A new perspective on the fictitious space lemma

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Introduction

In the present contribution we propose a new proof of the so-called fictitious space lemma. For the proof, we exhibit an explicit expression for the inverse of additive Schwarz preconditioners in terms of Moore-Penrose pseudo inverse of the map associated to the decomposition over the subdomain partition.

We will first briefly recall the definition of the pseudo-inverse of a matrix and some of its remarkable properties. We will then explain how this concept can be used to reformulate the fictitious space lemma in a very compact form. We will then give an alternative proof of the fictitious space lemma. As a remarkable feature, this proof does not rely on Cauchy-Schwarz inequality, as opposed to previous proofs provided by Nepomnyaschikh [5, 4], Griebel and Oswald [2], Dolean, Jolivet and Nataf [1], see also [6]. The present proof applies directly in the infinite dimensional case.

1 Moore-Penrose pseudo-inverse

Assume given Hilbert spaces $H$ (resp. $V$) equipped with the norms $\| \cdot \|_H$ (resp. $\| \cdot \|_V$) and consider a surjective map $R : V \to H$. Define $R^{-1}(\{y\}) := \{x' \in H, \; Rx' = y\}$. The Moore-Penrose pseudoinverse of this map, denoted $R^\dagger : H \to V$ is defined, for all $y \in H$, by

$$R \cdot R^\dagger y = y \quad \text{and} \quad \|R^\dagger y\|_V = \inf_{x \in R^{-1}(\{y\})} \|x\|_V. \quad (1)$$

The property above directly implies that $R^\dagger$ is injective. Let us denote $V_R = \text{Ker}(R)$. For any $y \in H$, since the restricted operator $R|_{V_R} : V_R \to H$ is a bijection, there exists a unique $x \in V_R$ such that $Rx = y$. Besides, if $x' \in V$ is another element satisfying $Rx' = y$ then $x - x' \in \text{Ker}(R)$ so that $(x, x - x')_V$ and thus, by Pythagore’s rule,

$$\|x\|_V^2 \leq \|x\|_V^2 + \|x - x'\|_V^2 = \|x'\|_V^2 \quad (2)$$

As a consequence $x \in V$ solves the minimization problem (1) i.e. $x = R^\dagger y$. From this discussion we conclude that $R^\dagger = (R|_{V_R})^{-1}$.

The property $RR^\dagger = \text{Id}$ implies that $(R^\dagger R)^2 = R^\dagger (RR^\dagger)R = R^\dagger R$ i.e. $R^\dagger R$ is a projector. Because $R^\dagger$ is injective we obtain $\text{Ker}(R^\dagger R) = \text{Ker}(R)$. Besides for any $x \in V_R$ satisfying
\( Rx = y \), we have seen that \( x = R^\dagger y = R^\dagger R x \) which implies \( V_R = \text{Im}(R^\dagger R) \). As a conclusion, since \( \text{Ker}(R) \) and \( V_R \) are orthogonal by definition, we conclude that \( R^\dagger R \) is an orthogonal projection, which rewrites
\[
(R^\dagger R x, y)_V = (x, R^\dagger R y) \quad \forall x, y \in V.
\]

### 2 Weighted pseudo-inverse

Keeping the notations from the previous section, consider continuous operator \( B : V \to V \), and assume this operator is self-adjoint so that it induces a scalar product \((x, y)_B := (B x, y)_V\) and a norm \(\|x\|_B := \sqrt{(x, x)_B}\). To each such \( B \) can be associated a so-called "weighted pseudo-inverse" \( R^\dagger_B : H \to V \) defined, for all \( y \in H \) by
\[
R \cdot R^\dagger_B y = y \quad \text{and} \quad \|R^\dagger_B y\|_B = \inf_{x \in \text{R}^{-1}(\{y\})} \|x\|_B.
\]

The operator \( R^\dagger_B \) satisfies the same properties as \( R^\dagger \) except that \((\ , \ )_V\) is this times replaced by \((\ , \ )_B\). In particular \ref{eq:adjoint} rewrites \( (R^\dagger_B R x, y)_B = (x, R^\dagger_B R y)_B \) for all \( x, y \in V \). Taking account of the expression of \((\ , \ )_B\) this is equivalent to
\[
B R^\dagger_B R = (R^\dagger_B R)^* B
\]
where, for any continuous linear operator \( M : V \to V \) we denote \( M^* \) its adjoint with respect to \((\ , \ )_V\) defined by \((M x, y)_V = (x, M^* y)_V\) for all \( x, y \in V \). Property \ref{eq:adjoint} leads to a lemma.

**Lemma 2.1.**
\[
RB^{-1} R^* = ((R^\dagger_B)^* BR^\dagger_B)^{-1}
\]

**Proof:** Since \( RR^\dagger_B = \text{Id} \) by construction, the lemma is a consequence of \ref{eq:adjoint} through direct calculation \((RB^{-1} R^*) \cdot ((R^\dagger_B)^* BR^\dagger_B) = RB^{-1}(R^\dagger_B R^*)^* BR^\dagger_B = R(B^{-1} B)R^\dagger_B R R^\dagger_B = (RR^\dagger_B)^2 = \text{Id}.\)

### 3 Re-interpretation of the fictitious space lemma

In this section, we provide a new proof of the fictitious space lemma relying on the concept weighted pseudo-inverse. As a preliminary, let us recall a classical characterisation of extremal eigenvalues of self-adjoint operators (see e.g. theorem 1.2.1 and theorem 1.2.3 in \cite{[3]}).

