MCKAY’S OBSERVATION AND VERTEX OPERATOR ALGEBRAS GENERATED BY TWO CONFORMAL VECTORS OF CENTRAL CHARGE 1/2

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Abstract. This paper is a continuation of [33] at which several coset subalgebras of the lattice VOA $V_{\sqrt{2}E_8}$ were constructed and the relationship between such algebras with the famous McKay observation on the extended $E_8$ diagram and the Monster simple group were discussed. In this article, we shall provide the technical details. We completely determine the structure of the coset subalgebras constructed and show that they are all generated by two conformal vectors of central charge 1/2. We also study the representation theory of these coset subalgebras and show that the product of two Miyamoto involutions is in the desired conjugacy class of the Monster simple group if a coset subalgebra $U$ is actually contained in the Moonshine VOA $V^\natural$. The existence of $U$ inside the Moonshine VOA $V^\natural$ for the cases of $1A, 2A, 2B$ and $4A$ is also established. Moreover, the cases for $3A, 5A$ and $3C$ are discussed.

1. Introduction

This paper is a continuation of the authors’ work [33] at which several coset subalgebras of the lattice VOA $V_{\sqrt{2}E_8}$ were constructed and the relationship between such algebras with the famous McKay observation [2, 38] on the extended $E_8$ diagram

\[
\begin{array}{cccccccccc}
3C & \frac{1}{2\pi} \\
1A & 2A & 3A & 4A & 5A & 6A & 4B & 2B \\
\frac{1}{4} & \frac{1}{32} & \frac{13}{2\pi} & \frac{1}{2\pi} & \frac{3}{2\pi} & \frac{5}{2\pi} & \frac{1}{2\pi} & 0
\end{array}
\]  

(1.1)

and the Monster simple group were discussed.

In this article, we shall provide the technical details. We shall determine the structure of the coset subalgebras and show that they are all generated by two conformal vectors of central charge 1/2. We also study the representation theory of these coset subalgebras and show that the product of two Miyamoto involutions is in the desired conjugacy class of the Monster simple group if a coset subalgebra $U$ is actually contained in the Moonshine

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The existence of $U$ inside the Moonshine VOA $V^\natural$ for the cases of $1A, 2A, 2B$ and $4A$ is also established. Moreover, the cases for $3A, 5A$ and $3C$ are discussed.

The organization of the article is as follows. In Section 2 we shall review some important notation and terminology from [33]. We review certain conformal vectors in the lattice VOA $V_{\sqrt{2}E_8}$, where $E$ is a root lattice of type $A, D$, or $E$ (cf. [7]). We then consider the sublattice $L$ of $E_8$ and define the coset subalgebra $U$ and two conformal vectors $\hat{e}$ and $\hat{f}$ of central charge $1/2$. A canonical automorphism $\sigma$ of order $n = |E_8/L|$ induced by the quotient group $E_8/L$ is also discussed. In Section 3, we study the structure of $U$ in each of the nine cases corresponding to the McKay’s diagram. We also study the representation theory of these coset subalgebras and show that the product of two Miyamoto involutions is in the desired conjugacy class of the Monster simple group if a coset subalgebra $U$ is actually contained in the Moonshine VOA $V^\natural$. The existence of $U$ inside the Moonshine VOA $V^\natural$ for the cases of $1A, 2A, 2B$ and $4A$ is also established. Moreover, the cases for $3A, 5A$ and $3C$ are discussed. Appendix contains the classification of conformal vectors in $U$, calculations of certain characters which are used in Section 3 and the classification of irreducible modules for $5A$ and $3C$ cases.

The authors thank Masahiko Miyamoto and Masaaki Kitazume for stimulating discussions and Kazuhiro Yokoyama for helping them to compute the conformal vectors for the cases of $5A$ and $6A$ by a computer algebra system Risa/Asir. In Appendix C we study an extension of a simple rational VOA by an irreducible module which is not a simple current module. A similar extension is also considered in [46]. The authors thank Kenichiro Tanabe for valuable discussions concerning it. Part of the work was done while the third author (H. Yamauchi) was visiting the National Center for Theoretical Science of Taiwan in August, 2004. He thanks the center for the hospitality during the stay.

2. Preliminary

In this section, we shall recall the notation and the constructions of certain coset subalgebras of $V_{\sqrt{2}E_8}$ and their automorphisms from [33]. We shall mainly deal with lattice VOAs introduced by [17]. Let $V_N$ be a lattice VOA associated with any positive definite even lattice $N$. By [34, Theorem 3.1], there is a unique symmetric invariant bilinear form $\langle \cdot, \cdot \rangle$ on $V_N$ such that $\langle 1, 1 \rangle = 1$. That the form $\langle \cdot, \cdot \rangle$ is invariant, i.e.,

$$\langle Y(u, z)v, w \rangle = \langle v, Y(e^{zL(1)}(-z^{-2})L(0)u, z^{-1})w \rangle \quad \text{for } u, v, w \in V_N.$$ 

For $u, v \in (V_N)_2$ with $L(1)u = 0$, the invariance of the form implies that $\langle u, v \rangle = \langle 1, u_3v \rangle$ which induces a bilinear form on $V_2$. It is also well known that $V_N$ possesses a positive definite invariant hermitian form $(\cdot, \cdot)$. Indeed, let $V_{N,R}$ be the $\mathbb{R}$-form of $V_N$ defined as in [17, Section 12.4]. Then $V_{N,R}$ is invariant under the automorphism $\theta$, where $\theta$ is a lift of $-1$ isometry of the lattice $N$. Let $V^\pm_{N,R}$ be the eigenspaces for $\theta$ with eigenvalues $\pm 1$. Then
⟨ · , · ⟩ is positive definite on $V_{N_+}^+, V_{N_+}^-$ and negative definite on $V_{N_-}^+, V_{N_-}^-$. Moreover, $\langle V_{N_+}^+, V_{N_-}^- \rangle = 0$. Hence $\langle · , · ⟩$ is positive definite on the $\mathbb{R}$-vector space $\tilde{V}_{N,\mathbb{R}} = V_{N_+}^+ + \sqrt{-1} V_{N_-}^-$. Clearly, $V_N = \mathbb{C} \otimes \mathbb{R} \tilde{V}_{N,\mathbb{R}}$ and so $\tilde{V}_{N,\mathbb{R}}$ is an $\mathbb{R}$-form of $V_N$. Define a hermitian form $( · , · )$ on $V_N$ by $(\lambda u, \mu v) = \lambda \mu (u, v)$ for $\lambda, \mu \in \mathbb{C}$ and $u, v \in \tilde{V}_{N,\mathbb{R}}$. Then $( · , · )$ is positive definite on $V_N$. Furthermore, it is $\tilde{V}_{N,\mathbb{R}}$-invariant, that is,

$$(Y(u, z)v, w) = (v, Y(e^{L(1)}(-z^{-2})L(0))u, z^{-1}w)$$

for $u \in \tilde{V}_{N,\mathbb{R}}$ and $v, w \in V_N$, where $L(0) = \omega_1$ and $L(1) = \omega_2$ with $\omega$ being the Virasoro element of $V_N$. These two forms $\langle · , · ⟩$ and $( · , · )$ for the case $N = \sqrt{2}E_8$ will be used in Section 2 and 3.

### 2.1. Conformal vectors

We shall now review the construction of certain conformal vectors in the lattice VOA $V_{\sqrt{2}\mathbb{R}}$ from [7], where $R$ is a root lattice of type $A_n$, $D_n$, or $E_n$.

Let $\Phi$ be the root system of $R$ and $\Phi^+$ and $\Phi^-$ the set of all positive roots and negative roots, respectively. Then $\Phi = \Phi^+ \cup \Phi^-$, and

$$\omega = \omega(\Phi) = \frac{1}{2h} \sum_{\alpha \in \Phi^+} \alpha(-1)^2 \cdot 1,$$

where $h$ is the Coxeter number of $\Phi$. Now define

$$s = s(\Phi) = \frac{2}{(h+2)} \sum_{\alpha \in \Phi^+} \left( \alpha(-1)^2 \cdot 1 - 2(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}) \right),$$

$$\tilde{\omega} = \tilde{\omega}(\Phi) = \omega - s.$$  \hspace{1cm} (2.1)

It is shown in [7] that $\tilde{\omega}$ and $s$ are mutually orthogonal conformal vectors and the central charge of $\tilde{\omega}$ is $2n/(n+3)$ if $R$ is of type $A_n$, 1 if $R$ is of type $D_n$ and 6/7, 7/10 and 1/2 if $R$ is of type $E_6$, $E_7$ and $E_8$, respectively. In this article, we denote by $\text{Vir}(x)$ the Virasoro sub VOA generated by a conformal vector $x$ of $V$.

Now let $\text{Aut}(\Phi)$ be the automorphism group of $\Phi$. For any element $g \in \text{Aut}(\Phi)$, $g$ induces an automorphism on the lattice $R$ and hence also defines an automorphism of the VOA $V_{\sqrt{2}\mathbb{R}}$ by

$$g(u \otimes e^{\sqrt{2}\alpha}) = gu \otimes e^{\sqrt{2}\alpha} \quad \text{for} \quad u \otimes e^{\sqrt{2}\alpha} \in M(1) \otimes e^{\sqrt{2}\alpha} \subset V_{\sqrt{2}\mathbb{R}}.$$ 

Note that both $s$ and $\tilde{\omega}$ are fixed by $\text{Aut}(\Phi)$ and thus also fixed by the Weyl group $W(\Phi)$ of $\Phi$.

Let $R^* = \{ \alpha \in \mathbb{Q} \otimes \mathbb{Z} \mid \langle \alpha, R \rangle \subset \mathbb{Z} \}$ be the dual lattice of $R$. Then we have the following Proposition.
Proposition 2.1 (cf. Section 2 of [33]). Let $\gamma + R$ be a coset of $R$ in $R^*$ and $k = \min\{\langle \alpha, \alpha \rangle | \alpha \in \gamma + R\}$. Define

$$v = \sum_{\frac{\alpha \in \gamma + R}{\langle \alpha, \alpha \rangle = k}} e^{\sqrt{2}\alpha} \in V_{\sqrt{2}(\gamma + R)}.$$ 

Then $v$ is a highest weight vector of highest weight $(0,k)$ in $V_{\sqrt{2}(\gamma + R)}$ with respect to $\text{Vir}(s) \otimes \text{Vir}(\omega)$, that is, $s_j v = \omega_j v = 0$ for all $j \geq 2$, $s_1 v = 0$, and $\omega_1 v = kv$. In other words, with respect to $s$, there is always a highest weight vector of weight 0 in $V_{\sqrt{2}(\gamma + R)}$.

2.2. Extended $E_8$ diagram and coset subalgebras of $V_{\sqrt{2}E_8}$. Next, we shall review the construction some coset VOAs $U$ using the extended $E_8$ diagram. In each case, $U$ contains some conformal vectors of central charge $1/2$ and the inner products among these conformal vectors are the same as the numbers given in the McKay diagram 1.1. First, we shall consider certain sublattices of the root lattice $E_8$ by using the extended $E_8$ diagram

$$\begin{array}{cccccccc}
\alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\
\end{array}$$

(2.2)

where $\alpha_1, \alpha_2, \ldots, \alpha_8$ are the simple roots of $E_8$ and

$$\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 + 3\alpha_8 = 0.$$ (2.3)

Then $\langle \alpha_i, \alpha_i \rangle = 2$, $0 \leq i \leq 8$. Moreover, for $i \neq j$, $\langle \alpha_i, \alpha_j \rangle = -1$ if the nodes $\alpha_i$ and $\alpha_j$ are connected by an edge and $\langle \alpha_i, \alpha_j \rangle = 0$ otherwise. Note that $-\alpha_0$ is the highest root.

For any $i = 0, 1, \ldots, 8$, let $L(i)$ be the sublattice generated by $\alpha_j, 0 \leq j \leq 8, j \neq i$. Then $L(i)$ is a rank 8 sublattice of $E_8$. Note $L(i)$ is the lattice associated with the Dynkin diagram obtained by removing the corresponding node $\alpha_i$ in the extended $E_8$ diagram (2.2) and the index $|E_8/L(i)|$ is equal to $n_i$, where $n_i$ is the coefficient of $\alpha_i$ in the left hand side of (2.3). Actually,

$$\begin{align*}
L(0) & \cong E_8, \\
L(1) & \cong A_1 \oplus E_7, \\
L(2) & \cong A_2 \oplus E_6, \\
L(3) & \cong A_3 \oplus D_5, \\
L(4) & \cong A_4 \oplus A_4, \\
L(5) & \cong A_5 \oplus A_2 \oplus A_1, \\
L(6) & \cong A_7 \oplus A_1, \\
L(7) & \cong D_8, \\
L(8) & \cong A_8.
\end{align*}$$ (2.4)

Now let us explain the details of our construction. First, we fix $i \in \{0, 1, \ldots, 8\}$ and denote $L(i)$ by $L$. In each case, $|E_8/L| = n_i$ and $\alpha_i + L$ is a generator of the quotient group $E_8/L$. Hence we have

$$E_8 = L \cup (\alpha_i + L) \cup (2\alpha_i + L) \cup \cdots \cup ((n_i - 1)\alpha_i + L).$$ (2.5)

Let $\lambda = \sqrt{2}\alpha_i$. Then

$$\sqrt{2}E_8 = \sqrt{2}L \cup (\lambda + \sqrt{2}L) \cup (2\lambda + \sqrt{2}L) \cup \cdots \cup ((n_i - 1)\lambda + \sqrt{2}L)$$

$$\cup \sqrt{2}E_8.$$
and the lattice VOA $V_{\sqrt{2}E_8}$ can be decomposed as
\[
V_{\sqrt{2}E_8} = V_{\sqrt{2}L} \oplus V_{\lambda+\sqrt{2}L} \oplus \cdots \oplus V_{(n_i-1)\lambda+\sqrt{2}L},
\]
where $V_{j\lambda+\sqrt{2}L}$, $j = 0, 1, \ldots, n_i - 1$, are irreducible modules of $V_{\sqrt{2}L}$ (cf. [4]).

**Remark 2.2.** The abelian group $E_8/L$ actually induces an automorphism $\sigma$ of $V_{\sqrt{2}E_8}$ such that
\[
\sigma(u) = \xi^j u \quad \text{for any } u \in V_{j\lambda+\sqrt{2}L},
\]
where $\xi = e^{2\pi\sqrt{-1}/n_i}$ is a primitive $n_i$-th root of unity. More precisely, let
\[
a = \begin{cases}
\alpha_1 & \text{if } i = 0, \\
\frac{1}{i+1}(\alpha_0 + 2\alpha_1 + \cdots + i\alpha_{i-1}) & \text{if } 1 \leq i \leq 5, \\
\frac{\sqrt{2}}{i}(\alpha_0 + 2\alpha_1 + \cdots + 6\alpha_5 + 7\alpha_8) & \text{if } i = 6, \\
\frac{\sqrt{2}}{5}(\alpha_0 + \alpha_8) & \text{if } i = 7, \\
\frac{\sqrt{2}}{3}(\alpha_0 + 2\alpha_1 + \cdots + 8\alpha_7) & \text{if } i = 8.
\end{cases}
\]
Then $\langle a, \alpha_j \rangle \in \mathbb{Z}$ for $0 \leq j \leq 8$ with $j \neq i$ and $\langle a, \alpha_i \rangle \equiv -1/n_i$ (mod $\mathbb{Z}$). The automorphism $\sigma : V_{\sqrt{2}E_8} \to V_{\sqrt{2}E_8}$ is in fact defined by
\[
\sigma = e^{-\pi\sqrt{-1}(\beta(0))} = \exp(-\sqrt{-2}\pi a(0)) \quad \text{with } \beta = \sqrt{2}a.
\]
For $u \in M(1) \otimes e^\alpha \subset V_{\sqrt{2}E_8}$, we have $\sigma(u) = e^{-\pi\sqrt{-1}(\beta,\alpha)}u$. Note that $a + R$ is a generator of the group $R^* / R$ for the cases $i \neq 0, 7$, where $R$ is an indecomposable component of the lattice $L$ of type $A$.

For any lattice VOA $V_N$ associated with a positive definite even lattice $N$, there is a natural involution $\theta$ induced by the isometry $\alpha \to -\alpha$ for $\alpha \in N$. If $N = \sqrt{2}E_8$, which is doubly even, we may define $\theta : V_{\sqrt{2}E_8} \to V_{\sqrt{2}E_8}$ by
\[
\alpha(-n) \to -\alpha(-n) \quad \text{and} \quad e^\alpha \to e^{-\alpha}
\]
for any $\alpha \in \sqrt{2}E_8$ (cf. [17]). Then $\theta \sigma \theta = \sigma^{-1}$ and the group generated by $\theta$ and $\sigma$ is isomorphic to a dihedral group of order $2n_i$.

Next we shall recall the definition of certain coset subalgebras from [33]. Let $R_1, \ldots, R_l$ be the indecomposable components of the lattice $L$ and $\Phi_1, \ldots, \Phi_l$ the corresponding root systems of $R_1, \ldots, R_l$ (cf. (2.4)). Then $L = R_1 \oplus \cdots \oplus R_l$ and
\[
V_{\sqrt{2}L} \cong V_{\sqrt{2}R_1} \otimes \cdots \otimes V_{\sqrt{2}R_l}.
\]
By (2.1) in Section 2, one obtains $2l$ mutually orthogonal conformal vectors
\[
s^k = s(\Phi_k), \quad \tilde{\omega}^k = \tilde{\omega}(\Phi_k), \quad 1 \leq k \leq l
\]
such that the Virasoro element $\omega$ of $V_{\sqrt{2}L}$, which is also the Virasoro element of $V_{\sqrt{2}E_8}$, can be written as a sum of these conformal vectors
\[
\omega = s^1 + \cdots + s^l + \tilde{\omega}^1 + \cdots + \tilde{\omega}^l.
\]
Now we define $U$ to be a coset (or commutant) subalgebra

$$U = \{ v \in V_{\sqrt{2}E_8} \mid (s^k)_i v = 0 \text{ for all } k = 1, \ldots, l \}. \quad (2.10)$$

Note that $U$ is a VOA with the Virasoro element $\omega' = \tilde{\omega}^1 + \cdots + \tilde{\omega}^l$ and the automorphism $\sigma$ defined by (2.6) induces an automorphism of order $n_i$ on $U$. By abuse of notation, we shall denote it by $\sigma$ also.

**Remark 2.3.** In [32], it is shown that \( \{ v \in V_{\sqrt{2}A_n} \mid s(A_n)_1 v = 0 \} \) is isomorphic to a parafermion algebra $W_{n+1}(2n/(n+3))$ of central charge $2n/(n+3)$. Thus, if $L$ has some indecomposable component of type $A_n$, then $U$ will contain some subalgebra isomorphic to a parafermion algebra. It is well known [49] that the parafermion algebra $W_{n+1}(2n/(n+3))$ processes a certain $\mathbb{Z}_{n+1}$ symmetry among its irreducible modules. The automorphism $\sigma$ is actually related to such a symmetry. More details about the relation between coset subalgebra $U$ and the parafermion algebra $W_{n+1}(2n/(n+3))$ can be found in Appendix B.

Next we shall recall the definition of two conformal vectors of central charge $1/2$ from [33]. Note that

$$\hat{e} = \frac{1}{16} \omega + \frac{1}{32} \sum_{\alpha \in \Phi^+(E_8)} (e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}) \quad (2.11)$$

is a conformal vector of central charge $1/2$. Let $\sigma$ be the automorphism defined in Remark 2.2. Then we have

**Theorem 2.4 (cf. [33]).** Let $\hat{e}$ be defined as above and $\hat{f} = \sigma \hat{e}$. Then $\hat{e}, \hat{f} \in U$ and we have

$$\langle \hat{e}, \hat{f} \rangle = \begin{cases} 
1/4 & \text{if } i = 0, \\
1/32 & \text{if } i = 1, \\
13/2^{10} & \text{if } i = 2, \\
1/2^7 & \text{if } i = 3, \\
3/2^9 & \text{if } i = 4, \\
5/2^{10} & \text{if } i = 5, \\
1/2^8 & \text{if } i = 6, \\
0 & \text{if } i = 7, \\
1/2^8 & \text{if } i = 8. 
\end{cases} \quad (2.12)$$

In other words, the values of $\langle \hat{e}, \hat{f} \rangle$ are exactly the values given in McKay’s diagram (1.1).

### 3. The coset subalgebra $U$ and Miyamoto’s $\tau$-involutions

This section is the main part of this article. We shall study the structure of the coset subalgebra $U$ defined by (2.10) for each of the nine cases. Except for the case of $4A$, we shall show that the subalgebra $U$ always contain a set of mutually orthogonal conformal
inequivalent irreducible L conformal vectors are all coming from the unitary series $B$.

Such a conformal vector generates a simple Virasoro VOA isomorphic to $L(c_m, 0)$ inside $U$. The irreducible modules of $L(c_m, 0)$ are of the form $L(c_m, h_{r,s}^m)$, where

$$h_{r,s}^m = \frac{(r(m+3) - s(m+2))^2 - 1}{4(m+2)(m+3)}, \quad 1 \leq r \leq m+1, \quad 1 \leq s \leq m+2. \quad (3.2)$$

Note that $h_{r,s}^m = h_{m+2-r,m+3-s}^m$ and that $L(c_m, h_{r,s}^m), 1 \leq s \leq r \leq m+1$ are all the inequivalent irreducible $L(c_m, 0)$-modules.

For the $4A$ case, we shall show that $U$ is isomorphic to the fixed point subalgebra $V_N^+$ of $\theta$ for some rank two lattice $N$.

Furthermore, we shall discuss the relation between Miyamoto’s $\tau$-involutions and the structure of the coset subalgebra $U$. We shall show that for any VOA $V$ which contains a subalgebra isomorphic to $U$, the product of the Miyamoto involutions $\tau_\ell$ and $\tau_j$ naturally defines an automorphism of order $n_i$ or $n_i/2$ on $V$. If $U$ is actually contained in the Moonshine VOA $V^2$, then we shall show that $\tau_\ell \tau_j$ is of the desired conjugacy class of the Monster simple group mentioned in the McKay diagram. The existence of $U$ inside the Moonshine VOA $V^2$ will also be established for the cases $1A, 2A, 2B, 3A$, and $4A$.

As in Section 2.2, $L = L(i), i = 0, 1, \ldots, 8$ denotes the lattice associated with the Dynkin diagram obtained by removing the $i$-th node $\alpha_i$ in the extended $E_8$ diagram. The coset decomposition of $E_8$ by $L$ is given in (2.5). For $j = 1, \ldots, n_i - 1$, we define

$$X^j = \sum_{\alpha \in \mathbb{Z}^2, (\alpha, \alpha) = 2} e^{\sqrt{2} \alpha}. \quad (3.3)$$

Clearly $X^j$ is of weight 2. By Proposition 2.1, it is easy to see that $X^j \in U$ for all $j = 1, \ldots, n_i - 1$.

Recall the positive definite invariant hermitian form $(\cdot, \cdot)$ on lattice VOAs mentioned in Introduction. We shall consider the form $(\cdot, \cdot)$ for the lattice VOA $V_{\sqrt{2}E_8}$. Let

$$Y^{j+,} = \frac{1}{2}(X^j + X^{n_i-j}), \quad Y^{j-,} = \frac{1}{2}\sqrt{-1}(X^j - X^{n_i-j}), \quad 1 \leq j \leq [n_i/2]$$

and set $B_\mathbb{R} = \text{span}_\mathbb{R}\{\tilde{\omega}^1, \ldots, \tilde{\omega}^l, Y^{1,\pm}, \ldots, Y^{[n_i/2],\pm}\}$. Then $B_\mathbb{R}$ is contained in $V_{\sqrt{2}E_8}$ and so the form $(\cdot, \cdot)$ is $B_\mathbb{R}$-invariant. It is clear that the conformal vectors $\hat{e}$ and $\hat{f}$ defined by (2.11) are contained in $B_\mathbb{R}$. In the following argument we use the fact that $V_{\sqrt{2}E_8}$ possesses a positive definite hermitian form which is $B_\mathbb{R}$-invariant.

Now let us study the structure of $U$ and the Miyamoto involutions $\tau_\ell$ and $\tau_j$ associated with the conformal vectors $\hat{e}$ and $\hat{f}$ defined by (2.11) in each of the nine cases. First we
shall note that
\[ \tau_e \tau_f = e^{2\pi \sqrt{-1} \theta(0)} \] as an automorphism of \( V_{\sqrt{\tau_{E_8}}} \). (cf. [33, Section 4]).

3.1. 1A case. In this case, \( L = L(0) \cong E_8, n_0 = |E_8/L| = 1, \) and \( U \cong L(1/2, 0) \). The conformal vector \( \omega^1 \in V_{\sqrt{\tau_{E_8}}} \) defined by (2.9) is the only conformal vector in \( U \). Its central charge is 1/2. Moreover, \( \hat{e} = \hat{f} = \omega^1 \) and \( U = V_{\omega^1} \subset V^c \). Thus we have \( \tau_e \tau_f = 1 \).

3.2. 2A case. In this case, \( L = L(1) \cong A_1 \oplus E_7, n_1 = |E_8/L| = 2, \) and the conformal vectors \( \omega^1 \in V_{\sqrt{\tau_{A_1}}} \) and \( \omega^2 \in V_{\sqrt{\tau_{E_7}}} \) defined by (2.9) are of central charge 1/2 and 7/10, respectively.

**Proposition 3.1.** The vector \( X^1 \) defined by (3.3) is a highest weight vector of highest weight \((1/2, 3/2)\) with respect to \( V(\omega^1) \otimes V(\omega^2) \). Thus as a module of \( V(\omega^1) \otimes V(\omega^2) \),
\[ U \cong L(1, 0) \oplus L(7, 0) \oplus L(1, 1/2) \oplus L(7, 3/2) \]

This VOA has been well studied in [24, 30]. In fact, \( U \) has exactly three conformal vectors of central charge 1/2, namely \( \hat{e}, \hat{f} \) and \( w = \omega^1 \). The automorphism group \( \text{Aut} \) \( U \) of \( U \) is a symmetric group \( S_3 \) of degree 3. Note that \( U \) is generated by \( \hat{e}, \hat{f} \) and they are both fixed by \( \theta \). Thus, \( U \subset V^c \).

