A class of quintic Hermite interpolation curve and the free parameters selection

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Abstract
The classical $C^2$ quintic Hermite interpolation curve not only needs the positions and derivatives but also needs the second-order derivatives as input. For most applications, one has to estimate the second-order derivatives in advance. In addition, the classical $C^2$ quintic Hermite interpolation curve is unique once the input data are fixed, the shape of the curve will not be modified when the interpolation is poor. In this paper, a class of $C^2$ quintic Hermite interpolation curve with free parameters that only needs the points and tangent vectors as input is presented. Due to the self-contained free parameters, the shapes of the proposed interpolation curve can be controlled. Moreover, the free parameters can be chosen reasonably so that the interpolation curve can meet some certain geometric requirements. Some numerical experiments show the feasibility of the proposed methods.

Keywords: Hermite interpolation curve, $C^2$ continuity, Free parameters, Free parameters selection

1. Introduction

It is known that the $C^1$ parametric Hermite interpolation curve has been used in computer aided design (CAD) and related application fields. Given a series of points $p_i$, together with the corresponding tangent vectors $m_i$, $i = 1, 2, \ldots, n$, the parametric cubic Hermite interpolation curve can be expressed as

$$H_{i,i+1}(t) = B^0(t)p_i + B^1(t)\left(\frac{1}{3}m_i\right) + B^2(t)\left(\frac{1}{3}m_{i+1}\right) + B^3(t)p_{i+1}, \quad i = 1, 2, \ldots, n-1,$$  \hspace{1cm} (1)

where $0 \leq t \leq 1$, $B^j(t)$ ($j = 0, 1, 2, 3$) are the cubic Bernstein polynomials. It is clear that $H_{i,i}(0) = p_i$, $H_{i,i+1}(1) = p_{i+1}$, $H'_{i,i}(0) = m_i$, $H'_{i,i+1}(1) = m_{i+1}$, and $H_{i,i+1}(t)$ satisfy $C^1$ continuity. Generally, the $C^1$ parametric cubic Hermite interpolation curve needs the points $p_i$ and tangent vectors $m_i$ as input.

However, the $C^2$ parametric Hermite interpolation curve also needs to be considered in some practical problems. In order to construct the $C^2$ parametric cubic Hermite interpolation curve, one might add two boundary conditions to obtain the $C^2$ continuity conditions and then solve an equation system to reconfirm the tangent vectors. The two boundary conditions can be chosen in many forms (see e.g. Behforooz and Papamichel, 1979; Boor, 1978; Micula and Micula, 1999). Alternatively, one might add the second-order derivatives instead of prescribing only points and tangent vectors to obtain $C^2$ parametric Hermite interpolation curve, and the lowest order Hermite polynomial to interpolate these data is of degree five (Farin, 2002a). Then given a series of points $p_i$ together with the corresponding tangent vectors $m_i$ and second-order derivatives $s_i$, $i = 1, 2, \ldots, n$, the parametric quintic Hermite interpolation curve can be expressed as

$$H_{i,i+1}(t) = B^0(t)p_i + B^1(t)\left(\frac{1}{5}m_i\right) + B^2(t)\left(\frac{2}{5}m_i + \frac{1}{20}s_i\right) +$$
\[ B_i^j(t) \left( p_{i+1} - \frac{2}{5} m_{i+1} + \frac{1}{20} s_{i+1} \right) + B_i^j(t) \left( p_{i-1} - \frac{1}{5} m_{i-1} \right), \quad i = 1, 2, \cdots, n-1, \tag{2} \]

where \( 0 \leq t \leq 1 \), \( B_i^j(t) \) \((j = 0, 1, \cdots, 5)\) are the quintic Bernstein polynomials. We can verify that \( H_{5,i}(0) = p_i \), \( H_{5,i}(1) = p_{i+1} \), \( H_{5,i}'(0) = m_i \), \( H_{5,i}'(1) = m_{i+1} \), \( H_{5,i}''(0) = s_i \), \( H_{5,i}''(1) = s_{i+1} \), and \( H_{5,i}(t) \) satisfy \( C^2 \) continuity. This method provides an effective way to construct \( C^2 \) parametric Hermite interpolation curve, but one will have to estimate the second-order derivatives in advance for most applications. In addition, we can see that the quintic Hermite interpolation curve expressed in (2) would be unique once the data are fixed, we can not modify the shapes of the interpolation curve even when the interpolation effect is poor. This shortcoming brings some inconvenience to practical application.

