Convex order, quantization and monotone approximations of ARCH models

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Abstract

We are interested in proposing approximations of a sequence of probability measures in the convex order by finitely supported probability measures still in the convex order. We propose to alternate transitions according to a martingale Markov kernel mapping a probability measure in the sequence to the next and dual quantization steps. In the case of ARCH models and in particular of the Euler scheme of a driftless Brownian diffusion, the noise has to be truncated to enable the dual quantization step. We analyze the error between the original ARCH model and its approximation with truncated noise and exhibit conditions under which the latter is dominated by the former in the convex order at the level of sample-paths. Last, we analyse the error of the scheme combining the dual quantization steps with truncation of the noise according to primal quantization.

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1 Introduction

For $d \in \mathbb{N}^*$, and $\mu, \nu$ in the set $\mathcal{P}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$, we say that $\mu$ is smaller than $\nu$ in the convex order and denote $\mu \leq_{cvx} \nu$ if

$$\forall \varphi : \mathbb{R}^d \to \mathbb{R} \text{ convex }, \int_{\mathbb{R}^d} \varphi(x) \mu(dx) \leq \int_{\mathbb{R}^d} \varphi(y) \nu(dy),$$  

(1.1)

when the integrals make sense (since any real valued convex function is bounded from below by an affine function $\int_{\mathbb{R}^d} \varphi(x) \mu(dx)$ makes sense in $\mathbb{R} \cup \{+\infty\}$ as soon as $\int_{\mathbb{R}^d} |x| \mu(dx) < +\infty$). We then also write $X \leq_{cvx} Y$ for $X$ and $Y$ random vectors respectively distributed according to $\mu$ and $\nu$.

For $p \geq 1$, we denote by $\mathcal{P}_p(\mathbb{R}^d) = \{\mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^p \mu(dx) < +\infty\}$ the Wasserstein space with index $p$ over $\mathbb{R}^d$. When $\mu, \nu \in \mathcal{P}_1(\mathbb{R}^d)$, according to the Strassen theorem [42], $\mu \leq_{cvx} \nu$ if and only if there exists a martingale coupling between $\mu$ and $\nu$ that is a probability measure $M(dx, dy)$ on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals $\int_{y \in \mathbb{R}^d} M(dx, dy)$ and $\int_{x \in \mathbb{R}^d} M(dx, dy)$ equal to $\mu(dx)$ and $\nu(dy)$ respectively such that $M(dx, dy) = \mu(dx)m(x, dy)$ for some Markov kernel $m$ with the martingale property:

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∀x ∈ ℜd, ∫ℜd |y| m(x, dy) < +∞ and ∫ℜd y m(x, dy) = x. If (X, Y) is distributed according to M, then X and Y are respectively distributed according to μ and ν and E(Y | X) = X.

In this paper, we are interested in constructing approximations of a sequence (µk)k=0;n ∈ (P1(ℜd))1+n in increasing convex order (∀k = 0 : n − 1, µk ≤ cvx µk+1) by a sequence (µ̂k)k=0;n of probability measures with finite supports still in the convex order. A possible motivation comes from mathematical finance when one wants to price exotic options written on d assets with price evolution $(S_t = (S^1_t, \ldots, S^d_t))_{t \geq 0}$. Suppose for simplicity zero interest rate and let $(T_k)_{k=0;n}$ be the maturities indexed in increasing order of the vanilla options written on these assets of the exotic option with payoff $c((S_{Tk})_{k=0;n})$. The trader typically picks up her favourite model, then calibrates it to vanilla options prices and uses this calibrated model $(S_t)_{t \geq 0}$ to compute the price $E[c((\tilde{S}_{Tk})_{k=0;n})]$ of this exotic option. A natural way for the bank to evaluate the model risk is to compute the range of prices of this exotic option in all models compatible with the marginal distributions (µk)k=0;n of $(S_{Tk})_{k=0;n}$ which are calibrated to the vanilla option prices. This approach is formalized by the Martingale Optimal Transport (MOT) problem introduced in [6] which has received recently a great attention in the financial mathematics literature. In particular, the structure of martingale transport couplings [7, 9, 11, 17, 22], continuous time formulations [12, 16, 21], links with the Skorokhod embedding problem [5], numerical methods [11, 10, 19, 20] and stability properties [3, 27, 13] have been investigated.

By absence of arbitrage opportunities, the marginal distributions are in increasing convex order and the range is $[C((\mu_k)_{k=0:n}), \tilde{C}((\mu_k)_{k=0:n})]$ with

$$C((\mu_k)_{k=0:n}) = \inf_{\mu \in \mathcal{M}((\mu_k)_{k=0:n})} \int_{(\mathbb{R}^d)^{n+1}} c((x_k)_{k=0:n}) \mu(dx_k)_{k=0:n}$$

and

$$\tilde{C}((\mu_k)_{k=0:n}) = \sup_{\mu \in \mathcal{M}((\mu_k)_{k=0:n})} \int_{(\mathbb{R}^d)^{n+1}} c((x_k)_{k=0:n}) \mu(dx_k)_{k=0:n}$$

where the set

$$\mathcal{M}((\mu_k)_{k=0:n}) = \left\{ \mu \in P_1((\mathbb{R}^d)^{n+1}) : \forall k = 0 : n \text{ and } B \in \text{Bor}(\mathbb{R}^d), \mu((\mathbb{R}^d)^k \times B \times (\mathbb{R}^d)^{n-k}) = \mu_k(B) \right\}$$

∀k = 0 : n − 1 and $\varphi : (\mathbb{R}^d)^{k+1} \to \mathbb{R}^d$ meas. bounded, $\int_{(\mathbb{R}^d)^{n+1}} \varphi((x_{\ell})_{\ell=0:k}),(x_{k+1} - x_k)\mu(dx_{\ell})_{\ell=0:n} = 0$

of martingale couplings between the marginals is non empty according to Strassen’s theorem [12]. The dual formulation of these optimization problems and its interpretation in terms of sub and super-hedging strategies are investigated in [6, 8]. One may approximate the above interval by $[C((\tilde{\mu}_k)_{k=0:n}), \tilde{C}((\tilde{\mu}_k)_{k=0:n})]$. This approach can be compared to [24, 25, 26] which also deal with robust pricing (and hedging) of various classes of path-dependent options.

If for $k = 0 : n$, $\tilde{\mu}_k = \sum_{i=1}^{N_k} \rho_k^i \delta_{x^k_i}$ with distinct elements $x^k_i$ of $\mathbb{R}^d$, then $C((\tilde{\mu}_k)_{k=0:n})$ (resp. $\tilde{C}((\tilde{\mu}_k)_{k=0:n})$) is the value of the linear programming problem which consists in minimizing (resp. maximizing)

$$\sum_{i_0=1,N_0} \cdots \sum_{i_n=1,N_n} p_{i_0,\ldots,i_n} c((x^k_{i_k})_{k=0:n})$$
These finite-dimensional linear programming problems can be solved using solvers like e.g. GLPK\(^1\)

Even for smooth payoff functions \(c\), the stability of the infimum and the supremum with respect to
the marginal distributions i.e. the continuity of \(\mathcal{C}\) and \(\hat{\mathcal{C}}\) which would give a theoretical ground to
this approach is still an open question when \(d \geq 2\) or \(n \geq 2\). When \(d = n = 1\), Backhoff-Veraguas
and Pammer \([3]\) prove that \(\mathcal{C}(\mu, \nu)\) converges to \(\mathcal{C}(\mu, \nu)\) as \(\ell \to +\infty\) when \((\mu_{\ell})_{\ell \geq 1}\) and \((\nu_{\ell})_{\ell \geq 1}\)
are two sequences in \(\mathcal{P}_1(\mathbb{R})\) respectively converging to \(\mu\) and \(\nu\) for the Wasserstein distance with
index one such that for each \(\ell \geq 1\), \(\mu_{\ell} \leq_{\text{cvx}} \nu_{\ell}\) and \(c : \mathbb{R}^2 \to \mathbb{R}\) is a continuous function such that
\(\sup_{(x,y) \in \mathbb{R}^2} \frac{|c(x,y)|}{1 + |x| + |y|} < +\infty\). Their result even applies to payoffs \(c_{\ell}\) depending on \(\ell\) and converging
uniformly to \(c\) as above when \(\ell \to +\infty\). See also \([33]\) for further results in that direction but still
restricted to the case \(d = n = 1\).

To our best knowledge, few studies consider the problem of preserving the convex order while
approximating a sequence of probability measures. We mention the thesis of Baker \([4]\) who
proposes the following construction in dimension \(d = 1\). Let for \(u \in (0, 1)\), \(F_{\mu_k}^{-1}(u) = \inf\{x \in \mathbb{R} :
\mu_k([-(\infty, x]) \geq u\}\) be the quantile of \(\mu_k\) of order \(u\). Let \((N_k)_{k=0:n}\) be a sequence of elements of \(\mathbb{N}^*\)
such that for \(k = 0 : n - 1\), \(N_{k+1}/N_k \in \mathbb{N}^*\) sOne has \(\hat{\mu}_0 \leq_{\text{cvx}} \hat{\mu}_1 \leq_{\text{cvx}} \ldots \leq_{\text{cvx}} \hat{\mu}_n\) for the choice
\[
\hat{\mu}_k = \frac{1}{N_k} \sum_{i=1}^{N_k} \delta_{\frac{N_k}{N_k} F_{\mu_k}^{-1}(u)} , \quad k = 0, \ldots, n.
\]

**Dual (or Delaunay) quantization** introduced by Pagès and Wilbertz \([36]\) and further studied in \([37]\, [38]\, [39]\) gives another way to preserve the convex order in dimension \(d = 1\) (see the remark after
Proposition 10 in \([37]\)) when \(\mu_n\) is compactly supported.

In two recent papers \([1]\, [2]\), Alfonsi, Corbetta and Jourdain propose to restore the convex
ordering from any finitely supported approximation \((\mu_k)_{k=0:n}\) of \((\mu_k)_{k=0:n}\). In dimension \(d = 1\),
one may define the increasing (resp. decreasing) convex order by adding the constraint that the
test function \(\varphi\) is non-decreasing (resp. non-increasing) in \([1, 1]\). Moreover, according to \([2]\),
this can be performed by forward (resp. backward) induction on \(k\) by setting \(\hat{\mu}_0 = \hat{\mu}_0\) (resp. \(\hat{\mu}_n = \hat{\mu}_n\))
and computing \(\hat{\mu}_k\) as the supremum between \(\hat{\mu}_{k-1}\) (resp. infimum between \(\hat{\mu}_{k+1}\)) and \(\hat{\mu}_k\) for
the increasing convex order when \(\int_{\mathbb{R}} x \hat{\mu}_k(dx) \leq \int_{\mathbb{R}} x \hat{\mu}_{k-1}(dx)\) (resp. \(\int_{\mathbb{R}} x \hat{\mu}_k(dx) \geq \int_{\mathbb{R}} x \hat{\mu}_{k+1}(dx)\)) and
the decreasing convex order when \(\int_{\mathbb{R}} x \hat{\mu}_k(dx) \geq \int_{\mathbb{R}} x \hat{\mu}_{k-1}(dx)\) (resp. \(\int_{\mathbb{R}} x \hat{\mu}_k(dx) \leq \int_{\mathbb{R}} x \hat{\mu}_{k+1}(dx)\)).

For a general dimension \(d\), \([1]\) suggests to set \(\hat{\mu}_n = \hat{\mu}_n\) and compute by backward induction on
\(k = 0 : n - 1\), \(\hat{\mu}_k\) as the projection of \(\hat{\mu}_k\) on the set of probability measures dominated by \(\hat{\mu}_{k+1}\)
for the quadratic Wasserstein distance by solving a quadratic optimization problem with linear
constraints.

For general dimensions \(d\) but with only two marginals \((n = 1)\) and \(\mu_1\) compactly supported,
the convex order is preserved by defining \(\hat{\mu}_0\) as a stationary primal (or Voronoi) quantization

\(^1\) https://www.gnu.org/software/glpk/
of \( \mu_0 \) on \( N_0 \) points and \( \hat{\mu}_1 \) as a dual (or Delaunay) quantization of \( \mu_1 \) on \( N_1 \) points. We will prove in Section 2.3 that when these quantizations are optimal and \( N_0 \) and \( N_1 \) go to infinity, then \( C(\mu_0, \hat{\mu}_1) \) and \( \hat{C}(\mu_0, \hat{\mu}_1) \) respectively converge to \( C(\mu_0, \mu_1) \) and \( \hat{C}(\mu_0, \mu_1) \) for continuous payoffs \( c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) with polynomial growth.

Dual quantization of a probability measure with bounded support yields an approximation by a probability measure which is larger for the convex order and has a finite support. In the present paper, taking advantage of both properties, we are going to propose a quantization-based spatial discretization scheme still valid for \( n, d \geq 2 \) when \( (\mu_k)_{k=0:n} \) is the sequence of marginals of an ARCH model evolving inductively according to

\[
X_{k+1} = X_k + \vartheta_k(X_k)Z_{k+1}, \quad k = 0, \ldots, n - 1
\]

with \( (Z_k)_{k=1:n} \) an \( \mathbb{R}^q \)-valued white noise independent of \( X_0 \) and, for \( k = 0, \ldots, n - 1 \), \( \vartheta_k \) goes from \( \mathbb{R}^d \) to the space \( M_{d,q} \) of real matrices with \( d \) rows and \( q \) columns.

The main problem especially to be solved in presence of several times steps \( (n \geq 2) \) is to control at every time \( k \) the (finite) size of the support of the approximation of \( X_k \) while preserving the martingale property i.e. the convex order. Preserving the last feature by simply spatially discretizing the white noise \( (Z_k)_{k=1:n} \), leads to an explosion of the support of the \( X_k \): indeed if \( X_0 = x_0 \in \mathbb{R}^d \) and the \( Z_k \) are replaced by \( \hat{Z}_k \) taking e.g. \( N \) values, then \( X_n \) will take \( N^n \) values which is totally unrealistic as soon as \( N = 2 \) if \( n = 20 \). Combining alternatively a Voronoi quantization step of the white noise and a dual quantization step of the ARCH will provide a tractable answer to this question with an a priori control of the induced quadratic error. The aim of the next sections of this paper is to investigate this approach in a step-by-step manner.

In the second section, we first prove that an optimal quadratic primal quantization of \( \mu \in \mathcal{P}_2(\mathbb{R}^d) \) on \( N \) points is a quadratic Wasserstein projection of \( \mu \) on the set of probability measures with support restricted to \( N \) points and smaller than \( \mu \) in the convex order. We next introduce the dual (Delaunay) quantization. We then prove the above mentioned stability property of \( C \) and \( \hat{C} \) when \( n = 1 \). We last introduce a theoretical approximation preserving the convex order for the marginals \( (\mu_k)_{k=0:n} \) of a martingale Markov chain: it consists in alternating dual quantization steps with conditional evolution according to the current Markov transition.

The third section is dedicated to ARCH models. When the support of \( X_0 \) is bounded and the functions \( (\vartheta_k)_{k=0:n-1} \) are locally bounded, the replacement of the white noise \( (Z_k)_{k=1:n} \) by a truncated bounded white noise yields an approximation \( (\hat{X}_k)_{k=0:n} \) of \( (X_k)_{k=0:n} \) where each random vector is compactly supported a condition necessary to undergo a dual quantization step. We analyze the resulting quadratic error. We then give conditions on the functions \( (\vartheta_k)_{k=0:n-1} \) ensuring the convex ordering \( (\hat{X}_k)_{k=0:n} \leq_{cwx} (X_k)_{k=0:n} \) of the whole paths whatever the white noise is in dimension \( d = 1 \) and with the r.v. \( Z_k \) radially distributed in higher dimensions.

In Section 4 the theoretical approximation proposed at the end of Section 2 is made more practical in the case of ARCH models. In dimension one, we show that, for some distributions of the white noise, a deterministic optimization can be implemented, based on some closed form formulas, without quantizing it. This includes the case of the Euler scheme of a Brownian diffusion. In higher dimension, the white noise \( (Z_k)_{k=1:n} \) in ARCH models can be replaced by an approximate white noise \( (\hat{Z}_k)_{k=1:n} \) where \( \hat{Z}_k \) only takes \( N_k^2 \) values in the martingale Markov transitions which

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2. By white noise we mean here a sequence of independent square integrable centered random vectors with identity covariance matrix.
alternate with the dual quantization steps. We analyze the resulting quadratic error in particular when, for each \( k = 0 : n \), \( \tilde{Z}_k \) is a stationary primal quantization of \( Z_k \). These multidimensional results still include the Euler scheme.

**Definitions and notations.**

- The space of real matrices with \( d \) rows and \( q \) columns is denoted by \( \mathbb{M}_{d,q} \).
- \(|·|\) denotes the canonical Euclidean norm on \( \mathbb{R}^d \).
- When \( \mathbb{R}^d \) and \( \mathbb{R}^q \) are endowed with the canonical Euclidean norms, the operator norm of a matrix \( A \in \mathbb{M}_{d,q} \) is denoted \( ||A|| \).
- If \( A : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{M}_{d,q} \), we denote by \( \|A\|^p = \left[ \mathbb{E}\|A||^p \right]^{\frac{1}{p}} \).
- \( \text{conv}(A) \) denotes the (closed) convex hull of \( A \subset \mathbb{R}^d \) and \( \text{card}(A) \) or \( |A| \) its cardinality (depending on the context).
- For every integer \( N \geq 1 \), we denote by \( \mathcal{P}(\mathbb{R}^d, N) \) the set of distributions on \( \mathbb{R}^d \) whose support contains at most \( N \) points.
- The symbol \( \perp \) denotes the independence of random variables or vectors.
- A white noise is a sequence of independent square integrable centered \( \mathbb{R}^q \)-valued random vectors with identity covariance matrix \( I_q \).

