \geq 1, \\
    AS_0^a + C & m = 0.
\end{cases}
\]
Thus, $\sum S_m$ is the solution to the Frobenius equation $x = (A + B)x^a + C$, which proves (10).

7.2 Growth of recursive Frobenius equations

We begin by proving some basic properties of $\mathcal{O}^{\infty}_{E_c}$, which are used throughout the rest of the article.

**Lemma 7.2.** Let $r, c > 0$ and let $\omega \in \mathcal{O}_L$ satisfy $v_q(\omega) = 1/r$.

(i) If $x \in \mathcal{O}^{r,c}_{E_\infty}$, then $\omega x \in \mathcal{O}^{r-1,c}_{E_\infty}$ and $x^\sigma \in \mathcal{O}^{r,c}_{E_\infty}$.

(ii) If $y = \omega x \in \mathcal{O}^{r,c}_{E_\infty}$ and $w_0(x) > 0$, then $x \in \mathcal{O}^{r,qc}_{E_\infty}$ and $y^{1+\sigma+\ldots+\sigma^n} \in \mathcal{O}^{r,qc}_{E_\infty}$.

(iii) Let $x$ and $y$ be as in part (ii) and let $z \in \mathcal{O}^{r,c}_{E_\infty}$. Then $yz^\sigma \in \mathcal{O}^{r,qc}_{E_\infty}$.

Proof. Parts (i)–(ii) follow from the definition of $\mathcal{O}^{r,c}_{E_\infty}$. To prove part (iii), note that $xz^\sigma \in \mathcal{O}^{r,qc}_{E_\infty}$.

**Corollary 7.3.** Let $y = \omega x \in \mathcal{O}^{r,c}_{E_\infty}$ and assume that $w_0(x) > 0$. If $z \in \mathcal{O}^{r,c}_{E_\infty}$, then $R(y, z) \in \mathcal{O}^{r,qc}_{E_\infty}$.

7.2.1 Approximating Frobenius equations. For this section, we fix $c > 0$ and $c > c_1 > 0$.

**Lemma 7.4.** There exists $N_0 > 0$, depending only on $c$ and $c_1$, such that:

(i) if $x, y \in \mathcal{O}^{1,c}_{E_\infty}$ and $q^{N_0}|x, y$, then $xy^\sigma \in \mathcal{O}^{1,c_1}_{E_\infty}$.

(ii) let $x \in \mathcal{O}^{1,c}_{E_\infty} \cap q\mathcal{O}_E$ and let $y \in \mathcal{O}^{1,c_1} \cap q^{N_0}\mathcal{O}_E$; then $xy^\sigma \in \mathcal{O}^{1,c_1} \cap q^{N_0}\mathcal{O}_E$.

Proof. Parts (i)–(ii) follow from the definition of $\mathcal{O}^{1,c}_{E_\infty}$. To prove part (iii), note that $xz^\sigma \in \mathcal{O}^{1,qc}_{E_\infty}$.

For the remainder of this subsection, we fix $N_0 > 0$ as in Lemma 7.4.

**Lemma 7.5.** Let $x \in \mathcal{O}^{1,c}_{E_\infty} \cap \mathcal{O}^{1,c}_{E_\infty}$. Then, for some $a \geq 0$, we may write $x = y + z$, where $y \in \mathcal{O}_{E_n}((T)) \cap \mathcal{O}^{1,c}_{E_\infty}$ and $z \in \mathcal{O}^{1,c_1}_{E_\infty} \cap q^{N_0}\mathcal{O}_E$.

Proof. Write $x = y + z$, where $z \in \mathcal{O}^{1,c_1}_{E_\infty} \cap q^{N_0}\mathcal{O}_E$ and $y$ is a Laurent series with a finite pole. As $\cup \mathcal{O}_{E_n}^1$ is dense in $\mathcal{O}_{E_\infty}^1$, we may take $y$ to lie in $\mathcal{O}_{E_n}((T))$ for $a$ sufficiently large. Furthermore, we know $y \in \mathcal{O}^{1,c}_{E_\infty}$, as both $x$ and $z$ lie in $\mathcal{O}^{1,c}_{E_\infty}$.

**Lemma 7.6.** Let $A$ be a matrix such that $qA \in M_{d \times d}(\mathcal{O}^{r,c}_{E_\infty})$ and $w_0(A) \geq 0$. Let $C \in M_{d \times 1}(\mathcal{O}^{1,c}_{E_\infty})$ and $Y_1$ (respectively, $Y_2$) be an $d \times d$ (respectively, $d \times 1$) matrix with entries in $\mathcal{O}^{1,c_1}_{E_\infty} \cap q^{N_0}\mathcal{O}_E$. Then,
\[
R(qA + Y_1, C) = R(qA, C) \mod \mathcal{O}^{1,c_1}_{E_\infty} \cap q^{N_0}\mathcal{O}_E, \quad (11)
\]
\[
R(qA, Y_2 + C) = R(qA, C) \mod \mathcal{O}^{1,c_1}_{E_\infty} \cap q^{N_0}\mathcal{O}_E. \quad (12)
\]
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Proof. Let $Z_n = (qA)^{(1-\sigma^n)/(1-\sigma)}Y^\sigma^n$. We have $Z_n \in \mathcal{O}_{E_\infty}^{1,1} \cap q^{-N_0}\mathcal{O}_{E_\infty}$ by Lemma 7.4 and the relation $Z_{n+1} = AZ_n^s$. Then (12) follows from (8) and (9). To prove (11), by Lemma 7.1 it is enough to show $R(qA:Y_1, \ldots, Y_i, C)$ has entries in $\mathcal{O}_{E_\infty}^{1,1} \cap q^{-N_0}\mathcal{O}_{E_\infty}$ for $m \geq 1$. This follows from Lemma 7.4.

\[ \square \]

7.3 Spaces of recursive Frobenius solutions

A tuple $\lambda$ will be taken to mean a finite tuple of negative integers $\lambda = (m_1, \ldots, m_r)$. We define $\text{len}(\lambda)$ to be $r$, i.e. the length of the tuple. For any $c \in \mathbb{Z}_{>0}$ we let $c\lambda$ denote the tuple obtained by scalar multiplication. We say $\lambda$ is $q^a$-prime if $q^a \mid m_i$ for each $i \geq 1$. If $\text{len}(\lambda) \geq 1$, we define

$S_a(\lambda) = R_a(q^{a}; T^{m_1}, \ldots, T^{m_r})$.

and if $\text{len}(\lambda) = 0$ we set $S_a(\lambda) = 1$. For $\mu_0, \mu_1 \in \mathbb{Z}_{<0}$ we have the following relations:

\[
S_a(\mu_0, \lambda) = T^{\mu_0} \cdot S_a(q^a\lambda) + q^aS_a(q^a\mu_0, q^a\lambda),
\]

\[
S_a(\mu_0, \mu_1, \lambda) = S_a(\mu_0 + q^a\mu_1, q^a\lambda) + q^aS_a(\mu_0, q^a\mu_1, q^a\lambda),
\]

which follow from (7). Finally, we define the following $E_a[T^{-1}]$-modules:

$N_a = \left\{ \sum_{i=1}^{n} b_i S_a(\lambda_i) \mid b_i \in E_a[T^{-1}] \text{ and the } \lambda_i \text{ are tuples} \right\},$

$N_a^{(p)} = \left\{ \sum_{i=1}^{n} b_i S_a(\lambda_i) \mid b_i \in E_a[T^{-1}] \text{ and the } \lambda_i \text{ are } q^a\text{-prime tuples} \right\}.$

Lemma 7.7. We have $N_a^{(p)} = N_a$.

