Flat trace statistics of the transfer operator of a random partially expanding map

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Abstract

We consider the skew-product of an expanding map $E$ on the circle $\mathbb{T}$ with an almost surely $C^k$ random perturbation $\tau = \tau_0 + \delta \tau$ of a deterministic function $\tau_0: F : \mathbb{T} \times \mathbb{R} \rightarrow \mathbb{T} \times \mathbb{R}$, $\tau(x) = (E(x), y + \tau(x))$. The associated transfer operator $L: u \in C^k(\mathbb{T} \times \mathbb{R}) \mapsto u \circ F$ can be decomposed with respect to frequency in the $y$ variable into a family of operators acting on functions on the circle: $L_\xi: C^k(\mathbb{T}) \rightarrow C^k(\mathbb{T})$, $u \mapsto e^{i\xi \tau} u \circ E$. We show that the flat traces of $L_n^\xi$ behave as normal distributions in the semi classical limit $n, \xi \rightarrow \infty$ up to the Ehrenfest time $n \lesssim c_1 \log \xi$.

Keywords: partially expanding map, skew product, thermodynamic formalism, random roof function, convergence in law

Mathematics Subject Classification numbers: 37D30 partially hyperbolic systems and dominated splittings, 37E10 Maps of the circle, 60F05 Central limit and other weak theorems, 37C30 Zeta functions, (Ruelle-Frobenius) transfer operators, other functional analytic techniques in dynamical systems.

(Some figures may appear in colour only in the online journal)

1. Introduction

This paper focuses on the distribution of the flat traces of iterates of the transfer operator of a simple example of partially expanding map. It is motivated by the Bohigas–Gianonni–Schmidt [BGS84] conjecture in quantum chaos (see below).

In chaotic dynamics, the transfer operator is an object of first importance linked to the asymptotics of the correlations. The collection of poles of its re solvent, called Ruelle–Pollicott spectrum, can be defined as the spectrum of the transfer operator in appropriate Banach spaces (see [Rue76] for analytic expanding maps, [Kit99], [BKL02], [BT07], [BT08], [GL06], [FRS08] for the construction of the spaces for Anosov diffeomorphisms.)

The study of the Ruelle spectrum for Anosov flows is more difficult because of the flow direction that is neither contracting nor expanding. Dolgopyat has shown in particular in
[Dol98] the exponential decay of correlations for the geodesic flow on negatively curved surfaces, and Liverani [Liv04] generalized this result to all $C^4$ contact Anosov flows. His method involved the construction of anisotropic Banach spaces in which the transfer operator is quasi compact, and no longer relies on symbolic dynamics that prevented from using advantage of the smoothness of the flow. Tsujii [Tsu10] constructed appropriate Hilbert spaces for the transfer operator of $C^r$ contact Anosov flows, $r \geq 3$ and gave explicit upper bounds for the essential spectral radii in terms of $r$ and the expansion constants of the flow. Butterley and Liverani [BL07] and later Faure and Sjöstrand [FS11] constructed good spaces for Anosov flows, without the contact hypothesis. Weich and Bonthondeau defined in [BW17] Ruelle spectrum for geodesic flow on negatively curved manifolds with a finite number of cusps. Dyatlov and Guillarmou [DG16] handled the case of open hyperbolic systems. A simple example of Anosov flow is the suspension of an Anosov diffeomorphism, or the suspension semi-flow of an expanding map. Pollicott showed exponential decay of correlations in this setting under a weak condition in [Pol85] and Tsujii constructed suitable spaces for the transfer operator and gave an upper bound on its essential spectral radius in [Tsu08].

In this article we study a closely related discrete time model, the skew product of an expanding map of the circle. It is a particular case of compact group extension [Dol02], which are partially hyperbolic maps, with compact leaves in the neutral direction that are isometric to each other. Dolgopyat showed in [Dol02] that in this case the correlation decrease generically rapidly. In our setting of skew-product of an expanding map of the circle, Faure [Fau11] has shown using semi-classical methods an upper bound on the essential spectral radius of the transfer operator under a condition shown to be generic by Nakano Tsujii and Wittsten [NTW16]. Arnoldi, Faure, and Weich [AFW17] and Faure and Weich [FW17] studied the case of some open partially expanding maps, iteration function schemes, for which they found an explicit bound on the essential spectral radius of the transfer operator in a suitable space, and obtained a Weyl law (upper bound on the number of Ruelle resonances outside the essential spectral radius). Naud [Nau16] studied a model close to the one presented in this paper, in the analytic setting, in which the transfer operator is trace-class, and used the trace formula, in the deterministic and random case to obtain a lower bound on the spectral radius of the transfer operator. In the more general framework of random dynamical systems in which the transfer operator changes randomly at each iteration, for the skew product of an expanding map of the circle, Nakano and Wittsten [NW15] showed exponential decay of correlations.

Semiclassical analysis describes the link between quantum dynamics and the associated classical dynamics in a symplectic manifold. The transfer operator happens to be a Fourier integral operator and the semiclassical approach has thus shown to be useful. The famous Bohigas–Giannoni–Schmidt [BGS84] conjecture of quantum chaos states that for quantum systems whose associated classical dynamic is chaotic, the spectrum of the Hamiltonian shows the same statistics as that of a random matrix (GUE, GOE or GSE according to the symmetries of the system) (see also [Gut13] and [GVZJ91]). We are interested analogously in investigating the possible links between the Ruelle–Pollicott spectrum and the spectrum of random matrices/operators. At first we try to get informations about the spectrum using a trace formula. More useful results could follow from the use of a global normal form as obtained by Faure–Weich in [FW17].

### 1.1. Expanding map

Let us consider a smooth orientation preserving expanding map $E : T \to T$ on the circle $T = \mathbb{R}/\mathbb{Z}$, that is, satisfying $E' > 1$, of degree $l$, and let us call

$$m := \inf E' > 1$$
and
\[ M := \sup E'. \]

### 1.2. Transfer operator

Let us fix a function \( \tau \in C^k(T) \) for some \( k \geq 0 \). We are interested in the partially expanding dynamical system on \( T \times \mathbb{R} \) defined by

\[
F(x, y) = (E(x), y + \tau(x))\
\]

We introduce the transfer operator

\[
\mathcal{L}_\tau : \left\{ C^k(T \times \mathbb{R}) \rightarrow C^k(T \times \mathbb{R}) \quad u \mapsto u \circ F \right\}.
\]

### 1.3. Reduction of the transfer operator

Due to the particular form of the map \( F \), the Fourier modes in \( y \) are invariant under \( \mathcal{L}_\tau \): if for some \( \xi \in \mathbb{R} \) and some \( v \in C^k(T) \),

\[
u(x, y) = v(x)e^{i\xi y},
\]

then

\[
\mathcal{L}_\tau u(x, y) = e^{i\xi\tau(x)}v(E(x))e^{i\xi y}.
\]

Given \( \xi \geq 0 \) and a function \( \tau \), let us consequently consider the transfer operator \( \mathcal{L}_{\xi, \tau} \) defined on functions \( v \in C^k(T) \) by

\[
\forall x \in T, \quad \mathcal{L}_{\xi, \tau} v(x) := e^{i\xi\tau(x)}v(E(x)).
\]

### 1.4. Spectrum and flat trace

In appropriate spaces, the transfer operator has a discrete spectrum outside a small disk, the eigenvalues are called Ruelle resonances. It is in general not trace-class, but one can define its flat trace (see appendix C for a more precise discussion about Ruelle resonances, flat trace and their relationship).

**Lemma 1.1** (Trace formula, [AB67], [G77]). For any \( C^0 \) function \( \tau \) on \( T \), the flat trace of \( \mathcal{L}_{\xi, \tau}^n \) is well defined and

\[
\text{Tr}^\flat \mathcal{L}_{\xi, \tau}^n = \sum_{x, E^k(x) = x} \frac{e^{i\xi\tau^n_x}}{(E^\xi y(x) - \Gamma)}.
\]

where \( \tau^n_x \) denotes the Birkhoff sum: For a function \( \phi \in C(T) \) and a point \( x \in T \) we define

\[
\phi^n_x := \sum_{k=0}^{n-1} \phi(E^k(x)).
\]
1.5. Gaussian random fields

We define our random functions on the circle by means of their Fourier coefficients. We are only interested in $C^0$ functions. We will denote by $\mathcal{N}(0, \sigma^2)$ (respectively $\mathcal{N}_C(0, \sigma^2)$) the real (respectively complex) centered Gaussian law of variance $\sigma^2$, with respective densities
\[
\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} \quad \text{and} \quad \frac{1}{\sigma\pi} e^{-\frac{|z|^2}{\sigma^2}}.
\]

With these conventions, a random variable of law $\mathcal{N}_C(0, \sigma^2)$ has independent real and imaginary parts of law $\mathcal{N}(0, \frac{\sigma^2}{2})$, and the variance of its modulus is consequently $\sigma^2$.

**Definition 1.2.** We will call centered stationary Gaussian random fields on $\mathbb{T}$ the real random distributions $\tau$ whose Fourier coefficients $\left(\hat{c}_p(\tau)\right)_{p \geq 1}$ are independent complex centered Gaussian random variables, with variances growing at most polynomially, such that $c_0(\tau)$ is a real centered Gaussian variable independent of the $c_p(\tau), p \geq 1$. The negative coefficients are necessarily given by
\[
c_{-p}(\tau) = \overline{c_p(\tau)}.
\]

The Gaussian fields are in general defined as distributions if their Fourier coefficients have variances with polynomial growth and the decay of the variances of the coefficients gives sufficient conditions for the regularity of the field.

**Lemma 1.3.** If $\mathbb{E}[|c_p(\tau)|^2]$ has a polynomial growth, $\tau = \sum_p c_p(\tau) e^{2i\pi p}$ defines almost surely a distribution: almost surely
\[
\forall \phi = \sum_p c_p(\phi) e^{2i\pi p} \in \mathcal{C}^\infty(\mathbb{T}), \quad \langle \tau, \phi \rangle := \sum_p \overline{c_p(\tau)} c_p(\phi) < \infty.
\]

Let $k \in \mathbb{N}$. If for some $\eta > 0$
\[
\mathbb{E} \left[ |c_p(\tau)|^2 \right] = O \left( \frac{1}{p^{2k+2+\eta}} \right),
\]
(1.4)

Then $\tau$ is almost surely $\mathcal{O}^k$.

