DUALITY AND STRINGS, SPACE AND TIME

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Duality symmetries in M–theory and string theory are reviewed, with particular emphasis on the way in which string winding modes and brane wrapping modes can lead to new spatial dimensions. Brane world-volumes wrapping around Lorentzian tori can give rise to extra time dimensions and in this way dualities can change the number of time dimensions as well as the number of space dimensions. This suggests that brane wrapping modes and spacetime momenta should be on an equal footing and M–theory should not be formulated in a spacetime of definite dimension or signature.

1 String Theory, M-Theory and Duality

String theory is defined as a perturbation theory in the string coupling constant $g_s$, which is valid when $g_s$ is small. The fundamental quanta are the excitations of relativistic strings moving in spacetime and comprise of a finite set of massless particles plus an infinite tower of massive particles with the scale of the mass set by the string tension $T = 1/l_s^2$, expressed in terms of a string length scale $l_s$. If the spacetime has some circular dimensions, or more generally has some non-contractible loops, the spectrum will also include winding modes in which a closed string winds around a non-contractible loop in spacetime. These have no analogue in local field theories and are responsible for some of the key differences between string theories and field theories. Physical quantities are calculated through a path integral over string histories, which can be calculated perturbatively in $g_s$ using stringy Feynman rules, with string world-sheets of genus $n$ contributing terms proportional to $g_s^n$. In the supersymmetric string theories, these contributions are believed to be finite at each order in $g_s$, giving a perturbatively finite quantum theory of gravity unified with other forces.

There are five distinct perturbative supersymmetric finite string theories, all in 9 + 1 dimensions (i.e. nine space and one time), the type I, type IIA and type IIB string theories, and the two heterotic string theories with gauge groups $SO(32)$ and $E_8 \times E_8$. The massless degrees of freedom of each of these theories are governed by a 10-dimensional supergravity theory, which is the low-energy effective field theory. It has been a long-standing puzzle as to why there should be five such theories of quantum gravity rather than one, and this has now been resolved. It is now understood that these are all equivalent non-perturbatively and that these distinct perturbation theories arise as different perturbative limits of a single underlying theory. We do not have an intrinsic formulation of this underlying non-perturbative theory yet, but the relationships between the string theories has been understood through the discovery of dualities linking them. A central role in the non-perturbative theory is played by the $p$-branes. These
are p-dimensional extended objects, so that a 0-brane is a particle, a 1-brane is a string, a 2-brane is a membrane and so on. In the perturbative superstring theories there is a 1-brane which is the fundamental string providing the perturbative states of the theory, while the other branes arise as solitons or as D-branes, which are branes on which fundamental strings can end. The type II string theories have a fundamental string and a solitonic 5-brane and a set of Dp-branes, where \( p = 0, 2, 4, 6, 8 \) for the IIA string theory and \( p = 1, 3, 5, 7, 9 \) for the IIB string theory.

There are duality symmetries of string theories that relate brane degrees of freedom to fundamental quanta, so that all the branes are on the same footing. If some of the spacetime dimensions are wrapped into some compact space \( K \), so that the spacetime is \( M \times K \) for some \( M \), then branes can wrap around homology cycles of \( K \) and these give extra massive states in the compactified theory on \( M \). For example, a \( p \)-brane wrapping around an \( n \)-cycle with \( n \leq p \) gives a \( p - n \) brane in the compactified theory. These brane wrapping modes generalise the string winding modes and are related to the perturbative states by U-dualities, and play an important role in the duality symmetries, as we shall see.

One of the best-understood dualities is T-duality, which relates string theory on a spacetime \( S^1 \times M \) with a circular dimension of radius \( R \) to a string theory on \( S^1 \times M \) where the circular dimension is now of radius

\[
\tilde{R} = \frac{l_s^2}{R}
\]

so that the radii \( R, \tilde{R} \) are inversely proportional. For bosonic and heterotic string theories, T-duality is a self-duality, so that heterotic (bosonic) string theory on a large circle is equivalent to heterotic (bosonic) string theory on a small circle, while it maps the type IIA string theory to the type IIB theory, with the result that type IIA string theory on a large circle is equivalent to type IIB string theory on a small circle. T-duality relates perturbative states to perturbative states, as does mirror symmetry which relates a superstring theory compactified on a Calabi-Yau manifold \( K \) to a superstring theory compactified on a topologically distinct Calabi-Yau manifold, the mirror \( \tilde{K} \) of \( K \).

There are also non-perturbative dualities. For example the type IIA string theory compactified on \( K^3 \) is equivalent to the heterotic string theory compactified on the 4-torus \( T^4 \), while the type I theory with string coupling \( g_s \) is equivalent to the \( SO(32) \) heterotic string theory with string coupling \( \tilde{g}_s = 1/g^2 \). This is an example of a strong-weak coupling duality relating the strong-coupling regime of one theory to the weak-coupling regime of another. Such dualities are important as they allow the description of strong-coupling physics in terms of a weakly-coupled dual theory.

M-theory arises as the strong-coupling limit of the IIA string theory. The IIA string is interpreted as an 11-dimensional theory compactified on a circle of radius \( R = l_s g_s \). Then at strong coupling, the extra dimension decompactifies to give a theory in 11 dimensions which has 11 dimensional supergravity as a low-energy limit. We will refer to this 10+1 dimensional theory as M-theory. Duality transformations relate this to each of the five string theories, and the string theories and M-theory can all be thought of as arising as different limits of a single underlying theory. The IIA string theory is obtained by compactifying M-theory on a circle, the IIB string is obtained from the IIA by T-duality or directly from compactifying M-theory on a 2-torus and taking the limit in which it shrinks of zero size, the \( E_8 \times E_8 \) heterotic string is obtained by modding out M-theory on a circle by a \( \mathbb{Z}_2 \) symmetry or equivalently from compactifying M-theory
on a line interval, the type I theory is obtained from the IIB string by orientifolding (modding out by world-sheet parity), and the $SO(32)$ heterotic string is the strong coupling limit of this. The type I theory and the $SO(32)$ heterotic string (as well as the type I' string) can be obtained directly from M–theory compactified on a cylinder as on a $T^2$ bundle over a circle.

