Dephasing of Electrons on Helium by Collisions with Gas Atoms

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I. INTRODUCTION

The damping of quantum effects in the transport properties of electrons deposited on a surface of liquid helium is studied. It is found that due to vertical motion of the helium vapour atoms the interference of paths of duration $t$ is damped by a factor $\exp\left(-t/\tau_v\right)^3$. An expression is derived for the weak-localization lineshape in the case that damping occurs by a combination of processes with this type of cubic exponential damping and processes with a simple exponential damping factor.

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1. INTRODUCTION

The damping of quantum effects in a system coupled to external degrees of freedom is a fundamental problem of atomic physics, condensed matter physics and quantum optics. There is great interest in understanding and controlling such damping in well characterised systems. Here we study the damping of quantum effects in the transport properties of a two-dimensional electron gas deposited on the surface of a pool of liquid helium.

Electrons on the surface of helium are vertically confined by their image charges and an (optional) applied holding field. They constitute a two-dimensional electron gas similar to those in semiconductor devices but with different scattering and damping mechanisms. Electrons may scatter off ripples on the surface of the helium pool ("ripplons") or off helium vapour atoms above the liquid surface. Above 1 K, gas atom scattering dominates, and we concentrate on this regime. On the electronic time-scale, the helium vapour atoms are almost stationary and hence similar to impurities in a metal film or a semiconductor device. Thus there are quantum interference corrections to the resistivity at low temperature familiar from studies of transport in metals and semiconductors. These corrections result from constructive interference between closed electron paths and their time-reversed counterparts, leading to a small enhancement of the resistance ("weak-localization correction").

The slow movement of helium atoms leads to damping of weak-localization. There is an important distinction between the effect of vertical and horizontal motion of helium atoms. Roughly, horizontal movement produces damping by scrambling the phase of the interfering paths; vertical movement, by reducing the weight of contributing paths of long duration. The effect of horizontal movement has been analysed previously; it is the purpose of this paper to study the effect of vertical motion.

The central result is that due to vertical motion of the helium atoms the interference contribution of paths of duration $t$ is reduced by a factor $\exp\left(-t/\tau_v\right)^3$. Thus paths of duration greater than the damping time, $\tau_v$, are effectively cutoff. An interesting feature is that damping due to both vertical and horizontal movement of helium atoms is not a simple exponential; it cuts off more sharply as the exponential of $t^3$. In contrast, electron-electron and electron-phonon interactions in metals and semiconductors are supposed to produce simple exponential damping. Damping in atomic physics and nuclear magnetic resonance is also commonly a simple exponential; this is indicated by the Lorentzian shape of spectral and magnetic resonance lines. As emphasized by Afonin et al. in context of quantum transport, the form of damping can be probed by measuring the magnetic field dependence of the weak-localization correction ("weak-localization lineshape"). In section III we exhibit some lineshapes corresponding to different forms of damping.

Weak-localization has been observed in a related system, electrons on a surface of solid hydrogen. In this system helium vapour was deliberately introduced above the solid hydrogen to scatter electrons; thus gas atom damping is relevant to this type of experiment. More recently, Karakurt et al. have systematically studied the dependence of the damping rate on various experimental parameters (electron density; gas vapour pressure, controlled via temperature; and holding field) for electrons on helium. In this way they have obtained quantitative information on the contributions of different mechanisms to the damping rate. It is the experiment of Karakurt et al. that prompted us to carry out the present investigation.

For orientation it is useful to recall some typical parameters for the experiment of Karakurt et al. In the absence of a holding field, the electron is bound to the surface by its image. The charge of the image is reduced from the bare charge of the electron by a factor $(\epsilon - 1)/(\epsilon + 1) = 7 \times 10^{-3}$; thus the vertical scale of the electronic wavefunction is 76 Å. The lowest vertical subband wavefunction is of the Fang-Howard form, $\phi(z) \propto z \exp(-z/b)$, at zero holding field; this form re-

