ONE-SIDED FRACTIONAL DERIVATIVES, FRACTIONAL LAPLACIANS, AND WEIGHTED SOBOLEV SPACES

PABLO RA´UL STINGA AND MARY VAUGHAN

Abstract. We characterize one-sided weighted Sobolev spaces $W^{1,p}(\mathbb{R}, \omega)$, where $\omega$ is a one-sided Sawyer weight, in terms of a.e. and weighted $L^p$ limits as $\alpha \to 1^−$ of Marchaud fractional derivatives of order $\alpha$. Similar results for weighted Sobolev spaces $W^{2,p}(\mathbb{R}^n, \nu)$, where $\nu$ is an $A_p$-Muckenhoupt weight, are proved in terms of limits as $s \to 1^−$ of fractional Laplacians $(-\Delta)^s$. These are Bourgain–Brezis–Mironescu-type characterizations for weighted Sobolev spaces. We also complement their work by studying a.e. and weighted $L^p$ limits as $\alpha, s \to 0^+$. 

1. Introduction and main results

G. Leibniz introduced the notation
\[
\frac{d^n}{dt^n} u(t)
\]
for derivatives of integer order $n \geq 1$ of a function $u = u(t) : \mathbb{R} \to \mathbb{R}$. In 1695, G. L'Hôpital posed Leibniz the question:

What if $n = 1/2$?

Since then, many “derivatives of fractional order” have been defined. Historical names are Lacroix, Fourier, Liouville, Riemann, Riesz, Weyl and, more recently, Chapman, Marchaud, Caputo, Jumarie, Grunwald and Letnikov, among others, see for instance [16]. In our opinion, any reasonable definition of derivative $D^\alpha$ of fractional order $0 < \alpha < 1$ should at least satisfy the relations $D^\alpha[D^\beta u](t) = D^{\alpha+\beta} u(t)$, 
\[
\lim_{\alpha \to 1^-} D^\alpha u(t) = u'(t) \quad \text{and} \quad \lim_{\alpha \to 0^+} D^\alpha u(t) = u(t)
\]
whenever $u$ is a sufficiently smooth function.

By looking at the various definitions of fractional derivatives [16], one notices that most of them have a one-sided nature. For example, the Marchaud left fractional derivative, given by
\[
(D_{\text{left}})^\alpha u(t) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^t \frac{u(\tau) - u(t)}{(\tau - t)^{1+\alpha}} d\tau
\]
where $\Gamma$ denotes the Gamma function, takes into account the values of $u$ to the left of $t$ (the past). Similarly, the Marchaud right fractional derivative
\[
(D_{\text{right}})^\alpha u(t) = \frac{1}{\Gamma(-\alpha)} \int_t^\infty \frac{u(\tau) - u(t)}{(\tau - t)^{1+\alpha}} d\tau
\]
looks at $u$ only to the right of $t$ (the future). These were first introduced by André Marchaud in his 1927 dissertation [11] (see also, for example, [1, 2, 3, 4, 16] for theory and applications). It is clear that if $u$ is a Schwartz class function, then

$$\lim_{\alpha \to 1^-} (D_{\text{left}})^\alpha u(t) = u'(t) \quad \text{and} \quad \lim_{\alpha \to 0^+} (D_{\text{left}})^\alpha u(t) = u(t).$$

In this paper, we study characterizations of Sobolev spaces by limits of fractional derivatives in the almost everywhere and $L^p$ senses. Of course, an obvious class of functions $u$ to work with is the classical Sobolev space $W^{1,p}(\mathbb{R})$. Instead, given the one-sided structure of fractional derivatives, we believe that a more natural, general class of functions to consider is the weighted Sobolev space $W^{1,p}(\mathbb{R}, \omega)$, $1 \leq p < \infty$, but where $\omega$ is now a one-sided Sawyer weight in $A_p^-(\mathbb{R})$ (for left-sided fractional derivatives) or in $A_p^+(\mathbb{R})$ (for right-sided fractional derivatives). These spaces are defined as

$$W^{1,p}(\mathbb{R}, \omega) = \{ u \in L^p(\mathbb{R}, \omega) : u' \in L^p(\mathbb{R}, \omega) \}$$

with the norm

$$\|u\|_{W^{1,p}(\mathbb{R}, \omega)}^p = \|u\|_{L^p(\mathbb{R}, \omega)}^p + \|u'\|_{L^p(\mathbb{R}, \omega)}^p$$

for $1 \leq p < \infty$. The Sawyer weights $\omega \in A_p^-(\mathbb{R})$ are the good weights for the original one-sided Hardy–Littlewood maximal function [9, p. 92]:

$$M^- u(t) = \sup_{h>0} \frac{1}{h} \int_{t-h}^t |u(\tau)| \, d\tau.$$  

Indeed, $M^-$ is bounded in $L^p(\mathbb{R}, \omega)$ if and only if $\omega \in A_p^-(\mathbb{R})$, $1 < p < \infty$, see [17], and $M^-$ is bounded from $L^1(\mathbb{R}, \omega)$ into weak-$L^1(\mathbb{R}, \omega)$ if and only if $\omega \in A_1^-(\mathbb{R})$, see [14]. It is clear that $A_p^-(\mathbb{R})$ is a larger family than the classical class of Muckenhoupt weights $A_p(\mathbb{R})$. In particular, any decreasing function is in $A_p^-(\mathbb{R})$, but there are decreasing functions that are not in $A_p(\mathbb{R})$. For instance, $\omega(t) = e^{-t}$ belongs to $A_p^-(\mathbb{R})$ but not to $A_p(\mathbb{R})$ because it is not a doubling weight. Similar considerations hold for right-sided weights in $A_p^+(\mathbb{R})$. See Section 2 for more details.

We find appropriate one-sided distributional spaces in which fractional derivatives have sense. Then we show that in such a setting one can always define $(D_{\text{left}})^\alpha u$ as a distribution for any function $u \in L^p(\mathbb{R}, \omega), \omega \in A_p^-(\mathbb{R})$. It turns out then that our weighted Sobolev spaces can be characterized by limits of one-sided left fractional derivatives.

**Theorem 1.1** ($W^{1,p}(\mathbb{R}, \omega)$ and limits of left fractional derivatives). Let $u \in L^p(\mathbb{R}, \omega)$, where $\omega \in A_p^-(\mathbb{R}), 0 \leq p < \infty$.

(a) If $u \in W^{1,p}(\mathbb{R}, \omega)$, then the distribution $(D_{\text{left}})^\alpha u$ coincides with a function in $L^p(\mathbb{R}, \omega)$ and

$$\tag{1.3}(D_{\text{left}})^\alpha u(t) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^t \frac{u(\tau) - u(t)}{(t-\tau)^{1+\alpha}} \, d\tau \quad \text{for a.e. } t \in \mathbb{R}$$

with

$$\tag{1.4}\|(D_{\text{left}})^\alpha u\|_{L^p(\mathbb{R}, \omega)} \leq C_{p, \omega} \left( \|u\|_{L^p(\mathbb{R}, \omega)} + \|u'\|_{L^p(\mathbb{R}, \omega)} \right)$$

for some constant $C_{p, \omega} > 0$. Moreover,

$$\tag{1.5} \lim_{\alpha \to 1^-} (D_{\text{left}})^\alpha u = u' \quad \text{in } L^p(\mathbb{R}, \omega) \text{ and a.e. in } \mathbb{R}$$

and

$$\tag{1.6} \lim_{\alpha \to 0^+} (D_{\text{left}})^\alpha u = u \quad \text{a.e. in } \mathbb{R}.$$
Furthermore, the limit in (1.6) holds also in $L^p(\mathbb{R}, \omega)$ when $1 < p < \infty$, and in weak-
$L^1(\mathbb{R}, \omega)$ when $p = 1$.

(b) Conversely, suppose that $(D_{\text{left}})^{\alpha}u \in L^p(\mathbb{R}, \omega)$ and that $(D_{\text{left}})^{\alpha}u$ converges in $L^p(\mathbb{R}, \omega)$
as $\alpha \to 1^-$. Then $u \in W^{1,p}(\mathbb{R}, \omega)$ and (1.5) holds.

(c) Alternatively, suppose that $(D_{\text{left}})^{\alpha}u \in L^p(\mathbb{R}, \omega)$ and that $(D_{\text{left}})^{\alpha}u$ converges in $L^p(\mathbb{R}, \omega)$
as $\alpha \to 0^+$. Then (1.6) holds and, as a consequence, $(D_{\text{left}})^{\alpha}u \to u$ in $L^p(\mathbb{R}, \omega)$ as $\alpha \to 0^+$.

Though we established Theorem 1.1 for the left fractional derivative, all the arguments carry on by replacing $D_{\text{left}}$ by $D_{\text{right}}$ and $A_p^-(\mathbb{R})$ by $A_p^+(\mathbb{R})$. Hence, for the rest of the paper, we will only consider the case of $D_{\text{left}}$ and left-sided Sawyer weights.

The one-sided $L^p(\mathbb{R}, \omega)$ spaces, with $\omega \in A_p^- (\mathbb{R})$, are also natural for the Marchaud left
fractional derivative in the sense of the Fundamental Theorem of Fractional Calculus. Indeed,
let $u \in L^p(\mathbb{R}, \omega)$ and consider the left-sided Weyl fractional integral [16]

$$(D_{\text{left}})^{-\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} \frac{u(\tau)}{(t-\tau)^{1+\alpha}} d\tau.$$ It was proved in [4] that $(D_{\text{left}})^{\alpha}(D_{\text{left}})^{-\alpha}u(t) = u(t)$ in $L^p(\mathbb{R}, \omega)$ and for a.e. $t \in \mathbb{R}$, for any $0 < \alpha < 1$. Our Theorem 1.1 complements this result.

The second question we address in this paper is the almost everywhere and $L^p$ characterization of weighted Sobolev spaces by the limits

$$\lim_{s \to 1^-} (-\Delta)^s u = -\Delta u \quad \text{and} \quad \lim_{s \to 0^+} (-\Delta)^s u = u$$where $(-\Delta)^s$ is the fractional Laplacian of order $0 < s < 1$ on $\mathbb{R}^n$, $n \geq 1$. Both limits hold whenever $u$ is a Schwartz class function. Up to the best of our knowledge, they have notbeen studied for the case of weighted $L^p$ spaces. We will consider the weighted Sobolev space $W^{2,p}(\mathbb{R}^n, \nu)$ defined by

$$W^{2,p}(\mathbb{R}^n, \nu) = \{ u \in L^p(\mathbb{R}^n, \nu) : \nabla u, D^2 u \in L^p(\mathbb{R}^n, \nu) \}$$with the norm

$$\|u\|_{W^{2,p}(\mathbb{R}^n, \nu)}^p = \|u\|_{L^p(\mathbb{R}^n, \nu)}^p + \|\nabla u\|_{L^p(\mathbb{R}^n, \nu)}^p + \|D^2 u\|_{L^p(\mathbb{R}^n, \nu)}^p$$where $\nu$ is a weight in the Muckenhoupt class $A_p(\mathbb{R}^n)$ (see Section 4), for $1 \leq p < \infty$. We recall that the $A_p(\mathbb{R}^n)$ Muckenhoupt weights are the good weights for the classical Hardy–Littlewood maximal function $M$ on $\mathbb{R}^n$. In the following statement, $\{e^{t\Delta}\}_{t \geq 0}$ denotes the heat semigroup generated by the Laplacian on $\mathbb{R}^n$.

**Theorem 1.2** ($W^{2,p}(\mathbb{R}^n, \nu)$ and limits of fractional Laplacians). Let $u \in L^p(\mathbb{R}^n, \nu)$, where $\nu \in A_p(\mathbb{R}^n)$, for $1 \leq p < \infty$.

(a) If $u \in W^{2,p}(\mathbb{R}^n, \nu)$, then the distribution $(-\Delta)^s u$ coincides with a function in $L^p(\mathbb{R}^n, \nu)$
and

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_{0}^{\infty} (e^{t\Delta} u(x) - u(x)) \frac{dt}{t^{1+s}} \quad \text{for a.e. } x \in \mathbb{R}^n.$$In addition,

$$(-\Delta)^s u(x) = c_{n,s} \lim_{\varepsilon \to 0} \int_{|x-y| > \varepsilon} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \quad \text{for a.e. } x \in \mathbb{R}^n \text{ and in } L^p(\mathbb{R}^n, \nu)$$with

$$\|(-\Delta)^s u\|_{L^p(\mathbb{R}^n, \nu)} \leq C_{n,p,\nu} \left( \|u\|_{L^p(\mathbb{R}^n, \nu)} + \|\Delta u\|_{L^p(\mathbb{R}^n, \nu)} \right)$$
for some constant $C_{n,p,\nu} > 0$. Moreover,
\begin{equation}
\lim_{s \to 1^{-}} (-\Delta)^s u = -\Delta u \quad \text{in } L^p(\mathbb{R}^n, \nu) \text{ and a.e. in } \mathbb{R}^n
\end{equation}
and
\begin{equation}
\lim_{s \to 0^+} (-\Delta)^s u = u \quad \text{a.e. in } \mathbb{R}^n.
\end{equation}
Furthermore, the limit in (1.11) holds also in $L^p(\mathbb{R}^n, \nu)$ when $1 < p < \infty$, and in weak-$L^1(\mathbb{R}^n, \nu)$ when $p = 1$.

(b) Conversely, suppose that $(-\Delta)^s u \in L^p(\mathbb{R}^n, \nu)$ and that $(-\Delta)^s u$ converges in $L^p(\mathbb{R}^n, \nu)$ as $s \to 1^-$. If $1 < p < \infty$ then $u \in W^{2,p}(\mathbb{R}^n, \nu)$ and (1.10) holds. If $p = 1$, then $D^2 u \in \text{weak-}L^1(\mathbb{R}^n, \nu)$.

(c) Alternatively, suppose that $(-\Delta)^s u \in L^p(\mathbb{R}^n, \nu)$ and that $(-\Delta)^s u$ converges in $L^p(\mathbb{R}^n, \nu)$ as $s \to 0^+$. Then (1.11) holds and, as a consequence, $(-\Delta)^s u \to u$ in $L^p(\mathbb{R}^n, \nu)$ as $s \to 0^+$.

Our Theorems 1.1 and 1.2 are rather nontrivial, nonisotropic weighted versions of the famous results by Bourgain–Brezis–Mironescu [5]. Indeed, [5] gives a characterization of $\|(-\Delta)^s\|_{L^p(\mathbb{R}^n, \nu)}$ and that $(-\Delta)^s u$ converges in $L^p(\mathbb{R}^n, \nu)$ as $s \to 1^-$. If $1 < p < \infty$ then $u \in W^{2,p}(\mathbb{R}^n, \nu)$ and (1.10) holds. If $p = 1$, then $D^2 u \in \text{weak-}L^1(\mathbb{R}^n, \nu)$.