In $D$-dimensional general relativity or supergravity, a spacetime with a large circle $S^1$ is physically distinct from one with a small circle $\tilde{S}^1$, and a spacetime $M \times K$ is physically distinct from the mirror spacetime $M \times \tilde{K}$, but in string theory these dual pairs of spacetimes define the same string theory and so define the same physics. The heterotic string on $M \times T^4$ is equivalent to the type IIA string on $M \times K3$, even though $T^4$ and $K3$ are very different spaces with different properties (e.g. they have different topologies and different curvatures) and there is no invariant answer to the question: what is the spacetime manifold? In the same way that spacetimes related by diffeomorphisms are regarded as equivalent, so too must spacetimes related by dualities, and the concept of spacetime manifold should be replaced by duality equivalence classes of spacetimes (or, more generally, duality equivalence classes of string or M–theory solutions).

In the usual picture, the five superstring theories and the 11-dimensional theory arising as the strong coupling limit of the IIA string (referred to as M–theory here) are depicted as being different corners of the moduli space of the mysterious fundamental theory underpinning all of these theories (sometimes also referred to as M–theory, although we shall resist this usage here). More precisely, compactifying string theory or M–theory gives a theory depending on the moduli of metrics and antisymmetric tensor gauge fields on the compactification space. Each modulus gives rise to a scalar field in the compactified theory and the expectation value of any of the scalar fields can be used to define a coupling constant. One can then examine the perturbation theory in that constant. For some choices it will give a field theory, for others it will give a perturbative string theory and different perturbative string theories will correspond to different choices of coupling. The string theories and M–theory are each linked to each other by chains of dualities and so there is only one basic theory.

More recently, other ‘corners’ corresponding to particular limits of the theory have been understood to correspond to field theories without gravity. For example the IIB string theory in the background given by the product of 5-dimensional anti-de Sitter space and a 5-sphere is equivalent to $N = 4$ supersymmetric Yang-Mills theory in four dimensions, with similar results for theories in other anti-de Sitter backgrounds, and certain null compactifications are equivalent to matrix models.

Many dualities have now been found which can relate theories with different gauge groups, different spacetime dimensions, different spacetime geometries and topologies, different amounts of supersymmetry, and even relate theories of gravity to gauge theories. Thus many of the concepts that had been thought absolute are now understood as relative: they depend on the ‘frame of reference’ used, where the concept of frame of reference is generalised to include the values of the various coupling constants. For example, the description of a given system when a certain coupling is weak can be very different from the description at strong coupling, and the two regimes can have different spacetime dimension, for example. However, in all this, one thing that has remained unchanged is the number of time dimensions; all the theories considered are formulated in a Lorentzian signature with one time coordinate, although the number of spatial dimensions can change. Remarkably, it turns out that dualities can change the number
of time dimensions as well, giving rise to exotic spacetime signatures. The resulting picture is that there should be some underlying fundamental theory and that different spacetime signatures as well as different dimensions can arise in various limits. The new theories are different real forms of the complexification of the original M-theory and type II string theories, perhaps suggesting an underlying complex nature of spacetime.

We will now proceed to examine some of these dualities in more detail, and in particular to focus on the way in which extra spacetime dimensions can emerge from brane wrapping modes.

2 Branes and Extra Dimensions

2.1 Compactification on $S^1$

For a field theory compactified from $D$ dimensions on a circle $S^1_R$ of radius $R$, the momentum $p$ in the circular dimension will be quantised with $p = n/R$ for some integer $n$. In the limit $R \to 0$ this becomes divergent, so that finite-momentum states must move in the remaining $D-1$ dimensions and are described by the dimensionally reduced theory in $D-1$ dimensions. For finite $R$, the states carrying internal momentum can be interpreted as states in $D-1$ dimensions with mass (taking the $D$-dimensional field theory to be massless for simplicity)

$$M = |n|/R$$

(2)

The set of all such states for all $n$ gives the ‘Kaluza-Klein tower’ of massive states arising from the compactification. If the original field theory includes gravity, there will be an infinite tower of massive gravitons, and if the theory is supersymmetric, then the tower fits into supersymmetry representations. In the limit $R \to 0$, the masses of all the states in these towers become infinite and they decouple, leaving the massless dimensionally reduced theory in $D-1$ dimensions. On the other hand, taking the decompactification limit $R \to \infty$, all the states in the tower become massless and combine with the massless $D-1$ dimensional fields to form the massless fields in $D$ dimensions. Such a tower becoming massless is often a signal of the decompactification of an extra dimension.

