On the Structure of the Solution Set of a Sign Changing Perturbation of the p-Laplacian under Dirichlet Boundary Condition

J. V. Goncalves  M. R. Marcial
Universidade Federal de Goiás
Instituto de Matemática e Estatística
74001-970 Goiânia, GO - Brasil
Email: goncalves.jva@gmail.com

Abstract

In a recent paper D. D. Hai showed that the equation
\[-\Delta_p u = \lambda f(u)\] in $\Omega$, under Dirichlet boundary condition, where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial \Omega$, $\Delta_p$ is the p-Laplacian, $f : (0, \infty) \rightarrow \mathbb{R}$ is a continuous function which may blow up to $\pm \infty$ at the origin, admits a solution if $\lambda > \lambda_0$ and has no solution if $0 < \lambda < \lambda_0$. In this paper we show that the solution set $S$ of the equation above, which is not empty by Hai's results, actually admits a continuum of positive solutions.

Mathematics Subject Classification: 35J25, 35J55, 35J70

1 Introduction

In this paper we establish existence of a continuum of positive solutions of

\[(P)_{\lambda} \quad \begin{cases} -\Delta_p u = \lambda f(u) + h & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega, \end{cases} \]

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial \Omega$, $\lambda > 0$ is a real parameter, $f : (0, \infty) \rightarrow \mathbb{R}$ is a continuous function which may blow up to $\pm \infty$ at the origin and $h : \Omega \rightarrow \mathbb{R}$ is a nonnegative $L^\infty$-function.

Definition 1.1 By a solution of $(P)_\lambda$ we mean a function $u \in W^{1,p}_0(\Omega)$ such that

\[
\int_{\Omega} |\nabla u|^{p-2}\nabla u.\nabla \varphi dx = \lambda \int_{\Omega} f(u)\varphi dx + \int_{\Omega} h\varphi dx, \quad \varphi \in W^{1,p}_0(\Omega). \tag{1.1}
\]

Definition 1.2 The solution set of $(P)_\lambda$ is

\[
S := \{(\lambda, u) \in (0, \infty) \times C(\overline{\Omega}) \mid u \text{ is a solution of } (P)_\lambda \}. \tag{1.2}
\]
It was shown by Hai [13] that there is a positive number $\lambda_0$ such that $(P)_\lambda$ admits: a solution if $\lambda > \lambda_0$ and no solution if $\lambda < \lambda_0$. Our aim is to investigate existence of connected components of $S$. By adapting estimates in [13] we succeeded in showing the existence of a continuum $\Sigma \subset S$ such that $\text{Proj}_R \Sigma = (\lambda_0, \infty)$.

The assumptions on $f$ are:

$$(f)_1 \quad f : (0, \infty) \to \mathbb{R} \text{ is continuous and } \lim_{u \to \infty} \frac{f(u)}{u^{p-1}} = 0,$$

$$(f)_2 \quad \text{there are positive numbers } a, \beta, A \text{ with } \beta < 1 \text{ such that }$

\begin{align*}
(i) \quad & f(u) \geq \frac{a}{u^\beta} \text{ for } u > A, \\
(ii) \quad & \limsup_{u \to 0} u^\beta |f(u)| < \infty.
\end{align*}

We give below a few examples of functions $f$ satisfying $(f)_1$, $(f)_2$. Those functions appear in several earlier works on existence of solutions, cf. section 2.,

\begin{align*}
a) \quad & u^q - \frac{1}{u^\beta}, \quad \beta > 0, \quad 0 < q < p - 1, \\
b) \quad & \frac{1}{u^\beta} - \frac{1}{u^\alpha}, \quad 0 < \beta < \alpha < 1, \\
c) \quad & a - \frac{1}{u^\alpha}, \quad a > 0, \quad 0 < \alpha < 1, \\
d) \quad & \frac{1}{u^\alpha} + u^q, \quad 0 < \alpha < 1, \quad 0 < q < p - 1, \\
e) \quad & \frac{1}{u^\alpha}, \quad 0 < \alpha < 1, \\
f) \quad & \ln u.
\end{align*}

The main results of this paper are,\n
**Theorem 1.1** Assume $(f)_1 - (f)_2$. Then there is a number $\lambda_* > 0$ and a connected subset $\Sigma$ of $[\lambda_*, \infty) \times C(\Omega)$ satisfying,

\begin{align*}
\Sigma \subset S, \\
\Sigma \cap (\{\lambda\} \times C(\Omega)) \neq \emptyset, \quad \lambda_* \leq \lambda < \infty. \quad (1.3) \quad (1.4)
\end{align*}

The prove of theorem [1.1] will be achieved by at first proving the following result.

**Theorem 1.2** Assume $(f)_1 - (f)_2$. Then there is a number $\lambda_* > 0$ and for each $\Lambda > \lambda_*$ there is a connected set $\Sigma_\Lambda \subset ([\lambda_*, \Lambda] \times C(\Omega)$ satisfying

\begin{align*}
\Sigma_\Lambda \subset S, \\
\Sigma_\Lambda \cap (\{\lambda_*\} \times C(\Omega)) \neq \emptyset, \quad (1.5) \quad (1.6) \\
\Sigma_\Lambda \cap (\{\Lambda\} \times C(\Omega)) \neq \emptyset. \quad (1.7)
\end{align*}

**Remark 1.1** The present work is motivated by Hai [13]. We will use $C, C_1, C_2, \tilde{C}$ to denote positive cumulative constants.
2 Background

The Dirichlet problem

\[- \Delta_p u = f(x,u) \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega, \tag{2.1}\]

where \( f : \Omega \times (0, \infty) \to \mathbb{R} \) is a function satisfying a condition like \( f(x,r) \to +\infty \) as \( r \to 0 \), referred to as singular at the origin has been extensively studied in the last years.

In the pioneering work [5], it was shown by Crandall, Rabinowitz & Tartar through the use of topological methods, e.g. Schauder Theory and Maximum Principles, that the problem

\[
\begin{align*}
-\Delta u &= u^{-\gamma} \text{ in } \Omega, \\
u &> 0 \text{ in } \Omega, \\
u &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

where \( \gamma > 0 \), admits a solution \( u \in C^2(\Omega) \cap C(\overline{\Omega}) \), (see also the references of [5]).

Subsequently, Lazer & McKenna in [14], established, among other results, the existence of a solution \( u \in C^{2+\alpha}(\Omega) \cap C(\Omega) \) \((0 < \alpha < 1)\) for the problem

\[
\begin{align*}
-\Delta u &= p(x)u^{-\gamma} \text{ in } \Omega, \\
u &> 0 \text{ in } \Omega, \\
u &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

where \( p \in C^\alpha(\overline{\Omega}) \) is a positive function.

Several techniques have been employed in the study of (2.1). In [26], by using lower and upper solutions, Zhang showed that there is some number \( \lambda \in (0, +\infty) \) such that the problem

\[
\begin{align*}
-\Delta u + \frac{1}{u^\alpha} &= \lambda u^p \text{ in } \Omega, \\
u &> 0 \text{ in } \Omega, \\
u &= 0 \text{ in } \partial \Omega,
\end{align*}
\]

where \( \alpha, p \in (0, 1) \), admits a solution \( u_\lambda \in C^{2+\gamma}(\Omega) \cap C(\overline{\Omega}) \cap H^1_0(\Omega) \) with \( u_\lambda^{-\alpha} \in L^1(\Omega) \) for each \( \lambda > \bar{\lambda} \) and no solution in \( C^2(\Omega) \cap C(\overline{\Omega}) \) for \( \lambda < \bar{\lambda} \). It was also shown that the problem above admits no solution in \( C(\overline{\Omega}) \cap H^1_0(\Omega) \) if \( \alpha \geq 1, \lambda > 0 \) and \( p > 0 \).

