Extending Hamiltonian Operators
to Get Bi-Hamiltonian Coupled KdV Systems

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Abstract
An analysis of extension of Hamiltonian operators from lower order to higher order of matrix paves a way for constructing Hamiltonian pairs which may result in hereditary operators. Based on a specific choice of Hamiltonian operators of lower order, new local bi-Hamiltonian coupled KdV systems are proposed. As a consequence of bi-Hamiltonian structure, they all possess infinitely many symmetries and infinitely many conserved densities.

1 Introduction
Bi-Hamiltonian structure has been identified as one of the basic mechanisms supplying conservation laws and commuting flows and even constructing multi-gap solutions for a given nonlinear system. From differential geometric point of view, bi-Hamiltonian structure can lead to an interesting application which yields a key to describe integrable surfaces embedded in pseudo-Euclidean spaces. For example, in many cases local bi-Hamiltonian structure provides additional information about some nonlocal Hamiltonian structures closely connected with metrics of constant curvatures. There exist other applications in handling quantization of Poisson brackets, and studying the problem of instability, for instance, in fluid mechanics. What we would like to develop in this paper is to establish more local bi-Hamiltonian structures which can engender hierarchies of coupled KdV systems.

Let \( u \) be a dependent variable \( u = (u^1, \cdots, u^q)^T \), where \( u^i, \ 1 \leq i \leq q \), depend on the spatial variable \( x = (x_1, \cdots, x_p)^T \) and on the temporal variable \( t \). We use \( \mathcal{A}^r (r \geq 1) \) to denote the space of \( r \) dimensional column function vectors depending on \( u \) itself and its derivatives with respect to the spatial variable \( x \). Sometimes we write \( \mathcal{A}^r(u) \) in order to specify the dependent variable \( u \).

A natural inner product over \( \mathcal{A}^r(u) \) is given by

\[
< \alpha, \beta > = \int_{\mathbb{R}^p} (\alpha)^T \beta \, dx, \ \alpha, \beta \in \mathcal{A}^r. \tag{1.1}
\]
Moreover on $\mathcal{A}^q(u)$ (note that the dimensions of both a function vector in $\mathcal{A}^q(u)$ and the dependent variable $u$ are the same), a Lie product can be defined as follows

$$[K, S] = \frac{\partial}{\partial \varepsilon} (K(u + \varepsilon S) - S(u + \varepsilon K)) \bigg|_{\varepsilon=0}, \quad K, S \in \mathcal{A}^q.$$  \hfill (1.2)

Two function vectors $K, S \in \mathcal{A}^q(u)$ are called to be commutative if $[K, S] = 0$. A fundamental conception that we need is Hamiltonian operators, which is shown in the following definition.

**Definition 1.1** A linear skew-symmetric operator $J(u) : \mathcal{A}^q(u) \to \mathcal{A}^q(u)$ is called a Hamiltonian operator or to be Hamiltonian, if the Jacobi identity

$$< \alpha, J'(u)[J(u)\beta]\gamma > + \text{cycle}(\alpha, \beta, \gamma) = 0 \hfill (1.3)$$

holds for all $\alpha, \beta, \gamma \in \mathcal{A}^q(u)$. A pair of operators $J(u), M(u) : \mathcal{A}^q(u) \to \mathcal{A}^q(u)$ is called a Hamiltonian pair if $J(u) + cM(u)$ is always Hamiltonian for any constant $c$.

Associated with a given Hamiltonian operator $J(u)$, the Poisson bracket is defined to be

$$\{\tilde{H}_1, \tilde{H}_2\}_{J} = \int \left( \frac{\delta \tilde{H}_1}{\delta u} \right)^T J \frac{\delta \tilde{H}_2}{\delta u} \, dx, \quad \tilde{H}_1, \tilde{H}_2 \in \tilde{\mathcal{A}}, \hfill (1.4)$$

where $\tilde{\mathcal{A}}$ consists of functionals $\tilde{H} = \int H \, dx, \quad H \in \mathcal{A}$. Two functionals $\tilde{H}_1, \tilde{H}_2$ are called to be commutative under the Poisson bracket associated with $J$, if $\{\tilde{H}_1, \tilde{H}_2\}_{J} = 0$. A Hamiltonian operator $J : \mathcal{A}^q \to \mathcal{A}^q$ has a nice property

$$[J \frac{\delta \tilde{H}_1}{\delta u}, J \frac{\delta \tilde{H}_2}{\delta u}] = J \left[ \frac{\delta \tilde{H}_1}{\delta u}, \frac{\delta \tilde{H}_2}{\delta u} \right]_{J}, \quad \tilde{H}_1, \tilde{H}_2 \in \tilde{\mathcal{A}}, \hfill (1.5)$$

which gives rise to an important relation between symmetries and conserved densities for a Hamiltonian system with the Hamiltonian operator $J$.

