ON COMPACT MANIFOLDS WITH HARMONIC CURVATURE AND POSITIVE SCALAR CURVATURE

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ABSTRACT. Let $M^n(n \geq 3)$ be an $n$-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. Assume that $M^n$ satisfies some integral pinching conditions. We give some rigidity theorems on compact manifolds with harmonic curvature and positive scalar curvature. In particular, Theorem 1.4, Corollary 1.6 and Theorem 1.9 are sharp for our conditions have the additional properties of being sharp. By this we mean that we can precisely characterize the case of equality.

1. INTRODUCTION AND MAIN RESULTS

Recall that an $n$-dimensional Riemannian manifold $(M^n, g)$ is said to be a manifold with harmonic curvature if the divergence of its Riemannian curvature tensor $Rm$ vanishes, i.e., $\delta Rm = 0$. In view of the second Bianchi identity, we know that $M$ has harmonic curvature if and only if the Ricci tensor of $M$ is a Codazzi tensor. When $n \geq 3$, by the Bianchi identity, the scalar curvature is constant. Thus, every Riemannian manifold with parallel Ricci tensor has harmonic curvature. Moreover, the constant curvature spaces, Einstein manifolds and the locally conformally flat manifolds with constant scalar curvature are also important examples of manifolds with harmonic curvature, however, the converse does not hold (see [2], for example). According to the decomposition of the Riemannian curvature tensor, the metric with harmonic curvature is a natural candidate for this study since one of the important problems in Riemannian geometry is to understand classes of metrics that are, in some sense, close to being Einstein or having constant curvature. The another reason for this study on the metric with harmonic curvature is the fact that a Riemannian manifold has harmonic curvature if and only if the Riemannian connection is a solution of the Yang-Mills equations on the tangent bundle [4]. The complete manifolds with harmonic curvature have been studied in literature (e.g., [5, 6, 9, 11, 14, 18, 21, 22, 25, 28, 29, 30]). Some isolation theorems of Weyl curvature tensor of positive Einstein manifolds are given in [7, 15, 16, 18, 28], when its $L^p$-norm is small. Some scholars [6, 18, 25, 30] classify conformally flat manifolds satisfying an $L^p$-pinching condition on the curvature. Recently, Xiao and the author [14] obtain some rigidities on complete manifolds with harmonic curvature satisfying an $L^p$-pinching condition on trace-free Riemannian curvature. The curvature pinching phenomenon plays an important role in global differential geometry. We are interested in $L^p$ pinching problems for compact Riemannian manifold with harmonic curvature and positive scalar curvature.

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Now we introduce the definition of the Yamabe constant. Given a compact Riemannian $n$-manifold $M$, we consider the Yamabe functional
\[ Q_g : C^\infty_c(M) \to \mathbb{R} : f \mapsto Q_g(f) = \frac{4(n-1)}{n-2} \int_M |\nabla f|^2 \, dv_g + \int_M R f^2 \, dv_g, \]
where $R$ denotes the constant scalar curvature of $M$. It follows that $Q_g$ is bounded below by Hölder inequality. We set
\[ \mu([g]) = \inf \{ Q_g(f) | f \in C^\infty_c(M) \}. \]
This constant $\mu([g])$ is an invariant of the conformal class of $(M, g)$, called the Yamabe constant. The important works of Schoen, Trudinger and Yamabe showed that the infimum in the above is always achieved (see [1, 23]). The Yamabe constant of a given compact manifold is determined by the sign of scalar curvature [1].

Throughout this paper, we always assume that $M$ is an $n$-dimensional complete Riemannian manifold with $n \geq 3$. In this note, we obtain the following rigidity theorems.

**Theorem 1.1.** Let $M$ be an $n$-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. Then
\[ \int_M |\tilde{Ric}|^2 \left[ R - \sqrt{(n-1)n} |\tilde{Ric}| - \frac{(n-2)(n-1)}{2} |W| \right] \leq 0 \]
and equality occurs if and only if
i) $M$ is a Einstein manifold;
ii) $M$ is covered isometrically by $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ with the product metric;
iii) $M$ is covered isometrically by $(\mathbb{S}^1 \times N^{n-1}, dt^2 + F^2(t) g_N)$, where $(N^{n-1}, g_N)$ is a compact Einstein manifold with positive scalar curvature and $F$ is a non-constant, positive, periodic function satisfying a precise ODE. This metric is called a rotationally symmetric Derdziński metric in [6].

**Theorem 1.2.** Let $M$ be an $n$-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. Then
\[ \int_M |\tilde{Rm}| [R - (n-1) C_1(n) |\tilde{Rm}|] \leq 0, \]
where $C_1(n)$ is defined in Lemma 2.1, and equality occurs if and only if $M$ is isometric to a quotient of the round $\mathbb{S}^n$.

**Corollary 1.3.** Let $M$ be a compact Riemannian $n$-manifold with harmonic curvature and positive scalar curvature. If
\[ |\tilde{Rm}| \leq \frac{R}{(n-1) C_1(n)}, \]
then $M$ is isometric to a quotient of the round $\mathbb{S}^n$.

**Theorem 1.4.** Let $(M^n, g)(n \geq 4)$ be an $n$-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If
\[ \left( \int_M |W + \frac{\sqrt{n}}{\sqrt{2(n-2)}} \tilde{Ric} \otimes g| \right)^{\frac{2}{n-2}} < \frac{\sqrt{2}}{\sqrt{2(n-2)(n-1)}} \mu([g]). \tag{1.1} \]
then $M$ is an Einstein manifold. In particular, if the pinching constant in (1.1) is weakened to $\frac{1}{2 C_2(n)} \mu([g])$, where $C_2(n)$ is defined in Lemma 2.4, then $M$ is isometric to a quotient of the round $\mathbb{S}^n$. 
Remark 1.5. Theorem 1.4 improves Theorems 1 and 2 in [18]. When \( n = 3 \), these manifolds with harmonic curvature are locally conformally flat. Theorem 1.4 in dimension 3 is discussed by Xiao and the author in [14]. The inequality (1.1) of this theorem is optimal.

The critical case is given by the following example. If \((S^1(t) \times S^{n-1}, g_t)\) is the product of the circle of radius \( t \) with \( S^{n-1} \), and if \( g_t \) is the standard product metric normalized such that \( \text{Vol}(g_t) = 1 \), we have \( W = 0 \), \( g_t \) is a Yamabe metric for small \( t \) (see [26]), and
\[
\left( \int_M |\mathring{Ric}|^2 \right)^{\frac{1}{2}} = \frac{\mu([g])}{\sqrt{\det(g_{0,n-1})}},
\]
which is the critical case of the inequality (1.1) in Theorem 1.4. We know that \((S^1(t) \times S^{n-1}, g_t)\) is not Einstein.

