Synchronization of dissipative dynamical systems driven by non-Gaussian Lévy noises *

Xianming Liu², Jinqiao Duan¹†, Jicheng Liu² and Peter E. Kloeden³

1. Department of Applied Mathematics
   Illinois Institute of Technology
   Chicago, IL 60616, USA
   E-mail: duan@iit.edu

2. School of Mathematics and Statistics
   Huazhong University of Science and Technology
   Wuhan 430074, China
   and

3. Institut für Mathematik
   Johann Wolfgang Goethe-Universität
   D-60054, Frankfurt am Main, Germany
   E-mail: kloeden@math.uni-frankfurt.de

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Abstract

Dynamical systems driven by Gaussian noises have been considered extensively in modeling, simulation and theory. However, complex systems in engineering and science are often subject to non-Gaussian

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†Corresponding author: duan@iit.edu
fluctuations or uncertainties. A coupled dynamical system under non-
Gaussian Lévy noises is considered. After discussing cocycle prop-
erty, stationary orbits and random attractors, a synchronization phe-
nomenon is shown to occur, when the drift terms of the coupled system
satisfy certain dissipativity and integrability conditions. The synchro-
nization result implies that coupled dynamical systems share a dy-
namical feature in some asymptotic sense.

**Key Words:** Synchronization; Lévy noise, Skorohod metric, ran-
don attractor, càdlàg random dynamical system, impact of noise.

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60G17

1 Introduction

Synchronization of coupled dynamical systems is an unbiqutous phenomenon
that has been observed in biology, physics and other areas. It concerns cou-
pled dynamical systems that share a dynamical feature in an asymptotic
sense. A descriptive account of its diversity of occurrence can be found in
the recent book [33]. Recently Caraballo and Kloeden [6, 7] have proved
that synchronization in coupled deterministic dissipative dynamical systems
persists in the presence of various Gaussian noises (in terms of Brownian mo-
tion), provided that appropriate concepts of random attractors and stochastic
stationary solutions are used instead of their deterministic counterparts.

In this paper we investigate a synchronization phenomenon for coupled
dynamical systems driven by non-Gaussian noises (in terms of Lévy motion).
We show that couple dissipative systems exhibits synchronization for a class
of Lévy motions.

Gaussian processes like Brownian motion have been widely used to model
fluctuations in engineering and science. The sample paths of a particle driven
by Brownian motion are continuous in time almost surely (i.e., no jumps), the
mean square displacement increases linearly in time (i.e., normal diffusion),
and the probability density function decays exponentially in space (i.e., light
tail or exponential relaxation) [20]. But some complex phenomena involve
non-Gaussian fluctuations, with peculiar properties such as anomalous dif-
fusion (mean square displacement is a nonlinear power law of time) [4] and
heavy tail (non-exponential relaxation) [35]. For instance, it has been argued
that diffusion in a case of geophysical turbulence [28] is anomalous. Loosely
speaking, the diffusion process consists of a series of “pauses”, when the particle is trapped by a coherent structure, and “flights” or “jumps” or other extreme events, when the particle moves in a jet flow. Moreover, anomalous electrical transport properties have been observed in some amorphous materials such as insulators, semiconductors and polymers, where transient current is asymptotically a power law function of time \cite{26, 12}. Finally, some paleoclimatic data \cite{9} indicates heavy tail distributions and some DNA data \cite{28} shows long range power law decay for spatial correlation.

Lévy motions are thought to be appropriate models for non-Gaussian processes with jumps \cite{25}. Let us recall that a Lévy motion $L(t)$, or $L_t$, is a non-Gaussian process with independent and stationary increments, i.e., increments $\Delta L(t, \Delta t) = L(t + \Delta t) - L(t)$ are stationary (therefore $\Delta L$ has no statistical dependence on $t$) and independent for any non overlapping time lags $\Delta t$. Moreover, its sample paths are only continuous in probability, namely, $\mathbb{P}(|L(t) - L(t_0)| \geq \delta) \to 0$ as $t \to t_0$ for any positive $\delta$. With a suitable modification \cite{1}, these paths may be taken as càdlàg, i.e., paths are continuous on the right and have limits on the left. This continuity is weaker than the usual continuity in time. In fact, a càdlàg function has finite or at most countable discontinuities on any time interval (see, e.g., p.118, \cite{1}). This generalizes the Brownian motion $B(t)$ or $B_t$, since $B(t)$ satisfies all these three conditions, but additionally, (i) almost every sample path of the Brownian motion is continuous in time in the usual sense, and (ii) the increments of Brownian motion are Gaussian distributed.

This paper is organized as follows. We first recall some basic facts about stochastic differential equations (SDEs) driven by Lévy noise in section 2, including a fact that the solution mappings of such SDEs generate random dynamical systems (RDS). In section 3, we formulate the problem of synchronization of stochastic dynamical systems driven by Lévy noises. The main result (Theorem \ref{thm:main}) and an example are presented in section 4.

2 Dynamical systems driven by Lévy noises

Dynamical systems driven by non-Gaussian Lévy motions have attracted much attention recently \cite{1, 27}. Under certain conditions, the SDEs driven by Lévy motion generate stochastic flows \cite{1} \cite{18}, and also generate random dynamical systems (or cocycles) in the sense of Arnold \cite{2}. Recently, exit time estimates have been investigated by Imkeller & Pavlyukevich \cite{13} \cite{14},
and Yang & Duan [34] for SDEs driven by Lévy motion. This shows some qualitatively different dynamical behaviors between SDEs driven by Gaussian and non-Gaussian noises.

