ON GAUSSIAN LIPSCHITZ SPACES AND THE BOUNDEDNESS OF FRACTIONAL INTEGRALS AND FRACTIONAL DERIVATIVES ON THEM

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Abstract. The main purpose of this paper is to study the boundedness of Gaussian fractional integrals and derivatives associated to Hermite polynomial expansions on Gaussian Lipschitz spaces $Lip\alpha(\gamma)$. To get these results we introduce formulas for these operators in terms of the Hermite-Poisson semigroup as well as the Gaussian Lipschitz spaces. This approach was originally developed for the classical Poisson integral. These proofs can also be extended to the case of Laguerre and Jacobi expansions. In subsequent papers we will study the same operators on Gaussian Besov-Lipschitz and Triebel-Lizorkin spaces.

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1. Preliminaries. Let us consider $\mathbb{R}^d$, with the Gaussian measure $d\gamma(x) = \frac{e^{-|x|^2}}{\pi^{d/2}} \, dx$, $x \in \mathbb{R}^d$ and the Ornstein-Uhlenbeck differential operator

$$L = \frac{1}{2} \triangle_x - \langle x, \nabla_x \rangle. \tag{1.1}$$

Let $\nu = (\nu_1, ..., \nu_d)$ be a multi-index, where $\nu_i$ is a non-negative integer, $1 \leq i \leq d$; let $\nu! = \prod_{i=1}^d \nu_i!$, $|\nu| = \sum_{i=1}^d \nu_i$, $\partial_i = \frac{\partial}{\partial x_i}$, for each $1 \leq i \leq d$ and $\partial^\nu = \partial_1^{\nu_1} ... \partial_d^{\nu_d}$.

Let $h_\nu$ be the Hermite polynomial of order $\nu$, in $d$ variables,

$$h_\nu(x) = \frac{1}{(2^{|
u|} \nu!)^{1/2}} \prod_{i=1}^d (-1)^{\nu_i} e^{x_i^2} \frac{\partial^{\nu_i}}{\partial x_i^{\nu_i}}(e^{-x_i^2}), \tag{1.2}$$

then, it is well known, that the Hermite polynomials are eigenfunctions of $L$,

$$Lh_\nu(x) = -|\nu| h_\nu(x). \tag{1.3}$$
Given a function \( f \in L^1(\gamma) \) its \( \nu \)-Fourier-Hermite coefficient is defined by

\[
\hat{f}(\nu) = \langle f, h_\nu \rangle = \int_{\mathbb{R}^d} f(x) h_\nu(x) \, d\gamma(x). \tag{1.4}
\]

For each \( n \), a non negative integer, let \( C_n \) be the closed subspace of \( L^2(\gamma) \) generated by the linear combinations of \( \{h_\nu : |\nu| = n\} \). By the orthogonality of the Hermite polynomials with respect to \( \gamma \), it is easy to see that \( \{C_n\} \) is an orthogonal decomposition of \( L^2(\gamma) \),

\[
L^2(\gamma) = \bigoplus_{n=0}^{\infty} C_n
\]

which is called the Wiener chaos.

If \( J_n \) is the orthogonal projection of \( L^2(\gamma) \) onto \( C_n \), that is, for \( f \in L^2(\gamma) \)

\[
J_n f = \sum_{|\nu|=n} \hat{f}(\nu) h_\nu,
\]

then the Hermite expansion of \( f \) can be written as

\[
f = \sum_{n=0}^{\infty} J_n f.
\]

Let us consider the Ornstein-Uhlenbeck semigroup \( \{T_t\}_{t \geq 0} \), i.e.

\[
T_t f(x) = \frac{1}{(1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{\sqrt{t}(|x|+|y|)^2 - 2e^{-t} \langle x, y \rangle}{1 - e^{-2t}}} f(y) \, d\gamma(y)
= \frac{1}{\pi^{d/2} (1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y - e^{-t}x|^2}{1 - e^{-2t}}} f(y) dy, \tag{1.5}
\]

for \( t > 0 \) and \( T_0 f = f \).

The family \( \{T_t\}_{t \geq 0} \) is a strongly continuous positive symmetric diffusion semigroup, see [15] on \( L^p(\gamma) \), \( 1 \leq p < \infty \), with infinitesimal generator \( L \). Now, by Bochner subordination formula, see Stein [14], the Poisson-Hermite semigroup \( \{P_t\}_{t \geq 0} \) is defined as

\[
P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t/4u} f(x) \, du.
\]

In addition, changing the variable, we have the following representation

\[
P_t f(x) = \int_0^\infty T_s f(x) \mu_t^{(1/2)}(ds),
\]

where the measure

\[
\mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}} \frac{e^{-t^2/4s}}{s^{3/2}} ds = g(t, s)ds, \tag{1.6}
\]

is called the one-side stable measure on \((0, \infty)\) of order 1/2.
From (1.5) we obtain, after the change of variable $r = e^{-\frac{t^2}{4u}}$,

$$P_t f(x) = \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 t \frac{\exp(t^2/4 \log r) \exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r} f(y) dy$$

$$= \int_{\mathbb{R}^d} p(t,x,y) f(y) dy,$$  \hspace{1cm} (1.7)

with

$$p(t,x,y) = \frac{1}{2\pi^{(d+1)/2}} \int_0^1 t \frac{\exp(t^2/4 \log r) \exp\left(\frac{-|y-rx|^2}{1-r^2}\right)}{(1-r^2)^{d/2}} \frac{dr}{r}.$$  \hspace{1cm} (1.8)

The family $\{P_t\}_{t \geq 0}$ is also a strongly continuous positive symmetric diffusion semigroup on $L^p(\gamma)$, $1 \leq p < \infty$, with infinitesimal generator $-(-L)^{1/2}$. In particular $P_t1 = 1$.

Observe that by (1.3), $h_\nu$ are eigenfunctions of $\{T_t\}_{t \geq 0}$ and $\{P_t\}_{t \geq 0}$ we have that

$$T_t h_\nu(x) = e^{-t|\nu|} h_\nu(x),$$  \hspace{1cm} (1.9)

and

$$P_t h_\nu(x) = e^{-t\sqrt{|\nu|}} h_\nu(x)$$  \hspace{1cm} (1.10)

and therefore, if $f \in L^2(\gamma)$

$$T_t f = \sum_n e^{-nt} J_n f \quad \text{and} \quad P_t f = \sum_n e^{-\sqrt{\pi}t} J_n f.$$  

Since $\{P_t\}_{t \geq 0}$ is a strongly continuous semigroup we have

$$\lim_{t \to 0^+} P_t f(x) = f(x), \quad \text{a.e. } x,$$  \hspace{1cm} (1.11)

and we have,

$$\lim_{t \to \infty} P_t f(x) = \int_{\mathbb{R}^d} f(y)d\gamma(y), \quad \text{a.e. } x.$$  \hspace{1cm} (1.12)

In what follows we will need the following technical result about the $L^1$-norm of the derivatives of the kernel $p(t,x,y)$.

**Lemma 1.1.** If $p(t,x,y)$ is the Poisson-Hermite kernel, then

$$\int_{\mathbb{R}^d} \left| \frac{\partial p(t,x,y)}{\partial t} \right| dy \leq \frac{C}{t},$$  \hspace{1cm} (1.13)

where $C$ is a constant independent of $x$ and $t$. Moreover, for any positive integer $k$ we have

$$\int_{\mathbb{R}^d} \left| \frac{\partial^k p(t,x,y)}{\partial t^k} \right| dy \leq \frac{C}{t^k}.$$  \hspace{1cm} (1.14)
Proof. Let us first prove (1.13). Remember that

\[ p(t, x, y) = \frac{1}{2\pi^{(d+1)/2}} \int_0^1 \frac{t \exp \left( t^2/4 \log r \right) \exp \left( \frac{-|y-rx|^2}{1-r^2} \right) dr}{(1-r^2)^{d/2}}. \]

Therefore,

\[ \frac{\partial p(t, x, y)}{\partial t} = \frac{1}{2\pi^{(d+1)/2}} \int_0^1 \frac{t \exp \left( t^2/4 \log r \right) (1 + t^2/2 \log r) \exp \left( \frac{-|y-rx|^2}{1-r^2} \right) dr}{r (1-r^2)^{d/2}}. \]

