Gravitational Collapse of the Shells with the Smeared Gravitational Source in Noncommutative Geometry

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Abstract

We study the formation of the (noncommutative) Schwarzschild black hole from collapsing shell of the generalized matters containing polytropic and Chaplygin gas. We show that this collapsing shell depending on various parameters forms either a black hole or a naked singular shell with the help of the pressure. Furthermore, by considering the smeared gravitational sources, we investigate the noncommutative black holes formation. Though this mild noncommutative correction of matters cannot ultimately resolve the emergence of the naked singularity, we show that in some parameter region the collapsing shell evolves to a noncommutative black hole before becoming a naked singular shell.

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1 Introduction

One of the fascinating topics in general relativity is the dynamical formation of black holes from gravitationally collapsing matter. A remarkable breakthrough that emerged from such studies is mathematical theorem [1] dubbed “singularity theorem”. According to the theorem, if a trapped surface forms during the collapse of physically reasonable matter, the resultant spacetime geometry inevitably leads to the creation of a singularity, assuming that there are no closed timelike curves (CTCs). This fact, however, does not imply that the collapse process is necessarily accompanied with the formation of black holes since there is no reason that the trapped surface hides the singularity from distant observers. Thus it has been suggested that a black hole must be formed from gravitationally collapsing physical matter with generic initial data in order to preserve future predictability beyond singularity formation, at least for distant observers. This is called the “Cosmic Censorship Hypothesis” [2, 3]. The issue of whether or not this hypothesis is true remains unsolved in the context of the general relativity – there are some counter-examples against the hypothesis, producing a naked singularity [4].

On the other hand, in the three-dimensional Einstein’s gravity with a cosmological constant, it has been shown that the cosmic censorship violation happens under a broad variety of initial conditions from the gravitationally collapsing shells of diverse static [5] and the rotating [6] matter configurations. As the shell collapses its energy density (and pressure, if any) diverges in finite proper time. The interesting point is that even if the exterior space develops no curvature singularity (since the exterior three-dimensional spacetime has constant curvature), the energy-momentum tensor of the shell diverges in finite proper time, where the time-evolution of the shell’s equation of motion breaks down. As a consequence, this leads to a violation of cosmic censorship in that there is no definite way to evolve the equation of motion beyond this point and a Cauchy horizon appears if the singularity is not screened by an event horizon. This behavior also appears in the previous studies of the gravitationally collapsing dust ball in three-dimensions [7] and the topological black hole formation in four dimensions [8, 13, 14].

As seen in previous studies, dealing with a singularity might cause some troubles in studying physics in the context of, at least, general relativity. This is due to the fact that the theory of gravity we study is adhere to the Riemmanian geometry that has a drawback in handling a singular point. This is the reason why the quantum gravity is strongly required. There are two different prospects toward quantum gravity so far – string theory and loop quantum gravity, in both which there should be somewhat noncommutative modification of spacetime geometry that possesses a symplectic structure. In this sense, such a modification of gravity might resolve the singularity problem.

In this paper, we study the formation of Schwarzschild black hole from gravitationally collapsing shells of polytropic matter and Chaplygin gas. And we study the formation of noncommutative Schwarzschild black hole slightly modified by the smeared gravitational source and classify the possible scenarios for the black hole (or naked singular shell) formation in the various parameter regions. In Sec. 2 we present a shell formalism and an introduction of the smeared gravitational source that yields noncommutative Schwarzschild black hole. In Sec. 2.2 we investigate
gravitational collapse scenarios for collapsing polytropic and the Chaplygin gas shells, in which it can form either a black hole or a naked singularity, depending on its initial data. In Sec. 3 we investigate the formation of the noncommutative black hole from the collapsing shells, where new-type of matter (non-polytrope type) is needed to solve the junction equation. Finally, we shall summarize and discuss the results in Sec. 4.

2 Shell Collapse and Smeared Gravitational Source

2.1 Hypersurface Formalism with Smeared Source

Let us consider a hypersurface in four-dimensional manifolds which is described by a surface stress-energy tensor denoted by $S_{\mu\nu}$. Then the manifold is divided into two parts – exterior and interior regions. By introducing the Heaviside distribution function $H(\sigma)$, we have

$$g_{\mu\nu} = H(\sigma)g_{\mu\nu}^+ + H(-\sigma)g_{\mu\nu}^-,$$

where $\sigma$ is a geodesic coordinate. Differentiating this yields a smoothness condition for the metric,

$$[g_{\mu\nu}] = 0,$$

where $[A] = A^+ - A^-$, which is called the first junction condition.