\[\footnotesize{1\text{Recall that the Fourier transform of } e^{-|t|}\text{ is a Lorentzian.}}\]
mains an excellent variational ansatz with $b$ an adjustable parameter when a holding field is applied. Here $z$ denotes the distance of the electron above the helium pool. The subband spacing is 6 K; hence for sufficiently low temperatures and electron densities below $2 \times 10^{15} / m^2$ the surface electrons behave like a two-dimensional electron gas. Much of the data of Karakurt et al. is at temperatures around 2 K and at a typical density of $2 \times 10^{11} / m^2$ corresponding to a Fermi temperature of 0.6 mK. Note that their two-dimensional electron gas is therefore non-degenerate in contrast to the situation in metal films and typical semiconductor devices. Thus transport properties are not determined entirely by mono-energetic electrons on the Fermi surface; instead we must sum the Boltzmann-weighted contribution of electrons of all energies. The electron-atom collision time inferred from mobility measurements was typically a few ps. The longest relevant electronic time scale is $\tau_z$, the time taken by a thermal electron to move a distance $b$ (see eq 5 below). At 2 K and zero holding field $\tau_z = 80$ ps. In comparison, the atom-atom collision time is enormous, of the order of 10 ns.

II. ANALYSIS OF DAMPING

In this section we analyse the damping produced by the vertical motion of helium vapour atoms. First we analyse a simple model (model I) that captures some of the essential physics, but leads to the incorrect conclusion that the damping factor goes as the exponential of $t^2$ rather than $t^4$, a result obtained earlier by Stephen [11]. We then identify a shortcoming of model I and in the next subsection introduce and analyse an improved version (model II) that leads to the correct answer.

A. Model I

In this model we assume that the Helium atoms are able to scatter electrons only if they are within a certain distance (denoted $b$) from the liquid helium surface. It is also assumed that the scattering is independent of the precise height of the atom so long as it lies within the prescribed distance.

\[ p(t) = \exp \left( -\frac{t}{\sqrt{\pi \tau_z}} \right) \]  
\[ \gamma(t) = \exp \left( -\frac{t^2}{\sqrt{\pi \tau_z \tau_e}} \right) \]

Consider $p(t) = \text{probability that an atom will remain within the scattering distance for a time } t$. At first let us assume the probability decays exponentially,

\[ \gamma(t) = \exp \left( -\frac{2t^2}{\sqrt{\pi \tau_z \tau_e}} \right) \]

Here $\gamma$ is the damping factor; essentially this result is given in ref [11].

A second improvement is needed because eq (1) is incorrect. $p(t)$ is easily calculated and seen to not be exponential. Here we mention only the relevant features of $p(t)$; the details are relegated to appendix A. (i) As expected on dimensional grounds, $p(t)$ is a function of $t/\tau_z$ alone, where

\[ \tau_z = \frac{\sqrt{Mb^2}}{2kT} \]

Here $M$ = mass of a helium atom. Physically, $\tau_z$ is the time taken by a thermal atom to move a distance $b$. (ii) For short times, $t \ll \tau_z$, we find...
\[ p(t) \approx 1 - \frac{t}{\sqrt{\pi} \tau_z} \quad (6) \]

(iii) For long times, \( t \gg \tau_z \), \( p(t) \) vanishes in a manner not relevant to our purpose.

Now the probability that \( n \) atoms remain near the surface is

\[ [p(t)]^n = \exp[n \ln p(t)] = \exp \left[ n \ln \left( 1 - \frac{t}{\sqrt{\pi} \tau_z} + \ldots \right) \right] \approx \exp \left( -\frac{nt}{\sqrt{\pi} \tau_z} \right). \quad (7) \]

This shows that for large \( n \), \([p(t)]^n\) can be approximated as an exponential only for \( t \ll \tau_z/\sqrt{n} \); but since it becomes negligible in any case once \( t \gg \tau_z/n \), there is no significant error in taking \([p(t)]^n\) to be an exponential.

The upshot of this discussion is that although \( p(t) \) is far from exponential, \([p(t)]^n\) is a simple exponential under appropriate circumstances; eq (2) is valid, although eq (1) is not. Similarly, we see that eq (4) is also valid provided \( \tau_z \gg \tau_e \), a condition needed for weak-localization.