In particular, they apply to Ahlfors-regular metric spaces. On the other hand, a weighted Gagliardo-type fractional seminorm with power weights was defined in [7]. Nevertheless, neither are our weighted spaces Ahlfors-regular nor do our seminorms $\|(D_{\text{left}})^{\alpha} u\|_{L^p(\mathbb{R}^n, \nu)}$ and $\|(-\Delta)^s u\|_{L^p(\mathbb{R}^n, \nu)}$ correspond to those in [7], even for power weights. An added difficulty we need to overcome in our case is the lack of translation invariance of the one-sided weighted $L^p(\mathbb{R}, \omega)$ spaces. Moreover, since constants are not in our weighted Sobolev spaces, we are able to complement [5] by studying limits as $\alpha, s \to 0^+$.

In general, statements involving a.e. convergence are proved by considering the underlying maximal operators, see, for example, [8, Chapter 2]. One of the novelties of our paper is that we are able to deduce the pointwise inequalities
\begin{equation}
\sup_{0 < \alpha < 1} \|(D_{\text{left}})^{\alpha} u(t)\| \leq C(M^{-}(u')(t) + M^{-} u(t)) \quad \text{for any } u \in W^{1,p}(\mathbb{R}, \omega)
\end{equation}
and
\begin{equation}
\sup_{0 < s < 1} \|(-\Delta)^s u(x)\| \leq C_n \left(M(D^2 u)(x) + Mu(x)\right) \quad \text{for any } u \in W^{2,p}(\mathbb{R}^n, \nu),
\end{equation}
see Theorems 2.10 and 4.6, respectively. The constant $C > 0$ in (1.12) is universal while $C_n > 0$ in (1.13) depends only on dimension. Notice that the maximal operators are taken with respect to the orders of the fractional derivative and the fractional Laplacian, respectively. We believe these estimates are of independent interest.

The paper is organized as follows. Section 2 contains preliminary results on one-sided Sawyer weights, the new distributional setting for one-sided fractional derivatives, and the proof of the maximal estimate (1.12). Theorem 1.1 is proved in Section 3. The fractional Laplacian in weighted Lebesgue spaces is studied in detail in Section 4, where we also show the maximal estimate (1.13). Finally, Section 5 contains the proof of Theorem 1.2.

Along the paper, we denote by $S(\mathbb{R}^n)$ the class of Schwartz functions on $\mathbb{R}^n$. We always take $0 < \alpha, s < 1$. We will use the following inequality: for any fixed $\rho > 0$ there exists $C_\rho > 0$ such that, for every $r > 0$,
\begin{equation}
e^{-r\rho t} \leq C_\rho e^{-r/2}.
\end{equation}
For a measure space \((X, \mu)\), we define the space weak-\(L^1(X, \mu)\) as the set of measurable functions \(u : X \to \mathbb{R}\) such that the quasi-norm \(\|u\|_{\text{weak-}L^1(X,\mu)}\), defined by
\[
\|u\|_{\text{weak-}L^1(X,\mu)} = \sup_{\lambda > 0} \lambda \mu\{x \in X : |u(x)| > \lambda\},
\]
is finite. We will need the following result from real analysis.

**Lemma 1.3.** Let \(u_k\) be a sequence of measurable functions on a measure space \((X, \mu)\) such that \(u_k \to 0\) \(\mu\)-a.e. as \(k \to \infty\). If \(|u_k| \leq v\) for some \(v \in \text{weak-}L^1(X, \mu)\) and all \(k \geq 1\), then
\[
\lim_{k \to \infty} \|u_k\|_{\text{weak-}L^1(X,\mu)} = 0.
\]

2. Fractional derivatives and one-sided spaces

Let \(u = u(t) \in \mathcal{S}(\mathbb{R})\) and define
\[
D_{\text{left}}u(t) = \lim_{\tau \to 0^+} \frac{u(t) - u(t - \tau)}{\tau} \quad \text{and} \quad D_{\text{right}}u(t) = \lim_{\tau \to 0^+} \frac{u(t) - u(t + \tau)}{\tau}.
\]
Observe that \(D_{\text{left}}u = -D_{\text{right}}u = u'\). From the Fourier transform identities
\[
\widehat{D_{\text{left}}}u(\xi) = (i\xi)^\alpha \widehat{u}(\xi) \quad \text{and} \quad \widehat{D_{\text{right}}}u(\xi) = (-i\xi)^\alpha \widehat{u}(\xi),
\]
one can define
\[
(\widehat{D_{\text{left}}}^\alpha u)(\xi) = (i\xi)^\alpha \widehat{u}(\xi) \quad \text{and} \quad (\widehat{D_{\text{right}}}^\alpha u)(\xi) = (-i\xi)^\alpha \widehat{u}(\xi).
\]
Using the semigroup of translations, it is shown in [4], see also [16], that \((D_{\text{left}})^\alpha u(t)\) and \((D_{\text{right}})^\alpha u(t)\) are given by the pointwise formulas in (1.1) and (1.2), respectively.

2.1. Distributional setting. If \(u, \varphi \in \mathcal{S}(\mathbb{R})\), then
\[
\int_{-\infty}^{\infty} (D_{\text{left}})^\alpha u \varphi dt = \int_{-\infty}^{\infty} u (D_{\text{right}})^\alpha \varphi dt.
\]
We will use this identity to define \((D_{\text{left}})^\alpha u\) in the sense of distributions. Notice that if \(u \in \mathcal{S}'(\mathbb{R})\), then a natural definition would be
\[
((D_{\text{left}})^\alpha u)(\varphi) = u ((D_{\text{right}})^\alpha \varphi).
\]
Nevertheless, it is straightforward from (2.1) to see that, in general, \((D_{\text{right}})^\alpha \varphi \notin \mathcal{S}(\mathbb{R})\), so we need to consider a different space of test functions and distributions.

We define the class
\[
\mathcal{S}_- = \{\varphi \in \mathcal{S}(\mathbb{R}) : \text{supp} \varphi \subset (-\infty, A), \text{ for some } A \in \mathbb{R}\}.
\]
We denote by \(\mathcal{S}_-^\alpha\) the set of functions
\[
\varphi \in C^\infty(\mathbb{R}) \text{ such that } \text{supp} \varphi \subset (-\infty, A) \text{ and } \left| \frac{d^k}{dt^k} \varphi(t) \right| \leq \frac{C}{1 + |t|^{1+\alpha}}
\]
for all \(k \geq 0\), for some \(A \in \mathbb{R}\) and \(C > 0\).

**Lemma 2.1.** If \(\varphi \in \mathcal{S}_-\) then \((D_{\text{right}})^\alpha \varphi \in \mathcal{S}_-^\alpha\).

**Proof.** Clearly, if \(\varphi \in \mathcal{S}_-\) with \(\text{supp} \varphi \subset (-\infty, A)\), then \((D_{\text{right}})^\alpha \varphi\) also has support in \((-\infty, A]\), see (1.2). Since \((D_{\text{right}})^\alpha \frac{d^k}{dt^k} \varphi = \frac{d^k}{dt^k} (D_{\text{right}})^\alpha \varphi\), we know \((D_{\text{right}})^\alpha \varphi \in C^\infty(\mathbb{R})\) and only need to estimate \((D_{\text{right}})^\alpha \varphi\). If \(-1 < t < A\), the estimate holds because \(\varphi\) is smooth
and bounded. If \( t > A \), then the estimate holds trivially because \((D_{\text{right}})^{\alpha} \varphi(t) = 0\). Suppose 
\(-\infty < t < -1\) and write
\[
\int_{t}^{\infty} \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|^{1+\alpha}} \, d\tau = \int_{t}^{t/2} \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|^{1+\alpha}} \, d\tau + \int_{t/2}^{\infty} \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|^{1+\alpha}} \, d\tau = I + II.
\]
For \( I \), note that
\[
|\varphi(\tau) - \varphi(t)| \leq |\varphi'(\xi)||\tau - t| = |\varphi'(\xi)|(1 + |\xi|)^{3} \frac{(\tau - t)}{(1 + |\xi|)^{3}} \leq C|\varphi'(\tau)|
\]
where \( \xi \) is some point in between \( t \) and \( \tau \). Hence,
\[
I \leq \frac{C}{|t|^{3}} \int_{t}^{t/2} \frac{1}{(\tau - t)^{\alpha}} \, d\tau = \frac{C}{|t|^{3+\alpha}} \leq \frac{C}{1 + |t|^{1+\alpha}}.
\]
On the other hand, if \( \tau > t/2 \), then \( \tau - t > -t/2 > 0 \) and
\[
II \leq \int_{t/2}^{\infty} \frac{|\varphi(\tau)|}{|\tau - t|^{1+\alpha}} \, d\tau + |\varphi(t)| \int_{t/2}^{\infty} \frac{1}{(\tau - t)^{1+\alpha}} \, d\tau
\leq \frac{C}{|t|^{1+\alpha}} \|\varphi\|_{L^{1}(\mathbb{R})} + \frac{C}{|t|^{1+\alpha}} |t\varphi(t)| \leq \frac{C}{1 + |t|^{1+\alpha}}.
\]
Collecting all the terms, we get
\[
|\varphi(\tau) - \varphi(t)| = \int_{t}^{\infty} \frac{|\varphi(\tau) - \varphi(t)|}{|\tau - t|^{1+\alpha}} \, d\tau \leq \frac{C}{1 + |t|^{1+\alpha}}
\]
for all \( t \in \mathbb{R} \). Thus, \((D_{\text{right}})^{\alpha} \varphi \in S_{\alpha}^{\circ} \).

We endow \( S^{\circ}_{\alpha} \) with the families of seminorms
\[
\rho^{\alpha,k}_{\cdot}(\varphi) = \sup_{t \in \mathbb{R}} |t|^{k} \left| \frac{d^{k}}{dt^{k}} \varphi(t) \right| \quad \text{for } \ell, k \geq 0,
\]
and
\[
\rho^{\alpha,k}_{\cdot}(\varphi) = \sup_{t \in \mathbb{R}} (1 + |t|^{1+\alpha}) \left| \frac{d^{k}}{dt^{k}} \varphi(t) \right| \quad \text{for } k \geq 0,
\]
respectively. Let us denote by \((S^{\circ}_{\alpha})'\) and \((S^{\circ})'\) the corresponding dual spaces of \( S^{\circ}_{\alpha} \) and \( S^{\circ} \). Notice that \( S^{\circ}_{\alpha} \subset S^{\circ} \), so that \((S^{\circ}_{\alpha})' \subset (S^{\circ})'\). It turns out that \((S^{\circ})'\) is the appropriate class of distributions to extend the definition of the left fractional derivative.

**Definition 2.2.** For \( u \in (S^{\circ})' \), we define \((D_{\text{left}})^{\alpha} u\) as the distribution in \((S^{\circ})'\) given by
\[
((D_{\text{left}})^{\alpha} u)(\varphi) = u((D_{\text{right}})^{\alpha} \varphi) \quad \text{for any } \varphi \in S^{\circ}_{\alpha}.
\]

Consider next the class of functions given by
\[
L^{\alpha}_{\cdot} = \left\{ u \in L^{1}_{\text{loc}}(\mathbb{R}) : \int_{-\infty}^{A} \frac{|u(\tau)|}{1 + |\tau|^{1+\alpha}} \, d\tau < \infty, \text{ for any } A \in \mathbb{R} \right\}.
\]
We use the notation
\[
\|u\|_{A} = \int_{-\infty}^{A} \frac{|u(\tau)|}{1 + |\tau|^{1+\alpha}} \, d\tau \quad \text{for } A \in \mathbb{R}.
\]
Any function \( u \in L^{\alpha}_{\cdot} \) defines a distribution in \((S^{\circ}_{\alpha})'\) in the usual way, so that \((D_{\text{left}})^{\alpha} u\) is well defined as an object in \((S^{\circ})'\). The following result is proved similarly as in the case of the fractional Laplacian, see Silvestre [18], so the details are omitted.
Proposition 2.3. Let $u \in L^\alpha_\omega$. Assume that $u \in C^{\alpha+\varepsilon}(I)$ for some $\varepsilon > 0$ and some open set $I \subset \mathbb{R}$. Then $(D_{\text{left}})\alpha u \in C(I)$ and

$$(D_{\text{left}})\alpha u(t) = \frac{1}{\Gamma(-\alpha)} \int_{-\infty}^t \frac{u(\tau) - u(t)}{(t-\tau)^{1+\alpha}} d\tau \quad \text{for all } t \in I.$$ 

Remark 2.4. We have found that the one-sided class $L^\alpha_\omega$ is the appropriate space of locally integrable functions to define the left fractional derivative. This is a refinement with respect to the distributional definition presented in [4, Remark 2.6], which was two-sided in nature.

2.2. One-sided weighted spaces. A nonnegative, locally integrable function $\omega = \omega(\tau)$ defined on $\mathbb{R}$ is in the left-sided Sawyer class $A_p^{-}(\mathbb{R})$, for $1 < p < \infty$, if there exists $C > 0$ such that

$$\left( \frac{1}{h} \int_{a}^{a+h} \omega d\tau \right)^{1/p} \left( \frac{1}{h} \int_{a-h}^{a} \omega^{1-p'} d\tau \right)^{1/p'} \leq C$$

for all $a \in \mathbb{R}$ and $h > 0$, where $1/p + 1/p' = 1$. We then write $\omega \in A_p^{-}(\mathbb{R})$. By re-orienting the real line, one may similarly define the right-sided $A_p^{+}(\mathbb{R})$-condition: a weight $\tilde{\omega}$ belongs to $A_p^{+}(\mathbb{R})$ if there is a constant $C > 0$ such that

$$\left( \frac{1}{h} \int_{a}^{a-h} \tilde{\omega} d\tau \right)^{1/p} \left( \frac{1}{h} \int_{a}^{a+h} \tilde{\omega}^{1-p'} d\tau \right)^{1/p'} \leq C$$

for all $a \in \mathbb{R}$ and $h > 0$. In this way, $\omega \in A_p^{-}(\mathbb{R})$ if and only if $\omega^{1-p'} \in A_p^{+}(\mathbb{R})$.

