A NOTE ON SUBMAXIMAL OPERATOR SPACE STRUCTURES

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Abstract. In this note, we consider the smallest submaximal space structure \( \mu(X) \) on a Banach space \( X \). We derive a characterization of \( \mu(X) \) up to complete isometric isomorphism in terms of a universal property. Also, we show that an injective Banach space has a unique submaximal space structure and we explore some duality relations of \( \mu \)-spaces.

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1. Introduction and Preliminaries

An operator space consists of a Banach space \( X \) and an isometric embedding \( J : X \to B(\mathcal{H}) \), for some Hilbert space \( \mathcal{H} \). In contrast to the Banach space case, an operator space carries not just a complete norm on \( X \), but also a sequence of complete norms on \( M_n(X) \), the space of \( n \times n \) matrices on \( X \), for every \( n \in \mathbb{N} \). These matrix norms are obtained via the natural identification of \( M_n(X) \) as a subspace of \( M_n(B(\mathcal{H})) \approx B(\mathcal{H}^n) \), where \( \mathcal{H}^n \) is the Hilbert space direct sum of \( n \) copies of \( \mathcal{H} \). In 1988, Z-J. Ruan [16] characterized the sequence of matrix norms on a Banach space \( X \) that makes \( X \) an operator space, in terms of two properties of matrix norms, known as Ruan’s axioms.

An \( (\text{abstract}) \) operator space is a pair \((X,\{\|\cdot\|_n\}_{n\in\mathbb{N}})\) consisting of a linear space \( X \) and a complete norm \( \|\cdot\|_n \) on \( M_n(X) \) for every \( n \in \mathbb{N} \), such that \((R1)\) \( \|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\| \) for all \( \alpha, \beta \in M_n \) and for all \( x \in M_n(X) \), and \((R2)\) \( \|x \oplus y\|_{m+n} = \max\{\|x\|_m,\|y\|_n\} \) for all \( x \in M_m(X) \), and for all \( y \in M_n(X) \), where \( x \oplus y \) denotes the matrix \( \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \) in \( M_{m+n}(X) \) where \( 0 \) stands for zero matrices of appropriate orders. The sequence of matrix norms \( \{\|\cdot\|_n\}_{n\in\mathbb{N}} \) is called an operator space structure on the linear space \( X \). An operator space structure on a normed space \((X,\|\cdot\|)\) will usually mean a sequence of matrix norms \( \{\|\cdot\|_n\}_{n\in\mathbb{N}} \) as above, but with \( \|\cdot\|_1 = \|\cdot\| \) and in that case, we say...
$\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ is an admissible operator space structure on $X$. If $X$ is an abstract operator space, then there exists a linear complete isometry $\varphi : X \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Also, the matrix norms on an operator space induced via the embedding $M_n(X) \subset M_n(B(\mathcal{H}))$ satisfies the Ruan’s axioms. Thus Ruan’s characterization allows us to describe an operator space in an abstract way free of any concrete representation on a Hilbert space.

If $X$ is a Banach space, the closed unit ball $\{x \in X; \|x\| \leq 1\}$ is denoted by $\text{Ball}(X)$. If $X$ and $Y$ are operator spaces and $\varphi : X \to Y$ is a linear map, $\varphi^{(n)} : M_n(X) \to M_n(Y)$, given by $[x_{ij}] \to [\varphi(x_{ij})]$, with $[x_{ij}] \in M_n(X)$ and $n \in \mathbb{N}$, determines a linear map from $M_n(X)$ to $M_n(Y)$. The complete bound norm (in short $\text{cb-norm}$) of $\varphi$ is defined as $\|\varphi\|_{\text{cb}} = \sup\{\|\varphi^{(n)}\| ; n \in \mathbb{N}\}$. $\varphi$ is completely bounded if $\|\varphi\|_{\text{cb}} < \infty$. $\varphi$ is a complete isometry if each map $\varphi^{(n)} : M_n(X) \to M_n(Y)$ is an isometry. If $\varphi$ is a complete isometry, then $\|\varphi\|_{\text{cb}} = 1$. If $\|\varphi\|_{\text{cb}} \leq 1$, $\varphi$ is said to be a complete contraction. If $\varphi : X \to Y$ is a completely bounded linear bijection and if its inverse is also completely bounded, then $\varphi$ is said to be a complete isomorphism. Two operator spaces are considered to be the same if there is a complete isometric isomorphism from $X$ to $Y$. In that case, we write $X \approx Y$ completely isometrically.

