Decay properties of solutions to the Cauchy problem for the scalar conservation law with nonlinearly degenerate viscosity

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Abstract

In this paper, we study the decay rate in time to solutions of the Cauchy problem for the one-dimensional viscous conservation law where the far field states are prescribed. Especially, we deal with the case that the flux function which is convex and also the viscosity is a nonlinearly degenerate one ($p$-Laplacian type viscosity). As the corresponding Riemann problem admits a Riemann solution as the constant state or the single rarefaction wave, it has already been proved by Matsumura-Nishihara that the solution to the Cauchy problem tends toward the constant state or the single rarefaction wave as the time goes to infinity. We investigate that the decay rate in time of the corresponding solutions. Furthermore, we also investigate that the decay rate in time of the solution for the higher order derivative. These are the first result concerning the asymptotic decay of the solutions to the Cauchy problem of the scalar conservation law with nonlinear viscosity. The proof is given by $L^1$, $L^2$-energy and time-weighted $L^q$-energy methods.

Keywords: viscous conservation law, decay estimates, asymptotic behavior, nonlinearly degenerate viscosity, rarefaction wave

1. Introduction and main theorems

In this paper, we shall consider the asymptotic behavior of solutions for the one-dimensional scalar conservation law with a nonlinearly degenerate viscosity
\(p\)-Laplace type viscosity with \(p > 1\)

\[
\begin{aligned}
\partial_t u + \partial_x (f(u)) &= \mu \partial_x \left( |\partial_x u|^{p-1} \partial_x u \right) \quad (t > 0, x \in \mathbb{R}), \\
u(0, x) &= u_0(x) \quad (x \in \mathbb{R}), \\
\lim_{x \to \pm \infty} u(t, x) &= u_\pm \quad (t \geq 0).
\end{aligned}
\]  

(1.1)

Here, \(u = u(t, x)\) denotes the unknown function of \(t > 0\) and \(x \in \mathbb{R}\), the so-called conserved quantity, \(f = f(u)\) is the flux function depending only on \(u\), \(\mu\) is the viscosity coefficient, \(u_0\) is the given initial data, and constants \(u_\pm \in \mathbb{R}\) are the prescribed far field states. We suppose the given flux \(f = f(u)\) is a \(C^3\)-function satisfying \(f(0) = f'(0) = 0\), \(\mu\) is a positive constant and far field states \(u_\pm\) satisfy \(u_- < u_+\) without loss of generality.

At first, we shall motivate the physical meaning to the nonlinearly degenerate viscosity and review the related models concerning with the Cauchy problem (1.1). It is known that if \(p = 1\) and \(f(u) = \frac{1}{2}u^2\), the equation in our problem (1.1) becomes the viscous Burgers equation:

\[\partial_t u + u \partial_x u = \mu \partial_x^2 u.\]

In particular, the viscosity term \(\mu \partial_x^2 u\) stands for Newtonian fluid. The Newtonian fluid is what satisfies the relation between the strain rate \(\partial_x u_i + \partial_x u_j\) \((\partial_x u,\) for one-dimensional case) is linear, that is,

\[\tau = \mu \left( \partial_x u_i + \partial_x u_j \right)\]  \text{or}  \[\tau = \mu \partial_x u.\]

On the other hand, if a fluid satisfies the relation between the strain rate and the stress is nonlinear (for example, polymers, viscoelastic or viscoplastic flow), the fluid is non-Newtonian fluid, such as, blood, honey, butter, whipped cream, suspension, and so on. The typical nonlinearity in the non-Newtonian fluid is the power-law fluid (cf. [19]), that is,

\[\tau = \mu \left( \partial_x u_i + \partial_x u_j \right)^p\]  \text{or}  \[\tau = \mu \left( \partial_x u \right)^p.\]

Ladyženskaja [13] has proposed a new mathematical model for the incompressible Navier-Stokes equation with the power-law type nonlinear viscosity (see also [3]). The Ladyženskaja equation is the following:

\[\partial_t u_i + u_j \partial_x u_i = -\partial_x p + \partial_x \left( \mu_0 + \mu_1 \left( \sum_{i,j} \left( \partial_x u_i \right)^2 \right)^{\frac{r}{\gamma}} \partial_x u_i \right) + f_i\]

where \(i = 1, 2\), or \(i = 1, 2, 3\). In particular, if \(\mu_0 = 0, \mu_1 > 0\) and \(r > -1\), this model is said to be the Ostwald-de Waele model:

\[\partial_t u_i + u_j \partial_x u_i = -\partial_x p + \partial_x \left( \mu |\partial_x u|^{r} \partial_x u_i \right) + f_i\]
where \(|Du| := \left(\sum_{i,j} (\partial x_i u_j)^2 \right)^{\frac{1}{2}}\), and \(i = 1, 2,\) or \(i = 1, 2, 3\). In this sense, our viscosity \(\mu \partial_x \left( |\partial_x u|^{p-1} \partial_x u \right)\) should be called the Ostwald-de Waele type viscosity.

We are interested in the asymptotic behavior and its precise estimates in time of the global solution to our problem (1.1). It can be expected that the large-time behavior is closely related to the weak solution (“Riemann solution”) of the corresponding Riemann problem (cf. [15], [31]) for the non-viscous hyperbolic part of (1.1):

\[
\begin{align*}
\partial_t u + \partial_x (f(u)) &= 0 \quad (t > 0, x \in \mathbb{R}), \\
u(0, x) &= u_0^R(x) \quad (x \in \mathbb{R}),
\end{align*}
\]

where \(u_0^R\) is the Riemann data defined by

\[
u_0^R(x) = u_0^R(x; u_-, u_+) := \begin{cases} 
  u_- & (x < 0), \\
  u_+ & (x > 0).
\end{cases}
\]

In fact, for \(p = 1\) in (1.1), the usual linear viscosity case:

\[
\begin{align*}
\partial_t u + \partial_x (f(u)) &= \mu \partial_x^2 u \quad (t > 0, x \in \mathbb{R}), \\
u(0, x) &= u_0(x) \quad (x \in \mathbb{R}), \\
\lim_{x \to \pm \infty} u(t, x) &= u_{\pm} \quad (t \geq 0),
\end{align*}
\]

when the smooth flux function \(f\) is genuinely nonlinear on the whole space \(\mathbb{R}\), i.e., \(f''(u) \neq 0\ (u \in \mathbb{R})\), Il’in-Oleinik [11] showed the following: if \(f''(u) > 0\ (u \in \mathbb{R})\), that is, the Riemann solution consists of a single rarefaction wave solution, the global solution in time of the Cauchy problem (1.3) tends toward the rarefaction wave; if \(f''(u) < 0\ (u \in \mathbb{R})\), that is, the Riemann solution consists of a single shock wave solution, the global solution of the Cauchy problem (1.3) does the corresponding smooth traveling wave solution (“viscous shock wave”) of (1.3) with a spacial shift (cf. [10]). Hattori-Nishihara [8] also proved that the asymptotic decay rate in time, of the solution toward the single rarefaction wave, is \((1 + t)^{-\frac{1}{2}}(1 - \frac{1}{p})\) in the \(L^p\)-norm \((1 \leq p \leq \infty)\) for large \(t > 0\) (see also [4], [7], [20]). More generally, in the case of the flux functions which are not uniformly genuinely nonlinear, when the Riemann solution consists of a single shock wave satisfying Oleinik’s shock condition, Matsumura-Nishihara [22] showed the asymptotic stability of the corresponding viscous shock wave. Moreover, Matsumura-Yoshida [23] considered the circumstances where the Riemann solution generically forms a pattern of multiple nonlinear waves which consists of rarefaction waves and waves of contact discontinuity (refer to [10]), and investigated that the case where the flux function \(f\) is smooth and genuinely nonlinear (that is, \(f\) is convex function or concave function) on the whole \(\mathbb{R}\) except a finite
interval \( I := (a, b) \subset \mathbb{R} \), and linearly degenerate on \( I \), that is,
\[
\begin{cases}
  f''(u) > 0 & (u \in (-\infty, a] \cup [b, +\infty)), \\
  f''(u) = 0 & (u \in (a, b)).
\end{cases}
\tag{1.4}
\]
Under the conditions (1.4), they proved the unique global solution in time to the Cauchy problem (1.3) tends uniformly in space toward the multiwave pattern of the combination of the viscous contact wave and the rarefaction waves as the time goes to infinity. Yoshida \[32\] also obtained that the precise decay properties for the asymptotics toward the multiwave pattern. In fact, owing to the time goes to infinity. Yoshida \[32\] also obtained that the precise decay rate in time is \( (1 + t)^{-\frac{1}{4}(1 - \frac{1}{2}) + \epsilon} \) for any \( \epsilon > 0 \) in the \( L^\infty \)-norm if the initial perturbation from the corresponding asymptotics satisfies \( H^1 \). Furthermore, if the perturbation satisfies \( H^1 \cap L^1 \), the decay rate in time is \( (1 + t)^{-\frac{1}{4}(1 - \frac{1}{2}) + \epsilon} \) for any \( \epsilon > 0 \) in the \( L^p \)-norm \( (1 \leq p < +\infty) \) and \( (1 + t)^{-\frac{1}{2} + \epsilon} \) for any \( \epsilon > 0 \) in the \( L^\infty \)-norm.

For \( p > 1 \), there are few results for the asymptotic behavior for the problem (1.1) (the related problems are studied in \[3, 24, 25\] and so on). In the case where the flux function is genuinely nonlinear on the whole space \( \mathbb{R} \), Matsumura-Nishihara \[22\] proved that if the far field states satisfy \( u_- = u_+ =: \tilde{u} \), then the solution tends toward the constant state \( \tilde{u} \), and if the far field states \( u_- < u_+ \), then the solution tends toward a single rarefaction wave. In the case where the flux function satisfies (1.4), Yoshida \[33\] recently showed that the asymptotics which tends toward the multiwave pattern of the combination of the viscous contact wave constructed by the Barenblatt-Kompaneece-Zel’dovič solution (see also \[2, 3, 12\]) of the porous medium equation, and the rarefaction waves. However, the decay rate of any asymptotics of the problem (1.1) has been not known.

The aim of the present paper is to obtain the precise time-decay estimates for the asymptotics of the previous study in \[21\].

**Stability Theorem 1.1** (Matsumura-Nishihara \[21\]). Let the flux function \( f \in C^3(\mathbb{R}) \) satisfy \( f(0) = f'(0) = 0 \) and \( f''(u) > 0 \) \( (u \in \mathbb{R}) \), and the far field states \( u_- = u_+ = \tilde{u} \). Assume that the initial data satisfies \( u_0 - \tilde{u} \in L^2 \) and \( \partial_x u_0 \in L^{p+1} \). Then the Cauchy problem (1.1) with \( p > 1 \) has a unique global weak solution in time \( u = u(t, x) \) satisfying
\[
\begin{align*}
  u - \tilde{u} & \in C^0([0, \infty); L^2) \cap L^\infty(\mathbb{R}^+; L^2), \\
  \partial_x u & \in L^\infty(\mathbb{R}^+; L^{p+1}) \cap L^{p+1}(\mathbb{R}^+_t \times \mathbb{R}_x) \cap L^{p+2}(\mathbb{R}^+_t \times \mathbb{R}_x), \\
  \partial_x \left( |\partial_x u|^{p-1} \partial_x u \right) & \in L^2(\mathbb{R}^+_t \times \mathbb{R}_x)
\end{align*}
\]
and the asymptotic behavior
\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |u(t, x) - \tilde{u}| = 0.
\]
Moreover, \( \partial_x^2 u \in L^2(\mathbb{R}^+_t \times \mathbb{R}_x) \) when \( 1 < p \leq \frac{3}{2} \) provided that \( \partial_x u_0 \in L^{3-p} \).
Stability Theorem 1.2 (Matsumura-Nishihara [21]). Let the flux function $f \in C^3(\mathbb{R})$ satisfy $f(0) = f'(0) = 0$ and $f''(u) > 0$ ($u \in \mathbb{R}$), and the far field states $u_l < u_r$. Assume that the initial data satisfies $u_0 - u_0^R \in L^2$ and $\partial_x u_0 \in L^{p+1}$. Then the Cauchy problem (1.1) with $p > 1$ has a unique global weak solution in time $u = u(t, x)$ satisfying

\[
\begin{align*}
&u - u_0^R \in C^0([0, \infty); L^2) \cap L^\infty(\mathbb{R}^+; L^2), \\
&\partial_x u \in L^\infty(\mathbb{R}^+; L^{p+1}) \cap L^{p+1}(\mathbb{R}_t^+ \times \mathbb{R}_x) \cap L^{p+2}(\mathbb{R}_t^+ \times \mathbb{R}_x), \\
&\partial_x (|\partial_x u|^{p-1} \partial_x u) \in L^2(\mathbb{R}_t^+ \times \mathbb{R}_x)
\end{align*}
\]

and the asymptotic behavior

\[
\limsup_{t \to \infty} \sup_{x \in \mathbb{R}} \left| u(t, x) - u^*\left(\frac{x}{t}; u_-, u_+\right) \right| = 0,
\]

where, the rarefaction wave $u^*$ which connects the far field states $u_-$ and $u_+$ is explicitly given by

\[
u^* = u^*\left(\frac{x}{t}; u_-, u_+\right) = \begin{cases}
u_- & (x \leq \lambda(u_-) t), \\
(\lambda)^{-1}\left(\frac{x}{t}\right) & (\lambda(u_-) t \leq x \leq \lambda(u_+) t), \\
u_+ & (x \geq \lambda(u_+) t),
\end{cases}
\]

where $\lambda(u) := f'(u)$. Moreover, $\partial_x^2 u \in L^2(\mathbb{R}_t^+ \times \mathbb{R}_x)$ when $1 < p \leq \frac{4}{3}$ provided that $\partial_x u_0 \in L^{3-p}$.

