Learning-Based Adaptive Control for Stochastic Linear Systems with Input Constraints

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Abstract—We propose a certainty-equivalence scheme for adaptive control of scalar linear systems subject to additive, i.i.d. Gaussian disturbances and bounded control input constraints, without requiring prior knowledge of the bounds of the system parameters, nor the control direction. Assuming that the system is at-worst marginally stable, mean square boundedness of the closed-loop system states is proven. Lastly, numerical examples are presented to illustrate our results.

I. INTRODUCTION

Adaptive control is useful for stabilizing dynamical systems with known model structure but unknown parameters. However, a major problem that arises when deploying controllers is actuator saturation, which, when unaccounted for, can result in failure to achieve stability. Moreover, in some systems, large, stochastic disturbances may occasionally perturb the system, which needs to be considered in controller design. This motivates the need to develop controllers which can stabilize systems with unknown parameters, whilst simultaneously handling both control input constraints and additive, unbounded, stochastic disturbances.

In recent years, there has been renewed interest in discrete-time (DT) stochastic adaptive control. One reason is the recent successes in online model-based reinforcement learning — especially for the online linear quadratic regulation (LQR) task, where the goal is to apply controls on an unknown linear stochastic system to minimize regret with respect to the optimal LQR controller in a single trajectory (e.g. see [1], [2], [3]). However, input constraints are not considered in these works. Recent results in [4] address state and input constraints, but assume bounded disturbances, and require an a priori known controller guaranteeing stability and constraint satisfaction. DT stochastic extremum seeking (ES) results have shown promise, and recently been applied beyond steady-state input-output maps to stabilize open-loop unstable systems (see [5], [6]), but they also do not consider input constraints. Looking back to classic results in stochastic adaptive control such as [7], [8], [9], many challenges have been addressed, but stochastic stability results considering bounded control constraints with unbounded disturbances for marginally stable plants are missing, despite almost all real systems having actuator constraints.

Beyond DT stochastic adaptive control, various works consider input-constrained linear systems. Although seemingly simple, the stability analysis of controllers for input-constrained, marginally stable, linear plants, with unbounded disturbances, is non-trivial, due to their nonlinear, stochastic, closed-loop dynamics. Results reporting mean square boundedness with arbitrary positive input constraints for known systems were not available until after 2012 in [10] and [11]. The adaptive control of unknown, DT output-feedback linear systems subject to bounded control constraints and bounded disturbances is considered in [12], [13], [14], and [15]. These works derive deterministic guarantees on at least the boundedness of the output under various conditions, but require bounded disturbances.

Although adjacent settings have been considered, to the best of our knowledge, no works address the problem of adaptive control for DT linear systems subject to control input constraints and unbounded disturbances with proven stability guarantees. We move towards filling this gap by addressing this task specifically for scalar, at-worst marginally stable linear systems, with additive i.i.d. zero-mean Gaussian disturbances. Our main contributions are twofold:

Firstly, we propose a certainty-equivalence (CE) adaptive control scheme comprised of an ordinary least squares-based plant parameter estimator component, in connection with an excited, saturated deadbeat controller based on the estimated parameters. Our controller is capable of satisfying any positive upper bound constraint on the magnitude of the control input. Moreover, it does not assume prior knowledge of bounds for the system parameters, nor does it require knowledge of the control direction — that is, the sign of the control parameter, which is a common assumption in adaptive control.

Secondly, we establish the mean square boundedness of the closed-loop system when applying our proposed control scheme to the system of interest. Despite the restricted problem setting, it is still non-trivial since saturated controls render the system nonlinear, and in general CE control does not stabilize nonlinear systems [16]. We overcome this difficulty by showing that our control scheme satisfies sufficient excitation conditions required to establish an upper bound on the probability that the parameter estimate lies outside a small ball around the true parameter, making use of results in non-asymptotic learning from [17]. We subsequently show that this upper bound decays sufficiently fast, allowing us to prove via a novel analysis that mean square boundedness holds. Typically, persistence of excitation is assumed in the control literature to establish parameter convergence,
whereas we explicitly demonstrate satisfaction of excitation conditions, which is non-trivial even in the scalar case due to the nonlinear, stochastic nature of our system. Moreover, establishing mean square boundedness is difficult due to the unbounded nature of the disturbances and saturated controls, requiring specialized results from [18] in our analysis.

**Notation:** Let $\mathbb{N}$ denote the set of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathbb{R}$ denote the real numbers, and $\mathbb{R}_{\geq} := [0, \infty)$. For $x \in \mathbb{R}$, we define $x^+ := \max(0, x)$. It has the properties 1) $x = x^+ - (-x)^+$ and 2) $x^+ - (-x)^+ = 0$ for all $x \in \mathbb{R}$. Let $S^{d-1}$ denote the unit sphere in $\mathbb{R}^d$. For $r > 0$, we define the saturation function $\sigma_r : \mathbb{R} \to \mathbb{R}$ by $\sigma_r(x) := x$ if $|x| \leq r$, and $\sigma_r(x) := x/\|x\|$ if $|x| > r$. For a square matrix $A \in \mathbb{R}^{d \times d}$, let $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalue of $A$ respectively. For symmetric matrices $A, B \in \mathbb{R}^{d \times d}$, we denote $A \prec B$ ($\preceq$) if $A - B$ is negative definite (semi-definite). Let erf$^{-1}$ denote the error function, and $\text{erfc}$ denote the complementary error function. Consider a probability space $(\Omega, F, P)$, and a random variable $X : \Omega \to \mathbb{R}$, sub-sigma-algebra $\mathcal{G} \subseteq F$, and events $A, B \in F$ defined on this space. Let $\mathbb{E}[-]$ denote the expectation operator. We say $X|\mathcal{G}$ is $\Sigma^2$-sub-Gaussian if $\mathbb{E}[e^{\lambda X|\mathcal{G}}] \leq e^{\lambda^2 \Sigma^2/2}$ for all $\lambda \in \mathbb{R}$. For an event $A \in F$, we define the indicator function $1_A : \Omega \to \{0, 1\}$ as $1_A := 1$ on the event $A$, and $1_A := 0$ on the event $A^C$. If $X$ takes values in $\mathbb{R}_{\geq 0}$, then $X1_{A \cup B} = \max(X1_A, X1_B)$. $\dagger$ denotes the Moore-Penrose inverse.

**II. PROBLEM SETUP**

Consider the stochastic scalar linear system:

$$
X_{t+1} = aX_t + bU_t + W_t, \quad t \in \mathbb{N}_0, \quad X_0 = x_0
$$

(1)

where the random sequences $(X_t)_{t \in \mathbb{N}_0}$, $(U_t)_{t \in \mathbb{N}_0}$ and $(W_t)_{t \in \mathbb{N}_0}$ are the states, controls and disturbances taking values in $\mathbb{R}$, $x_0 \in \mathbb{R}$ is the initial state, and $\theta^* = (a, b) \in \mathbb{R}^2$ are the system parameters. Throughout the paper, all random variables are defined on a probability space $(\Omega, F, P)$. Moreover, denote the $i$-th moment of the disturbance as $S_i := \mathbb{E}[|W_t|^i]$ for $i \in \mathbb{N}$. We make the following assumptions on the system in [1].

**A1.** The disturbance sequence $(W_t)_{t \in \mathbb{N}_0}$ is sampled $W_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \Sigma_W)$ where $\Sigma_W > 0$ is the variance;

**A2.** The system parameters $(a, b)$ satisfy $|a| \leq 1$ and $b \neq 0$.

**Remark.** A1 is selected since Gaussian random variables practically model many types of disturbances due to their unbounded support, which can represent rare events that cause arbitrarily large disturbances in real systems. A2 ensures the existence of control policies with bounded control constraints that render the system mean square bounded, as proven in [10]. Heuristically speaking, this is because A2 ensures global null-controllability in the deterministic setting, which is intuitively important because A1 can cause arbitrarily large jumps in the state.

Our goal is to formulate an adaptive control policy $(\pi_t)_{t \in \mathbb{N}_0}$ such that $\pi_t$ is a mapping from current and past state and control input data $(X_0, \ldots, X_t, U_0, \ldots, U_{t-1})$ and a randomization term $V_t$ to $\mathbb{R}$ for $t \in \mathbb{N}_0$, where $(V_t)_{t \in \mathbb{N}_0}$ taking values in $\mathbb{R}$ is an i.i.d. random sequence. We allow for stochastic policies (i.e. dependence on $V_t$) to excite the system and facilitate parameter convergence. Moreover, we require as part of our design that $\pi_t$ does not depend on the system parameters $(a, b)$, and that the following requirements are satisfied on the closed-loop system with $U_t = \pi_t(X_0, \ldots, X_t, U_0, \ldots, U_{t-1}, V_t)$ for $t \in \mathbb{N}_0$:

**G1.** The magnitude of the control input remains bounded by a desired constraint level: $|U_t| \leq U_{\text{max}}$ for all $t \in \mathbb{N}_0$, where $U_{\text{max}} > 0$ is a user specified constraint level;

**G2.** For the stochastic process $(X_t)_{t \in \mathbb{N}_0}$, mean-square boundedness is achieved: $\exists \varepsilon > 0 : \sup_{t \in \mathbb{N}_0} \mathbb{E}[X_t^2] \leq \varepsilon$.

In practice, $U_{\text{max}}$ is chosen based on the maximum control input the actuator can provide.

**III. METHOD AND MAIN RESULT**

The control strategy we employ to achieve G1 and G2 is summarized in Algorithm 1.

**Algorithm 1 AICMSS (Adaptive Input-Constrained Mean Square Stabilization)**

1: **Inputs:** $U_{\text{max}} > 0$, $C < U_{\text{max}}$, $\hat{a}_{\text{init}} \in \mathbb{R}$, $\hat{b}_{\text{init}} \in \mathbb{R}\setminus\{0\}$
2: $D \leftarrow U_{\text{max}} - C$
3: Measure $X_0$
4: for $t = 0, 1, \ldots$ do
5: Sample $V_t \overset{i.i.d.}{\sim} \text{Uniform}([-C, C])$
6: Compute control $U_t$ following (2)
7: Apply $U_t$ to (1)
8: Measure $X_{t+1}$ from (1)
9: if $t \geq 1$ then Compute $(\hat{a}_t, \hat{b}_t)$ following (3)
10: end for

We now describe our strategy in greater detail. The sequence of control inputs $(U_t)_{t \in \mathbb{N}_0}$ is given by the policy:

$$
U_t := \sigma_D (G_tX_t) + V_t,
$$

(2)

for $t \in \mathbb{N}_0$, where $G_t := -\hat{a}_{\text{init}}/\hat{b}_{\text{init}}$ for $t \leq 1$ and $G_t := -\hat{a}_{t-1}/\hat{b}_{t-1}$ for $t \geq 2$ is the gain factor, and $V_t \overset{i.i.d.}{\sim} \text{Uniform}([-C, C])$ is an additive excitation term independent of $W_t$. Moreover, $\hat{\theta}_t = (\hat{a}_t, \hat{b}_t) \in \mathbb{R}^2$ is the parameter estimate at time $t \in \mathbb{N}$ obtained via least squares estimation:

$$
\hat{\theta}_t = (Z_t^\top Z_t)^{-1} Z_t^\top X_{t+1}
$$

$\in \arg \min_{\theta \in \mathbb{R}^2} \sum_{s=1}^{t} \|X_{s+1} - \theta^\top Z_s\|^2$,  

(3)

where $(Z_t)_{t \in \mathbb{N}_0} \subseteq \mathbb{R}^2$ is the state-input data sequence:

$$
Z_t := (X_t, U_t), \quad t \in \mathbb{N}_0,
$$

(4)

and $Z_t = [Z_1, \ldots, Z_t]^\top$, $X_{t+1} = [X_2, \ldots, X_{t+1}]^\top$. The initial parameter estimate $\hat{\theta}_{\text{init}} = (\hat{a}_{\text{init}}, \hat{b}_{\text{init}})$ is freely chosen by the designer in $\mathbb{R} \times \mathbb{R}\setminus\{0\}$. Here, $C$ is a user-specified excitation constant satisfying $0 < C < U_{\text{max}}$ which determines
the size of the excitation term, and \( D = \max C \) determines the certainty-equivalence component of the control policy.

