On the 2D Dirac oscillator in the presence of vector and scalar potentials in the cosmic string spacetime in the context of spin and pseudospin symmetries

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The Dirac equation with both scalar and vector couplings describing the dynamics of a two-dimensional Dirac oscillator in the cosmic string spacetime is considered. We derive the Dirac-Pauli equation and solve it in the limit of the spin and the pseudo-spin symmetries. We analyze the presence of cylindrical symmetric scalar potentials which allows us to provide analytic solutions for the resultant field equation. By using an appropriate ansatz, we find that the radial equation is a biconfluent Heun-like differential equation. The solution of this equation provides us with more than one expression for the energy eigenvalues of the oscillator. We investigate these energies and find that there is a quantum condition between them. We study this condition in detail and find that it requires the fixation of one of the physical parameters involved in the problem. Expressions for the energy of the oscillator are obtained for some values of the quantum number \( n \). Some particular cases which lead to known physical systems are also addressed.

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I. INTRODUCTION

The study of the relativistic quantum dynamics of particles including electromagnetic interactions is an usual framework for studying properties of various physical systems. The mechanism used to describe these systems is a natural generalization of the coupling used in classical nonrelativistic quantum theory [1]. This coupling is implemented, for charged particles with charge \( e \), through the so-called minimal coupling prescription, given in terms of the modification of the 4-momentum operator, \( p_{\mu} \rightarrow p_{\mu} - eA_{\mu} = (p_0 - eA_0, p - eA) \), where \( A_{\mu} = (V(r), -A) \) (with \( A \) being the vector potential and \( V(r) \) being the scalar potential) represents the 4-vector potential of the associated electromagnetic field. This transformation preserves the gauge invariance associated with the Maxwell’s equations. Another way to insert interaction in the dynamics of the particle is by including a scalar potential through a modification in the mass term as \( M \rightarrow M + S(r) \). In this realization, the potential \( S(r) \) is coupled like a scalar, different from the minimum prescription, where the potential is coupled as a time-like component of a 4-vector. Although there is some similarity between the scalar and vector couplings, they have different physical implications. Actually, the scalar coupling acts equally on particles and antiparticles. On the other hand, the vector coupling acts differently on particles and antiparticles. As a result, the energy of particle and antiparticle are not equals, so that bound states exist only for one of the two kinds of particles [2].

Interesting issues that should be investigated with the insertion of the couplings in the Dirac equation are the so-called the spin and the pseudo-spin symmetries [3]. Basically, these symmetries occur when the couplings are composed by a vector \( V(r) \) and a scalar \( S(r) \) potential, under the assumption that \( S(r) = V(r) \) (\( S(r) = -V(r) \)), which is the necessary condition for occurrence of exact spin (pseudo-spin) symmetry. The spin symmetry has been identified by studying heavy-light mesons [4], single antinucleon spectra [5] and dynamics of a light quark (antiquark) in the field of a heavy antiquark (quark) [3] while that the pseudo-spin symmetry occurs in the motion of nucleons [3, 6]. In recent studies, both the spin and the pseudo-spin symmetries appear in several aspects concerning, for instance, the supersymmetry [7, 8], the Hartree-Fock theory [9], the electrons in graphene [10] and the interaction with a class of scalar and vector potentials [11–18]. An important physical system that can be studied by including such terms of interactions in the Dirac equation is the Dirac oscillator [19]. In the nonrelativistic regime, the Dirac oscillator behaves as a harmonic oscillator with a strong spin-orbit coupling. It was verified experimentally for the first time in 2013 by J. A. Franco-Villafañe et al. [20]. It has been addressed in various branches of physics since then. For instance, it appears in mathematical physics [21–25], nuclear physics [26–28], quantum optics [29–32], supersymmetry [33–35], and noncommutativity [36–39].

In this work, we analyze in detail the solutions of the Dirac equation with both scalar and vector interactions under the spin and the pseudo-spin symmetry limits in the cosmic string spacetime [40]. Cosmic strings are topologically stable gravitational defects. According to the grand unified theories, these defects arise from a vacuum phase transition in the near universe. Recently, several studies have been developed in the theoretical context [41–46] and also by evidence of cosmic strings [47–50]. Cosmic strings are objects of studies of current interest because of the several important applications of...
topological features on physics systems in gravitation [51], condensed matter [40] and cosmology [52].

We organize the paper as follows: In Sec. II, we derive the equation that governs the dynamics of a Dirac particle with the minimal, nonminimal and the scalar couplings in the cosmic string spacetime. In Sec. III, we consider the Dirac equation written in terms of a set of coupled differential equations. We investigate the existence of particular solutions for the problem by assuming that the relativistic energy of the particle is its rest energy in both the spin and the pseudo-spin symmetries limits. In Sec. IV, we investigate the dynamics considering that the energy of the particle is different from its rest energy. To this end, we write down the Dirac equation in its quadratic form. We obtain the energies and the corresponded wave functions and discuss their physical validity. In Sec. V, we address some particular solutions and compare them with previous results in the literature. Finally, the conclusions are presented in Sec. VI.

II. THE EQUATION OF MOTION

The spacetime generated by a cosmic string in cylindrical coordinates is described by the line element

$$ds^2 = dt^2 - dr^2 - \alpha^2 r^2 d\varphi^2 - dz^2,$$

(1)

with $-\infty < (t, z) < \infty$, $r \geq 0$ and $0 \leq \varphi \leq 2\pi$. The parameter $\alpha$ is related to the linear mass density $\tilde{m}$ of the string by $\alpha = 1 - 4\tilde{m}$ and it runs in the interval $(0, 1)$ and corresponds to a deficit angle $\gamma = 2\pi(1 - \alpha)$. Geometrically, the metric in Eq. (1) corresponds to a Minkowski spacetime with a conical singularity [53].