Now we are ready to state our main results.

Theorem 1.1 (Main Theorem I). Under the same assumptions in Stability Theorem 1.1, the unique global solution in time $u$ of the Cauchy problem (1.1) satisfying

\[
\begin{align*}
&u - \tilde{u} \in C^0([0, \infty); L^2) \cap L^\infty(\mathbb{R}^+; L^2), \\
&\partial_x u \in L^\infty(\mathbb{R}^+; L^{p+1}) \cap L^{p+1}(\mathbb{R}_t^+ \times \mathbb{R}_x) \cap L^{p+2}(\mathbb{R}_t^+ \times \mathbb{R}_x), \\
&\partial_x (|\partial_x u|^{p-1} \partial_x u) \in L^2(\mathbb{R}_t^+ \times \mathbb{R}_x)
\end{align*}
\]

satisfies the following time-decay estimates

\[
\begin{align*}
&\|u(t) - \tilde{u}\|_{L^q} \leq C(p, q, u_0) (1 + t)^{-\frac{1}{q} - \frac{1}{p+1}}, \\
&\|u(t) - \tilde{u}\|_{L^\infty} \leq C(\epsilon, p, q, u_0, \partial_x u_0) (1 + t)^{-\frac{1}{q} - \frac{1}{p+1} + \epsilon}
\end{align*}
\]

for $q \in [2, \infty)$ and any $\epsilon > 0$.

Theorem 1.2 (Main Theorem II). Under the same assumptions in Theorem 1.1, if the initial data further satisfies $u_0 - \tilde{u} \in L^1$, then it holds that the unique
global solution in time $u$ of the Cauchy problem (1.1) satisfies the following time-decay estimates

\[
\begin{align*}
\| u(t) - \tilde{u} \|_{L^q} &\leq C(p, q, u_0) (1 + t)^{-\frac{q}{p}(1 - \frac{r}{q})}, \\
\| u(t) - \tilde{u} \|_{L^\infty} &\leq C(\epsilon, p, q, u_0, \partial_x u_0) (1 + t)^{-\frac{1}{2r} + \epsilon}
\end{align*}
\]

for $q \in [1, \infty)$ and any $\epsilon > 0$. Furthermore, the solution satisfies the following time-decay estimate for the higher order derivative

\[
\| \partial_x u(t) \|_{L^{p+1}} \leq C(\epsilon, p, r, u_0, \partial_x u_0) (1 + t)^{-\frac{p+2r}{2(p+1)}} + \epsilon
\]

for any $\epsilon > 0$.

**Theorem 1.3** (Main Theorem III). Under the same assumptions in Theorem 1.2, if the initial data further satisfies $\partial_x u_0 \in L^{r+1}(r > p)$, then it holds that the unique global solution in time $u$ of the Cauchy problem (1.1) satisfies the following time-decay estimate for the higher order derivative

\[
\| \partial_x u(t) \|_{L^{r+1}} \leq C(\epsilon, p, r, u_0, \partial_x u_0) (1 + t)^{-\frac{r+2}{2(r+1)}} + \epsilon
\]

for any $\epsilon > 0$.

**Theorem 1.4** (Main Theorem IV). Under the same assumptions in Stability Theorem 1.2, the unique global solution in time $u$ of the Cauchy problem (1.1) satisfying

\[
\begin{align*}
&u - u_0^R \in C^0([0, \infty); L^2) \cap L^\infty(R^+; L^2), \\
&\partial_x u \in L^\infty(R^+; L^{p+1}) \cap L^{p+1}(R^+_t \times R_x) \cap L^{p+2}(R^+_t \times R_x), \\
&\partial_x \left( |\partial_x u|^{p-1} \partial_x u \right) \in L^2(R^+_t \times R_x)
\end{align*}
\]

satisfies the following time-decay estimates

\[
\begin{align*}
\| u(t) - u^R(t, u_-, u_+) \|_{L^q} &\leq C(p, q, u_0) (1 + t)^{-\frac{q}{p}(1 - \frac{r}{q})}, \\
\| u(t) - u^R(t, u_-, u_+) \|_{L^\infty} &\leq C(\epsilon, p, q, u_0, \partial_x u_0) (1 + t)^{-\frac{1}{2r} + \epsilon}
\end{align*}
\]

for $q \in [2, \infty)$ and any $\epsilon > 0$.

**Theorem 1.5** (Main Theorem V). Under the same assumptions in Theorem 1.4, if the initial data further satisfies $u_0 - u_0^R \in L^1$, then it holds that the unique global solution in time $u$ of the Cauchy problem (1.1) satisfies the following time-decay estimates

\[
\begin{align*}
\| u(t) - u^R(t, u_-, u_+) \|_{L^q} &\leq C(p, q, u_0) (1 + t)^{-\frac{q}{p}(1 - \frac{r}{q})}, \\
\| u(t) - u^R(t, u_-, u_+) \|_{L^\infty} &\leq C(\epsilon, p, q, u_0, \partial_x u_0) (1 + t)^{-\frac{1}{2r} + \epsilon}
\end{align*}
\]
for \( q \in [1, \infty) \) and any \( \epsilon > 0 \). Furthermore, the solution satisfies the following time-decay estimates for the higher order derivative

\[
\left\| \partial_x u(t) \right\|_{L^{p+1}}, \left\| \partial_x u(t) - \partial_x u^r \left( \frac{t}{T}; u_-, u_+ \right) \right\|_{L^{p+1}} \leq \begin{cases} C(p, u_0, \partial_x u_0)(1 + t)^{-\frac{2p+1}{2(2p+1)(p+1)}} & \left( 1 < p < \frac{2 + \sqrt{22}}{6} \right), \\ C(\epsilon, p, u_0, \partial_x u_0)(1 + t)^{-\frac{3}{2(2p+1)(3p-2)} + \epsilon} & \left( \frac{2 + \sqrt{22}}{6} \leq p \right) \end{cases}
\]

for any \( \epsilon > 0 \).

**Theorem 1.6 (Main Theorem VI).** Under the same assumptions in Theorem 1.5, if the initial data further satisfies \( \partial_x u_0 \in L^{r+1} (r > p) \), then it holds that the unique global solution in time \( u \) of the Cauchy problem (1.1) satisfies the following time-decay estimates for the higher order derivative

\[
\left\| \partial_x u(t) \right\|_{L^{r+1}}, \left\| \partial_x u(t) - \partial_x u^r \left( \frac{t}{T}; u_-, u_+ \right) \right\|_{L^{r+1}} \leq \begin{cases} C(p, r, u_0, \partial_x u_0)(1 + t)^{-\frac{2p^2+r^2}{(3p-1)(3p+1)}} & \left( 1 < p < \frac{2 + \sqrt{22}}{6}, r > p > \frac{18p^3 - 17p^2 - 16p - 3}{2(2p+1)} \right), \\ C(\epsilon, p, r, u_0, \partial_x u_0)(1 + t)^{-\frac{p+2r}{2p+1} + \epsilon} & \left( \frac{2 + \sqrt{22}}{6} \leq p \right) \end{cases}
\]

for any \( \epsilon > 0 \).

This paper is organized as follows. In Section 2, we shall prepare the basic properties of the rarefaction wave. In Section 3, we reformulate the problem in terms of the deviation from the asymptotic state (similarly in [23, 32], [33]), that is, the single rarefaction wave. Following the arguments in [21], we also prepare some uniform boundedness and energy estimates of the deviation as the solution to the reformulated problem. In order to obtain the time-decay estimates (Theorem 1.4 and Theorem 1.5), in Section 4 and Section 5, we establish the uniform energy estimates in time by using a very technical time-weighted energy method. In Section 6, we prove the time-decay \( L^{r+1} \)-estimate for the higher order derivative, Theorem 1.6. We shall finally discuss the time-decay rates in our main theorems comparing with those for a Cauchy problem of the symplest \( p \)-Laplacian evolution equation without convective term in Section 7.

**Some Notation.** We denote by \( C \) generic positive constants unless they need to be distinguished. In particular, use \( C(\alpha, \beta, \cdots) \) or \( C_{\alpha, \beta, \cdots} \) when we
emphasize the dependency on $\alpha, \beta, \cdots$. Use $\mathbb{R}^+$ as $\mathbb{R}^+ := (0, \infty)$, and the symbol \( \lor \) as 
\[ \forall \in \mathbb{R}^+ := (0, \infty), \quad \sup\{ \forall \in \mathbb{R} \mid |x| \leq 1 \}, \quad \int_{-\infty}^{\infty} \forall(x) \, dx = 1, \]
and $\rho_{\delta} * f$ denote the convolution. For function spaces, $L^p = L^p(\mathbb{R})$ and $H^k = H^k(\mathbb{R})$ denote the usual Lebesgue space and $k$-th order Sobolev space on the whole space $\mathbb{R}$ with norms $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^k}$, respectively. We also define the bounded $C^m$-class $B^m$ as follows 
\[ f \in B^m(\Omega) \iff f \in C^m(\Omega), \quad \sup_{\Omega} \sum_{k=0}^{m} \| D^k f \| < \infty \]
for $m < \infty$ and 
\[ f \in B^\infty(\Omega) \iff \forall n \in \mathbb{N}, \quad f \in C^n(\Omega), \quad \sup_{\Omega} \sum_{k=0}^{n} \| D^k f \| < \infty \]
where $\Omega \subset \mathbb{R}^d$ and $D^k$ denote the all of $k$-th order derivatives.

2. Preliminaries

In this section, we shall arrange the two lemmas concerned with the basic properties of the rarefaction wave for accomplishing the proof of our main theorems. Since the rarefaction wave $u^r$ is not smooth enough, we need some smooth approximated one as in the previous results in [7], [18], [20], [23]. We start with the well-known arguments on $u^r$ and the method of constructing its smooth approximation. We first consider the rarefaction wave solution $w^r$ to the Riemann problem for the non-viscous Burgers equation

\[
\begin{cases}
\partial_t w + \partial_x \left( \frac{1}{2} w^2 \right) = 0 & (t > 0, \ x \in \mathbb{R}), \\
w(0,x) = w^R_0(x; w_-, w_+) := \begin{cases}
w_+ & (x > 0), \\
w_- & (x < 0),
\end{cases}
\end{cases}
\tag{2.1}
\]

where $w_{\pm} \in \mathbb{R} (w_- < w_+)$ are the prescribed far field states. The unique global weak solution $w = w^r \left( \frac{t}{\tau}; w_-, w_+ \right)$ of (2.1) is explicitly given by

\[
w^r \left( \frac{x}{\tau}; w_-, w_+ \right) := \begin{cases}
w_- & (x \leq w_- t), \\
\frac{x}{\tau} & (w_- t \leq x \leq w_+ t), \\
w_+ & (x \geq w_+ t),
\end{cases}
\tag{2.2}
\]
Next, under the condition $f''(u) > 0 \ (u \in \mathbb{R})$ and $u_- < u_+$, the rarefaction wave solution $u = u^r \left( \frac{x}{t} ; u_-, u_+ \right)$ of the Riemann problem (1.2) for hyperbolic conservation law is exactly given by

$$u^r \left( \frac{x}{t} ; u_-, u_+ \right) = (\lambda)^{-1} \left( w^r \left( \frac{x}{t} ; \lambda_- , \lambda_+ \right) \right) \quad (2.3)$$

which is nothing but (1.6), where $\lambda_{\pm} := \lambda(u_{\pm}) = f'(u_{\pm})$. We define a smooth approximation of $w^r \left( \frac{x}{t} ; w_-, w_+ \right)$ by the unique classical solution

$$w = w(t, x ; w_-, w_+) \in \mathcal{D}^\infty ( [0, \infty) \times \mathbb{R})$$

to the Cauchy problem for the following non-viscous Burgers equation

$$\begin{align*}
\partial_t w + \partial_x \left( \frac{1}{2} w^2 \right) &= 0 \quad \text{for} \ t > 0, \ x \in \mathbb{R}, \\
w(0,x) &= w_0(x) := \frac{w_- + w_+}{2} + \frac{w_+ - w_-}{2} \tanh x \quad \text{for} \ x \in \mathbb{R},
\end{align*} \quad (2.4)$$

By using the method of characteristics, we get the following formula

$$\begin{align*}
w(t, x) &= w_0(x_0(t, x)) = \frac{\lambda_- + \lambda_+}{2} + \frac{\lambda_+ - \lambda_-}{2} \tanh(x_0(t, x)), \\
x &= x_0(t, x) + w_0(x_0(t, x)) t.
\end{align*} \quad (2.5)$$

We also note the assumption of the flux function $f$ to be $\lambda'(u) \left( = \frac{d^2 f}{du^2}(u) \right) > 0$.

