OPEN MANIFOLDS WITH ASYMPTOTICALLY NONNEGATIVE RICCI CURVATURE AND LARGE VOLUME GROWTH

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Abstract. In this paper, we study the topology of complete noncompact Riemannian manifolds with asymptotically nonnegative Ricci curvature and large volume growth. We prove that they have finite topological types under some curvature decay and volume growth conditions. We also generalize it to the manifolds with $k$th asymptotically nonnegative Ricci curvature by using extensions of Abresch-Gromoll’s excess function estimate.

1. Introduction

A complete noncompact Riemannian manifold is said to have an asymptotically nonnegative Ricci curvature if there exist a base point $p$ and a positive nonincreasing function $\lambda$ such that $\int_0^{+\infty} s\lambda(s)ds < +\infty$, and the Ricci curvature of $M$ at any point $x$ satisfies

$$\text{Ric}(x) \geq -(n-1)\lambda(d_p(x)),$$

where $d_p$ is the distance to $p$. Abresch and Gromoll were the first to study this class, and they proved that such manifolds have finite topological type if the sectional curvatures are uniformly bounded and the diameter growth has order $o(s^{1/n})$ with respect to the base point $p$. Recall that a manifold is said to have finite topological type if there exists a compact domain $\Omega$ with boundary such that $M \setminus \Omega$ is homeomorphic to $\partial \Omega \times [0,\infty]$. In order to complete their theorems, Abresch and Gromoll established important excess function estimates, which are also used by Hu, Xu and by Mahaman to prove some topological rigidity results for manifolds with asymptotically nonnegative Ricci curvature. They are also used as important tools for many geometers to study manifolds with nonnegative Ricci curvature; see, etc.

Let $B(x,r)$ denote the geodesic ball of radius $r$ and center $x$ in $M$ and let $B(\overline{p},r)$ denote the similar metric ball in the simply connected noncompact complete manifold with sectional curvature $-\lambda(d_{\overline{p}}(\overline{p}))$ at the point $\overline{p}$, where $d_{\overline{p}}(\overline{p}) = d(\overline{p},\overline{p})$ is the distance from $\overline{p}$ to $\overline{p}$. From the volume comparison theorem, which was proved by Zhu for the base point and by Mahaman for any point, we know that the function $r \mapsto \frac{\text{vol}B(x,r)}{\text{vol}B(\overline{p},r)}$ is monotone decreasing. Define

$$\alpha_x = \lim_{r \to +\infty} \frac{\text{vol}B(x,r)}{\text{vol}B(\overline{p},r)}$$

and $\alpha_M = \inf_{x \in M} \alpha_x$.

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We say $M$ is large volume growth if $\alpha_M > 0$.

For any $r > 0$, let

$$k_x(r) = \inf_{M \setminus B(x, r)} K,$$

where $K$ is the sectional curvature of $M$, and the infimum is taken over all the sections at all points on $M \setminus B(x, r)$. It is easy to see that $k_x(r)$ is a monotone function of $r$.

For a complete open Riemannian manifold with nonnegative Ricci curvature and large volume growth $\alpha_M > 0$, assume that $k_x(r) \geq -\frac{C}{(1+r)^{\alpha}}$ for some $x \in M$ and all $r$, where $C > 0$ and $\alpha \in [0, 2]$ are constants. Xia \cite{Xia} proved that $M$ has finite topological type if there is a constant $\epsilon = \epsilon(n, C, \alpha) > 0$, such that

$$\limsup_{r \to +\infty} \left\{ \left( \frac{\text{vol} B(x, r)}{\omega_n r^n} - \alpha_M \right) r^{(n-2+\frac{\alpha}{n})(1-\frac{2}{n})} \right\} \leq \epsilon_0 M.$$ 

The main purpose of this note is to extend the above result to manifolds with asymptotically nonnegative Ricci curvature. We have the following:

**Theorem 1.1.** Let $M$ be an $n$-dimensional $(n \geq 3)$ complete noncompact Riemannian manifold with

$$\text{Ric}(x) \geq -(n-1)\lambda(d_p(x))$$

and $K(x) \geq -\frac{C}{d_p(x)^\alpha}$,

where $C(\lambda) = \int_0^{+\infty} s\lambda(s)ds < +\infty$ and $C > 0$, $0 \leq \alpha \leq 2$. If $\alpha_p > 0$, then there exists a constant $\epsilon = \epsilon(n, \lambda, C, \alpha) > 0$, such that $M$ has finite topological type, provided that

$$\limsup_{r \to +\infty} \left\{ \left( \frac{\text{vol} B(x, r)}{\text{vol} B(\bar{x}, r)} - \alpha_p \right) r^{(n-2+\frac{1}{n})(1-\frac{2}{n})} \right\} \leq \epsilon_0.$$

