INFINITE MEASURE PRESERVING TRANSFORMATIONS
WITH RADON MSJ

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ABSTRACT
We introduce concepts of Radon MSJ and Radon disjointness for infinite Radon measure preserving homeomorphisms of the locally compact Cantor space. We construct an uncountable family of pairwise Radon disjoint infinite Chacon like transformations. Every such transformation is Radon strictly ergodic, totally ergodic, asymmetric (not isomorphic to its inverse), has Radon MSJ and possesses Radon joinings whose ergodic components are not joinings.

0. Introduction

This paper (inspired by recent progress in [JaRoRu]) is about the property of minimal self-joinings (MSJ) for infinite measure preserving transformations. The concept of MSJ for probability preserving maps was introduced in [Rud] as a powerful tool to construct systems with certain prescribed dynamical properties (see also [dJRud], [Ru], [Da2] and references therein for further developments). Some subsequent works are devoted to the extension of MSJ to infinite measure

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preserving and nonsingular systems: [RudSi], [SiWi], [dJSi], [JaRoRu], etc. However, such extensions are only partial due to principal obstacles such as the following ones:

— given a continuous transformation of a Polish space with a recurrent point, there exist uncountably many ergodic pairwise orthogonal nonsingular (as well as infinite invariant) measures [Gli], [Ef],
— ergodic components of a non-ergodic joining need not be joinings [Aa1],
— given an ergodic probability preserving transformation $S$ and $\lambda \in (0, 1)$, there is an ergodic 2-fold self-joining of $S$ of Krieger’s type $III_\lambda$ [RudSi].

To bypass these obstacles the authors of the aforementioned papers have to select special classes of measures invariant or quasi-invariant under Cartesian powers of the transformations under question. For instance, only boundedly finite infinite measures were considered in [JaRoRu] and mostly rational nonsingular joinings were studied in [RudSi]. We also follow this way.

We first develop elements of the general theory of joinings for ergodic measure preserving homeomorphisms $T$ of a locally compact non-compact Cantor space $X$ endowed with an infinite Radon measure $\mu$. This is not a restriction because every ergodic infinite (non-atomic) $\sigma$-finite measure preserving transformation on a standard Borel space is measure theoretically isomorphic to a Radon strictly ergodic homeomorphism on $X$ [Yu]. Considering only Radon joinings and self-joinings of ergodic Radon dynamical systems, we introduce notions of Radon MSJ and Radon disjointness. Using the topological nature of $X$, $T$ and $\mu$ and the Hopf ratio ergodic theorem we can define generic points for the system.

In the remaining part of the paper we present examples of transformations with Radon MSJ. First we consider the infinite Chacon transformation $T$. It was introduced in 1997 by Adams, Friedman and Silva (see [AdFrSi, Section 2]) as an infinite measure preserving counterpart of the classical Chacon map. In a recent paper [JaRoRu] Janvresse, Roy and de la Rue showed that this transformation has MSJ in the class of so-called boundedly finite (in fact, Radon) measures$^1$ and that there exist ergodic infinite $T^{\times d}$-invariant measures whose marginals are singular with respect to the original $T$-invariant measure. Such measures are called weird in [JaRoRu]. We provide different (shorter and more algebraic) proofs of these results in Theorem 5.2. While in [AdFrSi] and [JaRoRu], $T$ is a

$^1$ This extends the well known result that the classical Chacon has MSJ [dJRaSw].
Borel bijection (with countably many discontinuities) of a co-countable subset of the infinite interval $[0, \infty)$ furnished with Lebesgue measure, in this paper $T$ appears as a Radon strictly ergodic homeomorphism of $X$. In addition to what was done in [JaRoRu] we

- describe explicitly the recurrent points for $T \times T$ (Proposition 4.3),
- describe the weird $T^{\times d}$-invariant measures (for each $d > 1$) as quasi-graph measures (see Section 3.2),
- introduce an uncountable family of infinite Chacon like Radon strictly ergodic transformations $T^\omega$, $\omega \in \{0, 1\}^\mathbb{N}$ (in Section 6),
- show that each $T^\omega$ is totally ergodic, has Radon MSJ and is not isomorphic to the inverse $(T^\omega)^{-1}$ (Theorem 6.1, Corollary 6.2),
- prove that $T^\omega$ and $T^{\omega'}$ are isomorphic if the infinite sequences $\omega$ and $\omega'$ are tail equivalent and that $T^\omega$ and $T^{\omega'}$ are Radon disjoint otherwise (Theorem 6.1),
- give an example of nonergodic Radon 2-fold joining of $T$ such that almost all ergodic components of this joining are conservative quasi-graphs whose coordinate projections are singular to the original $T$-invariant measure (Example 5.4).

The outline of the paper is as follows. In Section 1 we introduce some basic notions for Radon dynamical systems such as recurrent points and generic points and discuss ergodic decompositions. Radon joinings and Radon disjointness for general Radon dynamical systems are studied in Section 2. In Section 3 we recall the $(C, F)$-construction of rank-one actions for Abelian groups. We introduce their quasi-graph invariant measures and study their properties. The $(C, F)$-construction is used in Section 4 to define the infinite Chacon transformation $T$. The recurrent points for the Cartesian square of $T$ are explicitly described there. In Section 5 we show that $T$ has Radon MSJ. As a byproduct we describe all $T^{\times d}$-invariant measures on $X^d$. An example of a Radon joining of $T$ whose ergodic components are not joinings is also given in this section. In Section 6 we introduce and study the infinite Chacon like transformations. In the final Section 7 we discuss possible generalizations of the class of dynamical systems to which the main results of this paper extend. Some open problems are also stated there.

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1. Basic definitions: recurrent points, generic points, ergodic decomposition

Let $X$ be a locally compact non-compact Cantor space and let $T$ be an aperiodic homeomorphism of $X$.

**Definition 1.1:** A point $x \in X$ is called $T$-**recurrent** if for each neighborhood $U$ of $x$ there is an integer $n \neq 0$ such that $T^n x \in U$.

We are interested in $T$-invariant (non-negative) $\sigma$-finite Borel measures on $X$. Since $X$ is non-compact, $T$-invariant probability measures on $X$ need not exist. On the other hand, it follows from [Gli] and [Sc] that if there exists a $T$-recurrent point, then there are uncountably many non-equivalent infinite $T$-invariant $\sigma$-finite Borel measures on $X$. In this work our main concern is Radon measures. We recall that a Borel measure $\lambda$ on $X$ is called **Radon** if it is finite on every compact subset of $X$. Of course, every Radon measure is $\sigma$-finite. The converse is not true. Denote by $\mathcal{M}_{Ra}(X)$ the cone of all Radon measures on $X$. It is a Polish space in the $*$-weak topology. Let $\mathcal{M}_{Ra}(X,T)$ stand for the subset of $T$-invariant Radon measures on $X$ and let $\mathcal{M}^e_{Ra}(X,T)$ denote the subset of ergodic $T$-invariant Radon measures on $X$. The two subsets are Borel. Given a pair of measures $\lambda, \lambda' \in \mathcal{M}^e_{Ra}(X,T)$, we have that either $\lambda = c\lambda'$ for some real $c > 0$ or $\lambda \perp \lambda'$. If $\mathcal{M}_{Ra}(X,T)$ consists of a single ray we say that $T$ is **Radon uniquely ergodic**. If, in addition, $T$ is minimal then we say that $T$ is **Radon strictly ergodic**.

**Definition 1.2:** Let $\lambda \in \mathcal{M}^e_{Ra}(X,T)$. A point $x \in X$ is called **generic** for $\lambda$ if for each pair of compact open subsets $K_1, K_2 \subset X$ such that $\lambda(K_2) > 0$, the following limit exists and is given by

$$
\lim_{n \to \infty} \frac{\sum_{a_n \leq j \leq b_n} 1_{K_1}(T^j x)}{\sum_{a_n \leq j \leq b_n} 1_{K_2}(T^j x)} = \frac{\lambda(K_1)}{\lambda(K_2)}
$$

whenever $a_n \leq b_n$ and $b_n - a_n \to +\infty$.

If $\lambda$ is non-atomic (i.e., the dynamical system $(X,\lambda, T)$ is conservative) then every generic point is recurrent. Denote by $\mathcal{G}(\lambda)$ the set of all generic points for $\lambda$. Then $\mathcal{G}(\lambda)$ is a Borel subset of $X$ and $TG(\lambda) = \mathcal{G}(\lambda)$ (pointwise, not only almost everywhere). By the Hopf ratio ergodic theorem [Aa2], $\lambda$-almost every
point of $X$ is generic for $\lambda$. If $\lambda'$ is another ergodic $T$-invariant Radon measure, then $G(\lambda) = G(\lambda')$ if $\lambda' = c\lambda$ for some $c > 0$ and $G(\lambda) \cap G(\lambda') = \emptyset$ otherwise.

Let $\eta \in \mathcal{M}_{\text{Ra}}(X,T)$. Then there is a probability measure $\kappa$ on $\mathcal{M}_{\text{Ra}}^e(X,T)$ such that $\eta = \int_{\mathcal{M}_{\text{Ra}}(X,T)} \lambda d\kappa(\lambda)$. This integral is called an ergodic decomposition of $\eta$. It is not unique.

2. Radon self-joinings and Radon disjointness

Let $X$ be a locally compact Cantor space, $G$ a discrete countable infinite Abelian group, $T = (T_g)_{g \in G}$ a continuous action of $G$ on $X$ and $\mu$ a $T$-invariant Radon measure on $X$. We call the triple $(X,T,\mu)$ a Radon dynamical system. From now on we assume that $\mu$ is ergodic.

Definition 2.1: Let $d > 1$. A Radon $((T_g)^{x\times d})_{g \in G}$-invariant measure on $X^d$ whose coordinate projections (marginals) are all equivalent to $\mu$ is called a $d$-fold Radon self-joining of $T$. The set of all Radon $d$-fold self-joinings of $T$ will be denoted by $J_{d,\text{Ra}}(T)$. The subset of ergodic Radon $d$-fold self-joinings of $T$ will be denoted by $J_{d,\text{Ra}}^e(T)$.

For instance, the product $\mu \otimes \cdots \otimes \mu$ ($d$ times) belongs to $J_{d,\text{Ra}}(T)$.

Denote by $C(T)$ the centralizer of $T$, i.e., the group of all invertible $\mu$-nonsingular transformations commuting with $T_g$ for each $g \in G$. If $S_1, \ldots, S_{d-1} \in C(T)$,

then there are constants $c_i > 0$ such that

$$\mu \circ S_i = c_i \mu$$

for all $i = 1, \ldots, d$. The measure $\mu_{S_1,\ldots,S_{d-1}}$ on $X^d$, given by

$$\mu_{S_1,\ldots,S_{d-1}}(A_1 \times \cdots \times A_d) := \mu(A_1 \cap S_1^{-1}A_2 \cap \cdots \cap S_{d-1}^{-1}A_d)$$

for all Borel subsets $A_1, \ldots, A_d \subset X$, belongs to $J_{d,\text{Ra}}^e(T)$. Since $\lambda_{S_1,\ldots,S_{d-1}}$ is supported on the graph of the map $X \ni x \mapsto (S_1x,\ldots,S_{d-1}x) \in X^{d-1}$, it is called a graph-joining of $T$.

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2 This theorem was stated there only for conservative systems. It is also true for dissipative (ergodic, Radon) ones with a trivial proof.

3 We note that these projections are not necessarily $\sigma$-finite.

4 Because $T$ is ergodic.
Definition 2.2: If for each $\lambda \in J^d_{Ra}(T)$ there is a partition of $\{1, \ldots, d\}$ into subsets $J_1, \ldots, J_k$ such that $\lambda$ splits into a direct product of its marginals $\lambda_i$ on $X^{J_i}$, each $\lambda_i$ is (up to a multiplicative constant) a graph-joining of $T$ if $\#J_i > 1$ or $\lambda_i = \mu$ if $\#J_i = 1$, $i = 1, \ldots, k$, and $C(T) = \{T_g \mid g \in G\}$, then we say that $T$ has the property of Radon MSJ$_d$. If $T$ has Radon MSJ$_d$ for each $d > 1$, then we say that $T$ has Radon MSJ.

