Einstein’s equation is incompatible with a Galilean limit in non-Euclidean spatial classes of topologies

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We show that the Galilean limit of Lorentzian structures applied to a solution of the Einstein equation only exists if the topological class (i.e. Thurston’s geometric structure) of the spatial sections of the spacetime 4-manifold is Euclidean. We argue that this result might be a signature of an incompleteness of the Einstein equation, which should feature an additional “topological term” linked to the spatial topological class. This term would imply that the limit exists in any topological class. However, the term is only known at the leading order of the Galilean limit, and remains unconstrained at full order. We propose strategies to determine this topological term. If indeed such a term is a missing feature of general relativity, the Einstein equation in its current form would be physically incorrect if the Universe has a non-Euclidean spatial topology.

I. MOTIVATIONS

Modern cosmology studies the dynamics of objects inside the Universe and the dynamics of the Universe itself. Such a description was made possible with the born of general relativity where the Universe is described by a 4-manifold \( \mathcal{M} \) equipped with a Lorentzian structure \((g, \nabla)\). As spacetime acquires a dynamics, through the Lorentzian structure, expansion of the intrinsic volume of space is now possible. Another property of the Universe that general relativity allows us to study is the topology, which is a property of the 4-manifold \( \mathcal{M} \). In certain conditions, the knowledge of the spatial curvature induced by the Lorentzian metric allows us to draw conclusions on this property. This is the case for the \( \Lambda \)CDM model, where the topology of the spatial sections is either Euclidean, spherical or hyperbolic (see section II A for a precise definition of these terms) as function of the (purely) trace of the spatial Ricci tensor.

These two properties of the Universe, i.e. expansion and topology, where thought until recently to be described only by general relativity. Newton’s theory were only able to describe expansion in an effective way with the usual introduction of a Hubble flow [e.g. 2, 13]. In this description, expansion corresponds to the expansion of a fluid in an absolute space, which contrasts with expansion in general relativity corresponding to the intrinsic spatial volume expansion. In Vigneron [17], we showed that by using Galilean structures within the Newton-Cartan formalism in its most general form, we where able to retrieve classical Newton’s equations in the presence of expansion. However, in that case this phenomenon corresponds to the intrinsic expansion of space, as in general relativity. Therefore the result of Vigneron [17] suggests that defining Newton’s theory with Galilean structures seems to be the most fundamental way of defining that theory, but also that general relativity has not anymore the monopole of the description of expansion as a fundamental field in the Universe.

While the Newton-Cartan equations impose the spatial Ricci tensor to be zero (as expected in Newton’s theory) hence imposing a Euclidean topology, the definition of the Galilean structure alone, does not constrain the spatial topology. This suggests that extending Newton’s theory to non-Euclidean topologies is possible, and therefore describing that aspect of cosmology with a Newtonian theory. Such a theory is called a non-Euclidean Newtonian theory (NEN theory), and was proposed using Galilean structures and a minimal modification of the Newton-Cartan equations by Vigneron [18]. More than being a theoretical curiosity, this theory is actually a powerful tool to study the effect of cosmic topology on the gravitational field and on structure formation [20], mainly because it is much easier to use than general relativity, while still encoding non-linearities in other topologies than the usual 3-torus. This theory might also allow us to study the effect of inhomogeneities on the global expansion (called cosmological backreaction) in different topologies [19]. Both of these effects might provide us with new observational probes for cosmic topology in our Universe.

However, to this day general relativity remains the genuine theory of gravity. Therefore any non-relativistic theory of gravitation should be retrieved with a non-relativistic limit of the Einstein equation to be qualified as a physical theory. This holds in particular for the NEN theory of Vigneron [18] which has yet to be derived from such a limit, and therefore which remains still heuristic.

The goal of this paper is to study whether or not general relativity allows for NEN theories, by performing the Galilean limit of Einstein’s equation. Such a limit was already performed by [e.g. 3, 10] with some restrictive assumptions on the energy-momentum tensor, but its importance with respect to topology has never been
considered, which is done in the present paper. After defining the context concerning cosmic topology in section II.A, we will detail the properties of the Galilean limit in section III. Then, we will show in section IV that such a limit is only possible (assuming either vacuum or a matter energy-momentum tensor) if the spatial topology is Euclidean, implying that no NEN theory defined through Galilean structures is compatible with general relativity. In section V, we will show that this result can be interpreted as a signature of an incompleteness of the Einstein equation, which should feature an additional “topological term” linked to the spatial topological class. Strategies to find this term are proposed in section VI.

II. NEWTONIAN THEORY AND TOPOLOGY

A. What we mean by Euclidean and non-Euclidean topologies

We will always consider 4-manifolds \( M \) of the form \( \mathbb{R} \times \Sigma \) where \( \Sigma \) is a closed 3-manifold. This closedness condition is required for a cosmological model featuring inhomogeneities. Furthermore, as shown in Dautcourt [4], Vigneron [17], the Poisson equation, essential in Newton’s theory, can only be retrieved from the Newton-Cartan equations (without additional assumptions) if \( \Sigma \) is closed or is \( \mathbb{R}^3 \) with fall-off conditions at infinity.\(^1\)

Therefore, the topology of the manifold \( \Sigma \) is described by Thurston’s decomposition which says that the topology of any closed differentiable 3-manifold can be decomposed into a connected sum of pieces that each has one of the eight Thurston topologies. We call them classes of topologies. Each corresponds to an ensemble of multi-connected\(^2\) topological spaces. The most common ones in cosmology are the Euclidean, spherical and hyperbolic topologies. The topology of a 3-manifold is said to be Euclidean, respectively spherical and hyperbolic, if the universal cover of the 3-manifold is homeomorphic to \( \mathbb{E}^3 \) (i.e. \( \mathbb{R}^3 \) equipped with a flat metric), respectively to the 3-sphere \( S^3 \) and to \( \mathbb{H}^3 \) (i.e. \( \mathbb{R}^3 \) equipped with a metric having a purely negative trace Ricci tensor), and whose fundamental group is a discrete subgroup of \( \mathbb{R}^3 \times SO(3) \), respectively of \( SO(4) \), and of \( PSL(2, \mathbb{C}) \) [see 11, for the description of the other five classes of topologies]. We call a non-Euclidean topology any topological space which does not purely lie in the Euclidean class (i.e. either it belongs to another class or it is a connected sum of multiple classes).

