The k-sat problem is a prototypical constraint satisfaction problem. There are many algorithms to study k-sat problem, BP algorithm is famous one of them. But BP algorithm does not converge when \( \alpha \) (constraint density) is bigger than some threshold value. In this paper we use CCCP (Concave Convex Procedure) algorithm to study 3-sat problem and we get better results than BP algorithm that CCCP algorithm still converges when BP algorithm does not converge. Our work almost builds on recent results by Yuille [1] who apply the CCCP algorithm to Bethe and Kikuchi free energies and obtained two algorithms on 2D and 3D spin glasses. Our implementation of CCCP algorithm on 3-sat problem is some different from his implementation and we have some different views about CCCP algorithm’s some properties. Some difference of these maybe because of CCCP algorithm have different properties and implementation process on different problem and some others of these are related to the CCCP algorithm itself. Our work indicates that CCCP algorithm has more learning and inference applications.

Key words: Bethe free energy, 3-sat problem, BP equation, CCCP algorithm
Kikuchi.

Yedidia et al’s result[7] proved that the fixed points of BP correspond to the extrema points of the Bethe free energy functions. They also developed a generalized belief propagation (GBP) algorithm whose fixed points correspond to the extrema points of the Kikuchi free energy. In practice, when BP and GBP converge they go to low energy minima of the Bethe/Kikuchi free energies. In general, GBP gives more accurate results than BP since Kikuchi is a better approximation than Bethe. For example, empirically, GBP converged more close to the true solution (Yedidia et al 2000) than BP algorithm on 2D spin glasses. We can get the SGBP[14, 15] algorithm by simplifying the GBP algorithm, and which convergence very quickly does than GBP.

The BP algorithm does not always converge, it is not convergent when constraint density bigger than some threshold value for the k-sat problem. So we search for other algorithms to minimize the Bethe/Kikuchi free energies, CCCP algorithm is one of them.

The research on the belief propagation algorithm still continues until to now[16–19], and we can get more precise results and more information about the system that we considered using by RSB theory. There are a lot of work[20–25] done in this field.

It’s main idea of the algorithms of developed using a Concave Convex Procedure (CCCP) is decomposing the free energy into concave and convex parts, and then construct discrete iterative rules which guaranteed to decrease the free energy monotonically. This procedure builds on results developed when studying mean field theory[20–28].

Yedidia (Yedidia et al 2000) using concave convex procedure (CCCP) principle to get extrema point of Bethe and Kikuchi free energies, this algorithm guaranteed to converge to extrema point of the Bethe and Kikuchi free energy. Although this algorithm some like the BP/GBP algorithm which estimate “beliefs” by propagating “messages”, but CCCP algorithm propagate messages (lagrange multipliers) by current estimates of beliefs which must be re-estimated periodically.

This algorithm (CCCP algorithm) starts by decomposing the free energy into concave and convex parts. From this decomposition we can get discrete update rules which decrease the free energy at each iteration step (but the free energy and entropy values are not very meaningful in first several steps when constraint density larger than one it is because of we have lagrange terms to add to convex free energy parts which effects the free energies property, except all the constraint and normalization conditions be satisfied). But the free energy is meaningful at each iteration steps if we initialize the messages and lagrange multipliers appropriately.

Yuille tested the algorithm on the 2D and 3D spin glasses in regular graph. In his way this algorithm guaranteed to monotonically decrease the free energy and the difference of beliefs. They randomly initialize the vertical and horizontal potentials from gaussian distribution, and randomly initialize the lagrange multipliers.

We tested CCCP principle on the 3-sat problem in random regular graph. In this case in order to converge we must add another lagrange multiplier (marginal normalization multiplier) to the lagrange parts (A.L.Yuille didn’t). This algorithm converges very rapidly if we implement it parallel update way which gives incorrect results, and it doesn’t converge very quickly if we implement it in unparallel update way which gives correct results. In our case this algorithm guaranteed to minimize the free energy and difference of beliefs monotonically at each iteration steps. Our results are very similar to the BP algorithm when BP algorithm can converged. This algorithm still converge when BP algorithm can’t converge, but this gives strange (not meaningful) results when ω is(constraint density) bigger than some threshold value (this value depend on the converge precision and the size of system).

The structure of this paper is as follows. Section (II) describes 3-sat problem and Bethe free energy, in section (III) we introduce the Bethe equation. Section (IV) we introduce CCCP principles and we apply CCCP algorithm to 3-sat problem to determine the final form of this double loop equation. Section (V) implementation CCCP algorithm and compare this algorithm with BP algorithm, in section (VI) we will discuss some properties of CCCP algorithm, in the last section (VII) we give our conclusion.

