Stable pure state quantum tomography from five orthonormal bases

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Abstract – For any finite-dimensional Hilbert space, we construct explicitly five orthonormal bases such that the corresponding measurements allow for efficient tomography of an arbitrary pure quantum state. This means that such measurements can be used to distinguish an arbitrary pure state from any other state, pure or mixed, and the pure state can be reconstructed from the outcome distribution in a feasible way. The set of measurements we construct is independent of the unknown state, and therefore our results provide a fixed scheme for pure state tomography, as opposed to the adaptive (state-dependent) scheme proposed by Goyeneche et al. (Phys. Rev. Lett., 115 (2015) 090401). We show that our scheme is robust with respect to noise, in the sense that any measurement scheme which approximates these measurements well enough is equally suitable for pure state tomography. Finally, we present two convex programs which can be used to reconstruct the unknown pure state from the measurement outcome distributions.

Introduction. – The aim of quantum tomography is to reconstruct the unknown state of a quantum system by performing suitable measurements on it. Tomography is a vital routine in quantum information, where it is used to characterize output states and test processing devices. However, quantum tomography is a consuming task: in order to obtain enough information for state reconstruction of a $d$-level system, it is necessary to perform measurements of $d + 1$ different orthonormal bases, or a generalized measurement with at least $d^2$ outcomes. This poor scaling has led to the search for more efficient methods which allow for a reduction of resources in specific cases.

Recent focus has been put on the identification of unknown pure (or more generally low-rank) states [1–7]. Any two pure states can be distinguished with a measurement having just $\sim 4d$ outcomes [1] or, when restricting to projective measurements, with only four orthonormal bases [3,7,8]. The drawback of these approaches is that the measurements they provide cannot distinguish pure states from all states, implying that one needs to know that the state is pure prior to the measurement in order not to confuse it with mixed states having the same measurement outcome distributions. Moreover, none of the approaches allows an efficient recovery algorithm, mainly since the non-convex nature of the problem renders usual techniques from convex optimization useless.

In [9], a scheme involving five orthonormal bases along with a reconstruction algorithm was proposed and experimentally demonstrated. Remarkably, such a scheme allows to certify the purity assumption on the state directly from the measurement outcomes. However, this method is adaptive in the sense that the outcome distribution of the first measurement affects the choice of the subsequent ones. Therefore, if one requires the procedure to work for all pure states the overall number of required measurement settings is considerably larger than five.

At the cost of a slightly higher number $O(d \ln d)$ of measurement outcomes, tomographic procedures based on compressed sensing were proposed in [10–12]. This approach allows for the stable recovery of pure quantum
states, as well as it satisfies the requirement of distinguishing pure states from all states [13]. However, rather than providing a functioning measurement set-up, compressed sensing techniques guarantee that, with high probability, any state can be reconstructed by using sufficiently many randomly drawn measurement settings. From a practical point of view, however, a deterministic approach which provides an explicit measurement set-up may be favourable.

In this letter we overcome these drawbacks by constructing five orthonormal bases such that every pure state can be efficiently reconstructed from the corresponding measurements. For any dimension, our set of measurements is fixed and, therefore, there is no need for data processing between the measurements. We show that these measurements distinguish pure states from all states, and this therefore shows that the scaling \( \sim 5d \) in the total number of outcomes is the same as without the constraint of having projective measurements [14]. More importantly, we prove that the presented set-up is robust with respect to noise. Finally, we provide reconstruction algorithms for the practical retrieval of the unknown state from the measurement data. We remark that, as compared to the practical point of view, however, a deterministic approach provides a functioning measurement set-up, compressed means that for any pure state represented by a unit vector \( \psi \), and any density matrix \( \varrho \), the equalities

\[
\langle v_j^\ell | \psi \rangle^2 = \langle v_j^\ell | \varrho | v_j^\ell \rangle, \quad \text{for all } v_j^\ell \in B^\ell \text{ and } \ell = 0, \ldots, 4,
\]

imply that \( \varrho = |\psi\rangle \langle \psi| \). The construction is an extension of [8] where, based on the properties of Hermite polynomials, four orthonormal bases \( B^1, \ldots, B^4 \) capable of distinguishing any two pure states were presented. That construction generalizes easily to any sequence of orthogonal polynomials as explained in [3]. Remarkably, by adding the canonical basis \( B^0 = \{e_0, \ldots, e_{d-1}\} \) to this set, we obtain the five bases with the desired property.

