Mini-Superspace Universality and No-Scale Quantum Cosmology

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We prove that, at the mini superspace level, and for an arbitrary Brans-Dicke parameter, one cannot tell traditional Einstein-Hilbert gravity from local scale invariant Weyl-Dirac gravity. Both quantum mechanical cosmologies are governed by the one and the same time-independent single-variable Hartle-Hawking wave function. It is only that its original argument, the cosmic scale factor $a$, is replaced by $a \phi$ ($\phi$ being the dilaton field) to form a Dirac in-scalar. The Weyl vector enters quantum cosmology only in the presence of an extra dimension, where its fifth component, serving as a 4-dim Kaluza-Klein in-scalar, governs the near Big Bang behavior of the wave function. The case of a constant Kaluza-Klein in-radius is discussed in some detail.

Introduction

The mini-superspace approximation $[1]$, while being premeditatedly naive and simple by construction, is still one of the best available theoretical tools to probe the quantum cosmology. The prototype mini-superspace Hartle-Hawking model is economical in its ingredients. They include: (i) The Einstein-Hilbert action, (ii) A pos-

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pend, using Dirac language, on in-scalars. Unfortunately,

the no-scale $C^2$ conformal cosmology is empty, leaving

the stage, as already mentioned earlier, to Weyl-Dirac

cosmology and two scalar gravity-anti-gravity cosmology
$[11]$. For more complicated cases, one may invoke Kaluza-

Klein reduced higher dimensional local scale symmetric

theories $[12]$. This allows the Weyl vector to enter the

game, and even govern the wave function behavior $[13]$ near Big Bang.

Weyl-Dirac preliminaries

Let our starting point be the Brans-Dicke theory, sup-

plemented by a quartic potential for the dilaton scalar

field $\phi(x)$, described by the action

$$I_{BD} = \int d^4x \sqrt{-g} \left( \phi^2 \mathcal{R} - 4 \omega g^\mu\nu \phi_{,\mu} \phi_{,\nu} - 2 \Lambda \phi^4 \right). \quad (1)$$

The theory enjoys \textit{global} scale invariance ($\Omega = \text{arbitrary constant}$) under

$$g_{\mu\nu} \to e^{2\Omega} g_{\mu\nu}, \quad \phi \to e^{-\Omega} \phi, \quad (2)$$

and furthermore exhibits the much reacher \textit{local} scale invariance ($\Omega(x) = \text{arbitrary function}$) for the critical case of $\omega = -\frac{3}{2}$. However, as prescribed by Dirac, local scale invariance can be extended to accompany any Brans-Dicke $\omega$-theory. The corresponding Weyl-Dirac gravity is field theoretically formulated by the action

$$I = \int d^4x \sqrt{-g} \left( \phi^2 \mathcal{R}^* - 4 \omega g^\mu\nu \phi_{,\mu} \phi_{,\nu} \right. \left. - \frac{1}{4} g^{\mu\nu} \mathcal{R} \mathcal{R}^* - 2 \Lambda \phi^4 \right). \quad (3)$$

The Ricci scalar $\mathcal{R}$, known to govern the Einstein-Hilbert action, has been consistently supplemented by two terms which involve the Weyl vector field $K^\mu$ and its divergence, such that the generalized (stared) curvature

$$\mathcal{R}^* = \mathcal{R} + g^\mu\nu \left( 6 K_{,\mu,\nu} - 6 K_{,\nu} K^\nu \right) \quad (4)$$

transforms as a co-scaler under the local scale symmetry, that is $\mathcal{R}^* \to e^{-2\Omega(x)} \mathcal{R}^*$. Respectively, the (stared) covariant dilaton derivative, defined by

$$\phi_{*\mu} = \phi_{,\mu} + K_{,\mu} \phi, \quad (5)$$

transforms ala $\phi_{*\mu} \to e^{-\Omega(x)} \phi_{*\mu}$ in accord with Eq. (2).

The last mandatory ingredient in the prescription is of course the Weyl Maxwell-like gauge transformation

$$K_{,\mu} \to K_{,\mu} + \Omega_{,\mu}, \quad (6)$$

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Introduction

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But is the exact $\psi_{HH}(a)$ a unique fingerprint of the underlying theory of general relativity? We first prove that the answer to this question is negative. One cannot tell, at least at the mini-superspace level, the Einstein-Hilbert gravity from the Weyl-Dirac gravity $[8]$ (not to be confused with $C^2$-gravity $[9]$, where $C_{\mu\nu,\lambda\sigma}$ stands for the Weyl tensor). Furthermore, the conclusion holds for an arbitrary Brans-Dicke parameter $[10]$.