II. BETHE FREE ENERGY AND 3-SAT PROBLEM

II.1. Bethe free energy

The Bethe free energy (Domb and Green 1972, Yedidia et al 2000) is a variational technique from statistical physics. It’s main idea is to replace an hard to calculating problem that we can’t calculate by an approximation which is solvable. This approximation is using joint distributions between variables which interacting each other. It can give good results when standard mean field theory only gives poor results. Consider a graph with nodes i=1,...,N. different problem will determine different connections node between the nodes. If we just consider about two body interaction situations, then We will only list connections between ij for node pairs i, j which are connected. The state of node is denoted by xi. So we can write the joint probability distribution function as

\[ P(x_1, \ldots, x_N) = \frac{1}{Z} \prod_{i,j:i>j} \psi_{ij}(x_i, x_j) \prod_i \psi_i(x_i) \] (1)

where \( \psi_i(x_i) = e^{-\beta E_i(x_i)} \) is the factor of the node i, Z is a normalization constant, and \( \psi_{ij}(x_i, x_j) = e^{-\beta E_{ij}(x_i, x_j)} \) is the factor of the interaction between nodes i and j. We use the convention i > j to avoid double counting (Attention: if node i and node j not connected, then we do not have the term \( \psi_{ij} \)). And then we can write the Bethe free energy as(Yedidia et al 2000)
\[ F_\beta(\{b_{ij}, b_i\}) = \sum_{i,j>i} \sum_{x_i, x_j} b_{ij}(x_i, x_j) \log \frac{b_{ij}(x_i, x_j)}{\phi_{ij}(x_i, x_j)} \]
\[ - \sum_i (n_i - 1) \sum_{x_i} b_i(x_i) \log \frac{b_i(x_i)}{\psi_i(x_i)} \]  

(2)

where \(\phi_{ij}(x_i, x_j) = \psi_i(x_i) \psi_j(x_j) \psi_{ij}(x_i, x_j)\), \(n_i\) is connectivity of the nodes \(i\) or variable degree of node \(i\) in other words, the marginal probability \(\{b_i(x_i)\}\), and joint probability \(\{b_{ij}(x_i, x_j)\}\) must satisfy the linear consistency constraints.

\[ \sum_{x_i, x_j} b_{ij}(x_i, x_j) = 1, \forall i, j : i > j \sum_{x_i} b_i(x_i) = 1, \forall i, \]  

(3)

\[ \sum_{x_i} b_{ij}(x_i, x_j) = b_j(x_j), \forall j, x_j, \sum_{x_j} b_{ij}(x_i, x_j) = b(x_i), \forall i, x_i. \]  

(4)

So we can write the lagrange parts of the free energy as

\[ \sum_{ij:i>j} \gamma_{ij} \left\{ \sum_{x_i, x_j} b_{ij}(x_i, x_j) - 1 \right\} \]
\[ + \sum_{ij:i>j} \sum_{x_i} \lambda_{ij \rightarrow j}(x_j) \left\{ \sum_{x_i} b_{ij}(x_i, x_j) - b_j(x_j) \right\} \]  

(5)

\[ + \sum_{ij:i>j} \sum_{x_i} \lambda_{ij \rightarrow i}(x_i) \left\{ \sum_{x_j} b_{ij}(x_i, x_j) - b_i(x_i) \right\} \]

If we consider many body interaction system, in the same way we can write the joint probability distribution function as

\[ P(x_1, ..., x_N) = \frac{1}{Z} \prod_\alpha \psi_\alpha(x_\alpha) \prod_i \psi_i(x_i) \]  

(6)

where The state of node \(i\) is denoted by \(x_i\), and the state of interaction \(\alpha\) is denoted by \(x_\alpha\). for example if we consider about the \(n\) body interaction system, the interaction has \(k^n\) (\(k\) is the possible state number of one variable node) possible states. \(\psi_i(x_i) = e^{-\beta E_i(x_i)}\) is the factor of the node \(i\), \(Z\) is a normalization constant, and \(\psi_\alpha(x_\alpha) = e^{-\beta E_\alpha(x_\alpha)}\) is the factor of the interaction \(\alpha\). And then we can write the Bethe free energy as(Yedidia et al 2000)

\[ F_\beta(\{b_\alpha, b_i\}) = \sum_\alpha \sum_{x_\alpha} b_\alpha(x_\alpha) \log \frac{b_\alpha(x_\alpha)}{\phi_\alpha(x_\alpha)} \]
\[ - \sum_i (n_i - 1) \sum_{x_i} b_i(x_i) \log \frac{b_i(x_i)}{\psi_i(x_i)} \]  

(7)

where \(\phi_\alpha(x_\alpha) = \psi_\alpha(x_\alpha) \prod_{\gamma \in \alpha} \psi_\gamma(x_\gamma)\), \(n_i\) is the variable degree of node \(i\), the marginal probability \(\{b_i(x_i)\}\), and joint probability \(\{b_\alpha(x_\alpha)\}\) must satisfy the linear consistency constraints.

\[ \sum_{x_\alpha} b_\alpha(x_\alpha) = 1, \forall \alpha, \sum_{x_i} b_i(x_i) = 1, \forall i, \]  

(8)

\[ \sum_{x_\alpha \in x_i} b_\alpha(x_\alpha) = b_i(x_i), \forall i \in \alpha \]  

(9)

So we can write the lagrange parts of the free energy as

\[ \sum_\alpha \gamma_\alpha \left\{ \sum_{x_\alpha} b_\alpha(x_\alpha) - 1 \right\} \]
\[ + \sum_\alpha \sum_{x_i} \sum_{x_\alpha \notin x_i} \lambda_{\alpha \rightarrow i}(x_i) \left\{ \sum_{x_\alpha \in x_i} b_\alpha(x_\alpha) - b_i(x_i) \right\} \]

(10)

### II. 3-sat problem

A 3-sat formula contain \(N\) variable nodes and \(M\) constraint nodes. Every variable nodes only has two states \(\{+1, -1\}\), every constraint nodes connect with 3 variable nodes and every connection represent one requirements (request the variable must be +1, or -1)to the connected variables, the constraint be satisfied when at least one of these three requirements be satisfied. So we can write the energy function of the 3-sat problem as

\[ E = \sum_{\alpha=1}^{M} \prod_{i \in \alpha} \left( 1 - \frac{J_i x_i}{2} \right) \]  

(11)

where the \(J_i\) is the requirement of the constraint node \(\alpha\) to the variable node \(i\), if at least one variable of three variables which connected by one constraint node satisfy the requirement, then the energy of the constraint node equal to zero. Otherwise the energy of the constraint node equal to one.