2.1 Lévy processes
A Lévy process or motion on $\mathbb{R}^d$ is characterized by a drift parameter $\gamma \in \mathbb{R}^d$, a covariance $d \times d$ matrix $A$ and a non-negative Borel measure $\nu$, defined on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and concentrated on $\mathbb{R}^d \setminus \{0\}$, which satisfies

$$\int_{\mathbb{R}^d \setminus \{0\}} (y^2 \wedge 1) \nu(dy) < \infty,$$

or equivalently

$$\int_{\mathbb{R}^d \setminus \{0\}} \frac{y^2}{1 + y^2} \nu(dy) < \infty.$$

This measure $\nu$ is the so-called the Lévy jump measure of the Lévy process $L(t)$. Moreover Lévy process $L_t$ has the following Lévy-Itô decomposition

$$L_t = \gamma t + B_t + \int_{|x|<1} x \tilde{N}(t, dx) + \int_{|x|\geq 1} x N(t, dx)$$

(1)

where $N(dt, dx)$ is Poisson random measure,

$$\tilde{N}(dt, dx) = N(dt, dx) - \nu(dx)dt$$

(2)

is the compensated Poisson random measure of $L_t$, and $B_t$ is an independent Brownian motion on $\mathbb{R}^d$ with covariance matrix $A$ (see [1, 25, 24, 34]). We call $(A, \nu, \gamma)$ the generating triplet.

The next useful lemma provides states some important pathwise properties of $L_t$ with two-sided time $t \in \mathbb{R}$. Here $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^d$.

**Lemma 1.** (*Pathwise boundedness and convergence*)
Let $L_t$ be a two-sided Lévy motion on $\mathbb{R}^d$ for which $E|L_1| < \infty$ and $E L_1 = \gamma$. Then

(i) $\lim_{t \to \pm \infty} \frac{1}{t} L_t = \gamma$.
(ii) The integrals \( \int_{-\infty}^{t} e^{-\lambda(t-s)} \, dL_s(\omega) \) are pathwisely uniformly bounded in \( \lambda > 0 \) on finite time intervals \([T_1, T_2]\) in \( \mathbb{R} \).

(iii) The integrals \( \int_{T_1}^{t} e^{-\lambda(t-s)} \, dL_s(\omega) \to 0 \) as \( \lambda \to \infty \), pathwise on finite time intervals \([T_1, T_2]\) in \( \mathbb{R} \).

Proof. (i) This convergence result comes from the law of large numbers, in [25], Theorem 36.5.

(ii) Due to the continuous of function \( h(t) = e^{-\lambda t} \), on integrating by parts we obtain

\[
\int_{-\infty}^{t} e^{-\lambda(t-s)} \, dL_s(\omega) = L_t(\omega) - \lambda \int_{-\infty}^{t} e^{-\lambda(t-s)} L_s(\omega) \, ds.
\]

Then we use (i) to conclude (ii).

(iii) Integrating again by parts, it follows that

\[
\int_{T_1}^{t} e^{-\lambda(t-s)} \, dL_s(\omega) = (L_t - L_{T_1})e^{-\lambda(t-T_1)} + \lambda \int_{T_1}^{t} e^{-\lambda(t-s)}(L_t(\omega) - L_s(\omega)) \, ds,
\]

from which the result follows. \( \square \)

Remark 1. The assumptions on \( L_t \) in the above lemma are satisfied by a wide class of Lévy processes, for instance, the \( \alpha \)-stable symmetric Lévy motion on \( \mathbb{R}^d \) with \( 1 < \alpha < 2 \). Indeed, in this case, we have \( \int_{|x|>1} |x| \nu(dx) < \infty \), and then \( E|L_1| < \infty \); see [25] Theorem 25.3.

Let us introduce the canonical sample space for Lévy processes, the space \( \Omega = D(\mathbb{R}, \mathbb{R}^d) \) of càdlàg functions, i.e., continuous on the right and have limits on the left, defined on \( \mathbb{R} \) and taking values in \( \mathbb{R}^d \).

If we use the usual compact-open metric, \( D(\mathbb{R}, \mathbb{R}^d) \) is not separable. However, it is complete and separable when endowed with the Skorohod metric \([3, 29]\), in which case we call \( D(\mathbb{R}, \mathbb{R}^d) \) a Skorohod space. The Skorohod metric on \( D(\mathbb{R}, \mathbb{R}^d) \) is defined as

\[
d(x, y) := \sum_{m=1}^{\infty} \frac{1}{2^m} (1 \wedge d_m^o(x^m, y^m)) \quad \text{for all } x, y \in D
\]

where \( x^m(t) := g_m(t)x(t), \ y^m(t) := g_m(t)y(t) \) with

\[
g_m(t) := \begin{cases} 
1, & \text{if } |t| \leq m - 1 \\
m - t, & \text{if } m - 1 \leq |t| \leq m, \\
0, & \text{if } |t| \geq m 
\end{cases}
\]
and
\[
d^\circ_m(x, y) := \inf_{\lambda \in \Lambda} \left\{ \sup_{-m \leq s < t \leq m} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \vee \sup_{-m \leq t \leq m} |x(t) - y(\lambda(t))| \right\},
\]
where \( \Lambda \) denotes the set of strictly increasing, continuous functions from \( \mathbb{R} \) to itself.

Similarly, we can define a Skorohod space on a bounded time interval \( D([T_1, T_2], \mathbb{R}^d) \). Then, in particular, the metric \( d_1^\circ \) is the Skorohod metric on \( D([-1, 1], \mathbb{R}^d) \).

We recall the following compactness result (see [3], Page 116) in \( D([T_1, T_2], \mathbb{R}^d) \).

**Lemma 2.** (Ascoli-Arzelà theorem in \( D([T_1, T_2], \mathbb{R}^d) \))

For \( S \subset [T_1, T_2] \), let
\[
w_x(S) = \sup \{|x(s) - x(t)| : s, t \in S\}
\]
and for \( 0 < \delta < 1 \) define
\[
w'_x(\delta) = \inf \max_{t_i \in \{t_i\}} w_x([t_{i-1}, t_i]),
\]
where the infimum is taken over all the finite sets \( \{t_i\} \) of points satisfying \( T_1 = t_0 < t_1 < \ldots < t_r = T_2 \) with \( t_i - t_{i-1} < \delta \) for \( i = 1, 2, \ldots, r \).