This brings us to the more general topic of no-scale quantum cosmology, where the underlying gravitational theory exhibits local scale invariance. The latter local symmetry is translated into an additional constraint (on top of the Hamiltonian constraint), and may have a far reaching impact on the mini-superspace. The associated no-scale wave function of the Universe can only depend, using Dirac language, on in-scalars. Unfortunately, the no-scale $C^2$ conformal cosmology is empty, leaving the stage, as already mentioned earlier, to Weyl-Dirac cosmology and two scalar gravity-anti-gravity cosmology $[11]$. For more complicated cases, one may invoke Kaluza-Klein reduced higher dimensional local scale symmetric theories $[12]$. This allows the Weyl vector to enter the game, and even govern the wave function behavior $[13]$ near Big Bang.
which leaves invariant the associated anti-symmetric co-tensor $K_{\mu\nu} = K_{\mu
u} - K_{\nu\mu} = K_{\mu\nu} - K_{\nu\mu}$.

**No-scale quantum cosmology**

To keep track of the Hartle-Hawking model, we stick to an open universe ($\kappa > 0$), and a positive cosmological constant ($\Lambda > 0$) in the Einstein frame. The cosmological FRW line element

$$ds^2 = -n^2(t)dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right)$$

is accompanied by $K_\mu dx^\mu = v(t)dt$ and $\phi(t)$.

At the mini-superspace level, and for an arbitrary Brans-Dicke parameter, the Weyl-Dirac action Eq. [3] reduces after spatial integration to $\int {\mathcal{L}} dt$, with the mini-Lagrangian taking the form

$${\mathcal{L}} = n a (6 \kappa - 2 \Lambda a^2 \dot{\phi}^2) \phi^2 + \frac{2a}{n} ((3 + 2 \omega) a^2 (v \phi + \phi') - 3(\phi' a + \phi a')^2)$$

(to be compared with the corresponding Hartle-Hawking mini-Lagrangian

$${\mathcal{L}}_{HH} = na(6\kappa - 2\Lambda a^2) - \frac{6a}{n} n \phi^2$$

The canonical variables $n$ and $v$ are non-dynamical, and as such one is not allowed to prefix (= fix at the level of the Lagrangian) their values. Prefixing $n$ would kill the Hamiltonian constraint, and by the same token, prefixing $v$ means throwing away the scale invariance constraint. To be a bit more specific, the two associated primary constraints are

$$p_v = \frac{\partial {\mathcal{L}}}{\partial \dot{\phi}} \approx 0, \quad p_n = \frac{\partial {\mathcal{L}}}{\partial \dot{n}} \approx 0$$

they are clearly first class, exhibiting vanishing Poisson brackets

$$\{ p_n, p_v \} = 0$$

The two left over non-trivial momenta are given by

$$p_n = -\frac{12a\phi}{n} (a\phi)'$$

and

$$p_\phi = \frac{4a^2}{n} ((3 + 2 \omega) a (v \phi + \phi') - 3(a \phi')')$$

For non-critical scale invariance, that is $3 + 2 \omega \neq 0$, one can now inversely calculate the velocities

$$a' = v a - n \frac{3 \phi p_\phi + 2\omega a p_n}{12(3 + 2 \omega) a^2 \phi^2}$$

and

$$\phi' = -v \phi + n \frac{\phi p_\phi - a p_n}{4(3 + 2 \omega) a^2 \phi}$$

to be substituted into $\mathcal{H} = \sum p_i \dot{q}_i - S$, and eventually arrive at the Weyl-Dirac mini-superspace Hamiltonian

$$\mathcal{H} = v(a p_\phi - \phi p_n)$$

$${\mathcal{L}} = -\frac{n}{a} \left( \frac{p_n^2}{24} - \frac{(a p_\phi - \phi p_n)^2}{8(3 + 2 \omega) a^2 \phi^2} + 2(3\kappa - \Lambda a^2 \phi^2) a^2 \phi^2 \right).$$

(16)

