Linear smoothed extended finite element method

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Summary
The extended finite element method was introduced in 1999 to treat problems involving discontinuities with no or minimal remeshing through appropriate enrichment functions. This enables elements to be split by a discontinuity, strong or weak, and hence requires the integration of discontinuous functions or functions with discontinuous derivatives over elementary volumes. A variety of approaches have been proposed to facilitate these special types of numerical integration, which have been shown to have a large impact on the accuracy and the convergence of the numerical solution. The smoothed extended finite element method (XFEM), for example, makes numerical integration elegant and simple by transforming volume integrals into surface integrals. However, it was reported in the literature that the strain smoothing is inaccurate when non-polynomial functions are in the basis. In this paper, we investigate the benefits of a recently developed Linear smoothing procedure which provides better approximation to higher-order polynomial fields in the basis. Some benchmark problems in the context of linear elastic fracture mechanics are solved and the results are compared with existing approaches. We observe that the stress intensity factors computed through the proposed linear smoothed XFEM is more accurate than that obtained through smoothed XFEM.

KEYWORDS
extended finite element method, fracture mechanics, linear smoothing, numerical integration, smoothed finite element method

1 | INTRODUCTION

The major difficulty associated with solving problems involving evolving discontinuities is the meshing and remeshing required as the discontinuities evolve in time. This difficulty is exacerbated when singularities are also present, as is the case in crack growth simulations. These difficulties are somewhat alleviated by the introduction of enrichment functions to represent the discontinuities and the singularities at the patch level, in finite elements or meshfree methods. A first approach to treat discontinuities without an explicit meshing was proposed as early as 1995 in Oliyer. A much more versatile approach was presented a few years later in the form of the extended finite element method (XFEM) by exploiting the partition of unity property identified by Melenk and Babuška. Partition of unity enrichment for problems with discontinuous solutions is now widely used both in academia and in industrial practice and is known under various names, including the generalized finite element method and the XFEM. The approach has also been widely used in the form of enriched meshfree methods.

Another problem associated with partition of unity methods involving non-polynomial basis functions is to integrate the resulting fields accurately. These enriched methods, also carry along the element mapping involved in building the system matrices. The regularity and positive definiteness of the isoparametric mapping poses a number of restrictions on the allowable shapes of
the finite elements: For example, the element should be convex. Meshfree methods also have to face such problems associated
with the regularity, distortion, and clustering in the point cloud. Under large distortions, meshfree methods face numerical insta-

bilities and low accuracy. Nodal integration also leads to instabilities in cases where higher-order shape functions are used.
This is due to the fact that, in the meshfree methods, each node would be associated with a support domain. And the shape func-
tions derivatives would be integrated in this support domain. This implies that each integration domain would be associated
with only 1 integration point (ie, the node). Hence, when only 1 integration is point is considered for higher-order functions (other
than constant strain) the integration would be similar to the inadequate reduced integration which in turn causes instabilities.

To alleviate these instabilities, the strain smoothing concept was introduced for meshfree methods. The basic idea of strain
smoothing is to transform numerical integration over volumes to integration over surfaces, thereby removing instabilities due to
the evaluation of the shape function derivatives at the nodes. This approach was later extended to finite element methods by Liu
et al. The resulting method was coined the smoothed finite element method (SFEM), discussed in a number of papers, and
applied to a wide variety of problems. The SFEM reduces the mesh sensitivity to some extent by avoiding the necessity of

evaluating the Jacobian. Since the derivatives are not needed, the isoparametric mapping is also avoided.

The SFEM involves computation of a smoothed strain from the standard compatible strain field. The smoothed strain is
evaluated as a spatial average of the standard strain field over smoothing subcells which cover the domain and that can be
chosen independently from the mesh structure. These smoothing cells are typically constructed from the mesh following different
approaches. This gives rise to a number of methods including cell-based SFEM, node-based SFEM, edge-based SFEM, and face-based SFEM. Liu and co-workers by modifying the basis functions, developed singular elements based on the edge-based SFEM.

The cell-based SFEM was later extended to solve problems with field discontinuities, both strong and weak, by Bordas et al. This was achieved by extending strain smoothing to the partition of unity framework. Though the smoothed FEM did make the integration procedure more elegant, it was also reported in Bordas et al that the error levels are higher due to the inaccurate approximation of the near-tip singular fields. Similar errors were also encountered in higher-order elements and polygons.

It is noteworthy that similar difficulties are also present in meshfree methods, as addressed in Duan et al by using the discrete divergence consistency requirement by establishing a compatibility relation between the shape function and its derivatives. This was recently extended for the FEM in Francis et al and named linear smoothing (LS) scheme. It was also reported that the LS scheme provides an improved accuracy compared with the standard constant-based smoothing of the SFEM.