In [9], Giacomoni, Schindler & Takac employed variational methods to investigate the problem

\[
\begin{align*}
-\Delta'_p u &= \frac{\lambda}{u^\delta} + u^q \text{ in } \Omega, \\
u &> 0 \text{ in } \Omega, \\
u &= 0 \text{ on } \partial \Omega,
\end{align*}
\]

where \( 1 < p < \infty, p - 1 < q < p* - 1, \lambda > 0 \) and \( 0 < \delta < 1 \) with \( p* = \frac{Np}{n-p} \) if \( 1 < p < N, p* \in (0, \infty) \) large if \( p = N \) and \( p* = \infty \) if \( p > N \). Several results were shown in that paper, among them existence, multiplicity and regularity of solutions.
In [20], Perera & Zhang used variational methods to prove existence of solution for the problem
\[
\begin{cases}
-\Delta_p u = a(x)u^{-\gamma} + \lambda f(x,u) & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where \(1 < p < \infty, \gamma, \lambda > 0\) are numbers, \(a \geq 0\) is a measurable, not identically zero function and \(f: \Omega \times [0, \infty) \rightarrow \mathbb{R}\) is a Carathéodory satisfying
\[
\sup_{(x,t) \in \Omega \times [0,T]} |f(x,t)| < \infty
\]
for each \(T > 0\).

There is a broad literature on singular problems and we further refer the reader to Gerghu & Radulescu [8], Goncalves, Rezende & Santos [11], Hai [12, 13], Mohammed [17], Shi & Yao [21], Hoang Loc & Schmitt [16], Montenegro & Queiroz [18] and their references.

## 3 Some Auxiliary Results

We gather below a few technical results. For completeness, a few proofs will be provided in the Appendix. The Euclidean distance from \(x \in \Omega\) to \(\partial \Omega\) is
\[
d(x) = \text{dist}(x, \partial \Omega).
\]
The result below derives from Gilbarg & Trudinger [10], Vázquez [25].

**Lemma 3.1** Let \(\Omega \subset \mathbb{R}^N\) be a smooth, bounded, domain. Then
\begin{enumerate}
\item[(i)] \(d \in \text{Lip}(\Omega)\) and \(d\) is \(C^2\) in a neighborhood of \(\partial \Omega\),
\item[(ii)] if \(\phi_1\) denotes a positive eigenfunction of \((-\Delta_p, W^{1,p}_0(\Omega))\) one has,
\[
\phi_1 \in C^{1,\alpha}(\overline{\Omega}) \text{ with } 0 < \alpha < 1, \quad \frac{\partial \phi_1}{\partial \nu} < 0 \text{ on } \partial \Omega,
\]
and there are positive constants \(C_1, C_2\) such that
\[
C_1 d(x) \leq \phi_1(x) \leq C_2 d(x), \quad x \in \Omega.
\]
\end{enumerate}

The result below is due to Crandall, Rabinowitz & Tartar [5], Lazer & McKenna [14] in the case \(p = 2\) and Giacomoni, Schindler & Takac [9] in the case \(1 < p < \infty\).

**Lemma 3.2** Let \(\beta \in (0,1)\) and \(m > 0\). Then the problem
\[
\begin{cases}
-\Delta_p u = \frac{m}{u^\beta} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(3.1)
admits an only weak solution \(u_m \in W^{1,p}_0(\Omega)\). Moreover \(u_m \geq \epsilon_m \phi_1\) in \(\Omega\) for some constant \(\epsilon_m > 0\).
Remark 3.1 By the results in [15, 9], there is $\alpha \in (0, 1)$ such that $u_m \in C^{1,\alpha}(\Omega)$.

The result below, which is crucial in this work, and whose proof is provided in the Appendix, is basically due to Hai [13].

Lemma 3.3 Let $g \in L^\infty_{\text{loc}}(\Omega)$. Assume that there is $\beta \in (0, 1)$ and $C > 0$ such that

$$|g(x)| \leq \frac{C}{d(x)^\beta}, \quad x \in \Omega. \quad (3.2)$$

Then there is an only weak solution $u \in W^{1,p}_0(\Omega)$ of

$$\begin{cases} 
-\Delta_p u = g & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega.
\end{cases} \quad (3.3)$$

In addition, there exist constants $\alpha \in (0, 1)$ and $M > 0$, with $M$ depending only on $C, \beta, \Omega$ such that $u \in C^{1,\alpha}(\Omega)$ and $||u||_{C^{1,\alpha}(\Omega)} \leq M$.

Remark 3.2 The solution operator associated to (3.3) is: let

$$\mathcal{M}_{\beta,\infty} = \{g \in L^\infty_{\text{loc}}(\Omega) \mid |g(x)| \leq \frac{C}{d(x)^\beta}, \quad x \in \Omega\},$$

$$S : \mathcal{M}_{\beta,\infty} \to W^{1,p}_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega}), \quad S(g) := u.$$

Notice that

$$||S(g)||_{C^{1,\alpha}(\overline{\Omega})} \leq M,$$

for all $g \in \mathcal{M}_{C,d,\beta,\infty}$ with $M$ depending only on $C, \beta, \Omega$.

Corollary 3.1 Let $g, \tilde{g} \in L^\infty_{\text{loc}}(\Omega)$ with $g \geq 0$, $g \not= 0$ satisfying (3.2). Then, for each $\epsilon > 0$, the problem

$$\begin{cases} 
-\Delta_p u_\epsilon = g \chi_{\{d>\epsilon\}} + \tilde{g} \chi_{\{d<\epsilon\}} & \text{em } \Omega; \\
u_\epsilon = 0 & \text{em } \partial \Omega
\end{cases} \quad (3.4)$$

admits an only solution $u_\epsilon \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. In addition, there is $\epsilon_0 > 0$ such that

$$u_\epsilon \geq \frac{u}{2} \quad \text{in } \Omega \quad \text{for each } \epsilon \in (0, \epsilon_0),$$

where $u$ is the solution of (3.3).

A proof of the Corollary above will be included in the Appendix.
4 Existence of Lower and Upper Solutions

In this section we present two results, essentially due to Hai [13], on existence of lower and upper solutions of $(P)_\lambda$. At first some definitions.

**Definition 4.1** A function $u \in W^{1,p}_0(\Omega)$ with $u > 0$ in $\Omega$ such that
\[
\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx \leq \lambda \int_\Omega f(u) \varphi \, dx + \int_\Omega h \varphi \, dx, \quad \varphi \in W^{1,p}_0(\Omega), \; \varphi \geq 0
\]
is a lower solution of $(P)_\lambda$.

**Definition 4.2** A function $\pi \in W^{1,p}_0(\Omega)$ with $\pi > 0$ in $\Omega$ such that
\[
\int_\Omega |\nabla \pi|^{p-2}\nabla \pi \cdot \nabla \varphi \, dx \geq \lambda \int_\Omega f(\pi) \varphi \, dx + \int_\Omega h \varphi \, dx, \quad \varphi \in W^{1,p}_0(\Omega), \; \varphi \geq 0.
\]
is an upper solution of $(P)_\lambda$.

**Theorem 4.1** Assume $(f)_1 - (f)_2$. Then there exist $\lambda^* > 0$ and a non-negative function $\psi \in C^{1,\alpha}(\overline{\Omega})$, with $\psi > 0$ in $\Omega$, $\psi = 0$ on $\partial\Omega$, $\alpha \in (0,1)$ such that for each $\lambda \in [\lambda^*, \infty)$, $u = \lambda^r \psi$ with $r = 1/(p + \beta - 1)$, is a lower solution of $(P)_\lambda$.

**Proof of Theorem 4.1** By $(f)_2(i)-(ii)$ there is $b > 0$ such that
\[
f(s) > -\frac{b}{s^\beta} \text{ for } s > 0. \tag{4.1}
\]
By lemma 3.2 there are both a function $\phi \in C^{1,\alpha}(\overline{\Omega})$, with $\alpha \in (0,1)$, such that
\[
\begin{cases}
-\Delta_p \phi = \frac{1}{\phi^{\beta}} \text{ in } \Omega, \\
\phi > 0 \text{ in } \Omega, \\
\phi = 0 \text{ on } \partial\Omega,
\end{cases} \tag{4.2}
\]
and a constant $C_1 > 0$ such that $\phi \geq C_1 d$ in $\Omega$. Take $\delta = a^{\frac{p-1}{\beta}}$ and $\gamma = 2\beta b_0^{\frac{\beta}{p-1}}$, where $a$ is given in $(f)_2(i)$.

