Fractional Charge and Quantized Current in the Quantum Spin Hall State

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Abstract

A profound manifestation of topologically non-trivial states of matter is the occurrence of fractionally charged elementary excitations. The quantum spin Hall insulator state is a fundamentally novel quantum state of matter that exists at zero external magnetic field. In this work, we show that a magnetic domain wall at the edge of the quantum spin Hall insulator carries one half of the unit of electron charge, and we propose an experiment to directly measure this fractional charge on an individual basis. We also show that as an additional consequence, a rotating magnetic field can induce a quantized dc electric current, and vice versa.
Soon after the theoretical proposal of the intrinsic spin Hall effect\cite{1,2} in doped semiconductors, the concept of a time-reversal invariant spin Hall insulator\cite{3} was introduced. In the extreme quantum limit, a *quantum* spin Hall (QSH) insulator state has been proposed for various systems\cite{4,5,6}. The QSH insulators are time-reversal invariant and have a bulk charge-excitation gap. However, this system also possesses topologically protected gapless edge states that lie inside the bulk insulating gap. The edge states of the QSH insulator state differ from the quantum Hall effect and have a distinct helical property: two states with opposite spin-polarization counter-propagate at a given edge\cite{4,7,8}. The edge states come in Kramers’ doublets, and time reversal symmetry ensures the crossing of their energy levels at special points in the Brillouin zone. Because of this level crossing, the spectrum of a QSH insulator cannot be adiabatically deformed into a topologically trivial insulating state; therefore, in this precise sense, the QSH insulators represent a topologically distinct new state of matter.

Recently, the QSH effect has been theoretically proposed\cite{6} and experimentally observed\cite{9} in HgTe quantum wells. In this experiment, an applied gate voltage can tune the carrier type from n-type doping to p-type doping, passing through a nominally insulating state. A residual charge conductance approaching $2e^2/h$ has been measured in this insulating regime. Furthermore, the residual charge conductance is independent of the width of the sample, indicating that it is due to the helical edge state channels of the QSH insulator.

Given the exciting theoretical and experimental development in this field, one central question remains unanswered - what is the direct experimental manifestation of this topologically non-trivial state of matter? In the case of the quantum Hall (QH) effect, it is the quantization of the Hall conductance and the fractional charge of the elementary excitations which are a result of non-trivial topological structure. The $Z_2$ topological invariant gives a correct mathematical characterization of the QSH state\cite{10}; however, unlike the TKNN quantum numbers\cite{11} of the quantum Hall (QH) state, it is not directly measurable experimentally. In this work, we show that for the QSH state a magnetic domain wall induces an elementary excitation with half the charge of an electron. We also show that a rotating magnetic field can induce a quantized dc electric current, and vice versa. Both of these physical phenomena are direct and experimentally observable consequences of the non-trivial topology of the QSH state. The idea of fractional charge induced at a domain wall goes back to the Su-Schrieffer-Heeger (SSH) model\cite{12}. For spinless fermions, a charge domain wall
induces an elementary excitation with one-half charge. However, for a real material such as polyacetylene, two spin orientations are present for each electron, and because of this doubling, a domain wall in polyacetylene only carries integer charge; the beautiful proposal of SSH, and its counter-part in field theory, the Jackiw-Rebbi model [13], have never been experimentally realized. Conventional one dimensional electronic systems have four basic degrees of freedom, \textit{i.e.} right and left movers with each spin orientation. However, a helical liquid at a given edge of the QSH insulator has only two: a spin up (or down) right mover and a spin down (or up) left mover. Therefore, the helical liquid has half the degrees of freedom of a conventional one-dimensional system, and thus avoids the doubling problem. Because of this fundamental topological property of the helical liquid, a domain wall carries charge $e/2$. We propose a Coulomb blockade experiment to observe this fractional charge. As a temporal analog of the fractional charge effect, the pumping of a quantized charge current during each periodic rotation of a magnetic field is also proposed. This provides a direct realization of Thouless’s topological pumping [14].

**Theoretical description.** In the absence of time-reversal symmetry (TRS) breaking we can express the effective theory for the edge states of a non-trivial QSH insulator as

$$H_0 = v_F \int dx (\psi_R^{\dagger} i \partial_x \psi_R + \psi_L^{\dagger} i \partial_x \psi_L) = v_F \int dx \Psi^{\dagger} i \sigma^3 \partial_x \Psi$$

where $\pm$ indicate members of a Kramers’s doublet, $L/R$ indicate left- or right-movers, and $\Psi = (\psi_R, \psi_L)^T$ [7, 8]. Note that these helical fermion states only have two degrees of freedom since the spin-polarization is correlated with the direction of motion.

