On the Realization of Hidden Markov Models and Tensor Decomposition

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Abstract: The minimum realization problem of hidden Markov models (HMM’s) is a fundamental question of stationary discrete-time processes with a finite alphabet. It was shown in the literature that tensor decomposition methods give the hidden Markov model with the minimum number of states generically. However, the tensor decomposition approach does not solve the minimum HMM realization problem when the observation is a deterministic function of the state, which is an important class of HMM’s not captured by a generic argument. In this paper, we show that the reduction of the number of rank-one tensors necessary to decompose the third-order tensor constructed from the probabilities of the process is possible when the reachable subspace is not the whole space or the null space is not the zero space. In fact, the rank of the tensor is not greater than the dimension of the effective subspace or the rank of the generalized Hankel matrix.

Keywords: hidden Markov models, realization, reachable space, null space, tensor decomposition.

1. INTRODUCTION

A hidden Markov model (HMM) produces a finite-valued process as the output of a finite-state Markov process. Because of the ability to model various kinds of signals, HMM’s have been exploited to solve many real-world problems such as speech processing (Rabiner (1989)) and computational biology (Krogh et al. (1994)).

The realization problem of HMM’s is to derive a finite-state Markov model with an observation map given the statistics of the stochastic process. This direction of research started in Blackwell and Koopmans (1957) and Gilbert (1959), where they considered whether a finite-state Markov model could be uniquely identified from the statistics of the output process. The studies by Anderson (1999) and Vidyasagar (2011) provide a comprehensive overview and original results on the realization problem.

As is discussed in Anderson (1999), the realization problem is closely related to the so-called generalized Hankel matrix whose entries are probabilities of strings occurring in the process arranged in a certain way just as the conventional Hankel matrix formed from impulse response coefficients. The finite-rank property of the generalized Hankel matrix is a necessary condition for the steady-state process to have a finite-state HMM realization. Still, the converse does not hold in general. See the discussion and the example in Vidyasagar (2011). The difficulty of characterizing the existence of a finite-state Markov model is the non-negativity of the transition matrix. A realization where the non-negativity constraint is relaxed is called a quasi-realization or a pseudo-realization and studied in Ito et al. (1992); Anderson (1999); Vidyasagar (2011). Recently, Huang et al. (2016) studied the minimal realization problems for HMM’s and showed that a minimal quasi-HMM-realization and a minimal HMM-realization could be efficiently solved generically for almost all HMM’s. The main tool for the calculation of a minimal HMM-realization is third-order tensor decomposition. It was shown that the rank of the tensor is equal to the minimum degree of an HMM realization using a sufficient condition for the uniqueness of tensor decomposition studied in Kruskal (1977) (see also the survey paper by Kolda and Bader (2009)). The paper assumes that any two columns of the observation matrix are linearly independent. If this is not the case, there are two identical columns. However, unlike the claim in Huang et al. (2016), two states cannot be merged to give an equivalent HMM realization of small order because the HMM-realization problem is nontrivial even if the observation is a deterministic function of the state.

In this note, we consider the uniqueness of third-order tensor decomposition when the observation is a deterministic function of the state and show that the tensor has low-rank decomposition if the generalized Hankel matrix has lower rank than the number of states. The study by Ito et al. (1992) introduced the notion of the reachable space and the null space of HMM’s. Notice that the condition implies that either the reachable space is not the whole space or the null space is nontrivial. Hence the tensor decomposition approach does not solve the minimum HMM realization problem when the observation is a deterministic function of the state, which is an important class of HMM’s not captured generically.

The rest of the paper is organized as follows. In Section 2, the realization problem of HMM’s is described when a stationary process is given. Section 3 reviews the notion
of the reachable space and the null space of HMM's and derives representations of these subspaces. Section 4 considers rank reduction of tensor decomposition when the reachable space is not the whole space, and the null space is not the zero subspace.

We use the following notations; Z denotes the set of integers, R denotes the set of real numbers, R^m denotes the set of real vectors of size m, and R^{m×n} denotes the set of real matrices of size m × n. For a matrix A ∈ R^{m×n} or a vector a ∈ R^n, A^T or a^T denotes the transpose of the matrix or the vector. If A ∈ R^{m×m} is invertible, A^{-T} denotes the inverse of A^T.

