Higher order asymptotic expansion of solutions to abstract linear hyperbolic equations

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Received: 9 October 2019 / Revised: 27 December 2019 / Published online: 3 February 2020
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Abstract
The paper concerned with higher order asymptotic expansion of solutions to the Cauchy problem of abstract hyperbolic equations of the form \( u'' + Au + u' = 0 \) in a Hilbert space, where \( A \) is a nonnegative selfadjoint operator. The result says that by assuming a suitable regularity of initial data, asymptotic profiles (of arbitrary order) are explicitly written by using the \( C_0 \)-semigroup \( e^{-tA} \) generated by \( -A \). To prove this, a kind of maximal regularity for \( e^{-tA} \) is used.

Mathematics Subject Classification Primary 35L90 · 35B40; Secondary 34G10 · 35E15

1 Introduction
Let \( H \) be a Hilbert space over \( \mathbb{R} \) with the inner product \( (\cdot, \cdot) \) and the norm \( \| \cdot \| \). In this paper we consider higher order asymptotic expansion of solutions to abstract hyperbolic equations of the form

\[
\begin{cases}
  u''(t) + Au(t) + u'(t) = 0, & t > 0, \\
  (u, u')(0) = (u_0, u_1),
\end{cases}
\]

(1.1)

where \( u : [0, \infty) \to H \) is an unknown function and \( A \) is a nonnegative selfadjoint operator in \( H \). We denote \( D(A) \) as a domain of \( A \). The pair \( (u_0, u_1) \in D(A^{1/2}) \times H \) is given.
The problem (1.1) is motivated as the generalization of the damped wave equation

\[
\begin{aligned}
\partial_t^2 u(x,t) - \Delta u(x,t) + \partial_t u(x,t) &= 0, \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \\
(u, u')(x, 0) &= (u_0(x), u_1(x)), \quad x \in \mathbb{R}^N.
\end{aligned}
\] (1.2)

with \((u_0, u_1) \in H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)\). The abstract framework as (1.1) is firstly introduced by Ikehata–Nishihara [6].

Equation (1.2) has been considered as a model of the phenomenon of heat conduction with finite propagation property (see Cattaneo [2] and Vernotte [21]). Therefore the solution of (1.2) is expected to have a similar profile of solutions to the heat equation

\[
\begin{aligned}
\partial_t v(x,t) - \Delta u(x,t) &= 0, \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \\
v(x,0) &= v_0(x), \quad x \in \mathbb{R}^N.
\end{aligned}
\] (1.3)

Actually it is known that under the suitable assumption, the solution \(u\) of (1.2) behaves like the solution \(v\) of (1.3) with \(v_0 = u_0 + u_1\). This phenomenon is now so-called diffusion phenomenon for the damped wave equation. There are many investigation dealing with diffusion phenomena for (1.2) (see e.g., Hsiao–Liu [4], Nishihara [11,12], Karch [9], Yang–Milani [24], Volkmer [22], Said-Houari [16] including some generalized problems like quasilinear systems). In the case of the damped wave equation in the exterior domain, Fourier analysis does not work well, and therefore, energy methods via integration by parts are often used (see e.g., Ikehata–Matsuyama [7], Ikehata [5] and Ikehata–Saeki [8]).

Another example of the problem (1.1) is the damped beam equation

\[
\begin{aligned}
\partial_t^2 u(x,t) + \partial_t^4 u(x,t) - \alpha \partial_t^2 u(x,t) + \partial_t u(x,t) &= \partial_x f(\partial_x u(x,t)), \quad (x, t) \in \mathbb{R} \times (0, \infty), \\
(u, u')(x, 0) &= (u_0(x), u_1(x)), \quad x \in \mathbb{R}^N.
\end{aligned}
\] (1.4)

The global existence and asymptotic behavior of solutions to (1.4) are studied in Takeda–Yoshikawa [18–20] and and Takeda [17]. In these papers, the asymptotic behavior of solutions of (1.4) with \(f \equiv 0\) plays a crucial role to consider global existence of nonlinear problem. Actually, in [18] the asymptotic behavior of solutions to the nonlinear problem can be found as solutions of a linear problem.

For the further analysis, as an improvement of diffusion phenomena, the problem of higher order asymptotic expansion should be naturally considered. In 2001, Orive–Zuazua–Pazoto [13], they considered the following general problem of (1.2):

\[
\begin{aligned}
\rho(x) \partial_t^2 u(x,t) - \text{div}(a(x) \nabla u(x,t)) + a_0 \rho(x) \partial_t u(x,t) &= 0, \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \\
(u, u')(x, 0) &= (u_0(x), u_1(x)), \quad x \in \mathbb{R}^N,
\end{aligned}
\] (1.5)

where \(\rho(x)\), \(\rho(x)^{-1}\) and \(a(x) = (a_{jk}(x))_{jk}\) are bounded and spatially periodic and the matrix \((a_{jk})_{jk}\) is uniformly and positively determined. In [13], higher order asymptotic expansion of solutions to (1.5) via the use of Bloch wave decomposition which is valid for spatial periodic coefficients. Later, Takeda [17] gave higher order asymptotic expansion by using the usual heat semigroup \(e^{t\Delta}\) for (1.2) via the Fourier multiplier.
Higher order asymptotic expansion of solutions... theory with asymptotic expansion of symbol of evolution operator with respect to the variable of Fourier spaces $\xi$. An asymptotic expansion for wave part can be found in a recent paper Michihisa [10].

Instead of the various results in the previous works, in this paper the interest is how to systematically determine the asymptotic profile of arbitrary order. To discuss this problem we then consider an abstract hyperbolic Eq. (1.1) which is included three situations (1.2) ($A = -\Delta$), (1.4) ($A = \frac{d^4}{dx^4} - \alpha \frac{d^2}{dx^2}$) and (1.5) ($Au = \rho^{-1} \text{div}(a \nabla u)$).

Concerning the previous works of the abstract setting, Ikehata–Nishihara [6] introduced the problem (1.1) and observed the asymptotic behavior of solutions in the following way:

$$\|u(t) - e^{-tA}(u_0 + u_1)\| \leq C(1 + t)^{-1} \left( \log(e + t) \right)^{1/2 + \varepsilon},

(u_0, u_1) \in D(A) \times D(A^{1/2}).$$

After that Chill–Haraux [3] discussed the same problem and succeeded in removing the logarithmic correction of the above inequality, which is conjectured in Ikehata–Nishihara [6]. Radu–Todorova–Yordanov [14] studied also diffusion phenomena with respect to stronger norms $\| \cdot \|_{D(A^k)}$ (a similar analysis for a linear hyperbolic equation in Hilbert spaces with time-dependent damping term $b(t)u'$ can be found in Yamazaki [23]). Radu–Todorova–Yordanov [15] discussed a higher order approximation of solutions to $Bu'' + Au + u' = 0$ ($B$ is bounded, selfadjoint and positively definite); however, their framework is only valid for semigroup in metric measure spaces $L^2(\Omega, \mu)$ with an extra maximal $L^p-L^q$ regularity.