One starts by considering the free Dirac equation, i.e., in the absence of interactions. The interaction will be included later. So, we have

$$(i\gamma^\mu \partial_\mu - M) \Psi = 0,$$

(2)

where $\Psi$ is a four-component spinorial wave function. In order to work out in the curved spacetime, we must write the Dirac gamma matrices $\gamma^\mu$ in the Minkowskian spacetime (written in terms of local coordinates) in terms of global coordinates and subsequently include the spinor affine connection $\Gamma_\mu$. In other words, we must contract $\gamma^\mu$ with the inverse tetrad,

$$\gamma^\mu = e^a_\mu \gamma^a,$$

(3)

satisfying the generalized Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}.$$  

(4)

The tetrad components $e^a_\mu$ in Eq. (3) obey the relation

$$e^a_\mu e^b_\nu = 2g_{\mu\nu},$$

(5)

where $(\mu, \nu) = (0, 1, 2, 3)$ are tensor indices and $(a, b) = (0, 1, 2, 3)$ are tetrad indices. The matrices $\gamma^a = (\gamma^0, \gamma^1)$ in Eq. (3) are the standard Dirac matrices in Minkowski spacetime, with

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad (i = 1, 2, 3)$$

(6)

where $\sigma^i$ are the standard Pauli matrices and $I$ is the $2 \times 2$ identity matrix. As we are interested on in a cosmic string, we need to write down the generalized Dirac equation in the curved spacetime background with a minimal coupling. Therefore, the relevant equation is

$$[i\gamma^\mu (\partial_\mu + \Gamma_\mu) - e\gamma^\mu A_\mu - M] \Psi = 0,$$

(7)

where $e$ is the electric charge and $A_\mu$ denotes the vector potential associated with the electromagnetic field. The spinor affine connection is often written as [54]

$$\Gamma_\mu = \frac{1}{8}\omega_{\mu ab} [\gamma^a, \gamma^b],$$

(8)

where $\omega_{\mu ab}$ is the spin connection, given by

$$\omega_{\mu ab} = \eta_{ac} e^\nu_b \Gamma^\nu_\mu - \eta_{ac} e^\nu_c \partial_\mu e^\nu_b.$$  

(9)

In (9), $\Gamma^\nu_\mu$ are the Christoffel symbols and $\eta^{ab}$ is the metric tensor. By the means of the spin connection, we can construct a local frame using a basis tetrad which gives the spinors in the curved spacetime. Here, the basis tetrad $e^a_\mu$ is chosen to be [55]

$$e^\mu_a (r, \varphi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi & 0 \\ 0 & -\sin \varphi / \alpha r & \cos \varphi / \alpha r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

(10)

satisfying the condition

$$e^\mu_a e^\nu_b \eta^{ab} = g^{\mu\nu}.$$  

(11)

Using (10), the matrices $\gamma^\mu$ in Eq. (7) are written more explicitly as

$$\gamma^0 = e^0_0 \gamma^0 \equiv \gamma^0,$$

$$\gamma^1 = e^0_1 \gamma^0 \equiv \gamma^1,$$

$$\gamma^1 = e^1_1 \gamma^1 \equiv \gamma^1,$$

$$\gamma^2 = e^2_2 \gamma^0 \equiv \gamma^2,$$

$$\gamma^2 = e^2_1 \gamma^1 \equiv \gamma^2,$$

$$\gamma^3 = e^3_3 \gamma^0 \equiv \gamma^3,$$

$$\gamma^3 = e^3_2 \gamma^1 \equiv \gamma^3.$$

(12-17)

Given the fact that the matrices in the curved space satisfy the condition $\nabla_\mu \gamma^\mu = 0$, i.e., they are covariantly constant, for the specific basis tetrad (10), the affine spin connection is found to be

$$\Gamma = (0, 0, 0, 0).$$

(18)
with the non-vanishing element given by
\[ \Gamma_\varphi = \frac{1}{2} (1 - \alpha) \gamma_1 \gamma_2. \]  
(19)

We are interested on including potentials with cylindrical symmetry, in such a way the resulting system will have trans-lational invariance along the \( z \) direction. Then, we can discard the third direction and thus consider the Dirac oscillator in two spacial dimensions [19] (see also Ref. [56]); assuming \( p_z = 0 \) [57]. This assumption allows us to reduce the four-component Dirac equation (7) to a two-component spinor equation. Moreover, according to the tetrad postulated [54], the \( \gamma^s \) matrices could be any set of constant Dirac matrices. Thus, a convenient representation is the following [58–60]
\[ \gamma^0 = \sigma^z, \quad \beta \gamma^1 = \sigma^1, \quad \beta \gamma^2 = s \sigma^2, \]  
(20)
where the parameter \( s \), which is twice the spin value, can be introduced to characterize the two spin states, with \( s = +1 \) for spin “up” and \( s = -1 \) for spin “down”. In the representation (20), the matrices (12), (14) and (16) assume the following form:
\[ \gamma^0 = \beta = \sigma^z, \quad \beta \gamma^r = \sigma^r = \begin{pmatrix} 0 & e^{-i s \varphi} \\ e^{i s \varphi} & 0 \end{pmatrix}, \]  
(21)
\[ \beta \gamma^\varphi = s \sigma^\varphi = \frac{s}{\alpha r} \begin{pmatrix} 0 & -i e^{-i s \varphi} \\ i e^{i s \varphi} & 0 \end{pmatrix}, \]  
(22)
and Eq. (19) becomes
\[ \Gamma_\varphi = -\frac{i s}{2} (1 - \alpha) \sigma^z. \]  
(24)

Now, let us include the interactions into the Dirac equation (7). We consider the effective potential [61, 62]
\[ M \omega \sigma^z (\beta \gamma \cdot \mathbf{r}) + \frac{1}{2} (I + \sigma^z) \Sigma(r) + \frac{1}{2} (I - \sigma^z) \Delta(r), \]  
(25)
with
\[ \Delta(r) = V(r) - S(r), \]  
(26)
\[ \Sigma(r) = V(r) + S(r), \]  
(27)
where
\[ V(r) = V_1(r) + V_2(r) = \frac{\eta c_1}{r} + \eta L_1 r, \]  
(28)
\[ S(r) = S_1(r) + S_2(r) = \frac{\eta c_2}{r} + \eta L_2 r, \]  
(29)
are cylindrically symmetric scalar and vector potentials. The first term in Eq. (25) represents the Dirac oscillator. In this manner, the time-independent Dirac equation (7) with energy \( E \) can be written as
\[ H_D \psi = E \psi, \]  
(30)
where \( \psi \) is a two-component spinor,
\[ H_D = \beta \gamma \cdot (p_\alpha - i \Gamma - i M \omega \beta r) + \frac{1}{2} (I + \beta) \Sigma(r) \]  
\[- \frac{1}{2} (I - \beta) \Delta(r) + \beta M, \]  
(31)
is the Dirac Hamiltonian and
\[ p_\alpha = -i \nabla_\alpha = -i \left( \frac{\partial}{\partial r^\alpha} + \frac{1}{\alpha r} \frac{\partial}{\partial \varphi} \right), \]  
(32)
is the planar spatial part of the gradient operator in the metric (1).