Since the three Pauli spin matrices $\sigma_{1,2,3}$ are odd under time-reversal, a mass term, being proportional to the Pauli matrices, can only be introduced in the Hamiltonian by coupling to a T-breaking external field such as a magnetic field or aligned magnetic impurities. To the leading order in perturbation theory, a magnetic field generates the mass terms

$$H_M = \int dx \Psi^{\dagger} \sum_{a=1,2,3} m_a(x,t) \sigma_a \Psi = \int dx \Psi^{\dagger} \sum_{a,i} t_{ai} B_i(x,t) \sigma_a \Psi$$

where the model dependent coefficient matrix $t_{ai}$ is determined by the coupling of the edge states to the magnetic field. According to the work of Goldstone and Wilczek [15], at zero temperature the ground-state charge density and current in a background field $m_a(x,t)$ is given by

$$j_\mu = \frac{1}{2\pi} \frac{1}{\sqrt{m_\alpha m_\beta}} \epsilon^{\mu\nu} \epsilon_{\alpha\beta} m_\alpha \partial_\nu m_\beta, \ \alpha, \beta = 1, 2.$$
with $\mu, \nu = 0, 1$ corresponding to the time and space components, respectively. Note that $m_3$ does not enter the long-wavelength charge-response equation. If we parameterize $m_1 = m \cos \theta$, $m_2 = m \sin \theta$, then the response equation is simplified to

$$\rho = \frac{1}{2\pi} \partial_x \theta(x, t), \quad j = -\frac{1}{2\pi} \partial_t \theta(x, t).$$

(3)

Such a response is “topological” in the sense that the net charge $Q$ in a region $[x_1, x_2]$ at time $t$ depends only on the boundary values of $\theta(x, t)$, i.e., $Q = [\theta(x_2, t) - \theta(x_1, t)]/2\pi$. In particular, a half-charge $\pm e/2$ is carried by an anti-phase domain wall of $\theta$, as shown in Fig. 1A. Similarly, the charge pumped by a purely time-dependent $\theta(t)$ field in a time interval $[t_1, t_2]$ is $\Delta Q_{\text{pump}}[t_1^2] = [\theta(t_2) - \theta(t_1)]/2\pi$. When $\theta$ is rotated from 0 to $2\pi$ adiabatically, a quantized charge $e$ is pumped through the 1d system, as shown in Fig. 1B.

From the linear relation $m_a = t_{ai} B_i$, the angle $\theta$ can be determined for a given magnetic field $B_i$ as $\theta(x, t) = \theta(B(x, t)) = \text{Im} \log (t_1 \cdot B(x, t) + it_2 \cdot B(x, t))$, in which $t_{1(2)i}$ is the 3d vector with components $t_{1(2)i}$, respectively. Since $\theta(B) = \theta(-B) + \pi$, the charge localized on an anti-phase magnetic domain wall of magnetization field is always $\pm e/2$. For the pumping effect, the winding number of $\theta(t)$ is given by the winding number of the $B$-vector around the axis $t_1 \times t_2$. The conditions for these effects to be observed are $k_B T, \hbar \omega \ll E_g$, where $\omega$ is the pumping frequency, and $E_g = \sqrt{(t_1 \cdot B)^2 + (t_2 \cdot B)^2}$ is the energy gap of the helical edge state generated by the magnetic field.

After the general analysis, we now discuss more details of the experimental realization of such topological effects. First, to gain intuition about the energy scales in this problem, we consider the coefficients $t_{ai}$ for HgTe/CdTe quantum wells which can be obtained numerically. By solving the four band effective model given in Ref. [6] with an open boundary we obtain the wavefunctions of the two edge states $|k; \pm \rangle$. The effective $2 \times 2$ Hamiltonian of the edge states including magnetic field effects is obtained by standard perturbation theory, from which the coefficients $t_{ai}$ are extracted. For a quantum well with thickness $d = 70\text{Å}$ and an edge along the $y$-direction, we obtain $t_1 = (-0.3, 0, 0) \text{meV/T}$ and $t_2 = (0, -0.3, -3.1) \text{meV/T}$. [16, 17] (Here and below the $z$ direction is the quantum well growth direction.) Thus, the gap induced by an in-plane field $B_x = 1\text{T}$ is $E_{gx} \simeq 0.3\text{meV}$, while the perpendicular component $B_z = 1\text{T}$ produces a much larger gap $E_{gz} \simeq 3.1\text{meV}$. (Such a large anisotropy between in-plane and perpendicular magnetic field agrees well with the experimental observations in Ref. [9].) Consequently, the charge fractionalization effect can
be observed at temperature $T \ll 35\text{K}$ on a domain wall with a perpendicular magnetic field, while the adiabatic pumping effect can only be observed at much lower temperatures $T \ll 3.5\text{K}$ since it depends partially on the gap from in-plane fields.