We note that if $\lambda$ is a non-ergodic Radon self-joining of $T$, then the ergodic components of $\lambda$ need not be self-joinings of $T$. For instance, there exist Radon strictly ergodic homeomorphisms $R$ of $X$ with a $T$-invariant Radon measure $\mu$ such that the system $(X \times X, \mu \times \mu, R \times R)$ is totally dissipative [Da1, Theorem 0.1(4)], i.e., almost every ergodic component of the Radon 2-fold self-joining $\mu \times \mu$ of $R$ is an $(R \times R)$-orbit equipped with the “counting” purely atomic measure. Purely atomic measures are not self-joinings of $R$ because their coordinate projections are also atomic and hence singular to $\mu$ which is continuous.

We also need the following analogue of the Furstenberg concept of disjointness for probability preserving systems [Fu].

Definition 2.3: Let $(X, (T_g)_{g \in G}, \mu)$ and $(Y, (S_g)_{g \in G}, \nu)$ be two ergodic Radon dynamical systems. We say that they are Radon disjoint if $\mu \times \nu$ is the only (up to a multiplicative constant) ergodic Radon $(T_g \times S_g)_{g \in G}$-invariant measure on $X \times Y$ whose coordinate projections are equivalent to $\mu$ and $\nu$ respectively.

Of course, if $T$ and $S$ are Radon disjoint then they are not isomorphic.

3. $(C,F)$-construction of rank-one Abelian actions

3.1. $(C,F)$-nomenclature. For a detailed exposition of the $(C,F)$-construction we refer the reader to [Da1] and [Da3].

Let $G$ be a discrete countable infinite Abelian group. Let $(F_n)_{n \geq 0}$ and $(C_n)_{n \geq 1}$ be two sequences of finite subsets in $G$ such that for each $n > 0$,

(I) $F_0 = \{0\}$, $\#C_n > 1$,

(II) $F_0 \subset F_1 \subset \cdots$ with $\bigcup_{n \geq 0} F_n = G$,

(III) $F_n + F_n + C_{n+1} \subset F_{n+1}$,

(IV) $(F_n + c) \cap (F_n + c') = \emptyset$ for all $c, c' \in C_{n+1}$ with $c \neq c'$.

We let $X_n := F_n \times C_{n+1} \times C_{n+2} \times \cdots$ and endow this set of infinite sequences with the infinite product topology. Then $X_n$ is a compact Cantor (i.e., totally disconnected) set.
disconnected perfect metric) space. The mapping

\[ X_n \ni (f_n, c_{n+1}, c_{n+2}, \ldots) \mapsto (f_n + c_{n+1}, c_{n+2}, \ldots) \in X_{n+1} \]

is a topological embedding of \( X_n \) into \( X_{n+1} \). From now on we can consider \( X_n \) a subset of \( X_{n+1} \). Let \( X \) denote the inductive limit of the sequence \( (X_n)_{n \geq 0} \) furnished with these embeddings. Then \( X \) is a well defined locally compact Cantor space. We call it the \((C, F)\)-space associated with the sequence \((C_n, F_{n-1})_{n \geq 1}\). Given a subset \( A \subset F_n \), we let

\[ [A]_n := \{ x = (f_n, c_{n+1}, \ldots) \in X_n \mid f_n \in A \} \]

and call this set an \( n \)-cylinder in \( X \). It is open and compact in \( X \). It is easy to verify that

\[ [A]_n \cap [B]_n = [A \cap B]_n, \quad [A]_n \cup [B]_n = [A \cup B]_n \quad \text{and} \quad [A]_n = [A + C_{n+1}]_{n+1} \]

for all \( A, B \subset F_n \) and \( n \geq 0 \). For brevity, we will write \([f]_n\) for \([\{f\}]_n\), \( f \in F_n \).

We note that the collection of all cylinders coincides with the family of all compact open subsets in \( X \).

Let \( \mathcal{R} \) denote the tail equivalence relation on \( X \). This means that the restriction of \( \mathcal{R} \) to \( X_n \) is the tail equivalence relation on \( X_n \) for each \( n \geq 0 \). We note that \( \mathcal{R} \) is minimal, i.e., the \( \mathcal{R} \)-class of every point is dense in \( X \). There exists a unique \( \sigma \)-finite \( \mathcal{R} \)-invariant Borel measure \( \mu \) on \( X \) such that \( \mu(X_0) = 1 \). It is Radon. Moreover, \( \mu \) is strictly positive on every non-empty open subset.

We note that the \( \mathcal{R} \)-invariance\(^5\) of \( \mu \) is equivalent to the following property:

\[ \mu([f]_n) = \mu([f']_n) \quad \text{for all } f, f' \in F_n, n \geq 0. \]

This and the normalization condition on \( \mu \) imply that

\[ \mu([A]_n) = \frac{\#A}{\#C_1 \cdots \#C_n} \quad \text{for each subset } A \subset F_n, \ n > 0. \]

We call \( \mu \) the \((C, F)\)-measure associated with \((C_n, F_{n-1})_{n \geq 1}\). It is infinite if and only if\(^6\)

\[ (3-1) \quad \lim_{n \to \infty} \frac{\#F_n}{\#C_1 \cdots \#C_n} = \infty. \]

\(^5\) \( \mu \) is called \( \mathcal{R} \)-invariant if \( \mu \) is invariant under each Borel transformation whose graph is contained in \( \mathcal{R} \).

\(^6\) In view of (I)–(IV), the sequence \( \frac{\#F_n}{\#C_1 \cdots \#C_n} \) is non-decreasing and bounded by 1 from below.
It is easy to see that $\mu$ is $\mathcal{R}$-ergodic, i.e., each Borel $\mathcal{R}$-saturated subset of $X$ is either $\mu$-null or $\mu$-conull. We now define an action of $G$ on $X$. Given $g \in G$, let

$$X^g_n := \{(f_n, c_{n+1}, c_{n+2}, \ldots) \in X_n \mid g + f_n \in F_n\}.$$ 

Then $X^g_n$ is a compact open subset of $X_n$ and $X^g_n \subset X^{g+1}_n$. In view of (III) and (II), $X_n \subset X^g_{n+1}$ eventually in $n$. Hence, given $x \in X$ and $g \in G$, there is $n > 0$ such that if we consider an expansion $x = (f_n, c_{n+1}, \ldots)$ of $x$ in $X_n$, then $g + f_n \in F_n$. We now let

$$T_g x := (g + f_n, c_{n+1}, \ldots) \in X_n \subset X.$$ 

It is standard to verify that

(i) the map $T_g : X \ni x \mapsto T_g x \in X$ is a homeomorphism of $X$ and

(ii) $T_g T_{g'} = T_{g+g'}$ for all $g, g' \in G$.

Thus $T := (T_g)_{g \in G}$ is a continuous action of $G$ on $X$. We call it the $(C, F)$-action of $G$ associated with the sequence $(C_n, F_n)_{n \geq 1}$. It is free. The $T$-orbit equivalence relation is $\mathcal{R}$. Indeed, if $(x, x') \in \mathcal{R}$ then there is $n > 0$ such that $x, x' \in X_n$. Consider now the expansions $x = (f_n, c_{n+1}, \ldots)$ and $x' = (f'_n, c'_{n+1}, \ldots)$ of $x$ and $x'$ in $X_n$. Then $x' = T_g x$, where

$$g := f'_n - f_n + \sum_{j>n} (c'_j - c_j).$$

It follows that $T$ preserves $\mu$. We call $(X, \mu, T)$ the dynamical system associated with $(C_n, F_{n-1})_{n \geq 1}$. It is Radon strictly ergodic.$^7$

We will need the following standard lemma (see [Da4, Lemma 1.1(ii)] for the proof of a more general fact).

**Lemma 3.1:** Let $g \in G$. If there is $\delta > 0$ such that for each $n \in \mathbb{N}$ and $v, w \in F_n$, there is a subset $A \subset [v]_n$ and $m \in \mathbb{Z}$ such that $T_{mg} A \subset [w]_n$ and $\mu(A) \geq \delta \mu([v]_n)$, then the transformation $T_g$ is ergodic.

**3.2. Quasi-graph invariant Radon measures.** In this subsection we introduce an important class of Radon $((T_g)^d)_{g \in G}$-invariant measures on $X^d$ which are “close” to graph-joinings but whose marginals can be singular to $\mu$. Such

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$^7$ We also note that $T$ has rank one along the sequence $(F_n)_{n=1}^\infty$ (in the measure theoretical sense, see [Da1]).
measures can appear as ergodic components of non-ergodic Radon $d$-fold self-joinings of $T$ (see §5 and §6). Throughout this subsection we will assume that the following condition holds:

(V) $F_n + C_{n+1} + F_n - F_n \subset F_{n+1}$ for each $n > 0$.

This condition is stronger than (III). Let $J$ be a subset of $\mathbb{N}$. For each $n \notin J$, fix an element $\alpha_n \in C_n$. Thus we obtain a sequence $\alpha := (\alpha_n)_{n \notin J}$. We now define a new $(C,F)$-sequence $(C_n^\alpha,F_n^\alpha)_{n \geq 1}$ by setting $F_n^\alpha := F_n$ and

$$C_n^\alpha := \begin{cases} C_n, & \text{if } n \in J, \\ \{\alpha_n\}, & \text{if } n \notin J. \end{cases}$$

Then $(C_n^\alpha,F_n^\alpha)_{n \geq 1}$ satisfies (I)–(V) except for the fact that the cardinality of $C_n^\alpha$ can be 1. This happens if and only if $n \notin J$. Nevertheless, the associated $(C,F)$-action $T^\alpha = (T_g^\alpha)_{g \in G}$ on a locally compact $(C,F)$-space $X^\alpha$ equipped with an invariant $\sigma$-finite measure $\mu^\alpha$ is well defined (via the same inductive construction process described in §3.1). The space $X^\alpha$ is totally disconnected. It is either perfect (i.e., Cantor) or purely discrete and countable. The latter happens if and only if $J$ is finite. We note that $\mu^\alpha$ is finite if and only if $\mu$ is finite and $J$ is cofinite.\footnote{I.e., the complement $\mathbb{N} \setminus J$ is finite.}

Given $A \subset F_m$, the corresponding cylinder in $X^\alpha$ will be denoted by $[A]^\alpha_m$ (in order to not confuse it with the cylinder $[A]_m$ in $X$). We now define an embedding $\phi_\alpha$ of $X^\alpha$ into $X$ by setting for each $n > 0$ and a point $x = (f_n,c_{n+1},c_{n+2},\ldots) \in X^\alpha_n \subset X^\alpha$,

$$\phi_\alpha(x) := (f_n,c_{n+1},c_{n+2},\ldots) \in X_n \subset X.$$

It is straightforward to verify that $\phi_\alpha$ is a well-defined, one-to-one continuous equivariant\footnote{I.e., $\phi^\alpha \circ T_g^\alpha = T_g \circ \phi^\alpha$ for each $g \in G$.} map. Hence the image $\phi^\alpha(X^\alpha)$ of $X^\alpha$ is a $T$-invariant dense $F_\sigma$-subset of $X$. Now given $m > 0$ and a subset $A \subset F_m$, take a point $x \in [A]^\alpha_m \cap \phi_\alpha(X^\alpha)$. If we write the expansion $x = (f_m,c_{m+1},c_{m+2},\ldots)$ of $x$ in $X_m$, then $f_m \in A$ and $c_l = \alpha_l$ for all $l \notin J$ that are greater than some $n > m$. It follows that

$$[A]^\alpha_m \cap \phi_\alpha(X^\alpha) = \bigsqcup_{g \in \sum_{j \geq m}(C_l - \alpha_l)} T_g \phi_\alpha([A]^\alpha_m).$$

We have the following trichotomy:
• if $J$ is cofinite, then $\mu_\alpha \circ \phi_\alpha^{-1}$ is a multiple of $\mu$ and hence $\phi_\alpha$ is an isomorphism (both topological and measure theoretical) of $(X_\alpha, \mu_\alpha, T_\alpha)$ onto $(X, \mu, T)$;

• if $J$ is finite, then $\mu_\alpha \circ \phi_\alpha^{-1} \perp \mu$ and $(X_\alpha, \mu_\alpha, T_\alpha)$ is an ergodic totally dissipative dynamical system;

• if $J$ is neither finite nor cofinite, then $\mu_\alpha \circ \phi_\alpha^{-1} \perp \mu$ and $(X_\alpha, \mu_\alpha, T_\alpha)$ is an ergodic conservative dynamical system.

