KINETIC LIMIT FOR A HARMONIC CHAIN WITH A CONSERVATIVE ORNSTEIN-UHLENBECK STOCHASTIC PERTURBATION

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Abstract. We consider a one dimensional infinite chain of harmonic oscillators whose dynamics is weakly perturbed by a stochastic term conserving energy and momentum and whose evolution is governed by an Ornstein-Uhlenbeck process. We prove the kinetic limit for the Wigner functions corresponding to the chain. This result generalizes the results of [7] obtained for a random momentum exchange that is of a white noise type. In contrast with [7] the scattering term in the limiting Boltzmann equation obtained in the present situation depends also on the dispersion relation.

1. Introduction. In the present paper we investigate the kinetic limit of the energy density for a one dimensional harmonic chain with a random mechanism of momentum exchange that ensures its conservation. The system is described by a stochastically perturbed discrete linear wave equation in a one dimensional integer lattice

\begin{align}
\frac{dq_y}{dt} &= \partial_{p_y} H(p, q), \\
\frac{dp_y}{dt} &= -\partial_{q_y} H(p, q) + \sqrt{\epsilon} \dot{\zeta}_y(t), \quad y \in \mathbb{Z}.
\end{align}

Here \((p, q) = ((p_y, q_y))_{y \in \mathbb{Z}}\), where the component labelled by \(y\) corresponds to the one dimensional momentum \(p_y\) and position \(q_y\). The Hamiltonian is (formally) given by

\[ H(p, q) := \frac{1}{2} \sum_{y \in \mathbb{Z}} p_y^2 + \frac{1}{2} \sum_{y, y' \in \mathbb{Z}} \alpha_{y-y'} q_y q_{y'} \]

The assumptions made about the coupling constants \((\alpha_y)_{y \in \mathbb{Z}}\) (see (a1)–(a3) made in Section 2.2) ensure that the potential energy \(q \mapsto \sum_{y, y' \in \mathbb{Z}} \alpha_{y-y'} q_y q_{y'}\) is non-negative and the interaction between the oscillators is local in space.

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The stochastic perturbation \( \sqrt{\epsilon} \xi_y(t) \) describes the momentum exchange mechanism that we superimpose on the Hamiltonian system. It is assumed here to be weak, as we shall later take \( \epsilon \ll 1 \). Its particular form is determined by the stipulation that it should conserve the momentum and be local, i.e. depend only on neighbouring sites. We shall assume therefore that it is the stochastic differential of an Ornstein-Uhlenbeck process \( \zeta := (\zeta(t))_{t \geq 0} \) of the form
\[
\zeta_y(t) := \sqrt{\epsilon} \sum_{x=-1,0,1} (Y_{y+z}y) \zeta_{y+z}(t), \quad y \in \mathbb{Z}. \tag{3}
\]
Vector fields \( Y_y \) that describe the momentum exchange are defined by
\[
Y_y := (p_y - p_{y+1}) \partial_{p_{y-1}} + (p_{y+1} - p_{y-1}) \partial_{p_y} + (p_{y-1} - p_y) \partial_{p_{y+1}}, \quad y \in \mathbb{Z}.
\]
Note that they are tangent to the circle described by the system of equation
\[
p_{y-1}^2 + p_y^2 + p_{y+1}^2 \equiv \text{const} \tag{4}
\]
and
\[
p_{y-1} + p_y + p_{y+1} \equiv \text{const} \tag{5}
\]
guaranteeing that the exchange of momenta, taking place in the chain, conserves locally the kinetic energy and the momentum. The process \((\xi(t))_{t \in \mathbb{R}}\), appearing in (3), is a zero mean Gaussian with the covariance given by
\[
\mathbb{E}[\xi_y(t) \xi_y(s)] = \int_T \epsilon^{-\gamma(k)|t-s|-2\pi ik(y-y')} \sigma(k) dk, \quad \forall t, s \in \mathbb{R}, \ y, y' \in \mathbb{Z}. \tag{6}
\]
Here \( T \) is the one dimensional unit torus, defined throughout this paper as the interval \([-1/2, 1/2]\) with identified endpoints. We assume that the time mixing rate \( \gamma(k) \) and spatial spectral density \( \sigma(k) \) are non-negative (detailed assumptions are given in Section 2.3) and \( \gamma(k) \geq \gamma^*, \ k \in \mathbb{T} \) for some \( \gamma^* > 0 \).

The main result of the present paper, see Theorem 2.5 below, concerns the asymptotics of the Wigner function \( W^*(t, x, k) \), that describes the resolution of the energy in spatial and frequency mode coordinates \((x, k) \in \mathbb{R} \times T\), see (34) below. We prove that at time \( t \sim 1/\epsilon \) the expectation of \( W^*(t, x, k) \) tends, as \( \epsilon \to 0 \), to the solution \( U(t, x, k) \) of a linear Boltzmann equation
\[
\partial_t U(t, x, k) + \frac{1}{2\pi} \omega'(k) \partial_x U(t, x, k) = L U(t, x, k), \quad (t, x, k) \in \mathbb{R}_+ \times \mathbb{R} \times T, \tag{7}
\]
where the transport coefficient is determined from the dispersion relation \( \omega(k) = \sqrt{\alpha(k)} \) (see (18)) and \( \alpha(k) \) is the Fourier transform of the sequence \( \alpha := (\alpha_y)_{y \in \mathbb{Z}} \).

It is real and non-negative thanks to the assumption on positive definiteness of the quadratic potential energy. In addition, since we assume that \( \alpha \) is a rapidly decaying, real valued sequence (see assumptions (a1)-(a3) below) the function \( \tilde{\alpha}(-\cdot) \) is even and \( C^\infty \) smooth.

The scattering operator \( L \) is of the form
\[
LU(t, x, k) := \int_T R(k, k') [U(t, x, k') - U(t, x, k)] dk', \tag{8}
\]
with the scattering kernel given by
\[
R(k, k') := \frac{2\sigma(k + k') \gamma(k + k') R_+(k, k')}{{\gamma^2}(k + k') + [\omega(k') + \omega(k)]^2} + \frac{2\sigma(k - k') \gamma(k - k') R_-(k, k')}{{\gamma^2}(k - k') + [\omega(k') - \omega(k)]^2}. \tag{9}
\]
The shape of functions $R_{\pm}(k, k')$ is determined by the type of the momentum exchange considered. In our case they equal

$$R_{\pm}(k, k') := 16 \sin^2(\pi k) \sin^2(\pi k') \sin^2(\pi (k \mp k')),$$

$k, k' \in \mathbb{T}$.

It is clear from the formula (9) that $R(k, k') = R(k', k)$ for all $k, k' \in \mathbb{T}$.

The kinetic limit in the case of a harmonic chain with a random momentum exchange based on a white noise in time has been considered in [7], see also [19], where the compensated wave rather than Wigner function has been studied and [18], where the long time, large scale asymptotics have been obtained using probabilistic representation of the solution of the kinetic equation (7). It has been assumed in [7] that $(\xi_y(t))$ is the cylindrical Wiener process. The limit of the Wigner functions is described then by a linear kinetic equation of the form (8) with the scattering kernel given by

$$R(k, k') = R_+(k, k') + R_-(k, k').$$

Note that we can think of the white noise model of [7] as a limiting case of the situation considered here, when both the mixing rate and spectral density are constants $\gamma(k) = \gamma$, $\sigma(k) = 2\gamma$ and $\gamma \to +\infty$. Then, the limit of the respective kernels (9) is given by (10).

A novel feature of our model, when compared with the one considered in [7], is the fact that the dispersion relation $\omega(k)$ (that determines the transport term in (7)) appears also in the formula for the scattering kernel, see (9). This fact may better reflect the coupling between scattering and transport, due to nonlinearity in non-linear chains, e.g. as in the case of a Fermi-Pasta-Ulam chain (see e.g. [21] for necessary definitions). Such a situation does not occur for white noise in time exchange models where the transport and scattering are fully decoupled.

In addition, we believe that the kinetic limit might be valid for a broader family of random perturbations that need not be uniformly mixing for all wavelengths (i.e. the assumption $\gamma(k) \geq \gamma_*, > 0, k \in \mathbb{T}$ need not hold). The proof of such a result may require a different approach though, e.g. one could use the Duhamel series expansion for the the Wigner function obtained from the equation system (62) and (65) below and a subsequent analysis of the Feynman diagrams that arise after taking the expectation of the resulting multiple products of Gaussians. As a result, upon a suitable choice of the mixing rate function e.g. of the form $\gamma(k) = |\sin(\pi k)|$ (implying long time correlations for the proces $\xi$), one could, in principle, observe a different type of asymptotics of the solutions to the linear Boltzmann equation (7) from the one seen in [14], even in the case when $\sigma(k) \equiv 1$, i.e. when the sequence $(\xi_y(t))_{y \in \mathbb{Z}}$ is i.i.d. for a fixed $t$. In particular it seems possible to have a situation when the total scattering kernel satisfies $R(k) = \int_{\mathbb{T}} R(k, k') dk' = +\infty$, for all $k \neq 0$. It would correspond to the situation when the solution of the kinetic equation has a smoothing property. A somewhat similar phenomenon takes place in the case of the kinetic limit for the Wigner function of the solution of the Schrödinger equation with a random potential, see [10].

Another interesting point is to understand how the particular form of the scattering kernel, as in (9), might influence the long time, large space scale asymptotics of the phonon and also the hydrodynamic limit of the energy distribution in the case when no weak coupling assumption is made for the system (1). Such limits have been considered for harmonic chains with the white noise exchange of momenta and some other closely related models in [9, 13, 14, 15]. It follows from the results of [1, 14, 23] that in the case of an acoustic chain, i.e. when $\omega(0) = 0$, the
limit of the solution of the linear Boltzmann equation (7), under the macroscopic scaling $t = Nt', x = N^{2/3}x'$, $k = k'$, as $N \to +\infty$, is described by the solution of the fractional heat equation $\partial_t \bar{U}(t, x) = -|\Delta x|^{3/4} \bar{U}(t, x)$ (see also [5] for a two dimensional result). The mode coupling argument of [26] (see Appendix 1) seems to suggest that a similar limit should occur in the case of the present model.

Comparing our method of proof with the argument used in [7] we note first that the equations governing the dynamics of the Wigner functions considered here (see (69) and (74) below) differ substantially from their analogues in [7], see (42) and (44) ibid. In particular, both (69) and (74) contain scattering terms that are of apparent order of magnitude $\epsilon^{-1/2}$, as $\epsilon \ll 1$. The corresponding terms in (42) and (44) of [7] are of order 1. It is due to the fact that in our model the dynamics of the wave function, see (29) below, is given by an ordinary differential equation containing a random perturbation of the Hamiltonian part (describing the scattering of waves) that is also of order $\epsilon^{-1/2}$, as $\epsilon \ll 1$, while the respective term in the corresponding equation in [7], see (14), rewritten in the kinetic time scale $t/\epsilon$, is of order 1. This explains why the equations (42) and (44) for the Wigner functions in [7] do not contain large scattering terms.

For this reason we are prevented to adopt the approach of [7] to the present model. To deal with the large scattering term we resort instead to a technique used in the averaging theory called the perturbed test function method, see e.g. [11]. It has been applied to derive the kinetic limit for the Wigner transform of the solution of the Schrödinger equation with a random, time dependent, Markovian potential by G. Bal et al. in [4]. In a nutshell (and with some degree of oversimplification) the method can be described as follows. To obtain the asymptotics of $\mathbb{E} \langle W(\cdot), G \rangle$, as $\epsilon \ll 1$, for a given (smooth) test function $G : \mathbb{R} \times T \to \mathbb{C}$, where $\langle \cdot, \cdot \rangle$ is the appropriate duality pairing, we replace the test function by a random field of the form $\tilde{G}(t, x) := G(t, x) + \sqrt{\epsilon} G^{1, \epsilon}(t, x) + \epsilon G^{2, \epsilon}(t, x)$, see (95) below (in fact we need three such fields corresponding to three Wigner function considered here). Then, using the temporal dynamics of the random perturbation process, we can find such fields $G^{j, \epsilon}$, $j = 1, 2$ (called correctors) that are both of order 1 (to prove this fact we use the spectral gap assumption $\gamma(k) \geq \gamma_\ast$) and the time derivative $\partial_t \mathbb{E} \langle W(\cdot), G \rangle$ does not contain terms of large magnitude. In the process we also identify the limiting equation satisfied by $\mathbb{E} \langle W(\cdot), G \rangle$. Finally, we mention here that an analogous model can be formulated in an arbitrary spatial dimension. One can obtain then a similar result concerning the kinetic limit of the Wigner functions. The only complication occurs on the level of notation (which is already a bit heavy for a one dimensional case).

The organization of the paper is as follows. In Section 2 we introduce the preliminaries and formulate the main result of our paper. The dynamics of the Wigner function, and the accompanying it set of functions needed to close the equations are presented in Section 3. The proof of the main result is contained in Section 5. Some additional results that are needed to make our presentation self-contained are presented in the Appendix.

2. Preliminaries and formulation of the main result.

2.1. Some basic notation. Let $\ell_p(\mathbb{Z})$ be the complex Banach space of all complex sequences $f = (f_y)_{y \in \mathbb{Z}}$ equipped with the norm $\|f\|_{\ell_p}^p := \sum_{y \in \mathbb{Z}} |f_y|^p$. Define the Fourier transform $\hat{f}$ of the sequence $f \in \ell_1$ by letting
\[ \hat{f}(k) := \sum_y f_y e^{-2\pi i y k}, \quad k \in \mathbb{T}. \]  

(11)

We denote by \( \| \cdot \|_{L^p(T)} \) the respective \( L^p \) norm. By the Plancherel theorem we have \( \| \hat{f} \|_{L^2(T)} = \| f \|_{L^2} \).

For a given \( m \in \mathbb{R} \) we define the the Hilbert space \( H^m \), made of those sequences, for which

\[ \| f \|_{H^m} := \left( \sum_{y \in \mathbb{Z}} (\langle y \rangle_2^m | f_y |^2 \right)^{1/2}, \]

where \( \langle y \rangle := (1+y^2)^{1/2} \). Its Fourier transform image shall be denoted by \( H^m(T) \). In what follows we shall also use the Sobolev spaces \( W^{m,p}(T) \) made of those functions that possess \( m \) generalized derivatives that are \( L^p \) integrable, for \( p \in [1, +\infty) \), or have finite essential supremum, if \( p = +\infty \), equipped with the respective standard norm.

Denote by \( \mathcal{S}(\mathbb{R}) \), \( \mathcal{S}'(\mathbb{R}) \) the spaces of all Schwartz functions on \( \mathbb{R} \) and the corresponding space of tempered distributions. By \( \mathcal{S}(\mathbb{R} \times \mathbb{T}) \) we denote the space of complex valued functions on \( \mathbb{R} \times \mathbb{T} \) that are \( C^\infty \) smooth and such that for any non-negative integers \( l, m, n \) we have

\[ \sup_{(x,k) \in \mathbb{R} \times \mathbb{T}} (x)_n | \partial_x^l \partial_k^m G(x,k) | < +\infty. \]

Let \( \tilde{G}(p,k) \) and \( \tilde{G}(x,y) \) be the Fourier and the inverse Fourier transforms of \( G \), respectively in the first and second variable, i.e.

\[ \tilde{G}(p,k) := \int_{\mathbb{R}} dx \ e^{-2\pi i px} G(x,k) \]  

(12)

and

\[ \tilde{G}(x,y) := \int_{\mathbb{T}} dk \ e^{2\pi i ky} G(x,k), \quad (x,y) \in \mathbb{R} \times \mathbb{Z}. \]  

(13)

Let \( \mathcal{A} \) be the completion of the space \( \mathcal{S}(\mathbb{R} \times \mathbb{T}) \) in the norm

\[ \| G \|_{\mathcal{A}} := \int_{\mathbb{R}} \left( \sup_k | \tilde{G}(p,k) | \right) dp. \]

We let \( (\mathcal{A}', \| \cdot \|_{\mathcal{A}'}) \) be the dual of \( \mathcal{A} \).

For any two sequences \( J^{(j)} = (J^{(j)}_{y,y'})_{y,y' \in \mathbb{Z}}, \ j = 1, 2 \) of elements of \( L^2(T) \) we let

\[ \langle J^{(1)}, J^{(2)} \rangle := \sum_{y,y'} \int_{\mathbb{T}} J^{(1)}_{y,y'}(k)(J^{(2)}_{y,y'})^*(k) dk. \]  

(14)

By \( \mathcal{S} \) we denote the space made of those sequences \( J = (J_{y,y'}(k))_{y,y' \in \mathbb{Z}} \), for which \( J_{y,y'}(\cdot) \in L^1(\mathbb{T}) \) and for any non-negative integer \( m \) we have

\[ \sup_{y,y'} (\langle y \rangle_2 \langle y' \rangle_2)^m \left\{ |\tilde{J}_{y,y'}(y' - y)| + |\tilde{J}_{y,y'}(y - y')| \right\} < +\infty. \]  

(15)

For given \( p \geq 1, \ m \in \mathbb{R} \) we consider \( \mathcal{L}_p \) and \( \mathcal{H}_m \) – the completions of \( \mathcal{S} \) in the respective norms defined by

\[ \| J \|_{\mathcal{L}_p} := \left\{ \sum_{y,y'} \left( |\tilde{J}_{y,y'}(y' - y)|^p + |\tilde{J}_{y,y'}(y - y')|^p \right) \right\}^{1/p} < +\infty \]  

(16)
\[ \|J\|_{L^2} := \left\{ \sum_{y, y'} \left( |\langle J_{y, y'} \rangle (y')|^2 + |\langle J_{y, y'} \rangle (y - y')|^2 \right) \right\}^{1/2} < +\infty. \] (17)

2.2. Hypotheses about the coupling constants. Concerning the sequence \( \alpha = (\alpha_y)_{y \in \mathbb{Z}} \), appearing in (2), we assume, that it is real valued and satisfies the following:

(a1) there exists \( C > 0 \) such that \( |\alpha_y| \leq Ce^{-|y|/C} \), \( y \in \mathbb{Z} \),

(a2) it is even, i.e. \( \alpha_y = \alpha_{-y} \), \( y \in \mathbb{Z} \),

(a3) its Fourier transform \( \hat{\alpha}(\cdot) \) is nonnegative on \( \mathbb{T} \), and \( \hat{\alpha}(k) > 0 \) for \( k \neq 0 \).

We define the dispersion relation

\[ \omega(k) := \sqrt{\hat{\alpha}(k)}, \quad k \in \mathbb{T}. \] (18)

Thanks to the assumptions (a1)–(a3) it belongs to \( C^\infty(T \setminus \{0\}) \), in the acoustic case, i.e. when \( \hat{\alpha}(0) = 0 \), and belongs to \( C^\infty(T) \) in the pinned case, i.e. \( \hat{\alpha}(0) \neq 0 \).

2.3. Ornstein-Uhlenbeck perturbation. Let \( \xi = (\xi_y(t))_{(t, y) \in \mathbb{R} \times \mathbb{Z}} \) be a stationary Gaussian random field defined over some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). The covariance function of the field is given by (6). The functions \( \gamma(\cdot) \) and \( \sigma(\cdot) \) appearing in (6) are nonnegative and belong to \( C^2(T) \). We suppose furthermore that they are both even, so that the field is real valued, and that

\[ \gamma^* \leq \gamma(k) \leq \frac{1}{\gamma^*}, \quad \forall k \in \mathbb{T} \] (19)

for some \( \gamma^* \in (0, 1) \). Let us fix \( m > 1/2 \). From an elementary regularity theory of Gaussian processes, see e.g. Theorem 3.4.1, p. 60 of [2], the above assumptions guarantee that \( (\xi(t))_{t \in \mathbb{R}} \) has continuous trajectories in the Hilbert space \( \mathcal{E} := h^{-m} \).

