On the Space of KdV Fields

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Abstract

The space of functions $A$ over the phase space of KdV-hierarchy is studied as a module over the ring $D$ generated by commuting derivations. A $D$-free resolution of $A$ is constructed by Babelon, Bernard and Smirnov by taking the classical limit of the construction in quantum integrable models assuming a certain conjecture. We propose another $D$-free resolution of $A$ by extending the construction in the classical finite dimensional integrable system associated with a certain family of hyperelliptic curves to infinite dimension assuming a similar conjecture. The relation of two constructions is given.

1 Introduction

In [1] Babelon, Bernard and Smirnov (BBS), by considering the classical limit of a model in two dimensional integrable quantum field theory, have studied the space of KdV fields $A = \mathbb{C}[u, u', ...]$ as a module over the ring of commuting derivations $D = \mathbb{C}[\partial_1, \partial_3, ...]$, where $\partial_i$ acts on $u$ according as the KdV-hierarchy (see section 2):

$$\partial_i u = S_{i+1}(u), \quad S_{i+1}(u) \in A.$$  

Assuming the conjecture that $A$ is generated over $D$ by $\mathbb{C}[S_2, S_4, ...]$ they have constructed a $D$-free resolution of $A$. In particular all $D$-linear relations among monomials of $\{S_{2i}\}$ are determined. They are called null vectors in [1]. For example the first two non-trivial null vectors are

$$\partial_3 S_2 - \partial_1 S_4 = 0, \quad \partial_1^2 S_2 - 4S_4 + 6S_2^2 = 0,$$

which give the KdV equation for $S_2$:

$$\partial_3 S_2 = \frac{1}{4} \partial_1^3 S_2 + 3S_2 \partial_1(S_2).$$

In [11] the affine ring $A_g$ of the affine Jacobian of a hyperelliptic curve of genus $g$ is studied as a module over the ring $D_g = \mathbb{C}[\partial_1, \partial_3, ..., \partial_{2g-1}]$ of invariant vector fields on the Jacobian. Assuming some conjecture a $D_g$-free resolution of $A_g$ has been constructed. Although the conjecture is verified only for $g \leq 3$ [9] up to now, this construction exhibits a remarkable consistency with other results. For example it recovers the character of $A_g$ [11] and the cohomologies of the affine Jacobian [7]. Since the $g \to \infty$ limit of $A_g$ is identified with $A$, in the present paper we directly construct a $D$-free resolution of $A$ extending the construction for $A_g$.  

*Department of Mathematics, Kyushu University, Ropponmatsu 4-2-1, Fukuoka 810-8560, Japan, e-mail: 6vertex@math.kyushu-u.ac.jp, Mathematics Subject Classifications: 37K20,17B69,17B37, Key words: KdV hierarchy, boson-fermion correspondence, space of fields, D-module.
Let $\tau(t)$ be the tau function of the KdV-hierarchy and $\zeta_{i_1 \ldots i_n} = \partial_{i_1} \cdots \partial_{i_n} \log \tau(t)$, $\partial_i = \partial/\partial t_i$. Then the generators of $A$ of this construction are given by the set of functions

$$1, \ (i_1 \ldots i_n; j_1 \ldots j_n) := \det(\zeta_{i_k j_l})_{1 \leq k, l \leq n}, \ n \geq 1,$$

where $i_1 < \cdots < i_n, j_1 < \cdots < j_n$. The $D$-linear relations among them are, for example,

$$\partial_{j_1}(i_1 i_2 : j_2 j_3) - \partial_{j_2}(i_1 i_2 : j_1 j_3) + \partial_{j_3}(i_1 i_2 : j_1 j_2) = 0.$$

In a sense these are trivial relations since they hold if $\tau(t)$ is replaced by an arbitrary function of $t_1, t_3, \ldots$. Notice that $S_{2n} = \partial_1 \partial_{2n-1}$ and the null vectors of BBS implies the bilinear form of the KdV equation for $\tau(t)$. Thus two free resolutions of $A$ give quite different generators and relations. However we prove that two constructions are equivalent by showing the equivalence of two conjectures.

The construction of [1] is related with the quantum groups at root of unity [8, 6]. While the construction of the present paper is directly related with the geometry of Jacobian varieties and the analysis of abelian functions [9, 10]. Thus the present result opens the way to study the latter subjects in terms of the representation theory. It is a quite interesting problem to extend conjectures and results of [11] to the case of more general algebraic curves than that of hyperelliptic curves based on this viewpoint. Toward this direction the results of [12, 13, 4] are important.