Consider now another sequence $\beta = (\beta_n)_{n \notin J}$ with $\beta_n \in C_n$ for each $n \notin J$. Let $(X_\beta, \mu_\beta, T_\beta)$ be the dynamical system associated with $(C_\beta, F_\beta)$, where $C_\beta$ and $F_\beta$ are defined in a similar way to $C_n$ and $F_n$. Then there is a natural measure preserving topological equivariant isomorphism $\phi_{\alpha, \beta}$ of $(X_\alpha, \mu_\alpha, T_\alpha)$ onto $(X_\beta, \mu_\beta, T_\beta)$. To define $\phi_{\alpha, \beta}$ we take a point $x \in X_{\alpha-1}^\beta$ for some $n > 0$ and write the expansion of $x$ in

$$X_\alpha : x = (f_n, c_{n+1}, c_{n+2}, \ldots) \in F_n \times C_{n+1}^\alpha \times C_{n+2}^\alpha \times \cdots.$$

Then $f_n \in F_{n-1} + C_n^\alpha$. We now set

$$\phi_{\beta, \alpha}(x) := (\tilde{f}_n, \tilde{c}_{n+1}, \tilde{c}_{n+2}, \ldots),$$

where $\tilde{f}_n := f_n + \sum_{j \not\in J \leq n} (\beta_j - \alpha_j)$ and for each $l > n$,

$$\tilde{c}_l := \begin{cases} c_l, & \text{if } l \in J, \\ \beta_l, & \text{if } l \notin J. \end{cases}$$

We note that if $n \in J$, then $\sum_{j \not\in J \leq n} (\beta_j - \alpha_j) \in F_{n-1} - F_{n-1}$ and hence

$$\tilde{f}_n \in f_n + F_{n-1} - F_{n-1} \subset F_{n-1} + C_n^\alpha + F_{n-1} - F_{n-1}.$$

If $n \not\in J$, then $\sum_{j \not\in J \leq n} (\beta_j - \alpha_j) \in \beta_n - \alpha_n + F_{n-1} - F_{n-1}$ and hence

$$\tilde{f}_n \in F_{n-1} + C_n^\alpha + \beta_n - \alpha_n + F_{n-1} - F_{n-1} = F_{n-1} + C_n^\beta + F_{n-1} - F_{n-1}.$$

Therefore in each of the two possible cases

$$\tilde{f}_n \in F_n$$

in view of (V). Hence $\phi_{\alpha, \beta}(x) \in X_\beta$. Thus $\phi_{\alpha, \beta}$ is well defined. Moreover, for each subset $A \subset F_{n-1} + C_n^\alpha$,

$$\phi_{\beta, \alpha}(A^\alpha) = [\tilde{A}]_\beta, \quad \text{where } \tilde{A} := A + \sum_{j \not\in J \leq n} (\beta_j - \alpha_j).$$
It is straightforward to verify that $\phi_{\beta, \alpha}$ is as claimed. It is easy to see that the converse $\tilde{\phi}_{\beta, \alpha}^{-1}$ to $\phi_{\beta, \alpha}$ is $\phi_{\alpha, \beta}$. We now define a map $\psi_{\alpha, \beta}$ from $X^\alpha$ to $X \times X$ by setting

$$\psi_{\alpha, \beta}(x) := (\phi_{\alpha}(x), \phi_{\beta}(\phi_{\alpha}(x))).$$

It is one-to-one, continuous and $\psi_{\alpha, \beta} \circ T_g^\alpha = (T_g \times T_g) \circ \psi_{\alpha, \beta}$ for all $g \in G$.

**Theorem 3.2:** Suppose that for each $l \not\in J$,

$$\text{(3-4)} \quad (C_l - \alpha_l + F_{l-1}^\bullet) \cap (C_l - \beta_l + F_{l-1}^\bullet) = F_{l-1}^\bullet,$$

where $F_{l-1}^\bullet := F_{l-2} + C_{l-1} - F_{l-2} - C_{l-1}$. Then the following are satisfied:

(i) The subset $\psi_{\alpha, \beta}(X^\alpha)$ is closed in $X \times X$. Hence $\psi_{\alpha, \beta}(X^\alpha)$ is locally compact in the induced topology.

(ii) The topological dynamical system $(\psi_{\alpha, \beta}(X^\alpha), (T_g \times T_g)_{g \in G})$ is Radon strictly ergodic.

(iii) $\mu^\alpha \circ \psi_{\alpha, \beta}^{-1}$ is the only (up to scaling) $T_1 \times T_1$-invariant Radon measure supported on $\psi_{\alpha, \beta}(X^\alpha)$.

(iv) Given $a, b \in F_m$, represent the cylinder $[a]_m \times [b]_m$ in $X \times X$ as the infinite product $\{(a, b)\} \times (C_{m+1} \times C_{m+1}) \times (C_{m+2} \times C_{m+2}) \times \cdots$. Then

$$\mu^\alpha \circ \psi_{\alpha, \beta}^{-1} | ([a]_m \times [b]_m) = \mu^\alpha([a]_m^\alpha) \cdot \delta_{a', b'} \otimes \bigotimes_{n>m} \gamma_n,$$

where $\gamma_n$ is a probability on $C_m \times C_m$ given by

$$\gamma_n := \begin{cases} \frac{1}{\# C_n} \sum_{c \in C_n} \delta_c \times \delta_c, & \text{if } n \in J, \\ \delta_{(\alpha_n, \beta_n)}, & \text{if } n \not\in J, \end{cases}$$

$$a' = a + \sum_{j \not\in J \leq m} \alpha_j \text{ and } b' = b + \sum_{j \not\in J \leq m} \beta_j.$$\(^{10}\)

**Proof.** (i) If $J$ is cofinite, then it is straightforward to verify that $\psi_{\alpha, \beta}(X_\alpha)$ is the the graph of $T_h$, where $h := \sum_{j \not\in J}(\beta_j - \alpha_j)$. The graph is closed in $X \times X$.

Consider now another particular case where $J$ is finite. Then it is straightforward to verify that $\psi_{\alpha, \beta}(X_\alpha)$ is the $(T_g \times T_g)_{g \in G}$-orbit of a point $(x, y) \in X \times X$ such that $x = (f_m, \alpha_{m+1}, \alpha_{m+2}, \ldots) \in X_m$ and $y = (r_m, \beta_{m+1}, \beta_{m+2}, \ldots) \in X_m$ for some $m \geq 0$ and elements $f_m, r_m \in F_m$. It suffices to show that given $l > m$

\(^{10}\) Here and below $\delta_c$ or $\delta_{(a, b)}$ denotes the probability measure supported at a single point $\{c\}$ or $\{(a, b)\}$ respectively while $\delta_{a', b'}$ denotes the Kronecker delta.
and subsets $A, B \subset F_l$, the intersection of the orbit of $(x, y)$ with the cylinder $[A]_l \times [B]_l$ is finite. Indeed, $(Tgx, Tgy) \in [A]_l \times [B]_l$ if and only if
\[
g \in A - f_m - \alpha_{m+1} - \cdots - \alpha_l + \sum_{i > l} (C_i - \alpha_i) \quad \text{and} \quad g \in B - r_m - \beta_{m+1} - \cdots - \beta_l + \sum_{i > l} (C_i - \beta_i).
\]
Hence $g \in F^*_{l+1} + \sum_{i > l+1} (C_i - \alpha_i)$ and $g \in F^*_{l+1} + \sum_{i > l+1} (C_i - \beta_i)$. It follows from (3-4) that $(F^*_{l+1} + \sum_{i > l+1} (C_i - \alpha_i)) \cap (F^*_{l+1} + \sum_{i > l+1} (C_i - \beta_i)) = F^*_{l+1}$. It remains to note that $F^*_{l+1}$ is finite.

Suppose now that $J$ is neither finite nor cofinite. Fix $n > 0$ and subsets $A \subset F_{n-1} + C_n$ and $B \subset F_{n-1} + C_n^\beta$. Applying (3-2) we obtain that
\[
\psi^{-1}_{\alpha, \beta}([A]_n \times [B]_n) \subset \phi^{-1}_{\alpha}([A]_n) \cap \phi^{-1}_{\beta, \alpha}(\phi^{-1}_{\beta}([B]_n))
\]
\[
= \left( \bigcup_{g \in \sum_{J \ni l > n}(C_l - \alpha_l)} T^\alpha_{g}[A]_l^\alpha \right) \cap \left( \bigcup_{g \in \sum_{J \ni l > n}(C_l - \beta_l)} T^\beta_{g}[B]_l^\beta \right)
\]
\[
= \left( \bigcup_{J \ni l > n} [A + C_{n+1} + \cdots + C_l]_l^\alpha \right) \cap \left( \bigcup_{J \ni l > n} [B + C_{n+1} + \cdots + C_l]_l^\beta \right)
\]
\[
= \bigcup_{J \ni l > n} ([A + C_{n+1} + \cdots + C_l]_l^\alpha \cap \phi_{\alpha, \beta}([B + C_{n+1} + \cdots + C_l]_l^\beta)).
\]

Given $l \not\in J$, let $l'$ be the least integer such that $l' > l$ and $l' \not\in J$. Then $C_l^\alpha = C_l^\beta = C_k$ if $l < k < l'$, $C_l^\alpha = \{\alpha_{l'}\}$, $C_l^\beta = \{\beta_{l'}\}$ and hence
\[
[A + C_{n+1} + \cdots + C_l]_l^\alpha = [A' + \alpha_{l'}]_l^\alpha \quad \text{and} \quad [B + C_{n+1} + \cdots + C_l]_l^\beta = [B' + \beta_{l'}]_l^\beta,
\]
where
\[
A' := A + C_{n+1} + \cdots + C_{l'-1} \subset F_{l'-1} \quad \text{and} \quad B' := B + C_{n+1} + \cdots + C_{l'-1} \subset F_{l'-1}.
\]
Utilizing this and (3-3) we obtain that\(^{11}\)
\[
\psi^{-1}_{\alpha, \beta}([A]_n \times [B]_n) \subset \bigcup_{J \ni l > n} \left( [A' + \alpha_{l'}]_l^\alpha \cap \left[ B' + \beta_{l'} + \sum_{J \ni j \leq l'} (\alpha_j - \beta_j) \right]_l^\alpha \right)
\]
\[
= \bigcup_{J \ni l > n} \left( \left[ \left( A' \cap \left[ B' + \sum_{J \ni j \leq l} (\alpha_j - \beta_j) \right] \right) + \alpha_{l'} \right]_l^\alpha \right).
\]
\(^{11}\) In the formulas below, $l' = l'(l)$. 

It follows from (3-4) that for all subsets \( D, D' \subset F_n \),
\[
\left( D + \sum_{j=n+1}^{l'-1} C_j - \sum_{n<j<l'} \alpha_j \right) \cap \left( D' + \sum_{j=n+1}^{l'-1} C_j - \sum_{n<j<l'} \beta_j \right) = (D \cap D') + \sum_{n<j<l'} C_j.
\]
Therefore we have
\[
A' \cap \left( B' + \sum_{j \not\in J \leq l} (\alpha_j - \beta_j) \right)
\]
\[
= \left( \left( A + \sum_{j=n+1}^{l'-1} C_j - \sum_{j<l'} \alpha_j \right) \cap \left( B + \sum_{j=n+1}^{l'-1} C_j - \sum_{j<l'} \beta_j \right) \right) + \sum_{j \not\in J \leq l} \alpha_j
\]
\[
= \left( \left( \hat{A} + \sum_{j=n+1}^{l'-1} C_j - \sum_{n<j<l'} \alpha_j \right) \cap \left( \hat{B} + \sum_{j=n+1}^{l'-1} C_j - \sum_{n<j<l'} \beta_j \right) \right) + \sum_{j \not\in J \leq l} \alpha_j
\]
\[
= (\hat{A} \cap \hat{B}) + \sum_{j \leq n, n<j<l'} C_j + \sum_{j \not\in J \leq l} \alpha_j + \sum_{n<j<l'} \alpha_j
\]
\[
= (A \cap \tilde{B}) + \sum_{n<j<l'} C_j^\alpha,
\]
where
\[
\hat{A} := A - \sum_{j \not\in J \leq n} \alpha_j \subset F_{n-1} + C_n - F_{n-1} - C_n = F_n, \]
\[
\hat{B} := B - \sum_{j \not\in J \leq n} \beta_j \subset F_n \quad \text{and}
\]
\[
\tilde{B} := B + \sum_{j \not\in J \leq n} (\alpha_j - \beta_j).
\]
Hence
\[
\psi_{\alpha,\beta}^{-1}(A_n \times B_n) \subset \bigcup_{J \not\in J \leq n} \left( (A \cap \tilde{B}) + \sum_{n<j<l'} C_j\right)^\alpha_{l'} = [A \cap \tilde{B}]_{n}^\alpha.
\]
On the other hand, it is straightforward to verify that the converse inclusion holds. Therefore,
\[
(3-5) \quad \psi_{\alpha,\beta}^{-1}(A_n \times B_n) = [A \cap \tilde{B}]_{n}^\alpha.
\]
It follows that the \( \psi_{\alpha,\beta} \)-preimage of \( [A]_n \times [B]_n \) is compact in \( X^\alpha \). Hence \( [A]_n \times [B]_n \) does not intersect the set \( \psi_{\alpha,\beta}(X^\alpha) \setminus \psi_{\alpha,\beta}(X_\alpha) \). Since \( J \) is not
finite, then we take \( A := B := F_{n-1} + C_n \) for \( n \in J \) and obtain that the union

\[
\bigcup_{n \in J} ([F_{n-1} + C_n]_n \times [F_{n-1} + C_n]_n) = \bigcup_{n \in J} (X_{n-1} \times X_{n-1}) = X \times X
\]
does not intersect \( \overline{\psi_{\alpha, \beta}(X_\alpha)} \setminus \psi_{\alpha, \beta}(X_\alpha) \). Hence \( \psi_{\alpha, \beta}(X_\alpha) \) is closed.