Is there an effective method for constructing the \( C^2 \) Hermite interpolation curve that does not require the second-order derivatives and can realize the shape adjustment? In order to achieve this goal, we present a class of \( C^2 \) quintic Hermite interpolation curve with free parameters that only needs the points and tangent vectors as input. Furthermore, we discuss how to choose the free parameters reasonably so that the interpolation curve can meet some certain specific requirements. The remainder of this paper is organized as follows. The quintic Hermite interpolation curve with two free parameters is presented in Section 2. The free parameters selection schemes and some numerical experiments are proposed in Section 3. A short conclusion is given in Section 4.

2. The quintic Hermite interpolation curve with free parameters

Let us first give the definition of the quintic Hermite interpolation curve with free parameters as follows.

**Definition 1.** Given a series of points \( p_i \) together with the corresponding tangent vectors \( m_i \), \( i = 1, 2, \cdots, n \). For \( 0 \leq t \leq 1 \), the curve

\[
B_i(t) = B_i^0(t)p_i + B_i^1(t)\left( p_{i+1} - \frac{2}{5} m_{i+1} + \frac{1}{20} s_{i+1} \right) + B_i^2(t)\left( p_{i-1} - \frac{1}{5} m_{i-1} \right) + B_i^3(t)\left( \frac{\alpha + 10}{10} p_i + \frac{\beta + 4}{10} m_i \right) + B_i^4(t)\left( \frac{10 + \alpha}{10} p_{i+1} + \frac{\beta - 4}{10} m_{i+1} \right) + B_i^5(t)\left( p_{i+1} - \frac{1}{5} m_{i+1} \right) + B_i^6(t)p_{i+1}, \quad i = 1, 2, \cdots, n-1, \tag{3} \]

is called quintic Hermite interpolation curve with free parameters, where \( B_i^j(t) \) \((j = 0, 1, \cdots, 5)\) are quintic Bernstein polynomials, \( \alpha \) and \( \beta \) are two real free parameters.

For convenience, we name the quintic Hermite interpolation curve expressed in (3) as \( \alpha \beta \)-Hermite curve for short in the following discussion.

**Theorem 1.** The \( \alpha \beta \)-Hermite curve interpolates the points \( p_i \) and the tangent vectors \( m_i \), and the curves satisfy \( C^2 \) continuity for any free parameters \( \alpha \) and \( \beta \).

Proof. By a deduction from (3), we obtain

\[
B_i(0) = p_i, \quad B_i(1) = p_{i+1}, \quad B_i'(0) = m_i, \quad B_i'(1) = m_{i+1}, \tag{4} \]

\[
B_i''(0) = 2\alpha p_i + 2\beta m_i, \quad B_i''(1) = 2\alpha p_{i+1} + 2\beta m_{i+1}. \tag{5} \]

Expression (4) shows that the \( \alpha \beta \)-Hermite curve interpolates the points \( p_i \) and the tangent vectors \( m_i \). From (4) and (5), we have

\[
B_i^{(k)}(1) = B_{i+1}^{(k)}(0), \quad k = 0, 1, 2, \quad i = 1, 2, \cdots, n-2. \tag{6} \]

Expression (6) shows that the \( \alpha \beta \)-Hermite curve satisfies \( C^2 \) continuity for any free parameters \( \alpha \) and \( \beta \).

Theorem 1 shows that the \( \alpha \beta \)-Hermite curve has the similar properties with the classical quintic Hermite interpolation curve expressed in (2). We can see that the \( \alpha \beta \)-Hermite curve only needs the points and tangent vectors as input without the second-order derivatives as input, which is a feature that the classical quintic Hermite interpolation curve does not has. Moreover, the shapes of the \( \alpha \beta \)-Hermite curve can be adjusted by altering the values of the two free parameters, which is another feature that the classical quintic Hermite interpolation curve does not has.
It is clear that the influence of the two free parameters on the αβ-Hermite curve is global. Fig. 1 shows the αβ-Hermite curve for the same points and tangent vectors but different free parameters, where the points and tangent vectors are taken as \( p_1 = (0,0) \), \( p_2 = (-2,0) \), \( p_3 = (-2,8) \), \( p_4 = (0,8) \), \( p_5 = (-1,12) \), \( m_1 = (-2,-2) \), \( m_2 = (-2,4) \), \( m_3 = (2,4) \), \( m_4 = (2,2) \), \( m_5 = (-1,2) \), \( m_6 = (0,-1) \).

