Zero-brane approach to study of particle-like solitons in classical and quantum Liouville field theory

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Abstract

The effective p-brane action approach is generalized for arbitrary scalar field and applied for the Liouville theory near a particle-like solution. It was established that this theory has the remarkable features discriminating it from the theories studied earlier. Removing zero modes we obtain the effective action describing the solution as a point particle with curvature, quantize it as the theory with higher derivatives and calculate the quantum corrections to mass.

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I. INTRODUCTION

Appeared more than a century ago [1] the Liouville model have a wide application both in conventional physics [2] and in such new-fashioned regions of modern physics as Chern-Simons theory and gauge gravity [3], string and conformal field theory [4], Wess-Zumino-Witten model [5], superstring and M-theory [6], Seiberg-Witten approach [7], to mention just few examples and references.

The aim of this paper is to study the Liouville theory in the neighborhood of the particle-like solution within the frameworks of the brane approach, which consists in the constructing of the effective action where the non-minimal terms (first of all, depending on the world-volume curvature) are induced by the field fluctuations. Then the required action evidently arises after nonlinear reparametrization of an initial theory when excluding zero field oscillations.

The paper is arranged as follows. In Sec. II we study the Liouville particle-like soliton on the classical level. In Sec. III we generalize and enhance the approach developed by Kapustnikov and Pashnev [8], for arbitrary scalar fields and apply it for the model in question. Sec. IV is devoted to quantization of the obtained zero-brane action as a constrained theory with higher derivatives. In result we obtain the Schrödinger wave equation effectively describing wave function and mass spectrum of a point particle with curvature, calculate the null and first excited levels to derive the quantum corrections to the mass of the studied soliton. Conclusions are made in Sec. V.

II. PARTICLE-LIKE SOLUTIONS

We start with the action

\[ S[\varphi] = \frac{1}{2} \int d^2 x \left[ (\partial_m \varphi)^2 + me^{2\beta \varphi} + \zeta \right], \]  

(2.1)

where \( m, \beta \) and \( \zeta \) are constants. The equation of motion in the class of solitary waves is

\[ \varphi_{\rho \rho} + \frac{m}{2} e^{2\beta \varphi} = 0, \]  

(2.2)

\( \rho = \gamma(x - vt), \gamma = 1/\sqrt{1 - v^2}. \) Its general solution has the form

\[ \varphi_{\text{sol}}(\rho) = \frac{1}{\beta} \text{arcsinh} \left[ \frac{C - \cosh \left( \sqrt{mC} \beta \rho \right)}{2 \sqrt{C} \cosh \left( \sqrt{mC} \beta \rho \right)} \right], \]  

(2.3)

where the \( C \neq 0 \) is another integration constant, all the constants are assumed to be such that the potential is real everywhere on the axis. The corresponding energy density,

\[ \varepsilon_{\text{sol}}(x, t) = \frac{m}{2} \left[ C - \zeta/m - 2C \text{sech}^2 \left( \sqrt{mC} \beta \rho \right) \right], \]  

(2.4)

turns to be localized if we suppose

\[ C = \zeta/m. \]  

(2.4)
Then the total energy is finite despite the potential diverges at the infinity, and can be rewritten in the explicit relativistic point-particle form

\[ E_{\text{class}} = \int_{-\infty}^{\infty} \varepsilon(x, t) \, dx = \gamma \mu, \quad \mu = \frac{2 \sqrt{\zeta}}{\beta}, \]  

(2.5)

hence it is interesting to note that the additive constant \( \zeta \) (in spite of it does not appear in the equation of motion) affects on the most important properties of the particle-like solitons \((2.3)\) by virtue of the localization requirement \((2.4)\). This is the key difference of the Liouville theory from those studied earlier, viz., \( \varphi^4 \) \([8]\), \( \varphi^3 \) \([9]\) and sine-Gordon. Among the other features one can mention the initial \( \varphi \rightarrow -\varphi \) symmetry breaking (like \( \varphi^3 \)-model) and that the theory does not admit the trivial vacuum state \( \varphi = 0 \).

