SPECTRUM AND COMBINATORICS OF RAMANUJAN TRIANGLE COMPLEXES

KONSTANTIN GOLUBEV AND ORI PARZANCHEVSKI

ABSTRACT. Ramanujan graphs have extremal spectral properties, which imply a remarkable combinatorial behavior. In this paper we compute the high-dimensional Laplace spectrum of Ramanujan triangle complexes, and show that it implies a combinatorial expansion property. For this purpose we prove a Cheeger-type inequality of independent interest, which generalizes the one in [PRT12].

CONTENTS

1. Introduction 1
2. Complexes and buildings 4
3. Iwahori-Hecke boundary maps 6
4. Analysis of the principal series 10
5. Unitary Iwahori-spherical representations 13
6. From spectral to combinatorial expansion 15
References 19

1. INTRODUCTION

A k-regular graph is said to be Ramanujan if its nontrivial Laplace spectrum (see (2.1)) is contained in the interval \( I_k = [k - 2\sqrt{k-1}, k + 2\sqrt{k-1}] \). The reason for these particular values is the following theorem, which says that asymptotically, they are the best that one can hope for:

**Theorem** (Alon-Boppana, [LPS88, Nil91, GŻ99, Li01]). Let \( \{G_i\} \) be a collection of k-regular graphs, \( \text{Spec}(G_i) \) the nontrivial Laplace spectrum of \( G_i \), and \( \lambda(G_i), \Lambda(G_i) \) the minimum and maximum of \( \text{Spec}(G_i) \), respectively.

1. If there are infinitely many different \( G_i \) then \( \lim \inf_i \lambda(G_i) \leq k - 2\sqrt{k-1} \).
2. If \( \{\text{girth}(G_i)\} \) is unbounded then \( \bigcup_i \text{Spec}(G_i) \) is dense in \( I_k \), and in particular \( \lim \sup_i \Lambda(G_i) \geq k + 2\sqrt{k-1} \).

Bounds on the Laplace spectrum of a graph are extremely useful. The so-called “discrete Cheeger inequalities” relate \( \lambda(G) \) to combinatorial expansion, and the “expander mixing lemma” connect spectral concentration (i.e. \( \Lambda(G) - \lambda(G) \)) to pseudo-randomness.

*Date*: June 26, 2014.
Thus, Ramanujan graphs are excellent expanders: we refer to [HLW06, Tao11, Lub12] for detailed surveys on expander graphs, their properties and applications.

The name Ramanujan graphs comes from the first constructions of such graphs [LPS88, Mar88], which use the Ramanujan–Petersson conjecture for GL₂, due to Eichler and Deligne in characteristic zero, and to Drinfeld in positive characteristic (the latter being used in [Mor94]). The constructed graphs are certain quotients of the Bruhat-Tits building associated with PGL₂ over a nonarchimedean local field. This building is a $k$-regular tree, and this gives the real reason for the values which appear in the Alon-Boppana theorem: the interval $I_k$ is precisely the $L^2$-spectrum of the Laplacian on a $k$-regular tree, by a classic result of Kesten [Kes59]. Thus, Ramanujan graphs are those whose nontrivial spectrum is contained in that of their universal cover. For the complete story of Ramanujan graphs we refer the reader to the monographs [Lub94, Sar90].

In recent years, several authors turned to study higher dimensional Ramanujan complexes, which are quotients of the Bruhat-Tits buildings associated with PGLₙ. This includes the works [Bal00, CSZ03, Li04, LSV05a, LSV05b, Sar07, KLW10, KL14], and a recent survey of the field appears in [Lub14]. This surge of interest was prompted by advances in the theory of automorphic representations, especially the resolution of the generalized Ramanujan conjecture for GLₙ in positive characteristic by Lafforgue [Laf02]. While Ramanujan complexes form natural generalizations of Ramanujan graphs, it is not clear what are the best generalizations of graph theoretic notions such as Laplace spectrum, Cheeger constant, pseudo-randomness, girth and so on. Until now, most of the research has focused on the spectral theory of the so-called Hecke operators, which act again on the vertices of the complex, and sum up to the graph adjacency operator. Our interest lies in the the spectrum of the simplicial Laplace operators, also known as combinatorial Laplacians. These originate in [Eck44], and form natural analogues of the Hodge-Laplace operators acting on the spaces of differential forms on a Riemannian manifold. The $i$-th Laplacian, denoted $\Delta^+_i$, acts on the simplicial $i$-forms of the complex, which are skew-symmetric functions on oriented cells of dimension $i$. The 0-forms are just the functions on vertices, and $\Delta^+_0$ is the standard graph Laplacian.

An Alon-Boppana theorem holds here as well:

**Theorem 1** ([Li04, PR12]). If $\{X_i\}$ is a family of quotients of the Bruhat-Tits building $\mathcal{B}_d$ with unbounded injectivity radius\(^{(1)}\), then $\bigcup_i \text{Spec } \Delta^+_i(X_i)$ is dense in the $L^2$-spectrum of $\Delta^+_j(\mathcal{B}_d)$.

(The statement in [Li04] addresses the simultaneous spectrum of the aforementioned Hecke operators, and this implies the theorem for $\Delta^+_0$; [PR12] handles the Laplacians in higher dimensions.)

But what is the spectrum of $\Delta^+_i$ on $\mathcal{B}_d$ and its Ramanujan quotients? It is well known (see e.g. [Mac79, Li04, LSV05a]) that the spectrum of $\Delta^+_0$ is concentrated (though not as good as of Ramanujan graphs). For higher dimension, a celebrated local-to-global argument of Garland [Gar73] establishes lower bounds on the $\Delta^+_i$-spectra (for any complex, not necessarily $\mathcal{B}_d$ and its quotients). While Garland’s argument can be adjusted to give an upper bound as well (see [Pap08, GW12]), the local picture of $\mathcal{B}_d$ gives trivial bounds for the higher end of the spectra of its quotients.

\(^{(1)}\)Note that the injectivity radius of a quotient of a tree is half its girth.
The purpose of this paper is twofold: to compute the Laplace spectrum of Ramanujan triangle complexes, and to relate the simplicial Laplace spectra to combinatorial properties of the complex. The major part of this paper is dedicated to Theorem 4, which determines explicitly the spectrum of $\Delta_{k_1}^+$ for Ramanujan triangle complexes. Our approach is inspired by [Lub94, LSV05a], and especially [KLW10], but it is different: rather then computing the Laplacians directly, we show (in Proposition 5) that certain elements of the Iwahori-Hecke algebra of $\text{PGL}_3$ act on appropriate representations as simplicial boundary and coboundary maps (see §2). We then invoke the classification of unitary, Iwahori-spherical representations of $\text{PGL}_3$ which is computed explicitly in [KLW10] (using results from [Bor76, Zel80, Cas80, Tad86]), to obtain the complete spectrum. While much of the work should be useful for higher dimensions as well, the relevant classification problem becomes harder and we do not pursue it here.

Our computations reveal that the picture for $\Delta_{k_1}^+$ is more intricate than that in dimension zero: most of the spectrum is concentrated in the strip $I_{k_1}$, there $k_1$ is the degree of edges in the complex. However, a small fraction of the spectrum is concentrated in a narrow strip around $2k_1$ (see Theorem 4). This gives, in particular, an explanation for the failure of Garland’s method to achieve spectral concentration.

As in the case of Ramanujan graphs, we would like to infer combinatorial expansion and pseudo-randomness from the spectral theory of the complex. For this purpose we prove a Cheeger-type inequality, which uses the concentration of $\text{Spec} \Delta_{k_1}^+$, and the lower bound on $\text{Spec} \Delta_{k_1}^+$ to obtain an isoperimetric bound:

**Theorem 2.** Let $X$ be a triangle complex on $n$ vertices, with nontrivial $\Delta_{k_1}^+$-spectrum contained in $[k_0 - \mu_0, k_0 + \mu_0]$, and nontrivial $\Delta_{k_1}^+$-spectrum bounded below by $\lambda_1$. Then for any partition of the vertices of $X$ into nonempty sets $A, B, C$, one has

$$\frac{|T(A, B, C)| n^2}{|A| |B| |C|} \geq \lambda_1 \left( k_0 - \mu_0 \left( 1 + \frac{10 n^3}{9 |A| |B| |C|} \right) \right),$$

where $T(A, B, C)$ is the set of triangles with one vertex in each of $A, B$ and $C$.

For complexes with a complete skeleton (underlying graph) this inequality reduces to the one which appears in [PRT12]. It can also be generalized to higher dimensions - this is done in Theorem 7. For an interpretation in terms of a Cheeger-type constant, see (6.1) and (6.2).

In a previous paper [Par13] we have shown that simultaneous concentration of the Laplace spectra in all dimensions implies pseudo-randomness. Since in Ramanujan triangle complexes the edge spectrum is not concentrated (having eigenvalues around both $k_1$ and $2k_1$), this result cannot be applied directly. Nevertheless, the fact that the bulk part of the spectrum is concentrated (see Theorem 4) gives hope to establish similar results in the future, by projecting orthogonally to the eigenforms which correspond to the $2k_1$ strip.

**Acknowledgement.** We would like to thank Alex Lubotzky for providing guidance and inspiration, and to Uriya First and Dima Trushin for patiently introducing us to the representation theory of $p$-adic groups. We are grateful to Shai Evra, Alex Kontorovich, Winnie Li and Doron Puder for helpful discussions and suggestions. The first author was supported by the ERC and by the Israel Science Foundation. The second author was supported by The Fund for Math, and is grateful for the hospitality of the Institute for Advanced Study.
In this section we recall the basic elements of simplicial Hodge theory, and the definition of affine Bruhat-Tits buildings. For a more relaxed exposition of the former we refer to [PRT12, §2], and for the latter to [Li04, LSV05a, Lub14].

Let $X$ be a finite simplicial complex of dimension $d$, and denote by $X^i$ the set of (unoriented) cells of dimension $i$ in $X$ ($i$-cells). The degree of an $i$-cell is the number of $(i + 1)$-cells which contain it. We denote by $\Omega^i = \Omega^i (X)$ the space of $i$-forms, namely skew-symmetric complex functions on the oriented $i$-cells, equipped with the inner product $\langle f, g \rangle = \sum_{\sigma \in X^i} f (\sigma) g (\sigma)$. The $i$-th boundary map $\partial_i : \Omega^i \to \Omega^{i-1}$ is defined by $\langle \partial_i f, \sigma \rangle = \sum_{\nu \sqsubset \sigma} f (\nu)$, and its dual is the $i$-th coboundary map $\delta_i = \partial^*_i : \Omega^{i-1} \to \Omega^i$, which is given by $\langle \delta_i f, \sigma \rangle = \sum_{i=0}^d (-1)^i f (\sigma \setminus \sigma_i)$. We use the standard notation $Z_i = \ker \partial_i$, $Z^i = \ker \delta_i$, $B_i = \text{im} \partial_{i+1}$ and $B^i = \text{im} \delta_i$ for cycles, cocycles, boundaries and coboundaries.