The purpose of the present paper is to discuss higher order asymptotic expansion of solutions (1.1) by using the $C_0$-semigroup $e^{-tA}$ generated by $-A$. Moreover, the main topic is to give a way how to construct higher order asymptotic profiles of arbitrary order.

To begin with, we state existence of solutions to the problem (1.1). The following proposition follows from the standard theory for $C_0$-semigroups on product spaces (see e.g., Ikehata–Nishihara [6] and also Brezis [1, Section X] when $A = -\Delta$).

**Proposition 1.1** Let $A$ be a nonnegative selfadjoint operator in $H$ with domain $D(A)$. Then the operator

$$A = \begin{pmatrix} 0 & -1 \\ A & 1 \end{pmatrix},$$

in $\mathcal{H} = D(A^{1/2}) \times H$ with domain $D(A) = D(A) \times D(A^{1/2})$ is quasi-$m$-accretive in $\mathcal{H}$. Namely, $-A$ generates a $C_0$-semigroup in $\mathcal{H}$.

In particular, the solution $u$ of (1.1) uniquely exists in the following sense:

(1) For $(u_0, u_1) \in D(A^{1/2}) \times D(H)$, one has $(u(t), u'(t)) = e^{-tA}(u_0, u_1)$ which implies

$$u \in C^2([0, \infty); D(A^{1/2})) \cap C^1([0, \infty); H) \cap C([0, \infty); D(A^{1/2})).$$
and the Eq. (1.1) is satisfied in \([D(A^{1/2})]^*\) (the dual space of \(D(A^{1/2})\)), that is,

\[
\langle u''(t), \varphi \rangle_{[D(A^{1/2})]^*, D(A^{1/2})} + (A^{1/2}u(t), A^{1/2}\varphi) + (u'(t), \varphi) = 0, \quad \forall \varphi \in D(A^{1/2})
\]

(2) For \((u_0, u_1) \in D(A^{m+1/2}) \times D(A^{m+1/2})\) for some \(m \in \mathbb{N}\cup\{0\}\), one has \((u(t), u'(t)) = e^{-tA}(u_0, u_1)\) which implies

\[
u \in \bigcap_{k=0}^{n+1} C^k \left([0, \infty); D \left(A^{\frac{n-k}{2}}\right)\right).
\]

Now we are in a position to give the result for higher order asymptotic expansion for abstract hyperbolic Eq. (1.1), which is the main result of the present paper.

**Theorem 1.2** Assume that \((u_0, u_1) \in D(A^{m+1/2}) \times D(A^{m+1/2})\) for some \(m \in \mathbb{N}\cup\{0\}\). Let \(u\) be a unique solution of (1.1). Define \(\nu_0 = u_0 + u_1\) and for \(\ell \in \mathbb{N}\cup\{0\}\),

\[
\nu_0(t) = e^{-tA}\nu_0, \\
\nu_\ell(t) = A^\ell \left(\sum_{j=0}^{\ell} \frac{(2\ell - 1)}{(\ell + j - 1)} \frac{(-tA)^j}{j!} e^{-tA}\nu_0 + \sum_{k=0}^{\ell-1} \frac{(2\ell - 1)}{\ell + k} \frac{(-tA)^k}{k!} e^{-tA}\nu_1\right).
\]

Then there exists a positive constant \(C\) such that for every \(t \geq 0\),

\[
\|u(t) - \sum_{\ell=0}^{m} \nu_\ell(t)\| \leq C(1 + t)^{-m-1/2}.
\]

**Remark 1.1** Especially, if 0 is an eigenvalue of \(A\) and \(\varphi_0\) is the corresponding eigenvector, then \(e^{-tA}\varphi_0 = \varphi_0\). Therefore there is no possibility to improve the upper bound (1.6) in the general setting. Since the asymptotic profiles \(\{\nu_\ell\}\) satisfy

\[
\|\nu_\ell(t)\| \leq C_\ell (1 + t)^{-\ell}, \quad (1.6)
\]

the assertion in Theorem 1.2 can be understood as a higher order asymptotic expansion of solution \(u\) of (1.1). Of course in the several situation such as \(-\Delta\) in \(\mathbb{R}^N\), we have \(\|e^{-tA}(u_0 + u_1)\| \leq C(1 + t)^{-\alpha}\) for some \(\alpha > 0\) under some assumption on \(u_0 + u_1\). In this case we need to do some effort to determine especially the regularity of initial data. We will not touch the details of specialized cases.

**Remark 1.2** If \(m = 0\), then Theorem 1.2 is weaker than those of Ikehata–Nishihara [6] and also Chill–Haraux [3]. Since the main topic of our result is to give a higher order asymptotic profiles, the optimality of decay rates of remainder terms is not precisely discussed.

**Remark 1.3** In the case (1.2), we choose \(H = L^2(\mathbb{R}^N)\) and \(A = -\Delta\) endowed with domain \(D(A) = H^2(\mathbb{R}^N)\). Takeda [17] obtained the same asymptotic expansion with a different expression.
Higher order asymptotic expansion of solutions...

\[ \overline{u}_\ell(t) = \frac{1}{2} \sum_{j=0}^{\ell} \alpha_{j,k} (-t)^j (-\Delta)^j \ell^j e^{t\Delta} u_0 + \sum_{0 \leq k_1 + k_2 \leq \ell} \alpha_{\ell-k_1-k_2,k_1} \beta_{k_2} (-t)^{\ell-k_1-k_2} (-\Delta)^{2\ell-k_1-k_2} e^{t\Delta} \left( \frac{1}{2} u_0 + u_1 \right), \]

where

\[ \alpha_{j,k} = \frac{1}{j! k!} \frac{d^k}{dr^k} \phi_j \bigg|_{r=0}, \quad \phi_j(r) = \left( \frac{1}{2} + \sqrt{\frac{1}{4} - r} \right)^{-2j}, \]

\[ \beta_k = \frac{1}{k!} \frac{d^k}{dr^k} \psi \bigg|_{r=0}, \quad \psi(r) = \frac{1}{2} \left( \frac{1}{4} - r \right)^{-1/2}. \]

This result is valid for the whole space case \( \mathbb{R}^N \) and the effect of high-frequency part is clearly written. In the case of damped wave equation in exterior domain, such a higher order asymptotic expansion of solutions is not known so far. Theorem 1.2 asserts that the same asymptotic expansion is valid for every nonnegative selfadjoint operator in a Hilbert space.

