Holomorphic Equivariant Cohomology via a Transversal Holomorphic Vector Field *

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Abstract In this paper an analytic proof of a generalization of a theorem of Bismut ([Bis1, Theorem 5.1]) is given, which says that, when \( v \) is a transversal holomorphic vector field on a compact complex manifold \( X \) with a zero point set \( Y \), the embedding \( j : Y \rightarrow X \) induces a natural isomorphism between the holomorphic equivariant cohomology of \( X \) via \( v \) with coefficients in \( \xi \) and the Dolbeault cohomology of \( Y \) with coefficients in \( \xi|_Y \), where \( \xi \rightarrow X \) is a holomorphic vector bundle over \( X \).

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1 Introduction

For a compact complex manifold \( X \), let \( T_CX \) denote the complexification of the real tangent bundle \( T_RX \) of \( X \). Then \( T_CX \) splits canonically as \( T_CX = TX \oplus \overline{TX} \), where \( TX \) and \( \overline{TX} \) are the holomorphic and the anti-holomorphic tangent bundle of \( X \), respectively. Let \( \xi \rightarrow X \) be a holomorphic vector bundle over \( X \). There exists a natural \( \mathbb{Z} \)-grading in \( \Lambda(T^*_C X) \otimes \xi \) defined by the following decomposition:

\[
\Lambda(T^*_C X) \otimes \xi = \bigoplus_{-\dim X \leq r \leq \dim X} \Lambda^{(r)} (T^*_C X) \otimes \xi,
\]

where

\[
\Lambda^{(r)} (T^*_C X) \otimes \xi = \bigoplus_{q-p=r} \Lambda^q(T^*X) \otimes (\Lambda^p(T^*X) \otimes \xi).
\]

We will use \( \Lambda^{(r)} (T^*_C X) \otimes \xi \) to denote \( \Lambda(T^*_C X) \otimes \xi \) with this \( \mathbb{Z} \)-grading. Let \( \Omega^{(r)}(X, \xi) \) (resp. \( \Omega^{(r)}(X, \xi) \)) be the complex vector space of smooth sections of \( \Lambda^{(r)}(T_C X) \otimes \xi \) (resp.

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Then \( (\Omega^{(\cdot)}(X, \xi) = \bigoplus_r \Omega^{(r)}(X, \xi), \partial^{X}) \) is a complex and the cohomology groups \( H^{(r)}(X, \xi) \) associated to this complex are direct sums:

\[
H^{(r)}(X, \xi) = \bigoplus_{p+q=r} H^{p,q}(X, \xi),
\]

where \( H^{p,q}(X, \xi) \) are the usual Dolbeault cohomology groups of \( X \) with coefficients in the holomorphic vector bundle \( \xi \).

For any holomorphic vector field \( v \) on \( X \), set

\[
\bar{\partial}^{X}_v = \bar{\partial}^{X} + i(v) : \Omega^{(\cdot)}(X, \xi) \to \Omega^{(\cdot+1)}(X, \xi),
\]

where \( i(v) \) is the standard contraction operator defined by \( v \). The consideration of operators \( \bar{\partial}^{X} \) and \( i(v) \) together goes back to [Bot]. Clearly, \( (\Omega^{(\cdot)}(X, \xi), \bar{\partial}^{X}_v) \) is also a \( \mathbb{Z} \)-graded complex. Denote the cohomology groups associated to this complex by \( H^{(\cdot)}_{v}(X, \xi) \), which are called the **holomorphic equivariant cohomology** groups of \( X \) via \( v \) with coefficients in \( \xi \) (cf. [L]).

In [CL], Carrell and Lieberman discussed the relation between Dolbeault cohomology of a connected compact Kähler manifold \( X \) and the zero point set \( Y \) of a holomorphic vector field \( v \) on \( X \) by using Deligne degeneracy criterion and proved that \( H^{(r)}(X, \mathbb{C}) \) vanished for all \( |r| > \dim Y \) if \( Y \neq \emptyset \). Since \( \dim H^{(1)}_{v}(X, \mathbb{C}) = \dim H^{(1)}(X, \mathbb{C}) \) in this case (see [L, Theorem 1.3] and [CL]), the corresponding vanishing results for \( H^{(1)}_{v}(X, \mathbb{C}) \) are also valid. For a general compact complex manifold \( X \) and a transversal holomorphic vector field \( v \) on \( X \) (see a definition in [Bis1, Sect.5.1]), Liu in [L] proved that \( \dim H^{(1)}_{v}(Y, \mathbb{C}) \leq \dim H^{(1)}(X, \mathbb{C}) \) by constructing an injective homomorphism \( \alpha_{v} : H^{(1)}_{v}(Y, \mathbb{C}) \to H^{(1)}_{v}(X, \mathbb{C}) \). Moreover, Liu got a counting formula for \( \dim H^{(1)}_{v}(X, \mathbb{C}) \) in terms of the multiplicities of the zero points of \( v \) if the zero points of \( v \) are discrete. Under the assumption that the zero point of \( v \) is nodegenerate, motivated by Witten’s deformation idea ([W]), he also sketched an analytic proof of his formula in [L, Sect.7] by examining the behavior of a natural deformation \( D^{X}_T \) of the Riemann-Roch operator on \( X \) as \( T \to \infty \).

A more general result in this direction is due to Bismut. By using the technique of spectral sequences in [Bis1, Theorem 5.1], Bismut proves that, when \( v \) is a transversal holomorphic vector field on \( X \) with a zero point set \( Y \), the embedding \( j : Y \to X \) induces naturally a quasi-isomorphism \( j^{*} : (\Omega^{(\cdot)}(X, \mathbb{C}), \bar{\partial}^{X}) \to (\Omega^{(\cdot)}(Y, \mathbb{C}), \bar{\partial}^{Y}) \).

In this paper, we will give an analytic proof of the following fairly straightforward generalization of the Bismut’s theorem [Bis1, Theorem 5.1]:

\[
\Lambda^{(\cdot)}(T_{\xi}X) \otimes \xi).
\]
Theorem 1.1 Let \( v \) be a transversal holomorphic vector field on a compact complex manifold \( X \) with a zero point set \( Y \). Then

\[
j^*: (\Omega^1(X, \xi), \bar{\partial}_v^X) \rightarrow (\Omega^1(Y, \xi|_Y), \bar{\partial}^Y)
\]

is a quasi-isomorphism.

Following Witten’s deformation idea ([W]) as in [L, Sect.7], we will also work with a deformation \( D^X_T \) of the twisted Riemann-Roch operator on \( X \) by \( \xi \) but the whole proof now is heavily based on the analytic localization techniques developed by Bismut and Lebeau (cf. [BL]), since the analysis involved here is much complicated than the situation in [L, Sect.7]. A key point in our proof is to express \( D^X_T \) and to trivialize the bundle on which \( D^X_T \) acts by using the Bismut connection \( \nabla^{-B} \) (see [Bis2, II. b]). We should point out that if the trivialization is made by a lifting of the holomorphic Hermitian connection \( \nabla^{TX} \) on \( TX \), then an extra term coming from the torsion of \( \nabla^{TX} \) will enter to the final operator on \( Y \) in the Bismut-Lebeau localization process of \( D^X_T \) as \( T \rightarrow \infty \). But it is not clear to us that the extra term is zero for a general complex manifold \( X \). Consequently, we could only obtain an equality related to the involved operators at the index level.