Then, a set \( B \) has compact closure in the Skorohod space \( D([T_1, T_2], \mathbb{R}^d) \) if and only if \( \sup_{x \in B} \sup_t |x(t)| < \infty \) and \( \lim_{\delta \to 0} \sup_{x \in A} w'_x(\delta) = 0 \).

**2.2 SDE driven by Lévy processes**

We consider the following stochastic differential equation (SDE) driven by Lévy motion, which has continuous drift and Brownian motion components, namely
\[
dY(t) = b(Y(t-))dt + \sigma(Y(t-))dB_t + \int_{|x|<c} F(Y(t-), x)\tilde{N}(dt, dx)
\]
\[
+ \int_{|x|\geq c} G(Y(t-), x)N(dt, dx)
\]
(3)
where \( \tilde{N}(dt, dx) \) and \( N(dt, dx) \) are defined above, and the coefficients \( b, \sigma, F, G \) are all assumed to be measurable. Here \( F \) and \( G \) may be different, while the positive parameter \( c \) may be different from 1, which allows greater generality.

We introduce the \( d \times d \) matrix

\[
a(x, y) = \sigma(x)\sigma(y)^T, \quad x, y \in \mathbb{R}^d, \quad (4)
\]

and define

\[
\|a(x, y)\| = \sum_{i=1}^{d} |a_{i,i}(x, y)|.
\]

We make the following general assumptions for the SDE (3):

**A.1** There exits \( K_1 > 0 \) such that, for all \( y_1, y_2 \in \mathbb{R}^d \),

\[
|b(y_1) - b(y_2)|^2 + \|a(y_1, y_1) - 2a(y_1, y_2) + a(y_2, y_2)\|
\]

\[
+ \int_{|x|<1} |F(y_1, x) - F(y_2, x)|^2 \nu(dx) \leq K_1 |y_1 - y_2|^2.
\]

**A.2** There exits \( K_2 > 0 \) such that, for all \( y \in \mathbb{R}^d \)

\[
|b(y)|^2 + \|a(y, y)\| + \int_{|x|<1} |F(y, x)|^2 \nu(dx) \leq K_2 |1 + y|^2.
\]

**A.3** There exits \( \delta > 2 \) and \( K_3 > 0 \) such that, for all \( y_1, y_2 \in \mathbb{R}^d \),

\[
\int_{|x|<1} |F(y_1, x) - F(y_2, x)|^p \nu(dx) \leq K_3 |y_1 - y_2|^p,
\]

for all \( 2 \leq p \leq \delta \).

From [1], Theorem 6.23, page 304, we have the following existence and uniqueness result for solutions of such SDE driven by Lévy motion

**Lemma 3.** Suppose that conditions (A1) and (A2) are satisfied and that the mapping \( y \to G(y, x) \) be continuous for all \( |x| \geq c \). Then there exists a unique global càdlàg adapted solution of the SDE (3).
Note that a càdlàg solution process has finite or at most countable discontinuities on any time interval (see, e.g., p.118, [1]). For more details about SDEs driven by Lévy motions, see [11, 19, 18]. Due to the Lévy-Itô decomposition (1), the following SDE, which we consider in the sequel,

\[ dY(t) = f(Y(t-)) dt + g(Y(t-)) dL_t \]

is a special case of (3).

**Remark 2.** The reason to take the left limit in \( Y(t-) \) in the equation (3) is to make sure that the càdlàg solution process \( Y \) is predictable and unique [21]. For typographical convenience, however, we will write \( Y(t) \) instead of \( Y(t-) \) for the rest of this paper. Moreover, in the case of additive noise, i.e., if the noise intensity \( g(\cdot) \) does not depend on the state \( Y \), the distinction for left limit or not is not necessary, when we consider the integral form of the equation (3), as \( \int_t^T f(Y(t-)) dt = \int_t^T f(Y(t)) dt \) for continuous \( f \). In this case \( f(Y(t-)) \) has only countable discontinuous points and is thus Riemann and Lebesgue integrable.

**Remark 3.** The above global assumptions do not hold for SDE, which we consider in the sequel, with a nonlinear dissipative drift \( f \) term such as \( x^T f(x) \leq K - l|x|^2 \) for some constants \( K \geq 0 \) and \( l > 0 \). However analogous global existence and uniqueness results also hold in this case since the dissipativity condition prevents explosions and hence ensures otherwise local existence is global. See [29] for more details.

### 2.3 Random dynamical systems

Following Arnold [2], a random dynamical system (RDS) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) consists of two ingredients: A driving flow \( \theta_t \) on the probability space \( \Omega \), i.e., \( \theta_t \) is a deterministic dynamical system; and a cocycle mapping \( \varphi : \mathbb{R} \times \Omega \times \mathbb{R}^d \to \mathbb{R}^d \), namely, \( \varphi \) satisfies the conditions:

\[ \varphi(0, \omega) = \mathbb{I}_{\mathbb{R}^d}, \quad \varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega), \]

for all \( \omega \in \Omega \) and all \( s, t \in \mathbb{R} \). This cocycle is required to be at least measurable from the \( \sigma \)-field \( \mathcal{B}(\mathbb{R}) \times \mathcal{F} \times \mathcal{B}(\mathbb{R}^d) \) to the \( \sigma \)-field \( \mathcal{B}(\mathbb{R}^d) \).

For random dynamical systems driven by Lévy noise we take \( \Omega = D(\mathbb{R}, \mathbb{R}^d) \) with the Skorohod metric as the canonical sample space and denote by \( \mathcal{F} := \mathcal{B}(D(\mathbb{R}, \mathbb{R}^d)) \) the associated Borel \( \sigma \)-field. Let \( \mu_L \) be the (Lévy) probability
measure on $\mathcal{F}$ which is given by the distribution of a two-sided Lévy process with paths in $D(\mathbb{R}, \mathbb{R}^d)$.

