Critical connectedness of thin arithmetical discrete planes

Timo Jolivet
Université Paris Diderot, France
University of Turku, Finland

Joint work with Valérie Berthé, Damien Jamet, Xavier Provençal
(Powered by ANR KIDICO)

DGCI 2013
El 22 de marzo de 2013
Universidad de Sevilla
Discrete planes $\mathcal{P}_{v,\omega}$

Plane: $x \in \mathbb{R}^3$ such that $\langle x, v \rangle = 0$

Discrete plane: $x \in \mathbb{Z}^3$ such that $\langle x, v \rangle \in [0, \omega[$

$v =$ normal vector $\quad \omega =$ thickness
Discrete planes $\mathcal{P}_{v,\omega}$

**Plane:** $x \in \mathbb{R}^3$ such that $\langle x, v \rangle = 0$

**Discrete plane:** $x \in \mathbb{Z}^3$ such that $\langle x, v \rangle \in [0, \omega[$

$v = (1, \sqrt{2}, \pi) \quad \omega = 4$
Discrete planes $\mathcal{P}_{v,\omega}$

**Plane:** $x \in \mathbb{R}^3$ such that $\langle x, v \rangle = 0$

**Discrete plane:** $x \in \mathbb{Z}^3$ such that $\langle x, v \rangle \in [0, \omega]$.

$v = (1, \sqrt{2}, \pi) \quad \omega = 0.2$
Discrete planes $\mathcal{P}_{v,\omega}$

**Plane:** $x \in \mathbb{R}^3$ such that $\langle x, v \rangle = 0$

**Discrete plane:** $x \in \mathbb{Z}^3$ such that $\langle x, v \rangle \in [0, \omega[$

$v = (1, \sqrt{2}, \pi)$ $\quad \omega = 0.5$
Discrete planes $\mathcal{P}_{v,\omega}$

**Plane:** $x \in \mathbb{R}^3$ such that $\langle x, v \rangle = 0$

**Discrete plane:** $x \in \mathbb{Z}^3$ such that $\langle x, v \rangle \in [0, \omega[$

$v = (1, \sqrt{2}, \pi) \quad \omega = 1$
Discrete planes $\mathcal{P}_{v,\omega}$

**Plane:** $x \in \mathbb{R}^3$ such that $\langle x, v \rangle = 0$

**Discrete plane:** $x \in \mathbb{Z}^3$ such that $\langle x, v \rangle \in [0, \omega[$

$v = (1, \sqrt{2}, \pi) \quad \omega = 1.5$
Discrete planes $\mathcal{P}_{v,\omega}$

**Plane:** $x \in \mathbb{R}^3$ such that $\langle x, v \rangle = 0$

**Discrete plane:** $x \in \mathbb{Z}^3$ such that $\langle x, v \rangle \in [0, \omega[$

$v = (1, \sqrt{2}, \pi) \quad \omega = 2.5$
Discrete planes $\mathcal{P}_{\mathbf{v}, \omega}$

Plane: $\mathbf{x} \in \mathbb{R}^3$ such that $\langle \mathbf{x}, \mathbf{v} \rangle = 0$

Discrete plane: $\mathbf{x} \in \mathbb{Z}^3$ such that $\langle \mathbf{x}, \mathbf{v} \rangle \in [0, \omega[$

$\mathbf{v} = (1, \sqrt{2}, \pi)$  \hspace{1cm} $\omega = 4$
Discrete planes $\mathcal{P}_{v, \omega}$

**Plane:** $x \in \mathbb{R}^3$ such that $\langle x, v \rangle = 0$

**Discrete plane:** $x \in \mathbb{Z}^3$ such that $\langle x, v \rangle \in [0, \omega[$

$v = (1, \sqrt{2}, \pi) \quad \omega = 6$
Discrete planes $\mathcal{P}_{v,\omega}$

