Processes with Photons and Gravitons

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Abstract. In this contribution to the memorial workshop in the honour of Victor Manuel Villanueva Sandoval, I give an overview over my long-term collaboration with Victor, which centered on effective actions, amplitudes and physical processes involving photons, gravitons and external fields.

1. Introduction
Victor and I met first in the year 2000 at the Fradkin Memorial conference in Moscow, and were surprised to find out that we were actually faculty colleagues at the Institute of Physics and Mathematics of the UMSNH, where I had just got hired in his absence. Practically on the same day, we started a collaboration that lasted until his grave illness and untimely passing away in 2013. Our joint work started with an analysis of the QED \( N \)-photon amplitudes in the low-energy limit [1], and later on extended to the mixed photon-graviton case, including also external fields. Among other things, we achieved the first calculation of the photon-graviton conversion process in a magnetic field at the one-loop level [2, 3]. On a more technical level, we solved a long-standing problem in the so-called “string-inspired formalism” [4, 5, 6] by finding an optimized integration-by-parts algorithm for the \( N \)-photon/gluon amplitudes [7]. Here I will summarize our joint work, as well as some closely related issues.

2. QED \( N \)-photon amplitudes from the Euler-Heisenberg Lagrangian
My first collaboration with Victor concerned the one-loop \( N \)-photon amplitudes in QED. Presently, the calculation of one-loop amplitudes is generally still a very hard problem beyond the four-point level, unless restrictions are imposed on the momenta and/or polarizations. For the case at hand, the four-photon amplitude was obtained by Karplus and Neuman already in 1950 in a landmark calculation [10], but the six-photon one only quite recently [11], and only for the massless case. In the massless case it is further known that, for \( N > 4 \), the one-loop photon amplitudes vanish if all or all but one helicities are equal [12].

In our work with L. Martin [1], we studied the low-energy limit of the \( N \)-photon amplitudes, defined by photon energies much smaller than the electron mass. In this limit, the full information on the amplitudes is contained in the QED effective Lagrangian for a constant external field, i.e. the famous Euler-Heisenberg Lagrangian, obtained by Heisenberg and Euler in 1936 [13] in terms of the following well-known proper-time representation:
\[ \mathcal{L}_{\text{spin}}(F) = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2T} \left[ \frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2 - b^2)T^2 - 1 \right]. \] (2.1)

Here, \( a, b \) are the Maxwell invariants, \( a^2 - b^2 = B^2 - E^2 \), \( ab = \mathbf{E} \cdot \mathbf{B} \).

Although at the four-photon level it is a standard textbook exercise to extract from this representation the four-photon amplitude in the low-energy limit (see, e.g., [14]), a generalization to higher points was not available in the literature yet. Using the spinor helicity technique together with a suitable choice of variables, we were able to arrive at the following closed-form expression for the low-energy \( N \) - photon amplitude with arbitrary helicity assignments:

\[ \Gamma^{(1)(EH)}_{\text{spin}}[\varepsilon_1^+; \ldots; \varepsilon_K^+; \varepsilon_{K+1}^-; \ldots; \varepsilon_N^-] = -\frac{m^4}{8\pi^2} \left( \frac{2ie}{m^2} \right)^N (N - 3)! \sum_{k=0}^{K} \sum_{l=0}^{N-K} (-1)^{N-K-l} B_{k+l} B_{N-k-l} \varepsilon_{k+l}^{+} \varepsilon_{N-K-l}^{-}. \] (2.2)

Here \( K \) denotes the number of positive helicities, the \( B_n \) are Bernoulli numbers, and we have introduced variables \( \chi_{\pm} \) constructed from the spinor products underlying the spinor helicity formalism:

\[ \chi_{K}^{+} = \frac{(K)!}{2^K} \left\{ [12]^2 [34]^2 \cdots [(K-1)K]^2 + \text{all permutations} \right\}, \]
\[ \chi_{N-K}^{-} = \frac{(N-K)!}{2^{N-K}} \left\{ (1(K+1)(K+2))^2 (3(K+3)(K+4))^2 \cdots (N-1)(N)^2 + \text{all permutations} \right\}. \] (2.3)

3. The string-inspired worldline formalism in flat space

Let us start with the basics of Feynman’s worldline representation of the (one-loop) effective action \( \Gamma(A) \) in QED [8, 9]. For Scalar QED, it can be written (in modern notation)

\[ \Gamma_{\text{scal}}(A) = \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(T)=x(0)} Dx(T) e^{- \int_0^T d\tau \frac{1}{2} \left( \frac{\dot{x}^2}{T^2} + i e \dot{x}^\mu A_\mu(x(\tau)) \right)}. \] (3.1)

Here \( T \) is the proper-time and \( m \) the mass of the scalar in the loop. For fixed \( T \), one has to perform a path integral over the closed trajectories in (euclidean) spacetime with periodicity \( x(T) = x(0) \), representing a relativistic scalar particle interacting with the external Maxwell field \( A(x) \).

