Combinatorial Invariants from Four Dimensional Lattice Models: II

Danny Birmingham

Universiteit van Amsterdam, Instituut voor Theoretische Fysica,
1018 XE Amsterdam, The Netherlands

Mark Rakowski

Yale University, Center for Theoretical Physics,
New Haven, CT 06511, USA

Abstract

We continue our analysis of the subdivision properties of certain lattice gauge theories based on the groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$, in four dimensions. We prove that the partition function for a closed four dimensional manifold is unity, at the special subdivision invariant points. For the case of manifolds with boundary, we show that Alexander type 2 and 3 subdivision of a bounding simplex is equivalent to the insertion of an operator which is equal to a delta function on trivial bounding holonomies.

YCTP-P11-93
May 1993

---

1Supported by Stichting voor Fundamenteel Onderzoek der Materie (FOM)
Email: Dannyb@phys.uva.nl

2Email: Rakowski@yalph2.bitnet
1 Introduction

In [1, 2], a class of lattice gauge theories was introduced which enjoyed certain topological features. An analytic proof of the invariance of the Boltzmann weights under all type $(k, l)$ subdivision moves was presented in [2]. The fundamental identity established there specified that a certain product of six Boltzmann weights was unity. Triviality of the invariant for closed 4-manifolds, which we establish in this paper, is based upon the observation that one can always realize a closed four dimensional complex as the base of a five dimensional cone. The Boltzmann weight for the boundary of such a cone is precisely 1 by this identity, and the result immediately follows. While the theory essentially reduces to behavior on the 3-boundary, the novelty of the construction is that gauge invariance actually requires a 4-dimensional perspective.

With this knowledge in hand, we turn our attention to the case of four manifolds with boundary. The central problem is to determine how the partition function behaves with respect to subdivision of the boundary, and we analyze the theory under the Alexander moves of type 2 and 3. We show that subdivisions of these types are equivalent to the insertion of an operator in the partition function which is a delta function on trivial bounding holonomies. While it is intriguing that subdivision of the underlying lattice naturally induces these constraints, the construction of an object which is fully subdivision invariant will not be resolved here.

In the following section, we briefly recall the definition of the Boltzmann weight, together with the statement of the fundamental identity. The triviality for closed four manifolds is then established. Considerations of boundary subdivision follow, and we present our analysis of the Alexander moves of type 2 and 3. We close with a few remarks.

2 Cobordism Independence

We show in this section that the four dimensional models that we have been considering actually reduce to behavior on the boundary. In particular, the partition function is identically 1 for all closed 4-manifolds.
Let us first recall that these models are defined by a Boltzmann weight $W$, which is a product of 15 distinct factors $B$; the generic structure of $B$ is given by:

$$B[v_0, v_1, v_2, v_3, v_4] = \exp[\beta (U - U^{-1})_{v_0v_1v_2}(U - U^{-1})_{v_3v_4}] ,$$  

for the $Z_3$ theory, and by

$$B[v_0, v_1, v_2, v_3, v_4] = \exp[\beta (U - 1)_{v_0v_1v_2}(U - 1)_{v_3v_4}] ,$$  

for the $Z_2$ theory. Here, $U_{v_0v_1v_2} = U_{v_0v_1}U_{v_1v_2}U_{v_2v_0}$ is the holonomy combination. We refer to [1] for the precise normalization.

To establish the triviality for closed 4-manifolds, consider the identity that was established in [2], namely that the Boltzmann weights of the $Z_2$ and $Z_3$ theories satisfy:

$$1 = W[0, 1, 2, 3, 4] W[0, 1, 2, 4, 5] W[0, 1, 2, 5, 3] W[1, 0, 3, 4, 5] W[2, 1, 3, 4, 5] W[0, 2, 3, 4, 5] ,$$  

at the special values of the coupling parameter. Written in this way, one can recognize that the 4-simplices in this identity are actually the boundary of a 5-simplex $[0, 1, 2, 3, 4, 5]$;

$$\partial [0, 1, 2, 3, 4, 5] = [1, 2, 3, 4, 5] - [0, 2, 3, 4, 5] + [0, 1, 3, 4, 5] - [0, 1, 2, 4, 5] + [0, 1, 2, 3, 5] - [0, 1, 2, 3, 4] .$$  