It is important to note that this notion of “Euclidean” and “non-Euclidean” is not to be confused with the term Euclidean and non-Euclidean geometries generally used in cosmology and general relativity to refer to the presence or not of a non-zero Ricci curvature tensor. Especially, while by definition a Euclidean geometry requires a zero Ricci tensor, a 3-manifold with a Euclidean topology can have a non-zero Ricci tensor, and we have the following property:

\[ R_{ab} = 0 \Rightarrow \text{Euclidean topology,} \quad (1) \]

or equivalently

\[ \text{non-Euclidean topology } \Rightarrow R_{ab} \neq 0, \]

where \( R_{ab} \) is any Ricci tensor defined on the 3-manifold.\(^3\)

Because of property (1), Newton’s theory, for which the spatial Ricci tensor is zero, is only defined for Euclidean topologies. Therefore a non-Euclidean Newtonian theory, such as the one defined in [18] for spherical and hyperbolic topologies, requires the spatial Ricci tensor to be non-zero. The validity of general relativity, Newton’s theory and NEN theory in [18, 20], as a function of Thurston topological class is summarised in Table I.

B. What limit?

The Newtonian limit is generally defined as a weak field limit of general relativity, where the Lorentzian metric \( g \) is a perturbation of the Minkowski metric \( \eta \) as \( g_{\alpha \beta} = \eta_{\alpha \beta} + f_{\alpha \beta} \) where \( |f_{\alpha \beta}| \ll 1 \) and \( |\partial_\gamma f_{\alpha \beta}| \ll 1 \). The Einstein equation is then linearised at first order in \( f_{\alpha \beta} \) and its derivatives. This approach to the Newtonian limit has many drawbacks. First, it is not a well-defined limit of a 1-parameter family of solutions of the Einstein equations. The limit is also built such that the Poisson equation is recovered. However, as shown in Vigneron [17], the complete Newton theory should fundamentally feature the cosmological Poisson equation (i.e. with density deviation), and the classical Poisson equation is only valid for isolated systems in \( \mathbb{R}^3 \). So the classical Newtonian limit does not allow us to recover the full version of Newton’s theory as given by the Newton-Cartan equations. Furthermore, because the perturbation is defined with respect to the Minkowski metric, the limit is also necessarily Euclidean for the spatial sections. One way to allow for non-Euclidean spatial sections would be to replace \( \eta \) by an FLRW metric, which is the approach of the standard perturbation theory in cosmology [1]. However, this would imply by definition the spatial expansion to be necessarily given by the Friedmann equations. This

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\(^1\) The origin of this result comes from the need of a Scalar-Vector-Tensor decomposition of a symmetric spatial tensor, essential to obtain the Poisson equation from the spacetime Newton-Cartan equations [17]. This decomposition is only uniquely defined for a closed topology or for \( \mathbb{R}^3 \) with fall-off conditions at infinity [21].

\(^2\) Only the 3-sphere \( S^3 \) is simply connected.

\(^3\) Throughout this paper, we denote indices running from 0 to 3 by Greek letters and indices running from 1 to 3 by Roman letters.
Table I. This table shows, as function of Thurston’s topological class, which theory among general relativity, Newton’s theory and NEN theory in [18, 20] are defined. Only general relativity is defined for any possible spatial topology. The, still heuristic, NEN theory of Vigneron [18] is only defined for the Euclidean, spherical and hyperbolic cases (in the Euclidean case it is equivalent to Newton’s theory). The goal of this paper is to determine if general relativity naturally defines a NEN theory, i.e. “fills the gaps” of this table.

| Covering space (topological class) | E³ | S³ | H³ | R × S² | R × H² | SL₂R | Sol | Nil |
|-----------------------------------|----|----|----|--------|--------|------|-----|-----|
| General relativity                | ×  | ×  | ×  | ×      | ×      | ×    | ×   | ×   |
| Newton’s theory                   | ×  | ×  | ×  | ×      | ×      | ×    | ×   | ×   |
| NEN theory of [18, 20]            | ×  | ×  | ×  | ×      | ×      | ×    | ×   | ×   |

is not appropriate as the expansion law should not be an assumption but a result coming from Newton’s equations as shown in Buchert & Ehlers [2], Vigneron [17]. Furthermore, the standard perturbation approach is a linear approximation. Therefore the full Newton theory, and by extension its non-Euclidean versions, cannot be retrieved as it features non-linearities.

As argued in section 4.1 of [18], the best way of defining a “Newtonian theory” whatever the topology is to use Galilean structures. They already allow for a formulation of Newton’s theory close to general relativity with the Newton-Cartan equations. In this formalism the physical equation relates the energy content of a (spacetime) 4-manifold to its Galilean (spacetime) curvature, similarly to general relativity with Lorentzian structures. Furthermore, this formulation of Newton’s theory allows for a better conceptual description of expansion, as briefly presented in the first section of the present paper [see also 19]. Therefore, the Newtonian limit to consider should also use Galilean structures. Such a limit was developed by Künzle [10] and is called the Galilean limit of Lorentzian structures (see next section). This limit does not have the problems listed above for the usual Newtonian limits: it is a covariant limit; there is no assumptions on the curvature of the spatial sections; there is no assumed background expansion; non-linearities for the fluid dynamics are present at the limit.

Remark. The frame theory of Ehlers [5, 6], which uses Galilean structures and encompasses general relativity and the Newton-Cartan theory, was explicitly constructed to obtain Newton’s theory. Therefore it is not useful for the purpose of this paper as we want to perform a general (i.e. for any topological class) Galilean limit of the Einstein equation.

III. GALILEAN LIMIT OF LORENTZIAN STRUCTURES

We present in this section the definition of Galilean structures and the Galilean limit. As this limit is applied to a Lorentzian structure which is not necessarily solution of the Einstein equation, for now we will not apply the limit on this equation. This will be done in section IV.