From the definition, one should note that, for $\omega \in A_p^{-}(\mathbb{R})$, there exist $-\infty \leq a < b \leq \infty$ such that $\omega = \infty$ in $(-\infty, a)$, $0 < \omega < \infty$ in $(a, b)$, $\omega = 0$ in $(b, \infty)$, and $\omega \in L^1_{\text{loc}}((a, b))$. For simplicity and without loss of generality, we will assume $(a, b) = \mathbb{R}$, so that $0 < \omega < \infty$ in $\mathbb{R}$.

The one-sided Hardy–Littlewood maximal functions $M^{-}$ and $M^{+}$ are defined by

$$M^{-} u(t) = \sup_{h > 0} \frac{1}{h} \int_{t-h}^{t} |u(\tau)| d\tau \quad \text{and} \quad M^{+} u(t) = \sup_{h > 0} \frac{1}{h} \int_{t}^{t+h} |u(\tau)| d\tau$$

respectively. If $1 < p < \infty$, then $M^\pm$ is bounded on $L^p(\mathbb{R}, \omega)$ if and only if $\omega \in A_p^{\pm}(\mathbb{R})$, see [17]. When $p = 1$, $M^\pm$ is bounded from $L^1(\mathbb{R}, \omega)$ into weak-$L^1(\mathbb{R}, \omega)$ if and only if $\omega \in A_1^{\pm}(\mathbb{R})$, namely, there exists $C > 0$ such that

$$M^\mp \omega(t) \leq C \omega(t) \quad \text{for a.e. } t \in \mathbb{R}$$

see [14]. We refer to [10, 12, 13, 14, 15, 17] for these and more properties of one-sided weights. For a measurable set $E \subset \mathbb{R}$, we denote

$$\omega(E) = \int_E \omega d\tau.$$

An important property that we will use is the following.

Lemma 2.5 (See [10, Theorem 3]). Let $\eta = \eta(t) \geq 0$ be a integrable function with support in $[0, \infty)$ and nonincreasing in $[0, \infty)$. Then, for any measurable function $u : \mathbb{R} \to \mathbb{R}$ and for almost all $t \in \mathbb{R}$, we have

$$|u \ast \eta(t)| \leq M^{-} u(t) \int_{0}^{\infty} \eta(\tau) d\tau.$$

By changing the orientation of the real line, the analogue conclusion holds for nondecreasing $\eta$ supported in $(-\infty, 0]$ with $M^{+}$ in place of $M^{-}$.
Lemma 2.6 (See [15, Theorem 1]). If $\omega \in A_p^\infty (\mathbb{R})$, $1 \leq p < \infty$, then there exist $C, \delta > 0$ such that

$$\frac{\omega(E)}{\omega((a, c))} \leq C \left( \frac{|E|}{b-a} \right)^{\delta}$$

for all $a < b < c$ and all measurable subsets $E \subset (b, c)$.

Lemma 2.7. If $\omega \in A_1^\infty (\mathbb{R})$, then there is a constant $C > 0$ such that, for any $0 < a < b$,

$$\frac{\omega((-a, -a + (b - a)))}{2(b - a)} \leq C \inf_{-b < t < -a} \omega(t).$$

Proof. Let $t \in (-b, -a)$. Since $(-a, -a + (b - a)) \subset (t, t + 2(b - a))$, then, by the $A_1^\infty (\mathbb{R})$-condition, we get

$$C \omega(t) \geq M^+ \omega(t) \geq \frac{1}{2(b-a)} \int_t^{t+2(b-a)} \omega(\tau) d\tau \geq \frac{1}{2(b-a)} \int_{-a}^{-a+(b-a)} \omega(\tau) d\tau = \frac{\omega((-a, -a + (b - a)))}{2(b - a)}$$

for almost every $t \in \mathbb{R}$. \qed

The following result says that $(D_{\text{left}})^{\alpha} u$ is well defined as a distribution in $(S_-)'$ whenever $u \in L^p(\mathbb{R}, \omega)$, for $\omega \in A_p^\infty (\mathbb{R})$, $1 \leq p < \infty$.

Proposition 2.8. If $\omega \in A_p^\infty (\mathbb{R})$, $1 \leq p < \infty$, then $L^p(\mathbb{R}, \omega) \subset L^\alpha_\omega, \alpha \geq 0$, and, for any $A \in \mathbb{R}$, there is a constant $C = C_{A, \omega, p} > 0$ such that

$$\|u\|_A \leq C \|u\|_{L^p(\mathbb{R}, \omega)}.$$ 

In particular, $L^p(\mathbb{R}, \omega) \subset L^1_{\text{loc}}(\mathbb{R})$.

Proof. Let $u \in L^p(\mathbb{R}, \omega)$ and fix any $A \in \mathbb{R}$.

We first let $1 < p < \infty$. By Hölder’s inequality,

$$\|u\|_A = \int_{-\infty}^{A} \frac{|u(\tau)|}{1 + |\tau|^{1+\alpha}} d\tau = \int_{-\infty}^{A} |u(\tau)| \omega(\tau)^{1/p} \frac{\omega(\tau)^{-1/p}}{1 + |\tau|^{1+\alpha}} d\tau \leq \|u\|_{L^p(\mathbb{R}, \omega)} \left( \int_{-\infty}^{A} \frac{\omega(\tau)^{-p'/p}}{(1 + |\tau|)^{p'}} d\tau \right)^{1/p'} = \|u\|_{L^p(\mathbb{R}, \omega)} \cdot (I_A)^{1/p'}.$$ 

Observe that $\hat{\omega}(\tau) = \omega(\tau)^{-p'/p} = \omega(\tau)^{1-p'} \in A_{p'}^\infty (\mathbb{R})$. To conclude, it is enough to recall that

$$I = \int_{-\infty}^{0} \frac{\hat{\omega}(\tau)}{(1 + |\tau|)^{p'}} d\tau < \infty,$$

see [12, Lemma 4].

Now let $p = 1$. For convenience with the notation, we let $A = 0$ (the general case follows the same lines). First observe that, by the $A_1^\infty (\mathbb{R})$-condition,

$$\int_{-1}^{0} \frac{|u(\tau)|}{1 + |\tau|^{1+\alpha}} d\tau \leq \int_{-1}^{0} |u(\tau)| \omega(\tau) \omega(\tau)^{-1} d\tau \leq \|u\|_{L^1(\mathbb{R}, \omega)} \sup_{t \in (-1,0)} \omega(t)^{-1} = \|u\|_{L^1(\omega)} \left( \inf_{t \in (-1,0)} \omega(t) \right)^{-1} \leq \|u\|_{L^1(\omega)} \frac{C}{\omega((-1,0))} < \infty.$$
On the other hand, by Lemma 2.7,
\[ \int_{-\infty}^{-1} \frac{|u(\tau)|}{1 + |\tau|^{1+\alpha}} d\tau \leq \sum_{k=0}^{\infty} \int_{-2^{k+1}}^{-2^k} \frac{|u(\tau)|}{|\tau|} d\tau \]
\[ \leq \sum_{k=0}^{\infty} \frac{1}{2^k} \int_{-2^{k+1}}^{-2^k} |u(\tau)| \omega(\tau)\omega^{-1}(\tau) d\tau \]
\[ \leq \|u\|_{L^1(\mathbb{R}, \omega)} \sum_{k=0}^{\infty} \frac{1}{2^k} \left( \inf_{-2^{k+1} < t < -2^k} \omega(t) \right)^{-1} \]
\[ \leq C \|u\|_{L^1(\mathbb{R}, \omega)} \sum_{k=0}^{\infty} \frac{1}{\omega((-2^k, 0))}. \]

Lemma 2.6 implies that there exist \( C, \delta > 0 \) such that
\[ \frac{\omega((-1, 0))}{\omega((-2^k, 0))} \leq C \left( \frac{1}{2^k} \right)^\delta. \]

Whence,
\[ \int_{-\infty}^{-1} \frac{|u(\tau)|}{1 + |\tau|^{1+\alpha}} d\tau \leq \frac{C}{\omega((-1, 0))} \|u\|_{L^1(\mathbb{R}, \omega)} \sum_{k=0}^{\infty} \left( \frac{1}{2^k} \right)^\delta < \infty. \]

Thus, \( u \in L^\alpha_\omega \) with the corresponding estimate. \( \Box \)

2.3. Density of smooth functions in \( W^{1,p}(\mathbb{R}, \omega) \). The proof of the following statement is similar to that of Lorente [10, Theorem 3]. Indeed, the idea is to bound \( \psi \in C_c^\infty([0, \infty)) \) by a measurable function \( \eta \) supported in \([0, \infty)\) which is nonincreasing in \([0, \infty)\), and follow the steps of the proof in [10].

**Proposition 2.9.** Let \( \omega \in A_p^-(\mathbb{R}) \) and \( u \in L^p(\mathbb{R}, \omega) \) for \( 1 \leq p < \infty \). Let \( \psi \in C_c^\infty([0, \infty)) \) such that \( \int_0^\infty \psi(t) dt = 1 \). Define \( \psi_\varepsilon(t) = \frac{1}{\varepsilon} \psi \left( \frac{t}{\varepsilon} \right) \). Then the following hold.

1. \( |u * \psi_\varepsilon(t)| \leq CM^{-u(t)} \) for almost every \( t \in \mathbb{R} \).
2. \( \|u * \psi_\varepsilon\|_{L^p(\mathbb{R}, \omega)} \leq C \|u\|_{L^p(\mathbb{R}, \omega)}. \)
3. \( \lim_{\varepsilon \to 0^+} u * \psi_\varepsilon(t) = u(t) \) for almost every \( t \in \mathbb{R} \).
4. \( \lim_{\varepsilon \to 0^+} \|u * \psi_\varepsilon - u\|_{L^p(\mathbb{R}, \omega)} = 0. \)

It follows that \( C^\infty(\mathbb{R}) \cap L^p(\mathbb{R}, \omega) \) and \( C_c^\infty(\mathbb{R}) \) are dense in \( L^p(\mathbb{R}, \omega) \) for \( \omega \in A_p^-(\mathbb{R}) \), \( 1 \leq p < \infty \). Additionally, notice that if \( \psi \) is as in Proposition 2.9 and \( u \in W^{1,p}(\mathbb{R}, \omega) \), then
\[ (u * \psi_\varepsilon)'(t) = \int_{-\infty}^{\infty} u'(\tau) \psi_\varepsilon(t - \tau) d\tau = (u' * \psi_\varepsilon)(t). \]

Hence \( u * \psi_\varepsilon \to u \) as \( \varepsilon \to 0^+ \) in \( W^{1,p}(\mathbb{R}, \omega) \), so that \( C^\infty(\mathbb{R}) \cap W^{1,p}(\mathbb{R}, \omega) \) and \( C_c^\infty(\mathbb{R}) \) are dense in \( W^{1,p}(\mathbb{R}, \omega) \) for \( \omega \in A_p^-(\mathbb{R}) \), \( 1 \leq p < \infty \).

2.4. The maximal estimate (1.12).

**Theorem 2.10.** There exists a universal constant \( C > 0 \) such that for any \( u \in W^{1,p}(\mathbb{R}, \omega) \), \( \omega \in A_p^-(\mathbb{R}) \), \( 1 \leq p < \infty \), we have
\[ \sup_{0 < \alpha < 1} \left| \frac{1}{\Gamma(-\alpha)} \int_0^\infty (u(t) - u(t)) \frac{d\tau}{\tau^{1+\alpha}} \right| \leq C \left( M^{-u'}(t) + M^{-u}(t) \right) \]
for a.e. \( t \in \mathbb{R} \).
Proof. We begin by writing

\begin{equation}
I_\alpha + II_\alpha := \frac{1}{\Gamma(-\alpha)} \int_0^1 (u(t-\tau) - u(t)) \frac{d\tau}{\tau^{1+\alpha}} + \frac{1}{\Gamma(-\alpha)} \int_1^\infty (u(t-\tau) - u(t)) \frac{d\tau}{\tau^{1+\alpha}}.
\end{equation}

To study \(I_\alpha\), notice that

\begin{equation}
\int_0^1 |u(t-\tau) - u(t)| \frac{d\tau}{\tau^{1+\alpha}} \leq \int_0^1 \tau \int_0^\tau |u'(t-r\tau)| \frac{d\tau}{\tau^{1+\alpha}} \frac{d\tau}{r} \\
\leq \int_0^1 \int_0^r |u'(t-\tau)| \frac{d\tau}{\tau^{1+\alpha}} r \frac{d\tau}{r} \\
\leq \frac{1}{\alpha} \int_0^1 |u'(t-\tau)| \frac{d\tau}{\tau^{1+\alpha}}.
\end{equation}

Then, if we let \(\eta(t) = t^{-\alpha} \chi_{(0,1)}(t)\), by Lemma 2.5,

\[ |I_\alpha| \leq \frac{1}{|\Gamma(1-\alpha)|} M^- u(t) \int_0^\infty \eta(\tau) d\tau = C_1 M^- u(t) \]

where

\[ C_1 = \frac{1}{\Gamma(2-\alpha)} \]

Considering now the second integral in (2.2), we observe that

\[ II_\alpha = \frac{1}{\Gamma(-\alpha)} \int_1^\infty u(t-\tau) \frac{d\tau}{\tau^{1+\alpha}} + \frac{1}{\Gamma(1-\alpha)} u(t). \]

For the first term, we estimate using Lemma 2.5 with \(\eta(t) = \chi_{(0,1)}(t) + t^{-1-\alpha} \chi_{(1,\infty)}(t)\),

\[ \left| \frac{1}{\Gamma(-\alpha)} \int_1^\infty u(t-\tau) \frac{d\tau}{\tau^{1+\alpha}} \right| \leq \frac{1}{|\Gamma(-\alpha)|} |u(\cdot) \ast \eta(t)| \leq C_2 M^- u(t) \]

where

\[ C_2 = \frac{1 + \alpha}{\Gamma(1-\alpha)} \]

which is bounded independently of \(\alpha\). Therefore,

\[ |II_\alpha| \leq C_2 M^- u(t) + C_3 |u(t)| \leq (C_2 + C_3) M^- u(t) \]

where

\[ C_3 = \frac{1}{|\Gamma(1-\alpha)|}. \]

The result follows.
3. Proof of Theorem 1.1

3.1. Proof of Theorem 1.1(a). The proof of part (a) is organized as follows. We first show that the formula in the right hand side of (1.3) is well-defined as a function in $L^p(\mathbb{R}, \omega)$. It is then shown that the distribution $(D_{b\alpha f})^\alpha u$ is indeed given by such pointwise formula using the fact that $C_0^\infty(\mathbb{R})$ is dense in $W^{1,p}(\mathbb{R}, \omega)$. The $L^p(\mathbb{R}, \omega)$ estimate in (1.4) follows immediately from these steps of the proof. Next, we show that the limit in (1.5) holds in $L^p(\mathbb{R}, \omega)$ for $u \in C_0^\infty(\mathbb{R})$ and then use a density argument to show the result for $u \in W^{1,p}(\mathbb{R}, \omega)$. The a.e. convergence of (1.5) is proved by showing that the set of functions in $W^{1,p}(\mathbb{R}, \omega)$ such that (1.5) holds a.e. is closed in $L^p(\mathbb{R}, \omega)$. The a.e. convergence of (1.6) follows similarly. Finally, the maximal estimate allows us to prove that (1.6) holds in $L^p(\mathbb{R}, \omega)$, $1 < p < \infty$.

