Abelian and Non-Abelian Monopole Configuration in Condensed Matters

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We discuss the Abelian and non-Abelian monopoles which could exist in condensed matters. We show how the Dirac monopole can be regularized by the charge screening, and argue that the Dirac monopole of mass of hundred meVs could exist in dielectric condensed matters. Moreover, we generalize this result to non-Abelian condensed matters to show the existence of the non-Abelian monopole configuration in two-gap condensed matters, and present explicit monopole solutions.

Keywords: topological objects in condensed matters, regularization of Dirac monopole in condensed matters, non-Abelian monopole in two-gap condensed matters

I. INTRODUCTION

Topological objects have played important roles in physics. Late nineteen century Kelvin first suggested that the atoms could be viewed as knots whose stability could be explained by the topology [1]. Later Dirac introduced the Dirac monopole based on the non-trivial U(1) topology [2]. With this topology and topological objects have played fundamental roles in physics. In particular, the monopole has become an obsession in theoretical and experimental physics [3–11].

After the Dirac monopole, we have had the Wu-Yang monopole, the ’tHooft-Polyakov monopole, and grand unification monopole [3–7]. But the most realistic monopole which could exist in nature is the electroweak monopole in the standard model which can be viewed as a hybrid between the ’tHooft-Polyakov monopole and Dirac monopole [12–14]. The importance of this monopole comes from the following facts. First, it appears as the electroweak generalization of Dirac monopole. So it is this monopole, not the Dirac monopole, which should exist in nature. Second, unlike the Dirac monopole which is optional, the electroweak monopole must exist if the standard model is correct. This means that the discovery of this monopole, not the Higgs particle, should be viewed as the final test of the standard model.

Moreover, it has deep implications in cosmology. As the only heavy and stable particle in the early universe it could generate the primordial black holes which could account for the dark matter, and become the seeds of the large scale structures of the universe. Most importantly, when discovered, it will become the first magnetically charged and stable topological elementary particle in the history of physics. For this reason MoEDAL and ATLAS at LHC, IceCube at south pole, and similar experiments are actively searching for the monopole [9–11].

Topological objects have also played important roles in condensed matter physics. The best known topological object in condensed matter is the Abrikosov vortex in superconductor made of quantized magnetic flux, which comes from the π1(S^1) topology of the Abelian U(1) gauge theory [15]. Similar string-like topological objects could also exist in non-Abelian two-component Bose-Einstein condensates and ^3He superfluids [16, 17]. Moreover, recently it has been shown that two-gap superconductors could have non-Abrikosov type magnetic vortices, D-type or N-type, with integer or fractional magnetic flux [18].

While these are certainly interesting topological objects, a more interesting object should be the monopole which has the π2(S^2) topology. There have been serious efforts to search for monopole-like objects in the condensed matter systems. One is to identify the monopole as emergent excitations of point charges of dipole moment in spin ice [19]. Another is to identify the monopole as point-like topological defects in spinor Bose-Einstein condensates [20]. A third approach is to treat angulon quasi-particles of rotating molecules in superfluid helium as an effective non-Abelian monopole [21].

However, these monopoles are collective phenomena

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exhibiting the $\pi_2(S^2)$ topology, which may not be viewed as a genuine magnetic monopole. The purpose of this paper is to argue that a genuine magnetic monopole could exist in dielectric and/or two-gap condensed matters, at least theoretically. We first show that a regularized Dirac monopole of the mass hundreds meV could exist in ordinary condensed matters. Next, we discuss under what condition a monopole-like configuration can exist in non-Abelian condensed matters, and construct explicit monopole solutions which could exist in a realistic two-gap condensed matter.

In specific, we show how the charge screening in dielectric condensed matters can be applied to make the energy of Dirac monopole finite. Moreover, we generalize the Dirac monopole to non-Abelian monopole and present explicit solutions of monopole and dyon which carry a Dirac-type singularity in two-gap condensed matters, and show that this singularity can be regularized by the same mechanism, by a non-vacuum electric permittivity that mimics the quantum correction of charge renormalization by virtual electron-positron pair production.

To generalize the Dirac monopole to a non-Abelian monopole, we need to understand the relation between the Abelian and non-Abelian gauge theories. A best way to understand this is the Abelian decomposition of non-Abelian gauge theory \cite{22, 23}. It decomposes the non-Abelian gauge potential into the restricted Abelian part and the non-Abelian valence part gauge independence. Moreover, it shows that the restricted Abelian gauge potential is made of two parts, the naive Abelian (electric) potential and the topological monopole (magnetic) potential.

More importantly, the restricted potential retains the full non-Abelian gaugedegrees of freedom and thus inherits the full non-Abelian topology of the theory, in spite of the fact that it describes the Abelian sub-dynamics of the non-Abelian gauge theory. This means that we can construct the restricted gauge theory with the restricted gauge potential which has the full non-Abelian gauge symmetry but describes the simpler Abelian sub-dynamics of the non-Abelian gauge theory \cite{22, 23}.

In comparison, the non-Abelian valence potential transforms covariantly under the gauge transformation and thus play the role of a gauge covariant source of the restricted potential. Thus the Abelian decomposition teaches us how the Abelian structure is embedded in the non-Abelian gauge theory and shows how to separates the Abelian structure from the non-Abelian gauge theory gauge independently. Moreover, it allows us to express the non-Abelian gauge theory in an Abelian form which is very useful for the condensed matter physics, as we will show in the following.

The Abelian decomposition has played a crucial role for us to clarify the complicated non-Abelian dynamics in QCD \cite{24, 25}. It decomposes the gluon to the color neutral neuron and colored chromon, and separates the topological monopole potential gauge independently. Moreover, it allows us to prove the monopole condensation necessary for the color confinement in QCD.

In this paper, we show that the Abelian decomposition can also play important role in condensed matter physics. This is because it separates the topological structure of the non-Abelian gauge theory gauge independently, and allows us to construct the topological objects more easily. Moreover, it shows how the Abelian gauge theory is related to the non-Abelian gauge theory by “abelianizing” the non-Abelian gauge theory gauge independently. This is very important, because condensed matters are often described by QED which is Abelian.