A. Galilean structures

A Galilean structure defined on a differentiable 4-manifold M is a set (τ, h, ∇), where τ is an exact 1-form, h is a symmetric positive semi-definite (2,0)-tensor of rank 3, i.e. its signature is (0,+,+,”), with h^{αμ}_{μ}τ_μ = 0, and ∇ is a connection compatible with τ and h, called a Galilean connection:

\[ \nabla_α τ_β = 0 \quad ; \quad \nabla_γ h^{αβ} = 0. \] (2)

In this framework, a vector u is called a timelike vector if \( u^μτ_μ \neq 0 \) or a unit timelike vector if \( u^μτ_μ = 1 \), and a \((n,0)\)-tensor T is called spatial if \( τ_μT^{−μ} = 0 \) for all \( α \in [1,n] \). The exact 1-form τ defines a foliation \( \{Σ_t\}_{t∈R} \) in M, where \( Σ_t \) are spatial hypersurfaces in M defined as the level surfaces of the scalar field \( t \), with \( τ = dt \). The tensor h naturally defines a (spatial) Riemannian metric on \( \{Σ_t\}_{t∈R} \). Finally, the coefficients \( Γ^σ_{αβ} \) of the Galilean connection define a (spacetime) Riemann tensor \( R^σ_{αβγ} \) on M with the usual formula

\[ R^σ_{αβγ} := 2 \partial\left[Γ^σ_{αβγ} \right] + 2 Γ^σ_{μβ}Γ^μ_{αγ}. \] (3)

Additional properties of Galilean structures are presented in A. This type of structures is invariant under local Galilean transformations [9], and the time τ and space h metrics define a preferred foliation in M, hence corresponding to the Newtonian picture of spacetime.

Remark. A condition for the existence of a Galilean structure given a 4-manifold M is the existence of a time and space orientable Lorentzian structure in this manifold, or in other words M must be orientable and non-compact [see section 3 of 9], which is in agreement with our choice \( M = R × Σ \).

B. The limit

The initial hypothesis behind the Galilean limit is to consider a 4-manifold M and a 1-parameter family of smooth Lorentzian metrics \( \{g^λ\}_{λ>0} \) on M that depends smoothly on \( λ \), such that

\[ g^{αβ} = h^{αβ} + λ g^{αβ}_{L} + λ^2 \tilde{g}^{αβ} + O(λ^3), \] (4)

where \( h \) is a positive semi-definite \((2,0)\)-tensor of rank 3 on M, and \( \tilde{g}^{αβ} \) with \( n \in N \) is the \( n \)-th order of the
components \( \hat{g}^{\alpha \beta} \) (i.e. the notation \( \hat{g}^{\alpha \beta} \) does not mean that \( \lambda = 1 \)). From there, Künzle [10] showed that the covariant components of the metrics \( \hat{g} \) take the form

\[
\hat{g}_{\alpha \beta} = -\frac{1}{\lambda} \tau_\alpha \tau_\beta + \hat{g}_{\alpha \beta} + \lambda \hat{g}_{\alpha \beta} + \mathcal{O}(\lambda^2),
\]

(5)

where the 1-form \( \tau \) lies in the kernel of \( h \), i.e. \( \tau_\mu h^{\mu \alpha} = 0 \), and such that \( \tau_\mu \tau_\nu \hat{g}^{\mu \nu} = -1 \). An important property is that the \( n \)-th order of the covariant components are not the inverse of the \( n \)-th order of the covariant components, i.e. \( \hat{g}^{\alpha \mu} \hat{g}_{\mu \beta} \neq \delta_\beta^\alpha \). Instead we have by definition \( \hat{g}^{\alpha \beta} \hat{g}_{\mu \beta} = \delta_\beta^\alpha \), whose zeroth and first orders give:

\[
- \hat{g}^{\alpha \mu} \tau_\mu \tau_\beta + h^{\alpha \mu} \hat{g}_{\mu \beta} = \delta_\alpha^\beta, \\
- \hat{g}^{\alpha \mu} \tau_\mu \tau_\beta + \hat{g}^{\alpha \mu} \hat{g}_{\mu \beta} + h^{\alpha \mu} \hat{g}_{\mu \beta} = 0.
\]

(6)

Equation (6) leads to

\[
h^{\alpha \mu} \hat{g}_{\mu \beta} = \delta_\beta^\alpha - \tau_\beta B^\alpha,
\]

(8)

where we denote \( B^\alpha := -\tau_\mu \hat{g}^{\mu \alpha} \). Then we can write \( \hat{g}_{\alpha \beta} \) as

\[
\hat{g}_{\alpha \beta} = B b_{\alpha \beta} - 2\phi \tau_\alpha \tau_\beta,
\]

(9)

where \( \phi \) is an arbitrary scalar and \( b_{\alpha \beta} \) is defined in equation (A3) and corresponds to the projector orthogonal to \( B \) with respect to the tensors \( \tau \) and \( h \). Then, computing the Levi-Civita connection of \( \hat{g} \), we find

\[
\hat{\Gamma}^\gamma_{\alpha \beta} = \frac{1}{\lambda} \lambda^{\gamma \mu} \left( \tau_\alpha \partial_\mu \tau_\beta + \tau_\beta \partial_\mu \tau_\alpha \right) + \left( \phi h^{\gamma \mu} + \hat{g}^{\gamma \mu} \right) \left( \tau_\alpha \partial_\mu \tau_\beta + \tau_\beta \partial_\mu \tau_\alpha \right) + h^{\gamma \mu} \left( \partial_\alpha B_{\beta \mu} - \frac{1}{2} \partial_\mu B_{\alpha \beta} \right) + B^\gamma \partial_\alpha \tau_\beta + \tau_\alpha \tau_\beta h^{\gamma \mu} \partial_\mu \phi + \mathcal{O}(\lambda). 
\]

(10)

If the 1-form \( \tau \) is exact, which we will suppose from now, the connection \( \hat{\Gamma}^\gamma_{\alpha \beta} \) has a regular limit for \( \lambda \to 0 \), which corresponds to a Galilean connection compatible with \( \tau \) and \( h \):\n
\[
\hat{\Gamma}^\gamma_{\alpha \beta} = \hat{\Gamma}^\gamma_{\alpha \beta} + \mathcal{O}(\lambda) \\
= \hat{\Gamma}^\gamma_{\alpha \beta} + \tau_\alpha \tau_\beta h^{\gamma \mu} \partial_\mu \phi + \mathcal{O}(\lambda),
\]

(11)

where \( B\hat{\Gamma}^\gamma_{\alpha \beta} \) is defined in equation (A2). This Galilean connection is not general as the 2-form \( \kappa_{\alpha \beta} \) [defined in equation (A1)] necessarily takes the form \( \kappa_{\alpha \beta} = \gamma_\alpha \partial_\beta \phi \).

In conclusion, the Lorentzian structures \( \hat{g}, \hat{\Gamma}^\gamma_{\alpha \beta} \) become at leading order the Galilean structure \( (\tau, h, \hat{\Gamma}^\gamma_{\alpha \beta}) \). This is an explicitly covariant limit, as it does not require to be performed in a specific coordinate system.

