A NEW TYPE OF FACTORIAL EXPANSIONS

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ABSTRACT. We construct a new type of convergent asymptotic representations, dyadic factorial expansions. Their convergence is geometric and the region of convergence can include Stokes rays, and often extends down to $0^+$. For special functions such as Bessel, Airy, Ei, Erfc, Gamma and others, this region is $\mathbb{C}$ without an arbitrarily chosen ray effectively providing uniform convergent asymptotic expansions for special functions.

We prove that relatively general functions, Écalle resurgent ones possess convergent dyadic factorial expansions. We show that dyadic expansions are numerically efficient representations.

The expansions translate into representations of the resolvent of self-adjoint operators in series in terms of the associated unitary evolution operator evaluated at some prescribed points (alternatively, in terms of the generated semigroup for positive operators).

1. Introduction

A classical rising factorial expansion (factorial series) as $x \to \infty$ is a series of the form

$$
(x)_k := x(x+1) \cdots (x+k-1) = \frac{\Gamma(x+k)}{\Gamma(x)}
$$

is known as the Pochhammer symbol, or rising factorial.

Factorial series have a long history going back to Stirling, Jensen, Landau, Nörlund and Horn (see, e.g. [22], [14], [17], [19], [13]). Excellent introductions to the classical theory of factorial series and their application to solving ODEs can be found in the books by Nörlund [19] and Wasow [23]; see also [20] Ch.4.

Since $(x)_{k+1}$ behaves like $k!$ for large $k$, in certain conditions the factorial expansion of a function converges even when its asymptotic series in powers of $1/x$ has empty domain of convergence; we elaborate more on this phenomenon in §7.

Recent use of factorial expansions to tackle divergent perturbation series in quantum mechanics and quantum field theory (see e.g. [13]) triggered considerable renewed interest and substantial literature. An excellent account of new developments is [24]; see also [10, 8, 25, 16] and references therein.

1.1. Drawbacks of classical factorial expansions. Most often, the classical factorial expansions arising in ODEs and physics have two major limitations: (1) slow convergence, at best power-like; (2) a limited (for the function, unnaturally) domain of convergence: a half plane which cannot be centered on the asymptotically important Stokes ray $\{x: x \omega \geq 0\}$ see §7. As a result they are not suitable for the study of Stokes phenomena ([15], [4]). One aim of the present work is to address and overcome these limitations.

1.2. Organization of the paper. For clarity of presentation, we start with examples. In §2 we first find a geometrically convergent "dyadic" factorial expansion for Ei in $\mathbb{C} \setminus i\mathbb{R}^-$, a region containing the Stokes ray. In §3 we establish a dyadic decomposition of the Cauchy kernel which we then use in §4 to obtain a somewhat simpler and more efficient expansion of Ei in $\mathbb{C} \setminus \mathbb{R}^+$. In §5 we make a first step towards

1 A Stokes ray of a function $f$ is a direction in the Borel $p$ plane along which its Borel (i.e. formal inverse Laplace) transform $F$ has singularities. If $\omega$ is a singularity of $F$ then the ray in the $x$ plane $\{x: x \omega \geq 0\}$ is sometimes also called a Stokes ray, and it is the direction where a small exponential is collected in the transseries of $f$. An antistokes ray is a direction where the small exponential becomes classically visible (purely oscillatory).
generalization and obtain dyadic factorial expansions for Airy and Bessel functions. Further examples and useful identities are given in §9.

In §6 we develop the general theory of constructing geometrically convergent dyadic expansions for typical Écalle resurgent functions. Since, by definition, resurgent divergent series are Écalle-Borel summable (to resurgent functions, cf. footnote 1), such series are also resummable in terms of dyadic expansions.

Our theory extends naturally to transseriable functions, but we do not pursue this in the present paper.

In §8 we develop the general theory of constructing geometrically convergent dyadic expansions for Airy and Bessel functions. Further examples (to resurgent functions, cf. footnote 1), such series are also resummable in terms of dyadic expansions.

In the process, we develop a general theory of decomposition of resurgent functions into simpler resurgent functions, "resurgent elements".

2. Dyadic factorial expansions of \( Ei \) in the Stokes sector

Let

\[
e^{-x}Ei^+(x) = \int_0^{\infty} \frac{e^{-px}}{1-p} dp
\]

where + refers to the intended direction of \( x \), one in the first quadrant, and by analytic continuation on the Riemann surface of the log. Note that \( \mathbb{R}^+ \) is a Stokes ray for \( e^{-x}Ei^+(x) \).

The following identity holds in \( \mathbb{C} \setminus \{1\} \) (see Corollary 6 below):

\[
\frac{1}{1-p} = -\pi i e^{-i\pi p} + \pi i \sum_{k=1}^{\infty} \frac{e_k}{2^k e^{-\pi k p} + e_k} \quad \text{where} \quad e_k = e^{-i\pi 2^{-k}}, \quad r_k = i\pi 2^{-k}
\]

Let \( x \) be in the first quadrant. We choose the path of integration in (2) as the vertical segment \([0, -i]\) followed by the horizontal half-line \(-i + \mathbb{R}^+\). Since on this path \(|e_k|/|e^{-r_k p} + e_k| < (1 - e^{-\pi/2})^{-1}\), the functions multiplying \( 1/2^k \) are uniformly bounded and we can Laplace transform the sum term by term. After rescaling \( p \) by \( 2^k \) we get

\[
e^{-x}Ei^+(x) = -i \int_0^\infty \frac{e^{-px/\pi}}{e^{-ip} + 1} dp + i \sum_{k=1}^{\infty} \int_0^\infty \frac{e_k e^{-\frac{2^k p x}{\pi}}}{e_k + e^{-ip}} dp
\]

Let \( x = i\pi y \). After one integration by parts (see also (33) for changes of variable motivating the way integration by parts is done) (4) becomes

\[
e^{-x}Ei^+(x) = -\frac{1}{2y} - \frac{i}{y} \int_0^\infty \frac{e^{-ip(y+1)}}{(e^{-ip} + 1)^2} dp + \sum_{k=1}^{\infty} \frac{e_k}{2^k y (e_k + 1)} + i \sum_{k=1}^{\infty} \frac{e_k}{2^k y} \int_0^{\infty} \frac{e^{-ip(2^k y + 1)}}{(e_k + e^{-ip})^2} dp
\]

and \( n - 1 \) successive integrations by parts yield

\[
e^{-x}Ei^+(x) = -\sum_{m=1}^{n-1} \frac{\Gamma(m)}{2m(y)_m} + R_n + \sum_{k=1}^{\infty} \left( \sum_{m=1}^{n-1} \frac{\Gamma(m) e_k}{(1 + e_k)^m (2^k y)_m} + R_{nk} \right)
\]

where

\[
R_n = -\frac{i\Gamma(n)}{(y)_{n-1}} \int_0^\infty e^{-ip(y+n-1)} dp, \quad \text{and} \quad R_{nk} = \frac{e_k \Gamma(n)}{(2^k y)_{n-1}} \int_0^\infty e^{-ip(2^k y + n-1)} (e_k + e^{-ip})^n
\]

where the integrals are defined for \( y \) in the second quadrant, and the remainders are analytically continued on the Riemann surface of the log.
1. Size of terms in the successive series on the Stokes ray with the formula (7).

