Recursion relations for Unitary integrals,
Combinatorics and the Toeplitz Lattice

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In a discussion in spring 2001, Alexei Borodin showed us recursion relations for the Toeplitz determinants going with the symbols $e^{t(z+z^{-1})}$ and $(1 - \xi z)^\alpha (1 - \xi z^{-1})^\beta$. Borodin obtained these relations using Riemann-Hilbert methods; see the recent work of Borodin [5] and also Borodin-Deift [6] and Baik [4]. The nature of Borodin’s recursion relations pointed towards the Toeplitz lattice and its Virasoro algebra, introduced by us in [3]. In this paper, we take the Toeplitz and Virasoro approach for a fairly large class of symbols, leading to a systematic way of generating recursion relations. The latter are very naturally expressed in terms of the $L$-matrices appearing in the Toeplitz lattice equations. As a surprise, we find, compared to Borodin’s, a different set of relations, except for the 3-step relations associated with the symbol $e^{t(z+z^{-1})}$.

0 Introduction and main results

The weight

$$\rho(z) := e^{P_1(z) + P_2(z^{-1})} z^\gamma (1 - d_1 z)^{\nu_1} (1 - d_2 z)^{\nu_2} (1 - d_1^{-1} z^{-1})^{\nu_1'} (1 - d_2^{-1} z^{-1})^{\nu_2'}$$  

(0.0.1)

with

$$P_1(z) := \sum_{i=1}^{N_1} \frac{u_i z^i}{i} \text{ and } P_2(z) := \sum_{i=1}^{N_2} \frac{u_{-i} z^i}{i},$$  

(0.0.2)
has a natural involution
\[ \tilde{\cdots} : z \leftrightarrow z^{-1}, \quad (0.0.3) \]
which induces an involution on the following quantities:
\[ \tilde{\cdots} : P_1(z) \leftrightarrow P_2(z^{-1}), \quad u_i \leftrightarrow u_{-i}, \quad N_1 \leftrightarrow N_2, \quad \gamma \leftrightarrow -\gamma, \quad d_i \leftrightarrow d_i^{-1}, \quad \gamma_i' \leftrightarrow \gamma_i''. \quad (0.0.4) \]
The multiple integral below is known to be expressible, both, as the determinant of a Toeplitz matrix and as an integral over the group \( U(n) \),
\[
I_n^{(c)} := \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left( \frac{z_k^\epsilon \rho(z_k)}{2\pi i z_k} \right) \frac{dz_k}{2\pi i z_k} \]
\[
= \det \left( \int_{S^1} z^{i-j} \rho(z) \frac{dz}{2\pi i z} \right)_{1 \leq i,j \leq n} \]
\[
= \int_{U(n)} \det(U^{\epsilon} \rho(U)) \, dU, \quad (0.0.5) \]
which for some special choices of \( \rho \) has an interesting interpretation in terms of random permutations; for that matter, look at the examples in section 4. Consider the basic variables, with \( I_n := I_n^{(0)} \), \( I_n^\pm = I_n^{(\pm 1)} \),
\[
x_n = (-1)^n \frac{I_n^+}{I_n}, \quad y_n = (-1)^n \frac{I_n^-}{I_n} \quad \text{and} \quad v_n = 1 - x_n y_n = \frac{I_n-1}{I_n^2}, \quad (0.0.6) \]
See section 1.1 for explanations. Then the basic object
\[
I_n = \int_{U(n)} \det(\rho(U)) \, dU = \det \left( \int_{S^1} z^{i-j} \rho(z) \frac{dz}{2\pi i z} \right)_{1 \leq i,j \leq n}, \]
which appears in several problems of random words and permutations, is obtained from the \( x_n, y_n \) and \( I_1 \), by means of the formula
\[
I_n = I_1^n \prod_{i=1}^{n-1} (1 - x_i y_i)^{n-i}. \quad (0.0.7) \]
The following matrices, intimately related to the Toeplitz lattice, will play an important role in this work:

\[
L_1 := 
\begin{pmatrix}
-x_1y_0 & 1 - x_1y_1 & 0 & 0 \\
-x_2y_0 & -x_2y_1 & 1 - x_2y_2 & 0 \\
-x_3y_0 & -x_3y_1 & -x_3y_2 & 1 - x_3y_3 \\
-x_4y_0 & -x_4y_1 & -x_4y_2 & -x_4y_3 \\
\vdots \\
\end{pmatrix}
\]

and

\[
L_2 := 
\begin{pmatrix}
-x_0y_1 & -x_0y_2 & -x_0y_3 & -x_0y_4 \\
1 - x_1y_1 & -x_1y_2 & -x_1y_3 & -x_1y_4 \\
0 & 1 - x_2y_2 & -x_2y_3 & -x_2y_4 \\
0 & 0 & 1 - x_3y_3 & -x_3y_4 \\
\vdots \\
\end{pmatrix}
\]

The Toeplitz lattice and its relation to the Toda lattice will be discussed in Section 1.1.

Define the matrices, depending on the positive integer \(n \geq 1\), and the exponents \(\gamma, \gamma', \gamma''\) in (0.0.1),

\[
L_1(n) := (aI + bL_1 + cL_2)P'(L_1) + c(n + \gamma' + \gamma_2 + \gamma)L_1 \\
L_2(n) := (cI + bL_2 + aL_1^2)P'(L_2) + a(n + \gamma'' + \gamma_2' - \gamma)L_2,
\]

and depending on arbitrary parameters \(a, b, c\). The involution \(\tilde{\cdot}\), defined in (0.0.3) and (0.0.4) induces involutions

\[
I_n \leftrightarrow I_n, \quad I_n^+ \leftrightarrow I_n^-, \quad x_n \leftrightarrow y_n, \quad a \leftrightarrow c, \quad b \leftrightarrow b,
\]

and so \(L_1 \leftrightarrow L_2^\top, \quad L_1^{(n)} \leftrightarrow L_2^{(n)\top}\).

Also note that (self-dual case)

\[
\rho(z) = \rho(z^{-1}) \text{ implies } x_n = y_n, \quad L_1 = L_2^\top, \quad L_1^{(n)} = L_2^{(n)\top}.
\]

Given a matrix \(A(n)\) containing explicitly the parameter \(n\), the “discrete derivative” \(\partial_n\) is defined as

\[
\partial_n A(n)_{nn} := A(n + 1)_{n+1,n+1} - A(n)_{nn}.
\]
Rational relations: In the Theorem 0.1, we show the polynomial relationships between consecutive \((x_i, y_i)\)'s. When the degrees of \(P_1\) and \(P_2\) differ by at most one, they actually lead to inductive rational relations, as is stated in Theorem 0.2. These relations are obtained by observing that the multiple integral (0.0.5) satisfy the Toeplitz lattice and an SL(2,\(\mathbb{Z}\))-set of Virasoro relations in the \(u_i\)-variables; see [3].

**Theorem 0.1** For the weight (0.0.1), the vectors \((x_k)_{k\geq 1}\) and \((y_k)_{k\geq 1}\) satisfy two finite difference relations, involving a finite number of steps:

- **Case 1.** When \(d_1, d_2, d_1 - d_2, |\gamma'_1| + |\gamma''_1|, |\gamma'_2| + |\gamma''_2| \neq 0\) in the weight (0.0.1), then the relations are

\[
\partial_n (\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)})_{n,n} + (cL_1 - aL_2)_{nn} = 0 \quad (0.0.14)
\]

\[
\partial_n (v_n\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)})_{n+1,n} + (cL_2^2 + bL_1)_{n+1,n+1} - C = 0, \quad (0.0.15)
\]

for all \(n \geq 1\), and where

\[a = 1, \ b = -d_1 - d_2, \ c = d_1d_2.\]

\(C\) is a constant independent of \(n\), thus expressible in terms of the initial value:

\[C := (v_1\mathcal{L}_1^{(1)} - \mathcal{L}_2^{(1)})_{2,1} + (cL_2^2 + bL_1)_{1,1}. \quad (0.0.16)\]

Only the first relation is self-dual for the involution \(\tilde{}\).

- **Case 2.** When \(d_1 \neq 0, |\gamma'_1| \neq 0, \gamma'_2 = \gamma''_1 = \gamma''_2 = d_2 = 0\), we may rescale \(z\) so that \(d_1 = -1\) and so

\[\rho(z) = z^\gamma(1 + z)^{\gamma'_1} e^{P_1(z)} + P_2(z^{-1}).\]

Then the same equations (0.0.14) and (0.0.15) are satisfied, where \(a, b, c\) can be chosen in two different ways, one being the dual of the other, namely

\[(a, b, c) = (1, 1, 0) \quad \text{or} \quad (a, b, c) = (0, 1, 1).\]

- **Case 3.** \(d_1 = d_2, \gamma'_1 = \gamma'_2 = \gamma''_1 = \gamma''_2 = 0\). Then

\[\rho(z) := z^\gamma e^{P_1(z)} + P_2(z^{-1})\]
and $a, b, c$ can be chosen totally arbitrary. Here it will be more advantageous to pick different relations, both of which are polynomials, dual to each other,

\[
\begin{align*}
\Gamma_n(x, y) &:= \frac{v_n}{y_n} \left( -(L_1 P_1(L_1))_{n+1,n+1} - (L_2 P_2(L_2))_{n,n} + (P'_1(L_1))_{n+1,n} + (P'_2(L_2))_{n,n+1} \right) + nx_n = 0 \\
\tilde{\Gamma}_n(x, y) &:= \frac{v_n}{x_n} \left( -(L_1 P_1(L_1))_{n,n} - (L_2 P_2(L_2))_{n+1,n+1} + (P'_1(L_1))_{n+1,n} + (P'_2(L_2))_{n,n+1} \right) + ny_n = 0
\end{align*}
\]

(0.0.17)

**Theorem 0.2** Requiring $N_1 = N_2$ or $N_2 \pm 1$ in the weight (0.0.14), the $x_n$ and $y_n$’s can be expressed rationally in terms of lower $x$’s and $y$’s, and thus, from (0.0.7), $I_n$ can be expressed in the terms of the $x$’s and $y$’s. To be precise,

- **Case 1** leads to two inductive rational $N_1 + N_2 + 4$-step relations
  \[
  x_n = F_n(x_{n-1}, y_{n-1}, \ldots, x_{n-N_1-N_2-3}, y_{n-N_1-N_2-3}) \\
y_n = G_n(x_{n-1}, y_{n-1}, \ldots, x_{n-N_1-N_2-3}, y_{n-N_1-N_2-3})
  \]

- **Case 2** leads to two inductive rational $N_1 + N_2 + 3$-step relations (0.0.14) and (0.0.13), such that\(^1\)

  when $N_1 = N_2$ or $N_1 = N_2 + 1$, use $(a, b, c) = (1, 1, 0)$

  when $N_2 = N_1$ or $N_2 = N_1 + 1$, use $(a, b, c) = (0, 1, 1)$.

Thus, we find rational functions $F_n$ and $G_n$:

\[
\begin{align*}
x_n &= F_n(x_{n-1}, y_{n-1}, \ldots, x_{n-N_1-N_2-2}, y_{n-N_1-N_2-2}) \\
y_n &= G_n(x_{n-1}, y_{n-1}, \ldots, x_{n-N_1-N_2-2}, y_{n-N_1-N_2-2})
\end{align*}
\]

- **Case 3** leads to two inductive $N_1 + N_2 + 1$-step rational relations

\[
\begin{align*}
x_n &= F_n(x_{n-1}, y_{n-1}, \ldots, x_{n-N_1-N_2}, y_{n-N_1-N_2}) \\
y_n &= G_n(x_{n-1}, y_{n-1}, \ldots, x_{n-N_1-N_2}, y_{n-N_1-N_2})
\end{align*}
\]

\(^1\)Both solutions can be used, when $N_1 = N_2$. 
Corollary 0.3 For the self-dual weight
\[ \rho(z) = e^{\sum_i u_i (z^i + z^{-i})}, \]
the polynomial in \( x_{k-N}, x_{k-N+1}, \ldots, x_k, \ldots, x_{k+N}, \)
\[ \Gamma_k := kx_k - \frac{v_k}{x_k} \left( \left( \sum_{i=1}^N u_i L_1^{(i)} \right)_{k+1,k+1} + \left( \sum_{i=1}^N u_i L_1^{(i)}_{k,k} \right) - 2 \left( \sum_{i=1}^N u_i L_1^{(i-1)} \right)_{k+1,k} \right) \]
leads to recurrence relations
\[ x_n = F_n(x_{n-1}, \ldots, x_{n-2N}). \]

Remark: In the self-dual case, i.e., when \( \rho(z) = \rho(z^{-1}) \), the first equation \((0.0.14)\) vanishes identically and the two equations in \((0.0.17)\) become identical. Only one equation is required, since all \( x_n = y_n \).

Remark: By the duality \( a \leftrightarrow c, b \leftrightarrow b, L_1 \leftrightarrow L_2^\top \), equations \((0.0.14)\) and \((0.0.13)\) map into
\[
\begin{align*}
\partial_n(L_1^{(n)} - L_2^{(n)})_{n,n} + (cL_1 - aL_2)_{n,n} &= 0 \\
\partial_n(L_1^{(n)} - v_n L_2^{(n)})_{n,n+1} - (aL_2^2 + bL_2)_{n+1,n+1} &= C',
\end{align*}
\]

where \( C' \) is now:
\[ C' := (L_1^{(1)} - v_1 L_2^{(1)})_{1,2} - (aL_2^2 + bL_2)_{1,1}. \]

An invariant manifold: The relations appearing in Theorem 0.1 for each of the cases happen to define an invariant manifold for the first Toeplitz flow. We shall do this here for case 3, where
\[ \rho(z) := z^\gamma e^{P_1(z) + P_2(z^{-1})}. \]
This case has the extra feature that the relations themselves \((0.0.17)\) satisfy an interesting system of differential equations. These statements will be established in section 3, as an immediate consequence of the Virasoro relations satisfied by the multiple integrals.

\[ \text{2The matrix } L_1 \text{ appearing in } (0.0.18) \text{ is the matrix } (0.0.8), \text{ with } y_i = x_i. \]
Theorem 0.4 Let $x_n = x_n(t_1, s_1)$ and $y_n = y_n(t_1, s_1)$ flow according to the differential equations ($v_n := 1 - x_n y_n$)

\[
\begin{align*}
\frac{\partial x_k}{\partial t_1} &= v_k x_{k+1} & \frac{\partial y_k}{\partial t_1} &= -v_k y_{k-1} \\
\frac{\partial x_k}{\partial s_1} &= v_k x_{k-1} & \frac{\partial y_k}{\partial s_1} &= -v_k y_{k+1}.
\end{align*}
\]  

(Toeplitz Lattice)  \ (0.0.21)

and the $u_i$, appearing in the polynomials $P_1(z)$ and $P_2(z)$, according to

\[
\begin{align*}
\frac{\partial u_k}{\partial t_1} &= \delta_{k,1} & \frac{\partial u_k}{\partial s_1} &= -\delta_{k,-1}.
\end{align*}
\]

Then:
(i) The polynomial recurrence relations $\Gamma_n$ and $\tilde{\Gamma}_n$, defined in (0.0.17), satisfy the differential equations

\[
\begin{align*}
\frac{\partial \Gamma_n}{\partial \{t_1, s_1\}} &= v_n \Gamma_{n+1} + x_{n+1} \left( x_n \tilde{\Gamma}_n - y_n \Gamma_n \right) \\
\frac{\partial \tilde{\Gamma}_n}{\partial \{t_1, s_1\}} &= -v_n \tilde{\Gamma}_{n+1} + y_{n+1} \left( x_n \Gamma_n - y_n \tilde{\Gamma}_n \right).
\end{align*}
\]  \ (0.0.22)

(ii) The locus $\mathcal{M}$ is an invariant manifold for the $t_1$ and $s_1$-flows above, where

\[
\mathcal{M} := \bigcap_{n \geq 0} \left\{ (x_k, y_k)_{k \geq 0} \text{ such that } \Gamma_n(x, y) = 0 \text{ and } \tilde{\Gamma}_n(x, y) = 0 \right\}.
\]  \ (0.0.23)

Corollary 0.5 Let $x_n = x_n(t)$ flow according to the differential equations ($v_n := 1 - x_n^2$)

\[
\frac{\partial x_n}{\partial t} = v_n (x_{n+1} - x_{n-1}),
\]  \ (0.0.24)
which is obtained by taking the linear combination \( \frac{\partial}{\partial t} - \frac{\partial}{\partial s_1} \) of the Toeplitz vector fields above and setting all \( x_k = y_k \). Let the \( u_i \), appearing in the self-dual weight

\[
\rho(z) = e^{\sum_{i=1}^{N} u_i (z^i + z^{-i})},
\]

flow according to

\[
\frac{\partial u_k}{\partial t} = \delta_{k,1}.
\]

Then:

(i) The polynomial recurrence relations \( \Gamma_n \), as in (0.0.18), satisfy the differential equations

\[
\frac{\partial \Gamma_n}{\partial t} = v_n (\Gamma_{n+1} - \Gamma_{n-1}).
\] (0.0.25)

(ii) The locus \( \mathcal{M} \) is an invariant manifold for the \( t \)-flow (0.0.24) above, where

\[
\mathcal{M} := \bigcap_{n \geq 0} \{ (x_k)_{k \geq 0} , \text{ such that } \Gamma_n(x, x) = 0 \}.
\] (0.0.26)

**Singularity confinement:** For the self-dual weight

\[
\rho(z) = e^{\sum_{i=1}^{N} u_i (z^i + z^{-i})},
\]

the polynomial relations (remember Corollary 0.3) in \( x_{k-N}, x_{k-N+1}, \ldots, x_k, \ldots, x_{k+N} \),

\[
0 = k x_k - \frac{v_k}{x_k} \left( \sum_{i=1}^{N} u_i L_{i,k+1,k} \right)_{k+1,k+1} + \left( \sum_{i=1}^{N} u_i L_{1,i,k} \right)_{k,k} - 2 \left( \sum_{i=1}^{N} u_i L_{1,i}^{-1} \right)_{k+1,k} \]

lead, in effect, to rational recurrence relations in the \( x_i \),

\[
x_k = F_k(x_{k-1}, \ldots, x_{k-2N}; u_1, \ldots, u_N),
\] (0.0.27)

depending rationally on the coefficients \( u_1, \ldots, u_N \) appearing in the weight \( \rho(z) \).

