Fractional Quantum Hall States of Dipolar Gases in Chern Bands

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We study fermions and hardcore bosons in topological checkerboard lattices with long range dipolar interactions at fractional fillings. By exact diagonalization of finite two-dimensional systems, we find clear signatures of fractional quantum Hall (FQH) states at filling factors 1/3 and 1/5 for fermions (1/2 and 1/4 for bosons) in the lowest Chern band with a robust spectrum gap at moderate dipolar interaction strength. When the dipolar interaction decreases, the fermionic FQH states turn into normal states, and the bosonic 1/4-FQH state turns into a superfluid state. The bosonic 1/2-FQH state survives even in the absence of the dipolar interaction, but vanishes when the hard core becomes a soft core with a critical onsite repulsion. In the thin torus limit, the static density structure factors indicates that the FQH state turns into a commensurate charge density wave (CDW) state.

I. INTRODUCTION

In a seminal paper, Haldane proposed a time-reversal breaking honeycomb lattice model as a zero magnetic field version of the integer quantum Hall effect [1], which is a classic example of an integer Chern insulator. Chern insulators host topological order in the band structure with a topological invariant characterized by the Chern number [2]. Theoretical studies suggest that the 1/3-FQH states is the ground state at $\nu = \frac{1}{3}$ filling of the lowest Chern band with a strong nearest neighbor interaction [3–5], which has the same Hall conductance as that of the 1/3-FQH state of two-dimensional electron gas described by the Laughlin wavefunction. The generic wavefunctions for fractional Chern insulators can be constructed from Laughlin wave functions by one to one correspondence between the continuous cylindrical Landau level wave function and Wannier functions [6]. These fascinating FQH states in Chern bands have been proposed by engineering triangular optical flux lattice [7], square/honeycomb optical lattice using laser-induced transitions [8], and checkerboard lattice reduced from dipolar spin system [9] in ultra-cold atomic or molecular gas, and Floquet Chern insulator [10]. Experimentally, the Haldane honeycomb insulator has been achieved from periodically modulating system [11].

Numerical evidences for the 1/2 and 1/3-FQH states are found with large spectrum gaps induced by nearest neighbor repulsions [3–5, 12, 13]. It has been observed that the composite fermion state at $\nu = 2/3$ is destructed upon lowering the nearest-neighbor repulsion [14], and fermionic 1/3-FQH state for the honeycomb lattice model would undergo phase transitions into a Fermi liquid by tuning down the nearest-neighbor repulsion [15]. Lower filling fraction 1/4 and 1/5-FQH states become robust only when large next-nearest neighbor interaction is included [16, 17], signifying the importance of long-range interaction. The role played by long-range interaction on these FQH states is worth investigation. In two-dimensional lattice with incommensurate magnetic flux, the spectral gap of bosonic FQH states can be enhanced by the dipole-dipole interaction [16]. In the one dimensional superlattice, fractional topological states emerge as the interplay of nontrivial topological band and dipolar long-range interactions, but not for the case with only short-range interactions even in the strongly interacting limit [17]. In a dispersionless Chern band, electrons with Coulomb repulsion exhibit a hierarchy of fractional Chern insulators [18, 19]. Recently, it is pointed out that screened Coulomb interaction may drive fractional Chern insulator into stripe CDW order of fractional charge [20]. It is suggested that for bosons on the lowest Hofstadter subband with strong on-site interaction, weak off-site long-range dipolar interactions can enhance the robustness of non-Abelian fractional Chern insulator states [21, 22] with adiabatic continuity between Hofstadter and Chern insulator states [23].

Ultracold fermionic [24] and bosonic [25] dipolar gases have been realized in experiments, providing promising candidates for realizing fractional Chern insulators. In a square optical lattice, antiferromagnetic and topogical superfluid states are proposed to exist in dipolar fermi gases [26]. In this paper we focus our interest on effects of the long range dipolar interaction on the stability of Abelian FQH states in checkerboard lattice models, and find convincing evidence for Abelian 1/2, 1/3, 1/4, 1/5-FQH states of dipolar quantum gas in the lowest Chern band. In contrast, rotating quasi-2D dipolar Fermi gas with dipole-dipole interaction is predicted to exhibit 1/3-FQH state in Landau level and crystalline order at lower fillings [27]. We show that the many-body ground states at these fractional fillings are FQH states with moderate dipolar interaction. The topological properties of these states are characterized by (i) fractional quantized topological invariants related to Hall conductance, (ii) degenerate ground state manifolds under the adiabatic insertion of flux quanta, and (iii) the quasihole statistics. For weak interaction strength, we find quantum phase transitions from FQH states to other states, such as superfluid state of bosonic systems and Fermi liquid state at

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of fermionic systems. We also consider the role played by aspect ratio of two dimensional torus geometry, and find that in thin torus limit, the ground state turns into a crystal phase.