Proof. We proceed by induction on $r = \text{len}(\lambda)$. When $r = 1$ the result follows from (13). Let $r \geq 1$ and assume the result holds for all $k < r$. Write $\lambda = (\mu_0, \mu_1, \lambda_0)$, where $\text{len}(\lambda_0) = r - 2$. By our inductive hypothesis we may assume that $\langle \mu_1, \lambda_0 \rangle$ is $q^a$-prime. If $q^a \mid \mu_0$, from (13) and (14) we obtain

$S_a(\mu_0, \mu_1, \lambda_0) = S_a(\mu_0 + q^a\mu_1, q^a\lambda_0) - T^{\mu_0}S_a(\mu_1, q^a\lambda_0) + S_a\left(\frac{\mu_0}{q^a}, \mu_1, \lambda_0\right).$

Both $S_a(\mu_0 + q^a\mu_1, q^a\lambda_0)$ and $T^{\mu_0}S_a(\mu_1, q^a\lambda_0)$ are contained in $N_a^{(p)}$ by our inductive hypothesis, so it suffices to prove $S_a(\mu_0/q^a, \mu_1, \lambda_0) \in N_a^{(p)}$. Repeating this argument proves the lemma. \[ \square \]

Lemma 7.8. Let $x \in E_\infty$ and let $a, b$ be integers with $b = a \cdot c$. We have

$R_a(q; x) = \sum_{i=0}^{c-1} q^{ia}R_b(q^b; x^{q^i}).$

Lemma 7.9. Let $C \in M_{d \times 1}(E_\infty)$ have entries in $N_a$ and let $B \in T^{-1}M_{d \times d}(E_a[T^{-1}])$. Then $R(q1_d; BC)$ has entries in $N_a$.

Corollary 7.10. Let $B_1, \ldots, B_m \in T^{-1}M_{d \times d}(E_a[T^{-1}])$ and $C \in T^{-1}M_{d \times 1}(E_a[T^{-1}])$. Then we have $R(q1_d; B_1, \ldots, B_m, C)$ has entries in $N_a$.

Proposition 7.11. Let $A \in q1_d + q^2T^{-1}M_{d \times d}(O_{E_\infty})$ and $C \in qT^{-1}M_{d \times 1}(O_{E_\infty})$. Let $c_1 > 0$ and let $N_0$ be a positive integer. Then, for a sufficiently large, there exists $Z_{c_1} \in M_{d \times 1}(E_\infty)$ with

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entries in \( \mathcal{N}_a^{(p)} \) such that
\[
R(qA, C) \equiv Z_{c_1} \mod \mathcal{O}^{1,c_1}_E \cap q^{N_0} \mathcal{O}_E.
\]

**Proof.** Write \( A \) as \( qA_1 + q^2 A_1 \). There exists \( c > c_1 \) such that \( qA_1 \) and \( C \) have entries in \( \mathcal{O}^{1,c}_E \). After increasing \( N_0 \), we may assume that the results of \S 7.2.1 hold. By Lemma 7.5 and Lemma 7.6, we may also assume the entries of \( qA_1 \) and \( C \) lie in \( T^{-1} \mathcal{O}_E [T^{-1}] \cap \mathcal{O}^{1,c}_E \). Then, by Corollary 7.3, we know that
\[
R(qA_1, \ldots, qA_1, C)
\]
is contained in \( M_{d \times 1} (\mathcal{O}^{1,c}_E) \) for all \( m \). Thus, for \( m_0 \) sufficiently large, we know that
\[
R(qA_1, \ldots, qA_1, C) = q^m R(qA_1, \ldots, qA_1, C)
\]
has entries in \( \mathcal{O}^{1,c_1}_E \cap q^{N_0} \mathcal{O}_E \) for all \( m > m_0 \). From Lemma 7.1 we obtain
\[
R(A, C) \equiv \sum_{m=0}^{m_0} R(qA_1, \ldots, qA_1, C_0) \mod \mathcal{O}^{1,c_1}_E \cap q^{N_0} \mathcal{O}_E.
\]
The right-hand side of this equivalence has entries in \( \mathcal{N}_a^{(p)} \) by Lemma 7.7 and Corollary 7.10. □

### 7.4 Stable growth for solutions of Frobenius equations

In this subsection, we study the growth of certain solutions to recursive Frobenius equations.

**Definition 7.12.** Let \( S \) be a finite subset of \( \mathcal{O}_E \). Let \( a \) be a positive integer and let \( s = \nu_p(q^a) \). We say that \( S \) has \( a \)-stable growth if for any \( k \in [0,s] \), there exists \( m_k \) and \( b_k \) such that
\[
\min_{x \in S} (w_k(x)) = m_k q^a + b_k
\]
for \( n \gg 0 \). We say that \( S \) has stable growth if \( S \) has \( a \)-stable growth for some \( a \).

**Lemma 7.13.** For \( i = 1, 2 \), let \( S_i \subset \mathcal{O}_E \) have \( a_i \) stable growth. Then \( S_1 \cup S_2 \) has \( \text{lcm}(a_1,a_2) \)-stable growth.

**Proof.** Observe that a set with \( a \)-stable growth has \( ad \)-stable growth for \( d \in \mathbb{Z}_{\geq 1} \). □

To state our main result of this subsection, we define the following spaces:
\[
\mathcal{M}_a = \mathcal{N}_a \otimes_{E_a[T^{-1}]} E_a((T))
\]
\[
\mathcal{M} = \bigcup_{a=1}^{\infty} \mathcal{M}_a.
\]

**Proposition 7.14.** Let \( S \subset \mathcal{M} \) and assume there exists \( c_1 > 1 \) such that
\[
\min_{x \in S} (w_k(x)) \leq -c_1 p^k
\]
for all \( k \gg 0 \). Then \( S \) has stable growth.

The proof of Proposition 7.14 is broken into several lemmas.
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Lemma 7.15. Let $a \in \mathbb{Z}_{\geq 1}$ and $s = v_p(q^a)$. Fix $x \in \mathcal{O}_{E_\infty}$ and $k \geq 0$. Assume that $q^a \nmid w_k(x)$ and that
\[ w_{k+sn}(x) > q^anw_k(x) \tag{15} \]
for $n \in \mathbb{Z}_{\geq 1}$. Then for $n \in \mathbb{Z}_{\geq 0}$ we have
\[ w_{k+sn}(R_a(q^a, x)) = q^anw_k(R_a(q^a, x)). \]

Proof. We proceed by induction on $n$. The case where $n = 0$. Set $n \geq 1$. Let $z = R_a(q^a, x)$, so that $z = q^az^{\sigma} + x$. Note that $w_k(z) \leq w_k(x)$, because $q^a \nmid w_k(x)$ and $q^a | w_k(q^az^{\sigma})$. We have
\[
\begin{align*}
w_{k+sn}(z) & \geq \min(w_{k+sn}(q^az^{\sigma}), w_{k+sn}(x)) \\
& = \min(q^aw_{k+sn}(z), w_{k+sn}(x)) \\
& = \min(q^anw_k(z), w_{k+sn}(x)).
\end{align*}
\]
By (15) we know that $q^anw_k(z) < w_{k+sn}(x)$, which proves the result. \qed

Lemma 7.16. Let $a$ and $s$ be as in Lemma 7.15. Let $\beta_1, \ldots, \beta_r \in \mathcal{O}_{E_a}$ and let $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ be a set of $q^a$-prime tuples. We set $x = \sum \beta_i S_a(\lambda_i)$. Then for any $k \in [0, s]$, there exists $m_k$ such that
\[ w_{k+sn}(x) = m_kq^an \tag{16} \]
for $n$ sufficiently large.