**Proof.** See appendix B.

In what follows we will always assume that (1.4) is satisfied, at least for $k = 0$, so that our random fields are random variables on $\mathcal{C}^0(\mathbb{T})$. This will ensure the existence of flat traces.

1.6. Result

If $x$ is a periodic point, let us write its prime period
\[
I_x := \min \{ k \geq 1, \mathbb{E}^k(x) = x \}.
\]

Let us define for every $n \in \mathbb{N}$:
\[
A_n := \left( \sum_{\mathbb{E}^n(x) = x} \frac{I_x}{((\mathbb{E}^n)(x) - 1)^2} \right)^{-\frac{1}{2}}.
\]
(1.5)
Theorem 1.4. Let $k \in \mathbb{N}$. Let $\tau_0 \in C^k(\mathbb{T})$. Let

$$\delta \tau = \sum_{p \in \mathbb{Z}} c_p e^{2i\pi p}$$

be a centered Gaussian random field, such that $\mathbb{E}[|c_p|^2] = O(p^{-2-\nu})$ for some $\nu > 0$. This way, $\delta \tau$ is as $C^0$. If

$$\exists \epsilon > 0, \ \exists C > 0, \ \forall p \in \mathbb{Z}^*, \ \mathbb{E}\left[|c_p|^2\right] \geq \frac{C}{p^{2k+2+\epsilon}},$$

(1.6)

then one has the convergence in law of the flat traces

$$A_n \text{Tr}^d \left( L_{\xi_0 + \delta \tau}^n \right) \rightarrow \mathcal{N}_\mathbb{C}(0, 1)$$

(1.7)

as $n$ and $\xi$ go to infinity, under the constraint

$$\exists 0 < c < 1, \ \forall n, \xi, \ n \leq c \frac{\log \xi}{\log \left(l + (k + \frac{1}{2} + \frac{\epsilon}{2}) \log M\right)}$$

(1.8)

Note that condition (1.6) can allow $\tau$ to be $C^k$ by lemma 1.3.

Remark 1.5. The statement implies that the convergence still holds if we multiply $\delta \tau$ by an arbitrarily small number $\eta > 0$. For instance for $\tau_0 = 0$,

$$A_n \text{Tr}^d \left( L_{\xi_0}^n \right) \rightarrow \infty$$

at exponential speed, uniformly in $\xi$, but if $\delta \tau$ is an irregular enough Gaussian field in the sense of (1.6), then for any $\eta > 0$ and $c < 1$ holds

$$A_n \text{Tr}^d \left( L_{\xi_0 + \delta \tau}^n \right) \rightarrow \mathcal{N}_\mathbb{C}(0, 1)$$

under condition (1.8).

Remark 1.6. Condition (1.8) means that time $n$ is smaller than a constant times the Ehrenfest time $\log \xi$, and this constant decreases with the regularity $k$ of the field $\delta \tau$.

1.7. Sketch of proof

The proof is based around the following arguments:

(a) Note first that the convergence (1.7) is satisfied if all the phases appearing in (1.2) are independent and uniformly distributed.

Remark 1.7. For sake of simplicity, in this sketch of proof, we will state pairwise independence for the phases in (1.2), while in fact we must pack them by orbits, since Birkhoff sums $\phi^k_n$ are the same on all the orbit, but this changes little to the problem. For instance this
simplification would remove the factor $l_s$ in the definition (3.22) of $A_n$ corresponding to this multiplicity.

The convergence can be deduced from the standard proof of the central limit theorem showing point wise convergence of the characteristic function. However, here, since the periodic points are dense in $\mathbb{T}$, requiring independence of the values $(\delta \tau(x))_{x \in T} = 1$ would lead to very bad regularity of the field (it is not hard to see that it would be almost surely nowhere locally bounded).

(b) We fix a Gaussian field $\delta \tau = \sum c_p e^{2\pi i p \cdot}$ fulfilling the hypothesis of theorem 1.4 and start by constructing an auxiliary field with the same law and show that it satisfies the convergence (1.7). This is sufficient since the convergence in law only involves the law of the random field.

(c) For each $j \geq 1$, we construct a smooth random field $\delta \tau_j$, such that for any pair of periodic points $x \neq y$ of period $j$, $\delta \tau_j(x)$ and $\delta \tau_j(y)$ are independent. Since by (1.2) $\text{Tr}(L^\xi_{\delta \tau_j})$ only involves points of period $n$, the phases appearing at time $n$, for the function $\delta \tau_n$, in $L^\xi_{\delta \tau_n}$ are consequently all independent random variables on $S^1$. If moreover $\xi$ is large enough, the variables $\xi(\delta \tau_n)$ are Gaussian with large variances, so $\xi(\delta \tau_n) \mod 2\pi$ (and therefore the phases $e^{i(\delta \tau_n)}$) are close to be uniform. Thus, the convergence (1.7) should hold for $\text{Tr}(L^\xi_{\delta \tau_n})$ under a certain relation between $n$ and $\xi$ that will be explained in number [8].

(d) An important point is that if the phases $(e^{i(\delta \tau_n)})_{x \in \mathbb{T}}$ are independent and close to be uniform, then adding to $\delta \tau_n$ an independent field will not change this fact, as the following lemma suggests:

**Lemma 1.8.** Let $X, X'$ be real independent random variables such that $e^{iX}, e^{iX'}$ are uniform on $S^1$. Let $Y, Y'$ be real random variables such that $X$ and $X'$ are independent of both $Y$ and $Y'$. Then $e^{i(X+Y)}$ and $e^{i(X'+Y')}$ are still independent uniform random variables on $S^1$.

Note that no independence between $Y$ and $Y'$ is needed. See appendix D for the proof.

(e) Using this analogy, if the fields $\delta \tau_j$ are chosen independent, it should follow that the convergence (1.7) holds for $\text{Tr}(\sum_{j \geq 1} e^{i\delta \tau_j})$ for large $\xi$.

(f) The fields $\delta \tau_j$ are almost surely smooth. However, because the distance between periodic points decreases as $M^{-j}$ according to lemma A.1, if we want to be sure that $\sum_j \delta \tau_j$ is $C^2$, and $E[\delta \tau_j(x) \delta \tau_j(y)] = 0$ for all $x \neq y$ of period $j$, let us see that we need to impose an exponential decay of the standard deviation (independent of the point $x$):

$$\sqrt{E[(\delta \tau_j(x))^2]} \approx M^{-j(k+{1/2}+\varepsilon)}$$

for some $\varepsilon > 0$. This can be deduced heuristically from the fact (see definition 3.1 below) that

$$E[\delta \tau_j(x) \delta \tau_j(y)] = \sum_p E[|c_p(\delta \tau_j)|^2] e^{2i\pi \varepsilon} =: K_j(x-y)$$

and the uncertainty principle: a localisation of $K_j$ at a scale $M^{-j}$ implies non negligible coefficients $E[|c_p(\delta \tau_j)|^2]$ for $p$ of order $M^j$. Let us for instance assume that the Fourier
coefficients $E[|c_p(\delta \tau_j)|^2]$ of $K_j$ write

$$E[|c_p(\delta \tau_j)|^2] = \alpha_j^2 f \left( \frac{p}{M_j} \right)^2$$

(1.11)

for some amplitudes $\alpha_j$ to determine and some positive Schwartz function $f : \mathbb{R} \to \mathbb{R}$. Then, since

$$\delta \tau_j = \sum_p \sqrt{E[|c_p(\delta \tau_j)|^2]} \zeta_{jp} e^{2\pi ip}$$

for i.i.d. $\mathcal{N}(0, 1)$ random variables $\zeta_{jp}$, roughly,

$$\sup |\delta \tau_j^{(k)}| \approx \alpha_j \sum_p |p|^k f \left( \frac{p}{M_j} \right)$$

$$= \alpha_j M_j^{k+1} \frac{1}{M_j} \sum_p |p|^k M_j^k f \left( \frac{p}{M_j} \right)$$

$$\sim C\alpha_j M_j^{k+1}.$$

(The second line involved a Riemann sum.) Consequently, with those approximations, choosing $\alpha_j = M_j^{-\kappa(k+1)+\varepsilon}$ gives a $C^k$ function $\sum_{j \geq 1} \delta \tau_j$. Then,

$$E[|\delta \tau_j(x)|^2] = \sum_p \frac{E[|c_p(\delta \tau_j)|^2]}{1 + \rho_p}$$

$$= \sum_p \alpha_j^2 f \left( \frac{p}{M_j} \right)^2$$

$$= \alpha_j^2 M_j^{k+1} \sum_p \frac{1}{M_j} f \left( \frac{p}{M_j} \right)^2$$

$$\sim C\alpha_j^2 M_j^{k+1} = M_j^{-\kappa(k+1)+2\varepsilon}$$

as announced.

(g) This condition, together with (1.6) can easily be shown to imply that the Fourier coefficients $\tilde{c}_p$ of $\sum_{j \geq 1} \delta \tau_j$ satisfy

$$E[|\tilde{c}_p|^2] \leq CE[|c_p|^2].$$

This allows us to define a field $\delta \tau_0$, that we chose independent from the other $\delta \tau_j$, by

$$E[|c_p(\delta \tau_0)|^2] = CE[|c_p|^2] - E[|\tilde{c}_p|^2],$$

so that $\frac{1}{\xi} \sum_{j \geq 1} \delta \tau_j$ has the same law as $\delta \tau$ and still satisfies the convergence (1.7) for $\xi$ large enough from (d) of this sketch.
(h) To get an idea of the origin of the relation (1.8) between \( n \) and \( \xi \), let us assume that we want all the arguments \( \xi(\delta \tau_n)_n \) to go uniformly to infinity in order to get approximate uniformity of the phases and thus convergence towards a Gaussian law. Note that for any \( x \),

\[
P \left[ \frac{|(\delta \tau_n)_n|}{\sqrt{E[|\delta \tau_n|_n^2]}} \leq \epsilon \right] = O(\epsilon). \tag{1.12}
\]

Let \((C_n)\) be a sequence going to infinity. (1.12) implies

\[
P \left[ \bigcap_{E^\mu(x) = x} \left\{ \xi(\delta \tau_n)_n > C_n \right\} \right] = 1 - \mathbb{P} \left[ \exists x, E^\mu(x) = x, \xi(\delta \tau_n)_n \leq C_n \right] \geq 1 - \sum_{E^\mu(x) = x} \mathbb{P} \left[ \xi(\delta \tau_n)_n \leq C_n \right] = 1 - (\mu - 1)\mathbb{P} \left[ \xi(\delta \tau_n)_n \leq C_n \right] \geq \frac{C_n}{1 - (\mu - 1)\mathbb{P} \left[ \xi(\delta \tau_n)_n \leq C_n \right]} \tag{1.12}
\]

if \( x \) denotes any point and \( \xi \gg \frac{C_n}{\sqrt{E[|\delta \tau_n|_n^2]}} \). By independence

\[
\sqrt{E[|\delta \tau_n|_n^2]} = \left( \sum_{k=0}^{n-1} E[|\delta \tau_n(E^\mu(x))]^2 \right)^{\frac{1}{2}} \approx \sqrt{nM^{k + \frac{1}{2} + \varepsilon}} \tag{1.9}
\]

for some \( \varepsilon > 0 \). Thus

\[
P \left[ \xi(\delta \tau_n)_n \rightarrow \infty \right. \text{ uniformly w.r.t.} \ x \ s.t. \ E^\mu(x) = x \left. \right] \rightarrow 1
\]

for \( \xi \gg \mu M^{k + \frac{1}{2}} \), which gives (1.8).