This paper is organized as follows. In Sec. II we give an overview of the checkerboard lattice model with topological invariant. In Sec. III we present numerical results of Abelian FQH states by exact diagonalization at fillings $\nu = 1/2, 1/3, 1/4, 1/5$, and discuss the properties of these ground states. Finally, in Sec. IV we summarize our results and discuss the experimental prospect of investigating nontrivial topological states in ultracold dipolar gases.

II. THE CHECKERBOARD MODEL

We consider the spinless fermions or hardcore bosons on the periodic checkerboard lattice with $N_s = N_x \times N_y$ unit cells. Each unit cell consists of two inequivalent lattice sites $A$ and $B$ separated at a distance $(a/2, a/2)$ where $a = 1$ is lattice constant. The Hamiltonian is given by

$$H_0 = \sum_{r,r'} \left[ t_{1} e^{i\phi(r,r')} a_{1}^r b_{1}^{r'} + t_{a}(r,r') a_{a}^r a_{a}^{r'} + t_{b}(r,r') b_{b}^r b_{b}^{r'} + h.c. \right] + \sum_{r} M(a_{a}^r b_{b}^r - b_{b}^r a_{a}^r),$$

where $r = (x,y)$ the lattice vector, $a_{a}$ and $b_{b}$ are annihilation operators on sublattice $A$ and $B$ respectively, $M$ is an on-site staggered potential, the nearest hopping phase is given by $\phi(r,r') = \phi \times \text{Sign}[(x' - x)(y' - y)]$, the next-nearest hopping parameters are given by $t_a(r,r') = t_2(1)^{x'-x} \text{for sublattice } A$ and $t_b(r,r') = t_2(1)^{x'-x} \text{for sublattice } B$ as in Ref. [13], and the next-nearest hopping $t_3$ is the same for both sublattices. In $k$-space, $a_k = \sum_{r} e^{ikr} a_{a}^r / \sqrt{N_x}$, $b_k = \sum_{r} e^{ikr} b_{b}^r / \sqrt{N_x}$, the Hamiltonian is given by

$$H_0 = \sum_{k} \psi_{k}^\dagger (h_{a}^k + i\xi_k \sigma_y + h_{b}^k \sigma_y + h_{c}^k \sigma_z) \psi_k,$$

where $\psi_k^\dagger = (a_{a}^k \dagger , b_{b}^k \dagger , h_{a}^k \dagger , h_{b}^k \dagger , h_{c}^k \dagger )$, $h_{a}^k = -2t_3[\cos(k_x + k_y) + \cos(k_x - k_y)]$, $h_{b}^k = M + 2t_2[\cos(k_x - k_y)]$, $h_{c}^k = 4t_1 \cos \phi \cos \frac{k_x}{2} \cos \frac{k_y}{2}$, $h_{a}^k = 4t_1 \sin \phi \sin \frac{k_x}{2} \sin \frac{k_y}{2}$. The lowest Chern band energy is given by $E_k = h_{b}^k - \xi_k$ with $\xi_k = \sqrt{(h_{a}^k)^2 + (h_{b}^k)^2 + (h_{c}^k)^2}$, and the eigenstate given by

$$\chi_k = \left( -e^{-i\phi_k} \sin \frac{\theta_k}{2} \right) \left( \cos \frac{\theta_k}{2} \right),$$