The driving system $\theta = (\theta_t, t \in \mathbb{R})$ on $\Omega$ is defined by the shift
\[
(\theta_t \omega)(s) := \omega(t + s) - \omega(t).
\]
(5)
The map $(t, \omega) \rightarrow \theta_t \omega$ is continuous, thus measurable (\cite{2} page 545), and the (Lévy) probability measure is $\theta$-invariant, i.e.
\[
\mu_L(\theta_t^{-1}(A)) = \mu_L(A)
\]
for all $A \in \mathcal{F}$, see \cite{1}, page 325.

**Lemma 4.** \textit{(RDS generated by SDEs driven by Lévy motion)}

Suppose in addition to the assumptions of Lemma 3 that condition (A3) is satisfied. Then there exists a unique càdlàg adapted solution to (3), and the solution mapping defines a RDS, which is continuous in $x$ but càdlàg in time.

**Proof.** Let $\Phi_{s,t}$ satisfy (3) with initial condition $\Phi_{s,s}(y) = y$, i.e.
\[
d\Phi_{s,t}(y) = b(\Phi_{s,t}(y)) \, dt + \sigma(\Phi_{s,t}(y)) \, dB_t + \int_{|x|<c} F(\Phi_{s,t}(y)), x \right) \tilde{N}(dt,dx) \\
+ \int_{|x| \geq c} G(\Phi_{s,t}(y)), x \right) N(dt,dx).
\]
(6)
By Theorem 6.4.2 on page 322 and Corollary 6.4.11 on page 327 of \cite{1}, $\Phi$ is a Lévy flow, and satisfies
\[
\Phi_{0,s+t}(y,\omega) = \Phi_{0,t}(\Phi_{0,s}(y),\theta_s(\omega)).
\]
We define $\varphi : \mathbb{R} \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^d$ by
\[
\varphi(t, y, \omega) = \Phi_{0,t}(y, \omega).
\]
(7)
It follows that
\[
\varphi(t + s, y, \omega) = \varphi(t, \varphi(s, y, \omega), \theta_s(\omega)).
\]
Moreover, we note that $\varphi(t, y, \omega)$ is continuous in $y$, measurable in $\omega$ and càdlàg in $t$. (cf.\cite{1}, page 336). It follows that the mapping $\varphi$ is measurable from $\mathbb{R} \times \mathbb{R}^d \times \Omega$ to $\mathbb{R}^d$.

With this we only need Theorem 1.3.2 and Remark 1.3.3 in \cite{2} (pages 17-20) to complete the proof. \hfill $\square$
Remark 4. In view of Remark 3 and analogous result holds for SDE with a nonlinear dissipative drift term [7]. Note that the perfection of crude discontinuous cocycles is considered in [32].

We say that a family \( \hat{A} = \{ A(\omega), \omega \in \Omega \} \) of non-empty measurable compact subsets \( A(\omega) \) of \( \mathbb{R}^d \) is invariant for a RDS \((\theta, \varphi)\), if \( \varphi(t, \omega, A(\omega)) = A(\theta_t \omega) \) for all \( t > 0 \) and that it is a random attractor if in addition it is pathwise pullback attracting in the sense that

\[
H_d^*(\varphi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)), A(\omega)) \to 0 \quad \text{as} \quad t \to \infty
\]
for all suitable families (called the attracting universe) of \( \hat{D} = \{ D(\omega), \omega \in \Omega \} \) of non-empty measurable bounded subsets \( D(\omega) \) of \( \mathbb{R}^d \), where \( H_d^* \) is the Hausdorff semi-distance on \( \mathbb{R}^d \).

The following result about the existence of a random attractor may be proved similarly as in [30, 8, 31, 17].

Lemma 5. (Random attractor for càdlàg RDS)
Let \((\theta, \varphi)\) be an RDS on \( \Omega \times \mathbb{R}^d \) and let \( \varphi \) be continuous in space, but càdlàg in time. If there exits a family \( \hat{B} = \{ B(\omega), \omega \in \Omega \} \) of non-empty measurable compact subsets \( B(\omega) \) of \( \mathbb{R}^d \) and a \( T_{\hat{D}, \omega} \geq 0 \) such that

\[
\varphi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)) \subset B(\omega), \quad \forall t \geq T_{\hat{D}, \omega},
\]
for all families \( \hat{D} = \{ D(\omega), \omega \in \Omega \} \) in a given attracting universe, then the RDS \((\theta, \varphi)\) has a random attractor \( \hat{A} = \{ A(\omega), \omega \in \Omega \} \) with the component subsets defined for each \( \omega \in \Omega \) by

\[
A(\omega) = \bigcap_{s > 0} \bigcup_{t \geq s} \varphi(t, \theta_{-t} \omega, B(\theta_{-t} \omega))
\]

Furthermore if the random attractor consists of singleton sets, i.e \( A(\omega) = \{ X^*(\omega) \} \) for some random variable \( X^* \), then \( X_t^*(\omega) = X^*(\theta_t \omega) \) is a stationary stochastic process.

We also need the following Gronwall’s lemma from [22].

Lemma 6. Let \( x(t) \) satisfy the differential inequality

\[
\frac{d}{dt} x \leq g(t)x + h(t)
\]
where \( \frac{d}{dt} x := \lim_{t \to 0^+} \frac{x(t+h) - x(t)}{h} \) is right-hand derivative of \( x \). Then

\[
x(t) \leq x(0) \exp \left[ \int_0^t g(r) dr \right] + \int_0^t \exp \left[ \int_s^t g(r) dr \right] h(s) ds.
\]
3 Dissipative synchronization

