On external backgrounds and linear potential in three dimensions

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For a three-dimensional theory with a coupling $\phi \varepsilon^{\mu \nu \lambda} v_\mu F_{\nu \lambda}$, where $v_\mu$ is an external constant background, we compute the interaction potential within the structure of the gauge-invariant but path-dependent variables formalism. While in the case of a purely timelike vector the static potential remains Coulombic, in the case of a purely spacelike vector the potential energy is the sum of a Bessel and a linear potentials, leading to the confinement of static charges. This result may be considered as another realization of the known Polyakov’s result.

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I. INTRODUCTION

It is presently widely accepted that quantitative understanding of confinement remains as the major challenge in QCD. In this respect, the distinction between the apparently related phenomena of screening and confinement have been of considerable importance in order to gain further insight and theoretical guidance into this problem. It is worth recalling at this stage that field theories which yield a linear potential are relevant to particle physics, since those theories may be used to understand the confinement of quarks and be considered as effective theories of QCD. According to this viewpoint, a simple effective theory in which confining potentials are obtained when a scalar field $\phi$ is coupled to gauge fields via the 3 + 1 dimensional interaction term

$$L_I = \frac{g}{8} \phi \varepsilon^{\mu \alpha \beta} F_{\mu \nu} F_{\alpha \beta},$$

(1)

has been investigated in Ref.[1]. The phenomenology of this model has shown that in the case of a constant electric field strength expectation value the static potential remains Coulombic, while in the case of a constant magnetic field strength the potential energy is the sum of a Yukawa and a linear potential, leading to the confinement of static charges. More interestingly, similar results have been obtained in the context of the dual Ginzburg-Landau theory [2], as well as for a theory of antisymmetric tensor fields that results from the condensation of topological defects as a consequence of the Julia-Toulouse mechanism [3]. Thus, from a phenomenological point of view, there is a class of models which are good candidates for effective theories of QCD.

In order to give continuity to this program it would be interesting to verify the above scenario when a scalar field $\phi$ is coupled to gauge fields via the 2 + 1 dimensional interaction term

$$L_I = \frac{g}{2} \phi \varepsilon^{\mu \nu \lambda} v_\mu F_{\nu \lambda},$$

(2)

where $v_\mu$ is an external constant background. Our motivation is mainly to explore the effects of the external background constant on the confining and screening nature of the potential that could be useful for later developments in more realistic theories, such as describing low-energy properties in condensed matter physics [4]. Here we should mention that the interaction term (2) arises from the dimensional reduction of Maxwell electrodynamics with the (Lorentz-violating) Carroll-Field-Jackiw term [5]. The Lorentz violating theories have been extensively discussed in the last few years. For example, in connection with ultra-high energy cosmic rays [6], with space-time varying coupling constants [7], and in supersymmetric Lorentz-violating extensions [8]. In this Letter, however, we examine the effects of the Lorentz violating background on the interaction energy along the lines of Refs.[1, 3]. It must now be observed that this approach provides a physically-based alternative to the usual Wilson loop approach, where in the latter the usual qualitative picture of confinement in terms of an electric flux tube linking quarks emerges naturally. As we shall see, the static potential remains Coulombic for a purely timelike vector $v^\mu$. On the other hand, adopting a purely spacelike vector $v^\mu$, the potential energy is the sum of a Bessel and a linear potential, that is, the confinement between static charges is obtained. In this last respect we recall that, almost twenty years ago, Polyakov [10] showed that compact Maxwell theory in 2 + 1 dimensions confines permanently electric test charges. In this way

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our calculation provides a complementary phenomenological picture of the known Polyakov’s result, in the hope that this will be helpful to understand better effective gauge theories in $2 + 1$ dimensions.