Plane: $x \in \mathbb{R}^3$ such that $\langle x, v \rangle = 0$

Discrete plane: $x \in \mathbb{Z}^3$ such that $\langle x, v \rangle \in [0, \omega[$

$v = (1, \sqrt{2}, \pi) \quad \omega = 10$
Critical thickness
Critical thickness

\[ \omega = \max(v) \quad \text{(no 2D hole, “naive” plane)} \]
Critical thickness

$$\omega = \max(v) + \max_2(v) \quad \text{(no 1D hole)}$$
Critical thickness

\[ \omega = \mathbf{v}_0 + \mathbf{v}_1 + \mathbf{v}_2 \]  
(no 0D hole, “standard” plane)
## Critical thickness

| “Holes” critical behaviour | [Andres-Acharya-Sibata 97] |
|----------------------------|---------------------------|


Critical thickness

Today: we look at $k$-connectedness

0-connected:

1-connected:

2-connected:
Critical thickness

\[ \omega = \text{too small for 0-connectedness} \]
Critical thickness

\[ \omega = \text{too small for 1-connectedness} \]
Critical thickness

\[ \omega = \text{too small for 2-connectedness} \]
Critical thickness

\[ \omega = \text{enough for 2-connectedness} \]
Today: \( \omega = \Omega(v) := \inf \{ \omega > 0 \text{ such that } P_{v,\omega} \text{ is 2-connected} \} \)
Critical thickness

Today: \( \omega = \Omega(v) := \inf\{\omega > 0 \text{ such that } \mathcal{P}_{v,\omega} \text{ is 2-connected}\} \)

\( \max(v) \leq \Omega(v) \leq v_0 + v_1 + v_2 \)
Critical thickness

Today: \( \omega = \Omega(v) := \inf\{ \omega > 0 \text{ such that } \mathcal{P}_{v, \omega} \text{ is 2-connected} \} \)

- \( \max(v) \leq \Omega(v) \leq v_0 + v_1 + v_2 \)
- Obviously, \( \forall \varepsilon > 0, \mathcal{P}_{v, \Omega(v) - \varepsilon} \) is not 2-connected
  \( \mathcal{P}_{v, \Omega(v) + \varepsilon} \) is 2-connected
Critical thickness

Today: \( \omega = \Omega(v) := \inf\{\omega > 0 \text{ such that } \mathcal{P}_{v,\omega} \text{ is 2-connected}\} \)

- \( \max(v) \leq \Omega(v) \leq v_0 + v_1 + v_2 \)
- Obviously, \( \forall \varepsilon > 0, \mathcal{P}_{v,\Omega(v)-\varepsilon} \) is not 2-connected
- \( \mathcal{P}_{v,\Omega(v)+\varepsilon} \) is 2-connected

- Is \( \mathcal{P}_{v,\Omega(v)} \) 2-connected?
Critical thickness

Today: \( \omega = \Omega(v) := \inf\{\omega > 0 \text{ such that } \mathcal{P}_{v,\omega} \text{ is 2-connected}\} \)

- \( \max(v) \leq \Omega(v) \leq v_0 + v_1 + v_2 \)
- Obviously, \( \forall \varepsilon > 0, \mathcal{P}_{v,\Omega(v)-\varepsilon} \) is not 2-connected
  \( \mathcal{P}_{v,\Omega(v)+\varepsilon} \) is 2-connected
- Is \( \mathcal{P}_{v,\Omega(v)} \) 2-connected?

Theorem [Berthé-Jamet-J-Provençal]

Yes and no.
Critical thickness

**Today:** \( \omega = \Omega(v) := \inf\{\omega > 0 \text{ such that } P_{v,\omega} \text{ is } 2\text{-connected}\} \)

- \( \max(v) \leq \Omega(v) \leq v_0 + v_1 + v_2 \)
- Obviously, \( \forall \varepsilon > 0, \ P_{v,\Omega(v)-\varepsilon} \text{ is not } 2\text{-connected} \)
- \( P_{v,\Omega(v)+\varepsilon} \text{ is } 2\text{-connected} \)
- **Is** \( P_{v,\Omega(v)} \) **2-connected?**

**Theorem** [Berthé-Jamet-J-Provençal]

Yes and no.

