Reasoning on Multi-Relational Contextual Hierarchies via Answer Set Programming with Algebraic Measures

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Abstract
Dealing with context dependent knowledge has led to different formalizations of the notion of context. Among them is the Contextualized Knowledge Repository (CKR) framework, which is rooted in description logics but links on the reasoning side strongly to logic programs and Answer Set Programming (ASP) in particular. The CKR framework caters for reasoning with defeasible axioms and exceptions in contexts, which was extended to knowledge inheritance across contexts in a coverage (specificity) hierarchy. However, the approach supports only this single type of contextual relation and the reasoning procedures work only for restricted hierarchies, due to non-trivial issues with model preference under exceptions. In this paper, we overcome these limitations and present a generalization of CKR hierarchies to multiple contextual relations, along with their interpretation of defeasible axioms and preference. To support reasoning, we use ASP with algebraic measures, which is a recent extension of ASP with weighted formulas over semirings that allows one to associate quantities with interpretations depending on the truth values of propositional atoms. Notably, we show that for a relevant fragment of CKR hierarchies with multiple contextual relations, query answering can be realized with the popular asprin framework. The algebraic measures approach is more powerful and enables e.g. reasoning with epistemic queries over CKRs, which opens interesting perspectives for the use of quantitative ASP extensions in other applications.

KEYWORDS: Defeasible Knowledge, Description Logics, ASP, Algebraic Measures, Justifiable Exceptions

1 Introduction
Representing and reasoning with context dependent knowledge is a fundamental theme in AI, with proposals dating back to the works of McCarthy (1993) and Giunchiglia and Serafini (1994). It has gained increasing attention for the Semantic Web as knowledge resources must be interpreted with contextual information from their metadata. Several approaches for contextual reasoning, most based on description logics, were developed (Straccia et al. 2010, Klarman 2013, Serafini and Homola 2012).

A rich framework among them are Contextualized Knowledge Repositories (CKR) (Serafini and Homola 2012). CKR knowledge bases (KBs) are 2-layered structures with a global context, which contains context-independent global knowledge and meta-knowledge about the structure
of the KB, and local contexts containing knowledge about specific situations (e.g., a region in space, a site of an organization). Notably, the global knowledge is propagated to local contexts, where inherited axioms may be defeasible, meaning that instances can be “overridden” on an exceptional basis (Bozzato et al. 2018a). Reasoning from CKRs strongly links to logic programming, as the KBs are over a Horn-description logic and the working of defeasible axioms was inspired by conflict handling in inheritance logic programs (Buccafurri et al. 1999). Furthermore, answering instance and conjunctive queries over a CKR is possible via a uniform ASP program that employs a materialization calculus akin to the one by Krotzsch (2010).

For modeling and analyzing complex scenarios where global regulations (e.g., laws, environmental regulations, access control rules) can be refined by more specific situations (e.g., time-bounded events, geographical areas, groups of users), the CKR model was extended (Bozzato et al. 2018b) to cater for defeasible axioms in local contexts and knowledge inheritance across hierarchies, based on a coverage contextual relation (Serafini and Homola 2012).

This approach, however, is limited to reason only on hierarchies based on this single type of contextual relation. In practice, defeasible inheritance may be necessary under different contextual relations. For example, along a location hierarchy, we may prefer axioms encoding regional laws overriding state-level regulations, while preferring newer rules over older laws along a temporal dimension. A further limitation is that even for a single coverage relation, it is challenging to encode the induced preference relation over CKR interpretation using ASP because the relation may not be transitive and thus not a strict partial order, as assumed e.g. in the popular asprin framework for preferences in ASP (Brewka et al. 2015). Instead, a specialized implementation for preferential reasoning was introduced (Bozzato et al. 2019), which however needs to consider all answer sets of a program to single out a preferred CKR model.

In this paper, we overcome these limitations and make the following contributions:

1. We generalize single-relational CKRs to multi-relational CKRs, where axioms are not defeasible in general but merely with regard to individual relations of hierarchies. By a combination of preferences over the distinct individual relations, we obtain an overall preference over the models of a CKR. While intuitive, the technical condition has pitfalls and needs care.

2. We show how to model multi-relation CKRs in ASP. Specifically, we use to this end ASP with algebraic measures (Eiter and Kiesel 2020), which is a foundation to express many quantitative reasoning problems. Here, weighted logic formulas (Droste and Gastin 2005) measure values associated with an interpretation \( I \) by performing a computation over a semiring, whose outcome depends on the truth of the propositional variables in \( I \). Such measures can be used for e.g. weighted model counting, probabilistic reasoning and, as in our case, preferential reasoning.

3. While asprin is a powerful tool for modeling preferences in ASP, it appears to be ill-suited for expressing multi-relational CKR. The reason are eval-expressions in CKRs, which propagate predicate extensions from one local context to another. We show, however, that under a well-behaved use of such expressions according to a syntactic disconnectedness condition, multi-relational CKRs can be expressed in asprin. This enables us to use the asprin solver to evaluate preferences for CKRs, which is showcased in a prototype implementation.

4. Furthermore, ASP with algebraic measures opens the possibility of reasoning tasks for CKRs beyond asprin’s capability, even in absence of eval-expression. As examples we consider obtaining preferred CKR models by overall weight queries and epistemic reasoning, which for description logics is specifically needed in aggregate queries (Calvanese et al. 2008).
In conclusion, ASP extended with preferences or algebraic computations is a valuable tool to express CKR extensions and reasoning on them, with a promising perspective for further research.

2 Preliminaries

Description Logics and SROIQ-RL. We follow the common presentation of description logics (DLs) (Baader et al. 2003) and the definition of the logic SROIQ (Horrocks et al. 2006).

A DL vocabulary \( \Sigma \) consists of the mutually disjoint countably infinite sets NC of atomic concepts, NR of atomic roles, and NI of individual constants. Complex concepts are recursively defined as the smallest sets containing all concepts that can be inductively constructed using the operators of the considered DL language \( \mathcal{L}_\Sigma \). A DL knowledge base \( \mathcal{K} = \langle T, R, A \rangle \) consists of: a TBox \( T \) which can contain general concept inclusion axioms \( C \sqsubseteq D \), where \( C \) and \( D \) are concepts; an RBox \( R \) which contains role inclusion axioms \( S \sqsubseteq R \), where \( S \) and \( R \) are roles, and role properties axioms; and an ABox \( A \) which contains assertions of the forms \( D(a) \), \( R(a,b) \), where \( a \) and \( b \) are any individual constants.

A DL interpretation is a pair \( \mathcal{I} = \langle \Delta^I, \cdot^I \rangle \) where \( \Delta^I \) is a non-empty set called domain and \( \cdot^I \) is the interpretation function which provides the interpretation for language elements: \( a^I \in \Delta^I \), for \( a \in \text{NI} \); \( A^I \subseteq \Delta^I \), for \( A \in \text{NC} \); \( R^I \subseteq \Delta^I \times \Delta^I \), for \( R \in \text{NR} \). The interpretation of complex concepts and roles is defined by the evaluation of their DL operators (see the paper by Horrocks et al. (2006) for SROIQ). An interpretation \( \mathcal{I} \) satisfies an axiom \( \phi \), denoted \( \mathcal{I} \models \phi \), if it verifies the respective semantic condition, in particular: for \( \phi = D(a) \), \( a^I \in D^I \); for \( \phi = R(a,b) \), \( \langle a^I, b^I \rangle \in R^I \); for \( \phi = C \sqsubseteq D \), \( C^I \subseteq D^I \) (resp. for role inclusions). \( \mathcal{I} \) is a model of \( \mathcal{K} \), denoted \( \mathcal{I} \models \mathcal{K} \), if it satisfies all axioms of \( \mathcal{K} \). We adopt w.l.o.g. the standard name assumption (SNA) in the DL setting, i.e., every element in \( \mathcal{I} \) is reachable via a distinct constant. We denote by \( \text{NI}_S \subseteq \text{NI} \) the set of all such constants, called standard names, which are uniform for all interpretations; see the papers by Eiter et al. (2008) and de Bruijn et al. (2008) for more details.

Most of the following definitions for simple CRK are independent from the DL used as representation language inside contexts: however, as in the paper by Bozzato et al. (2018a), we take as reference language a restriction of the SROIQ syntax called SROIQ-RL which corresponds to OWL-RL. We restrict as follows left-side concepts \( C \) and right-side concepts \( D \):

\[
C := A \mid \{a\} \mid C \cap C \mid C \cup C \mid \exists R.C \mid \exists R.\top \\
D := A \mid \neg C \mid D \cap D \mid \exists R,\{a\} \mid \forall R.D \mid \leq nR.\top
\]

where \( A \in \text{NC} \), \( R \in \text{NR} \) and \( n \in \{0, 1\} \). SROIQ-RL TBox axioms can only take the form \( C \sqsubseteq D \), where \( C \) is a left-side and \( D \) is a right-side or \( E \equiv F \), where \( E \) and \( F \) are both left- and right-side concepts. A SROIQ-RL RBox can contain role inclusions \( R \sqsubseteq S \) (with possibly left role composition), role disjointness, irreflexivity, symmetry, asymmetry and transitivity. SROIQ-RL ABox concept assertions can only be of form \( D(a) \), where \( D \) is a right-side concept. We remark that SROIQ-RL basically defines a restriction of SROIQ to axioms that are expressible as Horn rules (cf. FO translation provided by Bozzato et al. (2018a)).

Normal Programs and Answer Sets. We use function-free normal (datalog) rules with (default) negation under answer sets semantics (Gelfond and Lifschitz 1991) and gather them in ASP programs. A normal (datalog) rule \( r \) is an expression of the form:

\[
a \leftarrow b_1, \ldots, b_k, \text{not} b_{k+1}, \ldots, \text{not} b_m, \quad 0 \leq k \leq m,
\]

also written \( H(r) \leftarrow B(r) \) where \( a, b_1, \ldots, b_m \) are function-free FO-atoms and not is negation as
failure (NAF). We allow that $a$ is missing (constraint), viewing $a$ as logical constant for falsity. A (datalog) program $P$ is a finite set of rules. An atom (rule etc.) is ground, if no variables occur in it. A fact $H$ is a ground rule $r$ with $m = 0$. The grounding of a rule $r$, $\text{grnd}(r)$, is the set of all ground instances of $r$, and the grounding of a program $P$ is $\text{grnd}(P) = \bigcup_{r \in P} \text{grnd}(r)$.

For any program $P$, we denote by $U_P$ its Herbrand universe and by $B_P$ its Herbrand base; an (Herbrand) interpretation is any subset $I \subseteq B_P$ of $B_P$. An atom $a$ is true in $I$, denoted $I \models a$, if $a \in I$. Given a rule $r \in \text{grnd}(P)$, we say that $B(r)$ is true in $I$, denoted $I \models B(r)$, if (i) $I \models b$ for each $b \in B(r)$ and (ii) $I \not\models b$ for each $\lnot b$ in $B(r)$. A rule $r$ is satisfied in $I$, denoted $I \models r$, if either $I \models H(r)$ or $I \not\models B(r)$. An interpretation $I$ is a model of $P$, denoted $I \models P$, if $I \models r$ for each $r \in \text{grnd}(P)$; moreover, $I$ is minimal, if $I \not\models P$ for each subset $I' \subset I$. Furthermore, $I$ is an answer set of $P$, if $I$ is a minimal model of the (Gelfond-Lifschitz) reduct $G_I(P)$ of $P$ w.r.t. $I$, which results from $\text{grnd}(P)$ by removing (i) every rule $r$ such that $I \models l$ for some $\lnot l \in B(r)$, and (ii) all formulas $\lnot b$ from the remaining rules. The set of answer sets of $P$ is denoted $\mathcal{A}_S(P)$.

Semirings and Weighted Logic. A semiring $R = (R, +, \cdot, e_R, e_\otimes)$ is a set $R \neq \emptyset$ equipped with binary operations $+ \cdot$ (called addition and multiplication, such that (i) $(R, +)$ is a commutative monoid with identity element $e_R$, (ii) $(R, \cdot)$ is a monoid with identity element $e_\otimes$, (iii) multiplication left and right distributes over addition, and (iv) multiplication by $e_\otimes$ annihilates $R$, i.e. $\forall r \in R : r \cdot e_\otimes = e_\otimes = e_\otimes \cdot r$. Examples are the natural number semiring $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ with addition and multiplication, the powerset semiring $\mathcal{P}(A) = (2^A, \cup, \cap, \emptyset, A)$, with union and intersection, the Boolean semiring $\mathbb{B} = (\{t, f\}, \lor, \land, f, t)$, with disjunction and conjunction, and the tropical semiring $\mathcal{T}_{\text{trop}} = (\mathbb{Q} \cup \{\infty\}, \min, +, \infty, 0)$, with minimum and addition.

Weighted formulas over a semiring $R$ and an Herbrand base $B$ allow us to assign an interpretation $I$ a semiring value, depending on the truth of propositional variables w.r.t. $I$. Their syntax is:

$$\alpha ::= k \mid v \mid \lnot v \mid \alpha + \alpha \mid \alpha \cdot \alpha$$

where $k \in R$ and $v \in B$. The semantics $[\alpha]_R(I)$ of $\alpha$ over $R$ w.r.t. $I$ is:

$$[k]_R(I) = k \text{ for } k \in R$$

$$[v]_R(I) = \begin{cases} 1 & \text{if } I \models v \\ 0 & \text{otherwise} \end{cases}$$

$$[\alpha]_R(I) = [\alpha_1]_R(I) \cdot [\alpha_2]_R(I)$$

$$[\alpha_1 + \alpha_2]_R(I) = [\alpha_1]_R(I) \oplus [\alpha_2]_R(I)$$

$$[\alpha_1 \cdot \alpha_2]_R(I) = [\alpha_1]_R(I) \odot [\alpha_2]_R(I)$$

3 Multi-relational simple CKR

We generalize the definition of simple CKR (sCKR) introduced by Bozato et al. [2018b, 2019] from single- to multi-relational contextual hierarchies. As in the original formulation of CKR by Bozato et al. [2018a, 2013] a simple CKR is still a two layered structure, but the upper layer is simply a poset with multiple orderings, corresponding to different contextual relations. Simple CKRs define a core fragment of CKR allowing us to provide lean definitions on contextual hierarchies: the presented results, however, can be easily generalized to the full CKR.

We provide definitions for multi-relational simple CKRs with a general set of context relations and consider the case for 2-relational sCKR based on temporal and coverage relations.

Syntax. Consider a nonempty set $N \subseteq \text{NI}$ of context names. A contextual relation is any strict order $\prec \subseteq N \times N$ over contexts. We may use the non-strict relation $c_1 \preceq c_2$ to indicate that either $c_1 \prec c_2$ or $c_1$ and $c_2$ are the same context. We consider two contextual relations, namely coverage $\prec_c$ and temporal precedence $\prec_t$. Here, $c_1 \prec_c c_2$ (resp. $c_1 \prec_t c_2$) means that $c_1$ is more specific
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Figure 1. Context hierarchy of multi-relational example sCKR, with axioms per context.

(sub resp. newer) than $c_2$. More specific means that $c_1$ represents a portion of the world covered by the one referred to by $c_2$, as in the paper by Serafini and Homola (2012). We generalize the definition of defeasible axiom w.r.t. contextual relations:

Definition 1 (r-defeasible axiom)
Given a set $\mathcal{R}$ of contextual relations over $N$ and a description language $\mathcal{L}_\Sigma$, an r-defeasible axiom is any expression of the form $D_r(\alpha)$, where $\alpha$ is an axiom of $\mathcal{L}_\Sigma$ and $\prec_r \in \mathcal{R}$.

Thus, we identify coverage-defeasible axioms as $D_c(\alpha)$ and temporal-defeasible axioms as $D_t(\alpha)$.

We allow for the use of r-defeasible axioms in the local language of contexts:

Definition 2 (contextual language)
Given a set of context names $N$, for every description language $\mathcal{L}_\Sigma$ we define $\mathcal{L}_{\Sigma,N}$ as the extension of $\mathcal{L}_\Sigma$ where: (i) $\mathcal{L}_{\Sigma,N}$ contains the set of r-defeasible axioms in $\mathcal{L}_\Sigma$; (ii) $\text{eval}(X,c)$ is a concept (resp. role) of $\mathcal{L}_{\Sigma,N}$ if $X$ is a concept (resp. role) of $\mathcal{L}_\Sigma$ and $c \in N$.

Using these definitions, multi-relational simple CKRs are defined as follows:

Definition 3 (multi-relational simple CKR)
A multi-relational simple CKR (sCKR) over $\Sigma$ and $N$ is a structure $\mathcal{K} = \langle C, K_N \rangle$ where:

- $C$ is a structure $\langle N, \prec_1, \ldots, \prec_m \rangle$ where each $\prec_i$ is a contextual relation over $N$, and
- $K_N = \{ K_c \}_{c \in N}$ for each context name $c \in N$, $K_c$ is a DL knowledge base over $\mathcal{L}_{\Sigma,N}$.