The upper, lower and full $i$-Laplacians are $\Delta^+_i = \delta_{i+1} \delta_i$, $\Delta^-_i = \delta_i \delta_i$ and $\Delta_i = \Delta^+_i + \Delta^-_i$, respectively. For our purposes the upper Laplacian is the most important one, and as we have remarked, $\Delta^+_0$ is the classical Laplacian of graph theory:

$$(\Delta^+_0 f) (v) = \deg (v) f (v) - \sum_{w \sim v} f (w).$$  (2.1)

Since $(\Omega^*, \delta_*)$ is a cochain complex, $\ker \delta_i$ is always in $\ker \Delta^+_i$, and the corresponding zeros in $\text{Spec} \Delta^+_i$ are considered the trivial spectrum. The nontrivial spectrum (in dimension $i$) is thus

$$\text{Spec} \Delta^+_i|_{Z_i} = \text{Spec} \Delta^+_i|_{\ker \partial_i} = \text{Spec} \Delta^+_i|_{(\text{im} \delta_i)^\perp},$$  (2.2)

and $0$ belongs to the nontrivial spectrum if and only if the $i$-th cohomology of $X$ over $\mathbb{C}$ is nontrivial. We should remark that in the theory of Ramanujan graphs, the Laplacian eigenvalue $2k$ (for a $k$-regular graph) is also considered trivial. We return to this point later on.

Moving to buildings, let $F$ be a nonarchimedean local field (e.g. $\mathbb{Q}_p$ or $\mathbb{F}_p ((t))$) with ring of integers $\mathcal{O}$, uniformizer $\pi$, and residue field $\mathcal{O}/\mathcal{O} \pi$ of size $q$, which we shall denote by $\mathbb{F}_q$. The affine Bruhat-Tits building associated with $\text{PGL}_d (F)$ is an infinite contractible complex of dimension $d - 1$, denoted $\mathcal{B} = \mathcal{B}_d (F)$. Denoting $G = \text{PGL}_d (F)$ and $K = \text{PGL}_d (\mathcal{O})$, the vertices of $\mathcal{B}$ are in correspondence with the left $K$-cosets in $G$, and they admit a coloring in $\mathbb{Z}/d\mathbb{Z}$, defined by $\text{col} (gK) = \text{ord}_\pi (\det g) + d\mathbb{Z}$. To every vertex $gK$ we associate the $\mathcal{O}$-lattice $g\mathcal{O}^d$, which is only defined up to scaling in $F^\times$, since $g \in \text{PGL}_d (F) = \text{GL}_d (F)/F^\times$. A collection of vertices $\{g_i K\}_{i=0,..,r}$ forms an $r$-cell if there exist representatives $g'_i \in \text{GL}_d (F)$ for $g_i$, such that the corresponding lattices satisfy

$$\pi g'_0 \mathcal{O}^d < g'_1 \mathcal{O}^d < g'_{r-1} \mathcal{O}^d < \ldots < g'_r \mathcal{O}^d < g'_0 \mathcal{O}^d.$$  (2.3)

This relation turns out to be equivalent to having an edge between every $g_i K$ and $g_j K$ (i.e., representatives $g'_i, g'_j \in \text{GL}_d (F)$ with $\pi g'_i \mathcal{O}^d < g'_j \mathcal{O}^d < g'_0 \mathcal{O}^d$). In other words, $\mathcal{B}$ is a clique (or “flag”) complex.

The action of $G$ on $\mathcal{O}/K$ defines an action on the complex $\mathcal{B}$, as it clearly preserves (2.3). Given a cocompact, torsion-free lattice $\Gamma \leq G$, the quotient $\Gamma \backslash \mathcal{B}$ is a finite complex. For $d = 2$, $\mathcal{B}_2$ is a $(q + 1)$-regular tree, and its quotients by lattices in $G$ are $(q + 1)$-regular graphs. Using the generalized Peterson-Ramanujan conjecture for $\text{GL}_2 (F)$, the following is obtained:
Theorem ([LPS88, Mar88]). If $\Gamma$ is a torsion-free arithmetic lattice in $G$, then then $\Gamma \setminus B_2$ is a Ramanujan graph.

As we have remarked, some of these graphs have the eigenvalue $2k$ in their spectrum (with the rest of the nontrivial spectrum in $I_k$). This is the origin of this eigenvalue: recall that $B^0$ is colored, and note that the action of $G$ does not respect the coloring, since $\text{col}(gv) = \text{col}(v) + \text{ord}_r(\det g)$. The following are equivalent:

- $\Gamma \setminus B_2$ is bipartite.
- $2k = 2(q + 1)$ appears in $\text{Spec } \Delta_0^+(\Gamma \setminus B_2)$, corresponding to $f(v) = (-1)^{\text{col}v}$.
- The action of $\Gamma$ preserves the color of vertices.
- $\text{col } \Gamma \equiv 0$ (i.e. $\text{ord}_r(\det \gamma) \in 2\mathbb{Z}$ for all $\gamma \in \Gamma$).

Together with [Mor94], the mentioned papers establish that if $k - 1$ is a prime power, then there exist infinitely many $k$-regular non-bipartite Ramanujan graphs, and infinitely many bipartite ones. For general $k$, the existence of infinitely many bipartite Ramanujan graphs was only settled in [MSS13] (by a non-constructive argument), and it is still open for the non-bipartite case.

Let us continue to dimension two. The Bruhat-Tits building $B_3$ is a regular triangle complex, with vertex and edge degrees

$$k_0 = 2(q^2 + q + 1), \quad \text{and} \quad k_1 = q + 1$$

respectively. There are several plausible ways to define what are the Ramanujan quotients of $B_3$, and these are discussed in [CSŻ03, Li04, LSV05a, KLW10]. Luckily, they all coincide for $\text{PGL}_q$ (see [KLW10]), though not in general (see [FLW13]). The definition we use is the following (the notions in the definition are explained throughout §3):

Definition 3. A quotient of $B_3$ by a torsion-free cocompact lattice $\Gamma$ in $G = \text{PGL}_3(F)$ is Ramanujan if every Iwahori-spherical representation of $G$ which appears in $L^2(\Gamma \setminus G)$ is either finite-dimensional or tempered.

The papers [Li04, LSV05b, Sar07] give several constructions of Ramanujan complexes, some of which are clique complexes of Cayley graphs.

We begin the spectral study of these complexes at the higher end of the spectrum. We say that a $d$-complex is $(d + 1)$-partite if there exists a $(d + 1)$-coloring of its vertices such that no two vertices in a $d$-cell are of the same color. If $X$ is a $(d + 1)$-partite $d$-complex whose $(d - 1)$-cells have degree $k$, then $(d + 1)k \in \text{Spec } \Delta^+_{d-1}$. Unlike the graph case, the converse may fail: rather than $(d + 1)$-partiteness, the $(d + 1)k$ eigenvalue indicates disorientability (see [PR12]), the existence of an orientation of the $d$-cells so that they agree on the orientation of their intersections.

In our case, all quotients of $B_3$ by a lattice in $G$ are disorientable, i.e. have $3k_1 \in \text{Spec } \Delta^+_1$, and a similar situation holds in higher dimensions (see [Pap08]). If we define the color of a directed edge to be $\text{col } ([v, w]) = \text{col } (w) - \text{col } (v)$ (in $\mathbb{Z}/2\mathbb{Z}$), then the action of $G$ does preserve edge colors, so that the eigenform $f(e) = (-1)^{\text{col } e}$ always factors modulo $\Gamma$, giving $3k_1 \in \text{Spec } \Delta^+_1$.

If $\Gamma$ also preserves vertex coloring, i.e. $\text{col } \Gamma \equiv 0$, then $X = \Gamma \setminus B_3$ is naturally tripartite. The converse also holds: if $\text{col } \Gamma \neq 0$ then $\Gamma^0 = \{ \gamma \in \Gamma \mid \text{col } \gamma K = 0 \}$ is a normal subgroup of $\Gamma$ of index three, and $\tilde{X} = \Gamma^0 \setminus B_3$ is a tripartite three-cover of $X$, with the coloring $\text{col } = \text{ord}_r \det$. But any two edges in $\tilde{X}$ are connected by a chain of triangles, so that
\(\hat{X}\) admits a unique 3-coloring (up to renaming of colors). Thus, any three-coloring of \(X\) must lift to \(\text{col}\), which must therefore be \(\Gamma\)-invariant, contradicting \(\text{col} \Gamma \neq 0\).

Having dealt with the “colored” part of the spectrum\(^{(1)}\), we can finally move on to its interesting part:

**Theorem 4.** Let \(X\) be a non-tripartite Ramanujan triangle complex on \(n\) vertices.

1. The nontrivial spectrum of \(\Delta_0^+\) is contained within \([k_0 - 6q, k_0 + 3q]\).
2. The nontrivial spectrum of \(\Delta_1^+\) consists of:
   (a) \(n\) \((q^2 + q - 2) + 2\) eigenvalues in the strip \(I = [k_1 - 2\sqrt{q}, k_1 + 2\sqrt{q}]\).
   (b) For every nontrivial \(\lambda \in \text{Spec} \Delta_0^+\), the eigenvalues \(\frac{3k_1}{2} \pm \sqrt{\left(\frac{3k_1}{2}\right)^2 - \lambda}\).
   This amounts to \(n - 1\) eigenvalues in each of the strips
      \[I_- = \left[\frac{3k_1}{2} - \sqrt{\left(\frac{3k_1}{2}\right)^2 - 8q}, k_1 + 1\right]\]
      \[I_+ = \left[2k_1 - 1, \frac{3k_1}{2} + \sqrt{\left(\frac{3k_1}{2}\right)^2 + 8q}\right].\]
   (c) The disorientation eigenvalue \(3k_1\), corresponding to \(f(e) = (-1)^{\text{col} e}\).

If \(X\) is tripartite (Ramanujan):

1. The nontrivial spectrum of \(\Delta_1^+\) consists of:
   (a) \(n - 3\) eigenvalues of \(\Delta_0^+\) in \([k_0 - 6q, k_0 + 3q]\).
   (b) The eigenvalue \(\frac{3k_1}{2}\), twice, corresponding to \(f(v) = \exp\left(\pm \frac{2\pi i}{3} \text{col} v\right)\).
2. The nontrivial spectrum of \(\Delta_1^+\) consists of:
   (a) \(n\) \((q^2 + q - 2) + 6\) eigenvalues in \(I\).
   (b) \(n - 3\) eigenvalues in each of \(I_{\pm}\), corresponding to \(\frac{3k_1}{2} \pm \sqrt{\left(\frac{3k_1}{2}\right)^2 - \lambda}\) for the eigenvalues of \(\Delta_0^+\) in (a) above.
   (c) As before, \(3k_1\).

Let us make a few remarks:

1. These bounds can not be improved: by Theorem 1, a sequence of quotients with unbounded injectivity radius (as constructed in [LM07]) has Laplace spectra which accumulate to any point in these intervals.
2. Note that there are no nontrivial zeros in the spectra of \(\Delta_0^+\) and \(\Delta_1^+\), so that the zeroth and first (torsion-free) homology of \(X\) vanish, in accordance with [Gar73, Cas74].
3. The strip \(I_-\) is contained in \(I\), and both of the strips \(I_{\pm}\) are highly concentrated:
   \[I_- \subseteq [k_1 - 8, k_1 + 1], \quad I_+ \subseteq [2k_1 - 1, 2k_1 + 8].\]

We return to the concentration of the spectrum in §6. The proof of Theorem 4 occupies the next three sections.