The following Theorem tells us -roughly speaking- that the recurrence relations (0.0.27) for special initial condition leads to a solution, where one \( x_n \) blows up and all other \( x_k \) are finite. This is a kind of *discrete Painlevé property*, called “singularity confinement”; see Grammaticos, Nijhoff and Ramani [9], who define this to be discrete Painlevé recursion relations. For these
recurrence equations (0.0.27), the precise analytical statement of this phenomenon is stated in Corollary 0.7, claiming there is a *generic* solution with the kind of singularity above. The technique used here to prove Corollary 0.7 is to deform the variables $x_k$ and $y_k$ by means of the Toeplitz lattice; part (i) of Theorem 0.6 below shows that the Toeplitz lattice has a generic solution $x_0, x_1, \ldots$, with all $x_k, \ k \neq n$ finite and one $x_n$ blowing up. This is reminiscent of the Painlevé property of *algebraic integrable systems*, which originates in the work of S. Kowalewski; see [1] and references within. Part (ii) of Theorem 0.6 shows that these series can be made to stay within the locus $\Omega$, by restricting the free parameters. The proof of Theorem 0.6 and Corollary 0.7, which will be given in a subsequent paper, uses heavily the ideas of Theorem 0.4 and Corollary 0.5.

**Theorem 0.6 (i)*** Consider the system of differential equations (with boundary condition $x_0 = 1$)

$$\frac{\partial x_k}{\partial t} = (1 - x_k^2)(x_{k+1} - x_{k-1}), \text{ for } k = 0, 1, 2, \ldots, \tag{0.0.28}$$

and for a fixed, but arbitrary integer $n > 0$, let

$$\ldots, \alpha_{n-3}, \alpha_{n-2}, c, d, \alpha_{n+2}, \alpha_{n+3}, \ldots \tag{0.0.29}$$

be free parameters. Then the system (0.0.28) has a unique “formal” Laurent solution, with $x_n$ and only $x_n$ blowing up, having the form:

$$x_k(t) = \alpha_k + \ldots, \ \text{ for } |k - n| \geq 2$$

$$x_{n-1}(t) = \pm 1 + ct + \ldots$$

$$x_n(t) = \frac{1}{t}(\mp \frac{1}{2} + \frac{c-d}{8}t + \ldots) \tag{0.0.30}$$

$$x_{n+1}(t) = \mp 1 + dt + \ldots .$$

The coefficients in the series (0.0.30) are polynomials in the free parameters (0.0.29). This solution is generic, since

$$\#\{\text{free parameters}\} + 1 = \#\{\text{variables}\},$$

with the “1” accounting for the t-parameter.
(ii) Given the $2N - 1$ free parameters $\alpha_{n-2N}, \ldots, \alpha_{n-2}$, the series above are “formal” Laurent solutions to the recurrence relations

$$x_k = F_k(x_{k-1}, \ldots, x_{k-2N}; u_1 + t, \ldots, u_N), \quad (0.0.31)$$

with the remaining free parameters $c, d, \alpha_i$ for $i \leq n - 2N - 1$ or $i \geq n - 2$, being rational functions of $\alpha_{n-2N}, \ldots, \alpha_{n-2}$ and the parameters $u = (u_1, \ldots, u_N)$.

This Theorem leads to the “Painlevé singularity confinement” property for the recursive equations (0.0.27); the precise statement goes as follows:

**Corollary 0.7 (Singularity confinement)** Given arbitrary initial data

$$(x_{n-2N}, \ldots, x_{n-2}) = (x_{n-2N}^{(0)}, \ldots, x_{n-2}^{(0)}) =: \gamma$$

and setting

$$x_{n-1} = \pm 1 + \varepsilon, \quad (0.0.32)$$

the recurrence relations (0.0.27), namely

$$x_k = F_k(x_{k-1}, \ldots, x_{k-2N}; u_1, \ldots, u_N),$$

have a “generic” formal series solution in $\varepsilon$ of the form (i.e., depending on $2N - 1$ degrees of freedom)

$$x_{n-1}(t) = \pm 1 + \varepsilon$$

$$x_n(t) = \frac{1}{\varepsilon} \left( x_n^{(0)}(\gamma, u) + O(\varepsilon) \right)$$

$$x_{n+1}(t) = \mp 1 + O(\varepsilon)$$

$$x_k(t) = x_k^{(0)}(\gamma, u) + O(\varepsilon), \quad \text{for } k \geq n + 2,$$

with all coefficients of the $\varepsilon$-series depending rationally on $\gamma = (x_{n-2N}^{(0)}, \ldots, x_{n-2}^{(0)})$ and $u := (u_1, \ldots, u_N)$.

**Remark:** The initial condition (0.0.32) is the most general initial condition leading to blow-up at the $n$th step.
Theorem 0.6 and Corollary 0.7 will be established elsewhere, as well as analogous statements that can be made for the non-symmetric weight 

\[ \rho(z) = e^{\sum_N \left( \frac{u_i z^i + u^{-i} z^{-i}}{i} \right)} \]

**Examples:** Several examples will be discussed in section 4. It is also interesting to point out that each of the examples discussed in that section are related to random permutations, random words and point processes. They also admit a representation as a Fredholm determinant of an interesting kernel; concerning the latter, see Borodin and Okounkov [7].

## 1 The Toeplitz lattice and its Virasoro algebra

### 1.1 The Toeplitz lattice

Consider the inner-product on the circle

\[ \langle f(z), g(z) \rangle_{t,s} := \oint_{S^1} \frac{dz}{2\pi iz} f(z) \overline{g(z)} e^{\sum_{1}^\infty (t_i z^i - s_i z^{-i})} \]  

(1.1.1)

the associated moments \( \mu_{k-\ell}(t, s) := \langle y^k, z^\ell \rangle_{t,s} \), and the determinants (\(\tau\)-functions)

\[ \tau_n(t, s) := \det ( \mu_{k-\ell}(t, s) )_{0 \leq k, \ell \leq n-1} \]

\[ = \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^{2} \prod_{k=1}^{n} \left( e^{\sum_{1}^\infty (t_i z^i_k - s_i z^{-i}_k)} \frac{dz_k}{2\pi iz_k} \right) \]

\[ = \int_{U(n)} e^{\sum_{1}^\infty \text{Tr}(t_i M^i - s_i \overline{M}^i)} dM \]  

(1.1.2)

and

\[ \tau_n^\pm(t, s) := \det ( \mu_{k-\ell \pm 1}(t, s) )_{0 \leq k, \ell \leq n-1} \]

\[ = \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^{2} \prod_{k=1}^{n} \left( z_{k}^{\pm 1} e^{\sum_{1}^\infty (t_i z^i_k - s_i z^{-i}_k)} \frac{dz_k}{2\pi iz_k} \right) \]

\[ = \int_{U(n)} (\det M)^{\pm 1} e^{\sum_{1}^\infty \text{Tr}(t_i M^i - s_i \overline{M}^i)} dM \]  

(1.1.3)
The following \( \tau \)-function expressions are actually polynomials.

\[
p^{(1)}_n(t, s; u) = u^n \frac{\tau_n(t - [u^{-1}], s)}{\tau_n(t, s)}
\]

\[
= \frac{1}{n! \tau_n} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left( (u - z_k)e^{\sum_{i=1}^{\infty} (t_i z_k - z_i z_k^{-1})} \frac{dz_k}{2\pi i z_k} \right)
\]

\[
= \frac{1}{\tau_n(t, s)} \int_{U(n)} \det(uI - M)e^{\sum_{i=1}^{\infty} \text{Tr}(t_i M' - s_i M')} dM
\]

(1.1.4)

\[
p^{(2)}_n(t, s; u) = u^n \frac{\tau_n(t, s + [u^{-1}])}{\tau_n(t, s)}
\]

\[
= \frac{1}{n! \tau_n} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left( (u - z_k^{-1})e^{\sum_{i=1}^{\infty} (t_i z_k^{-1} - z_i z_k^{-1})} \frac{dz_k}{2\pi i z_k} \right)
\]

\[
= \frac{1}{\tau_n(t, s)} \int_{U(n)} \det(uI - \bar{M})e^{\sum_{i=1}^{\infty} \text{Tr}(t_i M' - s_i \bar{M})} dM
\]

(1.1.5)

and are bi-orthogonal for the inner-product above

\[
\langle p^{(1)}_n, p^{(2)}_m \rangle_{t, s} = \delta_{nm} h_n(t, s), \quad \text{with } h_n := \frac{\tau_{n+1}}{\tau_n},
\]

(1.1.6)

Define\footnote{For \( \alpha \in \mathbb{C} \), define \( [\alpha] := (\alpha, \alpha^2/2, \alpha^3/3, \ldots) \in \mathbb{C}^\infty \).}

\[
x_n(t, s) := p^{(1)}_n(t, s; 0)
\]

\[
= \frac{(-1)^n}{n! \tau_n} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left( z_k e^{\sum_{i=1}^{\infty} (t_i z_k - z_i z_k^{-1})} \frac{dz_k}{2\pi i z_k} \right)
\]

\[
= \frac{p_n(-\partial_t) \tau_n(t, s)}{\tau_n(t, s)} = (-1)^n \frac{\tau_n^+(t, s)}{\tau_n(t, s)}
\]

\[
y_n(t, s) := p^{(2)}_n(t, s; 0)
\]

\[
= \frac{(-1)^n}{n! \tau_n} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left( z_k^{-1} e^{\sum_{i=1}^{\infty} (t_i z_k^{-1} - z_i z_k^{-1})} \frac{dz_k}{2\pi i z_k} \right)
\]

\[
= \frac{p_n(\partial_t) \tau_n(t, s)}{\tau_n(t, s)} = (-1)^n \frac{\tau_n^-(t, s)}{\tau_n(t, s)}.
\]

(1.1.7)
Throughout the paper, set
\[ v_n := 1 - x_n y_n = 1 - p_n^{(1)}(t, s; 0) p_n^{(2)}(t, s; 0) = \frac{h_n}{h_{n-1}} = \frac{\tau_{n+1} \tau_{n-1}}{\tau_n^2} \quad (1.1.8) \]

In [3], it was pointed out that the quantities \( x_n \) and \( y_n \) satisfy the following integrable Hamiltonian system

\[
\begin{align*}
\frac{\partial x_n}{\partial t_i} &= (1 - x_n y_n) \frac{\partial H_i^{(1)}}{\partial y_n} & \frac{\partial y_n}{\partial t_i} &= -(1 - x_n y_n) \frac{\partial H_i^{(1)}}{\partial x_n} \\
\frac{\partial x_n}{\partial s_i} &= (1 - x_n y_n) \frac{\partial H_i^{(2)}}{\partial y_n} & \frac{\partial y_n}{\partial s_i} &= -(1 - x_n y_n) \frac{\partial H_i^{(2)}}{\partial x_n},
\end{align*}
\]

(Toeplitz lattice)

with initial condition \( x_n(0, 0) = y_n(0, 0) = 0 \) for \( n \geq 1 \) and boundary condition \( x_0(t, s) = y_0(t, s) = 1 \). This fact will be established in Proposition 1.1 below. The traces

\[
H_i^{(k)} = -\frac{1}{i} \text{Tr} L_i^k, \quad i = 1, 2, 3, \ldots, \quad k = 1, 2 \quad (1.1.10)
\]

of the matrices \( L_i \) below are integrals in involution with regard to the symplectic structure

\[
\omega := \sum_{i=1}^{\infty} \frac{dx_k \wedge dy_k}{1 - x_k y_k},
\]

where \( L_1 \) and \( L_2 \) are given by the “rank 2” semi-infinite matrices

\[
L_1 := \begin{pmatrix}
-x_1 y_0 & 1 - x_1 y_1 & 0 & 0 \\
-x_2 y_0 & -x_2 y_1 & 1 - x_2 y_2 & 0 \\
-x_3 y_0 & -x_3 y_1 & -x_3 y_2 & 1 - x_3 y_3 \\
-x_4 y_0 & -x_4 y_1 & -x_4 y_2 & -x_4 y_3 \\
& & & \ddots
\end{pmatrix}
\]

5By computing \( (p_{n+1}^{(1)}(u) - u p_n^{(1)}(u), p_{n+1}^{(2)}(u) - u p_n^{(2)}(u)) \) in two different ways, in a straightforward way and in another way, using

\[
\begin{align*}
p_{n+1}^{(1)}(u) - u p_n^{(1)}(u) &= p_{n+1}^{(1)}(0) u^{n} p_n^{(2)}(u^{-1}) \\
p_{n+1}^{(2)}(u) - u p_n^{(2)}(u) &= p_{n+1}^{(2)}(0) u^{n} p_n^{(1)}(u^{-1}).
\end{align*}
\]

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and

\[ L_2 := \begin{pmatrix} -x_0 y_1 & -x_0 y_2 & -x_0 y_3 & -x_0 y_4 \\ 1 - x_1 y_1 & -x_1 y_2 & -x_1 y_3 & -x_1 y_4 \\ 0 & 1 - x_2 y_2 & -x_2 y_3 & -x_2 y_4 \\ 0 & 0 & 1 - x_3 y_3 & -x_3 y_4 \end{pmatrix}. \]

Written out, the differential equations (1.1.9) read

\[
\frac{\partial x_n}{\partial t_i} = (1 - x_n y_n) \frac{\partial H^{(1)}_i}{\partial y_n} \\
= \frac{h_n}{h_{n-1}} \frac{\partial}{\partial y_n} \left( \frac{1}{i} \text{Tr} L_i \right), \text{ using } 1 - x_n y_n = \frac{h_n}{h_{n-1}} \\
= \frac{h_n}{h_{n-1}} \left\langle L_i^{i-1}, -\frac{\partial L_i}{\partial y_n} \right\rangle \\
= \frac{h_n}{h_{n-1}} \left\langle L_i^{i-1}, \begin{pmatrix} O & \vdots & 0 \\ 0 & x_n & O \\ \vdots & x_{n+1} & x_{n+2} \\ \vdots & \vdots & \vdots \end{pmatrix} \right\rangle \\
\]

\[ = \frac{h_n}{h_{n-1}} \sum_{j \geq n} (L_i^{i-1})_{n+1,j} x_j, \quad (1.1.11) \]

and similarly

\[ \langle A, B \rangle = \text{tr} AB^\top, \]

which differentiated behaves as

\[ \frac{\partial}{\partial x} \text{tr} A^n = \langle n A^{n-1}, \frac{\partial A}{\partial x} \rangle. \]
\[
\frac{\partial x_n}{\partial s_i} = (1 - x_n y_n) \frac{\partial H^{(2)}_i}{\partial y_n} \\
= \frac{h_n}{h_{n-1}} \left\langle L^{i-1}_2, -\frac{\partial L_2}{\partial y_n} \right\rangle \\
= \frac{h_n}{h_{n-1}} \left\langle L^{i-1}_2, \begin{pmatrix} n & \downarrow & x_0 & \vdots & x_n & O \\ 0 & 0 & \vdots & \end{pmatrix} \right\rangle \\
= \frac{h_n}{h_{n-1}} \sum_{j=1}^{n+1} \left( L^{i-1}_2 \right)_{n,j} x_{j-1} \\
= \frac{h_n}{h_{n-1}} \sum_{j=n-i+1}^{n+1} \left( L^{i-1}_2 \right)_{n,j} x_{j-1} ,
\]

(1.1.12)

using in the last identity the obvious fact that
\[
(L^\alpha_2)_{ij} = 0, \text{ unless } j \geq i - \alpha . 
\]

(1.1.13)

By the duality, mentioned in (1.1.15) below, one reads off the differential equations for the \( y_n \)'s.

Setting
\[
\hat{L}_1 := hL_1 h^{-1} \text{ and } \hat{L}_2 := L_2 ,
\]
the “rank 2”-structure of \( \hat{L}_1 \) and \( \hat{L}_2 \) is preserved by the equations
\[
\frac{\partial \hat{L}_i}{\partial t_n} = \left[ \left( \hat{L}_1^n \right)_+, \hat{L}_i \right] \text{ and } \frac{\partial \hat{L}_i}{\partial s_n} = \left[ \left( \hat{L}_2^n \right)_-, \hat{L}_i \right] \text{ i = 1, 2 and } n = 1, 2, \ldots 
\]

(Two-Toda Lattice) (1.1.14)

The Toda and toeplitz lattices have an involution, compatible with the involution \( - \), introduced in (0.0.3),
\[
x_n \leftrightarrow y_n, \ t_n \leftrightarrow -s_n, \ L_1 \leftrightarrow L_2^\top.
\]

(1.1.15)
The Proposition below was merely taken for granted in [3]. Here we give a complete proof.

