On matrices with simple spectra arising from tensor products

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Introduction.

Throughout all matrices are square with the real or complex entries. The purpose of this note is an application of topological methods to a problem on simultaneous approximation of several matrices by a finite family of diagonalizable matrices satisfying an additional spectral condition. That is the simplicity for all eigenvalues of a matrix product.

Problems of approximating tuples of matrices and linear operators by finite collections of those elements with some extra nice properties have a considerable history. They are intimately related to the very interesting questions in linear algebra, functional analysis, algebraic geometry and topology (see, e.g., [O’MCV], [DS] and the references therein). We mention the result established in 1955 by Motzkin and Taussky ([MT], Theorem 5) who proved that any pair of complex commuting matrices is approximately simultaneously diagonalizable. Moreover, those problems continue to find new applications, e.g., to the study of phylogenetic invariants in biomathematics [AR],[O’MCV, Ch. 6].

Originally we were motivated by our interest ([GG],[G1],[G2]) in approximating elements in the tensor products of Banach spaces and applications of parameterized families of polynomials to topological groups, particularly to perturbations in the matrix groups. Matrix tuples and products of their elements arise naturally in several contexts in the study of tensors, or multi-dimensional arrays, especially in connection with bounds on tensor ranks (see [AS],[AL],[TB],[S],[SMS],[T]). A part of the motivation for our work comes from the article [T] where some nontrivial tensor ranks estimations for inverse matrices are presented.

Preliminaries.

Let $\mathbb{F}$ denote either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$. The linear space of all square $n \times n$ matrices $M_n(\mathbb{F})$ over the field $\mathbb{F}$ is endowed with an arbitrary norm $\| \cdot \|$ which generates the same
metric topology. The space $M_n(\mathbb{F})$ with the ordinary matrix multiplication is a Banach algebra. As usual, the set of all invertible matrices in $M_n(\mathbb{F})$ is denoted by $GL_n(\mathbb{F})$. It is called the general linear group of degree $n$. An eigenvalue of a matrix is said to be simple if its algebraic multiplicity equals one. The spectrum of a matrix is said to be simple provided that all eigenvalues of a given matrix are simple. In other words, if all eigenvalues are pairwise distinct. Sometimes the matrices with simple spectra are called generic. Certainly, such matrices are diagonalizable.

In the following theorem we list necessary well known facts about topological properties of matrix sets.

**Theorem.** The following properties are fulfilled in the space $M_n(\mathbb{F})$:

- The set $GL_n(\mathbb{F})$ is dense and open.
- The general linear group is a locally compact group.
- The set of all invertible matrices with simple spectra is dense and open.

Recall that a mapping $f : X \to Y$ between two topological spaces is said to be open, if for any open set $O$ in $X$ the image $f(O)$ is open in $Y$. If $f : GL_n(\mathbb{F}) \to GL_n(\mathbb{F})$ is a surjective continuous homomorphism then it is open ([HR], Theorem 5.29).

**Results.**

In this section we prove several propositions concerning perturbations of matrix tuples by finite collections of matrices with simple spectra. To do this we make use of the facts from Theorem.

**Proposition 1.** Let $f$ and $g$ be self-mappings of the general linear group of degree $n$ over the field $\mathbb{F}$. Suppose that at least one of those mappings is open. Let $A$ and $B$ be matrices in $M_n(\mathbb{F})$. Then for any $\varepsilon > 0$ there exists a pair of invertible matrices $A_\varepsilon$ and $B_\varepsilon$ with simple spectra such that

$$\|A - A_\varepsilon\| < \varepsilon, \|B - B_\varepsilon\| < \varepsilon$$

and the product matrix $f(A_\varepsilon)g(B_\varepsilon)$ has a simple spectrum.

**Proof.** For definiteness, we assume that the mapping $f : GL_n(\mathbb{F}) \to GL_n(\mathbb{F})$ is open. Fix an arbitrary real number $\varepsilon > 0$. 

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Using Theorem, we choose two invertible matrices $\tilde{A}$ and $B_\varepsilon$ with simple spectra satisfying the inequalities

$$\|A - \tilde{A}\| < \frac{\varepsilon}{2}, \quad \|B - B_\varepsilon\| < \varepsilon.$$  

Further we consider the right multiplication by the element $g(B_\varepsilon)$:

$$R : GL_n(\mathbb{F}) \longrightarrow GL_n(\mathbb{F}) : X \longmapsto Xg(B_\varepsilon)$$

which is an open mapping.