We begin the study of the particle motion by looking for first order solutions of the Eq. (30). For this purpose, we write the Eq. (30) as follows,
\[ i e^{-is\varphi} \left[ -\frac{\partial}{\partial r} + M \omega r + \frac{is}{\alpha r} \frac{\partial}{\partial \varphi} - \frac{(1 - \alpha)}{2\alpha r} \right] \psi_2 = \]  
\[ [E - M - \Sigma(r)] \psi_1 \]  
(33a)
\[ i e^{+is\varphi} \left[ -\frac{\partial}{\partial r} - M \omega r - \frac{is}{\alpha r} \frac{\partial}{\partial \varphi} - \frac{(1 - \alpha)}{2\alpha r} \right] \psi_1 = \]  
\[ [E + M - \Delta(r)] \psi_2, \]  
(33b)
and we consider the solutions as
\[ \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \left( \sum_m \frac{f(r)}{i q(r)} e^{i(m+s)\varphi} \right), \]  
(34)
with \( m = 0, \pm 1, \pm 2, \pm 3, \ldots \) being the quantum angular momentum number. The substitution of (34) into (33a) and (33b) gives the following set of coupled differential equations:
\[ \left( \frac{d}{dr} + \frac{s J^z_\alpha}{r} - M \omega r \right) g_m = [E - M - \Sigma(r)] f_m, \]  
(35)
\[ \left( -\frac{d}{dr} + \frac{s J^+_\alpha}{r} - M \omega r \right) f_m = [E + M - \Delta(r)] g_m. \]  
(36)
where
\[ J^\pm_\alpha = \frac{1}{\alpha} \left[ m + \left( \frac{1 \mp 1}{2} \right) s + \frac{s}{2} (1 - \alpha) \right]. \]  
(37)
The reason why we are using superscripts (\( \pm \)) in Eq. (37) will be clarified in the next section. If we consider that \( \Delta(r) = 0 \) and \( E = -M \) or \( \Sigma(r) = 0 \) and \( E = +M \), the solutions of Eqs. (35) and (36) represent a particular solution for the problem, which is excluded from the Sturm-Liouville problem. In other words, such solutions would not be part of those obtained by solving the second-order differential equation obtained from Eq. (30). The procedure of imposing that either \( \Delta(r) = 0 \) or \( \Sigma(r) = 0 \) in Eqs. (35) and (36), respectively, is known in the literature as the exact limits of spin and pseudo-spin symmetries [3]. These conditions are taken into account in the next section.

III. PARTICULAR SOLUTIONS AND THE ANALYSIS OF THE SPIN AND THE PSEUDO-SPIN SYMMETRIES

As mentioned above, the exact limit of the spin symmetry occurs when \( \Delta(r) = 0 \) \( (V(r) = S(r) \) in Eq. (26), while
that the exact limit of the pseudospin symmetry is achieved by setting \( \Sigma(r) = 0 \) (\( V(r) = -S(r) \) in Eq. (27)). In what follows, the superscript \((+)\) holds for the spin symmetry and \((-)\) holds for the pseudo-spin symmetry. In these limits, the solutions are related to the up and down components of the spinor in Eq. (34), respectively.

In order to obtain the particular solutions, let us look for the bound state solutions which obey the following normalization condition,

\[
\int_0^\infty (|f_m(r)|^2 + |g_m(r)|^2) r dr = 1 .
\]

We assume \( E = \pm M \), as it was mentioned above.

### A. The exact spin symmetry

Here, the particular solutions for the bound states are obtained by considering \( \Delta(r) = 0 \) [63] along with the assumption \( E = -M \) in both Eqs. (35) and (36). Therefore, we have

\[
\left( \frac{d}{dr} + s \frac{J^-_r}{r} - M \omega r \right) g_m(r) = -2 [M + S(r)] f_m(r) ,
\]

\[
\left( -\frac{d}{dr} + s \frac{J^+_r}{r} - M \omega r \right) f_m(r) = 0 .
\]

Their solutions are written as

\[
f_m(r) = a_1 r^{s J^+_r} e^{-\frac{1}{2} M \omega r^2} ,
\]

\[
g_m(r) = r^{-s J^-_r} e^{\frac{1}{2} M \omega r^2} \times \left[ a_1 (M \omega)^{-\frac{1}{2} (J^+_r + J^-_r)} - \frac{1}{2} \Gamma(a,b,c) + a_2 \right] ,
\]

with

\[
\Gamma(a,b,c) = \eta_C (M \omega)^{\frac{1}{2}} \Gamma(a) + \eta_L (M \omega)^{\frac{1}{2}} \Gamma(b) + M^2 \omega \Gamma(c) ,
\]

where

\[
\Gamma(a) = \Gamma \left[ \frac{1}{2} s (J^+_r + J^-_r) , M \omega r^2 \right] ,
\]

\[
\Gamma(b) = \Gamma \left[ \frac{1}{2} s (J^+_r + J^-_r) + 1 , M \omega r^2 \right] ,
\]

\[
\Gamma(c) = \Gamma \left[ \frac{1}{2} s (J^+_r + J^-_r) + \frac{1}{2} , M \omega r^2 \right] ,
\]