If $t$ is any 5-simplex, we can write this compactly as:

$$W[\partial t] = 1 .$$  

Let $K$ denote a simplicial complex which models a 4-manifold, possibly with boundary. For our purposes, the 4-simplices $\{s_i\}$ in $K$ are most important,

$$K = \sum_i s_i .$$  

Now consider the abstract simplicial complex called the cone over $K$, which is obtained by adding a new vertex $x$ to the simplicial complex $K$, and
linking it to all other vertices; we denote this simplicial complex by \( x \ast K \). Computing the boundary of that complex, one sees

\[
\partial (x \ast K) = K - x \ast \partial K.
\]  \hfill (7)

Given that the Boltzmann weight of the left hand side is just 1, we have then

\[
W[K] = W[x \ast \partial K],
\]  \hfill (8)

where we mean, more precisely, that

\[
W[K] = \prod_i W[s_i].
\]  \hfill (9)

The simplicity of equation (8) is striking; the implication is that the Boltzmann weight of any four dimensional simplicial complex \( K \) is identical to that of the cone over its boundary. Essentially, the cone construction is giving a canonical presentation - or framing - of the boundary of \( K \). Moreover, if \( \partial K \) has several disjoint components \( M_\alpha \), then \( K \) is a cobordism connecting them, and we immediately have that

\[
Z[K] = \prod_\alpha Z[M_\alpha].
\]  \hfill (10)

This is one of the axioms for a topological field theory [4].

Having established that we are dealing with a four dimensional gauge theory which essentially reduces to something on the boundary, it is natural to wonder about its interpretation as an intrinsically three dimensional theory. As a gauge theory, we can gauge fix the links on any maximal tree, and one such tree is given by the links which spew from the vertex \( x \). The value of the partition function is independent of how we fix them, so we could always set those link variables to 1 say. However one chooses to gauge fix these link variables, we can consider the result to be a three dimensional lattice theory. If there is any residual gauge invariance left, then it is not at all manifest, but this is also reminiscent of the continuum Chern-Simons theories [5].

### 3 Behavior Under Boundary Subdivision

In this section, we will undertake an analysis of the theory when a bounding simplex is subdivided by an Alexander move of type 2 or 3. In the later case, a
new vertex is added to the center of the 3-simplex and new links are joined to
the old vertices, yielding an assembly of four tetrahedrons. We will find that
the Boltzmann weight is not generally invariant under this move, but we will
be able to prove that this is so when we restrict to field configurations which
have trivial holonomy around all four faces of the 3-simplex in question.

Consider what happens to the Boltzmann weight of the theory when
a bounding 3-simplex \([0,1,2,3]\) is subdivided. Let \(x\) be the cone vertex
discussed in the previous section, so that the partition function would have
the factor \(W[0,1,2,3,x]\). Under subdivision where we add a new vertex
\(c\) to the center of the tetrahedron, we would then consider a new set of
Boltzmann weights which are unchanged except that we would replace the
factor \(W[0,1,2,3,x]\) by the quantity,

\[
W[c,1,2,3,x] W[0,c,2,3,x] W[0,1,c,3,x] W[0,1,2,c,x]
\]

in the partition function, and sum over the new link variables. However, the
main identity that we established in [2], namely (3), says that this product
of four weights is equal to

\[
W[0,1,2,3,x] W^{-1}[0,1,2,3,c] .
\]

Here, \(W^{-1}\) denotes the inverse value which, for the Boltzmann weights in
our construction, is equivalent to simply an odd permutation of the vertices;
\(W^{-1}[0,1,2,3,4] = W[0,1,2,4,3]\). We see then that the subdivided
Boltzmann weights represented by (11) are precisely equivalent to having in-
troduced an extra factor \(W^{-1}[0,1,2,3,c]\) into the original assembly of Boltz-
mann weights. It is then crucial to understand how the theory behaves under
insertions of the kind:

\[
I[0,1,2,3] = \frac{1}{|G|^4} \sum_{U_{ci}} W^{-1}[0,1,2,3,c] ,
\]

where the sum is over the four link variables connected to \(c\), and \(|G|\) is the
order of the gauge group. At the trivial points where (3) holds, namely \(s(2) = 1\) or \(s(3) = 1\), this quantity is manifestly 1. We are interested in
investigating the nontrivial roots of unity.

