Note on Rotating Charged Black Holes in
Einstein-Maxwell-Chern-Simons Theory

Alikram N. Aliev and Dilek K. Çiftçi
Feza Gürsey Institute, P. K. 6 Çengelköy, 34684 Istanbul, Turkey
(Dated: February 16, 2009)

Abstract

We show that the general solution of Chong, Cvetič, Lu and Pope for nonextremal rotating charged black holes in five-dimensional minimal gauged supergravity, or equivalently in the Einstein-Maxwell-Chern-Simons theory with a negative cosmological constant and with the Chern-Simons coefficient \( \nu = 1 \), admits a simple description in a Kerr-Schild type framework with two scalar functions. Next, assuming this framework as an ansatz, we obtain new analytic solutions for slowly rotating charged black holes in the Einstein-Maxwell-Chern-Simons theory with \( \nu \neq 1 \). Using a covariant superpotential derived from Noether identities within the Katz-Bićák-Lynden-Bell approach, we calculate the mass and angular momenta for the general supergravity solution as well as for the slowly rotating solution with two independent rotation parameters. For the latter case, we also calculate the gyromagnetic ratios and obtain simple analytic formulas, involving both the parameters of the black holes and the Chern-Simons coefficient.
I. INTRODUCTION

The first study of higher-dimensional black hole solutions traces back to the work of Tangherlini [1], who found an exact metric for static charged black holes, generalizing the Reissner-Nordström solution of four-dimensional Einstein-Maxwell theory to all higher dimensions. The Tangherlini metric is also a solution to the Einstein-Maxwell-Chern-Simons theory for any value of the Chern-Simons (CS) coefficient [2]. However, the construction of rotating charged black hole solutions in the higher-dimensional Einstein-Maxwell theory turned out to be a rather complicated problem. As is known, one of the most useful methods for constructing such solutions in four dimensions is based on the use of the Kerr-Schild form for the spacetime metric [3]. The remarkable property of the Kerr-Schild form is that the exact metric looks like its linearized approximation around the flat background spacetime. From the mathematical point of view, this property results in reduction of the Einstein-Maxwell equations to linear equations, thereby providing a useful technique for solving these equations. However, in 1986 Myers and Perry showed that in higher dimensions this technique works only in the uncharged case. They presented the general asymptotically flat solution for rotating (non-charged) black holes in all spacetime dimensions [4]. Recently, the authors of work [5] have shown that the previously-known Kerr-de Sitter metrics in four and five dimensions [6, 7] can also be put in the Kerr-Schild form, where a “linearization” occurs around de Sitter spacetime. Generalizing this framework to higher dimensions, they were able to present the general Kerr-de Sitter metrics in arbitrary spacetime dimensions.

A new attempt to employ the Kerr-Schild framework for constructing the rotating charged black hole solutions in higher-dimensional Einstein-Maxwell theory was undertaken in [8, 9]. It was shown that this approach enables one to obtain the higher-dimensional charged solutions only in the limit of slow rotation. Thus, the use of the Kerr-Schild framework to obtain the exact metrics for rotating black holes in pure Einstein-Maxwell theory becomes restricted (at least, in its canonical form) to four spacetime dimensions. For this reason, the higher-dimensional counterpart of the Kerr-Newman solution still remains unknown.

Remarkably, the exact solutions for rotating charged black holes are known in the Einstein-Maxwell theory with a Chern-Simons term. The addition to the theory of the Chern-Simons term with a particular value of the CS coefficient $\nu = 1$, extends its symmetries, facilitating the search for the exact solutions. Furthermore, for this particular value of


2
the CS coefficient, the Einstein-Maxwell-CS theory with a negative cosmological constant in five-dimensions appears to be equivalent to five-dimensional minimal gauged supergravity. In the latter case, the rotating charged black hole solution with two equal angular momenta was constructed in [10], while the general solution was found in [11]. It is worth noting that these solutions were found in the absence of solution-generating techniques. The authors have employed only a procedure of trial and error guided by their physical intuition. Later on, the rotating charged black hole solutions with zero cosmological constant have also been obtained within generating techniques for solutions of five-dimensional minimal gauged supergravity [12, 13].

The purpose of the present paper is two-fold: (i) to present a simple Kerr-Schild type framework for rotating charged black holes in five-dimensional minimal gauged supergravity, (ii) to use this framework for constructing new analytic black hole solutions to the Einstein-Maxwell-CS theory, when the CS coefficient $\nu \neq 1$. It is known that for $\nu = 1$ there exists a certain balance in distributions of the energy and angular momentum of these black holes, which is provided by supersymmetry [2]. One can expect that going beyond $\nu = 1$ will violate this balance that may result in new important features of these black holes, such as non-uniqueness and instability. In this respect, the study of the rotating charged black holes with $\nu \neq 1$ is of particular interest.