Under this control strategy, the states of the closed-loop system evolve as,

\[
X_{t+1} = aX_t + b(G_tX_t) + V_t + W_t.
\]

The intuition behind our control strategy is the following; we estimate the system parameters \((a, b)\) from past data in (3), and use this estimate for certainty-equivalent control in (2). The presence of the injected excitation term \( V_t \) is to ensure the data \( Z_t \) is exciting so that convergence of parameter estimates \((\hat{a}, \hat{b})\) occurs, which is a key component of our analysis later in Theorem 1. This is in contrast to \( W_t \), which is an external system disturbance.

The control input (2) always satisfies (G1) since for all \( t \in \mathbb{N}_0 \), \( |\sigma_D(G_tX_t)| \) is bounded by \( D \), \( |V_t| \) is bounded by \( C \), and \( U_{\max} = C + D \). On the other hand, Theorem 1 states that the closed-loop system under our control law following Algorithm 1 satisfies mean-square boundedness (G2). We now provide Theorem 1.

**Theorem 1.** Assuming A1-A2 hold and \( x_0 \in \mathbb{R} \), the states \((X_t)_{t \in \mathbb{N}_0} \) corresponding to the closed-loop system (5) under our control strategy in Algorithm 1 satisfies

\[
\exists \epsilon > 0 : \sup_{t \in \mathbb{N}_0} \mathbb{E}[X_t^2] < \epsilon.
\]

**IV. PROOF OF MAIN RESULT**

We start by providing the main ideas behind the proof of Theorem 1. We then state supporting lemmas and a sketch of their proofs, before providing the formal proof of Theorem 1.

**A. Proof Idea for Theorem 1**

Firstly, let us define the block martingale small-ball (BMSB) condition.

**Definition 1.** (Martingale Small-Ball [17, Definition 2.1]) Given a process \((Z_t)_{t \geq 1}\) taking values in \( \mathbb{R}^2 \), we say that it satisfies the \((k, \Gamma_{ab}, p)\)-martingale small-ball (BMSB) condition for \( k \in \mathbb{N} \), \( \Gamma_{ab} > 0 \), and \( p > 0 \), if, for any \( \zeta \in \mathcal{S}^1 \) and \( \gamma > 0 \), \( \frac{1}{n} \sum_{i=1}^{k} P(|(i, Z_{i+j})| > \sqrt{k} \Gamma_{ab} \gamma) > p \) holds. Here, \((F_t)_{t \geq 1}\) is any filtration which \((\zeta, Z_t)_{t \geq 1}\) is adapted to.

This condition is related to the ‘excitability’ of some random sequence \((Z_t)_{t \geq 1}\) — intuitively, that is, given past observations of the sequence \( Z_1, \ldots, Z_j \), how spread out is the conditional distribution of future observations. Result 1 in Lemma 2 establishes that our state-input data sequence \((Z_t)_{t \in \mathbb{N}}\) satisfies the \((1, \Gamma_{ab}, p)\)-BMSB for some parameters \( p, \Gamma_{ab} \), which in turn implies a high probability lower bound on \( \lambda_{\min} \left(\sum_{i=1}^{k} Z_i Z_i^\top\right) \) holds [17]. Moreover, 2) in Lemma 2 provides a high-probability upper bound on \( \lambda_{\max} \left(\sum_{i=1}^{k} Z_i Z_i^\top\right) \). These bounds are important for deriving a high probability upper bound on the parameter estimation error when applying least squares estimation to a general time-series with linear responses (Theorem 2.4 in [17]). We provide the specialization to the case of covariates in \( \mathbb{R}^2 \) and responses in \( \mathbb{R} \) in Proposition 5. We subsequently rely on this result to provide Lemma 4, which gives an upper bound on the probability that the parameter estimate \( \hat{\theta}_t \) lies outside a ball of size \( d > 0 \) centered at the true parameter \( \theta_0 \) for sufficiently small \( d \) for \( t \in \mathbb{N} \). Finally, Lemma 5 says that so long as the aforementioned probability decays sufficiently fast, then our certainty-equivalent control strategy in (2) results in uniform boundedness of the mean squared states of the closed-loop system. Our proof of Theorem 1 concludes by showing that the probability upper bound established in Lemma 4 decays sufficiently fast, satisfying the premise of Lemma 2.

**Remark.** Larger \( C \) is related to improved ‘learnability’ properties via a larger existent \( p \) and \( \Gamma_{ab} \) in Lemma 2 contributing to a larger \( c_3 \) in Lemma 4 and hence faster exponential decay for the upper bound on \( P(||\hat{\theta}_t - \theta_0||_2 > d) \).

Only proof sketches that capture the main ideas are provided for the lemmas. Readers are referred to the Supplementary Materials for lengthy proofs of supporting results.

**B. Supporting Results**

**Lemma 2.** Suppose A1-A2 hold on the closed-loop system (5) and \( x_0 \in \mathbb{R} \). The following results hold on the sequence \((Z_t)_{t \in \mathbb{N}_0}\) from (4):

1) There exist \( p > 0 \) and \( \Gamma_{ab} > 0 \) such that \((Z_t)_{t \in \mathbb{N}}\) satisfies the \((1, \Gamma_{ab}, p)\)-BMSB condition;

2) \( P \left( \sum_{i=1}^{T} Z_i Z_i^\top \leq \frac{1}{2} i((i||b|| U_{\max} + \Sigma W) + |x_0|)^2 + U_{\max}^2 I \right) \leq \delta \) holds for all \( \delta > 0 \) and \( p, \Gamma_{ab} \).

**Proof Sketch.** For 1), let \((F_t)_{t \geq 0}\) be the natural filtration of \((Z_t)_{t \in \mathbb{N}_0}\). Let \( \gamma > 0 \) satisfy \( \mathbb{E}[|(\zeta^\top Z_{i+j})|] > \gamma \) for all \( \zeta \in \mathcal{S}^1 \) and \( t \in \mathbb{N}_0 \), where the existence of satisfactory values is established in Lemma 6 with the aid of a computer algebra system (CAS). For all \( \zeta = (\zeta_1, \zeta_2) \in \mathcal{S}^1 \) and \( t \in \mathbb{N}_0 \), \( \text{Var}(\zeta^\top Z_{t+1} | F_t) \leq 2(\sqrt{2} U_{\max}^2 + U_{\max}^2) \). Following an improvement of the Paley-Zygmund Inequality via the Cauchy-Schwarz inequality and making use of Jensen’s inequality,

\[
P \left( (\zeta^\top Z_{t+1})^2 > \sqrt{\gamma^2 \psi^2 \gamma^2 I} \right) \leq \frac{1}{\gamma^2 \psi^2} \mathbb{E}[|\zeta^\top Z_{t+1} | F_t]^{-1} - 1.
\]

For 2), suppose \( i \in \mathbb{N} \). Using A1-A2, the summed trace of the expected covariates can be bounded as \( \sum_{i=1}^{T} \text{tr} \left( \mathbb{E} [Z_i Z_i^\top] \right) \leq i((i||b|| U_{\max} + \Sigma W) + |x_0|)^2 + U_{\max}^2 I \). Next, supposing \( \delta > 0 \), Markov’s inequality is used to derive the upper bound \( P \left( \sum_{i=1}^{T} Z_i Z_i^\top \leq \frac{1}{2} i((i||b|| U_{\max} + \Sigma W) + |x_0|)^2 + U_{\max}^2 I \right) \leq \delta \) for the probability upper bound on \( ||\hat{\theta}_t - \theta_0||_2 > d \).
Proposition 3. [17, Theorem 2.4] Fix $\delta \in (0, 1), i \in \mathbb{N}$ and $0 < t_{sb} \leq \bar{t}$. Suppose $(Z_{i}, Y_{i})_{i=1}^{\infty} \in (\mathbb{R}^{2} \times \mathbb{R})^{i}$ is a random sequence such that (a) $Y_{i} = \theta_{T}^{T} Z_{i} + \eta_{i}$ for $t \leq i$, where $\eta_{i} \mid F_{i-1}$ is mean-zero and $\Sigma_{sb}$-sub-Gaussian with $F_{i}$ denoting the sigma-algebra generated by $\eta_{0}, \eta_{1}, \ldots, \eta_{i-1}, Z_{i}, \ldots, Z_{r}$, (b) $Z_{1}, \ldots, Z_{i}$ satisfies the $(k, \Gamma_{sb}, p)$-BMSB condition, and (c) $P(\sum_{i=1}^{\infty} I_{Z_{i}, Z_{i}^{T}} \leq i \Gamma_{sb}) \leq \delta$ holds. Then if

$$i \geq \frac{10k}{p^{2}} \left( \log \left( \frac{1}{\delta} \right) + 4 \log(10/p) + \log \det(T_{\Gamma_{sb}^{-1}}) \right),$$

we have

$$P\left( \left\| \hat{\theta}_{i} - \theta_{*} \right\|_{2} > \frac{90 \Sigma_{W}}{p} \times \sqrt{1 + 2 \log \frac{10}{p} + \log \det(T_{\Gamma_{sb}^{-1}}) + \log \left( \frac{2}{\delta} \right)} \right) \leq 3\delta,$$

where

$$\hat{\theta}_{i} = (Z_{i}^{T} Z_{i})^{-1} Z_{i} Y_{i} \in \text{arg \ min}_{\theta \in \mathbb{R}^{2}} \sum_{i=1}^{\infty} \left| Y_{i} - \theta^{T} Z_{i} \right|_{2}^{2}, \text{ and } Z_{i} = \left[ Z_{1}, \ldots, Z_{i} \right]^{T}, Y_{i} = \left[ Y_{1}, \ldots, Y_{i} \right]^{T}.$$

Lemma 4. Consider $\Gamma_{sb} > 0$, $p > 0$ and $q > 0$, $(Z_{i})_{i \in \mathbb{N}}$ from (4), and $(\hat{\theta}_{i})_{i \in \mathbb{N}}$ from (3). Suppose A1 holds on the closed-loop system (5). If the following are true:

1) $(Z_{i})_{i \in \mathbb{N}}$ satisfies the $(1, \Gamma_{sb}, p)$-BMSB condition;
2) $P(\sum_{i=1}^{\infty} I_{Z_{i}, Z_{i}^{T}} \leq 3a^{2} q I) \leq \delta$ holds for all $i \geq 1$ and $\delta \in (0, 1)$;
then for all $d \in \left( 0, \frac{90 \Sigma_{W}}{\sqrt{10 \lambda_{\text{max}}(\Gamma_{sb})}} \right)$ and $i \geq M(d, p, q, \Gamma_{sb})$,

$$P\left( \left\| \hat{\theta}_{i} - \theta_{*} \right\|_{2} > \frac{d}{2} \right) \leq i \frac{\delta}{3} e^{-c_{3}(d, p, \Gamma_{sb})} c_{4}(p, q, \Gamma_{sb}).$$

Here, $c_{1}, c_{2}, c_{3}, c_{4}$ are defined as

$$c_{1}(q, \Gamma_{sb}) := q + \lambda_{\text{max}}(\Gamma_{sb}), c_{2}(p) := 1 + 2 \log (10/p), c_{3}(d, p, \Gamma_{sb}) := \frac{\lambda_{\text{min}}(\Gamma_{sb}) d^{2} p^{2}}{3(90 \Sigma_{W})^{2}}, c_{4}(p, q, \Gamma_{sb}) := 3c_{1}(q, \Gamma_{sb}) e^{\frac{1}{3} \left( c_{2}(p) + \log(\det(\Gamma_{sb}^{-1})) \right)},$$

and $M(d, p, q, \Gamma_{sb}), M'(d, p, q, \Gamma_{sb})$ are defined as

$$M(d, p, q, \Gamma_{sb}) := \max \left\{ \left( \frac{p^{2}}{10} - \frac{\lambda_{\text{min}}(\Gamma_{sb}) d^{2} p^{2}}{(90 \Sigma_{W})^{2}} \right)^{-1} \times \left( 4 \log \frac{10}{p} - c_{2}(p) \right) \right\},$$

$$M'(d, p, q, \Gamma_{sb}) := \min \left\{ m \in \mathbb{N} \mid \left( \forall i \geq m \right) \left\{ \frac{1}{3} e^{-c_{3}(d, p, \Gamma_{sb})} c_{4}(p, q, \Gamma_{sb}) < 1 \right\} \right\},$$

for $d, p > 0$ and $\Gamma_{sb} > 0$.