This one-loop effective action is the generating functional of the photon amplitudes, and one can extract the contribution to the \( N \) - photon amplitude by expanding the interaction exponential \( e^{-ie\int_0^T dx^\mu A_\mu(x(\tau))} \) to \( N \)th order. Thus one has the equivalent of the Feynman diagrams depicted in Fig. 1, but in a form that involves first-quantized path integrals instead of momentum integrals (the diagrams involving the seagull vertex are not shown).

Feynman’s corresponding representation for the Spinor QED case [9] differs from (3.1) by an additional factor \( S[x, A] \) representing the electron spin.
fields” determinants and correlators in this auxiliary one-dimensional field theory. For the “coordinate if necessary, be manipulated into gaussian form, which reduces its calculation to the one of one-dimensional quantum field theory. The path integral will, by appropriate expansions evaluating this type of path integral. Here one treats all path integrals from the point of view

This Grassmann version is usually preferable if an analytical calculation of the effective action [15]

\[
S[x, A] = \text{tr}_\Gamma \mathcal{P} \exp \left[ i \frac{c}{4} [\gamma^\mu, \gamma^\nu] \int_0^T d\tau F_{\mu\nu}(x(\tau)) \right].
\] (3.2)

Here \( \text{tr}_\Gamma \) denotes the Dirac trace, \( F_{\mu\nu} \) is the field strength tensor corresponding to \( A \) and \( \mathcal{P} \) denotes the path ordering operator. Also the global normalization must be adjusted by a factor of \( -\frac{1}{2} \). Alternatively, this spin factor can be rewritten as the following Grassmann path integral

\[
S[x, A] = \int_{\psi(T)=-\psi(0)} \mathcal{D}\psi(\tau) \exp \left[ - \int_0^T d\tau \left( \frac{1}{2} \psi^\mu \cdot \psi_\mu - i e \psi^\mu F_{\mu\nu} \psi^\nu \right) \right].
\] (3.3)

The integration variables \( \psi^\mu(\tau) \) are anticommuting, antiperiodic Lorentz vectors:

\[
\begin{align*}
\psi(\tau_1)\psi(\tau_2) &= -\psi(\tau_2)\psi(\tau_1), \\
\psi(T) &= -\psi(0).
\end{align*}
\] (3.4)

This Grassmann version is usually preferable if an analytical calculation of the effective action is intended.

In this talk, I will focus on the “string-inspired” method [16, 4, 5, 6] (for a review, see [17]) for evaluating this type of path integral. Here one treats all path integrals from the point of view of one-dimensional quantum field theory. The path integral will, by appropriate expansions if necessary, be manipulated into gaussian form, which reduces its calculation to the one of determinants and correlators in this auxiliary one-dimensional field theory. For the “coordinate fields” \( x^\mu(\tau) \), first one must deal with the zero-mode of the path integral, which consists of the constant loops; on those the free worldline action in (3.1) vanishes, so that the kinetic operator is not invertible in the full space of trajectories. The most convenient way of doing this is to fix the center-of-mass of the loop, defined by

\[
x_0^\mu \equiv \frac{1}{T} \int_0^T d\tau x^\mu(\tau),
\] (3.5)

and to split the path integral as

\[
\int_{x(T) = x(0)} \mathcal{D}x(\tau) = \int d^D x_0 \int_{q(T) = q(0)} \mathcal{D}q(\tau),
\] (3.6)
where

\[ q^\mu(\tau) = x^\mu(\tau) - x_0^\mu. \]  

(3.7)

For the \( q \)-path integral, the kinetic operator is invertible, the inverse being given by the correlator \(^1\)

\[ \langle q^\mu(\tau_1)q^\nu(\tau_2) \rangle = - G_B(\tau_1, \tau_2) \eta^{\mu\nu} \]  

(3.8)

with the bosonic worldline Green’s function

\[ G_B(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{1}{T}(\tau_1 - \tau_2)^2. \]  

(3.9)

For the \( \psi \)-path integral there is no zero mode because of the antiperiodicity. The correlator is simply

\[ \langle \psi^\mu(\tau_1)\psi^\nu(\tau_2) \rangle = \frac{1}{2} G_F(\tau_1, \tau_2) \eta^{\mu\nu}. \]  

(3.10)

with

\[ G_F(\tau_1, \tau_2) = \text{sign}(\tau_1 - \tau_2). \]  

(3.11)

We start with the extraction of the \( N \)-photon amplitude from the effective action. Expanding \( \Gamma[A] \) to \( N \)th order in \( A \), choosing \( A \) as a sum of \( N \) plane waves,

\[ A^\mu(x(\tau)) = \sum_{i=1}^{N} \varepsilon_i^\mu e^{ik_i \cdot x(\tau)} \]  