(ii) It follows from (i) that \( \psi \) is an equivariant topological isomorphism of the dynamical system \((X_\alpha, T_\alpha)\) onto \((\psi_{\alpha, \beta}(X_\alpha), (T_g \times T_g)_{g \in G})\). Since the former system is Radon strictly ergodic, so is the latter.

(iii) is obvious now.

(iv) Let \( m, a, b \) be the same as in the statement of (iv). Suppose that \( J \) is infinite. Take \( n > m \) with \( n \in J \) and elements \( c_j, \hat{c}_j \in C_j \) whenever \( m < j \leq n \).

Then by (3-5),

\[
\psi_{\alpha, \beta}^{-1} \left( \left[ a + \sum_{m < j \leq n} c_j \right]_n \times \left[ b + \sum_{m < j \leq n} \hat{c}_j \right]_n \right)
\]

\[
= \left[ \left\{ a + \sum_{m < j \leq n} c_j \right\} \cap \left\{ b + \sum_{m < j \leq n} \hat{c}_j + \sum_{J \not\ni j \leq n} (\alpha_j - \beta_j) \right\} \right]_n.
\]

Therefore, by (3-4), the intersection in the above formula is non-empty if and only if \( a = b + \sum_{J \not\ni j \leq m} (\alpha_j - \beta_j) \), \( c_j = \hat{c}_j \) if \( J \ni j \) and \( c_j = \alpha_j \) and \( \hat{c}_j = \beta_j \) if \( J \not\ni j \). Hence

\[
\mu^\alpha \left( \psi_{\alpha, \beta}^{-1} \left( \left[ a + \sum_{j=m+1}^n c_j \right]_n \times \left[ b + \sum_{j=m+1}^n \hat{c}_j \right]_n \right) \right)
\]

\[
= \mu^\alpha ([a]_m^\alpha) \cdot \delta_{a', b'} \prod_{j=m+1}^n \gamma_j (c_j, \hat{c}_j),
\]

where \( a', b' \) and \( \gamma_j \) were introduced in the statement of (iv).

**Definition 3.3:** Given a subset \( J \subset \mathbb{N} \), two sequences \( \alpha = (\alpha_n)_{n \notin J} \) and \( \beta = (\beta_n)_{n \notin J} \) with \( \{\alpha_n, \beta_n\} \subset C_n \) for each \( n \notin J \) and an element \( h \in G \), we call the measure \((\mu^\alpha \circ \psi_{\alpha, \beta}^{-1}) \circ (I \times T_h^{-1})\) the \textbf{quasi-graph} measure.

It follows from Theorem 3.2 that if (3-4) is satisfied, then

\[
(\mu^\alpha \circ \psi_{\alpha, \beta}^{-1}) \circ (I \times T_h^{-1}) \in \mathcal{M}_{\text{Ra}}^G (X \times X, (T_g \times T_g)_{g \in G}).
\]

The projection of this measure to the first coordinate is equivalent to \( \mu^\alpha \) and the projection of this measure to the second coordinate is equivalent to \( \mu^\beta \). The conditional measures corresponding to the disintegrations of \((\mu^\alpha \circ \psi_{\alpha, \beta}^{-1}) \circ (I \times T_h^{-1})\)
over $\mu^\alpha$ (or over $\mu^\beta$) are delta-measures almost everywhere. The quasi-graph Radon measure is a 2-fold self-joining of $T$ if and only if $\mu^\alpha \sim \mu^\beta \sim \mu$. This happens if and only if $J$ is cofinite. In this case the quasi-graph measure is a graph-joining of $T$.

In a similar way we may define quasi-graph invariant Radon measures for higher Cartesian powers of $T$. Indeed, given $d \geq 1$ and sequences $\alpha^{(i)} = (\alpha^{(i)}_n)_{n \notin J}$, $i = 1, \ldots, d$, such that $\alpha^{(i)}_n \in C_n$ for all $i = 1, \ldots, d$, we define a map $\psi_{\alpha^{(1)}, \ldots, \alpha^{(d)}}$ from $X^{\alpha^{(1)}}$ to the $d$-th Cartesian power of $X$ by setting

$$
\psi_{\alpha^{(1)}, \ldots, \alpha^{(d)}}(x) := (\phi_{\alpha^{(1)}}(x), \phi_{\alpha^{(2)}}(\phi_{\alpha^{(2)}, \alpha^{(1)}}(x)), \ldots, \phi_{\alpha^{(d)}}(\phi_{\alpha^{(d)}, \alpha^{(1)}}(x))).
$$

This map is one-to-one, continuous and equivariant, i.e.,

$$
\psi_{\alpha^{(1)}, \ldots, \alpha^{(d)}} \circ (T^{\alpha^{(1)}})_1 = (T_1)^{\times d} \circ \psi_{\alpha^{(1)}, \ldots, \alpha^{(d)}}.
$$

The following theorem is a $d$-fold analogue of Theorem 3.2. It can be proved in a similar way. Therefore we leave the proof to the reader.

**Theorem 3.4:** Suppose that for each $l \notin J$, there are $u_l, v_l \in \{1, \ldots, d\}$ such that

$$
(C_l - \alpha^{(u_l)}_l + F^{\bullet}_{l-1}) \cap (C_l - \alpha^{(v_l)}_l + F^{\bullet}_{l-1}) = F^{\bullet}_{l-1},
$$

where $F^{\bullet}_{l-1}$ is same as in the statement of Theorem 3.2. Then the following are satisfied:

(i) The subset $\psi_{\alpha^{(1)}, \ldots, \alpha^{(d)}}(X^{\alpha^{(1)}})$ is closed in $X^d$. Hence it is locally compact in the induced topology.

(ii) The topological dynamical system $(\psi_{\alpha^{(1)}, \ldots, \alpha^{(d)}}(X^{\alpha^{(1)}}), (T_g)^{\times d})_{g \in G}$ is Radon strictly ergodic.

(iii) $\mu^{\alpha^{(1)}} \circ \psi^{-1}_{\alpha^{(1)}, \ldots, \alpha^{(d)}}$ is the only (up to scaling) $(T_g)^{\times d})_{g \in G}$-invariant Radon measure supported on $\psi_{\alpha^{(1)}, \ldots, \alpha^{(d)}}(X^{\alpha^{(1)}})$.

(iv) Given $a^{(1)}, \ldots, a^{(d)} \in F_m$, represent the cylinder $[a^{(1)}]_m \times \cdots \times [a^{(d)}]_m$ in $X^d$ as the infinite product $\{([a^{(1)}]_m \times \cdots \times [a^{(d)}]_m) \times C^{d}_{m+1} \times C^{d}_{m+2} \times \cdots\}$.  

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12 The quasi-graph invariant Radon measures which are not graph-joinings are analogues of the so-called weird measures considered in [JaRoRu] in the case when $T$ is an infinite Chacon transformation.
Then
\[ \mu^{(1)} \circ \psi^{-1} \cup \cdots \cup \mu^{(d)} \circ \psi^{-1} \restriction ([a(1)]_m \times \cdots \times [a(d)]_m) = \mu^{(1)} \left( [a]^{(1)}_m \right) \prod_{i=1}^{d} \delta_{a(i)} \otimes \bigotimes_{n>m} \gamma_n, \]
where \( \gamma_n \) is a probability on \( C_n \times C_n \) given by
\[ \gamma_n := \begin{cases} \frac{1}{\# C_n} \sum_{c \in C_n} \bigotimes_{i=1}^{d} \delta_{c(i)} & \text{if } n \in J, \\ \delta_{c_1 \otimes \cdots \otimes c_d} & \text{if } n \notin J, \end{cases} \]
\( a(i) := a(i) + \sum_{j \neq i \leq m} \alpha_j \) and elements \( c_i \in C_n, i = 1, \ldots, d, \) are such that \( c_i^{(u)} = \alpha_i^{(u)} \) and \( c_i^{(v)} = \alpha_i^{(v)} \) for each \( n > m. \)

By analogy with Definition 3.3 we now give the following definition.

**Definition 3.5**: Given \( \alpha^{(1)}, \ldots, \alpha^{(d)} \) as above and let \( h_1, \ldots, h_{d-1} \) be elements of \( G \). The measure \( \mu^{(1)} \circ \psi^{-1} \cup \cdots \mu^{(d)} \circ (I \times T_h^{-1} \times \cdots \times T_{h-1}^{-1}) \) is called a **quasi-graph** measure.

It follows from Theorem 3.4 that if (3-6) is satisfied for \( \alpha^{(1)}, \ldots, \alpha^{(d)}, \) then the corresponding quasi-graph measure is Radon, \( (T_g)^{\times d} \) \( g \in G \)-invariant and ergodic. In this case a quasi-graph Radon measure is purely atomic if and only if \( J \) is finite. On the other hand, a quasi-graph Radon measure is a \( d \)-fold Radon joining \( (d \)-fold graph-joining indeed) if and only if \( J \) is cofinite.

### 4. Infinite Chacon transformation and recurrent points for its Cartesian square

To define the infinite Chacon transformation we will use the \( (C, F) \)-construction. From now on \( G = \mathbb{Z} \). We first define recurrently a sequence of positive integers \( (h_n)_{n=0}^{\infty} \) by setting \( h_0 := 1 \) and \( h_{n+1} := 6h_n + 1 \) for \( n > 0. \)\(^{13}\) For each \( n \geq 0, \) we now let
\[ F_n := \left\{ \frac{h_n - 1}{2}, \frac{h_n - 3}{2}, \ldots, \frac{h_n - 1}{2} \right\} \]
and
\[ C_{n+1} := \{-h_n, 0, h_n + 1\}. \]

\(^{13}\) In [AdFrSi] and [JaRoRu], it was assumed that \( h_{n+1} = 6h_n + 2. \) To achieve the “symmetric” version of the infinite Chacon we use one spacer less at each step of the inductive construction. See Section 7 for a discussion about other possibilities to add extra spacers.
The sequence \((C_n, F_{n-1})_{n \geq 1}\) satisfies (I)–(IV) from Section 3 and (3-1). We also note that \(\#F_n = h_n\) and

\[(4-1)\quad F_{n-1} + C_n - F_{n-1} - C_n \subset F_n.\]

Let \((X, \mu, (T_n)_{n \in \mathbb{Z}})\) denote the dynamical system associated with \((C_n, F_{n-1})_{n \geq 1}\).

**Definition 4.1:** The transformation \(T_1\) (or, rather, the dynamical system \((X, \mu, T_1)\)) is called the infinite Chacon transformation.

For a (more common) cutting-and-stacking definition of the infinite Chacon transformation we refer to [AdFrSi], [JaRoRu] and Section 7 of the present paper.

We will need the following simple lemma.

**Lemma 4.2:** If \(x \in X_{n-1} \cap [f]_n\) and \(T_gx \in X_{n-1}\) for some \(g, f \in F_n\), then \(g + f \in F_n\).