### 3. Iwahori-Hecke boundary maps

Let us fix the fundamental vertex \(v_0 = K\) in \(\mathcal{B}^0 = G/K\). For a quotient \(X = \Gamma \backslash \mathcal{B}\) we denote by \(v\) the image of \(v \in \mathcal{B}^0\) in \(X^0\) as well, and the vertices of \(X\) are in correspondence with the double cosets in \(\Gamma \backslash G/K\). If we normalize the Haar measure \(\mu\) on \(G\) so that \(\mu(K) = 1\), then \(\mu(\Gamma \backslash G) = |\Gamma \backslash G/K| \mu(K) = n\). The measure induced by \(\mu\) on \(\Gamma \backslash G/K\)

\(^{(1)}\) We reserve the term “trivial” for the homological meaning - zeros obtained on coboundaries.
is the counting measure, so that there is a linear isometry $\Omega_0^1 (X) \cong L^2 (\Gamma \backslash G/K)$, given explicitly by $f (gv_0) = f (\Gamma g K)$. It will also be useful for us to identify $L^2 (\Gamma \backslash G/K)$ with $L^2 (\Gamma \backslash G)^K$, the space of $K$-fixed vectors in the $G$-representation $L^2 (\Gamma \backslash G)$, in the natural manner.

The element $\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G$ acts on $B$ by rotation on the triangle consisting of the vertices $v_0$, $\sigma v_0$, and $\sigma^2 v_0$ (note that $\sigma^3 = 1$ in $G$). We choose as a fundamental edge $e_0 = [v_0, \sigma v_0]$, and denote

$$E = \text{stab}_G e_0 = K \cap \sigma K \sigma^{-1} = \left\{ \begin{pmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \end{pmatrix} \in K \mid x, y \in \pi \mathcal{O} \right\}$$

(E is sometimes called a “parahoric subgroup”. ) Recall the coloring $\text{col} : B^1 \rightarrow \mathbb{Z}/3 \mathbb{Z}$ given by $\text{col} ([gK, g'K]) = \text{ord}_x \text{det} (g'g^{-1})$, which is invariant under $G$, and thus well defined on $X^1 = \Gamma \backslash B^1$. Every edge in $X$ (and $B$) has one orientation with color 1, and one with color 2 (by definition there are no edges of color 0). Furthermore, $G$ acts transitively on the nonoriented edges of $B$, so that the oriented edges of color 1 in $B$ can be identified with $G/E$ (as $\text{col} e_0 = 1$), and those of $X$ with $\Gamma \backslash G/E$, giving

$$\mu (E) = \frac{\mu (\Gamma \backslash G)}{|\Gamma \backslash G/E|} = \frac{n}{|X^1|} = \frac{1}{q^2 + q + 1},$$

and we can identify $\Omega^1 (X)$ with $L^2 (\Gamma \backslash G)^E$, by

$$f (ge_0) = f ([gv_0, g\sigma v_0]) = \sqrt{\mu (E)} f (\Gamma g), \quad f ([g\sigma v_0, gv_0]) = -\sqrt{\mu (E)} f (\Gamma g).$$

The scaling by $\sqrt{\mu (E)}$ is needed to make the isomorphism $\Omega_0^1 (X) \cong L^2 (\Gamma \backslash G)^E$ an isometry: if $g_1, \ldots, g_{n_{k_1}/2}$ is a set of representatives for $\Gamma \backslash G/E$, then for $f \in L^2 (\Gamma \backslash G)^E$ we have

$$\|f\|^2_{\Omega_0^1(X)} = \sum_{i=1}^{n_{k_1}/2} \|f(ge_0)\|^2 = \sum_{i=1}^{n_{k_1}/2} \mu (E) \|f(\Gamma g_i)\|^2 = \sum_{i=1}^{n_{k_1}/2} \int_E \|f(\Gamma g_i)\|^2 dg = \frac{1}{|\Gamma \backslash G|} \int_{\Gamma \backslash G} \|f(\Gamma g)\|^2 dg \leq \|f\|^2_{L^2(\Gamma \backslash G)}.$$  

We remark that this is simpler than the picture in higher dimensions: for example, $G = \text{PGL}_4 (F)$ acts transitively on the edges of color 1, and on the edges of color 3 in $B_3$. It also preserves their orientations, as it preserves edge coloring, and $\text{col} (vvw) = -\text{col} (vvv)$. However, $G$ acts transitively on the oriented edges of color 2, i.e. flipping their orientations as well (which is possible since $2 \equiv -2 \mod 4$). If $E_i$ is the stabilizer (as a cell) of some fundamental edge $e_i$ of color $i$, and $\xi : E_2 \rightarrow \pm 1$ is the character which indicates whether an element of $E_2$ fixes $e_2$ pointwise or flips it, then

$$\Omega_1 (\Gamma \backslash B_4) \cong L^2 (\Gamma \backslash G)^{E_1} \oplus L^2 (\Gamma \backslash G)^{E_2, \xi},$$

where $V^{E_2, \xi}$ denotes the $\xi$-isotypic component of a representation $V$ (this can be thought of as a higher “$E$-type”).

Coming back to $B_3$, we fix $t_0 = [v_0, \sigma v_0, \sigma^2 v_0]$ as a fundamental triangle. The pointwise stabilizer of $t_0$ is the Iwahori subgroup

$$I = K \cap \sigma K \sigma^{-1} \cap \sigma^2 K \sigma^{-2} = \left\{ \begin{pmatrix} \ast & \ast & \ast \\ \ast & \ast & \ast \end{pmatrix} \in K \mid x, y, z \in \pi \mathcal{O} \right\}.$$
As for edges, $G$ acts transitively on non-oriented triangles, and preserves triangle orientation (since a flip of a triangle would imply a flip of some edge). Thus, the stabilizer of $t_0$ as a cell (both oriented and non-oriented) is $T \overset{\text{def}}{=} \text{stab}_G t_0 = \langle \sigma \rangle I = I \mathfrak{I} \sigma I \mathfrak{I} \sigma^2 I$, and in particular $\langle \sigma \rangle$ and $I$, commute. Again, $f(gt_0) = \sqrt{\mu(T)} f(\Gamma g)$ gives a linear isometry $\Omega^2(X) \cong L^2(\Gamma \backslash G)^T$, and

$$
\mu(T) = \frac{\mu(\Gamma G)}{|\Gamma \backslash G / T|} = \frac{n}{|X^2|} = \frac{3!}{k_0 k_1} = \frac{3}{(q^2 + q + 1)(q + 1)}.
$$

Let us denote $K_0 = K$, $K_1 = E$, and $K_2 = T$. As $I \leq E \leq K$ and $I \leq T$, the three spaces $\Omega^i(X) \cong L^2(\Gamma \backslash G)^{K_i}$ are contained in $L^2(\Gamma \backslash G)^I$ - the subspace of Iwahori-fixed vectors. The Iwahori-Hecke algebra of $G$ is $H = C_c(I \backslash G / I)$, which stands for compactly supported bi-$I$-invariant functions on $G$. Multiplication in $H$ is given by convolution, and if $(\rho, V)$ is a representation of $G$, then $(\varpi, V^I)$ is a representation of $H$, by $\varpi(\eta)v = \int_G \eta(g) \rho(g) v \, dg$.

We proceed to construct elements in $H$ whose actions on the appropriate subspaces of $L^2(\Gamma \backslash G)$ coincide with the boundary and coboundary operators in the chain complex of simplicial forms on $X$. For a bi-$I$-invariant set $S \subseteq G$, let $\mathbb{I}_S$ denote the corresponding characteristic function, as an element of $H$.

**Proposition 5.** Let

$$
\partial_1 = \frac{1}{\sqrt{\mu(E)}} (\mathbb{1}_{K \sigma^2 E} - \mathbb{1}_K), \quad \partial_2 = \frac{1}{\sqrt{\mu(E)\mu(T)}} \mathbb{1}_{ET} \quad \delta_1 = \frac{1}{\sqrt{\mu(E)}} (\mathbb{1}_{\sigma K} - \mathbb{1}_K), \quad \delta_2 = \frac{1}{\sqrt{\mu(E)\mu(T)}} \mathbb{1}_{TE}.
$$

Then each $\partial_i \in H$ takes $L^2(\Gamma \backslash G)^{K_i}$ to $L^2(\Gamma \backslash G)^{K_{i-1}}$, and acts as the boundary operator $\partial_i : \Omega^i(X) \rightarrow \Omega^{i-1}(X)$ with respect to the identifications of $\Omega^i(X)$ with $L^2(\Gamma \backslash G)^{K_i}$. The analogue statements hold for $\delta_i \in H$ and $\delta_i : \Omega^{i-1} \rightarrow \Omega^i$ as well.

**Proof.** As $\partial_i$ is constant on right $K_{i-1}$ cosets, the image of its action on any $G$-representation $V$ is in $V^{K_{i-1}}$, and similarly for $\delta_i$ and $K_i$ (note that $\sigma K = E \sigma K = \sigma^{-1} E \sigma$ fixes $v_0$). Observe $V = L^2(\Gamma \backslash G)$ and $f \in V^E \cong \Omega^1(E)$, and denote by $\{k\}_{k \in K / E}$ an arbitrary choice of a left transversal for $E$ in $K$. For any $gv_0 \in X^0$ we have

$$
(\mathbb{1}_K f)(gv_0) = (\mathbb{1}_K f)(\Gamma g) = \int_G \mathbb{1}_K(x) (xf)(\Gamma g) \, dx = \int_K (xf)(\Gamma g) \, dx
$$

$$
= \int_K f(\Gamma gx) \, dx = \sum_{k \in K / E} \int_E f(\Gamma gke) \, de = \sum_{k \in K / E} \int_E f(\Gamma gk) \, de
$$

$$
= \mu(E) \sum_{k \in K / E} f(\Gamma gk) = \sqrt{\mu(E)} \sum_{k \in K / E} f(gke_0),
$$

where the first and last equalities are the identifications of $L^2(\Gamma \backslash G)^{K_i}$ with $\Omega^i(X)$. The group $K$ acts transitively on the $q^2 + q + 1$ edges in $B$ with origin $v_0$ and color 1, which correspond to the projective points in $\mathbb{P}^3 / \pi \cdot \mathbb{P}^3 \cong \mathbb{P}^3_q$. Thus, the color-1 edges with origin $gv_0$ (for any $g \in G$) are precisely $\{gke_0\}_{k \in K / E}$, giving

$$
\frac{1}{\sqrt{\mu(E)}} (\mathbb{1}_K f)(gv_0) = \sum_{k \in K / E} f(gke_0) = \sum_{\text{orig} \in gv_0 \atop \text{col} = 1} f(e) = - \sum_{\text{term} \in gv_0 \atop \text{col} = 2} f(e).
$$

(3.3)
In a similar manner, \( \sigma^2 e_0 = [\sigma^2 v_0, v_0] \) has color 1 and terminates in \( v_0 \), and \( K \) act transitively on such edges (which correspond to projective lines in \( \sigma^3 / \sigma \cdot \sigma^2 \)). Thus, the color 1 edges which terminate in \( g t_0 \) are \( \{ g k \sigma^2 e_0 \}_{k \in K} \), and if \( \{ k \sigma^2 \}_{k \sigma^2 E \in \sigma^2 E / E} \) is a left transversal of \( E \) in \( K \sigma^2 E \) then

\[
(1_{K \sigma^2 E}) (g t_0) = \int_{K \sigma^2 E} f (\Gamma g x) \, dx = \sum_{k \sigma^2 E \in \sigma^2 E / E} \mu (E) \sum_{f (k \sigma^2 e_0) = \mu (E) \sum_{\text{term}_e \in g t_0} f (e)} f (e).
\]

Together with (3.3), this implies that the diagram

\[
L^2 (\Gamma \backslash G)^K \xrightarrow{\partial_1 \in \mathcal{H}} L^2 (\Gamma \backslash G)^E \\
\Omega^0 (X) \xrightarrow{\partial_1} \Omega^1 (X)
\]

commutes (which should justify the abuse of notation). The reasoning for \( \partial_2 \) is similar, save for the fact that \( T \nsubseteq E \) (in fact, \( E \cap T = I \), as a common stabilizer of \( t_0 \) and \( e_0 \) must fix \( t_0 \) pointwise.) We observe that \( E \) acts transitively on \( (q + 1) \)-triangles containing \( e_0 \), which correspond to incidences of projective lines and points in \( \sigma^3 / \sigma \cdot \sigma^2 \). Therefore, for \( f \in V = L^2 (\Gamma \backslash G)^T \) and \( g e_0 \in X^1 \),

\[
\frac{1}{\sqrt{\mu (E) \mu (T)}} (1_{ET} f) (g e_0) = \frac{1}{\sqrt{\mu (E) \mu (T)}} (1_{ET} f) (\Gamma g) = \sqrt{\mu (T)} \sum_{\tau \in X^2 : e_0 \in \partial \tau} f (\tau),
\]

agreeing with \( \partial_2 : \Omega^2 \to \Omega^1 \).