To begin with, let us fix a sequence of orthogonal polynomials, that is, a sequence \( \{p_n\}_{n=0}^{\infty} \) of real polynomials such that \( p_n \) is of degree \( n \) and

\[
\langle p_j, p_i \rangle := \int_{-\infty}^{\infty} p_j(x)p_i(x)w(x)dx = \delta_{ij}
\]

for some positive weight function \( w \). For a \( d \)-dimensional system we will only need the first \( d+1 \) polynomials. To construct the first two bases, let \( x_0, \ldots, x_{d-1} \) be the zeros of \( p_d \), which are real and distinct numbers satisfying \( p_{d-1}(x_j) \neq 0 \) for all \( j \in \{0, \ldots, d-1\} \) ([15], sect. 3.3). Pick an \( \alpha \in \mathbb{R} \) such that \( e^{\gamma\alpha} \notin \mathbb{R} \) for all \( j \in \{1, \ldots, d-1\} \). Now for \( j = 0, \ldots, d-1 \), set

\[
v_j^1 := (p_0(x_j), p_1(x_j), \ldots, p_{d-1}(x_j)),
\]

\[
v_j^2 := \left( p_0(x_j), e^{i\alpha} p_1(x_j), \ldots, e^{i(d-1)\alpha} p_{d-1}(x_j) \right)
\]

and denote \( B^1 = \{v_j^1/\|v_j^1\| \mid j = 0, \ldots, d-1\} \) and \( B^2 = \{v_j^2/\|v_j^2\| \mid j = 0, \ldots, d-1\} \). The fact that these are actually orthonormal bases can be readily checked using the Christoffel-Darboux formula ([15], Theorem 3.2.2),

\[
\sum_{i=0}^{n} p_i(x)p_i(y) = \frac{k_n}{k_{n+1}} \frac{p_n+1(x)p_n(y) - p_n(x)p_{n+1}(y)}{x - y}
\]

where \( k_n \) is the leading coefficient of \( p_n \) (see [3,8] for more details). This formula evaluated at \( n = d-1 \) and \( x = y = x_j \) also yields the normalization factor

\[
\|v_j^1\|^2 = \|v_j^2\|^2 = \frac{k_{d-1}}{k_{d}} p_d(x_j)p_{d-1}(x_j).
\]

For the remaining two bases, let \( y_0, \ldots, y_{d-2} \) be the zeros of \( p_{d-1} \). As the polynomials \( p_d \) and \( p_{d-1} \) have no common zeros, the \( y_j \)'s are distinct from the \( x_j \)'s. For a similar reason, \( p_{d-2}(y_j) \neq 0 \) for all \( j = 0, \ldots, d-2 \). For \( j = 0, \ldots, d-2 \) define the non-zero vectors

\[
v_j^3 := (p_0(y_j), p_1(y_j), \ldots, p_{d-1}(y_j), 0),
\]

\[
v_j^4 := \left( p_0(y_j), e^{i\alpha} p_1(y_j), \ldots, e^{i(d-2)\alpha} p_{d-1}(y_j), 0 \right),
\]

and by setting \( v_j^{d-1} := e_d \) as well as \( v_j^{d+1} := e_{d-1} \) we have arrived at the two orthonormal bases \( B^3 = \{v_j^3/\|v_j^3\| \mid j = 0, \ldots, d-1\} \) and \( B^4 = \{v_j^4/\|v_j^4\| \mid j = 0, \ldots, d-1\} \). The normalization is now given by

\[
\|v_j^3\|^2 = \|v_j^4\|^2 = \frac{k_{d-2}}{k_{d-1}} p_d(y_j)p_{d-2}(y_j).
\]

**Theorem 1.** The five orthonormal bases \( B^0, \ldots, B^4 \) constructed above determine any pure state among all states.

**Proof.** Let \( \psi = \sum_{j=0}^{d-1} c_j v_j \) be a unit vector and let \( \varrho \) be an arbitrary state such that \( \langle v_j^\ell | \psi \rangle^2 = \langle v_j^\ell | \varrho | v_j^\ell \rangle \) for all \( v_j^\ell \in B^\ell \) and \( \ell = 0, \ldots, 4 \). From the standard basis \( B^0 \) we get \( g_{k,k} = |c_k|^2 \) for all \( k \). Let \( n \) denote the largest number such that \( g_{n,n} = |c_n|^2 \neq 0 \) so that by the positivity of \( \varrho \), \( g_{k,l} = g_{l,k} = 0 \) for all \( k > n \). By the definition of the bases and the equalities of the probabilities we then have