Step 1. The integral expression in (1.3) defines a function in $L^p(\mathbb{R}, \omega)$.

First let $1 < p < \infty$. By Theorem 2.10 and the boundedness of $M^{-\alpha}$ in $L^p(\mathbb{R}, \omega)$ for $\omega \in A^-_p(\mathbb{R})$, it is immediate that

$$
\| \int_0^\infty (u(t - \tau) - u(t)) \frac{d\tau}{\tau + \alpha} \|_{L^p(\mathbb{R}, \omega)} \leq C_\omega \left( \| u \|_{L^p(\mathbb{R}, \omega)} + \| u' \|_{L^p(\mathbb{R}, \omega)} \right).
$$

For $p = 1$, we consider the terms $I_\alpha$ and $II_\alpha$ as in (2.2). We use (2.3) to observe that

$$
\| I_\alpha \|_{L^1(\mathbb{R}, \omega)} \leq \frac{1}{\Gamma(1 - \alpha)} \int_0^\infty \int_0^1 |u'(t - \tau)| \frac{d\tau}{\tau + \alpha} \omega(t) \, dt
$$

$$
= \frac{1}{\Gamma(1 - \alpha)} \int_0^\infty \int_{t-1}^t |u'(\tau)| \frac{d\tau}{(\tau - \alpha)^\alpha} \omega(t) \, dt
$$

$$
= \frac{1}{\Gamma(1 - \alpha)} \int_0^\infty |u'(\tau)| \int_{\tau}^{\tau + 1} \frac{\omega(t)}{(t - \alpha)^\alpha} \, dt \, d\tau
$$

$$
= \frac{1}{\Gamma(1 - \alpha)} \int_0^\infty |u'(\tau)| \int_{t-1}^t \frac{\omega(t + \tau)}{t^\alpha} \, dt \, d\tau.
$$

Since $\omega \in A^-_1(\mathbb{R})$, for a.e. $\tau \in \mathbb{R}$ we can use Lemma 2.5 with $\eta(\tau) = |\tau|^{-\alpha} \chi_{(-1,0)}(\tau)$ to get

$$
\int_0^1 \frac{\omega(t + \tau)}{t^\alpha} \, dt = \int_0^1 \frac{\omega(t - \tau)}{|\tau|^{\alpha}} \, dt = (\omega * \eta)(\tau) \leq \frac{1}{(1 - \alpha)} \omega(\tau) \leq \frac{C}{(1 - \alpha)} \omega(\tau).
$$

Therefore,

$$
\| I_\alpha \|_{L^1(\mathbb{R}, \omega)} \leq C_\omega C_1 \| u' \|_{L^1(\mathbb{R}, \omega)}
$$

where $C_1$ is as in the proof of Theorem 2.10. Moving to the second term in (2.2), we write

$$
II_\alpha = \frac{1}{\Gamma(-\alpha)} \int_1^\infty u(t - \tau) \frac{d\tau}{\tau + \alpha} + \frac{1}{\Gamma(1 - \alpha)} u(t)
$$

and estimate

$$
\left\| \int_1^\infty u(t - \tau) \frac{d\tau}{\tau + \alpha} \right\|_{L^1(\mathbb{R}, \omega)} \leq \int_{-\infty}^\infty \int_1^\infty \frac{|u(t - \tau)|}{\tau + \alpha} \omega(t) \, dt \, d\tau
$$

$$
= \int_{-\infty}^\infty \int_{-\infty}^{t-1} \frac{|u(\tau)|}{(\tau - \alpha)^\alpha} \omega(t) \, dt \, d\tau
$$

$$
= \int_{-\infty}^\infty |u(\tau)| \int_{\tau + 1}^\infty \frac{\omega(t)}{(t - \alpha)^\alpha} \, dt \, d\tau
$$

$$
= \int_{-\infty}^\infty |u(\tau)| \int_1^\infty \frac{\omega(t + \tau)}{t^\alpha} \, dt \, d\tau.
$$


By using again the $A_1^p(\mathbb{R})$-condition and Lemma 2.5 with \( \eta(\tau) = \chi_{(-1,0)}(\tau) + |\tau|^{-1-\alpha} \chi_{(-\infty,-1)}(\tau) \), for a.e. \( \tau \in \mathbb{R} \),
\[
\int_1^\infty \frac{\omega(t+\tau)}{t^{1+\alpha}} \, dt = \int_{-\infty}^{-1} \frac{\omega(t-t)}{|t|^{1+\alpha}} \, dt \leq (\omega * \eta)(\tau) \leq \frac{1+\alpha}{\alpha} M^+\omega(\tau) \leq C \frac{1+\alpha}{\alpha} \omega(\tau).
\]

Therefore, by collecting terms,
\[
\|I_\alpha\|_{L^1(\mathbb{R},\omega)} \leq \left\| \frac{1}{\Gamma(-\alpha)} \int_1^{\infty} u(t-\tau) \frac{d\tau}{\tau^{1+\alpha}} \right\|_{L^1(\mathbb{R},\omega)} + \frac{1}{\Gamma(1-\alpha)} \|u\|_{L^1(\mathbb{R},\omega)} \leq C_\omega(C_2 + C_3) \|u\|_{L^1(\mathbb{R},\omega)} \]
where \( C_2, C_3 > 0 \) are as in the proof of Theorem 2.10. Thus,
\[
\|I_\alpha\|_{L^1(\mathbb{R},\omega)} \leq C_\omega \left( \|u\|_{L^1(\mathbb{R},\omega)} + \|u\|_{L^1(\mathbb{R},\omega)} \right).
\]

Hence, the integral in (1.3) is in \( L^p(\mathbb{R},\omega) \) for \( 1 \leq p < \infty \).

**Step 2.** The distribution \( (D_{\text{left}})^{\alpha}u \) coincides with the integral formula in (1.3). Therefore \( (D_{\text{left}})^{\alpha}u \) is in \( L^p(\mathbb{R},\omega) \) and, by (3.1) and (3.3), (1.4) holds.

To show (1.3), let \( u_k \in C_c^\infty(\mathbb{R}) \) such that \( u_k \to u \) in \( W^{1,p}(\mathbb{R}, \omega) \) as \( k \to \infty \). We may write
\[
(D_{\text{left}})^{\alpha}u_k(t) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty (u_k(t-\tau) - u_k(t)) \frac{d\tau}{\tau^{1+\alpha}}.
\]

Using (3.1) and (3.3), we can show that the formulas converge in norm. Indeed,
\[
\left\| \frac{1}{\Gamma(-\alpha)} \int_0^\infty (u_k(t-\tau) - u_k(t)) \frac{d\tau}{\tau^{1+\alpha}} - \frac{1}{\Gamma(-\alpha)} \int_0^\infty (u(t-\tau) - u(t)) \frac{d\tau}{\tau^{1+\alpha}} \right\|_{L^p(\mathbb{R},\omega)} \leq C \left( \|u_k - u\|_{L^p(\omega)} + \|u_k' - u'\|_{L^p(\omega)} \right) \to 0 \quad \text{as} \; k \to \infty.
\]

If \( \varphi \in C_c^\infty(\mathbb{R}) \) and \( A \) is such that \( \text{supp} \varphi \subset (-\infty, A) \), then \( \varphi \in S_- \) and \( (D_{\text{right}})^{\alpha}\varphi \in S_-^\omega \) with \( \text{supp}((D_{\text{right}})^{\alpha}\varphi) \subset (-\infty, A] \). Now, by Definition 2.2,
\[
((D_{\text{left}})^{\alpha}u)(\varphi) = \lim_{k \to \infty} \int_{-\infty}^{\infty} u(t) (D_{\text{right}})^{\alpha}\varphi(t) \, dt = \lim_{k \to \infty} \int_{-\infty}^{\infty} u_k(t) (D_{\text{right}})^{\alpha}\varphi(t) \, dt = \lim_{k \to \infty} \int_{-\infty}^{\infty} \left( \frac{1}{\Gamma(-\alpha)} \int_0^\infty (u_k(t-\tau) - u_k(t)) \frac{d\tau}{\tau^{1+\alpha}} \right) \varphi(t) \, dt.
\]

In the second identity above we used that, by Proposition 2.8,
\[
\left| \int_{-\infty}^{\infty} u_k(t) (D_{\text{right}})^{\alpha}\varphi(t) \, dt - \int_{-\infty}^{\infty} u(t) (D_{\text{right}})^{\alpha}\varphi(t) \, dt \right| \leq \int_{-\infty}^{A} |u_k(t) - u(t)| \left| (D_{\text{right}})^{\alpha}\varphi(t) \right| \, dt.
\]
\[
\leq C \int_{-\infty}^{A} \frac{|u_k(t) - u(t)|}{1 + |t|^{1+\alpha}} \, dt \leq C \|u_k - u\|_{L^p(R,\omega)} \rightarrow 0
\]
as \(k \rightarrow \infty\) and in the last equality we observed that
\[
\left| \int_{-\infty}^{\infty} (D_{\text{left}}^\alpha u_k(t)) \varphi(t) \, dt - \int_{-\infty}^{\infty} \left( \frac{1}{\Gamma(-\alpha)} \int_{0}^{\infty} (u(t - \tau) - u(t)) \frac{d\tau}{\tau^{1+\alpha}} \right) \varphi(t) \, dt \right|
\leq C \int_{-\infty}^{A} \left| \int_{0}^{\infty} (u_k(t - \tau) - u_k(t)) \frac{d\tau}{\tau^{1+\alpha}} - \int_{0}^{\infty} (u(t - \tau) - u(t)) \frac{d\tau}{\tau^{1+\alpha}} \right| \frac{1}{1 + |t|^{1+\alpha}} \, dt
\leq C \left\| \int_{0}^{\infty} (u_k(\cdot - \tau) - u_k(\cdot)) \frac{d\tau}{\tau^{1+\alpha}} - \int_{0}^{\infty} (u(\cdot - \tau) - u(\cdot)) \frac{d\tau}{\tau^{1+\alpha}} \right\|_{L^p(R,\omega)} \rightarrow 0
\]
as \(k \rightarrow \infty\). Therefore, since \(\varphi\) was arbitrary in (3.4),
\[
(D_{\text{left}}^\alpha u)(t) = \frac{1}{\Gamma(-\alpha)} \int_{0}^{\infty} (u(t - \tau) - u(t)) \frac{d\tau}{\tau^{1+\alpha}} \quad \text{a.e. in } R.
\]

**Step 3.** The limit as \(\alpha \rightarrow 1^–\) in (1.5) holds in \(L^p(R,\omega)\) for \(u \in C_c^\infty(R)\).

Suppose that \(u \in C_c^\infty(R)\) and write \((D_{\text{left}}^\alpha u)(t) = I_\alpha + I_\alpha\) as in (2.2). For \(1 < p < \infty\), we see from the proof of Theorem 2.10 that
\[
\|I_\alpha\|_{L^p(R,\omega)} \leq (C_2 + C_3) \|M^{-1}u\|_{L^p(R,\omega)}
\leq \left( \frac{1 + \alpha}{\Gamma(1 - \alpha)} + \frac{1}{\Gamma(1 - \alpha)} \right) C_\omega \|u\|_{L^p(R,\omega)} \rightarrow 0
\]
as \(\alpha \rightarrow 1^–\). For \(p = 1\), by (3.2) in Step 1, we similarly obtain
\[
\|I_\alpha\|_{L^1(\omega)} \leq C_\omega (C_2 + C_3) \|u\|_{L^1(\omega)} \rightarrow 0 \quad \text{as } \alpha \rightarrow 1^–.
\]

Next, observe that
\[
I_\alpha - u'(t) = \frac{1}{\Gamma(-\alpha)} \int_{0}^{1} \left( - \int_{0}^{\tau} u'(t - r) \, dr \right) \frac{d\tau}{\tau^{1+\alpha}} - u'(t)
= \frac{1}{\Gamma(-\alpha)} \int_{0}^{1} \int_{0}^{\tau} u'(t - \tau) - u'(t - r) \, dr \, d\tau \, u'(t)
= \frac{1}{\Gamma(-\alpha)} \int_{0}^{1} \int_{0}^{\tau} u''(t - \mu) \, d\mu \, d\tau \, u'(t).
\]
Let \(K\) be such that \(\text{supp } u''(\cdot - \mu) \subset [-K, K]\) for all \(\mu \in [0, 1]\). Then, for \(1 \leq p < \infty\),
\[
\|u''(\cdot - \mu)\|_{L^p(R,\omega)} \leq \|u''\|_{L^\infty(R)} \omega([0, 1])^{1/p} = c
\]
where \(c > 0\) is independent of \(\alpha\). Therefore,
\[
\|I_\alpha - u'\|_{L^p(R,\omega)} \leq \frac{1}{\Gamma(-\alpha)} \int_{0}^{1} \int_{0}^{\tau} \|u''(t - \mu)\|_{L^{p}(R,\omega)} \, d\mu \, d\tau \frac{d\tau}{\tau^{1+\alpha}} + \frac{\alpha}{\Gamma(2 - \alpha)} - 1 \|u'\|_{L^p(R,\omega)} \rightarrow 0 \quad \text{as } \alpha \rightarrow 1^–.
\]

Hence, \(\|D_{\text{left}}^\alpha u - u\|_{L^p(R,\omega)} \leq \|II_\alpha\|_{L^p(R,\omega)} + \|I_\alpha - u'\|_{L^p(R,\omega)} \rightarrow 0\), as \(\alpha \rightarrow 1^–\).

**Step 4.** The limit as \(\alpha \rightarrow 1^–\) in (1.5) holds in \(L^p(R,\omega)\) for \(u \in W^{1,p}(R,\omega)\).
Let \( u_k \in C_c^\infty(\mathbb{R}) \) such that \( u_k \to u \) in \( W^{1,p}(\mathbb{R}, \omega) \) as \( k \to \infty \). We just observe that, by the \( L^p \) estimate (1.4) (that was proved in Step 2), for \( 1 \leq p < \infty \),

\[
\|(D_{\text{left}})^\alpha u - u'\|_{L^p(\mathbb{R}, \omega)} \leq \|(D_{\text{left}})^\alpha (u - u_k)\|_{L^p(\mathbb{R}, \omega)} + \|(D_{\text{left}})^\alpha u_k - u'_k\|_{L^p(\mathbb{R}, \omega)} + \|u'_k - u'\|_{L^p(\mathbb{R}, \omega)}.
\]

Then take \( k \) large and choose \( \alpha \) close to \( 1^- \) (see Step 3).