(3.12)

and retaining only those terms in the result that involve each plane wave once, we obtain the following representation of the \( N \)-photon amplitude with momenta \( k_i \) and polarizations \( \varepsilon_i \):

\[ \Gamma[\{k_i, \varepsilon_i\}] = \int_0^\infty \frac{dT}{T} e^{-m^2T} \int_{x(T)=x(0)} Dx(\tau) e^{-\int_0^T d\tau \frac{x^2}{T}} V(k_1, \varepsilon_1) \cdots V(k_N, \varepsilon_N). \]  

(3.13)

Here we have introduced the “photon vertex operator”

\[ V(k, \varepsilon) \equiv \int_0^T d\tau \varepsilon \cdot \dot{x}(\tau) e^{ik \cdot x(\tau)}. \]  

(3.14)

The path integral is already gaussian. Using the formal exponentiation \( \varepsilon_i \cdot \dot{x}(\tau_i) = e^{\varepsilon_i \cdot \dot{x}(\tau_i)} \mid_{\text{lin}(\varepsilon_i)} \) it can be further simplified to a pure “completion-of-the-square”, which leads straightforwardly

\(^1\) with our conventions \( \eta^{\mu\nu} = \delta^{\mu\nu} \) in euclidean and \( \eta^{\mu\nu} = \text{diag}(--++) \) in Minkowski space.
to the following *Bern-Kosower Master Formula* for the one-loop \( N \)-photon amplitude in Scalar QED:

\[
\Gamma\{k_i, \varepsilon_i\} = (2\pi)^D \delta^D\left(\sum_{i=1}^N k_i\right) (-ie)^N \int_0^\infty \frac{dT}{T} (4\pi T)^{-\frac{D}{2}} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \\
\times \exp\left\{ \sum_{i,j=1}^N \left[ \frac{1}{2} G_{Bij} k_i \cdot k_j + i\tilde{G}_{Bij} k_i \cdot \varepsilon_j + \frac{1}{2} \tilde{G}_{Bij} \varepsilon_i \cdot \varepsilon_j \right] \right\} |\text{lin}(\varepsilon_1, \ldots, \varepsilon_N)| .
\]

(3.15)

Here a ‘dot’ means a derivative with respect to the first variable. The delta function for energy-momentum conservation comes from the integral over the “zero-mode” of the path integral, that is the constant loops. The factor \((4\pi T)^{-\frac{D}{2}}\) is the free path integral. Note that this formula is valid on- and off-shell. Originally, it was obtained by Bern and Kosower [4, 5] by an analysis of the field theory limit of certain string theory amplitudes, however this required the use of on-shell conditions (in string theory, the off-shell extension of amplitudes is often technically challenging).

We proceed right away to the Spinor QED case. The only difference is in the additional terms from the Grassmann path integral, and it is an interesting aspect of the work of Bern and Kosower that those can be inferred from the ones that are already there in the scalar case through the following “cycle replacement rule”: Expanding out the integrand in (3.15) leads to an expression

\[
\exp\left\{ \right\} |\text{multi-linear} = (-i)^N P_N(\tilde{G}_{Bij}, \tilde{G}_{Bij}) \exp\left[ \frac{1}{2} \sum_{i,j=1}^N G_{Bij} k_i \cdot k_j \right] \]

(3.16)

with a certain polynomial \( P_N \) that depends on the various \( \tilde{G}_{Bij} \)'s, \( \tilde{G}_{Bij} \)'s, as well as on the kinematic invariants. It is possible to remove all second derivatives \( \tilde{G}_{Bij} \) by integration by parts, leading to a new integrand where the prefactor of the exponential depends on the \( \tau_i \) only through the \( \tilde{G}_{Bij} \) and is homogeneous in the momenta:

\[
P_N(\tilde{G}_{Bij}, \tilde{G}_{Bij}) e^{\frac{1}{2} \sum_{i,j=1}^N G_{Bij} k_i \cdot k_j} \overset{\text{part.int.}}{\longrightarrow} Q_N(\tilde{G}_{Bij}) e^{\frac{1}{2} \sum_{i,j=1}^N G_{Bij} k_i \cdot k_j}. \]

(3.17)

All the terms from the Grassmann path integral, which make the difference between the scalar and spinor loop cases, can then be included simply by replacing every “closed cycle”

\[
\tilde{G}_{Bij_1} \tilde{G}_{Bij_2} \cdots \tilde{G}_{Bij_k} \]

in \( Q_N \) by the corresponding “super-cycle”, defined by

\[
\tilde{G}_{Bij_1} \tilde{G}_{Bij_2} \cdots \tilde{G}_{Bij_k} - G_{Fj_1j_2} G_{Fj_2j_3} \cdots G_{Fj_kj_1}. \]

(3.18)

