Coherent States, Dynamics and Semiclassical Limit on Quantum Groups

I.Ya. Aref’eva ∗, R. Parthasarathy †
K.S. Viswanathan ‡ and I.V. Volovich §
Department of Physics, Simon Fraser University
Burnaby, British Columbia, V5A 1S6, Canada

Abstract

Coherent states on the quantum group $SU_q(2)$ are defined by using harmonic analysis and representation theory of the algebra of functions on the quantum group. Semiclassical limit $q \to 1$ is discussed and the crucial role of special states on the quantum algebra in an investigation of the semiclassical limit is emphasized. An approach to $q$-deformation as a $q$-Weyl quantization and a relavence of contact geometry in this context is pointed out. Dynamics on the quantum group parametrized by a real time variable and corresponding to classical rotations is considered.

∗Permanent address: Steklov Mathematical Institute, Vavilov st. 42, GSP-1,117966, Moscow, Russia; E-mail : arefeva@qft.mian.su
†Permanent address: The Institute of Mathematical Sciences, Madras 600 113, India, E-mail: sarathy@imsc.ernet.in
‡E-mail: kviswana@sfu.ca
§Permanent address: Steklov Mathematical Institute, Vavilov st.42, GSP-1,117966, Moscow, Russia; E-mail: volovich@mph.mian.su
1 Introduction

Recently attempts at constructing field theoretic models with a quantum group playing the role of the gauge group have been made [1]-[11]. In contrast to applications of quantum groups [12]-[15] for solutions of standard models in field theory and statistical physics the point here is an attempt to build a new class of field theory models still preserving the standard Minkowski (or Riemannian) space-time. Such a development would be a natural mathematical application of the idea of quantum groups. As a possible physical motivation one notices a plausible mechanism of symmetry breaking which could be an alternative to the Higgs mechanism [1], a derivation of the Weinberg angle in terms of the parameter \( q \) [2] and a clarification of the special role of \( U(2) = SU(2) \times U(1) \) symmetry group [11]. Quantum group chiral fields, i.e. fields taking values in a quantum group have also been discussed [1, 3, 16, 17]. To give a rigorous meaning to such theories one still needs to clarify many questions even before quantization (one distinguishes \( q \)-deformation which corresponds to ‘classical’ theories on ‘quantum’ groups and ‘\( \hbar \)-deformations’ which involves quantization of such a theory, see [18]). In particular, one has to understand whether there exist nontrivial functions from classical space-time to the quantum groups and what it means for a variable to belong to a quantum group. See also a recent consideration of this issue by Frishman, Lukierski and Zakrzewski [17]. Another important issue is the question of dynamics on \( q \)- and \( \hbar \)-deformed phase space [18]-[24].

Recall that according to [12]-[13], a quantum group is considered, as it should be, as a ‘noncommutative manifold’ with a coordinate ring which is a Hopf algebra with a given set of generators. Therefore one thinks that there are only few points on a quantum group and it is not at all clear how one gets from this object the usual Lie group which is a smooth manifold. It is this question which we will discuss in this paper.

Our approach is the following. Consider the analogous question for the case of a simple quantum mechanical problem, like the harmonic oscillator. In quantum mechanics we also have an algebra with only a given set of fixed generators, say operators...
of position $\hat{x}$ and momentum $\hat{p}$. How can we get from these two fixed operators the classical phase space with two real variables $x$ and $p$? The answer of course is well known. To get correspondence with classical theory one should take the average of the operators $\hat{x}$ and $\hat{p}$ with respect to appropriate states, for example the coherent states. So the information about the classical variables is encoded into appropriate states. The same approach we will use for quantum groups. We define coherent states on quantum group algebra $SU_q(2)$ depending on an element $u = (u_{mn})$ of the classical group $SU(2)$ and show the correspondence in the sense that,

$$< g_{mn} \Psi(u) >_q \rightarrow u_{mn}, \quad (1)$$

when $q \rightarrow 1$. Here $g_{mn}$ are generators of the algebra of functions on the quantum group $SU_q(2)$, the operator $\Psi(u)$ defines a coherent state and the brackets in (1) stands for the Haar functional on $SU_q(2)$. The operator $\Psi(u)$ is

$$\Psi(u) = \sum_j (2j+1) \text{tr}(W^j \ast T^j(u)). \quad (2)$$

Here $T^j(u) = (D^j_{nm}(u))$ is a unitary representation of $SU(2)$ of spin $j$, matrices $W^j = (W^j_{mn})$ have entries from the algebra $A$ (see below). One has an analogous formula like (1) for an arbitrary polynomial $f(g)$ with respect to the generators $g_{ij}$,

