Ordered Products, $W_\infty$-Algebra, and Two-Variable, Definite-Parity, Orthogonal Polynomials

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Abstract

It has been shown that the Cartan subalgebra of $W_\infty$-algebra is the space of the two-variable, definite-parity polynomials. Explicit expressions of these polynomials, and their basic properties are presented. Also has been shown that they carry the infinite dimensional irreducible representation of the $su(1, 1)$ algebra having the spectrum bounded from below. A realization of this algebra in terms of difference operators is also obtained. For particular values of the ordering parameter $s$ they are identified with the classical orthogonal polynomials of a discrete variable, such as the Meixner, Meixner-Pollaczek, and Askey-Wilson polynomials. With respect to variable $s$ they satisfy a second order eigenvalue equation of hypergeometric type. Exact scattering states with zero energy for a family of potentials are expressed in terms of these polynomials. It has been put forward that it is the İnönü-Wigner contraction and its inverse that form bridge between the difference and differential calculus.
I. INTRODUCTION

The classical orthogonal polynomials of a discrete variable (the Hahn, Meixner, Krawtchouk, Charlier, Pollaczek, and Meixner-Pollaczek polynomials) which are the difference analogues of the classical orthogonal polynomials of the mathematical physics have been surfaced in various problems of the theoretical and mathematical physics, in group representation theory and in computational physics and techniques ([1–3], and references therein). Close relationships has been established between the generalized spherical harmonics for su(2) and the Krawtchouk polynomials, and between the Wigner 6j-symbols and the Racah polynomials which are the discrete analogues of the Jacobi polynomials on a quadratic lattice [1,4]. The Clebsch-Gordan coefficients and 6j-symbols for the su(1,1) can also be expressed in terms of the Hahn and Racah polynomials [4]. An important development in connection with these polynomials in the last decade was that the Hahn and Meixner polynomials can be analytically continued in the complex plane, both in variable and parameter, such that they become real polynomials which satisfy orthogonality relations with respect to continuous measure [5,2]. These polynomials are referred to as the continuous Hahn and Meixner, or, as the Hahn and Meixner polynomials of an imaginary argument (in fact, the second ones should be called the Meixner-Pollaczek polynomials [2]). The continuous Hahn polynomials are also closely related to the unitary irreducible representations of the Lorentz group SO(3,1).

Recent interest in these polynomials are mainly due to modifications and generalizations of them in connection with their q-analogues on non-uniform lattices ([6] and references therein). Another line of development, which has not attracted sufficient interest it deserves, was determination of their connections with the ordered products of the Heisenberg-Weyl (HW) algebra [6,8]. Connection between the ordered products of the HW-algebra and particular classes of the continuous Hanh polynomials was, for the first time, investigated in Ref. [7] to find a solution for the unitarity problem in the finite element approximation to quantum field equations. This connection was established when ordering rules that gives
Hermitian products are used \[7,8\]. In terms of ordering parameter \( s \in \mathbb{C} \), henceforth used in this paper, this corresponds to pure imaginary values of \( s \) of which the well-known Weyl ordering is a special \( s = 0 \) case. One of the purposes of this study is to revive interest in this direction by carrying out a systematic investigation in the most general framework.

In two recent studies \[9,10\], we managed to develop explicit expressions for the implicitly defined conventional \( s \)-ordered products widely used in the Weyl-Wigner-Groenevold-Moyal quantization and quantum optics \[9–11\]. In section II, we briefly review the ordered products, \( W_\infty \)-algebra and its Cartan subalgebra in their most general forms. Making use of these expressions, we firstly associate with each element of the Cartan subalgebra of the \( W_\infty \)-algebra in a general \( s \)-basis a two-variable, definite-parity, and (continuous and discrete) orthogonal polynomials which are two-variable generalization of those appeared in literature. Our variables are \( s \), and the \( c \)-number correspondence of the operator \( \hat{x} = (\hat{q}\hat{p} + \hat{p}\hat{q})/2 \) denoted by \( x \) (or, in some cases the variable \( u = \frac{x}{\hbar} - \frac{1}{2} \)). These polynomials solve a hypergeometric type differential equation with respect to \( s \), and a hypergeometric type difference equation with respect to \( x \). For generic values of \( s \) this difference equation is identical with that satisfied by a particular class of Meixner polynomials. Other basic properties, and difference- differential relations are also presented. We also show that they carry the irreducible representation of the \( su(1,1) \)-algebra and obtain its realization in terms of difference operators (section III). Secondly, for particular values of \( s \), we identify these polynomials with particular classes of the Meixner, Meixner- Pollaczek, Askey-Wilson, and Hahn polynomials, and give the discrete and continuous orthogonality relations for them (section IV). Thirdly, we show that exact zero-energy scattering states for a family of the Pösch-Teller type potentials are expressed in terms of these polynomials (section V). Finally, we conclude by pointing out the role played by the İnönü- Wigner contraction and its inverse transformation in connection with differential-difference calculus.
II. ORDERED PRODUCTS, $W_\infty$-ALGEBRA AND ITS CARTAN SUBALGEBRA

