FROM THE POINCARÉ–CARTAN FORM TO A GERSTENHABER ALGEBRA OF POISSON BRACKETS IN FIELD THEORY

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Abstract
We consider the generalization of the basic structures of classical analytical mechanics to field theory within the framework of the De Donder-Weyl (DW) covariant canonical theory. We start from the Poincaré-Cartan form and construct the analogue of the symplectic form – the polysymplectic form of degree \((n + 1)\), \(n\) is the dimension of the space-time. The dynamical variables are represented by differential forms and the polysymplectic form leads to a natural definition of the Poisson brackets on forms. The Poisson brackets equip the exterior algebra of dynamical variables with the structure of a "higher-order" Gerstenhaber algebra. We also briefly discuss a possible approach to field quantization which proceeds from the DW Hamiltonian formalism and the Poisson brackets of forms.

1. INTRODUCTION

In this communication I discuss the canonical structure underlying the so-called De Donder–Weyl (DW) Hamiltonian formulation in field theory and its possible application to a quantization of fields. The abovementioned structure was found in a recent paper of mine \(^1\), to which I refer both for further references and for additional details. In particular, I am going to show that the relationships between the Poincaré-Cartan form, the symplectic structure and the Poisson structure, which are well known in the mathematical formalism of classical mechanics, have their natural counterparts also in field theory within the framework of the DW canonical theory. This leads to the analogue of the symplectic structure, which I call polysymplectic, and to the analogue of the Poisson brackets which are defined on differential forms.

Recall that the Euler-Lagrange field equations may be written in the following form (see for instance Refs. 2-4)

\[
\frac{\partial p^i_a}{\partial x^i} = -\frac{\partial H}{\partial y^a}, \quad \frac{\partial y^a}{\partial x^i} = \frac{\partial H}{\partial p^i_a} \tag{1.1}
\]

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in terms of the variables

\[ p^i_a := \frac{\partial L}{\partial (\partial_i y^a)}, \quad (1.2) \]

\[ H := p^i_a \partial_i y^a - L, \quad (1.3) \]

which are to be referred to as the DW momenta and the DW Hamiltonian function respectively. Here \( L = L(y^a, \partial_i y^a, x^i) \) is the Lagrangian density, \( x^i, i = 1, ..., n \) are space-time coordinates and \( y^a, a = 1, ..., m \) are field variables. Eqs. (1) are reminiscent of Hamilton’s canonical equations in mechanics and, therefore, may be thought of as a specific covariant Hamiltonian formulation of field equations. We call Eqs. (1) the DW Hamiltonian field equations and the formulation of field theory in terms of the variables \( p^i_a \) and \( H \) above the DW Hamiltonian formulation. The formulation above originates from the works of De Donder (1935) and Weyl (1935) on the variational calculus of multiple integrals.

The mathematical structures underlying this formulation of field theory were considered earlier by several authors in the context of the so-called multisymplectic formalism\(^5\) which was recently studied in detail in Refs. 6-8. However, the possible analogues of the symplectic structure and the Poisson brackets, which are known to be so fruitful in the canonical formulation of classical mechanics, still are not properly understood within the DW canonical theory.

Our interest to this subject is motivated by the explicit covariance of the formulation above, in the sense that the space and time variables are not discriminated as usual, and its finite dimensionality, in the sense that the formulation refers to the finite dimensional analogue of the phase space namely, the space of variables \((y^a, p^i_a)\), as well as by the attempts to understand if or how it is possible to construct a formulation of quantum field theory which would be based on the DW Hamiltonian formulation. Clearly, the answer to the latter question requires the analogue of the Poisson brackets and the bracket representation of the equations of motion corresponding to the DW formulation.

The canonical formalism in classical mechanics is related to the variational principle of least action and it may be derived from the fundamental object of the calculus of variations – the Poincaré-Cartan (P-C) form (see e.g. Ref. 9). The corresponding construction leads to structures which are known to be important for quantization. Conventional generalization to field theory implies setting off the time dimension from other space-time dimensions and leads to the infinite dimensional functional version of the abovementioned construction. Here we are interested in the field theoretical generalization of these structures within the space-time symmetric DW formulation.