In summary, for model I the damping decays as the exponential of \( t^2 \). Provided \( \tau_z \gg \tau_e \), it is given by eq (4). The atomic time constant \( \tau_z \) is given by eq (5). Evidently, the three time scales are arranged in the hierarchy \( \tau_z > \tau_v > \tau_e \).

### B. Model II

The shortcoming of model I is the assumption stated in the first paragraph of the previous subsection. It is more realistic to assume that the ability of an atom to scatter electrons turns off smoothly as it moves away from the liquid helium surface.

If we treat the atoms as hard core potentials, the contribution of a closed path to the return amplitude is a product of the amplitude for the electron to go to atom 1, multiplied by the amplitude to scatter off atom 1, multiplied by the amplitude to go to atom 2, multiplied by the amplitude to scatter off atom 2, and so on around the loop.

Let \( A(z) \) be the amplitude to scatter from an atom at height \( z \) above the helium surface. Model I can be described as the case in which \( A(z) \) is a step function. Here we choose

\[
A(z) = \begin{cases} 
\frac{4\lambda z^2}{b^3} \exp \left( -\frac{2z}{b} \right) & \text{for } z > 0; \\
0 & \text{for } z < 0.
\end{cases}
\]

Here the average over vertical position is performed as \( \langle \ldots \rangle \). Hence eq (13) should be contrasted with \( q(t) \) above for model I.

To obtain the damping factor, roughly we must replace \( n \) in eq (13) by \( t/\tau_e \), the number of atoms encountered in a path of duration \( t \). Before that we must replace \( t^2 \) in eq (13) by \( t^2/3 \), its value averaged over the interval from 0 to \( t \) with uniform weight. This is to take into account the range in the difference of times at which an atom is encountered along the forward and reversed histories.

If the helium atoms are only allowed to move vertically the forward and backward paths remain in phase; however the interference contribution to the return probability is still modified because the forward and backward paths have different amplitudes to scatter from each atom. We must consider

\[ Q(t) = \langle A(z) A(z + vt) \rangle. \quad (9) \]

Here \( t \) is the difference in the times at which the atom is encountered on the forward and return path. The atom is assumed to move ballistically at vertical speed \( v \) for this time. \( \langle \ldots \rangle \) denotes an average over all possible configurations of the Helium atom (vertical position is assumed to be uniformly distributed and vertical speed is given by the Maxwell-Boltzmann formula).

Introduce the normalization factor \( R(t) \) defined by

\[ R(t) = \langle A(z) A(z + vt) \rangle. \quad (10) \]

Here the average over vertical position is performed as in eq (9) but the velocity distribution is assumed to be a delta function peaked about zero. \( R(t) \) is the value of \( Q(t) \) when the atoms don’t move. Let

\[ q(t) = Q(t)/R(t). \quad (11) \]

The contribution of paths of duration \( t \) is then reduced roughly by the factor \( q(t) \) raised to the power \( t/\tau_e \), the number of atoms encountered.

\( q(t) \) is analogous to \( p(t) \) for model I. Again on dimensional grounds, \( q(t) \) depends only on the ratio \( t/\tau_z \) and again we are interested only in the short time behaviour. This is evaluated in Appendix B. The difference from the previous case is that

\[ q(t) = 1 - \frac{t^2}{3\tau_z^2} + \ldots \quad (12) \]

for short times, \( t \ll \tau_z \). The behaviour is quadratic rather than linear (compare eq 6). Quadratic behaviour is generic; the linear behaviour for model I is an artifact of the discontinuous step in \( A(z) \). Hence \( q(t) \) raised to the power of \( n \) is approximately Gaussian rather than exponential

\[ [q(t)]^n \approx \exp \left( -\frac{nt^2}{3\tau_z^2} \right). \quad (13) \]

Eq (13) should be contrasted with eq (7) above for model I.
The result for the damping factor is
\[ \gamma(t) = \exp \left( -\frac{t^3}{\tau_e^3} \right) \] (14)

where \( \tau_e = (9\tau_c \tau_r^2)^{1/3} \). Eq (14) is the central result of this paper. It is valid provided \( \tau_c \gg \tau_e \).