**Step 5.** The limit as \( \alpha \to 1^- \) in (1.5) holds almost everywhere for \( u \in W^{1,p}(\mathbb{R}, \omega) \).

It follows from Theorem 2.10 and the properties of \( M^- \) that the operator \( T^* \) defined by

\[
T^*u(t) = \sup_{0 < \alpha < 1} (D_{\text{left}})^\alpha u(t) \quad \text{for } u \in W^{1,p}(\mathbb{R}, \omega)
\]

satisfies the estimates

\[
\|T^*u\|_{L^p(\mathbb{R}, \omega)} \leq C_{p,\omega} \|u\|_{W^{1,p}(\mathbb{R}, \omega)} \quad \text{for any } u \in W^{1,p}(\mathbb{R}, \omega), \quad 1 < p < \infty
\]

and

\[
\omega\{t \in \mathbb{R} : |T^*u(t)| > \lambda\} \leq \frac{C_{\omega}}{\lambda} \|u\|_{W^{1,1}(\mathbb{R}, \omega)} \quad \text{for any } u \in W^{1,1}(\mathbb{R}, \omega).
\]

In particular, \( T^* \) is bounded from \( W^{1,p}(\mathbb{R}, \omega) \) into weak-\( L^p(\mathbb{R}, \omega) \), for any \( 1 \leq p < \infty \). This in turn implies that the set

\[
E = \{u \in W^{1,p}(\mathbb{R}, \omega) : \lim_{\alpha \to 1^-} (D_{\text{left}})^\alpha u(t) = u'(t) \text{ a.e.}\}
\]

is closed in \( W^{1,p}(\mathbb{R}, \omega) \). Since \( C_c^\infty(\mathbb{R}) \subset E \), by density, we get \( E = W^{1,p}(\mathbb{R}, \omega) \).

**Step 6.** The limit as \( \alpha \to 0^+ \) in (1.6) holds almost everywhere for \( u \in W^{1,p}(\mathbb{R}, \omega) \).

As in Step 5, one can check that the set

\[
E' = \{u \in W^{1,p}(\mathbb{R}, \omega) : \lim_{\alpha \to 0^+} (D_{\text{left}})^\alpha u(t) = u(t) \text{ a.e.}\}
\]

is closed in \( W^{1,p}(\mathbb{R}, \omega) \). Since \( C_c^\infty(\mathbb{R}) \subset E' \), by density, we get \( E' = W^{1,p}(\mathbb{R}, \omega) \).

**Step 7.** The limit as \( \alpha \to 0^+ \) in (1.6) holds in \( L^p(\mathbb{R}, \omega) \), whenever \( 1 < p < \infty \), and in weak-\( L^1(\mathbb{R}, \omega) \) when \( p = 1 \).

By Theorem 2.10, for any \( 0 < \alpha < 1 \),

\[
|(D_{\text{left}})^\alpha u(t) - u(t)|^p \omega(t) \leq \left(C_n(M(u')(t) + Mu(t)) + |u(t)|\right)^p \omega(t) 
\leq C_{n,p} \left((M(u')(t))^p + (Mu(t))^p\right) \omega(t).
\]

Therefore, by Step 6 and the Dominated Convergence Theorem, (1.6) holds in \( L^p(\mathbb{R}, \omega) \) for \( 1 < p < \infty \) and, by Lemma 1.3, in weak-\( L^1(\mathbb{R}, \omega) \) when \( p = 1 \).

The proof of Theorem 1.1, part \((a)\), is completed. \( \square \)
3.2. **Proof of Theorem 1.1(b).** This is proved through a distributional argument.

Suppose that \((D_{\text{left}})^\alpha u \to v\) in \(L^p(\mathbb{R}, \omega)\) as \(\alpha \to 1^-\). Let \(\varphi \in C_0^\infty(\mathbb{R})\). Let \(A \in \mathbb{R}\) be such that \(\text{supp} \varphi \subset (-\infty, A]\), so that \(\varphi \in S_\alpha\) and \((D_{\text{right}})^\alpha \varphi \in S_\alpha^\infty\). By Proposition 2.8,

\[
\left| \int_{-\infty}^{\infty} v(t) \varphi(t) \, dt - \int_{-\infty}^{\infty} (D_{\text{left}})^\alpha u(t) \varphi(t) \, dt \right| \leq \int_{-\infty}^{A} |v(t) - (D_{\text{left}})^\alpha u(t)| \frac{C}{1 + |t|} \, dt
\]

as \(\alpha \to 1^-\). With this and the definition of \((D_{\text{left}})^\alpha u\) we can write

\[
\int_{-\infty}^{\infty} v \varphi \, dt = \lim_{\alpha \to 1^-} \int_{-\infty}^{\infty} (D_{\text{left}})^\alpha u \varphi \, dt
\]

Next, notice that, by Proposition 2.8,

\[
|u(t)| |(D_{\text{right}})^\alpha \varphi + \varphi'| \leq |u(t)| \frac{C_{\varphi}}{1 + |t|^{1+\alpha}} \chi_{(-\infty, A]}(t)
\]

\[
\leq C_{\varphi} \frac{|u(t)|}{1 + |t|} \chi_{(-\infty, A]}(t) \in L^1(\mathbb{R})
\]

Therefore, by the Dominated Convergence Theorem, as \((D_{\text{right}})^\alpha \varphi(t) \to -\varphi'(t)\) as \(\alpha \to 0^+\),

\[
\lim_{\alpha \to 1^-} \left| \int_{-\infty}^{\infty} u(t) (D_{\text{right}})^\alpha \varphi(t) \, dt + \int_{-\infty}^{\infty} u(t) \varphi'(t) \, dt \right|
\]

\[
\leq \int_{-\infty}^{\infty} \lim_{\alpha \to 1^-} |u(t)||((D_{\text{right}})^\alpha \varphi(t) + \varphi'(t)| \, dt = 0
\]

Whence,

\[
\int_{-\infty}^{\infty} v \varphi \, dt = \lim_{\alpha \to 1^-} \int_{-\infty}^{\infty} u (D_{\text{right}})^\alpha \varphi \, dt
\]

\[
= -\int_{-\infty}^{\infty} u \varphi' \, dt = \int_{-\infty}^{\infty} u \varphi \, dt.
\]

Therefore \(v = u'\) a.e. in \(\mathbb{R}\). Since \(u' = v \in L^p(\mathbb{R}, \omega)\), we get \(u \in W^{1,p}(\mathbb{R}, \omega)\), and by Theorem 1.1(a), the conclusion follows.

\[\square\]

3.3. **Proof of Theorem 1.1(c).** Using the exact same arguments as in part (b), we find that

\[
\int_{-\infty}^{\infty} v \varphi \, dt = \lim_{\alpha \to 0^+} \int_{-\infty}^{\infty} (D_{\text{left}})^\alpha u \varphi \, dt
\]

\[
= \lim_{\alpha \to 0^+} \int_{-\infty}^{\infty} u (D_{\text{right}})^\alpha \varphi \, dt = \int_{-\infty}^{\infty} u \varphi \, dt.
\]

Therefore \(v = u\) a.e. in \(\mathbb{R}\) and the conclusion follows.

\[\square\]

4. **Fractional Laplacians and Muckenhoupt Weights**

For \(u \in S(\mathbb{R}^n)\), the Fourier transform identity

\[
\widehat{(-\Delta)u}(\xi) = |\xi|^2 \hat{u}(\xi)
\]
is used to define the fractional Laplacian as

\[ (-\Delta)^s u(x) = |\xi|^{2s} \hat{u}(\xi) \quad \text{for } 0 < s < 1. \]

Using the heat diffusion semigroup \( \{e^{t\Delta}\}_{t \geq 0} \) generated by \(-\Delta\), it is shown in [19, 20] that the fractional Laplacian can be expressed using the semigroup formula (1.7) and that this is equivalent to the pointwise formula (1.8). Here, \( e^{t\Delta} \) is the operator

\[ e^{t\Delta} u(\xi) = e^{-|\xi|^2 t} \hat{u}(\xi). \]

It is well known that \( e^{t\Delta} u(x) = (W_t * u)(x) \) where \( W_t(x) \) is the Gauss–Weierstrass kernel

\[ W_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/(4t)} \quad \text{for } x \in \mathbb{R}^n, \ t > 0. \]

4.1. Distributional setting. The distributional setting for the fractional Laplacian was developed by Silvestre in [18]. Consider the function class

\[ \mathcal{S}_s = \left\{ \varphi \in C^\infty(\mathbb{R}^n) : |D^\gamma \varphi(x)| \leq \frac{C}{1 + |x|^{n+2s}}, \ \text{for all } \gamma \in \mathbb{N}_0^n, \ x \in \mathbb{R}^n, \ \text{for some } C > 0 \right\}. \]

We endow \( \mathcal{S}_s \) with the topology induced by the family of seminorms

\[ \rho_s^\gamma(\varphi) = \sup_{x \in \mathbb{R}^n} (1 + |x|^{n+2s}) |D^\gamma \varphi(x)|, \quad \text{for } \gamma \in \mathbb{N}_0^n. \]

Let \((\mathcal{S}_s)'\) be the dual space of \( \mathcal{S}_s \). Notice that \( \mathcal{S} \subset \mathcal{S}_s \), so that \((\mathcal{S}_s)' \subset \mathcal{S}'\). For \( u \in (\mathcal{S}_s)' \), \((-\Delta)^s u\) is defined as a distribution on \( \mathcal{S}' \) by

\[ ((-\Delta)^s u)(\varphi) = u((-\Delta)^s \varphi) \quad \text{for any } \varphi \in \mathcal{S}. \]

One can check that \( L_s \subset (\mathcal{S}_s)' \), where

\[ L_s = \left\{ u \in L^1_{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+2s}} \, dx < \infty \right\}. \]

**Proposition 4.1** (Silvestre [18]). Let \( \Omega \) be an open set in \( \mathbb{R}^n \) and \( u \in L_s \). If \( u \in C^{2s+\varepsilon}(\Omega) \) (or \( C^{1,2s+\varepsilon-1}(\Omega) \) if \( s \geq 1/2 \)) for some \( \varepsilon > 0 \), then \((-\Delta)^s u \in C(\Omega) \) and

\[ (-\Delta)^s u(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy \quad \text{for every } x \in \Omega. \]

Here (see [19, 20])

\[ c_{n,s} = \frac{4^s \Gamma(n/2 + s)}{\Gamma(-s)} \pi^{n/2} \sim s(1-s) \quad \text{as } s \to 0,1. \]

4.2. Muckenhoupt weights. A function \( \nu \in L^1_{\text{loc}}(\mathbb{R}^n) \), \( \nu > 0 \) a.e., is called an \( A_p(\mathbb{R}^n) \) Muckenhoupt weight, \( 1 < p < \infty \), if it satisfies the following condition: there exists \( C > 0 \) such that

\[ \left( \frac{1}{|B|} \int_B \nu \, dx \right)^{1/p} \left( \frac{1}{|B|} \int_B \nu^{1-p'} \, dx \right)^{1/p'} \leq C \]

for any ball \( B \subset \mathbb{R}^n \). If \( \nu \) satisfies (4.2), we write \( \nu \in A_p(\mathbb{R}^n) \). Observe that \( \nu \in A_p(\mathbb{R}^n) \) if and only if \( \nu^{1-p'} \in A_{p'}(\mathbb{R}^n) \). The Hardy–Littlewood maximal function is defined by

\[ Mu(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |u(y)| \, dy \]

where the supremum is taken over all balls \( B \subset \mathbb{R}^n \) containing \( x \). For \( 1 < p < \infty \), the operator \( M \) is bounded on \( L^p(\mathbb{R}^n, \nu) \) if and only if \( \nu \in A_p(\mathbb{R}^n) \). When \( p = 1 \), \( M \) is bounded
from $L^1(\mathbb{R}^n, \nu)$ into weak-$L^1(\mathbb{R}^n, \nu)$ if and only if $\nu \in A_1(\mathbb{R}^n)$, namely, there exists $C > 0$ such that

$$M \nu(x) \leq C \nu(x) \quad \text{for a.e. } x \in \mathbb{R}^n.$$  

For a measurable set $E \subset \mathbb{R}^n$ and a weight $\nu$, we denote

$$\nu(E) = \int_E \nu \, dx.$$ 

See [8] for more details about Muckenhoupt weights.

**Lemma 4.2** (See [8, Proposition 2.7]). Let $\eta = \eta(x)$ be a function that is positive, radial, decreasing (as a function on $(0, \infty)$) and integrable. Then for any measurable function $u : \mathbb{R}^n \to \mathbb{R}$ and for almost every $x \in \mathbb{R}^n$, we have

$$|u * \eta(x)| \leq \|\eta\|_{L^1(\mathbb{R}^n)} M \nu(x).$$

**Lemma 4.3** (See [8, Corollary 7.6]). If $\nu \in A_p(\mathbb{R}^n)$, $1 \leq p < \infty$, then there exists $\delta > 0$ such that given a ball $B$ and a measurable subset $S$ of $B$,

$$\frac{\nu(S)}{\nu(B)} \leq C \left( \frac{|S|}{|B|} \right)^{\delta}.$$ 

Our next result shows that for any function $u \in L^p(\mathbb{R}^n, \nu)$, $\nu \in A_p(\mathbb{R}^n)$, $1 \leq p < \infty$, the object $(-\Delta)^s u$ is well defined as a distribution in $S'$. 

**Proposition 4.4.** If $u \in L^p(\mathbb{R}^n, \nu)$, $\nu \in A_p(\mathbb{R}^n)$, $1 \leq p < \infty$, then $u \in L_s$, $s \geq 0$, and there is a constant $C = C_{n,p,\nu} > 0$ such that

$$\|u\|_{L_s} \leq C \|u\|_{L^p(\mathbb{R}^n, \nu)}.$$ 

In particular, $L^p(\mathbb{R}^n, \nu) \subset L^1_{\text{loc}}(\mathbb{R}^n)$.

