Uniform structures on differential spaces

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Abstract

The uniform structure on a differential space defined by a family of generators is considered.

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1 Introduction

This paper is the second of the series of publications concerning integration of differential forms and densities on differential spaces (the first one is [6]). It has the preliminary character. We recall basic facts from the theory of differential spaces and the theory of uniform structures. After that we describe uniform structures defined on a differential space by families of generators of its differential structure.

Section 2 of the paper contains basic definitions and the description of preliminary facts concerning theory of differential spaces. Foundations of theory of differential spaces can be find in [5]. In Section 3 we give basic definitions and describe the standard facts concerning theory of uniform spaces. We define (in a standard manner) the uniform structure given on a differential space by a family of generators of its differential structure. Section 4 contains basic facts concerning uniform (uniformly continuous) maps. In Section 5 we recall the definition of a complete uniform space and the standard construction of completion of a given uniform space. Here we introduce and investigate the notion of the extension of a differential structure.

Without any other explanation we use the following symbols: \( \mathbb{N} \)-the set of natural numbers; \( \mathbb{R} \)-the set of reals.

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2 Differential spaces

Let \( M \) be a nonempty set and let \( \mathcal{C} \) be a family of real valued functions on \( M \). Denote by \( \tau_\mathcal{C} \) the weakest topology on \( M \) with respect to which all functions of \( \mathcal{C} \) are continuous.

A base of the topology \( \tau_\mathcal{C} \) consists of sets:

\[
(\alpha_1, \ldots, \alpha_n)^{-1}(P) = \bigcap_{i=1}^{n} \{ p : a_i < \alpha_i(p) < b_i \},
\]

where \( n \in \mathbb{N}, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}, a_i < b_i, \alpha_1, \ldots, \alpha_n \in \mathcal{C}, P = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n; a_i < x_i < b_i, i = 1, \ldots, n \} \).

**Definition 2.1** A function \( f : M \to \mathbb{R} \) is called a local \( \mathcal{C} \)-function on \( M \) if for every \( p \in M \) there is a neighbourhood \( V \) of \( p \) and \( \alpha \in \mathcal{C} \) such that \( f|_V = \alpha|_V \). The set of all local \( \mathcal{C} \)-functions on \( M \) is denoted by \( \mathcal{C}_M \).

Note that any function \( f \in \mathcal{C}_M \) is continuous with respect to the topology \( \tau_\mathcal{C} \). In fact, if \( \{ V_i \}_{i \in I} \) is such an open (with respect to \( \tau_\mathcal{C} \)) covering of \( M \) that for any \( i \in I \) there exists \( \alpha_i \in \mathcal{C} \) satisfying \( f|_{V_i} = \alpha_i|_{V_i} \) and \( U \) is an open subset of \( \mathbb{R} \) then

\[
f^{-1}(U) = \bigcup_{i \in I} (\alpha_i|_{V_i})^{-1}(U).
\]

Since \( (\alpha_i|_{V_i})^{-1}(U) \) is open in \( V_i \) and \( V_i \in \tau_\mathcal{C} \) we obtain \( (\alpha_i|_{V_i})^{-1}(U) \in \tau_\mathcal{C} \) for any \( i \in I \). Hence \( f^{-1}(U) \in \tau_\mathcal{C} \). Bearing in mind that \( U \) is an arbitrary open set in \( \mathbb{R} \) we obtain that \( f \) is continuous with respect to \( \tau_\mathcal{C} \).

We have \( \mathcal{C} \subset \mathcal{C}_M \) which implies \( \tau_\mathcal{C} \subset \tau_{\mathcal{C}_M} \). On the other hand any element of \( \mathcal{C}_M \) is a function continuous with respect to \( \tau_\mathcal{C} \). Then \( \tau_{\mathcal{C}_M} \subset \tau_\mathcal{C} \) and consequently \( \tau_{\mathcal{C}_M} = \tau_\mathcal{C} \).

**Definition 2.2** A function \( f : M \to \mathbb{R} \) is called \( \mathcal{C} \)-smooth function on \( M \) if there exist \( n \in \mathbb{N}, \omega \in C^\infty(\mathbb{R}^n) \) and \( \alpha_1, \ldots, \alpha_n \in \mathcal{C} \) such that

\[
f = \omega \circ (\alpha_1, \ldots, \alpha_n).
\]

The set of all \( \mathcal{C} \)-smooth functions on \( M \) is denoted by \( \mathcal{C}_{sc} \).