Proof Sketch. Suppose $d \in \left( 0, \frac{90 \Sigma_{W}}{\sqrt{10 \lambda_{\text{max}}(\Gamma_{sb})}} \right)$, and $i \geq M(d, p, q, \Gamma_{sb})$. Let

$$\delta = \frac{1}{3} \frac{1}{2} e^{-c_{3}(d, p, \Gamma_{sb})} c_{4}(p, q, \Gamma_{sb}), \text{ and } \bar{\Gamma} = (1/\delta)^{2} c_{1}(q, \Gamma_{sb}) I.$$
Secondly, the uniform boundedness of \( \mathbb{E} \left[ (X_t - X_t^*)^2 \right] \) is derived by analyzing \( (X_t - X_t^*)^2 \). Define the time that \( \theta_t \) enters and remains in a ball of size \( d > 0 \) around \( \theta_* \) as

\[
T_d := \inf \left\{ t \in \mathbb{N} \mid \left\| \theta_t - \theta_* \right\|_2 < d \text{ for all } i \geq t \right\}.
\]  

(10)

Moreover, define \( d^* := \min(m/2, 1/2, |b|/2) \). From Lemma 10, we establish that \( \left( X_t - X_t^* \right)^2 \leq \max \left( 4b^2D^2(k+1)^2, 4 \left( \frac{|b| + d^*}{1-d^*} + 3b \right)^2D^2 \right) \) on the event \( \{ T_{d^*} = k \} \), which by the monotonicity of conditional expectation implies \( \mathbb{E} \left[ (X_t - X_t^*)^2 \mid T_{d^*} = k \right] \leq \max \left( 4b^2D^2(k+1)^2, 4 \left( \frac{|b| + d^*}{1-d^*} + 3b \right)^2D^2 \right) \) for \( k \in \mathbb{N} \). Using this upper bound, the law of total expectation, and the fact that \( P(T_{d^*} \geq k) \leq \sum_{i=k-1}^\infty P \left( \left\| \theta_i - \theta_* \right\|_2 > d^* \right) \) for \( k \in \mathbb{N} \), the upper bound

\[
\mathbb{E} \left[ (X_t - X_t^*)^2 \right] \leq \sum_{k=1}^{\infty} \max \left( 4b^2D^2(k+1)^2, 4 \left( \frac{|b| + d^*}{1-d^*} + 3b \right)^2D^2 \right)
+ 3b^2 \sum_{i=k-1}^\infty P \left( \left\| \theta_i - \theta_* \right\|_2 > d^* \right)
\]

holds for \( t \in \mathbb{N} \). The assumption that \( \sum_{k=1}^{\infty} k^2 \sum_{i=k-1}^\infty P \left( \left\| \theta_i - \theta_* \right\|_2 > d^* \right) < \infty \) for all \( d \in (0, m) \) from the premise implies that (11) is finite. Uniform boundedness of \( \mathbb{E} \left[ X_t^2 \right] \) then follows.

\( \square \)

C. Proof of Theorem 1

Proof. Let \( p > 0 \) and \( \Gamma_{\text{sb}} > 0 \) be such that \( (Z_t)_{t \in \mathbb{N}} \) satisfies the \((1, \Gamma_{\text{sb}}, p)\)-BMS condition in Definition 1 such that the existence of satisfactory \( p \) and \( \Gamma_{\text{sb}} \) is established in 1) in Lemma 2. Moreover, let \( q = (|b| U_{\max} + \Sigma_W + |x_0|)^2 + U_{\max}^2 \). Since \( \{ i(|b| U_{\max} + \Sigma_W) + |x_0| \}^2 + U_{\max}^2 \leq i^2q \) for \( i \in \mathbb{N} \), making use of 2) in Lemma 2, we follow that \( P \left( \sum_{i=1}^\infty Z_i^2 \leq t \right) \leq \frac{t}{i^2q} \) holds for all \( i \in \mathbb{N} \). With this, we have established that the premise of Lemma 4 is satisfied, and so we find

\[
P \left( \left\| \theta_i - \theta_* \right\|_2 > d^* \right) \leq i^2 \xi - c_i (d, p, \Gamma_{\text{sb}}) c_4(p, q, \Gamma_{\text{sb}})
\]

(12)

holds for all \( d \in \left( 0, \frac{90\xi}{\sqrt{10\Lambda_{\max} (\Gamma_{\text{sb}})}} \right) \) and \( i \geq M(d, p, q, \Gamma_{\text{sb}}) \).

Now, suppose \( d \in \left( 0, \frac{90\xi}{\sqrt{10\Lambda_{\max} (\Gamma_{\text{sb}})}} \right) \). For ease of readability, let us refer to the functions \( c_1, c_2, c_3, c_4 \) and \( M \) without their arguments, but with an implicit understanding of their dependence on \( d, p, q, \Gamma_{\text{sb}} \). The following holds:

\[
\sum_{k=1}^{\infty} k^2 \sum_{i=k-1}^{\infty} P \left( \left\| \theta_i - \theta_* \right\|_2 > d^* \right)
\leq \sum_{k=1}^{\infty} k^2 \left( 1 + c_4 \sum_{k=1}^{\infty} \sum_{i=\max(k-1, M)}^{\infty} i^2 \xi - c_i \right)
< \infty
\]

(13)

where (13) follows from (12), and (14) follows from

\[\sum_{k=1}^{\infty} k^2 \sum_{i=k-1}^{\infty} i = \sum_{k=1}^{M-1} M^2 (M - k + 1) < \infty, \text{ as well as}\]

(14)

Here, (15) follows since

\[
\sum_{k=1}^{M} k^2 \sum_{i=M-1}^{\infty} i^2 \exp \left( -c_i \right) = \sum_{k=1}^{M} k^2 \sum_{i=0}^{\infty} \frac{e^{-c_i (M-2)}}{(M-2)^2 M^2}
+ e^{c_3} (M+1) 2 M^2 - e^{3c_3} (M+1)^2 (4M^2 - 6M - 5)
+ e^{2c_3} (6M^4 - 5M^2 - 25M + 28)^2 + e^3 (4M^4 + 7M^2 - 24M + 9) < \infty.
\]

From (14), we have shown that

\[
\sum_{k=1}^{\infty} k^2 \sum_{i=k-1}^{\infty} P \left( \left\| \theta_i - \theta_* \right\|_2 > d^* \right) < \infty \text{ for all } \in \left( 0, \frac{90\xi}{\sqrt{10\Lambda_{\max} (\Gamma_{\text{sb}})}} \right).
\]

The premise of Lemma 5 is thus satisfied with \( m = \frac{90\xi}{\sqrt{10\Lambda_{\max} (\Gamma_{\text{sb}})}} \). The conclusion follows.

\( \square \)

V. Numerical Examples

To demonstrate the effectiveness of the control strategy in Algorithm 1, we tested it with \( U_{\max} = 1, \ C = 0.1 \) and \( \theta_{\text{init}} = (-1, -5) \) on three different systems with \( x_0 = 0 \):

- System 1: \( a = 0.7, b = -1, \Sigma_W = 1 \);
- System 2: \( a = -1, b = 2, \Sigma_W = 2 \);
- System 3: \( a = 1, b = 0.5, \Sigma_W = 1.5 \).

Fig. 1a shows the empirical ensemble average of \( X_t^2 \) for Systems 1, 2, and 3 respectively over 1000 runs. Our control strategy seemingly attains mean square boundedness for all three systems, matching the guarantee provided in Theorem 1. We simulate System 3 with no controls for comparison, which does not achieve mean square boundedness. Convergence of \( \hat{a}_t \) and \( \hat{b}_t \) over time are shown in Fig. 1b and 1c.
VI. Conclusion

We proposed a perturbed CE control scheme for adaptive control of stochastic, scalar, at-worst marginally stable linear systems subject to additive, i.i.d. Gaussian disturbances, with positive upper bound constraints on the control magnitude. Mean square boundedness of the closed-loop system is established, and demonstrated by numerical examples. It is possible to consider non-Gaussian stochastic processes in A1, and establish mean square boundedness. The most critical requirements are ensuring $W_t \mid F_{t-1}$ is mean-zero and $\Sigma^2$-sub-Gaussian, and proving Lemma 6. The latter requires careful inspection of the particular disturbance distribution, the excitation term, and the nonlinear saturation.

Our approach has a strong potential to be extended to higher dimensions. The core of our method is combining model-based control in Line 6 of Algorithm 1 with least squares parameter estimation in Line 9. Stability analysis follows by satisfying Lemma 2 to establish fast convergence of upper bounds on $P(|| \theta - \hat{\theta} || > d)$ in Lemma 4 and proving that fast convergence implies mean square boundedness in Lemma 5. This intuition generalizes to higher dimensions, but to make the jump analytically, some technical challenges remain to be solved. In particular, the careful analysis of 1) in Lemma 2 needs to be scaled up from the 1D case, and an equivalent result to Lemma 5 is required, since Lemma 10 for bounding $||X_t - \bar{X}_t||_2$ does not immediately hold in $n$ dimensions.

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APPENDIX

Lemma 6. Suppose A1-A2 hold on the closed-loop system (5) and $x_0 \in \mathbb{R}$. Let $(F_t)_{t \in \mathbb{N}_0}$ be the natural filtration of $(Z_t)_{t \in \mathbb{N}_0}$ from (4). There exists $\gamma > 0$ such that for all $\zeta \in S^3$ and $t \geq 0$, $E [|| \zeta^T Z_{t+1} || \mid F_t] \geq \gamma$.

Lemma 7. Consider the states $(X_t)_{t \in \mathbb{N}_0}$ from the closed-loop system (5). Suppose A1-A2 hold, $x_0 \in \mathbb{R}$, and $a \in (-1, 1)$. For all $\lambda \in \mathbb{C}$ and $t \in \mathbb{N}_0$, we have $E [X_t^2] \leq x_0^2 + \frac{\beta(\lambda)}{1 - \lambda}$, where $\beta(\lambda) := a^2 E(\lambda)^2 + 2 |a| E(\lambda) D_1 + D_2$, $E(\lambda) := |a| D_1 + \sqrt{a^2 D_1^2 + (\lambda - a^2)D_2}$, $D_1 := |b| U_{max} + S_1$, $D_2 := b^2 U_{max}^2 + 2 |b| U_{max} S_1 + S_2$.

Lemma 8. Consider the states $(X_t^x)_{t \in \mathbb{N}_0}$ from the reference system (9). Suppose A1-A2 hold, $x_0 \in \mathbb{R}$, and $a \in (-1, 1)$. There exists $e > 0$ such that for all $t \in \mathbb{N}_0$, $E [|| X_t^x ||^2] \leq e$.

Proposition 9. [18, Theorem 1] Let $(\xi_t)_{t \in \mathbb{N}_0}$ be a sequence of scalar random variables and let $(F_t)_{t \in \mathbb{N}_0}$ be any filtration to which $(\xi_t)_{t \in \mathbb{N}_0}$ is adapted. Suppose that there exist constants $\gamma > 0$ and $J, M < \infty$, such that $\xi_0 \leq J$, and for all $t$: $E [\xi_{t+1} - \xi_t \mid F_t] \leq -\gamma$ on the event $\{\xi_t > J\}$, then there exists a constant $c > 0$ such that $\sup_{t \in \mathbb{N}_0} E [|| \xi_t^2 ||^2] \leq c$. Then there exists a constant $c > 0$ such that $\sum_{t \in \mathbb{N}_0} E [|| \xi_t^2 ||^2] \leq c$.
where \( (19) \) holds since the existence of satisfactory values is established in Lemma \( \text{R} \) (with Combining \( (18), (20), \) and \( (21) \), we derive the following is illustrated in Figure 2.