![Fig. 1 The curves for the same data but different free parameters](image)

3. Schemes of the free parameters selection

Since the two free parameters have a significant influence on the αβ-Hermite curve when the points \( p_i \) together with the tangent vectors \( m_i \) are fixed, a natural idea drives us to choose them reasonably so that the αβ-Hermite curve can satisfy some certain geometric requirements. As we know, the arc length and the fairness are two important geometric features of a curve. Then we discuss how to determine the values of the free parameters to obtain the αβ-Hermite curve with the shortest arc length or the fairest αβ-Hermite curve.

(a) The αβ-Hermite curve with the approximate shortest arc length.

It is known that the arc length of a parametric curve \( r(t) \) is defined by

\[
I(r) = \int_0^1 \| r'(t) \| \, dt .
\]  

(7)

In order to simplify the calculation, we approximately express (7) as

\[
\hat{I}(r) = \int_0^1 [r'(t)]^2 \, dt .
\]  

(8)

According to (8), the arc length of the αβ-Hermite curve can be approximated by

\[
\hat{I}(B_i) = \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \| B'_i(t) \| \, dt .
\]  

(9)

Since we want to determine the values of the free parameters \( \alpha \) and \( \beta \) to obtain the αβ-Hermite curve with the approximate shortest arc length, we can obtain the following optimization problem from (9),

\[
\min I(\alpha, \beta) = \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} \| B'_i(t) \| \, dt .
\]  

(10)

We rewrite (3) as

\[
B_i(t) = P_i(t) \alpha + Q_i(t) \beta + R_i(t) ,
\]  

(11)
\[ P_i(t) = \frac{1}{20} \left[ B_i^2(t) p_i + B_i^1(t) p_{i+1} \right], \quad Q_i(t) = \frac{1}{20} \left[ B_i^2(t) m_i + B_i^1(t) m_{i+1} \right], \]
\[ R_i(t) = B_i^2(t) p_i + B_i^1(t) \left( p_i + \frac{2}{5} m_i \right) + B_i^0(t) \left( p_{i+1} - \frac{2}{5} m_i \right) + B_i^1(t) \left( p_{i+1} - \frac{1}{5} m_{i+1} \right) + B_i^2(t) p_{i+1}. \]

Then we have
\[ I(\alpha, \beta) = \sum_i \int_a^b \left[ P'_i(t) \alpha + Q'_i(t) \beta + R'_i(t) \right]^2 dt = a_i \alpha^2 + b_i \beta^2 + 2c_i \alpha \beta + 2d_i \alpha + 2e_i \beta + f_i, \]

where
\[ a_i = \sum_{k=1}^{n_i} \int_a^b \left\| P'_i(t) \right\|^2 dt = \frac{1}{630} \sum_{k=1}^{n_i} \left\| p_{i+1} \cdot p_{i+1} \right\|, \]
\[ b_i = \sum_{k=1}^{n_i} \int_a^b \left\| Q'_i(t) \right\|^2 dt = \frac{1}{630} \sum_{k=1}^{n_i} \left\| m_{i+1} \cdot m_{i+1} \right\|, \]
\[ c_i = \sum_{k=1}^{n_i} \int_a^b \left[ P'_i(t) \cdot Q'_i(t) \right] dt = \frac{1}{1260} \sum_{k=1}^{n_i} \left[ (2 p_{i+1} \cdot m_{i+1} + (p_i + 2 p_{i+1}) \cdot m_{i+1}) \right], \]
\[ d_i = \sum_{k=1}^{n_i} \int_a^b \left[ P'_i(t) \cdot R'_i(t) \right] dt = \frac{1}{420} \sum_{k=1}^{n_i} \left[ (2 p_{i+1} \cdot m_{i+1} - (2 m_{i+1} - 7 m_{i+1}) - 10 p_{i+1} \cdot m_{i+1} + 5 (p_{i+1} \cdot p_{i+1}) \right], \]
\[ e_i = \sum_{k=1}^{n_i} \int_a^b \left[ Q'_i(t) \cdot R'_i(t) \right] dt = \frac{1}{420} \sum_{k=1}^{n_i} \left[ (5 p_{i+1} - 5 p_{i+1} + 7 m_{i+1} + 7 m_{i+1}) \cdot (m_{i+1} - m_{i+1}) \right], \]
\[ f_i = \sum_{k=1}^{n_i} \int_a^b \left[ R'_i(t) \right] dt = \frac{1}{35} \sum_{k=1}^{n_i} \left[ 15 (p_{i+1} - p_{i+1}) \cdot (m_{i+1} + m_{i+1}) - m_{i+1} \cdot m_{i+1} - 100 p_{i+1} \cdot p_{i+1} + 50 (p_{i+1} \cdot p_{i+1}) + 8 (m_{i+1} \cdot m_{i+1}) \right]. \]