### III. EFFECTIVE ACTION

In this section we take into account the field fluctuations in the neighborhood of the solution \((2.3), (2.4)\) and construct the nonlinear effective zero-brane (non-minimal point particle) action. However, firstly, perfecting and generalizing the method \([8]\), we develop some general approach to such theories.

#### A. General formalism

Let us consider the action describing arbitrary one-scalar field

\[ S[\varphi] = \int L(\varphi) \, d^2 x, \]  

(3.1)

\[ L(\varphi) = \frac{1}{2} (\partial_n \varphi) (\partial^n \varphi) - U(\varphi). \]  

(3.2)

The corresponding equation of motion is

\[ \partial^a \partial_n \varphi + U_1(\varphi) = 0, \]  

(3.3)

where we defined

\[ U_1(\varphi) = \frac{\partial U(\varphi)}{\partial \varphi}, \quad U_2(\varphi) = \frac{\partial^2 U(\varphi)}{\partial \varphi^2}. \]

Suppose, we have a solution in the class of solitary waves

\[ \varphi_{\text{sol}}(\rho) = \varphi_{\text{sol}}(\gamma(x - vt)), \]  

(3.4)

having the localized energy density

\[ \varepsilon(\varphi) = \frac{\partial L(\varphi)}{\partial (\partial_0 \varphi)} \partial_0 \varphi - L(\varphi), \]  

(3.5)
and finite mass integral
\[
\mu = \int_{-\infty}^{+\infty} \varepsilon(\varphi_{\text{sol}}) \, d\rho = - \int_{-\infty}^{+\infty} L(\varphi_{\text{sol}}) \, d\rho < \infty, \quad (3.6)
\]
coinciding with the total energy up to the Lorentz factor \(\gamma\).

Let us change to the set of the collective coordinates \(\{\sigma_0 = s, \, \sigma_1 = \rho\}\) such that
\[
x^n = x^n(s) + e_{(1)}^n(s) \rho, \quad \varphi(x, t) = \tilde{\varphi}(\sigma),
\]
where \(x^n(s)\) turn to be the coordinates of a \((1+1)\)-dimensional point particle, \(e_{(1)}^n(s)\) is the unit spacelike vector orthogonal to the world line. Hence, the action (3.1) can be rewritten in new coordinates as
\[
S[\tilde{\varphi}] = \int L(\tilde{\varphi}) \Delta \, d^2 \sigma, \quad (3.8)
\]
\[
L(\tilde{\varphi}) = \frac{1}{2} \left[ \frac{(\partial_s \tilde{\varphi})^2}{\Delta^2} - (\partial_\rho \tilde{\varphi})^2 \right] - U(\tilde{\varphi}),
\]
where
\[
\Delta = \text{det} \left| \frac{\partial x^n}{\partial \sigma^l} \right| = \sqrt{\dot{x}^2 (1 - \rho k)},
\]
and \(k\) is the curvature of a particle world line
\[
k = \frac{\varepsilon_{ln} \dot{x}^l \dot{x}^n}{(\sqrt{\dot{x}^2})^3}. \quad (3.9)
\]
where \(\varepsilon_{ln}\) is the unit antisymmetric tensor. This new action contains the redundant degree of freedom which eventually leads to appearance of the so-called “zero modes”. To eliminate it we must constrain the model by means of the condition of vanishing of the functional derivative with respect to field fluctuations about a chosen static solution, and in result we will obtain the required effective action.

