Local Current Operators for Arbitrary Spin

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Abstract

Free current operators are constructed for massive particles with arbitrary spin \( j \). Such current operators are related to representations of the \( U(N, N) \) type groups and are covariant under the (extended) Poincaré group and charge conjugation, where the charge conjugation operation is defined as an automorphism on \( U(N, N) \) elements. The currents are also required to satisfy current conservation, hermiticity, and locality. The condition that the currents be local is shown to be equivalent to certain integral constraints on form factors. These constraints are satisfied by writing the currents in terms of free local spin \( j \) fields. It is shown that there are \((2j+1)\) different local currents for a spin \( j \) particle, each with an arbitrary form factor, generalizing the Dirac and Pauli currents for spin 1/2 particles. Static properties of the various currents are also given.

1 Introduction

Local current operators are usually defined in terms of bilinear operators made out of local fields. If the fields are free fields written in terms of creation and annihilation operators, the current operators will consist of a sum of four terms, of the form \( a(p, \sigma)a(p', \sigma'), b(p, \sigma)b(p', \sigma'), a(p, \sigma)b(p', \sigma'), \) and \( a(p, \sigma)b(p', \sigma') \). \( a(p, \sigma) \) is the annihilation operator for a particle of mass \( m \) \((p \cdot p = m^2)\) and spin \( j \) \((-j \leq \sigma \leq j)\) while \( b(p, \sigma) \) is the annihilation operator for the antiparticle. As will be shown, these four bilinears in creation and
annihilation operators form the Lie algebra of the $U(N, N)$ type groups. The goal of this paper is to construct current operators out of these Lie algebra elements that have the correct transformation properties under the extended Poincaré group, are conserved, hermitian, local and invariant under charge conjugation. To ensure locality, currents will be constructed from free spin $j$ local fields. It will be shown that there are two classes of currents that give a total of $2j + 1$ independent currents, generalizing the "$\gamma_\mu$" and anomalous currents for spin 1/2 particles. Moreover these features should persist even with arbitrary form factors for the particles.

The construction of currents out of spin $j$ local fields can be understood in the following way. Consider a $2(2j + 1)$ component field $\Psi(x)$ made out of two $(2j + 1)$ component fields $\phi(x)$ and $\chi(x)$, each a linear combination of creation and annihilation operators for particles and antiparticles. Under a Lorentz transformation $\Lambda$, $\phi(x)$ goes to $D^j(\Lambda^{-1})\phi(\Lambda x)$, where $D^j$ is the $(j, 0)$ representation of the Lorentz group, while $\chi(x)$ goes to $\bar{D}^j(\Lambda^{-1})\chi(\Lambda x)$, with $\bar{D}^j(\Lambda) = D^j(\Lambda^{-1})^\dagger$. Then $\Psi^\dagger(x)\beta\Psi(x)$ (where $\beta$ is the off diagonal identity matrix defined following Eq.111) is a Lorentz scalar and a Lorentz tensor can be defined by making use of the generalized gamma matrices defined by Weinberg [4]. These gamma matrices are themselves defined in terms of generalized sigma matrices, so that $\Psi^\dagger(x)\beta\gamma_{\mu_1...\mu_{2j}}\Psi(x)$ is of the form $\phi^\dagger(x)\sigma_{\mu_1...\mu_{2j}}\phi(x) + \chi^\dagger(x)\sigma_{\mu_1...\mu_{2j}}\chi(x)$, and is a Lorentz tensor, invariant under parity and charge conjugation.

There are now two ways to make currents from these tensors; the first generalizes the way currents are made from spin zero fields, namely by differentiating the fields to produce a four-vector operator. Since there are $2j$ components for the sigma tensors the fields must be differentiated sufficiently many times to contract with the sigma tensor indices. As shown in section 3 there are $j + 1$ different ways of doing this for integer spin, and $j + 1/2$ ways for half integer spin, resulting in $j + 1$ different currents for integer spin and $j + 1/2$ for half integer spin with electric, but no magnetic form factors. The second way generalizes the well known spin 1/2 current; in this case the fields are again differentiated sufficiently many times to contract with the sigma matrix tensor indices. However in this case one index is left uncontracted, which gives the four-vector current operator. The number of such currents is now $j$ for integer spin and $j + 1/2$ for half integer spin, resulting in the usual $2j + 1$ different possibilities for a spin $j$ system.

The motivation for constructing such current operators arises from point form relativistic quantum mechanics [2], where, for finite degree of freedom
systems, interactions are specified by mass operators which are matrix elements of vertices, while for infinite degree of freedom systems, interactions are given by four-momentum operators, obtained by integrating vertices over the forward hyperboloid. In both cases the electromagnetic interaction is formed by coupling the current operators described in this paper to the photon field, analyzed in detail in the next paper in this series [3], to form the electromagnetic vertex for arbitrary spin particles with arbitrary form factors. And if these currents are coupled to a classical field, it becomes possible to investigate some of the so-called acausal behavior of high spin particles in an external electromagnetic field [4]. More generally this series of papers begins an investigation of many-body relativistic theories in the point form, in which the interacting four-momentum operator is given solely in terms of creation and annihilation operators with arbitrary form factors.

Section 2 reviews properties of representations of the Poincaré group needed for the creation and annihilation operators, as well as the structure of point form quantum mechanics. The creation and annihilation operators are then written as bilinears to generate the $U(N, N)$ algebra. In this context it is shown that the charge and charge conjugation operators arise naturally from the algebra and its dual. The section concludes with a general representation for current operators in terms of arbitrary form factors that satisfies all the desired properties (such as charge conservation) except locality; conversely it is shown how locality imposes conditions on these form factors. In the following sections field theory is used as a guide to construct form factors that satisfy locality. In section 3 currents are constructed that generalize the way currents are made from spin zero fields and in section 4 the generalization of $\bar{\Psi} \gamma_\mu \Psi$ for spin 1/2 fields is given. In these sections heavy use is made of a paper by Weinberg [1] on local fields for arbitrary spin particles. Finally, section 5 shows how to generalize these results further by including form factors that keep the currents local.

2 Local Currents as Representations of $U(N, N)$

To have a relativistic quantum theory the commutation relations of the Poincaré algebra must be satisfied. In the point form of relativistic quantum mechanics [2] the commutation relations are to be satisfied by putting all interactions in the four-momentum operator and leaving the Lorentz generators...
kinematic. The commutation relations can then be written as
\[
[P_{\mu}, P_{\nu}] = 0 \tag{1}
\]
\[
U_{\Lambda} P_{\mu} U_{\Lambda}^{-1} = (\Lambda^{-1})_{\mu}^{\nu} P_{\nu}, \tag{2}
\]
where \( U_{\Lambda} \) is the unitary operator representing the Lorentz transformation \( \Lambda \). Eq.2 emphasizes the fact that since all Lorentz transformations are kinematic, it is easier to deal with global Lorentz transformations than with the infinitesimal generators of the Lorentz group. The four-momentum operator \( P_{\mu} \) is to be built out of creation and annihilation operators that themselves transform under irreducible representations of the Poincare group, in such a way that the components commute among themselves, which is Eq.1.

If \(|p, \sigma >\) is a state of four-momentum \( p = (p \cdot p = m^2) \) and spin projection \( \sigma (-j \leq \sigma \leq j) \), then under a Poincaré transformation,
\[
U_{a} |p, \sigma > = e^{i p \cdot a} |p, \sigma > \tag{3}
\]
\[
U_{\Lambda} |p, \sigma > = \sum a(p, \sigma', \sigma) (R_{W}(v, \Lambda)), \tag{4}
\]
where \( a \) is a four-translation, \( D_{\sigma \sigma'}^{j}(\cdot) \) is an \( SU(2) \) matrix element for spin \( j \) and \( R_{W}(v, \Lambda) \) is a Wigner rotation, an element of the rotation group \( SO(3) \) defined by
\[
R_{W}(v, \Lambda) = B^{-1}(\Lambda v) \Lambda B(v). \tag{5}
\]
\( B(v) \) is a boost, a Lorentz transformation satisfying \( p = B(v)p^{rest} \), with \( p^{rest} = (m, 0, 0, 0) \), \( v = p/m \). In this paper boosts are always canonical spin boosts, defined in the Appendix, Eq.70.