On the other hand, Shen-Wei \cite{ShenWei} studied manifolds with nonnegative $k$th Ricci curvature outside a geodesic ball $B(p, D)$ and weak bounded geometry, i.e. $K = \inf K > -\infty$, $v = \inf \text{vol} B(x, 1) > 0$. They proved that there is a constant $c = c(n, k, K, v, D) > 0$ such that $M$ has finite topological type, if the volume growth at a point $x \in M$ satisfies

$$\limsup_{r \to +\infty} \frac{\text{vol} B(p, r)}{r^1 \pi^{1/(k+1)}} < c.$$ 

Here we say the $k$th Ricci curvature of $M$, for some $1 \leq k \leq n-1$, satisfies $\text{Ric}(k)(x) \geq H$, at a point $x \in M$ if for all $(k+1)$-dimensional subspaces $V \subset T_xM$,

$$\sum_{i=1}^{k+1} \langle R(e_i, v)e_i \rangle \geq H$$

for all $v \in V$,

where $\{e_1, \cdots, e_{k+1}\}$ is any orthonormal basis for $V$. Stimulated by their methods, we can extend Theorem 1.1 to the case of $k$th asymptotically nonnegative Ricci curvature.

**Theorem 1.2.** Let $M$ be an $n$-dimensional $(n \geq 3)$ complete noncompact Riemannian manifold with

$$\text{Ric}(k)(x) \geq -k\lambda(d_p(x)),$$

for $2 \leq k \leq n-1$,
and

\[ K(x) \geq -\frac{C}{d_p(x)^\alpha}, \]

where \( C(\lambda) = \int_0^{+\infty} s\lambda(s)ds < +\infty \) and \( C > 0, \ 0 \leq \alpha \leq 2 \). If \( \alpha_p > 0 \), then there exists a constant \( \epsilon = \epsilon(n, k, \lambda, C, \alpha) > 0 \), such that \( M \) has finite topological type, provided that

\[
\limsup_{r \to +\infty} \left\{ \frac{\text{vol}B(x, r)}{\text{vol}B(\bar{x}, r)} - \alpha_p \right\} \leq \epsilon \alpha_p. 
\]

**Remark 1.3.** For \( k = n - 1 \), Theorem 1.2 is just the same as Theorem 1.1, while for \( k = 1 \), \( M \) has asymptotically nonnegative sectional curvature. It was proved by Abresch [1] that \( M \) always has finite topological type without any additional conditions.

Denote by \( \text{crit}_p \) the criticality radius of \( M \) at \( p \), i.e. \( \text{crit}_p \) is the smallest critical value for the distance function \( d_p(\cdot) \). Recall that a point \( x \neq p \) is called the critical point of \( d_p \), if for any \( v \) in the tangent space \( T_xM \) there is minimal geodesic \( \gamma \) from \( x \) to \( p \) forming an angle less than or equal to \( \pi/2 \) with \( \gamma'(0) \) (see [1]). In [12] Wang and Xia proved the following theorem.

**Theorem 1.4.** Given \( \beta \in [0, 2] \), positive numbers \( r_0 \) and \( C \), and an integer \( n \geq 2 \), there is an \( \epsilon = \epsilon(n, r_0, C, \beta) > 0 \) such that any complete Riemannian \( n \)-manifold \( M \) with Ricci curvature \( \text{Ric}_{M} \geq 0 \), \( \alpha_M > 0 \), \( \text{crit}_p \geq r_0 \), and

\[
k_p(r) \geq -\frac{C}{(1 + r)^\beta} \frac{\text{vol}B(p, r)}{\omega_n r^n} \leq \left( 1 + \frac{\epsilon}{r^{n(2+\frac{1}{n}) (1-\frac{\alpha}{2})}} \right) \alpha_M,
\]

for some \( p \in M \) and all \( r \geq r_0 \) is diffeomorphic to \( \mathbb{R}^n \).

In order to remove the condition of criticality radius, let us define the function

\[
\phi_\alpha(r) = \begin{cases} r^\alpha, & \text{for } r \geq 1, \\ r, & \text{for } r < 1. \end{cases}
\]

We will prove a more general result.