## 2 Quantization and convex order

This section is devoted to the connections between quantization modes, convex order and projections with respect to Wasserstein distances. Let \( d \in \mathbb{N}^* \) and \( p \in [1, +\infty) \) \((p \in (0, 1) \) should work as well by adapting some proofs as usual).

### 2.1 Primal (Voronoi) quantization

In this subsection which can be read independently of what follows, we make a connection between regular quantization and various projections (in the Wasserstein sense), including, in the quadratic case, with the one mentioned above in the introduction. Let us first recall the following basic facts about (primal) Voronoi quantization (see [18, 31, 34] among others):

- The \( L^p \)-quantization error modulus \( e_p(\Gamma, \mu) \) satisfies

\[
e^p_p(\Gamma, \mu)^p = \int_{\mathbb{R}^d} |x - \text{Proj}_\Gamma(x)|^p \mu(dx)
\]

where \( \text{Proj}_\Gamma \) denotes a Borel nearest neighbour projection on \( \Gamma \) (see (A.11) in Appendix A.1 for connections with the Voronoi diagrams).

- For any level \( N \geq 1 \), there exists an optimal grid or \( N \)-quantizer \( \Gamma^{(N)} \) such that

\[
e_{p,N}(\mu) := \inf \left\{ e_p(\Gamma, \mu) : \Gamma \subset \mathbb{R}^d, |\Gamma| \leq N \right\} = e_p(\Gamma^{(N)}, \mu).
\]
Moreover, when \( \text{supp}(\mu) \) contains at least \( N \) points, then \( \Gamma^{(N)} \) has exactly \( N \) pairwise distinct elements (see Appendix A.1).

- In the quadratic case \((p = 2)\), any optimal quantization grid \( \Gamma^{(N)} \) (possibly not unique) and its induced quantization \( \text{Proj}_{\Gamma}^{(N)}(X) \) satisfy a stationarity (or self-consistency) property (see (A.43) in Appendix A.1) that is, if \( X \sim \mu \) and \( \hat{X}^N = \text{Proj}_{\Gamma}^{(N)}(X) \sim \hat{\mu}^N \), then

\[
\mathbb{E}(X \mid \hat{X}^N) = \hat{X}^N
\]

so that

\[
\hat{\mu}^N \leq_{\text{cex}} \mu.
\]

**Proposition 2.1** (a) \( p \in [1, +\infty) \). Let \( \Gamma \subset \mathbb{R}^d \) be a finite set and \( \mathcal{P}(\Gamma) \) denote the subset of \( \Gamma \)-supported distributions. Let \( \mu \in \mathcal{P}_p(\mathbb{R}^d) \) and \( \nu \in \mathcal{P}(\Gamma) \) where \( \Gamma \subset \mathbb{R}^d \), \( \Gamma \) finite. Then

\[
W_p(\mu, \mathcal{P}(\Gamma)) := \inf_{\nu \in \mathcal{P}(\Gamma)} W_p(\mu, \nu) = e_p(\Gamma, \mu) := \|\text{dist}(\cdot, \Gamma)\|_{L^p(\mu)}
\]

and \( \hat{\mu}^\Gamma = \mu \circ \text{Proj}_{\Gamma}^{-1} \) is a projection of \( \mu \) on \( \mathcal{P}(\Gamma) \).

(b) Quadratic case \((p = 2)\). Let \( \Gamma^{(N)} \) be an optimal quadratic quantization grid at level \( N \geq 1 \). Then

\[
\hat{\mu}^N = \mu \circ \text{Proj}_{\Gamma}^{-1},
\]

is (the/) a projection of \( \mu \) on the set \( \mathcal{P}_{\leq \mu}(\mathbb{R}^d, N) \) of distributions dominated by \( \mu \) for the convex order whose support contains at most \( N \) elements.

**Proof.** (a) Let \( M \) be any distribution on \((\mathbb{R}^d \times \mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)\otimes \mathcal{B}(\mathbb{R}^d))\) with marginals \( \mu \) and \( \nu \in \mathcal{P}(\Gamma) \). Then

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p M(dx, dy) \geq \int \text{dist}(x, \Gamma)^p M(dx, dy) = \int \text{dist}(x, \Gamma)^p \mu(dx) = e_p(\Gamma, \mu)^p.
\]

Now let \( \hat{\mu}^\Gamma = \mu \circ \text{Proj}_{\Gamma}^{-1} \). It follows from (2.2) that

\[
W_p(\mu, \mathcal{P}(\Gamma)) \leq W_p^\Gamma(\mu, \hat{\mu}^\Gamma) \leq \int_{\mathbb{R}^d} |x - \text{Proj}_{\Gamma}(x)|^p \mu(dx) = e_p(\Gamma, \mu)^p.
\]

(b) One has by the stationarity property (2.3) that \( \hat{\mu}^N \in \mathcal{P}_{\leq \mu}(\mathbb{R}^d, N) \) and by (a)

\[
W_2(\mu, \hat{\mu}^N) = W_2(\mu, \Gamma^{(N)})) = e_2(\mu, \Gamma^{(N)})) = e_{2, N}(\mu) = W_2(\mu, \mathcal{P}(\mathbb{R}^d, N)) \leq W_2(\mu, \mathcal{P}_{\leq \mu}(\mathbb{R}^d, N))
\]

where the last inequality is in fact an equality since \( \hat{\mu}^N \in \mathcal{P}_{\leq \mu}(\mathbb{R}^d, N) \).

**Remark about uniqueness.** As a consequence of this proposition, it turns out that the uniqueness of \( W_p \)-projection on \( \mathcal{P}(\mathbb{R}^d, N) \) and that of the distribution \( \hat{\mu}^N \) of an optimal quantizer are equivalent. Thus in dimension \( d = 1 \) and \( p = 2 \) distributions with log-concave densities have a unique optimal \( N \)-quantization grid (see Kiefer [29]) hence this projection is unique. In higher dimension, a general result seems difficult to reach: indeed, the \( \mathcal{N}(0; I_d) \) distribution, being invariant under the action of \( \mathcal{O}(d, \mathbb{R}) \) (orthogonal transforms), so are the (hence infinite) sets of its optimal quantizers at levels \( N \geq 2 \).
2.2 Dual (Delaunay) quantization

We assume in this section that $\mu$ is compactly supported. Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ be a random vector lying in $L^\infty(\mathbb{P})$ with distribution $\mu$. Assume for simplicity that the support of $\mu$ spans $\mathbb{R}^d$ as an affine space ($\overline{\mu}$), or, equivalently, that it contains an affine basis of $\mathbb{R}^d$ or that its convex hull has a non-empty interior. It means that $d$ is the dimension of the state space of $X$. Otherwise one may always consider the affine space $A_\mu$ spanned by $\text{supp}(\mu)$ and reduce the problem to the former framework by combining a translation with a change of coordinates into an orthonormal basis of the vector space associated with $A_\mu$. Optimal dual (or Delaunay) quantization as introduced in [37] relies on the best approximation which can be achieved by a discrete random vector $\tilde{X}$ that satisfies a certain stationarity assumption on an extended probability space $(\Omega \times \overline{\mu}, \mathcal{A} \otimes \mathcal{A}, \mathbb{P} \otimes \mathbb{P})$ with $(\Omega_0, A_0, \mathbb{P}_0)$ supporting a random variable uniformly distributed on $[0, 1]$. To be more precise, we define, for $p \in [1, +\infty)$,

$$
d_{p,N}(X) = \inf_{\tilde{X}} \left\{ \|X - \tilde{X}\|_p : \tilde{X} : (\Omega \times \Omega_0, \mathcal{A} \otimes A_0, \mathbb{P} \otimes \mathbb{P}_0) \to \mathbb{R}^d, \right. $$

$$
\text{card}(\tilde{X}(\Omega \times \Omega_0)) \leq N \text{ and } \mathbb{E}(\tilde{X}|X) = X \}. $$

One checks that $d_{p,N}(X)$ only depends on the distribution $\mu$ of $X$ and can subsequently also be denoted $d_{p,N}(\mu)$. Moreover, for every level $N \geq d + 1$, there exists an $L^p$-optimal dual quantization grid $\Gamma^{(N)}$ (see [37]) i.e. satisfying $\text{supp}(\mu) \subset \text{conv}(\Gamma^{(N)}, \text{del})$ (hence with a non-empty interior) and

$$
d_{p,N}(X) = \|X - \hat{X}\|_p \quad \text{with} \quad \hat{X}(\Omega \times \Omega_0) = \Gamma^{(N)}, \text{del}. $$

One may always assume that $\Omega_0 = [0, 1]$ and define $\hat{X}$ as the dual projection on an appropriate Delaunay (hyper-)triangulation induced by $\Gamma^{(N), \text{del}}$ denoted $\text{Proj}_{\Gamma^{(N), \text{del}}}$ so that $\hat{X} = \text{Proj}_{\Gamma^{(N), \text{del}}}(X, U)$ with $U \perp X$, $U \sim \mathcal{U}([0, 1])$. Such a projection $\text{Proj}_{\Gamma^{\text{del}}} : \text{conv}(\Gamma) \times [0, 1] \to \Gamma$, also called a splitting operator, can be associated to any grid $\Gamma$ with non-empty interior satisfies, beyond measurability, the following stationarity property

$$
\forall y \in \text{conv}(\Gamma), \quad \int_0^1 \text{Proj}_{\Gamma^{\text{del}}}(y, u) du = y \quad \text{(2.5)}
$$

from which one derives the dual stationarity property for any $\text{conv}(\Gamma)$-valued random vector

$$
\mathbb{E} \left( \text{Proj}_{\Gamma^{\text{del}}}(X, U) \mid X \right) = X. \quad \text{(2.6)}
$$

This stationarity property is satisfied regardless of the optimality of the grid $\Gamma$ but of course it is in particular satisfied by the optimal dual grid $\Gamma^{(N)}$ so that

$$
\mathbb{E}(\hat{X} | X) = X. \quad \text{(2.7)}
$$

For more details on this dual projection, see Appendix A.2 see also [37, 38] where this notion has been introduced. When $\mu$ spans a lower dimensional affine space than $\mathbb{R}^d$, simply replace $\mathbb{R}^d$ by this affine space in what precedes.

3. i.e. $\{x_0 + \lambda_1(x_1 - x_0) + \cdots + \lambda_d(x_d - x_0), x_0, \ldots, x_d \in \text{supp}(\mu), \lambda_1, \ldots, \lambda_d \in \mathbb{R} \} = \mathbb{R}^d$. 

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Let \( \mathcal{P}_{\geq \mu}(N, \mathbb{R}^d) \) denote the set of distributions dominating \( \mu \) for the convex order and supported by at most \( N \) elements. By Lemma 2.22 in [28], we see that for each \( \nu \in \mathcal{P}_{\geq \mu}(N, \mathbb{R}^d) \) and each martingale coupling \( M \) between \( \mu \) and \( \nu \) there exists on \( (\Omega \times \Omega_0, \mathcal{A} \otimes \mathcal{A}_0, \mathbb{P} \otimes \mathbb{P}_0) \) a random vector \( \tilde{X} \) such that \( (X, \tilde{X}) \) is distributed according to \( M \). Hence
\[
d_{p,N}(\mu) = \inf_{\nu \in \mathcal{P}_{\geq \mu}(N, \mathbb{R}^d)} \inf_{\tilde{M} \in \mathcal{M}(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^p M(dx, dy),
\]
where \( \mathcal{M}(\mu, \nu) \) denotes the set of martingale couplings between \( \mu \) and \( \nu \). Note that, in the quadratic case \( p = 2 \), for each \( M \in \mathcal{M}(\mu, \nu) \), \( \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - x|^2 M(dx, dy) = \int_{\mathbb{R}^d} |y|^2 \nu(dy) - \int_{\mathbb{R}^d} |x|^2 \mu(dx) \) so that
\[
d_{2,N}(\mu) = \inf_{\nu \in \mathcal{P}_{\geq \mu}(N, \mathbb{R}^d)} \int_{\mathbb{R}^d} |y|^2 \nu(dy) - \int_{\mathbb{R}^d} |x|^2 \mu(dx).
\]
We consider, following [1], the Wasserstein-\( p \)-projection \( \tilde{\mu} \) of \( \mu \) on \( \mathcal{P}_{\geq \mu}(N, \mathbb{R}^d) \). It is clear from its very definition that
\[
d_{p,N}(\mu) \geq \inf_{\nu \in \mathcal{P}_{\geq \mu}(N, \mathbb{R}^d)} W_p(\mu, \nu).
\]
But this time, the converse inequality is not true as emphasized by the following counter-example.

**Counter-Example.** Let \( \mu(dx) = 2x1_{[0,1]}(x)dx \). We look for \( \nu \in \mathcal{P}_{\geq \mu}(3, \mathbb{R}) \) minimizing either \( \int_{\mathbb{R}} y^2 \nu(dy) \) to compute the law of the optimal quadratic quantization of \( \mu \) on \( N = 3 \) points or \( W^2_2(\mu, \nu) \) to compute \( \tilde{\mu} \). Since \( d = 1 \), \( W^2_2(\mu, \nu) \) is equal to the integral \( \int_0^1 (F^{-1}_\mu(u) - F^{-1}_\nu(u))^2 du \) of the squared difference between the quantile functions of \( \mu \) and \( \nu \). It is not difficult to check that it is equivalent to minimize over the set
\[
\{ \nu \in \mathcal{P}_{\geq \mu}(3, \mathbb{R}) : \nu([0,1]) = 1 \} = \left\{ \nu_u(dy) = \frac{u}{3} \delta_0(dy) + \frac{1 + \sqrt{u}}{3} \delta_\sqrt{u}(dy) + \frac{2 - \sqrt{u} - u}{3} \delta_1(dy) : u \in [0,1] \right\}.
\]
One has \( \int_{\mathbb{R}} y^2 \nu_u(dy) = 2 + \frac{u^{3/2} - \sqrt{u}}{3} \) and the infimum is attained for \( u = 1/3 \). On the other hand,
\[
W^2_2(\mu, \nu_u) = \int_{0}^{u/3} (0 - \sqrt{v})^2 dv + \int_{u/3}^{(1+\sqrt{u}+u)/3} (\sqrt{u} - \sqrt{v})^2 dv + \int_{(1+\sqrt{u}+u)/3}^{1} (1 - \sqrt{v})^2 dv
\]
\[
= -\frac{1}{6} + \frac{u^{3/2} - \sqrt{u}}{3} + \frac{1}{3} (1 - \sqrt{u})(1 + \sqrt{u} + u)^{3/2} + u^2.
\]
One easily checks that \( \frac{d}{du} W^2_2(\mu, \nu_u)|_{u=1/3} > 0 \) and that \( W^2_2(\mu, \nu_u) \) is minimal for \( u \approx 0.326 \).

### 2.3 Stability of MOT problems with two marginals under quantization approximation of these marginals

Let \( \mu_0, \mu_1 \in \mathcal{P}_1(\mathbb{R}^d) \) with \( \mu_0 \leq_{cvx} \mu_1 \) and let \( c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be a Borel cost function. Assume that there exists functions \( c_0, c_1 : \mathbb{R}^d \to \mathbb{R}_+ \) both Borel, such that
\[
\int_{\mathbb{R}^d} c_0(x_0) \mu_0(dx_0) + \int_{\mathbb{R}^d} c_1(x_1) \mu_1(dx_1) < +\infty \quad \text{and} \quad \forall (x_0, x_1) \in \mathbb{R}^d \times \mathbb{R}^d, \ c(x_0, x_1) \geq -c_0(x_0) - c_1(x_1).
\]
(2.8)
We recall that
\[ C(\mu_0, \mu_1) = \inf_{\mu \in \mathcal{M}(\mu_0, \mu_1)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1) \mu(dx_0, dx_1) \]
where the infimum is taken over the set \( \mathcal{M}(\mu_0, \mu_1) \) of martingale couplings between \( \mu_0 \) and \( \mu_1 \).

Setting \( \tilde{C}(\mu_0, \mu_1) = \sup_{\mu \in \mathcal{M}(\mu_0, \mu_1)} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1) \mu(dx_0, dx_1) \) when there exists functions \( c_0, c_1 : \mathbb{R}^d \to \mathbb{R}_+ \) both Borel, such that
\[
\int_{\mathbb{R}^d} c_0(x_0)\mu_0(dx_0) + \int_{\mathbb{R}^d} c_1(x_1)\mu_1(dx_1) < +\infty \quad \text{and} \quad \forall (x_0, x_1) \in \mathbb{R}^d \times \mathbb{R}^d, \ c(x_0, x_1) \leq c_0(x_0) + c_1(x_1),
\]
(2.9)

one has \( \tilde{C}(\mu_0, \mu_1, c) = -C(\mu_0, \mu_1, -c) \) when making explicit the dependence on the cost function \( c \).

