Polynomial identities of the adjoint Lie algebra of $M_{1,1}$

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Keywords: centre-by-metabelian Lie algebras, polynomial identities, $M_{1,1}$

AMS Subject Classification: 17B01, 17B60, 16W10, 16R10, 16R40

Abstract. We search an identity basis for the adjoint Lie algebra of the algebra $M_{1,1}(K)$ over a field, where $K$ is either the infinite generated Grassmann algebra $E$ or $E^1$, the variant of the algebra with 1. In particular, we prove that in the case of an infinite base field of characteristic different from two the identities of $M_{1,1}(E^1)$ are exactly all the consequences of the identity $[x, y, [z, t], v] = 0$. We also find an identity basis of $M_{1,1}(E)$ consisting of three identities.

Introduction

Let $F$ be a field and let $L(X)$ be the free linear $F$-algebra freely generated by the countable set $X = \{x_1, x_2, \ldots\}$. Suppose $A$ is a linear $F$-algebra. An element $f(x_1, x_2, \ldots)$ from $L(X)$ is called a polynomial identity of $A$ if $f(a_1, a_2, \ldots) = 0$ for all $a_1, a_2, \ldots$ from $A$. If one considers associative algebras, then $L(X)$ is the free associative algebra; if it is considered Lie algebras, then $L(X)$ is the free Lie algebra.

The Grassmann algebra $E$ is the associative $F$-algebra generated by a countable set of generators $e_1, e_2, \ldots$ with relations $e_i e_j = -e_j e_i$ and $e_i^2 = 0$ for all $i, j$. Let $E^1$ be the Grassmann algebra with identity element 1. We denote by $E_0$ the span of all elements from $E$ of even length, by $E_1$ the span of elements of odd length, and by $E_0^1$ the variant of $E_0$ with 1. Clearly, $E_0$ consists of central elements of $E$ and for any $x, y$ from $E_1$ we have $xy = -yx$ and $x^2 = 0$. The algebra of all matrices of the kind \[
\begin{pmatrix}
 a & b \\
 d & c
\end{pmatrix},
\] where $a, c \in E_0$ and $b, d \in E_1$, is denoted by $M_{1,1}(E)$. The algebra $M_{1,1}(E^1)$ consists of the same matrices with $a, c \in E_0^1$.

Each of the algebras generates a “small” so-called verbally prime varieties. Such varieties play a key role in the theory developed by A. Kemer for the solution of the long standing Specht problem (see, for example, [4]). Therefore, the identities of $M_{1,1}(E^1)$ and $M_{1,1}(E)$ were investigated by many authors from different points of view. A basis of identities of the algebras in the case of a zero characteristic base field are found by A. Popov [9]. The graded identities of $M_{1,1}(E^1)$ are investigated by O. Di Vincenzo [11]. The subvarieties of the variety generated by our algebra are studied by L. Samoilov [10]. Some other questions connected with verbally prime varieties and, in particular, with the algebras, are considered by P. Koshlukov with co-authors [11][3][5] and by other researchers.

*Partially supported by Russian Foundation for Basic Research (grants 17-01- 00551, 16-01-00795), by the Ministry of Education and Science of the Russian Federation (project 1.6018.2017/8.9) and the Competitiveness Enhancement Program of Ural Federal University; e-mail ob.finogenova@urfu.ru
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Our purpose is to study Lie identities of $M_{1,1}(E)$ and $M_{1,1}(E^1)$. Let us recall that with any associative algebra $(A, +, \cdot)$, the Lie algebra is associated in a natural way. The algebra is a so-called adjoint Lie algebra $(A, +, [\cdot, \cdot])$, where the multiplication $[\cdot, \cdot]$ is defined by letting $[a, b] = a \cdot b - b \cdot a$ for all $a, b \in A$. We consider the adjoint Lie algebras both of $M_{1,1}(E)$ and of $M_{1,1}(E^1)$ using the same notations for the Lie algebras as for corresponding associative ones. The context will make it clear which algebra, associative or adjoint Lie, is meant.

We write $[x_1, \ldots, x_n]$ for left-normalized product $[[\ldots[x_1, x_2], \ldots], x_{n-1}], x_n]$ and $[x, z^{(m)}]$ for $[x, z, \ldots, z]$. 