For a string theory the situation is very different, due to the presence of string winding modes which become light as the circle shrinks. A string can wind $m$ times around the circular dimension, and the corresponding state in the $D-1$ dimensional theory will have mass $mRT$ where $T$ is the string tension (into which a factor of $2\pi$ has been absorbed). The set of all such states for all $m$ forms a tower of massive states and in the limit $R \to 0$ these become massless, so that there is an infinite tower of states becoming massless (and fitting into supergravity multiplets, in the case of the superstring). This signals the opening up of a new circular dimension of radius

$$\tilde{R} = 1/TR$$

(3)

with the string winding mode around the original circle of mass

$$M = mRT = m/\tilde{R}$$

(4)

reinterpreted as a momentum mode in the dual circle of radius $\tilde{R}$. Similarly, the momentum modes on the original circle ($M = n/R$) can now be interpreted as string winding modes around the dual circle ($M = nT\tilde{R}$), and the new theory in $D$ dimensions is again
a string theory. A state with momentum \( n/R \) and winding number \( m \) will have mass

\[
M = \frac{n}{R} + mRT = \frac{n}{R} + \frac{m}{\tilde{R}}
\]

and this is clearly invariant under the T-duality transformation \( m \leftrightarrow n, R \leftrightarrow \tilde{R} \) interchanging the momentum and winding numbers and inverting the radius. Then the original string theory on \( M_{D-1} \times S^1 \) (with \( M_{D-1} \) some \( D-1 \)-dimensional spacetime) is equivalent to a string theory on \( M_{D-1} \times \tilde{S}^1 \) where \( \tilde{S}^1 \) has radius \( \tilde{R} \), with the momentum modes of one theory corresponding to the winding modes of the other, and this equivalence is known as a T-duality. In the limit \( R \to 0, \tilde{R} \to \infty \), the decompactified T-dual theory has full \( D \) dimensional Lorentz invariance. If the first string theory is a bosonic (heterotic) string, so is the second, while if one is a type IIA string theory, the other is a type IIB string theory. Then the type IIA string theory compactified on a circle of radius \( R \) is equivalent to the type IIB string theory compactified on a circle of radius \( \tilde{R} \). For 10-dimensional supergravity compactified on a circle of radius \( R \), taking \( R \to 0 \) will give a 9-dimensional supergravity theory while for a string theory a new dimension opens up to replace the one of radius \( R \) that has shrunk to zero size. If \( R \) is much larger than \( l_s \), the description in terms of string theory on \( S^1 \) is useful, while for \( R << l_s \), the T-dual description in terms of string theory on \( \tilde{S}^1 \) is more appropriate, with the light states having a conventional description in terms of momentum modes on \( \tilde{S}^1 \) instead of winding modes on \( S^1 \).

### 2.2 IIA String Theory and M-Theory

The type IIA string theory in 9+1 dimensions has D0-brane states, which are particle-like non-perturbative BPS states, with quantized charge \( n \) (for integers \( n \)) and mass

\[
M \sim \frac{|n|}{g_s l_s}
\]

The state of charge \( n \) can be thought of as composed of \( n \) elementary D0-branes. In the strong coupling limit \( g_s \to \infty \), these states all become massless. Moreover, the D0-brane states for a given \( n \) fit into a short massive supergravity multiplet with spins ranging from zero to two and so at strong coupling there is an infinite number of gravitons becoming massless. It was proposed in \( \text{[2]} \) that this tower of massless states should be interpreted as a Kaluza-Klein tower for an extra circular dimension of radius

\[
R_M = g_s l_s
\]

Then the strong coupling limit of the IIA string theory is interpreted as the limit in which \( R_M \to \infty \) so that the extra dimension decompactifies to give a theory in 10+1 dimensions, and this is M-theory. Moreover, for the IIA string theory in \( D = 10 \) Minkowski space, the strong coupling limit is invariant under the full 11-dimensional Lorentz group and the effective field theory describing the massless degrees of freedom of M–theory is 11-dimensional supergravity. The radius can be rewritten in terms of the 11-dimensional Planck length \( l_p \) as

\[
R_M = g_s^{2/3} l_p
\]

The IIA string theory is really only defined perturbatively for very small coupling \( g_s \). It can now be 'defined' at finite coupling \( g_s \) as M–theory compactified on a circle of radius \( R_M \), so that the problem is transferred to the one of defining M–theory.
However, at low energies we see that the non-perturbative IIA theory is described by 11-dimensional supergravity compactified on a circle, and this leads to important non-perturbative predictions, so that this viewpoint can be useful even though we still know rather little about M–theory.

The IIA string has D$p$-branes for all even $p$, while M–theory has a 2-brane or membrane and a 5-brane. All the branes of the IIA string theory have an M–theory origin. For example, an M–theory membrane will give the fundamental string of the IIA theory if it wraps around the circular dimension and the D2-brane if it does not.

2.3 Compactification on $T^2$

For a $D$ dimensional field theory compactified on a 2-torus there will be momentum modes with masses

$$M \sim \sqrt{\frac{p^2}{R_1^2} + \frac{q^2}{R_2^2}}$$

where $R_1, R_2$ are the radii of the circular dimensions, $p, q$ are integers and for simplicity we take the torus to be rectangular. These will decouple in the limit $R_1, R_2 \to 0$ leaving a theory in $D - 2$ dimensions. For example, for 11-dimensional supergravity, this limit will give the dimensionally reduced 9-dimensional maximal supergravity theory.

We now compare this with M–theory compactified on $T^2$. Consider first the circle of radius $R_2$, say. M–theory compactified on this circle is equivalent to the IIA string theory with coupling constant $g_s = (R_2/l_p)^{3/2}$, and so the limit $R_2 \to 0$ is the weak coupling limit of this IIA string theory. We now have the IIA string theory compactified on a circle of radius $R_1$, and by T-duality this is equivalent to the IIB string theory compactified on a circle of radius $\tilde{R}_1 = 1/TR_1$. Taking the limit $R_1 \to 0$ is then the limit in which $\tilde{R}_1 \to \infty$ and an extra circle opens up to give the IIB string theory in 9+1 dimensions. The IIA string winding modes provide the tower of states that become massless in the limit and which are re-interpreted as momentum modes on the circle of radius $\tilde{R}_1$. Moreover, these IIA string winding modes are M–theory membranes wrapped around the 2-torus. These membrane wrapping modes have mass

$$M \sim |n|T_2R_1R_2$$

where the membrane tension is $T_2 = 1/l_p^3$.

