IDEAL STRUCTURE OF CLIFFORD ALGEBRAS

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Abstract. The structures of the ideals of Clifford algebras which can be both infinite dimensional and degenerate over the real numbers are investigated.

1. Introduction

Clifford algebras have been playing an important role in describing electron spin, supersymmetry, and the fundamental representations of the orthogonal groups etc. We refer the readers to the introduction section of [7] for a discussion of the role played by Clifford algebras in quantum mechanics. Recently, there has been an increase of interest in the applications of Clifford algebras to other areas such as computational geometry and engineering. In these applications, the real Clifford algebras are usually called geometric algebras. It is shown in [7] that in general an infinite dimensional Clifford algebra over the real numbers contains all finite dimensional Clifford algebras over the real numbers as well as all finite dimensional Clifford algebras over the complex numbers. Therefore it is natural to study the structure of a real Clifford algebra under the general setting which allows both infinite dimensionality and degeneracy.

The structures of the finite dimensional Clifford algebras associated to nondegenerate quadratic forms have been well understood for a long period of time [2]. These Clifford algebras are either full matrix algebras or the direct sums of two full matrix algebras over the real numbers \( \mathbb{R} \), or the complex numbers \( \mathbb{C} \), or the quaternions \( \mathbb{H} \). The study of the idempotent structures of the finite dimensional Clifford algebras over the real numbers, including the degenerate cases, was carried out in [1] and [5]. Somewhat later, [7] and [8] investigated the structures of the infinite dimensional Clifford algebras over the real numbers. As pointed out in [7], the infinite dimensional Clifford algebras over the real number arise naturally in physics when an infinite dimensional real Hilbert space is considered, and these algebras are connected to

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the infinite dimensional orthogonal groups and spin groups. Although
in general an infinite dimensional Clifford algebra contains the finite
dimensional ones, the study of the structure of an infinite dimensional
Clifford algebra relies on the knowledge of the structures of the finite
dimensional ones. Our goal here is to answer some of the quest ions
about the ideal structures of these algebras under the assum ption that
the Clifford algebras can be finite or infinite dimensional as well as
degenerate.

2. Preliminaries

We recall the definition of a general Clifford algebra [7]. Let $M, P, Z \subseteq \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ and let $\Lambda = M \cup P \cup Z$. Let $q = |M|$, $p = |P|$, and $z = |Z|$. We allow $p, q$, and $z$ to be $\infty$. Let $C = C(p, q, z)$ be the Clif-{
ford algebra over the real numbers $\mathbb{R}$ generated by $\{e_\lambda : \lambda \in \Lambda\}$. That is, $C$ is the associative algebra (with 1) generated by the $e_\lambda$, $\lambda \in \Lambda$,
such that

$$e_\lambda e_\mu = -e_\mu e_\lambda, \quad \lambda \neq \mu, \quad (2.1)$$

$$e_\lambda^2 = \begin{cases} 
1, & \lambda \in P, \\
-1, & \lambda \in M, \\
0, & \lambda \in Z.
\end{cases} \quad (2.2)$$

By an ordered subset $\{i_1, i_2, \ldots\}$ of $\mathbb{Z}_+$ we mean that the condition $i_1 < i_2 < \cdots$ hold, and we shall denote an ordered subset of $\mathbb{Z}_+$ by $I = (i_1, i_2, \ldots)$. For a finite ordered subset $I = (i_1, \ldots, i_k)$ of $\mathbb{Z}_+$, let $e_I = e_{i_1} \cdots e_{i_k}$. If $I = \emptyset$, then $e_I = 1$. When we write $e_I e_J e_K$, we always assume that $I, J, K$ are finite ordered sets.

It follows from the definition that the set of elements

$$\{e_I e_J e_K : I \subseteq P, J \subseteq M, K \subseteq Z\} \quad (2.3)$$

form a basis of $C$.

We let $C(p, q)$ be the subalgebra generated by $\{e_\lambda : \lambda \in M \cup P\}$, and let $C(z)$ be the subalgebra generated by $\{e_\lambda : \lambda \in Z\}$.

If $z = 0$, then $C = C(p, q)$, and we divide the algebras into two cases: $p + q < \infty$ and $p + q = \infty$.

When $p + q = \infty$, $C$ is simple [7] and the idempotent structure is studied in [8]. In particular, it is proved in [8] that there is no primitive idempotent in $C$ and there exists an infinite set of pair-wise orthogonal idempotents.

If $p + q < \infty$, then $C$ is simple (a full matrix algebra) if $p - q \neq 1, 5$ modulo 8, and $C$ is a direct sum of two isomorphic simple ideals (two full matrix algebras) if $p - q = 1, 5$ modulo 8 [6, pp.132-133]. Therefore, in this case, the information about the ideals and idempotents of $C$ can
be readily obtained by using the corresponding results of matrices. For our convenience, if \( \mathcal{C} \) is not simple, we write

\[
\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2,
\]

where \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are the two simple ideals.