Now we are ready to formulate the main results of the paper.

**Theorem 1** If $\mathbb{F}$ is an infinite field of characteristic different from 2, then the polynomial identities of the Lie algebra $M_{1,1}(E^1)$ admit a basis consisting of the identity $[x, y, [z, t], u] = 0$.

**Theorem 2** If $\mathbb{F}$ is a zero characteristic field, then the polynomial identities of the Lie algebra $M_{1,1}(E)$ admit a basis consisting of the identity $[x, y, [z, t], u] = 0$.

**Theorem 3** Let $\mathbb{F}$ be a field, maybe finite, of characteristic $p > 2$. Then the polynomial identities of the Lie algebra $M_{1,1}(E)$ admit a basis consisting of the identities $[x, y, [z, t], u] = 0$, $[x, y, z, \ldots, z, t] = 0$, $[x, y, z, \ldots, z, x, \ldots, x, z] = 0$. 

The Theorem 1 shows that in the case of infinite field the algebra $M_{1,1}(E)$ generates the variety of all centre-by-metabelian Lie algebras. Another algebra as a “carrier” of the variety is offered by A. Krasnikov [6]. The questions of having finite identities bases for some Lie algebras and Lie rings are discussed there.

**Proofs of Theorem 1 and Theorem 2**

Let us recall some base notations. For a monomial $h$ and a variable $x$ denote by $deg_x(h)$ the degree of $h$ in the variable $x$, the number of times that $x$ occurs in $h$. If all monomials of a polynomial $f$ have the same degree $m$ in $x$, then $f$ is said to be homogeneous in $x$ of degree $m$ in $x$. In this case we write $deg_x(f) = m$. A multihomogeneous polynomial is the polynomial that is homogeneous with respect to all its variables. The length of monomials of such polynomial $f$ is said to be the degree of $f$.

The sum of all monomials of a fix degree in $x$ from $f$ is called a homogeneous in $x$ component of $f$. If one fixes degrees in all variables, then the sum of all monomials with such degrees is called a multihomogeneous component of $f$.

We write $f(\bar{x})$ instead of $f(x_1, x_2, \ldots)$ for brevity.

**Remark.** Let $f$ be an identity of an algebra. The following fact is well-known. If the maximal degree in $x$ of all monomials from $f$ is less than the base field order or the base field is infinite, then the identity $f$ is equivalent to the set of identities, obtained as homogeneous in $x$ components of $f$.

The purpose of the part is to prove the next two statements.

**Proposition 1** Let $\mathbb{F}$ be a field of characteristic different from two. Then every multilinear identity of the Lie algebra $M_{1,1}(E)$ or of the Lie algebra $M_{1,1}(E^1)$ is a consequence of the identity $[x, y, [z, t], u] = 0$.

**Proposition 2** Let $\mathbb{F}$ be a field of characteristic different from two. Then every multihomogeneous identity of the Lie algebra $M_{1,1}(E^1)$ is a consequence of the identity $[x, y, [z, t], u] = 0$.

The Propositions imply Theorems 1 and 2. Indeed, Theorem 1 is an evident consequence of Proposition 2 because of the remark above.

In the case of zero characteristic field identities of each algebra follow from multilinear ones. Hence, to show the truth of Theorem 2 is sufficient to notice that the algebra $M_{1,1}(E^1)$ satisfies
Remark. Let $F$ from $\mathbb{F}$ be a field. Suppose $g(x, y, z) = \sum_{ij} \alpha_{ijk}[x, y, z] + \sum_{ijk} \alpha_{ijk}[x, y, z, j, x, k]$. Then $g$ is a consequence of $f$ and $cm$.

Lemma 1. Let $f(x) = \sum_{ijk} \alpha_{ijk}[x, y, z] + [x, y, z, j, x, k]$ be a multilinear Lie polynomial with coefficients from $\mathbb{F}$. Suppose $g(x, y, z) = \sum_{ijk} \alpha_{ijk}[x, y, z] + \sum_{ijk} \alpha_{ijk}[x, y, z, j, x, k]$. Then $g$ is a consequence of $f$ and $cm$.