Therefore it is enough to deal with the infimum case in the proofs.

**Lemma 2.1** Let \( \mu_0, \mu_1 \in \mathcal{P}_1(\mathbb{R}^d) \) be such that \( \mu_0 \leq_{cvx} \mu_1 \). If \( c \) is lower semi-continuous and satisfies (2.8), then \( -\infty < C(\mu_0, \mu_1) \) and there exists \( \hat{\mu} \in \mathcal{M}(\mu_0, \mu_1) \) such that \( C(\mu_0, \mu_1) = \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)\hat{\mu}(dx_0, dx_1) \). If \( c \) is upper semi-continuous and satisfies (2.9), then \( \tilde{C}(\mu_0, \mu_1) < +\infty \) and there exists \( \tilde{\mu} \in \mathcal{M}(\mu_0, \mu_1) \) such that \( \tilde{C}(\mu_0, \mu_1) = \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)\tilde{\mu}(dx_0, dx_1) \)

Notice that, under (2.9) (resp. (2.8)), \( C(\mu_0, \mu_1) < +\infty \) (resp. \( -\infty < \tilde{C}(\mu_0, \mu_1) \)) inequalities which are not guaranteed under the assumptions of the Lemma.

**Proof.** Let \((\mu_0^m)_{m \in \mathbb{N}}\) and \((\mu_1^m)_{m \in \mathbb{N}}\) be two sequences in \( \mathcal{P}_1(\mathbb{R}^d) \) respectively weakly converging to \( \mu_0 \) and \( \mu_1 \) as \( m \to +\infty \) and such that \( \mu_0^m \leq_{cvx} \mu_1^m \) for each \( m \in \mathbb{N} \). Let also \( \mu^m \in \mathcal{M}(\mu_0^m, \mu_1^m) \) for each \( m \in \mathbb{N} \). The necessary condition in Prokhorov’s theorem ensures that the weakly converging sequences \((\mu_0^m)_{m \in \mathbb{N}}\) and \((\mu_1^m)_{m \in \mathbb{N}}\) are tight. We deduce that \((\mu^m)_{m \in \mathbb{N}}\) is tight. By continuity of the two canonical projections from \( \mathbb{R}^d \times \mathbb{R}^d \) onto \( \mathbb{R}^d \), the marginals of any weak limit of a subsequence of \((\mu^m)_{m \in \mathbb{N}}\) are \( \mu_0 \) and \( \mu_1 \). Since the martingale property is preserved by weak convergence, such a limit belongs to \( \mathcal{M}(\mu_0, \mu_1) \).

Let now \((\mu^m)_{m \in \mathbb{N}}\) be a sequence in \( \mathcal{M}(\mu_0, \mu_1) \) such that
\[
\lim_{m \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)\mu^m(dx_0, dx_1) = C(\mu_0, \mu_1).
\]
(2.10)

By the above argument, we may extract a subsequence, still denoted \((\mu^m)_m\) for notational convenience, converging weakly to some limit \( \tilde{\mu} \in \mathcal{M}(\mu_0, \mu_1) \). For \( a > 0 \) and \( \mu \in \mathcal{M}(\mu_0, \mu_1) \), we have
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \left(-a - c(x_0, x_1)\right)\mu(dx_0, dx_1) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} (c_0(x_0) + c_1(x_1) - a)\mu(dx_0, dx_1)
\]
\[
\leq \int_{\{c_0(x_0) + c_1(x_1) > a\}} (c_0(x_0) + c_1(x_1))\mu(dx_0, dx_1)
\]
\[
\leq \int_{\mathbb{R}^d \times \mathbb{R}^d} c_0(x_0)\left(\mathbf{1}_{\{c_0(x_0) > a/2\}} + \mathbf{1}_{\{c_0(x_0)\leq a/2, c_1(x_1) > a/2\}}\right)\mu(dx_0, dx_1)
\]
\[
+ \int_{\mathbb{R}^d \times \mathbb{R}^d} c_1(x_1)\left(\mathbf{1}_{\{c_1(x_1) > a/2\}} + \mathbf{1}_{\{c_1(x_1)\leq a/2, c_0(x_0) > a/2\}}\right)\mu(dx_0, dx_1)
\]
\[
\leq \int_{\mathbb{R}^d} c_0(x_0)\mathbf{1}_{\{c_0(x_0) > a/2\}}\mu_0(dx_0) + \frac{a}{2}\mu_1(\{x_1 : c_1(x_1) > a/2\})
\]
\[
+ \int_{\mathbb{R}^d} c_1(x_1)\mathbf{1}_{\{c_1(x_1) > a/2\}}\mu_1(dx_1) + \frac{a}{2}\mu_0(\{x_0 : c_0(x_0) > a/2\}).
\]
The right-hand side no longer depends on \( \mu \in \mathcal{M}(\mu_0, \mu_1) \) and goes to 0 as \( a \to +\infty \) by Lebesgue’s theorem. Since for \( a > 0 \), by the lower semi-continuity of \( c \) and the Portemanteau theorem,

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1) \leq \lim_{m \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1) \leq \lim_{m \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{- \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)} \, \mu^m(dx_0, dx_1),
\]

and using that \( c(x_0, x_1) \leq c(x_0, x_1) + (a - c(x_0, x_1))_+ \), we conclude that

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1) \mu(dx_0, dx_1) \leq \lim_{m \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1) \mu^m(dx_0, dx_1) = C(\mu, \mu_1). \tag{\text{\textbullet}}
\]

**Remark.** If \( c_0 \) and \( c_1 \) are themselves l.s.c., the second part of the proof is a straightforward application of Fatou’s lemma for weak convergence applied to the non-negative l.s.c. function \( c + c_0 + c_1 \).

**Proposition 2.1** Let \( \mu_0, \mu_1 \in \mathcal{P}(\mathbb{R}^d) \) be such that \( \mu_0 \leq_{cvx} \mu_1 \) with \( \mu_1 \) compactly supported and \( c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) be a continuous function with polynomial growth. Let \((N_m)_{m \in \mathbb{N}}\) and \((M_m)_{m \in \mathbb{N}}\) be two sequences of positive integers converging to \( \infty \) with \( m \) and \( \mu_0^m \) (resp. \( \mu_1^m \)) be an optimal primal (resp. dual) quantization of \( \mu_0 \) (resp. \( \mu_1 \)) on \( N_m \) (resp. \( M_m \)) points.  

(a) Then, \( C(\mu_0, \mu_1) = \lim_{m \to +\infty} C(\hat{\mu}_0^m, \hat{\mu}_1^m), \) \( C(\mu_0, \mu_1) = \lim_{m \to +\infty} C(\hat{\mu}_0, \hat{\mu}_1) \) and any sequence \( (\mu^m)_{m \in \mathbb{N}} \) with \( \mu^m \in \mathcal{M}(\hat{\mu}_0^m, \hat{\mu}_1^m) \) is tight.

(b) The weak limits of subsequences of \( c \)-minimal (resp. maximal) martingale couplings between \( \hat{\mu}_0^m \) and \( \hat{\mu}_1^m \), which exist for each \( m \in \mathbb{N} \), are \( c \)-minimal (resp. maximal) martingale couplings between \( \mu_0 \) and \( \mu_1 \).

**Remark.** For all the statements but the existence of \( \mu \) in \( \mathcal{M}(\mu_0, \mu_1) \)-polar sets has been studied by De March and Touzi, see \([11]\).

The proof of Proposition 2.1 relies on the next lemma which in turn crucially relies on the respective use of optimal primal and dual quantization to approximate the first and second marginals.

**Lemma 2.2** Let \( \mu_0, \mu_1, \hat{\mu}_0^m, \hat{\mu}_1^m \) be as in Proposition 2.1. For each \( \mu \in \mathcal{M}(\mu_0, \mu_1) \), there exists \( \hat{\mu}_m \in \mathcal{M}(\hat{\mu}_0^m, \hat{\mu}_1^m) \) such that

\[
\mathcal{W}_2^2(\hat{\mu}_m, \mu) \leq e_{2,N_{\mu}}^2(\mu_0) + d_{2,M_{\mu}}^2(\mu_1). \tag{\text{\textbullet}}
\]

**Proof.** Let \( \mu \in \mathcal{M}(\mu_0, \mu_1) \) and let \( q \) denote the martingale Markov kernel associated to this coupling in the sense that \( \mu(dx_0, dx_1) = \mu_0(dx_0)q(x_0, dx_1) \) (\( q(x_0, dx_1) \) is \( \mu_0(dx_0) \) a.e. unique). The image of \( \mu_0 \) by \( \mathbb{R}^d \ni x_0 \mapsto (\text{Proj}_{\Gamma_m}(x_0), x_0) \) is a martingale coupling between \( \mu_0 \) and \( \mu_0 \), optimal for the \( \mathcal{W}_2 \) distance: \( \mathcal{W}_2(\mu_0, \mu_0) = e_{2,N_{\mu}}(\mu_0) \). Let \( q_0^m(\hat{x}_0, dx_0) \) denote the associated martingale Markov kernel. For \( X_1 \) distributed according to \( \mu_1 \) and \( U \) an independent random variable uniformly distributed on \( [0,1] \), the law of \( (X_1, \text{Proj}_{\Gamma_m}(X_1, U)) \) is (a martingale) coupling between \( \mu_1 \) and \( \mu_1 \) so that \( \mathcal{W}_2^2(\mu_1, \mu_1) \leq \mathbb{E}[X_1 - \text{Proj}_{\Gamma_m}(X_1, U)]^2 = d_{2,M_{\mu}}^2(\mu_1) \). Let \( q_1^m(x_1, dx_1) \) denote the associated martingale Markov kernel. Then

\[
\hat{\mu}_m(dx_0, dx_1) = \hat{\mu}_0^m(dx_0) \int_{(x_0, x_1) \in \mathbb{R}^d \times \mathbb{R}^d} q_0^m(\hat{x}_0, dx_0)q(x_0, dx_1)q_1^m(x_1, dx_1) \in \mathcal{M}(\hat{\mu}_0^m, \hat{\mu}_1^m). \tag{\text{\textbullet}}
\]
Since
\[
\int_{(\tilde{x}_0, \tilde{x}_1) \in \mathbb{R}^d \times \mathbb{R}^d} \hat{\mu}^m_0 (d\tilde{x}_0) q^m_0 (\tilde{x}_0, dx_0)q(x_0, dx_1)q^m_1 (x_1, d\tilde{x}_1) = \mu(dx_0, dx_1),
\]
one has
\[
W_2^2 (\hat{\mu}, \mu) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d} (|\tilde{x}_0 - x_0|^2 + |\tilde{x}_1 - x_1|^2) \hat{\mu}^m_0 (d\tilde{x}_0) q^m_0 (\tilde{x}_0, dx_0)q(x_0, dx_1)q^m_1 (x_1, d\tilde{x}_1)
= c_{2,N_2}^2 (\mu_0) + d_{2,M}^2 (\mu_1). \quad \Box
\]

**Proof of Proposition 2.1.** Since \( \mu_1 \) is compactly supported \( \int_{\mathbb{R}^d} (|x| - K)_+ \mu_1(dx_1) = 0 \) for \( K \) large enough. Then, since \( \mathbb{R}^d \ni x \mapsto (|x| - K)_+ \) is convex and \( \mu_0 \leq \text{ess} \mu_1, \int_{\mathbb{R}^d} (|x| - K)_+ \mu_0(dx_1) = 0 \) and \( \mu_0 \) is also compactly supported. With Lemma 2.1 and the continuity and the polynomial growth of \( c \), we deduce that there exists \( \mu \in \mathcal{M}(\mu_0, \mu_1) \) be such that \( C(\mu, \mu_1) = \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1) \mu(dx_0, dx_1) \). By Lemma 2.2, there exists a sequence \( (\hat{\mu}^m)_{m \in \mathbb{N}} \) converging weakly to \( \mu \) as \( m \to +\infty \) with \( \hat{\mu}^m \in \mathcal{M}(\mu_0^m, \mu_1^m) \) for each \( m \in \mathbb{N} \). By continuity of \( c \) and uniform integrability deduced from the polynomial growth of \( c \) combined with Theorems A.1 and A.2 for primal and dual quantizations respectively,
\[
\lim_{m \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)\hat{\mu}^m(dx_0, dx_1) = \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)\mu(dx_0, dx_1) = C(\mu_0, \mu_1).
\]
On the other hand, by Lemma 2.1 there exists \( \mu^m \in \mathcal{M}(\mu_0^m, \mu_1^m) \) such that
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)\mu^m(dx_0, dx_1) = C(\hat{\mu}^m_0, \mu_1^m)
\]
so that \( \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)\mu^m(dx_0, dx_1) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)\hat{\mu}^m(dx_0, dx_1) \). By the first step in the proof of Lemma 2.1 like any sequence of elements of \( \mathcal{M}(\hat{\mu}^m_0, \mu_1^m) \), the sequence \( (\mu^m)_m \) is tight and the limit \( \mu \) of any weakly convergent subsequence still denoted by \( (\mu^m)_m \) for notational simplicity belongs to \( \mathcal{M}(\mu_0, \mu_1) \). Moreover, by the above arguments,
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)\mu(dx_0, dx_1) = \lim_{m \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)\mu^m(dx_0, dx_1)
\leq \lim_{m \to +\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} c(x_0, x_1)\hat{\mu}^m(dx_0, dx_1) = C(\mu_0, \mu_1)
\]
so that \( \mu \) is a \( c \)-minimal martingale coupling between \( \mu \) and \( \nu \). \( \Box \)

### 2.4 Monotone spatial approximation of a martingale dynamics

We consider a discrete time family of Markov transitions \( ((P_k(x, dy))_{x \in \mathbb{R}^d})_{k=0:n-1} \) satisfying a martingale property, namely
\[
\forall k \in \{0, \ldots, n-1\}, \quad \forall x \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} |y| P_k(x, dy) < +\infty \quad \text{and} \quad \int_{\mathbb{R}^d} y P_k(x, dy) = x. \quad (2.11)
\]
Equivalently, we may consider a Markov chain \( (X_k)_{k=0:n} \) with transitions \((P_k(x, dy))_{x \in \mathbb{R}^d}\) as above on the canonical space \( ((\mathbb{R}^d)^{n+1}, \text{Bor}(\mathbb{R}^d)^{n+1}, (\mathbb{P}_x)_{x \in \mathbb{R}^d}) \) so that \( \mathbb{E}(X_k | \mathcal{F}_{k-1}^X) = X_{k-1} \) where \( \mathcal{F}_{k-1}^X = \sigma((X_\ell)_{\ell=0:k-1}) \).
It is straightforward that in such a situation

\[ X_0 \leq_{\text{cvx}} X_1 \leq_{\text{cvx}} \cdots \leq_{\text{cvx}} X_n. \]

A natural question, closely connected with Martingale Optimal Transport (MOT, see [6, 21]) is to produce “tractable approximations” of the chain \((X_k)_{k=0:n}\) that still satisfy the above convex ordering.

### 2.4.1 A convex order monotone discretization based on dual quantization

For such a discrete time family of Markov transitions \((P_k(x,dy))_{x \in \mathbb{R}^d}\) satisfying (2.11), there exist measurable functions \((F_k: \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d)_{k=0:n-1}\) and independent \(\mathbb{R}^q\)-valued random vectors \((Z_k)_{k=1:n}\) independent from \(X_0\), such that the sequence \((X_k)_{k=0:n}\) defined inductively by

\[ X_{k+1} = G_k(X_k, Z_{k+1}), \quad k = 0, \ldots, n-1, \quad (2.12) \]

is a Markov chain with transition kernels \((P_k)_{k=0:n-1}\) (see e.g. Lemma 2.22 in [28] for the particular case \(q = 1\) and \((Z_k)_{k=1:n}\) uniformly distributed on \([0,1]\) for each \(k = 1, \ldots, n\)). Condition (2.11) then reads

\[ \forall k = 0, \ldots, n-1, \quad \forall x \in \mathbb{R}^d, \quad \mathbb{E}|G_k(x, Z_{k+1})| < +\infty \quad \text{and} \quad \mathbb{E}G_k(x, Z_{k+1}) = x \quad (2.13) \]

In what follows, this dynamical formulation as iterated random maps, which appears naturally in any application, will be the starting point of our investigations. We will make various assumptions on the functions \(G_k\) and the random vectors \(Z_k\). In most applications that follow we will suppose that

\[(X_k)_{k=0:n}; \text{ is an } \mathcal{F}^X_k\text{-martingale}, \]

which is an assumption more stringent than (2.13), since it also requires that \(\mathbb{E}|X_k| < +\infty \) for \(k = 0, \ldots, n\).

Keep in mind that, as a consequence of the dual stationarity property (2.6), for any random vector \((Y,U) \in L_{\mathbb{R}^d+1}^\infty(\Omega, \mathcal{A}, P), Y \perp \perp U, U \overset{d}{=} U([0,1]),\) and any grid \(\Gamma \subset \mathbb{R}^d,\)

\[ \mathbb{E}(\text{Proj}^\text{def}_{\Gamma}(Y,U) \mid Y) = Y, \]

where \(\text{Proj}^\text{def}_{\Gamma}\) is defined in Appendix A.2 (see (A.47)). The main geometric properties of dual quantization in connection with the (generalized) Delaunay triangulation and its optimization, when viewed as a function of the grid \(\Gamma\), are recalled in the Appendix.