III. LINESHAPE

Karakurt et al. observed damping by vapour atom motion at low electron density and by electron-electron interaction at high density [8]. At intermediate densities, damping by both mechanisms was substantial. Vapour atom scattering produces cubic exponential damping; electron-electron interaction is presumably simple exponential. Afonin et al. have pointed out that the weak-localization lineshape depends on the form of damping and they have given an expression for the lineshape in the extreme cases that the damping is entirely simple exponential or entirely cubic exponential. The purpose of this section is to study the lineshape in the intermediate regime and examine how it crosses over from one extreme form to the other.

For simplicity, first let us consider a degenerate electronic system. Assuming that the different damping mechanisms are independent the lineshape is given by
\[ \delta g(E, B) = \frac{1}{\pi} \frac{e^2}{h} \left( \frac{W}{L} \right) \phi(E, B); \]
\[ \phi(E, B) = \int_{\tau_e}^{\infty} dt \frac{4\pi e D B}{h} \frac{e^{-t/\tau_1} e^{-t^3/\tau_3^3}}{\sinh(4\pi e D t B/h)}. \] (15)

Here \( E \) is the Fermi energy; \( W \), the sample width; \( L \), the sample length; \( D \) the electron diffusion constant; \( 1/\tau_1 \), the simple exponential damping rate; and \( 1/\tau_3 \), the cubic exponential damping rate. Energy dependence enters the integrand in eq (15) through the diffusion constant \( D = E \tau_c / m \) and through the energy dependence (if any) of the time constants \( \tau_1 \) and \( \tau_3 \). The sinh factor in eq (15) may be recognised as the Fourier transform of the directed area distribution for closed random walks on a plane [12].

It is useful to manipulate eq (15) into a more revealing form. To this end introduce the dimensionless variable \( u = 8\pi e D t B/h \) to obtain
\[ \delta g = -\frac{1}{\pi} \frac{e^2}{h} \left( \frac{W}{L} \right) \int_{B/B_c}^{\infty} du \frac{\exp(-u \left( \frac{1}{2} + \frac{B_c}{B} \right) \exp(-u^3 \frac{B_3}{B_1})}{1 - e^{-u}}. \] (16)

where \( B_c = h/(8\pi e D \tau_c) \). Making use of the asymptotic formula
\[ \int_{\epsilon}^{\infty} du \frac{e^{-u}}{u} \approx \ln \frac{1}{\epsilon} + \gamma \] (17)

we obtain
\[ \delta g/\left( \frac{e^2 W}{h} \right) = -\frac{1}{2\pi} \left[ \ln \left( \frac{B_c}{B_1} + \ln \frac{B_3}{B_1} \right) + \frac{1}{\pi} F \left( \frac{B_1}{B}, \frac{B_3}{B} \right) \right] \] (18)

where \( B_1 = h/(8\pi e D \tau_1), B_2 = h/(8\pi e D \tau_3), \gamma = 0.577216 \ldots \) is Euler’s constant and the function
\[ F(x, y) = \frac{1}{2} \ln x + \frac{1}{2} \ln y + \gamma - \int_{0}^{\infty} du \left[ \frac{e^{-u}}{u} - \exp(-u (\frac{1}{x} + \frac{1}{y}) \exp(-u^3 y^3)) \right]. \] (19)

Eqs (18) and (19) constitute the generalisation of the standard weak-localisation lineshape to the case that both \( \tau_1 \) and \( \tau_3 \) damping are present. For the special case that there is no \( \tau_3 \) damping (hence \( y \to 0 \)) eqs (18,19) reduce to the familiar expression involving digamma functions by use of the integral representation [13]
\[ \int_{0}^{\infty} du \left( \frac{e^{-u}}{u} - \frac{e^{-\left( \frac{1}{x} + \frac{1}{y} \right) u}}{1 - e^{-u}} \right) = \psi(\frac{1}{2} + x). \] (20)

A significant feature revealed by eqs (18,19) is that the lineshape is universal: \( F \) does not depend on microscopic length scales. Note that the magnetic field dependence is entirely in the second term of eq (18); the first term is an additive constant. A practical advantage of eq (19) over eq (15) is that the integrand is well behaved for both large and small \( u \). In contrast, the integrand in eq (15) diverges at the lower end.