The paper is organised as follows. In Section II, we show how the Dirac monopole can be regularized in ordinary condensed matters by the non-trivial electromagnetic permittivity, and argue that the monopole with mass of hundreds meV could exist in ordinary condensed matters. In Section III we discuss a general framework which describes non-Abelian condensed matters and its mathematical structure, in particular, its topological structure, using the Abelian decomposition. In Section IV, we show how the singular monopole and dyon could exist in realistic two-gap condensed matters and present explicit solutions. In Section V, we show how to regularize the singular monopole and dyon solutions with the real electromagnetic permittivity which describes the electric charge screening, and show that in the Abelian limit the solution reduces to the regularized Dirac monopole. In Section VI, we compare our solutions with the monopole and dyon existing in the standard model. Finally in the last section we discuss the physical implications of our result.

**II. REGULARIZED DIRAC MONOPOLE IN ORDINARY CONDENSED MATTERS**

The topological objects in condensed matters can be of Abelian or non-Abelian. The best known topological object in Abelian gauge theory is the Dirac monopole. It is well known that the U(1) gauge theory which describes the Maxwell’s theory has no magnetic monopole. This is because the Maxwell’s equation, in particular the Bianchi identity, forbids the existence of the monopole. But in 1931 Dirac predicted the existence of the monopole generalizing the Maxwell’s equation and making the U(1) gauge theory non-trivial \cite{2}. He showed that the Maxwell’s theory can be generalized top admit the monopole, if we impose the charge quantization condition $e\gamma = 2\pi\gamma$. Since there is no explanation why the electric charges in nature are quantized, this charge quantization rule has often been used to argue the existence of the monopole. Later Schwinger generalized the monopole to dyon \cite{29}.
In spite of the huge efforts to try to discover the monopole experimentally, however, the monopole has not been discovered yet \[8\]. This was (at least) partly because there has been no information about the mass of the monopole. Since the Dirac monopole has infinite energy, there was no way to predict the mass. This has made most of the monopole experiments a blind search in the dark room, with no theoretical lead.

Now we show the existence of the regularized Dirac monopole which might have a finite mass of the order of meV in dielectric condensed matters. Consider the following Maxwell’s theory coupled to the neutral scalar field \( \rho \),

\[
\mathcal{L} = -\frac{1}{2} (\partial_\mu \rho)^2 - \frac{\lambda}{2} (\rho^2 - \rho_0^2)^2 - \frac{1}{4} \epsilon(\rho) \, F_{\mu\nu}^2, \tag{1}
\]

where we require \( \epsilon \) approaches to one asymptotically to make sure that the theory reduces to the Maxwell’s theory. We can interpret \( \rho \) as an emergent scalar field which represents the density of the polarization in condensed matters responsible for the charge screening. This is because effectively \( \epsilon \) in front of \( F_{\mu\nu} \) (when it becomes a constant) changes the gauge coupling \( \epsilon \) to the “running” coupling \( \epsilon = e/\sqrt{\epsilon} \), so that with the rescaling of \( A_\mu \) to \( A_\mu/e \), \( \epsilon \) changes to \( e/\sqrt{\epsilon} \). This makes it an ideal Lagrangian to describe the Dirac monopole in condensed matter.

To find the desired monopole solution choose the following ansatz for the Dirac monopole

\[
\rho = \rho(r), \quad A_\mu = -\frac{1}{e} (1 - \cos \theta) \partial_\mu \varphi. \tag{2}
\]

The ansatz has the well known Dirac string along the negative z-axis, but it can be gauged away and made unphysical when we make the U(1) bundle non-trivial and impose the charge quantization condition.

With this we have the equation of motion

\[
\ddot{\rho} + \frac{2}{r} \dot{\rho} = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho + \frac{e'}{2} \frac{1}{e^2} \dot{r}, \tag{3}
\]

where \( e' = d\epsilon/d\rho \). We can easily solve this with the boundary condition

\[
\rho(0) = 0, \quad \rho(\infty) = \rho_0, \tag{4}
\]

and obtain the finite energy monopole solution with

\[
\epsilon \simeq \left( \frac{\rho}{\rho_0} \right)^n. \tag{5}
\]

The solution for \( n = 6 \) is shown in Fig. 1. This tells that we can regularize the Dirac monopole, replacing the vacuum electromagnetic permittivity with a real electromagnetic permittivity. This is unexpected. As far as we understand, there has been no known way to regularize the Dirac monopole other than this. Exactly the same mechanism (the charge renormalization by the vacuum polarization) has been shown to make the mass of the electroweak monopole finite \[28\].

The monopole energy is given by

\[
E = 4\pi \int_0^\infty dr \left\{ \frac{\ddot{\epsilon}}{2e^2 r^2} + \frac{1}{2} (\dot{r}^2) \right\}, \tag{n}
\]

which, in the limit \( \lambda \) goes to zero, becomes

\[
E \simeq 0.25 \times \frac{4\pi}{e} \rho_0, \tag{7}
\]

where \( \rho_0 \) is in principle arbitrary. This has deep implication. It is natural to assume that \( \rho_0 \) in condensed matters to be of the order of the coherence length in ordinary superconductor, of the order of meV. In this case the Dirac monopole in ordinary superconductors could have mass about 1/\( \alpha \) times the coherence length, where \( \alpha \) is the fine structure constant. This is because the monopole mass is given by \( 4\pi/e\rho_0 \), while the coherence length is given by \( e\rho_0 \). This tells that the mass of the Dirac monopole in condensed matter could be of the order of hundred meV. This is really remarkable. In the following we will show that exactly the same regularized monopole could exist in two-gap condensed matters.

III. NON-ABELIAN STRUCTURE IN CONDENSED MATTERS: GENERAL FRAMEWORK

Now, we discuss the non-Abelian monopole in condensed matters. Consider a complex doublet \( \phi = (\phi_1, \phi_2) \) made of two condensates \( \phi_1 \) and \( \phi_2 \) which could represent two types of condensates similar to two different Cooper-pairs in two-gap superconductors or two types of...
states in two-component Bose-Einstein condensates [16–18]. The doublet could naturally accommodate the U(2) or SU(2)xU(1) gauge interaction described by

\[ \mathcal{L} = -|D_\mu \phi|^2 - \frac{\lambda}{2} (|\phi|^2 - \frac{\mu^2}{\lambda})^2 - \frac{1}{4} F_{\mu
u}^2 - \frac{1}{4} G_{\mu
u}^2, \]

\[ D_\mu \phi = (\partial_\mu - i \frac{g}{2} A_\mu - i \frac{g'}{2} \vec{\sigma} \cdot \vec{B}_\mu) \phi = (D_\mu - i \frac{g}{2} A_\mu) \phi, \]

\[ D_\mu = (\partial_\mu - i \frac{g'}{2} \vec{\sigma} \cdot \vec{B}_\mu) \phi, \] (8)

where \( A_\mu \) and \( \vec{B}_\mu \) are the overall U(1) and SU(2) gauge potentials, \( F_{\mu\nu} \) and \( G_{\mu\nu} \) are the corresponding field strengths, \( g \) and \( g' \) are the coupling constants.