Creation and annihilation operators with the same transformation properties as states generate multiparticle states from the Fock vacuum:
\[
|p, \sigma > = a^\dagger(p, \sigma)|0 >, \tag{6}
\]
\[
[a(p, \sigma), a^\dagger(p', \sigma')]_{\pm} = 2v_0 \delta^3(v - v') \delta_{\sigma, \sigma'} \tag{7}
\]
\[
U_{a} a(p, \sigma) U_{a}^{-1} = e^{-ip \cdot a} a(p, \sigma) \tag{8}
\]
\[
U_{\Lambda} a(p, \sigma) U_{\Lambda}^{-1} = \sum a(\Lambda p, \sigma') D_{\sigma \sigma'}^{j}(R_{W}(v, \Lambda))^{*}, \tag{9}
\]
where the \( \pm \) denotes commutator or anticommutator for bosons or fermions respectively. Note that the creation and annihilation operators are normalized in Eq.7 so that they are dimensionless. With these creation and annihilation operators the free four-momentum operator can be written as
\[
P_{\mu}(fr) = \sum \int \frac{d^3 v}{2v_0} p_{\mu} a^\dagger(p, \sigma) a(p, \sigma),
\]
\[
\]
and by virtue of the transformation properties of the creation and annihilation operators, Eq.9, satisfies the point form equations, Eqs.1,2.

As is well known locality requires the existence of antiparticles. If \( b^\dagger(p, \sigma) \) and \( b(p, \sigma) \) are the antiparticle creation and annihilation operators, then these operators will satisfy the same properties as the particle creation and annihilation operators, Eqs.6 through 9. The free four-momentum operator now is

\[
P_\mu(fr) = \sum \int \frac{d^3v}{2v_0} \mu(a^\dagger(p, \sigma)a(p, \sigma) + b^\dagger(p, \sigma)b(p, \sigma)), \tag{10}\]

and again satisfies the point form equations.

An interacting four-momentum operator is obtained by integrating a local scalar density over the forward hyperboloid. For the electromagnetic interaction this gives

\[
P_\mu(em) = \int d^4x \delta(x \cdot x - \tau^2) x_\mu J^\nu(x)A_\nu(x), \tag{11}\]

where \( J^\nu(x) \) is the current operator and \( A_\nu(x) \) the local photon field, discussed in reference [3]. \( P_\mu(em) \), as shown in reference [4], will not satisfy the point form equations unless the current operator is local.

Free current operators are operators bilinear in the four creation and annihilation operators. If the indices \((p, \sigma)\) are replaced by \(i\), the commutation relations of these bilinears can be written as

\[
[a^\dagger(i)a(j), b^\dagger(k)b(l)] = 0 \tag{12}
\]

\[
[a^\dagger(i)a(j), a^\dagger(k)b^\dagger(l)] = \delta_{j,k}a^\dagger(i)b^\dagger(l) \tag{13}
\]

\[
[a^\dagger(i)a(j), a(k)b(l)] = -\delta_{i,k}a(j)b(l) \tag{14}
\]

\[
[a(i)b(j), a^\dagger(k)b^\dagger(l)] = \delta_{i,k}\delta_{j,l} \pm \delta_{j,l}\delta_{i,k}a^\dagger(k)a(i) \pm \delta_{i,k}b^\dagger(l)b(j). \tag{15}
\]

It should be noted that only the last commutation relation, Eq.15 distinguishes between bosons and fermions. The commutation relations given above are those of the Lie algebra of the group \( U(N, N) \), defined by

\[
U(N, N) : = \{ g \in GL(2N, C)|g\tilde{\beta}g^\dagger = \tilde{\beta} \}, \tag{16}
\]

\[
\tilde{\beta} = diag(I, -I), \tag{17}
\]

where \( I \) is the \( N \) dimensional identity matrix.
The charge operator is defined as $Q := e \sum (a^\dagger(i)a(i) - b^\dagger(i)b(i))$ and the $U(1)$ group that it generates is dual to $U(N,N)$ in the sense defined in reference [3]. More generally, $U(N,N)$ is dual to $U(n)$ in that, on Fock space, the irreducible representations of $U(n)$ fix those of $U(N,N)$ and vice versa. Though internal symmetries are not considered in this paper, the theory of dual representations provides a natural setting for the $U(n)$ internal symmetries, a topic that will be taken up in a later paper. Also the charge conjugation operation is the element in the automorphism group of $U(N,N)$ given by $c = \beta$, where $\beta$ is the off diagonal identity given following Eq.111 with $I$ the same identity matrix as in Eq.17. It is easily checked that if $g$ is an element of $U(N,N)$, then so is $cgc$; thus $c$ maps the $U(N,N)$ Lie algebra into itself.

The discrete, infinite dimensional, unitary representations of the Lie algebra of $U(N,N)$ are specified by $a(i)b(j)|0 \rangle = 0$. In this paper $|0 \rangle$ will be taken to be the Fock vacuum, defined by the stronger condition, $a(i)|0 \rangle = b(j)|0 \rangle = 0$, for all $(i,j)$. There are however many other discrete series irreducible representations of $U(N,N)$ given for example, by the “vacuum” $|0 \rangle_n = a^\dagger(i_1)...a^\dagger(i_n)|0 \rangle$, for which the charge of the “vacuum” is not zero.

Returning to particle labels, free current operators are now defined as

$$J_\mu(0) = \sum \int \frac{d^3v_1 d^3v_2}{2v_1^0 2v_2^0} (F^a_\mu(v_1\sigma_1, v_2\sigma_2)a^\dagger(p_1\sigma_1)a(p_2\sigma_2)
+F^b_\mu(v_1\sigma_1, v_2\sigma_2)b^\dagger(p_1\sigma_1)b(p_2\sigma_2)
+F^c_\mu(v_1\sigma_1, v_2\sigma_2)a^\dagger(p_1\sigma_1)b(p_2\sigma_2)
+F^d_\mu(v_1\sigma_1, v_2\sigma_2)a(p_1\sigma_1)b(p_2\sigma_2))$$

$$= \sum \int dv_1dv_2(F^a_\mu(1,2)a^\dagger(1)a(2) + F^b_\mu(1,2)b^\dagger(1)b(2)
+F^c_\mu(1,2)a^\dagger(1)b^\dagger(2) + F^d_\mu(1,2)a(1)b(2)), \quad (18)$$

and the goal is to choose the $F_\mu(1,2)$ in such a way that the current operator is Poincaré covariant, conserved, local, hermitian, and invariant under charge conjugation.

Translational covariance is achieved by defining $J_\mu(x) := U_x J_\mu(0)U_x^{-1}$, where $U_x$ is defined in Eq.8. Lorentz covariance, charge conservation, parity and time-reversal covariance lead to conditions on the $F$’s discussed in reference [3]. In particular charge conservation is $[P^\mu(fr), J_\mu(0)] = 0$. Her-
miticity leads to the following relations on the $F_s$: $J_\mu(0)\dagger = J_\mu(0)$ means that $F_a^\dagger(2,1) = F_a(1,2)$, $F_b^\dagger(2,1) = F_b(1,2)$, and $F_d^\dagger(1,2) = F_c(1,2)^\dagger$.

Charge conjugation invariance is a bit more complicated. Following Weinberg [1], the unitary operator representing charge conjugation operates on the creation and annihilation operators in the following way:

$$U_c a(1) U_c^{-1} = \eta_c b(1), \quad U_c b(1) U_c^{-1} = \eta_c^* a(1), \quad U_c^\dagger U_c = 1,$$

with $\eta_c \eta_c^* = 1$. Further, $U_c Q U_c^{-1} = -Q$, where $Q := \int d^3 x J_\mu = 0(\vec{x})$ is the charge operator, which implies that charge conjugation anticommutes with the current operator. Putting these conditions together then gives the following conditions on the $F_s$: $F_b(1,2) = -F_a(1,2)$ and $F_c(2,1) = -F_c^\dagger(1,2)$.

The current operator can thus be written as

$$J_\mu(0) = \sum \int dv_1 dv_2 [F_a^\dagger(1,2)(a(1)a(2) - b(1)b(2)) + F_c(1,2)a(1)b(2) + F_c^\dagger(1,2)a^\dagger(1)b(2)],$$

and the charge operator formed from this $J_\mu(0)$ agrees with the charge operator given following Eq.17.