It is notably linear in $n, v$, and should be compared with its Hartle-Hawking analogue

$$\mathcal{H}_{HH} = -\frac{n}{a} \left( \frac{p_n^2}{24} + 6a^2 \phi^2 - 2\Lambda a^4 \right).$$

(17)

On self consistency grounds, the two associated first class constraints Eq. [10] must Poisson commute with the mini-Hamiltonian Eq. [16], thereby leading to

$$\{ p_v, \mathcal{H} \} = \phi p_\phi - a p_n = 0,$$

$$\{ p_n, \mathcal{H} \} = \frac{1}{a} \left( \frac{p_n^2}{24} + 6a^2 \phi^2 - 2\Lambda a^4 \right) = 0$$

(19)

These equations must be satisfied not only classically, but following Dirac prescription, quantum mechanically as well, thus giving rise to two Schrödinger Wheeler-deWitt equations for the time-independent quantum mechanical wave function $\psi(a, \phi)$.

Let $p = -i\hbar \frac{\partial}{\partial a}$, and as usual replace $qp$ by its symmetrized Hermitian version $\frac{1}{2}(qp + pq) = qp - \frac{1}{2}i\hbar$. Consequently, the first Schrödinger equation, reflecting the scale invariance constraint, takes the $a$-independent form

$$\left( a \frac{\partial}{\partial a} - \phi \frac{\partial}{\partial \phi} \right) \psi(a, \phi) = 0,$$

(20)

whose most general solution is given by

$$\psi(a, \phi) = \psi(a, \phi),$$

meaning a single variable wave function. This is of course not a coincidence. Recalling the local scale symmetries of the mini-Lagrangian, in particular $a(t) \rightarrow e^{\omega t} a(t)$ along with $\phi(t) \rightarrow e^{-\omega t} \phi(t)$, so that the product $b(t) = a(t) \phi(t)$ transforms like an in-scalar. This is just a simple example of the general rule: In no-scale mini-superspace cosmology, the wave function $\psi$ can only depend on in-tensors, carrying no length units.

The second Schrödinger equation, reflecting the Hamiltonian constraint, then becomes

$$-\frac{\hbar^2}{24} \frac{d^2 \psi(b)}{db^2} + (6\kappa b^2 - 2\Lambda b^4) \psi(b) = 0.$$

(22)

This is now an ordinary differential equation, recognized as the original Hartle-Hawking equation for the in-scalar

$$b = a \phi.$$

(23)

This simply means that, at the mini superspace level, one cannot really tell traditional Einstein-Hilbert gravity from the local scale invariant Weyl-Dirac gravity. In
other words, the wave function $\psi_{HH}(a)$ is traded for $\psi_{HH}(a\phi)$. The result being $\omega$-independent.

**Weyl-Dirac Kaluza-Klein reduction**

To introduce more ingredients into the no-scale cosmology, we consider a Kaluza-Klein reduction of Weyl-Dirac gravity in 5-dimensions, given explicitly by

$$I_5 = \int d^5x \sqrt{-G} \left( \phi^2 R_5 + 4\omega_5 G^{MN} \phi_{\ast M} \phi_{\ast N} \right) - \frac{1}{4} \phi^2 G^{MN} G^{PQ} W_{MP} W_{NQ} - 2\Lambda \phi^4 \right) .$$

(24)

Here, adjusting the coefficients and the various powers of $\phi$ to fit the 5-dim world, we also have

$$R_5 = R_5 + G^{MN} (8W_{MN} - 12 W_M W_N) ,$$

(25)

$$\phi_{\ast M} = \phi_{M} + \frac{3}{2} W_M \phi ,$$

(26)

where $W_M$ stands here for the 5-dim Weyl vector, and correspondingly $W_{MN} = W_{M;N} - W_{N;M}$. We further note that, in 5-dimensions, the critical value of the Brans-Dicke parameter is $\omega_5 = -\frac{4}{3}$.

Truly, Kaluza-Klein compactification does introduce a fundamental length scale into the theory, viz.

$$dx_5 = 5d\theta \quad (\Delta \theta = 2\pi) .$$

(27)

However, owing to the fact that $x_5$-independence is imposed on a classical scale invariant theory, the Kaluza-Klein radius $\ell$ can be fully absorbed within the redefinitions of the fields which constitute the 4-dim effective theory. It is to say that after integrating out the 5-th dimension, the effective 4-dim ground state stays locally scale symmetric (in the 4-dim language).