We have \( \mathcal{C} \subset \mathcal{C}_{sc} \) which implies \( \tau_\mathcal{C} \subset \tau_{\mathcal{C}_{sc}} \). On the other hand any superposition \( \omega \circ (\alpha_1, \ldots, \alpha_n) \) is continuous with respect to \( \tau_\mathcal{C} \) which gives \( \tau_{\mathcal{C}_{sc}} \subset \tau_\mathcal{C} \). Consequently \( \tau_{\mathcal{C}_{sc}} = \tau_\mathcal{C} \).

**Definition 2.3** A set \( \mathcal{C} \) of real functions on \( M \) is said to be a (Sikorski’s) differential structure if: (i) \( \mathcal{C} \) is closed with respect to localization i.e. \( \mathcal{C} = \mathcal{C}_M \); (ii) \( \mathcal{C} \) is closed with respect to superposition with smooth functions i.e. \( \mathcal{C} = \mathcal{C}_{sc} \).

In this case a pair \((M, \mathcal{C})\) is said to be a (Sikorski’s) differential space.
Proposition 2.1. The intersection of any family of differential structures defined on a set \( M \neq \emptyset \) is a differential structure on \( M \).

Proof. Let \( \{ C_i \}_{i \in I} \) be a family of differential structures defined on a set \( M \) and let \( C := \bigcap_{i \in I} C_i \). Then \( C \) is nonempty family of real-valued functions on \( M \) (it contains all constant functions). If \( n \in \mathbb{N}, \omega \in C^{\infty}(\mathbb{R}^n) \) and \( \alpha_1, \ldots, \alpha_n \in C \) then for any \( i \in I \) \( \alpha_1, \ldots, \alpha_n \in C_i \) and consequently \( \omega \circ (\alpha_1, \ldots, \alpha_n) \in C_i \). Hence \( \omega \circ (\alpha_1, \ldots, \alpha_n) \in C \) which means that \( scC = C \).

Since \( C \subset C_i \) for any \( i \in I \) we have \( \tau_C \subset \tau_{C_i} \). It means that any subset of \( M \) open with respect to \( \tau_C \) is open with respect to \( \tau_{C_i} \), for \( i \in I \).

Let \( \beta \in C_M \). Choose for any \( p \in M \) a set \( U_p \in \tau_C \) and a function \( \alpha_p \in C \) such that \( p \in U_p \) and \( \beta |_{U_p} = \alpha_p |_{U_p} \). Since \( \alpha_p \in C_i \) and \( U_p \in \tau_{C_i} \), we obtain \( \beta \in (C_i)_M = C_i \) for any \( i \in I \). Then \( \beta \in C \) and consequently \( C_M = C \).

Equalities \( C_M = C = scC \) means that \( C \) is a differential structure on \( M \). \( \square \)

Let \( \mathcal{F} \) be a set of real functions on \( M \). Then, by Proposition 2.1, the intersection \( \mathcal{C} \) of all differential structures on \( M \) containing \( \mathcal{F} \) is a differential structure on \( M \). It is the smallest differential structure on \( M \) containing \( \mathcal{F} \). One can easy prove that \( \mathcal{C} = (sc\mathcal{F})_M \) (see \([4]\)). This structure is called the differential structure generated by \( \mathcal{F} \). Functions of \( \mathcal{F} \) are called generators of the differential structure \( \mathcal{C} \). We have also \( \tau_{(sc\mathcal{F})_M} = \tau_{sc\mathcal{F}} = \tau_\mathcal{F} \) (see remarks after Definitions 2.1 and 2.2).

Let \((M, \mathcal{C})\) and \((N, \mathcal{D})\) be differential spaces. A map \( F : M \to N \) is said to be smooth if for any \( \beta \in \mathcal{D} \) the superposition \( \beta \circ F \in \mathcal{C} \). We will denote the fact that \( f \) is smooth writing

\[
F : (M, \mathcal{C}) \to (N, \mathcal{D}).
\]

If \( F : (M, \mathcal{C}) \to (N, \mathcal{D}) \) is a bijection and \( F^{-1} : (N, \mathcal{D}) \to (M, \mathcal{C}) \) then \( F \) is called a diffeomorphism.

