Integrable motion of two interacting curves, spin systems and the Manakov system

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Abstract

Integrable spin systems are an important subclass of integrable (soliton) nonlinear equations. They play important role in physics and mathematics. At present, many integrable spin systems were found and studied. They are related with the motion of 3-dimensional curves. In this paper, we consider a model of two moving interacting curves. Next, we find its integrable reduction related with some integrable coupled spin system. Then we show that this integrable coupled spin system is equivalent to the famous Manakov system.

1 Introduction

Among the integrable systems, the integrable spin systems in 1+1 and 2+1 dimensions play an important role in physics and mathematics [1]-[15]. In physics, they describe nonlinear dynamics of magnets. In differential geometry, they can reproduce some integrable classes of curves and surfaces [16]-[26]. The first and the most known representative of the integrable spin systems is the Heisenberg ferromagnetic equation (HFE) which has a form

\[ iA_t + \frac{1}{2}[A, A_{xx}] = 0, \] (1.1)

where \( A = (A_1, A_2, A_3) \) is a unit spin vector, \( A^2 = 1 \) and

\[ A = \begin{pmatrix} A_3 & A^- \\ A^+ & -A_3 \end{pmatrix}, \quad A^2 = I = \text{diag}(1,1), \quad A^\pm = A_1 \pm iA_2. \] (1.2)

The HFE is the Lakshmanan equivalent [7] to the nonlinear Schrödinger equation

\[ iq_t + q_{xx} + 2|q|^2q = 0. \] (1.3)

Also, it is well-known that these equations are gauge equivalent to each other [8]. At present, many integrable and nonintegrable spin systems were identified (see e.g. Refs.[27]-[40] and references therein). One of such spin systems is the Myrzakulov-LIII equation or shortly, the M-LIII equation [11]-[35]

\[ iA_t + \frac{1}{2}[A, A_{xx}] + iuA_x = 0, \] (1.4)

Here LIII \( \equiv 53 \) so that M-LIII \( \equiv M-53 \) and the M-LIII equation \( \equiv \) the M-53 equation.

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1Here LIII \( \equiv 53 \) so that M-LIII \( \equiv M-53 \) and the M-LIII equation \( \equiv \) the M-53 equation.
where \( u = u(t, x) \) is some real function (potential). The modified (inhomogeneous) M-LIII equation looks like

\[
iA_t + \frac{1}{2}A_{xx} + iuA_x + F = 0,
\]

(1.5)

where \( F \) is a matrix function. In this paper, we study the two-layer (“two-component”) generalization of the modified M-LIII equation (1.5) or in short, the coupled M-LIII equation.

The paper is organized as follows. In Sec. 2, the coupled M-LIII equation is introduced. In Sec. 3, we derived the Lakshmanan equivalent counterpart of the M-LIII equation, namely, the Manakov system. In Sec. 4, we present two types of Lax representations of the coupled M-LIII equation. In Sec. 5, the relation between the solutions of the Manakov system and the M-LIII equation is established. The gauge equivalent counterpart of the Manakov system is presented in Sec. 6. The relation between solutions of the coupled M-LIII equation and the \( \Gamma \)-spin system is considered in Sec. 7. At last, Sec. 8 is devoted to Conclusions.

2 The coupled M-LIII equation

Consider two spin vectors \( \mathbf{A} = (A_1, A_2, A_3) \) and \( \mathbf{B} = (B_1, B_2, B_3) \), where \( A^2 = B^2 = 1 \). Let these spin vectors satisfy the coupled Myrzakulov-LIII equation or the 2-layer Myrzakulov-LIII equation of the form \[39\]-\[46\]

\[
iA_t + \frac{1}{2}A_{xx} + iuA_x + F = 0,
\]

(2.1)

\[
iB_t + \frac{1}{2}B_{xx} + ivB_x + E = 0.
\]

(2.2)

Here \( u_k \) are real functions, \( F \) and \( E \) are matrix functions, \( B \) is the matrix form of the second spin vector

\[
B = \begin{pmatrix} B_3 & B^- \\ B^+ & -B_3 \end{pmatrix}, \quad F = \begin{pmatrix} F_3 & F^- \\ F^+ & -F_3 \end{pmatrix}, \quad E = \begin{pmatrix} E_3 & E^- \\ E^+ & -E_3 \end{pmatrix},
\]