Suppose we have two autonomous ordinary differential equations in \( \mathbb{R}^d \),

\[
\frac{dx}{dt} = f(x), \quad \frac{dy}{dt} = g(y)
\]

where the vector fields \( f \) and \( g \) are sufficiently regular (e.g., differentiable) to ensure the existence and uniqueness of local solutions, and additionally satisfy one-side dissipative Lipschitz conditions

\[
\max\{\langle x_1 - x_2, f(x_1) - f(x_2) \rangle, \langle x_1 - x_2, g(x_1) - g(x_2) \rangle\} \leq -l|x_1 - x_2|^2
\]

on \( \mathbb{R}^d \) for some \( l > 0 \). These dissipative Lipschitz conditions ensure existence and uniqueness of global solutions; see Remark 3 above. Each of the systems has a unique globally asymptotically stable equilibria, \( x \) and \( y \), respectively \[17\]. Then, the coupled deterministic dynamical system in \( \mathbb{R}^{2d} \)

\[
\frac{dx}{dt} = f(x) + \lambda(y - x), \quad \frac{dy}{dt} = g(x) + \lambda(x - y)
\]

with parameter \( \lambda > 0 \) also satisfies a one-sided dissipative Lipschitz condition and, hence, also has a unique equilibrium \((x^\lambda, y^\lambda)\), which is globally asymptotically stable \[17\]. Moreover, \((x^\lambda, y^\lambda) \to (\bar{x}, \bar{y})\) as \( \lambda \to \infty \), where \( \bar{y} \) is the unique globally asymptotically stable equilibrium of the “averaged” system in \( \mathbb{R}^d \)

\[
\frac{dz}{dt} = \frac{1}{2} (f(z) + g(z)).
\]

This phenomena is known as synchronization for the coupled deterministic system \[10\]. The parameter \( \lambda \) often appears naturally in the context of the problem under consideration. For example in control theory it is a control parameter which can be chosen by the engineer, whereas in chemical reactions in thin layers separated by a membrane it is the reciprocal of the thickness of the layers, see \[5\].

Caraballo et al. \[6, 7\] showed that this synchronization phenomenon persists under Gaussian Brownian noise, provided that asymptotically stable stochastic stationary solutions are considered rather than asymptotically stable steady state solutions. Recall that a stationary solution \( X^* \) of a SDE system may be characterized as a stationary orbit of the corresponding random dynamical system \((\theta, \varphi)(\text{defined by the SDE system})\), namely, \( \varphi(t, \omega, X^*(\omega)) = X^*(\theta_t \omega) \).
They considered a coupled system of stochastic differential equations (SDEs) in $\mathbb{R}^d$.

\[
\begin{aligned}
\begin{cases}
\, dX_t = [f(X_t) + \lambda(Y_t - X_t)] \, dt + \alpha \, dB^1_t, \\
\, dY_t = [g(Y_t) + \lambda(X_t - Y_t)] \, dt + \beta \, dB^2_t.
\end{cases}
\end{aligned}
\] (12)

where $\alpha, \beta \in \mathbb{R}^d$ are constant vectors with no components equal to zero, $B^1_t, B^2_t$ are independent two-sided scalar Brownian motions, and $f, g$ satisfy the one-side dissipative Lipschitz conditions (9). This coupled system has a unique stationary solution $(X^\lambda_t, Y^\lambda_t)$, which is pathwise globally asymptotically stable. Moreover, the coupled system (12) is synchronized to the “averaged” SDE in $\mathbb{R}^d$

\[
dZ_t = \frac{1}{2} \left[ f(Z_t) + g(Z_t) \right] \, dt + \frac{1}{2} \alpha \, dB^1_t + \frac{1}{2} \beta \, dB^2_t
\] (13)

in the sense that $(X^\lambda_t, Y^\lambda_t) \to (Z^\infty_t, Z^\infty_t)$ as $\lambda \to \infty$, where $Z^\infty_t$ is the unique pathwise globally asymptotically stable stationary solution of (13).

The aim of this paper is to investigate synchronization under non-Gaussian Lévy noise. In particular, we consider a coupled SDE system in $\mathbb{R}^d$, driven by non-Gaussian Lévy noise in $\mathbb{R}^{2d}$

\[
\begin{aligned}
\begin{cases}
\, dX_t = (f(X_t) + \lambda(Y_t - X_t)) \, dt + \alpha \, dL^1_t, \\
\, dY_t = (g(Y_t) + \lambda(X_t - Y_t)) \, dt + \beta \, dL^2_t,
\end{cases}
\end{aligned}
\] (14)

where $\alpha, \beta, f, g$ are as above, and $L^1_t, L^2_t$ are independent two-sided scalar Lévy processes satisfying conditions in Lemma 1. We assume that this coupled system defines a random dynamical system $\varphi$ (i.e., it satisfies the assumptions in Lemma 4 or some generalization of it).

In addition to the one-side Lipschitz dissipative condition (9) on the functions $f$ and $g$, as in [6] we further assume the following integrability condition: There exists $m_0 > 0$ such that for any $m \in (0, m_0]$, and any càdlàg function $u : \mathbb{R} \to \mathbb{R}^d$ with sub-exponential growth it follows

\[
\int_{-\infty}^{t} e^{ms} |f(u(s))|^2 \, ds < \infty, \quad \int_{-\infty}^{t} e^{ms} |g(u(s))|^2 \, ds < \infty.
\] (15)

Without loss of generality, we assume that the one-sided dissipative Lipschitz constant $l \leq m_0$. 

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In the next section we will show that the coupled system (14) has a unique stationary solution \((\overline{X}_t^\lambda, \overline{Y}_t^\lambda)\) which is pathwise globally asymptotically stable with \((\overline{X}_t^\lambda, \overline{Y}_t^\lambda) \rightarrow (Z_t^\infty, Z_t^\infty)\) in Skorohod metric as \(\lambda \rightarrow \infty\), pathwise on finite time-intervals \([T_1, T_2]\), where \(Z_t^\infty\) is the unique pathwise globally asymptotically stable stationary solution of the “averaged” SDE in \(\mathbb{R}^d\)

\[
dZ_t = \frac{1}{2} [f(Z_t) + g(Z_t)] \, dt + \frac{1}{2} \alpha \, dL^1_t + \frac{1}{2} \beta \, dL^2_t.
\]

### 4 Systems driven by Lévy noise

For the coupled system (14), we have the follow two lemmas about its stationary solutions.