2. Poincaré–Cartan Form, Classical Extremals and the Polysymplectic Form

In field theory, which is related to the variational problems with several independent variables, the analogue of the P-C form written in terms of the DW Hamiltonian variables (1.2), (1.3) reads

\[ \Theta = p^i_a dy^a \wedge \omega_i - H \omega, \quad (2.1) \]

where \( \omega := dx^1 \wedge ... \wedge dx^n \) and \( \omega_i := \partial_i \omega \). The equations of motion in the DW Hamiltonian form, Eqs. (1.1), may be shown to follow from the statement that the
classical extremals are the integral hypersurfaces of the multivector field of degree \( n \),
\[
\hat{X} := \frac{1}{n!}X^M(z) \partial_M,
\]  
(2.2)
where \( \partial_M := \partial_{M_1} \wedge ... \wedge \partial_{M_n} \), which annihilates the canonical \((n+1)\)-form
\[
\Omega_{DW} := d\Theta,
\]  
(2.3)
that is
\[
\hat{X} \int \Omega_{DW} = 0.
\]  
(2.4)
The integral hypersurfaces of \( \hat{X} \) are defined as the solutions of the equations
\[
\hat{X}^M(z) = N \partial(z^{M_1},...,z^{M_n}) \frac{\partial}{\partial(x^1,...,x^n)}
\]  
(2.5)
where a multiplier \( N \) depends on the chosen parametrization of a hypersurface and \( z^M := (x^i, y^a, p^i_a) \). The component calculations show that Eq. (2.4) specifies only a part of the components of \( \hat{X} \) and that the DW canonical equations (1.1) actually follow from the “vertical” components \( X^{vi_1...i_{n-1}} \). We call vertical the field and the DW momenta variables \( z^v := (y^a, p^i_a) \) and horizontal the space-time (independent) variables \( x^i \).

Introducing the notions of the vertical multivector field of degree \( p \):
\[
\hat{X}^V := \frac{1}{(p-1)!}X^{vi_1...i_{p-1}} \partial_{v_1...i_{p-1}},
\]  
(2.6)
the vertical exterior differential, \( d^V \), \( d^V... := dz^v \wedge \partial_v ... \), and the form
\[
\Omega := -dy^a \wedge dp^i_a \wedge \omega_i,
\]  
(2.7)
one may check that (2.4) is equivalent to
\[
\hat{X}^V \int \Omega = (-)^n d^V H,
\]  
(2.8)
if the parametrization in (2.5) is chosen such that
\[
\frac{1}{n!}X^{v_1...v_n} \partial_{v_1...v_n} \int \omega = 1.
\]
The form \( \Omega \) in (2.7) is to be referred to as polysymplectic.

The appearance of the DW field equations in the form of (2.8) suggests (cf. with mechanics!) that the polysymplectic form is a field theoretical analogue of the symplectic form, so that its properties should be taken seriously as a starting point for the canonical formalism.

As a generalization of (2.8), it is easy to see that the polysymplectic form maps in general the horizontal \( q \)-forms, \( \hat{F}^q \),
\[
\hat{F}^q := \frac{1}{q!}F^{i_1...i_q}(z)dx^{i_1...i_q},
\]  
(2.9)
where
\[
dx^{i_1...i_q} := dx^{i_1} \wedge ... \wedge dx^{i_q},
\]
to the vertical multivectors of degree \((n-q)\):
\[
\mathcal{X}^{n-q} \mathcal{J} \Omega = d^V \mathcal{F}
\]  
(2.10)
for all \(q = 0, \ldots, n-1\). Evidently, the horizontal forms play a role of dynamical variables within the present formalism. Henceforth we omit the superscripts \(V\) labelling the vertical multivectors.