2 A deformed twisted Riemann-Roch operator \( D^X_T \) and its local behavior near \( Y \)

This section is divided into three parts. In a) we introduce a deformation \( D^X_T \) of the twisted Riemann-Roch operator by \( \xi \) via \( v \), which has been used in [L, Sect.7] in the case of \( \xi = C \). In b) we recall the definition of the Bismut connection \( \nabla^{-B} \) (cf. [Bis2, II. b]) and express \( D^X_T \) in this connection by a direct application of [Bis2, Theorem 2.2]. In c) we study the local behavior of the deformed operator \( D^X_T \) near the submanifold \( Y \) following [BL, Sect.8], in which the Bismut connection will play an essential role.

a) A deformed twisted Riemann-Roch operator \( D^X_T \)

Let \( X \) be a compact complex manifold of \( C \)-dimension \( n \). For any \( T \in R \), we consider the following deformed operator

\[
\bar{\partial}^X_T = \bar{\partial} + Ti(v): \Omega^1(X, \xi) \rightarrow \Omega^{1}(X, \xi)
\]
and the deformed complex \((\Omega^{(\cdot)}(X, \xi), \overline{\partial}^X_\xi)\). One verifies easily as the proof of [L, Lemma 1.1] that the cohomologies associated to this deformed complex do not depend on \(T \neq 0\).

Let \(g^TX\) (resp. \(g^\xi\)) be a Hermitian metric on \(X\) (resp. \(\xi\)). By the standard procedure there is an induced Hermitian metric \(\langle \cdot, \cdot \rangle_{\Lambda^{(\cdot)}(T^*_C X) \otimes \xi}\) on \(\Lambda^{(\cdot)}(T^*_C X) \otimes \xi\). Let \(dv_X\) denote the Riemannian volume element of \((X, g^TX)\). Then for \(s_1, s_2 \in \Omega^{(\cdot)}(X, \xi)\),

\[
\langle \langle s_1, s_2 \rangle \rangle = \frac{1}{(2\pi)^{\dim C_X}} \int_X \langle s_1, s_2 \rangle_{\Lambda^{(\cdot)}(T^*_C X) \otimes \xi} dv_X
\]

defines an \(L^2\)-Hermitian inner product on \(\Omega^{(\cdot)}(X, \xi)\). Let \(\bar{\partial}^X\) and \(\bar{\partial}^X_\xi\) denote the formal adjoint operators of \(\partial^X\) and \(\partial^X_\xi\), respectively. For any \(U \in T^*_C X\), let \(U^*\) be an element in \(T^*_C X\) defined by \(g^TX(U, \cdot)\). Clearly, \(\bar{\partial}^*\) is a (1,0)-form on \(X\) and \(\bar{\partial}^*\wedge\) is the dual operator of \(i(v)\). Moreover, we have

\[
\bar{\partial}^X_\xi = \bar{\partial}^X + T \bar{\partial}^* \wedge : \Omega^{(\cdot)}(X, \xi) \to \Omega^{(-1)}(X, \xi).
\]

Set

\[
D^X = \sqrt{2}(\bar{\partial}^X + \bar{\partial}^X_\xi), \quad D^X_T = \sqrt{2}(\bar{\partial}^X + \bar{\partial}^X_\xi).
\]

Clearly, \(D^X\) is the usual twisted Riemann-Roch operator by the holomorphic bundle \(\xi\) and \(D^X_T\) is a deformation of \(D^X\) and interchanges \(\Omega^{(\text{even})}(X, \xi)\) and \(\Omega^{(\text{odd})}(X, \xi)\). From Hodge theory we have the following isomorphisms

\[
\ker (D^X_T)^2|_{\Omega^{(\cdot)}(X, \xi)} \cong H^{(r)}_\nu(X, \xi).
\]

**Lemma 2.1** For any open neighborhood \(U\) of \(Y\), there exist constants \(a > 0, b > 0\) and \(T_0 > 0\) such that for any \(s \in \Omega^{(\cdot)}(X, \xi)\) with \(\text{Supp} \ s \subset X \setminus U\) and any \(T \geq T_0\), one has the following estimate for Sobolev norms,

\[
\|D^X_T s\|^2_0 \geq a(\|s\|^2_1 + (T - b)\|s\|^2_0).
\]

**Proof.** An easy computation shows that

\[
D^X_T = D^X + 2T|v|^2 + 2T \left( (\bar{\partial}^X \bar{\partial}^* + i(\bar{\partial}^X \bar{\partial}^*)) \right),
\]

where \(i(\bar{\partial}^X \bar{\partial}^*)\) denotes the adjoint operator of \((\bar{\partial}^X \bar{\partial}^*)\). Note that \((\bar{\partial}^X \bar{\partial}^*)\) is a zero order operator and \(v \neq 0\) on \(X \setminus U\), the lemma follows from the well-known Garding’s inequality directly. 

\[
\square
\]
By Lemma 2.1 and Hodge theory, we can study $H^1_{v}(X, \xi)$ through the behavior of the operator $D_X^X$ near $Y$ for large $T$. Also by Lemma 2.1, it is an easy observation that, when $Y = \emptyset$, the cohomology group $H^1_{v}(X, \xi)$ vanishes. In the following we will always assume that $v$ is transversal and $Y \neq \emptyset$. Note that generally $Y$ consists of some connected components $Y_k$ with different $C$-dimensions $l_k$. When no confusion arises, we always drop the subscripts and simply denote them by $Y$ and $l$, respectively.

b) An expression of $D_X^X$ in the Bismut connection

We first recall the definition of the Bismut connection in [Bis2, II. b)]. For a complex manifold $X$ with a Hermitian metric $g^{TX}$, let $\nabla^{TX}$ be the holomorphic Hermitian connection on $TX$. Note that $\nabla^{TX}$ induces naturally an Euclidean connection on $T^R X$ which preserves the complex structure of $T^R X$. Let $T_X$ denote the torsion tensor of the connection $\nabla^{TX}$. Let $B_X$ be the antisymmetrization of the tensor $(U, V, W) \rightarrow \frac{1}{2}\langle T_X(U, V), W \rangle$ and let $S^{-B_X}$ denote the one form with values in antisymmetric elements of $\text{End}(T_R X)$ which is such that

$$\langle S^{-B_X}(U)V, W \rangle = -2B_X(U, V, W), \quad (2.8)$$

where $U, V, W \in T_R X$. Let $\nabla^{L_X}$ be the Levi-Civita connection on $T_R X$. Set

$$S_X = \nabla^{TX} - \nabla^{L_X}. \quad (2.9)$$

Then $S_X$ is also a one form with values in antisymmetric elements of $\text{End}(T_R X)$. The important thing here is that $S^{-B_X} - S_X$ preserves the complex structure of $T^R X$ (cf. [Bis2, (2.38)])). Now the Bismut connection $\nabla^{-B_X}$ on $T_R X$ is defined by (cf. [Bis2, (2.37)])

$$\nabla^{-B_X} = \nabla^{TX} + (S^{-B_X} - S_X) = \nabla^{L_X} + S^{-B_X}. \quad (2.10)$$

The Bismut connection $\nabla^{-B_X}$ also preserves the complex structure of $T_R X$ and so induces naturally a unitary connection on $TX$ and a unitary connection on $T^X$, which are still denoted by $\nabla^{-B_X}$. Note that when $X$ is Kähler, the Bismut connection $\nabla^{-B_X}$ coincides with the holomorphic Hermitian connection $\nabla^{TX}$. There is a unitary connection on $\Lambda'(TX)$ lifted canonically from $\nabla^{-B_X}$, which we still call the Bismut connection and denote by the same notation $\nabla^{-B_X}$.