We will be more specific.
Computing $\Omega(v)$

- We assume $v_0 \leq v_1 \leq v_2$
- Fully subtractive algo: $FS(v) = \text{sort}(v_0, v_1 - v_0, v_2 - v_0)$
Computing $\Omega(\mathbf{v})$

- We assume $\mathbf{v}_0 \leq \mathbf{v}_1 \leq \mathbf{v}_2$
- **Fully subtractive algo:** $FS(\mathbf{v}) = \text{sort}(\mathbf{v}_0, \mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0)$
- **Prop:** $\Omega(\mathbf{v}) = \Omega(FS(\mathbf{v})) + \mathbf{v}_0$
Computing $\Omega(v)$

- We assume $v_0 \leq v_1 \leq v_2$
- **Fully subtractive algo:** $FS(v) = \text{sort}(v_0, v_1 - v_0, v_2 - v_0)$
- **Prop:** $\Omega(v) = \Omega(FS(v)) + v_0$

**Algorithm to compute $\Omega(v)$ [Domenjoud-Jamet-Toutant]**

```python
def critical_thickness(v):
    if v[0]+v[1] <= v[2]:
        return max(v)
    else:
        return v[0] + critical_thickness(FS(v))
```

(Here we assume $v[0]$ is never 0 (i.e. $v_0, v_1, v_2$ are lin. ind. over $\mathbb{Q}$), but the algorithm can be modified to handle this.)

What if we never have $v[0]+v[1] \leq v[2]$?!?
Computing $\Omega(v)$

- We assume $v_0 \leq v_1 \leq v_2$
- **Fully subtractive algo**: $FS(v) = \text{sort}(v_0, v_1 - v_0, v_2 - v_0)$
- **Prop**: $\Omega(v) = \Omega(FS(v)) + v_0$

**Algorithm to compute $\Omega(v)$ [Domenjoud-Jamet-Toutant]**

```python
def critical_thickness(v):
    if v[0] + v[1] <= v[2]:
        return max(v)
    else:
        return v[0] + critical_thickness(FS(v))
```

- (Here we assume $v[0]$ is never 0 (i.e. $v_0, v_1, v_2$ are lin. ind. over $\mathbb{Q}$), but the algorithm can be modified to handle this.)
Computing $\Omega(v)$

- We assume $v_0 \leq v_1 \leq v_2$
- **Fully subtractive algo:** $FS(v) = \text{sort}(v_0, v_1 - v_0, v_2 - v_0)$
- **Prop:** $\Omega(v) = \Omega(FS(v)) + v_0$

Algorithm to compute $\Omega(v)$ [Domenjoud-Jamet-Toutant]

```python
def critical_thickness(v):
    if v[0]+v[1] <= v[2]:
        return max(v)
    else:
        return v[0] + critical_thickness(FS(v))
```

- (Here we assume $v[0]$ is never 0 (i.e. $v_0, v_1, v_2$ are lin. ind. over $\mathbb{Q}$), but the algorithm can be modified to handle this.)
- What if we **never** have $v[0]+v[1] \leq v[2]$ ?!
Computing $\Omega(v)$

Example. $v = (1, \sqrt{13}, \sqrt{17})$
Computing $\Omega(v)$

Example. $v = (1, \sqrt{13}, \sqrt{17})$

- $v^{(1)} = (1, \sqrt{13} - 1, \sqrt{17} - 1)$
Computing $\Omega(v)$

Example. $v = (1, \sqrt{13}, \sqrt{17})$

- $v^{(1)} = (1, \sqrt{13} - 1, \sqrt{17} - 1)$
- $v^{(2)} = (1, \sqrt{13} - 2, \sqrt{17} - 2)$
Computing $\Omega(v)$