**Remark 1.4** In the case (1.4), we choose \( H = L^2(\mathbb{R}) \) and \( A = \frac{d^4}{dx^4} - \alpha \frac{d^2}{dx^2} \) endowed with domain \( D(A) = H^4(\mathbb{R}) \). According to Theorem 1.2 with \( m = 0 \), we can see that

\[ \|u(t) - e^{-tA}(u_0 + u_1)\| \leq C(1 + t)^{-\frac{1}{2}}. \]

On the other hand, in Takeda–Yoshikawa [18], it is shown that the asymptotic behavior of the solution \( u \) is given by \( e^{-\alpha t A_0}(u_0 + u_1) \), where \( A_0 = -\frac{d^2}{dx^2} \). The observation in Theorem 1.2 does not have contradiction because of the following estimate

\[ \left\| e^{-tA}(u_0 + u_1) - e^{-\alpha t A_0}(u_0 + u_1) \right\| \leq \frac{C}{t} \|tA_0 e^{-\alpha t A_0}(u_0 + u_1)\|, \]

\( u_0 + u_1 \in H^2(\mathbb{R}) \).

However, if we choose the latter profile, then since the asymptotic expansion in Theorem 1.2 is written by the operator \( A (\neq A_0) \), their difference should be carefully analysed.

**Remark 1.5** In the case (1.5) (with bounded \( \rho, \rho^{-1} \)), we choose \( H = L^2(\mathbb{R}) \) but an inner product different from the usual one:

\[ (f, g)_H = \int_{\mathbb{R}^N} f g \rho \, dx. \]

Then we use \( Au = \rho(x)^{-1} \text{div}(a(x) \nabla u) \) endowed with domain \( D(A) = H^2(\mathbb{R}^N) \). Of course we can consider the periodic setting \( \mathbb{T}^N = (\mathbb{R}/\mathbb{Z})^N \) by choosing Sobolev...
spaces of periodic functions. Combining the strategy of Bloch wave decomposition, we can also deduce a similar result in [13].

Let us describe the strategy for the construction of asymptotic profiles. It is well known that the solution $u$ of (1.1) has the energy decreasing property

$$E(u, t) + 2 \int_0^t \|u'(s)\|^2 \, ds = E(u, 0) = \|u_1\|^2 + \|A^{1/2}u_0\|^2,$$

where the energy functional of the solution $u$ of (1.1) is defined as

$$E(u; t) = \|u'(t)\|^2 + \|A^{1/2}u(t)\|^2.$$

According to the experiences, we expect that the first order asymptotic profile of $u$ is $e^{-tA}(u_0 + u_1)$. On the other hand, if $w$ is the solution of (1.1), then by a direct computation we have

$$(1 + t)\|w'(t)\|^2 \leq (1 + t)E(w, t) + \|w(t)\|^2 \leq C.$$

Combining this concept, we consider the following auxiliary problem

$$\begin{cases}
U''_1 + AU_1 + U'_1 = Ae^{tA}(u_0 + u_1), & t > 0, \\
(U_1, U'_1)(0) = (0, -u_1). \tag{1.7}
\end{cases}$$

After some computation, we have $u(t) = e^{-tA}(u_0 + u_1) + U'_1(t)$, which is the first decomposition. It should be noticed that the function $U_1$ is completely the same as the function $Z$ in Ikehata–Nishihara [6], however, the treatment of $U_1$ is different from [6] in which $Z$ is understood as the solution of $Z' + AZ = u'$, $Z(0) = 0$. A definite difference is that $Z$ explicitly depends on the original solution. In contrast, $U_1$ can be determined independently. This provides a significant improvement in the analysis of asymptotic expansion. The idea of (1.7) can be understood as another use of the modified Morawetz method which is written in Ikehata–Matsuyama [7].

Then analysing the asymptotic profile of the solution $U_1$ and proceeding the similar argument as before, we successively obtain the sequence $U_\ell$ and their corresponding hyperbolic problems similar to (1.7). Finally, to obtain the desired decay property of error terms (such as $U'_1$), we use a kind of maximal regularity for the Cauchy problem of the parabolic equation $V' + AV = 0$. This consideration suggests that the solution $u$ of (1.1) can be decomposed by

$$u(t) = \sum_{\ell=0}^m d\ell V_\ell(t) + \frac{d^{m+1}}{dt^{m+1}} U_{m+1}(t).$$

The present paper is organized as follows. In Sect. 2, as a preliminary, we state and prove a kind of maximal regularity result the Cauchy problem of the parabolic equation $V' + AV = 0$, which we will use later. In Sect. 3, we provide a proof of the
simplest case \( m = 0 \) in Theorem 1.2 to clarify how to show the diffusion phenomena. The first part of Sect. 4 provides a construction of a family of functions which will be understood as higher order asymptotic profiles. Finally, at the rest of Sect. 4 we show higher order asymptotic expansion of solutions to (1.1) (Theorem 1.2 for \( m \in \mathbb{N} \)).

2 Preliminary result for the property of the \( C_0 \)-semigroup \( e^{-tA} \)

In this section, we give recall a kind of maximal regularity result for the abstract (selfadjoint) \( C_0 \)-semigroup \( e^{-tA} \). For the reader’s convenience, we provide a short proof.

**Lemma 2.1** If \( f \in H \), then for every \( n \in \mathbb{N} \cup \{0\} \) and \( t \geq 0 \),

\[
\|e^{-tA}f\|^2 + 2^n \frac{n!}{n!} \int_0^t s^n \|A^{\frac{n+1}{2}} e^{-sA} f\|^2 ds = \frac{\|f\|^2}{2}.
\]

Moreover, if \( f \in D(A^{n/2}) \), then there exists a positive constant \( C \) such that

\[
\int_0^t (1 + s)^n \|A^{\frac{n+1}{2}} e^{-sA} f\|^2 ds \leq C(\|f\|^2 + \|A^{n/2} f\|^2).
\]

**Proof** If \( f \in D(A^{n/2+1}) \), then \( w(t) = e^{-tA} f \) satisfies

\[
\frac{d}{dt} \|A^{k/2} w(t)\|^2 = -2(A^{k/2} w(t), A^{k/2} w'(t))
= -2(A^{k/2} w(t), A^{k/2+1} w(t))
= -2\|A^{k+1/2} w(t)\|^2.
\]

In view of the above computation, in particular, we have

\[
\frac{d}{dt} \left[ (k+1) \|A^{k+1/2} w(t)\|^2 \right] = (k + 1)t^k \|A^{k+1/2} w(t)\|^2 - 2t^{k+1} \|A^{k+2/2} w(t)\|^2
\]

and therefore

\[
\frac{d}{dt} \left[ \sum_{k=0}^{n-1} \frac{(2t)^{k+1}}{(k+1)!} \|A^{k+1/2} w(t)\|^2 \right]
= 2 \sum_{k=0}^{n-1} \frac{(2t)^k}{k!} \|A^{k+1/2} w(t)\|^2 - \frac{(2t)^{k+1}}{(k+1)!} \|A^{k+2/2} w(t)\|^2
= \|A^{1/2} w(t)\|^2 - \frac{(2t)^n}{n!} \|A^{n+1/2} w(t)\|^2.
\]

Integrating it over \([0, t]\), we obtain (2.1). \(\square\)
3 The first asymptotics

In this section, we will show the strategy how to find an asymptotic profile of solutions in the simplest situation $m = 0$ in Theorem 1.2.