Strassler [6] noted, that the same IBP procedure also goes a long way in making transparent the gauge invariance (transversality) of the photon/gluon amplitudes at an early stage of the calculation, before doing any of the parameter integrals. However, the systematics of the IBP procedure was worked out fully only quite recently [7].
For example, in scalar QED this leads to the following parameter integral representation of the four-photon amplitude:

\[
\Gamma_{\text{scal}}[k_1, \varepsilon_1; \ldots; k_4, \varepsilon_4] = \frac{e^4}{(4\pi)^2} \int_0^\infty \frac{dT e^{-m^2 T}}{T^{1+D/2}} \int_0^T d\tau_4 \cdots d\tau_1 \times \left( Q_4^4 + Q_4^3 + Q_4^2 + Q_4^{22} \right) e^{\sum_{i,j} G_{B_{ij}} k_i \cdot k_j},
\]

(3.19)

where

\[
Q_4^4 = \dot{G}(1234)Z_4(1234) + \text{perm.},
\]

\[
Q_4^3 = \dot{G}(123)Z_3(123)T_1(4) + \text{perm.},
\]

\[
Q_4^2 = \dot{G}(12)Z_2(12)T_2(34) + \text{perm.},
\]

\[
Q_4^{22} = \dot{G}(12)Z_2(12)\dot{G}(34)Z_2(34) + \text{perm.}
\]

(3.20)

and

\[
\dot{G}(i_1 i_2 \cdots i_n) = \dot{G}_{B_{i_1 i_2}} \dot{G}_{B_{i_2 i_3}} \cdots \dot{G}_{B_{i_n i_1}} \text{tr} (f_{i_1} f_{i_2} \cdots f_{i_n}),
\]

\[
T_1(i) = \frac{1}{k_i^2} \sum_{r=1}^4 \dot{G}_{B_{ir}} k_i \cdot f_i \cdot k_r,
\]

\[
T_2(ij) = \sum_{(r,s) \neq (i,j)}^4 \dot{G}_{B_{ir}} \dot{G}_{B_{js}} k_r \cdot H_{ij} \cdot k_s.
\]

\[
H_{ij}^{\mu\nu} \equiv \frac{k_i^\mu \epsilon_j^\nu (f_i f_j)^{\mu\nu} k_i \cdot k_j}{(k_i \cdot k_j)^2}.
\]

(3.21)

Here \( f_i \) denotes the field strength matrix associated to the \( i \)th photon,

\[
f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu.
\]

(3.22)

We note that, apart from its compactness, this representation has the following advantages:

- Manifest UV finiteness.
- Full permutation invariance.
- Gauge invariance at the parameter integral level: since now each polarization vector \( \varepsilon_i^\mu \) appears only in the corresponding field strength tensor \( f_i^{\mu\nu} \), a shift of \( \varepsilon_i^\mu \) by a multiple of \( k_i^\mu \) will have no effect. Usually, this property would be seen only after performing the parameter integrals, and summing over the six inequivalent Feynman diagrams.
- It gives the full amplitude: in terms of Feynman diagrams, it includes not only the diagram shown in Fig. 2 below, but also the ones related to it by permutations of the photons, or replacement of two cubic vertices by the quartic seagull-vertex.
The replacement rule (3.18) also generalizes in the obvious way.

For the one-loop $N$ - photon amplitude in the constant field $F_{\mu\nu}$ [19, 20, 21]. This requires a change of the vacuum worldline Green’s functions $G_B, G_F$ to generalized Green’s functions that take the external field into account,

$$ G_B(\tau_1, \tau_2) \to \mathcal{G}_B(\tau_1, \tau_2) = \frac{1}{2(\epsilon F)^2} \left( \frac{\epsilon F}{\sin(\epsilon FT)} e^{-ieFTG_{B12}} + ieFG_{B12} \right) , $$

$$ G_F(\tau_1, \tau_2) \to \mathcal{G}_F(\tau_1, \tau_2) = G_{F12} \frac{e^{-ieFT\mathcal{G}_{B12}}}{\cos(\epsilon FT)} , $$

and a change of the path integral determinant normalization, which also becomes field-dependent:

$$ (4\pi T)^{-\frac{D}{2}} \to (4\pi T)^{-\frac{D}{2}} \det^{-\frac{1}{2}} \left[ \frac{\sin(\epsilon FT)}{\epsilon FT} \right] \quad \text{(Scalar QED)} , $$

$$ (4\pi T)^{-\frac{D}{2}} \to (4\pi T)^{-\frac{D}{2}} \det^{-\frac{1}{2}} \left[ \frac{\tan(\epsilon FT)}{\epsilon FT} \right] \quad \text{(Spinor QED)} . $$

(3.24)