This plot can be used to determine the number of terms to be kept for a given accuracy. To get $10^{-5}$ accuracy, 10 terms of the first series plus 5 from the second and so on, and all terms from the fifth series on can be discarded.

As Proposition 1 below shows, the remainders go to zero when $n \to \infty$ and $x \in \mathbb{C} \setminus -i\mathbb{R}^+$ and we are left with a series which converges geometrically:

(7) $e^{-x}Ei^+(x) = -\sum_{m=1}^{\infty} \frac{\Gamma(m)}{2^m} \frac{1}{(y)_m} + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Gamma(m)e_k}{(1 + e_k)^m} \frac{1}{(2k)_m}$ \hspace{1cm} (y = -ix/\pi)

**Proposition 1.**

(i) For fixed $x \in \mathbb{C} \setminus -i\mathbb{R}^+$ and large $n$, $R_n = O(2^{-n}n^{-3x/\pi})$. For fixed $n$ and large $x$, $R_n = O(x^{-n})$.

(ii) For fixed $k$ and $x \in \mathbb{C} \setminus -i\mathbb{R}^+$, $R_{nk} = O(|1 + e_k|^{-n}n^{-2k3x/\pi})$. For fixed $n$ and large $2^kx$, $R_{nk} = O((2^kx)^{-n})$.

**Note 2.** The domain of convergence in Proposition 1 is clearly larger than the half plane of usual factorial expansions. In fact this domain is maximal for any convergent meromorphic expansion of $Ei^+$ since, due to the Stokes phenomenon, after a $2\pi$ rotation of $x$ its classical asymptotic behavior changes.

The numerical efficiency on the Stokes line $\mathbb{R}^+$, with respect to the number of terms to be kept from each of the infinitely many series in (7) can be determined from Fig. 1. Namely, after choosing a range of $x$ and a target accuracy, one can determine from the graphs the needed order of truncation in each individual series, as well as the number of series as described in Fig. 1.

In Fig. 3 we plot the relative error in calculating $Ei^+$ on the Stokes ray. Figure 4 below uses the same expansion (7) for $x$ on the two sides of $-i\mathbb{R}^+$; in the left picture $\Im e^{-x}Ei^+(x)$ is calculated for $x \in -i\mathbb{R} - 0.3$ and the right one is the graph of $\Im e^{-x}Ei^+(x)$ along $-i\mathbb{R} + 0.3$ (after multiplying by $e^{0.3}$ to adjust back the size). The oscillatory behavior is due to the exponential (with amplitude $2\pi i$) collected upon crossing the Stokes ray $\mathbb{R}^+$ arg $x = -\pi/2$ is an antistokes ray for $Ei^+$).

**Note 3.** There is a dense set of poles in (7) along $-i\mathbb{R}^+$ where the expansion breaks down. (This of course does not imply actual singularities of $Ei^+$.) Hence, in spite of eventual geometric convergence, near $-i\mathbb{R}^+$ more and more terms need to be kept for a given precision.

**Proof of Proposition 1.** The difference between the first integral in (4) and the first sum in (7) truncated to $n - 1$ terms is $R_n$ in (9).
Figure 2. $f(x) = e^{-x} \text{Ei}^+(x)$ on the Stokes line: $\Re f$, (green), $e^x \Im f$, (blue), $\ln(-\Im f)$, (red), from [7]. The small exponential is “born” on $\mathbb{R}^+$, with half of the residue, as expected by comparing with $\frac{1}{2} e^{-x} \left( \text{Ei}^+(x) + \text{Ei}^-(x) \right)$.

Figure 3. Numerical errors for $x \in [1, 14]$ for $e^{-x} \text{Ei}^+(x)$ along the Stokes line with the formula [7].

Figure 4. The classical Stokes transition of $\text{Ei}^+$ from asymptotically decaying to oscillatory.
With the notation $\tilde{x} = x/\pi$ (so $y = -i\tilde{x}$) formula (6) is

$$R_n = -\frac{i\Gamma(n)}{(-i\tilde{x})_{n-1}} \int_0^\infty \frac{e^{-p(\tilde{x}+(n-1))}}{(1 + e^{-ip})^n} \, dp, \quad R_{nk} = \frac{e_k\Gamma(n)}{(-i2^k\tilde{x})_{n-1}} \int_0^\infty \frac{e^{-p(2^k\tilde{x}+(n-1))}}{(e_k + e^{-ip})^n} \, dp,$$

(i) For large $n$ we rotate the contour of the integral in $R_n$ by $-\pi/2$ and change variables to $q = ip$; the integrand is majorized by $|e^{-q^2}|e^{-n[1+\ln(1+e^{-q})]}$. Since $q + \ln(1 + e^{-q})$ is increasing, Laplace’s method shows that the integral is $O(n^{-1}2^{-n})$ which combined with Stirling’s formula for the prefactor (since $y \not\in \mathbb{R}^-$) yields the stated estimate. The estimate of $R_{nk}$ is similar.

For fixed $n$ and large $x$ the integral in (8) is $O(\tilde{x}^{-1})$ by Watson’s Lemma, and its prefactor, a multiple of $1/(-i\tilde{x})_{n-1}$, is $O(\tilde{x}^{-n+1})$.

(ii) The proof is similar: for fixed $k$ and $x$ the integral in (8) is $O(n^{-1}(1 + e_k)^{-n})$, while the prefactor is estimated using Stirling’s formula.

**Note 4.** (i) A variation (12) of (3) allows for optimizing the rate of convergence of Proposition 1, see Note 7 part (3).

(ii) For numerical purposes, for those of the $k$ kept in the calculation which are large enough, the integrals can be evaluated by Berry hyperasymptotics [23].

3. Dyadic decompositions

**Lemma 5** (Dyadic decomposition). The following identity holds in $\mathbb{C} \setminus \{0\}$:

$$\frac{1}{p} = \frac{1}{1 - e^{-p}} - \sum_{k=1}^\infty \frac{2^{-k}}{1 + e^{-p/2^k}}$$

(The points $m\pi i$ are removable singularities of the right side: for any $R$, all the finite sums from a certain rank on are analytic in $\{p : |p| \in (0, R)\}$.)

The sum converges uniformly on any compact $K \subset \mathbb{C} \setminus \{0\}$.

**Proof.** The proof is elementary:

$$\frac{1}{1 - x} = \frac{2}{1 - x^2} - \frac{1}{x + 1} = \frac{4}{1 - x^4} - \frac{2}{x^2 + 1} - \frac{1}{x + 1} = \cdots = \frac{2^n}{1 - x^{2^n}} - \sum_{j=0}^{n-1} \frac{2^j}{1 + x^{2^j}},$$

which implies, with $x = e^{-p/2^n}$,

$$\frac{1}{2^n(1 - e^{-p/2^n})} = \frac{1}{1 - e^{-p}} - \sum_{k=1}^\infty \frac{2^{-k}}{e^{-p/2^k} + 1}.$$

Let now $K \subset \mathbb{C} \setminus \{0\}$ be compact. In $K$, the left side of (11) converges uniformly to $p^{-1}$ while the sum on the right side converges uniformly and absolutely to the series in (6) since for large $k$ (depending on $K$), $|e^{-p/2^k} + 1|^{-1} < 2$.