**Lemma 3.1.**

Assume \( H \) is an Hilbert space equipped with the scalar product \((\ , \ )_H\) and let \( T : H \to H \) be a bounded operator that is self-adjoint for \((\ , \ )_H\). Denoting \( \sigma(T) \) the spectrum of \( T \), we have
\[
\inf \sigma(T) = \inf_{x \in H \setminus \{0\}} (Tx, x)_H \quad \text{and} \quad \sup \sigma(T) = \sup_{x \in H \setminus \{0\}} (Tx, x)_H
\]

This lemma holds independently of the choice of the scalar product, provided that \( T \) be self-adjoint with respect to it. As a consequence of the previous lemma, if \( \alpha(\ , \ ) \) and \( \beta(\ , \ ) \) are two scalar products over \( H \) and \( T \) is self-adjoint with respect to both, then
\[
\inf_{x \in H \setminus \{0\}} \alpha(Tx, x)/\alpha(x, x) = \inf_{x \in H \setminus \{0\}} \beta(Tx, x)/\beta(x, x),
\]
and a similar result holds for the supremum.

Now we recall the fictitious space lemma, adopting the same formulation of this result as [1, Lemma 7.4] and [2, p.168].

**Lemma 3.2.** Let \(H\) and \(V\) be two Hilbert spaces equipped with the scalar products \((\cdot, \cdot)_H\) and \((\cdot, \cdot)_V\). Let \(A : H \to H\) (resp. \(B : V \to V\)) be a bounded operator that is positive definite self-adjoint with respect to \((\cdot, \cdot)_H\) (resp. \((\cdot, \cdot)_V\)), and denote \((u, v)_A := (Au, v)_H\) (resp. \((u, v)_B := (Bu, v)_V\)). Suppose that there exists a surjective bounded linear operator \(R : V \to H\), and constants \(c_{\pm} > 0\) such that

i) for all \(u \in H\) there exists \(v \in V\) with \(Ru = u\) and \(c_{-}(v, v)_B \leq (u, u)_A\),

ii) \((Ru, Rv)_A \leq c_{+}(v, v)_B\) for all \(v \in V\).

Then, denoting \(R^* : H \to V\) the linear map defined by \((Ru, v)_H = (u, R^*v)\) for all \(u \in V\), \(v \in H\), we have

\[
c_{-}(u, u)_A \leq (RB^{-1}R^*Au, u)_A \leq c_{+}(u, u)_A \quad \forall u \in H.
\]

In addition, if \(c_{\pm}\) are the optimal constants satisfying i)-ii) then the bounds in [3] are optimal as well.

**Proof:**

We simply reformulate i)-ii) by means of the weighted pseudo-inverse. If i) holds then, for any \(u \in H\) we have \(c_{-}\|v\|_B^2 \leq (u, u)_A\forall v \in R^{-1}(\{u\})\). Taking the infimum and using [4], we obtain \(c_{-}\|R_B^\dagger u\|_B \leq (u, u)_A\). On the other hand, it is clear that, if \(c_{-}\|R_B^\dagger u\|_B \leq (u, u)_A\forall u \in H\) then i) holds.

Next if ii) holds, then we have \((u, u)_A \leq c_{+}\|v\|_B\forall v \in R^{-1}(\{u\})\) and for all \(u \in H\). Taking the infimum over \(v \in R^{-1}(\{u\})\) and using [4], we conclude that \((u, u)_A \leq c_{+}\|R_B^\dagger u\|_B^2\forall u \in H\), and this is equivalent due to the optimality condition in [4]. To conclude we have just shown that conditions i)-ii) in Lemma 3.2 are actually equivalent to

\[
c_{-}(R_B^\dagger u, R_B^\dagger v)_B \leq (u, v)_A \leq c_{+}(R_B^\dagger u, R_B^\dagger v)_B \quad \forall u \in H.
\]

Next define \(S := (R_B^\dagger)^*BR_B^\dagger\), which is obviously bounded positive definite self-adjoint so it induces a scalar product \((u, v)_S := (Su, v)_H\) and a norm \(\|u\|_S := \sqrt{(u, u)_S}\). We can re-write \((R_B^\dagger u, R_B^\dagger v)_B = (u, u)_S\), and \((u, u)_A = (RB^{-1}R^*Au, u)_S\) according to Lemma 3.1. Hence (7) can be re-written

\[
c_{-}(u, u)_S \leq (RB^{-1}R^*Au, u)_S \leq c_{+}(u, u)_S \quad \forall u \in H.
\]

To conclude the proof there only remains to observe that, since \(RB^{-1}R^* = S^{-1}\), then \(RB^{-1}R^*A\) is self-adjoint with respect to both \((\cdot, \cdot)_S\) and \((\cdot, \cdot)_A\). As a consequence, Lemma 3.1 combined with (8) implies (9).
References

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