are upper incomplete Gamma functions [64], \( a_1 \) and \( a_2 \) are constants. Let us discuss the solutions (41) and (42). Since \( e^{-\frac{1}{2} M \omega r^2} \) dominates over \( r^{s J^+_r} \) for any value of \( s J^+_r \), the solution \( f_m(r) \) in Eq. (41) converges as \( r \to 0 \) and \( r \to \infty \). On the other hand, as the incomplete Gamma functions \( \Gamma(a,b,c) \) always diverge, so \( g_m(r) \) in (42) will only converge as \( r \to 0 \) if \( a_1 = 0 \), yielding \( f_m(r) = 0 \). The resulting solution are

\[
\begin{align*}
\begin{bmatrix} f_m(r) \\ g_m(r) \end{bmatrix} &= a_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} r^{-s J^-_r} e^{\frac{1}{2} M \omega r^2} ,
\end{align*}
\]

\[
\left\{ \begin{array}{l}
s = \pm 1 , \\
a_1 = 0 .
\end{array} \right.
\]

As \( M \omega > 0 \) in (47), there are no values of \( s J^-_r \) for which the functions are square-integrable. In this case, we can therefore conclude right away that for \( E = -M \) and exact spin symmetry there is no bound state solution.

### B. Exact pseudo-spin symmetry

In this section, we make \( \Sigma(r) = 0 \), with \( E = M \) in both Eqs. (35) and (36). This way, we obtain

\[
\left( \frac{d}{dr} + s \frac{J^-_r}{r} - M \omega r \right) g_m(r) = 0 ,
\]

\[
\left( -\frac{d}{dr} + s \frac{J^+_r}{r} - M \omega r \right) f_m(r) = 2 [M + S(r)] g_m(r) .
\]

Their solutions are given by

\[
f_m(r) = b_1 r^{s J^+_r} e^{-\frac{1}{2} M \omega r^2} \times \left[ \frac{1}{2} s (J^+_r - J^-_r) - \frac{1}{2} \Gamma(d,e,f) \right] ,
\]

\[
g_m(r) = b_2 r^{-s J^-_r} e^{\frac{1}{2} M \omega r^2} ,
\]

where \( b_1 \) and \( b_2 \) are constants, and

\[
\Gamma(d,e,f) = M^2 \omega \Gamma(d) - \eta_C (-M \omega)^{\frac{1}{2}} \Gamma(e) - \eta_L (-M \omega)^{\frac{1}{2}} \Gamma(f) ,
\]

with

\[
\Gamma(d) = \Gamma \left[ 1 - \frac{1}{2} s (J^+_r - J^-_r) , -M \omega r^2 \right] ,
\]

\[
\Gamma(e) = \Gamma \left[ \frac{1}{2} s (J^+_r - J^-_r) , -M \omega r^2 \right] ,
\]

\[
\Gamma(f) = \Gamma \left[ \frac{1}{2} s (J^+_r - J^-_r) , -M \omega r^2 \right] .
\]

Again, the incomplete Gamma functions \( \Gamma(d,e,f) \) in Eq. (50) always diverge, so that a normalized solution requires that \( b_2 = 0 \). In such a case, the function \( f_m(r) \) is square-integrable only for \( s J^+_r \geq 0 \). The physically acceptable solution is

\[
\begin{align*}
\begin{bmatrix} f_m(r) \\ g_m(r) \end{bmatrix} &= b_1 r^{s J^+_r} e^{-\frac{1}{2} M \omega r^2} \begin{bmatrix} 0 \\ 1 \end{bmatrix} ,
\end{align*}
\]

\[
\left\{ \begin{array}{l}
s J^+_r \geq 0 , \\
a_1 = 0 .
\end{array} \right.
\]

Therefore, we can conclude that for the case \( E = M \) along with the exact pseudo-spin symmetry there is a bound state solution.

### IV. THE DIRAC-PAULI EQUATION AND THE ANALYSIS OF BOTH THE SPIN AND THE PSEUDO-SPIN SYMMETRIES

In order to investigate the dynamics for \( E \neq \pm M \), we choose to work with the Eq. (30) in its quadratic form. After the application of the operator
\[ \beta \gamma \cdot (p_\alpha - i \Gamma - i M \omega \beta r) + \beta M + E + \frac{1}{2} (\beta - \Pi) \Sigma (r) - \frac{1}{2} (\Pi + \beta) \Delta (r), \]  

(57)

In Eq. (30), we find

\[ - \nabla^2 \psi - \frac{(1 - \alpha) s \sigma^z}{i \alpha^2 r^2} \frac{\partial}{\partial \varphi} + \frac{(1 - \alpha)^2}{4 \alpha^2 r^2} + M^2 \omega^2 r^2 \psi - 2 M \omega \left\{ \sigma^z + \frac{s}{\alpha} \left[ \frac{1}{i \partial} \sigma^x \cdot \frac{s}{2} (1 - \alpha) \sigma^z \right] \right\} \psi \]

- \Sigma (r) \Delta (r) \psi + (E + M) \Sigma (r) \psi + (E - M) \Delta (r) \psi + (M^2 - E^2) \psi - \frac{1}{2} i \sigma^x \left\{ \frac{d}{dr} \left[ \Sigma (r) + \Delta (r) \right] \right\} \psi

- \frac{1}{2} \sigma^z \left\{ \frac{d}{dr} \left[ \Sigma (r) - \Delta (r) \right] \right\} \psi = 0, \quad (58)

where \( \nabla^2_\alpha = \partial^2_x + (1/r) \partial_x + (1/\alpha^2 r^2) \partial_x^2 \) is the planar spatial part of the Laplace-Beltrami operator in the metric (1). By inserting the solutions (34) into Eq. (58), we obtain the following set of two coupled radial differential equations of second-order:

\[ - \frac{d^2 f (r)}{dr^2} - \frac{1}{r} \frac{df (r)}{dr} + \frac{(J^+ \alpha)^2}{r^2} f (r) + M^2 \omega^2 r^2 f (r) - 2 M \omega (s J^+ \alpha + 1) f (r) - \Sigma (r) \Delta (r) f (r) \]