The vacuum master formula (3.15) generalizes straightforwardly to the following master formula for the one-loop $N$ - photon amplitude in the constant field $F_{\mu\nu}$ in scalar QED:

$$ \Gamma_{\text{scal}}[\{k_i, \varepsilon_i\}, F_{\mu\nu}] = (-ie)^N \int_0^{\infty} \frac{dT}{T^2} (4\pi T)^{-\frac{D}{2}} e^{-m^2T} \det^{-\frac{1}{2}} \left[ \frac{\sin(\epsilon FT)}{\epsilon FT} \right] $$

$$ \times \prod_{i=1}^{N} \int_0^{T} d\tau_i \exp \left\{ \sum_{i,j=1}^{N} \left[ \frac{1}{2} k_i \cdot \mathcal{G}_{Bij} \cdot k_j - i\varepsilon_i \cdot \mathcal{G}_{Bij} \cdot k_j + \frac{1}{2} \varepsilon_i \cdot \mathcal{G}_{Bij} \cdot \varepsilon_j \right] \right\} |_{\text{lin}(\{\varepsilon_i\})} . $$

(3.25)

The replacement rule (3.18) also generalizes in the obvious way.
4. The worldline formalism in curved space

To include background gravity, naively one would simply replace the trivial Euclidean metric by a general one $g_{\mu\nu}$ in the kinetic term of the path integral (3.1):

$$S_0 = \frac{1}{4} \int_0^T d\tau \dot{x}^2 \to \frac{1}{4} \int_0^T d\tau \dot{x}^\mu g_{\mu\nu}(x(\tau)) \dot{x}^\nu.$$ (4.1)

As usual, the graviton field $h_{\mu\nu}(x)$ would then arise as a fluctuation of the metric around the flat metric,

$$g_{\mu\nu} = \delta_{\mu\nu} + \kappa h_{\mu\nu}.$$ (4.2)

However, it is easy to see that a straightforward analytical evaluation of the arising path integral will lead to ill-defined integrals involving $\delta(0), \delta^2(\tau_i - \tau_j), \ldots$. The reasons for this are, in principle, known already from nonrelativistic quantum mechanics [22].

The basic problem is the nontrivial measure of the path integral in curved space [23], which differs from the trivial measure by a factor

$$\prod_{0 \leq \tau < T} \sqrt{\det g_{\mu\nu}(x(\tau))}.$$ (4.3)

The only known practicable way of handling this factor in an analytical treatment of the path integral is to exponentiate it through the introduction of a set of worldline ghost fields $a^\mu, b^\mu, c^\mu$ [24]. However, this creates spurious UV divergences, which effectively make the path integral behave like a super-renormalizable (one-dimensional) field theory. The lack of manifest UV-finiteness means that the path integral has to be constructed choosing some regularization procedure, and correct results (i.e. results consistent with the known space-time physics) are only obtained by introducing a finite (small) number of regularization-dependent counterterms [28, 29, 30]. For the most common regularization schemes, these counterterms have already been calculated, once and for all [24].

Another subtlety with the curved-space path integral, or rather with its calculation in the string-inspired formalism, was discovered only relatively recently. This is that a naive application of the zero mode fixing procedure, as described above in the flat space case, leads to non-covariant total derivative terms in the effective action for gravity. Those can lead to inconsistencies when used in combination with Riemann normal coordinates, which are an essential tool in gravity. The solution is to covariantize these total derivative terms through the introduction of additional worldline ghosts of the Fadeev-Popov type [31]. Since those ghosts are only needed when using Riemann normal coordinates, we will disregard them in the following.

After the inclusion of those worldline ghost fields coming from the measure, the introduction of the graviton through (4.2) leads to the following vertex operator, describing the interaction of the graviton with a scalar:

$$V^h_{\text{scal}}[k, \epsilon] = \epsilon_{\mu\nu} \int_0^T d\tau \left[ \dot{x}^\mu(\tau) \dot{x}^\nu(\tau) + a^\mu(\tau)a^\nu(\tau) + b^\mu(\tau)c^\nu(\tau) ight. + 4\xi (\delta^\mu\nu k^2 - k^\mu k^\nu) \left. e^{ikx(\tau)} \right].$$ (4.4)
Here we have introduced the polarisation tensor $\varepsilon_{\mu\nu}$ of the graviton, and in adding the term with the constant $\xi$ we have now also taken into account that the coupling of a scalar to gravity is not unique [24]. For the graviton interacting with a spinor the vertex operator becomes

$$V_{\text{spin}}^{h}[k, \varepsilon] = \varepsilon_{\mu\nu} \int_{0}^{T} d\tau \left[ \dot{\phi}^{\mu}(\tau) \dot{\phi}^{\nu}(\tau) + a^{\mu}(\tau) a^{\nu}(\tau) + b^{\mu}(\tau) c^{\nu}(\tau) \right] e^{ik \cdot x(\tau)}$$

(4.5)

with another ghost field $\alpha^{\mu}(\tau)$.

