Adaptive Fuzzy Control for Nontriangular Stochastic High-Order Nonlinear Systems Subject to Asymmetric Output Constraints

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Abstract—In this article, an adaptive fuzzy control design strategy is presented for $p$-norm nontriangular stochastic high-order nonlinear systems with asymmetric output constraints and unknown nonlinearities. To prevent the violation of the asymmetric output constraint, a novel barrier Lyapunov function (BLF) is constructed. Then, combining the constructed BLF with adding a power integrator approach, the adaptive fuzzy control algorithm is developed by the backstepping technique. Simultaneously, the rigorous proof displays that the designed controller can ensure that all variables of the closed-loop system are bounded in probability with the achievement of the output constraint. Eventually, the theoretical result is further demonstrated via the simulation results.

Index Terms—Adaptive fuzzy control, adding a power integrator, asymmetric output constraints, barrier Lyapunov function (BLF), nontriangular stochastic nonlinear systems.

I. INTRODUCTION

FOR MANY of the past years, the control problem of stochastic nonlinear systems has always been of concern in control theory. In order to reach the control objectives of various stochastic systems, a lot of studies have been carried out, including backstepping [1]–[3]; adaptive control [4]; $\mathcal{H}_\infty$ control [5], [6]; adaptive filter algorithm [7], [8]; sliding-mode control [9], [10]; etc. Recently, the research of the stochastic systems with known or linearly parameterized nonlinearities has been extended to the stochastic nonlinear systems with completely unknown nonlinear functions. Applying the fuzzy-logic systems (FLSs) or neural networks (NNs), many control design schemes have been developed for different kinds of uncertain deterministic or/and stochastic nonlinear systems [11]–[18]. Specifically, using FLSs or NNs, references [11]–[14] have studied strict-feedback nonlinear systems possessing different characteristics, and Cheng et al. [15] and Niu et al. [16] have considered switched nonlinear systems. Meanwhile, the multi-input–multi-output (MIMO) stochastic nonlinear systems have also been addressed in [17] and [18].

It should be noted that most of the aforementioned systems have the triangular structure, which is a special class of nonlinear systems. In practice, however, the nontriangular system with a more general form widely exists. Due to the different structure from that of triangular systems, the control design of nontriangular systems will face more difficulties and cannot be solved by the approaches proposed for triangular systems. To overcome this obstacle, Huang and Xiang [19] and Wang et al. [20] have developed the fuzzy control design schemes by a variables separation technique which relies on the condition—the monotonously increasing property of bounding functions.

Later, Sui et al. [21] and Sun et al. [22] have removed the restriction condition and modified the above approaches by fully utilizing the property of the fuzzy basis function. However, the nontriangular nonlinear systems considered in [19]–[22] are all the systems with the fractional powers being equal to one, rather than $p$-norm nonlinear systems in which at least one of the fractional powers is greater than one.

In this article, we consider a class of stochastic high-order nonlinear systems with nontriangular structure

$$
\begin{align*}
\dot{x}_1 &= x_1^{q_1} dt + h_1(x) dt + \psi_1^T(x) d\omega \\
\dot{x}_2 &= x_2^{q_2} dt + h_2(x) dt + \psi_2^T(x) d\omega \\
&
\vdots
\end{align*}
$$

$$
y = x_1
$$

where $x = (x_1, \ldots, x_p)^T \in \mathbb{R}^p$, $u \in \mathbb{R}$, and $y \in \mathbb{R}$ are the system state vector, control input, and output, respectively, the fractional powers $q_i$’s satisfy $q_i \in R_{\text{odd}}^{\geq 1} \triangleq \{r | r \geq 1 \text{ is positive odd integers ratio}\}$, $\omega$ is an $N$-dimensional standard Wiener process, and $h_i : \mathbb{R} \to \mathbb{R}$ and $\psi_i : \mathbb{R} \to \mathbb{R}^N (i = 1, \ldots, n)$ are...
unknown smooth nonlinear functions satisfying $h_i(0) = 0$ and $\psi_i(0) = 0$. The system output is required to be kept in the following set:

$$\Gamma_1 = \{y(t) \in \mathbb{R} | -\varepsilon_1 < y(t) < \varepsilon_2\}$$

where $\varepsilon_1$ and $\varepsilon_2$ are known constants. System (1) is called $p$-norm stochastic nonlinear systems if there exists at least one $q_i > 1$.

In recent years, the $p$-norm nonlinear system has drawn growing attention since it possesses the inherent nonlinear property and can model many real systems, such as the under-actuated and weakly coupled mechanical system [23], and the coupled inverted double pendulums [24]. As is well known, the stability analysis and controller design have been well investigated for $p$-norm stochastic systems with known or linearly parameterized nonlinearities (see [23]–[25]). Nevertheless, there are very few related works [26]–[28] on the $p$-norm stochastic system with unknown nonlinearities. Specifically, Si et al. [26] and Duan and Min [27] have addressed the decentralized stability for large-scale $p$-norm stochastic nonlinear systems by utilizing the NNs, while Zhao et al. [28] have investigated the neural tracking control for switched $p$-norm stochastic nonlinear systems. It is noteworthy that the methods considered in [26]–[28] are all in lower triangular structural form, while the considered system (1) in this article is in nontriangular structural form.

Another noteworthy aspect is that the above-mentioned NNs control schemes for $p$-norm stochastic systems did not take output constraints into account. In reality, many systems may be affected by output constraints. Violating the constraint may result in system performance deterioration or even unexpected damage. For instance, the bending and torsion deformations, which are the outputs of the flapping-wing robotic aircraft, should be constrained in the prespecified ranges to avert the structural damage [29], [30]. For this reason, the research on output constraint has become an interesting topic. Particularly, the methods based on barrier Lyapunov functions (BLFs) have been proposed for deterministic and stochastic nonlinear systems [31], [32]. Then, many BLF-based schemes have been developed for various nonlinear systems with nonlinearities satisfying some growth conditions (see [33]–[35] and the references therein). On the basis, Niu et al. [36] and Yin et al. [37] have addressed the adaptive fuzzy control problem for stochastic nonlinear-switched systems with unknown nonlinearities and output constraints. Liu et al. [38] have further considered the fuzzy fault-tolerant control for the same stochastic system in [36]. Subsequently, Li and Tong [39] and Ma et al. [40] have investigated the adaptive fuzzy and neural-constrained control for MIMO nonstrict-feedback nonlinear systems, and a filter-based neural-constrained control strategy has been developed in [41]. It is worth noting that the systems considered in [36]–[41] are all strict-feedback stochastic systems (i.e., $q_i \equiv 1$), rather than $p$-norm systems.