In Sec.II we show that the spacetime of general rotating black holes in five-dimensional minimal gauged supergravity admits two specific congruences: a null congruence and a spacelike congruence. This allows one to establish for these black holes the Kerr-Schild type framework involving two scalar functions. In Sec.III we use the Kerr-Schild framework as an ansatz and obtain new analytic solutions for slowly rotating charged AdS black holes of the Einstein-Maxwell-CS theory with $\nu \neq 1$. In Sec.IV employing the covariant superpotential technique of Katz-Bičák-Lynden-Bell, we calculate the mass and angular momenta for the exact solution with $\nu = 1$ in the Kerr-Schild framework. Here we also calculate the mass, the angular momenta and the gyromagnetic ratios for the slowly rotating solution with two independent rotation parameters. In the Appendix we give the components of the Einstein field equations calculated for the single rotation parameter case.
II. THE KERR-SCHILDFRAMEWORK OF FIVE-DIMENSIONAL MINIMAL
SUPERGRAVITY

The general metrics for rotating black holes in higher-dimensional gravity with a cosmological constant belong to the Kerr-Schild class involving a single scalar function. It is also known that the general Kerr-Taub-NUT-de Sitter metrics in all higher dimensions admit a multi-Kerr-Schild structure. For instance, the Kerr-Taub-NUT-de Sitter spacetime in five dimensions contains two linearly independent and mutually orthogonal null geodesic congruences. This enables one to put the spacetime metric in a “double” Kerr-Schild form with two scalar functions. The extension of the Kerr-Schild framework to the case of general rotating charged black holes in five-dimensional minimal gauged supergravity shows that the spacetime metric admits two specific vector fields: a null vector field \( k \) and a spacelike vector field \( \ell \), which allow us to present the metric in the following form

\[
ds^2 = ds^2 + Hk \otimes k + V (k \otimes \ell + \ell \otimes k),
\]

where \( ds^2 \) is the background AdS spacetime, \( H \) and \( V \) are two scalar functions. In the spacetime coordinates \( x^\mu = \{t, r, \theta, \phi, \psi\} \), \( \mu = 0, \ldots, 4 \), the one-forms \( k \) and \( \ell \) are given by

\[
k = k_\mu dx^\mu = \left\{ \frac{\Delta_\theta}{\Xi_a \Xi_b} dt, 0, 0, -\frac{a \sin^2 \theta}{\Xi_a} d\varphi, -\frac{b \cos^2 \theta}{\Xi_b} d\psi \right\},
\]

\[
\ell = \ell_\mu dx^\mu = \left\{ \frac{\Delta_\theta}{\Xi_a \Xi_b}, \frac{ab}{l^2} dt, 0, 0, -\frac{b \sin^2 \theta}{\Xi_a} d\varphi, -\frac{a \cos^2 \theta}{\Xi_b} d\psi \right\}.
\]

and the metric (1) takes the form

\[
ds^2 = \left[ -\left( 1 + \frac{r^2}{l^2} \right) \frac{\Delta_\theta}{\Xi_a \Xi_b} dt^2 - 2dr \left( \frac{\Delta_\theta}{\Xi_a \Xi_b} dt - \frac{a \sin^2 \theta}{\Xi_a} d\varphi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right) + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \left( \frac{r^2 + a^2}{\Xi_a} \right) d\varphi^2 + \left( \frac{r^2 + b^2}{\Xi_b} \right) \cos^2 \theta d\psi^2 \right] + H \left( \frac{\Delta_\theta}{\Xi_a \Xi_b} dt - \frac{a \sin^2 \theta}{\Xi_a} d\varphi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right)^2
\]

\[
+ 2V \left( \frac{\Delta_\theta}{\Xi_a \Xi_b} dt - \frac{a \sin^2 \theta}{\Xi_a} d\varphi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right) \left( \frac{\Delta_\theta}{\Xi_a \Xi_b} \frac{ab}{l^2} dt - \frac{b \sin^2 \theta}{\Xi_a} d\varphi - \frac{a \cos^2 \theta}{\Xi_b} d\psi \right),
\]

where \( a \) and \( b \) are two independent rotation parameters, the expression in square brackets represents the AdS spacetime with a length scale parameter \( l \), which is determined by the negative cosmological constant \( l^2 = -\frac{6}{\Lambda} \). The metric functions

\[
\Delta_\theta = 1 - \frac{a^2}{l^2} \cos^2 \theta - \frac{b^2}{l^2} \sin^2 \theta,
\]

\[
\Sigma = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta.
\]
and
\[ \Xi_a = 1 - \frac{a^2}{l^2}, \quad \Xi_b = 1 - \frac{b^2}{l^2}. \] (6)

It is straightforward to show that the vectors \( k_\mu \) and \( \ell_\mu \) satisfy the relations
\[ k_\mu k^\mu = 0, \quad \ell_\mu \ell^\mu = \frac{a^2 \cos^2 \theta + b^2 \sin^2 \theta}{r^2} \] (7)
with respect to both the background AdS spacetime and the full metric (11). It follows that these two congruences are mutually orthogonal, but only one of them, namely the congruence of the vector field \( k \) is null. This is in contrast to the “double” Kerr-Schild metric form of the Kerr-Taub-NUT-de Sitter spacetime [15], where both geodesic vectors are null.