2. Numerical experiments

We consider an example with the nonlinear expanding map

\[
E(x) = 2x + 0.9/(2\pi) \sin(2\pi(x + 0.4)) \tag{2.1}
\]

plotted on figure 1. In figure 2, we have the histogram of the modulus \( S = |A_n \text{Tr}^\mu \left( \mathcal{L}_{\xi,\eta + \delta \tau} \right)| \) obtained after a sample of \( 10^4 \) random functions \( \delta \tau \). We compare the histogram with the
function $\text{CSexp}(-S^2)$ in red, i.e. the radial distribution of a Gaussian function, obtained from the prediction of theorem 1.4. We took $n = 11, \xi = 2 \times 10^6, \tau_0 = \cos(2\pi x)$. We also observe a good agreement for the (uniform) distribution of the arguments that is not represented here.
3. Proof of theorem 1.4

A stationary centered Gaussian random field is characterized by its covariance function:

**Definition 3.1.** Let \( \tau = \sum_{p \in \mathbb{Z}} c_p e^{2i\pi p \cdot} \) be a stationary centered Gaussian random field, satisfying

\[
\mathbb{E}[|c_p|^2] = O \left( \frac{1}{p^2 + \eta} \right)
\]

for some \( \eta > 0 \), so that \( \tau \) is almost surely \( C^0 \) according to lemma 1.3. Let us define its covariance function \( K \) by

\[
K(x) := \sum_p \mathbb{E}[|c_p|^2] e^{2i\pi px}.
\]

For any pair of points \((x, y) \in T^2\), we have

\[
\mathbb{E}[\tau(x)\tau(y)] = K(x - y).
\]

**Proof of the last statement.** Remark from appendix B that the condition \( \mathbb{E}[|c_p|^2] = O \left( \frac{1}{p^2 + \eta} \right) \) implies that \( \tau \) is almost surely equal to its Fourier series. Thus,

\[
\mathbb{E}[\tau(x)\tau(y)] = \sum_{p,q \in \mathbb{Z}} \mathbb{E}[|c_p|^2] e^{2i\pi px} e^{2i\pi qy} + \mathbb{E}[|c_p|^2] e^{2i\pi (px + qy)}
\]

from the independence relationships of the Fourier coefficients. Now,

\[
\mathbb{E}[|c_p|^2] = \mathbb{E}[(\text{Re}(c_p))^2] - \mathbb{E}[(\text{Im}(c_p))^2] + 2\mathbb{E}[(\text{Re}(c_p))\text{Im}(c_p)] = 0.
\]

\[\square\]

3.1. Definition of a Gaussian field satisfying theorem 1.4

Let us fix a random centered Gaussian field \( \delta \tau = \sum_{p \in \mathbb{Z}} c_p e^{2i\pi p \cdot} \) satisfying the hypothesis of theorem 1.4. Let us define the Gaussian fields mentioned in step (c) of the sketch of proof. Let \( K_{\text{init}} \in C^\infty_c(\mathbb{R}) \) be a smooth function supported in \([-\frac{1}{3}, \frac{1}{3}]\), with nonnegative Fourier transform, satisfying

\[
K_{\text{init}}(0) = 1.
\]

Let \( k \geq 0 \) be the integer involved in theorem 1.4 giving the regularity of the field. Let \( \epsilon > 0 \) be the constant appearing in theorem 1.4 and define for any integer \( j \geq 1 \)

\[
K_j(x) = \frac{1}{M^{2k+1+j}} K_{\text{init}}(M^j x).
\]

\[1\] To construct such a function, take a non zero even function \( g \in C^\infty_c(\mathbb{R}) \). \( g \) has a real Fourier transform. Then \( g * g \in C^\infty_c(\mathbb{R}) \) and its Fourier transform is \( \hat{g}^2 \geq 0 \). Moreover \( g * g(0) = \int \hat{g}^2 > 0 \).
The Fourier transform of $K_j$ is given by
\[
\hat{K}_j(\xi) = \frac{1}{M^{k+2+\epsilon}} \hat{K}_{\text{init}} \left( \frac{\xi}{M^j} \right) \geq 0
\] (3.5)

The functions $K_j$, for all $j \geq 1$, are supported in $[-\frac{1}{3}, \frac{1}{3}]$ and can then be seen as functions on the circle $\mathbb{T}$ by trivially periodizing them. Let $c_{p,j}$, for $p \geq 0, j \geq 1$ be independent centered Gaussian random variables of respective variances $\hat{K}_j(2\pi p)$, and let us write
\[
\delta \tau_j = \sum_{p} c_{p,j} e^{2i\pi p \cdot \cdot}.
\]

where $c_{-p,j} = c_{p,j}, p \geq 1$. Note that, since $K_j$ is smooth for all $j$, the variances $\hat{K}_j(2\pi p)$ of $c_{p,j}$ decay rapidly with $p$ (for fixed $j$), and therefore, each $\delta \tau_j$ is almost surely smooth by lemma B.1.

**Lemma 3.2.** \( \sum_{j \geq 1} \delta \tau_j \) is a centered Gaussian random field \( \sum \tilde{c}_{p} e^{2i\pi p} \) and
\[
\mathbb{E} \left[ |\tilde{c}_{p}|^2 \right] = O \left( \mathbb{E} \left[ |c_p|^2 \right] \right).
\]

**Proof.** We have seen in equation (3.5) that
\[
\hat{K}_j(\xi) = \frac{1}{M^{k+2+\epsilon}} \hat{K}_{\text{init}} \left( \frac{\xi}{M^j} \right).
\]

Since $K_{\text{init}}$ is smooth, there exists a constant $C > 0$ such that
\[
\forall \xi \in \mathbb{R} \quad \hat{K}_{\text{init}}(\xi) \leq \frac{C}{\langle \xi \rangle^{2k+2+\epsilon}}.
\]

with the usual notation $\langle \xi \rangle = \sqrt{1 + \xi^2} \geq |\xi|$. Thus,
\[
\mathbb{E} \left[ |c_{p,j}|^2 \right] = \frac{1}{M^{k+2+\epsilon}} \hat{K}_{\text{init}} \left( \frac{2\pi p}{M^j} \right) \leq C \frac{1}{M^{2+2+\epsilon}} \frac{1}{|2\pi p|^{2k+2+\epsilon}}.
\]

Consequently, since by independence
\[
\mathbb{E} \left[ |\tilde{c}_{p}|^2 \right] = \mathbb{E} \left[ \sum_{j \geq 1} c_{p,j} \right]^2 = \sum_{j \geq 1} \mathbb{E} \left[ |c_{p,j}|^2 \right],
\]
\[
\mathbb{E} \left[ |\tilde{c}_{p}|^2 \right] = O \left( \frac{1}{|2\pi p|^{2k+2+\epsilon}} \right) = O(\mathbb{E} \left[ |c_p|^2 \right]).
\]

Thus, fixing a constant $C$ such that
\[
CE \left[ |c_p|^2 \right] \geq \mathbb{E} \left[ |\tilde{c}_{p}|^2 \right],
\]

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we can define a random Gaussian field \( \delta \tau_0 = \sum_{p \in \mathbb{Z}} c_p \delta \tau_0 e^{2i\pi p} \) with coefficients \( c_{p,0} \) independent from the \( c_{p,j} \) such that
\[
\mathbb{E} \left[ |c_{p,0}|^2 \right] = C \mathbb{E} \left[ |c_p|^2 \right] - \mathbb{E} \left[ |c_p|^2 \right].
\]
This way \( \frac{1}{C} \sum_{j \geq 0} \delta \tau_j \) and \( \delta \tau \) have the same law. By this we mean that their Fourier coefficients have the same laws. By our hypothesis, the convergence of the Fourier series are almost surely normal, thus for any finite subset \( \{ x_k \} \) of \( T \), \( \left( \frac{1}{C} \sum_{j \geq 0} \delta \tau_j(x_k) \right) \) and \( (\delta \tau(x_k))_k \) have the same law.