The present paper aims at investigating how the use of LS, resolves the lack of accuracy observed when constant smoothing (CS) is used with non-polynomial basis functions. The paper is organized as follows. After presenting the governing equations for elasto-statics, a brief overview of the XFEM is given in Section 2. Section 3 presents the LS technique. A few standard benchmark problems in linear elastic fracture mechanics, solved by using the developed method, are presented; and the accuracy, convergence, and the efficiency of the proposed method are discussed in Section 4, followed by concluding remarks in the last section.

2 | THEORETICAL FORMULATION

2.1 | Governing equations for elastostatics

Consider a linear elastic body as shown in Figure 1, with a discontinuity. Let the domain be $\Omega \in \mathbb{R}^d$ bounded by $\Gamma$. In this case, the boundary has 3 parts, namely, $\Gamma_u$, where the displacement boundary conditions are applied; $\Gamma_t$, where the tractive boundary conditions are applied; and $\Gamma_c$, which is the traction-free surface representing the discontinuity, such that $\Gamma = \Gamma_u \cup \Gamma_t \cup \Gamma_c$ and $\Gamma_u \cap \Gamma_t = \emptyset$.

The governing equation to be solved is

$$\nabla \cdot \sigma + b = 0 \text{ in } \Omega. \quad (1)$$

The boundary conditions are as follows:

$$\sigma \cdot n = 0 \text{ on } \Gamma_c, \quad (2a)$$

$$\sigma \cdot n = \hat{t} \text{ on } \Gamma_t, \quad (2b)$$

$$u = \bar{u} \text{ on } \Gamma_u. \quad (2c)$$
where $\nabla$ is the gradient operator, $\sigma$ is the Cauchy stress tensor, $b$ is the body force per unit volume, $n$ is the unit outward normal, and $\hat{f}$ is the applied traction. For a body undergoing small displacements and strains, the strain displacement equation reads as

$$\varepsilon = \nabla u,$$

where $\nabla_s$ is the symmetric part of the gradient operator. By substituting the constitutive relations and the strain-displacement relations, the weak form of the above Equation 1 becomes Equation 4 in the absence of the body forces: Find $u \in U$ such that

$$\int_{\Omega} \varepsilon(u) : C : \varepsilon(v) \, d\Omega = \int_{\Gamma} \hat{f} v \, d\Gamma,$$

where $u$ and $v$ are the trial and the test functions, respectively. For the aforementioned problem, the function spaces includes functions that are discontinuous across $\Gamma_c$.

$$U := \{ u(x) \in [C^0(\Omega)]^d : u \in [W(\Omega)]^d \subseteq [H^1(\Omega)]^d, \quad u = \hat{u} \quad \text{on} \quad \Gamma_u \}$$

$$V := \{ v(x) \in [C^0(\Omega)]^d : v \in [W(\Omega)]^d \subseteq [H^1(\Omega)]^d, \quad v = 0 \quad \text{on} \quad \Gamma_u \}$$

where the space $W(\Omega)$ includes linear displacement fields. The domain is partitioned into elements $\Omega^h$, and on using shape functions $N_a$ that span at least the linear space, we substitute vector-valued trial and test functions $u^h = \sum a N_a u_a$ and $v^h = \sum b N_b v_b$, respectively, into Equation 4 and apply a standard Galerkin procedure to obtain the discrete weak form: Find $u^h \in U^h$ such that

$$\forall v^h \in V^h \quad a(u^h, v^h) = \ell(v^h),$$

which leads to the following system of linear equations:

$$K u^h = f,$$

$$K = \sum_h K^h = \sum_h \int_{\Omega^h} B^T C B \, d\Omega,$$

$$f = \sum_h f^h = \sum_h \left( \int_{\Omega^h} N^T b \, d\Omega + \int_{\Gamma^h} N^T \hat{f} \, d\Gamma \right),$$

where $K$ is the assembled stiffness matrix; $f$, the assembled nodal force vector; $u^h$, the assembled vector of nodal displacements; $N$, the matrix of shape functions; $C$, the constitutive matrix for an isotropic linear elastic material; and $B = \nabla_s N$, the strain-displacement matrix that is computed using the derivatives of the shape functions.