Then M–theory compactified on a general 2-torus of area $A$ and modulus $\tau$ is equivalent to the IIB string theory compactified on a circle of radius

$$R_B = \frac{\tau^3}{A}$$

with string coupling $g_s$ and axionic coupling $\theta$ (defined as the expectation value of the scalar field in the Ramond-Ramond sector) given by

$$\tau = \theta + i\frac{1}{g_s}$$

The states of the IIB string carrying momentum in the circular dimension arise from membranes wrapping the 2-torus while the $(p, q)$ string of the IIB theory winding round the circular dimension (with fundamental string charge $p$ and D-string charge $q$) arises from M–theory states carrying momentum $p/R_1$ and $q/R_2$ in the compact dimensions.
Then in the limit $A \to 0$, we lose two of the dimensions, as in the field theory, leaving a theory in $8+1$ dimensions, but a new spatial dimension opens up to give a theory in $9+1$ dimensions.

2.4 Compactification on $T^3$

For 11-dimensional supergravity compactified on a 3-torus, the limit in which the radii $R_1, R_2, R_3$ all tend to zero gives the dimensional reduction to the maximal supergravity in 8 dimensions. For M–theory on $T^3$, membranes can wrap any of the three 2-cycles of $T^3$. From the last section, we know that if $R_1$ and $R_2$ both shrink to zero while $R_3$ stays fixed, an extra dimension opens up with radius $\hat{R}_3 = l_p^3/R_1 R_2$ and the tower of membrane wrapping states is reinterpreted as a Kaluza-Klein tower for this extra dimension. The same picture applies to each of the three 2-cycles, and so if all 3 radii shrink, there are three extra dimensions opening up, with radii $\hat{R}_i$ given by

$$\hat{R}_i = \frac{l_p^3}{R_j R_k}, \quad i \neq j \neq k$$

(13)

Then when the original three torus shrinks to zero size, three dimensions are lost but three new ones emerge, so we are again back in 11 dimensions and the 11-dimensional theory is again M–theory. Thus M–theory compactified on a dual $T^3$ with radii $\hat{R}_1, \hat{R}_2, \hat{R}_3$ given by (13) is equivalent to M–theory compactified on the dual $T^3$ with radii $\tilde{R}_1, \tilde{R}_2, \tilde{R}_3$ given by (13).

2.5 Compactification on $T^4$

It is tempting to apply these arguments to higher tori. For example, $T^4$ has six 2-cycles, and membranes can wrap any of them. Compactifying $D = 11$ supergravity on a $T^4$ and taking all four radii $R_i \to 0$ gives a $D = 7$ field theory, but for M–theory there are six towers of states becoming massless in the limit arising from membranes wrapping each of the six shrinking 2-cycles. If each of these towers is interpreted as a Kaluza-Klein tower, this would give 6 extra dimensions in addition to the 7 original dimensions remaining, giving a total of 13 dimensions. However, there is no conventional supersymmetric theory in 13 dimensions, so it is difficult to see how such a theory could emerge. In fact the situation here is more complicated, and the 6 towers have a different interpretation here. The difference here is that there is also a string in the compactified theory arising from the M–theory 5-brane wrapped around the $T^4$ which becomes ‘light’ at the same time as the 6 towers of membrane wrapping modes. It turns out that M–theory on $T^4$ is dual to IIB string theory on $T^3$. The M–theory 5-brane wrapped around the $T^4$ gives the fundamental string of the IIB theory moving in 7 dimensions. Compactifying the IIB string on $T^3$ gives in addition three momentum modes and three winding modes, fitting into a 6 of the T-duality group $SO(3,3)$, and these correspond to the 6 towers of membrane wrapping modes, which themselves transform as a 6 of the torus group $SL(4) \sim SO(3,3)$. Thus only 3 of the 6 towers can be interpreted as momentum modes for an extra dimension, the other three being interpreted as string winding modes, and the spacetime dimension of the dual theory is 10, not 13. Note that there is no invariant way of choosing which three of the six correspond to spacetime dimensions, as T-duality transformations will relate momentum and winding modes and change one subset of three to another. In this case, taking the limit in which the $T^4$ on which the M–theory is compactified shrinks to zero size does not correspond to a decompactification limit of
the dual theory, but to the weak coupling limit in which the coupling constant $g_s$ of the compactified IIB string theory tends to zero.

3 Branes and Space and Time

We have seen that wrapped branes are associated with towers of massive states and that in some cases these can be interpreted as Kaluza-Klein towers for extra dimensions. In a limit in which such a tower becomes massless (e.g. $R_i \to 0$ for toroidal compactifications, or $g_s \to \infty$ for the IIA D0-branes), the corresponding dimension decompactifies and new dimensions unfold. The presence of an enlarged Lorentz symmetry puts the new braney dimensions on an equal footing with the other dimensions, and the full theory includes gravity in the enlarged space. The number of dimensions lost in the limit is not always the same as the number of extra dimensions, so that the total number of spacetime dimensions can change (as in the relations between 11-dimensional M–theory and 10-dimensional type II string theories considered in sections 2.2 and 2.3). We have also seen that, as in the case of M–theory on $T^4$, the towers of wrapped brane states cannot always be interpreted in terms of extra dimensions, and it is necessary to perform a more complete analysis to see what is going on.

In all of the above cases, branes were wrapped around spacelike cycles and the extra dimensions that arose were all spacelike. A brane world-volume can also wrap around timelike cycles, and we will see that in such cases the extra dimensions can be timelike, so that the signature of spacetime can change.