**Proof.** Suppose first that $1 < p < \infty$. By Hölder’s inequality,

$$\|u\|_{L_s} \leq \|u\|_{L^p(\mathbb{R}^n, \nu)} \left( C_n \int_{\mathbb{R}^n} \nu(x)^{1-p'} \left( 1 + |x|^n \right)^{p'} \, dx \right)^{\frac{1}{p'}}.$$ 

Let $\tilde{\nu}(x) = \nu(x)^{1-p'} \in A_1(\mathbb{R}^n)$. It is enough to show

$$\int_{\mathbb{R}^n} \frac{\tilde{\nu}(x)}{(1 + |x|)^{np'}} \, dx < \infty.$$ 

Let $f(x) = \chi_{B_1}(x)$. If $|x| \leq 1$, then $Mf(x) = 1$. If $|x| \geq 1$, then $B_1 \subset B(x, 2 |x|)$ and

$$Mf(x) \geq \frac{|B(0, 1)|}{|B(x, 2 |x|)|} = \frac{1}{(2 |x|)^n} \geq C_n (1 + |x|)^n.$$ 

Since $M$ is bounded on $L^p(\mathbb{R}^n, \tilde{\nu})$, for $\tilde{\nu} \in A_1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \frac{\tilde{\nu}(x)}{(1 + |x|)^{np'}} \, dx \leq C \int_{\mathbb{R}^n} \left( Mf(x) \right)^{p'} \tilde{\nu}(x) \, dx \leq C \int_{\mathbb{R}^n} \left( f(x) \right)^{p'} \tilde{\nu}(x) \, dx = C \int_{B_1} \tilde{\nu}(x) \, dx = C \nu^{1-p'}(B_1).$$ 

Therefore, $\|u\|_{L_s} \leq C \|u\|_{L^p(\mathbb{R}^n, \nu)} \nu^{1-p'}(B_1)$.

Now let $p = 1$. Observe that

$$\int_{|x| < 1} \frac{|u(x)|}{1 + |x|^{n+2s}} \, dx \leq \|u\|_{L^1(\mathbb{R}^n, \nu)} \sup_{x \in B_1} \nu(x)^{-1} \leq C_{n,\nu} \|u\|_{L^1(\mathbb{R}^n, \nu)}.$$
where in the last inequality we used that, since $\nu \in A_1(\mathbb{R}^n)$,
\[
\sup_{B_1} \nu^{-1} = \left(\inf_{B_1} \nu\right)^{-1} \leq C \left(\frac{\nu(B_1)}{|B_1|}\right)^{-1}.
\]

On the other hand, let $B_j = B_{2^j}(0)$, $j \geq 0$. By using the $A_1(\mathbb{R}^n)$-condition and Lemma 4.3 with $S = B_1$ and $B = B_j$,
\[
\int_{|x|>1} \frac{|u(x)|}{1+|x|^{n+2s}} \, dx \leq \sum_{j=0}^{\infty} \int_{B_{j+1}\setminus B_j} \frac{|u(x)|}{|x|^n} \, dx
\leq c_n \sum_{j=1}^{\infty} \frac{1}{(2j)^n} \int_{B_j} |u(x)| \, dx
\leq c_n \|u\|_{L^1(\mathbb{R}^n, \nu)} \sum_{j=1}^{\infty} \frac{1}{(2j)^n} \sup_{x \in B_j} \nu^{-1}(x)
\leq C \|u\|_{L^1(\mathbb{R}^n, \nu)} \sum_{j=1}^{\infty} \frac{1}{(2j)^n} \frac{|B_j|^{\delta}}{\nu(B_j)} |B_j|^{1-\delta}
\leq C \|u\|_{L^1(\mathbb{R}^n, \nu)} \sum_{j=1}^{\infty} \frac{1}{(2j)^n} \frac{|B_j|^{1-\delta}}{(2j)^n} \leq C_{n, \nu} \|u\|_{L^1(\mathbb{R}^n, \nu)}.
\]

The result for $p = 1$ follows by combining the previous estimates. \hfill \qed

4.3. **The heat semigroup on weighted spaces.** Recall the definition of the classical heat semigroup $\{e^{t\Delta}\}_{t \geq 0}$ on $\mathbb{R}^n$:
\[
e^{t\Delta}u(x) = \int_{\mathbb{R}^n} W_t(x - y) u(y) \, dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/(4t)} u(y) \, dy
\]
for $x \in \mathbb{R}^n$, $t > 0$. We believe that the following result belongs to the folklore, but we provide a proof for the sake of completeness.

**Theorem 4.5.** Let $\nu \in A_p(\mathbb{R}^n)$ and $u \in L^p(\mathbb{R}^n, \nu)$, $1 \leq p < \infty$. The following hold.

1. The integral defining $e^{t\Delta}u(x)$ in (4.3) is absolutely convergent for $x \in \mathbb{R}^n$, $t > 0$, and
\[
\sup_{t>0} |e^{t\Delta}u(x)| \leq Mu(x)
\]
for almost every $x \in \mathbb{R}^n$.
2. $e^{t\Delta}u(x) \in C^\infty((0, \infty) \times \mathbb{R}^n)$ and $\partial_t(e^{t\Delta}u) = \Delta(e^{t\Delta}u)$ in $\mathbb{R}^n \times (0, \infty)$.
3. $\|e^{t\Delta}u\|_{L^p(\mathbb{R}^n, \nu)} \leq C_{n,p,\nu} \|u\|_{L^p(\mathbb{R}^n, \nu)}$, where $C_{n,p,\nu} > 0$.
4. $\lim_{t \to 0^+} e^{t\Delta}u(x) = u(x)$ for almost every $x \in \mathbb{R}^n$.
5. $\lim_{t \to 0^+} \|e^{t\Delta}u - u\|_{L^p(\mathbb{R}^n, \nu)} = 0$.
6. If $u \in W^{2,p}(\mathbb{R}^n, \nu)$, then $e^{t\Delta}u = \Delta e^{t\Delta}u$.
7. $\lim_{\varepsilon \to 0^+} \int_{|x-y|<\varepsilon} W_t(x - y) u(y) \, dy \in L^p(\mathbb{R}^n, \nu)$.

**Proof.** Let $u \in L^p(\mathbb{R}^n, \nu)$, $\nu \in A_p(\mathbb{R}^n)$, for $1 \leq p < \infty$.

For (1), we apply Lemma 4.2 with $\eta(x) = W_t(x)$ and notice that $\|W_t\|_{L^1(\mathbb{R}^n)} = 1$, for each fixed $t > 0$. 


To prove (2), we recall that $W_t(x) \in C^\infty(\mathbb{R}^n \times (0, \infty))$, $\partial_t W_t = \Delta W_t$ in $\mathbb{R}^n \times (0, \infty)$ and that there exists $c > 0$ such that $|\partial_t W_t(x)| \leq c W_t(x)$ for each $t > 0$ and $x \in \mathbb{R}^n$. Thus, we can differentiate inside of the integral in (4.3) to find that $e^{t\Delta} u(x) \in C^\infty(\mathbb{R}^n \times (0, \infty))$ and solves the heat equation.

If $1 < p < \infty$, then part (1) and the boundedness of the maximal function $M$ show that $\|e^{t\Delta} u\|_{L^p(\mathbb{R}^n, \nu)} \leq C \|u\|_{L^p(\mathbb{R}^n, \nu)}^p$. If $p = 1$, as in part (1) and by using the $A_1(\mathbb{R}^n)$-condition,

$$\|e^{t\Delta} u\|_{L^1(\mathbb{R}^n, \nu)} \leq \int_{\mathbb{R}^n} |u(y)| \left( \int_{\mathbb{R}^n} W_t(x-y) \nu(x) \, dx \right) \, dy \leq \int_{\mathbb{R}^n} |u(y)| \, dM(y) \, dy \leq C \int_{\mathbb{R}^n} |u(x)| \nu(y) \, dy = C \|u\|_{L^1(\mathbb{R}^n, \nu)}.$$ 

Whence, (3) holds.

To verify the almost everywhere limit in (4), we only need to observe that $\lim_{t \to 0^+} e^{t\Delta} \varphi(x) = \varphi(x)$ for every $x \in \mathbb{R}^n$ whenever $\varphi \in C^\infty_c(\mathbb{R}^n)$, that $C^\infty_c(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n, \nu)$ and that, by part (1), the maximal operator

$$T^* u(x) = \sup_{t > 0} |e^{t\Delta} u(x)|$$

is bounded from $L^p(\mathbb{R}^n, \nu)$ into weak-$L^p(\mathbb{R}^n, \nu)$ (see, for instance, [8, Theorem 2.2]).

For (5), notice that if $\varphi \in C^\infty_c(\mathbb{R}^n)$, then, as in part (1),

$$|e^{t\Delta} \varphi(x) - \varphi(x)| = \left| \int_0^t \partial_s e^{s\Delta} \varphi(x) \, ds \right| \leq \int_0^t |e^{s\Delta} \varphi(x)| \, ds \leq C M(\Delta \varphi)(x) t.$$

For $1 < p < \infty,$

$$\|e^{t\Delta} \varphi - \varphi\|_{L^p(\mathbb{R}^n, \nu)} \leq C \|\Delta \varphi\|_{L^p(\mathbb{R}^n, \nu)} t \to 0$$

as $t \to 0^+$. If $p = 1$, then by part (3),

$$\|e^{t\Delta} \varphi - \varphi\|_{L^1(\mathbb{R}^n, \nu)} = \int_{\mathbb{R}^n} |e^{t\Delta} \varphi(x) - \varphi(x)| \, \nu(x) \, dx \leq \int_{\mathbb{R}^n} \int_0^t |e^{s\Delta} \varphi(x)| \, \nu(x) \, ds \, dx = \int_0^t \|e^{s\Delta} \varphi\|_{L^1(\mathbb{R}^n, \nu)} \, ds \leq \int_0^t C \|\Delta \varphi\|_{L^1(\mathbb{R}^n, \nu)} \, ds = C t \|\Delta \varphi\|_{L^1(\mathbb{R}^n, \nu)} \to 0$$

as $t \to 0^+$. We then use the density of $C^\infty_c(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n, \nu)$.

For (6), let $\varphi \in C^\infty_c(\mathbb{R}^n)$ and observe that

$$\int_{\mathbb{R}^n} \Delta e^{t\Delta} u(x) \varphi(x) \, dx = \int_{\mathbb{R}^n} e^{t\Delta} u(x) \Delta \varphi(x) \, dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} W_t(x-y) u(y) \Delta \varphi(x) \, dy \, dx = \int_{\mathbb{R}^n} W_t(z) \left[ \int_{\mathbb{R}^n} u(y) \Delta_\nu \varphi(x+y) \, dy \right] \, dx.$$
Lemma 4.2 and use (4.1) to get
\[
\int_{\mathbb{R}^n} W_t(z) \left[ \int_{\mathbb{R}^n} \Delta u(y) \varphi(x + y) \, dy \right] \, dx
= \int_{\mathbb{R}^n} e^{t\Delta} \Delta u(x) \varphi(x) \, dx.
\]
Then \( \Delta e^{t\Delta} u(x) = e^{t\Delta} \Delta u(x) \), for almost every \( x \in \mathbb{R}^n \).

Let us first estimate the second term. Take \( \eta \)
from which the statement follows. We write
\[
\text{We will show that there is a constant } C
\]
Proof. Define the operator
\[
\text{For } 1 < p < \infty, Mu \in \mathcal{L}^p(\mathbb{R}^n, \nu) \text{ so, by the Dominated Convergence Theorem,}
\]
\[
\lim_{\epsilon \to 0} \left\| \int_{|x-y|<\epsilon} W_t(x-y) |u(y)| \, dy \right\|_{\mathcal{L}^p(\mathbb{R}^n, \nu)}^p
= \int_{\mathbb{R}^n} \lim_{\epsilon \to 0} \left( \int_{|x-y|<\epsilon} W_t(x-y) |u(y)| \, dy \right)^p \nu(x) \, dx = 0.
\]
For \( p = 1 \),
\[
\left\| \int_{|x-y|<\epsilon} W_t(x-y) u(y) \, dy \right\|_{L^1(\mathbb{R}^n, \nu)} \leq \int_{\mathbb{R}^n} \left[ |u(y)| \int_{|x-y|<\epsilon} W_t(x-y) \nu(x) \, dx \right] \, dy
\]
and, by part (1),
\[
|u(y)| \int_{|x-y|<\epsilon} W_t(x-y) \nu(x) \, dx \leq |u(y)| M \nu(y) \leq C \nu(y) \, \nu(y) \in \mathcal{L}^1(\mathbb{R}^n)
\]
for a.e. \( y \in \mathbb{R}^n \). Therefore, (7) holds for \( p = 1 \) by the Dominated Convergence Theorem.

4.4. The maximal estimate (1.13).

Theorem 4.6. There exists a constant \( C_n > 0 \) such that for any \( u \in W^{2,p}(\mathbb{R}^n, \nu), \nu \in \mathcal{A}_p(\mathbb{R}^n), 1 \leq p < \infty \), we have
\[
\sup_{0<s<1} \sup_{\epsilon>0} \left| c_{n,s} \int_{|y|>\epsilon} \frac{u(x-y) - u(x)}{|y|^{n+2s}} \, dy \right| \leq C_n \left( M(D^2u)(x) + Mu(x) \right)
\]
for almost every \( x \in \mathbb{R}^n \).