+ (E + M) \Sigma (r) f (r) + (E - M) \Delta (r) f (r) + (M^2 - E^2) f (r) + \left[ \frac{d \Delta (r)}{dr} \right] g (r) = 0, \quad (59)

\[ - \frac{d^2 g (r)}{dr^2} - \frac{1}{r} \frac{dg (r)}{dr} + \frac{(J^+ \alpha)^2}{r^2} g (r) + M^2 \omega^2 r^2 g (r) - 2 M \omega (s J^+ \alpha - 1) g (r) - \Sigma (r) \Delta (r) g (r) \]

+ (E + M) \Sigma (r) g (r) + (E - M) \Delta (r) g (r) + (M^2 - E^2) g (r) - \left[ \frac{d \Sigma (r)}{dr} \right] f (r) = 0. \quad (60)

Notice that these two equations are coupled via the last terms and the spin and pseudospin symmetry limits uncouple them. So, here and henceforth we employ the following approach. For the spin symmetry limit, we solve the problem by considering the upper component of the spinor and denotes it by \( f^+ \) (i.e., + labels the spin symmetry solution) and for the pseudospin symmetry limit, we consider the lower component and denotes it by \( g^- \) (i.e., - labels the pseudospin symmetry solution).

A. The analysis of both the spin and the pseudo-spin symmetries

When we take into account the exact limits of spin and symmetries in Eqs. (59) and (60), each component of the spinor satisfy

\[ - \frac{d^2 f^+ (r)}{dr^2} - \frac{1}{r} \frac{df^+ (r)}{dr} + \frac{(J^+ \alpha)^2}{r^2} f^+ (r) + \varpi^2 r^2 f^+ (r) + \frac{a^+}{r} f^+ (r) + b^+ r f^+ (r) - \left( k^+ \right)^2 f^+ (r) = 0, \quad (61) \]

\[ - \frac{d^2 g^- (r)}{dr^2} - \frac{1}{r} \frac{dg^- (r)}{dr} + \frac{(J^+ \alpha)^2}{r^2} g^- (r) + \varpi^2 r^2 g^- (r) + \frac{a^-}{r} g^- (r) + b^- r g^- (r) - \left( k^- \right)^2 g^- (r) = 0, \quad (62) \]

where

\[ \left( k^\pm \right)^2 = E^2 - M^2 + 2 M \omega \left( s J^\pm \alpha \right), \quad (63) \]

\( \varpi = M \omega, a^\pm = 2 (E \pm M) \eta \) and \( b^\pm = 2 (E \pm M) \eta L \). The differential equations (61) and (62) can be placed in an convenient mode using, respectively, the following solutions:

\[ f^+ (x) = x^{J^+ \alpha} \left| e^{-\frac{i}{2} x^2} e^{-\frac{i}{2} \xi \xi_L x} y^+(x) \right|, \quad (64) \]

\[ g^- (x) = x^{J^+ \alpha} \left| e^{-\frac{i}{2} x^2} e^{-\frac{i}{2} \xi \xi_L x} y^-(x) \right|, \quad (65) \]

where \( x = \sqrt{\varpi} r \) and \( y^\pm (x) \) satisfies

\[ x \left[ y^\pm (x) \right]'' + \left[ J^\pm - 2 x^2 - \xi \xi_L x \right] \left[ y^\pm (x) \right]' + \left[ \Delta^\pm - J^\pm - 1 \right] x - \frac{1}{2} \left( J^\pm \xi \xi_L + 2 \xi \xi_L \right) \right] y^\pm (x) = 0, \quad (66) \]
where

$$\Delta^\pm = \left(\frac{\xi^\pm_L}{2}\right)^2 + \frac{(k^\pm)^2}{4\omega}$$  \hspace{1cm} (67)

$$J^\pm = 2|J^\pm_\alpha| + 1,$$  \hspace{1cm} (68)

$$\xi^\pm_C = a^\pm/\sqrt{\omega}$$ and $$\xi^\pm_L = b^\pm/\sqrt{\omega^3}.$$ Equation (66) is a homogeneous, linear, second-order, differential equations defined in the complex plane. The solutions of these equations are given in terms of the biconfluent Heun functions by [65, 66]

$$f^+(x) = c_l x^{2|J^+_\alpha|} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}ξ^+_L x} N^+ (2|J^+_\alpha|, ξ^+_L, Δ^+, 2ξ^+_C, x) + c_2 x^{-|J^+_\alpha|} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}ξ^+_L x} N^+ (-2|J^+_\alpha|, ξ^+_L, Δ^+, 2ξ^+_C, x),$$  \hspace{1cm} (69)

$$g^-(x) = c_l x^{2|J^-_\alpha|} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}ξ^+_L x} N^-(2|J^-_\alpha|, ξ^-_L, Δ^-, 2ξ^-_C, x) + c_2 x^{-|J^-_\alpha|} e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}ξ^-_L x} N^- (-2|J^-_\alpha|, ξ^-_L, Δ^-, 2ξ^-_C, x),$$  \hspace{1cm} (70)

where

$$N^\pm (2|J^\pm_\alpha|, ξ^\pm_L, Δ^\pm, 2ξ^\pm_C, x) = \sum_{q=0}^\infty \frac{A^\pm_q (2|J^\pm_\alpha|, ξ^\pm_L, Δ^\pm, 2ξ^\pm_C)}{(1 + 2|J^\pm_\alpha|)_q} q^q.$$  \hspace{1cm} (71)

The coefficients of the series are given by

$$A^+_0 = 1,$$  \hspace{1cm} (72)

$$A^\pm_1 = \frac{1}{2} (2ξ^\pm_L + ξ^\pm_L (1 + 2|J^\pm_\alpha|)),$$  \hspace{1cm} (73)

$$A^\pm_{q+2} = \left\{(q + 1)ξ^\pm_L + \frac{1}{2} [2ξ^\pm_L + ξ^\pm_L (1 + 2|J^\pm_\alpha|)]\right\} A^\pm_{q+1} - (q + 1) (q + 2|J^\pm_\alpha|) [Δ^\pm - 2|J^\pm_\alpha| - 2 - 2q] A^\pm_q,$$  \hspace{1cm} (74)

and

$$(1 + 2|J^\pm_\alpha|)_q = \frac{\Gamma (q + 2|J^\pm_\alpha| + 1)}{\Gamma (2|J^\pm_\alpha| + 1)}.$$  \hspace{1cm} (75)