A Banach space $Z$ is injective if for any Banach spaces $X$ and $Y$ where $Y$ contains $X$ as a closed subspace, and for any completely bounded linear map $\varphi : X \to Z$, there exists a bounded linear extension $\tilde{\varphi} : Y \to Z$ such that $\tilde{\varphi}|_X = \varphi$ and $\|\tilde{\varphi}\| = \|\varphi\|$. In a similar manner, an operator space $Z$ is injective [3] if for any operator spaces $X$ and $Y$ where $Y$ contains $X$ as a closed subspace, and for any completely bounded linear map $\varphi : X \to Z$, there exists a completely bounded extension $\tilde{\varphi} : Y \to Z$ such that $\tilde{\varphi}|_X = \varphi$ and $\|\tilde{\varphi}\|_{\text{cb}} = \|\varphi\|_{\text{cb}}$. An operator space $X$ is homogeneous [11] if each bounded linear operator $\varphi$ on $X$ is completely bounded with $\|\varphi\|_{\text{cb}} = \|\varphi\|$. More information about operator spaces and completely bounded mappings may be found in the papers [4], [5], [8] and [15] or in the books [6], [10] and [12].

2. Minimal and Maximal operator space structures

A given Banach space has in general many realizations as an operator space. A very basic question in operator space theory is to exhibit some particular operator space structures on a given Banach space $X$. In the most general situation, Blecher and Paulsen [2] achieved this by noting that the set of all
operator space structures admissible on a given Banach space $X$ admits a minimal and maximal element. These structures were further investigated in [8] and [9]. By Hahn-Banach theorem, it follows that any subspace of a minimal operator space is again minimal. Quotients of minimal operator spaces are called $Q$-spaces [14], and they need not be minimal. Also, the category of $Q$-spaces is stable under taking quotients and subspaces. An operator space $X$ is said to be submaximal [12] if it embeds completely isometrically into a maximal operator space $Y$. Generally, a submaximal space need not be maximal, but maximality passes to quotients [12]. Subspace structure of various maximal operator spaces were further studied in [7].

If $X$ is a Banach space, then there is a minimal operator space structure on $X$, denoted by $\text{Min}(X)$, and this quantization is characterized by the property that for any arbitrary operator space $Y$ and for any bounded linear map $\varphi : Y \rightarrow \text{Min}(X)$ is completely bounded and satisfies $\|\varphi : Y \rightarrow \text{Min}(X)\|_{cb} = \|\varphi : Y \rightarrow X\|$. An operator space $X$ is minimal if $\text{Min}(X) = X$. Also, an operator space is minimal if and only if it is completely isometric to a subspace of a commutative C*-algebra. If $X$ is a Banach space, there is a maximal way to consider it as an operator space. The matrix norms given by $\|[x_{ij}]\|_{n} = \sup \{\|\varphi(x_{ij})\| \mid \varphi \in \text{Ball}(B(X,Y))\}$ where the supremum is taken over all operator spaces $Y$ and all linear maps $\varphi \in \text{Ball}(B(X,Y))$, makes $X$ an operator space. This operator space is denoted by $\text{Max}(X)$ and is called the maximal operator space structure on $X$. For $[x_{ij}] \in M_{n}(X)$, we write $\|[x_{ij}]\|_{\text{Max}(X)}$ to denote its norm as an element of $M_{n}(\text{Max}(X))$. An operator space $X$ is maximal if $\text{Max}(X) = X$. By Ruan’s theorem, we also have $\|[x_{ij}]\|_{\text{Max}(X)} = \sup \{\|[\varphi(x_{ij})]\| \mid \varphi \in \text{Ball}(B(X,B(H)))\}$ where the supremum is taken over all Hilbert spaces $H$ and all linear maps $\varphi \in \text{Ball}(B(X,B(H)))$. By the definition of $\text{Max}(X)$, any operator space structure on $X$ is smaller than $\text{Max}(X)$. This maximal quantization of a normed space is characterized by the property that for any arbitrary operator space $Y$, any bounded linear map $\varphi : \text{Max}(X) \rightarrow Y$ is completely bounded and satisfies $\|\varphi : \text{Max}(X) \rightarrow Y\|_{cb} = \|\varphi : X \rightarrow Y\|$. If $X$ is any operator space, then the identity map on $X$ defines completely contractive maps $\text{Max}(X) \rightarrow X \rightarrow \text{Min}(X)$. For any Banach space $X$, we have the following duality relations [1]: $\text{Min}(X)^{*} \approx \text{Max}(X^{*})$ and $\text{Max}(X)^{*} \approx \text{Min}(X^{*})$ completely isometrically.