Computing the Riemann tensor leads to the Einstein tensors in both regions and the additional term from the shell’s edge

$$G_{\mu\nu} = \mathcal{H}(\sigma)G_{\mu\nu}^+ + \mathcal{H}(-\sigma)G_{\mu\nu}^- - \delta(\sigma)e_\alpha^a e_\beta^b ([K_{ab}] - g_{ab}[K]),$$

(2.1)

where $K_{ab} = e_\alpha^a e_\beta^b \nabla_\alpha n_\beta$ is the extrinsic curvature and $e_\alpha^a$ is a projection vector onto the hypersurface defined by $e_\alpha^a = \partial x^a / \partial x^\alpha$. Generically, assuming the energy-momentum tensor in the whole regions $T_{\mu\nu} = \mathcal{H}(\sigma)T_{\mu\nu}^+ + \mathcal{H}(-\sigma)T_{\mu\nu}^- + \delta(\sigma)S_{\mu\nu}$, then we have full equations of motion,

$$G_{\mu\nu}^\pm = \kappa^2 T_{\mu\nu}^\pm, \quad \text{(exterior and interior regions)}$$

(2.2)

$$\kappa^2 S_{ab} = -([K_{ab}] - g_{ab}[K]), \quad \text{(on the hypersurface)}$$

(2.3)

where $\kappa^2 = 8\pi G_N$. Note that Eq. (2.3) will determine the dynamical motion of the hypersurface. Here, the exterior and interior geometries are assumed to be vacuum (or other possible) geometries and the dynamical collapsing motion of the matter shell (hypersurface) will produce the final geometry of the exterior spacetime – this is the usual process of the gravitational shell-collapse. More precisely, the shell’s equation of motion can be reexpressed in terms of a point-like particles’ motion under the effective potential. The collapse scenarios of the shell can be analyzed by investigating the behavior of the effective potential.

On the other hand, when we consider the geometrical structure at the region with strong gravitational fields such as black hole horizon or curvature singularity, it is expected that the noncommutativity of spacetimes can re-

1 $H(\sigma)$ is 1 if $\sigma > 0$, 0 if $\sigma < 0$, and indeterminate if $\sigma = 0$. Its crucial properties are $\mathcal{H}^2(\sigma) = \mathcal{H}(\sigma)$, $\mathcal{H}(\sigma)\mathcal{H}(-\sigma) = 0$, and $d\mathcal{H}(\sigma)/d\sigma = \delta(\sigma)$, where $\delta(\sigma)$ is a delta function.

2 See [5, 6] for more detailed study of this formalism and analysis.
move the point-source structure due to the noncommutative effect of \( \Theta \), where \([x^\mu, x^\nu] = i \Theta^{\mu\nu} \) with \( \Theta^{\mu\nu} = \Theta \text{diag}(\epsilon_1, \cdots, \epsilon_D/2) \). Inspired by this philosophy, a simple mathematics tells us that the delta functional point source can be replaced by a smeared Gaussian distribution of the width \( \sqrt{\Theta} \), which implies that the energy density of the gravitational source can be chosen as

\[
\rho_{\Theta} = \frac{M}{(4\pi \Theta)^{3/2}} e^{-\frac{R^2}{4\Theta}}. 
\] (2.4)

The energy density and the conservation law for a static metric solution with a spherical symmetry tell us that the corresponding energy-momentum tensor is given by

\[
T_{\mu\nu} = \text{diag}(-\rho_{\Theta}, p_R, p_\perp, p_\perp),
\]

where the radial and the tangential components of the pressure are \( p_R = -\rho_{\Theta} \) and \( p_\perp = -\rho_{\Theta} - \frac{1}{2} R \partial_R \rho_{\Theta} \). Note that in Eq. (2.4) the energy-momentum tensor vanishes as \( R \to \infty \) or \( \Theta \to 0 \), which corresponds to the commutative limit. In the section 4, the black hole formation under the noncommutative effect \( \Theta \) will be investigated.

