Slices of motivic Landweber spectra

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1 Introduction

In this paper we show that a Conjecture of Voevodsky concerning the slices of the motivic Thom spectrum $\text{MGL}$ [13] implies a general statement about slices of motivic Landweber spectra.

A similar result is announced by Hopkins-Morel.
A proof of Voevodsky’s conjecture, to the author’s knowledge over fields of characteristic 0, is likewise announced by Hopkins-Morel.

In [11] Levine gives an unconditional computation of the slices of the algebraic $K$-theory spectrum $KGL$ over perfect fields yielding (shifted) motivic Eilenberg-MacLane spectra.

In [13] it is suggested that a Conner-Floyd type isomorphism

$$KGL_{**}(X) \cong MGL_{**}(X) \otimes_{MU_*} Z[u, u^{-1}]$$

for homotopy algebraic $K$-theory could yield a proof of the conjectures on the slices of $KGL$ in [13] assuming the conjectures about $MGL$.

Using the Conner-Floyd isomorphism for homotopy algebraic $K$-theory established in [7] and in [8] over fields in a slightly different form our result gives a positive answer to the strategy suggested in [13].

We point out that Voevodsky’s full Conjecture uses the motivic Eilenberg-MacLane spectrum, in particular the conjecture says that the zeroth slice of $MGL$ is the motivic Eilenberg-MacLane spectrum.

In our discussion we formulate the conjecture for all slices of $MGL$ relative to the zeroth slice.

The zeroth slice of the sphere spectrum is known to be motivic cohomology over perfect fields (see [12] for fields of characteristic 0 and [4] for perfect fields). By [11, Corollary 1.3] the zeroth’ slices of the sphere spectrum and $MGL$ agree.

The proof of the main result consists of two steps. In the first we show that the statement holds for Landweber exact spectra of the form $MGL \wedge E$, $E$ also a Landweber spectrum. The main ingredient is a topological result about the projective dimension of the $MU$-homology of an even topological Landweber spectrum, [2].

In the second step we use a cosimplicial resolution of the given Landweber spectrum in terms of spectra of the form from the first step.

In a last paragraph we show that the argument used here also shows that cohomological Landweber exactness holds for all compact spectra, not only for the strongly dualizable ones as shown in [6].

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2 Conventions

We fix a base scheme $S$ (Noetherian of finite Krull dimension) and denote the stable $\mathbb{A}^1$-homotopy category over $S$ by $\text{SH}(S)$. As in [6] we let $\text{SH}(S)_T$ be the full localizing triangulated category spanned by all Tate spheres $S^{p,q}$ which is also known as cellular spectra.

We let $\mathbf{1}$ be the motivic sphere spectrum.

Generalizing the notion of rigid homotopy groups of a spectrum $E$ given in [13] (i.e. $\pi_{p,q}^{\text{rig}}(E) = \pi_{p,q}(E)$) we set $\pi_{p,q}^{\text{rig}}(X_*):= \pi_{2p,p}(X_*)$ for an object $X_* \in \text{SH}(S)^\mathbb{Z}$.

3 Preliminaries on the slice filtration

As in [13] we denote the slices of a spectrum $X \in \text{SH}(S)$ by $s_i(X) \in \text{SH}(S)$.

The functor

$$s_*: \text{SH}(S) \to \text{SH}(S)^\mathbb{Z}$$

has good multiplicative properties, for a general treatment of that using the theory of model categories see [10]. It is in particular shown in loc. cit. that the functor $s_*$ preserves ring and module objects in a highly structured sense. For most of the paper we use these statements on the level of homotopy, see [10, page iv, (5)].

In [13, p. 5] it is observed that there are natural maps $s_i(E) \wedge s_j(F) \to s_{i+j}(E \wedge F)$ (the map in loc. cit. is written after taking sums over all $i$ resp. $j$). Assembling these maps in a graded way gives natural maps in $\text{SH}(S)^\mathbb{Z}$

$$\alpha_{E,F}: s_*(E) \wedge s_*(F) \to s_*(E \wedge F),$$

where the $\wedge$-product in $\text{SH}(S)^\mathbb{Z}$ is defined using the $\wedge$-product in $\text{SH}(S)$ and taking sums of $\wedge$-products of a fixed total degree.