(2.3)

where \( B^\pm = B_1 \pm iB_2 \), \( B^2 = I \), \( F^\pm = F_1 \pm iF_2 \), \( E^\pm = E_1 \pm iE_2 \). We now introduce two complex functions \( u \) and \( v \) as

\[
u = \frac{A^+}{1 + A_3}, \quad v = \frac{B^+}{1 + B_3}.
\]

(2.4)

Then these functions satisfy the following set of equations

\[
iu_t - u_{xx} + \frac{2u^2u_x^2}{1 + |u|^2} = F',
\]

(2.5)

\[
i v - v_{xx} + \frac{2v^2v_x^2}{1 + |v|^2} = E',
\]

(2.6)

where \( F' \) and \( E' \) are some complex functions

\[
F' = F'(u, v, u_x, v_x, ...), \quad E' = E'(u, v, u_x, v_x, ...).
\]

(2.7)

(2.8)

In this paper, we assume that \( F \) and \( E \) have the form

\[
F = v_1[\sigma_3, A], \quad E = v_2[\sigma_3, B],
\]

(2.9)

where \( v_j \) are some real functions (potentials). Then the coupled M-LIII equation (2.1)-(2.2) takes the form

\[
iA_t + \frac{1}{2}A_{xx} + iuA_x + v_1[\sigma_3, A] = 0,
\]

(2.10)

\[
iB_t + \frac{1}{2}B_{xx} + ivB_x + v_2[\sigma_3, B] = 0.
\]

(2.11)
Here $u_j$ and $v_j$ are coupling potentials and have the following forms

$$
u_1 = \frac{i(\hat{Z}B^+ - ZB^+)}{W(1 + B_3)},$$

$$v_1 = -\frac{|Z|^2}{2W(1 + A_3)^2},$$

$$u_2 = \frac{i(\hat{R}A^+ - RA^+)(1 + B_3)}{W(1 + A_3)^2},$$

$$v_2 = -\frac{|R|^2}{2W(1 + A_3)^2},$$

where

$$W = 2 + \frac{(1 + A_3)(1 - B_3)}{1 + B_3},$$

$$R = WA^+_x - MA^-,$$

$$Z = W[(1 + A_3)(1 + B_3)^{-1}B^-]_x - M[(1 + A_3)(1 + B_3)^{-1}B^-],$$

$$M = A_{3x}^{x} - A_{3x}^{x},$$

$$= \frac{A_{3x}^{x}(1 - B_3)}{1 + B_3} + \frac{(1 + A_3)B^+ B^-}{(1 + B_3)^2} - \frac{(1 + A_3)(1 - B_3)B_{3x}}{(1 + B_3)^2}.$$  