It is easy to show that if \( \mathcal{F} \) is a family of generators of the structure \( \mathcal{D} \) on the set \( N \) then \( F : (M, \mathcal{C}) \to (N, \mathcal{D}) \) iff for any \( f \in \mathcal{F} \) the superposition \( f \circ F \in \mathcal{C} \).

If \( A \) is a nonempty subset of \( M \) and \( \mathcal{C} \) is a differential structure on \( M \) then \( \mathcal{C}_A \) denotes the differential structure on \( A \) generated by the family of restrictions \( \{ \alpha |_A : \alpha \in \mathcal{C} \} \). The differential space \((A, \mathcal{C}_A)\) is called a differential subspace of \((M, \mathcal{C})\). One can easy prove the following

Proposition 2.2. Let \((M, \mathcal{C})\) and \((N, \mathcal{D})\) be differential spaces and let \( F : M \to N \). Then \( F : (M, \mathcal{C}) \to (N, \mathcal{D}) \) iff \( F : (M, \mathcal{C}) \to (F(M), F(M)_D) \).

If the map \( F : (M, \mathcal{C}) \to (F(M), F(M)_D) \) is a diffeomorphism then we say that \( F : M \to N \) is a diffeomorphism onto its range (in \((N, \mathcal{D})\)). In particular the natural embedding \( A \ni x \mapsto i(x) := x \in M \) is a diffeomorphism of \((A, \mathcal{C}_A)\) onto its range in \((M, \mathcal{C})\).

If \( \{ (M_i, \mathcal{C}_i) \}_{i \in I} \) is an arbitrary family of differential spaces then we consider the Cartesian product \( \prod_{i \in I} M_i \) as a differential space with the differential structure
\( \bigotimes_{i \in I} C_i \) generated by the family of functions \( F := \{ \alpha_i \circ pr_i : i \in I, \alpha_i \in C_i \} \), where \( \prod_{i \in I} M_i \ni (x_i) \mapsto pr_j((x_i)) =: x_j \in M_j \) for any \( j \in I \). The topology \( \tau_{\bigotimes_{i \in I} C_i} \) coincides with the standard product topology on \( \prod_{i \in I} M_i \).

A generator embedding of the differential space \((M, C)\) into the Cartesian space is a mapping \( \phi_F : (M, C) \rightarrow (R^F, C^\infty(R^F)) \) given by the formula

\[
\phi_F(p) = (\alpha(p))_{\alpha \in F}
\]

(for example if \( F = \{ \alpha_1, \alpha_2, \alpha_3 \} \) then \( \phi_F(p) = (\alpha_1(p), \alpha_2(p), \alpha_3(p)) \in R^3 \cong R^F \).

**Proposition 2.3.** Let \( F \) be a family of generators of the differential structure \( C \) on the set \( M \). If \( F \) separates points of \( M \) then the generator embedding defined by \( F \) is a diffeomorphism onto its image. On that image we consider a differential structure of a subspace of \((R^F, C^\infty(R^F))\).

**Proof.** Since \( F \) separates points of \( M \) it follows from the definition of differential embedding \( \phi_F \) that it is an one-to-one mapping onto its image in \( R^F \). Moreover for any \( f \in F \) we have \( pr_f \circ \phi_F = f \in C \). Since the differential structure \( C^\infty(R^F) \) is generated by the family \( \{ pr_g \}_{g \in C} \) we obtain that the map \( \phi_F \) is smooth with respect to \( C^\infty(R^F) \). It remains to show that the map \( \phi_F^{-1} \) is smooth.

For any \( f \in F \) we have

\[
f \circ \phi_F^{-1} = pr_f|_{F(M)}.
\]

It means that \( f \circ \phi_F^{-1} \in C^\infty(R^F)_{F(M)} \). Since the differential structure \( C \) is generated by \( F \) we obtain \( \phi_F^{-1} \) is smooth. \( \square \)

### 3 Uniform structures

Let \( X \) be a nonempty set.