The coboundary operators can be analyzed in a similar manner, but they also follow easily from the unitary structure: \( \mathcal{H} \) has a *-algebra structure induced by the group inversion, namely \( \mathcal{H}^* (g) = \mathcal{H}^* (g^{-1}) \). For any unitary representation \( \rho \) of \( G \), the induced representation \( \mathcal{P} \) is unitary, namely \( \mathcal{P} (\eta^*) = \mathcal{P} (\eta^*) \) (this uses unimodularity of \( G \)), and for \( V = L^2 (\Gamma \backslash G) \) this gives

\[
\partial^*_1 \quad \frac{1}{\sqrt{\rho (E)}} (1_\mathcal{K} - \mathcal{K}^1) = \frac{1}{\sqrt{\rho (E)}} (1_{E \mathcal{K}^1} - \mathcal{K}^1) = \frac{1}{\sqrt{\rho (E)}} (1_{\mathcal{K}^1} - \mathcal{K}^1)
\]

and similarly for \( \partial^*_2 \).

As \( \Gamma \) is cocompact, \( L^2 (\Gamma \backslash G) \) decomposes as a sum of irreducible unitary representations, \( L^2 (\Gamma \backslash G) = \bigoplus \omega W \alpha \), and \( \Omega^1 (X) \cong L^2 (\Gamma \backslash G)^{\mathcal{K}_I} = \bigoplus \omega W \alpha ^{\mathcal{K}_I} \leq \bigoplus \omega W \alpha ^{\mathcal{K}_I} \). Each \( W \alpha \cdot \mathcal{K}_I \) is a sub-\( \mathcal{H} \)-representation, so that the operators \( \partial_1, \partial_2 \) decompose with respect to this sum, and thus the Laplacians as well, giving \( \text{Spec} \Delta^\pm = \bigcup \omega \text{Spec} \Delta^\pm_{\mathcal{K}_I} \), with the correct multiplicities. To understand the spectra it is enough look at the \( \mathcal{K}_I \) which are Iwahori-spherical, namely, contain \( I \)-fixed vectors (for otherwise \( W \mathcal{K}_I = 0 \)). Furthermore, as \( \Delta \) act by elements in \( \mathcal{H} \), the isomorphism type of \( W \alpha \cdot \mathcal{K}_I \) as an \( \mathcal{H} \)-module already determines the spectrum of \( \Delta \) on \( W \mathcal{K}_I \). By [Cas80, Prop. 2.6], every Iwahori-spherical representation is isomorphic to a subrepresentation of a principal series representation. The principal series representation \( V \alpha \) with Satake parameters \( \beta = (z_1, z_2, z_3) \), where \( z_i \in \mathbb{C} \) and \( z_1 z_2 z_3 = 1 \),
is obtained as follows: The standard Borel group $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \in G \right\}$ admits a character
\[ \chi_3(b) = \prod_{i=1}^{3} z_i^{\text{ord}_i b_i} \] (here $z_1 z_2 z_3 = 1$ ensures that $\chi_3$ is well defined on PGL), and $V_3$ is the unitary induction of $\chi_3$ from $B$ to $G$, namely
\[ V_3 = \text{UInd}_B^G \chi_3 = \left\{ f : G \to \mathbb{C} \ \middle| \ \int_K |f(k)| \, dk < \infty \right\}, \]
where $\delta(b) = |b_{33}|^2 / |b_{11}|^2$ is the modular character of $B$. For obvious reasons, it is convenient to introduce the notation
\[ \tilde{\chi}_3(b) = \delta^{-\frac{1}{2}}(b) \chi_3(b) = \frac{|b_{11}|}{|b_{33}|} \prod_{i=1}^{3} z_i^{\text{ord}_i b_i} = \left( \frac{z_1}{q} \right)^{\text{ord}_1 b_{11}} z_2^{\text{ord}_2 b_{22}} (q z_3)^{\text{ord}_3 b_{33}}. \]

Let us stress the following point: having decomposed $L^2(\Gamma \setminus G) = \bigoplus_{\alpha} W_{\alpha}$, and found some $\Delta^\pm$-eigenform $f \in W^K_3 \leq \Omega^1(X)$, we can lift it to a $\Gamma$-periodic form $\tilde{f} \in \mathcal{I} \Omega^1(B)$, which is still a $\Delta^\pm$-eigenform (as $\Delta^\pm$ act in a local manner). For some Satake parameters $\delta$ there is an embedding of $G$-representations $\Psi : W_\alpha \to V_3$, and naturally $\Psi f \in V^K_3$. But as $V_3$ consists of complex functions on $G$, the elements of $V^K_3$ can also be seen as $i$-forms on $B$. Thus, both $\tilde{f}$ and $\Psi f$ are in $\Omega^1(B)$, and they are $\Delta^\pm$-eigenforms with the same eigenvalue (as $\Psi$ is $H$-equivariant). However, they are not the same, as $\tilde{f}$ attains finitely many values and $\Psi f$ infinitely many, in general.

While the forms $\tilde{f}$ and $\Psi f$ are different, their averages on $K$-orbits are the same: these averages correspond to the associated matrix coefficient of $W_\alpha \cong \Psi W_\alpha$. When these matrix coefficient are in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$, the representation $W_\alpha$ is called tempered.

The matrix coefficient $g \mapsto \langle gv, v \rangle$ for $v \in V^K_3$ is bi-$K$-invariant, and if $V$ is tempered it is an approximate $L^2$-eigenform in $\Omega^1(B)$, so that Spec $\Delta^\pm|_{W^K_3}$ is contained in the $L^2$-spectrum of $\Delta^\pm$ on $B$.

4. Analysis of the principal series

While in general an irreducible Iwahori-spherical representation $W$ is only isomorphic to a subrepresentation of $V_3$, it is simpler to consider the action of $\delta_0, \delta_1$ and $\Delta^\pm$ on $V_3$, and later find which eigenvalues are attained on $W$. By the Iwasawa decomposition $G = BK$, for $f \in V^K_3$ and $g = bk \in G$ one has $f(g) = f(bk) = \chi_3(b) f(1)$, so that $V^K_3$ is at most one-dimensional. In fact, it is one-dimensional, since $b \in B \cap K$ implies $b_{11}, b_{22}, b_{33} \in O^K$, so that $f^K(bk) := f_3(b)$ is well defined and spans $V^K_3$. Letting $w \in S_3$ stand for the image of the permutation matrix $(T)_{i,j} = \delta_{i,w(j)}$ in $G$, the so-called Iwahori-Bruhat decomposition $G = \bigcap_{w \in S_3} BwT$ shows that $V^K_3 = 6$, with basis $\{ f^I_w \}_{w \in S_3}$ defined by $f^I_w(w') = \delta_{w,w'}$ (note that $f^K = \sum_{w \in S_3} f^I_w$). Also, $(12) \in E$ (see (3.1)) implies $G = \bigcap_{w \in A_3} BwE$, and dim $V^K_E = 3$ with basis $\{ f^E_w \}_{w \in A_3}$, where $f^E_w := f^I_w + f^I_{w}(12)$ satisfy $f^E_w(w') = \delta_{w,w'}$ for $w, w' \in A_3$. For $T$ things are slightly more complicated, since $T \not\subset K$: here $G = BT \sqcup B (12) T$, so that dim $V^K_T = 2$. One can then define $f^T_w(w') := f^I_{w'}$ for $w, w' \in \{(12), (12)\}$, and this gives
\[ f^T_{(1)} = f^I_{(1)} + \frac{1}{q z_3} f^I_{(321)} + \frac{z_1}{q} f^I_{(123)}, \quad f^T_{(12)} = f^I_{(12)} + z_2 f^I_{(23)} + \frac{1}{q z_3} f^I_{(13)}; \] (4.1)
Indeed, if \( c_w \) is the coefficient of \( f_w^T \) in \( f_{(1)}^T \), then
\[
c_{(123)} = f_{(1)}^T ((1 2 3)) = f_{(1)}^T \left( \begin{pmatrix} \pi & 1 \\ 1 & 1 \end{pmatrix} \right) = \bar{\chi}_3 \left( \begin{pmatrix} \pi & 1 \\ 1 & 1 \end{pmatrix} \right) = \frac{z_1}{q},
\]
and the other coefficients are obtained similarly.

Now, let \( \Omega_3^1 (B) \) be the embedding of \( V^K_3 \) in \( \Omega_3 (B) \). Any \( f \in \Omega_3^0 (B) \) is determined by its value on \( v_0 \), namely \( f = f (v_0) \). Similarly, the value on \( e_0, e_1 = (3 2 1) \) and \( e_2 = (1 2 3) \) determine a unique element in \( \Omega_3^1 (B) \), and likewise for \( t_0, t_1 = (1 2) t_0 \) and \( \Omega_3^1 (B) \). As \( S_3 \leq K \), all of \( v_0, e_0, e_1, e_2, t_0, t_1 \) are in the star around \( v_0 \) in \( B \), and we can understand \( \partial_{(1)}|_{\Omega_3^1 (B)} \) and \( \delta_{(1)}|_{\Omega_3^0 (B)} \) by evaluation on star \( (v_0) \) alone (Figure 4.1).

![Figure 4.1. The star of \( v_0 \) in \( B \).](image)

Thus,
\[
\begin{align*}
(\delta_1 f^K)(e_0) &= f^K (\sigma v_0) - f^K (v_0) = f^K \left( \begin{pmatrix} 1 & \pi \\ 1 & 1 \end{pmatrix} \right) v_0 - 1 = \bar{\chi}_3 \left( \begin{pmatrix} 1 & \pi \\ 1 & 1 \end{pmatrix} \right) - 1 = q z_3 - 1 \\
(\delta_1 f^K)(e_1) &= f^K ((3 2 1) \sigma v_0) - f^K (v_0) = f^K \left( \begin{pmatrix} 1 & \pi \\ 1 & 1 \end{pmatrix} \right) v_0 - 1 = z_2 - 1 \\
(\delta_1 f^K)(e_2) &= f^K ((1 2 3) \sigma v_0) - f^K (v_0) = f^K \left( \begin{pmatrix} \pi & 1 \\ 1 & 1 \end{pmatrix} \right) v_0 - 1 = \frac{z_1}{q} - 1
\end{align*}
\]
gives
\[
\left[ \delta_{(1)}|_{\Omega_3^1 (B)} \right]_{\mathcal{B}^E} = \begin{pmatrix} q z_3 - 1 \\ q z_2 - 1 \\ \frac{z_1}{q} - 1 \end{pmatrix},
\]  
(4.2)

where \( \mathcal{B}^E \) denotes the ordered basis \( f_{(1)}^E, f_{(3 2 1)}^E, f_{(1 2 3)}^E \).