Just like every operator space embeds completely isometrically into $B(H)$ for
some Hilbert space $\mathcal{H}$, every submaximal space embeds completely isometrically into $\text{Max}(B(\mathcal{H}))$ for some Hilbert space $\mathcal{H}$. To see this, let $X \subset Y$, where $Y$ is a maximal operator space. Also, let $\iota : X \to B(\mathcal{H})$ be a complete isometric inclusion. Since $B(\mathcal{H})$ is injective, the inclusion $\iota : X \to B(\mathcal{H})$ extends to a complete contraction $\varphi : Y \to B(\mathcal{H})$. Since $Y$ is maximal, $\|\varphi : Y \to \text{Max}(B(\mathcal{H}))\|_{cb} \leq 1$. Let $\tilde{\iota} = \varphi|_X$, then $\|\tilde{\iota} : X \to \text{Max}(B(\mathcal{H}))\|_{cb} \leq 1$. If $\tilde{\iota}(X) = \tilde{X}$, by the definition of maximal operator spaces, we have $\|\tilde{\iota}^{-1} : \tilde{X} \to X\|_{cb} \leq 1$. Thus $\tilde{\iota} : X \to \tilde{X}$ is a completely isometric isomorphism.

In the following, we consider the smallest submaximal space structure on a Banach space $X$, namely the $\mu$-space structure which is denoted by $\mu(X)$. We prove that $\mu(X)$ will be homogeneous. We also derive a universal property of $\mu$-spaces which distinguishes it among other submaximal spaces. By making use of this property, we show that the class of $\mu$-spaces is stable under taking subspaces. Finally, we explore the duality relations of $\mu$-spaces.

3. Main results

Just like minimal and maximal operator space structures, we have a minimal and a maximal way to view a Banach space $X$ as a submaximal space, which we denote as $\text{Min}_S(X)$ and $\text{Max}_S(X)$ respectively. From the definition of a submaximal space it follows that $\text{Max}_S(X) = \text{Max}(X)$. T. Oikhberg [7] introduced the $\mu$-space structure on a Banach space $X$ and proved that $\text{Min}_S(X) = \mu(X)$. Suppose $X$ is a Banach space. Note that $\text{Min}(X)$ is the operator space structure on $X$ inherited by regarding $X \subset C(K)$, where $K = \text{Ball}(X^*)$, the closed unit ball of the dual space of $X$ with its weak* topology. Also, from [1], $\text{Max}(X)^* = \text{Min}(X^*)$, so that $\text{Max}(X)^* = (\text{Min}(X^*))^* = \text{Max}(X^{**})$ completely isometrically. Since $X \hookrightarrow X^{**}$, we have $\text{Max}(X) \hookrightarrow \text{Max}(X^{**})$ completely isometrically.

**Definition 3.1.** An operator space $X$ is a $\mu$-space if it embeds completely isometrically into $\text{Max}(C(K)^{**})$, where $K = \text{Ball}(X^*)$ the unit closed ball of the dual space of $X$ with its weak* topology.

A Banach space $X$, with the above defined $\mu$-space structure is denoted by $\mu(X)$ and the corresponding sequence of matrix norms by $\{\|\cdot\|_{\mu_n}\}_{n \in \mathbb{N}}$. Note
that the $\mu$-space structure on a Banach space $X$ is an admissible operator space structure on $X$.