If we have a Hamiltonian pair $J$ and $M$, one of which is invertible, for example $J$, then $\Phi := MJ^{-1}$ is a hereditary operator \[\Phi\]. Furthermore if the adjoint operator $\Psi := \Phi^\dagger = J^{-1}M$ of this hereditary operator $\Phi = MJ^{-1}$ maps a gradient vector $f_0 = \frac{\delta \tilde{H}_0}{\delta u}$ to another gradient vector $f_1 = \Psi f_0 = \frac{\delta \tilde{H}_1}{\delta u}$, then all vectors $\Psi^n f_0$, $n \geq 0$, are gradient, i.e. there exist functionals $\tilde{H}_i, \quad i \geq 2$, such that $\Psi^n f_0 = \frac{\delta \tilde{H}_i}{\delta u}$, $i \geq 2$ (see \[\Phi\] for more information). Then it follows from bi-Hamiltonian structure that all systems of evolution equations in the hierarchy $u_t = \Phi^n f_0$, $n \geq 0$, commute with each other, i.e. $[\Phi^m f_0, \Phi^n f_0] = 0, \quad m, n \geq 0$, and they have infinitely many common conserved densities being commutative under two Poisson brackets. Therefore Hamiltonian pairs can pave a way for constructing integrable systems. However the problem still exits and just turn to how to find Hamiltonian pairs.

In this paper, what we want to develop is to propose a possible way to generate Hamiltonian pairs. We are successful in constructing bi-Hamiltonian coupled KdV systems in such a way. The paper is organized as follows. We first concentrate on the techniques for extending Hamiltonian operators from lower order to higher order of matrix, motivated by the idea in \[\Phi\]. Then we will go on to analyze a class of Hamiltonian pairs which can yield hereditary operators and eventually new bi-Hamiltonian coupled KdV systems. A few of concluding remarks are given in the final section.
2 Extending Hamiltonian operators

Let us specify our dependent variables

\[ u_k = (u_k^1, \ldots, u_k^q)^T, \quad 1 \leq k \leq N, \]
\[ u = (u_1^T, \ldots, u_N^T)^T = (u_1^1, \ldots, u_1^q, \ldots, u_N^1, \ldots, u_N^q)^T, \]

and introduce a condition

\[ J'_k(u_k) = J'_l(u_l), \quad 1 \leq k, l \leq N, \quad (2.1) \]
on a set of given Hamiltonian operators \( J_k(u_k) : \mathcal{A}^q(u_k) \to \mathcal{A}^q(u_k), \quad 1 \leq k \leq N. \) This condition requires a kind of linearity property of the involved operators with regard to their dependent variables. Such sets of Hamiltonian operators \( J_k(u_k) \) do exist. For example, we can choose

\[ J_k(u_k) = -\frac{1}{4} \partial^3 + 2 \partial u_k \partial^{-1} + 2u_k, \quad \partial = \frac{\partial}{\partial x}, \quad 1 \leq k \leq N. \]

The problem that we want to handle here is how to generate Hamiltonian operators starting from a given set of \( J_k(u_k), \quad 1 \leq k \leq N. \) A simplest solution is to make a big operator to be \( J(u) = \text{diag}(J_1(u_1), \ldots, J_N(u_N)). \) What we want to develop below is to propose more general structure of Hamiltonian operators. To the end, we introduce the following structure of candidates for Hamiltonian operators

\[ J(u) = \left( \sum_{k=1}^{N} c_{ij}^k J_k(u_k) \right)_{N \times N}, \quad (2.2) \]

where \( \{ c_{ij}^k | i, j, k = 1, 2, \ldots N \} \) is a set of given constants. Obviously this big operator \( J(u) \) may be viewed as a linear operator

\[ J(u) : \mathcal{A}^{Nq}(u) = \mathcal{A}^q(u) \times \cdots \times \mathcal{A}^q(u) \to \mathcal{A}^{Nq}(u) = \mathcal{A}^q(u) \times \cdots \times \mathcal{A}^q(u), \]

where a vector function of \( \mathcal{A}^q(u) \) depends on all the dependent variables \( u_1, \ldots, u_N, \) not just certain dependent variable \( u_k, \) by defining

\[ J\alpha = \left( ((J\alpha)_1)^T, \ldots, ((J\alpha)_N)^T \right)^T, \quad (J\alpha)_i = \sum_{j,k=1}^{N} c_{ij}^k J_k(u_k) \alpha_j, \quad 1 \leq i \leq N, \quad (2.3) \]

where \( \alpha = (\alpha_1^T, \ldots, \alpha_N^T), \quad \alpha_i \in \mathcal{A}^q(u), \quad 1 \leq i \leq N. \) To guarantee the skew-symmetric property of the big operator \( J, \) a symmetric condition on \( \{ c_{ij}^k \} \)

\[ c_{ij}^k = c_{ji}^k, \quad 1 \leq i, j, k \leq N \quad (2.4) \]
suffices. The following theorem provides us with a sufficient condition for keeping the Jacobi identity \( (1.3). \)

**Theorem 2.1** If all \( J_k(u_k) : \mathcal{A}^q(u_k) \to \mathcal{A}^q(u_k), \quad 1 \leq k \leq N, \) are Hamiltonian operators having the linearity condition \( (2.1) \) and the constants \( c_{ij}^k, \quad 1 \leq i, j, k \leq N, \) satisfy the symmetric condition \( (2.4) \) and the following coupled condition

\[ \sum_{k=1}^{N} c_{ij}^k c_{kl}^n = \sum_{k=1}^{N} c_{lj}^k c_{ki}^n, \quad 1 \leq i, j, l, n \leq N, \quad (2.5) \]
then the operator $J(u) : \mathcal{A}^N(u) \to \mathcal{A}^N(u)$ defined by (2.2) and (2.3) is a Hamiltonian operator.