The edges of color 1 with origin \( v_0 \) are \( e_0, \left( \begin{pmatrix} 1 & \pi \\ 1 & 1 \end{pmatrix} \right) e_1 \) for \( x \in \mathbb{F}_q = \mathbb{O} / \pi \mathbb{O} \), and \( \left( \begin{pmatrix} 1 & \pi \\ 1 & 1 \end{pmatrix} \right) e_2 \) with \( x, y \in \mathbb{F}_q \). As \( \bar{\chi}_3 \) is trivial on \( U \), the group of upper-triangular unipotent matrices, every \( f \in \Omega_3^1 (B) \) is constant on the \( q \) translations of \( e_1 \), and on the \( q^2 \) translations of \( e_2 \) (in particular this implies that \( \oplus \mathcal{O} \Omega_3^1 (B) \) is a proper subspace of \( \Omega_3 (B) \)). Therefore, for \( f \in \Omega_3^1 (B) \) we have
\[
\left( \mu (E)^{-\frac{1}{2}} 1_K f \right)(v_0) = f (e_0) + q f (e_1) + q^2 f (e_2),
\]
\[\mu(E)^{-\frac{1}{2}} \mathbb{1}_{\Omega_1^E(B)} = (1 - q - q^2) \] (with respect to the bases \(\mathfrak{B}^E\) and \(\{f^K\}\)). The edges of color 1 which terminate in \(v_0\) are
\[\sigma^2 v_0, v_0 = (\pi \pi^{-1}) e_0 = (1 \pi \pi) (1 2 3) e_0 = (1 \pi \pi) e_2,\]
and similarly \((\pi \pi \pi \pi^{-1}) e_1\) and \((\pi \pi \pi \pi^{-1}) e_0 (x, y \in \mathbb{F}_q)\). Thus,
\[\left(\frac{1}{\sqrt{\mu(E)}} \mathbb{1}_{K \sigma^2 E f}\right) (v_0) = z_2 q z_3 f(e_2) + q \cdot z_1 z_3 f(e_1) + q^2 \cdot z_1 z_2 f(e_0),\]
and in total (see (3.2), and recall that \(z_1 z_2 z_3 = 1\))
\[\left[\partial_1|_{\Omega_1^E(B)}\right]_{\mathfrak{B}^E} = \left(\frac{x}{x_3} - 1 \quad \frac{y}{y_3} - q \quad \frac{z}{z_3} - q^2 \right).\] (4.3)

As \(\Delta_0^+ = \partial_1 \delta_1\) and \(\Delta_0^- = \delta_1 \partial_1\), we can now compute explicitly their action on the \(3\)-principal series. Denoting \(3 = \sum_{i=1}^3 (z_i + z_i^{-1})\), we have by (4.2) and (4.3)
\[\Delta_0^+|_{\Omega_1^E(B)} = (\lambda K)^{def} = (k_0 - q\delta),\] \(\Delta_0^-|_{\Omega_1^E(B)} = \left(\begin{array}{c}
\frac{q^2 q_3 - \frac{y}{y_3} + 1}{q z_3 + \frac{y}{y_3} + 1} - q^2 z_3 + \frac{e_3}{e_2} - \frac{e_2}{e_1} - q^2 z_3 + \frac{e_2}{e_1} + q^2 - \frac{e_2}{e_1} \\
- q z_2 + q_2 - \frac{y}{y_3} + 1 - q z_2 + q_2 - \frac{y}{y_3} + q - \frac{y}{y_3} + q_2 - \frac{y}{y_3} + 1
\end{array}\right).\] (4.4)

Note that \(\Delta_0^+\) agrees with the computation of the Hecke operator spectrum in [Mac79, Li04, LSV05a], as \(\Delta_0^+ = k_0 \cdot I - \sum_{i=1}^{d-1} A_i\) (here \(A_i\) is the “\(i\)-color adjacency” operator on \(E_d\), loc. cit.). We also understand \(\Delta_0^-\) well enough, as it has the eigenvalue \(\lambda K\) corresponding to \(\delta_1 \Omega_1^E(B)\) (since \(\Delta_0^+ \delta_1 f^K = \delta_1 \Delta_0^- f^K = \lambda K \delta_1 f^K\), and two zeros (which come from \(\partial_2 \Omega_1^E(B)\))). However, we can now proceed to compute as easily the edge/triangle spectrum: for \(f \in \Omega_1^E(B)\),
\[(\delta_2 f)(t_0) = \sum_{i=0}^2 f(\sigma^i e_0) = f(e_0) + f(\left(\begin{array}{c}
1 \\
1 \\
1
\end{array}\right) (3 2 1) e_0) + f(\left(\begin{array}{c}
1 \\
1 \\
1
\end{array}\right) (1 2 3) e_0)
= f(e_0) + q z_3 f(e_1) + z_2 q z_3 f(e_2)\]
\[(\delta_2 f)(t_1) = \sum_{i=0}^2 f(\left(\begin{array}{c}
1 2 \\
1 2 \\
1 2
\end{array}\right) \sigma^i e_0) = f(e_0) + f(\left(\begin{array}{c}
1 \\
1 \\
1
\end{array}\right) (1 2 3) e_0) + f(\left(\begin{array}{c}
1 \\
1 \\
1
\end{array}\right) (3 2 1) e_0)
= f(e_0) + q z_3 f(e_2) + z_1 z_3 f(e_1)\]

show that \(\left[\delta_2|_{\Omega_1^E(B)}\right]_{\mathfrak{B}^T} = \left(\begin{array}{c}
1 \\
1 \\
1
\end{array}\frac{q z_3}{z_1 z_3} \frac{q z_2 z_3}{z_1 z_3}
\end{array}\right),\) where \(\mathfrak{B}^T\) is the ordered basis \(f_1^T, f_2^T, \ldots, f_n^T\) (and at this point one can verify directly that \(\delta_2 \partial_1|_{\Omega_1^E(B)} = 0\)).

The triangles containing \(e_0\) are obtained by adjoining \(\sigma^2 v_0\) (which gives \(t_0\)) and \(\left(\begin{array}{c}
\pi \\
\pi \\
1
\end{array}\right) v_0\) \((x \in \mathbb{F}_q)\), giving \(\left(\begin{array}{c}
\pi \\
\pi \\
1
\end{array}\right) t_1\). This gives \((\partial_2 f)(e_0) = f(t_0) + q f(t_1)\), but for \(e_1, e_2\) we need
to work a little harder, and use (4.1):

\[
(\partial_2 f)(e_1) = (\partial_2 f)((3 2 1) e_0) = f((3 2 1) t_0) + \sum_{x \in F_q^*} f \left( \begin{array}{c} 3 2 1 \\ 1 1 \end{array} \right) t_1 \\
= \frac{1}{z_3^2} f(t_0) + f((2 3) t_0) + \sum_{x \in F_q^*} f \left( \begin{array}{c} \frac{x}{z_1} \\ 1 \end{array} \right) \left( \begin{array}{c} 3 2 1 \\ 1 1 \end{array} \right) t_0 \\
= \frac{1}{z_3^2} f(t_0) + z_2 f(t_1) + \sum_{x \in F_q^*} f((3 2 1) t_0) = \frac{1}{z_3} f(t_0) + z_2 f(t_1)
\]

\[
d(\partial_2 f)(e_2) = (\partial_2 f)((1 2 3) e_0) = f((1 2 3) t_0) + \sum_{x \in F_q^*} f \left( \begin{array}{c} 1 2 3 \\ 1 1 \end{array} \right) t_1 \\
= \frac{z_1}{q^2} f(t_0) + \sum_{x \in F_q^*} f \left( \begin{array}{c} 1 2 3 \\ 1 1 \end{array} \right) (13) t_0 = \frac{z_1}{q} f(t_0) + \frac{1}{z_3} f(t_1),
\]

so that \([\partial_2^*|_{\Omega_3^2(B)}]_{\mathfrak{g},E} = \left( \begin{array}{c} 1/z_3 \\ z_2 \\ z_1/q \end{array} \right) \), giving

\[
\left[ \Delta^+_1 |_{\Omega_3^2(B)} \right]_{\mathfrak{g},E} = \left( \begin{array}{ccc} q + 1 & \frac{q}{z_1} + q z_3 & q^2 z_3 + \frac{q}{z_1} \\
\frac{z_2 + 1}{z_3} & z_2 + 1 & \frac{q}{z_1} + q z_2 \\
\frac{\frac{q}{z_1} + q z_3}{z_2} & \frac{q}{z_1} + q z_3 & q + 1 \end{array} \right)
\]

\[
\left[ \Delta^-_2 |_{\Omega_3^2(B)} \right]_{\mathfrak{g},E} = \left( \begin{array}{ccc} q + 2 & \frac{q}{z_1} + q z_2 + q \\
\frac{1}{z_2} + z_1 + 1 & 2q + 1 \end{array} \right).
\]

Recalling that \(\lambda^K = k_0 - q_\delta = 2 (q^2 + q + 1) - q \left( \sum z_i + \frac{1}{z_1} \right)\), we obtain

\[
\text{Spec} \Delta^+_1 |_{\Omega_3^2(B)} = \{ \lambda^+_E\} \triangleleft \left\{ 0, \frac{3}{2} \left( q + 1 \right) \pm \frac{1}{2} \sqrt{(q + 1)^2 + 4q (2 + \delta)} \right\}
\]

\[
\text{Spec} \Delta^-_1 |_{\Omega_3^2(B)} = \{ \lambda^-_E\} \triangleleft \left\{ 0, \frac{3}{2} \left( q + 1 \right) \pm \sqrt{(\frac{3}{2} \left( q + 1 \right))^2 - \lambda^K} \right\},
\]

and again \(\text{Spec} \Delta^-_2 |_{\Omega_3^2(B)} = \{ \lambda^-_E\} \) as we have argued for \(\Delta^-_1\). For \(\Delta^+_1\), \(\lambda^+_E = 0\) is obtained on \(\delta_1 f^K\) (whose \(f^K\) coefficients were computed in (4.2)), and \(\lambda^-_E\) are obtained on

\[
f^E_{\pm} = \begin{pmatrix} 2 \left( z_3^2 + \frac{1}{z_1} \right) q^2 - 2 (z_3 + 1) q \\
1 - q z_1 + q \left( z_1 + \frac{z_2}{z_2} - \frac{1}{z_2} \right) \pm (q z_1 - 1) \sqrt{9 k_1^2 - 4 \lambda^K} \\
q z_1 \left( z_2^{-1} + 2 z_1 + 1 \right) - 2 \frac{z_1}{z_2} - z_1 - \frac{1}{z_2} \pm \left( -z_1 + \frac{z_2}{z_2} \right) \sqrt{9 k_1^2 - 4 \lambda^K} \end{pmatrix}^T \begin{pmatrix} f^E_{(1)} \\
f^E_{(1 2 3)} \end{pmatrix} \]

\[
= 2 q \left( 1 + z_2 + \frac{1}{z_1} \right) \partial_2 f^E_{(1)} + \left( q - 1 \pm \sqrt{9 k_1^2 - 4 \lambda^K} \right) \partial_2 f^E_{(1 2 3)}.
\]

5. Unitary Iwahori-spherical representations

In general, an irreducible Iwahori-spherical representation is only a subrepresentation of \(\mathfrak{V}_\delta\). Let us denote by \(W_\delta\) this subrepresentation (there is only one such). Tadic [Tad86] has classified the Satake triples \(\mathfrak{V}_\delta\) for which the representation \(W_\delta\) admits a unitary structure. In [KLW10] the possible \(\mathfrak{V}_\delta\) for \(\text{PGL}_3(F)\) are listed, and a basis for \(W_\delta \leq \mathfrak{V}_\delta\) is computed.
Apart from these there is the Steinberg color of edges, but not the color of vertices.)