Hence, at this stage, we know that, in order to dually quantize the chain \((X_k)_{k=0:n}\) we need exogenous i.i.d. random variables \(U_k \sim U([0,1]), k = 1:n,\) independent of \((Z_k)_{k=1:n}\). In this dual quantization of the chain note that the starting value \(X_0\) will have a special status and needs not to be dually quantized but simply spatially discretized by any tractable mean. A natural choice being then to perform on \(X_0\) a primal (optimal) quantization.

For every \(k = 0, \ldots, n,\) we consider grids \(\Gamma_k \subset \mathbb{R}^d\) satisfying the following inductive property:

\[(G_F) \quad \forall \Gamma \subset \Gamma_{k-1}, \quad G_{k-1}(x, Z_k(\Omega)) \subset \text{conv}(\Gamma_k), \quad k = 1:n.\]
It is clear that, by induction on \( k \), such \( n + 1 \)-tuples of grids satisfying \((G_F)\) exist as soon as the mappings \( G_k \) satisfy

\[
\forall k = 1 : n, \forall x \in \mathbb{R}^d, \quad G_{k-1}(x, Z_k(\Omega)) \text{ is bounded in } \mathbb{R}^d. \quad (2.14)
\]

Then, we may define by induction

\[
\begin{align*}
\hat{X}_0 &= g_0(X_0) \text{ for some Borel function } g_0 : \mathbb{R}^d \to \Gamma_0, \\
\hat{X}_k &= G_{k-1}(\hat{X}_{k-1}, Z_k) \quad \text{and} \quad \hat{X}_k = \text{Proj}_{\Gamma_k}^{\text{def}}(\hat{X}_k, U_k), \quad k = 1 : n. \quad (2.15)
\end{align*}
\]

where the sequence \((U_k)_{k=1:n}\) is i.i.d. uniformly distributed on the unit interval and independent of \((Z_k)_{k=1:n}\) and \(X_0\). This definition is consistent since, by induction, all the random vectors \( \hat{X}_k \) are bounded.

**Proposition 2.2** Assume \((G_F)\). Let \( \mathcal{F}_k = \sigma(X_0, (Z_\ell, U_\ell)_{\ell=1:k}), k = 0, \ldots, n \) and \( G_k = \sigma(X_0, (Z_\ell, U_\ell)_{\ell=1:k}, Z_k) \), \( k = 1 : n \). The sequences \((\hat{X}_k)_{k=0:n}\) and \((\hat{X}_k)_{k=1:n}\) defined by \((2.15)-(2.16)\) are respectively an \((\mathcal{F}_k)\)-martingale Markov chain and a \((G_k)\)-martingale Markov chain. Moreover,

\[
\hat{X}_0 \leq_{cvx} \hat{X}_1 \leq_{cvx} \hat{X}_1 \leq_{cvx} \cdots \leq_{cvx} \hat{X}_n \leq_{cvx} \hat{X}_n.
\]

**Proof.** The \((\mathcal{F}_k)\)-Markov property is clear since

\[
\hat{X}_k = \text{Proj}_{\Gamma_k}^{\text{def}}(G_{k-1}(\hat{X}_{k-1}, Z_k), U_k), \quad k = 1 : n.
\]

Likewise, since for \( k = 2 : n \), \( \hat{X}_k = G_{k-1}(\text{Proj}_{\Gamma_{k-1}}^{\text{def}}(\hat{X}_{k-1}, U_{k-1}), Z_k), \) \((\hat{X}_k)_{k=1:n}\) is a \((G_k)\)-Markov chain.

For \( k = 1 : n \), as \( U_k \perp \mathcal{G}_k \) and \( \hat{X}_k \) is \( \mathcal{G}_k \)-measurable,

\[
\mathbb{E}(\hat{X}_k | \mathcal{G}_k) = \mathbb{E}(\text{Proj}_{\Gamma_k}^{\text{def}}(\hat{X}_k, U_k) | \mathcal{G}_k) = \left[ \mathbb{E}(\text{Proj}_{\Gamma_k}^{\text{def}}(x, U)) \right] \bigg|_{x=\hat{X}_k} = \hat{X}_k. \quad (2.17)
\]

Moreover,

\[
\mathbb{E}(\hat{X}_k | \mathcal{F}_{k-1}) = \mathbb{E}(G_{k-1}(\hat{X}_{k-1}, Z_k) | \mathcal{F}_{k-1}) = \left[ \mathbb{E}(G_{k-1}(x, Z_k)) \right] \bigg|_{x=\hat{X}_{k-1}} = \hat{X}_{k-1}
\]

as \( \int_{\mathbb{R}^d} G_{k-1}(x, z) \mathbb{P}_Z(z) \, dz = x \) for every \( x \in \mathbb{R}^d \).

The convex ordering of the random vectors follows by Jensen’s inequality.

Since \( \mathcal{F}_{k-1} \subset \mathcal{G}_k \subset \mathcal{F}_k \), with the tower property of conditional expectation, we also deduce that

\[
\mathbb{E}(\hat{X}_k | \mathcal{F}_{k-1}) = \mathbb{E}(\mathbb{E}(\hat{X}_k | \mathcal{G}_k) | \mathcal{F}_{k-1}) = \mathbb{E}(\hat{X}_k | \mathcal{F}_{k-1}) = \hat{X}_{k-1}
\]

and, when \( k \leq n - 1 \),

\[
\mathbb{E}(\hat{X}_{k+1} | \mathcal{G}_k) = \mathbb{E}(\mathbb{E}(\hat{X}_{k+1} | \mathcal{F}_k) | \mathcal{G}_k) = \mathbb{E}(\hat{X}_k | \mathcal{G}_k) = \hat{X}_k.
\]

**Example: ARCH models with bounded innovation.** Let us consider the ARCH model

\[
X_{k+1} = X_k + \vartheta_k(X_k)Z_{k+1}, \quad X_0 \in L^2, \quad (2.18)
\]
where the (Borel) functions $\vartheta_k : \mathbb{R}^d \to \mathcal{M}_{d,q}$ are locally bounded and the r.v. $(Z_k)_{k=1:n}$ are square integrable, centered and mutually independent (when $\text{Cov}(Z_k) = I_q$ for every $k = 1 : n$, $(Z_k)_{k=1:n}$ is a white noise).

If the r.v. $Z_k$ all lie in $L^\infty_{\mathbb{R}^q}(\mathbb{P})$, then Assumption $(\mathcal{G}_F)$ is satisfied since $G_k(x,z) = x + \vartheta_k(x)z$.

The Euler scheme with Brownian increments of a martingale Brownian diffusion with diffusion coefficient $\vartheta(t,x)$ is an ARCH model corresponding to the choice $\vartheta_k(x) = \sqrt{T/n} \vartheta(t_k, x)$ with a Gaussian $\mathcal{N}(0; I_q)$-distributed white noise $(Z_k)_{k=0:n}$ since

$$
\bar{X}_{k+1} = \bar{X}_k + \vartheta(t_k, \bar{X}_k) \sqrt{\frac{T}{n}} Z_{k+1}, \quad k = 0, \ldots, n - 1
$$

(2.19)

where $t_0 = 0$, $t_k = \frac{kT}{n}$, $Z_k = \sqrt{T/n} (W_{t_k} - W_{t_{k-1}})$, $\mathcal{N}(0; I_q)$, $k = 1 : n$ ($W$ is a standard $q$-dimensional Brownian motion).

However, such a Gaussian white noise makes impossible assumption (2.14), and in turn $(\mathcal{G}_F)$ to hold true.

This can be fixed if the normalized Brownian increments are replaced by a $q$-dimensional Rademacher white noise or any other noise having a distribution with compact support in $\mathbb{R}$ like e.g. optimal primal (Voronoi) quantizations $\tilde{Z}_k^{\text{vor}}$ of $Z_k$, $k = 1 : n$ (see Section 3.1.1 further on). Then assumption $(\mathcal{G}_F)$ is fulfilled and the quantized scheme (2.15)-(2.16) can be designed.

### 2.4.2 Convergence of $(\bar{X}_k)_{k=0:n}$ toward $(X_k)_{k=0:n}$

We make an additional assumption on the mappings $G_k$ and the r.v. $Z_k$, namely a Lipschitz continuous property for the $L^2$-norm:

$$
\forall k = 0, \ldots, n - 1, \exists [G_k]_{\text{Lip}} < +\infty, \forall x, y \in \mathbb{R}^d, \|G_k(x, Z_{k+1}) - G_k(y, Z_{k+1})\|_2 \leq [G_k]_{\text{Lip}} |x - y|.
$$

(2.20)

**Remark.** Such an assumption is fulfilled by the above ARCH models (2.18) if the “diffusion” coefficients $\vartheta_k$ are Lipschitz continuous and the r.v. $Z_k$ are square integrable.

**Proposition 2.3 (Quadratic convergence rate)** Let $(X_k)_{k=0:n}$ be a martingale Markov chain defined by (2.13) such that the functions $G_k$ and the innovation sequence $(Z_k)_{k=1:n}$ satisfy (2.13), (2.14) and (2.20) and let $(\bar{X}_k)_{k=0:n}$ be defined by (2.15)-(2.16).

(a) For every $k \in \{0, \ldots, n\}$,

$$
\|\bar{X}_k - X_k\|_2 \leq \left( \sum_{\ell=0}^{k} [G_{\ell:k}]_{\text{Lip}}^2 \|\bar{X}_\ell - \bar{X}_\ell\|_2^2 \right)^{\frac{1}{2}},
$$

with the convention $\bar{X}_0 = X_0$ and where, for $0 \leq \ell \leq k$, $[G_{\ell:k}]_{\text{Lip}} = \prod_{i=\ell+1}^{k} [G_i]_{\text{Lip}}$ ($\prod_{\emptyset} = 1$).

(b) If $\bar{X}_0 = \text{Proj}_{\Gamma_0}^{\text{vor}}(X_0)$ with the grid $\Gamma_0$ $L^2$-Voronoi optimal and the grids $(\Gamma_k)_{k=1:n}$ are $L^2$-dually optimal, then, for every $k \in \{0, \ldots, n\}$,

$$
\|\bar{X}_k - X_k\|_2 \leq \left( [G_{0:k}]_{\text{Lip}}^2 (\tilde{C}_{2,d,q}^{\text{vor}})^2 \sigma_{2+\eta}^2 (X_0) + (\tilde{C}_{2,d,q}^{\text{del}})^2 \sum_{\ell=1}^{k} [G_{\ell:k}]_{\text{Lip}}^2 \sigma_{2+\eta}^2 (\bar{X}_\ell) N_{\ell}^{-\frac{3}{2}} \right)^{1/2},
$$

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where, for any \( \mathbb{R}^d \)-valued r.v. \( X \), \( \sigma_{2+\eta}(X) = \inf_{a \in \mathbb{R}^d} \|X - a\|_{2+\eta} \) is the \( L^{2+\eta} \)-pseudo-standard deviation of \( X \).

**Proof.** Let \( k \in \{0, \ldots, n-1\} \). Since \((\tilde{X}_{k+1}, X_{k+1})\) is \( \mathcal{G}_{k+1} \)-measurable and, by (2.17), \( \mathbb{E}(\tilde{X}_{k+1}|\mathcal{G}_{k+1}) = \tilde{X}_{k+1} \), the two terms in the right-hand side of the decomposition \( \tilde{X}_{k+1} - X_{k+1} = \tilde{X}_{k+1} - \tilde{X}_{k+1} + \tilde{X}_{k+1} - X_{k+1} \) are orthogonal. This implies

\[
\|\tilde{X}_{k+1} - X_{k+1}\|^2 = \|\tilde{X}_{k+1} - \tilde{X}_{k+1}\|^2 + \|G_k(\tilde{X}_{k}, Z_{k+1}) - G_k(X_{k}, Z_{k+1})\|^2 \\
\leq \|\tilde{X}_{k+1} - \tilde{X}_{k+1}\|^2 + \|G_k\|_{\text{Lip}}^2 \|\tilde{X}_k - X_k\|^2.
\]

A straightforward backward induction completes the proof of claim (a). (b) This follows from the non-asymptotic bounds for primally and dually optimized grids (Pierce’s lemma) recalled respectively in Theorem A.1 (b) in Appendix A.1 and Theorem A.2 (b) in Appendix A.2.

**Remark.** For this kind of Markov models, a control of the pseudo-\( L^{2+\eta} \)-standard deviation is established in Lemma 3.2 [33] and extended in the proofs of Propositions 2.2 and 2.4 [32] when, in (2.16), the dual quantization step is replaced by a primal quantization step still alternating with the transition step. In particular it holds for ARCH models.

### 3 ARCH models with truncated noise and convex order

An ARCH model evolving according to (2.18) with non-vanishing functions \( \vartheta_k \) satisfies (2.14) iff the noise \( (Z_k)_{k=1:n} \) is compactly supported. To be able to apply dual quantization to ARCH models with non compactly supported noise, we are first going to approximate them by ARCH models with truncated noise. In this section, we provide several examples of such ARCH approximations, analyse the resulting error and give conditions under which the whole path of the approximation is dominated by the path of the original ARCH model for the convex order.

#### 3.1 General ARCH models

To deal in a tractable way with general ARCH models satisfying the dynamics (2.18) with a sequence of locally bounded coefficients \( (\vartheta_k)_{k=0:n-1} \) and a general \( L^2 \)-noise \( (Z_k)_{k=0:n} \), a natural idea is simply to approximate the r.v. \( Z_k \) by \( \tilde{Z}_k \) which are bounded functions of \( Z_k \), namely

\[
\tilde{Z}_k = \varphi_k(Z_k) \text{ in such a way that } \begin{cases} (i) & \mathbb{E} \tilde{Z}_k = 0, \\
(ii) & \forall i, j \in \{1, \ldots, d\}, Z_k^i \perp \tilde{Z}_k^j \perp L^2 \tilde{Z}_k^i \text{ for } k = 1 : n. \end{cases} \tag{3.21}
\]

Note that, although the \( \tilde{Z}_k \) are independent by construction, the sequence \( (\tilde{Z}_k)_{k=1:n} \) is not a white noise – except if \( \tilde{Z}_k = Z_k \) a.s. – due to (ii) since \( \|Z_k\|^2 = \|Z_k - \tilde{Z}_k\|^2 + \|\tilde{Z}_k\|^2 \). In particular \( \|\tilde{Z}_k\|^2 \leq \|Z_k\|^2 \) with equality iff \( Z_k = \tilde{Z}_k \). A sequence satisfying (3.21) will be called a quasi-white noise in what follows and two canonical examples are given just below.

Then we define the ARCH model associated to \( (\tilde{Z}_k)_{k=1:n} \) still with the diffusion coefficients \( \vartheta_k \) by

\[
\tilde{X}_{k+1} = \tilde{X}_k + \vartheta_k(\tilde{X}_k)\tilde{Z}_{k+1}, \ k = 0, \ldots, n - 1, \quad \tilde{X}_0 = g_0(X_0) \tag{3.22}
\]
where \( g_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a bounded Borel function. It is clear, e.g., by mimicking the proof of Proposition 2.2 (with the same notations), that
\[
(\tilde{X}_k)_{k=0:n} \text{ is again an } \mathcal{F}_k\text{-martingale and an } \mathcal{F}_k\text{-Markov chain.}
\]

The aim of this section is to control the error induced by the substitution of \( Z_k \) by \( \tilde{Z}_k \) and to give conditions ensuring that if, for \( k = 1 : n \), \( \tilde{Z}_k \) is dominated by \( Z_k \) for the convex order (\( \tilde{Z}_k \leq_{cvx} Z_k \)) and \( g_0(X_0) \leq_{cvx} X_0 \), then \( \tilde{X}_{0:n} \) is dominated by \( X_{0:n} \). Below are two typical examples of bounded and dominated approximations of a white noise.

**Examples of interest.** ▶ **Truncated white noise.** Set
\[
\tilde{Z}_k = Z_k 1_{\{Z_k \in A_k\}}, \quad k = 1, \ldots, n
\]
where \( A_1, \ldots, A_n \) are compact sets such that
\[
\mathbb{E} Z_k 1_{\{Z_k \in A_k\}} = 0, \quad k = 1, \ldots, n.
\]
Such Borel sets \( A_k \) are easy to specify when the r.v. \( Z_k \) have symmetric (invariant by multiplication by \(-1\)) distributions since balls centered at 0 (or any symmetric set) are admissible. Notice that (3.21) \((ii)\) is satisfied whatever the choice of the sets \( A_k \) such that (3.25) holds.

▶ **Primal/Voronoi stationary quantization of the white noise.** We replace the white noise by a quantization, usually a Voronoi (primal) one, since the original white noise has no reason to be bounded. Then we set
\[
\tilde{Z}_k = \tilde{Z}_k^{\text{vor}} \Gamma_k
\]
where \( \Gamma_k \) is a stationary primal/Voronoi quantization grid with size \( N_k \geq 1 \). Conditions (3.21)(i)–(ii) follow from the (primal) stationarity property (2.3).

In both settings, defining \( (\tilde{X}_k)_{k=0:n} \) by (3.22), one obtains an approximation of \( (X_k)_{k=0:n} \) which is both non-decreasing for the convex order (as a martingale) and dominated by the original ARCH model, under some additional assumptions made precise in section 3.1.2.