$$< f(g) \Psi(u) >_q \rightarrow f(u), \quad q \rightarrow 1. \quad (3)$$

Coherent states in quantum mechanics are well known [26, 27]. q-deformed coherent states were considered in [28]-[31]. Coherent states on classical Lie groups are defined as [27],

$$| u > = T(u) | \phi >, \quad (4)$$

where $| \phi >$ is a vector in the space of the unitary representations of $T(u)$. It seems that formula (2) gives a natural generalization of coherent states (4). However there are two important differences between our definitions (1) and (2) and the formula (4). The first one is that one deals here with with a quantum group and so we have operators $W^j$ in (2). The second one is that formula (1) defines the vector $| \phi >$ as a pure state. One can talk about coherent vectors (not necessarily states) in this context. In (4) one has
really a state because the Haar functional as we will discuss below is nothing but the statistical partition function \[32\]. So in this case one really deals with a coherent state which is defined by means of a density matrix. In this interpretation the parameter \(q\) is equal to,

\[ q = \exp(-\beta), \]  

(5)

where \(\beta\) is the inverse temperature, \(\beta = 1/T\). One has an interpretation of the quantum group \(SU_q(2)\) as a model of Bose-gas at a temperature \(T = -1/\ln q\). Note here that this interpretation of the deformation parameter \(q\) as temperature is different from the considerations of q-Bosons under non-zero temperature \[33\].

We are not discussing here coherent states on the q-deformed universal enveloping algebras \[34, 35, 28-31, 36-38\]. We hope to clarify a relation of these with (1) and (2) in a forthcoming publication.

2 Representations of \(\text{Pol}(SU_q(2))\)

An element of the quantum group \(SU_q(2)\) for \(0 < q < 1\) is a \(2 \times 2\) matrix \(g = (g_{mn})\) which has the following canonical form

\[ g = (g_{mn}) = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}, \]  

(6)

and satisfies the unitarity conditions

\[ gg^* = g^*g = I. \]  

(7)

Here \(a\) and \(c\) are elements of some algebra with involution. Equations (7) are equivalent to the following known relations \[12, 14, 15\] for the elements \(a, c, a^*\) and \(c^*\):

\[ ac = qca, \quad ac^* = qc^*a, \quad cc^* = c^*c \]  

(8)

\[ aa^* + q^2cc^* = 1, \quad a^*a + c^*c = 1. \]  

(9)

The algebra \(A = \text{Pol}(SU_q(2))\) of polynomial functions on the quantum group \(SU_q(2)\) is generated as a C-algebra by elements \(a, c, a^*\) and \(c^*\) with the relations (5) and (3).
$A$ is a Hopf algebra with the standard coproduct,

$$\Delta : A \rightarrow A \otimes A$$

$$\Delta(g_{mn}) = \sum_k g_{mk} \otimes g_{kn} \quad (10)$$

with counit $\varepsilon : A \rightarrow \mathbb{C}$, $\varepsilon(g_{mn}) = \delta_{mn}$, involution $^*$ and antipode $S : A \rightarrow A$, $S g = g^*$. All representations of the algebra $A$ by operators in a Hilbert space were classified \cite{14, 39} and there are only the following series of unitary inequivalent nontrivial representations parametrized by a real parameter $\phi$, $0 \leq \phi < 2\pi$.

These are in the space $\mathcal{H} = \mathcal{L}^2(\mathbb{R})$ which is considered as the Fock space for the oscillator. One has a basis $\{e_n\}_{n=0}^{\infty}$ in $\mathcal{H}$ and operators $a = a_\phi$ and $c = c_\phi$ acting as,

$$ae_n = \sqrt{1 - q^{2n}}e_{n-1}, \quad ae_0 = 0; \quad ce_n = e^{i\phi}q^ne_n. \quad (11)$$

There is also a trivial representation : $a_\phi = e^{i\phi}$, $c_\phi = 0$. By introducing the standard creation and annihilation operators $b, b^*$ in $\mathcal{L}^2(\mathbb{R})$ satisfying $[b, b^*] = 1$, one can rewrite the representation (11) in the form

$$a = \sqrt{\frac{1 - q^{2(N+1)}}{N+1}}b, \quad c = e^{i\phi}q^N. \quad (12)$$

Here $N$ is the number operator, $N = b^*b$.