where $\theta_k = \text{arccos}(h_{c}^k / \xi_k)$, $\phi_k = \text{arctan}(h_{b}^k / h_{c}^k)$. The lowest Chern band becomes nearly flat under typical parameters such as $M = 0$, $t_2 = 0.3t_1$, $t_3 = -0.2t_1$, $\phi = \pi / 4$, with a small band width $W \simeq 0.1t_4$ and a large band gap $\Delta \simeq 2.5t_1$ separated from the upper Chern band. It is characterized by the Chern number given by $\nu = \int d^2k \nabla \times A(k) / (2\pi)$, where the Berry’s connection is given by $A(k) = i\chi_k \nabla \chi_k = (1 - \cos \theta_k) \nabla \phi_k / 2$ with singularities at $k_+ = (0, \pi)$ and $k_- = (\pi, 0)$. Using Cauchy integral, we have

$$\nu = \oint A(k) \cdot dk / 2\pi = -\frac{1}{4\pi}(\nu_+ \cos \theta_{k+} + \nu_- \cos \theta_{k-}),$$

where $\nu_{\pm} = \oint \nabla \phi_k / \Delta \pm 2\pi$. If $-4t_2 < M < 4t_2$, then $\cos \theta_{k+} = -1$, $\cos \theta_{k-} = +1$, the system is a Chern insulator with $\nu = 1$; if $|M| > 4t_2$, then $\nu = 0$, it is a trivial band insulator.

III. FRACTIONAL TOPOLOGICAL PHASES

We study the case that all the dipole moments are aligned in z-direction by a strong external field. The dipole-dipole interaction is given by $V(r - r') = d^2 / r - r'|^3$ where $d$ is the dipole moment. The dipolar interaction between nearest neighbors is the strongest given by $J = 2\sqrt{2}d^2 / (t_1 t_3^3)$. The model Hamiltonian

$$H = H_0 + \frac{1}{2} \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} V(\mathbf{r} - \mathbf{r'}) n_{\mathbf{r}} n_{\mathbf{r}}.$$

We explore the many-body ground state of $H$ by exactly diagonalizing a finite $N$-particle system at filling $\nu = N/N_s$ with the same lattice parameters $t_2 = 0.3t_1$, $t_3 = -0.2t_1$ as mentioned in Sec. III. Due to lattice translation symmetry, the total wavevector of the many-body state is conserved in the first Brillouin zone and we can classify the ground states in different total wavevector sectors. For convenience, we denote two dimensional wavevector $k = (k_x, k_y)$ in units of $2\pi / N_x a, 2\pi / N_y a$.

A. Degenerate Ground State Manifolds

As shown in Fig. I(a)-(b), for $N = 8$ dipolar bosons on lattice with $N_x = 4$ and $N_y = 4$ at $J = 0.5$, two nearly degenerate states with zero total wavevector $K_1 = K_2 = (0, 0)$ emerge, indicating the ground state degeneracy is two at half filling $\nu = 1/2$. Also for $N = 6$ bosons on lattice with $N_x = 3$, $N_y = 4$ at $J = 0.5$, we find two nearly degenerate states with total wavevector $K_1 = (0, 0)$ and $K_2 = (0, 2)$. These states are separated from the higher energy levels by a large gap $\Delta \simeq 0.4t_1$. The degeneracy of the ground states is equal to the inverse of the filling factor. For $N = 5$ bosons on a lattice with $N_x = 5$, $N_y = 4$ at $J = 0.5$, four nearly degenerate states with total wavevector $K_1 = (0, 0)$, $K_2 = (0, 1)$, $K_3 = (0, 2)$, and $K_4 = (0, 3)$ appear, protected by a smaller energy gap $\Delta \simeq 0.012t_1$ from higher-energy levels. Also for $N = 5$ bosons on lattice with $N_x = 2$, $N_y = 10$ at $J = 0.5$, four nearly degenerate states with energy gap $\Delta \simeq 0.023t_1$ and total wavevector $K_1 = K_2 = (0, 0)$ and $K_3 = K_4 = (0, 4)$ emerge.
In the fermion case, the degeneracy of the ground states is also given by the inverse of the filling factor. As shown in Fig. 1(c)-(d), for \( N = 5 \) fermions on a checkerboard lattice with \( N_x = 5, N_y = 3 \) at \( J = 0.5 \), three robust nearly degenerate states with total wavevector \( K_1 = (0, 0), K_2 = (0, 1), K_3 = (0, 2) \) emerge; for \( N = 6 \) fermions on lattice with \( N_x = 6, N_y = 3 \), three robust nearly degenerate states with total wavevector \( K_1 = K_2 = K_3 = (3, 0) \) emerge. These states are separated from the higher energy levels by a large gap \( \Delta \simeq 0.04t_1 \). For \( N = 4 \) particles on lattice with \( N_x = 4, N_y = 5 \) at \( J = 0.6 \), five robust nearly degenerate states with energy gap \( \Delta \simeq 0.002t_1 \) and total wavevector \( K_1 = (2, 0), K_2 = (2, 1), K_3 = (2, 2), K_4 = (2, 3), K_5 = (2, 4) \) emerge; for \( N = 4 \) particles on lattice with \( N_x = 2, N_y = 10 \) at \( J = 0.6 \), five robust nearly degenerate states with energy gap \( \Delta \simeq 0.007t_1 \) and total wavevector \( K_1 = (0, 0), K_2 = (0, 2), K_3 = (0, 4), K_4 = (0, 6), K_5 = (0, 8) \) emerge.

