Multi-adjoint concept lattices via quantaloid-enriched categories

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Abstract
With quantaloids carefully constructed from multi-adjoint frames, it is shown that multi-adjoint concept lattices, multi-adjoint property-oriented concept lattices and multi-adjoint object-oriented concept lattices are derivable from Isbell adjunctions, Kan adjunctions and dual Kan adjunctions between quantaloid-enriched categories, respectively.

Keywords: Multi-adjoint concept lattice, Formal concept analysis, Rough set theory, Quantaloid, Isbell adjunction, Kan adjunction

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1. Introduction
The theory of quantaloid-enriched categories [27, 32, 33, 34] has deeply impacted the study of formal concept analysis (FCA) [6, 7] and rough set theory (RST) [22, 23] by providing them with a general categorical framework [9, 12, 28, 29]. Explicitly, given a small quantaloid $Q$, a $Q$-distributor

$\varphi : X \longrightarrow Y$

between $Q$-categories $X$ and $Y$ may be thought of as a multi-typed and multi-valued relation that is compatible with the $Q$-categorical structures on $X$ and $Y$, and it induces three pairs of adjoint $Q$-functors between the (co)presheaf $Q$-categories of $X$ and $Y$:

1. the Isbell adjunction [29] $\varphi^! \dashv \varphi^* : \text{P}^! Y \longrightarrow \text{P} X$,
2. the Kan adjunction [29] $\varphi^* \dashv \varphi_* : \text{P} X \longrightarrow \text{P} Y$,
3. the dual Kan adjunction [28] $\varphi^! \dashv \varphi^* : \text{P}^! X \longrightarrow \text{P}^! Y$.

If we consider a $Q$-distributor $\varphi : X \longrightarrow Y$ as a multi-typed and multi-valued context in the sense of FCA and RST, then the complete $Q$-categories of fixed points of the above adjunctions, denoted by

$M_\varphi := \text{Fix}(\varphi^! \varphi_\uparrow)$, \quad $K_\varphi := \text{Fix}(\varphi_\uparrow \varphi^*)$ and \quad $K^!_\varphi := \text{Fix}(\varphi^! \varphi_\uparrow)$,

may be viewed as “concept lattices” of the context $(X, Y, \varphi)$; indeed, if we assume that the $Q$-categories $X$ and $Y$ consist of properties (also attributes) and objects, respectively, then $M_\varphi$, $K_\varphi$ and $K^!_\varphi$ present the categorical version of the formal concept lattice, the property-oriented concept lattice and the object-oriented concept lattice of $(X, Y, \varphi)$, respectively. The recent work [12] of Lai and Shen establishes a general framework for constructing various kinds of representation theorems of such “concept lattices”. In particular:

1. If $Q = 2$, the two-element Boolean algebra, and $\varphi$ is a binary relation between (crisp) sets $X$ and $Y$, then $M_\varphi$, $K_\varphi$ and $K^!_\varphi$ reduce to the formal concept lattice [7], the property-oriented concept lattice and the object-oriented concept lattice [35, 36] of the (crisp) context $(X, Y, \varphi)$ in the classical setting.

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and we carefully exhibit how

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\[ \text{such that} \]

in Propositions 2. Adjoint triples as quantaloids

The general framework of quantaloid-enriched categories, and the aim of this paper is to provide an a

These results, once again, illustrate the thesis of Lawvere that

We are able to apply the constructions of Isbell adjunctions, Kan adjunctions and dual Kan adjunctions to

out of a multi-adjoint frame, a multi-adjoint property-oriented frame and a multi-adjoint object-oriented frame \( \mathcal{L} \), respectively, in Propositions 3.1, 5.2 and 5.5. In each of the three cases, a context \((X, Y, \varphi)\) of the respective frame \( \mathcal{L} \) is expressed as a \( \mathcal{Q}_\mathcal{F}^\mathcal{L} \)-relation \( \varphi_F : X \twoheadrightarrow Y \), a \( \mathcal{Q}_\mathcal{P}^\mathcal{L} \)-relation \( \varphi_P : X \rightrightarrows Y \) and a \( \mathcal{Q}_\mathcal{O}^\mathcal{L} \)-relation \( \varphi_O : X \twoheadrightarrow Y \), respectively, in Propositions 3.3, 5.3 and 5.6. Therefore, with the necessary backgrounds of \( \mathcal{Q} \)-categories introduced in Section 4, we are able to apply the constructions of Isbell adjunctions, Kan adjunctions and dual Kan adjunctions to \( \varphi_F, \varphi_P \) and \( \varphi_O \), respectively, and obtain the following main results of this paper:

(1) The multi-adjoint concept lattice [20] of a context \((X, Y, \varphi)\) of a multi-adjoint frame \( \mathcal{L} \) is given by a fibre of the complete \( \mathcal{Q}_\mathcal{F}^\mathcal{L} \)-category \( \mathcal{M}_\mathcal{F} \) (Theorem 4.2).

(2) The multi-adjoint property-oriented concept lattice [16] of a context \((X, Y, \varphi)\) of a multi-adjoint property-oriented frame \( \mathcal{L} \) is given by a fibre of the complete \( \mathcal{Q}_\mathcal{P}^\mathcal{L} \)-category \( \mathcal{K}_\mathcal{P} \) (Theorem 5.4).

(3) The multi-adjoint object-oriented concept lattice [16] of a context \((X, Y, \varphi)\) of a multi-adjoint object-oriented frame \( \mathcal{L} \) is given by a fibre of the complete \( \mathcal{Q}_\mathcal{O}^\mathcal{L} \)-category \( \mathcal{K}_\mathcal{O} \) (Theorem 5.7).

These results, once again, illustrate the thesis of Lawvere that fundamental structures are themselves categories [14].

2. Adjoint triples as quantaloids

In this section we formalize adjoint triples, the cornerstone of the theory of multi-adjoint concept lattices, as a special kind of quantaloids.