Therefore, the laws of \( \text{Tr}^\flat \left( L^n_{\xi, \tau_0 + \delta \tau} \right) \) and \( \text{Tr}^\flat \left( L^n_{\xi, \tau_0 + \frac{1}{C} \sum \delta \tau_j} \right) \) are the same, and the convergence of theorem 1.4 is equivalent to
\[
A_n \text{Tr}^\flat \left( L^n_{\xi, \tau_0 + \delta \tau} \right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1) \tag{3.6}
\]
under condition (1.6). (The constant \( \frac{1}{C} \) can be ‘absorbed’ in \( \xi \) up to the replacement of \( \tau_0 \) by \( C \tau_0 \) that has no consequence.) In the rest of the paper we will show (3.6) and will write
\[
\tau := \tau_0 + \sum_{j \geq 0} \delta \tau_j \tag{3.7}
\]

3.2. New expression for \( \text{Tr}^\flat \left( L^n_{\xi, \tau} \right) \)

We will write the set of periodic orbits of (non primitive) period \( n \) as
\[
\text{Per}(n) := \left\{ \{ x, E(x), \ldots, E^{n-1}(x) \} \mid E^n(x) = x, x \in T \right\}, \tag{3.8}
\]
and the set of periodic orbits of primitive period \( n \) as
\[
\mathcal{P}_n := \left\{ \{ x, E(x), \ldots, E^{n-1}(x) \} \mid n = \min \{ k \in \mathbb{N}^\ast, E^k(x) = x \}, \quad x \in T \right\}. \tag{3.9}
\]
This way, \( \text{Per}(n) \) is the disjoint union
\[
\text{Per}(n) = \coprod_{m|n} \mathcal{P}_m. \tag{3.10}
\]
Let us rewrite the sum \( \text{Tr}^\flat \left( L^n_{\xi, \tau} \right) \), where \( \tau \) is given by (3.7). We know from (1.2) that
\[
\text{Tr}^\flat \left( L^n_{\xi, \tau} \right) = \sum_{E^n(x) = x} \frac{e^{i\xi \tau}}{(E^n)'(x) - 1}
\]
\[
= \sum_{E^n(x) = x} \frac{e^{J^n(x)}}{e^{J_{\mathcal{O}}^p} - 1},
\]
where \( J(x) = \log(E'(x)) > 0 \) and \( J_{\mathcal{O}}^p \) is the Birkhoff sum as defined in (1.3). If \( f_{\mathcal{O}}^n \) stands for the Birkhoff sum \( f_{\mathcal{O}}^{\mathcal{F}} \) for any \( x \in \mathcal{O} \), let us write
\[
\text{Tr}^\flat \left( L^n_{\xi, \tau} \right) = \sum_{m|n} m \sum_{\delta \in \mathcal{P}_m} \frac{e^{i\xi \tau_0}}{e^{J_{\mathcal{O}}^p} - 1}. \tag{3.11}
\]

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For $O \in \text{Per}(n)$, we can write
\[
\tau^n_O = (\delta\tau^n_O) + \sum_{j \neq n} (\delta\tau^n_j) + (\tau^n_0).
\]

Since the covariance function $K_n$ is supported in $[-\frac{1}{4m}, \frac{1}{4m}]$, we deduce from lemma A.1 and (3.2) that the values taken by $\delta\tau_n$ at different periodic points of period dividing $n$, which have law $\mathcal{N}(0, K_n(0))$ are independent random variables. Thus, for $n \in \mathbb{N}$, $m|n$ and $O \in \mathcal{P}_m$, $(\delta\tau^n_O) + (\tau^n_0)$ has variance $\left(\frac{m}{m}K_n(0) = mK_n(0)$.

Definition 3.3. We say that two families of real random variables $(X^n_O)_{n \geq 1}$ and $(Y^n_O)_{n \geq 1}$ satisfy condition (C) if
\[
\text{for every } m|n, O \in \mathcal{P}_m, X^n_O \text{ has law } \mathcal{N}(0, n^2mK_n(0)).
\]
\[
\text{for every } O' \neq O \in \text{Per}(n) \text{ and } O'' \in \text{Per}(n), X^n_O \text{ is independent of } X^n_{O'} \text{ and } Y^n_{O''}.
\]

Writing $X^n_O = (\delta\tau^n_O)$ and $Y^n_O = \sum_{j \neq n} (\delta\tau^n_j) + (\tau^n_0)$, we have obtained

Lemma 3.4. There exist families of random variables $(X^n_O)_{n \geq 1}$ and $(Y^n_O)_{n \geq 1}$ satisfying condition (C) of definition (3.3) such that for every $n \geq 1$ and $O \in \text{Per}(n)$
\[
\tau^n_O = X^n_O + Y^n_O. \quad (3.12)
\]

In order to adapt the proof of lemma 1.8, we want to show that for large $\xi$, the random variables $e^{i\xi(X^n_O + Y^n_O)}$, $O \in \text{Per}(n)$ are close to be independent and uniform on $S^1$.

Remark 3.5. We have
\[
\text{Tr}^i(L^n_{\xi,r}) = \sum_{m|n} \prod_{O \in \mathcal{P}_m} \frac{e^{i\xi(X^n_O + Y^n_O)}}{e^{r} - 1}. \quad (3.13)
\]

Our aim is to approximate the characteristic function of $A_n\text{Tr}^i(L^n_{\xi,r})$ which is the expectation of
\[
\exp \left(iA_n \left(\mu \text{Re}(\text{Tr}^i(L^n_{\xi,r})) + \nu \text{Im}(\text{Tr}^i(L^n_{\xi,r}))\right) \right)
\]
\[
= \prod_{m|n} \prod_{O \in \mathcal{P}_m} \exp \left[ -\frac{mA_n}{e^{r} - 1} \left(\mu \cos(\xi(X^n_O + Y^n_O)) + \nu \sin(\xi(X^n_O + Y^n_O))\right) \right] \quad (3.14)
\]

for fixed, $\mu, \nu \in \mathbb{R}$. The right-hand side of (3.14) can be written as
\[
\prod_{m|n} \prod_{O \in \mathcal{P}_m} \left(f_O \left(e^{i\xi(X^n_O + Y^n_O)}\right)\right)^m
\]
for some continuous functions \( f_O : S^1 \rightarrow \mathbb{C} \) (depending on \( \mu, \nu \)):

\[
f_O(z) = \exp \left[ i \frac{A_n}{e^{\mu} - 1} (\mu \text{Re}(z) + \nu \text{Im}(z)) \right].
\]

In the next lemma we first consider indicator functions on \( S^1 \) for \( f_O \).

**Lemma 3.6.** Let \( (X^O_n)_{n \geq 1} \) and \( (Y^O_n)_{n \geq 1} \) be two families of real random variables satisfying condition (C) of definition (3.3). Assume that \( n \) and \( \xi \) satisfy (1.8). Then there is a constant \( C > 0 \) such that for every \( n \in \mathbb{N} \) and every real numbers \( (\alpha^O)_{\in \text{Per}(n)}, (\beta^O)_{\in \text{Per}(n)} \) such that

\[
\forall O \in \text{Per}(n), \quad 0 < \beta^O - \alpha^O < 2\pi,
\]

for every complex numbers \( (\lambda^O_o)_{O \in \text{Per}(n)} \), \( A^O \) is a \( S^1 \subset \mathbb{C} \) and \( \mathbb{I}_{A^O} : S^1 \rightarrow \mathbb{C} \) is the characteristic function of \( A^O \), we have

\[
\left| \mathbb{E} \left[ \prod_{m|n} \prod_{O \in \text{Per}(m)} \lambda^O_o \mathbb{I}_{A^O} \left( e^{i(\lambda^O_o + y^O_n)} \right)^m \right] \prod_{m|n} \prod_{O \in \text{Per}(m)} \lambda^O_o \left( \frac{2\alpha^O - \beta^O}{2\pi} \right) - 1 \right| \leq C^{-1} n \xi \left( \frac{1}{t} \right).
\]

**Remark 3.7.** In this expression, we compare the law of the family of random variables \( \left( e^{i(\lambda^O_o + y^O_n)} \right)_{O \in \text{Per}(n)} \) to the uniform law on the torus of dimension \( \#\text{Per}(n) \). The proof of this lemma is given in the next subsection.

### 3.3. A normal law of large variance on the circle is close to uniform

We will need the following lemma, which evaluates how much the law \( N(0, 1)_{\text{mod } 1/2} \) differs from the uniform law on the circle \( \mathbb{R}/(1/2 \mathbb{Z}) \) for large values of \( t \) (figure 3).

**Lemma 3.8.** There exists a constant \( C > 0 \) such that for every real numbers \( \alpha, \beta \), such that \( 0 < \beta - \alpha < 2\pi \) and every real number \( t \geq 1 \),

\[
\left| \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbb{I}_{\alpha + 2\pi k \leq x \leq \beta + 2\pi k} e^{-x^2 \frac{1}{2\pi}} - e^{-y^2 \frac{1}{2\pi}} \right| dx \leq \frac{C}{t} (\beta - \alpha).
\]

**Proof.** By mean value inequality, if \( |x - y| \leq 1 \), then

\[
\left| e^{-x^2} - e^{-y^2} \right| \leq |x - y| f(y)
\]
Figure 3. As $t$ goes to infinity, the red area converges to $\frac{\beta - \alpha}{t}$ with speed $O\left(\frac{\alpha \beta}{t} t\right)$.

for the $L^1$ function

$$f(y) := \sup_{|u-y| \leq 1} |u|e^{-\frac{u^2}{2}}.$$

Let us then write for $u \in [\frac{\alpha}{t}, \frac{\beta}{t}]$, $u_k := u + \frac{2k\pi}{t}$ and $I_k := [u_k, u_{k+1}]$. We have just seen that for $t \geq 2\pi$, for all $y \in I_k$,

$$\left| e^{-\frac{u^2}{2}} - e^{-\frac{u_k^2}{2}} \right| \leq C f(y).$$

Integrating over $y \in I_k$ of length $\frac{2\pi}{t}$ and summing over $k \in \mathbb{Z}$ yields

$$\left| \frac{2\pi}{t} \sum_{k \in \mathbb{Z}} e^{-\frac{u_k^2}{2}} - \sqrt{2\pi} \right| \leq C.$$

(The value of the constant $C$ changes at each line, but it depends neither on $t$, nor on $\alpha, \beta$.) Averaging over $u \in [\frac{\alpha}{t}, \frac{\beta}{t}]$ gives

$$\left| \frac{2\pi}{\beta - \alpha} \sum_{k \in \mathbb{Z}} \int_{\frac{\alpha}{t}}^{\frac{\beta}{t}} \exp\left(-\frac{(u-2k\pi)^2}{2}\right) du - \sqrt{2\pi} \right| \leq C.$$
Consequently,
\[
\left| \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} \mathbb{1}_{\alpha + 2k \pi \leq x \leq \beta + 2k \pi} e^{\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} - \frac{\beta - \alpha}{2\pi} \right| \leq \frac{C}{I} (\beta - \alpha).
\]

\[\Box\]