**corollary 1.6.** Let \((M^4, g)\) be a 4-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If
\[
\int_M |W|^2 + 8 \int_M |\mathring{Ric}|^2 \leq \frac{1}{3} \int_M R^2,
\]
then i) \( M \) is isometric to a quotient of the round \( S^4 \);
ii) \( M \) is a compact positive Einstein manifolds which is either Kähler, or the quotient of a Kähler manifold by a free, isometric, anti-holomorphic involution;
iii) \( M \) is a \( CP^2 \) with the Fubini-Study metric;
iv) \( M \) is Einstein and self-dual.

In particular, if
\[
\int_M |W|^2 + 6 \int_M |\mathring{Ric}|^2 \leq \frac{1}{6} \int_M R^2,
\]
then i), ii) and iii) hold.

Remark 1.7. The pinching conditions (1.2) and (1.3) in Corollary 1.6 is equivalent to the following
\[
\int_M |W|^2 + \frac{1}{15} \int_M R^2 \leq \frac{128}{5} \pi^2 \chi(M)
\]
and
\[
\int_M |\mathring{Ric}|^2 \leq 16 \pi^2 \chi(M),
\]
where \( \chi(M) \) is the Euler-Poincaré characteristic of \( M \).

As we mentioned above, Theorem 1.4 is sharped. By this we mean that we can precisely characterize the case of equality:

**Theorem 1.8.** Let \((M^n, g) (n \geq 4)\) be an n-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If
\[
\left( \int_M |W| + \frac{\sqrt{n}}{\sqrt{2}(n-2)} R_{\mathring{g}} g |^2 \right)^{\frac{1}{2}} = \frac{\sqrt{2}}{\sqrt{n-2}(n-1)} \mu([g]),
\]
then i) \( M \) is Einstein manifold;
ii) \( M \) is covered isometrically by \( S^1 \times S^{n-1} \) with the product metric;
iii) \( M \) is covered isometrically by \( S^1 \times S^{n-1} \) with a rotationally symmetric Derdziński metric.

**Theorem 1.9.** Let \((M^n, g) (n \geq 4)\) be an n-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If
\[
C_2(n) \left( \int_M |W|^2 \right)^{\frac{1}{2}} + 2 \sqrt{\frac{n-1}{n}} \left( \int_M |\mathring{Ric}|^2 \right)^{\frac{1}{2}} < D(n) \mu([g]),
\]
where
\[
D(n) = \begin{cases} 
\frac{n^2-1}{n}, & n \leq 9; \\
\frac{n}{2}, & n \geq 10,
\end{cases}
\]
then \( M \) is isometric to a quotient of the round \( S^n \). Moreover, in dimension, \( 4 \leq n \leq 9 \), the same result holds only assuming the weak inequality.

**Remark 1.10.** When \( n \geq 10 \), the pinching condition of this theorem is optimal. The critical case is given by this example in Remark 1.5.

As we mentioned above, Theorem 1.9 is sharpened. By this we mean that we can precisely characterize the case of equality:

**Theorem 1.11.** Let \((M^n, g) (n \geq 10)\) be an \( n \)-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If
\[
C_2(n) \left( \int_M |W|^2 \right)^{\frac{1}{2}} + 2 \sqrt{\frac{n-1}{n}} \left( \int_M |\tilde{\text{Ric}}|^2 \right)^{\frac{1}{2}} = \frac{2}{\mu(g)},
\]
then i) \( M \) is Einstein manifold;
ii) \( M \) is covered isometrically by \( S^1 \times S^{n-1} \) with the product metric;
iii) \( M \) is covered isometrically by \( S^1 \times S^{n-1} \) with a rotationally symmetric Derdziński metric.

Based on Lemma 2.4, using the same argument as in the proof of Theorem 1.1, we can prove Theorem 1.12.

**Theorem 1.12.** Let \( M \) be an \( n \) (\( \geq 4 \))-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. Then
\[
\int_M |W| ||R - \sqrt{(n-1)n}|\tilde{\text{Ric}}| - \frac{C_2(n)n}{2}W|| \leq 0
\]
and equality occurs if and only if \( M \) is locally conformal flat.

**Remark 1.13.** G. Catino [6] classifies compact conformally at \( n \)-dimensional manifolds with constant positive scalar curvature and satisfying an optimal integral pinching condition.

**Remark 1.14.** When \( n = 4 \), we [13] extend to four-manifolds with harmonic Weyl tensor Theorems 1.4 and 1.8, and corollary 1.6.

We follow their methods [6, 7, 15, 19] to prove these theorems.

2. **Proofs of Lemmas**

In what follows, we adopt, without further comment, the moving frame notation with respect to a chosen local orthonormal frame.

Let \( M \) be a Riemannian manifold with harmonic curvature. The decomposition of the Riemannian curvature tensor into irreducible components yield
\[
R_{ijkl} = W_{ijkl} + \frac{1}{n-2} \left( R_{ik} \delta_{jl} - R_{il} \delta_{jk} + R_{jl} \delta_{ik} - R_{jk} \delta_{il} \right)
\]
\[- \frac{R}{(n-1)(n-2)} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk})
\]
\[
= W_{ijkl} + \frac{1}{n-2} (\hat{R}_{ik} \delta_{jl} - \hat{R}_{il} \delta_{jk} + \hat{R}_{jl} \delta_{ik} - \hat{R}_{jk} \delta_{il})
\]
\[+ \frac{R}{n(n-1)} (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}).
\]
where $R_{ijkl}$, $W_{ijkl}$, $R_{ij}$ and $\hat{R}_{ij}$ denote the components of $\hat{R}m$, the Weyl curvature tensor $W$, the Ricci tensor $\text{Ric}$ and the trace-free Ricci tensor $\hat{\text{Ric}} = \text{Ric} - \frac{\mathcal{R}}{n} g$, respectively, and $R$ is the scalar curvature.

