On the measurement of azimuthal anisotropies in nucleus–nucleus collisions

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Abstract

Azimuthal distributions of particles produced in nucleus-nucleus collisions are measured with respect to an estimated reaction plane which, because of finite multiplicity fluctuations, differs in general from the true reaction plane. It follows that the measured distributions do not coincide with the true ones. I propose a general method of reconstructing the Fourier coefficients of the true azimuthal distributions from the measured ones. This analysis suggests that the Fourier coefficients are the best observables to characterize azimuthal anisotropies because, unlike other observables such as the in-plane anisotropy ratio or the squeeze-out ratio, they can be reconstructed accurately.

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A characteristic aspect of collective behavior in nucleus–nucleus collisions is that the directions of the outgoing particles are correlated to the orientation of the impact parameter [1]: azimuthal distributions measured from the reaction plane (which is the plane containing the impact parameter and the beam axis) are not uniform. Quantitative studies of this effect should provide information on the nuclear equation of state [2]. Several experiments have measured azimuthal distributions of charged particles [3], identified protons and light nuclei [4, 5] and, charged [6] and neutral [7] pions, neutrons [8, 9] and Λ baryons [10]. In these analyses, the azimuthal angle is defined with respect to an estimated reaction plane which, because of finite multiplicity fluctuations, differs from the true reaction plane. It has been recently emphasized that a determination of nuclear equation of state from azimuthal anisotropies requires a good accuracy [11]. It is therefore important to correct the errors which are made in determining the reaction plane.

We propose a systematic procedure to reconstruct the true azimuthal distribution from the measured ones. Normalized azimuthal distributions can be expressed as Fourier series:

\[
\frac{dN}{d\phi} = \frac{1}{2\pi} \left( 1 + 2 \sum_{n \geq 1} c_n \cos n\phi \right)
\]

where \(c_n = \langle \cos n\phi \rangle\), the brackets denoting average values, and we assume that the azimuthal distributions are symmetric with respect to the reaction plane (i.e. even in \(\phi\)), which holds for spherical nuclei. The knowledge of all the Fourier coefficients \(\langle \cos n\phi \rangle\) allows to reconstruct the full distribution using Eq.(1).

In an actual experiment, the reaction plane is not known exactly. It is reconstructed event by event from the reaction products. The reconstructed plane differs in general from the true reaction plane by an error \(\Delta \phi\), which varies from one event to the other. Thus, the measured azimuthal angle \(\psi\) is related to the true azimuthal angle \(\phi\) by \(\psi = \phi - \Delta \phi\) (see Fig.1). Averaging over many events, assuming that \(\phi\) and \(\Delta \phi\) are statistically independent (this assumption will be discussed below), one obtains the following relation between the measured and true Fourier coefficients:

\[
\langle \cos n\psi \rangle = \langle \cos n\phi \rangle \langle \cos n\Delta \phi \rangle.
\]

From Eq.(2), we can reconstruct the true distribution once the correction factor \(\langle \cos n\Delta \phi \rangle\) is known.

Before we calculate \(\langle \cos n\Delta \phi \rangle\), let us briefly comment Eq.(2). This equation shows that the measured anisotropies are always smaller than the true ones: they are smeared by the error \(\Delta \phi\). More precisely, if the probability distribution of \(\Delta \phi\) has a typical width \(\delta\), i.e. if \(\delta\) denotes the typical error made in determining the reaction plane, \(\langle \cos n\Delta \phi \rangle\) will decrease with \(n\) and become small for \(n > 1/\delta\). This has two consequences. First, the higher order coefficients disappear in the measurement procedure. Indeed, all the distributions measured so far are well reproduced by keeping only the first two Fourier coefficients \(n = 1, 2\) in Eq.(1) [3, 6, 7, 8, 11]. However, higher order components might be sizeable in the true distributions. The second consequence is that observables involving higher order Fourier coefficients cannot be reconstructed accurately. This is unfortunately the case for widely used observables such
as the in-plane anisotropy \[ R_{\text{in-plane}} = \frac{(dN/d\phi)_{\phi=0}}{(dN/d\phi)_{\phi=180^\circ}} \] (3)