Recall that an ideal \( I \) of an algebra (associative) \( A \) is nil if for \( \forall x \in A \) there exists a positive integer \( n \) such that \( x^n = 0 \), and the nil radical \( \mathcal{R} \) of \( A \) is the sum of all nil ideals.

If \( z > 0 \), then the nil radical \( \mathcal{R} \) of \( \mathcal{C} \) is non-zero. By (2.3) and Proposition 2.1 below, we have that

\[
\mathcal{C} = \mathcal{C}(p, q) \oplus \mathcal{R}
\]

is a vector space direct sum. In this case, the structure of the idempotents of a finite dimensional \( \mathcal{C} \) is discussed in [1] by using the fact that the nil radical is idempotent-lifting.

The following proposition is given in [7], we provide a complete proof here.

**Proposition 2.1.** The nil radical \( \mathcal{R} \) is generated by \( \{e_\lambda : \lambda \in Z\} \).

**Proof.** Let \( N \) be the ideal generated by \( \{e_\lambda : \lambda \in Z\} \). Then by (2.3),

\[
\mathcal{C} = \mathcal{C}(p, q) \oplus N
\]

is a direct sum of vector spaces. Since the nil radical \( \mathcal{R} \) is the sum of all nil ideals, the ideal \( N \) is contained in \( \mathcal{R} \). If \( a \in \mathcal{R} \), write \( a = a_1 + a_2 \) such that \( a_1 \in \mathcal{C}(p, q) \) and \( a_2 \in N \) according to (2.6), then \( a_1 \in \mathcal{R} \). If \( a_1 \neq 0 \), then we can get a contradiction right away if the subalgebra \( \mathcal{C}(p, q) \) is simple. In the case that \( \mathcal{C}(p, q) \) is not simple, \( \mathcal{C}(p, q) \) is a full matrix algebra or a direct sum of two copies of a full matrix algebra. Thus by considering the matrices (say, use elementary matrices from both sides), we can see that there is an idempotent in the ideal generated by \( a_1 \), which also leads to a contradiction. So \( a_1 = 0 \) and \( N = \mathcal{R} \). \( \square \)

It follows from Proposition 2.1 that the elements \( e_I e_J e_K \), where \( I \subseteq P, J \subseteq M, K \subseteq Z \) such that \( K \neq \emptyset \), form a basis of \( \mathcal{R} \). The subalgebra \( \mathcal{C}(z) \) has a grading given by

\[
\mathcal{C}(z) = \bigoplus_{i=1}^{\infty} \mathcal{C}(z)_i,
\]

where \( \mathcal{C}(z)_i \) is the linear span of all the \( e_K \ (K \subseteq Z) \) such that \( |K| = i \). This grading induces a grading on \( \mathcal{R} \) given by

\[
\mathcal{R} = \bigoplus_{i=1}^{\infty} \mathcal{R}_i, \quad \mathcal{R}_i = \mathcal{C}(p, q) \mathcal{C}(z)_i.
\]
3. The ideal structure of $C$

In this section, we describe the ideals of $C$.

**Theorem 3.1.** Let $I$ be a non-trivial ideal of $C$. If $C(p, q)$ is simple, then $I \subseteq R$. If $C(p, q)$ is not simple and $I \not\subseteq R$, then $I$ is generated by $C_1$ or $C_2$ together with $I \cap R$. In fact, 

$$I = C_1 \oplus (I \cap R) \quad \text{or} \quad I = C_2 \oplus (I \cap R)$$

as vector space direct sums.

**Proof.** If $C(p, q)$ is simple and $I \not\subseteq R$, then there exists $a = b + c \in I$ such that $0 \neq b \in C(p, q)$ and $c \in R$. Since $C(p, q)bC(p, q) = C(p, q)$, from $b + c \in I$ we get $1 + x \in I$ for some $x \in R$. Since $x$ is nilpotent, $1 + x$ is invertible and $I = C$, which is a contradiction.