Recall that an adjoint triple [19, 20, 21] (\( \& \), \( \lor \), \( \setminus \)) with respect to posets \( L_1, L_2, P \) consists of maps

\[ \& : L_1 \times L_2 \longrightarrow P, \quad \lor : P \times L_2 \longrightarrow L_1, \quad \setminus : P \times L_1 \longrightarrow L_2 \]

such that

\[ x \geq x', \quad y \geq y', \quad z \leq z' \quad \Longrightarrow \quad x' \& y \leq x \& y, \quad z \lor y \leq z' \lor y', \quad z \setminus x \leq z' \setminus x' \quad (2.1) \]

and

\[ x \& y \leq z \quad \iff \quad x \leq z \lor y \quad \iff \quad y \leq z \setminus x \quad (2.2) \]
for all \( x, x' \in L_1, y, y' \in L_2, z, z' \in P \).

As the completeness of the posets under concern is necessary to construct concept lattices later on, it does no harm to restrict our discussion to adjoint triples with respect to complete lattices. From now on we always assume that \( L_1, L_2, P \) are complete lattices\(^1\). Hence, an adjoint triple \((\&, \land, \lor)\) with respect to \( L_1, L_2, P \) is uniquely determined by a map

\[
\&: L_1 \times L_2 \longrightarrow P
\]

that preserves joins on both sides, i.e.,

\[
\left( \bigvee_{i \in I} x_i \right) \land y = \bigvee_{i \in I} x_i \land y \quad \text{and} \quad x \land \left( \bigvee_{i \in I} y_i \right) = \bigvee_{i \in I} x \land y_i
\]

for all \( x, x_i \in L_1, y, y_i \in L_2 (i \in I) \); consequently, the maps \( \land : P \times L_2 \longrightarrow L_1, \lor : P \times L_1 \longrightarrow L_2 \) would be uniquely determined by the Galois connections

\[
L_1 \xrightarrow{\land} P \quad \text{and} \quad L_2 \xleftarrow{\lor} P
\]

induced by \( \land \) for all \( x \in L_1, y \in L_2 \), which necessarily satisfy Equations (2.i) and (2.ii).

It is then natural to regard \( L_1, L_2, P \) as hom-sets of a quantaloid of three objects as the following Proposition 2.1 reveals. To this end, let us recall that a quantaloid \( Q \) [27, 34] is a category whose hom-sets are complete lattices, such that the composition \( \circ \) of \( Q \)-arrows preserves arbitrary joins on both sides, i.e.,

\[
v \circ \left( \bigvee_{i \in I} u_i \right) = \bigvee_{i \in I} v \circ u_i \quad \text{and} \quad \left( \bigvee_{i \in I} v_i \right) \circ u = \bigvee_{i \in I} v_i \circ u
\]

for all \( u, u_i \in Q(p, q), v, v_i \in Q(q, r) (i \in I) \). Hence, the corresponding Galois connections induced by the compositions

\[
Q(q, r) \xrightarrow{\lor} Q(p, r) \quad \text{and} \quad Q(p, q) \xleftarrow{\land} Q(p, r)
\]

satisfy

\[
v \circ u \leq w \iff v \leq w / u \iff u \leq v \setminus w \quad (2.iii)
\]

for all \( u \in Q(p, q), v \in Q(q, r), w \in Q(p, r) \), where the operations / and \( \setminus \) are called left and right implications in \( Q \), respectively.

Let \( Q_0 \) denote the class of objects of a quantaloid \( Q \). For each \( p, q \in Q_0 \), we denote by \( \bot_{p,q} \) the bottom element of the hom-set \( Q(p, q) \), and by \( \text{id}_q \) the identity \( Q \)-arrow on \( q \). A quantaloid \( Q \) is non-trivial if

\[
\bot_{q,q} < \text{id}_q
\]

for all \( q \in Q_0 \), since \( \bot_{q,q} = \text{id}_q \) would force every hom-set \( Q(p, q) \) or \( Q(q, r) (p, r \in Q_0) \) to contain only one element, i.e., \( \bot_{p,q} \) or \( \bot_{q,r} \).

**Proposition 2.1.** Each adjoint triple \((\&, \land, \lor)\) with respect to \( L_1, L_2, P \) determines a non-trivial quantaloid \( Q_\& \) consisting of the following data:

- \((Q_\&)_0 = \{-1, 0, 1\} ;
- \( Q_\&(-1, 0) = L_1, Q_\&(0, 1) = L_2, Q_\&(-1, 1) = P ;
- \( Q_\&(i, j) = \{\bot_{i,j}, \text{id}_i\} \) for all \( i = -1, 0, 1 \), and \( Q_\&(i, j) = \{\bot_{i,j}\} \) whenever \( -1 \leq j < i \leq 1 ;

\(^1\)In fact, even if \( L_1, L_2, P \) are not complete, adjoint triples with respect to \( L_1, L_2, P \) may be extended to their Dedekind–MacNeille completions (see [18, Lemma 38]).
• compositions in $\mathcal{Q}_\&$ are given by
  \[ v \circ u = u \& v \]
  for all $u \in \mathcal{Q}_\&(-1, 0) = L_1$, $v \in \mathcal{Q}_\&(0, 1) = L_2$, and the other compositions are trivial;

• left and right implications in $\mathcal{Q}_\&$ are given by
  \[ w \leftarrow u = w \land u \quad \text{and} \quad v \rightarrow w = w \lor v \]
  for all $u \in \mathcal{Q}_\&(-1, 0) = L_1$, $v \in \mathcal{Q}_\&(0, 1) = L_2$, $w \in \mathcal{Q}_\&(-1, 1) = P$, and the other implications are trivial.