Indeed the $\alpha_{E,F}$ assemble to give $s_*$ the structure of a lax tensor functor by the following argument:

The slice $s_0(E)$ for an effective spectrum $E \in \text{SH}(S)^{\text{eff}}$ can be obtained by a left Bousfield localization of the triangulated category $\text{SH}(S)^{\text{eff}}$ along the subcategory $\Sigma_T \text{SH}(S)^{\text{eff}}$, see [10] for a model category version of this. In detail the functor $s_0$ restricted to effective objects is the composition

$$\text{SH}(S)^{\text{eff}} \to \text{SH}(S)^{\text{eff}}/\Sigma_T \text{SH}(S)^{\text{eff}} \to \text{SH}(S)^{\text{eff}},$$
where the first arrow is the quotient map and the second arrow the right adjoint to the quotient map which exhibits the quotient as a full subcategory of the first category. For the existence of the quotients see e.g. [3, par. 5.6]. Now $\Sigma T\text{SH}(S)^{\text{eff}}$ is a tensor ideal of $\text{SH}(S)^{\text{eff}}$, therefore there is an induced $\wedge$-product on the quotient $\text{SH}(S)^{\text{eff}}/\Sigma T\text{SH}(S)^{\text{eff}}$ and the quotient map $\text{SH}(S)^{\text{eff}} \to \text{SH}(S)^{\text{eff}}/\Sigma T\text{SH}(S)^{\text{eff}}$ is a tensor functor. Thus the right adjoint is a lax tensor functor, which gives $s_0: \text{SH}(S)^{\text{eff}} \to \text{SH}(S)^{\text{eff}}$ the structure of a lax tensor functor. By applying suitable shifts $\Sigma i$ this construction gives the functor $s_0^* : \text{SH}(S)^{\text{eff}} \to \text{SH}(S)^{\text{eff}}$ the structure of a lax tensor functor.

In the whole paper we will denote the spectrum $s_0(MGL)$ by $H$. By the above it is a ring spectrum and using the effectivity of $MGL$ ([11, Corollary 3.2]) it comes with a ring map $MGL \to H$.

We make the following assumption, which is called (SIMGL):

$s_i(MGL) \cong \Sigma T H \otimes \pi_{2i}(MU)$ in $\text{SH}(S)$ compatible with the homomorphism $MU_* \to MGL_{**}$ as in [13, Conjecture 5].

The assumption implies that shifted slices $\Sigma^{0-i}s_iM$ of a cellular $MGL$-module $M$ are in the localizing triangulated subcategory of $\text{SH}(S)$ generated by $H$. We call such spectra strictly $H$-cellular. We call a module $X_i \in \text{SH}(S)^{\mathbb{Z}}$ strictly $H$-cellular if for all $i$ the module $\Sigma^{0-i}X_i$ is strictly $H$-cellular.

4 Remarks on phantom maps

Throughout the paper we will use the notion of phantom map. We recall that in a compactly generated triangulated category with sums a map between two objects is called phantom if it induces the zero map between the represented cohomology theories on compact objects.

If the triangulated category has a compatible tensor product and if every compact object is strongly dualizable then this is the same that the corresponding homology transformation on the whole category, or equivalently on the compact objects, is zero.

This is the case e.g. for the categories $\text{SH}(S)_T$, $\text{SH}(S)_D$ (the last category is spanned by strongly dualizable objects, see [6, par. 4]).

Let $f: T \to S$ be a map between base schemes. Let $g$ be the right adjoint to the pullback functor $f^*: \text{SH}(S)_T \to \text{SH}(T)_T$. Then $g$ is a $\text{SH}(S)_T$-module functor (compare [6, Lemma 4.7]). Let $F: E \to F$ be a phantom map in $\text{SH}(S)_T$. Then $g$ applied to $f^*E \wedge K \to f^*F \wedge K$ yields $E \wedge g(K) \to F \wedge g(K)$. It follows that $f^*: \text{SH}(S)_T \to \text{SH}(T)_T$
preserves phantom maps. A similar argument shows that \( SH(S)_\tau \hookrightarrow SH(S)_D \) preserves phantoms.

Let \( F \) be as above. One can also see that for a smooth \( S \)-scheme \( X \) the transformation \( \text{Hom}([X], F) \) is zero. It is not clear to the author if \( F \) is necessarily phantom in \( SH(S) \).

## 5 Landweber spectra

We recall briefly some results from [6] which we shall need in this paper.