Proof. It is easy to see that modulo $\mathbb{F}$ the following holds:

$$f(x_1, \ldots, x_i, \ldots) \equiv \sum_{j, k \neq i} (\alpha_{ijk}[x, y, z] + \alpha_{2ijk}[x, y, z, j, x, k] + \alpha_{3ijk}[x, y, z, j, k, x]),$$

$$[x_i, x_j, z, x, y, z, x, k] = [x_i, x_j, z, x, y, z, x, k] + [x, y, z, j, x, k],$$

$$[x_i, x_j, y, z, x, k] = [x_i, x_j, y, z, x, k] + [x, y, z, j, x, k].$$

Hence, we have modulo $\mathbb{F}$

$$2g(x, y, z) \equiv \sum_{ijk} \alpha_{ijk}([x, y, z] + [x, y, z, j, x, k]) +$$

$$\sum_{ijk} \alpha_{ijk}([x, y, z] + [x, y, z, j, x, k]) =$$

$$\sum_{ijk} \alpha_{ijk}[x_i, x_j, y, z, x, k] + \sum_{ijk} \alpha_{ijk}([x, y, z] + [x, y, z, j, x, k]) +$$

$$\sum_{ijk} \alpha_{ijk}[x_i, x_j, y, z, x, k] + \sum_{ijk} \alpha_{ijk}[x, y, z, j, x, k, z, y] =$$

$$\sum_{ijk} \alpha_{ijk}[x_i, x_j, y, z, x, k] + \sum_{ijk} \alpha_{ijk}[x, y, z, j, x, k, z, y] +$$

$$\sum_{ijk} \alpha_{ijk}[x_i, x_j, y, z, x, k] + \sum_{ijk} \alpha_{ijk}[x, y, z, j, x, k, z, y] +$$

$$\sum_{ijk} \alpha_{ijk}[x_i, x_j, y, z, x, k] + [f(x_1, \ldots), z, y] =$$

$$\sum_{i} \sum_{j, k \neq i} (\alpha_{ijk}[x_i, y, z, x, j, x, k] + \alpha_{2ijk}[x_i, y, z, x, j, k, x, k] + \alpha_{3ijk}[x_i, y, z, x, j, k, x, k] + \alpha_{4ijk}[x_i, y, z, x, j, k, x, k]) +$$

$$+ [f(x_1, \ldots), z, y] \equiv \sum_{i} f(x_1, \ldots, x_i, y, z) + [f(x_1, \ldots), z, y].$$

Remark. Lemma 1 is true for the polynomials with the coefficients from a ring contained in $\mathbb{F}$.

Corollary 1 (Folklore) For each integer $k \geq 0$ and variables $t_1, t_2, \ldots$ the following identities are consequences of (1):

$$[x, y, t_1, \ldots, t_{2k}, z] + [y, z, t_1, \ldots, t_{2k}, x] + [z, x, t_1, \ldots, t_{2k}, y],$$

$$[x, y, t_1, \ldots, t_k, [u, v]] + (-1)^m[u, v, t_1, \ldots, t_k, [x, y]].$$
**Proof.** The identity \(\var{2}\) is an evident consequence of the Jacobi identity and Lemma \(\var{1}\). The identity \(\var{3}\) for an even integer \(k\) is obtained from \(\var{2}\) with the substitution \(z \mapsto [u, v]\), and for odd \(k\) — with the substitutions \(z \mapsto [u, v], y \mapsto t_k\), and \(x \mapsto [x, y]\).

**Lemma 2** The algebra \(M_{1,1}(E^1)\) does not satisfy the identity \([x, y, t^{(m)}, [u, v]] = 0\) for any integer \(m \geq 0\).

**Proof.** The result is true because of the substitution \(x = u = t = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}\), and \(v = \begin{pmatrix} 0 & 0 \\ e_2 & 0 \end{pmatrix}\). (Here \(e_1\) and \(e_2\) are generators of \(\mathbb{E}\).)

**Lemma 3** Let \(f = 0\) be a multihomogeneous identity of \(M_{1,1}(E^1)\) of degree \(m \geq 3\). If for some \(k \geq 1\)

\[
\begin{align*}
\var{2} &= \sum_{i,j \geq 2} \alpha_{ij}[x_1, x_i, \ldots, x_j] + \sum_{i \geq 1, i \neq k} \beta_i[x_k, x_i, \ldots, x_1]
\end{align*}
\]

then

(i) \(\alpha_{ij} = (-1)^m \alpha_{ji}\);

(ii) \(\beta_j = (-1)^{m+1} \sum_{i \geq 1} \alpha_{ij}\) for every \(j\) such that \(j \geq 2\) and \(j \neq k\);

(iii) \(\sum_i \beta_i = (-1)^m \sum_i \alpha_{ik}\) if \(k > 1\).

