A CONSTRUCTION OF N=2 AND CENTERLESS N=4
SUPERCONFORMAL FIELDS VIA AFFINE
SUPERALGEBRAS

Minoru Wakimoto
Graduate School of Mathematics, Kyushu University, Fukuoka 812-81,
Japan

Abstract
In this note we give a new construction of the N=2 superconformal
algebra using currents of the affine superalgebra \( \hat{sl}(2|1) \) and free
bosonic fields, and also the N=4 superconformal algebra without
central charge in terms of currents of \( \hat{sl}(2|2) \) and free bosonic fields.

0. Introduction
There seem to be several ways to construct the N=2 superconformal
algebra via affine Lie algebras or affine superalgebras by tensoring with
some fermionic or bosonic fields. In this note, first we give a simple
method to construct it from the affine superalgebra \( \hat{sl}(2|1) \) with the
help of two free bosonic fields and their dual fields. This construction
is achieved by a calculus of the inner product and the structure of root
systems. A very particular feature is that the similar construction does
not work for \( \hat{sl}(3,\mathbb{C}) \) to get a finite-rank algebra, but only gives an
algebra of infinite rank just like the famous Zamolodchikov’s W-algebra.
For \( \hat{sl}(2|1) \), however, the operator product is well-behaved and gives a
family of finite fields, namely the N=2 superconformal algebra. We see
in this construction that the existence of odd simple roots with zero
square length play a very important role. So our calculus is not de-
scribed only by the terminology of root systems, but the inner product
is concerned.

Since the construction works so well for the simplest superalgebra
\( \hat{sl}(2|1) \), one will naturally ask how to extend this method to get higher
rank superconformal algebras. It is not clear, however, whether super-
conformal algebras with fields of negative conformal weights are ex-
pected to be obtained in such a way associated to affine superalgebras.
It may be expected for physical superconformal algebras which, by def-
nition, do not admit fields of non-positive conformal weights and are
completely classified by \cite{3}. But actually the difficulty arises soon in the higher rank case to get a family of finite numbers of fields closed under the operator product. In this note, we propose a way to construct the N=4 superconformal algebra, although with zero central charge, associated to the affine superalgebra \( \hat{A}(1, 1) \), namely the affinization of the finite-dimensional simple superalgebra \( sl(2|2) \) divided by its center. We find a family of finite fields which are almost closed under the operator products, and an algebra of finite rank is obtained by factorizing this vertex algebra by an ideal with a single generator. This procedure of factorization, however, enforces us to retreat to the zero central charge.

In our construction, the N=2 superconformal algebra acts on the space of tensor product of an irreducible highest weight \( \hat{sl}(2|1) \)-module and the Clifford module, and also similar for the N=4 superconformal algebra. We find its invariant subspaces by using commutants. The analysis of their irreducibility and characters remains a problem.

1. Some formulas for operator products

Throughout this paper we make use of the physicists’ convention of the operator product

\[
a(z)b(w) = \sum_{j=1}^{n} \frac{c_j(w)}{(z - w)^j}
\]

to denote

\[
[a(z), b(w)] = \sum_{j=1}^{n} c_j(w) \frac{\partial^{j-1}}{(j-1)!} \delta(z - w).
\]

Let \( g \) be a finite-dimensional simple superalgebra with an even non-degenerate super-invariant super-symmetric bilinear form \( ( \mid \cdot \mid ) \), and \( \hat{g} \) be its affinization. Here “even” means, by definition, that the even part of \( g \) is orthogonal to the odd part with respect to \( ( \mid \cdot \mid ) \) (cf. \cite{2}). Let \( \Delta \) be the set of all roots of \( g \) with respect to its Cartan subalgebra \( h \), and \( \Delta^+ \) (resp. \( \Delta^- \)) the subset of \( \Delta \) consisting of all positive (resp. negative) roots. Also let \( \Delta_{even} \) (resp. \( \Delta_{odd} \)) denote the set of all even (resp. odd) roots. As usual, for a root \( \alpha \), \( X_\alpha \) is an element from the corresponding root space \( g_\alpha \), and \( H_\alpha \) is the element in \( h \) satisfying \( \alpha(h) = (H_\alpha|h) \) for all \( h \in h \).

For each \( \alpha \in \Delta^+ \), we introduce two fields of conformal weight \( \frac{1}{2} \):

\[
\psi_\alpha(z) = \sum_{n \in \mathbb{Z}} \psi_\alpha(n) z^{-n-\frac{1}{2}} = \sum_{n \in \frac{1}{2}+\mathbb{Z}} \psi_\alpha(n + \frac{1}{2}) z^{-n-1},
\]
and
\[ \psi^\alpha(z) = \sum_{n \in \mathbb{Z}} \psi^\alpha(n) z^{-n-\frac{1}{2}} = \sum_{n \in \mathbb{Z} + \frac{1}{2}} \psi^\alpha(n + \frac{1}{2}) z^{-n-1}. \]

These fields are fermionic or bosonic according as the associated root \( \alpha \) is even or odd, and assumed to satisfy the (anti-)commutation relations
\[
[\psi^\alpha(m), \psi^\beta(n)] = (-1)^{p(\alpha)} [\psi^\beta(n), \psi^\alpha(m)] = \delta_{\alpha,\beta} \delta_{m+n,0},
\]
where \( p(\alpha) \) denotes the parity of a root \( \alpha \), namely is equal to 0 if \( \alpha \) is even and 1 if \( \alpha \) is odd. We extend \( \psi^\alpha \) and \( \psi^{-\alpha} \) to negative roots by
\[
\psi^{-\alpha}(z) = (-1)^{p(\alpha)} \psi^{-\alpha}(z)
\]
for all roots \( \alpha \), where
\[
\epsilon(\alpha) := \begin{cases} 
1 & \text{if } \alpha \text{ is positive} \\
-1^{p(\alpha)} & \text{if } \alpha \text{ is negative}.
\end{cases}
\]

For a positive root \( \alpha \), the operators \( \psi^\alpha(m) \) with \( m < 0 \) (resp. \( m \geq 0 \)) and \( \psi^\alpha(n) \) with \( n \leq 0 \) (resp. \( n > 0 \)) are assumed to be creation (resp. annihilation) operators. Using the inner product
\[
(\psi^\alpha|\psi^\beta) = (\psi^\alpha|\psi^\beta) := \epsilon(\alpha) \delta_{\alpha+\beta,0},
\]
or equivalently
\[
(\psi^\alpha|\psi^\beta) = (-1)^{p(\alpha)} (\psi^\alpha|\psi^\beta) := \delta_{\alpha,\beta},
\]
the operator products of these fields are simply written as the following formula:
\[
\varphi(z) \psi(w) = \left( \frac{\varphi}{z-w} \right),
\]
where \( \varphi(z) \) and \( \psi(z) \) are elements from \( \psi^\alpha(z) \)'s and \( \psi_{\alpha}(z) \)'s.

For each element \( X \in \mathfrak{g} \), the field
\[ X(z) := \sum_{n \in \mathbb{Z}} X(n) z^{-n-1} \]
is called a current, where \( X(n) \)'s, \( n \in \mathbb{Z} \), are elements in the affinization \( \hat{\mathfrak{g}} \) of \( \mathfrak{g} \), satisfying the (anti-)commutation relation
\[
[X(m), Y(n)] = [X, Y](m+n) + m(X|Y) \delta_{m+n,0} K,
\]
which is the primary content of the document.
where $K$ is the canonical central element in $\hat{g}$.

