RICHARDSON VARIETIES HAVE KAWAMATA LOG TERMINAL SINGULARITIES

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Abstract. Let $X_w^v$ be a Richardson variety in the full flag variety $X$ associated to a symmetrizable Kac-Moody group $G$. Recall that $X_w^v$ is the intersection of the finite dimensional Schubert variety $X_w$ with the finite codimensional opposite Schubert variety $X^v$. We give an explicit $\mathbb{Q}$-divisor $\Delta$ on $X_w^v$ and prove that the pair $(X_w^v, \Delta)$ has Kawamata log terminal singularities. In fact, $-K_{X_w^v} - \Delta$ is ample, which additionally proves that $(X_w^v, \Delta)$ is log Fano.

We first give a proof of our result in the finite case (i.e., in the case when $G$ is a finite dimensional semisimple group) by a careful analysis of an explicit resolution of singularities of $X_w^v$ (similar to the BSDH resolution of the Schubert varieties). In the general Kac-Moody case, in the absence of an explicit resolution of $X_w^v$ as above, we give a proof that relies on the Frobenius splitting methods. In particular, we use Mathieu’s result asserting that the Richardson varieties are Frobenius split, and combine it with a result of N. Hara and K.-I. Watanabe relating Frobenius splittings with log canonical singularities.

1. Introduction

Let $G$ be any symmetrizable Kac-Moody group over $\mathbb{C}$ (or any algebraically closed field of characteristic zero) with the standard Borel subgroup $B$, the standard negative Borel subgroup $B^-$, the maximal torus $T = B \cap B^-$ and the Weyl group $W$. Let $X = G/B$ be the full flag variety. For any $w \in W$, we have the Schubert variety

$X_w := \overline{BwB/B} \subset G/B$

and the opposite Schubert variety

$X_w^o := \overline{B^-wB/B} \subset G/B$.

For any $v \leq w$, consider the Richardson variety $X_w^v$ which is defined to be the intersection of a Schubert variety and an opposite Schubert variety.

$X_w^v := X^v \cap X_w$

with the reduced subscheme structure. In this paper, we prove the following theorem.

Main Theorem (Theorem 3.2). With the notation as above, for any $v \leq w \in W$, there exists an effective $\mathbb{Q}$-divisor $\Delta$ on $X_w^v$ such that $(X_w^v, \Delta)$ has Kawamata log terminal (for short KLT) singularities.

Furthermore, $-K_{X_w^v} - \Delta$ is ample which proves that $(X_w^v, \Delta)$ is also log Fano.

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This divisor $\Delta$, described in Section 5, is built out of the boundary $\partial X^v_w$. As an immediate corollary of this result and Kawamata-Viehweg vanishing, we obtain the following cohomology vanishing (due to Brion–Lakshmibai in the finite case).

**Main Corollary** (Corollary 5.3). For a dominant integral weight $\lambda$, and any $v \leq w$,

$$H^i(X^v_w, L(\lambda)|_{X^v_w}) = 0, \quad \text{for all} \quad i > 0.$$  

Note, KLT singularities are a refinement of rational singularities. In particular, every KLT singularity is also a rational singularity, but not conversely except in the Gorenstein case. We note that, in the finite case, Richardson varieties have rational singularities [Bri05, Theorem 4.2.1], even in positive characteristics [KLS10, Appendix]. Indeed the singularities of generalizations of Richardson varieties has been a topic of interest lately [BiK10], [KLS10] and [KWY12].

On the other hand, KLT are the widest class of singularities for which the foundational theorems of the minimal model program over $\mathbb{C}$ are known to hold [KM98]. It is well known that toric varieties are KLT [CLST11, Section 11.4] and more generally V. Alexeev and M. Brion proved that spherical varieties are KLT [AB04]. Recently, D. Anderson and A. Stapledon proved that the Schubert varieties $X^v_w$ are log Fano and thus also KLT [AS12], also see [Hsi11].

The proof of our main result in the finite case is much simpler than the general Kac-Moody case and is given in Section 4. In this case we are able to directly prove that $(X^v_w, \Delta)$ is KLT through an explicit resolution of singularities of $X^v_w$ due to M. Brion (similar to the Bott-Samelson-Demazure-Hansen desingularization of the Schubert varieties).

In the general symmetrizable Kac-Moody case, we are not aware of an explicit resolution of singularities of $X^v_w$ to proceed as above. In the general case, we prove our main result by reduction to characteristic $p > 0$. In this case we use an unpublished result of O. Mathieu asserting that the Richardson varieties $X^v_w$ are Frobenius split compatibly splitting their boundary (cf. Proposition 5.3). This splitting together with results of N. Hara and K.-I. Watanabe relating Frobenius splittings and log canonical singularities (cf. Theorem 5.7) allow us to conclude that the pair $(X^v_w, \Delta)$ as above is KLT.

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## 2. Preliminaries and definitions

We follow the notation from [KM98, Notation 0.4]. We fix $X$ to be a normal variety over an algebraically closed field.

Suppose that $\pi : \tilde{X} \to X$ is a proper birational map with $\tilde{X}$ normal. For any $\mathbb{Q}$-divisor $\Delta = \sum_i \, d_i D_i$ on $X$, we let $\Delta' = \pi^* \Delta = \sum_i \, d_i D'_i$ denote the strict transform of $\Delta$ defined as the $\mathbb{Q}$-divisor on $\tilde{X}$, where $D'_i$ is the prime divisor on $\tilde{X}$ which is birational to $D_i$ under $\pi$. We let $\text{Exc}(\pi)$ of $\pi$ be the exceptional set of $\pi$; the closed subset of $\tilde{X}$ consisting of those $x \in \tilde{X}$ where $\pi$ is not biregular at $x$. We endow $\text{Exc}(\pi)$ with the reduced (closed) subscheme structure. An (integral) divisor $D = \sum_i n_i F_i$ is called a canonical divisor $K_X$ of $X$ if the restriction $D^\circ$ of $D$ to the smooth locus $X^\circ$ of $X$, represents the canonical line bundle $\omega_{X^\circ}$ of $X^\circ$.

Assume now that $K_{\tilde{X}} + \Delta$ is $\mathbb{Q}$-Cartier, i.e., some multiple $n(K_{\tilde{X}} + \Delta)$ (for $n \in \mathbb{N}$) is a Cartier divisor. We may choose $K_{\tilde{X}}$ that agrees with $K_X$ wherever $\pi$ is an isomorphism and thus it follows that there exists a (unique) $\mathbb{Q}$-divisor $E_\pi(\Delta)$ on $\tilde{X}$ supported in $\text{Exc}(\pi)$ such that

$$n(K_{\tilde{X}} + \Delta') = \pi^*(n(K_X + \Delta)) + nE_\pi(\Delta).$$
A $\mathbb{Q}$-divisor $D = \sum d_i D_i$ on a smooth variety $\tilde{X}$ is called a *simple normal crossing divisor* if each $D_i$ is smooth and they intersect transversally at each intersection point (in particular, this means that locally analytically the $D_i$ can be thought of as coordinate hyperplanes).