The potential one-form for the electromagnetic field of the spacetime (11) can be written in terms of the null one-form \( k \). We have
\[ A = \frac{\alpha}{\Sigma} k, \] (8)
where \( \alpha \) is an arbitrary constant. The electromagnetic field two-form \( F = dA \) is given by
\[ F = \frac{2\alpha r}{\Sigma^2} \left( \frac{\Delta_\theta}{\Xi_a \Xi_b} dt - \frac{a \sin^2 \theta}{\Xi_a} d\varphi - \frac{b \cos^2 \theta}{\Xi_b} d\psi \right) \wedge dr \]
\[ - \frac{\alpha \sin 2\theta}{\Sigma^2} \left[ \left( \frac{a^2 - b^2}{\Xi_a \Xi_b} \right) dt - \frac{a(r^2 + a^2)}{\Xi_a} d\varphi + \frac{b(r^2 + b^2)}{\Xi_b} d\psi \right] \wedge d\theta. \] (9)

In constructing the Kerr-Schild framework given in (11), we have used the known general solution for rotating charged black holes in five-dimensional minimal gauged supergravity [11]. Indeed, with
\[ H = \frac{2M}{\Sigma} - \frac{Q^2}{\Sigma^2}, \quad V = \frac{Q}{\Sigma}, \quad \alpha = \frac{\sqrt{3}}{2} Q, \] (10)
where \( M \) is the mass and \( Q \) is the electric charge parameters of the black hole, and by applying to the metric (11) the coordinate transformations
\[ dt = d\tau - \frac{(r^2 + a^2)(r^2 + b^2) + abQ}{\Delta, r^2} dr, \]
\[ d\varphi = d\phi + \frac{a}{l^2} d\tau - \frac{a(r^2 + b^2)(1 + r^2/l^2) + Qb}{\Delta, r^2} dr, \]
\[ d\psi = d\chi + \frac{b}{l^2} d\tau - \frac{b(r^2 + a^2)(1 + r^2/l^2) + Qa}{\Delta, r^2} dr, \] (11)
where
\[
\Delta_r = \frac{(r^2 + a^2)(r^2 + b^2)(1 + r^2/l^2) + 2abQ + Q^2}{r^2} - 2M,
\] (12)
we obtain the general solution of [11] in rotating at infinity Boyer-Lindquist coordinates (see [16]). It is also important to note that with the metric (4) and the potential one-form (8) one can easily solve the field equations of the Einstein-Maxwell-CS theory
\[
R^\nu_\mu - 2 \left( F_{\mu \lambda} F^{\nu \lambda} - \frac{1}{6} \delta^\nu_\mu F_{\alpha \beta} F^{\alpha \beta} \right) + \frac{4}{l^2} \delta^\nu_\mu = 0,
\] (13)
\[
\nabla_\nu F^{\mu \nu} + \frac{\nu}{2\sqrt{3}\sqrt{-g}} \epsilon^{\mu \alpha \beta \rho \tau} F_{\alpha \beta} F_{\rho \tau} = 0,
\] (14)
for the supergravity value \( \nu = 1 \), rederiving the solutions given in (10).

In the following, we shall use the metric form (4) as an ansatz for constructing the rotating charged AdS black hole solutions with the CS coefficient \( \nu \neq 1 \).

III. EINSTEIN-MAXWELL-CS BLACK HOLES

As we have mentioned above, the values of the CS coefficient make no difference to static Tangherlini type black holes of the Einstein-Maxwell-CS theory. However, this is not the case for stationary solutions. For instance, the value \( \nu = 1 \) turns out to be crucial for rotating charged black holes described by the metric (4). As far as we are aware, beyond this value of the CS coefficient, the exact rotating black hole solutions in five dimensions are not known. The authors of works [17, 18] have constructed numerical solutions for the Einstein-Maxwell-CS black holes, focusing on the case of spherical topology and two equal angular momenta. The numerical analysis uncovered the new interesting features of these black holes, such as the lack of uniqueness property and the existence of unstable solutions.