Remark 3.2. Suppose that $X$ and $Y$ are injective Banach spaces and $E$ and $F$ are isomorphic (isometric) closed subspaces of $X$ and $Y$ respectively. Let $\varphi : E \to F$ be an isomorphism. Since $Y$ is injective, there exists a map $\hat{\varphi} : \text{Max}(X) \to \text{Max}(Y)$ such that $\hat{\varphi}|_E = \varphi$ and $\|\hat{\varphi}\| = \|\varphi\|$. Since $X$ has maximal operator space structure, we have $\|\hat{\varphi}\|_{cb} = \|\varphi\|$. Thus, $\|\varphi\|_{cb} \leq \|\hat{\varphi}\|_{cb} = \|\varphi\|$. This shows that $\varphi$ is completely bounded. Similarly $\varphi^{-1}$ is also completely bounded, so that $\varphi$ is a complete isomorphism. Thus, $E$ and $F$ are completely (isometrically) isomorphic as operator subspaces of $\text{Max}(X)$ and $\text{Max}(Y)$ respectively. From this fact, it follows that $\mu$-spaces can also be described as a submaximal subspace of an injective commutative $C^*$-algebra, because the operator space structure is independent of the particular embedding.

Now we give a direct proof, different from [7] of the fact that $\mu(X)$ is the smallest submaximal operator space structure on a given Banach space $X$.

Theorem 3.3. Let $X$ be a Banach space. Then $\mu(X)$ is the smallest submaximal space structure on $X$.

Proof. Let $j$ be a complete isometric embedding of $X$ in $\text{Max}(C(K)^{**})$ described in the definition of $\mu$-spaces. Let $\varphi : X \to \text{Max}(Y)$ be a complete isometric embedding of $X$ into $\text{Max}(Y)$. Let the sequence of matrix norms on $X$ obtained via this embedding be denoted by $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$. Then for any $[x_{ij}] \in M_n(X)$,

$$\| [x_{ij}]_n^Y = \| [\varphi(x_{ij})] \|_{\text{Max}(Y)} = \sup \{\| u(\varphi(x_{ij})) \| : u \in \text{Ball}(B(Y, B(K))) \}$$

where the supremum is taken over all possible maps $u : Y \to B(K)$ and over all Hilbert spaces $K$. Also, we have

$$\| [x_{ij}]_n^H = \| [x_{ij}] \|_{\text{Max}(C(K)^{**})} = \sup \{\| v(x_{ij}) \| : v \in \text{Ball}(B(C(K)^{**}, B(H))) \}$$

where the supremum is taken over all possible maps $v : C(K)^{**} \to B(H)$ and over all Hilbert spaces $H$. Consider the following diagram.

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi} & \text{Max}(Y) \\
\downarrow{id} & & \downarrow u \\
X & \xrightarrow{j} & \text{Max}(C(K)^{**}) \\
& & \xrightarrow{v} B(H)
\end{array}
$$
Since $\varphi^{-1} : \varphi(X) \subset Y \rightarrow X$ is bounded, $w = j \circ id \circ \varphi^{-1} : \varphi(X) \subset Y \rightarrow C(K)^{**}$ is bounded and $\|w\| = 1$. Since $C(K)^{**}$ is injective as a Banach space, $w$ has a bounded extension $\tilde{w} : Y \rightarrow C(K)^{**}$ with $\|\tilde{w}\| = 1$. Therefore, for any map $v : C(K)^{**} \rightarrow B(H)$ with $\|v\| \leq 1$, $\tilde{v} = v \circ \tilde{w} : Y \rightarrow B(H)$ is a bounded map and is a completely bounded map with $\|\tilde{v}\|_{cb} \leq 1$ when regarded as a map from $Max(Y)$ to $B(H)$. Thus $\|[x_{ij}]^{\mu}_{n}\| \leq \|[x_{ij}]^{Y}_{n}\|$ for any $[x_{ij}] \in M_{n}(X)$. This shows that the $\mu$-space structure on $X$ is the smallest submaximal space structure on $X$. \hfill \Box

Maximal and minimal operator spaces are homogeneous, but in general, submaximal spaces need not be homogeneous. Now we show that $\mu$-spaces are homogeneous.