Proof. Define the operator \( T_{s,\epsilon} \) on \( W^{2,p}(\mathbb{R}^n, \nu) \) by
\[
T_{s,\epsilon} u(x) = c_{n,s} \int_{|y|>\epsilon} \frac{u(x-y) - u(x)}{|y|^{n+2s}} \, dy.
\]
We will show that there is a constant \( C = C_n > 0 \) such that
\[
|T_{s,\epsilon} u(x)| \leq C \left( M(D^2u)(x) + Mu(x) \right) \text{ for a.e. } x \in \mathbb{R}^n
\]
from which the statement follows. We write
\[
T_{s,\epsilon} u(x) = c_{n,s} \int_{\epsilon<|y|<1} \frac{u(x-y) - u(x)}{|y|^{n+2s}} \, dy + c_{n,s} \int_{|y|>1} \frac{u(x-y) - u(x)}{|y|^{n+2s}} \, dy = I + II.
\]
Let us first estimate the second term. Take \( \eta(x) = \chi_{\{|x|\leq 1\}}(x) + |x|^{-n-2s} \chi_{\{|x|>1\}}(x) \) in Lemma 4.2 and use (4.1) to get
\[
|II| \leq c_{n,s} \int_{|y|>1} \frac{|u(x-y)|}{|y|^{n+2s}} \, dy + c_{n,s} \int_{|y|>1} |u(y)| \frac{1}{|y|^{n+2s}} \, dy
\]
\[
\leq C_n s (1-s) \left( (|u| \ast \eta)(x) + \frac{|u(x)|}{s} \right)
\]
\[
\leq C_n s (1-s) \left( \frac{1 + 2s}{2s} M u(x) + \frac{|u(x)|}{s} \right)
\]
\[
\leq C_n M u(x).
\]

Consider now the first term, that we rewrite as

\[
I = c_{n,s} \int_{|y|<1} \frac{u(x-y) - u(x) + \nabla u(x) \cdot y}{|y|^{n+2s}} dy.
\]

Since \( u \in W^{2,p}(\mathbb{R}^n, \nu) \) and (4.1) holds, for a.e. \( x \in \mathbb{R}^n \) we can estimate

\[
|I| \leq c_{n,s} \int_{|y|<1} \frac{|u(x-y) - u(x) + \nabla u(x) \cdot y|}{|y|^{n+2s}} dy
\]

\[
\leq c_{n,s} \int_{|y|<1} \frac{|y|^2}{|y|^{n+2s}} \int_0^1 (1-t) \left| D^2 u(x - ty) \right| dt dy
\]

\[
= c_{n,s} \int_0^1 (1-t) \int_{|y|<1} \frac{\left| D^2 u(x - ty) \right|}{|y|^{n-2(1-s)}} dy dt
\]

\[
\leq c_{n,s} \int_0^1 (1-t) t^{-2(1-s)} \int_{|y|<t} \frac{\left| D^2 u(x - y) \right|}{|y|^{n-2(1-s)}} dy dt
\]

\[
\leq c_{n,s} \int_0^1 (1-t) t^{-2(1-s)} \sum_{k=0}^{\infty} \int_{2^{-k} t < |y| < 2^{-k-1} t} \frac{\left| D^2 u(x - y) \right|}{|y|^{n-2(1-s)}} dy dt
\]

\[
\leq c_{n,s} \int_0^1 (1-t) t^{-2(1-s)} \sum_{k=0}^{\infty} \frac{1}{(2^{-k} t)^{n-2(1-s)}} \int_{|y|<2^{-k} t} \left| D^2 u(x - y) \right| dy dt
\]

\[
\leq C_n s (1-s) 2^{n-2(1-s)} M(D^2 u)(x) \int_0^1 (1-t) \left[ \sum_{k=0}^{\infty} \frac{1}{(2^{1-s})^k} \right] dt
\]

\[
\leq C_n s (1-s) M(D^2 u)(x) \]

where in the last line we applied the estimate \( 4^{1-s} - 1 \geq c(1-s) \), for any \( 0 < s < 1 \). Therefore, \( |T_{s,c} u(x)| \leq |I| + |II| \leq C_n (M(D^2 u)(x) + M u(x)) \) for a.e \( x \in \mathbb{R}^n \). \qed

5. Proof of Theorem 1.2

5.1. Proof of Theorem 1.2 (a). The steps in the proof of part (a) are similar to the steps in the proof of Theorem 1.1 (a).

Step 1. The semigroup formula in (1.7) defines a function in \( L^p(\mathbb{R}^n, \nu) \).

Let us begin by writing

\[
\frac{1}{\Gamma(-s)} \int_0^\infty |e^{t \Delta} u(x) - u(x)| \frac{dt}{t^{1+s}} \geq I + II.
\]

(5.1)

\[
= \frac{1}{\Gamma(-s)} \int_0^1 |e^{t \Delta} u(x) - u(x)| \frac{dt}{t^{1+s}} + \frac{1}{\Gamma(-s)} \int_1^\infty |e^{t \Delta} u(x) - u(x)| \frac{dt}{t^{1+s}}
\]

\[
= I + II.
\]
To study $I$, recall Theorem 4.5 and observe for $t \in [0, 1]$ that
\[ \|e^{t\Delta}u - u\|_{L_p(\mathbb{R}^n, \nu)} \leq \int_0^t \|e^{r\Delta}(\Delta u)\|_{L_p(\mathbb{R}^n, \nu)} \, dr \leq C \|\Delta u\|_{L_p(\mathbb{R}^n, \nu)} \cdot t. \]
Therefore
\[ \|I\|_{L_p(\mathbb{R}^n, \nu)} \leq \frac{1}{\Gamma(-s)} \int_0^1 \|e^{t\Delta}u - u\|_{L_p(\mathbb{R}^n, \nu)} \, dt \frac{dt}{t^{1+s}} \]
(5.2)
\[ = \frac{C}{\Gamma(-s)} \|\Delta u\|_{L_p(\mathbb{R}^n, \nu)} \int_1^t t^{-s} \, dt = C \frac{s}{\Gamma(2 - s)} \|\Delta u\|_{L_p(\mathbb{R}^n, \nu)}. \]
For $II$, in view of Theorem 4.5,
\[ \|II\|_{L_p(\mathbb{R}^n, \nu)} \leq \frac{1}{\Gamma(-s)} \int_1^\infty \left( \|e^{t\Delta}u\|_{L_p(\mathbb{R}^n, \nu)} + \|u\|_{L_p(\mathbb{R}^n, \nu)} \right) \frac{dt}{t^{1+s}} \]
(5.3)
\[ \leq \frac{1}{\Gamma(-s)} \left( C \|u\|_{L_p(\mathbb{R}^n, \nu)} + \|u\|_{L_p(\mathbb{R}^n, \nu)} \right) \int_1^\infty \frac{dt}{t^{1+s}} \]
\[ = \frac{C(1 - s)}{\Gamma(2 - s)} \|u\|_{L_p(\mathbb{R}^n, \nu)}. \]
Therefore
\[ \left\| \frac{1}{\Gamma(-s)} \int_0^\infty \left( e^{t\Delta}u(x) - u(x) \right) \frac{dt}{t^{1+s}} \right\|_{L_p(\mathbb{R}^n, \nu)} \leq C \left( \|u\|_{L_p(\mathbb{R}^n, \nu)} + \|\Delta u\|_{L_p(\mathbb{R}^n, \nu)} \right) < \infty. \]

**Step 2.** The distribution $(-\Delta)^s u$ coincides with the semigroup formula in (1.7) for a.e. $x \in \mathbb{R}^n$. Therefore, $(-\Delta)^s u$ is in $L_p(\mathbb{R}, \nu)$ and, by (5.4), we see that (1.9) holds.

Since $C_c^{\infty}(\mathbb{R}^n)$ is dense in $W^{2,p}(\mathbb{R}^n, \nu)$ (see [21]), there exists a sequence $u_k \in C_c^{\infty}(\mathbb{R}^n)$ such that $u_k \to u$ in $W^{2,p}(\mathbb{R}^n, \nu)$. We consider the terms $I$ and $II$ as in (5.1) and, similarly,
\[ (-\Delta)^s u_k(x) = \frac{1}{\Gamma(-s)} \int_0^1 (e^{t\Delta}u_k(x) - u_k(x)) \frac{dt}{t^{1+s}} + \frac{1}{\Gamma(-s)} \int_1^\infty (e^{t\Delta}u_k(x) - u_k(x)) \frac{dt}{t^{1+s}} \]
\[ = I_k + II_k. \]
By (5.2),
\[ \|I_k - I\|_{L_p(\mathbb{R}^n, \nu)} \leq C \frac{s}{\Gamma(2 - s)} \|\Delta(u_k - u)\|_{L_p(\mathbb{R}^n, \nu)} \to 0 \quad \text{as } k \to \infty. \]
Similarly, by (5.3),
\[ \|II_k - II\|_{L_p(\mathbb{R}^n, \nu)} = \frac{C(1 - s)}{\Gamma(2 - s)} \|u_k - u\|_{L_p(\mathbb{R}^n, \nu)} \to 0 \quad \text{as } k \to \infty. \]
Therefore,
\[ (-\Delta)^s u_k(x) \to \frac{1}{\Gamma(-s)} \int_0^\infty (e^{t\Delta}u(x) - u(x)) \frac{dt}{t^{1+s}} \]
(5.5)
in $L_p(\mathbb{R}^n, \nu)$ as $k \to \infty$.
Next, let $\varphi \in C_c^{\infty}(\mathbb{R}^n)$ and note that $(-\Delta)^s \varphi \in \mathcal{S}$. By Proposition 4.4,
\[ \left| \int_{\mathbb{R}^n} u_k(x)(-\Delta)^s \varphi(x) \, dx - \int_{\mathbb{R}^n} u(x)(-\Delta)^s \varphi(x) \, dx \right| \]
\[ \leq C \int_{\mathbb{R}^n} \frac{|u_k(x) - u(x)|}{1 + |x|^{n+2s}} \, dx \]
In addition, by (5.5),
\[
\left| \int_{\mathbb{R}^n} (-\Delta)^su_k(x)\varphi(x) \, dx - \frac{1}{\Gamma(-s)} \int_{\mathbb{R}^n} \int_0^\infty \left( e^{t\Delta}u(x) - u(x) \right) \frac{dt}{t^{1+s}} \varphi(x) \, dx \right|
\leq C \int_{\mathbb{R}^n} \left| (-\Delta)^su_k(x) - \frac{1}{\Gamma(-s)} \int_0^\infty \left( e^{t\Delta}u(x) - u(x) \right) \frac{dt}{t^{1+s}} \right| \varphi(x) \, dx
\leq C \left\| (-\Delta)^su_k(x) - \frac{1}{\Gamma(-s)} \int_0^\infty \left( e^{t\Delta}u(x) - u(x) \right) \frac{dt}{t^{1+s}} \right\|_{L^p(\mathbb{R}^n,\nu)} \rightarrow 0
\]
as \( k \to \infty \). Therefore
\[
\int_{\mathbb{R}^n} (-\Delta)^su(x) \varphi(x) \, dx = \int_{\mathbb{R}^n} u(x) (-\Delta)^s \varphi(x) \, dx
= \lim_{k \to \infty} \int_{\mathbb{R}^n} u_k(x) (-\Delta)^s \varphi(x) \, dx
= \lim_{k \to \infty} \int_{\mathbb{R}^n} (-\Delta)^su_k(x) \varphi(x) \, dx
= \int_{\mathbb{R}^n} \left[ \frac{1}{\Gamma(-s)} \int_0^\infty \left( e^{t\Delta}u(x) - u(x) \right) \frac{dt}{t^{1+s}} \right] \varphi(x) \, dx,
\]
and so we obtain
\[
(-\Delta)^su(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left( e^{t\Delta}u(x) - u(x) \right) \frac{dt}{t^{1+s}} \quad \text{for a.e. } x \in \mathbb{R}^n.
\]

**Step 3.** The integral expression in (1.8) defines a function in \( L^p(\mathbb{R}^n,\nu) \) for all \( \nu > 0 \).

For \( \nu > 0 \), define the operator \( T_\epsilon \) on \( L^p(\mathbb{R}^n,\nu) \) by
\[
(5.6)
T_\epsilon u(x) = c_{n,s} \int_{|x-y|>\epsilon} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy.
\]
We claim that \( T_\epsilon u(x) \in L^p(\mathbb{R}^n,\nu) \) for all \( \epsilon > 0 \). Indeed, for \( 1 < p < \infty \) this is immediate by Theorem 4.6: there exists \( C > 0 \) such that
\[
\|T_\epsilon u\|_{L^p(\mathbb{R}^n,\nu)} \leq C \left( \|M(D^2u)\|_{L^p(\mathbb{R}^n,\nu)} + \|M u\|_{L^p(\mathbb{R}^n,\nu)} \right) < \infty.
\]
For \( p = 1 \), we write
\[
T_\epsilon u(x) = c_{n,s} \int_{|x-y|>\epsilon} \frac{1}{|x-y|^{n+2s}} \, dy + c_{n,s} \int_{|x-y|>\epsilon} \frac{u(y)}{|x-y|^{n+2s}} \, dy
= c_{n,s} \frac{C_n \epsilon^{-2s}}{2s} u(x) + c_{n,s} \int_{|x-y|>\epsilon} \frac{u(y)}{|x-y|^{n+2s}} \, dy.
\]
We only need to study the second term above. By applying Lemma 4.2 with \( \eta(y) = \chi_{\{|y|\leq\epsilon\}}(y) + |y|^{-n-2s} \chi_{\{|y|>\epsilon\}}(y) \) and the \( A_1(\mathbb{R}^n) \)-condition on \( \nu \), we find
\[
\left\| \int_{|x-y|>\epsilon} \frac{u(y)}{|x-y|^{n+2s}} \, dy \right\|_{L^1(\mathbb{R}^n,\nu)} \leq \int_{\mathbb{R}^n} |u(y)| \int_{|x-y|>\epsilon} \frac{\nu(x)}{|x-y|^{n+2s}} \, dx \, dy
\leq \int_{\mathbb{R}^n} |u(y)| (\nu \ast \eta)(y) \, dy
\]
The principal value in (1.8) converges in $L^p(\mathbb{R}^n, \nu)$ to the function $(-\Delta)^s u$.

\[
(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^1 \left( \int_{\mathbb{R}^n} W_t(x-y) (u(y) - u(x)) \, dy \right) \, \frac{dt}{t^{1+s}} + \frac{1}{\Gamma(-s)} \int_1^\infty \left( \int_{\mathbb{R}^n} W_t(x-y) (u(y) - u(x)) \, dy \right) \, \frac{dt}{t^{1+s}} \equiv I + II
\]

and, similarly,

\[
c_n \int_{|x-y| > \varepsilon} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \, dy = \frac{1}{\Gamma(-s)} \int_{|x-y| > \varepsilon} \left( u(y) - u(x) \right) \left( \int_0^\infty W_t(x-y) \, \frac{dt}{t^{1+s}} \right) \, dy
\]

\[
\leq C \int_1^\infty \left( \int_{|x-y| < \varepsilon} W_t(x-y) u(y) \, dy \right) \, \frac{dt}{t^{1+s}} + \int_{|x-y| < \varepsilon} W_t(z) \, dz \, \frac{dt}{t^{1+s}} \leq 0
\]

as $\varepsilon \to 0^+$. We next show $\|I - I_\varepsilon\|_{L^p(\mathbb{R}^n, \nu)} \to 0$ as $\varepsilon \to 0^+$ as well to conclude the proof. Indeed,

\[
\|I - I_\varepsilon\|_{L^p(\mathbb{R}^n, \nu)} = \left\| \frac{1}{\Gamma(-s)} \int_0^1 \left( \int_{|y| < \varepsilon} W_t(y) (u(x-y) - u(x)) \, dy \right) \, \frac{dt}{t^{1+s}} \right\|_{L^p(\mathbb{R}^n, \nu)}
\]