3. Contexts of a multi-adjoint frame as $\mathcal{Q}$-relations

The quantaloid constructed in Proposition 2.1 can be extended to characterize the notion of multi-adjoint frame [20]. Explicitly, a multi-adjoint frame is a tuple
\[ \mathcal{L} = (L_1, L_2, P, \&_1, \lor_1, \land_1, \ldots, \&_n, \lor_n, \land_n), \]
such that $(\&_i, \lor_i, \land_i)$ is an adjoint triple with respect to $L_1, L_2, P$ for all $i = 1, \ldots, n$, and it corresponds to a quantaloid of $n + 2$ objects:

**Proposition 3.1.** Each multi-adjoint frame $\mathcal{L} = (L_1, L_2, P, \&_1, \ldots, \&_n)$ gives rise to a non-trivial quantaloid $\mathcal{Q}_\mathcal{L}^\mathcal{F}$ consisting of the following data:

• $(\mathcal{Q}_\mathcal{L}^\mathcal{F})_0 = \{-1, 0, 1, \ldots, n\}$;

• $\mathcal{Q}_\mathcal{L}^\mathcal{F}(-1, 0) = L_1$, $\mathcal{Q}_\mathcal{L}^\mathcal{F}(0, i) = L_2$, $\mathcal{Q}_\mathcal{L}^\mathcal{F}(-1, i) = P$ for all $i = 1, \ldots, n$;

• $\mathcal{Q}_\mathcal{L}^\mathcal{F}(i, i) = \{\bot_i, \top_i, \land_i, \lor_i, \&_i, \lor_i, \land_i\}$ whenever $-1 \leq i \leq n$ or $0 < i < j \leq n$;

• compositions in $\mathcal{Q}_\mathcal{L}^\mathcal{F}$ are given by
  \[ v \circ u = u \&_i v \]
  for all $u \in \mathcal{Q}_\mathcal{L}^\mathcal{F}(-1, 0) = L_1$, $v \in \mathcal{Q}_\mathcal{L}^\mathcal{F}(0, i) = L_2$ $(i = 1, \ldots, n)$, and the other compositions are trivial;

• left and right implications in $\mathcal{Q}_\mathcal{L}^\mathcal{F}$ are given by
  \[ w \leftarrow u = w \land_i u \quad \text{and} \quad v \rightarrow w = w \lor_i v \]
  for all $u \in \mathcal{Q}_\mathcal{L}^\mathcal{F}(-1, 0) = L_1$, $v \in \mathcal{Q}_\mathcal{L}^\mathcal{F}(0, i) = L_2$, $w \in \mathcal{Q}_\mathcal{L}^\mathcal{F}(-1, i) = P$ $(i = 1, \ldots, n)$, and the other implications are trivial.

In what follows we will see that contexts of a multi-adjoint frame $\mathcal{L}$ may be considered as relations valued in the quantaloid $\mathcal{Q}_\mathcal{L}^\mathcal{F}$. As a preparation let us introduce the notion of $\mathcal{Q}$-relation for a small quantaloid $\mathcal{Q}$, in which $\mathcal{Q}_0$ is assumed to be a set instead of a proper class.

Given a (“base”) set $T$, a set $X$ equipped with a map $|\cdot| : X \rightarrow T$ is called a $T$-typed set, where the value $|x| \in T$ is the type of $x \in X$, and we write
\[ X_q := \{x \in X \mid |x| = q\} \]
for the fibre of $X$ over $q \in T$. Let $\mathcal{Q}$ be a small quantaloid and taking $\mathcal{Q}_0$ as the set of types, a $\mathcal{Q}$-relation (also $\mathcal{Q}$-matrix [10])
\[ \varphi : X \rightarrow Y \]
between $\mathcal{Q}_0$-typed sets is given by a family of $\mathcal{Q}$-arrows $\varphi(x, y) \in \mathcal{Q}(|x|, |y|)$ $(x \in X, y \in Y)$. With the pointwise local order
\[ \varphi \leq \varphi' : X \rightarrow Y \iff \forall x, y \in X : \varphi(x, y) \leq \varphi'(x, y) \text{ in } \mathcal{Q}(|x|, |y|) \]

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inherited from \( Q \), the category \( Q\text{-Rel} \) of \( Q_0 \)-typed sets and \( Q \)-relations becomes a (large) quantaloid in which

\[
\psi \circ \varphi : X \rightarrow Z, \quad (\psi \circ \varphi)(x, z) = \bigvee_{y \in Y} \psi(y, z) \circ \varphi(x, y), \quad (3.1)
\]

\[
\xi / \varphi : Y \rightarrow Z, \quad (\xi / \varphi)(y, z) = \bigwedge_{x \in X} \xi(x, z) / \varphi(x, y), \quad (3.ii)
\]

\[
\varphi \setminus \xi : X \rightarrow Y, \quad (\varphi \setminus \xi)(x, y) = \bigwedge_{z \in Z} \varphi(y, z) \setminus \xi(x, z) \quad (3.iii)
\]

for \( Q \)-relations \( \varphi : X \rightarrow Y, \, \psi : Y \rightarrow Z, \, \xi : X \rightarrow Z \), and

\[
\kappa_X : X \rightarrow X, \quad \kappa_X(x, y) = \begin{cases} \text{id}_{[x]}, & \text{if } x = y, \\ \downarrow_{[y]}, & \text{else} \end{cases}
\]

serves as the identity \( Q \)-relation on \( X \).

**Remark 3.2.** \( Q \)-relations between \( Q_0 \)-typed sets may be thought of as *multi-typed* and *multi-valued* relations. Indeed, a \( Q \)-relation \( \varphi : X \rightarrow Y \) may be decomposed into a family of \( Q(p, q) \)-valued relations

\[
\varphi_{p,q} : X_p \rightarrow Y_q \quad (p, q \in Q_0),
\]

i.e., a family of maps

\[
\varphi_{p,q} : X_p \times Y_q \rightarrow Q(p, q),
\]

where \( \varphi_{p,q} \) is the restriction of \( \varphi \) on the fibres \( X_p \) and \( Y_q \).

Recall that a context [20] of a multi-adjoint frame \( \mathcal{L} = (L_1, L_2, P, &_1, \ldots, &_n) \) is a \( P \)-valued relation \( \varphi : X \rightarrow Y \) (i.e., a map \( \varphi : X \times Y \rightarrow P \)) equipped with a map \( \|- : Y \rightarrow [1, \ldots, n] \), where \( X \) is interpreted as the set of *properties* (also *attributes*) and \( Y \) the set of *objects*. Therefore:

**Proposition 3.3.** Let \( \mathcal{L} = (L_1, L_2, P, &_1, \ldots, &_n) \) be a multi-adjoint frame and let \( Q^P_{\mathcal{L}} \) be the quantaloid determined by Proposition 3.1. Then a context \((X, Y, \varphi)\) of \( \mathcal{L} \) is exactly a \( Q^P_{\mathcal{L}} \)-relation \( \varphi_F : X \rightarrow Y \) between \( Q^P_{\mathcal{L}} \)-typed sets with

\[
|x| = -1, \quad |y| \in [1, \ldots, n] \quad \text{and} \quad \varphi_F(x, y) = \varphi(x, y)
\]

for all \( x \in X, \, y \in Y \).