### 2.2 Schwarzschild Black Hole from the Shell Collapse

In this section we first consider the Schwarzschild black hole formation from collapsing shells with the polytropic equation of state, including the perfect fluid and the Chaplygin gas. We assume that the metrics in both regions, \( V_+ \) (outside the shell) and \( V_- \) (inside the shell) are given by

\[
(ds)^2_{V_\pm} = -F_{\pm}(R)dT^2 + \frac{dR^2}{F_{\pm}(R)} + R^2(d\theta^2 + \sin^2 \theta d\phi^2),
\] (2.5)

where \( F_+ \) and \( F_- \) are exterior and interior metrics can be assumed to be a vacuum solution. The most general solution is given by the Schwarzschild black hole one, \( F_{\pm} = 1 - 2M_{\pm}/R \). If \( M_{\pm} = 0 \) it describes a flat space. Generically, there exist nine possible scenarios depending upon the value of the parameter \( M_{\pm} \) inside and outside the shell. We see that \( M_{\pm} > 0 \) describes black holes inside and outside the shell while \( M_{\pm} < 0 \) describes negative mass black holes in both sides [13]. However, the possibility of negative mass black holes will be discussed later.

Provided we employ a coordinate system \((t, \theta, \phi)\) on \( \Sigma \) at \( R = R(t) \), the induced metric on the shell becomes

\[
(ds)^2_\Sigma = -F_{\pm}dT^2 + \frac{dR^2}{F_{\pm}} + R(t)^2(d\theta^2 + \sin^2 \theta d\phi^2)
= -dt^2 + r_0^2 a(t)^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\] (2.6)

Continuity of the metric implies that \([g_{ij}] = 0\) or \( F_{\pm}^2 \dot{T}^2 = \dot{R}^2 + F_{\pm} \) and \( \dot{R}^2 = r_0^2 a(t)^2 \). However there exists a discontinuity in the extrinsic curvature of the shell, \([K_{ij}] \neq 0\), since nonvanishing surface stress-energy tensor exists. The extrinsic curvatures on \( V_{\pm} \) are

\[
K^t_{\pm} = \frac{d}{dR} \sqrt{\left( \frac{dR}{dt} \right)^2 + F_{\pm}}, \quad K^\theta_{\pm} = K^\phi_{\pm} = \frac{1}{R} \sqrt{\left( \frac{dR}{dt} \right)^2 + F_{\pm}}.
\] (2.7)
The surface stress-energy tensor is defined by

$$\kappa^2 S_{ab} = - ([K_{ab}] - [K] h_{ab}),$$

(2.8)

where \( h_{ab} \) is the induced metric on \( \Sigma \). The surface stress-energy tensor for a fluid of energy density \( \rho \) and pressure \( p \) is assumed to be

$$S_{ab} = \rho u^a u^b + p h_{ab},$$

(2.9)

where \( h_{ab} = g_{ab} + u^a u^b \) is an induced metric on \( \Sigma \), and \( u^a \) is the shell’s velocity. The surface stress-energy tensor can be straightforwardly evaluated

$$\kappa^2 S_t = \frac{2}{R} (\beta^+ - \beta^-), \quad \kappa^2 S_\theta = \frac{d}{dR} (\beta^+ - \beta^-) + \frac{1}{R} (\beta^+ - \beta^-),$$

(2.10)

where \( \beta^\pm \equiv \sqrt{\dot{R}^2 + F^\pm} \). Using Eqs. (2.9) and (2.10), we have two relations,

$$\frac{d}{dR} (\beta^+ - \beta^-) + \frac{1}{R} (\beta^+ - \beta^-) = 0,$$

(2.11)

$$\frac{d}{dR} (\beta^+ - \beta^-) + \frac{1}{R} (\beta^+ - \beta^-) - \kappa^2 p = 0.$$

(2.12)

These two relations can be easily reduced to a differential equation for the energy density

$$\frac{dp}{d\log R} + 2(\rho + p) = 0.$$

(2.13)

For the polytropic-type matter, the equation of state is given by

$$p = \omega \rho^{(n+1)/n},$$

(2.14)

where \( \omega \) is the equation of state parameter, which has been introduced and used in diverse astrophysical situations such as the Lane-Emden models [15]. In particular, Eq. (2.14) can be reparameterized as \( p = c \theta^{n+1} \) and \( \rho = \lambda \theta^n \), where \( c \) and \( \lambda \) are central pressure and central energy density, respectively, and \( n \) denotes the polytropic index. Many stellar models are led by some specific polytropic indices - neutron stars between \( n = 0.5 \) and \( n = 1 \), degenerate star cores of white dwarfs, brown dwarfs, and giant gaseous planets like Jupiter for \( n = 1.5 \), and main sequence stars such as Sun for \( n = 3 \). Moreover it can also describe a Chaplygin gas fluid by choosing \( n = -1/(\nu + 1) \) and \( \omega = -A \) where \( A > 0 \) [6, 16].