Now the $F_s$ are one-particle matrix elements of the current operator; for example

$$F_a^\dagger(v_1 \sigma_1, v_2 \sigma_2) = \langle p_1 \sigma_1 | J_\mu(0) | p_2 \sigma_2 \rangle = \langle 0 | a(p_1 \sigma_1) J_\mu(0) a^\dagger(p_2 \sigma_2) | 0 \rangle .$$

By exploiting the Poincaré tensor transformation properties of the current operator, reference [3] shows that the matrix element, Eq.23, can be written as a covariant (Clebsch-Gordan coefficient of the Poincare group) times an invariant (form factor). A simple way of deriving these results is to choose a standard frame (the Breit frame) and use invariance under z axis rotations, parity, and time reversal to get conditions on the standard (invariant form factor) matrix elements:

$$F_a^\dagger(Q^2)_{r_1r_2} = \langle p_1(st) | J_\mu(0) | p_2(st) r_2 \rangle \quad \text{and} \quad F_a^\dagger(Q^2)_{r_1r_2} = \langle p_1(st) | U_{R_z}^{-1} U_{R_x} J_\mu(0) U_{R_z} U_{R_x} | p_2(st) r_2 \rangle = A_{r_1}(Q^2) \delta_{r_1r_2};$$

$$\frac{\vec{Q}}{2}, r_1 | J_\mu = 0 | - \frac{\vec{Q}}{2}, r_2 > = G_C(Q^2) \sigma_{0\ldots0} + G_Q(Q^2) \sigma_{ij0\ldots0} \frac{Q_i Q_j}{2m^2} + \ldots .$$
where $A_{r_1}$ is diagonal as a matrix in the $r_1, r_2$ variables and the $r_1, r_2$ are invariant spin projection variables, ranging between $-j \leq r_1, r_2 \leq j$. The standard vectors can be written in several ways, all depending on the invariant momentum transfer only: $p_1(st) = (E(st), 0, 0, Q/2) = m(chx, 0, 0, shx)$, $p_2(st) = (E(st), 0, 0, -Q/2) = m(chx, 0, 0, -shx)$ where $q^2 = (p_1(st) - p_2(st))^2 = -4m^2sh^2x = -Q^2$ is the invariant momentum transfer. $E(st) = \sqrt{m^2 + (Q/2)^2} = mchx$ and $R_z$ is a rotation about the z axis through an angle $\phi$. Parity and time-reversal further restrict the $F^a_{\mu=0}$ as shown in reference [6].

It is possible to further decompose the unknown invariants $A_{r_1}(Q^2)$ into various moments, such as charge and quadrupole moments, by writing the spin indices in terms of powers of spin matrices. It is then natural to use the sigma tensors introduced by Weinberg [1]; the sigma tensors are themselves powers of spin matrices, with the property that for a particle of spin $j$ there are $(2j + 1)^2$ tensors. In Eq.25 the Breit frame matrix element is written in terms of charge ($G_C(Q^2)$), quadrupole ($G_Q(Q^2)$) and higher multipole terms times sigma matrices defined in the appendix, Eq.82; thus the sigma matrices provide a natural representation for spin indexed current matrix elements, which can be used to compare with the form factors obtained in sections 3 and 4. At momentum transfer $Q^2 = 0$ the various moments give the static properties of particles, such as their charge.

A similar analysis shows that the space components of the Breit frame current matrix elements are of the form

$$<\vec{Q}/2, r_1|J_{\mu=i}(0)| - \vec{Q}/2, r_2> = i\epsilon_{ijk}G_M(Q^2)\sigma_{k00...0}$$

$$+G_{M_2}(Q^2)\sigma_{klm00...0}Q_lQ_m + ... \quad (26)$$

where $G_M(Q^2)$ is the magnetic form factor and is the first in the series of higher magnetic form factors. Note that $\vec{Q}$ dotted into the Breit frame matrix element is zero, which is charge conservation.

To get the general form of the one-particle matrix element, Eq.23, let $\Lambda(v_1v_2)$ be the Lorentz transformation taking $p_1(st)$ to $p_1$ and $p_2(st)$ to $p_2$. Such a Lorentz transformation is a coset representative of the Lorentz group $SO(1, 3)$ with respect to the subgroup $SO(2)$ of z axis rotations, and is specified by five parameters. It can be written as

$$\Lambda(v_1v_2) = B(v)R(\hat{n}) \quad (27)$$
where \( v = \frac{p_1 + p_2}{2E(st)} \), the rotation \( R(\hat{n}) \) takes the unit vector along the z direction to the unit vector \( \hat{n} \) formed from \( B^{-1}(v)(p_1 - p_2) \), and the two length minus one space-like vectors \( w_i \) are fixed by Eq.27.

Inserting \( U_{\Lambda(v_1v_2)}^{-1} U_{\Lambda(v_1v_2)} \) before and after \( J_\mu(0) \) in Eq.24, and making use of the Lorentz transformation properties of single particle states, Eq.4, then gives the final result for \( F^a_\mu(v_1\sigma_1, v_2\sigma_2) \) in terms of covariants times invariants:

\[
F^a_\mu(v_1\sigma_1, v_2\sigma_2) = \sum \Lambda^\nu_\mu(v_1v_2)D^j_{\sigma_1r_1}(R_W(v_1(st), \Lambda(v_1v_2))) \quad (29)
\]

\[
F^a_\mu(Q^2)_{r_1r_2}D^j_{r_2\sigma_2}(R_W(v_2(st), \Lambda(v_1v_2))^{-1});
\]

the Wigner rotations appearing in Eq.29 will be worked out in detail in the following sections. Note that if \( v_1 \) and \( v_2 \) are chosen in their standard (Breit) frame, \( \Lambda(v_1v_2) \) is the identity element, as are the arguments of the two \( D^j \) functions. Charge conservation means \( (p_1 - p_2)^\mu F^a_\mu(1, 2) = 0 \) and is a consequence of the fact that \( (p_1 - p_2)^\mu \Lambda^\nu_\mu = 0 \) for \( \nu = 0, 1, 2; \) for \( \nu = 3 \), the invariant form factor is zero (see Eq.26).

A similar analysis holds for \( F^c_\mu(v_1\sigma_1, v_2\sigma_2) \); that is

\[
F^c_\mu(v_1\sigma_1, v_2\sigma_2) = < p_1\sigma_1, p_2\sigma_2 | J_\mu(0) | 0 > \quad (30)
\]

\[
= \sum \Lambda^\nu_\mu(v_1v_2)D^j_{\sigma_1r_1}(R_W(v_1(st), \Lambda(v_1v_2))) \quad (31)
\]

\[
D^j_{\sigma_2r_2}(R_W(v_2(st), \Lambda(v_1v_2)))F^c_\nu(Q^2)_{r_1r_2}
\]

\[
= \sum \Lambda^\nu_\mu(v_1v_2)D^j_{\sigma_1r_1}(R_W(v_1(st), \Lambda(v_1v_2))) \quad (32)
\]

\[
F^c_\nu(Q^2)_{r_1r_2}D^j_{r_2\sigma_2}(R_W(v_2(st), \Lambda(v_1v_2))^{-1}),
\]

with \( F^c_{\mu = 0}(Q^2)_{r_1r_2} \) antidiagonal in \( r_1 \) and \( r_2 \). The goal now is to relate \( F^a_\mu(1, 2) \) and \( F^c_\mu(1, 2) \) through locality.

Locality means that the commutator of the current with itself should vanish for space-like separation:

\[
[J_\mu(x), J_\nu(0)] = 0 \quad (33)
\]

for \( x \cdot x < 0 \). Though \( J_\mu(0) \) is not an operator in that it takes elements of the Fock space out of the Fock space, its matrix elements, \( (\Psi, J_\mu(0)\Psi) \), for elements \( \Psi \) in the Fock space are well defined. But in general products of such operators are not well-defined. Here however the underlying \( U(N, N) \)
structure guarantees that the commutators are well-defined, and so the problem to be solved is when the commutator is zero for $x$ spacelike. Using Eq.22 and translating by $x$ using Eq.8 makes it possible to calculate the commutator, $[J_\mu(x), J_\nu(0)]$; the condition that the commutator be zero for $x$ spacelike then reduces to the following conditions on the $F$’s:

\[ \sum \int dv_3 \left[ F^a_\mu(1, 3) F^a_\nu(3, 2) e^{-ip_3 \cdot x} + F^c_\mu(1, 3) F^c_\nu(3, 2)^* e^{ip_3 \cdot x} \right] = 0 \quad (34) \]

\[ \sum \int dv_3 \left[ F^a_\mu(1, 3) F^c_\nu(3, 2) e^{-ip_3 \cdot x} + F^c_\mu(1, 3) F^a_\nu(3, 2) e^{ip_3 \cdot x} \right] = 0. \quad (35) \]

\[ \sum \int dv_3 dv_4 \left[ F^c_\mu(3, 4)^* F^c_\nu(4, 3) e^{-i(p_3 + p_4) \cdot x} \right. \\
\left. + F^c_\mu(3, 4) F^c_\nu(4, 3)^* e^{i(p_3 + p_4) \cdot x} \right] = 0 \quad (36) \]

The question that now must be dealt with is how to choose $F^a_\mu$ and $F^c_\mu$ so that these constraints are satisfied. The next three sections use field theory as a guide to answer this question.