- \( |z_i| = 1 \) for \( i = 1, 2, 3 \). In this case \( V_j \) is irreducible, hence \( W_j = V_j \), and \( z \in [−3, 6] \) gives \( \lambda^K ∈ [k_0 − 6q, k_0 + 3q] \) and \( \lambda^E_± ∈ I^\pm_\pm \) (see (4.5) and (2.4)).

- \( z = (c^{-2}, cq^a, cq^{-a}) \) for some \(|c| = 1\) and \( 0 < a < \frac{1}{2} \). Here too \( V_j \) is irreducible.

- \( z = \left(\frac{c \sqrt{q}}{\sqrt{q}}, c \sqrt{q}, c^{-2}\right) \) for some \(|c| = 1\). In this case \( W_j^E \) is one-dimensional, and spanned by \( f_E^{±} \), which is proportional to \( qf_E^{(321)} − f_E^{(123)} \). It corresponds to

\[
\lambda^E = \frac{1}{2} \left( 3k_1 - \sqrt{k_1^2 + 8q + 4q \left( \frac{c}{\sqrt{q}} + c \sqrt{q} + c + c^{-2} + c^2 \right)} \right) = \frac{1}{2} \left( 3k_1 - \sqrt{q^2 + 8q \sqrt{q} (c) + 2q + 16qR(c)^2 + 1 + 8 \sqrt{q} R(c)} \right) = \frac{1}{2} \left( 3k_1 - (q + 4 \sqrt{q} R(c) + 1)) = k_1 - 2 \sqrt{q} R(c) \right)
\]

which lies in \([k_1 - 2 \sqrt{q}, k_1 + 2 \sqrt{q}]\). As \( f_E \) is not \( K \)-fixed, \( W_j^K = 0 \).

- \( z = (q, 1, \frac{1}{2}) \). In this case \( W_j \) is the trivial representation \( ρ : G → C^X \), and \( W_j^E = W_j^K \) are spanned by \( f^K = f_E \). Since \( f^K \) is constant and \( f_E \) is a “disorientation” (see §2) we get \( \lambda^K = 0 \) and \( \lambda^E = 3k_1 \) (this can also be verified directly using (4.4) and (4.5)).

- \( z = (ωq, ω, 1) \) where \( ω = e^{\frac{2πi}{3}} \) or \( ω = e^{-\frac{2πi}{3}} \). Here \( W_j \) is the one-dimensional representation \( ρ(γ) = ω^{\text{col}(γ)} \), and \( W_j^K = W_j^E = \langle f^K \rangle = \langle f_E \rangle \), giving \( \lambda^K = \frac{3k_0}{2} \).

Apart from these there is the Steinberg (Stn) representation \( z = (\frac{1}{2}, 1, \frac{1}{2}) \). It is not \( E \)-spherical, and \( W_j^T \) is spanned by \( f_0^T = qf_1^T − f_1^T \), which is always in \( \ker ξ = \ker ξ_2 \).

(In [KLM10] the twisted Steinberg representations \( z = (\varphi, ω, ωq) \) are considered as well, but these do not contribute to the simplicial theory as they have no \( K \), \( E \) or \( T \)-invariant vectors.)

Let \( X = \Gamma \backslash B \) be a non-tripartite Ramanujan complex with \( L^2(\Gamma \backslash G) ≅ \bigoplus \mathbb{C} W_j \), and denote by \( N_{(t)} \) the number of \( W_j \) of type \( (t) \). These are computed in [KLM10] for the tripartite case, and our arguments are very similar. By the Ramanujan assumption every Iwahori-spherical \( W_j \) is either tempered, which are the types (a), (c), and (Stn), or finite-dimensional (types (e), (f)), so that \( N_{(b)} = N_{(d)} = 0 \). The trivial representation (e) always appear once in \( L^2(\Gamma \backslash G) \) as the constant functions, so that \( N_{(c)} = 1 \).

The color representations (type (e)) correspond to \( f ∈ L^2(\Gamma \backslash G) \) satisfying \( f(Γg) = (g f)(Γ) = ω^{\text{col}(\varphi)} f(Γ) \), which is unique up to a scalar, and well defined iff \( \text{col} Γ ≡ 0 \). As we assume

\[ \text{(1)} \]
X to be non-tripartite, \( N(f) = 0 \) (and otherwise we would have had \( N(f) = 2 \)). Next,
\[
n = \dim \Omega^0 (X) = \sum_i \dim W_{ji}^K = N(a) + N(c) + N(f)
\]
\[
nk_0 / 2 = \dim \Omega^1 (X) = \sum_i \dim W_{ji}^E = 3N(a) + N(c) + N(e) + N(f)
\]
give \( N(a) = n - 1 \) and \( N(c) = n (q^2 + q - 2) + 2 \). This is summarized in Table 5.1, together with the tripartite case, and this also completes the proof of Theorem 4.

For completeness, we list in Table 5.1 what is \( W^T \) for each type. This gives \( \frac{n k_0 k_1}{6} = \dim \Omega^2 (X) = 2N(a) + N(c) + N(e) + N(Stn) \), from which follows that \( N(Stn) = \sum_{i=-1}^2 |X_i| = -\chi (X) \), the reduced Euler characteristic of \( X \) (as is shown in [KRW10, Prop. 1(1)] for the tripartite case).

| Type     | \( W^K \) | \( \Delta^+_0 \) e.v. | \( W^E \) | \( \Delta^+_1 \) e.v. | \( W^T \) | \( \text{mult. col } \Gamma \equiv 0 \) | \( \text{mult. col } \Gamma \not\equiv 0 \) |
|----------|-----------|----------------|---------|----------------|---------|----------------|----------------|
| (a)      | tempered  | \( f^K \)     | \( k_0 + q_3 \) | \( f^E \), \( f_0^E \) | \( \delta_1 f^E \) | \( 0, 3\frac{k_1}{2} \pm \sqrt{\frac{9k_1^2}{4} - \lambda K} \) | \( n - 3 \) | \( n - 1 \) |
| (b)      | tempered  | \( f^K \)     | \( k_0 + q_3 \) | \( f^E \), \( f_0^E \) | \( \delta_1 f_0^E \) | \( 0, 3\frac{k_1}{2} \pm \sqrt{\frac{9k_1^2}{4} - \lambda K} \) | \( 0 \) | \( 0 \) |
| (c)      | tempered  | \( 0 \)       | -        | \( f^E \)           | \( \delta_1 f^E \) | \( k_1 - 2\sqrt{\mu K} \) (c) | \( \delta_1 f_0^E \) | \( nq^2 + nq \) | \( nq^2 + nq \) |
| (d)      | tempered  | \( f^K \)     | \( k_0 + q_3 \) | \( f^E \), \( f_0^E \) | \( \delta_1 f_0^E \) | \( 0, 2k_1 + 2\sqrt{\mu K} \) (c) | \( \delta_1 f_0^E \) | \( 0 \) | \( 0 \) |
| (e)      | trivial   | \( f^K \)     | \( 0 \)   | \( f^E \)           | \( \delta_1 f^E \) | \( 0 \) | \( 2k_1 \) | \( \delta_1 f^E \) | \( 1 \) | \( 1 \) |
| (f)      | fin. dim. | \( f^K \)     | \( 3\frac{k_0}{2} \) | \( f^E \) | \( \delta_1 f_0^E \) | \( 0 \) | \( 3k_1 \) | \( \delta_1 f_0^E \) | \( -\chi (X) \) | \( -\chi (X) \) |
| (Stn)    | tempered  | \( 0 \)       | -        | \( f^E \)           | \( \delta_1 f_0^E \) | \( 0 \) | \( -\chi (X) \) | \( \delta_1 f_0^E \) | \( -\chi (X) \) | \( -\chi (X) \) |

Table 5.1. The representation appearing in \( L^2 (\Gamma \setminus G) \) for a Ramanujan complex \( X = \Gamma \setminus B \), with the corresponding Laplacian eigenvalues, and the multiplicity of appearance in the tripartite and in the non-tripartite cases.

6. From spectral to combinatorial expansion

We continue focusing on the non-tripartite Ramanujan case. By the results of the previous section, the nontrivial vertex spectrum (\( \text{Spec } \Delta^+_0 \mid _{Z_0} \)) is highly concentrated, lying in a \( k_0 \pm O (\sqrt{k_0}) \) strip. The nontrivial edge spectrum is “almost concentrated”: there are \( \approx nq^2 \) eigenvalues in a \( k_1 \pm O (\sqrt{k_1}) \) strip, but also \( n - 1 \) eigenvalues at \( 2k_1 \pm O (1) \) (and the eigenvalue \( 3k_1 \)). Nevertheless, having a concentrated vertex spectrum, and edge spectrum bounded away from zero is enough to prove the Cheeger-type inequality stated in Theorem 2. This can be rephrased it in terms of a Cheeger-type constant: for a fixed constant \( 0 \leq \theta \leq \frac{1}{4} \), define
\[
h_\theta (X) \overset{\text{def}}{=} \min_{\mathcal{V} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}} \frac{|T (A, B, C)| n^2}{|A| |B| |C|}
\]
(remark (3) below explains why sublinear sets must be avoided). Theorem 2 then implies that
\[
h_\theta (X) \geq \lambda_1 \left( k_0 - \mu_0 \left( 1 + \frac{10}{9\delta^2} \right) \right).
\]
(6.2)

For the Ramanujan triangle complexes, Theorem 4 gives \( \mu_0 = 6q \) and \( \lambda_1 = q + 1 - 2\sqrt{q} \), and therefore we have the following:
Corollary 6. Fix \( \vartheta > 0 \), and let \( X \) be a non-tripartite Ramanujan triangle complex with \( n \) vertices, vertex degree \( k_0 = 2 (q^2 + q + 1) \) and edge degree \( k_1 = q + 1 \). For any partition of the vertices of \( X \) into sets \( A, B, C \) of sizes at least \( \vartheta n \),

\[
\frac{|T(A, B, C)| n^2}{|A| |B| |C|} \geq (q + 1 - 2\sqrt{q}) \left( 2q^2 + 2q + 2 - 6q \left( 1 + \frac{10}{9\vartheta^2} \right) \right).
\]

In particular,

\[
h_{\vartheta} (X) \geq 2q^3 - O_\vartheta (q^{2.5}). \tag{6.3}
\]

Let us make a few remarks:

1. The inequality (6.3) corresponds to the pseudorandom intuition of expansion: \( X \) has \( \frac{1}{3} nk_0 k_1 \) triangles, so its triangle density is indeed \( \frac{1}{3} n \left( \frac{q^2 + q + 1}{n} \right) (q + 1) \approx \frac{2q^3}{n^2} \).