The present paper is organized in the following manner. In section 2 the space of KdV fields is defined. After reviewing the boson-fermion correspondence in section 3, the construction of the free resolution of $A$ due to Babelon, Bernard and Smirnov is reviewed in section 4. In section 5 another construction of the free resolution of $A$ is given. The relation of two constructions is given in section 6. Finally in section 7 concluding remarks are given.

## 2 The Space of KdV Fields

Let $A$ denote the differential algebra

$$A = \mathbb{C}[u, u', u'', \cdots]$$

generated by $u = u^{(0)}, u' = u^{(1)}, u'' = u^{(2)}, \ldots$ such that the derivation $'$ acts as $(u^{(m)})' = u^{(m+1)}$ for any $m$. The KdV hierarchy is the infinite number of compatible differential equations given by

$$\frac{\partial u}{\partial t_n} = S_{n+1}'(u), \quad n = 1, 3, 5, \ldots,$$

where $S_n(u)$ is the element of $A$ without constant term satisfying the equation

$$S_{n+2}'(u) = \frac{1}{4} S''(u) - u S'_n(u) - \frac{1}{2} u' S_n(u), \quad S_2(u) = -\frac{1}{2} u.$$

In particular $'$ is identified with $\partial/\partial t_1$. The KdV hierarchy defines the action of the commuting derivations $\partial_n$ on $A$ by

$$\partial_n(u^{(k)}) = S_{n+1}^{(k+1)}(u),$$

Thus $A$ is a $D$-module, where $D = \mathbb{C}[\partial_1, \partial_3, \cdots]$. 

2
3 Free Fermions and Fock Spaces

Let \( \psi_n, \psi_n^* \), \( n \in 2\mathbb{Z} + 1 \), satisfy the relations

\[
[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0, \quad [\psi_m^*, \psi_n]_+ = \delta_{m,n},
\]

where \([X,Y]_+ = XY + YX\). The vacuums \( <m|, |m> \), \( m \in 2\mathbb{Z} + 1 \) are defined by the conditions

\[
<m| \psi_n = 0 \quad \text{for} \quad n \leq m, \quad <m| \psi_n^* = 0 \quad \text{for} \quad n > m, \\
\psi_n|m> = 0 \quad \text{for} \quad n > m, \quad \psi_n^*|m> = 0 \quad \text{for} \quad n \leq m.
\]

They are related by

\[
\psi_n^*|m-2> = |m>, \quad <m-2|\psi_m = <m|.
\]

The Fock spaces \( H_m, H_m^* \) are constructed from \(|m>\) and \(<m|\) respectively by the equal number of \( \psi_k \) and \( \psi_k^* \). The pairing between \( H_m \) and \( H_m^* \) are defined by normalizing

\[
<m|m> = 1.
\]

Let us set

\[
h_{-2k} = \sum_{n \in 2\mathbb{Z} + 1} \psi_n \psi_{n+2k}^*, \quad T = \exp(-\sum_{k=1}^{\infty} \frac{1}{k} J_{2k} h_{-2k}),
\]

where \( J_{2k} \) are commutative variables. Notice that \( <m|T = <m| \) for any \( m \). The boson-fermion correspondence gives the isomorphism of bosonic and fermionic Fock spaces \( 3 \)

\[
H^*_{2m-1} \simeq \mathbb{C}[J_2, J_4, \ldots], \quad <2m-1|a \mapsto <2m-1|aT|2m-1>.
\]

4 Babelon-Bernard-Smirnov’s Construction

In this section we review the results of \( 1 \). Their discovery is that the fermionic description of the bosonic map \( ev_1 \) defined below \( 10 \) greatly simplifies the situation. Later it is understood that such structure is intimately related with quantum groups at a root of unity \( 8 \) \( 6 \).