4. Multi-adjoint concept lattices via Isbell adjunctions

It is time to exhibit the powerful arsenal of quantaloid-enriched categories [28, 29, 32, 33, 34] which, in particular, allows us to capture the categorial structure of multi-adjoint concept lattices.

4.1. \( Q \)-categories

From now on \( Q \) always denotes a small quantaloid. A \( Q \)-category (or, a *category enriched in \( Q \*)) [27, 32] is a \( Q_0 \)-typed set \( X \) equipped with a \( Q \)-relation \( 1^\#_X : X \rightarrow X \), such that

\[
\kappa_X \leq 1^\#_X \quad \text{and} \quad 1^\#_X \circ 1^\#_X \leq 1^\#_X
\]

in the quantaloid \( Q\text{-Rel} \); that is,

\[
\text{id}_{[x]} \leq 1^\#_X(x, x) \quad \text{and} \quad 1^\#_X(y, z) \circ 1^\#_X(x, y) \leq 1^\#_X(x, z)
\]

for all \( x, y, z \in X \). With morphisms of \( Q \)-categories given by \( Q \)-functors \( f : X \rightarrow Y \), i.e., maps \( f : X \rightarrow Y \) such that

\[
|x| = |fx| \quad \text{and} \quad 1^\#_Y(fx, fx') \leq 1^\#_Y(x, x')
\]
for all \( x, x' \in X \), we obtain a category

\[ \mathcal{Q}\text{-Cat}. \]

A pair of \( \mathcal{Q} \)-functors \( f : X \longrightarrow Y, \ g : Y \longrightarrow X \) forms an adjunction in \( \mathcal{Q}\text{-Cat} \), denoted by \( f \dashv g \), if

\[ 1^X_f(f(x), y) = 1^X_g(x, gy) \quad (4.1) \]

for all \( x \in X, \ y \in Y \). In this case, we say that \( f \) is the left adjoint of \( g \), and \( g \) is the right adjoint of \( f \).

A \( \mathcal{Q} \)-relation \( \varphi : X \leftrightarrow Y \) between \( \mathcal{Q} \)-categories becomes a \( \mathcal{Q} \)-distributor if

\[ 1^X_Y \circ \varphi \circ 1^X_X = \varphi; \]

that is,

\[ 1^Y_Y(y, y') \circ \varphi(x, y) \circ 1^X_X(x', x) \leq \varphi(x, y') \]

for all \( x, x' \in X, \ y, y' \in Y \). \( \mathcal{Q} \)-categories and \( \mathcal{Q} \)-distributors constitute a (large) quantaloid \( \mathcal{Q}\text{-Dist} \) in which compositions and implications are calculated as in \( \mathcal{Q}\text{-Rel} \); the identity \( \mathcal{Q} \)-distributor on each \( \mathcal{Q} \)-category \( X \) is given by \( 1^X_X : X \longrightarrow X \).

Each \( \mathcal{Q}_0 \)-typed set \( X \) is equipped with a discrete \( \mathcal{Q} \)-category structure, given by the identity \( \mathcal{Q} \)-relation \( s_X \). In particular, for each \( q \in \mathcal{Q}_0 \), \( \{q\} \) is a discrete \( \mathcal{Q} \)-category with only one object \( q \) with \( |q| = q \). It is obvious that each \( \mathcal{Q} \)-relation \( \varphi : X \leftrightarrow Y \) can be viewed as a \( \mathcal{Q} \)-distributor of discrete \( \mathcal{Q} \)-categories, and thus \( \mathcal{Q}\text{-Rel} \) is embedded in \( \mathcal{Q}\text{-Dist} \) as a full subquantaloid.

A presheaf with type \( q \) on a \( \mathcal{Q} \)-category \( X \) is a \( \mathcal{Q} \)-distributor \( \mu : X \leftrightarrow \{q\} \). Presheaves on \( X \) constitute a \( \mathcal{Q} \)-category \( \mathcal{P}^\dagger X \) with

\[ 1^X_\mathcal{P}^\dagger_X(\mu, \mu') := \mu'/\mu = \bigwedge_{x \in X} \mu'(x)/\mu(x) \]

for all \( \mu, \mu' \in \mathcal{P}^\dagger X \). Dually, the \( \mathcal{Q} \)-category \( \mathcal{P}^\dagger X \) of copresheaves on \( X \) consists of \( \mathcal{Q} \)-distributors \( \lambda : \{q\} \longrightarrow X \) as objects with type \( q \) (\( q \in \mathcal{Q}_0 \)), and

\[ 1^\mathcal{P}^\dagger_X^\dagger(\lambda, \lambda') := \lambda' \setminus \lambda = \bigwedge_{x \in X} \lambda'(x) \setminus \lambda(x) \]

for all \( \lambda, \lambda' \in \mathcal{P}^\dagger X \).

A \( \mathcal{Q} \)-category \( X \) is complete if the Yoneda embedding

\[ y : X \longrightarrow \mathcal{P}X, \ x \mapsto 1^X_X(-, x) \]

has a left adjoint \( \text{sup} : \mathcal{P}X \longrightarrow X \) in \( \mathcal{Q}\text{-Cat} \); that is,

\[ 1^X_X(\text{sup} \mu, -) = 1^\mathcal{P}^\dagger_X(\mu, y-) = 1^X_X(\mu, y) \]

for all \( \mu \in \mathcal{P}X \). It is well known that the completeness of \( X \) can also be characterized through the existence of a right adjoint \( \text{inf} : \mathcal{P}^\dagger X \longrightarrow X \) of the co-Yoneda embedding (see [32, Proposition 5.10])

\[ y^\dagger : X \longrightarrow \mathcal{P}^\dagger X, \ x \mapsto 1^\mathcal{P}^\dagger_X(x, -). \]

It follows from [32, Proposition 6.4] that for any \( \mathcal{Q} \)-category \( X \), both \( \mathcal{P}X \) and \( \mathcal{P}^\dagger X \) are complete \( \mathcal{Q} \)-categories.