2. If a triangle complex \( X \) has a complete skeleton, \( X^1 = \binom{V}{2} \), then \( \text{Spec} \Delta_+ | Z_0 = \{ n \} \), so that \( k_0 = n \) and \( \mu_0 = 0 \). Therefore, Theorem 2 reads \( \frac{|T(A, B, C)| n^2}{|A| |B| |C|} \geq \lambda_1 \cdot n \), which is precisely the Cheeger inequality which appears in [PRT12] for complexes with a complete skeleton (which includes the graph case as well).

3. For any infinite family of complexes with bounded edge degree (and thus noncomplete skeleton), one cannot go over all partitions: for example, taking \( A = \{ v \} \) and \( B = \{ u \} \) for some \( vu \notin X^1 \) gives immediately \( T(A, B, V \setminus \{ v, u \}) = \emptyset \). In fact, we must have \( |A|, |B|, |C| \) bounded linearly in \( n \), for the following reason: Assume \( X_i \) is a sequence of triangle complexes with \( n_i = |X^0_i| \to \infty \) and with globally bounded vertex degrees. If \( f(n) \) is any sub-linear function, one can take \( A \subseteq X^0_i \) to be any set of size \( f(n_i) \), \( B \) to be \( \partial A = \{ v \mid \text{dist}(v, A) = 1 \} \) (if \( |B| < f(n_i) \) enlarge it by adding any vertices), and \( C \) the rest of the vertices. Assuming \( i \) is large enough one has \( |A|, |B|, |C| \geq f(n_i) \), and \( T(A, B, C) = \emptyset \) since all triangles with a vertex in \( A \) have their other vertices in either \( A \) or \( B \).

4. Another Cheeger constant for complexes with non-complete skeleton was suggested in [PRT12], and related to the spectrum in [GS14]:

\[
\bar{h}(X) = \min_{V = A \sqcup B \sqcup C} \frac{|T(A, B, C)| \cdot n}{|T^\vartheta (A, B, C)|},
\]

where \( T^\vartheta (A, B, C) \) is the set of triangle-boundaries with one vertex in each of \( A, B \) and \( C \). This constant, however, is not interesting for Ramanujan complexes: these are clique complexes, so that \( \bar{h}(X) = n \) automatically.

We now proceed to the proof:

Proof of Theorem 2. Denote \( |A|, |B|, |C| \) by \( a, b, c \), respectively, and define \( f \in \Omega^3 \) by

\[
f(vw) = \begin{cases} c & v \in A, w \in B & -c & w \in A, v \in B \\ a & v \in B, w \in C & -a & w \in B, v \in C \\ b & v \in C, w \in A & -b & w \in C, v \in A \\ 0 & \text{else.} \end{cases}
\]
Let $f_B = \mathbb{P}_B f$ and $f_Z = \mathbb{P}_Z f$. Then
\[
|T (A, B, C)| n^2 = \sum_{t \in T} (\delta f)^2 (t) = \|\delta f\|^2 = \|\delta f_Z\|^2
\]
\[
= \langle \Delta^+ f_Z, f_Z \rangle \geq \lambda_1 \|f_Z\|^2 = \lambda_1 \left(\|f\|^2 - \|f_B\|^2\right).
\]
Let us denote $E = k_0 \mathbb{P}_B - \Delta \Gamma$. Any linear maps $T : V \to W$ and $S : W \to V$ satisfy
\[
\text{Spec} \, T S \{\emptyset\} = \text{Spec} \, S T \{\emptyset\},
\]
and thus
\[
\text{Spec} \, \Delta \Gamma |_{B^1} = \text{Spec} \, \Delta \Gamma \{\emptyset\} = \text{Spec} \, \Delta_0^+ \{\emptyset\} = \text{Spec} \, \Delta_0^+ |_{B_0},
\]
\[
\subseteq \text{Spec} \, \Delta_0^+ |_{Z_0} \subseteq [k_0 - \mu_0, k_0 + \mu_0].
\]
Together with $\Delta \Gamma |_{Z_1} = 0$ this implies $\|E\| \leq \mu_0$, so that
\[
\|f_B\|^2 = \langle \mathbb{P}_B f, \mathbb{P}_B f \rangle = \langle \mathbb{P}_B f, f \rangle \leq \frac{\|E f, f\| + \|\Delta \Gamma f, f\|}{k_0} \leq \frac{\mu_0 \|f\|^2 + \|\partial f\|^2}{k_0}.
\]
Using the expander mixing lemma\(^{(1)}\) for $E (A, B), E (B, C)$ and $E (C, A)$ we have
\[
\|f\|^2 = |E (A, B)| c^2 + |E (B, C)| a^2 + |E (C, A)| b^2
\]
\[
\geq \left(\frac{k_0}{n} ab - \mu_0 \sqrt{ab}\right) c^2 + \left(\frac{k_0}{n} bc - \mu_0 \sqrt{bc}\right) a^2 + \left(\frac{k_0}{n} ca - \mu_0 \sqrt{ca}\right) b^2
\]
\[
= k_0 abc - \mu_0 \left[\sqrt{abc}^2 + \sqrt{bca}^2 + \sqrt{acb}^2\right] \geq k_0 abc - \frac{\mu_0 n^3}{9},
\]
and we are left with the task of bounding $\|\partial f\|^2$. Let us begin with
\[
\sum_{\alpha \in A} (\partial f)^2 (\alpha) = \sum_{\alpha \in A} \left(c \sum_{\beta \in B} \delta_{\alpha \beta} - b \sum_{\gamma \in C} \delta_{\alpha \gamma}\right)^2
\]
\[
= c^2 \sum_{\beta, \beta' \in B} \sum_{\alpha \in A} \delta_{\alpha \beta} \delta_{\alpha \beta'} - 2bc \sum_{\alpha \beta \gamma} \delta_{\alpha \beta} \delta_{\alpha \gamma} + b^2 \sum_{\alpha \gamma \gamma'} \delta_{\alpha \gamma} \delta_{\alpha \gamma'}
\]
\[
= c^2 |P (B, A, B)| - 2bc |P (B, A, C)| + b^2 |P (C, A, C)|,
\]
where $P (S, T, R)$ is the sets of paths going from $S$ to $R$ through $T$. By [Par13, Lem. 1.3] with $\ell = 2$ and $j = 0$, we have (recall that $k_{-1} = n$ and $\varepsilon_{-1} = 0$):
\[
|P (B, A, B)| - \left(\frac{k_0}{n}\right)^2 b^2 a \leq 2k_0 \mu_0 b
\]
\[
|P (B, A, C)| - \left(\frac{k_0}{n}\right)^2 bac \leq 2k_0 \mu_0 \sqrt{bc}
\]
\[
|P (C, A, C)| - \left(\frac{k_0}{n}\right)^2 c^2 a \leq 2k_0 \mu_0 c.
\]
Therefore,
\[
\sum_{\alpha \in A} (\partial f)^2 (\alpha) \leq 2k_0 \mu_0 \left[c^2 b + 2 (bc)^{3/2} + b^2 c\right] = 2k_0 \mu_0 \sqrt{bc} + \sqrt{c}\right)^2,
\]
\(^{(1)}\)See e.g. [Par13] for a statement which does not assume regularity.
and repeating this for $\sum_{\beta \in B}$ and $\sum_{\gamma \in C}$ gives
\[
\|\delta f\|^2 \leq 2k_0\mu_0 \left[ b c \left( \sqrt{b} + \sqrt{c} \right)^2 + a c \left( \sqrt{a} + \sqrt{c} \right)^2 + a b \left( \sqrt{a} + \sqrt{b} \right)^2 \right] \leq k_0\mu_0 n^3.
\]
Combining everything now gives
\[
\frac{|T(A, B, C)| n^2}{abc} \geq \frac{\lambda_1}{abc} \left( \|f\|^2 - \|f_B\|^2 \right) \geq \frac{\lambda_1}{abc} \left( \frac{\|f\|^2}{2} \left( 1 - \frac{\mu_0}{k_0} \right) - \mu_0 n^3 \right) \geq \lambda_1 \left( k_0 - \mu_0 \left( 1 + \frac{10\mu_0 n^3}{9\mu_0 abc} \right) \right).
\]
This can be generalized to higher dimension. For a partition $A_0, \ldots, A_d$ of the vertices of a $d$-complex, let $F(A_0, \ldots, A_d)$ be the set of $d$-cells with one vertex in each $A_i$. We then have:

**Theorem 7.** Let $X$ be a $d$-complex on $n$ vertices, with $\text{Spec} \Delta^+|_{Z_i} \subseteq [k_i - \mu_i, k_i + \mu_i] \geq 0$. Then for any partition $V = \prod_{i=0}^{d} A_{i}$
\[
\frac{|F(A_0, \ldots, A_d)| n^d}{|A_0| \cdots |A_d|} \geq k_0 \cdot \ldots \cdot k_{d-2} \cdot \lambda_{d-1} \left( 1 - \frac{\mu_{d-2}}{k_{d-2}} - C_d \left( \frac{\mu_0}{k_0} + \ldots + \frac{\mu_{d-2}}{k_{d-2}} \right) \right),
\]
where $C_d$ depends only on $d$.

Thus, if we define
\[
h_{\theta}(X) = \min_{V(\theta) \in A_{\theta}} \frac{|F(A_0, \ldots, A_d)| n^d}{|A_0| \cdots |A_d|}
\]
(where $0 \leq \theta \leq \frac{1}{2\pi}$), then
\[
h_{\theta}(X) \geq k_0 \cdot \ldots \cdot k_{d-2} \cdot \lambda_{d-1} \left( 1 - \frac{\mu_{d-2}}{k_{d-2}} - C_d \left( \frac{\mu_0}{k_0} + \ldots + \frac{\mu_{d-2}}{k_{d-2}} \right) \right).
\]
For a complex with a complete skeleton, this reduces again to [PRT12, Thm. 1.2], as $k_1 = n$ and $\mu_i = 0$ for $0 \leq i \leq d - 2$.