So, the fluctuations of the field \(\tilde{\varphi}(\sigma)\) in the neighborhood of the static solution \(\varphi_{\text{sol}}(\rho)\) are given by the expression
\[
\tilde{\varphi}(\sigma) = \varphi_{\text{sol}}(\rho) + \delta \varphi(\sigma). \quad (3.10)
\]
Substituting them into eq. (3.8) and considering the static equation of motion (3.3) for \(\varphi_{\text{sol}}(\rho)\) we have
\[
S[\delta \varphi] = \int d^2 \sigma \left\{ \Delta \left[ L(\varphi_{\text{sol}}) + \frac{1}{2} \left( \frac{(\partial_s \delta \varphi)^2}{\Delta^2} - (\partial_\rho \delta \varphi)^2 \right) - U_2(\varphi_{\text{sol}}) \delta \varphi^2 \right] - k \sqrt{\dot{x}^2} \varphi_{\text{sol}}' \delta \varphi + O(\delta \varphi^3) \right\} + [\rho\text{-surface terms}], \quad (3.11)
\]
\[ L(\varphi_{\text{sol}}) = -\frac{1}{2} \varphi_{\text{sol}}'^2 - U(\varphi_{\text{sol}}), \]

where prime means the derivative with respect to \( \rho \). Extremalizing this action with respect to \( \delta \varphi \) one can obtain the system of equations in partial derivatives for field fluctuations:

\[ \left( \partial_s \Delta^{-1} \partial_s - \partial_\rho \Delta \partial_\rho \right) \delta \varphi + \Delta U_2(\varphi_{\text{sol}}) \delta \varphi + \varphi_{\text{sol}}' k \sqrt{x^2} = O(\delta \varphi^2), \tag{3.12} \]

which has to be the constraint removing a redundant degree of freedom. Supposing \( \delta \varphi(s, \rho) = k(s)f(\rho) \), in the linear approximations \( \rho k \ll 1 \) (which naturally guarantees also the smoothness of a world line at \( \rho \to 0 \)) and \( O(\delta \varphi^2) = 0 \) we obtain a system of the ordinary derivative equations

\[ \frac{1}{\sqrt{x^2}} \frac{d}{ds} \frac{1}{\sqrt{x^2}} \frac{dk}{ds} + ck = 0, \tag{3.13} \]

\[ -f'' + (U_2(\varphi_{\text{sol}}) - c) f + \varphi_{\text{sol}}' = 0, \tag{3.14} \]

where \( c \) is the constant of separation. Searching for a solution of the last equation in the form

\[ f = g + \frac{1}{c} \varphi_{\text{sol}}', \tag{3.15} \]

we obtain the homogeneous equation

\[ -g'' + \left( \frac{\varphi_{\text{sol}}''}{\varphi_{\text{sol}}} - c \right) g = 0. \tag{3.16} \]

Strictly speaking, the explicit form of \( g(\rho) \) is not significant for us, because we always can suppose integration constants to be zero thus restricting ourselves by the special solution. Nevertheless, the homogeneous equation should be considered as the eigenvalue problem for \( c \) (see below).

Substituting the found function \( \delta \varphi = kf \) back in the action (3.11), we can rewrite it in the explicit p-brane form

\[ S_{\text{eff}} = S_{\text{eff}}^{(\text{class})} + S_{\text{eff}}^{(\text{fluct})} = - \int ds \sqrt{x^2} \left( \mu + \alpha k^2 \right), \tag{3.17} \]

describing a point particle with curvature, where \( \mu \) was defined in (3.6), and

\[ \alpha = \frac{1}{2} \int_{-\infty}^{\infty} f \varphi_{\text{sol}}' d\rho + \frac{1}{2} \int_{-\infty}^{+\infty} (ff')' d\rho. \tag{3.18} \]

Further, from the static equation (3.3) we obtain the expression

\[ (\varphi_{\text{sol}}'' - U_1(\varphi_{\text{sol}})) \varphi_{\text{sol}}' = 0, \tag{3.19} \]

which can be rewritten as

\[ \varphi_{\text{sol}}'^2 = 2U(\varphi_{\text{sol}}(\rho)). \tag{3.20} \]
Considering eqs. (3.5), (3.6), (3.15), (3.19) and (3.20), the expression for $\alpha$ can be written in the simple form

$$\alpha = \frac{\mu}{2c} + \frac{1}{2c^2} \int_{-\infty}^{+\infty} U''(\varphi_{sol}(\rho)) \, d\rho,$$