### 2.2 Extended finite element method

With the regular FEM, the mesh topology has to conform to the discontinuous surface. The introduction of the XFEM has alleviated these difficulties by allowing the discontinuities to be independent of the underlying discretization. Within the framework of the XFEM, the trial functions take the following form:
\[ u^h(x) = \sum_{j \in \mathcal{N}_{\text{std}}} N_j(x) u_j + \sum_{j \in \mathcal{N}_{\text{hev}}} N_j(x) H(x) a_j + \sum_{k \in \mathcal{N}_{\text{tip}}} N_k(x) \left( \sum_{m=1}^{4} F_m(r, \theta) b^m_k \right), \]  

(8)

where \( \mathcal{N}_{\text{std}} \) is the set of all the nodes in the system; \( \mathcal{N}_{\text{hev}} \) is the set of nodes which are completely cut by the crack; \( \mathcal{N}_{\text{tip}} \) is the set of nodes which contain the crack tips as shown in Figure 2; \( N_j(x) \) are the standard shape functions associated with the standard \( df u_I \); \( H(x) \) is the Heaviside function associated with the enriched \( df a_J \) and \( F_m(r, \theta) \) are the tip enrichment functions associated with the \( df b^m_k \) that span the near-tip asymptotic fields:

\[ F_m(r, \theta) = \left\{ \sqrt{r} \sin \frac{\theta}{2}, \sqrt{r} \cos \frac{\theta}{2}, \sqrt{r} \sin \theta \sin \frac{\theta}{2}, \sqrt{r} \sin \theta \cos \frac{\theta}{2} \right\}. \]  

(9)

Following the Galerkin procedure, this modification to the trial function space leads to an enlarged problem to solve

\[ K u^h = F, \]  

(10)

where

\[ K = \begin{bmatrix} K_{uu} & K_{ua} & K_{ub} \\ K_{au} & K_{aa} & K_{ab} \\ K_{bu} & K_{ba} & K_{bb} \end{bmatrix}, \]  

(11)

where the superscript \( uu \) refers to standard FEM components, \( aa \) refers to the Heaviside enrichment terms, and \( bb \) refers to the asymptotic enrichment terms and other terms can be defined similarly. The augmentation of non-polynomial functions to the trial function space makes the numerical integration nontrivial. This has been of particular interest among research community, for example, equivalent polynomial approach by Ventura\(^3\) and Ventura et al.\(^3\) conformal mapping,\(^3\) Duffy transformation,\(^3\) generalized Gaussian quadrature,\(^3\) strain smoothing technique,\(^2\) exponentially convergent mapping,\(^3\) polar mapping,\(^3\) and very recently by using Euler homogeneous function theorem and Stoke theorem.\(^3\) In Bordas et al.\(^2\), the strain smoothing technique was combined with the XFEM, coined as smoothed XFEM (Sm-XFEM) to integrate over elements intersected with discontinuous surface. The main advantages of the Sm-XFEM are that no subdivision of the split elements is required and that the derivatives of the shape functions (including the enrichment functions) are not required. However, it was observed that the error level was greater when compared with the conventional XFEM, whilst yielding similar convergence rates.

**FIGURE 2** Extended finite element method discretization of a domain with internal discontinuity.
3 | LINEAR SMOOTHING IN THE XFEM

The strain smoothing was introduced in Chen et al. for the meshfree methods, which was later extended to the FEM by Liu and co-workers. The basic idea is to compute a strain field, referred to as “smoothed” strain field by evaluating the weighted average of the standard (or compatible) strain field. The support domain of the associated material point can be chosen based on surrounding cells, nodes, or edges. In this paper, we restrict our discussion only to the cell-based strain smoothed FEM. Within this framework, at a point \( x \), in element \( \Omega^h \) the smoothed strain is given below:

\[
\tilde{\varepsilon}^b_{ij}(x_c) = \int_{\Omega^h} \varepsilon^b_{ij}(x) \Phi(x) \, d\Omega.
\]  

(12)

In terms of the standard element shape function derivatives, \( N^b_{I,i}(x) \), the smoothed derivatives are given by

\[
\tilde{N}^b_{I,i}(x) = \int_{\Omega^h} N^b_{I,i}(x) \Phi(x) \, d\Omega,
\]  

(13)

where \( \Phi(x) \) is the smoothing function and \( i = x, y, z \). By invoking the divergence theorem, Equation 13 can be written as

\[
\int_{\Omega^h} N^b_{I,i}(x) \Phi(x) \, d\Omega = \int_{\Gamma^h} N^b_{I,i}(x) \Phi(x) n(x) \, d\Gamma.
\]  

(14)

This form of the strain has the following advantages:

- Domain integration is reduced to a boundary integration along the smoothing cells
- Does not require the derivatives of the shape functions and hence does not need the Jacobian
- Does not need isoparametric mapping there by giving a leverage on the distortion level of the mesh