Let

\[
\psi(z) = \sum_{n \in 2\mathbb{Z} + 1} \psi_n z^{-n}, \quad \psi^*(z) = \sum_{n \in 2\mathbb{Z} + 1} \psi_n^* z^n.
\]

Define two operators \( Q \) and \( C \) by

\[
Q = \int \frac{dz}{2\pi i} \nabla(z) \psi(z), \\
C = \int \frac{dz}{2\pi i} \psi(z) \frac{d\psi(z)}{dz} + \int \int |z_1|>|z_2| \frac{dz_1}{2\pi i} \frac{dz_2}{2\pi i} \log \left( 1 - \frac{z_1}{z_2} \right)^2 \nabla(z_1) \nabla(z_2) \psi(z_1) \psi(z_2).
\]
Here the simple integral signifies to take the coefficient of $z^{-1}$ and the double integral signifies to take that of $(z_1 z_2)^{-1}$ when the integrand is expanded at the region $|z_1/z_2| < 1$. These operators are maps of the following spaces

\[ Q : \mathcal{D} \otimes H_n^* \rightarrow \mathcal{D} \otimes H_{n+2}^*, \]
\[ C : \mathcal{D} \otimes H_n^* \rightarrow \mathcal{D} \otimes H_{n+4}^*. \]

They satisfy

\[ [Q, C] = 0, \quad Q^2 = 0. \quad (8) \]

Let us introduce new variables $\bar{S}$ by

\[ \exp(-\sum_{k=1}^{\infty} \frac{1}{k} J_{2k} z^{-2k}) = \sum_{n=0}^{\infty} \bar{S}_{2n} z^{-2n}. \]

By specifying the degrees as $\deg J_{2k} = 2k$, $\bar{S}_{2n}$ is a homogeneous polynomial of $J_{2k}$'s of degree $2k$ and has the form

\[ \bar{S}_{2n} = -\frac{J_{2n}}{2n} + \cdots, \]

where $\cdots$ part does not contain $J_{2n}$. In particular we have the isomorphism of polynomial rings

\[ \mathbb{C}[J_2, J_4, \ldots] \simeq \mathbb{C}[\bar{S}_2, \bar{S}_4, \ldots]. \quad (9) \]

The composition of (7) and (9) gives the isomorphism

\[ H_{-1}^* \simeq \mathbb{C}[\bar{S}_2, \bar{S}_4, \ldots]. \]

We identify these two spaces by this isomorphism. We define a map

\[ \text{ev}_1 : \mathcal{D} \otimes \mathbb{C}[\bar{S}_2, \bar{S}_4, \ldots] \rightarrow \mathcal{A}. \quad (10) \]

by

\[ P(\partial) \otimes \bar{S}_2^{n_2} \bar{S}_4^{n_4} \cdots \mapsto P(\partial)(\bar{S}_2^{n_2} \bar{S}_4^{n_4} \cdots). \]

Then Babelon, Bernard and Smirnov have proved

**Theorem 1** ([1, 2])

\[ Q(\mathcal{D} \otimes H_{-3}^*) + C(\mathcal{D} \otimes H_{-5}^*) \subset \text{Ker ev}_1. \]

They conjectured

**Conjecture 1** The map $\text{ev}_1$ is surjective.

We set

\[ B = \frac{\mathcal{D} \otimes H_{-1}^*}{Q(\mathcal{D} \otimes H_{-3}^*) + C(\mathcal{D} \otimes H_{-5}^*)}. \]
Since \(Q^2 = 0\) we have the complex
\[
\cdots \xrightarrow{Q} D \otimes H^*_{-5} \xrightarrow{Q} D \otimes H^*_{-3} \xrightarrow{Q} D \otimes H^*_{-1} \longrightarrow 0. \tag{11}
\]
Since \(C\) and \(Q\) commute, it induces the following complex;
\[
\cdots \xrightarrow{Q} \frac{D \otimes H^*_{-5}}{C(D \otimes H^*_{-5})} \xrightarrow{Q} \frac{D \otimes H^*_{-3}}{C(D \otimes H^*_{-3})} \xrightarrow{Q} \frac{D \otimes H^*_{-1}}{C(D \otimes H^*_{-1})} \longrightarrow 0, \tag{12}
\]
because \([Q, C] = 0\). Finally we have the complex
\[
\cdots \xrightarrow{Q} \frac{D \otimes H^*_{-5}}{C(D \otimes H^*_{-9})} \xrightarrow{Q} \frac{D \otimes H^*_{-3}}{C(D \otimes H^*_{-7})} \xrightarrow{Q} \frac{D \otimes H^*_{-1}}{C(D \otimes H^*_{-5})} \longrightarrow B \longrightarrow 0, \tag{13}
\]
where the map to \(B\) is the natural projection.

**Proposition 1**  
(i) For \(n \geq 0\)
\[
\frac{D \otimes H^*_{-2n-1}}{C(D \otimes H^*_{-2n-5})}
\]
is a free \(D\)-module.
(ii) The complex (13) is exact.