Example. $v = (1, \sqrt{13}, \sqrt{17})$

- $v^{(1)} = (1, \sqrt{13} - 1, \sqrt{17} - 1)$
- $v^{(2)} = (1, \sqrt{13} - 2, \sqrt{17} - 2)$
- $v^{(3)} = (\sqrt{13} - 3, 1, \sqrt{17} - 3)$
Computing $\Omega(v)$

Example. $v = (1, \sqrt{13}, \sqrt{17})$

- $v^{(1)} = (1, \sqrt{13} - 1, \sqrt{17} - 1)$
- $v^{(2)} = (1, \sqrt{13} - 2, \sqrt{17} - 2)$
- $v^{(3)} = (\sqrt{13} - 3, 1, \sqrt{17} - 3)$
- $v^{(4)} = (-\sqrt{13} + 4, -\sqrt{13} + \sqrt{17}, \sqrt{13} - 3)$
Computing $\Omega(v)$

Example. $v = (1, \sqrt{13}, \sqrt{17})$

- $v^{(1)} = (1, \sqrt{13} - 1, \sqrt{17} - 1)$
- $v^{(2)} = (1, \sqrt{13} - 2, \sqrt{17} - 2)$
- $v^{(3)} = (\sqrt{13} - 3, 1, \sqrt{17} - 3)$
- $v^{(4)} = (-\sqrt{13} + 4, -\sqrt{13} + \sqrt{17}, \sqrt{13} - 3)$
- $v^{(5)} = (\sqrt{17} - 4, 2\sqrt{13} - 7, -\sqrt{13} + 4)$: STOP
Computing $\Omega(v)$

Example. $v = (1, \sqrt{13}, \sqrt{17})$

$\triangleright v^{(1)} = (1, \sqrt{13} - 1, \sqrt{17} - 1)$

$\triangleright v^{(2)} = (1, \sqrt{13} - 2, \sqrt{17} - 2)$

$\triangleright v^{(3)} = (\sqrt{13} - 3, 1, \sqrt{17} - 3)$

$\triangleright v^{(4)} = (-\sqrt{13} + 4, -\sqrt{13} + \sqrt{17}, \sqrt{13} - 3)$

$\triangleright v^{(5)} = (\sqrt{17} - 4, 2\sqrt{13} - 7, -\sqrt{13} + 4) : \text{STOP}$