A sCKR that combines temporal and coverage orderings can be defined by $\mathcal{K} = (N, \prec_t, \prec_c)$. For simplicity, we assume that the priority for the combination of orderings is defined by the linear order in which they appear in $\mathcal{K}$: in the case above, we prioritize $\prec_t$ over $\prec_c$.

Example 1
We consider the following example to explain the expected behavior of defeasible axioms in the case of the combination of coverage and temporal relations. Let us consider $\mathcal{K}_{\text{org}} = \langle \mathcal{C}, K_N \rangle$ with $\mathcal{C} = (N, \prec_t, \prec_c)$ describing the organization of a corporation. The corporation has different policies with respect to its local branches, represented by coverage, and updates them along the time precedence. The structure of $\mathcal{C}$, together with the axioms at each context, is shown in Figure 1. We have a chain of three contexts (representing world, branch and local rules) in the direction of the coverage and three “time-slices” (2019, 2020 and 2021) along the time relation: thus, for example, we have $\mathcal{C}_{\text{world,2021}} \prec_c \mathcal{C}_{\text{branch,2021}}$ and $\mathcal{C}_{\text{branch,2020}} \prec_c \mathcal{C}_{\text{branch,2019}}$. The corporation is active in the fields of Electronics (E) and Robotics (R) and employs supervisors (S). In $\mathcal{C}_{\text{world,2019}}$, we state that, with respect to coverage, every Supervisor has to be applied by default to Electronics and that Electronics and Robotics are disjoint. In the lower context $\mathcal{C}_{\text{branch,2019}}$, we further specify that, with respect to time, Supervisors have to work by default OnSite (OS) (where
working OnSite and Remote (RE) are disjoint. In 2019’s local context $c_{local,2019}$ we assert that $i$ is a Supervisor. The previous defeasible statements are, however, contradicted by the ones in $c_{branch,2020}$, where Supervisors are applied to Robotics and work on Remote.

The interpretation of defeasible propagation and preferences, then, should define the interpretation of what is derivable in the local context in the three time-slices. In $c_{local,2019}$ no overriding takes place; then we should derive $E(i), OS(i)$. In $c_{local,2020}$ the more coverage-specific axiom in $c_{branch,2020}$ is preferred, thus we derive $R(i)$; the time-related defeasible axiom $D_i(S \sqsubseteq RE)$ is applied locally to the 2020 time-slice, thus we derive $RE(i)$. In the 2021 time-slice no new information is provided, thus the overriding preferences should enforce that the more specific and recent information is used: in $c_{local,2021}$ we expect to derive $R(i), RE(i)$.

Semantics. A sCKR interpretation gathers interpretations for the local contexts as follows.

Definition 4 (sCKR interpretation)
An interpretation for $\mathcal{L}_{\Sigma, N}$ is a family $\mathcal{I} = \{I(c)\}_{c \in N}$ of $\mathcal{L}_{\Sigma}$ interpretations, such that $\Delta I(c) = \Delta z(c')$ and $a I(c) = a I(c')$, for every $a \in NI$ and $c, c' \in N$.

The interpretation of concepts and role expressions in $\mathcal{L}_{\Sigma, N}$ is obtained by extending the standard interpretation to eval expressions: for every $c \in N$, $eval(X, c') I(c) = X I(c')$. We consider the definition of axiom instantiation provided by Bozzato et al. (2018a): given an axiom $\alpha \in \mathcal{L}_{\Sigma}$ with FO-translation $\forall x. \phi_x(x)$, the instantiation of $\alpha$ with a tuple $e$ of individuals in NI, written $\alpha(e)$, is the specialization of $\alpha$ to $e$, i.e., $\phi_x(e)$, depending on the type of $\alpha$.

For a structure $\mathcal{C} = (N, \prec_1, \ldots , \prec_m)$ and $1 \leq i \leq m$, we denote by $\preceq_{-i}$ the order obtained as the reflexive and transitive closure of $\bigcup_{j \neq i} \prec_j$, i.e., the union of all orders $\prec_j$ except for $\prec_i$. We denote by $\preceq$ the order obtained as the reflexive and transitive closure of the union of all $\prec_j$.

Definition 5 (clashing assumptions and sets)
A clashing assumption for a context $c$ and contextual relation $r$ is a pair $\langle \alpha, e \rangle$ such that $\alpha(e)$ is an axiom instantiation of $\alpha$, and $D_i(\alpha) \in K_c$ is a defeasible axiom of some $c' \geq_i c' \succ_i c$. A clashing set for $\langle \alpha, e \rangle$ is a satisfiable set $S$ of ABox assertions s.t. $S \cup \{\alpha(e)\}$ is unsatisfiable.

A clashing assumption $\langle \alpha, e \rangle$ represents that $\alpha(e)$ is not satisfiable in context $c$, and a clashing set $S$ provides a “justification” for the local assumption of overriding of $\alpha$ on $e$. CAS-interpretations include a set of clashing assumptions for each context and contextual relation:

Definition 6 (CAS-interpretation)
A CAS-interpretation is a structure $\mathcal{I}_{CAS} = \langle \mathcal{I}, \mathcal{X} \rangle$ where $\mathcal{I}$ is an interpretation and $\mathcal{X} = \{\mathcal{X}_1, \ldots , \mathcal{X}_m\}$ such that each $\mathcal{X}_i$, for $i \in \{1, \ldots , m\}$, maps every $c \in N$ to a set $\mathcal{X}_i(c)$ of clashing assumptions for context $c$ and context relation $\prec_i$.

Satisfaction of a sCKR $\mathcal{R}$ needs to consider the effect of the different relations:

Definition 7 (CAS-model)
Given a multi-relation sCKR $\mathcal{R}$, a CAS-interpretation $\mathcal{I}_{CAS} = \langle \mathcal{I}, \mathcal{X} \rangle$ is a CAS-model for $\mathcal{R}$ (denoted $\mathcal{I}_{CAS} \models \mathcal{R}$), if the following hold:

(i) for every $\alpha \in K_c$ (strict axiom), and $c' \preceq_i c$, $\mathcal{I}(c') \models \alpha$;
(ii) for every $D_i(\alpha) \in K_c$ and $c' \preceq_{-i} c$, $\mathcal{I}(c') \models \alpha$;

1 Here, it is important to ensure that (defeasible) axioms are correctly propagated w.r.t. any context relation $\prec_i$. 
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(iii) for every $D_i(\alpha) \in K_c$ and $c'' \prec_i c' \leq_i c$, if $(\alpha, d) \not\in \chi(c'')$, then $I(c'') \models \phi_d(d)$.

Intuitively: (i) strict axioms are propagated across the hierarchy structures over $\leq_i$ from higher to lower contexts; (ii) considering contexts that are related by relations other than $\leq_i$ (including the context in which axioms are declared), defeasible axioms $D_i(\alpha)$ are interpreted as strict axioms; (iii) over relation $\prec_i$, axioms $D_i(\alpha)$ are verified in context $c''$ only if applied to instances $d$ that are not in the clashing assumptions for $c''$ and relation $\prec_i$. Note that these propagation rules are applied for every contextual relation: however, the definition can be easily extended to assign different conditions for propagation and overriding for each of the orderings.

We provide a local preference on clashing assumption sets for each of the relations:

(LP). $\chi_i^1(c) > \chi_i^2(c)$, if for every $(\alpha_1, e) \in \chi_i^1(c) \setminus \chi_i^2(c)$ with $D_i(\alpha_1)$ at a context $c_1 \geq_i c_1b \succ_i c$, some $(\alpha_2, f) \in \chi_i^2(c) \setminus \chi_i^1(c)$ exists with $D_i(\alpha_2)$ at context $c_2 \geq_i c_2b \succ_i c_2b$.

Intuitively, $\chi_i^1(c)$ is preferred to $\chi_i^2(c)$ if $\chi_i^1(c)$ exchanges the “more costly” exceptions of $\chi_i^2(c)$ at more specialized contexts with “cheaper” ones at more general contexts. As above, multiple options for local preference can be adopted, e.g., $\chi_i^2(c) \supseteq \chi_i^1(c)$ for ranked hierarchies.

Two DL interpretations $I_1$ and $I_2$ are NI-congruent, if $c^{I_1} = c^{I_2}$ holds for every $c \in NI$. This extends to CAS interpretations $I_{CAS} = \langle I, \chi \rangle$ by considering all context interpretations $I(c) \in \chi$.

Definition 8 (justification)

We say that $(\alpha, e) \in \chi(c)$ is justified for a CAS model $I_{CAS}$, if some clashing set $S(\alpha, e, c)$ exists such that, for every $\chi^i_{CAS} = \langle I^i, \chi^i \rangle$ of $\mathfrak{R}$ that is NI-congruent with $I_{CAS}$, it holds that $I^i(c) = S(\alpha, e, c)$. A CAS model $I_{CAS}$ of a sCKR $\mathfrak{R}$ is justified, if every $(\alpha, e) \in \chi$ is justified in $\mathfrak{R}$.

We define a model preference by combining the preferences of the relations: it is a global lexicographical ordering on models where each $\prec_i$ defines the ordering at the $i$-th position.

(MP). $\chi^1_{CAS} = \langle I^1, \chi^1_1, \ldots, \chi^1_m \rangle$ is preferred to $\chi^2_{CAS} = \langle I^2, \chi^2_1, \ldots, \chi^2_m \rangle$ if:

(i) there exist $i \in \{1, \ldots, m\}$ and some $c \in NI$ s.t. $\chi^1_i(c) > \chi^2_i(c)$ and not $\chi^2_i(c) > \chi^1_i(c)$, and for no context $c' \neq c \in NI$ it holds that $\chi^1_i(c') < \chi^2_i(c')$ and not $\chi^2_i(c') < \chi^1_i(c')$.

(ii) for every $j < i \in \{1, \ldots, m\}$, it holds $\chi^1_j \approx \chi^2_j$ (i.e. (i) or its converse do not hold for $\prec_j$).

Then, CKR models are defined by taking into account justification and model preference.

Definition 9 (CKR model)

An interpretation $\mathfrak{I}$ is a CKR model of a sCKR $\mathfrak{R}$ (in symbols, $\mathfrak{I} \models \mathfrak{R}$) if: (i) $\mathfrak{R}$ has some justified CAS model $I_{CAS} = \langle I, \chi \rangle$; (ii) there exists no justified $I_{CAS} = \langle I', \chi' \rangle$ that is preferred to $I_{CAS}$.

Example 2

By considering the sCKR of Example[1] we can show how the preference for different relations influences the global model preference. In the case of $\mathfrak{R}_{org}$, we have 8 justified interpretations that are based on combinations of the following clashing assumption sets (for both relations) on contexts $c_{local,2020}$ and $c_{local,2021}$. For any CAS model $\chi(c_{local,2020}) = \chi^0(c_{local,2020}) = \{ S \subseteq OS,i \}$ and $\chi(c_{local,2021})$ is either $\chi^0(c_{local,2021}) = \{ S \subseteq E,i \}$ or $\chi^1(c_{local,2021}) = \{ S \subseteq R,i \}$.

For $c_{local,2021}$ we have that $\chi(c_{local,2021})$ is either $\chi^0(c_{local,2021}) = \{ S \subseteq OS,i \}$ or $\chi^2(c_{local,2021}) = \{ S \subseteq RE,i \}$. For $\chi(c_{local,2021})$ we have the same choices as for $\chi(c_{local,2020})$.

According to the (LP) definition, $\chi^0(c_{local,2020}) > \chi^1(c_{local,2020})$ since $D_i(S \subseteq E)$ occurs at a less specific context w.r.t. $\prec$ than $D_i(S \subseteq R)$. Similarly, $\chi^1(c_{local,2021}) > \chi^0(c_{local,2021})$.

Since we can choose the clashing assumptions per context independently, the clashing assumption map of CKR models is uniquely determined by (MP) as $\chi = \langle \chi_1, \chi_2 \rangle$ where $\chi_1 = \chi^0 \cup \chi^1$ and
$\chi_e = \chi_e^0 \cup \chi_e^2$. Indeed, this corresponds to the intuitive model where overridings over temporal relation occur on defeasible axioms in the “older” contexts and in the “higher” contexts with respect to the coverage relation.

**Reasoning and Complexity.** We consider the following reasoning tasks for sCKR:

- **c-entailment** $R \models c: \alpha$, denoting that axiom $\alpha$ is entailed in each CKR-model of $R$ at context $c$.
- **Boolean conjunctive query (BCQ)** answering $R \models \exists \gamma(y)$, where $\gamma(y) = \gamma_1 \land \cdots \land \gamma_m$ is an existentially closed conjunction of atoms $\gamma_i = c_i: \alpha_i(t_i)$ with context name $c_i$ and assertion $\alpha_i(t_i)$.

The complexity of reasoning with contextual hierarchies in sCKR was studied by Bozzato et al. (2018b; 2019). In particular, CKR satisfiability is NP-complete while CKR model checking is coNP-complete already for ranked hierarchies. This causes the complexity of c-entailment to increase in presence of hierarchies: for polynomial-time local preferences on overridings, c-entailment is $\Pi_2^p$-complete. In contrast, BCQ answering remains $\Pi_2^p$-complete as verifying a guess for a countermodel to the query remains in coNP. These results would carry over to multi-relational hierarchies: for combinations of polynomial-time preferences (like the global preference we considered), c-entailment and similarly BCQ answering would still be $\Pi_2^p$-complete.

### 4 Preferences with Algebraic Measures

The question rises how the reasoning problems above can be expressed and solved. Previously, in the case of sCKRs with a single relation the strategy was to encode the problem in ASP using a program whose stable models correspond to the least justified models of the sCKR. The preferred models, i.e. sCKR models, were then selected using weight constraints in the restricted case of ranked hierarchies (Bozzato et al. 2018a) or by using a dedicated algorithm for general hierarchies (Bozzato et al. 2019). The preference over models for multi-relational sCKRs is more complicated and thus not easily expressed with weight constraints: we can leverage the power of quantitative extensions of ASP to express model preferences induced by multi-relational sCKRs.

The recently introduced algebraic measures for ASP, which connect ASP with weighted formulas, were shown to be a general framework for specifying quantitative reasoning problems (Eiter and Kiesel 2020). Also preferential reasoning falls into this category, thus allowing us to use algebraic measures to specify a preference on the answer sets in such a way that the preferred answer sets correspond to the preferred least justified models. The concept is as follows.

**Definition 10 (Algebraic Measure)**

An algebraic measure $\mu = \langle \Pi, \alpha, R \rangle$ consists of an answer set program $\Pi$, a weighted formula $\alpha$, and a semiring $R$. The weight of an answer set $S \in AS(\Pi)$ is $\mu(S) = [\alpha]_R(S)$. And the overall weight of $\mu$ is defined as $\mu(\Pi) = \bigoplus_{S \in AS(\Pi)} \mu(S)$.

Intuitively, given $\mu = \langle \Pi, \alpha, R \rangle$ the program $\Pi$ specifies which interpretations are accepted and the weighted formula $\alpha$ measures some value associated with them. Using algebraic measures, we can not only assign answer sets a weight but also obtain some information from all answer sets by considering the overall weight $\mu(\Pi)$.

**Example 3**

Let $\Pi$ be some answer set program. Then, for example, for $\mu_1 = \langle \Pi, 1, \mathbb{N} \rangle$ the overall weight $\mu(\Pi)$ is the number of answer sets of $\Pi$. For $\mu_2 = \langle \Pi, (a_1 \ast 1 + \neg a_1) \ast \cdots \ast (a_n \ast 1 + \neg a_n), R_{\text{max}} \rangle$, where $R_{\text{max}} = \langle R \cup \{-\infty\}, \max, +, -\infty, 0 \rangle$, the weight $\mu(S)$ of an answer set $S$ is the number of
atoms $a_1, \ldots, a_n$ it satisfies. We need the additional term $-a_i$, since $a_i \cdot 1$ evaluates to $e_\leq = -\infty$ when $a_i$ is false and not to the desired value $e_\leq = 0$. Due to the usage of the semiring $\mathcal{R}_{\max}$, the overall weight $\mu(\Pi)$ is the maximum number of atoms from $a_1, \ldots, a_n$ that are satisfied in any answer set of $\Pi$.

A natural use case of algebraic measures is preferential reasoning. In the sequel, a preference relation is any asymmetric relation.

**Definition 11 (Preferred Answer Set)**

Given a measure $\mu = (\Pi, \alpha, \mathcal{R})$ and a preference relation $\succ$ on $\mathcal{R}$, an answer set $S \in \text{AS}(\Pi)$ is preferred w.r.t. $\mu$ and $\succ$ if no $S' \in \text{AS}(\Pi)$ exists such that $\mu(S') > \mu(S)$.