Let \(U_{ijk} = U_{ij} U_{jk} U_{ki}\) denote the holonomy through the three indicated
vertices and \(\delta(x)\) denote the usual delta function which is 1 for \(x = 0\) and
zero otherwise.
**Theorem 1:** The insertion $I[0,1,2,3]$ is equal to:

$$
\delta(U_{v_0v_1v_2} - 1) \delta(U_{v_0v_1v_3} - 1) \delta(U_{v_0v_2v_3} - 1) \delta(U_{v_1v_2v_3} - 1) \tag{14}
$$

at the points $s(2) = -1$, and $s(3) = \exp[\pm 2\pi i/3]$, in the $Z_2$ and $Z_3$ models respectively.

Thus, the insertion is 1 if all holonomies on the bounding 3-simplex are trivial, and zero otherwise. This identity is most easily established by computing both sides for all choices of boundary data and observing that they are equal; we have done this with a computer program.

It is also interesting to consider Alexander type 2 subdivision of a 2-simplex which belongs to the bounding 3-manifold. Since the bounding space is a manifold, a given 2-simplex, say $[0,1,2]$, will be shared by precisely two 3-simplices; we denote their sum by $[0,1,2,3] - [0,1,2,4]$. The requirement of the relative minus sign is dictated by the fact that we have a 3-manifold without boundary. Under type 2 subdivision, we add a new vertex $c$ to the center of the $[0,1,2]$ face, and link it to the other vertices,

$$
[0,1,2] \rightarrow [c,1,2] + [0,c,2] + [0,1,c] \ . \tag{15}
$$

Now, in the Boltzmann weights appropriate to the original complex, one will find the product,

$$
W[0,1,2,3,x] W^{-1}[0,1,2,4,x] \ . \tag{16}
$$

In the subdivided situation, each of these two factors will be replaced by a product of three Boltzmann weights according to the structure of (14). If we again use the identity (3), one finds that the subdivided situation is equivalent to the insertion of the following factor in the original product of Boltzmann weights:

$$
I'[0,1,2,3,4] = \frac{1}{|G|^5} \sum_{U_{ci}} W^{-1}[0,1,2,3,c] W[0,1,2,4,c] \ . \tag{17}
$$

Let $U_{ijkl} = U_{ij}U_{jk}U_{kl}U_{li}$ denote the holonomy through four vertices; we then have the following result.

**Theorem 2:** The quantity $I'[0,1,2,3,4]$ is equal to,

$$
\delta(U_{v_0v_1v_2} - 1) \delta(U_{v_0v_3v_4} - 1) \delta(U_{v_1v_3v_4} - 1) \delta(U_{v_2v_3v_4} - 1) \ , \tag{18}
$$
at the same points as in Theorem 1.

Notice that there is one 3-vertex holonomy around the 2-simplex \([0, 1, 2]\)
which is the face common to the two 3-simplices that have been glued to-
gether; the other 4-vertex holonomies are just products of the more element-
tary holonomies. Since any 2-simplex on the boundary is common to precisely
two bounding 3-simplices, the restriction imposed by type 2 subdivision is
actually equivalent to that from the type 3 move.

Again, the proof of Theorem 2 is most easily carried out with the aid of
a computer where one can simply compute both sides of the equation for all
values of boundary field configurations.