In components, the 2-layer M-LIII equation (2.10)-(2.11) reads as

$$iA_{1x}^+ + (A^+ A_{3xx}^+ - A_{3xx}^+ A_3) + iu_1 A_{1x}^+ - 2v_1 A^+ = 0,$$

$$iA_{1x}^- - (A^- A_{3xx}^- - A_{3xx}^- A_3) + iu_1 A_{1x}^- + 2v_1 A^- = 0,$$

$$iA_{3x}^+ + \frac{1}{2}(A^- A_{3xx}^- - A_{3xx}^- A^+) + iu_1 A_{3x} = 0,$$

$$iB_{1x}^+ + (B^+ B_{3xx}^+ - B_{3xx}^+ B_3) + iu_2 B_{1x}^+ - 2v_2 B^+ = 0,$$

$$iB_{1x}^- - (B^- B_{3xx}^- - B_{3xx}^- B_3) + iu_2 B_{1x}^- + 2v_2 B^- = 0,$$

$$iB_{3x}^+ + \frac{1}{2}(B^- B_{3xx}^- - B_{3xx}^- B^+) + iu_2 B_{3x} = 0$$

or

$$A_{1t} + A_{2} A_{3xx} - A_{3xx} A_3 + u_1 A_{1x} - 2v_1 A_2 = 0,$$

$$A_{2t} + A_{3} A_{1xx} - A_{1xx} A_1 + u_1 A_{2x} - 2v_1 A_1 = 0,$$

$$A_{3t} + A_{1} A_{2xx} - A_{1xx} A_2 + u_1 A_{4x} = 0,$$

$$B_{1t} + B_{2} B_{3xx} - B_{3xx} B_3 + u_2 B_{1x} - 2v_2 B_2 = 0,$$

$$B_{2t} + B_{3} B_{1xx} - B_{1xx} B_1 + u_2 B_{2x} - 2v_2 B_1 = 0,$$

$$B_{3t} + B_{1} B_{2xx} - B_{1xx} B_2 + u_2 B_{3x} = 0.$$

### 3 Lakshmanan equivalent counterpart of the coupled M-LIII equation

In this section, we present the Lakshmanan equivalent counterpart of the coupled M-LIII equation (2.10)-(2.11). To do that, let us rewrite the 2-layer M-LIII equation (2.10)-(2.11) in the vector form as \[39\]-\[40\]

$$A_{i} + A \wedge A_{i} + u_1 A_{i} - 2v_1 H \wedge A = 0,$$

$$B_{i} + B \wedge B_{i} + u_2 B_{i} + 2v_2 H \wedge B = 0,$$

where $H = (0, 0, 1)$ is the constant magnetic field. Now we consider two interacting 3-dimensional curves in $\mathbb{R}^n$. These curves are given by the following two basic vectors $e_k$ and $l_k$. The motion of these curves is defined by the following equations

$$\left( \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \right)_x = C \left( \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \right),$$

$$\left( \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \right)_t = D \left( \begin{array}{c} e_1 \\ e_2 \\ e_3 \end{array} \right)$$

$$(3.3)$$
and
\[
\begin{pmatrix}
  l_1 \\
  l_2 \\
  l_3
\end{pmatrix}_x = L \begin{pmatrix}
  l_1 \\
  l_2 \\
  l_3
\end{pmatrix}, \quad \begin{pmatrix}
  l_1 \\
  l_2 \\
  l_3
\end{pmatrix}_t = N \begin{pmatrix}
  l_1 \\
  l_2 \\
  l_3
\end{pmatrix}. \tag{3.4}
\]

Here \(e_1\), \(e_2\) and \(e_3\) are the unit tangent, normal and binormal vectors respectively to the first curve, \(l_1, l_2\) and \(l_3\) are the unit tangent, normal and binormal vectors respectively to the second curve, \(x\) is the arclength parametrising these both curves. The matrices \(C, D, L, N\) are given by
\[
C = \begin{pmatrix}
  0 & k_1 & 0 \\
  -k_1 & 0 & \tau_1 \\
  0 & -\tau_1 & 0
\end{pmatrix}, \quad G = \begin{pmatrix}
  0 & \omega_3 & -\omega_2 \\
  -\omega_3 & 0 & \omega_1 \\
  \omega_2 & -\omega_1 & 0
\end{pmatrix}, \tag{3.5}
\]
\[
L = \begin{pmatrix}
  0 & k_2 & 0 \\
  -k_2 & 0 & \tau_2 \\
  0 & -\tau_2 & 0
\end{pmatrix}, \quad N = \begin{pmatrix}
  0 & \theta_3 & -\theta_2 \\
  -\theta_3 & 0 & \theta_1 \\
  \theta_2 & -\theta_1 & 0
\end{pmatrix}. \tag{3.6}
\]

For the curvatures and torsions of curves we obtain
\[
k_1 = \sqrt{\omega_2^2} \quad \tau_1 = \frac{e_1 \cdot (e_{1x} \wedge e_{1xx})}{e_{1x}^2}, \tag{3.7}
\]
\[
k_2 = \sqrt{\omega_2^2} \quad \tau_2 = \frac{l_1 \cdot (l_{1x} \wedge l_{1xx})}{l_{1x}^2}. \tag{3.8}
\]