In this section, restricting ourselves to the regime of slow rotation, we give an analytic description of the Einstein-Maxwell-CS black holes with the CS coefficient \( \nu \neq 1 \). For this purpose, we take the metric (4) as a general ansatz for putative new solutions and begin with calculating the contravariant components of the electromagnetic field tensor. For convenience, we take the same value for the parameter \( \alpha \) as given in (10).
nonvanishing components, we obtain

\[ F_{01} = -\frac{\sqrt{3} Q [(r^2 + a^2)(r^2 + b^2) + abv \Sigma]}{r \Sigma^3} , \]

\[ F_{02} = \frac{\sqrt{3} Q (a^2 - b^2) \sin 2\theta}{2 \Sigma^3} , \]

\[ F_{13} = \frac{\sqrt{3} Q [a(r^2 + b^2)(1 + r^2/l^2) + bV \Sigma]}{r \Sigma^3} , \]

\[ F_{23} = \frac{-\sqrt{3} a \Delta \theta \cot \theta}{\Sigma^3} , \]

\[ F_{14} = \frac{\sqrt{3} Q [b(r^2 + a^2)(1 + r^2/l^2) + aV \Sigma]}{r \Sigma^3} , \]

\[ F_{24} = \frac{\sqrt{3} b \Delta \theta \tan \theta}{\Sigma^3} . \]  

(15)

Substituting these components along with those given by (9) in the Maxwell-CS field equations (14), we find that they reduce to a single differential equation for the function \( V \). Thus, we have the equation

\[ \frac{\partial V}{\partial r} - \frac{2r}{\Sigma} V + \frac{4Q\nu r}{\Sigma^2} = 0 , \]

which admits a simple solution of the form

\[ V = \frac{Q\nu}{\Sigma} . \]  

(17)

We note that this solution in general involves an arbitrary function of \( \theta \) as well. However, the latter is fixed to be zero from the Einstein equations (13). In the following, to make the description more transparent, we consider the solutions with single, two and equal rotation parameters separately.

A. The single rotation parameter solution with \( \nu \neq 1 \)

Taking \( b = 0 \) in the metric (14) as well as in the expressions (9) and (15) for the electromagnetic field tensor, we calculate all the components of the field equations in (13) by using Mathematica algebraic computing programme. Then, solving the \((X)_4^4\) equation, we find the solution

\[ H = \frac{2M}{\Sigma} - \frac{Q^2 (r^2 + a^2 \nu^2 \cos^2 \theta)}{\Sigma^3} . \]  

(18)

Substituting this solution in the remaining equations, we calculate explicitly their left-hand sides. The results are listed in the Appendix. From these expressions, it follows that for the supergravity value \( \nu = 1 \), the function \( H \) in (18) solves the complete set of equations in (13). Furthermore, these expressions show that, at the linear level in the rotation parameter \( a \), the function \( H \) satisfies the field equations for the value \( \nu \neq 1 \) as well. Thus, for \( \nu \neq 1 \),
to within terms of linear order in \( a \), we have the solutions

\[
H = \frac{2M}{r^2} - \frac{Q^2}{r^4}, \quad V = \frac{Q\nu}{r^2}. \tag{19}
\]

Using these expressions in the metric \( \text{[4]} \) and passing to the Boyer-Lindquist coordinates in the linear in \( a \) approximation

\[
dt = d\tau - \frac{r^2}{\Delta} dr,

d\varphi = d\phi + \frac{a}{l^2} d\tau - \frac{a(1 + r^2/l^2)}{\Delta} dr,

d\psi = d\chi - \frac{Qa\nu}{\Delta r^2} dr, \tag{20}
\]

where

\[
\Delta = r^2 \left(1 + \frac{r^2}{l^2}\right) + \frac{Q^2}{r^2} - 2M, \tag{21}
\]

we put the metric in the form

\[
ds^2 = -\frac{\Delta}{r^2} d\tau^2 + \frac{r^2}{\Delta} dr^2 - 2a \left(1 - \frac{\Delta}{r^2}\right) \sin^2 \theta d\phi d\tau - \frac{2Qa\nu}{r^2} \cos^2 \theta d\chi d\tau + r^2 d\Omega_3^2, \tag{22}
\]

where

\[
d\Omega_3^2 = d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\chi^2 \tag{23}
\]

and the potential one-form of the electromagnetic field is given by

\[
A = \frac{\sqrt{3}Q}{2r^2} \left(d\tau - a \sin^2 \theta d\phi\right). \tag{24}
\]

This metric describes slowly rotating charged Einstein-Maxwell-CS black holes with any value of the CS coefficient \( \nu \neq 1 \) within the linear in \( a \) approximation. For \( \nu = 0 \) and for zero cosmological constant, it recovers the metric obtained earlier in \[8, 9\] for rotating charged black holes in pure Einstein-Maxwell theory. (See also a recent paper \[19\]).