Let $X$ be an irreducible variety and $D$ a $\mathbb{Q}$-divisor on $X$. A *log resolution* of $(X, D)$ is a proper birational morphism $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is smooth, $\text{Exc}(\pi)$ is a divisor and $\text{Exc}(\pi) \cup \pi^{-1}(\text{Supp} D)$ is a simple normal crossing divisor. Log resolutions exist for any $(X, D)$ in characteristic zero by [Hir64].

Let $X$ be a proper scheme. Then a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $D$ is called *nef* (resp., *big*) if $D \cdot C \geq 0$, for every irreducible curve $C \subset X$ (resp., $ND$ is the sum of an ample and an effective divisor, for some $N \in \mathbb{N}$) (cf. [KM98] §§0.4 and 2.5). Recall that an ample Cartier divisor is nef and big.

**Definition 2.1.** Let $X$ be a normal irreducible variety over a field of characteristic zero and let $\Delta = \sum d_i D_i$ be a $\mathbb{Q}$-divisor with $d_i \in [0, 1)$. The pair $(X, \Delta)$ is called *Kawamata log terminal* (for short *KLT*) if the following two conditions are satisfied:

1. $K_X + \Delta$ is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, and
2. There exists a log resolution $\pi : \tilde{X} \to X$ of $(X, \Delta)$ such that the $\mathbb{Q}$-divisor $E = E(\pi) = \sum_i e_i E_i$, defined by (1), satisfies

   $\quad 1 < e_i$ for all $i$.

By [Deb01], Remarks 7.25, $(X, \Delta)$ satisfying (a) is KLT if and only if for every proper birational map $\pi' : Y \to X$ with normal $Y$, the divisor $E(\pi') = \sum_i f_j F_j$ satisfies (2), i.e., $-1 < f_j$ for all $j$. In fact, one may use this condition as a definition of KLT singularities in characteristic $p > 0$ (where it is an open question whether or not log resolutions exist).

For a normal irreducible variety $X$ of characteristic zero with a $\mathbb{Q}$-divisor $\Delta = \sum d_i D_i$ with $d_i \in [0, 1)$, the pair $(X, \Delta)$ is called *log canonical* if it satisfies the above conditions (a) and (b) with (2) replaced by

$\quad -1 \leq e_i$ for all $i$.

**Remark 2.2.** Let $X$ be a variety of characteristic zero. It is worth remarking that if $(X, \Delta)$ is KLT, then $X$ has rational singularities [Elk81], [KM98] §5.22. Conversely, if $K_X$ is Cartier and $X$ has rational singularities, then $(X, 0)$ is KLT [KM98] §5.24.

We conclude this section with one final definition.

**Definition 2.3.** If $X$ is projective, we say that a pair $(X, \Delta)$ is *log Fano* if $(X, \Delta)$ is KLT and $-K_X - \Delta$ is ample.

3. **Statement of the main result and its consequences**

3.1. **Notation.** Let $G$ be any symmetrizable Kac-Moody group over a field of characteristic zero with the standard Borel subgroup $B$, the standard negative Borel subgroup $B^-$, the maximal torus $T = B \cap B^-$ and the Weyl group $W$ (cf. [Kum02] Sections 6.1 and 6.2). Let $X = G/B$ be the full flag variety, which is a projective ind-variety. For any $w \in W$, we have the Schubert variety

   $X_w := BwB/B \subset G/B$

and the opposite Schubert variety

   $X^w := (B^-wB)^{-1} \subset G/B$.
Then, $X_w$ is a (finite dimensional) irreducible projective subvariety of $G/B$ and $X^v_w$ is a finite codimensional irreducible projective ind-subvariety of $G/B$ (cf. [Kum02] Section 7.1). For any integral weight $\lambda$ (i.e., any character $e^\lambda$ of $T$), we have a $G$-equivariant line bundle $L(\lambda)$ on $X$ associated to the character $e^{-\lambda}$ of $T$ (cf. [Kum02] Section 7.2] for a precise definition of $L(\lambda)$ in the general Kac-Moody case). In the finite case, recall that $L(\lambda)$ is the line bundle associated to the principal $B$-bundle $G \to G/B$ via the character $e^{-\lambda}$ of $B$ (any character of $T$ uniquely extends to a character of $B$), i.e.,

$$L(\lambda) = G \times^B \mathbb{C}_{-\lambda} \to G/B, \ [g,v] \mapsto gB,$$

where $\mathbb{C}_{-\lambda}$ is the one dimensional representation of $B$ corresponding to the character $e^{-\lambda}$ of $B$ and $[g,v]$ denotes the equivalence class of $(g,v) \in G \times \mathbb{C}_{-\lambda}$ under the $B$-action: $b \cdot (g,v) = (gb^{-1}, b \cdot v)$. Let $\{\alpha_1, \ldots, \alpha_\ell\} \subset \mathfrak{t}^*$ be the set of simple roots and $\{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subset \mathfrak{t}$ the set of simple coroots, where $\mathfrak{t} = \text{Lie } T$. When $\rho \in \mathfrak{t}^*$ is any integral weight satisfying

$$\rho(\alpha_i^\vee) = 1, \quad \text{for all} \quad 1 \leq i \leq \ell.$$

When $G$ is a finite dimensional semisimple group, $\rho$ is unique, but for a general Kac-Moody group $G$, it may not be unique.

For any $v \leq w \in W$, consider the Richardson variety

$$X^v_w := X^v \cap X_w,$$

and its boundary

$$\partial X^v_w := ((X^v \cap X_w) \cup (X^v \cap \partial X_w)),$$

both endowed with reduced subvariety structure, where $\partial X_w := X_w \setminus (BwB/B)$ and $\partial X^v := X^v \setminus (B^*vB/B)$.

Writing $\partial X^v_w = \cup_i X_i$ as the union of its irreducible components, the line bundle $L(2\rho)|_{X^v_w}$ can be written as a (Cartier) divisor (for justification, see §4.5; the proof of Theorem 3.2):

$$L(2\rho)|_{X^v_w} = O_{X^v_w} \left( \sum_i b_i X_i \right), \quad b_i \in \mathbb{N} := \{1, 2, 3, \ldots\}.$$  

Now, take a positive integer $N$ such that $N > b_i$ for all $i$, and consider the $\mathbb{Q}$-divisor on $X^v_w$:

$$\Delta = \sum_i \left( 1 - \frac{b_i}{N} \right) X_i.$$

The following theorem is the main result of the paper.