By this proposition (13) gives a \(D\)-free resolution of \(B\).

The statement (i) of this proposition is proved in [2]. We recall the change of fermions used there for further use. In the component form \(Q\) and \(C\) are written as
\[
Q = \sum_{n=1}^{\infty} \partial_{2n-1} \psi_{-(2n-1)},
\]
\[
C = \sum_{n=1}^{\infty} \left(2(2n-1)\psi_{2n-1} - \sum_{l=1}^{\infty} P_{n,l}(\partial)\psi_{2n-1-l} \right)\psi_{-(2n-1)},
\]
where we set
\[
P_{n,l}(\partial) = \sum_{i+j=l+1, j<n} \frac{1}{n-j} \partial_{2i-1}\partial_{2j-1}.
\]

We define
\[
\tilde{\psi}_{-(2n-1)} = \psi_{-(2n-1)} \quad \text{for } n \geq 1,
\]
\[
\tilde{\psi}_{2n-1} = 2(2n-1)\psi_{2n-1} - \sum_{l=1}^{\infty} P_{n,l}(\partial)\psi_{2n-1-2l} \quad \text{for } n \geq 1,
\]
Write these relations as
\[
\tilde{\psi}_i = \sum_{j \in 2\mathbb{Z}+1} d_{ij} \psi_j,
\]
and set \(D = (d_{ij})\) which is an invertible triangular matrix. Set
\[
D' = (d'_{ij}) = t(D^{-1}),
\]
\[
\tilde{\psi}_i = \sum_j d'_{ij} \tilde{\psi}_j.
\]
Then \( \{ \tilde{\psi}_i, \tilde{\psi}_j^* \} \) satisfy the canonical anti-commutation relations \( \textbf{[10]} \). Moreover the vacuums \( |m \rangle \) for \( \{ \psi_i, \psi_j^* \} \) become the vacuums for \( \{ \tilde{\psi}_i, \tilde{\psi}_j^* \} \). We denote the Fock spaces of \( \{ \tilde{\psi}_i, \tilde{\psi}_j^* \} \) by \( \tilde{H}_m, \tilde{H}_m^* \). Then

\[
Q = \sum_{n=1}^{\infty} \partial_{2n-1} \tilde{\psi}_{-(2n-1)}, \quad C = \sum_{n=1}^{\infty} \tilde{\psi}_{2n-1} \tilde{\psi}_{-(2n-1)},
\]

and we have isomorphisms

\[
\mathcal{D} \otimes H_{-2n-1}^* \simeq \mathcal{D} \otimes \tilde{H}_{-2n-1}^*,
\]

\[
C(\mathcal{D} \otimes H_{-2n-5}^*) \simeq \mathcal{D} \otimes \tilde{H}_{-2n-5}^*,
\]

for any integer \( n \). The statement (ii) follows from the following lemmas in a similar manner to Theorem 4.3 of \textbf{[8]}. The lemmas can also be proved similarly to Lemma 4.4 and 4.5 of \textbf{[8]}.

**Lemma 1** The complex \( (11) \) is exact at \( \mathcal{D} \otimes H_{-2n-1}^*, n \geq 1 \).

**Lemma 2** The map

\[
C : \tilde{H}_{-2m-1} \longrightarrow \tilde{H}_{-2m+3}
\]

is injective for \( m \geq 0 \).

**Corollary 1** If we assume Conjecture \textbf{[4]} then \( A \simeq B \) and \( (13) \) gives a \( \mathcal{D} \)-free resolution of \( A \),

\[
\cdots \longrightarrow Q \mathcal{D} \otimes \tilde{H}_{-5}^* \longrightarrow Q \mathcal{D} \otimes \tilde{H}_{-3}^* \longrightarrow Q \mathcal{D} \otimes \tilde{H}_{-1}^* \longrightarrow \mathcal{D} \otimes H_{-1}^* \longrightarrow \cdots \longrightarrow A \longrightarrow 0.
\]

**Proof.** Conjecture \textbf{[3]} implies that the map \( B \rightarrow A \) induced from \( \text{ev}_1 \) is surjective. Then the injectivity follows by comparing characters. For a graded vector space \( V = \oplus V_n \) with \( \text{dim} \ V_n < \infty \) we define the character of \( V \) by

\[
\text{ch} V = \sum q^n \text{dim} \ V_n.
\]