Intuitively, we use $\mu$ as an optimization function and take the preferred answer sets as those that achieve an optimal value.

**Example 4**

Reconsider the measure $\mu_2$ from Example[3]. If $a_1, \ldots, a_n$ are desired to be true, then we only want to consider those answer sets for which a maximal number of them is true. These are exactly the preferred answer sets with respect to the measure $\mu_2$ and the usual order over the reals.

We assume a program $PK(\mathfrak{R})$ (see Section[5]), which intuitively guesses a set of atoms $\text{ovr}(\phi, e, c, i)$, each corresponding to a clashing assumption $\langle \phi, e \rangle$ in $\mathcal{R}_i(c)$, and checks whether there is an CAS model $\mathfrak{I}_{\text{CAS}} = (\mathfrak{I}, \mathcal{X})$. The answer sets $I$ corresponds to the least CAS models with that property. Then we can introduce a measure $\mu_{\text{opt}}$ and order $\succ_{\text{opt}}$ to obtain those answer sets of $PK(\mathfrak{R})$ as preferred answer sets w.r.t. $\mu_{\text{opt}}$ and $\succ_{\text{opt}}$ that correspond to the preferred least justified models of $\mathfrak{R}$. Here, we do not require any restrictions on the $\mathfrak{R}$ at all.

We use the powerset semiring $P(CA)$ over the set $CA$, which contains the tuple $\langle \phi, e, c, i \rangle$ for each possible clashing assumption $\langle \phi, e \rangle$ that can occur at context $c$ w.r.t. relation $i$. The weighted formula of $\mu_{\text{opt}} = (PK(\mathfrak{R}), \alpha, P(CA))$ is given by $\alpha = \Sigma(\phi, e, c, i) \in CA \text{ovr}(\phi, e, c, i) \times \{\langle \phi, e, c, i \rangle\}$. It is easy to see that for each answer set $I$ of $PK(\mathfrak{R})$ it holds that $\langle \phi, e, c, i \rangle \in \mu_{\text{opt}}(I)$ iff $\text{ovr}(\phi, e, c, i) \in I$. Thus, we only need to define the order $\succ_{\text{opt}}$ on the semiring values $S \subseteq CA$ that correctly captures the ordering on the justified models. For this we let $S \subseteq CA$ and define $(\chi^{(S)}_i)_{i \in \mathcal{M}}$, the clashing assumption maps corresponding to $S$, by setting

$$\chi^{(S)}_i(c) = \{\langle \phi, e \rangle \mid \langle \phi, e, c, i \rangle \in S\}.$$

Then for $S, S' \subseteq CA$, we define $S \succ_{\text{opt}} S'$ iff

(i) there exists $i \in \{1, \ldots, m\}$ and some $c \in \mathcal{N}$ s.t. $\chi^{(S)}_i(c) > \chi^{(S')}_i(c)$ and not $\chi^{(S)}_i(c) > \chi^{(S')}_i(c)$, and for no context $c' \neq c \in \mathcal{N}$ it holds that $\chi^{(S)}_i(c') < \chi^{(S')}_i(c')$ and not $\chi^{(S)}_i(c') < \chi^{(S')}_i(c')$.

(ii) for every $1 \leq j < i \leq m$, we have $\chi^{(S)}_j \approx \chi^{(S')}_j$ (i.e. (i) or its converse is unprovable for $\prec_j$).

**Theorem 1**

Let $\mathfrak{R}$ be an sCKR and $PK(\mathfrak{R})$ as described above. Then the preferred answer sets w.r.t. $\mu_{\text{opt}}$ and $\succ_{\text{opt}}$ correspond to the least CKR models $(\mathfrak{I}, \mathcal{X})$ of $\mathfrak{R}$, i.e. those where $\mathfrak{I}$ is the $\subseteq$-minimal interpretation such that $\langle \mathfrak{I}, \mathcal{X} \rangle$ is a CKR model.

In the following we outline how such an ASP program $PK(\mathfrak{R})$ can be constructed. Furthermore, we show that for suitably restricted $\mathfrak{R}$, we can also express algebraic measures and preferential answer sets using asprin.
5 ASP Encoding of Reasoning Problems

ASP translation process. The ASP translation by Bozzato et al. (2018a) for instance checking (w.r.t. c-entailment, under UNA) in a SROIQ.Q-RL CKR can be extended to multi-relational sCKRs \( r = \langle C, K_M \rangle \), such that (1) a set of input rules \( I \) encode the contextual structure and local contents of contexts in \( r \) as facts; (2) uniform deduction rules \( P \) encode the interpretation of axioms; and (3) the instance query is encoded by output rules \( O \) as ground facts.

Formally, the CKR program \( PK(\bar{r}) = PG(c) \cup \bigcup_{c \in N} PC(c, \bar{r}) \) encodes the whole sCKR, where \( PG(c) = I_{glob}(c) \cup P_{glob} \) is the global program for \( c \) and \( PC(c, \bar{r}) = I_{loc}(K_c, c) \cup P_{loc} \) is the local program for \( c \in N \). Query answering \( \bar{r} \models c : \alpha \) is then achieved by testing whether the instance query, translated to \( O(\alpha, c) \), is a consequence of the preferred models of \( PK(\bar{r}) \), i.e., whether \( PK(\bar{r}) \cup P_{pref} \models O(\alpha, c) \) holds, where \( P_{pref} \) are the newly added rules for selection of preferred models. Analogously, this can be extended to conjunctive queries as shown by Bozzato et al. (2018a). The details of the translation rules are in the Appendix; in the following, we further discuss \( P_{pref} \).

asprin-based model selection. From the translation \( PK(\bar{r}) \) we obtain the least justified models of \( r \) as answer sets of an ASP program. In Section 4 we showed how to use algebraic measures for describing which answer sets correspond to preferred models. By suitably restricting the input CKR \( r \), we show that we can implement the preference already in the asprin framework (Brewka et al. 2015). The latter can not express sCKR preference relations in general as eval-expressions may cause non-transitive and even cyclic preference relations. We thus restrict the use of eval-expressions such that we can define an asprin preference relation \( > \) that has the same preferred answer sets as \( \mu_{opt} \) but is a strict partial order. For this, we consider a dependency graph.

Definition 12 (Dependency Graph)
The dependency graph of an sCKR \( r \) is the directed graph \( DEP(\bar{r}) = (V, E) \) is \( \bar{r} \), where:

- \( V = \{ X_c \mid X \) is a concept or role that occurs in \( K_c \} \), i.e., we have a vertex \( X_c \) for every combination of a concept or role \( X \) that occurs in \( \bar{r} \) and context \( c \in N \).
- \( (X_c, X'_c) \in E \) if either: (i) \( c = c' \), \( X \) is a complex concept or role and \( X' \) is a subexpression of \( X \); (ii) \( c = c' \) and \( X, X' \) co-occur in some (possibly defeasible) axiom; or (iii) \( X = \text{eval}(X', c') \).

Intuitively, a path connects two concepts/roles \( X_c, X'_c \) in \( DEP(\bar{r}) \) if the interpretations of \( X, X' \) at contexts \( c, c' \), respectively, may depend on each other. If there are no eval-expressions, then clearly there is no path between \( X_c, X'_c \) when \( c \neq c' \). In this case, we can choose the interpretations per context independently, which simplifies the choosing of preferred interpretations significantly. However, as the preference only refers to clashing assumptions caused by defaults, we can also use a weaker condition to a similar effect:

Definition 13 (eval-Disconnectedness)
Let \( r \) be an sCKR and \( X, X' \) two concepts or roles that occur in default axioms. Then \( X, X' \) are eval-disconnected if there is no path between \( X_c, X'_c \) in \( DEP(\bar{r}) \) for every \( c \neq c' \). Furthermore, \( r \) is eval-disconnected if every such \( X, X' \) are eval-disconnected.

In the following, we confine to eval-disconnected sCKR’s and define the preference in asprin as follows. We use so called “poset” preferences, which are specified using statements of the form:

\[
\#preference(p, \text{poset})\{ F_1 \gg F_2; F_3 \gg F_4; \ldots; F_{2n-1} \gg F_{2n} \}.
\]
Here each $F_i$ is a Boolean formula, and a partial order $\succ$ on such formulas is defined by the transitive closure of $\succ$. An interpretation $X$ is preferred over interpretation $Y$ w.r.t. $p$ (written $X \succ_p Y$) if (i) for some $i$, $X \models F_i$ and $Y \not\models F_i$, and (ii) for every $i$ s.t. $Y \models F_i$ and $X \not\models F_i$, some $j$ exists s.t. $F_j \succ F_i$ and $X \models F_j$ and $Y \not\models F_j$.

We then define the local preference w.r.t. context $c$ and relation $i$ by

\[
\#\text{preference}(\text{LocPref}(c,i), \text{poset}) \{
\neg\text{ovr}(\alpha, X, c, i) \Rightarrow \text{ovr}(\alpha, X, c, i);
\neg\text{ovr}(\alpha_2, Y, c, i) \Rightarrow \neg\text{ovr}(\alpha_1, X, c, i); \text{ for } c_1 \gtrsim_i c_{1b} \succ_i c \text{ and } c_2 \gtrsim_i c_{2b} \succ_i c \text{ and } c_{1b} \succ_i c_{2b} \text{ and } D_i(\alpha) \text{ in } K_c.
\}
\]

This encodes that, whenever possible, we prefer not to override a defeasible axiom $D_i(\alpha)$ (line 2); further, if we have to override some defeasible axiom, then we prefer to override the least specific one possible (line 3). Next, we emulate the preference definition (MP), where item (i) combines the local preferences into a preference per defeasibility relation and item (ii) states that the global ordering is the lexicographical combination of the preferences per relation.

Using asprin, we can combine existing preference orders into a new one. This is where eval-disconnectedness comes into play. While for general sCKRs this is not the case, for eval-disconnected sCKRs, the preferred models w.r.t. (i) are the pareto optimal models $X$, i.e., no model $Y$ exists that is strictly better than $X$ on one of the local preferences LocPref$(c,i)$ and at least as good on all the others. Thus, we use the pareto type to define the preference per relation $i$:

\[
\#\text{preference}(\text{RelPref}(i), \text{pareto})\{\#\text{LocPref}(C,i) : \text{context}(C)\}.
\]

Here, the condition context$(C)$ enforces that we take the pareto order over the orders LocPref$(C,i)$ for every context $C$. Finally, for (ii), we use asprin’s lexicographical preference over orders $(p_i)_{i \in [n]}$ with weights $(w_i)_{i \in [n]}$. When $w_i > w_j$ we may worsen $p_j$ to improve $p_i$.

\[
\#\text{preference}(\text{GlobPref}, \text{lexico})\{\#\text{RelPref}(I) : \text{rel}_w(I, W)\}.
\]

Similar to above, the condition rel$_w$(I, W) ensures that we obtain the lexicographical order over all preferences RelPref$(I)$, where $I$ is a relation with weight $w$; in our case, $W$ is its index.

**Correctness.** The presented encoding yields a sound and complete reasoning method for multi-relational sCKRs in SROIQ-RLD normal form, on time and coverage relations. SROIQ-RLD disallows defeasible SROIQ-RLD-axioms that introduce disjunctive information. The normal form of SROIQ-RLD due to [Bozato et al. (2018a)](Bozato2018) is summarized in the Appendix. Formally,

**Theorem 2**

Let $\mathcal{R}$ be a multi-relational sCKR that is eval-disconnected and in SROIQ-RLD normal form. Then under the unique name assumption (UNA),

(i) for every $\alpha$ and $c$ such that $O(\alpha,c)$ is defined, $\mathcal{R} \models c : \alpha$ iff $PK(\mathcal{R}) \cup \text{Pref} = O(\alpha,c)$;

(ii) for every BCQ $Q = \exists y \gamma(y)$ on $\mathcal{R}$, $\mathcal{R} \models Q$ iff $PK(\mathcal{R}) \cup \text{Pref} = O(Q)$.

Similarly to [Bozato et al. (2019), 2018b], the result is shown by proving a correspondence between the least CAS models of $\mathcal{R}$ and the answer sets of $PK(\mathcal{R})$, and then between preferred CAS models and answer sets, which are here selected by our asprin preference. For space reasons, we confine to a proof outline; more details are given in the Appendix.

Without loss of generality, we can restrict to named models, i.e., models $\mathcal{I}$ s.t. the interpretation of atomic concepts and roles belongs to $N^\mathcal{I}$ for some $N \subseteq NI \setminus NI_5$. This allows us to concentrate
on Herbrand models for \( \mathcal{A} \); in particular, w.r.t. a clashing assumption \( \mathcal{X} = (\chi, \chi_c) \), we have a least Herbrand model which we denote as \( \hat{\mathcal{I}}(\mathcal{X}) \).

Suppose \( \mathcal{I}_{\text{CAS}} = (\mathcal{I}, \mathcal{X}) \) is a justified named CAS-model. We can build from \( \mathcal{I}_{\text{CAS}} \) a corresponding Herbrand interpretation \( I(\mathcal{I}_{\text{CAS}}) \) for the program \( PK(\mathcal{A}) \). Along the lines of [Bozzato et al. (2018a), Lemma 6], we can then show that the answer sets of \( PK(\mathcal{A}) \) coincide with the sets \( I(\hat{\mathcal{I}}(\mathcal{X})) \) where \( \mathcal{X} \) is the clashing assumption of a named CAS model of \( \mathcal{A} \). With this in place, we show that in case of a multi-relational hierarchy, the answer sets of \( PK(\mathcal{A}) \cup P_{\text{pref}} \) found optimal by the asprin preference \( \text{GlobPref} \) (implementing \( P_{\text{pref}} \)) coincide with the sets \( I(\hat{\mathcal{I}}(\mathcal{X})) \) where \( \mathcal{X} \) is the clashing assumption of a named preferred CAS model (i.e. CKR model) of \( \mathcal{A} \).

**Prototype Implementation.** The ASP translation presented above is implemented as a proof-of-concept in the CKRew (CKR datalog rewriter) prototype ([Bozzato et al. 2018a]). CKRew is a Java-based command line application that builds on dlv. It accepts as input RDF files representing the contextual structure and local knowledge bases and produces as output a single .dlv text file with the ASP rewriting for the input CKR. The latest version of CKRew is available at [github.com/dkmfbk/ckrew/releases](https://github.com/dkmfbk/ckrew/releases) and includes sample RDF files for \( \mathcal{A}_\text{org} \) of Example 1.

## 6 Additional Possibilities with Algebraic Measures

We highlight further fruitful usages of algebraic measures for reasoning with sCKRs.

**Preferred Model as an Overall Weight.** First, we show another alternative way of obtaining a preferred model as the result of an overall weight query. Formally, we have the following:

**Theorem 3**

Let \( \mathcal{A} \) be a single-relational, eval-free sCKR. Then there exist a semiring \( R_{\text{one}}(\mathcal{A}) \) and weighted formula \( \alpha_{\text{one}} \) such that the overall weight of \( \mu_{\text{one}} = \langle PK(\mathcal{A}), \alpha_{\text{one}}, R_{\text{one}}(\mathcal{A}) \rangle \) is either \((I, \chi)\), where \( I \) is the minimum lexicographical preferred answer set of \( PK(\mathcal{A}) \) and \( \chi \) is the corresponding clashing assumption map, or \( 0 \) if there is no preferred answer set.

Here, the lexicographical order \( >_{\text{lex}} \) over answer sets is given by \( I >_{\text{lex}} I' \) iff there exists some \( b \in B_{PK(\mathcal{A})} \) such that \( b \in I \setminus I' \) and for all \( b' <_{\text{var}} b \) it holds that \( b' \in I \) iff \( b' \in I' \), where \( <_{\text{var}} \) is an arbitrary but fixed total order on \( B_{PK(\mathcal{A})} \).

Intuitively, we define \( R_{\text{one}}(\mathcal{A}) \) by the following strategy. The domain \( R \) is the set of all pairs \((I, \chi)\), where \( I \) is an interpretation of \( PK(\mathcal{A}) \) and \( \chi \) a possible clashing assumption map, and two constants \( 0, 1 \), which act as the zero and one of the semiring. The multiplication \( \otimes \) of \( R_{\text{one}}(\mathcal{A}) \) is (pointwise) union and can thus be used to build a representation of the interpretation \( I \) and its clashing assumption map \( \chi \). The addition \( \oplus \) corresponds to taking the “more preferred” interpretation or the one which is lexicographically smaller, in case of a tie.

Note that the restriction to eval-free sCKRs (or a similar fragment) is necessary: the strategy explained above is only viable if the preference relation over the models is transitive.

**Epistemic Reasoning using Overall Weight Queries.** Using asprin, we can enumerate preferred models. For obtaining all of them at once, we can use an overall weight query.