**Proposition 3.4.** Every $\mu$-space is homogeneous.

**Proof.** Let $\varphi : \mu(X) \rightarrow \mu(X)$ be a bounded linear map. Then $\varphi$ extends to a bounded linear map $\tilde{\varphi}$ on $C(K)^{**}$, and it is then completely bounded on $Max(C(K)^{**})$ and $\|\tilde{\varphi}\|_{cb} = \|\tilde{\varphi}\| = \|\varphi\|$. But $\|\varphi\|_{cb} \leq \|\tilde{\varphi}\|_{cb} = \|\varphi\|$. Hence $\mu(X)$ is homogeneous. \hfill \Box

Completely bounded Banach-Mazur distance between two operator spaces $X$ and $Y$ is defined as $d_{cb}(X, Y) = \inf\{\|\varphi\|_{cb} \|\varphi^{-1}\|_{cb} : \varphi : X \rightarrow Y$ is a complete isomorphism $\}$. If $X$ is a Banach space, then the $\mu$-space structure on $X$ lies between the operator space structures $Min(X)$ and $Max(X)$. Now we shall show that the cb distance between these spaces can be realized as the cb-norm of the identity mapping between them.

**Theorem 3.5.** For a Banach space $X$, we have:

$$d_{cb}(Min(X), \mu(X)) = \|id : Min(X) \rightarrow \mu(X)\|_{cb}$$

and

$$d_{cb}(\mu(X), Max(X)) = \|id : \mu(X) \rightarrow Max(X)\|_{cb}$$
Proof. Let $T : \mu(X) \to \text{Min}(X)$ be a complete isomorphism. Let $\tilde{T}$ denotes the same map regarded as a mapping from $\mu(X)$ to $\mu(X)$. Consider the following diagram.

\[
\begin{array}{ccc}
\mu(X) & \xrightarrow{T} & \text{Min}(X) \\
\uparrow \Phi & \downarrow \text{id} & \downarrow \text{id} \\
\mu(X) & \xleftarrow{T^{-1}} & \text{Min}(X)
\end{array}
\]

Here $\text{id}$ denotes the formal identity mapping regarded as a mapping from $\text{Min}(X)$ to $\mu(X)$. From the diagram, we get

\[
\|\text{id} : \text{Min}(X) \to \mu(X)\|_{cb} = \left\|\tilde{T} \circ T^{-1}\right\|_{cb} \leq \left\|\tilde{T}\right\|_{cb} \left\|T^{-1}\right\|_{cb}. 
\]

Since $\mu(X)$ is homogeneous (by above proposition 3.4), $\left\|\tilde{T}\right\|_{cb} = \left\|\tilde{T}\right\| = \left\|T\right\| = \left\|T\right\|_{cb}$, where the last equality is because of the minimal operator space structure of the range space of $T$. Thus we have $\|\text{id} : \text{Min}(X) \to \mu(X)\|_{cb} \leq \|T\|_{cb} \left\|T^{-1}\right\|_{cb}$. This shows that $d_{cb}(\text{Min}(X), \mu(X)) = \|\text{id} : \text{Min}(X) \to \mu(X)\|_{cb}$. Similarly the other case follows.

We show that among submaximal spaces, the $\mu$-spaces are characterized by the following universal property.

**Theorem 3.6.** A submaximal space $X$ is a $\mu$-space up to complete isometric isomorphism if and only if for any submaximal space $Y$, any bounded linear map $\varphi : Y \to X$ is completely bounded with $\|\varphi\|_{cb} = \|\varphi\|$.