Let $\beta \neq 0$. The linear affine transformation $p \to \beta p - \beta s$ gives:

**Corollary 6** (Dyadic decomposition of the Cauchy kernel).

$$\frac{1}{s - p} = -\frac{\beta e^{-\beta s}}{e^{-\beta s} - e^{-\beta p}} + \sum_{k=1}^\infty \frac{\beta 2^{-k}e^{-2^{-k}\beta s}}{e^{-2^{-k}\beta s} + e^{-2^{-k}\beta p}}.$$

The series converges uniformly for $p$ in compact sets in $\mathbb{C} \setminus \{s\}$.

See Notes 7 and 10 for the importance of the choice of the parameters $s$ and $\beta$.

**Note 7.** (1) The dyadic factorial expansion (3) of $Ei$ is in fact obtained using the general formula (12) with $\beta = 1/2$, $s = 2\pi i$ and $2\pi ip$ instead of $p$. 


(2) The choice of direction \( s \in i \mathbb{R}^+ \) is crucial for the result in Proposition 1.

(3) As mentioned, there is some arbitrariness in the choice of \( \beta \) and \( s \) in (12). In particular the choice above leads to diminishing the effective variable from \( x \) to \( y = -ix/\pi \), see (7). Depending on the range of \( x \), other choices of \( \beta \) would lead to better convergence rates, cf. Proposition 8.

4. \( \text{Ei} \) away from the Stokes ray, in \( \mathbb{C} \setminus \mathbb{R}^+ \)

In §2 we used dyadic expansions to obtain geometrically convergent expansions for \( \text{Ei} \) in \( \mathbb{C} \setminus -i \mathbb{R}^+ \), a region containing the Stokes ray \( \mathbb{R}^+ \). In this section we revisit the problem of obtaining somewhat simpler and more efficient expansions (faster than \( 2^{-m} \)) away from the Stokes ray.

There is substantial literature on classical factorial series representations of the exponential integral in the left-half plane, which is the sector opposite to the Stokes ray. For an excellent account of the literature see the recent paper [24]. See also [9](6.10) for extensive references.

Rotating the line of integration in (2) clockwise by an angle \( \pi^- \) while rotating \( x \) clockwise, we see that the study of \( \text{Ei} \) in \( \mathbb{C} \setminus -i \mathbb{R}^+ \) is equivalent to the study for \( 0 \neq x \in \mathbb{C} \), \( \arg x \neq \pi \) of the function

\[
e^{-xp} \text{Ei}(-x) = \int_0^\infty \frac{e^{-xp}}{p+1} \, dp
\]

**Proposition 8.** The following identity holds for all \( 0 \neq x \in \mathbb{C} \), \( \arg x \neq \pi \):

\[
e^{-xp} \text{Ei}(-x) = \int_0^\infty \frac{e^{-xp}}{e-e^{-p}} \, dp - \sum_{k=1}^{\infty} \int_0^\infty \frac{e^{2-k}e^{-2kq}}{e^{2-k}+e^{-q}} \, dq = \Phi(e^{-1}, 1, x-1) - \sum_{k=1}^{\infty} \Phi(-e^{-2-k}, 1, 2^k x)
\]

where \( \Phi \) is the Lerch Phi transcendent. Relation (13) implies

\[
e^{-xp} \text{Ei}(-x) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}e\Gamma(m)}{(e-1)m^m(x)} - \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Gamma(m)e^{-2k}}{(e^{2-k}+1)^m(2^k x)^m}
\]

The remainders, defined as in (5), satisfy: For fixed \( y \in \mathbb{C} \setminus \mathbb{R}^- \), \( R_n = O((e-1)^{-n}n^{-R_2}) \). For fixed \( n \) and large \( y \), \( R_n = O(x^{-n}) \). For fixed \( x \in \mathbb{C} \setminus \mathbb{R}^- \), \( R_{kn} = O(2^{-k}(1+e^{-2k})^{-n}n^{-2kR_2}) \). For fixed \( k, n \) and large \( x \), \( R_{kn} = O(2^{-k}x^{-n}) \).

**Note 9.** The effective variable, \( 2^k x \), gets rapidly large for large \( k \) and not many terms of the double sum are needed in practice. Even for \( x = 0.1 \) the first sum above requires 20 terms to give \( 10^{-5} \) relative errors.

**Proof of Proposition 8.** Taking \( \beta = 1 \) and \( s = -1 \) in (12), applying the Laplace transform, and then using dominated convergence we obtain (13). See [9](25.14.5) for the integral representation of the Lerch function.

Repeated integration by parts as in §2 yields the expansion (14); now, the remainders are:

\[
R_n = \frac{(-1)^n 1^\Gamma(n)}{(x)_{n-1}} \int_0^\infty \frac{e^{-p(x+n-1)}}{(e-e^{-p})^n} \, dp = \frac{(-1)^n e\Gamma(n)\Gamma(x)}{\Gamma(x+n-1)} \int_0^\infty \frac{e^{-p(x+n-1)}}{(e-e^{-p})^n} \, dp
\]

and

\[
R_{nk} = -e^{-2k} \Gamma(n) \int_0^\infty \frac{e^{-2k(x+n-1)}}{(e^{2-k}+e^{-q})^n} \, dq = -e^{2-k} \Gamma(n)\Gamma(2^k x) \int_0^\infty \frac{e^{-2k(x+n-1)}}{(e^{2-k}+e^{-q})^n} \, dq
\]

The remainders are estimated as in the proof of Proposition 1.

5. Dyadic expansions for Airy and Bessel functions

To our knowledge, the first systematic study of classical factorial series for Bessel function is [10]; see [11] for subsequent developments.
5.1. Dyadic expansions for the Airy function Ai. Again, to keep the logic simple, we analyze in some detail the Airy function Ai, as the general Bessel functions are dealt with similarly, as explained in §5.2.

After normalization, described in §10.2, the asymptotic series of the Airy function is Borel summable:

\[ h(x) = \int_0^\infty e^{-px} F(p) dp \]

where \( F(p) = 2F_1(1/6, 5/6; 1, -p) = P_{-1/6}(1 + 2p) \) is analytic except for a logarithmic singularity at \(-1\), see (64) and (65) below. The decay of \( F \) for large \( p \) is relatively slow, \( O(p^{-1/6}) \), and we integrate once by parts to improve it:

\[ h(x) = \frac{F(0)}{x} + \frac{1}{x} \int_0^\infty e^{-px} F'(p) dp \]

and use Cauchy’s formula (needed for applying the derivative of (12))

\[ F'(p) = \frac{1}{2\pi i} \int_{|p-s|<r} \frac{F(s)}{(s-p)^2} ds = \frac{1}{2\pi i} \int_{-\infty}^{-1} \frac{\Delta F(s)}{(p-s)^2} ds \]

where we pushed the contour to infinity so that a subsequent Laplace contour, \( \mathbb{R}^+ \) does not intersect the \( s \) integral, see Note 10 below. We are left with a Hankel contour around \(-1\); \( \Delta F \) is the jump of \( F \) across the cut \((-\infty, -1)\).