The motivic Thom spectrum \( MGL \) is a commutative monoid in \( SH(S) \). By the construction of [9, 2.1] there is a strictly commutative model as symmetric \( T \)-spectrum, \( T \) the Tate object \( A^1/(A^1 \setminus \{0\}) \).

We let \( \text{BAb} \) be the abelian category of bigraded abelian groups.

For a Landweber exact \( MU_* \)-module \( M_* \) (which we always consider to be evenly graded in the usual topological grading, but we adopt the convention that we regrade by dividing by 2) one looks at the functor

\[
SH(S) \rightarrow \text{BAb} \\
X \mapsto MGL_*(X) \otimes_{MU_*} M_*.
\]

Here \( MU_* \) and \( M_* \) are considered as bigraded (more precisely Adams graded graded) abelian groups via the diagonal \( Z(2,1) \) (for more precise statements see [6]). By the results of [6] this functor is a homology theory on \( SH(S) \) and representable by a cellular (or Tate-) spectrum \( E \). There is a choice of that spectrum which is canonical up to isomorphism (which is canonical up to a possible phantom map in Tate-spectra) by requiring that \( E \) is the pullback of a Tate-spectrum representing the same theory over the integers.

A refined version of this statement gives a representing object as highly structured \( MGL \)-module.

Let \( \mathcal{D}_{MGL} \) be the derived category of (highly structured) \( MGL \)-modules. Then the functor

\[
\mathcal{D}_{MGL} \rightarrow \text{BAb} \\
X \mapsto X_* \otimes_{MU_*} M_*.
\]

is a homology theory and representable by a cellular \( MGL \)-module.

We let \( \mathcal{D}_{MGL,T} \) be the subcategory of cellular \( MGL \)-modules.
6 Slices of Landweber exact theories

Theorem 6.1. Suppose (SlMGL) is fulfilled. Let $M_*$ be a Landweber exact $\text{MU}_*$-module and let $E_Z$ be the corresponding Landweber exact motivic spectrum in $\text{SH}(\mathbb{Z})$ given by [6, Theorem 9.7]. Let $E$ be its pullback to $S$. Then $s_*(E) \cong \Sigma_i^* H \otimes M_i$ (here $M_i$ is the $2i$-th homotopy group of the corresponding topological Landweber spectrum) compatible with the homomorphism $M_* \to E_{**}$.

In the above $H \otimes A$ for a torsion free abelian group $A$ is the spectrum $H \otimes (S_{\text{Top}} \otimes A)$, where the first $\otimes$ is the exterior action of the stable topological homotopy category and $S_{\text{Top}} \otimes A$ is the sphere spectrum with $A$-coefficients, i.e. a spectrum representing the homology theory $X \mapsto X_0 \otimes A$ on the topological stable homotopy category. $S_{\text{Top}} \otimes A$ is well defined up to possible phantom maps.

Corollary 6.2. Suppose (SlMGL) is fulfilled. Then $s_i(KGL) \cong \Sigma_i^* H$ compatible with the natural map $\mathbb{Z} \to \pi_{2i,i} KGL$.

Proof. The spectrum $KGL$ is Landweber exact for the $\text{MGL}$-algebra $\mathbb{Z}[u, u^{-1}]$ classifying the mutliplicative formal group law over $\mathbb{Z}[u, u^{-1}]$, see [7, Theorem 1.2]. The result follows from Theorem 6.1. □

Lemma 6.3. Let $R$ be a motivic ring spectrum (i.e. a commutative monoid in $\text{SH}(S)$), $A$ a torsion free abelian group, $M$ a $R$-module and $\varphi: A \to \pi_{0,0} M$ a map. Then there is a map $R \otimes A \to M$ which is an $R$-module map and which induces $\varphi$ via $A \to \pi_{0,0}(R \otimes A) \to \pi_{0,0} M$. Moreover it is well defined up to phantoms in $\text{SH}(S)$.