The VOA \( U \) is rational and it has exactly eight inequivalent irreducible modules \( M^j, W^j, j \in \{0, a, b, c\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). As \( V(\omega^1) \otimes V(\omega^2) \)-modules, they are of the following form,

\[
M^0 \cong [0, 0] \oplus \left[\frac{1}{2}, \frac{3}{2}\right] , \quad W^0 \cong [0, \frac{3}{5}] \oplus \left[\frac{1}{2}, \frac{1}{10}\right], \\
M^a \cong M^b \cong \left[\frac{1}{16}, \frac{7}{16}\right] , \quad W^a \cong W^b \cong \left[\frac{1}{16}, \frac{3}{80}\right], \\
M^c \cong \left[\frac{1}{2}, 0\right] \oplus [0, \frac{3}{2}], \quad W^c \cong \left[\frac{1}{2}, \frac{3}{5}\right] \oplus [0, \frac{1}{10}],
\]

where \([h_1, h_2]\) denotes \( L(1/2, h_1) \otimes L(7/10, h_2) \).

It is known that the fusion rules among irreducible \( U \)-modules have a symmetry of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \). For any VOA \( V \) containing a subalgebra isomorphic to \( U \), there are three automorphisms of order 2 or 1 associated with \( U \) (cf. [30]). They are given by

\[
\tau^a = \begin{cases} 
1 & \text{on } U^0, U^a, \\
-1 & \text{on } U^b, U^c,
\end{cases} \quad \tau^b = \begin{cases} 
1 & \text{on } U^0, U^b, \\
-1 & \text{on } U^c, U^a,
\end{cases} \quad \tau^c = \begin{cases} 
1 & \text{on } U^0, U^c, \\
-1 & \text{on } U^a, U^b,
\end{cases}
\]

where \( U^j \) is the sum of all irreducible \( U \)-submodules of \( V \) which are isomorphic to either \( M^j \) or \( W^j \) for \( j = 0, a, b, c \). Actually,

\[
\tau^a = \tau_w, \quad \tau^b = \tau_{\hat{e}}, \quad \tau^c = \tau_f,
\]

and we have \( \tau_e \tau_f = \tau^b \tau^c = \tau^a = \tau_w \).
If $V$ is the Moonshine VOA $V^\natural$, then we have the following theorem.

**Theorem 3.2.** As automorphisms of $V^\natural$, $\tau_e \tau_j = \tau_w$ and thus $\tau_e \tau_j$ is of class $2A$.

3.3. 3A case. In this case, $L = L(2) \cong A_2 \oplus E_6$, $n_2 = |E_8/L| = 3$, and the conformal vectors $\tilde{\omega}^1 \in V_{\sqrt{A_2}}$ and $\tilde{\omega}^2 \in V_{\sqrt{E}_6}$ defined by (2.9) are of central charge $4/5$ and $6/7$, respectively. \cite{26, 31}, we know that

$$U^1 = \{ u \in V_{\sqrt{A_2}} | (s^1)_1 u = 0 \} \cong L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3),$$

$$U^2 = \{ u \in V_{\sqrt{E}_6} | (s^2)_1 u = 0 \} \cong L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5).$$

Hence,

$$U \supset U^1 \otimes U^2 \cong \left( L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3) \right) \otimes \left( L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5) \right).$$

**Proposition 3.3.** Both of the vectors $X^1$ and $X^2$ defined by (3.3) are highest weight vectors of highest weight $(2/3, 4/3)$ with respect to $\text{Vir}(\tilde{\omega}^1) \otimes \text{Vir}(\tilde{\omega}^2)$.

**Proof.** First, note that $X^1$ and $X^2$ are highest weight vectors of weight 2 with respect to the Virasoro element $\omega' = \tilde{\omega}^1 + \tilde{\omega}^2$ of $U$ and that

$$\tilde{\omega}^1 = \frac{1}{15} \sum_{\alpha \in \Phi^+(A_2)} \alpha(-1)^2 \cdot 1 + \frac{1}{5} \sum_{\alpha \in \Phi^+(A_2)} (e^{\sqrt{2} \alpha} + e^{-\sqrt{2} \alpha})$$

$$= \frac{2}{5} \omega' + \frac{1}{5} \sum_{\alpha \in \Phi^+(A_2)} (e^{\sqrt{2} \alpha} + e^{-\sqrt{2} \alpha}).$$

Clearly, $(\tilde{\omega}^1)_3 X^1 = (\tilde{\omega}^1)_2 X^1 = 0$ and

$$(\tilde{\omega}^1)_1 X^1 = \left( \frac{2}{5} \omega' + \frac{1}{5} \sum_{\alpha \in \Phi^+(A_2)} (e^{\sqrt{2} \alpha} + e^{-\sqrt{2} \alpha}) \right) X^1$$

$$= \left( \frac{2}{5} \times \frac{2}{3} + \frac{1}{5} \times 2 \right) X^1 = \frac{2}{3} X^1.$$ 

Hence, $X^1$ is a highest weight vector of highest weight $(2/3, 4/3)$ with respect to $\text{Vir}(\tilde{\omega}^1) \otimes \text{Vir}(\tilde{\omega}^2)$. Similarly, $X^2$ is also a highest weight vector of highest weight $(2/3, 4/3)$ with respect to $\text{Vir}(\tilde{\omega}^1) \otimes \text{Vir}(\tilde{\omega}^2)$.

Since $U^1 \otimes U^2$ and $L(4/5, 2/3) \oplus L(6/7, 4/3)$ are the only irreducible modules of $U^1 \otimes U^2$ which have integral weights (cf. \cite{26, 29, 31}), by comparing dimensions of the homogeneous subspaces of small weights, we have the following proposition.

**Proposition 3.4.** As a module of $\text{Vir}(\tilde{\omega}^1) \otimes \text{Vir}(\tilde{\omega}^2)$,

$$U \cong \left( L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3) \right) \otimes \left( L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5) \right)$$

$$\oplus L(\frac{4}{5}, \frac{2}{3}) \otimes L(\frac{6}{7}, \frac{4}{3}) \oplus L(\frac{4}{5}, \frac{2}{3}) \otimes L(\frac{6}{7}, \frac{4}{3}).$$
By using Appendix A, we know that there are four distinct pairs of mutually orthogonal conformal vectors of central charge 4/5 and 6/7, namely, \( \{ \tilde{\omega}^1, \tilde{\omega}^2 \}, \{ x^j, y^j \}, j = 0, 1, 2 \), where
\[
x^j = \frac{1}{16} \tilde{\omega}^1 + \frac{7}{8} \tilde{\omega}^2 - \frac{1}{48} \xi^j X^1 - \frac{1}{48} \xi^{2j} X^2,
\]
and \( \xi = e^{2\pi\sqrt{-1}/3} \) is a primitive cubic root of unity.

**Lemma 3.5.** Let \( u = 135\tilde{\omega}^1 - 126\tilde{\omega}^2 - 13(X^1 + X^2) \) and \( v = X^1 - X^2 \). Then \( u \) and \( v \) are highest weight vectors of highest weight \( (2/3, 4/3) \) and \( (13/8, 3/8) \) with respect to \( \text{Vir}(x^0) \otimes \text{Vir}(y^0) \), respectively.

**Proof.** We have
\[
(x^0)_1 X^1 = (\frac{1}{16} \tilde{\omega}^1 + \frac{7}{8} \tilde{\omega}^2 - \frac{1}{48} X^1 - \frac{1}{48} X^2)_1 X^1
\]
\[= (\frac{1}{16} \times \frac{2}{3} + \frac{7}{8} \times \frac{4}{3}) - \frac{20}{48} X^2 - \frac{1}{48} (135\tilde{\omega}^1 + 252\tilde{\omega}^2)
\]
\[= \frac{29}{24} X^1 - \frac{5}{12} X^2 - \frac{1}{16} (45\tilde{\omega}^1 + 84\tilde{\omega}^2).
\]
Similarly,
\[
(x^0)_1 X^2 = \frac{29}{24} X^2 - \frac{5}{12} X^1 - \frac{1}{16} (45\tilde{\omega}^1 + 84\tilde{\omega}^2).
\]
Thus \( (x^0)_1 v = \frac{13}{8} v \) and hence \( v \) is a highest weight vector of highest weight \( (13/8, 3/8) \).

Furthermore,
\[
(x^0)_1 u = 135\left(\frac{1}{8} \tilde{\omega}^1 - \frac{1}{48} \times \frac{2}{3} (X^1 + X^2)\right) - 126\left(\frac{7}{4} \tilde{\omega}^2 - \frac{1}{48} \times \frac{4}{3} (X^1 + X^2)\right)
\]
\[= 13\left(\frac{19}{24} (X^1 + X^2) - \frac{1}{8} (45\tilde{\omega}^1 + 84\tilde{\omega}^2)\right)
\]
\[= \frac{2}{3} (135\tilde{\omega}^1 - 126\tilde{\omega}^2 - 13(X^1 + X^2)) = \frac{2}{3} u.
\]
Thus \( u \) is a highest weight vector of highest weight \( (2/3, 4/3) \). \( \square \)

**Proposition 3.6.** As a module of \( \text{Vir}(x^0) \otimes \text{Vir}(y^0) \),
\[
U \cong L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0) \oplus L(\frac{4}{5}, 3) \otimes L(\frac{6}{7}, 5) \oplus L(\frac{4}{5}, \frac{2}{3}) \otimes L(\frac{6}{7}, \frac{4}{3})
\]
\[\oplus L(\frac{4}{5}, \frac{13}{8}) \otimes L(\frac{6}{7}, \frac{3}{8}) \oplus L(\frac{4}{5}, \frac{1}{8}) \otimes L(\frac{6}{7}, \frac{23}{8}).
\]

**Proof.** The fixed point subalgebra \( U^+ \) of \( \theta \) contains \( \text{Vir}(x^0) \otimes \text{Vir}(y^0) \). Moreover, \( u \in U^+ \) and so \( U^+ \) contains a submodule of the form \( L(4/5, 2/3) \otimes L(6/7, 4/3) \) by Lemma 3.5. Hence comparing the first several terms of the characters, we know that
\[
U^+ \cong L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0) \oplus L(\frac{4}{5}, 3) \otimes L(\frac{6}{7}, 5) \oplus L(\frac{4}{5}, \frac{2}{3}) \otimes L(\frac{6}{7}, \frac{4}{3})
\]
\[\oplus L(\frac{4}{5}, \frac{13}{8}) \otimes L(\frac{6}{7}, \frac{3}{8}) \oplus L(\frac{4}{5}, \frac{1}{8}) \otimes L(\frac{6}{7}, \frac{23}{8}).
\]
as a module of \( \text{Vir}(x^0) \otimes \text{Vir}(y^0) \). Note that \( U^+ \) has the same form as a module of \( \text{Vir}(\tilde{\omega}^1) \otimes \text{Vir}(\tilde{\omega}^2) \). Lemma 3.5 also implies that \( U^- \) contains a submodule of the form \( L(4/5, 13/8) \otimes L(6/7, 3/8) \) since \( v \in U^- \). Thus

\[
U^- \cong L(\frac{4}{5}, \frac{13}{8}) \otimes L(\frac{6}{7}, \frac{3}{8}) \cong L(\frac{4}{5}, \frac{1}{15}) \otimes L(\frac{6}{7}, \frac{23}{8})
\]

and we have the desired result.

The VOA \( U \) has been constructed and studied by Sakuma and Yamauchi [44] (see also Miyamoto [42]). It is known that the automorphism group \( \text{Aut} U \) of \( U \) is isomorphic to the symmetric group \( S_3 \) and \( U \) is generated by two conformal vectors of central charge 1/2, namely, \( \hat{e} \) and \( \hat{f} \).

There are exactly six irreducible modules of \( U \) (cf. [44]), namely,

\[
M^0 \otimes A^0 \oplus M^1 \otimes A^1 \oplus M^2 \otimes A^2, \quad M^0 \otimes B^0 \oplus M^1 \otimes B^1 \oplus M^2 \otimes B^2,
\]

\[
M^0 \otimes C^0 \oplus M^1 \otimes C^1 \oplus M^2 \otimes C^2, \quad W^0 \otimes A^0 \oplus W^1 \otimes A^1 \oplus W^2 \otimes A^2,
\]

\[
W^0 \otimes B^0 \oplus W^1 \otimes B^1 \oplus W^2 \otimes B^2, \quad W^0 \otimes C^0 \oplus W^1 \otimes C^1 \oplus W^2 \otimes C^2,
\]

where

\[
M^0 \cong L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3), \quad M^1 \cong L(\frac{4}{5}, \frac{2}{3})^+, \quad M^2 \cong L(\frac{4}{5}, \frac{4}{3})^-,
\]

\[
W^0 \cong L(\frac{4}{5}, \frac{2}{5}) \oplus L(\frac{4}{5}, \frac{7}{5}), \quad W^1 \cong L(\frac{4}{5}, \frac{1}{15})^+, \quad W^2 \cong L(\frac{4}{5}, \frac{1}{15})^-.
\]

are the irreducible modules of \( L(4/5, 0) \oplus L(4/5, 3) \) and

\[
A^0 \cong L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5), \quad A^1 \cong L(\frac{6}{7}, \frac{4}{3})^+, \quad A^2 \cong L(\frac{6}{7}, \frac{4}{3})^-,
\]

\[
B^0 \cong L(\frac{6}{7}, 1) \oplus L(\frac{6}{7}, \frac{22}{7}), \quad B^1 \cong L(\frac{6}{7}, \frac{10}{21})^+, \quad B^2 \cong L(\frac{6}{7}, \frac{10}{21})^-,
\]

\[
C^0 \cong L(\frac{6}{7}, \frac{5}{7}) \oplus L(\frac{6}{7}, \frac{12}{7}), \quad C^1 \cong L(\frac{6}{7}, \frac{1}{21})^+, \quad C^2 \cong L(\frac{6}{7}, \frac{1}{21})^-.
\]

are the irreducible modules of \( L(6/7, 0) \oplus L(6/7, 5) \).

Both of \( L(4/5, 0) \oplus L(4/5, 3) \) and \( L(6/7, 0) \oplus L(6/7, 5) \) are rational VOAs and the fusion rules among their irreducible modules have been determined in [41] and [31]. There are two \( \mathbb{Z}_3 \)-symmetries given as follows.

\[
\tau_1 = \begin{cases} 
1 & \text{on } M^0, W^0, \\
e^{2\pi\sqrt{-1}/3} & \text{on } M^1, W^1, \\
e^{4\pi\sqrt{-1}/3} & \text{on } M^2, W^2,
\end{cases} \quad \tau_2 = \begin{cases} 
1 & \text{on } A^0, B^0, C^0, \\
e^{2\pi\sqrt{-1}/3} & \text{on } A^1, B^1, C^1, \\
e^{4\pi\sqrt{-1}/3} & \text{on } A^2, B^2, C^2.
\end{cases}
\]

If \( V \) is a VOA which contains \( U \) as a subalgebra, then both \( \tau_1 \) and \( \tau_2 \) induce automorphisms of \( V \). Moreover, as automorphisms of \( V \), \( \tau_1 = \tau_2 \) and we have that \( \tau_2 \tau_2 = \tau_1 \) is of order 3.
Remark 3.7. Recall that $\hat{e}$ and $\hat{f}$ are fixed by the Weyl group $W(\Phi) = W(A_2) \times W(E_6)$ of the root system $\Phi = A_2 \oplus E_6$ of $L$. Since $U$ is generated by $\hat{e}$ and $\hat{f}$, $W(\Phi)$ leaves every element of $U$ invariant. Let $g$ be an element of order 3 in $W(\Phi)$ which acts fixed-point-freely on $\sqrt{2}E_8$. Then $g$ induces a fixed-point-free action on the Leech lattice $\Lambda$ also (cf. [25]). Since every element of $U$ is fixed by $g$, $U$ is actually contained in the Moonshine VOA $V^2$ by the $\mathbb{Z}_3$-orbifold construction of $V^2$ given by Dong and Mason [11]. Thus, $\tau_{\hat{e}}\tau_{\hat{f}} = \tau_1$ is of class 3A (cf. [25, 41]).

In [42], Miyamoto showed that for any two conformal vectors $e$ and $f$ of central charge $1/2$ with $\langle e, f \rangle = 13/2^{10}$ in the Moonshine VOA $V^2$, the vertex subalgebra $W$ generated by $e$ and $f$ must contain a subalgebra of the form $L(4/5, 0) \otimes L(6/7, 0)$. In fact, Sakuma and Yamauchi [44] showed that the algebra $W$ must be isomorphic to $U$. This gives a proof that $U$ is contained in $V^2$.

3.4. 4A case. In this case, $L = L(3) = A_3 \oplus D_5$, $n_3 = |E_8/L| = 4$, and the conformal vectors $\tilde{\omega}^1 \in V_{\sqrt{2}A_3}$ and $\tilde{\omega}^2 \in V_{\sqrt{2}D_5}$ defined by (2.9) are both of central charge 1. Let $\epsilon_1, \ldots, \epsilon_8 \in \mathbb{R}^8$ be such that $\langle \epsilon_i, \epsilon_j \rangle = 2\delta_{ij}$ for any $i, j \in \{1, \ldots, 8\}$. Then

$$\sqrt{2}E_8 = \left\{ \sum_{i=1}^{8} a_i \epsilon_i \mid \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \frac{1}{2} + \mathbb{Z} \right\}.$$  

Let

$$K = \left\{ \sum_{i=1}^{8} a_i \epsilon_i \mid \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \frac{1}{2} + \mathbb{Z} \right\}.$$  

Then $\sqrt{2}E_8 \subset K$ and $|K/\sqrt{2}E_8| = 2$.

Now we consider two automorphisms $\theta$ and $\psi$ of $V_K$ defined by

$$\theta(\alpha(-1) \cdot 1) = -\alpha(-1) \cdot 1 \quad \text{and} \quad \theta(e^\alpha) = e^{-\alpha},$$

$$\psi(u \otimes e^\beta) = (-1)^{\langle \epsilon_1 + \cdots + \epsilon_8, \beta \rangle/2} u \otimes e^\beta$$

for any $\alpha, \beta \in K$ and $u \in M(1)$. Note that $V_{\sqrt{2}E_8} = (V_K)^\psi$. It is well known (cf. [17, Chapter 10]) that $\theta$ and $\psi$ are conjugate in $\text{Aut} V_K$. Thus, we have

$$V_{\sqrt{2}E_8} = (V_K)^\psi \cong V_K^+.$$  

By using the same argument as in Dong et al. [8, 9], one can show that the VOA $V_{\sqrt{2}A_3}$ contains a subalgebra isomorphic to $V_{\sqrt{2}A_2}^+ \otimes V_{\mathbb{Z}_7^3}^+$ and the VOA $V_{\sqrt{2}D_5}$ contains a subalgebra isomorphic to $V_{\sqrt{2}A_4}^+ \otimes V_{\mathbb{Z}_7^5}^+$ where $\sqrt{2}A_2 \cong \text{span}_{\mathbb{Z}} \{-\epsilon_1 + \epsilon_2, -\epsilon_2 + \epsilon_3\}$, $\gamma_3 = \epsilon_1 + \epsilon_2 + \epsilon_3$, $\sqrt{2}A_4 \cong \text{span}_{\mathbb{Z}} \{-\epsilon_4 + \epsilon_5, -\epsilon_5 + \epsilon_6, -\epsilon_6 + \epsilon_7, -\epsilon_7 + \epsilon_8\}$ and $\gamma_5 = \epsilon_4 + \epsilon_5 + \epsilon_6 + \epsilon_7 + \epsilon_8$. Moreover, $U$ contains a subalgebra isomorphic to $V_{\mathbb{Z}_7^3}^+ \otimes V_{\mathbb{Z}_7^5}^+$. Define

$$N = \{ x \in K \mid \langle x, -\epsilon_j + \epsilon_{j+1} \rangle = 0 \text{ for } j = 1, 2, 4, 5, 6, 7 \}.$$
Then 
\[ V_\mathcal{N} = \{ v \in V_K \mid u_n v = 0 \text{ for any } u \in V_{\sqrt{2}A_2} \text{ or } V_{\sqrt{3}A_4} \text{ and } n \geq 0 \}. \]
Since \( U = (V_\mathcal{N})^\psi \), it is now easy to see that \( U \cong V_\mathcal{N}^+. \) Note that 
\[ \mathcal{N} = \left\{ \sum_{i=1}^{8} a_i e_i \mid \text{all } a_i \in \mathbb{Z} \text{ or all } a_i \in \frac{1}{2} + \mathbb{Z} \text{ with } a_1 + a_2 + a_3 = a_4 + a_5 + a_6 + a_7 + a_8 = 0 \right\}. \]
It is of rank two and generated by the elements 
\[ \xi_1 = \frac{1}{2}(\gamma_3 + \gamma_5), \quad \xi_2 = \frac{1}{2}(-\gamma_3 + \gamma_5). \]

**Proposition 3.8.** The VOA \( V_\mathcal{N}^+ \) is generated by its weight 2 subspace \((V_\mathcal{N}^+)_2\).

**Proof.** Recall that \( V_\mathcal{N}^+ \) contains a subalgebra isomorphic to \( V_{\gamma_3}^+ \otimes V_{\gamma_5}^+ \). Moreover, 
\[ V_\mathcal{N}^+ = (V_{\gamma_3}^+ \otimes V_{\gamma_5}^-) \oplus (V_{\gamma_3}^- \otimes V_{\gamma_5}^+) \oplus (V_{\gamma_3/2+\gamma_5}^+ \otimes V_{\gamma_5/2+\gamma_5}^-) \oplus (V_{\gamma_3/2+\gamma_5}^- \otimes V_{\gamma_5/2+\gamma_5}^+) \]
Note that \((V_\mathcal{N}^+)_2\) is of dimension 5 and it has a basis consisting of the following five elements. 
\[ \beta_1(-1)^2 \cdot 1, \quad \beta_2(-1)^2 \cdot 1, \quad \beta_1(-1)\beta_2(-1) \cdot 1, \quad e^{\xi_1} + e^{-\xi_1}, \quad e^{\xi_2} + e^{-\xi_2}, \]
where \( \beta_1 = \frac{1}{\sqrt{6}} \gamma_3 \) and \( \beta_2 = \frac{1}{\sqrt{10}} \gamma_5 \). Let \( W \) be the subalgebra generated by \((V_\mathcal{N}^+)_2\). We want to show that \( W = V_\mathcal{N}^+ \). Since the minimal weights of \( V_{\gamma_3}^+ \otimes V_{\gamma_5}^- \), \( V_{\gamma_3/2+\gamma_5}^+ \otimes V_{\gamma_5/2+\gamma_5}^- \), and \( V_{\gamma_3/2+\gamma_5}^- \otimes V_{\gamma_5/2+\gamma_5}^+ \) are all equal to 2 and since they are irreducible \( V_{\gamma_3}^+ \otimes V_{\gamma_5}^- \)-modules, it suffices to show that \( V_{\gamma_3}^+ \otimes V_{\gamma_5}^- \subset W \).

For any rank one even lattice \( \mathbb{Z}a \) with \( \langle \alpha, \alpha \rangle = 2k \), it was shown in [5] that the VOA \( V_{\mathbb{Z}a}^+ \) is generated by three elements 
\[ \omega = \frac{1}{2} \beta(-1)^2 \cdot 1, \]
\[ J = \beta(-1)^4 \cdot 1 - 2\beta(-3)\beta(-1) \cdot 1 + \frac{3}{2} \beta(-2)^2 \cdot 1, \quad (3.5) \]
\[ E = e^\alpha + e^{-\alpha}, \]
where \( \beta = \frac{1}{\sqrt{2k}} \alpha \).

Let \( J_1 \) and \( J_2 \) be the elements obtained by replacing \( \beta \) with \( \beta_1 \) and \( \beta_2 \) in the element \( J \) of (3.5), respectively. Likewise, let \( E_1 = e^{\gamma_3} + e^{-\gamma_3} \) and \( E_2 = e^{\gamma_5} + e^{-\gamma_5} \). We only need to show that \( W \) contains \( J_1, J_2, E_1, \) and \( E_2 \).

By direct computation, we have 
\[ (\beta_1(-1)\beta_2(-1) \cdot 1)_{-1}(\beta_1(-1)\beta_2(-1) \cdot 1) \]
\[ = \beta_1(-3)\beta_1(-1) \cdot 1 + \beta_2(-3)\beta_2(-1) \cdot 1 + \beta_1(-1)^2\beta_2(-1)^2 \cdot 1 \]
and 
\[ (\beta_1(-1)^2 \cdot 1)_{-1}(\beta_2(-1)^2 \cdot 1) = \beta_1(-1)^2\beta_2(-1)^2 \cdot 1. \]
Thus, \( \beta_1(-3)\beta_1(-1) \cdot 1 + \beta_2(-3)\beta_2(-1) \cdot 1 \in W \) and
\[
\beta_1(-3)\beta_1(-1) \cdot 1 = \frac{1}{8}(\beta_1(-1)^2 \cdot 1)_1(\beta_1(-3)\beta_1(-1) \cdot 1 + \beta_2(-3)\beta_2(-1) \cdot 1)
\]
is also contained in \( W \). Moreover,
\[
(\beta_1(-1)^2 \cdot 1)_1\beta_1(-1) \cdot 1 = \beta_1(-1)^4 \cdot 1 + 4\beta_1(-3)\beta_1(-1) \cdot 1,
\]
\[
((\beta_1(-1)^2 \cdot 1)_0)^2(\beta_1(-1)^2 \cdot 1) = 16\beta_1(-3)\beta_1(-1) \cdot 1 + 8\beta_1(-2)^2 \cdot 1.
\]
Hence, we have \( \beta_1(-1)^4 \cdot 1, \beta_1(-2)^2 \cdot 1 \in W \) and thus \( J_1 \in W \). Similarly, \( J_2 \in W \).

