REMARK ON AN ELASTIC PLATE INTERACTING WITH A GAS IN A SEMI-INFINITE TUBE: PERIODIC SOLUTIONS

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Abstract. We consider a conservative system consisting of an elastic plate interacting with a gas filling a semi-infinite tube. The plate is placed on the bottom of the tube. The dynamics of the gas velocity potential is governed by the linear wave equation. The plate displacement satisfies the linear Kirchhoff equation. We show that this system possesses an infinite number of periodic solutions with the frequencies tending to infinity. This means that the well-known property of decaying of local wave energy in tube domains does not hold for the system considered.

Recently there was great interest in the study of long-time dynamics of elastic plates interacting with a flow of gas (see, e.g., [3, 5, 6, 7, 8, 10, 11] and the literature cited in those sources). A corresponding model has the form

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + \gamma u_t + f(u) &= \left( \partial_t U \partial_x \right) \phi \mid_{x_3=0} \quad \text{in } \Omega \times \mathbb{R}_+, \\
u(0) &= u_0; \quad u_t(0) = u_1, \\
u &= \Delta \nu = 0 \quad \text{on } \partial \Omega \times \mathbb{R}_+,
\end{align*}
\]

(1)

where \( \phi(x_1, x_2, x_3; t) \) solves the problem in \( \mathbb{R}_+^3 \equiv \{ x = (x_1; x_2; x_3) : x_3 > 0 \} \):

\[
\begin{align*}
(\partial_t + U \partial_x)^2 \phi &= \Delta \phi \quad \text{in } \mathbb{R}_+^3 \times \mathbb{R}_+, \\
\phi(0) &= \phi_0; \quad \phi_t(0) = \phi_1 \quad \text{in } \mathbb{R}_+^3, \\
\partial_{x_1} \phi &= \begin{cases} (\partial_t + U \partial_x) u(x_1, x_2) & \text{on } \Omega \times \mathbb{R}_+, \\
0 & \text{on } (\mathbb{R}^2 \setminus \Omega) \times \mathbb{R}_+.
\end{cases}
\end{align*}
\]

(2)

Here above \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^2 \) identified with \( \{(x_1; x_2; 0) : (x_1; x_2) \in \Omega \} \subset \mathbb{R}_+^2 \).

The term \( f(u) \) describes a nonlinear force which can be (a) von Karman type (like as in [5, 10]), or (b) Berger type (like as in [1, 11]), or even (c) generated by some Nemytskii operator (see, e.g., [4]). The unknown function \( u = u(x_1, x_2; t) \) measures the transverse displacement of the plate at the point \( (x_1; x_2) \) and time \( t \). The boundary conditions for \( u \) means that the plate is hinged on its edge. The function \( \phi(x, t) = \phi(x_1, x_2, x_3; t) \) is the velocity potential of the gas filling the domain \( \mathbb{R}_+^3 \). In (1)–(2) we deal with interaction of a plate with a gas flow moving with the speed \( U \) in the direction of the axis \( x_1 \). The aerodynamical pressure of the gas on the plate is given by the term \( p(x, t) = \mu (\phi_t + U \phi_{x_1}) \mid_{x_3=0}, \) the parameter \( \mu > 0 \)

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characterizes the intensity of the interaction between the gas and the plate. The transverse displacement \( u(x,t) \) has influence on the gas via boundary condition in (2).

For recent surveys of mathematical and applied aspects of the model above we refer to [3]. Here we only mention the convergence results in the subsonic case \((0 \leq U < 1)\) which were established in [10, 11] (see also [5] for related facts). In these works the full flow-plate dynamics stabilize to stationary states of the system as \( t \to \infty \), under some conditions concerning the initial data of the gas velocity potential. The corresponding argument critically requires positivity of the damping parameter \( \gamma \) and utilizes the gradient-type structure of the full flow-plate system in the case considered.

Following the above analyses, the question arises (see [2] and [5, p.694]) whether it is possible to obtain a similar stabilization result in the absence \((\gamma = 0)\) of the internal damping in the plate. This conjecture is based on the well-known property of the local energy decay for the wave equation in \( \mathbb{R}^3 \) and some other unbounded domains (see also Proposition 2.1 below).

Our main goal in this note is to show that unboundness of the wave domain \( \mathcal{O} \) is not sufficient to guarantee stabilization of solutions to equilibria (in the local sense). For this we consider a linear plate model without any damping \((\gamma = 0)\), coupled to the flow via matching velocities. The unperturbed flow velocity parameter \( U \) is taken to be zero for simplicity. We show that in this scenario, periodic solutions may exist. In the case of a specific tubular domain they are explicitly constructed. Whether the same result holds for nonlinear plate is an open question. However, the model indicates the necessity of introducing mechanical damping in the plate model if one expects a strong convergence to equilibria of the full flow-structure system.