**Proof.** Indeed, consider expansions \(x = (f_{n-1}, c_n, c_{n+1}, \ldots) \in X_{n-1}\) and \(T_gx = (f'_{n-1}, c'_n, c'_{n+1}, \ldots) \in X_{n-1}\) of \(x\) and \(T_gx\) respectively. Then there is \(m \geq n\) such that \(g = (f'_{n-1} - f_{n-1}) + \sum_{l=n}^{m}(c'_l - c_l)\). Moreover, if \(m > n\) we can additionally assume that \(c_m \neq c'_m\). It follows from (III) and (4-1) that \(g \in F_{m-1} + c'_m - c_m\). Since \(g \in F_n \subset F_{m-1}\), we get a contradiction with (IV).

Hence \(m = n\). Since \(x \in [f]_n\), it follows that \(f = f_{n-1} + c_n\). This implies that \(g = f'_{n-1} + c_n - f \in F_n - f\), as desired. \(\square\)

We first describe the recurrent points of the Cartesian square of \(T_1\).

**Proposition 4.3:** Let \((x, x') \in X_n \times X_n\) for some \(n > 0\). Expand \(x\) and \(x'\) as \(x = (f_n, c_{n+1}, \ldots)\) and \(x' = (f'_n, c'_{n+1}, \ldots)\).

(i) If there is \(L > n\) such that \(\{c_l, c'_l\} = \{-h_{l-1}, h_{l-1} + 1\}\) for each \(l > L\), then the \((T_1 \times T_1)\)-orbit of \((x, x')\) intersects each compact subset of \(X \times X\) at most finitely many times.

(ii) Otherwise \((x, x')\) is \((T_1 \times T_1)\)-recurrent.

**Proof.** (i) Let \(J := \{1, \ldots, L\}\). We now let \(\alpha_l := c_l\) and \(\beta_l := c'_l\) for all \(n \notin J\).

It is easy to see that (3-4) holds. Then by Theorem 3.2(i), the subset \(\psi_{\alpha, \beta}(X^\alpha)\) is closed in \(X \times X\). However, this subset is a single \((T_1 \times T_1)\)-orbit and the point \((x, x')\) belongs to this orbit. This follows directly from the definition of \(\psi_{\alpha, \beta}\) (see §3).
(ii) Let \( J := \{ l > 0 \mid \{ c_l, c'_l \} \neq \{ -h_{l-1}, h_{h-1} + 1 \} \} \). We set \( \alpha_l := c_l \) and \( \beta_l := c'_l \) for each \( l \not\in J \). Since \( J \) is infinite, \( X^\alpha \) is a perfect space. It follows now from Theorem 3.2(i) that \( \psi_{\alpha,\beta} \) is a perfect subset of \( X \times X \). Of course, \( (x, x') \in \psi_{\alpha,\beta}(X^\alpha) \). In view of Theorem 3.2(ii), the \((T_1 \times T_1)\)-orbit of \((x, x')\) is dense in \( \psi_{\alpha,\beta}(X^\alpha) \). Hence \((x, x')\) is \((T_1 \times T_1)\)-recurrent.

Remark 4.4: In a similar way one can prove the following generalization of Proposition 4.3(i). Let \( d > 1 \) and let \((x^{(1)}, \ldots, x^{(d)}) \in X^d \). If there is \( L > 0 \) such that \( \{ c^{(1)}_l, \ldots, c^{(d)}_l \} \supset \{ -h_{l-1}, h_{h-1} + 1 \} \) for all \( l > L \), then the \((T_1)^d\)-orbit of \((x^{(1)}, \ldots, x^{(d)})\) intersects each compact subset of \( X^d \) at most finitely many times. Hence the point \((x^{(1)}, \ldots, x^{(d)})\) is not \((T_1)^d\)-recurrent. All the other points of \( X^d \) are \((T_1 \times T_1)\)-recurrent.

Corollary 4.5: The counting measure\(^{14}\) on the \((T_1 \times T_1)\)-orbit of a point \((x, x') \in X \times X \) is a Radon measure on \( X \times X \) if and only if the condition of Proposition 4.3(i) is satisfied.

We also show that \( T_1 \) is totally ergodic.

Proposition 4.6: If \( m > 1 \) then the transformation \( T_m \) is ergodic.

Proof. Let \( k > 0 \). We set \( h := h_k + h_{k+1} + \cdots + h_{k+m-1} \). Take \( f, f' \in F_k \). There is \( j \in \{0, 1, \ldots, m - 1\} \) such that \( f' - f + h + j \) is divisible by \( m \). Take \( f, f' \in F_k \). Then \([f - h_{k+j} - \cdots - h_{k+m-1}]_{k+m} \subset [f]_k\).

\[
T_{h+j} f' - f [f - h_{k+j} - \cdots - h_{k+m-1}]_{k+m} = [f' + (h_k + 1) + \cdots + (h_{k+j-1} + 1)]_{k+m} \subset [f']_k
\]

and \( \mu([f - h_{k+j} - \cdots - h_{k+m-1}]_{k+m}) = 3^{-m} \mu([f]_k) \). It remains to apply Lemma 3.1. \( \blacksquare \)

5. Radon invariant measures for Cartesian powers of the infinite Chacon transformation

We will need the following general lemma about splitting of Radon measures into direct products (cf. with [RudSi, Lemma 3.1.1]). It is, in fact, equivalent to [JaRoRu, Lemma A.1]. We provide a short proof of it.

\(^{14}\) This measure is infinite, \( \sigma \)-finite, \((T_1 \times T_1)\)-invariant, ergodic but non-conservative.
Lemma 5.1: Let $T$ and $S$ be homeomorphisms of locally compact Cantor spaces $X$ and $Y$ respectively. If $\lambda \in \mathcal{M}_{Ra}^{e}(X \times Y, T \times S)$ and $\lambda \circ (\text{Id} \times S) = \lambda$, then there exist $\mu \in \mathcal{M}_{Ra}^{e}(X, T)$ and $\nu \in \mathcal{M}_{Ra}^{e}(Y, S)$ such that $\lambda = \mu \times \nu$.

Proof. Let $\tilde{\mu}$ be a probability measure on $(X, \mathcal{B})$ which is equivalent (i.e., it has the same ideal of Borel subsets of $0$-measure) to the projection of $\lambda$ to $X$. Then $\tilde{\mu}$ is quasi-invariant and ergodic under $T$. Consider the disintegration of $\lambda$ with respect to $\tilde{\mu}$:

$$\lambda = \int_X \delta_x \times \lambda_x \, d\tilde{\mu}(x),$$

where the map $X \ni x \mapsto \lambda_x$ is the corresponding system of $\sigma$-finite measures on $Y$. Since $\lambda \in \mathcal{M}_{Ra}(X \times Y)$, it follows that $\lambda_x \in \mathcal{M}_{Ra}(X)$ for $\tilde{\mu}$-a.e. $x$. We have that

$$\lambda \circ (T \times S) = \int_X \delta_{T^{-1}x} \times \lambda_x \circ S \, d\tilde{\mu}(x) = \int_X \delta_x \times \frac{d\tilde{\mu} \circ T}{d\tilde{\mu}}(x) \cdot \lambda_{T^x} \circ S \, d\tilde{\mu}(x).$$

In view of the uniqueness of the disintegration we obtain that $\lambda$ is invariant under $T \times S$ if and only if $\lambda_x = \frac{d\tilde{\mu} \circ T}{d\tilde{\mu}}(x) \cdot \lambda_{T^x} \circ S$ for $\tilde{\mu}$-a.e. $x$. In a similar way, $\lambda$ is invariant under $\text{Id} \times S$ if and only if $\lambda_x = \lambda_x \circ S$ for $\tilde{\mu}$-a.e. $x$. Hence $\lambda_x = \frac{d\tilde{\mu} \circ T}{d\tilde{\mu}}(x) \cdot \lambda_{T^x}$ for $\tilde{\mu}$-a.e. $x$. Since $T$ is ergodic and $\lambda_x$ is a Radon measure for $\tilde{\mu}$-a.e. $x \in X$, it follows that there is a single Radon measure $\nu$ on $Y$ and a measurable map $X \ni x \mapsto a(x) \in \mathbb{R}_+^*$ such that $\lambda_x = a(x)\nu$ for $\mu$-a.e. $x$. We now obtain that $\nu \circ S = \nu$ and $a(x) = \frac{d\tilde{\mu} \circ T}{d\tilde{\mu}}(x)a(Tx)$ for $\tilde{\mu}$-a.e. $x$. Define a $\sigma$-finite measure $\mu$ on $X$ by setting $\frac{d\mu}{d\tilde{\mu}}(x) = a(x)$ for all $x \in X$. Then $\mu$ is invariant under $T$. Moreover, $\lambda = \mu \times \nu$ and hence $\mu \in \mathcal{M}_{Ra}^{e}(X, T)$ and $\nu \in \mathcal{M}_{Ra}^{e}(Y, S)$.

We now state one of the main results of the paper.

Theorem 5.2: The infinite Chacon transformation has Radon MSJ.

Proof. Let $(X, \mu, T_1)$ denote the infinite Chacon transformation as above. We fix $d > 1$ and take $\lambda \in \mathcal{M}_{Ra}^{e}(X^d, (T_1)^d)$. Let $z = (x^{(1)}, \ldots, x^{(d)}) \in G(\lambda)$. We recall that $X = \bigcup_{n \geq 0} X_n$ and $X_n := F_n \times C_{n+1} \times \cdots$. If $m$ is large so that $z \in (X_m)^d$, we consider the expansion $x^{(j)} = (f^{(j)}_m, c^{(j)}_{m+1}, c^{(j)}_{m+2}, \ldots) \in X_m$ of $x^{(j)}$ for each $j = 1, \ldots, d$. We call $z$ extreme if for each sufficiently large $n > m$, either $\{-h_{n-1}, h_{n-1} + 1\} \subset \{c^{(1)}_n, \ldots, c^{(d)}_n\}$ or $c^{(1)}_n = \cdots = c^{(d)}_n$.

$^{15}$ Such a measure can be obtained as the projection (to $X$) of a probability measure on $X \times Y$ equivalent to $\lambda$. 
We first show that if $z$ is not extreme, then $\lambda$ splits into a direct product of its two marginals. Indeed, there is an infinite subset $J \subset \{1, \ldots, d\}$ such that $c_n^{(j)} = 0$ if $j \in J$ and either $c_n^{(j)} = -h_{n-1}$ for all $j \notin J$ or $c_n^{(j)} = h_{n-1} + 1$ for all $j \in \{1, \ldots, d\} \setminus J$ for each $n \in \mathcal{N}$. We consider only the former case since the latter one is similar. Let $A^{(1)}, \ldots, A^{(d)}$ be compact open subsets in $X$. Then there is $r > 0$ such that these subsets are $r$-cylinders. Hence for each $n \geq r$ and each $j = 1, \ldots, d$, there is a subset $A_n^{(j)} \subset F_n$ such that $A^{(j)} = [A_n^{(j)}]_n$. It follows from Lemma 4.2 that

\begin{equation}
(5.1) \quad \sum_{i \in F_n} 1_{A^{(i)}_n \times \cdots \times A^{(d)}_n}((T_1 \times \cdots \times T_d)^i z) = \# \left( F_n \cap \bigcap_{j=1}^d (A^{(j)}_n - f^{(j)}_n) \right).
\end{equation}

Since we may assume without loss of generality that $A^{(j)} \cup \{x^{(j)}\} \subset X_{n-1}$, it follows from (4.1) that

\begin{equation}
(5.2) \quad A_n^{(j)} - f_n^{(j)} \subset F_n \quad \text{for each } j = 1, \ldots, d.
\end{equation}

By the definition of a generic point, for each $\epsilon > 0$ and each sufficiently large $n$, we have (in view of (5.1) and (5.2)) that

\begin{equation}
(5.3) \quad \frac{\#(\bigcap_{j=1}^d (A^{(j)}_n - f_n^{(j)}))}{\#(\bigcap_{j=1}^d (F_{r,n} - f_n^{(j)}))} = \frac{\lambda(A^{(1)} \times \cdots \times A^{(d)})}{\lambda(X^d)} \pm \epsilon
\end{equation}

where $F_{r,n} := F_r + C_{r+1} + \cdots + C_n$. Now choose $n$ so that $n + 1 \in \mathcal{N}$. Then

$$f_{n+1}^{(j)} = f_n^{(j)} + c_{n+1}^{(j)} = \begin{cases} f_n^{(j)}, & \text{if } j \in J, \\ f_n^{(j)} - h_n, & \text{if } j \notin J. \end{cases}$$