More recently, Fang et al. [42], [43] have investigated the controller design and finite-time stability for $p$-norm stochastic nonlinear systems with output constraints. Furthermore, the full-state-constrained fuzzy control has been addressed, respectively, for $p$-norm deterministic and stochastic nonlinear systems in [44] and [45], while Chen and Sun [46] and Fang et al. [47] have studied the control problem in the situation of asymmetric output constraints. Observing the above results for $p$-norm nonlinear systems, two issues can be summarized in the following. One is that the asymmetric output constraint has only been considered for the system with known nonlinearities (e.g., [46] and [47]). Another is that most of the $p$-norm-constrained systems are with the lower triangular structure (see [42]–[44]). In other words, the control problem of system (1) has not been solved. Besides, it can be seen that all the above-mentioned approaches are not applicable for system (1) since the constructed BLFs in [42]–[47] are restricted by the intrinsic structures.

Inspired by the above discussions, we will investigate the adaptive control problem for system (1). A novel asymmetric BLF will be proposed to handle the output constraint, and subsequently, an adaptive fuzzy control scheme will be developed. The contributions of this article mainly contain the following two aspects.

1) An asymmetric BLF is first presented to deal with the asymmetric output constraint for $p$-norm stochastic systems. Although [36]–[47] are also concerned with output constraints, these BLFs used in above-related references cannot be directly applied to system (1) due to some inherent limitations (see Remark 3). Different from these existing BLFs, the proposed BLF is an asymmetric function only depending on the fractional powers $q_i$’s, which enables the proposed BLF to be applicable for system (1). What is more, the proposed BLF is still effective even if the output constraint vanishes.

2) Combining the proposed asymmetric BLF with adding a power integrator, an adaptive fuzzy control design strategy is developed. It should be pointed out that the system nonlinearities are in nontriangular structural forms and completely unknown without any assumptions. The FLSs are adopted to cope with the unknown functions, which makes the design of the virtual and real controllers become easy. More important, the designed controller can ensure not only that all variables are bounded in probability but also that the system output remains in a given set in the sense of probability.

II. PROBLEM AND PRELIMINARIES

A. Problem Statement

This article aims to construct an adaptive fuzzy controller $u$ for system (1), which can guarantee that not only all variables of the closed-loop system are bounded in probability but also the output constraint is not violated. To achieve this goal, some concepts and lemmas are given in this section.

Consider the stochastic system as follows:

$$dx = h(x)dt + \psi(x)d\omega$$

where $x$ and $\omega$ are defined the same as that in system (1), and $h(\cdot)$ and $\psi(\cdot)$ are locally Lipschitz functions with $h(0) = \psi(0) = 0$. 

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Definition 1 [2]: For any given \( V(x) \in C^2 \) related to system (2), the differential operator \( \mathcal{L} \) is defined as follows:

\[
\mathcal{L}V = \frac{\partial V}{\partial x} h(x) + \frac{1}{2} tr \left\{ \psi^T(x) \frac{\partial^2 V}{\partial x^2} \psi(x) \right\}.
\]

Definition 2 [31]: Suppose that \( \Pi \) is an open set containing the origin and \( V : \Pi \rightarrow \mathbb{R} \) is positive definite and continuously differentiable. Then, for system \( \dot{x} = g(x), V(x(t)) \) is called a BLF if for each solution \( x(t) \) starting from \( x(t_0) \in \Pi, V(x(t)) \rightarrow \infty \) as \( t \rightarrow t_0 \) and for some \( \tau \in \mathbb{R}^+ \).

Lemma 1 [18]: For a given constant \( \nu \geq 1 \) and any \( s_1, s_2 \in R \), one has

\[
|s_1 - s_2| \leq (2^{\nu - 2} + 2)|s_1 - s_2| |s_1 - s_2|^{\nu - 1} + |s_1 - s_2|^\nu.
\]

Lemma 3 [30]: Let \( v_1, v_2, c, \) and \( \varphi \) be positive real numbers. For any \( s_1, s_2 \in R \), the following inequality holds:

\[
\varphi|s_1|^v|s_2|^\nu \leq \frac{\varphi v_1}{v_1 + v_2} |s_1|^{v_1 + v_2} + \frac{\varphi v_2}{v_1 + v_2} |s_2|^{v_1 + v_2}.
\]

Lemma 4 [13]: With any \( a_0 > 0, c_0 > 0, \) and \( \chi(t) > 0, \) for \( \dot{\xi}(t) = a_0 \chi(t) - c_0 \xi(t), \) if \( \xi(0) \geq 0 \) can be satisfied, then \( \xi(t) \geq 0 \) for all \( t \geq 0. \)

B. Fuzzy-Logic Systems

In this article, the nonlinearities of system (1) are all unknown and will be approximated by the FLSs. What follows is a brief introduction of the FLS. The knowledge base for FLS includes a great quantity of fuzzy rules in IF-THEN form which can generally be expressed as follows:

IF \( x_1 \) is \( X_1^k \) and \( x_2 \) is \( X_2^k \) and \( \cdots \) and \( x_n \) is \( X_n^k \)

THEN \( y \) is \( D^k, k = 1, 2, \ldots, m \)

where \( x = (x_1, x_2, \ldots, x_n)^T \in X \subset \mathbb{R}^n \) and \( y \in D \subset \mathbb{R} \) are, respectively, the input and output of FLS, \( X_j^k (j = 1, 2, \ldots, n) \) and \( D^k \) are fuzzy sets in \( X \) and \( D \), respectively, and \( m \) is the number of fuzzy rules.

In terms of [48], the output can be written as

\[
y(x) = \frac{\sum_{k=1}^{m} \lambda_k \left( \prod_{j=1}^{n} \psi_j^k(x_j) \right)}{\sum_{k=1}^{m} \left( \prod_{j=1}^{n} \psi_j^k(x_j) \right)} \tag{4}
\]

where \( \psi_j^k(x_j) \) represents the membership function of \( x_j \) to \( X_j^k \), and \( \lambda_k \) is the point at which the membership function \( \Psi_j^k(\lambda_k) \) reaches its maximum value. For \( k = 1, 2, \ldots, m \), the fuzzy basis functions are defined as

\[
\phi_k(x) = \frac{\prod_{j=1}^{n} \psi_j^0(x_j)}{\sum_{k=1}^{m} \left( \prod_{j=1}^{n} \psi_j^k(x_j) \right)}
\]

where the membership function \( \psi_j^0(x_j) \) is usually selected as a Gaussian function, that is

\[
\psi_j^0(x_j) = e^{-\left(\frac{|x_j - \mu_j|^2}{2\sigma_j^2}\right)}
\]

with \( \mu_j \) and \( \sigma_j \) being the center vector placed on a regular lattice and the standard deviation of the Gaussian function, respectively.