Recall that it is the left representation of the Hopf algebra $A = Pol(G)$ which corresponds to the (right) representation of the group $G$. Let $T : G \rightarrow GL(V)$ be a representation of the group $G$, on a vector space $V$,

$$T(uu') = T(u)T(u'), \quad u, u' \in G. \quad (13)$$

Choose a basis $\{\xi_i\}$ in $V$ and suppose that the corresponding matrix elements are $W_{ij} \in Pol(G)$. Then one can rewrite (13) as,

$$\Delta(W_{ij}) = \sum_k W_{ik} \otimes W_{kj}. \quad (14)$$

Hence we have a linear map $\rho : V \rightarrow Pol(G) \otimes V$ defined by,

$$\rho(e_n) = \sum_m D_{nm} \otimes e_m, \quad (15)$$
satisfying,

\[(\Delta \circ \text{id}) \circ \rho = (\text{id} \circ \rho) \circ \rho\]

\[(\varepsilon \otimes \text{id}) \circ \rho = \text{id}.\]  \hspace{1cm} (16)

A linear space \(V\) is called a left corepresentation for the Hopf algebra \(A\) if there exists a linear map \(\rho : V \rightarrow A \otimes V\) satisfying (15).

3 Harmonic analysis on \(SU_q(2)\) and q-Weyl quantization.

We present now results on the representation theory and harmonic analysis (Fourier transform) on \(SU_q(2)\), see [14], [39]-[42]. First recall that for the group \(SU(2)\) for every dimension \(2j + 1\), where spin \(j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\) there is one irreducible unitary representation \(T^j(u)\). On \(SU(2)\) there exists an invariant Haar measure \(du\), i.e.

\[\int f(u)du = \int f(u')du = \int f(uu')du; \quad \int du = 1\]  \hspace{1cm} (17)

Matrix elements \(D^j_{mn}(u)\) of \(T^j(u)\) taken with respect to an orthonormal basis,

\[D^j_{mn}(u) = (e_m, T^j e_n)\]  \hspace{1cm} (18)

are orthogonal:

\[\int D^j_{mn}(u) D^{j'*}_{m'n'}(u)du = \frac{1}{2j + 1} \delta_{jj'} \delta_{mm'} \delta_{nn'},\]  \hspace{1cm} (19)

and any function \(f(g)\) on the group \(SU(2)\) can be expanded into Fourier series

\[f(u) = \sum_{j \in \mathbb{N}/2} (2j + 1) \sum_{m,n=-j}^j \tilde{f}^j_{mn} D^j_{mn}(u)\]  \hspace{1cm} (20)

where

\[\tilde{f}^j_{mn} = \int f(v) D^j_{mn}(v)dv\]

One can represent an elements \(u\) of the group \(SU(2)\) in the form

\[
\begin{pmatrix}
\alpha & -\gamma^* \\
\gamma & \alpha^*
\end{pmatrix}, \quad \alpha^* \alpha + \gamma^* \gamma = 1,
\] \hspace{1cm} (21)
or using the Euler angles

\[
\alpha = e^{-\frac{1}{2}i(\phi + \psi)} \cos \frac{1}{2} \theta, \quad \gamma = e^{-\frac{1}{2}i(\phi - \psi)} \sin \frac{1}{2} \theta,
\]

(22)

\[0 \leq \phi, \psi \leq 4\pi, \quad 0 \leq \theta \leq \pi\] and the Haar measure is \[du = \frac{1}{2} \sin \theta d\theta d\phi d\psi/4\pi^2.

The representation \(T^j(u)\) can be realized as a representation in the space of all homogeneous polynomials of degree \(2j\) of two complex variables. If \(f(z_1, z_2)\) is such polynomial, then the operator \(T^j(u)\) acts as follows

\[
T(u)f(z_1, z_2) = f(\alpha z_1 + \gamma z_2, -\gamma^* z_1 + \alpha^* z_2)
\]

(23)

One can take the following basis in the spin \(j\) representation:

\[
e_m = \frac{\sqrt{2j!}}{(j+m)!(j-m)!} z_1^{j-m} z_2^{j+m},
\]

where \(m \in I_j = \{-j, -j+1, ..., j\}\). Then one has

\[
T^j(u)e_m = \sum_n D^j_{mn}(u)e_n
\]

(24)

and

\[
D^j_{mn}(u) = \frac{(j+m)!(j-m)!}{(j+n)!(j-n)!} \sum_{\mu} \binom{j+n}{j-m} \binom{j-n}{\mu} \binom{j-\mu-m}{j-\mu} \cdot \alpha^{j-n-\mu} (\alpha^*)^{j-m-\mu} \gamma^\mu (-\gamma^*)^{m-n+\mu}.
\]

(25)