Proof. First note that $1 \otimes A$ has such a universal property by using the adjunction $\text{SH} \to \text{SH}(S)$, $\text{SH}$ the topological stable homotopy category and the corresponding universal property of $S_{\text{Top}} \otimes A$. Tensoring the resulting map $1 \otimes A \to M$ with $R$ and composing with the module structure map gives the required map. It is unique up to phantoms since on the level of cohomology theories on compacts it is well defined. □

6.1 Slices of Landweber spectra of the form $\text{MGL} \wedge E$

One idea is to use resolutions of the $\text{MU}_*$-module $M_*$ by free or projective $\text{MU}_*$-modules. Let $M_*$ be the coefficients of a Landweber spectrum $\text{MU} \wedge \text{E}_{\text{Top}}$ for $\text{E}_{\text{Top}}$ also Landweber. Here we induce the $\text{MU}_*$-module structure from the first factor in $\text{MU} \wedge \text{E}_{\text{Top}}$. We let $E_Z$
be the \(\text{MGL}_\mathbb{Z}\)-module representing the theory for the module \(E^\top_*\). Hence \(\text{MGL}_\mathbb{Z} \wedge E\mathbb{Z}\) represents the theory corresponding to \(M_*\).

By [2, 2.12 and 2.16] there exists a 2-term resolution of \(M_*\) by projective \(\text{MU}_*\)-modules

\[
0 \to P_* \overset{\phi}{\to} Q_* \to M_* \to 0, \tag{1}
\]

where \(P_*\) and \(Q_*\) come by construction as retracts of free \(\text{MU}_*\)-modules (see [1, Lemma 4.6] which is cited in the proof of [2, 2.14]), say of \(\bigoplus_i \text{MU}_*(n_i)\) and \(\bigoplus_j \text{MU}_*(m_j)\).

As \(\text{MU}_*\)-module \(M_*\) is flat. We shall not need this fact in this paragraph, it will become relevant in the last paragraph.

For any Landweber exact \(\text{MU}_*\)-module \(N_*\) (in particular for any projective \(\text{MU}_*\)-module) we denote by \(h_{M_*}\) the corresponding homology theory on \(\mathcal{D}_{\text{MGL}_\mathbb{Z}}\) given by \(X \mapsto (X_* \otimes_{\text{MU}_*} N_*)_0\). Any \(\text{MU}_*\)-module map between such modules gives rise to a transformation between the homology theories.

Hence we get the sequence

\[
0 \to h_{P_*} \to h_{Q_*} \to h_{M_*} \to 0 \tag{2}
\]

of homology theories.

This is short exact since by the flatness of \(M_*\) as quasi coherent sheaf over the moduli stack of formal groups with trivialized constant vector fields for any \(X \in \mathcal{D}_{\text{MGL}_\mathbb{Z}}\) the map \(h_{P_*}(X) \to h_{Q_*}(X)\) is a mono (*). Now lift \(h_{\phi}: h_{P_*} \to h_{Q_*}\) to a map between cellular \(\text{MGL}_\mathbb{Z}\)-modules \(\Phi: M_P \to M_Q\). (\(P_*\) and \(Q_*\) are projective so this is easy, one can also invoke that \(\mathcal{D}_{\text{MGL}_\mathbb{Z}, T}\) is Brown.)

Let \(C_\mathbb{Z}\) be the cofiber of \(\Phi\). By (*) the sequence of homology theories associated to the exact triangle

\[
M_P \to M_Q \to C_\mathbb{Z} \to M_P[1] \tag{3}
\]

is isomorphic to the sequence [2], in particular the homology theory associated to \(C_\mathbb{Z}\) is canonically isomorphic to \(h_{M_*}\). Hence \(C_\mathbb{Z}\) is isomorphic to \(\text{MGL}_\mathbb{Z} \wedge E\mathbb{Z}\) since \(\mathcal{D}_{\text{MGL}_\mathbb{Z}, T}\) is Brown.

We now look at the triangle

\[
s_*(M_{P,S}) \to s_*(M_{Q,S}) \to s_*(C_S) \to s_*(M_{P,S})[1] \tag{4}
\]

in \(\text{SH}(S)^\mathbb{Z}, M_{P,S}, M_{Q,S}, C_S\) the pullbacks of \(M_P, M_Q, C_\mathbb{Z}\) to \(S\).
Since we have maps $P_\ast \to M_{P,S,\ast}$, $Q_\ast \to M_{Q,S,\ast}$, $M_\ast \to C_{S,\ast}$ we get maps

$$P_\ast \to \pi_{\ast}^{rig} s_\ast (M_{P,S}),$$

likewise for $Q_\ast$ and $M_\ast$. These are $MU_\ast$-module maps ($s_\ast X$ has the structure of an $s_\ast (\text{MGL})$-module, $X$ a $\text{MGL}$-module).