Letting

\[
\Phi(x) = (\phi_1(x), \ldots, \phi_m(x))^T, \Lambda = (\lambda_1, \ldots, \lambda_m)^T
\]

then the FLS formula (4) can be rewritten as

\[
y(x) = \Lambda^T \Phi(x). \tag{5}\]

Lemma 5 [48]: Let \( G(x) \) be a continuous function defined on a compact set \( \Gamma \). Then, for any given constant \( \theta > 0 \), there exists an FLS (5) such that

\[
\sup_{x \in \Gamma} |G(x) - \Lambda^T \Phi(x)| \leq \theta.
\]

Remark 1: According to Lemma 5, it is obvious that any continuous function \( G(x) \) can be approximated by an FLS, that is

\[
G(x) = \Lambda^T \Phi(x) + \mu(x)
\]

where \( \mu(x) \) is the FLS approximation error with the bound \( \theta > 0. \)

Remark 2: In the FLS, the Gaussian membership functions are determined by two parameters, that is, the central point and the standard deviation, which are usually chosen according to the following criteria.

1) The central points are usually distributed uniformly over the universe of discourse.
2) The standard deviation affects the sensitivity of a controller, that is, a lower standard deviation results in the higher sensitivity of the controller. In practical application, it is often chosen as \( \sqrt{2} \) in a membership function.

III. MAIN RESULTS

In this section, we will construct a new asymmetric BLF and clearly display the controller design for system (1).

A. New Asymmetric BLF

First, the following transformation is chosen for the state variables of system (1):

\[
\eta_1 = x_1, \quad \eta_i = x_i - \gamma_{i-1}, i = 2, \ldots, n \tag{6}
\]

where \( \gamma_i \in \mathbb{R}, i = 1, \ldots, n, \) are the virtual controllers which will be given in the design steps.
Based on the above transformation, a new asymmetric BLF is constructed to deal with the asymmetric output constraint as follows:

$$V_B(\eta_1) = \frac{(\varepsilon_1 \cdot \varepsilon_2)^{q-q_1+4} \eta_1^{q-q_1+4}}{(q - q_1 + 4)\varepsilon_2 - \eta_1(\varepsilon_1 + \eta_1)^{q-q_1+4}}$$

where $q = \max_{1 \leq i \leq 2}(q_i)$, $\eta_1 = x_1 \in \Gamma_1$, and $\varepsilon_1$ and $\varepsilon_2$ are the given constants. It follows from direct calculation that:

$$\frac{\partial V_B}{\partial \eta_1} = H_1(\eta_1) \eta_1^{q-q_1+3}$$

$$\frac{\partial^2 V_B}{\partial \eta_1^2} = (q - q_1 + 3)H_1(\eta_1) \eta_1^{q-q_1+2} + \frac{2}{\eta_1^2 + \varepsilon_1^2} + \frac{(q - q_1 + 5)(2\eta_1 - \varepsilon_2 + \varepsilon_1)}{(\varepsilon_2 - \eta_1)(\varepsilon_1 + \eta_1)}$$

where

$$H_1(\eta_1) = \frac{\varepsilon_1 \cdot \varepsilon_2}{(\varepsilon_2 - \eta_1)(\varepsilon_1 + \eta_1)^{q-q_1+5}}$$

is a positive smooth function on $\Gamma_1$.

Since $q - q_1 + 4 \geq 4$ and the numerator of $q - q_1 + 4$ is even, it is clear that $V_B(\eta_1)$ is $C^2$ and positive definite on $\Gamma_1$. Meanwhile, it is easily obtained that $V_B(\eta_1) \to \infty$ as $x_1 \to -\varepsilon_1$ or $x_1 \to \varepsilon_2$ by the definition of $V_B(\eta_1)$. Thus, $V_B(\eta_1)$ is an asymmetric BLF for system (1).

Remark 3: It is worth noting that $V_B(\eta_1)$ is first proposed and differs from those existing tan-type and log-type BLFs in [36]–[41]. As a matter of fact, these BLFs adopted in [36]–[41] are all in fixed forms and independent from the fractional powers $q_i$'s. Therefore, these BLFs can only be employed to the strict-feedback stochastic systems (i.e., $q_i \equiv 1$).

On the other hand, although some kinds of tan-type and log-type BLFs have been proposed for $p$-norm nonlinear systems in the references [42]–[47], limited by their intrinsic structures, these BLFs in [42]–[45] can only handle symmetric output constraints. To handle the asymmetric constraint by using these BLFs, a natural way as stated in [46] is to transmute the asymmetric constraint into the symmetric one. Nevertheless, such a transmutation may cause the proposed schemes in [42]–[45] no longer be effective for the stabilization problem. In addition, based on the nonlinear growth condition, Chen and Sun [46] and Fang et al. [47] have proposed two asymmetric BLFs. Unlike the nonlinearities of system (1) are totally unknown, which implies that the BLFs proposed in [46] and [47] are also not applicable for system (1). On the contrary, $V_B(\eta_1)$ is an asymmetric function only depending on the fractional powers $q_i$'s, which guarantees that $V_B(\eta_1)$ could be employed to deal with the asymmetric constraint in system (1).

Remark 4: If the considered output constraint vanishes, that is, $\varepsilon_1 = \varepsilon_2 \to \infty$, one obtains

$$\lim_{\varepsilon_1 \to \infty, \varepsilon_2 \to \infty} V_B(\eta_1) = \frac{q - q_1 + 4}{\eta_1^{q-q_1+4}} \triangleq \bar{V}_B.$$

As we know, the function $\bar{V}_B$ is always applied to address the adaptive fuzzy control for $p$-norm stochastic nonlinear systems without any constraints (see [26]–[28]). In other words, the proposed strategy based on the new $V_B(\eta_1)$ could simultaneously solve the problem of the adaptive fuzzy control for system (1) with and without output constraints avoiding changing the controller structure.

B. Fuzzy Control Design and Stability Analysis

In the following, the design of a smooth state-feedback controller will be explicitly shown for system (1) and rigorous stability will be strictly proved.

For simplicity, the following notations are given:

$$\vec{x}_i = (x_1, \ldots, x_i)^T; \Gamma_n = \{x \in \mathbb{R}^n | -\varepsilon_1 < x_i < \varepsilon_2\}$$

$$Q_i = \left(\begin{array}{cc} q_i - 2 & 2q_i \\ 2 & q_i \end{array} \right), i = 1, \ldots, n$$

$$\rho_j = \frac{q - q_j + 3}{q + 3} \left[Q_j(\varepsilon_1)^{q_j} \eta_1^{q_j-1} \varepsilon_1 \eta_1 \delta_j \eta_1 + \frac{1}{\delta_j} \right]$$

where $\delta_j > 0$ and $f_j(.) > 0$ are, respectively, the parameters and positive smooth gain functions required to be designed later.