Clearly, \( W \) contains
\[
E_1 = e^{\gamma_3} + e^{-\gamma_3} = (e^{\xi_1} + e^{-\xi_1})_0(e^{\xi_2} + e^{-\xi_2}).
\]
Furthermore, \( W \) contains the following three elements.
\[
(\gamma_3(-1)^2 \cdot 1)_1E_1 = \gamma_3(-1)^2E_1 + 12\gamma_3(-2)(e^{\gamma_3} - e^{-\gamma_3}),
\]
\[
(\gamma_5(-1)^2 \cdot 1)_1E_1 = \gamma_5(-1)^2E_1,
\]
\[
(\gamma_3(-1)\gamma_5(-1) \cdot 1)_1E_1 = \gamma_3(-1)\gamma_5(-1)E_1 + 6\gamma_5(-2)(e^{\gamma_3} - e^{-\gamma_3}).
\]
Then \( W \) also contains
\[
(\gamma_3(-1)\gamma_5(-1) \cdot 1)_1(\gamma_3(-1)\gamma_5(-1)E_1 + 6\gamma_5(-2)(e^{\gamma_3} - e^{-\gamma_3}))
\]
\[
= 10(\gamma_3(-1)^2E_1 + 12\gamma_3(-2)(e^{\gamma_3} - e^{-\gamma_3})) + 6\gamma_5(-1)^2E_1 + 6\gamma_5(-2)(e^{\gamma_3} - e^{-\gamma_3}).
\]
Hence, we have \( \gamma_5(-2)(e^{\gamma_3} - e^{-\gamma_3}) \in W \) and so
\[
\gamma_3(-2)(e^{\gamma_3} - e^{-\gamma_3}) = \frac{1}{20}(\gamma_3(-1)\gamma_5(-1) \cdot 1)_1(\gamma_5(-2)(e^{\gamma_3} - e^{-\gamma_3}))
\]
is also contained in \( W \). Therefore, \( \gamma_3(-1)^2E_1, \gamma_5(-1)^2E_1, \) and \( \gamma_5(-1)\gamma_3(-1)E_1 \) are contained in \( W \).

Finally, \( W \) contains
\[
(e^{\xi_1} + e^{-\xi_1})_{-2}(e^{\xi_2} + e^{-\xi_2})
\]
\[
= E_2 + \frac{1}{2}\xi_1(-1)^2E_1 + \frac{1}{2}\xi_1(-2)(e^{\gamma_3} - e^{-\gamma_3})
\]
\[
= E_2 + \frac{1}{8}(\gamma_3 + \gamma_5)(-1)^2E_1 + \frac{1}{4}(\gamma_3 + \gamma_5)(-2)(e^{\gamma_3} - e^{-\gamma_3})
\]
and thus \( E_2 \in W \).

**Theorem 3.9.** (1) The Griess algebra \( U_2 \) of \( U \) is generated by \( \hat{e} \) and \( \hat{f} \).
(2) The coset subalgebra \( U \) is generated by \( \hat{e} \) and \( \hat{f} \).

**Proof.** We only need to show the first assertion. Note that \( U_2 = \text{span}_C\{\tilde{\omega}^1, \tilde{\omega}^2, X^1, X^2, X^3\} \)
is of dimension 5 and that
\[
\hat{e} = \tilde{\omega}^1 + \tilde{\omega}^2 + X^1 + X^2 + X^3,
\]
\[
\hat{f} = \tilde{\omega}^1 + \tilde{\omega}^2 + \sqrt{-1}X^1 - X^2 - \sqrt{-1}X^3.
\]
Let $G$ be the Griess subalgebra generated by $\hat{e}$ and $\hat{f}$. Then $\hat{e}_1 \hat{f}, \hat{e}_1 (\hat{e}_1 \hat{f}),$ and $\hat{f}_1 (\hat{e}_1 \hat{f})$ are also in $G$. By direct computation, it is easy to see that $\hat{e}, \hat{f}, \hat{e}_1 \hat{f}, \hat{e}_1 (\hat{e}_1 \hat{f}),$ and $\hat{f}_1 (\hat{e}_1 \hat{f})$ are linearly independent. Thus $G = U_2$. 

\textbf{Theorem 3.10.} The automorphism group $\text{Aut } U$ of $U$ is a dihedral group of order 8.

\textbf{Proof.} There are exactly four conformal vectors of central 1/2 in $U_2$, namely,

$$e_j = \sigma^j \hat{e} = \frac{3}{16} \hat{\omega}^1 + \frac{5}{16} \hat{\omega}^2 + \frac{1}{32} ((\sqrt{-1})^j X^1 + (-1)^j X^2 + (-\sqrt{-1})^j X^3), \quad 0 \leq j \leq 3.$$

Since $U$ is generated by $\hat{e}$ and $\hat{f}$, we can consider $\text{Aut } U$ as a subgroup of the permutation group on the set $\{e_0, e_1, e_2, e_3\}$. Now let $g \in \text{Aut } U$. Then $g$ also preserves the inner product and thus

$$\langle ge_i, ge_j \rangle = \langle e_i, e_j \rangle = \begin{cases} 1/2^7 & \text{if } i + j \text{ is odd,} \\ 0 & \text{if } i + j \text{ is even.} \end{cases}$$

Hence, $g$ will either keep $\{e_0, e_2\}$ and $\{e_1, e_3\}$ invariant or $g$ will map $\{e_0, e_2\}$ to $\{e_1, e_3\}$. Thus $|\text{Aut } U| \leq 8$. On the other hand, $\theta$ and $\sigma$ defined in Remark 2.2 generate a subgroup isomorphic to a dihedral group of order 8 inside $\text{Aut } U$. Hence the assertion holds. \qed

Note that $\langle \xi_1, \xi_1 \rangle = \langle \xi_2, \xi_2 \rangle = 4$, $\langle \xi_1, \xi_2 \rangle = -1$ and the Leech lattice $\Lambda$ does contain some sublattice isomorphic to $\mathcal{N}$. Therefore, $U \subset V_\Lambda^+ \subset V^4$.

In [1], the fusion rules for $V_{17}^+$ and $V_{27}^+$ are determined. It is known that there are $\mathbb{Z}_4$-symmetries among the irreducible modules of $V_{17}^+$ and also among the irreducible modules of $V_{27}^+$. By direct computation, it is easy to verify that the automorphism $\tau_{\hat{e}} \tau_{\hat{f}}$ agrees with the $\mathbb{Z}_4$-symmetries of $V_{17}^+$ and $V_{27}^+$ and thus $\tau_{\hat{e}} \tau_{\hat{f}}$ is of class $4A$ (cf. [37]). An explicit construction of $4A$-elements as automorphisms of $V^4$ has already been obtained by Shimakura [45].

3.5. 5A case. In this case, $L = L(4) \cong A_4 \oplus A_4$, $n_4 = |E_8/L| = 5$, and the conformal vectors $\tilde{\omega}^1 \in V_{\sqrt{5}A_4}$ and $\tilde{\omega}^2 \in V_{\sqrt{2}A_4}$ defined by (2.9) are both of central charge $8/7$.

\textbf{Lemma 3.11 (Lemma A.7).} The coset subalgebra $U$ contains a set of three mutually orthogonal conformal vectors of central charge $1/2, 25/28,$ and $25/28$, respectively, namely,

$$u = \hat{e} = \frac{7}{32}(\hat{\omega}^1 + \hat{\omega}^2) + \frac{1}{32}(X^1 + X^2 + X^3 + X^4),$$

$$v = \frac{15}{64} \hat{\omega}^1 + \frac{35}{64} \hat{\omega}^2 - \frac{3}{64} (X^1 + X^4) + \frac{1}{64} (X^2 + X^3),$$

$$w = \frac{35}{64} \hat{\omega}^1 + \frac{15}{64} \hat{\omega}^2 + \frac{1}{64} (X^1 + X^4) - \frac{3}{64} (X^2 + X^3).$$

By the above lemma, $U$ contains $\text{Vir}(u) \otimes \text{Vir}(v) \otimes \text{Vir}(w) \cong L(1/2, 0) \otimes L(25/28, 0) \otimes L(25/28, 0)$. All irreducible modules of $\text{Vir}(u) \otimes \text{Vir}(v) \otimes \text{Vir}(w)$ are known (cf. [13, 48]).
They are of the form \( L(1/2, h_1) \otimes L(25/28, h_2) \otimes L(25/28, h_3) \). Among them, the highest weights \((h_1, h_2, h_3)\) of the irreducible modules which have integral weights are as follows.

\[
(0, 0, 0), \left( \frac{1}{16}, \frac{5}{32}, \frac{57}{32} \right), \left( \frac{1}{16}, \frac{57}{32}, \frac{5}{32} \right), \left( \frac{1}{2}, \frac{3}{4}, \frac{3}{4} \right), (0, \frac{3}{4}, \frac{3}{4}), (0, \frac{13}{4}, \frac{3}{4}), \left( \frac{1}{16}, \frac{57}{32}, \frac{165}{32} \right), \left( \frac{1}{16}, \frac{165}{32}, \frac{57}{32} \right), \left( \frac{1}{2}, \frac{13}{4}, \frac{13}{4} \right), \left( \frac{1}{2}, 0, \frac{15}{2} \right), \left( \frac{1}{2}, \frac{15}{2}, 0 \right), (0, \frac{15}{2}, \frac{15}{2}).
\]

The following lemma can be proved by direct computation.

**Lemma 3.12.** Let

\[
a = (\tilde{\omega}^1 - \tilde{\omega}^2) - \frac{1}{35}(X^1 + X^4) + \frac{1}{35}(X^2 + X^3),
\]

\[
b^1 = 3(X^1 - X^4) + 2(X^2 - X^3),
\]

\[
b^2 = 2(X^1 - X^4) - 3(X^2 - X^3).
\]

Then \(a, b^1\) and \(b^2\) are highest weight vectors of highest weight \((\frac{1}{2}, \frac{3}{4}, \frac{3}{4}), (\frac{1}{16}, \frac{57}{32}, \frac{5}{32})\) and \((\frac{1}{16}, \frac{57}{32}, \frac{5}{32})\) with respect to \(\text{Vir}(u) \otimes \text{Vir}(v) \otimes \text{Vir}(w)\), respectively.

We denote the irreducible module \(L(1/2, h_1) \otimes L(25/28, h_2) \otimes L(25/28, h_3)\) by \([h_1, h_2, h_3]\) for simplicity of notation. By using the theory of characters (cf. Appendix B), we actually have the following decomposition of \(U\) into a direct sum of \([h_1, h_2, h_3]\)'s.

**Theorem 3.13** (Theorem B.7). As a module of \(\text{Vir}(u) \otimes \text{Vir}(v) \otimes \text{Vir}(w)\),

\[
U \cong [0, 0, 0] \oplus \left[ \frac{1}{16}, \frac{5}{32}, \frac{57}{32} \right] \oplus \left[ \frac{1}{16}, \frac{57}{32}, \frac{5}{32} \right] \oplus \left[ \frac{1}{2}, \frac{3}{4}, \frac{3}{4} \right] \\
\oplus \left[ 0, \frac{3}{4}, \frac{13}{4} \right] \oplus \left[ 0, \frac{13}{4}, \frac{3}{4} \right] \oplus \left[ \frac{1}{16}, \frac{57}{32}, \frac{165}{32} \right] \oplus \left[ \frac{1}{16}, \frac{165}{32}, \frac{57}{32} \right] \\
\oplus \left[ \frac{1}{2}, \frac{13}{4}, \frac{13}{4} \right] \oplus \left[ \frac{1}{2}, 0, \frac{15}{2} \right] \oplus \left[ \frac{1}{2}, \frac{15}{2}, 0 \right] \oplus \left[ 0, \frac{15}{2}, \frac{15}{2} \right].
\]

Next, we shall discuss the generators of \(U\).

**Theorem 3.14.** The coset subalgebra \(U\) is generated by its weight 2 subspace \(U_2\).

We shall divide the proof into several steps. By direct computation, we can verify the following lemma.

**Lemma 3.15.** Let

\[
y^1 = a_{-1}a - \frac{192}{343}u_{-3} \cdot 1 - \frac{128}{343}u_{-1}u - \frac{192}{455}v_{-3} \cdot 1 - \frac{128}{325}v_{-1}v - \frac{192}{455}w_{-3} \cdot 1 - \frac{128}{325}w_{-1}w,
\]

\[
y^2 = (b^1)_{-1}b^2 + 105u_{-1}a - \frac{735}{2}v_{-1}a + \frac{5355}{8}v_0v_0a - \frac{735}{2}w_{-1}a + \frac{1715}{8}w_0w_0a
\]

\[+ \frac{665}{2}u_0v_0a - \frac{245}{2}u_0w_0a - \frac{4655}{12}v_0w_0a.
\]

Then \(y^1\) and \(y^2\) are non-zero singular vectors for \(\text{Vir}(u) \otimes \text{Vir}(v) \otimes \text{Vir}(w)\), that is, \(u_ny^j = v_ny^j = w_ny^j = 0, j = 1, 2,\) for any \(n \geq 2\).
Now let $W$ be the subalgebra of $U$ generated by $U_2$.

**Lemma 3.16.** There are highest weight vectors of highest weight $(0, \frac{3}{4}, \frac{13}{4})$ and $(0, \frac{13}{4}, \frac{3}{4})$

with respect to $\text{Vir}(u) \otimes \text{Vir}(v) \otimes \text{Vir}(w)$ in $W$.

**Proof.** By fusion rules, we know that

\[a_{-1}a \in [0, 0, 0]_4 \oplus [0, \frac{3}{4}, \frac{13}{4}]_4 \oplus [0, \frac{13}{4}, \frac{3}{4}]_4,\]

\[\langle b^1 \rangle - b^2 \in \left[\frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right]_4 \oplus [0, \frac{3}{4}, \frac{13}{4}]_4 \oplus [0, \frac{13}{4}, \frac{3}{4}]_4.\]

Since both $y^1$ and $y^2$ are singular vectors of weight 4, we have

\[y^1, y^2 \in [0, \frac{3}{4}, \frac{13}{4}] \oplus [0, \frac{13}{4}, \frac{3}{4}]_4.\]

Moreover, by direct computation, one can show that $\langle y^1, y^2 \rangle = \langle a_{-1}a, \langle b^1 \rangle - b^2 \rangle = 0$. Note that $a, b^1$, and $b^2$ are highest weight vectors and $\langle a, b^1 \rangle = \langle a, b^2 \rangle = \langle b^1, b^2 \rangle = 0$. Hence $y^1$ and $y^2$ are linearly independent. Thus the assertion holds. \(\square\)

**Proof of Theorem 3.14.**

First, we note that the coset subalgebra $U$ is simple. Then the subalgebra $U^{\tau_u}$ consisting of the fixed points of the Miyamoto involution $\tau_u$ associated with $u$ in $U$ is also simple. As a module of $\text{Vir}(u) \otimes \text{Vir}(v) \otimes \text{Vir}(w)$,

\[U^{\tau_u} \cong [0, 0, 0]_4 \oplus [0, \frac{3}{4}, \frac{13}{4}]_4 \oplus [0, \frac{13}{4}, \frac{3}{4}]_4 \oplus [0, \frac{15}{4}, \frac{15}{4}]_4 \oplus \left[\frac{1}{2}, \frac{3}{4}, \frac{3}{4}\right]_4 \oplus \left[\frac{1}{2}, \frac{13}{4}, \frac{13}{4}\right]_4 \oplus \left[\frac{1}{2}, 0, \frac{15}{2}\right]_4 \oplus \left[\frac{1}{2}, \frac{1}{2}, 0\right].\]

Moreover, we can define an automorphism $\sigma_u$ on $U^{\tau_u}$ (cf. Miyamoto [39]) by

\[\sigma_u = \begin{cases} 1 & \text{on } L(\frac{1}{2}, 0), \\ -1 & \text{on } L(\frac{1}{2}, \frac{1}{2}). \end{cases}\]

Then the subalgebra $(U^{\tau_u})^{\sigma_u}$ consisting of the fixed points of $\sigma_u$ in $U^{\tau_u}$ is again simple. As a module of $\text{Vir}(u) \otimes \text{Vir}(v) \otimes \text{Vir}(w)$,

\[(U^{\tau_u})^{\sigma_u} \cong [0, 0, 0]_4 \oplus [0, \frac{3}{4}, \frac{13}{4}]_4 \oplus [0, \frac{13}{4}, \frac{3}{4}]_4 \oplus [0, \frac{15}{4}, \frac{15}{4}]_4.\]

By the previous lemma, we know that $W$ contains $[0, 0, 0], [0, 3/4, 13/4], [0, 13/4, 3/4], [1/2, 3/4, 3/4], [1/16, 5/32, 57/32]$ and $1/16, 57/32, 5/32]$. Hence, $(W^{\tau_u})^{\sigma_u}$ must contain an irreducible module isomorphic to $[0, 15/2, 15/2]$; otherwise $(W^{\tau_u})^{\sigma_u} \cong [0, 0, 0]_4 \oplus [0, 3/4, 13/4]_4 \oplus [0, 13/4, 3/4]_4$ and the orthogonal complement of $(W^{\tau_u})^{\sigma_u}$ in $(U^{\tau_u})^{\sigma_u}$ with respect to a positive definite invariant hermitian form is isomorphic to $[0, 15/2, 15/2]$. However, the orthogonal complement is a module for $(W^{\tau_u})^{\sigma_u}$ because of the invariance of the form, which is impossible by the fusion rules. Thus $(W^{\tau_u})^{\sigma_u} = (U^{\tau_u})^{\sigma_u}$. Then we also have $W^{\tau_u} = U^{\tau_u}$ since $U^{\tau_u}$ is a direct sum of two irreducible $(W^{\tau_u})^{\sigma_u}$-modules and
$W'_{\nu} \neq (W'_{\nu})_{\nu}$. Hence, $W$ contains all the simple current $\text{Vir}(u) \otimes \text{Vir}(v) \otimes \text{Vir}(w)$-modules. Now it is easy to see that $W = U$ by the fusion rules.

**Theorem 3.17.** (1) The Griess algebra $U_2$ of $U$ is generated by $\hat{e}$ and $\hat{f}$.

(2) The coset subalgebra $U$ is generated by $\hat{e}$ and $\hat{f}$.

**Proof.** By Theorem 3.14, it suffices to show the first assertion. Let $G$ be the Griess subalgebra generated by $\hat{e}$ and $\hat{f}$. Then by direct computation, we can verify that $\hat{e}, \hat{f}, \hat{e}_1 \hat{f}, \hat{f}_1 (\hat{e}_1 \hat{f}), \hat{e}_1 (\hat{f}_1 (\hat{e}_1 \hat{f})))$ are linearly independent. Thus $G = U_2$, since $\dim U_2 = 6$.

**Theorem 3.18.** The automorphism group $\text{Aut } U$ of $U$ is a dihedral group of order 10.

**Proof.** Recall that $\sigma$ and $\theta$ defined in Remark 2.2 generate a subgroup isomorphic to a dihedral group of order 10 in $\text{Aut } U$. By Lemma A.6, there are exactly five conformal vectors of central charge $1/2$ in $U$, namely, $e_j = \sigma^j \hat{e}, 0 \leq j \leq 4$. Since $U$ is generated by $\hat{e}$ and $\hat{f} = \sigma \hat{e}$, $\text{Aut } U$ can be considered as a subgroup of the permutation group on the five elements set $\{e_0, e_1, e_2, e_3, e_4\}$. In fact, $U$ is generated by any 2 distinct elements in the set $\{e_0, e_1, e_2, e_3, e_4\}$. Thus $\text{Aut } U$ can not contain any 2-cycle nor 3-cycle since such automorphisms must fix at least two elements in the set and thus fix the whole $U$. Hence there is no element of order 6 either. Now let $(i_0, i_1, i_2, i_3, i_4)$ be a 4-cycle. Then

$$(i_0, i_1, i_2, i_3, i_4) = (i_0, i_1, i_2, i_3, i_4)(i_0, i_1, i_2, i_3, i_4).$$

Since $(i_0, i_1, i_2, i_3, i_4) \in \text{Aut } U$ but $(i_0, i_1, i_2, i_3, i_4) \notin \text{Aut } U$, there is no 4-cycle in $\text{Aut } U$. Therefore, the only possible elements are 5-cycles and the products of two disjoint 2-cycles. Thus the assertion holds.

**Theorem 3.19.** There are exactly nine irreducible modules $U(i, j), i, j = 1, 3, 5$ for $U$. As $\text{Vir}(u) \otimes \text{Vir}(v) \otimes \text{Vir}(w)$-modules, they are of the following form.

$$U(i, j) \cong [0, h_{i,1}, h_{j,1}] \oplus [0, h_{i,3}, h_{j,3}] \oplus [0, h_{i,5}, h_{j,5}] \oplus [0, h_{i,7}, h_{j,7}]$$

$$\oplus \left[ \frac{1}{2}, h_{i,1}, h_{j,7} \right] \oplus \left[ \frac{1}{2}, h_{i,3}, h_{j,3} \right] \oplus \left[ \frac{1}{2}, h_{i,5}, h_{j,5} \right] \oplus \left[ \frac{1}{2}, h_{i,7}, h_{j,1} \right]$$

$$\oplus \left[ \frac{1}{16}, h_{i,2}, h_{j,4} \right] \oplus \left[ \frac{1}{16}, h_{i,4}, h_{j,2} \right] \oplus \left[ \frac{1}{16}, h_{i,6}, h_{j,4} \right] \oplus \left[ \frac{1}{16}, h_{i,4}, h_{j,6} \right],$$

where $h_{r,s} = h_{r,s}^5$ is defined by (3.2).

The proof of this theorem will be given at the Appendix C. We shall first note that $U = U(1, 1)$ and the lattice VOA $V_{\sqrt{2}E_8}$ can be decomposed as follows (cf. Appendix B.2).

$$V_{\sqrt{2}E_8} \cong \bigoplus_{1 \leq k, k_1 \leq j \leq k + \frac{1}{2}} L(c_1, h_{k_0, k_1}) \otimes L(c_1, h_{\ell_0, \ell_1}) \otimes \cdots \otimes L(c_4, h_{k_3, k_4}) \otimes L(c_4, h_{\ell_3, \ell_4}) \otimes U(k_4, \ell_4),$$

where $c_m$ and $h_{r,s}^m$ are defined by (3.1) and (3.2), respectively.
Recall that \( \tau_\varepsilon \tau_f = e^{2\pi \sqrt{-1} \beta(0)} \) as automorphisms of \( V_{\sqrt{2}E_8} \) (cf. Eq. 3.4). It thus induces a natural action on each of the \( U(i, j), i, j = 1, 3, 5 \). Hence if a VOA \( V \) contains a subalgebra isomorphic to \( U \), then \( \tau_\varepsilon \tau_f \) will define an automorphism of order 5 on \( V \). The subalgebra \( U^{\tau_\varepsilon \tau_f} \) consisting of the fixed points of \( \tau_\varepsilon \tau_f \) in \( U \) is of the form

\[
U^{\tau_\varepsilon \tau_f} \cong W_5(8/7) \otimes W_5(8/7),
\]

where \( W_5(8/7) \) is a parafermion algebra of central charge 8/7 (cf. [32]). It is well known that \( W_5(8/7) \) possesses a \( \mathbb{Z}_5 \)-symmetry (cf. [6, 49]). The automorphism \( \tau_\varepsilon \tau_f \) in fact agrees with this symmetry.

**Remark 3.20.** Recall that \( \hat{e} \) and \( \hat{f} \) are fixed by the Weyl group \( W(\Phi) = W(A_4) \times W(A_4) \) of the root system \( \Phi = A_4 \oplus A_4 \) of \( L \). Since \( U \) is generated by \( \hat{e} \) and \( \hat{f} \), \( W(\Phi) \) acts trivially on \( U \). There is an element \( \psi \) of order 5 in \( W(\Phi) \) such that it induces a fixed-point-free action on \( \sqrt{2}E_8 \) and on the Leech lattice \( \Lambda \). Therefore, if the conjectured \( \mathbb{Z}_5 \)-orbifold construction of the Moonshine VOA \( V_\tau \) holds, then one can prove that \( U \) is contained in \( V_\tau \) by using \( \psi \) and that as an automorphism of \( V_\tau \), \( \tau_\varepsilon \tau_f \) is of class 5A (cf. [37]).