**Theorem 3.2.** For any $v \leq w \in W$, the pair $(X^v_w, \Delta)$ defined above is KLT.

In fact, we will show in Lemma 4.4 that $O_{X^v_w}(-N(K_{X^v_w} + \Delta)) \cong L(2\rho)|_{X^v_w}$ is ample, which proves that $(X^v_w, \Delta)$ is log Fano.

We postpone the proof of this theorem until the next two sections. But we derive the following consequence proved earlier in the finite case (i.e., in the case when $G$ is a finite dimensional semisimple group) by Brion-Lakshmibai (see [BL03 Proposition 1]).

**Corollary 3.3.** For a dominant integral weight $\lambda$, and any $v \leq w$,

$$H^i(X^v_w, L(\lambda)|_{X^v_w}) = 0, \quad \text{for all} \quad i > 0.$$
Proof. By (the subsequent) Lemma 4.4, \( N(K_X + \Delta) \) is a Cartier divisor corresponding to the line bundle \( L(-2\rho)|_{X_v} \). Since \( \lambda \) is a dominant weight, the \( \mathbb{Q} \)-Cartier divisor \( D \) is nef and big, where \( ND \) is the Cartier divisor corresponding to the ample line bundle \( L(N\lambda + 2\rho)|_{X_v} \). Thus, the divisor \( K_X + \Delta + D \) is Cartier and corresponds to the line bundle \( L(\lambda)|_{X_v} \). Hence, the corollary follows from the Logarithmic Kawamata-Viehweg vanishing theorem which we state below (cf. [Deb01, Theorem 7.26] or [KM98, Theorem 2.70]). □

Theorem 3.4. Let \((X, \Delta)\) be a KLT pair and let \( D \) be a nef and big \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor on \( X \) such that \( K_X + \Delta + D \) is a Cartier divisor. Then, we have

\[ H^i(X, K_X + \Delta + D) = 0, \quad \text{for all} \quad i > 0. \]

4. Proof of Theorem 3.2: Finite case

In this section, except where otherwise noted, we assume that \( G \) is a finite dimensional semisimple simply-connected group. We refer to this as the finite case.

We first give a proof of Theorem 3.2 in the finite case. In this case, the proof is much simpler than the general (symmetrizable) Kac-Moody case proved in the next section. Unlike the general case, the proof in the finite case given below does not require any use of characteristic \( p > 0 \) methods.

Before we come to the proof of the theorem, we need some preliminaries on Bott-Samelson-Demazure-Hansen (for short BSDH) desingularization of Schubert varieties.

4.1. BSDH desingularization. For any \( w \in W \), pick a reduced decomposition as a product of simple reflections:

\[ w = s_{i_1} \ldots s_{i_n} \]

and let \( m_w : Z_w \to X_w \) be the BSDH desingularization (cf. [BK05 §2.2.1]), where \( w \) is the word \((s_{i_1}, \ldots, s_{i_n})\). This is a \( B \)-equivariant resolution, which is an isomorphism over the cell \( C_w := BwB/B \subset X_w \).

Similarly, there is a \( B^- \)-equivariant resolution

\[ m^\circ : Z^\circ \to X^\circ, \]

obtained by taking a reduced word \( \hat{w} = (s_{j_1}, \ldots, s_{j_m}) \) for \( w_0v \), i.e., \( w_0v = s_{j_1} \ldots s_{j_m} \) is a reduced decomposition, where \( w_0 \in W \) is the longest element. Now, set

\[ Z^\circ = Z_{\hat{w}}, \]

which is canonically a \( B \)-variety. We define the action of \( B^- \) on \( Z^\circ \) by twisting the \( B \)-action as follows:

\[ b^- \cdot z = (\hat{w}_0b^-\hat{w}_0^{-1}) \cdot z, \quad \text{for} \quad b^- \in B^- \quad \text{and} \quad z \in Z^\circ, \]

where \( \hat{w}_0 \) is a lift of \( w_0 \) in the normalizer \( N(T) \) of the torus \( T \). (Observe that this action does depend upon the choice of the lift \( \hat{w}_0 \) of \( w_0 \).) Moreover, define the map

\[ m^\circ : Z^\circ \to X^\circ = \hat{w}_0^{-1}X_{w_0v} \quad \text{by} \quad m^\circ(z) = \hat{w}_0^{-1}(m_0(z)), \quad \text{for} \quad z \in Z^\circ. \]

Clearly, \( m^\circ \) is a \( B^- \)-equivariant desingularization.
4.2. Desingularization of Richardson varieties. We recall the construction of a desingularization of Richardson varieties communicated to us by M. Brion (also see [Bal11, Section 1]). It is worked out in detail in any characteristic in [KLS10, Appendix]. We briefly sketch the construction in characteristic zero. Consider the fiber product morphism
\[ m^v : m^v \times m_w : Z^v \times Z_w \to X^v \times X_w = X_w^v, \]
which is a smooth desingularization. It is an isomorphism over the intersection \( C^v_w := C^v \cap C_w \) of the Bruhat cells, where \( C^v := B^\vee B/B \subset G/B \) and (as earlier) \( C_w := BwB/B \). Moreover, the complement of \( C^v_w \) inside \( Z^v_w \), considered as a reduced divisor, is a simple normal crossing divisor. (To prove these assertions, observe that by Kleiman’s transversality theorem [Har77, Theorem 10.8, Chap. III], the fiber product \( Z^v \times gZ_w \) is smooth for a general \( g \in G \) and hence for some \( g \in B^\vee B \). But since \( Z_w \) is a \( B \)-variety and \( Z^v \) is a \( B^\vee \)-variety,
\[ Z^v \times gZ_w \cong Z^v \times Z_w. \]
Moreover, \( Z^v \times Z_w \) is irreducible since each of its irreducible components is of the same dimension equal to \( \ell(w) - \ell(v) \) and the complement of \( C^v_w \) in \( Z^v \times Z_w \) is of dimension \(< \ell(w) - \ell(v).\)

**Lemma 4.3.** With the notation as above (still in characteristic zero), for any \( v \leq w \), the Richardson variety \( X^v_w \) is irreducible, normal and Cohen-Macaulay.

This is proven in the finite case in [Bri02, Lemma 2] and [BL03, Lemma 1]. The same result (with a similar proof as in [Bri02]) also holds in the Kac-Moody case (cf. [Kum12, Proposition 6.5]). Also see [KLS10] for some discussion in characteristic \( p > 0 \).