By corollary 3.1 there is a constant $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$, the problem
\[
\begin{cases}
-\Delta_p \psi = \delta \phi^{-\beta} \chi_{d > \epsilon} - \gamma \phi^{-\beta} \chi_{d < \epsilon} \text{ in } \Omega, \\
\psi > 0 \text{ in } \Omega, \\
\psi = 0 \text{ on } \partial\Omega,
\end{cases} \tag{4.3}
\]
admits a solution $\psi \in C^{1,\alpha}(\overline{\Omega})$ satisfying
\[
\psi \geq (\delta^{1/(p-1)}/2) \phi. \tag{4.4}
\]
Set \( \underline{u} = \lambda r \psi \) where \( r = 1/(p + \beta - 1) \) and \( \lambda > 0 \). Take \( \lambda_* = [2A/(C_1 \epsilon \delta \frac{1}{p-1})]^\frac{1}{r} \), with \( \epsilon \in (0, \epsilon_0) \) and \( A \) given by \((f)_2\).

**Claim** \( \underline{u} \) is a lower solution of \((P)_\lambda \) for \( \lambda \geq \lambda_* \).

Indeed, take \( \xi \in W^{1,p}_0(\Omega), \xi \geq 0 \). Using \((4.3)\) we have

\[
\int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \xi dx = \lambda \int_\{d>\epsilon\} \frac{\xi}{\phi_\beta^\delta} dx - \lambda \int_\{d<\epsilon\} \frac{\xi}{\phi_\beta^\delta} dx. \tag{4.5}
\]

We distinguish between two cases.

**Case 1** \( d > \epsilon \)

For each \( \lambda \geq \lambda_* \) we have by using \((4.4)\),

\[
\underline{u} = \lambda r \psi \geq \lambda r \frac{\delta \frac{1}{p-1}}{2} \phi \geq \lambda r \frac{\delta \frac{1}{p-1}}{2} C_1 d > \lambda r \frac{\delta \frac{1}{p-1}}{2} C_1 \epsilon > A.
\]

So \( \underline{u}(x) > A \) for each \( \lambda \geq \lambda_* \) with \( d(x) > \epsilon \). By \((4.2)\) and \((4.3)\),

\[
-\Delta_p \delta \frac{1}{p-1} \phi = \frac{\delta}{\phi_\beta^\delta} \geq - \Delta_p \psi. \tag{4.6}
\]

It follows by the weak comparison principle that

\[
\delta \frac{1}{p-1} \phi \geq \psi \quad \text{in } \Omega. \tag{4.6}
\]

Using \((f)_2(i)\) and \((4.6)\) we have,

\[
\lambda \int_{d>\epsilon} f(\underline{u}) \xi dx \geq \lambda a \int_{d>\epsilon} \frac{\xi}{\underline{u}^\beta} dx = \lambda a \int_{d>\epsilon} \frac{\xi}{\psi^\beta} dx \geq \lambda r^{(p-1)} \frac{4}{\delta \frac{1}{p-1}} \int_{d>\epsilon} \frac{\xi}{\phi_\beta^\delta} dx = \lambda r^{(p-1)} \delta \int_{d>\epsilon} \frac{\xi}{\phi_\beta^\delta} dx. \tag{4.7}
\]

**Case 2** \( d < \epsilon \).

Using \((4.1)\) and \((4.4)\) we have

\[
\lambda \int_{\{d<\epsilon\}} f(\underline{u}) \xi dx \geq -\lambda b \int_{\{d<\epsilon\}} \frac{\xi}{\underline{u}^\beta} dx = -\lambda 1-r \beta \int_{d<\epsilon} \frac{\xi}{\psi^\beta} dx \geq -\lambda r^{(p-1)} \frac{2}{\delta \frac{1}{p-1}} \int_{d<\epsilon} \frac{\xi}{\phi_\beta^\delta} dx = -\lambda r^{(p-1)} \gamma \int_{d<\epsilon} \frac{\xi}{\phi_\beta^\delta} dx. \tag{4.8}
\]

Using \((4.7)-(4.8)\) we get

\[
\lambda \int_\Omega f(\underline{u}) \xi dx + \int_\Omega h \xi dx \geq \int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \xi dx,
\]

showing that \( \underline{u} = \lambda r \psi \) is a lower solution of \((P)_\lambda \) for each \( \lambda \geq \lambda_* \), ending the proof of theorem 4.1.

Next, we show existence of an upper solution.
Theorem 4.2 Assume $(f)_1-(f)_2$ and take $\Lambda > \lambda_*$ with $\lambda_*$ as in theorem 4.1. Then for each $\lambda \in [\lambda^*, \Lambda]$, $(P)_\lambda$ admits an upper solution $\overline{u} = \overline{u}_\lambda = M\phi$ where $M > 0$ is a constant and $\phi$ is given by (4.2).

Proof of Theorem 4.2 Choose $\bar{\epsilon} > 0$ such that
\[
\Lambda\bar{\epsilon}\|\phi\|_{\infty}^{p-1+\beta} < \frac{1}{4}.
\] (4.9)

By $(f)_1$ and $(f)_2$ there are $A_1 > 0$ and $C > 0$ such that
\[
|f(u)| \leq \bar{\epsilon}u^{p-1} \text{ for } u > A_1
\] (4.10)

and
\[
|f(u)| \leq \frac{C}{u^\beta} \text{ for } u \leq A_1.
\] (4.11)

Choose
\[
M \geq \left\{ \Lambda^r \delta^{\frac{1}{p-1+\beta}}, (4\Lambda C)^{\frac{1}{p-1+\beta}}, (4\|h\|_\infty\|\phi\|_\infty^{\frac{1}{p-1+\beta}}) \right\}.
\] (4.12)

Using (4.9) and (4.12) we get
\[
\Lambda\bar{\epsilon}(M\|\phi\|_\infty)^{p+\beta-1} + \Lambda C \leq \frac{M^{p+\beta-1}}{4} + \frac{M^{p+\beta-1}}{4} = \frac{M^{p+\beta-1}}{2}.
\] (4.13)

Let $\overline{\pi} = M\phi$. Using (4.10)-(4.11) and picking $\lambda \leq \Lambda$ we have
\[
\lambda f(\overline{\pi}) \leq \lambda |f(\overline{\pi})|
\]
\[
\leq \lambda \left[ \bar{\epsilon} \overline{\pi}^{p-1} \chi(\overline{\pi} > A_1) + \frac{C}{\overline{\pi}^\beta} \chi(\overline{\pi} \leq A_1) \right]
\]
\[
\leq \lambda \left[ \bar{\epsilon} \overline{\pi}^{p-1} \chi(\overline{\pi} > A_1) + \bar{\epsilon} \overline{\pi}^{\beta-1} \chi(\overline{\pi} \leq A_1) + \frac{C}{\overline{\pi}^\beta} \chi(\overline{\pi} \leq A_1) + \frac{C}{\overline{\pi}^\beta} \chi(\overline{\pi} > A_1) \right]
\]
\[
= \lambda \left[ \bar{\epsilon} \overline{\pi}^{p-1} + \frac{C}{\overline{\pi}^\beta} \right].
\] (4.14)

Thus
\[
\lambda f(M\phi) \leq \lambda \left[ \frac{\overline{\pi}(M\|\phi\|_\infty)^{p+\beta-1} + C}{[M\phi]^\beta} \right]
\]
\[
\leq \Lambda \frac{\overline{\pi}(M\|\phi\|_\infty)^{p+\beta-1}}{[M\phi]^\beta} + \Lambda \frac{C}{[M\phi]^\beta}.
\] (4.15)