2. REALIZATION PROBLEM

Suppose {y_t} is a stationary discrete-time random process taking values in a finite set {1, ..., d}. For t, s ∈ Z, let u^t_s = (u_s, ..., u_t) ∈ {1, ..., d}^{t-s+1} be an array of length |t-s|+1. Note that the array is in ascending order if s < t and in descending order if s > t. Let the random vector y^t_s = (y_s, y_{t+1}, ..., y_t) be defined similarly. Let P(y^t_s = u_s^t) denote the probability of the event y^t_s = u_s^t. From the stationarity, P(y_{s+1}^t = u_s^t) = P(y_{t+1}^{s+1} = u_s^t) for any s, t, r ∈ Z and u_s^t ∈ {1, ..., d}^{t-s+1}.

Suppose {x_t} is a stationary Markov chain taking values in a state space {1, ..., k} with the state transition matrix Q = (q_{ij}) ∈ R^{k×k} such that

P(x_{t+1} = i | x_t = j) = q_{ij}, \quad i, j ∈ {1, ..., k}.

Note that Q is nonnegative and column-stochastic, i.e., the sum of every column is equal to one. Suppose O = (o_{ij}) ∈ R^{d×k} is nonnegative and column-stochastic. Construct a discrete-time process {z_t} to satisfy

P(z_{t+1} = i | z_t = j) = o_{ij}, \quad i ∈ {1, ..., d}, \quad j ∈ {1, ..., k}.

The matrix O is called the observation matrix. If {y_t} and {z_t} have the same law, we say (O, Q) is an HMM realization of order k.

When there exists a function φ : {1, ..., k} → {1, ..., d} such that y_t = φ(x_t), then the observation matrix can be selected as

o_{ij} = \begin{cases} 1 & \text{if } φ(j) = i, \\ 0 & \text{if } φ(j) ≠ i. \end{cases} \quad (1)

In this case, we say the observation is a deterministic function of the state. Note that each column of the matrix O has exactly one nonzero entry.

The HMM-realization problem is to find a realization (O, Q) given the probabilities P(y^t_s = u_s^t) for any s, t ∈ Z and u_s^t ∈ {1, ..., d}^{t-s+1}.

3. REACHABLE AND NULL SPACES

In this section, we review the reachable subspace and the null space for an HMM introduced in Ito et al. (1992) and derive representations of these subspaces for the later discussion.

Let Q ∈ R^{k×k} be a matrix having a maximum modulus eigenvalue at one with the left eigenvector e^T ∈ R^{1×k} and the right eigenvector ρ ∈ R^k where e is the vector whose elements are all one. Let φ : {1, ..., k} → {1, ..., d} be a map and define O = (o_{ij}) ∈ R^{d×k} by (1). Note that Q needs not to be a nonnegative matrix but the sum of each column is one. Let I_n ∈ R^{k×k} (u = 1, ..., d) be the diagonal matrix whose (i, i)th element is one if φ(i) = u and zero otherwise.

We define the reachable subspace V_R by

V_R = span \{ I_nQ, I_nQ^2I_nρ, ..., I_nQ^{t-s+1}I_nρ : u_0^t ∈ {1, ..., d}^{t-s+1}, n = 0, 1, 2, ..., \} \quad (2)

and the null space V_N by

V_N = \bigcap_{n=0,1,2,...} \{ v : e^TI_nQ, ..., I_nQ^{t-s+1}I_nρv = 0 \} \quad (3)

Define V_{R,u} = I_nV_R (u = 1, ..., d). Then it follows that

V_{R,u} ⊂ V_R, \quad V_R = \bigoplus_{u=1}^d V_{R,u}. \quad (4)

From the definitions (2) and (3), the following result is immediate.

Proposition 1. The reachable subspace V_R is the smallest subspace which is Q as well as I_n-invariant (u = 1, ..., d) and contains ρ. The null subspace V_N is the largest subspace which is Q as well as I_n-invariant (u = 1, ..., d) and contained in ker e^T.