One might wonder what is the motivation to consider the above Lagrangian. Consider first the overall U(1) gauge interaction coupled to the two different Cooper pairs \( \phi_1 \) and \( \phi_2 \). In the absence of the SU(2) gauge interaction this Lagrangian describes a very interesting non-Abelian two-gap superconductor which admits two types of non-Abelian Abrikosov vortex [18]. It has D-type magnetic vortex which has no concentration of the condensate at the core and N-type magnetic vortex which has a non-trivial profile of the condensate at the core. They are described by the non-Abelian topology \( \pi_2(S^2) \) and \( \pi_1(S^1) \), as well as the Abelian topology \( \pi_1(S^1) \). And they can carry both integer and fractional magnetic flux [18]. This is because the doublet \( \phi \) naturally introduce the non-Abelian structure. This justifies the overall U(1) gauge interaction.

Moreover, in the presence of the doublet \( \phi \) it becomes natural (and logical) to introduce the interaction between \( \phi_1 \) and \( \phi_2 \). This justifies the SU(2) gauge interaction in [8]. But one might still wonder how can one justify the two off-diagonal charged gauge bosons, since condensed matters do not appear to have a place for them. Actually, in condensed matters the two spin-half electrons could form a charged spin-one bound state which could be represented by the off-diagonal gauge bosons, so that they could play important role in condensed matters. In any case, we could always remove the off-diagonal gauge bosons if necessary, as we will see later. With this understanding we can say that the above U(2) gauge theory could describe a wide class of non-Abelian condensed matters.

To proceed we express the complex \( \phi \) with the scalar field \( \rho \) and the unit doublet \( \xi \) by

\[ \phi = \frac{1}{\sqrt{2}} \rho \xi, \quad (\xi_1 \xi = 1), \] (9)

and have

\[ \mathcal{L} = -\frac{1}{2}(\partial_\mu \rho)^2 - \frac{\rho^2}{2} |D_\mu \xi|^2 - \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2 - \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{4} G_{\mu\nu}^2, \] (10)

where \( \rho_0 = \sqrt{2 \mu^2 / \lambda} \) is the vacuum expectation value of the complex doublet field. To simplify this further we need the Abelian decomposition of the Lagrangian [8, 22, 23].

Consider the SU(2) gauge field \( \vec{B}_\mu \) first. Let \( (\hat{n}_1, \hat{n}_2, \hat{n}_3 = \hat{n}) \) be an arbitrary right-handed orthonormal SU(2) basis, and choose \( \hat{n} \) to be the Abelian direction at each space-time point. Imposing the condition on the gauge potential \( \vec{B}_\mu \),

\[ D_\mu \hat{n} = 0, \] (11)

we can project out the restricted potential \( \hat{B}_\mu \) which describes the Abelian subdynamics of the non-Abelian gauge theory [22, 23]

\[ \hat{B}_\mu = \hat{B}_\mu + \hat{C}_\mu, \]

\[ \hat{B}_\mu = B_\mu \hat{n}, \quad (B_\mu = \hat{n} \cdot \vec{B}_\mu), \quad \hat{C}_\mu = \frac{1}{g} \hat{n} \times \partial_\mu \hat{n}. \] (12)

Notice that the restricted potential is precisely the potential which leaves \( \hat{n} \) invariant under parallel transport (which makes \( \hat{n} \) covariantly constant). Remarkably it has a dual structure, made of two potentials \( \hat{B}_\mu \) and \( \hat{C}_\mu \).

With this we obtain the gauge independent Abelian decomposition of the SU(2) gauge field adding the valence part \( \vec{W}_\mu \) which was excluded by the isometry [22, 23]

\[ \vec{B}_\mu = \hat{B}_\mu + \vec{W}_\mu, \quad \vec{W}_\mu = W_\mu^1 \hat{n}_1 + W_\mu^2 \hat{n}_2. \] (13)

Under the (infinitesimal) gauge transformation

\[ \delta \hat{B}_\mu = \frac{1}{g'} D_\mu \hat{\alpha}, \quad \delta \hat{n} = -\hat{\alpha} \times \hat{n}, \] (14)

we have

\[ \delta B_\mu = \frac{1}{g'} \hat{n} \cdot \partial_\mu \hat{\alpha}, \]

\[ \delta \hat{B}_\mu = \frac{1}{g'} \hat{\alpha} \times \vec{W}_\mu. \] (15)

This tells that \( \hat{B}_\mu \) by itself describes an SU(2) connection which enjoys the full SU(2) gauge degrees of freedom. Furthermore the valence potential \( \vec{W}_\mu \) forms a gauge covariant vector field. But what is really remarkable is that this decomposition is gauge independent. Once \( \hat{n} \) is chosen, the decomposition follows automatically, regardless of the choice of gauge.

The restricted field strength \( \hat{G}_{\mu\nu} \) inherits the dual structure of \( \hat{B}_\mu \), which can also be described by two
Abelian potentials \( B_\mu \) and \( C_\mu \),

\[
\begin{align*}
\tilde{G}_{\mu\nu} &= \partial_\mu \tilde{B}_\nu - \partial_\nu \tilde{B}_\mu + g \tilde{B}_\mu \times \tilde{B}_\nu = G_{\mu\nu}^\prime \hat{n}, \\
G_{\mu\nu} &= G_{\mu\nu}^\prime + \mu B_\mu B_\nu - \partial_\mu \partial_\nu B_\mu, \\
G_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\
H_{\mu\nu} &= -\frac{1}{g} \hat{n} \cdot (\partial_\mu \hat{n} \times \partial_\nu \hat{n}) = \partial_\mu C_\nu - \partial_\nu C_\mu, \\
C_\mu &= -\frac{2i}{g} \xi \gamma^\mu \xi = -\frac{1}{g} \hat{n}_1 \cdot \partial_\mu \hat{n}_2, \\
B_\mu &= B_\mu + C_\mu.
\end{align*}
\]

Notice that the potential \( C_\mu \) for \( H_{\mu\nu} \) is determined uniquely up to the \( U(1) \) gauge freedom which leaves \( \hat{n} \) invariant.