Replacing (4.12) and (4.13) in (4.15),
\[
\lambda f(M\phi) \leq \frac{M^{p+\beta-1}}{2[M\phi]^\beta} = \frac{M^{p-1}}{2\phi^\beta}.
\]
It follows from (4.12) that
\[ h \leq \frac{M^{p-1}}{2\phi^\beta} \leq \frac{M^{p-1}}{2\phi^\beta}. \]
Thus
\[ \lambda f(\bar{u}) + h \leq \frac{M^{p-1}}{\phi^\beta}. \]
Taking \( \eta \in W^{1,p}_0(\Omega) \) with \( \eta \geq 0 \) we have by using (4.2),
\[
\lambda \int_\Omega f(\bar{u})\eta dx + \int_\Omega h\eta dx \leq M^{p-1} \int_\Omega \frac{\eta}{\phi^\beta} dx
\]
\[
= M^{p-1} \int_\Omega |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla \eta dx
\]
\[
= \int_\Omega |\nabla (M\phi)|^{p-2} \nabla (M\phi) \cdot \nabla \eta dx
\]
\[
= \int_\Omega |\nabla u|^{p-2} \nabla \bar{u} \cdot \nabla \eta dx,
\]
showing that \( \bar{u} = M\phi \) is an upper solution of \((P)_\lambda\) for \( \lambda \in [\lambda_*, \Lambda]. \)

5 Proofs of the Main Results

At first we introduce some notations, remarks and lemmas. Take \( \Lambda > \lambda_* \) and set
\( I_\Lambda := [\lambda_*, \Lambda]. \) For each \( \lambda \in I_\Lambda. \) By theorem 4.1
\[
\bar{u} = u_\lambda = \lambda^r \psi
\]
is a lower solution of \((P)_\lambda. \) Pick \( M = M_\Lambda \geq \Lambda^{p-1} \delta r^{\frac{1}{p-1}}. \) By theorem 4.2
\[ u = u_\lambda = M_\Lambda \phi \]
is an upper solution of \((P)_\lambda. \) It follows by (4.6) that
\[ u = \lambda^r \psi \leq \Lambda^{p-1} \delta r^{\frac{1}{p-1}} \phi \leq M\phi = \bar{u}. \] (5.1)
The convex, closed subset of \( I_\Lambda \times C(\overline{\Omega}), \) defined by
\[ G_\Lambda := \{ (\lambda, u) \in I_\Lambda \times C(\overline{\Omega}) \mid \lambda \in I_\Lambda, u \leq u \leq \bar{u} \text{ and } u = 0 \text{ on } \partial\Omega \} \]
will play a key role in this work.
For each \( u \in C(\overline{\Omega}) \) define
\[ f_\Lambda(u) = \chi_{S_1}f(u) + \chi_{S_2}f(u) + \chi_{S_3}f(\bar{u}), \quad x \in \Omega, \] (5.2)
where
\[ S_1 := \{ x \in \Omega \mid u(x) < \underline{u}(x) \}, \]
\[ S_2 := \{ x \in \Omega \mid \underline{u}(x) \leq u(x) \leq \bar{u}(x) \}, \]
\[ S_3 := \{ x \in \Omega \mid \bar{u}(x) < u(x) \}, \]
and \( \chi_{S_i} \) is the characteristic function of \( S_i \).

**Lemma 5.1** For each \( u \in C(\Omega) \), \( f_\Lambda(u) \in L^\infty_{\text{loc}}(\Omega) \) and there are \( C > 0, \beta \in (0,1) \) such that
\[
|f_\Lambda(u)(x)| \leq \frac{C}{d(x)^\beta}, \quad x \in \Omega. \tag{5.3}
\]

**Proof** Indeed, let \( K \subset \Omega \) be a compact subset. Then both \( \underline{u} \) and \( \bar{u} \) achieve a positive maximum and a positive minimum on \( K \). Since \( f \) is continuous in \((0, \infty)\) then \( f_\Lambda(u) \in L^\infty_{\text{loc}}(\Omega) \).

**Verification of (5.3):** Since \( \Omega = \bigcup_{i=1}^3 S_i \) it is enough to show that
\[
|f(u(x))| \leq \frac{C}{d(x)^\beta}, \quad x \in S_i, \quad i = 1, 2, 3.
\]

At first, by \((f)_{2(ii)}\) there are \( C, \delta > 0 \) such that
\[
|f(s)| \leq \frac{C}{s^\beta}, \quad 0 < s < \delta.
\]

Let
\[
\Omega_\delta = \{ x \in \Omega \mid d(x) < \delta \}.
\]

Recalling that \( \underline{u} \in C^1(\overline{\Omega}) \), let
\[
D = \max_{\underline{u}} d(x), \quad \nu_\delta := \min_{\underline{u}} d(x), \quad \nu^\delta := \max_{\underline{u}} d(x),
\]
and notice that both \( 0 < \nu_\delta \leq \nu^\delta \leq D < \infty \) and \( f([\nu_\delta, \nu^\delta]) \) is compact.

On the other hand, applying theorems [4.1, 4.2] lemmas [3.1, 3.2] and inequality [4.4] we infer that
\[
0 < \lambda_*^\psi \underline{u} \leq \bar{u} = M \phi \text{ in } \Omega
\]
and
\[
\frac{1}{\underline{u}^\beta}, \quad \frac{1}{\bar{u}^\beta}, \leq \frac{1}{(\lambda_*^\psi(x))}\beta \leq \frac{C}{d(x)^\beta}, \quad x \in \Omega_\delta.
\]

To finish to proof, we distinguish among three cases:

(i) \( x \in S_1 \): in this case,
\[
f_\Lambda(u(x)) = f(u(x)).
\]

If \( x \in S_1 \cap \Omega_\delta \) we infer that
\[
|f_\Lambda(u(x))| \leq \frac{C}{u(x)^\beta} \leq \frac{C}{d(x)^\beta}.
\]
If \( x \in S_1 \cap \Omega_\delta^c \). Pick positive numbers \( d_i, i = 1, 2 \) such that
\[
d_1 \leq u(x) \leq d_2, \ x \in \Omega_\delta^c.
\]
Hence
\[
|f(x)| \leq \frac{C}{d(x)^\beta}, \ x \in \Omega.
\]

(ii) \( x \in S_2 \): in this case,
\[
0 < \lambda \psi \leq u \leq M \phi.
\]
and as a consequence,
\[
|f(x)| \leq \frac{C}{u(x)^\beta}, \ x \in \Omega_\delta.
\]
Hence, there is a positive constant \( \tilde{C} \) such that
\[
|f(x)| \leq \tilde{C}, \ x \in \Omega_\delta.
\]
Thus
\[
|f(x)| \leq \begin{cases} \tilde{C} & \text{if } x \in \Omega_\delta^c, \\ \frac{C}{d(x)^\beta} & \text{if } x \in \Omega_\delta. \end{cases}
\]

On the other hand,
\[
\frac{1}{D^\beta} \leq \frac{1}{d(x)^\beta}, \ x \in \Omega_\delta^c,
\]
and therefore there is a constant \( C > 0 \) such that
\[
|f(x)| \leq \begin{cases} \frac{C}{D^\beta} & \text{if } x \in \Omega_\delta^c, \\ \frac{C}{d(x)^\beta} & \text{if } x \in \Omega_\delta. \end{cases}
\]
Therefore,
\[
|f(x)| \leq \frac{C}{d(x)^\beta}, \ x \in S_2, \ u \in \mathcal{G}_\Lambda.
\]

Case \( x \in S_3 \): in this case
\[
f_\Lambda(u(x)) = f(u(x)).
\]
If \( x \in S_3 \cap \Omega_\delta \) we infer that
\[
|f_\Lambda(u(x))| \leq \frac{C}{\overline{u}(x)^\beta} \leq \frac{C}{d(x)^\beta}.
\]
If \( x \in S_3 \cap \Omega_\delta^c \). Pick positive numbers \( d_i, i = 1, 2 \) such that
\[
d_1 \leq \overline{u}(x) \leq d_2, \ x \in \Omega_\delta^c.
\]
Hence
\[
|f_\Lambda(u(x))| \leq \frac{C}{d(x)^\beta}, \ x \in \Omega.
\]
This ends the proof of lemma 5.1.
Remark 5.1 By lemmas 3.3, 5.1 and remark (3.2), for each \( v \in C(\overline{\Omega}) \) and \( \lambda \in I_\Lambda \),
\[
(\lambda f_\Lambda(v) + h) \in L^\infty_{\text{loc}}(\Omega) \text{ and } |(\lambda f_\Lambda(v) + h)| \leq \frac{C_\Lambda}{d^\beta(x)} \text{ in } \Omega \tag{5.4}
\]
where \( C_\Lambda > 0 \) is a constant independent of \( v \) and \( \beta \in (0, 1) \). So for each \( v \),
\[
\left\{ \begin{array}{l}
-\Delta_p u = \lambda f_\Lambda(v) + h \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega
\end{array} \right. \tag{5.5}
\]
admits an only solution \( u = S(\lambda f_\Lambda(v) + h)) \in W^{1,p}_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega}) \).