Next, the Lyapunov function and the adaptive controller will be constructed step by step.

Step 1: Based on (6), one obtains

$$d\eta_1 = dx_1 = (x_1^{(2)} + h_1(x))dt + \psi_1^T(x)dw.$$ (10)

Choosing the following Lyapunov function:

$$V_1 = V_B + \frac{\bar{\beta}_1^2}{2a_1}$$ (11)

where $a_1 > 0$ is a design parameter, $\bar{\beta}_1 = \beta_1 - \bar{\beta}_1$ is the parameter error, and $\bar{\beta}_1$ is the estimation of the unknown parameter $\beta_1$, whose definition will be given later.

In light of (3), (8), and (9), it can be obtained that

$$L V_1 = \frac{\partial V_B}{\partial \eta_1} \left(\begin{array}{c} \varepsilon_1 \varepsilon_2 + \eta_1 \varepsilon_2 + h_1(x) \end{array} \right) + \frac{1}{2} \frac{\partial^2 V_B}{\partial \eta_1^2} \psi_1^T(x) \psi_1(x) - \frac{1}{a_1} \bar{\beta}_1^2$$

$$= H_1(\eta_1) \eta_1^{q-q_1+3} \left(\begin{array}{c} \varepsilon_1 \varepsilon_2 + \eta_1 \varepsilon_2 + h_1(x) \end{array} \right) - \frac{1}{a_1} \bar{\beta}_1^2$$

$$+ \frac{q - q_1 + 3}{2} H_1(\eta_1) \eta_1^{q-q_1+2} \|\psi_1(x)\|^2$$

$$+ H_1(\eta_1) \eta_1^{q-q_1+3} \|\psi_1(x)\|^2$$

$$+ H_1(\eta_1) \eta_1^{q-q_1+3} \left(\begin{array}{c} q - q_1 + 5(2\eta_1 - \varepsilon_2 + \varepsilon_1) \\
(2\varepsilon_1 \varepsilon_2 - \eta_1 \varepsilon_1) \eta_1 \varepsilon_1 \end{array} \right) \|\psi_1(x)\|^2.$$

Following from the complete squares formula gives:

$$q - q_1 + 3 \leq \frac{(q - q_1 + 3)^2}{8} H_1^2(\eta_1) \|\psi_1(x)\|^4 \eta_1^{q-2} + \frac{1}{2}.$$ (12)
Then, one obtains

$$\mathcal{L}V_1 \leq H_1(\eta_1)\eta_1^{\frac{q}{q-3}}q_1^{\frac{1}{q-3}}G_1(\xi_1) - c_1V_B - \rho_1\eta_1 + \frac{1}{2}\beta_1\dot{\beta}_1 + \frac{1}{2}$$

(13)

where $\xi_1 = x$ and

$$G_1(\xi_1) = h_1(x) + \frac{(q - q_1 + 3)^2}{8}H_1(\eta_1)\|\phi_1(x)\|^2 + \frac{1}{\eta_1^2 + \epsilon_1^2} + \frac{q_1}{q - q_1 + 3}\left(\|H_1(\eta_1)\| + \beta_1\right)\|\phi_1(x)\| + 1.$$  

Note that $h_1(x)$ and $\phi_1(x)$ are smooth functions. This, together with the definition and property of $H_1(\eta_1)$, infers that the unknown nonlinear function $G_1(\xi_1)$ is at least continuous on $\Gamma_n$. According to Lemma 5, an FLS $\Lambda_1^T \Phi_1(\xi_1)$ can be used to approximate $G_1(\xi_1)$, that is

$$G_1(\xi_1) = \Lambda_1^T \Phi_1(\xi_1) + \mu_1(\xi_1)$$

where $\Phi_1(\xi_1)$ and $\Lambda_1$ are, respectively, the vector of fuzzy membership functions and the corresponding optimal parameter vector, $|\mu_1(\xi_1)| \leq \theta_1$ is the FLS approximation error, and $\theta_1 > 0$ is a given constant.

Then, using Lemma 3 and the property $0 < \|\Phi_1(\xi_1)\|^2 = \Phi_1^T(\xi_1)\Phi_1(\xi_1) \leq 1$, one can obtain

$$H_1(\eta_1)\eta_1^{\frac{q}{q-3}}G_1(\xi_1) = H_1(\eta_1)\eta_1^{\frac{q}{q-3}}[\Lambda_1^T \Phi_1(\xi_1) + \mu_1(\xi_1)]$$

$$\leq H_1(\eta_1)\eta_1^{\frac{q}{q-3}}[\Lambda_1^{\frac{q}{q-3}}\Phi_1(\xi_1) + \theta_1]$$

$$\leq \frac{q - q_1 + 3}{q - 3}\|\Phi_1(\xi_1)\|_1 \|\Lambda_1\|_1 + \frac{q_1}{q - q_1 + 3}\theta_1$$

$$+ \frac{q - q_1 + 3}{q - 3}H_1(\eta_1)\|\phi_1(x)\|^2 + \frac{q}{q - 3}\|H_1(\eta_1)\| + \frac{q_1}{q - q_1 + 3}\theta_1$$

(14)

where $\beta_1 = \|\Lambda_1\|_1\|\phi_1(x)\|^2$ and $\delta_11 > 0$ is a design parameter.

Substituting (14) into (13) yields

$$\mathcal{L}V_1 \leq H_1(\eta_1)\eta_1^{\frac{q}{q-3}}q_1^{\frac{1}{q-3}}G_1(\xi_1) - c_1V_B - \rho_1\eta_1 + \frac{1}{2}\beta_1\dot{\beta}_1 + \frac{1}{2}$$

$$+ \frac{q - q_1 + 3}{q - 3}H_1(\eta_1)\|\phi_1(x)\|^2 + \frac{q_1}{q - q_1 + 3}\|H_1(\eta_1)\| + \frac{1}{2}\beta_1\dot{\beta}_1$$

$$+ \beta_1\left[\frac{q - q_1 + 3}{q - 3}H_1(\eta_1)\|\phi_1(x)\|^2 + \frac{q_1}{q - q_1 + 3}\|H_1(\eta_1)\| + \frac{1}{2}\beta_1\dot{\beta}_1\right]$$

$$- \rho_1\eta_1 + c_1V_B + \frac{q}{q - 3}\beta_1\dot{\beta}_1 + \frac{q_1}{q - q_1 + 3}\beta_1\dot{\beta}_1.$$

(15)

Let $S_1(x_1, \beta_1) \geq (q - q_1 + 3)/(q + 3)H_1(\eta_1)q_1^{(q_1)/(q - q_1 + 3)}\|\phi_1(x_1)\| + 1$ be a smooth function. Designing the virtual controller $\gamma_1$ and the adaptive law of $\gamma_1$ as

$$\gamma_1 = -\frac{1}{S_1(x_1, \beta_1)}\dot{x}_1 \eta_1 \equiv -f_1(x_1, \beta_1)\eta_1$$

(16)

and

$$\dot{\beta}_1 = \frac{q - q_1 + 3}{q + 3}\|\phi_1(x_1)\|\delta_11\eta_1 + c_1\beta_1$$

(17)

with $c_1 > 0$ being a design parameter.