### 3.6 6A case.
In this case, \( L = L(5) \cong A_2 \oplus A_1 \oplus A_5 \), \( n_5 = |E_8/L| = 6 \), and the conformal vectors \( \hat{\omega}^1 \in V_{\sqrt{2}A_2}, \hat{\omega}^2 \in V_{\sqrt{2}A_1}, \) and \( \hat{\omega}^3 \in V_{\sqrt{2}A_5} \) defined by (2.9) are of central charge 4/5, 1/2, and 5/4, respectively. Let \( K = \text{span}_\mathbb{Z}\{\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_8\} \cong A_1 \oplus A_5, \)

\( J = \text{span}_\mathbb{Z}\{\alpha_6, \alpha_7\} \cong A_2, \) and \( \hat{\alpha} = \alpha_3 + 2\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8. \) Then \( \mathbb{Z}\hat{\alpha} + K = K \cup (\hat{\alpha} + K). \) We have \( \langle \alpha_j, \hat{\alpha} \rangle = 0 \) for \( j = 0, 1, 3, 4, 6, 7 \), \( \langle \alpha_2, \hat{\alpha} \rangle = \langle \alpha_8, \hat{\alpha} \rangle = -1 \), \( \langle \alpha_i, \hat{\alpha} \rangle = 2, \) and \( \alpha_0 + \alpha_4 + \alpha_5 + 2\alpha_1 + 2\alpha_3 + 2\hat{\alpha} + 3\alpha_6 = 0. \) Hence \{\( \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \hat{\alpha}, \alpha_8 \)} forms an extended \( E_6 \) diagram and so \( K \cup (\hat{\alpha} + K) \cong E_6. \) Moreover, \( L = J \oplus K \) and \( \langle J, \mathbb{Z}\hat{\alpha} + K \rangle = 0. \) Therefore, we have isometric embeddings

\[
A_2 \oplus A_1 \oplus A_5 \subset A_2 \oplus E_6 \subset E_8.
\]

Then we obtain a conformal vector

\[
\hat{\omega}(E_6) = \frac{1}{168} \sum_{\alpha \in \Phi^+(E_6)} \alpha(-1)^2 \cdot 1 + \frac{1}{14} \sum_{\alpha \in \Phi^+(E_6)} (e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha})
\]

of central charge 6/7 in \( V_{\sqrt{2}E_6} \). By our embeddings, we also know that

\[
V_{\sqrt{2}E_6} \cong \{ v \in V_{\sqrt{2}E_3} \mid (s^1)_1 v = (\hat{\omega}^1)_1 v = 0 \}.
\]

If \( \alpha \in 3\alpha_5 + L \) satisfies \( \langle \alpha, \alpha \rangle = 2, \) then \( \alpha \in \hat{\alpha} + K \) since \( 3\alpha_5 + L = \hat{\alpha} + L \) and \( \langle J, \hat{\alpha} + K \rangle = 0. \) Thus \( \Phi(E_6) = \Phi(K) \cup \{ \alpha \in 3\alpha_5 + L \mid \langle \alpha, \alpha \rangle = 2 \}. \) Note also that the Virasoro element of \( V_{\sqrt{2}E_6} \) coincides with that of \( V_{\sqrt{2}K}. \) Now we can verify that

\[
\hat{\omega}(E_6) = \frac{2}{7} \hat{\omega}^2 + \frac{4}{7} \hat{\omega}^3 + \frac{1}{14} X^3 \in U.
\]
Let \( w^1 = \tilde{\omega}^1 \), \( w^2 = \tilde{\omega}(E_6) \), and \( w^3 = \tilde{\omega}^2 + \tilde{\omega}^3 - w^2 \). Then \( \{w^1, w^2, w^3\} \) is a set of mutually orthogonal conformal vectors of central charge \( 4/5, 6/7, \) and \( 25/28 \), respectively and the Virasoro element \( \omega' = \tilde{\omega}^1 + \tilde{\omega}^2 + \tilde{\omega}^3 \) of \( U \) is a sum of \( w^1, w^2, w^3 \). Note that

\[
s(E_6) = \frac{1}{28} \sum_{\alpha \in \Phi^+(E_6)} \left( \alpha(-1)^2 \cdot 1 - 2(e^{i\sqrt{2}x} + e^{-i\sqrt{2}x}) \right)
\]

is a linear combination of \( s^2 \), \( s^3 \), and \( w^3 \). Recall that \( s^1 = s(A_2) \), \( s^2 = s(A_1) \), and \( s^3 = s(A_5) \) are defined by (2.9).

Now let \( U' = \{ v \in V_{\sqrt{2}E_8} \mid (s^1)_1v = s(E_6)_1v = 0 \} \). Then

\[
U' = \{ v \in V_{\sqrt{2}E_8} \mid (s^1)_1v = (s^2)_1v = (s^3)_1v = (w^3)_1v = 0 \}
= \{ u \in U \mid (w^3)_1u = 0 \}.
\]

Hence \( U' \subset U \) and by using the results in the \( 3A \) case, we know that

\[
U' \cong \left( L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3) \right) \otimes \left( L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5) \right)
\]

\[
\oplus L(\frac{4}{5}, \frac{2}{3}) \otimes L(\frac{6}{7}, \frac{4}{3}) \oplus L(\frac{4}{5}, \frac{2}{3}) \otimes L(\frac{6}{7}, \frac{4}{3}).
\]

Note that \( U' \otimes L(25/28, 0) \), \( U'(5/7) \otimes L(25/28, 9/7) \), and \( U'(1/7) \otimes L(25/28, 34/7) \) are the only irreducible \( U' \otimes L(25/28, 0) \)-modules which have integral weights (cf. [44]), where

\[
U'(\frac{1}{7}) \cong \left( L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3) \right) \otimes \left( L(\frac{6}{7}, \frac{1}{7}) \oplus L(\frac{6}{7}, \frac{22}{7}) \right)
\]

\[
\oplus L(\frac{4}{5}, \frac{2}{3}) \otimes L(\frac{6}{7}, \frac{10}{21}) \oplus L(\frac{4}{5}, \frac{2}{3}) \otimes L(\frac{6}{7}, \frac{10}{21}),
\]

\[
U'(\frac{5}{7}) \cong \left( L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3) \right) \otimes \left( L(\frac{6}{7}, \frac{5}{7}) \oplus L(\frac{6}{7}, \frac{12}{7}) \right)
\]

\[
\oplus L(\frac{4}{5}, \frac{2}{3}) \otimes L(\frac{6}{7}, \frac{1}{21}) \oplus L(\frac{4}{5}, \frac{2}{3}) \otimes L(\frac{6}{7}, \frac{1}{21}).
\]

By direct computation, it is straightforward to verify that \( U \) is generated by its weight 2 subspace and it contains some highest weight vectors of weight \( 5/7 \) and \( 1/7 \) with respect to \( U' \) (cf. see Appendix B.3.1 for details). Thus we have the following theorem.

**Theorem 3.21.** As a module of \( U' \otimes L(25/28, 0) \),

\[
U \cong U' \otimes L(\frac{25}{28}, 0) \oplus U'(\frac{5}{7}) \otimes L(\frac{25}{28}, \frac{9}{7}) \oplus U'(\frac{1}{7}) \otimes L(\frac{25}{28}, \frac{34}{7}).
\]

Moreover, \( U \) is generated by its weight 2 subspace \( U_2 \).

**Theorem 3.22.** (1) The Griess algebra \( U_2 \) of \( U \) is generated by \( \hat{e} \) and \( \hat{f} \).

(2) The coset subalgebra \( U \) is generated by \( \hat{e} \) and \( \hat{f} \).

**Proof.** Let \( \mathcal{G} \) be the Griess subalgebra generated by \( \hat{e} \) and \( \hat{f} \). By (A.7), it is easy to verify that \( \hat{e}, \hat{f}, \hat{e}_1 \hat{f}, \hat{e}_1 (\hat{e}_1 \hat{f}) \), \( \hat{f}_1 (\hat{e}_1 \hat{f}) \), \( \hat{e}_1 (\hat{f}_1 (\hat{e}_1 \hat{f})) \), \( \hat{e}_1 (\hat{f}_1 (\hat{e}_1 \hat{f})) \) are linearly
independent. Thus $G = U_2$, since $\dim U_2 = 8$. The second assertion follows from the preceding theorem. \qed

**Theorem 3.23.** The automorphism group $\Aut U$ of $U$ is a dihedral group of order 12.

**Proof.** By Lemma A.9, there are exactly seven conformal vectors of central charge $1/2$ in $U$, namely, $\tilde{\omega}^2$ and $e_j = \sigma^j \hat{e}, 0 \leq j \leq 5$. Moreover, we have $\langle \tilde{\omega}^2, e_j \rangle = 1/32$ for any $0 \leq j \leq 5$ and

$$
\langle e_i, e_j \rangle = \begin{cases} 
1/4 & \text{if } i = j, \\
1/32 & \text{if } i - j \equiv 3 \mod 6, \\
13/2^{10} & \text{if } i - j \equiv 2 \mod 6, \\
5/2^{10} & \text{if } i - j \equiv 1 \mod 6.
\end{cases}
$$

(3.6)

Since $U$ is generated by $\hat{e}$ and $\hat{f} = \sigma \hat{e}$, $\Aut U$ can be considered as a subgroup of the permutation group on the set $\{ \tilde{\omega}^2, e_0, e_1, \ldots, e_5 \}$. Now let $g \in \Aut U$. Then $g$ must preserve the inner product and so $g$ fixes $\tilde{\omega}^2$. Let $ge_j = e_{\nu(j)}, 0 \leq j \leq 5$. Then by (3.6), $\nu(i) - \nu(j) \equiv \pm (i - j) \mod 6$ for any $i, j = 0, 1, \ldots, 5$. Hence there are only 12 possible choices for $\nu$. Thus the assertion holds since $\sigma$ and $\theta$ in Remark 2.2 generate a subgroup of $\Aut U$ isomorphic to a dihedral group of order 12. \qed

There is a $\mathbb{Z}_2$ symmetry among the irreducible modules of $L(25/28,0)$. It is given by

$$
\rho = \begin{cases} 
1 & \text{on } L(25/28, h_{r,s}) \text{ for odd } s, \\
-1 & \text{on } L(25/28, h_{r,s}) \text{ for even } s,
\end{cases}
$$

where $h_{r,s} = h_{r,s}^5$ is defined by (3.2). On the other hand, $U'$ defines an automorphism $\tau$ of order 3 (cf. the 3A case). If $V$ is a VOA which contains a subalgebra isomorphic to $U$, then there is an automorphism of order 6 defined by $\tau \rho$. In fact, $\tau_\theta \tau_f = \tau \rho$ in this case.

**Theorem 3.24.** If the Moonshine VOA $V^2$ contains a subalgebra isomorphic to $U$, then as an automorphism of $V^2$, $\tau_\theta \tau_f$ is of class 6A.

**Proof.** Since $\tau_\theta \tau_f = \tau \rho$, we have $(\tau_\theta \tau_f)^2 = \tau$ is of class 3A and $(\tau_\theta \tau_f)^3 = \rho$ is of class 2A. Thus $\tau_\theta \tau_f$ is of class 6A. \qed

3.7. 4B case. In this case, $L = L(6) \cong A_1 \oplus A_7$, $n_6 = |E_8/L| = 4$, and the conformal vectors $\tilde{\omega}^1 \in V_{\sqrt{2}A_1}$ and $\tilde{\omega}^2 \in V_{\sqrt{2}A_7}$ defined by (2.9) are of central charge $1/2$ and $7/5$, respectively. Let $\tilde{\alpha} = 2\alpha_6$. Then $E_7 = A_7 \cup (\tilde{\alpha} + A_7)$. Thus we obtain a conformal vector $\tilde{\omega}(E_7)$ of central charge $7/10$. Let $w^1 = \tilde{\omega}^1, w^2 = \tilde{\omega}(E_7)$, and $w^3 = \tilde{\omega}^2 - w^2$. Then $\{w^1, w^2, w^3\}$ is a set of mutually orthogonal conformal vectors of central charge $1/2, 7/10$, and $7/10$, respectively. Actually,

$$
w^1 = \tilde{\omega}^1, \quad w^2 = \frac{1}{2} \tilde{\omega}^2 + \frac{1}{20} X^2, \quad w^3 = \frac{1}{2} \tilde{\omega}^2 - \frac{1}{20} X^2.
$$
Lemma 3.25. Let \( u = X^1 + X^3 \) and \( v = X^1 - X^3 \). Then \( u \) and \( v \) are highest weight vectors of highest weight \( (1/2, 3/2, 0) \) and \( (1/2, 0, 3/2) \) with respect to \( \text{Vir}(w^1) \otimes \text{Vir}(w^2) \otimes \text{Vir}(w^3) \), respectively.

Proof. It follows from (A.10) that
\[
(w^2)_1 X^1 = \frac{3}{4} X^1 + \frac{3}{4} X^3, \quad (w^2)_1 X^3 = \frac{3}{4} X^1 + \frac{3}{4} X^3.
\]
Hence we have \( (w^2)_1 u = \frac{3}{2} u \) and \( (w^2)_1 v = 0 \). Similarly, \( (w^3)_1 u = 0 \) and \( (w^3)_1 v = \frac{3}{2} v \). □

For simplicity of notation, we denote \( L(1/2, h_1) \otimes L(7/10, h_2) \otimes L(7/10, h_3) \) by \([h_1, h_2, h_3]\).

The following proposition is an immediate consequence of the above lemma.

Proposition 3.26. As a module of \( \text{Vir}(w^1) \otimes \text{Vir}(w^2) \otimes \text{Vir}(w^3) \),
\[
U \cong [0, 0, 0] \oplus \left[ \frac{1}{2}, \frac{3}{2}, 0 \right] \oplus \left[ \frac{1}{2}, 0, \frac{3}{2} \right] \oplus \left[ 0, \frac{3}{2}, \frac{3}{2} \right].
\]

The coset subalgebra \( U \) contains four more sets \( \{ x^j, y^j, z^j \} \), \( 0 \leq j \leq 3 \), of three mutually orthogonal conformal vectors of central charge \( 1/2, 7/10, 7/10 \), respectively (cf. Appendix A). They are given by
\[
x^j = \frac{1}{8} \tilde{\omega}^1 + \frac{5}{16} \tilde{\omega}^2 + \frac{1}{32} \left( (\sqrt{-1})^j X^1 + (-1)^j X^2 + (-\sqrt{-1})^j X^3 \right),
\]
\[
y^j = \frac{1}{2} \tilde{\omega}^2 - (-1)^j \frac{1}{20} X^2,
\]
\[
z^j = \frac{7}{8} \tilde{\omega}^1 + \frac{3}{16} \tilde{\omega}^2 - \frac{1}{32} \left( (\sqrt{-1})^j X^1 - \frac{3}{5} (-1)^j X^2 + (-\sqrt{-1})^j X^3 \right).
\]

Note that \( x^0 = \hat{e} \). The following proposition can be verified by direct computation.

Proposition 3.27. As a module of \( \text{Vir}(x^j) \otimes \text{Vir}(y^j) \otimes \text{Vir}(z^j) \),
\[
U \cong [0, 0, 0] \oplus \left[ \frac{1}{2}, 0, \frac{3}{2} \right] \oplus \left[ \frac{1}{16}, \frac{3}{2}, \frac{7}{16} \right]
\]
if \( j = 0, 2 \) and
\[
U \cong [0, 0, 0] \oplus \left[ \frac{1}{2}, \frac{3}{2}, 0 \right] \oplus \left[ \frac{1}{16}, \frac{7}{16}, \frac{3}{2} \right]
\]
if \( j = 1, 3 \).

Theorem 3.28. (1) The Griess algebra \( U_2 \) of \( U \) is generated by \( \hat{e} \) and \( \hat{f} \).

(2) The coset subalgebra \( U \) is generated by \( \hat{e} \) and \( \hat{f} \).

Proof. Let \( G \) be the Griess subalgebra generated by \( \hat{e} \) and \( \hat{f} \). By (A.10), it is easy to verify that \( \hat{e}, \hat{f}, \hat{e}_1 \hat{f}, \hat{e}_1 (\hat{e}_1 \hat{f}) \) are linearly independent. Thus \( G = U_2 \), since \( \text{dim} \ U_2 = 5 \). By the structure of \( U \), it is easy to show that \( U \) is generated by \( U_2 \). Hence the second assertion holds. □

Theorem 3.29. The automorphism group \( \text{Aut} \ U \) of \( U \) is a dihedral group of order 8.
Proof. There are exactly five conformal vectors of central charge 1/2 in $U$, namely, $\tilde{\omega}^1$ and $e_j = \sigma^j \hat{e}$, $0 \leq j \leq 3$. Moreover, we have $\langle \tilde{\omega}^1, e_j \rangle = 1/32$ for any $j = 0, 1, 2, 3$, and

$$
\langle e_i, e_j \rangle = \begin{cases} 
1/32 & \text{if } i + j \equiv 0 \pmod{2}, \\
1/2^8 & \text{if } i + j \equiv 1 \pmod{2}.
\end{cases}
$$

Since $U$ is generated by $\hat{e}$ and $\hat{f} = \sigma \hat{e}$, Aut $U$ can be considered as a subgroup of the permutation group on the set $\{\tilde{\omega}^1, e_0, e_1, e_2, e_3\}$. Furthermore, Aut $U$ must preserve the inner product $\langle \cdot, \cdot \rangle$ so that Aut $U$ fixes $\tilde{\omega}^1$ and $|\text{Aut } U| \leq 8$. Since $\sigma$ and $\theta$ generate a subgroup isomorphic to a dihedral group of order 8 in Aut $U$, we have the assertion. □

The set of all irreducible modules of $U$ can be classified easily by using the same method as in [27, 29, 47]. They are given by

$$
[0, 0, 0] \oplus [0, \frac{3}{2}, \frac{3}{2}] \oplus [\frac{1}{2}, \frac{3}{2}, 0] \oplus [\frac{1}{2}, 0, \frac{3}{2}], \quad [0, 0, \frac{3}{2}] \oplus [0, \frac{3}{2}, 0] \oplus [\frac{1}{2}, 0, 0] \oplus [\frac{1}{2}, \frac{3}{2}, \frac{3}{2}],
$$

$$
[0, 0, \frac{3}{2}] \oplus [0, \frac{3}{2}, \frac{3}{2}] \oplus [\frac{1}{2}, \frac{3}{2}, 0] \oplus [\frac{1}{2}, 0, \frac{3}{2}], \quad [0, \frac{3}{2}, 0] \oplus [0, \frac{1}{2}, \frac{3}{2}] \oplus [\frac{1}{2}, \frac{3}{2}, \frac{3}{2}] \oplus [\frac{1}{2}, \frac{1}{2}, 0],
$$

$$
[0, \frac{3}{2}, \frac{3}{2}] \oplus [0, \frac{1}{2}, \frac{1}{2}] \oplus [\frac{1}{2}, \frac{1}{2}, \frac{3}{2}] \oplus [\frac{1}{2}, \frac{3}{2}, \frac{1}{2}], \quad [0, \frac{3}{2}, \frac{3}{2}] \oplus [0, \frac{1}{2}, \frac{1}{2}] \oplus [\frac{1}{2}, \frac{1}{2}, \frac{3}{2}] \oplus [\frac{1}{2}, \frac{3}{2}, \frac{1}{2}],
$$

and

$$
[\frac{1}{16}, \frac{7}{16}, \frac{7}{16}] \otimes Q, \quad [\frac{1}{16}, \frac{7}{16}, \frac{3}{80}] \otimes Q, \quad [\frac{1}{16}, \frac{3}{80}, \frac{7}{16}] \otimes Q, \quad [\frac{1}{16}, \frac{3}{80}, \frac{3}{80}] \otimes Q,
$$

where $Q$ is the unique 2-dimensional irreducible module of the quaternion group of order 8.

The fixed point subalgebra $U^{\tau \sigma \tau_f}$ of $\tau \sigma \tau_f$ in $U$ is isomorphic to $[0, 0, 0] \oplus [0, 3/2, 3/2]$.

The fusion rules among irreducible modules of $L(7/10, 0) \otimes L(7/10, 0) \otimes L(7/10, 3/2) \otimes L(7/10, 3/2)$ can be computed easily. There is a $\mathbb{Z}_4$ symmetry given as follows:

$$
\tau = \begin{cases} 
1 & \text{on } [0, 0] \oplus [\frac{3}{2}, \frac{3}{2}] \oplus [\frac{3}{2}, \frac{1}{10}], \\
 & \quad \oplus [\frac{1}{10}, \frac{3}{2}] \oplus [\frac{3}{2}, 0], \\
\sqrt{-1} & \text{on } [\frac{7}{10}, \frac{7}{10}]^+, \quad [\frac{7}{10}, \frac{3}{80}]^+, \\
 & \quad [\frac{3}{80}, \frac{7}{16}]^+, \quad [\frac{3}{80}, \frac{3}{80}]^+, \\
-1 & \text{on } [0, \frac{3}{2}] \oplus [\frac{3}{2}, 0], \quad [\frac{3}{2}, \frac{3}{80}] \oplus [0, \frac{1}{10}], \\
 & \quad \oplus [\frac{1}{10}, \frac{3}{2}] \oplus [\frac{1}{10}, 0] \oplus [\frac{3}{2}, \frac{3}{2}], \\
-\sqrt{-1} & \text{on } [\frac{7}{10}, \frac{7}{10}]^-, \quad [\frac{7}{10}, \frac{3}{80}]^-, \\
 & \quad [\frac{3}{80}, \frac{7}{10}]^-, \quad [\frac{3}{80}, \frac{3}{80}]^-,
\end{cases}
$$

where $[h_1, h_2]$ denotes the irreducible module $L(7/10, h_1) \otimes L(7/10, h_2)$ of $L(7/10, 0) \otimes L(7/10, 0)$. Suppose $U$ is contained in a VOA $V$. Then all $\tau, \tau_e$, and $\tau_f$ are well defined.
automorphisms of \( V \). In this case, \( \tau = \tau_\varepsilon \tau_f \) and \( \tau^2 = \tau_w \). Thus we have the following theorem.

**Theorem 3.31.** If the Moonshine VOA \( V^2 \) contains a subalgebra isomorphic to \( U \), then as an automorphism of \( V^2 \), \( \tau_\varepsilon \tau_f \) is of class 4B.

### 3.8 2B case

In this case, \( L = L(7) \cong D_8 \), \( n_7 = |E_8/L| = 2 \), and the conformal vector \( \bar{\omega}^1 \in V_{\sqrt{2}D_8} \) defined by (2.9) is of central charge 1. The Virasoro element \( \omega' \) of \( U \) is equal to \( \bar{\omega}^1 \).

Let \( \epsilon_1, \epsilon_2, \ldots, \epsilon_8 \in \mathbb{R}^8 \) be such that \( \langle \epsilon_i, \epsilon_j \rangle = 2\delta_{ij} \) for any \( i, j \). Then

\[
\sqrt{2}D_8 = \{ a_1\epsilon_1 + \cdots + a_8\epsilon_8 \mid a_i \in \mathbb{Z}, a_1 + \cdots + a_8 \equiv 0 \pmod{2} \}.
\]

Let \( \gamma = \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_8) \). Then \( \sqrt{2}E_8 \cong \sqrt{2}D_8 \cup (\gamma + \sqrt{2}D_8) \). Denote

\[
U^0 = \{ u \in V_{\sqrt{2}D_8} \mid s(D_8)u = 0 \}, \quad U^1 = \{ u \in V_{\gamma + \sqrt{2}D_8} \mid s(D_8)u = 0 \},
\]

where \( s(D_8) \) is defined as in (2.1). By [8, 9], we know that \( U^0 \cong V^+_{Z2\gamma} \) and \( U^1 \cong V^+_{\gamma + Z2\gamma} \).

Note that \( \mathbb{Z}\gamma = \mathbb{Z}2\gamma \cup (\gamma + \mathbb{Z}2\gamma) \) and hence we have \( U = U^0 \oplus U^1 \cong V^+_{Z2\gamma} \). Since \( \langle \gamma, \gamma \rangle = 4 \), it is well known that

\[
V^+_{Z2\gamma} \cong L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0).
\]

In fact, \( \frac{1}{16}\gamma(-1)^2 \cdot 1 + \frac{1}{4}(e^\gamma + e^{-\gamma}) \) and \( \frac{1}{16}\gamma(-1)^2 \cdot 1 - \frac{1}{4}(e^\gamma + e^{-\gamma}) \) are the two mutually orthogonal conformal vectors of central charge \( 1/2 \) in \( V^+_{Z2\gamma} \) (cf. [12, 39, 40]).

The following theorem is clear from the structure of \( U \).

**Theorem 3.32.** The coset subalgebra \( U \) is generated by \( \hat{e} \) and \( \hat{f} \). Moreover, the automorphism group \( \text{Aut} U \) of \( U \) is of order 2.

Both of \( \hat{e} \) and \( \hat{f} \) are fixed by \( \theta \) and thus \( U \) is contained in \( V^+_{A_8} \subset V^2 \). In this case, \( \hat{e} \) and \( \hat{f} \) are mutually orthogonal and so \( \tau_\varepsilon \) and \( \tau_f \) are commutative. Hence, \( |\tau_\varepsilon \tau_f| = 2 \) and as an automorphism of \( V^2 \), \( \tau_\varepsilon \tau_f \) is of class 2B (see also Shimakura [45]).

### 3.9 3C case

In this case, \( L = L(8) \cong A_8 \), \( n_8 = |E_8/L| = 3 \), and the conformal vector \( \bar{\omega}^1 \in V_{\sqrt{2}A_8} \) defined by (2.9) is of central charge 16/11. The Virasoro element \( \omega' \) of \( U \) is equal to \( \bar{\omega}^1 \). The following lemma can be easily verified.

**Lemma 3.32.** Let

\[
x = \frac{11}{32}\bar{\omega}^1 + \frac{1}{32}(X^1 + X^2), \quad y = \frac{21}{32}\bar{\omega}^1 - \frac{1}{32}(X^1 + X^2).
\]

Then \( x \) and \( y \) are mutually orthogonal conformal vectors of central charge \( 1/2 \) and \( 21/22 \), respectively. Moreover, \( \bar{\omega}^1 = x + y \).

The above lemma implies that \( U \) contains \( \text{Vir}(x) \otimes \text{Vir}(y) \cong L(1/2, 0) \otimes L(21/22, 0) \).

**Lemma 3.33.** Let \( v = X^1 - X^2 \). Then \( v \) is a highest weight vector of highest weight \((1/16, 31/16)\) with respect to \( \text{Vir}(x) \otimes \text{Vir}(y) \).
Proof. By (A.12), we have
\[ x_1 v = \left( \frac{11}{32} \varpi^1 + \frac{1}{32} (X^1 + X^2) \right) (X^1 - X^2) = \frac{1}{16} (X^1 - X^2). \]
Hence the assertion holds. \qed

Similarly, we obtain the following lemma by direct computation.

**Lemma 3.34.** Let
\[ u = 4v - 45x_3 \cdot 1 - 9x_1 x - 682x_1 y - 1089y_1 y - 165y_3 \cdot 1. \]
Then \( u \) is a highest weight vector of highest weight \((1/2, 7/2)\) with respect to \( \Vir(x) \otimes \Vir(y) \) and \( \langle u, u \rangle = 1280076 \).

Note that \( L(1/2, h_1) \otimes L(21/22, h_2) \) with \((h_1, h_2) = (0, 0), (0, 8), (1/2, 7/2), (1/2, 45/2), (1/16, 31/16) \) and \((1/16, 175/16)\) are the only irreducible \( L(1/2, 0) \otimes L(21/22, 0) \)-modules which are integrally graded. For simplicity of notation, we shall denote \( L(1/2, h_1) \otimes L(21/22, h_2) \) by \([h_1, h_2]\). The following theorem can be obtained by using the characters (cf. Appendix B).