Then, by Tonelli’s theorem, using the fact that

\[ \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} \frac{\exp \left( \frac{-|y-rx|^2}{1-r^2} \right)}{(1-r^2)^{d/2}} dy = 1, \]

we get

\[ \int_{\mathbb{R}^d} \left| \frac{\partial p(t, x, y)}{\partial t} \right| dy \]

\[ \leq \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 \frac{t \exp \left( t^2/4 \log r \right) (1 + t^2/2 \log r) \exp \left( \frac{-|y-rx|^2}{1-r^2} \right) dr}{r (1-r^2)^{d/2}} dy \]

\[ = \frac{1}{2\pi^{(d+1)/2}} \int_0^1 \frac{t \exp \left( t^2/4 \log r \right) (1 + t^2/2 \log r) \int_{\mathbb{R}^d} \frac{\exp \left( \frac{-|y-rx|^2}{1-r^2} \right)}{(1-r^2)^{d/2}} dy dr}{r}. \]

Thus what we need to prove is

\[ \int_0^1 \frac{t \exp \left( t^2/4 \log r \right) (1 + t^2/2 \log r) dr}{r} \leq C. \tag{1.15} \]

Making the change of variable \( s = -\log r \) we get

\[ \int_0^1 \frac{t \exp \left( t^2/4 \log r \right) (1 + t^2/2 \log r) dr}{r} = \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}} \left| 1 - \frac{t^2}{2s} \right| ds \]

\[ \leq \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}} ds + \int_0^\infty \frac{e^{-t^2/4s} t^2}{2s} ds. \]

Now, making the change of variable \( v = \frac{t^2}{4s} \), \( ds = -\frac{t^2}{4v^2} dv \), we get

\[ \int_0^\infty \frac{e^{-t^2/4s}}{s^{3/2}} ds = \int_0^\infty e^{-v \left( \frac{t^2}{4v} \right)^{3/2}} \frac{t^2}{4v^2} dv \]

\[ = \int_0^\infty e^{-v \left( \frac{4v}{v^3} \right)^{3/2}} \frac{t^2}{4v^2} dv = \frac{C}{t} \int_0^\infty e^{-v} v^{-1/2} dv \]

\[ = \frac{CT(1/2)}{t} = \frac{C'}{t}. \]
and
\[
\int_0^\infty e^{-t^2/s} \frac{t^2}{4s} ds = 2 \int_0^\infty e^{-v} \frac{v^{3/2}}{4v^2} t^2 dv \\
= 2 \int_0^\infty e^{-v} \frac{(4v)^{3/2}}{t^3} v^2 dv = C t \int_0^\infty e^{-v} v^{1/2} dv \\
= \frac{CT(3/2)}{t} = C'.
\]

For the proof of the general case (1.14) we use induction. Since the case \(k = 1\) is already proved let us assume that (1.14) holds for certain \(k\) and prove that it also holds for \(k + 1\). By the semigroup property and taking \(u = t + s\), we have

\[
\frac{\partial^{k+1} p(u, x, y)}{\partial t^{k+1}} = \frac{\partial}{\partial s} \frac{\partial^k}{\partial t^k} p(t + s, x, y) \\
= \frac{\partial}{\partial s} \frac{\partial^k}{\partial t^k} \left[ \int_{\mathbb{R}^d} p(s, x, v)p(t, v, y) dv \right] \\
= \int_{\mathbb{R}^d} \frac{\partial p(s, x, v)}{\partial s} \frac{\partial^k p(t, v, y)}{\partial t^k} dv.
\]

Therefore,

\[
\int_{\mathbb{R}^d} \left| \frac{\partial^{k+1} p(u, x, y)}{\partial t^{k+1}} \right| dy \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| \frac{\partial p(s, x, v)}{\partial s} \right| \left| \frac{\partial^k p(t, v, y)}{\partial t^k} \right| dv dy \\
\leq \int_{\mathbb{R}^d} \left| \frac{\partial p(s, x, v)}{\partial s} \right| \int_{\mathbb{R}^d} \left| \frac{\partial^k p(t, v, y)}{\partial t^k} \right| dy dv \\
\leq \frac{CC}{s t^k}.
\]

Finally, taking \(s = t = u/2\) the case \(k + 1\) is proved. \(\square\)

As usual in what follows \(C\) represents a constant that is not necessarily the same in each occurrence.

2. Gaussian Lipschitz spaces. We want to define Lipschitz spaces associated to the Gaussian measure. First, observe that the spaces \(L^p(\gamma)\) are not in general closed under the action of the classical translation operator \(\tau_y f(x) = f(x + y)\), see [11]. In the classical case, the Poisson semigroup provides an alternative characterization of the Lipchitz spaces, see [14], we will follow this approach, using the Poisson-Hermite semigroup, to define Gaussian Lipchitz spaces. First we need the following key result,
Proposition 2.1. Suppose $f \in L^\infty(\gamma)$ and $\alpha > 0$. Let $k$ and $l$ be two integers both greater than $\alpha$. The two conditions
\[ \| \frac{\partial^k P_t f}{\partial t^k} \|_{\infty} \leq A_{\alpha,k} t^{-k+\alpha} \] (2.1)
and
\[ \| \frac{\partial^l P_t f}{\partial t^l} \|_{\infty} \leq A_{\alpha,l} t^{-l+\alpha}, \] (2.2)
are equivalent. Moreover, the smallest $A_{\alpha,k}$ and $A_{\alpha,l}$ holding in the above inequalities are comparable.

Proof. It suffices to prove that if $k > \alpha$,
\[ \| \frac{\partial^k P_t f}{\partial t^k} \|_{\infty} \leq A_{\alpha,k} t^{-k+\alpha} \] (2.3)
and
\[ \| \frac{\partial^{k+1} P_t f}{\partial t^{k+1}} \|_{\infty} \leq A_{\alpha,k+1} t^{-(k+1)+\alpha}, \] (2.4)
are equivalent.

Let us assume (2.3). Since by the semigroup property if $t = t_1 + t_2$, $P_t f = P_{t_1} (P_{t_2} f)$ then using the hypothesis and Lemma 1.1,
\[ \| \frac{\partial^{k+1} P_t f}{\partial t^{k+1}} \|_{\infty} = \| \frac{\partial^k P_{t_1} f}{\partial t^k} \|_{\infty} \leq \| \frac{\partial^k P_{t_2} f}{\partial t^k} \|_{\infty} \int_{\mathbb{R}^d} \left| \frac{\partial p(t_1, \cdot, y)}{\partial t_1} \right| dy \]
\[ \leq A_{\alpha,k} t^{-k+\alpha} C t_1^{-1}. \]
For $t_1 = t_2 = t/2$ we get (2.4).

Let us now assume (2.4). Observe that, again by Lemma 1.1,
\[ \| \frac{\partial^k P_t f}{\partial t^k} \|_{\infty} \leq \| f \|_{\infty} \int_{\mathbb{R}^d} \left| \frac{\partial p(t, x, \cdot)}{\partial t^k} \right| dy \leq C \| t \| \| f \|_{\infty}, \]
so $\frac{\partial^k P_t f}{\partial t^k} \to 0$ as $t \to \infty$, and then by hypothesis
\[ \| \frac{\partial^k P_t f}{\partial t^k} \|_{\infty} \leq \int_t^{\infty} \| \frac{\partial^{k+1} P_s f}{\partial s^{k+1}} \|_{\infty} ds \leq A_{\alpha,k+1} t^{-k+\alpha} = C t^{-k+\alpha}. \]
□

Now, we can define the Gaussian Lipschitz spaces as follows,

Definition 2.1. For $\alpha > 0$ let $n$ be the smallest integer greater than $\alpha$. The Gaussian Lipschitz space $Lip_\alpha(\gamma)$ is defined as the set of $L^\infty$ functions for which there exists a constant $A_\alpha(f)$ such that
\[ \| \frac{\partial^n P_t f}{\partial t^n} \|_{\infty} \leq A_\alpha(f) t^{-n+\alpha}. \] (2.5)
The norm of \( f \in \text{Lip}_\alpha(\gamma) \) is defined as

\[
\|f\|_{\text{Lip}_\alpha(\gamma)} := \|f\|_\infty + A_\alpha(f),
\]

(2.6)

where \( A_\alpha(f) \) is the smallest constant \( A \) appearing in (2.5).