(3.21)

where the second term can be integrated as a full derivative, and vanishes when $|\varphi'_{sol}(\rho)| \leq O(1)$ at infinity. Even if it does not happen, we always can include this term into the surface terms of the action (3.11). Thus, we obtain the final form of the effective p-brane action of the theory

$$S_{\text{eff}} = -\mu \int ds \sqrt{\dot{x}^2} \left( 1 + \frac{1}{2c^2} k^2 \right).$$

(3.22)

It is straightforward to derive the corresponding equation of motion in the Frenet basis

$$\frac{1}{\sqrt{\dot{x}^2}} \frac{d}{ds} \frac{1}{\sqrt{\dot{x}^2}} \frac{dk}{ds} + \left( c - \frac{1}{2} k^2 \right) k = 0,$$

(3.23)

hence one can see that eq. (3.13) was nothing but this equation in the linear approximation $k \ll 1$, as was expected.

Thus, the only problem which yet demands on the resolving is the determination of eigenvalue $c$. It turns to be the Stourm-Liouville problem for eq. (3.16) provided some chosen boundary conditions. If one supposes, for instance, the finiteness of $g$ at infinity then the $c$ spectrum turns to be discrete. Moreover, it often happens that $c$ has only one or two admissible values [8,9]. In any case we are needed in the exact value of $c$ hence for each concrete case eq. (3.16) should be resolved as exactly as possible.

**B. Application for Liouville model**

We will suppose the final p-brane action (3.22) and will determine its parameters. We already have $\mu$ derived in (2.5), and the eigenvalue $c$ remains to be the only unknown parameter. As a boundary condition for the Stourm-Liouville problem (3.16), we require

$$g(+\infty) - g(-\infty) = O(1),$$

(3.24)

whereas (3.16) reads

$$g'' + \left( c + \frac{\zeta \beta^2}{\cosh^2 \left( \sqrt{\zeta \beta} \rho \right)} \right) g = 0.$$

(3.25)

According to the proven theorem (see Appendix A), the only admissible non-zero $c$ is

$$c = -\zeta \beta^2,$$

(3.26)

hence the effective zero-brane action of the Liouville model near particle-like solution (2.3), (2.4) with fluctuational corrections is

$$S_{\text{eff}} = -\mu \int ds \sqrt{\dot{x}^2} \left( 1 - \frac{k^2}{2\zeta \beta^2} \right).$$

(3.27)

In the next section we will quantize it to obtain the quantum corrections to the mass of the solution.
IV. QUANTIZATION

In the previous section we obtained a classical effective action for the model in question. Thus, to quantize it we must consecutively construct the Hamiltonian structure of dynamics of the point particle with curvature \([10]\). From the p-brane action (3.22) and definition of the world-line curvature one can see that we have the theory with higher derivatives \([11]\). Hence, below we will treat the coordinates and momenta as the canonically independent coordinates of phase space. Besides, the Hessian matrix constructed from the derivatives with respect to accelerations,

\[
M_{ab} = \left| \begin{array}{c}
\frac{\partial^2 L_{\text{eff}}}{\partial \ddot{x}^a \partial \ddot{x}^b} \\
\end{array} \right|,
\]

appears to be singular that says about the presence of the constraints on phase variables of the theory.