If $C(p, q)$ is not simple and $I \not\subseteq R$, then there exists $a = b + c + d \in I$ with $b \in C_1$, $c \in C_2$, and $d \in R$, such that at least one of $b$ and $c$ is non-zero. Let $1 = e_1 + e_2$ be the idempotent decomposition according to the decomposition of (2.4). Then $e_1 a = b + e_1 d \in I$. Thus, if $b \neq 0$, then $e_1 + d_1 \in I$ for some $d_1 \in R$. Similarly, if $c \neq 0$, then $e_2 + d_2 \in I$ for some $d_2 \in R$. Thus if both $b$ and $c$ are non-zero (or for some $a$, $b \neq 0$, and for another $a$, $c \neq 0$), we would have 

$$e_1 + d_1 + e_2 + d_2 = 1 + d_1 + d_2 \in I.$$

Now the fact that $d_1 + d_2 \in R$ implying that $1 + d_1 + d_2$ is invertible would lead to a contradiction. Hence without loss of generality, we can assume that $b \neq 0$ and $c = 0$ for all $a \in I \setminus R$. Then from $e_1 + d_1 \in I$, we have 

$$e_1 (e_1 + d_1) e_1 = e_1 + e_1 d_1 e_1 \in I.$$

Let $y = -e_1 d_1 e_1$. Then $y \in R$, $e_1 y = y = ye_1$, and $e_1 - y \in I$. Take a positive integer $m$ such that $y^m = 0$, then 

$$e_1 = (e_1 - y)(e_1 + y + y^2 + \cdots + y^{m-1}) \in I.$$

This implies that $C_1 \subseteq I$. Now the second statement of the theorem follows from (2.5). \[\Box\]

This theorem reduces the study of the nontrivial ideals of $C$ to the study of the ones contained in the nil radical $R$. We shall give a description of the nilpotent ideals of $C$ towards the end of this section.

Recall that an ideal $I$ of a noncommutative ring $R$ is said to be prime if $I \neq R$ and for any ideal $A, B \subseteq R$, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$. The following theorem describes the prime ideals of $C$. 

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Theorem 3.2. If the subalgebra $\mathcal{C}(p,q)$ is simple, then $\mathcal{C}$ has only one prime ideal, namely $\mathcal{R}$. If the subalgebra $\mathcal{C}(p,q)$ is not simple, then there are two prime ideals:

$$I = \mathcal{C}_1 \oplus \mathcal{R} \quad \text{and} \quad I = \mathcal{C}_2 \oplus \mathcal{R}.$$  

Proof. Let $I$ be a prime ideal of $\mathcal{C}$. Since $e_i^2 = 0$ for $i \in \mathbb{Z}$, $I \supseteq \mathcal{R}$. If $\mathcal{C}(p,q)$ is simple, then $\mathcal{R}$ is a prime ideal. If $\mathcal{C}(p,q) = \mathcal{C}_1 \oplus \mathcal{C}_2$, then $\mathcal{R}$ is not prime since $\mathcal{C}_1 \mathcal{C}_2 = (0)$, and in this case, $\mathcal{C}_i \oplus \mathcal{R}$, $i = 1,2$, are the two prime ideals. \qed

Recall that in a ring $R$, we have the implications: maximal ideal $\Rightarrow$ primitive ideal $\Rightarrow$ prime ideal (see Ch. 3 of [3]). Theorem 3.2 immediately implies the following corollary:

Corollary 3.1. In the algebra $\mathcal{C}$, maximal ideal = primitive ideal = prime ideal.

By using the descriptions of the Jacobson radical of a ring on p. 196 of [4], we also obtain the following corollary:

Corollary 3.2. The Jacobson radical $J(\mathcal{C})$ is equal to $\mathcal{R}$.

Although the fact that there exists an infinite set of pair-wise orthogonal idempotents in $\mathcal{C}$ when $p + q = \infty$ implies that $\mathcal{C}$ is not left Noetherian (hence not left Artinian), we do have the following theorem (note that the proof also shows that if $z = \infty$, then $\mathcal{C}$ is not left Noetherian even if $p + q < \infty$).

Theorem 3.3. The algebra $\mathcal{C}$ is Artinian (Noetherian) if and only if $p + q < \infty$ and $z < \infty$.

Proof. If $z = \infty$, we can assume that $Z = \mathbb{Z}_+$ and construct an infinite descending chain of ideals by letting $I_i = (e_0 e_1 \cdots e_i) \ (i \in \mathbb{Z}_+)$. Thus in this case, the algebra is not Artinian. It is also not Noetherian since the ascending chain of ideals $I_i \text{ generated by } \{e_0, \ldots, e_i\} \ (i \in \mathbb{Z}_+)$ does not terminate.

Suppose that $z$ is finite. There is nothing to prove if $z = 0$, so we assume that $z > 0$. We prove the statement that $\mathcal{C}$ is Artinian, the proof for Noetherian is similar. Since $\mathcal{C}(p,q)$ is either simple or a direct sum of two simple ideals, $\mathcal{C}/\mathcal{R}$ is Artinian. So to prove that $\mathcal{C}$ is Artinian, we only need to prove that $\mathcal{R}$ is Artinian. We use induction on $z$. If $z = 1$, then $\mathcal{R} = \mathcal{C}(p,q)e_0$, where $e_0$ is the only generator of $\mathcal{C}(z)$, is a simple ideal or is a direct sum of two simple ideals

$$\mathcal{R} = \mathcal{C}_1 e_0 \oplus \mathcal{C}_2 e_0,$$
and hence is Artinian. Suppose $\mathcal{C}$ is Artinian for $z = k \geq 1$, consider the case $z = k + 1$. Let $Z = \{0, 1, \ldots, k\}$. We claim that the ideal $(e_0)$ generated by $e_0$ is Artinian.