Let us consider the HW-algebra: \([\hat{q}, \hat{p}] = i\hbar \hat{I}\), where \(\hbar, \hat{I}, \hat{q}\) and \(\hat{p}\) are the Planck’s constant, the identity operator and the Hermitian position and momentum operators, respectively. Here and henceforth operators and functions of operators acting in a Hilbert space \(\mathcal{H}\) are denoted by a \(^\hat{\;}\) over letters. We define the s-ordered product \(\hat{t}^{(s)}_{nm} \equiv \{(\hat{q})^n(\hat{p})^m\}_s\), containing \(n\) factors of \(\hat{q}\) and \(m\) factors of \(\hat{p}\), by the following explicit equivalent formulas

\[
\hat{t}^{(s)}_{nm} = 2^{-n} \sum_{j=0}^{n} \binom{n}{j}(1 + s)^j (1 - s)^{n-j} \hat{q}^j \hat{p}^m \hat{q}^{n-j},
\]

where \(s = \pm 1\), \(\hat{t}^{(1)}_{nm} = \hat{q}^n \hat{p}^m\); \(\hat{t}^{(-1)}_{nm} = \hat{p}^m \hat{q}^n\) and for \(s = 0\), \(\hat{t}^{(0)}_{nm} = 2^{-m} \sum_{k=0}^{m} \binom{m}{k}(1 - s)^k (1 + s)^{m-k} \hat{p}^k \hat{q}^n \hat{p}^{m-k}\). While the first two of these expressions exhibit the standard and antistandard rules of ordering, respectively, that corresponding to \(s = 0\) are two well-known expressions of the Weyl, or symmetrically ordered products. It is possible to write many equivalent forms of the above relations, but, for later use only two of them have been written. Although, there are not any known physical applications apart from the three principle ones corresponding to \(s = 1, 0, -1\), embedding orderings in a continuum provides a natural context for viewing their differences and interrelationships in a continuous manner and enable us to carry out the related analyses in their most general forms.

An arbitrary s-ordered product can be expressed in terms of a polynomial in \(s'\)-ordered product as follows

\[
\hat{t}^{(s)}_{nm} = \sum_{k=0}^{n} \binom{n}{k} \binom{m}{k} k! [i\hbar (s - s')]^k \hat{t}^{(s')}_{n-k, m-k},
\]

where \((n, m)\) denotes the smaller of the integers \(n\) and \(m\), \(\binom{n}{k} = n![(n-k)!k!]^{-1}\) is a binomial coefficient, and \(s'\) is also arbitrary complex number. Note that \(i\hbar\) in Eq. (2) is the sign of the commutator of the corresponding operators there. Thus, the relations (1) and (2) can be used for any pair of the operators \(\hat{A}, \hat{B}\) of any algebra satisfying the commutation relation \([\hat{A}, \hat{B}] = i\lambda, \lambda \in \mathbb{C}\). From (1) easily follows that
\[ [\hat{t}^{(s)}_{nm}], \hat{t}^{(-\bar{s})}_{nm} = 0, \]  

where \( \dagger \) stands for the Hermitian conjugation and \( \bar{a} \) is the complex conjugation of \( a \). That is, for general \( n, m \) integers, \( \hat{t}^{(s)}_{nm} \) are Hermitian if and only if \( \bar{s} = -s \). In particular, the Weyl ordered products \( \hat{t}^{(0)}_{nm} \) are Hermitian. For general \( s, \alpha \in \mathbb{C} \) one can form combinations such as \( \hat{\kappa}^{(s)}_{nm} = \alpha \hat{t}^{(s)}_{nm} + \bar{\alpha} \hat{t}^{(-\bar{s})}_{nm} \) that are Hermitian.