Now we summarize the results for the smooth approximation $w(t, x; w_-, w_+)$ in the next lemma. Since the proof is given by the direct calculation as in [20], we omit it.

**Lemma 2.1.** Assume that the far field states satisfy $w_- < w_+$. Then the classical solution $w(t, x) = w(t, x; w_-, w_+)$ given by (2.4) satisfies the following properties:

1. $w_- < w(t, x) < w_+$ and $\partial_x w(t, x) > 0 \quad (t > 0, \ x \in \mathbb{R})$.
2. For any $1 \leq q \leq \infty$, there exists a positive constant $C_q$ such that

$$\| \partial_x w(t) \|_{L^q} \leq C_q (1 + t)^{-1 + \frac{1}{q}} \quad (t \geq 0),$$
$$\| \partial_x^2 w(t) \|_{L^q} \leq C_q (1 + t)^{-1} \quad (t \geq 0).$$

3. $\lim_{t \to \infty} \sup_{x \in \mathbb{R}} \left| w(t, x) - w^r \left( \frac{x}{t} \right) \right| = 0$.

We define the approximation for the rarefaction wave $u^r \left( \frac{x}{t} ; u_-, u_+ \right)$ by

$$U^r(t, x ; u_-, u_+) := (\lambda)^{-1} \left( w(t, x ; \lambda_- , \lambda_+) \right). \quad (2.6)$$

Then we have the next lemma as in the previous works (cf. [7], [18], [20], [23]).
Lemma 2.2. Assume that the far field states satisfy $u_- < u_+$, and the flux function $f \in C^3(\mathbb{R})$, $f''(u) > 0$ ($u \in [u_- , u_+]$). Then we have the following properties:

(1) $U^r(t, x)$ defined by (2.6) is the unique $C^2$-global solution in space-time of the Cauchy problem

\[
\begin{aligned}
\partial_t U^r + \partial_x \left( f(U^r) \right) &= 0 \quad (t > 0, x \in \mathbb{R}), \\
U^r(0, x) &= (\lambda)^{-1} \left( \frac{\lambda_- + \lambda_+}{2} + \frac{\lambda_+ - \lambda_-}{2} \tanh x \right) \quad (x \in \mathbb{R}), \\
\lim_{x \to \pm \infty} U^r(t, x) &= u_\pm \quad (t \geq 0).
\end{aligned}
\]

(2) $u_- < U^r(t, x) < u_+$ and $\partial_x U^r(t, x) > 0$ ($t > 0, x \in \mathbb{R}$).

(3) For any $1 \leq q \leq \infty$, there exists a positive constant $C_q$ such that

\[
\begin{aligned}
\| \partial_t U^r(t) \|_{L^q} &\leq C_q (1 + t)^{-1 + \frac{1}{q}} \quad (t \geq 0), \\
\| \partial^2_t U^r(t) \|_{L^q} &\leq C_q (1 + t)^{-1} \quad (t \geq 0).
\end{aligned}
\]

(4) $\lim_{t \to \infty} \sup_{x \in \mathbb{R}} \left| U^r(t, x) - u^r \left( \frac{x}{t} \right) \right| = 0$.

(5) For any $\epsilon \in (0, 1)$, there exists a positive constant $C_\epsilon$ such that

\[
\left| U^r(t, x) - u_+ \right| \leq C_\epsilon (1 + t)^{-1 + \epsilon} e^{-\epsilon|x-x_+|} \quad (t \geq 0, x \geq \lambda_+ t).
\]

(6) For any $\epsilon \in (0, 1)$, there exists a positive constant $C_\epsilon$ such that

\[
\left| U^r(t, x) - u_- \right| \leq C_\epsilon (1 + t)^{-1 + \epsilon} e^{-\epsilon|x-x_-|} \quad (t \geq 0, x \leq \lambda_- t).
\]

(7) For any $\epsilon \in (0, 1)$, there exists a positive constant $C_\epsilon$ such that

\[
\left| U^r(t, x) - u^r \left( \frac{x}{t} \right) \right| \leq C_\epsilon (1 + t)^{-1 + \epsilon} \quad (t \geq 1, \lambda_- t \leq x \leq \lambda_+ t).
\]

(8) For any $(\epsilon, q) \in (0, 1) \times [1, \infty]$, there exists a positive constant $C_{\epsilon,q}$ such that

\[
\left\| U^r(t, \cdot) - u^r \left( \frac{\cdot}{t} \right) \right\|_{L^q} \leq C_{\epsilon,q} (1 + t)^{-1 + \frac{1}{q} + \epsilon} \quad (t \geq 0).
\]

Because the proofs of (1) to (4) are given in [20], (5) to (7) are in [23] and (8) is in [32], we omit the proofs here.

3. Reformulation of the problem

In this section, we reformulate our Cauchy problem (1.1) in terms of the deviation from the asymptotic state, the single rarefaction wave. We first should note by Lemma 2.2, the asymptotic state $u^r \left( \frac{x}{t} ; u_-, u_+ \right)$ can be replaced by

\[
U^r(t, x ; u_-, u_+).
\]
In fact, from Lemma 2.1 (especially (8)), it follows that for any $\epsilon > 0$
\[
\left\| U^r(t;\cdot;u_-,u_+) - u^r\left(\frac{\partial}{\partial t};u_-,u_+\right)\right\|_{L^q}
\leq C_{\epsilon,q}(1+t)^{-\frac{1}{2}+\epsilon} \quad (t \geq 0; 1 \leq q \leq \infty).
\]
Then it is noted that $U^r$ is monotonically increasing and approximately satisfies
the equation of (1.1) as
\[
\partial_t U^r + \partial_x (f(U^r)) = 0. \tag{3.1}
\]
Now putting
\[
u(t, x) = U^r(t, x) + \phi(t, x) \tag{3.2}
\]
and using (3.1), we can reformulate the problem (1.1) in terms of the deviation
$\phi$ from $U^r$ as
\[
\left\{
\begin{array}{l}
\partial_t \phi + \partial_x (f(U^r + \phi) - f(U^r)) \\
- \mu \partial_x \left( |\partial_x U^r + \partial_x \phi|^{p-1} (\partial_x U^r + \partial_x \phi) - |\partial_x U^r|^{p-1} \partial_x U^r \right) \\
\phi(0, x) = \phi_0(x) := u_0(x) - U^r(0, x) \quad (x \in \mathbb{R}).
\end{array}\right. \tag{3.3}
\]
Then we look for the global solution in time
\[
\phi \in C^0\left( [0, \infty); L^2 \right) \cap L^\infty\left( \mathbb{R}^+; L^2 \right)
\]
with
\[
\partial_x \phi \in L^\infty\left( \mathbb{R}^+; L^{p+1} \right) \cap L^{p+1}\left( \mathbb{R}_t^+ \times \mathbb{R}_x \right).
\]
Here we note that the assumptions on $u_0$ and the fact $\partial_x U^r(0, \cdot) \in L^{p+1}$ imply
$\phi_0 \in L^2$ with $\partial_x \phi_0 \in L^{p+1}$. Then the corresponding our main theorems for $\phi$
should prove are as follows.

**Theorem 3.1.** Assume that the flux function $f \in C^4([0, \infty))$ satisfies $f(0) = f'(0) = 0$
and $f''(u) > 0$ ($u \in \mathbb{R}$), the far field states $u_- < 0 < u_+$, and the
initial data $\phi_0 \in L^2$ and $\partial_x u_0 \in L^{p+1}$. Then, the unique global solution in time
$\phi$ of the Cauchy problem (3.3) satisfying
\[
\left\{
\begin{array}{l}
\phi \in C^0\left( [0, \infty); L^2 \right) \cap L^\infty\left( \mathbb{R}^+; L^2 \right), \\
\partial_x \phi \in L^\infty\left( \mathbb{R}^+; L^{p+1} \right) \cap L^{p+1}\left( \mathbb{R}_t^+ \times \mathbb{R}_x \right),
\end{array}\right.
\]

\[
\left\{
\begin{array}{l}
\partial_x (U^r + \phi) \in L^\infty\left( \mathbb{R}^+; L^{p+1} \right) \cap L^{p+1}\left( \mathbb{R}_t^+ \times \mathbb{R}_x \right) \cap L^{p+2}\left( \mathbb{R}_t^+ \times \mathbb{R}_x \right), \\
\partial_x \left( |\partial_x (U^r + \phi)|^{p-1} \partial_x (U^r + \phi) \right) \in L^2\left( \mathbb{R}_t^+ \times \mathbb{R}_x \right)
\end{array}\right.
\]

and
\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |\phi(t, x)| = 0
\]
satisfies the following time-decay estimates

\[
\begin{align*}
\| \phi(t) \|_{L^q} &\leq C(p, q, \phi_0) (1 + t)^{-\frac{1}{p}(1 - \frac{3}{q})}, \\
\| \phi(t) \|_{L^\infty} &\leq C(\epsilon, p, q, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{p} + \epsilon}
\end{align*}
\]

for \( q \in [2, \infty) \) and any \( \epsilon > 0 \).

**Theorem 3.2.** Under the same assumptions in Theorem 3.1, if the initial data further satisfies \( \phi_0 \in L^1 \), then it holds that the unique global solution in time \( \phi \) of the Cauchy problem (3.3) satisfies the following time-decay estimates

\[
\begin{align*}
\| \phi(t) \|_{L^q} &\leq C(p, q, \phi_0) (1 + t)^{-\frac{1}{p}(1 - \frac{3}{q})}, \\
\| \phi(t) \|_{L^\infty} &\leq C(\epsilon, p, q, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{p} + \epsilon}
\end{align*}
\]

for \( q \in [1, \infty) \) and any \( \epsilon > 0 \). Furthermore, the solution satisfies the following time-decay estimates for the higher order derivative

\[
\| \partial_x u(t) \|_{L^{p+1}}, \quad \| \partial_x \phi(t) \|_{L^{p+1}}
\]

\[
\begin{align*}
&\leq \left\{ 
\begin{array}{ll}
C(\epsilon, p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{p} + \epsilon} & \left( 1 < p \leq \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{(p + 1)(3p - 2)}{3} \epsilon} \right), \\
C(\epsilon, p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{3}{2r + 1} + \epsilon} & \left( \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{(p + 1)(3p - 2)}{3} \epsilon} < p \right)
\end{array}\right.
\]

for any \( 0 < \epsilon \ll 1 \).

**Theorem 3.3.** Under the same assumptions in Theorem 3.2, if the initial data further satisfies \( \partial_x u_0 \in L^{r+1} (r > p) \), then it holds that the unique global solution in time \( \phi \) of the Cauchy problem (3.3) satisfies the following time-decay estimates for the higher order derivative

\[
\| \partial_x u(t) \|_{L^{r+1}}, \quad \| \partial_x \phi(t) \|_{L^{r+1}}
\]

\[
\begin{align*}
&\leq \left\{ 
\begin{array}{ll}
C(\epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^{-\frac{2p^3 + 3p^2 + r}{(3p + 2)(r + 1)}} \epsilon, \, r > p > \frac{18p^3 - 17p^2 - 16p - 3}{2(2p + 1)}, \\
C(\epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^{-\frac{p^2 + 2p}{2p(3p - 2)(r + 1)}} + \epsilon & \left( \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{p(3p - 2)(r + 1)}{2(r - p + 1)}} \epsilon < p \right)
\end{array}\right.
\]

for any \( 0 < \epsilon \ll 1 \).
In order to accomplish the proofs of Theorem 3.1, Theorem 3.2 and Theorem 3.3, we will need some estimates about boundedness of the perturbation $\phi$ and $u$. We shall arrange some lemmas for them.