**Proof of lemma 3.6.** Let us denote by \( E \) the expectation
\[
E := E \left[ \prod_{m/n \in \mathcal{P}_m} \lambda_{O} \mathbb{1}_{A_{O}} \left( e^{j\lambda_{O} x_{O} + \lambda_{O} y_{O}} \right) \right]^{m}.
\]
If we write respectively \( \mathbb{P}_X \), \( \mathbb{P}_Y \) and \( \mathbb{P}_{X,Y} \) the probability laws of the variables \((\xi^a_{O})|_{O \in \text{Per}(n)}, (\xi^b_{O})|_{O \in \text{Per}(n)}\) and \((\xi^a_{O})|_{O \in \text{Per}(n)} \cup (\xi^b_{O})|_{O \in \text{Per}(n)}\) respectively, then condition (C) of definition (3.3) implies
\[
d\mathbb{P}_{X,Y}|_{(\xi^a_{O})|_{O \in \text{Per}(n)}, (\xi^b_{O})|_{O \in \text{Per}(n)}} = \prod_{m/n \in \mathcal{P}_m} e^{-\frac{\alpha^2}{\sigma_{n,\xi}}} \frac{dx_{O}}{\sigma_{n,\xi} \sqrt{2\pi}} \otimes d\mathbb{P}_Y|_{(\xi^b_{O})|_{O \in \text{Per}(n)}}.
\]
with the variance \( \sigma_{n,\xi}^2 := \xi^2 \frac{m}{n} K_{\alpha}(0) \). We have
\[
E = \int_{\mathbb{R}^{\#\text{Per}(n)}} \prod_{O \in \text{Per}(n)} \left( \sum_{k \in \mathbb{Z}} \lambda_{O}^{m} \mathbb{1}_{\alpha + 2k \pi \leq x_{O} + 2k \pi + \beta + 2k \pi} (x_{O} + \beta + y_{O}) \right) d\mathbb{P}_{X,Y}|_{(\xi^a_{O})|_{O \in \text{Per}(n)}, (\xi^b_{O})|_{O \in \text{Per}(n)}}.
\]
Thus, writing \( u_{O} = \frac{\alpha_{O}}{\sigma_{n,\xi}} \) for \( O \in \mathcal{P}_m \),
\[
E = \int_{\mathbb{R}^{\#\text{Per}(n)}} \prod_{m/n \in \mathcal{P}_m} \prod_{O \in \text{Per}(n)} \left( \int_{\mathbb{R}} \lambda_{O}^{m} \mathbb{1}_{\alpha + 2k \pi \leq x_{O} + 2k \pi + \beta + 2k \pi} (u_{O}) e^{-\frac{\alpha_{O}^2}{2\sigma_{n,\xi}^2}} d\mu_{O} \right) d\mathbb{P}_Y|_{(\xi^b_{O})|_{O \in \text{Per}(n)}}.
\]
Let us write for \( O \in \text{Per}(n) \)
\[
I_{O} = \int_{\mathbb{R}} \lambda_{O}^{m} \mathbb{1}_{\alpha + 2k \pi \leq x_{O} + 2k \pi + \beta + 2k \pi} (u_{O}) e^{-\frac{\alpha_{O}^2}{2\sigma_{n,\xi}^2}} d\mu_{O},
\]
Lemma 3.8 yields
\[
I_{O} = \lambda_{O}^{m} \frac{\beta_{O} - \alpha_{O}}{2\pi} (1 + \epsilon_{O}),
\]
where
\[
\exists C > 0, |\epsilon_{O}| \leq \frac{C}{\sigma_{n,\xi}} \leq \frac{C}{\xi \sqrt{nK_{\alpha}(0)}}.
\]
Let us remark that for every finite family \( \{x_k\}_{k \subset \mathbb{R}} \), the expansion of the product and factorization after triangular inequality give

\[
\left| \prod_k (1 + x_k) - 1 \right| \leq \prod_k (1 + |x_k|) - 1.
\]

Thus,

\[
\left| \prod_{m,n \in \mathbb{Z}} \prod_{O \in P_m} I_O \left( \frac{2\pi - 2\alpha}{2\pi} \right) \right| = \left| \prod_{m,n \in \mathbb{Z}} \prod_{O \in P_m} (1 + \epsilon_O) - 1 \right|
\]

\[
\leq \left( 1 + \frac{C}{\xi(nK_a(0))^{1/2}} \right)^{\#\text{Per}(n)} - 1.
\]

From lemma A.1 we have \( \#\text{Per}(n) \leq \ell^n \).

Using hypothesis (1.8) we can bound the pre factor:

\[
\left( 1 + \frac{C}{\xi(nK_a(0))^{1/2}} \right)^{\ell^n} - 1 \leq \exp \left( \ell^n \frac{CM^{\rho(\frac{1}{2} + \frac{\xi}{\sqrt{n}})} - \xi \sqrt{n} \right) - 1
\]

\[
\leq C' \ell^n \frac{CM^{\rho(k + \frac{1}{2} + \frac{\xi}{\sqrt{n}})}}{\xi \sqrt{n}}
\]

for some \( C' > 0 \) for \( n \) and \( \xi \) large enough and satisfying (1.8) since

\[
\ell^n \frac{CM^{\rho(\frac{1}{2} + \frac{\xi}{\sqrt{n}})}}{\xi \sqrt{n}} \leq C' C^{-1} n^{-\frac{1}{2}} \nrightarrow 0.
\] (3.19)

\[\square\]

3.4. End of proof

We can now easily extend the lemma 3.6 from characteristic functions to step functions.

**Corollary 3.9.** Assume that \( n \) and \( \xi \) satisfy (1.8). For any families \( (X^n_O)_{n \geq 1} \) and \( (Y^n_O)_{n \geq 1} \) of real random variables satisfying condition (C) of definition (3.3), there exists
\[ C > 0 \text{ such that, if } (f_{n,O})_{n \geq 1} \text{ is a family of step functions } S^1 \to \mathbb{R}, \text{ then} \]
\[
\mathbb{E}\left[ \prod_{m/n} \prod_{O \in \mathcal{P}_m} f_{n,O}^m(e^{i(X^O_n + Y^O_n)}) \right] - \prod_{m/n} \prod_{O \in \mathcal{P}_m} \int f_{n,O}^m d\text{Leb} \leq C \xi^{-1} n^{-\frac{1}{2}} \prod_{m/n} \prod_{O \in \mathcal{P}_m} \int |f_{n,O}^m| d\text{Leb}.
\]

\[(3.20)\]

**Proof.** Let us write each \( f_{n,O} \) as
\[
f_{n,O} = \sum_{q=1}^{p_{n,O}} \lambda_{n,O,q} \mathbb{I}_{A_{n,O,q}},
\]
where the \( \lambda_{n,O,q} \) are complex numbers and the \( A_{n,O,q}, 1 \leq q \leq p_{n,O} \) are disjoint intervals. We develop (3.20), we use lemma 3.6 and factorize the result:
\[
E := \mathbb{E}\left[ \prod_{m/n} \prod_{O \in \mathcal{P}_m} f_{n,O}^m(e^{i(X^O_n + Y^O_n)}) \right] = \sum_{(q_O) \in \prod_{m/n} \prod_{O \in \mathcal{P}_m} \{1, \ldots, p_{n,O}\}} \mathbb{E}\left[ \prod_{m/n} \prod_{O \in \mathcal{P}_m} \lambda_{n,O,q_O} \mathbb{I}_{A_{n,O,q_O}}(e^{i(X^O_n + Y^O_n)}) \right].
\]

Consequently,
\[
\mathbb{E}\left[ \prod_{m/n} \prod_{O \in \mathcal{P}_m} \int f_{n,O}^m d\text{Leb} \right] \leq \sum_{(q_O) \in \prod_{m/n} \prod_{O \in \mathcal{P}_m} \{1, \ldots, p_{n,O}\}} \mathbb{E}\left[ \prod_{m/n} \prod_{O \in \mathcal{P}_m} \lambda_{n,O,q_O} \mathbb{I}_{A_{n,O,q_O}}(e^{i(X^O_n + Y^O_n)}) \right] - \prod_{m/n} \prod_{O \in \mathcal{P}_m} \lambda_{n,O,q_O} \text{Leb}(A_{n,O,q_O})
\]
\[
\leq C \xi^{-1} n^{-\frac{1}{2}} \sum_{(q_O) \in \prod_{m/n} \prod_{O \in \mathcal{P}_m} \{1, \ldots, p_{n,O}\}} \prod_{m/n} \prod_{O \in \mathcal{P}_m} |\lambda_{n,O,q_O}|^m \text{Leb}(A_{n,O,q_O})
\]
from the previous lemma.

Hence,
\[
\mathbb{E}\left[ \prod_{m/n} \prod_{O \in \mathcal{P}_m} \int f_{n,O}^m d\text{Leb} \right] \leq C \xi^{-1} n^{-\frac{1}{2}} \prod_{m/n} \prod_{O \in \mathcal{P}_m} \int |f_{n,O}^m| d\text{Leb}.
\]

We can use this result in order to estimate the characteristic function of \( \text{Tr}^\flat(L^n_{\xi, \tau}) \), using remark (3.5).
Corollary 3.10. Assume that $n$ and $\xi$ satisfy (1.8). Let $(X^n_O)_{n \geq 1}$ and $(Y^n_O)_{n \geq 1}$ be two families of real random variables satisfying condition (C) of definition (3.3). There exists $C > 0$ such that for all $(\mu_O, \nu_O) \in \mathbb{R}^2 \# \text{Per}(n)$,

$$\left| \mathbb{E} \left[ \prod_{m/n \in \mathcal{P}_n} e^{im\mu_O \cos(\xi(X^n_O + Y^n_O) + im\nu_O \sin(\xi(X^n_O + Y^n_O)))} \right] - \prod_{m/n \in \mathcal{P}_n} \int_0^{2\pi} e^{im\mu_O \cos \theta + im\nu_O \sin \theta} \frac{d\theta}{2\pi} \right| \leq C \xi^{-1} n^{-\frac{1}{2}}.$$  

Proof. Let $C$ be the constant from corollary 3.9. For $O \in \text{Per}(n)$, let $f_O$ be the function defined on $S^1$ by

$$f_O(e^{i\theta}) = e^{i\mu_O \cos \theta + i\nu_O \sin \theta}.$$  

Each $f_O$ is bounded by 1, we can consequently find for each $O \in \text{Per}(n)$ a family $(f_j, O)$ of step functions uniformly bounded by 1 converging point wise towards $f_O$. We have for $n$ fixed, by dominated convergence

$$E_j := \mathbb{E} \left[ \prod_{m/n \in \mathcal{P}_n} f_j^m(e^{i(\xi(X^n_O + Y^n_O))}) \right] \rightarrow E := \mathbb{E} \left[ \prod_{m/n \in \mathcal{P}_n} f_j^m(e^{i(\xi(X^n_O + Y^n_O))}) \right]$$

as well as

$$I_j := \prod_{m/n \in \mathcal{P}_n} \int_0^{2\pi} f_j^m(e^{i\theta}) \frac{d\theta}{2\pi} \rightarrow I := \prod_{m/n \in \mathcal{P}_n} \int_0^{2\pi} f_j^m(e^{i\theta}) \frac{d\theta}{2\pi}.$$  

It is thus possible to find an integer $j_0$ such that both

$$|E - E_j| \leq C \xi^{-1} n^{-\frac{1}{2}}$$

and

$$|I - I_j| \leq C \xi^{-1} n^{-\frac{1}{2}}$$

hold.