4.2. The underlying order of \( \mathcal{Q} \)-categories

Every \( \mathcal{Q} \)-category \( X \) admits a natural underlying (pre)order, given by

\[ x \leq y \iff |x| = |y| = q \text{ and } \text{id}_q \leq 1^X_X(x, y) \]

for all \( x, y \in X \). We write \( x \equiv y \) if \( x \leq y \) and \( y \leq x \). A \( \mathcal{Q} \)-category \( X \) is separated if its underlying order is a partial order; that is, \( x \equiv y \) implies \( x = y \) for all \( x, y \in X \).
4.3. Multi-adjoint concept lattices as fixed points of Isbell adjunctions

Recall that a Q-closure operator \([29] \ c : X \to X\) on a Q-category \(X\) is a Q-functor satisfying

\[
1_X \leq c \quad \text{and} \quad cc \equiv c,
\]

and it follows from \([29], \text{Propositions } 3.3 \text{ and } 3.5\) that if \(X\) is a complete Q-category, then

\[
\text{Fix}(c) := \{ x \in X \mid cx \equiv x \}
\]

is also complete with the inherited Q-category structure from \(X\). In particular, every pair of adjoint Q-functors

\[
X \xleftarrow{f} Y \quad \text{induces a Q-closure operator } gf : X \to X \quad \text{(see } [29, \text{Example } 3.2]).
\]

Each Q-distributor \(\varphi : X \to Y\) of Q-categories induces a pair of adjoint Q-functors

\[
\begin{array}{c}
\text{PX} \xleftarrow{\varphi} \text{P}^\dagger Y \\
\text{PX} \xrightarrow{\varphi^\dagger} \text{P}^\dagger Y
\end{array}
\]

in \(\text{Q-Cat}\), called the Isbell adjunction (see \([29, \text{Proposition } 4.1]\)), given by

\[
\varphi_! \mu = \varphi / \mu \quad \text{and} \quad \varphi^! \lambda = \lambda \setminus \varphi,
\]

for all \(\mu \in \text{PX}, \lambda \in \text{P}^\dagger Y\). In elementary words,

\[
(\varphi_! \mu)(y) = \bigvee_{x \in X} \varphi(x, y) / \mu(x) \quad \text{and} \quad (\varphi^! \lambda)(x) = \bigwedge_{y \in Y} \lambda(y) \setminus \varphi(x, y)
\]

for all \(\mu \in \text{PX}, y \in Y, \lambda \in \text{P}^\dagger Y, x \in X\). The induced Q-closure operator \(\varphi^! \varphi_! : \text{PX} \to \text{PX}\) generates a complete Q-category

\[
\text{M}_\varphi := \text{Fix}(\varphi^! \varphi_!) = \{ \mu \in \text{PX} \mid \varphi^! \varphi_! \mu = \mu \},
\]

where “\(\equiv\)” is replaced by “\(=\)” due to the separatedness of \(\text{PX}\).
Remark 4.1. Isbell adjunctions between quantaloid-enriched categories set up a very general framework of formal concept analysis (FCA).

If \( Q = \mathbb{2} \) is the two-element Boolean algebra, then a 2-distributor \( \varphi : X \longrightarrow Y \) between discrete 2-categories is just a binary relation between (crisp) sets, and \( M \varphi \) is the concept lattice of the (crisp) context \((X, Y, \varphi)\).

If \( Q \) has only one object, i.e., \( Q = \mathbb{Q} \) is a unital quantale [26], then a \( \mathbb{Q} \)-distributor \( \varphi : X \longrightarrow Y \) between discrete \( \mathbb{Q} \)-categories is a fuzzy relation between (crisp) sets (i.e., \( \varphi \) is a map \( X \times Y \to \mathbb{Q} \)). Considering \((X, Y, \varphi)\) as a fuzzy context of (crisp) sets \( X \) and \( Y \), its concept lattice is also given by \( M \varphi \) (cf. [1, 13, 30]).

If \( Q = \mathbb{D} \mathbb{Q} \) is the quantaloid of diagonals (cf. [11, 25, 34]) of a quantale \( \mathbb{Q} \), then a \( \mathbb{Q} \)-distributor \( \varphi : X \longrightarrow Y \) between discrete \( \mathbb{Q} \)-categories is a fuzzy relation between fuzzy sets (cf. [9, Definition 2.3]), and the induced \( M \varphi \) is the concept lattice of the fuzzy context \((X, Y, \varphi)\) of fuzzy sets \( X \) and \( Y \) [9, 28, 31].

Now let us return to the \( Q^L \)-relation \( \varphi_F : X \longrightarrow Y \) obtained from a context \((X, Y, \varphi)\) of a multi-adjoint frame \( L = (L_1, L_2, P, \&,...) \) in Proposition 3.3. Since \(|x| = -1\) and \(|y| \in \{1, \ldots, n\}\) for all \( x \in X \), \( y \in Y \), considering \( X \) and \( Y \) as discrete \( Q^L \)-categories we have

\[
\begin{align*}
(PX)_0 &= Q^L_X(1, 0)^X = L^X_1 \quad \text{and} \quad (P^Y)_0 = \prod_{i \in I} L^Y_i = L^Y_i
\end{align*}
\]

Hence, the restriction of the Isbell adjunction \((\varphi_F)_!* \dashv (\varphi_F)^!\) on the 0-fibres of \( P\!X \) and \( P^Y \)

\[
\begin{array}{ccc}
(PX)_0 & \xrightarrow{\varphi} & (P^Y)_0 \\
\downarrow & & \downarrow \\
(PX)_0 & \xrightarrow{(\varphi_F)_*} & (P^Y)_0
\end{array}
\]

exactly reproduces the Galois connection obtained in [20, Proposition 7], which satisfies

\[
\begin{align*}
((\varphi_F)_! \mu)(y) &= \bigwedge_{x \in X} \varphi_F(x, y) / \mu(x) = \bigwedge_{x \in X} \varphi(x, y) \wedge \mu(x) \\
((\varphi_F)^| \lambda)(x) &= \bigwedge_{y \in Y} \lambda(y) \vee \varphi_F(x, y) = \bigwedge_{y \in Y} \varphi(x, y) \vee \mu(y)
\end{align*}
\]

for all \( \mu \in (PX)_0 \) and \( \lambda \in (P^Y)_0 \).