**Proof:** We only need to prove that $J(u)$ satisfies the Jacobi identity (1.3), because the linearity property and the skew-symmetric property of $J$ have already been shown. Noting that $\mathcal{A}^q(u)$ is composed of column function vectors, we suppose for $\alpha, \beta, \gamma \in \mathcal{A}^N(u)$ that

$$\alpha = (\alpha^T_1, \cdots, \alpha^T_N)^T, \beta = (\beta^T_1, \cdots, \beta^T_N)^T, \gamma = (\gamma^T_1, \cdots, \gamma^T_N)^T, \alpha_i, \beta_i, \gamma_i \in \mathcal{A}^q(u), 1 \leq i \leq N.$$

Moreover we will utilize a convenient notation $(X)_i = X_i, X_i \in \mathcal{A}^q(u), 1 \leq i \leq N$, when a function vector $X \in \mathcal{A}^N(u)$ itself is complicated.

First from the definition (2.2), we have

$$J'(u)[J\beta] = \sum_{k=1}^N c_{ij}^k J_k'(u_k)[(J\beta)_k]_{N \times N},$$

$$(J'(u)[J\beta]_\gamma)_i = \sum_{j,k=1}^N c_{ij}^k J_k'(u_k)[(J\beta)_k]_{\gamma_j}, 1 \leq i \leq N.$$

Then taking advantage of the concrete definition (2.3), the linearity condition (2.1) and the coupled condition (2.3), we can make the following computation

$$\begin{align*}
\langle \alpha, J'(u)[J\beta]_\gamma \rangle & > +\text{cycle}(\alpha, \beta, \gamma) \\
= & \sum_{i,j,k=1}^N c_{ij}^k < \alpha_i, J_k'(u_k)[(J\beta)_k]_{\gamma_j} > +\text{cycle}(\alpha, \beta, \gamma) \\
= & \sum_{i,j,k=1}^N c_{ij}^k < \alpha_i, J_k'(u_k)[\sum_{l,n=1}^N c_{kl}^n J_n(u_n)\beta_l]_{\gamma_j} > +\text{cycle}(\alpha, \beta, \gamma) \\
= & \sum_{i,j,k,l,n=1}^N c_{ij}^k c_{kl}^n < \alpha_i, J_k'(u_k)[J_n(u_n)\beta_l]_{\gamma_j} > +\text{cycle}(\alpha, \beta, \gamma) \\
= & \sum_{i,j,l,n=1}^N \sum_{k=1}^N (c_{ij}^k c_{kl}^n) < \alpha_i, J_k'(u_k)[J_n(u_n)\beta_l]_{\gamma_j} > +\text{cycle}(\alpha, \beta, \gamma) \\
= & \sum_{i,j,l,n=1}^N \sum_{k=1}^N (c_{ij}^k c_{kl}^n) < \alpha_i, J_n'(u_n)[J_n(u_n)\beta_l]_{\gamma_j} > +\text{cycle}(\alpha, \beta, \gamma) \\
= & \sum_{i,j,l,n=1}^N \sum_{k=1}^N (c_{ij}^k c_{kl}^n) < \alpha_i, J_n'(u_n)[J_n(u_n)\beta_l]_{\gamma_j} > \\
& + \sum_{i,j,l,n=1}^N \sum_{k=1}^N (c_{ij}^k c_{kl}^n) < \beta_i, J_n'(u_n)[J_n(u_n)\gamma_l]_{\alpha_j} > \\
& + \sum_{i,j,l,n=1}^N \sum_{k=1}^N (c_{ij}^k c_{kl}^n) < \gamma_i, J_n'(u_n)[J_n(u_n)\alpha_l]_{\beta_j} > \\
= & \sum_{i,j,l,n=1}^N \sum_{k=1}^N (c_{ij}^k c_{kl}^n) \left( < \alpha_i, J_n'(u_n)[J_n(u_n)\beta_l]_{\gamma_j} > +\text{cycle}(\alpha_i, \beta_l, \gamma_j) \right).
\end{align*}$$
Recall each \( J_i(u_i) \) is Hamiltonian and it follows from the above equality that the big operator \( J \) defined by (2.2) and (2.3) satisfies the Jacobi identity (1.3). The proof is completed.

