An application of the reduction method to Sutherland type many-body systems

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Abstract

We study Hamiltonian reductions of the free geodesic motion on a non-compact simple Lie group using as reduction group the direct product of a maximal compact subgroup and the fixed point subgroup of an arbitrary involution commuting with the Cartan involution. In general, we describe the reduced system that arises upon restriction to a dense open submanifold and interpret it as a spin Sutherland system. This dense open part yields the full reduced system in important special examples without spin degrees of freedom, which include the $BC_n$ Sutherland system built on 3 arbitrary couplings for $m < n$ positively charged and $(n - m)$ negatively charged particles moving on the half-line.
1 Introduction

One of the most popular approaches to integrable classical mechanical systems is to realize systems of interest as reductions of higher dimensional “canonical free systems”. The point is that the properties of the reduced systems can be understood in elegant geometric terms. This approach was pioneered by Olshanetsky and Perelomov [1] and by Kazhdan, Kostant and Sternberg [2] who interpreted the celebrated rational Calogero and hyperbolic/trigonometric Sutherland systems as Hamiltonian reductions of free particles moving on Riemannian symmetric spaces. As reviewed in [3, 4, 5], these integrable many-body systems possess important generalizations based on arbitrary root systems and elliptic interaction potentials. They also admit relativistic deformations, extensions by “spin” degrees of freedom and generalizations describing interactions of charged particles. The Hamiltonian reduction approach to many of these systems was successfully worked out in the past (see e.g. [3, 6] and their references), but in some cases its discovery still poses us interesting open problems.

In a recent joint work with V. Ayadi [7], we enlarged the range of the reduction method to cover the $BC_n$ Sutherland system of charged particles defined by the following Hamiltonian:

$$H = \frac{1}{2} \sum_{j=1}^{n} p_j^2 - \sum_{1 \leq j < m < k \leq n} \left( \frac{\kappa^2}{\cosh^2(q_j - q_k)} + \frac{\kappa^2}{\cosh^2(q_j + q_k)} \right) + \sum_{1 \leq j < k \leq m} \left( \frac{\kappa^2}{\sinh^2(q_j - q_k)} + \frac{\kappa^2}{\sinh^2(q_j + q_k)} \right) + \sum_{m < j < k \leq n} \left( \frac{\kappa^2}{\sinh^2(q_j - q_k)} + \frac{\kappa^2}{\sinh^2(q_j + q_k)} \right)$$

$$+ \frac{1}{2} \sum_{j=1}^{n} (x_0 - y_0)^2 + \frac{1}{2} \sum_{j=1}^{m} \frac{x_0 y_0}{\sinh^2(q_j)} - \frac{1}{2} \sum_{j=m+1}^{n} \frac{x_0 y_0}{\cosh^2(q_j)}.$$  \hspace{1cm} (1)

Here $m$ and $n$ are positive integers subject to $m < n$, while $\kappa$, $x_0$ and $y_0$ are real coupling parameters satisfying the conditions $\kappa \neq 0$ and $(x_0^2 - y_0^2) \neq 0$, which permit to consistently restrict the dynamics to the domain where

$$q_1 > q_2 > \cdots > q_m > 0 \quad \text{and} \quad q_{m+1} > q_{m+2} > \cdots > q_n > 0.$$ \hspace{1cm} (2)

If $x_0y_0 > 0$, then we can interpret the Hamiltonian (1) in terms of attractive-repulsive interactions between $m$ positively charged and $(n - m)$ negatively charged particles influenced also by their mirror images and a positive charge fixed at the origin.

The derivation [7] of the Hamiltonian (1) relied on reducing the free geodesic motion on the group $Y := SU(n, n)$ using as symmetry group $Y_+ \times Y^+$, where $Y_+ < Y$ is a maximal compact subgroup and $Y^+ < Y$ is the (non-compact) fixed point subgroup of an involution of $Y$ that commutes with the Cartan involution fixing $Y_+$. This allowed us to cover the case of 3 arbitrary couplings, extending the previous derivation [8] of 2-parameter special cases of the system. The $m = 0$ special case was treated in [9] by applying the symmetry group $Y_+ \times Y_+$.