**Note 10.** When using (9), to be able to interchange summation and integration in a contour integral, we need of course to ensure that each term and not merely the sum in (9) is analytic on the contour. If \( s \) parametrizes the curve, then in particular the curve must avoid the half-lines \( s = p + k\pi i/\beta \). If \( F \) has only one singularity, then the Cauchy formula contour can be deformed into a Hänkel-like curve towards \(-\infty\) for a suitable \( \beta \).

After the change of variables \( s = -1 - t \) we get

\[ h(x) = \frac{F(0)}{x} - \frac{1}{2\pi x} \int_0^\infty e^{-xp} \int_0^\infty \frac{F(t)}{(1 + p + t)^2} dt \]

since \( \Delta F(-1 - t) = -iF(t) \), see (66) below.

To use the expansion (12) in (20) we first differentiate (12) in \( p \) and take \( \beta = 1 \) obtaining

\[
\frac{1}{(s-p)^2} = \frac{e^{-s-p}}{(e^{-s} - e^{-p})^2} \sum_{k=1}^\infty 4^{-k} \frac{e^{2^{-k}(-p-s)}}{(e^{2^{-k}s} + e^{2^{-k}p})} \]

which for \( s = -1 - t \) yields

\[ \int_0^\infty \frac{F(t) dt}{(1 + p + t)^2} = \int_0^\infty \frac{e^{-1-p+t} F(t) dt}{(e^{1+t} - e^-p)^2} + \sum_{k=1}^\infty \int_0^\infty \frac{4^{-k} e^{2^{-k}(1-p+t)} F(t) dt}{(e^{2^{-k}(1+t)} + e^{-2^{-k}p})^2} \]

yielding

\[ h(x) = \frac{F(0)}{x} - \frac{1}{2\pi x} \int_0^\infty e^{-p(x+1)} dp \int_0^\infty \frac{e^{1+t} F(t) dt}{(e^{1+t} - e^-p)^2} \]

\[ - \frac{2^{-k} e^{2^{-k}}}{2\pi x} \int_0^\infty e^{-q(2^k x+1)} dq \sum_{k=1}^\infty \int_0^\infty \frac{e^{2^{-k}l F(t) dt}}{(e^{2^{-k}(1+t)} + e^{-q})^2} \]

The dyadic factorial series is obtained, as before for \( \text{Ei} \), by repeated integration by parts, integrating the exponentials. This yields the dyadic factorial expansion

\[ h(x) = \frac{F(0)}{x} - \sum_{m=2}^\infty \frac{(-1)^m \Gamma(m)}{2\pi(x)^m} d_m - \sum_{k=1}^\infty 2^{-k} e^{2^{-k}} \sum_{m=2}^\infty \frac{\Gamma(m)}{2\pi(2^k x)^m} d_{km} \]
where

$$d_m := \int_0^\infty \frac{F(t)e^{t+1}dt}{(e^{t+1} - 1)^m}; \quad d_{km} := \int_0^\infty \frac{e^{2-kt}F(t)dt}{(e^{2-k(1+t)} + 1)^m}$$

Unlike in the case of Ei however, the coefficients $d_m$ do not have a simple closed form expression, nor of course can this be expected in general. The integrals can be evaluated numerically, or by power series. Alternatively, they can be calculated in the $x$ domain. Indeed, with $\varphi = LF$,

$$\int_0^\infty \frac{F(t)e^{t+1}dt}{(e^{t+1} - 1)^{m+2}} = \sum_{j=0}^\infty e^{-m-j-2} \int_0^\infty e^{-(m+j+1)t}F(t)dt = e^{-m-1} \sum_{j=1}^\infty e^{-j} \binom{m+j}{j} \varphi(m+j)$$

and for Airy, $\varphi = h$.

Fig. 5 shows the numerical results from (23) using Mathematica in machine precision to evaluate the integrals in the $d_m, d_{km}$.

### 5.2. General Bessel functions.**

There are few and relatively minor adaptations needed to deal with $K_\nu$ for more general $\nu$. After normalization, explained in §10.2, $F(p)$ is now the Legendre function $P_{\nu-1/2}(1 + 2p)$ for which the branch jump at $-1$ is $\Delta F(-1 - p) = -2i \cos(\pi\nu)F(p)$ (see (66)) and the leading behavior at infinity is $O(p^{\Re\nu-1/2})$. The steps followed in the Airy case apply after integrating by parts $k$ times until $|\Re\nu| - 1/2 - k < -1$. Alternatively, one can apply the general transformation in §17 that ensures exponential decay. For $J_\nu, Y_\nu$ the procedure is the same, except that the singularity is now on the imaginary line. For $J_\nu$ the singularity is on $\mathbb{R}^+$ and a choice of $\beta$ as for $\text{Ei}^+$ needs to be made.

### 6. Dyadic resolvent identities

Dyadic decompositions translate into representations of the resolvent of a self-adjoint operator in a series involving the unitary evolution operator at specific discrete times:

**Proposition 11.** (i) Let $\mathcal{H}$ be a Hilbert space, and $A$ a bounded or unbounded self-adjoint operator. Let $U$ be the unitary evolution operator generated by $A$, $U_t = e^{-itA}$. If $\lambda \in \mathbb{R}^+$, then

$$\begin{aligned}
(A - i\lambda)^{-1} &= i(1 - e^{-\lambda U_1})^{-1} - i \sum_{k=1}^\infty \frac{1}{2^k}(1 + e^{-\lambda/2^k}U_{2^{-k}})^{-1} \\
&= i \sum_{j=0}^\infty e^{-j\lambda}U_j - i \lim_{\ell \to \infty} \sum_{k=1}^\ell \sum_{j=0}^\infty (-1)^je^{-j\lambda/2^k}U_{j2^{-k}}
\end{aligned}$$

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Convergence holds in the strong operator topology. For \( \lambda < 0 \) one simply complex conjugates \( (26) \). (The limits cannot, generally, be interchanged.)

(ii) Assume \( A \) is a positive operator (thus self-adjoint) and \( 0 \notin \sigma(A) \). Let \( T_t \) be the semigroup generated by \( A \), \( T_t = e^{-tA} \). Then

\[
A^{-1} = (1 - T_1)^{-1} - \sum_{k=1}^{\infty} 2^{-k}(1 + T_1/2^k)^{-1} = \sum_{j=1}^{\infty} T_j - \lim_{\ell \to \infty} \sum_{k=1}^{\ell} \sum_{j=1}^{\infty} 2^{-k}(-1)^j T_1/2^k
\]

where now convergence is in operator norm. More generally, for \( s < 1 \), \( s \notin \mathbb{Z} \),

\[
\pi A^{s-1} = \Gamma(s) \sin(\pi s) \left[ \text{Li}_s(T_1) - \sum_{k=1}^{\infty} 2^{-k(1-s)} \text{Li}_s\left(-T_1/2^k\right) \right]
\]

in operator norm. Here, for \( |z| < 1 \), the polylog is defined by

\[
\text{Li}_s(z) = \sum_{k=1}^{\infty} k^{-s} z^k
\]