Proof. Since \sum_{u=1}^d I_u = 1, ρ = \sum_{u=1}^d I_uρ ∈ V_R. If v ∈ V_R, then Qv = \sum_{u=1}^d I_uQv ∈ V_R. Since I_nI_u = I_u and I_nI_u = 0 if u ≠ v, each V_{R,u} is I_n-invariant and so is V_R. Conversely, if V' is Q-invariant as well as I_n-invariant and contains ρ, then I_nQ^{t-s+1}Q ... I_nQρ ∈ V' for any (u_0, ..., u_n) ∈ {1, ..., d}^{n+1} (n = 0, 1, 2, ...). Since V' is a subspace, it contains any linear combination of this form. Hence V_R ⊂ V'. The proof for the null space is similar and is omitted.

Assumption 2. The subspaces V_{R,u} (u = 1, ..., d) satisfy V_{R,u} ⊄ ker e^T.

Now, we give a basis of the reachable subspace.

Proposition 3. Suppose Assumption 2 holds. Let \hat{k}_R = \dim V_R. There exist a full-column rank matrix T_R ∈ (I \cup{R}) ∈ R^{k×k} and a map \hat{φ} : \{1, ..., \hat{k}_R\} → \{1, ..., d\} satisfying

\begin{align*}
\hat{t}_{R,i} & = 0 & \text{if } \hat{φ}(i) ≠ \hat{φ}(j), \\
V_R &= \text{ran } T_R, \\
e^TT_R &= e^T_R.
\end{align*} \quad (5, 6, 7)

where e_R ∈ R^{k×k} is the vector whose elements are all one.

Proof. Choose a basis of V_{R,u}. Then a basis of V_R is constructed by collecting these bases. Define \hat{φ}_R by \hat{φ}_R(i) = j if \sum_{u=1}^j \hat{k}_u < i ≤ \sum_{u=1}^{j+1} \hat{k}_u. Let \hat{k}_R = \sum_{u=1}^{d-1} \hat{k}_u = \dim V_R. Arrange the basis to form a matrix T_R ∈ R^{k×k} so that the ith columns of T_R (i ∈ \hat{φ}_R^{-1}(u)) are a basis of V_{R,u}. From the construction, T_R satisfies (5). From Assumption 2, we can select those columns to satisfy e^TT_R = e^T_R by perturbing a little bit and rescaling.
Proposition 4. Let \( \hat{k}_N = k - \dim V_N \). There exist a full-row rank matrix \( T_N = (t_{N,i,j}) \in \mathbb{R}^{k \times k} \) and a map \( \phi_N : \{1, \ldots, kN\} \rightarrow \{1, \ldots, d\} \) satisfying

\[
t_{N,i,j} = 0 \text{ if } \phi_N(i) \neq \phi(j),
\]

\[
V_N = \ker T_N,
\]

\[
e^T = \hat{e}_N^T T_N,
\]

where \( \hat{e}_N \) is the vector whose elements are all one.

Remark 5. Note that the \( Q \)-invariance of \( V_N \) and (6) imply that there exists \( Q_R \in \mathbb{R}^{k \times k} \) such that \( QT_N = T_R Q_R \).

From (19), we have

\[
e^T \hat{Q}_R = e^T T_R \hat{Q}_R = e^T Q_T R = e^T T_R = e^T T_R,
\]

\[
\hat{Q}_R \hat{Q}_R \hat{Q}_R = QT_R \hat{Q}_R = Q_R = \hat{Q}_R \hat{Q}_R = \hat{\rho}_R,
\]

which means \( \hat{Q}_R \) has an eigenvalue at one with the left eigenvector \( \hat{e}_R \) and the right eigenvector \( \hat{\rho}_R \).

Similarly, the \( Q \)-invariance of \( V_N \) and (9) imply that there exists \( Q_R \in \mathbb{R}^{k \times k} \) such that \( Q_N T_N = T_N Q_N \).

Proposition 3. Define \( \hat{Q}_R \in \mathbb{R}^{k \times k} \), \( \hat{Q}_R \in \mathbb{R}^{d \times k} \), \( \hat{\rho}_R \),

\[
\hat{\rho}_R = QT_R \hat{Q}_R \hat{Q}_R = Q_R = \hat{Q}_R \hat{Q}_R = \hat{\rho}_R,
\]

where \( \hat{I}_{R,u} \) is the diagonal matrix whose \( (i, i) \)th element is one if \( \phi_R(i) = u \) and zero otherwise.

Then \( M = A \otimes B \otimes C = \hat{A}_R \otimes \hat{B}_R \otimes \hat{C}_R \) holds.