Set
\[
F_\Lambda(u)(x) = f_\Lambda(u(x)), \quad u \in C(\overline{\Omega}).
\]

and consider the operator
\[
T : I_\Lambda \times C(\overline{\Omega}) \to W^{1,p}_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega}),
\]

\[
T(\lambda, u) = S(\lambda F_\Lambda(u) + h)) \text{ if } \lambda_* \leq \lambda \leq \Lambda, \quad u \in C(\overline{\Omega}).
\]

Notice that if \( (\lambda, u) \in I_\Lambda \times C(\overline{\Omega}) \) satisfies \( u = T(\lambda, u) \) then \( u \) is a solution of
\[
\left\{ \begin{array}{l}
-\Delta_p u = \lambda f_\Lambda(u) + h \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega
\end{array} \right. \tag{5.6}
\]

Lemma 5.2 If \( (\lambda, u) \in I_\Lambda \times C(\overline{\Omega}) \) and \( u = T(\lambda, u) \) then \( (\lambda, u) \in G_\Lambda \).

**Proof** Indeed, let \( (\lambda, u) \in I_\Lambda \times C(\overline{\Omega}) \) such that \( T(\lambda, u) = u \). Then
\[
\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla \varphi dx = \lambda \int_\Omega f_\Lambda(u) \varphi dx + \int_\Omega h \varphi dx, \quad \varphi \in W^{1,p}_0(\Omega).
\]

We claim that \( u \geq u \). Assume on the contrary, that \( \varphi := (u - u)^+ \neq 0 \). Then
\[
\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla \varphi dx = \int_{u < u} |\nabla u|^{p-2}\nabla u.\nabla \varphi dx
\]
\[
= \lambda \int_{u < u} f_\Lambda(u) \varphi dx + \int_{u < u} h \varphi dx
\]
\[
= \lambda \int_{u < u} f(u) \varphi dx + \int_{u < u} h \varphi dx
\]
\[
\geq \int_{u < u} |\nabla u|^{p-2}\nabla u.\nabla \varphi dx
\]
\[
= \int_\Omega |\nabla u|^{p-2}\nabla u.\nabla \varphi dx.
\]
Hence
\[ \int_\Omega \left[ |\nabla u|^{p-2}\nabla u - |\nabla u|^p \right] : \nabla (u - \bar{u}) \, dx \leq 0. \]

It follows by lemma 6.1 that
\[ \int_\Omega |\nabla (u - \bar{u})|^p \, dx \leq 0, \] contradicting \( \varphi \not\equiv 0. \) Thus, \((u - \bar{u})^+ = 0\), that is, \( \bar{u} - u \leq 0 \), and so \( \bar{u} \leq T(\lambda, u) \).

We claim that \( u \geq \bar{u}. \) Assume on the contrary that \( \varphi := (u - \bar{u})^+ \not\equiv 0. \) We have
\[
\int_\Omega |\nabla u|^{p-2}\nabla u, \nabla \varphi \, dx = \int_{\pi < u} |\nabla u|^{p-2}\nabla u, \nabla \varphi \, dx
= \lambda \int_{\pi < u} f(\pi) \varphi \, dx + \int_{\pi < u} h \varphi \, dx
\leq \int_{\pi < u} |\nabla \pi|^{p-2}\nabla \pi, \nabla \varphi \, dx
= \int_\Omega |\nabla \pi|^{p-2}\nabla \pi, \nabla \varphi \, dx,
\]

Therefore,
\[
\int_\Omega \left[ |\nabla u|^{p-2}\nabla u - |\nabla \pi|^{p-2}\nabla \pi \right] : \nabla (u - \pi) \, dx \leq 0.
\]

contradicting \( \varphi \not\equiv 0. \) So \((u - \bar{u})^+ = 0\) so that \( u - \bar{u} \leq 0 \), which gives \( \bar{u} \geq T(\lambda, u) \).

As a consequence of the arguments above \( u \in G_\Lambda \), showing lemma 5.2.

**Remark 5.2** By the definitions of \( f_\Lambda \) and \( G_\Lambda \), for each \((\lambda, u) \in G_\Lambda \)
\[
f_\Lambda(u) = f(u), \quad x \in \Omega. \tag{5.7}
\]

**Remark 5.3** By remark 3.2, there is \( R_\Lambda > 0 \) such that \( G_\Lambda \subseteq B(0, R_\Lambda) \subseteq C(\bar{\Omega}) \) and
\[
T \left( I_\Lambda \times B(0, R_\Lambda) \right) \subseteq B(0, R_\Lambda).
\]

Notice that, by (5.7) and lemma 5.2, if \((\lambda, u) \in I_\Lambda \times C(\bar{\Omega})\) satisfies \( u = T(\lambda, u) \) then \((\lambda, u)\) is a solution of \((P)_\lambda\). By remark 5.2, to solve \((P)_\lambda\) it suffices to look for fixed points of \( T \).

**Lemma 5.3** \( T : I_\Lambda \times B(0, R_\Lambda) \to B(0, R_\Lambda) \) is continuous and compact.

**Proof** Let \( \{(\lambda_n, u_n)\} \subseteq I_\Lambda \times B(0, R_\Lambda) \) be a sequence such that
\[
\lambda_n \to \lambda \text{ and } u_n \to u \text{ in } C(\bar{\Omega}).
\]

Set
\[
\nu_n = T(\lambda_n, u_n) \text{ and } \nu = T(\lambda, u)
\]
so that 
\[ v_n = S(\lambda_n F_A(u_n) + h) \quad \text{and} \quad v = S(\lambda F_A(u) + h). \]

It follows that 
\[ \int_{\Omega} \left[ |\nabla v_n|^{p-2} \nabla v_n - |\nabla v|^{p-2} \nabla v \right] \nabla (v_n - v) \, dx = \lambda_n \int_{\Omega} (f_A(u_n) - f_A(u))(v_n - v) \, dx \]
\[ \leq C \int_{\Omega} |f_A(u_n) - f_A(u)| \, dx. \]

Since 
\[ |f_A(u_n) - f_A(u)| \leq \frac{C}{d(x)^\beta} \in L^1(\Omega) \]
and 
\[ f_A(u_n(x)) \to f_A(u(x)) \ \text{a.e.} \ x \in \Omega, \]
it follows by Lebesgue’s Theorem that 
\[ \int_{\Omega} |f_A(u_n) - f_A(u)| \, dx \to 0. \]

Therefore \( v_n \to v \) in \( W^{1,p}_0(\Omega) \).

On the other hand, since \( u_n \overset{C(\overline{\Omega})}{\to} u \), by the proof of lemma 5.1,
\[ (\lambda_n f_A(u_n) + h) \in L^1_{\text{loc}}(\Omega) \] and 
\[ |(\lambda_n f_A(u_n) + h)| \leq \frac{C_A}{d(x)^\beta} \in \Omega. \]

By lemma 5.3 there is a constant \( M > 0 \) such that
\[ ||v_n||_{C^{1,\alpha}(\overline{\Omega})} \leq M \]
so that \( v_n \overset{C(\overline{\Omega})}{\to} v \). This shows that \( T : I_A \times \overline{B(0,R_A)} \to \overline{B(0,R_A)} \) is continuous.

The compactness of \( T \) follows from the arguments in the five lines above.

5.1 Proof of Theorem 1.2

Some notations and technical results are needed. At first, we recall the Leray-Schauder Continuation Theorem (see [2],[3]).