**Theorem 1.5.** Let \( M \) be an \( n \)-dimensional \( (n \geq 3) \) complete noncompact Riemannian manifold with

\[
\text{Ric}(k)(x) \geq -k\lambda(d_p(x)), \text{ for } 2 \leq k \leq n - 1,
\]

and

\[
K(x) \geq -\frac{C}{d_p(x)^\alpha},
\]

where \( C(\lambda) = \int_0^{+\infty} s\lambda(s)ds < +\infty \) and \( C > 0, \ 0 \leq \alpha \leq 2 \). If \( \alpha_p > 0 \), then there exists a constant \( \epsilon = \epsilon(n, k, \lambda, C, \alpha) > 0 \), such that \( M \) is diffeomorphic to \( \mathbb{R}^n \), provided that

\[
\frac{\text{vol}B(p, r)}{\text{vol}B(\bar{p}, r)} \leq \left( 1 + \frac{\epsilon (\phi \frac{1}{\pi n} (\frac{n}{n-1} + 1))^{\frac{n-1}{n}} r^n}{\omega_n} \right) \alpha_p, \text{ for all } r > 0.
\]
In Section 2, we will give some Abresch-Gromoll excess function estimates for manifolds with $k$th asymptotically nonnegative Ricci curvature. In Section 3, we will show that manifolds with suitable ray density growth conditions and curvature decay conditions are diffeomorphic to $\mathbb{R}^n$ or have finite topological type, and then use this to prove Theorem 1.2 and Theorem 1.5.

2. Preliminaries

Let $M$ be an $n$-dimensional Riemannian manifold. For $p, q \in M$, the excess function $e_{pq}$ is defined by

$$e_{pq}(x) = d_p(x) + d_q(x) - d(p, q).$$

Let $\gamma$ be a minimal geodesic from $p$ to $q$. If $Ric_{(k)}(x) \geq 0$ on all $x$ of $M$, Abresch-Gromoll [2] (for $k = n - 1$) and Shen [8] (for any $k$) proved that

$$e_{pq}(x) \leq S \left( \frac{s^{k+1}}{r} \right)^{1/k},$$

where $s = d(x, \gamma), r = \min\{d(p, x), d(q, x)\}$.

For a manifold with $k$th asymptotically nonnegative Ricci curvature, we will also give an estimate for $e_{pq}(x)$. First we need

**Lemma 2.1.** Let $M$ be complete and $q, x \in M$. Suppose that $x$ is not on the cut locus of $q$, and

$$Ric_{(k)}(x) \geq -k\lambda(d_p(x)), \text{ with } C_0 = \int_0^{+\infty} s\lambda(s)ds < +\infty$$

along the minimal geodesic $\gamma$ from $x$ to $q$. Then for any orthonormal set $\{e_1, \cdots, e_{k+1}\}$ in $T_xM$ with $\gamma(0) \in \text{span}\{e_i\}$,

$$\sum_{i=1}^{k+1} \nabla^2 d_q(e_i, e_i) \leq \left\{ \begin{array}{ll}
\frac{1+\sqrt{1+8C_0}}{2} \cdot \frac{k}{d(p, x)}, & \text{for } q = p, \\
\frac{k\sqrt{C_0}}{d(p, q) - d(q, x)} + \frac{k}{d(q, x)}, & \text{for } q \neq p, d(q, x) < d(p, q).
\end{array} \right.$$

**Proof.** For $q = p$, let $\gamma(s) : [0, r] \to \mathbb{R}$ be the minimal normal geodesic from $x$ to $p$. Since $\gamma(0) \in \text{span}\{e_i\}$, without loss of generality, we may assume that $\text{grad}_{dd_p}(x) = \gamma(0) = e_1$ and along $\gamma(t)$ have an orthonormal frame such that $e_i(r) = e_i$, for $i = 1, \cdots, k + 1$. Put $N = \text{grad}_{dd_p}$; from [3], we have

$$\sum_{i=1}^{k+1} \langle R(e_i, N)N, e_i \rangle = \sum_{i=2}^{k+1} \langle (\nabla e_i \nabla N - \nabla N \nabla e_i - \nabla_{[e_i, N]}N)N, e_i \rangle = -\sum_{i=2}^{k+1} N \langle \nabla e_i, e_i \rangle - \sum_{i=2}^{k+1} \sum_{j=2}^{n} \langle \nabla e_i, N, e_j \rangle \langle \nabla e_j, N, e_i \rangle = -\left( \sum_{i=2}^{k+1} h_{ii} \right) - \sum_{i=2}^{k+1} \sum_{j=2}^{n} h_{ij}^2,$$
where $h_{ij} = \langle \nabla e_i, N, e_j \rangle$ is the second fundamental form of the distance sphere from $p$. From the Schwarz inequality and $k$th asymptotically nonnegative Ricci curvature condition, we have

$$-k(s) \lambda(s) \leq -\left( \sum_{i=2}^{k+1} h_{ii} \right)' - \frac{1}{k} \left( \sum_{i=2}^{k+1} h_{ii} \right)^2.$$

