Coherence for vectorial waves and majorization

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We show that majorization provides a powerful approach to the coherence conveyed by partially polarized transversal electromagnetic waves. Here we present the formalism, provide some examples and compare with standard measures of polarization and coherence of vectorial waves.

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I. INTRODUCTION

Coherence is a basic physical property that emerges in very different contexts, from classical optics to quantum mechanics. Therefore the efforts to find a proper measure of coherence seem justified, specially after its identification as a resource [1].

In this work we focus on the assessment of the coherence conveyed by pairs of partially polarized electromagnetic beams in classical optics. Coherence has two extreme physical manifestations: interference, as coherence when superimposing beams with the same vibrational state, and polarization, as coherence when superimposing beams with orthogonal vibration states. This makes specially attractive the analysis of coherence in the superposition of partially polarized waves. The complexity of the subject has motivated the introduction of complementary approaches focusing on different perspectives [2–4]. Among them, it makes sense to address the idea of global coherence embracing both interference and polarization at once [5, 6]. Parallels can also be drawn to its quantum counterpart through the idea of state purity [7–9].

Besides standard measures, entropies are in general good candidates to assess statistical properties. Entropy has already been used to measure polarization [8, 10], being specially useful for situations beyond Gaussian or second-order statistics. A rather attractive feature when dealing with entropies is the emergence of majorization [11]. As a proper precedent we can mention the application of majorization to the analysis of polarization of three-dimensional fields [13]. The conclusions will be contrasted to other accounts of coherence more focused on the ideas of field correlation or fringe visibility.

II. SETTINGS: FIELD STATES AND COHERENCE MEASURES

For definiteness let us focus on two vectorial electric fields \( \mathbf{E} \) at two spatial points \( r_{1,2} \) with just two non-vanishing components at each point, say \( E_{x,y} \). This can be the transverse electric field at the two pinholes of a Young interferometer. The complete system is made of four scalar electric fields that we will consider in the space-frequency domain \( E_\ell (r_j, \omega) \) with \( j = 1, 2, \ell = x, y \), and the temporal frequency \( \omega \) will be omitted from now on. Their statistics will be completely accounted for by the second-order field correlations gathered by the cross-spectral density tensor, that in our case is a \( 4 \times 4 \) non-negative matrix \( \Gamma \), with

\[
\Gamma = \begin{pmatrix}
\Gamma_{1,1} & \Gamma_{1,2} \\
\Gamma_{2,1} & \Gamma_{2,2}
\end{pmatrix},
\]

(2.1)

where \( \Gamma_{j,k} \) are \( 2 \times 2 \) correlation matrices with \( \Gamma_{j,k} = \Gamma_{k,j}^\dagger \),

\[
\Gamma_{j,k} = \langle \mathbf{E}(r_j) \mathbf{E}^\dagger(r_k) \rangle, \quad \mathbf{E}(r_j) = \begin{pmatrix} E_x(r_j) \\
E_y(r_j) \end{pmatrix}
\]

(2.2)

with \( \dagger \) representing Hermitian conjugation. Moreover, we can collect the four components into a single four-dimensional vector \( \mathbf{E} \) with components \( E_1 = E_x(r_1) \), \( E_2 = E_y(r_1) \), \( E_3 = E_x(r_2) \), \( E_4 = E_y(r_2) \), so that \( \Gamma_{j,k} = \langle E_j E_k^\dagger \rangle \), for \( j,k = 1, \ldots, 4 \). Under these conditions every account of coherence and polarization must be a function of these matrices \( \Gamma \) and \( \Gamma_{j,k} \).

Scalar interferometric coherence. – For completeness we recall the standard measure of coherence \( \mu \) in the simplest case of two scalar fields \( E_{1,2} \)

\[
\mu = \frac{\langle E_1^* E_2 \rangle}{\sqrt{\langle |E_1|^2 \rangle \langle |E_2|^2 \rangle}}.
\]

(2.3)

We refer to this as interferometric coherence in the sense of being the key factor controlling the visibility of the interference fringes obtained when superimposing \( E_{1,2} \).