The equations (3.3) and (3.4) are compatible if
\[
C_t - G_x + [C, G] = 0, \tag{3.9}
\]
\[
L_t - N_x + [L, N] = 0, \tag{3.10}
\]
respectively. In elements these equations take the form
\[
k_{1t} = \omega_3 + \tau_1\omega_2, \tag{3.11}
\]
\[
\tau_{1t} = \omega_1 - k_1\omega_2, \tag{3.12}
\]
\[
\omega_{2x} = \tau_1\omega_3 - k_1\omega_1 \tag{3.13}
\]
and
\[
k_{2t} = \theta_3 + \tau_2\theta_2, \tag{3.14}
\]
\[
\tau_{2t} = \theta_1 - k_2\theta_2, \tag{3.15}
\]
\[
\theta_{2x} = \tau_2\theta_3 - k_2\theta_1, \tag{3.16}
\]
respectively. Our next step is the following identifications:
\[
A \equiv e_1, \quad B \equiv l_1. \tag{3.17}
\]

We also assume that
\[
F = F_1 e_1 + F_2 e_2 + F_3 e_3, \quad E = E_1 l_1 + E_2 l_2 + E_3 l_3, \tag{3.18}
\]
where
\[
F = 2 v_1 H \wedge A, \quad E = 2 v_2 H \wedge B. \tag{3.19}
\]

Then we obtain
\[
k_1 = \sqrt{A_2^2}, \tag{3.20}
\]
\[
\tau_1 = \frac{A \cdot (A_x \wedge A_{xx})}{A_2^2}, \tag{3.21}
\]
\[
k_2 = \sqrt{B_2^2}, \tag{3.22}
\]
\[
\tau_2 = \frac{B \cdot (B_x \wedge B_{xx})}{B_2^2}. \tag{3.23}
\]

4
We now can write the equations for \( k_j \) where

\[
\begin{align*}
\omega_1 &= -\frac{k_{1xx} + F_2 \tau_1 + F_3}{k_1} + (\tau_1 - u_1) \tau_1, \\
\omega_2 &= k_{1x} + F_3, \\
\omega_3 &= k_1(\tau_1 - u_1) - F_2, \\
\theta_1 &= -\frac{k_{2xx} + E_2 \tau_2 + E_3}{k_2} + (\tau_2 - u_2) \tau_2, \\
\theta_2 &= k_{2x} + E_3, \\
\theta_3 &= k_2(\tau_2 - u_2) - E_2 \\
\end{align*}
\]

with

\[
F_1 = E_1 = 0.
\]

We now can write the equations for \( k_j \) and \( \tau_j \). They look like

\[
\begin{align*}
k_{1t} &= 2k_{1x} \tau_1 + k_1 \tau_{1x} - (u_1 k_1)_x - F_2x + F_3 \tau_1, \\
\tau_{1t} &= \left( -\frac{k_{1xx} + F_2 \tau_1 + F_3}{k_1} + (\tau_1 - u_1) \tau_1 - \frac{1}{2} k_1 \right)_x - F_3 k_1, \\
k_{2t} &= 2k_{2x} \tau_2 + k_2 \tau_{2x} - (u_2 k_2)_x - E_2x + E_3 \tau_2, \\
\tau_{2t} &= \left( -\frac{k_{2xx} + E_2 \tau_2 + E_3}{k_2} + (\tau_2 - u_2) \tau_2 - \frac{1}{2} k_2 \right)_x - E_3 k_2.
\end{align*}
\]

Let us now introduce new four real functions \( \alpha_j \) and \( \beta_j \) as

\[
\begin{align*}
\alpha_1 &= 0.5 k_1 \sqrt{1 + \zeta_1}, \\
\beta_1 &= \tau_1 (1 + \zeta_1), \\
\alpha_2 &= 0.5 k_2 \sqrt{1 + \zeta_2}, \\
\beta_2 &= \tau_2 (1 + \zeta_2),
\end{align*}
\]