**B. The two rotation parameters solution with \( \nu \neq 1 \)**

The generalization of the solution \[22\] to include two independent rotation parameters is straightforward. Indeed, checking the field equations \[13\] with the solutions \[19\] in the
regime of slow rotation we find that they are satisfied. Thus, at the linear level in $a$ and $b$, the metric (4) with the functions $H$ and $V$ given in (19) is also a solution to the Einstein-Maxwell-CS theory with $\nu \neq 1$. Again, passing to the Boyer-Lindquist coordinates

\[
\begin{align*}
    dt &= d\tau - \frac{r^2}{\Delta} dr, \\
    d\varphi &= d\phi + \frac{a}{l^2} d\tau - \frac{ar^2 (1 + r^2/1^2) + Qb\nu}{\Delta r^2} dr, \\
    d\psi &= d\chi + \frac{b}{l^2} d\tau - \frac{br^2 (1 + r^2/1^2) + Qa\nu}{\Delta r^2} dr,
\end{align*}
\]

we obtain the metric in the form

\[
\begin{align*}
    ds^2 &= -\frac{\Delta}{r^2} d\tau^2 + \frac{r^2}{\Delta} dr^2 - 2 \left(1 - \frac{\Delta}{r^2}\right) \left(a \sin^2 \theta \, d\phi + b \cos^2 \theta \, d\chi\right) d\tau \\
    &\quad - \frac{2Q\nu}{r^2} \left(b \sin^2 \theta \, d\phi + a \cos^2 \theta \, d\chi\right) d\tau + r^2 d\Omega_3^2
\end{align*}
\]

in which, the electromagnetic field is described by the potential one-form

\[
A = \frac{\sqrt{3}Q}{2r^2} \left(d\tau - a \sin^2 \theta \, d\phi - b \cos^2 \theta \, d\chi\right).
\]

For $\nu = 0$ and $l \to \infty$, this metric agrees with that obtained in [8].

C. The solution with equal rotation parameters

We now discuss an example of another class of solutions with $\nu \neq 1$, which, unlike the solutions given above, exhibits the singularity in the regime of slow rotation. Following the same strategy as in the previous subsections, we first write down explicitly the complete set of the Einstein field equations (13), using the metric ansatz (4) and the electromagnetic field components given in (9) and (15) along with (17). Next, solving the $(X)^{2}$ equation, we find

\[
H = \frac{2M}{\Sigma} + \frac{Q^2}{\Sigma^2} \left(1 - 2\nu^2\right) \left[1 + \frac{2(1 - \nu^2)}{1 - 2\nu^2} \frac{\Sigma}{\Sigma - r^2} \ln \left(\frac{r^2}{2\Sigma}\right)\right].
\]

Substituting this solution in the remaining equations, we obtain that they are satisfied only for slow rotation with equal rotation parameters. Thus, the solution subject to the system of the Einstein-Maxwell-CS equations with $\nu \neq 1$ is given by

\[
\begin{align*}
    H &= \frac{2M}{r^2} - \frac{Q^2}{r^4} \left[1 - 2 \ln \left(1 - \nu^2\right) \left(1 - \frac{r^2}{a^2}\right)\right],
\end{align*}
\]
which is obtained from (29) by taking \(a = b\) and expanding it in powers of the rotation parameter up to the linear order. We note that the expression for the function \(V\) is the same as that given in (19). Using the coordinate transformations

\[
\begin{align*}
    dt &= d\tau - \frac{dr}{f(r)}, \\
    d\phi &= d\phi + \frac{a}{l^2} d\tau - \frac{a}{r^2} \frac{dr}{g(r)}, \\
    d\psi &= d\chi + \frac{a}{l^2} d\tau - \frac{a}{r^2} \frac{dr}{g(r)},
\end{align*}
\]

where

\[
    f(r) = 1 + \frac{r^2}{l^2} - H, \quad g(r) = f(r) \left(1 + \frac{r^2}{l^2} + V\right)^{-1},
\]

we obtain that the spacetime metric in the Boyer-Lindquist coordinates takes the form

\[
    ds^2 = -f(r) d\tau^2 + \frac{dr^2}{f(r)} - 2a \left[1 - f(r) + V\right] \left(\sin^2 \theta d\phi + \cos^2 \theta d\chi\right) d\tau + r^2 d\Omega^2_3,
\]

where \(H\) is given by (30). The potential one-form is given by (28) with \(a = b\). We see that for \(a \to 0\) the metric components, unlike those in (27), become divergent. That is, the metric exhibits the singularity (instability) with respect to slow rotation. We conclude that the analytical examples of metrics given above explicitly show that the physics of rotating black holes in the Einstein-Maxwell-CS theory crucially depends on the value of the CS coefficient. These results are in qualitative agreement with the numerical analysis of works \([17, 18]\).