**Theorem 3.35 (Theorem B.5).** As a module of \( \Vir(x) \otimes \Vir(y) \),
\[ U \cong [0, 0] \oplus [0, 8] \oplus \frac{1}{2} [7 \cdot 2] \oplus \frac{1}{2} [45 \cdot 2] \oplus \frac{1}{16} [31 \cdot 16] \oplus \frac{1}{16} [175 \cdot 16]. \]

**Theorem 3.36.** (1) The Griess algebra \( U_2 \) of \( U \) is generated by \( \hat{e} \) and \( \hat{f} \).

(2) The coset subalgebra \( U \) is generated by \( \hat{e} \) and \( \hat{f} \).

**Proof.** The first assertion is clear since \( \dim U_2 = 3 \). Let \( W \) be the subalgebra generated by \( U_2 \). We want to show that \( W = U \). By Lemmas 3.33 and 3.34, \( W \) contains submodules of the form \([0, 0], [1/2, 7/2], \) and \([1/16, 31/16]\).

We shall show that there is a highest weight vector of highest weight \((0, 8)\) with respect to \( \Vir(x) \otimes \Vir(y) \) in \( W \). Suppose \( W \) does not contain a highest weight vector of highest weight \((0, 8)\). Then the fixed point subalgebra \( W^{\tau_x} \) of the Miyamoto involution \( \tau_x \) in \( W \) is of the form \( W^{\tau_x} \cong [0, 0] \oplus [1/2, 7/2] \). Let \( w \) be a highest weight vector of highest weight \((1/2, 7/2)\). We normalize \( w \) so that \( \langle w, w \rangle = 1 \). For example, we may take \( w = u/\sqrt{\langle u, u \rangle} \).

Then by fusion rules, \( w_1 w \in [0, 0] \). It is well known (cf. [14] [15]) that there is an explicit construction of the vertex operator superalgebra \( L(1/2, 0) \oplus L(1/2, 1/2) \) by using one free fermionic field and we can find an orthonormal basis of \( L(1/2, 0) \oplus L(1/2, 1/2) \) (cf. [23]). By the assumption \( W^{\tau_x} \cong [0, 0] \oplus [1/2, 7/2] \) is \( \mathbb{Z}_2 \)-graded and so we can use the orthonormal basis of \( L(1/2, 0) \oplus L(1/2, 1/2) \) to make a computation of the inner product \( \langle w_1 w, w_1 w \rangle \) easier. As a result, we can obtain
\[ \langle w_1 w, w_1 w \rangle = \frac{231091005602}{134258427}. \]
Note that this value is deduced based on the assumption that \([0, 8]\) is not contained in \(W\) as a \(\text{Vir}(x) \otimes \text{Vir}(y)\)-submodule.

On the other hand, we can also compute \(\langle w_{-1}w, w_{-1}w \rangle\) directly by using the definition of \(w\) and the Jacobi identity. In this case, we have

\[
\langle w_{-1}w, w_{-1}w \rangle = \langle w, (w_{-1}w)_{\tau}w \rangle = \sum_{n \geq 0} (-1)^n \binom{-1}{n} \langle w, (w_{-1-n}w_{\tau+n} + w_{6-n}w)w \rangle = \langle w, w_{-1}w_{\tau}w \rangle + \sum_{n=0}^{7} \langle w, w_{6-n}w_n w \rangle = 2\langle w, w \rangle^2 + \sum_{n=0}^{5} \langle w_n w, w_n w \rangle.
\]

Then by computing \(\langle w_{5-m}w, w_{5-m}w \rangle, m = 0, 1, \ldots, 5,\) inductively, we obtain another value

\[
\langle w_{-1}w, w_{-1}w \rangle = \frac{1036050}{589} \left( \neq \frac{231091005602}{134258427} \right).
\]

Note that this value is deduced from the structure of the Griess algebra \(U_2\) and independent of the shape of \(W\) as a \(\text{Vir}(x) \otimes \text{Vir}(y)\)-module. Thus this contradiction comes from the assumption that \(W\) does not contain a highest weight vector of highest weight \((0, 8)\). Hence we conclude that there is a highest weight vector of highest weight \((0, 8)\) in \(W\). By the computation above, we also note that \(w_{-1}w\) contains a singular vector with highest weight \((0, 8)\) as a non-trivial summand.

Then, \(W\) also contains a highest weight vector of highest weight \((1/2, 45/2)\); otherwise, \(W^{\tau_x} \cong [0, 0] \oplus [1/2, 7/2] \oplus [0, 8]\) and the orthogonal complement of \(W^{\tau_x}\) in \(U^{\tau_x}\) with respect to a positive definite invariant hermitian form is isomorphic to \([1/2, 45/2]\). But the orthogonal complement must be a module for \(W^{\tau_x}\), which is impossible by the fusion rules. Thus \(W^{\tau_x} = U^{\tau_x}\).

Since the fusion product of \([1/2, 45/2]\) and \([1/16, 31/16]\) is \([1/16, 175/16]\), \(W\) contains a highest weight vector of highest weight \((1/16, 175/16)\) also, and hence \(W = U\) as desired.

\[\Box\]

**Theorem 3.37.** The automorphism group \(\text{Aut } U\) of \(U\) is a symmetric group \(S_3\) of degree 3.

**Proof.** By Lemma A.14, there are exactly three conformal vectors of central charge \(1/2\) in \(U\), namely, \(\sigma^j \hat{e}, j = 0, 1, 2\). Thus \(\text{Aut } U\) can be considered as a subgroup of \(S_3\). Since \(\sigma\) and \(\theta\) already generate a subgroup of order 6 in \(\text{Aut } U\), we conclude that \(\text{Aut } U \cong S_3\). \[\Box\]
Theorem 3.38. There are exactly five irreducible $U$-modules $U(2k), 0 \leq k \leq 4$. In fact, $U(0) = U$ and as $\text{Vir}(x) \otimes \text{Vir}(y)$-modules,

$$U(2) \cong [0, \frac{13}{11}] \oplus [0, \frac{35}{11}] \oplus [\frac{1}{2}, \frac{15}{22}] \oplus [\frac{1}{2}, \frac{301}{16}, \frac{21}{176}] \oplus [\frac{1}{2}, \frac{1}{16}, \frac{901}{176}],$$

$$U(4) \cong [0, \frac{6}{11}] \oplus [0, \frac{50}{11}] \oplus [\frac{1}{2}, \frac{1}{22}] \oplus [\frac{1}{2}, \frac{155}{16}, \frac{85}{176}] \oplus [\frac{1}{2}, \frac{1}{16}, \frac{261}{176}],$$

$$U(6) \cong [0, \frac{1}{11}] \oplus [0, \frac{111}{11}] \oplus [\frac{1}{2}, \frac{35}{22}] \oplus [\frac{1}{2}, \frac{57}{22}] \oplus [\frac{1}{2}, \frac{5}{16}, \frac{533}{176}],$$

$$U(8) \cong [0, \frac{20}{11}] \oplus [0, \frac{196}{11}] \oplus [\frac{1}{2}, \frac{7}{22}] \oplus [\frac{1}{2}, \frac{117}{22}] \oplus [\frac{1}{2}, \frac{133}{16}, \frac{1365}{176}].$$

We shall again give a proof at Appendix C. Note that the lattice VOA $V_{\sqrt{2}E_8}$ can be decomposed as follows (cf. [28, 32]).

$$V_{\sqrt{2}E_8} \cong \bigoplus_{0 \leq k_j \leq l+1 \atop k_j \equiv 0 \bmod 2 \atop j=0,1,\ldots,8} L(c_1, h_{k_0+1,k_1+1}) \otimes \cdots \otimes L(c_8, h_{k_7+1,k_8+1}) \otimes U(k_8),$$

where $c_m$ and $h^m_{r,s}$ are given by (3.1) and (3.2). Moreover, as an automorphism of $V_{\sqrt{2}E_8}$, $\tau_e \tau_f = e^{2\pi \sqrt{-1} \beta(0)}$ is of order 3 (cf. Remark 3.4) and it induces a natural action on each of $U(2k), 0 \leq k \leq 4$. Hence, for any VOA $V$ which contains a subalgebra isomorphic to $U$, $\tau_e \tau_f$ defines an automorphism of order 3 on $V$.

Remark 3.39. In Miyamoto[42], it is shown that if $e$ and $f$ are two conformal vectors of central charge $1/2$ in the Moonshine VOA $V^*$ such that $\tau_e \tau_f$ is of order 3 and $\langle e, f \rangle = 1/2^8$, then the subalgebra $W$ generated by $e$ and $f$ must contain a subalgebra isomorphic to $L(1/2,0) \otimes L(21/22,0)$ and the weight 2 subspace $W_2$ of $W$ is of dimension 3. In fact, $W \cong U$ as $L(1/2,0) \otimes L(21/22,0)$-modules and $\tau_e \tau_f$ is of class $3C$.

Appendix A. Conformal Vectors in $U$

In this appendix, we shall compute the conformal vectors of the coset subalgebra $U$ defined by (2.10) in each of the nine cases. Except for the cases of $5A$ and $6A$, the computation was done by Maple 7. The cases for $5A$ and $6A$ were computed by Kazuhiro Yokoyama of Kyushu University using a computer algebra system Risa/Asir.

As in Section 2.2, $L = L(i)$ denotes the lattice associated with the Dynkin diagram obtained by removing the $i$-th node $\alpha_i$ in the extended $E_8$ diagram (2.2) and $n_i = |E_8/L|$. Let

$$B = \text{span}_\mathbb{C}\{\tilde{\omega}^1, \ldots, \tilde{\omega}^l, X^1, \ldots, X^{n_i-1}\},$$

where $\tilde{\omega}^j$ and $X^j$ are defined by (2.9) and (3.3). Then $\dim B = l + n_i - 1$. Actually, $B$ is equal to the Griess algebra $U_2$ of $U$. Note that

$$(\tilde{\omega}^j)_1 \tilde{\omega}^k = 2\tilde{\omega}^j \delta_{j,k}, \quad \langle \tilde{\omega}^j, \tilde{\omega}^k \rangle = c\delta_{j,k}/2, \quad \langle \tilde{\omega}^j, X^k \rangle = 0,$$

$$\langle X^j, X^k \rangle = 0 \quad \text{if} \quad j + k \equiv 0 \mod n_i,$$

(A.1)
where $c$ denotes the central charge of $\tilde{\omega}^j$.

1A case. In this case, $L = L(0) \cong E_8$, $n_0 = 1$, and $\tilde{\omega}^1 \in V_{\sqrt{2}E_8}$ is the only conformal vector in $B$, whose central charge is $1/2$.

2A case. In this case, $L = L(1) \cong A_1 \oplus E_7$, $n_1 = 2$, and the conformal vectors $\tilde{\omega}^1 \in V_{\sqrt{2}A_1}$ and $\tilde{\omega}^2 \in V_{\sqrt{2}E_7}$ are of central charge $1/2$ and $7/10$, respectively. The product and the inner product in $B$ are given by (A.1) and

$$(\tilde{\omega}^1)_1 X^1 = \frac{1}{2} X^1, \quad (\tilde{\omega}^2)_1 X^1 = \frac{3}{2} X^1, \quad (X^1)_1 X^1 = 224\tilde{\omega}^1 + 480\tilde{\omega}^2, \quad \langle X^1, X^1 \rangle = 112.$$

Lemma A.1. Let $w = a\tilde{\omega}^1 + b\tilde{\omega}^2 + cX^1$ be a conformal vector in $B$. Then $(a, b, c)$ satisfies the following system of equations.

$$a^2 + 112c^2 - a = 0, \quad b^2 + 240c^2 - b = 0, \quad ac + 3bc - 2c = 0. \quad (A.2)$$

The central charge of $w$ is given by $\frac{1}{2}a^2 + \frac{7}{10}b^2 + 224c^2$.

The solutions $(a, b, c)$ of Equation (A.2) are as follows.

Central charge $1/2$: $(1, 0, 0), (\frac{1}{8}, \frac{5}{8}, \frac{1}{32}), (\frac{1}{8}, \frac{5}{8}, -\frac{1}{32})$.

Central charge $7/10$: $(0, 1, 0), (\frac{7}{8}, \frac{3}{8}, \frac{1}{32}), (\frac{7}{8}, \frac{3}{8}, -\frac{1}{32})$.

Central charge $6/5$: $(1, 1, 0)$.

3A case. In this case, $L = L(2) \cong A_2 \oplus E_6$, $n_2 = 3$, and the conformal vectors $\tilde{\omega}^1 \in V_{\sqrt{2}A_2}$ and $\tilde{\omega}^2 \in V_{\sqrt{2}E_6}$ are of central charge $4/3$ and $6/7$, respectively. The product and the inner product in $B$ are given by (A.1) and

$$(\tilde{\omega}^1)_1 X^1 = \frac{2}{3} X^1, \quad (\tilde{\omega}^1)_1 X^2 = \frac{2}{3} X^2, \quad (\tilde{\omega}^2)_1 X^1 = \frac{4}{3} X^1, \quad (\tilde{\omega}^2)_1 X^2 = \frac{4}{3} X^2,$$

$$(X^1)_1 X^1 = 20X^2, \quad (X^2)_1 X^2 = 20X^1, \quad (X^1)_2 X^2 = 135\tilde{\omega}^1 + 252\tilde{\omega}^2,$$

$$\langle X^1, X^2 \rangle = 81.$$

Lemma A.2. Let $w = a\tilde{\omega}^1 + b\tilde{\omega}^2 + cX^1 + dX^2$ be a conformal vector in $B$. Then $(a, b, c, d)$ satisfies the following system of equations.

$$a^2 + 135cd - a = 0, \quad b^2 + 252cd - b = 0, \quad 15d^2 + 2ac + 4bc - 3c = 0, \quad 15c^2 + 2ad + 4bd - 3d = 0. \quad (A.3)$$

The central charge of $w$ is given by $\frac{4}{3}a^2 + \frac{6}{7}b^2 + 324cd$.

The solutions $(a, b, c, d)$ of Equation (A.3) are as follows, where $\xi = e^{2\pi \sqrt{-1}/3}$ is a primitive cubic root of unity.

Central charge $1/2$: $(\frac{5}{32}, \frac{7}{16}, \frac{1}{32} \xi^j, \frac{1}{18} \xi^{2j}), j = 0, 1, 2$.

Central charge $4/5$: $(1, 0, 0, 0), (\frac{1}{16}, \frac{7}{8}, -\frac{1}{48} \xi^j, -\frac{1}{48} \xi^{2j}), j = 0, 1, 2$. 
Central charge 6/7: \((0,1,0,0), \; (\frac{15}{16}, \frac{1}{8}, \frac{1}{48} \xi^j, \frac{1}{16} \xi^{2j}), \; j = 0, 1, 2\).

Central charge 81/70: \((\frac{27}{32}, \frac{9}{16}, -\frac{1}{32} \xi^j, -\frac{1}{32} \xi^{2j}), \; j = 0, 1, 2\).

Central charge 58/35: \((1,1,0,0)\).

**Lemma A.3.** Let \(w = a\tilde{w}^1 + b\tilde{w}^2 + cX^1 + dX^2 + eX^3\) be a conformal vector in \(B\). Then, \((a,b,c,d,e)\) satisfies the following system of equations.

\[
\begin{align*}
 a^2 + 60d^2 + 96ce - a &= 0, \\
 b^2 + 60d^2 + 160ce - b &= 0, \\
 3ac + 5bc + 60de - 4c &= 0, \\
 8c^2 + 8e^2 + ad + bd - d &= 0, \\
 3ae + 5be + 60cd - 4e &= 0. \\
\end{align*}
\]

The central charge of \(w\) is given by \(a^2 + b^2 + 256ce + 120d^2\).

The solutions \((a,b,c,d,e)\) of Equation (A.4) are as follows, where \(\xi = e^{\pi \sqrt{-1}/4}\) is a primitive 8-th root of unity.

Central charge 1/2: \((\frac{3}{16}, \frac{5}{16}, \frac{1}{32} (\sqrt{-1})^j, \frac{1}{32} (-1)^j, \frac{1}{32} (-\sqrt{-1})^j), \; 0 \leq j \leq 3\).

Central charge 1:

\[
\begin{align*}
 & (1,0,0,0,0), \; (0,1,0,0,0), \\
 & ((1 + \sqrt{1 - 240d^2})/2, (1 - \sqrt{1 - 240d^2})/2, 0, 0, 0) , \\
 & ((1 - \sqrt{1 - 240d^2})/2, (1 + \sqrt{1 - 240d^2})/2, 0, 0, 0) , \text{ where } d \in \mathbb{C} \setminus \{0\}. \\
\end{align*}
\]

Central charge 3/2: \((\frac{13}{16}, \frac{11}{16}, \frac{1}{32} (\sqrt{-1})^j, -\frac{1}{32} (-1)^j, \frac{1}{32} (\sqrt{-1})^j), \; 0 \leq j \leq 3\).

Central Charge 6/7: \((\frac{1}{7}, \frac{5}{7}, \frac{1}{28} \xi^{2j+1}, 0, \frac{1}{28} \xi^{-(2j+1)}) \; 0 \leq j \leq 3\).

Central Charge 8/7: \((\frac{6}{7}, \frac{2}{7}, \frac{1}{28} \xi^{2j+1}, 0, \frac{1}{28} \xi^{-(2j+1)}) \; 0 \leq j \leq 3\).

Central Charge 2: \((1,1,0,0,0)\).
5A case. In this case, $L = L(4) \cong A_4 \oplus A_4$, $n_4 = 5$, and the conformal vectors $	ilde{\omega}^1 \in V_{\sqrt{2}A_4}$ and $	ilde{\omega}^2 \in V_{\sqrt{2}A_4}$ are both of central charge $8/7$. The product and the inner product in $\mathcal{B}$ are given by (A.1) and

\[
(X^i)_1(X^j) = \begin{cases} 
12X^{i+j} & \text{if } i + j \not\equiv 0 \mod 5, \\
70\tilde{\omega}^1 + 105\tilde{\omega}^2 & \text{if } i = 5 - j = 1 \text{ or } 4, \\
105\tilde{\omega}^1 + 70\tilde{\omega}^2 & \text{if } i = 5 - j = 2 \text{ or } 3,
\end{cases}
\]

\[
\langle X^1, X^4 \rangle = \langle X^2, X^3 \rangle = 50.
\]

Lemma A.4. Let $w = a\tilde{\omega}^1 + b\tilde{\omega}^2 + cX^1 + dX^2 + eX^3 + fX^4$ be a conformal vector in $\mathcal{B}$. Then $(a, b, c, d, e, f)$ satisfies the following system of equations.

\[
a^2 + 70cf + 105de - a = 0, \quad b^2 + 105cf + 70de - b = 0,
\]

\[
4ac + 6bc + 30e^2 + 60df - 5c = 0, \quad 6ad + 4bd + 30c^2 + 60cf - 5d = 0,
\]

\[
6ae + 4be + 30f^2 + 60cd - 5e = 0, \quad 4af + 6bf + 30d^2 + 60ce - 5f = 0.
\]

(A.5)

The central charge of $w$ is given by

\[
\frac{8}{7}(a^2 + b^2) + 200(cf + de).
\]

(A.6)

In order to solve the above system of equations, we treat $a, b, c, d, e, f$ as variables. Let $\mathbb{C}[a, b, c, d, e, f]$ be the polynomial algebra with variables $a, b, c, d, e, f$ and $\mathcal{I}$ the ideal generated by the six polynomials which appear on the left hand side of (A.5). We then compute the primary decomposition of the ideal $\mathcal{I} = \cap P_i$ over the field $\mathbb{Q}$ of rational numbers and solve the system corresponding to each prime ideal $P_i$. By using the computer algebra system Risa/Asir, we found that $\mathcal{I}$ is an intersection of 21 prime ideals and there are 63 different nontrivial solutions of (A.5). Thus we have the following lemma.

Lemma A.5. There are exactly 63 conformal vectors in $\mathcal{B}$.

The central charges of those conformal vectors are easily calculated by (A.6). We verified that there are only 43 conformal vectors whose central charges are rational numbers. Their central charges are shown in Table 1.

| c.c. | $1/2$ | $25/28$ | $8/7$ | $16/7$ | $25/14$ | $39/28$ |
|------|-------|--------|------|------|--------|--------|
| number | 5     | 10     | 12   | 1    | 5      | 10     |

The following two lemmas were verified by computer.
Lemma A.6. There are exactly five conformal vectors of central charge $1/2$ in $\mathcal{B}$, namely, $\sigma^j \hat{e}$, $0 \leq j \leq 4$. Note that
\[
\sigma^j \hat{e} = \frac{7}{32} (\hat{\omega}^1 + \hat{\omega}^2) + \frac{1}{32} (\xi^j X^1 + \xi^{2j} X^2 + \xi^{3j} X^3 + \xi^{4j} X^4),
\]
where $\xi = e^{2\pi \sqrt{-1}/5}$ is a primitive 5-th root of unity. Moreover, the inner product is $\langle \sigma^i \hat{e}, \sigma^j \hat{e} \rangle = 3/512$ for any $i \neq j$.

Lemma A.7. There is a triple $(u, v, w)$ of mutually orthogonal conformal vectors in $\mathcal{B}$ such that the central charges of $u, v, w$ are $1/2, 25/28, 25/28$, respectively and $u + v + w$ is equal to the Virasoro element $\hat{\omega}^1 + \hat{\omega}^2$ of $U$. For example,
\[
\begin{align*}
u = \hat{e} &= \frac{7}{32} (\hat{\omega}^1 + \hat{\omega}^2) + \frac{1}{32} (X^1 + X^2 + X^3 + X^4), \\
v &= \frac{15}{64} \hat{\omega}^1 + \frac{35}{64} \hat{\omega}^2 - \frac{3}{64} (X^1 + X^4) + \frac{1}{64} (X^2 + X^3), \\
w &= \frac{35}{64} \hat{\omega}^1 + \frac{15}{64} \hat{\omega}^2 + \frac{1}{64} (X^1 + X^4) - \frac{3}{64} (X^2 + X^3).
\end{align*}
\]

6A case. In this case, $L = L(5) \cong A_2 \oplus A_1 \oplus A_5$, $n_5 = 6$, and the conformal vectors $\hat{\omega}^1 \in V_{\sqrt{5} A_1}$, $\hat{\omega}^2 \in V_{\sqrt{5} A_1}$, and $\hat{\omega}^3 \in V_{\sqrt{5} A_5}$ are of central charge $4/5$, $1/2$, and $5/4$, respectively. The product and the inner product in $\mathcal{B}$ are given by (A.1) and
\[
\begin{align*}
(\hat{\omega}^1)_1 X^1 &= \frac{2}{3} X^1, & (\hat{\omega}^2)_1 X^1 &= \frac{1}{2} X^1, & (\hat{\omega}^3)_1 X^1 &= \frac{5}{6} X^1, \\
(\hat{\omega}^1)_1 X^2 &= \frac{2}{3} X^2, & (\hat{\omega}^2)_1 X^2 &= 0, & (\hat{\omega}^3)_1 X^2 &= \frac{4}{3} X^2, \\
(\hat{\omega}^1)_1 X^3 &= 0, & (\hat{\omega}^2)_1 X^3 &= \frac{1}{2} X^3, & (\hat{\omega}^3)_1 X^3 &= \frac{3}{2} X^3, \\
(\hat{\omega}^1)_1 X^4 &= \frac{2}{3} X^4, & (\hat{\omega}^2)_1 X^4 &= 0, & (\hat{\omega}^3)_1 X^4 &= \frac{4}{3} X^4, \\
(\hat{\omega}^1)_1 X^5 &= \frac{2}{3} X^5, & (\hat{\omega}^2)_1 X^5 &= \frac{1}{2} X^5, & (\hat{\omega}^3)_1 X^5 &= \frac{5}{6} X^5, \\
(X^1)_1 X^1 &= 8X^2, & (X^1)_1 X^2 &= 9X^3, & (X^1)_1 X^3 &= 8X^4, \\
(X^1)_1 X^4 &= 10X^5, & (X^1)_1 X^5 &= 60\hat{\omega}^1 + 72\hat{\omega}^2 + 48\hat{\omega}^3, & (X^2)_1 X^2 &= 12X^4, \\
(X^2)_1 X^3 &= 10X^5, & (X^2)_1 X^4 &= 75\hat{\omega}^1 + 96\hat{\omega}^3, & (X^2)_1 X^5 &= 10X^3, \\
(X^3)_1 X^3 &= 80\hat{\omega}^2 + 96\hat{\omega}^3, & (X^3)_1 X^4 &= 10X^1, & (X^3)_1 X^5 &= 8X^2, \\
(X^4)_1 X^4 &= 12X^2, & (X^4)_1 X^5 &= 9X^3, & (X^5)_1 X^5 &= 8X^4, \\
\langle X^1, X^5 \rangle &= 36, & \langle X^2, X^4 \rangle &= 45, & \langle X^3, X^3 \rangle &= 40.
\end{align*}
\]
Lemma A.8. Let \( w = a \tilde{\omega}^1 + b \tilde{\omega}^2 + c \tilde{\omega}^3 + dX^1 + eX^2 + fX^e + gX^4 + hX^5 \) be a conformal vectors in \( B \). Then \((a, b, c, d, e, f, g, h)\) satisfies the following system of equations.

\[
\begin{align*}
  a^2 + 60dh + 75eg - a &= 0, \\
  b^2 + 40f^2 + 72dh - b &= 0, \\
  c^2 + 48f^2 + 48dh + 96eg - c &= 0, \\
  4ad + 3bd + 5cd + 60eh + 60fg - 6d &= 0, \\
  12d^2 + 18g^2 + 2ae + 4ce + 24fh - 3e &= 0, \\
  bf + 3cf + 18de + 18gh - 2f &= 0, \\
  18e^2 + 12h^2 + 2ag + 4cg + 24df - 3g &= 0, \\
  4ah + 3bh + 5ch + 60dg + 60ef - 6h &= 0. \\
\end{align*}
\]

(A.8)

The central charge of \( w \) is given by

\[
\frac{4}{5}a^2 + \frac{1}{2}b^2 + \frac{5}{4}c^2 + 80f^2 + 144dh + 180eg.
\]

(A.9)

Again we treat \( a, b, c, d, e, f, g, h \) as variables and let \( I \) be the ideal in \( \mathbb{C}[a, b, c, d, e, f, g, h] \) generated by the eight polynomials which appear on the left hand side of (A.8). By using the computer algebra system Risa/Asir, we found that the ideal \( I \) is an intersection of 112 prime ideals over the field \( \mathbb{Q} \) of rational numbers and that there are totally 256 conformal vectors in \( B \). The number of conformal vectors whose central charges are rational numbers less than 1 are listed in Table 2.