Note that

$$\sum_{i=2}^{k+1} h_{ii}(s) \sim \frac{k}{s}, \quad \text{as } s \to 0.$$

Consider the Riccati equation

$$v'(s) + v^2(s) - \lambda(s) = 0,$$

satisfying

$$v(s) \to \frac{1}{s}, \quad \text{as } s \to 0.$$

A standard comparison argument yields

$$\frac{1}{k} \sum_{i=2}^{k+1} h_{ii}(s) \leq v(s).$$

From Lemma 3.4 in [2], we get

$$v(r) \leq \frac{1 + \sqrt{1 + 8C_0}}{2r},$$

so we have

$$\sum_{i=1}^{k+1} \nabla^2 d_q(e_i, e_i) = \sum_{i=2}^{k+1} h_{ii}(r) \leq \frac{1 + \sqrt{1 + 8C_0}}{2} \cdot \frac{k}{d(p, x)}.$$

For $q \neq p$ and $d(q, x) < d(p, q)$, using a similar argument and Lemmas 3.2, 3.3 in [2], we have

$$\sum_{i=1}^{k+1} \nabla^2 d_q(e_i, e_i) \leq \frac{k \sqrt{2C_0}}{d(p, q) - d(q, x)} + \frac{k}{d(q, x)}.$$

\[\square\]

**Lemma 2.2.** Let $M$ be an $n$-dimensional ($n \geq 3$) complete Riemannian manifold and let $\gamma$ be a minimal geodesic joining the base point $p$ and another point $q \in M$; $x \in M$ is a third point such that $s < \min\{d(p, x), d(q, x), d(p, q) - d(q, x)\}$, where $s = d(x, \gamma)$. Suppose $C_0 = \int_0^{+\infty} s \lambda(s) ds < +\infty$ and

$$\text{Ric}(k)(x) \geq -k \lambda(d_p(x)), \quad \text{for } 2 \leq k \leq n - 1.$$

Then

$$e_{pq}(x) \leq \frac{2k}{k-1} \frac{d(p, x) - s}{\sqrt{2C_0}s} \sinh \frac{\sqrt{2C_0}s}{d(p, x) - s} \left( \frac{C_2(s)}{2(k+1)} \right)^{1/k}.$$

where $C_2(s) = \frac{1 + \sqrt{1 + 8C_0}}{2} \cdot \frac{k}{d(p, x) - s} + \frac{k \sqrt{2C_0}}{d(p, q) - d(q, x) - s} + \frac{k}{d(q, x) - s}$. 

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**Proof.** The argument uses modification methods of \([2]\) and \([10]\). Denote \(S_\kappa(t) = \frac{\sinh\sqrt{-\kappa t}}{\sqrt{-\kappa}}\) for \(\kappa < 0\). Take any \(s < R < \min\{d(p, x), d(q, x), d(p, q) - d(q, x)\}\) and \(C > C_2(R)\). Define \(f : \overline{B(x, R)} \to \mathbb{R}\) as

\[
f(y) = C \Phi_R(d_x(y)) - e_{pq}(y), y \in \overline{B(x, R)},
\]

where

\[
\Phi_R(\rho) = \int \int_{\rho \leq t \leq R} \left( \frac{\dot{S}_\kappa(\tau)}{S_\kappa(t)} \right)^k d\tau dt.
\]

Notice that from Lemma 3.3 in \([2]\), the lower bound \(\kappa\) on the Ricci curvature in the ball \(B(x, R)\) can be controlled by

\[
\text{Ric}(y) \geq -(n - 1) \frac{2C_0}{(d(p, x) - R)^2}, \quad \forall y \in B(x, R).
\]

Following the proof of Proposition 2.3 in \([2]\) and using Lemma 2.1, we can show that for any \(y \in \overline{B(x, R)} \setminus \{x\}\), there is an orthonormal set \(\{e_1, \cdots, e_{k+1}\}\) in \(T_yM\) such that the following inequality holds in a generalized sense:

\[
\sum_{i=1}^{k+1} \nabla^2 f(e_i, e_i) = C \left( \Phi_R^k \sum_{i=1}^{k+1} |\nabla e_i| d_x^2 + \Phi_R^k \sum_{i=1}^{k+1} \nabla^2 d_x(e_i, e_i) \right)
\]

\[
- \sum_{i=1}^{k+1} \nabla^2 d_p(e_i, e_i) - \sum_{i=1}^{k+1} \nabla^2 d_q(e_i, e_i)
\]

\[
\geq C - C_2(R) > 0.
\]