The trace-free Riemannian curvature tensor $\hat{R}m$ is

$$
\hat{R}_{ijkl} = R_{ijkl} - \frac{\mathcal{R}}{n(n-1)} (\delta_i \delta_j \delta_k - \delta_i \delta_j \delta_k)
$$

Then the following equalities are easily obtained from the properties of curvature tensor:

(2.2) $g^{ij} \hat{R}_{ijkl} = \hat{R}_{jl}$,

(2.3) $\hat{R}_{ijkl} + \hat{R}_{ilkj} + \hat{R}_{iklj} = 0$,

(2.4) $\hat{R}_{ijkl} = \hat{R}_{klij} = -\hat{R}_{jikl} = -\hat{R}_{ijkl}$,

(2.5) $|\hat{R}m|^2 = |W|^2 + \frac{4}{n-2} |\hat{\text{Ric}}|^2$.

Moreover, by the assumption of harmonic curvature, we compute

(2.6) $\hat{R}_{ijkl,m} + \hat{R}_{ijml,k} + \hat{R}_{ijkm,l} = 0$,

(2.7) $\hat{R}_{ijkl,l} = 0$,

(2.8) $\hat{W}_{ijkl,m} + \hat{W}_{ijml,k} + \hat{W}_{ijkm,l} = 0$

and

(2.9) $\hat{W}_{ijkl,l} = 0$.

Now, we compute the Laplacian of $|\hat{R}m|^2$, $|\hat{\text{Ric}}|^2$ and $|\hat{W}|^2$, respectively.

**Lemma 2.1.** Let $M$ be a complete Riemannian $n$-manifold with harmonic curvature and positive scalar curvature. Then

$$
\Delta |\hat{R}m|^2 \geq 2|\nabla \hat{R}m|^2 - 2C_1(n) |\hat{R}m|^2 + 2 \frac{(n-2)\mathcal{R}}{n(n-1)} |W|^2 + 2 \frac{\mathcal{R}}{n-1} |\hat{R}m|^2,
$$

where $C_1(n) = 2 \left[ \frac{2(n^2+4n-4)}{n(n+1)(n+2)} + \frac{n^2-n-4}{n-2(n-1)n(n+1)} + \sqrt{\frac{(n-2)(n+1)}{4n}} \right]$.

**Remark 2.2.** Although Lemma 2.1 has been proved in [14], we give its proof. When $M$ is a complete locally conformally flat Riemannian $n$-manifold, it follows from (2.12) that

$$
\Delta |\hat{\text{Ric}}|^2 \geq 2|\nabla \hat{\text{Ric}}|^2 - 2 \frac{n}{\sqrt{n(n-1)}} |\hat{\text{Ric}}|^3 + 2 \frac{\mathcal{R}}{n-1} |\hat{\text{Ric}}|^2.
$$

By the Kato inequality $|\nabla \hat{\text{Ric}}|^2 \geq \frac{n+2}{n} |\hat{\text{Ric}}|^2$, we obtain (see [25][30])

$$
|\hat{\text{Ric}}|_\Delta |\hat{\text{Ric}}|^2 \geq 2 \frac{n}{\sqrt{n(n-1)}} |\hat{\text{Ric}}|^3 + 2 \frac{\mathcal{R}}{n-1} |\hat{\text{Ric}}|^2.
$$
Proof. By the Ricci identities, we obtain from (2.1)-(2.7)

\[
\Delta |\tilde{R}m|^2 = 2|\nabla \tilde{R}m|^2 + 2\langle \nabla m, \Delta \tilde{R}m \rangle = 2|\nabla \tilde{R}m|^2 + 2\tilde{R}_{ijkl} \tilde{R}_{ijkl,mn} \\
= 2|\nabla \tilde{R}m|^2 + 2\tilde{R}_{ijkl}(\tilde{R}_{jkm,lm} + \tilde{R}_{jml,km}) \\
= 2|\nabla \tilde{R}m|^2 + 4\tilde{R}_{ijkl} \tilde{R}_{jkm,lm} \\
= 2|\nabla \tilde{R}m|^2 + 4\tilde{R}_{ijkl}(\tilde{R}_{jkm,ml} + \tilde{R}_{hk,mn}) + \tilde{R}_{hkm} \tilde{R}_{jlm} + \tilde{R}_{hlm} \tilde{R}_{jkm} + \tilde{R}_{jlm} \tilde{R}_{hkm} \\
= 2|\nabla \tilde{R}m|^2 + 4\tilde{R}_{ijkl}(\tilde{R}_{jkm,ml} + \tilde{R}_{hk,mn}) + \tilde{R}_{hkm} \tilde{R}_{jlm} + \tilde{R}_{hlm} \tilde{R}_{jkm} + \tilde{R}_{jlm} \tilde{R}_{hkm} \\
+ \frac{4R}{n(n-1)} \tilde{R}_{ijkl}(\tilde{R}_{ijkl} + \tilde{R}_{ijlk} + \tilde{R}_{ikjl}) \\
+ \frac{4R}{n} |\tilde{R}m|^2. \\
(2.12)
\]

We consider \( \tilde{R}m \) as a self adjoint operator on \( \wedge^2 V \) and \( S^2 V \), respectively. By the algebraic inequality for \( m \)-trace-free symmetric two-tensors \( T \), i.e., \( tr(T^3) \leq \frac{m^2 - 2}{m(m-1)} |T|^3 \) and equality holds if and only if \( T \) can be diagonalized with \( (n-1) \)-eigenvalues equal to \( \lambda \) and one eigenvalue equals to \( -(n-1)\lambda \), and the eigenvalues \( \lambda_i \) of \( T \) satisfy \( |\lambda_i| \leq \sqrt{\frac{m-2}{m}} |T| \) in [20], we obtain

\[
|\tilde{R}_{ijkl}(2\tilde{R}_{jkm} \tilde{R}_{hl} + \frac{1}{2} \tilde{R}_{hl} \tilde{R}_{jkm})| \\
\leq \left[ \frac{2(n^2 + n - 4)}{(n-1)(n+1)(n+2)} + \frac{n^2 - n - 4}{2 \sqrt{(n-2)(n-1)n(n+1)}} \right] |\tilde{R}m|^3, \\
(2.13)
\]

and

\[
|\tilde{R}_{ijkl} \tilde{R}_{ijkl}| \leq \sqrt{\frac{n-1}{n}} |\text{Ric}||\tilde{R}m|^2. \\
(2.14)
\]

From (2.5), we have

\[
|\text{Ric}|^2 \leq \frac{n-2}{4} |\tilde{R}m|^2. \\
(2.15)
\]

Combining with (2.12)-(2.15), we obtain that

\[
\Delta |\tilde{R}m|^2 \geq 2|\nabla \tilde{R}m|^2 + \frac{2(n-2)R}{n(n-1)} |W|^2 + \frac{2R}{n-1} |\tilde{R}m|^2 - 4\sqrt{\frac{(n-2)(n-1)}{4n}} |\tilde{R}m|^3. \\
\]