**Observation 2.1.** By Proposition 2.1 the definition of \( \text{Lip}_\alpha(\gamma) \) does not depend on which \( k > \alpha \) is chosen and the resulting norms are equivalent.

**Observation 2.2.** Condition (2.5) is of interest for \( t \) near zero, since the inequality

\[
\|\frac{\partial^n P_t f}{\partial t^n}\|_\infty \leq A t^{-n},
\]

(2.7)

which is stronger away from zero, follows for \( f \in L^\infty \) immediately from (1.14),

\[
\|\frac{\partial^n P_t f}{\partial t^n}\|_\infty \leq \int_{\mathbb{R}^d} \left| \frac{\partial^n p(t, x, y)}{\partial t^n} \right| |f(y)| dy \leq \frac{C}{t^n} \|f\|_\infty.
\]

These spaces can be also obtained using abstract interpolation theory using the Poisson-Hermite semigroup, see [16] 1.6.5.

Also there are inclusion relations among the Gaussian Lipschitz spaces,

**Proposition 2.2.** If \( 0 < \alpha_1 < \alpha_2 \) the inclusion \( \text{Lip}_{\alpha_2}(\gamma) \subset \text{Lip}_{\alpha_1}(\gamma) \) holds.

**Proof.** Take \( f \in \text{Lip}_{\alpha_2}(\gamma) \) and \( n \geq \alpha_2 \), then

\[
\|\frac{\partial^n P_t f}{\partial t^n}\|_\infty \leq A_\alpha(f) t^{-n+\alpha_2}.
\]

If \( 0 < t < 1 \) then \( t^{-n+\alpha_2} \leq t^{-n+\alpha_1} \) and therefore

\[
\|\frac{\partial^n P_t f}{\partial t^n}\|_\infty \leq A_\alpha(f) t^{-n+\alpha_1}.
\]

Now if \( t \geq 1 \) then we know from (2.7) that

\[
\|\frac{\partial^n P_t f}{\partial t^n}\|_\infty \leq A_\alpha(f) t^{-n}
\]

and as \( t^{-n+\alpha_1} > t^{-n} \) we get also in this case

\[
\|\frac{\partial^n P_t f}{\partial t^n}\|_\infty \leq A_\alpha(f) t^{-n+\alpha_1}
\]

since \( n > \alpha_1 \); then \( f \in \text{Lip}_{\alpha_1}(\gamma) \).

**Proposition 2.3.** If \( f \in \text{Lip}_\alpha(\gamma) \), \( 0 < \alpha < 1 \), then

\[
\|P_t f - f\|_\infty \leq A_\alpha(f) t^\alpha.
\]

(2.8)
Proof. Applying the fundamental theorem of Calculus,
\[
\|P_t f - f\|_\infty = \| \int_0^t \frac{\partial P_s f}{\partial s} ds \|_\infty \leq \int_0^t \| \frac{\partial P_s f}{\partial s} \|_\infty ds
\leq A_\alpha(f) \int_0^t s^{-1+\alpha} ds = A_\alpha(f) t^\alpha.
\]

3. Boundedness of fractional integrals and fractional derivatives on \(\text{Lip}_\alpha(\gamma)\). For \(\beta > 0\), the Bessel potential of order \(\beta > 0\), \(J_\beta\), associated to the Gaussian measure is defined formally as
\[
J_\beta = (I + \sqrt{-L})^{-\beta},
\]
meaning that for the Hermite polynomials we have,
\[
J_\beta h_\nu(x) = \frac{1}{(1 + \sqrt{\nu})^\beta} h_\nu(x).
\]
By linearity this definition can be extended to any polynomial and P.A. Meyer’s theorem allows us to extend Bessel potentials to a continuous operator on \(L^p(\gamma)\), \(1 < p < \infty\). Additionally, it is easy to see that \(J_\beta\) is a bijection over the set of polynomials \(\mathcal{P}\). Alternatively the Bessel potentials can be defined as
\[
J_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} P_s f(x) ds, f \in L^p(\gamma).
\]
For more details see [4]. Moreover \(\{J_\beta\}_\beta\) is a strongly continuous semigroup on \(L^p(\gamma), 1 \leq p < \infty\), with infinitesimal generator \(\frac{1}{2} \log(I - L)\).

We will study the action of the Bessel potentials on the Gaussian Lipschitz spaces \(\text{Lip}_\alpha(\gamma)\)

**Theorem 3.1.** Let \(\alpha > 0\) and \(\beta > 0\) then \(J_\beta\) is bounded from \(\text{Lip}_\alpha(\gamma)\) to \(\text{Lip}_{\alpha+\beta}(\gamma)\).

**Proof.** Let \(f \in \text{Lip}_\alpha(\gamma)\) and consider a fixed integer \(n > \alpha + \beta\), then
\[
\| \frac{\partial^n P_t f}{\partial t^n} \|_\infty \leq A_\alpha(f) t^{-n+\alpha}, \quad t > 0.
\]
Using (3.2), the fact that \(f \in L^\infty\), and consequently \(P_{t+s}f \in L^\infty\), we obtain
\[
P_t(J^\beta f)(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} e^{-s} P_{t+s} f(x) ds,
\]
and therefore
\[
\|P_t(J^\beta f)\|_\infty \leq \|f\|_\infty.
\]
i.e. $P_t(J_\beta f) \in L^\infty$.

Now we want to verify the Lipschitz condition. Differentiating (3.3), we get

$$\frac{\partial^n P_t(J_\beta f)(x)}{\partial t^n} = \frac{1}{\Gamma(\beta)} \int_0^{t+s} s^{\beta-1} e^{-s} \frac{\partial^n P_t f(x)}{\partial t^n} ds$$

and this implies

$$\left\| \frac{\partial^n P_t(J_\beta f)}{\partial t^n} \right\|_\infty \leq \frac{1}{\Gamma(\beta)} \int_0^{t} s^{\beta-1} e^{-s} \left\| \frac{\partial^n P_t f}{\partial (t+s)^n} \right\|_\infty ds$$

$$+ \frac{1}{\Gamma(\beta)} \int_0^{\infty} s^{\beta-1} e^{-s} \left\| \frac{\partial^n P_t f}{\partial (t+s)^n} \right\|_\infty ds = (I) + (II).$$

Since $\beta > 0$ and $t + s > t$,

$$(I) \leq \frac{A_\alpha(f)}{\Gamma(\beta)} \int_0^t s^{\beta-1} (t+s)^{-n+\alpha} e^{-s} ds$$

$$\leq \frac{A_\alpha(f)}{\Gamma(\beta)} t^{-n+\alpha} s^{\beta-1} ds(\gamma) \leq Ct^{-n+\alpha+\beta} \|f\|_{\text{Lip}_\alpha(\gamma)}.$$

On the other hand, since $n > \alpha + \beta$, and $t + s > s$

$$(II) \leq \frac{A_\alpha(f)}{\Gamma(\beta)} \int_t^\infty s^{\beta-1} e^{-s} (t+s)^{-n+\alpha} ds$$

$$\leq \frac{A_\alpha(f)}{\Gamma(\beta)} \int_t^\infty s^{\beta-1} e^{-s} s^{-n+\alpha} ds$$

$$\leq \frac{A_\alpha(f)}{\Gamma(\beta)} \int_t^\infty s^{-n+\alpha+\beta-1} ds = CA_\alpha(f)t^{-n+\alpha+\beta}.$$ 

Therefore,

$$\left\| \frac{\partial^n P_t(J_\beta f)}{\partial t^n} \right\|_\infty \leq CA_\alpha(f)t^{-n+\alpha+\beta}, \quad t > 0.$$ 