In terms of the operator product, the formula (1.6a) is written as

$$X(z)Y(w) = \frac{(X|Y)K}{(z-w)^2} + \frac{[X,Y](w)}{z-w}. \quad (1.6b)$$

In this note, $a(z), b(z), c(z), \cdots, X(z), Y(z), \cdots$ always stand for currents, and $\psi(z), \varphi(z), \cdots$ free fields with product (1.5), and $p(\ )$ denotes the parity of a field. We note that

$$p(\psi^\alpha) = p(\psi_\alpha) = p(\alpha) + 1 \mod 2\mathbb{Z}. \quad (1.7)$$

A very important tool in our operator calculus is the Wick theorem in particular for non-commutative fields, a clear explanation on which is given, e.g., in Section 3.3 of [4]. Here is a list of formulas obtained from the non-commutative Wick theorem, which will be used in this paper.

$$: a(z)b(z) :: c(w) := -\frac{([a,b]|c)K}{(z-w)^3} + (b|c)K \left\{ \frac{a(w)}{(z-w)^2} + \frac{\partial a(w)}{z-w} \right\}$$
$$+( -1 )^{p(b)p(c)}(a|c)K \left\{ \frac{b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} \right\} + (-1)^{p(a)p(b)} \frac{[b, [a, c]](w)}{(z-w)^3} \quad (1.8a)$$
$$+ \frac{1}{z-w} \left\{ : a(w)[b, c](w) : +(-1)^{p(a)p(b)} : b(w)[a, c](w) : \right\},$$

$$: a(z) :: b(w)c(w) := \frac{(a|[b, c])K}{(z-w)^3}$$
$$+ \frac{1}{(z-w)^2} \left\{ (a|b)K \cdot c(w) + (-1)^{p(a)p(b)}(a|c)K \cdot b(w) + [[a, b], c](w) \right\} \quad (1.8b)$$
$$+ \frac{1}{z-w} \left\{ : [a, b](w)c(w) : +(-1)^{p(a)p(b)} : b(w)[a, c](w) : \right\},$$

$$(-1)^{p(\varphi)p(Y)} : X(z)\varphi(z) :: Y(w)\psi(w) := (X|Y)(\varphi|\psi) \frac{K}{(z-w)^3}$$
$$+( \varphi|\psi ) \frac{[X,Y](w)}{(z-w)^2} + (X|Y)K \left\{ : \varphi(w)\psi(w) : + \frac{\partial \varphi(w)\psi(w)}{z-w} \right\} \quad (1.9)$$
$$+ \frac{1}{z-w} \left\{ [x, y](w) : \varphi(w)\psi(w) : + (\varphi|\psi) : X(w)Y(w) : \right\},$$
\begin{align}
(-1)^{p(X)p(\varphi)} : X(z)\varphi(z) &:: Y(w)Z(w) : \\
&= ([X,Y][Z])K \left\{ \frac{\varphi(w)}{(z-w)^3} + \frac{\partial \varphi(w)}{(z-w)^2} + \frac{1}{2}\partial^2 \varphi(w) \right\} \\
&+ (X|Y)K \left\{ \frac{\varphi(w)Z(w)}{(z-w)^2} + \frac{\partial \varphi(w)Z(w)}{z-w} \right\} \\
&+ (-1)^{p(X)p(Y)}(X|Z)K \left\{ \frac{\varphi(w)Y(w)}{(z-w)^2} + \frac{\partial \varphi(w)Y(w)}{z-w} \right\} \\
&+ \frac{\varphi(w)[X,Y,Z](w)}{(z-w)^2} + \frac{\partial \varphi(w)[X,Y,Z](w)}{z-w} \\
&+ \frac{1}{z-w} \varphi(w) : [X,Y](w)Z(w) : + \frac{(-1)^{p(X)p(Y)}}{z-w} \varphi(w) : Y(w)[X,Z](w) : ,
\end{align}

\begin{align}
(-1)^{p(Y)p(\varphi_1)+p(\varphi_2)} &:: X(z)\varphi_1(z)\varphi_2(z) :: Y(w)\psi(w) :
\end{align}

\begin{align}
&= (X|Y)K(\varphi_2|\psi) \left\{ \frac{\varphi_1(w)}{(z-w)^3} + \frac{2\partial \varphi_1(w)}{(z-w)^2} + \frac{3}{2}\partial^2 \varphi_1(w) \right\} \\
&+ (-1)^{p(\varphi_2)p(\psi)}(X|Y)K(\varphi_1|\psi) \left\{ \frac{\varphi_2(w)}{(z-w)^3} + \frac{2\partial \varphi_2(w)}{(z-w)^2} + \frac{3}{2}\partial^2 \varphi_2(w) \right\} \\
&+ (X|Y)K \left\{ \frac{\varphi_1(w)\varphi_2(w)\psi(w)}{(z-w)^2} + \frac{\partial(\varphi_1(w)\varphi_2(w))\psi(w)}{z-w} \right\} \\
&+ (\varphi_2|\psi) \left\{ \frac{[X,Y](w)\varphi_1(w)}{(z-w)^2} + \frac{[X,Y](w)\partial \varphi_1(w)}{z-w} + \frac{X(w)Y(w) : \varphi_1(w)}{z-w} \right\} \\
&+ (-1)^{p(\varphi_2)p(\psi)}(\varphi_1|\psi) \left\{ \frac{[X,Y](w)\varphi_2(w)}{(z-w)^2} + \frac{[X,Y](w)\partial \varphi_2(w)}{z-w} + \frac{X(w)Y(w) : \varphi_2(w)}{z-w} \right\} \\
&+ \frac{[X,Y](w) : \varphi_1(w)\varphi_2(w)\psi(w)}{z-w}.
\end{align}

In particular from (1.8) and (1.9) one deduces the following:

Lemma 1.1. Let $\alpha, \beta \in \Delta$ such that $\alpha + \beta \neq 0$, then

\begin{align}
X_\alpha(z)\psi^\alpha(z) &:: X_\beta(w)\psi^\beta(w) := (-1)^{p(\alpha)+1+p(\beta)} |X_\alpha, X_\beta|(w) : \psi^\alpha(w)\psi^\beta(w) :,
\end{align}

Lemma 1.2. Let $H \in h$ and $X_\alpha \in g_\alpha$ and $X_{-\alpha} \in g_{-\alpha}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$; then
1) \[ H(z) : X_\alpha(w)X_{-\alpha}(w) := \frac{(H_a|H)K}{(z-w)^3} + \alpha(H) \frac{H_a(w)}{(z-w)^2}, \]

2) \[ X_\alpha(z)X_{-\alpha}(z) : H(w) = -\frac{(H_a|H)K}{(z-w)^3} + \alpha(H) \left\{ \frac{H_a(w)}{(z-w)^2} + \frac{\partial H\alpha(w)}{z-w} \right\}, \]

3) \[ X_\alpha(z)\psi^\alpha(z) :: X_{-\alpha}(w)\psi^{-\alpha}(w) : \]
\[
= \epsilon(\alpha) \frac{(X_a|X_{-\alpha})K}{(z-w)^3} + \epsilon(\alpha) \frac{(X_a|X_{-\alpha})}{(z-w)^2} \left\{ (X_a|X_{-\alpha})K : \psi^\alpha(w)\psi_\alpha(w) : + H\alpha(w) \right\} \\
+ \frac{\epsilon(\alpha)}{z-w} \left\{ (X_a|X_{-\alpha})K : \psi^\alpha(w)\psi_\alpha(w) : + (X_a|X_{-\alpha})K : \partial\psi^\alpha(w)\psi_\alpha(w) : \right\}. \]