To understand the meaning of \( C_\mu \), let

\[
\xi = \exp(-i\gamma) \left( \begin{array}{c} \sin \frac{\alpha}{2} \\
-\cos \frac{\alpha}{2} \end{array} \right), \\
\hat{n} = -\xi^\dagger \partial_\xi = \left( \begin{array}{c} \sin \alpha \cos \beta \\
\sin \alpha \sin \beta \end{array} \right),
\]

\[
C_\mu = \frac{2i}{g} \xi^\dagger \partial_\xi = -\frac{1}{g} \hat{n}_1 \cdot \partial_\mu \hat{n}_2, \quad B_\mu = B_\mu + C_\mu.
\]

But here we can always put \( \gamma = 0 \) without loss of generality, because \( [8] \) has the overall \( U(1) \) gauge degrees of freedom. With this we have

\[
\begin{align*}
C_\mu &= -\frac{1}{g^\prime} (1 - \cos \alpha) \partial_\mu \beta, \\
\tilde{C}_\mu &= -\frac{1}{g^\prime} \hat{n} \times \partial_\mu \hat{n} = \frac{1}{g^\prime} \left( \hat{n}_1 \sin \alpha \partial_\mu \beta - \hat{n}_2 \partial_\mu \alpha \right), \\
\hat{n}_1 &= \left( \begin{array}{c} \cos \alpha \cos \beta \\
\cos \alpha \sin \beta \end{array} \right), \\
\hat{n}_2 &= \left( \begin{array}{c} -\sin \beta \\
\cos \beta \end{array} \right).
\end{align*}
\]

So when \( \hat{n} = \hat{r} \), the potential \( C_\mu \) describes the Dirac monopole and \( \tilde{C}_\mu \) describes the Wu-Yang monopole \([3, 6]\). This justifies us to call \( B_\mu \) and \( C_\mu \) the electric and magnetic potential. Moreover, this exercise tells that physically the Wu-Yang monopole is nothing but the non-Abelian realization of the Dirac monopole.

Since \( \tilde{B}_\mu \) has the full SU(2) gauge degrees of freedom, it inherits all topological characteristics of the original non-Abelian potential. First, it has the non-Abelian monopole described by the \( \pi_3(S^2) \) topology of \( \hat{n} \). Second, it retains the topologically distinct vacua characterized by the Hopf invariant \( \pi_3(S^2) \simeq \pi_3(S^3) \) of \( \hat{n} \) \([20, 31]\). This means that we can construct the restricted gauge theory which has the full SU(2) gauge invariance with the restricted potential \( \tilde{B}_\mu \).

Moreover, with \([13]\) we have

\[
\begin{align*}
\tilde{G}_{\mu\nu} &= \tilde{G}_{\mu\nu}^\prime + \tilde{D}_\mu \tilde{W}_\nu - \tilde{D}_\nu \tilde{W}_\mu + g' \tilde{W}_\mu \times \tilde{W}_\nu, \\
\tilde{D}_\mu &= \partial_\mu + g' \tilde{B}_\mu \times, \quad \tilde{D}_\mu = \partial_\mu + g \tilde{B}_\mu \times.
\end{align*}
\]

so that the SU(2) Lagrangian is decomposed into the restricted part and the valence part gauge independently,

\[
\begin{align*}
\mathcal{L}_{\text{SU}(2)} &= -\frac{1}{4} \tilde{G}_{\mu\nu}^2 - \frac{1}{4} \tilde{G}_{\mu\nu} - \frac{1}{4} (\tilde{D}_\mu \tilde{W}_\nu - \tilde{D}_\nu \tilde{W}_\mu)^2 \\
&= -g' \tilde{G}_{\mu\nu}^2 \cdot (\tilde{W}_\mu \times \tilde{W}_\nu) - \frac{g'^2}{4} (\tilde{W}_\mu \times \tilde{W}_\nu)^2.
\end{align*}
\]

This is the Abelian decomposition of the SU(2) gauge theory known as the Cho decomposition, Cho-Duan-Ge (CDG) decomposition, or Cho-Faddeev-Niemi (CFN) decomposition \([32, 33]\). In this form, the theory can be interpreted as the restricted gauge theory which has the valence part as a gauge covariant source.

An important advantage of the Abelian decomposition is that it can actually "abelianize" the non-Abelian gauge theory gauge independently \([22, 23]\). Indeed with \( W_n = (W_n^a + iW_n^3)/\sqrt{2} \), we can put \([20]\) in the Abelian form

\[
\begin{align*}
\mathcal{L}_{\text{SU}(2)} &= -\frac{1}{4} G_{\mu\nu}^2 - \frac{1}{4} (\tilde{D}_\mu \tilde{W}_\nu - \tilde{D}_\nu \tilde{W}_\mu)^2 \\
&+ i g' G_{\mu\nu} \tilde{W}_{\mu}^{\tau} \tilde{W}_{\nu} + \frac{g'^2}{4} (\tilde{W}_{\mu}^{\tau} \tilde{W}_{\nu} - \tilde{W}_{\nu} \tilde{W}_{\mu}^{\tau})^2, \\
D_\mu &= \partial_\mu + ig' B'_\mu.
\end{align*}
\]

One might wonder how the non-Abelian structure disappears in this Abelianization. Actually the non-Abelian structure has not disappeared but hidden. To see this notice that the potential \( B'_\mu \) in the Abelian formalism is dual, given by the sum of the electric and magnetic potentials \( B_\mu \) and \( C_\mu \). Clearly \( C_\mu \) represents the topological degrees of the non-Abelian symmetry which does not exist in the naive Abelianization that one obtains by fixing the gauge, choosing \( \hat{n} = (0, 0, 1) \) \([22, 23]\).

The Abelian decomposition has played a crucial role in QCD for us to prove the monopole condensation create the mass gap to generate the color confinement \([27, 28]\). But now it must be clear that it could also play an important role in condensed matter physics, because it teaches us how the Abelian gauge theory can be embedded in non-Abelian gauge theory and how the electromagnetic interaction can arise from the non-Abelian gauge interaction.

With the Abelian decomposition, the Lagrangian \((1)\) has two Abelian potentials, \( A_\mu \) and \( B'_\mu \), and we need to clarify the physical meaning of them. Clearly we can identify \( A_\mu \) as the electromagnetic potential if the two condensates \( \phi_1 \) and \( \phi_2 \) have the same charge, but \( B'_\mu \) as the electromagnetic potential if they have opposite charge. However, since this depends on what kind of materials we have at hand, it is better to identify the electromagnetic potential to be a linear combination of two potentials.