To begin with, we prove the following lemma.

Lemma 3.1 Assume that $(u_0, u_1) \in \mathcal{H}$ and let $u$ be a unique solution of (1.1). Then there exists a positive constant $C > 0$ such that for every $t \geq 0$,

$$(1 + t)E(u, t) + \|u(t)\|^2 + \int_0^t \left( (1 + s)\|u'(s)\|^2 + \|A^{1/2}u(s)\|^2 \right) ds \leq C\left( E(u, 0) + \|u_0\|^2 \right).$$

Proof By taking a suitable approximation of $(u_0, u_1)$, we may assume $(u_0, u_1) \in D(A)$ without loss of generality. Then we see by a direct computation with the Eq. (1.1) that

$$\frac{d}{dt} E(u; t) = 2(u', u'') + 2(A^{1/2}u', A^{1/2}u)$$
$$= 2(u', u'' + Au)$$
$$= -2\|u'\|^2.$$

This yields

$$\|u'(t)\|^2 + \|A^{1/2}u\|\|^2 + 2 \int_0^t \|u'(s)\|^2 ds = \|u_1\|^2 + \|A^{1/2}u_0\|^2.$$

On the other hand, using the Eq. (1.1), we also have the following two identities:

$$\frac{d}{dt} \left[ (3 + t)E(u; t) \right] = \|u'\|^2 + \|A^{1/2}u\|^2 - 2(3 + t)\|u'\|^2,$$
$$\frac{d}{dt} E_*(u; t) = 2\|u'\|^2 + 2(u, u'' + u')$$
$$= 2\|u'\|^2 - 2\|A^{1/2}u\|^2.$$

Summing up the above identities, we deduce

$$\frac{d}{dt} \left[ (3 + t)E(u; t) + E_*(u; t) \right] = -(3 + 2t)\|u'\|^2 - \|A^{1/2}u\|^2. \quad (3.1)$$

Noting that the differentiated function in (3.1) can be rewritten as

$$(3 + t)E(u; t) + E_*(u; t) = (1 + t)E(u, t) + 2\|A^{1/2}u\|^2$$
$$+ 2 \|u' + \frac{1}{2} u\|^2 + \frac{1}{2}\|u\|^2.$$
we obtain the desired inequality.

The next lemma enables us to divide the solution \( u \) of (1.1) into two factors: one reflects the diffusion phenomenon and the other reflects an effect of hyperbolicity. This treatment is the difference between the one in Ikehata–Nishihara [6] and ours.

**Lemma 3.2** Let \((u_0, u_1) \in D(A^{1/2}) \times D(A^{1/2})\) and let \(U_1\) be the unique solution of the problem

\[
\begin{align*}
U_1'' + AU_1 + U_1' &= Ae^{-tA}(u_0 + u_1), \quad t > 0, \\
(U_1, U_1')(0) &= (0, -u_1).
\end{align*}
\] (3.2)

Then the function \(e^{-tA}(u_0 + u_1) + U_1'\) coincides with the solution of (1.1).

**Proof** Take \((u_{0\varepsilon}, u_{1\varepsilon}) \in D(A^2) \times D(A^2)\) such that \(u_{0\varepsilon} \to u_0\) and \(u_{1\varepsilon} \to u_1\) in \(D(A^{1/2})\) as \(\varepsilon \to 0\) and consider

\[
\begin{align*}
U_{1\varepsilon}'' + AU_{1\varepsilon} + U_{1\varepsilon}' &= Ae^{-tA}(u_{0\varepsilon} + u_{1\varepsilon}), \quad t > 0, \\
(U_{1\varepsilon}, U_{1\varepsilon}')(0) &= (0, -u_{1\varepsilon}).
\end{align*}
\] (3.3)

Since \(Ae^{tA}(u_{0\varepsilon} + u_{1\varepsilon}) \in C([0, \infty); D(A))\), this problem has a unique solution

\[U_{1\varepsilon} \in C^2([0, \infty); D(A^{1/2})) \cap C^1([0, \infty); D(A)) \cap C([0, \infty); D(A^{3/2}))\].

Moreover, by using Duhamel’s principle we can see that the solution \(U_{1\varepsilon}\) can be represented by

\[
(U_{1\varepsilon}(t), U_{1\varepsilon}'(t)) = e^{-tA}(0, -u_{1\varepsilon}) + \int_0^t e^{-(t-s)A} \left(0, Ae^{-sA}(u_{0\varepsilon} + u_{1\varepsilon})\right) ds
\]

which converges to the solution of (3.2) in \(\mathcal{H}\) as \(\varepsilon \to 0\):

\[
e^{-tA}(0, -u_1) + \int_0^t e^{-(t-s)A}(0, Ae^{-sA}(u_{0\varepsilon} + u_{1\varepsilon})) ds = (U_1(t), U_1'(t)).
\]

Setting \(v_\varepsilon(t) = e^{-tA}(u_{0\varepsilon} + u_{1\varepsilon})\) and \(w_\varepsilon(t) = v_\varepsilon(t) + U_{1\varepsilon}'(t)\), we have

\[w_\varepsilon(0) = u_{0\varepsilon} + u_{1\varepsilon} - u_{1\varepsilon} = u_{0\varepsilon}\]

and

\[
w_\varepsilon'(t) = v_\varepsilon'(t) + U_{1\varepsilon}''(t) \\
= v_\varepsilon'(t) + Au_\varepsilon(t) - AU_{1\varepsilon}(t) - U_{1\varepsilon}'(t) \]

\[= -AU_{1\varepsilon}(t) - U_{1\varepsilon}'(t).
\]
This yields
\[
\lim_{t \to 0} w'_e(t) = -AU_{1e}(0) - U'_{1e}(0) = u_{1e}
\]
and also \( w'_e \in C^1((0, \infty); H) \) with
\[
w''_e(t) = -AU'_{1e}(t) - (w_e(t) - v_e(t))' = -A(w_e(t) - v_e(t)) - w'_e(t) + v'_e(t) = -Aw_e(t) - w'_e(t).
\]
This implies \((w_e(t), w'_e(t)) = e^{-tA}(u_{0e}, u_{1e})\) and therefore \((w_e(t), w'_e(t)) \to (u(t), u'(t))\) in \( \mathcal{H} \) as \( \varepsilon \to 0 \). Consequently, noting that \( v_e(t) \to e^{-tA}(u_0 + u_1) \) in \( D(A^{1/2}) \) as \( \varepsilon \to 0 \) we have
\[
u(t) = \lim_{\varepsilon \to 0} w_e(t) = e^{-tA}(u_0 + u_1) + U'_1(t).
\]
The proof is complete. \( \square \)

Observe that a decaying property of \( e^{-tA}(u_0 + u_1) \) cannot be expected as mentioned in Remark 1.1. In contrast, the other factor \( U' \) decays like \((1 + t)^{-1/2}\) under a suitable assumption. To prove the decay property we proceed a similar way in the proof of Lemma 3.1, however, we need Lemma 2.1 which is a kind of maximal regularity result for \( e^{-tA} \).

