Zeeman Effect in Phase Space

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Abstract

The two-dimensional hydrogen atom in an external magnetic field is considered in the context of phase space. Using solution of the Schrödinger equation in phase space the Wigner function related to the Zeeman effect is calculated. For this purpose, the Bohlin mapping is used to transform the Coulomb potential into a harmonic oscillator problem. Then it is possible to solve the Schrödinger equation easier by using the perturbation theory. The negativity parameter for this system is realised.
I. INTRODUCTION

Since the early years of the development of quantum mechanics its formulation in phase space has been troublesome. The seminal paper by Wigner in 1932 addressed such a problem in an attempt to deal with the superfluidity of Helium [1]. The Wigner function, \( f_W(q,p) \), was introduced as a Fourier transform of the density matrix, \( \rho(q,q') \). Then the phase space manifold (\( \Gamma \)), which has a symplectic structure, is described by the coordinates \((q,p)\) [1–4]. The Wigner function is identified as a quasi-probability density since \( f_W(q,p) \) is real but it is not positive defined. However, the integrals \( \rho(q) = \int f_W(q,p) \, dp \) and \( \rho(p) = \int f_W(q,p) \, dq \) are distribution functions.

In the Wigner formalism, each operator, \( A \), defined in the Hilbert space, \( \mathcal{H} \), is associated with a function, \( a_W(q,p) \), in \( \Gamma \). This procedure is precisely specified by a mapping \( \Omega_W : A \rightarrow a_W(q,p) \), such that, the associative algebra of operators defined in \( \mathcal{H} \) leads to be an algebra in \( \Gamma \), given by \( \Omega_W : AB \rightarrow a_W \star b_W \), where the star-product, \( \star \), is defined by

\[
a_W \star b_W = a_W(q,p) \exp \left[ \frac{ih}{2} \left( \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q} \right) \right] b_W(q,p). \tag{1}
\]

As a consequence a non-commutative structure in \( \Gamma \) is obtained, that has been explored in different ways [2]–[25]. Recently [28, 29, 48–50] unitary representations of symmetry Lie groups have been obtained on a symplectic manifold, exploring the noncommutative nature of the star-product and using the mapping \( \Omega_W \) [28, 29, 48]. As a result, using a specific representation of a Galilei group, Schrödinger equation in phase space is obtained. On the other hand, the scalar representation of Lorentz group for spin 0 and spin 1/2, leads to the Klein-Gordon and Dirac equations in phase space. In relativist and non-relativistic approach, the wave functions are closely associated with the Wigner function [28, 29]. This provides a fundamental ingredient for the physical interpretation of the formalism showing its advantage in relation to other attempts.

In recent years the two-dimensional physical systems have been investigated due to both experimental and theoretical interests. For example, it is possible to cite the fractional Hall effect in a tilted magnetic field [30, 31], superconductivity in two-dimensional organic conductors induced by magnetic field [32], investigations of graphene [33, 34], etc. Particulary, two-dimensional model of hydrogen atom was considered in several contexts, for instance it is possible to describe highly three-dimensional anisotropic crystals [35], semiconductor heterostructures [36, 38] and astrophysical applications [39, 41]. In addition, the hydrogen atom in an uniform magnetic field can present a non-classical behavior with the increase of strength of magnetic field [42, 43]. In this paper, the objective is to investigate the two-dimensional hydrogen atom in a constant magnetic field in
phase space picture, i.e., the Schrödinger equation in phase space is used to study the Zeeman effect. Zeeman effect is applied in a variety of systems, including intense laser lights, cosmic rays, and the study of intergalactic and interstellar medium [46, 47]. The importance to study the Wigner function for such an effect is in order to obtain information about the chaotic nature of such systems.

In Section II, a review about the formalism of quantum mechanics in phase space and its connection with Wigner function is presented. In Section III, the hamiltonian of hydrogen atom in an external magnetic field and Bohlin mapping are discussed. The solution for the Schrödinger equation for Zeeman effect in phase space is solved by perturbative method in Section IV. In Section V, a summary and concluding remarks are presented.

II. SYMPLECTIC QUANTUM MECHANICS

In this section, the representation of the Galilei group in \( H(\Gamma) \) is presented. This procedure leads us to the Schrödinger equation in phase space. Then a connection between this representation and the Wigner formalism is established.