The representations of \(SU_q(2)\) are described in a similar way. We will use the explicit construction of these representations by Masuda, Mimachi, Nagakami, Noumi and Ueno [40]. Let \(V^{(j)}\) be a \(C\)-linear space with the basis

\[
\zeta^{(j)}_m = \left[\begin{array}{c} 2j \\ j+m \end{array}\right]^{1/2} a^{j-m} c^{j+m} q^m
\]

(26)

where \(j \in N/2, m \in I_j, a\) and \(c\) are generators of the algebra \(A\), and \(\left[\begin{array}{c} m \\ n \end{array}\right]_q\) are the Gauss \(q\)-binomial coefficients:

\[
\left[\begin{array}{c} m \\ n \end{array}\right]_q = \frac{(q; q)_m}{(q; q)_n(q; q)_{m-n}} \quad (p; q)_m = \prod_{k=0}^{m-1} (1 - pq^k), \quad (p; q)_0 = 1.
\]

(27)
There exists a unique linear functional $h$ on $A$ such that

$$\Delta(\zeta_m^{(j)}) = \sum_{n \in I_j} \zeta_m^{(j)} \otimes \zeta_n^{(j)}$$

$$\Delta(W_m^{(j)}) = \sum_{n \in I_j} W_m^{(j)} \otimes W_n^{(j)}.$$

There exists an explicit representation for the elements $W_m^{(j)}$ in terms of the little $q$-Jacobi polynomials [40] as follows:

If $m + n \leq 0, m \geq n$:

$$W_m^{(j)} = q^{(j+n)(j-m)} \left[ \begin{array}{c} j + m \\ j - n \end{array} \right]^{1/2} \left[ \begin{array}{c} j - n \\ m - n \end{array} \right]^{1/2} a^{-m-n} c^{m-n} P_{j+n}^{m-n,-m-n}(c^*; q^2),$$

If $m + n \leq 0, n \geq m$:

$$W_m^{(j)} = q^{(j+m)(m-n)} \left[ \begin{array}{c} j - m \\ n - m \end{array} \right]^{1/2} \left[ \begin{array}{c} j + n \\ n - m \end{array} \right]^{1/2} a^{-m-n} c^{m-n} P_{j+n}^{m-n,-m-n}(c^*; q^2),$$

If $m + n \geq 0, n \geq m$:

$$W_m^{(j)} = q^{(n-m)(n-j)} \left[ \begin{array}{c} j - m \\ n - m \end{array} \right]^{1/2} \left[ \begin{array}{c} j + n \\ n - m \end{array} \right]^{1/2} P_{j-n}^{n-m,m+n}(-c^*; q^2)(-c^*)^{-m-n}(a^*)^{m+n},$$

If $m + n \geq 0, m \geq n$:

$$W_m^{(j)} = q^{(m-n)(m-j)} \left[ \begin{array}{c} j + m \\ m - n \end{array} \right]^{1/2} \left[ \begin{array}{c} j - n \\ m - n \end{array} \right]^{1/2} P_{j+n}^{n-m,-m-n}(c^*; q^2)c^{m-n}(a^*)^{n+m}.$$

Here the little $q$-Jacobi polynomials are defined by

$$P_{n}^{(\alpha,\beta)}(z; q) = \sum_{r=0}^{\infty} \frac{(q^{-n}; q)_r(q^{\alpha+\beta+n+1}; q)_r}{(q; q)_r(q^{n+1}; q)_r} (qz)^r.$$

There exists a unique linear functional $h : A \rightarrow C$ with $h(f^*f) \geq 0$ for all $f \in A$ and $h(1) = 1$ which is invariant, i.e. it satisfies the condition

$$(h \otimes id) \circ \Delta = e \circ h = (id \otimes h) \circ \Delta.$$

The functional $h$ on $A$ is the quantum Haar functional. We will denote it

$$h(f) = [f]_q, \quad f \in A$$
By using the representation (12) for \( a \) and \( c \) operators it is defined as

\[
< f >_q = \frac{\text{Tr} f e^{-\beta H}}{\text{Tr} e^{-\beta H}},
\]

(37)

where

\[
\text{Tr} f = \sum_{n=0}^{\infty} \frac{1}{2\pi} \int_{0}^{2\pi} \langle n | f | n \rangle d\phi,
\]

(38)

\[
H = 2N,
\]