Combining with (2.12)-(2.15), we obtain that

\[
\Delta |\tilde{R}m|^2 \geq 2|\nabla \tilde{R}m|^2 + \frac{2(n-2)R}{n(n-1)} |W|^2 + \frac{2R}{n-1} |\tilde{R}m|^2 - 4\sqrt{\frac{(n-2)(n-1)}{4n}} |\tilde{R}m|^3. \\
\]
This completes the proof of this Lemma. \( \square \)

**Lemma 2.3.** Let \( M \) be a complete Riemannian \( n \)-manifold with harmonic curvature. Then

\[
\triangle |\hat{\text{Ric}}|^2 \geq 2|\nabla \hat{\text{Ric}}|^2 - 2 \sqrt{\frac{n-2}{2(n-1)}} |\hat{W}| |\hat{\text{Ric}}|^2 - 2 \sqrt{\frac{n}{n-1}} |\hat{\text{Ric}}|^3 + 2 \frac{R}{n-1} |\hat{\text{Ric}}|^2.
\]

**Proof.** We compute

\[
\triangle |\hat{\text{Ric}}|^2 = 2|\nabla \hat{\text{Ric}}|^2 + 2 \langle \hat{\text{Ric}}, \triangle \hat{\text{Ric}} \rangle = 2|\nabla \hat{\text{Ric}}|^2 + 2 \hat{R}_{ij} \hat{R}_{jikl}.
\]

Since the traceless Ricci tensor is Codazzi, by the Ricci identities, we obtain

\[
\hat{R}_{jikk} = \hat{R}_{ikjk} = \hat{R}_{ikl} R_{lkjk} + \hat{R}_{kl} R_{lijk}
\]

which gives

\[
\triangle |\hat{\text{Ric}}|^2 = 2|\nabla \hat{\text{Ric}}|^2 + 2 \hat{R}_{ij} \hat{R}_{jikk} = 2|\nabla \hat{\text{Ric}}|^2 + 2 \hat{R}_{ij} R_{ik} R_{jlk} + 2 \hat{R}_{ij} \hat{R}_{kl} R_{lij}.
\]

We compute

\[
\triangle |\hat{\text{Ric}}|^2 = 2|\nabla \hat{\text{Ric}}|^2 + 2 W_{ijkl} \hat{R}_{ij} \hat{R}_{kl} + 2 \frac{n}{n-2} \hat{R}_{ij} \hat{R}_{kl} \hat{R}_{lij} + 2 \frac{R}{n-1} |\hat{\text{Ric}}|^2.
\]

By the inequality \( |W_{ijkl} \hat{R}_{ij} \hat{R}_{kl}| \leq \sqrt{\frac{n-2}{2(n-1)}} |\hat{W}| |\hat{\text{Ric}}|^2 \) given in \([20]\), from (2.20) we get

\[
\triangle |\hat{\text{Ric}}|^2 \geq 2|\nabla \hat{\text{Ric}}|^2 - 2 \sqrt{\frac{n-2}{2(n-1)}} |\hat{W}| |\hat{\text{Ric}}|^2 - 2 \sqrt{\frac{n}{n-1}} |\hat{\text{Ric}}|^3 + 2 \frac{R}{n-1} |\hat{\text{Ric}}|^2.
\]

This completes the proof of this Lemma. \( \square \)

**Lemma 2.4.** Let \( M \) be a complete Riemannian \( n \)-manifold with harmonic curvature. Then

\[
\triangle |W|^2 \geq 2|\nabla W|^2 - 2 C_2(n) |W|^3 - 4 \sqrt{\frac{n-1}{n}} |W|^2 |\hat{\text{Ric}}| + \frac{4R}{n} |W|^2,
\]

where

\[
C_2(n) = \begin{cases} \frac{26}{8 \sqrt{10}}, & n = 4 \\ \frac{6}{4 \sqrt{10} (n^2 + n)}, & n = 5 \\ \frac{4(n^2 + n - 4)}{\sqrt{(n-1)3(n+1)(n+2)}}, & n \geq 6 \end{cases}
\]
Lemma 2.5.

Proof. By the Ricci identities, we obtain form (2.8) and (2.9)

\[ (2.22) \]

\[ \Delta |W|^2 = 2|\nabla W|^2 + 2(W, \Delta W) = 2|\nabla W|^2 + 2W_{ijkl}W_{ijkl}^{\mu\nu} \]

\[ = 2|\nabla W|^2 + 2W_{ijkl}(W_{ijkm}L_m + W_{ijml}K_m) \]

\[ = 2|\nabla W|^2 + 4W_{ijkl}W_{ijkl} \]

\[ = 2|\nabla W|^2 + 4W_{ijkl}(W_{ijkm}L_m + W_{ijkm}R_{kilm} + W_{ijlm}R_{jikm} + W_{ijkl}R_{hkln}) \]

\[ = 2|\nabla W|^2 + 4W_{ijkl}(W_{hjkm}R_{hilm} + W_{ihkm}R_{jhlm} + W_{ijhm}R_{hklm} + W_{ijkl}R_{hjlm}) \]

\[ = 2|\nabla W|^2 + 4W_{ijkl}(W_{hjkm}W_{hilm} + W_{ihkm}W_{jhlm} + W_{ijhm}W_{hkln} + W_{ijkl}W_{hjlm}) \]

\[ + \frac{4}{n-2}W_{ijkl}(\hat{R}_{ijlm} - \delta_{ij}\delta_{lm} + \hat{R}_{ik}\delta_{jl} - \hat{R}_{il}\delta_{jk} - \hat{R}_{jk}\delta_{il} - \hat{R}_{jl}\delta_{ik}) \]

\[ + W_{ihkm}(\hat{R}_{jklm} - \delta_{jk}\delta_{lm} + \hat{R}_{jk}\delta_{lm} + \hat{R}_{jl}\delta_{km}) + \frac{4R}{n(n-1)}W_{ijkl}(W_{ijkl} + W_{iklj} + W_{ijlk} + \frac{4R}{n}|W|^2) \]

\[ = 2|\nabla W|^2 + 4W_{ijkl}(2W_{hjkm}W_{hilm} - \frac{1}{2}W_{ijhm}W_{klhm}) + \frac{4R}{n}|W|^2 - 4\sqrt{\frac{n-1}{n}}|W|^2|Ric|, \]

and equality holds if and only if \( W = 0 \) or \( Ric \) can be diagonalized with \((n-1)-\)eigenvalues equal to \( \lambda \) and one eigenvalue equals to \( -(n-1)\lambda \) and \((W_{ijkl}, \ldots, W_{ijkm})\) correspondingly takes the value \((- (n-1)\lambda, 0, \ldots, 0)\).