The hierarchy of maps (2.10) may be viewed as a local consequence of the hierarchy of “graded canonical symmetries”
\[
\mathcal{L}^p \mathcal{X} \mathcal{J} \Omega = 0,
\]  
(2.11)
p = 1, \ldots, n, which are formulated in terms of the generalized Lie derivatives with respect to the vertical multivector fields. By definition, for any form \(\mu\)
\[
\mathcal{L}^p_{\mathcal{X}} \mu := \mathcal{X} \mathcal{J} d^V \mu - (-1)^p d^V (\mathcal{X} \mathcal{J} \mu).
\]  
(2.12)

Now, by analogy with the terminology known from mechanics, I call the vertical multivector fields fulfilling (2.11) \(\text{locally Hamiltonian}\) and those fulfilling (2.10) (globally) \(\text{Hamiltonian}\). Correspondingly, the forms to which the Hamiltonian multivector fields can be associated through the map (2.10) are referred to as the \(\text{Hamiltonian forms}\).

The notion of a Hamiltonian form implies certain restriction on the dependence of its components on the DW momenta. For example, the components of the vector field \(X_F := X^a \partial_a + X^i_a \partial_t^a\) associated through the map \(X_F \mathcal{J} \Omega = d^V F\) with the \((n-1)\)-form \(F := F^i \omega_i\) are given by
\[
X^i_a = \partial_a F^i, \quad -X^a \delta^i_j = \partial_t^a F^i.
\]  
(2.13)
The latter relation restricts the admissible \((n-1)\)-forms to those which have a simple dependence on the DW momenta namely, \(F^i(y, p, x) = f^a(y, x) p^a + g^i(y, x)\).

Note also that the Hamiltonian multivector field associated with a form through the map (2.10) is actually defined up to an addition of \(\text{primitive}\) fields which annihilate the polysymplectic form
\[
\mathcal{X}_0 \mathcal{J} \Omega = 0.
\]  
(2.14)
Therefore, the image of a Hamiltonian form under the map (2.10) given by the polysymplectic form is rather the equivalence class of Hamiltonian multivector fields of corresponding degree modulo an addition of primitive fields.

3. THE POISSON BRACKETS ON FORMS AND A GERSTENHABER ALGEBRA

It is natural to define the bracket of two locally Hamiltonian multivector fields as follows:
\[
[X_1, X_2] \mathcal{J} \Omega := \mathcal{L}^p_{X_1} (X_2 \mathcal{J} \Omega).
\]  
(3.1)
From the definition it follows that
\[
deg([X_1, X_2]) = p + q - 1,
\]  
(3.2)
\[
[X_1, X_2] = -(-1)^{(p-1)(q-1)}[X_2, X_1],
\]  
(3.3)
\[
(-1)^{g_1 g_2} [X, [X, X]] + (-1)^{g_1 g_2} [X, [X, X]]
\]  
(3.4)
where \( g_1 = p - 1 \), \( g_2 = q - 1 \) and \( g_3 = r - 1 \).

These properties allow us to identify the bracket in (3.1) with the vertical (i.e. taken w.r.t. the vertical variables) Schouten–Nijenhuis (SN) bracket of multivector fields and to conclude that the space of LH fields is a graded Lie algebra with respect to the (vertical) SN bracket.

For two Hamiltonian multivector fields one obtains

\[
[\mathring{X}, \mathring{X}']_\Omega = \mathcal{L}_{\mathring{X}} d V \mathring{F}'_2
\]

\[
= (-1)^{p+1} d V (\mathring{X} d V \mathring{F}'_2)
\]

\[
=: -d V \{\mathring{F}_1, \mathring{F}'_2\},
\]

where \( r = n - p \) and \( s = n - q \). From (3.5) it follows that the SN bracket of two Hamiltonian fields is a Hamiltonian field (as in mechanics). In (3.6) one defines the bracket operation on Hamiltonian forms which is induced by the vertical SN bracket of multivector fields associated with them.