Thus \(f\) has no locally maximal point in \(B(x, R)\). Since \(f|_{S(x, R)} \leq 0\) and \(f|_{\overline{B(x, s)} \cap \gamma} > 0\), we know that

\[
e_{pq}(x) \leq \min_{0 \leq \rho \leq R} \left\{ \min_{y \in S(x, \rho)} e_{pq}(y) + 2\rho \right\}
\]

\[
\leq \min_{0 \leq \rho \leq R} \left\{ C \Phi_R(\rho) + 2\rho \right\}
\]

\[
\leq \min_{0 \leq \rho \leq R} \left\{ 2\rho + C(S_0R^2(1))^{k} \left[ \frac{2R^{k+1}}{k-1} \left( \rho^{1-k} - R^{1-k} \right) + \rho^2 - R^2 \right] \right\}
\]

\[
\leq \frac{2k}{k-1} \frac{d(p, x) - R}{\sinh \frac{\sqrt{2C_0R}}{d(p, x) - R} \left( \frac{CR^{k+1}}{2(k+1)} \right)^{1/k}}.
\]

Letting \(R \to s\) and \(C \to C_2(s)\), we get (2.4). \(\square\)

Using Lemma 2.2 and an easy argument, we can get an excess estimate for a manifold with \(k\)th asymptotically nonnegative Ricci curvature, which can be considered as an extended estimate of Abresch-Gromoll and Shen.

**Lemma 2.3.** Suppose

\[
\text{Ric}_{(k)}(x) \geq -k\lambda(d_p(x)), \quad \text{for } 2 \leq k \leq n - 1.
\]

Then

\[
e_{pq}(x) \leq 8(1 + 8C_0)^{\frac{1+k}{2k}} \left( \frac{s^{k+1}}{r} \right)^{\frac{1}{k}}.
\]

where \(s = d(x, \gamma), r = \min\{d(p, x), d(q, x)\}\).
3. Proof of the Theorems

Let $R_p$ denote the set of all rays issuing from $p$ and let

$$H(p,r) = \max_{x \in S(p,r)} d(x, R_p).$$

For a manifold with quadratic sectional curvature decay, Wang and Xia [12] proved that there exists a constant $\epsilon$ such that if $H(p,r) < \epsilon r$, then it is diffeomorphic to $\mathbb{R}^n$. We will extend it to the following.

**Theorem 3.1.** Given $C > 0$ and $\alpha \in [0, 2]$, suppose that $M$ is an $n$-dimensional complete noncompact Riemannian manifold with $K(x) \geq -\frac{C}{d^2(x)^\alpha}$. Then there exists a positive constant $\epsilon = \epsilon(\alpha, C)$ such that if $H(p,r) < \epsilon \phi_\omega^2(r)$, then $M$ is diffeomorphic to $\mathbb{R}^n$.

**Proof.** Let $\delta$ be a solution of the inequality $\cosh(2^{2+\alpha} \sqrt{C} \epsilon) - \cosh^2(\frac{3}{2} 2^{2+\alpha} \sqrt{C} \epsilon) < 0$ and take $\epsilon = \min\{\frac{1}{2} \delta, \frac{3}{2} \delta \}$. From the Disk Theorem (cf. [4]), it suffices to show that $d_p$ has no critical point other than $p$. Take an arbitrary point $x(\neq p) \in M$ and let $r = d(p, x)$. Since $R_p$ is closed, there exists a ray $\gamma$ issuing from $p$ such that $s = d(x, \gamma)$. From our condition, we have

$$s \leq \epsilon \phi_\omega^2(r).$$

(3.1)