Proof. Define $L(\Lambda)$ to be $\max\{\text{len}(\lambda_i)\}$. We proceed by induction on $L(\Lambda)$. When $L(\Lambda)$ is 1, then $x = R_a(q^a, y)$ with $y \in \mathcal{O}_{E_a}[T^{-1}] \cap \mathcal{E}^0(q^a)$ and the lemma follows from Lemma 7.15. Now assume the proposition holds for all collections of $q^a$-prime tuples $\Lambda'$ such that $L(\Lambda') \leq m$. Let $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ be a collection of $q^a$-prime tuples with $L(\Lambda) = m + 1$. There exist tuples $\Lambda' = \{\lambda'_1, \ldots, \lambda'_{r'}\}$ and negative integers $\{\kappa_1, \ldots, \kappa_{r'}\}$ such that $L(\Lambda') = m$ and $\lambda_i = (\kappa_i, \lambda'_i)$. For each $d < 0$ we define
\[ y_d = \begin{cases} 0 & \text{if } d \neq \kappa_i \text{ for every } i, \\ \sum \kappa_i = d, \beta_i S_a(\lambda'_i) & \text{otherwise.} \end{cases} \]
Let $d_1, \ldots, d_t$ be the values of $d$ for which $y_d \neq 0$. Then we have
\[
x = \sum_{i=1}^{r} R_a(q^{a}; \beta_i T^{\kappa_i} S_a(\lambda'_i)) = \sum_{j=1}^{t} R_a(q^{a}; T^{d_j} y_{d_j}).
\]
By our inductive assumption, there exists $m_{j,k}$ such that $w_{k+sn}(y_{d_j}) = q^am_{j,k}$ for $n \gg 0$. Without loss of generality, we may assume $d_1 + q^a m_{1,k} < \min_{j \geq 2} (d_j + q^a m_{j,k})$ for $n \gg 0$. This holds because the $d_j$ are distinct. Thus, $w_{k+sn}(x) = d_1 + q^a m_{1,k}$, for large $n$. The proposition follows from Lemma 7.15. \qed

Proof of Proposition 7.14. By Lemma 7.13, it is enough to prove the proposition for $S = \{x\} \subset \mathcal{M}_a$. After multiplying $x$ by a large power of $qT$ we may write $x = \sum_{i=1}^{r} b_i S_a(\lambda_i)$, where $b_i \in \mathcal{O}_{E_a}[T]$ and $\lambda_i$ are $q^a$-prime. Let $A \subset \mathcal{O}_{E_a}$ be the $\mathcal{O}_{E_a}$-module generated by $S_a(\lambda_1), \ldots, S_a(\lambda_r)$. For $N$ sufficiently large, we know $p^N A \subset \mathcal{O}_{E_a}^{[1]}$. Let $X = \{x_1, \ldots, x_t\} \subset A$ be a system of
representatives of $A/p^N A$. Then

$$x = \sum_{i=0}^{\infty} y_i T^i \mod O_{E_\alpha}^{1, c_1},$$

where $y_i \in A$ and $z_i \in X$ with $y_i \equiv z_i \mod p^N A$. After reorganizing (17), we obtain

$$x \equiv \sum_{j=1}^{t} x_j f_j(T) \mod O_{E_\alpha}^{1, c_1},$$

where $f_j(T) \in T^{d_j} + T^{d_j+1} O_{E_\alpha}[[T]]$ and the $d_j$ are distinct. By Lemma 7.16, there exists $m_{j,k}$ such that for $n$ large we have

$$w_{k+sn}(x_j f_j(T)) = d_j + m_{j,k} q^{an}.$$

For $n$ sufficiently large, the values $d_j + m_{j,k} q^{an}$ are distinct. Without loss of generality, we may assume that $d_1 + m_{1,k} q^{an} < \min_{j \geq 2} (d_j + m_{j,k} q^{an})$, which implies $w_{k+sn}(x) = d_1 + m_{1,k} q^{an}$ for $n$ large.

\section{8. Growth properties of the slope filtration}

\subsection{8.1 Local setup}

Let $M$ be a rank-$n$ object of $F - \text{Isoc}^\dagger(F) \otimes L$ with slope filtration

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_d = \nu(M),$$

where each graded piece $gr_i(M) = M_i/M_{i-1}$ is isoclinic of slope $\alpha_i$ and has rank $n_i$ (see § 4.2.1.4). After replacing $L$ with a finite ramified extension, we may assume there exists $\omega_i \in L$ with $v_q(\omega_i) = \alpha_i$. From (18), we know that there is a Frobenius matrix $A_0$ of $\nu(M)$ of the form

$$A_0 = \begin{pmatrix}
\omega_1 A_1 & * & * & \cdots & * \\
0 & \omega_2 A_2 & * & \cdots & * \\
0 & 0 & \omega_3 A_3 & \cdots & * \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \omega_d A_d
\end{pmatrix},$$

where $\omega_i A_i$ is the Frobenius structure of $gr_i(M)$. As $gr_i(M) \otimes L(\omega_i)$ is unit-root, we may assume that $A_i$ has entries in $O_E$. This follows from the construction of the Frobenius structure from the corresponding Galois representation (see, e.g., [Kat73, § 4]).

\subsection{8.2 The shape of the Frobenius structure of $M$}

\subsubsection{8.2.1 Logarithmic growth of the slope filtration}

We now show that $M_1$ has a Frobenius matrix with log-decay entries. The rate of log-decay depends on the difference of the first two slopes.

\begin{lemma}
Let $A$ be a Frobenius matrix of $M$. Let $N > \alpha_d$ and let $r = 1/(\alpha_2 - \alpha_1)$. There exists $C \in M_{n \times n}(O_{E^r})$ such that $CAC^{-\sigma}$ is of the form

$$\begin{pmatrix}
\omega_1 A_{1,1} & q^N A_{1,2} \\
0 & \omega_2 A_{2,2}
\end{pmatrix},$$

where $A_{1,1}$, $A_{1,2}$ and $A_{2,2}$ have entries in $O_{E^r}$ and $A_{1,1}$ is an $n_1 \times n_1$ matrix.
\end{lemma}
Proof. After replacing $M$ with $M \otimes L(\omega_1)$, we may assume $\alpha_1 = 0$ and $\omega_1 = 1$. Consider the Frobenius matrix $A_0$ of $\iota^1(M)$ from (19). We may conjugate $A_0$ by a matrix with powers of $q$ along the diagonal, so that each $*$ in the upper-right is divisible by $q^N$. Next, we skew-conjugate (19) by a matrix with powers of $T$ along the diagonal so that $w_0(A_0^{-1}) \geq 0$. Let $A$ be a Frobenius matrix of $M$. There exists $B \in \text{GL}_n(\mathcal{O}_\mathcal{E})$ such that $BAB^{-\sigma} = A_0$. By approximating $B$ with some $B_1 \in \text{GL}_n(\mathcal{O}_\mathcal{E}^1)$ and skew-conjugating $A$, we may assume that

$$A = \begin{pmatrix} A_{1,1} & q^N A_{1,2} \\ q^N A_{2,1} & \omega_2 A_{2,2} \end{pmatrix},$$

where $A_{1,1} \equiv A_1 \mod q^N$. For $c$ sufficiently large, the entries of $A_{1,1}, q^N A_{1,2}, q^N A_{2,1},$ and $\omega_2 A_{2,2}$ are contained in $\mathcal{O}_\mathcal{E}^{r,c}$. We show inductively that there exists $C_k = \begin{pmatrix} 1_n \cdot 1_n \cdot \cdots \cdot 1_n \end{pmatrix}$ such that:

(i) $A_k = C_k A C_k^{-\sigma}$ is of the form $\begin{pmatrix} A_{1,1,k} & q^N A_{1,2} \\ \omega_2 A_{2,1,k} \omega_2 A_{2,2,k} \end{pmatrix}$;

(ii) the entries of $\omega_2^1 A_{2,1,k}, \omega_2 A_{2,2,k}$, and $A_{1,1,k}$ are contained in $\mathcal{O}_\mathcal{E}^{r,c}$;

(iii) for all $k$ we have $C_k \equiv C_{k-1} \mod \omega_2^k$.

The result follows by taking $C = \lim_{k \to \infty} C_k$. When $k = 1$, this follows from (20). Now let $k \geq 1$ and assume $C_k$ exists. We define $D_k = \begin{pmatrix} 1_n \cdot 1_n \cdot \cdots \cdot 1_n \end{pmatrix}$ and set $C_{k+1} = D_k C_k$. It is immediate that parts (i) and (iii) are satisfied. We verify part (ii) using Lemma 7.2.  