**Proof.** For $f \in \Omega^{d-1}(X)$ defined as
\[
f([\sigma_0 \sigma_1 \ldots \sigma_{d-1}]) = \begin{cases} 
\text{sgn} \pi |A_{\pi(d)}| & \exists \pi \in \text{Sym}(0, d) \text{ with } \sigma_i \in A_{\pi(i)} \text{ for } 0 \leq i \leq d - 1 \\
0 & \text{else, i.e. } \exists k, i \neq j \text{ with } \sigma_i, \sigma_j \in A_k,
\end{cases}
\]
it is shown in [PRT12, §4.1] that $\|\delta f\|^2 = |F(A_0, \ldots, A_d)| n^2$ (this part does not use the complete skeleton assumption.) For $f_B = \mathbb{P}_{B^{d-1}} f$, this gives again $|F(A_0, \ldots, A_d)| n^2 \geq \lambda_{d-1} \left( \|f\|^2 - \|f_B\|^2 \right)$. Denoting $\mathcal{K} = k_0 \cdot \ldots \cdot k_{d-2}$ and $\mathcal{E} = \frac{\mu_0}{k_0} + \ldots + \frac{\mu_{d-2}}{k_{d-2}}$, we have by [Par13, Thm. 1.1]
\[
\|f\|^2 \geq \sum_{i=0}^{d} |F(A_0, \ldots, \widehat{A_i}, \ldots, A_d)| |A_i| \geq \sum_{i=0}^{d} \left[ \frac{K}{n^{d-1} \prod_{j \neq i} |A_j| - c_{d-1} K \mathcal{E} \max_{j \neq i} |A_j|} \right] |A_i|^2 \geq \frac{K}{n^{d-2} \prod_{i=0}^{d} |A_i|} \left[ (d + 1) c_{d-1} K \mathcal{E} n^3 \right] \geq K \left( n^{2-d} \prod_{i=0}^{d} |A_i| - (d + 1) c_{d-1} \mathcal{E} n^3 \right).
\]
Turning to $f_B$, we have $\|f_B\|^2 \leq \frac{n^{d-2}}{k_{d-2}} \|f\|^2 + \frac{1}{k_{d-2}} \|\partial f\|^2$ in a similar way to the triangle case, and we note that $\partial f$ is supported on $(d - 2)$-cells with vertices in distinct blocks of the partition $\{A_i\}$. For a sequence of sets $B_0, \ldots, B_\ell$, we denote by $F^j (B_0, \ldots, B_\ell)$ the set of $j$-galleries in $B_0, \ldots, B_\ell$, namely, sequences of $j$-cells $\tau_i \in F (B_i, \ldots, B_{i+j})$ such that $\tau_i$ and $\tau_{i+1}$ intersect in a $(j - 1)$-cell. Arguing similarly to (6.4),

$$\|\partial f\|^2 = \sum_{0 \leq i < j \leq d} \sum_{\tau \in F (A_0, \ldots, A_i, \ldots, A_d)} (\partial f)(\tau)^2$$

$$= \sum_{0 \leq i < j \leq d} \left| A_j \right|^2 F^{d-1} (A_i, A_0, \ldots, \hat{A}_i, A_i, \ldots, A_d, A_j) - 2 |A_i| |A_j| F^{d-1} (A_i, A_0, \ldots, \hat{A}_i, A_i, \ldots, A_d, A_j) + |A_i|^2 F^{d-1} (A_j, A_0, \ldots, \hat{A}_i, A_i, \ldots, A_d, A_j). \tag{6.5}$$

Proposition 3.1 in [Par13] estimates of the number of $j$-galleries in $B_0, \ldots, B_\ell$ when each $j + 1$ tuple $B_i, B_{i+1}, \ldots, B_{i+j+1}$ consists of disjoint sets, giving

$$F^{d-1} (A_i, A_0, \ldots, \hat{A}_i, \ldots, A_d, A_j) - \frac{K k_{d-2}}{n^{d-2}} \left| A_i \right| \prod_{k \neq j} \left| A_k \right| \leq c_{d-2, d} K \mathcal{E} k_{d-2} \max_{k \neq j} \left| A_k \right|,$$

and similarly for the other terms in (6.5). Again, the main terms cancel out and the error terms result in $\|\partial f\|^2 \leq 4 \left( \frac{d+1}{2} \right) c_{d-2, d} K \mathcal{E} k_{d-2} n^3$. In total,

$$\left| \frac{F (A_0, \ldots, A_d)}{|A_0| \ldots |A_d|} \right| n^d \geq \frac{\lambda_{d-1} \nu^{d-2}}{|A_0| \ldots |A_d|} \left( 1 - \frac{\mu_{d-2}}{k_{d-2}} \right) \left( \frac{1}{k_{d-2}} \|f\|^2 - \frac{1}{k_{d-2}} \|\partial f\|^2 \right)$$

$$\geq \mathcal{K} \lambda_{d-1} \left( \left( \frac{1}{1 - \frac{\mu_{d-2}}{k_{d-2}}} \right) \left( \frac{\nu^{d-2}}{2 \prod |A_i|} \right) - 4 \left( \frac{d+1}{2} \right) c_{d-2, d} \mathcal{E} \frac{\nu^{d+1}}{\prod |A_i|} \right)$$

$$\geq \mathcal{K} \lambda_{d-1} \left( \left( \frac{1}{1 - \frac{\mu_{d-2}}{k_{d-2}}} \right) \left( \frac{\nu^{d-2}}{2 \prod |A_i|} \right) - 4 \left( \frac{d+1}{2} \right) c_{d-2, d} \mathcal{E} \frac{\nu^{d+1}}{\prod |A_i|} \right)$$

and the theorem follows.

References

[Bal00] C.M. Ballantine, *Ramanujan type buildings*, Canadian Journal of Mathematics 52 (2000), no. 6, 1121–1148.

[Bor76] A. Borel, *Admissible representations of a semi-simple group over a local field with vectors fixed under an Iwahori subgroup*, Inventiones mathematicae 35 (1976), no. 1, 233–259.

[Cas74] W. Casselman, *On a p-adic vanishing theorem of Garland*, Bulletin of the American Mathematical Society 80 (1974), no. 5, 1001–1004.

[Cas80] , *The unramified principal series of p-adic groups. I. The spherical function*, Compositio Mathematica 40 (1980), no. 3, 387–406.

[CSŽ03] D.I. Cartwright, P. Solé, and A. Zuk, *Ramanujan geometries of type $\tilde{A}_n$*, Discrete Mathematics 269 (2003), no. 1, 35–43.

[Eck44] B. Eckmann, *Harmonische funktionen und randwertaufgaben in einem komplex*, Commentarii Mathematici Helvetici 17 (1944), no. 1, 240–255.
[FLW13] Y. Fang, W.C.W. Li, and C.J. Wang, *The zeta functions of complexes from Sp(4)*, International Mathematics Research Notices 2013 (2013), no. 4, 886–923.

[Gar73] H. Garland, *p-adic curvature and the cohomology of discrete subgroups of p-adic groups*, The Annals of Mathematics 97 (1973), no. 3, 375–423.

[GS14] A. Gundert and M. Szell, *Higher dimensional Cheeger inequalities*, Annual Symposium on Computational Geometry (New York, NY, USA), SOCG’14, ACM, 2014, pp. 181:181–181:188.

[GW12] A. Gundert and U. Wagner, *On Laplacians of random complexes*, Proceedings of the 2012 symposium on Computational Geometry, SOCG’12, ACM, 2012, pp. 151–160.

[GŽ99] R.I. Grigorchuk and A. Žuk, *On the asymptotic spectrum of random walks on infinite families of graphs*, Random walks and discrete potential theory (Cortona, 1997), Sympos. Math 39 (1999), 188–204.

[HLW06] S. Hoory, N. Linial, and A. Wigderson, *Expander graphs and their applications*, Bulletin of the American Mathematical Society 43 (2006), no. 4, 439–562.

[Kes59] H. Kesten, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. 92 (1959), 336–354. MR 0109367 (22 #253)

[KL14] M.H. Kang and W.C.W. Li, *Zeta functions of complexes arising from PGL (3)*, Advances in Mathematics 256 (2014), 46–103.

[KLW10] M.H. Kang, W.C.W. Li, and C.J. Wang, *The zeta functions of complexes from PGL(3): a representation-theoretic approach*, Israel Journal of Mathematics 177 (2010), no. 1, 335–348.

[Laf02] L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Inventiones mathematicae 147 (2002), no. 1, 1–241.

[Li01] W.C.W. Li, *On negative eigenvalues of regular graphs*, Comptes Rendus de l’Académie des Sciences-Series I-Mathematics 333 (2001), no. 10, 907–912.

[Li04] [La02], *Ramanujan hypergraphs*, Geometric and Functional Analysis 14 (2004), no. 2, 380–399.

[LM07] A. Lubotzky and R. Meshulam, *A Moore bound for simplicial complexes*, Bulletin of the London Mathematical Society 39 (2007), no. 3, 353–358.

[LPS88] A. Lubotzky, R. Phillips, and P. Sarnak, *Ramanujan graphs*, Combinatorica 8 (1988), no. 3, 261–277.

[LSV05a] A. Lubotzky, B. Samuels, and U. Vishne, *Ramanujan complexes of type ˜A_d*, Israel Journal of Mathematics 149 (2005), no. 1, 267–299.

[LSV05b] [La05b], *Explicit constructions of Ramanujan complexes of type ˜A_d*, Eur. J. Comb. 26 (2005), no. 6, 965–993.

[Lub94] A. Lubotzky, *Discrete groups, expanding graphs and invariant measures*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 1994, With an appendix by Jonathan D. Rogawski.

[Lub12] [La12], *Expander graphs in pure and applied mathematics*, Bull. Amer. Math. Soc 49 (2012), 113–162.

[Lub14] [La14], *Ramanujan complexes and high dimensional expanders*, Takagi lectures 11 (2014), 29–58.

[Mac79] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Clarendon Press Oxford, 1979.

[Mar88] G.A. Margulis, *Explicit group-theoretical constructions of combinatorial schemes and their application to the design of expanders and concentrators*, Problemy Peredachi Informatsii 24 (1988), no. 1, 51–60.

[Mor94] M. Morgenstern, *Existence and explicit constructions of q+1 regular Ramanujan graphs for every prime power q*, Journal of Combinatorial Theory, Series B 62 (1994), no. 1, 44–62.

[MSS13] A. Marcus, D.A. Spielman, and N. Srivastava, *Interlacing families I: Bipartite Ramanujan graphs of all degrees*, arXiv preprint arXiv:1304.4132 (2013).
[Nil91] A. Nilli, *On the second eigenvalue of a graph*, Discrete Mathematics 91 (1991), no. 2, 207–210.

[Pap08] M. Papikian, *On eigenvalues of p-adic curvature*, manuscripta mathematica 127 (2008), no. 3, 397–410.

[Par13] O. Parzanchevski, *Mixing in high-dimensional expanders*, arXiv preprint arXiv:1310.6477 (2013).

[PR12] O. Parzanchevski and R. Rosenthal, *Simplicial complexes: spectrum, homology and random walks*, arXiv preprint arXiv:1211.6775 (2012).

[PRT12] O. Parzanchevski, R. Rosenthal, and R.J. Tessler, *Isoperimetric inequalities in simplicial complexes*, Combinatorica, To appear (2012), arXiv:1207.0638.

[Sar90] P. Sarnak, *Some applications of modular forms*, vol. 99, Cambridge University Press, 1990.

[Sar07] A. Sarveniazi, *Explicit construction of a Ramanujan (n_1, n_2, ..., n_d−1)-regular hypergraph*, Duke Mathematical Journal 139 (2007), no. 1, 141–171.

[Tad86] M. Tadic, *Classification of unitary representations in irreducible representations of general linear group (non-archimedean case)*, Annales scientifiques de l’Ecole normale supérieure 19 (1986), no. 3, 335–382.

[Tao11] T. Tao, *Basic theory of expander graphs*, http://terrytao.wordpress.com/2011/12/02/245b-notes-1-basic-theory-of-expander-graphs/, 2011.

[Zel80] A.V. Zelevinsky, *Induced representations of reductive p-adic groups II. On irreducible representations of GL(n)*, Annales Scientifiques de l’Ecole Normale Supérieure 13 (1980), no. 2, 165–210.