Carrying out the Kaluza-Klein reduction procedure, the 5-dim line element dictionary reads

$$ds_5^2 = S^{-\frac{1}{2}} ds_4^2 + \frac{3}{2} \ell^2 (d\theta + A_{\mu} dx^\mu)^2 ,$$

(28)

with the algebraic advantage that $R_5 \sqrt{-G} = R_4 \sqrt{-g} + ...$

Similarly, the 5-dim Weyl vector field can be decomposed into

$$W_M dx^M = 2sK_{\mu} dx^\mu + s\ell (d\theta + A_{\mu} dx^\mu) ,$$

(29)

where the two 4-dim vector fields, the Weyl vector field $K_{\mu}$ and the Maxwell vector field $A_{\mu}$ have been normalized and fully diagonalized (imitating a $U(1) \otimes U(1)$ gauge theory). To this we add the simple yet powerful relation

$$\omega_5 = \omega_4 + \frac{1}{6} ,$$

(30)

so that $\frac{4}{3}(4 + 3\omega_5) = 2(3 + 2\omega_4)$, thereby assuring that the critical 5-dim Brans-Dicke theory properly reduces, as it should, down to the critical 4-dim Brans-Dicke theory.

After integrating out the Kaluza-Klein circle, that is $\int L_5 d^5x \rightarrow 2\pi \ell \int L_4 d^4x$, it becomes mandatory to redefine the various scalars floating around by adjusting their length units to fit the 4-dim language. Indeed, uniquely redefining according to

$$\phi \rightarrow \ell^{-\frac{1}{2}} \phi ,$$

$$S \rightarrow \ell^{-2} S ,$$

$$s \rightarrow \ell^{-1} s ,$$

(31)

will leave the action $\ell$-independent, meaning that local scale invariance has been restored. The interplay among these scalars, reflecting their specific charges under scale symmetry, will be discussed soon. Crucial for the forthcoming discussion is the identification of the in-scalars involved.

Altogether, the effective 4-dim theory is described by the Lagrangian

$$L = \phi^2 R_4 - 4\omega_4 g^{\mu \nu} \phi_{\ast \mu} \phi_{\ast \nu} - \frac{1}{4} \phi^2 SF^{\mu \nu} F_{\mu \nu}$$

$$- \frac{1}{4} (\phi^2 S)^2 \left( \frac{2}{5} K_{\mu} + sF_{\mu \nu} \right) \left( \frac{2}{5} K_{\mu} + sF_{\mu \nu} \right) - \frac{1}{6} \phi^2 g^{\mu \nu} \left( \phi^2 S \right)_{\ast \mu} \left( \phi^2 S \right)_{\ast \nu} - \frac{\phi^2}{(\phi^2 S)^2} g^{\mu \nu} s_{\ast \mu} s_{\ast \nu}$$

$$- \left( \frac{9}{2} \left( 3 + 2\omega_4 \right) \frac{s^2}{(\phi^2 S)^2} + \frac{2\Lambda}{(\phi^2 S)^2} \right) \phi^4 ,$$

(32)

where the 4-dim star derivatives are given explicitly by

$$\phi_{\ast \mu} = \phi_{\mu} + K_{\mu} \phi ,$$

$$\left( \phi^2 S \right)_{\ast \mu} = \left( \phi^2 S \right)_{\mu} ,$$

$$s_{\ast \mu} = s_{\mu} .$$

(33)

The detailed derivation has been carried out elsewhere, but in any case, Eq. (32) can have life of its own. While the above 4-dim Lagrangian may look a bit messy, it is nevertheless invariant under the local Maxwell gauge transformations

$$A_{\mu} \rightarrow A_{\mu} + \chi_{\mu} \; , \; K_{\mu} \rightarrow K_{\mu} ,$$

(34)

and in particular under the 4-dim local Weyl scale transformations

$$K_{\mu} \rightarrow K_{\mu} + \Omega_{\mu} \; , \; A_{\mu} \rightarrow A_{\mu} \; ,$$

$$g_{\mu \nu} \rightarrow e^{2\Omega} g_{\mu \nu} \; , \; \phi \rightarrow e^{-\Omega} \phi \; , \; S \rightarrow e^{2\Omega} S \; , \; s \rightarrow s .$$

(35)

We can thus finally identify the in-scalars of the theory, relevant for quantum cosmology. They are the built-in $s$ and the product $\phi^2 S$. In turn, any ‘ordinary’ 4-dim scalar field, having conventional units of $(length)^{-1}$, must be of the form $\phi^p (\phi^2 S)^q$ for arbitrary $p,q$.