Let \( \xi^*_y(t) := \xi_y(t/\epsilon) \). The field \( \xi^*_y(t) := (\xi^*_y(t))_{(t, y) \in \mathbb{R} \times \mathbb{Z}} \) is also Gaussian and homogeneous in \( (t, y) \). Therefore, see e.g. Theorem 5.2 of [3], for any \( \ell, N > 0 \) we have

\[ \bar{C}_{\ell, N} := \mathbb{E}|C^N_{\ell, \epsilon}(t, \epsilon) < +\infty, \] (20)

where \( \mathbb{E} \) is the expectation with respect to measure \( \mathbb{P} \) and

\[ C_{\ell}(t, \epsilon) := \sup_{y} \left| \frac{\xi^*_y(t)}{\langle y \rangle^\ell} \right|, \quad t \in \mathbb{R}. \] (21)

Due to stationarity of the process, the left hand side of (20) does not depend on \( t \).

2.4. Initial data. We assume that the initial data is random and distributed according to a probability law \( \mu_\epsilon \) supported in \( \ell_2(\mathbb{Z}) \). Moreover we require that macroscopically the amount energy per unit length stays finite, i.e. there exists finite \( K > 0 \) such that

\[ K := \sup_{\epsilon \in (0, 1]} \epsilon \langle \mathcal{H}(p, q) \rangle_{\mu_\epsilon} < +\infty, \] (22)

where \( \mathcal{H}(p, q) \) is given by (2).
2.5. Wave function and its dynamics. Suppose that \((q, p)\) belongs to \((\ell_2(\mathbb{Z}))^2\). The wave function (see [7]) is defined as
\[
\psi_y := (\hat{\omega} * q)_y + ip_y, \quad y \in \mathbb{Z}. \tag{23}
\]
Here \(\hat{\omega}_y\) are the Fourier coefficients of the dispersion relation (18). Since \(\omega' \in L^\infty(\mathbb{R})\) we conclude that \((\hat{\omega}_y) \in \ell_1(\mathbb{Z})\). The convolution \((\hat{\omega} * q)_y\) is given by \(\sum_{y'} \hat{\omega}_{y-y'} q_{y'}\) and belongs to \(\ell_2(\mathbb{Z})\). In fact, when \(\sum_y \hat{\omega}_y = 0\) it suffices only to assume that the sequence \((q_y - q_{y-1})_{y \in \mathbb{Z}}\) belongs to \(\ell_2(\mathbb{Z})\) in order for \((\hat{\omega} * q)\) to be defined as an element of \(\ell_2(\mathbb{Z})\).

Taking into account that the size of the stochastic perturbation is of order \(\sqrt{\epsilon}\) we expect that its effects shall be significant on the time scale \(t/\epsilon\) (as it is the scale on which its fluctuations are macroscopically of order \(O(1)\)). Adjusting the time variable to the macroscopic scale we define rescaled wave function, that corresponds to \((q(\cdot), p(\cdot))\) – the solution of (1) – as
\[
\psi^\epsilon_y(t) := \psi_y(t/\epsilon). \tag{24}
\]
From (1) we obtain the following equation
\[
\frac{d\psi^\epsilon_y(t)}{dt} = -i\frac{\epsilon}{\epsilon} (\hat{\omega} * \psi^\epsilon(t))_y + \frac{1}{2\sqrt{\epsilon}} \sum_{|z| \leq 2} \theta^\epsilon_{y+z}(t) \left(\psi^\epsilon_{y+z}(t) - (\psi^\epsilon)^*_y(t)\right), \tag{25}
\]
where \(\theta^\epsilon_{y,0}(t) \equiv 0\) and
\[
\theta^\epsilon_{y,\pm 2}(t) := \pm \xi^\epsilon_{y+1}(t), \quad \theta^\epsilon_{y,\pm 1}(t) := \pm \left(\xi^\epsilon_y(t) + \xi^\epsilon_{y\pm 1}(t)\right), \quad \text{and} \quad \xi^\epsilon_y(t) := \xi_y(t/\epsilon). \tag{26}
\]
Using Theorem A.2 below we conclude that the above Cauchy problem has a unique solution in the sense explained in Appendix A (see (149)), and the \(\ell_2\)-norm of the solution is preserved in time, i.e.
\[
\sum_{y \in \mathbb{Z}} |\psi^\epsilon_y(t)|^2 = \sum_{y \in \mathbb{Z}} |\psi_y(y)|^2, \quad t \geq 0, \quad \text{for a.s. realization of} \ \xi^\epsilon. \tag{27}
\]
From (22) we get
\[
\sup_{\epsilon \in (0, 1], t \geq 0} \sum_{y \in \mathbb{Z}} \epsilon |\psi^\epsilon_y(t)|^2 \mu_\epsilon = K, \quad \text{for a.s. realization of} \ \xi^\epsilon. \tag{28}
\]
By calculating the Fourier transform of the both sides in (25) we obtain the equation for the evolution of \(\hat{\psi}^\epsilon(t)\), which reads
\[
\frac{d\hat{\psi}^\epsilon(t, k)}{dt} = -i\frac{\omega(k)}{\epsilon} \hat{\psi}^\epsilon(t, k)
+ \frac{i}{\sqrt{\epsilon}} \int_{\mathbb{R}} r(k, k') \left(\hat{\psi}^\epsilon(t, k - k') - \hat{\psi}^\epsilon(t, k')\right) \hat{V}^\epsilon(t, dk'), \tag{29}
\]
\[
\hat{\psi}^\epsilon(0, k) = \hat{\psi}(k).
\]
Here \(\hat{V}^\epsilon(t, dk)\) – the spectral measure of the field \(\xi^\epsilon\) – is an \(H^{-m}(\mathbb{R})\)-valued process
\[
\hat{V}^\epsilon(t, dk) := \sum_{y \in \mathbb{Z}} \xi^\epsilon_y(t) \exp \{-2\pi i ky\} dk. \tag{30}
\]
The kernel \( r(k, k') \) is given by the formula
\[
r(k, k') := 4s(k)s(k - k')s(2k - k'), \quad k, k' \in \mathbb{T}
\]
and
\[
s(k) := \sin(\pi k), \quad c(k) := \cos(\pi k).
\]

**Remark 2.1.** To understand the integral appearing in the right hand side of (29) note that
\[
\int_{\pi} r(k, k') \hat{\psi}^{(\epsilon)}(t, k-k') \hat{V}^{(\epsilon)}(t, dk') = \sum_{j=1}^{4} \left( \hat{V}^{(\epsilon)}(t, \cdot) * (r_{j}^{(1)} \hat{\psi}^{(\epsilon)}) \right)(t, k) r_{j}^{(2)}(k), \tag{33}
\]
where \( r(k, k') = \sum_{j=1}^{4} r_{j}^{(1)}(k - k') r_{j}^{(2)}(k) \) and
\[
\begin{align*}
r_{1}^{(1)}(k) &= r_{2}^{(2)}(k) = 1, \\
r_{2}^{(1)}(k) &= r_{3}^{(2)}(k) := s(2k), \\
r_{3}^{(1)}(k) &= r_{4}^{(2)}(k) := s(k), \\
r_{4}^{(1)}(k) &= r_{4}^{(1)}(k') := c(k). \quad k, k' \in \mathbb{T}.
\end{align*}
\]
The stochastic measure \( \hat{V}^{(\epsilon)}(t, \cdot) \) belongs to \( H^{-m}(\mathbb{T}) \). Therefore, its convolutions with \( r_{j}^{(1)} \hat{\psi}^{(\epsilon)} \in L^{2}(\mathbb{T}) \) also belong to \( H^{-m}(\mathbb{T}) \) and, as a result, the right hand side of (33) is in that space as well. In particular, the time derivative of \( \hat{\psi}^{(\epsilon)}(t) \) appearing on the right hand side of (29) exists in the strong sense in \( H^{-m}(\mathbb{T}) \). Likewise the derivative appearing on the left hand side of (25) exists strongly in \( h^{-m} \).

### 2.6. Definition of the Wigner functions.

To describe the energy transport on the lattice in large space-time scales we use the lattice Wigner functions corresponding to \( \psi^{\epsilon}(t) \), see [22]. We define them as the distributions:
\[
(W^{\epsilon}(t), G) := \int_{\mathbb{R}} \hat{W}^{\epsilon}(t, p, k) \hat{G}^{\ast}(p, k) dp \tag{34}
\]
and
\[
(Y^{\epsilon}(t), G) := \int_{\mathbb{R}} \hat{Y}^{\epsilon}(t, p, k) \hat{G}^{\ast}(p, k) dp, \quad G \in \mathcal{A}. \tag{35}
\]

Here \( \hat{W}^{\epsilon}(t, p, k) \) and \( \hat{Y}^{\epsilon}(t, p, k) \) – the respective Fourier-Wigner functions – are given by
\[
\begin{align*}
\hat{W}^{\epsilon}(t, p, k) &:= \frac{\epsilon}{2} \left\langle \left( \hat{\psi}^{\epsilon} \right) \ast \left( t, k - \frac{cp}{2} \right) \hat{\psi}^{\epsilon} \left( t, k + \frac{cp}{2} \right) \right\rangle_{\mu_{\epsilon}}, \tag{36}
\hat{Y}^{\epsilon}(t, p, k) &:= \frac{\epsilon}{2} \left\langle \hat{\psi}^{\epsilon} \left( t, -k - \frac{cp}{2} \right) \hat{\psi}^{\epsilon} \left( t, k + \frac{cp}{2} \right) \right\rangle_{\mu_{\epsilon}}. \tag{37}
\end{align*}
\]
for \( (p, k) \in \mathbb{R} \times \mathbb{T} \) and \( t \geq 0 \). The averaging \( \left\langle \cdot \right\rangle_{\mu_{\epsilon}} \) is performed with respect to the initial distribution \( \mu_{\epsilon} \) so the expressions in (34) and (35) are random, since they depend on the realizations of \( \xi^{\epsilon} \). We can write
\[
\hat{W}^{\epsilon}(t, p, k) = \sum_{y, y' \in \mathbb{Z}} \mathcal{W}^{\epsilon}_{y, y'}(t, k) e^{-\pi i p (y + y')}, \tag{38}
\]
and
\[
\hat{Y}^{\epsilon}(t, p, k) = \sum_{y, y' \in \mathbb{Z}} \mathcal{Y}^{\epsilon}_{y, y'}(t, k) e^{-\pi i p (y + y')}, \tag{39}
\]
where
\[
\begin{align*}
\mathcal{W}^{\epsilon}_{y, y'}(t, k) &:= \frac{\epsilon}{2} \left\langle \psi^{\epsilon}_{y}(t) \psi^{\epsilon}_{y'}(t) \right\rangle_{\mu_{\epsilon}} e^{2i\pi k (y' - y)}, \tag{40}
\mathcal{Y}^{\epsilon}_{y, y'}(t, k) &:= \frac{\epsilon}{2} \left\langle \psi^{\epsilon}_{y}(t) \psi^{\epsilon}_{y'}(t) \right\rangle_{\mu_{\epsilon}} e^{2i\pi k (y' - y)}, \quad (y, y', k) \in \mathbb{Z}^{2} \times \mathbb{T}. \tag{41}
\end{align*}
\]
To close the equation for the dynamics of $W^\varepsilon(t)$ and $Y^\varepsilon(t)$ we shall also need $Y^\pm_y(k) := Y_y^\varepsilon(-k)^*$. It is clear that $W^\varepsilon$ and $Y^\varepsilon\pm$ belong to $L_1$, cf (16). Here $Y^\pm := Y_y^\varepsilon$.

Using Cauchy-Schwarz inequality we conclude that
\[
\sup_{t,\omega} \|\hat{W}^\varepsilon(t, p, \cdot)\|_{L^1(T)} \leq K, \quad \sup_{t,\omega} \|\hat{Y}^\varepsilon(t, p, \cdot)\|_{L^1(T)} \leq K, \quad \text{a.s.,}
\] (42)
where $K$ is as in (28). Directly from (42) we infer that
\[
\sup_{t,\omega} \|W^\varepsilon(t)\|_{\mathcal{A}'} \leq K, \quad \sup_{t,\omega} \|Y^\varepsilon(t)\|_{\mathcal{A}'} \leq K, \quad \text{a.s.}
\] (43)
where $K$ is the constant defined in (22).

2.7. The statement of the main result. We formulate the following Cauchy problem for the linear kinetic equation
\[
\begin{aligned}
\partial_t U(t, x, k) + \frac{\omega'(k)}{2\pi} \partial_x U(t, x, k) &= \mathcal{L}U(t, x, k), \quad (t, x, k) \in \mathbb{R}_+ \times \mathbb{R} \times T, \\
U(0, x, k) &= U_0(x, k),
\end{aligned}
\] (44)
with an appropriate initial condition $U_0$. The scattering operator $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ has the form
\[
\hat{\mathcal{L}}G(p, k) := \int_T R(k, k') [\hat{G}(p, k') - \hat{G}(p, k)] dk', \quad G \in \mathcal{A},
\] (45)
with the scattering kernel $R(k, k')$ given by (9).

Denote by $M_+(\mathbb{R} \times T)$ the space of all finite Borel measures on $\mathbb{R} \times T$ and by $L^1_+(\mathbb{R} \times T)$ the subset of all non-negative elements of $L^1(\mathbb{R} \times T)$. We shall write $\langle \mu, J \rangle := \int_{\mathbb{R} \times T} J^* d\mu$, where $J$ belongs to $B_b(\mathbb{R} \times T)$ - the space of all bounded, Borel measurable functions on $\mathbb{R} \times T$.

**Definition 2.2.** Suppose that $U_0 \in M_+(\mathbb{R} \times T)$. By a solution of equation (44) we understand a function
\[
[0, +\infty) \ni t \to U(t) \in M_+(\mathbb{R} \times T)
\]
satisfying the following:

i) for any $T > 0$ we have
\[
\sup_{t \in [0, T]} U(t; \mathbb{R} \times T) < +\infty,
\] (46)

ii) for each $G \in \mathcal{S}(\mathbb{R} \times T)$ the function $t \mapsto \langle U(t), G \rangle$ is measurable,

iii) for every continuous and differentiable $G(\cdot) \in C^1([0, +\infty); \mathcal{S}(\mathbb{R} \times T))$ we have:
\[
\langle U(t), G(t) \rangle - \langle U_0, G(0) \rangle = \int_0^t ds \langle U(s), \partial_s G(s) \rangle
\] (47)
\[
+ \frac{1}{2\pi} \int_0^t ds \langle U(s), \omega' \partial_x G(s) \rangle + \int_0^t ds \langle U(s), \mathcal{L}G(s) \rangle.
\]

We have the following result concerning the existence and uniqueness of solutions of (44).

**Theorem 2.3.** Suppose that $U_0 \in M_+(\mathbb{R} \times T)$. Then, equation (44) has a solution $U(\cdot)$ in the aforementioned sense. The solution is unique if the dispersion relation...
satisfies $\omega \in C^\infty(\mathbb{T})$. In addition, in the acoustic case ($\hat{\alpha}(0) = 0$) if $U_0 \in M_+(\mathbb{R} \times \mathbb{T})$ satisfies
\begin{equation}
\lim_{\delta \to 0+} U_0(\mathbb{R} \times \{ |k| < \delta \}) = 0 \tag{48}
\end{equation}
then the solution is unique. We also have
\begin{equation}
U(t;\mathbb{R} \times \mathbb{T}) = U_0(\mathbb{R} \times \mathbb{T}) \quad \text{for all} \quad t \geq 0. \tag{49}
\end{equation}
If $U_0 \in L^1_+(\mathbb{R} \times \mathbb{T})$ then $U(t) \in L^1_+(\mathbb{R} \times \mathbb{T})$ for every $t \in [0, +\infty)$.

The proof of the above theorem is presented in Section C of the Appendix.

The goal of the present paper is to show that the average of the Wigner function converges, as $\epsilon \to 0+$, to the solution of (44). We consider both the case of an acoustic and pinned chain and i.e. when, respectively
\begin{enumerate}
   \item[(a4)] we have $\hat{\alpha}(0) = 0$ and $\hat{\alpha}''(0) > 0,$ or
   \item[(a4')] $\hat{\alpha}(0) > 0$.
\end{enumerate}
In the case (a4) we shall need an additional condition, namely:
\begin{equation}
\lim_{\delta \to 0+} \limsup_{\epsilon \to 0+} \left\{ \epsilon \int_{\{|k| \leq \delta\}} \langle |\hat{\psi}(k)|^2 \rangle_{\mu_\epsilon} dk \right\} = 0. \tag{50}
\end{equation}

**Remark 2.4.** The above condition appeared in [7], see condition (b4) on p. 177. It can be interpreted as the absence of energy at the longest wavelengths in the acoustic chain at time $t = 0$. This component of energy corresponds to the macroscopic profiles of the elongation $r_y := q_y - q_{y-1}$ and momentum $p_y$, see [16].

Our main result can be stated as follows.

**Theorem 2.5.** Suppose that the sequence of coupling constants $\alpha$ satisfies (a1)-(a3), and either (a4) or (a4'). Assume that the initial distributions ($\mu_\epsilon$) satisfy (22), and in case (a4) holds, we suppose also that (50) is in force. Furthermore, assume that there exists $U_0 \in M_+(\mathbb{R} \times \mathbb{T})$ for which
\begin{equation}
\lim_{\epsilon \to 0+} \langle W^\epsilon(0), G \rangle = \langle U_0, G \rangle, \quad G \in \mathcal{A}. \tag{51}
\end{equation}
Then, for each $t \geq 0$ fixed, $E(W)(t)$ converges as $\epsilon \to 0+$ in the $*$-weak topology of $\mathcal{A}'$ to $U(t)$, defined by the solution of the Cauchy problem (44).

### 3. Evolution of the Wigner functions.
In the present section we formulate the equations governing the evolution of the Wigner functions $W^\epsilon(t)$ and $Y^\epsilon(t)$, see (69) and (74) below. Note that the right hand side of equation (69) below contains terms that are of order of magnitude $\epsilon^{-1/2}$. This stands in stark contrast with (42) of [7] where we find no large terms (in $\epsilon$) in its right hand side.

We start with the following.

**Proposition 3.1.** Suppose that $J \in \mathcal{H}_m$ for some $m > 0$ (see (17)). Then for any $\epsilon > 0$ we have
\begin{equation}
\frac{d}{dt} \langle W^\epsilon(t), J \rangle = \langle \partial_t W^\epsilon(t), J \rangle \tag{51}
\end{equation}
and
\begin{equation}
\frac{d}{dt} \langle Y^\epsilon(t), J \rangle = \langle \partial_t Y^\epsilon(t), J \rangle, \quad \text{for a.e. } t \geq 0, \text{ a.s. in } \xi^\epsilon. \tag{52}
\end{equation}
Proof. We only prove equality (51), as the argument for (52) is analogous. The equality in question follows, provided we can prove that:

$$\sum_{y,y'} \int_0^T \mathbb{E} \left\{ \left| \int_T \partial_t \mathcal{W}^*(t,k) J_{y,y'}^* (k) \, dk \right| \right\} dt < +\infty, \quad T \geq 0. \quad (53)$$

Indeed, by the Lebesgue dominated convergence theorem we conclude then that

$$\langle \mathcal{W}^*(T), J \rangle - \langle \mathcal{W}^*(0), J \rangle$$

$$= \sum_{y,y'} \int_0^T \left\{ \int_T \partial_t \mathcal{W}^*(t,k) J_{y,y'}^* (k) \, dk \right\} dt, \quad T \geq 0, \text{ a.s. in } \xi^\epsilon$$

and formula (51) follows from the differentiation of both sides of (54).