4.1 Reduction using reachable subspace

If the reachable space \( V_R \) is not the whole space \( \mathbb{R}^k \), then the number of the rank-one tensors in (12) can be reduced to \( k = \dim V_R \).

Theorem 6. Suppose \( \hat{k}_R = \dim V_R < k \). Define \( T_R \in \mathbb{R}^{k \times k} \) and \( \phi_R : \{1, \ldots, \hat{k}_R\} \rightarrow \{1, \ldots, d\} \) as

\[
\hat{Q}_R \hat{Q}_R \hat{Q}_R = QT_R \hat{Q}_R = Q_R = \hat{Q}_R \hat{Q}_R = \hat{\rho}_R,
\]

Proposition 3. Define \( \hat{Q}_R \in \mathbb{R}^{k \times k} \), \( \hat{Q}_R \in \mathbb{R}^{d \times k} \), \( \hat{\rho}_R \),

\[
\hat{\rho}_R = QT_R \hat{Q}_R \hat{Q}_R = Q_R = \hat{Q}_R \hat{Q}_R = \hat{\rho}_R,
\]

where \( \hat{I}_{R,u} \) is the diagonal matrix whose \( (i, i) \)th element is one if \( \phi_R(i) = u \) and zero otherwise. Then

\[
M = A \otimes B \otimes C = \hat{A}_R \otimes \hat{B}_R \otimes \hat{C}_R
\]

holds.

Proof. From \( QT_R = T_R \hat{Q}_R \) and \( I_N T_R = T_R \hat{I}_{R,u} \), we have

\[
A_{L(u^T)} T_R = e^T I_{u_1} Q \cdots I_{u_d} Q T_R
\]

\[
= e^T I_{u_1} Q \cdots I_{u_d} T_R \hat{Q}_R
\]

\[
= e^T I_{u_1} Q \cdots T_R \hat{I}_{R,u_1} Q_R
\]

\[
\vdots
\]

\[
\hat{B}_{R,L(u^T)} T_R \hat{T}_R = \hat{\rho}_R \hat{I}_{R,u_1} Q \cdots \hat{I}_{R,u_d} Q R
\]

\[
= \hat{\rho}_R \hat{I}_{R,u_1} Q \cdots \hat{I}_{R,u_d} T_R \hat{Q}_R
\]

\[
= \hat{\rho}_R \hat{I}_{R,u_1} Q \cdots \hat{I}_{R,u_d} \hat{Q} \hat{T}_R
\]

\[
= \hat{\rho}_R \hat{I}_{R,u_1} Q \cdots \hat{I}_{R,u_d} \hat{T}_R
\]

\[
\vdots
\]

\[
M = A \otimes B \otimes C = \hat{A}_R \otimes \hat{B}_R \otimes \hat{C}_R
\]

Denote the column vectors of \( \hat{A}_R \), \( \hat{B}_R \), and \( \hat{C}_R \) by \( \hat{a}_1, \hat{b}_1 \), and \( \hat{c}_i \) (\( i = 1, \ldots, \hat{k}_R \)), respectively, and the column vectors of \( A, B, \) and \( C \) by \( a_i, b_i \), and \( c_i \) (\( i = 1, \ldots, k \)), respectively. From (19), we have

\[
AT_R = \hat{A}_R,
\]

\[
B = \hat{B}_R T_R^T,
\]

\[
CT_R = \hat{C}_R
\]
\[ A \otimes B \otimes C = \sum_{i=1}^{k} a_i \otimes b_i \otimes c_i \]
\[ = \sum_{i=1}^{k} a_i \otimes \left( \sum_{j=1}^{k} t_{R,i,j} \hat{b}_j \right) \otimes c_i \]
\[ = \sum_{j=1}^{k} \sum_{i \in \Phi^{-1}(\hat{\phi}(j))} t_{R,i,j} a_i \otimes \hat{b}_j \otimes c_i \]
\[ = \sum_{j=1}^{k} \sum_{i \in \Phi^{-1}(\hat{\phi}(j))} t_{R,i,j} a_i \otimes \hat{b}_j \otimes \hat{c}_j \]
\[ = \sum_{j=1}^{k} \hat{a}_j \otimes \hat{b}_j \otimes \hat{c}_j = \hat{A}_R \otimes \hat{B}_R \otimes \hat{C}_R. \]

### 4.2 Reduction using null space

If the null space \( V_N \) is not the zero subspace, then the number of the rank-one tensors in (12) can be reduced to \( \hat{k} = k - \dim V_N \).