C. Galilean limit and topology

Under the framework of this limit and the manifold \( M \) is unchanged by the procedure \( \lambda \to 0 \). This contrasts with perturbation approaches where a 1-parameter family of 4-manifolds \( \{ M \}_{\lambda > 0} \) is sometimes defined [e.g. 1, 7, 8]. Therefore, the topology of \( M \), and especially the topology of \( \Sigma \) (since we consider \( M = \mathbb{R} \times \Sigma \), is unchanged when taking the Galilean limit. This is a very important property that will be considered when taking the limit of the Einstein equation.

D. Interpretation of \( \lambda \)

The Minkowski metric, in coordinates where \( x^0 \) has the dimension of a time, can be written as \( \eta_{\alpha \beta} = \text{diag}(-c^2, 1, 1, 1) \), where \( c \) is the speed of light. Its inverse is \( \eta^{\alpha \beta} = \text{diag}(-c^{-2}, 1, 1, 1) \). Then assuming \( c \to \infty \) these two matrices become

\[
\eta_{\alpha \beta} \sim \text{diag}(-c^2, 0, 0, 0), \\
\eta^{\alpha \beta} \sim \text{diag}(0, 1, 1, 1),
\]

These leading orders for \( c \to \infty \) have the same form as the leading orders of \( \hat{g} \) for \( \lambda \to 0 \): in coordinates adapted to the foliation given by \( \tau \) (i.e. \( \tau_\alpha = \delta_\alpha^0 \) and \( h^{\alpha \beta} = h^{ab} \)), we have \( \hat{g}_{\alpha \beta} = h^{ab} \) and \( \hat{g}^{\alpha \beta} = \text{diag}(1/\lambda, 0, 0, 0) \). This shows that the limit \( \lambda \to 0 \) can be seen as a limit where \( c \to \infty \) with \( \lambda = c^{-2} \).

Therefore, the speed of light related to a Lorentzian metric \( \hat{g} \) of the family \( \{ \hat{g} \}_{\lambda > 0} \) depends on \( \lambda \) and is given by

\[
e_\lambda = \lambda^{-1/2}.
\]

(12)

This means that the family \( \{ \hat{g} \}_{\lambda > 0} \) of Lorentzian metrics defines a family of light-cones at each point of \( M \). The light-cone related to a metric \( \hat{g} \) is more open than the one related to a metric \( \hat{g} \) if \( \lambda_1 < \lambda_2 \). This is represented in figure 1.

If one wants to set the speed of light to be 1, which corresponds to choosing a coordinate system such that \( x^0 = ct \), this is only possible for one Lorentzian metric \( \hat{g} \). For all the other metrics, the speed of light in this coordinate system will differ from 1. This property is really important as it tells us that when we will consider equations (and their Galilean limit) which should feature the speed of light, we cannot take \( c = 1 \), and are obliged to take \( c = \lambda^{-1/2} \). This will be the case for the norm of timelike vectors (see section III.E) and for the Einstein equation (see section IV).
can compute the covariant components of a vector which is g-timelike for the Lorentzian family of light-cones is this hypersurface. The green vector is the slice $\Sigma^t(X)$ of vectors. Because the speed of light related to $g$ cannot be set to 1 for all $\lambda$, then “unit” means that the norm of $u$ is $c_\lambda$.

E. Limit of g-timelike and lightlike vectors

Because the notion of timelike vectors is defined in both structures, but is not equivalent, to distinguish between the two, we will call g-timelike vectors the ones related to a Lorentzian structure, and $\tau$-timelike vectors the ones related to the Galilean structure. This notation will also be applied for spacelike vectors.

Let us consider a vector $u$ which is a “unit” g-timelike vector for every member of the family $\{g\}_{\lambda>0}$, i.e. $u^{\mu}u^{\nu}\tilde{g}_{\mu\nu} = -1/\lambda$. If we assume that this vector does not depend on $\lambda$, then for a sufficiently small $\lambda$ we have $(u^{\mu}\tau_\mu)^2 = 1$. As $u$ is any unit g-timelike vector, this cannot be possible. So, $u$ needs to depend on $\lambda$. We write it $\hat{u}$ (more precisely we define a family $\{\hat{u}\}_{\lambda>0}$ of vectors). Because it is unit for all $g$, then we have

$$\hat{u}^\alpha = \delta^\alpha + \lambda \lambda^\alpha + O(\lambda^2),$$

with $\hat{u}^{\mu}\tau_\mu = 1$, and $\hat{u}^{\mu} \tau_\mu = \frac{1}{2} \hat{u}^{\mu} \hat{u}^{\nu} \tilde{g}_{\mu\nu}$. In addition we can compute the covariant components of $\hat{u}$, defined as

$$\hat{u}_\alpha := \hat{u}^{\mu} \tilde{g}_{\mu\alpha},$$

which become

$$\hat{u}_\alpha = -\frac{1}{\lambda} \tau_\alpha + \hat{u}_\alpha + O(\lambda),$$

where $\hat{u}_\alpha = \hat{u}^{\mu} \tilde{g}_{\mu\alpha} - \frac{1}{2} \hat{u}^{\mu} \hat{u}^{\nu} \tilde{g}_{\mu\nu} \tau_\alpha$. Note that $\hat{u}_\alpha \neq \hat{u}^{\mu} \tilde{g}_{\mu\alpha}$. Because $\hat{u}^{\mu} \tau_\mu = 1$, this implies that a unit g-timelike vector $\hat{u}$ for all members of $\{\hat{g}\}_{\lambda>0}$ corresponds to a unit $\tau$-timelike vector at the limit $\lambda \to 0$. However, the reverse is not possible: a $\tau$-timelike vector can never correspond to a g-timelike vector for all $\lambda$.

Finally, let us consider the case of lightlike vectors. We define a family of vectors $\hat{u}$ such that $\hat{u}^{\mu} \hat{u}^{\nu} \tilde{g}_{\mu\nu} = 0$ for all $\lambda$. Then if we assume that $\hat{u}$ can be decomposed into a Taylor series of $\lambda$, then each term in this series would be zero. This means that we cannot define a family of vectors $\{\hat{u}\}_{\lambda>0}$ having a Taylor expand with respect to $\lambda$ and which is lightlike for all members of $\{\hat{g}\}_{\lambda>0}$. This can be easily understood because from Galilean structures no equivalent of the lightlike vectors of Lorentzian structures can be defined.