**Table 2.** Central charge (c.c.) and number of conformal vectors

| c.c.     | 1/2 | 7/10 | 4/5 | 6/7 | 25/28 | 11/12 | 14/15 | 21/22 |
|----------|-----|------|-----|-----|-------|-------|-------|-------|
| number   | 7   | 9    | 7   | 14  | 5     | 6     | 6     | 6     |

The following three lemmas were verified by computer.

Lemma A.9. There are exactly seven conformal vectors of central charge 1/2 in \( B \), namely, \( \tilde{\omega}^2 \) and \( \sigma_j \hat{e} \), \( 0 \leq j \leq 5 \). Note that

\[
\sigma_j \hat{e} = \frac{5}{32} \tilde{\omega}^1 + \frac{1}{8} \tilde{\omega}^2 + \frac{1}{4} \tilde{\omega}^3 + \frac{1}{32} (\xi^j X^1 + \xi^{2j} X^2 + \xi^{3j} X^3 + \xi^{4j} X^4 + \xi^{5j} X^5),
\]

where \( \xi = e^{\pi \sqrt{-1}/3} \) is a primitive 6-th root of unity. Moreover, the inner product among \( \sigma_j \hat{e}, 0 \leq j \leq 5 \) are

\[
\langle \sigma_i \hat{e}, \sigma_j \hat{e} \rangle = \begin{cases} 
5/2^{10} & \text{if } i - j \equiv \pm 1 \mod 6, \\
13/2^{10} & \text{if } i - j \equiv \pm 2 \mod 6, \\
1/32 & \text{if } i - j \equiv 3 \mod 6.
\end{cases}
\]
We are mainly interested in mutually orthogonal conformal vectors whose sum is the Virasoro element $\omega' = \tilde{\omega}^1 + \tilde{\omega}^2 + \tilde{\omega}^3$ of $U$.

**Lemma A.10.** There are exactly 10 triples $(u, v, w)$ of mutually orthogonal conformal vectors in $B$ such that the central charge of $u, v, w$ are $4/5, 6/7, 25/28$, respectively and $u + v + w = \omega'$.

**Lemma A.11.** There are exactly 6 triples $(u, v, w)$ of mutually orthogonal conformal vectors in $B$ such that the central charge of $u, v, w$ are $7/10, 11/12, 14/15$, respectively and $u + v + w = \omega'$.

4B case. In this case, $L = L(6) \cong A_1 \oplus A_7$, $n_6 = 4$, and the conformal vectors $\tilde{\omega}^1 \in V_{\sqrt{2}A_1}$ and $\tilde{\omega}^2 \in V_{\sqrt{2}A_7}$ are of central charge $1/2$ and $7/5$ respectively. The product and the inner product in $B$ are given by (A.1) and

\[
\begin{align*}
(\tilde{\omega}^1)_1X^1 &= \frac{1}{2}X^1, & (\tilde{\omega}^1)_1X^2 &= 0, & (\tilde{\omega}^1)_1X^3 &= \frac{1}{2}X^3, \\
(\tilde{\omega}^2)_1X^1 &= \frac{3}{2}X^1, & (\tilde{\omega}^2)_1X^2 &= 2X^2, & (\tilde{\omega}^2)_1X^3 &= \frac{3}{2}X^3, \\
(X^1)_1X^1 &= 12X^2, & (X^1)_1X^2 &= 15X^3, & (X^1)_1X^3 &= 112\tilde{\omega}^1 + 120\tilde{\omega}^2, \\
(X^2)_1X^2 &= 200\tilde{\omega}^2, & (X^2)_1X^3 &= 15X^1, & (X^3)_1X^3 &= 12X^2, \\
\langle X^1, X^3 \rangle &= 56, & \langle X^2, X^2 \rangle &= 70. 
\end{align*}
\]

**Lemma A.12.** Let $w = a\tilde{\omega}^1 + b\tilde{\omega}^2 + cX^1 + dX^2 + eX^3$ be a conformal vector in $B$. Then $(a, b, c, d, e)$ satisfies the following system of equations.

\[
\begin{align*}
& a^2 + 112ce - a = 0, & b^2 + 100d^2 + 120ce - b = 0, \\
& ac + 3be + 30de - 2e = 0, & 6c^2 + 6e^2 + 2bd - d = 0, \\
& ae + 3be + 30cd - 2e = 0. & 
\end{align*}
\]

The central charge of $w$ is given by $\frac{1}{2}a^2 + \frac{7}{9}b^2 + 224ce + 140d^2$.

The solutions $(a, b, c, d, e)$ of Equation (A.11) are as follows, where $\xi = e^{\pi\sqrt{-1}/4}$ is a primitive 8-th root of unity.

Central charge 1/2: $(1, 0, 0, 0, 0), (\frac{1}{8}, \frac{5}{16}, \frac{1}{32}(\sqrt{-1})^j, \frac{1}{16}, (-1)^j, \frac{1}{32}(-\sqrt{-1})^j), 0 \leq j \leq 3$.

Central charge 7/10: $(0, \frac{1}{2}, 0, \pm \frac{1}{20}, 0), (\frac{2}{7}, \frac{3}{8}, \frac{1}{32}(\sqrt{-1})^j, \frac{3}{160}(-1)^j, \frac{1}{32}(-\sqrt{-1})^j), 0 \leq j \leq 3$.

Central charge 21/22: $\left(\frac{7}{11}, \frac{5}{11}, \frac{1}{22}\xi^{2j+1}, 0, \frac{1}{22}\xi^{-(2j+1)}\right), 0 \leq j \leq 3$.

Central charge 52/55: $(\frac{1}{11}, \frac{6}{11}, \frac{1}{22}\xi^{2j+1}, 0, \frac{1}{22}\xi^{-(2j+1)}), 0 \leq j \leq 3$.

Central charge 6/5: $(1, \frac{1}{2}, 0, \pm \frac{1}{20}, 0), (\frac{1}{8}, \frac{13}{16}, \frac{1}{32}(\sqrt{-1})^j, -\frac{3}{160}(-1)^j, \frac{1}{32}(-\sqrt{-1})^j), 0 \leq j \leq 3$.

Central charge 7/5: $(0, 1, 0, 0, 0), (\frac{7}{8}, \frac{11}{16}, \frac{1}{32}(\sqrt{-1})^j, -\frac{1}{32}(-1)^j, \frac{1}{32}(-\sqrt{-1})^j), 0 \leq j \leq 3$.

Central charge 19/10: $(1, 1, 0, 0, 0)$.
2B case. In this case, $L = L(7) \cong D_8$, $n_7 = 2$, and the conformal vector $\tilde{\omega}^1 \in V_{\sqrt{2}D_8}$ is of central charge 1. The product and the inner product in $\mathcal{B}$ are given by (A.1) and

$$(\tilde{\omega}^1)_1X^1 = 2X^1, \quad (X^1)_1X^1 = 512\tilde{\omega}^1, \quad \langle X^1, X^1 \rangle = 128.$$ 

Lemma A.13. Let $w = a\tilde{\omega}^1 + bX^1$ be a conformal vector in $\mathcal{B}$. Then we have $a = 1/2$ and $b = \pm 1/32$, and the central charge of $w$ is 1/2. In other words, there are exactly two conformal vectors in $\mathcal{B}$ and both of them are of central charge 1/2.

3C case. In this case, $L = L(8) \cong A_8$, $n_8 = 3$, and the conformal vector $\tilde{\omega}^1 \in V_{\sqrt{2}A_8}$ is of central charge 16/11. The product and the inner product in $\mathcal{B}$ are given by (A.1) and

$$(\tilde{\omega}^1)_1X^1 = 2X^1, \quad (\tilde{\omega}^1)_1X^2 = 2X^2,$$

$$(X^1)_1X^1 = 20X^2, \quad (X^2)_1X^1 = 20X^1, \quad (X^1)_1X^2 = 231\tilde{\omega}^1, \quad \langle X^1, X^2 \rangle = 84.$$ 

Lemma A.14. Let $w = a\tilde{\omega}^1 + bX^1 + cX^2$ be a conformal vector in $\mathcal{B}$. Then $(a, b, c)$ satisfies the following system of equations.

$$a^2 + 924bc - a = 0, \quad 2ab + 10c^2 - b = 0,$$

$$2ac + 10b^2 - c = 0.$$ 

(A.13)

The central charge of $w$ is given by $\frac{16}{11}a^2 + 336bc$.

The solutions $(a, b, c)$ of Equation (A.13) are as follows, where $\xi = e^{2\pi\sqrt{-1}/3}$ is a primitive cubic root of unity.

Central charge 1/2: $\left(\frac{11}{32}, \frac{1}{32}\xi^j, \frac{1}{32}\xi^2j\right)$, $j = 0, 1, 2$.

Central charge 21/22: $\left(\frac{31}{32}, -\frac{1}{32}\xi^j, -\frac{1}{32}\xi^2j\right)$, $j = 0, 1, 2$.

Central charge 16/11: $(1, 0, 0, 0)$.

Appendix B. Characters of $W$-algebras and the structure of $U$ for the cases of 3C and 5A

In this appendix, we shall determine the structure of the coset subalgebra $U$ defined by (2.10) for the cases of 3C and 5A. The main tool is the character of parafermion algebras. First, let us recall a construction of parafermion algebras from [6].

Let $\Lambda_0$ and $\Lambda_1$ be the fundamental weights of the affine Lie algebra $\hat{sl}_2(\mathbb{C})$. For any positive integer $\ell$ and $0 \leq j \leq \ell$, let $\mathcal{L}(\ell, j)$ be the irreducible highest weight module of $\hat{sl}_2(\mathbb{C})$ with the highest weight $(\ell - j)\Lambda_0 + j\Lambda_1$. Note that $\mathcal{L}(\ell, 0)$ has a natural VOA structure and $\{\mathcal{L}(\ell, j) | 0 \leq j \leq \ell\}$ is the set of all inequivalent irreducible modules of $\mathcal{L}(\ell, 0)$ (cf. [18]).

Now let $A^\ell_1 = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_\ell$ be an even lattice with $\langle \epsilon_i, \epsilon_j \rangle = 2\delta_{i,j}$ and $V_{A^\ell_1}$ the lattice VOA associated with $A^\ell_1$. Then $V_{A^\ell_1} \cong (V_{A_1})^{\otimes \ell} \cong \mathcal{L}(1, 0)^{\otimes \ell}$. 
Set \( H(\ell) = \epsilon_1(-1) \cdot 1 + \cdots + \epsilon_\ell(-1) \cdot 1, \) \( E(\ell) = e^{\epsilon_1} + \cdots + e^{\epsilon_\ell}, \) and \( F(\ell) = e^{-\epsilon_1} + \cdots + e^{-\epsilon_\ell}. \) Then \( \mathbb{C}H(\ell) + \mathbb{C}E(\ell) + \mathbb{C}F(\ell) \) forms a simple Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \) inside the weight one subspace of \( V_{A_1}. \) Moreover, the subVOA generated by \( \{H(\ell), E(\ell), F(\ell)\} \) is isomorphic to \( \mathcal{L}(\ell, 0) \) (cf. [6]).

Let \( \gamma = \epsilon_1 + \cdots + \epsilon_\ell. \) Then \( \gamma(-1) \cdot 1 = H(\ell) \) and it is easy to verify that

\[
e^\gamma = \frac{1}{\ell!}((E(\ell)_{-1})^{\ell-1}E(\ell).
\]

Thus \( \mathcal{L}(\ell, 0) \) contains a subalgebra isomorphic to the lattice VOA \( V_{\mathbb{Z}\gamma}. \)

Let \( A = \langle e^\gamma \rangle \) be the subgroup of \( \mathbb{C}\{A_1\} \) generated by \( e^\gamma \) and denote by \( \Omega^A_{\mathcal{L}(\ell, j)} \) the set of all highest weight vectors for \( V_{\mathbb{Z}\gamma} \) in \( \mathcal{L}(\ell, j), 0 \leq j \leq \ell. \) It is shown in [6] that \( \Omega^A_{\mathcal{L}(\ell, 0)} = \bigoplus_{k=\ell/2}^{\ell-1} \Omega^A_{\mathcal{L}(\ell, 0)}k \) is a generalized VOA and \( \Omega^A_{\mathcal{L}(\ell, j)} \) are irreducible \( \Omega^A_{\mathcal{L}(\ell, 0)} \)-modules. Note that \( \langle \Omega^A_{\mathcal{L}(\ell, 0)} \rangle \) itself is a VOA, which we shall denote by \( W_\ell. \)

Now let

\[
\mathcal{L}(\ell, j) = \bigoplus_{k=0}^{2\ell-1} V_{(k/2)\gamma+\mathbb{Z}\gamma} \otimes W_\ell(j, k)
\]

be the decomposition of \( \mathcal{L}(\ell, j) \) as a \( V_{\mathbb{Z}\gamma} \otimes W_\ell \)-module, where \( W_\ell(j, k) \) is the multiplicity of \( V_{k\gamma/2\ell+\mathbb{Z}\gamma} \) in \( \mathcal{L}(\ell, j). \) By [6], \( W_\ell(j, k) = 0 \) if \( j + k \equiv 1 \mod 2 \) and so

\[
\mathcal{L}(\ell, j) = \begin{cases} 
\bigoplus_{k=0}^{\ell-1} V_{(k/\ell)\gamma+\mathbb{Z}\gamma} \otimes W_\ell(j, 2k) & \text{if } j \text{ is even}, \\
\bigoplus_{k=0}^{\ell-1} V_{((2k+1)/2)\gamma+\mathbb{Z}\gamma} \otimes W_\ell(j, 2k+1) & \text{if } j \text{ is odd}.
\end{cases}
\]

(B.1)

**Proposition B.1** (cf. Dong and Lepowsky [6]). All \( W_\ell(j, k), 0 \leq j \leq \ell, 0 \leq k \leq 2\ell-1, j \equiv k \mod 2, \) are irreducible \( W_\ell \)-modules.

By (B.1), we can actually compute the character of \( W_\ell(j, k) \) by using the character of \( \mathcal{L}(\ell, j) \) and \( V_{(k/2)\gamma+\mathbb{Z}\gamma}. \) Recall that the character \( \text{ch}_M(q) \) of a module \( M \) of a VOA \( V \) is defined by \( \text{ch}_M(q) = \text{tr}_M q^L(0), \) where \( L(0) = \omega_1 \) with \( \omega \) being the Virasoro element of \( V. \) In Kac and Raina [23], the following formula is proved.

\[
\text{tr}_{\mathcal{L}(\ell, j)} \sigma(-1/2z\gamma) q^{L(0)-\ell/8(\ell+2)} = \frac{\theta_{j+1,\ell+2}(\tau, z) - \theta_{j-1,\ell+2}(\tau, z)}{\theta_{1,2}(\tau, z) - \theta_{-1,2}(\tau, z)},
\]

(B.2)

where \( q = e^{2\pi\sqrt{-1}\tau}, \) \( \sigma(z\gamma) = e^{2\pi\sqrt{-1}z\gamma(0)} \) for \( z \in \mathbb{Q}, \) and

\[
\theta_{n,m}(\tau, z) = \sum_{j \in n/2m+\mathbb{Z}} e^{2\pi\sqrt{-1}mzj} q^{mj^2}.
\]
By using the \( q \)-integers \([n]_q = (q^n - q^{-n})/(q - q^{-1}) = q^{n-1} + q^{n-2} + \cdots + q - 1\) for \( n \in \mathbb{Z} \), we can rewrite the formula (B.2) in the following form.

\[
\text{tr}_{L(\ell,0)}(\sigma(-1/2)\eta)q^{L(0)} = \frac{1 + \sum_{m \in \mathbb{Z}\setminus\{0\}} [2(\ell + 2)m + 1] e^{\sqrt{-1} \tau} \cdot q^{(\ell+2)m^2+m}}{1 + \sum_{n \in \mathbb{Z}\setminus\{0\}} [4n + 1] e^{\sqrt{-1} \tau} \cdot q^{2n^2+n}}.
\]

(B.3)

The character of \( V_{(k/2\ell)\gamma + \mathbb{Z}\gamma} \) is also well known (cf. [17]). It is given by

\[
\text{ch}_{V_{(k/2\ell)\gamma + \mathbb{Z}\gamma}}(q) = \text{tr}_{V_{(k/2\ell)\gamma + \mathbb{Z}\gamma}} q^{L(0)} = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^{\theta_{(k/2\ell)\gamma + \mathbb{Z}\gamma}}},
\]

where \( \theta_{(k/2\ell)\gamma + \mathbb{Z}\gamma}(q) = \sum_{\alpha \in (k/2\ell)\gamma + \mathbb{Z}\gamma} q^{(\alpha,\alpha)/2} \).

Next, we consider another construction of the VOA \( W_\ell = W_\ell(0,0) \) given in [32] by using the lattice VOA \( V_{\sqrt{2}A_{\ell-1}} \), namely,

\[
W_\ell \cong \{ u \in V_{\sqrt{2}A_{\ell-1}} | s(A_{\ell-1})u = 0 \},
\]

where

\[
s(A_{\ell-1}) = \frac{1}{2(\ell + 2)} \sum_{\alpha \in \Phi^+(A_{\ell-1})} \left( \alpha(-1)^2 \cdot 2(e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}) \right)
\]

and \( \Phi^+(A_{\ell-1}) \) is the set of all positive roots of the lattice of type \( A_{\ell-1} \) (cf. (2.1)).

Let \( N = \text{span}_\mathbb{Z}\{\epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, \ldots, \epsilon_{\ell-1} - \epsilon_\ell\} \subset A_1^\ell \). Then \( N \cong \sqrt{2}A_{\ell-1} \). Moreover,

\[
A_1^\ell = \bigcup_{k=0}^{\ell-1} (k\eta + N + \mathbb{Z}\gamma) = \bigcup_{k=0}^{\ell-1} \left((k\eta + N) + (\frac{k}{\ell}\gamma + \mathbb{Z}\gamma)\right),
\]

where \( \gamma = \epsilon_1 + \cdots + \epsilon_\ell \) and \( \eta = \frac{1}{\ell^2}(-\epsilon_1 - \cdots - \epsilon_{\ell-1} + (\ell - 1)\epsilon_\ell) \). Indeed, \( |A_1^\ell/(N + \mathbb{Z}\gamma)| = \ell \) and \( \eta + \gamma/\ell = \epsilon_\ell \). Note that \( \langle \eta, \gamma \rangle = 0 \) and \( \langle N, \gamma \rangle = 0 \).

By the above argument, we obtain the following proposition.

**Proposition B.2.** For any positive integer \( \ell \) and \( k = 0, \ldots, \ell - 1 \),

\[
W_\ell(0,2k) \cong \{ u \in V_{k\eta+N} | s(A_{\ell-1})u = 0 \}.
\]

**B.1. The case for 3C.** We consider an embedding of a \( \sqrt{2}E_8 \) lattice into \( \mathbb{R}^9 \) by using \( N \) and \( \eta \) for the case \( \ell = 9 \). Indeed, \( \sqrt{2}E_8 \) can be realized as \( \sqrt{2}E_8 = \epsilon_{j+1} - \epsilon_{j+2} \), where \( \alpha_0, \alpha_1, \ldots, \alpha_8 \) are the nine nodes in the extended \( E_8 \) diagram (2.2), can be realized as \( \sqrt{2}E_8 = \epsilon_{j+1} - \epsilon_{j+2} \), where \( \alpha_0, \alpha_1, \ldots, \alpha_8 \) are the nine nodes in the extended \( E_8 \) diagram (2.2), can be realized as \( \sqrt{2}E_8 = \epsilon_{j+1} - \epsilon_{j+2} \), where \( \alpha_0, \alpha_1, \ldots, \alpha_8 \) are the nine nodes in the extended \( E_8 \) diagram (2.2), can be realized as \( \sqrt{2}E_8 = \epsilon_{j+1} - \epsilon_{j+2} \). Then \( N = \sqrt{2}L(8) \), where \( L(8) \) is the sublattice spanned by \( \alpha_0, \alpha_1, \ldots, \alpha_7 \) (cf. Section 3). Hence \( \sqrt{2}E_8 = N \cup (3\eta + N) \cup (6\eta + N) \). This implies that

\[
V_{\sqrt{2}E_8} = V_{\sqrt{2}A_8} \oplus V_{3\eta+\sqrt{2}A_8} \oplus V_{6\eta+\sqrt{2}A_8}.
\]

Furthermore, by the definition (2.10) of the coset subalgebra, \( U = \{ u \in V_{\sqrt{2}E_8} | s(A_8)u = 0 \} \). So by Proposition B.2, we have the following result.
Proposition B.3. As a module of $W_9$,

$$U \cong W_9(0,0) \oplus W_9(0,6) \oplus W_9(0,12).$$  \hfill (B.5)

Remark B.4. It is easy to see that $W_9(0,6)$ and $W_9(0,12)$ have the same character by their construction. In fact, $W_9(0,6)$ is the dual module of $W_9(0,12)$. Thus by (B.5), $\text{ch}_U(q) \equiv \text{ch}_{W_9}(q) \mod 2$.

As discussed in Subsection 3.9, the coset subalgebra $U$ can be decomposed into a direct sum of irreducible modules of $L(1/2,0) \otimes L(21/22,0)$ in the following form.

$$U \cong m_1 L(\frac{1}{2},0) \otimes L(\frac{21}{22},0) \oplus m_2 L(\frac{1}{2},0) \otimes L(\frac{21}{22},8)$$

$$\oplus m_3 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{7}{2}) \oplus m_4 L(\frac{1}{2}, \frac{1}{2}) \otimes L(\frac{21}{22}, \frac{45}{2})$$

$$\oplus m_5 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{31}{16}) \oplus m_6 L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{21}{22}, \frac{175}{16}),$$  \hfill (B.6)

where $m_j \in \mathbb{Z}$ denotes the multiplicity of each summand. We want to show that $m_j = 1$ for $j = 1, \ldots, 6$.

By (B.1), $L(9,0) = \bigoplus_{k=0}^{8} V_{(k/9)\gamma + z\gamma} \otimes W_9(0,2k)$. Let $z = j/9$ in (B.2). Then $\sigma(-\frac{j}{18}\gamma)$ acts on $V_{(k/9)\gamma + z\gamma}$ as a scalar $e^{-2\pi\sqrt{-1}kj/9}$ and acts on $W_9(0,2k)$ as the identity. Hence

$$\text{tr}_{L(9,0)}\sigma(-\frac{j}{18}\gamma)q^{L(0)} = \sum_{k=0}^{8} e^{-2\pi\sqrt{-1}kj/9} \text{ch}_{V_{(k/9)\gamma + z\gamma}}(q) \text{ch}_{W_9(0,2k)}(q)$$

and thus

$$\text{ch}_{V_{(k/9)\gamma + z\gamma}}(q) \text{ch}_{W_9(0,2k)}(q) = \frac{1}{9} \sum_{j=0}^{8} \text{tr}_{L(9,0)}\sigma(-\frac{j}{18}\gamma)q^{L(0)} e^{2\pi\sqrt{-1}kj/9}.$$  \hfill (B.7)

Now, consider the case for $k = 0, 3,$ and $6$. By (B.2) and (B.4), we can calculate that

$$\text{ch}_{W_9(0,0)}(q) = 1 + q^2 + 2q^3 + 4q^4 + 6q^5 + 11q^6 + 16q^7 + 27q^8$$

$$+ 40q^9 + 62q^{10} + 90q^{11} + 137q^{12} + 194q^{13} + 284q^{14}$$

$$+ 400q^{15} + 569q^{16} + 788q^{17} + 1102q^{18} + 1504q^{19}$$

$$+ 2066q^{20} + 2792q^{21} + 3776q^{22} + 5046q^{23} + \cdots,$$

$$\text{ch}_{W_9(0,6)}(q) = \text{ch}_{W_9(0,12)}(q) = q^2 + q^3 + 3q^4 + 5q^5 + 9q^6 + 14q^7$$

$$+ 25q^8 + 36q^9 + 58q^{10} + 86q^{11} + \cdots.$$  \hfill (B.8)

Let $[h_1, h_2]_n$ be the weight $n$ subspace of $L(1/2, h_1) \otimes L(21/22, h_2)$. Its dimension for small $n$ is as follows.
dim[0, 0]_2 = 2, \quad \dim[0, 0]_4 = 5, \quad \dim[\frac{1}{16}, \frac{16}{16}]_4 = 4, \\
\dim[0, 0]_8 = 27, \quad \dim[\frac{1}{16}, \frac{31}{16}]_8 = 36, \quad \dim[\frac{1}{2}, \frac{7}{2}]_8 = 13, \\
\dim[0, 0]_{11} = 75, \quad \dim[\frac{1}{16}, \frac{31}{16}]_{11} = 130, \quad \dim[\frac{1}{2}, \frac{7}{2}]_{11} = 51, \\
\dim[0, 8]_{11} = 5, \quad \dim[0, 0]_{23} = 3073, \quad \dim[\frac{1}{16}, \frac{31}{16}]_{23} = 7040, \\
\dim[\frac{1}{2}, \frac{7}{2}]_{23} = 3510, \quad \dim[0, 8]_{23} = 946, \quad \dim[\frac{1}{16}, \frac{175}{16}]_{23} = 490.

Comparing the characters of (B.5) and (B.6), we have

Furthermore, the following proposition holds by the definition (2.10) of the coset sub-

B.2. The case for 5A. In this case we consider an embedding of a $\sqrt{2}E_8$ lattice into $\mathbb{R}^{10}$
by using two $N$’s and two $\eta$’s for the case $\ell = 5$. Let $\epsilon_i, \epsilon'_i \in \mathbb{R}^{10}, 1 \leq i \leq 5$, be such that

$N = \text{span}_\mathbb{Z}\{\epsilon_i - \epsilon_{i+1}; 1 \leq i \leq 4\}$, \quad $N' = \text{span}_\mathbb{Z}\{\epsilon'_i - \epsilon'_{i+1}; 1 \leq i \leq 4\}$,

$\eta = \frac{1}{5}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 + 4\epsilon_5)$, \quad $\eta' = \frac{1}{5}(\epsilon'_1 - \epsilon'_2 - \epsilon'_3 - \epsilon'_4 + 4\epsilon'_5)$.