For this reason we define the electromagnetic potential \( A_{(\text{em})}^\mu \) and another potential \( X_\mu \) with the mixing an-
ggle θ by

\[
(A_{\mu}^{(\text{em})})_X/\sqrt{g^2 + g'^2} \begin{pmatrix} g' & g \\ -g & g' \end{pmatrix} (A_{\mu} B'_\mu)
\]

\[
= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} A_{\mu} \\ B'_\mu \end{pmatrix}.
\]

(22)

Notice that when θ is zero (i.e., with g=0) \(A_{\mu}\) becomes \(A_{\mu}^{(\text{em})}\), but when \(\theta\) becomes π/2 (i.e., with \(g' = 0\), \(B'_\mu\) describes the electromagnetic potential. But in general when \(gg' \neq 0\), \(A_{\mu}^{(\text{em})}\) is given by a linear combination of \(A_{\mu}\) and \(B'_\mu\).

From this we have the identity

\[
D_{\mu} \xi = -i\frac{g}{2} (B'_\mu \hat{n} + \hat{\vec{W}}_\mu) \cdot \vec{\sigma} \xi,
\]

\[
|D_{\mu} \xi|^2 = \frac{g^2}{4} (B'^2 + \hat{\vec{W}}^2),
\]

\[
|D_{\mu} \xi|^2 = \frac{g^2 + g'^2}{8} X^2 + \frac{g^2}{4} \hat{\vec{W}}^2.
\]

(23)

With this we can remove the doublet \(\phi\) completely from the Lagrangian [8] and abelianize it gauge independently in the following form

\[
\mathcal{L} = -\frac{1}{2} (\partial_{\mu} \rho)^2 - \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2 - \frac{1}{4} F_{\mu \nu}^{(\text{em})} - \frac{1}{4} \hat{\vec{W}}_{\mu \nu} - \frac{g^2}{4} (g W_{\mu}^* W_{\nu} + \frac{g^2 + g'^2}{2} X_{\mu}^2)
\]

\[
- \frac{1}{2} |(D_{\mu}^{(\text{em})} + ie g' X_{\mu}) W_{\nu} - (D_{\nu}^{(\text{em})} + ie g' X_{\nu}) W_{\mu}|^2
\]

\[
+ ie (F_{\mu \nu}^{(\text{em})} + \frac{g'}{g} X_{\mu \nu}) W_{\mu}^* W_{\nu}
\]

\[
+ \frac{g'^2}{4} (W_{\mu}^* W_{\nu} - W_{\nu}^* W_{\mu})^2,
\]

(24)

where \(D_{\mu}^{(\text{em})} = \partial_{\mu} + ie A_{\mu}^{(\text{em})}\) and \(e\) is the electric charge

\[
e = \frac{gg'}{\sqrt{g^2 + g'^2}} = g' \sin \theta = g \cos \theta.
\]

(25)

In this form the Lagrangian describes QED coupled to two gauge bosons, the charged \(W_{\mu}\) boson and the neutral \(X_{\mu}\) boson, which acquire the mass

\[
M_W = \frac{g'}{2} \rho_0, \quad M_X = \frac{\sqrt{g^2 + g'^2}}{2} \rho_0,
\]

(26)

through the Higgs mechanism. The popular interpretation of this is that the gauge bosons acquire the mass by “the spontaneous symmetry breaking” of U(2) down to the electromagnetic U(1) through the Higgs vacuum. But we emphasize that here the gauge bosons acquire the mass without any symmetry breaking, spontaneous or not. All we have to do is the reparametrize the fields which does not involve any symmetry breaking.

Notice that we can easily switch off the X boson if necessary. In this case the Lagrangian reduces to

\[
\mathcal{L}_1 = -\frac{1}{2} (\partial_{\mu} \rho)^2 - \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2
\]

\[
- \frac{1}{4} F_{\mu \nu}^{(\text{em})} - \frac{1}{2} |D_{\mu}^{(\text{em})} W_{\nu} - D_{\nu}^{(\text{em})} W_{\mu}|^2
\]

\[
- \frac{g^2}{4} (W_{\mu}^* W_{\nu} + i e F_{\mu \nu}^{(\text{em})}) W_{\mu} W_{\nu}
\]

\[
+ \frac{g'^2}{4} (W_{\mu}^* W_{\nu} - W_{\nu}^* W_{\mu})^2.
\]

(27)

This describes QED coupled to a charged vector field \(W_{\mu}\) as the source whose mass comes from the Higgs mechanism. Moreover, when we switch off \(F_{\mu \nu}^{(\text{em})}\) and \(W_{\mu}\), the Lagrangian reduces to

\[
\mathcal{L}_2 = -\frac{1}{2} (\partial_{\mu} \rho)^2 - \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2
\]

\[
- \frac{1}{4} X_{\mu \nu} - \frac{g^2 + g'^2}{8} \rho^2 X_{\mu}^2.
\]

(28)

This shows that the Lagrangian [8] can describe a variety of non-Abelian condensed matters.

**IV. MONOPOLE CONFIGURATION IN TWO-GAP CONDENSED MATTERS**

It is well known that QED has the Dirac monopole, and SU(2) gauge theory has the ’tHooft-Polyakov monopole [23][4]. If so, it is quite likely that the above Lagrangian has a monopole-like topological object. In this section we show that indeed [8] has a very interesting monopole which can be generalized to a dyon. To show this we choose the following ansatz in the spherical coordinates \((t, r, \theta, \varphi)\),

\[
\rho = \rho(r), \quad \xi = i \begin{pmatrix} \sin(\theta/2) e^{-i\varphi} \\ -\cos(\theta/2) \end{pmatrix},
\]

\[
A_{\mu} = \frac{1}{g} A(r) \partial_{\mu} \hat{t} - \frac{1}{g} (1 - \cos \theta) \partial_{\mu} \varphi,
\]

\[
\vec{B}_{\mu} = \frac{1}{g} B(r) \partial_{t} \hat{r} + \frac{1}{g} (f(r) - 1) \hat{r} \times \partial_{\mu} \hat{r},
\]

\[
\hat{n} = -\xi \slashed{\partial} \xi = \hat{r}.
\]

(29)

It has the following features. First, when \(A(r) = B(r) = 0\), \(A_{\mu}\) describes the Dirac-type Abelian monopole and \(\vec{B}_{\mu}\) describes the ’tHooft-Polyakov monopole. So the ansatz is a hybrid between Dirac and ’tHooft-Polyakov. Second, it has the Coulombic part \(A(r)\) and \(\vec{B}(r)\) which could add the electric charge to the monopole to make it a dyon. To see this, notice that in terms of the physical fields the
ansatz \(29\) can be expressed by

\[
A_{\mu}^{(\text{em})} = e \left( \frac{A(r)}{g^2} + \frac{B(r)}{g'^2} \right) \partial_{\mu} t - \frac{1}{e} (1 - \cos \theta) \partial_{\mu} \varphi, \\
X_{\mu} = \frac{e}{gg'} (B(r) - A(r)) \partial_{\mu} t, \\
W_{\mu} = \frac{i}{g' \sqrt{2}} e^{i \varphi} (\partial_{\mu} \theta + i \sin \theta \partial_{\mu} \varphi). \tag{30}
\]

This clearly shows that the ansatz is for a real electromagnetic dyon.