Case 1. When \( n = 4 \), it was proved in [20] that

\[ |W_{ijkl}W_{hjkm}W_{hilm}| + \frac{1}{2}W_{ijkl}W_{ijmk}W_{hklm}| \leq \frac{\sqrt{5}}{4}|W|^3. \]

Case 2. When \( n = 5 \), Jack and Parker [24] have proved that \( W_{ijkl}W_{hjkm}W_{hilm} = 4W_{ijkl}W_{hjkm}W_{hilm}. \)

We consider \( W \) as a self adjoint operator on \( \wedge^2 V \), and obtain

\[ |W_{ijkl}W_{hjkm}W_{hilm}| + \frac{1}{2}W_{ijkl}W_{ijmk}W_{hklm}| = |W_{ijkl}W_{hjkm}W_{hilm}| \leq \frac{4\sqrt{10}}{15}|W|^3. \]

Case 3. When \( n \geq 6 \), considering \( W \) as a self adjoint operator on \( S^2 V \), we have

\[ |W_{ijkl}W_{hjkm}W_{hilm}| + \frac{1}{2}W_{ijkl}W_{ijmk}W_{hklm}| \leq 2|W_{ijkl}W_{hjkm}W_{hilm}| + \frac{1}{2}|W_{ijkl}W_{ijmk}W_{hklm}| \]

\[ \leq \left[ \frac{2(n^2 + n - 4)}{\sqrt{(n-1)n(n+1)(n+2)}} + n^2 - n - 4 \sqrt{(n-2)(n-1)n(n+1)} \right]|W|^3. \]

From (2.22) and Cases 1, 2 and 3, we complete the proof of this Lemma. \( \square \)

Lemma 2.5. On every \( n \)-dimensional Riemannian manifold the following estimate holds

\[ \left| -W_{ijkl}\hat{R}_{ij} + \frac{n}{n-2}\hat{R}_{ij}\hat{R}_{ik}\hat{R}_{kl} \right| \leq \left( \frac{n-2}{2(n-1)} \right)^\frac{1}{2} \left( |W|^2 + \frac{2n}{n-2}|Ric|^2 \right)^\frac{1}{2} |Ric|^2. \]

Remark 2.6. We follow these proofs of Proposition 2.1 in [7] and Lemma 4.7 in [3] to prove this lemma.
3. Proofs of Theorems

Proof of Theorem 1.1. By the Kato inequality $|\nabla \bar{\Ric}|^2 \geq \frac{n+2}{n}|\nabla |\bar{\Ric}|^2$ and Lemma 2.3, we obtain

\begin{equation}
|\bar{\Ric}|^2 \geq \frac{2}{n} |\nabla |\bar{\Ric}|^2 - \sqrt{\frac{n-2}{2(n-1)}} |\nabla |\bar{\Ric}|^2 - \sqrt{\frac{n}{n-1}} |\bar{\Ric}|^3 + \frac{R}{n-1} |\bar{\Ric}|^2
\end{equation}

in the sense of distributions. Using (3.1), we compute

\begin{equation}
|\bar{\Ric}|^2 \triangle |\bar{\Ric}|^2 = |\bar{\Ric}|^2 \left( -\frac{2(n-2)}{n^2} |\bar{\Ric}|^2 + \frac{n}{n^2} |\bar{\Ric}|^2 \triangle |\bar{\Ric}|^2 \right) = -\frac{2(n-2)}{n^2} |\nabla |\bar{\Ric}|^2|^2 + \frac{n-2}{n} |\nabla |\bar{\Ric}|^2|\triangle |\bar{\Ric}|^2|
\end{equation}

\begin{equation}
\geq \frac{n-2}{n} \left( -\sqrt{\frac{n}{n-1}} |\nabla |\bar{\Ric}|^2|^{2+1} + \frac{R}{n-1} |\nabla |\bar{\Ric}|^2|^{2} - \sqrt{\frac{n-2}{2(n-1)}} |W||\nabla |\bar{\Ric}|^2|^{2}\right),
\end{equation}

in the sense of distributions. From (3.2), we get

\begin{equation}
\Delta |\bar{\Ric}|^2 \geq \frac{n-2}{n} \left( -\sqrt{\frac{n}{n-1}} |\nabla |\bar{\Ric}|^2|^{2+1} + \frac{R}{n-1} |\nabla |\bar{\Ric}|^2|^{2} - \sqrt{\frac{n-2}{2(n-1)}} |W||\nabla |\bar{\Ric}|^2|^{2}\right).
\end{equation}

Integrating (3.3) over $M$, we obtain

\begin{equation}
\int_M \left( R - \sqrt{n(n-1)} |\bar{\Ric}| - \sqrt{\frac{(n-2)(n-1)}{2}} |W| \right) |\bar{\Ric}|^2 \leq 0.
\end{equation}

If the equality holds in (3.4), all inequalities leading to (3.1) become equalities. Hence at every point, either $\bar{\Ric}$ is null, i.e., $M$ is Einstein, or it has an eigenvalue of multiplicity $(n-1)$ and another of multiplicity 1. Since $M$ has harmonic curvature, and by the regularity result of DeTurck and Goldschmidt [12], $M$ must be real analytic in suitable (harmonic) local coordinates.

Suppose that the Ricci tensor has an eigenvalue of multiplicity $(n-1)$ and another of multiplicity 1. If the Ricci tensor is parallel, by the de Rham decomposition Theorem [10], $M$ is covered isometrically by the product of Einstein manifolds. We have $R = \sqrt{n(n-1)} |\bar{\Ric}|$. From (3.4), we get $W = 0$, i.e. is conformally flat. Since $M$ has positive scalar curvature, then the only possibility is that $M$ is covered isometrically by $S^1 \times S^{n-1}$ with the product metric.

On the other hand, if the Ricci tensor is not parallel, by the classification result of Derdziński (see Theorem 10 of [11], see also Theorem 3.2 of [6]), this concludes the proof of Theorem 1.1.