Proof. Assume that $X = \mu(X)$. By definition of $\mu$-spaces, $X = \mu(X) \subset \text{Max}(C(K)^{**})$, where $K = \text{Ball}(X^*)$. Since $Y$ is submaximal, we have $Y \subset \text{Max}(Z)$ for some operator space $Z$. Now, $\varphi : Y \to \mu(X)$ can be regarded as a map $Y$ to $\text{Max}(C(K)^{**})$. Since bidual of $C(K)$ is injective, there exists $\tilde{\varphi} : Z \to \text{Max}(C(K)^{**})$ with $\|\tilde{\varphi}\| = \|\varphi\|$ and $\tilde{\varphi}|_Y = \varphi$. Considering $\tilde{\varphi} : \text{Max}(Z) \to \text{Max}(C(K)^{**})$, we see that $\tilde{\varphi}$ is completely bounded and $\|\tilde{\varphi}\|_{cb} = \|\tilde{\varphi}\|$. But $\|\varphi\|_{cb} \leq \|\tilde{\varphi}\|_{cb} = \|\tilde{\varphi}\| = \|\varphi\|$. This shows that $\|\varphi\|_{cb} = \|\varphi\|$. Conversely, take $Y = \mu(X)$ and $\varphi = \text{id} : \mu(X) \to X$, the formal identity mapping. Then by assumption, $\|\text{id}\|_{cb} = \|\varphi\| = 1$. Also, from the above part, $\|\text{id}^{-1}\|_{cb} = \|\text{id} : X \to \mu(X)\|_{cb} = \|\text{id}\| = 1$. Thus $\text{id} : X \to \mu(X)$ is a complete isometric isomorphism.

**Remark 3.7.** We know that if $X$ has minimal operator space structure, then every bounded linear map defined on another operator space with values in $X$ is
completely bounded. Also, we have shown that if $X$ has the $\mu$-space structure, then any bounded linear map from a submaximal space to $X$ is completely bounded. Now, let $X$ be endowed with any operator space structure $\{\|\cdot\|_n\}_{n \in \mathbb{N}}$ such that $\|\begin{bmatrix} x_{ij} \end{bmatrix}\|_n \leq \|\begin{bmatrix} x_{ij} \end{bmatrix}\|_\mu^n$ for every $\begin{bmatrix} x_{ij} \end{bmatrix} \in M_n(X)$, and for all $n \in \mathbb{N}$. In this case also, any bounded linear map $\varphi$ from a submaximal space $Y$ to $X$ is completely bounded with $\|\varphi\|_{cb} = \|\varphi\|$. Here, the operator space structure on $X$ is not submaximal, since the $\mu$-space structure is the smallest submaximal structure on any normed space. To see this, consider the following diagram.

\[
\begin{array}{c}
Y \\ \varphi \downarrow \\ X \\
\downarrow \tilde{\varphi} \\
\mu(X)
\end{array}
\]

Since the identity mapping $id : \mu(X) \to X$ is a complete contraction, using theorem 3.6, we have: $\|\varphi\|_{cb} \leq \|id\|_{cb} \|\tilde{\varphi}\|_{cb} \leq \|\tilde{\varphi}\| = \|\varphi\|$. Thus, $\|\varphi\|_{cb} = \|\varphi\|$. 

Now we make use of the universal property of $\mu$-spaces to show that the class of $\mu$-spaces is stable under taking subspaces.

**Corollary 3.8.** Let $Y \subset \mu(X)$. Then $Y$ is also a $\mu$-space.

**Proof.** Let $Z$ be any submaximal space. Then any bounded linear map $\varphi : Z \to Y$ can be regarded as a map from $Z$ to $\mu(X)$. By the universal property of $\mu$-spaces, we see that $\|\varphi\|_{cb} = \|\tilde{\varphi}\|$. This shows that $Y$ is a $\mu$-space. □

The following theorem gives a more general characterization of $\mu$-spaces up to complete isomorphism.

**Theorem 3.9.** A submaximal space $X$ is completely isomorphic to a $\mu$-space if and only if for any submaximal space $Y$, any completely bounded linear bijection $\varphi : X \to Y$ is a complete isomorphism.

**Proof.** Let $\psi : X \to \mu(Z)$ be a complete isomorphism. Then for any completely bounded linear bijection $\varphi : X \to Y$, by theorem 3.6, $\psi \circ \varphi^{-1} : Y \to \mu(Z)$ is completely bounded and $\|\psi \circ \varphi^{-1}\|_{cb} = \|\psi \circ \varphi^{-1}\|$. Therefore, $\|\varphi^{-1}\|_{cb} = \|\psi^{-1} \circ \psi \circ \varphi^{-1}\|_{cb} \leq \|\psi^{-1}\|_{cb} \|\psi \circ \varphi^{-1}\|_{cb} \leq \infty$, showing that $\varphi$ is a complete isomorphism. For the converse, take $Y$ as $\mu(X)$ and $\varphi$ as the formal identity map $id : X \to \mu(X)$. □
Now we look at the case when the domain is endowed with the $\mu$-space structure.