Performing the integral over $k$ we can rewrite the series in (53) in the form

$$\sum_{y,y'} \int_0^T \frac{\epsilon}{2} \mathbb{E} \left\{ \left| \partial_t \langle \psi^\epsilon_y (t) (\psi^\epsilon_{y'})^* (t) \rangle_{\mu_\epsilon} J_{y,y'}^* (y - y') \right| \right\} dt. \quad (55)$$

From (25) we get

$$\partial_t \langle \psi^\epsilon_y (t) (\psi^\epsilon_{y'})^* (t) \rangle_{\mu_\epsilon} = \frac{i}{\epsilon} \left[ \langle \psi^\epsilon_y (t) (\tilde{\omega} * \psi^\epsilon_{y'})^* (t) \rangle_{\mu_\epsilon} - \langle (\psi^\epsilon_{y'})^* (t) \tilde{\omega} * \psi^\epsilon_y (t) \rangle_{\mu_\epsilon} \right]$$

$$+ \frac{1}{2 \sqrt{\epsilon}} \sum_{|z| \leq 2} \theta^\epsilon_{y,z} (t) \left( \langle \psi^\epsilon_{y+z} (t) - (\psi^\epsilon_y)^{y+z} (t) (\psi^\epsilon_{y'})^* (t) \rangle_{\mu_\epsilon} \right)$$

$$+ \frac{1}{2 \sqrt{\epsilon}} \sum_{|z| \leq 2} \theta^\epsilon_{y'+z} (t) \left( \langle (\psi^\epsilon)^{y'+z} (t) - \psi^\epsilon_{y'+z} (t) \psi^\epsilon_y (t) \rangle_{\mu_\epsilon} \right). \quad (56)$$

Substituting into (55) we obtain an expression that can be written as the sum of the series corresponding to each term appearing in the right hand side of (56). We shall prove that

$$\sum_{y,y'} \int_0^T \mathbb{E} \left\{ \left| \langle \psi^\epsilon_y (t) (\tilde{\omega} * \psi^\epsilon_{y'})^* (t) \rangle_{\mu_\epsilon} J_{y,y'}^* (y - y') \right| \right\} dt < +\infty \quad (57)$$

and

$$\sum_{y,y',|z| \leq 2} \int_0^T \mathbb{E} \left\{ \left| \langle \theta^\epsilon_{y,z} (t) \psi^\epsilon_{y+z} (t) (\psi^\epsilon_{y'})^* (t) \rangle_{\mu_\epsilon} J_{y,y'}^* (y - y') \right| \right\} dt < +\infty, \quad (58)$$

as the remaining terms can be dealt with in a similar fashion.

The expression in (57) can be estimated by the Cauchy-Schwarz inequality and we infer that it can be bounded from above by

$$\| J \|_{\mathcal{L}_1} \int_0^T \mathbb{E}_x \left( \| \psi^\epsilon (t) \|_{\ell_2} \| \tilde{\omega} * \psi^\epsilon (t) \|_{\ell_2} \right) dt. \quad (59)$$

Here $\mathbb{E}_x$ is the expectation with respect to the product measure $\mathbb{P} \otimes \mu_\epsilon$. The assumptions about regularity of the dispersion relation imply that $\omega \in H^1 (T)$, therefore, in particular $(\tilde{\omega})_{x \in \mathbb{Z}} \in \ell_1$ and by Young’s inequality

$$\| \tilde{\omega} * \psi^\epsilon (t) \|_{\ell_2} \leq \| \tilde{\omega} \|_{\ell_1} \| \psi^\epsilon (t) \|_{\ell_2}, \quad t \geq 0, \epsilon > 0 \quad (60)$$

thus the series (57) is finite thanks to (28).
Concerning (58) observe that for a fixed \( \epsilon > 0 \) the expression appearing there can be estimated, using stationarity of \( \xi \), by (cf the definition (20))

\[
\int_0^T \mathbb{E} \left[ C_m(t; \epsilon) \sum_{y,y'} \langle (y \langle y' \rangle)^m \rangle | \mathcal{F}_t \rangle \langle \psi_{y'}^\epsilon(t) | \psi_y^\epsilon(t) \rangle_{\mu_\epsilon} \right],
\]

where \( C_m(t; \epsilon) \) is defined in (21). Using Cauchy-Schwarz inequality we can estimate the expression by

\[
\| J \|_{\mathcal{H}_m} \int_0^T \mathbb{E} \left[ C_m(t; \epsilon) \langle \| \psi^\epsilon(t) \|_{L_2}^2 \rangle_{\mu_\epsilon} \right] dt = T \| J \|_{\mathcal{H}_m} \tilde{C}_{m,1} \langle \| \psi^\epsilon(0) \|_{L_2}^2 \rangle_{\mu_\epsilon} < +\infty.
\]

In the last equality we have used the time invariance of \( \| \psi^\epsilon(t) \|_{L_2}^2 \) (cf (28)).

In what follows we shall adopt the shorthand notation

\[
\begin{align*}
    r^+_{y,y'}(k,k') &:= r(k,k') e^{2\pi ik'y'}, \\
    r^-_{y,y'}(k,k') &:= r(k,k') e^{2\pi ik'y'}, \\
    r^{\prime+}_{y,y'}(k,k') &:= r^+(y,y') + r^-(y,y').
\end{align*}
\]

Using Proposition 3.1 and formula (56) we conclude that for any \( J \in \mathcal{S} \) we have

\[
\frac{d}{dt} \langle \mathcal{W}^\epsilon(t), J \rangle = -\frac{1}{\epsilon} \langle \mathcal{W}^\epsilon(t), \mathcal{D} J \rangle - \frac{i}{\sqrt{\epsilon}} \langle \mathcal{W}^\epsilon(t), K_{\xi(t)}^o J \rangle + \frac{i}{\sqrt{\epsilon}} \sum_{i=\pm} \langle \mathcal{W}^\epsilon(t), K_{\xi(t)}^i J \rangle
\]

for a.e. \( t \geq 0 \) and a.s. realization of \( \xi^\epsilon \). Here, for any \( J \in \mathcal{S} \) we have defined:

\[
\begin{align*}
    \mathcal{D} J_{y,y'}(k) &:= i \sum_{z,z'} J_{z,z'}(k) \int_{\mathbb{T}^2} e^{2\pi i p(z-y)} e^{2\pi i p'(z'-y')} \left[ \omega(k+p') - \omega(k-p) \right] dpdp', \\
    K_{\xi}^- J_{y,y'}(k) &:= \sum_{z,z'} J_{z,z'}(k) \int_{\mathbb{T}^2} e^{2\pi i p(z-y)} e^{2\pi i p'(z'-y')} r^-_{y,y'}(k+k'; f) dpdp', \\
    K_{\xi}^+ J_{y,y'}(k) &:= \sum_{z,z'} J_{z,z'}(k) \int_{\mathbb{T}^2} e^{2\pi i p(z-y)} e^{2\pi i p'(z'-y')} r^{\prime+}_{y,y'}(k+k'; f) dpdp',
\end{align*}
\]

\( \pi \) a.s. \( f \in \mathcal{E} \) and

\[
K^o := K^- + K^+.
\]

A stationary Gaussian random field \((r^\epsilon_{y,y'}))_{y,y' \in \mathbb{Z}}\) is given by

\[
r^\epsilon_{y,y'}(k; f) := \int_{\mathbb{Z}^2} r^\epsilon_{y,y'}(k+k') \bar{V}(dk'; f), \quad (y,y', f) \in \mathbb{Z}^2 \times \mathcal{E}, \quad \epsilon \in \{-,0,\}.
\]

Analogously, we obtain

\[
\frac{d}{dt} \langle \mathcal{Y}^\epsilon(t), J \rangle = -\frac{i}{\epsilon} \langle \mathcal{Y}^\epsilon(t), \Theta J \rangle - \frac{i}{\sqrt{\epsilon}} \langle \mathcal{Y}^\epsilon(t), K_{\xi(t)}^o J \rangle + \frac{i}{\sqrt{\epsilon}} \langle \mathcal{Y}^\epsilon(t), K_{\xi(t)}^+ J \rangle,
\]

where

\[
J_{z,z'}^\pm(k) := J_{z,z'}(k) + J_{z',z}(-k)
\]

and

\[
\Theta J_{y,y'}(k) := \sum_{z,z'} J_{z,z'}(k) \int_{\mathbb{T}^2} e^{2\pi i p(z-y)} e^{2\pi i p'(z'-y')} \left[ \omega(k+p') + \omega(k-p) \right] dpdp'.
\]

For \( G \in \mathcal{S}(\mathbb{R} \times \mathbb{T}) \) we denote, cf (11),

\[
G^\epsilon_{y,y'}(k) := G(\epsilon(y+y')/2, k) \quad \text{and} \quad G^\epsilon_{y,y'} := \tilde{G}(\epsilon(y+y')/2, y - y'), \quad y,y' \in \mathbb{Z}.
\]

(67)
Note that \( G^* \in \mathcal{S} \). Thanks to (38) for any \( \epsilon > 0 \) we have

\[
\frac{d}{dt} \langle W^*(t), G \rangle = \langle \partial_t W^*(t), G^* \rangle, \quad t \geq 0
\]

for a.s realization of \( \xi^*(\cdot) \). From (68) and (62) we obtain, after a straightforward calculation, that

\[
\frac{d}{dt} \langle W^*(t), G \rangle = -\langle W^*(t), \mathcal{D}_* G^* \rangle - \frac{i}{\sqrt{\epsilon}} \langle W^*(t), \mathcal{K}_{\xi^*(t)}^\circ G^* \rangle + \frac{i}{\sqrt{\epsilon}} \sum_{i=\pm} \langle Y^{\epsilon,i}, \mathcal{K}_{\xi^*(t)}^i G^* \rangle.
\]

Here

\[
\mathcal{D}_* G_{y,y'}^{*}(k) := \frac{1}{\epsilon} \mathcal{D} G_{y,y'}^{*}(k)
\]

\[
= \frac{i}{\epsilon} \sum_z G(\varepsilon z/2, k) \int_T e^{2\pi ip(z-y-y')}[\omega(k + p) - \omega(k - p)]dp,
\]

\[
\mathcal{K}^{-}_{f} G_{y,y'}^{*}(k) = \sum_z G(\varepsilon z/2, k) \int_T e^{2\pi ip(z-y-y')}p_{y,y'}^-(k + p; f)dp,
\]

\[
\mathcal{K}^{+}_{f} G_{y,y'}^{*}(k) = \sum_z G(\varepsilon z/2, k) \int_T e^{2\pi ip(z-y-y')}p_{y,y'}^+(k + p; f)dp,
\]

\( \pi \) a.s. in \( f \in \mathcal{E} \).

A simple calculation shows that

\[
\langle W^*(t), \mathcal{D}_* G^* \rangle = -i \int_{-1/(2\varepsilon)}^{1/(2\varepsilon)} \int_T \delta_{\epsilon}(k) p \hat{W}^\varepsilon(t, p, k) \hat{G}^*(p, k)dpdk + O(t, \epsilon),
\]

where \( \lim_{\varepsilon \to 0^+} \sup_{t \in [0, T]} |O(t, \epsilon)| = 0 \) for any \( T > 0 \) and

\[
\delta_{\epsilon}(k) := \frac{1}{\epsilon} \left[ \omega \left( k + \frac{\epsilon p}{2} \right) - \omega \left( k - \frac{\epsilon p}{2} \right) \right].
\]

As a result we conclude that

\[
\langle W^*(t), \mathcal{D}_* G^* \rangle = -i \int_T \omega(k) \hat{W}^\varepsilon(t, p, k) \hat{G}^*(p, k)dpdk + O(t, \epsilon).
\]

Analogously, we obtain

\[
\frac{d}{dt} \langle Y^*(t), G \rangle = -\frac{i}{\epsilon} \langle Y^*, \Theta G^* \rangle - \frac{i}{\sqrt{\epsilon}} \langle Y^*(t), \mathcal{K}_{\xi^*(t)}^\circ G^* \rangle + \frac{i}{\sqrt{\epsilon}} \langle W^*(t), \mathcal{K}_{\xi^*(t)}^+ G^* \rangle,
\]

where \( G_{y,y'}^{*}(k) := G(\epsilon(y + y')/2, k) + G(\epsilon(y + y')/2, -k) \) and

\[
\Theta G_{y,y'}^{*}(k) = \sum_z G(\varepsilon z/2, k) \int_T e^{2\pi ip(z-y-y')}[\omega(k + p) + \omega(k - p)]dp.
\]

4. Properties of the stochastic perturbation.

4.1. Markov property of the process \( \xi \). The process \( \xi \), described in Section 2.3, is Markovian and reversible in the following sense. Denote by \( \pi \) the law of \( \xi(0) \) on \( (\mathcal{E}, B(\mathcal{E})) \), with \( B(\mathcal{E}) \) the Borel \( \sigma \)-algebra on \( \mathcal{E} \). Let \( (\mathcal{F}_t)_{t \geq 0} \) be the natural filtration associated with the process. Using a fairly standard argument, relying on the second quantization (see e.g. Chapter 4 of [12]), one can show that there exists a strongly continuous semigroup \( (P_t)_{t \geq 0} \) of non-negative, symmetric contractions on \( L^2(\pi) \), the transition probability operators, such that \( P_1 \mathbf{1} = \mathbf{1} \) and
(i) (Markov property) for any $F \in L^2(\pi)$

$$
E[F(\xi(t + s))|\mathcal{F}_t] = E[F(\xi(t + s))|\xi(t)] = P_t F(\xi(t)), \quad \forall t, s \geq 0,
$$

(ii) (spectral gap property) for any $F \in L^2(\pi)$ such that $\int_{\mathbb{T}} F d\pi = 0$ we have

$$
\|P_t F\|_{L^2(\pi)} \leq e^{-\gamma t}\|F\|_{L^2(\pi)}, \quad \forall t \geq 0,
$$

with $\gamma$ as in (19).

Suppose that $y \in \mathbb{Z}$ and $V_y : E \to \mathbb{R}$ is the coordinate mapping on $E$ given by

$$
V_y(f) := f_y, \quad f \in h^{-m}.
$$

Then, $V = (V_y)_{y \in \mathbb{Z}}$ is a Gaussian field whose co-variance equals

$$
\langle V_y, V_{y'} \rangle_{L^2(\pi)} = \int_{\mathbb{T}} e^{2\pi i k(y-y')} \sigma(k) dk, \quad \forall y, y' \in \mathbb{Z},
$$

with $\langle \cdot, \cdot \rangle_{L^2(\pi)}$ the scalar product in $L^2(\pi)$. Its spectral measure $\hat{V}(dk)$, cf (30), is the $L^2(\pi)$-valued Borel measure on $\mathbb{T}$ satisfying

$$
\hat{V}(A) := \sum_{z \in \mathbb{Z}} V_z \int_A e^{-2\pi i k z} dk, \quad A \in \mathcal{B}(\mathbb{T}).
$$

For any $\psi(\cdot)$ that belongs to $L^2(\mathbb{T};\sigma)$ - the space of complex valued functions on $\mathbb{T}$, square integrable with respect to $\sigma(k)dk$ - we can define the stochastic integral with respect to the spectral measure as

$$
\mathcal{I}(\psi) = \int_{\mathbb{T}} \psi(k) \hat{V}(dk) := \sum_{z \in \mathbb{Z}} V_z \int_{\mathbb{T}} e^{-2\pi i k z} \psi(k) dk.
$$

The series on the right hand side of (75) is convergent in $L^2(\pi)$. The measure $\hat{V}$ satisfies $\hat{V}^\ast(dk) = \hat{V}(-dk)$ and its structure measure equals

$$
E[\hat{V}(dk)\hat{V}(dk')] = \delta(k + k')\sigma(k)dk.
$$

Denote by $\mathfrak{h}_1$ the closed subspace of $L^2(\pi)$ spanned by the stochastic integrals (75) (the first degree Hermite polynomials in the chaos expansion over Gaussian measure $\pi$). From the construction of the semigroup, via the second quantization (see p. 45 of [12]), each $P_t$ leaves $\mathfrak{h}_1$ invariant for any $t \geq 0$. For any $\psi_1, \psi_2 \in L^2(\mathbb{T};\sigma)$ we conclude from (6) that

$$
E[\mathcal{I}(\psi_1;\xi(t))\mathcal{I}(\psi_2;\xi(0))] = \langle S_t \psi_1, \psi_2 \rangle_{L^2(\mathbb{T};\sigma)} = E[\mathcal{I}(S_t \psi_1;\xi(0))\mathcal{I}(\psi_2;\xi(0))],
$$

where $\langle \cdot, \cdot \rangle_{L^2(\mathbb{T};\sigma)}$ the scalar product in $L^2(\mathbb{T};\sigma)$ and $S_t \psi_1(k) := e^{-\gamma(k)t} \psi_1(k)$. Thanks to the invariance of $\mathfrak{h}_1$ under $P_t$ we conclude from the above that $P_t \mathcal{I}(\psi_1) = \mathcal{I}(S_t \psi_1)$ for all $t \geq 0$ and, as a result,

$$
\mathfrak{Q} \mathcal{I}(\psi_1) = -\mathcal{I}(\gamma \psi_1),
$$

where $\mathfrak{Q}$ is the $L^2(\pi)$ generator of the semigroup $(P_t)_{t \geq 0}$. In addition, using formula (4.3), p. 45 of [12], we may conclude that

$$
\mathfrak{Q}(\mathcal{I}(\psi_1)\mathcal{I}(\psi_2)) = -\left(\mathcal{I}(\gamma \psi_1)\mathcal{I}(\psi_2) + \mathcal{I}(\psi_1)\mathcal{I}(\gamma \psi_2)\right)
+ 2 \int_{\mathbb{T}} \gamma(k)\sigma(k)\psi_1(k)\psi_2(-k) dk.
$$
Remark 4.1. Suppose that $\sigma(k) \equiv \gamma/2$, and $\gamma(k) \equiv \gamma$, where $\gamma > 0$ is some constant. The respective processes $(\xi(t))_{t \geq 0}$ converge in law, as $\gamma \to +\infty$, to a white noise process $(\dot{w}(t))_{t \geq 0}$ on $L^2(\mathbb{Z})$, i.e. $\dot{w}(t) = (\dot{w}_y(t))_{y \in \mathbb{Z}}$ where $(\dot{w}_y)_{y \in \mathbb{Z}}$ are i.i.d Gaussian, $S'(\mathbb{R})$-valued processes satisfying

$$E(\dot{w}_y, J_1) = 0, \quad E[(\dot{w}_y, J_1)(\dot{w}_{y'}, J_2)] = \delta_{yy'}(J_1, J_2)_{L^2(\mathbb{R})}$$

for all $y, y' \in \mathbb{Z}$, $J_1, J_2 \in S(\mathbb{R})$. The model of harmonic oscillators with the conservative noise introduced in [6] is therefore the limiting case of the present model, as the mixing rate (spectral gap size) $\gamma$ of the Ornstein-Uhlenbeck process tends to $+\infty$.