**Lemma 3.3** If \((u_0, u_1) \in D(A^{1/2}) \times D(A^{1/2})\), then
\[
(1 + t)E(U_1; t) + \|U_1(t)\|^2 \in L^\infty((0, \infty)), \quad (1 + t)\|U'_1(t)\|^2
\]
\[
+ \|A^{1/2}U_1(t)\|^2 \in L^1((0, \infty)).
\]

**Proof** By using the same approximation as in the proof of Lemma 3.2, we may assume \( U_1 \in C^2([0, \infty); D(A^{1/2})) \cap C^1([0, \infty); D(A)) \cap C([0, \infty); D(A^{3/2})) \). Here we deduce an energy inequality for \( U_1 \). By the Eq. (3.2), we have
\[
\frac{d}{dt} E_*(U_1; t) = 2\|U'_1\|^2 + 2(U_1, -AU_1 + Ae^{-tA}(u_0 + u_1))
\]
\[
= 2\|U'_1\|^2 - 2\|A^{1/2}U_1\|^2 + 2(U_1, Ae^{-tA}(u_0 + u_1))
\]
\[
\leq 2\|U'_1\|^2 - 2\|A^{1/2}U_1\|^2 + \|A^{1/2}e^{-tA}(u_0 + u_1)\|^2
\]
and
\[
\frac{d}{dt} \left[ (6 + t)E(U_1; t) \right] = \|U'_1\|^2 + \|A^{1/2}U_1\|^2
\]
\[
+ 2(6 + t)(U'_1, -U'_1 + Ae^{-tA}(u_0 + u_1))
\]
\[
\leq \|U'_1\|^2 + \|A^{1/2}U_1\|^2 - (6 + t)\|U'_1\|^2
\]
\[
+ (6 + t)\|Ae^{-tA}(u_0 + u_1)\|^2.
\]
In view of Lemma 2.1 with \( f = u_0 + u_1 \in D(A^{1/2}) \), we have

\[
\frac{d}{dt} \left[ (6 + t) E(U_1; t) + 2 E_{\ast}(U_1; t) \right] + (1 + t) \| U_1' \|^2 + \| A^{1/2} U_1 \|^2 \\
\leq (6 + t) \| e^{-tA} (u_0 + u_1) \|^2 + 2 \| A^{1/2} e^{-tA} (u_0 + u_1) \|^2
\]

which is integrable with respect to \( t \in (0, \infty) \). Integrating the above estimate over \([0, t]\), we obtain all desired estimates. \qed

Combining Lemmas 3.1, 3.2 and 3.3, we obtain the following proposition which is the same as the assertion of Theorem 1.2 with \( m = 0 \).

**Proposition 3.4** Assume \((u_0, u_1) \in D(A^{1/2}) \times D(A^{1/2})\). Then there exists a positive constant \( C > 0 \) such that for every \( t \geq 0 \),

\[
\| u(t) - e^{-tA} (u_0 + u_1) \| \leq C (1 + t)^{-\frac{1}{2}}.
\]

### 4 Higher order asymptotic expansion

In this section, we prove Theorem 1.2 with \( m \in \mathbb{N} \) which gives higher order asymptotic expansion of solutions to (1.1). The philosophy of the proof is essentially the same as the previous section.

#### 4.1 Construction of family of functions for asymptotics

To consider higher order asymptotic expansions, we define the following functions.

**Definition 4.1** For \( m \in \mathbb{N} \) and \((u_0, u_1) \in \mathcal{H}\), define \( v_0 = u_0 + u_1 \) and

\[
V_0^{(1)}(t) = e^{-tA} v_0, \\
V_m^{(1)}(t) = (-1)^m \sum_{j=1}^m \binom{m}{j-1} \frac{(-tA)^j}{j!} e^{-tA} v_0, \quad m \in \mathbb{N}, \\
V_0^{(2)}(t) = 0, \\
V_m^{(2)}(t) = (-1)^m \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{(-tA)^k}{k!} e^{-tA} u_1, \quad m \in \mathbb{N}
\]

and \( V_m(t) = V_m^{(1)}(t) + V_m^{(2)}(t) \).

The family \( \{V_m\} \) will provide higher order asymptotic profiles; this will be explained after stating Lemma 4.4 below.

The function \( V_m \) can be successively found by using the following structure.
Lemma 4.1 Assume \( u_1 \in D(A^{1/2}) \). Then for \( m \in \mathbb{N} \cup \{0\} \), \( V_{m+1} \) is a unique solution of

\[
\begin{cases}
V'_{m+1} + AV_{m+1} = -V_m', & t > 0, \\
V_{m+1}(0) = (-1)^{m+1}u_1.
\end{cases}
\]

Proof We use the Duhamel principle. The solution of

\[
\begin{cases}
w'_{m+1} + Aw_{m+1} = -V_m', & t > 0, \\
w_{m+1}(0) = (-1)^{m+1}u_1.
\end{cases}
\]

can be written by the formula

\[
w_{m+1}(t) = e^{-tA}(-1)^{m+1}u_1 - \int_0^t e^{-(t-s)A}V_m'(s)\,ds.
\]

For the case \( m = 0 \), we see from the semigroup property that

\[
w_1(t) = e^{-tA}[-u_1] - \int_0^t e^{-(t-s)A}V_0'(s)\,ds
= -e^{-tA}u_1 - \int_0^t e^{-(t-s)A}(-Ae^{-sA}v_0)\,ds
= -e^{-tA}u_1 + tAe^{-tA}v_0
= V^{(1)}_1(t) + V^{(2)}_1(t).
\]