(39)

and \( | n \rangle \) are \( n \)-particle oscillator states, \( N | n \rangle = n | n \rangle \). Therefore the quantum Haar functional is the thermodynamic average with the Hamiltonian (39) (this is noted in [32]). In particular the partition function is

\[
Z = \text{Tr} e^{-2\beta N} = \sum_{n=0}^{\infty} e^{-\beta 2n} = \frac{1}{1 - e^{-2\beta}}.
\]

(40)

The Hopf algebra \( A = \text{Pol}(SU_q(2)) \) has an orthogonal decomposition

\[
A = \bigoplus_{j \in \mathbb{N}/2} W^j
\]

(41)

with respect to \( < . , . >_q \), where \( W^j \) is spanned by matrix elements \( W^j_{mn} \). The matrix elements \( W^j_{mn} \) satisfy the following orthogonality relations

\[
< W^j_{mn} (W^j_{m'n'})^* >_q = \delta^{jj'} \delta_{mm'} \delta_{nn'} \frac{q^{-2n}}{[2j + 1]_q},
\]

(42)

where

\[
[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.
\]

One can consider Fourier transformation on \( SU_q(2) \):

\[
\mathcal{F} : A = \text{Pol}(SU_q(2)) \to \text{Mat}(C)
\]

\[
\mathcal{F}(f) = (\tilde{f}^{(j)})_{j \in \mathbb{N}/2}, \quad \tilde{f}^{(j)}_{mn} = < f W^j_{mn} >_q.
\]

(43)

The inversion formula is given by

\[
f = \sum_{j \in \mathbb{N}/2} [2j + 1]_q \text{Tr}_q (\tilde{f}^{(j)} W^j ),
\]

(44)

where the \( q \)-trace \( \text{Tr}_q M \) of a \( (2j + 1) \times (2j + 1) \) matrix \( M \) is given by

\[
\text{Tr}_q M = \sum_{k \in I_j} q^{2k} M_{kk}.
\]

(45)
Let us recall that the standard Weyl quantization is based on the Fourier analysis. If one has a function \( f(x, p) \) on the classical phase space admitting the Fourier representation

\[
f(x, p) = \int \tilde{f}(\zeta, \eta) e^{-i(x\zeta + p\eta)} \, d\zeta \, d\eta \tag{46}
\]

then one defines a corresponding Weyl operator as

\[
\hat{f}(X, P) = \int \tilde{f}(\zeta, \eta) e^{-i(X\zeta + P\eta)} \, d\zeta \, d\eta, \tag{47}
\]

where \( X \) and \( P \) are the usual position and momentum operators.

Analogously for a function \( f(u) \) on \( SU(2) \) one has the Fourier representation (20), then a corresponding q-Weyl operator is

\[
\hat{f} = Wf = \sum_{jmn} (2j + 1) \tilde{f}_{mn}^j W_{mn}^j, \tag{48}
\]

which gives an element of the algebra \( A \). In particular

\[
Wu = g.
\]

One can develop now a q-analog of the theory of pseudodifferential operators and the Wigner-Moyal approach [44] to quantum group.

4 Coherent States on \( SU_q(2) \)

We define a coherent state operator on \( SU_q(2) \) as

\[
\Psi(u) = \sum_{j\in\mathbb{N}/2} \sum_{m,n\in I_j} (2j + 1)W_{mn}^{j*}D_{mn}^j(u) = \sum_{j\in\mathbb{N}/2} (2j + 1)\text{Tr}W_{mn}^{j*}D_{mn}^j(u^{-1}). \tag{49}
\]

Here \( W_{mn}^j \) are matrix elements of the representation of \( SU_q(2) \) of spin \( j \) (i.e. the corepresentation of \( A = Pol(SU_q(2)) \)) see (30)–(33) and \( D_{mn}^j(u) \) are the \( D \)-functions (25), \( u \in SU(2) \). Instead of \( (2j + 1) \) one can put another constant \( \lambda_j \) depending on \( q \) such that \( \lambda_j \to (2j + 1) \) when \( q \to 1 \). We are considering \( \Psi(u) \) as an analog of the operator \( \exp ib^*z \) creating the standard coherent states from the Fock vacuum, where \( b^* \) is the creation operator and \( z \) is a complex number.
Now we consider the "classical" limit for the coherent states. Let us prove that one has the following limiting formula

\[
\lim_{q \to 1} <f(g)\Psi(u)>_q = f(u).
\] (50)

Here \(f(g)\) is an element from \(A = Pol(SU_q(2))\), i.e. a polynomial in the generators \(a^*, a, c^*\) and \(c\) with coefficients independent of \(q\). For definiteness we assume the following ordering of the arguments of the polynomial \(f(a^*, a, c^*, c)\). In each monomial let us put first from the left \(a^*\) of some degree, then \(a\) and after that \(c^*\) and \(c\). One can use also the "normal" ordering.