Therefore, we obtain
\[ \begin{align*}
\frac{\partial I}{\partial \alpha} &= 2a_i \alpha + 2c_i \beta + 2d_i, \\
\frac{\partial I}{\partial \beta} &= 2c_i \alpha + 2b_i \beta + 2e_i.
\end{align*} \]

Then, for minimizing \( I(\alpha, \beta) \), we obtain the equation system,\[ \begin{align*}
a_i \alpha + c_i \beta &= -d_i, \\
c_i \alpha + b_i \beta &= -e_i.
\end{align*} \]

Note that \( a_i, b_i, c_i, d_i, e_i \) are constants once the points and tangent vectors are given. It can be easily checked that \( a_i b_i - c_i^2 \neq 0 \). Hence, systems (13) has the unique solution expressed as
\[ \alpha = \frac{c_i e_i - b_i d_i}{a_i b_i - c_i^2}, \quad \beta = \frac{c_i d_i - a_i e_i}{a_i b_i - c_i^2}. \]

(b) The approximate fairest \( \alpha \beta \)-Hermite curve.

Although the smoothness of a curve is difficult to be expressed in a quantitative way, the strain energy (also called bending energy) minimization or the curvature variation energy minimization is adopted to construct fair curves in most cases (see e.g. Ahn et al, 2014; Jaklič and Žagar, 2011a, 2011b; Li et al, 2012; Li, 2018; Lu, 2015a, 2015b; Lu et al, 2017; Xu et al, 2011). Here, we determine the values of the free parameters by minimizing curvature variation energy to obtain the fairest \( \alpha \beta \)-Hermite curve.

The curvature variation energy of a parametric curve \( r(t) \) is defined by (Farin, 2008b)
\[ s(r) = \int_0^1 \left| \kappa'(t) \right|^2 dt. \]
where \( \kappa(t) = \left\| \frac{r'(t) \times r''(t)}{\left\| r'(t) \right\|} \right\| \).

In order to simplify the calculation, expression (11) can be approximated by (Lu, 2015a)

\[
\hat{s}(r) = \int_0^1 \left\| r''(t) \right\|^2 \, dt.
\]  

(16)

According to (16), we can approximately express the curvature variation energy of the \( \alpha\beta \)-Hermite curve as

\[
\hat{s}(B) = \sum_{i=0}^{n-1} \int_0^1 \left\| B''(t) \right\|^2 \, dt.
\]  

(17)

Since we want to determine the values of the free parameters \( \alpha \) and \( \beta \) to obtain the approximate fairest \( \alpha\beta \)-Hermite curve, we can obtain the following optimization problem from (17),

\[
\min J(\alpha, \beta) = \sum_{i=0}^{n-1} \int_0^1 \left\| B''(t) \right\|^2 \, dt.
\]  

(18)

From (11) we have

\[
J(\alpha, \beta) = \sum_{i=0}^{n-1} \int_0^1 \left\| P''(t) \alpha + Q''(t) \beta + R''(t) \right\| dt = a_2 \alpha^2 + b_2 \beta^2 + c_2 \alpha \beta + 2 e_2 \beta + f_2,
\]  

(19)

where

\[
a_2 := \sum_{i=0}^{n-1} \int_0^1 \left\| P''(t) \right\|^2 \, dt = \sum_{i=0}^{n-1} \left( 9 \left\| p_i \right\|^2 - 6 p_i \cdot p_{i+1} + 9 \left\| p_{i+1} \right\|^2 \right),
\]

\[
b_2 := \sum_{i=0}^{n-1} \int_0^1 \left\| Q''(t) \right\|^2 \, dt = \sum_{i=0}^{n-1} \left( 9 \left\| m_i \right\|^2 - 6 m_i \cdot m_{i+1} + 9 \left\| m_{i+1} \right\|^2 \right),
\]