Using these vertex operators together with gaussian path integration yields a highly efficient formalism for one-loop calculations in Einstein-Maxwell theory, as has been demonstrated in a number of state-of-the-art calculations. In the remainder of my talk, I will sketch a few of those calculations, with an emphasis on Victor’s contributions.

5. Photon-graviton conversion in a magnetic field

Einstein-Maxwell theory contains the following tree level vertex for the conversion of photons into gravitons, or vice versa, in a background electromagnetic field:

$$\frac{1}{2} \kappa h_{\mu\nu} \left( F_{\mu\alpha} f_{\alpha} + f_{\alpha} F_{\mu\alpha} \right) - \frac{1}{4} \kappa h_{\mu} F_{\alpha\beta} f_{\alpha\beta}.$$  

(5.1)

Here $h_{\mu\nu}$ denotes the graviton, $f_{\mu\nu}$ the photon, $F_{\mu\nu}$ the external field, and $\kappa$ the gravitational coupling constant.

This process is of interest for cosmologists, since it leads to photon-graviton oscillations similar to the better known neutrino-axion oscillations [32, 33]. The true eigenstates of propagation are then found by diagonalization of the photon-graviton mixing matrix. At the one-loop level, the non-diagonal contribution to this matrix is the photon - graviton vacuum polarization diagram shown in Fig. 3:

![Figure 3. One-loop photon-graviton diagram in a constant field.](image)

In [34, 2] we achieved the first calculations of this amplitude at the one-loop level, for both the scalar and spinor loop cases. In the present formalism, they can be written in terms of Wick contractions of vertex operators:

$$\varepsilon_{\mu\nu} \Pi_{\text{scal}}^{\mu\nu,\alpha}(k) \varepsilon_{\alpha} = \frac{i e \kappa}{4} \int_{0}^{\infty} \frac{dT}{T} e^{-m^{2}T(4\pi T)^{-1/2} \text{det}^{-\frac{1}{2}}} \left[ \frac{\sin(eFT)}{eFT} \right] \left\langle V_{\text{scal}}^{h}[k, \varepsilon^{h}] V_{\text{scal}}^{A}[-k, \varepsilon^{A}] \right\rangle,$$

$$\varepsilon_{\mu\nu} \Pi_{\text{spin}}^{\mu\nu,\alpha}(k) \varepsilon_{\alpha} = \frac{-i e \kappa}{2} \int_{0}^{\infty} \frac{dT}{T} e^{-m^{2}T(4\pi T)^{-1/2} \text{det}^{-\frac{1}{2}}} \left[ \frac{\tan(eFT)}{eFT} \right] \left\langle V_{\text{spin}}^{h}[k, \varepsilon^{h}] V_{\text{spin}}^{A}[-k, \varepsilon^{A}] \right\rangle.$$  

(5.2)
The Wick contractions have to be performed using the generalized Green’s functions (3.23). The result is too lengthy [2] to be given here, but let us discuss here the most important features of the amplitude. Those are the following:

As expected by power counting, the amplitude has an UV divergence. Using dimensional regularization (this is now the usual dimensional regularization in space-time, not the one-dimensional dimensional regularization mentioned above as a scheme for well-defining the path integral) it becomes, e.g. for the scalar loop case,

$$\Pi_{\text{scal,div}}^{\mu\nu,\alpha}(k) = \frac{i\epsilon^2}{3(4\pi)^2} \frac{1}{D-4} C_{\mu\nu,\alpha}.$$  (5.3)

Here the tensor $C_{\mu\nu,\alpha}$ is the tree-level vertex in momentum space,

$$C_{\mu\nu,\alpha} = (F\cdot k)^{\alpha} \delta^{\mu\nu} + F^{\mu\alpha} k^\nu + F^{\nu\alpha} k^\mu - (F\cdot k)^{\mu} \delta^{\nu\alpha} - (F\cdot k)^{\nu} \delta^{\mu\alpha}.$$  (5.4)

so that the amplitude can be renormalized multiplicatively.

The photon and graviton polarizations are best chosen adapted to the field. Generically, by a Lorentz transformation one can assume the electric and magnetic field to be collinear. For the photons we then can use polarizations $\varepsilon_{\perp}, \varepsilon_{\parallel}$ that are perpendicular resp. parallel to the joint field direction. The same vectors can then also be used to construct suitable polarization tensors for the graviton:

$$\varepsilon_{\oplus \mu\nu} = \varepsilon_{\perp \mu} \varepsilon_{\perp \nu} - \varepsilon_{\parallel \mu} \varepsilon_{\parallel \nu},$$  
$$\varepsilon_{\otimes \mu\nu} = \varepsilon_{\perp \mu} \varepsilon_{\parallel \nu} + \varepsilon_{\parallel \mu} \varepsilon_{\perp \nu}.$$  (5.5)

This basis is extremely convenient, since it can be shown that CP invariance implies the following selection rules for this amplitude:

(i) For a purely magnetic field $\varepsilon_{\oplus}$ couples only to $\varepsilon_{\perp}$ and $\varepsilon_{\otimes}$ only to $\varepsilon_{\parallel}$.
(ii) For a purely electric field $\varepsilon_{\oplus}$ couples only to $\varepsilon_{\parallel}$ and $\varepsilon_{\otimes}$ only to $\varepsilon_{\perp}$.