It is natural to ask whether it makes sense to consider compact time. There are many classical solutions of gravity, supergravity, string and M–theories with compact time and it is of interest to investigate their properties. Compact time does not appear to be a feature of our universe, but almost all spacetimes that are studied are also unrealistic. The presence of closed timelike loops means that the physics in such spaces is unusual, but it has often been fruitful in the past to study solutions that have little in common with the real world. An important issue with these solutions (as with many others) is whether a consistent quantum theory can be formulated in such backgrounds. If time were compact but with a huge period, it is not clear how that would manifest itself.

With a compact time, it is straightforward to solve classical field equations, imposing periodic boundary conditions in time instead of developing Cauchy data. Can quantum theory make sense with compact time? There is no problem in solving Schrödinger or wave equations with periodic boundary conditions, but it is difficult to formulate any concept of measurement or collapse of a wave-function, as these would be inconsistent with periodic time: if a superposition of states collapsed to an eigenstate of an observable in some measurement process, it must already have been in that eigenstate from the last time it was measured. In string theory, it is straightforward to study the solutions of the physical state conditions, but there are new issues that arise from string world-sheets (and brane world-volumes) wrapping around the compact time. It has proved very fruitful to consider such compactifications in string theory. For example, the compactification of all 25+1 dimensions in bosonic string theory on a special Lorentzian torus played a central role in the work of Borcherds on the construction of vertex algebras and their application to the monster group.

4 Compactification on Lorentzian Tori and Signature Change
4.1 Compactification on a Timelike Circle

Consider spacetimes of the form $M_{D-1} \times S^1$ where $S^1$ is a timelike circle of radius $R$ and $M_{D-1}$ is a Riemannian space. The time component of momentum is quantized

$$p^0 = \frac{n}{R}$$

and in the limit $R \to 0$, only the states with $p^0 = 0$ survive. For a field theory, the result is a dimensional reduction to a Euclidean field theory in $D - 1$ dimensions, on $M_{D-1}$.

For example, dimensionally reducing $D = 11$ supergravity on a timelike circle gives a supergravity theory in 10 Euclidean dimensions, denoted the IIA_E supergravity theory in $\mathbb{R}^{10,1}$. Timelike reductions of supergravity theories have been considered in $\mathbb{R}^{11,1}$. The field theory resulting from such a timelike reduction will in general have fields whose kinetic terms have the wrong sign. For example, the $D$ dimensional graviton will give a graviton, a scalar and a vector field in $D - 1$ dimensions on reducing on a circle, and if the circle is timelike, then the vector field will have a kinetic term of the wrong sign. Then the action for the physical matter fields of the reduced theory in $D - 1$ Euclidean dimensions will not be positive. This apparent problem is the result of the truncation to $p^0 = 0$ states. If this truncation is not made and if the full Kaluza-Klein towers of states with $p^0 = n/R$ for all $n$ are kept, then the theory is the full unitary $D$ dimensional theory on a particular background, and the $D$ dimensional gauge invariance can be used to choose a physical gauge locally with a positive action and states with positive norm. In general such a gauge choice cannot be made globally and there will be zero-mode states for which the action will not be positive. For example, the states with $p^0 = 0$ are governed by the non-positive dimensionally reduced action in $D - 1$ dimensions. For a Yang-Mills theory reduced on a timelike circle, the time component of the vector potential $A_0$ gives a scalar field in $D - 1$ dimensions with a kinetic term of the wrong sign. For the full $D$ dimensional theory compactified on the timelike circle, the negative-norm $A_0$ can be brought to a constant by $D$-dimensional gauge transformations, but one cannot gauge away the degrees of freedom associated with Wilson lines winding around the compact time dimension. (The fields with kinetic terms of the wrong sign can be handled in the path integral in the same way as the negative-action gravitational conformal mode is sometimes dealt with, namely by analytic continuation so that the offending field becomes imaginary.)

In a string theory, however, there will be winding modes in which the 1+1 dimensional string world-sheet winds around the compact time dimension, giving a spacelike ‘world-line’ in the compactified theory in $D - 1$ dimensions. As in the spacelike case, as $R \to 0$, a dual circle opens up with radius $\tilde{R} = 1/TR$, and the new circle is again timelike. The winding number becomes the $p^0$ of the dual theory, and in this way a superstring theory in 9+1 dimensions compactified on a timelike circle of radius $R$ is T-dual to a superstring theory in 9+1 dimensions compactified on a timelike circle of radius $\tilde{R}$. Such timelike T-dualities were considered for the bosonic and heterotic strings in e.g. $\mathbb{R}^{10,1}$, and they take the bosonic string theory to the bosonic string theory and the heterotic string theory to the heterotic string theory. However, for type II theories there is a surprise. It is straightforward to see that timelike T-duality cannot take the IIA string theory to either the IIB string or the IIA string, but must take it to a ‘new’ theory, denoted the IIB* string theory in $\mathbb{R}^{10,1}$. Similarly, timelike T-duality takes the IIB string to a IIA* string theory in $\mathbb{R}^{10,1}$.

The IIA* and IIB* strings are taken into each other by T-duality on a spacelike
circle, and the $IIA^*$ ($IIB^*$) theory is obtained from the IIA (IIB) string theory by acting with $(i)^{F_L}$ (where $F_L$ is the left-handed fermion number). The supergravity limits of the $IIA^*$ and $IIB^*$ have non-positive actions for the matter fields (the kinetic terms for the fields in the R-NS and R-R sectors have the wrong sign) so that the low-energy field theories are non-unitary, but the $IIA^*$ and $IIB^*$ string theories compactified on a timelike circle are equivalent to the IIA and IIB string theories on the dual timelike circle. Then, at least when on a timelike circle, the $IIA^*$ and $IIB^*$ string theories are precisely the timelike compactifications of the usual IIA and IIB string theories, albeit written in dual variables. The supergravity limit for the IIA or IIB variables is the conventional one, while the supergravity limit for the dual variables is non-unitary. A physical gauge can then be chosen locally for the $IIA^*$ and $IIB^*$ string theories on a timelike circle, and any lack of unitarity or positivity is due to zero-modes.