The emergence of system (1) as reduced system required to impose very special constraints on the free motion. Thus it is natural to enquire about the reduced systems that would arise under other moment map constraints. In fact, the main purpose of this contribution is to characterize the reduced systems in a general case, where $Y$ will be taken to be an arbitrary non-compact simple Lie group, $Y_+ \times Y^+$ will have similar structure as mentioned above, and the moment map constraint will be chosen arbitrarily.
In Section 2, we study reductions of the geodesic system on $Y$ restricting all considerations to a dense open submanifold consisting of regular elements. In general, we shall interpret the reduced system as a spin Sutherland type system. In exceptional cases, the initial restriction to regular elements is immaterial in the sense that the moment map constraint enforces the same restriction. This happens in the reduction that yields the spinless system (1), as will be sketched in Section 3. Finally, we shall present a short conclusion in Section 4.

2 Spin Sutherland type systems from reduction

We need to fix notations and recall an important group theoretic result before turning to the reduction of our interest.

2.1 Generalized Cartan decomposition

Let $Y$ be a non-compact connected simple real Lie group with Lie algebra $\mathcal{Y}$. Equip $\mathcal{Y}$ with the scalar product $\langle \ , \ \rangle$ given by a positive multiple of the Killing form. Suppose that $\Theta$ is a Cartan involution of $Y$ (whose fixed point set is a maximal compact subgroup) and $\Gamma$ is an arbitrary involution commuting with $\Theta$. The corresponding involutions of $\mathcal{Y}$, denoted by $\theta$ and $\gamma$, lead to the orthogonal decomposition

$$\mathcal{Y} = \mathcal{Y}_+^+ + \mathcal{Y}_-^- + \mathcal{Y}_+^- + \mathcal{Y}_-^+,$$

where the subscripts $\pm$ refer to eigenvalues $\pm 1$ of $\theta$ and the superscripts to the eigenvalues of $\gamma$. We may also use the associated projection operators

$$\pi^\pm : \mathcal{Y} \to \mathcal{Y}_\pm^\pm,$$

as well as $\pi_+ = \pi_+^+ + \pi_+^-$ and $\pi^- = \pi_-^+ + \pi_-^-$. We choose a maximal Abelian subspace

$$\mathcal{A} \subset \mathcal{Y}_-^-,$$

and define

$$C := \text{Cent}_\mathcal{Y}(\mathcal{A}) = \{ \eta \in \mathcal{Y} \mid [\eta, \alpha] = 0 \ \forall \alpha \in \mathcal{A} \}.$$

An element $\alpha \in \mathcal{A}$ is called regular if its centralizer inside $\mathcal{Y}$ is precisely $C$. The connected subgroup $\mathcal{A} < Y$ associated with $\mathcal{A}$ is diffeomorphic to $\mathcal{A}$ by the exponential map. For later use, we fix a connected component $\tilde{\mathcal{A}}$ of the set of regular elements of $\mathcal{A}$, and introduce also the open submanifold

$$\tilde{\mathcal{A}} := \exp(\tilde{\mathcal{A}}) \subset A.$$

The restriction of the scalar product to $C$ is non-degenerate and thus we obtain the orthogonal decomposition

$$\mathcal{Y} = C + C^\perp.$$

According to (3), any $X \in \mathcal{Y}$ can be written uniquely as $X = X_C + X_{C^\perp}$. Equation (3) induces also the decomposition

$$C = C_+^+ + C_-^- + C_+^+ + C_-^-,$$

and similarly for $C^\perp$. 
Let \( Y_+ \) and \( Y^+ \) be the fixed point subgroups of \( \Theta \) and \( \Gamma \), respectively, possessing as their Lie algebras
\[
\mathfrak{y}_+ = \mathfrak{y}_+^+ + \mathfrak{y}_-^+ \quad \text{and} \quad \mathfrak{y}^+ = \mathfrak{y}_+^+ + \mathfrak{y}_-^+.
\]
Consider the group
\[
Y^+_+ := Y_+ \cap Y^+
\]
and its subgroup
\[
M := \text{Cent}_{Y^+_+} (A).
\]
(6)
Pretending that we deal only with matrix Lie groups, the elements \( m \in M \) have the defining property \( mam^{-1} = \alpha \) for all \( \alpha \in A \). Note that \( \mathfrak{c}_+^+ \) is the Lie algebra of \( M \).