It is also convenient to normalize this one-loop amplitude by the tree level amplitude:

$$\hat{\Pi}_{\text{scal,spin}}^{Aa}(\hat{\omega}, \hat{B}, \hat{E}) \equiv \text{Re} \left( \frac{\Pi_{\text{scal,spin}}^{Aa}(\hat{\omega}, \hat{B}, \hat{E})}{C_{Aa}} \right).$$  (5.6)

where $A = \oplus, \otimes$ and $a = \perp, \parallel$, $\hat{\omega} = \frac{\omega}{m}$, $\hat{B} = \frac{eB}{m^2}$, $\hat{E} = \frac{eE}{m^2}$. The ”bar” on $\Pi^{Aa}$ means that it is already renormalized.

Phenomenologically, photon-graviton conversion is presently mainly of interest for astrophysics, and here in most cases only the magnetic field is relevant, so that I will restrict myself to $\hat{E} = 0$ in the following. Even so, the parameter integrals are still not amenable to an analytical calculation (they are similar in structure to the ones arising in the case of the photon propagator in a constant magnetic field, well-known to be intractable analytically). Nevertheless, there are a number of special cases that admit an explicit or numerical calculation:

- For photon energies $\omega$ below the pair creation threshold $\omega_{cr}$ the parameter integrals are suitable for a straightforward numerical evaluation.
• In the limit of a weak magnetic field, but arbitrary photon energy, the integrals can be reduced to integrals over Airy functions which are again well-suited to numerical integration.

• In the large magnetic field strength limit one finds a logarithmic dependence on the field strength:

\[
\hat{\Pi}_{\text{scal}}^{Aa}(\hat{\omega}, \hat{B}) \xrightarrow{\hat{B} \to \infty} -\frac{\alpha}{12\pi} \ln(\hat{B}) ,
\]

\[
\hat{\Pi}_{\text{spin}}^{Aa}(\hat{\omega}, \hat{B}) \xrightarrow{\hat{B} \to \infty} -\frac{\alpha}{3\pi} \ln(\hat{B}) .
\]

(5.7)

The coefficients in these leading asymptotic terms relate to the corresponding UV counterterms in the same way as had been found by Ritus in 1975 for the photon propagator in a magnetic field [25].

• Finally, in the limit of vanishing photon-graviton energy \(\omega\) the amplitudes can be related to the Euler-Heisenberg, resp. to its scalar QED equivalent due to Weisskopf [26]:

\[
\hat{\Pi}_{\text{scal,spin}}^{\oplus \perp}(\hat{\omega} = 0, \hat{B}) = -\frac{2\pi\alpha}{m^4} \left( \frac{1}{B} \frac{\partial}{\partial B} + \frac{\partial^2}{\partial B^2} \right) L_{\text{EH}}^{\text{scal,spin}}(\hat{B}) ,
\]

\[
\hat{\Pi}_{\text{scal,spin}}^{\otimes \parallel}(\hat{\omega} = 0, \hat{B}) = -\frac{4\pi\alpha}{m^4} \frac{1}{B} \frac{\partial}{\partial B} L_{\text{EH}}^{\text{scal,spin}}(\hat{B}) .
\]

(5.8)

Of the rich phemomenology of these amplitudes, let us mention here only one interesting effect, namely dichroism. The rate of conversion of photons to gravitons in the magnetic field turns out to depend on the photon polarization, and thus leads to a rotation of the polarization vector of the incident electromagnetic wave. This effect is better known for the photon-axion conversion, where dichroism arises already at the tree-level, and has been considered as a means for detecting the production of axions by lasers [27]. The photon-graviton amplitude is different in that, although it also exists already at tree-level, it is only at one-loop that polarization dependence kicks in. Moreover, although the corresponding dichroism is a very small effect - being a loop correction to an amplitude that involves the gravitational coupling constant - a systematic study of dichroism in the standard model, including standard gravity, has come to the astonishing conclusion [27] that this tiny effect is still larger than any other source of dichroism in the standard model!