Since $A^{(j)}_{n+1} = A^{(j)}_n + C_{n+1}$, we obtain that

$$\bigcap_{j=1}^d (A^{(j)}_{n+1} - f_{n+1}^{(j)}) = \bigcap_{j \in J} (A^{(j)}_n - f_n^{(j)} + C_{n+1}) \cap \bigcap_{j \notin J} (A^{(j)}_n - f_n^{(j)} + C_{n+1} + h_n)$$

$$= \bigcap_{j \in J} (A^{(j)}_n - f_n^{(j)} + \{0, h_n + 1\}) \cap \bigcap_{j \notin J} (A^{(j)}_n - f_n^{(j)} + \{0, h_n\})$$

$$= \bigcap_{j=1}^d (A^{(j)}_n - f_n^{(j)}) \cup \left( h_n + \bigcap_{j \in J} (A^{(j)}_n - f_n^{(j)} + 1) \cap \bigcap_{j \notin J} (A^{(j)}_n - f_n^{(j)}) \right).$$
We define a map $S : X^d \rightarrow X^d$ by setting $S(x^{(1)}, \ldots, x^{(d)}) = (y^{(1)}, \ldots, y^{(d)})$, where $y^{(j)} := x^{(j)}$ if $j \in J$ and $y^{(j)} := T_1 x^{(j)}$ if $j \not\in J$. Of course, $S$ is a homeomorphism of $X^d$. It commutes with $T_1^{x^d}$. Let

$$g^{(j)} := \begin{cases} 
0, & \text{if } j \in J, \\
1, & \text{if } j \not\in J.
\end{cases}$$

Then

$$\# \left( \bigcap_{j=1}^d (A_{n+1}^{(j)} - f_{n+1}^{(j)}) \right) = \# \left( \bigcap_{j=1}^d (A_n^{(j)} - f_n^{(j)}) \right) + \# \left( \bigcap_{j=1}^d (A_n^{(j)} + g^{(j)} - f_n^{(j)}) \right).$$

In a similar way we obtain that

$$\# \left( \bigcap_{j=1}^d (F_{r,n+1} - f_{n+1}^{(j)}) \right) = \# \left( \bigcap_{j=1}^d (F_{r,n} - f_n^{(j)}) \right) + \# \left( \bigcap_{j=1}^d (F_{r,n} + g^{(j)} - f_n^{(j)}) \right).$$

Thus we obtain that the left-hand side of (5-3) (with $n+1$ in place of $n$) equals

$$(5-4) \quad \frac{\#(\bigcap_{j=1}^d (A_n^{(j)} + g^{(j)} - f_n^{(j)})) + \#(\bigcap_{j=1}^d (A_n^{(j)} - f_n^{(j)}))}{\#(\bigcap_{j=1}^d (F_{r,n} + g^{(j)} - f_n^{(j)})) + \#(\bigcap_{j=1}^d (F_{r,n} - f_n^{(j)}))}.$$ 

It is straightforward to verify that $\lambda(S(X_r)^d)/\lambda((X_r)^d) = 1 \pm \epsilon$ if $r$ is large enough. Therefore it follows from (5-3) and (5-4) that

$$\frac{\lambda(A^{(1)} \times \cdots \times A^{(d)})}{\lambda(X_r^{d})} \pm \epsilon \geq \frac{\lambda \circ S(A^{(1)} \times \cdots \times A^{(d)})}{\lambda(X_r^{d})} \pm \frac{\lambda(A^{(1)} \times \cdots \times A^{(d)})}{\lambda(X_r^{d})} \pm 2\epsilon.$$ 

Hence $\lambda = (\lambda + \lambda \circ S)/2$, i.e., $\lambda \circ S = \lambda$. Lemma 5.1 yields that $\lambda = \lambda_1 \times \lambda_2$, where $\lambda_1 \in \mathcal{M}_{Ra}^c(X^J, T_1^J)$ and $\lambda_2 \in \mathcal{M}_{Ra}^c(X^{\{1,\ldots,d\}\setminus J}, T_1^{\{1,\ldots,d\}\setminus J})$. Continuing this way several times we obtain finally a splitting of $\lambda$ into a direct product of its marginals whose generic points are all extreme.

Thus to complete the proof of the theorem it suffices to prove the following fact: if every $\lambda$-generic point is extreme, then either there is $s \in \{1, \ldots, d\}$, $d > 1$, such that the projection of $G(\lambda)$ to the $s$-coordinate is $\mu$-negligible or there exist $n_1, \ldots, n_{d-1} \in \mathbb{Z}$ such that $\lambda = \mu_{T^{n_1}, \ldots, T^{n_{d-1}}}$.\textsuperscript{16} Thus we fix $z \in G(\lambda)$ as above and find $m > 0$ such that for each $n > m$, either

$$\{-h_n, h_n + 1\} \subset \{c_n^{(1)}, \ldots, c_n^{(d)}\} \quad \text{or} \quad c_n^{(1)} = \cdots = c_n^{(d)}.$$

\textsuperscript{16} We omit the case where $d = 1$ because it is trivial. We recall that every $(C, F)$-action is Radon strictly ergodic.
Put
\[ I := \{ i > m \mid c_i^{(1)} = \cdots = c_i^{(d)} \}. \]

We can now describe explicitly the set \( \mathcal{G}(\lambda) \). Namely, we claim that
\[
\mathcal{G}(\lambda) \cap ([f_m^{(1)}]_m \times \cdots \times [f_m^{(d)}]_m) = \bigcap_{l > m, j_1, j_2=1}^d \{ \tilde{z} \in (X_m)^d \mid \tilde{c}_i^{(j_1)} - \tilde{c}_i^{(j_2)} = c_i^{(j_1)} - c_i^{(j_2)} \},
\]
where the coordinates \( \tilde{c}_i^{(j)} \) are taken from the expansion
\[
\tilde{z} = ((\tilde{f}_m^{(1)}), \tilde{c}_m^{(1)}, \tilde{c}_m^{(2)}, \ldots), \ldots, (\tilde{f}_m^{(d)}, \tilde{c}_m^{(1)}, \tilde{c}_m^{(2)}, \ldots)
\]
of \( \tilde{z} \). Indeed, let \( A^{(1)}, \ldots, A^{(d)} \) be arbitrary \( r \)-cylinders with \( r \geq m \). Then, as above, we have
\[
\sum_{i \in F_{r+1}} 1_{A^{(1)} \times \cdots \times A^{(d)}}((T^{\times d})^i z) = \frac{\#(\bigcap_{j=1}^d (A^{(j)}_{r+1} - f^{(j)}_{r+1}))}{\#(\bigcap_{j=1}^d (F_{r+1} - f^{(j)}_{r+1}))}.
\]

Consider now two cases. If \( r + 2 \in I \), then
\[
\bigcap_{j=1}^d (A^{(j)}_{r+2} - f^{(j)}_{r+2}) = \bigcap_{j=1}^d (A^{(j)}_{r+1} - f^{(j)}_{r+1} + C_{r+2} - c^{(j)}_{r+2}) = \bigcup_{c \in C_{r+2} - c^{(1)}_{r+2}} \bigcap_{j=1}^d (A^{(j)}_{r+2} - f^{(j)}_{r+2} + c)
\]
and hence \( \#((\bigcap_{j=1}^d (A^{(j)}_{r+2} - f^{(j)}_{r+2})) = 3\#((\bigcap_{j=1}^d (A^{(j)}_{r+1} - f^{(j)}_{r+1}))) \). In a similar way,
\[
\#((\bigcap_{j=1}^d (F_{r+2} - f^{(j)}_{r+2})) = 3\#((\bigcap_{j=1}^d (F_{r+1} - f^{(j)}_{r+1}))).
\]

If \( r + 2 \notin I \), then
\[
\bigcap_{j=1}^d (A^{(j)}_{r+2} - f^{(j)}_{r+2}) = \bigcap_{j=1}^d (A^{(j)}_{r+1} - f^{(j)}_{r+1} + C_{r+2} - c^{(j)}_{r+2}) = \bigcap_{j=1}^d (A^{(j)}_{r+1} - f^{(j)}_{r+1})
\]
and hence \( \#((\bigcap_{j=1}^d (A^{(j)}_{r+2} - f^{(j)}_{r+2})) = \#((\bigcap_{j=1}^d (A^{(j)}_{r+1} - f^{(j)}_{r+1}))) \). In a similar way,
\[
\#((\bigcap_{j=1}^d (F_{r+2} - f^{(j)}_{r+2})) = \#((\bigcap_{j=1}^d (F_{r+1} - f^{(j)}_{r+1}))).
\]

Thus in each of the two cases we obtain that
\[
\frac{\#((\bigcap_{j=1}^d (A^{(j)}_{r+1} - f^{(j)}_{r+1})))}{\#((\bigcap_{j=1}^d (F_{r+1} - f^{(j)}_{r+1})))} = \frac{\#((\bigcap_{j=1}^d (A^{(j)}_{r+2} - f^{(j)}_{r+2})))}{\#((\bigcap_{j=1}^d (F_{r+2} - f^{(j)}_{r+2})))}
\]
for each $r \geq m$. Moreover, if we assume additionally that $A^{(j)} \subset X_{r-1}$ or, equivalently, $A^{(j)} \subset F_{r-1} + C_r$ for each $j = 1, \ldots, d$, then the same argument as above yields that (5-7) holds also if we replace $G$ with $\lambda$ large. Thus, (5-9) is another $\lambda$-generic point with $\tilde{z} \in [f_{m}^{(1)}]_m \times \cdots \times [f_{m}^{(d)}]_m$, then it is extreme and therefore (5-8) holds with $\tilde{f}_{r}^{(j)}$ in place of $f_{r}^{(j)}$, $j = 1, \ldots, d$. This yields

$$\frac{\#(\bigcap_{j=1}^{d} (A_{r}^{(j)} - f_{r}^{(j)}))}{\#(\bigcap_{j=1}^{d} (F_{r} - f_{r}^{(j)}))} = \frac{\#(\bigcap_{j=1}^{d} (A_{r}^{(j)} - \tilde{f}_{r}^{(j)}))}{\#(\bigcap_{j=1}^{d} (F_{r} - \tilde{f}_{r}^{(j)}))}.$$  

We obtain that $\bigcap_{j=1}^{d} (A_{r}^{(j)} - f_{r}^{(j)}) \neq \emptyset$ if and only if $\bigcap_{j=1}^{d} (A_{r}^{(j)} - \tilde{f}_{r}^{(j)}) \neq \emptyset$. Substituting $A^{(j)} := \{f_{r}^{(j)}\}$, we obtain that $f_{r}^{(j_1)} - f_{r}^{(j_2)} = f_{r}^{(j_1)} - \tilde{f}_{r}^{(j_2)}$ for each $1 \leq j_1 < j_2 \leq d$. Since $r$ is arbitrary, we conclude that $c_{l}^{(j_1)} - c_{l}^{(j_2)} = \tilde{c}_{l}^{(j_1)} - \tilde{c}_{l}^{(j_2)}$ for each $1 \leq j_1 < j_2 \leq d$ and all sufficiently large $l$. This proves the “⊂”-part of the equality in (5-5). The “⊃”-part of this equality follows easily from (5-9) and (5-6). Thus (5-5) is proved.

Consider now two cases. If $N \setminus I$ is finite, then we have $x^{(j)} = T_{g_{j}} x^{(1)}$ for $g_{j} := f_{m}^{(j)} - f_{m}^{(1)} + \sum_{n > m} (c_{n}^{(j)} - c_{n}^{(1)})$, $j = 2, \ldots, d$. Hence for each sufficiently large $r$ and each $j \in \{1, \ldots, d\}$, we have that $f_{r}^{(j)} = g_{j} + f_{r}^{(1)}$. Then (5-7) implies that

$$\frac{\lambda(A^{(1)} \times \cdots \times A^{(d)})}{\lambda(X^{(d)})} = \frac{\#(\bigcap_{j=1}^{d} (A_{r}^{(j)} - g_{j}))}{\#(\bigcap_{j=1}^{d} (F_{r} - g_{j}))} = \frac{\mu_{T_{g_{2}} \cdots T_{g_{d}}}(A^{(1)} \times \cdots \times A^{(d)})}{\mu_{T_{g_{2}} \cdots T_{g_{d}}}(X^{(d)})}.$$  

Hence $\lambda$ is a multiple of $\mu_{T_{g_{2}} \cdots T_{g_{d}}}$. Thus $\lambda$ is a graph-joining. Conversely, if $\lambda$ is a graph-joining then $N \setminus I$ is finite.