4.2. Multiple stochastic integrals. Using the spectral measure $\hat{V}(dk)$ we can define a multiple spectral integral

$$\mathcal{I}_n(\psi) = \int_{\mathcal{T}^n} \psi(k_1, \ldots, k_n)\hat{V}(dk_1, \ldots, dk_n).$$

(78)

Computations using objects of this type appear in the proofs of Lemma 5.1 (see steps 2 and 3) and Lemma 5.2 below. We let $\mathcal{I}_n$ be a linear mapping defined on the space $\mathcal{L}_n^{(0)}$ of finite linear combinations of the tensor products $\psi_1 \otimes \ldots \otimes \psi_n$, where $\psi_1, \ldots, \psi_n \in C(T)$ are complex valued, and

$$\mathcal{I}_n(\psi_1 \otimes \ldots \otimes \psi_n) := \prod_{j=1}^n \mathcal{I}(\psi_j).$$

Here $\mathcal{I}(\psi_j)$ is the stochastic integral given by (75). By a density argument we can extend $\mathcal{I}_n$ to the space $\mathcal{L}_n^2(\sigma)$ obtained as the completion of $\mathcal{L}_n^{(0)}$ with respect to the Hilbert pseudo-norm

$$\|\psi\|^2_{\mathcal{L}_n^2(\sigma)} := \sum_{\mathfrak{F}} \int_{\mathbb{R}^{2n}} \psi(k_1, \ldots, k_n)\psi^*(k_{n+1}, \ldots, k_{2n}) \times \prod_{(j, j') \in \mathfrak{F}} \sigma(k_j)\delta(k_j + k_{j'})dk_1 \ldots dk_{2n}.$$

The summation extends over all possible pairings $\mathfrak{F}$ made between elements of the set $\{1, \ldots, 2n\}$. The multiple stochastic integral in (78) is then defined on $\mathcal{L}_n^2(\sigma)$ as the resulting extension of $\mathcal{I}_n$, see appendix of [17] for details of this construction.

Given $n \geq 1$ the subspace $\mathfrak{p}_n$ of $L^2(\pi)$ is defined as the closure of the linear span over all $\mathcal{I}_m(\psi), \psi \in \mathcal{L}_m^2(\sigma)$, where $m \leq n$. It is called the space of the polynomials of $n$-th degree, of Def. 2.1, p. 17 of [12]. It is clear that $\mathfrak{p}_n \subset \mathfrak{p}_{n+1}$ for each $n \geq 1$, with $\mathfrak{p}_0 := \text{span}(1)$.

4.3. Pseudogenerator. We recall the notion of the pseudogenerator of a process following Section 2.2 p. 38 of [11] (see also [20]). Suppose that $\eta = (\eta(t))_{t \geq 0}$ is a complex valued stochastic process over $(\Omega, \mathcal{F}, \mathbb{P})$ that is progressively measurable with respect to a filtration $(\mathcal{F}_t)_{t \geq 0}$ and such that

$$\sup_{t \in [0, T]} E|\eta(t)| < +\infty, \quad \forall T > 0 \quad \text{and} \quad \lim_{h \to 0+} E|\eta(t+h) - \eta(t)| = 0, \quad \forall t \geq 0. \quad (79)$$

We say that the process $\mathfrak{L}\eta = (\mathfrak{L}\eta(t))_{t \geq 0}$ is the pseudogenerator of $\eta$ if it is progressively measurable, subordinated with respect to $(\mathcal{F}_t)_{t \geq 0}$, satisfies (79) and
It is well known, see Theorem 2.2.1, p. 39 of [11], that then the process \( Q \) is a martingale.

Consider the processes \( \Phi \) and \( \Psi \) equal to \( \xi(t) \) and \( \sqrt{y,y} \), respectively. The proof of this result is presented in Section B of the Appendix.

Proposition 4.2. Under the hypotheses made above the processes \( (\Phi^\epsilon(t)) \) and \( (\Psi^\epsilon(t)) \) satisfy (79) for each \( \epsilon > 0 \). In addition, the respective pseudo-generators equal

\[
\mathfrak{L}\Phi^\epsilon(t) = \frac{1}{\epsilon} \langle \mathcal{W}^\epsilon(t), \mathcal{D} J^\epsilon(t) \rangle - \frac{1}{\epsilon} \langle \mathcal{W}^\epsilon(t), \mathcal{D} J^\epsilon(t) \rangle
\]

\[
- \frac{i}{\sqrt{\epsilon}} \langle \mathcal{W}^\epsilon(t), \mathcal{K}_{\xi^\epsilon(t)}^0 J^\epsilon(t) \rangle + \frac{i}{\sqrt{\epsilon}} \sum_{i=\pm} \langle \mathcal{W}^\epsilon(t), \mathcal{K}_{\xi^\epsilon(t)}^i J^\epsilon(t) \rangle;
\]

\[
\mathfrak{L}\Psi^\epsilon(t) = \frac{1}{\epsilon} \langle \mathcal{W}^\epsilon(t), \mathcal{D} J^\epsilon(t) \rangle - \frac{i}{\sqrt{\epsilon}} \langle \mathcal{W}^\epsilon(t), \mathcal{D} J^\epsilon(t) \rangle - \frac{i}{\sqrt{\epsilon}} \langle \mathcal{W}^\epsilon(t), \mathcal{K}_{\xi^\epsilon(t)}^0 J^\epsilon(t) \rangle
\]

\[
+ \frac{i}{\sqrt{\epsilon}} \langle \mathcal{W}^\epsilon(t), \mathcal{K}_{\xi^\epsilon(t)}^+ J^\epsilon(t) \rangle, \quad t \geq 0.
\]

The proof of this result is presented in Section B of the Appendix.
5. Proof of Theorem 2.5.

5.1. Outline of the proof of Theorem 2.5. We would like to sketch briefly and somewhat informally the method of the proof of the theorem. It follows from (69) that for any test function $G \in \mathcal{S}(\mathbb{R} \times T)$ we can write that

$$\frac{d}{dt}(W^\epsilon(t), G) = -\langle W^\epsilon(t), \omega'(k)G \rangle$$

where $\omega'(k) = \langle \mathcal{R}_x(t)G^\epsilon(t) \rangle + \frac{i}{\sqrt{\epsilon}} \langle \mathcal{K}^\circ_x(t)G^\epsilon(t) \rangle + \frac{i}{\sqrt{\epsilon}} \sum_{j=\pm} \langle Y^\circ_j(t), \mathcal{K}^\circ_{x_j}(t)G^\epsilon(t) \rangle + o(1),$

where $o(1) \to 0+$, as $\epsilon \to 0+$, $G^\epsilon$ is given by (67) and $\mathcal{K}^\iota$, $\iota \in \{-, +, o\}$ are the random scattering operators introduced in Section 3. It is worthwhile to compare this equation with the corresponding one for $\partial_t(W^\epsilon(t), G)$ appearing in [7], see (42) p. 187. All terms in the right hand side of (42) are of order 1. The elimination of the terms containing $Y^\iota(t)$ is then possible due to the fact that from the equation describing the dynamics of $(Y^\iota(t), G)$, see (44) ibid., we can easily conclude that $|\langle Y^\iota(t), G \rangle| \to 0$, as $\epsilon \to 0+$, provided $G(x, k)/\omega(k)$ stays bounded. The identification of the limit of $(W^\epsilon(t), G)$ follows then from a fairly direct calculation, see Section 4.3.1 of [7].

This argument cannot be applied in the present case since the terms appearing in the right hand side of (88), except the one corresponding to the dispersion, are of order of magnitude $\epsilon^{-1/2}$. To deal with the large terms we use the perturbed test method that comes from the stochastic averaging theory, see [20, 11]. Let us describe briefly this technique. To keep our discussion as simple as possible we shall omit in our sketch the terms in (88) containing $Y_{\epsilon, \pm}(t)$.

The main idea is to replace the test function $G$ in (88) by a field $\tilde{G}^\epsilon : E \to S$, cf (15), such that

$$\tilde{G}^\epsilon(\xi(t)) = G^\epsilon + \sqrt{\epsilon}G^{1, \epsilon}(\xi(t)) + \epsilon G^{3, \epsilon}(\xi(t)),$$

where for a given $G \in \mathcal{S}(\mathbb{R} \times T)$ (deterministic) test function $G^\epsilon$ is defined by (67) and $G^{j, \epsilon} : E \to S$, $j = 1, 2$ (called correctors) are yet to be determined. Substituting into (88) the expression for $\tilde{G}^\epsilon(\xi(t))$ and using the Markov property of $\xi$, see Section 4.1, we obtain that

$$\partial_t(W^\epsilon(t), \tilde{G}^\epsilon(\xi(t))) \approx \frac{1}{\sqrt{\epsilon}} \psi^1_\epsilon(t) + \psi^2_\epsilon(t) + \sqrt{\epsilon} \psi^3_\epsilon(t),$$

where,

$$\psi^1_\epsilon(t) := \left\langle \mathcal{W}^\epsilon(t), i\mathcal{K}^G_{\xi(t)}G^\epsilon(\xi(t)) + \left(\Omega - \mathcal{D}\right)G^{1, \epsilon}(\xi(t)) \right\rangle,$$

$$\psi^2_\epsilon(t) := \left\langle \mathcal{W}^\epsilon(t), i\mathcal{K}^\circ_{\xi(t)}G^{1, \epsilon}(\xi(t)) + \left(\Omega - \mathcal{D}\right)G^{2, \epsilon}(\xi(t)) \right\rangle,$$

$$\psi^3_\epsilon(t) := -i \left\langle \mathcal{W}^\epsilon(t), \mathcal{K}^\circ_{\xi(t)}G^{2, \epsilon}(\xi(t)) \right\rangle.$$

Here $\Omega$ is the generator of the Ornstein-Uhlenbeck process introduced in Section 4.1. The actual calculation performed in Section 5.2 below, (see Step 1 of the proof of Lemma 5.1) involves a bit more complex random processes $\Psi^j_\epsilon(t)$, $j = 1, 2, 3, 4$, since we also have to account for the terms $Y_{\epsilon, \pm}(t)$.

Next, cf (99) below, we choose $G^{1, \epsilon}$ in such a way that $\psi^1_\epsilon(t) \equiv 0$ (this choice eliminates the large term in (89)). This is achieved, by solving in explicit terms,
the corrector equation \((\Omega - \mathcal{D})G^{1,\epsilon} = -iK^0 G^\epsilon\). The solution exists, due to the fact that \(\int_{\mathbb{T}} K^0 G^\epsilon d\pi = 0\). To see this, note that from the definition of \(\Omega - \mathcal{D}\) constants are in the null space of its adjoint, so the latter condition is indeed necessary for solvability of the corrector equation. Thanks to the spectral gap assumption (ii) (that holds due to \(\gamma(\cdot) \geq \gamma_* > 0\)) the operator \(\Omega - \mathcal{D}\) is invertible on the subspace of \(\pi\)-zero mean functions so the solution of the corrector equation exists and is unique in that space.

Having chosen \(G^{1,\epsilon}\) we turn our attention to the term \(\psi_2^2(t)\), which is stochastic and of order of magnitude \(O(1)\), as \(\epsilon \to 0^+\). It can be replaced by a deterministic term, if we recenter the random scattering operator, i.e. choose \(G^{2,\epsilon}\) in such a way that

\[-i\tilde{K}^0_{\xi(t)}G^{1,\epsilon} = (\Omega - \mathcal{D})G^{2,\epsilon},\]

where

\[\tilde{K}^0_{\xi(t)}G^{1,\epsilon} := K^0_{\xi(t)}G^{1,\epsilon} - \int_{\mathbb{E}} K^0_{\xi(t)}G^{1,\epsilon} d\pi.\]

It turns out, see Step 2 of the proof of Lemma 5.1 and Section 5.1, that

\[-i\int_{\mathbb{E}} K^0_{\xi(t)}G^{1,\epsilon} d\pi = \tilde{\mathcal{L}}_\epsilon G^\epsilon,
\]

where

\[\lim_{\epsilon \to 0^+} \left[\langle W^\epsilon(t), \tilde{\mathcal{L}}_\epsilon G^\epsilon \rangle - \langle W^\epsilon(t), \mathcal{L} \rangle \right] = 0,\]

with \(\tilde{\mathcal{L}}_\epsilon\) and \(\mathcal{L}\) the approximate and limiting scattering operators described in (97) and (8) respectively. This allows us to conclude the proof of Theorem 2.5.

5.2. **Approximate kinetic equation for the Wigner function.** We introduce the following shorthand notation: for given a bounded measurable \(g : \mathbb{T}^4 \to \mathbb{C}\) we define the (random) operators \(\mathcal{K}^{(\pm)}_g\) that generalize the operators \(\mathcal{K}^{(\pm)}\) introduced in (63). Let

\[\mathcal{K}^{+}_{g,f,J_{y,y'}}(k) := \sum_{z,z'} J_{z,z'}(k) \int_{\mathbb{T}^3} e^{2\pi ip(z-y)} e^{2\pi i p'(z'-y')} r^+_{y,y'}(k + p', k') \times g(k, k', p, p') dpdp' \hat{V}(dk'; f),\]

\[\mathcal{K}^{-}_{g,f,J_{y,y'}}(k) := \sum_{z,z'} J_{z,z'}(k) \int_{\mathbb{T}^3} e^{2\pi ip(z-y)} e^{2\pi i p'(z'-y')} r^-_{y,y'}(-k + p, k') \times g(k, k', p, p') dpdp' \hat{V}(dk'; f),\]

for \(\pi\) a.s. \(f\) and \(J \in \mathcal{S}\).

Let also

\[\tilde{\mathcal{L}}_\epsilon G^\epsilon = \tilde{\mathcal{L}}^+_\epsilon G^\epsilon + \tilde{\mathcal{L}}^-_\epsilon G^\epsilon,\]

where

\[\left(\tilde{\mathcal{L}}^+_\epsilon G^\epsilon\right)_{y,y'}(k) := \sum_{\varepsilon = \pm} \int_{\mathbb{R} \times \mathbb{T}} \hat{G}(q, k) e^{\varepsilon \pi i q(y+y')} \sigma(k') dqdk' \times \{\gamma(k') + i\{\omega(k - \epsilon eq/2) - \omega(k - k' + \epsilon eq/2)\}\}^{-1}\]

\[\{e^{2\pi i k'(y'-y)} r(k + eq/2, k') r(k - eq/2, k') - r^2(k + eq/2, k')\}\]
and

\[
\left(\tilde{L}_\epsilon G^\epsilon\right)_{y,y'}(k) := -\sum_{i=\pm} \int_{\mathbb{R} \times \mathbb{T}} \tilde{G}(q, k) e^{\pi i q(y+y')} \sigma(k') dqdk' \\
\times \left\{ \gamma(k') + i\epsilon [\omega(k + i\epsilon q/2) + \omega(k - k' - i\epsilon q/2)] \right\}^{-1} r^2(k - i\epsilon q/2, k') \\
+ \sum_{i=\pm} \int_{\mathbb{R} \times \mathbb{T}} \tilde{G}(q, -k) e^{\pi i q(y+y')} e^{2\pi i k'(y'-y)} \sigma(k') dqdk' \\
\times \left\{ \gamma(k') + i\epsilon [\omega(k - i\epsilon q/2) + \omega(k - k' + i\epsilon q/2)] \right\}^{-1} r(k + i\epsilon q/2, k') r(k - i\epsilon q/2, k').
\]

**Lemma 5.1.** Suppose that \( G \in \mathcal{S}(\mathbb{R} \times T) \) then

\[
(W^\epsilon(t), G^\epsilon) = \langle W^\epsilon(0), G^\epsilon \rangle + \int_0^t ds \langle W^\epsilon(s), \tilde{L}_\epsilon G^\epsilon - \mathcal{D}_t G^\epsilon \rangle + \mathcal{R}^\epsilon(t), \ t \geq 0,
\]

a.s. in \( \xi^\epsilon(\cdot) \). Here \( G^\epsilon \) is given by (67). In addition, for every \( T > 0 \)

\[
\lim_{\epsilon \to 0+} \sup_{t \leq T} \| \mathbb{E} \mathcal{R}^\epsilon(t) \| = 0.
\]

**Proof.** The proof of the lemma is divided into three steps. We start with the two scale expansion scheme for the test function in the formula for the pseudo-generator, see (87). The requirement that no large terms arise as a result of this expansion leads to equations for the resulting terms (whose solutions we call the correctors). In step 2 we solve these equations and in consequence compute quite explicitly the terms of the expansion and the scattering operator \( \tilde{L}_\epsilon \) appearing in (94). Finally, in step 3 of the proof we prove the validity of the expansion claimed in (94).

**Step 1 of the proof: The corrector expansion.** Let \( (G_{j,y,y'}^\epsilon)_{y,y' \in \mathbb{Z}^2} \), where \( j \in \{-,0,+\} \), be random fields defined over the probability space \( (\tilde{E}, \mathcal{B}(\tilde{E}), \pi) \) of the form

\[
G_0^\epsilon := G^\epsilon + \sqrt{\epsilon} G_0^{1,\epsilon} + \epsilon G_0^{2,\epsilon}, \quad G_\pm^\epsilon := \sqrt{\epsilon} G_\pm^{1,\epsilon} + \epsilon G_\pm^{2,\epsilon},
\]

with the fields \( G_i^{1,\epsilon}, i = 1,2, j \in \{-,0,+,\} \) to be determined later on. Let

\[
\Psi_\epsilon(t) := \langle W^\epsilon(t), G_0^\epsilon(\xi^\epsilon(t)) \rangle + \sum_{i=\pm} \langle W^\epsilon(t), G_i^\epsilon(\xi^\epsilon(t)) \rangle.
\]

To simplify the notation we shall write \( \tilde{G}^\epsilon(t) \) instead of \( \tilde{G}^\epsilon(\xi^\epsilon(t)) \). A similar convention shall be used for the other functions appearing in (95). We also use the following convention: suppose that \( K \) is any one of the operators \( K^\iota, \iota \in \{-,+,\} \) and \( J \) is a random \( \mathcal{S} \)-valued element. We let

\[
\mathcal{K}_\iota J := \int_\tilde{E} \mathcal{K}_\iota J d\pi \quad \text{and} \quad \tilde{\mathcal{K}}_\iota J := \mathcal{K}_\iota J - \mathcal{K}_\iota J.
\]

To calculate the pseudo-generator of the process \( \Psi_\epsilon(t) \) we use Proposition 4.2. Here we do it somewhat formally (without verifying the assumptions (83)-(85)).

The remark on the applicability of the proposition is made in Section 5.3, after determining formulas for the fields \( G_i^{1,\epsilon}, i = 1,2, j \in \{-,0,+,\} \). Using formula (87) we obtain then

\[
\mathcal{L}_\epsilon \Psi_\epsilon(t) = \frac{1}{\sqrt{\epsilon}} \Psi_\epsilon^1(t) + \Psi_\epsilon^2(t) + \sqrt{\epsilon} \Psi_\epsilon^3(t),
\]

(96)
where we have grouped terms that stand by the respective powers of $\epsilon^{j/2}$, $j = -1, 0, 1$. We have

$$
\Psi_1^1(t) := \left\langle \mathcal{W}(t), i\mathcal{K}_\epsilon^0 G^\epsilon + (\mathcal{Q} - \mathcal{D}) G^\epsilon_0(t) \right\rangle 
+ \sum_{i = \pm} \left\langle \mathcal{Y}_i(t), -i\mathcal{K}_\epsilon^i G^\epsilon + (\mathcal{Q} - i\mathcal{\Theta}) G^\epsilon_i(t) \right\rangle,
$$

Next $\Psi_2^1(t) = \Psi_1^2(t) + \Psi_2^1(t)$, with

$$
\Psi_2^1(t) := \left\langle \mathcal{W}(t), i\mathcal{K}_\epsilon^0 G^\epsilon_0(t) - \sum_{i = \pm} \mathcal{K}_\epsilon^i G^\epsilon_i(t) + (\mathcal{Q} - \mathcal{D}) G^\epsilon_0(t) \right\rangle 
+ \sum_{i = \pm} \left\langle -\mathcal{K}_\epsilon^i G^\epsilon_i(t) - i\mathcal{K}_\epsilon^i G^\epsilon_0(t) + (\mathcal{Q} + i\mathcal{\Theta}) G^\epsilon_i(t) \right\rangle,
$$

and $\Psi_2^2(t) := \sum_{j=1}^2 \Psi_j^2(t)$, with

$$
\Psi_1^2(t) := \left\langle \mathcal{W}(t), \mathcal{L}_\epsilon G^\epsilon - \mathcal{D}_\epsilon G^\epsilon \right\rangle,
\Psi_2^2(t) := \sum_{i = \pm} \left\langle \mathcal{Y}_i(t), \mathcal{L}_\epsilon G^\epsilon \right\rangle.
$$

Let also

$$
\mathcal{L}_\epsilon G^\epsilon := i\mathcal{K}_\epsilon^0 G^\epsilon_0(t) - \sum_{i = \pm} \mathcal{K}_\epsilon^i G^\epsilon_i(t),
$$

$$
\mathcal{L}_\epsilon G^\epsilon := i\mathcal{K}_\epsilon^0 G^\epsilon_0(t) - \sum_{i = \pm} \mathcal{K}_\epsilon^i G^\epsilon_i(t), \quad i \in \{\pm\}.
$$

Finally,

$$
\Psi_3^1(t) := i \left\langle \mathcal{W}(t), \sum_{i = \pm} \mathcal{K}_\epsilon^i(t) G^\epsilon_i(t) - \mathcal{K}_\epsilon^0(t) G^\epsilon_0(t) \right\rangle
+ \sum_{i = \pm} \left\langle \mathcal{Y}_i(t), \mathcal{K}_\epsilon^i(t) G^\epsilon_i(t) - \mathcal{K}_\epsilon^0(t) G^\epsilon_0(t) \right\rangle.
$$

Here $G_{i,0,0,y,y'}^\epsilon(k) := G_{i,0,0,y,y'}^\epsilon(k) + G_{i,0,0,y,y'}^\epsilon(-k)$.