Obviously, the virtual controller $\gamma_1$ is a compound function. Analyzing the property of each component in $\gamma_1$ can verify that it is smooth, which means that all of the virtual controllers to be mentioned are also smooth.

Remark 5: Applying Lemma 4 to (17), one obtains $\beta(t) \geq 0$, for $\forall t \geq 0$. In each step of the design procedure, this characteristic will be always used.

Remark 6: The gain function of controller (16) contains the term $H_1(\eta_1)$, which increases along with the state $x_1$ approaching the boundaries $x_1 = -\epsilon_1$ or $x_1 = \epsilon_2$. This property forces the output variable to stay in the constrained region.

By means of (16) and (17), (15) can be rewritten as

$$\mathcal{L}V_1 \leq -c_1V_B - \frac{c_1}{2\alpha_1}\beta_1^2 - \rho_1\eta_1^{\frac{q}{q-3}} + K_1 + H_1(\eta_1)\|\phi_1(x)\| + \frac{q_1}{q - q_1 + 3}\beta_1\dot{\beta}_1$$

Furthermore, the following inequality can be easily obtained:

$$\frac{c_1}{2\alpha_1}\beta_1 + \frac{c_1}{2\alpha_1}\beta_1 \leq \frac{c_1}{2\alpha_1}\beta_1^2 + \frac{c_1}{2\alpha_1}\beta_1^2$$

(19)

In view of (18) and (19), one obtains

$$\mathcal{L}V_1 \leq -c_1V_B - \frac{c_1}{2\alpha_1}\beta_1^2 - \rho_1\eta_1^{\frac{q}{q-3}} + K_1 + H_1(\eta_1)\|\phi_1(x)\| + \frac{q_1}{q - q_1 + 3}\beta_1\dot{\beta}_1$$

(20)

where $K_1 = (1/2) + (q_1/(q + 3))\delta_11\|\phi_1(x)\|^2 + (q_1/(q + 3))\|\phi_1(x)\|^2$. 

Remark 7: The purpose of designing the negative term $-\rho_1\eta_1^{\frac{q}{q-3}}$ in (20) is to provide a sufficient stable region to counteract the positive term containing $\eta_1$ which will appear in step 2. In the following steps, the terms $-\rho_1\eta_1^{\frac{q}{q-3}}$ will appear with the same purpose.

Step 2: From (6) and Itô's formula, one obtains

$$d\eta_2 = (\xi_2^2 + h_2(x) - \mathcal{L}\gamma_1)dt + (\psi_2(x) - \frac{\partial\gamma_1}{\partial x_1}\psi_1(x))d\omega$$

(21)

where $\mathcal{L}\gamma_1 = (\partial\gamma_1/\partial x_1)\xi_2 + h_1(x) + (\partial\gamma_1/\partial \beta_1)\dot{\beta}_1 + (1/2)(\partial^2\gamma_1/\partial x_1^2)\psi_1^2(x)$.

Combining the definition of $\gamma_1$ and the properties of $h_1(x)$ and $\psi_1(x)$, implies that $\mathcal{L}\gamma_1$ is valid and smooth.

Choosing the Lyapunov function as

$$V_2 = V_1 + U_2$$

(22)

with

$$U_2 = \frac{q_2^{q-q_2+4}}{q - q_2 + 4} + \frac{\beta_2^2}{2a_2}$$

(23)

where $a_2 > 0$ is the design parameter, $\beta_2 = \beta - \hat{\beta}$ is the parameter error, and $\hat{\beta}_2$ is the estimation of the unknown parameter $\beta_2$ whose definition will be given later.
Applying (3), (21), and (23), it is not difficult to obtain that
\[
\mathcal{L}U_2 = \eta_2^{-q-\eta_2^3} (x_1^q + h_2(x) - \mathcal{L}y_1) - \frac{1}{a_2} \tilde{\beta}_2 \bar{\beta}_2
\]
\[
+ \frac{q - q_2 + 3}{2} \eta_2^{-q-\eta_2^3} \eta_2 \left[ \frac{\partial y_1}{\partial x_1} \psi_1(x) \right]^2.
\]
Using the complete squares formula, one has
\[
\frac{q - q_2 + 3}{2} \eta_2^{-q-\eta_2^3} \left[ \frac{\partial y_1}{\partial x_1} \psi_1(x) \right]^2 \leq \frac{(q - q_2 + 3)^2}{8} \left[ \frac{\partial y_1}{\partial x_1} \psi_1(x) \right]^4 \eta_2^{-q-\eta_2^3} + \frac{1}{2}.
\]
Then, it can be deduced that
\[
\mathcal{L}U_2 \leq \eta_2^{-q-\eta_2^3} \left( x_3 - y_2^q \right)
\]
\[
+ \frac{q - q_2 + 3}{2} \eta_2^{-q-\eta_2^3} \eta_2 \left[ \frac{\partial y_1}{\partial x_1} \psi_1(x) \right]^2
\]
\[
+ \frac{1}{2} - \frac{1}{a_2} \tilde{\beta}_2 \bar{\beta}_2.
\]
where \( \bar{\beta}_2 \) is a design parameter. By means of (28) and (29),
\[
\bar{\beta}_2 \geq \eta_2^3 \left[ \frac{\delta_2{\beta}_2}{(q + 3) \| \Phi_2(\bar{x}_2, \bar{\beta}_1) \|} - \frac{1}{a_2} \right]
\]
\[
+ \frac{q}{q + 3} \delta_2{\beta}_2 + \frac{q}{q + 3} \frac{q - q_2}{a_2} + \frac{1}{2}.
\]
On the other hand, it follows from Lemmas 2 and 3 that:
\[
H_1(\eta_1 \eta_1)^q \left| q - q_1^q \right|
\]
\[
\leq H_1(\eta_1) \left| q - q_1^q \right|
\]
\[
\leq H_1(\eta_1) \left| \eta_1 \right| q - q_1^q \| Q_1 \| x_2 - y_1 \left[ |x_2 - y_1|^{q_1^q - 1} + |y_1|^{q_1^q - 1} \right]
\]
\[
\leq Q_1 H_1(\eta_1) \left[ |\eta_1| q - q_1^q \| \eta_2 \| | \eta_1 \| q - q_1^q \| \eta_1 \| q - q_1^q \| \right]
\]
\[
\leq q_1 + \frac{1}{q} + \frac{1}{3} \delta_2 \eta_2^3 + \frac{1}{q} \eta_1 \eta_1^q
\]
where \( \delta_2 \geq 0 \) is a design parameter.