Let $q = \gamma(2r)$ and let $\sigma_1$ and $\sigma_2$ be geodesics joining $x$ to $p$ and $q$ respectively. Set $\tilde{p} = \sigma_1(4\epsilon \phi_\omega^2(r)); \tilde{q} = \sigma_2(4\epsilon \phi_\omega^2(r))$. Consider the triangle $\Delta(x, \tilde{p}, \tilde{q})$. If $y$ is a point on this triangle, then

$$d(p, y) \geq d(p, x) - d(x, y) \geq d(p, x) - d(\tilde{p}, x) - d(\tilde{p}, y)$$

and

$$d(p, y) \geq d(p, x) - d(x, y) \geq d(p, x) - d(\tilde{q}, x) - d(\tilde{q}, y),$$

which means $d(p, y) \geq r - 8\epsilon \phi_\omega^2(r) > \frac{r}{2}$. Hence the triangle $\Delta(x, \tilde{p}, \tilde{q}) \subset M \setminus B(p, \frac{r}{4})$. Applying the Toponogov Theorem to the triangle $\Delta(x, \tilde{p}, \tilde{q})$ we have

$$\cosh\left(2^{2+\alpha} \sqrt{C} \frac{d(\tilde{p}, \tilde{q})}{r^\frac{\alpha}{2}}\right) \leq \cosh^2\left(2^{2+\alpha} \sqrt{C} \frac{d(\tilde{p}, x)}{r^\frac{\alpha}{2}}\right) - \sinh^2\left(2^{2+\alpha} \sqrt{C} \frac{d(\tilde{p}, x)}{r^\frac{\alpha}{2}}\right) \cos \theta,$$

where $\theta = \angle \sigma_1^0(0), \sigma_2^0(0)$. Since $e_{pq}(x) \leq 2s \leq 2\epsilon \phi_\omega^2(r)$, from triangle inequality, we have

$$d(\tilde{p}, \tilde{q}) \geq d(p, q) - d(p, x) + d(\tilde{p}, x) - d(x, q) + d(\tilde{q}, x)$$

$$\geq 8\epsilon \phi_\omega^2(r) - e_{pq}(x)$$

$$\geq 6\epsilon \phi_\omega^2(r).$$

(3.3)

From (3.2), (3.3), and (3.1) we have

$$\sinh^2\left(2^{2+\alpha} \sqrt{C} \frac{d(\tilde{p}, x)}{r^\frac{\alpha}{2}}\right) \cos \theta \leq \cosh^2\left(2^{2+\alpha} \sqrt{C} \frac{e \phi_\omega^2(r)}{r^\frac{\alpha}{2}}\right) - \cosh\left(2^{2+\alpha} \sqrt{C} \frac{3\epsilon \phi_\omega^2(r)}{2r^\frac{\alpha}{2}}\right)$$

$$\leq \cosh^2\left(2^{2+\alpha} \sqrt{C} \epsilon\right) - \cosh\left(\frac{3}{2} 2^{2+\alpha} \sqrt{C} \epsilon\right)$$

$$< 0,$$
so
\[ \theta > \frac{\pi}{2}, \]
which shows that \( x \) is not a critical point of \( d_p \) and Theorem 3.1 follows. \( \square \)

Before proving Theorem 1.2, we need the following lemma, which can be considered as a generalization of Lemma 3.1 in [13].

**Lemma 3.2.** Let \( M \) be an \( n \)-dimensional \( (n \geq 3) \) complete noncompact Riemannian manifold with
\[
\text{Ric}_{(k)}(x) \geq -k\lambda(d_p(x)), \quad \text{for} \quad 2 \leq k \leq n - 1,
\]
and
\[
K(x) \geq -\frac{C}{d_p(x)^{\alpha}},
\]
where \( C(\lambda) = \int_0^{+\infty} s\lambda(s)ds < +\infty \) and \( C > 0, \quad 0 \leq \alpha \leq 2 \). There exists a constant \( \epsilon' = \epsilon'(k, \lambda, C, \alpha) > 0 \), such that \( M \) has finite topological type, provided that
\[
\text{(3.4)} \quad \limsup_{r \to +\infty} \frac{H(p, r)}{r^{\frac{k}{k+1} + 1}} \leq \epsilon.
\]

**Proof.** Let \( \delta \) be the solution of the inequality
\[
\text{(3.5)} \quad \cosh^2 \left( 2^\alpha \sqrt{C}\delta \right) - \cosh \left( \frac{3}{2} 2^\alpha \sqrt{C}\delta \right) < 0,
\]
and take \( \epsilon' \) to be
\[
\epsilon' = \left( \frac{\delta}{16(1 + 8C_0)^{\frac{k+\alpha}{2}}} \right)^{\frac{k}{k+1}}.
\]
From (3.4), we can find a constant \( r_0 > 1 \) such that
\[
\text{(3.7)} \quad H(p, r) \leq \epsilon' r^{\frac{k}{k+1} + 1}, \quad \forall r \geq r_0.
\]