Let $\sqrt{2}\alpha_j = \epsilon_{j+1} - \epsilon_{j+2}, 0 \leq j \leq 3$, $\sqrt{2}\alpha_7 = \epsilon'_1 - \epsilon'_2$, $\sqrt{2}\alpha_6 = \epsilon'_2 - \epsilon'_3$, $\sqrt{2}\alpha_5 = \epsilon'_3 - \epsilon'_4$, $\sqrt{2}\alpha_8 = \epsilon'_4 - \epsilon'_5$, $\sqrt{2}\alpha_4 = \eta + 2\eta' + (\epsilon'_4 - \epsilon'_5) \in \eta + 2\eta' + N'$. Then $\alpha_0, \alpha_1, \ldots, \alpha_8$ are the

nine nodes in the extended $E_8$ diagram (2.2) and $N \oplus N' = \sqrt{2}L(4)$, where $L(4)$ is the

sublattice spanned by $\alpha_j, 0 \leq j \leq 8, j \neq 4$ (cf. Section 3). Hence $|\sqrt{2}E_8/N \oplus N'| = 5$ and

$\sqrt{2}E_8 = N \oplus N' \cup (\eta + N) \oplus (2\eta' + N') \cup (2\eta + N) \oplus (4\eta' + N')$

$\cup (3\eta + N) \oplus (\eta' + N') \cup (4\eta + N) \oplus (3\eta' + N')$.

This implies that

$V_{\sqrt{2}E_8} = (V_{\sqrt{2}A_4} \otimes V_{\sqrt{2}A_4}) \oplus (V_{\eta + \sqrt{2}A_4} \otimes V_{2\eta + \sqrt{2}A_4}) \oplus (V_{2\eta + \sqrt{2}A_4} \otimes V_{4\eta + \sqrt{2}A_4})$

$\oplus (V_{3\eta + \sqrt{2}A_4} \otimes V_{\eta + \sqrt{2}A_4}) \oplus (V_{4\eta + \sqrt{2}A_4} \otimes V_{3\eta + \sqrt{2}A_4})$.

Furthermore, the following proposition holds by the definition (2.10) of the coset sub-

algebra $U$ and Proposition B.2.
Proposition B.6. As a module of $W_5 \otimes W_5$,
\[ U \cong (W_5(0, 0) \otimes W_5(0, 0)) \oplus (W_5(0, 2) \otimes W_5(0, 4)) \oplus (W_5(0, 4) \otimes W_5(0, 8)) \oplus (W_5(0, 6) \otimes W_5(0, 2)) \oplus (W_5(0, 8) \otimes W_5(0, 6)). \quad (B.7) \]

Note that $\text{ch}_{V_{\gamma + \sqrt{3}A_4}}(q) = \text{ch}_{V_{2\gamma + \sqrt{3}A_4}}(q)$ and $\text{ch}_{V_{2\gamma + \sqrt{3}A_4}}(q) = \text{ch}_{V_{\gamma + \sqrt{3}A_4}}(q)$. Thus by the construction, it is also easy to see that

\[ \text{ch}_{W_5(0,2) \otimes W_5(0,4)}(q) = \text{ch}_{W_5(0,4) \otimes W_5(0,8)}(q) = \text{ch}_{W_5(0,6) \otimes W_5(0,2)}(q) = \text{ch}_{W_5(0,8) \otimes W_5(0,6)}(q). \]

Hence

\[ \text{ch}_U(q) = \text{ch}_{W_5(0,0)}(q)^2 + 4\text{ch}_{W_5(0,2)}(q)\text{ch}_{W_5(0,4)}(q). \quad (B.8) \]

By (B.1), $L(5, 0) = \bigoplus_{k=0}^4 V_{(k/5)\gamma + Z\gamma} \otimes W_5(0, 2k)$. Let $z = j/5$ in (B.2). Then $\sigma(-\frac{j}{10})$ acts on $V_{(k/5)\gamma + Z\gamma}$ as a scalar $e^{-2\pi\sqrt{-1}kj/5}$ and acts on $W_5(0, 2k)$ as the identity. Then arguing as in the case for $3C$, we get

\[ \text{ch}_{V_{(k/5)\gamma + Z\gamma}}(q)\text{ch}_{W_5(0,2k)}(q) = \frac{1}{5} \sum_{j=0}^4 e^{2\pi\sqrt{-1}kj/5} \text{tr}_{L(5,0)}(\sigma(-\frac{j}{10})q)J(0). \]

Now using (B.2) and (B.4), we can calculate that

\[ \text{ch}_{W_5(0,0)}(\tau) = 1 + q^2 + 2q^3 + 4q^4 + 6q^5 + 10q^6 + 14q^7 + 23q^8 + 32q^9 + 48q^{10} + 66q^{11} + 96q^{12} + 130q^{13} + 183q^{14} + 246q^{15} + \cdots, \]

\[ \text{ch}_{W_5(0,2)}(\tau) = q^{4/5}(1 + q + 2q^2 + 3q^3 + 6q^4 + 8q^5 + 14q^6 + 20q^7 + 31q^8 + 43q^9 + 64q^{10} + 87q^{11} + 125q^{12} + 169q^{13} + 234q^{14} + 313q^{15} + \cdots), \]

\[ \text{ch}_{W_5(0,4)}(\tau) = q^{6/5}(1 + q + 3q^2 + 4q^3 + 7q^4 + 10q^5 + 17q^6 + 23q^7 + 36q^8 + 50q^9 + 73q^{10} + 100q^{11} + 142q^{12} + 191q^{13} + 265q^{14} + 353q^{15} + \cdots). \]

In Subsection 3.5 we have shown that $U$ contains a subalgebra isomorphic to $L(1/2, 0) \otimes L(25/28, 0) \otimes L(25/28, 0)$. We also know all irreducible modules $L(1/2, h_1) \otimes L(25/28, h_2) \otimes L(25/28, h_3)$ with integral weights. Note that $U$ is a direct sum of these irreducible modules. Comparing the characters of those irreducible modules and (B.8), we can verify the following theorem.

Theorem B.7. As a module of $L(1/2, 0) \otimes L(25/28, 0) \otimes L(25/28, 0)$,
\[ U \cong L(\frac{1}{2}, 0) \otimes L(\frac{25}{28}, 0) \otimes L(\frac{25}{28}, 0) \oplus L(\frac{1}{2}, \frac{1}{10}) \otimes L(\frac{25}{28}, \frac{5}{72}) \otimes L(\frac{25}{28}, \frac{57}{72}) \]
\[ \oplus L(\frac{1}{2}, \frac{1}{10}) \otimes L(\frac{25}{28}, \frac{57}{72}) \otimes L(\frac{25}{28}, \frac{5}{72}) \oplus L(\frac{7}{12}, \frac{1}{7}) \otimes L(\frac{25}{28}, \frac{13}{4}) \otimes L(\frac{25}{28}, \frac{1}{4}) \]
\[ \oplus L(\frac{1}{2}, 0) \otimes L(\frac{25}{28}, \frac{3}{4}) \otimes L(\frac{25}{28}, \frac{13}{4}) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{25}{28}, \frac{13}{4}) \otimes L(\frac{25}{28}, \frac{1}{4}) \]
\[ \oplus L(\frac{1}{2}, \frac{1}{10}) \otimes L(\frac{25}{28}, \frac{57}{72}) \otimes L(\frac{25}{28}, \frac{165}{32}) \oplus L(\frac{1}{2}, \frac{1}{10}) \otimes L(\frac{25}{28}, \frac{165}{32}) \otimes L(\frac{25}{28}, \frac{57}{32}) \]
\[ \oplus L(\frac{1}{2}, \frac{1}{10}) \otimes L(\frac{25}{28}, \frac{13}{4}) \otimes L(\frac{25}{28}, \frac{13}{4}) \oplus L(\frac{1}{2}, \frac{1}{10}) \otimes L(\frac{25}{28}, 0) \otimes L(\frac{25}{28}, \frac{15}{2}) \]
\[ \oplus L(\frac{1}{2}, \frac{1}{10}) \otimes L(\frac{25}{28}, \frac{57}{72}) \otimes L(\frac{25}{28}, 0) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{25}{28}, \frac{15}{2}) \otimes L(\frac{25}{28}, \frac{15}{2}). \]
B.3. Character of the 6A-algebra. Let $U_{6A}$ be the 6A-algebra, the coset subalgebra of $V_{\sqrt{2}E_8}$ constructed from the coset decomposition $E_8/(A_5 \oplus A_2 \oplus A_1)$. We show here that

$$U_{6A} \simeq U_{3A}(0) \otimes L(25/28,0) \bigoplus U_{3A}(5/7) \otimes L(25/28,9/7) \bigoplus U_{3A}(1/7) \otimes L(25/28,34/7),$$

(B.9)

where $U_{3A}(0)$ denotes the 3A-algebra and $U_{3A}(h)$, $h = 5/7, 1/7$, denote the irreducible $U_{3A}(0)$-modules whose top weights are equal to $h$. We show the above isomorphism by computing the $q$-character of $U_{6A}$. Recall the notion of $W_q$-algebras in Appendix B. By considering an isometric embedding $\sqrt{2}A_5 \oplus \sqrt{2}A_2 \oplus \sqrt{2}A_1 \rightarrow A_1^0 \oplus A_3^1 \oplus \sqrt{2}A_1$, one can easily verify that

$$U_{6A} \simeq W_6(0,0) \otimes W_3(0,0) \otimes L(1/2,0) \bigoplus W_6(0,4) \otimes W_3(0,4) \otimes L(1/2,1/2) \bigoplus W_6(0,8) \otimes W_3(0,4) \otimes L(1/2,0) \bigoplus W_6(0,10) \otimes W_3(0,2) \otimes L(1/2,1/2),$$

where we have used the fact that the lattice VOA $V_{\sqrt{2}A_1}$ is isomorphic to the code VOA associated to a binary code $\{(00), (11)\}$. Since $W_3(0,0) \simeq L(4/5,0) \oplus L(4/5,3)$ and $W_3(0,2) \simeq W_3(0,4) \simeq L(4/5,2/3)$ as $L(4/5,0)$-modules, we only need to compute the characters of $W_6(0,2s)$, $0 \leq s \leq 5$. However, by noticing the dual module relations, we know that $ch_{W_6(0,2)}(q) = ch_{W_6(0,10)}(q)$ and $ch_{W_6(0,4)}(q) = ch_{W_6(0,8)}(q)$. Therefore, we should compute characters of $W_6(0,2s)$ for $s = 0, 1, 2, 3$. By a method in the preceding, we can obtain the following results.

$$ch_{W_6(0,0)} = 1 + q^2 + 2q^3 + 4q^4 + 6q^5 + 11q^6 + \cdots,$n
$$ch_{W_6(0,2)} = q^{5/6} + q^{11/6} + 2q^{17/6} + 3q^{23/6} + 6q^{29/6} + 9q^{35/6} + \cdots,$n
$$ch_{W_6(0,4)} = q^{4/3} + q^{7/3} + 3q^{10/3} + 4q^{13/3} + 8q^{16/3} + \cdots,$n
$$ch_{W_6(0,6)} = q^{3/2} + q^{5/2} + 3q^{7/2} + 5q^{9/2} + 8q^{11/2} + \cdots.$n

Then by comparing characters, we can establish the desired isomorphism (B.9).

B.3.1. Highest weight vector of weight $(0, 1/7, 34/7)$. In this section, we prove that the 6A-algebra is generated by its weight two subspace as a vertex operator algebra. We use the notation as in section 5.6 of our preprint. Let $w^1 = \tilde{\omega}(A_2)$, $w^2 = \tilde{\omega}(E_6)$ and $w^3 = \tilde{\omega}(A_1) + \tilde{\omega}(A_5) - w^2$. Then $w^1, w^2, w^3$ are mutually orthogonal conformal vector of central charges $4/5, 6/7$ and $25/28$, respectively, and the sum $w^1 + w^2 + w^3$ is the Virasoro vector of $U_{6A}$ and the sum $w^1 + w^2$ is the Virasoro vector of $U_{3A}(0)$ in the decomposition (B.9). We can write down $w^i$ explicitly as follows.

$$w^1 = \tilde{\omega}(A_2), \quad w^2 = \frac{2}{7}\tilde{\omega}(A_1) + \frac{4}{7}\tilde{\omega}(A_5) + \frac{1}{14}X^3, \quad w^3 = \frac{5}{7}\tilde{\omega}(A_1) + \frac{3}{7}\tilde{\omega}(A_5) - \frac{1}{14}X^3.$n

Let $W$ be the subalgebra of $U_{6A}$ generated by its weight two subspace. It is clear that $W$ contains both $U_{3A}(0) \otimes L(25/28,0)$ and $U_{3A}(5/7) \otimes L(25/28,9/7)$. So we only have
to show that $W$ contains an irreducible $\text{Vir}(w^1) \otimes \text{Vir}(w^2) \otimes \text{Vir}(w^3)$-module of highest weight $(0, 1/7, 34/7)$. Let $v$ be a highest weight vector for $\text{Vir}(w^1) \otimes \text{Vir}(w^2) \otimes \text{Vir}(w^3)$ with highest weight $(0, 5/7, 9/7)$. Since such a vector is unique up to linearity, we may take $v$ as follows.

$$v = 400\bar{\omega}(A_1) - 16\bar{\omega}(A_5) + X^3.$$ 

Then by the fusion rules for the unitary Virasoro VOAs, we know that $v_{-2}v$ is contained in the weight 5 subspace of

$$L(4/5, 0) \otimes L(6/7, 5/7) \otimes L(25/28, 9/7) \oplus L(4/5, 0) \otimes L(6/7, 1/7) \otimes L(25/28, 34/7).$$

Assume that $W$ is a proper subalgebra. Then $W$ is isomorphic to $U_{3A}(0) \otimes L(25/28, 0) \oplus U_{3A}(5/7) \otimes L(25/28, 9/7)$ and $v_{-2}v$ is in the weight 5 subspace of

$$L(4/5, 0) \otimes L(6/7, 5/7) \otimes L(25/28, 9/7).$$

In this case, by computing inner products, we must have the following equality:

$$v_{-2}v = \frac{132}{7}w_1^2w_2^2v + \frac{1870}{49}w_2^3v - \frac{80}{21}w_2w_3^3v + \frac{1128}{49}w_2^2w_2v + \frac{288}{23}w_0^2w_3^2v$$

$$- \frac{40}{23}w_0^2w_3^3v + \frac{440}{19}w_1^2w_3^3v - \frac{198}{19}w_0^2w_2w_3^3v.$$ 

However, the squared length of the right hand side is $1913313600/437$, whereas that of the left hand side is $10209600$. This means that there is a highest weight vector of weight $(0, 1/7, 34/7)$ in $v_{-2}v$, and we can understand that the difference

$$10209600 - 1913313600/437 = 2548281600/437$$

is the squared length of the highest weight vector. Thus the 6A-algebra $U_{6A}$ is generated by its weight two subspace.

**Appendix C. Classification for the Irreducible Modules of the 3C and 5A Algebras**

In this appendix, we shall give the classification of all the irreducible modules for the 3C and 5A algebras. For convenience, we introduce the following notation. Let $V$ be a VOA and $M$ its module. For subsets $A \subset V$ and $B \subset M$, we set

$$A \cdot B = \text{span}\{a_nv \mid a \in A, v \in B, n \in \mathbb{Z}\}.$$ 

It is shown in Lemma 3.12 of [36] that for $a, b \in V$, $v \in M$ and $p, q \in \mathbb{Z}$, there are $m, n \geq 0$ such that

$$a_pb_qv = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{p-n}{i} \binom{n}{j} (a_{p-n-i+j}b)_{q+n+i-j}v. \quad (C.1)$$

In particular, $V \cdot (V \cdot v) = V \cdot v$ for any $v \in M$ so that $V \cdot v$ is a submodule of $M$. 
Recall the notion of the fusion products (cf. [22] [35]). We denote by $M^1 \boxtimes_W M^2$ the fusion product of $V$-modules $M^1$ and $M^2$. The basic result (loc. cit.) is that the fusion product exists if $V$ is rational.

We shall study an extension of a rational VOA by an irreducible module which is not a simple current module. Our settings are as follows.

Let $V$ be a simple rational VOA and $W$ an irreducible $V$-module such that

$$\dim I_V \left( \begin{array}{c} M^2 \\ W \\ M^1 \end{array} \right) \leq 1$$

(C.2)

for any irreducible $V$-modules $M^1$ and $M^2$, where $I_V \left( \begin{array}{c} M^2 \\ W \\ M^1 \end{array} \right)$ denotes the space of $V$-intertwining operators of type $\left( \begin{array}{c} M^2 \\ W \\ M^1 \end{array} \right)$. Assume that the space $\tilde{V} = V \oplus W$ has a simple VOA structure $(\tilde{V}, \tilde{Y}(\cdot, z))$ which is an extension of $V$ such that for any $u, v \in W$,

$$\tilde{Y}(u, z)v = (\mathcal{I}(u, z) + \mathcal{J}(u, z))v,$$

where

$$\mathcal{I}(\cdot, z) \in I_V \left( \begin{array}{c} V \\ W \\ W \end{array} \right) \quad \text{and} \quad \mathcal{J}(\cdot, z) \in I_V \left( \begin{array}{c} W \\ W \end{array} \right)$$

are non-zero intertwining operators. Note that the simplicity of $\tilde{V}$ implies that $V$ and $W$ are inequivalent $V$-modules. For, if $V$ and $W$ are isomorphic, then the isomorphic image of the vacuum vector of $V$ in $W$ is a vacuum-like vector (cf. [34]). Since the vertex operator of a vacuum-like vector commutes with all the vertex operators on $\tilde{V}$ and $\tilde{V}$ is simple, every vacuum-like vector is a scalar multiple of the vacuum vector. Thus $V$ and $W$ are inequivalent. Then it follows from (C.1) that $\tilde{V} = W \cdot W$. By fixing one VOA structure on $\tilde{V}$, we shall show that the module structure of certain types of irreducible $\tilde{V}$-modules are uniquely determined by their $V$-module structures.

**Lemma C.1.** Let $M$ be an irreducible $\tilde{V}$-module. Assume that $M$ contains an irreducible $V$-submodule $M^0$ such that $M$ is a direct sum $nM^0$ of $n$ copies of $M^0$ as a $V$-module. Then $M = M^0$, i.e., $n = 1$.

**Proof.** By (C.1), we know that $W \cdot M^0$ is a $V$-submodule of $M$ which is a direct sum of some copies of $M^0$. Since $\tilde{V}$ is simple, it is clear that $W \cdot M^0 \neq 0$ (cf. Proposition 11.9 of [6]). By the universal property of the fusion product (cf. [22] [35]), there exists a $V$-epimorphism from $W \boxtimes_V M^0$ onto $W \cdot M^0$. Since $\dim I_V \left( \begin{array}{c} M^0 \\ W \end{array} \right) = 1$ by (C.2), $W \boxtimes_V M^0$ contains a $V$-submodule isomorphic to $M^0$ with multiplicity one. Therefore, $W \cdot M^0$ is an irreducible $V$-submodule isomorphic to $M^0$. Similarly, $W \cdot (W \cdot M^0)$ is also an irreducible $V$-submodule of $M$. If $W \cdot M^0 = M^0$, then $\tilde{V} \cdot M^0 = (V \cdot M^0) + (W \cdot M^0) = M^0$ as a $V$-module so that $M = M^0$ and we are done. Assume that $W \cdot M^0 \neq M^0$. In this case $M^0 + (W \cdot M^0) = M^0 \oplus (W \cdot M^0)$ since both $M^0$ and $W \cdot M^0$ are irreducible $V$-submodules. Then by the irreducibility we have $M = \tilde{V} \cdot M^0 = M^0 \oplus (W \cdot M^0)$. Consider $(W \cdot W) \cdot M^0$. 

As we have seen, \( W \cdot W = V \oplus W = \tilde{V} \) so that \((W \cdot W) \cdot M^0 = M\). By the associativity formula
\[
(a_m b)_n = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} \{a_{m-i} b_{n+i} - (-1)^m b_{m+n-i} a_i\}, \tag{C.3}
\]
we see that \((W \cdot W) \cdot M^0 \subset W \cdot (W \cdot M^0)\). Since \( W \cdot (W \cdot M^0) \) is an irreducible \( V \)-submodule, so is \((W \cdot W) \cdot M^0 = \tilde{V} \cdot M^0\). But this is a contradiction. Hence \( W \cdot M^0 = M^0 \) and \( M = M^0 \).

**Lemma C.2.** Let \( M \) be an irreducible \( \tilde{V} \)-module which is also irreducible as a \( V \)-module. Then there is exactly one irreducible \( \tilde{V} \)-module structure on \( M \) up to isomorphism.

**Proof.** Suppose that \((M, Y_M(\cdot, z))\) and \((M, Y'_M(\cdot, z))\) are two irreducible \( \tilde{V} \)-module structures on an irreducible \( V \)-module \( M \). Without loss, we may assume that \( Y_M(u, z) = Y'_M(u, z) \) for all \( u \in V \). Since \( M \) is irreducible and \( \dim I_V(W^M_M) \leq 1 \) by (C.2), there exists \( \lambda \neq 0 \) such that
\[
Y'_M(a, z)v = \lambda Y_M(a, z)v \quad \text{for any } a \in W, \; v \in M.
\]

By the associativity, for any \( a, b \in W \) and \( v \in M \), there exists \( k > 0 \) such that
\[
(z_0 + z_2)^k Y_M(a, z_0 + z_2) Y_M(b, z_2)v = (z_0 + z_2)^k Y_M(Y_{\tilde{V}}(a, z_0)b, z_2)v = (z_0 + z_2)^k Y_M(I(a, z_0)b + J(a, z_0)b, z_2)v. \tag{C.4}
\]

Similarly, we have
\[
(z_0 + z_2)^k Y'_M(a, z_0 + z_2) Y'_M(b, z_2)v = (z_0 + z_2)^k Y'_M(Y_{\tilde{V}}(a, z_0)b, z_2)v = (z_0 + z_2)^k Y'_M(I(a, z_0)b + J(a, z_0)b, z_2)v.
\]

and thus
\[
\lambda^2 (z_0 + z_2)^k Y_M(a, z_0 + z_2) Y_M(b, z_2)v = (z_0 + z_2)^k (Y_M(I(a, z_0)b, z_2)v + \lambda Y_M(J(a, z_0)b, z_2)v). \tag{C.5}
\]

By (C.4) and (C.5), we have
\[
(z_0 + z_2)^k Y_M((\lambda^2 - 1)I(a, z_0)b + (\lambda^2 - \lambda)J(a, z_0)b, z_2)v = 0. \tag{C.6}
\]

We note that
\[
Y_M((\lambda^2 - 1)I(a, z_0)b + (\lambda^2 - \lambda)J(a, z_0)b, z_2)v \in M(((z_0))[[z_2, z_2^{-1}]]).
\]

Thus (C.6) implies that
\[
(\lambda^2 - 1)I(a, z_0)b = 0 \quad \text{and} \quad (\lambda^2 - \lambda)J(a, z_0)b = 0.
\]

Therefore, we have \( \lambda^2 = \lambda = 1 \). Hence there is only one irreducible \( \tilde{V} \)-module structure on \( M \). \( \square \)
Lemma C.3. Let \( M \) be an irreducible \( \tilde{V} \)-module. Assume that there are two inequivalent irreducible \( V \)-submodules \( M^0 \) and \( M^1 \) of \( M \) such that \( M \) is isomorphic to \( mM^0 \oplus nM^1 \) with \( m, n > 0 \) as a \( V \)-module. If \( \dim I_V(M^0_0) = 0 \), then \( m = n = 1 \) and there is exactly one \( \tilde{V} \)-module structure on \( M \) up to isomorphism.

Proof. First, we shall show that \( W \cdot M^0 = M^1 \) and \( M = W \cdot M^1 = M^0 \oplus M^1 \). It is clear from (C.1) that both \( W \cdot M^0 \) and \( W \cdot M^1 \) are \( V \)-submodules of \( M \) which are non-zero by Proposition 11.9 of [6]. By the universal property of the fusion product, we have a \( V \)-epimorphism from \( W \otimes_V M^0 \) onto \( W \cdot M^0 \). By our setting (C.2) and the assumption \( \dim (C.3) = 0 \), \( W \cdot M^0 \) is an irreducible \( V \)-submodule isomorphic to \( M^1 \). Thus \( M^0 + (W \cdot M^0) \) is a direct sum. Since \( M \) is an irreducible \( \tilde{V} \)-module, \( M = \tilde{V} \cdot M^0 = (V \cdot M^0) + (W \cdot M^0) = M^0 \oplus (W \cdot M^0) \). This proves \( W \cdot M^0 = M^1 \) and \( M = M^0 \oplus M^1 \). Then \( W \cdot M^1 = W \cdot (W \cdot M^0) = (W \cdot W) \cdot M^0 = \tilde{V} \cdot M^0 = M \) by (C.1) and (C.3).