Let $Y$ be a complex submanifold of $X$. Let $\pi : N \rightarrow Y$ be the normal bundle of $Y$ in $X$. We identify $N$ with the sub-bundle of $TX|_Y$ orthogonal to $TY$ with respect to the restriction metric $g^{TX|_Y}$ on $TX|_Y$ by $g^{TX}$. So we have the identification of $C^\infty$ bundles $TX|_Y = TY \oplus N$. Let $g^{TY}$ (resp. $g^N$) be the induced metric on $TY$ (resp. $N$) from $g^{TX|_Y}$. Let $P^{TrX}, P^{N_R}$ be the orthogonal projection operators from $T_R X|_Y$ onto $T_R Y$ and $N_R$.
respectively. Let $j$ denote the embedding of $Y$ into $X$. Then $j^*\nabla^{-B_x}$ is a connection on $T_{\mathbb{R}X}|_Y$ preserving the metric $g^{TX|_Y}$ and the complex structure of $T_{\mathbb{R}X}|_Y$. Moreover, $p^T_{\mathbb{R}Y}(j^*\nabla^{-B_x}) p^T_{\mathbb{R}Y}$ is exactly the Bismut connection on $T_{\mathbb{R}Y}$ associated to the induce metric $g^{TY}$, and $p^N_{\mathbb{R}}(j^*\nabla^{-B_x}) p^N_{\mathbb{R}}$ is a connection on $N_{\mathbb{R}}$ preserving the metric $g^N$ and the complex structure of $N_{\mathbb{R}}$. Set

$$\nabla^{-B_x,\oplus} = \nabla^{-B_Y} \oplus \left( p^N_{\mathbb{R}}(j^*\nabla^{-B_x}) p^N_{\mathbb{R}} \right),$$

(2.11)

$$A = j^*\nabla^{-B_x} - \nabla^{-B_x,\oplus}.$$  

(2.12)

Clearly, $\nabla^{-B_x,\oplus}$ is also a connection on $T_{\mathbb{R}X}|_Y$ preserving the metric $g^{TX|_Y}$ and the complex structure of $T_{\mathbb{R}X}|_Y$, and $A$ is the second fundamental form of the Bismut connection $\nabla^{-B_x}$. We still use the same notation $\nabla^{-B_x,\oplus}$ to denote its restriction on $T^X|_Y$ as well as its lifting on $\Lambda(T^*X|_Y)$.

Now we return to our situation and express the deformed twisted Riemann-Roch operator $D^X_T$ in the Bismut connection by applying [Bis2, Theorem 2.2]. To do this we still need a holomorphic Hermitian connection on the bundle $\Lambda(T^*X) \otimes \xi$. Since our problem does not depend on the metrics, we can and will choose a special metric on the holomorphic bundle $\Lambda(T^*X)$ to simplify the analysis.

Let $L_\nu : TX|_Y \rightarrow TX|_Y$ be the holomorphic Lie homomorphism defined by $L_\nu(u) = [v, u]$ for any $u \in TX|_Y$. Denote $L_\nu(TX|_Y)$ by $N$. Since $v$ is transversal, $TX|_Y$ splits holomorphically into $TY \oplus N$ and $L_\nu$ induces an isomorphism from $N$ to $\tilde{N}$, which we still denote by $L_\nu$. We introduce a new Hermitian metric $\tilde{g}^N$ on $\tilde{N}$ by requiring that $L_\nu : N \rightarrow \tilde{N}$ is unitary fiberwisely. Consequently, we get a new Hermitian metric $\tilde{g}^{TX|_Y} = \tilde{g}^{TY} \oplus \tilde{g}^N$ on $TX|_Y$. We can and we will extend $\tilde{g}^{TX|_Y}$ to a Hermitian metric $\tilde{g}^{TX}$ on $TX$. We will denote $TX$ (resp. $TX|_Y$) with the metric $\tilde{g}^{TX}$ (resp. $\tilde{g}^{TX|_Y}$) by $T\tilde{X}$ (resp. $T\tilde{X}|_Y$) to distinguish the same bundle with different metrics. Let $\nabla^{T\tilde{X}}$, $\nabla^{T\tilde{X}|_Y}$, resp. $\tilde{N}$ be the holomorphic Hermitian connection on $T\tilde{X}$ (resp. $T\tilde{X}|_Y$, resp. $\tilde{N}$). We have the following standard fact:

$$j^*\nabla^{T\tilde{X}} = \nabla^{T\tilde{X}|_Y} = \nabla^{TY} \oplus \nabla^{\tilde{N}}.$$  

(2.13)

We lift the holomorphic Hermitian connection $\nabla^{T\tilde{X}}$ (resp. $\nabla^{TY}$) to the holomorphic Hermitian connection $\nabla^A(T^*\tilde{X})$ (resp. $\nabla^A(T^*Y)$) on $\Lambda(T^*\tilde{X})$ (resp. $\Lambda(T^*Y)$).

Let $g^\xi$ be a Hermitian metric on $\xi$ and let $\nabla^\xi$ be the holomorphic Hermitian connection on $\xi$. Set $\nabla^{\xi|_Y} = j^*\nabla^\xi$, which is the holomorphic Hermitian connection on $\xi|_Y$. So $\nabla^A(T^*\tilde{X}) \otimes \xi^\oplus = \nabla^A(T^*\tilde{X}) \otimes 1 + 1 \otimes \nabla^\xi$ (resp. $\nabla^A(T^*Y) \otimes \xi|_Y$) is a
holomorphic Hermitian connection on \( \Lambda^\vee(T^*X) \otimes \xi \) (resp. \( \Lambda^\vee(T^*Y) \otimes \xi|_Y \)). Therefore,

\[
\nabla^{B,X} = \nabla^{-B_X} \otimes 1 + 1 \otimes \nabla^{\Lambda^\vee(T^*X) \otimes \xi},
\]

\[
\nabla^{B,Y} = \nabla^{-B_Y} \otimes 1 + 1 \otimes \nabla^{\Lambda^\vee(T^*Y) \otimes \xi|_Y}
\]

are unitary connections on the Hermitian vector bundle \( \Lambda^\vee(T^*C_X) \otimes \xi \), \( \Lambda^\vee(T^*C_Y) \otimes \xi|_Y \), respectively.