**Lemma 4.4.** For any symmetrizable Kac-Moody group \( G \), and any \( v \leq w \in W \), the canonical divisor \( K^v_w \) of \( X^v_w \) is given by:
\[ K^v_w = O_{X^v_w}[-\partial X^v_w], \]
where \( \partial X^v_w \) is considered as a reduced divisor.

**Proof.** The finite case can be found in [Bri05, Theorem 4.2.1(i)]. The detailed proof for the general symmetrizable Kac-Moody group can be found in [Kum12, Lemma 8.5]. We give a brief idea here.

Since \( X^v_w \) is Cohen-Macaulay by [Kum12, Proposition 5.6] (in particular, so is \( X_w \)) and the codimension of \( X^v_w \) in \( X_w \) is \( \ell(v) \), the dualizing sheaf
\[ \omega_{X^v_w} \cong \mathcal{E}xt^{\ell(v)}_{O_{X_w}}(O_{X_w}, \omega_{X_w}), \]
(cf. [E95, Theorem 21.15]). Observing that \( \text{depth}(\omega_{X_w}) = \text{depth}(O_{X_w}) \), as \( O_{X_w} \)-modules, (cf. [E95, Theorem 21.8])
\[ \mathcal{E}xt^{\ell(v)}_{O_{X_w}}(O_{X_w}, \omega_{X_w}) \cong \mathcal{E}xt^{\ell(v)}_{O_{\tilde{X}}}(O_{\tilde{X}}, \omega_{\tilde{X}}), \]
where \( \tilde{X} \) is the ‘thick’ flag variety. By [GK08, Proposition 2.2], as \( T \)-equivariant sheaves,
\[ \omega_{X_w} \cong \mathbb{C}(-\rho) \otimes L(-\rho) \otimes O_{X_w}(-\partial X_w). \]
Similarly, by [Kum12, Theorem 10.4] (due to Kashiwara),
\[ \omega_{X^v} \cong \mathbb{C}(-\rho) \otimes L(-\rho) \otimes O_{X^v}(-\partial X^v). \]
Similar to the identity (6), we also have
\[ \omega_{X^v} \cong \mathcal{E}xt^{\ell(v)}_{O_{\tilde{X}}}(O_{X^v}, \omega_{\tilde{X}}). \]
Since $\omega_\chi \approx L(-2\rho)$, combining the isomorphisms (6) - (10), we get
\[ \omega_{X_w} \cong O_{X_w}(-\partial X_w) \otimes O_{X_w}(-\partial X_w). \]
Now, the lemma follows since all the intersections $X_v \cap X_w$, $(\partial X_v) \cap X_w$, $X_v \cap X_w$, and $\partial(X_v) \cap X_w$ are proper. In fact, we need the corresponding local Tor vanishing result (cf. [Kum12, Lemma 5.5]).

We are now ready to prove Theorem 3.2 in the finite case. The basic strategy is similar to the proof that toric varieties have KLT singularities and in fact are log Fano [CLS11, Section 11.4].

4.5. Proof of Theorem 3.2 in the finite case. Let us denote $Z_v \times X_w$ by $Z_{vw}$. Consider the desingularization $m_{vw} : Z_{vw} \to X_{vw}$,

as in § 4.2.

By [BK05, Proposition 2.2.2], the canonical line bundle of $Z_w$ is isomorphic with the line bundle $L_w(-\rho) \otimes O_{Z_w}[\partial Z_w]$,

where $L_w(-\rho)$ is the pull-back of the line bundle $L(-\rho)$ to $Z_w$ via the morphism $m_w$ and $\partial Z_w$ is the reduced divisor $Z_w \setminus C_w$.

Similarly, the canonical line bundle of $Z_v$ is isomorphic with $L_v(-\rho) \otimes O_{Z_v}[\partial Z_v]$,

where $L_v(-\rho)$ is the pull-back of the line bundle $L(-\rho)$ to $Z_v$ via $m_v$ and $\partial Z_v$ is the reduced divisor $Z_v \setminus C_v$.

Thus, by adapting the proof of [Bri02, Lemma 1],

\[ \omega_{Z_{vw}} \equiv \omega_{Z_v} \otimes_{O_{X_v}} \omega_{Z_w} \otimes_{O_{X_w}} (m_{vw}^*)^\omega_{X_{vw}}^{-1} \]
\[ = O_{Z_{vw}}[-\partial Z_{vw}], \]

(11)

where $\partial Z_{vw}$ is the reduced divisor

\[ (\partial Z_v \times_X Z_w) \cup (Z_v \times_X \partial Z_w). \]

Consider the desingularization

\[ m_{vw}^p : Z_v^p \to X_w^p. \]

Note that $m_{vw}^p$ is an isomorphism outside of $\partial Z_{vw}$.

By Lemma 4.4

\[ K_{X_v} = - \sum X_i, \]

where we have written $\partial X_v = \cup X_i$ as the union of prime divisors. Thus, by (5),

\[ K_{X_v} + \Delta = -\frac{1}{N} \sum b_i X_i, \]

(12)

which is a $\mathbb{Q}$-Cartier divisor by (4).

We next calculate $\text{Exc}(m_{vw}^p)$ and the proper transform $\Delta'$ of $\Delta$ under the desingularization $m_{vw}^p : Z_v^p \to X_w^p$.

Any irreducible component $X_i$ of $\partial X_v^p$ is of the form $X_{v'}$ or $X_{w'}$ for some $v \to v'$ and $w' \to w$, where the notation $w' \to w$ means that $\ell(w) = \ell(w') + 1$ and $w = s_\alpha w'$, for some reflection $s_\alpha$.
through a positive (not necessarily simple) root \( \alpha \). Conversely, any \( X_v' \) and \( X_{w'} \) (for \( v \to v' \) and \( w' \to w \)) is an irreducible component of \( \partial X_v' \). Thus, we have the prime decomposition

\[
\partial X_v' = (\cup_{v \to v'} X_v') \cup (\cup_{w' \to w} X_{w'}). 
\]

We define \( Z_i \) as the prime divisor (of the resolution \( Z_{w}^o \)) \( Z^o \times_X Z_w \) if \( X_i = X_v' \) or \( Z^o \times_X Z_{w'} \) if \( X_i = X_{w'}' \). Thus, the strict transform of \( \Delta \) can be written as

\[
\Delta' = \sum_i \left(1 - \frac{b_i}{N}\right)Z_i.
\]