The choice of the smoothing function and the integration order used, decide the accuracy of the smoothed strain field. If a CS function is used, the method is termed the SFEM. It was shown in Bordas et al and Natarajan et al that the gradient term which ensures consistency between the shape functions and their derivatives. This modified equation was termed the divergence consistency. It was also shown that such consistency requirement is implicitly satisfied, if linear field is used. It can be seen that Equation 15 would reduce to Equation 14, if \( \Phi \) is a constant.

\[
\int_{\Omega^h} N^b_{I,i}(x) \Phi(x) \, d\Omega = \int_{\Gamma^h} N^b_{I,i}(x) \Phi(x) n(x) \, d\Gamma - \int_{\Omega^h} N^b_{I,i}(x) \Phi'(x) \, d\Omega,
\]  

(15)

where \( \Gamma^h \) is the contour of the smoothing cell. Here, the domain integral term vanishes as the smoothing function is constant over the domain. Since we assumed linear displacement functions, the strain would be a constant and a unique value can be computed using a single equation, hence requiring just 1 interior Gauß point. This can be written as

\[
\tilde{N}^b_{I,i}(x_c) = \frac{1}{A_c} \int_{\Gamma^h} N^b_{I,i}(x) n(x) \, d\Gamma,
\]

(16)

\[
\begin{align*}
\tilde{N}^b_{I,i}(x_c) &= \frac{1}{V^c} \sum_{b=1}^{nb} \left\{ \begin{array}{ccc}
N^b_{I,i}(x^C_b) & n_x & 0 \\
0 & N^b_{I,i}(x^C_b) & n_y \\
0 & 0 & N^b_{I,i}(x^C_b) \\
N^b_{I,i}(x^C_b) & n_x & 0 \\
0 & N^b_{I,i}(x^C_b) & n_y \\
0 & 0 & N^b_{I,i}(x^C_b) \\
N^b_{I,i}(x^C_b) & n_x & 0 \\
N^b_{I,i}(x^C_b) & n_y & 0 \\
N^b_{I,i}(x^C_b) & n_z & 0 \\
\end{array} \right\} A^b_c,
\end{align*}
\]

where \( V^c \) is the volume of the subcell, \( n_b \) is the number of boundary surfaces of the subcell, \( A^c_b \) and \( x^C_b \) are the area and Gauß point of the boundary surface \( b \), respectively. The smoothing technique has been very efficient for polyhedral elements since the polyhedrons can be divided into number of subcells (usually tetrahedrons), and the stiffness matrix is summed up over each subcell. It can be seen in Equation 16 that the derivatives of the shape functions are not needed to evaluate the strains. Hence, the computation of Jacobian is avoided. This also avoids the associated isoparametric mapping. The stiffness matrix is evaluated...
FIGURE 3 Subdivision of a hexahedral elements into tetrahedral elements. This subdivision is solely for the purpose of numerical integration. A smoothed strain field is computed over each subdivision depending on the choice of smoothing function as in the regular finite element method by replacing the terms in the strain gradient matrix with the terms evaluated above and summing it up over the subcells. The constant smoothing technique when applied to elements other than constant strain elements (3-node triangles and 4-node tetrahedrons) yields results which are bounded by the results of reduced integration procedure (smoothing over 1 subcell) and full integration procedure (smoothing over infinite number of subcells). The method is hence not variationally consistent for any number of subcells other than 1 and $\infty$ whereas the LS procedure is variationally consistent. The CS and LS schemes differ in the choice of the smoothing function. In the LS scheme, the basis function used is $\Phi(x) = [1 \ x \ y \ z \ xy \ yz \ zx \ xyz]^T$ in case of hexahedral subcells and $\Phi(x) = [1 \ x \ y \ z]^T$ if tetrahedral subcells are used. Figure 3 shows one possible division of hexahedral elements into tetrahedral elements (also referred as subcells in the literature) for the purpose of numerical integration. The number of terms in the basis function should be consistent with the number of Gauss points to obtain a unique solution. Since a linear basis function is being used the domain integral term which results as a consequence of the divergence consistency does not vanish, and hence, it has to be computed by using the appropriate order of Gaussian integration. In the case of tetrahedral subcells, the system of equations for a linear basis would be

\[
\int_{\Omega^b} N_{I,a}^b(x) \, d\Omega = \int_{\Gamma^b} N_f^b(x) n_i^b \, d\Gamma, \tag{17a}
\]

\[
\int_{\Omega^b} N_{I,a}^b(x) y \, d\Omega = \int_{\Gamma^b} N_f^b(x) y n_i^b \, d\Gamma - \int_{\Omega^b} N_f^b(x) y \, d\Omega, \tag{17b}
\]

\[
\int_{\Omega^b} N_{I,a}^b(x) z \, d\Omega = \int_{\Gamma^b} N_f^b(x) z n_i^b \, d\Gamma, \tag{17c}
\]

\[
\int_{\Omega^b} N_{I,a}^b(x) z \, d\Omega = \int_{\Gamma^b} N_f^b(x) z n_i^b \, d\Gamma \tag{17d}
\]

for $N_{I,a}^b(x)$. 