For \( U \in (TX, g^{TX}) \), set

\[
c(U) = \sqrt{2} U^* \wedge, \quad c(\bar{U}) = -\sqrt{2} \bar{U}(U);
\]

and for \( U \in (T\bar{X}, g^{TX}) \), set

\[
c(\bar{U}) = -\sqrt{-2} \bar{U}^* \wedge.
\]

We extend the map \( c \) (resp. \( \bar{c} \)) by \( \mathbb{C} \) linearity into the Clifford action of \( T_CX \) (resp. \( T_{\bar{C}}X \)) on \( \Lambda^\vee(T^*X) \) (resp. \( \Lambda^\vee(T^*X) \)).

Let \( \{e_1, \ldots, e_{2n}\} \) be an orthonormal basis \( T_{\mathbb{R}}X \). Set

\[
c(B_X) = \frac{1}{6} \sum_{i,j,k=1}^{2n} B_X(e_i, e_j, e_k) c(e_i) c(e_j) c(e_k).
\]

Note that

\[
\frac{1}{4} \sum_{i,j,k=1}^{2n} \langle S^{-B_X}(e_i) e_j, e_k \rangle c(e_i) c(e_j) c(e_k) = -3c(B_X).
\]

Now recall the definition (2.10) and apply [Bis2, Theorem 2.2] directly, we obtain the following expressions of \( D^Y \) and \( D^X_T \) with respect to the orthonormal basis \( \{e'_1, \ldots, e'_l\} \) for \( T_{\mathbb{R}}Y \) and \( \{e_1, \ldots, e_{2n}\} \) for \( T_{\mathbb{R}}X \), respectively:

\[
D^Y = \sum_{i=1}^{2l} c(e'_i) \nabla^{B,Y}_{e'_i} + 2c(B_Y),
\]

\[
D^X_T = \sum_{i=1}^{2n} c(e_i) \nabla^{B,X}_{e_i} + 2c(B_X) + \sqrt{-1} T(\bar{c}(v) + \bar{c}(\bar{v})).
\]

c) The local behavior of the deformed operator \( D^X_T \) near \( Y \)
For $y \in Y$ and $Z \in N_{R,y}$, let $t \in \mathbb{R} \to x_t = \exp_{\frac{t}{y}}(tZ) \in X$ be the geodesic in $X$ with respect to the Levi-Civita connection $\nabla^{L_X}$, such that $x_0 = y$, $dx/dt|_{t=0} = Z$. For $\epsilon > 0$, let $B_\epsilon = \{ Z \in N_{\mathbb{R}} \mid |Z| < \epsilon \}$. Since $X$ and $Y$ are compact, there exists an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the map $(y, Z) \in N_{\mathbb{R}} \to \exp_{\frac{t}{y}}(Z) \in X$ is a diffeomorphism from $B_\epsilon$ to a tubular neighborhood $U_\epsilon$ of $Y$ in $X$. From now on, we will identify $B_\epsilon$ with $U_\epsilon$ and use the notation $x = (y, Z)$ instead of $x = \exp_{\frac{t}{y}}(Z)$.

We will make use of the trivialization of $(\Lambda^1(T^*_C Y) \otimes \xi)|_{U_{\epsilon_0}}$ by the parallel transport of $(\Lambda^1(T^*_C Y) \otimes \xi)|_{Y}$ with respect to the connection $\nabla^{B,Y}$ along the geodesic $t \mapsto (y, tZ)$. The key point here is that this trivialization preserves the metric and the $\mathbb{Z}$-grading since the Bismut connection is a unitary connection and preserves the complex structure of $T_RX$. By using the trivialization of $\Lambda^1(T^*_C Y) \otimes \xi$ over $U_{\epsilon_0}$, we can and will make the identification of $(\Lambda^1(T^*_C Y) \otimes \xi)|_{U_{\epsilon_0}}$ with $\pi^*((\Lambda^1(T^*_C Y) \otimes \xi)|_{Y})|_{B_{\epsilon_0}}$, and so we can consider $\nabla^{B,Y}$ as a unitary connection on the Hermitian vector bundle $\pi^*((\Lambda^1(T^*_C Y) \otimes \xi)|_{Y})|_{B_{\epsilon_0}}$ with the obviously induced metric. Note that there exists another unitary connection $\nabla^{B,Y,\oplus}$ on $\pi^*((\Lambda^1(T^*_C Y) \otimes \xi)|_{Y})$ defined by

$$\nabla^{B,Y,\oplus} = \pi^* \left( \nabla^{B,Y} \otimes 1 + 1 \otimes j^* \nabla^{\Lambda^1(T^*_C Y) \otimes \xi} \right).$$

Let $dv_Y$ (resp. $dv_N$) denote the Riemannian volume element of $(Y, g^Y)$ (resp. the fibres of $(N, g^N)$). We define a smooth positive function $k(y, Z)$ on $B_{\epsilon_0}$ by the equation $dv_X(y, Z) = k(y, Z)dv_Y(y)dv_{N,y}(Z)$ and an $L^2$-Hermitian inner product on $E$ by

$$\langle f, g \rangle = \int_Y \int_{N_{R,y}} \langle f, g \rangle(y, Z)dv_{N,y}(Z)dv_Y(y),$$

for any $f, g \in E$ with compact support, where $E$ denotes the set of smooth sections of $\pi^*((\Lambda^1(T^*_C Y) \otimes \xi)|_{Y})$ on $N_{R}$. Clearly, $k(y) = k(y, 0) = 1$ on $Y$ and $k(y, Z)$ has a positive lower bound on $U_{\epsilon_0/2}$. If $f \in E$ has compact support in $B_{\epsilon_0}$, we can identify $f$ with an element in $\Omega^1(X, \xi)$ which has compact support in $U_{\epsilon_0}$.

Let $TN_{R} = T^H N_{R} \oplus N_{R}$ be the splitting of $TN_{R}$ induced by the Euclidean connection $\nabla^{L_N} = P^{N_R}(j^* \nabla^{L_X}) P^{N_R}$ on $N_{R}$, where $T^HN_{R}$ denotes the horizontal part of $TN_{R}$. If $U \in T_RY$, let $U^H \in T^HN_{R}$ denote the horizontal lift of $U$ in $T^HN_{R}$, so that $\pi_* U^H = U$. Let

$$\{ e_1, \ldots, e_{2l}, e_{2l+1}, \ldots, e_{2n} \}$$

be an orthonormal basis of $T_RX|_Y$ with $\{ e_1, \ldots, e_{2l} \}$ an orthonormal basis of $T_RY$ and $\{ e_{2l+1}, \ldots, e_{2n} \}$ an orthonormal basis of $N_{R}$. 