or the squeeze-out ratio \[ R_{\text{squeeze}} = \frac{[(dN/d\phi)_{\phi=90^\circ} + (dN/d\phi)_{\phi=270^\circ}]}{[(dN/d\phi)_{\phi=0^\circ} + (dN/d\phi)_{\phi=180^\circ}]} \] (4)

which both involve an infinite number of Fourier coefficients (see Eq.(1)). Furthermore, Fourier coefficients, which are integrated quantities, are also easier to evaluate in theoretical models (especially in Monte-Carlo calculations) than observables using the value of the distribution at a specific point.

We now recall how the orientation of the reaction plane is estimated from the reaction products: in high energy collisions \((E/A > 100 \text{ MeV})\), the projectile (target) fragments are deflected away from the target (projectile). Therefore, the vector obtained by summing all the transverse momenta of the particles produced in the projectile (target) rapidity region is parallel (antiparallel) to the impact parameter. More generally, one constructs a vector \( \mathbf{Q} \) [12]:

\[ \mathbf{Q} = \sum_{k=1}^{N} w_k \mathbf{u}_k \] (5)

where the sum runs over all the detected particles in the event. \( \mathbf{u}_k \) is the unit vector parallel to the transverse momentum of the particle, and \( w_k \) is a weight which may depend on the type of particle, its rapidity and transverse momentum. The choice of \( w_k \) is to a large extent arbitrary. Danielewicz and Odyniec [12] choose \( w_k = p_T \) for \( y > 0.3 \), \( w_k = -p_T \) if \( y < -0.3 \) and \( w_k = 0 \) if \(|y| < 0.3\). Many alternative definitions have been used [13, 14], some of which do not require particle identification. This method of determining the reaction plane is commonly referred to as the transverse momentum method.

For an ideal system with infinite multiplicity \( N \), \( \mathbf{Q} \) lies in the true reaction plane, and azimuthal distributions can be measured from \( \mathbf{Q} \). In an actual experiment, the multiplicity is finite, which has two effects. First, because of statistical fluctuations, there is a deviation \( \Delta \phi \) between the true reaction plane and \( \mathbf{Q} \). Another effect of the finite multiplicity is that when one measures the azimuthal angle of a particle with respect to \( \mathbf{Q} \), there is a correlation if the particle is included in the sum in Eq.(5). This can be avoided by constructing a new vector \( \mathbf{Q} \) obtained by summation over the \( N-1 \) remaining particles [12]. Then, it is reasonable to assume, as we have done, that \( \phi \) and \( \Delta \phi \) are statistically independent. However, this is not quite true for small nuclei where the constraint of global momentum conservation creates important correlations. A method to subtract these correlations is described in [15].

We now proceed to evaluate \( \langle \cos n\Delta \phi \rangle \). We first show that the distribution of \( \Delta \phi \) is a universal function of a single real parameter \( \chi \), which scales like \( \sqrt{N} \). Then we express \( \langle \cos n\Delta \phi \rangle \) as a function of \( \chi \). Finally, we explain how to extract \( \chi \) from the data.

We consider a large sample of events having the same magnitude of impact parameter. Experimentally, this can be done approximately by selecting events having the same
multiplicity, or the same transverse energy, or the same energy in a zero degree calorimeter. The number $N$ of particles entering the definition of the vector $\mathbf{Q}$ in Eq. (5) is usually much larger than unity. Then the central limit theorem shows that, for given magnitude and orientation of the true impact parameter, the fluctuations of $\mathbf{Q}$ around its average value, $\langle \mathbf{Q} \rangle$, are gaussian. We choose the direction of impact parameter as the $x$-axis. Then $\mathbf{Q} = Q (e_x \cos \Delta \phi + e_y \sin \Delta \phi)$ and $\langle \mathbf{Q} \rangle = Q e_x$, and the two dimensional distribution of $\mathbf{Q}$ takes the form