**Proposition 1.1** The two-Toda lattice flows (1.1.14) are equivalent to the Hamiltonian Toeplitz lattice flows (1.1.9).

**Proof:** From the general 2-Toda theory, as related to biorthogonal polynomials, we know that the vector (see [2])

\[ \Psi_1 := (\Psi_{1,n})_{n \geq 0} = e^{\sum \Sigma_1^\infty t_i z_i} (p_n^{(1)}(t,s;z))_{n \geq 0} \]  

(1.1.16)

with

\[ \Psi_{1,n} = e^{\sum \Sigma_1^\infty t_i z_i} z^n \frac{\tau_n(t - [z^{-1}], s)}{\tau_n(t,s)} = e^{\sum \Sigma_1^\infty t_i z_i} p_n^{(1)}(t,s;z), \]  

(1.1.17)

is an eigenvector for the matrix \( L_1 \),

\[ L_1 \Psi_1 = z \Psi_1, \]

and (1.1.16) satisfies the differential equations

\[ \frac{\partial \Psi_1}{\partial t_n} = (\hat{L}_1^n)_+ \Psi_1 \quad \text{and} \quad \frac{\partial \Psi_1}{\partial s_n} = (\hat{L}_2^n)_- \Psi_1. \]  

(1.1.18)

So, from (1.1.16), (1.1.17) and (1.1.18), it follows that the vector \( p^{(1)}(z) := (p_n^{(1)}(t,s;z))_{n \geq 0} \) satisfies the differential equations

\[ \frac{\partial p^{(1)}(z)}{\partial t_n} = (\hat{L}_1^n)_+ p^{(1)}(z) - z^n p^{(1)}(z) \]

\[ \frac{\partial p^{(1)}(z)}{\partial s_n} = (\hat{L}_2^n)_- p^{(1)}(z). \]  

(1.1.19)

Since (1.1.7) implies \( p^{(1)}(t,s;0) = (x_0, x_1, \ldots)^\top \), the differential equations (1.1.19) evaluated at \( z = 0 \) read

\[ \frac{\partial x}{\partial t_i} = (\hat{L}_1^i)_+ x \quad \text{and} \quad \frac{\partial x}{\partial s_i} = (\hat{L}_2^i)_- x, \]  

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yielding componentwise (remember $\hat{L}_1 = hL_1h^{-1}$)

$$\frac{\partial x_n}{\partial t_i} = h_n \sum_{j \geq n} (L^i_1)_{n+1,j+1} \frac{x_j}{h_j}$$  \hfill (1.1.20)

$$\frac{\partial x_n}{\partial s_i} = \sum_{1 \leq j \leq n} (L^i_2)_{n+1,j} x_{j-1} = \sum_{j=n+1-i}^{n} (L^i_2)_{n+1,j} x_{j-1}. \quad \hfill (1.1.21)$$

The point of Proposition 1.1 is to show that the equations (1.1.21), obtained
via the 2-Toda lattice, are equivalent to the equations (1.1.11), (1.1.12),
coming from the Toeplitz lattice, i.e., we must show, for $n \geq 0$,

$$\sum_{j \geq n} (L^i_1)_{n+1,j+1} \frac{x_j}{h_j} = \frac{1}{h_{n-1}} \sum_{j \geq n} (L^{i-1}_1)_{n+1,j} x_j \quad \hfill (1.1.22)$$

$$\sum_{j=n-i+1}^{n} (L^i_2)_{n+1,j} x_{j-1} = \frac{h_n}{h_{n-1}} \sum_{j=n-i+1}^{n+1} (L^{i-1}_2)_{nj} x_{j-1} \quad \hfill (1.1.23)$$

by duality, it suffices to show equations just for the $x_n$-variables.

To show (1.1.22), compute, using $x_j y_j = 1 - h_j/h_{j-1}$:

$$\sum_{j \geq n} (L^i_1)_{n+1,j+1} \frac{x_j}{h_j} = \sum_{r \geq j \geq n} (L^{i-1}_1)_{n+1,r} (L_1)_{r,j+1} \frac{x_j}{h_j}$$

$$= \sum_{r \geq j \geq n} (L^{i-1}_1)_{n+1,r} (\delta_{r,j} - x_r y_j) \frac{x_j}{h_j}$$

$$= \sum_{j \geq n} (L^{i-1}_1)_{n+1,j} \frac{x_j}{h_j} - \sum_{r \geq n} (L^{i-1}_1)_{n+1,r} x_r \left( \frac{1}{h_j} - \frac{1}{h_{j-1}} \right)$$

$$= \sum_{j \geq n} (L^{i-1}_1)_{n+1,j} \frac{x_j}{h_j} - \sum_{r \geq n} (L^{i-1}_1)_{n+1,r} x_r \left( \frac{1}{h_r} - \frac{1}{h_{n-1}} \right)$$

$$= \frac{1}{h_{n-1}} \sum_{r \geq n} (L^{i-1}_1)_{n+1,r} x_r$$

Next establish (1.1.23), using

$$(L^i_2)_{n+1,n} = h_n/h_{n-1} = 1 - x_n y_n \quad \hfill (1.1.24)$$

$$(L^i_2)_{n+1,r} x_{j-1} = x_n (L_2)_{jr} \quad \hbox{provided } r > n \quad \hbox{and } r > j - 1 \quad \hfill (1.1.25)$$
Indeed, setting \( i = k + 1 \), one computes

\[
\sum_{j=n-k}^{n} (L_2^{k+1})_{n+1,j} x_{j-1} - \frac{h_n}{h_{n-1}} \sum_{j=n-k}^{n+1} (L_2^k)_{nj} x_{j-1}
\]

\[
= \sum_{j=n-k}^{n} \sum_{r=n}^{j+k} (L_2)_{n+1,r} (L_2^k)_{rj} x_{j-1} - \frac{h_n}{h_{n-1}} \sum_{j=n-k}^{n+1} (L_2^k)_{nj} x_{j-1}
\]

\[
= \frac{h_n}{h_{n-1}} \sum_{j=n-k}^{n} (L_2^k)_{nj} x_{j-1} - \frac{h_n}{h_{n-1}} \sum_{j=n-k}^{n+1} (L_2^k)_{nj} x_{j-1}
\]

\[
\quad \quad \quad \quad + \sum_{j=n-k}^{n} \sum_{r=n+1}^{j+k} (L_2)_{n+1,r} (L_2^k)_{rj} x_{j-1} \quad \text{using (1.1.24)}
\]

\[
= x_n \left( -(L_2)_{n+1,n} (L_2^k)_{n,n+1} + \sum_{j=n-k}^{n} \sum_{r=n+1}^{j+k} (L_2)_{jr} (L_2^k)_{rj} \right)
\]

\[
= 0, \quad (1.1.26)
\]

using in \( \ast \) formulas (1.1.24), (1.1.25) and the inequalities \( n - k - 1 \leq j - 1 \leq n - 1 < n + 1 \leq r \leq j + k \).

As a last step, we show that indeed the last expression (1.1.26) vanishes. The first expression in the bracket equals

\[
(L_2)_{n+1,n} (L_2^k)_{n,n+1}
\]

\[
= (L_2)_{n+1,n} \sum_{n \leq \beta_1 + 1 \leq \beta_2 + 2 \leq \beta_3 + 3 \leq \ldots \leq \beta_{k-1} + k-1 \leq n + k + 1} (L_2)_{\beta_1 \beta_2 \ldots \beta_{k-1} n+1},
\]

\[
(1.1.27)
\]

whereas the second expression in the bracket of (1.1.26) equals, upon setting
\[ \alpha_0 = r, \alpha_k = j: \]
\[
\sum_{j=n-k}^{n} \sum_{r=n+1}^{j+k} (L_2)_{jr}(L^k_{2rj})
\]
\[
= \sum_{r=n+1}^{n+k} \sum_{j=r-k}^{n} (L_2)_{jr}(L^k_{2rj})
\]
\[
= \sum_{n+1 \leq \alpha_0 \leq \alpha_1+1 \leq \alpha_2+2 \leq \ldots \leq \alpha_{k-1}+k-1 \leq \alpha_k+k \leq n+k} (L_2)_{\alpha_0,\alpha_1}(L_2)_{\alpha_1,\alpha_2} \ldots (L_2)_{\alpha_{k-1},\alpha_k}(L_2)_{\alpha_k,\alpha_0}
\]
\[ (1.1.28) \]

Each term in this sum is a product of \( k+1 \) entries of \( L_2 \), with indices \( \alpha_i + i \), squeezed between \( n+1 \) and \( n+k \). Therefore, we must have for some \( 1 \leq j \leq k \) that
\[
\alpha_j + j = \alpha_{j+1} + j + 1 = n + j + 1,
\]
and so for that \( (\alpha_j, \alpha_{j+1}) \), we have
\[
(L_2)_{\alpha_j,\alpha_{j+1}} = (L_2)_{n+1,n},
\]
which appears in every term of the sum \((1.1.28)\); it can therefore be taken out, leaving sums of products of \( k \) terms. The inequalities under the summation sign of \((1.1.28)\) can then be written as follows
\[
\begin{align*}
n+j+1 & \leq \alpha_0 + k \leq \alpha_1 +1 \leq \ldots \leq \alpha_{j-1}+j-1 \leq \alpha_j+j = \alpha_{j+1}+j+1 \leq \alpha_{j+2}+j+2 \leq \ldots \leq \alpha_k+k \leq n+k
\end{align*}
\]

Add \( k-j \) to the sequence above, from \( n+1 \) up to including \( \alpha_j+j = n+j+1 \) and \( -j-1 \) to the sequence above, starting with \( \alpha_{j+1}+j+1 = n+j+1 \), up to \( n+k \), yielding the two sequences
\[
\begin{align*}
n+k-j+1 & \leq \alpha_0 + k-j \leq \ldots \leq \alpha_{j-1}+k-1 \leq n+k+1
\end{align*}
\]
and
\[
\begin{align*}
n & \leq \alpha_{j+2}+1 \leq \ldots \leq \alpha_k + k-j-1 \leq n+k-j-1
\end{align*}
\]
Since obviously \( n+k-j-1 < n+k-j+1 \), we have the inequalities appearing in the summation of formula \((1.1.27)\), but with \( \beta \)'s replaced by \( \alpha \)'s. This ends the proof of Proposition 1.1.
According to [3], we also have \((h := \text{diag}(h_0, h_1, \ldots) = \text{diag}(\frac{\alpha}{r_0}, \frac{\alpha}{r_1}, \ldots))\)

\[
\begin{align*}
x_{n+1}y_{n+1} &= 1 - \frac{h_{n+1}}{h_n} \\
x_{n+1}y_n &= -\frac{\partial}{\partial t_i} \log h_n \\
x_{n+1}y_{n-1} &= -\frac{h_{n-1}}{h_n} \left(\frac{\partial}{\partial t_i}\right)^2 \log \tau_n
\end{align*}
\]

\[
\begin{align*}
y_{n+1}x_{n+1} &= 1 - \frac{h_{n+1}}{h_n} \\
y_{n+1}x_n &= \frac{\partial}{\partial s_i} \log h_n \\
y_{n+1}x_{n-1} &= -\frac{h_{n-1}}{h_n} \left(\frac{\partial}{\partial s_i}\right)^2 \log \tau_n
\end{align*}
\] (1.1.29)

\[
\begin{align*}
x_{n+1}y_{n-k} &= -\frac{h_{n-k} p_{k+1} (\overline{\partial}_i) \tau_{n-k+1}^1 \circ \tau_n}{h_n} \\
y_{n+1}x_{n-k} &= -\frac{h_{n-k} p_{k+1} (-\overline{\partial}_i) \tau_{n-k+1}^1 \circ \tau_n}{h_n}, \quad k \geq 0. \quad (1.1.30)
\end{align*}
\]

**Lemma 1.2** The \(t_i\)- and \(s_i\)-derivatives of \(x_n\) and \(y_n\) can be expressed in terms of the elements on the main diagonal and one above and below the main diagonal of \(L^i_k\) and \(L^i_{k-1}\):

\[
\begin{align*}
\frac{x_n y_n}{v_n} \sum_{i \geq 1} \left( \alpha_i \frac{\partial}{\partial t_i} - \beta_i \frac{\partial}{\partial s_i} \right) \log x_n &= y_n \frac{\partial}{\partial y_n} \sum_{i \geq 1} \left( \alpha_i H^{(1)}_i - \beta_i H^{(2)}_i \right) \\
&= \sum_{i \geq 1} \left\{ \alpha_i \left( -(L^i_1)_{n+1,n+1} + (L^{i-1}_1)_{n+1,n} \right) + \beta_i \left( (L^i_2)_{n,n} - (L^{i-1}_2)_{n,n+1} \right) \right\} \\
\frac{x_n y_n}{v_n} \sum_{i \geq 1} \left( \alpha_i \frac{\partial}{\partial t_i} - \beta_i \frac{\partial}{\partial s_i} \right) \log y_n &= -x_n \frac{\partial}{\partial x_n} \sum_{i \geq 1} \left( \alpha_i H^{(1)}_i - \beta_i H^{(2)}_i \right) \\
&= \sum_{i \geq 1} \left\{ \alpha_i \left( (L^i_1)_{n,n} - (L^{i-1}_1)_{n+1,n} \right) - \beta_i \left( (L^i_2)_{n+1,n+1} - (L^{i-1}_2)_{n,n+1} \right) \right\}
\end{align*}
\] (1.1.31)
In particular, we have

\[
\frac{x_n y_n}{v_n} \left( e \frac{\partial}{\partial t_1} - a \frac{\partial}{\partial s_1} \right) \log x_n = y_n \frac{\partial}{\partial y_n} (c H_1^{(1)} - a H_1^{(2)}) = -a x_n y_n + c x_{n+1} y_n = a(L_2)_{nn} - c(L_1)_{n+1,n+1}
\]

\[
\frac{x_n y_n}{v_n} \left( e \frac{\partial}{\partial t_1} - a \frac{\partial}{\partial s_1} \right) \log y_n = -x_n \frac{\partial}{\partial x_n} (c H_1^{(1)} - a H_1^{(2)}) = ax_n y_{n+1} - c x_n y_{n-1} = -a(L_2)_{n+1,n+1} + c(L_1)_{nn} \quad (1.1.32)
\]

**Proof:** The first equality in each of the identities above follows immediately from the Hamiltonian vector fields (1.1.9), with Hamiltonians (1.1.10). Note the following

\[
\begin{pmatrix}
-x_n y_0 & -x_n y_1 & \ldots & -x_n y_{n-1} & -x_n y_n + x_n y_n & 0 & \ldots \\
-0 & -x_n y_1 & \ldots & -x_n y_{n-1} & -x_n y_n + x_n y_n & 0 & \ldots \\
-x_n y_1 & -x_n y_1 & \ldots & -x_n y_{n-1} & -x_n y_n + x_n y_n & 0 & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
x_n y_n & x_n y_n & \ldots & x_n y_n & x_n y_{n+1} & 0 & \ldots \\
0 & 0 & \ldots & 0 & x_n y_{n+1} & 0 & \ldots \\
O & O & \ldots & O & 0 & 0 & \ldots
\end{pmatrix}
\]

and
\[
\left( x_n \frac{\partial}{\partial x_n} - y_n \frac{\partial}{\partial y_n} \right) L_2 = \begin{pmatrix}
\vdots \\
x_0 y_n \\
0 & \cdots & \cdots & 0 & x_n y_n - x_n y_{n+1} - x_n y_{n+2} & \cdots \\
O & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}^{n} \\
\downarrow \\
\begin{pmatrix}
x_n - y_n \frac{\partial}{\partial y_n} \\
0 & \cdots & \cdots & 0 & 0 & \cdots \\
O & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}^{n+1} \rightarrow \begin{pmatrix}
x_n - y_n \frac{\partial}{\partial y_n} \\
0 & \cdots & \cdots & 0 & 0 & \cdots \\
O & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}^{n+1}
\]

We shall also need the following trivial identities:

\[
(L_i^n)_{nn} = \begin{cases}
(L_1 L_1^{i-1})_{nn} = -x_n \sum_{j=1}^{n+1} y_{j-1} (L_1^{i-1})_{jn} + (L_1^{i-1})_{n+1,n} \\
(L_1^{i-1} L_1)_{nn} = -y_{n-1} \sum_{j=n-2}^{n+i-2} x_{j+1} (L_1^{i-1})_{n,j+1} + (L_1^{i-1})_{n,n-1}
\end{cases}
\]

and

\[
(L_i^n)_{nn} = \begin{cases}
(L_2 L_2^{i-1})_{nn} = -x_{n-1} \sum_{j=n-2}^{n+i-2} y_{j+1} (L_2^{i-1})_{j+1,n} + (L_2^{i-1})_{n-1,n} \\
(L_2^{i-1} L_2)_{nn} = -y_n \sum_{j=1}^{n+1} x_{j-1} (L_2^{i-1})_{n,j} + (L_2^{i-1})_{n,n+1}
\end{cases}
\]

(1.1.33)

(1.1.34)
We now have (see the definition (1.1.10) of $H^{(k)}_i$)

\[
y_n \frac{\partial}{\partial y_n} \sum_{i \geq 1} \left( \alpha_i H^{(1)}_i - \beta_i H^{(2)}_i \right)
\]

\[
= -y_n \frac{\partial}{\partial y_n} \sum_{i \geq 1} \text{tr} \left( \frac{\alpha_i}{i} L_1^i - \frac{\beta_i}{i} L_2^i \right)
\]

\[
= \left\langle \sum_{i \geq 1} \alpha_i L_1^{i-1}, -y_n \frac{\partial}{\partial y_n} L_1 \right\rangle - \left\langle \sum_{i \geq 1} \beta_i L_2^{i-1}, -y_n \frac{\partial}{\partial y_n} L_2 \right\rangle
\]

\[
= \sum_{i \geq 1} \alpha_i \left( y_n \sum_{j=n-1}^{n+i-1} x_{j+1} (L_1^{i-1})_{n+1,j+1} \right) + \sum_{i \geq 1} \beta_i \left( -y_n \sum_{j=1}^{n+1} x_{j-1} (L_2^{i-1})_{n,j} \right)
\]

using (1.1.33)

\[
= \sum_{i \geq 1} \alpha_i \left( -(L_1^i)_{n+1,n+1} + (L_1^{i-1})_{n+1,n} \right) + \sum_{i \geq 1} \beta_i \left( (L_2^i)_{nn} - (L_2^{i-1})_{n,n+1} \right),
\]

using in the last equality identities (1.1.34).