8.2.2 Approximating the Frobenius structure by diagonal matrices.

Lemma 8.2. Let $N > \alpha_d$. After replacing $F$ with a finite extension, we may assume that $M$ has a Frobenius matrix of the form

$$A \equiv \begin{pmatrix} \omega_1 1_n & 0 & 0 & \cdots & 0 \\ 0 & \omega_2 1_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_d 1_{r_d} \end{pmatrix} \mod q^N. \tag{21}$$

Proof. Let $\rho_i : G_F \to \text{GL}_n(\mathcal{O}_L)$ denote the representation corresponding to $gr_i(M) \otimes L(\omega_i)$. After replacing $F$ with a finite separable extension, we may assume that $\rho_i(G_F) \subseteq 1_n + q^N M_{n \times n}(\mathcal{O}_L)$ for each $i$. In particular, there exists a Frobenius matrix $A_i$ of $gr_i(M) \otimes L(\omega_i)$ such that

$$A_i \equiv 1_n \mod q^N. \tag{22}$$

As in the proof of Lemma 8.1, we may assume the $*$ in (19) are divisible by $q^N$. Then (22) gives

$$A_0 \equiv \begin{pmatrix} \omega_1 1_n & 0 & 0 & \cdots & 0 \\ 0 & \omega_2 1_n & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_d 1_{r_d} \end{pmatrix} \mod q^N. \tag{21}$$

Let $A$ be a Frobenius matrix of $M$. There exists a matrix $B \in \text{GL}_n(\mathcal{E})$ such that $BAB^{-\sigma} = A_0$. The lemma follows by taking $B_0 \in \text{GL}_n(\mathcal{E}^1)$ whose entries are sufficiently close to $B$. \qed
The case of integer slopes. We now assume that \( \alpha_i = i - 1 \) and fix \( N > \alpha_d \). Assume that \( M \) has a Frobenius matrix as in Lemma 8.2. We may write

\[
A = \begin{pmatrix}
A_{1,1} & q^N A_{1,2} & q^N A_{1,3} & \cdots & q^N A_{1,d} \\
q^N A_{2,1} & q A_{2,2} & q^N A_{2,3} & \cdots & q^N A_{2,d} \\
q^N A_{3,1} & q^N A_{3,2} & q^2 A_{3,3} & \cdots & q^N A_{3,d} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
q^N A_{d,1} & q^N A_{d,2} & q^N A_{d,3} & \cdots & q^{d-1} A_{d,d}
\end{pmatrix},
\]

where \( A_{i,j} \in M_{n_i \times n_j}(\mathcal{O}_E) \) and \( q^{i-1}A_{i,i} \equiv q^{j-1}A_{i,j} \mod q^N \). In particular, there exists \( c > 0 \) such that \( A \) and \( A_{1,1}^{-1} \) have entries in \( \mathcal{O}_E^{1,c} \).

Proposition 8.3. Let \( c > c_1 > 0 \) and let \( N_0 > N > \alpha_d \). There is a Frobenius matrix \( A \) of \( \eta^* M \) satisfying:

(i) the congruence (21) holds and \( A \) has entries in \( \mathcal{O}_E^{1,c} \cap \mathcal{O}_E^{1,c} \);
(ii) (bottom left) for any \( i > j \), the matrix \( q^N A_{i,j} \) is divisible by \( q^{N_0} \);
(iii) (first column) for \( i > 2 \), the matrix \( q^N A_{i,1} \) has entries in \( \mathcal{O}_E^{1,c} \);
(iv) (first row) for \( j \geq 1 \), the matrix \( A_{1,j} \) has entries in \( \mathcal{O}_E^{(q)} \);
(v) the matrix \( A_{2,1} \) has entries \( T^{-1}\mathcal{O}_E^{1,c} \); the matrix \( A_{i,i} \) lies in \( 1_{n_i} + qT^{-1}M_{n_i \times n_i}(\mathcal{O}_E^{1,c}) \) for \( i = 1, 2 \).

Proof. The Frobenius matrix (23) already satisfies property (i). We prove properties (ii)–(v) in two steps. We must ensure that the second step does not undo the first step. To this end, we increase \( N_0 \) so that Lemma 7.4 is satisfied.

**Step 1.** Consider the matrix

\[
B = \begin{pmatrix}
1_{n_1} & 0 & 0 & \cdots & 0 \\
-q^N A_{2,1} & 1_{n_2} & 0 & \cdots & 0 \\
-q^N A_{3,1} & -q^N A_{3,2} & 1_{n_3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-q^N A_{d,1} & -q^N A_{d,2} & -q^N A_{d,3} & \cdots & 1_{n_d}
\end{pmatrix}.
\]

By Lemma 7.2, we know \( BAB^{-\sigma} \) has entries in \( \mathcal{O}_E^{1,c} \). In addition, we have

\[
BAB^{-\sigma} \equiv \begin{pmatrix}
A_{1,1} & q^N \ast & q^N \ast & \cdots & q^N \ast \\
0 & q A_{2,2} & q^N \ast & \cdots & q^N \ast \\
0 & 0 & q^2 A_{3,3} & \cdots & q^N \ast \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & q^{d-1} A_{d,d}
\end{pmatrix} \mod q^{N+1}.
\]

Furthermore, from Lemma 7.2 we see that the \((j,1)\)-block matrix has entries in \( \mathcal{O}_E^{1,q^{-1}c} \) for \( j > 2 \). After repeating this finitely many times we obtain a Frobenius matrix satisfying properties (i)–(iii).

**Step 2.** By the previous step, we may assume properties (i)–(iii) hold. Let \( m > 0 \) be large enough so that \( A \) has entries in \( \mathcal{O}_E^{1,m} \) (see §3.3). Assume that \( A \) satisfies properties (iv) and (v) modulo \( q^c \). We find a matrix \( B \) such that:

(a) \( B \in M_{n \times n}(\mathcal{O}_E^{1,m}) \) and \( B \equiv 1_n \mod q^c \).
(b) \( BAB^{-\sigma} \) satisfies properties (i)–(iii) and properties (iv) and (v) modulo \( q^{c+1} \).
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The proposition follows by taking the limit. This limit exists because $\mathcal{O}_{E_\infty}^{1,m}$ is $p$-adically closed.

For $j > 1$ (respectively, $j = 1$) we write $q^N A_{1,j} = C_j + D_j^q$ (respectively, $A_{1,1} = 1_{d_1} + C_1 + D_1^q$), where $C_j$ has entries in $\mathcal{O}_{E_\infty}^{(q)}$. We have $v \geq v_q(D_j)$ for each $j$ by our inductive assumption. Consider the matrix

$$B_0 = \begin{pmatrix}
1_{n_1} - D_1 & -D_2 & \cdots & -D_d \\
0 & 1_{n_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1_{n_d}
\end{pmatrix}.$$ 

Note that $B_0 A B_0^{-\sigma}$ satisfies properties (i)–(iii) (we are using Lemma 7.4 to verify property (iii)). Furthermore, $B_0^{-\sigma} \in M_{n \times n}(\mathcal{O}_{E_\infty}^{1,m})$, because each $D_j^q$ has entries in $\mathcal{O}_{E_\infty}^{1,m}$. The top row of $B_0 A B_0^{-\sigma}$ is equivalent to $[1 + C_1 \ C_2 \ \cdots \ C_d]$ modulo $q^{v+1}$.