\[
\]
\[
\int_{\Omega^b} N_{I_S}^h(x) d\Omega = \int_{\Gamma^b} N_I^h(x)n_I d\Gamma ,
\] (18a)
\[
\int_{\Omega^b} N_{I_S}^h(x) n_I d\Omega = \int_{\Gamma^b} N_I^h(x)n_I d\Gamma ,
\] (18b)
\[
\int_{\Omega^b} N_{I_S}^h(x) y_I d\Omega = \int_{\Gamma^b} N_I^h(x)y_I d\Gamma \quad - \int_{\Omega^b} N_I^h(x) d\Omega ,
\] (18c)
\[
\int_{\Omega^b} N_{I_S}^h(x) z_I d\Omega = \int_{\Gamma^b} N_I^h(x)z_I d\Gamma
\] (18d)

for \( N_{I_S}^h(x) \).

\[
\int_{\Omega^b} N_{I_S}^h(x) n_I d\Omega = \int_{\Gamma^b} N_I^h(x)n_I d\Gamma ,
\] (19a)
\[
\int_{\Omega^b} N_{I_S}^h(x) n_I d\Omega = \int_{\Gamma^b} N_I^h(x)n_I d\Gamma ,
\] (19b)
\[
\int_{\Omega^b} N_{I_S}^h(x) y_I d\Omega = \int_{\Gamma^b} N_I^h(x)y_I d\Gamma ,
\] (19c)
\[
\int_{\Omega^b} N_{I_S}^h(x) z_I d\Omega = \int_{\Gamma^b} N_I^h(x)z_I d\Gamma \quad - \int_{\Omega^b} N_I^h(x) d\Omega
\] (19d)

for \( N_{I_S}^h(x) \). Here, \( N_I \) represents the shape function associated with the \( I \)th node of the parent element. It is independent of the subcell. The location of the Gauß points for the boundary integration and domain integration in a tetrahedral subcell are shown.

**FIGURE 4** The location of the Gauß points for the boundary and the domain integration over a tetrahedron subcell of a hexahedral element. * represents the domain integration points, and o represents integration points on the boundary.
in Figure 4. Let the natural coordinates of the \( m \)-th interior Gauss point of a subcell be \( \mathbf{p}_m = (x_m, y_m, z_m) \) and its associated Gauss weight be \( w_m \); coordinates of the \( k \)-th boundary of the subcell be \( \mathbf{c}^k = (x^k, y^k, z^k) \) and the associated weights be \( v^k \). The unit outward normal associated with the \( g \)-th Gauss point of the \( k \)-th boundary of the subcell is denoted by \( \mathbf{n}^k = (n^x_k, n^y_k, n^z_k) \). The smoothed derivatives are computed numerically as follows:

\[
\mathbf{W}_d = \mathbf{f}_i, \quad \text{where} \quad i = x, y, z. \tag{20}
\]

\[
\mathbf{W} = \begin{pmatrix}
    w_1 & w_2 & w_3 & w_4 \\
    w_1 x_1 & w_2 x_2 & w_3 x_3 & w_4 x_4 \\
    w_1 y_1 & w_2 y_2 & w_3 y_3 & w_4 y_4 \\
    w_1 z_1 & w_2 z_2 & w_3 z_3 & w_4 z_4
\end{pmatrix} \tag{21}
\]

\[
\mathbf{f}_i = \sum_{k=1}^{4} \sum_{g=1}^{4} N_T(c^k) n_i^k v^k - \sum_{m=1}^{4} N_T(p_m) w_m
\] \tag{22}

\[
\mathbf{f}_x = \sum_{k=1}^{4} \sum_{g=1}^{4} N_T(c^k) x_k n_i^k v^k - \sum_{m=1}^{4} N_T(p_m) w_m
\] \tag{23}

\[
\mathbf{f}_y = \sum_{k=1}^{4} \sum_{g=1}^{4} N_T(c^k) y_k n_i^k v^k - \sum_{m=1}^{4} N_T(p_m) w_m
\] \tag{24}