If time is compact and the physics is periodic in time, the requirements for a sensible theory are not the same as in Minkowski space. A theory that is unstable in Minkowski space (perhaps due to negative energy configurations) need not be pathological if time is compact: the periodic boundary conditions forbid any runaway solutions and the system will always return to its starting point after a period. A nonunitary theory in Minkowski space will not conserve probability, but with periodic time, any probability that is lost will always come back, as the solutions of the wave equations are required to be periodic. This suggests that the timelike compactifications of the $IIA^*$ and $IIB^*$ string theories should be consistent, although the question remains as to the status of the decompactification limit in which the radius of the timelike circle becomes infinite. Similar considerations will apply to the other new theories of $II^*$ described in this section. See [25] for further discussion of the type $II^*$ theories.

4.2 Compactification on $T^{1,1}$

Consider now compactification on the Lorentzian torus $T^{1,1}$ with one spacelike circle and one timelike one. (We will use the notation $T^{s,t}$ for a torus with $s$ spacelike circles and $t$ timelike ones.) For 11-dimensional supergravity, the limit $R_s, R_t \to 0$ gives a 9-dimensional Euclidean supergravity theory. For M–theory on a Euclidean torus $T^2$, we saw in section 2.3 that in the limit in which the torus shrank to zero size, one new spacelike dimension opened up to give the IIB string theory in 9+1 dimensions. Here we expect something similar to happen. Considering first the compactification on the spacelike circle of radius $R_s$, when $R_s$ is small we obtain the IIA string theory with coupling constant $g_s = (R/l_p)^{3/2}$. The compactification of this on a timelike circle of radius $R_t$ is T-dual to the $IIB^*$ string theory compactified on a timelike circle of radius

$$\tilde{R}_t = \frac{1}{TR_t}$$

Then taking the limit $R_t \to 0$, we obtain a theory in the expected 9 spacelike dimensions together with a new time dimension which opens up, the T-dual of the original timelike dimension. The membranes wrapping around $T^{1,1}$ have become the modes carrying the time component of momentum $p^0$ of the dual $IIB^*$ theory, and M–theory compactified on $T^{1,1}$ with radii $R_s, R_t$ is dual to the $IIB^*$ string theory compactified on a timelike circle of radius $l_p^2 / R_s R_t$, as was shown in [13].
4.3 Compactification on $T^{2,1}$

We have seen in section 2.3 that M-theory compactified on a Euclidean 2-torus $T^2$ gains a new spatial dimension in the limit in which the 2-torus shrinks to zero size, replacing the two which have disappeared, so that the original theory in $(10,1)$ dimensions becomes a theory in $(9,1)$ dimensions: $(9,1) = (10,1) - (2,0) + (1,0)$. Similarly, we have seen in section 4.1 that M-theory compactified on a Lorentzian 2-torus $T^{1,1}$ gains a new time dimension in the limit in which the 2-torus shrinks to zero size, replacing the $(1,1)$ dimensions which have disappeared so that the original theory in $(10,1)$ dimensions again becomes a theory in $(9,1)$ dimensions: $(9,1) = (10,1) - (1,1) + (0,1)$. Thus a shrinking $T^2$ is associated with an extra space dimension while a shrinking $T^{1,1}$ is associated with an extra time dimension.

For M-theory on a shrinking Euclidean $T^3$, an extra space dimension emerges for each of the three shrinking 2-cycles, so that the three toroidal dimensions which are lost are replaced by three new spatial dimensions, and we end up back in M-theory in $(10,1)$ dimensions: $(10,1) = (10,1) - 3 \times (1,0) + 3 \times (1,0)$.

Consider now the compactification on a Lorentzian 3-torus $T^{2,1}$ with two spacelike and one timelike circles. In the limit in which the torus shrinks to zero size, $2+1$ dimensions are lost leaving 8 Euclidean dimensions and reducing 11-dimensional supergravity on $T^{2,1}$ indeed gives a supergravity in $(8,0)$ dimensions. In M-theory, if the discussion above applies here, we expect an extra space dimension for every shrinking $T^2$ and an extra time dimension for every shrinking $T^{1,1}$. The torus $T^{2,1}$ has two Lorentzian 2-cycles and one Euclidean one, so that this suggests there should be an extra two time dimensions and one space dimension that open up in this limit, giving a theory in 11 dimensions with two-timing signature $(9,2) = (8,0) + (1,0) + 2 \times (0,1)$. If all the towers of wrapped membranes give extra dimensions, this must be the result, but we have seen that in some cases towers of wrapped brane states can have other meanings. A more careful analysis shows that this interpretation is indeed correct and taking M-theory on a shrinking $T^{2,1}$ gives a new theory in 9+2 dimensions, and it has an effective field theory which is a new supergravity theory in 9+2 dimensions. M-theory compactified on $T^{2,1}$ is equivalent to $M^*$ theory compactified on a two-time torus $T^{1,2}$, with the sizes of the circles related by a formula similar to \( (13) \).

Then dualities can change the number of time dimensions as well as the number of space dimensions. This new theory in 9+2 dimensions was referred to as the $M^*$ theory in \([13]\), and it has an effective field theory which is a new supergravity theory in 9+2 dimensions. M-theory compactified on $T^{2,1}$ is equivalent to $M^*$ theory compactified on a two-time torus $T^{1,2}$, with the sizes of the circles related by a formula similar to \( (13) \).