The \( W_\infty \)-algebra is the infinite algebra generated by the ordered products \( \hat{t}^{(s)}_{nm} \). Up to a trivial central element it is the universal enveloping algebra of the HW-algebra. In the most general basis the structure constants of the \( W_\infty \)-algebra can be read off from

\[
[\hat{t}^{(s)}_{kl}, \hat{t}^{(s)}_{nm}] = -\sum_{j=0}^{j_{\text{max}}} \frac{j_j!}{j_{\text{max}}!} \sum_{r=0}^{r_{\text{max}}} (s_j)^r \sum_{j'=0}^{j'} f_{srj} a_{nmkl,rj} \hat{t}^{(s)}_{n+k-j,m+l-j},
\]

where the prime over the second summation indicates that the maximum value that \( r \) may take is \( r_{\text{max}} = (m, k) \) and

\[
a_{nmkl,rj} = \frac{n!m!k!!}{(n + r - j)!(m - r)!(k - r)!(l + r - j)!}.
\]

The restrictions imposed on summations also follows from the expression of \( a_{nmkl,rj} \). In relation (4) \( f_{srj} = (s^-)^r(-s^+)^{j-r} - (s^-)^{j-r}(-s^+)^r \), \( s^\pm = \hbar (1 \pm s)/2 \), is the only factor depending on the chosen rule of ordering. Here we observe that anti-commutator of the ordered products is given by the same relation as (4) only provided that the ordering factor is replaced by \( f_{srj}^+ = (s^-)^r(-s^+)^{j-r} + (s^-)^{j-r}(-s^+)^r \). Relation (4), which was first reported in Ref. [10], for \( s = 0, \pm 1 \) coincides with those appeared in literature [14], and generalize them for arbitrary values of \( s \).

\( W_\infty \) has some finite and infinite dimensional subalgebras. But, for the purpose of this work we will be concerned only with the infinite abelian subalgebra consisting of the generators \( \hat{H}^{(s)}_n \equiv \hat{t}^{(s)}_{nn} \). The commutativity of the generators \( [\hat{H}^{(s)}_n, \hat{H}^{(s')}_{k}] = 0 \) follows from (4). But, for our purpose we prove this, and \( [\hat{H}^{(s)}_n, \hat{H}^{(s')}_k] = 0 \) by showing that for any values of \( n \geq 0 \) and \( s \in \mathbb{C} \) all the ordered products of the form \( \hat{H}^{(s)}_n \equiv \hat{t}^{(s)}_{nn} \) can be expressed in terms of single operator \( \hat{x} \equiv (\hat{q}\hat{p} + \hat{p}\hat{q})/2 \). To see this, let us first consider \( \hat{H}^{(1)}_n = \hat{q}_n^{(1)} \hat{p}_n^{(1)} \) which can be written as
\[ \hat{H}_n^{(1)} = \hat{q}^{n-1} \hat{q} \hat{p}^{n-1} = \hat{q}^{n-1} \hat{p}^{n-1} [\hat{x} + \hat{c}(2n - 1)] \]
\[ = \hat{q}^{n-2} \hat{p}^{n-2} [\hat{x} + \hat{c}(2n - 3)] [\hat{x} + \hat{c}(2n - 1)]. \]

Hence, by induction, we have

\[ \hat{H}_n^{(1)} = \prod_{j=1}^{n} [\hat{x} + \hat{c}(2j - 1)], \quad \hat{H}_n^{(-1)} = \prod_{j=1}^{n} [\hat{x} - \hat{c}(2j - 1)], \]

(6)

where \( \hat{c} \equiv i\hbar \hat{I}/2 \) and the relations \( \hat{q} \hat{p} = \hat{x} + \hat{c} \), \( [\hat{x}, \hat{p}^k] = 2\hat{c} \hat{p}^k \) are used. The second relation is written by making use of (3).