Polarization. – Polarization expresses the coherence between two components, say \( E_{x,y} \) of a transverse field vector at a given spatial point as

\[
P^2 = 2 \frac{\text{tr} \left( \Gamma^2 \right)}{(\text{tr} \Gamma)^2} - 1,
\]

(2.4)
where here $\Gamma$ refers to the $2 \times 2$ coherency matrix with matrix elements $\Gamma_{j,k} = \langle E_j E_k^\dagger \rangle$, $j, k = x, y$. It is worth noting that this is the maximum interferometric coherence [2.3] that can be reached between any two field components $E_{1,2}$ obtained from $E_{x,y}$ by a unitary $2 \times 2$ matrix $U \in U(2)$, this is to say $\mu(U) \leq P$. After computing $P$ for $E(r_{1,2})$ the two degrees of polarization $P_{1,2}$ can be be combined to provide a single value, for example as in Refs. [14].

**Vectorial interferometric coherence.**—The interferometric account of coherence $\mu$ becomes more complex when the interfering fields are vectorial. Accordingly, different definitions have been proposed as different generalizations of Eq. (2.3), such as [2.4]:

$$\mu_{KW} = \frac{\text{tr} \Gamma_{1,2}}{\sqrt{\text{tr} \Gamma_{1,1} \text{tr} \Gamma_{2,2}}},$$

(2.5)

$$\mu_{TSF}^2 = \frac{\text{tr} \left( \Gamma_{1,2} \Gamma_{1,2}^\dagger \right)}{\text{tr} \Gamma_{1,1} \text{tr} \Gamma_{2,2}},$$

(2.6)

and, when the corresponding inverses exist,

$$\mu_{S,I} = \text{singular values of } \Gamma_{1,1}^{-1/2} \Gamma_{1,2} \Gamma_{2,2}^{-1/2},$$

(2.7)

where $\mu_{S,I}$ are real and $\mu_S \geq \mu_I \geq 0$.

**Global coherence.—**In this work we are mostly interested in regarding the four field components as a whole, asking for the global coherence conveyed by the complete field. This should comprise both the polarization and interferometric contributions, and should be expressed by the whole $\Gamma$ instead of its sub-matrices $\Gamma_{j,k}$.

For another perspective, we may say that polarization and interferometric coherence depends on a choice of field modes, which is the equivalent of the basis dependence in quantum mechanics. Thus we can attempt a mode-independent approach where coherence may be referred to as intrinsic, per se, or global [1]. This would be analogous to the role played by the degree of polarization (2.4) versus the degree of coherence (2.3) regarding two scalar electric fields.

A convenient generalization of the degree of polarization (2.4) to four-dimensional fields can be

$$\mu_g^2 = \frac{4}{3} \frac{\text{tr} \left( \Gamma^2 \right)}{\left( \text{tr} \Gamma \right)^2} - \frac{1}{3} = \frac{4}{3} \lambda^2 - \frac{1}{3},$$

(3.1)

where $\lambda$ is a four-dimensional vector containing the eigenvalues of $\Gamma/\text{tr} \Gamma \geq 0$. An alternative assessment along this line can be found in Ref. [5]. In the next section we show that $\mu_g$ is actually a case of Rényi entropy.

**III. MAJORIZATION AND GLOBAL DEGREE OF COHERENCE**

Next we recall the idea of majorization and its relation with coherence measures as functions of the $N \times N$ coherency matrix $\Gamma$. The main idea is that coherence is reflected in the dispersion of the eigenvalues of $\Gamma/\text{tr} \Gamma$, that are real, positive, and normalized $\sum j=1^N \lambda_j = 1$. We have two clear extremes. We have full coherence for those $\Gamma_p$ with only one eigenvalue different from zero, say $\lambda_1 = 1, \lambda_{j\neq 1} = 0$. On the other extreme, there is total lack of coherence for those $\Gamma_j$ where all the eigenvalues are equal $\lambda_j = 1/N$, so that $\Gamma_j$ is proportional to the identity. In between, the degree of coherence may be assessed using many possible functions of $\lambda$. Most of them are of entropic nature, such as the Rényi entropies [12]

$$R_q(\lambda) = \frac{1}{1-q} \ln \left( \sum_{j=1}^N \lambda_j^q \right),$$

(3.1)

where $q > 0$ is an index labeling different entropies. The limiting case $q \to 1$ is the Shannon entropy $R_1 = -\sum_{j=1}^N \lambda_j \ln \lambda_j$. For example, we have that $\mu_g$ in Eq. (2.8) is essentially $R_2$ as

$$\mu_g^2 = \frac{4}{3} R_2 - \frac{1}{3},$$

(3.2)