By the previous paragraph, we may assume $A$ satisfies properties (i)–(iii) and (iv) modulo $q^{v+1}$. Write $q^N A_{2,1} = X_{2,1} + Y_{2,1}$ with $X_{2,1} \in T^{-1} \mathcal{O}_{E_\infty}^{1,c}$ and $Y_{2,1} \in \mathcal{O}_E[[T]] \otimes \mathcal{O}_{E_\infty}$. Similarly, we write $A_{i,i} = X_{i,i} + Y_{i,i}$ with $X_{i,i} \in T^{-1} \mathcal{O}_{E_\infty}^{1,c}$ and $Y_{i,i} \in \mathcal{O}_E[[T]] \otimes \mathcal{O}_{E_\infty}$ for $i = 1, 2$. There exists $Z_i$ with entries in $\mathcal{O}_E[[T]] \otimes \mathcal{O}_{E_\infty}$ such that $Z_i^q - Z_i = Y_{i,i}$ (note that this is only true after tensoring by $\mathcal{O}_{E_\infty}$). Consider the matrix

$$B_1 = \begin{pmatrix}
1_{n_1} + Z_1 & 0 & 0 & \cdots & 0 \\
-Y_{2,1} & 1_{n_2} + Z_2 & 0 & \cdots & 0 \\
0 & 0 & 1_{n_3} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1_{n_d}
\end{pmatrix}.$$ 

The matrix $B_1 A B_1^{-\sigma}$ still satisfies property (iv) modulo $q^{v+1}$ and properties (i)–(iii) (again, we use Lemma 7.4 to verify property (iii)). Furthermore, we see that $B_1 A B_1^{-\sigma}$ satisfies property (v) modulo $q^{v+1}$ by construction. \hfill $\square$

Remark 8.4. In the proof Proposition 8.3, it was sufficient to remain in $\mathcal{O}_{E_\infty}^{1,c}$ for properties (i)–(iv). It is only necessary to go to the geometric fiber $\eta^* M$ for property (v).

8.3 The Frobenius structure of the unit-root subcrystal

We continue with the setup from the beginning of §8.2.3, with the additional assumption that $M^{\text{unit}}$ has rank one (see §4.2.1.4). This subsection is dedicated to proving the following proposition.

PROPOSITION 8.5. There exists a maximal Frobenius $\lambda \in \mathcal{O}_{E_\infty}$ of $\eta^* M^{\text{unit}}$ such that the following is satisfied: for any $c_1 > 0$, there exists $\lambda_{c_1} \in \mathcal{M}$ with

$$\lambda \equiv \lambda_{c_1} \mod \mathcal{O}_{E_\infty}^{1,c_1}.$$ 

Let $N_0$ be sufficiently large so that the results of §7.2.1 hold. Let $A$ be a Frobenius matrix of $\eta^* M$ satisfying the properties in Proposition 8.3 and write $A$ as a block matrix as in (23). Let $e = (e_1, \ldots, e_n)$ be the basis of $\eta^* M$ such that $\varphi(e^T) = Ae^T$ and let $u \in \eta^* M^{\text{unit}}$. After normalizing $u$ we have

$$u = e_1 + e_2 e_2 + \cdots + e_n e_n,$$

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where $e_i \in \mathcal{E}_\infty$. The Frobenius structure of $\eta^* M_{\text{unit}}^\lambda$ is given by $\lambda \in \mathcal{E}_\infty$ satisfying $\varphi(u) = \lambda u$.

Write $a_{i,j}$ for the $(i,j)$ entry of $A$ and let $b_{i,j} = a_{i,j}/a_{1,1}$. We obtain the following equations:

\[
\begin{aligned}
\lambda &= a_{1,1} + a_{1,2} e_2^\sigma + \cdots + a_{1,n} e_n^\sigma, \\
\epsilon_i &= b_{i,1} + b_{i,2} x_2^\sigma + \cdots + b_{i,n} x_n^\sigma - (b_{1,2} e_2^\sigma \epsilon_i + \cdots + b_{1,n} e_n^\sigma \epsilon_i).
\end{aligned}
\]

For each $i > n_2 + 1$, we have $b_{i,1} \in \mathcal{O}_{\mathcal{E}_\infty}^{1,c} \cap q^{N_0} \mathcal{O}_{\mathcal{E}_\infty}$.

By Lemma 8.3(iv), we know $\lambda \in \mathcal{O}_{\mathcal{E}_\infty}^{(q)}$ and by Lemma 6.6, we know $\lambda$ is maximal. Define the matrices

\[
B = \begin{bmatrix} b_{2,2} & \ldots & b_{2,n} \\
\vdots & \ddots & \vdots \\
b_{n,2} & \ldots & b_{n,n} \end{bmatrix} \quad \text{and} \quad \mathbf{x} = [x_2 \ldots x_n],
\]

where $x_i = b_{i,1} - \sum_{j=2}^n b_{i,j} e_j^\sigma \epsilon_i$. It is convenient to write $B$ as the block matrix $[B_{1,1} \ B_{2,2}]$, where $B_{1,1} = a_{1,1}^{-1} A_{2,2}$. The vector $\epsilon = [\epsilon_2 \ldots \epsilon_n]$ satisfies the recursive equation

\[
\epsilon^T = B \epsilon^{\sigma T} + \mathbf{x}^T.
\]

From Proposition 8.3 and Lemma 7.4 we may deduce the following lemma about $B$.

**Lemma 8.6.** We have the following.

(i) The entries of $B$ lie in $\mathcal{O}_{\mathcal{E}_\infty}^{1,c} \cap \mathcal{O}_{\mathcal{E}_\infty}^{1,c}$. 

(ii) The block matrix $B_{2,1}$ has entries in $\mathcal{O}_{\mathcal{E}_\infty}^{1,c} \cap q^{N_0} \mathcal{O}_{\mathcal{E}_\infty}$. 

(iii) For $i > n_2 + 1$, we have $b_{i,1} \in \mathcal{O}_{\mathcal{E}_\infty}^{1,c} \cap q^{N_0} \mathcal{O}_{\mathcal{E}_\infty}$. 

(iv) The matrix $B_{1,1}$ is of the form $ql_{n_2} + q^2 C$, where $q^2 C$ has entries in $T^{-1} \mathcal{O}_{\mathcal{E}_\infty}^{1,c}$. 

**Lemma 8.7.** For each $i$ we have $\epsilon_i \in \mathcal{O}_{\mathcal{E}_\infty}^{1,c} \cap q^{N_0} \mathcal{O}_{\mathcal{E}_\infty}$. 

**Proof.** We first prove that $v_q(\epsilon) \geq N_0$. Let $k = v_q(\epsilon)$. By the definition of $\mathbf{x}$ and the fact that $q^{N_0}|b_{i,1}$ we see $v_q(\mathbf{x}) \geq \min(k + 1, N_0)$. Then from (25) we see that $v_q(\epsilon) \geq \min(k + 1, N_0)$. Thus, $v_q(\epsilon) \geq N_0$. Next, we show that $\epsilon$ has entries in $\mathcal{O}_{\mathcal{E}_\infty}^{1,c} + q^k \mathcal{O}_{\mathcal{E}_\infty}$ for every $k$. For $k = N_0$ this is immediate. Let $k \geq N_0$ and assume $\epsilon$ has entries in $\mathcal{O}_{\mathcal{E}_\infty}^{1,c} + q^k \mathcal{O}_{\mathcal{E}_\infty}$. Then $\epsilon_i = u_i + q^k v_i$, where $u_i \in \mathcal{O}_{\mathcal{E}_\infty}^{1,c}$. As $q|B$, we know

\[
Be^{\sigma T} \equiv B \begin{bmatrix} u_2^\sigma & \ldots & u_n^\sigma \end{bmatrix}^T \mod q^{k+1}.
\]

By Lemma 7.2, the right-hand side of (26) is contained in $\mathcal{O}_{\mathcal{E}_\infty}^{1,c}$. Thus, $B \epsilon^{\sigma T}$ has entries in $\mathcal{O}_{\mathcal{E}_\infty}^{1,c} + q^{k+1} \mathcal{O}_{\mathcal{E}_\infty}$. Similarly, observe $\mathbf{x}$ has entries in $\mathcal{O}_{\mathcal{E}_\infty}^{1,c} + q^{k+1} \mathcal{O}_{\mathcal{E}_\infty}$. By (25), we know $\epsilon$ has entries in $\mathcal{O}_{\mathcal{E}_\infty}^{1,c} + q^{k+1} \mathcal{O}_{\mathcal{E}_\infty}$. 