We have shown through the discussions above that the fractional charge and adiabatic pumping effects are experimentally feasible. In the following paragraphs we propose detailed experimental settings needed to observe these two topological effects.

**Observation of fractional charge on the domain wall.** Recently, a novel device has been developed to measure the charge of a confined region: the single-electron transistor (SET)\cite{18, 19, 20}. When applying a gate voltage $V$ on top of a confined region (e.g., quantum dot or wire) with capacitance $C$, the Coulomb energy is given by $E_c(Q) = Q^2/2C + VQ$. The number of charges trapped in the confined region is quantized as $Q = Ne$, with the integer $N$ determined by minimizing the energy $E_c(Ne) = \min_{n \in \mathbb{Z}} E_c(ne)$. (Here and below $\mathbb{Z}$ stands for the set of all integers.) Thus an additional electron entering the confined region will cost an energy $\Delta E = E_c((N + 1)e) - E_c(Ne)$. A straightforward calculation shows that $\Delta E = e^2/2C$ for $CV/e \in \mathbb{Z}$, while $\Delta E = 0$ for $CV/e - 1/2 \in \mathbb{Z}$. Consequently, the two-terminal conductance of the device is governed by an activation behavior $G = G(V) \propto e^{-\Delta E/k_B T}$ and shows oscillations with period $\Delta V = e/C$ and peak positions at $V = (n + 1/2)e/C$, $n \in \mathbb{Z}$. In other words, the conductance peak appears when the net charge of the confined region is a half-odd integer times the electron charge. This allows one to sensitively measure charges comparable to or even smaller than the electronic charge\cite{20, 21}.

The fractional charge created by a magnetic domain wall on the QSH edge is confined in the region between the two magnetic domains separated by the wall. This confined charge can be measured by designing a magnetic SET experiment. A schematic picture of such a device is shown in Fig. 2. Two magnetic islands can trap the electrons between them, just like a quantum wire trapped between two potential barriers\cite{19}. In such a device, the conductance oscillations can be observed as in usual Coulomb blockade measurements. The background charge in the confined region consists of two parts: $Q_b = Q_e + Q_c$, with $Q_e$ the contribution of the lowest subband electrons and $Q_c$ that of higher energy bands and nuclei. When the field direction in one of the magnetic domains is switched, $Q_c$ remains invariant but $Q_e$ will change by $e/2$ which demonstrates the half-charge associated to the antiphase domain wall. Consequently, if we use a top gate on the confined region and measure the conductance oscillations $G(V)$, there will be a half-period phase shift between the oscillation
pattern of parallel and anti-parallel magnetic domains, as shown in Fig. 2.

Experimentally, magnetic islands can be deposited on top of semiconductor heterostructures creating a hybrid ferromagnet-semiconductor device [22, 23]. The magnetic islands can be polarized via magnetic field and locally switched using a coercive field and conventional magnetic-force microscopy techniques (see e.g. [24]). The quantum well will be locally exposed to the fringe fields of the ferromagnetic islands. To observe the conductance oscillations, several conditions should be satisfied by the magnetic field configuration:

1. The oscillation period $\Delta V = e/C$ should be much smaller than the bulk gap scale $V = E_g/e$ so that several periods of oscillation can be observed before the gate voltage is so high that the bulk states are activated. This leads to the requirement $C \gg e^2/E_g$.

2. The ratio of the minimal conductance to the maximal conductance is estimated by

$$G_{\text{min}}/G_{\text{max}} \simeq \exp \left[ -\frac{e^2/2C}{k_B T} \right]$$

with $e^2/2C$ the maximal charge activation gap. For the oscillations to be observable this ratio should be reasonably smaller than one, which leads to the condition $e^2/2C \geq k_B T$ or $C \leq e^2/2k_B T$.

3. The domain wall state trapped between the magnetic domains has an exponential tail as shown in Fig. 2. To make the conductance measurement, the size of the magnetic islands should be comparable to the exponential tail length $\xi_M \simeq \hbar v_F / E_g$ with $E_g$ the magnetic field-induced gap, so that the localized state is well-confined, but tunnelling through the barrier is still strong enough to support observable transport.