For a $MU_\ast$-module $N_\ast$ which is torsionfree as abelian group we informally denote by $s_\ast (\text{MGL}) \otimes_{MU_\ast} N_\ast$ the module in $\text{SH}(S)^2$ which has $\Sigma^q_{\ast} H \otimes N_q$ in the $q$-th component, similarly for maps between such $MU_\ast$-modules. By Lemma (6.3) the module $s_\ast (\text{MGL}) \otimes_{MU_\ast} N_\ast$ has the weak universal property that for a given map of $MU_\ast$-modules $\phi: N_\ast \to \pi_{\ast}^{rig} s_\ast (N')$, $N'$ a $\text{MGL}$-module, there is an induced map

$$s_\ast (\text{MGL}) \otimes_{MU_\ast} N_\ast \to s_\ast (N'),$$

compatible with $\phi$ unique up to possible phantoms.

Thus we get maps

$$\psi_P: s_\ast (\text{MGL}) \otimes_{MU_\ast} P_\ast \to s_\ast (M_{P,S}),$$

similarly $\psi_Q$ and $\psi_M$ for $Q_\ast$ and $M_\ast$. The maps $\psi_P$ and $\psi_Q$ are isomorphisms by the assumption (SlMGL) and since $P_\ast$ and $Q_\ast$ are retracts of free $MU_\ast$-modules. Via these isomorphisms the map

$$s_\ast (M_{P,S}) \to s_\ast (M_{Q,S})$$

represents the map

$$s_\ast (\text{MGL}) \otimes_{MU_\ast} (P_\ast \to Q_\ast).$$

Now since $M_\ast$ is torsionfree the cofiber of the last map is $s_\ast (\text{MGL}) \otimes_{MU_\ast} M_\ast$ (this is already so for the cofibers of the maps $S^{\text{Top}} \otimes (P_q \to Q_q)$ in the topological stable homotopy category).

This shows that the map $\psi_M: s_\ast (\text{MGL}) \otimes_{MU_\ast} M_\ast \to s_\ast (C_S)$ is an isomorphism. This is the content of the following proposition.

**Proposition 6.4.** Theorem (6.1) holds for Landweber spectra of the form $\text{MGL} \wedge E$ for $E$ Landweber.

**Remark 6.5.** Consider the base change of the boundary map $C_Z \to M_P[1]$ of the triangle (3) to the spectrum $S$ of a subfield of $C$. It is phantom in $\text{SH}(S)_T$ since the corresponding homology theories yield a short exact sequence. In general it is non-trivial since after topological realization we recover the original sequence $P_\ast \to Q_\ast \to M_\ast$ as coefficients, and $M_\ast$ is in general not projective.
7 Cosimplicial resolutions

In this section we prove theorem (6.1).

Let $\triangle$ be the simplicial category, $\triangle^*$ the category of the ordered pointed sets $[n]_* = \{0, \ldots, n\} \amalg \{*\}$ for $n \in \{-1, 0, 1, \ldots\}$ pointed by $*$ and order preserving pointed maps. An extension of a cosimplicial diagram to $\triangle^*$ corresponds to a ‘contraction’ to the value at $[-1]_*$. For example the homotopy limit of a cosimplicial diagram which is the restriction of a $\triangle^*$-diagram in a model category is weakly equivalent to the value at $[-1]_*$. We shall only need the following strict version of the assertion.

**Lemma 7.1.** Let $\psi_\bullet : A_\bullet \to B_\bullet$ be a map between $\triangle^*$-diagrams in a category. Suppose $\psi_\bullet$ is an isomorphism on the objects $[i]_*$ of $\triangle^*$ for $i \geq 0$. Then $\psi_{-1}$ is also an isomorphism.

**Proof.** We let $g : A_{-1} \to A_0$, $h : A_0 \to A_{-1}$ be the maps induced by the unique maps in $\triangle^*$, $f,e : A_0 \to A_1$ the maps induced by the maps $[0]_* \to [1]_*$ which send 0 to 0 resp. 1, $k : A_1 \to A_0$ the map induced by the map $[1]_* \to [0]_*$ sending 0 to 0 and 1 to $*$. It is easily seen that these maps furnish a split equalizer. Hence $A_{-1}$ is the limit of $A_\bullet |_{\Delta}$, likewise for $B_\bullet$. The result follows.