By using the maximum principle (cf. [10], [11]), we first have the following uniform boundedness of the perturbation $\phi$ (and also $u$), that is,

**Lemma 3.1 (uniform boundedness).** It holds that

$$\sup_{t \in [0, \infty), x \in \mathbb{R}} |\phi(t, x)| \leq \|\phi_0\|_{L^\infty} + 2 \left( |u_-| + |u_+| \right),$$

(3.4)

$$\sup_{t \in [0, \infty), x \in \mathbb{R}} |u(t, x)| \leq \|\phi_0\|_{L^\infty} + 2 \left( |u_-| + |u_+| \right) + |u_-| \vee |u_+| =: \tilde{C}.$$  

(3.5)

Secondly, we also have the uniform estimates of $\phi$ as follows (for the proof of it, see [21]).

**Lemma 3.2 (uniform estimates).** The unique global solution in time $\phi$ of the Cauchy problem (3.3) satisfies the following uniform energy inequalities

$$\|\phi(t)\|_{L^2}^2 + \int_0^\infty \|\partial_x \phi(t)\|_{L^{p+1}}^{p+1} \, dt \leq C_p \left( \|\phi_0\|_{L^2} \right),$$

(3.6)

$$\|\partial_x u(t)\|_{L^{p+1}}^{p+1} + \int_0^\infty \int_{-\infty}^{\infty} \left| \partial_x u(t) \right|^{2(p-1)} \left( \partial_x^2 u(t) \right)^2 \, dx \, dt \leq C_p \left( \|\phi_0\|_{L^2}, \|\partial_x u_0\|_{L^{p+1}} \right),$$

(3.7)

$$\int_0^\infty \|\partial_x u(t)\|_{L^{p+2}}^{p+2} \, dt \leq C_p \left( \|\phi_0\|_{L^2}, \|\partial_x u_0\|_{L^{p+1}} \right)$$  

(3.8)

for $t \in [0, \infty)$.

4. **Time-decay estimates with $2 \leq q \leq \infty$**

In this section, we show the time-decay estimates with $2 \leq q \leq \infty$ (not assuming $L^1$-integrability to the initial perturbation), that is, Theorem 3.1. To do that, we shall obtain the time-weighted $L^q$-energy estimates to $\phi$ with $2 \leq q < \infty$ (cf. [32]).

**Proposition 4.1.** Suppose the same assumptions in Theorem 3.1. For any $q \in [2, \infty)$, there exist positive constants $\alpha$ and $C_{\alpha,p,q}$, such that the unique
Lemma 4.2. Assume \( (1) \) that for any \( \varphi \), any \( q > 1 \), any \( \alpha, p, q, \phi \), and any \( t > 0 \), there exists a positive constant \( C \) such that

\[
(1 + t)^\alpha \| \varphi(t) \|_{L^q}^q + \int_0^t (1 + \tau)^\alpha \int_\infty^{-\infty} |\phi|^q \partial_x U^r \, dx \, d\tau
\]

\[
\quad + \int_0^t (1 + \tau)^\alpha \int_\infty^{-\infty} |\phi|^q (\partial_x \phi)^2 \left( |\partial_x \phi|^{p-1} + |\partial_x U^r|^{p-1} \right) \, dx \, d\tau
\]

\[
\quad + \int_0^t (1 + \tau)^\alpha \int_\infty^{-\infty} |\phi|^q \left| \partial_x \phi + \partial_x U^r \right|^{p-1} - |\partial_x U^r|^{p-1} \right| \times (\partial_x \phi + \partial_x U^r)^2 - (\partial_x U^r)^2 \, dx \, d\tau
\]

\[
\leq C_{\alpha,p,q} \| \varphi_0 \|_{L^q}^p + C(\alpha, p, q, \phi_0) (1 + t)^{\alpha - \frac{2}{pq+1}} \quad (t \geq 0).
\]

The proof of Proposition 4.1 is provided by the following two lemmas.

Lemma 4.1. For any \( 2 \leq q < \infty \), there exist positive constants \( \alpha \) and \( C_q \) such that

\[
(1 + t)^\alpha \| \varphi(t) \|_{L^q}^q + q (q - 1) \int_0^t (1 + \tau)^\alpha
\]

\[
\quad \times \int_\infty^{-\infty} \int_\infty^{-\infty} \left( \lambda(\tilde{U} + \eta) - \lambda(\tilde{U}) \right) |\eta|^q - \partial_x U^r \, dx \, d\tau
\]

\[
\quad + C_q \int_0^t (1 + \tau)^\alpha \int_\infty^{-\infty} |\phi|^q (\partial_x \phi)^2 \left( |\partial_x \phi|^{p-1} + |\partial_x U^r|^{p-1} \right) \, dx \, d\tau
\]

\[
\quad + C_q \int_0^t (1 + \tau)^\alpha \int_\infty^{-\infty} |\phi|^q \left| \partial_x \phi + \partial_x U^r \right|^{p-2} - |\partial_x U^r|^{p-1} \right| \times (\partial_x \phi + \partial_x U^r)^2 - (\partial_x U^r)^2 \, dx \, d\tau
\]

\[
\leq \| \varphi_0 \|_{L^q}^q + \alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \varphi(\tau) \|_{L^p}^p \, d\tau
\]

\[
\quad + \mu \int_0^t (1 + \tau)^\alpha \| \varphi(\tau) \|_{L^p}^{p-1}
\]

\[
\quad \times \left| \partial_x \left( |\partial_x U^r|^{p-2} \partial_x U^r \right) \right| \, d\tau \quad (t \geq 0).
\]

Lemma 4.2. Assume \( p > 1 \) and \( 2 \leq q < \infty \). We have the following interpolation inequalities.

1. For any \( 2 \leq r < \infty \), there exists a positive constant \( C_{p,q,r} \) such that

\[
\| \varphi(t) \|_{L^r} \leq C_{p,q,r} \left( \int_{-\infty}^{\infty} |\phi|^r \, dx \right)^{\frac{p-1}{(p+1)r}}
\]

\[
\quad \times \left( \int_{-\infty}^{\infty} |\phi|^q \left| \partial_x \phi \right|^{p+1} \, dx \right)^{\frac{p-2}{(p+1)r}} \quad (t \geq 0).
\]
(2) There exists a positive constant $C_{p,q}$ such that
\[
\| \phi(t) \|_{L^\infty} \leq C_{p,q} \left( \int_{-\infty}^{\infty} |\phi|^2 \, dx \right)^{\frac{p}{2p+q-1}} \times \left( \int_{-\infty}^{\infty} |\phi|^{q-2} |\partial_x \phi|^{p+1} \, dx \right)^{\frac{1}{2p+q-1}} \quad (t \geq 0).
\]

In what follows, we first prove Lemma 4.1 and Lemma 4.2, and finally give the proof of Proposition 4.1.

**Proof of Lemma 4.1.** Multiplying the equation in (3.3) by $|\phi|^{q-2} \phi$ with $2 \leq q < \infty$, we obtain the divergence form
\[
\partial_t \left( \frac{1}{q} |\phi|^q \right) + \partial_x \left( |\phi|^{q-2} \phi (f(U^r + \phi) - f(U^r)) \right) + \partial_x \left( (q-1) \int_0^\phi (f(U^r + \eta) - f(U^r)) |\eta|^{q-2} \, d\eta \right) + \partial_x \left( -\mu |\phi|^{q-2} \phi \times \left( |\partial_x U^r + \partial_x \phi|^{p-1} (\partial_x U^r + \partial_x \phi) - |\partial_x U^r|^{p-1} (\partial_x U^r) \right) \right)
\]
\[
+ (q-1) \int_0^\phi (\lambda(U^r + \eta) - \lambda(U^r)) |\eta|^{q-2} \, d\eta \left( \partial_x U^r \right)
\]
\[
+ \mu (q-1) |\phi|^{q-2} \partial_x \phi \times \left( |\partial_x U^r + \partial_x \phi|^{p-1} (\partial_x U^r + \partial_x \phi) - |\partial_x U^r|^{p-1} (\partial_x U^r) \right)
\]
\[
= \mu |\phi|^{q-2} \phi \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right).
\]

Integrating (4.3) with respect to $x$, we have
\[
\frac{1}{q} \frac{d}{dt} \| \phi(t) \|_{L^q}^q
\]
\[
+ \int_{-\infty}^{\infty} (q-1) \int_0^\phi \left( \lambda(U^r + \eta) - \lambda(U^r) \right) |\eta|^{q-2} \, d\eta \left( \partial_x U^r \right) \, dx
\]
\[
+ \mu (q-1) \int_{-\infty}^{\infty} |\phi|^{q-2} \partial_x \phi \times \left( |\partial_x U^r + \partial_x \phi|^{p-1} (\partial_x U^r + \partial_x \phi) - |\partial_x U^r|^{p-1} (\partial_x U^r) \right) \, dx
\]
\[
= \mu \int_{-\infty}^{\infty} |\phi|^{q-2} \phi \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right) \, dx.
\]
Thus, multiplying the inequality by \((1 + t)x\) with \(\alpha > 0\) and integrating over \((0, t)\) with respect to the time, we complete the proof of Lemma 4.1.

**Proof of Lemma 4.2.** Noting that \(\phi(t, \cdot) \in L^2\) and \(\partial_t \phi(t, \cdot) \in L^{p+1}\) imply \(\lim_{x \to \pm \infty} \phi(t, x) = 0\) for \(t \geq 0\), we have

\[
|\phi| \leq s \int_{-\infty}^{\infty} |\phi|^{\alpha-1} \left| \partial_x \phi \right| dx. \tag{4.7}
\]

By the Cauchy-Schwarz inequality, we have

\[
|\phi| \leq s \left( \int_{-\infty}^{\infty} \left| \phi \right|^{\alpha-1} \left| \partial_x \phi \right|^2 dx \right)^{\frac{1}{\alpha}} \tag{4.8}
\]

Taking \(s = \frac{3p+q-1}{p+1}\), we get

\[
\| \phi \|_{L^{\frac{3p+q}{p+1}}} \leq \frac{3p+q-1}{p+1} \left( \int_{-\infty}^{\infty} |\phi|^2 dx \right)^{\frac{1}{p+1}} \times \left( \int_{-\infty}^{\infty} |\phi|^{q-2} \left| \partial_x \phi \right|^{p+1} dx \right)^{\frac{1}{p+1}}, \tag{4.9}
\]
and
\[ \| \phi \|_{L^r} \leq \| \phi \|_{L^\infty}^2 \| \phi \|_{L^2}^2 \]
\[ \leq \left( \frac{3p + q - 1}{p + 1} \right)^{\frac{(p+1)(r-2)}{3p+q-1}} \left( \int_{-\infty}^{\infty} |\phi|^2 \, dx \right)^{\frac{p+r+q-1}{3p+q-1}} \times \left( \int_{-\infty}^{\infty} |\phi|^{q-2} |\partial_x \phi|^{p+1} \, dx \right)^{\frac{r-2}{3p+q-1}}. \] (4.10)

Thus, we complete the proof of Lemma 4.2.

**Proof of Proposition 4.1.** By using Lemma 4.1 and Lemma 4.2, we shall estimate the second term, the third term and the fourth term on the right-hand side of (4.2) as follows: for any \( \epsilon > 0 \),
\[
\alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \phi(\tau) \|_{L^q}^q \, d\tau \\
\leq C_{\alpha,p,q} \int_0^t (1 + \tau)^{\alpha - 1} \left( \int_{-\infty}^{\infty} |\phi|^{q-2} |\partial_x \phi|^{p+1} \, dx \right)^{\frac{q-2}{3p+q-1}} \times \left( \int_{-\infty}^{\infty} |\phi|^2 \, dx \right)^{\frac{p+r+q-1}{3p+q-1}} \, d\tau \\
\leq \int_0^t \left( (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\phi|^{q-2} |\partial_x \phi|^{p+1} \, dx \right)^{\frac{q-2}{3p+q-1}} \times C_{\alpha,p,q} (1 + \tau)^{\alpha - 1 - \frac{\alpha(q-2)}{3p+q-1}} \| \phi(\tau) \|_{L^2}^{\frac{2(p+q+q-1)}{3p+q-1}} \, d\tau \\
\leq \epsilon \int_0^t (1 + \tau)^\alpha \left( \int_{-\infty}^{\infty} |\phi|^{q-2} |\partial_x \phi|^{p+1} \, dx \right) \, d\tau \\
+ C_{\alpha,p,q}(\epsilon) \int_0^t (1 + \tau)^{\alpha - \frac{3p+q-1}{3p+1}} \| \phi(\tau) \|_{L^2}^{\frac{2(p+q+q-1)}{3p+1}} \, d\tau,
\] (4.11)
Substituting (4.11) and (4.12) into (4.2), we have

\[ q \int_0^t (1 + \tau)^\alpha \| \phi(\tau) \|_{L^p}^{q-1} \left| \int_{-\infty}^\infty |\phi|^{q-2} \phi \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right) \right| dx \, d\tau \]

\[ \leq C_{p,q} \int_0^t (1 + \tau)^\alpha \left( \int_{-\infty}^\infty |\phi|^{q-1} \left| \partial_x \phi \right|^{p+1} dx \right) \]

\[ \times \left( \int_{-\infty}^\infty |\phi|^2 dx \right)^{\frac{q(q-1)}{2(p+q-1)}} \left\| \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right)(\tau) \right\|_{L^1} \, d\tau \]

\[ \leq \epsilon \int_0^t (1 + \tau)^\alpha \left( \int_{-\infty}^\infty |\phi|^{q-1} \left| \partial_x \phi \right|^{p+1} dx \right) \, d\tau \]

\[ + C_{p,q}(\epsilon) \int_0^t (1 + \tau)^\alpha \| \phi(\tau) \|_{L^2}^{\frac{2(q-1)}{p+q-1}} \left\| \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right)(\tau) \right\|_{L^1} \, d\tau. \]