As was mentioned, the phase space consists of the two pairs of canonical variables:

\[
x_m, \quad p_m = \frac{\partial L_{\text{eff}}}{\partial q^m} - \dot{\Pi}_m, \\
q_m = \dot{x}_m, \quad \Pi_m = \frac{\partial L_{\text{eff}}}{\partial \dot{q}^m},
\]

hence we have

\[
p^n = -e^n_{(0)} \mu \left[ 1 - \frac{1}{2c} \right] + \mu \frac{e^n_{(1)} e_k}{c \sqrt{q^2}}, \quad (4.3)
\]

\[
\Pi^n = -\frac{\mu e^n_{(1)}}{c \sqrt{q^2}} k, \quad (4.4)
\]

where the components of the Frenet basis are

\[
e^m_{(0)} = \frac{\dot{x}^m}{\sqrt{\dot{x}^2}}, \quad e^m_{(1)} = -\frac{1}{\sqrt{\dot{x}^2}} \frac{\dot{e}^m_{(0)}}{k},
\]

There exist the two primary constraints of first kind

\[
\Phi_1 = \Pi^m q_m \approx 0, \quad (4.5)
\]

\[
\Phi_2 = p^m q_m + \sqrt{q^2} \left[ \mu + \frac{c}{2\mu} q^2 \Pi^2 \right] \approx 0, \quad (4.6)
\]

besides we should add the proper time gauge condition,

\[
G = \sqrt{q^2} - 1 \approx 0, \quad (4.7)
\]

to remove the non-physical gauge degree of freedom. Then, when introducing the new variables,

\[
\rho = \sqrt{q^2}, \quad v = \text{arctanh} \left( \frac{p_{(1)}}{p_{(0)}} \right), \quad (4.8)
\]
the constraints can be rewritten in the form
\[
\Phi_1 = \rho \Pi, \\
\Phi_2 = \rho \left[ -\sqrt{p^2 \cosh v + \mu} - \frac{c}{2\mu} \left( \Pi_v^2 - \rho^2 \Pi^2 \right) \right], \\
G = \rho - 1,
\]

hence finally we obtain the constraint
\[
\Phi_2 = -\sqrt{p^2 \cosh v + \mu} - \frac{c}{2\mu} \Pi_v^2 \approx 0,
\]

which in the quantum theory \((\Pi_v = -i\partial/\partial v)\) yields
\[
\hat{\Phi}_2 |\Psi\rangle = 0.
\]

As was shown in Ref. [8], the constraint \(\Phi_2\) on the quantum level admits several coordinate representations that, generally speaking, lead to different nonequivalent theories, therefore, the choice between the different forms of \(\hat{\Phi}_2\) should be based on the physical relevance. Then the physically admissible equation determining quantum dynamics of the quantum kink and bell particles has the form:
\[
[\hat{H} - \varepsilon] \Psi(\zeta) = 0, 
\]

\[
\hat{H} = -\frac{d^2}{d\zeta^2} + \frac{B^2}{4} \sinh^2 \zeta - B \left( S + \frac{1}{2} \right) \cosh \zeta,
\]

where
\[
\zeta = v/2, \quad \sqrt{p^2} = \mathcal{M}, \\
B = 8 \sqrt{\frac{\mu \mathcal{M}}{c}}, \\
\varepsilon = \frac{8\mu^2}{c} \left( 1 - \frac{\mathcal{M}}{\mu} \right),
\]

and \(S = 0\) in our case.

As was established in the works [12,13], SU(2) has to be the dynamical symmetry group for this Hamiltonian which can be rewritten in the form of the spin Hamiltonian
\[
\hat{H} = -S_z^2 - BS_x,
\]

where the spin operators,
\[
S_x = S \cosh \zeta - \frac{B}{2} \sinh^2 \zeta - \sinh \zeta \frac{d}{d\zeta}, \\
S_y = i \left\{ -S \sinh \zeta + \frac{B}{2} \sinh \zeta \cosh \zeta + \cosh \zeta \frac{d}{d\zeta} \right\}, \\
S_z = \frac{B}{2} \sinh \zeta + \frac{d}{d\zeta},
\]
satisfy with the commutation relations

\[ [S_i, S_j] = i \epsilon_{ijk} S_k, \]

besides

\[ S_x^2 + S_y^2 + S_z^2 \equiv S(S+1). \]

In this connection it should be noted that though the reformulation of some interaction concerning the coordinate degrees of freedom in terms of spin variables is widely used (e.g., in the theories with the Heisenberg Hamiltonian, see Ref. \[14\]), it has to be just the physical approximation as a rule, whereas in our case the spin-coordinate correspondence is exact.