The term $\Psi_1^1(t)$ corresponds to an expression in (96) of an apparent order of magnitude $O(\epsilon^{-1/2})$, see (96). To make it vanish we assume that $G^\epsilon_i$ are chosen in such a way that

$$
(\mathcal{Q} - \mathcal{D}) G^\epsilon_0 = -i\mathcal{K}_\epsilon^0 G^\epsilon, 
(\mathcal{Q} - i\mathcal{\Theta}) G^\epsilon_0 = i\mathcal{K}_\epsilon^0 G^\epsilon, \quad i \in \{\pm\}.
$$

Having solved the above system for $G^\epsilon_i$, our next step is to eliminate the term $\Psi_3^1(t)$ that corresponds to random operators $\mathcal{K}_\epsilon$. We stipulate therefore that $G^\epsilon_i$, $i \in \{-, +, 0\}$ are such that

$$
(\mathcal{Q} - \mathcal{D}) G^\epsilon_0 = -i\mathcal{K}_\epsilon^0 G^\epsilon_0 + \sum_{i = \pm} \mathcal{K}_\epsilon^i G^\epsilon_i, 
(\mathcal{Q} - i\mathcal{\Theta}) G^\epsilon_0 = i\mathcal{K}_\epsilon^0 G^\epsilon_0 - i\mathcal{K}_\epsilon^i G^\epsilon_i, \quad i \in \{\pm\}.
$$

Concerning the remaining terms we let

$$
\mathcal{R}_\epsilon(t) := \int_0^t \Psi_2^2(s) ds + \sqrt{\epsilon} \Psi_3^3(t).
$$
and prove that
\[
\lim_{\epsilon \to 0^+} \sup_{t \in [0,T]} \left| \int_0^t \mathbb{E} \mathcal{V}_{2,\epsilon}^2(s) ds \right| = 0 \tag{100}
\]
and
\[
\limsup_{\epsilon \to 0^+} \sup_{t \in [0,T]} |\mathbb{E} \mathcal{V}_{3}^3(t)| < +\infty, \quad \forall T > 0. \tag{101}
\]

The above facts combined allow us to conclude (94).

**Step 2: Computation of the corrector terms and scattering operators.**

Since the right hand sides of the equations (99) are fields of the form
\[
H = \left( \int_{\mathbb{T}} H_{y,y'}(k,k') \tilde{V}(dk') \right)_{(y,y',k) \in \mathbb{Z}^2 \times \mathbb{T}},
\]
where \(H_{y,y'}(k,k')\) are some deterministic functions, we are looking for solutions that take the form
\[
J = \left( \int_{\mathbb{T}} J_{y,y'}(k,k') \tilde{V}(dk') \right)_{(y,y',k) \in \mathbb{Z}^2 \times \mathbb{T}}.
\]
for some deterministic \(J_{y,y'}(k,k')\). Note that then
\[
((\mathcal{Q} - \mathfrak{D}) J)_{y,y'}(k) = \sum_{z,z'} \int_{\mathbb{T}^3} e^{2\pi i p (z - y)} e^{2\pi i p' (z' - y')} \times \left[ \gamma(k') + i \left( \omega(k + p') - \omega(k + k' - p) \right) \right] J_{z,z'}(k,k') dp dp' \tilde{V}(dk').
\]
Comparing the left and right hand sides of the first equation of (99) we obtain

\[
G_{1,0}^{1,\epsilon} = i \sum_{\iota = \pm} \mathcal{K}_{\iota}^\epsilon G^\iota,
\]
with
\[
u_- := \left[ \gamma(k') + i \left( \omega(k + p') - \omega(k + k' - p) \right) \right]^{-1},
\]
\[
u_+ := \left[ \gamma(k') + i \left( \omega(k - k' + p') - \omega(k - p) \right) \right]^{-1}. \tag{103}
\]

Similarly,

\[
G_{1,0}^{1,\epsilon} = i \sum_{\iota = \pm} \mathcal{K}_{\iota}^\epsilon G^\iota \quad \text{and} \quad G_{1,\iota}^{1,\epsilon} = -i \mathcal{K}_{\iota}^\epsilon G^\iota, \quad \iota = \pm, \tag{104}
\]

where
\[
v_+ := \left[ \gamma(k') + i \left( \omega(k - k' + p') + \omega(k - p) \right) \right]^{-1},
\]
\[
v_- := \left[ \gamma(k') - i \left( \omega(k + p') + \omega(k + k' - p) \right) \right]^{-1}.
\]

Concerning functions \(G_{1,\iota}^{2,\epsilon}\) we conclude that they are of the form
\[
G_{0,y,y'}^{2,\epsilon}(k) = - \sum_{z,z'} \int_{\mathbb{T}^4} e^{2\pi i k_3 (z - y)} e^{2\pi i k_4 (z' - y')} \left( \sum_{j=1}^{8} \mathcal{J}_j \right) \tilde{V}(dk_1,dk_2,dk_3,dk_4).
\]
Each $\mathcal{J}_j$ is of the form
\[
\mathcal{J}_j := G \left( \frac{z + z'}{2}, 2, 2, \sigma_1 k \right) e^{2\pi i (\rho_1 k_1 + \rho_2 k_2) y} e^{2\pi i ((1 - \rho_1) k_1 + (1 - \rho_2) k_2) y'} \times \prod_{\ell = 1}^2 r \left( \sum_{j=1}^4 \rho_j^{(\ell)} k_j, k_\ell \right) \] (105)
\times \left\{ \gamma(k_2) + i\sigma_2 \left[ \omega \left( k + \sum_{j=1}^4 \rho_j^{(3)} k_j \right) + \sigma_3\omega \left( k + \sum_{j=1}^4 \rho_j^{(4)} k_j \right) \right] \right\}^{-1}
\times \left\{ \gamma(k_1) + \gamma(k_2) + i\sigma_4 \left[ \omega \left( k + \sum_{j=1}^4 \rho_j^{(5)} k_j \right) + \sigma_5\omega \left( k + \sum_{j=1}^4 \rho_j^{(6)} k_j \right) \right] \right\}^{-1},
\]
where $\sigma_j \in \{-1, 1\}$, $j = 1, \ldots, 5$, $\rho_j^{(\ell)} \in \{-1, 0, 1\}$, $j = 1, \ldots, 4$, $\ell = 1, \ldots, 6$ and $\rho_j \in \{0, 1\}$, $j = 1, 2$. The terms $G^{2,\epsilon}_0$, $\epsilon = \pm$ are of an analogous form, except that they can be represented as sums of four terms, each of the form given by (105).

The operator $\hat{\mathcal{L}}_\epsilon$ defined in the first line of (97) coincides with the one given by formula (92). In addition, for $\epsilon \in \{-, +\}$ we have
\[
\hat{\mathcal{L}}_{\epsilon, t} G^{\epsilon}_{\gamma, y'}(k) = \int_{\mathbb{R} \times \mathbb{T}} \hat{G}(q, k) e^{\pi \epsilon i q(y + y')}
\times \left\{ \gamma(k') + i\epsilon \left[ \omega(k + \epsilon q/2) + \omega(k - k' - \epsilon q/2) \right] \right\}^{-1}
\times r^2(k - \epsilon q/2; k') - e^{2\pi i k'(y' - y)} r(k - \epsilon q/2; k') r(k + \epsilon q/2; k') \sigma(k') dq dk'
+ \int_{\mathbb{R} \times \mathbb{T}} \hat{G}(q, k) e^{\pi \epsilon i q(y + y')} \sigma(k') dq dk'
\times \left\{ \gamma(k') + i\epsilon \left[ \omega(k - k' - \epsilon q/2) - \omega(k + \epsilon q/2) \right] \right\}^{-1} r^2(k - \epsilon q/2; k')
\times e^{2\pi i k'(y' - y)} \left\{ \gamma(k') - i\epsilon \left[ \omega(k - k' + \epsilon q/2) - \omega(k - \epsilon q/2) \right] \right\}^{-1}
\times r(k + \epsilon q/2; k') r(k - \epsilon q/2; k').
\]

Step 3: The proof of the limits (100) – (101).

5.2.1. Proof of (101). We shall explain only the fact that
\[
\lim_{\epsilon \to 0^+} \sup_{t \in [0, T]} \left\| \mathbb{E} \left\langle W^\epsilon(t), \mathcal{K}^{\epsilon}_{+}(t) G^{2,\epsilon}_0(t) \right\rangle \right\| < +\infty, \quad \forall T > 0
\]
(107)
as the remaining terms arising in the proof of (101) can be dealt with in a similar fashion. The random field $\mathcal{K}^{\epsilon} G^{2,\epsilon}_0$ is of the form
\[
\mathcal{K}^{\epsilon}_{+} G^{2,\epsilon}_0(k, \gamma, y') = \sum_{\ell = 1}^8 \int_{\mathbb{R} \times \mathbb{T}^3} \hat{G}(q, k) e^{\pi \epsilon i q(y + y') \prod_{j=1}^3 e^{2\pi i k_j (\rho_j^{(\ell)} y + (1 - \rho_j^{(\ell)}) y')}}
\times f_\ell(k, k_1, k_2, k_3, \epsilon q/2) dV(\rho_1, \rho_2, \rho_3),
\]
(108)
where $\rho_j^{(\ell)} \in \{0, 1\}$ and each $f_\ell$ is a complex valued function that belongs to $W^{1,\infty}(\mathbb{T}^4 \times \mathbb{R})$ (i.e. it is bounded with all its first partial derivatives). The latter is the result of the fact that $\gamma(k) \geq \gamma^*$ for all $k \in \mathbb{T}$. Therefore the term under
the expectation in (107) can be written as the sum of 8 terms of the form

\[
\mathbb{E} \left[ \int_{\mathbb{R}^3} \hat{W}^\epsilon \left( t, q, k + \sum_{j=1}^{3} \sigma_j^{(\ell)} k_j \right) \hat{G}^* \left( q - \frac{1}{\epsilon} \sum_{j=1}^{3} k_j, k \right) \right]
\]

(109)

\[
\times \hat{f}_\ell(k, k_1, k_2, k_3, \epsilon q/2) dqdk \hat{V}(dk_1, dk_2, dk_3; \xi(t))
\]

for some \( \sigma_j^{(\ell)} \in \{-1/2, 1/2\} \), \( \hat{f}_\ell \in W^{1,\infty}(\mathbb{T}^4 \times \mathbb{R}) \) and \( \ell = 1, \ldots, 8 \). Changing variables \( \tilde{k} := k + \sum_{j=1}^{3} \sigma_j^{(\ell)} k_j \) and using the bound (42) we arrive at the estimate of (109) by the following expression

\[
K \int_{\mathbb{R}} dq \int_{\mathcal{E}} d\pi \left[ \sup_{k \in \mathbb{T}} \left| \int_{\mathbb{T}} \hat{G}^* \left( q, k - \sum_{j=1}^{3} \sigma_j^{(\ell)} k_j \right) \right| \right]
\]

(110)

\[
\times \hat{f}_\ell(k - \sum_{j=1}^{3} \sigma_j^{(\ell)} k_j, k_1, k_2, k_3, \epsilon q/2 + 1/2 \sum_{j=1}^{3} k_j) \hat{V}(dk_1, dk_2, dk_3)
\]

Thanks to the Sobolev embedding we can further estimate this expression by

\[
CK \int_{\mathbb{R}} dq \left( \int_{\mathbb{T}} dk \int_{\mathcal{E}} d\pi \left[ \int_{\mathbb{T}} \partial_k \hat{G}^* \left( q, k - \sum_{j=1}^{3} \sigma_j^{(\ell)} k_j \right) \right] \right)
\]

(111)

\[
\times \hat{f}_\ell(k - \sum_{j=1}^{3} \sigma_j^{(\ell)} k_j, k_1, k_2, k_3, \epsilon q/2 + 1/2 \sum_{j=1}^{3} k_j) \hat{V}(dk_1, dk_2, dk_3) \right)^{2^{1/2}}
\]

for some constant \( C > 0 \) independent of \( \epsilon > 0 \). The expectation of this expression stays bounded for \( \epsilon \to 0^+ \), so (107) follows.

5.2.2. Proof of (100). Taking into account formula (106) we conclude that the expression appearing in (100) is a sum of terms of the form

\[
\int_{0}^{t'} \mathbb{E}(\mathcal{Y}^{\epsilon, \pm}(s), \Theta J) ds,
\]

where each \( J \) is given by: either

\[
J_{y,y'}(k) := \int_{\mathbb{R} \times \mathbb{T}} \hat{G}(q, k) H \left( \frac{\epsilon q}{2}, k, k' \right) e^{\epsilon \pi i q (y + y')} dq dk',
\]

(112)

or

\[
J_{y,y'}(k) := \epsilon_1 \int_{\mathbb{R} \times \mathbb{T}} \hat{G}(q, k) \tilde{H} \left( \frac{\epsilon q}{2}, k, k' \right) e^{\epsilon \pi i q (y + y')} e^{2\pi i k' (y' - y)} dq dk', \quad y, y' \in \mathbb{Z},
\]

(113)

and \( G, \tilde{G} \in \mathcal{S}(\mathbb{R} \times \mathbb{T}) \). Here

\[
H (q, k, k') := \epsilon_1 r^2 (k - \epsilon_2 q, k') \sigma(k')
\]

(114)

\[
\{\omega(k + q) + \omega(k - q)\}^{-1} \{\gamma(k') + \epsilon_2 \epsilon_3 i \omega(k + \epsilon_2 q) + \epsilon_3 \omega(k - k' - \epsilon_2 q)\}^{-1}
\]

and

\[
\tilde{H} (q, k, k') := \epsilon_1 r (k - q, k') r(k + q, k') \sigma(k')
\]

(115)

\[
\{\omega(k + q) + \omega(k - q)\}^{-1} \{\gamma(k') + \epsilon_2 \epsilon_3 i \omega(k + \epsilon_2 q) + \epsilon_3 \omega(k - k' - \epsilon_2 q)\}^{-1}
\]
for \( t_1, t_2, t_3 \in \{-, + \} \). We claim that
\[
\lim_{c \to 0^+} \sup_{t \in [0, T]} \left| \int_0^t \mathbb{E}(\mathcal{Y}^\tau^+(s), \Theta J) ds \right| = 0
\]  
(116)
for any \( T > 0 \). Consider only the terms corresponding to \( J \) of the form (113), since the ones corresponding to \( J \) as in (112) can be dealt with similarly.

Using (65) we obtain
\[
\left| \int_0^t \mathbb{E}(\mathcal{Y}^\tau^+(s), \Theta J) ds \right| \leq I_\epsilon + II_\epsilon + III_\epsilon, 
\]  
(117)
where
\[
I_\epsilon := \epsilon \sup_{t \in [0, T]} \left| \mathbb{E}(\mathcal{Y}^\tau^+(t), J) \right|, 
\]
\[
II_\epsilon := \sqrt{\epsilon} \int_0^T \left| \mathbb{E}(\mathcal{Y}^\tau^+(t), \mathcal{K}_{\xi^\tau(t)}^\epsilon J) \right| dt, 
\]  
(118)
\[
III_\epsilon := \sqrt{\epsilon} \int_0^T \left| \mathbb{E}(\mathcal{W}^\tau^+(t), \mathcal{K}_{\xi^\tau(t)}^{\epsilon^\tau} J^\nu) \right| dt.
\]
Here \( J^\nu \) is given by (66).

We have
\[
\tilde{J}_{y, y'}(y' - y) = \int_{\mathbb{R} \times \mathbb{T}^2} \hat{G}(q, k) \hat{H} \left( \frac{eq}{2}, k, k + k' \right) 
\]
\[
\times e^{i\pi q(y + y')} e^{2\pi i k'(y' - y)} dq dk dk', 
\]
\( y, y' \in \mathbb{Z} \).

We have
\[
\left| \mathbb{E}(\mathcal{Y}^\tau^+(t), J) \right| \leq \frac{\epsilon}{2} \sum_{y, y'} \mathbb{E} \left[ |\psi_y'(t)| |\psi_{y'}'(t)| \right] \left| \tilde{J}_{y, y'}(y' - y) \right| 
\]
\[
\leq \frac{\epsilon}{4} \sum_{y} (R_{y, \epsilon} + R'_{y, \epsilon}) \langle |\psi_y|^2 \rangle_{\mu_\epsilon} \leq \frac{K}{2} \sup_y (R_{y, \epsilon} + R'_{y, \epsilon}) 
\]  
(119)
where \( K \) is as in (28) and
\[
R_{y, \epsilon} := \sum_{y'} |\tilde{J}_{y, y'}(y' - y)|, \quad R'_{y, \epsilon} := \sum_{y'} |\tilde{J}_{y, y'}(y' - y)|.
\]
Note that for any \( y' \neq y \) we have
\[
\tilde{J}_{y, y'}(y' - y) = -\frac{1}{[2\pi(y' - y)]^2} \int_{\mathbb{R} \times \mathbb{T}^2} \hat{G}(q, k) 
\]
\[
\times \partial_{q'}^2 \hat{H} \left( \frac{eq}{2}, k, k + k' \right) e^{i\pi q(y + y')} e^{2\pi i k'(y' - y)} dq dk dk' 
\]
\[
+ \frac{1}{[2\pi(y' - y)]^2} \sum_{l = \pm} \int_{\mathbb{R} \times \mathbb{T}^2} \hat{G}(q, k) 
\]
\[
\times e^{i\pi q(y + y')} e^{-i2\pi i l(q - q')} \Delta \partial_{q'}^2 \hat{H} \left( \frac{eq}{2}, k, -l, \frac{eq}{2} \right) dq dk, 
\]  
(120)
where
\[
\Delta \partial_{q'}^2 \hat{H} (q, k, q') := \partial_{q'}\hat{H} (q, k, q') + \partial_{q'}\hat{H} (q, k, q'). 
\]
Therefore, using the fact that \( \gamma(k) \) is bounded away from 0, we conclude that
\[
\sup_{c > 0, y} (R_{y, \epsilon} + R'_{y, \epsilon}) < +\infty, 
\]
The only thing yet to be shown is that for some coefficients $c$, where $\bar{\epsilon} \in J$, we conclude, see the proof of Proposition 3.1, that

$$\mathbb{E}(\chi_{n^+}(t), K^\varepsilon_{\xi}(t)) = J_1^\varepsilon(t) + J_2^\varepsilon(t)$$

where

$$J_1^\varepsilon(t) := \frac{\varepsilon}{2} \sum_{y,y'} \sum_{|z_1|,|z_2|,|z_3| \leq 2} c_{z_1,z_2,z_3} \hat{J}_{y,y'}^{z_2} (y' - y + z_3) \mathbb{E} [\xi_{y'}^{z_2} (t) \psi_{y'}^{z_2} (t) \psi_{y'}^{z_2} (t)]$$

$$J_2^\varepsilon(t) := \frac{\varepsilon}{2} \sum_{y,y'} \sum_{|z_1|,|z_2|,|z_3| \leq 2} c'_{z_1,z_2,z_3} \hat{J}_{y,y'}^{z_2} (y' - y + z_3) \mathbb{E} [\xi_{y'}^{z_2} (t) \psi_{y'}^{z_2} (t) \psi_{y'}^{z_2} (t)]$$

for some coefficients $c_{z_1,z_2,z_3}, c'_{z_1,z_2,z_3}$ and $|z_1|,|z_2|,|z_3| \leq 2$, where $z_j$-s are integers. We show that

$$\lim_{\varepsilon \to 0^+} \sqrt{\varepsilon} \sup_{t \in [0, T]} J_j^\varepsilon(t) = 0, \quad j = 1, 2. \quad (122)$$

We prove (122) for $j = 1$, as the argument for $j = 2$ follows analogously.