So $\omega = -\sqrt{13} + 8$
Computing $\Omega(v)$

Example. $v = (1, \sqrt[3]{10}, \pi)$
Computing $\Omega(v)$

Example. $v = (1, \sqrt[3]{10}, \pi)$

1. $(1, \sqrt[3]{10} - 1, \pi - 1)$
2. $(\sqrt[3]{10} - 2, 1, \pi - 2)$
3. $(\sqrt[3]{10} - 2, -\sqrt[3]{10} + 3, \pi - \sqrt[3]{10})$
4. $(\sqrt[3]{10} - 2, -2\sqrt[3]{10} + 5, \pi - 2\sqrt[3]{10} + 2)$
5. $(\sqrt[3]{10} - 2, -3\sqrt[3]{10} + 7, \pi - 3\sqrt[3]{10} + 4)$
6. $(\sqrt[3]{10} - 2, -4\sqrt[3]{10} + 9, \pi - 4\sqrt[3]{10} + 6)$
7. $(\sqrt[3]{10} - 2, -5\sqrt[3]{10} + 11, \pi - 5\sqrt[3]{10} + 8)$
8. $(-6\sqrt[3]{10} + 13, \sqrt[3]{10} - 2, \pi - 6\sqrt[3]{10} + 10)$
9. $(-6\sqrt[3]{10} + 13, 7\sqrt[3]{10} - 15, \pi - 3)$
10. $(13\sqrt[3]{10} - 28, \pi + 6\sqrt[3]{10} - 16, -6\sqrt[3]{10} + 13)$
11. $(13\sqrt[3]{10} - 28, \pi - 7\sqrt[3]{10} + 12, -19\sqrt[3]{10} + 41)$
12. $(13\sqrt[3]{10} - 28, \pi - 20\sqrt[3]{10} + 40, -32\sqrt[3]{10} + 69)$
13. $(13\sqrt[3]{10} - 28, \pi - 33\sqrt[3]{10} + 68, -45\sqrt[3]{10} + 97)$
14. $(13\sqrt[3]{10} - 28, \pi - 46\sqrt[3]{10} + 96, -58\sqrt[3]{10} + 125)$
15. $(13\sqrt[3]{10} - 28, \pi - 59\sqrt[3]{10} + 124, -71\sqrt[3]{10} + 153)$
16. $(13\sqrt[3]{10} - 28, \pi - 72\sqrt[3]{10} + 152, -84\sqrt[3]{10} + 181)$
17. $(13\sqrt[3]{10} - 28, \pi - 85\sqrt[3]{10} + 180, -97\sqrt[3]{10} + 209)$
18. $(\pi - 98\sqrt[3]{10} + 208, 13\sqrt[3]{10} - 28, -110\sqrt[3]{10} + 237)$
19. $(-\pi + 111\sqrt[3]{10} - 236, -\pi - 12\sqrt[3]{10} + 29, \pi - 98\sqrt[3]{10} + 208) : \text{STOP}$

So $\omega = 2\pi - 98\sqrt[3]{10} + 208$
Computing $\Omega(v)$: an infinite loop

- If $FS(v) = (v_1 - v_0, v_2 - v_0, v_0)$
  then $FS(v) = Mv$ where $M = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
Computing $\Omega(v)$: an infinite loop

- If $FS(v) = (v_1 - v_0, v_2 - v_0, v_0)$
  then $FS(v) = Mv$ where $M = \begin{pmatrix}
-1 & 1 & 0 \\
-1 & 0 & 1 \\
1 & 0 & 0 
\end{pmatrix}$

- Let $v$ such that $Mv = \alpha v$
  $v = (1, \alpha + 1, \alpha^2 + \alpha + 1) = (1, 1.54\ldots, 1.84\ldots)$
  where $\alpha = 0.54\ldots$ is the real root of $x^3 + x^2 + x - 1$
Computing $\Omega(v)$: an infinite loop

- If $FS(v) = (v_1 - v_0, v_2 - v_0, v_0)$
  then $FS(v) = Mv$ where $M = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

- Let $v$ such that $Mv = \alpha v$
  $v = (1, \alpha + 1, \alpha^2 + \alpha + 1) = (1, 1.54\ldots, 1.84\ldots)$
  where $\alpha = 0.54\ldots$ is the real root of $x^3 + x^2 + x - 1$

- So: $v_0^{(n)} + v_1^{(n)} > v_2^{(n)}$ for all $n \geq 0$: loop!
Computing $\Omega(v)$

$$\mathcal{F}_3 := \{v \text{ such that } v_0(n) + v_1(n) > v_2(n) \text{ for all } n \geq 0\}$$
$$= \{v \text{ such that the algorithm loops forever}\}$$

Uncountable set of Lebesgue measure zero
Computing $\Omega(v)$

$\mathcal{F}_3 := \{ v \text{ such that } v_0^{(n)} + v_1^{(n)} > v_2^{(n)} \text{ for all } n \geq 0 \}$

$= \{ v \text{ such that the algorithm loops forever} \}$

Uncountable set of Lebesgue measure zero

**Algorithm to compute $\Omega(v)$, continued**

- If $v \notin \mathcal{F}_3$: run the algorithm until it halts
- If $v \in \mathcal{F}_3$: we can prove that $\Omega(v) = \frac{v_0 + v_1 + v_2}{2}$
Computing $\Omega(v)$

\[ F_3 := \{ v \text{ such that } v_0^{(n)} + v_1^{(n)} > v_2^{(n)} \text{ for all } n \geq 0 \} = \{ v \text{ such that the algorithm loops forever} \} \]