These ground states at total wavevector \( K_i = (K_{ix}, K_{iy}) \) and filling \( \nu = 1/m \) can be qualitatively understood from the generalized Pauli principle \( 23, 24 \), i.e., no more than one particle is allowed occupy within any consecutive \( m \)-orbitals with one dimensional orbital index \( \lambda = k_x + N_\lambda k_y \). For \( N_\lambda \) in multiples of \( m \), this yields \( K_{ix} = N(N_x - 1)/2 \) (mod \( N_x \)) and \( K_{iy} = [N(i - 1) + N(N_y - m)/2] \) (mod \( N_y \)). In Sec. III, we are going to show that these degenerate ground states are FQH states.

The FQH ground states are more robust under a larger dipolar interaction with a smaller ratio of ground state energy splitting to the spectrum gap. As shown in Fig. 2(a), these FQH states are rather robust for large \( J \) but may not survive for small \( J \) where the spectrum gap collapses. The only exception is \( \nu = 1/2 \) boson case due to infinite onsite repulsion. At higher fillings, for example at \( \nu = 2/3 \), for \( N = 10 \) dipolar fermions in a checkerboard lattice with \( N_x = 5, N_y = 3 \) at \( J = 0.5 \), we do not find three robust and nearly degenerate states. The cases with lower fillings will be discussed in the last section. The existence of FQH state is also affected by the band parameters, i.e., the next-nearest hopping \( t_2 \) determining the topology of band structure and the next-nearest hopping \( t_3 \) contributing crucial to the flatness of the lowest Chern band. FQH states do not always exist as ground states when tuning \( t_2 \) and \( t_3 \). We confirm that the flatter the topological band is, the more robust FQH states are.

**B. Phase transitions induced by dipolar interactions**

As mentioned above, except for \( \nu = 1/2 \) bosonic case with infinite onsite repulsion, all FQH states vanish at some dipolar interactions with collapse of the spectrum.

![FIG. 1.](image1) (Color online) Low-energy spectrum for bosonic 1/2, 1/4 and fermionic 1/3, 1/5 FQH states. (a) \( N = 8, 6 \) bosons with \( N_x = N/2, N_y = 4 \) at \( J = 0.5 \). (b) \( N = 5 \) bosons with \( N_x = 5, N_y = 4 \) and \( N = 4 \) particles with \( N_x = 2, N_y = 8 \) at \( J = 0.5 \). (c) \( N = 6, 5 \) fermions with \( N_x = N, N_y = 3 \) at \( J = 0.5 \). (d) \( N = 4 \) fermions with \( N_x = 4, N_y = 5 \) and \( N = 4 \) fermions with \( N_x = 2, N_y = 10 \) at \( J = 0.6 \). The energies are measured relative to the lowest energy for each system.