$$
\frac{dN}{Q dQ d\Delta \phi} = \frac{1}{\pi \sigma^2} \exp \left( -\frac{\mid \mathbf{Q} - \langle \mathbf{Q} \rangle \mid^2}{\sigma^2} \right) = \frac{1}{\pi \sigma^2} \exp \left( -\frac{Q^2 + \bar{Q}^2 - 2Q\bar{Q} \cos \Delta \phi}{\sigma^2} \right).
$$

(6)

We have assumed that the fluctuations $\sigma$ have the same magnitude in the $x$ and $y$ directions. This assumption will be justified later. Note that $\bar{Q}$ scales like $N$ while $\sigma$ scales like $\sqrt{N}$.

Eq.(6) can be easily integrated over $Q$ to yield the distribution of $\Delta \phi$:

$$
\frac{dN}{\Delta \phi} = \frac{1}{\pi} \exp(-\chi^2) \left\{ 1 + z \sqrt{\pi} \left[ 1 + \text{erf}(z) \right] \exp(z^2) \right\}.
$$

(7)

where $z = \chi \cos \Delta \phi$ and erf($x$) is the error function. This distribution depends on $\bar{Q}$ and $\sigma$ only through the dimensionless parameter $\chi \equiv Q / \sigma$, which scales like $\sqrt{N}$. Typically, $\Delta \phi$ is of order $1/\chi$ (see Fig.2). The Fourier coefficients are most easily calculated by integrating Eq.(6) first over $\Delta \phi$ and then over $\mathbf{Q}$:

$$
\langle \cos n \Delta \phi \rangle = \frac{\sqrt{\pi}}{2} \chi e^{-\chi^2/2} \left[ I_{n+\frac{1}{2}} \left( \frac{\chi^2}{2} \right) + I_{n-\frac{1}{2}} \left( \frac{\chi^2}{2} \right) \right]
$$

(8)

where $I_k$ is the modified Bessel function of order $k$. The variations of the first coefficients with $\chi$ is displayed in Fig.3. As expected, $\cos n \Delta \phi$ decreases with increasing $n$, and becomes vanishingly small for $n \gg \chi$. Experiments report values of $\langle \cos \Delta \phi \rangle$ ranging from 0.35 for light nuclei ($A \simeq 20$) to 0.94 for the heaviest ones ($A \simeq 200$), corresponding to values of $\chi$ between 0.4 and 2.2 respectively. Fig.3 shows that the corrections are important in this range.

If $\chi \gg 1$, The distribution of $\Delta \phi$ becomes approximately gaussian

$$
\frac{dN}{d\Delta \phi} = \frac{\chi}{\sqrt{\pi}} \exp \left( -\chi^2 \Delta \phi^2 \right)
$$

(9)

and the Fourier coefficients are given by

$$
\langle \cos n \Delta \phi \rangle = \exp(-n^2/4\chi^2).
$$

(10)

In the limit $\chi \ll 1$, the $n^{th}$ Fourier coefficient is of order $\chi^n$:

$$
\langle \cos n \Delta \phi \rangle = \frac{\sqrt{\pi}}{2^n \Gamma \left( \frac{n+1}{2} \right)} \chi^n
$$