**Proof.** Denote by \(A\) the commutator \([x, y]\) and by \(B\) the commutator \([u, v]\). To prove (i) substitute in the identity \(f = 0\) the sum \(x_i + A\) instead of \(x_i\) and \(x_j + B\) instead of \(x_j\), if \(i \neq j\), and \(x_i + A + B\) instead of \(x_i\), if \(i = j\). Then one expands and denotes by \(g\) the sum of all summands which are not consequences of \(cm\) and contain both \(A\) and \(B\). It is clear that every summand from \(g\) contains both \(A\) and \(B\) exactly once and the equality \(g = 0\) is also an identity of \(M_{1,1}(E^1)\) for any field \(\mathbb{F}\). We have \(g = \alpha_{ij}[x_1, A, \ldots, B] + \alpha_{ji}[x_1, B, \ldots, A]\). Let us use \(\var{3}\) to obtain

\[
(\alpha_{ij} + (-1)^{m+1} \alpha_{ji})[x_1, A, \ldots, B] = 0.
\]

The latest equation and Lemma \(\var{2}\) provide (i).

For the case (ii) we repeat the arguments above with the substitutions \(x_1 \mapsto x_1 + A\) and \(x_j \mapsto x_j + B\). To prove (iii) we substitute \(x_1 + A\) instead \(x_1\) and \(x_k + B\) instead \(x_k\) and also repeat the arguments of the previous paragraph.

**Proof of Proposition \(\var{1}\)**

Let \(f(x_1, x_2, \ldots, x_n) = 0\) be a multilinear identity of \(M_{1,1}(E)\). Then modulo \(cm\) and the Jacobi identity the polynomial \(f\) can be written in the following form:

\[
f = \sum_{i \neq j, i,j \geq 2} \alpha_{ij}[x_1, x_i, \ldots, x_j].
\]

Firstly, suppose that \(n\) is odd. Then \(n \geq 3\). By Lemma \(\var{3}(i)\) we have \(\alpha_{ji} = -\alpha_{ij}\). Using \(\var{2}\) one obtains

\[
f = \sum_{2 \leq i \neq j \leq n} \alpha_{ij}([x_1, x_i, \ldots, x_j] - [x_1, x_j, \ldots, x_i]) = \sum_{2 \leq i < j \leq n} \alpha_{ij}[x_j, x_i, \ldots, x_1].
\]

The Jacobi identity provides that \(f = \sum_{i=3}^n \beta_i[x_2, x_i, \ldots, x_1]\). Hence, by Lemma \(\var{3}(i)\) we have \(\beta_i = 0\) for any \(i \geq 3\). This means that \(f = 0\) is a consequence of \(cm\).

Now we assume that \(m \geq 4\) is even. By Lemma \(\var{3}(i)\)

\[
f = \sum_{2 \leq i < j \leq n} \alpha_{ij}([x_1, x_i, \ldots, x_j] + [x_1, x_j, \ldots, x_i]).
\]
Notice that if \( i > 2 \) and \( j > 2 \) we have

\[
[x_1, x_i, \ldots, x_2, x_j] + [x_1, x_j, \ldots, x_2, x_i] = 0
\]

\[
- [x_2, x_1, \ldots, x_j, x_j] - [x_1, x_2, \ldots, x_1, x_j] + [x_1, x_j, \ldots, x_2, x_i] =
\]

\[
[x_1, x_2, \ldots, x_i, x_j] + [x_2, x_1, \ldots, x_2, x_j] =
\]

\[
[x_1, x_j, \ldots, x_2, x_i] + [x_2, x_j, \ldots, x_1, x_i] + [x_1, x_j, \ldots, x_i, x_2].
\]

Hence, \( f \) can be written in the following form

\[
f = \sum_{i=3}^{n} \delta_i [x_1, x_2, \ldots, x_i] + \sum_{i=3}^{2} \gamma_i [x_1, x_i, \ldots, x_2] + \sum_{i=3}^{n} \beta_i [x_2, x_i, \ldots, x_1].
\]

By Lemma 3 (ii) we have \( \gamma_i = \delta_i \) and \( \beta_i = -\delta_i \) for every \( i \geq 3 \) and by Lemma 3 (iii) \( \sum_i \beta_i = \sum_i \gamma_i \).