The ansatz reduces the equations of motion to

\[
\ddot{\rho} + \frac{2}{r} \dot{\rho} - \frac{f^2}{r^2} \rho + \frac{1}{4} (B - A)^2 \rho = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho, \\
\ddot{f} - \frac{f^2 - 1}{r^2} f = \left( \frac{g^2}{4} \rho^2 - B^2 \right) f, \\
\ddot{A} + \frac{2}{r} A = - \frac{g^2}{4} \rho^2 (B - A), \\
B + \frac{2}{r} B - \frac{2 f^2}{r^2} B = \frac{g^2}{4} \rho^2 (B - A). \tag{31}
\]

Obviously this has a trivial solution

\[
\rho = \rho_0 = \sqrt{2 \mu^2 / X}, \quad f = 0, \quad A = B = 0, \tag{32}
\]

which describes the Dirac type point monopole

\[
A_{\mu}^{(\text{em})} = - \frac{1}{e} (1 - \cos \theta) \partial_{\mu} \varphi. \tag{33}
\]

This monopole has two remarkable features. First, this is not the Dirac’s monopole. It has the electric charge \(4\pi/e\), not \(2\pi/e\) \[12\]. Second, this monopole naturally admits a non-trivial dressing of W boson. Indeed we can integrate \[31\] with \(A = B = 0\) and the boundary conditions

\[
\rho(0) = 0, \quad \rho(\infty) = \rho_0, \\
\dot{f}(0) = 1, \quad f(\infty) = 0. \tag{34}
\]

This gives the dressed monopole solution shown in Fig. \ref{fig:2}. Clearly this can be viewed as a hybrid between the Dirac monopole and the t’Hooft-Polyakov monopole.

Moreover, we can extend it to a dyon solution with the following boundary conditions

\[
\rho(0) = 0, \quad f(0) = 1, \quad A(0) = a_0, \quad B(0) = 0, \\
\rho(\infty) = \rho_0, \quad f(\infty) = 0, \quad A(\infty) = B(\infty) = A_0, \tag{35}
\]

and can show that the equation \[31\] admits a family of solutions labeled by the real parameter \(A_0\) lying in the range

\[
0 < A_0 < \min \left( e \rho_0, \frac{g'}{2} \rho_0 \right). \tag{36}
\]

Near the origin it has the following behavior,

\[
\rho \simeq \alpha_1 r^{\delta_-}, \quad f \simeq 1 + \beta_1 r^2, \\
A \simeq a_0 + a_1 r^{2 \delta_+}, \quad B \simeq b_1 r, \tag{37}
\]

where \(\delta_{\pm} = (\sqrt{3} \pm 1)/2\). Asymptotically we have

\[
\rho \simeq \rho_0 + \rho_1 \exp(-\sqrt{2} \mu r), \quad f \simeq f_1 \exp(-\omega r), \\
A \simeq B + A_1 \exp(-\nu r), \quad B \simeq A_0 + B_1 \exp(-\nu r), \tag{38}
\]

where \(\omega = \sqrt{(g \rho_0)^2 / 4 - A_0^2}\), and \(\nu = \sqrt{(g^2 + g''^2) \rho_0 / 2}\).

This tells that \(M_H, \sqrt{1 - (A_0/M_W)^2} \approx M_W\), and \(M_X\) determine the exponential damping of the Higgs boson, W boson, and X boson to their vacuum expectation values asymptotically. The dyon solution is shown in Fig. \ref{fig:3}.

The dyon has the following electromagnetic charges

\[
q_e = - \frac{8\pi}{e} \sin^2 \theta \int_0^\infty f^2 B dr = \frac{4\pi}{e} A_1, \\
q_m = \frac{4\pi}{e}. \tag{39}
\]

Moreover, the dyon equation \[31\] is invariant under the reflection

\[
A \to -A, \quad B \to -B. \tag{40}
\]
This means that, for a given magnetic charge, there are two dyon solutions which carry opposite electric charges ±εe. We can also have the anti-monopole or in general anti-dyon solution, the charge conjugate state of the dyon which has the magnetic charge qm = −4π/e.

V. REGULARIZATION OF MONOPOLE IN TWO-GAP CONDENSED MATTERS

As we have pointed out the above monopole is a hybrid between the Dirac and 'tHooft-Polyakov monopoles, so that it has an infinite energy. Of course classically there is nothing wrong with this. Nevertheless one might wonder if we can regularize the monopole to a finite energy. One might think there is no reason to regularize the monopole, because in real condensed matters we have a natural cut-off given by the atomic size which can make the monopole energy finite. This might be so, but from the theoretical point of view, it would be nice to make the monopole energy finite without any cut-off. Now, we show that the quantum correction at short distance could make the monopole energy finite.

Consider the following effective Lagrangian

$$
\mathcal{L} = -\frac{1}{2} (\partial_\mu \Theta \rho)^2 - \frac{\lambda}{8} (\rho^2 - \rho_0^2)^2 - \frac{1}{4} \epsilon F_{\mu \nu}^{(em)} F^{\mu \nu} - \frac{1}{4} \left( g^2 W_\mu^* W_\mu + \frac{g^2 + g'^2}{2} X_\mu^2 \right) - \frac{1}{2} \left( (D_\mu^{(em)})^2 + i e \frac{g'}{g} X_\mu W_\nu - (D_\nu^{(em)})^2 + i e \frac{g'}{g} X_\nu W_\mu \right) \right| ^2
$$

$$
+ i e (F_{\mu \nu}^{(em)} + \frac{g'}{g} X_{\mu \nu}) W_\mu^* W_\nu
+ \frac{g'^2}{4} (W_\mu^* W_\nu - W_\nu^* W_\mu)^2,
$$

where ϵ is the real non-vacuum electromagnetic permittivity of the condensed matter. It retains all symmetries of the Lagrangian [24]. The only difference is that here we have ε in front of $F_{\mu \nu}^{(em)}$ which effectively changes the electromagnetic coupling constant $\epsilon$ to a running coupling $\epsilon/\sqrt{\epsilon}$.