For \( H_n^* \) and \( \tilde{H}_n^* \) we assign

\[
\deg \psi_n = n, \quad \deg \psi_n^* = -n, \quad \deg <2m-1| = m^2, \quad \deg \partial_i = i.
\]

Then

\[
\text{ch} H_{-2m+1}^* = \text{ch} \tilde{H}_{-2m+1}^* = \frac{q^{m^2}}{\prod_{i=1}^{\infty} (1-q^{2i})}.
\]

For \( A \) we define \( \deg u^{(i)} = 2 + i \). Then

\[
\text{ch} A = \frac{1 - q}{\prod_{i=1}^{\infty} (1-q^i)}
\]

and the map \( \text{ev}_1 \) preserves grading. Using the free resolution \( (13) \) of \( B \) we have

\[
\text{ch} B = \frac{1 - q}{\prod_{i=1}^{\infty} (1-q^i)} = \text{ch} A,
\]

which completes the proof. \( \blacksquare \)
5 Another Construction of Free Resolution

In this section we shall generalize the construction of [11] to the case of infinite degrees of freedom.

Let \( \tau(t) = \tau(t_1, t_3, \ldots) \) be the tau function of the KdV-hierarchy [3]. We set

\[
\zeta_i = \partial_i \log \tau(t), \quad \zeta_{ij} = \partial_i \partial_j \log \tau(t), \quad \partial_i = \frac{\partial}{\partial t_i}.
\]

Notice that \( \zeta_{ij} \) can be expressed as a differential polynomial of \( u = \zeta_{11} \) and thereby is contained in \( A \). Then

\[
d\zeta_i = \sum_{j: \text{odd}} \zeta_{ij} dt_j
\]

is a 1-form with the coefficients in \( A \).

Let

\[
\alpha_{2n-1} = \tilde{\psi}_{-(2n-1)}, \quad \beta_{2n-1} = \tilde{\psi}_{2n-1}, \quad n \geq 1.
\]

Then

\[
Q = \sum_{n=1}^{\infty} \partial_{2n-1} \alpha_{2n-1}, \quad C = \sum_{n=1}^{\infty} \beta_{2n-1} \alpha_{2n-1}.
\]

For \( N \geq 1 \)

\[
\tilde{H}^*_{-1}(N) = \sum_{k=0}^{N} \sum C < -2N - 1 \alpha_{i_N-k} \cdots \alpha_{i_1} \beta_{j_k} \cdots \beta_{j_1},
\]

where the second summation is over all odd integers satisfying

\[
2N + 1 > i_{N-k} > \cdots > i_1 \geq 1, \quad j_k > \cdots > j_1 \geq 1.
\]

Set \( H^*_0(0) = C < -1 \). We use the notation like

\[
\alpha_I = \alpha_{i_{N-k}} \cdots \alpha_{i_1},
\]

for \( I = (i_{N-k}, \ldots, i_1) \).

For \( N < N' \) we have the inclusion

\[
\tilde{H}^*_{-1}(N) \subset \tilde{H}^*_{-1}(N'),
\]

\[
< -2N - 1 \alpha \mapsto < -2N' - 1 \alpha_{2N'-1} \alpha_{2N'-3} \cdots \alpha_{2N+1} \alpha.
\]

Thus \( \{ \tilde{H}^*_{-1}(N) \} \) defines an increasing filtration of \( H^*_1 \):

\[
\tilde{H}^*_{-1} = \bigcup_{N=0}^{\infty} \tilde{H}^*_{-1}(N).
\]

We define a map of \( D \)-modules

\[
ev_2 : D \otimes \tilde{H}^*_{-1} \rightarrow A,
\]

as follows.
Let
\[ \text{vol} = \cdots \wedge dt_5 \wedge dt_3 \wedge dt_1, \quad \Omega_\mathbb{C} = \mathbb{C} \text{vol}, \]
and \( \Omega_\mathbb{C}^{-p} \) be the vector space generated by differential forms which are obtained from \( \text{vol} \) by removing \( p \) \( dt_i \)'s. We define the action of \( D \) on \( A \otimes \Omega_\mathbb{C} \) by
\[ P(F \text{vol}) = P(F) \text{vol}, \quad P \in D, \quad F \in A, \]
where we omit the tensor symbol for simplicity. Then we have the isomorphism of \( D \)-modules
\[ A \otimes \Omega_\mathbb{C} \cong A, \quad F \text{vol} \mapsto F. \quad (16) \]