So we can write the distribution of one constraint nodes as

\[ \psi_{ijk}(x_i, x_j, x_k) = e^{-\beta E_\alpha} = e^{-\beta \prod_{i \in \alpha} (\frac{1-J_i x_i}{2})} \]
\[ = e^{-\beta (\frac{1-J_i x_i}{2}) (\frac{1-J_j x_j}{2}) (\frac{1-J_k x_k}{2})} \]  

(12)

Because of there is no external field so

\[ E_i = 0, \psi_i(x_i) = 1 \]  

(13)

\[ \phi_{ijk} = \psi_i \psi_j \psi_k \psi_{ijk} = \psi_{ijk} \]  

(14)

According to the equation (7) we can write the Bethe free energy for 3-sat problem as
\[ F_0(b_{ijk}, b_i) = \sum_{x_i, x_j, x_k} b_{ijk}(x_i, x_j, x_k) \log \frac{b_{ijk}(x_i, x_j, x_k)}{\phi_{ijk}(x_i, x_j, x_k)} \]

\[- \sum_{i} (n_i - 1) \sum_{x_i} b_i(x_i) \log \frac{b_i(x_i)}{\phi_i(x_i)} \]  

(15)

And according to the equation (10), the lagrange multipliers part can be written as

\[ \sum_{ijk} \gamma_{ijk} \{ \sum_{x_i, x_j, x_k} b_{ijk}(x_i, x_j, x_k) - 1 \} \]

\[ + \sum_{i,j,k} \lambda_{ijk \to k}(x_k) \{ \sum_{x_i, x_j} b_{ijk}(x_i, x_j, x_k) - b_k(x_k) \} \]

\[ + \sum_{i,j,k} \lambda_{ijk \to j}(x_j) \{ \sum_{x_i, x_k} b_{ijk}(x_i, x_j, x_k) - b_j(x_j) \} \]

\[ + \sum_{i,j,k} \lambda_{ijk \to i}(x_i) \{ \sum_{x_j, x_k} b_{ijk}(x_i, x_j, x_k) - b_i(x_i) \} \]

(16)

\[ \text{III. BP EQUATION} \]

Directly compute the partition function is a hard task in large system, but this complexity can be reduced when the underlying factor graph has some special structure. In factor graph N variables can be expressed by N variable nodes (circle empty nodes), M interactions can be expressed by function nodes (square filled nodes). So we can write the partition function as

\[ Z = \prod_{\alpha} \left( \prod_{a=1}^{\alpha-1} \psi_i(\sigma_i) \right) \prod_{(j,a) \in G} \delta(\sigma_i^a, \sigma_j) \]  

(17)

where the \( \sigma \) represents all the configurations of the N variables system, \( \sigma_\alpha \) represents all the configurations of the interaction \( \alpha \). \( \psi_i(x_i) = e^{-\beta E_i(\sigma_i)} \) represents the external field factor of the variable node \( i \), \( \psi_a(\sigma_\alpha) = e^{-\beta E_a(\sigma_\alpha)} \) represents the interaction field factor of the clause node \( a \).

Now we introduce the edge Auxiliary probability function to the partition function, so

\[ Z = \prod_{\alpha} \left( \prod_{a=1}^{\alpha-1} \psi_i(\sigma_i) \right) \prod_{(j,a) \in G} \delta(\sigma_i^a, \sigma_j) \]

\[ \times \prod_{(k,b) \in G} \frac{\delta(\sigma_i^b, \sigma_k)}{q_{k \to a}(\sigma_k)} \]  

(18)

in there, the edge Auxiliary probability function satisfy the normalization condition.

And we introduce the another normalized edge Auxiliary probability function to the partition function, so

\[ Z = \prod_{\alpha} \left( \prod_{a=1}^{\alpha-1} \psi_i(\sigma_i) \right) \prod_{(j,a) \in G} \delta(\sigma_i^a, \sigma_j) \]

\[ \times \prod_{(k,b) \in G} \frac{\delta(\sigma_i^b, \sigma_k)}{q_{k \to a}(\sigma_k)} \]  

\[ \prod_{(k,b) \in G} \frac{\delta(\sigma_i^b, \sigma_k)}{q_{k \to a}(\sigma_k)} \]  

(19)

Eventually we can write the partition function as

\[ Z = Z_0(1 + \sum_{g \in G} L_g) \]  

(20)

in there, the edge Auxiliary probability function satisfy the following iterative equations, and then these distributions of the loopy factor graphs (Marc Mezard, Andrea Montanari. 2009) are equal to zero, then we can write the partition function as

\[ Z = Z_0 \]  

(25)

In this case we can easily to calculate all the thermodynamical functions.