II. INTERACTION ENERGY

As mentioned above, our principal purpose is to calculate explicitly the interaction energy between static pointlike sources for a model one which contains the term $\frac{1}{4}g^2\epsilon^{\mu\nu\lambda}v_\mu F_{\nu\lambda} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2$, this model is similar to the studied in Ref. [1]. To this end we will compute the expectation value of the energy operator $\mathcal{H}$ in the physical state $|\Phi\rangle$ describing the sources, which we will denote by $\langle H \rangle_\Phi$. The Abelian gauge theory we are considering is defined by the following generating functional in three-dimensional spacetime:

$$Z = \int \mathcal{D}\phi\mathcal{D}A \exp \left\{ i \int d^3x \mathcal{L} \right\},$$

where the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \frac{g^2}{8}\epsilon^{\mu\nu\lambda}v_\mu F_{\nu\lambda} + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2,$$

and $m$ is the mass for the scalar field $\phi$. As in [1] we restrict ourselves to static scalar fields, a consequence of this is that one may replace $\Delta \phi = -\nabla^2\phi$, with $\Delta \equiv \partial_\mu\partial^\mu$. It also implies that, after performing the integration over $\phi$ in $Z$, the effective Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{g^2}{8}\epsilon^{\mu\nu\lambda}v_\mu F_{\nu\lambda} \frac{1}{\sqrt{V^2 - m^2}}\epsilon^{\sigma\gamma\beta}v_\sigma F_{\gamma\beta}.$$

By introducing $V^{\nu\lambda} \equiv \epsilon^{\mu\nu\lambda}v_\mu$, expression (5) then becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{g^2}{8}V^{\nu\lambda}F_{\nu\lambda} \frac{1}{\sqrt{V^2 - m^2}}V_{\gamma\beta}F_{\gamma\beta}.$$

At this stage we note that (6) has the same form as the corresponding effective Lagrangian density in four-dimensional spacetime. This common feature provides the starting point for the examination of the effects of the Lorentz violating background on the interaction energy.

A. Spacelike background case

We now proceed to obtain the interaction energy in the $V^{0i} \neq 0$ and $V^{ij} = 0$ ($v_0 = 0$) case (referred to as the spacelike background in what follows), by computing the expectation value of the Hamiltonian in the physical state $|\Phi\rangle$. Lagrangian density (6) then becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 - \frac{g^2}{2}V^{0i}F_{0\mu} \frac{1}{\sqrt{V^2 - m^2}}V^{0k}F_{0k} - A_0 J^0,$$

where $J^0$ is the external current, $(\mu, \nu = 0, 1, 2)$ and $(i, k = 1, 2)$.

Once this is done, the canonical quantization in the manner of Dirac yields the following results. The canonical momenta are

$$\Pi^0 = 0,$$

and

$$\Pi_i = D_{ij}E_j,$$

where $E_i \equiv F_{0i}$ and $D_{ij} \equiv \left( \delta_{ij} - g^2V_{0i} \frac{1}{\sqrt{V^2 - m^2}}V_{0j} \right)$. Since $D$ is a nonsingular matrix $(\det D = 1 - g^2V^2 \neq 0)$ with $V^2 \equiv V^{0i}V^{i0}$, there exists the inverse of $D$ and from Eq. (6) we obtain

$$E_i = \frac{1}{\det D} \left\{ \delta_{ij} \det D + g^2V_{0i} \frac{1}{\sqrt{V^2 - m^2}}V_{0j} \right\} \Pi_j.$$
The corresponding canonical Hamiltonian is thus

$$H_C = \int d^2x \left\{ -A_0 \left( \partial_t \Pi^i - J^0 \right) + \frac{1}{2} \Pi^2 + \frac{g^2}{2} \frac{(V \cdot \Pi)^2}{(\nabla^2 - M^2)} + \frac{1}{2} B^2 \right\}, \quad (11)$$

where $M^2 \equiv m^2 + g^2 V^2$ and $B$ is the magnetic field. Requiring the primary constraint $\Psi(x) \equiv \partial_0 \Pi^i - J^0 = 0$, we discard extended Hamiltonian that generates translations in time then read $\Pi^i_0(x) = c_0(x) \Pi^i_0(x) + c_1(x) \Gamma_1(x)$, where $c_0(x)$ and $c_1(x)$ are the Lagrange multiplier fields. Since $\Pi^i_0 = 0$ for all time and $A_0^0 = c_0(x)$, which is completely arbitrary, we discard $A_0(x)$ and $\Pi^i_0(x)$ because they add nothing to the description of the system. Then, the Hamiltonian takes the form

$$H = \int d^2x \left\{ \frac{1}{2} \Pi^2 + \frac{g^2}{2} \frac{(V \cdot \Pi)^2}{(\nabla^2 - M^2)} + \frac{1}{2} B^2 + c(x) \left( \partial_0 \Pi^i - J^0 \right) \right\}, \quad (12)$$

where $c(x) = c_1(x) - A_0(x)$.