**Lemma 7. (Existence of stationary solutions)**

If the Assumption (15) holds, and \(f\) and \(g\) satisfy the one-side Lipschitz dissipative conditions (9), then the coupled stochastic system (14) has a unique stationary solution.

**Proof.** First, the stationary solutions of the Langevin equations

\[
\begin{align*}
    dX_t &= -\lambda X_t \, dt + \alpha \, dL^1_t, \\
    dY_t &= -\lambda Y_t \, dt + \beta \, dL^2_t
\end{align*}
\]

are given by

\[
\begin{align*}
    \overline{X}_t^\lambda &= \alpha e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} \, dL^1_s, \\
    \overline{Y}_t^\lambda &= \beta e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda s} \, dL^2_s
\end{align*}
\]

The differences of the solutions of (14) and these stationary solutions satisfy a system of random ordinary differential equations, with right-hand derivative in time:

\[
\begin{align*}
    \frac{d}{dt^+} (X_t - \overline{X}_t^\lambda) &= f(X_t) + \lambda (Y_t - X_t) + \lambda \overline{X}_t^\lambda, \\
    \frac{d}{dt^+} (Y_t - \overline{Y}_t^\lambda) &= g(Y_t) + \lambda (X_t - Y_t) + \lambda \overline{Y}_t^\lambda
\end{align*}
\]

The equations (19) are equivalent to

\[
\begin{align*}
    \frac{d}{dt^+} U_t^\lambda &= f(X_t) + \lambda (V_t^\lambda - U_t^\lambda) + \lambda \overline{Y}_t^\lambda, \\
    \frac{d}{dt^+} V_t^\lambda &= g(Y_t) + \lambda (U_t^\lambda - V_t^\lambda) + \lambda \overline{X}_t^\lambda
\end{align*}
\]
where $U_t^\lambda = X_t - \bar{X}_t^\lambda$ and $V_t^\lambda = Y_t - \bar{Y}_t^\lambda$. Thus,

$$\frac{1}{2} \frac{d}{dt} (|U_t^\lambda|^2 + |V_t^\lambda|^2) = (U_t^\lambda, f(U_t^\lambda + \bar{X}_t^\lambda) - f(\bar{X}_t^\lambda)) + (V_t^\lambda, g(V_t^\lambda + \bar{Y}_t^\lambda) - g(\bar{Y}_t^\lambda))$$

$$+ (U_t^\lambda, f(\bar{X}_t^\lambda) + \lambda \bar{Y}_t^\lambda) + (V_t^\lambda, g(\bar{Y}_t^\lambda) + \lambda \bar{X}_t^\lambda) - \lambda |U_t^\lambda - V_t^\lambda|^2$$

$$\leq - \frac{1}{2} (|U_t^\lambda|^2 + |V_t^\lambda|^2) + \frac{2}{l} |f(\bar{X}_t^\lambda) + \lambda \bar{Y}_t^\lambda|^2 + \frac{2}{l} |g(\bar{Y}_t^\lambda) + \lambda \bar{X}_t^\lambda|^2$$

Hence, by Lemma 6,

$$|U_t^\lambda|^2 + |V_t^\lambda|^2 \leq (|U_{t_0}^\lambda|^2 + |V_{t_0}^\lambda|^2) e^{-\frac{1}{2l}(t-t_0)} + 4e^{-\frac{1}{2l}} \int_{t_0}^t e^{\frac{1}{2l} s} \left[ |f(\bar{X}_t^\lambda) + \lambda \bar{Y}_t^\lambda|^2 + |g(\bar{Y}_t^\lambda) + \lambda \bar{X}_t^\lambda|^2 \right] ds$$

This means that $|U_t^\lambda|^2 + |V_t^\lambda|^2$ is pathwise absorbed by the family $\hat{B}_{2d}^\lambda = \{B_{2d}^\lambda(\omega), \omega \in \Omega\}$ of closed balls in $\mathbb{R}^{2d}$ centred on the origin and of radius $R_{\lambda}(\omega)$, where the $|R_{\lambda}(\omega)|^2$ is given by

$$1 + 4e^{-\frac{1}{2l}} \int_{-\infty}^t e^{\frac{1}{2l} s} \left[ |f(\bar{X}_t^\lambda(\theta, \omega)) + \lambda \bar{Y}_t^\lambda(\theta, \omega)|^2 + |g(\bar{Y}_t^\lambda(\theta, \omega)) + \lambda \bar{X}_t^\lambda(\theta, \omega)|^2 \right] ds$$

Hence, by Lemma 6 the coupled system has a random attractor $\hat{A}^\lambda = \{A_\lambda(\omega), \omega \in \Omega\}$ with $A_\lambda(\omega) \subset B_{2d}^\lambda(\omega)$.

However, the difference $(\Delta X_t, \Delta Y_t) = (X_t^1 - X_t^2, Y_t^1 - Y_t^2)$ of any pair of solutions satisfies the system of random ordinary differential equations

$$\frac{d}{dt} \Delta X_t = f(X_t^1) - f(X_t^2) + \lambda (\Delta Y_t - \Delta X_t),$$

$$\frac{d}{dt} \Delta Y_t = g(Y_t^1) - g(Y_t^2) - \lambda (\Delta Y_t - \Delta X_t),$$

so

$$\frac{d}{dt} (|\Delta X_t|^2 + |\Delta Y_t|^2) = 2(\Delta X_t, f(X_t^1) - f(X_t^2)) + 2(\Delta Y_t, g(Y_t^1) - g(Y_t^2)) - 2\lambda |\Delta X_t - \Delta Y_t|^2$$

$$\leq -2l (|\Delta X_t|^2 + |\Delta Y_t|^2)$$

from which we obtain

$$|\Delta X_t|^2 + |\Delta Y_t|^2 \leq (|\Delta X_0|^2 + |\Delta Y_0|^2) e^{-2lt}$$

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which means all solutions converge pathwise to each other as $t \to \infty$. Thus the random attractor consists of a singleton set formed by an ordered pair of stationary processes $(\overline{X}_t^\lambda(\omega), \overline{Y}_t^\lambda(\omega))$.