The edge state trapped between magnetic domains has the linear size $\xi \times d \times L$ where $\xi \simeq \hbar v_F / M$ is the penetration depth of the edge state, $d$ is the thickness of quantum well, and $L$ is the distance between the two magnetic regions. Under the one-dimensional approximation $L \gg d$, $\xi$ we obtain the approximate form of the capacitance $C \simeq 4\pi \epsilon_0 L / [\log (L^2/d\xi) - 2]$. For $d = 70\text{Å}$ we find the condition $1\mu\text{m} \ll L \ll 100\mu\text{m}$. For a magnetic field $B_z = 1T$ the size of each magnetic island is $r \sim 120$ nm.

So far in our discussions, the magnetic domain wall structure is externally imposed. However, it is also possible that the system can spontaneously generate such magnetic
domain walls. In Ref. [7,8], it was shown that two-particle backscattering interactions are allowed in the helical liquid. In the strong coupling limit, such a process can lead to a spontaneous breaking of TRS, and spontaneous generation of a magnetic moment at the edge. This symmetry breaking is described by an Ising like $Z_2$ order parameter, which at any finite temperature in 1d leads to a finite density of magnetic domain walls. Our work shows that such domain walls will carry fractional charge $\pm e/2$, which are the elementary excitations of the system.

We remark that the phenomenon of fractional charge associated with a magnetic domain wall is an example of “electro-magnetic duality” in 1d. Here, a domain wall is a point-like object, and is dual to a point particle. In our particular case, a magnetic domain wall induces an electric point charge $\frac{e}{2\pi}\Delta\theta$. In 3d, a natural magnetic point singularity is a magnetic monopole, and Witten showed [25] that it can induce an electric point charge $\frac{e}{2\pi}\theta$, where $\theta$ is the vacuum angle of quantum-chromodynamics. The duality between a magnetic point charge and an electric point charge in the helical liquid of the QSH state has many other profound consequences which we shall demonstrate in future publications.

Observation of the quantized charge current. If the magnetic moment of one domain in the proposed SET device is rotated continuously by a full period while the other one remains static, the conductance peak position will shift relatively by a full period, as shown in Fig. 2B. Such a shift of the conductance peaks shows the change of background charge by $e$ in the confined region, which is a consequence of the topological pumping effect. Another device to measure the pumping current directly is shown in Fig. 3. One strongly-pinned and one easy-plane magnetic island are deposited above one of each of the two arms of the device, respectively. When applying a small rotating external field with frequency $\omega$, the magnetization of the easy-plane island will be rotated while the pinned one remains static. Consequently, a quantized charge current is pumped, given by the formula $I = e\omega/2\pi$ under the adiabatic approximation. To observe such an effect the magnetic field-induced gap must be much larger than the temperature, which will require Mn doping as discussed earlier.

Conclusion and more discussions. In conclusion, based on both the general theoretical discussions and quantitative numerical calculations, we have shown the feasibility of two striking topological effects in the QSH state. Using a single-electron transistor-like sensor we proposed an experimental setting to create and observe the fractionally charged domain wall. Such topological phenomena, if observed, not only provide the first experimental re-
alization of the fractional charge in one-dimensional systems, but also introduce a physical and operational definition of the two-dimensional topological (QSH) insulator. An (infinitesimal) magnetic field domain wall configuration can be used as a sensor to characterize two-dimensional insulators. If such a detection device induces a localized fractional charge response on the sample edge, then the system is defined to be a topological insulator. Such a definition is experimentally meaningful since it is based on the response of the system to some physical external field, and is completely analogous to the definition of the quantum Hall insulator as system which produces a quantized Hall response to an external electric field.

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FIG. 1: A. Schematic picture of the half-charge domain wall. The blue arrows show a magnetic domain wall configuration and the purple line shows the mass kink. The red curve shows the charge density distribution. B. Schematic picture of the pumping induced by the rotation of magnetic field. The blue circle with arrow shows the magnetic field rotation trajectory.

\[ \rho = \partial_x \theta / 2\pi \]

\[ j = -\partial_t \theta / 2\pi \]
FIG. 2: A. The SET device with parallel and anti-parallel magnetic domains. The blue and grey rectangles are the magnetic domains and voltage probes, respectively. (In all the figures, the magnetic domain on the bottom arm is always pinned, which is designed to block the transport of that arm so that the conductance of the SET device on the upper arm can be measured.) The conductance peak shift is shown on the right. The inset shows a schematic picture of the bound state wavefunction. B. Schematic picture of the conductance peak positions $G_{\text{peak}}$ being shifted by continuous rotation of the magnetic field. (Here and in Fig. 3, the actual direction of the rotating magnetic field should be in the plane perpendicular to the edge.)
FIG. 3: Schematic picture of the device for measuring the quantized charge current. A and B are source and drain without an applied voltage bias between them.