(11)

where $\Gamma$ is the Euler function.
We now turn to the determination of $\chi$. The most widely used method \([6,7,8,9]\) to estimate the accuracy of the reaction plane determination is to divide each event randomly into two subevents containing half of the particles each, and to construct $Q$ for the two subevents \([12]\). One thus obtains two vectors $Q_I$ and $Q_{II}$. The distributions of $Q_I$ and $Q_{II}$ are given by an equation similar to Eq.(12). However, since each subevent contains only $N/2$ particles, the corresponding average value and fluctuations must be scaled: $\bar{Q}_I = \bar{Q}_{II} = \bar{Q}/2$, $\sigma_I = \sigma_{II} = \sigma/\sqrt{2}$, and therefore $\chi_I = \chi_{II} = \chi/\sqrt{2}$. The distribution of the relative angle $\Delta \phi_R \equiv |\Delta \phi_I - \Delta \phi_{II}|$ can be calculated analytically (see the Appendix of \([10]\) and the note added in proof)

$$
\frac{dN}{d\Delta \phi_R} = \frac{e^{-\chi_I^2}}{2} \left\{ \frac{2}{\pi} (1 + \chi_I^2) + z[I_0(z) + L_0(z)] + \chi_I^2 [I_1(z) + L_1(z)] \right\}
$$

where $z = \chi_I^2 \cos \Delta \phi_R$ and $L_0$ and $L_1$ are modified Struve functions \([18]\). This distribution is normalized to unity between 0 and $\pi$.

The value of $\chi$ can be obtained directly from the fraction of events for which $\Delta \phi_R > 90^\circ$, which is calculated by integrating Eq.(12) over $\Delta \phi_R$:

$$
\frac{N(90^\circ < \Delta \phi_R < 180^\circ)}{N(0^\circ < \Delta \phi_R < 180^\circ)} = \frac{\exp(-\chi_I^2)}{2} = \frac{\exp(-\chi^2/2)}{2}.
$$

Alternatively, one can obtain $\chi$ by measuring

$$
\langle \cos \Delta \phi_R \rangle = \langle \cos \Delta \phi_I \rangle \langle \cos \Delta \phi_{II} \rangle = \frac{\pi}{8} \chi^2 e^{-\chi^2/2} \left[ I_0 \left( \chi^2/4 \right) + I_1 \left( \chi^2/4 \right) \right]^2
$$

where we have used Eq.(8) with $n = 1$ and $\chi$ replaced by $\chi_I = \chi/\sqrt{2}$. The variation of $\langle \cos \Delta \phi_R \rangle$ with $\chi$ is displayed in Fig.4. Eq.(12) is more reliable than Eq.(13) if $\chi$ is large, for in this case the ratio in Eq.(13) is very small and is therefore subject to relatively large statistical fluctuations. Other methods to measure $\chi$ are described in \([10,12]\).

We finally justify the hypothesis that was made in writing Eq.(6), namely that the fluctuations have the same magnitude in both $x$ and $y$ directions. If this assumption is released, Eq.(6) is replaced by

$$
\frac{dN}{Q dQ d\phi} = \frac{1}{\pi \sigma_x \sigma_y} \exp \left[ -\frac{(Q \cos \Delta \phi - \bar{Q})^2}{\sigma_x^2} - \frac{Q^2 \sin^2 \Delta \phi}{\sigma_y^2} \right].
$$

The quantities $\bar{Q}$, $\sigma_x$ and $\sigma_y$, entering this distribution are related to the azimuthal distribution of particles in the following way. Assuming for simplicity that the multiplicity $N$ is the same for all events in the sample (it is at least approximately true since the impact parameter is fixed), we get from Eq.(9)

$$
Q = \langle Q \cdot e_x \rangle = N \langle w \cos \phi \rangle
$$

where $\phi$ is the true azimuthal angle of particles, and the last average involves all the detected particles of all events. Similarly, the fluctuations in the $x$ and $y$ directions are given by

$$
\sigma_x^2 = 2 \mathbb{E}[(Q \cdot e_x)^2] - \bar{Q}^2 = 2N \left[ \langle w^2 \cos^2 \phi \rangle - \langle w \cos \phi \rangle^2 \right].
$$
\[ \sigma_y^2 = 2\langle (Q \cdot e_y)^2 \rangle = 2N \langle w^2 \sin^2 \phi \rangle. \]  

We define the average fluctuation \( \sigma \) by
\[ \sigma^2 = \frac{1}{2}(\sigma_x^2 + \sigma_y^2) = \langle Q^2 \rangle - \bar{Q}^2 = N \left[ \langle w^2 \rangle - \langle w \cos \phi \rangle^2 \right] \]  
and the anisotropy of the fluctuations by
\[ \frac{1}{2}(\sigma_x^2 - \sigma_y^2) = N \left[ \langle w^2 \cos 2\phi \rangle - \langle w \cos \phi \rangle^2 \right]. \]  

Three cases must be distinguished, depending on the relative magnitudes of \( \bar{Q}, \sigma_x \) and \( \sigma_y \).

