THE Oracle estimator is sub-optimal for global minimum variance portfolio optimisation *

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ABSTRACT
A common misconception is that the Oracle eigenvalue estimator of the covariance matrix yields the best realized portfolio performance. In reality, the Oracle estimator simply modifies the empirical covariance matrix eigenvalues so as to minimize the Frobenius distance between the filtered and the realized covariance matrices. This leads to the best portfolios only when the in-sample eigenvectors coincide with the out-of-sample ones. In all the other cases, the optimal eigenvalue correction can be obtained from the solution of a Quadratic-Programming problem. Solving it shows that the Oracle estimators only yield the best portfolios in the limit of infinite data points per asset and only in stationary systems.

Keywords Covariance cleaning · Portfolio Optimization · Random Matrix Theory

1 Introduction
Covariance filtering is essential in multivariate Finance [1] and in all scientific fields (see [2] for a recent review). A widely applied method is linear shrinkage [3]. More recently, several methods of non-linear shrinkage (NLS) have been introduced [4, 5, 6, 7]. They all belong to the same family of estimators, known as Rotationally Invariant Estimators (RIEs): they filter the covariance matrix eigenvalues while keeping its eigenvectors untouched. A useful benchmark is provided by the Oracle eigenvalues that use information not available in the calibration window where the covariance matrix is estimated, such as future information about the realized covariance, or information about the true covariance matrix. In a stationary setting, the RIE that uses the Oracle eigenvalues can be shown to minimise the Frobenius distance between the filtered covariance matrix and the true one, or, in a dynamical context, the realized covariance matrix. A truly remarkable result shows how to build an approximation of the Oracle eigenvalues with data from the calibration window only that converges to the Oracle estimator if the system is very large, stationary, and has not too heavy-tailed data [4, 5, 6, 7]. We denote this estimator by osRIE (optimal stationary RIE).

Thus, for any application of covariance filtering in large and stationary systems, it is natural to expect that the osRIE is the best RIE possible. This is a commonly shared assumption in portfolio optimisation nowadays [8, 9, 10, 11, 12, 13, 14]. The state-of-the-art combination of dynamic conditional covariance and NLS also uses the osRIE [15]. This results from the implicit belief that Frobenius-optimal eigenvalues are also optimal for portfolio optimization, or equivalently that the Oracle eigenvalues inevitably provide the best (or almost the best) eigenvalues for portfolio optimization; since the osRIE is the best approximation to the Oracle eigenvalues, it is logical to use the osRIE. This point of view only holds for large and stationary systems. We recently showed that a long-term average of the Oracle eigenvalues outperforms the osRIE in strongly nonstationary systems such as US and Hong Kong equity markets [16]. In this work, we numerically compute the RIE that yields the GMV portfolio and show that it outperforms even the Oracle eigenvalues except in stationarized data and in the non-problematic limit of large in-sample and out-of-sample windows at fixed small number of assets.

* Citation: Authors. Title. Pages.... DOI:000000/11111.
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2 Problem Statement

Consider two \( n \) price return time-series of length \( (\delta_{in}, \delta_{out}) \) and let us denote the in-sample (empirical) covariance matrix by \( \Sigma_{in} \) and the out-of-sample (realized) covariance matrix by \( \Sigma_{out} \). According to the spectral theorem, the in-sample covariance matrix can be decomposed into a sum of terms involving its eigenvalues \( \lambda_k \) and their associated eigenvectors components \( \mathbf{v}_i \) as

\[
\Sigma_{in} = \sum_{k=1}^{n} \lambda_k \mathbf{v}_i \mathbf{v}_j.
\]

An RIE estimator uses filtered eigenvalues, which yields the spectral decomposition

\[
\Xi(\lambda^*_k) = \sum_{k=1}^{n} \lambda_k^* \mathbf{v}_i \mathbf{v}_j
\]

where \( \lambda^*_k \) are a set of filtered eigenvalues obtained with some procedure and \( \mathbf{v}_i \) are still the eigenvectors of \( \Sigma_{in} \).

We aim to compare the filtered eigenvalues from the osRIE and from the ones that are optimal for portfolio optimization.

2.1 Oracle Eigenvalues

The Oracle eigenvalues are obtained from the out-of-sample covariance matrix from

\[
\lambda_{Oracle}^k = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{v}_i \Sigma_{out} \mathbf{v}_j
\]

with \( \mathbf{v}_i \) the in-sample eigenvectors.

Such an eigenvalue correction provably produces an estimator that minimizes the Frobenius norm \([4, 7]\)

\[
||\Xi(\lambda_{Oracle}) - \Sigma_{out}||_F = \sum_{i=1}^{n} \sum_{j=1}^{n} [\Xi(\lambda_{Oracle})_{ij} - \Sigma_{out}]^2.
\]

2.2 Optimal RIEs for GMV Portfolios

The simplest portfolio optimization problem (and the most relevant one to assess covariance filtering methods) is Global Minimum Variance (GMV) portfolios. We denote the fraction of capital assigned to each possible asset \( i = 1, \ldots, n \) by \( w \in \mathbb{R}^n \). GMV portfolios aim to minimize the realized portfolio variance \( \Sigma_{out} \) at fixed net leverage \( \sum_i w_i = 1 \).