Proof. (i) We recall the projector-valued measure spectral theorem for self-adjoint operators. If \( \mathcal{H} \) and \( A \) are as above and \( g : \mathbb{R} \to \mathbb{R} \) is a Borel function (or a complex one, by writing \( g = g_1 + ig_2 \)), then \( g(A) = \int_{-\infty}^{\infty} g(q) dP_q \) where \( \{P_1\} \) are the projector-valued measures induced by \( A \) on \( \mathcal{H} \) (see \[21\] Theorem VIII.6 p. 263). The spectral theorem together with \( (11) \) for \( p = \lambda + i \varepsilon \) give

\[
(1 - e^{-\lambda U_1})^{-1} - \sum_{k=1}^{n} 2^{-k}(1 + e^{-2^{-k}\lambda U_2})^{-1} = \varepsilon(1 - e^{-\lambda e^{-i\varepsilon A}})^{-1} = \int_{\mathbb{R}} \frac{\varepsilon dP_q}{1 - e^{-\varepsilon(\lambda + i\varepsilon)}}
\]

where \( \varepsilon_n := \varepsilon = 2^{-n} \). An elementary calculation shows that the modulus of the integrand is uniformly bounded by \( \lambda^{-1} \). Since the integrand converges pointwise to \( (\lambda + i\varepsilon)^{-1} \) as \( \varepsilon \to 0 \), dominated convergence shows that the integral converges to \( (\lambda + iA)^{-1} \). Dominated convergence also shows that the integrand, seen as a multiplication operator, converges in the strong operator topology, implying the result.

(ii) The proof, based on the same argument as in (i), is simpler and we omit it. For \( (28) \) we combine this argument with Lemma \( (26) \) below. The sums in \( (27) \) are manifestly convergent in the operator norm since \( \|T_t\| < 1 \) and \( T_t > 0 \).

\( \square \)

7. When do classical factorial series converge geometrically?

Here we motivate the treatment of general resurgent functions in \[8\] and explain why expansions of the form \( (12) \) yield to geometrically convergent factorial expansions. The conclusions are summarized in Note \( 16 \).

The connection of Horn factorial expansions to Borel summation is made already in \[19\]. Assume \( f \) is the Borel sum of a series, that is

\[
f(x) = \int_{0}^{\infty} F(p)e^{-px} \, dp
\]

where \( F \) is analytic in an open sector containing \( \mathbb{R}^+ \) and exponentially bounded at infinity. The asymptotic series for large \( x \) follows from Watson’s lemma \[23\] or, in this case, simply by integration by parts: for \( x \) large enough we have

\[
f(x) = x^{-1} F(0) + x^{-2} F'(0) + \cdots + x^{-n} F^{(n-1)}(0) + x^{-n} \int_{0}^{\infty} F^{(n)}(p)e^{-px} \, dp
\]

Integration by parts results in a growing power of \( \frac{d}{dp} \) and thus, by Cauchy’s theorem leads to factorial divergence of the asymptotic series, unless \( F \) is entire (rarely the case in applications). Nörlund notices
however that the simple change of variables $\varphi(s) = F(-\ln s)$ brings the representation of $f$ to the form

$$f(x) = \int_0^1 s^{x-1} \varphi(s) \, ds$$

(33)

Now integration by parts gives the factorial expansion

$$f(x) = \varphi(1) \frac{1}{x} - \varphi'(1) \frac{1}{(x)^2} + \cdots + \frac{(-1)^{n-1}}{(x)^n} \varphi^{(n-1)}(1) + \frac{(-1)^n}{(x)^n} \int_0^1 s^{x+n-1} \varphi^{(n)}(s) \, ds$$

(34)

or, without remainder, we have the factorial series (Horn expansion)

$$\tilde{f}(x) = \sum_{k=0}^{\infty} (-1)^k \frac{\varphi^{(k)}(1)}{(x)_{k+1}}$$

(35)

Note 12. Since $F$ is analytic at zero, $\varphi$ is analytic at one. Using Stirling’s formula in (1), we see that, for large $k$, the $k + 1$th term of the expansion (34) behaves like

$$(-1)^k \Gamma(x) \frac{\varphi^{(k)}(1)}{k!} k^{-x}$$

(36)

Due to the $1/k!$ factor in (36) the series $\tilde{f}(x)$ can converge even if (32) is factorially divergent.

Note 13. For $\tilde{f}$ to converge, (36) shows that $\varphi$ needs to be analytic in a disk of radius one centered at $s = 1$, which translates in analyticity of $F$ in the region $S = \{p = p_1 + ip_2 | |p_2| < \arccos(\frac{1}{2} e^{-p_1}) \}; S$ is a strip-like region of width $\pi$ centered on $\mathbb{R}_+$ (see [23] p. 328).

If $F$ is analytic in a strip $\omega S$ (for some $\omega \neq 0$) and $F$ has at most exponential growth, then replacing $p$ by $\omega p$ and $x$ by $x/\omega$ then, in the new variables, the conditions mentioned before are necessary and sufficient for convergence of (35).

Note 14. (i) We also note that if $F$ is exponentially bounded in a strip $S$ then $\tilde{f}(x)$ converges in a half-$x$-plane, and no more. A rigorous proof based on Hadamard’s theory of order on the circle of convergence is given in [19] pp. 45-59. Heuristically, if $|F(p)| \leq e^{\beta p}$ in $S$ then $|\varphi(s)| \leq |s^{-\beta}|$ near $s = 0$ and then, by Cauchy’s formula, $|\varphi^{(k)}(1)/k!| \leq k^{-\beta}$ hence convergence requires $\Re x > \beta$.

(ii) Conversely, if a factorial series $f(x) = \sum c_k \frac{1}{(x)_k}$ converges in a half-plane, since

$$\left[ \mathcal{L}^{-1} \frac{1}{(x)_k} \right] (p) = \frac{(1 - e^{-p})^k}{k!}$$

(37)

(see §10.1) we have

$$\left[ \mathcal{L}^{-1} f \right] (p) = \sum_{k=0}^{\infty} c_k \frac{(1 - e^{-p})^k}{k!} := F(p)$$

with $F$ as in (i). (See also [19] p.188.)

Note 15. As mentioned, $F$ is required to be analytic on $\mathbb{R}^+$; hence $\mathbb{R}^+$ cannot be a Stokes line (a line containing Borel plane singularities). Because of this, classical factorial series (34) are not suitable for the study of Stokes phenomena (15, 2).

Note 16. Convergence of $\tilde{f}(x)$ is typically slow, generally at most power-like. The theorems in [19] and [23] are too general to allow for more precise estimates of the rate. In the specific case of Bessel functions of order $\nu$, Lutz and Dunster [10] showed that the $m$th term in the series is $m!/\nu(z)_{m+1} O(m^{-1} \ln^{\nu-5/2} m)$ for $\Re z > \epsilon > 0$, implying that the rate of convergence is $O(m^{-\Re z} \ln m^{\nu-5/2})$. In general, by (36), we see that convergence is geometric only if $\varphi$ is analytic in a disk of radius $1$, in particular at $s = 0$, or analytic in $s^{\beta}$ by changing $x$ to $2 + (x - 2)/\beta$, or, more generally, a sum of such functions for various $\beta$’s. Therefore convergence cannot be geometric unless $F$ is analytic in $e^{-\beta p}$ at infinity (for some $\beta$), or a sum of such functions.
8. Dyadic series of general resurgent functions

In a nutshell, a resurgent function in the sense of Écalle is a function which is endlessly continuable and has suitable exponential bounds at infinity \[12\]. The singularities are typically assumed to be regular, in the sense of having convergent local Puiseux series possibly mixed with logs.