**III. TWO-VARIABLE, DEFINITE PARITY POLYNOMIALS**

Now, evaluating (2) for \( s' = \pm 1 \) by making use of (6), and then replacing \( \hat{x} \) in results by the c-number variable \( x \), we obtain two-variable polynomials \( P_n(s, x) \) for each element of the Cartan subalgebra. Here the variables are the (dimensionless) ordering parameter \( s \in \mathbb{C} \) and \( x \in \mathbb{R} \) which has dimension of angular momentum (since \( \hat{x} \) is an hermitian operator we consider \( x \) as a real variable). Two equivalent, explicit expressions for these polynomials are as follows

\[ P_n(s, x) = \sum_{k=0}^{n} \binom{n}{k}^2 k! [c(1 - s)]^k \prod_{j=1}^{n-k} [x + \hat{c}(2j - 1)] \]
\[ = \sum_{k=0}^{n} \binom{n}{k}^2 k! [c(1 + s)]^k \prod_{j=1}^{n-k} [x - \hat{c}(2j - 1)], \]

(7)

where \( c = i\hbar/2 \). From these relations it is obvious that under the action of two dimensional parity transformation in \( \mathbb{C}x\mathbb{R} \), they transform as

\[ P_n(-s, -x) = (-1)^n P_n(s, x), \]

(8)

that is, they have the same parity with \( n \). The following recursion relation can also be verified

\[ P_{n+1}(s, x) = [x + \hat{c}(2n + 1)s] P_n(s, x) + c^2 (1 - s^2) n^2 P_{n-1}(s, x), \]

(9)

The easiest way of obtaining this relation may be first noting the relation
\[ [\hat{x}, \hat{t}_{nm}^{(s)}]_+ = 2[\hat{t}_{n+1,m+1}^{(s)} - cs(m + n + 1)\hat{t}_{nm}^{(s)} - c^2nm(1 - s^2)\hat{t}_{n-1,m-1}^{(s)}], \]

where \([,]_+\) stands for the anticommutator. They also obey the following derivatives and difference relations

\[
\partial_s^k P_n(s, x) = c^k\left[\frac{n!}{(n-k)!}\right]^2 P_{n-k}(s, x), \tag{10}
\]

\[(x \pm c) P_n(s, x \pm 2c) = [x \pm c(2n + 1)] P_n(s, x) + 2c^2n^2(1 \mp s) P_{n-1}(s, x). \tag{11}\]

with respect to \(s\) and \(x\), respectively. This last property immediately results by first noting the relations

\[(x \pm c) P_n(\pm 1, x \pm 2c) = [x \pm c(2n + 1)] P_n(\pm 1, x), \tag{12}\]

which easily result from (6).

From (9) and (10) we see that, with respect to \(s\) the lowest order differential equation obeyed by these polynomials is the following hypergeometric type differential equation

\[
\left\{(1 - s^2)\partial_s^2 + \left[\frac{x}{c} + (2n - 1)s\right]\partial_s - n^2\right\} P_n(s, x) = 0. \tag{13}\]

The difference relations (12) can also be recast in a form in which the discrete differences are in a more conventional form by using the central first and second differences;

\[
D_h f(x) = \frac{f(x + h) - f(x - h)}{2h}, \quad D_h^2 f(x) = \frac{f(x + h) - 2f(x) + f(x - h)}{h^2}, \tag{14}\]

which approximate the usual first and the second derivatives on a lattice with the constant mesh \(\Delta x = h\) up to second order in \(h\) \[1,8\]. For this purpose we take the sum and difference of two relations given by (12)

\[
(x + c) P_n(s, x + 2c) - (x - c) P_n(s, x - 2c) = c(4n + 2) P_n(s, x) - 4sc^2n^2 P_{n-1}(s, x), \tag{15}\]

\[
(x + c) P_n(s, x + 2c) + (x - c) P_n(s, x - 2c) = 2xP_n(s, x) + 4c^2n^2P_{n-1}(s, x), \tag{16}\]

which, in terms of \(D_h\) and \(D_h^2\) can be rewritten as follows

\[
(c^2D_h^2 + xD_h - n) P_n(s, x) = -s cn^2 P_{n-1}(s, x), \tag{17}\]

\[
(xD_h^2 + D_h) P_n(s, x) = n^2 P_{n-1}(s, x), \tag{18}\]

7
where, \( h = 2c \) and \( D_h \) denotes the partial difference operation with respect to \( x \). Note that all the difference operations are with respect to variable \( x \) and all the derivatives are with respect to the ordering variable \( s \).