Using the star-operator, \( \hat{A} = a^* \), the position and momentum operators, respectively, are defined by

\[
\hat{Q} = q^* = q + \frac{i\hbar}{2} \partial_p,
\]

\[
\hat{P} = p^* = p - \frac{i\hbar}{2} \partial_q.
\]

The operators given in Eqs. (2-3) satisfy the Heisenberg commutation relation,

\[
[\hat{Q}, \hat{P}] = i\hbar.
\]

In addition, the following operators are introduced,

\[
\hat{K} = m\hat{Q}_i - t\hat{P}_i,
\]

\[
\hat{L}_i = \epsilon_{ijk}\hat{Q}_j\hat{P}_k,
\]

\[
\hat{H} = \frac{\hat{P}^2}{2m} = \frac{1}{2m}(\hat{P}_1^2 + \hat{P}_2^2 + \hat{P}_3^2).
\]
From this set of unitary operators, after simple calculations, the following set of commutation relations are obtained:

\[
\begin{align*}
\hat{L}_i, \hat{L}_j &= i\hbar\epsilon_{ijk}\hat{L}_k, \\
\hat{L}_i, \hat{K}_j &= i\hbar\epsilon_{ijk}\hat{K}_k, \\
\hat{L}_i, \hat{P}_j &= i\hbar\epsilon_{ijk}\hat{P}_k, \\
\hat{K}_i, \hat{P}_j &= i\hbar m\delta_{ij}\mathbf{1}, \\
\hat{K}_i, \hat{H} &= i\hbar \hat{P}_i, \\
\end{align*}
\]

with all other commutation relations being null. This is the Galilei-Lie algebra with a central extension characterized by \(m\). The operators defining the Galilei symmetry \(\hat{P}, \hat{K}, \hat{L}\) and \(\hat{H}\) are then generators of translations, boost, rotations and time translations, respectively.

Defining the operators

\[
\overline{Q} = q\mathbf{1} \quad \text{and} \quad \overline{P} = p\mathbf{1},
\]

for boost, operators \(\overline{Q}\) and \(\overline{P}\), transform as

\[
\exp(-iv\frac{\hat{K}}{\hbar})2\overline{Q}\exp(iv\frac{\hat{K}}{\hbar}) = 2\overline{Q} + vt\mathbf{1},
\]

and

\[
\exp(-iv\frac{\hat{K}}{\hbar})2\overline{P}\exp(iv\frac{\hat{K}}{\hbar}) = 2\overline{P} + mv\mathbf{1}.
\]

This shows that, \(\overline{Q}\) and \(\overline{P}\) transform as position and momentum variables, respectively. These operators satisfy \([\overline{Q}, \overline{P}] = 0\). Then \(\overline{Q}\) and \(\overline{P}\) cannot be interpreted as observables. Nevertheless, they can be used to construct a Hilbert space framework in phase space. Then we define the functions \(\phi(q, p)\) in \(\mathcal{H}(\Gamma)\), that satisfy the condition

\[
\int dq dp \phi^*(q, p)\phi(q, p) < \infty.
\]
The wave function $\psi(q, p, t) = \langle q, p | \psi(t) \rangle$ associated with the state of the system is defined, but do not have the content of the usual quantum mechanics state.

The time evolution equation for $\psi(q, p, t)$ is derived by using the generator of time translations, i.e.

$$\psi(t) = e^{-\frac{iH}{\hbar}} \psi(0).$$

Then this leads to

$$i\hbar \partial_t \psi(q, p; t) = \hat{H}(q, p) \psi(q, p; t),$$
or

$$i\hbar \partial_t \psi(q, p; t) = H(q, p) * \psi(q, p; t),$$

which is the Schrödinger equation in phase space [28].

The association of $\psi(q, p, t)$ with the Wigner function is given by [28],

$$f_W(q, p) = \psi(q, p, t) \star \psi^\dagger(q, p, t).$$

This function satisfies the Liouville-von Neumann equation [28].

III. TWO-DIMENSIONAL HYDROGEN ATOM IN AN EXTERNAL MAGNETIC FIELD AND BOHLIN MAPPING

The Hamiltonian for the two dimensional hydrogen atom in a constant and uniform magnetic field $B = B\hat{z}$ is given as [51, 53]

$$H = \frac{(P - eA)^2}{2m} - \frac{k}{(x^2 + y^2)^{1/2}},$$

where $m$ and $e$ represent the electron mass and charge, respectively, $A$ is the magnetic potential vector and $k$ is a constant. In two-dimensional case Eq. (10) is written as

$$H = \frac{P^2}{2m} - \frac{k}{(x^2 + y^2)^{1/2}} + \frac{m\omega}{2}(x^2 + y^2) + \omega L_z,$$

where $\omega = eB/2mc$ is the frequency and $L_z$ is angular momentum in the $z$-direction. Here the constant $\omega L_z$ will be neglected once the energy is defined up to a constant.