**Definition 3.1.** A set \( \Delta = \{(x, x) : x \in X \} \) is said to be the diagonal of the product \( X \times X \). A set \( V \subset X \times X \) is called a neighbourhood of the diagonal if \( \Delta \subset V \) and \( V = -V \), where \( -V = \{(x, y) : (y, x) \in V \} \).

A family of all neighborhoods of the diagonal is denoted by \( D_X \).

**Definition 3.2** If for \( x, y \in X \) and \( V \in D_X \) we have \( (x, y) \in V \), then we say that \( x \) and \( y \) are distant less then \( V (|x - y| < V) \). We say that the diameter of a set \( A \subset X \) is less then \( V (\delta(A) < V) \) if for all \( x, y \in A \) we have \( |x - y| < V \). A ball with the center at \( x_0 \in X \) and the radius \( V \) is a set \( K(x_0, V) = \{ x \in X : |x_0 - x| < V \} \). The set

\[
2V := \{(x, y) \in X \times X : \exists z \in X [(x, z) \in V \land (x, z) \in V]\}.
\]
**Definition 3.3** A uniform structure $U$ on $X$ is a subfamily of $D_X$ satisfying the following conditions:

1) $(V \in U \land V \subset W \in D_X \Rightarrow (W \in U))$;

2) $(V_1, V_2 \in U) \Rightarrow (V_1 \cap V_2 \in U)$;

3) $\forall V \in U \exists W \in U [2W \subset V]$;

4) $\bigcap U = \Delta$.

If $U$ is a uniform structure on $X$ then the pair $(X, U)$ is called a uniform space.

**Definition 3.4** A base of a uniform structure $U$ in $X$ is a family $B \subset U$ such that for all $V \in U$ there exists $W \in B$ satisfying $W \subset V$.

Each base $B$ has following properties:

B1) $(V_1, V_2 \in B) \Rightarrow (\exists V \in B [V \subset V_1 \cap V_2])$;

B2) $\forall V \in B \exists W \in B [2W \subset V]$;

B3) $\bigcap B = \Delta$.

On the other hand it can be easy proved that if a family $B$ of neighbourhoods of the diagonal of a set $X$ fulfills conditions (B1)-(B3) then there exists exactly one uniform structure $U$ on $X$ such that $B$ is a base of $U$.

Every neighbourhood $V \in D_X$ of the diagonal defines the covering $P(V) = \{K(x, V)\}_{x \in X}$ of the set $X$. If $U$ is a uniform structure in $X$ then every covering $O$ of $X$ for which there exists $V \in U$ such that $P(V)$ is a refinement of $O$ is said to be a uniform covering (with respect to $U$).

Each uniform structure on $X$ defines a topology on $X$. In other words each uniform space $(X, U)$ defines a topological space $(X, \Theta)$.

**Theorem 3.1** If $U$ is a uniform structure on $X$, then a family $\Theta = \{G \subset X : \forall x \in G \exists V \in U \cdot [K(x, V) \subset G]\}$ is a topology in $X$ and $(X, \Theta)$ is $T_1$-space. A topology $\Theta$ is said to be a topology given in $X$ by uniform structure $U$ and is denoted by $\tau_U$.

For the proof see [1] or [2].

It can be proved that a topology $\tau$ on a topological space $X$ is given by some uniform structure on $X$ if and only if $X$ is a Tichonov space (see [2]).
Let \( \varrho \) be a pseudometric on a uniform space \((X, \mathcal{U})\). If for every \( \varepsilon > 0 \) there is \( V \in \mathcal{U} \) such that if \(|x - y| < V\) then \( \varrho(x, y) < \varepsilon \), then \( \varrho \) is called a uniform pseudometric (with respect to \( \mathcal{U} \)).

We can defined a uniform structure on three different ways: (i) if we give a base; (ii) if we give a family of uniform coverings or (iii) if we give a family of pseudometrics (see [2]).