**Theorem 7.** Suppose \( \hat{k}_N = k - \dim V_N < k \). Define \( T_N \in \mathbb{R}^{k_N \times k_N} \) and \( \hat{\phi}_N : \{1, \ldots, \hat{k}_N\} \to \{1, \ldots, d\} \) as in Proposition 4. Define \( \hat{Q}_N \in \mathbb{R}^{k_N \times k_N}, \hat{O}_N \in \mathbb{R}^{d \times k_N}, \hat{e}_N, \) and \( \hat{\rho}_N \) by \( \hat{Q}_N T_N = T_N \hat{Q}_N, \hat{O}_N T_N = O, \hat{e}_N^T T_N = e^T, \) and \( \hat{\rho}_N = T_N \rho_N \), respectively. Define \( \hat{A}_N \in \mathbb{R}^{d \times k_N}, \hat{B}_N \in T_N \mathbb{R}^{d \times k_N}, \) and \( \hat{C}_N \in \mathbb{R}^{d \times k_N} \) by
\[
\begin{align*}
\hat{A}_{N,L(u^0)} &= e^T \hat{I}_{N,u} \hat{Q}_N \cdots \hat{I}_{N,u} \hat{Q}_N, \\
\hat{B}_{N,L(u^0)} &= \hat{p}^T \hat{I}_{N,u} \hat{Q}_N^T \cdots \hat{I}_{N,u} \hat{Q}_N, \\
\hat{C}_N &= \hat{O}_N,
\end{align*}
\]
where \( \hat{I}_{N,u} \) is the diagonal matrix whose \((i, i)\)th element is one if \( \hat{\phi}_N(i) = u \) and zero otherwise. Then
\[ M = A \otimes B \otimes C = \hat{A}_N \otimes \hat{B}_N \otimes \hat{C}_N \]
holds.

**Proof.** From \( \hat{Q}_N T_N = T_N \hat{Q}_N \) and \( \hat{I}_{N,u} T_N = T_N \hat{I}_{N,u} \), we have
\[
\hat{A}_N T_N = A, \quad \hat{B}_N = B \hat{T}_N^T, \quad \hat{C}_N T_N = C.
\]
The rest of the proof is similar to the proof of Theorem 6 and is omitted.

### 4.3 Reduction using effective space

When the reachable subspace is not the whole space and the null space is not the zero subspace, we may have the situation where both Theorems 6 and 7 can be applied. In fact, by using the notion of the effective space in Ito et al. (1992), we can reduce the number of rank-one tensors in the decomposition (12).

The effective space is defined as
\[ (V_R + V_N) / V_N, \]
and its dimension is given by
\[ \hat{k} = \dim (V_R + V_N) / V_N \]
\[ = \dim (V_R + V_N) - \dim V_N \]
\[ = \dim V_R - \dim (V_R \cap V_N). \]

**Proposition 8.** Select \( T_R \) and \( T_N \) as in Propositions 3 and 4, respectively. Then \( \hat{k} = \text{rank } T_N T_R \). There exist a full row rank matrix \( T = (t_{ij}) \in \mathbb{R}^{k \times k} \), and a map \( \hat{\phi} : \{1, \ldots, \hat{k}\} \to \{1, \ldots, d\} \) satisfying
\[
\begin{align*}
t_{ij} &= 0 \text{ if } \hat{\phi}(i) \neq \hat{\phi}(j), \\
\ker T &= \ker T_N T_R, \\
\hat{e}_R^T T &= \hat{c}_R^T T, \\
\hat{Q}_R \ker T &= \ker T.
\end{align*}
\]

**Proof.** Because \( T_N \) and \( T_R \) have the block structure, we can select \( \hat{k} \) linearly independent rows of \( T_N T_R \) to construct \( T = (t_{ij}) \in \mathbb{R}^{k \times k} \), satisfying (23) and (24). The condition (25) is satisfied by scaling and perturbation if necessary. From \( T_N T_R \hat{Q}_R = T_N T_R \hat{Q}_N T_N T_R \), \( \ker T = \ker T_N T_R \) is \( \hat{Q}_R \) invariant.