2. THE N=2 SUPERCONFORMAL ALGEBRA ASSOCIATED TO \( \hat{sl}(2|1) \)

In this section we start our discussion under a general situation, where \( g \) is a simple Lie superalgebra with an even non-degenerate super-invariant super-symmetric bilinear form \( (\ | \ ) \). One can choose \( H\alpha \) and \( X_\alpha \) such that
\[
(X_a|X_{-\alpha}) = (-1)^{p(\alpha)}\epsilon(\alpha), \tag{2.1} \]
\[
H\alpha = (-1)^{p(\alpha)}\epsilon(\alpha)\alpha. \tag{2.2a} \]

Note that
\[
H_{-\alpha} = -(-1)^{p(\alpha)}H\alpha. \tag{2.2b} \]

Given a pair \((\alpha, \beta)\) of roots such that
\[
(\text{the linear span of } \alpha \text{ and } \beta) \cap \Delta = \{ \pm\alpha, \pm\beta, \pm(\alpha + \beta) \}, \tag{2.3} \]
we introduce the following fields:
\[
G_+(z) := X_\alpha(z)\psi^\alpha(z) + X_{-\beta}(z)\psi^{-\beta}(z),
\]
\[
G_-(z) := X_\beta(z)\psi^{\beta}(z) + X_{-\alpha}(z)\psi^{-\alpha}(z),
\]
\[
A(z) := K\{(-1)^{p(\alpha)}\epsilon(\alpha) : \psi^\alpha(z)\psi_\alpha(z) : -(-1)^{p(\beta)}\epsilon(\beta) : \psi^{\beta}(z)\psi_\beta(z) : \} \\
+ H\alpha(z) - H_\beta(z),
\]
\[
B(z) := K\{(-1)^{p(\alpha)}\epsilon(\alpha) : \partial\psi^\alpha(z)\psi_\alpha(z) : -(-1)^{p(\beta)}\epsilon(\beta) : \psi^{\beta}(z)\partial\psi_\beta(z) : \} \\
+ : X_\alpha(z)X_{-\alpha}(z) : + : X_\beta(z)X_{-\beta}(z) : -\partial H_\beta(z) \]
\[ H_\alpha(z) : \psi^\alpha(z)\psi_\alpha(z) : + H_\beta(z) : \psi^\beta(z)\psi_\beta(z) : \\
+ (-1)^{p(\alpha)p(\beta)} c_{\alpha,\beta}\{((-1)^p : X_{\alpha+\beta}(z)\psi^\alpha(z)\psi^\beta(z) : \\
- (-1)^p \epsilon(\alpha)\epsilon(\beta) : X_{-\alpha-\beta}(z)\psi_\alpha(z)\psi_\beta(z) : \}. \]

In the actual calculation of operator products of fields, one needs the following relations among structure constants, which are deduced from the Jacobi identity:

\[
\begin{align*}
c_{-\alpha,-\beta} &= -(1)^{p(\alpha+\beta)} c_{\alpha,\beta}, \\
c_{\alpha,-\alpha-\beta} &= -(1)^{p(\alpha)+1} p(\beta) \epsilon(\alpha) c_{\alpha,\beta}, \\
c_{\beta,-\alpha-\beta} &= -(1)^{p(\alpha)} \epsilon(\beta) c_{\alpha,\beta}, \\
c_{-\alpha,\alpha+\beta} &= -(1)^{p(\alpha)p(\beta)} \epsilon(\alpha) c_{\alpha,\beta}, \\
c_{-\beta,\alpha+\beta} &= -\epsilon(\beta) c_{\alpha,\beta}.
\end{align*}
\]

Then, by Lemmas 1.1 and 1.2, one easily has the following:

**Lemma 2.1.**

1) \[ G_+(z)G_+(w) = G_-(z)G_-(w) = 0, \]

2) \[ G_+(z)G_-(w) = \left\{ (-1)^{p(\alpha)} \epsilon(\alpha) + (-1)^{p(\beta)} \epsilon(\beta) \right\} \frac{K}{(z-w)^3} + \frac{A(w)}{(z-w)^2} + \frac{B(w)}{z-w}. \]

And the following is shown by a simple calculation using (1.8),(1.9) and (1.10):

\[ (2.5a) \quad A(z) : X_\alpha(w)\psi^\alpha(w) : \\
= \{\epsilon(\alpha)K + (-1)^{p(\alpha)} \epsilon(\alpha)(\alpha|\alpha) - (-1)^{p(\beta)} \epsilon(\beta)(\alpha|\beta)\} \frac{X_\alpha(w)\psi^\alpha(w)}{z-w}, \]

\[ (2.5b) \quad A(z) : X_\beta(w)\psi^{-\beta}(w) : \\
= \{\epsilon(\beta)K - (-1)^{p(\alpha)} \epsilon(\alpha)(\alpha|\beta) + (-1)^{p(\beta)} \epsilon(\beta)(\beta|\beta)\} \frac{X_\beta(w)\psi^{-\beta}(w)}{z-w}, \]

\[ (2.6a) \quad A(z) : X_\beta(w)\psi^\beta(w) : \\
= \{-\epsilon(\beta)K + (-1)^{p(\alpha)} \epsilon(\alpha)(\alpha|\beta) - (-1)^{p(\beta)} \epsilon(\beta)(\beta|\beta)\} \frac{X_\beta(w)\psi^\beta(w)}{z-w}, \]
(2.6b) \[ A(z) : X_{-\alpha}(w)\psi^{-\alpha}(w) : \]
\[ = \{-\epsilon(\alpha)K - (-1)^{p(\alpha)}\epsilon(\alpha)(\alpha|\alpha) + (-1)^{p(\beta)}\epsilon(\beta)(\alpha|\beta)\} \frac{X_{-\alpha}(w)\psi^{-\alpha}(w)}{z - w}. \]

From these, one has the following:

**Lemma 2.2.** Assume that \((\alpha|\alpha) = (\beta|\beta) = 0\), \((\alpha|\beta) = 1\), and \(\epsilon(\alpha) = \epsilon(\beta) = 1\). Then

1) \[ A(z)G_+(w) = \frac{K + 1}{z - w}G_+(w), \]

2) \[ A(z)G_-(w) = \frac{-(K + 1)}{z - w}G_-(w). \]

We now proceed to compute the operator product of \(B(z)\) with \(G_\pm(z)\). After a long and tiresome calculation using (1.9), (1.10) and (1.11), one arrives at the following:

(2.7a) \[ B(z) : X_\alpha(w)\psi^\alpha(w) : \]
\[ = \epsilon(\alpha)\{K + (\alpha|\alpha)\} \left\{ \frac{X_\alpha(w)\psi^\alpha(w)}{(z - w)^2} + \frac{\partial(X_\alpha(w)\psi^\alpha(w))}{z - w} \right\} \]
\[ + (-1)^{(p(\alpha)+1)p(\beta)}\epsilon(\beta) \left\{ \frac{X_\alpha(w)\psi^\alpha(w)}{(z - w)^2} + \frac{\partial(X_\alpha(w)\psi^\alpha(w))}{z - w} \right\} \]
\[ + (-1)^{p(\alpha)}\epsilon(\beta)(\alpha|\beta) \frac{X_{-\alpha}(w)\psi^{-\alpha}(w)}{(z - w)^2} \]
\[ + (-1)^{p(\alpha)+p(\beta)+1} \left\{ \frac{X_{-\beta}(w)\psi^{-\beta}(w)}{(z - w)^2} + \frac{\partial(X_{-\beta}(w)\psi^{-\beta}(w))}{z - w} \right\} \]
\[ + (-1)^{p(\alpha)+p(\beta)}\epsilon(\alpha)\epsilon(\beta) c_{\alpha\beta} \frac{X_\alpha(w)X_{-\alpha}(w)\psi^{-\beta}(w)}{z - w} : \]
\[ + (-1)^{p(\alpha)}\epsilon(\alpha)\frac{(\alpha|\alpha)}{z - w} : X_\alpha(w)\psi^\alpha(w)\psi^\alpha(w) : \]
\[ + (-1)^{p(\beta)}\epsilon(\beta)\frac{(\beta|\beta)}{z - w} : X_\alpha(w)\psi^\alpha(w)\psi^\alpha(w) : \]
\[ + (-1)^{p(\alpha)}\epsilon(\beta)\frac{\epsilon(\beta)}{z - w} : X_{-\beta}(w)\psi^{-\beta}(w)\psi^\alpha(w) : \]
\[ + (-1)^{p(\alpha)+p(\beta)}\epsilon(\beta)\frac{\epsilon(\beta)}{z - w} : X_{-\beta}(w)\psi^{-\beta}(w)\psi^\alpha(w) :, \]