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**Definition 2.2** Let $D^H$, $D^N$ be the operators acting on $E$

\[
D^H = \sum_{i=1}^{2l} c(e_i)\nabla_{e_i}^{B,X,\oplus} + 2c(B_Y), \quad D^N = \sum_{\alpha=2l+1}^{2n} c(e_\alpha)\nabla_{e_\alpha}^{B,X,\oplus}. \tag{2.25}
\]

Clearly, $D^N$ acts along the fibres $N_{R,y}$ as the operator $\sqrt{2}(\bar{\partial}N_y + \bar{\partial}N_y^*)$. Note that $D^H$, $D^N$ are self-adjoint with respect to the Hermitian inner product (2.23).

Now we turn to Taylor expansions of $v$ near $Y$ along the geodesic $(y, tZ)$ for $y \in Y$ and $Z \in N_{R,y}$. Let \{w_1, \ldots, w_l, w_{l+1}, \ldots, w_n\} be a unitary basis for $TX|_Y$ and let $(z^{l+1}, \ldots, z^n)$ denote the associated holomorphic coordinate system on $N_y$ with $w_\alpha = \sqrt{2}\partial/\partial z^\alpha$ for $l + 1 \leq \alpha \leq n$. Note that $L_v : N \to \tilde{N}$ is unitary fiberwisely. Set

\[
\tilde{w}_\alpha = L_v(w_\alpha), \quad \tilde{\bar{w}}_\alpha = L_v(\bar{w}_\alpha), \quad l + 1 \leq \alpha \leq n. \tag{2.26}
\]

Hence, \{w_1, \ldots, w_l, \tilde{w}_{l+1}, \ldots, \tilde{w}_n\} is a unitary basis for $\tilde{T}X|_Y$. We use $\tilde{w}$ to denote the parallel transport of $\bar{w}$ with respect to the holomorphic Hermitian connection $\nabla^{\tilde{T}X}$ along the geodesic $(y, tZ)$. We write $v$ on $U_\epsilon$ as

\[
v(y, Z) = \frac{1}{\sqrt{2}} \left( \sum_{i=1}^{l} v^i(y, Z)w_i^\tau + \sum_{\alpha=l+1}^{n} v^\alpha(y, Z)\tilde{w}_\alpha^\tau \right) \tag{2.27}
\]

for some smooth functions $v^i$ and $v^\alpha$. Set

\[
v_{Y,1} = \frac{1}{\sqrt{2}} \sum_{i=1}^{l} \sum_{\alpha=l+1}^{n} \frac{\partial v^i}{\partial z^\alpha}(y)z^\alpha w_i, \quad v_{N,1} = \frac{1}{\sqrt{2}} \sum_{\alpha,\beta=l+1}^{n} \frac{\partial v^\alpha}{\partial z^\beta}(y)z^\beta \tilde{w}_\alpha, \tag{2.28}
\]

\[
v_{Y,2} = \frac{1}{2\sqrt{2}} \sum_{i=1}^{l} \sum_{\alpha,\beta=l+1}^{n} \frac{\partial^2 v^i}{\partial z^\alpha \partial z^\beta}(y)z^\alpha z^\beta w_i, \quad v_{N,2} = \frac{1}{2\sqrt{2}} \sum_{\alpha,\beta,\gamma=l+1}^{n} \frac{\partial^2 v^\alpha}{\partial z^\beta \partial z^\gamma}(y)z^\beta z^\gamma \tilde{w}_\alpha. \tag{2.29}
\]

Since $v$ is transversal, we get by the definition (2.26),

\[
v_{Y,1} = 0, \quad v_{N,1} = -\frac{1}{\sqrt{2}} \sum_{\alpha=l+1}^{n} z^\alpha \tilde{w}_\alpha, \tag{2.30}
\]

and so

\[
v(y, Z) = v_{N,1}(y, Z) + v_{Y,2}(y, Z) + v_{N,2}(y, Z) + O(|Z|^3). \tag{2.31}
\]
Define
\[ D_T^N = D^N + \sqrt{-1}T(\hat{c}(v_{N,1}) + \hat{c}(\bar{v}_{N,1})). \]  
(2.32)

A direct and easy computation shows that
\[ (D_T^N)^2 = -4 \sum_{\alpha = l+1}^{n} \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\alpha} + T^2|Z|^2 - \sqrt{-1}T \sum_{\alpha = l+1}^{n} (c(\bar{w}_\alpha)\hat{c}(\bar{w}_\alpha) + c(w_\alpha)\hat{c}(\bar{w}_\alpha)). \]  
(2.33)

Set
\[ \theta = \sum_{\alpha = l+1}^{n} w_\alpha^* \wedge \bar{w}_\alpha^*. \]  
(2.34)

Clearly, \( \theta \) is a well-defined smooth section of \( \Lambda^{(\cdot)}(\bar{N}_y \otimes \bar{N}_y) \) over \( Y \) of the degree 0. Now we have the following analogue of [BL, Proposition 7.3]:

**Lemma 2.3** Take \( T > 0 \). Then for any \( y \in Y \), the operator \( (D_T^N)^2 \) acting on \( \Gamma(\pi^*\Lambda^{(\cdot)}(\bar{N}_y \oplus \bar{N}_y)) \) over \( N_y \) is nonnegative with the kernel \( C\{\beta_y\} \), where
\[ \beta_y = \exp (\theta_y - \frac{T}{2}|Z|^2), \quad |\exp \theta_y|_{\Lambda^{(\cdot)}(\bar{N}_y \oplus \bar{N}_y)} = 2^{(n-l)/2}. \]  
(2.35)

Moreover, the nonzero eigenvalues of \( (D_T^N)^2 \) are all \( \geq TA \) for some positive constant \( A \) which can be chosen to be independent of \( y \).

**Proof.** The proof of the lemma is standard (cf. [BL, Sect.7, (7.10)–(7.13)]; also cf. [Z1, Chapter 4, Sect. 4.5]). \( \square \)

For any \( y \in Y \), \( Z \in N_{R,y} \), let \( \tau U \) denote the parallel transport of \( U \in T_{R,y}X \) with respect to the Levi-Civita connection \( \nabla^{L_x} \) along the geodesic \( (y, tZ) \). Note that we have identified the bundle \( \pi^*(\Lambda^{(\cdot)}(T^*_C X) \otimes \xi)|_{B_{\alpha_0}} \) with the bundle \( \Lambda^{(\cdot)}(T^*_C X) \otimes \xi|_{U_{\alpha_0}} \) by trivializing the later bundle along the geodesic \( (y, tZ) \) by using the connection \( \nabla^{B_x} \). The Clifford action of \( c((\tau U)(y, tZ)) \) on \( \Lambda^{(\cdot)}(T^*_C X) \otimes \xi \) is generally not constant along the geodesic \( (y, tZ) \). This is different from the situation in [BL, Sect.8], where the connection on the related bundle is the lifting of the Levi-Civita connection on \( TX \) since the manifold \( X \) is Kähler. Hence, to obtain an analogue of [BL, Theorem 8.18], we need to work out the difference between \( c((\tau U)(y, Z)) \) and the constant Clifford action \( c(U) \) on \( \pi^*(\Lambda^{(\cdot)}(T^*_C X) \otimes \xi)|_{B_{\alpha_0}} \). Since \( \nabla^{-B_x} = \nabla^{L_x} + S^{-B_x} \) is unitary, we know that
\[ ([\nabla^{B_x}_Z, c((\tau U)(y, tZ))]|_{t=0} = c(\nabla^{B_x}_Z (\tau U)(y, tZ))|_{t=0} = c(S^{-B_x}(Z)U), \]  
(2.36)
thus
\[ c((\tau U)(y, Z)) = c(U) + c(S^{-B_X}(Z)U) + O(|Z|^2). \] (2.37)