Thus $J_\beta f \in \text{Lip}_{\alpha+\beta}(\gamma)$ and moreover,

$$\|J_\beta f\|_{\text{Lip}_{\alpha+\beta}(\gamma)} = \|J_\beta f\|_\infty + A_\alpha(J_\beta f)$$

$$\leq \|f\|_\infty + CA_\alpha(f) \leq C\|f\|_{\text{Lip}_\alpha(\gamma)}.$$

□
For $\beta > 0$ the Riesz fractional integral or Riesz potential of order $\beta$, $I_\beta$, with respect to the Gaussian measure is defined formally as

$$I_\beta = (-L)^{-\beta/2}\Pi_0,$$

(3.4)

where, $\Pi_0 f = f - \int_{\mathbb{R}^d} f(y)\gamma(dy)$, for $f \in L^2(\gamma)$. That means that for the Hermite polynomials $\{h_\nu\}$, with $|\nu| > 0$,

$$I_\beta h_\nu(x) = \frac{1}{|\nu|^{\beta/2}}h_\nu(x),$$

(3.5)

and for $\nu = (0,0,\cdots,0)$, $I_\beta (h_\nu) = 0$. By linearity can be extended to any polynomial. If $f$ is a polynomial with $\int_{\mathbb{R}^d} f(y)\gamma(dy) = 0$,

$$I_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty s^{\beta-1}P_s f(x) \, ds.$$  (3.6)

By P.A. Meyer’s multiplier theorem, $I_\beta$ admits a continuous extension to $L^p(\gamma_d)$, $1 < p < \infty$, and (3.6) can be extended for $f \in L^p(\gamma)$, see [13]. In addition, if $f \in C^2_B(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} f(y)\gamma(dy) = 0$, then

$$I_\beta f = -\frac{1}{\beta\Gamma(\beta)} \int_0^\infty s^{\beta-1}\partial P_s f \, ds,$$  (3.7)

see [10].

The Riesz potentials are not bounded operators on $L^\infty(\gamma_d)$ nor on $Lip_\alpha(\gamma)$.

Following the classical case, the Riesz fractional derivate of order $\beta > 0$ with respect to the Gaussian measure $D^\beta$, is defined formally as

$$D^\beta = (-L)^{\beta/2},$$

which means that for the Hermite polynomials, we have

$$D^\beta h_\nu(x) = |\nu|^{\beta/2}h_\nu(x).$$

(3.8)

In the case of $0 < \beta < \alpha < 1$ we have the following integral representation,

$$D^\beta f = \frac{1}{c_\beta} \int_0^\infty s^{-\beta-1}(P_s - I) f \, ds,$$  (3.9)

for $f \in Lip_\alpha(\gamma)$, where $c_\beta = \int_0^\infty u^{-\beta-1}(e^{-u} - 1) du$, which is finite as $0 < \beta < 1$.

The action of Riesz fractional derivates on Lipchitz spaces is as follows,

**Theorem 3.2.** For $0 < \beta < \alpha < 1$, the Riesz fractional derivate of order $\beta$, $D^\beta : Lip_\alpha(\gamma) \to Lip_{\alpha-\beta}(\gamma)$ is bounded.
Proof. Take \( f \in \text{Lip}_\alpha(\gamma) \) i.e. \( f \in L^\infty \) and \( \| \frac{\partial P_t f}{\partial r} \|_\infty \leq A_\alpha(f)t^{-1+\alpha} \). Let us observe that using representation (3.9), Proposition 2.1 and (2.8) we get,

\[
|D^\beta f(x)| \leq \frac{1}{c_\beta} \int_0^1 s^{-\beta-1} |P_s f(x) - f(x)| ds + \frac{1}{c_\beta} \int_1^\infty s^{-\beta-1} |P_s f(x) - f(x)| ds \\
\leq \frac{1}{c_\beta} \int_0^1 s^{-\beta-1} \|P_s f - f\|_\infty ds + \frac{2\|f\|_\infty}{c_\beta} \int_1^\infty s^{-\beta-1} ds \\
\leq \frac{A_\alpha(f)}{c_\beta} \int_0^1 s^{\alpha-\beta-1} ds + \frac{2\|f\|_\infty}{\beta c_\beta} \int_1^\infty s^{-\beta-1} ds \\
= \frac{A_\alpha(f)}{c_\beta(\alpha-\beta)} + \frac{2\|f\|_\infty}{\beta c_\beta} \leq C_{\alpha,\beta}\|f\|_{\text{Lip}_\alpha(\gamma)},
\]

thus \( D^\beta f \in L^\infty \).

Now we want to verify the Lipchitz condition. Fixing \( t \) and using again representation (3.9), we have

\[
\frac{\partial}{\partial t}(P_t D^\beta f(x)) = \frac{1}{c_\beta} \frac{\partial}{\partial t} \left[ \int_0^\infty s^{-\beta-1}(P_{t+s} f(x) - P_t f(x)) ds \right] \\
= \frac{1}{c_\beta} \int_0^\infty s^{-\beta-1} \left[ \frac{\partial P_{s+t} f(x)}{\partial t} - \frac{\partial P_t f(x)}{\partial s} \right] ds \\
= \frac{1}{c_\beta} \int_0^t s^{-\beta-1} \left[ \frac{\partial P_{s+t} f(x)}{\partial t} - \frac{\partial P_t f(x)}{\partial t} \right] ds \\
\quad + \frac{1}{c_\beta} \int_t^\infty s^{-\beta-1} \left[ \frac{\partial P_{s+t} f(x)}{\partial t} - \frac{\partial P_t f(x)}{\partial t} \right] ds \\
= (I) + (II).
\]

By Proposition 2.2 we have

\[
\| \frac{\partial^2 P_t f}{\partial r^2} \|_\infty \leq A r^{\alpha-2}, \quad (3.10)
\]

and using the fundamental theorem of Calculus we get, for \( s > 0 \)

\[
\left| \frac{\partial P_{t+s} f(x)}{\partial t} - \frac{\partial P_t f(x)}{\partial t} \right| \leq \int_t^{t+s} \left| \frac{\partial^2 P_r f(x)}{\partial r^2} \right| dr \leq A \int_t^{s+t} r^{\alpha-2} dr \\
\leq A t^{\alpha-2} s. \quad (3.11)
\]

Therefore,

\[
|I| \leq \frac{1}{c_\beta} \int_0^t s^{-\beta-1} \left| \frac{\partial P_{t+s} f(x)}{\partial t} - \frac{\partial P_t f(x)}{\partial t} \right| ds \\
\leq A \frac{t^{\alpha-2}}{c_\beta} \int_0^t s^{-\beta} ds = C_{\alpha,\beta} t^{-1+\alpha-\beta}.
\]
On the other hand, using (2.7),

\[ |(II)| \leq \frac{1}{c_{\beta}} \int_{t}^{\infty} s^{-\beta-1} \left[ |\partial P_{t+s}f(x)| + |\partial P_{t}f(x)| \right] ds \]

\[ \leq \frac{C}{c_{\beta}} \int_{t}^{\infty} s^{-\beta-1}[(t+s)^{-1+\alpha} + t^{-1+\alpha}] ds \]

\[ \leq C t^{-1+\alpha} \int_{t}^{\infty} s^{-\beta-1} ds = C_{\alpha,\beta} t^{-1+\alpha-\beta}. \]

Thus,

\[ \left\| \frac{\partial}{\partial t}(P_{t}D^{\beta} f) \right\|_{\infty} \leq C_{\alpha,\beta} t^{\alpha-\beta-1}, \]

which implies \( D^{\beta} f \in \operatorname{Lip}_{\alpha-\beta} \). □

We can also consider a Bessel fractional derivative \( D^{\beta} \), defined formally as

\[ D^{\beta} = (I + \sqrt{-L})^{\beta}, \]

which means that for the Hermite polynomials, we have

\[ D^{\beta} h_{\nu}(x) = (1 + |\nu|)^{\beta} h_{\nu}(x). \quad (3.12) \]

In the case of \( 0 < \beta < 1 \) we have the following integral representation,

\[ D^{\beta} f = \frac{1}{c_{\beta}} \int_{0}^{\infty} t^{-\beta-1}(e^{-t}P_{t} - I) f dt, \quad (3.13) \]

where, as before, \( c_{\beta} = \int_{0}^{\infty} u^{-\beta-1}(e^{-u} - 1) du \).

We want to study of the action of the Bessel fractional derivative \( D^{\beta} \) on the Gaussian Lipschitz spaces.

**Theorem 3.3.** For \( 0 < \beta < \alpha < 1 \), the Bessel fractional derivative of order \( \beta \), \( D^{\beta} : \operatorname{Lip}_{\alpha}(\gamma) \to \operatorname{Lip}_{\alpha-\beta}(\gamma) \) is bounded.