Mathematically, the problem can be written as the function of the weights as

\[
\min_w \sum_{i=1}^{n} \sum_{j=1}^{n} w_i \Sigma_{out} w_j \quad \text{with} \quad \sum_{k=1}^{N} w_k = 1.
\]

This problem is readily solved if the future is known: the optimal GMV weights are given by

\[
w_{opt} = (\Sigma_{out}^{-1} e) e' (\Sigma_{out}^{-1} e).
\]

where \( e = 1, \ldots, 1 \) is a \( n \)-dimensional vector of ones.

The main contribution of this paper is to show how not optimal the osRIE is for GMV portfolio optimization in a practical context. To this end, we compute the GMV-optimal RIE, which constrains the weights \( w \) to be written as a function of \( \Xi(\lambda^*) \) instead of \( \Sigma_{out} \), the optimization variables being \( \Xi \)’s eigenvalues. The optimal weights are now

\[
w_k = \frac{\sum_{j=1}^{n} \Xi^{-1}_{ij} \mathbf{v}_i \mathbf{v}_j}{\sum_{i=1}^{n} \sum_{j=1}^{n} \Xi^{-1}_{ij}},
\]

where \( \Xi^{-1} \) is the inverted covariance matrix RIE whose spectral decomposition is

\[
\Xi^{-1}_{ij} = \sum_{k=1}^{n} \frac{1}{\lambda_k} \mathbf{v}_i \mathbf{v}_j.
\]
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It is important to point out that the denominator of Eq. (7) is a normalization factor which ensures that the sum of the weights equals one. Equation (7) is thus equivalent to

\[
\begin{align*}
    w_k &= \sum_{j=1}^{n} \Xi_{kj}^{-1} ; \quad k = 1, \ldots, n \\
    \sum_{k=1}^{n} w_k &= 1.
\end{align*}
\]  (9)

Another important point is that the optimal GMV portfolio obtained from (9) and (8) does not depend on the scale of the weights equals one. Equation (7) is thus equivalent to

\[
\begin{align*}
    \zeta_k &:= \frac{1}{\lambda_k^*} \geq 0; \quad \text{for } k = 1, \ldots, n.
\end{align*}
\]  (10)

Finally, the full QP problem is expressed as

\[
\begin{align*}
    \min_{\textbf{w}, \Xi^{-1}, \zeta} & \quad \sum_{i=1}^{n} w_i \Sigma_{ij} \text{w} j \\
    \text{subject to} & \quad w_k = \sum_{j=1}^{n} \Xi_{kj}^{-1} ; \quad k = 1, \ldots, n \\
    & \quad \sum_{k=1}^{n} w_k = 1 \\
    & \quad \Xi_{ij} = \sum_{k=1}^{n} \zeta_k \nu_{ik} \nu_{jk} ; i, j = 1, \ldots, n \\
    & \quad \zeta_k \geq 0; \quad k = 1, \ldots, n \\
    & \quad \zeta_k \geq \zeta_{k-1}; \quad k = 2, \ldots, n.
\end{align*}
\]  (11)

Eq. (11) defines a convex Quadratic Programming problem, which can be solved by numerical methods. In this QP problem formulation the variables \textbf{w} and \Xi^{-1} are slack variables that will be identified by the optimization algorithm. In total, the QP has \(n(n+5)/2\) variables: \(n\) for \textbf{w}, \(n(n+1)/2\) for \Xi^{-1}, as it is the inverted covariance symmetric, and \(n\) for \zeta which are of interest here. The number of constraints are \((n^2 + 5n + 2)/2\). The resulting optimal \zeta_k can be then normalized to have a set of eigenvalues whose sum equals expected volatility.

The procedure described above does not guarantee an ordered sequence of optimal eigenvalues. Let us therefore add \(n - 1\) ordering constraints to Eq. (11), which yields

\[
\begin{align*}
    \min_{\textbf{w}, \Xi^{-1}, \zeta} & \quad \sum_{i=1}^{n} w_i \Sigma_{ij} \text{w} j \\
    \text{subject to} & \quad w_k = \sum_{j=1}^{n} \Xi_{kj}^{-1} ; \quad k = 1, \ldots, n \\
    & \quad \sum_{k=1}^{n} w_k = 1 \\
    & \quad \Xi_{ij} = \sum_{k=1}^{n} \zeta_k \nu_{ik} \nu_{jk} ; i, j = 1, \ldots, n \\
    & \quad \zeta_k \geq 0; \quad k = 1, \ldots, n \\
    & \quad \zeta_k \geq \zeta_{k-1}; \quad k = 2, \ldots, n.
\end{align*}
\]  (12)

These constraints necessarily imply a less optimal solution with respect to the unsorted case. A Python implementation of both QP problems is available at [17].