Similarly, one computes

\[
-x_n \frac{\partial}{\partial x_n} \sum_{i \geq 1} \left( \alpha_i H^{(1)}_i - \beta_i H^{(2)}_i \right)
\]

\[
= x_n \frac{\partial}{\partial x_n} \sum_{i \geq 1} \text{tr} \left( \frac{\alpha_i}{i} L_1^i - \frac{\beta_i}{i} L_2^i \right)
\]

\[
= \left\langle \sum_{i \geq 1} \alpha_i L_1^{i-1}, x_n \frac{\partial}{\partial x_n} L_1 \right\rangle - \left\langle \sum_{i \geq 1} \beta_i L_2^{i-1}, x_n \frac{\partial}{\partial x_n} L_2 \right\rangle
\]

\[
= \sum_{i \geq 1} \alpha_i \left( -x_n \sum_{j=1}^{n+1} y_{j-1} (L_1^{i-1})_{j,n} \right) + \sum_{i \geq 1} \beta_i \left( x_n \sum_{j=n-1}^{n+i-1} y_{j+1} (L_2^{i-1})_{j+1,n+1} \right)
\]

\[
= \sum_{i \geq 1} \alpha_i \left( (L_1^i)_{nn} - (L_1^{i-1})_{n+1,n} \right) + \sum_{i \geq 1} \beta_i \left( -(L_2^i)_{n+1,n+1} + (L_2^{i-1})_{n,n+1} \right)
\]
The last couple of relations (1.1.32) follow from specializing (1.1.31) to $i = 1$, thus ending the proof of Lemma 1.2.

Lemma 1.3

\[
\begin{align*}
\left( \frac{\partial L_i}{\partial t_1} \right)_{nn} &= v_n(L^i_1)_{n+1,n} - v_{n-1}(L^i_1)_{n,n-1} \\
\left( \frac{\partial L^i_2}{\partial t_1} \right)_{nn} &= (L^i_2)_{n+1,n} - (L^i_2)_{n,n-1} \\
\left( \frac{\partial L^i_2}{\partial s_1} \right)_{nn} &= (L^i_1)_{n-1,n} - (L^i_1)_{n,n+1} \\
\left( \frac{\partial L^i_2}{\partial s_1} \right)_{nn} &= v_{n-1}(L^i_2)_{n-1,n} - v_n(L^i_2)_{n,n+1}
\end{align*}
\]

Proof: From the Toda equations (1.1.14) for $\hat{L}_i$, one computes the $t_1$-flow for $\hat{L}_1$, where $h := \text{diag}(h_1, h_2, \ldots)$ and where $A_+, A_{++}$ and $A_0$ denote the upper-triangular, strictly upper-triangular and diagonal part of the matrix $A$, (remember $L_1 = h^{-1}\hat{L}_1 h$)

\[
\frac{\partial L^i_1}{\partial t_1} = \frac{\partial}{\partial t_1}(h^{-1}\hat{L}^i_1 h)
\]

\[
= h^{-1}\frac{\partial \hat{L}^i_1}{\partial t_1} h - \frac{\partial \log h}{\partial t_1} h^{-1}\hat{L}^i_1 h + h^{-1}\hat{L}^i_1 h \frac{\partial \log h}{\partial t_1}
\]

\[
= [h^{-1}(\hat{L}^i_1)_+ h, h^{-1}\hat{L}^i_1 h] - \left[ \frac{\partial \log h}{\partial t_1}, h^{-1}\hat{L}^i_1 h \right]
\]

\[
= [(L^i_1)_{++}, L^i_1], \quad \text{using (1.1.29)}
\]

\[
= \text{diag}(v_1, v_2, \ldots) \Lambda, L^i_1).
\]

Hence, setting $v_n = h_n/h_{n-1} = 1 - x_n y_n$

\[
\left( \frac{\partial L^i_1}{\partial t_1} \right)_0 = \text{diag} (v_1(L^i_1)_{21}, v_2(L^i_1)_{32} - v_1(L^i_1)_{21}, \ldots).
\]

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In particular,
\[
\left(\frac{\partial L_i}{\partial t_1}\right)_{n,n} = v_n (L_i^1)_{n+1,n} - v_{n-1} (L_i^1)_{n,n-1}
\]
\[
\left(\frac{\partial L_i^1}{\partial t_1}\right)_{n,n} = -v_n x_{n+1} y_{n-1} + v_{n-1} x_n y_{n-2}.
\]

We also need the \(t_1\)-derivative of \(L_2^i\),
\[
\frac{\partial L_2^i}{\partial t_1} = [(\hat{L}_1), L_2^i] = \begin{pmatrix}
-x_1 y_0 & 1 & & \\
-x_2 y_1 & 1 & & \\
& -x_3 y_2 & 1 & \\
& & & \ddots
\end{pmatrix}, L_2^i
\]
hence
\[
\left(\frac{\partial L_2^i}{\partial t_1}\right)_0 = \text{diag} ((L_2^i)_{21}, (L_2^i)_{32} - (L_2^i)_{21}, (L_2^i)_{43} - (L_2^i)_{32}, \ldots ),
\]
leading to
\[
\left(\frac{\partial L_2^i}{\partial t_1}\right)_{n,n} = (L_2^i)_{n+1,n} - (L_2^i)_{n,n-1}
\]
\[
\left(\frac{\partial L_2^i}{\partial t_1}\right)_{n,n} = x_{n-1} y_{n-1} - x_n y_n,
\]
while the latter relations of Lemma 1.3 are obtained from the first two by the 2-Toda involution \(1.1.15\). This ends the proof of Lemma 1.3. ■

### 1.2 Virasoro constraints

According to \(\text{[3]}\), the (vector) vertex operator\(\text{[3]}\)
\[
X_{12}(t, s; u, v) = \Lambda^{-1} e^{\sum_i^{\infty} (t_i u^i - s_i v^i)} e^{-\sum_i^{\infty} (u_i \frac{\partial}{\partial t_i} - v_i \frac{\partial}{\partial s_i})} \chi(uv),
\]
\(\text{[3]}\)For \(v = (v_0, v_1, \ldots)^\top\), \((\Lambda v)_n = v_{n+1}\), \((\Lambda^\top v)_n = v_{n-1}\), and \(\chi(z) := (1, z, z^2, \ldots)\).
acting on vectors of functions of \( t \) and \( s \), interacts with the operators \( \mathcal{J}_k^{(i)}(t) = \left( \mathcal{J}_{k,n}^{(i)}(t, n) \right)_{n \geq 0} \), as follows: (for definitions, see the appendix 1)
\[
\begin{align*}
\mathcal{X}_{12}(t, s; u, v) &= \left[ \mathcal{J}_k^{(1)}(t), \mathcal{X}_{12}(t, s; u, v) \right] \\
\frac{\partial}{\partial u} \mathcal{X}_{12}(t, s; u, v) &= \left[ \mathcal{J}_k^{(2)}(t), \mathcal{X}_{12}(t, s; u, v) \right].
\end{align*}
\] (1.2.2)

A similar statement can be made, upon replacing the operators \( \mathcal{X}_{12} \) and \( \frac{\partial}{\partial u} \) by \( \mathcal{V}_k^{(i)} \) and \( \frac{\partial}{\partial v} \), and upon using \( \mathcal{J}_k^{(i)}(s) = \mathcal{J}_k^{(i)}(-s) \).

Also consider the vertex operator, integrated over the unit circle and depending on an integer \( \gamma \),
\[
\mathcal{V}^\gamma(t, s) = \int_{S^1} \frac{du}{2\pi i u} u^\gamma \mathcal{X}_{12}(t, s; u, u^{-1}).
\] (1.2.3)

and the vector Virasoro constraint \( \mathcal{V}_k(t, s) := (\mathcal{V}_{k,n}^\gamma)_{n \geq 0} \)
\[
\mathcal{V}_k := \mathcal{J}_k^{(2)}(t) - \mathcal{J}_k^{(2)}(-s) - (k - \gamma) \left( \theta \mathcal{J}_k^{(1)}(t) + (1 - \theta) \mathcal{J}_k^{(1)}(-s) \right)
\] (1.2.4)
depending on a free parameter \( \theta \).

**Theorem 1.4** (Adler-van Moerbeke [3]) *The multiple integrals over the unit circle \( S^1 \),
\[
\tau_n^\gamma(t, s) = \frac{1}{n!} \int_{(S^1)^2} |\Delta_n(z)|^2 \prod_{k=1}^n z_k^{\gamma} e^{\sum_{k=1}^n \left( t_k z_k - s_k z_k^{-1} \right)} \frac{d\zeta_k}{2\pi i \zeta_k}, \quad n > 0,
\] (1.2.5)
with \( I_0 = 1 \), satisfy an \( SL(2, \mathbb{Z}) \)-algebra of Virasoro constraints:
\[
\mathcal{V}_{k,n}^\gamma \tau_n(t, s) = 0, \quad \text{for } \left\{ \begin{array}{l}
k = -1, \quad \theta = 0 \\
k = 0, \quad \theta \text{ arbitrary} \\
k = 1, \quad \theta = 1
\end{array} \right\} \text{ only.}
\] (1.2.6)

Working out the Virasoro equations of Appendix 1 (for \( \beta = 1/2 \))
\[
\mathcal{V}_{k,n}^\gamma := \mathcal{J}_{k,n}^{(2)}(t, n) - \mathcal{J}_{k,n}^{(2)}(-s, n) - (k - \gamma) \left( \theta \mathcal{J}_{k,n}^{(1)}(t, n) + (1 - \theta) \mathcal{J}_{k,n}^{(1)}(-s, n) \right)
\] (1.2.7)
one finds Virasoro constraints for the integral $\tau_n^\gamma$:

\[
\mathcal{V}_{-1,n}^\gamma \tau_n^\gamma = \left( \sum_{i \geq 1} (i+1)t_{i+1} \frac{\partial}{\partial t_i} - \sum_{i \geq 2} (i-1)s_{i-1} \frac{\partial}{\partial s_i} + nt_1 + (n-\gamma)nt_1 \right) \tau_n^\gamma = 0
\]

\[
\mathcal{V}_{0,n}^\gamma \tau_n^\gamma = \sum_{i \geq 1} \left( it_i \frac{\partial}{\partial t_i} - is_i \frac{\partial}{\partial s_i} \right) \tau_n^\gamma + \gamma n \tau_n^\gamma = 0
\]  \hspace{1cm} (1.2.8)

\[
\mathcal{V}_{1,n}^\gamma \tau_n^\gamma = \left( - \sum_{i \geq 1} (i+1)s_{i+1} + \sum_{i \geq 2} (i-1)t_{i-1} \frac{\partial}{\partial t_i} + ns_1 + (n+\gamma)nts_1 \right) \tau_n^\gamma = 0.
\]

The theorem is based on two lemmas:

**Lemma 1.5** The following commutation relations hold:

\[
u^{-\gamma} u \frac{d}{du} u^{k+\gamma} \mathcal{X}_{12}(t, s; u, u^{-1}) = [\mathcal{V}_k^\gamma, \mathcal{X}_{12}(t, s; u, u^{-1})], \quad (1.2.9)
\]

and

\[
[\mathcal{Y}^\gamma, \mathcal{V}_k^\gamma] = 0. \quad (1.2.10)
\]

**Proof:** Using (1.2.2), a standard computation shows

\[
u \frac{d}{du} u k \mathcal{X}_{12}(t, s; u, u^{-1})
\]

\[
= \left( u^{k+1} \frac{d}{du} u k \mathcal{X}_{12}(t, s; u, u^{-1}) \right)
\]

\[
= \left( u^{k+1} \frac{\partial}{\partial u} - v^{1-k} \frac{\partial}{\partial v} + ku \mathcal{X}_{12}(t, s; u, v) \right) \bigg|_{v=u^{-1}}
\]

\[
= \left( \frac{\partial}{\partial u} u^{k+1} - \frac{\partial}{\partial v} v^{1-k} - ku \mathcal{X}_{12}(t, s; u, v) \right) \bigg|_{v=u^{-1}}
\]

\[
= \left[ \mathcal{Y}_{k}^{(2)}(t) - \mathcal{Y}_{-k}^{(2)}(-s) - k \left( \theta \mathcal{Y}_{k}^{(1)}(t) + (1-\theta) \mathcal{Y}_{-k}^{(1)}(-s) \right) \right. \mathcal{X}_{12}(t, s; u, u^{-1})
\]

\[
= \left[ \mathcal{V}_{k}^{(0)}, \mathcal{X}_{12}(t, s; u, u^{-1}) \right], \quad (1.2.12)
\]

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from which (1.2.9) follows, for \( \gamma = 0 \). More generally, we compute

\[
\begin{align*}
    u^{-\gamma} u \frac{d}{du} u^{k+\gamma} X_{12}(t, s; u, u^{-1}) &= (u^{k+1} + ku) X_{12}(t, s; u, u^{-1}) + \gamma u^k X_{12}(t, s; u, u^{-1}) \\
    &= \left[ \mathcal{V}_k^{(0)} X_{12}(t, s; u, u^{-1}) \right] + \gamma \left( \theta u^k + (1 - \theta) v^{-k} \right) \bigg|_{v=u^{-1}} X_{12} \\
    &= \left[ \mathcal{V}_k^{(0)} + \gamma \left( \theta \mathcal{J}_k^{(1)}(t) + (1 - \theta) \mathcal{J}_k^{(1)}(-s) \right), X_{12}(t, s; u, u^{-1}) \right] \\
    &= \left[ \mathcal{V}_k, X_{12}(t, s; u, u^{-1}) \right] 
\end{align*}
\]

from which (1.2.9) follows. We then have

\[
\begin{align*}
    \left[ \mathcal{V}_k^\gamma, \mathcal{Y}_\gamma(t, s) \right] &= \left[ \mathcal{V}_k^\gamma, \int_{S^1} X_{12}(t, s; u, u^{-1}) u^\gamma \frac{du}{2\pi i u} \right] \\
    &= \int_{S^1} \left[ \mathcal{V}_k^\gamma, X_{12}(t, s; u, u^{-1}) \right] u^\gamma \frac{du}{2\pi i u} \\
    &= \int_{S^1} \frac{du}{2\pi i} \frac{d}{du} u^{k+\gamma} X_{12}(t, s; u, u^{-1}) \\
    &= 0,
\end{align*}
\]

leading to (1.2.10).

**Lemma 1.6** The vector \( I := (I_n)_{n \geq 0} \), with \( I_n = n! \tau^\gamma_n \), is a fixed point for the vertex operator \( \mathcal{Y}_\gamma \),

\[
\mathcal{Y}_\gamma(t, s) I(t, s) = I(t, s).
\]
Proof: Setting $\rho(dz) = z^\gamma dz$, one computes, for $n \geq 1$,

\[
I_n(t, s) = n! \tau_n(t, s) = \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left( e^{\sum_{i=1}^\infty (t_i z_k^{-i} - s_i z_k^{-i})} \rho(dz_k) \right) \frac{2\pi i z_k}{2\pi}
\]

\[
= \int_{S^1} \frac{\rho(du)}{2\pi i u} e^{\sum_{i=1}^\infty (t_i u^{-i} - s_i u^{-i})} u^{n-1} u^{-n+1} \int_{(S^1)^{n-1}} \Delta_{n-1}(z) \bar{\Delta}_{n-1}(z) \prod_{k=1}^{n-1} \left( 1 - \frac{z_k}{u} \right) \frac{1}{u} \sum_{i=1}^\infty \left( \frac{1}{u_i - \bar{u}_i} \frac{1}{v_i - \bar{v}_i} \right) e^{\sum_{i=1}^\infty (t_i z_k^{-i} - s_i z_k^{-i})} \rho(dz_k) \frac{2\pi i z_k}{2\pi}
\]

\[
= \int_{S^1} \frac{\rho(du)}{2\pi i u} e^{\sum_{i=1}^\infty (t_i u^{-i} - s_i u^{-i})} e^{\sum_{i=1}^\infty \left( \frac{1}{u_i - \bar{u}_i} \frac{1}{v_i - \bar{v}_i} \right)} \prod_{k=1}^{n-1} \Delta_{n-1}(z) \bar{\Delta}_{n-1}(z) I_{n-1}(t, s)
\]

\[
= \left( \mathbb{Y}^\gamma(t, s) I(t, s) \right)_n, \tag{1.2.16}
\]

from which (1.2.15) follows.