\[
\sum_{k,l=0}^{n} (g_{k,l} - c_k\overline{c_l}) p_k(\overline{z})p_l(\overline{z}) = 0, \quad (1)
\]

\[
\sum_{k,l=0}^{n} (g_{k,l} - c_k\overline{c_l}) e^{i(l-k)\alpha} p_k(\overline{z})p_l(\overline{z}) = 0, \quad (2)
\]

for all \( z = x_j \) and \( z = y_j \), but since the polynomials have degree at most \( 2n \leq 2d - 2 \) and they vanish on \( 2d - 1 \)
distinct points, they must be identically zero. In other words, the above equalities must hold for all \( z \in \mathbb{R} \). Let us denote \( t_{k,l} = \varrho_{k,l} - c_k c_l \) so that \( t_{k,k} = t_{k,l} = 0 \). By looking at the highest degree terms in (1) and (2) we get \( \text{Re} (t_{n,n-1}) = \text{Re} (e^{i(n-1)\alpha} t_{n,n-1}) = 0 \), which imply that \( t_{n,n-1} = 0 \). In other words, the matrix elements of the two states coincide on the diagonal and the bottom right \((d-n+1) \times (d-n+1)\)-block. We now proceed by induction.

Firstly, whenever the two states coincide on some bottom right \((d-r) \times (d-r)\)-block, with \( 1 \leq r \leq n-1 \), we have \( t_{k,l} = 0 \) for \( k \geq r \) and \( l \geq r \). But then the highest-degree terms in (1) and (2) give \( \text{Re} (t_{n,r-1}) = \text{Re} (e^{i(r-n-1)\alpha} t_{n,r-1}) = 0 \), which yield \( t_{n,r-1} = 0 \), that is \( \varrho_{n,r-1} = c_n \overline{c_{r-1}} \). Secondly, using this and the positivity of \( \varrho \) we can calculate for all \( r < k < n \)

\[
0 \leq \begin{vmatrix}
\varrho_{r-1,r-1} & \varrho_{r-1,k} & \varrho_{r-1,n} \\
\varrho_{k,r-1} & \varrho_{k,k} & \varrho_{k,n} \\
\varrho_{n,r-1} & \varrho_{n,k} & \varrho_{n,n}
\end{vmatrix}
= \frac{|c_{r-1}|^2}{|c_{n}|^2} \begin{vmatrix}
\varrho_{r-1,k} & \varrho_{r-1,n} \\
|c_{k}|^2 & |c_{k}|^2
\end{vmatrix}
= -|c_{n}|^2 |\varrho_{r-1,k} - c_{r-1} \overline{c_{k}}|^2
\]

which is satisfied if and only if the right-hand side is zero. Since \( c_n \neq 0 \), this gives \( \varrho_{r-1,k} = c_{r-1} \overline{c_k} \). The two states therefore coincide on a larger bottom right block. By induction, the states must be equal. \( \square \)

To give an example of the previously explained construction of five bases, we take the Chebyshev polynomials of the second kind \((U_n)_{n=0}^\infty\). These are the unique polynomials such that \([15], p. 3\)

\[
U_n(\cos \theta) = \frac{\sin ((n+1)\theta)}{\sin \theta}
\]

holds for all \( n = 0, 1, \ldots \) and \( \theta \in [0, 2\pi) \). The \( n \) roots of \( U_n \) are given by

\[
\cos \left( \frac{j+1}{n+1} \pi \right), \quad j = 0, \ldots, n-1,
\]

and its leading coefficient is \( k_n = 2^n \). Hence, the normalized vectors of the first and the third basis are

\[
v_j = \sqrt{\frac{2}{d+1}} \left( \sin \left( \frac{1}{d+1} \pi \right), \ldots, \sin \left( \frac{j}{d+1} \pi \right) \right),
\]

\[
v_j = \sqrt{\frac{2}{d}} \left( \sin \left( \frac{1}{d} \pi \right), \ldots, \sin \left( \frac{j}{d} \pi \right), 0 \right)
\]

with \( v_{j+1} \) and \( v_j \) are similar.