In this paper we restrict to functions which appear in generic meromorphic ODEs and difference equations. More precisely, a resurgent function is a function \( F \) (in Borel plane) which: has a finite number of arrays of singularities; in each array the singularities are regular and equally spaced; and \( F \) is exponentially bounded at infinity (away from the singular arrays). For details see \[6\] and \[5\]. By abuse of language, the Laplace transform of a resurgent function is often also called “resurgent”.

Using Lemma \[5\] we show that modulo simple, algorithmic transformations, resurgent functions can be written in the form \( \sum_j F_j(e^{-\beta_j p}) \), where the sum converges geometrically, \( F_j(e^{-\beta_j p}) \) are also resurgent, and \( F_j(z) \) are analytic at zero. Thus the factorial series of \( LF_j(e^{-\beta_j p}) \) are geometrically convergent (see Note \[16\]) in a cut plane, thus allowing for the study of Stokes phenomena. Due to rapid convergence, their associated factorial series are also suitable for precise and efficient numerical calculations.

8.1. Elementary resurgent functions. We define resurgent “elements” to be resurgent functions with only one regular singularity, and with algebraic decay at infinity. There are two main properties of resurgent elements which do not hold for general resurgent functions: decay at infinity in \( p \) and the property of having only one singularity. However, the following decomposition holds:

**Theorem 17.** The Laplace transform of a resurgent function as described at the beginning of §8 can be written, modulo a convergent series at infinity and translations of the variable, as a sum of Laplace transforms of resurgent elements.

**Proof.** The proof is given in §8.2. □

**Note 18.** The exponential integral and the \( \Psi \) function treated in §9.1 are examples of elements with nonramified singularities. Airy and Bessel functions treated in §5 are examples of elements with ramified singularities, treated via the Cauchy kernel decomposition. The incomplete gamma function and the error function treated in §9.4 have power-ramified singularities for which a polylog dyadic expansion (Lemma \[26\]) gives more explicit decompositions. Theorem 17 extends these techniques to general resurgent functions.

8.2. Proof of Theorem 17. In this section we describe how a general resurgent function can be decomposed into resurgent elements. To avoid cumbersome details and keep the presentation clear, we present the essential steps in the case where the resurgent function is the Laplace transform of a solution of a generic meromorphic ODE.

Denoting the singularities of the resurgent function \( F \) by \( \omega_i \), we thus assume:

(a) Each \( \omega_i \) is of the form \( j \lambda_k \), with \( j \in \mathbb{Z}^+ \) and \( \lambda_k \in \{\lambda_1, \ldots, \lambda_n\} \) (the eigenvalues of the linearization at \( \omega_i \) of the ODE assumed to be linearly independent over \( \mathbb{Z} \) and of different complex arguments);

(b) there is a \( \nu \) such that

\[
\|F\|_\nu := \sup_{p \in A} |F(p)e^{-\nu |p|}| < \infty
\]

where \( A \) is the complement of the union of thin half-strips \( S_i \) containing exactly one singularity \( \omega_i \). We let \( C_i = \partial S_i \), (see Fig. 6). \( C_i \) are non-intersecting Hankel contours around the \( \omega_i \), going vertically if \( \omega_i \) belongs to a singularity ray in the open right half plane and towards \( \infty \) in the left half plane otherwise; \( C_i \) are traversed anticlockwise.

Let

\[
G(p) = F(p) - \sum_{\omega_i} \frac{\exp(\mu_i p)}{2\pi i} \int_{C_i} \frac{F(s) \exp(-\mu_i s)}{s-p} \, ds
\]

where the sum is over the \( \omega_i \) and the integral is over the contour \( C_i \).
where:

(a) $|\mu_i| = \mu > \nu$,

(b) $\arg(\mu_i)$ is the negative of the angle of the contour $C_i$, i.e., $\mu_i s \in \mathbb{R}^+$ for large $s$. Note that the set $\{\arg(\mu_i)\}$ is finite, since there are only finitely many rays with singularities.

**Lemma 19.** In any compact set in $\mathcal{A}$, the sum in (38) converges at least as fast as $\sum_{j \in \mathbb{Z}^+, k=1,...,n} e^{-j|\lambda_k|(|\mu-\nu|)}$.

**Proof.** Let $\omega_i = j\lambda_k$ and

$$F_i(p) = \frac{\exp(\mu_i p)}{2\pi i} \int_{C_i} \frac{F(s) \exp(-\mu_i s)}{s-p} ds$$

For some $a$ depending on the width of the contour, the distance to the singularity (these two parameters can be chosen to be the same for all contours), on the position and diameter of the compact set (also the same for all $i$), we have

$$|F_i| \leq a \|F\| e^{-j|\lambda_k|(|\mu-\nu|)}$$

**Lemma 20.** On the first Riemann sheet, each $F_i$ in (39) has precisely one singularity, namely at $\omega_i$. Furthermore $F - F_i$ is analytic at $\omega_i$.

**Proof.** Let $p \neq \omega_i$. If $p$ is outside $C_i$ then function $F_i$ is manifestly analytic at $p$. To analytically continue in $p$ to the interior of $C_i$ it is convenient to first deform $C_i$ past $p$, collecting the residue. We get

$$F_i(p) = \frac{\exp(\mu_i p)}{2\pi i} \left[ \int_{\tilde{C}_i} \frac{F(s) \exp(-\mu_i s)}{s-p} ds + 2\pi i F(p) \exp(-\mu_i p) \right] = F(p) + \frac{\exp(\mu_i p)}{2\pi i} \int_{\tilde{C}_i} \frac{F(s) \exp(-\mu_i s)}{s-p} ds$$

where now $p$ sits inside $\tilde{C}_i$, and the new integral is again manifestly analytic.

Thus $F_i$ is singular only at $p = \omega_i$, and $F - F_i$ is analytic at $\omega_i$.\hfill $\square$
Lemma 21. The function

\[ G(p) = F(p) - \sum_{i} F_i \]

is entire and \( \|G\|_{\mu'} < \infty \) for any \( \mu' > \mu \).

Proof. Analyticity follows from the monodromy theorem, since \( G \) has analytic continuation along any ray in \( \mathbb{C} \). The bound follows easily from the previous lemmas. \( \square \)

Lemma 22. \( g = L \) \( G \) has a convergent asymptotic series at infinity, and is equal to the sum of the series.

Proof. Cauchy estimates show in a straightforward way that \( G(n)(0) \approx \mu' n / n! \). Watson’s lemma shows convergence of the series. The function \( h(z) = g(1/z) \) is bounded at zero and single-valued, as is seen by deformation of contour (since \( G \) is exponentially bounded and entire). Thus \( h \) is analytic at zero, and therefore the sum of its asymptotic (=Taylor) series at zero. \( \square \)

Lemma 23. Each function \( e^{-\mu_i p} F_i \) decays like \( 1/p \) as \( p \to \infty \).