Now, by making use of the recursion relation (9), and (17), (18) we have

\[
\begin{align*}
J_- P_n(s, x) &= n^2 P_{n-1}(s, x), \\
J_+ P_n(s, x) &= P_{n+1}(s, x), \\
J_0 P_n(s, x) &= (n + \frac{1}{2}) P_n(s, x),
\end{align*}
\]

where

\[
\begin{align*}
J_- &= [x D_h^2 + D_h] = \left[ \frac{Ax}{h} \sin^2 \left( \frac{h}{2} \partial_x \right) + \frac{1}{h} \sin(h \partial_x) \right], \\
J_+ &= c^2(1 + s^2) J_- + 2cs \left( c^2 D_h^2 + xD_h \right) + (x + cs) \\
&= c^2(1 + s^2) J_- + 2cs \left[ - \sin^2 \left( \frac{h}{2} \partial_x \right) + \frac{x}{h} \sin(h \partial_x) \right] + (x + cs), \\
J_0 &= c(c + sx) D_h^2 + (x + cs) D_h + \frac{1}{2} \\
&= c(c + sx) \frac{4}{h^2} \sin^2 \left( \frac{h}{2} \partial_x \right) + (x + cs) \frac{1}{h} \sin(h \partial_x) + \frac{1}{2}.
\end{align*}
\]

It is not hard to verify that by their actions on an arbitrary function, the generators \( (J_0, J_\pm) \) obey the standart defining relations of the \( su(1, 1) \) algebra

\[
[J_+, J_-] = -2J_0, \quad [J_0, J_\pm] = \pm J_\pm.
\]

The value of the Casimir operator \( J^2 = -J_- J_+ + J_0^2 + J_0 \) is found to be \( J^2 P_n(s, x) = -\frac{1}{4} P_n(s, x) \). Thus, we have obtained a realization of the \( su(1, 1) \) algebra in terms of the difference operators, and the polynomials we have found carry the infinite dimensional irreducible representation of this algebra having the spectrum bounded from below which is designated by \( D^+_{-\frac{1}{2}} \).

Finally, in this section we write down explicit expressions of the first five polynomials;

\[
\begin{align*}
P_0(s, x) &= 1, \\
P_1(s, x) &= x + cs,
\end{align*}
\]
\[ P_2(s, x) = x^2 + 4csx + c^2(2s^2 + 1), \]
\[ P_3(s, x) = x^3 + 9csx^2 + c^2(18s^2 + 5)x + 3c^3s(2s^2 + 3), \]
\[ P_4(s, x) = x^4 + 16csx^3 + 2c^2(36s^2 + 7)x^2 + 16c^3s(6s^2 + 5)x + 3c^4(8s^4 + 24s^2 + 3). \]  

(24)

The values of these polynomials at \((s, x = 0)\) and at \((s = 0, x = 0)\) can be easily obtained from (7) \[16,17\]. Because of (3), these are real valued for pure imaginary \(s\). Note that the relation (2), which enables us to write \(P_n(s, x)\) in terms of \(P_n(s', x)\), in view of (10), is nothing more than a Taylor expansion.

\section*{IV. CONNECTIONS WITH THE MEIXNER, MEIXNER-POLLACZEK AND CONTINUOUS HAHN POLYNOMIALS}

To compare the polynomials we have found with the classical orthogonal polynomials of a discrete variable we use

\[ u = \frac{x}{2c} - \frac{1}{2}. \]  

(25)

In that case we have the following difference relations

\[ (u + 1)P_n(s, u + 1) = (u + n + 1)P_n(s, u) + c(1 - s)n^2P_{n-1}(s, u), \]

(26)

\[ uP_n(s, u - 1) = (u - n)P_n(s, u) + c(1 + s)n^2P_{n-1}(s, u). \]

(27)

Eliminating the last terms between these two relations and writing out the result in terms of the difference operators

\[ \Delta f(u) = f(u + 1) - f(u), \quad \nabla f(u) = f(u) - f(u - 1), \]  

(28)

we arrive at

\[ \{u\Delta \nabla - \left[ \frac{2}{1-s}u + \frac{1+s}{1-s}\right] \Delta + \frac{2n}{1-s}\}P_n(s, u) = 0. \]  

(29)

This is the same type of difference equation satisfied by the Meixner polynomials \(m_n^{(\gamma, \mu)}(u)\):

\[ \{u\Delta \nabla + [(\mu - 1)u + \mu \gamma] \Delta + \lambda_n\}m_n^{(\gamma, \mu)}(u) = 0. \]  

(30)
Comparing with Eq. (29) we have
\[ \mu = \frac{s + 1}{s - 1}, \quad \gamma = 1, \quad \lambda_n = \frac{2n}{1 - s}. \] (31)

The Meixner polynomials are normalized as follows
\[ \Delta_n m_n^{(\gamma, \mu)}(u) = n! \left( \frac{\mu - 1}{\mu} \right)^n. \]