The smoothed derivative of the \( i \)-th shape function evaluated at the 4 interior Gauss points of a tetrahedral subcell is given by

\[
d_i = [d^1_i \quad d^2_i \quad d^3_i \quad d^4_i]^T, \quad \text{where} \quad i = x, y, z. \tag{25}
\]

The same procedure is to be repeated for all the shape functions of the parent element. For the \( m \)-th interior Gauss point of a subcell of a \( n \)-sided polygon, the smoothed strain displacement matrix is given by

\[
\mathbf{B}(\mathbf{p}_m) = \begin{bmatrix}
    \mathbf{B}_{11}(\mathbf{p}_m) & \mathbf{B}_{12}(\mathbf{p}_m) & \cdots & \mathbf{B}_{1n}(\mathbf{p}_m)
\end{bmatrix}, \quad \text{where} \quad m = 1, 2, 3, 4. \tag{26}
\]

\[
\mathbf{B}_{ix}(\mathbf{p}_m) = \begin{bmatrix}
    d^1_i & 0 & 0 \\
    0 & d^2_i & 0 \\
    0 & 0 & d^3_i \\
    d^1_i & d^2_i & d^3_i
\end{bmatrix} \tag{27}
\]
For the displacement approximation given by Equation 8, the compatible strain field is given by

$$
e^h(x) = [B_{\text{fem}} \ B_{\text{hev}} \ B_{\text{tip}}] q^T,$$

(28)

where $q = \{u \ a b\}$ is the vector of degrees of freedom, $B_{\text{fem}}$, $B_{\text{hev}}$, and $B_{\text{tip}}$, contains the strain displacement terms corresponding to the regular finite element part, Heaviside-enriched part, and the tip-enriched part, respectively. The components of the compatible strain field are

$$B_{\text{fem}} = LN_f,$n
$$B_{\text{hev}} = LN_f (H(x) - H(x_f)),$n
$$B_{\text{tip}} = LN_K \left( \sum_{m=1}^{d} (F_m(x) - F_m(x_K)) \right).$$

(29)

where

$$L = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}.$$n

The smoothed counterpart of the above compatible strain field can be obtained by following the procedure outlined earlier. The elements that are intersected by the discontinuous surface is divided into tetrahedra, and a linear smoothing basis, $f(x) = \{1 \ x \ y \ z\}$, is chosen to evaluate the smoothed strain.

**Remark 1.** In case of 2 dimensions, the subcell is a triangle and the smoothing procedure can be derived from the linear basis

$$f(x) = \{1 \ x \ y\}$$

(30)

with derivative

$$f_j(x) = \{0 \ \delta_{xj} \ \delta_{yj}\}.$$ 

(31)

## 4 | NUMERICAL EXAMPLES

In this section, the accuracy and the convergence properties of the proposed formulation is numerically studied within the framework of linear elastic fracture mechanics in both two and three dimensions. The domain is discretized with 4-noded quadrilateral and 8-noded hexahedral elements in two and three dimensions, respectively. The numerical results from the present formulation is compared with the conventional XFEM and the SmXFEM. The following convention is adopted whilst discussing the results:

- **XFEM** - conventional XFEM. The numerical integration is performed by sub-dividing the elements intersected by the discontinuity.
- **Sm-XFEM** - Smoothed XFEM. A constant smoothing function is employed to compute the corrected derivative.
- **LSm-XFEM** - Linear smoothed XFEM. A linear smoothing function is employed to compute the corrected derivative. The elements intersected by the discontinuity is sub-divided into triangles and three integration points are used per triangle.

For the conventional XFEM, the elements that are intersected by the discontinuous surface is triangulated and a higher-order triangular quadrature scheme is adopted. Table 1 depicts the number of integration points used for the Sm-XFEM and the LSm-XFEM. To estimate the error and to study the convergence properties, the $L^2$ norm and the $H^1$ semi-norm are used.
TABLE 1  Number of sub-cells used to compute the stiffness matrix for the constant smoothed XFEM (Sm-XFEM) and the linear smoothed XFEM (LSm-XFEM). In case of Sm-XFEM, the smoothing function is chosen as \( \Phi(x) = 1 \), whilst in case of LSm-XFEM, a complete set of polynomials is chosen. For example, \( \Phi(x) = \{1, xy\} \) for 2 dimensions and \( \Phi(x) = \{1, xy, yz\} \) for 3 dimensions as smoothing function

| Type of element       | Sm-XFEM | LSm-XFEM |
|-----------------------|---------|----------|
| 2 dimensions          |         |          |
| Standard elements     | 4       | 1        |
| Tip-enriched elements | 5       | 5        |
| Split-enriched elements | 8     | 8        |
| 3 dimensions          |         |          |
| Standard elements     | 6       | 1        |
| Tip-enriched elements | 24      | 24       |
| Split-enriched elements | 12    | 12       |