**Theorem 3.10.** Let $X$ be an operator space. Then the formal identity map $id: \mu(X) \to X$ is completely bounded if and only if for every submaximal space $Y$, every bounded linear map $\varphi: Y \to X$ is completely bounded. Moreover, we have: $\|id: \mu(X) \to X\|_{cb} = \sup\{\|u\|_{cb}\} \|u\|$, where the supremum is taken over all bounded non zero linear maps $u: Y \to X$ and all submaximal spaces $Y$.

**Proof.** Assume that $id: \mu(X) \to X$ is completely bounded with $\|id\|_{cb} = C$. Let $Y$ be a submaximal space and $u: Y \to X$ be a bounded linear map. Let $\tilde{u}$ denotes the same map $u$ regarded as a map from $Y$ to $\mu(X)$. Then by the universal property of $\mu$-spaces, $\tilde{u}$ is completely bounded and $\|\tilde{u}\|_{cb} = \|u\|$. Since $u = id \circ \tilde{u}$, we have $\|u\|_{cb} \leq \|id\|_{cb}\|u\| = C\|u\| < \infty$. Thus $u$ is completely bounded. For the converse, take $Y$ as $\mu(X)$, and $u$ as the identity map. Also, from the above inequality, it follows that $\|id: \mu(X) \to X\|_{cb} = \sup\{\|u\|_{cb}\}$, where the supremum is taken over all bounded non zero linear maps $u: Y \to X$ and all submaximal spaces $Y$. □

The following theorem shows that an injective Banach space $X$ has a unique submaximal space structure, or in other words $Min_S(X) = Max_S(X)$, if $X$ is an injective Banach space.

**Theorem 3.11.** If $X$ is an injective Banach space, then $\mu(X)$ is completely isometrically isomorphic to $Max(X)$.

**Proof.** Consider the formal identity map $id: \mu(X) \to Max(X)$. By definition of $\mu$-spaces, $\mu(X) \subset Max(C(K)^{**})$ and since $X$ is injective as a Banach space, $id$ extends to a bounded linear map $\tilde{id}: Max(C(K)^{**}) \to Max(X)$ with $\|\tilde{id}\| = \|id\| = 1$. Since domain has maximal operator space structure, we have $\|\tilde{id}\|_{cb} = 1$ and hence $\|id\|_{cb} = 1$. Also, $\|id^{-1}\|_{cb} = 1$, showing that $id: \mu(X) \to Max(X)$ is a complete isometric isomorphism. □

We know that the converse of the above theorem is not true. For example, the space $\ell_2^1$ has a unique operator space structure [12], but it is not an injective Banach space. The following theorem describes some equivalent conditions for the uniqueness of the submaximal space structures.

**Theorem 3.12.** For a Banach space $X$, the following are equivalent.

1. $X$ has a unique submaximal space structure.
(2) $\mu(X) = \text{Max}(X)$ completely isometrically.
(3) Any bounded linear map $\varphi : X \rightarrow B(H)$ admits a bounded extension $\tilde{\varphi} : \text{C}(\text{Ball}(X^*)) \rightarrow B(H)$ with $\|\tilde{\varphi}\| = \|\varphi\|.$

**Proof.** It is clear from the definition that (1) $\Leftrightarrow$ (2). Now to prove (2) $\Rightarrow$ (3), regard $\varphi$ as a map from $\mu(X) \rightarrow B(H)$, we see that $\varphi$ is completely bounded and $\|\varphi\|_c = \|\varphi\|.$ Since $\mu(X) \rightarrow \text{Max}(\text{C}($Ball$(X^*)))$ completely isometrically and since $B(H)$ is injective, there exists an extension $\tilde{\varphi} : \text{Max}(\text{C}($Ball$(X^*))) \rightarrow B(H)$ with $\|\tilde{\varphi}\|_c = \|\varphi\|_c.$ Thus $\tilde{\varphi} : (\text{C}(\text{Ball}(X^*)) \rightarrow B(H)$ satisfies $\|\tilde{\varphi}\| = \|\varphi\|_c = \|\varphi\|_c.$