(2.7b) \[ B(z) : X_{-\beta}(w)\psi^{-\beta}(w) : \]
\[ = \epsilon(\beta)\{K + (\beta|\beta)\} \left\{ \frac{X_{-\beta}(w)\psi^{-\beta}(w)}{(z - w)^2} + \frac{\partial(X_{-\beta}(w)\psi^{-\beta}(w))}{z - w} \right\} \]
\[ + (-1)^{(p(\alpha)+1)p(\beta)} \left\{ \frac{X_\alpha(w)\psi^\alpha(w)}{(z-w)^2} + \frac{X_\alpha(w)\partial\psi^\alpha(w)}{z-w} \right\} \]

\[ + \epsilon(\beta)(\beta|\beta) \left\{ 1 - (-1)^{p(\beta)} \right\} \frac{X_{-\beta}(w)\psi^{-\beta}(w)}{(z-w)^2} \]

\[ - (-1)^{p(\alpha)+p(\beta)} \epsilon(\beta) \frac{c_{\alpha,\beta}}{z-w} : X_\alpha(w)X_{-\alpha-\beta}(w)\psi_\beta(w) : \]

\[ + \epsilon(\beta) \frac{c_{\alpha,\beta}}{z-w} : X_{\alpha+\beta}(w)X_{-\beta}(w)\psi^\alpha(w) : \]

\[ - (-1)^{p(\beta)} \frac{(\beta|\beta)}{z-w} : X_{-\beta}(w)\psi^\beta(w)\psi_\beta(w) : \]

\[ - (-1)^{p(\alpha)} \epsilon(\alpha) \epsilon(\beta) \frac{(\alpha|\beta)}{z-w} : X_{-\beta}(w)\psi_\beta(w)\psi^\alpha(w)\psi_\alpha(w) : \]

\[ + (-1)^{(p(\alpha)+1)p(\beta)} \frac{1}{z-w} : X_\alpha(w)\psi^\alpha(w)\psi^\beta(w)\psi_\beta(w) : , \]

\[ (2.8a) \quad B(z) : X_\beta(w)\psi^\beta(w) : \]

\[ = \epsilon(\beta) \left\{ K + (\beta|\beta) \right\} \left\{ \frac{X_\beta(w)\psi^\beta(w)}{(z-w)^2} + \frac{\partial(X_\beta(w)\psi^\beta(w))}{z-w} \right\} \]

\[ + (-1)^{p(\alpha)p(\beta)+1} \epsilon(\alpha) \left\{ \frac{X_\beta(w)\psi^\beta(w)}{(z-w)^2} + \frac{\partial X_\beta(w)\psi^\beta(w)}{z-w} \right\} \]

\[ + \epsilon(\beta) \left\{ K + (-1)^{p(\beta)}(\beta|\beta) \right\} \frac{X_\beta(w)\psi^\beta(w)}{(z-w)^2} \]

\[ + (-1)^{(p(\alpha)+1)p(\beta)} \left\{ \frac{X_{-\alpha}(w)\psi^{-\alpha}(w)}{(z-w)^2} + \frac{\partial(X_{-\alpha}(w)\psi^{-\alpha}(w))}{z-w} \right\} \]

\[ + (-1)^{(p(\alpha)+1)p(\beta)+1} \epsilon(\alpha) \epsilon(\beta) \frac{c_{\alpha,\beta}}{z-w} : X_\beta(w)X_{-\alpha-\beta}(w)\psi_\alpha(w) : \]

\[ - (-1)^{p(\alpha)+p(\beta)+1} \frac{c_{\alpha,\beta}}{z-w} : X_{\alpha+\beta}(w)X_{-\alpha}(w)\psi^\beta(w) : \]

\[ + (-1)^{p(\alpha)} \epsilon(\alpha) \frac{(\alpha|\beta)}{z-w} : X_\beta(w)\psi^\beta(w)\psi^\alpha(w)\psi_\alpha(w) : \]

\[ + (-1)^{p(\beta)} \epsilon(\beta) \frac{(\beta|\beta)}{z-w} : X_\beta(w)\psi^\beta(w)\psi_\beta(w)\psi^\beta(w) : \]

\[ - (-1)^{(p(\alpha)+1)p(\beta)} \epsilon(\beta) \frac{(\beta|\beta)}{z-w} : X_{-\alpha}(w)\psi_\alpha(w)\psi^\beta(w)\psi_\beta(w) : , \]

\[ (2.8b) \quad B(z) : X_{-\alpha}(w)\psi^{-\alpha}(w) : \]

\[ = \epsilon(\alpha) \left\{ K + (\alpha|\alpha) \right\} \left\{ \frac{X_{-\alpha}(w)\psi^{-\alpha}(w)}{(z-w)^2} + \frac{\partial(X_{-\alpha}(w)\psi^{-\alpha}(w))}{z-w} \right\} \]
\begin{align*}
+ (-1)^{p(\alpha)p(\beta)+1} \left\{ \frac{X_\beta(w)\psi^\beta(w)}{(z-w)^2} + \frac{X_\beta(w)\partial\psi^\beta(w)}{z-w} \right\} \\
+ \{\epsilon(\alpha)(K + (\alpha|\alpha)) - (-1)^{p(\beta)}\epsilon(\beta)(\alpha|\beta)\} \frac{X_{-\alpha}(w)\psi^{-\alpha}(w)}{(z-w)^2} \\
- (-1)^{p(\alpha)+1(p(\beta)+1)}\epsilon(\alpha) \frac{c_{\alpha,\beta}}{z-w} : X_\beta(w)X_{-\alpha-\beta}(w)\psi_\alpha(w) : \\
- (-1)^{p(\alpha)}\epsilon(\alpha) \frac{c_{\alpha,\beta}}{z-w} : X_{-\alpha}(w)\psi^\alpha_\alpha(w)\psi_\alpha(w) : \\
- (-1)^{p(\beta)}\epsilon(\alpha)\epsilon(\beta) \frac{(\alpha|\beta)}{z-w} : X_{-\alpha}(w)\psi^\alpha_\alpha(w)\psi_\beta(w) : \\
+ (-1)^{p(\alpha)p(\beta)+1} \frac{1}{z-w} : X_\beta(w)\psi^\beta(w)\psi^\alpha_\alpha(w) :.
\end{align*}

Looking at the above formulas, one sees that all of the extra terms cancel out and disappear if the conditions \((\alpha|\alpha) = (\beta|\beta) = 0\), \((\alpha|\beta) = 1\), and \(\epsilon(\alpha) = \epsilon(\beta) = 1\) and \(c_{\alpha,\beta} = 1\) are satisfied, and has the following:

**Lemma 2.3.** Assume that \((\alpha|\alpha) = (\beta|\beta) = 0\), \((\alpha|\beta) = 1\), and \(\epsilon(\alpha) = \epsilon(\beta) = 1\) and \(c_{\alpha,\beta} = 1\). Then

1) \(B(z)G_+(w) = (K + 1) \left\{ \frac{G_+(w)}{(z-w)^2} + \frac{\partial G_+(w)}{z-w} \right\}\),

2) \(B(z)G_-(w) = (K + 1) \left\{ \frac{2G_-(w)}{(z-w)^2} + \frac{\partial G_-(w)}{z-w} \right\}\).