This gives the assertion with \( m = 0 \). For the case \( m \in \mathbb{N} \), observe that

\[
\frac{d}{dt}V^{(1)}_m(t) = (-1)^m \left[ \sum_{j=1}^m \left( \begin{array}{c} m-1 \\ j-1 \end{array} \right) \frac{(-tA)^{j-1}}{(j-1)!} (-A)e^{-tA}v_0 \\
+ \sum_{j=1}^m \left( \begin{array}{c} m-1 \\ j-1 \end{array} \right) \frac{(-tA)^j}{j!} (-A)e^{-tA}v_0 \right]
= (-1)^m \left[ \sum_{j=1}^m \left( \begin{array}{c} m-1 \\ j-1 \end{array} \right) \frac{(-tA)^{j-1}}{(j-1)!} (-A)e^{-tA}v_0 \\
+ \sum_{j=2}^{m+1} \left( \begin{array}{c} m-1 \\ j-2 \end{array} \right) \frac{(-tA)^{j-1}}{(j-1)!} (-A)e^{-tA}v_0 \right]
= (-1)^m \sum_{j=1}^{m+1} \left( \begin{array}{c} m \\ j-1 \end{array} \right) \frac{(-tA)^{j-1}}{(j-1)!} (-A)e^{-tA}v_0.
\]
Therefore

\[
\int_0^t e^{-(t-s)A} \left( \frac{d}{dt} V_m^{(1)}(s) \right) ds = \int_0^t e^{-(t-s)A} \left( (-1)^m \sum_{j=1}^{m+1} \left( \begin{array}{c} m \\ j-1 \end{array} \right) \frac{(-sA)^{j-1}}{(j-1)!} (-A)e^{-sA}v_0 \right) ds
\]

\[
= (-1)^m \sum_{j=1}^{m+1} \left( \begin{array}{c} m \\ j-1 \end{array} \right) \left( \int_0^t s^{j-1} \frac{d}{ds} \right) ds \left( -A \right)^j e^{-tA}v_0
\]

\[
= (-1)^m \sum_{j=1}^{m+1} \left( \begin{array}{c} m \\ j-1 \end{array} \right) \frac{(-tA)^j}{j!} e^{-tA}v_0
\]

\[
= -V_m^{(1)}(t).
\]

Similarly, we have

\[
\frac{d}{dt} V_m^{(2)}(t) = (-1)^m \sum_{k=1}^{m-1} \left( \begin{array}{c} m-1 \\ k \end{array} \right) \frac{(-tA)^{k-1}}{(k-1)!} (-A)e^{-tA}u_1
\]

\[
+ \sum_{k=0}^{m-1} \left( \begin{array}{c} m-1 \\ k \end{array} \right) \frac{(-tA)^k}{k!} (-A)e^{-tA}u_1
\]

\[
= (-1)^m \sum_{k=1}^{m-1} \left( \begin{array}{c} m-1 \\ k \end{array} \right) \frac{(-tA)^{k-1}}{(k-1)!} (-A)e^{-tA}u_1
\]

\[
+ \sum_{k=1}^{m} \left( \begin{array}{c} m-1 \\ k-1 \end{array} \right) \frac{(-tA)^{k-1}}{(k-1)!} (-A)e^{-tA}u_1
\]

\[
= (-1)^m \sum_{k=1}^{m} \left( \begin{array}{c} m \\ k \end{array} \right) \frac{(-tA)^{k-1}}{(k-1)!} (-A)e^{-tA}u_1
\]

and then

\[
\int_0^t e^{-(t-s)A} \left( \frac{d}{dt} V_m^{(2)}(s) \right) ds = (-1)^m \sum_{k=1}^{m} \left( \begin{array}{c} m \\ k \end{array} \right) \frac{(-tA)^k}{k!} e^{-tA}u_1
\]

\[
= -V_m^{(1)}(t) + (-1)^{m+1} e^{-tA}u_1.
\]

Combining the above two identities, we obtain the assertions for \( m \in \mathbb{N} \). \qed

We prepare the formula of higher order derivative of the functions \( V_m \).

**Lemma 4.2** Let \( V_m \) be as in Definition 4.1. For every \( m, \ell \in \mathbb{N} \), one has

\[
\frac{d^\ell}{dt^\ell} V_m(t) = (-A)^\ell V_{m,\ell}(t),
\]
where

\[
V_{m,\ell}(t) = (-1)^m \left[ \sum_{j=0}^{m} \binom{\ell + m - 1}{\ell + j - 1} \frac{(-tA)^j}{j!} e^{-tA} v_0 \right. \\
+ \sum_{k=0}^{m-1} \binom{\ell + m - 1}{\ell + k} \left( \frac{(-tA)^k}{k!} e^{-tA} u_1 \right) \right].
\]

**Proof** In Lemma 4.1, the following inequalities are already proved:

\[
d\frac{d}{dt} V_{m}^{(1)}(t) = (-1)^m (-A) \left( \sum_{j=0}^{m} \binom{1 + (m - 1)}{1 + (j - 1)} \frac{(-tA)^j}{j!} e^{-tA} v_0 \right),
\]

\[
d\frac{d}{dt} V_{m}^{(2)}(t) = (-1)^m (-A) \left( \sum_{k=0}^{m-1} \binom{1 + (m - 1)}{1 + k} \frac{(-tA)^k}{k!} e^{-tA} u_1 \right).
\]

As the same manner as in these proofs, we can directly check the desired assertion. \qed

The following lemma gives integrability and boundness of functions $V_m$ with respect to $t \in (0, \infty)$.

**Lemma 4.3** Let $(u_0, u_1) \in D(A^{\ell+1/2}) \times D(A^{\ell+1/2})$. Then for every $m, \ell \in \mathbb{N} \cup \{0\}$,

\[
(1 + t)^{2\ell} \left\| \frac{d}{dt} V_{m}(t) \right\|^2 \in L^\infty((0, \infty)), \tag{4.1}
\]

\[
(1 + t)^{2\ell+1} \left\| \frac{d}{dt} A^{\ell+1/2} V_{m}(t) \right\|^2 + (1 + t)^{2\ell} \left\| A^{\ell+1/2} V_{m,\ell+1}(t) \right\|^2 \in L^1((\infty)). \tag{4.2}
\]

**Proof** By Lemma 4.2 and the boundedness of operators $(tA)^n e^{-\frac{t}{2}A}$ for $n \in \mathbb{N}$, we have

\[
(1 + t)^{2\ell+1} \left\| \left( \frac{d}{dt} \right)^{\ell+1} V_{m}(t) \right\|^2 \\
\leq C(1 + t)^{2\ell+1} \left( \left\| A^{\ell+1/2} e^{-\frac{t}{2}A} v_0 \right\|^2 + \left\| A^{\ell+1/2} e^{-\frac{t}{2}A} u_1 \right\|^2 \right),
\]

\[
(1 + t)^{2\ell} \left\| A^{\ell+1/2} V_{m,\ell+1}(t) \right\|^2 \\
\leq C(1 + t)^{2\ell} \left( \left\| A^{\ell+1/2} e^{-\frac{t}{2}A} v_0 \right\|^2 + \left\| A^{\ell+1/2} e^{-\frac{t}{2}A} u_1 \right\|^2 \right).
\]

Applying Lemma 2.1 with $n = 2\ell + 1$, we obtain the desired estimate. \qed
4.2 Proof of higher order asymptotic expansion

Here we provide the relation between the solution \( u \) of (1.1) and the family \( \{ V_m \} \).

**Lemma 4.4** Assume \( (u_0, u_1) \in D(A^{1/2}) \times D(A^{1/2}) \). Let \( \{ U_m \}_{m \in \mathbb{N}} \) be a weak solution of

\[
\begin{aligned}
U''_m + AU_m + U'_m &= -V'_m, \quad t > 0, \\
(U_m, U'_m)(0) &= (0, (-1)^{m+1} u_1).
\end{aligned}
\]  

Then \( U_m = V_m + U'_m \).