**Lemma 8.8.** Let $B_0$ denote the $(n-1) \times (n-1)$ matrix $[B_{1,1} \ 0 \ 0]$ and let

\[
\mathbf{x}_0 = \begin{bmatrix} b_{2,1} & \ldots & b_{n_2+1,1} & 0 & \ldots & 0 \end{bmatrix}.
\]

Then

\[
R(B, \mathbf{x}^T) \equiv R(B_0, \mathbf{x}_0^T) \mod \mathcal{O}_{\mathcal{E}_\infty}^{1,c} \cap q^{N_0} \mathcal{O}_{\mathcal{E}_\infty}.
\]

**Proof.** First, note that $b_{i,j} e_j^\sigma \epsilon_i \in \mathcal{O}_{\mathcal{E}_\infty}^{1,c} \cap q^{N_0} \mathcal{O}_{\mathcal{E}_\infty}$. This follows from Lemmas 7.4, 8.6, and 8.7. Furthermore, for $i > n_2 + 1$ we know from Lemma 8.6 that $b_{i,1} \in \mathcal{O}_{\mathcal{E}_\infty}^{1,c} \cap q^{N_0} \mathcal{O}_{\mathcal{E}_\infty}$. This means
\( \mathbf{x} - \mathbf{x}_0 \) has entries in \( \mathcal{O}^{1,\mathbf{c}_1}_\mathcal{E}_\infty \cap q^N \mathcal{O}_\mathcal{E}_\infty \). By Lemma 7.6 we have
\[
R(B, \mathbf{x}) \equiv R(B, \mathbf{x}_0) \mod \mathcal{O}^{1,\mathbf{c}_1}_\mathcal{E}_\infty \cap q^N \mathcal{O}_\mathcal{E}_\infty.
\]
Let \( B_1 = \begin{bmatrix} 0 & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix} \) and set \( S_m = R(B_0; B_1, \ldots, B_1, \mathbf{x}_0) \). By Lemma 7.1, it suffices to prove \( S_m \)
has entries in \( \mathcal{O}^{1,\mathbf{c}_1}_\mathcal{E}_\infty \cap q^N \mathcal{O}_\mathcal{E}_\infty \) for \( m \geq 1 \). Write \( \mathbf{z}_0 = [z_2 \ldots z_n] \). By Corollary 7.3 we know \( z_i \in \mathcal{O}^{1,\mathbf{c}_1}_\mathcal{E}_\infty \cap q^N \mathcal{O}_\mathcal{E}_\infty \). In addition, \( z_i = 0 \) for \( i > n_2 + 1 \). Thus, by Lemmas 7.4 and 8.6, the entries of \( B_1 \mathbf{z}_0 \) are contained in \( \mathcal{O}^{1,\mathbf{c}_1}_\mathcal{E}_\infty \cap q^N \mathcal{O}_\mathcal{E}_\infty \). The result follows from Lemmas 7.4 and 7.6.

**Proof of Proposition 8.5.** Let \( [y_2 \ldots y_n]^T = R(B_0, \mathbf{x}_0^T) \) and note that \( y_i = 0 \) for \( i > n_2 + 1 \). By (24) and Lemmas 7.4 and 8.8, we see that
\[
\lambda \equiv a_1,1 + a_1,2 y_2^{\sigma} + \cdots + a_1,n_2+1 y_{n_2+1}^{\sigma} \mod \mathcal{O}^{1,\mathbf{c}_1}_\mathcal{E}_\infty \cap q^N \mathcal{O}_\mathcal{E}_\infty. \tag{27}
\]
Let \( \mathbf{y}_0 = [y_2 \ldots y_{n_2+1}]^T \) and note that \( \mathbf{y}_0 = R(B_1, [b_{2,1} \ldots b_{n_2+1,1}]^T) \). By Lemma 8.6, the hypothesis of Proposition 7.11 is satisfied. Therefore, there exist \( a > 0 \) and a column matrix \( z = [z_2, \ldots, z_{n_2+1}]^T \) with entries in \( \mathcal{N}_a \) (see § 7.3 for the definition of this space) such that
\[
\mathbf{y}_0 \equiv z \mod \mathcal{O}^{1,\mathbf{c}_1}_\mathcal{E}_\infty \cap q^N \mathcal{O}_\mathcal{E}_\infty.
\]
By Lemma 7.5, after increasing \( a \) we may assume that \( a_{1,i} = u_i + v_i \), where \( u_i \in \mathcal{O}_{\mathcal{E}_a}((T)) \) and \( v_i \in \mathcal{O}^{1,\mathbf{c}_1}_\mathcal{E}_\infty \cap q^N \mathcal{O}_\mathcal{E}_\infty \). Then we set
\[
\lambda_{\mathbf{c}_1} = u_1 + u_2 z_2^{\sigma} + \cdots + u_{n_2+1} z_{n_2+1}^{\sigma},
\]
which is contained in \( \mathcal{M}_a \) (see § 7.4 for the definition of this space). From Lemma 7.4 we have
\[
\lambda \equiv \lambda_{\mathbf{c}_1} \mod \mathcal{O}^{1,\mathbf{c}_1}_\mathcal{E}_\infty \cap q^N \mathcal{O}_\mathcal{E}_\infty,
\]
which proves the proposition.

\[ \square \]

## 9. Main results

### 9.1 Local results

**Theorem 9.1.** Adopt the notation from § 8.1. Let \( r_i = 1/(\alpha_{i+1} - \alpha_i) \). Then \( M_i \) has \( r_i \)-log-decay.

**Proof.** We first show that \( M_i \) has \( r_i \)-log-decay for \( T \). The general result will follow by the fact that the slope filtration is functorial. Note that \( M_i \) has \( r_i \)-log-decay for \( T \) if and only if \( \det(M_i) \) has \( r_i \)-log-decay for \( T \). This follows from a standard exterior power trick ([Kat79, Theorem 2.4.2] or [Kra21, proof of Proposition 6.1]). Thus, by replacing \( M \) with \( \wedge^{\text{rank}(M_i)} M \), it suffices to prove the result when \( i = 1 \) under the assumption that \( M_1 \) has rank one. Let \( A \) be a Frobenius matrix of \( M \) and let \( C \) be the corresponding connection matrix. By Lemma 8.1, there exists \( B \in \text{GL}_n(\mathcal{O}_{\mathcal{E}_i}) \) such that
\[
BAB^{-\sigma} = \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}.
\]

The connection matrix after this change of basis is
\[
B^{-1}(\delta_T(B) + CB) = \begin{pmatrix} Q & R \\ S & U \end{pmatrix},
\]
whose entries are contained in \( \mathcal{O}_{\mathcal{E}_i} \). From (28) we see that \( M \otimes \mathcal{O}_{\mathcal{E}_i} \) has a sub-\( \sigma \)-module \( M_i' \) of rank one. We prove that \( S = 0 \). This implies that \( M_i' \) is fixed by \( \nabla \), which means \( M_i' \) is a \( (\sigma, \nabla) \)-module over \( \mathcal{E}_i \). The uniqueness of the slope filtration implies \( M_i' = M_1 \).
The compatibility between the Frobenius and the connection gives the relation

\[ SX = \frac{dT^\sigma}{dT} Z S^\sigma. \]

Then \( v_p(SX) = v_p(S) \), because \( X \) is a \( p \)-adic unit. In addition, note that \( v_p((dT^\sigma/dT)Z) > 0 \) and \( v_p(S) = v_p(S^\sigma) \). In particular, if \( v_p(S) < \infty \), then \( v_p((dT^\sigma/dT)Z S^\sigma) = \infty \). It follows that \( v_p(S) = \infty \). \( \square \)

**Remark 9.2.** The proof of Theorem 9.1 uses an exterior power trick to reduce to the case of \( i = 1 \) and \( M_1 \) has rank one. It is natural to ask whether the same trick can be applied to Conjecture 4.11. The answer is no (at least without some additional work). An issue arises if \( \wedge^\text{rank}(M_i) M \) is not irreducible. It could happen that \( \wedge^\text{rank}(M_i) M \) has an irreducible subobject \( N \) such that \( \text{det}(N) \) is the first step in the slope filtration for \( N \). It is not clear that the second smallest slope of \( N \) will be the second smallest slope of \( \wedge^\text{rank}(M_i) M \).

**Corollary 9.3.** Keep the notation from Theorem 9.1. If \( \alpha_{i+1} - \alpha_i > 1 \), then \( M_i \) is overconvergent.