We shall study the reductions of a free particle moving on \( Y \) utilizing the symmetry group
\[
G := Y_+ \times Y^+ < Y \times Y.
\]
It is a well-known group theoretic result (see e.g. [10]) that every element \( y \in Y \) can be written in the form
\[
y = y_l a y_r \quad \text{with} \quad y_l \in Y_+, \ y_r \in Y^+, \ a \in A.
\]
(7)
This is symbolically expressed as the set-equality
\[
Y = Y_+ A Y^+.
\]
(8)
Furthermore, the subset of regular elements given by
\[
\hat{Y} := Y_+ \hat{A} Y^+
\]
(9)
is open and dense in \( Y \). The decomposition of \( y \in \hat{Y} \) in the form (7) is unique up to the replacement \((y_l, y_r) \to (y_l m, m^{-1} y_r)\) with any \( m \in M \). The product decomposition (8) is usually referred to as a generalized Cartan decomposition since it reduces to the usual Cartan decomposition in the case \( \gamma = \theta \). This decomposition will play crucial role in what follows.

### 2.2 Generic Hamiltonian reduction

We wish to reduce the Hamiltonian system of a free particle moving on \( Y \) along geodesics of the pseudo-Riemannian metric associated with the scalar product \( \langle \ , \ \rangle \). To begin, we trivialize \( T^*Y \) by right-translations, identify \( \mathcal{Y} \) with \( \mathcal{Y}' \) (and similarly for \( \mathcal{Y}_+ \) and \( \mathcal{Y}^+ \)) by the scalar product, and choose an arbitrary coadjoint orbit
\[
\mathcal{O} := \mathcal{O}_l \times \mathcal{O}_r
\]
of the symmetry group \( G = Y_+ \times Y^+ \). We then consider the phase space
\[
P := T^*Y \times \mathcal{O} \simeq Y \times \mathcal{Y} \times \mathcal{O}_l \times \mathcal{O}_r = \{(y, J, \xi^l, \xi^r)\}
\]
endowed with its natural symplectic form \( \omega \) and the free Hamiltonian \( \mathcal{H} \),
\[
\mathcal{H}(y, J, \xi^l, \xi^r) := \frac{1}{2} \langle J, J \rangle.
\]
The form $\omega$ can be written symbolically as $\omega = d(J, (dy)y^{-1}) + \Omega$, where $\Omega$ is the canonical symplectic form of the orbit $O$.

The action of $(g_l, g_r) \in G$ on $P$ is defined by

$$\Psi_{(g_l, g_r)} : (y, J, \xi^l, \xi^r) \mapsto (g_ly g^{-1}_r, g_lJg^{-1}_l, g_l \xi^l g^{-1}_l, g_r \xi^r g^{-1}_r).$$

This Hamiltonian action is generated by the moment map $\Phi = (\Phi^l, \Phi^r) : P \to \mathcal{Y}_+ \times \mathcal{Y}^+$ whose components are

$$\Phi^l(y, J, \xi^l, \xi^r) = \pi_+ (J) + \xi^l, \quad \Phi^r(y, J, \xi^l, \xi^r) = -\pi_+ (y^{-1}Jy) + \xi^r.$$

We restrict our attention to the “big cell” $\tilde{P}_{\text{red}}$ of the full reduced phase space

$$P_{\text{red}} := \Phi^{-1}(0)/G$$

that arises as the symplectic reduction of the dense open submanifold

$$\tilde{P} := T^*\tilde{Y} \times O \subset P.$$

In other words, we wish to describe the set of $G$-orbits,

$$\tilde{P}_{\text{red}} := \tilde{P}_{c}/G,$$

in the constraint surface

$$\tilde{P}_c := \Phi^{-1}(0) \cap \tilde{P}.$$ (12)