Let \( v \in \tilde{H}^*_{-1}(N) \) be of the form
\[ v = \langle -2N - 1 | \alpha_I \beta_J, \quad I = (i_{N-1}, \ldots, i_1), \quad J = (j_k, \ldots, j_1), \]
and \( P \in D \). We define
\[ ev_2(P \otimes v) \text{vol} = P(\cdots \wedge dt_{2N+3} \wedge dt_{2N+1} \wedge dt_I \wedge d\zeta_J), \]
where \( dt_I = dt_{i_{N-1}} \wedge \cdots \wedge dt_{i_1} \), etc. We can write \( ev_2 \) more explicitly using certain determinants. Write
\[ \cdots \wedge dt_{2N+3} \wedge dt_{2N+1} \wedge dt_I \wedge d\zeta_J = F_{IJ} \text{vol}, \quad F_{IJ} \in A. \]

Let
\[ I^c = \{1, 3, \ldots, 2N - 1\} \setminus I = \{l_k > \cdots > l_1\} \]
and \( \text{sgn}(I, I^c) \) be the sign of the permutation
\[ (2N - 1, \ldots, 3, 1) \longrightarrow (I, I^c). \]

Then
\[ ev_2(P \otimes v) = P(F_{IJ}), \quad F_{IJ} = \text{sgn}(I, I^c) \det(\zeta_{l_a j_b})_{1 \leq a, b \leq k}. \]

One can immediately check, using (15), that the definition of \( ev_2 \) does not depend on the choice of \( N \) such that \( v \in \tilde{H}^*_{-1}(N) \).

**Proposition 2** (i) \( ev_2\left(Q(D \otimes \tilde{H}^*_{-3})\right) = 0 \).
(ii) \( ev_2\left(C(D \otimes \tilde{H}^*_{-5})\right) = 0 \).

**Proof.** (i) We define
\[ d : A \otimes \Omega_\mathbb{C}^{-p} \longrightarrow A \otimes \Omega_\mathbb{C}^{-p+1}, \]
by
\[ d(F \otimes w) = \sum_{n=1}^{\infty} \partial_{2n-1} F \otimes w \wedge dt_{2n-1}, \quad F \in A, \quad w \in \Omega_\mathbb{C}^{-p}. \]
Then
\[ d^2 = 0, \quad d \Omega^{\mathbb{H}^{-p}} = 0, \]
\[ d(w_1 \wedge w_2) = w_1 \wedge dw_2 + (-1)^q dw_1 \wedge w_2, \quad w_1 \in A \otimes \Omega^{\mathbb{H}^{-p-q}}, \quad w_2 \in A \otimes \Omega^q, \]
where \( \Omega^q \) is the space of \( q \)-forms of \( dt_1, dt_3, \ldots \) and \( dw_2 \) is defined in an obvious manner.

For \( I = (i_k, \ldots, i_1) \) we set \( |I| = k \). Let
\[ v = -2N - 1 |\alpha_I \beta_J \in \tilde{H}^*_3, \quad |I| + |J| = N - 1, \]
and \( P \in \mathcal{D} \). Then
\[ Q(P \otimes v) = \sum_{n=1}^{\infty} \partial_{2n-1} P \otimes -2N - 1 |\alpha_I \beta_J \alpha_{2n-1}, \]
and
\[ ev_2 \left( Q(P \otimes v) \right)_{\text{vol}} = \sum_{n=1}^{\infty} \partial_{2n-1} P (\cdots \land dt_{2n+1} \land dt_I \land d\zeta_J \land dt_{2n-1}) \]
\[ = P \left( d(\cdots \land dt_{2n+1} \land dt_I \land d\zeta_J) \right) \]
\[ = P \left( (-1)^{N-1} d(\cdots \land dt_{2n+1} \land dt_I \land d\zeta_J + \cdots \land dt_{2n+1} \land d(dt_I \land d\zeta_J) \right) \]
\[ = 0. \]

(ii) Let
\[ v = P \otimes -2N - 1 |\alpha_I \beta_J \in \mathcal{D} \otimes \tilde{H}^*_5. \]
Then
\[ ev_2(Cv)_{\text{vol}} = ev_2(-P \otimes \sum_{n=1}^{\infty} -2N - 1 |\alpha_I \beta_J \alpha_{2n-1} \beta_{2n-1}) \]
\[ = -P (\cdots \land dt_{2n+1} \land dt_I \land d\zeta_J \land \sum_{n=1}^{\infty} dt_{2n-1} \land d\zeta_{2n-1}) \]
\[ = 0, \]
since
\[ \sum_{n=1}^{\infty} dt_{2n-1} \land d\zeta_{2n-1} = \sum_{n,m=1}^{\infty} (\partial_{2m-1} \partial_{2n-1} \log \tau) dt_{2n-1} \land dt_{2n-1} = 0. \]