### 3 Local Currents with Electric Form Factors

The simplest way to generate local currents from spin $j$ fields is to generalize the way currents are obtained from spin zero fields. That is, the current operator $J_\mu(x) = i(\phi^\dagger(x) \frac{\partial}{\partial x^\mu} \phi(x) - (\frac{\partial}{\partial x^\mu} \phi^\dagger(x))\phi(x))$, where $\phi(x)$ is the field defined in Eq.99 for $j = 0$, can be generalized to

\[ J_\mu(x) = i(\Psi^\dagger(x) \beta \frac{\partial}{\partial x^\mu} \Psi(x) - (\frac{\partial}{\partial x^\mu} \Psi^\dagger(x))\beta \Psi(x)), \quad (37) \]

where $\Psi(x)$ is the $2(2j + 1)$ component field defined in Eq.110. However, such a current will have only a charge, and no other moments; the goal of this section is to further generalize the current in Eq.37 so that it also has higher electric moments. Then in the next section currents that also have magnetic moments are analyzed.

The idea is to use higher derivatives in conjunction with the gamma tensors defined in the appendix, Eq.111 to form more complex local current operators out of local fields. To that end define the current operator as

\[ J_\mu(x) : = i((\frac{\partial}{\partial x_{\mu_1}} \ldots \frac{\partial}{\partial x_{\mu_k}})\Psi^\dagger(x))\beta \gamma_{\mu_1 \ldots \mu_2}(\frac{\partial}{\partial x_{\mu_{k+1}}} \ldots \frac{\partial}{\partial x_{\mu_2}})\frac{\partial}{\partial x^\mu} \Psi(x) \quad (38) \]
plus three other terms that are needed to guarantee hermiticity and charge conservation; such a current is conserved by virtue of the fields satisfying the Klein-Gordan equation. It also reduces to Eq.37 if \( k = 0 \) by virtue of the generalized Dirac equation, Eq.109. \( \beta \) is the generalization of the four dimensional matrix for spin 1/2 and is defined after Eq.111.

Since the free fields defined in Eqs.99,106 are sums of creation and annihilation operators, the four terms in the current operator, Eq.38 will be of the desired form, Eq.22; moreover, the \( F \)’s can then be obtained by taking matrix elements of the current operator:

\[
F_{\mu}^a(1, 2) = \langle 0 | a(1) J_\mu(0) a(2) | 0 \rangle
\]

\[
= (-1)^{k+1} (p_1 + p_2) \mu [D^i(B(v_1))\bar{\sigma}_{\mu_1...\mu_k\mu_{k+1}...\mu_2} D^j(B(v_2))
+ D^i(B(v_1))\bar{\sigma}_{\mu_1...\mu_k\mu_{k+1}...\mu_2} D^j(B(v_2))]
\]

\[
v_1^{\mu_1}...v_1^{\mu_k} v_2^{\mu_{k+1}}...v_2^{\mu_2} + hc
\]

\[
= (-1)^{k+1} (p_1 + p_2) \mu [D^i(B(v_1))\bar{\sigma}_k(1, 2) D^j(B(v_2))
+ D^j(B(v_1))\sigma_k(1, 2) D^i(B(v_2))],
\]

where \( \sigma_k(1, 2) := \sigma_{\mu_1...\mu_k\mu_{k+1}...\mu_2} (v_1^{\mu_1}...v_1^{\mu_k} v_2^{\mu_{k+1}}...v_2^{\mu_2}) + v_2^{\mu_1}...v_2^{\mu_k} v_1^{\mu_{k+1}}...v_1^{\mu_2} \) and is symmetric under the interchange of 1 and 2. To show that the matrix element \( F_{\mu}^a(1, 2) \) has the general form given by Eq.29, that part of Eq.40 in square brackets is designated as A and manipulated in the following way:

\[
\begin{align*}
A & = D^i(B(v_1))\bar{\sigma}_k(1, 2) D^j(B(v_2)) + D^j(B(v_1))\sigma_k(1, 2) D^i(B(v_2)) \\
& = D^j(B^{-1}(v_1)) [D^i(B(v_1)) B(v_1)] \sigma_k(1, 2)
+ \sigma_k(1, 2) D^j(B^{-1}(v_2)) B^{-1}(v_2)] D^i(B(v_2)) \\
& = D^j(B^{-1}(v_1)) \Lambda(1, 2)] D^j(\Lambda(1, 2)) D^i(B(v_1) B(v_1)) D^j(\Lambda^{-1}(1, 2))
+ D^j(\Lambda(1, 2)) \Lambda(1, 2) D^i(\Lambda(1, 2) + D^j(\Lambda^{-1}(1, 2)) \sigma_k(v_1 v_2) D^j(\Lambda^{-1}(1, 2))
+ D^j(\Lambda(1, 2)) \Lambda(1, 2) D^i(B^{-1}(v_2)) B^{-1}(v_2)] D^j(\Lambda^{-1}(1, 2) B(v_2)) \\
& = D^j(R_W(v_1, \Lambda(1, 2)) \Lambda(1, 2) \Lambda(1, 2)) D^j(R_W^{-1}(v_2, \Lambda(1, 2)),
\end{align*}
\]

which is of the form given in Eq.29 when \( F_{\mu=3}(Q^2) \) is zero; that is, there are only electric and no magnetic form factors, as predicted. If the matrix element, Eq.40 is evaluated in the standard (Breit) frame, the two Wigner rotations are the identity and what remains is exactly the invariant electric form factor, namely

\[
F_{\mu=0}^a(Q^2)_{r_1 r_2} = (-1)^{k+1} (\sigma_k(v_1(st) v_2(st)) + \bar{\sigma}_k(v_1(st) v_2(st)))_{r_1 r_2}.
\]
It is of course possible to take linear combinations of these invariant form factors, summing over the k variable, which then gives the most general class of electric form factors.

Similarly, the matrix element $F^c_{\mu}(1, 2)$ is given by

$$F^c_{\mu}(1, 2) = < p_1\sigma_1, p_2\sigma_2 | J_{\mu}(0) | 0 >$$

$$= (-p_1 + p_2)_{\mu} [(-1)^{J_1} J_0(B(v_1))\bar{\sigma}(1, 2)J_0(B(v_2))$$

$$+ D(B(v_1))\sigma(1, 2)D(B(v_2))]|C^{-1},$$

and can also be manipulated into the form of Eq.32. C is the conjugation matrix defined after Eq.102. If the expressions for $F^a_{\mu}$ in Eq.40 and $F^c_{\mu}$ in Eq.44 are substituted into Eqs.34,35,36 which expresses the locality of the current operator, it is straightforward but tedious to show that locality is satisfied; this of course is not surprising, since these matrix elements came from currents that were given in terms of local fields.

Finally, some examples of invariant form factors for spin 1/2, 1, and 3/2 are given by the following expressions:

$$F^a_{\mu=0}(Q^2) = \sigma(1, 2) + \bar{\sigma}(1, 2)$$

$$= (\sigma_{\mu_1...\mu_k\mu_{k+1}...\mu_2} + \bar{\sigma}_{\mu_1...\mu_k\mu_{k+1}...\mu_2})$$

$$= (v^1_1(st)v^1_2(st)v^1_2(st)v^1_2(st)$$

$$+ v^1_2(st)v^1_2(st)v^1_2(st)v^1_2(st))$$

$$j = 1/2 : F^a_{\mu=0}(Q^2) = (\sigma + \bar{\sigma})(v^1_1(st) + v^1_2(st))$$

$$= (\sigma_0 + \bar{\sigma}_0)2ch\alpha$$

$$= 4ch\alpha I;$$

$$j = 1 : F^a_{\mu=0}(Q^2) = (\sigma_{\mu_1\mu_2} + \bar{\sigma}_{\mu_1\mu_2})(v^1_1(st)v^1_2(st) + v^1_2(st)v^1_2(st))$$

$$= 2\sigma_{00}(2ch^2\alpha) + 2\sigma_{33}(2sh^2\alpha)$$

$$= 4 \begin{bmatrix} ch\alpha & 1 \\ 1 & ch\alpha \end{bmatrix}, (k = 2j = 2);$$

$$j = 1 : F^a_{\mu=0}(Q^2) = (\sigma_{\mu_1\mu_2} + \bar{\sigma}_{\mu_1\mu_2})(v^1_1(st)v^1_2(st))$$