Next, note that the following inequality holds for all \( \psi \in (0,1) \):

\[
P \left( \left| \zeta^\top Z_{t+1} > \sqrt{\text{Var} (\psi^2 \gamma^2 I) \zeta} \right| \mathcal{F}_t \right) = P \left( \left| \zeta^\top Z_{t+1} > \psi \gamma \right| \mathcal{F}_t \right) 
\]

\[
\geq P \left( \left| \zeta^\top Z_{t+1} > \psi \mathcal{E} \left[ \left| \zeta^\top Z_{t+1} \right| \mathcal{F}_t \right] \mathcal{F}_t \right) 
\]

\[
\geq P \left( \left| \zeta^\top Z_{t+1} > \psi \mathcal{E} \left[ \left| \zeta^\top Z_{t+1} \right| \mathcal{F}_t \right] \mathcal{F}_t \right)^2 
\]

\[
\geq 1 + P \left( \left| \zeta^\top Z_{t+1} > \psi \mathcal{E} \left[ \left| \zeta^\top Z_{t+1} \right| \mathcal{F}_t \right] \mathcal{F}_t \right)^2 
\]

\[
\geq 1 + P \left( \left| \zeta^\top Z_{t+1} > \psi \mathcal{E} \left[ \left| \zeta^\top Z_{t+1} \right| \mathcal{F}_t \right] \mathcal{F}_t \right)^2 
\]

where \( (23) \) holds since \( \zeta^\top \zeta = 1 \), \( (24) \) follows from \( \mathcal{E} \left[ \left| \zeta^\top Z_{t+1} \right| \mathcal{F}_t \right] \geq \gamma \), \( (25) \) follows from an improvement of the Paley-Zygmund inequality via the Cauchy-Schwarz inequality, \( (26) \) holds since for any random variable \( X \) taking values in \( \mathbb{R} \), \( \text{Var} (\{X\}) \leq \text{Var} (\{X\}) \) is true making use of Jensen’s inequality. Finally, \( (27) \) follows from \( \mathcal{E} \left[ \left| \zeta^\top Z_{t+1} \right| \mathcal{F}_t \right] \geq \gamma \) and \( (22) \). Fixing \( \psi \in (0,1) \), and setting \( \Gamma_{\psi} = \psi^2 \gamma^2 I \) and \( p = (1 + 2(2(2 \psi^2 + 2) - (1 - \psi)^2 \gamma^2))^{-1} \), result 1) then follows.

We now prove 2). We start this by establishing that \( \sum_{t=1}^{i} \mathbb{E} \left[ U_{t}^2 \right] \leq i U_{t_{max}}^2 \). Using the fact that for all \( t \in \mathbb{N} \), \( U_{t}^2 \leq U_{t_{max}}^2 \), we have,

\[
\sum_{t=1}^{i} \mathbb{E} \left[ U_{t}^2 \right] \leq \sum_{t=1}^{i} U_{t_{max}}^2 = i U_{t_{max}}^2. 
\]

Next, we prove that \( \sum_{t=1}^{i} \mathbb{E} \left[ X_{t}^2 \right] \leq i(i(\psi U_{t_{max}} + \Sigma W) + |x_0|)^2 \). For all \( t \in \mathbb{N} \), we have,

\[
\mathbb{E} \left[ X_{t}^2 \right] = \mathbb{E} [(aX_t + bU_t + W_t)]^2 
\]

\[
= a^2 \mathbb{E} [X_t^2] + 2ab \mathbb{E} [X_t U_t] + b^2 \mathbb{E} [U_t^2] + \Sigma W 
\]

\[
\leq \mathbb{E} [X_t^2] + 2 |b| U_{t_{max}} \sqrt{\mathbb{E} [X_t^2]} + b^2 U_{t_{max}}^2 + \Sigma W 
\]

\[
= \sqrt{\mathbb{E} [X_t^2]} + |b| U_{t_{max}} + \Sigma W. 
\]

where \( (29) \) follows from A2 and the Cauchy-Schwarz inequality, and \( (30) \) follows via quadratic factorization. Taking the square root of both sides, we then have,

\[
\sqrt{\mathbb{E} \left[ X_{t}^2 \right]} \leq \sqrt{\left( \sqrt{\mathbb{E} [X_t^2]} + |b| U_{t_{max}} + \Sigma W \right) + \Sigma W}. 
\]

By iteratively applying \( (31) \) and noting that \( \mathbb{E} \left[ X_{0}^2 \right] = x_0^2 \), we have for \( t \geq 1 \),

\[
\sqrt{\mathbb{E} \left[ X_{t}^2 \right]} \leq \sum_{s=0}^{t-1} \left( |b| U_{s_{max}} + \Sigma W \right) + |x_0| 
\]

\[
= t(|b| U_{t_{max}} + \Sigma W) + |x_0|. 
\]
Squaring both sides, we then have
\[ E[X_t^2] \leq (t(|b| U_{\max} + \Sigma_W) + |x_0|)^2. \]
Summing from \( t = 1 \) to \( i \), we have,
\[
\sum_{t=1}^{i} E[X_t^2] \\
\leq \sum_{t=1}^{i} (t(|b| U_{\max} + \Sigma_W) + |x_0|)^2 \\
\leq \sum_{t=1}^{i} (i(|b| U_{\max} + \Sigma_W) + |x_0|)^2 \\
= i(i(|b| U_{\max} + \Sigma_W) + |x_0|)^2. \quad (32)
\]
Next, note that the following holds:
\[
\sum_{t=1}^{i} \text{tr} \left( E[Z_t Z_t^\top] \right) \\
= \sum_{t=1}^{i} E[X_t^2] + \sum_{t=1}^{i} E[U_t^2] \\
\leq i(i(|b| U_{\max} + \Sigma_W) + |x_0|)^2 + U_{\max}^2 \quad (33)
\]
where (33) follows from (25) and (32). Finally, fix \( \delta \in (0, 1) \). We find that
\[
P\left( \sum_{t=1}^{i} Z_t Z_t^\top \not\leq \frac{1}{\delta} i(|i(|b| U_{\max} + \Sigma_W) + |x_0|)^2 + U_{\max}^2 \right) \\
= P\left( \lambda_{\max}\left(\left( i(|i(|b| U_{\max} + \Sigma_W) + |x_0|)^2 + U_{\max}^2 \right) \right)^{-1} \\
\times \sum_{t=1}^{i} Z_t Z_t^\top \geq \frac{1}{\delta} \right) \quad (34) \\
\leq \delta E \left[ \lambda_{\max}\left(\left( i(|i(|b| U_{\max} + \Sigma_W) + |x_0|)^2 + U_{\max}^2 \right) \right)^{-1} \\
\times \sum_{t=1}^{i} Z_t Z_t^\top \right] \quad (35) \\
\leq \delta \text{tr} \left( E \left[ \left( i(|i(|b| U_{\max} + \Sigma_W) + |x_0|)^2 + U_{\max}^2 \right)^{-1} \\
\times \sum_{t=1}^{i} Z_t Z_t^\top \right] \right) \quad (36) \\
= \delta \left( i(|i(|b| U_{\max} + \Sigma_W) + |x_0|)^2 + U_{\max}^2 \right)^{-1} \\
\times \sum_{t=1}^{i} \text{tr} \left( E[Z_t Z_t^\top] \right) \quad (37)
\]
where (34) follows from the definition of \( \preceq \), (35) follows from Markov’s inequality, and (36) holds since for a matrix \( M \in \mathbb{R}^{d \times d} \), \( \lambda_{\max}(M) \leq \text{tr}(M) \), and (37) follows from (33). Thus, result 2) has been established.

**Proof of Lemma 6.** For all \( \zeta = (\zeta_1, \zeta_2) \in \mathcal{S}^1 \) and \( t \geq 0 \), we have,
\[
E[\zeta^\top Z_{t+1} | F_t] \\
= E[\zeta_1 X_{t+1} + \zeta_2 V_{t+1} | F_t] \\
= E[\zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1}) + V_{t+1}) | F_t] \quad (38) \\
= E[\zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})) + \zeta_2 V_{t+1} | F_t] \quad (39)
\]
where (38) follows from (2). A lower bound can be derived for (39) as follows:
\[
E[\zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})) + \zeta_2 V_{t+1} | F_t] \\
= E[E[\zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})) | X_{t+1}, F_t] | F_t] \quad (40) \\
\geq E[E[\zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})) | X_{t+1}, F_t] | F_t] \quad (41) \\
= E[E[\zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})) | X_{t+1}, F_t] | F_t] \quad (42) \\
= E[|\zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1}))| | F_t] \quad (43) \\
= E[|A_{t+1}(\zeta)| | F_t] \quad (44)
\]
where (40) follows from the tower property, (41) follows from Jensen’s inequality and the monotonicity of conditional expectation, (42) follows from the independence of \( V_{t+1} \) and \( X_{t+1}, F_t \), and (43) follows since \( \zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})) \) is \( X_{t+1}, F_t \)-measurable, and (44) follows by defining
\[
A_{t+1}(\zeta) := \zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})), \quad t \in \mathbb{N}_0. \quad (45)
\]
Similarly,
\[
E[\zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})) + \zeta_2 V_{t+1} | F_t] \\
\geq E[E[\zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})) | V_{t+1}, F_t] | F_t] \quad (46) \\
\geq E[E[\zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})) | V_{t+1}, F_t] | F_t] \quad (47) \\
= E[|B_{t+1}(\zeta)| | F_t] \quad (48)
\]
where (46) follows from the tower property, Jensen’s inequality and the monotonicity of conditional expectation, and (47) follows since \( V_{t+1} \) is \( V_{t+1}, F_t \)-measurable and \( \zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})) \) is independent of \( V_{t+1} \), and (48) follows by defining
\[
B_{t+1}(\zeta) := E[\zeta_1 X_{t+1} + \zeta_2 (\sigma_D(G_{t+1}X_{t+1})) | F_t] + \zeta_2 V_{t+1}. \quad (49)
\]
for \( t \in \mathbb{N}_0 \). The lower bounds from (44) and (48) are then combined to obtain \( E[|\zeta^\top Z_{t+1} | F_t] \geq \max \{ E[|A_{t+1}(\zeta)| | F_t], E[|B_{t+1}(\zeta)| | F_t] \} \).

From Lemma 11, we have that \( E[|A_t(\zeta)| | F_t] \geq \)
Lemma 11. Suppose A1-A2 hold on the closed-loop system [5] and $x_0 \in \mathbb{R}$. Let $(F_t)_{t \in \mathbb{N}_0}$ be the natural filtration of $(Z_t)_{t \in \mathbb{N}_0}$ from [4]. Then, $E[|\xi_{t+1}(\zeta_1)| | F_t] \geq f(\zeta_1, \zeta_2)$ for all $\zeta \in \mathcal{S}_1$ and $t \in \mathbb{N}_0$, where $A_{t+1}$ is from [45] and $f$ is from [50].

Proof. Suppose $t \geq 0$, and $\zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2$. Define a new random variable $Q_{t+1}$ taking values in $\mathbb{R}$, satisfying

$$Q_{t+1} \in \arg \min_{q \in [-D, D]} |\zeta_1 X_{t+1} + \zeta_2 q|.$$

When $\zeta \neq 0$ and $\zeta_2 \neq 0$, $Q_{t+1}$ satisfies

$$\begin{align*}
Q_{t+1} &= \begin{cases}
-\frac{\zeta_1}{\zeta_2} X_{t+1}, & -D \leq -\frac{\zeta_1}{\zeta_2} X_{t+1} \leq D \\
-D, & -\frac{\zeta_1}{\zeta_2} X_{t+1} < -D \\
D, & -\frac{\zeta_1}{\zeta_2} X_{t+1} > D
\end{cases}
\end{align*}$$

When $\zeta_1 = 0$ or $\zeta_2 = 0$, a satisfactory choice of $Q_{t+1}$ is $Q_{t+1} = 0$.

Since $\sigma_D(G_{t+1}X_{t+1})$ takes values in $[-D, D]$, it follows that

$$E[|\zeta_{t+1} + \zeta_2 Q_{t+1}| | F_t] \geq E[|\zeta_{t+1} + \zeta_2 Q_{t+1}| | F_t].$$

When $\zeta_1 = 0$ and $\zeta_2 \neq 0$, $|\zeta_{t+1} + \zeta_2 Q_{t+1}| = 0$. Thus, $E[|\zeta_{t+1} + \zeta_2 Q_{t+1}| | F_t] = 0$.

