Abstract—In this work, we provide a global framework analysis of a multi-hop relaying systems wherein the transmitter (TX) communicates with the receiver (RX) through a set of intermediary relays deployed either in series or in parallel. Regenerative based relaying scheme is assumed such as the repetition-coded decoded-and-forward (DF) wherein the decoding is threshold-based. To reflect a wide range of fading, we introduce the generalized $H$-function (also termed as Fox-$H$ function) distribution model which enables the modeling of radio-frequency (RF) fading like Weibull and Gamma, as well as the free-space optic (FSO) such as the Double Generalized Gamma and Málaga fading. In this context, we introduce various power and rate adaptation policies based on the channel state information (CSI) availability at TX and RX. Finally, we address the effects of relaying topology, number of relays and fading model, etc, on the performance reliability of each link adaptation policy.

Index Terms—power and rate adaptation, serial and parallel relaying, Fox $H$-function, decode-and-forward.

I. INTRODUCTION

Driven by the need for statistical models that better characterize fading, shadowing and atmospheric turbulences, the past few decades have seen a rise in the interest to develop more generalized statistical model that reflects wide range of distributions. This is often achieved by adding more parameters to already existing models which involves elementary as well as complicated mathematical functions. Mathematically speaking, generalized fading model is equivalent in deriving a general mathematical function that represents all possible elementary and complicated functions. Elliptic integrals, Zeta, Beta, Gamma, ERF, Mathieu & Spherical, and Bessel are known to be complicated and involved in many but limited number of probability distributions. To circumvent this limitation, Hypergeometric family has been proposed as more general functions that can represent the previous mathematical functions, in particular, Meijer G-function [1]. Meijer G-function, although introduced back to 1936 by Cornelis Simon Meijer, has been recently resurrected to address the free space optical (FSO) fading models that are known to be complicated. However, the Fox $H$-function, introduced by Charles Fox in 1961, is a generalization of the Meijer G-function and therefore any statistical distributions can be expressed by Fox $H$-function [2]. In the literature, complicated fading distributions are derived in relaying networks and in particular for mixed radio frequency (RF)/FSO systems. In addition, relaying networks can cover different scenarios for regenerative such as repetition coded decode-and-forward [3], [4] and non-regenerative like amplify-and-forward for fixed and variable relaying gains [5], [6]. In this work, we consider multihop regenerative relaying for serial and parallel topologies wherein all-active and selective relaying are considered as parallel deployment. We further consider the Fox $H$-function fading to represent the common RF and FSO fading models. Capitalizing on this, we evaluate the mutual information for different rate and power adaptation schemes. The remainder of the paper is organized as follows: Section II discusses the different topologies along with the derivations of the cumulative distribution function (CDF). Sections III and IV present the mutual information for different rate and power adaptation schemes relative to the availability of CSIR and CSIT. Section V presents the numerical simulation along with the analysis while the conclusive summaries and future direction are reported in Section VI.

II. SYSTEM MODEL

In this section, we consider the analysis of serial and parallel relaying topologies wherein we derive the channel state information (CSI) statistics as a function of the number of relays $N$.

A. Serial Relaying Topology

![Fig. 1: Serial multihop relaying system consisting of a source $S$ communicating with a destination $D$ through the intermediary of $N$ relays. For $N$ serial relays, $S$ reaches out to $D$ through $N + 1$ successive hops.](https://example.com/serial_relaying_system)

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about the performance of regenerative relaying which outperforms the non-regenerative scheme, however, regenerative relaying requires a threshold to decode the received signal. If the decoding threshold was not satisfied, the DF relay fails to forward the signal resulting in outage.