FIGURE 5  An infinite plate with a centre crack subject to far-field tensile stress

4.1 Infinite plate with centre crack under far-field tension

Consider an infinite plate with a centre crack subjected to far-field tension under plane strain condition as shown in Figure 5. A small section ABCD has been solved. The effect of the far-field stress has been accounted by prescribing equivalent displacements as given by following closed-form solution (Equation 32) in polar coordinates centred at the crack tip.

\[
\sigma_r(r, \theta) = \frac{2(1 + \nu)}{E} \sqrt{\frac{\pi}{2}} \frac{K_I}{r} \cos \frac{\theta}{2} \left( 2 - 2\nu - \cos^2 \frac{\theta}{2} \right),
\]

\[
\sigma_\theta(r, \theta) = \frac{2(1 + \nu)}{E} \sqrt{\frac{\pi}{2}} \frac{K_I}{r} \sin \frac{\theta}{2} \left( 2 - 2\nu - \cos^2 \frac{\theta}{2} \right),
\]

where \( K_I = \sigma \sqrt{\pi a} \), the mode I stress intensity factor; \( \nu \) is the Poisson ratio; and \( E \) is the Young's modulus. The dimension has been chosen as 10x10 mm; \( a \) is the half crack length. The convergence of the relative error in the displacement (\( L^2 \) norm) and the stress intensity factor is shown in Figure 6. It can be seen that in general, all the three methods yields a rate of convergence of 1 in the \( L^2 \) norm and 0.5 in the convergence of the relative error in the stress intensity factor. For a given \( df \), the conventional XFEM yields slightly accurate results than the Sm-XFEM or the LSm-XFEM, but the errors are within the same order. Moreover, it is noted that in the XFEM, 13 integration points per triangle (for the tip element) are used when compared with 3 integration
FIGURE 6 Convergence of the relative error in the displacement and in the stress intensity factor with mesh refinement for an infinite plate with a centre crack subjected to uniform tensile stress

points in case of LSm-XFEM and 1 integration point in case of Sm-XFEM. The suboptimal rate of convergence is due to the fact that we are using topological enrichment scheme as opposed to geometric enrichment.

4.2 Finite plate with an edge crack subjected to tensile and shear stresses

Next, consider a finite plate with an edge crack subjected to tensile and shear stresses as shown in Figure 7. A consistent system of units is used for the analysis.

4.2.1 Plate subjected to tensile stress

In this case, the dimension of the plate is 1 × 2 units. The Young modulus, E, and Poisson ratio, ν, are taken as 1000 and 0.3, respectively. A state of plane strain condition is assumed. The crack width is taken as 0.5 units. The obtained SIF is compared with the reference empirical solution:

\[ K_{ref} = f(\alpha) = \frac{\sigma}{\sqrt{\pi a}} \]

where \( f(\alpha) = 1.12 - 0.231a + 10.55a^2 - 21.72a^3 + 30.39a^4 \), \( a = a/W \) is the crack width ratio, \( a \) is the half-crack width, and \( w \) is the plate width. The convergence of the relative error in the stress intensity factor is shown in Figure 8. It can be seen that all the three methods converge at almost identical rates (≈ 0.5). The results of LS scheme are better than the CS scheme and are almost equal with the conventional XFEM.
4.2.2 Plate subjected to shear stresses

In this case, the dimensions of the plate are taken as $W = 7$ units and $L = 16$ units. The plate is subjected to shear stress on the top edges, while the displacements are constrained at the bottom edge. The crack width is taken as 3.5 units. The Young modulus, $E$, and Poisson ratio, $\nu$, are taken as $3 \times 10^7$ and 0.25, respectively. Plane strain condition is assumed. The reference SIF is taken from Ewalds and Wanhill, which is $K_I = 34$ units, $K_{II} = 4.55$ units. The convergence of the $K_I$ and the $K_{II}$ are presented in Figure 9. It is again seen that all the three methods have similar convergence rates. The LS scheme is also more accurate than the CS scheme with a very minor additional computational expense. It is again recalled that the additional integration points still require only the shape function values which can be obtained by linear interpolation along the boundary. The error can be attributed to the inadequate approximation space in the local crack tip region, ie, the asymptotic fields are approximated by a linear field.