4.4 Compactifications of $M^*$ Theory

We can now investigate the compactifications of $M^*$ theory on various tori. Compactlyfying the $M^*$ theory on a timelike circle gives the $IIA^*$ string theory in 9+1 dimensions, while compactifying on a spacelike circle gives a new IIA-like string theory in 8+2 dimensions. Next consider the compactification on 2-tori in the limit in which they shrink to zero size. For $T^{0,2}$ this gives the $IIB$ string (compactification on the first circle gives the $IIA^*$ theory and the second then gives its T-dual on a timelike circle), for $T^{1,1}$ it gives the $IIA^*$ theory and for $T^{2,0}$ it gives a new IIB-like theory in 7+3 dimensions. Thus a shrinking $T^{0,2}$ gives an extra time dimension, a shrinking $T^{1,1}$ gives an extra space dimension and a shrinking $T^{2,0}$ gives an extra time dimension. This can now be used to find the results of compactification on a shrinking three-torus. For $T^{1,2}$ there are two $T^{1,1}$ cycles and one Euclidean $T^2$ cycle giving a theory in $(9,2) - (1,2) + 2 \times (1,0) + (0,1) = (10,1)$ dimensions and we are back in M-theory, for $T^{2,1}$ there are two $T^2$ cycles and one $T^{1,1}$
cycle giving a theory in \((9, 2) - (1, 2) + 2 \times (0, 1) + (1, 0) = (9, 2)\) dimensions and we are back in \(M^*\) theory, while for \(T^{3,0}\) there are three Euclidean \(T^2\) cycles giving a theory in \((9, 2) - (3, 0) + 3 \times (0, 1) = (6, 5)\) dimensions, giving a new theory in 6+5 dimensions. This theory was denoted the \(M'\) theory in \[\text{4}\], and \(M^*\) theory compactified on \(T^{3,0}\) is equivalent to the \(M'\) theory compactified on a dual \(T^{0,3}\).

The above analysis can then be repeated for this new \(M'\) theory, and it turns out that only 11-dimensional theories that arise are the \(M, M^*\) and \(M'\) theories, with signatures \((10,1), (9,2)\) and \((6,5)\), together with the mirror theories in signatures \((1,10), (2,9)\) and \((5,6)\). Reduction on circles gives IIA-like theories in signatures \(10+0, 9+1, 8+2, 6+4\) and \(5+5\) while reducing on 2-tori gives IIB-like theories in signatures \(9+1, 7+3, \text{and} 5+5\). (There are of course also mirror string theories in the signatures \(1+9, 2+8\) etc with space and time interchanged.)

In each of these 10 and 11 dimensional cases there is a corresponding supergravity limit and it is a non-trivial result that these supergravities exist, and it is unlikely that there are maximal supergravities in signatures outside this list. These theories are linked to each other by an intricate web of dualities, some of which have been outlined above, and in particular all are linked by dualities to \(M\)-theory.

Each of these theories has a set of branes of various world-volume signatures. For the \(M\)-type theories, \(M\)-theory has branes of world-volume signature \(2+1\) and \(5+1\) (the usual M2 and M5 branes), \(M^*\) theory has branes of world-volume signature \(3+0, 1+2\) and \(5+1\) while \(M'\) theory has branes of world-volume signature \(2+1, 0+3, 5+1, 3+3\) and \(1+5\).

5 Discussion

In a field theory, compactification and then shrinking the internal space \(K\) to zero size gives a dimensionally reduced field theory in lower dimensions. In compactified string theory or \(M\)-theory, however, new dimensions can emerge when the internal space shrinks, with the Kaluza-Klein towers for the new dimensions corresponding to the brane wrapping modes in which branes wrap around cycles of \(K\). In some cases (e.g. toroidal compactifications of string theory or \(M\)-theory on \(T^3\)) the number of new dimensions equals the number that are lost and one regains the original spacetime dimension, while in others (such as \(M\)-theory compactified on \(T^2\)) the number of new dimensions is different from the number that are lost and so the dimension of spacetime changes (for \(M\)-theory on \(T^2\) it changes from 11 to 10).

Clearly, the notion of what is a spacetime dimension is not an invariant concept, but depends on the ‘frame of reference’, in the sense that it will depend on the values of various moduli. A given tower of BPS states could have a natural interpretation as a Kaluza-Klein tower associated with momentum in a particular compact spacetime dimension for one set of parameters, but could have an interpretation as a tower of brane wrapping modes for other values, and we have seen many examples of this in the preceding sections. We are used to considering field theories in spacetimes of given dimension and signature, but any attempt to formulate \(M\)-theory or string theory as a theory in a given spacetime dimension or signature will be misleading. In particular, the theory underpinning all these theories has a limit which behaves like a theory in 10+1 dimensions with a supergravity limit and systematic corrections, but cannot at the fundamental level be a theory in 10+1 dimensions, as it has some limits which live 9+1 dimensions and others that live in 9+2 or 6+5 dimensions.
The supersymmetry algebra in 10+1 dimensions is

\[ \{Q, Q\} = C \left( \Gamma^M P_M - \frac{1}{2!} \Gamma^{M_1, M_2} Z_{M_1, M_2} - \frac{1}{5!} \Gamma^{M_1, \ldots, M_5} Z_{M_1, \ldots, M_5} \right), \]

(16)

where \( C \) is the charge conjugation matrix, \( P_M \) is the energy-momentum 11-vector and \( Z_{M_1, M_2} \) and \( Z_{M_1, \ldots, M_5} \) are 2-form and 5-form charges, associated with brane charges. There are 11+55+462=528 charges on the right-hand-side, which can be assembled into a symmetric bi-spinor \( X_{\alpha\beta} \). Compactifying and then dualising, one finds that some of the brane charges become momenta of the dual theory and some of the momenta become brane charges of the dual theory, so that the split of the bi-spinor \( X \) into an 11-momentum and brane charges changes under duality.