\[
c_2 := \sum_{i=0}^{n-1} \int_0^1 \left( P''(t) \cdot Q''(t) \right) dt = \sum_{i=0}^{n-1} \left[ (9 p_i - 3 p_{i+1}) \cdot m_i + (9 p_{i+1} - 3 p_i) \cdot m_{i+1} \right],
\]

\[
d_2 := \sum_{i=0}^{n-1} \int_0^1 \left( P''(t) \cdot R''(t) \right) dt = \sum_{i=0}^{n-1} \left[ (36 p_i - 24 p_{i+1}) \cdot m_i + (24 p_{i+1} - 36 p_i) \cdot m_{i+1} - 120 p_i \cdot p_{i+1} + 60 \left( \left\| p_i \right\|^2 + \left\| p_{i+1} \right\|^2 \right) \right],
\]

\[
e_2 := \sum_{i=0}^{n-1} \int_0^1 \left( Q''(t) \cdot R''(t) \right) dt = 12 \sum_{i=0}^{n-1} \left[ (5 p_i - 5 p_{i+1} + 3 m_i + 3 m_{i+1}) \cdot (m_i - m_{i+1}) \right],
\]

\[
f_2 := \sum_{i=0}^{n-1} \left\| R''(t) \right\|^2 \, dt = \sum_{i=0}^{n-1} \left[ 1520 (p_i - p_{i+1}) \cdot (m_i + m_{i+1}) + 336 m_i \cdot m_{i+1} - 1440 p_i \cdot p_{i+1} + 720 \left( \left\| p_i \right\|^2 + \left\| p_{i+1} \right\|^2 \right) + 192 \left( \left\| m_i \right\|^2 + \left\| m_{i+1} \right\|^2 \right) \right].
\]

Therefore, we obtain

\[
\begin{aligned}
\frac{\partial J}{\partial \alpha} &= 2 a_2 \alpha + 2 c_2 \beta + 2 d_2, \\
\frac{\partial J}{\partial \beta} &= 2 c_2 \alpha + 2 b_2 \beta + 2 e_2.
\end{aligned}
\]

Then, for minimizing \( J(\alpha, \beta) \), we obtain the equation system,

\[
\begin{aligned}
a_2 \alpha + c_2 \beta &= -d_2, \\
c_2 \alpha + b_2 \beta &= -e_2.
\end{aligned}
\]  

(20)

Note that \( a_2, b_2, c_2, d_2, e_2 \) are constants once the points and tangent vectors are given. It can be easily checked that \( a_2 b_2 - c_2^2 \neq 0 \). Hence, systems (20) has the unique solution expressed as
\[ \alpha = \frac{c_2d_1 - b_1d_2}{a_1b_2 - c_1^2}, \quad \beta = \frac{c_2d_2 - a_1c_2}{a_1b_2 - c_1^2}. \]  

(21)

Now, let us present some numerical experiments to illustrate the feasibility of the proposed methods.

We first present the performance of the proposed method by using the experimental data in Fig. 1. For the data in Fig. 1, the free parameters of the \( \alpha \beta \)-Hermite curve with the shortest arc length calculated by (14) are \((\alpha, \beta) = (-1.3776, 1.4159)\). The \( \alpha \beta \)-Hermite curve with the approximate shortest arc length (solid lines) and the \( \alpha \beta \)-Hermite curve with other free parameters (dashed lines represent \((\alpha, \beta) = (-1, -1)\), dotted lines represent \((\alpha, \beta) = (1, 1)\)) are shown in Fig. 2.

![Fig. 2](image)

For the data in Fig. 1, the free parameters of the fairest \( \alpha \beta \)-Hermite curve calculated by (21) are \((\alpha, \beta) = (-0.8193, 0.3295)\). The approximate fairest \( \alpha \beta \)-Hermite curve (solid lines) and the \( \alpha \beta \)-Hermite curve with other free parameters (dashed lines represent \((\alpha, \beta) = (-1, -1)\) and dotted lines represent \((\alpha, \beta) = (1, 1)\)) are shown in Fig. 3.