From the definition in (3.6) it follows

\[
\{\mathring{F}_1, \mathring{F}'_2\} = (-1)^{(n-r)} X_1 d V \mathring{F}'_2 = (-1)^{(n-r)} X_1 \mathcal{J} X_2 \mathcal{J} \Omega
\]

and

\[
\deg \{\mathring{F}_1, \mathring{F}'_2\} = r + s - n + 1,
\]

\[
\{\mathring{F}_1, \mathring{F}'_2\} = (-1)^{g_1 g_2} \{\mathring{F}_2, \mathring{F}'_1\},
\]

\[
(-1)^{g_1 g_3} \{\mathring{F}, \{\mathring{F}, \mathring{F}'\}\} + \]

\[
(-1)^{g_1 g_2} \{\mathring{F}, \{\mathring{F}, \mathring{F}'\}\} + (-1)^{g_2 g_3} \{\mathring{F}, \{\mathring{F}, \mathring{F}'\}\} = 0,
\]

\[
\{\mathring{F}, \mathring{F} \wedge \mathring{F}'\} = \{\mathring{F}, \mathring{F}'\} \wedge \mathring{F} + (-1)^{q(n-p-1)} \mathring{F} \wedge \{\mathring{F}, \mathring{F}'\}
\]

\[
+ \ \text{higher-order corrections},
\]

where \( g_1 = n - p - 1 \), \( g_2 = n - q - 1 \) and \( g_3 = n - r - 1 \).

The algebraic construction which satisfies the axioms (3.9), (3.10) and (3.11) without higher-order corrections, together with the familiar properties of the exterior product, is known as the Gerstenhaber algebra. Higher-order corrections in Eq. (3.11) are composed of the terms like

\[
\frac{1}{(n-p-1)!} X^{v_1 \ldots v_{n-p-1}} (\partial_{v_1} \ldots \partial_{v_{n-p-1}} d V \mathring{F}) \wedge \partial_{i_{s+1}} \ldots \partial_{i_{s+2}} \ldots \partial_{i_{n-p-1}} \mathring{F},
\]

with \( s = 1, \ldots, n - p - 1 \), and are similar to the last term in the "Leibniz rule" for, say, the second derivative: \((fg)'' = f''g + fg'' + 2f'g'\). They appear due to the fact that the multivector field \( \mathring{X} \) associated with the form \( \mathring{F} \) does not act on exterior forms as a graded derivation, but rather as a graded differential operator of order \(-(n-p)\) which is composed of the subsequent actions of graded derivations of order \(-1\). The latter are given by the vector fields \( \partial_v \) and \( \partial_{i_s} \), \( s = 1, \ldots, n-p-1 \), which constitute the vertical multivector \( X^{n-p} \). The algebraic structure given by Eqs. (3.9)–(3.11) may be called the higher-order Gerstenhaber algebra, but in the following we will continue to refer to it as a Gerstenhaber algebra, for short.

Remark: Strictly speaking, the space of Hamiltonian forms is not closed with respect to the exterior product, so that the full justification of the Leibniz rule (3.11)
requires a generalization of the above construction which admits arbitrary horizontal forms as the dynamical variables (see Ref. 12). Higher-order corrections in (3.11) were overlooked in previous communications (cf. Refs. 1,12).

4. EQUATIONS OF MOTION IN THE BRACKET FORM

By analogy with mechanics, one can expect that the equations of motion are given by the bracket with the DW Hamiltonian function. Indeed, for the bracket of $H$ with the \((n-1)\)-form $F := F^a \omega_i$ one obtains

$$\{[H,F] = X_F \int d^V H = X_F^a \partial_a H + X_F^i \partial_i H$$

$$= \partial_i F^j \partial_j F^i + \partial_i y^k \partial_k F^i,$$

where we have used (2.3) and (1.1). Introducing the total (i.e. evaluated on extremals) exterior differential $d$ of a horizontal form of degree $p$,

$$dp^F := \partial_i z^M dx^i \wedge \partial_M F = \partial_i z^v dx^i \wedge \partial_v F + dx^i \wedge \partial_i F = d^V F + d^{hor} F,$$

one can write the equation of motion of Hamiltonian \((n-1)\)-form $F$ as (by definition, $\ast^{-1} \omega := 1$)

$$\ast^{-1} dF = \{[H,F] \} + \partial_i F^i. \tag{4.1}$$

The bracket of a $p$–form with $H$ vanishes for $p < n-1$. The equations of motion of arbitrary forms may be written in terms of the bracket with the $n$–form $H \omega$. This implies a certain extension of the construction in Section 2. Namely, we map $H \omega$ to a vector-valued form $\tilde{X} := \tilde{X}^i dx^i \otimes \partial_v$ by

$$\tilde{X} \bigwedge \Omega = d^V H \omega, \tag{4.2}$$

where $\tilde{X} \bigwedge \Omega := X^v_i dx^i \wedge (\partial_v \bigwedge \Omega)$ is the Fröhlicher-Nijenhuis inner product. From (4.2) it follows

$$\tilde{X}^i_k = \partial_k H, \quad \tilde{X}^i_k \delta^k_i = -\partial_a H. \tag{4.3}$$