**Proof.** Assume that \( \alpha_{i+1} - \alpha_i > 1 \). Let \( g \in \Theta \). From Theorem 9.1 we know that for \( \text{det}(M_i^g) \) has \( r_i \)-log-decay, where \( r_i < 1 \). By Corollary 6.17, we see that \( \text{det}(M_i) \) is overconvergent. The fully faithfulness of \( i^g \) (see [Ked04]) implies \( \text{det}(M_i) \) is a subobject of \( \wedge^\text{rank}(M_i) M \) in \( F - \text{Isoc}^1(\text{Spec}(F)) \otimes L \). The result follows from the same exterior product argument used in Theorem 9.1. \( \square \)

**Corollary 9.4.** Let \( M \) be an irreducible object of \( F - \text{Isoc}^1(\text{Spec}(F)) \) with integral slopes. Then Conjecture 4.11 holds.

**Proof.** We know that \( M_i \) has 1-log-decay by Theorem 9.1. If there exists \( r < 1 \) such that \( M_i \) has \( r \)-log-decay, then \( \text{det}(M_i) \) also has \( r \)-log-decay. From Corollary 6.17, we find that \( \text{det}(M_i) \) is overconvergent. Then Kedlaya’s fully faithfulness theorem tells us that \( M \) is not irreducible. \( \square \)

**Theorem 9.5.** Adopt the notation from Theorem 9.1 with the additional assumption that \( M \) is irreducible. Let \( \rho_i \) denote a Galois representation associated to \( \text{det}(M_i) \) (see § 4.2.1.3). The following hold:

(i) \( \rho_i \) has \( r_i \)-bounded monodromy;
(ii) if \( M \) has integer slopes, then \( \rho_i \) has stable monodromy.

**Proof.** The first statement follows from Corollary 6.17 and Proposition 9.1. Assume the slopes of \( M \) are integers. It suffices to prove this result for \( i = 1 \) under the assumption that \( M_1 \) is unit-root and has rank one. After replacing \( F \) with a finite ramified extension we may assume that \( \rho_1 \) is harshly ramified (see Corollary 5.6). We may also assume Proposition 8.5 is satisfied. These base changes do not change the result by Corollary 5.7. As \( \rho_1 \) is harshly ramified, there exists \( c_1 \) such that \( s_k(\rho_1) > c_1 p^k \) for all \( k \geq 0 \). By Proposition 8.5, for each \( g \in \Theta \) there exists a maximal Frobenius \( \lambda_g \) of \( M_i^g \) with \( \lambda_g = \lambda_{g,c_1} + z_{g,c_1} \), where \( \lambda_{g,c_1} \in M \) and \( z_{g,c_1} \in O_L^{1,c_1} \). From Corollary 6.16, we know

\[ s_k(\rho_1) = \max_{g \in \Theta} \{-w_k(\lambda_{g,c_1})\} \]

for all \( k \). The theorem follows from Proposition 7.14. \( \square \)
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**Corollary 9.6.** Let \( \psi : V \to \text{Spec}(F) \) be a smooth proper morphism. Then \( M = R^i\psi_*\mathcal{O}_{V,\text{cris}} \) is an object of \( \mathbf{F} - \text{Isoc}(\text{Spec}(F)) \). Assume that \( R^i\psi_*\mathcal{O}_{V,\text{cris}} = L \) and let \( \rho \) be the Galois representation corresponding to the lisse étale sheaf \( \det(R^i\psi_*\mathcal{O}_{V,\text{cris}}) \).

(i) Let \( \alpha \) denote the first non-zero slope of \( M \) and let \( r = 1/\alpha \). Then \( \rho \) has \( r \)-bounded monodromy.

(ii) Assume \( M \) has integer slopes. Then either \( \rho \) has finite monodromy or \( \rho \) has stable monodromy.

**Proof.** By Theorem A.1 we know \( R^i\psi_*\mathcal{O}_{V,\text{cris}} \) is an object of \( \mathbf{F} - \text{Isoc}(\text{Spec}(F)) \). The corollary follows from Theorem 9.5, as \( M^{\text{unit}} \) corresponds to \( R^i\psi_*\mathcal{O}_{V,\text{cris}} \).

**9.2 Global results**

**Theorem 9.7** (Drinfeld–Kedlaya). Let \( C \) be a smooth curve and let \( M \) be an irreducible object of \( \mathbf{F} - \text{Isoc}(C) \otimes L \) or \( \mathbf{F} - \text{Isoc}(C) \otimes L \). The differences between the consecutive generic slopes of \( M \) are bounded by one.

**Proof.** Assume that \( M \) has two consecutive generic slopes whose difference is greater than one. By taking exterior powers and twisting we may assume that the first two generic slopes of \( M \) are 0 and \( \hat{\alpha} \), the smooth compactification of \( \mathbf{C} \) and let \( \rho \) be the Galois representation corresponding to the lisse étale sheaf \( \det(R^i\psi_*\mathcal{O}_{V,\text{cris}}) \).

(i) Let \( \alpha \) denote the first non-zero slope of \( M \) and let \( r = 1/\alpha \). Then \( \rho \) has \( r \)-bounded monodromy.

(ii) Assume \( M \) has integer slopes. Then either \( \rho \) has finite monodromy or \( \rho \) has stable monodromy.

**Remark 9.8.** Our proof of Theorem 9.7 is somewhat perpendicular to the work of Drinfeld and Kedlaya. In [DK17], they shrink \( C \) until the Newton polygons is the same at each point of \( C \). They then prove that certain \( \text{Ext} \) groups vanish, using ideas that trace back to Kedlaya’s thesis (see [Ked00, 5.2.1]). From more advanced faithfulness results of Kedlaya, Shih, and de Jong, it follows from general nonsense that \( M \) decomposes into the direct sum of two \( \mathbf{F} \)-isocrystals. In contrast, we use Corollary 6.17 and Theorem 9.1 to prove the Newton polygon at each \( x \in C \) agrees with the generic Newton polygon.

**Theorem 9.9.** Let \( U \) be a smooth curve over \( k \) and let \( C \) be its smooth compactification. Let \( M \) be an irreducible object of \( \mathbf{F} - \text{Isoc}(U) \). After replacing \( U \) with a dense open subset, there is a slope filtration \( 0 = M_0 \subset M_1 \subset \cdots \subset M_d = \pi_1^{\text{et}}(U) \) where each graded piece \( M_{i+1}/M_i \) is isoclinic of slope \( \alpha_i \). Let \( \rho_i \) be a \( p \)-adic character of \( \pi_1^{\text{et}}(U) \) corresponding to \( \det(M_i) \).
Corollary 9.10. Let $M$ be a smooth proper morphism. Let $M = R^i\psi_*\mathcal{O}_{V,\text{cris}}$ and assume that the generic slopes of $M$ are integers. Then the $p$-adic character of $\pi_1^\dagger(U)$ corresponding to $\det(R^i\psi_*\mathbb{Q}_p^{\text{et}})$ is genus pseudo-stable.

**Proof.** Let $X$ be the smooth compactification of $U$. By Theorem A.1 applied to each $x \in X - U$, we know that $M$ is an object of $\mathbb{F} - \mathcal{Isoc}(\Spec(F))$. The $p$-adic character of $\pi_1^\dagger(U)$ corresponding to $\det(R^i\psi_*\mathbb{Q}_p^{\text{et}})$ is the same as $\rho_1$ from Theorem 9.9.

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Appendix A. A local Berthelot’s conjecture for constant sheafs

Berthelot’s conjecture states that the higher direct images of an overconvergent $F$-isocrystal along a smooth proper morphism are again overconvergent $F$-isocrystals. In this appendix, we prove a local version of this conjecture for the higher direct images of the constant sheaf. This is originally due to Kedlaya in [Ked00, Chapter 7], as an application of a fully faithfulness result. However, to the best of our knowledge there is no published proof. We follow closely the proof in [Ked00, Chapter 7].