F. Limit of the Lorentzian Riemann tensor

For the purpose of taking the limit of the Einstein equation, we present in this section the limit of the Riemann tensor coefficients $\tilde{R}^\alpha{}_{\beta\gamma\delta}$ (14) in series of $\lambda$ we easily see that the expansion of $\tilde{R}^\alpha{}_{\beta\gamma\delta}$ in series of $\lambda$ takes the form

$$\lim_{\lambda \to 0} \tilde{R}^\alpha{}_{\beta\gamma\delta} = \delta^\alpha{}_{\beta\gamma\delta} + O(\lambda),$$

where $\tilde{R}^\alpha{}_{\beta\gamma\delta}$ is the Riemann tensor associated with the Galilean connection coefficients $\hat{\Gamma}^\alpha{}_{\beta\gamma}$. Consequently, the leading order of the Ricci tensor coefficients $\tilde{R}_{\alpha\beta}$ corresponds to the Ricci tensor of the Galilean connection:

$$\lim_{\lambda \to 0} \tilde{R}_{\alpha\beta} = \hat{R}_{\alpha\beta} + O(\lambda).$$

Because Galilean structures do not feature non-degenerate metrics, raising and lowering indices is not possible. Therefore the Riemann and Ricci tensors of Galilean connections are only defined as, respectively, a (1,3)-tensor and a (0,2)-tensor. Consequently, identities (for instance Bianchi identities) where the coefficients $\tilde{R}^\alpha{}_{\beta\gamma\delta}$ are raised or lowered by $\hat{g}$ have no equivalent in Galilean structures. However, the limit of these identities
gives additional constraints on the (Galilean) Riemann tensor \( \hat{R}^\gamma_{\gamma\alpha\beta} \).

From the leading order of the interchanged symmetry \( \hat{R}^\gamma_{\gamma\beta\sigma} = \hat{R}^\beta_{\beta\sigma\gamma} \), we obtain

\[
h^{\mu\beta} \hat{R}^\alpha_{\alpha\mu\sigma} - h^{\mu\alpha} \hat{R}^\beta_{\beta\sigma\gamma} = 0, \tag{15}\]

and from the contracted second Bianchi identity \( \hat{\nabla}_\mu \hat{R}_{\nu\alpha} - \frac{1}{2} \hat{\nabla}_\alpha \hat{R}_{\mu\nu} = 0 \), we obtain

\[
h^{\mu\nu} \hat{\nabla}_\mu \hat{R}_{\nu\alpha} - \hat{\nabla}_\alpha \frac{h^{\mu\nu} \hat{R}_{\mu\nu}}{2} = 0. \tag{16}\]

Both of these equations are geometrical constraints on the Galilean structures. They are part of the Newton-Cartan system of equations. In particular, equation (15), called the Trautman-Kähnele condition, is essential to have an irrotational gravitational field [see e.g. 6, 17], as expected for a Newtonian theory.

Apart for the leading orders of the Bianchi identities, no other constraints on the Galilean structure appears. In particular, from this limit alone, the spatial Ricci curvature \( h^{\gamma\beta} \hat{R}^\mu_{\mu\gamma\beta} \) is totally free, and therefore the spatial topology. Constraints on this tensor can only appear once we assume the Lorentzian structures to be solutions of the Einstein equation.

IV. GALILEAN LIMIT OF THE EINSTEIN EQUATION

A. Limit of the energy-momentum tensor

The energy-momentum tensor \( \hat{T}^{\alpha\beta} \) of a general fluid of 4-velocity \( \hat{u}^\alpha \) can be written as

\[
\hat{T}^{\alpha\beta} = \hat{\epsilon} \hat{u}^\alpha \hat{u}^\beta + 2 \hat{\lambda} \hat{u}^\alpha \hat{u}^\beta + \hat{\pi} \hat{u}^\alpha \hat{u}^\beta + \hat{\pi}^{\alpha\beta}, \tag{17}\]

where \( \hat{u}^{\alpha\beta} := \hat{\lambda} \hat{u}^\alpha \hat{u}^\beta + \lambda \hat{u}_\alpha \hat{u}_\beta \) is the projector orthogonal to \( \hat{u} \). The expansion series of \( \hat{u} \) is detailed in B. As \( \hat{T} \) has the dimension of an energy and \( \hat{u} \) the geometrical dimension of a velocity, then \( \hat{\epsilon} \) has the dimension of a mass density. Therefore in the case of a dust fluid, \( \hat{\epsilon} \) is a zeroth order term given by the mass density of the fluid: \( \hat{\epsilon} = \rho \) for all \( \lambda \). For a more general matter fluid, where the fluid elements have internal energy, \( \hat{\epsilon} \) will have higher order terms [16]. Using equations (13) and (B4) on the (Lorentzian) conservation law \( \hat{\nabla}_\mu \hat{T}^{\mu\alpha} = 0 \), we can obtain the Newtonian mass conservation law, i.e. \( \hat{\nabla}_\mu (\hat{\rho} \hat{u}^\mu) = 0 \), only if [10]

\[
\hat{\rho} = O(1) ; \quad \hat{\lambda} = O(\lambda) \quad \hat{\pi} = O(1). \tag{18}\]

That is, the pressure \( \hat{\rho} \), the heat flux \( \hat{\lambda} \) and the anisotropic stress \( \hat{\pi} \) are all regular for \( \lambda \to 0 \). Therefore, once we consider that there is no non-relativistic process for which mass is not conserved, i.e. once we require to have \( \hat{\nabla}_\mu (\rho \hat{u}^\mu) = 0 \) in a Newtonian theory, then the conditions (18) must hold. In that case, \( \hat{T}^{\alpha\beta} \) has a regular limit, and we have

\[
\hat{T}^{\alpha\beta} = \frac{1}{\lambda^2} [\rho \tau_{\alpha\tau\beta}] + O(\lambda^{-1}), \tag{19}\]

\[
\hat{T}_{\alpha\beta} - \frac{1}{2} \hat{T}^{\mu\nu} \hat{g}_{\mu\nu} \hat{g}_{\alpha\beta} = \frac{1}{\lambda^2} \left[ \frac{\rho}{\tau_{\alpha\tau\beta}} \right] + O(\lambda^{-1}). \tag{20}\]

Remark. The regularity of \( \hat{T}^{\alpha\beta} \) for \( \lambda \to 0 \) is generally assumed in papers using the Galilean limit, in order to obtain the Newton-cartan equations [e.g. 3, 10]. The present section aimed at giving some physical insights supporting this hypothesis, without requiring a priori the Newton-Cartan equations to hold.