**Theorem 4**

Let \( \mathcal{A} \) be a single-relational, eval-free sCKR. Then there exists a semiring \( R_{\text{all}}(\mathcal{A}) \) and weighted formula \( \alpha_{\text{all}} \) such that the overall weight of \( \mu_{\text{all}} = \langle PK(\mathcal{A}), \alpha_{\text{all}}, R_{\text{all}}(\mathcal{A}) \rangle \) is \((A_c)_{c \in \mathbb{N}} \) and the set of CKR models corresponds to \( \{(I(c))_{c \in \mathbb{N}} \mid \text{for each } c \in \mathbb{N} : (I(c), \chi(c)) \in A_c\} \).
The definition of $\mathcal{R}_{all}(\mathcal{R})$ is similar to that of $\mathcal{R}_{one}(\mathcal{R})$. However, instead of pairs $(I, \chi)$ the semiring values here are sets of pairs $(I, \chi)$. Given such sets $A, B$, addition and multiplication select the preferred pairs in the result of the union $A \cup B$ and the “Cartesian” union $\{(S_1 \cup S_2, \chi_1 \cup \chi_2) \mid (S_1, \chi_1) \in A, (S_2, \chi_2) \in B\}$, respectively.

We can use the overall weight $\mu_{all}(PK(\mathcal{R}))$ not only to single out all preferred models but also for further advanced tasks. E.g., the cautious and brave consequences at context $c$ are obtained by

$$\bigcap \{I(c) \mid I(c) \in \mu_{all}(PK(\mathcal{R}))\} \quad \text{respectively} \quad \bigcup \{I(c) \mid I(c) \in \mu_{all}(PK(\mathcal{R}))\}.$$

Apart from this, we can also use the result to evaluate epistemic aggregate queries, akin to the ones defined by Calvanese et al. (2008), of the form

$$q((x, \alpha(y))) \leftarrow K\,x, y, z, \phi, [\psi],$$

where $\phi$ and $\psi$ are conjunctions of possibly non-ground atoms and $x, y, z$ are sequences of variables that occur in $\phi$, such that $z$ is distinct from $x$ and $y$. Furthermore, $\alpha$ is an aggregation function. The meaning of this expression given a knowledge base $KB$ is intuitively as follows. For each assignment to $\alpha$, we aggregate over all values $y$ using $\alpha$, subject to the constraint that for every model $D$ of $KB$ the assignment to $x, y$ can be completed to an assignment $\gamma$ to all the variables in $\phi$ and $\psi$ such that (i) $\phi$ and $\psi$ are satisfied by $D$ w.r.t. $\gamma$, and (ii) for every model $D'$ of $KB$ it holds that $\gamma$ restricted to $x, y, z$ is a certain answer for the query $(*) aux_q(x, y, z) \leftarrow \phi, \psi$. Then, $q(x, y)$ is an answer of the above epistemic aggregate query if it is the result of the query in every model $D$. For formal details, we refer to the paper by Calvanese et al. (2008).

Calvanese et al. showed that for “restricted” queries, the value of the aggregate is obtained by

$$q_0((x, y, z^0)) \leftarrow \text{Cert}(aux_q, K)(x, y, z).$$

Here $z^0$ are the variables of $z$ that occur in $\phi$ and $\text{Cert}(aux_q, K)(x, y, z)$ refers to the certain answers of the query $(*)$. Unfortunately, we cannot use ASP alone to compute the certain answers in the presence of defeasible axioms and preferences in sCKRs. However, the overall weight $\mu^*(PK(\mathcal{R}))$ contains the information necessary to conclude what the certain answers are. These in turn can then be used to evaluate epistemic aggregates over sCKRs.

7 Discussion and Conclusions

We considered the application of ASP with algebraic measures for expressing preferences of defeasibility in multi-relational CKRs. The problem of representing notions of defeasibility in DLs has led to many proposals and is still an active area of research (Giordano et al. 2011; Bonatti et al. 2015; Pensel and Turhan 2018; Britz et al. 2021). A detailed comparison of justifiable exceptions with other definitions of non-monotonicity in DLs and contextual systems can be found in the papers by Bozzato et al. (2018a, 2019). Our work on CKRs with multiple contextual relations was influenced by approaches dealing with exceptions under different relations or diverse definitions of normality. One of the latest in this direction is the work by Giordano and Dupré (2020), where the notion of typicality in DLs is extended to a “concept-aware multi-preference semantics”: the domain elements are organized in multiple preference orderings $\leq_C$ to represent their typicality w.r.t. a concept $C$; models are then ordered by a global preference combining the concept-related preferences. Similar to our approach, entailment is encoded in ASP using a fragment of Krötzsch’s (2010) materialization calculus and representing combination of preferences in asprin. Gil (2014) earlier studied the effects of adding multiple preferences to a typicality extension of $ALC$. 
Concerning semirings for general quantitative specifications, several works used semirings to define quantitative generalisations of well-known qualitative problems. For example, Semiring-based Constraint Satisfaction Problems (SCSP) (Bistarelli et al. 1999) allow for quantitative semantics of CSP’s and capture other quantitative extensions of CSP’s (weighted CSP) as special cases for some specific semiring. Semiring Provenance (Green et al. 2007), generalizes the bag semantics and other definitions of provenance for relational algebra to semirings: this allows one to capture existing quantitative semantics, but also to introduce additional novel capabilities to obtain the provenance lineage of a query. Moreover, algebraic ProbLog (Kimmig et al. 2011) introduced an algebraic semantics of logic programs by facilitating semirings. Intuitively, their approach can be seen as a fragment of ASP with algebraic measures allowing only a restricted use of negation in programs and no arbitrary recursive sums and products in the weighted formulas.

The parametrization of semantics with a semiring allows for flexible and highly general quantitative frameworks: in particular, algebraic measures allow for an intuitive specification of computations depending on the truth of propositional variables. Building on ASP, they offer an appealing specification language for quantitative reasoning problems like preferential reasoning.

**Outlook.** In the direction of using the capabilities of algebraic measures for comparing models, we plan to further study the possibilities for epistemic reasoning on DLs as introduced in previous sections. With respect to contextual reasoning, a possible continuation of this work can consider a refinement of the definitions of preference and knowledge propagation across different contextual relations, possibly by considering a motivating real-world application.

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Appendix A Single-relational Example

We also give an example of a single-relational sCKR.

**Example 5**

We consider a single-relation hierarchy on coverage by reviewing the example from (Bozzato et al. 2019; Bozzato et al. 2018b). Let us consider a sCKR $\mathcal{R}_{\text{org}} = (\mathcal{C}, K_N)$ with $\mathcal{C} = (N, \prec_C)$ describing the organization of a corporation. The corporation wants to define different policies in the fields of Musical instruments ($M$), Electronics ($E$) and Robotics ($R$). A supervisor ($S$) can be assigned to manage only one of these fields. Defeasible axioms in contexts in $K_N$ define the assignment of local supervisors to their field:

\[
\mathcal{C}: \{c_{br1} \prec_C c_{world}, c_{br2} \prec_C c_{world}, c_{br1} \prec_C c_{br2}, c_{local1} \prec_C c_{br1}, \}
\]

\[
c_{world}: \{M \sqcap E \sqsubseteq \bot, M \sqcap R \sqsubseteq \bot, E \sqcap R \sqsubseteq \bot, D(S \sqsubseteq E)\}
\]

\[
c_{br1}: \{D_i(S \sqsubseteq M)\} c_{br2}: \{D_i(S \sqsubseteq R)\} c_{local1}: \{S(i)\}
\]

In $c_{world}$ we say that supervisors are assigned to Electronics, while in the sub-context for $c_{br2}$ we contradict this by assigning all local supervisors to the Robotics area and in $c_{br1}$ we further specialize this by assigning supervisors to the Musical instruments area. In the context $c_{local1}$ for a local site we have information about an instance $i$. Note that different assignments of areas for $i$ are possible by instantiating the defeasible axioms: intuitively, we want to prefer the interpretations that override the higher defeasible axioms in $c_{world}$ and $c_{br2}$.

Observe that different justified CAS models are possible, depending on the different assignments of the individual $i$ in $c_{local1}$ to the alternative areas denoted by defeasible axioms. We have three possible clashing assumptions sets for context $c_{local1}$:

\[
\chi_1^1(c_{local1}) = \{(S \sqsubseteq E, i), (S \sqsubseteq R, i)\} \quad \chi_2^2(c_{local1}) = \{(S \sqsubseteq M, i), (S \sqsubseteq R, i)\}
\]

\[
\chi_3^3(c_{local1}) = \{(S \sqsubseteq M, i), (S \sqsubseteq E, i)\}
\]

By the ordering on clashing assumption sets, in particular $\chi_1^1(c_{local1}) > \chi_2^2(c_{local1})$, $\chi_1^1(c_{local1}) > \chi_3^3(c_{local1})$ and $\chi_2^2(c_{local1}) > \chi_3^3(c_{local1})$. Thus, $\mathcal{R}_{\text{org}}$ has one preferred model which corresponds to $\chi_1^1$: it corresponds to the intended interpretation in which the defeasible axiom $D(S \sqsubseteq M)$ associated to $c_{br1}$ wins over the more general rules asserted in $c_{br2}$ and $c_{world}$.

Appendix B ASP Translation and Rule Set Tables

We provide further details on the ASP encoding introduced in Section 5. The ASP translation is defined by adapting the encoding presented in (Bozzato et al. 2019; Bozzato et al. 2018b) (which, in turn, is based on the translation introduced in (Bozzato et al. 2018a) to the manage the interpretation of multiple relations in simple CKRs.

The ASP translation is defined for $\mathcal{SROIQ}$-RLD multi-relational simple CKRs of the form $\mathcal{R} = (\mathcal{C}, K_N)$ with $\mathcal{C} = (N, \prec_C, \prec_r)$, i.e. over time and coverage contextual relations.

The language of $\mathcal{SROIQ}$-RLD (Bozzato et al. 2018a) restrict the form of $\mathcal{SROIQ}$-RL expressions in defeasible axioms: in defeasible axioms, $D \sqcap D$ can not appear as a right-side concept and each right-side concept $\forall R. D$ has $D \in NC$. We consider the $\mathcal{SROIQ}$-RLD normal form transformation proposed in (Bozzato et al. 2018a) for the formulation of the rules (considering axioms that can appear in simple CKRs) and we assume again the Unique Name Assumption. For ease of reference, the form of (strict and defeasible) axioms in normal form is presented in
Table B 1. \( SROIQ \)-RLD normal form for axioms in \( L_\Sigma \)

**Strict axioms:** for \( A, B \in NC, R, S, T \in NR, a, b \in NI, c \in N: 

\[
\begin{align*}
A(a) & \quad R(a, b) & a = b & a \neq b \\
A \sqsubseteq B & \quad \{a\} \sqsubseteq B & A \sqcap B \sqsubseteq C \\
\exists R A \sqsubseteq B & \quad A \sqsubseteq \exists R \{a\} & A \sqsubseteq \forall R B & A \sqsubseteq \sqsubseteq R. \top \\
R \sqsubseteq T & \quad R \circ S \sqsubseteq T & \text{Dis}(R, S) & \text{Inv}(R. S) & \text{Irr}(R) \\
ev(A, c) \sqsubseteq B & \quad ev(R, c) \sqsubseteq S \\
\end{align*}
\]

**Defeasible axioms:** for \( A, B \in NC, R, S \in NR, a \in NI, rel \in \{t, c\}: 

\[
\begin{align*}
D_{rel}(A \sqsubseteq B) & \quad D_{rel}(A \sqcap B \sqsubseteq C) & D_{rel}(\exists R A \sqsubseteq B) \\
D_{rel}(A \sqsubseteq \exists R \{a\}) & \quad D_{rel}(A \sqsubseteq \forall R B) & D_{rel}(A \sqsubseteq \sqsubseteq R. \top) \\
D_{rel}(R \sqsubseteq S) & \quad D_{rel}(R \circ S \sqsubseteq T) & D_{rel}(\text{Dis}(R, S)) & D_{rel}(\text{Inv}(R. S)) & D_{rel}(\text{Irr}(R)) \\
\end{align*}
\]

Note that we further simplified the normalization of defeasible class and role assertions and negative assertions as they can be easily represented using defeasible class and role inclusions with auxiliary symbols.

As in the original formulation (inspired by the materialization calculus in (Krötzsch 2010)), the translation includes sets of *input rules* \( I \) (which encode DL axioms and signature as facts), *deduction rules* \( P \) (normal rules providing instance level inference) and *output rules* \( O \) (that encode in terms of a fact the ABox assertion to be proved).

The sets of rules for the proposed translation are presented in tables in the following pages. The input rules \( I_t \) and deduction rules \( P_t \) for \( SROIQ \)-RLD axioms are shown in Table B 2. Table B 3 shows input rules \( I_{glob} \) and deduction rules \( P_{glob} \) for the translation of the contextual structure in \( \Sigma \) local input rules \( I_{eval} \) and deduction rules \( P_{eval} \) for managing \( eval \) expressions, and output rules \( O \) for encoding the output instance query. Input rules \( I_D \) in Table B 4 provide the encoding of defeasible axioms. Deduction rules in \( P_D \) manage the interpretation of defeasible axioms and knowledge propagation. Table B 5 shows rules defining the overriding of axioms. Rules for the inheritance of strict axioms are shown in Table B 6, while rules in Table B 7 define defeasible inheritance. Table B 8 shows rules for the propagation of defeasible axioms on a relation \( rel \) over the other relation. Auxiliary test rules in \( P_B \) are shown in Table B 9. Finally, rules and directives in \( P_{pref} \) define the aspirin preference: the definition of aspirin local and global preferences is shown in Table B 11, while rules in Table B 10 provide auxiliary rules.

Given a multi-relational \( cSKR \) \( \mathcal{R} = (\Sigma, K^P) \) in \( SROIQ \)-RLD normal form with \( \Sigma = (\mathbb{N}, \prec_t, \prec_c) \), a program \( PK(\mathcal{R}) \) that encodes \( \mathcal{R} \) is obtained as follows:

1. the *global program* for \( \Sigma \) is built as: \( PG(\Sigma) = I_{glob}(\Sigma) \cup P_{glob} \)
2. for each \( c \in \mathbb{N} \), we define each local program for context \( c \) as: \( PC(c, \mathcal{R}) = I_{loc}(K_c, c) \cup P_{loc} \), where \( I_{loc}(K_c, c) = I_t(K_c, c) \cup I_{eval}(K_c, c) \cup I_D(K_c, c) \) and \( P_{loc} = P_t \cup P_{eval} \cup P_B \)
3. The *CKR program* \( PK(\mathcal{R}) \) is defined as: \( PK(\mathcal{R}) = PG(\Sigma) \cup \bigcup_{c \in \mathbb{N}} PC(c, \mathcal{R}) \)

Query answering \( \mathcal{R} \models c : \alpha \) is then obtained by testing whether the instance query, translated to ASP by \( O(\alpha, c) \), is a consequence of the *preferred* models of \( PK(\mathcal{R}) \), i.e., whether \( PK(\mathcal{R}) \cup \)
Table B 2. \textit{SROIQ-RL} input and deduction rules

| \textbf{SROIQ-RL input translation} $\mathcal{I}_f(S,c)$ |
|--------------------------------------------------------|
| (irl-nom) $a \in N \rightarrow \{\text{nom}(a,c)\}$      |
| (irl-clas) $A \in NC \rightarrow \{\text{cls}(A,c)\}$ |
| (irl-rol) $R \in NR \rightarrow \{\text{rol}(R,c)\}$  |
| (irl-subr) $\text{subr} \rightarrow \{\text{subr}(R,c)\}$ |
| (irl-subcnj) $\text{subcnj} \rightarrow \{\text{subcnj}(R,c)\}$ |
| (irl-supforall) $\text{supforall} \rightarrow \{\text{supforall}(R,c)\}$ |
| (irl-supex) $\text{supex} \rightarrow \{\text{supex}(R,c)\}$ |
| (irl-subc) $\text{subc} \rightarrow \{\text{subc}(R,c)\}$ |
| (irl-leqone) $\text{leqone} \rightarrow \{\text{leqone}(R,c)\}$ |
| (irl-forall) $\text{forall} \rightarrow \{\text{forall}(R,c)\}$ |
| (irl-inv) $\text{inv} \rightarrow \{\text{inv}(R,c)\}$ |
| (irl-dis) $\text{dis} \rightarrow \{\text{dis}(R,c)\}$ |
| (irl-eq) $\text{eq} \rightarrow \{\text{eq}(a,b,c)\}$ |
| (irl-subcex) $\text{subcex} \rightarrow \{\text{subcex}(R,c)\}$ |
| (irl-subconj) $\text{subconj} \rightarrow \{\text{subconj}(R,c)\}$ |
| (irl-subex) $\text{subex} \rightarrow \{\text{subex}(R,c)\}$ |
| (irl-leq) $\text{leq} \rightarrow \{\text{leq}(R,c)\}$ |
| (irl-supone) $\text{supone} \rightarrow \{\text{supone}(R,c)\}$ |
| (irl-subsub) $\text{subsub} \rightarrow \{\text{subsub}(R,c)\}$ |
| (irl-subtriple) $\text{subtriple} \rightarrow \{\text{subtriple}(R,c)\}$ |
| (irl-subtripled) $\text{subtripled} \rightarrow \{\text{subtripled}(R,c)\}$ |
| (irl-triplea) $\text{triplea} \rightarrow \{\text{triplea}(R,c)\}$ |
| (irl-tripleb) $\text{tripleb} \rightarrow \{\text{tripleb}(R,c)\}$ |
| (irl-triplec) $\text{triplec} \rightarrow \{\text{triplec}(R,c)\}$ |
| (irl-tripled) $\text{tripled} \rightarrow \{\text{tripled}(R,c)\}$ |
| (irl-top) $\top(\alpha) \rightarrow \{\text{top}(\alpha,c)\}$ |
| (irl-bot) $\bot(\alpha) \rightarrow \{\text{bot}(\alpha,c)\}$ |
| (irl-rol) $\text{rol} \rightarrow \{\text{rol}(R,c)\}$ |
| (irl-end) $\text{end} \rightarrow \{\text{end}(R,c)\}$ |