![FIG. 2.](image2) (Color online) (a) The spectrum gap \( \Delta \) versus dipolar interaction strength \( J \) for hardcore bosonic 1/2, 1/4 and fermionic 1/3, 1/5-FQH states. At fillings \( \nu = 1/3, 1/4, 1/5 \), by decreasing the dipolar interaction, degenerate ground state manifolds eventually collapse. At \( \nu = 1/2 \), a quite large gap is present even without dipolar interaction, indicating the robustness of 1/2-FQH state. (b) The superfluid density \( \rho_s \) in the ground state at fillings \( \nu = 1/2, 1/4 \) as a function of \( J \) for hardcore boson. In FQH states, \( \rho_s \) vanishes up to a precision of \( 10^{-3} \). (c) The ground state wave function fidelity susceptibility for hardcorbosonic 1/2, 1/4-FQH states \( \chi_J \) under the variation of \( J \). The dashed vertical curves indicate quantum phase transition point. (d) DMRG results of the spectral gap \( \Delta \) and ground energy splitting \( \Delta E \) for softcore bosonic system with \( N = 6, N_x = 3, N_y = 4 \) as a function of onsite repulsion \( U/t_1 \), without dipolar interaction.
As shown in Fig. 2(a), we find that in the fermion case, the critical dipolar interaction strength is \( J = 0.18 \) at filling \( \nu = 1/3 \) and \( J = 0.47 \) at filling \( \nu = 1/5 \). In the boson case, the critical dipolar interaction strength is \( J = 0.24 \) at filling \( \nu = 1/4 \). Below these critical interactions, the fermion systems may be in the normal liquid state and the boson systems are in the superfluid states.

In the boson case, we can also study the phase transition at critical interaction by examining the superfluid density. In the superfluid state, there is a symmetry breaking in the order parameter \( \langle n_\ell \rangle \propto \exp(x \phi(r)) \), where \( \phi(r) \) is the superfluid phase. When twisted boundary condition \( \psi(r + N_y a) = e^{i\theta} \psi(r) \) with very small \( \theta \), the superfluid density \( \rho_s \) can be determined from the energy relation \( E(\theta) - E(0) \approx \rho_s \theta^2 / a^2 \), where \( E(\theta) \) is the ground state energy. Fig. 2(b) shows the evolution of \( \rho_s \) in the lowest ground state as a function of \( J \). Consistent with Fig. 2(a), \( \nu = 1/2 \), the superfluid density is zero and the system is a FQH state for any value of \( J \). However, at \( \nu = 1/4 \), a small but finite \( \rho_s \) emerges when the dipolar interaction strength is reduced to \( J = 0.24 \), signalling the quantum phase transition from the FQH phase into a superfluid phase.

we also calculate the ground-state fidelity susceptibility \( \chi_J \), which measures the change of the ground state wave function \( \psi(J) \) under a small change of the interaction strength, defined by

\[
\chi_J = 2 \left( 1 - \frac{F(J, \delta J)}{(\delta J)^2} \right) \tag{6}
\]

where the fidelity \( F(J, \delta J) = |\langle \psi(J) | \psi(J + \delta J) \rangle| \) measures the overlap of the ground-state wavefunctions between \( J \) and \( J + \delta J \) with \( \delta J \to 0 \). Inside a given quantum phase, the value of \( \chi_J \) remains analytic and small. However, near a quantum phase transition point, the fidelity susceptibility diverges in the thermodynamic limit [30], which serves as a signal of quantum phase transitions. As shown in Fig. 2(c), for bosonic \( \nu = 1/2 \), there does not exist any peak in the fidelity susceptibility; for bosonic \( \nu = 1/4 \), a peak near \( J = 0.24 \) marks the phase transition between the Bose superfluid and the FQH state.

Finally, to clarify that 1/2-FQH state origins from on-site repulsion only, we consider the Hubbard model for softcore Bosons in a finite system with \( N = 6, N_x = 3, N_y = 4 \), and the model Hamiltonian is given by \( H = H_0 + \sum_{r} U n_{r}(n_{r} - 1)/2 \). In the DMRG approach, we keep up to \( m = 500 \) basis states in DMRG block, test the performance by comparing three lowest energy states \( E_1, E_2, E_3 \) with exact results in opposite limits \( U = 0 \) and \( U = \infty \), and obtain the accurate energies with energy deviations of the order \( 10^{-4} \sim 10^{-3}t_1 \) and the maximum truncation error less than \( 10^{-5} \). In Fig. 2(d), we show the evolutions of the energy spectral gap \( \Delta = E_3 - E_2 \) of two ground states \( E_1, E_2 \) separated from the third excited state \( E_3 \) and the energy splitting \( \Delta E = E_2 - E_1 \) of the gapped two ground states. For \( U/t_1 < (U/t_1)_c \approx 0.5 \), the 1/2-FQH state collapses due to the softening of \( \Delta \) and the ground states on flatband are highly degenerate when \( U = 0 \).