We say that $\Gamma$ majorizes $\hat{\Gamma}$, which will be expressed as $\Gamma \prec \hat{\Gamma}$, when the following relation between all the ordered partial sums $S_n$ of their corresponding eigenvalues holds,

$$S_n \left( \Gamma \right) = \sum_{j=1}^n \lambda_j^\downarrow \leq \sum_{k=1}^n \lambda_k^\downarrow = S_n \left( \hat{\Gamma} \right),$$

(3.3)

for all $n = 1, 2 \ldots, N$, where the superscript $\downarrow$ denotes the same $\lambda_j$ but arranged in decreasing order

$$\lambda_1^\downarrow \geq \lambda_2^\downarrow \geq \ldots \geq \lambda_N^\downarrow.$$

(3.4)

Throughout we will say that $\Gamma$ and $\hat{\Gamma}$ are comparable if one majorizes the other.

Next, some interesting facts about majorization. Majorization holds if and only if $\lambda = M \lambda$ where $M$ is a doubly stochastic matrix, so that $\hat{\Gamma}$ is more uniform or

![FIG. 1: Relation between partial ordered sums $S_n$ when the majorization $\Gamma \prec \hat{\Gamma}$ holds.](image)
more mixed than $\Gamma$. For all $\Gamma$ we have $\Gamma_1 \prec \Gamma \prec \Gamma_p$ [13]. Whenever two $\Gamma$ are comparable, the result is respected by all Schur-concave functions that include the Rényi entropies: if $\Gamma \prec \Gamma$ then $R_q(\Gamma) > R_q(\Gamma)$ for all $q$.

After all these facts we may say if $\Gamma \prec \Gamma$ then $\Gamma$ is more coherent than $\Gamma$. Majorization is a partial ordering relation, so that there are incomparable states: this is neither $\Gamma \prec \Gamma$ nor $\Gamma \prec \Gamma$. In such a case the ordered sums in Fig. 1 will intersect and the entropies will provide contradictory conclusions, such that $R_q(\Gamma) > R_q(\Gamma)$ while $R_q(\Gamma) < R_q(\Gamma)$ for different entropies $p \neq q$.

The case of $2 \times 2$ matrices $\Gamma$ is rather trivial since after normalization the spectrum $\lambda$ depends on a single parameter. Thus any two $\Gamma$ are comparable and there is no room for ambiguities or discrepancies between measures. The case of $3 \times 3$ matrices $\Gamma$ has been completely addressed in a recent work [13] regarding polarization and interferometric coherence are.

IV. UNPOLARIZED BEAMS OF THE SAME INTENSITY

In this case by means of suitable $U(2)$ transformations the $\Gamma$ matrix can be arranged so that

$$\Gamma_{1,1} = \Gamma_{2,2} = I \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \quad \Gamma_{1,2} = I \left( \begin{array}{cc} \mu_s & 0 \\ 0 & \mu_I \end{array} \right),$$

(4.1)

where $I$ represents the intensity of each component, that will play no role on the following.

The corresponding values of the above measures of polarization and interferometric coherence are: $P_1 = P_2 = 0$,

$$\mu_{KW} = \frac{1}{2} (\mu_s + \mu_I), \quad \mu_{TSF}^2 = \frac{1}{4} (\mu_s^2 + \mu_I^2),$$

(4.2)

while the global coherence is

$$\mu_g^2 = \frac{1}{6} (\mu_s^2 + 2 \mu_I^2).$$

(4.3)

The properly ordered eigenvalues of $\Gamma/\text{tr} \Gamma$ in decreasing order are

$$\lambda_1 = \frac{1}{4} (1 + \mu_s), \quad \lambda_2 = \frac{1}{4} (1 + \mu_I),$$

$$\lambda_3 = \frac{1}{4} (1 - \mu_s), \quad \lambda_4 = \frac{1}{4} (1 - \mu_I),$$