Proof of Theorem 1.4: From (1.2.10), we have

\[
0 = (\mathbb{V}_k + \mathbb{Y}^\gamma)^n I_n = (\mathbb{V}_k^\gamma)^n I - (\mathbb{Y}^\gamma)^n \mathbb{V}_k^\gamma I_n = (\mathbb{V}_k^\gamma I - (\mathbb{Y}^\gamma)^n \mathbb{V}_k^\gamma I)_n.
\]

Taking the $n^{th}$ component and taking into account the presence of $\Lambda^{-1}$ in $X(t, s; u, u^{-1})$, we find

\[
0 = (\mathbb{V}_k^\gamma I - \mathbb{V}_k^\gamma \mathbb{V}_k^\gamma I)_n
\]

\[
= \mathbb{V}_k^\gamma I_n - \int_{S^1} \frac{du}{2\pi i u} e^{\sum_{i=1}^\infty (t_i u^{-i} - s_i u^{-i})} e^{\sum_{i=1}^\infty \left( \frac{u_i}{\bar{u}_i} \frac{v_i}{\bar{v}_i} \right)} \mathbb{V}_k^\gamma I_0.
\]
Remember from (1.2.7), $\mathcal{V}_k^r(t,s)$ has the following form

$$\begin{align*}
\mathcal{V}_k^r(t,s) &= \frac{1}{2} \left( J_k^{(2)}(t) - J_{-k}^{(2)}(-s) + (2n + k + 1)J_k^{(1)}(t) - (2n - k + 1)J_{-k}^{(1)}(-s) \right) \\
&\quad - (k - \gamma) \left( \theta J_k^{(1)}(t) + (1 - \theta)J_{-k}^{(1)}(-s) \right) + \gamma n \delta_{k,0}.
\end{align*}$$

and one checks immediately that, given $\tau_0 = 1$,

$$\mathcal{V}_k^r(t,s)\tau_0 = 0 \quad \text{only for} \quad \begin{cases} 
  k = -1, \; \theta = 0 \\
  k = 0, \; \theta \; \text{arbitrary} \\
  k = 1, \; \theta = 1
\end{cases},$$

ending the proof of Theorem 1.4. \hfill \blacksquare

\section{Rational recursion relations}

\subsection{Weights}

\begin{lemma}

Lemma 2.1

\end{lemma}

\begin{align*}
\mathcal{L} &= \left\{ \begin{array}{l}
  i t_i = i t_i^{(0)} := \begin{cases} 
    u_i - (\gamma'_1 d_1^i + \gamma'_2 d_2^i), & \text{for } 1 \leq i \leq N_1 \\
    - (\gamma'_1 d_1^i + \gamma'_2 d_2^i), & \text{for } N_1 + 1 \leq i < \infty
  \end{cases} \\
  i s_i = i s_i^{(0)} := \begin{cases} 
    - u_i + (\gamma''_1 d_1^{-i} + \gamma''_2 d_2^{-i}), & \text{for } 1 \leq i \leq N_2 \\
    (\gamma''_1 d_1^{-i} + \gamma''_2 d_2^{-i}), & \text{for } N_2 + 1 \leq i < \infty
  \end{cases}
\end{array} \right. \\
&= (2.1.1)
\end{align*}

Then, setting $\gamma_1 = \gamma'_1 + \gamma'_1$ and $\gamma_2 = \gamma'_2 + \gamma''_2$, we have

$$e^{\sum_{i=1}^{\infty} (t_i z^{i-1} s_i z^{-i})} \Big|_{\mathcal{L}} = e^{P_1(z) + P_2(z^{-1})} (1 - d_1 z)^{\gamma'_1} (1 - d_2 z)^{\gamma'_2} (1 - d_1^{-1} z^{-1})^{\gamma''_1} (1 - d_2^{-1} z^{-1})^{\gamma''_2}$$

$$= k z^{-\gamma''_1 - \gamma''_2} e^{P_1(z) + P_2(z^{-1})} (1 - d_1 z)^{\gamma_1} (1 - d_2 z)^{\gamma_2} \quad (2.1.2)$$

with a constant $k$ and

$$P_1(z) := \sum_{i=1}^{N_1} \frac{u_i z^i}{i} \quad \text{and} \quad P_2(z^{-1}) := \sum_{i=1}^{N_2} \frac{u_i z^{-i}}{i}. \quad (2.1.3)$$

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Moreover, there exist \(a, b, c\) such that
\[
a(i + 1)t_{i+1}^{(0)} + bit_i^{(0)} + c(i - 1)t_{i-1}^{(0)} = 0 \quad \text{for all } i \geq N_1 + 2 \\
a(i - 1)s_{i-1}^{(0)} + bis_i^{(0)} + c(i + 1)s_{i+1}^{(0)} = 0 \quad \text{for all } i \geq N_2 + 2. \quad (2.1.4)
\]

Then
\[
a(i + 1)t_{i+1}^{(0)} + bit_i^{(0)} + c(i - 1)t_{i-1}^{(0)} = a\nu_{i+1} + bu_i + cu_{i-1} + c\delta_{i1}(\gamma_1' + \gamma_2'), \\
\quad \text{for } 1 \leq i \leq N_1 + 1
\]
\[
a(i - 1)s_{i-1}^{(0)} + bis_i^{(0)} + c(i + 1)s_{i+1}^{(0)} = -cu_{-i-1} - bu_{-i} - au_{-i+1} - a\delta_{i1}(\gamma_1'' + \gamma_2'') \\
\quad \text{for } 1 \leq i \leq N_2 + 1. \quad (2.1.5)
\]

upon setting \(u_0 = u_{N_1+1} = u_{N_2+2} = u_{-N_2-1} = u_{-N_2-2} = 0\).

- **Case 1.** \(d_1, d_2, d_1 - d_2 \neq 0\) and \(|\gamma_1'| + |\gamma_1''|, |\gamma_2'| + |\gamma_2''| \neq 0\). Then the unique solution to (2.1.4) is given by
  \[
a = 1, \ b = -(d_1 - d_2), \ c = d_1d_2.
\]

- **Case 2.** \(d_1 \neq 0, \gamma_1' \neq 0\) arbitrary, \(d_2 = 0, \gamma_2' = \gamma_1'' = \gamma_2'' = 0\). Then there exist two solutions
  \[
  (a, b, c) = (1, -d_1, 0) \quad \text{and} \quad (a, b, c) = (0, 1, -d_1)
  \]
such that (2.1.4) holds.

- **Case 3.** \(d_1 = d_2 = 0, \gamma_1' = \gamma_2' = \gamma_1'' = \gamma_2'' = 0\). Then \(a, b, c\) may be taken arbitrary.

**Proof:** Formula (2.1.2) follows immediately from \(1 - x = \exp\left(-\sum_1^\infty x^i/i\right)\), while (2.1.4) and (2.1.5) are obvious.

**Remark:** The locus \(L\), defined in (2.1.1), provides the only example where (2.1.4) holds.
2.2 Rational recursion relations

Considering the $t$-dependent basic variables,

\[ x_n(t) = (-1)^n \frac{\tau_+^n(t)}{\tau_n(t)} \quad \text{and} \quad y_n(t) = (-1)^n \frac{\tau_-^n(t)}{\tau_n(t)}, \quad (2.2.1) \]

where $\tau_n$ is the integral

\[ \tau_n^\pm = \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left( z_k^{\gamma \pm 1} e^{\sum_{i=1}^{\infty} (t_i z_i^+ - s_i z_i^-)} \frac{dz_k}{2\pi i z_k} \right) \]

\[ \tau_n = \frac{1}{n!} \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left( z_k^{\gamma} e^{\sum_{i=1}^{\infty} (t_i z_i^+ - s_i z_i^-)} \frac{dz_k}{2\pi i z_k} \right). \]

Along the locus $L$, defined in (2.1.1), these integrals and variables reduce to the original integrals and variables (0.0.5) and (0.0.6). In the statement below, we deal with the variables $x_n(t)$ and $y_n(t)$, without restricting to the locus $L$.

**Theorem 2.2** Set $v_n := 1 - x_n y_n$,

\[ \alpha_i(t) := a(i+1) t_{i+1} + b t_i + c(i-1) t_{i-1} + c(n+\gamma) \delta_{i1} \]

\[ \beta_i(s) := a(i-1) s_i - b s_i + c(i+1) s_{i+1} - a(n-\gamma) \delta_{i1}, \]

and

\[ L_1^{(n)} = \sum_{i \geq 1} \alpha_i(t) L_1^i \quad \text{and} \quad L_2^{(n)} = -\sum_{i \geq 1} \beta_i(t) L_2^i. \quad (2.2.2) \]

Then the following holds:

- **CASES 1 and 2**: $a, c$ not both $= 0$:

  \[ \partial_n (L_1^{(n)} - L_2^{(n)})_{nn} + (c L_1 - a L_2)_{nn} = 0 \quad (2.2.3) \]

  and

  \[ \partial_n^2 \left( v_{n-1} L_1^{(n-1)} - L_2^{(n-1)} \right)_{n,n-1} + \partial_n \left( c(L_1^2)_{nn} + b(L_1)_{nn} \right) = 0. \quad (2.2.4) \]
CASE 3: $a = c = 0, \ b = 1$:

$$
\begin{align*}
\sum_{i \geq 1} \left\{ \alpha_i \left( (L_1^i)_{n+1,n+1} - (L_1^{i-1})_{n+1,n} \right) - \beta_i \left( (L_2^i)_{nn} - (L_2^{i-1})_{n,n+1} \right) \right\} \\
\sum_{i \geq 1} \left\{ \alpha_i \left( (L_2^i)_{nn} - (L_1^{i-1})_{n+1,n} \right) - \beta_i \left( (L_2^i)_{n+1,n+1} - (L_2^{i-1})_{n,n+1} \right) \right\}
\end{align*}
$$

$$
- n \frac{x_n y_n}{v_n} = 0
$$

(2.2.5)

Remark: Written out, the equations (2.2.3) and (2.2.4) take on the following form

$$(\mathcal{L}_1 - \mathcal{L}_2 + aL_2 - cL_1)_{nn} - (\mathcal{L}_1 - \mathcal{L}_2 - aL_2 + cL_1)_{n+1,n+1} = 0$$

(2.2.6)

and

$$2(v_n \mathcal{L}_1 - \mathcal{L}_2)_{n+1,n} - (v_{n-1} \mathcal{L}_1 - \mathcal{L}_2)_{n,n-1} - (v_{n+1} \mathcal{L}_1 - \mathcal{L}_2)_{n+2,n+1}$$

$$+ a(v_{n+1} - v_{n-1}) + b(x_{n+1} y_n - x_n y_{n-1})$$

$$+ c(2y_n x_{n+2} v_{n+1} - 2x_n y_{n-2} v_{n-1} + x_n^2 y_{n-1}^2 - y_n^2 x_{n+1}^2) = 0$$

(2.2.7)

Proof: At first, compute the first and second difference in (2.2.3) and (2.2.4),

$$\partial_n (L_1^{(n)} - L_2^{(n)})_{n,n}$$

$$= (L_1^{(n)} - L_2^{(n)})_{n+1,n+1} - (L_1^{(n)} - L_2^{(n)})_{n,n} + (cL_1 - aL_2)_{n+1,n+1}$$

(2.2.8)

$$\partial_n^2 (v_{n-1} L_1^{(n)} - L_2^{(n)})_{n,n-1}$$

$$= (v_{n+1} L_1^{(n)} - L_2^{(n)})_{n+2,n+1} + (v_{n-1} L_1^{(n)} - L_2^{(n)})_{n,n-1} - 2(v_n L_1^{(n)} - L_2^{(n)})_{n+1,n}$$

$$+ 2c (v_{n+1} (L_1)_{n+2,n+1} - v_n (L_1)_{n+1,n}) - 2a ((L_2)_{n+2,n+1} - (L_2)_{n+1,n})$$

$$= (v_{n+1} L_1^{(n)} - L_2^{(n)})_{n+2,n+1} + (v_{n-1} L_1^{(n)} - L_2^{(n)})_{n,n-1} - 2(v_n L_1^{(n)} - L_2^{(n)})_{n+1,n}$$

$$+ 2\partial_n (c v_n (L_1)_{n+1,n} - a (L_2)_{n+1,n})$$

(2.2.9)
In order to obtain the first identity, namely (2.2.3), form arbitrary linear combinations of the Virasoro equations (1.2.8),

\[
\frac{1}{\tau_n^{\gamma+\epsilon}} \left( a\mathcal{V}^{-\epsilon}_{-1,n} + b\mathcal{V}^{\gamma+\epsilon}_{0,n} + c\mathcal{V}^{\gamma+\epsilon}_{1,n} \right) \tau_n^{\gamma+\epsilon} = \sum_{i \geq 1} \left( (a(i+1)t_{i+1} + bit_i + c(i-1)t_{i-1}) \frac{\partial}{\partial t_i} - (a(i-1)s_{i-1} + bis_i + c(i+1)s_{i+1}) \frac{\partial}{\partial s_i} + c(n + \gamma) \frac{\partial}{\partial t_1} + a(n - \gamma) \frac{\partial}{\partial s_1} \right) \log \tau_n^{\gamma+\epsilon} + \epsilon \left( -a \frac{\partial}{\partial s_1} + c \frac{\partial}{\partial t_1} \log \tau_n^{\gamma+\epsilon} + bn \right) + b\gamma n.
\]

Subtracting the \(\epsilon = \pm 1\) contribution from the \(\epsilon = 0\) contribution and omitting the lower index \(n\) in \(\mathcal{V}^{\gamma+\epsilon}_{k,n}\), leads to

\[
0 = \frac{1}{\tau_n^{\gamma+\epsilon}} \left( a\mathcal{V}^{-\epsilon}_{-1,n} + b\mathcal{V}^{\gamma+\epsilon}_{0,n} + c\mathcal{V}^{\gamma+\epsilon}_{1,n} \right) \tau_n^{\gamma+\epsilon} - \frac{1}{\tau_n^{\gamma}} \left( a\mathcal{V}^{-\epsilon}_{-1,n} + b\mathcal{V}^{\gamma}_{0,n} + c\mathcal{V}^{\gamma}_{1,n} \right) \tau_n^{\gamma} = \sum_{i \geq 1} \left( \alpha_i(t) \frac{\partial}{\partial t_i} - \beta_i(s) \frac{\partial}{\partial s_i} \right) \log \frac{\tau_n^{\gamma+\epsilon}}{\tau_n^{\gamma}} + \epsilon \left( c \frac{\partial}{\partial t_1} - a \frac{\partial}{\partial s_1} \log \tau_n^{\gamma+\epsilon} + bn \right)
\]

(2.2.10)

Set, for brevity, \(\tau_n = \tau_n^{\gamma}\) and \(\tau_n^{\pm} = \tau_n^{\gamma}\), \(V = \mathcal{V}^{\gamma}\), \(\mathcal{V}^\pm = \mathcal{V}^{\gamma+\epsilon}\), with \(\epsilon = \pm 1\).

**CASE 1 and 2:** when not both \(a = c = 0\), the terms \(\frac{\partial}{\partial t_1} \log \tau_n^{\gamma+\epsilon}\) and \(\frac{\partial}{\partial s_1} \log \tau_n^{\gamma+\epsilon}\) are present; they cannot be easily expressed in terms of \(x_n\) and \(y_n\).
But adding the +contribution to the −contribution eliminates those terms:

\[
0 = \frac{x_n y_n}{v_n} \left\{ \frac{1}{\tau_n^+} (a \mathcal{V}_1^+ + b \mathcal{V}_0^+ + c \mathcal{V}_1^+)^{\tau_n^+} + \frac{1}{\tau_n^-} (a \mathcal{V}_1^- + b \mathcal{V}_0^- + c \mathcal{V}_1^-)^{\tau_n^-} \right\} \\
= \frac{x_n y_n}{v_n} \left( \sum_{i \geq 1} \left( \alpha_i(t) \frac{\partial}{\partial t_i} - \beta_i(s) \frac{\partial}{\partial s_i} \right) \log x_n y_n + \left( c \frac{\partial}{\partial t_1} - a \frac{\partial}{\partial s_1} \right) \log x_n \right).
\]

\[
= \frac{x_n y_n}{v_n} \left( \sum_{i \geq 1} (\alpha_i \frac{\partial}{\partial t_i} - \beta_i \frac{\partial}{\partial s_i}) (\log x_n + \log y_n) + (c \frac{\partial}{\partial t_1} - a \frac{\partial}{\partial s_1}) (\log x_n - \log y_n) \right)
\]

\[
= (\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)}) + aL_2 - cL_1)_{n,n} - (\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)} - aL_2 + cL_1)_{n+1,n+1},
\]

using (2.2.1), (1.1.31) and the definition (2.2.2) of \(\mathcal{L}_i^{(n)}\),

\[
= -\partial_n (\mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)})_{n,n} + (aL_2 - cL_1)_{n,n}
\]

using (2.2.8). This establishes the first relation, namely (2.2.3).