Now consider the second case, where $N \setminus I$ is infinite. Then it follows from (5-5) that there exists $s \in \{1, \ldots, d\}$ and an infinite subset $I_{0} \subset \{m+1, m+2, \ldots\} \setminus I$ such that for every $\tilde{z} \in G(\lambda) \cap ([f_{m}^{(1)}]_{m} \times \cdots \times [f_{m}^{(d)}]_{m})$, we have that $\tilde{c}_{j}^{(s)} \neq 0$ whenever $j \in I_{0}$. However, $\mu(\{x = (f_{m}, c_{m+1}, \ldots) \in X_{m} \mid c_{j} 
eq 0 \text{ for all } j \in I_{0}\}) = 0$. Thus the projection of the subset $G(\lambda) \cap ([f_{m}^{(1)}]_{m} \times \cdots \times [f_{m}^{(d)}]_{m}) \subset X^{d}$ to the $s$-th coordinate is $\mu$-negligible. Since the projection is equivariant and $\lambda([f_{m}^{(1)}]_{m} \times \cdots \times [f_{m}^{(d)}]_{m}) > 0$, it follows that the projection of the entire $G(\lambda)$ to the $s$-th coordinate is also $\mu$-negligible. Hence $\lambda \not\subset J_{d,Ra}((T_{1})^{\times d})$, as desired. Thus, $T_{1}$ has Radon MSJ.  

We note that not only the Radon MSJ property for $T_1$ was established in the proof of Theorem 5.2, but also a complete description of the set of ergodic $(T_1)^d$-invariant Radon measures on $X^d$ was, in fact, obtained. Namely, the following theorem was indeed proved:

**Theorem 5.3:** Let $\lambda \in \mathcal{M}_e^{Ra}(X^d, (T_1)^d)$. Then there is a partition of $\{1, \ldots, d\}$ into subsets $J_1, \ldots, J_k$ such that $\lambda$ splits into a direct product of its marginals $\lambda_i$ on $X^{J_i}$, each $\lambda_i$ is (up to a multiplicative constant) either $\mu$ if $\#J_i = 1$ or an invariant quasi-graph Radon measure if $\#J_i > 1$.

We now show that there are non-ergodic 2-fold Radon self-joinings $\rho$ of $T_1$ such that almost every ergodic component of $\rho$ is not a joining of $T$: the ergodic components of $\rho$ are conservative quasi-graph invariant Radon measures which are not graph-joinings.

**Example 5.4:** Let $A := \{-1, 0, 1\}$. Given a sequence $a := (a_n)_{n=1}^{\infty} \in A$, we define a measure $\lambda_a$ on $X^2_0$ considered as the infinite product

\[
\{(0, 0)\} \times C^2_1 \times C^2_2 \times \cdots
\]

by setting

(5.10) \hspace{1cm} \lambda_a = \delta_{(0,0)} \otimes \bigotimes_{n>0} \gamma_n^a,

where $\gamma_n$ is a probability on $C^2_n$ given by

(5.11) \hspace{1cm} \gamma_n^a := \begin{cases} 
\delta_{(-h_{n-1}, h_{n-1}+1)} & \text{if } a_n = -1, \\
\delta_{(h_{n-1}+1, -h_{n-1})} & \text{if } a_n = 1, \\
\frac{1}{3} \sum_{c \in C_n} \delta_{(c, c)} & \text{if } \alpha_n = 0.
\end{cases}

We recall that $C_n = \{-h_{n-1}, 0, h_{n-1} + 1\}$. Let $J_a := \{n \mid a_n \neq 0\}$. We now set for each $n \not\in J_a$,

\[
\alpha_n^a := \begin{cases} 
-h_{n-1}, & \text{if } a_n = -1, \\
h_{n-1} + 1, & \text{if } a_n = 1;
\end{cases} \quad \beta_n^a := \begin{cases} 
h_{n-1} + 1, & \text{if } a_n = -1, \\
-h_{n-1}, & \text{if } a_n = 1.
\end{cases}
\]

It is easy to see that the sequences $\alpha^a := (\alpha_n^a)_{n \not\in J}$ and $\beta^a := (\beta_n^a)_{n \not\in J}$ satisfy (3.4). Then it follows from Theorem 3.2(iv) that $\lambda_a = \mu^{\alpha^a} \circ \psi^{-1}_{\alpha^a, \beta^a} \upharpoonright X^2_0$ and that $\mu^{\alpha^a} \circ \psi^{-1}_{\alpha^a, \beta^a}$ is a $(T_1 \times T_1)$-invariant quasi-graph Radon measure.\textsuperscript{17} For

\textsuperscript{17} For the definition of $\mu^\alpha$ and $\psi_{\alpha, \beta}$ we refer to §3.2.
each \( n > 0 \), let \( \kappa_n \) be a probability on \( \{-1, 0, 1\} \) such that \( \kappa_n(-1) = \kappa_n(1) = \frac{1}{2^n} \) and \( \kappa_n(0) = 1 - \frac{1}{n} \). We now set \( \kappa := \bigotimes_{n>0} \kappa_n \) and \( \lambda := \int_A \lambda_a \, d\kappa(a) \). Then

\[
\lambda = \left( \int_A \mu^{\alpha^a} \circ \psi_{\alpha^a, \beta^a}^{-1} \, d\kappa(a) \right) \upharpoonright X_0^2.
\]

We claim that the Radon measure

(5-12)

\[ \rho := \int_A \mu^{\alpha^a} \circ \psi_{\alpha^a, \beta^a}^{-1} \, d\kappa(a) \]

is a 2-fold self-joining of \( T_1 \). For that we have to verify that the two coordinate projections of this measure are equivalent to \( \mu \). It suffices to show that the two coordinate projections of \( \lambda \) are equivalent to \( \mu \upharpoonright X_0 \). A straightforward computation shows that the projection of \( \lambda \) to the first coordinate is the following measure:

\[
\delta_0 \otimes \bigotimes_{n>0} \left( \kappa_n(-1)\delta_{-h_{n-1}} + \kappa_n(1)\delta_{h_{n-1}+1} + \frac{\kappa_n(0)}{3} \sum_{c\in C_n} \delta_c \right).
\]

It is equivalent to the measure \( \delta_0 \otimes \bigotimes_{n>0} \left( \frac{1}{3} \sum_{c\in C_n} \delta_c \right) \) (i.e., to \( \mu \upharpoonright X_0 \)) by the Kakutani theorem on equivalence of infinite product measures [Ka]. In a similar way one can prove that the projection of \( \rho \) to the second coordinate is also equivalent to \( \mu \). Therefore \( \rho \in J_{2,Ra}(T_1) \). On the other hand,

\[
0 = \kappa(\{a \mid J_a \text{ is cofinite}\}) = \kappa(\{a \mid \mu^{\alpha^a} \circ \psi_{\alpha^a, \beta^a}^{-1} \text{ is a 2-fold joining of } T_1\}),
\]

\[
0 = \kappa(\{a \mid J_a \text{ is finite}\}) = \kappa(\{a \mid \mu^{\alpha^a} \circ \psi_{\alpha^a, \beta^a}^{-1} \text{ is purely atomic}\}).
\]

We note that (5-12) is the ergodic decomposition of \( \rho \) and \( \kappa \) is the corresponding measure on the space of ergodic components of \( \rho \). Hence almost all ergodic components of \( \rho \) are conservative Radon dynamical systems whose coordinate projections are singular to \( \mu \).

6. Uncountable family of pairwise Radon disjoint infinite Chacon like transformations

Take an infinite sequence \( \omega \in \{0,1\}^\mathbb{N} \). Define a sequence of positive integers \((h_n)_{n=0}^\infty\) by setting \( h_0 := 1 \) and \( h_{n+1} := 6h_n + 1 \) for \( n > 0 \). For each \( n \geq 0 \), we

\[\text{We use (5-10) and (5-11) to obtain this.}\]
The sequence \((C_n^\omega, F_n)_{n \geq 1}\) satisfies (3-1) and (I)–(IV) from Section 3. We also note that \(\#F_n = h_n\) and
\[
F_n + C_{n+1}^\omega = F_n - C_{n+1}^\omega \subset F_{n+1}.
\]
Let \((X^\omega, \mu^\omega, T^\omega)\) denote the associated \((C,F)\)-dynamical system. It is Radon strictly ergodic. We call it an **infinite Chacon like** transformation. The infinite Chacon transformation corresponds to the case \(\omega(n) = 1\) for all \(n\).

If \(\omega, \omega' \in \{0,1\}^\mathbb{N}\), we write \(\omega \sim \omega'\) if the pair \((\omega, \omega')\) belongs to the tail equivalence relation on \(\{0,1\}^\mathbb{N}\), i.e., there is \(N > 0\) such that \(\omega(n) = \omega'(n)\) for all \(n > N\).

**Theorem 6.1:**
(i) For each \(\omega \in \{0,1\}^\mathbb{N}\), the dynamical system \((X^\omega, \mu^\omega, T^\omega_1)\) is totally ergodic. It has Radon MSJ; \(C(T^\omega_1) = \{T^\omega_n \mid n \in \mathbb{Z}\}\).

(ii) If \(\omega \sim \omega'\), then \((X^\omega, \mu^\omega, T^\omega_1)\) is isomorphic to \((X^\omega', \mu^\omega', T^\omega_1')\).

(iii) If \(\omega \not\sim \omega'\), then \((X^\omega, \mu^\omega, T^\omega_1)\) is Radon disjoint with \((X^\omega', \mu^\omega', T^\omega_1')\).

**Sketch of the proof.** (i) is proved in the same way as Theorem 5.2 and Proposition 4.6.

(ii) Let \(\omega(i) = \omega'(i)\) for all \(i > N\). Given \(x \in X^\omega\), let \(x = (f_n, c_{n+1}, \ldots) \in (X^\omega)_n\) for some \(n > N\). We now define \(\phi : X^\omega \to X^\omega'\) by setting
\[
\phi(x) := (f_n, c_{n+1}, \ldots) \in (X^\omega')_n.
\]
Then \(\phi\) is an isomorphism of \((X^\omega, \mu^\omega, T^\omega_1)\) onto \((X^\omega', \mu^\omega', T^\omega_1')\).

(iii) Let \(\lambda\) be an ergodic \((T^\omega_1 \times T^\omega_1')\)-invariant measure on \(X^\omega \times X^\omega'\) whose marginals are equivalent to \(\mu^\omega\) and \(\mu^\omega'\) respectively. Fix a generic point \((x, x')\) of \(\lambda\). Find \(m\) such that \(x \in (X^\omega)_m\) and \(x' \in (X^\omega')_m\). Consider expansions \(x = (f_m, c_{m+1}, \ldots)\) and \(x' = (f_m', c_{m+1}', \ldots)\) of \(x\) and \(x'\) in \((X^\omega)_m\) and \((X^\omega')_m\) respectively. Let
\[
J := \{i \in \mathbb{N} \mid \omega(i) \neq \omega'(i)\}.
\]
This set is infinite. One of the following cases takes place.

(A) The subset \(J_A := \{j \in J \mid \{c_j, c'_j\} = \{h_{j-1} + 1, h_{j-1}\}\}\) is infinite.

(B) The subset \(J_B := \{j \in J \mid \{c_j, c'_j\} = \{-h_{j-1}, -h_{j-1} - 1\}\}\) is infinite.

(C) The subset \(J_C := \{j \notin J \mid 0 \in \{c_j, c'_j\} \neq \{0\}\}\) is infinite.