**Proof.** The proof of this result is essentially analogous to the proof of Theorem 3.2. Let \( f \in \operatorname{Lip}_{\alpha}(\gamma) \) i.e. \( f \in L^{\infty} \) such that \( \left\| \frac{\partial P_{t} f}{\partial t} \right\|_{\infty} \leq A_{\alpha}(f)t^{-1+\alpha} \). Using the representation (3.13), (2.8) and Proposition 2.1, we get,

\[ |D^{\beta} f(x)| \leq \frac{1}{c_{\beta}} \int_{0}^{\infty} s^{-\beta-1}|e^{-s}P_{s}f(x) - f(x)| ds \]

\[ = \frac{1}{c_{\beta}} \int_{0}^{1} s^{-\beta-1}|e^{-s}P_{s}f(x) - f(x)| ds \]

\[ + \frac{1}{c_{\beta}} \int_{1}^{\infty} s^{-\beta-1}|e^{-s}P_{s}f(x) - f(x)| ds \]

\[ = (I) + (II). \]
Now

\[(I) \leq \frac{1}{c_\beta} \int_0^1 s^{-\beta-1}e^{-s}|P_s f(x) - f(x)|ds + \frac{1}{c_\beta} \int_0^1 s^{-\beta-1}|e^{-s} - 1||f(x)|ds\]

\[
\leq \frac{1}{c_\beta} \int_0^1 s^{-\beta-1}||P_s f - f||_\infty ds + \frac{1}{c_\beta} \int_0^1 s^{-\beta-1}|e^{-s} - 1||f(x)|ds
\]

\[
\leq \frac{A(\alpha)}{c_\beta} \int_0^1 s^{\alpha-\beta-1}ds + \|f\|_\infty \int_0^1 s^{-\beta-1}(1 - e^{-s})ds
\]

\[
= \frac{A(\alpha)}{c_\beta(\alpha - \beta)} + \frac{C\|f\|_\infty}{c_\beta} \leq C_{\alpha, \beta}\|f\|_{Lip_\alpha(\gamma)}
\]

and

\[(II) \leq \frac{1}{c_\beta} \int_1^\infty s^{-\beta-1}[e^{-s}|P_s f(x)| + |f(x)|]ds\]

\[
\leq \frac{2\|f\|_\infty}{c_\beta} \int_1^\infty s^{-\beta-1}ds \leq \frac{2\|f\|_\infty}{\beta c_\beta} \leq C_{\alpha, \beta}\|f\|_{Lip_\alpha(\gamma)},
\]

thus $D^\beta f \in L^\infty$.

Now we want to verify the Lipchitz condition. By Observation 2.2 it is enough to consider the case $0 < t < 1/2$. Using again representation (3.13) we have

\[
\frac{\partial}{\partial t}(P_tD^\beta f(x)) = \frac{1}{c_\beta} \frac{\partial}{\partial t} \left[ \int_0^\infty s^{-\beta-1}(e^{-s}P_{t+s}f(x) - P_t f(x))ds \right]
\]

\[
= \frac{1}{c_\beta} \int_0^\infty s^{-\beta-1}[e^{-s}\frac{\partial P_{t+s}f(x)}{\partial t} - \frac{\partial P_t f(x)}{\partial t}]ds
\]

\[
= \frac{1}{c_\beta} \int_0^t s^{-\beta-1}[e^{-s}\frac{\partial P_{t+s}f(x)}{\partial t} - \frac{\partial P_t f(x)}{\partial t}]ds
\]

\[
+ \frac{1}{c_\beta} \int_t^\infty s^{-\beta-1}[e^{-s}\frac{\partial P_{t+s}f(x)}{\partial t} - \frac{\partial P_t f(x)}{\partial t}]ds
\]

\[
= (III) + (IV).
\]

Using (3.10) we have, for $0 < r < 1$,

\[
\left| \frac{\partial}{\partial t}(P_tD^\beta f(x)) \right| \leq | - e^{-r}\frac{\partial P_r f(x)}{\partial t} | + | e^{-r}\frac{\partial^2 P_r f(x)}{\partial t^2} |
\]

\[
\leq A(\alpha)e^{-r}r^{\alpha-1} + Ae^{-r}r^{\alpha-2} < Ce^{-r}r^{\alpha-2},
\]

and then, by the fundamental theorem of Calculus, for $s < t$

\[
|e^{-(t+s)}\frac{\partial P_{t+s}f(x)}{\partial t} - e^{-t}\frac{\partial P_t f(x)}{\partial t}| \leq \int_t^{t+s} \left| \frac{\partial}{\partial r}(e^{-r}\frac{\partial P_r f(x)}{\partial r}) \right| dr
\]

\[
\leq C \int_t^{t+s} e^{-r}r^{\alpha-2}dr \leq Ce^{-t}t^{\alpha-2}s. \quad (3.14)
\]
Thus
\[|(III)| \leq \frac{e^t}{c_\beta} \int_0^t s^{-\beta-1} \left| e^{-s} \frac{\partial P_{t+s} f(x)}{\partial t} - e^{-t} \frac{\partial P_t f(x)}{\partial t}\right| ds \]
\[\leq \frac{Ce^t}{c_\beta} \int_0^t s^{-\beta-1} e^{-t} t^{\alpha-2} s ds = Ct^{\alpha-2} \int_0^t s^{-\beta} ds = C_{\alpha,\beta} t^{1+\alpha-\beta}.\]

On the other hand, as \(t + s > t\) we have
\[|(IV)| \leq \frac{1}{c_\beta} \int_t^\infty s^{-\beta-1} [e^{-s} \frac{\partial P_{t+s} f(x)}{\partial t} - \frac{\partial P_t f(x)}{\partial t}] ds \]
\[\leq \frac{1}{c_\beta} \int_t^\infty s^{-\beta-1} \left[ s^{-1} (t + s)^{1+\alpha} + t^{1+\alpha}\right] ds \]
\[\leq Ct^{\alpha-1} \int_t^\infty s^{-\beta-1} ds = C_{\alpha,\beta} t^{1+\alpha-\beta}.\]

Therefore,
\[\|\frac{\partial}{\partial t} (P_t D^\beta f)\|_\infty \leq C_{\alpha,\beta} t^{1+\alpha-\beta},\]
which implies \(D^\beta f \in Lip_{\alpha-\beta}(\gamma)\).

Moreover, if \(\beta \geq 1\), let \(k\) be the smallest integer such that \(\beta < k\), then the Riesz
fractional derivative \(D^\beta\) can be represented as
\[D^\beta f = \frac{1}{c_\beta^k} \int_0^\infty s^{-\beta-1} (P_s - I)^k f ds, \tag{3.15}\]
and the Bessel fractional derivative \(\mathcal{D}^\beta\) can be represented as
\[\mathcal{D}^\beta f = \frac{1}{c_\beta^k} \int_0^\infty s^{-\beta-1} (e^{-s} P_s - I)^k f ds, \tag{3.16}\]
where in both cases \(c_\beta^k = \int_0^\infty u^{-\beta-1} (e^{-u} - 1)^k du < \infty\).

Observe that (3.15) and (3.16) are the right formulas since it is easy to prove
form that for any Hermite polynomial \(h_\nu\),
\[D^\beta h_\nu = \nu^{\beta/2} h_\nu, \quad \text{and} \quad \mathcal{D}^\beta h_\nu = (1 + \sqrt{\nu})^{\beta} h_\nu.\]

In this general case we also want to study the action of \(D^\beta\) and \(\mathcal{D}^\beta\) on the
Gaussian Lipschitz spaces,

**Theorem 3.4.** Given \(1 \leq \beta < \alpha\), then

i) The Riesz fractional derivate of order \(\beta\), \(D^\beta : Lip_\alpha(\gamma) \to Lip_{\alpha-\beta}(\gamma)\) is bounded.
The Bessel fractional derivative of order \( \beta \), \( D^\beta : \text{Lip}_\alpha(\gamma) \to \text{Lip}_{\alpha - \beta}(\gamma) \) is bounded.