It suffices to show that \( d_p \) has no critical point in \( M \setminus B(p, r_0) \). To show this, take an arbitrary point \( x \in M \setminus B(p, r_0) \) and let \( r = d(p, x) \). From our condition, there exists a ray \( \gamma \) issuing from \( p \) such that \( s = d(x, \gamma) \) and
\[
\text{(3.8)} \quad s \leq \epsilon' r^{\frac{k}{k+1} + 1} < r.
\]
Let \( q = \gamma(2r) \) and let \( \sigma_1 \) and \( \sigma_2 \) be geodesics joining \( x \) to \( p \) and \( q \) respectively. Set \( p' = \sigma_1(\delta r^{\frac{k}{2}}); \quad q' = \sigma_2(\delta r^{\frac{k}{2}}) \). As in the proof of Theorem 3.1, we know that the triangle \( \Delta(x, p', q') \subset M \setminus B(p, \frac{r}{2}) \) and
\[
\text{(3.9)} \quad d(p', q') \geq 2\delta r^{\frac{k}{2}} - e_{pq}(x).
\]
Using (2.5), (3.6), and (3.8) we have
\[
e_{pq}(x) \leq 8(1 + 8C_0)^{\frac{k}{2}} \left( \frac{s^{k+1}}{r} \right)^{\frac{k}{k+1}} \leq 8(1 + 8C_0)^{\frac{k}{2}} \left( \frac{(\epsilon')^{k+1} r^{k\alpha/2+1}}{r} \right)^{\frac{k}{k+1}} = \delta r^{\frac{k}{2}}.
\]
So we have

\begin{equation}
(3.10) \quad d(p', q') \geq 2\delta r^\frac{\alpha}{2} - \frac{1}{2}\delta r^\frac{\alpha}{2} = \frac{3}{2}\delta r^\frac{\alpha}{2}.
\end{equation}

Applying the Toponogov Theorem to the triangle $\Delta(x, p', q')$, we have

\begin{align*}
\sinh^2 \left( \frac{2\alpha \sqrt{C}}{r^\frac{\alpha}{2}} d(p', x) \right) \cos \theta & \leq \cosh^2 \left( \frac{2\alpha \sqrt{C}}{r^\frac{\alpha}{2}} d(p', x) \right) - \cosh \left( \frac{2\alpha \sqrt{C}}{r^\frac{\alpha}{2}} d(p', q') \right) \\
& \leq \cosh^2 \left( 2\alpha \sqrt{C} \delta \right) - \cosh \left( \frac{3}{2} 2\alpha \sqrt{C} \delta \right) \\
& < 0,
\end{align*}

so

\[ \theta > \frac{\pi}{2}, \]

which shows that $x$ is not a critical point of $d_p$ and Lemma 3.2 follows. \hfill \Box

**Proof of Theorem 1.2.** Take the number $\epsilon$ in Theorem 1.2 to be

\begin{equation}
(3.11) \quad \epsilon = \left( \epsilon' \right)^{n-1} \frac{1}{18n e^{3(n-1)C_0}},
\end{equation}

where $\epsilon' = \epsilon'(k, \lambda, C, \alpha)$ is as in Lemma 3.2. From (1.4), we can find a constant $r_0 > 1$ such that

\begin{equation}
(3.12) \quad \frac{\text{vol} B(x, r)}{\text{vol} B(\bar{x}, r)} - \alpha_p \leq \epsilon \alpha_p r^{-(\frac{n-1}{k+1})(1-\frac{\alpha}{2})}, \quad \forall r \geq r_0.
\end{equation}

From Lemma 3.2, we only need to show that for any arbitrary point $x \in M \setminus B(p, r_0)$ and a ray $\gamma$ issuing from $p$, set $r = d(p, x)$ and $s = d(x, \gamma)$.

Then

\begin{equation}
(3.13) \quad s \leq \epsilon' r^{\frac{1}{k+1}} \left( \frac{k}{2} + 1 \right).
\end{equation}

To prove this, let $\Sigma_p(\infty)$ be the set of unit vectors $v \in S_p M$ such that the geodesic $\gamma(t) = \exp_p(tv)$ is a ray and $\Sigma_p^c(\infty) = S_p \setminus \Sigma_p(\infty)$. We have

\[ B(x, s) \subset B_{\Sigma_p(\infty)}(p, r + s) \setminus B(p, r - s), \]

which means

\begin{equation}
(3.14) \quad \text{vol} B(x, s) \leq \text{vol} B_{\Sigma_p(\infty)}(p, r + s) - \text{vol} B(p, r - s).
\end{equation}
By the Relative Comparison Theorem for asymptotically nonnegative Ricci curvature (see [6]), we have