\textbf{SROIQ-RL deduction rules} $\mathcal{P}_d$

| (prl-instd) $\text{instd}(x,z,c) \rightarrow \{\text{insta}(x,z,c)\}$ |
| (prl-tripled) $\text{tripled}(x,r,y,c) \rightarrow \{\text{triplea}(x,r,y,c)\}$ |
| (prl-eq) $\text{eq}(x,y,c) \rightarrow \{\text{eq}(x,y,c)\}$ |
| (prl-bot) $\text{bot}(x,c) \rightarrow \{\text{bot}(x,c)\}$ |
| (prl-subc) $\text{subc}(x,z,c) \rightarrow \{\text{subc}(x,z,c)\}$ |
| (prl-subconj) $\text{subconj}(x,z,c) \rightarrow \{\text{subconj}(x,z,c)\}$ |
| (prl-subex) $\text{subex}(x,z,c) \rightarrow \{\text{subex}(x,z,c)\}$ |
| (prl-supone) $\text{supone}(x,z,c) \rightarrow \{\text{supone}(x,z,c)\}$ |
| (prl-subsub) $\text{subsub}(x,z,c) \rightarrow \{\text{subsub}(x,z,c)\}$ |
| (prl-subtriple) $\text{subtriple}(x,z,c) \rightarrow \{\text{subtriple}(x,z,c)\}$ |
| (prl-subtripled) $\text{subtripled}(x,z,c) \rightarrow \{\text{subtripled}(x,z,c)\}$ |
| (prl-triplea) $\text{triplea}(x,z,c) \rightarrow \{\text{triplea}(x,z,c)\}$ |
| (prl-tripleb) $\text{tripleb}(x,z,c) \rightarrow \{\text{tripleb}(x,z,c)\}$ |
| (prl-triplec) $\text{triplec}(x,z,c) \rightarrow \{\text{triplec}(x,z,c)\}$ |
| (prl-tripled) $\text{tripled}(x,r,y,c) \rightarrow \{\text{tripled}(x,r,y,c)\}$ |
| (prl-top) $\top(\alpha) \rightarrow \{\text{top}(\alpha,c)\}$ |
| (prl-bot) $\bot(\alpha) \rightarrow \{\text{bot}(\alpha,c)\}$ |
| (prl-rol) $\text{rol}(x,c) \rightarrow \{\text{rol}(x,c)\}$ |
| (prl-end) $\text{end}(x,c) \rightarrow \{\text{end}(x,c)\}$ |

$P_{\text{pref}} \models O(\alpha,c)$ holds. This can be extended to conjunctive queries $Q$ by applying the output rules to its atoms and checking if $PK(R) \cup P_{\text{pref}} \models O(Q)$ holds.
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Table B 3. Global, local and output rules

| Global input rules $I_{glob}(\mathcal{C})$ |
|---------------------------------------------|
| (ggl-ctx) $c \in \mathbb{N} \rightarrow \{\text{context}(c)\}$ |
| (ggl-rel-t) $\sqrt{t} \in \mathcal{C} \rightarrow \{\text{relation}(\text{time})\}$ |
| (ggl-rel-c) $\prec_c \in \mathcal{C} \rightarrow \{\text{relation}(\text{covers})\}$ |
| (ggl-covers-t) $c_1 \prec_c c_2 \rightarrow \{\text{prec}(c_1,c_2,\text{time})\}$ |
| (ggl-covers-c) $c_1 \prec_c c_2 \rightarrow \{\text{prec}(c_1,c_2,\text{covers})\}$ |

| Global deduction rules $P_{glob}$ |
|-----------------------------------|
| (pgl-preceq1) $\text{preceq}(c_1,c_2,\text{rel}) \leftarrow \text{prec}(c_1,c_2,\text{rel})$. |
| (pgl-preceq2) $\text{preceq}(c_1,c_1,\text{rel}) \leftarrow \text{context}(c_1),\text{relation}(\text{rel})$. |
| (pgl-preceqexc1) $\text{preceq}_{\text{except}}(c_1,c_2,\text{rel}) \leftarrow \text{relation}(\text{rel}),\text{preceq}(c_1,c_3,\text{rel}_1)$. |
| (pgl-preceqexc2) $\text{preceq}_{\text{except}}(c_1,c_2,\text{rel}) \leftarrow \text{relation}(\text{rel}),\text{preceq}(c_2,c_1,\text{rel}_2),\text{rel} \neq \text{rel}_1,\text{rel} \neq \text{rel}_2$. |

| Local eval input rules $I_{eval}(S,c)$ |
|---------------------------------------|
| (ilc-subevalat) $\text{eval}(A,c_1) \subseteq B \rightarrow \{\text{subEval}(A,c_1,B,c)\}$ |
| (ilc-subevalr) $\text{eval}(R,c_1) \subseteq T \rightarrow \{\text{subEval}(R,c_1,T,c)\}$ |

| Local eval deduction rules $P_{eval}$ |
|---------------------------------------|
| (plc-subevalat) $\text{instd}(x,b,c,t) \leftarrow \text{subEval}(a,c_1,b,c),\text{instd}(x,a,c_1,t)$. |
| (plc-subevalr) $\text{tripled}(x,s,y,c,t) \leftarrow \text{subEval}(r,c_1,s,c),\text{tripled}(x,r,y,c_1,t)$. |
| (plc-subevalatp) $\text{instd}(x,b,c,t) \leftarrow \text{subEval}(a,c_1,b,c),\text{instd}(x,a,c_1,t),\text{prec}(c_1,c_2,\text{rel}_1)\rightarrow\text{preceq}(c_1,c_2,\text{rel}_2),\text{rel} \neq \text{rel}_1$. |
| (plc-subevalpp) $\text{tripled}(x,s,y,c,t) \leftarrow \text{subEval}(r,c_1,s,c),\text{tripled}(x,r,y,c_1,t),\text{prec}(c_1,c_2,\text{rel}_1)\rightarrow\text{preceq}(c_1,c_2,\text{rel}_2),\text{rel} \neq \text{rel}_2$. |

| Output translation $O(\alpha,c)$ |
|----------------------------------|
| (o-concept) $A(a) \rightarrow \{\text{instd}(a,A,c,\text{main})\}$ |
| (o-role) $R(a,b) \rightarrow \{\text{tripled}(a,R,b,c,\text{main})\}$ |

Table B 4. Input rules $I_{2}(S,c)$ for defeasible axioms

| (id-subc) $D_{\text{rel}}(A \sqsubseteq B) \rightarrow \{\text{def_subclass}(A,B,c,\text{rel}).\}$ |
| (id-subcn) $D_{\text{rel}}(A_1 \sqcap A_2 \sqsubseteq B) \rightarrow \{\text{def_subcn}(A_1,A_2,B,c,\text{rel}).\}$ |
| (id-subex) $D_{\text{rel}}(\exists A \sqsubseteq B) \rightarrow \{\text{def_subex}(R,A,B,c,\text{rel}).\}$ |
| (id-supex) $D_{\text{rel}}(A \sqsubseteq \exists R \{a\}) \rightarrow \{\text{def_supex}(A,R,a,c,\text{rel}).\}$ |
| (id-forall) $D_{\text{rel}}(A \sqsubseteq \forall R B) \rightarrow \{\text{def_supforall}(A,R,B,c,\text{rel}).\}$ |
| (id-leqone) $D_{\text{rel}}(A \sqsubseteq \sqsubseteq 1 R T) \rightarrow \{\text{def_supleqone}(A,R,c,\text{rel}).\}$ |
| (id-subr) $D_{\text{rel}}(R \sqsubseteq S) \rightarrow \{\text{def_subr}(R,S,c,\text{rel}).\}$ |
| (id-subrc) $D_{\text{rel}}(R \sqcap S \sqsubseteq T) \rightarrow \{\text{def_subrc}(A_1,A_2,B,c,\text{rel}).\}$ |
| (id-di) $D_{\text{rel}}(\exists R \sqsubseteq S) \rightarrow \{\text{def_dis}(R,S,c,\text{rel}).\}$ |
| (id-inv) $D_{\text{rel}}(\sqsubseteq R,S) \rightarrow \{\text{def_inv}(R,S,c,\text{rel}).\}$ |
| (id-irr) $D_{\text{rel}}(\sqsubseteq R) \rightarrow \{\text{def_irr}(R,c,\text{rel}).\}$ |

Appendix C Translation correctness: more details

Let us consider a CAS-interpretation $\mathcal{J}_{\text{CAS}} = \langle \mathcal{J}, \mathcal{X} \rangle$ with $\mathcal{X} = \langle \mathcal{X}_d, \mathcal{X}_c \rangle$. We construct its set of atoms corresponding to its overriding assumptions as:

$$OVR(\mathcal{J}_{\text{CAS}}) = \{ \text{ovr}(p(e),c,\text{rel}) \mid (\alpha,e) \in \mathcal{X}_{\text{rel}}(c), I_2(\alpha,c_1) = p \}.$$
Table B 5. Deduction rules $\mathcal{P}_o$ for defeasible axioms: overriding rules

| Rule Descriptions | Rule Formulations |
|-------------------|-------------------|
| (ovr-subc) | $\text{ovr}(\text{subClass}, x, y, z, c_1, c, rel1) \leftarrow \text{def}_{\text{subclass}}(y, z, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{instd}(x, y, c, \text{main}), \text{not test fails}(\text{mlit}(x, z, c))$. |
| (ovr-cnj) | $\text{ovr}(\text{subConj}, x, y_1, y_2, z, c_1, c, rel1) \leftarrow \text{def}_{\text{subcnj}}(y_1, y_2, z, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{instd}(x, y_1, c, \text{main}), \text{instd}(x, y_2, c, \text{main}), \text{not test fails}(\text{mlit}(x, z, c))$. |
| (ovr-subex) | $\text{ovr}(\text{subEx}, x, y, z, c_1, c, rel1) \leftarrow \text{def}_{\text{subex}}(y, z, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{tripled}(x, r, w, c, \text{main}), \text{instd}(w, y, c, \text{main}), \text{not test fails}(\text{mlit}(x, z, c))$. |
| (ovr-supex) | $\text{ovr}(\text{supEx}, x, y, r, w, c_1, c, rel1) \leftarrow \text{def}_{\text{supex}}(y, r, w, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{instd}(x, y, c, \text{main}), \text{not test fails}(\text{arel}(x, r, w, c))$. |
| (ovr-suprc) | $\text{ovr}(\text{supForall}, x, y, z, r, w, c_1, c, rel1) \leftarrow \text{def}_{\text{supforall}}(z, r, w, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{instd}(x, z, c, \text{main}), \text{tripled}(x, r, w, c, \text{main}), \text{not test fails}(\text{mlit}(y, w, c))$. |
| (ovr-lexone) | $\text{ovr}(\text{supLexOne}, x, x_1, x_2, z, r, c_1, c, rel1) \leftarrow \text{def}_{\text{suplexone}}(z, r, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{instd}(x, z, c, \text{main}), \text{tripled}(x, r, x_1, c, \text{main}), \text{tripled}(x, r, x_2, c, \text{main})$. |
| (ovr-subr) | $\text{ovr}(\text{subRole}, x, y, r, s, c_1, c, rel1) \leftarrow \text{def}_{\text{subrc}}(r, s, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{tripled}(x, r, y, c, \text{main}), \text{not test fails}(\text{nrel}(x, s, y, c))$. |
| (ovr-subrc) | $\text{ovr}(\text{subRChain}, x, y, z, r, s, t, c_1, c, rel1) \leftarrow \text{def}_{\text{subrc}}(r, s, t, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{tripled}(x, r, y, c, \text{main}), \text{tripled}(y, s, z, c, \text{main}), \text{not test fails}(\text{nrel}(x, t, z, c))$. |
| (ovr-dis) | $\text{ovr}(\text{dis}, x, y, r, s, c_1, c, rel1) \leftarrow \text{def}_{\text{dis}}(r, s, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{tripled}(x, r, y, c, \text{main}), \text{tripled}(x, s, y, c, \text{main})$. |
| (ovr-inv1) | $\text{ovr}(\text{inv}, x, y, r, s, c_1, c, rel1) \leftarrow \text{def}_{\text{inv}}(r, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{tripled}(x, r, y, c, \text{main}), \text{not test fails}(\text{nrel}(x, s, y, c))$. |
| (ovr-inv2) | $\text{ovr}(\text{inv}, x, y, r, s, c_1, c, rel1) \leftarrow \text{def}_{\text{inv}}(r, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{tripled}(y, s, x, c, \text{main}), \text{not test fails}(\text{nrel}(x, r, y, c))$. |
| (ovr-irr) | $\text{ovr}(\text{irr}, x, r, c_1, c, rel1) \leftarrow \text{def}_{\text{irr}}(r, c_1, \text{rel1})$, $\text{precc}(c_2, \text{rel1}), \text{preceq}(c_2, \text{rel1}, \text{rel2}), \text{rel1} \neq \text{rel2}$, $\text{tripled}(x, r, c, \text{main})$. |

We can build\textsuperscript{2} from its components a corresponding Herbrand interpretation $\mathcal{I}(\mathcal{J}_{\text{CAS}})$ of the program $\mathcal{P}K(\mathbb{R})$ as the smallest set of literals containing:

- all facts of $\mathcal{P}K(\mathbb{R})$;
- $\text{instd}(a, A, c, \text{main})$, if $\mathcal{I}(c) \models A(a)$;
- $\text{tripled}(a, R, b, c, \text{main})$, if $\mathcal{I}(c) \models R(a, b)$;
- each ovr-literal from $\text{OVR}(\mathcal{J}_{\text{CAS}})$.

\textsuperscript{2} See similar construction in \cite{Bozzato2018} Section A.5.2 for further details.
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Table B6. Deduction rules $P_3$ for defeasible axioms: strict inheritance rules

| Rule          | Definition                                                                 |
|--------------|-----------------------------------------------------------------------------|
| (props-inst) | $\text{inst}(x,z,c,\text{main}) \leftarrow \text{inst}(x,z,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |
| (props-triple)| $\text{tripled}(x,r,y,c,\text{main}) \leftarrow \text{triple}(x,r,y,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |
| (props-sub)| $\text{inst}(x,z,c,\text{t}) \leftarrow \text{subClass}(y,z,c_1), \text{inst}(x,y,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |
| (props-cnj)| $\text{inst}(x,z,c,\text{t}) \leftarrow \text{subConj}(y_1,y_2,z,c_1), \text{inst}(x,y_1,c_1), \text{inst}(x,y_2,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |
| (props-subex)| $\text{inst}(x,z,c,\text{t}) \leftarrow \text{subEx}(y,v_1,z,c_1), \text{triple}(x,v_1,c_1), \text{inst}(x,v_1,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |
| (props-supex)| $\text{unsat}(t) \leftarrow \text{supEx}(y,v,x_1,c_1), \text{inst}(x,y,c_1), \text{triple}(x,v,x_1,c_1), \text{inst}(x,v,x_1,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |
| (props-subr)| $\text{triple}(x,w,x_1,c_1) \leftarrow \text{subRule}(v,w,c_1), \text{triple}(x,v,x_1,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |
| (props-subrc)| $\text{triple}(x,w,y_1,c_1) \leftarrow \text{subRule}(y_1,w,c_1), \text{triple}(x,y_1,y_1,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |
| (props-dis)| $\text{unsat}(t) \leftarrow \text{dis}(u,v,c_1), \text{triple}(x,u,c_1), \text{triple}(x,v,y_1,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |
| (props-inv1)| $\text{triple}(y,v,x,c_1) \leftarrow \text{inv}(u,v,c_1), \text{triple}(x,u,y_1,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |
| (props-inv2)| $\text{triple}(x,u,y_1,c_1) \leftarrow \text{inv}(u,v,c_1), \text{triple}(y,v,x,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |
| (props-irr)| $\text{unsat}(t) \leftarrow \text{irr}(u,c_1), \text{triple}(x,u,x_1,c_1), \prec(c_{c_2},\text{rel}1), \text{preceq}(c_{c_2},c_1,\text{rel}2), \text{rel}1 \neq \text{rel}2.$ |

- each literal $l$ with environment $t \neq \text{main}$, if $\text{test}(t) \in I(\mathcal{I}_{\text{CAS}})$ and $l$ is in the head of a rule $r \in \text{grnd}(PK(\mathcal{R}))$ with $B(r) \subseteq I(\mathcal{I}_{\text{CAS}})$:
- $\text{test}(t)$, if $\text{test}_{\text{fails}}(t)$ appears in the body of an overriding rule $r$ in $\text{grnd}(PK(\mathcal{R}))$ and the head of $r$ is an over literal in $\text{OVR}(\mathcal{I}_{\text{CAS}})$:
- $\text{unsat}(t) \in I(\mathcal{I}_{\text{CAS}})$, if adding the literal corresponding to $t$ to the local interpretation of its context $c$ violates some axiom of the local knowledge $K_c$;
- $\text{test}_{\text{fails}}(t)$, if $\text{unsat}(t) \notin I(\mathcal{I}_{\text{CAS}})$.