The energy density \( \rho \) assumed to be positive is given by a function of \( R \) by solving Eq. (2.13)

$$\rho = \left[ (\omega + \rho_0^{-1/n}) \left( \frac{R}{R_0} \right)^{2/n} - \omega \right]^{-n} \quad \text{(finite n)}$$

(2.15)

$$\rho = \rho_0 \left( \frac{R_0}{R} \right)^{2\omega+2} \quad \text{(perfect fluid)}$$

(2.16)

3In three-dimensional theory [17], the polytropic index \( n \) describes many sorts of known fluids – for example, it describes constant energy density for \( n = 0 \), nonrelativistic degenerate fermions for \( n = 1 \), nonrelativistic matter or radiation pressure for \( n = 2 \), linear (perfect) fluid for \( n \to \infty \).
Note that for finite and positive $n$’s, the energy density diverges at

$$\mathcal{R}_s \equiv \mathcal{R}_0 \left( \frac{\omega}{\omega + \rho_0^{-1/n}} \right)^{n/2}. \quad (2.17)$$

Moreover, for even $n$’s, the density always has positive values while for odd $n$’s, they are negative when $\mathcal{R} < \mathcal{R}_s$ and positive when $\mathcal{R} > \mathcal{R}_s$, respectively. In this relation, $\mathcal{R}_0$ is an arbitrary position of the shell and $\rho_0$ is the energy density of the matter located at $\mathcal{R} = \mathcal{R}_0$. Combining Eqs. (2.11) and (2.12) leads to an equation of motion for the shell’s edge as

$$\dot{\mathcal{R}}^2 + V_{eff}(\mathcal{R}) = 0, \quad (2.18)$$

where the effective potential $V_{eff}$ is given by

$$V_{eff}(\mathcal{R}) = \frac{1}{2} (F_+ + F_-) - \frac{(F_+ - F_-)^2}{(\kappa^2 \rho \mathcal{R})^2} - \frac{1}{16}(\kappa^2 \rho \mathcal{R})^2. \quad (2.19)$$

By defining $x \equiv \mathcal{R}/\mathcal{R}_0$, $\tau \equiv t/\mathcal{R}_0$, and using Eq. (2.15), the effective potential can be simplified to

$$\dot{x}^2 + V_{eff}(x) = 0, \quad (2.20)$$

where the effective potential for finite $n$’s is

$$V_{eff}(x) = 1 - \frac{\mu_+}{x} - \frac{\mu_- \rho_0^2}{4x^4}(x^{2/n} - c_n)^2 - \frac{\rho_0^2 x^2}{(x^{2/n} - c_n)^2n}. \quad (2.21)$$

with

$$\mu_{\pm} = \frac{M_+ \pm M_-}{\mathcal{R}_0}, \quad c_n = \frac{\omega}{\omega + \rho_0^{-1/n}}, \quad \rho_0 = \frac{4(\omega + \rho_0^{-1/n})^n}{\kappa^2 \mathcal{R}_0}. \quad (2.22)$$

For perfect fluids, it is given by

$$V_{eff}(x) = 1 - \frac{\mu_+}{x} - \frac{\rho_0^2}{x^{2+4\omega}} - \frac{\mu_- x^{4\omega}}{4\rho_0^2}, \quad (2.23)$$

where

$$\rho_0^2 = \frac{1}{16} \kappa^2 \rho_0^2 \mathcal{R}_0^{2-8\omega}. \quad (2.24)$$