### 3.1.1 Convergence rate approximation of ARCH models

Now we need to estimate the error induced by replacing the original ARCH dynamics (2.18) driven by a true white noise \( (Z_k)_{k=1:n} \) by this ARCH model (3.22) driven by a quasi-white noise. To this end, we will first make precise some vector and matrix notions.

We equip the space \( \mathcal{M}_{d,q} \) with the operator norm \( \|B\| = \sup_{|x| \leq 1} |Bx| \) where \( |\cdot| \) denotes the canonical Euclidean norm. Then for an \( \mathcal{M}_{d,q}\)-random variable \( M \) we denote in short \( \|M\|_2 \) for \( \|M\| \). Then we will denote by \([\vartheta]_{\text{Lip}}\) the Lipschitz coefficient (if finite) of \( \vartheta : (\mathbb{R}^d, |\cdot|) \rightarrow (\mathcal{M}_{d,q}, \|\cdot\|) \). We will also make use of the Fröbenius norm \( \|B\|_{\text{Fr}} = \sqrt{\text{Tr}(BB^T)} \), \( B \in \mathcal{M}_{d,q} \) which satisfies \( \|B\| \leq \|B\|_{\text{Fr}} \leq \sqrt{d \wedge q \|B\|} \).

For a Lipschitz continuous function \( \vartheta : \mathbb{R}^d \rightarrow \mathbb{R}^d \), we define
\[
c(\vartheta) = \sup_{x \in \mathbb{R}^d} \frac{\|\vartheta(x)\|^2}{1 + |x|^2} \leq 2(\|\vartheta(0)\|^2 + [\vartheta]_{\text{Lip}}^2) < +\infty
\]
\[
c_{\text{Fr}}(\vartheta) = \sup_{x \in \mathbb{R}^d} \frac{\|\vartheta(x)\|_{\text{Fr}}^2}{1 + |x|^2} \leq 2(\|\vartheta(0)\|_{\text{Fr}}^2 + [\vartheta]_{\text{Fr,Lip}}^2) < +\infty
\]
where \([\vartheta]_{\text{Lip}}\) and \([\vartheta]_{\text{Fr},\text{Lip}}\) denote the Lipschitz coefficients of \(\vartheta\) with respect to the operator and the Fröbenius norms respectively.

Thus, using that, if \(\zeta \in L^2_{\mathbb{R}^d}(\mathbb{P})\) is centered with \(L^2\)-orthogonal components \(\zeta^i\), then \(\mathbb{E}|A\zeta|^2 = \sum_{i=1}^d AA_i^* \mathbb{E}(\zeta^i)^2\), we straightforwardly derive the following inequality

\[
\mathbb{E}|X_{k+1}|^2 = \mathbb{E}|X_k|^2 + \mathbb{E}|\vartheta_k(X_k)Z_{k+1}|^2 = \mathbb{E}|X_k|^2 + \mathbb{E}|\text{Tr}(\vartheta_k^*(X_k))| \leq (1 + c_{\text{Fr}}(\vartheta_k))\mathbb{E}|X_k|^2 + c_{\text{Fr}}(\vartheta_k).
\]

Standard induction then yields

\[
\|X_k\|^2 = \mathbb{E}|X_k|^2 \leq \left[ \prod_{\ell=0}^{k-1} (1 + c_{\text{Fr}}(\vartheta_\ell)) \right] (\mathbb{E}|X_0|^2 + 1) - 1. \tag{3.26}
\]

**Proposition 3.1** Assume that all the functions \(\vartheta_k : \mathbb{R}^d \to M_{d,q}, \ k = 0 : n - 1\), are Lipschitz continuous and that \(X_0 \in L^2_{\mathbb{R}^d}(\mathbb{P})\) (e.g. because \(X_0 = x_0 \in \mathbb{R}^d\)). Then, for every \(k = 0, \ldots, n\),

\[
\|X_k - \bar{X}_k\|^2 \leq \|X_0 - \bar{X}_0\|^2 \prod_{\ell=0}^{k-1} (1 + q[\vartheta_\ell]_{\text{Lip}}^2) + (1 + \|X_0\|^2) \sum_{\ell=1}^{k-1} (1 + c_{\text{Fr}}(\vartheta_\ell)) \sum_{i=0}^{k-1} c(\vartheta_i) \|ar{Z}_\ell - \bar{Z}_\ell\|_2^2 \tag{3.27}
\]

where \(C(\vartheta) = (q[\vartheta]_{\text{Lip}}^2) \vee c_{\text{Fr}}(\vartheta)\). Moreover,

\[
\text{max}_{k=0:n} \|X_k - \bar{X}_k\|_2 \leq 4\|X_n - \bar{X}_n\|_2.
\]

**Proof.** Using successively the martingale property of \((X_k - \bar{X}_k)_{k=0:n}\), then (3.21)(ii) and last \(\|\bar{Z}_{k+1}\|^2 = q\), one obtains that, for every \(k = 0, \ldots, n - 1\),

\[
\|\bar{X}_{k+1} - X_{k+1}\|^2 = \|\bar{X}_k - X_k\|^2 + \|\vartheta_k(\bar{X}_k)\bar{Z}_{k+1} - \vartheta_k(X_k)Z_{k+1}\|^2 = \|\bar{X}_k - X_k\|^2 + \|\vartheta_k(X_k)\|^2 \|\bar{Z}_{k+1}\|^2 + \|\vartheta_k^*(X_k)\|^2 \|Z_{k+1} - \bar{Z}_{k+1}\|^2 \leq \|\bar{X}_k - X_k\|^2 + q[\vartheta_k]_{\text{Lip}}^2 \|\bar{X}_k - X_k\|^2 + \|\vartheta_k(X_k)\|^2 \|Z_{k+1} - \bar{Z}_{k+1}\|^2.
\]

Then, using that \(\|\vartheta_k(X_k)\|^2 \leq c(\vartheta_k)(1 + \|X_k\|^2)\) and (3.26), one concludes by a discrete Gronwall’s Lemma. The last inequality follows from Doob’s Inequality.

**Remark 3.1** If, furthermore, the \(\bar{Z}_k\) have diagonal covariance matrices, then, for \(A \in M_{d,q}\),

\[
\mathbb{E}|A\bar{Z}_k|^2 = \sum_{i=1}^q (AA^*)_{ii} \mathbb{E}(\bar{Z}_k^i)^2 \leq \|A\|^2 \text{Fr} \max_{i=1:q} \mathbb{E}(\bar{Z}_k^i)^2 \leq \|A\|^2 \text{Fr}.
\]

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Moreover, for \( i \neq j \) and \( k = 1, \ldots, n \),
\[
\mathbb{E}[(\hat{Z}_k^i - Z_k^i)(\hat{Z}_k^j - Z_k^j)] = \mathbb{E}((\hat{Z}_k^i - Z_k^i)\hat{Z}_k^j) + \mathbb{E}((\hat{Z}_k^j - Z_k^j)\hat{Z}_k^i) + \mathbb{E}[Z_k^i\hat{Z}_k^i - Z_k^i\hat{Z}_k^j] = 0,
\]
by (3.21) and since the covariance matrices of both \( Z_k \) and \( \hat{Z}_k \) are diagonal. Hence, for \( A \in \mathbb{M}_{d,q} \),
\[
\mathbb{E} |A(Z_k - \hat{Z}_k)|^2 \leq \|A\|_{Fr}^2 \max_{i=1,q} \mathbb{E} (Z_k^i - \hat{Z}_k^i)^2. 
\]
Consequently
\[
\mathbb{E} |\vartheta(X_k)\hat{Z}_{k+1} - \vartheta(X_k)Z_{k+1}|^2 \leq \mathbb{E} \|\vartheta(X_k) - \vartheta_k(X_k)\|^2_{Fr} + \|\vartheta_k(X_k)\|^2_{Fr} \max_{i=1,q} \mathbb{E} (Z_k^i - \hat{Z}_k^i)^2.
\]
Hence (3.27) holds with \( C(\vartheta) = [\vartheta]^2_{Fr,Lip} \cup c_{Fr}(\vartheta), \vartheta q[\vartheta]^2_{Lip} \) and \( c(\vartheta)\|Z_k - \hat{Z}_k\|^2_{2} \) replaced by \( [\vartheta]^2_{Fr,Lip}, c(\vartheta)\|Z_k - \hat{Z}_k\|^2_{2} \) respectively.

- Let us assume that the vectors \( Z_k \) have independent coordinates. Then, the diagonal covariance matrix condition can be achieved by choosing \( \hat{Z}_k^i = \varphi_{k,i}(Z_k^i) \) in such a way that (3.21)(i) is satisfied and (3.21)(ii) holds for \( i = j \). Then, for \( i \neq j \), (3.21)(ii) follows from the independence property.

As for truncation, this can be done by considering sets of the form \( A_k = \prod_{i=1}^d A_k^i \) such that
\[
\mathbb{E} Z_k^i 1_{\{Z_k^i \in A_k^i\}} = 0, \ i = 1 : q.
\]
As for the quantization based approach, first note that optimal Voronoi quantization usually does not satisfy this property. But this can be achieved by calling upon product quantization, by considering product grids \( \Gamma_Z = \prod_{1 \leq i \leq q} \Gamma^i \) where \( \Gamma^i \) is a Voronoi stationary grid of the i'th marginal \( Z^i \) so that \( \mathbb{E} (Z_k^i | (\hat{Z}_k^j)_{1 \leq j \leq q}) = \mathbb{E} (Z_k^i | \hat{Z}_k^i) = \hat{Z}_k^i \) where \( \hat{Z}_k^i \) is the projection of \( Z_k^i \) onto \( \hat{Z}_k^i \). One easily derives from this componentwise stationarity property that (3.21) holds true for \( \hat{Z}_k = (\hat{Z}_k^j)_{i=1}^d = \hat{Z}_k^i \). Finally note that, when the marginal quantizations \( \hat{Z}_k^i \) are \( L^2 \)-optimal, then \( \hat{Z}_k^i \) is rate optimal and satisfies the universal non-asymptotic upper-bound provided by Pierce's Lemma (see the remark following Theorem \( A.1 \) in Appendix \( A.1 \)).

**Truncation of the Euler scheme with Gaussian increments**

Let \( h := T/n \) denote the step of the Euler scheme defined by (2.19). Assume that the diffusion function \( \vartheta(t,x) \) is Lipschitz continuous in \( x \) with constant \( [\vartheta]_{Lip} \) uniformly in \( t \in [0,T] \). Then
\[
\vartheta_k(x) = \sqrt{h} \vartheta(\frac{kT}{n},x), \ k = 0, \ldots, n-1 \text{ and } Z_t \sim \mathcal{N}(0;I_q).
\]
We set \( A_k = B(0;a) \) for every \( k \geq 1 \) and some \( a > 0 \). Then
\[
c(\vartheta_k) = hc(\vartheta) \quad \text{and} \quad [\vartheta_k]_{Lip}^2 = h[\vartheta]_{Lip}^2 \quad k = 0, \ldots, n-1.
\]
so that \( C(\vartheta_k) = hC(\vartheta) \) from which we derive
\[
\|X_k - \hat{X}_k\|^2 \leq \|X_0 - \hat{X}_0\|^2 \left( 1 + \frac{T}{n} [\vartheta]_{Lip}^2 \right)^k + (1 + \|X_0\|^2) \left( 1 + \frac{T}{n} C(\vartheta) \right)^k \sum_{c=1}^k \|Z_t 1_{\{|Z_t| \geq a\}}\|^2_{2},
\]
\[
\leq \|X_0 - \hat{X}_0\|^2 e^{[\vartheta]_{Lip}^2 \frac{kT}{n}} + (1 + \|X_0\|^2) e^{C(\vartheta)T/n} c(\vartheta) \frac{kT}{n} \|Z_t 1_{\{|Z_t| \geq a\}}\|^2_{2}, \quad k = 0, \ldots, n,
\]
so that, by Doob's Inequality, with obvious notations
\[
\left\| \max_{k=0:n} \|X_k - \hat{X}_k\|^2 \right\|_{2}^2 \leq 4\|X_0 - \hat{X}_0\|^2 e^{[\vartheta]_{Lip}^2 T} + 4T(1 + \|X_0\|^2) e^{C(\vartheta)T} c(\vartheta) \|Z_t 1_{\{|Z_t| \geq a\}}\|^2_{2},
\]

Choice of \( a = a(n) \).

- If \( q = 1 \), the tail expectation can be estimated by a straightforward integration by parts which: for every \( a > 0 \)
  \[
  \mathbb{E} |Z_1|^2 \mathbf{1}_{\{|Z_1| \geq a\}} \leq \sqrt{\frac{2}{\pi}} \left( a + \frac{1}{a} \right) e^{-\frac{a^2}{2}}, \quad a > 0.
  \]

If we set \( a = a_n \geq \sqrt{c \log n} \) for some \( c > 0 \), then

\[
\mathbb{E} |Z_1|^2 \mathbf{1}_{\{|Z_1| \geq a\}} = O \left( \frac{\sqrt{\log n}}{n^{\frac{d}{2}}} \right) \to 0.
\]

- If \( q \geq 2 \), a simple, though sub-optimal, approach is the following: we start from the obvious
  \[
  \mathbb{E} |Z_1|^2 \mathbf{1}_{\{|Z_1| \geq a\}} \leq e^{-\lambda a^2} \mathbb{E} |Z_1|^2 e^{\lambda |Z_1|^2} = e^{-\lambda a^2} q \mathbb{E} \zeta^2 e^{\lambda \zeta^2} \times \left( \mathbb{E} e^{\lambda \zeta^2} \right)^{d-1} = q \frac{e^{-\lambda a^2}}{(1 - 2\lambda) \frac{d}{2} + \frac{d+2}{2}} \quad \text{where} \quad \zeta \sim \mathcal{N}(0; 1).
  \]

As soon as \( a > \sqrt{d+2} \), the function \( \lambda \mapsto -\lambda a^2 - \frac{d+2}{2} \log(1 - 2\lambda) \) is minimum at \( \lambda(a) = \frac{1}{2} \left( 1 - \frac{d+2}{a^2} \right) \in (0, \frac{1}{2}) \). Hence

\[
\mathbb{E} |Z_1|^2 \mathbf{1}_{\{|Z_1| \geq a\}} \leq q e^{-\frac{a^2}{2}} \left( \frac{ea^2}{d+2} \right)^{1+\frac{d}{2}}
\]

and, if \( a = a_n \geq \sqrt{c \log n} \) for some \( c > 0 \), then

\[
\mathbb{E} |Z_1|^2 \mathbf{1}_{\{|Z_1| \geq a\}} = O \left( \frac{(\log n)^{1+\frac{d}{2}}}{n^{\frac{d}{2}}} \right) \to 0.
\]

\[\text{Voronoi/primal Quantization of the increments of the Euler scheme}\]

As \( \mathcal{N}(0; I_q) \) has \( 2+\eta \)-moment for any \( \eta > 0 \), it follows from Zador’s Theorem (see Theorem A.1) that, if \( \hat{Z}_k \) are either optimal quantizations of \( Z_k \) at level \( N_z \) or (like in the above remark) a product quantization of optimal quantizations of the marginal, in both cases

\[
\|Z_k - \hat{Z}_k\|_2 = e_{2,N_Z}(\mathcal{N}(0; I_q)) \leq C_{q,\eta} \sigma_{2+\eta}(\mathcal{N}(0; I_q)) N_z^{-1/d},
\]

where \( \sigma_{2+\eta}(\mathcal{N}(0; I_q)) = \left( \frac{2^{\eta/2}}{\pi^{\eta/2}} S_q - 1 \Gamma \left( \frac{q+\eta}{2} + 1 \right) \right)^{\frac{1}{2+\eta}} \) with \( \Gamma(.) \) denoting the Euler \( \Gamma \) function and \( S_q \) the area of the unit sphere of dimension \( q-1 \).

3.1.2 Convex ARCH models: Domination of \( (\hat{X}_k)_{k=0,n} \) by \( (X_k)_{k=0,n} \) for the convex order

When the functions \( \vartheta_k \) are convex in an appropriate sense and the variables \( Z_{k+1} \) have radial distributions, then the ARCH model (2.18) dominates all its approximations with truncated white noise as established in Propositions 3.3 and 3.5. We start with two lemmas giving conditions ensuring convex ordering between two r.v..
Lemma 3.1 (Truncation) Let \( Z \in L^1_{\mathbb{R}^q}(\Omega, \mathcal{A}, \mathbb{P}) \) be a centered random vector. For any Borel set \( A \), let \( Z^A = Z1_{\{Z \in A\}} \). If \( \mathbb{E} Z^A = 0 \), then
\[
Z^A \leq_{\text{cvx}} Z.
\]

**Proof.** One may restrict to convex functions \( \varphi : \mathbb{R}^q \to \mathbb{R} \) with linear growth (see e.g. [1]) for which \( \varphi(Z) \in L^1 \). We may assume w.l.o.g. that \( \mathbb{P}(Z \notin A) > 0 \) (otherwise the result is trivial). Then
\[
\mathbb{E} \varphi(Z) - \mathbb{E} \varphi(Z^A) = \mathbb{E} \varphi(Z)1_{\{Z \notin A\}} - \varphi(0)\mathbb{P}(Z \notin A)
\]
\[
= \mathbb{P}(Z \notin A)\left( \mathbb{E} (\varphi(Z) | Z \notin A) - \varphi(0) \right) \geq 0
\]
owing to Jensen’s Inequality and \( \mathbb{E} (Z | Z \notin A) = 0 \). \( \square \)

Lemma 3.2 Let \( Z \) be an integrable \( \mathbb{R}^q \)-valued r.v. and. For \( i = 1 : q \), denote by \( Z_{-i} \) the subvector obtained by removing the \( i \)-th coordinate \( Z_i \) from \( Z \).