(D) The subset \(J_A \cup J_B \cup J_C\) is finite.
Applying (6-1) we now obtain that

\[ \sum_{i \in F_n} 1_{A \times A'}(T_{i}^{\omega} \times T_{i}^{\omega'}) = \mathcal{L}(A - f_n) \cap (A' - f'_n). \]

Therefore for each \( \epsilon > 0 \) and each sufficiently large \( n \), we have that

\[ \frac{\mathcal{L}(A \times A')}{\mathcal{L}(X_r)} = \frac{\lambda(A \times A')}{\lambda(X_r)} \mp \epsilon. \]

This follows from the fact that \((x, x') \in G(T_{r,n}^{\omega} \times T_{r,n}^{\omega'})\). Now choose \( n \) so that \( n+1 \in J_A \). Then \( c_{n+1} = h_{n+1} \) and \( c'_{n+1} = h_n \) (or \( c_{n+1} = h_n \) and \( c'_{n+1} = h_n + 1 \), which is considered in a similar way) and hence \( f_{n+1} = f_n + h_n + 1 \) and \( f'_{n+1} = f'_n + h_n \). We now have

\[ (A_{n+1} - f_{n+1}) \cap (A'_{n+1} - f'_{n+1}) \]

\[ \subseteq (A_n - f_n + C_n^{\omega} - h_n) \cap (A_n - f_n + C_n^{\omega'} - h_n). \]

Since \( C_n^{\omega} = \{-h_n, 0, h_n + 1\} \) and \( C_n^{\omega'} = \{-h_n - 1, 0, h_n\} \), it follows that

\[ \#((A_{n+1} - f_{n+1}) \cap (A'_{n+1} - f'_{n+1})) = \mathcal{L}(A_n - f_n) \cap (A'_{n} - f'_{n}) \]

\[ \quad + \#((A_n - f_n) \cap (A'_{n} - f'_{n})) \]

\[ \quad + \#((A_n - f_n) \cap (A'_{n} - f'_{n})). \]

In a similar way we obtain that

\[ \#((F_{r,n+1} - f_{n+1}) \cap (F_{r,n+1} - f'_{n+1})) \]

\[ = 2\#((F_{r,n} - f_n) \cap (F_{r,n} - f'_{n})) + \#((F_{r,n} - f_n) \cap (F_{r,n} - f'_{n})). \]

Applying (6-1) we now obtain that

\[ \frac{\lambda(A \times A')}{\lambda(X_r)} = \frac{\lambda(A \times A')}{\lambda(X_r)} + \frac{2\lambda(T_{r,n}^{\omega} \times \text{Id})(A \times A')}{\lambda(X_r)} \pm 3\epsilon. \]

This yields that \( \lambda \circ (T_{r,n}^{\omega} \times \text{Id}) = \lambda \). By Lemma 5.1, \( \lambda \) equals \( \mu \times \mu' \) (up to a multiplicative constant).

Case (B) is analogous to Case (A).

Case (C) was considered, in fact, in the first part of the proof of Theorem 5.2. In this case we also obtain that \( \lambda = q \cdot \mu \times \mu' \) for some \( q > 0 \).

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19 Because \( h_n + 1 \in C_{n+1}^{\omega} \) and \( h_n \in C_{n+1}^{\omega'} \).
Case (D). We will show that in this case one of the two marginals of the set \( G(\lambda) \) is of null measure. For that we will argue as in the final part of the proof of Theorem 5.2. Without loss of generality we may assume that

\[
\{c_n, c'_n\} \cap (J_A \cup J_B \cup J_C) = \emptyset \quad \text{for each } n > m.
\]

We claim that

\[
(6-2) \quad G(\lambda) \cap ([f_m]_m \times [f'_m]_m)
\]

where the coordinates \( \tilde{c}_l \) and \( \tilde{c}'_l \) are taken from the expansion

\[
\tilde{x} = (\tilde{f}_m, \tilde{c}_{m+1}, \tilde{c}_{m+2}, \ldots), \quad \tilde{x}' = (\tilde{f}'_m, \tilde{c}'_{m+1}, \tilde{c}'_{m+2}, \ldots)
\]

of \( \tilde{x} \) and \( \tilde{x}' \). We let

\[
\alpha_n := \#((A_n - f_n) \cap (A'_n - f'_n)).
\]

One can verify (as in the proof of Theorem 5.2) that

\[
(6-3) \quad \alpha_{n+1} = \begin{cases} 
3\alpha_n & \text{if } c_{n+1} = c'_{n+1}, \\
\alpha_n & \text{if } 0 \notin \{c_{n+1}, c'_{n+1}\} \text{ but } c_{n+1} \neq c'_{n+1}, \\
2\alpha_n & \text{if } 0 \in \{c_{n+1}, c'_{n+1}\} \text{ but } c_{n+1} \neq c'_{n+1}.
\end{cases}
\]

This implies

\[
\frac{\#((A_{r+1} - f_{r+1}) \cap (A'_{r+1} - f'_{r+1}))}{\#((F_{r+1} - f_{r+1}) \cap (F'_{r+1} - f'_{r+1}))} = \frac{\#((A_{r+2} - f_{r+2}) \cap (A'_{r+2} - f'_{r+2}))}{\#((F_{r+2} - f_{r+2}) \cap (F'_{r+2} - f'_{r+2}))}.
\]

Slightly modifying the argument in the proof of Theorem 5.2 we deduce (6-2). If either the second or the third condition from (6-3) is satisfied for infinitely many \( n \), then one can easily deduce from (6-2) and (6-3) that one of the coordinate projection of \( G(\lambda) \) is of 0 measure. If the second and the third condition are satisfied for only finitely many \( n \), then the first condition in (6-3) is satisfied for all but finitely many \( n \). Hence it is satisfied for infinitely many elements of \( J \). However, if \( \tilde{c}_n = \tilde{c}'_n \) for \( n \in J \), then \( \tilde{c}_n = \tilde{c}'_n = 0 \). Therefore the two marginals of \( G(\lambda) \) are of 0 measure.

We also note that one can construct,\(^20\) for each \( \omega \), a non-ergodic Radon 2-fold self-joining of \( T_1^\omega \) whose ergodic components are not joinings because their coordinate projections are singular to \( \mu^\omega \).

\(^20\) Via a slight modification of Example 5.4.
Given $\omega \in \{0,1\}^\mathbb{N}$, we define $\omega^* \in \{0,1\}^\mathbb{N}$ by setting $\omega^*(i) = 1 - \omega(i)$. It is easy to see that $T_1^{\omega^*}$ is conjugate to the inverse to $T_1^{\omega}$.

**Corollary 6.2:** For each $\omega$, the transformation $T_1^{\omega^*}$ is not conjugate to its inverse. Moreover, $T_1^{\omega}$ and its inverse are Radon disjoint.

### 7. Further generalizations and some open problems

It is easy to see that the above argument works almost verbally for a more general class of rank-one transformations. To specify this class we first give an alternative (but equivalent) description of the infinite Chacon transformation. For that we will use the classical language of cutting-and-stacking construction [Fr]. The initial 0-th tower consists of a single interval $[0,1)$. We now describe the inductive procedure of passing from the $n$-th tower consisting of $h_n$ levels of width $\frac{1}{3^n}$ to the $(n+1)$-th tower. For that we cut the $n$-th tower into 3 subtowers (called copies) of equal width. Then we place the second copy over the first one, add an additional level (called spacer) over the second copy and put the third copy over this spacer. Next we put $[1.5h_n]$ spacers over the top of the third copy and $[1.5h_n] + 1$ spacers under the bottom of the first copy. Here $[\cdot]$ stands for the integer part. We thus obtain the $(n+1)$-tower consisting of $h_{n+1} = 6h_n + 1$ levels of width $\frac{1}{3^{n+1}}$. The transformation $T$ moves each (except for the highest one) level of the tower one level up. The transformation is not defined on the highest level of the tower. However, in the limit we obtain a well defined transformation on $\mathbb{R}$ (which is the union of all levels of all towers) endowed with Lebesgue measure. It is measure theoretically isomorphic to the infinite Chacon transformation described above via the $(C,F)$-construction in Section 4.

We can now obtain a family of transformations using almost the same cutting-and-stacking algorithm but adding “more” spacers. I mean the following: Let $(\alpha_n)_{n \in \mathbb{N}}$ and $(\beta_n)_{n \in \mathbb{N}}$ be two arbitrary sequences of nonnegative integers. When constructing the $(n+1)$-th tower, we first do verbally what we did in the above construction of the infinite Chacon transformation and after that we put $\alpha_n$ additional spacers on the top and $\beta_n$ spacers under the bottom of the tower. In the limit of the inductive construction we obtain a transformation that has Radon MSJ. The proof is almost the same as for the infinite Chacon transformation.
We conclude the paper with a list of open problems.

— Are there two ergodic Radon dynamical systems which are Radon disjoint but which admit a non-ergodic Radon joining?

— Does the infinite Chacon transformation have invariant sub-$\sigma$-algebras admitting equivalent $\sigma$-finite invariant measures?

— More generally, given a $(C,F)$-transformation $T$ with an infinite invariant Radon measure $\mu$ such that all ergodic $T \times T$-invariant Radon measures are either quasi-graphs or $\mu \times \mu$, does it have factors admitting equivalent $\sigma$-finite invariant measures?

References

[Aa1] J. Aaronson, The intrinsic normalizing constants of transformations preserving infinite measures, Journal d’Analyse Mathématique 49 (1987), 239–270.

[Aa2] J. Aaronson, An Introduction to Infinite Ergodic theory, Mathematical Surveys and Monographs, Vol. 50, American Mathematical Society, Providence, RI, 1997.

[AdFrSi] T. Adams, N. Friedman and C. E. Silva, Rank-one weak mixing for nonsingular transformations, Israel Journal of Mathematics 102 (1997), 269–281.

[Da1] A. I. Danilenko, Funny rank-one weak mixing for nonsingular Abelian actions, Israel Journal of Mathematics 121 (2001), 29–54.

[Da2] A. I. Danilenko, On simplicity concepts for ergodic actions, Journal d’Analyse Mathématique 102 (2007), 77–118.

[Da3] A. I. Danilenko, $(C,F)$-actions in ergodic theory, in Geometry and Dynamics of Groups and Spaces, Progress in Mathematics, Vol. 265, Birkhäuser, Basel, 2008, pp. 325–351.

[Da4] A. I. Danilenko, Finite ergodic index and asymmetry for infinite measure preserving actions, Proceedings of the American Mathematical Society 144 (2016), 2521–2532.

[dJRaSw] A. del Junco, M. Rahe and L. Swanson, Chacon’s automorphism has minimal self-joinings, Journal s’Analyse Mathématique 37 (1980), 276–284.

[dJRud] A. del Junco and D. Rudolph, On ergodic actions whose self-joinings are graphs, Ergodic Theory and Dynamical Systems 7 (1987), 531–557.

[dJSi] A. del Junco and C. E. Silva, On factors of non-singular Cartesian products, Ergodic Theory and Dynamical Systems 23 (2003), 1445–1465.

[Ef] E. G. Effros, Transformation groups and $C^*$-algebras, Annals of Mathematics 81 (1965), 38–55.

[Fr] N. Friedman, Introduction to Ergodic Theory, Van Nostrand Reinhold Mathematical Studies, Vol. 29, Van Nostrand Reinhold, New York–Toronto, ON–London, 1970.

[Fu] H. Furstenberg, Disjointness in ergodic theory, minimal sets and diophantine approximation, Mathematical Systems Theory 1 (1967), 1–49.

[Gli] J. Glimm, Locally compact transformation groups, Transactions of the American Mathematical Society 101 (1961), 124–138.
[JaRoRu] E. Janvresse, E. Roy and T. de la Rue, Invariant measures for Cartesian powers of Chacon infinite transformation, Israel Journal of Mathematics 224 (2018), 1–37.

[Ka] S. Kakutani, On equivalence of infinite product measures, Annals of Mathematics 49 (1948), 214–224.

[Rud] D. J. Rudolph, An example of a measure preserving map with minimal self-joinings, and applications, Journal d’Analyse Mathématique 35 (1979), 97–122.

[RudSi] D. J. Rudolph and S. E. Silva, Minimal self-joinings for nonsingular transformations, Ergodic Theory and Dynamical Systems 9 (1989), 759–800.

[Ru] T. de la Rue, Joinings in ergodic theory, in Encyclopedia of Complexity and Systems Science, Springer, New York, 2009, pp. 5037–5051.

[Sc] K. Schmidt, Infinite invariant measures on the circle, in Convegno sulle Misure su Gruppi e su Spazi Vettoriali, Convegno sui Gruppi e Anelli Ordinati, INDAM, (Rome, 1975), Symposia Mathematica, Vol. 21, Academic Press, London, 1977, pp. 37–43.

[SiWi] C. E. Silva and D. Witte, On quotients of nonsingular actions whose self-joinings are graphs, International Journal of Mathematics 5 (1994), 219–237.

[Yu] H. Yuasa, Uniform sets for infinite measure-preserving systems, Journal d’Analyse Mathématique 120 (2013), 333–356.