An auxiliary symplectic reduction of the orbit $(O, \Omega)$ by the group $M$ (8) will appear in our final result. Notice that $M$ acts naturally on $O$ by its diagonal embedding into $Y_+ \times Y^+$, i.e., by the symplectomorphisms

$$\psi_m : (\xi^l, \xi^r) \mapsto (m \xi^l m^{-1}, m \xi^r m^{-1}), \quad \forall m \in M.$$ (13)

This action has its own moment map $\phi : O \to (C^+_+)^* \simeq C^+_+$ furnished by

$$\phi : (\xi^l, \xi^r) \mapsto \pi^+_+ (\xi^l + \xi^r),$$

defined by means of equations (4) and (5). The reduced orbit

$$O_{\text{red}} := \phi^{-1}(0)/M$$ (14)

is a stratified symplectic space in general [11]. In particular, $O_{\text{red}}$ contains a dense open subset which is a symplectic manifold and its complement is the disjunct union of lower dimensional symplectic manifolds. Accordingly, when talking about the reduced orbit $(O_{\text{red}}, \Omega_{\text{red}})$, $\Omega_{\text{red}}$ actually denotes a collection of symplectic forms on the various strata of $O_{\text{red}}$.

The key result for the characterization of $\tilde{P}_{\text{red}}$ (11) is encapsulated by the following proposition, whose formulation contains the functions

$$w(x) := \frac{1}{\sinh(x)} \quad \text{and} \quad \chi(x) := \frac{1}{\cosh(x)}.$$ (15)
Proposition 1. Every \( G \)-orbit in the constraint surface \( \hat{P}_c \) (12) possesses representatives of the form \((e^q, J, \xi^l, \xi^r)\), where \( q \in \hat{A} \), \( p \in A \), \( \phi(\xi^l, \xi^r) = 0 \) and \( J \) is given by the formula

\[
J = p - \xi^l - w(ad_{q}) \circ \pi^+_{\xi} + \text{coth}(ad_{q}) \circ \pi^+_{\xi} - \text{coth}(ad_{q}) \circ \pi^+_{\xi} - \text{coth}(ad_{q}) \circ \pi^+_{\xi}.
\]

(16)

Every element \((e^q, J, \xi^l, \xi^r)\) of the above specified form belongs to \( \hat{P}_c \), and two such elements belong to the same \( G \)-orbit if and only if they are related by the action of the subgroup \( M_{\text{diag}} < Y_+ \times Y^+ \), under which \( q \) and \( p \) are invariant and the pair \((\xi^l, \xi^r)\) transforms by \((13)\). Consequently, the space of orbits \( \hat{P}_c \) can be identified as

\[
\hat{P}_c \simeq (\hat{A} \times A) \times \mathcal{O}_{\text{red}}.
\]

This yields the symplectic identification \( \hat{P}_c \simeq T^*\hat{A} \times \mathcal{O}_{\text{red}} \), i.e., the reduced (stratified) symplectic form \( \omega_{\text{red}} \) of \( \hat{P}_c \) can be represented as

\[
\omega_{\text{red}} = d\langle p, dq \rangle + \Omega_{\text{red}}.
\]

(17)

Here, \( T^*\hat{A} \) is identified with \( \hat{A} \times A = \{(q, p)\} \) and \((\mathcal{O}_{\text{red}}, \Omega_{\text{red}})\) is the reduced orbit \((14)\).

Proposition 1 is easily proved by solving the moment map constraint after “diagonalizing” \( y \in Y \) utilizing the generalized Cartan decomposition \((9)\). The expression \((17)\) of \( \omega_{\text{red}} \) follows by evaluation of the original symplectic form \( \omega \) on the “overcomplete set of representatives” \\{\((e^q, J, \xi^l, \xi^r)\)\} of the \( G \)-orbits in \( \hat{P}_c \). The operator functions of \( ad_{q} \) that appear in \((16)\) are well-defined since \( q \in \hat{A} \) is regular. Indeed, \( ad_{q} \) in \((16)\) always acts on \( C_{\perp} \), where it is invertible.

Now the formula of the reduced “kinetic energy” \( \mathcal{H} = \frac{1}{2}\langle J, J \rangle \) is readily calculated.