C. Flux Insertion

The question whether or not these ground states are FQH states can be answered by their topological invariants, i.e. the Chern numbers. With twisted boundary conditions, \( \psi(r + N_x a) = e^{i\theta} \psi(r) \), \( \psi(r + N_y a) = e^{i\theta} \psi(r) \) where \( \theta_{x,y} \) are the twisted angle, the Chern number is given by \( \nu = \int d\theta_x d\theta_y F(\theta_x, \theta_y)/2\pi \) where the Berry curvature is given by \( F(\theta_x, \theta_y) = \text{Im}(\partial_{\theta_x} \psi | \partial_{\theta_y} \psi - \langle \partial_{\theta_y} \psi | \partial_{\theta_x} \psi \rangle) \). By calculating the Berry curvatures shown in Fig. 3, we obtain the Chern numbers of two gapped ground states for \( \nu = 1/2 \) bosonic system with \( N = 6, N_x = 3, N_y = 4 \) at \( J = 0.5 \) and found \( \nu_1 \approx 0.51 \) and \( \nu_2 \approx 0.49 \); for fermionic system with \( N = 5, N_x = 5, N_y = 3 \) at \( J = 0.5 \), the Chern numbers of three gapped ground states at \( \nu = 1/3 \) filling are \( \nu_1 \approx 0.32, \nu_2 \approx 0.32, \) and \( \nu_3 \approx 0.35 \). The total Chern number of all the ground states of each system are unity, \( \sum_{i=1}^{m} \nu_i = 1 \). Similarly, we obtain \( \nu \approx 0.25 \) for the \( K = (0,0) \) ground state of bosonic system with \( N = 5, N_x = 4, N_y = 5 \) and \( \nu \approx 0.20 \) for the \( K = (2,0) \) ground state of fermionic system with \( N = 4, N_x = 4, N_y = 5 \). We find that the fractional Chern number holds true even in thin torus case, i.e. \( \nu \approx 0.24 \) for the \( K = (0,0) \) ground state of bosonic system with \( N = 5, N_x = 2, N_y = 10 \). By expanding the many body wave function in the lowest Chern band \( \psi = \sum_{\{k_l\}} \psi^{(\{k_l\})} \prod_{i=1}^{N} \chi_{k_i} \), the Chern number can be further written as

\[
\nu = \frac{i}{2\pi} \int d^2k n_{k_l} \nabla_k \times (\chi_k \nabla_k \chi_k)
\]

\[
= -\frac{1}{4\pi} \left[ n_{k_+} \cos \theta_{k_+} \nu_+ + n_{k_-} \cos \theta_{k_-} \nu_- \right] \tag{7}
\]

The momentum occupation number of the many body ground state in the lowest Chern band is given by \( n_{k_l} = \langle \psi | \delta_{k_l} | \psi \rangle = \sum_{\{k_l\}} \psi^{2(\{k_l\})} \) where \( \delta_{k_l} \) is the annihilation operator in the lowest Chern band and the summation is only over the configuration with \( k \)-state occupied. In the thermodynamic limit the orbital occupation should be uniform, \( n_{k_+} = n_{k_-} = 1/m \), consistent with the \( \nu = 1/m \) FQH state.

In addition, topological degeneracy of the ground state manifold should be \( m \)-fold for \( \nu = 1/m \) filling. By varying \( \theta_y \) from 0 to \( 2\pi \) which is equivalent to the adiabatic insertion of \( m \) flux quanta into the system [30-31], these quasi-degenerate ground states evolve into each other without mixing with exited levels during the spectral flow. As shown in Fig. 3 the ground states of our system clearly show the robustness of topological degeneracy. Adiabatically inserting \( m \) flux quanta changes the Berry phase associated with braiding a quasiparticle around the circle by \( \Delta \theta = 2\pi mq \) [32], where \( q \) is the charge of the quasiparticle. Since the final state is in the same topological sector with the initial state, \( \Delta \theta \) must be an integer multiple of \( 2\pi \), thus the smallest charge of
the quasiparticle should be \( q = 1/m \). One of the hallmarks of the FQH state is the existence of quasihole excitations which carry fractional charge obeying the fractional statistics. Removing one particle or adding \( m \)-flux quanta would create \( m \) quasiholes, each with charge \( 1/m \).