We now calculate \( E = E_{m^o_w}(\Delta) \). By definition (cf. (1)),

\[
E = (K_{Z_w^o} + \Delta') - \frac{1}{N}(m^o_w)^*(N(K_{X_v^o} + \Delta)).
\]

Consider the prime decomposition of the reduced divisor \( \partial Z_{w}^o \):

\[
\partial Z_{w}^o = (\cup_i Z_i) \cup (\cup_j Z'_j),
\]

where \( Z'_j \) are the irreducible components of \( \partial Z_{w}^o \) which are not of the form \( Z_i \). The line bundle \( \mathcal{L}(2\rho)_{X_v^o} \) has a section vanishing exactly on the set \( \partial X_v' \). To see this, consider the Borel-Weil isomorphism

\[
\beta : V(\rho)^* \to H^0(G/B, \mathcal{L}(\rho)), \quad \beta(\chi)(gB) = [g, (g^{-1}\chi)_{C_{v'}}],
\]

where \( V(\rho) \) is the irreducible \( G \)-module with highest weight \( \rho \) and \( v_+ \in V(\rho) \) is a highest weight vector. Let \( \chi_v \) be the unique (up to a scalar multiple) vector of \( V(\rho)^* \) with weight \( -1\rho \). Now, take the section \( \beta(\chi_v' \cdot \beta(\chi_w) \cdot \beta(\chi_v)) \) of the line bundle \( \mathcal{L}(2\rho)_{X_v^o} \). Then, it has the zero set precisely equal to \( \partial X_v' \), since the zero set \( Z(\beta(\chi_v' \cdot \beta(\chi_w)) \cdot \beta(\chi_v)) \) is given by

\[
Z(\beta(\chi_v' \cdot \beta(\chi_w))) = \{gB \in X_v^o : \chi'_v(gv_+) = 0\}
\]

\[
= \bigcup_{\nu' \supset \nu} B^{-\nu'}/B
\]

\[
= \partial X_v'.
\]

We fix \( H = \sum_i b_iX_i \) to be the divisor corresponding to the section \( \beta(\chi_v') \cdot \beta(\chi_w) \) of the line bundle \( \mathcal{L}(2\rho)_{X_v^o} \) as in (4). Observe that the coefficients of \( m^o_w \cdot H = \sum_i b_iZ_i + \sum_j d_jZ'_j \) are all strictly positive integers.

Thus, by combining the identities (4), (11)–(15), we get

\[
E = -\sum_i \frac{b_i}{N}Z_i - \sum_j Z'_j + \frac{1}{N}m^o_w \cdot (H)
\]

\[
= -\sum_i \frac{b_i}{N}Z_i - \sum_j Z'_j + \frac{1}{N} \sum_i b_iZ_i + \frac{1}{N} \sum d_jZ'_j,
\]

\[
= \sum_j \left(\frac{d_j}{N} - 1\right)Z'_j,
\]

for some \( d_j \in \mathbb{N} \) (since the zero set of a certain section of \( \mathcal{L}(2\rho)_{X_v^o} \) is precisely equal to \( \partial X_v' \) and all the \( Z'_j \) lie over \( \partial X_v' \)). Thus, the coefficient \( e_j \) of \( Z'_j \) in \( E \) satisfies \(-1 < e_j\).
Finally, observe that \( \text{Exc}(m_w^n) + \Delta' \) is a \( \mathbb{Q} \)-divisor with simple normal crossings since \( \text{Supp}(\text{Exc}(m_w^n) + \Delta') \subset \partial Z_{w}^n \) and the latter is a simple normal crossing divisor since
\[
\partial Z_{w}^n = Z_{w}^n \setminus C_w^n,
\]
(cf. §4.2).

This completes the proof of Theorem 3.2 in the finite case. \( \Box \)

**Remark 4.5.** The above proof crucially uses the explicit BSDH type resolution of the Richardson varieties \( X_{v}^w \) given in §4.2. This resolution is available in the finite case, but we are not aware of such an explicit resolution in the Kac-Moody case. This is the main reason that we need to handle the general Kac-Moody case differently.

## 5. Proof of Theorem 3.2 in the Kac-Moody case

Our proof of Theorem 3.2 in the general Kac-Moody case is more involved. It requires the use of characteristic \( p \) methods; in particular, the Frobenius splitting.

For the construction of the flag variety \( X = G/B \), Schubert subvarieties \( X_w \), opposite Schubert subvarieties \( X^c \) (and thus the Richardson varieties \( X_{v}^w \)) associated to any Kac-Moody group \( G \) over an algebraically closed field \( k \), we refer to [T81], [T82], [T85], [Mat88], and [Mat89].

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \( Y : Y_0 \subset Y_1 \subset \ldots \) be an ind-variety over \( k \) and let \( O_Y \) be its structure sheaf (cf. [Kum02, Definition 4.1.1]). The absolute Frobenius morphism
\[
F_Y : Y \longrightarrow Y
\]
is the identity on the underlying space of \( Y \), and the \( p \)-th power map on the structure sheaf \( O_Y \).

Consider the \( O_Y \)-linear Frobenius map
\[
F^\#: O_Y \rightarrow F_*O_Y, \ f \mapsto f^p.
\]

Identical to the definition of Frobenius split varieties, we have the following definition for ind-varieties.

**Definition 5.1.** An ind-variety \( Y \) is called **Frobenius split** (or just **split**) if the \( O_Y \)-linear map \( F^\# \) splits, i.e., there exists an \( O_Y \)-linear map
\[
\varphi : F_*O_Y \longrightarrow O_Y
\]
such that the composition \( \varphi \circ F^\# \) is the identity of \( O_Y \). Any such \( \varphi \) is called a **splitting**.

A closed ind-subvariety \( Z \) of \( Y \) is **compatibly split** under the splitting \( \varphi \) if
\[
\varphi(F_*I_Z) \subseteq I_Z,
\]
where \( I_Z \subset O_Y \) is the ideal sheaf of \( Z \).

Clearly, a splitting of \( Y \) is equivalent to a sequence of splittings \( \varphi_n \) of \( Y_n \) such that \( \varphi_n \) compatibly splits \( Y_{n-1} \) inducing the splitting \( \varphi_{n-1} \) on \( Y_{n-1} \).