3 Results: real Global Minimal Variance portfolios

We apply both estimators to a data set of adjusted daily returns of the most capitalized US equities spanning the 1995-2017 period.

The experiments are carried out in the following way. We randomly select two contiguous time intervals \([t - \delta_{in}, t]\) and \([t, t + \delta_{out}]\) from the whole period, remove all the stocks which have more than 20\% of missing values or zero returns, and discard any two stocks with an in-sample correlation larger than 0.95. From the remaining assets, we randomly select \(n = 50\) stocks and we compute \Sigma_{in} and \Sigma_{out} from the two intervals respectively. From the eigenvector basis of \Sigma_{in} and \Sigma_{out} we compute the optimal eigenvalues, the sorted optimal eigenvalue and the Oracle ones. Finally, we compute the out-of-sample (realized volatility) of GMV portfolios for each RIE.

In the upper panels of Fig. 1 we show the average annualized volatility over random 10,000 portfolios at random times. The performance ranking is always the same one: applying Oracle correction is always better than using only the past \(\Sigma_{in}\), and the weights from the QP problem are always better than the Oracle RIE.

One notices a large gap between the Optimal and the Optimal sorted weights; when \(\delta_{out} < n + 1\), this gap comes from the fact that when \(\delta_{out} < n + 1\), the out-of-sample covariance matrix is not positively-defined. This means that there are \(d = n + 1 - \delta_{out}\) null eigenvalues which imply an eigenspace of dimension \(d\) from which every portfolio will have null variance. This is a purely mechanical effect that cannot be exploited from in-sample data only. This gap disappears when the monotonicity of the filtered eigenvalues is imposed. Interestingly, when \(\delta_{out} > n + 1\) the
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We finally note that a RIE does not filter the noise in the eigenvectors which contain useful additional but noisy information with a pseudo-inverse [18]. Averages over 10,000 portfolios of size (1000 days, i.e., about 4 years of data). The phenomenon occurs when the in-sample window size is very large at fixed out-of-sample length. The similarity between the RIE and the realized matrix is not the right cost function.

This paper showed that the remaining margin for improvement is substantial and that minimizing the Frobenius norm is an open question. The point is that any improvement will mechanically improve on the state-of-the art DCC+NLS scheme [15]. The simplest route is to keep improving RIEs for which many exact asymptotic results are known [2].

To confirm this hypothesis, we stationarize the in-sample and out-of-sample time-series. The idea is simply to shuffle the in-sample and out-of-sample days of each subperiod $[t - \delta_{in}, t + \delta_{out}]$, in such a way that both the in-sample and out-of-sample holds a similar proportion of past and future days. In the lower panels of Fig. [1] we show that increasing both $\delta_{in}$ and $\delta_{out}$ on two stationary time-series reduces substantially the bias. This comes from the fact that in these conditions, the in-sample and out-of-sample eigenvector bases tend to be very similar. If either of the two time-window lengths are reduced, the similarity between the two eigenvector bases decreases.

The difference of performance between all the methods remains approximately constant until $\delta_{out}$ reaches very large values (1000 days, i.e., about 4 years of data). The phenomenon occurs when the in-sample window size $\delta_{in}$ increases from 200 to 2000. This effect comes from the non-stationarity of the financial data: very large calibration or test periods produce out-of-date eigenvector bases.

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4 Conclusion

Covariance filtering for finance, while having much progressed in the last decades, still needs improvements. The problem lies in the nonstationary nature of dependence in financial markets, which conflicts with one of the main assumptions of the optimal stationary RIE. Thus, finding the optimal covariance cleaning scheme for equity markets is an open question. The point is that any improvement will mechanically improve on the state-of-the art DCC+NLS scheme [15]. The simplest route is to keep improving RIEs for which many exact asymptotic results are known [2].

This paper showed that the remaining margin for improvement is substantial and that minimizing the Frobenius norm between the RIE and the realized matrix is not the right cost function.

From the application to US equities, one sees that the performance gap between the Oracle estimator and the optimal correction decreases when the out-of-sample time-horizon is very large at fixed $n$. However, in practical applications from portfolio management the typical time-horizon ranges from a month to an year.

We finally note that a RIE does not filter the noise in the eigenvectors which contain useful additional but noisy structures. A way to filter the latter is provided for example by ansätze such as hierarchical clustering [19] or probabilistic hierarchical clustering [20] [21], which outperform the optimal stationary RIE for GMV portfolios when $\delta_{in} < 2n$.
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Acknowledgments

This work was performed using HPC resources from the “Mésocentre” computing center of CentraleSupélec and École Normale Supérieure Paris-Saclay supported by CNRS and Région Île-de-France (http://mesocentre.centralesupelec.fr/)

Funding

This publication stems from a partnership between CentraleSupélec and BNP Paribas.

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