(4.12)

Substituting (4.11) and (4.12) into (4.2), we have

\[ (1 + t)^\alpha \| \phi(t) \|_{L^q}^q + \int_0^t (1 + \tau)^\alpha \int_{-\infty}^\infty |\phi|^q \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right) \, dx \, d\tau \]

\[ + \int_0^t (1 + \tau)^\alpha \int_{-\infty}^\infty |\phi|^{q-2} \left( \left| \partial_x \phi \right|^{p+1} + \left| \partial_x U^r \right|^{p-1} \right) \, dx \, d\tau \]

\[ + \int_0^t (1 + \tau)^\alpha \int_{-\infty}^\infty |\phi|^{q-2} \left| \left| \partial_x \phi + \partial_x U^r \right|^{p-1} - \left| \partial_x U^r \right|^{p-1} \right| \]

\[ \times \left( \left| \partial_x \phi + \partial_x U^r \right|^2 - \left| \partial_x U^r \right|^2 \right) \, dx \, d\tau \]

\[ \leq C_{\alpha,p,q} \| \phi_0 \|_{L^q}^q + C_{\alpha,p,q} \int_0^t (1 + \tau)^{\alpha - \frac{3p+q-1}{p+q+1}} \| \phi(\tau) \|_{L^2}^{\frac{2(q-1)}{p+q-1}} \, d\tau \]

\[ + C_{p,q} \int_0^t (1 + \tau)^\alpha \| \phi(\tau) \|_{L^2}^{\frac{2(q-1)}{p+q-1}} \]

\[ \times \left\| \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right)(\tau) \right\|_{L^1} \, d\tau. \]

(4.13)

By using the $L^2$-boundedness of $\phi$, (3.6), and

\[ \left\| \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right)(\tau) \right\|_{L^1} \leq p \| \partial_x U^r(t) \|_{L^\infty}^{p-1} \| \partial_x^2 U^r(t) \|_{L^1} \]

\[ \leq C_p (1 + t)^{-p}, \]

(4.14)
we estimate the each terms on the right-hand side of (4.13) as follows:

\[ C_{\alpha,p,q} \int_0^t (1 + \tau)^{\alpha - \frac{3p+q-1}{3p+1}} \| \phi(\tau) \|^\frac{2p+q}{3p+1} \|_{L^2} \, d\tau \]

\leq C_{\alpha,p,q} \left( C_p(\phi_0) \right)^\frac{2p+q}{3p+1} \int_0^t (1 + \tau)^{\alpha - \frac{3p+q-1}{3p+1}} \, d\tau \tag{4.15}

\leq C_{\alpha,p,q} \left( C_p(\phi_0) \right)^\frac{2p+q}{3p+1} (1 + t)^{\alpha - \frac{3p+q-1}{3p+1}},

\[ C_{p,q} \int_0^t (1 + \tau)^{\alpha} \| \phi(\tau) \|_{L^2}^{\frac{2}{p(q-1)}} \left| \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right)(\tau) \right|^{\frac{3p+q-1}{3p}} \, d\tau \]

\leq C_{p,q} \left( C_p(\phi_0) \right)^\frac{1}{2(q-1)} \int_0^t (1 + \tau)^{\alpha - \frac{3p+q-1}{3}} \, d\tau \tag{4.16}

\leq C_{p,q} \left( C_p(\phi_0) \right)^\frac{1}{2(q-1)} (1 + t)^{\alpha - \frac{3p+q-4}{3}}.

Substituting (4.15) and (4.16) into (4.13), we get (4.1). Thus the proof of Proposition 4.1 is complete. In particular, it follows that

\[ \| \phi(t) \|_{L^q} \leq C(p, q, \phi_0) (1 + t)^{- \frac{1}{3p} + \alpha - \frac{2}{3}} \] \tag{4.17}

for \( 2 \leq q < \infty \).

**Proof of Theorem 3.1.** We already have proved the decay estimate of \( \| \phi(t) \|_{L^q} \) with \( 2 \leq q < \infty \). Therefore we only show the \( L^\infty \)-estimate. We first note by Lemma 2.2 that

\[ \left| \left| \partial_x \phi(t) \right| \right|_{L^{p+1}}^{p+1} \leq \left| \left| \partial_x u(t) \right| \right|_{L^{p+1}}^{p+1} + \left| \left| \partial_x U^r(t) \right| \right|_{L^{p+1}}^{p+1} \]

\leq C \left( \| \phi_0 \|_{L^2}, \| \partial_x u_0 \|_{L^{p+1}} \right) + C_p (1 + t)^{-p}. \tag{4.18}

We use the following Gagliardo-Nirenberg inequality:

\[ \| \phi(t) \|_{L^\infty} \leq C_{q,\theta} \| \phi(t) \|_{L^p}^{1-\theta} \| \partial_x \phi(t) \|_{L^{p+1}}^\theta \] \tag{4.19}

for any \( (q, \theta) \in [1, \infty) \times (0, 1] \) satisfying

\[ \frac{p}{p+1} \theta = (1 - \theta) \frac{1}{q} \]

Substituting (4.17) and (4.18) into (4.19), we have

\[ \| \phi(t) \|_{L^\infty} \leq C(p, q, \theta, \phi_0, \partial_x u_0) (1 + t)^{- \frac{1}{3p} + \alpha - \frac{2}{3} \theta} \]

\[ \leq C(p, \theta, \phi_0, \partial_x u_0) (1 + t)^{- \frac{1}{3p} + \frac{\theta}{3p+1}} \] \tag{4.20}

for \( \theta \in (0, 1] \). Consequently, we do complete the proof of Theorem 3.1.
5. Time-decay estimates with $1 \leq q \leq \infty$

In this section, we show the time-decay estimates with $1 \leq q \leq \infty$ and time-decay estimate for the higher order derivative in the $L^{p+1}$-norm, in the case where $\phi_0 \in L^1 \cap L^2$ with $\partial_x u_0 \in L^{p+1}$, that is, Theorem 3.2. Then, we first establish the $L^1$-estimate to the solution $\phi$ of the reformulated Cauchy problem (3.3). To do that, we use the Friedrichs mollifier $\rho_\delta$, where, $\rho_\delta(\phi) := \frac{1}{\delta} \rho \left(\frac{\phi}{\delta}\right)$ with $\rho \in C^\infty_0(\mathbb{R})$, $\rho(\phi) \geq 0$ ($\phi \in \mathbb{R}$), $\text{supp}\{\rho\} \subset \{\phi \in \mathbb{R} | |\phi| \leq 1\}$, $\int_{-\infty}^{\infty} \rho(\phi) \, d\phi = 1$.

Some useful properties of the mollifier are as follows.

**Lemma 5.1.**
(i) $\lim_{\delta \to 0} (\rho_\delta * \text{sgn}) (\phi) = \text{sgn}(\phi)$ ($\phi \in \mathbb{R}$),
(ii) $\lim_{\delta \to 0} \int_0^\phi (\rho_\delta * \text{sgn}) (\eta) \, d\eta = |\phi|$ ($\phi \in \mathbb{R}$),
(iii) $(\rho_\delta * \text{sgn}) \big|_{\phi=0} = 0$,
(iv) $\frac{d}{d\phi} (\rho_\delta * \text{sgn}) (\phi) = 2 \rho_\delta(\phi) \geq 0$ ($\phi \in \mathbb{R}$),

where

$$(\rho_\delta * \text{sgn}) (\phi) := \int_{-\infty}^{\infty} \rho_\delta(\phi - y) \, \text{sgn}(y) \, dy \quad (\phi \in \mathbb{R})$$

and

$$\text{sgn}(\phi) := \begin{cases} -1 & (\phi < 0), \\ 0 & (\phi = 0), \\ 1 & (\phi > 0). \end{cases}$$

Making use of Lemma 5.1, we obtain the following $L^1$-estimate.

**Proposition 5.1.** Assume that the same assumptions in Theorem 3.2. For any $p > 1$, the unique global solution in time $\phi$ of the Cauchy problem (3.3) satisfies the following $L^1$-estimate

$$\| \phi(t) \|_{L^1} \leq \| \phi_0 \|_{L^1} + C_p (1 + t)^{-(p-1)} \quad (t \geq 0). \quad (5.1)$$

**Proof of Proposition 5.1.** Multiplying the equation in the problem (3.3) by $(\rho_\delta * \text{sgn}) (\phi)$, we obtain the divergence form
\[
\partial_t \left( \int_0^\phi (\rho_\delta \ast \text{sgn})(\eta) \, d\eta \right) \\
+ \partial_x \left( (\rho_\delta \ast \text{sgn})(\phi) \left( f(U^r + \phi) - f(U^r) \right) \right) \\
+ \partial_x \left( - \int_0^\phi (f(U^r + \eta) - f(U^r)) \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\eta) \, d\eta \right) \\
+ \partial_x \left( - \mu (\rho_\delta \ast \text{sgn})(\phi) \right) \\
\times \left( |\partial_x U^r + \partial_x \phi|^{p-1}(\partial_x U^r + \partial_x \phi) - |\partial_x U^r|^{p-1}(\partial_x U^r) \right) \\
+ \int_0^\phi (\lambda(U^r + \eta) - \lambda(U^r)) \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\eta) \, d\eta \left( \partial_x U^r \right) \\
+ \mu \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\phi) \partial_x \phi \\
\times \left( |\partial_x U^r + \partial_x \phi|^{p-1}(\partial_x U^r + \partial_x \phi) - |\partial_x U^r|^{p-1}(\partial_x U^r) \right) \\
= \mu (\rho_\delta \ast \text{sgn})(\phi) \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right). \\
\]

(5.2)

By using (4.5) and integrating (5.2) with respect to \( x \) and \( t \), we have

\[
\int_{-\infty}^\infty \int_{-\infty}^{\phi(t)} (\rho_\delta \ast \text{sgn})(\eta) \, d\eta \, dx \\
+ \int_0^t \int_{-\infty}^{\infty} \int_0^\phi (\lambda(U^r + \eta) - \lambda(U^r)) \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\eta) \, d\eta \left( \partial_x U^r \right) \, dx \, d\tau \\
+ \frac{\mu}{2} \int_0^t \int_{-\infty}^{\infty} \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\phi) \left( \partial_x \phi \right)^2 \left( |\partial_x \phi + \partial_x U^r|^{p-1} + |\partial_x U^r|^{p-1} \right) \, dx \, d\tau \\
+ \frac{\mu}{2} \int_0^t \int_{-\infty}^{\infty} \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\phi) \left| \partial_x \phi + \partial_x U^r \right|^{p-1} \left| \partial_x \phi + \partial_x U^r \right|^{p-1} \left( \partial_x \phi \right)^2 \left( \partial_x U^r \right)^2 \, dx \, d\tau \\
= \int_{-\infty}^\infty \int_0^{\phi_0} (\rho_\delta \ast \text{sgn})(\eta) \, d\phi \, dx \\
+ \mu \int_0^t \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn})(\phi) \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right) \, dx \, d\tau.
\]

(5.3)
By using Lemma 5.1, we note that for $t \in [0, \infty)$,
\[
\int_{-\infty}^{\infty} \int_{0}^{\phi(t)} \left( \lambda(U^r + \eta) - \lambda(U^r) \right) \frac{d(\rho_\delta * \text{sgn})}{d\phi}(\eta) d\eta \left( \partial_x U^r \right) dx \\
\geq 2 \left( \min_{|u| \leq C} \mathcal{X}(u) \right) \int_{-\infty}^{\infty} \left| \int_{0}^{\phi(t)} \eta \rho_\delta(\eta) d\eta \right| \left( \partial_x U^r \right) dx \geq 0,
\]
and we get
\[
\left| \int_{0}^{\phi} (\rho_\delta * \text{sgn}) (\eta) d\eta \right| \leq (\rho_\delta * \text{sgn}) (|\phi|) |\phi| \leq |\phi|.
\]
\[
\lim_{\delta \to 0} \int_{-\infty}^{\infty} \int_{0}^{\phi(t)} (\rho_\delta * \text{sgn}) (\eta) d\eta = \| \phi(t) \|_{L^1},
\]
By using (4.14), we can also get
\[
\left| \int_{-\infty}^{\infty} (\rho_\delta * \text{sgn}) (\phi) \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right) dx \right| (t) \\
\leq \left( \int_{-\infty}^{\infty} |\text{sgn}(\phi)| \left| \partial_x \left( |\partial_x U^r|^{p-1} \partial_x U^r \right) \right| dx \right) (t) \\
\leq C_p (1 + t)^{-p} \quad (t \geq 0).
\]
Then, substituting (5.8) into (5.7), we have the desired $L^1$-estimate (5.1).