From the recursion relation (74), the function

$$N^\pm (2|J^\pm_\alpha|, ξ^\pm_L, Δ^\pm, 2ξ^\pm_C, x)$$

becomes a polynomial of degree $n$, if and only if, the two following conditions are imposed [66, 67]:

$$Δ^\pm - 2 (1 + |J^\pm_\alpha|) = 2n, \quad n = 0, 1, 2, \ldots,$$  \hspace{1cm} (76)

$$A^\pm_{n+1} = 0.$$  \hspace{1cm} (77)

In this case, the $(n + 1)$th coefficient in the series expansion is a polynomial of degree $n$ in $2ξ^\pm_C$. When $2ξ^\pm_C$ is a root of this polynomial, the $(n + 1)$th and subsequent coefficients cancel and the series truncates, resulting in a polynomial form of degree $n$ for the solution $N^\pm (2|J^\pm_\alpha|, ξ^\pm_L, Δ^\pm, 2ξ^\pm_C, x)$. From the condition (76), we extract the following expressions involving the energy $E^+_{nm}$:

$$(E^+_{nm})^2 - M^2 = 2M\omega \left[ n + \frac{1}{\alpha} |m - \frac{s}{2}(1 - \alpha)| + 1 \right]$$

$$- 2M\omega \left( \frac{s}{\alpha} |m - \frac{s}{2}(1 - \alpha)| + 1 \right)$$

$$- \frac{\eta^2}{M^2\omega^2} (E^+_{nm} + M)^2,$$  \hspace{1cm} (78)

$$(E^-_{nm})^2 - M^2 = 2M\omega \left[ n + \frac{1}{\alpha} |m + \frac{s}{2}(1 - \alpha)| + 1 \right]$$

$$- 2M\omega \left( \frac{s}{\alpha} |m + \frac{s}{2}(1 - \alpha)| - 1 \right)$$

$$- \frac{\eta^2}{M^2\omega^2} (E^-_{nm} - M)^2.$$  \hspace{1cm} (79)

We notice in Eqs. (78) and (79) the absence of the parameter $\eta_C$. This stems from the fact that these expressions do not represent the energies of the system in its present form. Actually, the condition (77) allows us to establish a quantum condition that links the energy and others physical quantities, including $\eta_C$ [62, 68, 69]. As a result, it is possible to express the energy in terms of all the physical parameters involved in the problem, namely, $\eta_C, \eta_L, M$, and $\omega$. We emphasize that that, $a priori$, we are free to choose which parameter we want to fix. Here, such a quantum condition is established through the frequency $\omega$ of the system. Therefore, we now label $\omega$ as $\omega_{nm}$. Before performing the procedure, let us consider the so-
The energies in Eqs. (86a) and (86b) now depend on all the physical parameters involved in the problem. In Figs. 1 and 2, we plot the profile of these energies as a function of the parameter \( \alpha \). In both plots we clearly see that the energy levels cross each other. Moreover, there is no channel that allows the spontaneous creation of particles because none of the lines of the spectrum cross each other.

V. PARTICULAR CASES

We now return to Eq. (66). The first particular case is when \( \eta_L = 0 \), resulting in the dynamics of a two-dimensional Dirac oscillator interacting with the potential \( \eta_C/r \). In this case, the
A quantum number and we need to solve the pseudo-spin symmetry limit (Eq. (86b)) as a function of the parameter $\alpha$. Fig. (a) refers to $s = 1$ while (b) to $s = -1$ for $M = 1$, $\eta_C = 1$ and $\eta_L = 1$.

solutions are still given in terms of the Heun functions,

$$
\tilde{f}^+(x) = \tilde{c}_1 x|J^+_\alpha|e^{-\frac{1}{2}x^2}\tilde{N}^+ (2|J^+_\alpha|, 0, \Delta^+, 2\xi_\alpha^+, x) \\
+ \tilde{c}_2 x^{-1}|J^-_\alpha|e^{-\frac{1}{2}x^2}\tilde{N}^- (-2|J^-_\alpha|, 0, \Delta^+, 2\xi_\alpha^+, x),
$$

$$
\tilde{g}^-(x) = \tilde{c}_1 x|J^-_\alpha|e^{-\frac{1}{2}x^2}\tilde{N}^- (2|J^-_\alpha|, 0, \Delta^-, 2\xi_\alpha^-, x) \\
+ \tilde{c}_2 x^{-1}|J^+_\alpha|e^{-\frac{1}{2}x^2}\tilde{N}^+ (-2|J^+_\alpha|, 0, \Delta^-, 2\xi_\alpha^-, x).
$$

Then, using the condition (76), we find the energies

$$
\langle \tilde{E}^+_nm \rangle^2 - M^2 = 2M (n + |J^+_\alpha| - sJ^+_\alpha) \tilde{\omega}^+_nm,
$$

$$
\langle \tilde{E}^-nm \rangle^2 - M^2 = 2M (n + |J^-_\alpha| - sJ^-_\alpha + 2) \tilde{\omega}^-nm.
$$

Moreover, from condition (77), we consider again $A^\pm_{n+1} = 0$ for $n = 0$, and solve it for $\tilde{\omega}^\pm_{0m}$. One can thus verify that it is not possible to extract a physically acceptable expression for $\tilde{\omega}^\pm_{0m}$. Consequently, $n = 0$ is not an allowed value for the quantum number and we need to solve $A^\pm_{n+1} = 0$ for $n = 1$.