Let \((M, \mathcal{C})\) be a differential space such that \( \mathcal{C} = \text{scF}_M \) and \((M, \tau_c)\) is a Hausdorff space (the last is true iff the family \( \mathcal{C} \) separates points in \( X \) iff the family \( \mathcal{F} \) separates points in \( X \)). On the set \( M \) the family \( \mathcal{F} \) defines the uniform structure \( \mathcal{U}_F \) such that the base \( \mathcal{B} \) of \( \mathcal{U}_F \) is given as follows:

\[
\mathcal{B} = \{ V(f_1, \ldots, f_k, \varepsilon) \subset M \times M; k \in \mathbb{N}; f_1, \ldots, f_k \in \mathcal{F}, \varepsilon > 0 \},
\]

where

\[
V(f_1, \ldots, f_k, \varepsilon) = \{ (x, y) \in M \times M : \forall 1 \leq i \leq k \quad |f_i(x) - f_i(y)| < \varepsilon \}.
\]

**Proposition 3.1** The family \( \mathcal{B} \) satisfies on \( M \) conditions B1 - B3.

**Proof.** (B1) Let: \( V_1 = V(f_1, \ldots, f_k, \varepsilon_1) \in \mathcal{B}, V_2 = V(g_1, \ldots, g_m, \varepsilon_2) \in \mathcal{B} \) and \( \varepsilon = \min(\varepsilon_1, \varepsilon_2) \). Then

\[
V := V(f_1, \ldots, f_k, g_1, \ldots, g_n, \varepsilon) =
\]

\[
\{ (x, y) \in M \times M : \forall 1 \leq i \leq k \quad |f_i(x) - f_i(y)| < \varepsilon \} \land \forall 1 \leq j \leq n \quad |g_j(x) - g_j(y)| < \varepsilon \} \in \mathcal{B}
\]

and \( V \subset V_1 \cap V_2 \).

(B2) Let \( V = V(f_1, \ldots, f_k, \varepsilon) \in \mathcal{B} \). Then \( W := V(f_1, \ldots, f_k, \frac{\varepsilon}{2}) \in \mathcal{B} \) and

\[
2W =
\]

\[
\{ (x, y) \in M \times M : \exists z \in M \quad \forall 1 \leq i \leq k \quad |f_i(x) - f_i(z)| < \frac{\varepsilon}{2} \land |f_i(z) - f_i(y)| < \frac{\varepsilon}{2} \}
\]

\[
\subset \{ (x, y) \in M \times M : \forall 1 \leq i \leq k \quad |f_i(x) - f_i(y)| < \varepsilon \} = V.
\]

(B3) Since for any \( V \in \mathcal{B} \) there is \( \Delta \subset V \) we have

\[
\Delta \subset \bigcap \mathcal{B}.
\]

On the other hand

\[
\bigcap \mathcal{B} \subset \bigcap_{f \in \mathcal{F}, \varepsilon > 0} V(f, \varepsilon) =
\]

\[
\{ (x, y) \in M \times M : \forall f \in \mathcal{F} \quad \forall \varepsilon > 0 \quad |f(x) - f(y)| < \varepsilon \} =
\]

\[
\{ (x, y) \in M \times M : \forall f \in \mathcal{F} \quad |f(x) = f(y)| \} =
\]

\[
\{ (x, x) \in M \times M \} = \Delta. \quad \square
\]
The uniform space \((M, \mathcal{U}_F)\) is said to be the uniform space given by the family of function \(F\).

If we have two different families \(F_1\) and \(F_2\) of generators of differential space \((M, C)\) then the uniform structures \(\mathcal{U}_{F_1}\) and \(\mathcal{U}_{F_2}\) can be different too.

**Example 3.1** Let \(M = \mathbb{R}, C = \mathcal{C}^\infty(\mathbb{R}), F_1 = \{\text{id}_\mathbb{R}\}\) and \(F_2 = \{\text{id}_\mathbb{R}, f\}\), where
\[
\text{id}_\mathbb{R}(x) = x, \quad \text{and} \quad f(x) = x^2, \quad x \in \mathbb{R}.
\]
Then does not exists \(\varepsilon > 0\) such that \(V(\text{id}_\mathbb{R}, \varepsilon) \subset V(f, 1)\). Hence \(V(f, 1) \notin \mathcal{U}_{F_1}\) and \(\mathcal{U}_{F_1} \neq \mathcal{U}_{F_2}\). □

### 4 Uniform continuous mapping

Let \((X, \mathcal{U}), (Y, \mathcal{V}), (X, \mathcal{U}), (Y, \mathcal{V})\) be uniform spaces.