IV. THE MASSES, ANGULAR MOMENTA AND THE GYROMAGNETIC RATIOS

The most simple way of calculation the mass and angular momenta of black holes in asymptotically flat spacetime is achieved within the Komar approach \([20]\). However, this approach must be used with care for rotating AdS black holes, where it gives an ambiguous result for the mass. Therefore, the mass of these black holes has been calculated by integrating the first law of thermodynamics \([21]\) as well as in the framework of other approaches based on the use of conservation laws derived from the symmetries of the system \([22, 23, 24]\). The mass of rotating charged black holes in five-dimensional minimal gauged supergravity
has also been calculated using both the first law of thermodynamics \[11\] and the conformal definition of Ashtekar, Magnon and Das \[25\]. Here we wish to calculate the mass of these black holes using the Kerr-Schild framework (1) for the spacetime metric and employing the covariant superpotential technique of Katz, Bičák and Lynden-Bell (KBL) \[26\]. (See also a recent review \[27\]). Using the language of differential forms, we can write the integral of the KBL superpotential in the form

\[
K = -\frac{1}{16\pi} \oint d(\delta \hat{\xi}) - \frac{1}{8\pi} \oint (\delta S),
\]

(34)

where

\[
S = \frac{1}{2} \xi[\mu,\nu\rho] dx^\mu \wedge dx^\nu, \quad \kappa^\mu = g^{\mu\nu} \delta \Gamma^\nu_{\rho\lambda} - g^{\alpha\beta} \delta \Gamma^\mu_{\alpha\beta}
\]

(35)

and \(\delta \hat{\xi}\) denotes the difference between the Killing isometries of the original spacetime (1) and its reference background. The latter is obtained by taking \(M = 0\) and \(Q = 0\) in the original spacetime. Similarly, \(\delta \Gamma^\lambda_{\sigma\rho}\) stands for the difference between the Christoffel symbols of the original spacetime and those of its reference background. The difference between the corresponding metric determinants \(\delta g = 0\).

Evaluating the integral in (34) with respect to the timelike Killing vector \(\partial_t\) and over a 3-sphere at spatial infinity, we obtain the mass \(M' = K[\partial_t]\) of the spacetime (1). Indeed, using for the integrands the asymptotic expansions at \(r \to \infty\)

\[
\delta \xi^{t;\tau}_{(t)} = \frac{2}{\Xi_a \Xi_b} \left[ \frac{M \left( 2 \Xi_b \sin^2 \theta + 2 \Xi_a \cos^2 \theta - \Xi_a \Xi_b \right)}{r^3} + 2 Q a b l^{-2} \Delta_0 \right] + \mathcal{O} \left( \frac{1}{r^5} \right),
\]

(36)

\[
\xi^{t;\kappa}_{(r)} = -\frac{M}{r^3} + \mathcal{O} \left( \frac{1}{r^5} \right),
\]

(37)

and performing the integration procedure, we find the expression

\[
M' = \pi M \left( 2 \Xi_a + 2 \Xi_b - \Xi_a \Xi_b \right) + 2 \pi Q a b l^{-2} \left( \Xi_a + \Xi_b \right) -\frac{4 \Xi_a^2 \Xi_b^2}{\Xi_a^2 \Xi_b^2},
\]

(38)

which is precisely the same as that obtained earlier within other approaches \[11, 25\].

The angular momenta of the spacetime (1) are obtained by evaluating the integral in (34) with respect to the rotational Killing vectors \(\partial_\phi\) and \(\partial_\psi\). Namely, we have \(J'_a = -K[\partial_\phi]\) and \(J'_b = -K[\partial_\psi]\). Substituting the asymptotic expansions at \(r \to \infty\)

\[
\delta \xi^{t;\tau}_{(\phi)} = \frac{2 \sin^2 \theta}{\Xi_a} \left[ \frac{2aM + Q b(2 - \Xi_a)}{r^3} \right] + \mathcal{O} \left( \frac{1}{r^5} \right),
\]

(39)

\[
\delta \xi^{t;\tau}_{(\psi)} = \frac{2 \cos^2 \theta}{\Xi_b} \left[ \frac{2bM + Q a(2 - \Xi_b)}{r^3} \right] + \mathcal{O} \left( \frac{1}{r^5} \right),
\]

(40)
in (34) and taking the integrals over a 3-sphere at spatial infinity, we obtain that the angular
momenta are given by
\[ J'_a = \frac{\pi}{4} \frac{2aM + Qb(2 - \Xi_a)}{\Xi_a^2 b} , \] (41)
\[ J'_b = \frac{\pi}{4} \frac{2bM + Qa(2 - \Xi_b)}{\Xi_b^2 a} . \] (42)
These expressions also agree with those obtained in [11, 25].