**More general measurements and stability.** A realistic measurement is affected by noise and, therefore, cannot be described simply by an orthonormal basis. Even more, an optimal measurement for a given task might not even be related to an orthonormal basis. For these reasons, one needs to have a wider mathematical framework for measurements. A general measurement in quantum mechanics can be modelled by a positive-operator-valued measure (POVM) \([16]\), which is a function \( j \mapsto P(j) \) from a finite set of measurement outcomes \( \{1, \ldots, m\} \) to the linear space of \( d \times d \) Hermitian matrices \( H(d) \) such that \( P(j) \geq 0 \) and \( \sum_{j=1}^m P(j) = 1 \). In practice one might want to measure more than one POVM. For instance, a noisy measurement of each orthonormal basis can be described by a separate POVM. By a measurement scheme we mean a set \( Q := \{P_1, \ldots, P_r\} \) of POVMs. It is not restrictive to assume that all POVMs in a given measurement scheme have the same set of outcomes \( \{1, \ldots, m\} \). A measurement scheme \( Q \) therefore induces a linear map \( M_Q \) from the real vector space \( H(d) \) to the set of real \( l \times m \) matrices \( M_{lm}(\mathbb{R}) \) via

\[
M_Q(X)_{i,j} = \text{tr} [XP_i(j)].
\]

The image of a state \( \varrho \) is the real matrix whose \( i \)-th row contains the outcome probabilities corresponding to \( P_i \). Analogously to the case of projective measurements, we say that the measurement scheme \( Q \) determines any pure state among all states if for any pure state \( \sigma = |\psi \rangle \langle \psi| \) and any state \( \varrho \), the equality \( M_Q(\sigma) = M_Q(\varrho) \) implies \( \varrho = \sigma \). By adapting the argument of \([2]\), Theorem 1, we obtain the following characterization.

**Proposition 1.** A measurement scheme \( Q \) determines any pure state among all states if and only if every non-zero element of \( \ker M_Q \) has at least two positive eigenvalues.

**Proof.** The measurement scheme \( Q \) does not determine pure states among all states if and only if \( M_Q(\sigma - \varrho) = 0 \) for some states \( \sigma \) and \( \varrho \) such that \( \sigma \) is pure and \( \sigma - \varrho \neq 0 \). This implies that \( \sigma - \varrho \in \ker M_Q \), and \( \sigma - \varrho \) has at most one positive eigenvalue by Weyl’s inequalities \([17]\, \text{Theorem III.2.1}\). Conversely, if \( \sigma \in \ker M_Q \) is non-zero and has at most one positive eigenvalue, then it has exactly one positive eigenvalue since \( \text{tr} [X] = \sum_j M_Q(X)_{i,j} = 0 \). Hence, its positive part \( X_+ \) has rank 1. Defining the states \( \sigma = X_+/\text{tr} [X_+] \) and \( \varrho = (X_r - X)/\text{tr} [X_+] \), we have that \( \sigma \) is pure, \( \sigma - \varrho = X/\text{tr} [X_+] \neq 0 \) and \( M_Q(\sigma - \varrho) = M_Q(X)/\text{tr} [X_+] = 0 \). \( \square \)

With this framework of measurement schemes we are now prepared to discuss the noise robustness of the result stated in Theorem 1. First, we will need to have a notion of closeness of two measurement schemes, and for this reason we fix norms on the real vector spaces \( H(d) \) and \( M_{lm}(\mathbb{R}) \). Since these are finite-dimensional vector spaces, all norms are equivalent and the choice is not important for our purposes. Typical choices are, e.g., the trace norm \( \|X\| = \text{tr} [\|X\|] \) on \( H(d) \), and on \( M_{lm}(\mathbb{R}) \) the supremum of
the $\ell^1$-norm over all lines, i.e.,

$$\|M\| = \sup_i \sum_j |M_{i,j}|.$$ 

The inequality $\|M(\varrho) - M(\varrho')\| \leq \epsilon$ then means that the measurement outcome distributions of all the POVMs in $\mathcal{Q}$ and $\mathcal{Q}'$ measured on the same state $\varrho$ are uniformly close in the total variation norm. We will say that two measurement schemes $\mathcal{Q}$ and $\mathcal{Q}'$ are $\epsilon$-close if $\|M(\varrho) - M(\varrho')\|_\infty < \epsilon$, where $\|\cdot\|_\infty$ is the uniform operator norm in the chosen norms of $H(d)$ and $M_{lm}(\mathbb{R})$.

**Theorem 2** (Stability). If a measurement scheme $\mathcal{Q}$ determines any pure state among all states, then there is an $\epsilon > 0$ such that every measurement scheme $\mathcal{Q}'$ which is $\epsilon$-close to $\mathcal{Q}$ has this same property.