**Theorem A.1.** Let $\psi : V \to \Spec(F)$ be a smooth proper morphism. Then $M = R^i\psi_*\mathcal{O}_{V,\text{cris}} \otimes \mathbb{Q}_p$, which a priori is an object of $\mathbb{F} - \mathcal{Isoc}(\Spec(F))$, is overconvergent.

We begin with two lemmas.

**Lemma A.2.** Let $\overline{\psi} : \overline{V} \to \Spec(\mathcal{R}[[T]])$ be a proper morphism whose special fiber is reduced with strict normal crossings. Let $\overline{\psi}$ be the generic fiber of $\overline{\psi}$. Let $A$ be the standard fine log structure associated to $\Spec(\mathcal{R}[[T]])$ and let $B$ be the fine log structure associated to the special divisor of $\overline{V}$. In particular, $\overline{\psi} : (\overline{V}, B) \to (\Spec(\mathcal{R}[[T]]), A)$ is a smooth map of log schemes. Then $R^i\overline{\psi}_*(\mathcal{O}_{V,\text{cris}}) \otimes \mathbb{Q}_p$ is an object of $\mathbb{F} - \mathcal{Isoc}(\Spec(F))$.

**Proof.** First note that $R^i\overline{\psi}_*(\mathcal{O}_{V,\text{cris}}) \otimes \mathbb{Q}_p$ and $R^i\overline{\psi}_*(\mathcal{O}_{V,\text{cris}}) \otimes \mathbb{Q}_p$ give rise to the same element of $\mathbb{F} - \mathcal{Isoc}(\Spec(F))$. That is, the restriction of $R^i\overline{\psi}_*(\mathcal{O}_{V,\text{cris}}) \otimes \mathbb{Q}_p$ to the generic point of $\Spec(\mathcal{R}[[T]])$ is the same as $R^i\psi_*\mathcal{O}_{V,\text{cris}} \otimes \mathbb{Q}_p$. Thus, it suffices to prove the result for
R^i\psi_*(O_{V,cris}^{\log}) \otimes \mathbb{Q}_p$. First, we use \(O_L[[T]]\) as a test object in the log-crystalline site over \(\mathcal{R}[[T]]\). Thus, \(R^i\psi_*(O_{V,cris}^{\log}) \otimes \mathbb{Q}_p\) gives rise to a free \(O_L[[T]] \otimes L\)-module with a connection and semilinear Frobenius map. From [HK94, Proposition 2.24] we deduce that the Frobenius structure must be an isomorphism. As \(O_L[[T]] \otimes L \subset \mathcal{E}^1\), we see that the \((\sigma, \nabla)\)-module over \(\mathcal{E}\) associated to \(R^i\psi_*(O_{V,cris}^{\log}) \otimes \mathbb{Q}_p\) descends to \(\mathcal{E}^1\).

**Lemma A.3.** Let \(\psi : X \to \text{Spec}(F)\) and \(\eta : Y \to \text{Spec}(F)\) smooth, irreducible, proper varieties over \(\text{Spec}(F)\). Let \(f : X \to Y\) be a surjective morphism. Then the induced map \(f^* : R^i\eta_*(O_{Y,cris}) \to R^i\psi_*(O_{X,cris})\) is injective and there exists a projector mapping \(R^i\psi_*(O_{V,cris})\) to the image of \(f\).

**Proof.** Crystalline cohomology for varieties over \(\text{Spec}(F)\) is a Weil cohomology theory (see, e.g., [Ber74]). The injectivity then follows from [Kle68, Proposition 1.2.4]. Furthermore, from the proof of [Kle68, Proposition 1.2.4] we see that \(f_*f^*\) is injective. The projector is then \((1/d)f^*f_*\), where \(d\) is the generic degree of \(f\).

**Corollary A.4.** Let \(X\) and \(Y\) be as in Lemma A.3. If \(R^i\psi_*(O_{X,cris})\) is an object of \(\mathbf{F} - \mathbf{Isoc}^1(\text{Spec}(F))\), then \(R^i\eta_*(O_{Y,cris})\) is an object of \(\mathbf{F} - \mathbf{Isoc}^1(\text{Spec}(F))\).

**Proof.** Recall from § 4.2 that \(\iota^!\) is fully faithful. This means that the projector from Lemma A.3, which is a morphism in \(\mathbf{F} - \mathbf{Isoc}(\text{Spec}(F))\), must be a morphism in \(\mathbf{F} - \mathbf{Isoc}^1(\text{Spec}(F))\). The corollary follows.

**Lemma A.5.** Let \(\phi : \text{Spec}(F_0) \to \text{Spec}(F)\) be a finite field extension. Let \(M\) be an object of \(\mathbf{F} - \mathbf{Isoc}(\text{Spec}(F))\). If \(\phi^*M\) is overconvergent, then \(M\) is overconvergent.

**Proof.** For separable extensions this follows from the fully faithfulness of \(\iota^!\) and the fact that \(M\) is a direct summand of \(\phi_*\phi^*M\). For the inseparable case, it suffices to prove the lemma for the degree-p case. Thus, we may assume \(F = \mathcal{R}((T))\) and \(F_0 = \mathcal{R}((T^{1/p}))\). We may view \(\mathcal{E}_{F,L}\) as a subfield of \(\mathcal{E}_{F_0,L}\) and note that \(\sigma\) extends to a Frobenius endomorphism of \(\mathcal{E}_{F_0,L}\) sending \(T^{1/p}\) to \(T\). It is known that \(\sigma^*\) induces an equivalence of categories between \((\sigma, \nabla)\)-modules over \(\mathcal{E}_{F_0,L}\) and \(\mathcal{E}_{F,L}^1\). Explicitly, let \(M\) be a \((\sigma, \nabla)\)-module over \(\mathcal{E}_{F_0,L}\), and let \(A\) be a Frobenius matrix of \(M\). We may view \(\varphi(\varphi(M))\) as \(A^r\varphi(M)\), so that the Frobenius structure descends to \(\mathcal{E}_{F,L}^1\) (one deduces the connection in a similar manner).

**Proof of Theorem A.1.** First assume that \(X\) is projective over \(F\). Take an embedding \(X \hookrightarrow \mathbb{P}_p^n\), and let \(\overline{X}\) be the Zariski closure of \(X\) in \(\mathbb{P}_p^n\). By [dJ96, Theorem 6.5], there exists a finite extension \(\mathcal{R}'(\mathbb{Z}[u]) \to \mathcal{R}([T])\), an alteration \(\phi_1 : X_1 \to X\) where \(X_1\) is a \(\mathcal{R}'(\mathbb{Z}[u])\)-variety, and an open immersion \(j_1 : X_1 \to X_1\) such that the pair \((\overline{X}_1, X_1\setminus X_1)\) is semi-stable. Furthermore, we may assume that \(\overline{X}_1\setminus X_1\) is equal to the special fiber. Let \(F_1 = \mathcal{R}'((u))\) and consider the map \(\psi_1 : \overline{X}_1 \times_{\text{Spec}(\mathcal{R}'[\mathbb{Z}[u]])} \text{Spec}(F_1) \to \text{Spec}(F_1)\). By Lemma A.2 we know that \(R^i(\psi_{1*})\) is an object of \(\mathbf{F} - \mathbf{Isoc}^1(\text{Spec}(F_1))\). Then by applying Corollary A.4 to the map of \(\text{Spec}(F_1)\)-varieties \(X_1 \times_{\text{Spec}(\mathcal{R}'[\mathbb{Z}[u]])} \text{Spec}(F_1) \to X \times_{\text{Spec}(F)} \text{Spec}(F_1)\) induced by \(\phi\), we see that \(R^i(\phi \times \text{Spec}(F_1))\) is an object of \(\mathbf{F} - \mathbf{Isoc}^1(\text{Spec}(F_1))\). By base change for crystalline cohomology (see, e.g., [BO78, Corollary 7.12]) and Lemma A.5, this implies \(R^i\phi_*O_{X,cris}\) is overconvergent.
Now consider general $V$. By [dJ96, Theorem 4.1] we know that there is an alteration $V_1 \to V$ that is projective and that $V_1$ will be smooth over a finite extension of $F$. The result follows by repeating an argument from the previous paragraph. □

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