![Fig. 3](image)

The second example let us consider the closed curve. Suppose the data are taken as \( p_1 = p_6 = (-1, 0), \ p_2 = (0, 1), \ p_3 = (1, 0), \ p_4 = (0, -1), \ m_1 = m_6 = (0, 1), \ m_2 = (1, 0), \ m_3 = (0, -1), \ m_4 = (-1, 0) \), then the free parameters of the \( \alpha \beta \)-Hermite curve with the shortest arc length calculated by (14) are \((\alpha, \beta) = (-10.5, 0)\). The \( \alpha \beta \)-Hermite curve with the approximate shortest arc length (solid lines) and the \( \alpha \beta \)-Hermite curve with other free parameters (dashed lines represent \((\alpha, \beta) = (-5, -5)\), dotted lines represent \((\alpha, \beta) = (5, 5)\)) are shown in Fig. 4.
Fig. 4  The closed $\alpha\beta$-Hermite curve with the approximate shortest arc length (solid lines)

For the data in Fig. 4, the free parameters of the fairest $\alpha\beta$-Hermite curve calculated by (21) are $(\alpha, \beta) = (-4, 0, 0)$.

The approximate fairest $\alpha\beta$-Hermite curve (solid lines) and the $\alpha\beta$-Hermite curve with other free parameters (dashed lines represent $(\alpha, \beta) = (-5, -5)$, dotted lines represent $(\alpha, \beta) = (5, 5)$) are shown in Fig. 5.

Fig. 5  The closed approximate fairest $\alpha\beta$-Hermite curve (solid lines)

4. Conclusion

In this paper, we present a class of quintic $C^2$ Hermite interpolation curve with free parameters. We can alter the values of the free parameters to achieve the shape adjustment of the proposed interpolation curve. By choosing the free parameters reasonably, we can obtain the interpolation curve that meets the approximate shortest arc length or the approximate fairness. Compared with the classical quintic $C^2$ Hermite interpolation curve, the proposed curve has the following advantages,

(a) Only needs the positions and derivatives as input, not the second-order derivatives.

(b) Contains two free parameters that can be used to adjust the shape of the curve and to obtain curves that satisfy some certain geometric requirements.

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References
Ahn, Y. J., Hoffmann, C. and Rosen, P. Geometric constraints on quadratic Bézier curves using minimal length and energy, Journal of Computational and Applied Mathematics, Vol.255 (2014), pp.887–897.

Behforooz, G. H. and Papamichel, N., End conditions for cubic spline interpolation, Applied Mathematics and Computation, Vol.29, No.3 (1979), pp.231-244.

Boor, C. D., A practical guide to splines (1978), Springer-Verlag.

Farin, G., Curves and surfaces for CAGD: a practical guide (2002a), Academic Press.

Farin, G., Geometric Hermite interpolation with circular precision, Computer Aided Geometric Design, Vol.40, No.4 (2008), pp.476-479.

Jaklič, G. and Žagar, E., Planar cubic $G^1$ interpolatory splines with small strain energy, Journal of Computational and Applied Mathematics, Vol.235, No.8 (2011a), pp.2758–2765.

Jaklič, G. and Žagar, E., Curvature variation minimizing cubic Hermite interpolants, Applied Mathematics and Computation, Vol.218, No.7 (2011b), pp.3918–3924.

Li, P. P., Zhang C. M., Li X. M. and Li W. T., Discussion on minimal curvature variation in cubic Hermite curve construction, Journal of Advanced Mechanical Design, Systems, and manufacturing, Vol.6, No.3 (2012), pp.366-375.

Li, J. C., Planar T-Bézier curve with approximate minimum curvature variation, Journal of Advanced Mechanical Design, Systems, and manufacturing, Vol.12, No.1 (2018), DOI: 10.1299/jamdsm.2018jamdsm0029.

Lu, L. Z., A note on curvature variation minimizing cubic Hermite interpolants, Applied Mathematics and Computation, Vol.259 (2015a), pp.596–599.

Lu, L. Z., Planar quintic $G^2$ Hermite interpolation with minimum strain energy, Journal of Computational and Applied Mathematics, Vol.274 (2015b), pp.109–117.

Lu, L. Z., Jiang, C. K. and Hu, Q. Q., Planar cubic $G^1$ and quintic $G^2$ Hermite interpolations via curvature variation minimization, Computers & Graphics, Vol. 70 (2017), pp.92-98.

Micula, G. and Micula, S., Handbook of splines, Mathematics and its Applications, Vol. 462, No.8 (1999), pp.790-790.

Xu, G. Wang, G. Z. and Chen, W. Y., Geometric construction of energy-minimizing Bézier curves, Science China Information Science, Vol.54, No.7 (2011), pp.1395–1406.