Let \( B \) be the standard Borel subgroup of any Kac-Moody group \( G \) over an algebraically closed field \( k \) of characteristic \( p > 0 \) and \( T \subset B \) the standard maximal torus. For any real root \( \beta \), let \( U_{\beta} \) be the corresponding root subgroup. Then, there exists an algebraic group isomorphism \( \varepsilon_{\beta} : \mathbb{G}_a \rightarrow U_{\beta} \) satisfying
\[
t \varepsilon_{\beta}(z)^{-1} = \varepsilon_{\beta}(\beta(t)z),
\]
for \( z \in G_w \) and \( t \in T \). For any \( B \)-locally finite algebraic representation \( V \) of \( B \), \( v \in V \) and \( z \in G_w \),

\[
\varepsilon^m_{\beta}(z)v = \sum_{m \geq 0} z^m (\varepsilon^m_{\beta} \cdot v),
\]

where \( \varepsilon^m_{\beta} \) denotes the \( m \)-th divided power of the root vector \( e_\beta \), which is, by definition, the derivative of \( \varepsilon_{\beta} \) at 0.

Now, we come to the definition of \( B \)-canonical splittings for ind-varieties (cf. [BK05], Section 4.1) for more details in the finite case).

**Definition 5.2.** Let \( Y \) be a \( B \)-ind-variety, i.e., \( B \) acts on the ind-variety \( Y \) via ind-variety isomorphisms. Let \( \text{End}_F(Y) := \text{Hom}(F, O_Y, O_Y) \) be the additive group of all the \( O_Y \)-module maps \( F, O_Y \to O_Y \). Recall that \( F, O_Y \) can canonically be identified with \( O_Y \) as a sheaf of abelian groups on \( Y \); however, the \( O_Y \)-module structure is given by \( f \circ g := f^p g \), for \( f, g \in O_Y \). Since \( Y \) is a \( B \)-ind-variety, \( B \) acts on \( \text{End}_F(Y) \) by

\[(b * \psi)s = b(\psi(b^{-1}s)), \quad \text{for } b \in B, \psi \in \text{End}_F(Y) \text{ and } s \in F_*O_Y,\]

where the action of \( B \) on \( F_*O_Y \) is defined to be the standard action of \( B \) on \( O_Y \) under the identification \( F_*O_Y = O_Y \) (as sheaves of abelian groups). We define the \( k \)-linear structure on \( \text{End}_F(Y) \) by

\[(z * \psi)s = \psi(zs) = z^{1/p} \psi(s),\]

for \( z \in k, \psi \in \text{End}_F(Y) \) and \( s \in O_Y \).

A splitting \( \phi \in \text{End}_F(Y) \) is called a \( B \)-canonical splitting if \( \phi \) satisfies the following conditions:

(a) \( \phi \) is \( T \)-invariant, i.e.,

\[t * \phi = \phi, \quad \text{for all } t \in T.\]

(b) For any simple root \( \alpha_i, 1 \leq i \leq \ell \), there exist \( \phi_{i,j} \in \text{End}_F(X), 0 \leq j \leq p - 1 \), such that

\[
\varepsilon_{\alpha_i}(z) * \phi = \sum_{j=0}^{p-1} z^j * \phi_{i,j}, \quad \text{for all } z \in G_w.
\]

The definition of \( B^- \)-canonical is of course parallel.

Before we come to the proof of Theorem 3.2 for the Kac-Moody case, we need the following results.

The following result in the symmetrizable Kac-Moody case is due to O. Mathieu (unpublished). (For a proof in the finite case, see [BK05], Theorem 2.3.2.) Since Mathieu’s proof is unpublished, we briefly give an outline of his proof contained in [Mat11].

**Proposition 5.3.** Consider the Richardson variety \( X_w^v(k) \) (for any \( v \leq w \)) over an algebraically closed field \( k \) of characteristic \( p > 0 \). Then, \( X_w^v(k) \) is Frobenius split compatibly splitting its boundary \( \partial X_w^v \).

**Proof.**

**Assertion I:** The full flag variety \( X = X(k) \) admits a \( B \)-canonical splitting.

For any \( w \in W \) and any reduced decomposition \( w \) of \( w \), consider the BSDH desingularization \( Z_w = Z_w(k) \) of the Schubert variety \( X_w \) and the section \( \sigma \in H^0(Z_w, O_{Z_w}[\partial Z_w]) \) with the associated divisor of zeroes \( (\sigma)_0 = \partial Z_w \). Clearly, such a section is unique (up to a nonzero scalar multiple). Take the unique, up to nonzero multiple, nonzero section \( \theta \in H^0(Z_w, L_w(\rho)) \) of weight \(-\rho\). (Such a section exists since \( H^0(Z_w, L_w(\rho)) \to H^0([1], L_w(\rho)[1]) \) is surjective by [BK05] Theorem 3.1.4], where \( 1 := [1, \ldots, 1] \in Z_w \) and \( [1, \ldots, 1] \) denotes the \( B^{(w)} \)-orbit of \( (1, \ldots, 1) \) as
in [BK05, Definition 2.2.1]. Moreover, such a section is unique up to a scalar multiple since $H^0(Z_\theta, L_\theta(\rho))^* = H^0(X_\theta, L_\theta(\rho)) \hookrightarrow V(\rho)$. By the above, the section $\theta$ does not vanish at the base point 1. Thus, by [BK05, Proposition 1.3.11 and Proposition 2.2.2], $(\sigma \theta)^{p-1}$ provides a splitting $\hat{\sigma}_w$ of $Z_\theta$ compatibly splitting $\partial Z_\theta$. Since the Schubert variety $X_\theta$ is normal, the splitting $\hat{\sigma}_w$ descends to give a splitting $\hat{\sigma}_w$ of $X_\theta$ compatibly splitting all the Schubert subvarieties of $X_\theta$.

Now, the splitting $\hat{\sigma}_w$ is $B$-canonical and it is the unique $B$-canonical splitting of $Z_\theta$ (cf. [BK05, Exercise 4.1.E.2]; even though this exercise is for finite dimensional $G$, the same proof works for the Kac-Moody case). We claim that the induced splittings $\hat{\sigma}_w$ of $X_\theta$ are compatible to give a splitting of $X = \cup_\theta X_\theta$. Take $v, w \in W$ and choose $u \in W$ with $v \leq u$ and $w \leq u$. Choose a reduced word $\tilde{u}$ of $u$. Then there is a reduced subword $\tilde{v}$ (resp. $\tilde{w}$) of $\tilde{u}$ corresponding to $v$ (resp. $w$). The $B$-canonical splitting $\hat{\sigma}_\tilde{u}$ of $Z_{\tilde{u}}$ (by the uniqueness of the $B$-canonical splittings of $Z_{\tilde{u}}$) restricts to the $B$-canonical splitting $\hat{\sigma}_v$ of $Z_v$ and $\hat{\sigma}_w$ of $Z_w$). In particular, the splitting $\hat{\sigma}_u$ of $X_u$ restricts to the splitting $\hat{\sigma}_v$ of $X_v$ (and $\hat{\sigma}_w$ of $X_w$). This proves the assertion that the splittings $\hat{\sigma}_u$ of $X_u$ are compatible to give a $B$-canonical splitting $\hat{\sigma}$ of $X$. By the same proof as that of [BK05, Proposition 4.1.10], we obtain that the $B$-canonical splitting $\hat{\sigma}$ of $X$ is automatically $B^\circ$-canonical.