4 Concluding Remarks

As we have seen, achieving invariance with respect to boundary subdivision
naturally leads to the insertion of operators which are delta functions on
trivial bounding holonomies. One might wonder if these models are somehow
related to the discrete Chern-Simons type theories of [6, 7], but we shall leave
that issue open.

In [1], the unrestricted partition function (where one includes contribu-
tions from all bounding field configurations) was evaluated for a simplicial
complex which consisted of a single 4-simplex, thereby modelling the 4-disk.
In addition, it was shown that the results obtained from such a computation
were invariant under particular subdivisions (single Alexander moves) of this
particular complex. For the case of the \(Z_2\) and \(Z_3\) models, the partition func-
tion took the value \(Z = 1/|G|^3\). However, one should note that this value
is not generally invariant for arbitrary complexes which model the 4-disk. It
suffices to consider a simple example.

Let us begin with a 4-simplex \([0, 1, 2, 3, 4]\), and first perform an Alexander
move of type 4. This involves placing a new vertex 5 in the center of the
simplex, and joining it to all remaining vertices. The resulting complex takes
the form:

\[
[5, 1, 2, 3, 4] + [0, 5, 2, 3, 4] + [0, 1, 5, 3, 4] + [0, 1, 2, 5, 4] + [0, 1, 2, 3, 5].
\] (19)

A boundary subdivision is now induced by performing two Alexander moves
of type 3; these are effected by introducing two additional vertices, 6 and 7, in the center of the 3-simplices \([1, 2, 3, 4]\) and \([0, 1, 3, 4]\), respectively. The final complex is given by:

\[
K = [5, 6, 2, 3, 4] + [5, 1, 6, 3, 4] + [5, 1, 2, 6, 4] + [5, 1, 2, 3, 6] \\
  + [0, 5, 2, 3, 4] + [7, 1, 5, 3, 4] + [0, 7, 5, 3, 4] + [0, 1, 5, 7, 4] \\
  + [0, 1, 5, 3, 7] + [0, 1, 2, 5, 4] + [0, 1, 2, 3, 5] ,
\] (20)

It is straightforward to compute the unrestricted partition function for \(K\), and the \(Z_2\) model yields the result: \(Z = 1/2^5\).

Given our analysis of boundary subdivisions, the central problem is to construct an object related to the original partition function which is fully subdivision invariant. This may involve, for example, restricting the field configurations on the boundary. Alternatively, one might hope that the subdivision moves eventually lead to a stable value for the original partition function.

Another avenue which should be interesting to investigate concerns modifications of the holonomy factors in our action. Since the \((U - 1)\) combination essentially measures deviations from trivial holonomy, one can simply consider replacing this by an angle, keeping the bow-tie structure of the Boltzmann weight intact. This is in the spirit of the Manton/Villain type actions. One would expect this simplification to streamline the general analysis of these theories for all abelian groups.

On a different course, it is well known that \(Z_2\) lattice gauge theory of the “\(F^2\)” type is dual to the Ising model in 3 dimensions. It would be interesting to investigate the duality properties of the models we have been considering.

References

[1] D. Birmingham and M. Rakowski, Subdivision Invariant Models in Lattice Gauge Theory, YCTP-P4-93 preprint, February 1993.

[2] D. Birmingham and M. Rakowski, Combinatorial Invariants from Four Dimensional Lattice Models, YCTP-P6-93 preprint, March 1993.
[3] J. Munkres, *Elements of Algebraic Topology*, Addison-Wesley, Menlo Park, 1984.

[4] M.F. Atiyah, *Topological Quantum Field Theories*, Publ. Math. IHES 68 (1988) 175.

[5] E. Witten, *Quantum Field Theory and the Jones Polynomial*, Commun. Math. Phys. 121 (1989) 351.

[6] R. Dijkgraaf and E. Witten, *Topological Gauge Theories and Group Cohomology*, Commun. Math. Phys. 129 (1990) 393.

[7] D. Altschuler and A. Coste, *Quasi-Quantum Groups, Knots, Three-Manifolds, and Topological Field Theory*, Commun. Math. Phys. 150 (1992) 83.