\qed

Proof of Theorem 1.2. By the Kato inequality $|\nabla \bar{\Rm}|^2 \geq |\nabla |\bar{\Rm}|^2$ and Lemma 2.2, we get

\begin{equation}
\frac{1}{2} \Delta |\bar{\Rm}|^2 \geq |\nabla |\bar{\Rm}|^2|^{2} + \frac{(n-2)R}{m(n-1)} |W|^2 + \frac{R}{n-1} |\bar{\Rm}|^2
\end{equation}

in the sense of distributions.

For small $\epsilon > 0$, define $\Omega_\epsilon = \{x \in M||\bar{\Rm}| \geq \epsilon\}$, and

\begin{equation}
f_\epsilon(x) = \begin{cases}
|\bar{\Rm}|(x) & x \in \Omega_\epsilon \\
\epsilon & x \in M \setminus \Omega_\epsilon.
\end{cases}
\end{equation}
Multiplying both sides of (3.5) by $f^{-1}_\varepsilon$ and then integrating over $M$, we obtain

\[
0 \geq \frac{1}{2} \int_M \triangle \hat{Rm}^2 f^{-1}_\varepsilon + \int_M |\nabla \hat{Rm}|^2 f^{-1}_\varepsilon \\
+ \int_M [-C_1(n)]|\hat{Rm}_\varepsilon|^3 + \frac{(n-2)R}{n(n-1)}|W|^2 + \frac{R}{n-1} |\hat{Rm}_\varepsilon|^2 f^{-1}_\varepsilon \\
= \frac{1}{2} \int_M \langle \nabla \hat{Rm}_\varepsilon, \nabla f^{-1}_\varepsilon \rangle + \int_M |\nabla \hat{Rm}_\varepsilon|^2 f^{-1}_\varepsilon \\
+ \int_M [-C_1(n)]|\hat{Rm}_\varepsilon|^3 + \frac{(n-2)R}{n(n-1)}|W|^2 + \frac{R}{n-1} |\hat{Rm}_\varepsilon|^2 f^{-1}_\varepsilon \\
= -\int_M \langle \nabla \hat{Rm}_\varepsilon, \nabla f^{-1}_\varepsilon \rangle|\hat{Rm}_\varepsilon|^2 f^{-1}_\varepsilon + \int_M |\nabla \hat{Rm}_\varepsilon|^2 f^{-1}_\varepsilon \\
(3.6) + \int_M [-C_1(n)]|\hat{Rm}_\varepsilon|^3 + \frac{(n-2)R}{n(n-1)}|W|^2 + \frac{R}{n-1} |\hat{Rm}_\varepsilon|^2 f^{-1}_\varepsilon \\
= -\int_M |\nabla f^{-1}_\varepsilon|^2 f^{-1}_\varepsilon + \int_M |\nabla \hat{Rm}_\varepsilon|^2 f^{-1}_\varepsilon \\
+ \int_M [-C_1(n)]|\hat{Rm}_\varepsilon|^3 + \frac{(n-2)R}{n(n-1)}|W|^2 + \frac{R}{n-1} |\hat{Rm}_\varepsilon|^2 f^{-1}_\varepsilon \\
\geq \int_M [-C_1(n)]|\hat{Rm}_\varepsilon|^3 + \frac{(n-2)R}{n(n-1)}|W|^2 + \frac{R}{n-1} |\hat{Rm}_\varepsilon|^2 f^{-1}_\varepsilon \\
\geq \int_M [-C_1(n)]|\hat{Rm}_\varepsilon|^3 + \frac{R}{n-1} |\hat{Rm}_\varepsilon|^2 f^{-1}_\varepsilon.
\]

It follows from the proof of (3.6) that

\[
\int_M [R - C_1(n)(n-1)|\hat{Rm}_\varepsilon||\hat{Rm}_\varepsilon|^2 f^{-1}_\varepsilon \leq 0.
\]

Now, taking the limit as $\varepsilon \to 0$, we get from the above inequality

(3.7) \[
\int_M [R - C_1(n)(n-1)|\hat{Rm}_\varepsilon||\hat{Rm} \leq 0.
\]

If the equality holds in (3.7), all inequalities leading to (3.5) become equalities. Hence we get $W = 0$, i.e., $M$ is a compact conformally flat manifold with constant positive scalar curvature. By (2.5) and (3.7), we have

(3.8) \[
\int_M [R - C_1(n)(n-1)\sqrt{\frac{4}{n-2}}|\hat{Ric}_\varepsilon||\hat{Ric} = 0.
\]

Since $M$ is a compact conformally flat manifold with constant positive scalar curvature, based on (2.11), proceeding as in the proof of (3.7), we obtain

(3.9) \[
\int_M [R - \sqrt{n(n-1)}|\hat{Ric}_\varepsilon||\hat{Ric} \leq 0.
\]

By comparing (3.8) with (3.9), we obtain $\hat{Ric} = 0$, i.e., $M$ is a Einstein manifold. Hence $M$ is isometric to a quotient of the round $\mathbb{S}^n$.

Proof of corollary 1.3. When $\hat{Rm} < \frac{R}{(n-1)|\hat{Ric}|}$, by (2.10) and the maximum principle, we get $\hat{Rm} = 0$. Thus $M$ is isometric to a quotient of the round $\mathbb{S}^n$. \[\square\]
When $\tilde{R}n = \frac{R}{(n-1)c_1(g)}$, by Theorem 1.2, we have that $M$ is isometric to a quotient of the round $S^n$.

\[ \square \]

**Proof of Theorem 1.4.** By (2.20) and Lemma 2.5, we get

\[
\frac{1}{2} \triangle |\tilde{Ric}|^2 \geq \frac{n+2}{n} |\nabla |\tilde{Ric}| |^2 - \sqrt{\frac{n-2}{2(n-1)}} \left( |W|^2 + \frac{2n}{n-2} |\tilde{Ric}|^2 \right) \frac{1}{\gamma} |\tilde{Ric}|^2 + \frac{R}{n-1} |\tilde{Ric}|^2.
\]

We rewrite (3.10) as

\[
|\tilde{Ric}|^2 |\tilde{Ric}|^2 \geq \frac{2}{n} |\nabla |\tilde{Ric}| |^2 - \sqrt{\frac{n-2}{2(n-1)}} \left( |W|^2 + \frac{2n}{n-2} |\tilde{Ric}|^2 \right) \frac{1}{\gamma} |\tilde{Ric}|^2 + \frac{R}{n-1} |\tilde{Ric}|^2
\]

in the sense of distributions.