Now let us return to the \( Q^L \)-relation \( \varphi_F : X \longrightarrow Y \) obtained from a context \((X, Y, \varphi)\) of a multi-adjoint frame \( L = (L_1, L_2, P, \&,...) \) in Proposition 3.3. Since \(|x| = -1\) and \(|y| \in \{1, \ldots, n\}\) for all \( x \in X \), \( y \in Y \), considering \( X \) and \( Y \) as discrete \( Q^L \)-categories we have

\[
\begin{align*}
(PX)_0 &= Q^L_X(-1, 0)^X = L^X_1 \quad \text{and} \quad (P^Y)_0 = \prod_{i \in I} Q^L_Y(0, i)^Y = \prod_{i \in I} L^Y_i = L^Y_i
\end{align*}
\]

Hence, the restriction of the Isbell adjunction \((\varphi_F)_!* \dashv (\varphi_F)^!\) on the 0-fibres of \( P\!X \) and \( P^Y \)

\[
\begin{array}{ccc}
(PX)_0 & \xrightarrow{\varphi} & (P^Y)_0 \\
\downarrow & & \downarrow \\
(PX)_0 & \xrightarrow{(\varphi_F)_*} & (P^Y)_0
\end{array}
\]

exactly reproduces the Galois connection obtained in [20, Proposition 7], which satisfies

\[
\begin{align*}
((\varphi_F)_! \mu)(y) &= \bigwedge_{x \in X} \varphi_F(x, y) / \mu(x) = \bigwedge_{x \in X} \varphi(x, y) \wedge \mu(x) \\
((\varphi_F)^| \lambda)(x) &= \bigwedge_{y \in Y} \lambda(y) \vee \varphi_F(x, y) = \bigwedge_{y \in Y} \varphi(x, y) \vee \mu(y)
\end{align*}
\]

for all \( \mu \in (PX)_0 \) and \( \lambda \in (P^Y)_0 \).

Since the multi-adjoint concept lattice of \((X, Y, \varphi)\) is the complete lattice of fixed points of the Galois connection (4.iv) (cf. [20, Definition 8]), it is obviously given by the 0-fibre of \( M \varphi_F \).

Theorem 4.2. The multi-adjoint concept lattice of a context \((X, Y, \varphi)\) of a multi-adjoint frame \( L \) is isomorphic to the complete lattice \((M \varphi)_0 \), where \( M \varphi \) is the complete \( Q^L \)-category of fixed points of the Isbell adjunction (4.iii) induced by the \( Q^L \)-relation \( \varphi_F : X \longrightarrow Y \) in Proposition 3.3.

5. Multi-adjoint property-oriented and object-oriented concept lattices via Kan adjunctions

Multi-adjoint object-oriented and property-oriented concept lattices introduced in [16] can also be realized through adjoint functors enriched in quantaloids, and it is the goal of this section.

5.1. Kan adjunctions

Each \( Q \)-distributor \( \varphi : X \longrightarrow Y \) of \( Q \)-categories induces another two pairs of adjoint \( Q \)-functors in \( Q\text{-Cat} \): one is the Kan adjunction (see [29, Proposition 5.1])

\[
\begin{array}{ccc}
PY & \xrightarrow{\varphi^*} & PX \\
\downarrow & & \downarrow \\
PY & \xrightarrow{\varphi} & PX
\end{array}
\]

given by

\[
\varphi^* \lambda = \lambda \circ \varphi \quad \text{and} \quad \varphi_* \mu = \mu / \varphi,
\]

8
which are calculated as
\[(\varphi^\ast, \lambda)(x) = \bigvee_{y \in Y} \lambda(y) \circ \varphi(x, y) \quad \text{and} \quad (\varphi^\ast \mu)(y) = \bigwedge_{x \in X} \mu(x) / \varphi(x, y)\]
for all \(\lambda \in PY, x \in X, \mu \in PX, y \in Y\); the other is the dual Kan adjunction (see [28, Proposition 6.2.1])
\[
P^\ast Y \xrightarrow{\varphi^\ast} P^\ast X
\]
given by
\[\varphi^\ast \lambda = \varphi \setminus \lambda \quad \text{and} \quad \varphi^\ast \mu = \varphi \circ \mu,\]
which are calculated as
\[(\varphi^\ast \lambda)(x) = \bigwedge_{y \in Y} \varphi(x, y) \setminus \lambda(y) \quad \text{and} \quad (\varphi^\ast \mu)(y) = \bigvee_{x \in X} \varphi(x, y) \circ \mu(x)\]
for all \(\lambda \in P^\ast Y, x \in X, \mu \in P^\ast X, y \in Y\). The induced \(Q\)-closure operators \(\varphi, \varphi^* : PY \longrightarrow PY\) and \(\varphi^\ast \varphi_1 : P^\ast Y \longrightarrow P^\ast Y\) give rise to complete \(Q\)-categories
\[K_Q := \text{Fix}(\varphi, \varphi^*) = \{\lambda \in PY \mid \varphi, \varphi^* \lambda = \lambda\} \quad \text{and} \quad K_1^Q := \text{Fix}(\varphi^\ast \varphi_1) = \{\lambda \in P^\ast Y \mid \varphi^\ast \varphi_1 \lambda = \lambda\}.\]

**Remark 5.1.** The complete \(Q\)-categories \(K_Q\) and \(K_1^Q\) present a categorical extension of concept lattices based on rough set theory (RST). In the case of \(Q = 2\), considering \(X\) as the (discrete) set of properties and \(Y\) as the (discrete) set of objects, \(K_Q\) and \(K_1^Q\) are respectively the property-oriented concept lattice and the object-oriented concept lattice of the (crisp) context \((X, Y, \varphi)\) introduced in [35, 36], which have also been generalized to those of fuzzy contexts of (crisp) sets [8, 13, 24, 30] and fuzzy contexts of fuzzy sets [9, 28].