Using the definition of \(\Psi(u)\) (44) and the Fourier expansion (44) for \(f(g)\) one sees that the proof of (50) is reduced to proving the relation

\[
\lim_{q \to 1} <f(g)W^j_{mn}>_q = \int f(u)D^j_{mn}(u)du.
\]

First prove that

\[
\lim_{q \to 1} <(a^*)^k_1(a)^{k_2}(c^*)^{k_3}(c)^{k_4}>_q = \int (a^*)^{k_1}(a)^{k_2}(c^*)^{k_3}(c)^{k_4}du,
\] (51)

where \(k_i\) are natural integers. Let us note that the left and the right hand sides of (51) vanish identically if \(k_1 \neq k_2\) and \(k_3 \neq k_4\). So we need only to consider

\[
<(a^*a)^{k_1}(c^*c)^{k_3}>_q
\] (52)

Using (44) this is equal to

\[
<(1-c^*c)^{k_1}(c^*c)^{k_3}>_q,
\] (53)

i.e. one needs to consider only \(<(c^*c)^k>_q\). It is equal to

\[
<(c^*c)^k>_q = (1-q^2)\sum_{n=0}^{\infty} q^{2(k+1)n} = \frac{q^{-k}}{[k+1]_q}.
\] (54)

For the corresponding classical expression, by using (22) one has

\[
\int (\gamma^*\gamma)^k du = \frac{1}{2} \int_0^\pi (\sin \frac{1}{2} \theta)^{2k} \sin \theta d\theta = \frac{1}{k+1}.
\] (55)

Comparing (54) and (53) one gets (51).
Now let us prove that
\[
\lim_{q \to 1} < (a^*)^{k_1}(a)^{k_2}(c^*)^{k_3}(c)^{k_4} W^j_{mn} >_q = \int (\alpha^*)^{k_1}(\alpha)^{k_2}(\gamma^*)^{k_3}(\gamma)^{k_4} D^j_{mn}(u) du.
\]
Since for any monomial one has the limiting relation (51) it is enough to prove the relation
\[
\lim_{q \to 1} W^j_{mn}(u) = D^j_{mn}(u),
\]
where in \( W^j_{mn}(u) \) we mean the expression (30)-(34) in which the quantum generators \( a^*, a, c^* \) and \( c \) are replaced by their classical counterparts \( \alpha^*, \alpha, \gamma^* \) and \( \gamma \).

Let us discuss the case \( m + m' \geq 0, m \geq m' \). In this case
\[
W^j_{mm'}(u) = q^{(m-j)(m-m')} \left[ \begin{array}{c} j + m \\ m - n \end{array} \right]^{1/2} \left[ \begin{array}{c} j - n \\ m - n \end{array} \right]^{1/2} q^2 \]
\[
(\alpha^*)^{m+m'}(\gamma)^{m-m'} P^{(m-m',m+m')}_{j-m'; m-m'}(\gamma^* \gamma : q^2).
\]
As one can expect, from explicit formulae (34) it follows that the little q-Jacobi polynomials go to Jacobi polynomials when \( q \to 1 \):
\[
\lim_{q \to 1} P^{s,t}_{n}(z; q) = P^{s,t}_{n}(1 - 2z) / P^{s,t}_{n}(1),
\]
and
\[
P^{s,t}_{n}(1) = \frac{(s + 1)n}{n!}.
\]
Here the Jacobi polynomials are defined by the formula
\[
P^{s,t}_{n}(z) = \frac{(s + 1)n}{n!} F(-n, n + s + t + 1; s + 1; \frac{1 - z}{2})
\]
\[
= \frac{(-1)^n 2^n n!}{(1 - z)^{-s} (1 + z)^{-t}} \frac{d^n}{dz^n}[(1 - z)^{n+s}(1 + z)^{n+t}],
\]
where \( F \) is the hypergeometric function. Therefore one has
\[
\lim_{q \to 1} W^j_{mm'}(u) = \frac{(j + m')!(j - m')!}{(j + m)!(j - m)!} 1^{1/2} \frac{1}{(m - m')!} \]
\[
(-\gamma^*)^{m-m'}(\alpha^*)^{m+m'} F(m - j, j + m + 1, m - m' + 1, \gamma^* \gamma).
\]
One can show that the expression (61) is equal to (25). To this end one needs to use the equality

\[ F(m - j, j + m + 1, m - m' + 1, \gamma^* \gamma) = \]

\[ \sum_{\mu} (1 - \gamma^* \gamma)^{j - m - \mu} (-\gamma^* \gamma)^\mu \frac{(j + m')!}{\mu!(j + m - \mu)!} \frac{(j - m)!}{(j - \mu + m)!} \frac{(m - m')!}{(m - m' + \mu)!} \]  