The other products \(A(z)A(w)\), \(A(z)B(w)\) and \(B(z)B(w)\) are obtained from the above Lemmas by using the Borcherds-Jacobi identity (see, e.g., the formula (4.6.7) in [1]):

\[ [a_\alpha, b_\beta] = \sum_{j \geq 0} \binom{m}{j} a_{\alpha(j)} b_{\alpha(m+n-j)}, \quad (2.9) \]

for mutually local fields \(a(z) = \sum_{n \in \mathbb{Z}} a_\alpha z^{-n-1}\) and \(b(z) = \sum_{n \in \mathbb{Z}} b_\alpha z^{-n-1}\). Actually the calculation goes as follows. Put

\[ A(z) = \sum_{n \in \mathbb{Z}} A_\alpha z^{-n-1}, \]

\[ B(z) = \sum_{n \in \mathbb{Z}} B_\alpha z^{-n-1}, \]
and
\[ G_\pm(z) = \sum_{n \in \mathbb{Z}} G_\pm(n) z^{-n-1}. \]

Then the above lemmas, together with skew-symmetry, give the following:

\[ G_{+(n)} G_- = \begin{cases} B & \text{if } n = 0, \\ A & \text{if } n = 1, \\ -2K & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases} \]

\[ G_{-(n)} G_+ = \begin{cases} B - \partial A & \text{if } n = 0, \\ -A & \text{if } n = 1, \\ -2K & \text{if } n = 2, \\ 0 & \text{if } n \geq 3, \end{cases} \]

\[ A_{(n)} G_\pm = -G_\pm(n) A = \begin{cases} \pm(K + 1)G_\pm & \text{if } n = 0, \\ 0 & \text{if } n \geq 1, \end{cases} \]

\[ B_{(n)} G_+ = \begin{cases} (K + 1)\partial G_+ & \text{if } n = 0, \\ (K + 1)G_+ & \text{if } n = 1, \\ 0 & \text{if } n \geq 2, \end{cases} \]

\[ G_{+(n)} B = \begin{cases} 0 & \text{if } n = 0 \text{ or } n \geq 2, \\ (K + 1)G_+ & \text{if } n = 1, \end{cases} \]

\[ B_{(n)} G_- = G_{-(n)} B = \begin{cases} (K + 1)\partial G_- & \text{if } n = 0, \\ 2(K + 1)G_- & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases} \]

So, by (2.9) applied to \( a = G_+ \) and \( b = G_- \), one has
\[ [G_{+(0)}, G_{-(n)}] = B_{(n)}, \]
\[ [G_{+(1)}, G_{-(n)}] = B_{(n+1)} + A_{(n)}, \]
and, using these, can compute \( A_{(n)} A, A_{(n)} B, B_{(n)} A \) and \( B_{(n)} B \); e.g.,
\[ B_{(n)} A = G_{+(0)} G_{-(n)} A + G_{-(n)} G_{+(0)} A = \begin{cases} (K + 1)\partial A & \text{if } n = 0, \\ (K + 1)A & \text{if } n = 1, \\ 2K(K + 1) & \text{if } n = 2, \\ 0 & \text{if } n \geq 3. \end{cases} \]
Translating into the terminology of operator products, one has the following:

Lemma 2.4.
\[ A(z)A(w) = \frac{-4K(K + 1)}{(z - w)^2}, \]
\[ A(z)B(w) = (K + 1) \left\{ \frac{-2K}{(z - w)^2} + \frac{A(w)}{(z - w)^2} \right\}, \]
\[ B(z)A(w) = (K + 1) \left\{ \frac{2K}{(z - w)^2} + \frac{A(w)}{(z - w)^2} + \frac{\partial A(w)}{z - w} \right\}, \]
\[ B(z)B(w) = (K + 1) \left\{ \frac{2B(w)}{(z - w)^2} + \frac{\partial B(w)}{z - w} \right\}. \]

Summing up the above, one obtains the following:

Theorem 2.1. Let \( \{\alpha_1, \alpha_2\} \) (resp. \( \{\alpha^\vee_1, \alpha^\vee_2\} \)) be the set of simple roots (resp. simple coroots) of the Lie superalgebra \( sl(2|1) \) such that
\[
(\langle \alpha^\vee_i, \alpha_j \rangle)_{i,j=1,2} = (\langle \alpha^\vee_i | \alpha^\vee_j \rangle)_{i,j=1,2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]
where \( (\ | \ ) \) is the super-invariant super-symmetric bilinear form. Choose \( X_\alpha \in g_\alpha \), for each root \( \alpha \), satisfying the conditions (2.1), (2.2) and \( c_{\alpha_1, \alpha_2} = 1 \), where \( H_\alpha = \alpha_i^\vee \). Then the fields
\[
G'(z) := X_{\alpha_1}(z)\psi^{\alpha_1}(z) + X_{-\alpha_2}(z)\psi^{-\alpha_2}(z),
\]
\[
G^{*s}(z) := X_{-\alpha_1}(z)\psi^{-\alpha_1}(z) + X_{\alpha_2}(z)\psi^{\alpha_2}(z),
\]
\[
A(z) := K \{- : \psi^{\alpha_1}(z)\psi_{\alpha_1}(z) : + : \psi^{\alpha_2}(z)\psi_{\alpha_2}(z) : \} + (H_{\alpha_1} - H_{\alpha_2})(z),
\]
\[
B(z) := K \{- : \partial\psi^{\alpha_1}(z)\psi_{\alpha_1}(z) : + : \psi^{\alpha_2}(z)\partial\psi_{\alpha_2}(z) : \}
+ \sum_{i=1,2} \{ : X_{\alpha_i}(z)X_{-\alpha_i}(z) : + : H_\alpha(z)\psi^{\alpha_i}(z)\psi_{\alpha_i}(z) : \}
- \partial H_{\alpha_2}(z) + : X_{\alpha_1+\alpha_2}(z)\psi^{\alpha_1}(z)\psi^{\alpha_2}(z) : - : X_{-\alpha_1-\alpha_2}(z)\psi_{\alpha_1}(z)\psi_{\alpha_2}(z) : \}
\]
satisfy the following product formulas:
\[
G'(z)G'(w) = G^{*s}(z)G^{*s}(w) = 0,
\]
\[
G'(z)G^{*s}(w) = \frac{-2K}{(z - w)^2} + \frac{A(w)}{(z - w)^2} + \frac{B(w)}{z - w}, \]
\[
A(z)G'(w) = \frac{K + 1}{z - w}G'(w), \]
\[
A(z)G^{*s}(w) = \frac{-(K + 1)}{z - w}G^{*s}(w),
\]

12
\[ B(z)G'(w) = (K + 1) \left\{ \frac{G'(w)}{(z - w)^2} + \frac{\partial G'(w)}{z - w} \right\}, \]
\[ B(z)G^{*}(w) = (K + 1) \left\{ \frac{2G^{*}(w)}{(z - w)^2} + \frac{\partial G^{*}(w)}{z - w} \right\}, \]
\[ A(z)A(w) = \frac{-4K(K + 1)}{(z - w)^2}, \]
\[ A(z)B(w) = (K + 1) \left\{ \frac{-2K}{(z - w)^2} + \frac{A(w)}{(z - w)^2} \right\}, \]
\[ B(z)A(w) = (K + 1) \left\{ \frac{2K}{(z - w)^2} + \frac{A(w)}{(z - w)^2} + \frac{\partial A(w)}{z - w} \right\}, \]
\[ B(z)B(w) = (K + 1) \left\{ \frac{2B(w)}{(z - w)^2} + \frac{\partial B(w)}{z - w} \right\}. \]