1. Model. Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^2 \). We consider the following problem

\[
\begin{align*}
\partial_t^2 u + \Delta^2 u - \mu_t [\phi|_{x_3=0}] &= 0, \quad x = (x_1; x_2) \in \Omega, \quad t > 0, \\
u|_{\partial \Omega} &= \Delta u|_{\partial \Omega} = 0, \quad t > 0, \\
u|_{t=0} &= u_0(x), \quad \partial_t u|_{t=0} = u_1(x), \quad x = (x_1; x_2) \in \Omega,
\end{align*}
\]

We denote by \( \phi|_{x_3=0} \) the trace of a function \( \phi(x_1, x_2, x_3, t) \) in the semi-infinite tubular domain \( \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \) which solves the problem

\[
\begin{align*}
\partial_t \phi &= \Delta \phi, \quad x \in \mathcal{O}_+ \equiv \{x = (x_1; x_2; x_3) : (x_1; x_2) \in \Omega, \ x_3 > 0\}, \\
\partial_{x_3} \phi &= \partial_t u(x_1, x_2, t), \quad (x_1; x_2) \in \Omega, \ x_3 = 0, \ t > 0, \\
\phi &= 0, \quad (x_1; x_2) \in \partial \Omega, \ x_3 > 0, \ t > 0, \\
\phi|_{t=0} &= \phi_0(x), \quad \partial_t \phi|_{t=0} = \phi_1(x), \quad x \in \mathcal{O}_+.
\end{align*}
\]

The function \( u = u(x,t) \) is the transverse displacement of the hinged plate. The function \( \phi(x,t) = \phi(x_1, x_2, x_3; t) \) is velocity potential of the gas filling the tube \( \mathcal{O}_+ \). The pressure of the gas on the plate is given by the term \( p(x,t) = \mu \cdot \phi|_{x_3=0} \), where \( \mu > 0 \).

In the case of bounded domains \( \mathcal{O} \) systems like (3) and (4) have been studied by many authors (see the discussion and the references in [5] and [9]).
Below we use the notation $H^s(D)$ for the Sobolev space of order $s$ on a domain $D$ in $\mathbb{R}^d$, $d = 2, 3$.

We start with the following assertion.

**Proposition 1.1** (Well-posedness). Assume that

$$\phi_0 \in H^1(\Omega_+), \phi_1 \in L_2(\Omega_+), u_0 \in (H^2 \cap H^1_0)(\Omega), \ u_1 \in L_2(\Omega).$$

Then there exists a unique couple \(\{u; \phi\}\) of function

$$u \in C^1(\mathbb{R}_+; L_2(\Omega)) \cap C(\mathbb{R}_+; (H^2 \cap H^1_0)(\Omega))$$

and

$$\phi \in C^1(\mathbb{R}_+; L_2(\Omega_+)) \cap C(\mathbb{R}_+; H^1(\Omega_+))$$

solving (3) and (4) in the sense of distributions. Moreover this solution satisfies the energy preservation law of the form

$$E(t) \equiv E_{\Omega_+}^{gas}(\phi(t), \phi(t)) + \mu^{-1}E_{\Omega}^{plate}(u(t), u(t)) = E(0), \text{ for all } t > 0,$$

where we use the notations

$$E_D^{gas}(\phi, \phi) = \frac{1}{2} \int_D \left[|\phi_t(x,t)|^2 + |\nabla \phi(x,t)|^2\right] dx$$

for every $D \subseteq \Omega_+ \subset \mathbb{R}^3$ and

$$E_{\Omega}^{plate}(u, u) = \frac{1}{2} \int_\Omega \left[|u_t(x,t)|^2 + |\Delta u(x,t)|^2\right] dx.$$ 

**Proof.** We can apply well-known general result reported in [9], see also [5, Chapter 6], where nonlinear versions of similar problems are discussed. However, we can also give a more direct argument based on some symmetry of this linear problem and involving the separation of variables. We sketch the corresponding argument below.