**Theorem 5.1** Let \( D \) be an open bounded subset of the Banach space \( X \). Let \( a, b \in \mathbb{R} \) with \( a < b \) and assume that \( T : [a, b] \times \overline{D} \to X \) is compact and continuous. Consider \( \Phi : [a, b] \times \overline{D} \to X \) defined by \( \Phi(t, u) = u - T(t, u) \). Assume that

(i) \( \Phi(t, u) \neq 0 \), \( t \in [a, b], \ u \in \partial D \), (ii) \( \text{deg}(\Phi(t, \cdot), D, 0) \neq 0 \) for some \( t \in [a, b] \).

and set
\[ S_{a,b} = \{(t, u) \in [a, b] \times \overline{D} \mid \Phi(t, u) = 0\}. \]
Then, there is a connected compact subset \( \Sigma_{a,b} \) of \( S_{a,b} \) such that
\[
\Sigma_{a,b} \cap \{a\} \times D \neq \emptyset
\]
and
\[
\Sigma_{a,b} \cap \{b\} \times D \neq \emptyset.
\]
The Leray-Schauder Theorem above will be applied to the operator \( T \) in the settings of Section 5. Remember that \( T \) continuous, compact and \( T(I_{\Lambda} \times \overline{B(0, R_{\Lambda})}) \subset B(0, R_{\Lambda}) \). Consider \( \Phi : I_{\Lambda} \times \overline{B(0, R)} \rightarrow B(0, R) \) defined by
\[
\Phi(\lambda, u) = u - T(\lambda, u).
\]

**Lemma 5.4** \( \Phi \) satisfies:

(i) \( \Phi(\lambda, u) \neq 0 \) \( (\lambda, u) \in I_{\Lambda} \times \partial B(0, R_{\Lambda}) \),

(ii) \( \deg(\Phi(\lambda, .), B(0, R_{\Lambda}), 0) \neq 0 \) for each \( \lambda \in I_{\Lambda} \),

**Proof** The verification of (i) is straightforward since \( T(I_{\Lambda} \times \overline{B(0, R_{\Lambda})}) \subset B(0, R_{\Lambda}) \).

To prove (ii), set \( R = R_{\Lambda} \), take \( \lambda \in I_{\Lambda} \) and consider the homotopy
\[
\Psi_{\lambda}(t, u) = u - tT(\lambda, u), \quad (t, u) \in [0, 1] \times \overline{B(0, R)}.
\]

It follows that \( 0 \notin \Psi_{\lambda}(I \times \partial B(0, R)) \). By the invariance under homotopy property of the Leray-Schauder degree
\[
\deg(\Psi_{\lambda}(t, .), B(0, R), 0) = \deg(\Psi_{\lambda}(0, .), B(0, R), 0) = 1, \quad t \in [0, 1].
\]

Setting
\[
\Phi(\lambda, u) = u - T(\lambda, u), \quad (\lambda, u) \in I_{\Lambda} \times \overline{B(0, R)},
\]
we also have
\[
\deg(\Phi(\lambda, .), B(0, R), 0) = 1, \quad \lambda \in I_{\Lambda}.
\]

Set
\[
S_{\Lambda} = \{(\lambda, u) \in I_{\Lambda} \times \overline{B(0, R)} \mid \Phi(\lambda, u) = 0 \} \subset G_{\Lambda}.
\]

By the Leray-Schauder Continuation Theorem, there is a connected component \( \Sigma_{\Lambda} \subset S_{\Lambda} \) such that
\[
\Sigma_{\Lambda} \cap \{\lambda_{*}\} \times \overline{B(0, R)} \neq \emptyset
\]
and
\[
\Sigma_{\Lambda} \cap \{\Lambda\} \times \overline{B(0, R)} \neq \emptyset.
\]
We point out that $S_{\Lambda}$ is the solution set of the auxiliary problem

$$\begin{cases}
  -\Delta p u &= \lambda f_{\Lambda}(u) + h \quad \text{in} \quad \Omega, \\
  u &= 0 \quad \text{on} \quad \partial \Omega
\end{cases}$$

and since $\Sigma_{\Lambda} \subset S_{\Lambda} \subset G_{\Lambda}$ it follows using the definition of $f_{\Lambda}$ that

$$\begin{cases}
  -\Delta p u &= \lambda f(u) + h \quad \text{in} \quad \Omega, \\
  u &= 0 \quad \text{on} \quad \partial \Omega
\end{cases}$$

for $(\lambda, u) \in \Sigma_{\Lambda}$, showing that $\Sigma_{\Lambda} \subset S$. This ends the proof of theorem 1.2.

5.2 Proof of Theorem 1.1

We shall employ topological arguments to construct a suitable connected component of the solution set $S$ of $(P)_{\lambda}$. To this aim some notations are needed.

Let $M = (M, d)$ be a metric space and denote by $\{\Sigma_{n}\}$ be a sequence of connected components of $M$. The upper limit of $\{\Sigma_{n}\}$ is defined by

$$\lim \Sigma_{n} = \{ u \in M \mid \text{there is } (u_{n}) \subseteq \bigcup \Sigma_{n} \text{ with } u_{n} \in \Sigma_{n}, \text{ and } u_{n} \to u \}. $$

Remark 5.4 $\lim \Sigma_{n}$ is a closed subset of $M$.

We shall apply theorem 2.1 in Sun & Song [23], stated below for the reader’s convenience.

**Theorem 5.2** Let $M$ be a metric space and $\{\alpha_{n}\}, \{\beta_{n}\} \in \mathbb{R}$ be sequences satisfying

$$\cdots < \alpha_{n} < \cdots < \alpha_{1} < \beta_{1} < \cdots < \beta_{n} < \cdots$$

with

$$\alpha_{n} \to -\infty \quad \text{and} \quad \beta_{n} \to \infty.$$ 

Assume that $\{\Sigma_{n}^{*}\}$ is a sequence of connected subsets of $\mathbb{R} \times M$ satisfying,

(i) $\Sigma_{n}^{*} \cap (\{\alpha_{n}\} \times M) \neq \emptyset$,

(ii) $\Sigma_{n}^{*} \cap (\{\beta_{n}\} \times M) \neq \emptyset$,

for each $n$. For each $\alpha, \beta \in (-\infty, \infty)$ with $\alpha < \beta$,

(iii) $\left( \bigcup \Sigma_{n}^{*} \right) \cap ([\alpha, \beta] \times M)$ is a relatively compact subset of $\mathbb{R} \times M$.

Then there is a connected component $\Sigma^{*}$ of $\lim \Sigma_{n}^{*}$ such that

$$\Sigma^{*} \cap (\{\lambda\} \times M) \neq \emptyset \quad \text{for each } \lambda \in (\lambda_{*}, \infty).$$
Proof of Theorem 1.1 (finished) Consider $\Lambda$ as introduced in Section 5 and take a sequence $\{\Lambda_n\}$ such that $\lambda_* < \Lambda_1 < \Lambda_2 < \cdots$ with $\Lambda_n \to \infty$. Set $\beta_n = \Lambda_n$ and take a sequence $\{\alpha_n\} \subset \mathbb{R}$ such that $\alpha_n \to -\infty$ and $\cdots < \alpha_n < \cdots < \alpha_1 < \lambda_*$. Following the notations of Section 5 consider the sequence of intervals $I_n = [\lambda_*, \Lambda_n]$. Set $M = C(\Omega)$ and let

$$G_{\Lambda_n} := \{(\lambda, u) \in I_n \times \overline{B}_{R_n} \mid u \leq u, u = 0 \text{ on } \partial \Omega\},$$

where $R_n = R_{\Lambda_n}$. Consider the sequence of compact operators

$$T_n : [\lambda_*, \Lambda] \times \overline{B}_{R_n} \to \overline{B}_{R_n}$$

defined by

$$T_n(\lambda, u) = S(\lambda F_{\Lambda_n}(u) + h)) \text{ if } \lambda_* \leq \lambda \leq \Lambda_n, \ u \in \overline{B}_{R_n}.$$ 

Next consider the extension of $T_n$, namely $\tilde{T}_n : \mathbb{R} \times \overline{B}_{R_n} \rightarrow \overline{B}_{R_n}$ defined by

$$\tilde{T}_n(\lambda, u) = \begin{cases} T_n(\lambda_* , u) & \text{if } \lambda \leq \lambda_*, \\ T_n(\lambda, u) & \lambda_* \leq \lambda \leq \Lambda_n, \\ T_n(\Lambda_n, u) & \lambda \geq \Lambda_n. \end{cases}$$

Notice that $\tilde{T}_n$ is continuous, compact.