Further, at \( S \geq 0 \) there exists an irreducible \((2S+1)\)-dimensional subspace of the representation space of the su(2) Lie algebra, which is invariant with respect to these operators. Determining eigenvalues and eigenvectors of the spin Hamiltonian in the matrix representation which is realized in this subspace, one can prove that the solution of eq. (4.11) is the function

\[ \Psi(\zeta) = \exp \left( -\frac{B}{2} \cosh \zeta \right) \sum_{\sigma=-S}^{S} \frac{c_{\sigma}}{\sqrt{(S-\sigma)!} (S+\sigma)!} \exp (\sigma \zeta), \tag{4.16} \]

where the coefficients \( c_{\sigma} \) are the solutions of the system of linear equations

\[ \left( \varepsilon + \sigma^2 \right) c_{\sigma} + \frac{B}{2} \left[ \sqrt{(S-\sigma)(S+\sigma+1)} c_{\sigma+1} + \sqrt{(S+\sigma)(S-\sigma+1)} c_{\sigma-1} \right] = 0, \]

\[ c_{S+1} = c_{-S-1} = 0, \quad \sigma = -S, -S+1, \ldots, S. \]

However, it should be noted that these expressions give only the finite number of exact solutions which is equal to the dimensionality of the invariant subspace (this is the so-called QES, quasi-exactly solvable, system). Therefore, for the spin \( S = 0 \) we can find only the ground state wave function and eigenvalue:

\[ \Psi_0(\zeta) = C_1 \exp \left( -\frac{B}{2} \cosh \zeta \right), \quad \varepsilon_0 = 0. \tag{4.17} \]

Hence, we obtain that the ground-state mass of the quantum particle with curvature coincides with the classical one,

\[ M_0 = \mu, \tag{4.18} \]

as was expected (strictly speaking, it coincides up to the sign which is insufficient, see below).

Further, in absence of exact wave functions for more excited levels one can find the first (small) quantum correction to mass in the approximation of the quantum harmonic oscillator. It is easy to see that at \( B \geq 1 \) the (effective) potential

\[ V(\zeta) = \left( \frac{B}{2} \right)^2 \sinh^2 \zeta - \frac{B}{2} \cosh \zeta \tag{4.19} \]
has the single minimum

\[ V_{\text{min}} = -B/2 \text{ at } \zeta_{\text{min}} = 0. \]

Then following to the \( \hbar \)-expansion technique we shift the origin of coordinates in the point of minimum (to satisfy \( \varepsilon = \varepsilon_0 = 0 \) in absence of quantum oscillations), and expand \( V \) in the Taylor series to second order near the origin thus reducing the model to the oscillator of the unit mass, energy \( \varepsilon/2 \) and oscillation frequency

\[ \omega = \frac{1}{2} \sqrt{B(B-1)}. \]

Therefore, the quantization rules yield the discrete spectrum

\[ \varepsilon = \sqrt{B(B-1)}(n + 1/2) + O(h^2), \quad n = 0, 1, 2, ..., \]  
(4.20)

and the first quantum correction to particle masses will be determined by the lower energy of oscillations:

\[ \varepsilon = \frac{1}{2} \sqrt{B(B-1)} + O(h^2), \]  
(4.21)

that gives the algebraic equation for \( \mathcal{M} \) as a function of \( m \) and \( \mu \).