Uncountable set of Lebesgue measure zero

**Algorithm to compute $\Omega(v)$, continued**

- If $v \notin F_3$: run the algorithm until it halts
- If $v \in F_3$: we can prove that $\Omega(v) = \frac{v_0 + v_1 + v_2}{2}$

**BUT:** in general, given $v$, how to ensure that $v \in F_3$?
### Computing $\Omega(v)$

$$\mathcal{F}_3 := \{v \text{ such that } v_0^{(n)} + v_1^{(n)} > v_2^{(n)} \text{ for all } n \geq 0\}$$

$$= \{v \text{ such that the algorithm loops forever}\}$$

Uncountable set of Lebesgue measure zero

### Algorithm to compute $\Omega(v)$, continued

- If $v \notin \mathcal{F}_3$: run the algorithm until it halts
- If $v \in \mathcal{F}_3$: we can prove that $\Omega(v) = \frac{v_0 + v_1 + v_2}{2}$

**BUT:** in general, given $v$, how to ensure that $v \in \mathcal{F}_3$?  
(This is a nice open problem.)
Computing $\Omega(v)$

$$\mathcal{F}_3 := \{ v \text{ such that } v_0^{(n)} + v_1^{(n)} > v_2^{(n)} \text{ for all } n \geq 0 \}$$
$$= \{ v \text{ such that the algorithm loops forever} \}$$

Uncountable set of Lebesgue measure zero

© Pierre Arnoux and Štěpán Starosta
Computing $\Omega(v)$

$F_3 := \{ v \text{ such that } v_0^{(n)} + v_1^{(n)} > v_2^{(n)} \text{ for all } n \geq 0 \}$

$= \{ v \text{ such that the algorithm loops forever} \}$

Uncountable set of Lebesgue measure zero

Nice, unexpected links

Natural disc.
geom. problem
$\Omega(v)$

Fully sub. algorithm
FS

Strange fractal set
$F_3$

©Pierre Arnoux and Štěpán Starosta
Main result

**Theorem** [Berthé-Jamet-J-Provençal]

1. \( v \notin \mathcal{F}_3 \implies \mathcal{P}_{v,\Omega(v)} \) is not 2-connected (left picture)
2. \( v \in \mathcal{F}_3 \implies \mathcal{P}_{v,\Omega(v)} \) is 2-connected (right picture)
$v \in \mathcal{F}_3 \implies \mathcal{P}_{v,\Omega(v)}$ is 2-connected

We construct a new set $T_v$ such that

1. $T_v$ is 2-connected
2. $\mathcal{P}_{v,\text{max}(v)} \subseteq T_v \subseteq \mathcal{P}_{v,\Omega(v)}$
$v \in \mathcal{F}_3 \implies \mathcal{P}_{v, \Omega(v)}$ is 2-connected

We construct a new set $T_v$ such that

1. $T_v$ is 2-connected
2. $\mathcal{P}_{v, \max(v)} \subseteq T_v \subseteq \mathcal{P}_{v, \Omega(v)}$

So:

$\mathcal{P}_{v, \Omega(v)} \ni x_1 \quad x_2 \in \mathcal{P}_{v, \Omega(v)}$
$v \in \mathcal{F}_3 \implies \mathcal{P}_{v,\Omega(v)}$ is 2-connected

We construct a new set $T_v$ such that

1. $T_v$ is 2-connected
2. $\mathcal{P}_{v,\text{max}(v)} \subseteq T_v \subseteq \mathcal{P}_{v,\Omega(v)}$