To prove the second relation (2.2.4), we take the \(t_1\)-derivative of the first relation. Using

\[
\frac{\partial \alpha_i}{\partial t_1} = 0 \quad \text{for } i \geq 3 \\
= c \quad \text{for } i = 2 \\
= b \quad \text{for } i = 1
\]

\[
\frac{\partial \beta_i}{\partial t_1} = 0 \quad \text{for } i \geq 1
\]

we obtain

\[
\frac{\partial (\mathcal{L}_1 - \mathcal{L}_2)}{\partial t_1} = \sum_{i \geq 1} \alpha_i(t) \frac{\partial L_1^i}{\partial t_1} + \sum_{i \geq 1} \beta_i(t) \frac{\partial L_2^i}{\partial t_1} + cL_2 + bL_1
\]

Using the above and

\[
(L_1^2)_{n,n} = x_n^2 y_{n-1}^2 - x_n y_{n-2} v_{n-1} - x_{n+1} y_{n-1} v_n
\]

\[
(L_2^2)_{n,n} = y_n^2 x_{n-1}^2 - y_n x_{n-2} v_{n-1} - y_{n+1} x_{n-1} v_n
\]

\[
(L_2^2)_{n,n-1} = -v_{n-1} (x_{n-1} y_n + x_{n-2} y_{n-1})
\]

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one computes, using (2.2.2) and Lemma 1.3,

\[
0 = \frac{\partial}{\partial t_1}(L_1 - L_2 + aL_2 - cL_1)_{n,n} - \frac{\partial}{\partial t_1}(L_1 - L_2 + aL_2 - cL_1)_{n+1,n+1}
\]

\[
= \left( \frac{\partial(L_1 - L_2)}{\partial t_1} \right)_{n,n} - \left( \frac{\partial(L_1 - L_2)}{\partial t_1} \right)_{n+1,n+1}
\]

\[
+ a \left( \frac{\partial L_2}{\partial t_1} \right)_{n,n} + \left( \frac{\partial L_2}{\partial t_1} \right)_{n+1,n+1}
\]

\[
- c \left( \frac{\partial L_1}{\partial t_1} \right)_{n,n} - \left( \frac{\partial L_1}{\partial t_1} \right)_{n+1,n+1}
\]

\[
\sum_{i \geq 1} \alpha_i(t) \left( 2v_n(L_1^i)_{n+1,n} - v_{n-1}(L_1^i)_{n,n-1} - v_{n+1}(L_1^i)_{n+2,n+1} \right)
\]

\[
+ \sum_{i \geq 1} \beta_i(t) \left( 2(L_1^i)_{n+1,n} - (L_2^i)_{n,n-1} - (L_2^i)_{n+2,n+1} \right)
\]

\[
+ c \left( (L_1^2)_{nn} - (L_1^2)_{n+1,n+1} - v_{n+1}(L_1)_{n+2,n+1} + v_{n-1}(L_1)_{n,n-1} \right)
\]

\[
+ a \left( (L_2)_{n+2,n+1} - (L_2)_{n,n-1} \right) + b \left( (L_1)_{nn} - (L_1)_{n+1,n+1} \right)
\]

\[
= 2v_n(L_1)_{n+1,n} - v_{n-1}(L_1)_{n,n-1} - v_{n+1}(L_1)_{n+2,n+1}
\]

\[
-2(L_2)_{n+1,n} + (L_2)_{n,n-1} + (L_2)_{n+2,n+1}
\]

\[
+ c \left( (L_1^2)_{nn} - (L_1^2)_{n+1,n+1} - v_{n+1}(L_1)_{n+2,n+1} + v_{n-1}(L_1)_{n,n-1} \right)
\]

\[
+ a \left( (L_2)_{n+2,n+1} - (L_2)_{n,n-1} \right) + b \left( (L_1)_{nn} - (L_1)_{n+1,n+1} \right)
\]

\[
= 2(v_nL_1 - L_2)_{n+1,n} - (v_{n-1}L_1 - L_2)_{n,n-1} - (v_{n+1}L_1 - L_2)_{n+2,n+1}
\]

\[
+ a(v_{n+1} - v_{n-1}) + b(x_{n+1}y_n - x_{n}y_{n-1})
\]

\[
+ c(2y_nx_{n+2}v_{n+1} - 2x_{n}y_{n+2}v_{n-1} + x_{n}^2y_{n-1}^2 - y_{n}^2x_{n+1}^2)
\]

\[
= 2(v_nL_1 - L_2)_{n+1,n} - (v_{n-1}L_1 - L_2)_{n,n-1} - (v_{n+1}L_1 - L_2)_{n+2,n+1}
\]

\[
+ \partial_n \left( \begin{array}{c}
\alpha \left( (L_2)_{n,n-1} + (L_2)_{n+1,n} \\
- (c(L_1^2)_{nn} + b(L_1)_{nn}) \\
- c(v_n(L_1)_{n+1,n} + v_{n-1}(L_1)_{n,n-1})
\end{array} \right)
\]

\[
= -\partial_n^2(v_{n-1}L_1 - L_2)_{n,n-1} + 2\partial_n (cv_n(L_1)_{n+1,n} - a(L_2)_{n+1,n})
\]

\[
+ \partial_n \left( \begin{array}{c}
\alpha \left( (L_2)_{n,n-1} + (L_2)_{n+1,n} \\
- (c(L_1^2)_{nn} + b(L_1)_{nn}) \\
- c(v_n(L_1)_{n+1,n} + v_{n-1}(L_1)_{n,n-1})
\end{array} \right)
\]
\[ -\partial^2_n (v_{n-1} \mathcal{L}_1 - \mathcal{L}_2)_{n,n-1} \\
+ \partial_n \left( a ((L_2)_{n,n-1} - (L_2)_{n+1,n}) \\
- (c(L^2_1)_{nn} + b(L_1)_{nn}) \\
+ c(v_n(L_{n+1,n} - v_{n-1}(L_1)_{n,n-1}) \right) \\
= -\partial^2_n (v_{n-1} \mathcal{L}_1^{(n)} - \mathcal{L}_2^{(n)})_{n,n-1} \\
- \partial_n (c(L^2_1)_{nn} + b(L_1)_{nn}) \\
= -\partial_n (\partial_n (v_{n-1} \mathcal{L}_1^{(n-1)} - \mathcal{L}_2^{(n-1)})_{n,n-1} - \partial_n (c(L^2_1)_{nn} + b(L_1)_{nn}) \right). \\
\]

using (??) in **, ending the proof of identity (2.2.4). Equality * leads to the expression in the remark after the statement of the Theorem.

• **CASE 3**: when both \( a = c = 0 \), the terms \( \frac{\partial}{\partial t_i} \log \tau_n^{\gamma+\varepsilon} \) and \( \frac{\partial}{\partial s_i} \log \tau_n^{\gamma+\varepsilon} \) are absent in (2.2.10). So, using again (1.1.31), setting \( \alpha_i(t) = it_i, \beta_i(s) = is_i \) and \( b = 1 \), leads to the polynomials

\[ 0 = -x_n \left( \sum_{i \geq 1} \left( \alpha_i(t) \frac{\partial}{\partial t_i} - \beta_i(s) \frac{\partial}{\partial s_i} \right) \log x_n + bn \right) \\
= \frac{v_n}{y_n} \sum_{i \geq 1} \left\{ \alpha_i ((L^1_1)_{n+1,n+1} - (L^{i-1}_1)_{n+1,n}) - \beta_i ((L^2_i)_{nn} - (L^{i-1}_2)_{n,n+1}) \right\} \\
- n x_n \]

\[ 0 = y_n \left( \sum_{i \geq 1} \left( \alpha_i(t) \frac{\partial}{\partial t_i} - \beta_i(s) \frac{\partial}{\partial s_i} \right) \log y_n - bn \right) \\
= \frac{v_n}{x_n} \sum_{i \geq 1} \left\{ \alpha_i ((L^1_1)_{nn} - (L^{i-1}_1)_{n+1,n}) - \beta_i ((L^2_i)_{n+1,n+1} - (L^{i-1}_2)_{n,n+1}) \right\} \\
- n y_n. \quad (2.2.11) \]

ending the proof of Theorem 2.2.
2.3 Proof of main Theorem

Proof of Theorem 0.1: Remember the locus, defined in (2.1.1),

\[
\mathcal{L} = \begin{cases}
    i_{ti} = i_{t_{i}^{(0)}} & : \begin{cases}
        u_{i} - (\gamma_{1}'d_{1}^{i} + \gamma_{2}'d_{2}^{i}), & \text{for } 1 \leq i \leq N_{1} \\
        - (\gamma_{1}'d_{1}^{i} + \gamma_{2}'d_{2}^{i}), & \text{for } N_{1} + 1 \leq i < \infty
    \end{cases} \\
    i_{si} = i_{s_{i}^{(0)}} & : \begin{cases}
        -u_{-i} + (\gamma_{1}''d_{1}^{-i} + \gamma_{2}''d_{2}^{-i}), & \text{for } 1 \leq i \leq N_{2} \\
        (\gamma_{1}''d_{1}^{-i} + \gamma_{2}''d_{2}^{-i}), & \text{for } N_{2} + 1 \leq i < \infty
    \end{cases}
\end{cases} \tag{2.3.1}
\]

From (2.1.3) in Lemma 2.1, for all \(i \geq 1\),

\[
\alpha_{i}(t^{(0)}) := au_{i+1} + bu_{i} + cu_{i-1} + c(n + \gamma_{1}' + \gamma_{2}' + \gamma)\delta_{i1},
\]

\[
\beta_{i}(s^{(0)}) := -au_{-i+1} - bu_{-i} - cu_{-i-1} - a(n + \gamma_{1}' + \gamma_{1}' + \gamma_{2}' - \gamma)\delta_{i1}.
\]

Then the locus (2.3.1) can equally be described by

\[
\mathcal{L} = \begin{cases}
    \text{all } \alpha_{i}(t) = 0, \text{ for } i \geq N_{1} + 2 \text{ and all } \beta_{i}(t) = 0, \text{ for } i \geq N_{2} + 2, \\
    \alpha_{i}(t) = au_{i+1} + bu_{i} + c(u_{i-1} + \delta_{i1}(n + \gamma_{1}' + \gamma_{2}' + \gamma)) \text{ and } \\
    \beta_{i}(s) = -cu_{-i-1} - bu_{-i} - a(u_{-i+1} + \delta_{i1}(n + \gamma_{1}' + \gamma_{2}' - \gamma)), \\
    \text{otherwise}
\end{cases}, \tag{2.3.2}
\]

The \(\mathcal{L}_{i}\)-matrices (2.2.2) now and only now are finite sums and so have the form, setting \(u_{0} = u_{N_{1}+1} = u_{N_{1}+2} = u_{-N_{2}-1} = u_{-N_{2}-2} = 0\),

\[
\mathcal{L}_{1} = \sum_{1}^{N_{1}+1} \alpha_{i}(t^{(0)})L_{1}^{i}
\]

\[
= \sum_{1}^{N_{1}+1} (au_{i+1} + bu_{i} + cu_{i-1})L_{1}^{i} + c(n + \gamma_{1}' + \gamma_{2}' + \gamma)L_{1}
\]

\[
= (aI + bL_{1} + cL_{1}^{2})\sum_{1}^{N_{1}} u_{i}L_{1}^{i-1} + c(n + \gamma_{1}' + \gamma_{2}' + \gamma)L_{1} - au_{1}I
\]

\[
= (aI + bL_{1} + cL_{1}^{2})P_{1}(L_{1}) + c(n + \gamma_{1}' + \gamma_{2}' + \gamma)L_{1} - au_{1}I
\]

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\[ \mathcal{L}_2 = - \sum_{i=1}^{N_2+1} \beta_i(s^{(0)}) L_i^2 \]

\[ = \sum_{i=1}^{N_2+1} \left( cu_{-i-1} + bu_{-i} + au_{-i+1} \right) L_i^2 + a(n + \gamma''_1 + \gamma''_2 - \gamma) L_2 \]

\[ = \left( cI + bL_2 + aL_2^2 \right) \sum_{i=1}^{N_2} u_{-i} L_2^{i-1} + a(n + \gamma''_1 + \gamma''_2 - \gamma) L_2 - cu_{-1} I \]

\[ = \left( cI + bL_2 + aL_2^2 \right) P'_2(L_2) + a(n + \gamma''_1 + \gamma''_2 - \gamma) L_2 - cu_{-1} I, \]

which are precisely the expressions \( \mathcal{L}_i^{(n)} \), introduced in (0.0.10), in the introduction, but modulo the identity pieces. Still, the identities (2.2.3), (2.2.4) and (2.2.5) remain valid, upon evaluating along the locus \( \mathcal{L} \); i.e., with \( \alpha_i(t^{(0)}) \) and \( \beta_i(s^{(0)}) \) as in (2.3.2) and \( t \) and \( s \) replaced by \( t^{(0)} \) and \( s^{(0)} \) in the variables \( x_n(t, s) \) and \( y_n(t, s) \).

- **CASE 1 and 2:** not both \( a = c = 0 \). Thus, the first identity (2.2.3) holds and the second identity (2.2.4) expresses the fact that a difference \( \partial_n \) of an expression vanishes; therefore the expression equals that same expression at the origin. This ends the proof of identities (0.0.14) and (0.0.15) in Theorem 0.1, upon observing the identity pieces in \( \mathcal{L}_1 \) and \( \mathcal{L}_2 \) above make no contribution.

- **CASE 3:** both \( a = c = 0 \) and \( b = 1 \). From (2.1.1) or (2.3.1), setting \( a = c = 0 \) and \( b = 1 \), we have

\[ \alpha_i(t^{(0)}) = u_i \quad \text{and} \quad \beta_i(t^{(0)}) = -u_{-i} \]

Remember from (0.0.2), \( P_1(z) := \sum_{i=1}^{N_1} \frac{u_i z^i}{i} \) and \( P_2(z) := \sum_{i=1}^{N_2} \frac{u_{-i} z^i}{i} \), and so we have

\[ \sum_{i \geq 1} \alpha_i(t^{(0)}) L_1^{i-1} = \sum_{i \geq 1} u_i L_1^{i-1} = P'_1(L_1) \]

\[ \sum_{i \geq 1} \beta_i(t^{(0)}) L_2^{i-1} = -\sum_{i \geq 1} u_{-i} L_2^{i-1} = -P'_2(L_2), \]

and so (2.2.5) leads immediately to (0.0.17), ending the proof of Theorem 0.1. \[ \square \]
3 Invariant manifolds for the first Toeplitz flow

Proof of Theorem 0.4: The weight here is
\[ \rho(z) := z^\gamma e^{P_1(z)+P_2(z^{-1})}, \]
corresponding to case 3 of Theorem 0.1. From the latter, the variables \( x_n, y_n \) satisfy recurrence relations
\[
\begin{align*}
\Gamma_n(x, y) &:= \frac{v_n}{y_n} \left( -(L_1 P'_1(L_1))_{n+1,n+1} - (L_2 P'_2(L_2))_{n,n} \right) + n x_n = 0 \\
\tilde{\Gamma}_n(x, y) &:= \frac{v_n}{x_n} \left( -(L_1 P'_1(L_1))_{n,n} - (L_2 P'_2(L_2))_{n+1,n+1} \right) + n y_n = 0,
\end{align*}
\]
which by virtue of (2.2.11) and the nature of the locus (2.3.1), can be written
\[
\begin{align*}
\Gamma_n &= v_0 x_n + n x_n \\
\tilde{\Gamma}_n &= -v_0 y_n + n y_n,
\end{align*}
\]
where
\[ v_0 := \sum_{i \geq 1} \left( u_i \frac{\partial}{\partial t_i} + u_{-i} \frac{\partial}{\partial s_1} \right), \]
in terms of the Toeplitz vector fields. Recall from section 6 (Appendix 2) the form of the first vector field: \( (v_n := 1 - x_n y_n) \)
\[
\begin{align*}
\frac{\partial x_n}{\partial \begin{cases} t_1 \\ s_1 \end{cases}} &= v_n x_{n \mp 1} \\
\frac{\partial y_n}{\partial \begin{cases} t_1 \\ s_1 \end{cases}} &= -v_n y_{n \mp 1}.
\end{align*}
\]
Also in statement of Theorem 0.4, we assume the \( u_i \), appearing in the polynomials \( P_1(z) \) and \( P_2(z) \), flow according to
\[
\begin{align*}
\frac{\partial u_k}{\partial t_1} &= \delta_{k,1} \\
\frac{\partial u_k}{\partial s_1} &= -\delta_{k,-1}.
\end{align*}
\]
Noticing that, from (3.0.3),
\[
\left[ \frac{\partial}{\partial t_1}, \mathcal{V}_0 \right] = \frac{\partial}{\partial t_1}, \quad \left[ \frac{\partial}{\partial s_1}, \mathcal{V}_0 \right] = -\frac{\partial}{\partial s_1},
\]
we compute
\[
\frac{\partial}{\partial t_1} \Gamma_n = \mathcal{V}_0 \frac{\partial x_n}{\partial t_1} + \frac{\partial x_n}{\partial t_1} + n \frac{\partial x_n}{\partial t_1} = \mathcal{V}_0 (v_n x_{n+1}) + (n + 1) v_n x_{n+1} = v_n (\mathcal{V}_0 x_{n+1} + (n + 1) x_{n+1}) + x_{n+1} \mathcal{V}_0 (v_n) = v_n \Gamma_{n+1} - x_{n+1} (x_n \mathcal{V}_0 (y_n) + y_n \mathcal{V}_0 (x_n)) = v_n \Gamma_{n+1} + x_{n+1} \left( x_n \tilde{\Gamma}_n - y_n \Gamma_n \right), \tag{3.0.4}
\]
Similarly, one shows
\[
\frac{\partial}{\partial s_1} \Gamma_n = \mathcal{V}_0 \frac{\partial x_n}{\partial s_1} - \frac{\partial x_n}{\partial s_1} + n \frac{\partial x_n}{\partial s_1} = \mathcal{V}_0 (v_n x_{n-1}) + (n - 1) v_n x_{n-1} = v_n \Gamma_{n-1} + x_{n-1} \left( x_n \tilde{\Gamma}_n - y_n \Gamma_n \right) \tag{3.0.5}
\]
and so by the duality
\[
\frac{\partial}{\partial t_1} \tilde{\Gamma}_n = -v_n \tilde{\Gamma}_{n-1} + y_{n-1} \left( x_n \tilde{\Gamma}_n - y_n \Gamma_n \right),
\]
\[
\frac{\partial}{\partial s_1} \tilde{\Gamma}_n = -v_n \tilde{\Gamma}_{n+1} + y_{n+1} \left( x_n \tilde{\Gamma}_n - y_n \Gamma_n \right), \tag{3.0.6}
\]
thus establishing (0.0.22). Setting as in (0.0.23),
\[
\mathcal{M} := \bigcap_{n \geq 0} \left\{ (x_k, y_k)_{k \geq 0} \mid \text{ such that } \Gamma_n(x, y) = 0 \text{ and } \tilde{\Gamma}_n(x, y) = 0 \right\},
\]
the differential equations (3.0.4), (3.0.5) and (3.0.6) imply at once that, along the locus \(\mathcal{M}\), defined in (0.0.23),
\[
\frac{\partial}{\partial t_1} \Gamma_n \bigg|_{\mathcal{M}} = \frac{\partial}{\partial t_1} \tilde{\Gamma}_n \bigg|_{\mathcal{M}} = \frac{\partial}{\partial s_1} \Gamma_n \bigg|_{\mathcal{M}} = \frac{\partial}{\partial s_1} \tilde{\Gamma}_n \bigg|_{\mathcal{M}} = 0,
\]
showing the locus \(\mathcal{M}\) is an invariant manifold for these flows. This ends the proof of Theorem 0.4.  \(\blacksquare\)
Proof of Corollary 0.5: In the self-dual case,
\[ \Gamma_n = \tilde{\Gamma}_n, \]
since in (3.0.2) all \( u_n = u_{-n}, x_n = y_n \) and so \( \partial x_n / \partial t_i = -\partial y_n / \partial s_i, \partial x_n / \partial s_i = -\partial y_n / \partial t_i \) for all \( n, i \).