First of all, observe that, using the binomial theorem and the semigroup property, we have

\[
(P_t - I)^k f(x) = \sum_{j=0}^{k} \binom{k}{j} P^j_t (-I)^j f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j P^k_t f(x)
\]

\[
= \sum_{j=0}^{k} \binom{k}{j} (-1)^j P^{k-j}_t f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j u(x, (k-j)t)
\]

\[
= \Delta^k_t (u(x, \cdot), 0),
\]

where as usual, \( u(x, t) = P_t f(x) \), and

\[
\Delta^k_s (f, t) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(t + (k-j)s)
\]

is the \( k \)-th order forward difference of \( f \) starting at \( t \) with increment \( s \). We will need some technical results about forward differences that will be used later. These are well known results in forward differences’ theory, see for instance [3], but for the sake of completeness, their proofs will be given in an appendix.

**Lemma 3.1.** The forward differences have the following properties:

i) For any positive integer \( k \),

\[
\Delta^k_s (f, t) = \Delta^{k-1}_s (\Delta_s (f, \cdot), t) = \Delta_s (\Delta^{k-1}_s (f, \cdot), t).
\]

ii) For any positive integer \( k \),

\[
\Delta^k_s (f, t) = \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} \int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k) dv_k dv_{k-1} \cdots dv_2 dv_1.
\]

iii) For any positive integer \( k \),

\[
\frac{\partial}{\partial s} (\Delta^k_s (f, t)) = k \Delta^{k-1}_s (f', t + s),
\]

and for any integer \( j > 0 \),

\[
\frac{\partial^j}{\partial t^j} (\Delta^k_s (f, t)) = \Delta^k_s (f^{(j)}, t).
\]

For the proof of Theorem 3.4 we will need estimates analogous to (2.8), (3.11), and those will follow from the next result.
Proposition 3.1. Let \( \delta \) a real number and \( k \) a positive integer such that \( \delta < k \). Let \( f \) be a function such that for some integer \( k \):

\[
|f^{(k)}(r)| \leq C r^{-k+\delta}, \tag{3.22}
\]

then

\[
|\Delta^k_s(f, t)| \leq C s^k t^{-k+\delta}. \tag{3.23}
\]

Proof. The proof is immediate from (3.19), since as \( \delta < k \)

\[
|\Delta^k_s(f, t)| \leq \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} \int_{v_{k-1}}^{v_{k-1}+s} |f^{(k)}(v_k)| \, dv_k \, dv_{k-1} \cdots dv_2 \, dv_1 \leq C \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} \int_{v_{k-1}}^{v_{k-1}+s} v_k^{-k+\delta} \, dv_k \, dv_{k-1} \cdots dv_2 \, dv_1 \leq C s^k t^{-k+\delta}. \]

The following result is a generalization of Proposition 2.3,

Proposition 3.2. We have the following estimates:

i) If \( f \in L^\infty \), for any positive integer \( k \)

\[
|| (P_t - I)^k f ||_\infty \leq 2^k || f ||_\infty. \tag{3.24}
\]

ii) Let \( \alpha > 1 \) and \( n \) be the smallest integer bigger than \( \alpha \). If \( f \in Lip_\alpha(\gamma) \) then

\[
|| (P_t - I)^n f ||_\infty \leq A_\alpha(f) t^{\alpha}. \tag{3.25}
\]

Proof.

i) We already know from (3.17) that

\[
(P_t - I)^k f(x) = \Delta^n_t(u(x, \cdot), 0).
\]

Then for any \( k \) inequality (3.24) is immediate,

\[
|| (P_t - I)^k f ||_\infty \leq \sum_{j=0}^{k} \binom{k}{j} \| P_{(k-j)t} f \|_\infty = 2^k \| f \|_\infty.
\]

ii) Now to prove (3.25) observe \( \alpha - 1 < n - 1 \) and condition (2.5) can be rewritten as

\[
\| \frac{\partial^n}{\partial t^n} u(\cdot, t) \|_\infty = \| \frac{\partial^{n-1}}{\partial t^{n-1}} u'(\cdot, t) \|_\infty \leq A_\alpha(f) t^{-n+1+(\alpha-1)},
\]
i.e. condition (3.22) is satisfied for $\delta = \alpha - 1$, then using (3.20) and then (3.21) with $t = s = r$,

$$
| (P_t - I)^n f(x) | \leq \int_0^t \left| \frac{\partial}{\partial r} (\Delta_r^n (u(x, \cdot), 0)) \right| dr
$$

$$
= n \int_0^t |(\Delta_r^{n-1} (u'(x, \cdot), r))| dr
$$

$$
\leq n A_\alpha \int_0^t r^{n-1} r^{-n+1+(\alpha-1)} dr = C \int_0^t r^{\alpha-1} dr = Ct^{\alpha}. \quad \square
$$

Finally, let us prove Theorem 3.4.

i) Take $f \in \text{Lip}_\alpha(\gamma)$, then $f \in L^\infty$ and $\| \frac{\partial^n u(\cdot,t)}{\partial r^n} \|_\infty \leq A_\alpha(t)^{-n+\alpha}$. Remember that $\beta < \alpha$, $k$ is the smallest integer bigger than $\beta$ and let $n$ be the smallest integer bigger than $\alpha$, note $k \leq n$. Using representation (3.15), and then inequalities (3.24) and (3.25),

$$
|D^\beta f(x)| \leq \frac{1}{c_\beta} \int_0^\infty s^{-\beta-1} |(P_s - I)^k f(x)| ds
$$

$$
= \frac{1}{c_\beta} \int_0^1 s^{-\beta-1} |(P_s - I)^k f(x)| ds + \frac{1}{c_\beta} \int_1^\infty s^{-\beta-1} |(P_s - I)^k f(x)| ds
$$

$$
= (I) + (II).
$$

Now, let us assume $k < n$, if $k = n$ the argument is straightforward. Let $\varepsilon > 0$ such that $\beta + \varepsilon < k$, by Proposition 2.2 $\text{Lip}_\alpha(\gamma) \subset \text{Lip}_{\beta+\varepsilon}(\gamma)$, using (3.25),

$$
(I) \leq \frac{1}{c_\beta} \int_0^1 s^{-\beta-1} \| (P_s - I)^k f \|_\infty ds \leq \frac{A_{\beta+\varepsilon}(f)}{c_\beta^k} \int_0^1 s^{\beta+\varepsilon-\beta-1} ds = \frac{A_{\beta+\varepsilon}(f)}{c_\beta^{k-\varepsilon}}.
$$

On the other hand

$$
(II) \leq \frac{1}{c_\beta} \int_1^\infty s^{-\beta-1} \| (P_s - I)^k f \|_\infty ds \leq \frac{2 k \| f \|_\infty}{c_\beta^k} \int_1^\infty s^{-\beta-1} ds = C_\beta \| f \|_\infty.
$$

Thus $D^\beta f \in L^\infty$.

Now we want to verify the Lipchitz condition. Observe that by the semigroup property,

$$
P_t [(P_s - I)^k f(x)] = P_t (\Delta_s^k (u(x, \cdot), 0)) = P_t \left( \sum_{j=0}^{k} \binom{k}{j} (-1)^j P_{(k-j)s} f(x) \right)
$$

$$
= \sum_{j=0}^{k} \binom{k}{j} (-1)^j P_{(k-j)s} f(x) = \Delta_s^k (u(x, \cdot), t).
$$
Fixing $t > 0$, using again representation (3.15) and (3.21), we have

$$
\frac{\partial^n (P_t D^\beta f(x))}{\partial t^n} = \frac{1}{c^n_{\beta}} \frac{\partial^n}{\partial t^n} \left[ \int_0^{\infty} s^{-\beta-1} P_t [(P_s - I)^k f(x)] ds \right]
$$

$$
= \frac{1}{c^n_{\beta}} \int_0^{\infty} s^{-\beta-1} \frac{\partial^n}{\partial t^n} [\Delta^k_s (u(x, \cdot), t)] ds
$$

$$
= \frac{1}{c^n_{\beta}} \int_0^t s^{-\beta-1} [\Delta^k_s (u^{(n)}(x, \cdot), t)] ds
$$

$$
+ \frac{1}{c^n_{\beta}} \int_0^{\infty} s^{-\beta-1} [\Delta^k_s (u^{(n)}(x, \cdot), t)] ds
$$

$$
= (III) + (IV).
$$

Now, by Proposition 2.1 we have from (2.5),

$$
\| \frac{\partial^k}{\partial t^k} (u^{(n)}(\cdot, t)) \|_\infty = \| \frac{\partial^{n+k}}{\partial t^{n+k}} (u(\cdot, t)) \|_\infty \leq At^{(n+k)+\alpha} = At^{-k+(\alpha-n)},
$$

then by (3.23)

$$
(III) \leq \frac{1}{c^n_{\beta}} \int_0^t s^{-\beta-1} |\Delta^k_s (u^{(n)}(x, \cdot), t)| ds
$$

$$
\leq \frac{A t^{-k+(\alpha-n)}}{c^n_{\beta}} \int_0^t s^{-\beta+k-1} ds = C_{\alpha, \beta} t^{-k+\alpha-n-\beta+k} = C_{\alpha, \beta} t^{n+\alpha-\beta}.
$$