Let us fix a Landweber exact $\text{MGL}$-module $E$ giving rise to a Landweber homology theory for the $\text{MU}_*\text{-module } M_*$. Let $E^{\text{Top}}$ be the topological Landweber spectrum.

The cosimplicial resolution $\text{MGL}^{\wedge \bullet} \wedge E$ of $E$ extends to a functor $\triangle^* \to \mathcal{D}_{\text{MGL}}$ using the $\text{MGL}$-module structure on $E$. The wedge $\text{MGL}^{\wedge i} \wedge E$ is regarded as $\text{MGL}$-module via the last factor.

We have natural maps

$$\pi_{2j}(\text{MU}^{\wedge i} \wedge E^{\text{Top}}) \to \pi_{j}^{\text{rig}} s_* (\text{MGL}^{\wedge i} \wedge E)$$

which induce maps

$$\Sigma^j_T H \otimes \pi_{2j}(\text{MU}^{\wedge i} \wedge E^{\text{Top}}) \to s_j (\text{MGL}^{\wedge i} \wedge E)$$

which are unique up to possible phantoms.

These maps are also functorial in $i$ up to possible phantom maps. More precisely we have a $\triangle^*$-diagram $\Sigma^j_T H \otimes \pi_{2j}(\text{MU}^{\wedge \bullet} \wedge E^{\text{Top}})$ in $\text{SH}(S)$ modulo phantoms and a transformation of $\triangle^*$-diagrams

$$\Sigma^j_T H \otimes \pi_{2j}(\text{MU}^{\wedge \bullet} \wedge E^{\text{Top}}) \to s_j (\text{MGL}^{\wedge \bullet} \wedge E),$$
again well defined up to possible phantoms.

This induces a transformation of diagrams of cohomology theories defined on compact objects of $\text{SH}(S)$

$$\text{Hom}(-, \Sigma^j_T H \otimes \pi_{2j}(\text{MU}^\bullet \wedge \text{ETop})) =$$

$$\text{Hom}(-, \Sigma^j_T H) \otimes \pi_{2j}(\text{MU}^\bullet \wedge \text{ETop}) \rightarrow \text{Hom}(-, s_j(\text{MGL}^\bullet \wedge \text{E})).$$

By Proposition 6.4 we know that this is an isomorphism on the subcategory of $\Delta_*$ spanned by the objects $\{[0]*, [1]*, \ldots\}$. By Lemma (7.1) it follows that it is also an isomorphism on $[-1]*$, which is Theorem (6.1).

Remark 7.2. One can try to streamline the argument in the second step by showing that $H$ can be realized as an $E_\infty$-algebra. First note that $s_0$ can be obtained by colocalization among all $\{\Sigma^p\Sigma^q X | q \geq 0\}$ and then localization along the maps $S = \{\Sigma^p\Sigma^q X \rightarrow 0 | q > 0\}$. There is the problem that the colocalization might not be cofibrantly generated, hence we cannot apply the techniques available to pursue the further localization. Instead one looks at the full $\infty$-subcategory of the $\infty$-category associated to the semimodel category of $E_\infty$-ring spectra whose underlying objects are effective. This is presentable in the sense of [2] and thus one should be able to find a left proper combinatorial model. Then one can directly localize this model category of effective $E_\infty$-ring spectra along the free $E_\infty$-maps generated by $S$. Alternatively one can try to localize the $\infty$-category directly.

A local model with respect to this localization yields $H$ as an $E_\infty$-algebra under $\text{MGL}$.

Having this one can form the derived category of $H$-modules $\mathcal{D}_H$ and using in the arguments of this paragraph that a map between strictly $H$-cellular objects in $\mathcal{D}_H$ (with the definition of being strictly $H$-cellular altered to be generated by $H$ inside $\mathcal{D}_H$) is an isomorphism if it is so on the $\pi_{i,0}$, $i \in \mathbb{Z}$.

8 Coherent Landweber Exactness

We start again with a topological evenly graded Landweber spectrum $E^{\text{Top}}$ and let $M_* = E^{\text{Top}}$ be the coefficients. Let $E \in \mathcal{D}_{\text{MGL}}$ be the corresponding Landweber module. It is well defined up to phantoms in $\mathcal{D}_{\text{MGL,T}}$. We also denote by $E$ the underlying spectrum in $\text{SH}(S)_T$ with the $\text{MGL}$-module structure in $\text{SH}(S)$.