Next, we show the time-weighted $L^q$-energy estimates to $\phi$.

**Proposition 5.2.** Suppose the same assumptions in Theorem 3.2. For any $q \in [1, \infty)$, there exist positive constants $\alpha$ and $C_{\alpha,p,q}$, such that the unique global solution in time $\phi$ of the Cauchy problem (3.3) satisfies the following $L^q$-energy estimate
\[
(1 + t)^\alpha \| \phi(t) \|_{L^q}^q + \int_{0}^{t} (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\phi|^q \partial_x U^r dx d\tau \\
+ \int_{0}^{t} (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\phi|^{q-2} (\partial_x \phi)^2 \left( |\partial_x \phi|^{p-1} + |\partial_x U^r|^{p-1} \right) dx d\tau \\
+ \int_{0}^{t} (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\phi|^{q-2} \left| \partial_x \phi + \partial_x U^r \right|^{p-1} \left| \partial_x U^r \right|^{p-1} dx d\tau \\
\times \left| \left( \partial_x \phi + \partial_x U^r \right)^2 - \left( \partial_x U^r \right)^2 \right| dx d\tau \\
\leq C_{\alpha,p,q} \| \phi_0 \|_{L^q}^q + C (\alpha, p, q, \phi_0) (1 + t)^{\alpha - \frac{q}{p}} \quad (t \geq 0).
\]
The proof of Proposition 5.2 is given by the following two lemmas.

**Lemma 5.2.** For any $1 \leq q < \infty$, there exist positive constants $\alpha$ and $C_q$ such that

\[
(1 + t)^\alpha \|\phi(t)\|_{L^q}^q + q(q - 1) \int_0^t (1 + \tau)^\alpha \\
\times \int_0^\infty \int_0^\phi (\lambda(U^r + \eta) - \lambda(U^r)) \|\eta\|^{q - 2} d\eta \left( \partial_x U^r \right) dxd\tau \\
+ C_q \int_0^t (1 + \tau)^\alpha \int_0^\infty |\phi|^{q - 2} \left( |\partial_x \phi|^{p - 1} + |\partial_x U^r|^{p - 1} \right) dxd\tau \\
+ C_q \int_0^t (1 + \tau)^\alpha \int_0^\infty |\phi|^{q - 2} \left| \partial_x \phi + \partial_x U^r \right|^{p - 1} - |\partial_x U^r|^{p - 1} \right| \\
\times \left( (\partial_x \phi + \partial_x U^r)^2 - (\partial_x U^r)^2 \right) dxd\tau \\
\leq \|\phi_0\|_{L^q}^q + \alpha \int_0^t (1 + \tau)^{\alpha - 1} \|\phi(\tau)\|_{L^p}^p d\tau \\
+ \mu \int_0^t (1 + \tau)^\alpha \|\phi(\tau)\|_{L^\infty}^{p - 1} \\
\times \left\| \partial_x \left( |\partial_x U^r|^{p - 1} C_q \partial_x U^r \right) \right\|_{L^1} d\tau \quad (t \geq 0).
\]

(5.10)

**Lemma 5.3.** Assume $p > 1$ and $1 \leq q < \infty$. We have the following interpolation inequalities.

1. For any $1 \leq r < \infty$, there exists a positive constant $C_{p,q,r}$ such that

\[
\|\phi(t)\|_{L^r} \leq C_{p,q,r} \left( \int_{-\infty}^{\infty} |\phi| \, dx \right)^{r \frac{p + q - 1}{2(p + q - 1)r}} \\
\times \left( \int_{-\infty}^{\infty} |\phi|^{q - 2} \left| \partial_x \phi \right|^{p + 1} \, dx \right)^{\frac{1}{2(p + q - 1)r}} \quad (t \geq 0).
\]

2. There exists a positive constant $C_{p,q}$ such that

\[
\|\phi(t)\|_{L^\infty} \leq C_{p,q} \left( \int_{-\infty}^{\infty} |\phi| \, dx \right)^{\frac{q}{2(p + q - 1)q - r}} \\
\times \left( \int_{-\infty}^{\infty} |\phi|^{q - 2} \left| \partial_x \phi \right|^{p + 1} \, dx \right)^{\frac{1}{2(p + q - 1)r}} \quad (t \geq 0).
\]

The proofs of Lemma 5.2, Lemma 5.3 and Proposition 5.2 are given in the same way as those of Lemma 4.1, Lemma 4.2 and Proposition 4.1, so we omit them.
We particularly note that we have by Proposition 5.2
\[ \| \phi(t) \|_{L^q} \leq C(p, q, \phi_0)(1 + t)^{-\frac{1}{2}\left(1 - \frac{1}{q}\right)} \] (5.11)
for \( 1 \leq q < \infty \).

We shall finally obtain the time-decay estimates for the higher order derivatives, that is, \( \partial_x \phi \) and \( \partial_x u \), and also get the \( L^\infty \)-estimate for \( \phi \).

**Proposition 5.3.** Suppose the same assumptions in Theorem 3.2. There exist positive constants \( \alpha \) and \( C_{\alpha, p} \), such that the unique global solution in time \( \phi \) of the Cauchy problem (3.3) satisfies the following \( L^{p+1} \)-energy estimate
\[
(1 + t)^\alpha \| \partial_x u(t) \|_{L^{p+1}}^{p+1} + \int_0^t (1 + \tau)^\alpha \int_{-\infty}^\infty \left| \partial_x u \right|^2 (\partial_x^2 u)^2 \, dx \, d\tau \\
+ \int_0^t (1 + \tau)^\alpha \| \partial_x u(\tau) \|_{L^{p+2}}^{p+2} \, d\tau \\
\leq C_{\alpha, p} \| \partial_x u_0 \|_{L^{p+1}}^{p+1} + C(\alpha, p, \phi_0, \partial_x u_0)(1 + t)^{\alpha - \frac{1}{p+1}} \quad (t \geq 0). \tag{5.12}
\]

To obtain Proposition 5.3, we first show the following.

**Lemma 5.4.** It follows that
\[
(1 + t)^\alpha \| \partial_x u(t) \|_{L^{p+1}}^{p+1} \\
+ \mu p^2 (p + 1) \int_0^t (1 + \tau)^\alpha \int_{-\infty}^\infty \left| \partial_x u \right|^2 (\partial_x^2 u)^2 \, dx \, d\tau \\
+ p \int_0^t (1 + \tau)^\alpha \int_{\partial_x u \geq 0} f''(u) \left| \partial_x u \right|^{p+2} \, dx \, d\tau \\
= \| \partial_x u_0 \|_{L^{p+1}}^{p+1} + \alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \partial_x u(\tau) \|_{L^{p+1}}^{p+1} \, d\tau \\
+ p \int_0^t (1 + \tau)^\alpha \int_{\partial_x u < 0} f''(u) \left| \partial_x u \right|^{p+2} \, dx \, d\tau \quad (t \geq 0). \tag{5.13}
\]

**Proof of Lemma 5.4.** Multiplying the equation in the problem (1.1), that is,
\[ \partial_t u + \partial_x (f(u)) = \mu \partial_x \left( \left| \partial_x u \right|^{p-1} \partial_x u \right) \]
by
\[ -\partial_x \left( \left| \partial_x u \right|^{p-1} \partial_x u \right), \]
we obtain the divergence form
\[ \partial_t \left( \frac{1}{p+1} |\partial_x u|^{p+1} \right) + \partial_x \left( - |\partial_x u|^{p-1} \partial_x u \cdot \partial_t u \right) \]
\[ + \partial_x \left( - \frac{p}{p+1} f'(u) |\partial_x u|^{p+1} \right) \]
\[ + \frac{p}{p+1} f''(u) |\partial_x u|^{p+1} \partial_x u \cdot \partial_t u + \mu \rho \partial_x u \right)^2 \right) = 0. \]  
(5.14)

Integrating the divergence form (5.14) with respect to \( x \), we have
\[ \frac{1}{p+1} \frac{d}{dt} \| \partial_x u(t) \|_{L^{p+1}} + \mu p^2 \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \, dx \]
\[ + \frac{p}{p+1} \int_{-\infty}^{\infty} f''(u) |\partial_x u|^{p+1} \partial_x u \, dx = 0. \]  
(5.15)

We separate the integral region to the third term on the left-hand side of (5.15) as
\[ \int_{-\infty}^{\infty} f''(u) |\partial_x u|^{p+1} \partial_x u \, dx \]
\[ = \int_{\partial_x u \geq 0} + \int_{\partial_x u < 0} \]
\[ = \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{p+2} \, dx - \int_{\partial_x u < 0} f''(u) |\partial_x u|^{p+2} \, dx. \]  
(5.16)

Substituting (5.16) into (5.15), we get the following equality
\[ \frac{1}{p+1} \frac{d}{dt} \| \partial_x u(t) \|_{L^{p+1}} + \mu p^2 \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \, dx \]
\[ + \frac{p}{p+1} \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{p+2} \, dx = \frac{p}{p+1} \int_{\partial_x u < 0} f''(u) |\partial_x u|^{p+2} \, dx. \]  
(5.17)

Multiplying (5.17) by \( (1+t)\alpha \) with \( \alpha > 0 \) and integrating over \( (0,t) \) with respect to the time, we complete the proof of Lemma 5.4.

**Proof of Proposition 5.3.** We use the following important results (cf. [33]).

**Lemma 5.5.** For any \( s \geq 0 \), there exists a positive constant \( C_s \) such that
\[ \int_{\partial_x u < 0} f''(u) |\partial_x u|^s \, dx \leq C_s \int_{\partial_x u < 0} |\partial_x \phi|^s \, dx. \]  
(5.18)

In fact, taking care of the relation by using Lemma 2.2
\[ \partial_x u = \partial_x U + \partial_x \phi < 0 \iff \partial_x \phi < 0, \partial_x U < |\partial_x \phi|, \]  
(5.19)
we immediately have

$$\int_{\partial_x u < 0} f''(u) |\partial_x u|^p \, dx$$

$$\leq 2^q \left( \max_{|u| \leq C} f''(u) \right) \int_{\partial_x \phi < 0, \partial_x U \leq |\partial_x \phi|} |\partial_x \phi|^p \, dx. \quad (5.20)$$

Since $\partial_x u$ is absolutely continuous, we first note that for any $x \in \{ x \in \mathbb{R} \mid \partial_x u < 0 \}$, there exists $x_k \in \mathbb{R} \cup \{-\infty\}$ such that

$$\partial_x u(x_k) = 0, \quad \partial_x u(y) < 0 \quad (y \in (x_k, x)).$$

Therefore, it follows that for such $x$ and $x_k$ with $q \geq p \ (p > 1)$,

$$|\partial_x u|^q = (-\partial_x u)^q = q \int_{x_k}^x (-\partial_x u)^{q-1} (-\partial_x^2 u) \, dy \quad (5.21)$$

By using the Cauchy-Schwarz inequality, we have

**Lemma 5.6.** It holds that

$$\int_{\partial_x u < 0} |\partial_x u|^{p+2} \, dx$$

$$\leq C_p \left( \int_{\partial_x u < 0} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \, dx \right)^{3/(p+1)} \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{3p/(3p+2)}. \quad (5.22)$$

By using Young’s inequality to (5.22), we also have

**Lemma 5.7.** It follows that for any $\epsilon > 0$, there exists a positive constant $C_p(\epsilon)$ such that,

$$\int_{\partial_x u < 0} |\partial_x u|^{p+2} \, dx$$

$$\leq \epsilon \int_{\partial_x u < 0} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \, dx + C_p(\epsilon) \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{3p/(3p+2)}. \quad (5.23)$$
By using Lemma 5.5, Lemma 5.6 and Lemma 5.7 with \( \epsilon = \frac{\mu p^2 (p+1)}{2} \), we have

\[
(1 + t)^\alpha \| \partial_x u(t) \|_{L^{p+1}}^{p+1} \\
+ \frac{\mu p^2 (p+1)}{2} \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\partial_x u| \left( \partial_x^2 u \right)^2 \, dx \, d\tau \\
+ p \int_0^t (1 + \tau)^\alpha \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{p+1} \, dx \, d\tau \\
\leq \| \partial_x u_0 \|_{L^{p+1}}^{p+1} \\
+ \alpha \int_0^t (1 + \tau)^{\alpha - 1} \left( \| \partial_x \phi(\tau) \|_{L^{p+1}}^{p+1} + \| \partial_x U'(\tau) \|_{L^{p+1}}^{p+1} \right) \, d\tau \\
+ C_p \int_0^t (1 + \tau)^\alpha \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{1}{p+1}} \, d\tau.
\]

(5.24)

By using Proposition 5.2, we get the following time-decay estimates.