$$= 2\sigma_{00}ch^2\alpha - 2\sigma_{33}sh^2\alpha$$

$$= 2ch^2\alpha I - 2(2S_z^2 - I)sh^2\alpha$$

$$= 2 \begin{bmatrix} 1 & ch2\alpha \\ ch2\alpha & 1 \end{bmatrix}, (k = 1);$$

$$j = 1 : F^a_{\mu=0}(Q^2) = (\sigma_{\mu_1\mu_2} + \bar{\sigma}_{\mu_1\mu_2})(v^1_1(st)v^1_2(st))$$

$$= 2\sigma_{00}ch^2\alpha - 2\sigma_{33}sh^2\alpha$$

$$= 2ch^2\alpha I - 2(2S_z^2 - I)sh^2\alpha$$

$$= 2 \begin{bmatrix} 1 & ch2\alpha \\ ch2\alpha & 1 \end{bmatrix}, (k = 1);$$
\[ j = 3/2 : F_{\mu=0}^a(Q^2) = (\sigma_{\mu_1\mu_2\mu_3} + \bar{\sigma}_{\mu_1\mu_2\mu_3})(v_1^{\mu_1}(st)v_1^{\mu_2}(st)v_1^{\mu_3}(st) + 1 < - > 2) \]
\[ = 4\sigma_{000}ch^3\alpha + 4\sigma_{330}sh^2\alpha ch\alpha \]
\[ = 4 \begin{bmatrix}
  ch\alpha \\
  \alpha \\
  ch\alpha \\
  ch3\alpha 
\end{bmatrix}, (k = 2j = 3); \quad (50) \]

\[ j = 3/2 : F_{\mu=0}^a(Q^2) = (\sigma_{\mu_1\mu_2\mu_3} + \bar{\sigma}_{\mu_1\mu_2\mu_3}) \]
\[ (v_1^{\mu_1}(st)v_1^{\mu_2}(st)v_1^{\mu_3}(st)) + v_1^{\mu_1}(st)v_1^{\mu_2}(st)v_2^{\mu_3}(st)) \]
\[ = 4\sigma_{000}ch^3\alpha - 4\sigma_{330}sh^2\alpha ch\alpha \]
\[ = 4ch\alpha \begin{bmatrix}
  1 \\
  1 + \frac{2}{3}sh^2\alpha \\
  1 + \frac{2}{3}sh^2\alpha \\
  1 
\end{bmatrix}, (k = 2). \quad (51) \]

As discussed after Eq.25 the momentum transfer squared is given by \( \frac{Q^2}{4m^2} = sh^2\alpha \). For each value of \( j, k \) goes from 0 to 2\( j \); but since \( \sigma_k(v_1 v_2) \) is symmetric under the interchange of 1 and 2, there are \( j + 1 \) different possibilities for integer spin and \( j + 1/2 \) possibilities for half integer spin. For example, there are two different electric form factors for both \( j = 1 \) and \( j = 3/2 \), as seen in Eqs.48 through 51.

4 Local Currents with Electric and Magnetic Form Factors

The goal of this section is to construct local currents that generalize the well-known \( \bar{\Psi}\gamma_\mu\Psi \) construction for spin 1/2. To that end use is again made of the generalized gamma matrices defined in the appendix, Eq.111, to define the following current operator for arbitrary spin:

\[ J_\mu(x) : = \left( \frac{\partial}{\partial x_{\mu_1}} ... \frac{\partial}{\partial x_{\mu_k}} \right) \bar{\Psi}(x)\beta\gamma_{\mu_1 ... \mu_k ... \mu_{2j-1}} \]
\[ \left( \frac{\partial}{\partial x_{\mu_{k+1}}} ... \frac{\partial}{\partial x_{\mu_{2j-1}}} \right) \Psi(x) + ... \quad (52) \]

with the extra terms needed to satisfy current conservation and hermiticity. These terms will be added on after the expression in Eq.52 has been decomposed into the appropriate matrix elements. As in section 3 matrix elements
of Eq.52 are evaluated at \( x = 0 \):

\[
F^a_\mu(1,2) = < p_1 \sigma_1 | J_\mu(0) | p_2 \sigma_2 > 
\]

\[
= (-1)^{k+1}(D^j(B(v_1)))\sigma_k(1,2)_\mu D^j(B(v_2)) 
= \bar{D}^j(B(v_1))\sigma_k(1,2)_\mu \bar{D}^j(B(v_2)),
\]

(53)

where now \( \sigma_k(1,2)_\mu := \sigma_{\mu\mu_1...\mu_k\mu_{k+1}...\mu_{2j-1}}(v_1^{\mu_1}...v_1^{\mu_k}v_2^{\mu_{k+1}}...v_2^{\mu_{2j-1}} + v_2^{\mu_1}...v_2^{\mu_k}v_1^{\mu_{k+1}}...v_1^{\mu_{2j-1}}) \). The same sort of manipulations used in Eq.41 in section 3 can be used to show that the matrix element, Eq.54, can be brought to the form, Eq.29. In this case however, the space parts of the invariant form factor are not zero, indicating there are magnetic as well as electric form factors. By working out several examples it can be seen that the magnetic form factors have the general structure given in Eq.26. The invariant form factor is gotten from Eq.54 by evaluating the matrix element in the standard (Breit) frame:

\[
F^a_\mu(Q^2) = D^j(B(v_1(st)))\bar{\sigma}_k(1,2)_\mu D^j(B(v_2(st))) 
= \bar{D}^j(B(v_1(st)))\sigma_k(1,2)_\mu \bar{D}^j(B(v_2(st)))
\]

\[
F^a_{\mu=0}(Q^2) = \bar{\sigma}_0 + \sigma_0 
= 2I;
\]

\[
F^a_{\mu=3}(Q^2) = 2i(\hat{z} \times \bar{\sigma})i_{sh\alpha}.
\]

For \( j = 1/2 \) this gives

\[
F^a_\mu(Q^2) = D^{1/2}(B(v_1(st)))\bar{\sigma}_\mu D^{1/2}(B(v_2(st))) 
+ D^{1/2}(B(v_1(st)))\sigma_\mu D^{1/2}(B(v_2(st)))
\]

\[
F^a_{\mu=0}(Q^2) = \bar{\sigma}_0 + \sigma_0 
= 2I;
\]

\[
F^a_{\mu=1}(Q^2) = 2i(\hat{z} \times \bar{\sigma})_{i_{sh\alpha}}.
\]

For \( j = 1 \) there still is only one form factor, of the form

\[
F^a_\mu(Q^2) = (D^1(B(v_1(st)))\bar{\sigma}_{\mu\mu}D^1(B(v_2(st))) 
+ D^1(B(v_1(st)))\sigma_{\mu\mu}D^1(B(v_2(st)))(v_1^\nu(st) + v_2^\nu(st))
\]

\[
= (D^1(B(v_1(st)))\bar{\sigma}_{\mu 0}D^1(B(v_2(st))) 
+ D^1(B(v_1(st)))\sigma_{\mu 0}D^1(B(v_2(st))))2ch\alpha
\]

\[
F^a_{\mu=0}(Q^2) = 4ch\alpha I;
\]

\[
F^a_{\mu=1}(Q^2) = 4i(\hat{z} \times \bar{S})_{i_{sh\alpha}}.
\]

(61)
Starting with $j = \frac{3}{2}$ there will be several possibilities, depending on the value of $k$; here only the $j = \frac{3}{2}$ form factors are given:

$$F^a_\mu(Q^2) = (D^{3/2}(B(v_1(st)))\bar{\sigma}_{\mu\nu_1\mu_2}D^{3/2}(B(v_2(st))) + D^{3/2}(B(v_1(st)))\sigma_{\mu\nu_1\mu_2}D^{3/2}(B(v_2(st)))(v_1^{\mu_1}v_1^{\mu_2} + v_2^{\mu_1}v_2^{\mu_2})$$

$$F^a_{\mu=0}(Q^2) = 4ch^2\alpha I + 4\sigma_{\mu\nu_1\mu_2}D^{3/2}(B(v_1(st)))\bar{\sigma}_{\mu\nu_1\mu_2}D^{3/2}(B(v_2(st)))(v_1^{\mu_1}v_1^{\mu_2} + v_2^{\mu_1}v_2^{\mu_2})$$

$$F^a_{\mu=1}(Q^2) = (D^{3/2}(B(v_1(st)))\bar{\sigma}_{i00}D^{3/2}(B(v_2(st))) + \bar{D}^{3/2}(B(v_1(st)))\sigma_{i00}\bar{D}^{3/2}(B(v_2(st))))2ch^2\alpha$$

$$+ (D^{3/2}(B(v_1(st)))\bar{\sigma}_{i33}D^{3/2}(B(v_2(st))) + \bar{D}^{3/2}(B(v_1(st)))\sigma_{i33}\bar{D}^{3/2}(B(v_2(st))))2sh^2\alpha$$

$$= 4i(\hat{z} \times \vec{S})_i sh\alpha + \frac{8}{9}i[(\hat{z} \times \vec{S})_i + 2S^z(\hat{z} \times \vec{S})_i, S^z$$

$$+ 2S^2(\hat{z} \times \vec{S})_i + 2(\hat{z} \times \vec{S})_i, S^2] sh^3\alpha, (63)$$

for $k = 2$. For $k = 1$ there are similar expressions for the electric and magnetic form factors; because $v_1^{\mu_1}v_2^{\mu_2}$ replaces $v_1^{\mu_1}v_1^{\mu_2} + v_2^{\mu_1}v_2^{\mu_2}$, the factor 8 in Eq.62 is replaced by $-3$ and the factor 8 in Eq.63 by $-4$.