Let \(\{\epsilon_k\}\) be an orthonormal basis in $L_2(\Omega)$ consisting of eigenvectors of the problem

$$\Delta w + \lambda w = 0, \ w|_{\partial \Omega} = 0,$$

and $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ the corresponding eigenvalues. We are looking for solutions to problem (3) and (4) in the following form

$$u(x_1, x_2, t) = \sum_{k=1}^{\infty} u_k(t)\epsilon_k(x_1, x_2), \ \phi(x_1, x_2, x_3, t) = \sum_{k=1}^{\infty} \phi_k(t, x_3)\epsilon_k(x_1, x_2).$$

It is clear that the function $\phi_k(t, z)$ solves the problem$^1$

$$\begin{cases} 
\partial_t \phi_k - \partial_z \phi_k + \lambda_k \phi_k = 0, & z > 0, \ t > 0, \\
\partial_z \phi_k = u_k(t), & z = 0, \ t > 0,
\end{cases}$$

$$\phi_k|_{t=0} = \phi_{0k}(z), \ \partial_t \phi_k|_{t=0} = \phi_{1k}(z), \ z > 0,$$

where $u_k(t)$ satisfies the equation

$$\dot{u}_k + \lambda_k^2 u_k - \mu \partial_t \phi_k(0, t) = 0, \ u_k|_{t=0} = u_{0k}, \ \dot{u}_k|_{t=0} = u_{1k}.$$

It is easy to show that for each $k$ problem (8) and (9) has a unique solution $(\phi_k(t), u_k(t))$ for which we have the following energy balance relation

$$E_k(t) = E_k(s), \ t \geq s,$$

$^1$Here, and in what follows, we identify $z$ with $x_3$ for simplicity.
where the energy $E_k$ of the $k$-mode has the form
\[
E_k(t) = \frac{1}{2} \int_0^\infty \left[ |\partial_t \phi_k(t, z)|^2 + |\partial_z \phi_k(t, z)|^2 + \lambda_k |\phi_k(t, z)|^2 \right] \, dz
\]
\[+ \frac{1}{2\mu} \left[ |\dot{u}_k(t)|^2 + \lambda_k^2 |u_k(t)|^2 \right].
\]
These observations allow us to obtain appropriate a priori estimates and conclude the proof by the standard compactness method.

\[\square\]

2. Dynamics. We start with the following assertion that shows a local energy decay in the case when the bottom $\Omega$ of the cylinder $O_+$ is rigid. This means that we consider the wave dynamics only. This dynamics is described by the following equations
\[
\begin{cases}
\partial_t \phi = \Delta \phi, & x \in O_+, \\
\partial_{x_3} \phi = 0, & (x_1; x_2) \in \Omega, \ x_3 = 0, \ t > 0, \\
\phi = 0, & (x_1; x_2) \in \partial \Omega, \ x_3 > 0, \ t > 0, \\
\phi|_{t=0} = \phi_0(x), & \partial_t \phi|_{t=0} = \phi_1(x), \ x \in O_+.
\end{cases}
\tag{10}
\]

**Proposition 2.1** (Local energy decay). Assume that $\phi_0 \in H^1(O_+)$ and $\phi_1 \in L_2(O_+).$ Then problem (10) has a unique variational solution $\phi$ which belongs to the class
\[\phi \in C^1(\mathbb{R}_+; L_2(O_+)) \cap C(\mathbb{R}_+; H^1(O_+)).\]
This solution satisfies the energy preservation law of the form
\[E_{O_+}^{gas}(\phi(t), \phi(t)) = E_{O_+}^{gas}(\phi_1, \phi_0), \text{ for all } t > 0.
\]
Moreover, we have decaying of $\phi$ as $t \to \infty$ in the local energy norm, i.e.,
\[\lim_{t \to +\infty} E_{O_{+R}}^{gas}(\phi(t), \phi(t)) = 0 \text{ for every } R > 0,
\]
where $E_{O_{+R}}^{gas}$ is given by (5) with
\[O_{+R} = \{ x = (x_1; x_2; x_3) : (x_1; x_2) \in \Omega, \ 0 < x_3 < R \}.
\]

**Proof.** The existence and uniqueness of solutions to (10) is obvious (we can use the same idea as in Proposition 1.1, for instance). It is also clear that the energy relation is satisfied. Thus we only need to establish the property in (11).

Extending the initial data to even functions in the variable $x_3$ on the whole $x_3$-axis we can consider the wave equation in the domain
\[O = \{ x = (x_1; x_2; x_3) : (x_1; x_2) \in \Omega, \ -\infty < x_3 < +\infty \},
\]
with the Dirichlet boundary conditions on $\partial O.$ Now we can separate variables as above apply the same idea as in [12] to prove (11) for localized initial data (having compact support) $\phi_0$ and $\phi_1.$ In fact, the article [12] contains exactly this statement for the case when $\Omega = (0, \pi) \times (0, \pi).$ The method suggested in [12] relies on the presentation of the solution $\phi$ in the form
\[\phi(x_1, x_2, x_3, t) = \sum_{m,n=1}^{\infty} \phi_{mn}(t, x_3)e_{mn}(x_1, x_2),
\]
where $e_{mn}(x_1, x_2) = 2\pi^{-1} \sin(mx_1) \sin(nx_2)$ are solutions to the spectral problem (6) for $\Omega = (0, \pi) \times (0, \pi)$ and $\phi_{mn}(t, z)$ solves the equation

$$\partial_{tt}\phi_{mn} - \partial_{zz}\phi_{mn} + (m^2 + n^2)\phi_{mn} = 0, \quad z \in \mathbb{R}, \ t > 0.$$  