If and only if the edge Auxiliary probability function satisfy the following iterative equations, and then these distributions of the loopy factor graphs are equal to zero.

\[ q_{i \to a}(\sigma) = \frac{1}{z_{i \to a}} \psi_i(\sigma) \prod_{b \in \partial i \setminus a} p_{b \to i}(\sigma) \]  

(26)

\[ p_{a \to i}(\sigma) = \frac{1}{z_{a \to i}} \psi_a(\sigma) \prod_{j \in \partial a \setminus i} q_{j \to a}(\sigma_j) \]  

(27)

These equations called as Belief Propagation equations(abbrivation BP equations).

In this equations

\[ z_{i \to a} = \sum_{\sigma} \psi_i(\sigma) \prod_{b \in \partial i \setminus a} p_{b \to i}(\sigma) \]  

(28)
\[ z_{a \rightarrow i} = \sum_{\sigma_{\partial a}} \psi_{q}(\sigma_{\partial a}) \prod_{j \in \partial a \setminus i} q_{j \rightarrow a}(\sigma_{j}) \]  

(29)

So if the factor graph has tree like structure (or no contain loopy structure), we can calculate the partition function with very simplest form by iterating the BP equations and getting the stable points of Auxiliary probability functions. But in the 3-sat problem the BP equations are not convergent when constraint density greater than 3.86 in big system.

IV. CONCAVE CONVEX PROCEDURE (CCCP) ALGORITHM

The main idea of the CCCP algorithm is that first step decomposing the free energy into two parts respectively convex part and concave part. In the second step we add the constraint condition to the convex part. Find the minimum point of the bethe free energy by dynamical programming procedure. The algorithm iterates by matching points on the two curves (convex and concave) that have the same tangent vectors.

Below we simplest way to recall the process of the derivation of CCCP algorithm:

IV.1. the minimum point of the Bethe free energy without constraints

Theorem 1. Consider an energy function \( E(\vec{z}) \) (bounded below) of form \( E(\vec{z}) = E_{\text{vex}}(\vec{z}) + E_{\text{cave}}(\vec{z}) \) where \( E_{\text{vex}}(\vec{z}) \) and \( E_{\text{cave}}(\vec{z}) \) are convex and concave functions of \( \vec{z} \) respectively. Then the discrete iterative algorithm \( \vec{z}^t \rightarrow \vec{z}^{t+1} \) given by:

\[ \nabla E_{\text{vex}}(\vec{z}^{t+1}) = -\nabla E_{\text{cave}}(\vec{z}^t), \]

we can prove that this equation guaranteed to minimize the free energy.

Proof: corresponding to the property of concavity and convexity functions, we can write following inequation:

\[ E_{\text{vex}}(\vec{z}_2) \geq E_{\text{vex}}(\vec{z}_1) + (\vec{z}_2 - \vec{z}_1) \cdot \nabla E_{\text{vex}}(\vec{z}_1) \]  

(30)

\[ E_{\text{cave}}(\vec{z}_4) \leq E_{\text{cave}}(\vec{z}_3) + (\vec{z}_4 - \vec{z}_3) \cdot \nabla E_{\text{cave}}(\vec{z}_3), \]  

(31)

now we set \( \vec{z}_1 = \vec{z}^{t+1}, \vec{z}_2 = \vec{z}^t, \vec{z}_3 = \vec{z}^t, \vec{z}_4 = \vec{z}^{t+1} \) using upper three equation we obtain that:

\[ E_{\text{vex}}(\vec{z}^{t+1}) + E_{\text{cave}}(\vec{z}^{t+1}) \leq E_{\text{vex}}(\vec{z}^t) + E_{\text{cave}}(\vec{z}^t), \]  

(32)

which prove the claim.

IV.2. the minimum point of the Bethe free energy with constraints

Theorem 2. Consider a function \( E(\vec{z}) = E_{\text{vex}}(\vec{z}) + E_{\text{cave}}(\vec{z}) \) subject to k linear constraints \( \vec{\phi}^\mu \cdot \vec{z} = c^\mu \) where \( \{c^\mu : \mu = 1,...,k\} \) are constants. Then the algorithm \( \vec{z}^t \rightarrow \vec{z}^{t+1} \) given by:

\[ \nabla E_{\text{vex}}(\vec{z}^{t+1}) = -\nabla E_{\text{cave}}(\vec{z}^t) - \sum_{\mu=1}^{k} \alpha^\mu \vec{\phi}^\mu, \]  

(33)

where the parameters \( \{\alpha^\mu\} \) are chosen to ensure that \( \vec{z}^{t+1} = \vec{c}^\mu \) for \( \mu = 1,...,k \).

Proof: first we define orthogonal unit vectors \( \{\vec{\psi}^\nu : \nu = 1,...,n-k\} \), which span the space orthogonal to the constraints \( \{\vec{\phi}^\mu : \mu = 1,...,k\} \). Let \( \vec{y}(\vec{z}) = \sum_{\nu=1}^{n-k} \vec{\psi}^\nu(\vec{z},\vec{\psi}^\nu) \) (project the z to different vectors or on the other hand express the z in \{\vec{\psi}^\nu\} space). Define function \( E_{\text{vex}}(\vec{y}), E_{\text{cave}}(\vec{y}) \), which ensure that \( E_{\text{vex}}(\vec{y}(\vec{z})) = E_{\text{vex}}(\vec{z}), E_{\text{cave}}(\vec{y}(\vec{z})) = E_{\text{cave}}(\vec{z}) \).