When $\zeta_1 \neq 0$ and $\zeta_2 = 0$, $|\zeta_{t+1} + \zeta_2 Q_{t+1}| = |\zeta_{t+1}|$. It follows that

$$\begin{align*}
E[|\zeta_{t+1} + \zeta_2 Q_{t+1}| | F_t] &= E[|\zeta_{t+1} + \zeta_2 Q_{t+1}| | F_t] \\
&= |\zeta_{t+1}| E[|X_{t+1}| | F_t] \\
&= |\zeta_{t+1}| \left(\mathbb{E}[|X_{t+1}| | F_t] \right) \\
&= |\zeta_{t+1}| \left(\mathbb{E}[|X_{t+1}| | F_t] \right) \\
&= |\zeta_{t+1}| \left(\mathbb{E}[|X_{t+1}| | F_t] \right)
\end{align*}$$

where $\left(51\right)$ is due to the fact that $X_{t+1} = \theta_X^* Z_t + W_t$, so $X_{t+1} \sim X_{t+1} \sim N(\theta_X^* Z_t, \Sigma_W)$, and hence $E[|X_{t+1}| | F_t] = E[|X_{t+1}| | Z_t]$ is the mean of the corresponding folded normal distribution. $\left(52\right)$ follows since $\left(51\right)$ is minimised at $\theta_X^* Z_t = 0$.

When $\zeta_1 \neq 0$ and $\zeta_2 \neq 0$, we have

$$|\zeta_{t+1} + \zeta_2 Q_{t+1}| = \begin{cases}
0, & -\frac{\zeta_1}{\zeta_2} X_{t+1} \leq D, \\
|\zeta_{t+1} + \zeta_2 Q_{t+1}|, & -\frac{\zeta_1}{\zeta_2} X_{t+1} < -D, \\
|\zeta_{t+1} + \zeta_2 Q_{t+1}|, & -\frac{\zeta_1}{\zeta_2} X_{t+1} > D
\end{cases}$$

The conditional expectation is then given by

$$E[|\zeta_{t+1} + \zeta_2 Q_{t+1}| | F_t]$$
\[ \begin{align*}
&= \mathbb{E} \left[ \mathbf{1}_{\left\{ -\frac{\zeta_2}{\Sigma_1} X_{t+1} \leq D \right\}} \left( \zeta_1 X_{t+1} + \zeta_2 Q_{t+1} \right) \right] \\
&\quad + \mathbb{E} \left[ \mathbf{1}_{\left\{ -\frac{\zeta_2}{\Sigma_1} X_{t+1} > -D \right\}} \left( \zeta_1 X_{t+1} + \zeta_2 Q_{t+1} \right) \mid F_t \right] \\
&\quad + \mathbb{E} \left[ \mathbf{1}_{\left\{ -\frac{\zeta_2}{\Sigma_1} X_{t+1} > D \right\}} \left( \zeta_1 X_{t+1} + \zeta_2 Q_{t+1} \right) \mid F_t \right]
\end{align*} \]
\[ (53) \]

We further split \( \zeta_2 \neq 0 \) into two cases, where \( \zeta_2 < 0 \), and \( \zeta_2 > 0 \). Let us start with \( \zeta_1 \neq 0 \) and \( \zeta_2 < 0 \). Evaluating the conditional expectation in (53) and (54), we have
\[ \begin{align*}
&= \mathbb{E} \left[ \mathbf{1}_{\left\{ -\frac{\zeta_2}{\Sigma_1} X_{t+1} < -D \right\}} \left( \zeta_1 X_{t+1} - \zeta_2 D \right) \mid F_t \right] \\
&\quad + \mathbb{E} \left[ \mathbf{1}_{\left\{ -\frac{\zeta_2}{\Sigma_1} X_{t+1} > D \right\}} \left( \zeta_1 X_{t+1} + \zeta_2 D \right) \mid F_t \right]
\end{align*} \]
\[ (54) \]

Now, we focus on the case where \( \zeta_1 
eq 0 \) and \( \zeta_2 > 0 \). Evaluating the conditional expectation in (55) and (54), we have
\[ \begin{align*}
&= \mathbb{E} \left[ \mathbf{1}_{\left\{ -\frac{\zeta_2}{\Sigma_1} X_{t+1} < -D \right\}} \left( \zeta_1 X_{t+1} - \zeta_2 D \right) \mid F_t \right] \\
&\quad + \mathbb{E} \left[ \mathbf{1}_{\left\{ -\frac{\zeta_2}{\Sigma_1} X_{t+1} > D \right\}} \left( \zeta_1 X_{t+1} + \zeta_2 D \right) \mid F_t \right]
\end{align*} \]
\[ (55) \]

where (55) follows from the fact that \( \zeta_1 X_{t+1} - \zeta_2 D \mid Z_t = z_t \sim \mathcal{N} \left( \zeta_1 \theta^*_t z_t - \zeta_2 D, \zeta_1^2 \mathbf{I} + \zeta_2^2 \frac{\zeta_1^2}{\Sigma_1^2} \right) \) and \( \zeta_1 X_{t+1} + \zeta_2 D \mid Z_t = z_t \sim \mathcal{N} \left( \zeta_1 \theta^*_t z_t + \zeta_2 D, \zeta_1^2 \mathbf{I} + \zeta_2^2 \frac{\zeta_1^2}{\Sigma_1^2} \right) \). Additionally, we denote the conditional expectation by the function \( h_1 \) for ease of notation. Taking the partial derivative of \( h_1 \) with respect to \( \hat{x} \), we have
\[ \begin{align*}
\frac{\partial h_1(\zeta_1, \zeta_2, \hat{x})}{\partial \hat{x}} &= \frac{1}{2} \left| \zeta_1 \right| \left( \text{erf} \left( \frac{D \zeta_2 + \zeta_1 \hat{x}}{\sqrt{2} \Sigma_1} \right) + \text{erf} \left( -\frac{D \zeta_2 + \zeta_1 \hat{x}}{\sqrt{2} \Sigma_1} \right) \right).
\end{align*} \]
\[ (56) \]

This partial derivative was symbolically computed using a CAS. Next, suppose \( \zeta_1 \neq 0 \) and \( \zeta_2 < 0 \). When \( \hat{x} = 0 \), \( \frac{\partial h_1(\zeta_1, \zeta_2, \hat{x})}{\partial \hat{x}} = 0 \). When \( \hat{x} = 0 \), \( \frac{\partial h_1(\zeta_1, \zeta_2, \hat{x})}{\partial \hat{x}} > 0 \). When \( \hat{x} < 0 \), \( \frac{\partial h_1(\zeta_1, \zeta_2, \hat{x})}{\partial \hat{x}} < 0 \). Thus, \( h_1(\zeta_1, \zeta_2, \hat{x}) \) is minimised at \( \hat{x} = 0 \). We use this to lower bound (56) for all \( \zeta_1 \neq 0 \) and \( \zeta_2 < 0 \):
\[ h_1(\zeta_1, \zeta_2, \theta^*_t Z_t) \geq \exp \left( -\frac{D^2 \zeta_2^2}{2 \Sigma_1^2} \right) \sqrt{\frac{2}{\pi}} \Sigma W^2 \left| \zeta_1 \right| \right) + D \zeta_2 \left( 1 + \exp \left( \frac{D \zeta_2}{\sqrt{2} \Sigma W \left| \zeta_1 \right|} \right) \right)
\]
\[ (57) \]

Combining (56) and (57), we have that for all \( \zeta_1 \neq 0 \) and \( \zeta_2 < 0 \),
\[ \begin{align*}
&\mathbb{E} \left[ \mathbf{1}_{\left\{ -\frac{\zeta_2}{\Sigma_1} X_{t+1} < -D \right\}} \left( \zeta_1 X_{t+1} - \zeta_2 D \right) \mid F_t \right] \\
&\quad + \mathbb{E} \left[ \mathbf{1}_{\left\{ -\frac{\zeta_2}{\Sigma_1} X_{t+1} > D \right\}} \left( \zeta_1 X_{t+1} + \zeta_2 D \right) \mid F_t \right]
\end{align*} \]
\[ (58) \]

where (58) follows similarly to (56) using the fact that \( \zeta_1 X_{t+1} - \zeta_2 D \mid Z_t = z_t \sim \mathcal{N} \left( \zeta_1 \theta^*_t z_t - \zeta_2 D, \zeta_1^2 \mathbf{I} + \zeta_2^2 \frac{\zeta_1^2}{\Sigma_1^2} \right) \) and \( \zeta_1 X_{t+1} + \zeta_2 D \mid Z_t = z_t \sim \mathcal{N} \left( \zeta_1 \theta^*_t z_t + \zeta_2 D, \zeta_1^2 \mathbf{I} + \zeta_2^2 \frac{\zeta_1^2}{\Sigma_1^2} \right) \). Additionally, we denote the conditional expectation by the function \( h_2 \) for ease of notation. Taking the partial derivative of \( h_2 \) with respect to \( \hat{x} \), we arrive at
\[ \frac{\partial h_2(\zeta_1, \zeta_2, \hat{x})}{\partial \hat{x}} = \frac{1}{2} \left| \zeta_1 \right| \left( \text{erf} \left( -\frac{D \zeta_2 + \zeta_1 \hat{x}}{\sqrt{2} \Sigma_1} \right) + \text{erf} \left( \frac{D \zeta_2 + \zeta_1 \hat{x}}{\sqrt{2} \Sigma_1} \right) \right).
\]

Suppose \( \zeta_1 \neq 0 \) and \( \zeta_2 > 0 \). When \( \hat{x} = 0 \), \( \frac{\partial h_2(\zeta_1, \zeta_2, \hat{x})}{\partial \hat{x}} = 0 \). When \( \hat{x} > 0 \), \( \frac{\partial h_2(\zeta_1, \zeta_2, \hat{x})}{\partial \hat{x}} > 0 \). When \( \hat{x} < 0 \), \( \frac{\partial h_2(\zeta_1, \zeta_2, \hat{x})}{\partial \hat{x}} < 0 \). Thus, \( h_2(\zeta_1, \zeta_2, \hat{x}) \) is minimised at \( \hat{x} = 0 \). We use this to lower bound (59) for all \( \zeta_1 \neq 0 \), \( \zeta_2 > 0 \) and \( \hat{x} \in \mathbb{R} \):
\[ h_2(\zeta_1, \zeta_2, \theta^*_t Z_t) \]
\[ \geq \exp \left( -\frac{D^2 \zeta_2}{2 \Sigma_W W} \right) \sqrt{\frac{2}{\pi} \Sigma_W |\zeta_1| - D\zeta_2 \text{erfc} \left( \frac{D\zeta_2}{\sqrt{2 \Sigma_W W}} \right) } \]

Therefore, for all \( \zeta_1 \neq 0 \) and \( \zeta_2 > 0 \), the following holds.

\[
\mathbb{E} \left[ 1 \left\{ \frac{\zeta_1}{\Sigma} X_{t+1} < -D \right\} | \zeta_1 X_{t+1} - \zeta_2 D | / \mathcal{F}_t \right] \\
+ \mathbb{E} \left[ 1 \left\{ \frac{\zeta_1}{\Sigma} X_{t+1} > D \right\} | \zeta_1 X_{t+1} + \zeta_2 D | / \mathcal{F}_t \right]
\geq \exp \left( -\frac{D^2 \zeta_2}{2 \Sigma_W W} \right) \sqrt{\frac{2}{\pi} \Sigma_W |\zeta_1|} - D\zeta_2 \text{erfc} \left( \frac{D\zeta_2}{\sqrt{2 \Sigma_W W}} \right).
\]

The conclusion follows by observing that \( \mathbb{E} \left[ A_{t+1} (\zeta) | F_t \right] \geq f(\zeta_1, \zeta_2) \) holds for all \( (\zeta_1, \zeta_2) \in \mathbb{R}^2 \).

**Lemma 12.** Let \((\mathcal{F}_t)_{t \in \mathbb{N}_0}\) be the natural filtration of \((Z_t)_{t \in \mathbb{N}_0}\) from \([4]\). Suppose A1-A2 hold on the closed-loop system \([5]\) and \( x_0 \in \mathbb{R} \). Then, \( \mathbb{E} \left[ |B_{t+1}(\zeta)| | \mathcal{F}_t \right] \geq \frac{|\zeta_2| C}{2} \), where \( B_{t+1} \) is from \([49]\).