In order to solve the Schrödinger equation for this Hamiltonian, the Bohlin mapping is used.
A. Bohlin mapping

Bohlin mapping is defined by [54–56]

\[ x + iy = (q_1^2 - q_2^2) + i(2q_1q_2), \quad (12) \]

or

\[ x = q_1^2 - q_2^2, \quad (13) \]

and

\[ y = 2q_1q_2. \quad (14) \]

Defining

\[ P_x + iP_y = \frac{p_1 + ip_2}{2(q_1 + iq_2)}, \quad (15) \]

leads to

\[ P_x = \frac{p_1q_1 + p_2q_2}{2(q_1^2 + q_2^2)}, \quad (16) \]

and

\[ P_y = \frac{p_2q_1 - p_1q_2}{2(q_1^2 + q_2^2)}. \quad (17) \]

Substituting Eqs. (13), (14), (16) and (17) in Eq. (11), leads to the Hamiltonian

\[ H = \frac{1}{2} \left( \frac{p_1^2 + p_2^2}{(q_1^2 + q_2^2)} \right) - \frac{k}{(q_1^2 + q_2^2)} + \frac{B^2}{8} (q_1^2 + q_2^2)^3. \quad (18) \]

Using \( h = \omega = e = m = 1 \) and taking the hyper-surface given by \( H = E \), leads to

\[ \frac{1}{2} (p_1^2 + p_2^2) + \frac{B^2}{8} (q_1^2 + q_2^2)^3 - E(q_1^2 + q_2^2) - k = 0, \quad (19) \]

which is the Hamiltonian to be used in the next section. It should be noted that Bohlin transformation is a canonical transformation [57].

IV. ZEEMAN EFFECT IN PHASE SPACE

Using Eq. (19), the equation is written as

\[ \left[ \frac{1}{2} (p_1^2 + p_2^2) + \frac{B^2}{8} (q_1^2 + q_2^2)^3 - E(q_1^2 + q_2^2) - k \right] \ast \psi(q_1, p_1, q_2, p_2) = 0. \quad (20) \]
It should be noted that the above equation is obtained from the classical Hamiltonian by means of the star product. Thus the Bohlin mapping that lead to Eq. (19) is a classical transformation. This equation is analyzed by perturbation theory. The equation in phase space is defined as

\[
(\hat{H}_0 + \hat{H}_1)\psi(q_1, p_1, q_2, p_2) = k\psi(q_1, p_1, q_2, p_2),
\]

(21)

where \( \hat{H}_0 = \frac{1}{2}(p_1^2 + p_2^2) \ast -E(q_1^2 + q_2^2) \ast \) and \( \hat{H}_1 = \frac{B^2}{8} (q_1^2 + q_2^2)^3 \ast \).

The equation for \( \hat{H}_0 \) has the form

\[
\hat{H}_0\psi(0)(q_1, p_1, q_2, p_2) = k(0)\psi(0)(q_1, p_1, q_2, p_2),
\]

(22)

where \( \psi(0)(q_1, p_1, q_2, p_2) \) and \( k(0) \) represent, respectively, the eigenfunction and eigenvalue of the unperturbed hamiltonian.

Defining the operators

\[
\hat{a} = \left( \sqrt{\frac{W}{2}} q_1 \ast + i \sqrt{\frac{1}{2W}} p_1 \ast \right),
\]

(23)

\[
\hat{a}^\dagger = \left( \sqrt{\frac{W}{2}} q_1 \ast - i \sqrt{\frac{1}{2W}} p_1 \ast \right),
\]

(24)

\[
\hat{b} = \left( \sqrt{\frac{W}{2}} q_2 \ast + i \sqrt{\frac{1}{2W}} p_2 \ast \right),
\]

(25)

\[
\hat{b}^\dagger = \left( \sqrt{\frac{W}{2}} q_2 \ast - i \sqrt{12W} p_2 \ast \right),
\]