It is easy to check that the effective potential negatively diverges at $x = x_s \equiv c_n^{n/2}$ and $x = 0$ ($c_n > 0$) or only at $x = 0$ ($c_n < 0$). When $x \to \infty$ it approaches $V_{eff}(x) \to v_f \equiv 1 - \mu_+ \rho_0^2/4$ for finite $n$’s, which is depicted in Fig. 1. Whereas for an infinite $n$ (perfect fluid case), the behaviors of the potential are classified by four options depending on the value of the equation of state parameter $\omega$, which is shown in Fig. 2. Note that the energy density diverges at $x = x_s$ and $x = 0$ ($x = 0$ only) leads to the divergence of the effective potential at those points for finite $n$ (infinite $n$). So the intrinsic Ricci scalar on the shell

$$R^\mu_{\mu}[\Sigma] = \frac{2}{x^2} \left[ 2x \dot{x} + \dot{x}^2 + \frac{1}{\mathcal{R}_0^2} \right] = \frac{2}{x^2} \left[ -x \frac{d}{dx} V_{eff}(x) - V_{eff}(x) + \frac{1}{\mathcal{R}_0^2} \right]. \quad (2.25)$$
$V_{\text{eff}} = x s v_f \equiv 1 - \mu^2 - \bar{\kappa}^2_0 / 4$ for finite $v_f > 0$,
$v_f = 0$
$v_f < 0$

Figure 1: A Cartoon view of the effective potential for collapsing polytropic shell with finite $n$’s to either the Schwarzschild black hole to a naked singularity.

Figure 2: A Cartoon view of the effective potential for collapsing perfect fluid shell to either the Schwarzschild black hole to a naked singularity.
is also singular at the same points.

For perfect fluids, there are four options in the collapse scenario, depending on the equation-of-state parameter and shell’s initial data. First, as seen in Fig. 2(a), for \( \omega > 0 \) or \( \omega < -1/2 \), the effective potential is convex up, so the collapse scenario has three choices: (i) there are two roots – if the shell’s initial position is in the left-hand side, it always collapses to a point at \( x = 0 \), creating a curvature singularity while the shell’s position is in the right-hand side, it expands indefinitely, (ii) there is one degenerate root – the shell is in the metastable root, which leads to either collapse or expansion, (iii) there are no roots – the shell either collapses to a point or expands indefinitely, depending on its initial data. In above all cases, if we assume the black hole in the exterior spacetime and flat interior spacetimes, the black hole can be formed by collapsing perfect fluid shells.

For \( -1/2 < \omega < 0 \), the effective potential negatively diverges at \( x = 0 \) while it diverges positively at \( x \to \infty \), which implies that the shell always collapses to a point regardless of its initial condition – i.e., the black hole can be formed all the time, regardless of its initial data. For the values of \( \omega = 0 \) and \( \omega = -1/2 \), the behavior of the potential has the same pattern in that it diverges at \( x = 0 \) and approaches a finite value at asymptotic region. Therefore, if there is one root, the shell always collapses to a point regardless of its initial condition while if not, the shell either collapses or expands indefinitely, depending on the initial data, which implies the critical phenomenon of the black hole formation in that this behavior depends on the magnitude between black hole masses and the shell’s initial masses.

On the other hand, for the case of finite \( n \)’s, one of interesting features is that the singularity at \( x = 0 \) is shifted to finite position of \( x = x_s \), so the spherical singular shell is formed after the collapse when \( c_n > 0 \) while for \( c_n < 0 \), the behavior of the potential is similar to Fig. 2(b) and (d).

As seen in Eq. (2.18), the equation of motion is highly nonlinear to obtain an exact solution but its numerical solution can be easily obtained since it is the first-order differential equation however the effective potential is extremely complicated. Some numerical behaviors for a few parameter sets are shown in Fig. 3 by using the fourth order Runge-Kutta method.

The previous analysis for the Chaplygin gas shell also shows the similar behaviors as studied before. The crucial point of this collapsing behavior is whether or not the matter is pressureless. As shown in three-dimensional case \([5, 6]\), the pressure of collapsing matter can stretch the singularity to finite size since the curvature and the energy density on the shell diverges at that point. This singularity can be screened by an event horizon for an appropriate initial data while it is equivalently possible to be exposed outside the geometry, depending on initial condition, which might require a certain modification of formalism beyond the classical theory of gravity.

\(^4\)For \( c_n = 0 \), we shall not discuss this case here since it describes the case of dust shell (\( \omega = 0 \)).
Figure 3: Some behaviors of numerical solutions of collapsing shells (LHS: $c_n = 1$ & RHS: $c_n = -1$) and specific parameters are set to be $M_+ = 1$, $\rho_0 = 1$, $n = 1$. The red dot-line is for the interior black hole ($M_+ = 0.5$) while the blue dot-line is for the interior flat geometries ($M_+ = 0.0$).