By Taylor’s Remainder Theorem and (1.14),

\[
\left| \int_{|y| < \varepsilon} W_t(y) (u(x-y) - u(x)) \, dy \right|
\]

\[
\leq \int_{|y| < \varepsilon} W_t(y) |y|^2 \left( \int_0^1 (1-r) |D^2u(x-ry)| \, dr \right) \, dy
\]

\[
\leq C \varepsilon \int_{|y| < \varepsilon} W_2(y) \left( \int_0^1 (1-r) |D^2u(x-ry)| \, dr \right) \, dy
\]

Step 4. The principal value in (1.8) converges in $L^p(\mathbb{R}^n, \nu)$ to the function $(-\Delta)^s u$. We write the semigroup formula (1.7) as

\[
(-\Delta)^s u(x) = \frac{1}{\Gamma(-s)} \int_0^1 \left( \int_{\mathbb{R}^n} W_t(x-y) (u(y) - u(x)) \, dy \right) \, \frac{dt}{t^{1+s}} + \frac{1}{\Gamma(-s)} \int_1^\infty \left( \int_{\mathbb{R}^n} W_t(x-y) (u(y) - u(x)) \, dy \right) \, \frac{dt}{t^{1+s}} \equiv I + II
\]

and, similarly,
\[
= Ct \int_0^1 (1 - r) \left( \int_{|y| < \varepsilon} W_{2r}(y) \left| D^2 u(x - ry) \right| \, dy \right) \, dr
\]
\[
= Ct \int_0^1 (1 - r) \left( \int_{|y| < r\varepsilon} W_{2r^2}(y) \left| D^2 u(x - y) \right| \, dy \right) \, dr.
\]

In particular, since \( D^2 u \in L^p(\mathbb{R}^n, \nu) \), by Theorem 4.5,

\[
(5.7) \quad \int_{|y| < \varepsilon} W_t(y) (u(x - y) - u(x)) \, dy \to 0 \quad \text{as} \ \varepsilon \to 0^+
\]
a.e. in \( \mathbb{R}^n \). We continue estimating by

\[
\left| \int_{|y| < \varepsilon} W_t(y) (u(x - y) - u(x)) \, dy \right| \leq Ct \int_0^1 (1 - r) \left( \int_{\mathbb{R}^n} W_{2r^2}(y) \left| D^2 u(x - y) \right| \, dy \right) \, dr
\]
\[
\leq CtM(D^2 u)(x) \int_0^1 (1 - r) \, dr = CtM(D^2 u)(x).
\]

Whence, for \( 1 < p < \infty \), we have

\[
|I - I_\varepsilon| \leq CM(D^2 u)(x) \int_0^1 t \frac{dt}{t^{1+s}} \leq CM(D^2 u)(x) \in L^p(\mathbb{R}^n, \nu)
\]

where \( C > 0 \) is independent of \( \varepsilon \). Thus, by the Dominated Convergence Theorem and (5.7),

\[
\lim_{\varepsilon \to 0^+} \| I - I_\varepsilon \|_{L^p(\mathbb{R}^n, \nu)} = 0.
\]

When \( p = 1 \), by following the computations above and by Theorem 4.5, we get

\[
\| I - I_\varepsilon \|_{L^1(\mathbb{R}^n, \nu)}
\]
\[
\leq C \int_{\mathbb{R}^n} \int_0^1 \int_0^1 t \, (1 - r) \left( \int_{|y| < \varepsilon} W_{2r^2}(x - y) \left| D^2 u(y) \right| \, dy \right) \frac{dt}{t^{1+s}} \nu(x) \, dx
\]
\[
= C \int_0^1 \int_0^1 (1 - r) \left( \int_{|y| < \varepsilon} W_{2r^2}(x - y) \left| D^2 u(y) \right| \, dy \right) \left( \int_{\mathbb{R}^n} W_{2r^2}(x - y) \nu(x) \, dx \right) \frac{dt}{t^{1+s}}
\]
\[
\leq C \int_0^1 \int_0^1 (1 - r) \left( \int_{|y| < \varepsilon} W_{2r^2}(x - y) \left| D^2 u(y) \right| \nu(y) \, dy \right) \frac{dt}{t^{1+s}}
\]
\[
\leq C \int_0^1 \int_0^1 (1 - r) \left( \int_{|y| < \varepsilon} W_{2r^2}(x - y) \left| \nu(y) \right| \, dy \right) \frac{dt}{t^{1+s}}
\]
\[
= C \int_{|y| < \varepsilon} \left| D^2 u(y) \right| \nu(y) \, dy \to 0 \quad \text{as} \ \varepsilon \to 0^+.
\]

**Step 5.** The principal value in (1.8) converges almost everywhere in \( \mathbb{R}^n \) to \((-\Delta)^s u\).

It follows from Theorem 4.6 and the properties of \( M \) that the operator \( T^* \) defined by

\[
T^* u(t) = \sup_{\varepsilon > 0} \left| T_\varepsilon u(x) \right| \quad \text{for} \ u \in W^{2,p}(\mathbb{R}^n, \nu),
\]

where \( T_\varepsilon \) is defined as in (5.6), satisfies the estimates

\[
\| T^* u \|_{L^p(\mathbb{R}^n, \nu)} \leq C \| u \|_{W^{2,p}(\mathbb{R}^n, \nu)} \quad \text{for} \ u \in W^{2,p}(\mathbb{R}^n, \nu), \ 1 < p < \infty
\]
and

$$\nu(\{x \in \mathbb{R}^n : |T^* u(x)| > \lambda\}) \leq \frac{C}{\lambda} \|u\|_{W^2,1(\mathbb{R}^n, \nu)} \quad \text{for any } u \in W^{2,1}(\mathbb{R}^n, \nu), \lambda > 0$$

where $C > 0$ is independent of $u$. In particular, $T^*$ is bounded from $W^{2,p}(\mathbb{R}^n, \nu)$ into weak-$L^p(\mathbb{R}^n, \nu)$, for any $1 \leq p < \infty$. With these estimates, as in Step 5 of the proof of Theorem 1.1(a), we find that the set

$$E = \left\{ u \in W^{2,p}(\mathbb{R}^n, \nu) : \lim_{\varepsilon \to 0^+} T_\varepsilon u(x) = (-\Delta)^s u(x) \text{ a.e.} \right\}$$

is closed in $W^{2,p}(\mathbb{R}^n, \nu)$. Since $C^\infty_c(\mathbb{R}^n) \subset E$, by density, we obtain $E = W^{2,p}(\mathbb{R}^n, \nu)$.

**Step 6.** The limit as $s \to 1^-$ in (1.10) holds in $L^p(\mathbb{R}, \nu)$.

Fix $\varepsilon > 0$. By Theorem 4.5, there exists $\delta > 0$ such that

$$\|e^{t\Delta} \Delta u - \Delta u\|_{L^p(\mathbb{R}^n, \nu)} < \varepsilon \quad \text{when } |t| < \delta.$$

We write

$$(-\Delta)^s u(x) = \frac{1}{\Gamma(1-s)} \int_0^\delta \partial_t e^{t\Delta} u(x) - u(x)) \frac{dt}{t^{1+s}} + \frac{1}{\Gamma(-s)} \int_{\delta}^\infty (e^{t\Delta} u(x) - u(x)) \frac{dt}{t^{1+s}}$$

$$= I_\delta + I_{\delta}.$$

Looking at the second term, by Theorem 4.5,

$$\|I_{\delta}\|_{L^p(\mathbb{R}^n, \nu)} \leq \frac{1}{\Gamma(-s)} \int_{\delta}^\infty \left( \|e^{t\Delta} u\|_{L^p(\mathbb{R}^n, \nu)} + \|u\|_{L^p(\mathbb{R}^n, \nu)} \right) \frac{dt}{t^{1+s}}$$

$$\leq \frac{C \|u\|_{L^p(\mathbb{R}^n, \nu)}}{\Gamma(-s)} \int_{\delta}^\infty t^{-1-s} \, dt = C \|u\|_{L^p(\mathbb{R}^n, \nu)} \delta^{-s} \left(\frac{1}{\Gamma(2-s)} + 1\right) \to 0$$

as $s \to 1^-$. Next,

$$\|I_{\delta}(-\Delta) u\|_{L^p(\mathbb{R}^n, \nu)}$$

$$= \left\| \frac{1}{\Gamma(-s)} \int_0^\delta \int_0^t \partial_r e^{r\Delta} u(x) dr \frac{dt}{t^{1+s}} + \Delta u(x) \right\|_{L^p(\mathbb{R}^n, \nu)}$$

$$= \left\| \frac{1}{\Gamma(-s)} \int_0^\delta \int_0^t e^{r\Delta} u(x) dr \frac{dt}{t^{1+s}} + \Delta u(x) \right\|_{L^p(\mathbb{R}^n, \nu)}$$

$$= \left\| \frac{1}{\Gamma(-s)} \int_0^\delta \int_0^t (e^{r\Delta} u(x) - \Delta u(x)) dr \frac{dt}{t^{1+s}} + \left(\frac{-s}{\Gamma(2-s)} + 1\right) \Delta u(x) \right\|_{L^p(\mathbb{R}^n, \nu)}$$

$$\leq \frac{1}{\Gamma(-s)} \int_0^\delta \int_0^t \|e^{r\Delta} u - \Delta u\|_{L^p(\mathbb{R}^n, \nu)} dr \frac{dt}{t^{1+s}} + \left(\frac{-s}{\Gamma(2-s)} + 1\right) \|\Delta u\|_{L^p(\mathbb{R}^n, \nu)}$$

$$\leq \varepsilon \delta^{1-s} \frac{s}{\Gamma(2-s)} + \left(\frac{-s}{\Gamma(2-s)} + 1\right) \|\Delta u\|_{L^p(\mathbb{R}^n, \nu)} \to \varepsilon \quad \text{as } s \to 1^-.$$

Since $\varepsilon > 0$ was arbitrary, (1.10) follows in $L^p(\mathbb{R}, \nu)$.

**Step 7.** The limits as $s \to 1^-$ in (1.10) and as $s \to 0^+$ in (1.11) hold a.e. in $\mathbb{R}^n$.

This is proved as in Step 5, by noticing that sup$_{0 < s < 1} (|(-\Delta)^s u(x)|)$ can be bounded by means of Theorem 4.6 and that lim$_{s \to 1^-} (-\Delta)^s u(x) = -\Delta u(x)$ and lim$_{s \to 0^+} (-\Delta)^s u(x) = u(x)$ for all $x \in \mathbb{R}^n$, for any $u \in C^\infty_c(\mathbb{R}^n)$. 


**Step 8.** The limit as $s \to 0^+$ in (1.11) holds in $L^p(\mathbb{R}^n, \nu)$, when $1 < p < \infty$, and in weak-$L^1(\mathbb{R}^n, \nu)$ when $p = 1$.

By Theorem 4.6, for any $0 < s < 1$,
\[
|(-\Delta)^s u(x) - u(x)|^p \nu(x) \leq (C_n(M(D^2u)(x) + Mu(x)) + |u(x)|)^p \nu(x) \\
\leq C_{n,p} ((M(D^2u)(x))^p + (Mu(x))^p) \nu(x).
\]

Therefore, by Step 7 and the Dominated Convergence Theorem, (1.11) holds in $L^p(\mathbb{R}^n, \nu)$ for $1 < p < \infty$ and, by Lemma 1.3, in weak-$L^1(\mathbb{R}^n, \nu)$ when $p = 1$.

This completes the proof of Theorem 1.2, part (a). □

5.2. **Proof of Theorem 1.2(b).** Suppose $(-\Delta)^s u \to v$ in $L^p(\mathbb{R}^n, \nu)$ as $s \to 1^-$. Let $\varphi \in C^\infty_c(\mathbb{R}^n)$ and observe that
\[
\int_{\mathbb{R}^n} v \varphi \, dx = \lim_{s \to 1^-} \int_{\mathbb{R}^n} (-\Delta)^s u \varphi \, dx \\
= \lim_{s \to 1^-} \int_{\mathbb{R}^n} u(-\Delta)^s \varphi \, dx \\
= \int_{\mathbb{R}^n} u(-\Delta) \varphi \, dx = (-\Delta u)(\varphi).
\]

In the first line we used that, by Proposition 4.4 and the fact that $\varphi \in C^\infty_c(\mathbb{R}^n)$,
\[
|\int_{\mathbb{R}^n} v(x) \varphi(x) \, dx - \int_{\mathbb{R}^n} (-\Delta)^s u(x) \varphi(x) \, dx| \leq \int_{\mathbb{R}^n} |v(x) - (-\Delta)^s u(x)| \frac{C_\varphi}{1 + |x|^n} \, dx \\
\leq C_{\varphi,n,p,\nu} \|v - (-\Delta)^s u\|_{L^p(\mathbb{R}^n, \nu)} \to 0
\]
as $s \to 1^-$, while in the second to last identity we used the Dominated Convergence Theorem, the fact that $(-\Delta)^s \varphi \in S_s$, and Proposition 4.4 in the case of $L_0$.

Therefore, $v = -\Delta u$ a.e. in $\mathbb{R}^n$. Since $v \in L^p(\mathbb{R}^n, \nu)$, we get that $\Delta u \in L^p(\mathbb{R}^n, \nu)$. Now we apply the weighted Calderón–Zygmund estimates (see [8]). Hence, if $1 < p < \infty$, then $u \in W^{2,p}(\mathbb{R}^n, \nu)$ and, as a consequence of part (a), (1.10) holds. On the other hand, if $p = 1$, then $D^2 u \in \text{weak-}L^1(\mathbb{R}^n, \nu)$.

\[\square\]

5.3. **Proof of Theorem 1.2(c).** Suppose $(-\Delta)^s u \to v$ in $L^p(\mathbb{R}^n, \nu)$ as $s \to 0^+$, and let $\varphi \in C^\infty_c(\mathbb{R}^n)$. Using the exact same arguments as in part (b), we find that
\[
\int_{\mathbb{R}^n} v \varphi \, dx = \lim_{s \to 0^+} \int_{\mathbb{R}^n} (-\Delta)^s u \varphi \, dx \\
= \lim_{s \to 0^+} \int_{\mathbb{R}^n} u(-\Delta)^s \varphi \, dx = \int_{\mathbb{R}^n} u \varphi \, dx.
\]

Therefore, $u = v = \lim_{s \to 0^+} (-\Delta)^s u$ a.e. in $\mathbb{R}^n$ and the result follows. □

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Department of Mathematics, Iowa State University, 396 Carver Hall, Ames, IA 50011, USA
E-mail address: stinga@iastate.edu, maryo@iastate.edu