Set with respect to the basis (2.24):
\[ M = \frac{1}{2} \sum_{i,j}^{2l} \sum_{\alpha=2l+1}^{2n} (A(e_i)e_j, e_\alpha) c(e_i)c(e_j)c(e_\alpha) - \frac{1}{2} \sum_{\alpha=2l+1}^{2n} (e_\alpha k)c(e_\alpha), \] (2.38)
\[ c(B'(y)) = \frac{1}{2} \sum_{i,j=1}^{2l} \sum_{\alpha=2l+1}^{2n} B_X(e_i, e_j, e_\alpha)c(e_i)c(e_j)c(e_\alpha) + \frac{1}{6} \sum_{\alpha,\beta,\gamma=2l+1}^{2n} B_X(e_\alpha, e_\beta, e_\gamma)c(e_\alpha)c(e_\beta)c(e_\gamma), \] (2.39)
\[ c(B''(y)) = \frac{1}{2} \sum_{i=1}^{2l} \sum_{\alpha,\beta=2l+1}^{2n} B_X(e_i, e_\alpha, e_\beta)c(e_i)c(e_\alpha)c(e_\beta). \] (2.40)

One verifies easily that
\[ c(B_X(y)) = c(B_Y(y)) + c(B'(y)) + c(B''(y)). \] (2.41)

Now we have the following analogue of [BL, Theorem 8.18], which describes the local behavior of \( D_X^T \) near \( Y \). Comparing to [BL, Theorem 8.18, (8.58)], some new terms enter into the following theorem.

**Theorem 2.4** As \( T \to +\infty \), then
\[ k^{1/2} D_T^X k^{-1/2} = D^H + D_T^N + M_T + c(B'') + T\sqrt{-1}c(v_{Y,2} + \bar{v}_{Y,2}) + S + R_T, \] (2.42)
where
\[ M_T = M + c(B') + T\sqrt{-1}c(v_{N,2} + \bar{v}_{N,2}), \] (2.43)
\[ S = -\sum_{i=1}^{2n} c(e_i) \nabla^{B_X e_i}_{p^n_{N,S}-B_X(Z)e_i}, \] (2.44)
\[ R_T = O(|Z|\partial^H + |Z|^2\partial^N + |Z| + T|Z|^3), \] (2.45)
and \( \partial^H, \partial^N \) represent horizontal and vertical differential operators, respectively.
Proof. Let \( \{ \tau e_1, \ldots, \tau e_{2n} \} \) be the parallel transport of the basis (2.24) with respect to the Levi-Civita connection \( \nabla^{L^X} \) along the geodesic \((y, tZ)\) for \( y \in Y \) and \( Z \in \mathcal{N}_{R,y} \). From (2.21), we have

\[
D_T^X = \sum_{i=1}^{2n} c(\tau e_i) \nabla^{B,X}_{\tau e_i} + 2c(B_X) + \sqrt{-1}T(\hat{c}(v) + \hat{c}(\bar{v})).
\]  

(2.46)

We identify \( T_T^X \) with \( \pi^*(T^X|_Y) \) over \( U_\alpha \) by trivializing \( T_T^X \) with respect to the Bismut connection \( \nabla^{-B_X} \) along the geodesic \((y, tZ)\) and set

\[
\Gamma = \nabla^{-B_X} - \nabla^{-B_X: \oplus}.
\]  

(2.47)

Let \( \Gamma^\wedge \) denote the lifting action of \( \Gamma \) on \( \pi^*(\Lambda^*(T^*X)|_Y) \). For any \( y \in Y \), we find by (2.12)

\[
\sum_{i=1}^{2n} c(e_i) \Gamma^\wedge_y(e_i) = \frac{1}{2} \sum_{i,j}^{2l} \sum_{\alpha=2l+1}^{2n} \langle A_y(e_i)e_j, c(e_i)c(e_j)c(e_\alpha) \rangle.
\]  

(2.48)

Furthermore, recall (2.13) and then we get

\[
k^{1/2}D_T^X k^{-1/2} = \sum_{i=1}^{2n} c(\tau e_i) \nabla^{B,X: \oplus}_{\tau e_i} + 2c(B_X) + \sqrt{-1}T(\hat{c}(v) + \hat{c}(\bar{v}))
\]

\[
+ \sum_{i=1}^{2n} c(e_i) \Gamma^\wedge_y(e_i) - \frac{1}{2} \sum_{\alpha=2l+1}^{2n} (e_\alpha k)(y)c(e_\alpha) + O(|Z|).
\]

Note that (2.31), (2.38), (2.41), (2.43) and (2.48), we have

\[
k^{1/2}D_T^X k^{-1/2} = \sum_{i=1}^{2n} c(\tau e_i) \nabla^{B,X: \oplus}_{\tau e_i} + 2c(B_Y) + T\sqrt{-1}T(\hat{c}(v_{N,1} + \hat{v}_{N,1})
\]

\[
+ M_T + c(B''') + T\sqrt{-1}T(\hat{c}(v_{Y,2} + \hat{v}_{Y,2}) + O(|Z| + T|Z|^3).
\]

By (2.37) and the expansion of \( \tau e_i \) along \((y, tZ)\) in the proof of [BL, Theorem 8.18], especially [BL, (8.80), (8.84)], we have

\[
\sum_{i=1}^{2n} c(\tau e_i) \nabla^{B,X: \oplus}_{\tau e_i} = \sum_{i=1}^{2l} c(e_i) \nabla^{B,X: \oplus}_{e_i} + \sum_{\alpha=2l+1}^{2n} c(e_\alpha) \nabla^{B,X: \oplus}_{e_\alpha}
\]

\[
+ \sum_{i=1}^{2n} c(S^{-B_X}(Z)e_i) \nabla^{B,X: \oplus}_{e_i} + O(|Z|\partial^H + |Z|^2\partial^N),
\]

12
and then by the definition of $D^H$ and $D^N$,

$$k^{1/2}D^X_T k^{-1/2} = D^H + D^N_T + M_T + c(B'' + T\sqrt{-1}\bar{c}(\nu_Y - 2 + \bar{\nu}_Y)$$

$$+ \sum_{i=1}^{2n} c(S^{-B_X}(Z)e_i)\nabla_{e_i}^{B,X,\oplus} + O(|Z|\partial^H + |Z|^2\partial^N + |Z| + T|Z|^3).$$

But

$$\sum_{i=1}^{2n} c(S^{-B_X}(Z)e_i)\nabla_{e_i}^{B,X,\oplus} = -\sum_{i=1}^{2n} c(e_i)\nabla_{S^{-B_X}(Z)e_i}^{B,X,\oplus}$$

$$= -\sum_{i=1}^{2n} c(e_i)\nabla_{P^N_R S^{-B_X}(Z)e_i}^{B,X,\oplus} - \sum_{i=1}^{2n} c(e_i)\nabla_{P^R Y S^{-B_X}(Z)e_i}^{B,X,\oplus}$$

$$= S + O(|Z|\partial^H),$$

from which we complete the proof of the theorem. \hfill \Box

## 3 The proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using Bismut-Lebeau’s techniques in [BL, Sect.9, Sect.10, a)].