where

\[
\begin{align*}
\zeta_1 &= \frac{2 [WA_x^2 - MA^2]^2}{W^2 (1 + A_3)^2 A_x^2} - 1, \\
\zeta_2 &= \frac{2 [W [(1 + A_3) (1 + B_3)^{-1} B^{-1}]_x - M [(1 + A_3) (1 + B_3)^{-1} B^{-1}]^2 ]_x^2}{W^2 (1 + A_3)^2 B_x^2} - 1, \\
\xi_1 &= \frac{\bar{R}_x R - \bar{R} R_x - \frac{4 i |R|^2 \nu_x}{2 \alpha_1^2 W^2 (1 + A_3)^2 \tau_1}}{1 - \frac{2 \alpha_2^2 W^2 (1 + A_3)^2 \tau_1}{1}}, \\
\xi_2 &= \frac{\bar{Z}_x Z - \bar{Z} Z_x - \frac{4 i |Z|^2 \nu_x}{2 \alpha_2^2 W^2 (1 + A_3)^2 \tau_2}}{1} - 1.
\end{align*}
\]

Here

\[
\nu = \partial_x^{-1} \left[ \frac{A_1 A_{2x} - A_{1x} A_2}{(1 + A_3) W} - \frac{(1 + A_3) (B_{1x} B_2 - B_1 B_{2x})}{(1 + B_3)^2 W} \right].
\]

We now ready to write the equations for the functions \( \alpha_i \) and \( \beta_j \). They satisfy the following four equations

\[
\begin{align*}
\alpha_{1t} - 2 \alpha_{1x} \beta_1 - \alpha_1 \beta_{1x} &= 0, \\
\beta_{1t} + \left( \frac{\alpha_{1xx}}{\alpha_1} - \beta_1^2 + 2(\alpha_1^2 + \alpha_2^2) \right)_x &= 0, \\
\alpha_{2t} - 2 \alpha_{2x} \beta_2 - \alpha_2 \beta_{2x} &= 0, \\
\beta_{2t} + \left( \frac{\alpha_{2xx}}{\alpha_2} - \beta_2^2 + 2(\alpha_1^2 + \alpha_2^2) \right)_x &= 0.
\end{align*}
\]
Let us now introduce new two complex functions using by the A-transformation. The A-transformation reads as (see e.g. [39]-[46])

\[ q_1 = \alpha_1 e^{-i\partial_x^{-1} \beta_1}, \]  
\[ q_2 = \alpha_2 e^{-i\partial_x^{-1} \beta_2}. \]  

Sometime we use the following explicit form of the A-transformation

\[ q_1 = 0.5k_1 \sqrt{1 + \zeta_1} e^{-i\partial_x^{-1} |\tau_1(1+\xi_1)|}, \]  
\[ q_2 = 0.5k_2 \sqrt{1 + \zeta_2} e^{-i\partial_x^{-1} |\tau_2(1+\xi_2)|}. \]  

It is not difficult to verify that these new complex functions \( q_j \) satisfy the following system of equations (see e.g. [38])

\[ iq_{1t} + q_{1xx} + 2(|q_1|^2 + |q_2|^2)q_1 = 0, \]  
\[ iq_{2t} + q_{2xx} + 2(|q_1|^2 + |q_2|^2)q_2 = 0. \]

It is nothing but the Manakov system. So we proved that the Manakov system (3.52)-(3.53) is the Lakshmanan equivalent counterpart of the 2-layer M-LIII equation (2.10)-(2.11) or in the vector form (3.1)-(3.2). Finally we note that if \( \zeta_1 = \xi_1 = 0 \) then the A-transformation (3.48)-(3.49) or (3.50)-(3.51) reduces to the Hasimoto transformation

\[ q_1 = 0.5\kappa_1 e^{-i\partial_x^{-1} \tau_1}, \]  
\[ q_2 = 0.5\kappa_2 e^{-i\partial_x^{-1} \tau_2}. \]

4 Lax representation of the coupled M-LIII equation hierarchy

In the previous section we have shown that the coupled M-LIII equation is the Lakshmanan equivalent to the Manakov system. This means that the coupled M-LIII equation is integrable by the IST method since its equivalent counterpart - the Manakov system is integrable. In turn, it means that the M-LIII equation admits all ingredients of integrable systems like Lax representation (LR), infinite number of commuting integrals of motion, n-soliton solutions etc. Below we present two possible versions of the LR for the M-LIII equation hierarchy.