(4.4)

leading to the following ordered partial sums

$$S_1 = \frac{1}{4} (1 + \mu_s), \quad S_2 = \frac{1}{4} (2 + \mu_s + \mu_I),$$

$$S_3 = \frac{1}{4} (3 + \mu_s), \quad S_4 = 1.$$

(4.5)

After these expressions for $S_n$ majorization is actually determined just by the two first conditions in Eq. [3.3] so that $\Gamma \prec \Gamma$ if and only if the following two conditions are satisfied

$$\mu_s \geq 0, \quad \mu_s + \mu_I \geq 0, \quad \mu_s + \mu_I \geq \mu_s + \mu_I.$$

(4.6)

Since the degree of polarization of $E (r_j)$ vanishes one might ask whether the majorization conditions [4.6] are equivalent to any of the above degrees of interferometric coherence in Eq. [4.2] or the global coherence in Eq. [4.3]. The result is negative. To show this the picture in Fig. 2 may be useful. This is a $\mu_{TSF}$ plane where any $\Gamma$ is represented by a point in the region of the first quadrant below the bisecting line. This is to respect the condition $\mu_s \geq 0, \mu_I \geq 0$. The dotted line represents the condition $\mu_s + \mu_I \geq \mu_s$, while the dashed line represents the condition $\mu_s + \mu_I \geq \mu_s + \mu_I$. Therefore relations [4.6] define three regions:

$\alpha$: Points $\Gamma$ with $\Gamma \prec \Gamma$ are to the left of the dotted line below the dashed line.

$\beta$: Points $\Gamma$ with $\Gamma \prec \Gamma$ are to the right of the dotted line and above the dashed line.

$\gamma$: Points $\Gamma$ incomparable with $\Gamma$.

After this it is clear from Eqs. [4.2] and [4.6] that $\mu_{KW} \geq \mu_{KW}$ is just a necessary but not sufficient condition for $\Gamma \prec \Gamma$. On the other hand, both $\mu_{TSF}$ and $\mu_g$ depend just on the distance of the point $\Gamma$ to the origin.

A simple example of incomparable states is provided by $\mu_s = 1, \mu_I = 0$ and $1 \geq \mu_s \geq \mu_I \geq 1/2$. In this case we have always $\mu_{KW} \geq \mu_{KW}$. In Fig. 3 we have represented $R_3(\Gamma) - R_2(\Gamma)$ and $R_1(\Gamma) - R_1(\Gamma)$. It can be appreciated that for $0.78 > \mu_s > 0.71$ the two Rényi entropies provide contradictory conclusions: this is $R_3(\Gamma) > R_3(\Gamma)$ versus $R_3(\Gamma) < R_3(\Gamma)$. We have plotted as well $\mu_{TSF} - \mu_{TSF}$ around this same region showing regions where this measure of interferometric coherence contradicts the entropies. Moreover, in Fig. 4 we plot the ordered partial sums $S_n(\Gamma)$ for $\mu_s = 0.75$ showing the lack of majorization.

V. FULLY POLARIZED VERSUS PARTIALLY POLARIZED BEAMS

Let us combine a fully polarized beam linearly polarized along then axis $x$ with a partially polarized beam with

$$\Gamma_{1,1} = I \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \quad \Gamma_{2,2} = I \left( \begin{array}{cc} 1 & 0 \\ 0 & \delta \end{array} \right),$$

$$\Gamma_{1,2} = I \left( \begin{array}{cc} \mu_s & 0 \\ 0 & 0 \end{array} \right),$$

(5.1)
where we will assume $\delta \geq 0$, and $I$ represents the intensity of each component that will play no role on the following.

The corresponding values of the measures of polarization and interferometric coherence are:

$$P_1 = 1, \quad P_2 = \frac{|1 - \delta|}{1 + \delta}, \quad (5.2)$$

$$\mu_{T SF}^2 = \mu_{KW}^2 = \frac{\mu_S^2}{1 + \delta}, \quad (5.3)$$

and

$$\mu_S^2 = \frac{4 + 8 \mu_S^2 + 3 \delta^2 - 4 \delta}{3 (2 + \delta)^2}. \quad (5.4)$$

The dependence on $\mu_S$ is the expected one: larger $\mu_S$ implies both larger interferometric coherence and larger global coherence. On the other hand, the dependence on $\delta$ is more complicated. The degree of polarization has a minimum at $\delta = 1$, while interferometric coherence always decreases when increasing $\delta$. Regarding global coherence we have that for fixed $\mu_S$ it has a minimum at $\delta = 1 + \mu_S^2$. Thus this example provides an interesting competition between polarization and interferometric coherence. From now on we will focus on the dependence on $\delta$ for fixed $\mu_S$.