Proposition 2. The reduction of the free Hamiltonian \( \mathcal{H} \) is given by the following \( M \)-invariant function, \( \mathcal{H}_{\text{red}} \), on \( T^*\hat{A} \times \phi^{-1}(0) \):

\[
2\mathcal{H}_{\text{red}}(q, p, \xi^l, \xi^r) = \langle p, p \rangle + \langle \xi^l, \xi^l \rangle + \langle \pi_{\perp}(\xi^l), \pi_{\perp}(\xi^l) \rangle
- \langle w^2(ad_{q}) \circ \pi^{+}(\xi^l), \pi^{+}(\xi^l) \rangle - \langle w^2(ad_{q}) \circ \pi^{+}(\xi^l), \pi^{+}(\xi^l) \rangle
+ \langle \chi^2(ad_{q}) \circ \pi^{+}(\xi^l), \pi^{+}(\xi^l) \rangle + \langle \chi^2(ad_{q}) \circ \pi^{+}(\xi^l), \pi^{+}(\xi^l) \rangle
- 2\langle (w^2\chi^{-1})(ad_{q}) \circ \pi^{+}(\xi^l), \pi^{+}(\xi^l) \rangle + 2\langle (w^2\chi^{-1})(ad_{q}) \circ \pi^{+}(\xi^l), \pi^{+}(\xi^l) \rangle,
\]

(18)

where the notations \((15)\) and \( \chi^{-1}(x) := \cosh(x), w^{-1}(x) := \sinh(x) \) are applied.

In the special case \( \gamma = \theta \), studied in \([3]\), the formulae simplify considerably. Indeed, in this case \( \pi^{+}_{\perp} = \pi^{+} = 0 \), and thus the second line of equation \((16)\) and all terms in the last two lines of \((18)\) except the one containing \( w^2\chi^{-1} \) disappear. (This term can be recast in a more friendly form by the identity \( (w^2\chi^{-1})(x) = \frac{1}{2}w^2\left(\frac{x}{2}\right) - w^2(x) \).) Although such simplification does not occur in general, we can interpret \( \mathcal{H}_{\text{red}} \) as a spin Sutherland type Hamiltonian. This means that we view the components of \( q \) as describing the positions of point particles moving on the line, whose interaction is governed by hyperbolic functions of \( q \) and “dynamical coupling parameters” encoded by the “spin” degrees of freedom represented by \( \mathcal{O}_{\text{red}} \).

\footnote{For example, the action of \( w(ad_{q}) \) in \((16)\) is defined by expanding \( w(x) \) as \( x^{-1} \) plus a power series in \( x \), and then substituting \( (ad_{q} \mid_{C_{\perp}})^{-1} \) for \( x^{-1} \).}
3 A spinless example

We now recall the special case \[7\] whereby the previously described general construction leads to the $BC_n$ Sutherland system \([11]\). We start by fixing positive integers $1 \leq m < n$. We then prepare the matrices

\[
Q_{n,n} := \begin{bmatrix} 0 & 1_n \\ 1_n & 0 \end{bmatrix} \in \mathfrak{gl}(2n, \mathbb{C}), \quad I_m := \text{diag}(1_m, -1_{n-m}) \in \mathfrak{gl}(n, \mathbb{C}),
\]

where $1_n$ denotes the $n \times n$ unit matrix, and introduce also

\[
D_m := \text{diag}(I_m, I_m) = \text{diag}(1_m, -1_{n-m}, 1_m, -1_{n-m}) \in \mathfrak{gl}(2n, \mathbb{C}).
\]

We realize the group $Y := SU(n, n)$ as

\[
SU(n, n) = \{ y \in SL(2n, \mathbb{C}) \mid y^\dagger Q_{n,n} y = Q_{n,n} \},
\]

and define its involutions $\Theta$ and $\Gamma$ by

\[
\Theta(y) := (y^\dagger)^{-1}, \quad \Gamma(y) := D_m \Theta(y) D_m, \quad \forall y \in Y.
\]