Putting
\[ L(z) := \frac{1}{K + 1} \left\{ B(z) - \frac{1}{2} \partial A(z) \right\}, \quad J(z) := \frac{1}{K + 1} A(z), \]
\[ G(z) := \frac{1}{K + 1} G'(z), \quad G^{*}(z) := G'^{*}(z), \]
\[ K' := \frac{-2K}{K + 1}, \]

one can rewrite the above into the the standard product formulas of the N=2 superconformal algebra as follows:

**Corollary 2.1.**

\[ L(z)L(w) = \frac{1}{2} K' \frac{1}{(z - w)^4} + \frac{2L(w)}{(z - w)^2} + \frac{\partial L(w)}{z - w}, \]
\[ L(z)J(w) = \frac{J(w)}{(z - w)^2} + \frac{\partial J(w)}{z - w}, \quad J(z)L(w) = \frac{J(w)}{(z - w)^2}, \]
\[ L(z)G(w) = \frac{3}{2} \frac{G(w)}{(z - w)^2} + \frac{\partial G(w)}{z - w}, \quad G(z)L(w) = \frac{3}{2} \frac{G(w)}{(z - w)^2} + \frac{1}{2} \frac{\partial G(w)}{z - w}, \]
\[ L(z)G^{*}(w) = \frac{3}{2} \frac{G^{*}(w)}{(z - w)^2} + \frac{\partial G^{*}(w)}{z - w}, \quad G^{*}(z)L(w) = \frac{3}{2} \frac{G^{*}(w)}{(z - w)^2} + \frac{1}{2} \frac{\partial G^{*}(w)}{z - w}, \]
\[ J(z)J(w) = \frac{2K'}{(z - w)^2}, \]
\[ J(z)G(w) = \frac{G(w)}{z - w}, \quad G(z)J(w) = \frac{-G(w)}{z - w}. \]
\[
J(z)G^\ast(w) = \frac{-G^\ast(w)}{z-w}, \quad G^\ast(z)J(w) = \frac{G^\ast(w)}{z-w},
\]
\[
G(z)G^\ast(w) = \frac{K'}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{L(w) + \frac{1}{2}\partial J(w)}{z-w},
\]
\[
G^\ast(z)G(w) = \frac{K'}{(z-w)^3} - \frac{J(w)}{(z-w)^2} + \frac{L(w) - \frac{1}{2}\partial J(w)}{z-w},
\]
\[
G(z)G(w) = G^\ast(z)G^\ast(w) = 0.
\]

We note that this superconformal algebra \( \mathfrak{A} \) has important commuta
tants. The following theorem is easily shown from (1.5) and (1.6b):

**Theorem 2.2.** Introduce two fields:

\[
d_1(z) := X_{\alpha_1+\alpha_2}(z)+:\psi_{\alpha_1}(z)\psi_{\alpha_2}(z):
\]

and

\[
d_2(z) := X_{-\alpha_1-\alpha_2}(z)-:\psi^{\alpha_1}(z)\psi^{\alpha_2}(z):
\]

Then,

\[
d_i(z)G(w) = d_i(z)G^\ast(w) = 0 \quad \text{for } i = 1, 2.
\]

This implies, by the Borcherds-Jacobi identity (2.9), that

\[
[d_{i(m)}, G_{(n)}] = [d_{i(m)}, G^\ast_{(n)}] = 0 \quad \text{(2.10)}
\]

for all \( m, n \in \mathbb{Z} \) and \( i = 1, 2 \).

Let \( V \) be the tensor product of a highest weight \( \hat{\mathfrak{sl}}(2|1) \)-module and the symmetric algebra over \( \{\psi^{\alpha_i}(n); \quad i = 1, 2 \text{ and } n < 0\} \). Then \( V \) is an \( \mathfrak{A} \)-module and, by (2.10), all simultaneous eigenspaces of \( d_{i(m)} \), \( i = 1, 2 \) and \( m \in \mathbb{Z} \), are stable under the action of \( G_{(n)} \) and \( G^\ast_{(n)} \), and so are \( \mathfrak{A} \)-submodules.

### 3. The Centerless N=4 Superconformal Algebra Via \( \hat{A}(1,1) \)

Let us consider the superalgebra \( \mathfrak{sl}(2|2) \) with simple roots \( \Pi = \{\alpha_1, \alpha_2, \alpha_3\} \) and Chevalley generators \( (e_i, f_i, \alpha_i^\vee)_{i=1,2,3} \), satisfying

\[
(\langle \alpha^\vee_i, \alpha_j \rangle)_{i,j=1,2,3} = \begin{pmatrix}
0 & 1 & 0 \\
1 & -2 & 1 \\
0 & 1 & 0
\end{pmatrix}.
\]
The algebra $sl(2|2)$ has a super-invariant super-symmetric bilinear form $(\ | \ )$ such that

$$((\alpha^\gamma_i|\alpha^\gamma_j))_{i,j=1,2,3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  \ (3.2)

From this, one sees that $sl(2|2)$ is not simple, since the Cartan matrix (3.1) is singular of rank 2, or in other words, the inner product defined by (3.2) is degenerate, and that $\alpha^\gamma_i - \alpha^\gamma_j$ spans the center. The quotient superalgebra $sl(2|2)/\mathbb{C} \cdot (\alpha^\gamma_i - \alpha^\gamma_j)$ is called $A(1,1)$. Its Cartan subalgebra $\mathfrak{h}$ is, therefore, 2-dimensional, and each $\alpha_i$ naturally defines a linear form on $\mathfrak{h}$, which is also denoted by $\alpha_i$ using the same characters. Under this notation, the induced inner product on the dual space $\mathfrak{h}^*$ satisfies

$$((\alpha_i|\alpha_j))_{i,j=1,2,3} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$  \ (3.3)

For simplicity, we write $(m_1,m_2,m_3)$ in place of $\sum_{i=1}^3 m_i \alpha_i$, and also $-(m_1,m_2,m_3)$ in place of $-\sum_{i=1}^3 m_i \alpha_i$. Then the sets $\Delta^+$, $\Delta_{\text{even}}^+$ and $\Delta_{\text{odd}}^+$ of roots of $A(1,1)$ are given as follows:

$$\Delta_{\text{even}}^+ = \{(0,1,0), \ (1,1,1)\},$$

$$\Delta_{\text{odd}}^+ = \{(1,0,0), \ (0,0,1), \ (1,1,0), \ (0,1,1)\},$$

$$\Delta^+ = \Delta_{\text{even}}^+ \cup \Delta_{\text{odd}}^+.$$  

We note that the square length of a root $\alpha$ is equal to $-2$ if $\alpha = \pm \alpha_2$, 2 if $\alpha = \pm (1,1,1)$, and 0 if $\alpha$ is odd. For each root $\alpha = (m_1,m_2,m_3)$ we choose a root vector $X_\alpha = X_{(m_1,m_2,m_3)}$ in such a way that

$$e_{(1,1,0)} := [e_1, e_2], \quad e_{(0,1,1)} := [e_2, e_3], \quad e_{(1,1,1)} := [e_1, e_{(0,1,1)}] = [e_{(1,1,0)}, e_3],$$

$$f_{(1,1,0)} := [f_1, f_2], \quad f_{(0,1,1)} := [f_2, f_3], \quad f_{(1,1,1)} := [f_1, f_{(0,1,1)}] = [f_{(1,1,0)}, f_3].$$