**Proof** The strategy of the proof is essentially the same as Lemma 3.2. By taking a suitable approximation of \( (u_0, u_1) \), we can assume that \( U_m \) is smooth enough without loss of generality. Putting \( w = V_m + U'_m \), we see from Lemma 4.1 that

\[
w(0) = V_m(0) + U'_m(0) = (-1)^m u_1 + (-1)^{m+1} u_1 = 0
\]

and

\[
w'(t) = V'_m(t) + U''_m(t) = -AU_m(t) - U'_m(t).
\]

This gives

\[
\lim_{t \to 0} w'(t) = -AU_m(0) - U'_m(0) = (-1)^m u_1
\]

and also

\[
w''(t) = -AU'_m(t) + (V_m(t) - w(t))' = -A(w(t) - V_m(t)) + V'_m(t) = -Aw(t) - w'(t) + (AV_m(t) + V'_m(t)) = -Aw(t) - w'(t) - V'_{m-1}(t),
\]

where we have used Lemma 4.1 at the last line. Noting the uniqueness of solution to (4.3), we obtain \( w = U_m \).

Formally speaking, from the viewpoint of Lemmas 3.2 and 4.4 we can see that the solution \( u \) of (1.1) can be decomposed as

\[
\begin{aligned}
(um+1) + AU_{m+1} + U'_{m+1} &= -V'_m, \quad t > 0, \\
(U_{m+1}, U'_{m+1})(0) &= (0, (-1)^{m+1} u_1).
\end{aligned}
\]
\[ u(t) = V_0(t) + u(t) - V_0(t) \\
= V_0(t) + U'_1(t) \\
= V_0(t) + V_1(t)' + (V_1(t) - U_1(t))' \\
= V_0(t) + V'_1(t) + U''_2(t) \\
\vdots \\
= \sum_{\ell=0}^{m} \frac{d^\ell}{dt^\ell} V_\ell(t) + \frac{d^{m+1}}{dt^{m+1}} U_{m+1}(t). \]

By virtue of (4.1) in Lemma 4.3, we already have for every $\ell = 0, \ldots, m$,

\[ \left\| \frac{d^\ell}{dt^\ell} V_\ell(t) \right\| \leq C(1 + t)^{-\ell}. \]

To ensure that the above description is the asymptotic expansion, we finally prove the following proposition.

**Proposition 4.5** Assume that $(u_0, u_1) \in \mathcal{D}(A^{m+1/2}) \times \mathcal{D}(A^{m+1/2})$. Then

\[ \left\| \frac{d^{m+1}}{dt^{m+1}} U_{m+1}(t) \right\| \leq (1 + t)^{-m-\frac{1}{2}}. \]

The first step of the proof of Proposition 4.5 is to show the following lemma.

**Lemma 4.6** If $(u_0, u_1) \in \mathcal{D}(A^{1/2}) \times \mathcal{D}(A^{1/2})$, then

\[ (1 + t) E(U_{m+1}; t) + \| U_{m+1}(t) \|^2 \in L^\infty, \quad (1 + t) \| U'_{m+1}(t) \|^2 \]

\[ + \| A^{1/2} U_{m+1}(t) \|^2 \in L^1 \]

**Proof** The strategy is similar to Lemma 3.3. In view of Lemma 4.4, we compute

\[
\frac{d}{dt} \left[ (6 + t) E(U_{m+1}, t) + 2 E_\pi(U_{m+1}, t) \right] \\
= \| U'_{m+1} \|^2 + \| A^{1/2} U_{m+1} \|^2 + 2(6 + t)(U'_{m+1}, -U''_{m+1} - V''_m) \\
+ 4\| U'_{m+1} \|^2 + 4(U_{m+1}, -A U_{m+1} - V'_m) \\
= -(7 + 2t)\| U'_{m+1} \|^2 - 3\| A^{1/2} U_{m+1} \|^2 \\
- 2(6 + t)(U'_{m+1}, V'_m) + 4(U_{m+1}, A V_{m+1}) \\
= -(1 + t)\| U'_{m+1} \|^2 - \| A^{1/2} U_{m+1} \|^2 - (6 + t)\| V'_m \|^2 + 2\| A^{1/2} V_{m+1} \|^2.
\]

By applying the second estimate (4.2) in Lemma 4.3 with $\ell = 0$, the proof is complete. \qed

To give an extra decay property of $\frac{d^\ell}{dt^\ell} U_{m+1}$ via induction, we use the following lemma.
Lemma 4.7 Assume \((\tilde{u}_0, \tilde{u}_1) \in D(A^{1/2}) \times D(A^{1/2})\) and \(F \in C([0, \infty); D(A))\). Let \(w\) be the solution of

\[
\begin{aligned}
&w'' + Aw + w' = AF, \quad t > 0, \\
&(w, w')(0) = (\tilde{u}_0, \tilde{u}_1).
\end{aligned}
\] (4.4)

If \(w\) and \(F\) satisfy additional condition

\[
(1 + t)^{2\ell + 1}\|AF(t)\|^2 + (1 + t)^{2\ell}\|A^{1/2} F(t)\|^2 + (1 + t)^{2\ell - 1}\|w(t)\|^2 \in L^1((0, \infty)),
\]

then

\[
(1 + t)^{2\ell + 1} E(w, t) + (1 + t)^{2\ell}\|w\|^2 \in L^\infty((0, \infty)),
\]

\[
(1 + t)^{2\ell + 1}\|w'(t)\|^2 \in L^1((0, \infty)).
\]

Proof Let \(t_0 \geq 1\) be a constant determined later. Since we can use an approximation \(w_\varepsilon(t) = J_\varepsilon w(t)\) with \(J_\varepsilon = (I + \varepsilon A)^{-1}\), which satisfies

\[
\begin{aligned}
w''_\varepsilon + A w_\varepsilon + w'_\varepsilon = J_\varepsilon AF, \quad t > 0, \\
(w, w')(0) = (J_\varepsilon \tilde{u}_0, J_\varepsilon \tilde{u}_1) \in D(A^{3/2}) \times D(A^{3/2}),
\end{aligned}
\] (4.5)

we may assume \(w \in C^2([0, \infty); D(A^{1/2})) \cap C^1([0, \infty); D(A)) \cap C([0, \infty); D(A^{3/2}))\).