**Definition 4.1** A mapping \(f : X \rightarrow Y\) is said to be uniform with respect to uniform structures \(\mathcal{U}\) and \(\mathcal{V}\) if
\[
\forall V \in \mathcal{V} \exists U \in \mathcal{U} \forall x, x' \in X \ |x - x'| < U \Rightarrow |f(x) - f(x')| < V.
\]
In other words for every \(V \in \mathcal{V}\) there is \(U \in \mathcal{U}\) such that \(U \subset (f \times f)^{-1}(V)\).

We denote it by \(f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})\).

It is easy to prove that:
(i) any uniform mapping \(f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})\) is continuous with respect to topologies \(\tau_{\mathcal{U}}\) and \(\tau_{\mathcal{V}}\);
(ii) a superposition of uniform mappings is a uniform mapping.

We can give criteria of the uniformity:

**Theorem 4.1** Let \(f : X \rightarrow Y\) and let \(\mathcal{U}\) and \(\mathcal{V}\) be uniform structures on \(X\) and \(Y\) respectively. Then the following conditions are equivalent:

(a) \(f : (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})\).

(b) If \(\mathcal{B}\) and \(\mathcal{D}\) are bases of \(\mathcal{U}\) and \(\mathcal{V}\) respectively then for each \(V \in \mathcal{D}\) there exists \(U \in \mathcal{B}\) such that \(U \subset (f \times f)^{-1}(V)\).

(c) For every covering \(\mathcal{A}\) of \(Y\) uniform with respect to \(\mathcal{V}\), a covering \(\{f^{-1}(A)\}_{A \in \mathcal{A}}\) of \(X\) is uniform with respect to \(\mathcal{U}\).
(d) For every pseudometric \( \rho \) on \( Y \) uniform with respect to \( V \), a pseudometric \( \sigma \) on \( X \) given by the formula

\[
\sigma(x, y) = \rho(f(x), f(y)) \quad x, y \in X
\]

is uniform with respect to the uniform structure \( U \).

For the proof see [2].

A mapping \( f \), that is a uniform with respect to uniform structures \( U \) and \( V \) could be not uniform with respect to uniform structures \( \overline{U} \) and \( \overline{V} \).

Example 4.1. Let \( M = \mathbb{R}, C = C^\infty(\mathbb{R}), F_1 = \{id_\mathbb{R}\}, F_2 = \{id_\mathbb{R}, f\} \), where \( f(x) = x^2, x \in \mathbb{R} \).

Here \( f \) is the uniform mapping with respect to \( F_2 \), but it is not uniform with respect to \( F_1 \). In fact, the set \( V = \{(x, y): |f(x) - f(y)| = x^2 - y^2 < \varepsilon\} \) is an element of \( D \) and does not exists \( U \in B \) such that \( U \subset (f \times f)^{-1}(V) \) (see Example 3.1).

**Definition 4.2** A bijective mapping \( f : (X, U) \to (Y, V) \) is a uniform homeomorphism if \( f^{-1} \) is a uniform mapping. Then we say that \( (X, U) \) and \( (Y, V) \) are uniformly homeomorphic.

By (i) it is obvious that if \( f : (X, U) \to (Y, V) \) is a uniform homeomorphism then \( f \) is a homeomorphism of the topological spaces \( (X, \tau_U) \) and \( (Y, \tau_V) \).

5 Complete uniform spaces and extensions of differential structure

**Definition 5.1** Let \( X \) be a nonempty set, \( x \in X \) and \( V \in \mathcal{D}_X \) (see Definition 3.1). A set \( U \subset X \) is said to be small of rank \( V \) if \( \exists x \in V \ [U \subset K(x, V)] \) (see Definition 3.2).