Thus, one can obtain
\[
\mathcal{L}V_2 = \mathcal{L}V_1 + \mathcal{L}U_2
\]
\[
\leq -c_1 V_2 + \frac{c_1}{2 a_1} \bar{\beta}_2^2 + \frac{q}{q + 3} \delta_2 \eta_2^3 + I_1
\]
\[
+ \frac{q - q_2 + 3}{2} \eta_2^{-q-\eta_2^3} + \frac{q - q_2 + 3}{2} \eta_2^{-\eta_2^3} + \frac{1}{a_2} \tilde{\beta}_2 \bar{\beta}_2.
\]
where \( \bar{\beta}_2 \) is a design parameter. By means of (28) and (29),
\[
\bar{\beta}_2 \geq \eta_2^3 \left[ \frac{\delta_2{\beta}_2}{(q + 3) \| \Phi_2(\bar{x}_2, \bar{\beta}_1) \|} - \frac{1}{a_2} \right]
\]
\[
+ \frac{q}{q + 3} \delta_2{\beta}_2 + \frac{q}{q + 3} \frac{q - q_2}{a_2} + \frac{1}{2}.
\]
In addition, it is obvious that
\[ \frac{c_2}{a_2} \beta_2^2 \leq - \frac{c_2}{2a_2} \beta_2^2 + \frac{c_2}{2a_2} \beta_2^2. \] (31)
Substituting (31) into (30), one obtains
\[ L \delta_\zeta \leq -c_1 \beta_i \psi, \]
where
\[ K_2 = (1/2) \left( q_2/(q+3) \right) \delta_\zeta (q_2/(q+3))/q \]
and
\[ \eta_i(q_2) = (q_2/(q+3))/q \]
In particular, using (6) and Itô’s formula, one obtains
\[ d\delta_\zeta = (\beta_i - \hat{\beta}_i) \delta_\zeta dt + \eta_i \delta_\zeta \psi_T(x) \delta_\zeta \]
Choosing the Lyapunov function
\[ V_i = V_{i-1} + U_i \] (34)
with
\[ U_i = \frac{\eta_i(q_2)}{q_2} \beta_i^2 + \frac{\beta_i^2}{2a_i} \] (35)
where \( a_i > 0 \) is a design parameter, \( \beta_i = \beta_i - \hat{\beta}_i \) is the parameter error, and \( \hat{\beta}_i \) is the estimation of the unknown parameter \( \beta_i \).

Applying (3), (33), and (35), one can verify that
\[ L \delta_i \leq -c_1 \beta_i \psi_i - \eta_i \eta_i \psi_i - \frac{1}{a_i} \beta_i \]
where
\[ \beta_i = \beta_i - \hat{\beta}_i \] and \( \hat{\beta}_i \) is the estimation of the unknown parameter \( \beta_i \).

Then, one obtains
\[ L \delta_i \leq \frac{\eta_i(q_2)}{q_2} \beta_i^2 + \frac{\beta_i^2}{2a_i} \] (36)

where
\[ G_i(\theta) = h_i(x) - L \gamma_i - \frac{c_i \eta_i}{q - q_i + 4} + \rho \eta_i \eta_i \psi_i \]
with \( \gamma_i = (x^T, \beta_i^T) \) and \( \hat{\beta}_i \) is the FLS approximation error, and \( \theta_i > 0 \) is a design parameter.

Similarly, to the first two steps, it can be proved that \( G_i(\theta) \) is at least continuous on \( \Gamma_n \). Thus, \( G_i(\theta) \) can also be approximated by an FLS \( \Lambda_i \Phi_i(\theta) \) as
\[ G_i(\theta) = \Lambda_i \Phi_i(\theta) + \mu_i(\theta) \]
where \( \Phi_i(\theta) \) and \( \Lambda_i \Phi_i(\theta) \) are, respectively, the vector of fuzzy membership functions and the corresponding optimal parameter vector, and \( \mu_i(\theta) \) is the FLS approximation error, and \( \theta_i > 0 \) is a given constant. This, together with Lemma 3 and the property \( \| \Phi_i(\theta) \|^2 = \Phi_i^T(\theta) \Phi_i(\theta) \leq 1 \), yields
\[ \eta_i(q_2) \eta_i \psi_i - \frac{1}{a_i} \beta_i \]
where \( \beta_i = \| \Lambda_i \| \| \Phi_i(\theta) \| + \delta_i(\theta) \)
and \( \delta_i > 0 \) is a design parameter.

Substituting (37) into (36) obtains
\[ L \delta_i \leq \frac{\eta_i(q_2)}{q_2} \beta_i^2 + \frac{\beta_i^2}{2a_i} \] (38)

By Lemma 3, it is easily obtained that
\[ \frac{q - q_i + 3}{2} \psi_i(x) - \sum_{j=1}^{i-1} \frac{\delta_i \psi_i(x)}{a_i} \]
By Lemmas 2 and 3, one can also obtain that
\[ \eta_i(q_2) \beta_i^2 \]
where \( \delta_2 > 0 \) is a design parameter.
be a smooth function, where $\tilde{\beta}_i = (\hat{\beta}_1, \ldots, \hat{\beta}_n)^T$. Then, we design the $i$th smooth virtual controller $\gamma_i$ and the adaptive law for $\hat{\beta}_i$ as

$$
\gamma_i = -\left(S_i(\tilde{x}_i, \tilde{\beta}_i)\right)^\frac{1}{2}\eta_i \triangleq -F_i(\tilde{x}_i, \tilde{\beta}_i)\eta_i
$$

(40)

and

$$
\dot{\hat{\beta}}_i = \frac{(q - q_i + 3)\delta_{1i}a_i}{(q + 3)\|\Phi_i(\tilde{x}_i, \tilde{\beta}_i)\|} - c_i\hat{\beta}_i
$$

(41)

where $c_i > 0$ is a design parameter.