(26)

where \( W^2/2 = -E \), the star operators \( q_i \ast \) and \( p_i \ast \) are given by

\[
q_i \ast = q_i + \frac{i}{2} \frac{\partial}{\partial p_i},
\]

(27)

\[
p_i \ast = p_i - \frac{i}{2} \frac{\partial}{\partial q_i},
\]

(28)

and perturbed hamiltonian is

\[
\hat{H} = W(\hat{a}\hat{a}^\dagger + \hat{b}\hat{b}^\dagger + 1) + \frac{B^2}{8} (\hat{a} + \hat{a}^\dagger)^2 + (\hat{b} + \hat{b}^\dagger)^2 \ast \]

(29)

The unperturbed hamiltonian is defined as

\[
\hat{H}_0 = W(\hat{a}\hat{a}^\dagger + \hat{b}\hat{b}^\dagger + 1).
\]

(30)
Then, the perturbed part is
\[
\hat{H}_1 = \frac{B^2}{8} (|\hat{a} + \hat{a}^\dagger|^2 + |\hat{b} + \hat{b}^\dagger|^2)^3.
\] (31)

The equation that is to be analyzed is given as
\[
H \ast \psi(q_1, p_1, q_2, p_2) = k \psi(q_1, p_1, q_2, p_2).
\] (32)

The unperturbed equation is
\[
H_0 \ast \psi_{n_1, n_2}(q_1, p_1, q_2, p_2) = k_n(q_1, p_1) \psi_{n_1, n_2}(q_1, p_1, q_2, p_2),
\] (33)

The unperturbed part, \(\hat{H}_0\), has solutions given by
\[
\psi_{n_1, n_2}(q_1, p_1, q_2, p_2) = \phi_{n_1}(q_1, p_1) \Gamma_{n_2}(q_2, p_2),
\] (34)

where \(\phi_{n_1}(q_1, p_1)\) and \(\Gamma_{n_2}(q_2, p_2)\) are solutions. The eigenvalue equations are given by
\[
\hat{a} \phi_{n_1} = \sqrt{n_1} \phi_{n_1-1},
\] (35)

\[
\hat{a}^\dagger \phi_{n_1} = \sqrt{n_1 + 1} \phi_{n_1+1},
\] (36)

\[
\hat{b} \Gamma_{n_2} = \sqrt{n_2} \Gamma_{n_2-1},
\] (37)

\[
\hat{b}^\dagger \Gamma_{n_2} = \sqrt{n_2 + 1} \Gamma_{n_2+1}.
\] (38)

Using the relations \(\hat{a} \phi_0 = 0\) and \(\hat{b} \Gamma_0 = 0\), the ground-state solution is,
\[
\psi_{0,0}(q_1, p_1, q_2, p_2) = \mathcal{N} e^{-(Wq_1^2 + p_1^2)} L_{n_1}(Wq_1^2 + p_1^2) e^{-(Wq_2^2 + p_2^2)} L_{n_2}(Wq_2^2 + p_2^2),
\] (39)

where \(L_{n_1}\) and \(L_{n_2}\) are Laguerre polynomials; and \(\mathcal{N}\) is a normalization constant. The eigenvalues solutions, given in Eq. (33), are
\[
k_{n_1, n_2}^{(0)} = (n_1 + n_2 + 1)W.
\] (40)

The excited states are obtained from Eq. (39) using operators given in Eqs. (35-37).

Then, the solution for the first order perturbed Hamiltonian is given by
\[
\psi_{n_1, n_2}^{(1)}(q_1, p_1, q_2, p_2) = \psi_{n_1, n_2}^{(0)}(q_1, p_1, q_2, p_2)
+ \sum_{m_1 \neq n_1; m_2 \neq n_2} \left( \int \psi_{m_1, m_2}^{*}(q_1, p_1, q_2, p_2) \hat{H}_1 \psi_{m_1, m_2}^{(0)}(q_1, p_1, q_2, p_2) dq_1 dp_1 dq_2 dp_2 \right)
\times \psi_{m_1, m_2}^{(0)}(q_1, p_1, q_2, p_2).
\] (41)
It is to be noted that the following integral needs to be solved

\[
I = \int \psi^{(0)}_{m_1,m_2}(q_1,p_1,q_2,p_2) \frac{B^2}{8} [((\hat{a} + \hat{a}^\dagger)^2 + (\hat{b} + \hat{b}^\dagger)^2)]^3 \psi^{(0)}_{n_1,n_2}(q_1,p_1,q_2,p_2) dq_1 dp_1 dq_2 dp_2 ,
\]