Set $u = |\tilde{Ric}|$. By (3.11), we compute

\[
u^\gamma \Delta u^\gamma = u^\gamma \left( \gamma (\gamma - 1) u^{-2\gamma} |\nabla u|^2 + \gamma u^{-2\gamma} \Delta u \right)
\]

\[
\gamma \frac{\gamma - 1}{\gamma} |\nabla u|^2 + \gamma u^{2\gamma - 2} \Delta u
\]

\[
(3.12)
\]

Integrating (3.12) by parts over $M^n$, it follows that

\[
0 \geq \left( 2 - \frac{n-2}{n\gamma} \right) \int_M |\nabla u|^2 + \gamma \frac{R}{n-1} \int_M u^{2\gamma} - \sqrt{\frac{n-2}{2(n-1)}} \gamma \int_M \left( |W|^2 + \frac{2n}{n-2} u^2 \right)^{\frac{1}{\gamma}} u^{\gamma} + \frac{R}{n-1} \gamma u^{2\gamma}.
\]

For $2 - \frac{n-2}{n\gamma} > 0$, by the definition of Yamabe constant and (3.13), we get

\[
0 \geq \left( 2 - \frac{n-2}{n\gamma} \right) \frac{n-2}{4(n-1)} \mu([g]) \left( \int_M u^{2\gamma} \right)^{\frac{\gamma}{2}} - \sqrt{\frac{n-2}{2(n-1)}} \gamma \int_M \left( |W|^2 + \frac{2n}{n-2} u^2 \right)^{\frac{1}{\gamma}} u^{\gamma} + \frac{4n\gamma + \frac{1}{\gamma}(n-2)^2 - 2n(n-2)}{4n(n-1)} R \int_M u^{2\gamma}.
\]

By Hölder inequality, we obtain

\[
0 \geq \left( 2 - \frac{n-2}{n\gamma} \right) \frac{n-2}{4(n-1)} \mu([g]) - \sqrt{\frac{n-2}{2(n-1)}} \gamma \left( \int_M \left( |W|^2 + \frac{2n}{n-2} u^2 \right)^{\frac{1}{\gamma}} \right)^{\frac{\gamma}{2}} \left( \int_M u^{2\gamma} \right)^{\frac{\gamma}{2}} + \frac{4n\gamma + \frac{1}{\gamma}(n-2)^2 - 2n(n-2)}{4n(n-1)} R \int_M u^{2\gamma}.
\]

Taking $\gamma = \frac{(n-2)(1+\sqrt{\frac{n-2}{4}})}{4}$, from (3.15), we get

\[
0 \geq \left( 2 - \frac{n-2}{n\gamma} \right) \frac{n-2}{\sqrt{(n-2)(n-1)}} \mu([g]) - \left( \int_M \left( |W|^2 + \frac{2n}{n-2} u^2 \right)^{\frac{1}{\gamma}} \right)^{\frac{\gamma}{2}} \left( \int_M u^{2\gamma} \right)^{\frac{\gamma}{2}}.
\]
Since $W$ is totally trace-free, one has
\[ |W + \frac{\sqrt{n}}{\sqrt{2(n-2)} R_{\text{ic}} \otimes g}|^2 = |W|^2 + \frac{2n}{n-2}|R_{\text{ic}}|^2 \]
and the pinching condition (1.1) implies that $M$ is Einstein.

In particular, we choose $\gamma$ such that $\frac{4\gamma}{3n + (n-2)^2 - 2n(n-2)} > 0$, and
\[ 0 \geq \left[ \frac{2}{C_2(n)n} \mu([g]) - \left( \int_M |W|^2 + \frac{2n}{n-2} \mu^2 \right) \right] \left( \int_M \mu^{\frac{2n}{2n-4}} \right) . \]
From the above, the pinching condition (1.1) implies that $M$ is Einstein. Hence, the pinching condition (1.1) implies
\[ (3.17) \quad \left( \int_M |W|^2 \right)^{\frac{2}{n}} < \frac{2}{C_2(n)n} \mu([g]) . \]

By the rigidity result for positively curved Einstein manifolds (see Theorem 1.1 of [16]), (3.17) implies that $M$ is isometric to a quotient of the round $\mathbb{S}^n$.

\textbf{Proofs of corollary 1.6 and Remark 1.7.} To prove Corollary 1.6, we need the following lower bound for the Yamabe invariant on compact four-dimensional manifolds which was proved by M. J. Gursky (see [7, 17]):
\[ \int_M R^2 - 12 \int_M |\text{Ric}|^2 \leq \mu([g]) \left( \int_M |W|^2 \right) , \]
the inequality is strict unless $(M^4, g)$ is conformally Einstein. From this inequality, we get
\[ \int_M |W|^2 + 4 \int_M |\text{Ric}|^2 - \frac{1}{3} \mu([g]) \leq \int_M |W|^2 + 8 \int_M |\text{Ric}|^2 - \frac{1}{3} \int_M R^2 \]
and
\[ \int_M |W|^2 + 4 \int_M |\text{Ric}|^2 - \frac{1}{6} \mu([g]) \leq \int_M |W|^2 + 6 \int_M |\text{Ric}|^2 - \frac{1}{6} \int_M R^2 \]
Moreover, the two inequalities are strict unless $(M^4, g)$ is conformally Einstein. In the first case “<”, Theorem 1.4 immediately implies Corollary 1.6. In the second case “=”, $g$ is conformally Einstein. Since $g$ has constant scalar curvature, $g$ is Einstein from the proof of Obata theorem (see [23]). Hence $\frac{1}{6} \int_M R^2 = \frac{1}{6} \mu([g])$, by Corollary 1.16 in [13], we complete the proof of this corollary.