From corollary 3.9, we know that for all $n \in \mathbb{N}$

$$|E_{j_0} - I_{j_0}| \leq C \xi^{-1} n^{-\frac{1}{2}} \sup |f_{j_0}|.$$  

Thus,

$$|E - I| \leq |E - E_{j_0}| + |E_{j_0} - I_{j_0}| + |I - I_{j_0}| \leq (C + 2) \xi^{-1} n^{-\frac{1}{2}}.$$  

We can now prove the final proposition:
Proposition 3.11. Let \((X^n_0)_{n \geq 1}^{\infty}\) and \((Y^n_0)_{n \geq 1}^{\infty}\) be two families of real random variables satisfying condition (C) of definition (3.3). If condition (1.8) is satisfied then we have the following convergence in law

\[
T_{n,\xi} := A_n \sum_{m/n} m \sum_{O \in P_m} \frac{e^{i\xi (X^n_0 + Y^n_0)}}{\partial^n_0 - 1} \xrightarrow{n \xi \to \infty} \mathcal{N}(0, 1),
\]

with the amplitude \(A_n\) defined in (1.5) by

\[
A_n = \left( \sum_{m/n} m^2 \sum_{O \in P_m} \frac{1}{(\partial^n_0 - 1)^2} \right)^{-\frac{1}{2}}.
\]

Proof. Let us fix two real numbers \(\xi_1\) and \(\xi_2\) and let \(\phi_n\) be the characteristic function of \(T_{n,\xi}^n\):

\[
\phi_n(\xi_1, \xi_2) := \mathbb{E} \left[ \exp \left( i A_n \left( \xi_1 \sum_{m/n} m \sum_{O \in P_m} \cos(\xi (X^n_0 + Y^n_0)) \frac{\partial^n_0 - 1}{\partial^n_0} + \xi_2 \sum_{m/n} m \sum_{O \in P_m} \sin(\xi (X^n_0 + Y^n_0)) \frac{\partial^n_0 - 1}{\partial^n_0} \right) \right) \right].
\]

We compute the limit of \(\phi_n(\xi_1, \xi_2)\) as \(n\) goes to infinity. Corollary (3.10) yields

\[
\left| \phi_n(\xi_1, \xi_2) - \prod_{m/n} \prod_{O \in P_m} \int_0^{2\pi} e^{i \frac{\xi_1 \cos \theta + \xi_2 \sin \theta}{\partial^n_0 - 1}} \frac{d\theta}{2\pi} \right| \leq C \xi c^{-1} n^{-\frac{1}{2}} \to 0 \quad (3.23)
\]

under the assumption (1.8).

Let

\[
\psi(\xi_1, \xi_2) := \int_0^{2\pi} e^{i \xi_1 \cos \theta + \xi_2 \sin \theta} \frac{d\theta}{2\pi}.
\]

We have the following Taylor’s expansion in 0:

\[
\psi(\xi_1, \xi_2) = 1 - \frac{1}{4}(\xi_1^2 + \xi_2^2) + o(\xi_1^2 + \xi_2^2).
\]

In order to apply this to equation (3.23), we need to check that

Lemma 3.12.

\[
n A_n \sup_{O \in P(n)} \frac{1}{\partial^n_0 - 1} \xrightarrow{n \to \infty} 0.
\]

Proof. See appendix E.3
Figure 4. Histogram of $S = \left| A_n \text{Tr}^2 \left( \mathcal{L}_{t_0}^a \right) \right|$ for a sample of $10^4$ random values of $\xi$ uniformly distributed in $[\xi_0, \xi_0 + 10]$ with $\xi_0 = 2 \times 10^6$ and $n = 11$ corresponding to a fraction of the Ehrenfest time $C_x := n \log_2 \xi_0 = 0.5$. It is well fitted by the red curve $S \mapsto C \exp(-S^2)$.

We can now state that

$$\prod_{m/n} \prod_{O \in \mathcal{P}_m} \int_0^{2\pi} e^{i \frac{m}{n} \left( \xi_1 \cos \theta + \xi_2 \sin \theta \right)} \frac{d\theta}{2\pi} = \prod_{m/n} \prod_{O \in \mathcal{P}_m} \psi \left( \frac{\xi_1 mA_n}{e^\xi_0 - 1}, \frac{\xi_2 mA_n}{e^\xi_0 - 1} \right)$$

$$= \prod_{m/n} \prod_{O \in \mathcal{P}_m} \left( 1 - \frac{\xi_1^2 + \xi_2^2}{4} \frac{(mA_n)^2}{(e^\xi_0 - 1)^2} + o \left( \frac{(mA_n)^2}{(e^\xi_0 - 1)^2} \right) \right)$$

$$= \exp \left( \sum_{m/n} \sum_{O \in \mathcal{P}_m} \log \left( 1 - \frac{\xi_1^2 + \xi_2^2}{4} \frac{(mA_n)^2}{(e^\xi_0 - 1)^2} + o \left( \frac{(mA_n)^2}{(e^\xi_0 - 1)^2} \right) \right) \right)$$

$$= \exp \left( \sum_{m/n} \sum_{O \in \mathcal{P}_m} -\frac{\xi_1^2 + \xi_2^2}{4} \frac{(mA_n)^2}{(e^\xi_0 - 1)^2} + o \left( \frac{(mA_n)^2}{(e^\xi_0 - 1)^2} \right) \right)$$

$$= \exp \left( \text{3.22} \right) + o(1).$$

We deduce that

$$\phi_n(\xi_1, \xi_2) \xrightarrow{n \to \infty} e^{-\frac{\xi_1^2 + \xi_2^2}{4}}$$
which is the characteristic function of a Gaussian variable of law $\mathcal{N}(0, 1)$. □

4. Discussion

In this paper we have considered a model where the roof function $\tau$ is random. However, the numerical experiments suggest a far stronger result: for a fixed function $\tau$ and a semi classical parameter $\xi$ chosen according to a uniform random distribution in a small window at high frequencies, the result seems to remain true, as shown in the following figures for $\tau(x) = \sin(2\pi x)$. The moduli also seem to become uniform (figure 4).

It would be interesting to understand what informations about the Ruelle resonances can be recovered from the convergence (1.7). We know from the Weyl law from [AFW17] established in a similar context that the number of resonances of $L_{\xi, \tau}$ outside the essential spectral radius, for a given $\tau$, are of order $O(\xi)$. A complete characterization would thus require a knowledge of the traces of $L_{\xi, \tau}^n$ up to times of order $O(\xi)$, while we only have information for $n = O(\log \xi)$ (figure 5).

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Appendix A. Proof of lemma A.1

Lemma A.1. For every integer \( n \), \( E^n \) has \( n - 1 \) fixed points. The distance between two distinct periodic points is bounded from below by \( \frac{1}{M_{n-1}} \).

Proof. \( E \) is topologically conjugated to the linear expanding map of same degree \( x \mapsto lx \mod 1 \), (see [KH97], p 73). Thus \( E^n \) has \( n - 1 \) fixed points. Let \( \tilde{E} : \mathbb{R} \to \mathbb{R} \) be a lift of \( E \), \( x \neq y \) be two fixed points of \( E^n \) and \( \tilde{x}, \tilde{y} \in \mathbb{R} \) be representatives of \( x \) and \( y \) respectively. Note that

\[
d(x, y) = \inf |\tilde{x} - \tilde{y}|
\]

where the infimum is taken over all couples of representatives \( (\tilde{x}, \tilde{y}) \). Since \( E^n(x) = x \) and \( E^n(y) = y \), \( \tilde{E}^n(\tilde{y}) - \tilde{E}^n(\tilde{x}) - (\tilde{y} - \tilde{x}) \) is an integer, different from 0 because \( \tilde{E}^n \) is expanding. Thus,

\[
|\tilde{E}^n(\tilde{y}) - \tilde{E}^n(\tilde{x}) - (\tilde{y} - \tilde{x})| \geq 1,
\]

that is

\[
\left| \int_{\tilde{x}}^{\tilde{y}} \left( \tilde{E}^n(y) - 1 \right) dy \right| \geq 1.
\]

Finally,

\[
|\tilde{y} - \tilde{x}|(M_{n-1}) \geq 1.
\]

Taking the infimum gives the result. \( \square \)

Appendix B. Proof of lemma 1.3 on the link between regularity of a Gaussian field and variance of the Fourier coefficients

Let us recall the following classical estimate:

Lemma B.1. If \( (X_p)_{p \in \mathbb{Z}} \) is a family of independent centered Gaussian random variables of variance 1, then, almost surely,

\[
\forall \delta > 0, \quad X_p = o(p^\delta).
\]

Proof. Let \( \delta > 0 \). Let us use Borel–Cantelli lemma:

\[
\forall p \in \mathbb{Z}, \quad P(|X_p| > p^\delta) = \int_{|x| > p^\delta} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}}
\]

Now, we have the upper bound

\[
p^\delta \int_{p^\delta}^{+\infty} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} \leq \int_{p^\delta}^{+\infty} xe^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = \frac{e^{-\frac{p^2\delta}{2}}}{\sqrt{2\pi}}.
\]

Thus,

\[
\forall p \in \mathbb{Z}^+, \quad P(|X_p| > p^\delta) \leq \frac{2}{\sqrt{2\pi}p^\delta} e^{-\frac{p^2\delta}{2}}.
\]
Consequently,
\[ \sum_p \mathbb{P}(|X_p| > p^\delta) < \infty \]
and by Borel–Cantelli, almost surely,
\[ \# \{ p \in \mathbb{Z} : |X_p| > p^\delta \} < \infty. \]

With this in mind, we can see that if a real random function \( \tau \) has random Fourier coefficients \((c_p)_{p \in \mathbb{Z}}\), pairwise independent (for non-negative values of \( p \)), with variance
\[ \sigma_p^2 := \mathbb{E}[|c_p|^2] = O \left( \frac{1}{p^{k+\eta}} \right), \]
for some \( \eta > 0 \), then by the previous lemma, almost surely, for all \( \delta > 0 \),
\[ \frac{c_p}{\sigma_p} = o(p^\delta), \]
and thus for \( \delta = \frac{\eta}{2} \),
\[ c_p = O \left( \frac{1}{p^{k+\frac{\eta}{2}}} \right) \text{ a.s.} \]
As a consequence,
\[ \sum_p c_p (2i\pi p)^k e^{2i\pi px} \]
converges normally and thus \( \tau \) is almost surely \( C^k \).