**Remark 5.** Using Lemma 1, it can be shown that the random compact absorbing balls $B^\lambda_2(\omega)$ are contained in the common compact ball for $\lambda \geq 1$.

**Lemma 8.** (A property of stationary solutions) The stationary solutions of the coupled stochastic system (14) have the following asymptotic behavior:

$$\overline{X}_t^\lambda(\omega) - \overline{Y}_t^\lambda(\omega) \to 0 \quad \text{as } \lambda \to \infty$$

pathwise on any bounded time-interval $[T_1, T_2]$ of $\mathbb{R}$.

**Proof.** Since

$$d(\overline{X}_t^\lambda - \overline{Y}_t^\lambda) = \left(-2\lambda(\overline{X}_t^\lambda - \overline{Y}_t^\lambda) + f(\overline{X}_t^\lambda) - g(\overline{Y}_t^\lambda)\right) dt + \alpha dL_t^1 - \beta dL_t^2,$$

we have

$$d(D_t^\lambda e^{2\lambda t}) = e^{2\lambda t} \left(f(\overline{X}_t^\lambda) - g(\overline{Y}_t^\lambda)\right) + \alpha e^{2\lambda t} dL_t^1 - \beta e^{2\lambda t} dL_t^2,$$

for with $D_t^\lambda = \overline{X}_t^\lambda - \overline{Y}_t^\lambda$, so pathwise

$$|D_t^\lambda| \leq e^{-2\lambda(t-T_1)}|D_{T_1}^\lambda| + \int_{T_1}^{t} e^{-2\lambda(t-s)} \left(|f(\overline{X}_s^\lambda)| + |g(\overline{Y}_s^\lambda)|\right) ds$$

$$\quad + |\alpha| \left|\int_{T_1}^{t} e^{-2\lambda(t-s)} dL_t^1\right| + |\beta| \left|\int_{T_1}^{t} e^{-2\lambda(t-s)} dL_t^2\right|.$$ 

By Lemma 1 we see that the radius $R_\lambda(\theta(\omega))$ is pathwise uniformly bounded on each bounded time-interval $[T_1, T_2]$, so we see that the right hand of above inequality converge to 0 as $\lambda \to \infty$ pathwise on the bounded time-interval $[T_1, T_2]$. 

We now present the main result of this paper.
Theorem 1. (Synchronization under non-Gaussian Lévy noise)
Suppose that the coupled stochastic system in \( \mathbb{R}^d \)
\[
\begin{align*}
    dX_t &= (f(X_t) + \lambda(Y_t - X_t)) \, dt + \alpha \, dL^1_t, \\
    dY_t &= (g(Y_t) + \lambda(X_t - Y_t)) \, dt + \beta \, dL^2_t,
\end{align*}
\]
defines a random dynamical system \( (\theta, \phi) \). In addition, assume that \( f \) and \( g \) satisfy the integrability condition \( (15) \) as well as the one-side Lipschitz dissipative condition \( (9) \).
Then the coupled stochastic system \( (21) \) is synchronized to a single averaged SDE in \( \mathbb{R}^d \)
\[
    dZ_t = \frac{1}{2} \left[ f(Z_t) + g(Z_t) \right] \, dt + \frac{1}{2} \alpha \, dL^1_t + \frac{1}{2} \beta \, dL^2_t,
\]
in the sense that the stationary solutions of \( (21) \) pathwise converge to that of \( (22) \), i.e. \( (\overline{X}_t^\lambda, \overline{Y}_t^\lambda) \to (Z_t^\infty, Z_t^\infty) \) in Skorohod metric on any bounded time-interval \( [T_1, T_2] \) as parameter \( \lambda \to \infty \).

Proof. It is enough to demonstrate the result for any sequence \( \lambda_n \to \infty \).
Define
\[
    Z_t^\lambda := \frac{1}{2} \left[ \overline{X}_t^\lambda + \overline{Y}_t^\lambda \right], \quad t \in \mathbb{R}.
\]
Note that \( Z_t^\lambda \) satisfies the equation
\[
    dZ_t^\lambda = \frac{1}{2} \left[ f(X_t^\lambda) + g(Y_t^\lambda) \right] \, dt + \frac{1}{2} \alpha \, dL^1_t + \frac{1}{2} \beta \, dL^2_t.
\]
Also we define
\[
    \overline{Z}_t := \overline{X}_t + \overline{Y}_t, \quad t \in \mathbb{R},
\]
where \( \overline{X}_t \) and \( \overline{Y}_t \) are the (stationary) solutions of the Langevin equations
\[
    dX_t = -X_t \, dt + \alpha dL^1_t, \quad dY_t = -Y_t \, dt + \beta dL^2_t,
\]
i.e.
\[
    \overline{X}_t = \alpha e^{-t} \int_{-\infty}^{t} e^s \, dL^1_s, \quad \overline{Y}_t = \beta e^{-t} \int_{-\infty}^{t} e^s \, dL^2_s.
\]
The difference \( Z_t^\lambda - \overline{Z}_t \) satisfies pathwise a random ordinary differential equation
\[
    \frac{d}{dt} (Z_t^\lambda - \overline{Z}_t) = \frac{1}{2} \left( f(X_t^\lambda) + g(Y_t^\lambda) \right) + \frac{1}{2} \left( \overline{X}_t + \overline{Y}_t \right).
\]
By Lemma 1, we obtain

\[
\left| \frac{d}{dt+} (Z^\lambda_t(\omega) - Z_t(\omega)) \right| \leq \frac{1}{2} |f(X^\lambda_t(\omega)) + g(Y^\lambda_t(\omega))| + \frac{1}{2} |X_t(\omega) + Y_t(\omega)|
\]