The quantization of the theory requires the removal of non-physical variables, which is done by imposing a gauge condition such that the full set of constraints becomes second class. A convenient choice is found to be $\Pi^i_0 = 0, \Pi^i_0 = 0$.

$$\Gamma_2(x) \equiv \int d\xi \Pi^i_0(\xi) \equiv \int d\lambda \partial^i \lambda A_i \left( \lambda x \right) = 0, \quad (13)$$

where $\lambda \left( 0 \leq \lambda \leq 1 \right)$ is the parameter describing the spacelike straight path $x^i = \xi^i + \lambda (x - \xi)^i$, and $\xi$ is a fixed point (reference point). There is no essential loss of generality if we restrict our considerations to $\xi^i = 0$. The Dirac brackets can now be obtained and the nontrivial Dirac bracket involving the field variables takes the form

$$\left\{ A_i(x), \Pi^j(y) \right\}^\star = \delta^i_0 \delta^{(2)}(x - y) - \partial_x^i \int_0^1 d\lambda \delta^{(2)}(\lambda x - y). \quad (14)$$

We are now in a position to evaluate the interaction energy between pointlike sources in the model under consideration, where a fermion is localized at $y'$ and an antifermion at $y$. From our above discussion, we see that $\langle H \rangle_{\Phi}$ reads

$$\langle H \rangle_{\Phi} = \langle \Phi | \int d^2x \left\{ \frac{1}{2} \Pi^2 + \frac{g^2}{2} \frac{(V \cdot \Pi)^2}{(\nabla^2 - M^2)} + \frac{1}{2} B^2 \right\} |\Phi \rangle. \quad (15)$$

Next, as was first established by Dirac [12], the physical state can be written as

$$|\Phi \rangle \equiv |\overline{\Pi}(y) \Psi(y')\rangle = \overline{\Psi}(y) \exp \left( i \frac{e}{h} d^2z A_i(z) \right) \psi(y') |0\rangle, \quad (16)$$

where $|0\rangle$ is the physical vacuum state and the line integral appearing in the above expression is along a spacelike path starting at $y'$ and ending at $y$, on a fixed time slice. From this we see that the fermion fields are now dressed by a cloud of gauge fields. From the foregoing Hamiltonian discussion, we first note that

$$\Pi_i(x) |\overline{\Pi}(y) \Psi(y')\rangle = \overline{\Psi}(y) \Psi(y') \Pi_i(x) |0\rangle + e \int_{y'}^y d^2z \delta^{(3)}(z - x) |\Phi \rangle. \quad (17)$$

Combining Eqs. (15) and (17), we have

$$\langle H \rangle_{\Phi} = \langle H \rangle_0 + V^{(1)} + V^{(2)}, \quad (18)$$

where $\langle H \rangle_0 = \langle 0 | H | 0 \rangle$, and the $V^{(1)}$ and $V^{(2)}$ terms are given by:

$$V^{(1)} = -\frac{e^2}{2} \int d^2x \int_{y'}^y d^2z' \delta^{(2)}(z - x') \frac{1}{\nabla^2 - M^2} \int_y^{y'} d^2z \delta^{(2)}(x - z). \quad (19)$$
and
\[ V^{(2)} = \frac{e^2 m^2}{2} \int d^2 x \int_{y}^{y'} dz'_i \delta^{(2)} (x - z') \frac{1}{\nabla z'_i - M^2} \int_{y}^{y'} \int y' \, dz'^2 \delta^{(2)} (x - z), \] (20)

where the integrals over \( z^i \) and \( z'_i \) are zero except on the contour of integration.