As seen from the general expression for the form factors, Eq.55, the number of different form factors for integer spin is $j$ while for half integer spin is $j + 1/2$. Combining these results with the electric form factors of section 3 then gives the correct number of total form factors, namely $2j + 1$.

### 5 Local Currents with Arbitrary Form Factors

In the two previous sections currents were constructed from spin $j$ fields by differentiating the bilinear fields sufficiently many times to contract with the gamma tensors to form a four-vector operator. In this section more derivatives will act on the fields to produce arbitrary scalar form factors, while maintaining the locality of the current operator. The idea is best illustrated by starting with a scalar field, for which the current is given before Eq.37;
such a current will not have any form factors, and the functions $F^a_{\mu}$ and $F^c_{\mu}$ are given by $(p_1 + p_2)\mu$ and $(p_1 - p_2)\mu$ respectively, in Eq.22. By taking the $n^{th}$ derivative of the fields, the following current results:

$$J^{(n)}_{\mu}(x) = i\left(\frac{\partial}{\partial x^{\nu_1}} \cdots \frac{\partial}{\partial x^{\nu_n}}\right)\phi^\dagger(x)(\frac{\partial}{\partial x^{\nu_1}} \cdots \frac{\partial}{\partial x^{\nu_n}})\frac{\partial}{\partial x^\mu}\phi(x)$$

(64)

minus a similar expression with the derivative of the free index, $\frac{\partial}{\partial x^\sigma}$ acting on the adjoint of the field, needed for charge conservation. If the fields are expanded in creation and annihilation operators, the current becomes

$$J^{(n)}_{\mu}(0) = \int dv_1 dv_2[(p_1 + p_2)_{\mu}(a^\dagger(p_1)a(p_2) - b^\dagger(p_1)b(p_2))(v_1 \cdot v_2)^n$$

$$+ (p_1 - p_2)_{\mu}(a^\dagger(p_1)b^\dagger(p_2) + a(p_1)b(p_2))(-v_1 \cdot v_2)^n],$$

(65)

from which it is clear that the matrix elements are $F^a_{\mu} = (p_1 + p_2)_{\mu}(v_1 \cdot v_2)^n$ and $F^c_{\mu} = (p_1 - p_2)_{\mu}(-v_1 \cdot v_2)^n$. If now each such current is multiplied by a constant and summed over $n$, these matrix elements become

$$F^a_{\mu}(1, 2) = (p_1 + p_2)_{\mu}f(v_1 \cdot v_2)$$

(66)

$$F^c_{\mu}(1, 2) = (p_1 - p_2)_{\mu}f(-v_1 \cdot v_2),$$

(67)

where $f(z) = \sum c_n z^n$ defines the power series for the function $f$ in terms of real coefficients, the first of which gives the charge. Notice that $v_1 \cdot v_2 = ch^2\alpha + sh^2\alpha = 1 + 2sh^2\alpha = 1 + \frac{\alpha^2}{\sqrt{2}m^2}$, and shows that $f$ depends only on the momentum transfer squared. If the expressions in Eqs.66,67 are substituted into Eqs.34,35,36 expressing locality, it should be possible to check the circumstances under which the current remains local. What is of particular interest is whether the current remains local when the power series defines an analytic function. This question will be taken up in a later work.

It is now possible to combine these results with the results of the previous two sections to obtain the most general current matrix elements for a particle of spin $j$:

$$F^a_{\mu}(1, 2) = (p_1 + p_2)_{\mu}\sum_k f^E(v_1 \cdot v_2)_k[D^j(B(v_1))\hat{\sigma}_k(1, 2)D^j(B(v_2))$$

$$+ \hat{D}^j(B(v_1))\sigma_k(1, 2)\hat{D}^j(B(v_2)]],$$

(68)

$$F^a_{\mu}(1, 2) = \sum_k f^M(v_1 \cdot v_2)_k[D^j(B(v_1))\hat{\sigma}_k(1, 2)D^j(B(v_2))$$

$$+ \hat{D}^j(B(v_1))\sigma_k(1, 2)\hat{D}^j(B(v_2)]],$$

(69)
where the first matrix element, Eq.68, corresponds to the most general electric form factors of section 3, while the second matrix element corresponds to the most general electric and magnetic form factor of section 4. Associated with these matrix elements are the corresponding $F^c_\mu$ type matrix elements, which together form the most general local current operator. The factors $(-1)^k$ have been ignored in these expressions, since they can all be absorbed in the $f_k$ coefficients.

6 Conclusion

This paper has shown how to construct local current operators in terms of creation and annihilation operators for particles (and antiparticles) of mass $m$ and spin $j$. Since the construction of such current operators involves arbitrary form factors, it is possible to view such particles either as fundamental, or as composites of more fundamental constituents.

These results have been obtained by thinking of current operators as elements of representations of the Lie algebra of the $U(N, N)$ type groups, where the basis elements are bilinears in the creation and annihilation operators. In such a description the charge conjugation operation is an automorphism acting on the group elements (or Lie algebra elements). The coefficients (one particle matrix elements) that multiply the bilinears are partially constrained by requiring that the current operators have the correct transformation properties under the extended Poincaré group, and be conserved, hermitian and invariant under charge conjugation. After these constraints have been satisfied there are two remaining matrix elements (see Eq.22) that themselves can be written as covariants times invariants (the Wigner-Eckardt theorem for the Poincaré group, see reference [1]). The invariants can be further decomposed into form factors for the various moments (charge, magnetic moment, quadrupole moment etc.) times the matrix tensors introduced by Weinberg [1]. The representation given for electric moments (Eq.25) and magnetic moments (Eq.26) generalizes well known results for spin 1/2 and 1 to arbitrary spin particles.

The requirement that the current operator be local involves further relations between the two remaining matrix elements. Since it is not clear how these relations might be satisfied (see Eqs.34, 35, 36), local fields for arbitrary spin were introduced, following work of Weinberg [1]. Bilinears in these local fields give current operators, the matrix elements of which will
satisfy-by construction-the locality requirements.

For a spin $j$ particle, there are $2j + 1$ different possible currents that can be built out of such local fields. They break into two classes, generalizing on the one hand currents made out of scalar particles, which have only electric moments (there are $j + 1$ such possibilities for integer spin and $j + 1/2$ for half-integer spin), while generalizing on the other hand currents for spin $1/2$ of the form $\bar{\Psi} \gamma_\mu \Psi$ (there are $j$ such possibilities for integer spin and $j + 1/2$ for half-integer spin). For each of these classes the one particle matrix elements were given, in section 3 for the scalar field generalization (see Eq.40) and in section 4 for the spin $1/2$ generalization (see Eq.54). Examples of these matrix elements were also given for low spin values (see Eqs.47 to 51, and 58 to 63) and it was also shown that the form of these matrix elements agrees with that given by the more general analysis given in section 2 (see Eqs.25,26).