**Definition 5.2** A nonempty family \( F \) of subsets of a set \( X \) is said to be a filter on \( X \) if:

(F1) \( (F \in \mathcal{F} \land F \subset U \subset X) \Rightarrow (U \in \mathcal{F}); \)

(F2) \( (F_1, F_2 \in \mathcal{F}) \Rightarrow (F_1 \cap F_2 \in \mathcal{F}); \)

(F3) \( \emptyset \notin \mathcal{F}. \)

**Definition 5.3** A filtering base on \( X \) is a nonempty family \( \mathcal{B} \) of subsets of \( X \) such that
(FB1) \( \forall A_1, A_2 \in \mathcal{B} \ \exists A_3 \in \mathcal{B} \ [A_3 \subset A_1 \cap A_2] \);

(FB2) \( \emptyset \notin \mathcal{B} \).

If \( \mathcal{B} \) is a filtering base on \( X \) then

\[ \mathcal{F} = \{ F \subset X : \exists A \in \mathcal{B} \ [A \subset F] \} \]

is a filter on \( X \). It is called the filter defined by \( \mathcal{B} \).

**Definition 5.4** Let \( X \) be a topological space. We say that a filter \( \mathcal{F} \) on \( X \) is convergent to \( x \in X \) (\( \mathcal{F} \rightarrow x \)) if for any neighbourhood \( U \) of \( x \) there exists \( F \in \mathcal{F} \) such that \( F \subset U \).

**Definition 5.5** Let \( (X, \mathcal{U}) \) be a uniform space. A filter \( \mathcal{F} \) on \( X \) is a Cauchy filter if

\[ \forall V \in \mathcal{U} \ \exists F \in \mathcal{F} \ [F \times F \subset V] \]

**Definition 5.6** A uniform space \( (X, \mathcal{U}) \) is said to be complete if each Cauchy filter on \( X \) is convergent in \( \tau_{\mathcal{U}} \).

Let \( (X, \mathcal{U}) \) be a uniform space, \( M \subset X \) and \( M \neq \emptyset \). Denote

\[ \mathcal{U}_M := \{ V \cap M : V \in \mathcal{U} \} \]

Then it is easy to show that \( \mathcal{U}_M \) is a uniform structure on \( M \). We call \( (M, \mathcal{U}_M) \) a uniform subspace of the uniform space \( (X, \mathcal{U}) \).

**Theorem 5.1** If \( (X, \mathcal{U}) \) is a complete uniform space and \( M \) is a closed subset of the topological space \( (X, \tau_{\mathcal{U}}) \) then a uniform space \( (M, \mathcal{U}_M) \) is complete. Conversely, If \( (M, \mathcal{U}_M) \) is a complete uniform subspace of some (not necessarily complete) uniform space \( (X, \mathcal{U}) \) then \( M \) is closed in \( X \) with respect to \( \tau_{\mathcal{U}} \).

For the proof see \[1\], \[2\] or \[3\].

Any uniform space can be treated as a uniform subspace of some complete uniform space. We have the following

**Theorem 5.2** For each uniform space \( (X, \mathcal{U}) \):

(i) there exists a complete uniform space \((\bar{X}, \bar{\mathcal{U}})\) and a set \( A \subset \bar{X} \) dense in \( \bar{X} \) (with respect to the topology \( \tau_{\bar{\mathcal{U}}} \)) such that \( (X, \mathcal{U}) \) is uniformly homeomorphic to \( (A, \mathcal{U}_A) \);

(ii) if the complete uniform spaces \((\bar{X}_1, \bar{\mathcal{U}}_1)\) and \((\bar{X}_2, \bar{\mathcal{U}}_2)\) satisfies condition of the point (i) then they are uniformly homeomorphic.

For the details of the proof see \[1\] or \[4\]. Here we only want to describe the construction of \((\bar{X}, \bar{\mathcal{U}})\).
Let \( \tilde{X} \) be the set of all minimal (with respect to the order defined by inclusion) Cauchy filters in \( X \). For every symmetric set \( V \in \mathcal{U} \) we denote by \( \tilde{V} \) the set of all pairs \( (\mathcal{F}_1, \mathcal{F}_2) \) of minimal Cauchy’s filters, which have a common element being a small set in rank \( V \). We define a family \( \mathcal{U} \) of subsets of set \( \tilde{X} \times \tilde{X} \) as the smallest uniform structure on \( X \) containing all sets from the family \( \{ \tilde{V} : V \in \mathcal{U} \} \).