4.1 LR type - I

The first type of LR for the coupled M-LIII equation hierarchy reads as

\[ Y_x = -i\lambda P Y, \]  
\[ Y_t = \sum_{j=1}^{N} \lambda^j V_j Y, \]

where \( \lambda \) is a spectral parameter and

\[ P = \frac{1}{2+K} \left( \begin{array}{c} 2A_3 - K \\ 2A^+ \\ \frac{2(1+A_3)B^-}{1+B_3} \\ \frac{2A^-}{1+B_3} \\ \frac{2(1+A_3)B^-}{1+B_3} \\ \frac{2A^-}{1+B_3} \end{array} \right), \]  
\[ K = (1+A_3)(1-B_3)(1+B_3)^{-1}. \]

The compatibility condition of this system gives the coupled M-LIII equation hierarchy. As the particular example, let us consider the case when \( N = 2 \). Then the set of equations (4.1)-(4.2) takes the form

\[ Y_x = -i\lambda P Y, \]  
\[ Y_t = (\lambda^2 V_2 + \lambda V_1) Y, \]  
\[ Y = \sum_{j=1}^{N} \lambda^j V_j Y. \]
where

\[ V_2 = -2iP, \quad V_1 = PP, \]

(4.6)

The compatibility condition of the equations (4.4)-(4.5) gives the 2-layer M-LIII equation (2.10)-(2.11) or (3.1)-(3.2) that is same.

### 4.2 LR type - II

The second type of LR for the coupled M-LIII equation hierarchy can be written in the following form

\[ Y_x = -i\lambda QY, \]

(4.7)

\[ Y_t = \sum_{j=1}^{N} \lambda^j W_j Y. \]

(4.8)

Here

\[ Q = Q_1 + Q_2, \]

(4.9)

where

\[
Q_1 = \begin{pmatrix}
0 & A_1 & A_2 & A_3 \\
-A_1 & 0 & A_3 & -A_2 \\
-A_2 & -A_3 & 0 & A_1 \\
-A_3 & A_2 & -A_1 & 0
\end{pmatrix}, \quad Q_2 = \begin{pmatrix}
0 & B_1 & B_2 & -B_3 \\
-B_1 & 0 & B_3 & B_2 \\
-B_2 & -B_3 & 0 & -B_1 \\
B_3 & -B_2 & B_1 & 0
\end{pmatrix}.
\]

(4.10)

From the compatibility condition of the set of equations (4.7)-(4.8) \( Y_{xt} = Y_{tx} \) we obtain the coupled M-LIII equation hierarchy.

### 5 Relation between solutions of the coupled M-LIII equation and the Manakov system

Let \( A_j \) and \( B_j \) be the solution of the coupled M-LIII equation (2.10)-(2.11). Then the solution of the Manakov system (3.52)-(3.53) is given by

\[ q_1 = \frac{Re e^{2i\nu}}{W(1 + A_3)} \]

(5.1)

\[ q_2 = \frac{Ze^{2i\nu}}{W(1 + A_3)}. \]

(5.2)

### 6 Gauge equivalence between the \( \Gamma \)-spin system and the Manakov system

Above, we have proved that the coupled M-LIII equation (2.10)-(2.11) and the Manakov system (3.52)-(3.53) is the Lakshmanan equivalent to each other. In this section, we want to present the another (gauge) equivalent counterpart of the Manakov system. It is well-known that the Lax representation of the Manakov equation (3.52)-(3.53) has the form (see e.g. [38])

\[ \Phi_x = U\Phi, \]

(6.1)

\[ \Phi_t = V\Phi. \]

(6.2)

Here

\[ U = -i\lambda \Sigma + U_0, \quad V = -2i\lambda^2 \Sigma + 2\lambda U_0 + V_0 \]

(6.3)
with
\[
\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad U_0 = \begin{pmatrix} 0 & q_1 & q_2 \\ -\bar{q}_1 & 0 & 0 \\ -\bar{q}_2 & 0 & 0 \end{pmatrix}, \quad V_0 = i \begin{pmatrix} |q_1|^2 + |q_2|^2 & q_{1x} & q_{2x} \\ \bar{q}_{1x} & -|q_1|^2 & -\bar{q}_1 q_{2x} \\ \bar{q}_{2x} & -q_2 q_{1x} & -|q_2|^2 \end{pmatrix}.
\]