4.3 Plate with an inclined centre crack subjected to tensile stresses

Next, to illustrate the efficacy of the formulation, SIFs in case of mixed-mode loading conditions are computed. Consider a finite plate with an inclined centre crack subjected to far-field tension (see Figure 10). The dimensions of the plate are taken as $20 \times 20$. The crack width, $2a$ is chosen as 4 units. A far-field uniform tensile stress, $1 \times 10^4$ units is applied with Young
FIGURE 9 Relative error in the modes I and II stress intensity factors for a plate with an edge crack subjected to shear stress

modulus, $E = 1 \times 10^7$, and Poisson ratio, $\nu = 0.3$. The accuracy of the numerically computed SIFs are compared with analytical SIFs given by

$$
K_I = \sigma \sqrt{\pi a \cos^2(\beta)},
$$
$$
K_{II} = \sigma \sqrt{\pi a \sin(\beta) \cos(\beta)},
$$

(34)

where $\beta$ is the inclination of the crack measured anti-clockwise from the $x$-axis. Based on a progressive refinement, it was observed that a structured mesh of $100 \times 100$ quadrilateral mesh is adequate. The influence of the crack angle and different modelling approaches, viz, XFEM, Sm-XFEM, LSm-XFEM on the SIFs are shown in Figure 11. It can be seen that the results from the proposed approach are accurate and comparable with the conventional XFEM and slightly more accurate than the Sm-XFEM.

4.4 Finite plate with a through-thickness edge crack subjected to tensile stresses

As a last example, the LS technique is extended to three-dimensional domain with a through-the-thickness edge crack subjected to uniform tensile stress as shown in Figure 12 with dimensions $W/a = 1$ and $H/W = 3$. The displacement at the bottom face is constrained in all directions and a uniform tensile stress $\sigma = 1 \times 10^4$ is applied on the top face. The material properties are Young modulus, $E = 1 \times 10^7$, and Poisson ratio, $\nu = 0.3$. The domain is discretized with structured 8-node hexahedral elements and the normalized SIF from Saputra et al\textsuperscript{40} is taken as the reference solution.
The smoothed strain field over a standard element is computed without any further subdivisions and with $f(x) = [1 \ x \ y \ z \ xy \ yz \ zx \ xyz]$ as a smoothing function. For the elements that are intersected by the discontinuous surface, the element is subdivided into tetrahedra and $f(x) = [1 \ x \ y \ z]$ is chosen as the smoothing function. In case of the LSm-XFEM, a total of 96 Gauß points are used in case of tip-enriched elements, whereas 300 Gauß points are used in the conventional XFEM. In the case of Heaviside-enriched elements, 48 Gauß points are used in case of LSm-XFEM and 60 Gauß points are used in case of the conventional XFEM. The convergence of the relative error in the normalized stress intensity factor is shown in Figure 13. It can be seen that the LSm-XFEM is more accurate than the Sm-XFEM and is in good agreement with the conventional XFEM.

**FIGURE 10** Plate with an inclined centre crack subjected to tensile stress: geometry and boundary conditions

**FIGURE 11** Influence of inclination of the crack on the modes I and II stress intensity factors for a plate with a centre crack subjected to far-field tensile stress
**FIGURE 12** Finite plate with a through-thickness edge crack subjected to tensile stress

**FIGURE 13** Relative error in the normalized SIF $\frac{K}{\sigma \sqrt{2\pi a}}$ for a three-dimensional domain with an edge crack subjected to uniform tensile stress
5 CONCLUSIONS

In this paper, the LS (second-order smoothing) was discussed and a method to couple it with the XFEM was presented. The developed method was used to solve problems with discontinuities and singularities in both two and three dimensions. The method also involves a rational integration procedure using the Green theorem. The performance of the LS scheme for enriched approximation space was studied. Through numerical examples, it was shown that the LS scheme is more accurate than its constant counterpart. The LS scheme yields improved results when compared with CS. Moreover, it requires fewer quadrature points.

The CS and the LS technique is extended to three dimensions for the first time. Although the presented example in three dimensions is for straight crack, it can be easily extended to other crack profiles. The superior accuracy of the LS technique is also obtained in the three-dimensional case. These results are attributed to the superior approximation properties of the LS compared with the constant strain smoothing, which is immediately apparent for problems involving complex, non-polynomial, enrichment functions. The remaining, incompressible, error level is attributed to the inadequate approximation space in the smoothed strain, ie, to the inability of a linear smoothed strain to approximate the singular strains provided by the enriched approximations. Future, ongoing work includes the enrichment of the smoothing space with suitable enrichment functions to investigate any additional accuracy improvements as well as the introduction of the approach in recently developed stable extended finite element schemes.41,42

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