3 Gravitational Collapse of the Smeared Source

In this section we consider the noncommutative Schwarzschild black hole formation from collapsing shells, including the new matter related to the noncommutativity as well as the polytropic matter on the shell. This is the generalization of the shell collapsing on the noncommutative background. Notice that to solve the junction equation we need new matter different with the polytropic matter on the shell, whose energy density depending on the noncommutative parameter $\Theta$ and the shell position $R$ will be calculated. Another interesting point of this section is to check whether the noncommutative effect of the background can get rid of the naked singular shell.

By introducing the smeared energy density $\rho_{\pm}$ to the bulk, we assume that the metrics in both regions, $\mathcal{V}_+$ (outside the shell) and $\mathcal{V}_-$ (inside the shell) are given by

$$ (ds)^2_{\mathcal{V}_\pm} = -F_\pm(R)dt^2 + \frac{dR^2}{F_\pm(R)} + R^2(d\theta^2 + \sin^2 \theta d\phi^2), $$

where $F_+$ and $F_-$ are exterior and interior metrics that can be assumed to be an appropriate solution of the Einstein equation, so we set

$$ F_\pm(R) = 1 - \frac{2m_\pm(R)}{R} = 1 - \frac{4M_\pm}{R\sqrt{\pi}} \gamma \left( \frac{3}{2}, \frac{R^2}{4\Theta} \right), $$

where $\gamma(3/2, x^2)$ is an incomplete gamma function defined as

$$ \gamma \left( \frac{1}{2}, x^2 \right) \equiv 2 \int_0^x e^{-t^2} dt = \sqrt{\pi} \text{erf}(x) $$

$$ \gamma(a + 1, x) = a\gamma(a, x) - x^a e^{-x}. $$

$^5$See Ref. [13] for more details on the gamma function and its properties.
Note that the metric (3.2) becomes the standard Schwarzschild metric in the limit of $R/\sqrt{\Theta} \to \infty$ (commutative limit). On the other hand, the surface energy-momentum tensor (2.9) should be modified by $\rho \to \rho_s + \rho_\theta$ and $p \to p_s + p_\perp$, where $\rho_s$ and $p_s$ are the energy density and pressure of the polytropic-type matter in the previous section. Then, the junction equation becomes

\[(\beta_+ - \beta_-) + \frac{\kappa^2}{2} \rho R = 0 \tag{3.4}\]
\[\frac{d}{dR}(\beta_+ - \beta_-) + \frac{1}{R}(\beta_+ - \beta_-) - \kappa^2 \rho = 0. \tag{3.5}\]

The equation for the energy density related to the noncommutativity is reduced to

\[\frac{d\rho_\theta}{d\log R} + 2(\rho_\theta + p_\perp) = 0. \tag{3.6}\]

Using the fact that the pressure $p_\perp$ can be rewritten as

\[p_\perp = -\left(1 - \frac{R^2}{4\Theta}\right) \rho_\theta, \tag{3.7}\]

the energy density becomes

\[\rho_\theta = \bar{\rho} e^{-\frac{\pi^2 - x^2}{\Theta}}, \tag{3.8}\]

where $\bar{\rho}$ is the value at $R = R_0$. Note that the above energy density is different with that of the polytropic matter, which means that for describing the shell collapse in the noncommutative background consistently we should introduce new matter smeared in the hypersurface. Interestingly, when $R/\sqrt{\Theta} \to 0$, this matter behaves like the cosmological constant. Plugging this into Eq. (3.4) yields the equation describing the shell’s motion

\[\dot{R}^2 + V_{eff}(R) = 0, \tag{3.9}\]

with the effective potential

\[V_{eff}(R) = \frac{1}{2}(F_+ + F_-) - \frac{(F_+ - F_-)^2}{(\kappa^2 R)^2} \rho^{-2} - \frac{1}{16}(\kappa^2 R)^2 \rho^2, \tag{3.10}\]

where $\rho$ is given by

\[\rho = \left[\left(\omega + \rho_0^{-1/n}\right) \left(\frac{R}{R_0}\right)^{2/n} - \omega\right]^{-n} + \bar{\rho} e^{-\frac{\pi^2 - x^2}{\Theta}}. \tag{3.11}\]

In the case of $n \to \infty$, the energy density becomes

\[\rho = \rho_0 + \bar{\rho} e^{-\frac{\pi^2 - x^2}{\Theta}}. \tag{3.12}\]