(i) If for \( i = 1 : q \), \( \mathbb{E}[Z_i|Z_{-i}] = 0 \) a.s. (vanishing conditional expectations assumption) and \( 0 \leq \lambda_i \leq \ell_i \), then \( \text{Diag}(\lambda_1, \ldots, \lambda_q)Z \leq_{\text{cvx}} \text{Diag}(\ell_1,\ldots,\ell_q)Z \) where \( \text{Diag}(\lambda_1,\ldots,\lambda_q) \in \mathbb{M}_{q,q} \) denotes the diagonal matrix with diagonal elements \( \lambda_1,\ldots,\lambda_q \).

(ii) If for each \( i = 1 : q \), the conditional laws of \( Z_i \) and \( -Z_i \) given \( Z_{-i} \) coincide a.s. (symmetric conditional laws assumption) and \( |\lambda_i| \leq |\ell_i| \), then \( \text{Diag}(\lambda_1,\ldots,\lambda_q)Z \leq_{\text{cvx}} \text{Diag}(\ell_1,\ldots,\ell_q)Z \).

(iii) If \( A, B \in \mathbb{M}_{d,q} \) and \( Z \) has a radial distribution i.e. for each orthogonal matrix \( O \in \mathbb{M}_{q,q} \), \( OZ \) has the same distribution as \( Z \), then \( AA^* \leq BB^* \Rightarrow AZ \leq_{\text{cvx}} BZ \). If moreover \( \mathbb{E}|Z|^2 \in (0, +\infty) \), then the converse implication holds.

**Remark 3.2** The radial distribution assumption implies the symmetric conditional law assumption (choose the orthogonal transformation which only changes the sign of the \( i \)-th coordinate) which, in turn, implies the vanishing conditional expectation assumption. On the other hand, the assumptions on the matrices multiplying \( Z \) get weaker from (i) to (iii).

When \( Z \) follows the radial distribution \( \mathcal{N}(0; I_q) \) and \( AA^* \leq BB^* \) then for \( \zeta \sim \mathcal{N}(0; BB^* - AA^*) \) independent of \( Z \), \( \mathbb{E}[AZ + \zeta |Z] = AZ \), so that \( AZ \leq_{\text{cvx}} AZ + \zeta \) and \( AZ + \zeta \sim \mathcal{N}(0; BB^*) \) so that \( AZ + \zeta \) has the same distribution as \( BZ \). Hence \( AZ \leq_{\text{cvx}} BZ \). This is a simple alternative argument to the one in [13] which has inspired the generalization to any radial distribution below.

**Proof.** (i) The function \( u \mapsto \mathbb{E} \psi(uX) \) is clearly convex and attains its minimum at \( u = 0 \) owing to Jensen’s inequality. Hence it is non-decreasing on \( \mathbb{R}_+ \) and non-increasing on \( \mathbb{R}_- \). Now for \( \varphi : \mathbb{R}^d \to \mathbb{R} \) convex with linear growth, repeatedly using the monotonicity property on \( \mathbb{R}_+ \), one obtains
\[
\mathbb{E} \varphi(\text{Diag}(\lambda_1,\ldots,\lambda_q)Z) = \mathbb{E} \mathbb{E}(\varphi(\text{Diag}(\lambda_1,\ldots,\lambda_q)Z)|Z_{-1}) \leq \mathbb{E} \mathbb{E}(\varphi(\text{Diag}(\ell_1,\lambda_2,\ldots,\lambda_q)Z)|Z_{-1})
\]
\[
= \mathbb{E} \mathbb{E}(\varphi(\text{Diag}(\ell_1,\ell_2,\ldots,\lambda_q)Z)|Z_{-2}) \leq \mathbb{E} \mathbb{E}(\varphi(\text{Diag}(\ell_1,\ell_2,\ldots,\ell_q)Z)|Z_{-2}) \leq \ldots \leq \mathbb{E} \varphi(\text{Diag}(\ell_1,\ldots,\ell_q)Z).
\]
By Lemma A.1 in [1] (see also Remark 1.1 p.2 in [23]), one concludes that \( \text{Diag}(\lambda_1,\ldots,\lambda_q)Z \leq_{\text{cvx}} \text{Diag}(\ell_1,\ldots,\ell_q)Z \).
(ii) Since, under the assumption, \( \text{Diag}(\lambda_1, \ldots, \lambda_q)Z \) and \( \text{Diag}(\ell_1, \ldots, \ell_q)Z \) respectively have the same distributions as \( \text{Diag}(|\lambda_1|, \ldots, |\lambda_q|)Z \) and \( \text{Diag}(|\ell_1|, \ldots, |\ell_q|)Z \), the conclusion follows from (i).

(iii) **Step 1.** For \( C \in \mathbb{M}_{q,q} \), the singular value decomposition of \( C \) writes \( C = ODV \) for matrices \( O, D, V \in \mathbb{M}_{q,q} \) with \( O, V \) orthogonal and \( D \) diagonal with nonnegative diagonal elements. One has \( \sqrt{CC^*} = OD^*O \) and if \( Z \) has a radial distribution, for any measurable and bounded function \( \varphi : \mathbb{R}^q \to \mathbb{R} \), \( E\varphi(CZ) = E\varphi(ODVZ) = E\varphi(ODO^*Z) = E\varphi(\sqrt{CC^*Z}) \) so that \( CZ \) and \( \sqrt{CC^*Z} \) share the same distribution.

**Step 2.** Let us now assume that the \( \mathbb{R}^q \)-valued r.v. \( Z \) has a radial distribution, \( AA^* \leq BB^* \) and \( d = q \). We set \( B_{\varepsilon} = \sqrt{BB^* + \varepsilon I_q} \). We have \( B_{\varepsilon}^{-1}AA^*(B_{\varepsilon}^{-1})^* \leq I_q \) for \( \varepsilon > 0 \). One deduces that \( \sqrt{B_{\varepsilon}^{-1}AA^*(B_{\varepsilon}^{-1})^*} = OD^*O \) for matrices \( O, D \in \mathbb{M}_{q,q} \) with \( O \) orthogonal and \( D \) diagonal with diagonal elements belonging to \([0, 1]\). For \( \varphi : \mathbb{R}^d \to \mathbb{R} \) convex with linear growth, the function \( \psi(x) = \varphi(B_{\varepsilon}Ox) \) is convex with linear growth and

\[
E\varphi(AZ) = E\psi(O^*B_{\varepsilon}^{-1}AZ) = E\psi(O^*ODO^*Z) = E\psi(DZ) \leq E\psi(Z) = E\varphi(B_{\varepsilon}OZ) = E\varphi(B_{\varepsilon}Z),
\]

where we used the definition of \( \psi \) for the first and fourth equalities, Step 1 for the second equality, the radial property of the distribution of \( Z \) for the third and fifth equalities and (i) for the inequality. One has \( \lim_{\varepsilon \to 0} B_{\varepsilon} = \sqrt{BB^*} \), so that by Lebesgue’s theorem and Step 1, \( \lim_{\varepsilon \to 0} E\varphi(B_{\varepsilon}Z) = E\varphi(\sqrt{BB^*}Z) = E\varphi(BZ) \). We deduce that \( E\varphi(AZ) \leq E\varphi(BZ) \) so that \( AZ \leq_{c_v} BZ \).

**Step 3.** Let us now assume that \( Z \) has a radial distribution, \( AA^* \leq BB^* \) and \( d < q \). Let \( A, B \in \mathbb{M}_{q,q} \) be defined by

\[
(A_{ij}, B_{ij}) = \begin{cases} (A_{ij}, B_{ij}) & \text{for } i = 1: d, j = 1: q \\ (0, 0) & \text{for } i = d + 1: q, j = 1: q \end{cases}.
\]

We have \( \tilde{A}A^* \leq BB^* \), so that, by Step 2, \( \tilde{A}Z \leq_{c_v} BZ \). For \( M \in \mathbb{M}_{d,q} \) with non-zero coefficients \( M_{i,i} = 1 \) for \( i = 1: q \), we have \( AZ = M\tilde{A}Z \) and \( BZ = M\tilde{B}Z \). Since for any convex function \( \varphi : \mathbb{R}^d \to \mathbb{R} \), \( \mathbb{R}^q \ni x \mapsto \varphi(Mx) \) is convex as the composition of a convex function with a linear function, we conclude that \( AZ \leq_{c_v} BZ \).

**Step 4.** Let us finally assume that \( Z \) has a radial distribution, \( AA^* \leq BB^* \) and \( d > q \). We have \( \text{Ker}B^* \subset \text{Ker}A^* \) so that \( \text{Im}A \subset \text{Im}B \). Let \( O \in \mathbb{M}_{d,q} \) be a matrix with orthogonal columns with norm one such that the first \( d \) columns \( \text{Im}B \) (we have \( \text{dim} \text{Im}B \leq q \)) columns form an orthonormal basis of \( \text{Im}B \). Then \( B = O^*B \) and \( A = O^*A \), \( O^*BB^*O \geq O^*AA^*O \) and, by Step 2, \( O^*AZ \leq_{c_v} O^*BZ \). Since for any convex function \( \varphi : \mathbb{R}^d \to \mathbb{R} \), \( \mathbb{R}^q \ni x \mapsto \varphi(Ox) \) is convex as the composition of a convex function with a linear function, we conclude that \( AZ = O^*AZ \leq_{c_v} O^*BZ = BZ \).

**Step 5.** Let us suppose that \( Z \) is square integrable with a radial distribution. Then \( E(Z_iZ_j) = 1_{(i=j)} \frac{|Z|^2}{q} \). If \( A, B \in \mathbb{M}_{d,q} \) are such that \( AZ \leq_{c_v} BZ \), then, for \( u \in \mathbb{R}^d \), the choice of the convex function \( \varphi : x \in \mathbb{R}^d \mapsto (u^*x)^2 \) in the inequality defining the convex order yields \( u^*AA^*u \frac{|Z|^2}{q} \leq u^*BB^*u \frac{|Z|^2}{q} \).

\[ \square \]

**Proposition 3.2 (Convex order: from the noise to the ARCH)** Let \( (Z_k)_{k=1:n} \) and \( (Z_k')_{k=1:n} \) be two sequences of \( \mathbb{R}^q \)-valued independent integrable and centered random vectors. Let \( (\vartheta_k)_{k=0:n-1} \)
and \((\vartheta'_k)_{k=0:n-1}\) be two sequences of \(M_{d,q}\)-valued functions with linear growth defined on \(\mathbb{R}^d\) such that: \(\forall x \in \mathbb{R}^d, \vartheta_k \vartheta_k^*(x) \leq \vartheta'_k(\vartheta'_k)^*(x)\) for \(k = 0, \ldots, n-1\) and, either for every \(k = 0, \ldots, n-1\),

\[
\forall x, y \in \mathbb{R}^d, \forall \alpha \in [0, 1], \exists O = O_{k,x,y,\alpha} \in M_{d,q} \text{ orthogonal such that } \\
\vartheta_k \vartheta_k^*(ax + (1 - \alpha)y) \leq (\alpha \vartheta_k(x) + (1 - \alpha)\vartheta_k(y))O(\alpha \vartheta_k(x) + (1 - \alpha)\vartheta_k(y))^*. \tag{3.28}
\]

and the r.v. \(Z_{k+1}\) has a radial distribution (we say that the assumption is satisfied by \((Z_{k+1}, \vartheta_k, \vartheta'_k)_{k=0:n-1}\)) or for each \(k = 0, \ldots, n-1, \tag{3.28}\) is satisfied with \(\vartheta_k\) replaced by \(\vartheta'_k\) and \(Z'_{k+1}\) has a radial distribution (we say that the assumption is satisfied by \((Z'_{k+1}, \vartheta'_k, \vartheta'_k)_{k=0:n-1}\)).

Let \(X_0\) and \(X'_0\) be integrable \(\mathbb{R}^d\)-valued r.v. independent of \((Z_k)_{k=1:n}\) and \((Z'_k)_{k=1:n}\) respectively. Denote by \((X_k)_{k=0:n}\) and \((X'_k)_{k=0:n}\) the two ARCH models respectively defined by \((2.18)\) and by \(X'_{k+1} = X'_k + \vartheta'_k(X'_k)Z'_{k+1}\) for \(k = 0, \ldots, n-1\).

If \(X_0 \leq_{\text{cev}} X'_0\) and \(Z_k \leq_{\text{cev}} Z'_k\) for every \(k = 1, \ldots, n\), then

\[
(X_k)_{k=0:n} \leq_{\text{cev}} (X'_k)_{k=0:n}.
\]

**Proof.** By the linear growth of the coefficients \(\vartheta_k\), the integrability of the initial conditions and the noises and the independence structure, one easily checks by forward induction that \(X_0: k\) and \(X'_0: k\) are integrable for every \(k = 0, \ldots, n\). According to Lemma A.1 in [1], it is enough to prove that \(\mathbb{E} \Phi_n(X_0:n) \leq \mathbb{E} \Phi_n(X'_0:n)\) for \(\Phi_n : (\mathbb{R}^d)^{n+1} \to \mathbb{R}\) convex with linear growth.

We proceed by successive backward inductions. We define the functions \(\Phi_k : (\mathbb{R}^d)^{k+1} \to \mathbb{R}, \ k = 0, \ldots, n-1, \) by backward induction as follows:

\[
\Phi_k(x_{0:k}) = \Psi_k(x_{0:k}, \vartheta_k(x_k)), \ k = 0, \ldots, n-1
\]

where, for every \((x_{0:k}, u) \in (\mathbb{R}^d)^{k+1} \times M_{d,q},\)

\[
\Psi_k(x_{0:k}, u) = \mathbb{E} \Phi_{k+1}(x_{0:k}, x_k + uZ_{k+1}), \ k = 0, \ldots, n-1.
\]

By backward induction, using the integrability of the random variables \(Z_{k+1}\) and the linear growth of the function \(\vartheta_k(x_k)\), one easily checks that the functions \(\Phi_k\) and \(\Psi_k\) all have linear growth and in particular that the expectation in the definition of \(\Psi_k\) makes sense. Notice that, since the law of \(Z_{k+1}\) is radial,

\[
\forall (x_{0:k}, u, O) \in (\mathbb{R}^d)^{k+1} \times M_{d,q} \times M_{d,q} \text{ with } O \text{ orthogonal } \Psi_k(x_{0:k}, u) = \Psi_k(x_{0:k}, uO). \tag{3.29}
\]

Starting from \(\Phi'_n = \Phi_n\), we define the functions \(\Phi'_k, \Psi'_k, \ k = 0, \ldots, n-1\) likewise using the sequence \((Z'_k)_{k=1:n}\) instead of \((Z_k)_{k=1:n}\). The processes \((X_k)_{k=0:n}\) and \((X'_k)_{k=0:n}\) are \((\mathcal{F}^Z_k = \sigma(X_0, (Z_\ell)_{\ell=1:k}))_{k=0:n}\) and \((\mathcal{F}^Z'_k = \sigma(X'_0, (Z'_\ell)_{\ell=1:k}))_{k=0:n}\)-Markov chains respectively. It is clear by backward induction that

\[
\Phi_k(X_{0:k}) = \mathbb{E}(\Phi_n(X_{0:n}) \mid \mathcal{F}^Z_k) \quad \text{and} \quad \Phi'_k(X_{0:k}) = \mathbb{E}(\Phi'_n(X'_{0:n}) \mid \mathcal{F}^Z'_k), \ k = 0, \ldots, n.
\]

Let us suppose that for each \(k = 0, \ldots, n-1, \tag{3.28}\) holds and \(Z_{k+1}\) has a radial distribution. We first check by backward induction that the functional \(\Phi_k\) are convex. The function \(\Phi_n\) is convex by assumption. If \(\Phi_{k+1}\) is convex, by the convexity of \(\mathbb{R}^d \ni w \mapsto \Phi_{k+1}(x_{0:k}, x_k + w)\) and Lemma 3.2 (iii),

\[
\forall x_{0:k} \in (\mathbb{R}^d)^{k+1}, \forall u, v \in M_{d,q} \text{ s.t. } uu^* \leq vv^*, \ \Psi_k(x_{0:k}, u) \leq \Psi_k(x_{0:k}, v). \tag{3.30}
\]
With (3.28) then the convexity of \( \Psi_k \) consequence of the one of \( \Phi_{k+1} \) and last (3.29), we deduce that for \( x_{0:k}, y_{0:k} \in (\mathbb{R}^d)^{k+1} \) and \( \alpha \in [0, 1], \)

\[
\Phi_k(\alpha x_{0:k} + (1 - \alpha)y_{0:k}) = \Psi_k(\alpha x_{0:k} + (1 - \alpha)y_{0:k}, \partial_k(\alpha x_k + (1 - \alpha)y_k)) \\
\leq \Psi_k(\alpha x_{0:k} + (1 - \alpha)y_{0:k}, \alpha \partial_k(x_k) + (1 - \alpha)\partial_k(y_k)O_k, x_k, y_k, \alpha) \\
\leq \alpha \Psi_k(x_{0:k}, \partial_k(x_k)) + (1 - \alpha)\Psi_k(y_{0:k}, \partial_k(y_k))O_k, x_k, y_k, \alpha) \\
= \alpha \Psi_k(x_{0:k}, \partial_k(x_k)) + (1 - \alpha)\Psi_k(y_{0:k}, \partial_k(y_k)) \\
= \alpha \Phi_k(x_{0:k}) + (1 - \alpha)\Phi_k(y_{0:k}).
\]

As a second step, let us prove that \( \Phi'_k \geq \Phi_k, k = 0, \ldots, n \), still by backward induction. This is true for \( k = n \) since \( \Phi_n = \Phi'_n \). Assume \( \Phi'_{k+1} \geq \Phi_{k+1} \). Then,

\[
\Psi'_{k}(x_{0:k}, u) \geq \mathbb{E} \Phi'_{k+1}(x_{0:k}, x_k + uZ'_{k+1}).
\]

Now, for every \( (x_{0:k}, u) \in (\mathbb{R}^d)^{k+1} \times \mathbb{M}_{d,q} \), the function \( z \mapsto \Phi_{k+1}(x_{0:k}, x_k + u z) \) is convex as the composition of a convex function with an affine function. The assumption \( Z_{k+1} \leq_{cvx} Z'_{k+1} \) implies that

\[
\Psi'_{k}(x_{0:k}, u) \geq \mathbb{E} \Phi'_{k+1}(x_{0:k}, x_k + uZ'_{k+1}) = \Psi_k(x_{0:k}, u)
\]

which in turn ensures, once composed with \( \partial'_k \), that \( \Phi'_k(x_{0:k}) \geq \Psi_k(x_{0:k}, \partial'_k(x_k)) \). With the condition \( \partial_k \partial'_k \leq \partial'_k(\partial'_k)^* \) and (3.30) we deduce that \( \Phi'_k \geq \Phi_k \). Since this inequality holds for every \( k \), one has in particular that \( \Phi'_0 \geq \Phi_0 \) so that

\[
\mathbb{E} \Phi'_n(X'_{0:n}) = \mathbb{E} \Phi'_0(X'_0) \geq \mathbb{E} \Phi_0(X'_0) \geq \mathbb{E} \Phi_0(X_0) = \mathbb{E} \Phi_n(X_{0:n}),
\]

where we used in the last inequality the assumption \( X_0 \leq_{cvx} X'_0 \) and the convexity of \( \Phi_0 \).