Moreover, it is not difficult to verify that

$$
\frac{c_i}{a_i}\hat{\beta}_i \leq -\frac{c_i}{2a_i}\hat{\beta}_i^2 + \frac{c_i}{2a_i}\hat{\beta}_i^2.
$$

(42)

Substituting (40) and (41) into (38), it can be inferred from (32), (39), and (42) that

$$
\mathcal{L}V_n = \mathcal{L}V_{n-1} + \mathcal{L}U_i
$$

\begin{align*}
\leq & -c_1 V_B - \sum_{j=2}^{i} c_j \eta_j^{q - q_{j+4}} - \sum_{j=1}^{i-1} \frac{c_j}{2a_j}\hat{\beta}_j^2 + c_i\hat{\beta}_i - \rho\eta_i^{q^{i+3}} + \eta_i^{q - q_{i+3}}(x_{i+1} - \eta_i^q) + \sum_{j=1}^{i-1} K_j \\
&+ \frac{q_i - q_{i+1}}{q + 3}\delta_{1i} \eta_i^{q^{i+3}} + \frac{q_i - q_{i+1}}{q + 3}\delta_{1i} \eta_i^{q^{i+3}} + \frac{1}{2} \\
&\leq -c_1 V_B - \sum_{j=2}^{i} \frac{c_j}{2a_j}\hat{\beta}_j^2 - \rho\eta_i^{q^{i+3}} + \sum_{j=1}^{i-1} K_j \\
&+ \eta_i^{q^{i+3}}(x_{i+1} - \eta_i^q)
\end{align*}

(43)

where $K_i = (1/2) + (q_i/(q_i + 3))\delta_i^{-(q_i - q_{i+3})/(q_i)} + (q_i/(q_i + 3))\delta_i^{-(q_i - q_{i+3})/(q_i)} + (c_i/2a_i)\hat{\beta}_i^2.$

Through the inductive discussion, one can obtain that (40), (41), and (43) hold for all $i = 3, \ldots, n$. This completes the inductive step.

**Step n:** According to the inductive step, it can be easily obtained that (43) holds for $i = n$ with $x_{n+1} = u$, and a series of virtual controllers and adaptive parameters (41) hold for $i = n$. Hence, the actual controller $u$ and the adaptive law of $\hat{\beta}_n$ can be constructed as

$$
u = \gamma_n = -S_n\eta_n \triangleq -F_n(x, \tilde{\beta}_n)\eta_n
$$

(44)

$$
\dot{\hat{\beta}}_n = \frac{(q - q_n + 3)\delta_{1n}a_n}{(q + 3)\|\Phi_n(x, \tilde{\beta}_{n-1})\|} \eta_n^{q^{i+3}} - c_n\hat{\beta}_n.
$$

(45)

Obviously, it can be further obtained that

$$
\mathcal{L}V_n \leq -c_1 V_B - \sum_{j=2}^{n} \frac{c_j}{2a_j}\hat{\beta}_j^2 - \sum_{j=1}^{n} K_j.
$$

Till now, the design of the adaptive fuzzy control has been completed by the backstepping technique. In what follows, a theorem will be presented to summarize our main result.

**Theorem 1:** For the $p$-norm stochastic nonlinear system (1) and any two given constants $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$, controller (44) with several parameter adaptive laws (17), (29), (41), and (45) can ensure that:

1) the given output constraint will not be violated in the sense of probability, that is, $P(-\varepsilon_1 < y(t) < \varepsilon_2) = 1$;
2) all variables in the closed-loop system will be bounded in probability.

**Proof:** For any initial state $x_0 = (x_1(0), \ldots, x_n(0))^T$ satisfying $x_1(0) \in \Gamma_1$, we have the fact that

$$
V_B(0) = V_B(\eta_1(0)) = V_B(x_1(0)) < \infty.
$$

(47)

From the definition of $V_n(t)$ and (47), one can obtain

$$
V_n(0) < \infty.
$$

Note that $V_n(t)$ is continuous and positive definite. Letting $\pi_0 = \min_{1 \leq j \leq n}(c_j)$ and $b_0 = \sum_{j=1}^{n} K_j$, it can be obtained from (46) that

$$
\mathcal{L}V_n \leq -\pi_0 V_n + b_0.
$$

(48)

According to Lemma 1, it directly follows from (48) that

$$
EV_n(t) \leq V_n(0)e^{-\pi_0 t} + \frac{b_0}{\pi_0} < \infty.
$$

(49)

Since $0 < e^{-\pi_0 t} < 1$, we have

$$
EV_n(t) \leq V_n(0) + \frac{b_0}{\pi_0} < \infty.
$$

(50)

1) From (50), it is clear that the mean of $V_n(t)$ is bounded, which implies that $V_n$ is bounded in probability. Hence, it can be further deduced from the fact $V_n = V_1 + \sum_{j=2}^{n} U_j = V_B + (1/2a_n)\hat{\beta}_n^2 + \sum_{j=2}^{n} U_j$ that

$$
P(V_B(t) < \infty) = P(V_B(\eta_1(t)) < \infty) = 1.
$$

Therefore, it is easy to verify that $P(-\varepsilon_1 < \eta_1(t) < \varepsilon_2) = 1$, which means that

$$
P(-\varepsilon_1 < y(t) < \varepsilon_2) = P(-\varepsilon_1 < x_1(t) < \varepsilon_2) = 1.
$$

In other words, the asymmetric output constraint of system (1) is not violated in the sense of probability.

2) Based on (48), one can easily derive that the sates $\eta_i(i = 1, \ldots, n)$ and $\hat{\beta}_i(i = 1, \ldots, n)$ are all bounded in probability by Lemma 1 and the definition of $V_n$. Since $\beta_i(i = 1, \ldots, n)$ are all constants and $\hat{\beta}_i = \hat{\beta}_i - \hat{\beta}_i$, it can be further obtained that $\hat{\beta}_i(i = 1, \ldots, n)$ are also bounded in probability. Meanwhile, notice that $\gamma_i$ is a continuous function of $\eta_i$ and $\hat{\beta}_i$, so it can be inferred from the property of the continuity of $\gamma_i$ that $\gamma_i$ is bounded in probability for $\forall i = 1, \ldots, n$. In addition, with $\eta_i = x_i - \gamma_i - 1$ in mind, it can be deduced that all signals $x_i, i = 1, \ldots, n$, are bounded in probability. Finally, noting that the control signal $u$ is the function of $x_1, \ldots, x_n$ and $\hat{\beta}_1, \ldots, \hat{\beta}_n$, it can be directly deduced
that the control signal $u$ is bounded in probability due to the boundedness of $x_i, \hat{\beta}_i (i = 1, \ldots, n)$. Consequently, all variables in the closed-loop system (1) are proved to be bounded in probability.