B. Limit of the Einstein equation

The Einstein equation features two constants: the cosmological constant \( \Lambda \) and the gravitational constant \( G \). The energy-momentum tensor \( \hat{T} \) having the dimension of an energy, it must appear as \( 8\pi G \Lambda \hat{\nabla}^2 \) in Einstein’s equation. However, from that equation alone the cosmological constant does not have, a priori, a preferred dimension. For instance, if we consider \( \Lambda \) to have the dimension of a curvature, then Einstein’s equation will feature the term \( \Lambda \hat{g}_{\alpha\beta} \). Instead, if we consider that \( \Lambda \) has the dimension of a time\(^{-2} \), there will be \( \Lambda \Lambda \hat{g}_{\alpha\beta} \).

The limit of the Einstein equation needs to be performed on the two covariant version, i.e. as a \((2,0)\)-tensor equation, featuring the Ricci tensor. Considering the \((2,0)\) or the \((1,1)\) versions, or the formulation featuring the Einstein tensor, give either no or redundant information at the limit. Therefore we have two possibilities:

\[
\hat{\nabla}_\mu \hat{\nabla}_\nu \hat{\nabla}_\rho \hat{\nabla}_\sigma \hat{\hat{R}}^{\rho\sigma}_{\nu\alpha\beta} = 8\pi G \Lambda \left( \hat{\hat{\nabla}}_{\alpha\beta} - \frac{1}{2} \hat{\hat{\nabla}}_{\gamma\delta} \right) + \Lambda \hat{g}_{\alpha\beta}, \tag{21}\]

or

\[
\hat{\nabla}_\mu \hat{\nabla}_\nu \hat{\nabla}_\rho \hat{\nabla}_\sigma \hat{\hat{R}}^{\rho\sigma}_{\nu\alpha\beta} = 8\pi G \Lambda \left( \hat{\hat{\nabla}}_{\alpha\beta} - \frac{1}{2} \hat{\hat{\nabla}}_{\gamma\delta} \right) + \Lambda \Lambda \hat{g}_{\alpha\beta}. \tag{22}\]

For the choice “\( \Lambda \hat{g}_{\alpha\beta} \)”, the leading order of (21) is \( \frac{1}{2} \Lambda \tau_{\alpha\tau\beta} + O(1) = 0 \), implying \( \Lambda = 0 \). Therefore, only the choice “\( \Lambda \Lambda \hat{g}_{\alpha\beta} \)” allows for a non-zero cosmological constant at the Galilean limit. Then the limit of (22) gives:

\[
\hat{\nabla}_\mu \hat{\nabla}_\nu \hat{\nabla}_\rho \hat{\nabla}_\sigma \hat{\hat{R}}^{\rho\sigma}_{\nu\alpha\beta} - (4\pi G \rho - \Lambda) \tau_{\alpha\tau\beta} + O(\lambda) = 0. \tag{23}\]

This equation along with the geometrical constraint (15) and the conservation law \( \hat{\nabla}_\mu (\hat{\rho} \hat{u}^\mu) = 0 \) corresponds to the Newton-Cartan system.
This system has been derived from the Einstein equation without any assumptions on the spatial topology or on the spatial Ricci curvature tensor $bR^{\alpha\beta}$ of the Galilean structure. Still, equation (23) necessarily implies $bR^{\alpha\beta} = 0$, which can be seen by projecting (23) twice along $h$. From property (1) this means that the topology of each hypersurface member of the foliation induced by the Galilean structure is a Euclidean topology. As explained in section III C, the Galilean limit does not change the topology of $M$. This means that the topology of $\Sigma$ needs to be Euclidean even before taking the limit. In other words, the Einstein equation has a Galilean limit if and only if the spatial topology is Euclidean. The only assumption on the energy-momentum tensor for this result to hold is that it describes a matter fluid whose mass is conserved at the limit. In any case, the result holds in vacuum.

Remark. This result can also be derived directly from the 1+3-Einstein system of equations related to the fluid velocity $\dot{u}$ [see chapter 4 in 16].

Remark. Künzle [10] also performed the Galilean limit of the Einstein equation, and found the same leading orders, especially without spatial Ricci curvature. However, in that paper, the author imposes the limit of the energy-momentum tensor so that it leads to Newton-Cartan equations, while in the present paper we show that these equations are always obtained (for a matter energy-momentum tensor or in vacuum), even with the presence of the cosmological constant.

V. POSSIBLE INTERPRETATIONS

A. No physical NEN theory

As general relativity, with Einstein’s equation, is considered to be the genuine theory of gravitation, the first interpretation of the above result is to say that any Newtonian theory which would be defined on a non-Euclidean topology would be non-physical. A consequence of this statement is that if our Universe has a non-Euclidean topology (e.g. spherical or hyperbolic like in the $\Lambda$CDM model), no Newtonian calculation could be performed because the relativistic description of the Universe would not have any Newtonian (Galilean) limit. This is a problem as this would prevent the use of Newton’s theory even for the solar system. And conversely, being able to use Newton’s theory to accurately describe gravity would imply the topology of the Universe to be Euclidean if we consider Einstein’s equation to hold for the relativistic regime.

An escape to this problem, could be to say that the relativistic solution has a Newtonian limit only in certain regions of the universe, and therefore the relation $bR^{\alpha\beta} = 0$ would only be valid in these regions, and not on the whole spatial sections. In that case, property (1) does not hold. However, this would prevent the derivation of the Poisson equation from the Newton-Cartan system, which requires the validity of this system (and therefore of $bR^{\alpha\beta} = 0$) on the whole 3-manifold $\Sigma$ [4, 17]. In other words we could not obtain the classical Newton equations.

B. Missing term in Einstein’s equation

It is always assumed that Einstein’s theory must contain Newton’s theory in some limit, and this because the latter works quite well in several situations. That is how, using the Newtonian limit, we can derive the constant $\frac{h^2c^2}{\pi G}$ in front of the energy-momentum tensor in the Einstein equation, or the form $T = \rho u \otimes u$ for a dust fluid. However the situations in which Newton’s theory was tested to be valid does not rule out NEN theories [for instance, the second NEN theory in 18] with spatial curvature radius big with respect to the size of the test domain. In this sense, general relativity should also contain any NEN theory in some limit. But as shown before, once we consider a matter energy-momentum tensor, or vacuum, the topology is necessarily Euclidean. So it seems that the only way to obtain a non-Euclidean version of Newton’s theory at the limit is to modify the Einstein equation.

Another argument in favour of such a modification is to say that Einstein’s equation was initially proposed so that it is, at least, compatible with Newton’s theory. But this equation was not constructed so that it is compatible with a non-Euclidean version of Newton’s theory, and in this sense it is not necessarily surprising that it is actually incompatible with such a theory.