We now consider the affinization $\tilde{A}(1,1)$, and the following fields:

$$G_1(z) := X_{(1,1,0)}(z) \psi^{\alpha_1}(z) - X_{(0,0,1)}(z) \psi^{-\alpha_3}(z),$$

$$G_1^*(z) := -X_{(1,1,0)}(z) \psi^{-\alpha_1}(z) + X_{(0,0,1)}(z) \psi^{\alpha_3}(z),$$

$$G_2(z) := X_{(1,0,0)}(z) \psi^{\alpha_1}(z) - X_{(0,1,1)}(z) \psi^{-\alpha_3}(z),$$

$$G_2^*(z) := -X_{(1,0,0)}(z) \psi^{-\alpha_1}(z) + X_{(0,1,1)}(z) \psi^{\alpha_3}(z),$$

$$A_1(z) := K \{ \psi^{\alpha_3}(z) \partial \psi_{\alpha_3}(z) : - : \psi^{\alpha_1}(z) \psi_{\alpha_1}(z) : - \alpha_2^\gamma(z),$$

$$A_2(z) := K \{ \psi^{\alpha_3}(z) \partial \psi_{\alpha_3}(z) : - : \psi^{\alpha_1}(z) \psi_{\alpha_1}(z) : + \alpha_2^\gamma(z),$$

$$B_1(z) := K \{ \psi^{\alpha_3}(z) \partial \psi_{\alpha_3}(z) : - : \partial \psi^{\alpha_1}(z) \psi_{\alpha_1}(z) : \}.$$
\begin{align*}
- : X_{(1,1,0)}(z)X_{-(1,1,0)}(z) & : + : X_{(0,0,1)}(z)X_{-(0,0,1)}(z) : \\
- : (\alpha_1^\vee + \alpha_2^\vee)(z)\psi^{\alpha_1}(z)\psi_{\alpha_1}(z) : - : \alpha_3^\vee(z)\psi^{\alpha_3}(z)\psi_{\alpha_3}(z) : \\
+ \partial\alpha_2^\vee(z) + : X_{(1,1,1)}(z)\psi^{\alpha_1}(z)\psi^{\alpha_3}(z) : + : X_{-(1,1,1)}(z)\psi_{\alpha_1}(z)\psi_{\alpha_3}(z) : , \\
B_2(z) & := K \left\{ \psi^{\alpha_3}(z)\partial\psi_{\alpha_3}(z) : - : \partial\psi_{\alpha_1}(z)\psi_{\alpha_1}(z) : \right\} \\
- : X_{(1,0,0)}(z)X_{-(1,0,0)}(z) & : - : X_{(0,1,1)}(z)X_{-(0,1,1)}(z) : \\
- : (\alpha_2^\vee + \alpha_3^\vee)(z)\psi^{\alpha_1}(z)\psi_{\alpha_3}(z) & : - : \alpha_1^\vee(z)\psi^{\alpha_1}(z)\psi_{\alpha_1}(z) : \\
+ \partial(\alpha_2^\vee + \alpha_3^\vee)(z) + : X_{(1,1,1)}(z)\psi^{\alpha_1}(z)\psi^{\alpha_3}(z) : \\
+ : X_{-(1,1,1)}(z)\psi_{\alpha_1}(z)\psi_{\alpha_3}(z) : . \\
\end{align*}

Then Theorem 2.1 applied to each quadruplet \((G_i, G^*_i, A_i, B_i), i = 1, 2,\) gives the following:

\begin{align*}
G_i(z)G_i(w) = G_i^*(z)G_i^*(w) & = 0, \quad (3.4a) \\
G_i(z)G_i^*(w) & = \frac{-2K}{(z-w)^3} + \frac{A_i(w)}{(z-w)^2} + \frac{B_i(w)}{z-w}, \quad (3.4b) \\
A_i(z)G_i(w) & = \frac{K + 1}{z-w}G_i(w), \quad (3.5a) \\
A_i(z)G_i^*(w) & = \frac{-(K + 1)}{z-w}G_i^*(w), \quad (3.5b) \\
B_i(z)G_i(w) & = (K + 1) \left\{ \frac{G_i(w)}{(z-w)^2} + \frac{\partial G_i(w)}{z-w} \right\}, \quad (3.6a) \\
B_i(z)G_i^*(w) & = (K + 1) \left\{ \frac{2G_i^*(w)}{(z-w)^2} + \frac{\partial G_i^*(w)}{z-w} \right\}. \quad (3.6b)
\end{align*}

One also has the following:

\begin{align*}
e_2(z)G_1(w) & = 0, \quad e_2(z)G_2(w) = \frac{-G_1(w)}{z-w}, \quad (3.7a) \\
e_2(z)G_1^*(w) & = \frac{G_2^*(w)}{z-w}, \quad e_2(z)G_2^*(w) = 0, \quad (3.7b) \\
f_2(z)G_1(w) & = \frac{G_2(w)}{z-w}, \quad f_2(z)G_2(w) = 0, \quad (3.8a) \\
f_2(z)G_1^*(w) & = 0, \quad f_2(z)G_2^*(w) = \frac{-G_1^*(w)}{z-w}. \quad (3.8b)
\end{align*}
Lemma 1.1 as follows:

\[ \alpha_2^\nu(z) G_1(w) = \frac{-G_1(w)}{z - w}, \quad \alpha_2^\nu(z) G_2(w) = \frac{G_2(w)}{z - w}. \] (3.9a)

\[ \alpha_2^\nu(z) G_1^*(w) = \frac{G_1^*(w)}{z - w}, \quad \alpha_2^\nu(z) G_2^*(w) = \frac{-G_2^*(w)}{z - w}. \] (3.9b)

and

\[ A_1(z) G_2(w) = \frac{K - 1}{z - w} G_2(w), \quad A_1(z) G_2^*(w) = \frac{1 - K}{z - w} G_2^*(w), \] (3.10a)

\[ A_2(z) G_1(w) = \frac{K - 1}{z - w} G_1(w), \quad A_2(z) G_1^*(w) = \frac{1 - K}{z - w} G_1^*(w). \] (3.10b)

The operator products \( G_i(z) G_j^*(w) \) for \( i \neq j \) are obtained from Lemma 1.1 as follows:

\[ G_1(z) G_2(w) = G_1^*(z) G_2^*(w) = 0, \] (3.11a)

\[ G_1(z) G_2^*(w) = -\frac{2e_2(w)}{(z-w)^2} - \frac{\partial e_2(w)}{z-w} \]

\[ + \frac{1}{z-w} \{ e_2(w) (: \psi^{\alpha_3}(w) \psi_{\alpha_3}(w) : - : \psi^{\alpha_1}(w) \psi_{\alpha_1}(w) :) \} \]

\[ - : X_{(1,1,0)}(w) X_{-(1,0,0)}(w) : - : X_{(0,1,1)}(w) X_{-(0,0,1)}(w) : \}. \] (3.11b)

\[ G_2(z) G_1^*(w) = \frac{2f_2(w)}{(z-w)^2} + \frac{\partial f_2(w)}{z-w} \]

\[ + \frac{1}{z-w} \{ f_2(w) (: \psi^{\alpha_1}(w) \psi_{\alpha_1}(w) : - : \psi^{\alpha_3}(w) \psi_{\alpha_3}(w) :) \} \]

\[ - : X_{(1,0,0)}(w) X_{-(1,1,0)}(w) : - : X_{(0,0,1)}(w) X_{-(0,1,1)}(w) : \}. \] (3.11c)