before a solution for Eq.(41). Using the orthogonality relations

\[
\int \phi^{\ast}_n(q_1,p_1) \phi_m(q_1,p_1) dq_1 dp_1 = \delta_{n,m},
\]

\[
\int \Gamma^{\ast}_n(q_2,p_2) \Gamma_m(q_2,p_2) dq_2 dp_2 = \delta_{n,m},
\]

the ground-state is

\[
\psi^{(1)}_{0,0}(q_1,p_1,q_2,p_2) = \psi^{(0)}_{0,0}(q_1,p_1,q_2,p_2) + \frac{B^2}{8W} \left[ (-21\sqrt{2} - 18 - 25\sqrt{10}) \psi^{(0)}_{2,0} + \left( -3\sqrt{2} - 3 \right) \psi^{(0)}_{2,2} + (-30\sqrt{21} - 3\sqrt{6}) \psi^{(0)}_{4,0} + 4\sqrt{3} \psi^{(0)}_{4,2} - 8\sqrt{115} \psi^{(0)}_{6,0} \right].
\]

And for excited states, the wave functions are

\[
\psi^{(1)}_{1,0} = \psi^{(0)}_{1,0} + \frac{B^2}{8W} \left[ 89, 30 \psi^{(0)}_{1,2} + 13, 47 \psi^{(0)}_{1,4} + 6, 32 \psi^{(0)}_{1,6} - 89, 43 \psi^{(0)}_{3,0} - 19, 33 \psi^{(0)}_{3,2} - 23, 51 \psi^{(0)}_{5,0} - 10, 31 \psi^{(0)}_{5,4} - 11, 83 \psi^{(0)}_{7,0} \right],
\]

and

\[
\psi^{(1)}_{0,1} = \psi^{(0)}_{0,1} + \frac{B^2}{8W} \left[ -89, 30 \psi^{(0)}_{2,1} - 13, 47 \psi^{(0)}_{2,3} - 6, 32 \psi^{(0)}_{6,1} + 89, 43 \psi^{(0)}_{0,3} + 19, 33 \psi^{(0)}_{2,3} + 23, 51 \psi^{(0)}_{5,0} + 10, 31 \psi^{(0)}_{5,4} + 11, 83 \psi^{(0)}_{7,0} \right].
\]

The Wigner function for the hydrogen atom in a constant magnetic field is given by

\[
f_W(q_1,p_1,q_2,p_2) = \psi^{(1)}_{n_1,n_2}(q_1,p_1,q_2,p_2) \ast \psi^{(1)}_{n_1,n_2}(q_1,p_1,q_2,p_2).
\]

It should be noted that all plots consider \(q_2 = p_2 = 1\) in order to show a 3D figure, thus \(q_1 = q\) and \(p_1 = p\). In Fig.\([1]\) and Fig.\([2]\), the behavior of Wigner function is presented for order zero with magnetic field assuming values \(B = 1\) and \(B = 0.1\), respectively with \(E = 1\). In Fig.\([3]\) and Fig.\([4]\), the behavior of Wigner function is shown for first order with magnetic field assuming values \(B = 1\) and \(B = 0.1\), respectively, with \(E = 10\).
Comparing the graphics given in Figs. (1)-(4), the negative part of the Wigner function increases with larger values of energy and magnetic field.

Wigner function to first order for magnetic field values $B = 1$ and $B = 0.1$ is shown in Figs. (5) and Fig.(6) for $E = 1$. The behavior of Wigner function to first order with magnetic field value $B = 0.5$ is shown in Figs. (7) and Fig.(8), for $E = 1$ and $E = 10$, respectively.

The correction of first order of energy value of Eq. (32) is given by

$$
\delta k_{n_1,n_2}^{(1)} = \int \psi_{n_1,n_2}^{(0)} \hat{H}_1 \psi_{n_1,n_2}^{(0)*} dq_1 dp_1 dq_2 dp_2.
$$

Performing calculations for $\hat{H}_1$ leads to
\[ \kappa_{n_1,n_2}^{(1)} = (n_1 + n_2 + 1)W + \frac{B^2}{8}\delta, \]  

(50)
FIG. 5: Wigner function first order - Zeeman effect, $E = 1$ and $B = 1$

FIG. 6: Wigner function first order - Zeeman effect, $E = 1$ and $B = 0.1$
FIG. 7: Wigner function first order - Zeeman effect, $E = 1$ and $B = 0.5$