Assertion III: $X^w$ is compatibly split under $\hat{\sigma}$.

Since $\hat{\sigma}$ is $B^\circ$-canonical, by [BK05, Proposition 4.1.8], for any closed ind-subvariety $Y$ of $X$ which is compatibly split under $\hat{\sigma}$, the $B^\circ$-orbit closure $B^\circ Y \subset X$ is also compatibly split. In particular, the opposite Schubert variety $X^w := B^\circ \hat{w} B/B$ is compatibly split.

Thus, we get that the Richardson varieties $X^w_v$ (for $v \leq w$) are compatibly split under the splitting $\hat{\sigma}$ of $X$. Since the boundary $\partial X^w_v$ is a union of other Richardson varieties, the boundary also is compatibly split. This proves the proposition.

We need the following general result. First we recall a definition.

Definition 5.4. Suppose that $X$ is a normal variety over an algebraically closed field of characteristic $p > 0$ and $D$ is an effective $\mathbb{Q}$-divisor on $X$. The pair $(X, D)$ is called sharply $F$-pure if, for every point $x \in X$, there exists an integer $e \geq 1$ such that $e$-iterated Frobenius map

$$O_{X,x} \to F^e_*(\mathcal{O}_{X,x}/((p^e - 1)D))$$

admits an $O_{X,x}$-module splitting. In fact, if there exists a splitting for one $e > 0$, by composing maps, we obtain a Frobenius splitting for all sufficiently divisible $e > 0$.

Note that by definition, if $O_X \to F^*_x O_X$ is split relative to a divisor $D$, then the pair $(X, \frac{1}{p^e-1} D)$ is sharply $F$-pure.

Note that being sharply $F$-pure is a purely local condition, unlike being $F$-split.
Proposition 5.5. Let $X$ be an irreducible normal variety over a field of characteristic $p > 0$ and $D = \sum_i D_i$, a reduced divisor in $X$. Assume further that $X$ is Frobenius split compatibly splitting Supp $D$. Then, the pair $(X, D)$ is sharply F-pure.

Proof. Note that we have a global splitting of $O_X(-D) \rightarrow F_*O_X(-D)$. Twisting both sides by $D$ and applying the projection formula gives us a global splitting of $O_X \rightarrow F_*O_X((p-1)D)$. We may localize this at any stalk and take $e = 1$. \hfill $\Box$

By Lemma 4.3, the Richardson varieties $X_w^v$ are normal in characteristic 0; in particular, they are normal in characteristics $p \gg 0$. Thus, combining Propositions 5.3 and 5.5 we get the following.

Corollary 5.6. With the notation as above, for any $v \leq w$, $(X_w^v, \partial X_w^v)$ is sharply F-pure in characteristics $p \gg 0$.

We also recall the following from [HW02, Theorem 3.7]. It should be mentioned that even though in loc. cit. the result is proved in the local situation, the same proof works for projective varieties. We sketch a proof for the convenience of the reader.

Theorem 5.7. Let $X$ be an irreducible normal variety over a field of characteristic 0 and let $D$ be an effective $\mathbb{Q}$-divisor on $X$ such that $K_X + D$ is $\mathbb{Q}$-Cartier and such that $\lceil D \rceil$ is reduced and effective (i.e., the coefficients of $D$ are in $[0, 1]$). If the reduction $(X_p, D_p)$ mod $p$ of $(X, D)$ is sharply F-pure for infinitely many primes $p$, then $(X, D)$ is log canonical.

Proof. (Sketch) Fix a log resolution $\pi : \tilde{X} \rightarrow X$ of $(X, D)$ and write

$$E_{p}(D) - D' = K_{\tilde{X}} - \frac{1}{n}(\pi^*(nK_X + nD))$$

for a choice of $K_{\tilde{X}}$ agreeing with $K_X$ wherever $\pi$ is an isomorphism as in (1). We need to show that the coefficients of $E_{p}(D)$ are $\geq -1$. We reduce the entire setup to some characteristic $p \gg 0$ where $(X_p, D_p)$ is sharply F-pure (for a discussion of this process, see [HW02] or see [BK05, Chapter 1.6] in the special case when the varieties are defined over $\mathbb{Z}$).

Fix $x \in X_p$. We have a Frobenius splitting $\phi$:

$$O_{X_p,x} \xleftarrow{} F_*^eO_{X_p,x} \xleftarrow{} F_*^eO_{X_p,x}(\lceil (p^e - 1)D_{p,x} \rceil) \xrightarrow{} O_{X_p,x},$$

for some $e \geq 1$. This splitting $\phi \in \text{Hom}(F_*^eO_{X_p,x}, O_{X_p,x})$ corresponds to a divisor $B_x \geq \lceil (p^e - 1)D_{p,x} \rceil$ on Spec $O_{X_p,x}$, which is linearly equivalent to $(1 - p^e)K_{X_p,x}$ as in [BK05, §1.3]. Set $\Delta = \frac{1}{p^e - 1}B_x$. Observe that $(p^e - 1)(K_{X_p,x} + \Delta)$ is linearly equivalent to 0; and thus $K_{X_p,x} + \Delta$ is $\mathbb{Q}$-Cartier. Since

$$\Delta = \frac{1}{p^e - 1}B_x \geq \frac{1}{p^e - 1}[\lceil (p^e - 1)D_{p,x} \rceil] \geq D_{p,x},$$

we know

$$E_{p}(D_{p,x}) \geq E_{p}(\Delta).$$

Therefore, it is sufficient to prove that the coefficients of $E_{p}(\Delta)$ are $\geq -1$. Note that it is possible that $\pi_p$ is not a log resolution for $\Delta$, but this will not matter for us.
We can factor the splitting \( \phi \) as follows (we leave this verification to the reader):

\[
\begin{align*}
F^*_s O_{\breve{X}_p} & \longrightarrow F^*_s O_{\breve{X}_p, k((p^e - 1)D_p, \Delta)} \longrightarrow F^*_s O_{\breve{X}_p, ((p^e - 1) - \Delta)} \psi \longrightarrow O_{\breve{X}_p}.
\end{align*}
\]

Let \( C \) be any prime exceptional divisor of \( \pi_p : \breve{X}_p \to X_p \) with generic point \( \eta \) and let \( O_{\breve{X}_p, \eta} \) be the associated valuation ring. Let \( a \in \mathbb{Q} \) be the coefficient of \( C \) in \( E_{\pi_p}(\Delta) \). There are two cases:

(i) \( a > 0 \)

(ii) \( a \leq 0 \).