**Proof.** Suppose \( \zeta = (\zeta_1, \zeta_2) \in \mathbb{R}^2 \) and \( t \in \mathbb{N}_0 \). When \( \zeta_2 = 0 \), using the monotonicity of conditional expectation we have

\[ \mathbb{E} \left[ |B_{t+1}(\zeta)| | Z_0 = z_0, \ldots, Z_t = z_t \right] \geq \mathbb{E} \left[ |B_{t+1}(\zeta)| | Z_0 = z_0, \ldots, Z_t = z_t \right]. \]

Next, let \( g_t \) be the mapping satisfying

\[
g_t(\zeta_1, \zeta_2, (z_0, \ldots, z_t)) = \mathbb{E} \left[ \left| \zeta_1 X_{t+1} + \zeta_2 D G_{t+1}(X_{t+1}) \right| | \mathcal{F}_t \right].
\]

Note that the distribution of \( B_{t+1}(\zeta) \) \( Z_0 = z_0, \ldots, Z_t = z_t \) is

\[
\text{Uniform}(\{g_t(\zeta_1, \zeta_2, (z_0, \ldots, z_t)) - |\zeta_2| C, g_t(\zeta_1, \zeta_2, (z_0, \ldots, z_t)) + |\zeta_2| C\}) \quad \text{for all}\quad z_0, \ldots, z_t \in \mathbb{R}^2.
\]

Using this fact and the law of the unconscious statistician, when \( \zeta_2 \neq 0 \), we have

\[
\mathbb{E} \left[ |B_{t+1}(\zeta)| | Z_0 = z_0, \ldots, Z_t = z_t \right] \geq \mathbb{E} \left[ |B_{t+1}(\zeta)| | Z_0 = z_0, \ldots, Z_t = z_t \right].
\]

**Lemma 13.** Consider the function \( f \) from \([50]\). The following properties hold:

1. \( f(\cos(\phi), \sin(\phi)) \) is continuous over \( \phi \in [-\pi, \pi] \);
2. \( f(\cos(\phi), \sin(\phi)) \) is strictly decreasing over \( \phi \in [-\pi, -\pi/2] \) and \( [0, \pi/2] \), and strictly increasing over \([-\pi/2, 0]\) and \([\pi/2, \pi]\);
3. \( f(\cos(\phi), \sin(\phi)) \geq 0 \) for all \( \phi \in [-\pi, \pi] \).

**Proof.** The proof of 1) follows from the fact that \( f \) is continuous over the domains \( \{(\zeta_1, \zeta_2) \in \mathbb{R}^2 | \zeta_1 \neq 0, \zeta_2 = 0\}, \{(\zeta_1, \zeta_2) \in \mathbb{R}^2 | \zeta_1 = 0, \zeta_2 \in \mathbb{R}\}, \{(\zeta_1, \zeta_2) \in \mathbb{R}^2 | \zeta_1 \neq 0, \zeta_2 < 0\} \), and \( \{(\zeta_1, \zeta_2) \in \mathbb{R}^2 | \zeta_1 \neq 0, \zeta_2 > 0\} \), alongside the fact that \( \text{lim}_{\phi \rightarrow \pi} f(\cos(\phi), \sin(\phi)) = f(\cos(\phi), \sin(\phi))) \) for all \( c \in \{-\pi, -\pi/2, 0, \pi/2, \pi\} \).

We now prove 2). Over the interval \( \phi \in (-\pi, -\pi/2) \cup (-\pi/2, 0) \), we find that \( \frac{d}{d\phi} f(\cos(\phi), \sin(\phi)) = -\frac{D \sin(\phi)}{\Sigma_W \sqrt{\cos(2\phi) + 1}} \frac{\partial f}{\partial \phi} \), using a CAS. On the interval \( \phi \in (\pi, \pi/2) \), \( \frac{d}{d\phi} f(\cos(\phi), \sin(\phi)) < 0 \) holds, and on the interval \( \phi \in (-\pi/2, 0) \), \( \frac{d}{d\phi} f(\cos(\phi), \sin(\phi)) > 0 \) holds. Thus, \( f(\cos(\phi), \sin(\phi)) \) is strictly decreasing and strictly increasing over the open intervals \( \phi \in (-\pi, -\pi/2) \) and \( \phi \in (-\pi/2, 0) \) respectively, and due to the continuity of \( f(\cos(\phi), \sin(\phi)) \) these same properties hold over their respective associated closed intervals. Similarly, over the interval \( \phi \in (0, \pi/2) \cup (\pi/2, \pi) \), we find that \( \frac{d}{d\phi} f(\cos(\phi), \sin(\phi)) = \frac{D \cos(\phi) \text{erfc} \left( \frac{D \sin(\phi)}{\Sigma_W \sqrt{\cos(2\phi) + 1}} \right)}{\Sigma_W \sqrt{\cos(2\phi) + 1}} \), using a CAS. On the interval \( \phi \in (0, \pi/2) \), \( \frac{d}{d\phi} f(\cos(\phi), \sin(\phi)) > 0 \) holds, and on the interval \( \phi \in (\pi/2, \pi) \), \( \frac{d}{d\phi} f(\cos(\phi), \sin(\phi)) < 0 \) holds. Thus, \( f(\cos(\phi), \sin(\phi)) \) is strictly decreasing and strictly increasing over the open intervals \( \phi \in (0, \pi/2) \) and \( \phi \in (\pi/2, \pi) \) respectively, and due to the continuity of \( f(\cos(\phi), \sin(\phi)) \) these same properties hold over their respective closed intervals. The proof of 2) is thus completed.

The proof of 3) follows from the fact that \( f(\zeta_1, \zeta_2) \geq 0 \) for all \( \zeta_1, \zeta_2 \in \mathbb{R} \).

**B. Analysis for Lemma 2**

We provide the proof of Lemma 2.

**Proof of Lemma 2** Suppose \( d \in \left( 0, \frac{90\Sigma_W}{\sqrt{10\lambda_{\max}(\Gamma_{ab})}} \right) \), and \( i \geq M(d, p, q, \Gamma_{ab}) \). Set

\[
\delta = i^2 e^{-\frac{\lambda_{\max}(\Gamma_{ab})}{10\Sigma_W}} \frac{1}{4} \left| C \right| \Gamma_{ab} \left| C \right| \Gamma_{ab} \left( \| \Gamma_{ab} \|_2 \right) \frac{1}{2} \log \left( \det \left( \Gamma_{ab}^{-1} \right) \right),
\]

(60)
and $\Gamma = \frac{1}{2} \gamma^2 c_1(q, \Gamma_{sb}) I$.

Now, we will establish that the sequence $\{Z_t, X_t\}_{t=1}^\infty$ satisfies the premise of Proposition 3. Let $F_t$ be the sigma-algebra generated by $W_0, \ldots, W_t, Z_1, \ldots, Z_t$ for $t \in \mathbb{N}$. Note that (a) is satisfied since $X_{t+1} = \theta_t^\Gamma Z_t + W_t$ holds for $t \in \mathbb{N}$, with $W_t \perp_{F_t} Z_t \perp_{F_{t-1}}$ due to $A1$. Moreover, $F_0, Z_1, \ldots, Z_t$ satisfies the $(1, \Gamma, p)$-BMSB condition due to (1) in the premise, establishing (b). We are left to prove that (c) $P \left( \sum_{i=1}^t Z_i Z_i^\top \not\preceq \Gamma_i \right) \leq \delta$, $\delta \in (0, 1)$, $\Gamma_{sb} \succeq \Gamma$, and (6) holds with $k = 1$.

We start by proving that $\delta \in (0, 1)$. Since $i \geq M(d, p, q, \Gamma_{sb})$, then $i \geq M(d, p, q, \Gamma_{sb})$, and so

$$
i \geq e^{-\frac{3M(d, \gamma)(\gamma)^2}{\lambda_{min}(\Gamma_{sb})}} c_1(q, \Gamma_{sb}) \frac{\sqrt{i}}{2} [c_2(p) + \log(\det(\Gamma_{sb})^{-1})]$$

holds by definition in (5). Thus, $\delta = i \geq e^{-\frac{3M(d, \gamma)(\gamma)^2}{\lambda_{min}(\Gamma_{sb})}} c_1(q, \Gamma_{sb}) \frac{\sqrt{i}}{2} [c_2(p) + \log(\det(\Gamma_{sb})^{-1})] \in (0, 1)$.

Next, since $\delta \in (0, 1)$ and $i \geq 1$, we find that $\Gamma = \frac{1}{2} \gamma^2 (q + \lambda_{max}(\Gamma_{sb})) > (q + \lambda_{max}(\Gamma_{sb})) > \Gamma_{sb}$ holds.

Now, we know from 2) in the premise that $P \left( \sum_{i=1}^t Z_i Z_i^\top \not\preceq \frac{\sqrt{i}}{2} \Gamma_i \right) \leq \delta$, which implies that $P \left( \sum_{i=1}^t Z_i Z_i^\top \not\preceq \frac{1}{2} \Gamma_i \right) = P \left( \sum_{i=1}^t Z_i Z_i^\top \not\preceq \frac{1}{2} \Gamma_i \right)$

where we rely on $\left( \frac{p^2}{10} - \frac{\lambda_{min}(\Gamma_{sb}) \gamma^2}{(90\Sigma W_p)} \right) > 0$ for $d \in \left( 0, \frac{90\Sigma W_p}{\sqrt{10 \lambda_{min}(\Gamma_{sb})}} \right)$. It follows from (60) and (61) that $\delta \geq i \geq e^{-\frac{3M(d, \gamma)(\gamma)^2}{\lambda_{min}(\Gamma_{sb})}} c_1(q, \Gamma_{sb}) \frac{\sqrt{i}}{2} [4 \log(10/p) + \log(\det(\Gamma_{sb})^{-1})]$. Rearranging the right hand side of this inequality, we find

$$\delta \geq e^{\frac{i}{\delta} \left( -\frac{p^2}{10} \gamma^2 + 2 \log(c_2(q, \Gamma_{sb})) + 4 \log(10/p) + \log(\det(\Gamma_{sb})^{-1}) \right) \log(\det(\Gamma_{sb})^{-1})}.$$
\(\lambda \in (a^2, 1)\), with \(\beta(\lambda)\) defined in Lemma 7. The conclusion follows by choosing \(\lambda \in (a^2, 1)\), and setting \(c = x_2^3 + \beta(\lambda) x\).

Case 2: Suppose \(a \in \{-1, 1\}\), and consider the process \((X_t)_{t \in \mathbb{N}_0}\). For all \(t \in \mathbb{N}_0\), \(\mathbb{E}[X_t^2]\) is upper bounded by

\[
\mathbb{E}[X_t^2] = \mathbb{E}\left[(X_t - X_t^* + X_t^*)^2\right] \\
\leq 2 \left(\mathbb{E}\left[(X_t - X_t^*)^2\right] + \mathbb{E}\left[(X_t^*)^2\right]\right),
\]

(67)

where (67) follows from \((a + b)^2 \leq 2(a^2 + b^2)\) for \(a, b \in \mathbb{R}\), linearity of expectation, and the definition of \(X_t^*\). From Lemma 8, we know that there exists \(\epsilon_1 > 0\) such that for all \(t \in \mathbb{N}_0\), \(\mathbb{E}\left[(X_t^*)^2\right] \leq \epsilon_1\).

Now, we aim to prove that there exists \(\epsilon_2 > 0\) such that for all \(t \in \mathbb{N}_0\), \(\mathbb{E}\left[(X_t - X_t^*)^2\right] \leq \epsilon_2\).