Here if we set $R/R_0 \equiv x$ and $t/R_0 \equiv \tau$ as before, then one finds the energy density

\[\rho = \left[(\omega + \rho_0^{-1/n})x^{2/n} - \omega\right]^{-n} + \bar{\rho} e^{-\frac{\pi^2}{x^2}(x^2-1)} \tag{3.13}\]
and the behavior of the effective potential with Eqs. (3.2) and (3.13) is similar but slightly modified by the noncommutative effect of $\Theta$. For the generic polytropic matters with $c_n > 0$, there exists a singular point at $x = x_s \equiv \omega^{n/2}/(\omega + \rho_0^{-1/n})^{n/2}$ while it is not the case for the perfect fluid of $\rho_s$. Due to the finiteness of the second term in Eq. (3.13), the noncommutative correction does not affect the typical shape of the effective potential coming from the polytropic matter. In other words, the noncommutative energy density cannot remove the singularity caused by the polytropic matters. The effective potential for some specific parameters and difference due to the noncommutative effect are depicted in Fig. 4.

![Figure 4](image.png)

**Figure 4:** Some behaviors of the effective potential in varying $\omega$ (LHS). The RHS is focused view for the inner convex of the effective potential with respect to the value of $\Theta$.

The main effect of $\Theta$ parameter (noncommutative effect of collapsing matter) is shown in RHS of Fig. 4. As seen in Fig. 2(a), it is found that the convex of potential in $0 < x < x_s$ has always negative value for some numerical checks in the commutative case. Provided we introduce the $\Theta$-effect, then the potential barrier in this region appears as shown in RHS of Fig. 4. Anyway, as mentioned previously, the $\Theta$-effect cannot alter the shell’s collapsing behavior. Though the noncommutative effect does not get rid of the singularity of the effective potential, it may be helpful to make a black hole solution before the shell touches the singular point. To see this, we first calculate the black hole horizon $x_h$ such that

\[
1 - \frac{4M}{x_h R_0 \sqrt{\pi}} \gamma \left( \frac{3}{2}, \frac{x_h^2 R_0^2}{4 \Theta} \right) = 0,
\]

in which the horizon depends on $\Theta$. For example, for some fixed numeric values (here $G = 1$, $M_+ = 1$, $M_- = 0$,
\( \rho_0 = 1, \omega = 1, n = 1 \), the position of the horizon is shifted as
\[
x_h = \begin{cases} 
0.3211830425 & \text{for } \Theta = 0.01 \\
0.6040244364 & \text{for } \Theta = 0.05 \\
1.2899484640 & \text{for } \Theta = 0.50.
\end{cases}
\] (3.15)

Since \( x_s = \omega^{n/2}/(\omega + \rho_0^{1/n})^{n/2} = 0.7071067810 \) for the parameters we are considering above, some collapsing scenarios are possible:

- For the choice of \( \Theta = 0.01 \), since the black hole horizon is only located between the origin and the potential wall, if the initial shell starts collapsing in \( x_h < x_0 < x_{wall} \), it always forms a noncommutative Schwarzschild geometry. However, the shell’s initial position is outside the wall, the collapsing shell will always encounter the naked singularity at \( x = x_s \).

- For the choice of \( \Theta = 0.05 \), since the black hole horizon is located at \( 0 < x_h < x_s \), the shell starting from \( x_h < x_0 < x_s \) either forms a black hole or encounters a singularity at \( x = x_s \) – this picture cannot be allowed since its initial geometry already includes a singular point in the manifold. Another option is that the shell’s initial position is located at \( x_s < x_0 < x_{max} \), for which the shell can form a naked singularity at the final stage of the collapse.

- For the choice of \( \Theta = 0.50 \) case, the black hole horizon is outside the singular point at \( x_s \), so the shell can form a black hole as the shell collapses. All these scenarios are depicted in Fig. 5.

![Figure 5: Some collapse scenarios with respect to noncommutative parameters and initial data of the shell's position.](image)

As shown in Fig. 5(c), as the noncommutative effect becomes strong, the black hole horizon has a large value so that the shell’s collapse can make the black hole before touching the singular point. The equation of motion Eq. (3.9) with Eqs. (3.10) and (3.13) is highly nonlinear and complicated even if it is simply the first-order differential equation. Instead of its analytic solution, we can estimate numerical solutions for some appropriate initial data and
fixed parameters. Fig. 6 shows that the collapsing shell of the polytropic matter and the smeared gravitational source can form a noncommutative Schwarzschild black hole.