Let us now consider the gauge transformation
\[
\Psi = g^{-1} \Phi, \quad g = \Phi_{\lambda=0}.
\]

Then \(\Psi\) obeys the equations
\[
\Psi_x = U'\Psi, \quad \Psi_t = V'\Psi,
\]
where
\[
U' = -i\lambda \Gamma, \quad V' = -2i\lambda^2 \Gamma + \frac{1}{2}\lambda[\Gamma, \Gamma_x].
\]

Elements of the \(\Gamma\) matrix satisfy some restrictions
\[
\Gamma_{33} = -(1 + \Gamma_{11} + \Gamma_{22}), \quad \Gamma_{ij} = \Gamma_{ji},
\]
and
\[
\Gamma_{ik}\Gamma_{kj} + \Gamma_{i(k+1)}\Gamma_{(k+1)j} + \Gamma_{i(k+2)}\Gamma_{(k+2)j} = 0, \quad (i \neq k \neq j),
\]
\[
\Gamma_{ik}\Gamma_{kj} + \Gamma_{i(k+1)}\Gamma_{(k+1)j} + \Gamma_{i(k+2)}\Gamma_{(k+2)j} = 1.
\]

The compatibility condition of the equations (5.6)-(5.7) gives
\[
i\Gamma_t + \frac{1}{2}[\Gamma, \Gamma_{xx}] = 0.
\]

We call this equation - the \(\Gamma\)-spin system. Thus the \(\Gamma\)-spin system (6.14) is the gauge equivalent counterpart of the Manakov system (3.52)-(3.53). It is the well-known result (see e.g. [33]). In terms of elements, the \(\Gamma\)-spin system (6.14) reads as
\[
i\Gamma_{11t} + \frac{1}{2}(\Gamma_{12}\Gamma_{21} + \Gamma_{13}\Gamma_{31} - \Gamma_{12xx}\Gamma_{21} - \Gamma_{13xx}\Gamma_{31}) = 0, \quad (6.15)
\]
\[
i\Gamma_{12t} + \frac{1}{2}(\Gamma_{11}\Gamma_{12} + \Gamma_{13}\Gamma_{32} - \Gamma_{12xx}\Gamma_{21} - \Gamma_{13xx}\Gamma_{32}) = 0, \quad (6.16)
\]
\[
i\Gamma_{13t} + \frac{1}{2}(\Gamma_{11}\Gamma_{13} + \Gamma_{12}\Gamma_{23} - \Gamma_{11xx}\Gamma_{13} - \Gamma_{12xx}\Gamma_{23} - \Gamma_{13xx}\Gamma_{11}) = 0, \quad (6.17)
\]
\[
i\Gamma_{21t} + \frac{1}{2}(\Gamma_{21}\Gamma_{11} + \Gamma_{22}\Gamma_{12} + \Gamma_{23}\Gamma_{31} - \Gamma_{21xx}\Gamma_{11} - \Gamma_{22xx}\Gamma_{21} - \Gamma_{23xx}\Gamma_{31}) = 0, \quad (6.18)
\]
\[
i\Gamma_{22t} + \frac{1}{2}(\Gamma_{21}\Gamma_{12} + \Gamma_{23}\Gamma_{32} - \Gamma_{21xx}\Gamma_{12} - \Gamma_{23xx}\Gamma_{32}) = 0, \quad (6.19)
\]
\[
i\Gamma_{23t} + \frac{1}{2}(\Gamma_{21}\Gamma_{33} + \Gamma_{22}\Gamma_{23} + \Gamma_{23}\Gamma_{33} - \Gamma_{21xx}\Gamma_{13} - \Gamma_{22xx}\Gamma_{23} - \Gamma_{23xx}\Gamma_{33}) = 0, \quad (6.20)
\]
\[
i\Gamma_{31t} + \frac{1}{2}(\Gamma_{31}\Gamma_{11} + \Gamma_{32}\Gamma_{21} + \Gamma_{33}\Gamma_{31} - \Gamma_{31xx}\Gamma_{11} - \Gamma_{32xx}\Gamma_{21} - \Gamma_{33xx}\Gamma_{31}) = 0, \quad (6.21)
\]
\[
i\Gamma_{32t} + \frac{1}{2}(\Gamma_{31}\Gamma_{12} + \Gamma_{32}\Gamma_{22} + \Gamma_{33}\Gamma_{32} - \Gamma_{31xx}\Gamma_{12} - \Gamma_{32xx}\Gamma_{22} - \Gamma_{33xx}\Gamma_{32}) = 0, \quad (6.22)
\]
\[
i\Gamma_{33t} + \frac{1}{2}(\Gamma_{31}\Gamma_{33} + \Gamma_{32}\Gamma_{23} - \Gamma_{31xx}\Gamma_{13} - \Gamma_{32xx}\Gamma_{23}) = 0. \quad (6.23)
\]
7 Relation between solutions of the coupled M-LIII equation and the $\Gamma$-spin system