This suggests that rather than trying to formulate the theory in 10+1 dimensions, all 528 charges should be treated in the same way. There seem to be at least two ways in which this might be done. The first would be a geometrical one in which all 528 charges were treated as momenta and there is an underlying spacetime of perhaps 528 dimensions. The duality symmetries could then act geometrically, and there would be perhaps some dynamical way of choosing 11 of the dimensions as the preferred ones, e.g. through the ‘world’ being an 11-dimensional surface in this space. For example, in considering T-duality between a string theory on a space \( M \times S^1 \) and one in the dual space \( M \times \tilde{S}^1 \), it is sometimes useful to consider models on \( M \times S^1 \times \tilde{S}^1 \) in which both the circle of radius \( R \) and the dual circle of radius \( \tilde{R} \) are present, with different projections or gaugings giving the two T-dual models; see [4] and references therein.

We have seen that different spacetimes related by dualities can define the same physics, so that the notion of spacetime geometry cannot be fundamental. This suggests that different degrees of freedom should be used, with spacetime emerging as a derived concept. An alternative ‘anti-geometrical’ formulation would be one in which none of the charges were geometrical, but instead an algebraic approach similar to that of matrix theory was used. For example, M–theory could be compactified to 0, 1 or 2 dimensions to give a theory that would be expected to have duality symmetry \( E_{11}, E_{10} \) or \( E_9 \) where \( E_9 \) is an affine \( E_8 \), \( E_{10} \) is a hyperbolic algebra discussed, for example, in [3] and \( E_{11} \) might be some huge algebraic structure associated with the \( E_{11} \) Dynkin diagram. In one dimension the theory might be some matrix quantum mechanics associated with \( E_{10} \) while in zero dimensions it would be some form of non-dynamical matrix theory. At special points in the moduli space, some of the charges would be associated with extra dimensions that are decompactifying. At different points, different numbers of space and time dimensions could emerge.

Such formulations might be related to the reformulations of 11-dimensional supergravity of [1, 2, 3] in which the tangent space group is enlarged so that some of the duality symmetries are manifest. For example, in the context of compactifications to 2+1 dimensions, the usual tangent space group \( SO(10, 1) \) is broken to \( SO(2, 1) \times SO(8) \) and then anti-symmetric tensor degrees of freedom were used in [2] to reformulate the theory with tangent space group \( SO(2, 1) \times SO(16) \), with the \( SO(16) \) associated with the usual local \( SO(16) \) invariance of 3-dimensional supergravity. These formulations show that there are alternatives to the usual formulation in 11 spacetime dimensions and it would be interesting to consider others.

The five superstring theories and M–theory are different corners of the moduli space of some as yet unknown fundamental theory and the dualities linking them all involve compactification on Riemannian spaces. If this is extended to include compactification on spaces with Lorentzian signature a richer structure emerges. The strong coupling limit
of the type IIA superstring is M–theory in 10+1 dimensions whose low energy limit is 11-dimensional supergravity theory. The type I, type II and heterotic superstring theories and certain supersymmetric gauge theories emerge as different limits of M–theory. The M–theory in 10+1 dimensions is linked via dualities to $M^*$ theory in 9+2 dimensions and $M'$-theory in 6+5 dimensions. Various limits of these give rise to IIA-like string theories in 10+0, 9+1,8+2,6+4 and 5+5 dimensions, and to IIB-like string theories in 9+1,7+3, and 5+5 dimensions. The field theory limits are supergravity theories with 32 supersymmetries in 10 and 11 dimensions with these signatures, many of which are new. Further dualities similar to those of $M^*$ relate these to supersymmetric gauge theories in various signatures and dimensions, such as 2+2, 3+1 and 4+0. These new string theories and M–type theories in various spacetime signatures can all be thought of as providing extra corners of the moduli space. Some corners are stranger than others, but in any case we can only live in one corner (perhaps M–theory compactified on the product of a line interval and a Calabi-Yau 3-fold) and there is no reason why other corners might not have quite unfamiliar properties.

Theories in non-Lorentzian signatures usually have many problems, such as lack of unitarity and instability. However, the theories considered here are related to M–theory via dualities and so are just the usual theory expressed in terms of unusual variables. For example, the $M^*$ theory in 9+2 dimensions compactified on $T^{1,2}$ is equivalent to M–theory compactified on $T^{2,1}$, and so the compactified $M^*$ theory will make sense provided M–theory compactified on a Lorentzian torus is a consistent theory. Then the problems with formulating a theory in 9+2 dimensions are in this case only apparent, as the theory can be rewritten as a theory in 10+1 dimensions using different variables, so that the extra time dimension is replaced by the degrees of freedom associated with branes wrapped around time.

There are several possible generalisations of the notion of a particle to general signatures. A physical particle or an observer in Lorentzian spacetime with signature $(S, 1)$ follows a timelike (or null) world-line while a tachyon would follow a spacelike one. In a space of signature $(S, T)$, one can again consider worldlines of signature $(0, 1)$, but other generalisations of particle might include branes with worldvolumes (‘time-sheets’) of signature $(0, t)$ with $t \leq T$, sweeping out some or all of the times. In a general signature $(S, T)$, it is natural to consider branes of arbitrary signature $(s, t)$ with $s \leq S$ and $t \leq T$, and the conditions on $(s, t)$ for these to be supersymmetric were given in [26].

In conclusion, we have reviewed part of the intricate web of duality symmetries linking many apparently different theories, but since the theories are all related in this way, they should all be regarded as corners of a single underlying theory. In particular, two dual theories can be formulated in spacetimes of different geometry, topology and even signature and dimension, and so all of these concepts must be relative rather than absolute, depending on the values of certain parameters or couplings, and such a relativity principle should be a feature of the fundamental theory that underlies all this.

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