5.2. **Multi-adjoint property-oriented concept lattices as fixed points of Kan adjunctions**

Recall that a multi-adjoint property-oriented frame [16] is a tuple
\[\mathcal{L} = (L_1, L_2, P, \&_1, \varpi^1, \&_1, \ldots, \&_n, \varpi^n, \&_n),\]
such that \((\&_i, \varpi^i, \&_i)\) is an adjoint triple with respect to \(P, L_2, L_1\) for all \(i = 1, \ldots, n\); that is, the maps
\[\&_i : P \times L_2 \longrightarrow L_1, \quad \varpi^i : L_1 \times L_2 \longrightarrow P, \quad \&_i : L_1 \times P \longrightarrow L_2\]
satisfy
\[z \&_i y \leq x \iff z \leq x \varpi^i y \iff y \leq x \&_i z\]
for all \(z \in P, y \in L_2, x \in L_1\). With a suitable modification of Proposition 3.1 we may construct a quantaloid \(Q_{\mathcal{L}}^P\) from a multi-adjoint property-oriented frame \(\mathcal{L}\):

**Proposition 5.2.** Each multi-adjoint property-oriented frame \(\mathcal{L} = (L_1, L_2, P, \&_1, \ldots, \&_n)\) gives rise to a non-trivial quantaloid \(Q_{\mathcal{L}}^P\) consisting of the following data:
- \((Q_{\mathcal{L}}^P)_0 = \{0, 1, \ldots, n, \infty\};\)
- \(Q_{\mathcal{L}}^P(0, i) = P, Q_{\mathcal{L}}^P(i, \infty) = L_2, Q_{\mathcal{L}}^P(0, \infty) = L_1\) for all \(i = 1, \ldots, n;\)
- \(Q_{\mathcal{L}}^P(i, i) = \{\perp_{i, i}, \operatorname{id}_i\}\) for all \(i = 0, 1, \ldots, n, \infty\), and \(Q_{\mathcal{L}}^P(i, j) = \{\perp_{i, j}\}\) whenever \(0 \leq j < i \leq \infty\) or \(0 < i < j < \infty;\)
- compositions in \(Q_{\mathcal{L}}^P\) are given by
  \[v \circ u = u \&_i v\]
  for all \(u \in Q_{\mathcal{L}}^P(0, i) = P, v \in Q_{\mathcal{L}}^P(i, \infty) = L_2 (i = 1, \ldots, n)\), and the other compositions are trivial;
left and right implications in \( Q^P_L \) are given by
\[
\begin{align*}
  w \land u &= w \land u \\
  v \lor w &= w \lor v
  
\end{align*}
\]
for all \( u \in Q^P_L(0, i) = P, v \in Q^P_L(i, \infty) = L_2, w \in Q^P_L(0, \infty) = L_1 \) for \( i = 1, \ldots, n \), and the other implications are trivial.

A context [16] of a multi-adjoint property-oriented frame \( \mathcal{L} = (L_1, L_2, P, \&_1, \ldots, \&_n) \) is also defined as a \( P \)-valued relation \( \varphi : X \to Y \) equipped with a map \( |\cdot| : Y \to \{1, \ldots, n\} \), where \( X \) is interpreted as the set of properties and \( Y \) the set of objects. Therefore:

**Proposition 5.3.** Let \( \mathcal{L} = (L_1, L_2, P, \&_1, \ldots, \&_n) \) be a multi-adjoint property-oriented frame and let \( Q^P_L \) be the quantaloid determined by Proposition 5.2. Then a context \((X, Y, \varphi)\) of \( \mathcal{L} \) is exactly a \( Q^P_L \)-relation \( \varphi_P : X \to Y \) between \((Q^P_L)_{\circ}\)-typed sets with
\[
[x] = 0, \quad |y| \in \{1, \ldots, n\} \quad \text{and} \quad \varphi_P(x, y) = \varphi(x, y)
\]
for all \( x \in X, y \in Y \).

Considering the \( Q^P_L \)-relation \( \varphi_P : X \to Y \) obtained in Proposition 5.3, we have
\[
(PX)_{\circ} = Q^P_L(0, \infty)^X = L_1^X \quad \text{and} \quad (PY)_{\circ} = \prod_{1 \leq i \leq n} Q^P_L(i, \infty)^Y = \prod_{1 \leq i \leq n} L_2^Y = L_1^Y.
\]
Hence, by restricting the Kan adjunction \((\varphi_P)^* \dashv (\varphi_P)_*\) on the \( \infty \)-fibre of \( PY \) and \( PX \)
\[
(PY)_{\circ} \xrightarrow{(\varphi_P)_*} (PX)_{\circ}
\]
we obtain the Galois connection given in [16, Section 4], which satisfies
\[
((\varphi_P)^* \lambda)(x) = \bigvee_{y \in Y} \lambda(y) \circ \varphi_P(x, y) = \bigvee_{y \in Y} \varphi(x, y) \&_{|y|} \lambda(y)
\]
\[
((\varphi_P)_* \mu)(y) = \bigwedge_{x \in X} \mu(x) / \varphi_P(x, y) = \bigwedge_{x \in X} \mu(x) \&_{|y|} \varphi(x, y)
\]
for all \( \lambda \in (PY)_{\circ} = L_2^Y, x \in X, \mu \in (PX)_{\circ} = L_1^X, y \in Y \).

Since the multi-adjoint property-oriented concept lattice of \((X, Y, \varphi)\) is the complete lattice of fixed points of the Galois connection (5.iii) (cf. [16, Section 4]), it is obviously given by the \( \infty \)-fibre of \( K_{\varphi_P} \):

**Theorem 5.4.** The multi-adjoint property-oriented concept lattice of a context \((X, Y, \varphi)\) of a multi-adjoint property-oriented frame \( \mathcal{L} \) is isomorphic to the complete lattice \((K_{\varphi_P})_{\circ}\), where \( K_{\varphi_P} \) is the complete \( Q^P_L \)-category of fixed points of the Kan adjunction (5.i) induced by the \( Q^P_L \)-relation \( \varphi_P : X \to Y \) in Proposition 5.3.