Proof. The function \( p F_i \) is manifestly bounded. \( \square \)

Lemma 24. The change of variable \( \tilde{x} = x - \mu_i \) leads to \( L[F_i](x) = L[\tilde{F}_i](\tilde{x}) \) where \( \tilde{F}_i \) decays like \( 1/p \) as \( p \to \infty \).

Combining these lemmas, Theorem 17 follows.

9. The \( \Psi \) Function

9.1. Dyadic factorial expansion for the \( \Psi \) function. Replacing \( p \) by \(-p\) in (9) we get

\[ \frac{1}{p} - \frac{1}{e^p - 1} = \sum_{k=1}^{\infty} \frac{e^{-\frac{p}{2^k}}}{2^k (e^{-\frac{p}{2^k}} + 1)} \]

On the other hand we have, see [7] eq. (4.61) p. 99,

\[ \frac{\Gamma'(x+1)}{\Gamma(x+1)} - \ln x = \int_{0}^{\infty} \left( \frac{1}{p} - \frac{1}{e^p - 1} \right) e^{-xp} dp \]

Thus, changing the variable of integration to \( q = p/2^k \) we get

\[ \Psi(x+1) = \ln x + \sum_{k=1}^{\infty} \int_{0}^{\infty} \frac{e^{-q(1+2^k x)}}{1 + e^{-q}} dq \]

and integrating by parts we obtain the dyadic factorial expansion

\[ \Psi(x+1) = \ln x - \sum_{k=1}^{\infty} \Phi(-1, 1, 2^k x + 1) = \ln x + \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{(j - 1)!}{2^j (2^k x + 1)^j} \]

9.2. Factorial expansion for differences of the \( \Psi \) function and a strange identity.

Proposition 25. We have

\[ \frac{1}{2} \Psi \left( \frac{x}{2} + \frac{1}{2} \right) - \frac{1}{2} \Psi \left( \frac{x}{2} \right) = \int_{0}^{1} \frac{t^{x-1}}{t + 1} dt = \frac{1}{2x} - \frac{1}{2^2 (x)_2} + \cdots + \frac{(-1)^{n-1} \Gamma(n)}{2^n (x)_n} + \cdots \]

Combining with (44) we get

\[ \Psi(x+1) = \ln x - \frac{1}{2} \sum_{k=0}^{\infty} \left[ \Psi \left( 2^k x + 1 \right) - \Psi \left( 2^k x + \frac{1}{2} \right) \right] \]
Proof. Consider the functional equation

\[(47) \quad f(x + 1) + f(x) = \frac{1}{x}\]

After Borel transform (i.e. substitution of (31)) it becomes \((e^{-p} + 1)F(p) = 1\) yielding

\[(48) \quad f(x) = \int_0^\infty \frac{e^{-px}}{e^{-p} + 1} dp = \int_0^\infty \sum_{n=0}^\infty (-1)^n e^{-p(x+n)} dp = \sum_{n=0}^\infty \frac{(-1)^n}{x+n}\]

where the interchange of summation and integration is justified, say, by the monotone convergence theorem applied to \(\sum_{n=0}^{2N}(-1)^n e^{-p(x+n)}\). Of course, the integral converges only for \(\Re x > 0\), but the series converges for all \(x \notin \{0, -1, -2, \ldots\}\). Therefore \(f(x)\) is meromorphic, having simple poles at \(x = -n, n \in \mathbb{N}\).

On the other hand \(f(x) = \frac{1}{2} \psi\left(\frac{x}{2} + \frac{1}{2}\right) - \frac{1}{2} \psi\left(\frac{x}{2}\right)\) which follows from integrating the identity

\[\psi'(z) = \sum_{n=0}^\infty (z + n)^{-2}\]

(see [1], (31) p. 200) between \(z = \frac{x}{2}\) and \(z = \frac{x+1}{2}\).

The integral representation in (45) then follows by substituting \(e^{-p} = t\) in (48) and the factorial expansion in (45) is then obtained as usual, by integration by parts.

\[\square\]

9.3. Duplication formulas and incomplete Gamma functions. The polylog \(\text{Li}_s(z) = \sum_{k=1}^\infty \frac{z^k}{k^s}\) has the integral representation

\[(49) \quad \text{Li}_s(z) = \frac{z}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - z} dx\]

and satisfies the general duplication formula

\[(50) \quad f(z) + f(-z) = 2^{1-s}f(z^2)\]

(see [18]; also, (49), (50) are easily checked directly).

**Lemma 26** (A ramified generalization of (9)). The following identity holds in \(\mathbb{C} \setminus \{0\}\) if \(s < 1\):

\[(51) \quad \pi p^{s-1} = \Gamma(s) \sin(\pi s) \left[ \text{Li}_s(e^{-p}) - \sum_{k=1}^\infty 2^{-k(1-s)}\text{Li}_s\left(-e^{-2^{-k}p}\right) \right]\]

which reduces to (9) if \(s = 0\).

**Proof.** Let \(s < 1\). As in the proof of Lemma 5 we iterate (50) \(n\) times, then replace \(z\) by \(e^{2^{-n}z}\) to obtain

\[(52) \quad \frac{\text{Li}_s(e^{z/2^n})}{2^n(1-s)} = \text{Li}_s(e^z) - \sum_{k=1}^n 2^{-k(1-s)}\text{Li}_s\left(-e^{2^{-k}z}\right)\]

\[(53) \quad 2^{-n(1-s)}\text{Li}_s\left(e^{-2^{-n}z}\right) = \text{Li}_s(e^{-z}) - \sum_{k=1}^n 2^{-k(1-s)}\text{Li}_s\left(-e^{-2^{-k}z}\right)\]

Using the identity [9](25.12.12)

\[\text{Li}_s(z) = \Gamma(1-s) \left(\frac{1}{z}\right)^{s-1} + \sum_{n=0}^\infty \zeta(s-n) \frac{(\ln z)^n}{n!}, \quad s \neq 1, 2, \ldots, |\ln z| < 2\pi\]
in (53) we get, in the limit \( n \to \infty \),
\[
z^{s-1} \Gamma(1 - s) = \text{Li}_s(e^{-z}) - \sum_{k=1}^{\infty} 2^{-k(1-s)} \text{Li}_s \left( -e^{-2^{-k}z} \right)
\]
from which (51) follows by using the reflection formula \( \Gamma(s)\Gamma(1 - s) = \pi / \sin(\pi s) \).

\[\square\]