Since we are trying to prove that \( a \geq -1 \), if we are in case (i), we are already done. Therefore, we may assume that \( a \leq 0 \). By tensoring \( \phi \) with the fraction field \( K(X_p) = K(\breve{X}_p) \), we obtain a map 
\( \phi_{K(X_p)} : F^*_s K(X_p) \to K(X_p) \).

**Claim 5.8.** By restricting \( \phi_{K(X_p)} \) to the stalk \( F^*_s O_{\breve{X}_p, \eta} \), we obtain a map \( \phi_\eta \) which factors as:

\[
\phi_\eta : F^*_s O_{\breve{X}_p, \eta} \hookrightarrow F^*_s O_{\breve{X}_p, \eta}( -a(p^e - 1)C ) \to O_{\breve{X}_p, \eta}.
\]

**Proof of the claim.** Indeed, similar arguments are used to prove Grauert-Riemenschneider vanishing for Frobenius split varieties [MvdK92, BK05 Theorem 1.3.14]. We briefly sketch the idea of the proof.

We identify \( \psi \) with a section \( s \in O_{\breve{X}_p, k((1 - p^e)(K_{\breve{X}_p, x} + \Delta))} \cong \text{Hom}_{O_{\breve{X}_p, x}}(F^*_s O_{\breve{X}_p, ((p^e - 1)\Delta)}, O_{\breve{X}_p, x}) \).

Recall that \( \pi^* (1 - p^e)(K_{\breve{X}_p, x} + \Delta) = (1 - p^e)K_{\breve{X}_p, x} - (1 - p^e)E_{\pi_p}(\Delta) + (1 - p^e)\Delta' \), where \( \Delta' \) is the strict transform of \( \Delta \). Thus we can pull \( s \) back to a section

\[
t := \pi^* s \in O_{\breve{X}_p, \eta}(\pi^*(1 - p^e)(K_{\breve{X}_p, x} + \Delta))
\]

\[
= O_{\breve{X}_p, \eta}(1 - p^e)K_{\breve{X}_p, x} - (1 - p^e)E_{\pi_p}(\Delta) + (1 - p^e)\Delta'
\]

\[
= O_{\breve{X}_p, \eta}(1 - p^e)K_{\breve{X}_p, x} - (1 - p^e)aC
\]

\[
\cong \text{Hom}_{O_{\breve{X}_p, \eta}}(F^*_s O_{\breve{X}_p, \eta}(-a(p^e - 1)C), O_{\breve{X}_p, \eta}).
\]

It is not hard to see that the homomorphism \( \psi_\eta : F^*_s O_{\breve{X}_p, \eta}(-a(p^e - 1)C) \to O_{\breve{X}_p, \eta} \) corresponding to \( t \) can be chosen to agree with \( \psi \) on the fraction field \( K(X) = K(\breve{X}) \), we leave this verification to the reader. It follows that \( \phi_\eta \) is the composition \( F^*_s O_{\breve{X}_p, \eta} \hookrightarrow F^*_s O_{\breve{X}_p, \eta}(-a(p^e - 1)C) \psi_\eta \to O_{\breve{X}_p, \eta} \). This concludes the proof of the claim. \( \square \)

Now we complete the proof of Theorem 5.7. Note that \( \phi_\eta \) is a splitting because \( \phi \) was a splitting and both the maps agree on the field of fractions. Therefore, \( 0 \leq -a(p^e - 1) \leq p^e - 1 \) since the splitting along a divisor can not vanish to order greater than \( p^e - 1 \). Dividing by \( (1 - p^e) \) proves that \( a \geq -1 \) as desired. \( \square \)

We also recall the following.

**Lemma 5.9.** Let \( X \) be an irreducible normal projective variety over \( \mathbb{C} \) and let \( D = \sum a_i D_i \) be an effective \( \mathbb{Q} \)-divisor on \( X \) such that \( X \setminus \text{Supp} D \) is smooth and \( (X, D) \) is log canonical. Now, consider a \( \mathbb{Q} \)-divisor \( \Delta = \sum c_i D_i \) with \( c_i \in [0, 1) \) and \( c_i < a_i \) for all \( i \), such that \( K_X + \Delta \) is \( \mathbb{Q} \)-Cartier. Then, \( (X, \Delta) \) is KLT.
Proof. We may choose a resolution of singularities $\pi : \tilde{X} \to X$ which is a log resolution for $(X, D)$ (and hence also for $(X, \Delta)$) by \cite{Hir64}. Furthermore, we may assume that $\pi$ is an isomorphism over $X \setminus \text{Supp}(D) \subseteq X \setminus \text{Supp}(\Delta)$. Therefore, we see that

$$E_{\pi}(\Delta) = K_{\tilde{X}} - \frac{1}{n} \pi^*(n(K_X + \Delta)) + \Delta' \geq K_{\tilde{X}} - \frac{1}{n} \pi^*(n(K_X + D)) + D' = E_{\pi}(D)$$

with strict inequality in every nonzero coefficient. Since every coefficient of $E_{\pi}(D)$ is $\geq -1$, we are done. \qed

We now come to the proof of Theorem 3.2 in the Kac-Moody case.

Proof of Theorem 3.2 We begin by reducing our entire setup to characteristic $p \gg 0$.

By Corollary 5.6 $(X_v^w(k), \partial X_v^w(k))$ is sharply $F$-pure for any algebraically closed field $k$ of characteristic $p \gg 0$. Moreover, $K_{X_v^w} + \partial X_v^w = 0$ by Lemma 3.4 in particular, it is Cartier.

Hence, returning now to characteristic zero, by Theorem 5.7 $(X_v^w, \partial X_v^w)$ is log canonical. Furthermore, the $\mathbb{Q}$-divisor $\Delta = \sum (1 - \frac{b_i}{N_i})X_i$, where $\partial X_v^w = \sum X_i$, clearly satisfies all the assumptions of Lemma 5.9 Thus, $(X_v^w, \Delta)$ is KLT, proving Theorem 3.2 in the Kac-Moody case as well. \qed

Remark 5.10. One can define KLT singularities in positive characteristic too by considering all valuations on all normal birational models. In particular, a similar argument shows that any normal Richardson variety is KLT in characteristic $p > 0$ as well. (It is expected that, for any symmetrizable Kac-Moody group, all the Richardson varieties $X_v^w$ are normal in any characteristic.)

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