Hence, \( 2 \sum_i \beta_i = 0 \) and \( \sum_i \beta_i = 0 \). It is clear that \( \beta_i = -\sum_{i=4}^{n} \beta_i \). Therefore, we can rewrite \( f \) in the following form:

\[
f = \sum_{i=4}^{n} \beta_i ([x_1, x_2, \ldots, x_i] + [x_1, x_i, \ldots, x_2] - [x_2, x_i, \ldots, x_1]) -
\]

\[
(\sum_{i=4}^{n} \beta_i) ([x_1, x_2, \ldots, x_3] + [x_1, x_3, \ldots, x_2] - [x_2, x_3, \ldots, x_1]) =
\]

\[
\sum_{i=4}^{n} \beta_i ([x_1, x_2, \ldots, x_3, x_i] + [x_1, x_3, x_i, \ldots, x_2] - [x_2, x_3, x_i, \ldots, x_1]).
\]

It remains to notice that for each \( i \) the sum of these three elements in the brackets is a consequence of the identity (2).

**Proof of Proposition 2**

Let \( f \) be a multihomogeneous polynomial of degree \( m \). One can assume that \( f \) is not multilinear due to Theorem 1. Let \( x_1 \) be a variable from \( f \) of degree greater than 1. Using the Jacobi identity one can write \( f \) in the following form

\[
f = \sum_{i,j \geq 2, i \neq j} \alpha_{ij} [x_1, x_i, \ldots, x_j] + \sum_{i \geq 2} \beta_i [x_1, x_i, \ldots, x_1].
\]

If \( m \) is odd then \( \alpha_{ij} = 0 \) because of (i) in Lemma 3. If \( m \) is even then \( [x_1, x_i, \ldots, x_1, x_i] = 0 \), provided by (3). This means that one can replace \( [x_1, x_i, \ldots, x_i] \) by \( [x_1, x_i, \ldots, x_1] \). Anyway one can assume that \( \alpha_{ij} = 0 \).

By (3) the equality \( [x_1, x_i, \ldots, x_1, x_j] = (-1)^{m+1} [x_1, x_j, \ldots, x_1, x_i] \) is a consequence of (1). Hence,

\[
[x_1, x_i, \ldots, x_1, x_j] = [x_1, x_i, \ldots, x_j, x_1] + (-1)^{m+1} [x_1, x_j, \ldots, x_1, x_i] + (-1)^{m+1} [x_1, x_j, \ldots, x_1, x_i].
\]

Therefore, we can assume that \( \alpha_{ij} = 0 \) if \( i \geq j \). Then by Lemma 3 (i) all \( \alpha_{ij} \) equal zero. Now it remains to see that \( \beta_i = 0 \) for all \( i \) because of Lemma 3 (ii). This means that \( f \) is a consequence of \( cm \).
Proof of Theorem 3

The following result shows that for $M_{1,1}(E)$ the situation with identities over a finite field is not too different from another one, when the base field is infinite.

Lemma 4 Let $\mathbb{F}$ be a finite field. Then the multihomogeneous components of every identity of $M_{1,1}(E)$ are also identities of the algebra.

Proof. Assume the contrary. Then there exists $f(x, \ldots)$, an identity of $M_{1,1}(E)$, equal to the sum of non-identities $g(x, \ldots)$ and $h(x, \ldots)$, where $g$ is homogeneous in $x$ of degree $n$ and each monomial of $h$ has degree in $x$ less than $n$. One can suppose that $n$ is the minimal integer greater than one with the condition. By Remark (see p. 2), we conclude that $n \geq p$, if $p$ is a characteristic of $\mathbb{F}$. Let $\tilde{f}$, $\tilde{g}$, $\tilde{h}$ are results of the linearization of the polynomials, that is we have for $f$