So we can write:

\[ \vec{\psi}^\nu \cdot \nabla_{\vec{z}} E_{\text{vex}}(\vec{z}^{t+1}) = \vec{\psi}^\nu \cdot \nabla_{\vec{z}} E_{\text{cave}}(\vec{z}^t), \]  

(34)

where \( \nu = 1,...,n-k, (\vec{\psi}^\nu \cdot \vec{\phi}^\nu = 0). \)

It follows from theorem2 that:

\[ E_{\text{vex}}(\vec{z}^{t+1}) = E_{\text{vex}}(\vec{z}^{t+1}) + \sum_{\mu} \alpha^\mu \{\vec{\phi}^\mu \cdot \vec{z}^{t+1} - c^\mu\}, \]  

(35)

So we need to impose the constraints only on convex terms.
IV.3. the solution of the cccp algorithm

Theorem 3. let $E_{\text{ex}}(z) = \sum_i z_i \log \frac{z_i}{\xi_i}$. Then the update equation of theorem 2 can be expressed as minimizing the convex energy function:

$$E^{t+1}(z^{t+1}) = z^{t+1, h} + \sum_i z_i^{t+1} \log \frac{z_i^{t+1}}{\xi_i} + \sum_{\mu} \alpha^\mu \{ \phi^\mu, z^{t+1} - e^\mu \}$$

(36)

Where $h = \nabla \tilde{E}_{\text{cave}}(z)$, and $z^{t+1, h} = \sum_i z_i^{t+1} \hat{h}_i$. The solution of the form is

$$z_i^{t+1}(\alpha) = \xi_i e^{-h_i} e^{-1 - \sum_{\mu} \alpha^\mu \phi^\mu}$$

(37)

Proof: insert the convex free energy function to the theorem 2 equation, then we can get

$$\log \frac{z^{t+1}_i}{\xi_i} + 1 + \sum_{\mu} \alpha^\mu \phi^\mu = -\hat{h}_i$$

(38)

Where $h_i = \nabla z_i E_{\text{cave}}(z)$

$$z_i^{t+1}(\alpha) = \xi_i e^{-h_i} e^{-1 - \sum_{\mu} \alpha^\mu \phi^\mu}$$

(39)

where the Lagrange multipliers $\{\alpha^\mu\}$ are constrained to maximize the dual energy

$$\hat{E}^{t+1}(\alpha) = -\sum_i z_i^{t+1}(\alpha) - \sum_{\mu} \alpha^\mu e^\mu$$

(40)

Moreover, maximizing $\hat{E}^{t+1}(\alpha)$ with respect to specific $\alpha^\mu$ enables us to satisfy the corresponding constraint exactly.

IV.4. convex and concave procedure (CCCP) algorithm for the 3-sat problem

Theorem 3 specifies a double loop algorithm where the outer loop is given by the solution equation and the inner loop equation is given by the normalization condition and constraint condition, and the inner loop determines the $\{\alpha^\mu\}$ by maximizing the dual energy.

$$F_{\text{ex}} = \sum_{ijk} \sum_{x_i, x_j, x_k} b_{ijk}(x_i, x_j, x_k) \log \frac{b_{ijk}(x_i, x_j, x_k)}{\psi_{ijk}(x_i, x_j, x_k)} + \sum_i \sum_{x_i} b_i(x_i) \log \frac{b_i(x_i)}{\psi_i(x_i)}$$

(41)

$$F_{\text{cave}} = -\sum_i n_i \sum_{x_i} b_i(x_i) \log \frac{b_i(x_i)}{\psi_i(x_i)}$$

(42)

The outer loop equations

$$b_{ijk}(x_i, x_j, x_k; t + 1) = \psi_{ijk}(x_i, x_j, x_k) e^{-1 - \lambda_{ijk \rightarrow i}(x_i)} e^{-\lambda_{ijk \rightarrow j}(x_j)} e^{-\lambda_{ijk \rightarrow k}(x_k)} e^{-\gamma_{ijk}}$$

(43)

$$b_i(x_i; t + 1) = \psi_i(x_i) e^{-1 - n_i [b_i(x_i; t)] n_i} e^{\sum_j \sum_k \lambda_{ijkl \rightarrow i}(x_i)}$$

(44)

The inner loop equations

$$e^{\gamma_{ijk}(t+1)} = \sum_{x_i, x_j, x_k} \psi_{ijk}(x_i, x_j, x_k) e^{-\lambda_{ijk \rightarrow i}(x_i; t)} e^{-\lambda_{ijk \rightarrow j}(x_j; t)} e^{-\lambda_{ijk \rightarrow k}(x_k; t)}$$

(45)

$$e^{2\lambda_{ijk \rightarrow k}(x_k; t+1)} = \sum_{x_i, x_j, x_k} \psi_{ijk}(x_i, x_j, x_k) e^{-\lambda_{ijk \rightarrow i}(x_i; t)} e^{-\lambda_{ijk \rightarrow j}(x_j; t)} e^{-\gamma_{ijk}(t)}$$