Substitution of the natural parametrization of $\tilde{X}$:

$$\tilde{X}^v_k = \frac{\partial z^v}{\partial x^k},$$

into (4.3) leads to the DW Hamiltonian equations (1.1).

Now, we define the bracket with $H \omega$ (cf. (3.7))

$$\{[H \omega,F^p] := \tilde{X}_{H \omega} \bigwedge d^V F \tag{4.4}$$

and find that

$$d^p F = \{[H \omega,F^p] \} + d^{hor} F. \tag{4.5}$$

Thus, we have shown that the bracket with the DW Hamiltonian $n$–form $H \omega$ is related to the exterior differential of a form.

Remark: The bracket which is naively defined in (4.4) does not satisfy in general the axioms of a Gerstenhaber algebra. The appropriate extension of a Gerstenhaber algebra structure to $n$–forms is a part of the generalization of the present construction to the forms which are not Hamiltonian according to the definition in Section 2 (see Ref. 12).
5. TOWARDS A QUANTIZATION

An appropriate quantization of a Gerstenhaber algebra of exterior forms, which generalizes to field theory the Poisson algebra of dynamical variables, may in principle lead to certain quantization procedure in field theory. The purpose of this section is to discuss briefly a possible heuristic approach to such a quantization.

We start from the observation that

\[ \{ [p_a, y^b] \} = \delta^b_a, \]  

where \( p_a := p_i^a \omega_i \) is the \((n-1)\)-form which may be considered as the momentum variable canonically conjugate to fields \( y^a \). Applying Dirac’s quantization prescription \([ , ]_\pm = i\hbar \{ , \}_\pm \), one obtains the canonical commutation relation for the operators corresponding to fields and the \((n-1)\)-form momenta

\[ [\hat{p}_a, \hat{y}^b] = i\hbar \delta^b_a. \]  

In the “\(y\)-representation” one finds the differential operator realization of \( \hat{p}_a \)

\[ \hat{p}_a = i\hbar \frac{\partial}{\partial y^a}. \]  

Based on the analogy with the quantization of classical mechanics in Schrödinger’s representation and the observation made in Section 4 that the exterior differential is related to the DW Hamiltonian \( n \)-form, one can conjecture the following form of the covariant “Schrödinger equation”

\[ i\hbar \frac{d}{dt} \Psi = (H \omega)^{op} \Psi \]  

for the “wave function” \( \Psi = \Psi(x^i, y^a) \), which depends on the space-time and field variables which form the analogue of a configuration space within the present formulation.

In the particular example of a system of scalar fields \( y^a \) interacting through the potential \( V(y) \), which is given by the Lagrangian

\[ L = -\frac{1}{2} \partial_i y^a \partial^i y_a - V(y), \]  

the DW Hamiltonian function takes the form

\[ H = -\frac{1}{2} p_i^a p_i^a + V(y). \]  

In terms of the \((n-1)\)-form momenta variables \( p_a \) the \( n \)-form \( H \omega \) may be written as

\[ H \omega = \frac{1}{2} * p_a \wedge p^a + V(y) \omega, \]  

where \( * p_a = -p_i^a dx_i \) (the Minkowski metric in the \(x\)-space is assumed). The realization of the operator corresponding to the non-Hamiltonian one-form \( * p_a \)

\[ \hat{* p}_a = * \hat{p}_a \]  

is suggested by the quantization of the bracket

\[ \{ [* p_a, y^b \omega_i ] \} = -\delta^b_a dx_i = * \{ [ p_a, y^b \omega_i ] \}, \]
which may be calculated either with the help of the Leibniz rule (3.10) or within a more
general scheme, where one associates arbitrary horizontal forms with the differential
operators on exterior algebra which are represented by the multivector-valued forms
instead of the multivectors as in the case of Hamiltonian forms.