6. Effective actions for the \(N\) photon - one graviton amplitudes

On the theoretical side, recently there has been much interest in graviton amplitudes and, in particular, in their relation to gauge theory amplitudes (see, e.g., [35] and refs. therein). The existence of such relations is not obvious from the standard field theory definitions of gravity and Yang-Mills theory, but follows naturally from the string-theoretical representation of such amplitudes in the field theory limit [36]. Most results so far are for massless amplitudes, and comparing photon or gluon amplitudes with graviton amplitudes. In work that is presently still ongoing [37], we use the effective action based approach of section 2 to study the mixed photon-graviton amplitudes in the low energy limit. This requires a generalization of the Euler-Heisenberg Lagrangian to Einstein-Maxwell theory. Its calculation at the one-loop level is similar to the amplitude calculations described in sections 4 and 5, only that for the effective action it will be computationally essential to achieve manifest gauge invariance as well as general coordinate
invariance. This can be implemented by combining Fock-Schwinger gauge with Riemann normal coordinate expansions:

\[ A_\mu(x_0 + y) = \frac{1}{2} y^\mu F_{\rho\mu} + \frac{1}{3} y^\rho g^{\rho\nu} D_\nu F_{\rho\mu} + ... \]

\[ g_{\mu\nu}(x_0 + y) = g_{\mu\nu}(x_0) + \frac{1}{3} R_{\mu\alpha\beta\nu}(x_0) y^\alpha y^\beta + ... \]  

(6.1)

\((x_0 \text{ denotes the loop center of mass})\). To linear order in the curvature one obtains, for the fermion loop case [38]:

\[ L_{\text{spin}} = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \det^{-1/2} \left[ \frac{\tan(FT)}{FT} \right] \]

\[ \times \left\{ 1 + \frac{iT^2}{8} F_{\mu\nu;\alpha\beta} G_{B11}^{\alpha\beta} \left( \dot{G}_{B11}^{\mu\nu} - 2 G_{F11}^{\mu\nu} \right) \right. \]

\[ + \frac{iT^2}{24} F_{\lambda\mu\nu} R_{\alpha\beta\mu\nu}^{\lambda} \left( \dot{G}_{B11}^{\nu\mu} \dot{G}_{B11}^{\alpha\beta} + \dot{G}_{B11}^{\mu\nu} \dot{G}_{B11}^{\alpha\beta} + \dot{G}_{B11}^{\nu\alpha} \dot{G}_{B11}^{\mu\beta} + 4 G_{F11}^{\mu\nu} \right) \]

\[ + \frac{1}{12} R_{\mu\alpha\beta\nu} \left( \dot{G}_{B11}^{\alpha\nu} \dot{G}_{B11}^{\mu\beta} + \dot{G}_{B11}^{\nu\mu} \dot{G}_{B11}^{\alpha\beta} + \left( \dot{G}_{B11}^{\mu\nu} - 2 g_{\mu\nu} \delta(0) \right) G_{B11}^{\alpha\beta} \right) \]

\[ + \frac{i}{6} T^2 F_{\alpha\beta\gamma} F_{\mu\nu;\delta} \int_0^1 d\tau \left( \dot{G}_{B12}^{\alpha\nu} \dot{G}_{B12}^{\beta\mu} + \dot{G}_{B12}^{\mu\nu} \dot{G}_{B12}^{\beta\alpha} \right) \dot{G}_{B12}^{\gamma\mu} \]

\[ + \frac{3}{2} \dot{G}_{B12}^{\gamma\delta} \dot{G}_{B12}^{\alpha\mu} \dot{G}_{B12}^{\beta\nu} \left\} \right. \]  

(6.2)

This effective Lagrangian holds the information on the one-loop amplitudes with a fermion loop, a single graviton and any numbers of photons, in the low energy limit. Expanding it out in powers of the field strength tensor \( F \), we get the amplitudes with one graviton and any fixed number of photons. In [38] we worked this out to order \( F^2 \), and showed that the result differed only by a total derivative from the well-known effective Lagrangian \( L_{\text{spin}}^{(DH)} \) obtained in 1980 by Drummond and Hathrell [39]:

\[ L_{\text{spin}}^{(DH)} = \frac{1}{180(4\pi)^2 m^2} \left( 5 R F_{\mu\nu}^2 - 26 R_{\mu\nu} F^{\mu\alpha} F_{\nu}^\alpha + 2 R_{\mu\alpha\beta\nu} F^{\mu\nu} F^{\alpha\beta} + 24 (\nabla^\alpha F_{\alpha\mu})^2 \right) \]  

(6.3)

In [40] we then worked out the next order, \( RF^3 \) (there is no order \( RF^3 \) for parity reasons):
This is already a new result, and one that would be difficult to achieve with other methods.

The next logical step in this program will be to study the one-loop amplitude with one graviton and four photons, pictured in Fig. 4.

Figure 4. One-graviton four-photon amplitude.

Using the above result for the effective action, we expect to be able to compute all the helicity components of this amplitude in the low energy limit. Beyond the four-point case, in the low-energy limit we expect to be further able to obtain the one graviton – \( N \) photon amplitudes with “all +” photon helicities.

During all these years, working with Victor was a pleasure. He had a deep interest in science, a constructive spirit and a knack for seeing the humorous side of things, properties that he was able to preserve to his last day.

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