The values of the Breit frame matrix elements at zero momentum transfer give the static moments of the particle. If a definite current is chosen the corresponding matrix element then fixes all the $2j + 1$ static moments of the particle; for example, for a spin $1/2$ particle, if the current is chosen to be of $\gamma_\mu$ type, then the matrix element is given in Eqs.58,59 and the magnetic moment has a $g$ factor of 2, as is usually obtained from the Dirac equation. Similarly, if the matrix elements for a spin 1 particle are given by Eqs.60,61 then the magnetic moment and quadrupole moment of the particle are fixed by the value of the matrix element at zero momentum transfer. But conversely, since there are $2j + 1$ different possible currents, and each of them can be multiplied by an arbitrary constant, using all the spin $j$ currents results in arbitrary values for all the static moments. These results are similar to analyses of higher spin wave equations [4], where the choice of some wave equation also implies that the static moments of the particle are fixed. But unlike wave equations, where additional constraints are imposed on the wave function solutions of the wave equations, extracting the static properties from matrix elements of current operators does not involve any constraints. As is well known these additional constraints imposed on wave functions lead to acausal properties for higher spin particles [4]. Further work is required to see what happens when the currents for higher spin particles given in this paper are coupled to an external electromagnetic field via the interaction given in Eq.11.

Aside from the current matrix elements extracted from the bilinears in free local fields given in sections 3 and 4, there is also the possibility of including arbitrary form factors while keeping the current operators local. This possi-
bility was discussed in section 5 by letting the number of derivatives acting on a current made out of fields be arbitrary. By allowing arbitrary constants in front of each power of derivatives, and then adding up the resulting current operator that results, locality is preserved. The question that is raised in such a procedure is if the number of derivatives go to infinity, what sorts of functions (if any) keep the currents local.

Assuming there are classes of functions for which locality is preserved, the current operator given in Eq.22 will be a local operator for this class of form factors and can be coupled to the photon field to give the electromagnetic four-momentum operator for arbitrary spin particles given in Eq.11. Since the current operator is required to transform as a four-vector under Lorentz transformations, it is also necessary that the photon field transform as a four vector, in order that the product transform as a scalar density. Such a photon field can be constructed as an induced representation of the Poincaré group, as discussed in the next paper in this series \[3\]. For infinite degree of freedom systems, it is crucial that both the current and photon operators be local operators, in order that the point form equations, Eqs.1,2 be satisfied. On the other hand, for finite degree of freedom systems, locality is not so important, and the general representation for current operators given in Eq.22 for arbitrary form factors can be used in the electromagnetic vertex to form the electromagnetic mass operator. In both cases the current operators for arbitrary spin particles can be used to produce an electromagnetic interaction in point form relativistic quantum mechanics.

7 Appendix: Finite Dimensional Representations of the Lorentz Group and Free Local Fields

Finite dimensional representations of the Lorentz group play a crucial role in the analysis of currents. Let $\Lambda$ denote an element of the (proper) Lorentz group, SO(1,3), with $\Lambda g\Lambda^T = g$, $g = diag(1, -1, -1, -1)$ and define the column vector $x := (t, \vec{x})$; then a Lorentz transformation $\Lambda$ sends $x$ to $x' = \Lambda x$. This can also be written in index notation as $x^\mu \rightarrow x'^\mu = \Lambda^\nu_\mu x^\nu$. The Lorentz invariant is $x \cdot x = x^T g x = x^\mu g_{\mu\nu} x^\nu$. Both the matrix and index notation will be used in this paper, depending on the context.

Any element of SO(1,3) can be decomposed into a boost times a rotation,
\[ \Lambda = B(v)R, \]
where \( B(v) \) is a boost, a coset representative of \( \text{SO}(1,3) \) with respect to the (proper) rotation group \( \text{SO}(3) \). In this paper \( B(v) \) will always be taken to be a canonical spin boost, namely

\[ B(v) = R(\hat{v})\Lambda_z(\alpha)R^{-1}(\hat{v}) \]  
\[ R(\hat{v}) = R_z(\phi)R_y(\theta) \]  
\[ \Lambda_z(\alpha) = \begin{bmatrix} \cosh \alpha & 0 & 0 & \sinh \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \alpha & 0 & 0 & \cosh \alpha \end{bmatrix}, \]

with \( \phi \) the azimuthal angle of the \( z \) axis rotation \( R_z(\phi) \) and \( \theta \) the polar angle of the \( y \) axis rotation \( R_y(\theta) \) forming the unit vector \( \hat{v} \). \( v \) is the four-velocity satisfying \( v \cdot v = 1 \) and \( \sinh \alpha = v^0 = v_0, \cosh \alpha = |\vec{v}| \). In particular

\[ B(v)p^{\text{rest}} = p = mv, \]

with \( p^{\text{rest}} = (m, \vec{0}) \).

A Lorentz transformation followed by a boost gives a Wigner rotation:

\[ \Lambda B(v) = B(\Lambda v)R_W(v, \Lambda) \]
\[ R_W(v, \Lambda) : = B^{-1}(\Lambda v)\Lambda B(v); \]
\[ R_W(v, R) = R, \]

where in Eq.76 use has been made of the property of canonical spin boosts that the Wigner rotation of a rotation is that rotation.

In this paper only those Lorentz representations commonly denoted by \((j, 0)\) and \((0, j)\) will be needed, rather than the more general \((j, j')\) representations. The representations \((j, 0)\) are the so-called holomorphic representations of \( \text{GL}(2,\mathbb{C}) \) and can be obtained as induced representations on Bargmann space \([3]\). From this analysis only the matrix elements

\[ D^{(j,0)}(\Lambda_z(\alpha)) = \text{diag}(e^{j\alpha}...e^{-j\alpha}) \]  
\[ D^{(j,0)}(R) = D^j(R) \]

will be needed. Combining the coset decomposition of a Lorentz transformation with the expression for the canonical spin boost, Eq.70, then gives the double coset decomposition of a Lorentz transformation, \( \Lambda = R\Lambda_z(\alpha)R' \), so that the representation of a Lorentz transformation is given by \( D^j(\Lambda) = D^j(R)D^j(\Lambda_z(\alpha))D^j(R') \) where \( j \) means \((j,0)\).
From the definition of canonical spin boost, Eq. 70, it follows that

$$D^j(B(v)) = D^j(R(\hat{v}))D^j(\Lambda_z(\alpha))D^j(R^{-1}(\hat{v}))$$

(79)

$$= D^j(B(v))^\dagger;$$

(80)

$$D^{1/2}(B(v)) = \frac{1}{\sqrt{2(v_0 + 1)}} \begin{bmatrix} v_1 + 1 & v_\perp^* \\ v_\perp & v_- + 1 \end{bmatrix}$$

(81)

with $v_\pm = v_0 \pm v_3$.

The generalized $\sigma$ matrices play an important role in the analysis of currents. They are defined by

$$\sigma_{\mu_1...\mu_2j} v^{\mu_1}...v^{\mu_2j} : = D^j(B(v)B(v))$$

(82)

and are symmetric (in all indices), traceless, matrix tensors; because of our different metric and sign conventions, we have chosen to denote them by $\sigma$ (or $\bar{\sigma}$), rather than $t$ (or $\bar{t}$) as originally defined by Weinberg in his Eq.A10 (Eq.A38). In spin indices the sigmas are $2j + 1$ by $2j + 1$ matrices, which is also the number of traceless symmetric tensors.

From the definition given in Eq. 82 it follows that

$$D^j(\Lambda)\sigma_{\mu_1...\mu_2j} v^{\mu_1}...v^{\mu_2j} D^j(\Lambda)^\dagger = D^j(\Lambda)D^j(B(v)B(v))D^j(\Lambda)^\dagger$$

$$= D^j(\Lambda B(v))D^j(\Lambda B(v))^\dagger$$

$$= D^j(B(\Lambda v)R_W)D^j(B(\Lambda v)R_W)^\dagger$$

$$= D^j(B(\Lambda v)B(\Lambda v))$$

$$= \sigma_{\mu_1...\mu_2j}(\Lambda v)^{\mu_1}...(\Lambda v)^{\mu_2j},$$

(83)

which implies that $\sigma$ transforms as a $2j^{th}$ rank tensor under Lorentz transformations. In particular, from Eq. 81

$$D^{1/2}(B(v)B(v)) = \begin{bmatrix} v_+ + 1 & v_\perp^* \\ v_\perp & v_- + 1 \end{bmatrix}$$

(84)

$$D^{1/2}(\Lambda)\sigma_{\mu} v^\mu D^{1/2}(\Lambda)^\dagger = \sigma_{\mu}(\Lambda v)^{\mu}$$

$$= \sigma_{\mu}\Lambda_{\nu}^\mu v^\nu.$$
matrix tensors defined by
\[ \bar{\sigma}_{\mu_1...\mu_2j} v^{\mu_1}...v^{\mu_2j} : = D^j(B(v)B(v)); \] (86)
\[ \bar{D}^j(\Lambda) \bar{\sigma}_{\mu_1...\mu_2j} v^{\mu_1}...v^{\mu_2j} \tilde{D}^j(\Lambda)^\dagger = \bar{\sigma}_{\mu_1...\mu_2j} (\Lambda v)^{\mu_1}...(\Lambda v)^{\mu_2j}. \] (87)