\[
\leq M_{T_1,T_2}(\omega) < \infty
\]

by the càdlàg property of the solutions and the fact that these solutions belong to the common compact ball. We can use Lemma 2 to conclude that for any sequence \(\lambda_n \to \infty\), there is a random subsequence \(\lambda_{n_j}(\omega) \to \infty\), such that \(Z_t^{\lambda_{n_j}}(\omega) - Z_t(\omega) \to Z_t^{\infty}(\omega) - Z_t(\omega)\) in Skorohod metric as \(j \to \infty\). Thus \(Z_t^{\lambda_{n_j}}(\omega) \to Z_t^{\infty}(\omega)\) in the Skorohod metric as \(j \to \infty\). Now,

\[
\frac{X_t^{\lambda_{n_j}}(\omega) - X_t^{\lambda_{n_j}}(\omega)}{2} \to 0,
\]

\[
\frac{Y_t^{\lambda_{n_j}}(\omega) - Y_t^{\lambda_{n_j}}(\omega)}{2} \to 0,
\]

as \(\lambda_{n_j} \to \infty\), so

\[
X_t^{\lambda_{n_j}}(\omega) = 2Z_t^{\lambda_{n_j}}(\omega) - Y_t^{\lambda_{n_j}}(\omega) \to Z_t^{\infty}(\omega),
\]

\[
Y_t^{\lambda_{n_j}}(\omega) = 2Z_t^{\lambda_{n_j}}(\omega) - X_t^{\lambda_{n_j}}(\omega) \to Z_t^{\infty}(\omega),
\]

as \(\lambda_{n_j} \to \infty\). Moreover,

\[
Z_t^\lambda - Z_t = Z_t^{\lambda_{T_1}} - Z_{T_1} + \frac{1}{2} \int_{T_1}^t f(X_s^\lambda) \, ds + \frac{1}{2} \int_{T_1}^t g(Y_s^\lambda) \, ds
\]

\[
+ \frac{1}{2} \int_{T_1}^t X_s \, ds + \frac{1}{2} \int_{T_1}^t Y_s \, ds,
\]

which converges pathwise to

\[
Z_t^{\infty} = Z_t^{\infty}_{T_1} + \frac{1}{2} \int_{T_1}^t f(X_s^{\infty}) \, ds + \frac{1}{2} \int_{T_1}^t g(Y_s^{\infty}) \, ds
\]

\[
+ Z_t - Z_{T_1} + \frac{1}{2} \int_{T_1}^t X_s \, ds + \frac{1}{2} \int_{T_1}^t Y_s \, ds
\]

\[
= Z_t^{\infty}_{T_1} + \frac{1}{2} \int_{T_1}^t f(X_s^{\infty}) \, ds + \frac{1}{2} \int_{T_1}^t g(Y_s^{\infty}) \, ds + \frac{\beta}{2} \int_{T_1}^t dL_s^1 + \frac{\beta}{2} \int_{T_1}^t dL_s^2,
\]

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on the interval \([T_1, T_2]\). Therefore, \(Z_t^\infty\) is a solution of the averaged SDE (22) for all \(t \in \mathbb{R}\). The drift of this SDE satisfies the dissipative one-side condition (9). It has a random attractor consisting of a singleton set formed by a stationary orbit, which must be equal to \(Z_t^\infty\).

Finally, we note that all possible subsequences of \(Z_{t_n}^{\lambda_n}\) have the same pathwise limit. Thus the full sequence \(Z_{t_n}^{\lambda_n}\) converges to \(Z_t^\infty\), as \(\lambda_n \to \infty\). This completes the proof.

\[\text{4.1 An example}\]
Consider two scalar SDEs:
\[
dx_t = -(X_t + 1)\, dt + dL_t^1, \quad dY_t = -(Y_t + 3)\, dt + 2\, dL_t^2,
\]
which we rewrite as
\[
dx_t = -X_t\, dt + dL_t^3, \quad dY_t = -Y_t\, dt + 2\, dL_t^4,
\]
where \(L_t^3 = 1 + L_t^1\) and \(L_t^4 = 3/2 + L_t^2\).

The corresponding coupled system (21) is
\[
\begin{cases}
   dX_t = -X_t\, dt + \lambda(Y_t - X_t)\, dt + dL_t^3, \\
   dY_t = -Y_t\, dt + \lambda(X_t - Y_t)\, dt + 2\, dL_t^4
\end{cases}
\]
with the stationary solutions
\[
\begin{align*}
   \overline{X}_t^{\lambda} &= \int_{-\infty}^{t} e^{-(\lambda+1)(t-s)} \cosh(\lambda(t-s)) \, dL_s^3 + 2 \int_{-\infty}^{t} e^{-(\lambda+1)(t-s)} \sinh(\lambda(t-s)) \, dL_s^4, \\
   \overline{Y}_t^{\lambda} &= \int_{-\infty}^{t} e^{-(\lambda+1)(t-s)} \sinh(\lambda(t-s)) \, dL_s^3 + 2 \int_{-\infty}^{t} e^{-(\lambda+1)(t-s)} \cosh(\lambda(t-s)) \, dL_s^4.
\end{align*}
\]
Let \(\lambda \to \infty\), then
\[\overline{X}_t^{\lambda}, \overline{Y}_t^{\lambda} \to (Z_t^\infty, Z_t^\infty),\]
where \(Z_t^\infty\), given by
\[
Z_t^\infty = \int_{-\infty}^{t} \frac{1}{2} e^{-(t-s)} \, dL_s^3 + \int_{-\infty}^{t} e^{-(t-s)} \, dL_s^4,
\]
is the stationary solution of the following averaged SDE
\[
dZ_t = -Z_t\, dt + \frac{1}{2} dL_t^3 + dL_t^4,
\]
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which is equivalent to the following SDE, in terms of the original Lévy motions $L^1$ and $L^2$:

\[ dZ_t = -(Z_t + 2) \, dt + \frac{1}{2} \, dL^1_t + dL^2_t. \]

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