For any $\mu \geq 0$, let $E^\mu$ (resp. $E^\mu$, resp. $F^\mu$) be the set of sections of $\Lambda^{(1)}(T^*_C X) \otimes \xi$ on $X$ (resp. of $\sigma^{*}((\Lambda^{(1)}(T^*_C X) \otimes \xi)|_Y)$ on the total space of $N$, resp. of $\Lambda^{(1)}(T^*_C Y) \otimes \xi|_Y$ on $Y$) which lie in the $\mu$-th Sobolev spaces. Let $\| \cdot \|_{E^\mu}$ (resp. $\| \cdot \|_{E^\mu}$, resp. $\| \cdot \|_{F^\mu}$) be the Sobolev norm on $E^\mu$ (resp. $E^\mu$, resp. $F^\mu$). We always assume that the norms $\| \cdot \|_{E^0}$ (resp. $\| \cdot \|_{E^0}$, resp. $\| \cdot \|_{F^0}$ is the norm associated with the scalar products on the corresponding bundles).

Let $\gamma : \mathbb{R} \to [0,1]$ be a smooth even function with $\gamma(a) = 1$ if $|a| \leq \frac{1}{2}$ and $\gamma(a) = 0$ if $|a| \geq 1$. For any $y \in Y, Z \in N_y$ and $\epsilon \in (0, \epsilon_0)$, where $\epsilon_0$ is chosen as in Section 2, c), set

$$\gamma_\epsilon(Z) = \gamma\left(\frac{|Z|}{\epsilon}\right), \quad \alpha_T = \int_{N_{R,y}} \gamma_\epsilon^2(Z)\exp \left(-T|Z|^2\right)\frac{d\nu_{N_y}(Z)}{(2\pi)^{\dim C_N}}.$$ \hfill (3.1)

Clearly, $\alpha_T$ does not depend on $y \in Y$ and $\alpha_T = O\left(\frac{1}{T^{n-1}}\right)$.

For $\mu \geq 0, T > 0$, define linear maps $I_T : F^\mu \to E^\mu$ and $J_T : F^\mu \to E^\mu$ by

$$I_T u = \left(\frac{1}{2^{n-1}\alpha_T}\right)^{1/2} \gamma_\epsilon(Z)(\pi^* u)\beta_y, \quad J_T u = k^{-1/2}I_T u, \quad \forall u \in F^\mu.$$ \hfill (3.2)
It is easy to see that \( I_T, J_T \) are isometries from \( F^0 \) onto their images. For \( \mu \geq 0, T > 0 \), let \( E^\mu_T \) (resp. \( E^\mu_T \)) be the image of \( F^\mu \) in \( E^\mu \) (resp. \( E^\mu \)) under \( I_T \) (resp. \( J_T \)) and let \( E^{0,\perp}_T \) (resp. \( E^{0,\perp}_T \)) be the orthogonal complement of \( E^0_T \) (resp. \( E^0_T \)) in \( E^0 \) (resp. \( E^0 \)) and let \( p_T, p_T^\perp \) (resp. \( \bar{p}_T, \bar{p}_T^\perp \)) be the orthogonal projection operators from \( E^0 \) (resp. \( E^0 \)) onto \( E^0_T, E^{0,\perp}_T \) (resp. \( E^0_T, E^{0,\perp}_T \)), respectively. Set

\[
E^{\mu,\perp} = E^\mu \cap E^{0,\perp}_T. \tag{3.3}
\]

Then \( E^0 \) splits orthogonally into

\[
E^0 = E^0_T \oplus E^{0,\perp}_T. \tag{3.4}
\]

Since the map \( s \in E^0 \to k^{-1/2}s \in E^0 \) is an isometry, we see that the map \( s \to k^{-1/2}s \) identifies the Hilbert space \( E^0_T \) and \( E^0_T \). Corresponding to the decomposition (3.4) we set:

\[
D_{T,1} = \bar{p}_T D_T^X \bar{p}_T, \quad D_{T,2} = \bar{p}_T D_T^X \bar{p}_T^\perp, \quad D_{T,3} = \bar{p}_T D_T^X \bar{p}_T, \quad D_{T,4} = \bar{p}_T D_T^X \bar{p}_T^\perp. \tag{3.5}
\]

Then

\[
D_T^X = D_{T,1} + D_{T,2} + D_{T,3} + D_{T,4}. \tag{3.6}
\]

We have the following analogue of [BL, Theorem 9.8].

**Lemma 3.1** The following formula holds on \( \Gamma(\Lambda^{(T)}(T^*_C Y) \otimes \xi|_Y) \) as \( T \to +\infty \)

\[
J_T^{-1} D_{T,1} J_T = D^Y + O(\frac{1}{\sqrt{T}}), \tag{3.7}
\]

where \( O(\frac{1}{\sqrt{T}}) \) is a first order differential operator with smooth coefficients dominated by \( C/\sqrt{T} \).

**Proof.** Note that the action of the operator \( M_T \) on \( \pi^*((\Lambda^{(T)}(T^*_C X) \otimes \xi)|_Y) \) interchanges \( \pi^*(\Lambda^{(even)}(\tilde{N}^* \oplus \tilde{N}^*)) \) and \( \pi^*(\Lambda^{(odd)}(\tilde{N}^* \oplus \tilde{N}^*)) \), we get

\[
\pi_T M_T p_T = 0. \tag{3.8}
\]

Note that \( B_X \) is antisymmetric and \( \langle c(e_\alpha)c(e_\beta)\beta_y, \beta_y \rangle = 0 \) for any \( \alpha, \beta \) with \( 2l + 1 \leq \alpha < \beta \leq 2n \), we obtain

\[
p_T c(B''_T)p_T = 0. \tag{3.9}
\]
Since $B_X$ is antisymmetric, by (2.8) we get $\langle S^{-B_X}(Z)e_i, Z \rangle = 0$ and so
\[
\nabla_{p_{NR}^S - B_X(Z)e_i}^B X (\gamma_\epsilon(Z) \exp (\theta_y - \frac{T|Z|^2}{2})) = \exp (\theta_y - \frac{T|Z|^2}{2}) (p_{NR}^S - B_X(Z)e_i) \gamma_\epsilon(Z).
\]
(3.10)

From the equality above we can prove easily the following estimate for some uniformly positive constant $C$ and any $s \in E^1$:
\[
\|p_T s\|_{E^0} \leq C \frac{\|s\|_{E^1}}{\sqrt{T}}.
\]
(3.11)

Since $\int_C e^{-T|z|^2} z^2 dz d\bar{z} = 0$ and $(\gamma_\epsilon^2 - 1)$ vanishes on a symmetric domain containing 0, we have for any $u \in F$ that
\[
I_T^{-1} p_T (\hat{c}(v_{Y,2}) + \hat{c}(\bar{v}_{Y,2})) \left( \frac{1}{2^{n-1} \alpha_T} \right)^{1/2} \gamma_\epsilon(Z) (\pi^* u) \beta_y = O(\frac{1}{T^{3/2}}).
\]
(3.12)

On the other hand, note that $\beta_y$ is of constant length on $Y$, we get for $1 \leq i \leq 2l$ that
\[
\langle \nabla_{e_i}^B X, \beta, \beta \rangle = 0
\]
and so
\[
I_T^{-1} p_T D^H p_T I_T = D^Y.
\]
(3.13)

One can then proceed as in [BL, Proof of Theorem 9.8] and use (3.8)–(3.12) to complete the proof of Lemma 3.1 easily.