Lemma 8.1. The functor $v: \mathcal{D}_{\text{MGL,T}} \rightarrow \text{SH}(S)_T$ preserves phantom maps.
Proof. For $X \in \text{SH}(S)_T$ and $E \in \mathcal{D}_{\text{MGL}}$ we have $\text{Hom}(X, vE) = \text{Hom}(\text{MGL} \wedge X, E)$. \hfill □

We want to exhibit a natural map

$$\alpha_{M^*,X}: \text{MGL}^{**}X \otimes_{\text{MU}^*} M^* \rightarrow E^{**}X$$

for any $X \in \text{SH}(S)$. As usual $M^* = M_{-\bullet}$.

Therefore let $a \in \text{MGL}^{p,q}X$ and $b \in M^i$. By smashing the map $a: \Sigma^{-p,-q}X \rightarrow \text{MGL}$ with $E$ and applying the module structure map we get a map $\Sigma^{-p,-q}X \wedge E \rightarrow E$. Composing with $b: 1^{-2i,-i} \rightarrow E$ we get a map $\Sigma^{-2i-p,-i-q}X \rightarrow E$. This defines the map $\alpha_{M^*,X}$.

Let $N^*$ be other Landweber coefficients and $M^* \rightarrow N^*$ a $\text{MU}^*$-map. Let $F$ be the motivic spectrum corresponding to $N^*$ derived from a $\text{MGL}$-module and $f: E \rightarrow F$ be a map of $\text{MGL}$-modules in $\text{SH}(S)$ corresponding to $M^* \rightarrow N^*$. It is unique up to possible phantoms in $\text{SH}(S)_T$.

From the definition of $\alpha_{M^*,X}$ and $\alpha_{N^*,X}$ it follows that these maps are natural in $M^* \rightarrow N^*$ and $f$.

It follows that we get a transformation of $\triangle$-diagrams

$$\alpha_{(E^{\text{Top}} \wedge \text{MU}^*)^{**},X}: \text{MGL}^{**}X \otimes_{\text{MU}^*} (E^{\text{Top}} \wedge \text{MU}^{\bullet})^* \rightarrow (E \wedge \text{MGL}^{\bullet})^{**}X.$$

**Lemma 8.2.** $\alpha_{(E^{\text{Top}} \wedge \text{MU}^*)^{**},X}$ is an isomorphism for compact $X$ and $\bullet > 0$.

**Proof.** Clearly it is sufficient to prove the statement for $\bullet = 1$. Let $N^* = (E^{\text{Top}} \wedge \text{MU})^*_*$ be the coefficients of $E \wedge \text{MGL}$. Here we view $N^*$ as $\text{MU}_s$-module via the last factor. As already remarked $N^*$ is flat as $\text{MU}_s$-module. This can be seen e.g. by considering $M^i$ as flat quasi coherent sheaf on the moduli stack of formal groups with trivialized constant vector fields. Then $N^*$ is just the pullback of this sheaf to $\text{Spec} (\text{MU}_s)$.

Let

$$0 \rightarrow P^* \xrightarrow{\phi} Q^* \rightarrow N^* \rightarrow 0,$$

be a resolution by projective $\text{MU}_s$-modules as in section (6.1).

Then

$$0 \rightarrow \text{MGL}^{**}X \otimes_{\text{MU}^*} P^* \rightarrow \text{MGL}^{**}X \otimes_{\text{MU}^*} Q^* \rightarrow \text{MGL}^{**}X \otimes_{\text{MU}^*} N^*$$
is again exact by the flatness of $N_*$. Moreover $\alpha_{P_*}$, $\alpha_{Q_*}$ are easily seen to be isomorphisms on compacts. Thus the map induced by $\phi$ on the targets of these maps is injective on compacts. Since this is part of the long exact cohomology sequence for the triangle corresponding to the resolution we deduce that the target of $\alpha_{N_*}$ is the cokernel of the above injection on compacts. This proves the claim.

**Corollary 8.3.** $\alpha_{M_*}$ is an isomorphism between cohomology theories defined on compact objects.

We also deduce the following uniqueness statement:

**Corollary 8.4.** The phantom maps in $\mathcal{SH}(S)_T$ coming from $\mathcal{D}_{\text{MGL}}(T)$ up to which the Landweber spectrum $E$ is well-defined are also phantom in $\mathcal{SH}(S)$.

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