Special attention should be devoted to the so-called Maxwell-Weyl kinetic mixing $sK^{\mu \nu} F_{\mu \nu}$, mediated by
the in-scalar \( s \). This is a unique feature which characterizes the Weyl-Dirac Kaluza-Klein interplay. Unfortunately, it does not play a direct role in no-scale cosmology, and as such, its remarkable aspects will be discussed in detail in a sequel publication.

**No-scale Kaluza-Klein quantum cosmology**

At the 4-dim mini-superspace level, cosmology can only tolerate the pure gauge configurations

\[ A_\mu = (A(t), 0, 0, 0) , \quad K_\mu = (\upsilon(t), 0, 0, 0) , \]

for which \( F_{\mu\nu} = K_{\mu\nu} = 0 \). While the various fields are neutral under the Kaluza-Klein \( U(1) \) symmetry, and as such do not couple to \( A_\mu \), the Weyl vector field \( K_\mu \) does enter the game via the star derivatives of the scalar fields.

It takes then some algebra, a bit lengthy but straightforward, to translate the mini-superspace version of the Lagrangian Eq. (32) into the Hamiltonian formalism. The result

\[
\mathcal{H} = v(pa - \phi p_\phi + 2SP_S) - \frac{n}{a} \left[ \frac{p_a^2}{24\phi^2} - \frac{3S^2 p_\phi^2}{2a^2 \phi^2} - \frac{S^2 p_\phi^2}{2a^2 \phi^2} - \frac{(pa - \phi p_\phi + 2SP_S)^2}{8(3 + 2\omega) a^2 \phi^2} + 6a^2 \phi^2 - \left( \frac{9(3 + 2\omega))}{2S^2} \right) a^4 \phi^4 \right]
\]

\[ \text{(37)} \]

is to be compared, term by term, with the simpler 4-dim Hamiltonian Eq. (16). As before, we face a Hamiltonian linear in \( v, n \), giving rise to two first class constraint. Since our interest primarily lies with quantum no-scale cosmology, we momentarily skip the classical equations of motion and their solutions, and proceed directly to the pair of Schrödinger equations. The coefficient of \( v \) in Eq. (37) is immediately recognized as the \( h \)-independent scale symmetry constraint, leading to

\[
\left( a \frac{\partial}{\partial a} - \phi \frac{\partial}{\partial \phi} + 2S \frac{\partial}{\partial S} \right) \psi(a, \phi, S, s) = 0 . \]

\[ \text{(38)} \]

This leaves the wave function to solely depend on Dirac’s in-scalars, e.g. on

\[ a^\alpha \phi^\beta S^\gamma \quad \text{for} \quad \alpha - \beta + 2\gamma = 0 . \]

\[ \text{(39)} \]

Without losing generality, the simplest choice would be

\[ \psi(a, \phi, S, s) = \psi(b, z, s) , \]

\[ \text{(40)} \]

where we have used the short hand in-scalar notations

\[ b = a\phi \ , \quad z = \log S \phi^2 . \]

\[ \text{(41)} \]

The Associated Hamiltonian constraint, identified as the coefficient of \( n \) in Eq. (37), eventually becomes the zero energy Schrödinger equation

\[ -\hbar^2 \frac{\partial^2 \psi}{\partial b^2} + 3\hbar^2 \frac{\partial^2 \psi}{\partial b^2} + \frac{e^{2z} \hbar^2}{2b^2} \frac{\partial^2 \psi}{\partial z^2} + \frac{e^{2z} \hbar^2}{2b^2} \frac{\partial^2 \psi}{\partial s} + V(b, z, s) \psi = 0 , \]

\[ \text{(42)} \]

with the accompanying potential being

\[ V(b, z, s) = 6\kappa b^2 - \left( \frac{9}{2} (3 + 2\omega_4)e^{-z} s^2 + 2e^{-\frac{1}{2} z} \Lambda \right) b^4 . \]

\[ \text{(43)} \]

The door is now widely open for a variety of spacial cases. Of particular interest is perhaps the \( \Lambda = 0 \) case, where the coefficient of \( b^4 \) can still be positive provided \( 3 + 2\omega_4 > 0 \). In some respects, the \( (3+2\omega) s^2 \)-term resembles the role of \( \Lambda_{eff}(s) \), and may even be tiny if the wave function is somehow concentrated around \( s^2 \ll 1 \) (this point will be sharpened soon). At any rate, on simplicity grounds, we choose to analyze in some detail only the special case of a constant Kaluza-Klein in-radius.