$$\tilde{f}(x, y, \ldots) = f(x + y, \ldots) - f(x, \ldots) - f(y, \ldots),$$

and the similar equalities for $\tilde{g}$ and $\tilde{h}$. As far as $n$ is supposed to be minimal we conclude that $\tilde{g}$ is an identity of $M_{1,1}(E)$. Indeed, if $n = 2$, then $\tilde{g}$ is an identity because then $\text{deg}_x(h) = 1$ and $\tilde{h}$ becomes zero. If $n > 2$, then for each $i = 1, \ldots, n - 1$ we denote by $\tilde{g}_i$, the homogeneous in $x$ component of degree $i$ in $x$ from $\tilde{g}$. The polynomial $\tilde{g}_{n-1}$ is an identity of our algebra because otherwise one can consider it as a “new” polynomial $g$ having a degree in $x$ less than $n$. Clearly, $\tilde{g}_i$ is also an identity because it is obtained from $g_{n-1}$ by the transposition of $x$ and $y$. Repeating the argument we show that all $\tilde{g}_i$ are identities of the algebra as well as $\tilde{g}$.

Anyway, $\tilde{g}$ is an identity of $M_{1,1}(E)$. This means that $g$ can be considered as a map acting on our algebra and linear in $x$. It is easy to see that each element from $M_{1,1}(E)$ is a sum of such elements $a_i$ that an arbitrary monomial $w$ with $\text{deg}_i(w) \geq p$ becomes zero after the substitution $t \mapsto a_i$, whatever the choice of $i$. As far as $n > p$ we have that $g$ becomes zero for every substitution of elements from our algebra. A contradiction. \hfill \blacksquare

Now we need some technical results and notations. Denote

$$[x, z, u^{(p)}, v]; \quad [x, y, x^{(p-1)}, y^{(p-1)}, v]. \quad (4, 5)$$

Lemma 5 If $\mathbb{F}$ is a field of characteristic $p > 2$, then for each positive integer $k$ the following identities are consequences of (4), (5), and (6):

$$[y, z_1, \ldots, z_{2k}, y^{(p)}]; \quad [x, y^{(p)}, z_1, \ldots, z_{2k-1}, t] + [x, t, z_1, \ldots, z_{2k+1}, y^{(p)}] - [x, y, t, z_1, \ldots, z_{2k+1}, y^{(p-1)}]; \quad (6, 7)$$

$$[x, y^{(p)}, z_1, \ldots, z_{2k-1}, x^{(p-1)}] - [y, x^{(p)}, z_1, \ldots, z_{2k-1}, y^{(p-1)}]. \quad (8)$$

Proof. We have

$$0 \equiv [z_2, z_1, y^{(p)}, \ldots, x] \quad (\text{Jacobi identity}) \equiv -[z_1, y, z_2, y^{(p-1)}, \ldots, x] - [y, z_2, z_1, y^{(p-1)}, \ldots, x] \quad (2)$$

$$- [z_1, y, z_2, y^{(p-1)}, \ldots, x] + [x, [y, z_2], y^{(p-1)}, \ldots, z_1] + [z_1, x, y^{(p-1)}, \ldots, [y, z_2]] =$$

$$-[z_1, y, z_2, y^{(p-1)}, \ldots, x] - [y, z_2, x, y^{(p-1)}, \ldots, z_1] +$$

$$[z_1, x, y^{(p-1)}, \ldots, y, z_2] = [z_1, x, y^{(p-1)}, \ldots, z_2, y] \quad (4, 4)$$

$$- [z_1, y, z_2, y^{(p-1)}, \ldots, x] - [y, z_2, x, y^{(p-1)}, \ldots, z_1] - [z_1, x, y^{(p-1)}, \ldots, z_2, y].$$

The second summand becomes zero modulo (1) and (4) after the substitution $x \mapsto y$. Hence, after the substitution we have modulo (1) and (4)

$$0 = 2[y, z_1, z_2, \ldots, y^{(p)}].$$

This proves that (3) follows from (1) and (4).
In the case of infinite field \( \mathbb{F} \) the identity (7) is a partial linearization of (6). But we obtain the identity for an arbitrary field. At first notice that due to the Jacobi identity the following equality holds modulo (4):

\[
[x, y^{(p)}, z, v] = [z, y^{(p)}, x, v].
\]

(9)