Applying theorem 5.1 to $\tilde{T}_n : [\alpha_n, \beta_n] \times \overline{B}_{R_n} \rightarrow \overline{B}_{R_n}$ we get a compact connected component $\Sigma_n^*$ of

$$\mathcal{S}_n = \{(\lambda, u) \in [\alpha_n, \beta_n] \times \overline{B}_{R_n} \mid \Phi_n(\lambda, u) = 0\},$$

where

$$\Phi_n(\lambda, u) = u - \tilde{T}_n(\lambda, u).$$

Notice that $\Sigma_n^*$ is also a connected subset of $\mathbb{R} \times M$. By theorem 5.2 there is a connected component $\Sigma^*$ of $\lim \Sigma_n^*$ such that

$$\Sigma^* \cap (\{\lambda\} \times M) \neq \emptyset \text{ for each } \lambda \in \mathbb{R}.$$ 

Set $\Sigma = ([\lambda_*, \infty) \times M) \cap \Sigma^*$. Then $\Sigma \subset \mathbb{R} \times M$ is connected and

$$\Sigma \cap (\{\lambda\} \times M) \neq \emptyset, \ \lambda_* \leq \lambda < \infty.$$ 

We claim that $\Sigma \subset \mathcal{S}$. Indeed, at first notice that

$$\tilde{T}_{n+1}|_{([\lambda_*, \Lambda_n] \times \overline{B}_{R_n})} = \tilde{T}_n|_{([\lambda_*, \Lambda_n] \times \overline{B}_{R_n})} = T_n. \quad (5.8)$$

If $(\lambda, u) \in \Sigma$ with $\lambda > \lambda_*$, there is a sequence $(\lambda_n, u_n) \in \cup \Sigma_n^*$ with $(\lambda_n, u_n) \in \Sigma_n^*$ such that $\lambda_n \to \lambda$ and $u_n \to u$. Then $u \in B_{R_N}$ for some integer $N > 1$. 

17
We can assume that \((\lambda_n, u_n) \in [\lambda_s, \Lambda_N] \times B_{R_N}\). On the other hand, by (5.8),
\[
u_n = T_n(\lambda_n, u_n) = T_N(\lambda_n, u_n).
\]
Passing to the limit we get
\[
u = T_N(\lambda, u)
\]
which shows that \((\lambda, u) \in \Sigma_N\) and so
\[(\lambda, u) \in S := \{ (\lambda, u) \in (0, \infty) \times C(\overline{\Omega}) \mid u \text{ is a solution of } (P)_\lambda \}.
\]
This ends the proof of theorem 1.1.

6 Appendix

In this section we present proofs of lemma 3.3, corollary 3.1 and recall some results referred to in the paper. We begin with the Browder-Minty Theorem, (cf. Deimling [6]). Let \(X\) be a real reflexive Banach space with dual space \(X^*\). A map \(F : X \to X^*\) is monotone if
\[
\langle Fx - Fy, x - y \rangle \geq 0, \quad x, y \in X,
\]
\(F\) is hemicontinuous if
\[
F(x + ty) \rightharpoonup Fx \text{ as } t \to 0,
\]
and \(F\) is coercive if
\[
\frac{\langle Fx, x \rangle}{|x|} \to \infty \text{ as } |x| \to \infty.
\]

**Theorem 6.1** Let \(X\) be a real reflexive Banach space and let \(F : X \to X^*\) be a monotone, hemicontinuous and coercive operator. Then \(F(X) = X^*\). Moreover, if \(F\) is strictly monotone then it is a homeomorphism.

The inequality below, (cf. [22], [19]), is very useful when dealing with the \(p\)-Laplacian.

**Lemma 6.1** Let \(p > 1\). Then there is a constant \(C_p > 0\) such that
\[
(|x|^{p-2}x - |y|^{p-2}y, x - y) \geq \begin{cases}
C_p |x - y|^p & \text{if } p \geq 2, \\
\frac{C_p |x - y|^p}{(1 + |x| + |y|)^{p-2}} & \text{if } p \leq 2,
\end{cases}
\]
where \(x, y \in \mathbb{R}^N\) and \((.,.)\) is the usual inner product of \(\mathbb{R}^N\).

The Hardy Inequality (cf. Brézis [3]) is:

**Theorem 6.2** There is a positive constant \(C\) such that
\[
\int_\Omega |\frac{u}{d}|^\beta \, dx \leq C \int_\Omega |\nabla u|^p, \quad u \in W_0^{1,p}(\Omega).
\]
Proof of lemma 3.3 By the H"older inequality,
\[ \int_\Omega |\nabla u|^{p-1} |\nabla v| dx \leq ||u||_{1,p'}||v||_{1,p}, \] (6.2)
where $1/p + 1/p' = 1$, and so the expression
\[ \langle -\Delta_p u, v \rangle := \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v dx, \quad u, v \in W^{1,p}_0(\Omega), \] (6.3)
defines a continuous, bounded (nonlinear) operator namely
\[ \Delta_p : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega) \]
\[ u \mapsto -\Delta_p u. \]

By (6.1), $-\Delta_p$ it is strictly monotone and coercive, that is
\[ \langle -\Delta_p u - (-\Delta_p v), u - v \rangle > 0, \quad u, v \in W^{1,p}_0(\Omega), \quad u \neq v \]
and
\[ \frac{\langle -\Delta_p u, u \rangle}{||u||_{1,p} \rightarrow \infty} \rightarrow \infty. \]

By the Browder-Minty Theorem, $\Delta_p : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is a homeomorphism. Consider
\[ F_g(u) = \int_\Omega gudx, \quad u \in W^{1,p}_0(\Omega). \]

Claim $F_g \in W^{-1,p'}(\Omega)$.

Assume for a while the Claim has been proved. Since $-\Delta_p : W^{1,p}_0(\Omega) \rightarrow W^{-1,p'}(\Omega)$ is a homeomorphism, there is an only $u \in W^{1,p}_0(\Omega)$ such that
\[ -\Delta_p u = F_g, \]
that is
\[ \langle -\Delta_p u, v \rangle = \int_\Omega gvdx, \quad v \in W^{1,p}_0(\Omega) \]

Verification of the Claim. Let $V$ be an open neighborhood of $\partial \Omega$ such that
\[ 0 < d(x) < 1 \quad \text{for} \quad x \in V \quad \text{so that} \]
\[ 1 < \frac{1}{d(x)^\beta} < \frac{1}{d(x)}, \quad x \in V. \]

Now, if $v \in W^{1,p}_0(\Omega)$ we have
\[ |F_g(v)| \leq \int_\Omega |g||v|dx = \int_{V^c} |g||v|dx + \int_V |g||v|dx \leq C||v||_{1,p} + \int_\Omega \frac{|v|}{d} dx. \]

19
Applying the Hardy Inequality in the last term above we get to,

\[ |F_g(v)| \leq C||v||_{1,p}, \]

showing that \( F_g \in W^{-1,p'}(\Omega) \), proving the Claim.

**Regularity of** \( u \): At first we treat the case \( p = 2 \). By [5] there is a solution \( v \) of

\[
\begin{cases}
-\Delta v = \frac{1}{v^\beta} & \text{in } \Omega, \\
v > 0 & \text{in } \Omega, \\
v = 0 & \text{on } \partial\Omega,
\end{cases}
\]

which belongs to \( C^1(\overline{\Omega}) \) and by the Hopf theorem \( \frac{\partial v}{\partial \nu} < 0 \) on \( \partial\Omega \). Since also \( d \in C^1(\overline{\Omega}) \) and \( \frac{\partial d}{\partial \nu} < 0 \) on \( \partial\Omega \) there a constant \( C > 0 \) such that

\[ v \leq Cd \text{ in } \Omega. \]

Moreover,

\[ -\Delta v = \frac{1}{v^\beta} \geq \frac{C}{d^\beta}. \]

Consider the problem

\[
\begin{cases}
-\Delta \tilde{u} = |g| & \text{in } \Omega, \\ 
\tilde{u} = 0 & \text{on } \partial\Omega.
\end{cases}
\]

By [9] theorem B.1,

\[ \tilde{u} \in C^{1,\alpha}(\overline{\Omega}) \text{ and } ||\tilde{u}||_{C^{1,\alpha}(\overline{\Omega})} \leq M_0, \]

for some positive constant \( M_0 \). By the Maximum Principle,

\[ \tilde{u} \leq v \leq Cd \text{ in } \Omega. \]

Setting \( \overline{u} = u + \tilde{u} \) we get

\[ -\Delta \overline{u} = g + |g| \geq 0 \text{ in } \Omega \]

and by the arguments above, \( \overline{u} \leq Cd \text{ in } \Omega \). Thus, as a consequence of [9] theorem B.1, the are \( \alpha \in (0, 1) \) and \( M_0 > 0 \) such that

\[ \overline{u}, \tilde{u} \in C^{1,\alpha}(\overline{\Omega}) \text{ and } ||\overline{u}||_{C^{1,\alpha}(\overline{\Omega})}, ||\tilde{u}||_{C^{1,\alpha}(\overline{\Omega})} \leq M_0, \]

ending the proof of lemma 3.3 in the case \( p = 2 \).