### 5 Dynamics on Quantum Groups

Dynamics on the classical group \( SU(2) \) is reduced to Euler rotations. For the classical matrix \( u = \begin{pmatrix} \alpha & -\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \in SU(2) \), one has three basic motions:

\[ \alpha(t) = \exp(-it) \alpha, \quad \gamma(t) = \gamma \]  

\[ \alpha(t) = \alpha, \quad \gamma(t) = \exp(-it) \gamma \]  

\[ \alpha(t) = \alpha \cos t + \gamma \sin t, \quad \gamma(t) = \gamma \cos t - \alpha \sin t \]  

Now we consider the corresponding "quantum", i.e. q-deformed motions. First note that if \( U_t \) is an unitary operator in \( L_2(R) \), then the matrix

\[ g(t) = (g_{mn}(t)) = \begin{pmatrix} a(t) & -qc^*(t) \\ c(t) & a^*(t) \end{pmatrix} \]  

belongs to \( SU_q(2) \). Here,

\[ a(t) = U_t \ a \ U_t^*, \quad c(t) = U_t \ c \ U_t^*. \]  

Therefore any unitary group operator \( U_t \) gives a group of automorphisms of \( SU_q(2) \). One can take,

\[ U_t = e^{itH}, \]  

where \( H \) is an arbitrary self adjoint operator depending on \( a, a^*, c \) and \( c^* \). The quantum dynamics is given by,

\[ g_{mn}(t) = e^{itH} g_{mn} e^{-itH}. \]  

We want to find such a Hamiltonian that in the semiclassical limit \( q \to 1 \) one could have,

\[ < g_{mn}(t) \Psi(u) >_q \to u_{mn}(t), \]
where $u_{mn}(t)$ is a classical rotation. For the infinitesimal rotation one has,

$$\delta g_{mn} = [iH, g_{mn}]\epsilon, \quad (71)$$

and,

$$< [iH, g_{mn}]\epsilon \Psi(u) > \rightarrow \delta u_{mn}. \quad (72)$$

To the first classical motion (63) corresponds the Hamiltonian $H = N$. Indeed, in this case

$$a(t) = e^{itN}ae^{-itN} = e^{-it}a; \quad c(t) = c. \quad (73)$$

and in the classical limit $q \rightarrow 1$ one gets,

$$< a(t)\Psi(u) > \rightarrow e^{-it} < a\Psi(u) > \rightarrow \alpha(t) = e^{-it}\alpha, \quad (74)$$

$$< c(t)\Psi(u) > \rightarrow \gamma(t) = \gamma. \quad (75)$$

To the classical motion (64) one has the corresponding ‘quantum’ dynamics,

$$a_{\phi}(t) = a_{\phi}, \quad c_{\phi}(t) = e^{-it}c_{\phi}. \quad (76)$$

In the semiclassical limit $q \rightarrow 1$ one gets,

$$< a_{\phi}(t)\Psi(u) > \rightarrow \alpha, \quad < c_{\phi}(t)\Psi(u) > \rightarrow e^{-it}\gamma. \quad (77)$$

Now let us discuss rotations (65). Take the following Hamiltonian,

$$H_{\phi} = \frac{1}{1-q} (ae^{-i\phi} + a^* e^{i\phi}). \quad (78)$$

Note the occurrence of the singular factor $\frac{1}{1-q}$. By using formulae (8) and (9) one gets for the infinitesimal ”rotations” of matrix $g$ the following answer,

$$g(\epsilon) = e^{-i\epsilon H} g e^{i\epsilon H} = g - i\epsilon[H, g] + ... =$$

$$\begin{pmatrix}
  a + i\epsilon(1 + q)cq^N & -q(c^* - i\epsilon(a^* q^N - q^N ae^{-2i\phi})) \\
  c + i\epsilon(q^N a - a^* q^N e^{2i\phi}) & a^* - i\epsilon(1 + q)q^N c^*
\end{pmatrix}. \quad (79)$$

Now consider the limit $q \rightarrow 1$. The operator of coherent states $\Psi(u)$ has the form of a sum over all spins

$$\Psi(u) = 1 + \Psi^{1/2}(u) + \Psi^1(u) + ..., \quad (79)$$
where
\[ \Psi^j(u) = (2j + 1) \sum W_{mn}^j D_{mn}^j(u). \] (80)