Assume that any bounded linear map $\varphi : X \rightarrow B(H)$ admits a bounded extension $\tilde{\varphi} : (\text{C}(\text{Ball}(X^*)) \rightarrow B(H)$ with $\|\tilde{\varphi}\| = \|\varphi\|.$

Clearly $\|\varphi(x_{ij})\|_{max} \leq \|\varphi\|_n$ for any $[x_{ij}] \in M_n(X).$ By definition of maximal spaces,

$\|\varphi(x_{ij})\|_{max} = \sup\{\|\varphi(x_{ij})\| : \varphi \in \text{Ball}(B(X,B(H)))\}$

where the supremum is taken over all possible bounded linear maps $\varphi : X \rightarrow B(H)$ and over all Hilbert spaces $H$. Also,

$\|\varphi(x_{ij})\|_n = \|\varphi(x_{ij})\|_{\text{Max}(\text{C}(\text{Ball}(X^*)))}$

$= \sup\{\|v(x_{ij})\| : v \in \text{Ball}(B(\text{C}(\text{Ball}(X^*)), B(H)))\}$

where the supremum is taken over all possible bounded linear maps $v : (\text{C}(\text{Ball}(X^*)) \rightarrow B(H)$ and over all Hilbert spaces $H$. By the assumed extension property of $X$, corresponding to any $u \in \text{Ball}(B(X,B(H)))$, we have an extended function $\tilde{u} \in \text{Ball}(B(\text{C}(\text{Ball}(X^*)), B(H)))$, so that $\|\varphi(x_{ij})\|_n \geq \|\varphi(x_{ij})\|_{max}$ for any $[x_{ij}] \in M_n(X).$

Thus $\mu(X) = \text{Max}(X)$, showing that (3) $\Rightarrow$ (2). \hfill $\Box$

**Remark 3.13.** Since every injective Banach space has a unique submaximal space structure, every injective Banach space $X$ has the above described extension property.

A $Q$-space is an (operator) quotient of a minimal space [14]. Note that if $X$ is a $Q$-space, then $X^*$ is a submaximal space. Conversely, the dual of a submaximal space is a $Q$-space. Eric Ricard [13] introduced the maximal $Q$-space structure on a Banach space $X$ denoted by $\text{Max}_Q(X)$, where the matrix norms are defined as:

$\|\varphi\| = \sup\{\|u(x_{ij})\|_{M_n(E)} : u : X \rightarrow E, E$ a $Q$-space and $\|u\| \leq 1 \}$
We now prove the duality relations between the $\mu(X)$ and $\text{Max}_Q(X)$.

**Theorem 3.14.** For any Banach space $X$, we have: $(\text{Max}_Q(X))^* = \mu(X^*)$ and $(\mu(X))^* = \text{Max}_Q(X^*)$.

**Proof.** Note that $(\text{Max}_Q(X))^*$ is a submaximal space structure on $X^*$, so that by theorem 3.3, the formal identity map $id : (\text{Max}_Q(X))^* \to \mu(X^*)$ is a complete contraction. Also, the embedding $X \subset (\mu(X^*))^*$ gives a $Q$-space structure on $X$. Hence the identity map $id : \text{Max}_Q(X) \to (\mu(X^*))^*$ is a complete contraction. Taking the duals, we see that $id : \mu(X^*) \to (\text{Max}_Q(X))^*$ is a complete contraction. Thus $(\text{Max}_Q(X))^* = \mu(X^*)$. The other part follows by duality. □

**Corollary 3.15.** An operator space $X$ is a $\mu$-space if and only if its bidual $X^{**}$ is a $\mu$-space.

**Proof.** Let $X = \mu(X)$. Then $X^* = \mu(X)^* = \text{Max}_Q(X^*)$, so that $X^{**} = (\text{Max}_Q(X^*))^* = \mu(X^{**})$. Thus $X^{**}$ is a $\mu$-space. The converse part follows from the fact that $X \subset X^{**}$ and from the Corollary 3.8 □

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