We can write

$$|J_1^\varepsilon(t)| \leq \frac{\varepsilon}{2} \max |c_{z_1,z_2,z_3}| \sum_{y,y'} \sum_{|z_1|,|z_2|,|z_3| \leq 2} |\hat{J}_{y,y'}^{z_2} (y' - y + z_3)| \mathbb{E} [\xi_{y'}^{z_2} (t)(|\psi_{y'}^{z_2} (t)|^2 + |\psi_{y'}^{z_2} (t)|^2)]. \quad (123)$$

We shall deal only with the term $z_1 = z_2 = z_3 = 0$. The other cases are similar. Suppose that $m \in (0, 1/2)$. The right hand side of (123) is then estimated by $K [\sup_y (\hat{R}_{y,e} + \hat{R}'_{y,e}) + \hat{R}''_{e}]$, with

$$\hat{R}_{y,e} := \sum_{y'} |\hat{J}_{y,y'}^{z_2} (y' - y)| \mathbb{E} [\xi_{y'}^{z_2} (t)],$$

$$\hat{R}'_{y,e} := \sum_{y'} |\hat{J}_{y,y'}^{z_2} (y' - y')| \mathbb{E} \left[ \sup_{|z| \leq \varepsilon^{-1-m}} |\xi_{y'}^{z_2} (t)| \right],$$

$$\hat{R}''_{e} := \mathbb{E} \left[ \sup_{|z| \geq \varepsilon^{-1-m}} \sum_{y'} |\hat{J}_{y,y'}^{z_2} (y' - y')| |\xi_{y'}^{z_2} (t)| \right].$$

The same argument as in the case of $I_e$ can be used to prove that

$$\sup_{e > 0, y} \hat{R}_{y,e} < +\infty.$$ 

We also have

$$\hat{R}'_{y,e} \leq \varepsilon^{-(1+m)\rho} \bar{C}_{\rho,1} \sum_{y'} |\hat{J}_{y,y'}^{z_2} (y' - y')|,$$

where $\bar{C}_{\rho,1}$ is given by (20) and $\rho$ is adjusted in such a way that $1 + m \rho \in (0, 1/2)$. Using again the argument as above, we conclude that

$$\lim_{\varepsilon \to 0^+} \sqrt{\varepsilon} \sup_y \hat{R}'_{y,e} = 0.$$ 

The only thing yet to be shown is

$$\lim_{\varepsilon \to 0^+} \sqrt{\varepsilon} \hat{R}''_{e} = 0. \quad (124)$$
Using formula (120) and integrating by parts in the $q$ variable we conclude that there exists $C > 0$ for which

$$
\tilde{R}''_\epsilon \leq \frac{C}{\epsilon} \mathbb{E} \left\{ \sup_{|y| \geq \epsilon^{-1-m}} \sum_{y'} \frac{|\xi_y^\epsilon(t)|}{(y - y')^2} \left[ \frac{1}{(y + y')^{1/2}} + \frac{1}{\langle y + y' - \xi_2(y' - y) \rangle} \right] \right\} = \sum_{j=1}^2 \tilde{R}_\epsilon''^{\prime},
$$

for $\epsilon > 0$. Here the terms $\tilde{R}_{\epsilon,1}''$ and $\tilde{R}_{\epsilon,2}''$ correspond to the summation over $|y' - y| \geq |y|/4$ and $|y' - y| < |y|/4$. Let $\rho \in (0, 1)$. We can write then that for some constants $C', C''$ (cf (20))

$$
\tilde{R}_{\epsilon,1}'' \leq \frac{C'}{\epsilon} \mathbb{E} \left\{ \sup_{|y| \geq \epsilon^{-1-m}} \sum_{|y'| \geq |y|/4} \frac{|\xi_y^\epsilon(t)|}{(y - y')^{2-\rho}} \right\} \leq C' \mathbb{E} \left\{ \sup_{y} \frac{|\xi_y^\epsilon(t)|}{(y)^\rho} \right\} \leq \frac{C''}{\epsilon} (1+m)(1-\rho) C_{\lambda,\rho}.
$$

Adjusting $\rho > 0$ in such a way that $(1+m)(1-\rho) > 1$ we conclude that $\lim_{\epsilon \to 0+} \tilde{R}_{\epsilon,1}'' = 0$.

Finally, we can write that for some constant $C > 0$ we have

$$
\tilde{R}_{\epsilon,2}'' \leq C \mathbb{E} \left\{ \sup_{|y| \geq \epsilon^{-1-m}} \frac{|\xi_y^\epsilon(t)|}{\epsilon(y)} \sum_{y'} \frac{1}{(y - y')^2} \right\}.
$$

Since $\epsilon(y) \geq \epsilon^{-m}$ we have $\epsilon(y) \geq \epsilon^q(y)^\rho$, when $\rho \in (0, 1/2)$. We obtain therefore that for some constant $C' > 0$ independent of $\epsilon$ we have

$$
\tilde{R}_{\epsilon,2}'' \leq \frac{C'}{\epsilon^q} \mathbb{E} \left\{ \sup_{y} \frac{|\xi_y^\epsilon(t)|}{(y)^\rho} \right\} = \frac{C'}{\epsilon^q} C_{\rho,1}.
$$

Hence $\lim_{\epsilon \to 0+} \sqrt{\epsilon} \tilde{R}_{\epsilon,2}'' = 0$. We have shown that

$$
\lim_{\epsilon \to 0+} \Pi_{\epsilon} = 0.
$$

The proof that also $\lim_{\epsilon \to 0+} \Pi_{\epsilon} = 0$ is analogous. Thus we conclude the proof of (100) and finish in this way the demonstration of Lemma 5.1.

5.3. Remarks about derivation of formula (96). Here we explain how to derive formula (96), used in step 1 of the proof of Lemma 5.1. The issue consists in verifying that the functionals given in (95) satisfy hypotheses (83)–(85) so we can apply Proposition 4.2. Let us focus our attention on $G^\epsilon_0$, as the arguments for the other functionals are very similar. In case of the term $G^\epsilon$ that appears in the definition of $G^\epsilon_0$ the verification is trivial because it is constant over $\mathcal{E}$, clearly takes values in $\mathcal{H}_m$ (as $G \in \mathcal{S}(\mathbb{R} \times \mathbb{T})$) and $\Omega G^\epsilon_{y,y'}(k) \equiv 0$ for each $y, y' \in \mathbb{Z}$ and $k \in \mathbb{T}$.

Consider therefore the term $G^\epsilon_{1, \epsilon}$. In fact we can take $\epsilon = 1$ (as in this case the value of $\epsilon$ is fixed) and omit writing the superscript $\epsilon$ in our notation. We can use formula (102) together with (70) to conclude that $G^\epsilon_{1, y,y'}(k)$ takes values in the space of the first degree polynomials $p_1$ (see Section 4.2) for each $y, y' \in \mathbb{Z}$ and $k \in \mathbb{T}$. Applying formula (76) we can verify that

$$
(P_t - I) \Omega G^\epsilon_{1} = i \sum_{i=\pm} K^\epsilon_{0,i} G,
$$

(125)
where \( \tilde{u}_- \) are given by
\[
\tilde{u}_- := \left[ e^{-\gamma(k)t} - 1 \right] \gamma(k) \gamma(k') + i \omega(k + p') - \omega(k - k' - p)) \right]^{-1},
\]
(126)
\[
\tilde{u}_+ := \left[ e^{-\gamma(k)t} - 1 \right] \gamma(k) \gamma(k') + i \omega(k - k' + p') - \omega(k - p) \right]^{-1}.
\]
Calculating the second moment of the \( \| \cdot \|_{L^2} \) norm of the first term on the right hand side of (125) we conclude that it is of the form \( f \in L^2(T) \). This would prove the condition (83). The verification of the remaining conditions (84) and (85) can be done in an analogous manner. Similarly, we can perform the calculations for the second degree polynomials \( G^2_{0,y,y'}(k), y, y' \in \mathbb{Z} \) and \( k \in \mathbb{T} \).

5.4. The evolution of the energy density in the frequency domain. Let \( \mathcal{E}^\epsilon(t, k) := (1/2) \langle |\tilde{\psi}^\epsilon(t, k)|^2 \rangle \), where \( \tilde{\psi}^\epsilon(t, k) \) is the Fourier transform of the wave function \( (\psi^\epsilon_y(t)) \). Recall that the scattering operator \( \mathcal{L} \) is defined by (45). We have the following analogue of Lemma 5.1.

**Lemma 5.2.** Suppose that \( J \in C^\infty(\mathbb{T}) \) then
\[
\langle \mathcal{E}^\epsilon(t), J \rangle_{L^2(\mathbb{T})} = \langle \mathcal{E}^\epsilon(0), J \rangle_{L^2(\mathbb{T})} + \int_0^t ds \langle \mathcal{E}^\epsilon(s), \mathcal{L} J \rangle_{L^2(\mathbb{T})} + \tilde{R}^\epsilon(t), \ t \geq 0,
\]
a.s. in \( \xi(\cdot) \). In addition, for every \( T > 0 \)
\[
\lim_{\epsilon \to 0^+} \sup_{t \in [0,T]} |\mathbb{E} \tilde{R}^\epsilon(t)| = 0.
\]
(128)
**Proof.** For a given \( \lambda > 0 \) we consider a function \( G^\lambda \in \mathcal{S}(\mathbb{R} \times \mathbb{T}) \) whose Fourier transform in the first variable equals
\[
\hat{G}^\lambda(q, k) := (2\pi \lambda)^{-1/2} e^{-|q|^2/\lambda} \hat{J}(k), \quad (q, k) \in \mathbb{R} \times \mathbb{T}.
\]
In the proof of Lemma 5.1 we can use function \( G^\lambda \) in place of \( G \) and obtain in this way the formula (94) for \( G^\lambda \). Letting \( \lambda \to 0^+ \), for a fixed \( \epsilon > 0 \), we obtain (127) with
\[
\tilde{R}^\epsilon(t) := \int_0^t |\mathbb{E}^3_{\epsilon,0}(s)| ds + \sqrt{\epsilon} |\mathbb{E}^3_{\epsilon,0}(t)|.
\]
Here \( \mathbb{E}^3_{\epsilon,0}(t) \) is obtained from (98) by letting \( q = 0 \) and collapsing the integration over \( q \). The term \( \mathbb{E}^3_{\epsilon,0}(t) \) is therefore a combination of a finite number of the terms of the form: either
\[
\int_{\mathbb{T}^4} J^*(k) dk \prod_{j=1}^{3} e^{2\pi ik_j (\rho y_j (1 - \rho) y')} f(k, k_1, k_2, k_3) \mathcal{E}^\epsilon(t, k) \hat{V}(dk_1, dk_2, dk_3),
\]
or
\[
\int_{\mathbb{T}^4} J^*(k) dk \prod_{j=1}^{3} e^{2\pi ik_j (\rho y_j (1 - \rho) y')} f(k, k_1, k_2, k_3) Y^\epsilon,\ell(t, k) \hat{V}(dk_1, dk_2, dk_3), \quad \ell = \pm,
\]
where functions \( f \) are complex valued and belong to \( W^{1,\infty}(\mathbb{T}^4) \) (due to the fact that \( \gamma(k) \geq \gamma_* \)). The argument used in Section 5.2.1 allows us to conclude that
\[
\limsup_{\epsilon \to 0^+} \sup_{t \in [0,T]} \left| \mathbb{E}^3_{\epsilon,0}(t) \right| < +\infty.
\]
On the other hand,
\[ \Psi_0^2(t) := \sum_{k=\pm} \left\langle Y^{\epsilon,t}(t), \tilde{\mathcal{L}}_0^G \right\rangle_{L^2(T)}, \]
(129)
with
\[ Y^{\epsilon,+}(t,k) := \frac{1}{2} (\hat{\psi}^{\epsilon}(t,k) \hat{\psi}^\ast(t,k)) \mu_\epsilon, \quad Y^{\epsilon,-}(t,k) := (Y^{\epsilon,+}(t,k))^\ast \]
and
\[ \tilde{\mathcal{L}}_0^G(k) = \int_T \frac{2\gamma(k' - k) \sigma(k' - k) \nu^2(k, k-k')}{\gamma^2(k' - k) + [\omega(k) + i\omega(k')]^2} (\hat{G}(k) - \hat{G}(k')) dk'. \]
(130)
Concerning the term given by (129), its expectation can be written as a sum of the terms in the form
\[ \int_0^t \mathbb{E}\langle Y^{\epsilon, \pm}(s), \Theta J' \rangle_{L^2(T)} ds, \]
where each \( J' \) is given by: either
\[ J'(k) := \int_T J(k) H(0, k, k') dk', \]
(131)
or
\[ J'(k) := \iota_1 \int_T J(k) \tilde{H}(0, k, k') e^{2\pi i k' (\rho - \nu)} dk' \]
(132)
and \( H, \tilde{H} \) are given by (114) and (115) respectively. Repeating the argument used Section 5.2.2 we conclude that
\[ \lim_{\epsilon \to 0+} \sup_{t \in [0,T]} \mathbb{E} \left[ \int_0^t \Psi_0^2(s) ds \right] = 0 \]
and (128) follows. \( \Box \)

**Corollary 5.3.** Suppose that condition (50) holds. Then,
\[ \lim\limsup_{\delta \to 0+} \sup_{t \in [0,T]} \epsilon \int_{|k| \leq \delta} \mathbb{E}_\epsilon \left[ |\hat{\psi}^{\epsilon}(t,k)|^2 \right] dk = 0, \quad \forall T \geq 0. \]
(133)
**Proof:** Suppose that \( \eta > 0 \) is arbitrary. We can find \( \delta > 0 \) such that for any \( J \in C^\infty(T) \) such that \( J(k) \equiv 1 \) for \( |k| \leq \delta, J(k) \equiv 0 \) for \( |k| \geq 2\delta \) and \( 0 \leq J(k) \leq 1 \) otherwise we have
\[ R_\ast(J) := \sup_{k' \in T} \int_T J(k) R(k,k') dk < \eta. \]
Using (127) and then (128) we obtain, cf (28),
\[ \limsup_{\epsilon \to 0+} \mathbb{E}\langle \mathcal{E}'(t), J' \rangle_{L^2(T)} \]
(134)
\[ \leq \limsup_{\epsilon \to 0+} \mathbb{E}\langle \mathcal{E}'(0), J' \rangle_{L^2(T)} + R_\ast(J) KT + \limsup_{\epsilon \to 0+} \sup_{t \in [0,T]} |\mathbb{E}\tilde{\mathcal{E}}^{\epsilon}(t)| \]
\[ \leq \limsup_{\epsilon \to 0+} \mathbb{E}\langle \mathcal{E}'(0), J' \rangle_{L^2(T)} + \eta KT. \]
Thanks to (50) we conclude that
\[ \lim\limsup_{\delta \to 0+} \sup_{t \in [0,T]} \mathbb{E}_\epsilon \left[ |\hat{\psi}^{\epsilon}(t,k)|^2 \right] dk \leq \eta KT \]
(135)
and, since \( \eta > 0 \) has been arbitrarily chosen, we conclude (133). \( \Box \)
5.5. The end of the proof of Theorem 2.5. Thanks to the bound (43) we know that \((W^n(t))\) is \(\ast\)-weakly compact in \(\mathcal{A}'\) for each \(t \geq 0\) fixed. Suppose that \(\epsilon_n \to 0^+\). We prove that \(\lim_{n \to +\infty} W^n(t) = U(t)\) \(\ast\)-weakly in \(\mathcal{A}'\), with \(U(t)\) described in the statement of the theorem. It suffices to show that any convergent subsequence of the sequence (which we denote for convenience by the same symbol) has \(U(t)\) as its limit. In fact, choosing an appropriate subsequence we may assume that \((W^n(t))\) converges for any non-negative rational \(t\). Suppose that \(G \in S(\mathbb{R} \times \mathbb{T})\). Condition (43) implies that for any \(T > 0\)

\[
\sup_{\epsilon \in [0, 1]} \sup_{s \in [0, T]} \left| \langle W^n(s), \hat{\mathcal{L}}_\epsilon G^n - \mathcal{D}_\epsilon G^n \rangle \right| < +\infty. \tag{136}
\]

Thanks to (5.1) the bound (136), in turn, implies that the sequence

\[
\langle W^n(t), G \rangle = \langle W^n(t), G^n \rangle, \quad t \geq 0
\]

is equicontinuous in \(C[0, T]\). Due to (43) it is also uniformly bounded in the \(t\) variable. Using the above one can easily argue that \((W^n(t))\) converges \(\ast\)-weakly in \(\mathcal{A}'\) for any \(t \geq 0\). We show that its limit \(U(t)\) is the solution of (44) in the sense of definition given in Section 2.7. Thanks to Theorem B. 4, p. 154 of [22] we have \(U(t) \in M_+(\mathbb{R} \times \mathbb{T})\) for each \(t \geq 0\). It is an immediate consequence of the construction of \(U(t)\) that conditions i) and ii) of Definition 2.2 are satisfied. To prove iii) it suffices only to show that

\[
\lim_{n \to +\infty} \langle W^n(s), \hat{\mathcal{L}}_\epsilon G^n - \mathcal{D}_\epsilon G^n \rangle = \langle U(s), \mathcal{L} G - \omega' G \rangle \tag{137}
\]

We consider first the acoustic case, i.e. when \((a4')\) holds (then condition (133) holds). Given \(\delta > 0\) we let \(J_0 \in C^\infty(\mathbb{T})\) be such that \(J_0(k) \equiv 1\) for \(|k| \leq \delta\), \(J_0(k) \equiv 0\) for \(|k| \geq 2\delta\) and \(0 \leq J_0(k) \leq 1\) otherwise. Let \(J_1(k) = 1 - J_0(k)\). We can write then

\[
\langle W^n(s), \hat{\mathcal{L}}_\epsilon G^n \rangle = \sum_{i_1, i_2 \in \{0, 1\}} \langle W^n(s), \mathcal{J}^{(n)}_{i_1i_2} \rangle, \tag{138}
\]

where

\[
\mathcal{J}_{i_1i_2}^{(n)}(q, k) := J_{i_2}(k) \sum_{i = \pm} \int_{\mathbb{T}} \hat{G}^*(q, k') \sigma(k - k') J_{i_1}(k') r(k' + \epsilon_n q/2, k' - k) \times \left(\gamma(k - k') - i\epsilon_n q/2 - \omega(k + \epsilon_n q/2)\omega(k + \epsilon_n q/2)\right)^{-1} dk' \\
- \sum_{i = \pm} \hat{G}^*(q, k) J_{i_1}(k) \int_{\mathbb{T}} \sigma(k - k') J_{i_2}(k') r^2(k + \epsilon_n q/2, k - k') \times \left(\gamma(k - k') - i\epsilon_n q/2 - \omega(k' + \epsilon_n q/2)\omega(k' + \epsilon_n q/2)\right)^{-1} dk'. \tag{139}
\]