From the differential equations (3.0.4) and (3.0.5), it follows that
\[ \frac{\partial}{\partial t_1} \Gamma_n = v_n \Gamma_{n+1} \quad \text{and} \quad \frac{\partial}{\partial s_1} \Gamma_n = v_n \Gamma_{n-1}, \]
and so for \( \frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} - \frac{\partial}{\partial s_1} \),
\[ \frac{\partial \Gamma_n}{\partial t} = v_n (\Gamma_{n+1} - \Gamma_{n-1}) \]
establishing equation (0.0.25). Therefore, along the locus \( \mathfrak{N} \), defined in (0.0.26),
\[ \frac{\partial \Gamma_n}{\partial t} \bigg|_{\mathfrak{N}} = 0 \]
and so the locus \( \mathfrak{N} \) is invariant with respect to the \( \partial / \partial t \) vector field, ending the proof of Corollary 0.5.

4 Rational relations for special weights

4.1 Weight \( e^{t(z + z^{-1})} \)

This weight comes up by considering the uniform probability \( P \) on the group \( S_k \) of permutations \( \pi_k \) and
\[ L(\pi_k) = \text{length of the longest (strictly) increasing subsequence of } \pi_k. \] (4.1.1)

Then, according to Gessel [10], the generating function below can be expressed as the determinant of a Toeplitz matrix and thus as a unitary matrix integral:
\[ \sum_{k=0}^{\infty} \frac{t^{2k}}{k!} P(L(\pi_k) \leq n) = E_U(n) e^{t \text{Tr}(M + \bar{M})} \] (4.1.2)
\[ = \frac{1}{n!} \int_{(S_1)^n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left( e^{t(z_k + \bar{z}_k)} \frac{dz_k}{2\pi i z_k} \right). \]
Then
\[
x_n = (-1)^n \frac{E_{U(n)}(\det M) e^{t \text{Tr}(M + \bar{M})}}{E_{U(n)} e^{t \text{Tr}(M + \bar{M})}}
\]
\[
= (-1)^n \int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left( z_k e^{t(z_k + \bar{z}_k)} \frac{dz_k}{2\pi iz_k} \right)
\]
\[
\int_{(S^1)^n} |\Delta_n(z)|^2 \prod_{k=1}^n \left( e^{t(z_k + \bar{z}_k)} \frac{dz_k}{2\pi iz_k} \right)
\]

This weight is a special case of the self-dual weight of Corollary 0.3, which lead to the relation (0.0.18). Thus we find
\[
0 = n x_n - v_n \left( t(L_1)_{n+1,n+1} + t(L_1)_{nn} \right)
\]
\[
= n x_n - v_n \left( -t x_{n+1} x_n - t x_n x_{n-1} \right)
\]
yielding the 3-step relation, found by Borodin [5], with highest (respectively, lowest) terms doubly (respectively, simply) underlined,
\[
n x_n + t(1 - x_n^2)(x_{n+1} + x_{n-1}) = 0. \tag{4.1.3}
\]

It is interesting to point out that this map (4.1.3) is the simplest instance of a family of area-preserving maps of the plane, having an invariant, as found by McMillan [8], and extended by Suris [12] to maps of the form \(\partial_n^2 x(n) = f(x(n))\), having an analytic invariant of two variables \(\Phi(y, z)\), i.e.,
\[
\Phi(x_{n+1}, x_n) = \Phi(x_n, x_{n-1}).
\]
The invariant in the case of the maps (4.1.3) is
\[
(1 - y^2)(1 - z^2) + ayz, \text{ with } a = -\frac{n}{t}.
\]
For more on this matter, see the review by B. Grammaticos, F. Nijhoff, A. Ramani [4].

### 4.2 Weight \(e^{t(z + z^{-1}) + s(z^2 + z^{-2})}\)

Consider instead the subgroups of odd permutations, with \(2^k k!\) elements
\[
S_{2k}^{\text{odd}} = \left\{ \pi_{2k} \in S_{2k}, \pi_{2k} : (-k, \ldots, -1, 1, \ldots, k) \cap \pi_{2k}(-j) = -\pi_{2k}(j), \text{ for all } j \right\} \subset S_{2k}
\]
\[ S_{2k+1}^{\text{odd}} = \left\{ \pi_{2k+1} \in S_{2k+1}, \pi_{2k} : (-k, \ldots, -1, 0, 1, \ldots, k) \cup \pi_{2k+1}(-j) = -\pi_{2k+1}(j), \text{ for all } j \right\} \subset S_{2k} \]

Then, according to Rains [11] and Tracy-Widom [13], the following generating functions, again involving the length of the longest necessary sequence, are related to matrix integrals:

\[ \sum_{0}^{\infty} \frac{(\sqrt{2}s)^{2k}}{k!} P(L(\pi_{2k}) \leq n) = E_{U(n)} e^{s\text{Tr}(M^2+\tilde{M}^2)} \]

\[ = \frac{1}{n!} \int_{(S')n} |\Delta_n(z)|^2 \prod_{k=1}^{n} \left( e^{s(z_k^2-z_k^{-2})} \frac{dz_k}{2\pi z_k} \right) \]

\[ \sum_{0}^{\infty} \frac{(\sqrt{2}s)^{2k}}{k!} P(L(\pi_{2k+1}) \leq n) = \frac{1}{4} \frac{\partial}{\partial t^2} \left( E_{U(n)} e^{t(M+\tilde{M})+s(M^2+\tilde{M}^2)} + E_{U(n)} e^{t(M+\tilde{M})-s(M^2+\tilde{M}^2)} \right) \bigg|_{t=0} \]

The weight \( e^{t(z^{-1})+s(z^2-z^{-2})} \) is a special case of the one in corollary 0.3, thus leading to (0.0.17). So, we find

\[ x_n = \left( -1 \right)^n \frac{E_{U(n)}(\det M) e^{t\text{Tr}(M+\tilde{M})+s\text{Tr}(M^2+\tilde{M}^2)}}{E_{U(n)} e^{t\text{Tr}(M+\tilde{M})+s\text{Tr}(M^2+\tilde{M}^2)}} \]

satisfies a 5-step relation, with highest (respectively, lowest) terms doubly (respectively, simply) underlined,

\[ 0 = nx_n + tv_n(x_{n-1} + x_{n+1}) + 2sv_n \left( x_{n+2}v_{n+1} + x_{n-2}v_{n-1} - x_n(x_{n+1} + x_{n-1})^2 \right). \]

(4.2.1)

Also here the map has a polynomial invariant

\[ \Phi(x, y, z, u) = nyz - (1 - y^2)(1 - z^2) \left( t + 2s(x(u - y) - z(u + y)) \right). \]

So we have for all \( n \),

\[ \Phi(x_{n-1}, x_n, x_{n+1}, x_{n+2}) = \Phi(x_n, x_{n+1}, x_{n+2}, x_{n+3}). \]
4.3 Weight \((1 + z)^{\alpha}e^{-sz^{-1}}\)

Let \(P\) be the uniform probability on

\[ S_{k,\alpha} = \{ \text{words } \pi_k \text{ of length } k \text{ from an alphabet of } \alpha \text{ letters} \}. \]

Then, if \(L\) denotes the same as in (4.1.1), but without ‘strictly’, we have, according to Tracy-Widom [14] (see also [3]),

\[
\sum_{k=0}^{\infty} \frac{(\alpha s)^k}{k!} P \{ \pi_k \in S_{k,\alpha} \mid L(\pi_k) \leq n \} = E_{U(n)} \det(I + M)^{\alpha}e^{-s \text{ tr} \bar{M}}.
\]

For this weight, \(P_1(z) = 0\) and \(P_2(z) = -sz\), so that \(N_1 = 0, \ N_2 = 1, \ u_{-1} = -s,\) and all other \(u_i = 0\). Also \(\gamma'_1 = \alpha, \ \gamma = \gamma'_2 = \gamma''_1 = \gamma''_2 = 0\). This is a special case of Case 2 of Theorems 0.1 and 0.2. Hence, we choose

\[ a = 0, \ b = c = 1, \]

for which one computes

\[
\mathcal{L}_1^{(n)} = (n + \alpha)L_1, \\
\mathcal{L}_2^{(n)} = s(I + L_2),
\]

and so (0.0.14) and (0.0.13) become

\[
\partial_n((n + \alpha)L_1 + sL_2)_{nn} + c(L_1)_{nn} = 0 \\
\partial_n((n - 1 + \alpha)v_{n-1}L_1 + sL_2)_{n,n-1} + (L^2_1 + L_1)_{n,n} = \text{same} \bigg|_{n=1}
\]

Thus spelled out, the variables

\[
\begin{align*}
x_n &= (-1)^n \frac{E_{U(n)} \ (\det M)^{\pm 1} \det(I + M)^{\alpha}e^{-s \text{ tr} \bar{M}}}{E_{U(n)} \ \det(I + M)^{\alpha}e^{-s \text{ tr} \bar{M}}} \\
y_n &= \frac{\text{same}}{\text{same}}
\end{align*}
\]

satisfy a 3-step and a 4-step relation, linear in \(x_{n+1}\) and \(y_{n+1}\):

\[
-(n + \alpha + 1)x_{n+1}y_n - sx_ny_{n+1} + (n + \alpha - 1)x_ny_{n-1} + sx_{n-1}y_n = 0
\]

\[
-v_n((n + \alpha + 1)x_{n+1}y_{n-1} - s) + v_{n-1}((n + \alpha - 2)x_{n}y_{n-2} - s)
\]

\[
+ x_ny_{n-1}(x_ny_{n-1} - 1) = v_1(s - (2 + \alpha)x_2) + x_1(x_1 - 1).
\]
4.4 Weight \( (1 - \xi z)^\alpha (1 - \xi z^{-1})^\beta \)

This weight, considered by Borodin \[5\] and Borodin-Deift \[6\] and coming up in point processes, is obtained by setting

\[
\gamma_1' = \alpha, \quad \gamma_2'' = \beta, \quad \gamma = \gamma_2' = \gamma'' = 0, \quad d_1 = \xi, \quad d_2 = \xi^{-1}, \quad \text{all} \ u_i = 0, \quad N_1 = N_2 = 0.
\]

\[\alpha = c = 1, \quad b = -\xi - \xi^{-1}\]

in the weight (0.0.1). We have that

\[
\alpha_i = (n + \alpha)\delta_{i1} \quad \text{and} \quad \beta_i = -(n + \beta)\delta_{i1}
\]

and so

\[
L_1^{(n)} = (n + \alpha)L_1 \quad \text{and} \quad L_2^{(n)} = -(n + \beta)L_2.
\]

Therefore (0.0.14) and (0.0.15) read

\[
((n + \alpha + 1)L_1 - (n + \beta + 1)L_2)_{n+1,n+1} - ((n + \alpha - 1)L_1 - (n + \beta - 1)L_2)_{nn} = 0
\]

leading to a 3-step relation and a 4-step relation in \(x_{n+1}\) and \(y_{n+1}\),

\[-(n + \alpha + 1)x_{n+1}y_n + (n + \beta + 1)y_{n+1}x_n + (n + \alpha - 1)y_{n-1}x_n - (n + \beta - 1)x_{n-1}y_n = 0.
\]

and

\[-v_n((n + \alpha + 1)x_{n+1}y_{n-1} + n + \beta) + v_{n-1}((n + \alpha - 2)x_ny_{n-2} + n + \beta - 1) + x_ny_{n-1}(x_ny_{n-1} + \xi + \xi^{-1})
\]

leading to

\[-v_1(x_2(\alpha + 2) + \beta + 1) + x_1(x_1 + \xi + \xi^{-1}).\]

So, all \(x_n\) and \(y_n\) are rational expressions in terms of \(x_1, x_2, y_1, y_2\). Note these relations are different from those found by Borodin \[5\].

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5 Appendix 1: Virasoro algebras

In [2], we defined a Heisenberg and Virasoro algebra of vector operators $\beta \mathbb{J}^{(i)}_k$, depending on a parameter $\beta > 0$:

$$\begin{align*}
\left(\beta \mathbb{J}^{(1)}_k\right)_n &= \beta J_k^{(1)} + nJ_k^{(0)} \quad \text{and} \quad \left(\mathbb{J}^{(0)}_k\right)_n = nJ_k^{(0)} = n\delta_{0k},
\end{align*}$$

and

$$\begin{align*}
\beta \mathbb{J}^{(2)}_k &= \beta \sum_{i+j=k} :\beta \mathbb{J}_i^{(1)} \beta \mathbb{J}_j^{(1)} : + (1 - \beta) \left((k + 1) \beta \mathbb{J}_k^{(1)} - k \mathbb{J}_k^{(0)}\right) \\
&= \left(\beta \beta J_k^{(2)} + (2n\beta + (k + 1)(1 - \beta)) \beta J_k^{(1)} + n(n\beta + 1 - \beta)J_k^{(0)}\right)_{n \in \mathbb{Z}}.
\end{align*}$$

The $\beta \mathbb{J}_k^{(2)}$'s satisfy the commutation relations: (see [2])

$$\begin{align*}
\left[ \mathbb{J}_k^{(1)} , \beta \mathbb{J}_\ell^{(1)} \right] &= \frac{k}{2\beta} \delta_{k,-\ell} \\
\left[ \beta \mathbb{J}_k^{(2)} , \beta \mathbb{J}_\ell^{(1)} \right] &= -\ell \beta \mathbb{J}_k^{(1)} + \frac{k(k+1)}{2} \left(\frac{1}{\beta} - 1\right) \delta_{k,-\ell} \\
\left[ \beta \mathbb{J}_k^{(2)} , \beta \mathbb{J}_\ell^{(2)} \right] &= (k - \ell) \beta \mathbb{J}_k^{(2)} + c \left(\frac{k^3 - k}{12}\right) \delta_{k,-\ell},
\end{align*}$$

with central charge

$$c = 1 - 6 \left(\beta^{1/2} - \beta^{-1/2}\right)^2.$$ 

In the expressions above,

$$\begin{align*}
\beta J_k^{(1)} &= \frac{\partial}{\partial t_k} \quad \text{for} \quad k > 0 \\
&= \frac{1}{2\beta}(-k)t_{-k} \quad \text{for} \quad k < 0 \\
&= 0 \quad \text{for} \quad k = 0
\end{align*}$$

$$\beta J_k^{(2)} = \sum_{i+j=k} \frac{\partial^2}{\partial t_i \partial t_j} + \frac{1}{\beta} \sum_{-i+j=k} it_i \frac{\partial}{\partial t_j} + \frac{1}{4\beta^2} \sum_{-i-j=k} it_i j t_j$$

(5.0.3)
In particular, for $\beta = 1/2$ and 1, the $\beta J^{(2)}_k$ take on the form:

$$
\beta J^{(2)}_k(t)_{\beta=1/2} = \frac{1}{2} \left( \beta J^{(2)}_k + (2n + k + 1) \beta J^{(1)}_k + n(n + 1)J^{(0)}_k \right)_{n \in \mathbb{Z}},
$$

(5.0.4)

$$
\beta J^{(2)}_k(t)_{\beta=1} = \left( \beta J^{(2)}_k + 2n \beta J^{(1)}_k + n^2 J^{(0)}_k \right)_{n \in \mathbb{Z}},
$$

(5.0.5)