Thus, we have

$$
\tilde{\omega}^+_n = \frac{2n^2_C}{M} (\xi^+_n + M)^2, \quad (93)
$$

$$
\tilde{\omega}^-_n = \frac{2n^2_C}{M} (\xi^-_n - M)^2. \quad (94)
$$

Substituting (93) and (94) into (91) and (92) and solving these equations for $\xi^\pm_n$, we find

$$
\langle \tilde{E}^+_nm \rangle_p = \frac{1 + 2n^2_C}{x} \left( 1 + \xi^+_n - 2sJ^+_\alpha \right) M, \quad (95a)
$$

$$
\langle \tilde{E}^-nm \rangle_{ap} = -M, \quad (95b)
$$

and

$$
\langle \tilde{E}^+_nm \rangle_p = \frac{1 + 2n^2_C}{x} \left( 1 + \xi^-_n - 2sJ^-_\alpha + 5 \right) M, \quad (96a)
$$

$$
\langle \tilde{E}^-nm \rangle_{ap} = \frac{1 + 2n^2_C}{x} \left( 1 - \xi^-_n - 2sJ^-_\alpha + 5 \right) M. \quad (96b)
$$
where the subscripts $p$ and $ap$ refer to the energies of the particle and antiparticle, respectively. As we are studying the dynamics for which $\mathcal{E}_{om}^\pm \neq \pm M$, the energies $[\mathcal{E}_{om}^+]_{ap}$ and $[\mathcal{E}_{om}^-]_{p}$ are not allowed energies for the particle. The profiles of the energies (95a) and (96b) are shown in Figs. 3 and 4, respectively. We can observe in Fig. 3(a) ($s = +1$) the presence of degeneracy for $m = -2, -1, 0$, while in Fig. 3(b) ($s = -1$), the degeneracy occurs for $m = 0, 1, 2$. We also investigate the energy profile of Eq. (96b), where we verify the presence of degeneracy for $m = -2$ and $m = -1$ (see Fig. 4(a)) for $s = +1$, and $m = 1$ and $m = 2$ (see Fig. 4(b)) for $s = -1$ for some values of $\alpha$.

The second particular case is when $\eta_C = 0$. In this case, the system consists of a Dirac oscillator interacting with a linear potential, $\eta_Lr$. Thus, the solutions of Eq. (66) is again given in terms of the Heun functions,

\[
\tilde{f}^+(x) = \tilde{c}_1 x |J_p| e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}x^2} N^+(2 |J_p^+|, \xi_L^+, \Delta^+, 0, x) + \tilde{c}_2 x |J_p| e^{-\frac{1}{2}x^2} e^{-\frac{1}{2}x^2} N^+(2 |J_p^+|, \xi_L^+, \Delta^+, 0, x),
\]

FIG. 4. (Color online) Energy in the pseudo-spin symmetry limit (Eq. (96b)) as a function of the parameter $\alpha$ for the particular case when $\eta_L = 0$. Fig. (a) refers to $s = +1$ and (b) to $s = -1$. In both plots we take $M = 1$ and $\eta_C = 1$.

\[\tilde{E}_{nm}^+ = 2M \tilde{\omega}(n + |J_p^+| - sJ_p^+) - \frac{\eta_L^2}{M^2 \tilde{\omega}^2} (\tilde{E}_{nm}^+ + M)^2, \quad (97)\]

and

\[\tilde{E}_{nm}^- = 2M \tilde{\omega}(n + |J_p^+| - sJ_p^+) - \frac{\eta_L^2}{M^2 \tilde{\omega}^2} (\tilde{E}_{nm}^- - M)^2. \quad (98)\]

Note that energies (97) and (98) are identical to those given in Eqs. (78) and (79). However, the frequency $\tilde{\omega}$ is not the same. The difference between them is just the imposition established by the condition (77). For $n = 0$, we obtain the frequencies

\[\tilde{\omega}_{om}^+ = 0, \quad (99)\]

\[\tilde{\omega}_{om}^- = 0. \quad (100)\]

By substituting (99) and (100) into the respective energies (97) and (98), we find

\[\tilde{E}_{om}^+ = -M \pm M, \quad (101)\]

\[\tilde{E}_{om}^- = M \pm M. \quad (102)\]

For $n = 1$, we have

\[\tilde{\omega}_{1m}^+ = \left[ \frac{\eta_L^2}{M} \left( 1 + \frac{1}{2} J_p^+ \right) \right] \tilde{E}_{1m}^+ + M, \quad (103)\]

\[\tilde{\omega}_{1m}^- = \left[ \frac{\eta_L^2}{M} \left( 1 + \frac{1}{2} J_p^- \right) \right] \tilde{E}_{1m}^- - M, \quad (104)\]

and the energies are given by

\[\tilde{E}_{1m}^+ = 2M \tilde{\omega}_{1m} (n + |J_p^+| - sJ_p^+) - \frac{\eta_L^2}{M^2 \tilde{\omega}_{1m}^2} (\tilde{E}_{1m}^+ + M)^2, \quad (105)\]

\[\tilde{E}_{1m}^- = 2M \tilde{\omega}_{1m} (n + |J_p^+| - sJ_p^+) - \frac{\eta_L^2}{M^2 \tilde{\omega}_{1m}^2} (\tilde{E}_{1m}^- - M)^2. \quad (106)\]

with $\tilde{\omega}_{1m}^\pm$ given in Eqs. (103) and (104). For this particular case, it is verified that both Eqs. (105) and (106) present four eigenvalues of energy. However, only two of them are physically acceptable. The profiles of these two energies are illustrated in Figs. 5 and 6, respectively. We also see that both particle and antiparticle belong to the same spectrum and contains no degeneracy.