(46)

$$e^{2\lambda_{ijk \rightarrow j}(x_j; t+1)} = \sum_{x_i, x_k} \psi_{ijk}(x_i, x_j, x_k) e^{-\lambda_{ijk \rightarrow i}(x_i; t)} e^{-\lambda_{ijk \rightarrow k}(x_k; t)} e^{-\gamma_{ijk}(t)}$$

(47)
\[ e^{2\lambda_{ijk}\rightarrow a} (x_i; t+1) = \frac{\sum_{x_j, x_k} \psi_{ijk}(x_i, x_j, x_k) e^{-\lambda_{ijk}\rightarrow a}(x_i; t) e^{-\lambda_{ijk}\rightarrow k}(x_k; t) e^{-\gamma_{ijk}(t)}}{e^{n_i(b_i(x_i; t))}} \]

BP style for outer loop

\[ b_{ijk}(x_i, x_j, x_k; t + 1) = \psi_{ijk}(x_i, x_j, x_k) e^{-1} e^{-\lambda_{ijk}\rightarrow a}(x_j) e^{-\lambda_{ijk}\rightarrow a}(x_i) e^{-\lambda_{ijk}\rightarrow h}(x_k) e^{-\gamma_{ijk}} \]

\[ b_i(x_i; t + 1) = \psi_i(x_i) e^{-1} e^{n_i(b(x_i; t))} \sum_{a \in \partial i} \lambda_{n\rightarrow a}(x_i) = \psi_i(x_i) e^{-1} e^{n_i(b(x_i; t))} \sum_{a \in \partial i} \lambda_{n\rightarrow a}(x_i), \]

where \( \lambda_{ij\rightarrow a}(x_j) = \lambda_{ij\rightarrow k}(x_j) = \lambda_{ij\rightarrow j}(x_j) \). We can see the CCCP algorithm updating data on unidirectional graph, but the BP algorithm do on bidirectional graph.

IV.5. Discussion and the final form of the double loop equation

Because of these equations can not guaranteed to satisfy the constraint condition in simulation, so the lagrange multipliers always converge to NAN, it leads to the marginal beliefs converge to NAN. In order to avoid this we must normalize the marginal beliefs by adding another lagrange multipliers.

\[ \sum_i \alpha_i \left( \sum_{x_i} b_i(x_i) - 1 \right) \]

Our outer loop equation

\[ b_{ijk}(x_i, x_j, x_k; t + 1) = \psi_{ijk}(x_i, x_j, x_k) e^{-1} e^{-\lambda_{ijk}\rightarrow a}(x_i) \times e^{-\lambda_{ijk}\rightarrow j}(x_j) e^{-\lambda_{ijk}\rightarrow h}(x_k) e^{-\gamma_{ijk}} \]

Our inner loop equation

\[ e^{\gamma_{ijk}(t+1)} = \sum_{x_i, x_j, x_k} \psi_{ijk}(x_i, x_j, x_k) e^{-1} e^{-\lambda_{ijk}\rightarrow a}(x_i; t) e^{-\lambda_{ijk}\rightarrow j}(x_j; t) e^{-\lambda_{ijk}\rightarrow h}(x_k; t) \]

\[ e^{2\lambda_{ijk}\rightarrow k}(x_k; t + 1) = \frac{\sum_{x_j, x_k} \psi_{ijk}(x_i, x_j, x_k) e^{-\lambda_{ijk}\rightarrow a}(x_i; t) e^{-\lambda_{ijk}\rightarrow j}(x_j; t) e^{-\gamma_{ijk}(t)}}{e^{n_i(b_k(x_k; t))}} \]

\[ e^{2\lambda_{ijk}\rightarrow a}(x_i; t + 1) = \frac{\sum_{x_j, x_k} \psi_{ijk}(x_i, x_j, x_k) e^{-\lambda_{ijk}\rightarrow j}(x_j; t) e^{-\lambda_{ijk}\rightarrow h}(x_k; t) e^{-\gamma_{ijk}(t)}}{e^{n_i(b_j(x_j; t))}} \]

In this case we update \( \lambda_{ijk} \) lagrange multipliers by using constraint condition and using normalization condition for the \( \gamma_{ijk}, \alpha_i \).

Attention: we can update the marginal normalization lagrange multipliers by the constraint condition \( b_i(x_i) = \sum_{x_j, x_k} b_{ijk}(x_i, x_j, x_k) \) when these beliefs guaranteed to satisfy the constraint condition. In this case we can update the marginal normalization lagrange multipliers by using the formula that indicated below

\[ e^{\alpha_i} = e^{n_i-1} \sum_{x_i} [b_i(x_i; t)] e^{\sum_{\alpha \neq i} \lambda_{\alpha pq\rightarrow i}(x_i)} \]

Unfortunately the CCCP algorithm can’t guaranteed to satisfy the constraint condition, so we can’t use this equation for updating.
IV.6. the form of the CCCP equations when zero temp

When temperature equal to zero (or inverse temperature is infinite), the CCCP dynamics quickly gets trapped to a local minimal region of the Free energy landscape.