In this case, since there is always a vanishing eigenvalue we have $S_3 = S_4 = 1$ and majorization is just determined by the first two partial sums in Eq. (3.3). Taking into account that $S_1 = \lambda_1^2$ and $S_2 = S_2 + \lambda_3^2 = 1$ we get a very simple relation already found in the three-dimensional problem considered in Ref. [13]. This is that $\bar{\Gamma} \prec \Gamma$ is equivalent to

$$\lambda_1^2 \geq \bar{\lambda}_1^2, \quad \lambda_3^2 \leq \bar{\lambda}_3^2. \quad (5.5)$$

Regarding the ordering of the eigenvalues $\lambda$, we can distinguish three cases depending on the relation between $\delta$ and $\mu_S$. The first one we consider is $\delta < 1 - \mu_S$, where the arrangement of $\lambda$ in decreasing order is

$$\lambda_1^2 = \frac{1 + \mu_S}{2 + \delta}, \quad \lambda_2^2 = \frac{1 - \mu_S}{2 + \delta}, \quad \lambda_3^2 = \frac{\delta}{2 + \delta}, \quad \lambda_4^2 = 0. \quad (5.6)$$

After Eqs. (5.5) and (5.6) for $\mu_S = \bar{\mu}_S$ we readily get

$$\bar{\Gamma} \prec \Gamma \leftrightarrow \delta \geq \delta. \quad (5.7)$$

This is a quite expected result since for $\delta < 1$ increasing $\delta$ means both lesser degree of polarization and lesser interferometric coherence.

The opposite situation holds for $\delta > 1 + \mu_S$. In this case we will have $\delta > 1$ so we may expect that global coherence should emerge of a suitable balance between the opposed behaviors displayed by the degree of polarization and the interferometric coherence. The ordering of eigenvalues is

$$\lambda_1^2 = \frac{\delta}{2 + \delta}, \quad \lambda_2^2 = \frac{1 + \mu_S}{2 + \delta}, \quad \lambda_3^2 = \frac{1 - \mu_S}{2 + \delta}, \quad \lambda_4^2 = 0. \quad (5.8)$$
After Eqs. (5.5) and (5.8) for $\mu_S = \tilde{\mu}_S$ we readily get
\[ \Gamma < \Gamma \leftrightarrow \delta \leq \delta. \quad (5.9) \]
Roughly speaking, when $\delta$ increases the purity of the state provided by polarization overwhelms the decrease of the interferometric coherence.

Finally, in the intermediate situation $1 + \mu_S > \delta > 1 - \mu_S$ and $\mu_S = \tilde{\mu}_S$ the states are always incomparable unless $\delta = \delta$. This is because
\begin{align*}
\lambda_1^2 &= \frac{1 + \mu_S}{2 + \delta}, \quad \lambda_2^2 = \frac{\delta}{2 + \delta}, \\
\lambda_3^2 &= \frac{1 - \mu_S}{2 + \delta}, \quad \lambda_4^2 = 0, \quad (5.10)
\end{align*}
so that the two conditions in Eq. (5.5) are incompatible
\[ \lambda_1^2 \geq \lambda_4^2 \leftrightarrow \delta \geq \delta, \quad \lambda_3^2 \leq \lambda_4^2 \leftrightarrow \delta \leq \delta. \quad (5.11) \]

**VI. CONCLUSIONS**

We have shown that majorization provides a powerful approach to the coherence conveyed by partially polarized transversal waves. This is because it can be regarded as a kind of meta-measure of global coherence whose conclusions are respected by entropic measures of polarization and coherence. Moreover, majorization allows us to draw many parallels with coherence in quantum physics.

We have illustrated the approach by means of some simple but meaningful examples. The results are contrasted to other measures of polarization and interferometric coherence for vectorial waves. The situation is particularly interesting when polarization and interference behave in opposite ways.