**Theorem 9.** Define \( T \in \mathbb{R}^{k \times k} \), and \( \hat{\phi} : \{1, \ldots, \hat{k}\} \to \{1, \ldots, d\} \) as in Proposition 8. Define \( \hat{Q} \in \mathbb{R}^{k \times k}, \hat{O} \in \mathbb{R}^{d \times k}, \hat{\rho} \) by \( \hat{Q} = T \hat{Q}_R, \hat{O} = \hat{O}_R, \hat{\rho}^T = \hat{\rho}_R \), and \( \hat{\rho} = T \hat{\rho}_R \), respectively. Define \( \hat{A} \in \mathbb{R}^{d \times k}, \hat{B} \in \mathbb{R}^{d \times k} \), and \( \hat{C} \in \mathbb{R}^{d \times k} \) by
\[
\begin{align*}
\hat{A}_{L(u^0)} &= e^T \hat{I}_{u} \hat{Q} \cdots \hat{I}_{u} \hat{Q}, \\
\hat{B}_{L(u^0)} &= \hat{p}^T \hat{I}_{u} \hat{Q}^T \cdots \hat{I}_{u} \hat{Q}, \\
\hat{C} &= \hat{O},
\end{align*}
\]
where \( \hat{I}_u \) is the diagonal matrix whose \((i, i)\)th element is one if \( \hat{\phi}(i) = u \) and zero otherwise. Then
\[ M = A \otimes B \otimes C = \hat{A} \otimes \hat{B} \otimes \hat{C} \]
holds.

**Proof.** We apply Theorem 7 to the system \( (\hat{O}, \hat{Q}_R) \).

**Remark 10.** The generalized Hankel matrix introduced in Picci (1978) takes the following form (see also Anderson (1999)).
\[
H = \begin{bmatrix}
H^{(00)} & H^{(01)} & H^{(02)} & \cdots \\
H^{(10)} & H^{(11)} & H^{(12)} & \cdots \\
H^{(20)} & H^{(21)} & H^{(22)} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix},
\]
where \( H^{(ij)} \) is a \( d^i \times d^j \) matrix whose \( \left( L(u_{ij}^{(-i-1)}), L(u_{ij}^t) \right) \)th element is given by
\[
H^{(ij)}_{L(u_{ij}^{(-i-1)}), L(u_{ij}^t)} = P \left\{ y_{ij}^{(-i-1)} = u_{ij}^t \right\}.
\]
Let \( (\hat{O}, \hat{Q}) \) be a realization of the HMM of order \( k \). Define
\[ \Theta = [\Theta^{(0)} \Theta^{(1)} \Theta^{(2)} \ldots], \]
\[ \Gamma = [\Gamma^{(0)} \Gamma^{(1)} \Gamma^{(2)} \ldots], \]
where \( \Theta^{(i)} \) is a \( k \times d^i \) matrix whose \( L(u_0^{(i-1)}) \)th column is given by
\[ \Theta^{(i)}_{L(u_0^{(i-1)})} = I_{u_0}Q \cdots I_{u_{i-1}}p, \]
and \( \Gamma^{(i)} \) is a \( k \times d^i \) matrix whose \( L(u_i) \)th column is given by
\[ \Gamma^{(i)}_{L(u_i)} = I_{u_i}Q^T \cdots I_{u_1}e. \]

Then, we have
\[ H = \Theta^\Gamma \]
holds, and thus rank \( H \) is at most \( k \). Notice that the columns of \( \Theta \) generate the reachable subspace (2) and the columns of \( \Gamma \) generate the orthogonal complement of the null space (3). Hence, rank \( H \) is equal to the dimension of the effective subspace.

### 4.4 Example

This example is modified from the example discussed in Vidyasagar (2011) (originally in Fox and Rubin (1968) and Dharmadhikari and Nadkarni (1970)). Let \( \lambda \in (0, 0.5] \) and \( \alpha = 2\pi/m \) for some \( m \in \{3, 4, \ldots\} \). Let \( \zeta = e^{\alpha}\). Note that the example was intended to illustrate the case where the generalized Hankel matrix has finite rank but it does not have a finite-state Markov model realization; in this case, \( \alpha \) is selected to be non-commensurate with \( \pi \). The example in this section assumes that \( \alpha \) and \( \pi \) are commensurate.