In what follows we treat the case \( p > 1 \). Let \( u \) be a solution of (3.3). It follows that

\[ -\Delta_p u = g \leq \frac{C}{d^\beta} \text{ and } -\Delta_p(-u) = (-1)^{p-1}g \leq \frac{C}{d^\beta}. \]

By lemma 3.2 the problem

\[
\begin{cases}
-\Delta_p v = \frac{C}{v^\beta} & \text{in } \Omega \\
v = 0 & \text{on } \partial\Omega
\end{cases}
\]
admits an only positive solution \( v \in W^{1,p}_0(\Omega) \cap C^{1,\alpha}(\overline{\Omega}) \) for some \( \alpha \in (0,1) \) with \( v \leq Cd \) in \( \Omega \). Hence,
\[
-\Delta_p(v) = \frac{C}{v^\beta} \geq \frac{1}{d^\beta} \text{ in } \Omega.
\]
Therefore,
\[
-\Delta_p |u| \leq \frac{C}{d^\beta} \leq -\Delta_p v.
\]
By the weak comparison principle,
\[
|u| \leq v \leq Cd \text{ in } \Omega,
\]
showing that \( u \in L^\infty(\Omega) \). Pick \( w \in C^{1,\alpha}(\overline{\Omega}) \) such that
\[
-\Delta w = g \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega.
\]
We have
\[
\text{div}(|\nabla u|^{p-2} \nabla u - \nabla w) = 0 \text{ in } \Omega
\]
in the weak sense. By Lieberman [15, theorem 1] the proof of lemma 3.3 ends.

**Proof of Corollary 3.1**

Existence of \( u_\epsilon \) follows directly by lemma 3.3. Moreover there are \( M > 0 \) and \( \alpha \in (0,1) \) such that
\[
||u||_{C^{1,\alpha}(\overline{\Omega})}, \quad ||u_\epsilon||_{C^{1,\alpha}(\overline{\Omega})} < M.
\]
By Vázquez [25, theorem 5], \( \frac{\partial u}{\partial \nu} < 0 \) on \( \partial \Omega \) and recalling that \( d \in C^1(\overline{\Omega}) \) and \( \frac{\partial d}{\partial \nu} < 0 \) on \( \partial \Omega \) it follows that
\[
u \geq Cd \text{ in } \Omega. \quad (6.4)
\]
Multiplying the equation
\[
-\Delta_p u - (-\Delta_p u_\epsilon) = g - \left( h\chi_{[d(x) > \epsilon]} + \tilde{g}\chi_{[d(x) < \epsilon]} \right)
\]
by \( u - u_\epsilon \) and integrating we have
\[
\int_{\Omega} (|\nabla u|^{p-2} \nabla u - |\nabla u_\epsilon|^{p-2} \nabla u_\epsilon) \cdot \nabla (u - u_\epsilon) \, dx \leq 2M \int_{d(x) < \epsilon} |g - \tilde{g}| \, dx.
\]
Using lemma 6.1, we infer that \( ||u - u_\epsilon||_{1,p} \to 0 \) as \( \epsilon \to 0 \). By the compact embedding \( C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega}) \) it follows that
\[
||u - u_\epsilon||_{C^{1,\alpha}(\overline{\Omega})} \leq C d,
\]
and using (6.4),
\[
u_\epsilon \geq u - \frac{C}{2} d \geq u - \frac{u}{2} = u.
\]
References

[1] A. Anane, *Simplicité et isolation de la premiè re valeur propre du p-Lapacien avec poids*, CRAS Paris Série I (1987) 725-728.

[2] L. Boccardo, F. Murat & J. P. Puel, *Résultats d’existence pour certains problèmes elliptiques quasilinéaires*, Annali Scuola Normale Superiore Pisa 2 (1984) 213-235.

[3] H. Brézis, *Functional Analysis, Sobolev Spaces and partial differential equations*. Springer (2011).

[4] D. G. Costa & J.V. Goncalves, *Existence and Multiplicity Results for a Class of Nonlinear Elliptic Boundary Value Problems at Resonance*, J. Math. Anal. Appl. 84 (1981) 328-337.

[5] M. G. Crandall, P. H. Rabinowitz & L. Tartar, *On a Dirichlet problem with a singular nonlinearity*, Comm. Partial Differential Equations 2 (1977) 193-222.

[6] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, (1985).

[7] E. DiBenedetto, *$C^{1+\alpha}$-local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. 7 (1983), 827-850.

[8] M. Ghergu & V. Radulescu, *Sublinear singular elliptic problems with two parameters*, J. Diff. Equations 195 (2003) 520-536.

[9] J. Giacomoni, I. Schindler & P. Takac, *Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 6 (1) (2007) 117-158.

[10] D. Gilbarg & N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, (1983).

[11] J.V. Goncalves, M.C. Rezende & C.A. Santos, *Positive solutions for a mixed and singular quasilinear problem*, Nonlinear Anal. 74 (2011) 132-140.

[12] D.D. Hai, *Singular boundary value problems for the p-Laplacian*, Nonlinear Anal. 73 (2010) 2876-2881.

[13] D. D. Hai, *On a class of singular p-Laplacian boundary value problems*, J. Math. Anal. Appl. 383 (2011) 619-626.

[14] A. C. Lazer & P. J. McKenna, *On a singular nonlinear elliptic boundary value problem*, Proceedings American Mathematical Society 111 (1991) 721-730.

[15] G. M. Liebermann, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. 12 (1988) 1203-1219.
[16] N. H. Loc & K. Schmitt, Boundary value problems for singular elliptic equations, Rocky Mountain J. Math. 41 (2011) 555-572.

[17] A. Mohammed, Positive solutions of the p-Laplace equation with singular nonlinearity, J. Math. Anal. Appl. 352 (2009) 234-245.

[18] M. Montenegro & O. S. de Queiroz, Existence and regularity to an elliptic equation with logarithmic nonlinearity, J. Differential Equations 246 (2009) 482-511.

[19] I. Peral, Multiplicity of Solutions for the p-Laplacian, Second School on Nonlinear Functional Analysis and Applications to Differential Equations - Trieste, Italy, (1997).

[20] K. Perera & Z. Zhang, Multiple positive solutions of singular p-Laplacian problems by variational methods, Boundary Value Problems (2005) 377-382.

[21] J. Shi & M. Yao, On a singular nonlinear semilinear elliptic problem, Proc. Roy. Soc. Edinburgh Sect. A 138 (1998) 1389-1401.

[22] J. Simon, Regularité de la solution d’une equation non linéaire dans $\mathbb{R}^N$, Lecture Notes in Mathematics # 665, Springer-Verlag, (1978).

[23] Jingxian Sun & Fumin Song, A property of connected components and its applications, Topology and its Applications 125 (2002) 553-560.

[24] P. Tolksdorff, Regularity for a more general class of quasilinear elliptic equations, J. Diff. Equations 51 (1984), 126-150.

[25] J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191-202.

[26] Z. Zhang, On a Dirichlet problem with a singular nonlinearity, J. Math. Anal. Appl. 194 (1995) 103-113.