- (A) Azimuthal anisotropies are small, i.e. the Fourier coefficients of the azimuthal distribution are much smaller than unity. Then \( \langle w \cos \phi \rangle^2 \) are both small compared to \( \langle w^2 \rangle \). We deduce from Eqs.(19) and (20) that \( \sigma_x \simeq \sigma_y \simeq \sigma \): our assumption is justified in this case.

- (B) There are situations where \( \langle w \cos \phi \rangle^2 \) cannot be neglected compared to \( \langle w^2 \rangle \). This is the situation when the flow is strong. Using Eqs.(17) and (18) and the fact that \( N \gg 1 \), we see that in this case \( \bar{Q} \gg \sigma_x, \sigma_y \): fluctuations are small and \( \Delta \phi \simeq Q \cdot e_y/\bar{Q} \ll 1 \) (see Fig.2). Therefore, the distribution of \( \Delta \phi \) involves only \( \sigma_y \), not \( \sigma_x \). Although \( \sigma_x \) and \( \sigma_y \) may differ in this case, one can replace \( \sigma_x \) by \( \sigma_y \) without altering the distribution of \( \Delta \phi \). Note that in this case, \( \chi = \bar{Q}/\sigma \gg 1 \), hence the distribution of \( \Delta \phi \) reduces to its asymptotic form, Eq.(9).

- (C) Finally, there is the case when \( \langle w^2 \cos 2\phi \rangle \) is of order \( \langle w^2 \rangle \) while \( \langle w \cos \phi \rangle^2 \) is much smaller. In this case, there is a strong anisotropy in the second Fourier component, which should be used to determine the reaction plane. Instead of \( Q \), one should construct the vector \( Q_2 \) defined as
\[ Q_2 = \sum_{k=1}^{N} w_k' (e_x \cos 2\phi_k + e_y \sin 2\phi_k) \]  
with the same notations as in Eq.(5), and \( w_k' \) is an appropriate weight. The azimuthal angle of the reaction plane is estimated as half the azimuthal angle of \( Q_2 \), hence it is defined modulo \( \pi \), i.e. one cannot distinguish \( \phi \) and \( \phi + \pi \). This method is equivalent to the diagonalisation of the transverse sphericity tensor, which has been claimed to be more efficient than the transverse momentum method at intermediate energies \( (E/A < 100 \text{ MeV}, \text{ see [20]} \) and in particular Eq.(12) of [21]) and at ultrarelativistic energies where the flow angle is very small due to increasing nuclear transparency \([16, 23]\). Then, the procedure described in this paper can be applied to reconstruct the azimuthal distributions, replacing \( Q \) by \( Q_2 \). However, since the azimuthal angle is defined modulo \( \pi \), only the even Fourier components can be reconstructed.

Before we conclude, let us briefly comment on corrections which have been applied, in previous works, to the measured azimuthal distributions. The analysis of Demoulin et al...
[17] is very similar to ours, although limited to \( n = 1 \) and \( n = 2 \). The main difference is that they do not use the central limit theorem to calculate the Fourier coefficient \( P_2 \), but extract it directly from the measured higher moments of the momentum distribution. The method proposed by Tsang et al. [21] [22] is based on an ansatz for the distributions of \( \phi \) and \( \Delta \phi \), which are assumed to be proportional to \( \exp(-\omega^2 \sin^2 \phi) \), where \( \omega \) is a fitted parameter. Our analysis is more general in the sense that it does not make any a priori hypothesis on the shape of azimuthal distributions.