The sigma tensors can be given explicitly by making use of their symmetric traceless properties; in particular, for \( v = (ch\alpha, 0, 0, sh\alpha) \) it is possible to write
\[ D^j(B(v)B(v)) = \text{diag}(e^{2j\alpha}...e^{-2j\alpha}) \]
\[ = \sigma_{0...0}(ch\alpha)^{2j} + 2j\sigma_{30...0}(ch\alpha)^{2j-1}sh\alpha \]
\[ + \frac{2j(2j-1)}{2}\sigma_{330...0}(ch\alpha)^{2j-2}(sh\alpha)^2 + ...; \] (88)
\[ \bar{D}^j(B(v)B(v)) = \text{diag}(e^{-2j\alpha}...e^{2j\alpha}) \]
\[ = \bar{\sigma}_{0...0}(ch\alpha)^{2j} + 2j\bar{\sigma}_{30...0}(ch\alpha)^{2j-1}sh\alpha + ... \] (89)

By differentiating Eq.88 sufficiently many times with respect to \( \alpha \) and then setting \( \alpha \) to zero, it follows that
\[ \sigma_{0...0} = I \] (90)
\[ \sigma_{30...0} = \frac{1}{j}S_z \] (91)
\[ \sigma_{330...0} = \frac{2}{j(2j-1)}S_z^2 - \frac{1}{2j-1}I \] (92)

with similar expressions for the higher order sigma matrices. The \( S_i \) are spin angular momentum matrices and \( I \) is the identity matrix (of dimension \( 2j + 1 \)). Combining the results for the special values of the sigma matrices given in Eqs.91,92 with the general form for symmetric traceless tensors then results in the following lower order expressions for the sigma matrices:
\[ \sigma_{i0...0} = \frac{1}{j}S_i \] (93)
\[ \sigma_{i_1i_20...0} = \frac{1}{j(2j-1)}(S_{i_1}S_{i_2} + S_{i_2}S_{i_1}) - \frac{1}{2j-1}\delta_{i_1i_2}I \] (94)

Also, differentiating Eq.88 with respect to alpha \( n \) times gives factors of \( (2S_z)^n \), while differentiating Eq.89 with respect to alpha \( n \) times gives factors of \( (-2S_z)^n \). Thus the two sigma tensors are related to one another by
\[ \bar{\sigma}_{i_1...i_n0...0} = \sigma_{i_1...i_n0...0}, n \text{ even} \] (95)
\[ \bar{\sigma}_{i_1...i_n0...0} = -\sigma_{i_1...i_n0...0}, n \text{ odd} \] (96)
Finally, to construct the local fields needed for making current operators, it is necessary to work out the properties of the discrete transformations, especially parity. The parity operation is defined by 

\[ P \cdot x = (x_0, -\vec{x}) = gx. \]

Further, on states, 

\[ U_P |p\sigma >= \eta |gp, \sigma >, \]

where \( \eta \) is the intrinsic parity. Also, from Eq.70

\[ B^{-1}(\vec{v}) = R(\vec{v}) \Lambda_z^{-1}(\alpha) R^{-1}(\vec{v}) \]
\[ = R(\vec{v}) R_x(\pi) \Lambda_z(\alpha) R_x(\pi) R^{-1}(\vec{v}) \]
\[ = R(-\vec{v}) \Lambda_z(\alpha) R^{-1}(-\vec{v}) \]
\[ = B(-\vec{v}). \] \hspace{1cm} (97)

\[ D^j(B(-\vec{v})) = D^j(B^{-1}(\vec{v})) \]
\[ = \tilde{D}^j(B(\vec{v})), \] \hspace{1cm} (98)

a result that is needed for the transformation properties of the fields under parity.

Following Weinberg \[ \text{II} \] local spin \( j \) fields are defined by

\[ \phi_n(x) = \sum \int dv D^j_{n\sigma}(B(v))[a(p\sigma)e^{-ip\cdot x} + C_{\sigma\sigma'}^{-1} b^\dagger(p\sigma')e^{ip\cdot x}] \] \hspace{1cm} (99)

\[ U_a \phi_n(x) U_a^{-1} = \phi_n(x + a) \] \hspace{1cm} (100)

\[ U_\Lambda \phi_n(x) U_\Lambda^{-1} = \sum D^j_{nn'}(\Lambda^{-1}) \phi_{n'}(\Lambda x), \] \hspace{1cm} (101)

with locality properties given by

\[ [\phi_n(x), \phi_{n'}^\dagger(0)]_\pm = \sum \int dv D^j_{nn'}(B(v)B(v))[e^{-ip\cdot x} \pm e^{ip\cdot x}] \]
\[ = \frac{1}{m^2 j} \sigma_{\mu_1...\mu_2} (i \frac{\partial}{\partial \mu_1} ... i \frac{\partial}{\partial \mu_2}) \int dv [e^{-ip\cdot x} \pm (-1)^{2j} e^{ip\cdot x}], \] \hspace{1cm} (102)

where the measure \( dv = \frac{d^3v}{2\nu_0} \) and \( C \) is the conjugation matrix defined by \( D^j(R)^* = C D^j(R) C^{-1} \). As shown by Weinberg such fields will be local only if \( \pm (-1)^{2j} = -1 \), which gives the usual connection between spin and statistics.

Under parity the \( 2j + 1 \) component field \( \phi \) transforms into a second \( 2j + 1 \) component field \( \chi \):

\[ U_P \phi_n(x) U_P^{-1} = \sum \int dv D^j_{n\sigma}(B(v))[\eta_P a(gp, \sigma)e^{-ip\cdot x} \]
\[ + \bar{\eta}_P C_{\sigma\sigma'}^{-1} b^\dagger(gp, \sigma')e^{ip\cdot x}] \] \hspace{1cm} (103)
\[ \chi_n(x) : = \sum \int dv D_j(B^{-1}(v)) [a(p\sigma)e^{-ip \cdot x} + (-1)^{2j} C^{-1}_{\sigma\sigma'} b^\dagger(p\sigma') e^{ip \cdot x}], \tag{106} \]

with the intrinsic parity of the antiparticle related to the intrinsic parity of the particle by \( \eta \bar{\eta} = (-1)^{2j} \).

The two fields are related by the sigma tensors in the following way:

\[ \tilde{\sigma}_{\mu_1...\mu_{2j}}(i \frac{\partial}{\partial x_{\mu_1}} ... i \frac{\partial}{\partial x_{\mu_{2j}}}) \phi(x) = m^{2j} \sum \int dv D_j(B^{-1}(v)B^{-1}(v)B(v)) [a(p\sigma)e^{-ip \cdot x} + (-1)^{2j} C^{-1}_{\sigma\sigma'} b^\dagger(p\sigma') e^{ip \cdot x}] = m^{2j} \chi(x); \tag{107} \]

\[ \sigma_{\mu_1...\mu_{2j}}(i \frac{\partial}{\partial x_{\mu_1}} ... i \frac{\partial}{\partial x_{\mu_{2j}}}) \chi(x) = m^{2j} \phi(x). \tag{108} \]

By combining the two fields into a \( 2(2j + 1) \) component field and forming a generalized \( \gamma \) matrix from the two sigma matrices, a generalized Dirac equation can be written as

\[ (-i \gamma_{\mu_1...\mu_{2j}}(i \frac{\partial}{\partial x_{\mu_1}} ... i \frac{\partial}{\partial x_{\mu_{2j}}}) + m^{2j}) \Psi(x) = 0, \tag{109} \]

where

\[ \Psi(x) : = \begin{bmatrix} \phi(x) \\ \chi(x) \end{bmatrix}, \tag{110} \]

\[ \gamma_{\mu_1...\mu_{2j}} : = -i(2j) \begin{bmatrix} 0 & \sigma_{\mu_1...\mu_{2j}} \\ \tilde{\sigma}_{\mu_1...\mu_{2j}} & 0 \end{bmatrix}, \tag{111} \]

which agrees with the usual Dirac equation when \( j = 1/2 \). Also, under parity, using Eq.103, \( U_P \Psi(x)U_P^{-1} = \eta \beta \Psi(gx) \), where \( \beta = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \) is the generalized \( \beta \) matrix and \( I \) the \( (2j + 1) \) dimensional identity matrix. The generalized Dirac equation, viewed as a wave equation presents difficulties connected with minimal substitution and acausal solutions; for a discussion of these points, see reference [7].
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