On the other hand, the polynomials \( P_n(s, u) \) obey the relation
\[ \Delta^n P_n(s, u) = (2c)^n n!. \]

(For a polynomial \( q_m(u) \) of degree \( m \) the \( m \)th differences and \( m \)th derivatives are equal: \( \Delta^m q_m(u) = \partial^m u q_m(u) \)). Thus, we obtain
\[ P_n(s, u) = [c(s + 1)]^n m_n^{(1, s + 1)}(u). \] (32)

If the conditions \( \gamma > 0 \), and \( 0 < \mu < 1 \) are satisfied the Meixner’s polynomials obey the discrete-orthogonality relation
\[ \sum_{u=0}^{\infty} m_n^{(\gamma, \mu)}(u) m_m^{(\gamma, \mu)}(u) \rho(u) = \delta_{nm} d_n^2. \] (33)

Thus, provided that \( s < -1 \) this relation in terms of \( P_n(s, u) \) is as follows
\[ \sum_{u=0}^{\infty} P_n(s, u) P_m(s, u) \rho'(u) = \delta_{nm} d_n^2, \] (34)

where, (for \( s < -1 \)) the weight \( \rho'(u) \), and the squared norm \( d_n^2 \) are found to be
\[ \rho'(u) = \left( \frac{s + 1}{s - 1} \right)^n \frac{\Gamma(u + 1)}{u!}, \quad d_n^2 = \frac{1}{2} (n!)^2 c^n (1 - s)(s^2 - 1)^n, \] (35)

Here \( \Gamma(z) \) is the gamma function. It is also easy to verify that, the three-term recursion relation (9) for the polynomials \( P_n(s, u) \) is the same as that obeyed by the Meixner’s polynomials
\[ \mu m_{n+1}^{(\gamma, \mu)}(u) = [\gamma \mu + (1 + \mu)n - (1 - \mu)u] m_n^{(\gamma, \mu)}(u) - n(n + \gamma - 1)m_{n-1}^{(\gamma, \mu)}(u), \] (36)

provided that relations given by (31) are satisfied.

In Ref. [5] by considering the analytic continuation of the orthogonality relation (33) in the parameter \( \mu = \exp(-2i\phi) \) the polynomials
\[ P_n^\lambda(\phi, t) = \frac{e^{-i\phi}}{n!} m_n^{(2\lambda, \mu)}(-\lambda + it), \] (37)
which, under the conditions \( \lambda > 0 \), and \( 0 < \phi < \pi \), obey the orthogonality relation

\[
\int_{-\infty}^{\infty} P_{n}^{\lambda}(\phi, t)P_{m}^{\lambda}(\phi, t)\rho_{P}(t)dt = \delta_{nm}\frac{\Gamma(2\lambda + n)}{n!}
\]

with respect to continuous measure are obtained. Here the weight \( \rho_{P} \) is as follows

\[
\rho_{P}(t) = \frac{1}{2\pi}(2\sin \phi)^{2\lambda}|\Gamma(\lambda + it)|^{2}\exp[(2\phi - \pi)t].
\]

(39)

These polynomials should be called the Meixner-Pollaczek polynomials [2]. In view of the relation (32), if we identify \( \hat{x}/\hbar \) with the real variable \( t \) we find that \( \lambda = 1/2 \) and \( s = i\cot \phi \). Thus, with help of (37), we obtain

\[
P_{n}(s, i\frac{x}{\hbar} - \frac{1}{2}) = n!\left(-\frac{\hbar}{2\sin \phi}\right)^{n}P_{n}^{1/2}(\phi, \frac{x}{\hbar}).
\]

(40)

In terms of \( P_{n}(s, u) \) the orthogonality relation (38) is as follows

\[
\int_{-\infty}^{\infty} P_{n}(s, i\frac{x}{\hbar} - \frac{1}{2})P_{m}(s, i\frac{x}{\hbar} - \frac{1}{2})\rho_{0}(x)dx = \delta_{nm}(n!)^{2}\left(\frac{\hbar}{2\sin \phi}\right)^{2n+1},
\]

(41)

with the weight

\[
\rho_{0}(x) = \frac{\exp[(2\phi - \pi)x]}{\cosh \frac{\pi x}{\hbar}}.
\]

(42)