When for each \( k = 0, \ldots, n - 1 \), (3.28) holds with \( \partial'_k \) replacing \( \partial_k \) and \( Z'_{k+1} \) has a radial distribution, then we check as above that, for each \( k = 0, \ldots, n - 1 \), \( \Phi'_k \) is convex and that

\[
\forall x_{0:k} \in (\mathbb{R}^d)^{k+1}, \forall u, v \in \mathbb{M}_{d,q} \text{ s.t. } uu^* \leq vv^*, \Psi'_k(x_{0:k}, u) \leq \Psi'_k(x_{0:k}, v). \tag{3.31}
\]

To deduce by backward induction that \( \Phi_k \leq \Phi'_k, k = 0, \ldots, n \), we assume \( \Phi_{k+1} \leq \Phi'_{k+1} \). Then,

\[
\Psi_k(x_{0:k}, u) \leq \mathbb{E} \Phi_{k+1}(x_{0:k}, x_k + uZ_{k+1}) \leq \mathbb{E} \Phi'_{k+1}(x_{0:k}, x_k + uZ'_{k+1}) = \Psi'_k(x_{0:k}, u),
\]

where we used the convexity of \( \Phi'_{k+1} \) and \( Z_{k+1} \leq_{cvx} Z'_{k+1} \) for the second inequality. By composing with \( \partial_k \) then using (3.31) with \( u = \partial_k(x_k) \) and \( v = \partial'_k(x_k) \) thanks to the condition \( \partial_k \partial'_k \leq \partial'_k(\partial'_k)^* \), we deduce that

\[
\Phi_k(x_{0:k}) \leq \Psi'_k(x_{0:k}, \partial_k(x_k)) \leq \Psi'_k(x_{0:k}, \partial'_k(x_k)) = \Phi'_k(x_{0:k}).
\]

One has in particular that \( \Phi_0 \leq \Phi'_0 \) so that

\[
\mathbb{E} \Phi_n(X'_{0:n}) = \mathbb{E} \Phi_0(X_0) \leq \mathbb{E} \Phi'_0(X'_0) \leq \mathbb{E} \Phi'_0(X_0) = \mathbb{E} \Phi_n(X'_{0:n}),
\]

where we used in the last inequality the assumption \( X_0 \leq_{cvx} X'_0 \) and the convexity of \( \Phi'_0 \). \( \square \)

This leads to the following result.
Proposition 3.3 (Domination) Let \((X_k)_{k=0:n}\) be an \(\mathbb{R}^d\)-valued ARCH model defined by (2.18) where the \(\mathbb{R}^q\)-valued white noise \((Z_k)_{k=1:n}\) is a sequence of integrable \(\mathbb{R}^q\)-valued r.v. with radial distributions, the initial random vector \(X_0\) is integrable and the \(\mathcal{M}_{d,q}\)-valued functions \(\vartheta_k\) for \(k = 1, \ldots, n\), are convex in the sense of (3.28) with linear growth. Assume that \(g_0(X_0) \leq_{\text{cvx}} X_0\).

(a) Truncation. Let \(\hat{Z}_k = Z_k^A\) be an \(n\)-tuple of Borel sets satisfying \(\mathbb{E}\{Z_k 1_{\{Z_k \in A_k\}}\} = 0\), \(k = 1, \ldots, n\), and \((\hat{X}_k^A)_{k=0:n}\) be the induced approximating ARCH process defined by (3.22). Then

\[
\hat{X}_{0:n}^A \leq_{\text{cvx}} X_{0:n}.
\]

(b) Quantization. Let \(\hat{Z}_k = \hat{Z}_k^{\text{var}}\) be a stationary (Voronoi) quantized approximation of the white noise \(Z_{1:n}\) and \(\hat{X}_{0:n}\) be the induced approximating ARCH process defined by (3.22). Then

\[
\hat{X}_{0:n} \leq_{\text{cvx}} X_{0:n}.
\]

When \(q = d\) and in particular in the one-dimensional case \(q = d = 1\), we can rely on points (i) and (ii) in Lemma 3.2 in addition to point (iii). This leads to the following relaxed assumption:

either for each \(k = 0, \ldots, n - 1\), one of the following conditions holds (we say that the assumption is satisfied by \((Z_{k+1}' \vartheta_k', \vartheta_k)_{k=0:n-1}\)):

- \(Z_{k+1}\) satisfies the vanishing conditional expectation assumption, \(\vartheta_k\) and \(\vartheta_k'\) are diagonal both with non-negative entries, the ones of \(\vartheta_k\) being moreover convex and \(\vartheta_k \vartheta_k^* \leq \vartheta_k' (\vartheta_k')^*\) (i.e. \((\vartheta_k)_{ii} \leq (\vartheta_k')_{ii}, i = 1 : d\)),

- \(Z_{k+1}\) satisfies the symmetric conditional distribution assumption, \(\vartheta_k\) and \(\vartheta_k'\) are diagonal with the entries of \(\vartheta_k\) convex and \(\vartheta_k \vartheta_k^* \leq \vartheta_k' (\vartheta_k')^*\) (i.e. \(|(\vartheta_k)_{ii}| \leq |(\vartheta_k')_{ii}|, i = 1 : d\)),

- \(Z_{k+1}\) has a radial distribution, \(\vartheta_k\) is convex in the the matrix-convexity sense (3.28) and \(\vartheta_k \vartheta_k^* \leq \vartheta_k' (\vartheta_k')^*\),

or (assumption satisfied by \((Z_{k+1}' \vartheta_k', \vartheta_k)_{k=0:n-1}\)) for each \(k = 0, \ldots, n - 1\):

one of the previous conditions holds with preservation of the inequalities between \(\vartheta_k\) and \(\vartheta_k'\) and replacement of \(Z_k\) and \(\vartheta_k\) by \(Z_k'\) and \(\vartheta_k'\) in the other assertions.

Notice that from one indent to the next, the assumption on the noise \(Z_{k+1}\) becomes stronger whereas the assumption on the coefficient \(\vartheta_k\) becomes weaker.

Proposition 3.4 (Convex order: \(q = d\)) Let \((Z_k)_{k=1:n}\) and \((Z_k')_{k=1:n}\) be two sequences of independent integrable and centered \(\mathbb{R}^d\)-valued random vectors. Let \((\vartheta_k)_{k=0:n-1}\) and \((\vartheta_k')_{k=0:n-1}\) be two sequences of \(\mathcal{M}_{d,d}\)-valued functions with linear growth defined on \(\mathbb{R}^d\). Let \(X_0\) and \(X_{0}'\) be integrable \(\mathbb{R}^d\)-valued r.v. independent of \((Z_k)_{k=1:n}\) and \((Z_k')_{k=1:n}\) respectively. Denote by \((X_k)_{k=0:n}\) and \((X_k')_{k=0:n}\) the two ARCH models respectively defined by (2.18) and by \(X_{k+1}' = X_k' + \vartheta_k' (X_k') Z_{k+1}'\) for \(k = 0, \ldots, n - 1\).

Under the assumption just before the proposition and if \(X_0 \leq_{\text{cvx}} X_0'\) and \(Z_k \leq_{\text{cvx}} Z_k'\) for every \(k = 1 : n\), then

\[
(X_k)_{k=0:n} \leq_{\text{cvx}} (X_k')_{k=0:n}.
\]

Proof. The proof is formally similar to that of Proposition 3.2 when \((Z_k)_{k=1:n}\) satisfies the radial distribution assumption and \(\vartheta_k\) the matrix-convexity assumption. So, we are simply going to explain how to adapt the backward induction steps when the assumption before the proposition is satisfied by \((Z_{k+1}, \vartheta_k, \vartheta_k')_{k=0:n-1}\) and \(Z_{k+1}\) satisfies either the vanishing conditional expectations
assumption or the conditional distribution assumption and the matrices $\vartheta_k$ and $\vartheta'_k$ are diagonal. Let $\Phi_n : (\mathbb{R}^d)^{n+1} \to \mathbb{R}$ be a convex function with linear growth and $\Phi'_n = \Phi_n$. We define by backward induction the sequence $(\Psi_k, \Phi_k, \Psi'_k, \Phi'_k)_{k=0:n-1}$ by using the formulas at the beginning of the proof of Proposition 3.2 when $Z_{k+1}$ has a radial distribution and otherwise by

$$
\Psi_k(x_{0:k}, u) = \mathbb{E} \Phi_{k+1}(x_{0:k}, x_k + \text{Diag}(u)Z_{k+1}) \quad \text{and} \quad \Phi_k(x_{0:k}) = \Psi_k(x_{0:k}, (\vartheta_k(x_k)\mathbf{1}))
$$

$$
\Psi'_k(x_{0:k}, u) = \mathbb{E} \Phi'_{k+1}(x_{0:k}, x_k + \text{Diag}(u)Z'_{k+1}) \quad \text{and} \quad \Phi'_k(x_{0:k}) = \Psi_k(x_{0:k}, (\vartheta'_k(x_k)\mathbf{1}))
$$

where $x_{0:k} \in (\mathbb{R}^d)^{k+1}$, $u \in \mathbb{R}^d$, $\text{Diag}(u) \in \mathbb{M}_{d,d}$ denotes the diagonal matrix with diagonal coefficients $\text{Diag}(u)_{ii} = u_i$, $i = 1:d$, and $\mathbf{1}$ is the vector in $\mathbb{R}^d$ with all coefficients equal to 1. Note that when $\vartheta_k$ is diagonal $\text{Diag}(\vartheta_k(x)\mathbf{1}) = \vartheta_k(x)$ for all $x \in \mathbb{R}^d$. If $\Phi_{k+1}$ is convex and $Z_{k+1}$ satisfies the vanishing conditional expectations assumption (or the stronger symmetric conditional distributions assumption), then by the convexity of $\mathbb{R}^d \ni w \mapsto \Phi_{k+1}(x_{0:k}, x_k + w)$ and Lemma 3.2 (i),

$$
\forall x_{0:k} \in (\mathbb{R}^d)^{k+1}, \forall u, v \in \mathbb{R}^d \text{ s.t. } u_i \leq v_i \text{ for } i = 1:d, \quad \Psi_k(x_{0:k}, u) \leq \Psi_k(x_{0:k}, v).
$$

(3.32)

For $u \in \mathbb{R}^d$, let us denote by $\text{abs}(u)$ the vector in $\mathbb{R}^d$ defined by $\text{abs}(u)_i = |u_i|$, $i = 1:d$. Assume moreover either that $\vartheta_k$ is diagonal with nonnegative and convex diagonal coefficients or $Z_{k+1}$ satisfies the symmetric conditional distributions assumption, $\vartheta_k$ is diagonal and the absolute values of its diagonal coefficients are convex functions. Then $\Psi_k(x_{0:k}, \vartheta_k(x_k)\mathbf{1}) = \Psi_k(x_{0:k}, \text{abs}(\vartheta_k(x_k)\mathbf{1}))$ with the coefficients of $\text{abs}(\vartheta_k(x_k)\mathbf{1})$ nonnegative and convex. With (3.32) then the convexity of $\Psi_k$ consequence of the one of $\Phi_{k+1}$, we deduce that for $x_{0:k}, y_{0:k} \in (\mathbb{R}^d)^{k+1}$ and $\alpha \in [0,1],$

$$
\Phi_k(\alpha x_{0:k} + (1-\alpha)y_{0:k}) = \Psi_k(\alpha x_{0:k} + (1-\alpha)y_{0:k}, \text{abs}(\vartheta_k(\alpha x_{0:k} + (1-\alpha)y_{0:k})\mathbf{1}))
$$

$$
\leq \Psi_k(\alpha x_{0:k} + (1-\alpha)y_{0:k}, \alpha \text{abs}(\vartheta_k(x_k)\mathbf{1}) + (1-\alpha)\text{abs}(\vartheta_k(y_k)\mathbf{1}))
$$

$$
\leq \alpha \Psi_k(x_{0:k}, \text{abs}(\vartheta_k(x_k)\mathbf{1})) + (1-\alpha)\Psi_k(y_{0:k}, \text{abs}(\vartheta_k(x_k)\mathbf{1}))
$$

$$
= \alpha \Phi_k(x_{0:k}) + (1-\alpha)\Phi_k(y_{0:k}).
$$

If $\Phi'_{k+1} \geq \Phi_{k+1}$, then,

$$
\Psi'_k(x_{0:k}, u) \geq \mathbb{E} \Phi'_{k+1}(x_{0:k}, x_k + \text{Diag}(u)Z'_{k+1}).
$$

Now, for every $(x_{0:k}, u) \in (\mathbb{R}^d)^{k+1} \times \mathbb{R}^d$, the function $z \mapsto \Phi_{k+1}(x_{0:k}, x_k + \text{Diag}(u) z)$ is convex as the composition of a convex function with an affine function. The assumption $Z_{k+1} \leq_{\text{cvx}} Z'_{k+1}$ implies that

$$
\Psi'_k(x_{0:k}, u) \geq \mathbb{E} \Phi_{k+1}(x_{0:k}, x_k + \text{Diag}(u)Z_{k+1}) = \Psi_k(x_{0:k}, u)
$$

which in turn ensures, once composed with $\vartheta'_k$, that

$$
\Phi'_k(x_{0:k}) \geq \Psi_k(x_{0:k}, \vartheta'_k(x_k)\mathbf{1}) = \Psi_k(x_{0:k}, \text{abs}(\vartheta'_k(x_k)\mathbf{1})).
$$

Since the absolute values of the diagonal coefficients of $\vartheta'_k$ are not smaller than the ones of $\vartheta_k$, we deduce with (3.32) that $\Phi'_k \geq \Phi_k$.

In the scalar $q = d = 1$ case, we deduce the following result.
Proposition 3.5 (Scalar setting: \(d = q = 1\)) Let \((X_k)_{k=0:n}\) be a scalar ARCH model defined by (2.18) where the white noise \((Z_k)_{k=1:n}\) is scalar but (possibly) not bounded and \(g_0(X_0) \leq_{cvx} X_0\) with \(X_0\) integrable. Assume that the functions \(|\vartheta_k|, k = 0, \ldots, n-1\) are convex with linear growth and that for each \(k = 0, \ldots, n-1\), either \(\vartheta_k\) is nonnegative or \(-Z_{k+1}\) has the same distribution as \(Z_{k+1}\).