**Proof.** For $i \in \{1, \ldots, d\}$, denote by $\lambda_i(X)$ the $i$-th greatest eigenvalue of a Hermitian matrix $X \in H(d)$. Let $K := \{X \in H(d) : \lambda_2(X) \leq 0, \|X\| = 1\}$ be the set of unit norm Hermitian matrices with at most one positive eigenvalue. Consider the map $\phi : H(d) \to \mathbb{R}^d$,

$$\phi(X) = (\lambda_1(X), -\lambda_2(X), \ldots, -\lambda_d(X))$$

and let $L := [0, +\infty)^d$. We have $K = \phi^{-1}(L) \cap H(d)_1$, where $H(d)_1$ is the unit sphere in $H(d)$. Since $H(d)_1$ is compact, $L$ is closed, and $\phi$ is continuous by Weyl’s perturbation theorem ([17], Corollary III.2.6), we conclude that $K$ is a compact set.

We claim that a measurement scheme $\mathcal{Q}$ determines pure states among all states if and only if $c := \min_{X \in K} \|M(\varrho)(X)\| > 0$. First, assume $c > 0$, and let $X \neq 0$ be such that $\lambda_2(X) \leq 0$. We have $X/\|X\| \in K$, hence

$$\|M(\varrho)(X)\| = \|X\|\|M(\varrho)(X/\|X\|)\| \geq \|X\|c \neq 0,$$

that is, $X \notin \ker M(\varrho)$. Therefore, $\mathcal{Q}$ determines any pure state among all states by Proposition 1. Conversely, suppose that $\mathcal{Q}$ has the latter property. By the compactness of $K$, there is $X \in K$ such that $c = \|M(\varrho)(X)\|$. Since Proposition 1 implies that every non-zero element of $\ker M(\varrho)$ has at least two positive eigenvalues, we have $M(\varrho)(X) \neq 0$ and thus $c \neq 0$.

Finally, if $\|M(\varrho) - M(\varrho')\|_\infty < \epsilon$, then

$$\min_{X \in K} \|M(\varrho)(X)\| \geq \min_{X \in K} \|M(\varrho)(X)\| - \|M(\varrho) - M(\varrho')(X)\| \geq c - \epsilon.$$

Hence, for any $\epsilon < c$, the measurement scheme $\mathcal{Q}'$ determines any pure state among all states.

**Pure-state quantum tomography.** -- The most notable practical feature of measurement schemes that determine pure states among all states is that they allow for a computationally efficient tomography of pure quantum states. Essentially, this is due to the fact that for every pure state $\sigma$, the unique solution to the feasibility problem

$$\text{find } X \quad \text{subject to } X \geq 0, \|M(Q(X) - M(\varrho))\| \leq \epsilon,$$

is given by $\sigma$. Indeed, since $\text{tr} [X] = \sum_{ij} M(Q(X)_{ij})$, the constraints imply that any solution is a state. Such a state must then coincide with $\sigma$, as the measurement scheme $\mathcal{Q}$ determines $\sigma$ among all states.

In practice, the state $\sigma$ might not be pure, but just well approximated by a pure state, the measurement $M(\varrho)$ might be affected by systematic errors and, furthermore, there is statistical noise. Because of that, in a realistic scenario, one has to reconstruct $\sigma$ from the perturbed measurement data $b := M(Q) + f$, where $f \in M_{lm}(\mathbb{R})$ is a small error term capturing all of these sources of error. In the remainder of this section we present two convex optimization problems which allow for a recovery of any pure state $\sigma$ from the noisy measurement data $b$ provided that the measurement scheme $\mathcal{Q}$ determines pure states among all states. Results in this direction have been reported also in [18].

First, consider the well-known [19] semi-definite program

$$\begin{align*}
\text{minimize } & \text{tr}(Y) \\
\text{subject to } & Y \geq 0, \|M(Q) - b\| \leq \epsilon,
\end{align*}$$

where $\epsilon > 0$ is an error scale which has to be fixed in advance. Then, as an easy consequence of [20], Theorem IV.1, we get the following recovery result (a simple proof is reported below).