We can easily resolve it in the approximation

\[ B \gg 1 \iff c/\mu^2 \to 0, \]  
(4.22)

which is admissible for the major physical cases, and obtain

\[ \varepsilon = \frac{B}{2} + O(h^2 c/\mu^2), \]  
(4.23)

that after considering of eqs. (4.13) and (4.18) yields

\[ (\mathcal{M} - \mu)^2 = \frac{c\mathcal{M}}{4\mu} + O(h^2 c/\mu^2). \]  
(4.24)

Then one can seek for mass in the form \( \mathcal{M} = \mu + \delta \) (\( \delta \ll \mu \)), and finally we obtain the mass of a particle with curvature (3.22) with first-order quantum corrections (considering that \( \mathcal{M} \) is always defined up to a sign, besides \( \mu \) is defined up to a sign as well)

\[ \text{sign}(\mathcal{M}) \mathcal{M} = \mu \pm \sqrt{\text{sign}(\mathcal{M})} \frac{c}{2} + O(h^2 c/\mu^2), \]

where \( \text{sign}(\mathcal{M}) \) is chosen such that after all we have a positive value

\[ \mathcal{M} = |\mu| \pm \sqrt{|c|} \frac{c}{2} + O(h^2 c/\mu^2), \]  
(4.25)

i. e., quantization procedure restores the positivity of mass.
The nature of the justified choice of the root sign before the second term is not so clear as it seems for a first look, because there exist the two interfering points of view. The first (physical) one is: if we apply this formalism for the one-scalar \( \phi^4 \) model \[8\] and compare the result with that obtained in other ways \[15\], we should suppose the sign “+” (or, at least, Rajaraman made no mentions upon the choice of signs). However, the second, mathematical, counterargument is as follows: the known exact spectra of the operators with the QES potentials like (4.11) are split, as a rule by virtue of radicals, hence the signs “±” might approximately represent such a bifurcation and thus should be unharmed. If it is really so, quantum fluctuations should divide the classically unified particle with curvature into several subtypes with respect to mass.

Let us apply these results for our case. One can see that \( \mu \) is indeed defined up to a sign. Therefore, considering eqs. (2.5), (3.27) and (4.25), the mass of the quantum particle-like solution (2.3) in the first approximation is

\[
\mathcal{M} = \sqrt{\zeta} \left( \frac{2}{\beta} \pm \frac{\beta}{2} \right),
\]  

(4.26)

where for beauty we have omitted the magnitude symbol but suppose the r.h.s. to be positive. The problem of the obtaining of further corrections appears to be the mathematically standard Stourm-Liouville problem for the Razavi potential, all the more so it is well-like on the whole axis and hence admits only the bound states with a discrete spectrum.

Finally we note that because of the Liouville model does not contain the vacuum state \( \varphi = 0 \), the obtained spectrum is nonperturbative and can not be derived by virtue of the standard perturbation theory starting from the vacuum sector.

V. CONCLUSION

Let us enumerate the main results obtained. It was shown that the Liouville field theory admits the particle-like field solution which (weakly) diverges at infinity but nevertheless have the localized energy. Besides, its mass depends on the constant which is additive to the Lagrangian and, therefore, does not appear in the equation of motion.

Further, considering field fluctuations in the neighborhood of this soliton we ruled out the action for it as a non-minimal point particle with curvature, thereby we have generalized and polished the procedure of obtaining of brane actions. When quantizing this action as the constrained theory with higher derivatives, it was shown that the resulting Schröedinger equation is the special case of the Razavi equation having SU(2) dynamical symmetry group in the ground state. Finally, we found the first quantum correction to mass of the solution in question which could not be calculated by means of series of the perturbation theory.

APPENDIX A: EIGENVALUE THEOREM

Theorem. The bound-state singular Stourm-Liouville problem

\[-f''(u) + \left( 1 - 2 \text{sech}^2 u \right) f(u) - cf(u) = 0,\]  

(A1)
\[ f(\infty) = f(-\infty) = O(1), \quad (A2) \]

has only the two sets of eigenfunctions and eigenvalues

\[ f_0 = K_0 \sech u, \quad c_0 = 0, \]
\[ f_1 = K_1 \tanh u, \quad c_1 = 1. \]

where \( K_i \) are arbitrary integration constants.