In the sequel analysis, we assume \( N \) relays deployed in series between \( S \) and \( D \), i.e., there are \( N+1 \) hops between \( S \) and \( D \). We denote the \( n \)-th CSI by \( \gamma_n \), \( n = 1, \ldots, N+1 \). The outage probability is defined as the probability when at least one CSI is less than a target threshold \( \tau \). Mathematically, it is more straightforward to formulate the coverage probability and then express the cumulative distribution function (CDF) or the outage probability (\( P_{\text{out}}(\text{SNR}, \tau) = 1 - P_{\text{coverage}}(\text{SNR}, \tau) \)). Consequently, the CDF is expressed by

\[
F_\gamma(\tau) = 1 - P[\gamma_1 \geq \tau, \gamma_2 \geq \tau, \ldots, \gamma_{N+1} \geq \tau] \tag{1}
\]

Assuming independence between the \( N+1 \) CSIs, the resulting CDF of the end-to-end CSI (\( \gamma \)) is expressed by

\[
F_\gamma(\tau) = 1 - \prod_{n=1}^{N+1} (1 - F_{\tau_n}(\tau)) \tag{2}
\]

where \( F_{\tau_n}(\cdot) \) is the CDF of the \( n \)-th CSI.

**B. Parallel Relaying Topology**

![Diagram of Parallel Relaying Topology]

Fig. 2: Dualhop relaying system consisting of a source \( S \) communicating with a destination \( D \) through the intermediary of \( N \) parallel relays.

Fig. 2 illustrates a wireless communication system wherein \( S \) and \( D \) communicates through the intermediary of \( N \) parallel relays. Unlike serial relaying topology, the signal can reach the destination through different ways which may decrease the outage occurrence in particular when some relays fail to decode the received signal. In the literature, parallel relaying topology has been widely investigated, in particular, relaying protocols to define the signal routing between \( S \) and \( D \). In this work, we consider two relaying protocols as follows.

1) All-Active Relaying: In this scenario, the signal reaches \( S \) through the \( N \) relays, i.e., the \( N \) relays are simultaneously active to forward to the destination. Although this protocol results in the highest CSI at \( D \), it requires not only huge power consumption but also it may arise the problem of synchronization at the receiver.

   Given that the relays are employing the DF scheme, the CSI of the \( n \)-th branch (\( n = 1, \ldots, N \)) is given

\[
\gamma_n = \min(\gamma_{1(n)}, \gamma_{2(n)}). \tag{3}
\]

Consequently, the output CSI at \( D \) is the sum of all the CSIs of the \( N \) links. The CDF of the end-to-end CSI is given by

\[
F_\gamma(\tau) = \mathbb{P} \left[ \sum_{n=1}^{N} \gamma_n \leq \tau \right]. \tag{4}
\]

2) Selective Relaying: In this scenario, the signal reaches the destination through one relay that is selected following a given protocol. In this context, we propose to select the relay/branch with the highest end-to-end CSI as follows

\[
\gamma = \max \left( \min \left( \gamma_{1(1)}, \gamma_{2(1)} \right), \ldots, \min \left( \gamma_{1(N)}, \gamma_{2(N)} \right) \right). \tag{5}
\]

After some mathematical manipulations, the CDF of the end-to-end SNR is given by

\[
F_\gamma(\tau) = \left( 1 - \prod_{n=1}^{N} (1 - F_{\tau_n(1)}(\tau)) \right) \times \left( 1 - \prod_{m=1}^{N} (1 - F_{\tau_{2(m)}(\tau)}) \right). \tag{6}
\]

**C. H-function fading model**

This fading is considered as a generalized model of the common fading channels since it involves the famous Fox’s \( H \)-function. It said that a given random variable \( \gamma \) follows the unified Fox’s \( H \)-function distribution if its PDF is expressed as follows [7] Sec. (4.1)]

\[
f_x(\gamma) = \kappa H^{m,n}_{p,q}(\gamma) \left( \delta \left| \begin{array}{c} (a_j, \ A_j)_{j=1:p} \\ (b_j, \ B_j)_{j=1:q} \end{array} \right|, \gamma > 0 \tag{7}\right)
\]

where the parameters \( \delta > 0 \), and the constant \( \kappa \) are chosen to satisfy \( \int_0^\infty f_x(\gamma)d\gamma = 1 \). The univariate \( H \)-function, \( H^{m,n}_{p,q}(\cdot) \), is defined by [8]

\[
H^{m,n}_{p,q}(x) = \frac{1}{2\pi i} \int_{C} ds \prod_{j=1}^{n} \Gamma(1-a_j-A_j s) \prod_{j=1}^{m} \Gamma(1-b_j - B_j s) x^{-s} ds, \tag{8}\]

\[
\times \prod_{j=n+1}^{p} \Gamma(a_j + A_j s) \prod_{j=m+1}^{q} \Gamma(1-b_j - B_j s) \tag{9}\]

Since Fox’s \( H \)-function is generalized, common fading models for RF and FSO channels can be expressed in terms of this function. Special cases of the \( H \)-function distribution are summarized in Table I.