**Remark 8:** It can be seen from (49) that a better convergence performance could be obtained by reducing $c_i$ or increasing $a_i$ and $\delta_{1i}$. In addition, noting that $\delta_{2i}$’s do not appear in (49), they only appear in the expression of the control variable $u$. That means, $c_i$ can be taken as a smaller value, $a_i$ and $\delta_{1i}$ can be chosen as some large values, while $\delta_{2i}$ can be selected freely. Nevertheless, based on (44) and (45), it is noted that if $a_i$’s, $\delta_{1i}$’s, and $\delta_{2i}$’s are all large, while $c_i$’s are all very small, then the adaptive parameters $\hat{\beta}_i (i)$’s and the attitude of the controller $u$ may be also very large. To avoid this situation, the parameters $a_i$’s, $\delta_{1i}$’s, and $\delta_{2i}$’s cannot be chosen too large or the parameters $c_i$’s cannot be chosen too small. Therefore, the design parameters should be carefully adjusted catering to the control performance in practical applications.

**Remark 9:** In view of the control design procedure, two limitations of the proposed control strategy should be mentioned as follows. On the one hand, when the state $x_1$ closes to the boundaries $x_1 = -\varepsilon_1$ or $x_1 = \varepsilon_2$, the control input may become very large. In other words, the proposed scheme may not be applicable if the input is saturated. However, the control signal remains bounded for all time. Thus, as stated in [31], the control signal can be kept in a desirable operating range by carefully selecting the parameters. On the other hand, repeated differentiation of virtual controls or other nonlinear functions may result in the explosion of the complexity problem. This problem can be solved by the dynamic surface control technique in which the virtual control is passed through a first-order filter to obtain the derivative at each step. In addition, it can also be addressed by the approach proposed in [41].

**IV. Simulation Example**

In this section, simulation results of the following system are provided to further verify the result of Theorem 1:

\[
\begin{align*}
\dot{x}_1 &= \frac{9}{2}x_1 dt + 4x_1^2 dt + x_1^2 \sin x_2 dw \\
\dot{x}_2 &= a_2 \sigma dt + x_1 x_2^2 dt + x_1^2 \sin x_2 dw \\
y &= x_1.
\end{align*}
\]

Assume $\varepsilon_1 = 1$ and $\varepsilon_2 = 2$. This also means that the output $y = x_1$ is constrained by $\Gamma_1 = \{y(t) \in \mathbb{R} | -1 < y(t) < 2\}$. According to the criterion given in Remark 2, the fuzzy membership functions can be chosen as

\[
\begin{align*}
\mu_{x_1}^k(x_1) &= e^{-0.5(x_1+k)^2} \\
\mu_{x_2}^k(x_2) &= e^{-0.5(x_2+k)^2} \\
\mu_{\hat{\beta}}^k(\hat{\beta}) &= e^{-0.5(\hat{\beta}_1+k)^2}, k = -5, -3, -1, 0, 1, 3, 5.
\end{align*}
\]

In light of the controller design in Section III, we can choose

\[
\begin{align*}
S_1 &= \frac{137}{82} H_1(\eta_1) \frac{45}{22} \left( \| \Phi_1(x_1) \|_{2m}^2 + \frac{137}{22} a_1 \delta_{11} \eta_1^2 - c_1 \hat{\beta}_1 \right) \\
\gamma_1 &= -S_1^2 \eta_1
\end{align*}
\]

where $\eta_1 = x_1, \eta_2 = x_2 - \gamma_1, H_1(\eta_1) = ((\varepsilon_1 \cdot \varepsilon_2)^{q-\eta_1+4}(\eta_1^2 + \varepsilon_1 \varepsilon_2))/[(\varepsilon_2 - \eta_1)(\varepsilon_1 + \eta_1)^{q-\eta_1+5})]$, and $\delta_{11}, \delta_{21}$, and $\delta_{22}$ are positive constants.

Furthermore, using (52), the controller $u$ and the adaptive law are, respectively, constructed as follows:

\[
\begin{align*}
u &= \gamma_2 = -S_2^2 \eta_2 \\
\hat{\beta}_1 &= \frac{137}{82} (H_1(\eta_1))^{\frac{182}{22}} \| \Phi_1(x_1) \|_{2m}^2 a_1 \delta_{11} \eta_1^2 - c_1 \hat{\beta}_1 \\
\hat{\beta}_2 &= \frac{15}{26} \| \Phi_2(\xi_2) \|_{2m}^2 a_2 \delta_{21} \eta_2^2 - c_2 \hat{\beta}_2
\end{align*}
\]

where $a_1$, $a_2$, $c_1$, and $c_2$ are positive design parameters.

Based on Remark 8 and to obtain a satisfactory performance, the related parameters are chosen as $\delta_{11} = 20, \delta_{21} = 20, \delta_{22} = 40, a_1 = 20, a_2 = 20, c_1 = (1/6),$ and $c_2 = (1/5)$. Next, we select the initial adaptive parameters as $[\hat{\beta}_1, \hat{\beta}_2]^T = [0.5, 0.5]^T$ and take two different values of the initial state $x_0$ as $[1.2, 7]^T$ and $[-0.9, 7]^T$. Figs. 1–8 show the simulation results.

Figs. 1 and 2 display the trajectory of $x_1(t)$ under the different initial values, which demonstrates the output constraint is not violated for any initial value $x_1(0)$ satisfying $-1 < x_1(0) < 2$. Meanwhile, the trajectory of $x_2(t)$ is also expressed in Figs. 3 and 4, and Figs. 5 and 6 show the curves of the adaptive parameter vector $\hat{\beta} = [\hat{\beta}_1, \hat{\beta}_2]^T$. Finally, time history of controller $u$ is provided in Figs. 7 and 8. It can be
clearly observed from these figures that all variables of the closed-loop system (51) are bounded under controller (53).

V. CONCLUSION

This article addressed the adaptive control problem of a class of $p$-norm uncertain stochastic nonlinear systems that are in nontriangular form and subjected to asymmetric output constraints. Compared with the existing related results, the considered system was with a more general form and two kinds of factors (output constraints and unknown nonlinearities) were considered simultaneously. The main contributions of this article focused on constructing a new asymmetric BLF and presenting an adaptive fuzzy control strategy. In addition, some challenges and issues still remain open.

1) The first challenge is to extend the range of $q_i$.

2) Another challenge is to solve the explosion of the complexity problem. The preliminary idea is combining the proposed scheme with the command filter used in [41].

3) The stability in the sense of ultimate boundedness is achieved for system (1). There are very few results on finite-time stability of system (1) with unknown nonlinearities. In the future, we will consider the finite-time control for $p$-norm stochastic nonlinear systems with output constraints and unknown nonlinearities.

4) In this article, the stochastic system (1) is built with respect to Itô’s approach, which is applicable for white noise and cannot be applied for colored noise. In the future, we will consider extending the results of this...
article to the situation of colored noise and solve the corresponding control problem in the sense of Stratonovich by the methods given in [7] and [8].

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