In conclusion, we have described a simple, general procedure to reconstruct the true azimuthal distributions from the measured ones by means of analytical formulae. Let us summarize this procedure: given a large sample of events in a restricted centrality interval, one measures the distribution of the relative angle between subevents \( \Delta \phi_R \), from which one extracts, using Eq.(13) or Eq.(14), the crucial parameter \( \chi \) which characterizes the accuracy of the reaction plane determination. Then one uses Eqs.(2) and (8) to reconstruct the Fourier coefficients of the true azimuthal distribution from the measured ones.

We have introduced a dimensionless parameter \( \chi \), which is a very convenient measure of the accuracy of the reaction plane determination. Unlike other quantities frequently used in the literature such as \( \langle \cos \Delta \phi \rangle \) [3] [7], or \( \langle \Delta \phi^2 \rangle^{1/2} \) [3] [6] [8] [10], it scales simply with the multiplicity like \( \sqrt{N} \). It has a simple physical interpretation, being the ratio of the average value of the flow vector \( \mathbf{Q} \) to the typical statistical fluctuation \( \sigma \). And furthermore, it can be directly deduced from measured quantities by simple expressions such as Eq.(13).

Our analysis suggests that the Fourier coefficients of the azimuthal distribution, which can be reconstructed accurately and are also easy to estimate in theoretical models, are the best observables to characterize azimuthal anisotropies. Note that we have chosen to measure azimuthal angles around the beam axis. It was argued in Ref.[24] that azimuthal distributions should rather be measured around the flow axis, determined by a sphericity tensor analysis [25]. But the fluctuations of the sphericity tensor, which is a \( 3 \times 3 \) matrix, are much more complex [26] than those of \( \mathbf{Q} \), and cannot be described in terms of a single parameter \( \chi \). There exists no simple procedure to subtract statistical errors in this case.

Let us recall that if the azimuthal angles of \( N \) particles are measured (summing over all events), the corresponding statistical error on the measured Fourier coefficient is \( 1/\sqrt{2N} \). Systematic errors can be removed, at least partly, using a mixed event technique. This allows to make accurate measurements even in situations where the reaction plane is poorly known, as is the case with light projectiles [17]. In experiments using heavy ions, it would be interesting to try to measure higher order Fourier coefficients with \( n \geq 3 \) which, although probably small, can be measured accurately.

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Figure captions

Fig. 1: Schematic picture of a semi-central nucleus-nucleus collision viewed in the transverse plane (the beam axis is orthogonal to the figure). \( b \) is the impact parameter oriented from the target to the projectile and \( Q \) is the vector defined by Eq.(5). A particle is emitted along the dashed arrow. Its azimuthal angle measured with respect to \( Q \) is \( \psi \), while the “true” azimuthal angle is \( \psi + \Delta \phi \).

Fig. 2: Schematic picture of the distribution of \( Q \), given by Eq.(6). The solid thick arrow indicates the average value \( \langle Q \rangle = \bar{Q}e_x \), which lies along the direction of the true impact parameter. \( Q \) fluctuates around this average value with a standard deviation \( \sigma \), so that a typical value of \( Q \) (dotted arrow) lies within the dotted circle with radius \( \sigma \). It is obvious from this figure that the typical magnitude of \( \Delta \phi \) is \( \sigma/\bar{Q} = 1/\chi \).

Fig. 3: Solid lines: variation of \( \langle \cos n\Delta \phi \rangle \) with the parameter \( \chi \), calculated from Eq.(8). The curves are labeled by the value of \( n \). Dotted lines: asymptotic behavior, Eq.(10).

Fig. 4: Variation of \( \langle \cos \Delta \phi_R \rangle \) with \( \chi \), given by Eq.(14).
Fig. 1
Fig. 2
Fig. 3
Fig. 4