**Remark.** Due to the limited number of pages for the letter, we provide the closed-form expressions of the coverage probability and the mutual information in the journal extension of this letter.

**III. Channel Capacity Under CSIR**

In this section, we consider the systems with knowledge of CSI at the receiver, i.e., the ergodic capacity and effective capacity.
| Fading Model                                   | Probability Density Function                                                                 |
|-----------------------------------------------|-----------------------------------------------------------------------------------------------|
| Exponential [9] Eq. (2.7)]                    | $\frac{1}{\gamma} H^{1.0}_{0.1} \left( \frac{\gamma}{(0, 1)} \right)$                     |
| Gamma [9] Eq. (2.21)]                        | $\frac{m}{\Gamma(m)} \frac{H^{1.0}_{0.1}}{\gamma^m} \left( \frac{1}{m, 1, 1} \right)$        |
| Weibull [9] Eq. (2.27)]                      | $\frac{\omega}{\gamma} H^{1.0}_{0.1} \left( \frac{\gamma}{1 - 1/\kappa, 1/\kappa} \right)$ |
| Generalized Gamma [10]                       | $\frac{\beta}{\Gamma(m)} H^{1.0}_{0.1} \left( \frac{\gamma}{m - 1/\xi, 1/\xi} \right)$       |
| Weibull-Gamma [11] Eq. (4)]                  | $\frac{\beta \gamma^{-\beta - 1}}{\Gamma(\alpha)} \left( \frac{1}{\xi} \right) \left( \frac{\gamma}{\xi} \right)^{\alpha - \frac{1}{\beta}} \Lambda + \kappa$ |
| Gamma-Gamma [12] Eq. (2)                     | $\frac{\xi^2 \left( \frac{\gamma}{\mu} \right)^{\alpha \gamma}}{\Gamma(\beta) \gamma} \left( \frac{\gamma}{\xi^2 + 1, 1/\gamma} \right)$ |
| Double Generalized Gamma [13] Eq. (32)]      | $\frac{p_{m+1/2} q_{m-1/2} (2\pi)^{1-2/\nu}}{\alpha^2 \Gamma(m_1) \Gamma(m_2) \gamma} \left( \frac{\gamma}{\mu} \right)^{\alpha \gamma}$ |
| Málaga [14] Eq. (51)]                        | $\sum_{a=1}^{A} a \left( \frac{\alpha \beta}{\gamma} \right) \left( \frac{\gamma}{\mu + \alpha + n, 2} \right) \left( \frac{\gamma}{\mu - \alpha - n, 2} \right)$ |

A. Optimal Rate Adaptation

This rate adaptation transmission policy assumes that the transmit power is kept constant which is also called the ergodic capacity which is defined as

$$I(\text{SNR}) = \frac{1}{2} \mathbb{E} [\log_2 (1 + \gamma)]$$ (10)

This expression can be transformed by integration by parts into

$$I(\text{SNR}) = \frac{1}{2 \log(2)} \int_0^{+\infty} \frac{1 - F_\gamma(\gamma)}{1 + \gamma} d\gamma$$ (11)

B. Effective Channel Capacity

Modern radio systems such as UMTS and LTE aim at supporting a set of services such as messaging, voice calls, videos sharing, etc which require a certain predefined quality-of-service (QoS) to be satisfied. Particularly, QoS metrics are commonly the delay, data rate and signal-to-interference-plus-noise ratio (SINR), etc. Authors in [53] introduced the concept of effective capacity as

$$I(\text{SNR}) = - \frac{1}{\delta} \log \left( \int_0^{+\infty} \frac{f_\gamma(\gamma)}{1 + \gamma} \frac{d\gamma}{\gamma} \right)$$ (12)

where $\delta = \phi BT f$, $\phi$, $B$ and $T$ being the QoS exponent, bandwidth and fading block/frame length, respectively. Smaller values of $\phi$ correspond to slow decaying rate and looser QoS constraints while larger values correspond to fast decaying rate with more stringent QoS constraints.