Note that $\text{unsat}(\text{main})$ is not included in $I(\mathcal{I}_{\text{CAS}})$. Moreover, as all the facts of $PK(\mathcal{R})$ are included in the set, also the atoms $\prec$ and $\text{preceq}$ defining the contextual relations of $\mathcal{R}$ are included in $I(\mathcal{I}_{\text{CAS}})$.

The correctness result provided by Theorem 2 in Section 5 is a consequence of the following Lemma 1 showing the correspondence between the minimal justified CKR-models of $\mathcal{R}$ and the answer sets of $PK(\mathcal{R})$, and Lemma 2 proving the correspondence between preferred models and answer sets selected by the aspirin preference in $P_{\text{pref}}$.

Lemma 1
Let $\mathcal{R}$ be a multi-relational sCKR in $\text{SROIQ-RLD}$ normal form, then:
(i). for every (named) justified clashing assumption $\mathcal{F}$, the interpretation $S = I(\hat{\mathcal{F}})$ is an answer set of $PK(\hat{\mathcal{R}})$;

(ii). every answer set $S$ of $PK(\hat{\mathcal{R}})$ is of the form $S = I(\hat{\mathcal{F}})$ with $\mathcal{F}$ a (named) justified clashing assumption for $\hat{\mathcal{R}}$.

Proof (Sketch)

Intuitively, we are interested in computing the correspondence with all (not necessarily preferred) answer sets of $PK(\hat{\mathcal{R}})$: we can show that the new form of rules for managing multiple contextual relations do not influence the construction of such answer sets, thus the result can be proved similarly to Lemma 6 in (Bozzato et al. 2018a) and its extension to hierarchies in (Bozzato et al. 2018b), Lemma 1).

Let us consider the interpretation $S = I(\hat{\mathcal{F}})$ as defined above and the reduct $G_S(PK(\hat{\mathcal{R}}))$
of \( PK(\mathfrak{R}) \) with respect \( S \). The lemma can then be proved by showing that the answer sets of \( PK(\mathfrak{R}) \) coincide with the sets \( S = I(\hat{\mathcal{FA}}) \) where \( \mathcal{FA} = (\mathcal{F}, \chi) \) is composed by justified clashing assumptions of \( \mathfrak{R} \).

(i). Assuming \( \mathcal{FA} = (\mathcal{F}, \chi) \) is justified, we show that \( S = I(\hat{\mathcal{FA}}) \) is an answer set of \( PK(\mathfrak{R}) \). We first prove that \( S \models G_3(\mathcal{FA}) \), that is for every rule instance \( r \in G_3(\mathcal{FA}) \) it holds that \( S \models r \). This is proved by examining the possible rule forms that occur in \( G_3(\mathcal{FA}) \). Here we show some representative cases (see also [Bozzato et al. 2018a]):

- (prl-instd): then \( \text{instd}(a,A,c) \in I(\hat{\mathcal{FA}}) \) and, by definition of the translation, \( A(a) \in K_c \). This implies that \( T(c) \models A(a) \) and thus \( \text{instd}(a,A,c,\text{main}) \) is added to \( I(\hat{\mathcal{FA}}) \).

- (prl-subc): then \( \{\text{subClass}(A,B,c),\text{instd}(a,A,c,t)\} \subseteq I(\hat{\mathcal{FA}}) \). By definition of the translation, we have \( A \sqsubseteq B \in K_c \). For the construction of \( I(\hat{\mathcal{FA}}) \), if \( t = \text{main} \) then \( T(c) \models A(a) \). This implies that \( T(c) \models B(a) \) and \( \text{instd}(a,B,c,\text{main}) \) is added to \( I(\hat{\mathcal{FA}}) \). Otherwise, if \( t \neq \text{main} \) then \( \text{instd}(a,B,c,t) \) is directly added to \( I(\hat{\mathcal{FA}}) \) by its construction.

- (ovr-subc): then \( \{\text{defSubClass}(B,c_1,rel1),\text{prec}(c_1,c_2,rel1),\text{preeq}(c_2,c_1,rel2),\text{instd}(a,A,c,\text{main})\} \subseteq I(\hat{\mathcal{FA}}) \). Since \( r \in G_3(\mathcal{FA}) \), then \( \text{test fails}(\text{nlit}(a,B,c)) \notin I(\hat{\mathcal{FA}}) \). By construction of \( I(\hat{\mathcal{FA}}) \), this implies that \( \text{unsat}(\text{nlit}(a,B,c)) \notin I(\hat{\mathcal{FA}}) \), meaning that \( T(c) \models \neg B(a) \). Thus, \( T(c) \) satisfies the clashing set \( \{A(a),\neg B(a)\} \) for the clashing assumption \( (A \sqsubseteq B,a) \) for \( rel1 \) in context \( c \). Consequently, \( (A \sqsubseteq B,a) \in \mathcal{F}_{rel1}(c) \) and by construction \( \text{ovr}(\text{subClass},a,B,c) \) is added to \( I(\hat{\mathcal{FA}}) \).

- (props-subc): then \( \{\text{subClass}(A,B,c_1),\text{instd}(a,A,c,t),\text{preeq}(c_2,c_1,rel2)\} \subseteq I(\hat{\mathcal{FA}}) \).
Table B 9. Deduction rules $R_D$ for defeasible axioms: test rules

| Rule | Description |
|------|-------------|
| (test-subc) | $\text{test}(\text{nlit}(x,z,c)) \leftrightarrow \text{def\_subclass}(y,z,c_1,rel1), \text{inst}(x,y,c,\text{main}), \text{prec}(c_2,rel1), \text{prec}(c_1,rel2), \text{rel} \neq \text{rel}2.$ |
| (constr-subc) | $\leftarrow \text{test\_fails}(\text{nlit}(x,z,c)), \text{ovr}(\text{subclass},x,y,z,c_1,rel).$ |
| (test-subcnj) | $\text{test}(\text{nlit}(x,z,c)) \leftrightarrow \text{def\_subcnj}(y_1,y_2,z,c_1,rel1), \text{inst}(x,y_1,c,\text{main}), \text{inst}(x,y_2,c,\text{main}), \text{prec}(c_2,rel1), \text{prec}(c_1,rel2), \text{rel} \neq \text{rel}2.$ |
| (constr-subcnj) | $\leftarrow \text{test\_fails}(\text{nlit}(x,z,c)), \text{ovr}(\text{subconj},x,y_1,y_2,z,c_1,rel).$ |
| (test-subex) | $\text{test}(\text{nlit}(x,z,c)) \leftrightarrow \text{def\_subex}(x,y,z,c_1,rel1), \text{tripled}(x,y,w,c,\text{main}), \text{inst}(w,y,c,\text{main}), \text{prec}(c_2,rel1), \text{prec}(c_1,rel2), \text{rel} \neq \text{rel}2.$ |
| (constr-subex) | $\leftarrow \text{test\_fails}(\text{nlit}(x,z,c)), \text{ovr}(\text{subex},x,y,z,c_1,rel).$ |
| (test-supex) | $\text{test}(\text{rel}(x,r,w,c)) \leftrightarrow \text{def\_supex}(x,r,w,c_1,rel1), \text{inst}(x,y,c,\text{main}), \text{prec}(c_2,rel1), \text{prec}(c_1,rel2), \text{rel} \neq \text{rel}2.$ |
| (constr-supex) | $\leftarrow \text{test\_fails}(\text{rel}(x,r,w,c)), \text{ovr}(\text{supex},x,r,y,w,c_1,rel).$ |
| (test-supforall) | $\text{test}(\text{nlit}(y,w,c)) \leftrightarrow \text{def\_supforall}(z,r,w,c_1,rel1), \text{inst}(x,z,c,\text{main}), \text{tripled}(x,y,z,c,\text{main}), \text{prec}(c_2,rel1), \text{prec}(c_1,rel2), \text{rel} \neq \text{rel}2.$ |
| (constr-supforall) | $\leftarrow \text{test\_fails}(\text{nlit}(y,w,c)), \text{ovr}(\text{supforall},x,y,z,r,w,c_1,rel).$ |
| (test-sub) | $\text{test}(\text{rel}(x,s,y,c)) \leftrightarrow \text{def\_sub}(x,s,c_1,rel1), \text{tripled}(x,y,c,\text{main}), \text{prec}(c_2,rel1), \text{prec}(c_1,rel2), \text{rel} \neq \text{rel}2.$ |
| (constr-sub) | $\leftarrow \text{test\_fails}(\text{rel}(x,s,y,c)), \text{ovr}(\text{sub},x,r,y,s,c_1,rel).$ |
| (test-subrc) | $\text{test}(\text{rel}(x,t,z,c)) \leftrightarrow \text{def\_subrc}(x,t,c_1,rel1), \text{tripled}(x,y,z,c,\text{main}), \text{prec}(c_2,rel1), \text{prec}(c_1,rel2), \text{rel} \neq \text{rel}2.$ |
| (constr-subrc) | $\leftarrow \text{test\_fails}(\text{rel}(x,t,z,c)), \text{ovr}(\text{subrc},x,r,y,z,t,c_1,rel).$ |
| (test-inv1) | $\text{test}(\text{rel}(x,s,y,c)) \leftrightarrow \text{def\_inv}(x,s,c_1,rel1), \text{tripled}(x,y,c,\text{main}), \text{prec}(c_2,rel1), \text{prec}(c_1,rel2), \text{rel} \neq \text{rel}2.$ |
| (test-inv2) | $\text{test}(\text{rel}(y,x,c)) \leftrightarrow \text{def\_inv}(x,t,c_1,rel1), \text{tripled}(x,y,c,\text{main}), \text{prec}(c_2,rel1), \text{prec}(c_1,rel2), \text{rel} \neq \text{rel}2.$ |
| (test-fails1) | $\text{test\_fails}(\text{nlit}(x,z,c)) \leftrightarrow \text{inst}(x,z,c), \text{nlit}(x,z,c), \text{not}\_\text{unsat}(\text{nlit}(x,z,c)).$ |
| (test-fails2) | $\text{test\_fails}(\text{rel}(x,s,y,c)) \leftrightarrow \text{tripled}(x,y,c), \text{not}\_\text{unsat}(\text{rel}(x,s,y,c)), \text{nlit}(x,z,c)$ |
| (test-add1) | $\text{inst}(x,z,c), \text{nlit}(x,z,c)) \leftrightarrow \text{test}(\text{nlit}(x,z,c)).$ |
| (test-add2) | $\text{tripled}(x,y,c), \text{nlit}(x,y,c)) \leftrightarrow \text{test}(\text{rel}(x,y,c)).$ |
| (test-copy1) | $\text{inst}(x_1,y_1,c,t) \leftrightarrow \text{inst}(x_1,y_1,c,\text{main}), \text{test}(t).$ |
| (test-copy2) | $\text{tripled}(x_1,y_1,c,t) \leftrightarrow \text{tripled}(x_1,y_1,c,\text{main}), \text{test}(t).$ |

$\text{prec}(c_2,\text{rel}1) \subseteq I(\bar{\chi}(\bar{\Phi})).$ By definition, $A \subseteq B \in K_{c_1}$ and, if $t = \text{main}$, $\overline{\mathcal{I}}(c) \models A(a).$ Thus, for the definition of CAS-model (condition (i) on strict axioms propagation), $\text{inst}(a,B,c,t)$ is added to $I(\bar{\chi}(\bar{\Phi})).$ If $t \neq \text{main}$, then $\text{inst}(a,B,c,t)$ is added to $I(\bar{\chi}(\bar{\Phi}))$ by construction.

- (propd-subc): then $\{\text{def\_subclass}(A,B,c_1,rel1), \text{inst}(a,A,c,t), \text{prec}(c_2,rel1), \text{prec}(c_1,rel2)\} \subseteq I(\bar{\chi}(\bar{\Phi})).$ Since $r \in G_{c_1}(\text{PK}(\bar{\Phi}))$, $\text{ovr}(\text{subclass},a,A,B,c_1,rel1) \notin OVR(\bar{\chi}(\bar{\Phi}))$ and hence $(A \subseteq B,a) \notin \chi_{rel1}(c).$ By definition, $D(A \subseteq B) \in K_{c_1}$ and, if $t = \text{main}$, $\overline{\mathcal{I}}(c) \models A(a).$ Thus, for the definition of CAS-model (condition (iii) on defeasible axioms propagation), $\text{inst}(a,B,c,t)$ is added to $I(\bar{\chi}(\bar{\Phi})).$ If $t \neq \text{main}$, then $\text{inst}(a,B,c,t)$ is added to $I(\bar{\chi}(\bar{\Phi}))$ by construction.

- (propd-subc): then $\{\text{def\_subclass}(A,B,c_1,rel1), \text{inst}(a,A,c,t), \text{prec}(c_1,rel2)\} \subseteq I(\bar{\chi}(\bar{\Phi})).$ By definition, $D(A \subseteq B) \in K_{c_1}$ and, if $t = \text{main}$, $\overline{\mathcal{I}}(c) \models A(a)$ with $c \prec_{rel2} c_1.$ Thus,
for the definition of CAS-model (condition (ii) on propagation of defeasible axioms over other relations), \(\text{instd}(a,B,c,t)\) is added to \(I(\tilde{\chi})\). If \(t \neq \text{main}\), then \(\text{instd}(a,B,c,t)\) is added to \(I(\tilde{\chi})\) by construction.

- (test-subc): then \{\(\text{def}_{\text{subclass}}(A,B,c_1,\text{rel1})\), \(\text{instd}(a,A,c,\text{main})\), \(\text{prec}(c_2,\text{rel1})\), \(\text{preceq}(c_2,\text{rel1})\) \} \subseteq I(\tilde{\chi})\). Thus \(D(A \subseteq B) \in K_c\) and \(L(c) = A(a)\) with \(c \sim_{\text{rel1}} c_2 \leq_{\text{rel2}} c_1\). By the construction of \(I(\tilde{\chi})\) we have that \(\text{test}_{\text{fail}}(a,B,c) \in I(\tilde{\chi})\).

Minimality of \(S = I(\tilde{\chi})\) w.r.t. the positive deduction rules of \(G_S(PK(\tilde{\mathcal{R}}))\) can then be motivated as in the original proof in [Bozzato et al. 2018a]: thus, \(I(\tilde{\chi})\) is an answer set of \(PK(\tilde{\mathcal{R}})\).

(ii). Let \(S\) be an answer set of \(PK(\tilde{\mathcal{R}})\). We show that there is some justified clashing assumption \(\tilde{\mathcal{X}}\) for \(\tilde{\mathcal{R}}\) such that \(S = I(\tilde{\chi})\) holds.