On the other hand,

$$
|\Delta^k_s (u^{(n)}(x, \cdot), t)| \leq \sum_{j=0}^k \binom{k}{j} |u^{(n)}(x, t + (k-j)s)|
$$

$$
\leq A_\alpha (f) \sum_{j=0}^k \binom{k}{j} |(t + (k-j)s)^{-n+\alpha}| \leq Ct^{-n+\alpha}
$$

and then,

$$
(IV) \leq \frac{1}{c^n_{\beta}} \int_0^{\infty} s^{-\beta-1} |\Delta^k_s (u^{(n)}(x, \cdot), t)| ds
$$

$$
\leq \frac{C t^{-n+\alpha}}{c^n_{\beta}} \int_t^{\infty} s^{-\beta-1} ds = C_{\alpha, \beta} t^{-n+\alpha-\beta}.
$$

Therefore,

$$
\| \frac{\partial^n (P_t D^\beta f)}{\partial t^n} \|_\infty \leq C t^{-n+\alpha-\beta},
$$

and since $\alpha - \beta < n$ by Proposition 2.1 this implies $D^\beta f \in Lip_{\alpha-\beta}(\gamma)$. 
ii) Take \( f \in \text{Lip}_\alpha(\gamma) \), then \( f \in L^\infty \) and \( \| \frac{\partial^n u(t)}{\partial t^n} \|_\infty \leq A_\alpha(f) t^{-n+\alpha} \). Remember that \( \beta < \alpha \), \( k \) is the smallest integer bigger than \( \beta \) and let \( n \) be the smallest integer bigger than \( \alpha \). Observe that from (3.24) and (3.25), we get,

\[
\| D^2 f(x) \| \leq \frac{1}{c_\beta} \int_0^\infty s^{-\beta-1} |(e^{-s} P_s - I)^k f(x)| ds \\
= \frac{1}{c_\beta} \int_0^1 s^{-\beta-1} |(e^{-s} P_s - I)^k f(x)| ds \\
+ \frac{1}{c_\beta} \int_1^\infty s^{-\beta-1} |(e^{-s} P_s - I)^k f(x)| ds \\
= (I) + (II).
\]

Now, let us assume \( k < n \), if \( k = n \) the argument is straightforward. Let \( 0 < \varepsilon < 1/2 \) such that \( \beta + \varepsilon < k \), by Proposition 2.2 \( \text{Lip}_\alpha(\gamma) \subset \text{Lip}_{\beta+\varepsilon}(\gamma) \subset \text{Lip}_{\beta-\varepsilon}(\gamma), j = 1, 2, \cdots, k - 1 \). Then, using the identity

\[
(e^{-s} P_s - I)^k f(x) = \sum_{j=0}^{k} \binom{k}{j} e^{-js}(P_s - I)^j (e^{-s} - 1)^{k-j} f(x)
\]

and (3.25), we get

\[
(I) \leq \frac{1}{c_\beta} \sum_{j=0}^{k} \binom{k}{j} \int_0^1 s^{-\beta-1} |e^{-js}(P_s - I)^j| ds \\
\leq \frac{1}{c_\beta} \sum_{j=0}^{k} \binom{k}{j} \int_0^1 s^{-\beta-1} |e^{-js}(P_s - I)^j f| ds \\
\leq \frac{1}{c_\beta} \sum_{j=0}^{k} \binom{k}{j} \int_0^1 s^{-\beta-1} s^{k-j} |(P_s - I)^j f| ds \\
\leq \frac{1}{c_\beta} \int_0^1 s^{k-\beta-1} ds \| f \|_\infty + \sum_{j=1}^{k-1} \binom{k}{j} \frac{A_{j-\varepsilon}(f)}{c_\beta} \int_0^1 s^{k-\beta-1} s^{j-\varepsilon} ds \\
+ \frac{A_{\beta+\varepsilon}(f)}{c_\beta} \int_0^1 s^{\beta+\varepsilon-\beta-1} ds \\
= \frac{C}{c_\beta(k-\beta)} \| f \|_\infty + \sum_{j=1}^{k-1} \binom{k}{j} \frac{A_{j-\varepsilon}(f)}{c_\beta(k-\beta-\varepsilon)} + \frac{A_{\beta+\varepsilon}(f)}{c_\beta \varepsilon} \\
\leq C_\beta \| f \|_{\text{Lip}_\alpha(\gamma)},
\]

and

\[
(II) \leq \frac{1}{c_\beta} \int_1^\infty s^{-\beta-1} \sum_{j=0}^{k} \binom{k}{j} |e^{-(k-j)s}| \| P_{(k-j)s} f \| \| f \|_\infty ds \\
\leq \frac{\| f \|_\infty}{c_\beta} \int_1^\infty s^{-\beta-1} (1 + e^{-s})^k ds \leq \frac{2^k \| f \|_\infty}{\beta c_\beta} \leq C_\beta \| f \|_{\text{Lip}_\alpha(\gamma)}.
\]
Thus $D^\beta f \in L^\infty$.

Now we want to verify the Lipchitz condition. By Observation 2.2 it is enough to consider the case $0 < t < 1$. Observe that by the semigroup property,

$$P_t[(e^{-s}P_s - I)^k f(x)] = P_t(\sum_{j=0}^k \binom{k}{j}(-1)^j e^{-(k-j)s} P_{(k-j)s} f(x))$$

$$= \sum_{j=0}^k \binom{k}{j}(-1)^j e^{-(k-j)s} P_{t+(k-j)s} f(x)$$

$$= \sum_{j=0}^k \binom{k}{j}(-1)^j e^{-(k-j)s} u(x, t + (k - j)s),$$

then

$$\frac{\partial^n}{\partial t^n} [P_t(e^{-s}P_s - I)^k f(x)] = \sum_{j=0}^k \binom{k}{j}(-1)^j e^{-(k-j)s} u^{(n)}(x, t + (k - j)s).$$

Therefore, using again representation (3.15),

$$\frac{\partial^n}{\partial t^n} (P_t D^\beta f(x)) = \frac{1}{c^k_\beta} \frac{\partial^n}{\partial t^n} \int_0^\infty s^{-\beta-1} P_t[(P_s - I)^k f(x)] ds$$

$$= \frac{1}{c^k_\beta} \frac{\partial^n}{\partial t^n} \int_0^t s^{-\beta-1} P_t[(P_s - I)^k f(x)] ds$$

$$+ \frac{1}{c^k_\beta} \frac{\partial^n}{\partial t^n} \int_t^\infty s^{-\beta-1} P_t[(P_s - I)^k f(x)] ds$$

$$= \frac{1}{c^k_\beta} \int_0^t s^{-\beta-1} \sum_{j=0}^k \binom{k}{j}(-1)^j e^{-(k-j)s} u^{(n)}(x, t + (k - j)s) ds$$

$$+ \frac{1}{c^k_\beta} \int_t^\infty s^{-\beta-1} \sum_{j=0}^k \binom{k}{j}(-1)^j e^{-(k-j)s} u^{(n)}(x, t + (k - j)s) ds$$

$$= (III) + (IV).$$

Using (3.21), we have

$$|(III)| = \frac{e^t}{c^k_\beta} \int_0^t s^{-\beta-1} \sum_{j=0}^k \binom{k}{j}(-1)^j e^{-t-(k-j)s} u^{(n)}(x, t + (k - j)s) ds$$

$$= \frac{e^t}{c^k_\beta} \int_0^t s^{-\beta-1} |\Delta^k_s (e^{-(n)} u(x, \cdot), t)| ds.$$
and then using the Leibnitz formula, and the fact that $0 < t < 1$