Furthermore, the classical identity $\omega = *1$ suggests that $\hat{\omega} = *$ and, therefore, one
can write

$$(H\omega)^{op} = *\left(-\frac{\hbar^2}{2} \triangle + V(y)\right) =: *H^{op}, \quad (5.9)$$

where $\triangle := \partial^a \partial_a$ is the Laplace operator in a field space. Thus, the Schrödinger
equation (5.4) may also be written as

$$i\hbar *^{-1} d\Psi = H^{op}\Psi. \quad (5.10)$$

Evidently, this equation makes sense only if the wave function $\Psi$ is a nonhomogeneous
horizontal form. In the simple case of the DW Hamiltonian operator (5.9), which does
not depend explicitly on the space-time coordinates, one can take into account only the
zero- and $(n-1)$-form contributions, so that

$$\Psi = \psi_0(x, y) + \psi^i(x, y) \omega_i. \quad (5.11)$$

Substituting (5.11) into (5.10), one obtains the component form of our Schrödinger
equation:

$$i\hbar \partial_i \psi^i = H^{op} \psi_0, \quad (5.12)$$

$$-i\hbar \partial_0 \psi_0 = H^{op} \psi_i. \quad (5.13)$$

The integrability condition of this set of equations is

$$\delta \Psi = 0. \quad (5.14)$$

By a straightforward calculation, one can derive from (5.12) and (5.13) the following
conservation law

$$\partial_i [\bar{\psi}_0 \psi^i + \psi_0 \bar{\psi}^i] = \frac{i\hbar}{2} \partial_a [\bar{\psi}_0 \partial_a \psi_0 - \bar{\psi}^i \partial_a \psi_i]. \quad (5.15)$$

If one assumes a sufficiently rapid decay of the wave function $\Psi(x, y)$ for large values
of fields $|y| \rightarrow \infty$, by Gauss’ theorem one obtains

$$\partial_i \int dy [\bar{\psi}_0 \psi^i + \psi_0 \bar{\psi}^i] = 0. \quad (5.16)$$

Thus, the current

$$j^i := \int dy [\bar{\psi}_0 \psi^i + \psi_0 \bar{\psi}^i]$$

is the conserved space-time current of the theory. It suggests the inner product of
nonhomogeneous forms $\Psi$, which one needs for the calculation of quantum theoretical
expectation values.

The covariant Schrödinger equation may be solved by means of the separation of
field and space-time variables. Namely, let us write

$$\Psi(x, y) = \Phi(x) f(y), \quad (5.17)$$

where $\Phi(x)$ is a nonhomogeneous form with the components depending on $x$:

$$\Phi(x) := \phi_0(x) + \phi^i(x) \omega_i \quad (5.18)$$
and $f(y)$ is a function of field variables. Substituting this Ansatz into the Schrödinger equation (5.10), we arrive at the eigenvalue problem for the DW Hamiltonian operator

$$H^{\text{op}} f = \kappa f,$$

(5.19)

and the equation on $\Phi(x)$:

$$i\hbar^{-1} d\Phi(x) = \kappa \Phi(x).$$

(5.20)

From the latter equation it follows that

$$\square \phi_0 = \frac{\kappa^2}{\hbar^2} \phi_0, \quad \phi_i = -\frac{i\hbar}{\kappa} \partial_i \phi_0.$$

(5.21)

The solutions of (5.19) and (5.21) provide us with a basis for decomposition of an arbitrary solution of the covariant Schrödinger equation.