By direct computation with the use of (4.4), we have

\[
\frac{d}{dt} \left[(t_0 + t)^{2\ell + 1} E(w, t) + (\ell + 2)(t_0 + t)^{2\ell} E_\varepsilon(w, t)\right]
\]

\[
= (2\ell + 1)(t_0 + t)^{2\ell} E(w, t) + 2(t_0 + t)^{2\ell + 1} (w', -w') + AF + (t_0 + t)^{2\ell} \|w\|^2
\]

\[
+ (t_0 + t)^{2\ell}(w, -Aw - AF)
\]

\[
\leq (t_0 + t)^{2\ell} \left[(4\ell + 5 + \frac{2\ell(\ell + 1)}{t_0 + t})\|w\|^2 - 2\|A^{1/2}w\|^2 - (t_0 + t)\|w\|^2\right]
\]

\[
+ (t_0 + t)^{2\ell + 1}\|AF\|^2 + (\ell + 2)^2 (t_0 + t)^{2\ell}\|A^{1/2} F\|^2
\]

\[
+ 4\ell(\ell + 1)(t_0 + t)^{2\ell - 1}\|w\|^2.
\]

Taking \(t_0 = 6 + 4\ell + 2\ell(\ell + 1)\), we obtain the desired estimate.

Here we prove Proposition 4.5.

Proof of Proposition 4.5 By Lemma 4.6, we have

\[
(1 + t)\|U'_{m+1}(t)\|^2 \in L^1((0, \infty)).
\]
Here $w = U_{m+1}'$ satisfies

$$
\begin{aligned}
\begin{cases}
  w'' + Aw + w' = -V''_m = -A^2 V_{m,2}, & t > 0, \\
  (w, w')(0) = ((-1)^{m+1} u_1, -A U_{m+1}(0) - V'_m(0)) = ((-1)^{m+1} u_1, A V_{m,1}(0))
\end{cases}
\end{aligned}
$$

(4.6)

in view of the compatibility condition. By virtue of Lemma 4.3 with $\ell = 1$, applying Lemma 4.7 with $\ell = 1$, we have

$$(1 + t)^3 \| U_{m+1}'(t) \|^2 = (1 + t)^3 \| w'(t) \|^2 \in L^1((0, \infty)).$$

Successively, we apply Lemma 4.7 with $w = \frac{d^k}{dt^k} U_{m+1}$ and $\ell = k+1 (k = 1, \ldots, m)$. Then finally, from Lemma 4.3 we deduce

$$(1 + t)^{2m+1} \left\| \frac{d^{m+1}}{dt^{m+1}} U_{m+1}(t) \right\|^2 \leq C (1 + t)^{2m+1} E \left( \frac{d^m}{dt^m} U_{m+1}, t \right) \in L^\infty((0, \infty));$$

note that we need the regularity $(u_0, u_1) \in D(A^{m+1/2}) \times D(A^{m+1/2})$ to verify all deductions. The proof is complete. \hfill \Box

To close the paper, we give a proof of Theorem 1.2.

**Proof of Theorem 1.2** Observing the decomposition (at the beginning of this section) and Lemma 4.2, we have

$$u(t) = \sum_{\ell=0}^m (-A)^\ell V_{\ell,\ell}(t) + \frac{d^{m+1}}{dt^{m+1}} U_{m+1}(t).$$

Combining the definition of $V_{\ell,\ell}$ and Proposition 4.5, we obtain the desired assertion. \hfill \Box

**Acknowledgements** This work is partially supported by JSPS KAKENHI Grant Number JP18K13445.

**References**

1. Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2011)
2. Cattaneo, C.: Sur une forme de léquation de la chaleur éliminant le paradoxe d’une propagation instantanée. Compt. Rendu. 247, 431–433 (1958)
3. Chill, R., Haraux, A.: An optimal estimate for the difference of solutions of two abstract evolution equations. J. Differ. Equ. 193, 385–395 (2003)
4. Hsiao, L., Liu, T.-P.: Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping. Commun. Math. Phys. 43, 599–605 (1992)
5. Ikehata, R.: Diffusion phenomenon for linear dissipative wave equations in an exterior domain. J. Differ. Equ. 186, 633–651 (2002)
6. Ikehata, R., Nishihara, K.: Diffusion phenomenon for second order linear evolution equations. Stud. Math. 158, 153–161 (2003)
7. Ikehata, R., Matsuyama, T.: $L^2$-behaviour of solutions to the linear heat and wave equations in exterior domains. Sci. Math. Jpn. 55, 33–42 (2002)
8. Saeki, A., Ikehata, R.: Remarks on the decay rate for the energy of the dissipative linear wave equations in exterior domains. SUT J. Math. 36, 267–277 (2000)
9. Karch, G.: Selfsimilar profiles in large time asymptotics of solutions to damped wave equations. Stud. Math. 143, 175–197 (2000)
10. Michihisa, H.: $L^2$ asymptotic profiles of solutions to linear damped wave equations, arXiv:1710.04870
11. Nishihara, K.: Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping. J. Differ. Equ. 137, 384–395 (1997)
12. Nishihara, K.: $L^p$-$L^q$ estimates of solutions to the damped wave equation in 3-dimensional space and their application. Math. Z. 244, 631–649 (2003)
13. Orive, R., Zuazua, E., Pazoto, A.F.: Asymptotic expansion for damped wave equations with periodic coefficients. Math. Models Methods Appl. Sci. 11, 1285–1310 (2001)
14. Radu, P., Todorova, G., Yordanov, B.: Diffusion phenomenon in Hilbert spaces and applications. J. Differ. Equ. 250, 4200–4218 (2011)
15. Radu, P., Todorova, G., Yordanov, B.: The generalized diffusion phenomenon and applications. SIAM J. Math. Anal. 48, 174–203 (2016)
16. Said-Houari, B.: Diffusion phenomenon for linear dissipative wave equations. Z. Anal. Anwend. 31, 267–282 (2012)
17. Takeda, H.: Higher-order expansion of solutions for a damped wave equation. Asymptot. Anal. 94, 1–31 (2015)
18. Takeda, H., Yoshikawa, S.: On the initial value problem of the semilinear beam equation with weak damping II: asymptotic profiles. J. Differ. Equ. 253, 3061–3080 (2012)
19. Takeda, H., Yoshikawa, S.: On the initial value problem of the semilinear beam equation with weak damping I: smoothing effect. J. Math. Anal. Appl. 401, 244–258 (2013)
20. Takeda, H., Yoshikawa, S.: On the decay property of solutions to the Cauchy problem of the semilinear beam equation with weak damping for large initial data, nonlinear dynamics in partial differential equations. Adv. Stud. Pure Math. Math. Soc. Jpn 64, 507–514 (2015)
21. Vernotte, P.: Les paradoxes de la théorie continue de l’équation de la chaleur. Comptes Rendus 246, 3154–3155 (1958)
22. Volkmer, H.: Asymptotic expansion of $L^2$-norms of solutions to the heat and dissipative wave equations. Asymptot. Anal. 67, 85–100 (2010)
23. Yamazaki, T.: Asymptotic behavior for abstract wave equations with decaying dissipation. Adv. Differ. Equ. 11, 419–456 (2006)
24. Yang, H., Milani, A.: On the diffusion phenomenon of quasilinear hyperbolic waves. Bull. Sci. Math. 124, 415–433 (2000)

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