$$\left| \frac{\partial^k [e^{-t}u^{(n)}(x,t)]]}{\partial t^k} \right| = \left| e^{-t} \sum_{j=0}^{k} \binom{k}{j} (-1)^j u^{n+(k-j)}(x,t) \right|$$

$$\leq e^{-t} \sum_{j=0}^{k} \binom{k}{j} |u^{n+(n-j)}(x,t)|$$

$$\leq Ce^{-t} \sum_{j=0}^{k} \binom{k}{j} t^{-(n+(k-j))}$$

$$= Ce^{-t} t^{-n+\alpha} \sum_{j=0}^{k} \binom{k}{j} t^{-(k-j)}$$

$$\leq Ce^{-t} t^{-n+\alpha} 2^k t^{-k} = Ce^{-t} t^{-(n+k)+\alpha}.$$

Then with a small variation of the argument of the proof of (3.23) we get

$$|\Delta_s^k (e^{-u^{(n)}}(x,\cdot),t)| \leq Ct^{-(n+k)+\alpha} e^{-t}s^k,$$

and therefore

$$|(III)| \leq \frac{C t^{-(n+k)+\alpha}}{c_\beta^k} \int_0^t s^{-\beta+k-1} ds = C_{\alpha,\beta} t^{-(n+k)-\alpha-\beta+k} = C_{\alpha,\beta} t^{-(n+k)-\beta}.$$  

On the other hand,

$$|(IV)| \leq \frac{1}{c_\beta^k} \int_t^\infty s^{-\beta-1} \sum_{j=0}^{k} \binom{k}{j} e^{-(k-j)s} |u^{(n)}(x, t + (k-j)s)| ds$$

$$\leq \frac{1}{c_\beta^k} \int_t^\infty s^{-\beta-1} 2^k (t + (k-j)s)^{-n+\alpha} ds$$

$$\leq Ct^{-n+\alpha} \int_t^\infty s^{-\beta-1} ds = C_{\alpha,\beta} t^{-n+\alpha-\beta}.$$

Therefore, we can conclude that

$$\| \frac{\partial}{\partial t} (P_t D^{\beta} f) \|_{\infty} \leq C_{\alpha,\beta} t^{-n+\alpha-\beta},$$

and again, since $\alpha - \beta < n$ by Proposition 2.1 this implies $D^{\beta} f \in Lip_{\alpha-\beta}(\gamma)$.

\[\]

**Observation 3.1.** Let us observe that the arguments given in the proofs of Theorems 2.1 and 2.2 are valid in the classical case taking the Poisson integral, and therefore they are alternative proofs of the ones given in [14].
Observation 3.2. Moreover, if instead of considering the Ornstein-Uhlenbeck operator \((1.1)\) and the Poisson-Hermite semigroup \((1.6)\) we consider the Laguerre differential operator in \(\mathbb{R}^d_+\).

\[
\mathcal{L}^\alpha = \sum_{i=1}^{d} \left[ x_i \frac{\partial^2}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial}{\partial x_i} \right],
\]

and the corresponding Poisson-Laguerre semigroup, or if we consider the Jacobi differential operator in \((-1, 1)^d,\)

\[
L^{\alpha, \beta} = -\sum_{i=1}^{d} \left[ (1 - x_i^2) \frac{\partial^2}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2) x_i) \frac{\partial}{\partial x_i} \right],
\]

and the corresponding Poisson-Jacobi semigroup (for details we refer to [17]), the arguments are completely analogous. That is to say, we can defined in analogous manner Laguerre-Lipschitz spaces and Jacobi-Lipschitz spaces and prove that the corresponding notions of fractional integrals and fractional derivatives, see [8], [1], behave similarly. In order to see this it is more convenient to use the representation \((1.6)\) of \(P_t\) in terms of the one-sided stable measure \(\mu_t^{(1/2)}(ds)\) and the write Lemma 1.1 in terms of it, see [11].

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4. Appendix.

Proof of Lemma 3.1.

i) Let us prove the first equality, the second one is totally analogous,

\[
\Delta_s^{k-1}(\Delta_s(f, \cdot), t) = \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j \Delta_s(f, t + (k - 1 - j)s) \\
= \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j f(t + (k - j)s) \\
- \sum_{j=0}^{k-1} \binom{k-1}{j} (-1)^j f(t + (k - 1 - j)s) \\
= f(t + ks) + \sum_{j=1}^{k-1} \binom{k-1}{j} (-1)^j f(t + (k - j)s) \\
+ \sum_{j=0}^{k-2} \binom{k-1}{j} (-1)^{(j+1)} f(t + (k - (j + 1))s) + (-1)^k f(t)
\]
\[
= f(t + ks) + \sum_{j=1}^{k-1} \binom{k-1}{j} (-1)^j f(t + (k-j)s) \\
+ \sum_{j=1}^{k-1} \binom{k-1}{j+1} (-1)^j f(t + (k-j)s) + (-1)^k f(t)
\]

ii) By induction in \( k \). For \( k = 1 \), using the fundamental theorem
\[
\Delta_s(f, t) = f(t + s) - f(t) = \int_t^{t+s} f'(v) dv.
\]

Let us assume that the identity is true for \( k - 1 \),
\[
\Delta_s^{k-1}(f, t) = \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} f^{(k-1)}(v_{k-1}) dv_{k-1} \cdots dv_2 dv_1,
\]
and let us prove it for \( k \). Using i) and the fundamental theorem, we get, after performing \( k - 1 \) change of variables,
\[
\Delta_s^k(f, t) = \Delta_s(\Delta_s^{k-1}(f, t), t) = \Delta_s^{k-1}(f, t + s) - \Delta_s^{k-1}(f, t)
\]
\[
= \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} f^{(k-1)}(v_{k-1}) dv_{k-1} \cdots dv_2 dv_1
\]
\[
- \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} f^{(k-1)}(v_{k-1}) dv_{k-1} \cdots dv_2 dv_1
\]
\[
= \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+2s} f^{(k-1)}(v_{k-1}) dv_{k-1} \cdots dv_2 dv_1
\]
\[
- \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} f^{(k-1)}(v_{k-1}) dv_{k-1} \cdots dv_2 dv_1
\]
\[
= \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+2s} f^{(k-1)}(v_{k-1}) dv_{k-1} \cdots dv_2 dv_1
\]
\[
- \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} f^{(k-1)}(v_{k-1}) dv_{k-1} \cdots dv_2 dv_1
\]
\[
= \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+2s} f^{(k-1)}(v_{k-1}) dv_{k-1} \cdots dv_2 dv_1
\]
\[
- \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} f^{(k-1)}(v_{k-1}) dv_{k-1} \cdots dv_2 dv_1
\]
\[
= \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+2s} [f^{(k-1)}(v_{k-1} + s) - f^{(k-1)}(v_{k-1})] dv_{k-1} \cdots dv_2 dv_1
\]
\[
= \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-2}}^{v_{k-2}+s} f^{(k)}(v_{k}) dv_{k} dv_{k-1} \cdots dv_2 dv_1.
\]
iii) Let us prove (3.20),
\[
\frac{\partial}{\partial s}(\Delta_k^s(f, t)) = D_s\left(\sum_{j=0}^{k}\binom{k}{j}(-1)^j f(t + (k-j)s)\right)
\]
\[
= \sum_{j=0}^{k}\binom{k}{j}(-1)^j \frac{\partial}{\partial s}(f(t + (k-j)s))
\]
\[
= \sum_{j=0}^{k-1}\binom{k}{j}(-1)^j (k-j)f'(t + (k-j)s)
\]
\[
= k \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j f'((t + s) + (k-1-j)s)
\]
\[
= k \Delta_{k-1}^s(f', t + s).
\]

Now, let us prove (3.21)
\[
\frac{\partial^j}{\partial t^j}(\Delta_k^s(f, t)) = \frac{\partial^j}{\partial t^j}\left(\sum_{j=0}^{k}\binom{k}{j}(-1)^j f(t + (k-j)s)\right)
\]
\[
= \sum_{j=0}^{k}\binom{k}{j}(-1)^j \frac{\partial^j}{\partial t^j}(f(t + (k-j)s))
\]
\[
= \sum_{j=0}^{k}\binom{k}{j}(-1)^j f^{(j)}(t + (k-j)s)
\]
\[
= \Delta_k^s(f^{(j)}, t).
\]

\[
\square
\]

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