We remark that if we consider the coefficients \( \{ c_{ij}^k \} \) to be the structural constants of a finite-dimensional algebra with a basis \( e_1, e_2, \ldots, e_N \) as follows

\[
e_i \ast e_j = \sum_{k=1}^{N} c_{ij}^k e_k, \quad 1 \leq i, j \leq N,
\]

then the symmetric condition (2.4) and the coupled condition (2.5) can become

\[
e_i \ast e_j = e_j \ast e_i, \quad 1 \leq i, j \leq N,
\]

\[
(e_i \ast e_j) \ast e_k = (e_k \ast e_i) \ast e_j, \quad 1 \leq i, j, k \leq N,
\]

respectively. They reflect two specific properties of the related algebra.

Apparently basic scalar Hamiltonian operators satisfying the linearity condition (2.1) can be the following set

\[
J_i(u_i) = c_i \partial^3 + d_i \partial + 2u_i x + 4u_i \partial, \quad 1 \leq i \leq N,
\]

where \( \partial = \partial/\partial x \) and \( c_i, d_i, 1 \leq i \leq N \), are arbitrary constants. Of course, matrix Hamiltonian operators having the linearity condition (2.1) may be chosen. Actually, such sets of Hamiltonian operators may be presented directly from the above operators by Theorem 2.1 or by perturbation around solutions as in Refs. [9] [10].

In what follows, we example applications of Theorem 2.1 to two specific choices of \( \{ c_{ij}^k \} \). The analysis below is quite similar to the one made for the extension of hereditary operators in Ref. [11].

**Example 1:** Let us first choose

\[
c_{ij}^k = \delta_{i+j,k-p}, \quad (2.10)
\]

where \( p \) is an integer and \( \delta_{kl} \) denotes the Kronecker symbol. The corresponding big operator formed by (2.2) becomes

\[
J(u) = \begin{bmatrix}
J_{p+2}(u_{p+2}) & J_{p+3}(u_{p+3}) & \cdots & J_{p+N+1}(u_{p+N+1}) \\
J_{p+3}(u_{p+3}) & \ddots & \vdots & J_{p+N+2}(u_{p+N+2}) \\
\vdots & \ddots & \ddots & \vdots \\
J_{p+N+1}(u_{p+N+1}) & J_{p+N+2}(u_{p+N+2}) & \cdots & J_{p+2N}(u_{p+2N})
\end{bmatrix}, \quad (2.11)
\]

where we accept that \( J_i(u_i) = 0 \) if \( i \leq 0 \) or \( i \geq N + 1 \).

For two cases of \(-2N + 1 \leq p \leq N\) and \(-1 \leq p \leq N - 2\), the coupled condition (2.5) can be satisfied, because we have

\[
\sum_{k=1}^{N} c_{ij}^k c_{ki}^n = \sum_{k=1}^{N} c_{il}^k c_{kj}^n = \begin{cases} 1, & \text{when } n - i - j - l = 2p, \\ 0, & \text{otherwise.} \end{cases}
\]
While proving the above equality, we should keep in mind that we have
\[ 1 \leq i + j + p = n - l - p \leq N, \quad 1 \leq i + l + p = n - j - p \leq N, \]
when \( n - i - j - l = 2p \). But for the case of \(-N < p < -1\), upon choosing \( i = -p + 1, j = -p - 1, n = l = 1\), we obtain
\[
\sum_{k=1}^{N} c_{ij}^{k} c_{kl}^{n} = 1, \quad \sum_{k=1}^{N} c_{il}^{k} c_{kj}^{n} = 0,
\]
and thus the coupled condition (2.3) can not be satisfied.

Note that when \( p \geq N - 1 \) or \( p \leq -2N \), the resulting operators are all zero operators. Therefore among the operators defined by (2.11), we can obtain only two sets of candidates for Hamiltonian operators
\[
J(u) = \begin{bmatrix}
0 & J_{p+N+1}(u_{p+N+1}) \\
J_{p+N+1}(u_{p+N+1}) & J_{p+N+2}(u_{p+N+2}) & \cdots & \cdots \\
J_{p+N+2}(u_{p+N+2}) & \cdots & \cdots & J_{p+2N}(u_{p+2N}) \\
\vdots & \vdots & \vdots & \vdots \\
J_{p+N+1}(u_{p+N+1}) & \cdots & 0
\end{bmatrix},
\]
\[ -2N + 1 \leq p \leq -N, \quad (2.12) \]
\[
J(u) = \begin{bmatrix}
J_{p+2}(u_{p+2}) & J_{p+3}(u_{p+3}) & \cdots & J_{p+N+1}(u_{p+N+1}) \\
J_{p+3}(u_{p+3}) & \cdots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
J_{p+N+1}(u_{p+N+1}) & \cdots & 0
\end{bmatrix},
\]
\[ -1 \leq p \leq N - 2, \quad (2.13) \]
where we still accept that \( \Phi(u_i) = 0 \) if \( i \leq 0 \) or \( i \geq N + 1 \). These two sets of operators can be changed to each other by a simple transformation \( (u_1, u_2, \cdots, u_N) \leftrightarrow (u_N, u_{N-1}, \cdots, u_1) \).