**Theorem 3** (Stable Recovery I). Let $\epsilon > 0$. There is a constant $C_0 > 0$ independent of $\epsilon$ such that for all pure states $\sigma$ and all error terms $f \in M_{lm}(\mathbb{R})$ with $\|f\| \leq \epsilon$, any minimizer $Y^*$ of (3) satisfies

$$\|Y^* - \sigma\| \leq C_0 \epsilon.$$

Secondly, consider the following convex program, which was also proposed in [21]:

$$\begin{align*}
\text{minimize } & \|M(Q) - b\| \\
\text{subject to } & Y \geq 0,
\end{align*}$$

(4)

Note that, differently from the program (3), there is no need to guess an error scale $\epsilon$ in advance, which might be desirable from a practical point of view. The next result then follows from [20], Lemma V.5 (see also below).

**Theorem 4** (Stable Recovery II). Let $\epsilon > 0$. There is a constant $C_0 > 0$ independent of $\epsilon$ such that for all pure states $\sigma$ and all error terms $f \in M_{lm}(\mathbb{R})$ with $\|f\| \leq \epsilon$, any minimizer $Y^*$ of (4) satisfies

$$\|Y^* - \sigma\| \leq C_0 \epsilon.$$
Proof of Theorems 3 and 4. Note that for both of the optimizations (3) and (4) the minimizer $Y^*$ satisfies $\|M_Q(Y^*) - M_Q(\sigma) - f\| \leq \epsilon$. Hence, in both cases we find

$$
\begin{align*}
\epsilon & \geq \|M_Q(Y^*) - M_Q(\sigma) - f\| \geq \|M_Q(Y^* - \sigma)\| - \|f\| \\
& \geq \|Y^* - \sigma\| \inf \limits_{X', \sigma' \geq 0, X' \neq \sigma', \text{rank } \sigma' = 1} \|M_Q(\sigma' - X)\| - \epsilon.
\end{align*}
$$

By Weyl’s inequalities,

$$
\left\{ \sigma' - X \middle| X, \sigma' \geq 0, X \neq \sigma' \text{ and rank } \sigma' = 1 \right\} \subseteq K,
$$

where $K := \{ X' \in H(d) : \lambda_2(X') \leq 0, \|X'\| = 1 \}$. (Actually, it is easy to see that the two sets are equal.) By the argument in the proof of Theorem 2, the set $K$ is compact. Since the measurement scheme $Q$ determines pure states among all states, we have $M_Q(X') \neq 0$ for all $X' \in K$ by Proposition 1, and hence

$$
c_Q := \min \limits_{X' \in K} \|M_Q(X')\| > 0.
$$

This, together with (5), implies

$$
\|Y^* - \sigma\| \leq \frac{2}{c_Q} \epsilon
$$

and hence we can choose $C_Q = 2/c_Q$. □

Note that in both Theorems 3 and 4, the constant $C_Q$ appearing in the stability bound might depend on all the parameters of $Q$. We do not know how to estimate $C_Q$ and hence we cannot make our stability results more explicit. Therefore, we have to rely on numerical simulations to evaluate whether the measurement schemes we constructed perform well enough in practice.

Numerical results. – For our simulations we choose the measurement schemes constructed from the Chebyshev polynomials of the second kind $(U_n)_{n=0}^\infty$. Moreover, we choose $\alpha = \pi/d$ and we use the Hilbert-Schmidt norm $\|\cdot\|_2$ on both $H(d)$ and $M_{lm}(\mathbb{R})$. For dimensions $d = 10, 20, \ldots, 60$, we ran the semi-definite program (3) for $10^5$ times, where we sampled the pure states and error terms $f \in M_{lm}(\mathbb{R})$ with $\|f\|_2 = \epsilon$ independently according to the respective Haar measures. The error scale was set to $\epsilon = 10^{-4}$.

Figure 1 shows the empiric probability density function of the reconstruction error for the dimensions $d = 10, 30, 50$. In all cases the distribution appears to be well located, indicating a good reconstruction for most signals.

Figure 2 shows the empiric 96%, 99% and 99.75% quantiles of the reconstruction error as well as its arithmetic mean. In the selected range of dimension the 99.75% quantile error does not exceed 60ε. This suggests that for most signals the reconstruction is feasible. Furthermore, all quantiles appear to scale sublinearly with the dimension.

Conclusion. – We have presented an explicit construction of five measurement settings which allow the efficient reconstruction of pure quantum states. Unlike earlier approaches, our method is deterministic and non-adaptive, meaning that the setting is fixed and works for all states. An important fact from the practical point of view is that the scheme is robust with respect to noise. Thus, state reconstruction from the measurement data can then be applied at the practical level by using the presented algorithms.

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