We now introduce the field

\[ D(z) := B_1(z) - B_2(z) + \partial \alpha_2^\nu(z) \]

\[ = : X_{(1,0,0)}(z) X_{-(1,0,0)}(z) : + : X_{(0,1,1)}(z) X_{-(0,1,1)}(z) : \]

\[ - : X_{(1,1,0)}(z) X_{-(1,1,0)}(z) : - : X_{(0,0,1)}(z) X_{-(0,0,1)}(z) : \] \hspace{1em} (3.12)

\[ + \alpha_2^\nu(z) (: \psi^{\alpha_3}(z) \psi_{\alpha_3}(z) : - : \psi^{\alpha_1}(z) \psi_{\alpha_1}(z) :). \]

Then one has

\[ D(z) e_2(w) = \frac{2}{z-w} \{ : X_{(1,1,0)}(w) X_{-(1,0,0)}(w) : + : X_{(0,1,1)}(w) X_{-(0,0,1)}(w) : \}

\[ + e_2(w) (: \psi^{\alpha_1}(w) \psi_{\alpha_1}(w) : - : \psi^{\alpha_3}(w) \psi_{\alpha_3}(w) :) \}, \] (3.13a)
\[ D(z)f_2(w) = \frac{2}{z-w}\{ :X_{(1,0,0)}(w)X_{(1,1,0)}(w) : + :X_{(0,0,1)}(w)X_{(0,1,1)}(w) : \]
\[ + f_2(w) (: \psi^{\alpha}(w)\psi_{\alpha}(w) : - : \psi^{\alpha}(w)\psi_{\alpha}(w) :) \}. \]  
(3.13b)

Let \( \mathfrak{A} \) be the vertex algebra generated by \( G_i, G_i^* \) \( (i = 1, 2) \), \( e_2, f_2 \), \( \alpha^\gamma \), and \( \mathfrak{J} \) its ideal generated by \( D \), and we consider the vertex algebra \( \mathfrak{A} := \mathfrak{A}/\mathfrak{J} \). Then, by (3.13a) and (3.13b), the third terms in the right-hand sides of (3.11b) and (3.11c) vanish in \( \mathfrak{A} \), and one has

\[ G_1(z)G_2^*(w) = - \left\{ \frac{2e_2(w)}{(z-w)^2} + \frac{\partial e_2(w)}{z-w} \right\}, \]  
(3.11'b)

\[ G_2(z)G_1^*(w) = \frac{2f_2(w)}{(z-w)^2} + \frac{\partial f_2(w)}{z-w}. \]  
(3.11'c)

As a cost of this ideal \( \mathfrak{J} \), we have to assume henceforward that \( K = 0 \), since otherwise \( G_i \) and \( G_i^* \) belong to \( \mathfrak{J} \) and the vertex algebra \( \mathfrak{A} \) collapses. Then

\[ A_1 = -A_2 = -\alpha^\gamma, \]  
(3.14)

and all other products are computed from (3.4) \( \sim (3.11') \) by the Borcherds-Jacobi identity (2.9) in a similar way as is explained in Section 2. And by putting

\[
L(z) := B_1(z) - \frac{1}{2}\partial A_1(z) = B_2(z) - \frac{1}{2}\partial A_2(z), \quad 
 h(z) := -\alpha^\gamma(z), \quad 
 e(z) := -e_2(z), \quad 
 f(z) := f_2(z),
\]

one sees that these fields satisfy the product formulas of the N=4 superconformal algebra:

**Theorem 3.1.**

\[
L(z)L(w) = \frac{2L(w)}{(z-w)^2} + \frac{\partial L(w)}{z-w},
\]

\[
L(z)X(w) = \frac{X(w)}{(z-w)^2} + \frac{\partial X(w)}{z-w}, \quad X(z)L(w) = \frac{X(w)}{(z-w)^2},
\]

\[
X(z)Y(w) = \frac{[X,Y](w)}{z-w} \quad \text{for } X, Y = h, e, \text{ or } f,
\]

\[
L(z)F(w) = \frac{3}{2}F(w) + \frac{\partial F(w)}{z-w},
\]

\[
F(z)L(w) = \frac{3}{2}F(w) + \frac{1}{2}\partial F(w) = \frac{z-w}{z-w} \quad \text{for } F = G_i, G_i^*,
\]

\[
h(z)G_1(w) = -G_1(z)h(w) = \frac{G_1(w)}{z-w}, \quad h(z)G_1^*(w) = -G_1^*(z)h(w) = \frac{-G_1^*(w)}{z-w},
\]
\[ h(z)G_2(w) = -G_2(z)h(w) = \frac{-G_2(w)}{z - w}, \quad h(z)G_2^*(w) = -G_2^*(z)h(w) = \frac{G_2^*(w)}{z - w}, \]
\[ e(z)G_1(w) = G_1(z)e(w) = 0, \quad e(z)G_1^*(w) = -G_1^*(z)e(w) = \frac{-G_2^*(w)}{z - w}, \]
\[ e(z)G_2(w) = -G_2(z)e(w) = \frac{G_1(w)}{z - w}, \quad e(z)G_2^*(w) = G_2^*(z)e(w) = 0, \]
\[ f(z)G_1(w) = -G_1(z)f(w) = \frac{G_2(w)}{z - w}, \quad f(z)G_1^*(w) = G_1^*(z)f(w) = 0, \]
\[ f(z)G_2(w) = G_2(z)f(w) = 0, \quad f(z)G_2^*(w) = -G_2^*(z)f(w) = \frac{-G_1^*(w)}{z - w}, \]
\[ G_i(z)G_j(w) = G_i^*(z)G_j^*(w) = 0, \]
\[ G_1(z)G_1^*(w) = G_2^*(z)G_2(w) = \frac{h(w)}{(z - w)^2} + \frac{L(w) + \frac{1}{2} \partial h(w)}{z - w}, \]
\[ G_2(z)G_2^*(w) = G_1^*(z)G_1(w) = \frac{-h(w)}{(z - w)^2} + \frac{L(w) - \frac{1}{2} \partial h(w)}{z - w}, \]
\[ G_1(z)G_2^*(w) = -G_2^*(z)G_1(w) = \frac{2e(w)}{(z - w)^2} + \frac{\partial e(w)}{z - w}, \]
\[ G_2(z)G_1^*(w) = -G_1^*(z)G_2(w) = \frac{2f(w)}{(z - w)^2} + \frac{\partial f(w)}{z - w}. \]

Commutants for \( \mathfrak{A} \) are given as follows in this case:

**Theorem 3.2.** Consider two fields:

\[ d_1(z) := X_{(1,1,1)}(z) + : \psi_{\alpha_1}(z) \psi_{\alpha_3}(z) : \]

and

\[ d_2(z) := X_{-(1,1,1)}(z) + : \psi^{\alpha_1}(z) \psi^{\alpha_3}(z) :. \]

Then

\[ d_i(z)G_j(w) = d_i(z)G_j^*(w) = 0 \quad \text{and} \quad d_i(z)X(w) = 0, \]

for \( i, j = 1, 2 \) and \( X = h, e, f \).

From this theorem, one has

\[ [d_{i(m)}, G_{j(n)}] = [d_{i(m)}, G_{j(n)}^*] = 0 \]

and

\[ [d_{i(m)}, X_{(n)}] = 0, \]

for \( i, j = 1, 2, X = h, e, f, \) and \( m, n \in \mathbb{Z} \). Then, by (3.12), all \( d_{i(m)} \)'s commute with \( D_{(n)} \) and so, when we consider representations, commute with the action of the vertex algebra \( \mathfrak{A} \). And all simultaneous eigenspaces of \( d_{i(m)}, \) \( i = 1, 2 \) and \( m \in \mathbb{Z} \), are \( \mathfrak{A} \)-submodules.
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