**Example 2:** Let us second choose
\[ c_{ij}^{k} = \delta_{kl}, \quad l = i + j - p \text{ (mod } N), \]  
(2.14)
where \( 1 \leq p \leq N \) is fixed and \( \delta_{kl} \) denotes the Kronecker symbol again. In this case, we have
\[
\sum_{k=1}^{N} c_{ij}^{k} c_{kl}^{n} = \sum_{k=1}^{N} c_{il}^{k} c_{kj}^{n} = \begin{cases} 1, & \text{when } i + j + l - n = 2p \text{ (mod } N), \\ 0, & \text{otherwise}, \end{cases}
\]  
(2.15)
which allows us to conclude that the coupled condition (2.3) holds, indeed. Therefore we
obtain a set of candidates for Hamiltonian operators

\[
J(u) = \begin{bmatrix}
J_{2-p}(u_{2-p}) & J_{3-p}(u_{3-p}) & \cdots & J_{N-p+1}(u_{N-p+1}) \\
J_{3-p}(u_{3-p}) & J_{N-p+2}(u_{N-p+2}) \\
\vdots & \vdots & \ddots & \vdots \\
J_{N-p+1}(u_{N-p+1}) & J_{N-p+2}(u_{N-p+2}) & \cdots & J_{2-p}(u_{2-p}) \\
\end{bmatrix},
\]  \quad (2.16)

where we accept \( J_i(u_i) = J_j(u_j) \) if \( i \equiv j \) \((\text{mod} \ N)\), while determining the operators involved, for example, \( J_{2-p}(u_{2-p}) = J_N(u_N) \) when \( p = 2 \). A special choice of \( p = 1 \) leads to a candidate of Hamiltonian operators

\[
J(u) = \begin{bmatrix}
J_1(u_1) & J_2(u_2) & \cdots & J_N(u_N) \\
J_2(u_2) & \ddots & J_1(u_1) \\
\vdots & \ddots & \ddots & \vdots \\
J_N(u_N) & J_1(u_1) & \cdots & J_{N-1}(u_{N-1}) \\
\end{bmatrix}.
\]  \quad (2.17)

This operator will be our starting point for constructing new integrable systems in next section.

### 3 New bi-Hamiltonian coupled KdV systems

From now on, we focus on the candidate of Hamiltonian operators defined by (2.17). Let us pick out the Hamiltonian operators formed by (2.17) under a choice of (2.9). Then the following Hamiltonian pair can be engendered

\[
J(u) = A \partial = \begin{bmatrix}
a_1 & a_2 & \cdots & a_N \\
a_2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
a_N & a_1 & \cdots & a_{N-1} \\
\end{bmatrix} \partial,
\]  \quad (3.1)

\[
M(u) = \begin{bmatrix}
M_1(u_1) & M_2(u_2) & \cdots & M_N(u_N) \\
M_2(u_2) & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
M_N(u_N) & M_1(u_1) & \cdots & M_{N-1}(u_{N-1}) \\
\end{bmatrix},
\]  \quad (3.2)

where \( \partial = \partial/\partial x \), \( u = (u_1, u_2, \cdots, u_N)^T \), \( a_i = \text{const.}, 1 \leq i \leq N \), and the operators \( M_i(u_i), 1 \leq i \leq N \), are given by

\[
M_i(u_i) = c_i \partial^3 + d_i \partial + 2u_{ix} + 4u_i \partial, \quad c_i, d_i = \text{const.}, 1 \leq i \leq N,
\]  \quad (3.3)

which are all Hamiltonian.
We assume that the constant matrix $A$ is invertible to guarantee the invertibility of $J$, and its inverse matrix is given by

$$B = \begin{bmatrix} b_1 & b_2 & \cdots & b_N \\ b_2 & \ddots & b_1 \\ \vdots & \ddots & \ddots & \vdots \\ b_N & b_1 & \cdots & b_{N-1} \end{bmatrix},$$

(3.4)

where $b_i$, $1 \leq i \leq N$, can be determined by solving a specific linear system

$$\begin{cases} a_1b_1 + a_2b_2 + \cdots + a_Nb_N = 1, \\ a_2b_1 + a_3b_2 + \cdots + a_Nb_{N-1} + a_1b_N = 0, \\ \vdots \\ a_Nb_1 + a_1b_2 + \cdots + a_{N-1}b_N = 0. \end{cases}$$

Now the resulting hereditary operator $\Phi = MJ^{-1}$ reads as

$$\Phi(u) = \left[ \begin{array}{c} M_1(u_1) \\ M_2(u_2) \\ \vdots \\ M_N(u_N) \end{array} \right] \left[ \begin{array}{c} \Phi_1(u_1) \\ \Phi_2(u_2) \\ \vdots \\ \Phi_N(u_N) \end{array} \right] = \left[ \begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_N \end{array} \right] \partial^{-1}$$