6 Appendix 2: Useful formulae about the Toeplitz lattice

The first Toeplitz vector field (1.1.9), corresponding to the Hamiltonians

$$
H^{(1)}_1 = -\text{Tr} \ L_1 = \sum_{i=0}^{\infty} x_{i+1}y_i, \quad H^{(2)}_1 = -\text{Tr} \ L_2 = \sum_{i=0}^{\infty} x_iy_{i+1},
$$

and

$$
H^{(1)}_2 = -\frac{1}{2} \text{Tr} \ L_1^2 = -\frac{1}{2} \sum_{i=0}^{\infty} x_{i+1}^2y_i^2 + \sum_{i=0}^{\infty} y_ix_{i+2}y_{i+1}
$$

$$
H^{(2)}_2 = -\frac{1}{2} \text{Tr} \ L_2^2 = -\frac{1}{2} \sum_{i=0}^{\infty} x_i^2y_{i+1}^2 + \sum_{i=0}^{\infty} x_iy_{i+2}y_{i+1}
$$

reads:

$$
\frac{\partial x_n}{\partial t_1} = v_nx_{n+1} \quad \frac{\partial y_n}{\partial t_1} = -v_ny_{n-1}
$$

$$
\frac{\partial x_n}{\partial s_1} = v_nx_{n-1} \quad \frac{\partial y_n}{\partial s_1} = -v_ny_{n+1}.
$$
\[
\begin{align*}
\partial x_n \frac{\partial H}{\partial y_n} &= v_n \left( x_{n+1}^2 y_n - x_{n+2}v_{n+1} + x_n x_{n+1} y_{n-1} \right) \\
\partial y_n \frac{\partial H}{\partial x_n} &= -v_n \left( x_y n_{y-1} - y_{y-2}v_{y-1} + y_{y-1}x_{y+1} \right) \\
\partial x_n \frac{\partial H}{\partial s_2} &= v_n \left( x_{y+1}^2 y_{n+1} - y_{y+2}v_{y+1} + y_{y+1}x_{y+1} \right) \\
\partial y_n \frac{\partial H}{\partial s_2} &= -v_n \left( x_y n_{y+1} - y_{y+2}v_{y+1} + y_{y+1}x_{y+1} \right)
\end{align*}
\]

7 Appendix 3: Proof of Theorem 0.2

Before giving the proof of Theorem 0.2, we need

Lemma 7.1

\[
\begin{align*}
(L_1^{N+1})_{nn} &= -x_{n+N}y_{n-1} \prod_{1}^{N} v_{n+i-1} + \ldots - x_{n}y_{n-N-1} \prod_{1}^{N} v_{n-i} \\
(L_2^{N+1})_{nn} &= -y_{n+N}x_{n-1} \prod_{1}^{N} v_{n+i-1} + \ldots - y_{n}x_{n-N-1} \prod_{1}^{N} v_{n-i} \\
v_n (L_1^{N+1})_{n+1,n} &= -x_{n+N+1}y_{n-1} \prod_{0}^{N} v_{n+i} + \ldots - x_{n+1}y_{n-N-1} \prod_{0}^{N} v_{n-i} \\
(L_2^{N+1})_{n+1,n} &= -y_{n+N}x_{n} \prod_{0}^{N-1} v_{n+i} + \ldots - y_{n-N}y_{n} \prod_{1}^{N} v_{n-i+1}.
\end{align*}
\]

The two highest and two lowest terms in

\[
L_1^{(n)} = \sum_{i=1}^{N_1+1} \alpha_i(t)L_1^i \quad \text{and} \quad L_2^{(n)} = -\sum_{i=1}^{N_2+1} \beta_i(t)L_2^i,
\]

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are the following:

\[
(L_{1}^{(n)})_{nn} = -\alpha_{N_{1}+1} \left( x_{n+N_{1}} y_{n-1} \prod_{i=1}^{N_{1}} v_{n+i-1} + \ldots + x_{n} y_{n-N_{1}-1} \prod_{i=1}^{N_{1}} v_{n-i} \right)
- \alpha_{N_{1}} \left( x_{n+N_{1}-1} y_{n-1} \prod_{i=1}^{N_{1}-1} v_{n+i-1} + \ldots + x_{n} y_{n-N_{1}} \prod_{i=1}^{N_{1}-1} v_{n-i} \right) + \ldots
\]

\[
(L_{2}^{(n)})_{nn} = \beta_{N_{2}+1} \left( y_{n+N_{2}} x_{n-1} \prod_{i=1}^{N_{2}} v_{n+i-1} + \ldots + y_{n} x_{n-N_{2}-1} \prod_{i=1}^{N_{2}} v_{n-i} \right)
+ \beta_{N_{2}} \left( y_{n+N_{2}-1} x_{n-1} \prod_{i=1}^{N_{2}-1} v_{n+i-1} + \ldots + y_{n} x_{n-N_{2}} \prod_{i=1}^{N_{2}-1} v_{n-i} \right) + \ldots
\]

\[
v_{n}(L_{1}^{(n)})_{n+1,n} = -\alpha_{N_{1}+1} \left( x_{n+N_{1}+1} y_{n-1} \prod_{i=0}^{N_{1}} v_{n+i-1} + \ldots + x_{n+1} y_{n-N_{1}-1} \prod_{i=0}^{N_{1}} v_{n-i} \right)
- \alpha_{N_{1}} \left( x_{n+N_{1}} y_{n-1} \prod_{i=0}^{N_{1}-1} v_{n+i-1} + \ldots + x_{n+1} y_{n-N_{1}} \prod_{i=0}^{N_{1}-1} v_{n-i} \right) + \ldots
\]

\[
(L_{2}^{(n)})_{n+1,n} = \beta_{N_{2}+1} \left( y_{n+N_{2}} x_{n} \prod_{i=0}^{N_{2}-1} v_{n+i} + \ldots + y_{n} x_{n-N_{2}} \prod_{i=0}^{N_{2}-1} v_{n-i+1} \right)
+ \beta_{N_{2}} \left( y_{n+N_{2}-1} x_{n} \prod_{i=0}^{N_{2}-2} v_{n+i} + \ldots + y_{n} x_{n-N_{2}+1} \prod_{i=0}^{N_{2}-2} v_{n-i+1} \right) + \ldots
\]

with

\[
\alpha_{N_{1}+1} = c \left( u_{N_{1}} + \delta_{N_{1},0}(n + \gamma_{1} + \gamma_{2}) \right)
\]

\[
\alpha_{N_{1}} = c \left( u_{N_{1}-1} + \delta_{N_{1},1}(n + \gamma_{1} + \gamma_{2}) \right) + bu_{N_{1}}
\]

\[
-\beta_{N_{2}+1} = a \left( u_{-N_{2}} + \delta_{N_{2},0}(n + \gamma_{1} + \gamma_{2}) \right)
\]

\[
-\beta_{N_{2}} = a \left( u_{-N_{2}+1} + \delta_{N_{2},1}(n + \gamma_{1} + \gamma_{2}) \right) + bu_{-N_{2}}
\]

**Proof of Theorem 0.2:**

**CASE 1:** \( a, c \neq 0 \): Equations (0.0.14) and (0.0.15) are two inductive equations, one having \((N_{1} + N_{2} + 4)\) steps and the other having \((N_{1} + N_{2} + 3)\) steps. In the equations below, we underline twice the highest terms and once the lowest ones. The exact equations (0.0.14) and (0.0.15) are denoted by \((0.0.14)_{n}\) and \((0.0.15)_{n}\).
• $N_1 = N_2 = N$: The two equations \((0.0.14)_{n+1}\) and \((0.0.15)_{n}\), form a system of two equations in the unknowns $x_{n+N+2}$ and $y_{n+N+2}$, the variable with the lowest index being $y_{n-N-1}$:

\[
(0.0.14)_{n+1} = -\left(\alpha_{N+1} x_{n+N+2} y_{n+1} + \beta_{N+1} y_{n+N+2} x_{n+1}\right) \prod_{i=1}^{N} v_{n+i+1} + \ldots
\]

\[
(0.0.15)_{n} = -\alpha_{N+1} x_{n+N+2} y_{n} \prod_{i=1}^{N} v_{n+i+1} + \ldots + \alpha_{N+1} y_{n-N-1} x_{n+1} \prod_{i=0}^{N} v_{n-i}
\]

• $N_1 = N, N_2 = N + 1$: The two equations \((0.0.14)_n\) and \((0.0.15)_n\), form a system of two equations in the unknowns $x_{n+N+2}$ and $y_{n+N+2}$, the variable with the lowest index being $x_{n-N-2}$:

\[
(0.0.14)_n = -\beta_{N+2} y_{n+N+2} x_{n} \prod_{i=1}^{N+1} v_{n+i} + \ldots + \beta_{N+2} y_{n-N-2} x_{n} \prod_{i=1}^{N+1} v_{n-i}
\]

\[
(0.0.15)_n = -\left(\alpha_{N+1} x_{n+N+2} y_{n} + \beta_{N+2} y_{n+N+2} x_{n+1}\right) \prod_{i=0}^{N} v_{n+i+1} + \ldots
\]

• $N_1 = N, N_2 = N - 1$: The two equations \((0.0.14)_n\) and the dual equation \((0.0.15)\,$\top$\,$_n\), via the involution $\tilde{}$, form a system of two equations in the unknowns $x_{n+N+1}$ and $y_{n+N+1}$, the variable with the lowest index being $y_{n-N-1}$:

\[
(0.0.14)_n = -\alpha_{N+1} x_{n+N+1} y_{n} \prod_{i=1}^{N} v_{n+i} + \ldots + \alpha_{N+1} y_{n-N-1} x_{n} \prod_{i=1}^{N} v_{n-i}
\]

\[
(0.0.15)_{n}^{\top} = -\left(\alpha_{N+1} x_{n+N+1} y_{n+1} + \beta_{N} y_{n+N+1} x_{n}\right) \prod_{i=0}^{N-1} v_{n+i+1} + \ldots
\]

CASE 2: All cases below lead to two inductive equations, one having $(N_1 + N_2 + 3)$ steps and the other having $(N_1 + N_2 + 2)$ steps:

• $N_1 = N, N_2 = N, a = 0, b = c = 1$: The two equations \((0.0.14)_n\) and \((0.0.15)_n\), form a system of two equations in the unknowns $x_{n+N+1}$ and
\( y_{n+N+1} \), the variable with the lowest index being \( y_{n-N-1} \):

\[
\begin{align*}
(0.0.14)_n &= -\alpha_{N+1} x_{n+N+1} y_n \prod_{1}^{N} v_{n+i} + \ldots + \alpha_{N+1} y_{n-N-1} x_n \prod_{1}^{N} v_{n-i} \\
(0.0.15)_{\tilde{n}} &= - \left( \alpha_{N+1} x_{n+N+1} y_{n+1} + \beta_N y_{n+N+1} x_n \right) \prod_{0}^{N-1} v_{n+i+1} + \ldots
\end{align*}
\]

- \( N_1 = N, \quad N_2 = N + 1, \quad a = 0, \quad b = c = 1 \): The two equations \((0.0.14)_{n+1}\) and \((0.0.15)_{\tilde{n}}\) form a system of two equations in the unknowns \( x_{n+N+2} \) and \( y_{n+N+2} \), the variable with the lowest index being \( y_{n-N-1} \):

\[
\begin{align*}
(0.0.14)_{n+1} &= - \left( \alpha_{N+1} x_{n+N+2} y_{n+1} + \beta_N y_{n+N+2} x_n \right) \prod_{1}^{N} v_{n+i+1} + \ldots \\
(0.0.15)_{\tilde{n}} &= - \beta_N y_{n+N+1} x_n \prod_{0}^{N-1} v_{n+i+1} + \ldots + \beta_N y_{n-N} y_{n+1} \prod_{0}^{N-1} v_{n-i}
\end{align*}
\]

- \( N_1 = N, \quad N_2 = N - 1, \quad c = 0, \quad a = b = 1 \): The two equations \((0.0.14)_{n+1}\) and the equation \((0.0.15)_{\tilde{n}}\) form a system of two equations in the unknowns \( x_{n+N+1} \) and \( y_{n+N+1} \), the variable with the lowest index being \( x_{n-N} \):

\[
\begin{align*}
(0.0.14)_{n+1} &= - \left( \alpha_{N} x_{n+N+1} y_{n+1} + \beta_N y_{n+N+1} x_n \right) \prod_{1}^{N-1} v_{n+i+1} + \ldots \\
(0.0.15)_{\tilde{n}} &= - \beta_N y_{n+N+1} x_n \prod_{0}^{N-1} v_{n+i+1} + \ldots + \beta_N x_{n-N} y_{n+1} \prod_{0}^{N-1} v_{n-i}
\end{align*}
\]

**CASE 3.** All cases below lead to two inductive equations, both having \( N_1 + N_2 + 1 \) steps. Using again Lemma 7.1, one searches for the highest and
lowest terms in the relations (0.0.17):

\[
0 = n x_n + \frac{v_n}{y_n} \left( - (L_1 P'_1(L_1))_{n+1,n+1} - (L_2 P'_2(L_2))_{n,n} \right) + \frac{v_n}{y_n} \left( + (P'_1(L_1))_{n+1,n} + (P'_2(L_2))_{n,n+1} \right)
\]

\[
= \frac{v_n}{y_n} \left( - u_{N_1} (L_1^{N_1})_{n+1,n+1} - u_{-N_2} (L_2^{N_2})_{n,n} + \cdots \right) + \frac{v_n}{y_n} \left( + u_{N_1} (L_1^{N_1-1})_{n+1,n} + u_{-N_2} (L_2^{N_2-1})_{n,n+1} + \cdots \right)
\]

\[
= u_{N_1} \frac{v_n}{y_n} \left( x_{n+N_1} y_n \prod_{i=1}^{N_1-1} v_{n+i} + \cdots + x_{n+1} y_{n-N_1+1} \prod_{i=1}^{N_1-1} v_{n+1-i} \right) - u_{N_1} \frac{v_n}{y_n} \left( \cdots + x_{n+1} y_{n-N_1+1} \prod_{i=1}^{N_1-2} v_{n-i} \right) + \cdots
\]

\[
= u_{N_2} \frac{v_n}{y_n} \left( y_{n+N_2} x_n \prod_{i=1}^{N_2-1} v_{n+i-1} + \cdots + y_n x_{n-N_2} \prod_{i=1}^{N_2-1} v_{n-i} \right) - u_{N_2} \frac{v_n}{y_n} \left( \cdots + y_{n+N_2-1} x_{n-1} \prod_{i=1}^{N_2-2} v_{n+i} + \cdots \right) + \cdots
\]

\[
= u_{N_1} \left( x_{n+N_1} \prod_{i=0}^{N_1-1} v_{n+i} + \cdots - x_{n+1} y_{n-N_1+1} x_n \prod_{i=0}^{N_1-2} v_{n-i} \right) - u_{N_2} \left( y_{n+N_2} x_n \prod_{i=0}^{N_2-2} v_{n+i} + \cdots - x_{n-N_2} y_{n-N_2+1} y_n \prod_{i=0}^{N_2-1} v_{n-i} \right) + \cdots
\]

and, by duality,

\[
0 = n y_n + \frac{v_n}{x_n} \left( - (L_1 P'_1(L_1))_{n,n} - (L_2 P'_2(L_2))_{n+1,n+1} \right) + \frac{v_n}{x_n} \left( + (P'_1(L_1))_{n+1,n} + (P'_2(L_2))_{n,n+1} \right)
\]

\[
= - u_{N_1} \left( x_{n+N_1-1} y_{n-1} y_n \prod_{i=0}^{N_1-2} v_{n+i} - y_n \prod_{i=0}^{N_1-1} v_{n-i} \right) + u_{N_2} \left( y_{n+N_2} \prod_{i=0}^{N_2-1} v_{n+i} + \cdots - y_{n+1} x_{n-N_2+1} y_n \prod_{i=0}^{N_2-2} v_{n-i} \right) + \cdots
\]

Here again, one uses different indices \(n\) for each of the cases:
• $N_1 = N, \ N_2 = N$

\[
\begin{align*}
(0.0.17)_n &= u_N x_{n+N} \prod_{0}^{N-1} v_{n+i} + \ldots + u_{-N} x_{n-N} \prod_{0}^{N-1} v_{n-i} + \\
(0.0.17)_{n+1} &= u_{-N} y_{n+N} \prod_{0}^{N} v_{n+i} + \ldots + u_{N} y_{n-N} \prod_{0}^{N-1} v_{n-i} + 
\end{align*}
\]

• $N_1 = N, \ N_2 = N + 1$

\[
\begin{align*}
(0.0.17)_{n+1} &= \left( u_N x_{n+N+1} - u_{-N-1} y_{n+N+1} x_{n+1} \right) \prod_{0}^{N-1} v_{n+i+1} + \\
&+ u_{-N-1} x_{n-N} \prod_{0}^{N} v_{n-i+1} + \\
(0.0.17)_n &= u_{-N-1} y_{n+N+1} \prod_{0}^{N} v_{n+i} + \\
&- \left( u_{-N-1} y_{n+1} x_{n-N} y_n - u_{N} y_{n-N} \right) \prod_{0}^{N-1} v_{n-i} + 
\end{align*}
\]

• $N_1 = N, \ N_2 = N - 1,$

\[
\begin{align*}
(0.0.17)_n &= u_N x_{n+N} \prod_{0}^{N-1} v_{n+i} + \\
&- \left( u_N x_{n+1} y_{n-N+1} x_n - u_{-N+1} x_{n-N+1} \right) \prod_{0}^{N} v_{n-i} + \\
(0.0.17)_{n+1} &= \left( u_{-N+1} y_{n+N} - u_{N} x_{n+N} y_{n+1} \right) \prod_{0}^{N-2} v_{n+i+1} + \\
&+ u_{N} y_{n-N+1} \prod_{0}^{N-1} v_{n-i+1} + 
\end{align*}
\]

This ends the proof of Theorem 0.2. 

\[\blacksquare\]
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