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[1] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying coherence, Phys. Rev. Lett. 113, 140401 (2014); S. Cheng and M. J. W. Hall, Complementarity relations for quantum coherence, Phys. Rev. A 92, 042101 (2015).
[2] B. Karczewski, Degree of coherence of the electromagnetic field, Phys. Lett. 5, 191–192 (1963); E. Wolf, Unified theory of coherence and polarization of random electromagnetic beams, Phys. Lett. A 312, 263–267 (2003).
[3] J. Tervo, T. Setälä, and A. T. Friberg, Degree of coherence for electromagnetic fields, Opt. Express 11, 1137–1143 (2003).
[4] P. Réfrégier and F. Goudail, Invariant degrees of coherence of partially polarized light, Opt. Express 13, 6051–6060 (2005).
[5] H. M. Ozaktas, S. Yüksel, and M. A. Kutay, Linear algebraic theory of partial coherence: discrete fields and measures of partial coherence, J. Opt. Soc. Am. A 19, 1563–1571 (2002).
[6] A. Luis, Degree of coherence for vectorial electromagnetic fields as the distance between correlation matrices, J. Opt. Soc. Am. A 24, 1063–1068 (2007); A. Luis, Overall degree of coherence for vectorial electromagnetic fields and the Wigner function, J. Opt. Soc. Am. A 24, 2070–2074 (2007); A. Luis, Coherence and visibility for vectorial light, J. Opt. Soc. Am. A 27, 1764–1769 (2010).
[7] A. Luis, Quantum-classical correspondence for visibility, coherence, and relative phase for multidimensional systems, Phys. Rev. A 78, 025802 (2008).
[8] O. Gamel and D. F. V. James, Measures of quantum state purity and classical degree of polarization Phys. Rev. A 86, 033830 (2012).
[9] J. J. Gil, Polarimetric characterization of light and media. Physical quantities involved in polarimetric phenomena, Eur. Phys. J. Appl. Phys. 40, 147 (2007); J. J. Gil, Interpretation of the coherency matrix for three-dimensional polarization states, Phys. Rev. A 90, 043858 (2014); J. J. Gil and R. Ossikovski, Polarized Light and the Mueller Matrix Approach (CRC Press, 2016).
[10] Ch. Brosseau, Fundamentals of Polarized Light: A Statistical Optics Approach (Wiley, 1998); A. Luis, Degree of polarization in quantum optics, Phys. Rev. A 66, 013806 (2002); A. Picozzi, Entropy and degree of polarization for nonlinear optical waves, Opt. Lett. 29, 1653–1655 (2004); Ph. Réfrégier, Polarization degree of optical waves with non-Gaussian probability density functions: Kullback relative entropy-based approach, Opt. Lett. 30, 1090–1092 (2005); Ph. Réfrégier and F. Goudail, Kullback relative entropy and characterization of partially polarized optical waves, J. Opt. Soc. Am. A 23, 671–678 (2006).
[11] A. W. Marshall and I. Olkin, Inequalities: Theory of Majorization and Its Applications (Academic Press, New York, 1980); J. Aczel and Z. Daroczy, On Measures of Information and Their Characterization (Academic Press, New York, 1975).
[12] A. Rényi, On the measures of entropy and information, Proc. 4th Berkeley Symp. on Mathematics and Statistical Probability (University of California Press, 1961). Vol. 1, pp. 547–561; Ch. Beck, Generalised information and entropy measures in physics, Contemporary Physics 50, 495–510 (2009).
[13] O. Gamel and D. F. V. James, Majorization and measures of classical polarization in three dimensions, J. Opt. Soc. Am. A 31, 1620–1626 (2014).
[14] G. Piquero, J. M. Movilla, P. M. Mejías, R. Martínez-herrero, Degree of polarization of non-uniformly partially polarized beams: a proposal, Opt. Quantum Electron. 31, 223–226 (1999); A. Luis, Coherence, polarization, and entanglement for classical light fields, Opt. Commun. 282, 3665–3670 (2009).

[15] I. Bengtsson and K. Życzkowski, Geometry of Quantum States: An Introduction to Quantum Entanglement, Cambridge University Press, Cambridge, 2006.