Note that as \(S\) is an answer set for the CKR program, all literals on \(\text{ovr}\) and \(\text{test}_{\text{fails}}\) in \(S\) are derivable from the reduce \(G_S(PK(\tilde{\mathcal{R}}))\). By the definition of \(I(\tilde{\chi})\) we can easily build

| Rule                        | Description                                      |
|-----------------------------|--------------------------------------------------|
| \(\text{prep-indv}\)       | \(\text{ind}(x) ← \text{nom}(x,c)\).             |
| \(\text{prep-ovr-subc}\)   | \(\text{ovr}(\text{subClass}(x,y,z),c,\text{rel}) ← \text{def}_{\text{subclass}}(y,z,c,\text{rel}),\) \(\text{ind}(x)\). |
| \(\text{prep-ovr-supfa}\)  | \(\text{ovr}(\text{supForall}(x,y,z,r,w),c,\text{rel}) ← \text{sup}_{\text{forall}}(x,y,z,r,w,c,\text{rel}),\) \(\text{ind}(x),\text{ind}(y)\). |
| \(\text{prep-ovr-subbr}\)  | \(\text{ovr}(\text{subRole}(x,y,r,s,t),c,\text{rel}) ← \text{def}_{\text{subrole}}(r,s,t,c,\text{rel}),\) \(\text{ind}(x),\text{ind}(y),\text{ind}(z)\). |
| \(\text{prep-ovr-dis}\)    | \(\text{ovr}(\text{dis}(x,y,r,s,t),c,\text{rel}) ← \text{def}_{\text{dis}}(r,s,t,c,\text{rel}),\) \(\text{ind}(x),\text{ind}(y),\text{ind}(z)\). |
| \(\text{prep-ovr-inv}\)    | \(\text{ovr}(\text{inv}(x,y,r,s,t),c,\text{rel}) ← \text{def}_{\text{inv}}(r,s,t,c,\text{rel}),\) \(\text{ind}(x),\text{ind}(y)\). |
| \(\text{act-ovr-subc}\)    | \(\text{ovr}(\text{subClass}(x,y,z),c_1,c,\text{rel}) ← \text{ovr}(\text{subClass}(x,y,z),c_1,c,\text{rel})\). |
| \(\text{act-ovr-subbr}\)   | \(\text{ovr}(\text{subRole}(x,y,r,s,t),c_1,c,\text{rel}) ← \text{ovr}(\text{subRole}(x,y,r,s,t),c_1,c,\text{rel})\). |
| \(\text{act-ovr-supfa}\)   | \(\text{ovr}(\text{supForall}(x,y,z,r,w),c_1,c,\text{rel}) ← \text{sup}_{\text{forall}}(x,y,z,r,w,c_1,c,\text{rel})\). |
| \(\text{act-ovr-subbr}\)   | \(\text{ovr}(\text{subRole}(x,y,r,s,t),c_1,c,\text{rel}) ← \text{ovr}(\text{subRole}(x,y,r,s,t),c_1,c,\text{rel})\). |

Table B 10. Rules in \(P_{\text{pref}}\) for preference definitions: preparation rules
Finally, $\mathcal{S} = (\mathcal{I}_c, \chi^S)$ from the answer set $S$ as follows: for every $c \in \mathbb{N}$, we build the local interpretation $\mathcal{I}_S(c) = (\Delta_c, \chi^S)$ as follows:

- $\Delta_c = \{ d \mid d \in \mathbb{N} \}$;
- $a^{\chi^S} = a$, for every $a \in \mathbb{N}$;
- $A^{\chi^S} = \{ d \in \Delta_c \mid S \models \text{instd}(d, A, c, \text{main}) \}$, for every $A \in \text{NC}$;
- $R^{\chi^S} = \{ (d, d') \in \Delta_c \times \Delta_c \mid S \models \text{tripled}(d, R, d', c, \text{main}) \}$, for $R \in \text{NR}$;

Finally, $\chi^S = (\chi^c, \chi^S)$ where $\chi^S_{\text{rel}}(c) = \{ (\alpha, e) \mid I_c(\alpha, c') = p, \text{ovr}(p(e), c, \text{rel}) \in S \}$. We have to show that $\mathcal{I}_S$ meets the definition of a least justified CAS-model for a multi-relational $\mathcal{A}$, that is:

(i) for every $\alpha \in K_c$ (strict axiom), and $c' \preceq_i c$, $\mathcal{I}_S(c') \models \alpha$;
(ii) for every $D_i(\alpha) \in K_c$ and $c' \preceq_i c$, $\mathcal{I}_S(c') \models \alpha$;
(iii) for every $D_i(\alpha) \in K_c$ and $c'' \preceq_i c'$, if $(\alpha, d) \notin \chi_i(c'')$, then $\mathcal{I}_S(c'') \models \phi_\text{d}(d)$.

Note that, since we are considering multi-relational CKRs based only on two relations (time and coverage), the relational closure $c' \preceq_i c$ can be read as $c' \preceq_j c$ with $j \neq i$: this corresponds to the conditions $\text{preceq}(c', c, \text{rel2})$ with $\text{rel1} \neq \text{rel2}$ in the formulation of the rules.

Item (i) should be proved in the local case where $c' = c$ and in the “strict propagation” case where $c' \preceq_c c$. The second case can be shown similarly to the local case, considering strict propagation rules in Table B.6. Thus, considering $c' = c$, we verify the condition by showing that, for every $K_c$, we have $\mathcal{I}_S(c) \models K_c$. This can be shown by cases considering the form of all of the (strict) axioms $\beta \in L_{c,N}$ that can occur in $K_c$. For example (the other cases are similar):

- Let $\beta = A(a) \in K_c$, then, by rule (prl-instd), $S \models \text{instd}(a, A, c, \text{main})$. This directly implies that $a^{\chi^S} \in A^{\chi^S}$.
- Let $\beta = A \sqcup B \in K_c$, then $S \models \text{subClass}(A, B, c)$. If $d \in A^{\chi^S}$, then by definition $S \models \text{instd}(d, A, c, \text{main})$. By rule (prl-subc) we obtain that $S \models \text{instd}(d, B, c, \text{main})$ and thus $d \in B^{\chi^S}$.

### Table B.11. asprin program $P_{\text{pref}}$ for preference definitions

| Preference Definition | Program |
|-----------------------|---------|
| # preference (LocPref(C, REL), poset) { | \begin{verbatim}
   ~ovr(A, Cp, C, REL) >> ovr(A, Cp, C, REL);
   ~ovr(A1, C1, C, REL) >> ovr(A2, C2, C, REL):
   preceq_except(C1b, C1, REL), preceq_except(C2b, C2, REL),
   prec(C, C2b, REL), prec(C2b, C1b, REL),
   p_ovr(A1, C1, REL), p_ovr(A2, C2, REL).
\end{verbatim} |
| (pref-local) | |
| # preference (RelPref(REL), pareto) { | \begin{verbatim}
   ** LocPref(C, REL) : context(C) |
\end{verbatim} |
| (pref-rel-local) | |
| # preference (GlobPref, lexico) { | \begin{verbatim}
   W : ** RelPref(REL) : relation_weight(REL, W) |
\end{verbatim} |
| (pref-global) | |
| # optimize (GlobPref). | |
Condition (ii) can be proved similarly, considering rules of Table [B.8]. In particular, assuming that $D_i(\beta) \in K_c$ with $c \preceq_{rel2} c'$ we can proceed by cases on the possible forms of $\beta$ and consider the (strict) propagation of defeasible axioms to $c$ along the “parallel” relations. For example:

- Let $\beta = A(a)$. Then, by definition of the translation, we have $S \models \text{def}_\text{insta}(a,A,c',rel1)$. Moreover, since $c \preceq_{rel2} c'$, we have $S \models \text{preeq}(c,c',rel2)$ with $rel1 \neq rel2$. By the corresponding instantiation of rule (propp-inst), it holds that $S \models \text{instd}(a,A,c,main)$. By definition, this means that $a^{Z(c)} \in A^{Z(c)}$.

- Let $\beta = A \sqsubset B$. Then, by definition of the translation, $S \models \text{def}_\text{subclass}(A,B,c',rel1)$. Since $c \preceq_{rel2} c'$, we have $S \models \text{preeq}(c,c',rel2)$ with $rel1 \neq rel2$. If $a^{Z(c)} \in A^{Z(c)}$, then by definition $S \models \text{instd}(a,A,c,main)$: by rule (propp-subc), we obtain that $S \models \text{instd}(a,B,c,main)$ and thus $a^{Z(c)} \in B^{Z(c)}$.

To prove condition (iii), let us assume that $D_i(\beta) \in K_c$ with $c \preceq_{rel} c'$. We proceed again by cases on the possible forms of $\beta$ as in the original proof in [Bozzato et al. 2018a], by considering the defeasible propagation to $c$ along the relation $i$. For example:

- Let $\beta = A(a)$. Then, by definition of the translation, we have that $S \models \text{def}_\text{insta}(a,A,c',rel1)$. Suppose that $\langle A(x),a \rangle \not\in X_{rel1}^S(c)$. Then by definition of the translation, $ovr(\text{insta},a,A,c',c,rel1) \not\in OVR(\tilde{\mathcal{X}})$.

- Let $\beta = A \sqsubset B$. Then, by definition of the translation, $S \models \text{def}_\text{subclass}(A,B,c',rel1)$. As above, we also have $S \models \text{preeq}(c,c',rel1)$ and $S \models \text{preeq}(c',c',rel2)$. Let us suppose that $b^{Z(c)} \in A^{Z(c)}$: then $S \models \text{instd}(b,A,c,main)$. Suppose that $\langle A \sqsubset B, b \rangle \not\in \chi_S(c)$: by definition of the translation, $ovr(\text{subclass},b,A,B,c',c,rel1) \not\in OVR(\tilde{\mathcal{X}})$. By the definition of the reduction, the corresponding instantiation of rule (propp-subc) has not been removed from $G_\Sigma(PK(\tilde{\mathcal{R}}))$: this implies that $S \models \text{instd}(b,B,c,main)$. Thus, by definition, this means that $b^{Z(c)} \in B^{Z(c)}$.

We have shown that $\mathcal{J}_S$ is a CAS-model of $\tilde{\mathcal{R}}$: using the same reasoning in the original proof in [Bozzato et al. 2018a], we can also prove the $\mathcal{J}_S$ corresponds to the least model and that $\tilde{\mathcal{X}}^S$ is justified, thus proving the result. □

**Lemma 2**

Let $\mathcal{R}$ be a multi-relational sCKR in $\mathcal{SROIQ}$-$RLD$ normal form. Then, $\tilde{\mathcal{J}}$ is a CKR model of $\mathcal{R}$ iff there exists a (named) justified clashing assumption $\tilde{\mathcal{X}}$ s.t. $I(\tilde{\mathcal{J}}(\tilde{\mathcal{X}}))$ is a preferred answer set of $PK(\tilde{\mathcal{R}}) \cup P_{\text{pref}}$.

For the proof we need the following result:

**Theorem 5**

Let $\mathcal{R}$ be an eval-disconnected sCKR and $\mathcal{J}_{\text{CAS}} = (\mathcal{J}, \mathcal{X}_1, \ldots, \mathcal{X}_m)$ a justified model of $\mathcal{R}$. Then $\mathcal{J}_{\text{CAS}}$ is preferred with respect to $P_{\mathcal{J}_1}$ defined by:

- $P_{\mathcal{J}_1} : (\mathcal{J}, \mathcal{X}_1, \ldots, \mathcal{X}_m), (\mathcal{J}', \mathcal{X}_1', \ldots, \mathcal{X}_m')$ if there exists some $c \in N$ s.t. $\mathcal{X}_i^c > \mathcal{X}_i^{c'}$ and not $\mathcal{X}_i^c > \mathcal{X}_i^{c'}$, and for all contexts $c' \neq c$ in $N$ it holds that $\mathcal{X}_i^c < \mathcal{X}_i^{c'}$ (and not $\mathcal{X}_i^{c'} < \mathcal{X}_i^c$).

If it is preferred with respect to $P_{\mathcal{J}_2}$ defined by:

- $P_{\mathcal{J}_2} : (\mathcal{J}, \mathcal{X}_1, \ldots, \mathcal{X}_m), (\mathcal{J}', \mathcal{X}_1', \ldots, \mathcal{X}_m')$ if there exists some $c \in N$ s.t. $\mathcal{X}_i^c > \mathcal{X}_i^{c'}$ and not $\mathcal{X}_i^c > \mathcal{X}_i^{c'}$, and for all contexts $c' \in N$ it holds that $\mathcal{X}_i^c > \mathcal{X}_i^{c'}$ (or $\mathcal{X}_i^{c'} = \mathcal{X}_i^c$).
Proof (sketch) of Theorem 2

$P_{2,i}(⟨I^1, X^1⟩, ⟨I^2, X^2⟩)$ implies $P_{1,i}(⟨I^1, X^1⟩, ⟨I^3, X^2⟩)$. So we consider the other direction.

Let $J_{CAS}$ be preferred with respect to $P_{2,i}$. Assume that there exists a justified model $J'_{CAS}$ of $\mathcal{R}$ such that $P_{1,i}(J'_{CAS}, J_{CAS})$ holds.

Let $J_{CAS} = \{\{I(c)\}_{c \in \mathbb{N}}, X\}$ and $J'_{CAS} = \{\{I'(c)\}_{c \in \mathbb{N}}, X'\}$. We know there exists some $c^* \in \mathbb{N}$ such that $\chi'(c^*) > \chi(c^*)$. This implies that some $D(\alpha) \in K_\mathcal{R}$ and $e$ exist such that $\langle \alpha, e \rangle \in \chi(c^*) \setminus \chi'(c^*)$. Let $C$ be the component of $DEP(\mathcal{R})$ that contains $X_{c^*}$, where $X$ is any concept or role appearing in $\alpha$. Note that $C$ is independent of the choice of $X$, since any two possible choices $X, X'$ satisfy that $X_{c^*}$ and $X'_{c^*}$ are reachable from one another.

We take $\mathcal{J}'_{CAS} = \{\{I''(c)\}_{c \in \mathbb{N}}, X''\}$ such that $X''(c) = X(c)$ for $X \not\in C$ and $X''(c) = X(c)$ otherwise, and we let $\chi''(c) = \chi(c)$ for $c \neq c^*$ and $\chi''(c) = \chi'(c)$ otherwise. That is, we take the original justified model $J_{CAS}$ and swap the interpretations of all the concepts and roles that were changed in order to satisfy $\alpha(e)$ at context $c^*$ by their changed interpretation in $J'_{CAS}$. The result, $\mathcal{J}'_{CAS}$, is still a model of $\mathcal{R}$, as we exchanged the interpretation for the whole component and therefore any relevant axioms stay satisfied, since they were satisfied in $J_{CAS}$. Furthermore, since $\mathcal{R}$ is eval-disconnected, $\chi''$ is justified because the default $D(\alpha)$ does not use any concept/role $X$ such that $X_{c^*}$ is connected to $X'_{c^*}$ such that $c \neq c^*$ and $X$ is used in another default $D(\beta)$. This implies that only the clashing assumptions for $c^*$ were changed.

Now, however know that $P_{2,i}(\mathcal{J}'_{CAS}, J_{CAS})$. This is a contradiction to our original assumption. Therefore, there cannot exist some $\mathcal{J}'_{CAS}$ such that $P_{1,i}(\mathcal{J}'_{CAS}, J_{CAS})$ and $J_{CAS}$ is preferred with respect to $P_{1,i}$.

Proof (sketch) of Lemma 2

Our definition of the preferences in $P_{pref}$ mirrors the definition of preference: both go from local preference on the clashing assumptions per context, i.e. $\chi_i(c)$, to per relation preference and finally to the global preference. We show that the definitions correspond for each step.

We start with the local preference. So let $X, Y$ be two interpretations of $PK(\mathcal{R})$, $c$ a context and $i$ a relation. Then it holds that $X >_{locPref(c,i)} Y$ iff:

- $X$ and $Y$ do not have the same clashing assumptions at $c$ w.r.t. relation $i$;
- for each $\neg ovr(\alpha_1, e, c_1, i)$ s.t. $X \not\models ovr(\alpha_1, e, c_1, i)$ and $Y \models ovr(\alpha_1, e, c_1, i)$ there exists $\neg ovr(\alpha_2, f, c_2, i) > ovr(\alpha_1, e, c_1, i)$ s.t. $X \models ovr(\alpha_2, f, c_2, i)$ and $Y \not\models ovr(\alpha_2, f, c_2, i)$.

or equivalently:

- $X$ and $Y$ do not have the same clashing assumptions at $c$ w.r.t. relation $i$;
- for each $\alpha_1, e$, where $\alpha_1$ is from context $c_1 \geq_{\neg} c_{1b} \geq_{i} c$, s.t. $X \models ovr(\alpha_1, e, c_1, i)$ and $Y \not\models ovr(\alpha_1, e, c_1, i)$ there exists $\alpha_2, f$, where $\alpha_2$ is from context $c_2 \geq_{\neg} c_{2b} \geq_{i} c$, s.t. $c_{1b} \geq_{i} c_{2b}$ and $X \not\models ovr(\alpha_2, f, c_2, i)$ and $Y \models ovr(\alpha_2, f, c_2, i)$.