1. Constraints on the missing term

If we require the Galilean limit to exist in non-Euclidean topologies, an additional term $\mathcal{T}$ must be added to the Einstein equation:

$$\hat{G}_{\alpha\beta} = 8\pi G\lambda^2\hat{T}_{\alpha\beta} - \lambda\hat{T}_{\alpha\beta} + \hat{T}_{\alpha\beta}, \quad \text{(24)}$$

such that

$$\hat{T}_{\alpha\beta} = \frac{1}{\lambda}\left(\frac{hR^{\alpha\beta}}{2} - h^{\alpha\beta}\right) + O(1), \quad \text{(25)}$$

$$\hat{T}_{\alpha\beta} = hR^{\alpha\beta} - h^{\alpha\beta} + O(\lambda), \quad \text{(26)}$$

allowing for $bR^{\alpha\beta} \neq 0$ at the limit. Apart for these two conditions, no other constraint is required on $\mathcal{T}$ to enable non-Euclidean topologies. Therefore, there is substantial freedom on the extrapolation at full order of this term, i.e. on its form in the Einstein equation before taking the limit. A major question is also raised: what is the physical interpretation and the mathematical origin of $\mathcal{T}$?
In the derivation of the result in section IVB, we assumed that the energy-momentum tensor was the one of classical matter, i.e. the leading order of the energy density \( \dot{\varepsilon} \) is the mass density \( \rho \). This suggests that \( \mathcal{T} \) might be part of the energy-momentum tensor, which consequently would describe a more general type of fluid. As the main reason for the topological constraint at the limit is the zero spatial curvature relation \( h R^{\alpha \beta} = 0 \), the term \( \mathcal{T} \) could be interpreted as coupling the matter with this curvature. In this interpretation, the variables characterising the fluid (i.e. \( \dot{\varepsilon}, \dot{\rho}, \dot{q}^h \) and \( \dot{\pi}^{a \beta} \)) would have the following leading orders

\[
\dot{\varepsilon} = \frac{1}{\lambda} \left[ \frac{h R}{2 + 8 \pi G} \right] + \mathcal{O}(\lambda), \tag{27}\]
\[
\dot{\rho} = \frac{1}{\lambda^2} \left[ - \frac{h R}{6 + 8 \pi G} \right] + \frac{1}{\lambda^2} \dot{\rho} + \mathcal{O}(\lambda), \tag{28}\]
\[
\dot{q}^h = \frac{1}{\lambda^2} \dot{q}^h + q^h + \mathcal{O}(\lambda), \tag{29}\]
\[
\dot{\pi}^{a \beta} = \frac{1}{\lambda^2} \left[ \frac{h R^{(a \beta)}}{8 \pi G} \right] + \frac{1}{\lambda^2} \pi^{a \beta} + \mathcal{O}(\lambda), \tag{30}\]

where \( \dot{\rho}, \dot{q}^h, \dot{\pi}^{a \beta} \) are unconstrained. In particular, \( \dot{\pi}^{a \beta} \) is not regular anymore.

The main problem with this line of interpretation is that it does not hold for vacuum. In that case, because \( \mathcal{T} \) would be a coupling term, it should vanish, and the limit of the (modified) Einstein equation would again impose the spatial topology to be Euclidean.

3. Topological term

So, if the Einstein equation is indeed missing a term \( \mathcal{T} \) which would allow for the Galilean limit in non-Euclidian topologies, it is more likely that this term is not linked to the matter, but only to the topology of \( \Sigma \), or more generally of \( \mathcal{M} \), i.e. \( \mathcal{T} \) is a topological term. As before, this term remains unknown at full order. In the following section, we propose two possible strategies which might provide us the full form of \( \mathcal{T} \).

Remark. A natural solution for \( \mathcal{T} \) could have been to consider the cosmological constant, because for certain solutions of the Einstein equation it can already be linked to topological properties [e.g. 15]. However, as shown in section IVB, the cosmological constant term is required to appear as \( \lambda \pi^{a \beta} \) for which the leading orders do not agree with equations (25) and (26).

VI. POSSIBLE STRATEGIES

A. Reference metric theory of Rosen [14]

While Lovelock [12] showed that the Einstein equation was the most general second order equation (in the derivatives of the metric) which could be built from the metric tensor, the initial introduction of the Einstein equation was made so that (in particular) it was compatible with Newton’s theory. To extrapolate the topological term at full orders, we can follow the same approach and say that this extrapolation should be consistent with a presupposed NEN theory. In this view, we would start from the NEN theory in section 4.5 of Vigneron [18] [see also 20], which features most of the properties we expect for a Newtonian theory (e.g. exact N-body calculation, space-time separation of the spatial metric, no gravitomagnetism). In this theory, the Newton-Cartan equation is modified to allow for non-zero spatial curvature as follows

\[
\dot{R}_{\alpha \beta} - \frac{R}{3} G_{b \alpha \beta} = (4 \pi G \rho - \Lambda) \tau_\alpha \tau_\beta, \tag{31}\]

where \( G \) is a reference observer, called a Galilean observer. The spatial topology can either be spherical or hyperbolic.

Because the additional term \( \frac{R}{3} G_{b \alpha \beta} \) in (31) is a second order term in the spatial metric, this suggests that the topological term (which would become \( \frac{R}{3} G_{b \alpha \beta} \) at leading order) in the Einstein equation should also be second order, and therefore, from Lovelock [12], cannot only be expressed as a function of the Lorentzian metric. In other words, an additional field must be introduced.