5.3. Multi-adjoint object-oriented concept lattices as fixed points of dual Kan adjunctions

Following the terminology of [16, Section 5], a multi-adjoint object-oriented frame is a tuple
\[
\mathcal{L} = (L_1, L_2, P, \&_1, \lor^1, \land^1, \ldots, \&_n, \lor^n, \land^n)
\]
such that \((\&_i, \lor^i, \land^i)\) is an adjoint triple with respect to \( L_1, P, L_2 \) for all \( i = 1, \ldots, n \); that is, the maps
\[
\&_i : L_1 \times P \to L_2, \quad \lor^i : L_2 \times P \to L_1, \quad \land^i : L_2 \times L_1 \to P
\]
satisfy
\[
x \&_i z \leq y \iff x \leq y \lor^i z \iff z \leq y \land^i x
\]
for all \( x \in L_1, z \in P, y \in L_2 \). Similarly as in Proposition 5.2 we may construct a quantaloid \( Q^O_L \).
Proposition 5.5. Each multi-adjoint object-oriented frame \( L = (L_1, L_2, P, &_1, \ldots, &_n) \) gives rise to a non-trivial quantaloid \( Q^O_L \) consisting of the following data:

- \( (Q^O_L)^0 = \{-1, 0, 1, \ldots, n\} \);
- \( Q^O_L(-1,0) = L_1, Q^O_L(0,i) = P, Q^O_L(-1,i) = L_2 \) for all \( i = 1, \ldots, n \);
- \( Q^O_L(i,i) = \{\bot, 1\} \) for all \( i = -1, 0, 1, \ldots, n \), and \( Q^O_L(i,j) = \{\bot, 1\} \) whenever \(-1 \leq j < i \leq n \) or \( 0 < i < j \leq n \);
- compositions in \( Q^O_L \) are given by
  \[
  v \circ u = u \&_i v
  \]
  for all \( u \in Q^O_L(-1,0) = L_1, v \in Q^O_L(0,i) = P \) \((i = 1, \ldots, n)\), and the other compositions are trivial;
- left and right implications in \( Q^O_L \) are given by
  \[
  w \backslash u = w \land_X u \quad \text{and} \quad v \lor w = w \lor_X v
  \]
  for all \( u \in Q^O_L(-1,0) = L_1, v \in Q^O_L(0,i) = P, w \in Q^O_L(-1,i) = L_2 \) \((i = 1, \ldots, n)\), and the other implications are trivial.

With a context [16] of a multi-adjoint object-oriented frame \( L = (L_1, L_2, P, &_1, \ldots, &_n) \) defined as a \( P \)-valued relation \( \varphi : X \twoheadrightarrow Y \) equipped with a map \( \lceil \cdot \rceil : Y \longrightarrow \{1, \ldots, n\} \), where elements in \( X \) and \( Y \) are properties and objects, respectively, we deduce the following parallel proposition of 5.3:

Proposition 5.6. Let \( L = (L_1, L_2, P, &_1, \ldots, &_n) \) be a multi-adjoint object-oriented frame and let \( Q^O_L \) be the quantaloid determined by Proposition 5.5. Then a context \( (X, Y, \varphi) \) of \( L \) is exactly a \( Q^O_L \)-relation \( \varphi_0 : X \twoheadrightarrow Y \) between \( (Q^O_L)^0 \)-typed sets with

\[
|\cdot| = 0, \quad |\cdot| \in \{1, \ldots, n\}
\]

and \( \varphi_0(x,y) = \varphi(x,y) \) for all \( x \in X, y \in Y \).

For the above \( Q^O_L \)-relation \( \varphi_0 : X \twoheadrightarrow Y \), it is easy to see that

\[
(P^1X)_{-1} = Q^O_L(-1,0)^X = L_1^X \quad \text{and} \quad (P^1Y)_{-1} = \prod_{1 \leq i \leq n} Q^O_L(-1,i)^Y = \prod_{1 \leq i \leq n} L_2^Y = L_2^Y.
\]

Consequently, by restricting the dual Kan adjunction \((\varphi_0)_+ \dashv (\varphi_0)^+\) on the \((1)-\)fibres of \( P^1Y \) and \( P^1X \)

\[
(P^1Y)_{-1} \xrightarrow{(\varphi_0)_+} (P^1X)_{-1}
\]

(5.iv)

we obtain the Galois connection given in [16, Section 5], which satisfies

\[
((\varphi_0)_+ \lambda)(x) = \bigvee_{y \in Y} \varphi_0(x,y) \backslash \lambda(y) = \bigvee_{y \in Y} \lambda(y) \varphi^\perp \varphi(x,y)
\]

\[
((\varphi_0)^+ \mu)(y) = \bigvee_{x \in X} \varphi_0(x,y) \circ \mu(x) = \bigvee_{x \in X} \mu(x) \backslash \lambda \varphi(x,y)
\]

for all \( \lambda \in (P^1Y)_{-1}, x \in X, \mu \in (P^1X)_{-1} = L_1^X, y \in Y \).

As the multi-adjoint object-oriented concept lattice of \((X, Y, \varphi)\) is the complete lattice of fixed points of the Galois connection (5.iv) (cf. [16, Section 5]), it is clearly given by the \((1)-\)fibre of \( K\varphi_0 \):

Theorem 5.7. The multi-adjoint object-oriented concept lattice of a context \((X, Y, \varphi)\) of a multi-adjoint object-oriented frame \( L \) is isomorphic to the complete lattice \((K^I\varphi_0)_{-1}\), where \((K^I\varphi_0)_{-1}\) is the complete \( Q^O_L \)-category of fixed points of the dual Kan adjunction (5.ii) induced by the \( Q^O_L \)-relation \( \varphi_0 : X \twoheadrightarrow Y \) in Proposition 5.6.
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