By (ii) of Proposition 2 we have a map of \( \mathcal{D} \)-modules
\[ \mathcal{D} \otimes \frac{\tilde{H}^*_1}{CH^*_5} \xrightarrow{ev_2} A. \tag{17} \]

Notice that the proof of Proposition 2 is much simpler than that of Theorem 1 in [1, 2].

The following theorem is proved in the next section.

**Theorem 2** If we assume Conjecture [3] then the map (17) is surjective. In particular if we replace \( ev_1 \) by \( ev_2 \) in (17) then it gives a \( \mathcal{D} \)-free resolution of \( A \).
6 Relation of Two Constructions

In this section we show

**Theorem 3** The map $ev_1$ is surjective if and only if the map $ev_2$ is surjective.

The remaining part of this section is devoted to the proof of this theorem. Let us set

$$
\omega_{n,m} = ev_1(\bar{\omega}_{n,m}), \quad \bar{\omega}_{n,m} = \langle -1|\psi_n\psi^*_m T| - 1 \rangle, \quad m, n \geq 1,
$$

$$
\omega_n = \sum_{m: \text{odd}} \omega_{n,m} dt_m.
$$

**Lemma 3** The 1-form $\omega_n$ is closed.

**Proof.** By Theorem 1

$$
\text{ev}_1 \left( \langle -1|\psi_n\psi^*_m \psi^*_m QT| - 1 \rangle \right) = 0.
$$

We substitute

$$
Q = \sum_{i: \text{odd}} \partial_1 \psi_{-i}
$$

into (18) and get

$$
\partial_{m_2} \omega_{n,m_1} - \partial_{m_1} \omega_{n,m_2} = 0
$$

which proves the lemma.

By the lemma $\omega_n$ should be written as $d\eta_n$ for some function $\eta_n$ which is not necessarily in $A$. We shall find the explicit form of $\eta_n$ and study its properties.

Let

$$
\Psi(z) = \frac{\tau(t - [z^{-1}])}{\tau(t)} e^{\xi(t,z)}, \quad \xi(t,z) = \sum_{n=1}^{\infty} t_{2n-1} z^{2n-1},
$$

be the wave function of the KdV-hierarchy [3, 2], where $[z^{-1}] = (z^{-1}, z^{-3}, z^{-5}, \ldots)$. Set

$$
S(z) = \sum_{n=0}^{\infty} S_{2n} z^{-2n}, \quad S_0 = 1.
$$

Notice that $S_{2n} = \partial_1 \partial_2 \log \tau(t)$. Then, by the bilinear identity for $\tau(t)$ [3], we have (see Remark 1, p391, in [2] for example)

$$
S(z) = \frac{\tau(t - [z^{-1}])\tau(t + [z^{-1}])}{\tau(t)^2}.
$$

Define $X(z)$ by (2) p388

$$
X(z) = -\frac{1}{2} \log S(z) + \log \Psi(z).
$$
Using (19) we have

\[ X(z) = \frac{1}{2} \left( \log \tau(t - [z^{-1}]) - \log \tau(t + [z^{-1}]) \right) + \xi(t, z). \] (20)

Let us set

\[ \eta(z) = z^{-1} \left( -X(z) + \xi(t, z) \right). \]

By (20) \( \eta(z) \) is expanded into negative even powers of \( z \),

\[ \eta(z) = \sum_{n=1}^{\infty} \eta_{2n-1} z^{-2n}. \]

Then

**Proposition 3** (i) \( d\eta_{2n-1} = \omega_{2n-1} \).

(ii) We have

\[ \eta_{2n-1} = \frac{1}{2n-1} \xi_{2n-1} + a_{2n-1}, \] (21)

for some \( a_{2n-1} \in A \).