\[
volB(x, \frac{s}{2}) \leq volB_{\Sigma_p(\infty)}(p, r + \frac{s}{2}) - volB_{\Sigma_\bar{p}(\infty)}(p, r - \frac{s}{2}) = \frac{volB_{\Sigma_p(\infty)}(p, r + \frac{s}{2})}{volB_{\Sigma_\bar{p}(\infty)}(p, r - \frac{s}{2})} - 1 \\
\leq volB_{\Sigma_p(\infty)}(p, r - \frac{s}{2}) \frac{volB(\bar{p}, r + \frac{s}{2})}{volB(\bar{p}, r - \frac{s}{2})} - 1 \\
\leq e^{(n-1)C_0} \left( \left( \frac{r + \frac{s}{2}}{r - \frac{s}{2}} \right)^n - 1 \right) volB_{\Sigma_p(\infty)}(p, r - \frac{s}{2})
\]

(3.15)

\[
\leq (3^n - 1) e^{(n-1)C_0} \frac{s}{r} volB_{\Sigma_p(\infty)}(p, r - \frac{s}{2})
\]

(3.16)

where the last two inequalities are in fact due to Mahaman by using the volume element estimate \(dv(t) \leq e^{(n-1)C_0} t^{n-1} dt \wedge dS_{n-1}\) in polar coordinates.

Now, from (3.14), Lemma 3.10 in [6], and (3.12), we have

\[
volB_{\Sigma_p(\infty)}(p, r - \frac{s}{2}) = volB(p, r - \frac{s}{2}) - \alpha_p volB(\bar{p}, r - \frac{s}{2}) \\
\leq volB(p, r - \frac{s}{2}) - \alpha_p volB(\bar{p}, r - \frac{s}{2}) \\
\leq \epsilon \alpha_p r^{-\frac{(n-1)k}{k+1}}(1 - \frac{a}{2}) volB(\bar{p}, r - \frac{s}{2}) \\
\leq \epsilon \omega_n \alpha_p e^{(n-1)C_0} r^{-\frac{(n-1)k}{k+1}}(1 - \frac{a}{2}) r^n.
\]

(3.17)

Substituting (3.17) in (3.14), we get

\[
volB(x, \frac{s}{2}) \leq (3^n - 1) \epsilon \omega_n \alpha_p e^{2(n-1)C_0 s r^{n-1} - \frac{(n-1)k}{k+1} (1 - \frac{a}{2})}.
\]

On the other hand, using (15) in [6], we know that

\[
volB(x, \frac{s}{2}) \geq \frac{\omega_n}{3^n} \alpha_p e^{-(n-1)C_0} \left( \frac{s}{2} \right)^n.
\]

(3.18)

From these two inequalities, we have

\[
s^{n-1} \leq \epsilon 18^n e^{3(n-1)C_0} r^{\frac{n-1}{k+1}}(1 - \frac{a}{2} + 1).
\]

Using (3.12), we obtain

\[
s \leq (\epsilon 18^n e^{3(n-1)C_0})^{-\frac{1}{n-1}} r^{\frac{n-1}{k+1}}(1 - \frac{a}{2} + 1) = \epsilon' r^{\frac{n-1}{k+1}}(1 - \frac{a}{2} + 1),
\]

which satisfies (3.13) and completes the proof of Theorem 1.2. \(\square\)

**Proof of Theorem 1.5.** We choose the number \(\epsilon\) in Theorem 1.5 to be

\[
\epsilon = \frac{(\epsilon')^{n-1}}{18^n e^{3(n-1)C_0}},
\]

where \(\epsilon' = \epsilon(k, \lambda, C, \alpha)\) is as in Lemma 3.2. Take any arbitrary point \(x \in M \setminus \{p\}\) and a ray \(\gamma\) issuing from \(p\), and set \(r = d(p, x)\) and \(s = d(x, \gamma)\). Using similar methods as in the proof of Theorem 1.2 and our condition,

\[
volB(x, \frac{s}{2}) \leq (3^n - 1) \epsilon \omega_n \alpha_p e^{2(n-1)C_0 s} \left( \phi_{\frac{1}{k+1}}(\frac{k+1}{k}(1 + r)) \right)^{n-1}.
\]

(3.22)
From (3.19) and (3.22), we obtain
\begin{equation}
(3.23) \quad s \leq \epsilon \phi_{\frac{1}{1+r}}(k\alpha^2 + 1)(r).
\end{equation}

Repeating the argument as in the proof of Theorem 3.1 and Lemma 3.2, we can show that \( x \) is not a critical point of \( d_p \) and Theorem 1.5 follows.

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