This, with the ansatz [30], gives the following equation of motion

$$
\ddot{\rho} + \frac{2}{r} \dot{\rho} - 2 f \dot{\rho}^2 \theta = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho - \frac{1}{4} (B - A)^2 \rho
$$

$$
+ \frac{\epsilon'}{2} \left( \frac{1}{\epsilon'^2 r^4} - \epsilon^2 (\frac{\dot{A}}{g^2} + \frac{\dot{B}}{g'^2})^2 \right),
$$

$$
\ddot{f} - \frac{f^2 - 1}{r^2} f = \left( \frac{g^2}{4} \rho^2 - B^2 \right) f,
$$

$$
\ddot{A} + \frac{2}{r} \dot{A} + \frac{2}{\epsilon} \rho^2 (\frac{\dot{A}}{g^2} + \frac{\dot{B}}{g'^2}) + \frac{2 e^2}{g^2} (1 - \frac{1}{\epsilon}) \frac{f^2}{r^2} B
$$

$$
= - \frac{g^2}{4} \rho^2 (B - A),
$$

$$
\ddot{B} + \frac{2}{r} \dot{B} + \frac{2}{\epsilon} \rho^2 (\frac{\dot{A}}{g^2} + \frac{\dot{B}}{g'^2}) - \frac{2 e^2}{g^2} (\frac{g'^2}{g^2} + \frac{1}{\epsilon}) \frac{f^2}{r^2} B
$$

$$
= \frac{g^2}{4} (B - A) \rho^2.
$$

Notice that if we switch off the W and X bosons, the above Lagrangian reduces to the Lagrangian [4], and the equations of motion (with $f = A = B = 0$) reduces to [4]. This confirms that the above effective Lagrangian could be interpreted as a non-Abelian generalization of [4].

To integrate it out and find a finite energy solution, we have to choose a proper boundary condition. To do this notice that with

$$
\epsilon = \epsilon_0 + \epsilon_1, \quad \epsilon_0 = \frac{g^2}{g^2 + g'^2}, \quad \epsilon_1 \simeq \left( \frac{\rho}{\rho_0} \right)^n,
$$

where g and g' are the W and X bosons of the Lagrangian. The only difference is that here we have ε in front of $F_{\mu \nu}^{(em)}$ which effectively changes the electromagnetic coupling constant $\epsilon$ to a running coupling $\epsilon/\sqrt{\epsilon}$. This, with the ansatz [30], gives the following equation of motion

$$
\ddot{\rho} + \frac{2}{r} \dot{\rho} - 2 f \dot{\rho}^2 \theta = \frac{\lambda}{2} (\rho^2 - \rho_0^2) \rho - \frac{1}{4} (B - A)^2 \rho
$$

$$
+ \frac{\epsilon'}{2} \left( \frac{1}{\epsilon'^2 r^4} - \epsilon^2 (\frac{\dot{A}}{g^2} + \frac{\dot{B}}{g'^2})^2 \right),
$$

$$
\ddot{f} - \frac{f^2 - 1}{r^2} f = \left( \frac{g^2}{4} \rho^2 - B^2 \right) f,
$$

$$
\ddot{A} + \frac{2}{r} \dot{A} + \frac{2}{\epsilon} \rho^2 (\frac{\dot{A}}{g^2} + \frac{\dot{B}}{g'^2}) + \frac{2 e^2}{g^2} (1 - \frac{1}{\epsilon}) \frac{f^2}{r^2} B
$$

$$
= - \frac{g^2}{4} \rho^2 (B - A),
$$

$$
\ddot{B} + \frac{2}{r} \dot{B} + \frac{2}{\epsilon} \rho^2 (\frac{\dot{A}}{g^2} + \frac{\dot{B}}{g'^2}) - \frac{2 e^2}{g^2} (\frac{g'^2}{g^2} + \frac{1}{\epsilon}) \frac{f^2}{r^2} B
$$

$$
= \frac{g^2}{4} (B - A) \rho^2.
$$
the energy is expressed by
\[
E = 4\pi \int_0^\infty dr \left\{ \frac{1}{2e^2r^2} \left( \epsilon_0(f^2 - 1)^2 + \epsilon_1 \right) + \frac{1}{2}(r\dot{\rho})^2 \right. \\
+ \frac{\lambda r^2}{8} \left( \rho^2 - \rho_0^2 \right)^2 + \frac{1}{g^2} f^2 + \frac{1}{4} f^2 \rho^2 + \frac{f^2A^2}{g^2} \\
+ \frac{r^2(\dot{A} - \dot{B})^2}{2(g^2 + g'^2)} + \frac{\epsilon e^2 r^2}{2} \left( \frac{\dot{A}}{g^2} + \frac{\dot{B}}{g'^2} \right)^2 \left. \right\}, \tag{44}
\]

This tells that we can make the energy finite imposing the boundary condition \( f(0) = 1 \), when \( n > 2 \). In other words we can regularize the monopole with the real electromagnetic permittivity \( \epsilon \) making \( \epsilon(0) \) finite.

To see that this regularization works, consider the monopole solution first. With \( A = B = 0 \) we can integrate \( \text{[42]} \) with the boundary condition \( f(0) = 1 \), and obtain the monopole solution shown in Fig. \( \text{[4]} \). We can generalize this to the dyon solution solving \( \text{[42]} \), and obtain the finite energy dyon solution shown in Fig. \( \text{[5]} \).

This confirms that we can indeed regularize monopole and dyon solutions with the real electromagnetic permittivity. Notice here that a non-vanishing value of \( \epsilon \) at the origin plays an essential role to yield the finite energy solution.

One might ask if we can have a regularized monopole solution without \( f \) (without the W boson). This is a relevant question because many of real condensed matters may not have the W boson (the charged spin-one two-electron bound state). Clearly this is possible, because with \( f = A = B = 0 \) the equation \( \text{[42]} \) reduces to \( \text{[3]} \) which has the regularized Dirac monopole solution shown in Fig. \( \text{[1]} \). The only thing is that here we must choose \( \epsilon = (\rho/\rho_0)^n \) to make the energy \( \text{[44]} \) finite. This tells that we can actually regularize the monopole without the W boson with the real electromagnetic permittivity. Moreover, this reconfirms that the above monopole is the non-Abelian generalization of the Dirac monopole.