Establishing appropriate bounds for $\phi_{mn}$ (see [12]) one can prove the desired result for $\Omega = (0, \pi) \times (0, \pi)$. The calculations given in [12] can be easily extended to the case of general domains $\Omega$.

Then using approximation procedure for initial data and the energy relation, we can obtain the result for every pair $(\phi_1; \phi_0) \in L^2(\mathcal{O}_+) \times H^1(\mathcal{O}_+)$. The decay property of the local wave energy demonstrated in Proposition 2.1 is not valid for the coupled system in (3) and (4). More precisely, we show that problem (3) and (4) possesses an infinite number of periodic solutions with different periods.

**Theorem 2.2.** Let $\{\lambda_k\}$ be the eigenvalues of problem (6) and $\{e_k\}$ be the corresponding eigen-basis. Then there exist sequences $\{\omega_k\}$ and $\{\alpha_k\}$ of positive numbers with properties

$$\lim_{k \to \infty} \left[ \omega_k^2 - \lambda_k \right] = 0 \quad \text{and} \quad \lim_{k \to \infty} \alpha_k \lambda_k = \mu \quad (12)$$

such that the functions

$$\varphi^k(x_1, x_2; t) = \left[ A_k \cos \omega_k t + B_k \sin \omega_k t \right] e^{-\alpha_k x_3} e_k(x_1, x_2) \quad (13)$$

and

$$u^k(x_1, x_2; t) = -\alpha_k \left[ A_k \sin \omega_k t - B_k \cos \omega_k t \right] e_k(x_1, x_2), \quad (14)$$

where $A_k$ and $B_k$ are arbitrary real numbers, solve problem (3) and (4) with appropriate initial data. Each trajectory $(\varphi^k; \varphi^k_t; u^k; u^k_t)$ is Lyapunov stable in the phase space $\mathcal{H} = H^1(\mathcal{O}_+) \times L^2(\mathcal{O}_+) \times H^1(\Omega) \times L^2(\Omega)$.

**Proof.** Let us look for solutions to (8) and (9) of the form

$$\phi_k(t, z) = e^{i\omega t} e^{-\alpha z}; \ u_k(t) = ae^{i\omega t}$$

with $\alpha > 0$ and $\omega, a \in \mathbb{C}$. The substitution in (8) and (9) gives us the relations

$$-\omega^2 - \alpha^2 + \lambda_k = 0,$$

$$\alpha = i\omega a,$$

$$a(-\omega^2 + \lambda_k^2) + i\mu \omega = 0.$$  

This implies that $a = -i\omega^{-1}$ and also

$$\omega^2 = \lambda_k - \alpha^2, \ \omega^2 = \frac{\alpha}{\alpha + \mu} \lambda_k^2. \quad (15)$$

Reducing to a cubic in $\alpha$, one can see for every $k$ there exists unique solution $(\omega_k^2, \alpha_k)$ to (15). It is also easy to find that

$$\left(\omega_k^2, \alpha_k\right) \sim (\lambda_k, \mu \lambda_k^{-1}) \quad \text{when} \ k \to +\infty$$

in the sense of (12). This implies the structure of a solution written in (13) and (14).

Stability properties of solutions follow from the energy preservation law. \qed
Theorem 2.2 shows that the elasticity of the bottom $\Omega$ of the cylinder $O_\bot$ destroys the local energy decay property which we observe in the case of a rigid bottom (see Proposition 2.1).

We conclude this note with several open questions which, we believe, are important for understanding of long-time dynamics of flow-structure systems.

Open questions:

- Can we show that the minimal subspace in $H$ that contains all solutions $(\varphi_k^k; \psi_k^k; u_k^k; u_k^k)$ is asymptotically stable? Is this subspace a global minimal attractor? If not, what is a real candidate on the role of global attractor for (3) and (4)?

- What can we say about stability and spectral properties of the generators of $C_0$ semigroups generated by (3) and (4) and its dissipative perturbation? For instance, is it possible to stabilize the system by introducing internal damping in the plate component only?

These questions are important not only for linear dynamics, but also for nonlinear perturbations of (3) and (4).

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