To study the crossover in lineshape we fix the damping rate \( 1/\tau_1 + 1/\tau_3 = 1/\tau_c \). Equivalently, we fix \( B_1 + B_3 = B_0 \). \( \delta g \) is plotted as a function of \( B \) for several values of the ratio \( B_1/B_0 \). Fig 1 shows that for the same damping rate the lineshape changes noticeably as damping shifts from simple exponential to cubic exponential.

Fig 2 shows the behaviour of the conductance minimum at \( B = 0 \) for a fixed damping rate. It is given by
\[ \delta g(B = 0) = -\frac{1}{\pi} \frac{e^2}{h} \left[ \ln \left( \frac{B_c}{B_1} + u \left( \frac{B_3}{B_1} \right) \right) \right] \] (21)

with the crossover function
\[ u(x) = \ln(1 + x) + \gamma + \int_{0}^{\infty} ds (1 + 3s^2 x^3) \ln s e^{-s} e^{-s^3 x^3}. \] (22)

As implied by eqs (18) and (19) the crossover depends only on the ratio \( B_3/B_1 \). \( u \) has the limiting values \( u(0) = 0 \) and \( u(\infty) = 2\gamma/3 \).

Under experimental conditions [18] the electron gas is non-degenerate. At finite temperature
\[ \delta g(T, B) = - \int_0^\infty dE \frac{\partial f}{\partial E} \delta g(E, B) \approx \frac{n \pi h^2}{m kT^2} \int_{E_c}^\infty dE \delta g(E, B) e^{-E/kT}. \] (23)

\(n\) is the area density of electrons. In the second line of eq (23) we have approximated the Fermi function by a Boltzmann factor and imposed a lower cutoff \(E_c\). Below the cutoff energy the electrons are presumed to be strongly localized and to make an insignificant contribution to the conductance. These finite temperature considerations make it more difficult to extract the form of damping from the lineshape.

**IV. CONCLUSION**

In summary, we have given a physical argument that due to vertical motion of helium atoms the interference of electron paths of duration \(t\) is damped by a factor \(\exp(-t/\tau)^3\). We have derived a formula for the universal magnetococonductance lineshape for the case that both \(\tau_1\) and \(\tau_3\) damping are present. It should be possible to rederive these results via impurity averaged diagrams; this is left open for future work.

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**APPENDIX A: ASYMPTOTICS OF \(P(T)\)**

We wish to calculate \(p(t)\), the probability that a vapour atom will remain within a vertical elevation \(b\) of the liquid surface for a time \(t\). We assume (i) the initial elevation of the atom is uniformly distributed between zero and \(b\); (ii) the vertical velocity is Maxwell-Boltzman distributed; (iii) the atom moves ballistically; and (iv) if the atom strikes the liquid surface it sticks and does not reflect \(\dagger\). Due to assumption (iii) the expression for \(p(t)\) that we derive is valid only for short times relative to the atom-atom collision time; however this is not a serious restriction since we are interested only in the short time behaviour of \(p(t)\).

Based on these assumptions we may write

\[ p(t) = \frac{1}{b} \int_0^b \int_0^{(b-z)/t} dv \sqrt{\frac{M}{2 \pi kT}} \exp \left( -\frac{M v^2}{2kT} \right) dz + \frac{1}{b} \int_0^b \int_{-z/t}^0 dv \sqrt{\frac{M}{2 \pi kT}} \exp \left( -\frac{M v^2}{2kT} \right). \] (A1)

The two contributions correspond to the atom moving up and down respectively.