\textbf{Appendix C. Ruelle resonances and flat trace}

\textbf{C.1. Ruelle spectrum}

If \( \tau \in C^k(\mathbb{T}) \), the operator \( \mathcal{L}_{\xi,\tau} \) can be extended to distributions \((C^k(\mathbb{T}))'\) by duality. We will denote \( H^s(\mathbb{T}) \) the Sobolev space of order \( s \in \mathbb{R} \).

\textbf{Theorem C.1} ([Rue86], [Bal18] theorem 2.15 and lemma 2.16). Let \( k \geq 1 \). If \( \tau \) belongs to \( C^k \), then for every \( 0 \leq s < k \), \( \mathcal{L}_{\xi,\tau} : H^{-s}(\mathbb{T}) \to H^{-s}(\mathbb{T}) \) is bounded and its essential spectral radius \( r_{ess} \) satisfies
\[ r_{ess} \leq e^{\Pr \left( -\frac{1}{2} J \right)} \left( \frac{1}{m^s} \right), \]
where \( m = \inf E', J(x) = \log E'(x) \) and \( \Pr \left( -\frac{1}{2} J \right) \) is defined in \textbf{E.1}. 

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The discrete set of eigenvalues of finite multiplicities outside a given disk of radius \( r \geq \frac{p}{2} \), and the associated Eigen spaces remain the same in every space \( H^{-s}(\mathbb{T}) \) for \( s' \geq s \). This can be deduced for example from the fact that these spectral elements give the asymptotic behaviour of the correlation functions: for any smooth functions \( f, g \) on \( \mathbb{T} \), for any \( s \) large enough, if \( \mathcal{L}_{\xi, \tau} : H^{-s}(\mathbb{T}) \to H^{-s}(\mathbb{T}) \) has no eigenvalue of modulus \( r \),

\[
\int \mathcal{L}_{\xi, \tau}^nf \cdot g = \sum_{\lambda \in \sigma(\mathcal{L}_{\xi, \tau})} \left\{ \lim_{\nu \to \infty} (\nu^\rho g) \right\} + O_{n \to \infty}(\nu^\rho),
\]

where \( \Pi_\lambda \) is the spectral projector associated to \( \lambda \). We are interested in the statistical properties of these eigenvalues, called Ruelle–Pollicott spectrum or Ruelle resonances, when \( \tau \) is a random function. One way to get informations about the spectrum of such operators is using a trace formula. Although \( \mathcal{L}_{\xi, \tau} \) is not trace-class, we can give a certain sense to the trace of \( \mathcal{L}_{\xi, \tau} \).

### C.2. Flat trace

This section is an adaptation of section 3.2.2 in [Bal18] In order to motivate the definition of flat trace, let us first recall the following fact:

**Lemma C.2** Let \( m > \frac{1}{2} \). (Then the Dirac distributions belong to \( H^{-m}(\mathbb{T}) \)). If \( T : H^{-m}(\mathbb{T}) \to H^m(\mathbb{T}) \) is a bounded operator, then it has a continuous Schwartz kernel \( K \) and

\[
K(x, y) = \langle \delta_x, T\delta_y \rangle.
\]

If moreover \( T \) is class-trace, then

\[
\text{Tr } T = \int_{\mathbb{T}} K(x, x)dx.
\]

Let \( \rho \) be a smooth compactly supported function such that \( \int_{\mathbb{R}} \rho = 1 \). For \( \epsilon > 0 \) and \( y \in \mathbb{T} \) we write

\[
\rho_{\epsilon, y}(t) = \frac{1}{\epsilon} \rho \left( \frac{t - y}{\epsilon} \right).
\]

Per iodizing this function gives rise to a smooth function \( \rho_{\epsilon, y} \) on \( \mathbb{T} \) satisfying

\[
\rho_{\epsilon, y} \xrightarrow{\epsilon \to 0} \delta_y
\]

as distributions.

**Definition C.3.** Let \( m \geq 0 \) and \( T : H^{-m}(\mathbb{T}) \to H^{-m}(\mathbb{T}) \) be a bounded operator extending to a continuous operator \( (C^0(\mathbb{T}))' \to (C^0(\mathbb{T}))' \). Then the formula

\[
K_\epsilon(x, y) := \langle \rho_{\epsilon, x}, T \delta_y \rangle
\]

defines for every \( \epsilon > 0 \) a continuous function on \( \mathbb{T}^2 \). Let
\[ \text{Tr}_\epsilon^\flat (T) := \int_T K_\epsilon(x,x) \, dx. \]

We say that \( T \) admits a flat trace \( \text{Tr}_\epsilon^\flat (T) \) if \( \text{Tr}_\epsilon^\flat (T) \to \text{Tr}_\epsilon^\flat (T) \) as \( \epsilon \) goes to zero, independently of the choice of the mollifying function \( \rho \).

Note that, for any \( n \in \mathbb{N}^*, \xi \in \mathbb{R}, \tau \in \mathcal{C}^0(\mathbb{T}) \), the transfer operator \( \mathcal{L}^n_{\xi,\tau} : (\mathcal{C}^0(\mathbb{T}))' \to (\mathcal{C}^0(\mathbb{T}))' \) is bounded.

**Lemma C.4** (Trace formula, [AB67], [G77]). Let \( \tau \in \mathcal{C}^k(\mathbb{T}), k \geq 0 \). For any integer \( n \geq 1 \), \( \mathcal{L}^n_{\xi,\tau} \) has a flat trace

\[ \text{Tr}_\epsilon^\flat \mathcal{L}^n_{\xi,\tau} = \sum_{x, E^n(y)=x} \frac{e^{i\xi E^n(y)}}{(E^n(y))'} \cdot (C.2) \]

**Proof.**

\[ \text{Tr}_\epsilon^\flat (\mathcal{L}^n_{\xi,\tau}) = \int_\mathbb{T} \langle \rho_{\epsilon,x}, \mathcal{L}^n_{\xi,\tau} \delta_x \rangle \, dx. \]

By definition of the action of \( \mathcal{L}^n_{\xi,\tau} \) on distributions,

\[ \langle \rho_{\epsilon,x}, \mathcal{L}^n_{\xi,\tau} \delta_x \rangle = (\mathcal{L}^n_{\xi,\tau})^* \rho_{\epsilon,x}(x), \]

where \( (\mathcal{L}^n_{\xi,\tau})^* \) is the \( L^2 \)-adjoint of \( \mathcal{L}^n_{\xi,\tau} \). Let us recall that, if \( \phi : \mathbb{T} \to \mathbb{T} \) is a local diffeomorphism, for every continuous functions \( u, v \) on \( \mathbb{T} \),

\[ \int u(\phi(y))v(y) \, dy = \int u(x) \sum_{\phi(y)=x} \frac{v(y)}{|\phi'(y)|} \, dx. \quad (C.3) \]

Thus,

\[ (\mathcal{L}^n_{\xi,\tau})^* v(x) = \sum_{E^n(y)=x} \frac{v(y)e^{i\xi E^n(y)}}{(E^n(y))'}. \]

Therefore

\[ \text{Tr}_\epsilon^\flat (\mathcal{L}^n_{\xi,\tau}) = \int_\mathbb{T} (\mathcal{L}^n_{\xi,\tau})^* \rho_{\epsilon,x}(x) \, dx \]

\[ = \int \sum_{E^n(y)=x} \frac{\rho_{\epsilon,y}(y - E^n(y))e^{i\xi E^n(y)}}{(E^n(y))'} \, dx \]

\[ = \int \rho_{\epsilon,y}(y - E^n(y))e^{i\xi E^n(y)} \, dy \]

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by the change of variables \(x = E^0(y)\). Now, since \(E\) is expansive, \(y \mapsto y - E^0(y)\) is a local diffeomorphism, so applying (C.3) once again gives

\[
\Tr^j \left( \mathcal{L}_{\xi,\tau}^n \right) = \int T \rho_{\epsilon,0}(z) \sum_{y \in E^0(y) = z} \frac{e^{i\xi^\tau}}{(\mathcal{E}^0(y) - 1)} \, dz \\
\xrightarrow{\epsilon \to 0} \sum_{y \in E^0(y)} \frac{e^{i\xi^\tau}}{(\mathcal{E}^0(y) - 1)}.
\]

\[\square\]

If \(E\) and \(\tau\) are analytic, it is well known that \(\mathcal{L}\) is trace-class and that \(\Tr^j \left( \mathcal{L}_{\xi,\tau} \right) = \Tr \left( \mathcal{L}_{\xi,\tau} \right)\) (see for instance [Jéz17]). In the smooth setting however the decay of the Ruelle–Pollicott spectrum can be arbitrarily slow ([Jéz17], proposition 1.10). The flat trace is however related to the Ruelle–Pollicott spectrum defined above in the following way (This is a consequence of theorem 3.5 in [Bal18] and theorem 2.4 in [Jéz17]):

**Proposition C.5.** Assume that \(\tau \in C^k(T)\) for some \(k \geq 1\). Let \(\xi \in \mathbb{R}, 0 \leq s < k, \text{ and } r > \frac{s}{m} \) be such that \(\mathcal{L}_{\xi,\tau} : H^{-s}(T) \to H^{-s}(T)\) has no eigenvalue of modulus \(r\), then

\[
\exists C > 0, \quad \forall n \in \mathbb{N}, \quad \left| \Tr^j \mathcal{L}_{\xi,\tau}^n - \sum_{\lambda \in \sigma(\mathcal{L}_{\xi,\tau})} \lambda^n \right| \leq Cr^n,
\]

where the eigenvalues are counted with multiplicity.