So:

\[
\begin{align*}
\mathcal{P}_{v,\Omega(v)} & \ni x_1 \\
\mathcal{P}_{v,\text{max}(v)} & \ni y_1 \\
\mathcal{P}_{v,\Omega(v)} & \ni x_2 \\
\mathcal{P}_{v,\text{max}(v)} & \ni y_2
\end{align*}
\]

2-adjacent
\( v \in F_3 \implies \mathcal{P}_{v, \Omega(v)} \text{ is 2-connected} \)

We construct a new set \( T_v \) such that

1. \( T_v \) is 2-connected
2. \( \mathcal{P}_{v, \max(v)} \subseteq T_v \subseteq \mathcal{P}_{v, \Omega(v)} \)

So:

\[
\begin{align*}
\mathcal{P}_{v, \Omega(v)} & \ni x_1 \\
\mathcal{P}_{v, \max(v)} & \ni y_1 \\
\mathcal{P}_{v, \max(v)} & \ni y_2 \\
\mathcal{P}_{v, \Omega(v)} & \ni x_2
\end{align*}
\]

2-adjacent

thanks to (1), (2)
$v \in \mathcal{F}_3 \implies \mathcal{P}_{v,\Omega(v)}$ is 2-connected

We construct a new set $T_v$ such that

1. $T_v$ is 2-connected
2. $\mathcal{P}_{v,\text{max}(v)} \subseteq T_v \subseteq \mathcal{P}_{v,\Omega(v)}$

So:

$\mathcal{P}_{v,\Omega(v)} \ni x_1 \quad \text{2-adjacent} \quad \mathcal{P}_{v,\text{max}(v)} \ni y_1 \quad \text{2-adjacent} \quad y_2 \in \mathcal{P}_{v,\text{max}(v)}$

$\mathcal{P}_{v,\Omega(v)} \ni x_2 \quad \text{2-adjacent} \quad \mathcal{P}_{v,\text{max}(v)} \ni y_2$

2-connected thanks to (1), (2)

- Proof of (1):
- Proof of $T_v \subseteq \mathcal{P}_{v,\Omega(v)}$:
- Proof of $\mathcal{P}_{v,\text{max}(v)} \subseteq T_v$:
\[ v \in \mathcal{F}_3 \implies \mathcal{P}_{v,\Omega(v)} \text{ is 2-connected} \]

We construct a new set \( T_v \) such that

1. \( T_v \) is 2-connected
2. \( \mathcal{P}_{v,\text{max}(v)} \subseteq T_v \subseteq \mathcal{P}_{v,\Omega(v)} \)

So:

\[ \mathcal{P}_{v,\Omega(v)} \ni x_1 \quad \text{2-adjacent} \quad \mathcal{P}_{v,\text{max}(v)} \ni y_1 \quad \text{2-connected} \quad y_2 \in \mathcal{P}_{v,\text{max}(v)} \]

Proof of (1): clever arithmetics

Proof of \( T_v \subseteq \mathcal{P}_{v,\Omega(v)} \): arithmetics

Proof of \( \mathcal{P}_{v,\text{max}(v)} \subseteq T_v \):
$v \in \mathcal{F}_3 \implies \mathcal{P}_{v,\Omega(v)}$ is 2-connected

We construct a new set $\mathbf{T}_v = \bigcup_{n \in \mathbb{N}} T_n$ such that

1. $\mathbf{T}_v$ is 2-connected
2. $\mathcal{P}_{v, \max(v)} \subseteq \mathbf{T}_v \subseteq \mathcal{P}_{v,\Omega(v)}$

So:

$\mathcal{P}_{v,\Omega(v)} \ni \exists x_1$

2-adjacent

$\mathcal{P}_{v,\max(v)} \ni \exists y_1 \quad 2$-adjacent

$\mathcal{P}_{v,\max(v)} \ni \exists y_2 \in \mathcal{P}_{v,\max(v)}$

2-connected

thanks to (1), (2)