The \( V^{(1)} \) term may look peculiar, but it is just the familiar Bessel interaction plus self-energy terms. In effect, expression (19) can also be written as
\[ V^{(1)} = \frac{e^2}{2} \int_{y}^{y'} \int y' \, dz'_i \delta^{(2)} (x - z') G(z', z), \] (21)

where \( G \) is the Green function
\[ G(z', z) = \frac{1}{2\pi} K_0 (M |z' - z|). \] (22)

Employing Eq. (22) and remembering that the integrals over \( z^i \) and \( z'_i \) are zero except on the contour of integration, expression (21) reduces to the familiar Bessel interaction after subtracting the self-energy terms, that is,
\[ V^{(1)} = -\frac{e^2}{2\pi} K_0 (M |y - y'|). \] (23)

The task is now to evaluate the \( V^{(2)} \) term, which is given by
\[ V^{(2)} = \frac{e^2 m^2}{2} \int_{y}^{y'} \int y' \, dz'^2 G(z', z). \] (24)

By using the integral representation of the Green function (22)
\[ K_0 (x) = \int_0^\infty \cos (x \sinh t) dt = \int_0^\infty \frac{\cos (xt)}{\sqrt{t^2 + 1}} dt, \] (25)

where \( x > 0 \), expression (24) can also be written as
\[ V^{(2)} = \frac{e^2 m^2}{2\pi M^2} \int_0^\infty \frac{dt}{t^2 + \frac{\sin^2 (MLt)}{\sqrt{t^2 + 1}}} (1 - \cos (MLt)), \] (26)

where \( L \equiv |y - y'| \).

Now let us calculate integral (26). For this purpose we introduce a new auxiliary parameter \( \varepsilon \) by making in the denominator of integral (26) the substitution \( t^2 \rightarrow t^2 + \varepsilon^2 \). Thus it follows that
\[ V^{(2)} = \lim_{\varepsilon \to 0} V^{(2)} = \lim_{\varepsilon \to 0} \frac{e^2 m^2}{2\pi M^2} \int_0^\infty \frac{dt}{t^2 + \varepsilon^2} \frac{1}{\sqrt{t^2 + 1}} (1 - \cos (MLt)). \] (27)

A direct computation on the \( t \)-complex plane yields
\[ V^{(2)} = \frac{e^2 m^2}{4M^2} \left( \frac{1 - e^{-ML\varepsilon}}{\varepsilon} \right) \frac{1}{\sqrt{1 - \varepsilon^2}}. \] (28)

Taking the limit \( \varepsilon \to 0 \), expression (28) then becomes
\[ V^{(2)} = \frac{e^2 m^2}{4M} |y - y'|. \] (29)

From Eqs. (23) and (29), the corresponding static potential for two opposite charges located at \( y \) and \( y' \) may be written as
\[ V(L) = -\frac{e^2}{2\pi} K_0 (ML) + \frac{e^2 m^2}{4M} L, \] (30)
where \( L \equiv |y - y'|. \)

It must now be observed that the rotational symmetry is restored in the resulting form of the potential, although the external background breaks the isotropy of the problem in a manifest way. It should be remarked that this feature is also shared by the corresponding four-dimensional spacetime interaction energy.

We further note that the result agrees with that of Polyakov based on the monopole plasma mechanism, in the short distance regime. In this way the above analysis reveals that, although both models are different, the physical content is identical in the short distance regime. This behavior is also obtained in the context of the condensation of topological defects.

\[ \text{III. FINAL REMARKS} \]

In summary, we have considered the confinement versus screening issue for a three-dimensional theory with a coupling \( \phi_\mu \nu \lambda v_\mu F_{\nu \lambda} \), when the external constant vector \( v_\mu \) is pure spacelike or timelike, respectively.
It was shown that in the case when the vector \( v_\mu \) is purely timelike no unexpected features are found. Indeed, the resulting static potential remains Coulombic. More interestingly, it was shown that when the vector \( v_\mu \) is purely spacelike the static potential displays a Bessel piece plus a linear confining piece. Effectively, therefore, the model studied leads to a confining potential between static charges for spacelike \( v_\mu \). An analogous situation in the four-dimensional spacetime case may be recalled \([1]\), where a constant expectation value for the gauge field strength \( \langle F_{\mu\nu} \rangle \) characterizes the external background constant \( v_\mu \). Also, a common feature of these models (three and four-dimensional) is that the rotational symmetry is restored in the resulting interaction energy.

We conclude by noting that our result can be considered as another physical realization of the Polyakov’s model. However, although both models lead to confinement, the above analysis reveals that the mechanism of obtaining a linear potential is quite different.

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