From Ref. [28], the number of quasihole states of \( N \) particles in \( N_x \) orbitals reads as

\[
N_q = \frac{N_x}{N} \left( N_x - \frac{(m-1)N-1}{N} \right)
\]

(8)

As shown in Fig. 3, we compute the spectrum of quasihole states which lies in a low-energy manifold (quasihole states) separated by a gap from higher states, and find their number matches that of the quasihole states in Eq. (8).

D. Density Structure Factor

Another important issue is whether or not there is a commensurate CDW state competing with FQH states [33]. The \( m \)-fold degeneracy of ground states alone does not resolve this issue, due to possible degenerate CDW states. One distinctive feature of the CDW state is the Bragg peak in the static density structure factor defined by

\[
S(\mathbf{q}) = \frac{1}{2N_e} \sum_{\mathbf{r},\mathbf{r}'} e^{i \mathbf{q} \cdot (\mathbf{r}-\mathbf{r}')} \left( \langle n_{\mathbf{r}} n_{\mathbf{r}'} \rangle - \langle n_{\mathbf{r}} \rangle \langle n_{\mathbf{r}'} \rangle \delta_{\mathbf{q},0} \right). 
\]

(9)

As shown in Fig. 4 in our numerical results of finite two-dimensional systems, there are no particular Bragg peaks in \( S(\mathbf{q}) \) at any finite wavevector. We also calculate the local densities of these degenerate states and they are almost uniform for both sublattice, \( \langle \hat{n}_{a\mathbf{r}}^a \rangle \approx \langle \hat{n}_{b\mathbf{r}}^b \rangle \approx \nu/2 = N/2N_x \) with error less than two percent. The intrasublattice structure factor \( S^{aa}(\mathbf{q}) = \sum_{\mathbf{r},\mathbf{r}'} e^{i \mathbf{q} \cdot (\mathbf{r}-\mathbf{r}')} \left( \langle n_{\mathbf{r}}^a n_{\mathbf{r}'}^a \rangle - \langle n_{\mathbf{r}}^a \rangle \langle n_{\mathbf{r}'}^a \rangle \delta_{\mathbf{q},0} \right) / N_x \) and the intersublattice structure factor \( S^{ab}(\mathbf{q}) = \sum_{\mathbf{r},\mathbf{r}'} e^{i \mathbf{q} \cdot (\mathbf{r}-\mathbf{r}')} \left( \langle n_{\mathbf{r}}^a n_{\mathbf{r}'}^b \rangle - \langle n_{\mathbf{r}}^a \rangle \langle n_{\mathbf{r}'}^b \rangle \delta_{\mathbf{q},0} \right) / N_x \) do not exhibit any peak as well. Thus we can rule out the possibility of CDW states as the competing ground state.

Nevertheless, in the one-dimensional thin torus limit, the ground states at rational filling fraction \( \nu = 1/q \) are not only \( q \)-fold degenerate, but also display peaks.

FIG. 3. (Color online) Berry curvatures \( F(\theta_x, \theta_y)/2\pi \) in: (a) the \( K = (0,0) \) ground state of bosonic system with \( N = 6, N_x = 3, N_y = 4 \); (b) the \( K = (0,0) \) ground state of bosonic system with \( N = 5, N_x = 4, N_y = 5 \); (c) the \( K = (0,0) \) ground state of fermionic system with \( N = 5, N_x = 5, N_y = 3 \); (d) the \( K = (2,0) \) ground state fermionic system with \( N = 4, N_x = 4, N_y = 5 \).

FIG. 4. (Color online) The spectral flow under flux insertion along the \( y \)-direction, which is equivalent to the adiabatic insertion of flux quanta, for (a) \( N = 6 \) bosons in a checkerboard lattice with \( N_x = 3, N_y = 4 \) at \( J = 0.5 \), (b) \( N = 5 \) bosons with \( N_x = 5, N_y = 4 \) at \( J = 0.5 \), (c) \( N = 5 \) fermions with \( N_x = 5, N_y = 3 \) at \( J = 0.5 \), (d) \( N = 4 \) fermions with \( N_x = 4, N_y = 5 \) at \( J = 0.6 \).