By Theorem, in the space $M_n(\mathbb{F})$ we can take the open ball $B(\tilde{A}; r)$ of radius $r > 0$ centered at the point $\tilde{A}$, i.e., the open set

$$B(\tilde{A}; r) = \{X \in M_n(\mathbb{F}) : \|\tilde{A} - X\| < r\}$$

such that the following two conditions are satisfied:

1. $r < \frac{\varepsilon}{2}$;
2. $B(\tilde{A}; r)$ consists of invertible matrices with simple spectra.

Since the composition of open mappings is open the image $R(f(B(\tilde{A}; r)))$ is a non-empty open set. It follows from Theorem that there exists an invertible matrix $C \in R(f(B(\tilde{A}; r)))$ whose spectrum is simple. Therefore the equality $C = f(A_\varepsilon)g(B_\varepsilon)$ holds for some matrix $A_\varepsilon \in B(\tilde{A}; r)$.

Obviously, the matrices $A_\varepsilon$ and $B_\varepsilon$ are the required perturbations of the given matrices $A$ and $B$. The proposition is proved.

Further, let $f$ be the identity mapping and $g$ be the inverse mapping on the group $GL_n(\mathbb{F})$. We have the following consequence of Proposition 1.

**Proposition 2.** Let $A$ and $B$ be elements of $M_n(\mathbb{F})$. Then for any $\varepsilon > 0$ there exists a pair of invertible matrices $A_\varepsilon$ and $B_\varepsilon$ with simple spectra such that

$$\|A - A_\varepsilon\| < \varepsilon, \quad \|B - B_\varepsilon\| < \varepsilon$$

and the product matrix $A_\varepsilon B_\varepsilon^{-1}$ has a simple spectrum.

There is an interesting application of Proposition 2 to estimating the tensor rank of the inverse matrix in [T] when the matrices $A$ and $B$ are the factors of the Kronecker (tensor) products. More precisely, let $X \in GL_{pq}(\mathbb{F})$ be a matrix whose tensor rank $tRank_{p,q}(X) = 2$. That is, a decomposition of $X$ into the sum of the Kronecker products of two matrices from $M_p(\mathbb{F})$ and $M_q(\mathbb{F})$ with a minimal possible number of summands consists of two terms, i.e., $X = A \otimes C + B \otimes D$ for some $A, B \in$...
$M_p(\mathbb{F})$ and $C, D \in M_q(\mathbb{F})$. Proposition 2 guarantees the possibility of approximating the matrix $X$ by tensor-product binomials with two factors satisfying the properties indicated in that assertion. This fact is involved in the proof of the following upper bound (see [T, p.3172]):

$$t\text{Rank}_{p,q}(X^{-1}) \leq \min\{p, q\}.$$ 

The matrix products of the forms $AB^{-1}$ and $A^{-1}B$ with different spectral conditions arise in studying the tensor ranks of 3-tensors (see, e.g., [AS],[AL],[TB],[S],[SMS]). In this case the slices of three-dimensional arrays serve as the matrices $A$ and $B$.

It is worth noting that the spectral properties of such matrix products play a significant role in a number of problems in the theory of matrix pencils (see, e.g., [I]).

In conclusion, we formulate the following statement which is a natural generalization of Proposition 1.

**Proposition 3.** Let $f_1, f_2, \ldots, f_k$ be a $k$-tuple of self-mappings for the general linear group of degree $n$ over the field $\mathbb{F}$. Suppose that at least one of those mappings is open. Let $A_1, A_2, \ldots, A_k$ be a $k$-tuple of matrices in $M_n(\mathbb{F})$. Then for any $\varepsilon > 0$ there exists a $k$-tuple $A_{1\varepsilon}, A_{2\varepsilon}, \ldots, A_{k\varepsilon}$ consisting of invertible matrices with simple spectra such that

$$\|A - A_{1\varepsilon}\| < \varepsilon, \|A - A_{2\varepsilon}\| < \varepsilon, \ldots, \|A - A_{k\varepsilon}\| < \varepsilon,$$

and the product matrix $f_1(A_{1\varepsilon})f_2(A_{2\varepsilon})\ldots f_k(A_{k\varepsilon})$ has a simple spectrum.

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