Note that the estimate (3.12) and proceed as the proof of Theorem 9.10, Theorem 9.11 and Theorem 9.14 in [BL, Sect.9], one can prove the following lemma without any new difficulty.

**Lemma 3.2** There exist $C_1 > 0$, $C_2 > 0$ and $T_0 > 0$ such that for any $T \geq T_0$, $s \in E_T^{1,1}$ and $s' \in E_T^1$, then
\[
\|D_{T,2} s\|_{E^0} \leq C_1 \left( \frac{\|s\|_{E^1}}{\sqrt{T}} + \|s\|_{E^0} \right),
\]
(3.14)
\[
\|D_{T,3} s'\|_{E^0} \leq C_1 \left( \frac{\|s'\|_{E^1}}{\sqrt{T}} + \|s'\|_{E^0} \right),
\]
(3.15)
\[
\|D_{T,4} s\|_{E^0} \geq C_2 (\|s\|_{E^1} + \sqrt{T} \|s\|_{E^0}).
\]
(3.16)
Let \( \text{Spec} (D^Y) \) denote the spectrum of \( D^Y \). Choose \( c > 0 \) such that \( \text{Spec} (D^Y) \cap [-2c, 2c] \subset \{0\} \). Let \( \delta = \{ \lambda \in \mathbb{C} : |\lambda| = c \} \). Let \( E_c(T) \) denote the direct sum of the eigenspaces of \( D^X_T \) with eigenvalues lying in \([-c, c]\). Then \( E_c(T) \) is a finite dimensional subspace of \( E^0 \). Let \( P_{T,c} \) denote the orthogonal projection from \( J_T(\ker (D^Y)) \) to \( E_c(T) \). By Lemma 3.1 and Lemma 3.2, we have the following analogue of [BL, (9.15 6)] (also see [TZ, Proposition 4.4] for a proof without using the norm in [BL, Sect.9, Definition 9.17] and the distance in [BL, Sect.9, Definition 9.22]):

**Theorem 3.3** There exist \( c > 0 \) and \( T_0 > 0 \) such that for any \( T \geq T_0 \), the projection

\[
P_{T,c} : J_T(\ker (D^Y)) \to E_c(T)
\]

is an isomorphism.

Now to prove Theorem 1.1 we only need to prove that when \( T \) large enough, \( D^X_T \) has no nonzero small eigenvalues or equivalently, to prove the following equality:

\[
E_c(T) = \ker (D^X_T).
\]

Let \( Q \) denote the orthogonal projection from \( \Omega^1(Y, \xi) \) to \( \ker (D^Y) \). Then we have the following analogue of [Z2, Theorem 1.10] (also see [BL, Theorem 10.1, (10.4)]):

**Theorem 3.4** There exist \( c > 0 \), \( C > 0 \), \( T_1 > 0 \) such that for any \( T \geq T_1 \), any \( \sigma \in \ker (D^Y) \),

\[
\| (2^{n-1} \alpha_T)^{1/2} Q j^* P_{T,c} J_T \sigma - \sigma \|_0 \leq \frac{C}{\sqrt{T}} \| \sigma \|_0.
\]

**Proof.** The proof of [Z2, Theorem 1.10], which is a modified version of the proof of [BL, Theorem 10.1, (10.4)], is carried out here with the identity [Z2,(1.34)] in the proof of the [Z2, Theorem 1.10] replaced by the following equality

\[
j^* \frac{1}{2\pi \sqrt{-1}} \int_{\delta} k^{-1/2} \gamma e^{(\pi^* \sigma) \beta_y / \lambda} d\lambda = \sigma.
\]

The proof of (3.20) is similar to that of the identity [BL, (10.29)]. \( \square \)

Note that \( j^* \beta_y = 1 \) is crucial in the proof of (3.19). It is no longer true for the case of the analytic proof of Morse inequalities of Witten ([W]) since in that case the contribution
of the bundle \( \Lambda(N^*) \) to the kernel of \( D^X_T \) is a pure \( p \)-form around each critical point of index \( p \) (cf. [Z1, Chapter 5, 6]) and its pull-back by \( j^* \) vanishes on \( Y \). Consequently, \( j^* \) can not be a quasi-isomorphism at all in that case.

**Proof of Theorem 1.1.** The proof of Theorem 1.1 now is similar to that in [Z2, Sect.1, e)]. Note that the trick used in Zhang’s proof and so ours is inspired by Braverman ([Br, Sect.3]). First of all, we know from Theorem 3.3 that

\[
P_{T,c}J_T : \ker (D^Y) \to E_c(T) \tag{3.21}
\]

is an isomorphism when \( T \) is very large. Take \( \alpha \in E_c(T) \). Then \( \bar{\partial}^X_T \alpha \in E_c(T) \). By the above discussion, there exists \( \sigma \in \ker (D^Y) \) such that

\[
\bar{\partial}^X_T \alpha = (2^{n-l} \alpha_T)^{1/2} P_{T,c}J_T \sigma. \tag{3.22}
\]

From (3.22) and that \( j^* \) is a quasi-homomorphism, i.e. \( j^* \bar{\partial}^X_T = \bar{\partial}^Y j^* \), we have

\[
(2^{n-l} \alpha_T)^{1/2} Q j^* P_{T,c}J_T \sigma = Q j^* \bar{\partial}^X_T \alpha = Q \bar{\partial}^Y j^* \alpha = 0. \tag{3.23}
\]

From (3.23) and (3.19), we get

\[
\| \sigma \|_0 \leq C \sqrt{T} \| \sigma \|_0, \tag{3.24}
\]

and so \( \sigma = 0 \) as \( T \) large enough. Thus, when \( T \) is large enough, we have that

\[
\bar{\partial}^X_T |_{E_c(T)} = 0. \tag{3.25}
\]

From (3.25) and Theorem 3.3, we have that when \( T \) is large enough,

\[
\dim \ker (D^X_T) = \dim E_c(T) = \dim \ker (D^Y). \tag{3.26}
\]

Now by Theorem 3.4,

\[
j^* : \ker (D^X_T) \to \ker (D^Y) \tag{3.27}
\]

is clearly an injective and so an isomorphism from (3.26).

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