Suppose that a stationary discrete-time random process \( \{y_i\} \) taking binary values \( \{1, 2\} \) has an HMM realization whose transition matrix \( Q \in \mathbb{R}^{(m+1) \times (m+1)} \) and observation matrix \( O \in \mathbb{R}^{2 \times (m+1)} \) are given by
\[
Q = \begin{bmatrix}
1 - \sum_{i=1}^{m-1} \frac{\lambda^i}{1 - \lambda^m} \sin^2 \left( \frac{i\pi}{m} \right) & 1 - \lambda & 0 & \cdots & 0 \\
\frac{\lambda}{1 - \lambda^m} \sin^2 \left( \frac{\pi}{m} \right) & 0 & 1 & \vdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{\lambda^{m-1}}{1 - \lambda^m} \sin^2 \left( (m-1)\pi \right) & 0 & 0 & 1 & 0 \\
\end{bmatrix},
\]
\[
O = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 1 \\
\end{bmatrix}.
\]
Let
\[
p_0 = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0 \\
\end{bmatrix}, \quad p_1 = \begin{bmatrix}
0 \\
1 \\
\vdots \\
\lambda^{m-1} \\
\end{bmatrix},
\]
\[
p_2 = \begin{bmatrix}
0 \\
1 \\
\lambda \zeta \\
\vdots \\
\lambda^{m-1} \zeta^{m-1} \\
\end{bmatrix}, \quad p_3 = \begin{bmatrix}
0 \\
1 \\
\lambda \zeta^{-1} \\
\vdots \\
\lambda^{m-1} \zeta^{-(m-1)} \\
\end{bmatrix}.
\]

Then we can show that \( \{p_0, p_1, (p_2 + p_3)/2, (p_2 - p_3)/2j\} \) is a basis of the reachable space \( V_R \). By scaling,
\[
T_R = [p_0 p_1 p_2 p_3] \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 1/2 & 1/2j \\
0 & 1 & 1/2 & 1/2j \\
\end{bmatrix}^{-1}
\times \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 - \lambda^m & 0 & 0 \\
0 & 0 & (1 - \lambda^m)(1 - \lambda \cos \alpha) & 0 \\
0 & 0 & 0 & \lambda(1 - \lambda^m) \sin \alpha \\
\end{bmatrix} \begin{bmatrix}
\frac{\eta}{\lambda} & 1 - \lambda \\
\frac{2(1 - \lambda)}{\lambda \cos \alpha (\lambda \cos \alpha - 1)} & 0 \\
\frac{2(1 - 2\lambda \cos \alpha + \lambda^2)}{\lambda^2 \sin^2 \alpha} & 0 \\
\frac{1 - 2\lambda \cos \alpha + \lambda^2}{\lambda^2 \sin^2 \alpha} & 0 \\
\lambda \cos \alpha & \lambda \cos \alpha & 1 - \lambda \cos \alpha & \lambda \cos \alpha \\
\end{bmatrix},
\]
\[
\hat{Q}_R = \begin{bmatrix}
\frac{\eta}{\lambda} & 1 - \lambda \\
\frac{2(1 - \lambda)}{\lambda \cos \alpha (\lambda \cos \alpha - 1)} & 0 \\
\frac{2(1 - 2\lambda \cos \alpha + \lambda^2)}{\lambda^2 \sin^2 \alpha} & 0 \\
\frac{1 - 2\lambda \cos \alpha + \lambda^2}{\lambda^2 \sin^2 \alpha} & 0 \\
\end{bmatrix} - \frac{1}{\lambda^2 \sin^2 \alpha} \begin{bmatrix}
\lambda \cos \alpha & \lambda \cos \alpha & 1 - \lambda \cos \alpha & \lambda \cos \alpha \\
\end{bmatrix},
\]
\[
\hat{O}_R = OT_R = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
\end{bmatrix},
\]
where
\[
\eta = 2 - 3\lambda - 3\lambda \cos \alpha + \lambda^2 + 5\lambda^2 \cos \alpha - 2\lambda^3.
\]
Let
\[
\hat{I}_{R,1} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad \hat{I}_{R,2} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix},
\]
\[
\hat{e}_R = \begin{bmatrix}
1 \\
1 \\
\vdots \\
1 \\
\end{bmatrix}, \quad \hat{\rho}_R = \begin{bmatrix}
\rho_1 \\
\rho_1 \lambda \\
\rho_1 (\lambda \cos \alpha - \lambda(\lambda \cos \alpha - 1)) \\
\frac{2(1 - \lambda \cos \alpha + \lambda^2)^2}{2(1 - 2\lambda \cos \alpha + \lambda^2)^2} \\
\frac{\rho_1 \lambda^2 \sin^2 \alpha}{\rho_1 (\lambda \cos \alpha - \lambda(\lambda \cos \alpha - 1))} \\
\frac{2(1 - 2\lambda \cos \alpha + \lambda^2)^2}{2(1 - 2(1 - \lambda \cos \alpha + \lambda^2))^2} \\
\end{bmatrix},
\]
where \( \rho_1 \) is selected to satisfy \( \hat{e}_R^T \hat{\rho}_R = 1 \). Define \( \hat{A}_R \in \mathbb{R}^{2 \times 4}, \hat{B}_R \in \mathbb{R}^{2 \times 4}, \) and \( \hat{C}_R \in \mathbb{R}^{2 \times 4} \) by (16), (17), and (18), respectively. Then \( \hat{A}_R \otimes \hat{B}_R \otimes \hat{C}_R \) is a sum of four rank-one tensors for the third-order tensor (11) which is originally written as a sum of \( (m+1) \) rank-one tensors. So, if \( m \geq 4 \), then we have a reduced number of rank-one tensors for the decomposition (12).