The fixed point subgroups $Y_+$ and $Y^+$ turn out to be isomorphic to $S(U(n) \times U(n))$ and $S(U(m, n-m) \times U(m, n-m))$, respectively. We choose the maximal Abelian subspace $\mathcal{A}$ as

\[
\mathcal{A} := \left\{ q := \begin{bmatrix} q & 0 \\ 0 & -q \end{bmatrix} : q = \text{diag}(q_1, \ldots, q_n), \quad q_k \in \mathbb{R} \right\}.
\]

Its centralizer is $\mathcal{C} = \mathcal{A} + \mathcal{M}$ with

\[
\mathcal{M} \equiv C^+_+ = \left\{ d := i \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} : d = \text{diag}(d_1, \ldots, d_n), \quad d_k \in \mathbb{R}, \quad \text{tr}(d) = 0 \right\}.
\]

In particular, now $C^-_+ = C^+_+ = \{0\}$. The “Weyl chamber” $\tilde{\mathcal{A}}$ can be chosen as those elements $q \in \mathcal{A}$ \([19]\) whose components satisfy Eq. \([2]\).

It is important for us that both $\mathcal{Y}_+$ and $\mathcal{Y}^+$ possess one-dimensional centres, whose elements can be viewed also as non-trivial one-point coadjoint orbits of $Y_+$ and $Y^+$. The centre of $\mathcal{Y}_+$ is generated by $C^l := iQ_{n,n}$, and the centre of $\mathcal{Y}^+$ is spanned by

\[
C^r := i \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}.
\]

These elements enjoy the property

\[
C^\lambda \in (C^\pm)^+_+ \quad \text{for} \quad \lambda = l, r.
\]

Taking non-zero real constants $\kappa$ and $x_0$, we choose the coadjoint orbit of $Y_+$ to be

\[
\mathcal{O}^l \equiv \mathcal{O}_{x_0} := \{ x_0 C^l + \xi(u) \mid u \in \mathbb{C}^n, \quad u^\dagger u = 2\kappa n \},
\]

where

\[
\xi(u) := \frac{1}{2} \begin{bmatrix} X(u) & X(u) \\ X(u) & X(u) \end{bmatrix} \quad \text{with} \quad X(u) := i \left( uu^\dagger - \frac{u^\dagger u}{n} 1_n \right).
\]

\[
(20)
\]
It is not difficult to see that the elements $\xi(u)$ in (20) constitute a minimal coadjoint orbit of an $SU(n)$ block of $Y_+ \simeq S(U(n) \times U(n))$. The orbit $O'$ of $Y^+$ is chosen to be $\{y_0 e^r\}$ with some $y_0 \in \mathbb{R}$, imposing for technical reasons that $(x_0^2 - y_0^2) \neq 0$.

With the above data, we proved that the full reduced phase space $P_{\text{red}}$ (10) is given by the cotangent bundle $T^* \mathcal{A}$, i.e., $P_{\text{red}} = P_{\text{red}}$. Moreover, the reduced free Hamiltonian turned out to yield precisely the $BC_n$ Sutherland Hamiltonian (1). The details can be found in [7].

It is an important feature of our example that $O'$ is a one-point coadjoint orbit that belongs to $(C^\perp)^\perp$. Notice that several terms of (18), including the unpleasant last term, disappear for any such orbit. An even more special feature of the example is that $O_{\text{red}}$ contains a single element, which means that no spin degrees of freedom are present. This can be traced back to the well-known fact that the reductions of the minimal coadjoint orbits of $SU(n)$ by the maximal torus, at zero moment map value, yield one-point spaces. This fact underlies all derivations of spinless Sutherland type systems from free geodesic motion that we are aware of, starting from the classical paper [2].

4 Conclusion

In this contribution, we described a general class of Hamiltonian reductions of free motion on a non-compact simple Lie group. All spin Sutherland type systems that we obtained are expected to yield integrable systems after taking into account their complete phase spaces provided by $P_{\text{red}}$ (10). It could be interesting to investigate the fine details of these reduced phase spaces and to also investigate their quantization. Because of their more immediate physical interpretation, the exceptional spinless members (like the system (11)) of the pertinent family of spin Sutherland type systems deserve closer attention, and this may motivate one to ask about the list of all spinless cases that can occur in the reduction framework.

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