Choose an arbitrary \(\rho > 0\). Thanks to (43) and (133) one can adjust \(\delta > 0\) in such a way that

\[
\lim_{n \to +\infty} \sup_{\epsilon \in [0, 1]} \left| \langle W^n(s), \mathcal{J}_{i_1i_2}^{(n)} \rangle \right| < \rho, \tag{140}
\]

provided \((i_1, i_2) \neq (1, 1)\). Due to the assumed \(C^2\)-regularity of \(\sigma(\cdot), \gamma(\cdot)\) and \(C^\infty\) regularity of \(\omega(\cdot)\) on \(\mathbb{T} \setminus \{0\}\) we conclude that \(\mathcal{J}_{1,1}^{(n)}\) converge strongly in \(\mathcal{A}\) to \(\mathcal{J}_{1,1}\).
as \( n \to +\infty \), where \( \mathcal{J}_{1,2} \) is defined by formula (139) in which \( \epsilon_n \) is set to be equal to 0. This proves that

\[
\limsup_{n \to +\infty} \left| \langle W^{\epsilon_n}(s), \mathcal{J}_{1,1}^{(n)} \rangle - \langle U(s), \mathcal{J}_{1,1} \rangle \right| = 0.
\]

Since \( \delta > 0 \) can be adjusted in such a way that

\[
|\langle U(s), \mathcal{J}_{1,1} \rangle| < \rho, \quad (t_1, t_2) \neq (1, 1)
\]

we conclude therefore that for any \( \rho > 0 \)

\[
\limsup_{n \to +\infty} \left| \langle W^{\epsilon_n}(s), \mathcal{L}^\pm G^{\epsilon_n} \rangle - \langle U(s), \mathcal{L}^\pm \rangle \right| < 6\rho,
\]

where

\[
\mathcal{L}^\pm G(x, k) := 2 \int_{T} \frac{\sigma(k + ik')\gamma(k + ik')r^2(k, k + ik')}{\gamma^2(k + ik') + [\omega(k') + i\omega(k)]^2} [G(x, k') - G(x, k)] dk',
\]

for \( G \in \mathcal{A}, \ i = \pm \). Thus,

\[
\lim_{n \to +\infty} \left| \langle W^{\epsilon_n}(s), \hat{\mathcal{L}}^\pm_{\epsilon_n} G^{\epsilon_n} \rangle - \langle U(s), \mathcal{L}^\pm \rangle \right| = 0, \ s \geq 0.
\]

Similarly, we show that

\[
\lim_{n \to +\infty} \left| \langle W^{\epsilon_n}(s), \hat{\mathcal{L}}^-_{\epsilon_n} G^{\epsilon_n} \rangle - \langle U(s), \mathcal{L}^- \rangle \right| = 0
\]

and, thanks to (73),

\[
\lim_{n \to +\infty} \left| \langle W^{\epsilon_n}(s), \mathcal{D}_{\epsilon_n} G^{\epsilon_n} \rangle - \langle U(s), \omega' \partial_x G \rangle \right| = 0, \ s \geq 0.
\]

This ends the demonstration of (137) concluding in this way the proof of the fact that \( (U(t)) \) is the solution of (44), thus identifying the limit of \( (W^{\epsilon_n}(t)) \).

In the case when the chain is pinned, i.e. (a4) holds, we can essentially repeat the argument for the convergence in (137) given above. In fact, things get simpler due to the fact that the dispersion relation is \( C^\infty(\mathbb{T}) \) smooth and we do not need to introduce the partition as in (138).

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**Appendix A. Solution of (25).** For simplicity we assume that \( \epsilon = 1 \) in equation (25) and omit the parameter from our notation throughout this section. Equation (25) reads then

\[
\frac{d\psi_y(t)}{dt} = -i (\tilde{\omega} + \psi(t))_y + \frac{1}{2} \sum_{|z| \leq 2} \theta_{y,z}(t) \left( \psi_{y+z}(t) - \psi^*_{y+z}(t) \right), \quad (t, y) \in \mathbb{R} \times \mathbb{Z}
\]

(143)

\[
\psi_y(0) = \psi_{0,y},
\]

where \( \theta_{y,z}(t) \) are given by (26). Let \( \mathcal{C} \) be the set of all compactly supported complex sequences \( (f_x) \), i.e. such that there exists \( M > 0 \) for which \( f_x = 0 \), when \( |x| \geq M \). Define \( A(t; \xi()) : \mathcal{C} \to \mathcal{C} \) as

\[
(A(t)f)_y := \frac{1}{2} \sum_{|z| \in \{1,2\}} \theta_{y,z}(t)f_{y+z}(t).
\]

(144)
For a given integer $N \geq 1$ let $\theta^N_{y,z}(t)$ be a sequence formed according to (26) from $\xi^N_y(t)$, given by

$$\xi^N_y(t) = \begin{cases} \xi_y(t) & \text{when } |y| \leq N \\ 0 & \text{when } |y| \geq N + 1. \end{cases}$$

We define $A^N(t; \xi(\cdot))$ using an analogue of (144), where $\theta_{y,z}(t)$ is replaced by $\theta^N_{y,z}(t)$. Then $A^N(t)f \in \mathcal{C}$ for $f \in \ell_2(\mathbb{Z})$ and $(A^N(t))_{t \in \mathbb{R}}$ are operators acting on $\ell_2(\mathbb{Z})$ that are uniformly bounded on compact intervals. One can easily check that

$$\langle A^N(t)h, g \rangle_{\ell_2} = -\langle h, A^N(t)g \rangle_{\ell_2}$$

for each $N \geq 1$, all realizations of $(\xi(t))$ and all $g, h \in \ell_2(\mathbb{Z})$, $t \in \mathbb{R}$.

**Lemma A.1.** We have

$$\omega_* := \sup_{y \in \mathbb{Z}} |\omega_y|^2 < \infty.$$  
(146)

**Proof.** For any $y \neq 0$ we have

$$\omega_y = \int_{\mathbb{T}} \omega(k) \exp\{2\pi iyk\} dk = -\frac{1}{2\pi iy} \int_{\mathbb{T}} \omega'(k) \exp\{2\pi iyk\} dk$$

$$= -\frac{1}{(2\pi iy)^2} \omega'(k) \exp\{2\pi iyk\} \bigg|_0^0 + \frac{1}{(2\pi iy)^2} \int_{-1/2}^{1/2} \omega''(k) \exp\{2\pi iyk\} dk - \frac{1}{(2\pi iy)^2} \int_{0}^{1/2} \omega''(k) \exp\{2\pi iyk\} dk$$

Relation (146) then follows.

As a consequence of the above lemma and Young’s inequality the operator $\Omega : \ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$, given by

$$\Omega f := -i\omega \ast f$$

is bounded. Using a standard theorem on the existence and uniqueness of solutions of ordinary differential equations with a Lipschitz right hand side we conclude that for any $N \geq 1$ and $\psi_0 \in \ell_2(\mathbb{Z})$ there exists a unique $\psi^N \in C(\mathbb{R}; \ell_2(\mathbb{Z}))$ that is differentiable in $\ell_2(\mathbb{Z})$, such that

$$\psi^N(t) = \Omega \psi^N(t) + A^N(t; \xi(\cdot))\psi^N(t) - (\psi^N(t))^*, \quad t \in \mathbb{R},$$

$$\psi^N(0) = \psi_0.$$  
(147)

Our main goal in this section is to show the following.

**Theorem A.2.** Given $\psi_0 \in \ell_2$ the sequence $(\psi^N(\cdot))$ converges, as $N \to +\infty$, uniformly on compact sets in $C(\mathbb{R}; \ell_2(\mathbb{Z}))$ to a certain process $(\psi(t))$ in the $L_2$ sense over the realizations of $(\xi(t))$. In addition, for almost sure realization of $(\xi(t))$ we have

$$\psi_y(t) = \psi_{0,y} - i \int_0^t (\tilde{\omega} \ast \psi(s))_y ds + \int_0^t (A(s)[\psi(s) - \psi^*(s)])_y ds$$

and

$$\|\psi(t)\|_{\ell_2} = \|\psi_0\|_{\ell_2}$$

for all $t \in \mathbb{R}$ and $y \in \mathbb{Z}$.
A.1. Some auxiliary evolution families. Consider the following linear system of equations on $\ell_2(\mathbb{Z})$

$$\frac{d\Psi^N(t)}{dt} = A^N(t)\Psi^N(t), \quad t \in \mathbb{R},$$

$$\Psi^N(s) = \Psi_0.$$  

Denote by $U^N(t, s; \xi(\cdot))\Psi_0 := \Psi^N(t)$. The following result is a simple consequence of the existence and uniqueness result for solutions of equations with Lipschitz coefficients and anti-symmetry of $A^N(t)$.

**Proposition A.3.** For a.s. realization of $(\xi(\cdot))$ and an arbitrary $N \geq 1$ we have

$$||U^N(t, s)\Psi||_{\ell_2} = ||\Psi||_{\ell_2}, \quad \forall \Psi \in \ell_2, t, s \in \mathbb{R}.$$  

In addition,

i) $U^N(t, t)$ is the identity map on $\ell_2$,

ii) $U^N(t, s)U^N(s, u) = U^N(t, u)$ for any $t, s, u \in \mathbb{R}$,

iii) the mapping

$$(t, s) \mapsto U^N(t, s)$$

is uniformly continuous.

From (150) we conclude that

$$(U^N(t, s)\Psi)_y = \Psi_y + \frac{1}{2} \sum_{|z| \in \{1, 2\}} \int_t^s \theta^N_{y, z}(\tau)(U^N(\tau, s)\Psi)_y + d\tau.$$  

(151)

Iterating the above equality we obtain the following Duhamel series representation of $U^N(t, s)\Psi$

$$(U^N(t, s)\Psi)_y = \Psi_y + \sum_{n=1}^{+\infty} (I^N_n(t, s))_y,$$  

(152)

where

$$(I^N_n(t, s))_y := \frac{1}{2^n} \sum_{|z_1|, \ldots, |z_n| \in \{1, 2\}} \int_{\Lambda_n(t, s)} d\tau_1, n \prod_{j=1}^n \theta^N_{y, z_1, j-1, z_j}(\tau_j)\Psi_{y, z_1, n}$$  

(153)

and $z_{1, 0} := 0$, $z_{1, j} := z_{1, j-1} + z_j$, $j \geq 1$,

$$\Lambda_n(t, s) := \{t \geq \tau_1 \geq \ldots \geq \tau_n \geq s\},$$

and $d\tau_{1, n} := \prod_{j=1}^n d\tau_j$. The series in (152) converges uniformly on compact sets in $(s, t)$ in $\ell_2(\mathbb{Z})$ for a.s. realization of $(\xi(t))$. In particular, we conclude also

$$U^N(t, s; \theta_h(\cdot)) = U^N(t + h, s + h; \xi(\cdot)), \quad t, s, h \in \mathbb{R}, \quad \text{a.s. in } \xi(\cdot),$$  

(154)

where $\theta_h(\cdot) := \xi(\cdot + h)$. Define

$$(\Psi(t, s))_y = \Psi_y + \sum_{n=1}^{+\infty} (I_n(t, s))_y,$$  

(155)

where

$$(I_n(t, s))_y := \frac{1}{2^n} \sum_{|z_1|, \ldots, |z_n| \in \{1, 2\}} \int_{\Lambda_n(t, s)} d\tau_1, n \prod_{j=1}^n \theta_{y, z_1, j-1, z_j}(\tau_j)\Psi_{y, z_1, n}, \quad y \in \mathbb{Z},$$  

(156)
Proposition A.4. Given $\Psi \in \ell_2(\mathbb{Z})$, the series in (155) converges uniformly on compact sets in $(s, t)$ in $\ell_2(\mathbb{Z})$ metric in the $L_1$ sense with respect to $P$. In addition, for any $R > 0$ we have

$$\lim_{N \to +\infty} \mathbb{E} \left[ \sup_{|s|, |t| \leq R} \| \Psi(t, s) - U^N(t, s)\Psi \|_{\ell_2} \right] = 0. \quad (157)$$

Proof. To prove the first statement it suffices to show that

$$\sum_{n=1}^{+\infty} \mathbb{E} \left[ \sup_{|s|, |t| \leq R} \| \mathcal{I}_n(t, s) \|_{\ell_2} \right] < +\infty. \quad (158)$$

By virtue of Jensen's inequality, applied twice, we get

$$\mathbb{E} \left[ \sup_{|s|, |t| \leq R} \| \mathcal{I}_n(t, s) \|_{\ell_2} \right] \leq \left\{ \sum_{y} \mathbb{E} \left[ \sum_{|z_1|, \ldots, |z_n| \in \{1, 2\}} \int_{\Lambda_n(R, -R)} d\tau_{1,n} \prod_{j=1}^{n} \theta_{y+z_1,j-1,z_j}^{\ell}(\tau_j) \right] \right\}^{1/2} \leq |\Lambda_n(R, -R)|^{1/2} \left\{ \sum_{y} \left[ \sum_{|z_1|, \ldots, |z_n| \in \{1, 2\}} \int_{\Lambda_n(R, -R)} d\tau_{1,n} \mathbb{E} \left[ \prod_{j=1}^{n} \theta_{y+z_1,j-1,z_j}^{\ell}(\tau_j) \right] \right] \right\}^{1/2}.$$

Here $|\Lambda_n(R, -R)|$ is the volume of the simplex, that equals $(2R)^n/n!$. The expectation of the mixed Gaussian $2n$-th moment can be expressed as a sum of $(2n-1)!!$ terms. Each is a product of $n$ suitable covariances between $\theta_{y+z_1,j-1,z_j}(\tau_j)$ and $\theta_{y+z_1,j',-1,z_j}(\tau_j)$ that can be estimated by $2\mathbb{E} \xi_0^2(0)$. Therefore, the expression in the right hand side of (159) can be estimated by

$$\left\{ \frac{(2R)^n}{n!} \sum_{|z_1|, \ldots, |z_n| \in \{1, 2\}} \int_{\Lambda_n(R, -R)} d\tau_{1,n} (2n-1)!! 2^n [\mathbb{E} \xi_0^2(0)]^n \sum_{y} \| \psi_{y+z_1,n} \|_{\ell_2}^2 \right\}^{1/2} \leq \left( \frac{4R}{n!} \right)^n \left( (2n-1)!! [\mathbb{E} \xi_0^2(0)]^n \right)^{1/2} \left\| \Psi \right\|_{\ell_2} \leq \left\| \Psi \right\|_{\ell_2} \frac{C^n}{\sqrt{n}!}, \quad (160)$$

for some constant $C > 0$ independent of $n$, and (158) follows.

Using the same argument one can also conclude that

$$\mathbb{E} \left[ \sup_{|s|, |t| \leq R} \| \mathcal{I}_n(t, s) \|_{\ell_2} \right] \leq \frac{C^n}{\sqrt{n}!}, \quad (161)$$

where the constant $C > 0$ does not depend on $N$. To prove (157) it suffices therefore to show that for a fixed $n$ we have

$$\lim_{N \to +\infty} \mathbb{E} \left[ \sup_{|s|, |t| \leq R} \| \mathcal{I}_n(t, s) - \mathcal{I}_n^N(t, s) \|_{\ell_2} \right] = 0. \quad (162)$$
From the definition of the truncated field $\theta_{y,z}^N$ it is clear that the expectation under the limit can be estimated by

$$
\sum_{|z_1|, \ldots, |z_n| \in \{1,2\}} \int_{\Lambda_n(R,-R)} d\tau_{t,n} \times \sum_{y} \mathbb{E} \left\{ \left[ \prod_{j=1}^{n} \theta_{y+z_1,j-1,z_j}(\tau_j) - \prod_{j=1}^{n} \theta_{y+z_1,j-1,z_j}(\tau_j) \right]^2 \right\} \leq \sum_{|z_1|, \ldots, |z_n| \in \{1,2\}} \int_{\Lambda_n(R,-R)} d\tau_{t,n} \\
\times \sum_{|y| \geq N-2n} \mathbb{E} \left\{ \left[ \prod_{j=1}^{n} \theta_{y+z_1,j-1,z_j}(\tau_j) - \prod_{j=1}^{n} \theta_{y+z_1,j-1,z_j}(\tau_j) \right]^2 \right\} \leq C \left\{ \sum_{|y| \geq N-4n} |\Psi_y|^2 \right\}^{1/2},
$$

where the constant $C > 0$ may depend on $n$. Letting $N \to +\infty$ we conclude that the expression on the utmost right hand side tends to 0 and (162) follows.

Corollary A.5. For a.s. realization of $\xi$ we can define a strongly continuous evolution family $U(t,s)$ on $\ell_2(\mathbb{Z})$, i.e.

i) $U(t,t)$ is an identity map,

ii) $U(t,s)U(s,u) = U(t,u)$ for any $t, s, u \in \mathbb{R}$,

iii) the mapping $(t,s) \mapsto U(t,s)$ is strongly continuous and satisfies $U(t,s)\Psi = \Psi(t,s)$ for all $t, s \in \mathbb{R}$,

iv) for $\mathbb{P}$ a.s. $\xi$ we have

$$
U(t+h, s+h; \xi) = U(t, s; \theta_h \xi), \quad \text{for any } t, s, h \in \mathbb{R}. \quad (163)
$$

In addition,

$$
\|U(t,s)\Psi\|_{\ell_2} = \|\Psi\|_{\ell_2}, \quad \forall \Psi \in \ell_2, \; t, s \in \mathbb{R} \quad (164)
$$

and

$$
\lim_{N \to +\infty} \sup_{|t|, |s| \leq R} \|U(t,s)\Psi - U^N(t,s)\Psi\|_{\ell_2} = 0, \quad \forall \Psi \in \ell_2, \; R > 0, \quad (165)
$$

for any $\Psi \in \ell_2(\mathbb{Z})$ a.s. in realizations of $\xi$.