FIG. 5. (Color online) The low-energy quasihole spectrum. Only ten lowest energies per momentum sector are displayed. (a) \( N = 5 \) bosons on lattice with \( N_x = 3, N_y = 4 \) at \( J = 0.5 \). The number of states below the red dashed line is 3 per momentum sector. (b) \( N = 4 \) fermions on lattice with \( N_x = 5, N_y = 3 \) at \( J = 0.5 \). The number of states below the red dashed line is 5 per momentum sector.
FIG. 6. (Color online) The static density structure factors $S(q)$ in: (a) $\nu = 1/2$ bosonic states at $J = 0.5$; (b) $\nu = 1/4$ bosonic states at $J = 0.5$; (c) $\nu = 1/3$ fermionic states at $J = 0.5$ ($J = 200$ for $N = 6, N_x = 2, N_y = 9$); (d) $\nu = 1/5$ fermionic states at $J = 0.6$. The structure factors are almost the same for the ground states in the same degenerate manifold. The Bragg peaks in structure factors are indications of CDW states.

In $S(q)$. We find that: (i) For $N = 4, 5$ bosons on lattice with $N_x = 2, N_y = 2N$ at $J = 0.5$ exhibiting one Bragg peak at $q = (0, N)$; (ii) for $N = 4$ fermions on lattice with $N_x = 2, N_y = 10$ at $J = 0.6$ exhibiting two Bragg peaks at $q_1 = (0, 4)$ and $q_2 = (0, 6)$; (iii) For strongly interacting $N = 6$ fermions on lattice with $N_x = 2, N_y = 9$ at $J = 200$ exhibiting three Bragg peaks at $q_1 = (0, 4)$, $q_2 = (0, 6)$ and $q_3 = (1, 0)$, where the $q_3 = (1, 0)$ peak comes from $S^{\text{off}}(q)$ and the other two come from both $S^{\text{on}}(q)$ and $S^{\text{off}}(q)$, implying the one dimensional crystalline order with the small aspect ratio $N_x/N_y$ and the strong interaction strength.

IV. SUMMARY AND DISCUSSIONS

In summary, we show that dipolar quantum gas could host FQH states at a partial filling $\nu = 1/m$ in the lowest Chern band, characterized by $m$-fold quasidegenerate ground states, with very small energy splitting owing to the finite-size effects. Numerically we have found convincing evidences of the fermionic $1/3, 1/5$ and the bosonic $1/2, 1/4$-FQH states, protected by a robust energy gap from higher excited states. The spectrum gap can be enhanced by increasing dipolar interaction strength, such that FQH states may be realizable by trapping dipolar gas in optical lattices with Chern bands.

We also find evidence for FQH states with lower fillings. Numerical studies on lattice with the same parameters and $J = 2$, indicate that: (i) six quasidegenerate gapped states with total wavevector $K_i = (0, i), i = 0 - 5$ separated from the higher energy levels by a gap $\Delta \simeq 0.0074t_1$, emerge for $N = 5$ dipolar bosons on lattice with $N_x = 5, N_y = 6$; (ii) seven quasidegenerate gapped states with total wavevector $K_i = (0, i), i = 0 - 6$ separated from the higher energy levels by a gap $\Delta \simeq 0.0055t_1$, emerge for $N = 5$ dipolar fermions on lattice with $N_x = 5, N_y = 7$. These results indicates that the checkerboard model should host robust bosonic $1/6$ and fermionic $1/7$-FQH states for stronger dipolar interaction strengths. With three body interactions polar molecules in this type of optical lattice may be in a fermionic non-Abelian FQH states similar to Moore-Read phases at $\nu = \frac{1}{2}$ filling.

In realistic situations, inhomogeneities, such as impurities and disorder, are expected. The total wavevector is no longer a good quantum number. Weak disordered onsite potential could cause level anti-crossing among these ground states under the adiabatic insertion of flux quanta. However, the spectral gap between these nearly degenerate states and the first excited state should remain open.

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