Let us consider two different uniform structures at the same differential space \((\mathbb{R}, C^\infty)\): \( \mathcal{U}_\mathcal{F} \) and \( \mathcal{U}_\mathcal{G} \), where \( \mathcal{F} = \{ id_{\mathbb{R}} \} \) and \( \mathcal{G} = \{ \arctgx \} \). Then \((\mathbb{R}, \mathcal{U}_\mathcal{F})\) is the complete space \((\mathbb{R} = \mathbb{R})\) whereas \((\mathbb{R}, \mathcal{U}_\mathcal{G})\). In this case we can identify \( \mathbb{R} \) with the interval \([-\frac{\pi}{2}; \frac{\pi}{2}]\).

Let \( N \) be a set, \( M \subseteq N \), \( M \neq \emptyset \), \( \mathcal{C} \) be a differential structure on \( M \).

**Definition 5.7.** The differential structure \( \mathcal{D} \) on \( N \) is an extension of the differential structure \( \mathcal{C} \) from the set \( M \) to the set \( N \) if \( \mathcal{C} = \mathcal{D}_M \) (if we get the structure \( \mathcal{C} \) by localization of the structure \( \mathcal{D} \) to \( M \)).

For the sets \( N, M \) and the differential structure \( \mathcal{C} \) on \( M \) we can construct many different extensions of the structure \( M \) to \( N \).

**Example 5.1.** If for each function \( f \in \mathcal{C} \) we assign the function \( f_0 \in \mathbb{R}^N \) such that \( f_0|_{M} = f \) and \( f_0|_{N \setminus M} = 0 \). Then the differential structure generated on \( N \) by the family of functions \( \{ f_0 \}_{f \in \mathcal{C}} \) is the extension of \( \mathcal{C} \) from \( M \) to \( N \). Similarly, if for each function \( f \in \mathcal{C} \) we assign the family of the functions \( \mathcal{F}_f := \{ g \in \mathbb{R}^N : g|_{M} = f \} \), then the differential structure on \( N \) generated the family of the functions \( \mathcal{F} := \bigcup_{f \in \mathcal{C}} \mathcal{F}_f \) is the extension of \( \mathcal{C} \) from \( M \) to \( N \). If the set \( N \setminus M \) contains at least two elements, then the differential structures generated by the families \( \{ f_0 \}_{f \in \mathcal{C}} \) and \( \mathcal{F} \) are different.

**Definition 5.8.** If \( \tau \) is a topology on the set \( N \), then the extension \( \mathcal{D} \) of the differential structure \( \mathcal{C} \) from \( M \) to \( N \) is continuous with respect to \( \tau \) if each function \( f \in \mathcal{D} \) is continuous in the topology \( \tau \) (\( \tau_D \subseteq \tau \)).

If on the set \( N \) there exists continuous (with respect to \( \tau \)) extension of the differential structure \( \mathcal{C} \) from the set \( M \subseteq N \), then the structure \( \mathcal{C} \) is said to be extendable from the set \( M \) to the topological space \((N, \tau)\).

**Example 5.2.** The differential structure \( C^\infty(\mathbb{R})_Q \) is extendable from the set of rationales to the set of reals. The continuous extensions are e.g. \( C^\infty(\mathbb{R}) \) and the structure \( \mathcal{D} \) generated on \( \mathbb{R} \) by the family of functions \( C^\infty(\mathbb{R}) \cup \{ f \} \), where \( f : \mathbb{R} \to \mathbb{R}, f(x) := |x - \sqrt{2}|, x \in \mathbb{R} \).

It is not difficult to show that if \( \mathcal{F} \) is a family of generators of a differential structure \( \mathcal{C} \) on a set \( M \) then the completion \( \bar{M} \) of \( M \) with respect to the uniform structure \( \mathcal{U}_\mathcal{F} \) can be identify with the closure of the range \( \phi_\mathcal{F}(M) \) of the generator embedding \( \phi_\mathcal{F} \) in the Cartesian product \( \mathbb{R}^\mathcal{F} \). In this case the differential structure \( C^\infty(\mathbb{R}^\mathcal{F})_{\phi_\mathcal{F}(M)} \) is a natural continuous extension of \( \mathcal{C} \) from \( M \) to \( \bar{M} \).

10
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