For \( \lambda = 1/2 \), and \( \phi = \pi/2 \) the Meixner-Pollaczek polynomials are related to the Askey-Wilson polynomials \( q_{n}^{(\alpha)}(x, \delta) \) as follows \( q_{n}^{(0)}(x, 1/2) = (1/2)_{n}P_{n}^{1/2}(\pi/2, x) \), where \( (a)_{n} = \Gamma(a + n)/\Gamma(a) \). Note that this particular case corresponds to \( s = 0 \) (Weyl ordering) and the weight is \( \rho_{0}(x) = \cosh(\pi x/\hbar) \). On the other hand, the Askey-Wilson polynomials are particular cases of the continuous Hahn polynomials \( h_{n}^{(\alpha, \beta)}(z, N) \):

\[
q_{n}^{(\alpha)}(x, \delta) = (-i)^{n}h_{n}^{(\alpha, \alpha)}(\frac{1}{2}ix - \frac{1}{2}(\alpha - \delta + 1), -\alpha + \delta).
\]

(43)

Making use of these relations we have

\[
P_{n}(s = 0, i\frac{x}{\hbar} - \frac{1}{2}) = n!(\frac{-\hbar}{2})^{n}[\frac{1}{2}]_{n}^{-1}q_{n}^{(0)}(\frac{x}{\hbar}, \frac{1}{2})
\]

\[
= n!(\frac{-\hbar}{2})^{n}[\frac{1}{2}]_{n}^{-1}(\frac{1}{i})^{n}h_{n}^{(0, 0)}(i\frac{x}{2\hbar} - \frac{1}{4}, \frac{1}{2}).
\]

(44)

As a last remark in this section we note that for particular values, or ranges of the variables, these polynomials can also be identified with particular cases of the Jacobi polynomials and the generalized spherical harmonics (the Bargmann functions) [3].
V. RODRIGUES FORMULA, GENERATING FUNCTION, AND EXACT ZERO-ENERGY SCATTERING STATES FOR A FAMILY OF POTENTIALS

For a given $x$, the polynomials solve the eigenvalue equation (13), which is a differential equation of the hypergeometric type. In order to uncover more properties of these polynomials we restrict the investigation to

$$s = iy, \quad \frac{x}{c} = -iv; \quad y, v \in \mathbb{R} \quad (45)$$

Note that in that case the corresponding ordered products are Hermitian. In terms of the new variables, Eq. (13) is as follows

$$\{(1 + y^2)\partial_y^2 + [v - (2n - 1)y]\partial_y + n^2\} P_n(y, v) = 0 \quad (46)$$

With the help of the function

$$\rho(y, v) = \frac{e^{\tan^{-1}y}}{(1 + y^2)^{n+1}}, \quad (47)$$

which is determined from

$$\partial_y[(1 + y^2)\rho] = [v - (2n - 1)y]\rho, \quad (48)$$

Eq. (46) can be written in the self-adjoint form

$$\{\partial_y[(1 + y^2)\rho(y, v)\partial_y] + \rho(y, v)n^2\} P_n(y, v) = 0. \quad (49)$$

Thus, we have found the following Rodrigues formula, integral representation, and the generating function for these polynomials:

$$P_n(y, v) = \frac{(-ic)^n}{\rho(y, v)} \partial_y^n \left[ \frac{e^{\tan^{-1}y}}{(1 + y^2)^{n+1}} \right], \quad (50)$$

$$P_n(y, v) = \frac{(-ic)^nn!}{2\pi i\rho(y, v)} \oint_C \frac{(1 + z^2)^n}{(z - y)^{n+1}}\rho(z, v)dz, \quad (51)$$

$$\Phi(y, v, u) = (1 - 4uy - 4u^2)^{-1/2} \frac{\rho(\xi, v)}{\rho(y, v)}, \quad (\xi = (2u)^{-1}[1 - (1 - 4uy - 4u^2)^{1/2}]). \quad (52)$$
In Eq. (51) \( C \) is a closed contour surrounding the point \( z = y \), and the expansion of the generating function \( \Phi(y, x, u) \) in power of \( u \) has, for sufficiently small \( |u| \), the form

\[
\Phi(y, v, u) = \sum_{n=0}^{\infty} \frac{P_n(y, v)}{n!} (-u/\imath c)^n.
\]