Finally, the last case we want to discuss in that in which $\eta_L = \eta_C = 0$ in Eq. (66). In this case, the solutions (69) and (70) take the form

\[f^+(x) = x |J_p| e^{-\frac{1}{2}x^2} F^+(x), \quad (107)\]

\[g^-(x) = x |J_p| e^{-\frac{1}{2}x^2} F^-(x), \quad (108)\]
where \( x = \sqrt{\omega r} \) and \( F^\pm(x) \) satisfies the Kummer differential equation [64, 65]

\[
(F^\pm)^\prime\prime(x) + \left( \frac{2|J_\alpha^+| + 1}{x} - 2x \right) (F^\pm)^\prime(x) + \left[ \tilde{\Delta}^\pm - (2|J_\alpha^+| + 2) \right] (F^\pm)(x) = 0,
\]

whose general solution is known to be

\[
F^\pm(x) = a_nM \left( \frac{1}{2} + \frac{|J_\alpha^+|}{2} - \frac{\tilde{\Delta}^\pm}{4} \right) \left[ 1 + |J_\alpha^+|, x^2 \right] + b_nx^{-2|J_\alpha^+|}M \left( \frac{1}{2} - \frac{|J_\alpha^+|}{2} - \frac{\tilde{\Delta}^\pm}{4} \right) \left[ 1 - |J_\alpha^+|, x^2 \right],
\]

(109)

In the above equations, \( M \) is the Kummer function [64, 65]. For this particular case, if we write the condition (76) in the form

\[
\frac{1}{2} + \frac{|J_\alpha^+|}{2} - \frac{\tilde{\Delta}^\pm}{4} = -n',
\]

(110)

with \( n' = 0, 1, 2, \ldots \), where \( \tilde{\Delta}^\pm = \left( \tilde{k}^\pm \right)^2/M\omega \) and \( \left( \tilde{k}^\pm \right)^2 = (\epsilon_{nm})^2 - M^2 - 2M\omega (sJ_\alpha^+ \pm 1) \), the energies of the oscillator are obtained. Since \( V(r) = S(r) = 0 \), spin and pseudo-spin symmetries are now absent, and signals \((\pm)\) in Eq. (110) are only used to represent the function \( f^+(x) \), \( g^-(x) \) (components of \( \psi \)) of Eq. (58) with positive and negative energy, respectively, of the particle. In this way, the eigenvalues of Eq. (109) are given by

\[
(\epsilon_{nm}^+)^2 - M^2 = M\omega \left[ 2n + J^+ + 1 \right] - 2M\omega (sJ_\alpha^+ + 1),
\]

(111)

\[
(\epsilon_{nm}^-)^2 - M^2 = M\omega \left[ 2n + J^- + 1 \right] - 2M\omega (sJ_\alpha^- - 1),
\]

(112)

and the unnormalized bound state wave functions are

\[
f^+(x) = x^{\left| J_\alpha^+ \right|}e^{-\frac{1}{2}x^2}M \left[ -n + 1 + \left| J_\alpha^+ \right|, x^2 \right],
\]

(113)

\[
g^-(x) = x^{\left| J_\alpha^- \right|}e^{-\frac{1}{2}x^2}M \left[ -n - 1 + \left| J_\alpha^- \right|, x^2 \right].
\]

(114)

The profile of the energies (111) and (112) are plotted in Figs. 7 e 8, respectively. In Fig. 7(a), we have degeneracy when
m = 1 e m = 2 for s = 1 while in Fig. 7(b) the degeneracy occurs when m = −2 and m = −1 for s = −1. On the other hand, in Fig. 8(a), only m = −2 is non-degenerate while in Fig. 8(b), only m = 2 is non-degenerate. This system has been studied in Ref. [70] for the particular case when the oscillator interacts with the Aharonov-Bohm potential.

VI. CONCLUSION

In this paper, we have studied the motion of a 2D Dirac oscillator interacting with cylindrically symmetric scalar and vector potentials in the space-time of the cosmic string. The problem was solved taking into account the spin and pseudospin symmetry exact limits through two stages. First we have solved the Dirac equation by looking for first order solutions. We used an appropriate ansatz for the Dirac equation and obtained a system of coupled first order differential equations. We investigated this system and verified that it admits physically acceptable particular solutions, i.e., bound states solutions, only at the pseudo-spin symmetry exact limit, Σ = 0 e E = M. In the second stage, we have constructed and solved the Dirac equation in its quadratic form, which discards E ≠ ±M from its solution. For this case, we shown that the resulting radial differential equation is biconfluent Heun equation. We studied the series solution of this equation as well as its asymptotic behavior at infinity and at origin and found two conditions (Eqs. (76) and (77)) to make the series a polynomial. The use of these two conditions allowed us to obtain expressions for the energies corresponding to the state n = 0, Eqs. (86a) and (86b). We have plotted these energies as a function of the parameter α for the allowed values of s. In all the graphs of energies of this work, we considered m = 0, ±1, ±2 for the angular momentum quantum number. For these values of angular momentum, we have not observed the presence of degeneracy in this first investigation.

We also investigated some special cases for the solution of the Eq. (58). In the first case, we have assumed that the vanishing of the linear potential by imposing ηL = 0. We obtained the spectrum (Eqs. (95a) and (96b)) and plotted it as function of the parameter α. In the energy profile (95a), we observed the presence of degeneracy for m = −2, −1, 0, 2 (Fig. 3(a)) for s = 1 and m = −2, 0, 1, 2 (Fig. 3(b)) for s = −1. In the energy profile (96b), we have degeneracy for
m = −2 and m = −1 (Fig. 4(a)) for s = +1, and m = 1 and m = 2 (Fig. 4(b)) for s = −1. In the second particular case investigated, we considered ηC = 0, and we have found that both Eqs. (105) and (106) have four energy eigenvalues, but only two of them are physically acceptable because of the requirement that E ≠ ±M. The energy profiles (Figs. 5 and 6 for s = ±1) showed no degeneracy. In the last particular case studied, we have assumed ηL = ηC = 0. For this system, the resulting radial equation was an equation Kummer differential equation type. We solved this equation and obtained its energy spectrum (Eqs. (111) and (112)). In the energy profile (111), we have verified degeneracy for m = 1 and m = 2 (Fig. 7(a)) for s = +1, and m = −2 and m = −1 (Fig. 7(b)) for s = −1. In the profile of the energy (112), only m = −2 (Fig 8(a)) for s = +1 and m = 2 (Fig. 8(b)) for s = −1 are non-dregerative. A feature present in all energy profiles, including the general case, is the absence of channel that allows creation of particles, and also no crossings of lines, which guarantees that particle and antiparticle belong to the same spectrum.

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Otherwise, we shall have an overall phase factor of the kind $e^{i\psi z}$ in the final wave function.