Moreover, we have

\[
0 = [x, y, y^{(p-2)}, z_1, \ldots, z_{2k-1}, [y, t]] + [y, t, y^{(p-2)}, z_1, \ldots, z_{2k-1}, [x, y]] = [x, y, \ldots, y, t] - [x, y, \ldots, t, y] + [t, x, \ldots, y] - [y, t, \ldots, y, x] = [z_1, y^{(p)}, x, \ldots, t] + [t, x, \ldots, y^{(p)}] + [y, t, \ldots, x, y] + [y, t, \ldots, x, y] + [z_1, y^{(p)}, t, \ldots, x, y] = 2z_1(y^{(p)}, x, \ldots, t) + [t, x, \ldots, y^{(p)}] + 2[y, t, \ldots, x, y] + [z_1, y^{(p)}, \ldots, [t, x]] = 2(z_1(y^{(p)}, x, \ldots, t) + [y, t, \ldots, x, y]) \tag{9}
\]

(\text{Jacobi identity})

Moreover, we have

\[
0 = [x, y, y^{(p-1)}, y^{(p-1)}, z_1, \ldots, z_{2k-1}] = [x, y, z_{2k-1}, x^{(p-1)}, y^{(p-1)}, z_1, \ldots, z_{2k-3}, x] - [z_{2k-1}, x^{(p)}, y^{(p)}, z_1, \ldots, z_{2k-3}, y] \tag{6}
\]

(9)

\[
0 = [x, y, y^{(p-1)}, x^{(p-1)}, z_1, \ldots, z_{2k-1}, x^{(p-1)}] - [y, x^{(p)}, z_1, \ldots, z_{2k-1}, y^{(p-1)}].
\]

\textbf{Lemma 6} Let \( \mathbb{F} \) be a field of characteristic \( p > 2 \). The algebra \( M_{1,1}(E) \) does not satisfy the following identities for any integer \( k \geq 0 \):

\[
[y, x, z_1, \ldots, z_{2k}, y^{(p)}] = 0;
\]

\[
[x, y, z_1, \ldots, z_{2k}, y] = 0.
\]

\textbf{Proof.} Let us put \( z_1 = \left( \begin{array}{c} a_1 \\ 0 \\ 0 \end{array} \right) \), \( x = \left( \begin{array}{c} b_0 \\ b_1 \\ 0 \end{array} \right) \), and \( y = \left( \begin{array}{c} c_0 \\ c_1 \\ c_2 \end{array} \right) \). Here \( a_i, b_0, c_0 \) are from \( E_0 \), and \( b_1, c_1, c_2 \) are from \( E_1 \). Then

\[
[y, x, z_1, \ldots, z_{2k}, y^{(p)}] = \left( \begin{array}{c} w \\ 0 \\ 0 \end{array} \right),
\]

where \( w = 2a_1 \ldots a_{2k} b_0 c_0^{p-1} c_1 c_2 \). Clearly, \( w \) is not an identity of \( E \). After the substitutions the element

\[
[x, y^{(p-1)}, x^{(p-1)}, z_1, \ldots, z_{2k}, y]
\]

turns into the diagonal matrix with the element \( a_1 \ldots a_{2k} b_0^{p-1} c_0^{p-1} c_1 c_2 \) at the diagonal. \( \blacksquare \)

\textbf{Proof of Theorem 8} It is easy to show that (11) and (12) are identities of \( M_{1,1}(E) \).

Now we want to make sure that every identity of \( M_{1,1}(E) \) follows from (11), (12) and (13). Due to Lemma 4 we can consider only multihomogeneous polynomial identities of our algebra. Let \( f \) be a such identity of degree \( m \). If some variable in \( f \) has a degree less than \( p \), then one can linearize \( f \) in the variable to obtain the polynomial which \( f \) follows from, and which is a sum of multihomogeneous identities. The variable includes in each such summand in degrees less than in \( f \). Thus, we can assume that every variable in \( f \) has either a degree equal to 1 or a degree greater than or equal to \( p \). By Proposition 1 one can assume that \( f \) is not multilinear. Let us denote by \( y \) the variable from \( f \) of the greatest degree and by \( x \) the greatest degree variable among the variables without \( y \). It is clear that the degree of \( y \) is greater than or equal to \( p \). We assume that \( f \) is not a consequence of the identities (11) and (12). This means in particular that the degree of \( y \) is
less than \( p + 2 \). Using the Jacobi identity and (1) one can write the polynomial \( f \) in the following form:

\[
f = \sum_i \alpha_i [x, y, y, \ldots, x_i] + \sum_i \beta_i [x, x_i, \ldots, y, y] + \gamma[x, y, \ldots, y],
\]

where the variables \( x_i \) don’t coincide with \( y \) but can coincide with \( x \).