**Lemma 5.8.** There exist positive constants \( \alpha \gg 1 \) and \( C_{\alpha, p, q} \), such that

\[
\int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\phi|^{q-2} \left( |\partial_x \phi|^2 + |\partial_x U'|^{p-1} \right) \, dx \, d\tau \\
\leq C (\alpha, p, q, \phi_0) (1 + t)^{\alpha - \frac{2q-1}{p}} \quad (t \geq 0).
\]

(5.25)

By using Lemma 5.8 with \( \alpha \to \alpha - 1 \gg 1 \) and \( q = 2 \), we have

\[
\alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \partial_x \phi(\tau) \|_{L^{p+1}}^{p+1} \, d\tau \leq C (\alpha, p, \phi_0) (1 + t)^{\alpha - \frac{2q-1}{p}}.
\]

(5.26)

We can also estimate by using Lemma 2.2 as

\[
\alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \partial_x U'(\tau) \|_{L^{p+1}}^{p+1} \, d\tau \leq C (\alpha, p) (1 + t)^{\alpha - p}.
\]

(5.27)

By using the uniform boundedness in Lemma 3.2, that is,

\[
\| \partial_x u(t) \|_{L^{p+1}} \leq C_p (\| \phi_0 \|_{L^2}, \| \partial_x u_0 \|_{L^{p+1}})
\]

and Lemma 5.8 with \( q = 2 \), we can estimate as

\[
C_p \int_0^t (1 + \tau)^\alpha \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{1}{p+1}} \, d\tau \\
\leq C_p \int_0^t (1 + \tau)^\alpha \int_{\partial_x u < 0} |\partial_x \phi|^{p+1} \, dx \cdot \| \partial_x u(\tau) \|_{L^{p+1}}^{2(p+1)} \, d\tau \\
\leq C (p, \phi_0, \partial_x u_0) \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\partial_x \phi|^2 \, dx \, d\tau \\
\leq C (\alpha, p, \phi_0, \partial_x u_0) (1 + t)^{\alpha - \frac{1}{p}}.
\]

(5.28)
Substituting (5.26), (5.27) and (5.28) into (5.24), we complete the proof of Proposition 5.3. In particular, we have
\[
\| \partial_x u(t) \|_{L^{p+1}}^{p+1} \, dt \leq C \left( p, \phi_0, \partial_x u_0 \right) (1 + t)^{-\frac{1}{2}} + \frac{3}{2p} \left( \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{(p+1)(3p-2)}{3} \epsilon} \right),
\]
and
\[
\| \partial_x \phi(t) \|_{L^{p+1}}^{p+1} \, dt \leq \| \partial_x u(t) \|_{L^{p+1}}^{p+1} + \| \partial_x U^\tau(t) \|_{L^{p+1}}^{p+1} \leq C \left( p, \phi_0, \partial_x u_0 \right) (1 + t)^{-\frac{1}{2}} + \frac{3}{2p} \left( \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{(p+1)(3p-2)}{3} \epsilon} \right)
\]
for \( 1 \leq q < \infty \).

**Proof of Theorem 3.2.** We already have proved the decay estimate of \( \| \phi(t) \|_{L^q} \) with \( 1 \leq q < \infty \). Therefore we only show the following time-decay estimate for the higher order derivative
\[
\left\| \partial_x u(t) \right\|_{L^{p+1}} \leq \left\{ \begin{array}{lcl}
C(\epsilon, p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{p}{p+1}} & & \\
\left( 1 < p \leq \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{(p+1)(3p-2)}{3} \epsilon} \right) & & \\
C(\epsilon, p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{3}{2p} \left( \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{(p+1)(3p-2)}{3} \epsilon} \right)} & & \left( \epsilon \leq p \right)
\end{array} \right.
\]
for any \( 0 < \epsilon \ll 1 \), and the \( L^\infty \)-estimate for \( \phi \).

We first prove (5.31). Substituting (5.29) into (5.28), we have
\[
C_p \int_0^t (1 + \tau)^{\alpha} \int_{\partial_x u < 0} |\partial_x \phi|^p \, dx \cdot \| \partial_x u(\tau) \|_{L^{p+1}}^{\frac{2(p+1)}{3p}} \, d\tau \leq C \left( p, \phi_0, \partial_x u_0 \right) \int_0^t (1 + \tau)^{\alpha - \frac{2}{3p}} \frac{2}{3p} \int_0^{\infty} |\partial_x \phi|^p \, dr \, d\tau.
\]

By using Lemma 5.8 with \( \alpha \mapsto \alpha - \frac{2}{3p} \gg 1 \) and \( q = 2 \), we also have
\[
\alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \partial_x \phi(\tau) \|_{L^{p+1}}^{p+1} \, d\tau \leq C \left( \alpha, p, \phi_0 \right) (1 + t)^{\alpha - \frac{2}{3p} - \frac{2}{3p}}.
\]
Substituting (5.33) into (5.24), we have

\[
\| \partial_x u(t) \|_{L^{p+1}} \quad \| \partial_x \phi(t) \|_{L^{p+1}} \\
\leq C(p, \phi_0, \partial_x u_0) \\
\times \left( (1 + t)^{-\frac{2p+1}{2p}} + (1 + t)^{-p} + (1 + t)^{-\left(\frac{1}{2p} + \frac{1}{4} + \frac{1}{4p} \right)} \right) \\
\leq \begin{cases} \\
C(p, \phi_0, \partial_x u_0) \left( (1 + t)^{-p} + (1 + t)^{-\left(\frac{1}{2p} + \frac{1}{4} + \frac{1}{4p} \right)} \right) & \quad \left( 1 < p \leq \frac{1 + \sqrt{3}}{2} \right), \\
C(p, \phi_0, \partial_x u_0) \left( (1 + t)^{-\frac{2p+1}{2p}} + (1 + t)^{-\left(\frac{1}{2p} + \frac{1}{4} + \frac{1}{4p} \right)} \right) & \quad \left( \frac{1 + \sqrt{3}}{2} < p \right) .
\end{cases}
\] (5.34)
Iterating \("\infty\)\)-times the above process, we will get
\[
\left\| \partial_x u(t) \right\|_{L^{p+1}}^{p+1}, \left\| \partial_x \phi(t) \right\|_{L^{p+1}}^{p+1} \leq \begin{cases} 
C(\epsilon, p, \phi_0, \partial_x u_0) 
(1 + t)^{-p} + (1 + t) \frac{1}{2p} \sum_{n=0}^{\infty} \left( \frac{2}{3p} \right)^n + \epsilon 
(1 < p \leq \frac{1 + \sqrt{3}}{2}) , 
\end{cases}
\]
\[
C(\epsilon, p, \phi_0, \partial_x u_0) 
(1 + t)^{-\frac{2p+1}{2p}} + (1 + t) \frac{1}{2p} \sum_{n=0}^{\infty} \left( \frac{2}{3p} \right)^n + \epsilon 
\left( \frac{1 + \sqrt{3}}{2} < p \right) ,
\]
\[
C(\epsilon, p, \phi_0, \partial_x u_0) 
(1 + t)^{-p} + (1 + t) \frac{1}{2p} \sum_{n=0}^{\infty} \left( \frac{2}{3p} \right)^n + \epsilon 
\left( \frac{1 + \sqrt{3}}{2} < p \right) ,
\]
\[
C(\epsilon, p, \phi_0, \partial_x u_0) 
(1 + t)^{-\frac{2p+1}{2p}} + (1 + t) \frac{1}{2p} \sum_{n=0}^{\infty} \left( \frac{2}{3p} \right)^n + \epsilon 
\left( \frac{1 + \sqrt{3}}{2} < p \right) ,
\]
\[
C(\epsilon, p, \phi_0, \partial_x u_0) (1 + t)^{-p} \left( 1 < p \leq \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{3p - 2}{3} \epsilon} \right) ,
\]
\[
C(\epsilon, p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{2p+1}{2p}} + \epsilon \left( \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{3p - 2}{3} \epsilon} < p \right) \)
\]
(5.35)

for any \(0 < \epsilon \ll 1\).

Thus, we get (5.31).

We finally show the \(L^\infty\)-estimate for \(\phi\) by using the Gagliardo-Nirenberg
inequality. Substituting (5.11) and (5.31) into (4.19), we get
\[
\| \phi(t) \|_{L^\infty} \leq C(\epsilon, p, \theta, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{2p} + \left(\frac{2p+1}{2p} - \frac{p}{p+1}\right) \theta}
\]
\[
\left(\begin{array}{l}
1 < p \leq \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{(p+1)(3p-2)}{3} \epsilon} \\
\frac{1}{3} + \sqrt{\frac{11}{18} - \frac{(p+1)(3p-2)}{3} \epsilon} < p
\end{array}\right)
\]
for \( \theta \in (0, 1] \) and any \( 0 < \epsilon \ll 1 \). Consequently, we do complete the proof of Theorem 3.2.

6. \( L^{r+1} \)-estimate for the higher order derivative with \( r > p \)

In this section, we show the time-decay estimates for the higher order derivative in the \( L^{r+1} \)-norm with \( r > p \), in the case where \( \phi_0 \in L^1 \cap L^2 \) with \( \partial_x u_0 \in L^{p+1} \cap L^{r+1} \), that is, Theorem 3.3.

**Proposition 6.1.** Suppose the same assumptions in Theorem 3.3. For any \( r > p \), there exist positive constants \( \alpha \) and \( C_{\alpha, p, r} \), such that the unique global solution in time \( \phi \) of the Cauchy problem (3.3) satisfies the following \( L^{r+1} \)-norm estimate.
energy estimate

\[(1 + t) \alpha \| \partial_x u(t) \|^2_{L^{r+1}} + \int_0^t (1 + \tau) \alpha \int_{-\infty}^{\infty} | \partial_x u |^{p^* - 2} (\partial_x^2 u)^2 \, dx \, d\tau + \int_0^t (1 + \tau) \alpha \| \partial_x u(\tau) \|^2_{L^{r+2}} \, d\tau \leq C_{\alpha, p, r} \| \partial_x u_0 \|^2_{L^{r+1}}.
\]

\[
\begin{cases}
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0)(1 + t)^{\alpha - \frac{2p^* - 2}{3p^*}} \\
+ \begin{cases}
(1 < p \leq \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{3p - 2}{3}} \epsilon, r > p > \frac{18p^3 - 17p^2 - 16p - 3}{2(2p + 1)}), \\
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0)(1 + t)^{\alpha - \frac{p^* + 2r}{2p^* - 2r} + \frac{2(r - p + 1)}{3p} \epsilon}\bigg(\frac{1}{3} + \sqrt{\frac{11}{18} - \frac{3p - 2}{3}} \epsilon < p\bigg)
\end{cases}
\end{cases}
\]

for $t \geq 0$ and any $0 < \epsilon \ll 1$.

The proof of Proposition 6.1 is given by the following three lemmas. Because the proofs of them are similar to those of Lemma 5.4, Lemma 5.5, Lemma 5.6 and Lemma 5.7, we state only here.