We consider only the contribution from \( \Psi^{1/2}(u) \), which is
\[ \Psi^{1/2}(u) = 2 \text{tr}(ug^*) = 2(a\alpha^* + c^*\gamma + qc\gamma^* + a^*\alpha). \] (81)

After a simple calculation one gets from (78)
\[ \langle [iH,a]\Psi^{1/2}(u) \rangle_q = -2i \frac{(1 + q)(1 - q^2)}{(1 - q^5)} \gamma \xrightarrow{q \to 1} -\frac{8}{5} \gamma, \] (82)
\[ \langle [iH,c]\Psi^{1/2}(u) \rangle_q = -2i \frac{(1 - q^2)^2}{(1 - q^3)(1 - q^5)} \alpha \xrightarrow{q \to 1} -\frac{8}{15} \alpha. \] (83)

Therefore for the infinitesimal rotations, one has
\[ \langle g(\epsilon)\Psi^{1/2}(u) \rangle_q \xrightarrow{q \to 1} u + i\epsilon \delta u, \] (84)

where
\[ \delta u = \frac{4}{5} \begin{pmatrix} \gamma & \frac{1}{3}\alpha^* \\ \frac{1}{3} \alpha & -\gamma^* \end{pmatrix}. \] (85)

Infinitesimal classical rotation corresponding to (88) is
\[ \delta u = \begin{pmatrix} \gamma & \alpha^* \\ \alpha & -\gamma^* \end{pmatrix}. \] (86)

The formulae (88) and (89) differ by numerical coefficients.

This consideration of quantum dynamics is not fully satisfactory. We would like to find such a quantum dynamics which in the limit \( q \to 1 \) leads precisely to classical dynamics.

6 Conclusions and Discussion

We have defined in this paper coherent states on the quantum group \( SU_q(2) \). One can generalize the operator of coherent states \( \Psi (2) \) to an arbitrary quantum group \( G_q \) in the form
\[ \Psi(g,u) = \sum m_{\lambda \chi \lambda}(W^\lambda(g)T^\lambda(u)). \] (87)
Here $\lambda$ enumerates all representations of $G_q$, $T^\lambda$ is a unitary representation of the classical Lie group $G$, $W^\lambda(g)$ is the corresponding representation of the quantum group $G_q$, $\chi_\lambda$ is the character of the representation $\lambda$ and $m_\lambda$ are numbers. It would be interesting to elaborate on such an approach for noncompact groups. We have noted the role of an appropriate choice of states in the consideration of the semiclassical limit $q \to 1$ and show the existence of classical limit for our coherent states. It would be interesting to find a full semiclassical expansion for $\langle f(g)\Psi(u) \rangle_q$ like we have in standard quantum mechanics.

There are considerations of $q$-deformed coherent states on quantum algebras, in particular on the $q$-deformation of the universal enveloping $U_q(su(2))$, see [28]-[31],[36]-[38]. Because of the duality between the quantum groups and the quantum algebra it is possible that there is a relation between our coherent states and the coherent states on quantum algebras. We postpone a discussion of this problem to a future publication.

The group $SU(2)$ is a 3-dimensional manifold and there is no symplectic structure on it. Therefore it seems that in passing to quantum group $SU_q(2)$ we should quantize not the Poisson brackets but another invariant geometrical structure on $SU(2)$. There is such a structure; it is the contact structure which is an analog of the symplectic structure on odd dimensional manifolds [13]. A contact structure on a manifold is a nondegenerate field of tangent hyperplanes. Such a field is given by a contact form. If we write a contact form on $SU(2)$ locally as $\omega = d\phi + pdx$ then the coordinate $\phi$ is not quantized in passing to $SU_q(2)$ and $p$ and $x$ after complexification turn out to be the operators $b$ and $b^*$ in (12). Note, parenthetically, that in fact in gauge theory one also deals with contact geometry since due to the constraints one has four coordinates $A_\mu$ and only three canonical momenta $p_i$ and a natural contact form is $\omega = dA_0 + p_i dA_i$.

Note that quantum groups from the point of view of geometric quantization are considered in [13]. Deformations of Poisson brackets on Lie groups and quantum duality are discussed in [16].

We have considered the semiclassical limit $q \to 1$ by using coherent states. In standard quantum mechanics there is an approach to quantization involving deformation theory using Wigner distribution and the Moyal brackets [44]. In such an approach
one can do semiclassical limit directly for operators. It would be very interesting to elaborate on an operator expansion corresponding to our coherent states.

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