**Case 1** \((\deg_y f = p + 1)\). It is evident that except for the last summand of \( f \), all of them are consequences of (1) and (3). Hence, one can assume that \( f = \gamma[x, y, \ldots, y] \). If \( m \) is even then \( f = 0 \) is a consequence of (3). If \( m \) is odd and the degree of \( x \) is equal to 1, then \( \gamma = 0 \) by Lemma 5. If \( m \) is odd and the degree of \( x \) is greater than 1, then \( f = 0 \) is an identity of \( \mathcal{V} \).

**Case 2** \((\deg_y f = p, \deg_x f = 1)\). By Lemma 6 \( f \) depends on greater than 2 variables.

Let us substitute \( x_i \mapsto A \), where \( A = [u, v] \). Then modulo identities (1) and (4) we have

\[
0 = \alpha_i [x, y, y, \ldots, A] + \beta_i [x, A, \ldots, y, y] \tag{3}
\]

\[
(-1)^m \alpha_i [A, y, \ldots, [x, y]] - \beta_i [A, x, \ldots, y, y] \tag{11}
\]

\[
(-1)^m \alpha_i [A, x, \ldots, y, y] - (-1)^m \alpha_i [A, y, \ldots, x, y] - \delta_i [A, x, \ldots, y, y] \tag{4}
\]

\[
((-1)^m \alpha_i - \beta_i) [A, x, \ldots, y, y]
\]

Hence, we have \((-1)^m \alpha_i = \beta_i \) provided by Lemma 5 for every \( i \).

If \( m \) is odd then

\[
f = \sum_i \alpha_i ([x, y, y, \ldots, x_i] - [x, x_i, \ldots, y, y]) + \gamma[x, y, \ldots, y] \tag{2} \tag{11}
\]

\[
\sum_i \alpha_i [x_i, y, y, \ldots, x] + \gamma[x, y, \ldots, y] \tag{J} \tag{10}
\]

\[
\delta[x, y, y, \ldots, x] + \gamma[x, y, \ldots, y].
\]

It remains to substitute \( x \mapsto y \) to obtain \( \delta = 0 \) provided by Lemma 5. Then this lemma guarantees that \( \gamma = 0 \).

If \( m \) is even then

\[
f = \sum_i \alpha_i ([x, y, y, \ldots, x_i] + [x, x_i, \ldots, y, y]) + \gamma[x, y, \ldots, y] \tag{9} \tag{10}
\]

\[
\sum_i \alpha_i [x, y, y, \ldots, y] + \gamma[x, y, \ldots, y] = \delta[x, y, \ldots, y].
\]

In this case we substitute \( x \mapsto [x, y] \) to use Lemma 6 and obtain \( \delta = 0 \).

**Case 3** \((\deg_y f = p, \deg_x f = 0)\).

First we notice that if \( x_i \neq x \) then \([x, y, \ldots, x_i] = 0 \) is a consequence of (5). Furthermore, by (3) we have

\[
[x, x_i, \ldots, y, y] = (-1)^m [x, y, \ldots, [x, x_i]] + [x, x_i, \ldots, y, y] =
\]

\[
(-1)^m [x, y, \ldots, x_i] - (-1)^m [x, y, \ldots, x_i, x] + [x, x_i, \ldots, y, y] \tag{5} \tag{5} \tag{5}
\]

\[
- (-1)^m [x, y, \ldots, x, x, x].
\]

These two facts above mean that

\[
f = \phi[x, y, \ldots, x] + \gamma[x, y, \ldots, y].
\]

If \( m \) is even then one can use (4) with \( t = x \) to obtain

\[
[x, y, \ldots, x] = [x, y, \ldots, y].
\]

The same result implies from (5) for odd \( m \). Anyway, \( f = (\phi + \gamma)[x, y, \ldots, y] \). By Lemma 5 we have \( \phi + \gamma = 0 \). Theorem 3 is proved.
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