By the Chern-Gauss-Bonnet formula (see Equation 6.31 of [2])
\[ \int_M |W|^2 - 2 \int_M |\text{Ric}|^2 + \frac{1}{6} \int_M R^2 = 32\pi^2 \chi(M) , \]
the right-hand sides can be written as
\[ \int_M |W|^2 + 8 \int_M |\text{Ric}|^2 - \frac{1}{3} \int_M R^2 = 5 \int_M |W|^2 + \frac{1}{3} \int_M R^2 - 128\pi^2 \chi(M) \]
and
\[ \int_M |W|^2 + 6 \int_M |\text{Ric}|^2 - \frac{1}{6} \int_M R^2 = 2 \int_M |\text{Ric}|^2 - 32\pi^2 \chi(M) . \]
This proves Remark 1.7.
Proof of Theorem 1.8. By (1.4), all inequalities leading to (3.16) become equalities. Hence
\[ n |W|^2 + \frac{n}{2} |\nabla |Ric| |^2 x = |Ric| |^2 x, \]
where \( c \) is constant, and at every point either \( Ric \) is null, i.e., \( M \) is Einstein, or it has an eigenvalue of multiplicity \( n - 1 \) and another of multiplicity 1 for \( |\nabla Ric|^2 = \frac{n}{n-1} |\nabla |Ric| |^2. \) Since \( M \) has harmonic curvature, \( M \) must be real analytic in suitable (harmonic) local coordinates. So we have \( W_{ij} R_j R_i = 0. \)
From (2.20), we get
\[
\triangle |Ric|^2 = 2 |\nabla |Ric| |^2 + 2 \frac{n}{n-2} R_i R_j \delta_i R_j + 2 \frac{R}{n-1} |Ric|^2.
\]
Based on the above equality, using the same argument as in the proof of (3.15), we can obtain that
\[
\left( \int_M |\tilde{Ric}|^2 \right)^{\frac{1}{2}} < \frac{1}{\sqrt{n(n-1)}} \mu(|g|),
\]
then \( M \) is Einstein.

Case 1 When \( W \neq 0 \), (1.4) implies that \( \left( \int_M |\tilde{Ric}|^2 \right)^{\frac{1}{2}} < \frac{1}{\sqrt{n(n-1)}} \mu(|g|) \). By the above result, \( M \) is Einstein.

Case 2 When \( W \equiv 0 \), \( M \) is locally conformally flat. Suppose that the Ricci tensor has an eigenvalue of multiplicity \( n - 1 \) and another of multiplicity 1. If the Ricci tensor is parallel, by the de Rham decomposition Theorem, \( M \) is covered isometrically by the product of Einstein manifolds. We have \( R = \sqrt{n(n-1)} |Ric| \). Since \( M \) has positive scalar curvature, then the only possibility is that \( M \) is covered isometrically by \( \mathbb{S}^1 \times \mathbb{S}^{n-1} \) with the product metric. On the other hand, if the Ricci tensor is not parallel, by the classification result of Derdziński, this concludes the proof of Theorem 1.8.

Proof of Theorem 1.9. Based on Lemma 2.4, using the same argument as in the proof of (3.15), we can obtain
\[
0 \geq \left[ (2 - \frac{1}{y}) \frac{n-2}{4(n-1)} \mu(|g|) \right] - C_2(n) \gamma \left( \int_M |W|^\frac{2}{y} \right)^{\frac{1}{2}} - 2 \left( \frac{n-1}{n} \right) \gamma \left( \int_M |Ric|^\frac{2}{y} \right)^{\frac{1}{2}} \left( \int_M |\tilde{Ric}|^{\frac{2}{y}} \right)^{\frac{1}{2}}
\]
\[+ \frac{8(n-1) \gamma + \frac{1}{2} n(n-2) - 2n(n-2)}{4n(n-1)} R \int_M |W|^2.\]

Case 1. \( 4 \leq n \leq 9 \), choose \( y = 1 \). It follows from (3.18) that
\[
0 \geq \left[ \frac{n-2}{4(n-1)} \mu(|g|) \right] - C_2(n) \left( \int_M |W|^2 \right)^{\frac{1}{2}} - 2 \left( \frac{n-1}{n} \right) \left( \int_M |Ric|^2 \right)^{\frac{1}{2}} \left( \int_M |\tilde{Ric}|^2 \right)^{\frac{1}{2}}
\[+ \frac{10n - 8 - n^2}{4n(n-1)} R \int_M |W|^2.\]

From (3.19), the pinching condition (1.5) implies that \( M \) is locally conformal flat. Moreover, we get the same conclusion if we assume just the weak inequality in (1.5). Hence, the pinching condition (1.5) implies
\[
\left( \int_M |\tilde{Ric}|^2 \right)^{\frac{1}{2}} < \frac{(n-2) \sqrt{n}}{8(n-1)^{\frac{1}{2}}} \mu(|g|) < \frac{1}{\sqrt{n(n-1)}} \mu(|g|).
\]
Case 2. \( n \geq 10 \), choose \( \frac{1}{y} = 1 + \frac{R(n-1)}{2n(n-2)} \). From (3.18), we get

\[
(3.21) \quad 0 \geq \left[ \frac{2}{n} \mu_{(g)} - C_2(n) \right] \left( \int_M |W|^2 \right)^{\frac{\gamma}{n}} - 2 \sqrt{\frac{n-1}{n}} \left( \int_M |\tilde{\nu}|^2 \right)^{\frac{\gamma}{n}} \left( \int_M |\tilde{\nu}|^2 \right)^{\frac{\gamma}{n}}.
\]

From (3.21), the pinching condition (1.5) implies that \( M \) is locally conformal flat. Hence, the pinching condition (1.5) implies

\[
(3.22) \quad \left( \int_M |\tilde{\nu}|^2 \right)^{\frac{\gamma}{n}} < \frac{1}{\sqrt{n(n-1)}} \mu_{(g)}.
\]

By the rigidity result for locally conformal flat manifolds (see Theorem 1.13 of [14]), noting that the difference between the Yamabe constants in [14] and this paper is \( \frac{4(n-1)}{n-1} \), (3.20) and (3.22) imply that \( M \) is isometric to a quotient of the round \( S^n \).

**Proof of Theorem 1.11.** By (1.6), all inequalities leading to (3.18) become equalities. Hence at every point, \( W \equiv 0 \) or either \( \tilde{\nu} \) is null, i.e., \( M \) is Einstein, or it has an eigenvalue of multiplicity \( (n-1) \) and another of multiplicity 1 for \( W_{ijkl} W_{ijkl} \tilde{\nu}_{kl} = \frac{1}{n} \mu_{(g)} |\tilde{\nu}| W^2 \).

By the same argument as in the proof of Theorem 1.8, we can obtain that if

\[
\left( \int_M |\tilde{\nu}|^2 \right)^{\frac{\gamma}{n}} < \frac{1}{\sqrt{n(n-1)}} \mu_{(g)},
\]

then \( M \) is Einstein.

Case 1 When \( W \equiv 0 \), (1.6) implies that \( \left( \int_M |\tilde{\nu}|^2 \right)^{\frac{\gamma}{n}} < \frac{1}{\sqrt{n(n-1)}} \mu_{(g)} \). By the above result, \( M \) is Einstein.

Case 2 When \( W \neq 0 \), \( M \) is locally conformally flat. (1.6) implies that (1.4) holds. By Theorem 1.8, this concludes the proof of Theorem 1.11.

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