IV. CHANNEL CAPACITY UNDER CSIT AND CSIR

In this section, we consider a relaying systems wherein we assume the full knowledge of CSI at the transmitter and the receiver.

A. Channel Inversion with Fixed Rate

This technique offers an easy implementation to maintain a constant SNR at the receiver with fixed rate modulation and fixed code design or channel inversion with fixed rate (CIFR). Under this policy, the rate is expressed as

$$I(\text{SNR}) = \frac{1}{2 \log(2)} \log \left( 1 + \left( \int_0^{+\infty} f_\gamma(\gamma) d\gamma \right)^{-1} \right)$$ (13)

Besides, the CIFR technique may struggle large rate penalties in severe fading conditions as compared to other techniques. To address this shortcoming, a modified inversion which inverts the channel fading above a predetermined truncated fade $\gamma_0$ (TCIFR scheme) is often used.

B. Truncated Channel Inversion with Fixed Rate

The channel capacity based on the truncated channel inversion with fixed rate (TCIFR) policy is defined as

$$I(\text{SNR}) = \frac{1}{2 \log(2)} \log \left( 1 + \left( \frac{1}{\gamma_0} \int_{\gamma_0}^{+\infty} f_\gamma(\gamma) d\gamma \right)^{-1} \right) \mathbb{P}[\gamma \geq \gamma_0]$$ (14)
C. Optimal Power and Rate Adaptation

The optimal power and rate adaptation (OPRA) method achieves the highest possible rate with CSI by employing a multiplexed multiple codebook design to match the transmission power and rate of the system. The channel capacity for a system employing the OPRA technique is defined by

\[
I(SNR) = \frac{1}{2} \log_2 \left( \frac{\gamma}{\gamma_0} \right) + \int_{\gamma_0}^{\infty} \frac{1}{\gamma} f_\gamma(\gamma) d\gamma \tag{15}
\]

where \(\gamma_0\) is the optimal cut-off SNR below which data transmission is not allowed. Consequently, \(\gamma_0\) must satisfy the following requirement

\[
\int_{\gamma_0}^{\infty} \frac{1}{\gamma} - F_\gamma(\gamma) d\gamma = 1 \tag{16}
\]

Using integration by parts, (15) becomes

\[
I(SNR) = \frac{1}{2} \log(2) + \int_{\gamma_0}^{\infty} \frac{1}{\gamma} d\gamma \tag{17}
\]

V. Numerical Results

In this section, we investigate and verify the validity of the analytical expression derived in the preceding Sections.

In Fig. 3, we investigate the capacity of the system under different adaptive transmission schemes. It can be deduced that the mutual information under OPRA scheme performs better than the rest of the schemes, especially at low SNR, and this is in line with the definition of OPRA capacity as the highest achievable mutual information.

In the case of effective capacity, Fig. 5 suggests that as the product between \(\phi\), \(B\), \(T_f\) (QoS exponent, bandwidth and fading block/frame length) decreases, the effective capacity increases to match the ergodic capacity.

VI. Conclusion

In this paper, we have revisited the performance analysis of multihop regenerative relaying systems under the Generalized Fox \(H\)-function fading presented for different adaptive transmission schemes. In particular, we presented...
the common fading models with the Fox $H$-function and in particular, we evaluated the mutual information for serial, all-active and selective relaying protocols and under for three FSO fading models; Gamma-Gamma, Málaga and Double Generalized Gamma, respectively. As a future direction, we plan to present the closed-form expressions of the coverage and mutual information for regenerative and non-regenerative multihop relaying under different power and rate adaptation schemes.

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