(3.5)

with

$$\Phi_1(u_i) = M_i(u_i)\partial^{-1} = c_i \partial^2 + d_i + 2u_{ix}\partial^{-1} + 4u_i, \quad 1 \leq i \leq N.$$  

(3.6)

This hereditary operator can be rewritten as a concise form

$$\Phi(u) = MJ^{-1} = \left( \sum_{k=1}^{N} b_{k-i+j}\Phi_k(u_k) \right),$$

(3.7)

where $b_i = b_j$ if $i \equiv j \pmod{N}$. It is also an example satisfying the extension scheme of hereditary operators in Ref. [11]; because upon setting $c_{ij}^k = b_{k-i+j}$ we have three equal sums for all $1 \leq i, j, l, n \leq N$:

$$\sum_{k=1}^{N} c_{ij}^k c_{kn} = \sum_{k=1}^{N} b_{k-i+j} b_{l-k+n} = \sum_{m=1}^{N} b_{m} b_{l+i-j+n-m},$$

$$\sum_{k=1}^{N} c_{ik}^l c_{kj}^n = \sum_{k=1}^{N} b_{l-i+k} b_{n-k+j} = \sum_{m=1}^{N} b_{m} b_{l+i-j+n-m},$$

$$\sum_{k=1}^{N} c_{lm}^k c_{kj} = \sum_{k=1}^{N} b_{k-l+n} b_{l-k+j} = \sum_{m=1}^{N} b_{m} b_{l+i-j+n-m},$$

$$\sum_{k=1}^{N} c_{ik}^l c_{kj} = \sum_{k=1}^{N} b_{l-i+k} b_{n-k+j} = \sum_{m=1}^{N} b_{m} b_{l+i-j+n-m},$$

$$\sum_{k=1}^{N} c_{lm}^k c_{kj} = \sum_{k=1}^{N} b_{k-l+n} b_{l-k+j} = \sum_{m=1}^{N} b_{m} b_{l+i-j+n-m},$$

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which are sufficient for $\Phi(u)$ to be hereditary (see [11]).

We now turn our attention to investigating the nonlinear systems which can be resulted from the above Hamiltonian pairs. Such first system can be the following

$$u_t = \Phi u_x = MJ^{-1}u_x,$$  \hfill (3.8)

which can be represented as

$$u_{it} = \sum_{k,j=1}^{N} b_{k-i+j}(c_ku_{jxx} + d_ku_{jx} + 2u_{kx}u_j + 4u_{k}u_{jx}), \quad 1 \leq i \leq N,$$  \hfill (3.9)

where we again accept $b_i = b_j$ if $i \equiv j \pmod{N}$. It is easy to find that

$$f_0 := J^{-1}u_x = Bu = \frac{\delta\hat{H}_0}{\delta u}, \quad \hat{H}_0 = \int H_0 dx, \quad H_0 = \frac{1}{2}u^T Bu. \hfill (3.10)$$

Go ahead to check whether or not the next vector defined by

$$f_1 := \Psi f_0 = \Phi^T f_0 = B \begin{bmatrix} \Psi_1(u_1) & \Psi_2(u_2) & \cdots & \Psi_N(u_N) \\ \Psi_2(u_2) & \cdots & & \Psi_1(u_1) \\ \vdots & \cdots & \cdots & \vdots \\ \Psi_N(u_N) & \Psi_1(u_1) & \cdots & \Psi_{N-1}(u_{N-1}) \end{bmatrix} Bu \hfill (3.11)$$

is a gradient vector, where $\Psi_i(u_i) = \Phi^T_i(u_i) = \partial^{-1}M_i(u_i), 1 \leq i \leq N$. That is true, indeed. Actually we have

$$f_1 = \Psi f_0 = \frac{\delta\hat{H}_1}{\delta u}, \quad \hat{H}_1 = \int H_1 dx,$$

$$H_1 = (Bu)^T \begin{bmatrix} \Theta_1(u_1) & \Theta_2(u_2) & \cdots & \Theta_N(u_N) \\ \Theta_2(u_2) & \cdots & \Theta_1(u_1) \\ \vdots & \cdots & \vdots \\ \Theta_N(u_N) & \Theta_1(u_1) & \cdots & \Theta_{N-1}(u_{N-1}) \end{bmatrix} Bu$$

$$= \sum_{j,k,l=1}^{N} \left( \sum_{i=1}^{N} b_{i+l-1}b_{k+j-1} \right) u_i \left( \frac{1}{2}c_ku_{jxx} + \frac{1}{2}d_ku_{jx} + u_ku_j \right), \hfill (3.12)$$

where the operators $\Theta_i(u_i), 1 \leq i \leq N$, are given by

$$\Theta_i(u_i) = \frac{1}{2}c_i\partial^2 + \frac{1}{2}d_i + \frac{2}{3}u_i + \frac{2}{3}\partial^{-1}u_i\partial, \quad 1 \leq i \leq N,$$  \hfill (3.13)

and $b_i = b_j$ if $i \equiv j \pmod{N}$. We can also choose the energy form for the functional $\hat{H}_1$:

$$\hat{H}_1 = \int H_1 dx, \quad H_1 = \sum_{j,k,l=1}^{N} \left( \sum_{i=1}^{N} b_{i+l-1}b_{k+j-1} \right) \left( -\frac{1}{2}c_ku_{jxx}u_{jx} + \frac{1}{2}d_ku_{jx}u_j + u_ku_{jx} \right). \hfill (3.14)$$
Therefore according to the Magri scheme [3], [4], [5], there exist other functionals \( \tilde{H}_n, n \geq 2 \), such that \( \Psi^n f_0 = \frac{\delta \tilde{H}_n}{\delta u}, n \geq 2 \). All such functionals can be generated by computing the following integrals

\[
\tilde{H}_n = \int \int_0^1 <(\Psi^n f_0)(\lambda u), u > d\lambda dx, n \geq 0.
\]  

(3.15)

Further we can obtain a hierarchy of bi-Hamiltonian equations

\[
u_t = K_n := (\Phi(u))^{n+1} u_x = J \frac{\delta \tilde{H}_{n+1}}{\delta u} = M \frac{\delta \tilde{H}_n}{\delta u}, n \geq 0,
\]

(3.16)

which includes the nonlinear system (3.9) as the first member. It follows that they have infinitely many commutative symmetries \( \{K_m\}_0^\infty \) and infinitely many commutative conserved densities \( \{H_m\}_0^\infty \). All systems of evolution equations can reduce to KdV equations once \( u_j = c_i = d_j = 0, j \neq 1 \), are selected. Therefore they are all \( N \)-component coupled KdV systems.

Let us now work out a concrete example for a choice of \( A \) with \( a_1 = 1, a_i = 0, 2 \leq i \leq N \).

In this case, we have \( b_1 = 1, b_i = 0, 2 \leq i \leq N \). Then the first Hamiltonian structure is given by

\[
\partial_t u_1 = \partial_x \frac{\delta \tilde{H}_1}{\delta u_1}, \partial_t u_k = \partial_x \frac{\delta \tilde{H}_1}{\delta u_{N+2-k}}, \partial_x = \frac{\partial}{\partial x}, 2 \leq k \leq N.
\]

(3.17)

This first Hamiltonian structure has the momentum:

\[
\tilde{P} = \tilde{H}_0 = \frac{1}{2} \int [u_1^2 + \sum_{k=2}^N u_k u_{N+2-k}] dx,
\]

(3.18)

which is of quadratic form with respect to its \( N \) Casimirs (annihilators of the first Poisson bracket) \( \tilde{F}_k = \int u_k dx, k = 1, \ldots, N \). The conservation law of momentum is

\[
\partial_t H_1 = \partial_x [\sum_{k=1}^N u_k \frac{\delta \tilde{H}_1}{\delta u_k} - F],
\]

(3.19)

where \( \tilde{H}_1 = \int H_1 dx, \partial_x F = \sum_{k=1}^N \frac{\delta \tilde{H}_1}{\delta u_k} u_{kx} \). For simplicity in the selection (3.6) we can put \( d_i = 0 \), because we can eliminate those constants in the expressions of \( \Phi_i \) by making shifts \( u_i \rightarrow u_i - d_i/4 \). At this moment our coupled KdV system reads as

\[
\partial_t u_m = \partial_x \left( \partial_x^2 \left( \sum_{k=1}^m c_{m+1-k} u_k + \sum_{k=m+1}^N c_{N+m+1-k} u_k \right) + 3 \sum_{k=1}^m u_k u_{m+1-k} + 3 \sum_{k=m+1}^N u_k u_{N+m+1-k} \right),
\]

(3.20)

which has the following Hamiltonian for the first Hamiltonian structure

\[
\tilde{H}_1 = \int H_1 dx
\]

(3.21)
\[ + \sum_{m=3}^{N} c_m \sum_{k=N+3-m}^{N} w_k w_{2N-m+3-k} + u_1^3 + 3u_1 \sum_{k=2}^{N} u_k u_{N+2-k} + \sum_{k=2}^{N-1} u_k \sum_{m=2}^{N+1-k} u_m u_{N+3-k-m} + \sum_{m=0}^{N-3} u_{m+3} \sum_{k=0}^{m} u_{N-k} u_{N+k-m}, \]

where \( w_k = \partial_x u_k \), \( 1 \leq k \leq N \), and \( N \) is assumed to be greater than two for producing nontrivial systems. The second Hamiltonian structure can be determined by its recursion operator

\[ \partial_t u_i = \sum_{k=1}^{i} \Phi_{i+1-k} \partial_x \frac{\delta \tilde{H}_0}{\delta u_k} + \sum_{k=i+1}^{N} \Phi_{N+1+i-k} \partial_x \frac{\delta \tilde{H}_0}{\delta u_k}, \quad 1 \leq i \leq N, \quad (3.22) \]

where \( \Phi_i = c_i \partial_x^2 + 2(2u_i + w_i \partial_x^{-1}) \), \( 1 \leq i \leq N \). The second Hamiltonian structure has the momentum \( F_i' = \int u_1 dx \) and the Hamiltonian \( \tilde{H}_0 \).