One might try to generalize this to a dyon solution. But with \( f = 0 \), we find that \( \text{[42]} \) has no solution with non-trivial \( A \) and \( B \). The results in this section confirm that we can regularize the monopole in condensed matters replacing the vacuum electric permittivity with a real electric permittivity, even in the absence of the W boson. Clearly the results enhance the possibility for a monopole to exist in real superconductors.

VI. COMPARISON WITH ELECTROWEAK MONOPOLE

One might have noticed that the monopole and dyon solutions discussed in Sections III and IV look very similar to the electroweak monopole and dyon solutions in the standard model as the Cho-Maison monopole and dyon \( \text{[12,14,30,38]} \). In fact we can easily confirm that mathematically they are identical, so that formally there is exactly one to one correspondence between the Lagrangian \( \text{[3]} \) and the Weinberg-Salam Lagrangian. The mixing angle \( \theta \) in \( \text{[22]} \) corresponds to the Weinberg angle, the W boson and X boson correspond to the W boson and Z boson in the standard model. So, the above monopole and dyon correspond exactly to the electroweak monopole and dyon in the standard model \( \text{[12,13,30,38]} \).

From the physical point of view, however, we emphasize that the two Lagrangians describe completely different physics. The standard model which unifies the electromagnetic interaction with the weak interaction is a fundamental theory of nature. The coupling constants \( g \) and \( g' \) in the standard model represent the fundamental constants which determine the electric charge and the Weinberg angle. Moreover, the Higgs particle, W boson, and Z bosons are the elementary particles of nature. And the Higgs vacuum \( \rho_0 \) sets the electroweak scale of the order of 100 GeV.

On the other hand, our Lagrangian \( \text{[3]} \) is an effective Lagrangian describing a low energy physics of condensed matters, not a fundamental interaction of nature. So here the two coupling constants and the mixing angle in general depend on the type of the condensed matter, so that they should not be viewed as fundamental constants of physics. Moreover, all fields here (except the photon) should be regarded as not elementary but emergent fields, so that the W and X bosons should be interpreted as composite (pseudo) particles. In particular, the Higgs doublet here describes the two condensates of the non-Abelian condensed matters, which has a typical energy scale of several meV.

Consequently, the two monopoles have totally different meanings. The electroweak monopole is a fundamental particle which exists in the standard model. So, when discovered, the monopole will be viewed as the first absolutely stable topological elementary particle in the history of physics. On the other hand, the above monopole in condensed matters is not an elementary particle, but an emergent particle existing in condensed matters. Hence, it may not be stable, even though it is topological.

This can be understood as follows. Consider the Abrikosov vortex. We can create it applying magnetic field on the superconductor. But it is not fundamental nor stable, although it is topological. When we switch off the magnetic field, it disappears. The monopole in condensed matters here should be similar. We could possibly create it imposing the monopole topology by brute force with an external magnetic field, but when we switch off the magnetic field, it probably will disappear.

Another difference is the mass and size of the monopole. The mass of the electroweak monopole is ex-
pected to be about 4 to 10 TeV, although the mass of
the other electroweak particles (W boson, Z boson, and
Higgs particle) is of the order of 100 GeV. This is because
basically the mass of the monopole comes from the same
Higgs mechanism which makes the W boson massive, ex-
cept that the coupling is magnetic (not e but 4π/e). This
means that the mass of the monopole should be heavier
than that of the W boson by the factor 1/α. This makes
the monopole mass hundred times heavier than the W
boson mass, around 10 TeV [13] 36 38.

The same logic applies to the monopole in condensed
matter. It is known that the scale of coherence length
of ordinary superconductors is roughly of the order of
(inverse of) meV, meaning that the mass of the W
boson, X boson, and Higgs scalar in condensed matters
are of the order of meV. This implies that the mass of
the monopole in condensed matters should be roughly
100 meV, namely, 1/α times heavier than the mass of
the Higgs scalar in superconductors. As for the size, the
size of the electroweak monopole is set by the electroweak
scale, the inverse of 100 GeV [12] [14], while the size of
the monopole in condensed matters should be of the order
of inverse of O(1) meV. This tells that the two monopoles
are very different, in energy scale by the factor of 10 13,
and in volume scale by the factor of 10 39. However, it is a
remarkable beauty that the mathematically same theory
can describe totally different physics.

VII. DISCUSSIONS

In this paper we have discussed the Abelian and
non-Abelian monopoles which could exist in condensed
matters. In particular, we have shown how the non-
vacuum electromagnetic permittivity could regularize the
Dirac monopole in dielectric condensed matters, and how
this regularization could allow the monopole with mass
around hundreds meV in ordinary condensed matters.
Moreover, we have shown how we can generalize this
Abelian monopole to a non-Abelian monopole in two-gap
condensed matters.

In doing this we have shown how a non-Abelian struc-
ture can emerge in multi-component condensed matters,
and how it can be reduced to an Abelian structure with
the Abelian decomposition. The necessary condition for
the non-Abelian structure in condensed matter is the ex-
istence of two or more independent bases, such as multi-
gap or multi-component states, which can be treated as a
non-Abelian multiplet. Once this requirement is fulfilled,
we can naturally introduce the non-Abelian gauge inter-
taction to the condensed matters. This condition is natu-
urally satisfied in a wide class of condensed matters, e.g.,
two-gap superconductors, two-component Bose-Einstein
condensates, and cold atoms with two dark states, etc.

Does this mean that we can have the monopole in
condensed matters? Definitely yes. But as we have em-
phasized in Section VI, these monopoles should be re-
garded as emergent objects, not elementary objects. So
it is unlikely the monopole exist in condensed matters
naturally. Just as we can create the Abrikosov vortex in
type II superconductor, we must create them by brute
force. Experimentally, this may not be easy, but our
analysis tells that it is not impossible. We hope that
our work could stimulate the search for the monopole in
condensed matters.

At this point one might wonder if the Lagrangian could
also describe a two-gap superconductor. This is a
very interesting question. As it is, the Lagrangian is
not likely to describe the superconductor, because the
photon remains massless in . On the other hand we
have already noticed that, when we remove the SU(2)
gauge interaction in the Lagrangian, it describes a
generalized Landau-Ginzburg theory of two-gap super-
conductor which has many interesting features. This
strongly implies that the Lagrangian, with some modifi-
cation, has a potential to describe a two-gap supercon-
ductor. But we need more discussion to show exactly
under what condition could describe a realistic two-
gap superconductor. This matter will be discussed in a
separate publication.

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