The introduction of the term \( \frac{R}{3} G_{b \alpha \beta} \) to define the NEN theory in Vigneron [18] also introduces a reference (Galilean) observer, defined by the \( \tau \)-timelike vector \( G \). This suggests that the topological term \( \mathcal{T} \) must also be defined with respect to a reference observer, or reference frame. In combination with the fact that \( \mathcal{T} \) must be second order or less, it appears that a promising solution is given by the bi-metric theory of Rosen [14]. In this theory, a reference metric \( \bar{g} \) is introduced on \( \mathcal{M} \), in addition to the physical metric \( g \). The Einstein equation is modified to be \(^5\)

\[
R_{\alpha \beta} - \bar{R}_{\alpha \beta} = 8 \pi G \Lambda^2 \left( T_{\alpha \beta} - \frac{1}{2} T_{\mu \nu} g_{\alpha \beta} \right) + \lambda \pi_{a \beta}, \tag{32}\]

where \( \bar{R} \) is the Ricci tensor associated to \( \bar{g} \). In this case

\[
T_{\alpha \beta} = \bar{T}_{\alpha \beta} - \frac{1}{2} \bar{R}_{\mu \nu} g^{\mu \nu} g_{\alpha \beta}. \tag{33}\]

By definition, \( \bar{R} \) being a reference non-dynamical field, it does not change when taking the limit. Then, \( \bar{R}_{\alpha \beta} \)

\(^5\) The Einstein-Hilbert action is also modified.
should directly lead to $\frac{\delta S}{\delta C_{\alpha\beta}}$ in equation (31). The choice of topology is made through the choice of reference Ricci tensor. For instance, taking $\bar{g}$ so that there exists a coordinate system where

$$\bar{g}_{\alpha\beta} = \text{diag}(-1;1;\sin^2\chi;\sin^2\chi\sin^2\theta)$$  \hspace{1cm} (34)

imposes $\Sigma$ to have a spherical topology.\footnote{This choice is different from the one made in Rosen [14] where $\bar{g}$ is the de Sitter metric.} This approach will be studied in a follow-up paper.

**B. Lagrangian multiplier**

As explained in section III C, the Galilean limit does not change the topology of the 4-manifold $\mathcal{M}$. If we consider Einstein’s equation as resulting from a variational principle, the same arises. The action $S$ is defined on a 4-manifold $\mathcal{M}$. Among all the equations possible on $\mathcal{M}$ for the physical variable (i.e. $g$), we consider that the physical one is the one which minimises the action, and is obtained with $\delta S = 0$. Therefore the variational principle is performed on the manifold $\mathcal{M}$ which is a priori defined. Because the topology is a property of $\mathcal{M}$ and not of the structure we define on $\mathcal{M}$, this means that, when performing the variation of the action, the topology should already be fixed. This implies that we are forbidden to consider paths for $g$ (when varying the action) for which the Riemann tensor is incompatible with the topology.

In classical mechanics, when constraints are applied on a variational principle, an additional term with Lagrange multiplier must be added in the action. The same might be needed for the topology in general relativity where we would have an action of the form:

$$S = \int \sqrt{g} \left( R_{\mu\nu} g^{\mu\nu} + \lambda \cdot C + L_{\text{matter}} \right),$$  \hspace{1cm} (35)

where $C = 0$ would be a tensorial equation constraining the topology of $\mathcal{M}$ and especially the topological class of $\Sigma$ in the case $\mathcal{M} = \mathbb{R} \times \Sigma$, and $\lambda$ would be the tensorial Lagrange multiplier associated to this constraint. The idea of the present approach is to say that the topological term in the Einstein equation would result from the variation of $\sqrt{g} \lambda \cdot C$, i.e. from the fixation of the topology in the variational principle.

While the proposition of section VIA gives a solution for $\mathcal{T}$, it does not explain its physical/mathematical origin: this question is shifted to the origin of $\bar{g}$. On the contrary, the Lagrange multiplier approach might explain this origin, but does not provide $\mathcal{T}$, as $C$ is unknown for now.

**Remark.** It might also be interesting to study whether or not the Gibbons–Hawking–York boundary term could provide, at the Galilean limit, the expected modified Newton-Cartan equation (31).

**VII. CONCLUSION**

This paper aimed at studying the possibility of obtaining a Newtonian theory on a non-Euclidean spatial topology (in the sense of Thurston topological classification) from the Einstein equation, using the Galilean limit of Lorentzian structures. Such a non-Euclidean Newtonian theory (NEN theory), best defined using Galilean structures, is essential to study the Newtonian regime in (for instance) spherical or hyperbolic model universes. While a NEN theory was proposed in Vigneron [18], there remained to obtain it from the Einstein equation for a full justification of this proposition.

We showed in the present paper that either for a general matter energy-momentum tensor where we impose mass conservation at the limit, or in vacuum, the Galilean limit of the Einstein equation imposes the spatial Ricci curvature tensor (at the limit) to be zero, and therefore the topology to be Euclidean. Because the topology is unchanged during the limiting process, this implies that the Galilean limit of the Einstein equation is only possible in Euclidean spatial topologies. We proposed two main interpretations of this result:

1. there is no extension of Newton’s theory to non-Euclidean topologies which can be considered physical,

2. or, the Einstein equation is incomplete.

The main drawback of the first possibility is that the Newtonian regime, and consequently Newtonian like calculation, would not exist in a universe with a non-Euclidean spatial topology, for instance spherical or hyperbolic. The second line of interpretation offers interesting perspectives. In that case, the Einstein equation misses a “topological term”, present even in vacuum, which would allow us to perform the Galilean limit in any topologies. However, this term is only known at leading order and remains to be extrapolated at full orders. For this, we proposed two promising strategies: using the bimetric theory of Rosen [14]; adding a Lagrange multiplier to fixe the topology in the Einstein-Hilbert action. They will be the subject of follow-up papers.

If indeed a topological term is a missing feature of general relativity, the Einstein equation in its current form would be physically incorrect if the Universe has a non-Euclidean topology. Consequences for physical solutions would only be significant at large scales, i.e. comparable to the (finite) size of the spatial sections. Therefore it is essential for cosmology to assess the reality of this term.
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Appendix A: Properties of Galilean connections

Unlike for Lorentzian structures, from the knowledge of $\tau$ and $h$, the Galilean connection $\nabla$ is not unique and is defined up to a unit $\tau$-timelike vector $B$ and a two form $\kappa$. Its coefficients $\Gamma_{\alpha\beta}^{\gamma}$ take the form

$$\Gamma_{\alpha\beta}^{\gamma} = B\Gamma_{\alpha\beta}^{\gamma} + 2\tau(\alpha\kappa_{\beta})_{\mu}h^{\mu\gamma}, \quad (A1)$$

where

$$B\Gamma_{\alpha\beta}^{\gamma} := h^{\gamma\mu}\left(\partial_{(\alpha} B_{\beta)\mu} - \frac{1}{2} \partial_{\mu} B_{\alpha\beta}\right) + B^{\gamma} \partial_{(\alpha} \tau_{\beta)}, \quad (A2)$$

and where $B_{\alpha\beta}$ is the projector orthonormal to the $\tau$-timelike vector $B$, defined with

$$B_{\alpha\beta} B^{\mu} := 0; \quad B_{\alpha\beta} h^{\mu\beta} := \delta_{\alpha\beta} - \tau_{\alpha} B^{\beta}. \quad (A3)$$

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