Let \(d^* := \min(m/2, 1/2, |b|/2)\). Using the law of total expectation and the definition of \(T_d\), for all \(t \in \mathbb{N}_0\), \(\mathbb{E}\left[(X_t - X_t^*)^2\right]\) can be upper bounded by

\[
\mathbb{E}\left[(X_t - X_t^*)^2\right] \leq \sum_{k=1}^{\infty} \mathbb{E}\left[(X_t - X_t^*)^2 \mid T_d= k\right] P(T_d= k). \quad (68)
\]

From Lemma 10, we find that for all \(k \in \mathbb{N}\) and \(t \in \mathbb{N}_0\), on the event \((T_d= k)\), \(|X_t - X_t^*| \leq \max(2|b| D(k + 1), 2 \left(\frac{|b| + d^*}{1 - d^*} + 3 |b|\right) D)\). Using the monotonicity property of expectation, it follows that

\[
\mathbb{E}\left[(X_t - X_t^*)^2 \mid T_d= k\right] \leq \mathbb{E}\left[\max\left(2|b| D(k + 1), 2 \left(\frac{|b| + d^*}{1 - d^*} + 3 |b|\right) D\right)^2 \mid T_d= k\right]
\]

\[= \max\left(4b^2D^2(k + 1)^2, 4 \left(\frac{|b| + d^*}{1 - d^*} + 3 |b|\right)^2 D^2\right). \quad (69)
\]

Next, using the union bound, we find that for all \(k \in \mathbb{N}\)

\[
P(T_d \geq k) = P\left(\exists i \geq k - 1 : \left\|\hat{\theta}_i - \theta_*\right\|_2 > d^*\right) \leq \sum_{i \geq k-1} P\left(\left\|\hat{\theta}_i - \theta_*\right\|_2 > d^*\right). \quad (70)
\]

Combining (68), (69) and (70) we find

\[
\mathbb{E}\left[(X_t - X_t^*)^2\right] \leq \sum_{k=1}^{\infty} \max\left(4b^2D^2(k + 1)^2, 4 \left(\frac{|b| + d^*}{1 - d^*} + 3 |b|\right)^2 D^2\right) \times \sum_{i \geq k-1} P\left(\left\|\hat{\theta}_i - \theta_*\right\|_2 > d^*\right) =: e_2
\]

(71)

for \(k \in \mathbb{N}\), where we introduce \(e_2\) to denote the infinite sum which uniformly bounds \(\mathbb{E}\left[(X_t - X_t^*)^2\right]\) for all \(t \in \mathbb{N}_0\).

Now, let \(N = \left\lceil \frac{|b| + d^*}{|b|} + 3 |b| \right\rceil - 1\). We find that

\[
\sum_{k=1}^{\infty} \max\left(4b^2D^2(k + 1)^2, 4 \left(\frac{|b| + d^*}{1 - d^*} + 3 |b|\right)^2 D^2\right) \times \sum_{i \geq k-1} P\left(\left\|\hat{\theta}_i - \theta_*\right\|_2 > d^*\right)
\]

\[= 4b^2D^2 \sum_{k=1}^{\infty} (k + 1)^2 \sum_{i \geq k-1} P\left(\left\|\hat{\theta}_i - \theta_*\right\|_2 > d^*\right) \quad (72)
\]

\[\leq 4b^2D^2 \sum_{k=1}^{\infty} (k + 1)^2 \sum_{i \geq k-1} P\left(\left\|\hat{\theta}_i - \theta_*\right\|_2 > d^*\right)
\]

\[\leq 8b^2D^2 \sum_{k=1}^{\infty} k^2 \sum_{i \geq k-1} P\left(\left\|\hat{\theta}_i - \theta_*\right\|_2 > d^*\right) \quad (73)
\]

where (72) follows since \(\max\left(4b^2D^2(k + 1)^2, 4 \left(\frac{|b| + d^*}{1 - d^*} + 3 |b|\right)^2 D^2\right) = 4b^2D^2 (k + 1)^2\) for \(k \leq N\), and (73) follows from the assumption that \(\sum_{k=1}^{N-1} k^2 \sum_{i \geq k-1} P\left(\left\|\hat{\theta}_i - \theta_*\right\|_2 > d\right) < \infty\) for all \(d \in (0, m)\) in the premise. Moreover, we have that

\[
\sum_{k=1}^{N-1} \max\left(4b^2D^2(k + 1)^2, 4 \left(\frac{|b| + d^*}{1 - d^*} + 3 |b|\right)^2 D^2\right) \times \sum_{i \geq k-1} P\left(\left\|\hat{\theta}_i - \theta_*\right\|_2 > d^*\right)
\]

\[= 4 \left(\frac{|b| + d^*}{1 - d^*} + 3 |b|\right)^2 D^2 \times \sum_{k=1}^{N-1} \sum_{i \geq k-1} P\left(\left\|\hat{\theta}_i - \theta_*\right\|_2 > d^*\right) \quad (74)
\]

\[\leq 4 \left(\frac{|b| + d^*}{1 - d^*} + 3 |b|\right)^2 D^2 \times \sum_{k=1}^{\infty} k^2 \sum_{i \geq k-1} P\left(\left\|\hat{\theta}_i - \theta_*\right\|_2 > d^*\right) \quad (75)
\]

where (74) follows since \(\max\left(4b^2D^2(k + 1)^2, 4 \left(\frac{|b| + d^*}{1 - d^*} + 3 |b|\right)^2 D^2\right) = 4 \left(\frac{|b| + d^*}{1 - d^*} + 3 |b|\right)^2 D^2\) for \(k < N\), and (75) follows from the premise. From (74), (73) and (71), it follows that \(e_2 < \infty\). Our conclusion follows by setting \(e = 2(e_1 + e_2)\).

Proof of Lemma 7: Suppose \(t \in \mathbb{N}_0\). Recall the closed-loop system from (5). Squaring this, we obtain

\[
X_{t+1}^2 = (|a| |X_t| + |b| U_t + |W_t|)^2 \\
\leq (|a| |X_t| + |b| U_{\max} + |W_t|)^2 \quad (76)
\]

\[= a^2X_t^2 + 2 |a| |X_t| C_t + C_t^2 \quad (77)
\]

where (76) holds since \(G_1\) is satisfied by construction, and (77) holds by the definition \(C_t := |b| U_{\max} + |W_t|\). Note that the first and second moments of \(C_t\) satisfy \(\mathbb{E}[C_t] = D_1\) and \(\mathbb{E}[C_t^2] = D_2\).
Now define \( K(\lambda) := \{ x \in \mathbb{R} : |x| \leq E(\lambda) \} \) for all \( \lambda \in (a^2, 1) \). Suppose \( \lambda \in (a^2, 1) \). On the event \( \{ X_t \notin K(\lambda) \} \), we have

\[
E[X_{t+1}^2 | X_t] \leq a^2 X_t^2 + 2 |a| |X_t| D_1 + D_2 \quad (78)
\]

\[
\leq \lambda X_t^2 \quad (79)
\]

where (78) follows from (77) since \( C_t \) is independent of \( X_t \), and (79) follows from the fact that on the event \( \{ X_t \notin K(\lambda) \} \), \( |X_t| > E(\lambda) \) holds (by definition), as well as the fact that \( |x| > E(\lambda) \implies \lambda x^2 \geq |a|^2 |x|^2 + 2 |a| |x| D_1 + D_2 \) (seen by applying the quadratic formula to solve for the set of \( |x| \) such that \( (\lambda - |a|^2) |x|^2 - 2 |a| |x| D_1 + D_2 \geq 0 \)). On the event \( \{ X_t \in K(\lambda) \} \), we have,

\[
E[X_{t+1}^2 | X_t] \leq a^2 X_t^2 + 2 |a| |X_t| D_1 + D_2
\]

\[
\leq a^2 E(\lambda)^2 + 2 |a| E D_1 + D_2 = \beta(\lambda) \quad (80)
\]

where (80) follows since on the event \( \{ X_t \in K(\lambda) \} \), \( |X_t| \leq E(\lambda) \) holds (from the definition of \( K(\lambda) \)). Finally, we find

\[
E[X_t^2] = E[E[X_t^2 | X_{t-1}]]
\]

\[
= E[E[X_t^2 | X_{t-1}] \mathbf{1}_{\{X_{t-1} \notin K(\lambda)\}}] + E[E[X_t^2 | X_{t-1}] \mathbf{1}_{\{X_{t-1} \in K(\lambda)\}}]
\]

\[
\leq E[\lambda X_{t-1}^2 \mathbf{1}_{\{X_{t-1} \notin K(\lambda)\}}] + \lambda E[X_{t-1}^2 \mathbf{1}_{\{X_{t-1} \in K(\lambda)\}}]
\]

\[
\leq \lambda \beta(\lambda) + \beta(\lambda)
\]

\[
\leq \lambda \beta(X_t^2 + \beta(\lambda) \sum_{k=0}^{t-1} \lambda^{t-1-k})
\]

\[
\leq a^2 \beta(\lambda) \quad (84)
\]

for all \( t \in \mathbb{N}_0 \), where (81) follows from conditions (79) and (80), (82) follows by iteratively applying (82), and (84) follows from \( X_t^2 = x_0^2 \) and the infinite sum of a geometric sequence.

Proof of Lemma 9 Define \( Y_t^* := a^t X_t^* \), \( t \in \mathbb{N}_0 \). Then, \( E[(X_t^*)^2] \) can be equivalently rewritten as follows for all \( t \in \mathbb{N}_0 \):

\[
E[(X_t^*)^2] = E[(Y_t^*)^2]
\]

\[
= E\left[(Y_t^*)^2 \right] + E\left[(-Y_t^*)^2 \right] \quad (85)
\]

where (85) follows from the properties of \((\cdot)^+\) and linearity of expectation. We will prove that there exists \( e_1 > 0 \) such that for all \( t \in \mathbb{N}_0 \), \( E\left[(Y_t^*)^2 \right] \leq e_1 \). This will be accomplished by showing that \( (Y^*)_{t \in \mathbb{N}_0} \) satisfies all of the conditions in Proposition 9. In particular, the conditions are satisfied with \( \gamma = |b| D, \ J = |b| D + |x_0|, \) and \( M = 8 (b^4 t^4_{\max} + S_4) \). Firstly, note that the condition \( Y_0^* \leq J \) is satisfied since \( Y_0^* = a^0 X_0^* \leq |b| D + |x_0| \).

Next, we verify condition (16). Note that

\[
Y_{t+1}^* = a^{t+1} X_{t+1}^*
\]

\[
= a^t Y_t^* + a^{t+1} b \sigma_D \left(-\frac{a}{b} X_t^* \right)
\]

\[
+ a^{t+1} \left( b V_t + W_t \right) \quad (86)
\]

\[
= Y_t^* - \sigma_{|b|D} (Y_t^*) + a^{t+1} \left( b V_t + W_t \right) \quad (87)
\]

where (86) holds from the closed-loop system (5) and the definition of \( Y_t^* \), and (87) is due to the following equality:

\[
a^{t+1} b \sigma_D \left(-\frac{a}{b} X_t \right)
\]

\[
\leq \begin{cases} 
  b \left( -\frac{a}{b} a^{-t+1} Y_t^* \right), & \left| -\frac{a}{b} a^{-t+1} Y_t^* \right| \leq D \\
  b \left( -\frac{a}{b} a^{-t+1} Y_t^* \right), & \left| -\frac{a}{b} a^{-t+1} Y_t^* \right| > D 
\end{cases} \quad (88)
\]

\[
= \begin{cases} 
 (\lambda^*), & \left| Y_t^* \right| \leq D \left| b \right| \\
 \left| \frac{\lambda^*}{\gamma} \right| D \left| b \right|, & \left| Y_t^* \right| > D \left| b \right| = \sigma_{|b|D} (Y_t^*) \quad (89)
\end{cases}
\]

where both (88) and (89) follow from the definition of \( \sigma_D(.) \). Let \( F_t \) be the natural filtration of the process \((Y_t^*)_{t \in \mathbb{N}_0}\). For all \( t \in \mathbb{N}_0 \), on the event \( \{ Y_t^* > |b| D + |x_0| \} \), we have

\[
E[Y_{t+1}^* - Y_t^* | F_t] = E[-\sigma_{|b|D} (Y_t^*) + a^{t+1} \left( b V_t + W_t \right) | F_t] \quad (90)
\]

\[
= -|b| D = -\gamma, \quad (91)
\]

where (90) holds due to (87), and (91) holds since \( \sigma_{|b|D} (Y_t^*) = |b| D \) when \( Y_t^* > |b| D \), and \( E[b V_t + W_t | F_t] = 0 \). Thus, condition (16) has been verified.