9.4. Dyadic factorial series for incomplete gamma functions and \( \text{erfc} \). The incomplete gamma function is defined by
\[
\Gamma(s, x) = \int_x^{\infty} t^{s-1} e^{-t} \, dt
\]
and has as a special case the error function,
\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} \, dt = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{1}{2}, x^2 \right)
\]
Noting that
\[
\int_0^{\infty} (1 + p)^{s-1} e^{-xp} \, dp = e^x x^{-s} \Gamma(s, x)
\]
we see that \( e^x x^{-s} \Gamma(s, x) \) is the Laplace transform of a function which has a ramified singularity if \( s \not\in \mathbb{Z} \).
In this case we apply Lemma 26 and obtain the expansion, for \( s < 1 \)
\[
(54) \quad \Gamma(1 - s) e^x x^{-s} \Gamma(s, x) = \mathcal{L} \text{Li}_s \left( e^{-p-1} \right) - \sum_{k=1}^{\infty} 2^{-k(1-s)} \mathcal{L} \text{Li}_s \left( -e^{-2^{-k}(p+1)} \right)
\]
and in particular
\[
(55) \quad \pi e^x x^{-1/2} \text{erfc} \left( \sqrt{x} \right) = \mathcal{L} \text{Li}_{1/2} \left( e^{-p-1} \right) - \sum_{k=1}^{\infty} 2^{-k/2} \mathcal{L} \text{Li}_{1/2} \left( -e^{-2^{-k}(p+1)} \right)
\]
From this point on, the dyadic expansions are obtained as in the previous examples. For example, the first Laplace transform in (55) has the factorial series
\[
\mathcal{L} \text{Li}_{1/2} \left( e^{-p-1} \right) = \int_0^{1} t^{x-1} \text{Li}_{1/2} \left( \frac{t}{e} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{e^{x}k!} \text{Li}_{1/2} \left( e^{-1} \right) := \sum_{k=0}^{\infty} \frac{c_k}{(x)k+1}
\]
with
\[
c_k = (-1)^k \sum_{j=0}^{k} s(k, j) \text{Li}_{1/2-j} \left( e^{-1} \right)
\]
where \( s(k, j) \) are the Stirling numbers of the first kind, where we used the formula (see §10.3 for details)
\[
(56) \quad \frac{d^k}{dz^k} \text{Li}_\nu(z) = z^{-k} \sum_{j=0}^{k} s(k, j) \text{Li}_{\nu-j}(z)
\]

10. Appendix

10.1. The rising factorial and the Laplace transform. Let \( \delta f(x) = f(x) - f(x + 1) \). We first note that
\[
(57) \quad (\mathcal{L}^{-1} \delta f)(p) = (1 - e^{-p}) (\mathcal{L}^{-1} f)(p)
\]
and that \( \delta(1/(x)_n) = n/(x)_{n+1} \), hence
\[
(58) \quad \delta_{n+1} \frac{1}{x} = \frac{n!}{(x)_n}
\]
Eq. (58) and (57) imply (37).
10.2. Normalized Airy and Bessel functions. The modified Bessel equation is
\( x^2 y'' + xy' + (\nu^2 - x^2) y = 0 \)  \hspace{1cm} (59)

The transformation \( y = e^{-x} x^{1/2} h(2x) \), \( u = 2x \) brings (59) to the normalized form
\[
\frac{d^2}{du^2} h'' - \frac{1}{u} \frac{d}{du} h' - \left( \frac{1}{u} - \frac{1}{4u^2} + \frac{\nu^2}{u^2} \right) h = 0
\]  \hspace{1cm} (60)

This normalized form is suitable for Borel summation since it admits a formal power series solution in powers of \( u^{-1} \) starting with \( u^{-1} \); it is further normalized to ensure that the Borel plane singularity is placed at \( p = -1 \). One way to obtain the transformation is to rely on the classical asymptotic behavior of Bessel functions and seek a transformation that formally leads to a solution as above.

The Airy equation
\[
f'' - xf = 0
\]  \hspace{1cm} (61)

can be brought to the Bessel equation with \( \nu = 1/3 \), as is well known. The normalizing transformation can be obtained directly by the recipe above, based on the asymptotic behavior at \( \infty \). With the change of variable
\[
f(x) = x^{5/4} e^{-2x^{3/4}} h(x); \quad x = (3u/4)^{2/3}
\]
the equation becomes
\[
\frac{d^2}{du^2} h'' - \frac{1}{u} \frac{d}{du} h' - \left( \frac{1}{u} - \frac{5}{36u^2} \right) h = 0
\]  \hspace{1cm} (62)

which is indeed (60) for \( \nu = 1/3 \). From this point, without notable algebraic complications we analyze (60).

The inverse Laplace transform of (60) is
\[
p(p+1) H''(p) + (2p+1) H'(p) + \left( \frac{1}{4} - \nu^2 \right) H(p) = 0
\]  \hspace{1cm} (63)

whose solution which is analytic at zero is (a constant multiple of)
\[
\text{\large \( \text{\_} F_1 \left( \frac{1}{2} + \nu, \frac{1}{2} - \nu; 1; -p \right) = P_{\nu - \frac{1}{2}}(1 + 2p) \)}
\]  \hspace{1cm} (64)

where \( \text{\large \( \text{\_} F_1 \)} \) is the usual hypergeometric function and \( P \) is the Legendre \( P \) function [9](14.3.1). On the first Riemann sheet, the solution has two regular singularities, \( p = -1 \) and \( p = \infty \). The behavior at zero is [9](15.2.1)
\[
\text{\large \( P_{\nu - \frac{1}{2}}(1 + 2p) = 1 + \left( \nu^2 - \frac{1}{4} \right) p + \frac{1}{64} (16\nu^4 - 40\nu^2 + 9) p^2 + \cdots \)}
\]  \hspace{1cm} (65)

At \( \infty \), the convergent series of the solution is [9](15.12.1(i))
\[
\text{\large \( P_{\nu - \frac{1}{2}}(1 + 2p) = \Gamma(-2\nu) \Gamma\left( \frac{1}{2} - \nu \right) \left( \frac{1}{2} - \nu \right)^{-2} p^{-\nu-1/2}[1 + o(1/p)] + \Gamma(2\nu) \Gamma\left( \nu + \frac{1}{2} \right)^{-2} p^{\nu-1/2}[1 + o(1/p)] \)}
\]  \hspace{1cm} (66)

At the singularity \( p = -1 \) we have (see [9](14.8.2) and (14.6.1))
\[
\text{\large \( P_{\nu - \frac{1}{2}}(1 + 2p) = -\pi^{-1} \cos(\pi\nu) \log(p + 1) A_1(1 + p) + A_2(1 + p) \)}
\]  \hspace{1cm} (67)

where \( A_1, A_2 \) are analytic and \( A_1(-1) = 1 \).

The difference between the analytic continuation of \( P_{\nu - \frac{1}{2}}(1 + 2p) \) below and above \((-\infty, -1)\) is (see [9](15.10), or directly, from (65) and the fact that the difference of two solutions of (63) is again a solution)
\[
\text{\large \( P_{\nu - \frac{1}{2}}(1 + 2p)^- - P_{\nu - \frac{1}{2}}(1 + 2p)^+ = -2i \cos(\pi\nu) P_{\nu - \frac{1}{2}}(-1 - 2p) \)}
\]  \hspace{1cm} (68)
10.3. **The derivatives of the polylogarithm.** For \( k = 0 \) we have \( s(0, 0) = 1 \). It is easy to check that \( \text{Li}'_s(z) = z^{-1}\text{Li}_{s-1}(z) \) confirming that \( s(1, 0) = 0 \) and \( s(1, 1) = 1 \). For higher \( k \) formula (56) is then checked by a simple induction, which leads to the recurrence relations

\[
s(k + 1, 0) = -ks(k, 0), \quad s(k + 1, k + 1) = s(k, k), \quad s(k + 1, j) = -ks(k, j) + s(k, j - 1)
\]

which are the recurrence relations satisfied by the Stirling numbers of the first kind, see [9] Sec.26.8.

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