By exchanging the order of integration we can perform the \(z\) integral first to obtain

\[ p(t) = \frac{2}{\sqrt{\pi}} \int_0^{1/\tau} du (1-u\tilde{\tau}) \exp(-u^2). \] (A2)

We have rescaled variables so that \(u = v/\sqrt{2kT/M}\) and \(\tilde{\tau} = t/\tau_z\). Note that \(p(t=0) = 1\) and as \(t \to \infty\), \(p(t) \to 0\). Eq (A2) is an exact expression for \(p(t)\). The small time, \(t \ll \tau_z\), asymptotic behaviour is

\[ p(t) \approx \left( 1 - \frac{\tilde{\tau}}{\sqrt{\tau_z}} + \ldots \right). \] (A3)

**APPENDIX B: ASYMPTOTICS OF \(Q(T)\)**

To calculate \(q(t)\) we assume that the initial elevation of the vapour atom is uniformly distributed between the liquid surface and an upper cutoff \(L\). Ultimately we shall take \(L \to \infty\). Aside from this we share the assumptions (ii), (iii) and (iv) of Appendix A.

Hence we obtain

\[ Q(t) = \langle A(z) A(z+vt) \rangle = \frac{1}{L} \frac{M}{2 \pi kT} \int_0^L dz \int_{-\infty}^\infty dv \exp \left( -\frac{M v^2}{2kT} \right) \langle A(z) A(z+vt) \rangle. \] (B1)

Using eq (8) for \(A(z)\) and rescaling we obtain

\[ Q(t) = \frac{16 \lambda^2}{\sqrt{\pi} b L} \int_0^\infty du e^{-u^2} e^{-2u\tilde{\tau}} \int_0^{L/b} d\zeta e^{-4\zeta^2 (\zeta + u\tilde{\tau})^2} + \frac{16 \lambda^2}{\sqrt{\pi} b L} \int_{-\infty}^0 du e^{-u^2} e^{-2u\tilde{\tau}} \int_{-u\tilde{\tau}}^{L/b} d\zeta e^{-4\zeta^2 (\zeta + u\tilde{\tau})^2} \] (B2)

Here \(\tilde{\tau} = t/\tau_z\), \(u = v/\sqrt{(2kT)/M}\) and \(\zeta = z/b\).

Performing the \(\zeta\) integral yields

\[ Q(t) = \frac{\lambda^2}{4 \sqrt{\pi} b L} \int_0^\infty du e^{-u^2} e^{-2u\tilde{\tau}} (3 + 6u\tilde{\tau} + 4u^2\tilde{\tau}^2) \] (B3)

and hence the normalization

\[ R(t) = Q(0) = \frac{3 \lambda^2}{8 b L}. \] (B4)

The exact reduction factor is then

\[ q(t) = \frac{Q(t)}{R(t)} = \frac{2}{3 \sqrt{\pi}} \int_0^\infty du e^{-u^2} e^{-2u\tilde{\tau}} (3 + 6u\tilde{\tau} + 4u^2\tilde{\tau}^2) \] (B5)

with the small time, \(t \ll \tau_z\), asymptotic behaviour
\[ q(t) \approx 1 - \frac{\tau^2}{3} + \frac{\tau^4}{2} + \ldots \] (B6)

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Figure 1. Lineshape crossover: The conductance of degenerate electrons is plotted in units of $e^2/h$ as a function of magnetic field. The damping rate is held fixed ($B_1 + B_3 = B_\phi$). Different curves correspond to different proportions of $\tau_1$ and $\tau_3$ damping, measured by the ratio $B_1/B_\phi$. $B_1/B_\phi = 1$ corresponds to pure $\tau_1$ damping; this is the conventional weak-localization lineshape. $B_1/B_\phi = 0$ corresponds to pure $\tau_3$ damping. The magnetic field is in arbitrary units such that $B_\phi = 1$. We take $B_e = 200$. 
Figure 2. Zero-field crossover: The change in conductance at zero field as the system varies from pure $\tau_1$ damping to pure $\tau_3$ damping for a fixed total damping rate. The horizontal axis is $B_3/B_1 = \tau_1/\tau_3$. $B_3/B_1 = 0$ corresponds to pure $\tau_1$ damping; $B_3/B_1 \to \infty$ corresponds to pure $\tau_3$ damping. The vertical axis is the conductance in units of $e^2/h$; the conductance at $B_3/B_1 = 0$ has been subtracted.