We now verify condition (17) as follows:

\[
E \left[ \left| Y_{t+1}^* - Y_t^* \right|^4 | Y_0^*, \ldots, Y_t^* \right] \leq E \left[ \left( |b| (D + C) + |W_t| \right)^4 | Y_0^*, \ldots, Y_t^* \right] \quad (92)
\]

\[
\leq 8 \left( b^4 U_{\max}^4 + E \left[ |W_t|^4 \right] \right) = M, \quad (93)
\]

where (92) follows from (87) and the definition of \( \sigma_{|b|D}(.) \), and (93) follows from \((a + b)^4 \leq 8(a^4 + b^4)\) for \( a, b \in \mathbb{R} \) and linearity of conditional expectation.

Therefore, since \( (Y_t^*)_{t \in \mathbb{N}_0} \) satisfies the conditions in Proposition 9, we find that there exists \( e_1 > 0 \) such that for all \( t \in \mathbb{N}_0 \), \( E\left[(Y_t^*)^2 \right] \leq e_1 \).

Following an analogous method, we are also able to establish that the process \(-Y_t^*)_{t \in \mathbb{N}_0} \) satisfies the conditions in Proposition 9 so there exists \( e_2 > 0 \) such that for all \( t \in \mathbb{N}_0 \), \( E\left[(-Y_t^*)^2 \right] \leq e_2 \). Setting \( c = e_1 + e_2 \), it follows that \( E\left[(Y_t^*)^2 \right] \leq c \), and therefore \( E\left[(X_t^*)^2 \right] \leq c \).

Proof of Lemma 10 For all \( t \in \mathbb{N}_0 \), the error \( X_{t+1} - X_{t+1}^* \) ...
where (94) follows from (5). Taking the absolute value, we find $|X_{t+1} - X_{t+1}^*|$ is upper bounded in terms of $|X_t - X_t^*|$ as follows

$$|X_{t+1} - X_{t+1}^*| = \left| a \left( X_t - X_t^* \right) + b \left( |\sigma_D(G_tX_t) - \sigma_D \left( -\frac{a}{b}X_t^* \right) | \right) \right| \leq |X_t - X_t^*| + |b| |\sigma_D(G_tX_t) - \sigma_D \left( -\frac{a}{b}X_t^* \right) | \leq |X_t - X_t^*| + |b| 2 D$$

By iteratively applying (95), the following then holds for $k \in \mathbb{N}_0$, and $t \leq k + 1$:

$$|X_t - X_t^*| \leq t |b| 2 D \leq (k + 1) |b| 2 D$$

We now move onto the case where $t \geq k + 1$. Firstly, define the processes $(Y_t)_{t\in\mathbb{N}_0}$ and $(Y_t^*)_{t\in\mathbb{N}_0}$ so $Y_t := a^t X_t$ and $Y_t^* := a^t X_t^*$. Their difference $Y_{t+1} - Y_{t+1}^*$ can be written as

$$Y_{t+1} - Y_{t+1}^* = a^t+1 (X_{t+1} - X_{t+1}^*) = a^t+1 \left( a \left( X_t - X_t^* \right) + b \left( |\sigma_D(G_tX_t) - \sigma_D \left( -\frac{a}{b}X_t^* \right) | \right) \right) = Y_t - Y_t^* + a^t+1 b \left( |\sigma_D(G_tX_t) - \sigma_D \left( -\frac{a}{b}X_t^* \right) | \right)$$

where (97) follows from (95).

Next, let $\Omega$ denote the underlying sample space. Suppose $d \in \left( 0, \min(1, |b|) \right)$. Let $H = (|b| + d)/(1 - d)$, and let $E_1 = \{ |Y_t| > HD \}$, $E_2 = \{ |Y_t^*| > HD \}$, $E_3 = \{ |Y_t| > (H + 2 |b|) D \}$, $E_4 = \{ |Y_t^*| > (H + 2 |b|) D \}$. Moreover, let $A_1 = \{ |Y_t| > HD, |Y_t^*| > HD \}$, $A_2 = \{ |Y_t| > (H + 2 |b|) D \}$, $A_3 = \{ |Y_t| \leq HD, |Y_t^*| > (H + 2 |b|) D \}$, and $A_4 = \{ |Y_t| \leq (H + 2 |b|) D \}$. The sample space $\Omega$ can be equivalently written as

$$\Omega = (E_1 \cap E_2) \cup (E_1 \cap E_3^C) \cup (E_2 \cap E_3^C) \cup (E_3 \cap E_2^C) = (E_1 \cap E_2) \cup (E_1 \cap E_2 \cap E_3) \cup (E_3 \cap E_2 \cap E_1) \cup (E_3 \cap E_2 \cap E_1^C) = (E_1 \cap E_2) \cup (E_1 \cap E_2 \cap E_3) \cup (E_3 \cap E_2 \cap E_1) \cup (E_3 \cap E_2 \cap E_1^C) = A_1 \cup A_2 \cup A_3 \cup A_4,$$

with (99) holding since $A_1 = E_1 \cap E_2$, $A_2 = E_3 \cap E_2^C$, $A_3 = E_3 \cap E_2$, and $(E_3 \cap E_2^C) \cup (E_3 \cap E_2) \cup (E_3 \cap E_2^C) \subseteq A_4 \subseteq \Omega$. Since $|Y_{t+1} - Y_{t+1}^*|$ takes values in $\mathbb{R}_{\geq 0}$, making use of the properties of the indicator function, we find

$$|Y_{t+1} - Y_{t+1}^*| = \max \left( |Y_{t+1} - Y_{t+1}^*|, 1_{A_1}, |Y_{t+1} - Y_{t+1}^*|, 1_{A_2}, |Y_{t+1} - Y_{t+1}^*|, 1_{A_3}, |Y_{t+1} - Y_{t+1}^*|, 1_{A_4} \right).$$

We now prove upper bounds for $|Y_{t+1} - Y_{t+1}^*|$ on the event $\{ T_d = k \} \cap A_i$ for $i \in \{ 1, \ldots, 4 \}$ and $t \geq k + 1$.

Case 1: Consider the event $\{ T_d = k \} \cap A_1$, and suppose $t \geq k + 1$. Since $|Y_t| > HD, |Y_t^*| > HD$ on $A_1$, we have

$$Y_{t+1} - Y_{t+1}^* = Y_t - Y_t^* + a^{t+1} b \left( |\sigma_D \left( -\frac{a}{b}X_t \right) | \right) \leq Y_t - Y_t^* + a^{t+1} b \left( \left| \frac{a}{b} X_t \right| D \right).$$

where (101) follows from (98) and (2), (102) follows from the definition of $\sigma_D(\cdot)$, and (103) follows from Lemma 14.

If $Y_t > HD$ and $Y_t^* > HD$, or $Y_t < -HD$ and $Y_t^* < -HD$, then $\frac{Y_t}{|Y_t^*|} - \frac{Y_t^*}{|Y_t|} = 0$ and so $Y_{t+1} - Y_{t+1}^* = Y_t - Y_t^*$. If $Y_t > HD$ and $Y_t^* < -HD$, then $\frac{Y_t}{|Y_t^*|} - \frac{Y_t^*}{|Y_t|} = 2$, and so $|Y_{t+1} - Y_{t+1}^*| = |Y_t - Y_t^*| + 2 |b| D \leq |Y_t - Y_t^*| + 2 |b| D \leq |Y_t - Y_t^*| - 2 |b| D \leq |Y_t - Y_t^*|$. If $Y_t < -HD$ and $Y_t^* > HD$, then $\frac{Y_t}{|Y_t^*|} - \frac{Y_t^*}{|Y_t|} = -2$, and so $|Y_{t+1} - Y_{t+1}^*| = |Y_t - Y_t^*| + 2 |b| D = (-1) (Y_t - Y_t^* + 2 |b| D) = (-1) (Y_t - Y_t^*) - 2 |b| D = |Y_t - Y_t^*| - 2 |b| D \leq |Y_t - Y_t^*|$. Thus, it follows that on the event $\{ T_d = k \} \cap A_1$, $|Y_{t+1} - Y_{t+1}^*| \leq |Y_t - Y_t^*|$ holds.

Case 2: Consider the event $\{ T_d = k \} \cap A_2$, and suppose $t \geq k + 1$. Since $|Y_t| \leq (H + 2 |b|) D$ and $|Y_t^*| \leq HD$ on
\[ Y_{t+1} - Y_{t*}^+ = Y_t - Y_t^* + a^{t+1}b \left( \sigma_D \left( -\frac{\hat{a}_{t-1}}{b_{t-1}} X_t \right) - \sigma_D \left( \frac{a}{b} X_t^* \right) \right) \]

(104)

\[ = Y_t - Y_t^* + a^{t+1}b \left( \frac{a}{b} X_t \right) \]

(105)

\[ = Y_t - Y_t^* + a^{t+1}b \left( \frac{a}{b} X_t \right) \]

(106)

where (104) follows from (98) and (2), (105) follows from the definition of \( \sigma_D(\cdot) \) and (106) follows from Lemma 14.

Consider when \( Y_t > (H + 2 |b|)D \). Then, \( Y_{t+1} - Y_{t*}^+ = Y_t - Y_t^* + |b| D + a^{t+1}b \left( -\sigma_D \left( \frac{a}{b} X_t^* \right) \right) \). Since \( HD > Y_t \), then \( Y_{t+1} - Y_t^* \geq (H + 2 |b|)D - HD = 2 |b| D \). Moreover, since \( a^{t+1}b \left( -\sigma_D \left( \frac{a}{b} X_t^* \right) \right) \leq |b| D \), then \( Y_{t+1} - Y_t^* \geq |b| D + a^{t+1}b \left( -\sigma_D \left( \frac{a}{b} X_t^* \right) \right) \). Therefore, \( Y_{t+1} - Y_t^* = Y_t - Y_t^* + |b| D + a^{t+1}b \left( -\sigma_D \left( \frac{a}{b} X_t^* \right) \right) \leq Y_t - Y_t^* + |b| D \).

Now, consider when \( Y_t < -(H + 2 |b|)D \). Then, \( Y_{t+1} = Y_t - Y_t^* + |b| D + a^{t+1}b \left( -\sigma_D \left( \frac{a}{b} X_t^* \right) \right) \).

\[ \text{Case 3: Consider the event } \{ T_d = k \} \cap A_2, \quad Y_{t+1} - Y_t^* \leq Y_t - Y_t^* \text{ holds}. \]

\[ \text{Case 4: Consider the event } \{ T_d = k \} \cap A_4, \text{ and suppose } t \geq k + 1. \]

The following holds:

\[ |Y_{t+1} - Y_t^*| \leq 2(\text{min}(1,|b|))D + |b|/2. \]

(107)

Iteratively applying (108), we find that \(|Y_t - Y_t^*| \leq \text{max} \left( |Y_{k+1} - Y_{k+1}^*|, 2(H + 3 |b|)D \right) \) for all \( t \geq k + 1 \).

Lemma 14. Consider the parameter estimates \((\hat{a}_t, \hat{b}_t)_{t \in \mathbb{N}}\) from (5). Suppose A1-A2 and \( \alpha \in (-1, 1) \) hold on the closed-loop system (5) and \( x_0 \in \mathbb{R} \). Then, for all \( d \in (0, \min(1, |b|)) \), \( k \in \mathbb{N} \), and \( t \geq k + 1 \), \( \hat{a}_{t-1} - |a|/|a| \) and \( \hat{b}_{t-1} - b_{t-1} = b/|b| \) hold on the event \( \{ T_d = k \} \) (with \( T_d \) defined in (10)).

Proof. Suppose \( k \in \mathbb{N} \), and \( t \geq k + 1 \). On the event \( \{ T_d = k \} \), we have \( |\hat{a}_{t-1} - a| \leq \|\hat{a}_{t-1} - \theta_a\|_2 \leq \|\hat{a}_{t-1} - \theta_a\|_\infty < 1 \), and so \( \hat{a}_{t-1} \in (a-1, a+1) \), which implies \( \hat{a}_{t-1}/|\hat{a}_{t-1}| = a/|a| \).

Additionally, on the event \( \{ T_d = k \} \) we have \( \|\hat{b}_{t-1} - b\|_2 \leq \|\hat{b}_{t-1} - \theta_b\|_\infty \leq \|\hat{b}_{t-1} - \theta_b\|_2 < |b| \), and so \( \hat{b}_{t-1} \in (b - |b|, b + |b|) \), which implies \( \hat{b}_{t-1}/|\hat{b}_{t-1}| = b/|b| \).

\[ \Box \]