**Remarks:**

1. The canonical bracket (5.1) belongs to the subalgebra of zero- and $(n-1)$-forms of a Gerstenhaber algebra of dynamical variables. The other canonical brackets from this subalgebra are

$$\{[p^i_a, y^b \omega_j]\} = \delta^b_a \delta^i_j,$$

(5.22)

$$\{[p^a, y^b \omega_i]\} = \delta^a_b \omega_i.$$

(5.23)

Quantization of these three brackets is a part of the problem of quantization of the center of a Gerstenhaber algebra, which is formed by the forms of the kind $p^i_a dx^{i-1} \ldots dx^0$ and $y^a dx^{n-1} \ldots dx^0$, where $dx^{n-1} \ldots dx^0$ denotes the basis elements of Grassmann algebra of horizontal forms. The question as to which subalgebra of a Gerstenhaber algebra of dynamical variables should or can be quantized remains open and deserves the same careful study as the similar question concerning the quantizable subalgebra of the Poisson algebra of observables in mechanics. The minimal subalgebra is that of $(n-1)$-forms and the canonical bracket from this subalgebra is given by (5.23). Its quantization rather than a quantization of (5.1) gives rise to the operator realization of $p_a$ in (5.3). The quantization of the subalgebra of zero- and $(n-1)$-forms with the canonical brackets (5.1), (5.22) and (5.23) leads to the problem of realization of the operator $\hat{p}_a$, which would be consistent with the realization of $\hat{p}_a$ and the requirement $\hat{p}_a = \hat{p}_a \circ \hat{\omega}_i$, as well as to the problem of the proper realization of the operation $\circ$ of the multiplication of quantum operators. When quantizing the (centre of the) Gerstenhaber algebra, the latter problem is that of the proper realization of the quantized wedge product, which is in this case a generalization of the Jordan symmetric product of operators in quantum mechanics.

2. The quantization of the bracket (5.1) leads to the operator realization of the $(n-1)$–form $p_a$ which is the 0-form. In general, the form degree of the operator corresponding to a dynamical variable is different from the classical form degree of the latter. This gives rise to an additional problem of which degree should define the graded products of operators which correspond to the exterior product and the quantized Poisson bracket respectively.

3. It is interesting to note that the realization (5.3) is not consistent with the classical property $dx^i \wedge p_a = (-)^{n-1} p_a \wedge dx^i$ which one may require to be also fulfilled on the quantum level. This may be achieved if the quantization prescription is modified in such a way that

$$[\ ,\ ]_{\pm} = \gamma \hbar \{\ ,\ \}_{\pm}$$

(5.24)
where \( \gamma \) denotes the imaginary unit corresponding to the Clifford algebra of the \( n \)-dimensional space-time over which a field theory under quantization is formulated. In Minkowski space-time \( \gamma := \gamma_0 \gamma_1 \gamma_2 \gamma_3 \). In the case of mechanics \((n = 1) \gamma = i \) and the above quantization prescription reduces to that of Dirac. The quantization according to (5.24) leads to the realization

\[
\hat{p}_a = \gamma \hbar \partial_a \quad \text{and} \quad \hat{dx}^i \wedge = \gamma^i \wedge,
\]

where \( \wedge \) on the right hand side denotes the graded symmetrized Clifford product. Correspondingly, the wave function may be considered as taking values in the Clifford–Kähler algebra of nonhomogeneous forms which corresponds to the \( n \)-dimensional space-time (see e.g. Ref. 13 and the references quoted there). The latter reduces to complex numbers in the case of mechanics. This quantization prescription leads to the same realization of the DW Hamiltonian operator as in (5.9). However, in general, it is not clear which quantization prescription is more appropriate both physically and mathematically for the quantization of the suitable “quantizable” subalgebra of a Gerstenhaber algebra of forms–dynamical–variables in field theory.

4. The elements of quantum theory presented above possess the basic features of a quantum description of dynamics and its connections with the structures of classical mechanics. These elements are easily seen to reduce to the corresponding elements of quantum mechanics at \( n = 1 \). In this sense at least our formulation may be viewed as an approach to the quantum description of fields. Establishing the possible links with the known approaches and results in quantum field theory and a physical interpretation of the present formulation poses many conceptual questions and needs a further study which we hope to communicate elsewhere. In particular, it would be interesting to understand a possible relation of our nonhomogeneous form-valued wave function \( \Psi(x, y) \) to the Schrödinger wave functional \( \Psi(t, [y(x)]) \) and of our covariant Schrödinger equation to the functional Schrödinger equation.

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