In fact, the ideal $R_0 = (e_1, \ldots, e_k)$ is Artinian by assumption, and

$$(e_0) = R_0e_0 \oplus \mathcal{C}(p, q)e_0,$$

so $(e_0)$ is Artinian since both $R_0e_0$ and $(e_0)/R_0e_0$ are. Now $R/(e_0) \simeq R_0$, implies that $\mathcal{R}$ is Artinian. \[\square\]

We now describe the nilpotent ideals of $\mathcal{C}$. For a basis element $e_I e_J e_K$ described in (2.3), let

$$N(e_I e_J e_K) = \{e_k : k \in K\}.$$ 

In general, for $u = \sum_{IJK} c_{IJK}e_I e_J e_K \in \mathcal{C}$, let

$$N(u) = \bigcup_{IJK} N(e_I e_J e_K).$$

That is, $N(u)$ is the set of the $e_\lambda$ ($\lambda \in Z$) that occur in the expression of $u$ via the basis described by (2.3). For a subset $S \subseteq Z$, let $I_S$ be the ideal of $\mathcal{C}$ generated by $\{e_\lambda : \lambda \in S\}$.

**Theorem 3.4.** Let $I \subseteq \mathcal{R}$ be an ideal of $\mathcal{C}$. Then $I$ is nilpotent if and only if there exists a finite subset $S \subseteq Z$ such that $I \subseteq I_S$.

**Proof.** We can assume that $z = \infty$, since the proof is only needed in this case. Since $I_S$ is nilpotent if $S$ is finite, the condition is sufficient.

Assume that $I$ is a nilpotent ideal of $\mathcal{C}$. We claim that if $I \not\subseteq I_S$ for any finite $S \subseteq Z$, then we can construct a sequence of elements $(u_1, u_2, \ldots)$ in $I$ such that the product $u_1u_2\cdots u_t \neq 0$ for any $t \geq 1$, and thus obtain a contradiction.

Start with $u_1 \neq 0$ in $I$, set $a_1$ to be a nonzero term in the expression of $u_1$ via the basis of (2.3). Since $N(u_1)$ is finite, $I \not\subseteq (N(u_1))$. Thus we can choose $u_2 \neq 0$ in $I$, such that if

$$u_2 = \bigoplus_{IJK} c_{IJK}e_I e_J e_K,$$

then there exists a nonzero term

$$a_2 = c_{IJK}e_I e_J e_K$$
with $N(e_K) \cap N(u_1) = \emptyset$. Now since $N(u_1) \cup N(u_2)$ is finite, we can choose $u_3 \neq 0$ in $I$ such that there exists a term

$$a_3 = d_{IJK} e_I e_J e_K$$

in the expression of

$$u_3 = \bigoplus_{IJK} d_{IJK} e_I e_J e_K$$

with

$$N(u_3) \cap (N(u_1) \cup N(u_2)) = \emptyset.$$  

Continuing this process, we get an infinite sequence of nonzero elements $(u_1, u_2, \ldots)$ in $I$. Note that if $\mathcal{C}(p, q)$ is simple, then $u_1 u_2 \cdots u_t \neq 0$ for any $t \geq 1$, and we are done. If $\mathcal{C}(p, q)$ is not simple, a product $u_1 u_2 \cdots u_t$ can be zero since for two terms

$$e_{I_i} e_{J_i} e_{K_i}, \quad i = 1, 2,$$

the $\mathcal{C}(p, q)$ parts $e_{I_i} e_{J_i}, i = 1, 2$, may belong to different simple ideals of $\mathcal{C}(p, q)$ (see (2.4)). Thus if $\mathcal{C}(p, q)$ is not simple, we need to further choose a subsequence $(v_1, v_2, \ldots)$ of $(u_1, u_2, \ldots)$ such that the projections of the $\mathcal{C}(p, q)$ parts of the corresponding $a_i$ (see above) to one of the simple ideals of $\mathcal{C}(p, q)$ are all nonzero to obtain the required sequence. This is possible since at least one of the projections of each of the $a_i$ to the two simple ideals of $\mathcal{C}(p, q)$ must be nonzero. □

Since $I_S$ is Noetherian for any finite subset $S \subseteq Z$ by Theorem 3.3, we have the following corollary:

**Corollary 3.3.** Every nilpotent ideal of $\mathcal{C}$ is finitely generated.

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