Let us suppose that \( (M, Y_M(\cdot, z)) \) and \( (M, Y'_M(\cdot, z)) \) are two irreducible \( \tilde{V} \)-module structures on the \( V \)-module \( M = M^0 \oplus M^1 \). Without loss, we may assume that \( Y_M(u, z) = Y'_M(u, z) \) for all \( u \in V \). Let \( a \in W, v^0 \in M^0 \) and \( v^1 \in M^1 \) be arbitrary. As we have shown, \( W \cdot M^0 = M^1 \) under both structures. Thus there is a non-zero \( V \)-intertwining operator \( T(\cdot, z) \) of type \( (M^1, M^0) \) and a scalar \( \gamma \neq 0 \) such that

\[
Y_M(a, z)v^0 = T(a, z)v^0 \quad \text{and} \quad Y'_M(a, z)v^0 = \gamma T(a, z)v^0.
\]

We have also shown that \( W \cdot M^1 = M^0 \oplus M^1 \) under both structures. Thus there exist non-zero \( V \)-intertwining operators \( I(\cdot, z) \in I_V(M^0_0), J(\cdot, z) \in I_V(M^1_0) \) and scalars \( \mu, \lambda \neq 0 \) such that

\[
Y_M(a, z)v^1 = I(a, z)v^1 + J(a, z)v^1 \quad \text{and} \quad Y'_M(a, z)v^1 = \mu I(a, z)v^1 + \lambda J(a, z)v^1.
\]

Then

\[
Y_M(u + a, z)(v^0 + v^1) = Y_M(u, z)v^0 + I(a, z)v^1 + T(a, z)v^0 + (Y_M(u, z) + J(a, z))v^1,
\]

where \( u \in V \). Note that \( Y_M(u, z)v^0 \in M^0(\langle z \rangle), I(a, z)v^1 \in M^0(\langle z \rangle), T(a, z)v^0 \in M^1(\langle z \rangle) \) and \( (Y_M(u, z) + J(a, z))v^1 \in M^1(\langle z \rangle) \). Hence the vertex operator \( Y_M(\cdot, z) \) can be written as the following matrix form.

\[
Y_M \left( \begin{bmatrix} u \\ a \end{bmatrix}, z \right) \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} = \begin{bmatrix} Y_M(u, z) & I(a, z) \\ T(a, z) & Y_M(u, z) + J(a, z) \end{bmatrix} \begin{bmatrix} v^0 \\ v^1 \end{bmatrix}.
\]

Likewise,

\[
Y'_M \left( \begin{bmatrix} u \\ a \end{bmatrix}, z \right) \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} = \begin{bmatrix} Y_M(u, z) & \mu I(a, z) \\ \gamma T(a, z) & Y_M(u, z) + \lambda J(a, z) \end{bmatrix} \begin{bmatrix} v^0 \\ v^1 \end{bmatrix}.
\]
For simplicity, we use the following notation. For \( f(z_0, z_2), g(z_0, z_2) \in M[[z_0, z_0^{-1}, z_2, z_2^{-1}]] \), \( f(z_0, z_2) \sim g(z_0, z_2) \) means that there exists \( k > 0 \) such that
\[
(z_0 + z_2)^k f(z_0, z_2) = (z_0 + z_2)^k g(z_0, z_2).
\]

Then, by the associativity on \( Y'_M(\cdot, z) \), for any \( a, b \in W \) we have the following system of equations:
\[
\begin{bmatrix}
Y_M(\mathcal{I}(a, z_0)b, z_2) & \mu I(\mathcal{J}(a, z_0)b, z_2) \\
\gamma T(\mathcal{J}(a, z_0)b, z_2) & Y_M(\mathcal{I}(a, z_0)b, z_2) + \lambda J(\mathcal{J}(a, z_0)b, z_2) \\
\end{bmatrix}
\sim \begin{bmatrix}
\mu \gamma I(a, z_0 + z_2)T(b, z_2) & \mu \lambda I(a, z_0 + z_2)J(b, z_2) \\
\lambda \gamma J(a, z_0 + z_2)T(b, z_2) & \mu \gamma T(a, z_0 + z_2)I(b, z_2) + \lambda^2 J(a, z_0 + z_2)J(b, z_2) \\
\end{bmatrix}.
\]

Similarly, by the associativity on \( Y_M(\cdot, z) \), we have
\[
\begin{bmatrix}
Y_M(\mathcal{I}(a, z_0)b, z_2) & I(\mathcal{J}(a, z_0)b, z_2) \\
T(\mathcal{J}(a, z_0)b, z_2) & Y_M(\mathcal{I}(a, z_0)b, z_2) + J(\mathcal{J}(a, z_0)b, z_2) \\
\end{bmatrix}
\sim \begin{bmatrix}
I(a, z_0 + z_2)T(b, z_2) & I(a, z_0 + z_2)J(b, z_2) \\
J(a, z_0 + z_2)T(b, z_2) & T(a, z_0 + z_2)I(b, z_2) + J(a, z_0 + z_2)J(b, z_2) \\
\end{bmatrix}.
\]

Then by a similar argument as in the proof of Lemma C.2, we have \( \mu \gamma = 1, \mu = \mu \lambda \) and \( \lambda \gamma = \gamma \), from which we conclude that \( \lambda = 1 \) and \( \mu = 1/\gamma \). Thus we obtain the following relation:
\[
Y'_M \left( \begin{bmatrix} u \\ a \end{bmatrix}, z \right) \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\gamma} I(a, z) \\
\gamma T(a, z) + Y_M(u, z) + J(a, z) \end{bmatrix} \begin{bmatrix} v^0 \\ v^1 \end{bmatrix}.
\]

Now define \( \psi : M^0 \oplus M^1 \to M^0 \oplus M^1 \) by
\[
\psi(v^0) = \frac{1}{\gamma} v^0 \quad \text{for } v^0 \in M^0, \quad \psi(v^1) = v^1 \quad \text{for } v^1 \in M^1.
\]
Then
\[
\psi \left( Y_M \left( \begin{bmatrix} u \\ a \end{bmatrix}, z \right) \begin{bmatrix} v^0 \\ v^1 \end{bmatrix} \right) = \begin{bmatrix} \frac{1}{\gamma}(Y_M(u, z)v^0 + I(a, z)v^1) \\
\gamma T(a, z)v^0 + Y_M(u, z)v^1 + J(a, z) \end{bmatrix} = \begin{bmatrix} Y'_M \left( \begin{bmatrix} u \\ a \end{bmatrix}, z \right) \begin{bmatrix} \psi(v^0) \\ \psi(v^1) \end{bmatrix} \right).
\]
Hence \( \psi \) induces a \( \tilde{V} \)-isomorphism from \((M, Y_M(\cdot, z))\) to \((M, Y'_M(\cdot, z))\).

C.1. Irreducible modules for the 3C algebra. First we shall classify all irreducible modules for the 3C-algebra \( U_{3C} \). Recall that
\[
U_{3C} \cong L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, 0\right) + L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{21}{22}, 8\right) \\
+ L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{7}{2}\right) + L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{21}{22}, \frac{45}{2}\right) \\
+ L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{31}{16}\right) + L\left(\frac{1}{2}, \frac{1}{16}\right) \otimes L\left(\frac{21}{22}, \frac{175}{16}\right).
\]
as a module of $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$. For simplicity, we shall use $[h_1, h_2]$ to denote the module $L(\frac{1}{2}, h_1) \otimes L(\frac{21}{22}, h_2)$.

Let $e \in U_{3C}$ be the Virasoro element of the VOA $L(\frac{1}{2}, 0)$ and $\tau_e$ the corresponding Miyamoto involution. Then the corresponding fixed point subalgebra of $\tau_e$ is

$$U_{3C}^\tau = [0, 0] \oplus [0, 8] \oplus \left[ \frac{1}{2}, \frac{7}{2} \right] \oplus \left[ \frac{1}{2}, \frac{45}{2} \right].$$

Set $V = [0, 0] \oplus \left[ \frac{1}{2}, \frac{45}{2} \right]$ and $W = [0, 8] \oplus \left[ \frac{1}{2}, \frac{7}{2} \right]$. Then $V$ is a subalgebra of $U_{3C}^\tau$ and $W$ is an irreducible $V$-submodule of $U_{3C}^\tau$. It is shown in the proof of Theorem 3.36 that $W \cdot W = V \oplus W = U_{3C}^\tau$. Therefore, we can use the representation theory for $V$ to classify irreducible $U_{3C}^\tau$-modules. Note that $V$ is a $\mathbb{Z}_2$-graded simple current extension of $[0, 0]$ so that $V$ is rational. Moreover, irreducible $V$-modules and their fusion rules are easily determined (cf. [27, 29, 47]).

**Lemma C.4** (cf. [27, 29, 47]). The irreducible modules for $V = [0, 0] \oplus \left[ \frac{1}{2}, \frac{45}{2} \right]$ are given as follows.

$$[0, h_{i,j}] \oplus \left[ \frac{1}{2}, h_{i,12-j} \right], \quad \left[ \frac{1}{16}, h_{i,k} \right] \oplus \left[ \frac{1}{16}, h_{i,12-k} \right], \quad \left[ \frac{1}{16}, h_{i,6} \right]^+ \text{ and } \left[ \frac{1}{16}, h_{i,6} \right]^-, $$

where $1 \leq i \leq 5$, $j = 1, 3, 5, 7, 9, 11$ and $k = 2, 4$.

Note that all irreducible modules are integrally graded, i.e., if $M$ is an irreducible module, then

$$M = \bigoplus_{n \in \mathbb{Z}} M_{\lambda+n} \quad \text{with} \quad L(0)|_{M_{\lambda+n}} = \lambda + n$$

for some $\lambda \in \mathbb{C}$. Thus, by using the fusion rules of $V$-modules and the integrally graded condition above, we can list up possible irreducible $U_{3C}^\tau$-modules as follows.

**Lemma C.5.** Let $M$ be an irreducible $U_{3C}^\tau$-module. Assume that $M$ does not contain $[\frac{1}{16}, h_{i,6}]$, $1 \leq i \leq 5$, as $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$-submodules. Then as a module of $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$, $M$ is isomorphic to one of the following.

$$[0, h_{i,1}] \oplus \left[ \frac{1}{2}, h_{i,5} \right] \oplus [0, h_{i,7}] \oplus \left[ \frac{1}{2}, h_{i,11} \right],$$

$$\left[ \frac{1}{2}, h_{i,1} \right] \oplus [0, h_{i,5}] \oplus \left[ \frac{1}{2}, h_{i,7} \right] \oplus [0, h_{i,11}],$$

$$\left[ \frac{1}{16}, h_{i,4} \right] \oplus \left[ \frac{1}{16}, h_{i,8} \right],$$

where $1 \leq i \leq 5$. Moreover, the $U_{3C}^\tau$-module structure on $M$ is uniquely determined by its $V$-module structure.

**Proof.** Let $M$ be an irreducible $U_{3C}^\tau$-module. Since $U_{3C}^\tau = V \oplus W$ and $V$ is rational, $M$ is a direct sum of irreducible $V$-modules. By the list of irreducible $V$-modules shown in Lemma C.4 and the fusion rules of Vir($e$)-modules, we have the following two cases: As a Vir($e$)-module, $M$ is a sum of $L(\frac{1}{2}, \frac{1}{16})$ or $M$ does not contain $L(\frac{1}{2}, \frac{1}{16})$. In the former case, by the fusion rules of $V$-modules and the integrally graded condition, we see that
all irreducible $V$-submodules of $M$ are mutually isomorphic and they are isomorphic to one of $[\frac{1}{16}, h_{i,4}] \oplus [\frac{1}{16}, h_{i,8}]$ with $1 \leq i \leq 5$. Then by Lemmas C.1 and C.2, $M$ is in fact irreducible as a $V$-module and its $U_{3C}^{\tau_e}$-module structure is uniquely determined by its $V$-module structure. Hence $M$ is as in the assertion.

If $M$ does not contain $L(\frac{1}{2}, \frac{1}{16})$ as Vir($e$)-submodules, then by the list of irreducible $V$-modules shown in Lemma C.4 and the fusion rules of $V$-modules, we are in the situation as in Lemma C.3 and the integrally graded condition leads to that $M$ is in the list of the assertion. In this case the uniqueness of $U_{3C}^{\tau_e}$-module structure is already shown in Lemma C.3. 

Finally, we have the following classification theorem.

**Theorem C.6.** There are exactly five irreducible $U_{3C}$-modules. As $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$-modules, they are of the following form:

1. $U(0) \cong [0,0] \oplus [0,8] \oplus [\frac{1}{2}, \frac{7}{2}] \oplus [\frac{1}{2}, \frac{45}{2}] \oplus [\frac{1}{16}, \frac{31}{16}] \oplus [\frac{1}{16}, \frac{175}{16}]$ ($= U_{3C}$),
2. $U(2) \cong [0, \frac{13}{11}] \oplus [0, \frac{35}{11}] \oplus [\frac{1}{2}, \frac{15}{22}] \oplus [\frac{1}{2}, \frac{301}{22}] \oplus [\frac{1}{16}, \frac{21}{16}] \oplus [\frac{1}{16}, \frac{901}{16}]$,
3. $U(4) \cong [0, \frac{6}{11}] \oplus [0, \frac{50}{11}] \oplus [\frac{1}{2}, \frac{22}{22}] \oplus [\frac{1}{2}, \frac{155}{22}] \oplus [\frac{1}{16}, \frac{85}{16}] \oplus [\frac{1}{16}, \frac{261}{16}]$,
4. $U(6) \cong [0, \frac{1}{11}] \oplus [0, \frac{111}{11}] \oplus [\frac{1}{2}, \frac{2}{22}] \oplus [\frac{1}{2}, \frac{57}{22}] \oplus [\frac{1}{16}, \frac{5}{16}] \oplus [\frac{1}{16}, \frac{533}{16}]$,
5. $U(8) \cong [0, \frac{20}{11}] \oplus [0, \frac{196}{11}] \oplus [\frac{1}{2}, \frac{7}{22}] \oplus [\frac{1}{2}, \frac{117}{22}] \oplus [\frac{1}{16}, \frac{133}{16}] \oplus [\frac{1}{16}, \frac{1365}{16}]$.

**Proof.** Set $U_{3C}^{-} = [\frac{1}{16}, \frac{31}{16}] \oplus [\frac{1}{16}, \frac{175}{16}]$. Then $\tau_e \in \text{Aut}(U_{3C})$ acts on $U_{3C}^{-}$ as $-1$ and $U_{3C} = U_{3C}^{\tau_e} \oplus U_{3C}^{-}$ is a $\mathbb{Z}_2$-graded extension of $U_{3C}^{\tau_e}$. Let $M$ be an irreducible $U_{3C}$-module. Denote by $M^0$ the sum of irreducible Vir($e$)-submodules of $M$ isomorphic to $L(\frac{1}{2}, 0)$ or $L(\frac{1}{2}, \frac{1}{2})$ and by $M^1$ the sum of irreducible Vir($e$)-submodules of $M$ isomorphic to $L(\frac{1}{2}, \frac{1}{16})$. Then $M = M^0 \oplus M^1$. By the fusion rules of $L(\frac{1}{2}, 0)$-modules, $M^0$ and $M^1$ are inequivalent irreducible $U_{3C}^{\tau_e}$-submodules and $M$ carries the following $\mathbb{Z}_2$-grading: $U_{3C}^{-} \cdot M^0 = M^1$ and $U_{3C}^{-} \cdot M^1 = M^0$. In Lemma C.5 we have classified irreducible $U_{3C}^{\tau_e}$-modules having no Vir($e$)-submodules isomorphic to $L(\frac{1}{2}, \frac{1}{16})$. Then by the integrally graded condition, we see that $M$ cannot contain $[\frac{1}{16}, h_{i,6}]$, $1 \leq i \leq 5$ as $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$-submodules. Now thanks to Lemma C.5 we can classify the possible pairs $(M^0, M^1)$ of irreducible $U_{3C}^{\tau_e}$-modules such that $M^0 \oplus M^1$ are integrally graded. As a result, we see that $M$ is isomorphic to one of $U(2i)$, $i = 0, 1, \ldots, 4$, as an $L(\frac{1}{2}, 0) \otimes L(\frac{21}{22}, 0)$-module. We already know that all $U(2i)$, $i = 0, 1, \ldots, 4$, appear as $U_{3C}$-submodules of $V_{\sqrt{2}E_8}$. Thus there exist irreducible $U_{3C}$-modules of the form $U(2i)$. It remains to show that there is only one irreducible $U_{3C}$-module structure on each $U(2i)$. Let $U(2i) = M^0 \oplus M^1$ be the decomposition above. It is clear that both $M^0$ and $M^1$ are self-dual $U_{3C}^{\tau_e}$-modules. By case by case verifications, we
can deduce the fusion rule \( \dim I_{U_{3C}^c}(U_{3C}^M)^1 = 1 \) from the fusion rules of \( L(\frac{1}{2},0) \otimes L(\frac{25}{28},0) \)-modules. Since both \( M^0 \) and \( M^1 \) are self-dual, we also have \( \dim I_{U_{3C}^c}(U_{3C}^M)^0 = 1 \). Then by a standard argument as in \([27, 29, 47]\) we can easily show that the \( K \) exists only one irreducible \( U_{3C} \)-module structure on \( U(2i) = M^0 \oplus M^1 \), since \( U_{3C} = U_{3C}^c \oplus U_{3C}^c \) is a \( \mathbb{Z}_2 \)-graded extension of \( U_{3C}^c \).

\[ \square \]

C.2. Irreducible modules for the 5A algebra. Next we shall classify the irreducible modules for the 5A-algebra \( U_{5A} \). The argument used here is exactly the same as in the case of \( U_{3C} \) so that we omit details in this case. Recall that

\[ U_{5A} \cong L(\frac{1}{2}, 0) \otimes L(\frac{25}{28}, 0) \otimes L(\frac{25}{28}, 0) \oplus L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{25}{28}, \frac{5}{32}) \otimes L(\frac{25}{28}, \frac{5}{32}) \]

\[ \oplus L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{25}{28}, \frac{5}{32}) \otimes L(\frac{25}{28}, \frac{5}{32}) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{25}{28}, \frac{13}{4}) \otimes L(\frac{25}{28}, \frac{13}{4}) \]

\[ \oplus L(\frac{1}{2}, 0) \otimes L(\frac{25}{28}, \frac{13}{4}) \otimes L(\frac{25}{28}, \frac{13}{4}) \oplus L(\frac{1}{2}, \frac{1}{16}) \otimes L(\frac{25}{28}, 0) \oplus L(\frac{1}{2}, 0) \otimes L(\frac{25}{28}, \frac{13}{4}) \otimes L(\frac{25}{28}, \frac{13}{4}) \]

Again, we shall use \( e \in U_{5A} \) to denote the Virasoro element of the VOA \( L(1/2, 0) \) and \( \tau_e \) the corresponding Miyamoto involution. We shall also use \([h_1, h_2, h_3]\) to denote the module of the form \( L(\frac{1}{2}, h_1) \otimes L(\frac{25}{28}, h_2) \otimes L(\frac{25}{28}, h_3) \).

Note that the fixed point subalgebra of \( \tau_e \) is as follows.

\[ U_{5A}^e \cong [0, 0, 0] \oplus [0, \frac{15}{2}, \frac{15}{2}] \oplus [\frac{1}{2}, \frac{15}{2}, 0] \oplus [\frac{1}{2}, 0, \frac{15}{2}] \]

\[ \oplus [0, 3, \frac{13}{4}] \oplus [0, \frac{13}{4}, \frac{13}{4}] \oplus [\frac{1}{2}, \frac{13}{4}, \frac{13}{4}] \oplus [\frac{1}{2}, 3, \frac{3}{4}] \]

It contains a subalgebra

\[ V = [0, 0, 0] \oplus [0, \frac{15}{2}, \frac{15}{2}] \oplus [\frac{1}{2}, \frac{15}{2}, 0] \oplus [\frac{1}{2}, 0, \frac{15}{2}] \]

which is a \((\mathbb{Z}_2 \times \mathbb{Z}_2)\)-graded simple current extension of \([0, 0, 0] \). Set

\[ W = [0, \frac{3}{4}, \frac{13}{4}] \oplus [0, \frac{13}{4}, \frac{3}{4}] \oplus [\frac{1}{2}, \frac{13}{4}, \frac{13}{4}] \oplus [\frac{1}{2}, 3, \frac{3}{4}] \]

Then \( W \) is an irreducible \( V \)-submodule of \( U_{5A}^e \) and we have a decomposition \( U_{5A}^e = V \oplus W \). It is shown in Lemma 3.16 that \( W \cdot W = V \oplus W = U_{5A}^e \). Therefore, we can use the representation theory of \( V \) to classify irreducible \( U_{5A} \)-modules.

**Lemma C.7** (cf. \([27, 29, 47]\)). The irreducible modules for \( V \) are given as follows.

\[ [0, h_{i,m}, h_{k,n}] \oplus [\frac{1}{2}, h_{i,s-m}, h_{k,n}] \oplus [\frac{1}{2}, h_{i,m}, h_{k,s-n}] \oplus [0, h_{i,s-m}, h_{k,s-n}] \]

\[ [\frac{1}{16}, h_{i,j}, h_{k,l}] \oplus [\frac{1}{16}, h_{i,j}, h_{k,l}] \oplus [\frac{1}{16}, h_{i,j}, h_{k,s-l}] \oplus [\frac{1}{16}, h_{i,j}, h_{k,l}] \]

\[ ([\frac{1}{16}, h_{i,j}, h_{k,l}] \oplus [\frac{1}{16}, h_{i,j}, h_{k,l}] \oplus [\frac{1}{16}, h_{i,j}, h_{k,l}] \oplus [\frac{1}{16}, h_{i,j}, h_{k,l}]) \]

\[ ([\frac{1}{16}, h_{i,j}, h_{k,l}] \oplus [\frac{1}{16}, h_{i,j}, h_{k,l}] \oplus [\frac{1}{16}, h_{i,j}, h_{k,l}]) \]

\[ [\frac{1}{16}, h_{i,j}, h_{k,l}] \otimes Q, \]
where $1 \leq i, k \leq 3$, $m, n = 1, 3, 5, 7$, $j, \ell = 2, 6$ and $Q$ is the unique 2-dimensional irreducible module of the quaternion group of order 8.

Now by using the fusion rules and the fact that the weights of irreducible modules are integrally graded, we have

**Lemma C.8.** Let $M$ be an irreducible module for $U_{5A}^{7}$. Assume that $M$ does not contain $[\frac{1}{16}, h_{p,4}, h_{q,4}]$, $1 \leq p, q \leq 3$, as $L(\frac{1}{2},0) \otimes L(\frac{25}{28},0) \otimes L(\frac{25}{28},0)$-submodules. Then, as an module of $L(\frac{1}{2},0) \otimes L(\frac{25}{28},0) \otimes L(\frac{25}{28},0)$, $M$ is isomorphic to one of the following, where $i, k = 1, 2$ or 3.

\[
[0, h_{i,1}, h_{k,1}] \oplus [\frac{1}{2}, h_{i,7}, h_{k,1}] \oplus [\frac{1}{2}, h_{i,1}, h_{k,7}] \oplus [0, h_{i,7}, h_{k,7}]
\]

\[
\oplus [0, h_{i,3}, h_{k,5}] \oplus [\frac{1}{2}, h_{i,3}, h_{k,3}] \oplus [\frac{1}{2}, h_{i,5}, h_{k,5}] \oplus [0, h_{i,5}, h_{k,3}],
\]

\[
[0, h_{i,1}, h_{k,7}] \oplus [\frac{1}{2}, h_{i,1}, h_{k,1}] \oplus [\frac{1}{2}, h_{i,7}, h_{k,7}] \oplus [0, h_{i,7}, h_{k,1}]
\]

\[
\oplus [0, h_{i,3}, h_{k,3}] \oplus [\frac{1}{2}, h_{i,3}, h_{k,5}] \oplus [\frac{1}{2}, h_{i,5}, h_{k,3}] \oplus [0, h_{i,5}, h_{k,5}],
\]

\[
[\frac{1}{16}, h_{i,2}, h_{k,6}] \oplus [\frac{1}{16}, h_{i,6}, h_{k,6}] \oplus [\frac{1}{16}, h_{i,6}, h_{k,2}] \oplus [\frac{1}{16}, h_{i,2}, h_{k,2}]
\]

\[
[\frac{1}{16}, h_{i,4}, h_{k,2}] \oplus [\frac{1}{16}, h_{i,4}, h_{k,6}] \oplus [\frac{1}{16}, h_{i,6}, h_{k,4}] \oplus [\frac{1}{16}, h_{i,2}, h_{k,4}].
\]

Moreover, the $U_{5A}^{7}$-module structure on $M$ is uniquely determined by its $V$-module structure.

Finally by using the same method for the $3C$ case, we have the following theorem.

**Theorem C.9.** There are exactly nine irreducible modules $U(i, j)$, $i, j = 1, 3, 5$ for $U_{5A}$. As $L(\frac{1}{2},0) \otimes L(\frac{25}{28},0) \otimes L(\frac{25}{28},0)$-modules, they are of the following form.

\[
U(i, j) \cong [0, h_{i,1}, h_{j,1}] \oplus [0, h_{i,3}, h_{j,3}] \oplus [0, h_{i,5}, h_{j,3}] \oplus [0, h_{i,7}, h_{j,7}]
\]

\[
\oplus [\frac{1}{2}, h_{i,1}, h_{j,7}] \oplus [\frac{1}{2}, h_{i,3}, h_{j,3}] \oplus [\frac{1}{2}, h_{i,5}, h_{j,5}] \oplus [\frac{1}{2}, h_{i,7}, h_{j,1}]
\]

\[
\oplus [\frac{1}{16}, h_{i,2}, h_{j,4}] \oplus [\frac{1}{16}, h_{i,4}, h_{j,2}] \oplus [\frac{1}{16}, h_{i,6}, h_{j,4}] \oplus [\frac{1}{16}, h_{i,4}, h_{j,6}].
\]

**Proof.** Let $M$ be an irreducible $U_{5A}$-module. By the integrally graded condition, we know that $M$ cannot contain $[\frac{1}{16}, h_{i,4}, h_{k,4}]$, $1 \leq i, k \leq 3$, as $L(\frac{1}{2},0) \otimes L(\frac{25}{28},0) \otimes L(\frac{25}{28},0)$-submodules. Then by Lemma C.8 and the integrally graded condition, $M$ is isomorphic to one of $U(i, j)$, $i, j = 1, 3, 5$, as an $L(\frac{1}{2},0) \otimes L(\frac{25}{28},0) \otimes L(\frac{25}{28},0)$-module. We already know that all $U(i, j)$, $i, j = 1, 3, 5$, appear as $U_{5A}$-submodules of $V_{\sqrt{2E}}$. Thus there exist irreducible $U_{5A}$-modules of the form $U(i, j)$. It remains to show the uniqueness of the irreducible $U_{5A}$-module structure on each $U(i, j)$. By a similar argument as in the case of $U_{3C}$-modules, we can establish the uniqueness. \qed
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