Proof. Suppose that $\mathcal{G} \subset \ell_2(\mathbb{Z})$ is dense and countable. For each $\Psi \in \ell_2(\mathbb{Z})$ we can find a set $N_\Psi$ such that $\mathbb{P}[N_\Psi] = 0$ and the family $\Psi(t,s)$ is defined by (155) for all $s, t \in \mathbb{R}$ outside $N_\Psi$. Using (157) we can adjust the set so that

$$
\lim_{N \to +\infty} \sup_{|t|, |s| \leq R} \|\Psi(t,s) - U^N(t,s)\Psi\|_{\ell_2} = 0, \quad \forall \Psi \in \ell_2, \; R > 0. \quad (166)
$$

outside $N_\Psi$. Hence, outside $\mathcal{N} := \bigcup_{\Psi \in \mathcal{G}} N_\Psi$ we can define

$$
U(t,s)\Psi := \Psi(t,s), \quad \forall t, s \in \mathbb{R}.
$$

Thanks to (166) we conclude that (164) holds for all $\Psi \in \mathcal{G}$ outside $\mathcal{N}$. Operators $U(t,s)$ can be therefore extended to the entire $\ell_2(\mathbb{Z})$ outside $\mathcal{N}$. In addition, (165) holds outside $\mathcal{N}$. This and (154) imply properties i)-iv) of $U(t,s)$. \qed
A.2. Proof of Theorem A.2. Observe that the solution of (147) satisfies
\[ \psi^N(t) = \phi^N(t), \]
where \((\phi^N(t), \chi^N(t))\) satisfies the following linear system
\[ \begin{align*}
\frac{d\phi^N(t)}{dt} &= \Omega \phi^N(t) + A^N(t)[\phi^N(t) - \chi^N(t)], \\
\frac{d\chi^N(t)}{dt} &= -\Omega \chi^N(t) - A^N(t)[\phi^N(t) - \chi^N(t)], \quad t \in \mathbb{R},
\end{align*} \tag{167} \]
\[ \phi^N(0) = \psi_0, \quad \chi^N(0) = \psi_0^*. \]
Introducing
\[ u^N(t) := \frac{1}{2} (\phi^N(t) - \chi^N(t)), \]
\[ v^N(t) = \frac{1}{2} (\phi^N(t) + \chi^N(t)) \]
we can rewrite (167) in the form
\[ \begin{align*}
\frac{du^N(t)}{dt} &= \Omega v^N(t) + 2A^N(t)u^N(t), \\
\frac{dv^N(t)}{dt} &= -\Omega u^N(t), \quad t \in \mathbb{R},
\end{align*} \tag{168} \]
\[ u^N(0) = i\text{Im} \psi_0, \quad v^N(0) = \text{Re} \psi_0. \]
Therefore
\[ \psi^N(t) = \frac{1}{2} (u^N(t) + v^N(t)), \quad t \in \mathbb{R}. \]
To prove the convergence part of the assertions made in Theorem A.2 it suffices to show that an analogous assertion holds for
\[ z^N(t) = \begin{bmatrix} u^N(t) \\ v^N(t) \end{bmatrix}. \]
Let \(z_0 = z^N(0)\). Note that \(z^N(t)\) satisfies the following mild formulation of (168)
\[ z^N(t) = \tilde{U}^N(t, s)z^N(s) + \int_s^t \tilde{U}^N(t, \tau)\tilde{\Omega}z^N(\tau)d\tau, \]
where \(\tilde{U}^N(t, \tau)\) is a strongly continuous evolution family of unitary maps on \(\ell_2(\mathbb{Z}) \times \ell_2(\mathbb{Z})\) given by the operator entry \(2 \times 2\) matrix
\[ \tilde{U}^N(t, \tau) := \begin{bmatrix} U^N(t, \tau) & 0 \\ 0 & I \end{bmatrix}. \]
We also let \(\tilde{\Omega}\) be a bounded linear operator on \(\ell_2(\mathbb{Z}) \times \ell_2(\mathbb{Z})\) given by
\[ \tilde{\Omega} = \begin{bmatrix} 0 & \Omega \\ -\Omega & 0 \end{bmatrix}, \]
with \(U^N(t, s)\) the evolution family corresponding to \(2A^N\). Iterating we obtain the Duhamel series expression for \(z^N(t)\)
\[ z^N(t) = \tilde{U}^N(t, s)z^N(s) + \sum_{n=1}^{\infty} \int_{\Lambda_n(t, s)} \tilde{U}^N(t, \tau_1)\tilde{\Omega}\tilde{U}^N(\tau_1, \tau_2)\tilde{\Omega}z^N(\tau_2)d\tau_1, n, \]
with \( \Lambda_n(t, s) := [s \leq \tau_n \leq \ldots \leq \tau_1 \leq t] \), and the relevant convergence result for \( z^N(t) \) follows from (165). In addition,

\[
z(t) = \bar{U}(t, s)z(s) + \sum_{n=1}^{\infty} \int_{\Lambda_n(t, s)} \bar{U}(t, \tau_1)\bar{U}(\tau_1, \tau_2)\cdots \bar{U}(\tau_{n-1}, \tau_n)\bar{\Omega}(s) d\tau_{1,n}
\]

with \( \bar{U}(t, s) \) the limit of \( \bar{U}^N(t, s) \), as \( N \to +\infty \).

The proof of (148) follows from the fact that for a fixed \( y \) we have \( \theta^N_{y,z}(t) = \theta_{y,z}(t) \) for \( N > |y| + 2 \) and the uniform convergence of \( \psi^N_{y+z}(t) \) to \( \psi_{y+z}(t) \) for \( z \in \{0, \pm 1, \pm 2\} \).

Appendix B. Proof of Proposition 4.2. In this section we restore the index \( \epsilon \) in the notation. Recall that \( (\mathcal{G}_t^\epsilon)_{t \geq 0} \) is the natural filtration corresponding to the process \( (\xi^\epsilon(t), p^\epsilon(t))_{t \geq 0} \). Note that

\[
\frac{1}{h} \left\{ \mathbb{E} \left[ \Phi_{\epsilon}(t+h)|\mathcal{G}_t^\epsilon \right] - \Phi_{\epsilon}(t) \right\} = \frac{1}{h} \int_t^{t+h} \mathbb{E} \left[ \frac{d}{ds} (W^\epsilon(s), J(\xi^\epsilon(t+h)))|\mathcal{G}_s^\epsilon \right] ds
\]

\[
+ \frac{1}{h} \int_0^{h} \mathbb{E} \left[ (W^\epsilon(t), P^\epsilon \Theta J(\xi^\epsilon(t)))|\mathcal{G}_t^\epsilon \right] ds.
\]

(170)

An analogous argument to the one used in Section 3 yields

\[
\frac{d}{ds} (W^\epsilon(s), J(\xi^\epsilon(t+h))) = \langle \partial_t W^\epsilon(s), J(\xi^\epsilon(t+h)) \rangle = -\langle W^\epsilon(s), \Theta J(\xi^\epsilon(t+h)) \rangle
\]

\[
- \frac{i}{\sqrt{\epsilon}} \langle W^\epsilon(s), K^\epsilon_{\xi^\epsilon}(s)J(\xi^\epsilon(t+h)) \rangle + \frac{i}{\sqrt{\epsilon}} \sum_{i=\pm} \langle Y^\epsilon_i(s), K^\epsilon_{\xi^\epsilon}(s)J(\xi^\epsilon(t+h)) \rangle
\]

and

\[
\frac{d}{ds} (Y^\epsilon(s), J(\xi^\epsilon(t+h))) = \langle \partial_t Y^\epsilon(s), J(\xi^\epsilon(t+h)) \rangle = -\frac{i}{\epsilon} \langle Y^\epsilon(s), \Theta J(\xi^\epsilon(t+h)) \rangle
\]

\[
- \frac{i}{\sqrt{\epsilon}} \langle Y^\epsilon(s), K^\epsilon_{\xi^\epsilon}(s)J(\xi^\epsilon(t+h)) \rangle + \frac{i}{\sqrt{\epsilon}} \langle W^\epsilon(s), K^\epsilon_{\xi^\epsilon}(s)J(\xi^\epsilon(t+h)) \rangle.
\]

To prove (80) and (81) it suffices to show that

\[
\sup_{t+h \geq 0} \sup_{t+h \leq T} \mathbb{E} |\langle \partial_t W^\epsilon(s), J(\xi^\epsilon(t+h)) \rangle| < +\infty,
\]

\[
\sup_{t+h \geq 0} \sup_{T \leq s \leq t+h} \mathbb{E} |\langle W^\epsilon(t), P^\epsilon \Theta J(\xi^\epsilon(t)) \rangle| < +\infty
\]

(171)

for any \( T > 0 \) and

\[
\lim_{h \to 0^+} \sup_{t \leq s \leq t+h} \mathbb{E} |\langle \partial_t W^\epsilon(s), J(\xi^\epsilon(t+h)) \rangle - \langle \partial_t W^\epsilon(t), J(\xi^\epsilon(t)) \rangle| = 0,
\]

(172)

\[
\lim_{h \to 0^+} \sup_{0 \leq s \leq h} \mathbb{E} |\langle W^\epsilon(t), P^\epsilon \Theta J(\xi^\epsilon(t)) \rangle - \langle W^\epsilon(t), \Theta J(\xi^\epsilon(t)) \rangle| = 0
\]

for any \( t \geq 0 \). We only verify the first equality in (172), as the others follow from a similar argument. The equality in question follows, provided we can prove that:

\[
\lim_{h \to 0^+} \sup_{t \leq s \leq t+h} \mathbb{E} |\langle \partial_t W^\epsilon(s), \delta_h J(\xi^\epsilon(t)) \rangle| = 0,
\]

(173)

and

\[
\lim_{h \to 0^+} \sup_{t \leq s \leq t+h} \mathbb{E} |\langle \partial_t W^\epsilon(s), J(\xi^\epsilon(t)) \rangle - \langle \partial_t W^\epsilon(t), J(\xi^\epsilon(t)) \rangle| = 0.
\]

(174)
Here
\[ \delta_h J(\xi^\varepsilon(t); k) := J(\xi^\varepsilon(t + h); k) - J(\xi^\varepsilon(t); k). \]
We also use the notation
\[ \delta_h \tilde{J}(\xi^\varepsilon(t); z) := J(\xi^\varepsilon(t + h); z) - \tilde{J}(\xi^\varepsilon(t); z), \quad z \in \mathbb{Z}. \]

To show (173) it suffices to prove that
\[ \lim_{h \to 0^+} \sup_{t \leq s \leq t + h} \sum_{y, y'} \mathbb{E} \left\{ \int_T^{\infty} \partial_s \mathcal{W}_{y, y'}(s, k) (\delta_h J_{y, y'})^*(\xi^\varepsilon(t); k) \, dk \right\} = 0. \tag{175} \]

Performing the integral over \( k \) we can rewrite the series above in the form
\[ \sum_{y, y'} \frac{\epsilon}{2} \mathbb{E} \left\{\left| \partial_s \left\langle \psi_y^\varepsilon(s)(\psi_y^\varepsilon)^*(s) \right\rangle_{\mu_s} (\delta_h \tilde{J}_{y, y'})^*(\xi^\varepsilon(t); y - y')\right| \right\}. \tag{176} \]
Substituting for \( \partial_s \left\langle \psi_y^\varepsilon(s)(\psi_y^\varepsilon)^*(s) \right\rangle_{\mu_s} \) from (56) we obtain an expression that can be written as the sum of the series corresponding to each term appearing in the right hand side of (56). We shall prove that
\[ \lim_{h \to 0^+} \sup_{t \leq s \leq t + h} \sum_{y, y'} \mathbb{E} \left\{ \left| \left\langle \psi_y^\varepsilon(s)(\tilde{\omega} \ast \psi_y^\varepsilon)^*(s) \right\rangle_{\mu_s} (\delta_h \tilde{J}_{y, y'})^*(\xi^\varepsilon(t); y - y') \right| \right\} = 0 \tag{177} \]
and
\[ \lim_{h \to 0^+} \sup_{t \leq s \leq t + h} \sum_{y, y' \mid |z| \leq 2} \mathbb{E} \left\{ \left| \left\langle \psi_y^\varepsilon(s)(\psi_y^\varepsilon)^*(s) \right\rangle_{\mu_s} (\delta_h \tilde{J}_{y, y'})^*(\xi^\varepsilon(t); y - y') \right| \right\} = 0, \tag{178} \]
as the remaining terms can be dealt with in a similar fashion.

The expression under the limit in (177) can be estimated using the Cauchy-Schwarz inequality by
\[
\mathbb{E} \left\{ \sum_{y, y'} \left| \left\langle \psi_y^\varepsilon(s)(\tilde{\omega} \ast \psi_y^\varepsilon)^*(s) \right\rangle_{\mu_s} (\delta_h \tilde{J}_{y, y'})^*(\xi^\varepsilon(t); y - y') \right| \right\} \leq \mathbb{E} \left[ \sum_{y, y'} |\psi_y^\varepsilon(s)|^2 |\tilde{\omega} \ast \psi_y^\varepsilon| \right]^{1/2} \|\delta_h J(\xi^\varepsilon(t))\|_{L^2} \leq \mathbb{E} [\|\psi^\varepsilon(s)\|_{L^2}^2 \|\tilde{\omega}\|_{L^1} \|\delta_h J(\xi^\varepsilon(t))\|_{L^2}]. \tag{179} \]
The last estimate follows from Young’s inequality. Due to Lemma A.1 we have \( \|\tilde{\omega}\|_{L^1} < +\infty \). Thanks to the conservation of energy, see (27), and stationarity of \( \xi^\varepsilon \) the expression in the utmost right hand side of (179) equals
\[ \|\tilde{\omega}\|_{L^1} \|\psi^\varepsilon(0)\|_{L^2} \mathbb{E} \|\delta_h J(\xi^\varepsilon(0))\|_{L^2}. \]
Hence (177) follows from the assumption (85). Concerning (178) observe that for fixed \( \epsilon, m > 0 \) the expression appearing there can be estimated by
\[
\mathbb{E} \left\{ \sup_{|z| \leq 2} \frac{|\theta_{y, y'}^\varepsilon(s)|}{(y')^{m}} \sum_{y, y'} \langle y \rangle \langle y' \rangle^m |\psi_y^\varepsilon(s)|^2 |\psi_y^\varepsilon| \right\} \leq \|\psi^\varepsilon(0)\|_{L^2}^2 \mathbb{E} \left[ C_m(s; \epsilon \|\delta_h J(\xi^\varepsilon(t))\|_{H^m}) \right].
\]
Using Hölder inequality, conditions (20) and (85) we conclude that the latter expression tends to 0 when \( h \to 0^+ \).

**Appendix C. Proof of Theorem 2.3.** With no loss of generality we may and shall assume that \( U_0 \) is a probability measure.

**C.1. Existence part.** Suppose that \((X_0, K_0)\) is a random vector distributed according to \( U_0 \). Let \( K_t \) be the process starting at \( K_0 \) with the generator \( \mathcal{L} \) and let \( U(t) \) be the law of the process of \((X_t, K_t)\), where \( X_t := X_0 + 1/(2\pi) \int_0^t \omega'(K_s) \, ds \).

It is well known, see e.g. [8], that for any \( J \in \mathcal{S} \) we have

\[
\frac{d}{dt} \langle U_t, J \rangle = \frac{d}{dt} \int_{\mathbb{R} \times \mathbb{T}} \mathbb{E} [J^*(X_t, K_t)] = \frac{d}{dt} \int_{\mathbb{R} \times \mathbb{T}} \mathbb{E} \left[ \left( \mathcal{L} + \frac{\omega'(K_t)}{2\pi} \partial_x \right) J^*(X_t, K_t) \right]
\]

Using Hölder inequality, conditions (20) and (85) we conclude that the latter expression tends to 0 when \( h \to 0^+ \).

**C.2. Uniqueness.** Suppose that \((X_0, K_0)\) is a random vector distributed according to \( U_0 \). Let \( K_t \) be the process starting at \( K_0 \) with the generator \( \mathcal{L} \) and let \( U(t) \) be the law of the process of \((X_t, K_t)\), where \( X_t := X_0 + 1/(2\pi) \int_0^t \omega'(K_s) \, ds \).

It is well known, see e.g. [8], that for any \( J \in \mathcal{S} \) we have

\[
\frac{d}{dt} \langle U_t, J \rangle = \frac{d}{dt} \int_{\mathbb{R} \times \mathbb{T}} \mathbb{E} [J^*(X_t, K_t)] = \frac{d}{dt} \int_{\mathbb{R} \times \mathbb{T}} \mathbb{E} \left[ \left( \mathcal{L} + \frac{\omega'(K_t)}{2\pi} \partial_x \right) J^*(X_t, K_t) \right]
\]

Using Hölder inequality, conditions (20) and (85) we conclude that the latter expression tends to 0 when \( h \to 0^+ \).

**Lemma C.1.** Under the assumptions spelled out above for any \( T > 0 \) we have

\[
\lim_{\delta \to 0^+} \sup_{t \in [0, T]} \mathbb{E} \left[ \left| J^*(t; \partial_x) \right| \right] \leq 0.
\]

**Proof.** Let

\[
u(t; \partial_x) := \int_{\mathbb{R}} U(t; \partial_x, \partial_x).
\]

and \( u_0(\partial_x) := u(0; \partial_x) \). It satisfies equation (44) with \( \omega'(k) \equiv 0 \). Thus, by virtue of (181) we obtain

\[
\int_0^T \mathbb{E} \left[ \left| J^*(t; \partial_x) \right| \right] dt = \int_0^T \mathbb{E} \left[ \left| J^*(t; \partial_x) \right| \right] dt.
\]

Choosing \( J(k) \equiv 1 \) on \( B_\delta := \{|k| < \delta\}, 0 \leq J \leq 1 \) and \( J(k) \equiv 0 \) on \( B_{2\delta}^c \) we can easily conclude from (48) that for any \( \rho > 0 \) we can adjust \( \delta > 0 \) in such a way that

\[
\mathbb{E} \left[ \left| J^*(t; \partial_x) \right| \right] \leq \int_T 1_{B_{2\delta}^c}(K_t) u_0(\partial_x) dt < \rho
\]
for all \( t \in [0, T] \). Thus (182) follows.

Suppose that \( \omega_h(k) \) is a family of functions from \( C^\infty(\mathbb{T}) \) such that \( \omega_h'(k) = \omega'(k) \) for \( |k| \geq h \) and

\[
\|\omega_h'\|_\infty \leq 2\|\omega'\|_\infty \quad \text{and} \quad \lim_{h \to 0^+} \omega_h'(k) = \omega'(k), \quad \forall k \neq 0.
\]

For any \( J \in \mathcal{S} \) and \( h > 0 \) define a function \( t \mapsto J_h(t) \) whose Fourier transform (in the first variable) equals

\[
\hat{J}_h(t, p, k) := \mathbb{E} \left[ \exp \left\{ 2\pi i p \int_0^t \omega_h(K_s(k)) \, ds \right\} \hat{J}(p, K_t(k)) \right].
\]

It belongs to \( C^1([0, +\infty), \mathcal{S}) \). A simple calculation, using (47), shows that

\[
\langle U(0), J_h(t) \rangle - \langle U(t), J_h(0) \rangle = \frac{1}{2\pi} \int_0^t \langle U(t-s), (\omega' - \omega_h') \partial_x J_h(s) \rangle \, ds.
\]

The absolute value of the expression on the right hand side of (183) can be estimated by

\[
\frac{1}{2\pi} \|J\| \int_0^t ds \int_{\mathbb{T}} |(\omega'(k) - \omega_h'(k))u(s, dk)| \leq \frac{1}{\pi} \|\omega'\|_\infty \|J\| \int_0^t u(s, B_h) \, ds,
\]

which clearly tends to 0, as \( h \to 0^+ \). As a result we again obtain equality (180), which identifies \( U(t) \).

In addition, (49) is a consequence of the fact that, according to the existence part, \( U(t) \) is the joint law of a random vector. If, on the other hand, \( U_0 \in L^1_1(\mathbb{R} \times \mathbb{T}) \) then

\[
U(t, x, k) = \mathbb{E} U_0 \left( x - \int_0^t \omega'(K_s(k)) \, ds, K_t(k) \right).
\]

Due to the invariance of the Lebesgue measure under the dynamics of \( K_t(k) \) we conclude that

\[
\int_{\mathbb{R} \times \mathbb{T}} |U(t, x, k)| \, dx \, dk \leq \int_{\mathbb{T}} dk \mathbb{E} \int_{\mathbb{R}} \left| U_0 \left( x - \int_0^t \omega'(K_s(k)) \, ds, K_t(k) \right) \right| \, dx
\]

\[
= \int_{\mathbb{T}} dk \mathbb{E} \int_{\mathbb{R}} |U_0 (x, K_t(k))| \, dx = \|U_0\|_{L^1(\mathbb{R} \times \mathbb{T})},
\]

which proves that \( U(t) \in L^1_1(\mathbb{R} \times \mathbb{T}) \).

\[\square\]

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