The outer loop equation

\[ b_{ijk}(x_i, x_j, x_k; t + 1) = \begin{cases} 0 & \text{if } E_{\alpha}(x_i, x_j, x_k) = 1, \\ \psi_{ijk}(x_i, x_j, x_k)e^{-1} \prod_{\alpha \in \partial x} e^{-\lambda_{\alpha} \cdot \alpha(x_i)} e^{-\gamma_{\alpha}} & \text{otherwise} \end{cases} \]

The inner loop equation

\[ e^{\gamma_{ijk}(t + 1)} = \sum_{x_i, x_j, x_k, E_{\alpha} \neq 1} \psi_{ijk}(x_i, x_j, x_k)e^{-1} - \lambda_{ijk - k(x_k; t)} e^{-\gamma_{ijk}(t)} \]

\[ e^{2\lambda_{ijk - k(x_k; t) + 1}} = \sum_{x_i, x_j, E_{\alpha} \neq 1} \psi_{ijk}(x_i, x_j, x_k)e^{-\lambda_{ijk - k(x_k; t)}} e^{-\gamma_{ijk}(t)} \]

\[ e^{2\lambda_{ijk - j(x_j; t) + 1}} = \sum_{x_k, x_i, E_{\alpha} \neq 1} \psi_{ijk}(x_i, x_j, x_k)e^{-\lambda_{ijk - k(x_k; t)}} e^{-\gamma_{ijk}(t)} \]

\[ e^{2\lambda_{ijk - i(x_i; t) + 1}} = \sum_{x_k, x_i, E_{\alpha} \neq 1} \psi_{ijk}(x_i, x_j, x_k)e^{-\lambda_{ijk - k(x_k; t)}} e^{-\gamma_{ijk}(t)} \]

\[ e^{n_i} = e^{n_i - 1} \sum_{x_i} [b_i(x_i)] e^{\sum_{\alpha \in \partial x_i} \lambda_{\alpha} \cdot \alpha(x_i)} \]

V. IMPLEMENTATION

We implement the 3-sat CCCP algorithm in different constraint density (a respectively equal to 0.5, 1.0, 2.0, 3.0, 4.0, 4.267), we compared CCCP algorithm with BP algorithm in the circumstance that BP algorithm converge, and give CCCP algorithm results alone in another case. We selected the 10000 variables system and we use binary state variables, each variable only have two states \( x_i \in \{-1, +1\} \).

Every clause connected with three variables randomly and the each connection have two states \( J^i_{jk} \in \{-1, +1\} \), the total clauses number changes with constraint density \( M = \alpha N \), where \( M \) is clauses number, \( N \) is variables number, \( \alpha \) is constraint density. We randomly choose one map from clause configurations (the total clause configurations number \( \sigma(J) = 2^{3M} \), where \( M \) is total clause number ) after the constraint density be fixed, in our case there is no external field potentials.

We derived the Bethe free energy for the 3-sat problem and implemented the CCCP and BP algorithm. We randomly initialized the lagrange multipliers, we use unary marginal beliefs \( b_i(x_i) \) and unary joint beliefs \( b_{ijk}(x_i, x_j, x_k) \). We used formula (58) for updating the marginal normalization lagrange multiplier \( \alpha_i \).

In the CCCP algorithm we used half parallel updating (parallel updating in one clause but unparallel updating between clauses) rule for the inner loop, it gives exact and stable results than parallel updating. Although parallel updating converge very quickly, but it gives poor results (marginal beliefs converge to 1 or 0, all the joint beliefs whose energy equal to zero converge to identical , and the last one joint belief whose energy equal to one converge to very small than other beliefs).

In the CCCP algorithm we used one iteration for the inner loop and one iteration for the outer loop in one step updating. This guaranteed to converge very quickly than more inner loop iterations and still gives good result. If we use more than one iteration for the inner loop it leads the algorithm converges very slowly, even it does not converge.

The CCCP algorithm guaranteed to satisfy the normalization condition of marginal beliefs and joint beliefs, but it can’t guaranteed to satisfy the constraint condition, this is why we add new lagrange multiplier \( \alpha_i \) to guaranteed to convergence of the CCCP algorithm.
In 2D and 3D spin glass (A.L. Yuille) the free energy and divergence that getting by CCCP algorithm is vibrating in the first some iterations (very small part of all iterations) and then monotonically decreasing until to converge. So the free energy almost decreased monotonically. But in our case the difference that getting by CCCP algorithm monotonically decreasing until to converge. The free energy decreased monotonically too when the constraint density smaller than 2.6, but it monotonically decreasing in forepart and coming to increase in the back end (but the increase very small almost no change) in other case, it is maybe caused by the lost of data (it is related to property of computer or we must simplify the updating formula) when programming. If we ignore the lost of data, then we can say the free energy always converge monotonically.

In the small system ($N < 600$) we can get solution of 3-sat problem by using CCCP algorithm, but in large system we can’t.

In order to avoid lost of data when we add up two different data we must avoid to add the biggest one to the smallest one as far as possible.

We implemented the BP algorithm in standard manner, we used a complete parallel update rule.

BP algorithm can’t guaranteed to decrease the free energy and divergence of beliefs at each step of iterations. So it is more unstable than CCCP algorithm. In our case the BP algorithm can’t converge when constraint density $\alpha_b$ bigger than 3.86, but the CCCP algorithm still converge in this case. The computation time very long along with constraint density, but we can promote the computation speed by using lower convergence precision (it gives almost same results with higher convergence precision, if $\varepsilon_1 = 0.0001, \varepsilon_2 = 0.000001$ this two converge precisions only leads 0.0001 and 0.000001 error respectively). The beliefs that getting by BP algorithm don’t satisfy the constraint condition too until to converge.