**Proof.** Performing the change \( z = \cosh^2 u \), we rewrite the conditions of the theorem in the form

\[ 2z(z-1)f_{zz} + (2z-1)f_z - \left( c - \frac{1}{z} \right)f = 0, \quad (A3) \]

\[ f(1) = 0, \quad f(\infty) = O(1), \quad (A4) \]

where \( c = 1 - c \). The general integral of eq. \((A3)\) can be expressed in terms of the hypergeometric functions

\[ f = \frac{C_1}{\sqrt{z}} F \left( -1 - \frac{\sqrt{c}}{2}, -1 + \frac{\sqrt{c}}{2}, -\frac{1}{2}; z \right) + C_2 z F \left( 1 - \frac{\sqrt{c}}{2}, 1 + \frac{\sqrt{c}}{2}, \frac{5}{2}; z \right). \]

Using the asymptotics of the hypergeometric functions in the neighborhood \( z = 1 \), it is straightforward to derive that the first from the conditions \((A4)\) will be satisfied if we suppose

\[ \frac{1}{C_1} f^{\text{reg}} = \frac{1}{\sqrt{z}} F \left( -1 - \frac{\sqrt{c}}{2}, -1 + \frac{\sqrt{c}}{2}, -\frac{3}{2}; z \right) - C^{\text{reg}} z F \left( \frac{3 - \sqrt{c}}{2}, \frac{3 + \sqrt{c}}{2}, \frac{7}{2}; z \right), \quad (A5) \]

where

\[ C^{\text{reg}} = \sqrt{c}(c - 1) \tan \left( \frac{\pi \sqrt{c}}{2} \right). \]

Further, to specify the parameters at which this function satisfies with the second condition \((A4)\) we should consider the asymptotical behavior of \( f^{\text{reg}} \) near infinity. We have

\[ \frac{1}{C_1} f^{\text{reg}}(z \to \infty) = \frac{2\tilde{\gamma}}{\pi^{3/2}} (-1)^{1+\sqrt{c}/2} \tan \left( \frac{\pi \sqrt{c}}{2} \right) \sin \left( \frac{\pi \sqrt{c}}{2} \right) z^{\sqrt{c}/2} \left[ 1 + O(1/z) \right], \quad (A6) \]

where

\[ \tilde{\gamma} = \Gamma(\sqrt{c}) \left[ i\sqrt{c}(c - 1)\Gamma(-1/2 - \sqrt{c}/2)\Gamma(\sqrt{c}/2) - 8 \Gamma(1 - \sqrt{c}/2)\Gamma(3/2 - \sqrt{c}/2) \right]. \]

From this expression it can easily be seen that \( f^{\text{reg}} \) diverges at infinity everywhere except perhaps the points:
\( \tilde{c} = (2n)^2 = 0, 4, 16, \ldots \), and \( \tilde{c} = 1 \),

which demand on an individual consideration. From eq. (A3) we have

\[
f_{\tilde{c}=0} = C_1 \sqrt{1 - \frac{1}{z}} + C_2 \left[ i - \sqrt{1 - \frac{1}{z}} \arcsin \sqrt{z} \right],
\]

\[
f_{\tilde{c}=1} = \frac{C_1}{\sqrt{z}} + C_2 \left[ \sqrt{1 - \frac{1}{z}} - i \frac{\arcsin \sqrt{z}}{\sqrt{z}} \right],
\]

\[
\tilde{f}_{\tilde{c}=4} = C_1 \sqrt{1 - \frac{1}{z}} (2z + 1) + C_2 z,
\]

\[
f_{\tilde{c}=16} = C_1 \sqrt{1 - \frac{1}{z}} \left( 24z^2 - 8z - 1 \right) + C_2 z (1 - 6z/5),
\]

and so on. By induction it is clear that at \( \tilde{c} \geq 4 \) there are no \( C_i \) at which \( f \) would satisfy with the requirements (A2).
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