**Constant Kaluza-Klein in-radius**

In the standard Kaluza-Klein theory, with the line element Eq. (28), the invariant 5-dim radius is given by \( S^{\frac{2}{3}} \ell \). In the earliest versions of the theory (the original works of Kaluza and Klein, separately), the scalar degree of freedom was in fact frozen \( S = 1 \). Such an ansatz, while quite welcome on mathematical simplicity grounds, does...
not make any sense once local scale symmetry is applied. The closest one can get is by freezing a tenable in-scalar (nothing to do with gauge fixing), with the obvious in-ansatz being

\[ S\phi^2 = 1. \]  (44)

This will allow us to focus on the special role (beyond Weyl-Maxwell mixing, which anyhow does not have any cosmological fingerprints) played by the Weyl 4-dim in-scalar \( s \). The corresponding ‘handicapped’ wave function \( \psi(b, s) \) obeys the Hartle-Hawking equation

\[
-\frac{h^2}{24} \frac{\partial^2 \psi}{\partial b^2} + \frac{h^2}{2b^2} \frac{\partial^2 \psi}{\partial s^2} + V(b, s)\psi = 0, \tag{45}
\]

\[
V(b, s) = 6k b^2 - \left( \frac{9}{2} (3 + 2\omega_4) s^2 + 2\Lambda \right) b^4. \tag{46}
\]

The critical case \( \omega_4 = \frac{-3}{2} \) is the easiest to handle. It allows for the separation of variables \( \psi(b, s) = f(b)g(s) \), with \( f(b) \) serving as a modified Hartle-Hawking wave function subject to the effective potential

\[
V_{\text{eff}}(b) = \frac{n h^2}{2 b^2} + 6k b^2 - 2\Lambda b^4. \tag{47}
\]

and where the constant \( \eta \) governs the equation

\[
g''(s) = \eta g(s). \tag{48}
\]

Depending on the sign of \( \eta \), new quantum phenomena is expected to arise near the Big Bang origin \( b \to 0 \). The three cases are:

(i) If \( \eta = 0 \), the Hartle-Hawking model is fully recovered, accompanied by \( g(s) = \text{const} \). The no-boundary proposal recovered for \( f(b) \sim b \).

(ii) If \( \eta > 0 \) then \( g(s) = e^{\pm \sqrt{\eta s}} \) is unbounded, presumably signaling a non-physical case.

(iii) If \( \eta < 0 \) then \( g(s) = e^{\pm \sqrt{|\eta| s}} \) is well behaved. If furthermore \( \eta h^2 A^2 + 16\eta^2 > 0 \), as evident from the shape acquired by the effective potential in this case, cosmic evolution undergoes a (classically allowed) embryonic era. The no-boundary proposal is not recovered. However, the bonus in this case is automatic deWitt initial conditions, as both solutions \( f_{1,2}(b) \sim b^{1/2} \) (with \( 0 < Re(\delta_{1,2}) < 1 \)) vanish asymptotically at the origin. For further details, see Fig. 1, upper plot.

The non-critical case, for comparison, is characterized by an effective cosmological constant

\[
\Lambda_{\text{eff}}(s) = \Lambda + \frac{9}{4} (3 + 2\omega_4) s^2. \tag{49}
\]

The supplemented term is positive for a super-critical Brans-Dicke parameter \( \omega_4 > -\frac{3}{2} \) (including, in particular, the ghost-free case \( \omega_4 \geq 0 \)). While the separation of variables method does not work any more, the structure of the Schrodinger equation is such that the behavior of the wave function near the Big Bang is not sensitive to the value of \( \omega_4 \). The larger \( \omega_4 \), the more concentrated is the wave function around \( s^2 \ll 1 \). For further details, see Fig. 1, lower plot.

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