In the previous sections we have shown that to the one and same set of equations - the Manakov system (3.52)-(3.53), correspond two spin systems: the coupled M-LIII equation (3.1)-(3.2) and the $\Gamma$-spin system (6.14). It tells us that between these two spin systems there must be some exact relation/correspondence. In other words, the 2-layer M-LIII equation (2.10)-(2.11) and the $\Gamma$-spin system (6.14) are equivalent to each other by some exact transformations. Below we will present these transformations.

7.1 Direct M-transformation

According to the M-transformation, in terms of the spin vectors $A$ and $B$, the elements of the $\Gamma$-spin system are expressed as

$$\Gamma = \frac{1}{2 + K} \begin{pmatrix} 2A_3 - K & 2A^- & \frac{2(1 + A_3)B^-}{1 + B_3} \\ 2A^+ & -(2A_3 + K) & \frac{2A^+ B^-}{1 + B_3} \\ \frac{2(1 + A_3)B^+}{1 + B_3} & \frac{2A^+ B^+}{1 + B_3} & K - 2 \end{pmatrix},$$

(7.1)

where

$$K = \frac{(1 + A_3)(1 - B_3)}{1 + B_3}.$$

(7.2)

This is the direct M-transformation. This M-transformation allows us to find solutions of the $\Gamma$-spin system (6.14) if we know the solutions of the coupled M-LIII equation (2.10)-(2.11).

7.2 Inverse M-transformation

According to the inverse M-transformation, solutions of the coupled M-LIII equation can be expressed by the components of the $\Gamma$-spin system as

$$A = \frac{1}{1 - \Gamma_{33}} \begin{pmatrix} \Gamma_{11} - \Gamma_{22} & 2\Gamma_{12} \\ 2\Gamma_{21} & \Gamma_{22} - \Gamma_{11} \end{pmatrix},$$

(7.3)

$$B = \frac{1}{1 - \Gamma_{22}} \begin{pmatrix} \Gamma_{11} - \Gamma_{33} & 2\Gamma_{13} \\ 2\Gamma_{31} & \Gamma_{33} - \Gamma_{11} \end{pmatrix}.$$

(7.4)

The transformations (7.3)-(7.4) is called the inverse M-transformation. Using the inverse M-transformation, we can find solutions of the coupled M-LIII equation (2.10)-(2.11), if we know the solutions of the $\Gamma$-spin system (6.14).

8 Conclusions

In this paper, we have shown how the dynamics of two interacting and moving curves in some space $\mathbb{R}^n$ can be related to the dynamics of the coupled spin systems, namely, the coupled M-LIII equation. Next, after some algebra we have proved that these two interacting and moving curves are related with the Manakov system. On the other hand, it is well-known that the Manakov system is equivalent to the $\Gamma$-spin system. Also, we have presented the transformations which established the relation between solutions of the $\Gamma$-spin system and the coupled M-LIII equation. Our results can also be generalized to higher dimensional spaces (see e.g. refs. [39]-[46]). Work along these lines is in progress.
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