BP algorithm and CCCP algorithm give same results when constraint density smaller than 3.86 where BP algorithm can converge. In this area we give simulation results only at $\alpha = 2.0$ and $\alpha = 3.0$, but in other circumstances our conclusion is still right. We only give CCCP algorithm simulation results alone when constraint density bigger than 3.86 where BP algorithm can’t converge, and we only give simulation results at $\alpha = 4.0$ and $\alpha = 4.267$ (transition point of 3-sat problem in 10000 points system) respectively, the CCCP algorithm still converge when $\alpha > 4.267$, but the result is not very meaningful.

VI. DISCUSSION

It is known that BP algorithm is guaranteed to converge extrema point of the Bethe free energy, although it gives exact solution on tree graphs, but it amazingly successful on the graphs with no loops. We can obtain the BP algorithm by using Bethe approximation with statistical technic. Although It’s converge speed very quick, but it can’t converge when constraint density bigger than some threshold value. In our case the BP algorithm can’t converge when constraint density bigger than 3.86, this is a big disadvantages of the BP algorithm.

The CCCP algorithm still guaranteed to converge to the extrema point of the Bethe free energy when constraint density bigger than 3.86, and it gives very similar results (the free energy, entropy, mean energy) with the BP algorithm in the area where BP algorithm converge. But these two algorithms have different changing rules.

![Figure 2. Compare results of CCCP algorithm and BP algorithm. Constraint density $\alpha = 2.0$ plots (Top Panels) and $\alpha = 3.0$ plots (Bottom Panels). Left to Right the vertical axis represent Total Free Energy, Total Mean Energy and Entropy Density respectively. The horizontal axis represent the Inverse Temperature. The CCCP algorithm give same results with BP algorithm in the BP algorithm converged area.](image1)

![Figure 3. Performance of CCCP algorithm. Constraint density $\alpha = 4.0$ plots (Top Panels) and $\alpha = 4.267$ plots (Bottom Panels). Left to Right the vertical axis represent Total Free Energy, Total Mean Energy and Entropy Density respectively. The horizontal axis represent the Inverse Temperature. The CCCP algorithm still converge when Constraint Density bigger than 3.86, and the Mean Energy still converge to zero when the Inverse Temperature bigger than some threshold value.](image2)
Figure 4. Performance of CCCP algorithm. Total Constraint difference plots (Top Panels) and Total Free energy plots (Bottom Panels). Left to Right the vertical axis represent The Constraint density \( \alpha = 2.0, 3.0, 4.0 \) respectively. The horizontal axis represent the number of iteration Step. The constraint condition is not satisfied until to converge for any constraint density, and the satisfiability of the constraint condition decline with the increase of the constraint density.

Figure 5. these are entropy, free energy and mean energy diagrams of CCCP algorithm at zero temperature, the Y axis respectively represent the entropy, free energy and mean energy, the X axis represent the constraint density \( (\alpha = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.26) \)

on difference and free energy in the whole convergence process. This property indicate that although these two algorithm converge to same extrema points of thermodynamic free energy functions, but the updating mechanism totally different from each other. The BP algorithm update data which only satisfy the normalization condition never consider about constraint condition, but the CCCP algorithm update data which satisfy all the condition theoretically (but it still doesn’t satisfy the constraint condition when programming, but it more and more close to satisfy the constraint condition at each inner loop iteration step).

VII. CONCLUSION

In this paper we use CCCP principle to get double loop algorithm for 3-sat problem, this algorithm gives very same results with BP algorithm where the BP algorithm can converge. The CCCP algorithm still converge when BP algorithm can’t converge.

The CCCP algorithm converge very stable than BP algorithm. In our case the divergence monotonically decrease in all iteration process for any constraint density. But the free energy monotonically decrease in the forepart and begin to increase (but this increase very small almost no changes this is related to how to set convergence precision) in the back end when constraint density bigger than 2.6, actually we can stop the iteration when free energy begin to increase and in this time, we can still get accurate results, so we can say the free energy (almost) always monotonically decrease.

Our CCCP double loop algorithm some different from Yuille’s double loop algorithm. We add marginal normalization lagrange multipliers to the double loop algorithm, but he doesn’t. If we don’t add this term to this algorithm, the double loop algorithm can’t converge. It is because of we can make up the error that leaded by inner loop equation(inner loop equation can’t guaranteed to satisfy the constraint condition and this directly affect the beliefs of double loop algorithm, eventually this leads NAN results of CCCP double loop algorithm) by adding this new variables. We use one iteration for inner loop and updating data by half parallel way. If we use parallel updating way, the CCCP algorithm can converge but gives wrong answer. Yuille take five iterations and he not to say he updating the data by parallel way or unparallel way. So our double loop algorithm has different updating style from Yuille’s double loop algorithm.

ACKNOWLEDGMENTS

I would like thanks to Prof Haijun Zhou for helpful discussion and KITPC(Beijing) for hospitality. This work was supported by the National Natural Science Foundation of China with Grants No....... (and the National Fundamental Research Program of China with Grants No...)
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