Next, we calculate the mass and angular momenta for the spacetime (27) of slowly rotating
charged black holes in the Einstein-Maxwell-CS theory with \( \nu \neq 1 \). Performing for this case
similar calculations of the integral in (34), we find
\[ M' = \frac{3\pi}{4} M , \quad J_a = \frac{\pi}{4} (2aM + Qb\nu) , \quad J_b = \frac{\pi}{4} (2bM + Qa\nu) . \] (43)
It follows that the mass of these black holes is not affected by the CS coefficient, whereas
their angular momenta depend on the CS coefficient. It is also interesting to know how
the CS coefficient affects the gyromagnetic ratios of these black holes. For \( \nu = 1 \), the
gyromagnetic ratios for the general metric (4) was calculated in [16]. Using the arguments
of this work for the metric (27), we can define the two gyromagnetic ratios
\[ g_a = \frac{2M'\mu'_a}{Q'J'_a} , \quad g_b = \frac{2M'\mu'_b}{Q'J'_b} , \] (44)
where
\[ \mu'_a = Q'a , \quad \mu'_b = Q'b \] (45)
are two magnetic dipole moments of the black hole associated with its two rotation 2-
planes and \( Q' \) is the physical charge determined by the Gauss law. Substituting in (44) the
expressions for the mass and angular momenta from (43), we find
\[ g_a = 3 \left( 1 - \frac{Qb\nu}{2aM + Qb\nu} \right) , \] (46)
\[ g_b = 3 \left( 1 - \frac{Qa\nu}{2bM + Qa\nu} \right) . \] (47)
It is easy to see that for \( \nu \to 0 \), the gyromagnetic ratio tends to its value [8] in the Einstein-
Maxwell theory, \( g \to 3 \). From these expressions, it also follows that for given parameters
of the black hole, the value of the gyromagnetic ratios tend to decrease with the growth
(within the linear approximation in rotation parameters) of the CS coefficient.
V. CONCLUSION

The Kerr-Schild framework played a profound role in classical general relativity, resulting in the discovery of the most important stationary black hole solution to the Einstein field equations, which nowadays is known as the Kerr metric. It turned out that the Kerr-Schild framework survives in higher-dimensional general relativity as well, thereby paving the way for the construction of the counterpart of the Kerr and Kerr-de Sitter metrics in all higher dimensions.

In this paper, we have shown that the Kerr-Schild framework can also be extended to five-dimensional minimal gauged supergravity, where the general metric for rotating charged black holes admits a Kerr-Schild type form with two scalar functions. It should be stressed that in this metric form only one of two congruences is null, the other one is spacelike. This framework ensures the simple derivation of the general rotating charged black hole solution in five-dimensional minimal gauged supergravity. Assuming the “double” Kerr-Schild form as an ansatz, we have found new analytic solutions for slowly rotating charged AdS black holes of the Einstein-Maxwell-CS theory with $\nu \neq 1$. These are, to our knowledge, the first examples of analytic solutions in the rotating case.

Using the covariant superpotential technique of Katz-Bičák-Lynden-Bell, we have calculated the mass and angular momenta for both the general black holes with $\nu = 1$ and the slowly rotating black hole with any value of the CS coefficient $\nu \neq 1$ within the linear approximation in slow rotation. We have also given simple analytic expressions for the gyromagnetic ratios of the slowly rotating Einstein-Maxwell-CS black holes.

VI. ACKNOWLEDGMENTS

A. N. thanks Teoman Turgut and Nihat Berker for their stimulating encouragements. D. K. thanks TÜBİTAK (BIDEB-2218) for financial support.

APPENDIX: THE FIELD EQUATIONS

Introducing for convenience the notation

$$M^{\nu}_{\mu} = F_{\mu\lambda}F^{\nu\lambda} - \frac{1}{6} \delta^{\nu}_{\mu} F_{\alpha\beta}F^{\alpha\beta}$$  \hspace{1cm} (A.1)
in the field equations (13) and using the solution of the \((X)^4\) equation given by (18), we obtain