**Lemma 6.1.** There exist positive constants $C_{p, r}$ and $C_{\alpha, p, r}$ such that

\[
(1 + t)\alpha \| \partial_x u(t) \|^2_{L^{r+1}} + \mu p r (r + 1) \int_0^t (1 + \tau) \alpha \int_{-\infty}^{\infty} | \partial_x u |^{p^* - 2} (\partial_x^2 u)^2 \, dx \, d\tau + r \int_0^t (1 + \tau) \alpha \int_{\partial_x u \geq 0} f''(u) | \partial_x u |^{r^* + 2} \, dx \, d\tau \leq \| \partial_x u_0 \|^2_{L^{r+1}}
\]

\[
\begin{cases}
C_{\alpha, p, r} \int_0^t (1 + \tau)^{\alpha - \frac{2p^* + 1}{3p^* + 1}} \left( \int_{-\infty}^{\infty} | \partial_x u |^{p^* + 1} \, dx \right)^{\frac{3p^* + 1}{3p^* + 1}} \, d\tau \\
+ C_{p, r} \int_0^t (1 + \tau)^{\alpha} \left( \int_{\partial_x u < 0} | \partial_x u |^{p^* + 1} \, dx \right)^{\frac{3p}{3p^* - 2}} \, d\tau (t \geq 0).
\end{cases}
\]

**Lemma 6.2.** Assume $p > 1$ and $r > p$. We have the following interpolation inequalities.
(1) There exists a positive constant $C_{p,r}$ such that
\[
\| \partial_x u(t) \|_{L^{r+1}} \leq C_{p,r} \left( \int_{-\infty}^{\infty} |\partial_x u(t)|^{r+2} \left( \partial_x^2 u \right)^2 \, dx \right)^{\frac{r-2}{2(p+r+1)(r+1)}} \times \left( \int_{-\infty}^{\infty} |\partial_x u(t)|^{p+1} \, dx \right)^{\frac{1}{2(p+r+1)(r+1)}} .
\]

(2) There exists a positive constant $C_{p,r}$ such that
\[
\| \partial_x u(t) \|_{L^{\infty}} \leq C_{p,r} \left( \int_{-\infty}^{\infty} |\partial_x u(t)|^{r+2} \left( \partial_x^2 u \right)^2 \, dx \right)^{\frac{1}{2(p+r+1)(r+1)}} \times \left( \int_{-\infty}^{\infty} |\partial_x u(t)|^{p+1} \, dx \right)^{\frac{1}{2(p+r+1)(r+1)}} .
\]

Lemma 6.3. Assume $p > 1$ and $r > p$. We have the following interpolation inequalities.

(1) There exists a positive constant $C_{p,r}$ such that
\[
\| \partial_x u(t) \|_{L^{r+1}(\{\partial_x u < 0\})} \leq C_{p,r} \left( \int_{\partial_x u < 0} |\partial_x u|^{r+2} \left( \partial_x^2 u \right)^2 \, dx \right)^{\frac{r-2}{2(p+r+1)(r+1)}} \times \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{1}{2(p+r+1)(r+1)}} .
\]

(2) There exists a positive constant $C_{p,r}$ such that
\[
\| \partial_x u(t) \|_{L^{\infty}(\{\partial_x u < 0\})} \leq C_{p,r} \left( \int_{\partial_x u < 0} |\partial_x u|^{r+2} \left( \partial_x^2 u \right)^2 \, dx \right)^{\frac{1}{2(p+r+1)}} \times \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{1}{2(p+r+1)}} .
\]

Proof of Proposition 6.1. By using (5.31), we estimate the each terms on
the right-hand side of (6.2) as

\[
C_{\alpha, p, r} \int_0^t (1 + \tau)^{\alpha - \frac{2p + r + 1}{3p + 1}} \left( \int_{-\infty}^{\infty} |\partial_x u|^{p+1} \, dx \right)^{\frac{p + 2r + 1}{3p + 1}} \, d\tau
\]

\[
\leq \begin{cases} 
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) \int_0^t (1 + \tau)^{\alpha - \frac{2p + r + 1}{3p + 1}} \left( \frac{p(p+2r+1) - 3p - 2 \epsilon}{3(3p+1)(3p-2) - 3} \right) \, d\tau & \left( \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{3p - 2 \epsilon}{3}} \epsilon < p \right) \\
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^{\alpha - \frac{2p + r + 1}{3p + 1}} & \left( 1 < p \leq \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{3p - 2 \epsilon}{3}} \right) 
\end{cases}
\]

(6.3)
\[ C_{p,r} \int_0^t (1 + \tau)^\alpha \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{p+2r+2}{sp}} \, d\tau \leq C_{p,r} \int_0^t (1 + \tau)^\alpha \left( \int_{-\infty}^{\partial_x u > 0} |\partial_x \phi|^{p+1} \, dx \right) \|\partial_x u(\tau)\|_{L^{p+1}}^{2(p+1)(r-p+1)} \, d\tau \]

\[
\leq \begin{cases} 
C(\alpha, \epsilon, p, r, \partial_x u_0) \int_0^t (1 + \tau)^\alpha \frac{2(r-p+1)}{p} \|\partial_x \phi(\tau)\|_{L^{p+1}}^{p+1} \, d\tau \\
\quad \quad \quad \left( 1 < p \leq \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{3p-2}{3} \epsilon} \right)
\end{cases}
\]

for any \(0 < \epsilon \ll 1\).

By using Lemma 5.8 with

\[
\alpha \mapsto \begin{cases} 
\alpha - \frac{2(r-p+1)}{3} & \left( 1 < p \leq \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{3p-2}{3} \epsilon} \right) \\
\alpha - \left( \frac{r-p+1}{p(3p-2)} - \frac{2(r-p+1)}{3p} \epsilon \right) & \left( \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{3p-2}{3} \epsilon < p} \right)
\end{cases}
\]

and \(q = 2\), we get

\[ C_{p,r} \int_0^t (1 + \tau)^\alpha \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{p+2r+2}{sp}} \, d\tau \leq \begin{cases} 
C(\alpha, \epsilon, p, r, \partial_x u_0) (1 + t)^\alpha \frac{4p(r-p)+4p+3}{sp} \\
\quad \quad \quad \left( 1 < p \leq \frac{1}{3} + \sqrt{\frac{11}{18} - \frac{3p-2}{3} \epsilon} \right)
\end{cases}
\]

(6.5)

for any \(0 < \epsilon \ll 1\).
Substituting (6.3) and (6.5) into (6.2), we have
\[
(1 + t)^\alpha \| \partial_x u(t) \|_{L^{r+1}}^{r+1} \\
+ \int_0^t (1 + \tau)^\alpha \int_0^\infty \left| \partial_x u \right|^{p+r-2} \left( \partial_x^2 u \right)^2 \, dx \, d\tau \\
+ \int_0^t (1 + \tau)^\alpha \int_{\partial_x u \geq 0} f''(u) \left| \partial_x u \right|^{r+2} \, dx \, d\tau \\
\leq C_{\alpha,p,r} \| \partial_x u_0 \|_{L^{r+1}}^{r+1}
\]

\[
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0)(1 + t)^\alpha \\
\times \left( (1 + t)^{-\frac{2p(p^2-1)}{2p+1}} + (1 + t)^{-\frac{4(p-r)+4p+3}{6p}} \right) \\
\times \left( 1 + \tau \right)^{-\frac{2q(r-p)+2q+r+3}{2q(q+1)(3p-2)}} + \frac{2q+2r+1}{3q+1} + \frac{2(r-p)+1}{3p} + \frac{4q(r-p)+4q+3}{6p} \\
\times \left( 1 + \tau \right)^{-\frac{p+2r-2q+1}{2p(q+1)(3p-2)}} + \frac{2q+2r+1}{3q+1} + \frac{4q(r-p)+4q+3}{6p}
\]

for any \( 0 < \epsilon \ll 1 \).

We also note the following: if \( 1 < p \leq \frac{1}{3} + \sqrt{\frac{11}{18} \frac{3p-2}{3} \epsilon} \), then \( r > p > \frac{18p^3 - 12p^2 - 16p - 3}{2(2p+1)} \) and
\[
(1 + t)^{-\frac{4(p-r)+4p+3}{6p}} \leq (1 + t)^{-\frac{2p+2p^2}{3p+1}},
\]
and if \( \epsilon < \frac{3(r-p)(q-1)(3p+1)}{(3p-2)(9p^2 - 2p - 2r - 2)} \), then
\[
(1 + t)^{-\frac{2q(r-p)+2q+r+3}{2q(q+1)(3p-2)}} + \frac{2q+2r+1}{3q+1} \leq (1 + t)^{-\frac{p+2r-2q+1}{2p(q+1)(3p-2)}} + \frac{2(r-p)+1}{3p} \quad (\forall p > 1, \forall r > p).
\]

Thus, we do complete the proof of Proposition 6.1.

\section{Discussion}

In this section, we discuss the time-decay rates in our main theorems. To do that, we recall the time-decay rates of solutions to a Cauchy problem for the
simplest $p$-Laplacian evolution equation without convective term:

\[
\begin{aligned}
\partial_t u - \mu \partial_x \left( \left| \partial_x u \right|^{p-1} \partial_x u \right) &= 0 \quad (t > 0, x \in \mathbb{R}), \\
u(0, x) &= u_0(x) \quad (x \in \mathbb{R}), \\
\lim_{x \to \pm \infty} u(t, x) &= 0 \quad (t \geq 0),
\end{aligned}
\]

(7.1)

where, $u = u(t, x)$ denotes the unknown function of $t > 0$ and $x \in \mathbb{R}$. The theorems concerning the time-decay estimates to the problem (7.1) are as follows (the proofs are similar to those in the previous sections).

**Theorem 7.1.** If the initial data satisfies $u_0 \in L^2$ and $\partial_x u_0 \in L^{p+1}$, then there uniquely exists a global solution in time $u$ of the Cauchy problem (7.1) satisfying

\[
\begin{aligned}
u &\in C^0([0, \infty); L^2) \cap L^\infty(\mathbb{R}^+; L^2), \\
\partial_x \left( \left| \partial_x u \right|^{p-1} \partial_x u \right) &\in L^2(\mathbb{R}_t^+ \times \mathbb{R}_x)
\end{aligned}
\]

and the following time-decay estimates

\[
\begin{aligned}
\| u(t) \|_{L^q} &\leq C(p, q, u_0) (1 + t)^{-\frac{1}{p} \left( 1 - \frac{1}{q} \right)}, \\
\| u(t) \|_{L^\infty} &\leq C(\epsilon, p, q, u_0, \partial_x u_0) (1 + t)^{-\frac{1}{p} + \epsilon}
\end{aligned}
\]

for $q \in [2, \infty)$ and any $\epsilon > 0$.

**Theorem 7.2.** If the initial data further satisfies $u_0 \in L^1 \cap L^2$ and $\partial_x u_0 \in L^{p+1}$, then it holds that the unique global solution in time $u$ of the Cauchy problem (7.1) satisfies the following time-decay estimates

\[
\begin{aligned}
\| u(t) \|_{L^q} &\leq C(p, q, u_0) (1 + t)^{-\frac{1}{p} \left( 1 - \frac{1}{q} \right)}, \\
\| u(t) \|_{L^\infty} &\leq C(p, q, u_0, \partial_x u_0) (1 + t)^{-\frac{1}{p}}
\end{aligned}
\]

for $q \in [1, \infty)$. Furthermore, the solution satisfies the following time-decay estimates for the higher order derivative

\[
\| \partial_x u(t) \|_{L^{r+1}} \leq C(p, u_0, \partial_x u_0) (1 + t)^{-\frac{rp+1}{r(p+1)}}.
\]

**Theorem 7.3.** If the initial data further satisfies $u_0 \in L^1 \cap L^2$ and $\partial_x u_0 \in L^{p+1} \cap L^{r+1}(r > p)$, then it holds that the unique global solution in time $u$ of the Cauchy problem (7.1) satisfies the following time-decay estimates for the higher order derivative

\[
\| \partial_x u(t) \|_{L^{r+1}} \leq C(p, r, u_0, \partial_x u_0) (1 + t)^{-\frac{rp+1}{rp+1}}.
\]
It is clear that the time-decay rates in the $L^q$-norm ($2 \leq q \leq \infty$ or $1 \leq q \leq \infty$) for the lower order $u - \bar{u}$ or $u - u^r$ in Theorems 1.1, 1.2, 1.4 and 1.5 are quite or almost the same as $u$ in Theorem 7.1 and 7.2. This shows that the affection to the time-decay from the formulation of the equation

$$\partial_t u - \mu \partial_x \left( \left| \partial_x u \right|^{p-1} \partial_x u \right) = 0$$

is stronger than those from the asymptotic states, the rarefaction wave $u^r$ (or $U^r$) or the constant states $\bar{u}$ (and also from the shape of the flux function $f$, see (4.2), (5.3) and (5.10)). In fact, the time-decay in (4.16) is faster than that in (4.15) without $\alpha \gg 1$ in Section 4. On the other hand, the time-decay rates in the $L^{p+1}$-norm or the $L^{r+1}$-norm ($r > p$) for the higher order $\partial_x u$ or $\partial_x u - \partial_x u^r$ in Theorems 1.2, 1.3, 1.5, 1.6, 7.2 and 7.3 are all different from with each other. The reason for the difference must arise from that the asumptions for the flux function $f$ and the far field states $u_{\pm}$, $\bar{u}$, and the characteristic properties of the asymptotic states ($u^r$ or $\bar{u}$) affect (in some sense) the strong nonlinearlity of the higher order $\partial_x u$ (not $u$), that is, $p$-Laplacian type viscosity (see (5.18), (5.19), (5.25), (5.26), (5.33) in Section 5 and (6.5) in Section 6). However, the optimality of the all time-decay rates still remains open.

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