The second item is equivalent to

for every $\eta = \langle \alpha_1, e \rangle \in \chi_1(c) \setminus \chi_2(c)$ with $D_i(\alpha_1)$ at a context $c_1 \geq_{\neg} c_{1b} \geq_{i} c$, there exist an $\eta' = \langle \alpha_2, f \rangle \in \chi_2(c) \setminus \chi_1(c)$ with $D_i(\alpha_2)$ at context $c_2 \geq_{\neg} c_{2b} \geq_{i} c$ such that $c_{1b} \geq_{i} c_{2b}$.

So, we see that the only difference between $>_{locPref(c,i)}$ and the order on the context $c$ is the first condition, i.e. that the clashing assumptions on $c$ must be different. However, this does not affect us, since the definition of preference for justified interpretations always uses $E = "\chi_1(c) < \chi_2(c)"$ and not $\chi_2(c) < \chi_1(c)"$. This is equivalent to "$X >_{locPref(c,i)} Y$ and not $Y >_{locPref(c,i)} X"$, since $E$ can only hold when the clashing assumption sets at $c$ w.r.t. relation $i$ are different.
Next, we consider the preference per relation. As we have shown in Theorem 5, the preferred models with respect to the original preference relation $P_1$ are the same as the preferred models with respect to the preference relation $P_{2,i}$. However, as can be easily seen from the definition, $P_{2,i}$ is the order that has the models that are pareto optimal with respect to the local preference orders $\text{LocPref}(c,i)$ per context as its optimal models. We see that $\text{RelPref}(i)$ correctly captures this, as it is the pareto combination of the orders $\text{LocPref}(c,i)$ for each context $c$.

Last but not least, we consider the global preference. In our definition, we say that we prioritize the preference on the clashing assumptions with respect to the relations with a lower index. This corresponds to the lexicographical combination of the orders $\text{LocPref}(c,i)$ for each relation $i$, when assigning the weight $i$ to relation $i$, when it is the preference relation with index $i$. □

Appendix D Proofs for Overall Weight Queries

Before we define the semiring, we ensure that the preference relation $\text{LocPref}(\text{rel})$ is transitive.

**Lemma 3**
The preference relation $\text{LocPref}(\text{rel})$ defined in Section 5 is transitive.

We use the transitivity of the local preference:

**Lemma 4**
Let $\chi_1^i(c) > \chi_2^i(c)$ and $\chi_2^i(c) > \chi_3^i(c)$. Then $\chi_1^i(c) > \chi_3^i(c)$.

**Proof**
Assume $\chi_1^i(c) > \chi_2^i(c)$, $\chi_2^i(c) > \chi_3^i(c)$ and $\langle \alpha_1, e \rangle \in \chi_1^i(c) \setminus \chi_3^i(c)$ with $D_i(\alpha_1)$ at a context $c_1 \geq_{-i} c_{1b} \succ_i c$.

Case 1: If $\langle \alpha_1, e \rangle \notin \chi_3^i(c)$ then since $\chi_1^i(c) > \chi_2^i(c)$ there exists $\langle \alpha_2, f \rangle \in \chi_2^i(c) \setminus \chi_1^i(c)$ with $D_i(\alpha_2)$ at context $c_2 \geq_{-i} c_{2b} \succeq_i c$ such that $c_{1b} \succ_i c_{2b}$.

Case 1.1: If $\langle \alpha_2, f \rangle \in \chi_3^i(c)$ we are done.

Case 1.2: Else, $\langle \alpha_2, f \rangle \in \chi_3^i(c) \setminus \chi_1^i(c)$. Then since $\chi_2^i(c) > \chi_3^i(c)$ there exists $\langle \alpha_3, g \rangle \in \chi_1^i(c) \setminus \chi_2^i(c)$ with $D_i(\alpha_3)$ at context $c_3 \geq_{-i} c_{3b} \succeq_i c$ such that $c_{2b} \succ_i c_{3b}$.

Case 1.2.1: If $\langle \alpha_3, g \rangle \notin \chi_1^i(c)$ we are done.

Case 1.2.2: Otherwise, $\langle \alpha_3, g \rangle \notin \chi_1^i(c) \setminus \chi_2^i(c)$. Note that this is the same situation as in case 1 except that $D_i(\alpha_3)$ is at context $c_{3b} \succ_i c$ such that $c_{1b} \succ_i c_{2b} \succ_i c_{3b}$. Since $\succ_i$ is a strict (partial) order and we only have finitely many contexts this can only occur finitely often. Since in all other cases below case 1 we have that $\chi_1^i(c) > \chi_3^i(c)$ we are done with case 1.

Case 2: If $\langle \alpha_1, e \rangle \in \chi_2^i(c)$ we are in a similar situation as in case 1.2 the statement follows by analogous reasoning. □

**Proof**
As we have seen, $\text{LocPref}(c, \text{rel})$ is transitive for each context $c$ and relation $\text{rel}$. Thus their pareto combination is also transitive. □

As the domain of the semiring we choose $R = \{(S, \chi) | S \in \mathbb{N}^B, \chi \text{ clashing assumption multiset-map}\}$. Here, we need $S$ to be a multiset and $\chi$ to map to multisets of clashing assumptions for technical reasons (namely so that our semiring satisfies the distributive law). We generalize the definition of the local preference to multisets by using
χ^1(c) > χ^2(c), if for every η = ⟨α, e⟩ s.t. the multiplicity of η in χ^1(c) is greater than its multiplicity in χ^2(c) with D_i(α_1) at a context c_1 ⊆ c, there exists an η' = ⟨α_2, f⟩ s.t. the multiplicity of η' in χ^3(c) is greater than its multiplicity in χ^4(c) with D_i(α_2) at context c_2 ⊇ c such that c_1b ⊇ c_2b.

With this in mind, we can define the semiring \( R_{\text{one}}(\mathcal{R}) = (R \cup \{0, 1\}, \oplus, \otimes, 0, 1) \) by letting

\[
a \oplus b = \begin{cases} 
  a & \text{if } a \rightarrow_{\text{LocPref}(\text{rel})} b \\
  b & \text{if } b \rightarrow_{\text{LocPref}(\text{rel})} a \\
  \text{lex-min}(a, b) & \text{otherwise.}
\end{cases}
\]

\[
(S_1, χ^{(1)}) \otimes (S_2, χ^{(2)}) = (S_1 + S_2, χ^{(1)} + χ^{(2)})
\]

Here, lex-min(a, b) takes the lexicographical minimum of a, b and the addition refers to pointwise multiset union, i.e., \((χ^{(1)} + χ^{(2)})(c) = χ^{(1)}(c) + χ^{(2)}(c)\).

Now we can define the following weighted formula:

\[
\alpha_{\text{one}} = \alpha_1 * \alpha_2
\]

\[
\alpha_1 = \Pi_{a \in B}(a * \{\{a\}, \emptyset\} + \neg a)
\]

\[
\alpha_2 = \Pi_{e \in C}(\Pi_{(\alpha, e, i) \in \text{pclash}(c)} \text{ovr}(\alpha, e, c, i) * \{c \mapsto \emptyset\}) + \neg \text{ovr}(\alpha, e, c, i)
\]

where B is the Herbrand base and pclash(c) = \{(\alpha, e, i) \mid (\alpha, e) \text{ is a possible clashing assumption for } c \text{ and } i\}. Intuitively, \(\alpha_1\) collects the atoms that are true in the given interpretation and \(\alpha_2\) builds the clashing assumption map, which is used to decide whether one interpretation is preferred over the other.

**Theorem 6**

\( R_{\text{one}}(\mathcal{R}) \) is a semiring and the overall weight of \( μ = ⟨PK(\mathcal{R}), \alpha_{\text{one}}, R_{\text{one}}(\mathcal{R})⟩ \) is \((I, χ)\), where I is the minimum lexicographical preferred model of \( \mathcal{R} \) and \( χ \) its clashing assumption map or \( \emptyset \) if there is no preferred model.

**Proof**

Associativity of \( \oplus \) follows from transitivity of \( \text{LocPref}(c, \text{rel}) \) and the lexicographical order. Commutativity of \( \oplus \) is clear, \( 0 \) and \( 1 \) are identities and annihilators of \( \otimes, \oplus \) by definition. Associativity of \( \otimes \) is clear.

It remains to prove that multiplication distributes over addition. So let \( A_i = (I_i, χ_i) \) for \( i = 1, 2, 3 \). Then, in the expression

\[
A_1 \otimes (A_2 \oplus A_3)
\]

Assume w.l.o.g. that \( (A_2 \oplus A_3) \) evaluates to \( A_2 \). If \( A_2 >_{\text{LocPref}(\text{rel})} A_3 \) then there exists a context \( c \) such that \( χ_2(c) >_{\text{LocPref}(c, \text{rel})} χ_3(c) \). Then it also holds that \( (χ_1 + χ_2)(c) >_{\text{LocPref}(c, \text{rel})} (χ_1 + χ_3)(c) \) and thus

\[
A_1 \otimes (A_2 \oplus A_3) = A_1 \otimes A_2 = A_1 \otimes A_2 \oplus A_1 \otimes A_3.
\]

If \( A_2 \not>_{\text{LocPref}(\text{rel})} A_3 \) this implies that \( A_2 \) is either equal to \( A_3 \) (in this case we are done) or that \( A_2 \) is smaller lexicographically. In the latter case the sum \( A_1 \otimes A_2 \) is however also lexicographically smaller than \( A_1 \otimes A_3 \) since we add \( A_3 \) both times.
Thus we have established that $R_{\text{one}}(\mathcal{R})$ is a semiring. For each answer set $I$ of $PK(\mathcal{R})$, we know that $I$ corresponds to a (least) CAS model. Thus,

$$\| \alpha_{\text{one}} \|_{R_{\text{one}}} (I) = (\mathcal{J}, \chi),$$

where $\mathcal{J} \in \{0, 1\}^B$ and $\chi$ only maps to multisets that can be interpreted as sets (i.e. each of their elements has at most multiplicity one). The lexicographical minimum CKR model $(\mathcal{J}', \chi')$ satisfies that $(\mathcal{J}, \chi) \odot (\mathcal{J}', \chi') = (\mathcal{J}', \chi')$ for all $(\mathcal{J}, \chi)$ that are the semantics of $\alpha_{\text{one}}$ w.r.t. some answer set of $PK(\mathcal{R})$. Therefore, if there exists a CKR model, the overall weight is $(\mathcal{J}', \chi')$. Otherwise, it is $0$. \qed

We continue with the $R_{\text{all}}$ semiring. Again, we need some additional lemma

**Lemma 5**

Let $\mathcal{R}$ be a single relational sCKR without eval expressions. Then a CAS model $((\mathcal{I}(c))_{c \in \mathcal{N}}, \chi)$ is a CKR model iff no CAS model $((\mathcal{I}'(c))_{c \in \mathcal{N}}, \chi')$ and $c \in \mathcal{N}$ exist such that $\chi'(c) > \chi(c)$.

Therefore, we can take the locally optimal models $\mathcal{I}(c)$ for each context $c$ and obtain the global optimal models as arbitrary combinations of locally preferred models.

In the following, we let $D$ be the Herbrand base.

Using this notation, we define the semiring $R_c = (R_c, \oplus_c, \ominus_c, e_\ominus, e_\oplus)$ that collects all locally optimal models $\mathcal{I}(c)$. Here,

$$R_c = \{ \text{opt}(B) | A \subseteq 2^D, B = \{(S, \chi) | S \in A, \chi \text{ multiset of clashing assumptions}\} \}$$

$$A \odot_c B = \text{opt}_c(A \cup B)$$

$$A \ominus_c B = \text{opt}_c(\{(S_1 \cup S_2, \chi_1 + \chi_2) | (S_1, \chi_1) \in A, (S_2, \chi_2) \in B\})$$

$$e_\ominus = \emptyset$$

$$e_\oplus = \{(\emptyset, \{\emptyset\}\}\}$$

$$\text{opt}_c(A) = \{(S, \chi) \in A | \forall (S', \chi') \in A : \neg(\chi'(c) > \chi(c))\}$$

We again have to use multisets for $\chi$ instead of sets. This is necessary because otherwise multiplication and addition do not satisfy the distributive law.

Then, we can define the measure $\mu_c = (PK(\mathcal{R}), \alpha_{\text{all}}, R_c)$, where

$$\alpha_{\text{all}} = \alpha_1 \ast \alpha_2$$

$$\alpha_1 = \Pi_{d \in D} d \ast (\{(d), \{\emptyset\}\}\} + \neg d$$

$$\alpha_2 = \Pi_{(\alpha, e, c) \in \text{pclash}(c)} \text{ovr}(\alpha, e, c) \ast (\emptyset, \{c \mapsto \{(\alpha, e)\}\}) \ast \neg \text{oov}(\alpha, e, c).$$

where $\text{pclash}(c)$ is the set of all possible clashing assumptions $\langle \alpha, e \rangle$ for $c$. We obtain

**Theorem 7**

$R_c$ is a semiring and the overall weight $\mu_c(PK(\mathcal{R}))$ is equal to the set containing for each locally optimal interpretation $\mathcal{I}(c)$ of $\mathcal{R}$ the pair $(\mathcal{I}(c), \chi_{\mathcal{I}(c)})$, where $\chi_{\mathcal{I}(c)}$ is the unique multiset containing each justified clashing assumption of $\mathcal{I}(c)$ once.

We take $R_{\text{all}}$ to be the crossproduct semiring $(R_c)_{c \in \mathcal{N}}$ defined by

$$(R_c)_{c \in \mathcal{N}} = ((R_c)_{c \in \mathcal{N}}, \oplus, \ominus, (\emptyset)_{c \in \mathcal{N}}, \{(\emptyset, \{\emptyset\}\})_{c \in \mathcal{N}}),$$

where

$$(A_c)_{c \in \mathcal{N}} \odot (B_c)_{c \in \mathcal{N}} = (A_c \odot_c B_c)_{c \in \mathcal{N}},$$

for $\odot \in \{\oplus, \ominus\}$

Using it, we can obtain the locally optimal interpretations for each context as the crossproduct of
measures $\mu^* = (\mu_c)_{c \in \mathbb{N}}$ which is a measure over the crossproduct semiring $(\mathcal{R}_c)_{c \in \mathbb{N}}$. As we have shown in Lemma 5, this gives us all the preferred models. Namely, let $\mu^*(\text{PK}(\mathcal{R})) = (A_c)_{c \in \mathbb{N}}$, then \{(I(c))_{c \in \mathbb{N}} : (I(c), \chi(c)) \in A_c\} is the set of preferred models.

Example 6
The sCKR $\mathcal{R}$ defined in Example 5 has five contexts $c_{\text{world}}$, $c_{\text{branch}1}$, $c_{\text{branch}2}$, $c_{\text{local}1}$, and $c_{\text{local}2}$. Therefore, the measure $\mu^*$ is a crossproduct of the five measures $\mu_{c_{\text{world}}}$, $\mu_{c_{\text{branch}1}}$, $\mu_{c_{\text{branch}2}}$, $\mu_{c_{\text{local}1}}$, and $\mu_{c_{\text{local}2}}$. Their overall weight is given by

$\mu_{c_{\text{world}}}(\text{PK}(\mathcal{R})) = \mu_{c_{\text{branch}1}}(\text{PK}(\mathcal{R})) = \mu_{c_{\text{branch}2}}(\text{PK}(\mathcal{R})) = \mu_{c_{\text{local}2}}(\text{PK}(\mathcal{R})) = \{(0, \{\})\}$

$\mu_{c_{\text{local}1}}(\text{PK}(\mathcal{R})) = \{(\{S(i), M(i)\}, \{\text{ovr}(S \subseteq R, i, c_{\text{branch}2}), \text{ovr}(S \subseteq E, i, c_{\text{world}})\})\}$

Accordingly, there is exactly one preferred model $(I(c))_{c \in \mathbb{N}}$, where

$I_{c_{\text{world}}} = I_{c_{\text{branch}1}} = I_{c_{\text{branch}2}} = \emptyset, I_{c_{\text{local}1}} = \{S(i), M(i)\}$

Theorem 8
Let $\mathcal{R}$ be a single-relational, eval-free sCKR, then $\mathcal{R}_{\text{all}}$ is a semiring and the overall weight of $\mu_{\text{all}} = (\text{PK}(\mathcal{R}), \alpha_{\text{all}}, \mathcal{R}_{\text{all}}(\mathcal{R}))$ is $(A_c)_{c \in \mathbb{N}}$ and the set of preferred models corresponds to \{(I(c))_{c \in \mathbb{N}} : (I(c), \chi(c)) \in A_c\} for each $c \in \mathbb{N}$.

Proof
The reasoning that $\mathcal{R}_{\text{all}}$ is a semiring is along the same lines as that for $\mathcal{R}_{\text{one}}$. The fact that the result is as desired can be clearly seen during the construction of the semiring. □