\[
\begin{align*}
R^0_0 - 2M^0_0 + \frac{4}{l^2} &= -\frac{4Q^2a^2(\nu^2 - 1)\cos^2\theta}{\Xi_a \Sigma^6} \left[ 4r^2\Delta_\theta(r^2 + a^2) + \Sigma a^2\sin^2\theta \left( 1 + \frac{2r^2}{l^2} \right) \right], \\
R^1_1 - 2M^1_1 + \frac{4}{l^2} &= -\frac{16Q^2a^2r^2(\nu^2 - 1)\cos^2\theta}{\Sigma^5}, \\
R^2_2 - 2M^2_2 + \frac{4}{l^2} &= \frac{4Q^2a^2(\nu^2 - 1)\cos^2\theta}{\Sigma^5} (r^2 + \Sigma), \\
R^3_3 - 2M^3_3 + \frac{4}{l^2} &= \frac{4Q^2a^2(\nu^2 - 1)\cos^2\theta}{\Xi_a \Sigma^6} \left\{ \Delta_\theta \left[ \Sigma a^2 + 2r^2(r^2 + a^2) \right] + 2a^2r^2\sin^2\theta \left( 1 + \frac{2r^2}{l^2} \right) \right\}, \\
R^4_4 &= \frac{Q^2a^3r(\nu^2 - 1)(3\Sigma - 8r^2)\Delta_\theta \sin^2\theta \sin 2\theta}{\Xi_a \Sigma^6}, \\
R^5_5 &= \frac{4Q^2a^2r^2(\nu^2 - 1)\Delta_\theta}{\Xi_a \Sigma^6} \left[ \Delta_\theta (r^2 \cos 2\theta + a^2 \cos^2\theta) + \Xi_a \Sigma \cos^2\theta + \frac{a^2\sin^2 2\theta}{2} \left( 1 + \frac{r^2}{l^2} \right) \right], \\
R^6_6 &= \frac{4Q^2a^4\nu(\nu^2 - 1)\sin^2 2\theta}{\Xi_a \Sigma^6}, \\
R^7_7 &= \frac{4Q^2a^3\nu(\nu^2 - 1)\Delta_\theta}{\Xi_a \Sigma^5} \left[ \frac{r^2}{l^2} + \left( 1 + \frac{r^2}{l^2} \right) \left( 1 + \frac{4r^2}{\Sigma} \right) \right], \\
R^8_8 &= \frac{Q^2a^3(\nu^2 - 1)\sin^2 2\theta}{\Xi_a \Sigma^5} \left[ r^2 + (r^2 + a^2) \left( 1 + \frac{4r^2}{\Sigma} \right) \right].
\end{align*}
\]

We note that the above expressions vanish identically for the values \(\nu = \pm 1\). Furthermore, we note that they involve only the square or higher powers of the rotation parameter \(a\). It follows that for any \(\nu \neq \pm 1\), to within the linear approximation in \(a\), these expressions vanish as well. That is, the solution in (18) satisfies the complete set of the field equations (13) in the limit of slow rotation.

[1] F. R. Tangherlini, Nuovo Cimento 27, 636 (1963).
[2] J. P. Gauntlett, R. C. Myers and P. K. Townsend, Class. Quant. Grav. 16, 1 (1999).
[3] R. P. Kerr and A. Schild, Proc. Symp. Appl. Math. 17, 199 (1965).
[4] R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) 172, 304 (1986).
[5] G. W. Gibbons, H. Lü, D. N. Page and C. N. Pope, Phys. Rev. Lett. 93, 171102 (2004).
[6] B. Carter, Commun. Math. Phys. 10, 280 (1968).
[7] S. W. Hawking, C. J. Hunter and M. M. Taylor-Robinson, Phys. Rev. D 59, 064005 (1999).
[8] A. N. Aliev, Phys. Rev. D 74, 024011 (2006).
[9] A. N. Aliev, Mod. Phys. Lett. A 21, 751 (2006).
[10] M. Cvetiˇ c, H. Lü and C. N. Pope, Phys. Lett. B 598, 273 (2004).
[11] Z-W. Chong , M. Cvetiˇ c, H. Lü and C. N. Pope, Phys. Rev. Lett. 95, 161301 (2005).
[12] A. Bouchareb, G. Clement, C. M. Chen, D. V. Gal’tsov, N. G. Scherbluk and T. Wolf, Phys. Rev. D 76, 104032 (2007); Erratum-ibid. D 78, 029901 (2008).
[13] H. Lu, J. Mei and C. N. Pope, arXiv:0806.2204 [hep-th].
[14] W. Chen and H. Lü, Phys. Lett. B 658, 158 (2008).
[15] Z-W. Chong , G. W. Gibbons, H. Lü and C. N. Pope, Phys. Lett. B 609, 124 (2005).
[16] A. N. Aliev, Phys. Rev. D 77, 044038 (2008).
[17] J. Kunz and F. Navarro-Lerida, Phys. Rev. Lett. 96, 081101 (2006).
[18] J. Kunz and F. Navarro-Lerida, Phys. Lett. B 643, 55 (2006).
[19] H. C. Kim and R. G. Cai, Phys. Rev. D 77, 024045 (2008).
[20] A. Komar, Phys. Rev. 113, 934 (1959).
[21] G. W. Gibbons, M. J. Perry and C.N. Pope, Class. Quant. Grav. 22,1503 (2005).
[22] N. Deruelle and J. Katz, Class. Quant. Grav. 22, 421 (2005).
[23] S. Deser, I. Kanik and B. Tekin, Class. Quantum Grav. 22, 3383 (2005).
[24] A. N. Aliev, Phys. Rev. D 75, 084041 (2007).
[25] W. Chen, H. Lu and C. N. Pope, Phys. Rev. D 73, 104036 (2006).
[26] J. Katz, J. Bičák and D. Lynden-Bell, Phys. Rev. D 55, 5957 (1997).
[27] A. N. Petrov, arXiv:0705.0019 [gr-qc].