**Proof.** We use the following lemma ([2] p392 Remark 3).

**Lemma 4** Let \( \nabla (w) = \sum_{n=1}^{\infty} \partial_{2n-1} w^{-2n} \). Then

\[ \nabla(w)X(z) = \frac{z}{w^2 - z^2} S(w) S(z), \quad |w| > |z|. \]

By this lemma we get

\[ \nabla(w)\eta(z) = \frac{1}{z^2 - w^2} \left( \frac{S(w)}{S(z)} - 1 \right), \quad |w| > |z|. \] (22)

Since the right hand side of this equation is regular at \( z^2 = w^2 \), (22) is valid at \( |z| > |w| \). Let \( \bar{S}(z) = \sum_{n=0}^{\infty} \bar{S}_{2n} z^{-2n} \) with \( \bar{S}_0 = 1 \). Using

\[ T\psi(z)T^{-1} = \bar{S}(z)\psi(z), \quad T\psi^*(z)T^{-1} = \bar{S}(z)^{-1}\psi^*(z), \]

\[ < -1|\psi(z)\psi^*(w)| - 1 > = \frac{z w}{z^2 - w^2}, \quad |z| > |w|, \]

we have

\[ < -1|\psi(z)\psi^*(w)T| - 1 > = \frac{\bar{S}(w)}{\bar{S}(z)} \frac{z w}{z^2 - w^2}, \quad |z| > |w|. \]

On the other hand, expanding in \( z \) and \( w \), we have

\[ < -1|\psi(z)\psi^*(w)T| - 1 > = \sum_{m,n=1}^{\infty} \bar{\omega}_{2n-1,2m-1} z^{-2n+1} w^{-2m+1} + \frac{z w}{z^2 - w^2}, \quad |z| > |w|. \]
Thus
\[
\frac{1}{z^2 - w^2} \left( \frac{\overline{S}(w)}{S(z)} - 1 \right) = (zw)^{-1} < -1 |\psi(z)\psi^*(w)| T - 1 > - \frac{1}{z^2 - w^2} = \sum_{m,n=1}^{\infty} \overline{\omega}_{2n-1,2m-1} z^{-2n} w^{-2m}.
\]

Taking $\text{ev}_1$ we have
\[
\partial_{2m-1} \eta_{2n-1} = \omega_{2n-1,2m-1}.
\]

**Proof of Theorem 3**

We denote by $\text{vol}(i_1, \ldots, i_k)$ the differential form which is obtained from $\text{vol}$ by removing $dt_{i_1}, \ldots, dt_{i_k}$.

Suppose that $\text{ev}_1$ is surjective. Notice that
\[
\text{ev}_1 : D \otimes H_{-1}^* \rightarrow A,
\]
is given by
\[
\text{ev}_1 \left( P \otimes < -1 |\psi_{2i_1-1} \cdots \psi_{2i_k-1} \psi_{2j_1}^* \cdots \psi_{2j_k}^* | \right) = P \left( \det(\omega_{2i_a-1,2j_b-1})_{1 \leq a,b \leq k} \right).
\]

Thus the space
\[
\left( \sum_{n=1}^{\infty} \frac{A}{\partial_{2n-1} A} \right) \text{vol}
\]
is generated by all elements of the form
\[
\text{vol}(i_1, \ldots, i_k) \wedge \omega_{2j_1-1} \wedge \cdots \wedge \omega_{2j_k-1},
\]
as a vector space over $\mathbb{C}$. We substitute (21) into (24). Then the space (23) is generated by the elements of the form
\[
\text{vol}(i_1, \ldots, i_k) \wedge d\zeta_{2j_1-1} \wedge \cdots \wedge d\zeta_{2j_k-1},
\]
since the terms containing $da_{2r-1}$ belong to the denominator $(\sum_{n=1}^{\infty} \partial_{2n-1} A)\text{vol}$. Since (23) generates $A$ over $D$, $\text{ev}_2$ is surjective. The converse is similarly proved. ■

As a corollary of Theorem 3, Theorem 2 in the previous section is proved.

### 7 Concluding Remarks

From the viewpoint of conformal field theories and their integrable deformations [15, 5, 14], the space $A$ corresponds to the descendents of the identity operator. It is possible to consider the spaces corresponding to descendents of other primary fields [11]. It is an interesting problem to construct their $D$-free resolutions. In the quantum case free resolutions are constructed for those spaces [11, 8]. In particular the spaces become free modules in “odd cases”. The classical and even the finite dimensional cases are expected to have similar structures. Geometrically, to consider non-identity primary fields corresponds to consider the spaces of sections of certain non-trivial flat line bundles over affine Jacobians in stead of affine rings [11].
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