FIG. 8: Wigner function first order - Zeeman effect, $E = 10$ and $B = 0.5$
where

\[
\delta = (n_1 + 1)(n_1 + 2)(n_1 + 3) \\
+ (n_1 + 1)(n_1 + 2)^2 + (n_1 - 1)n_1(n_1 + 1)^2 \\
+ (n_1 + 1)n_1(n_1 + 1) + \sqrt{n_1^3(n_1 + 1)^3} \\
+ (n_1 + 1)n_1^2 + (n_1 + 1)n_1(n_1 - 1) \\
+ n_1(n_1 - 1)^2 + n_1(n_1 - 1)(n_1 - 2) \\
+ 3(n_1 + 1)n_1(n_2 + 1) + 3(n_1 + 1)n_1n_2 \\
+ 3n_1^2(n_2 + 1) + 3n_1^2n_2 + 3(n_1 - 1)n_1(n_2 + 1) \\
+ 3(n_1 - 1)n_1n_2 + 3(n_1 + 1)n_2(n_1 + 1) \\
+ 3(n_2 + 1)n_2n_1 + 3n_2^2(n_1 + 1) + 3n_2^2n_1 \\
+ 3(n_2 - 1)n_2(n_1 + 1) + 3(n_2 - 1)n_2n_1 \\
+ (n_2 + 1)(n_2 + 2)(n_2 + 3) + (n_2 + 1)(n_2 + 2)^2 \\
+ (n_2 - 1)n_2(n_2 + 1)^2 + \sqrt{n_2^3(n_2 + 1)^3} \\
+ \sqrt{(n_2 + 1)^2n_2^4 + (n_2 + 1)n_2(n_2 - 1)} \\
+ n_2(n_2 - 1)^2 + n_2(n_2 - 1)(n_2 - 2).
\]

Then the eigenvalue is

\[
W = k_{n_1,n_2}^{(1)} - \frac{(B^2/8)\delta}{n_1 + n_2 + 1}.
\]

With the relation \(W^2/2 = -E\), the eigenvalues associated to the Zeeman effect in phase space are given by

\[
E_{n_1,n_2} = -\frac{1}{2} \left[ \frac{k_{n_1,n_2}^{(1)} - (B^2/8)\delta}{n_1 + n_2 + 1} \right]^2,
\]

and

\[
E_N = -\frac{1}{2} \left[ \frac{k_{n_1,n_2}^{(1)} - (B^2/8)\delta}{N} \right]^2,
\]

where \(N = n_1 + n_2 + 1\). Note that if \(B \to 0\), the known results are obtained [58].

Using the Wigner function, the negative parameter for the system is calculated. The results are presented in Tables (I-II). It is to be noted that when the magnetic field increases the negativity parameter also increases. In addition, for a given value of the magnetic field, the negativity
TABLE I: Negativity parameter, $B = 1$.

| $n_1, n_2$ | $\eta(\psi)$ |
|------------|---------------|
| 0, 0       | 0.14345       |
| 0, 1       | 0.32645       |
| 1, 0       | 0.32645       |
| 1, 1       | 0.45786       |
| 2, 0       | 0.45786       |
| 0, 2       | 0.45786       |

TABLE II: Negativity parameter, $B = 0.1$.

| $n_1, n_2$ | $\eta(\psi)$ |
|------------|---------------|
| 0, 0       | 0.0034        |
| 1, 0       | 0.0562        |
| 0, 1       | 0.0562        |
| 1, 1       | 0.0635        |
| 2, 0       | 0.0635        |
| 0, 2       | 0.0635        |

parameter increases when the sum $n_1 + n_2$ increases. This result is presented in the graphics above.

V. CONCLUDING REMARKS

The Zeeman effect in phase space for Schrödinger equation, which endows the Galilean symmetry, is analyzed. The Wigner function is calculated numerically and presented in the panels for several parameters. Such a function has a clear interpretation in the classical limit and can be projected in the momenta or coordinate space for experimental purpose. The modulus of the Wigner function is also finite that allows a calculation its negativity. The results are presented in Tables I-II. It indicates a direct relation between the magnetic field and the discrete parameter $N = n_1 + n_2$. The increase of the magnetic field is related to the departure from the classical behavior since the negativity parameter increases accordingly.

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Data Availability

No experimental data were used in this article.

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