Figure 6: Numerical estimates of the shell’s collapsing solution for the noncommutative black hole formation. The LHS is for $\omega = 0$ (dust) and the RHS is for $\omega = 1$ (polytropic pressure). Some parameters are set by $M_+ = 1$, $M_- = 0$, $G = 1$, $\rho_0 = 1$, $n = 1$, $\Theta = 0.5$, and $R_0 = 1$.

Note that the possibility of a negative energy density that violates the weak energy condition can be also taken into account. Indeed the study of such configurations has been done in Ref. [13], in which the freely falling dust cloud with a negative energy density collapses to the black hole with the negative ADM mass and different topological structure. More precisely, the exterior topological geometry is set to be the solution of Einstein equations with cosmological constant $\Lambda < 0$ with mass parameter $M > 0$ and topology of $R^2 \times H_g$, where $H_g$ is a two-surface of genus $g \geq 2$. Then a straightforward computation reveals $M_{ADM} = -M(g - 1) < 0$, which shows that the resulting geometry from freely falling dust cloud of negative energy density should be a negative mass black hole. See Refs. [8, 13, 14] for more details on the topological black holes including the negative mass black hole.

Concerning the negative energy density [13], the aspect seems to be unchanged comparing to the case of the positive energy density. Indeed, a short glance at the effective potential in Eqs. (2.21) and (2.23) shows that the initial energy density is quadratic, implying that the negative energy density does not alter the previous result of the case of $\rho_0 > 0$. However, the total energy system must be positive, which implies that

$$E_\Theta = \int dV \rho_\Theta = \int dV \frac{M}{(4\pi \Theta)^{3/2}} e^{-\frac{q^2}{4\Theta}} > 0,$$

(3.16)
clearly leading $M > 0$. The main reason that Ref. [13] can allow the negative mass is the only topological reason of the geometry since the system can produce the positive total energy with the combination of the negative mass and
the topological genus as mentioned above. For this reason, the system we are considering in this paper cannot accept the negative mass case.

4 Discussions

In this paper, we investigated the gravitational shell collapse problem with the smeared gravitational source in the context of the mild noncommutative modification of the space. The collapsing matters with the most generic polytropic equation of state and the Chaplygin gas were considered, which leads to formation of either black hole or naked singularity, depending on its initial data. Moreover, the existence of the nontrivial pressure yields the stretch of singularity, which produces the singular shell with a finite size, regardless of an event horizon. Provided it is cloaked by an event horizon, the collapsing shell can form a black hole while it forms a naked singular shell if not – this crucially depends on the initial condition. This feature has been already observed in the lower dimensional theories of gravity [5, 6]. When considering the noncommutative effect, we showed that though the noncommutativity cannot remove the singularity caused by the polytropic matter, it makes the collapsing shell becomes a black hole before touching the singular point.

As studied before, the introduction of the smeared gravitational source and the corresponding slight modification of the Schwarzschild geometry do not offer a crucial prescription for the treatment of naked singularities in the shell collapse formalism. Considering the smeared source in the presence of Θ-effect can be regarded as so-called “semi-noncommutative” approach since the matter part of Einstein’s equation has Θ-effect while the shell formalism from the geometry parts of Einstein’s equation has no such terms. Therefore, in order to deal with this issue completely, we, somehow, should modify the gravitational action and have a modified shell formalism. One of possible ways of doing this is to add some higher curvature terms to the Einstein-Hilbert action, which can alter the junction equation. This scenario can be viable for three- or five-dimensional cases because we can add the gravitational Chern-Simons (GCS) terms or Gauss-Bonnet terms to the Einstein-Hilbert action. In four dimensions, no viable expression of higher curvature terms is known due to the topological reason. In this sense, a recent suggestion of the Horava-Lifshits gravity [19, 20] might draw a great interest since the theory has highly-powered curvature terms in the action. Another way to exit the issue is to consider the another alternate theory of gravitation such as the scalar tensor theory of Brans-Dicke type, which clearly alters the junction equations on the shell. In these all possibilities, it is worthwhile to study whether or not the singularity issue is resolved, which is being in progress. However, above all possibilities might be temporal prescriptions of handling a singularity in that those corrections only hold in the context of the effective theory. The most fundamental solution to the singularity and the cosmic censorship should be suggested through the full quantum theory of gravity.
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