Finally in this section, we would like to interpret these polynomials as the solutions of the Schrödinger type equations. By transforming the dependent variable as

\[
\Psi_n(y, v) = [(1 + y^2) \rho(y, v)]^{1/2} P_n(y, v),
\]

the first derivative term in Eq. (46) vanishes and the equation

\[
\partial_y^2 \Psi_n(y, v) + \frac{y^2 + 2v(2n + 1)y + (2n + 1)^2 - (v^2 + 3)}{4(1 + y^2)^2} \Psi_n(y, v) = 0,
\]

results. \( \eta \) being a constant of dimension \((\text{length})^{-1}\), Eq. (55) is the time independent Schrödinger equation for the potential

\[
\frac{2m}{\eta^2 \hbar^2} V_n(y, v) = -\frac{y^2 + 2v(2n + 1)y + (2n + 1)^2 - (v^2 + 3)}{4(1 + y^2)^2}.
\]

In fact, here we have a family of potentials which are labelled by \( n \) and \( v \), and are similar to the Pösch-Teller type \((\propto - \cosh^{-2} y)\) potential holes. Thus, the wave functions given by (54) are the exact zero-energy scattering states for this family of potentials. Note that for \( v = 0 \), \( V(y, 0) \) is an even function, so admit the solutions \( \Psi_n(y, 0) \) with the same parity as \( n \).

**VI. CONCLUSION**

The main points of this study can be summarized as follows. (i) By using the explicit expression for the ordered products, which form a basis for the universal enveloping algebra of the HW-algebra, we have shown that, the infinite Cartan subalgebra is, in fact, the abelian algebra of two-variable polynomials. The variables are the ordering parameter \( s \) and the c-number correspondence of the squeeze operator \( \hat{x} \) which is an element of the symplectic algebra in two dimensions. We expect that these definite-parity polynomials will
play an important role in two dimensional physics in which tremendous developments are taking place in the recent years (see, for instance [13], and references therein). (ii) The realization of the $su(1,1)$ obtained in this report, is, since it contains a single variable, a difference analogue of generalized Gelfand-Dyson realization [18]. Furthermore, the analyses of section V show that this algebra play the role of “potential algebra” (i.e., an algebra whose generators connect the states of the same energy in different potential strengths [18]) for a class of potentials. Regarding the role of the $su(1,1)$ algebra in the exactly solvable problems of quantum mechanics, we expect that with its realization in terms of the difference operators found in this study, or, with appropriate generalizations, it is also the underlying algebra of the exactly solvable difference equations. This point, of course, requires further studies to be done. In particular, an algebraic approach to the polynomials of a discrete variable is now under investigation. (iii) Another particularly intriguing point requiring further studies is that the basic algebraic building block underlying the difference equations seems to be an expansion of the HW-algebra, which is the basic algebraic structure of both classical and quantum mechanics. More concretely, the algebra:

$$[x, D_h] = -\Delta_h, \quad [x, \Delta_h] = \hbar^2 D_h, \quad [D_h, \Delta_h] = 0,$$

satisfied by the fundamental difference operations \{\(x, D_h, 2\Delta_h f(x) = f(x+h) + f(x-h)\)\} is the two-dimensional Euclidean algebra \(e(2) = \{L, N_1, N_2 : [L, N_1] = N_2, [L, N_2] = -N_1, [N_1, N_2] = 0\}\) subjected to singular transformation (with respect to \(N_2\)): \(\{x = \hbar L, \Delta_h = \hbar N_1, D_h = N_2\}\). The \(\hbar \to 0\) (İnönü-Wigner) contraction of this algebra is the HW-algebra generated by

$$x, \quad \partial_x = \lim_{\hbar \to 0} D_h, \quad I = \lim_{\hbar \to 0} \Delta_h.$$ 

Thus, we expect that it is the İnönü-Wigner and its inverse transformation which form a bridge between the exactly solvable differential and difference equations of mathematical physics. Other results in this direction and more detailed properties of these polynomial will be given elsewhere.
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[16] For \( s = 0 \), and \( \hbar = 1 \) these polynomials coincide, up to factors of the form \( n!/2^n \), with that \( S_n(x) \) investigated in Ref. [7,8].

[17] For \( \hbar = 1 \), i.e., \( c^2 = -1/4 \) and \( s = 0 \) the last line of Eqs. (19) is the lattice analog of the differential equation satisfied by the Hermite polynomials \( He_n''(\xi) - 2\xi He_n'(\xi) + 2nHe_n(\xi) = 0 \) provided that \( \xi = \sqrt{2x} \). This point was first observed in reference [8]. Here we note that, for \( s = 0 \) or \( s \neq 0 \) we have two additional difference recursion relations which, as we show in the main text, reveal important properties of these polynomials.

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