### 5. CONCLUSION

This paper considered the minimum HMM realization problem using tensor decomposition methods. If the observation is deterministic, or the Kruskal rank of the observation matrix equals one, then the third-order tensor can be
decomposed to a sum of rank-one tensors whose number is not greater than the dimension of the effective space. Since the dimension of the effective space is equal to the rank of the generalized Hankel matrix, the tensor decomposition ends up giving a minimal quasi-realization. This means that determining the minimum number of states to realize a stationary stochastic process with a finite alphabet for the class of HMM’s with deterministic observation is yet unresolved.

REFERENCES
Anderson, B.D.O. (1999). The realization problem for hidden Markov models. Mathematics of Control, Signals, and Systems, 12(1), 80–120.
Blackwell, D. and Koopmans, L. (1957). On the identifiability problem for functions of finite Markov chains. The Annals of Mathematical Statistics, 28(4), 1011–1015.
Dharmadhikari, S.W. and Nadkarni, M.G. (1970). Some regular and non-regular functions of finite Markov chains. The Annals of Mathematical Statistics, 41(1), 207–213.
Fox, M. and Rubin, H. (1968). Functions of processes with Markovian states. The Annals of Mathematical Statistics, 39(3), 938–946.
Gilbert, E.J. (1959). On the identifiability problem for functions of finite Markov chains. The Annals of Mathematical Statistics, 30(3), 688–697.
Huang, Q., Ge, R., Kakade, S., and Dahleh, M. (2016). Minimal realization problems for hidden Markov models. IEEE Transactions on Signal Processing, 64(7), 1896–1904.
Ito, H., Amari, S.I., and Kobayashi, K. (1992). Identifiability of hidden Markov information sources and their minimum degrees of freedom. IEEE Transactions on Information Theory, 38(2), 324–333.
Kolda, T.G. and Bader, B.W. (2009). Tensor decompositions and applications. SIAM Review, 51(3), 455–500.
Krogh, A., Brown, M., Mian, I.S., Sjölander, K., and Haussler, D. (1994). Hidden Markov models in computational biology applications to protein modeling. Journal of Molecular Biology, 235, 1501–1531.
Kruskal, J. (1977). Three-wat arrays. Linear Algebra, 18, 95–138.
Picci, G. (1978). On the internal structure of finite-state stochastic processes. In R.R. Mohler and A. Ruberti (eds.), Recent developments in variable structure systems, economics and biology, volume 162, 288–304. Springer-Verlag. Lecture Notes in Economics and Mathematical Systems.
Rabiner, L.R. (1989). A tutorial on hidden Markov models and selected applications in speech recognition. Proceedings of the IEEE, 77(2), 257–286.
Vidyasagar, M. (2011). The complete realization problem for hidden Markov models: a survey and some new results. Mathematics of Control, Signals, and Systems, 23(1), 1–65.