The relationship between the gradients of \( \tilde{H}_k \) and \( \tilde{H}_{k+1} \) determined by recursion operator yields a possibility for constructing an infinite set of conservation laws and commuting flows by iterations. At each step we need to compute integrals to construct \( \tilde{H}_{k+1} \). They can be done by using the formula \( (3.15) \) in variational analysis.

Moreover we have an alternative way to construct an infinite set of conservation laws and commuting flows. Let us introduce an eigenfunction problem for the recursion operator

\[ [\Phi(u) - \lambda]u_\tau = 0, \quad (3.23) \]

i.e.

\[ \sum_{k=1}^{i} \Phi_{i+1-k} \partial_\tau u_k + \sum_{k=i+1}^{N} \Phi_{N+1+i-k} \partial_\tau u_k = \lambda \partial_\tau u_i, \quad 1 \leq i \leq N. \quad (3.24) \]

This eigenfunction problem can be equivalently rewritten as

\[ \begin{cases} 
\partial_\tau u_k = \partial_x v_k, \quad 1 \leq k \leq N, \\
\sum_{k=1}^{i} \Phi_{i+1-k} \partial_x v_k + \sum_{k=i+1}^{N} \Phi_{N+1+i-k} \partial_x v_k = \lambda \partial_x v_i, \quad 1 \leq i \leq N. 
\end{cases} \]

If we use formal series near \( \lambda \to \infty \)

\[ \partial_\tau = \partial_t + \frac{1}{\lambda} \partial_{t_1} + \frac{1}{\lambda^2} \partial_{t_2} + \cdots, \quad (3.25) \]

\[ v_i = v_i^{(0)} + \frac{1}{\lambda} v_i^{(1)} + \frac{1}{\lambda^2} v_i^{(2)} + \cdots, \quad (3.26) \]

we can obtain directly from the above eigenfunction problem an infinite set of commuting flows. Furthermore we can use formal series near \( \lambda \to 0 \). Then the first resulting nontrivial commuting flow is a coupled long wave equation

\[ \sum_{k=1}^{i} \Phi_{i+1-k} \partial_\tau u_k + \sum_{k=i+1}^{N} \Phi_{N+1+i-k} \partial_\tau u_k = 0, \quad 1 \leq i \leq N. \quad (3.27) \]

This is an \( N \)-component generalization of long wave equation (see [12] for more information about long wave equation), which commutes with the KdV equation.
4 Concluding remarks

We have proposed new coupled KdV systems possessing bi-Hamiltonian structures by extending Hamiltonian operators from lower order to higher order of matrix. Clearly we may make other choices of $J_i(u_i)$ to give more results based on Theorem 2.1.

Compared to the well-known coupled KdV systems (for example, see [13])

$$u_t = \Phi^n u_x, \quad \Phi(u) = \begin{bmatrix} 0 & \cdots & 0 & \Phi_1 \\ 1 & \cdots & 0 & \Phi_2 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & \Phi_N \end{bmatrix}, \quad n \geq 0,$$

and quite new couple KdV systems introduced in [14]

$$u_t = \Phi^n u_x, \quad \Phi(u) = \begin{bmatrix} b_N \Phi_1 \\ b_{N-1} \Phi_1 + b_N \Phi_2 & b_N \Phi_1 \\ \vdots & \ddots & \ddots \\ b_1 \Phi_1 + \cdots + b_N \Phi_N & \cdots & b_{N-1} \Phi_1 + b_N \Phi_2 & b_N \Phi_1 \end{bmatrix}, \quad n \geq 0,$$

where $u = (u_1, u_2, \ldots, u_N)^T$, $b_i, 1 \leq i \leq N$, are arbitrary constants except $b_N \neq 0$, $\Phi_i = \Phi_i(u_i), 1 \leq i \leq N$, are still defined by (3.6), our new coupled KdV systems (3.16) have similar nice integrable properties, for example, bi-Hamiltonian structures, dispersionless limits having bi-Hamiltonian structures (see [15] for the case of the well-known coupled KdV systems). But there exist differences among the structures of recursion operators corresponding to these three hierarchies of coupled KdV systems. The bi-Hamiltonian structure (see [14]) of the hierarchy (4.2) can similarly be derived from the Hamiltonian operators formed by (2.12) in the first example of the second section. This is why we didn’t deliver above a detailed analysis for constructing integrable systems starting from (2.12) or equivalently from (2.13). However we don’t know whether or not two new coupled KdV hierarchies (4.2) and (3.16) have other integrable properties, for example, Lax pairs like (4.1).

In terms of the existence of recursion operators, other integrable couple KdV systems, say, Jordan KdV systems, have also been derived (see for example [16] [17] [18] [19]). A natural question is whether there exist other hierarchies possessing bi-Hamiltonian structures among those coupled KdV systems. This will enrich the content of Hamiltonian theory for coupled KdV systems.

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