➤ **Proof of (1):** clever arithmetics

➤ **Proof of** $\mathbf{T}_v \subseteq \mathcal{P}_{v,\Omega(v)}$: arithmetics

➤ **Proof of** $\mathcal{P}_{v,\max(v)} \subseteq \mathbf{T}_v$:
\( \mathbf{v} \in \mathcal{F}_3 \implies \mathcal{P}_{v,\Omega(v)} \text{ is 2-connected} \)

We construct a new set \( T_v = \bigcup_{n \in \mathbb{N}} T_n \) such that

1. \( T_v \) is 2-connected
2. \( \mathcal{P}_{v,\text{max}(v)} \subseteq T_v \subseteq \mathcal{P}_{v,\Omega(v)} \)

So:

\[
\begin{align*}
\mathcal{P}_{v,\Omega(v)} & \ni x_1 & x_2 & \in \mathcal{P}_{v,\Omega(v)} \\
\text{2-adjacent} & & \text{2-adjacent} \\
\mathcal{P}_{v,\text{max}(v)} & \ni y_1 & \cdots & y_2 & \in \mathcal{P}_{v,\text{max}(v)} \\
\text{2-connected} & & & & \\
\text{thanks to (1), (2)}
\end{align*}
\]

- **Proof of (1):** clever arithmetics
- **Proof of** \( T_v \subseteq \mathcal{P}_{v,\Omega(v)} \): arithmetics
- **Proof of** \( \mathcal{P}_{v,\text{max}(v)} \subseteq T_v \): prove \( T_1, T_2, \ldots \) generate \( \mathcal{P}_{v,\text{max}(v)} \)
sort(\(v_0, v_1 - v_0, v_2 - v_0\))
Tools

\[ \text{sort}(v_0, v_1 - v_0, v_2 - v_0) \]
sort($v_0, v_1 - v_0, v_2 - v_0$)

$v_0, v_1 - v_0, v_2 - v_0$

$v_1 - v_0, v_0, v_2 - v_0$

$v_1 - v_0, v_2 - v_0, v_0$

$$
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 1 & 0 \\
1 & 1 & 1 \\
0 & 0 & 1 \\
\end{pmatrix}
$$

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
$$
sort\( (v_0, v_1 - v_0, v_2 - v_0) \)

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
1 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

| 1 | 2 | 3 |
|---|---|---|
| 1 | 21 | 31 |
| 1 | 12 | 32 |
| 1 | 13 | 23 |
$\text{sort}(v_0, v_1 - v_0, v_2 - v_0)$

$v_0, v_1 - v_0, v_2 - v_0$

\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

$1 \mapsto 1$

$2 \mapsto 21$

$3 \mapsto 31$

$v_1 - v_0, v_0, v_2 - v_0$

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix}
\]

$1 \mapsto 2$

$2 \mapsto 12$

$3 \mapsto 32$

$v_1 - v_0, v_2 - v_0, v_0$

\[
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\]

$1 \mapsto 3$

$2 \mapsto 13$

$3 \mapsto 23$
Tools
Tools
Tools
Tools
Tools
Tools
Tools
Tools
Tools
Tools
Tools
Tools
Tools
Conclusion and perspectives

- We fully understand the critical behavior at $\omega = \Omega(v)$.
- We can deduce the answer for 0- and 1-connectedness (in 3D).
Conclusion and perspectives

- We fully understand the critical behavior at $\omega = \Omega(v)$.
- We can deduce the answer for 0- and 1-connectedness (in 3D).
- **Question:** Higher dimensions?
- **Question:** Can we decide if $v \in \mathcal{F}_3$?
Conclusion and perspectives

- We fully understand the critical behavior at $\omega = \Omega(v)$.
- We can deduce the answer for 0- and 1-connectedness (in 3D).
- **Question:** Higher dimensions?
- **Question:** Can we decide if $v \in \mathcal{F}_3$?

Thank you for your attention