**Appendix D. Proof of lemma 1.8**

**Proof.** Let \(X, X', Y, \text{ and } Y'\) be as in the statement of the lemma real random variables such that \(e^{iX}, e^{iX'}\) are uniform on \(S^1\) and so that \(X \text{ and } X'\) are both independent of all three other random variables. Let us write \(P_Z\) the law of a random variable \(Z\). To show that \(e^{i(X+Y)}\) and \(e^{i(X'+Y')}\) are independent and uniform on \(S^1\), it suffices to show that for any continuous functions \(f, g : S^1 \to \mathbb{R}\),

\[
\mathbb{E} \left[ f(e^{i(X+Y)}) g(e^{i(X'+Y')}) \right] = \int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i\theta'}) \frac{d\theta}{2\pi} \frac{d\theta'}{2\pi}.
\]

\[
\mathbb{E} \left[ f(e^{i(X+Y)}) g(e^{i(X'+Y')}) \right] = \int_{S^1} f(e^{i\theta}) g(e^{i\theta'}) \text{d}P_{(X,Y,X',Y')}(x,y,x',y').
\]

By hypothesis,

\[
\text{d}P_{(X,Y,X',Y')}(x,y,x',y') = \frac{dx \, dx' \, dy' \, dy'}{2\pi} \text{d}P_{(Y,Y)}(y,y').
\]

Thus,
\[
E \left[ f(e^{i(X+Y)})g(e^{i(X'+Y')}) \right] = \int_{(S^1)^2} \left( \int_0^{2\pi} \int_0^{2\pi} f(e^{i(e^{\theta} + \theta')}) g(e^{i(e^{\theta'} + \theta')}) \frac{dx \, d\theta'}{2\pi} \right) dP(Y, Y')(y, y') \noalign{\vskip2.5pt} = \int_0^{2\pi} \int_0^{2\pi} f(e^{i\theta}) g(e^{i\theta'}) \frac{d\theta \, d\theta'}{2\pi}.
\]

□

Appendix E. Topological pressure

E.1. Definition

Definition E.1. Let \( \phi : T \to \mathbb{R} \) be a Hölder-continuous function. The limit

\[
\Pr(\phi) := \lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{E^n(x)=x} e^{n\phi} \right)
\]

exists and is called the topological pressure of \( \phi \) (see [KH97] proposition 20.3.3 p 630).

In other words

\[
\sum_{E^n(x)=x} e^{n\phi} = e^{n\Pr(\phi) + o(n)}.
\]

The particular case \( \phi = 0 \) gives the topological entropy \( \Pr(0) = h_{\text{top}} \).

Remark E.2. Note that the expression \( e^{n\Pr(\phi) + o(n)} \) describes a large class of sequences, since for instance for any \( k \in \mathbb{N} \),

\[
n^k e^{n\Pr(\phi)} = e^{n\Pr(\phi) + o(n)}.
\]

E.2. Variational principle

Another definition of the pressure is given by the variational principle. Let us denote by \( h(\mu) \) the entropy of a measure \( \mu \) invariant under \( E \) (see [KH97] section 4.3 for a definition of entropy). For the next theorem, see [KH97], sections 20.2 and 20.3. The last sentence comes from proposition 20.3.10.

Theorem E.3 (Variational principle). Let \( \phi : T \to \mathbb{R} \) be a Hölder function.

\[
\Pr(\phi) = \sup_{\mu \text{ E-invariant}} \left( \int \phi \, d\mu + h(\mu) \right).
\]

This supremum, taken over the invariant probability measures, is moreover attained for a unique \( E \)-invariant measure \( \mu \), called equilibrium measure. In addition, if we note \( J = \log E' \) and \( \mu_{\beta} \) the equilibrium measure of \( -\beta J \), \( \beta \mapsto \mu_{\beta} \) is one-to-one.
Corollary E.4. The function
\[
F : \left\{ \begin{array}{c}
\mathbb{R}_+^* \to \mathbb{R} \\
\beta \mapsto \frac{1}{\beta} \Pr(-\beta J)
\end{array} \right. \tag{E.3}
\]
is strictly decreasing.

Proof. Let \(\beta' > \beta > 0\). By the previous theorem, with the same notations,
\[
\int -\beta J \, d\mu_\beta + h(\mu_\beta) > \int -\beta' J \, d\mu_{\beta'} + h(\mu_{\beta'})
\]
and thus
\[
F(\beta) = \int -J \, d\mu_\beta + \frac{h(\mu_\beta)}{\beta} > \int -J \, d\mu_{\beta'} + \frac{h(\mu_{\beta'})}{\beta'} \geq \int -J \, d\mu_{\beta'} + \frac{h(\mu_{\beta'})}{\beta'} = F(\beta').
\]
\[\Box\]

E.3. Proof of lemma 3.12

Let \(\phi : \mathbb{T} \to \mathbb{R}\) be a \(C^1\) function. Let as before \(\phi^n\) be the Birkhoff sum (1.3). By sub additivity of the sequence \(\left\{\inf_{x \in \mathbb{T}} \phi^n\right\}\) and Fekete’s lemma we can define the following quantity:

Definition E.5. Let us define
\[
\phi_{\min} := \lim_{n \to \infty} \inf_{x \in \mathbb{T}} \frac{1}{n} \phi^n.
\tag{E.4}
\]

Lemma E.6. The infimum in (E.4) can be taken over periodic points:
\[
\phi_{\min} = \lim_{n \to \infty} \inf_{x \in \mathbb{T}, E^n(x) = x} \frac{1}{n} \phi^n.
\tag{E.5}
\]

Proof. By lifting the expanding map to \(\mathbb{R}\), we easily see that \(E\) has at least a fixed point \(x_0\). This point has \(n\) pre images by \(E^n\), defining \(P^n - 1\) intervals \(I_k^n\) such that for all \(1 \leq k \leq P^n - 1\)
\[
E^n : I_k^n \to \mathbb{T} \setminus \{x_0\}
\]
is a diffeomorphism. Thus, there exists \(C > 0\) such that for all \(k\), if \(x, y \in I_k^n\),
\[
\forall 0 \leq j \leq n, \quad d(E^j(x), E^j(y)) \leq \frac{C}{m^{n-j}}.
\]
with \(m = \inf |E'| > 1\). Each \(I_k^n\) contains moreover a periodic point \(y_{k,n}\) of period \(n\) given by \(E^n(y_{k,n}) = y_{k,n} + k\). Hence let \(n \in \mathbb{N}\), let \(x_n \in \mathbb{T}\) be such that
\[
\phi^n_{x_n} = \inf_{x \in \mathbb{T}} \phi^n_x,
\]
and suppose that \(x_n \in I_k^n\). We have
\[ |\phi_{\beta,n}^n - \phi_n| = \left| \sum_{j=0}^{n-1} \phi(E^j(x_n)) - \phi(E^j(y_{\lambda,n})) \right| \]
\[ \leq C \max |\phi'\sum_{k=0}^{\infty} \frac{1}{m^k} | \]

is bounded independently of \( n \). Consequently

\[ \lim_{n \to \infty} \inf_{x \in \mathbb{R}, E^1(x) = x} \frac{1}{n} \phi_n^x = \phi_{\min}. \]

**Lemma E.7.**

\( F(\beta) \to -\phi_{\min} \) as \( \beta \to +\infty \).

**Proof.** Let \( \beta > 0 \). Let us write

\[ F_n(\beta) = \frac{1}{n\beta} \log \left( \sum_{x \in E^1(x) = x} e^{-\beta \phi_n^x} \right), \]

so that

\[ F(\beta) = \lim_{n \to \infty} F_n(\beta). \]

Let \( \epsilon > 0 \). By definition of \( \phi_{\min} \), for \( n \) large enough,

\[ \forall x \in \text{Per}(n), \quad \phi_n^x \geq n(\phi_{\min} - \epsilon) \]

and

\[ \exists x \in \text{Per}(n), \quad \phi_n^x \leq n(\phi_{\min} + \epsilon). \]

Thus,

\[ e^{-\beta n(\phi_{\min} + \epsilon)} \leq \sum_{x \in E^1(x) = x} e^{-\beta \phi_n^x} \leq L e^{-\beta n(\phi_{\min} - \epsilon)} \]

and consequently

\[ -\phi_{\min} - \epsilon \leq F_n(\beta) \leq \frac{\log L}{\beta} - \phi_{\min} + \epsilon. \]

Hence, letting \( \epsilon \to 0 \), we get

\[ -\phi_{\min} \leq F(\beta) \leq \frac{\log L}{\beta} - \phi_{\min}. \]

When \( \beta \) goes to infinity, the result follows. \( \square \)
Proof of lemma 3.12. Now we take \( \phi = J = \log(E') \). By the definition of \( J_{\min} \)

\[
\inf_{O \in \text{Per}(n)} J_O^n = n J_{\min} + o(n),
\]

thus

\[
\sup_{O \in \text{Per}(n)} \frac{1}{e^{D_O} - 1} = e^{-n J_{\min} + o(n)} = e^{\lim_{n \to \infty} F(2) + o(n)}.
\]

(E.6)

We have

\[
\frac{1}{\sqrt{n}} \left( \sum_{m \mid n} \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{D_O} - 1)^2} \right)^{\frac{1}{2}} \leq A_n = (1.5) \left( \sum_{m \mid n} \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{D_O} - 1)^2} \right)^{\frac{1}{2}} \leq \left( \sum_{m \mid n} \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{D_O} - 1)^2} \right)^{\frac{1}{2}}.
\]

(E.7)

Since

\[
\sum_{m \mid n} \sum_{O \in \mathcal{P}_m} \frac{1}{(e^{D_O} - 1)^2} = \sum_{E^m(\lambda) = x} \frac{1}{(e^{r} - 1)^2} = \sum_{E^m(\lambda) = x} e^{-2r} \left( 1 + O(e^{-2r}) \right) = \left( \sum_{E^m(\lambda) = x} e^{-2r} \right) (1 + o(1)) = e^{\Pr(-2J) + o(n)}.
\]

Equation (E.7) gives

\[
\frac{1}{\sqrt{n}} e^{-\frac{1}{2} \Pr(-2J) + o(n)} \leq A_n \leq e^{-\frac{1}{2} \Pr(-2J) + o(n)}
\]

hence from remark E.2

\[
nA_n = e^{-\frac{1}{2} \Pr(-2J) + o(n)} = e^{-nF(2) + o(n)}.
\]

Finally,

\[
nA_n \sup_{O \in \text{Per}(n)} \frac{1}{e^{D_O} - 1} = e^{\lim_{n \to \infty} F - F(2) + o(n)} \to 0
\]

from corollary E.4. \(\square\)
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