(a) Truncation. Let \((\tilde{Z}_k = Z_k^A)_{k=1:n}\) with \((A_k)_{k=1:n}\) an \(n\)-tuple of Borel sets satisfying \(\mathbb{E}Z_k^1\{Z_k \in A_k\} = 0, k = 1, \ldots, n\) and \((X_k^A)_{k=0:n}\) be the induced approximating ARCH process defined by (3.22). Then

\[
(X_k^A)_{k=0:n} \leq_{cvx} (X_k)_{k=0:n}.
\]

(b) Voronoi Quantization. Let \((\hat{Z}_k = \hat{Z}_k^{vor})_{k=1:n}\) be a stationary (Voronoi) quantized approximation of the white noise \(Z_{1:n}\) and \(\hat{X}_{0:n}\) the induced approximating ARCH process defined by (3.22). Then

\[
(\hat{X}_k)_{k=0:n} \leq_{cvx} (X_k)_{k=0:n}.
\]

When \(g_0\) is nearest neighbour projection on a stationary Voronoi (primal) quantization grid for \(X_0\), the hypothesis \(g_0(X_0) \leq_{cvx} X_0\) is satisfied.

**Proof.** (a) Follows from the combination of Lemma 3.1 and Proposition 3.4 (b) Follows from the stationarity property which implies \(\hat{Z}_k^{vor} = \mathbb{E}(Z_k \mid \hat{Z}_k^{vor}) \leq_{cvx} Z_k, k = 1 : n\) and Proposition 3.4.

4 Dual quantization of ARCH models with truncated noise

To approximate the sequence \((\mu_k)_{k=0:n}\) of marginal distributions of the ARCH model (2.18)

\[
X_{k+1} = X_k + \vartheta_k(X_k)Z_{k+1}
\]

driven by a white noise \((Z_k)_{k=1:n}\) we adopt an ARCH approximation \((\tilde{X}_k)_{k=0:n}\) based on the dual quantization of the ARCH

\[
\tilde{X}_{k+1} = \tilde{X}_k + \vartheta_k(\tilde{X}_k)\tilde{Z}_{k+1}
\]

(4.33)

where \((\tilde{Z}_k)_{k=1:n}\) is a bounded quasi-white noise satisfying (3.21). To be more precise we start from any approximation \(\tilde{X}_0 = g_0(X_0)\) of \(X_0\) (usually a Voronoi quantization of \(X_0\)) supported by a grid \(\Gamma_0\) with \(N_0\) points and we assume that each \(\tilde{X}_{k+1}\) is obtained from \(\tilde{X}_k\) by applying a dual quantization using a grid \(\Gamma_k = \{x_i^k, i = 1 : N_k\}\) with \(N_k\) points after the above ARCH step with truncated noise \(\tilde{Z}_{k+1}\). The resulting dynamics, starting from \(\tilde{X}_0\) reads

\[
\tilde{X}_{k+1} = \tilde{X}_k + \vartheta_k(\tilde{X}_k)\tilde{Z}_{k+1}\quad \text{and}\quad \tilde{X}_{k+1} = \text{Proj}_{\Gamma_{k+1}}^\text{del}(\tilde{X}_{k+1}, U_{k+1}), k = 0, \ldots, n-1.
\]

(4.34)

where \((U_k)_{k=1:n}\) is a sequence of independent random variables uniformly distributed on \([0,1]\) independent from \((Z_k, \tilde{Z}_k)_{k=1:n}\) and \(X_0\). When the assumption in Proposition 3.2 (or \(q = d\) and the assumption before Proposition 3.4) is satisfied by \((Z_{k+1}, \vartheta_k, \vartheta_k)_{k=0:n-1}\) or \((Z_{k+1}, \vartheta_k, \vartheta_k)_{k=0:n-1}\) and \(\tilde{X}_0 \leq_{cvx} X_0\), then \((\tilde{X}_k)_{k=0:n} \leq_{cvx} (X_k)_{k=0:n}\). Since each dual quantization step is convex order increasing, \((\tilde{X}_k)_{k=0:n}\) is, in general, not comparable to the original ARCH \((X_k)_{k=0:n}\). More precisely, when the assumption in Proposition 3.2 (or \(q = d\) and the assumption before Proposition 3.4) is satisfied by \((\tilde{Z}_{k+1}, \vartheta_k, \vartheta_k)_{k=0:n-1}\) and \(\tilde{X}_0 \leq_{cvx} X_0\), then \((\tilde{X}_k)_{k=0:n} \leq_{cvx} (\tilde{X}_k)_{k=0:n}\). This can be
checked by adapting the proof of Proposition 3.2 or Proposition 3.4, the functions $\Phi_k, \Psi_k$ functions associated with the ARCH model (4.33) are still convex. Let $\Phi'_k, \Psi'_k$ be defined by the backward induction: $\Phi'_n = \Phi_n$ and

$$\Psi'_k(x_{0:k}, u) = \mathbb{E}\phi_{k+1}(x_{0:k}, \text{Proj}_{\Gamma_{k+1}}^c (x_k + u\hat{Z}_{k+1}, U_{k+1})), \quad \Phi'_k(x_{0:k}) = \Psi'_k(x_{0:k}, \vartheta_k(x_k)), k = 0, \ldots, n-1.$$ 

Since, except in the scalar case $d = 1$, the convex property is not necessarily preserved through the dual quantization step, the convexity of the functions $\Psi'_k, \Phi'_k$ is not clear. Nevertheless, when $\Phi'_k \geq \Phi_{k+1}$, by convexity of $\Phi_{k+1}$, independence of $U_{k+1}$ and $\hat{Z}_{k+1}$ and Jensen’s inequality,

$$\Psi'_k(x_{0:k}, u) \geq \mathbb{E}(\phi_{k+1}(x_{0:k}, \text{Proj}_{\Gamma_{k+1}}^c (x_k + u\hat{Z}_{k+1}, U_{k+1}))) \geq \mathbb{E} \phi_{k+1}(x_{0:k}, x_k + u\hat{Z}_{k+1})$$

from which we can deduce, like in the proof of Proposition 3.4, that $\Phi'_k \geq \Phi_k$. Finally, $\mathbb{E} \Phi_n(\hat{X}_{0:n}) \geq \mathbb{E} \Phi_n(\hat{X}_{0:n})$.

Note that the fact that truncation and dual quantization have opposite effects in terms of convex order is not so bad for numerical purposes: the errors coming from these two approximations should, at least partially, compensate.

The monotonicity of the sequence $(\mu_k)_{k=0:n}$ for the convex order is preserved by the approximation: by Proposition 2.2, we have $\hat{X}_0 \leq_{cvx} \hat{X}_1 \leq_{cvx} \ldots \leq_{cvx} \hat{X}_n$. But this monotonicity property is guaranteed only if the laws of the r.v. $\hat{X}_k$ are computed exactly.

In this perspective, it is possible to choose a quasi-white noise $(\hat{Z}_k)_{k=1:n}$ satisfying (3.21) with the additional condition that each $\hat{Z}_k$, $k = 1, \ldots, n$, takes finitely many values, say $N_{k}^Z$. Such a quasi-white noise may be obtained by primal quantization of the original white noise $(\hat{Z}_k)_{k=1:n}$. Then we may calculate the $N_k \times N_{k+1}^Z$ possible values of $\tilde{X}_{k+1}$ to compute its distribution and then the one of $\hat{X}_{k+1}$. More precisely, if $\hat{X}_k$ and $\hat{Z}_{k+1}$ are respectively distributed according to $\sum_{i=1}^{N_k} \tilde{p}_i \delta_{X_i}$ and $\sum_{j=1}^{N_{k+1}^Z} \hat{q}_j \delta_{Z_j}$, then $\tilde{X}_{k+1}$ and $\hat{X}_{k+1}$ are respectively distributed according to

$$\sum_{i=1}^{N_k} \sum_{j=1}^{N_{k+1}^Z} \tilde{p}_i \hat{q}_j \delta_{\tilde{X}_i + \vartheta_k(x_i^k)Z_j} \text{ and } \sum_{i=1}^{N_k} \sum_{j=1}^{N_{k+1}^Z} \tilde{p}_i \hat{q}_j \delta_{\tilde{X}_i + \vartheta_k(x_i^k)(Z_j)}$$

$$\sum_{\ell=1}^{N_{k+1}} \sum_{i=1}^{N_k} \tilde{p}_i \hat{q}_{j_k} \delta_{\tilde{X}_i + \vartheta_k(x_i^k)Z_j} \mathbb{P}(\text{Proj}_{\Gamma_{k+1}}^c (X_i^k + \vartheta_k(x_i^k)Z_j, U_{k+1}) = x_{k+1}^k) \delta_{\tilde{X}_i + \vartheta_k(x_i^k)(Z_j)}.$$ 

While preserving the monotonicity of the sequence $(\tilde{X}_k)_{k=0:n}$ for the convex order, the dual quantization steps by mapping $\tilde{X}_{k+1}$ to $\hat{X}_{k+1} = \text{Proj}_{\Gamma_{k+1}}^c (\tilde{X}_{k+1}, U_{k+1})$ make possible to control the size/cardinality $N_{k+1}$ of the support of the approximation of the law of $\tilde{X}_{k+1}$ hence avoiding its explosion with $k$: in contrast, the cardinality of the support of $\hat{X}_{k+1}$ can be equal to $\prod_{\ell=0}^{k+1} N_{\ell}$ when $X_0 = \hat{X}_0$.

In the scalar case $q = d = 1$, the finite support property of the truncated white noise is not needed to compute the laws of the r.v. $\hat{X}_k$. Let us now explain this for the truncated noise $Z_k 1_{\{Z_k \in [a_k, b_k]\}}$ before giving general error estimations.
4.1 Scalar setting $d = q = 1$

Assume for simplicity that all the $\vartheta_k$ are (strictly) positive. Assume that we have closed forms for the c.d.f. and the partial first moment of the white noise:

$$F_k(z) := \mathbb{P}(Z_k \leq z) \quad \text{and} \quad K_k(z) := \mathbb{E} Z_k 1\{Z_k \leq z\}, \quad k = 1, \ldots, n$$

and also for the starting value of the chain $X_0$, denoted by $F_0$ and $K_0$ respectively. Denote by $F_k(z-)$ the left-hand limit at point $z$ of the function $F_k$. We may proceed by (centered) truncation of the noise by considering, for every $k = 1, \ldots, n$

$$\hat{Z}_k = Z_k 1\{Z_k \in [\alpha_k, \beta_k]\}, \quad \alpha_k < 0 < \beta_k$$

(so that $\mathbb{E} Z_k 1\{Z_k \in [\alpha_k, \beta_k]\} = 0$). The transition kernels $\tilde{F}_k(x, dy)$ of the chain $(\hat{X}_k)_{k=0}^n$ have a c.d.f. $\hat{F}_k(x,u)$ and partial first moment $\hat{K}_k(x, u)$ functions given by

$$\tilde{F}_k(x, u) := \mathbb{P}(\hat{X}_{k+1} \leq u \mid \hat{X}_k = x) = 1_{\{x \leq u\}} (1 - F_{k+1}(\beta_{k+1}) - F_k(\alpha_{k+1} - )) + F_{k+1}(\alpha_{k+1} + \frac{u - x}{\vartheta_k(x)} \wedge \beta_{k+1}) - F_{k+1}(\alpha_{k+1} - ) \quad (4.35)$$

and

$$\hat{K}_k(x, u) := \mathbb{E}(\hat{X}_{k+1} 1\{\hat{X}_{k+1} \leq u\} \mid \hat{X}_k = x) = x \hat{F}_k(x, u) + \vartheta_k(x)(K_{k+1}(\alpha_{k+1} + \frac{u - x}{\vartheta_k(x)} \wedge \beta_{k+1}) - K_{k+1}(\alpha_{k+1})). \quad (4.36)$$

**Fixed grids.** For the Voronoi quantization at time $k = 0$ associated with the grid $(x_i^0)_{i=1}^{N_0}$, the weights are

$$\hat{p}_i^0 = \mathbb{P}(\hat{X}_0 = x_i^0) = F_0(x_{i+1/2}) - F_0(x_{i-1/2}), \quad i = 1 : N_0,$$

with $x_{1/2}^0 = -\infty$, $x_{N_0+1/2}^0 = +\infty$ and $x_{i+1/2}^0 = \frac{x_i^0 + x_{i+1}^0}{2}$, $i = 1 : N_0 - 1$.

Now, if the quantization grids $\Gamma_k = \{x_1^k, \ldots, x_{N_k}^k\}$, $k = 1, \ldots, n$, that dually quantize $\hat{X}_k$ (from time 1) are supposed to be fixed, then one may directly compute the transition weights of $\hat{X}_k$, $k = 0, \ldots, n$, using Equation (A.49) from the Appendix A.2, keeping in mind that we have above closed form formulas for $\tilde{F}_k$ and $\hat{K}_k$:

$$\hat{\pi}_{ij}^k = \mathbb{P}(\hat{X}_{k+1} = x_{j+1}^k \mid \hat{X}_k = x_i^k)$$

$$= \frac{\hat{K}_k(x_i^k, x_{j+1}^k) - \hat{K}_k(x_i^k, x_{j-1}^k) - x_{j+1} x_{j-1} (\tilde{F}_k(x_i^k, x_{j+1}^k) - \tilde{F}_k(x_i^k, x_{j-1}^k))}{x_{j+1} - x_{j-1}}$$

$$+ \frac{x_{j+1} (\tilde{F}_k(x_i^k, x_{j+1}^k) - \tilde{F}_k(x_i^k, x_{j+1}^k)) - (\hat{K}_k(x_i^k, x_{j+1}^k) - \hat{K}_k(x_i^k, x_{j+1}^k))}{x_{j+1} - x_{j-1}}$$

and

$$\hat{p}_{i}^{k+1} = \mathbb{P}(\hat{X}_{k+1} = x_{j+1}^k) = \sum_{i=1}^{N_k} \hat{p}_i^k \hat{\pi}_{ij}^k, \quad j = 1 : N_{k+1}.$$
4.2 Error estimation

Implementation of the dual quantization of dimensions are respectively discussed in Appendix A.1 and Appendix A.2. For a convenient implementation of the dual Lloyd procedure (A.50) called dual Lloyd procedure. In particular if we assume that the grid \( \Gamma_k = \{ x_1^k, \ldots, x_{N_k}^k \} \) that dually quantizes \( \hat{X}_k \) (into \( \hat{X}_k \)) has been optimized, then the c.d.f. \( F_{\hat{X}_{k+1}} \) and first partial moment function \( K_{\hat{X}_{k+1}} \) of \( \hat{X}_{k+1} \) are given by

\[
F_{\hat{X}_{k+1}}(u) = \sum_{j=1}^{N_k} \tilde{p}_j^k \tilde{F}_k(x_j^k, u) \quad \text{and} \quad K_{\hat{X}_{k+1}}(u) = \sum_{j=1}^{N_k} \tilde{p}_j^k \tilde{K}_k(x_j^k, u)
\]

respectively where \( \tilde{F}_k \) and \( \tilde{K}_k \) are given by (4.35) and (4.36). One defines the dual Lloyd mapping \( T_k \) by replacing \( F \) and \( K \) by \( F_{\hat{X}_{k+1}} \) and \( K_{\hat{X}_{k+1}} \) in (A.50) and implement the iterative procedure

\[
\Gamma_{k+1}^{[\ell+1]} = T_{k+1}(\Gamma_{k+1}^{[\ell]}), \quad \ell \geq 0, \quad \text{conv}(\Gamma_{k+1}^{[0]}) \supset \text{supp}(\mathbb{P}_{\hat{X}_{k+1}}).
\]

4.2 Error estimation

The algorithms permitting the optimization of primal and dual quantization grids in general dimensions are respectively discussed in Appendix A.1 and Appendix A.2. For a convenient implementation of the dual quantization of \( \hat{X}_k + \vartheta_k(X_k) \hat{Z}_{k+1} \), one needs to know the affine space spanned by the support of the distribution of this random vector. Let \( \hat{C}_{k+1} \) denote the covariance matrix of \( \hat{Z}_{k+1} \). If for each \( x \in \mathbb{R}^d \), the matrix \( \vartheta_k(x) C_{k+1} \vartheta_k^*(x) \) is positive definite (which implies that \( q \geq d \)), then this affine space is \( \mathbb{R}^d \).

**Theorem 4.1** Let \( (U_k)_{k=0:n} \) be a sequence of i.i.d. random variables uniformly distributed over \([0, 1]\) and let \( (Z_k, \hat{Z}_k)_{k=1:n} \) be an independent sequence of independent square integrable \( \mathbb{R}^{q \times q} \)-valued random vectors, independent of \( X_0 \) satisfying (3.21) and such that \( (Z_k)_{k=1:n} \) is a white noise. Let \( \hat{X}_0 = g_0(X_0) \) and let \( (\Gamma_k)_{k=1:n} \) be grids satisfying the consistency condition \( (G_F) \).

(a) General case. Then, for every \( k \in \{0, \ldots, n\} \),

\[
\| \hat{X}_k - X_k \|^2 \leq \prod_{i=1}^k (1 + q[\vartheta_{i-1}]_{\text{Lip}}^2) \| \hat{X}_0 - X_0 \|^2 + \sum_{\ell=1}^k \prod_{i=\ell+1}^k (1 + q[\vartheta_{i-1}]_{\text{Lip}}^2) \left( \| \vartheta_{\ell-1}(X_{\ell-1}) \|_2^2 \| Z_\ell - \hat{Z}_\ell \|_2^2 + \| \hat{X}_\ell - \hat{X}_\ell \|_2^2 \right).
\]

(b) Quantized innovation. Assume \( \hat{X}_0 = \text{Proj}_\Gamma^{\text{vor}}(X_0) \) with \( |\Gamma_0| = N_0 \) and \( \hat{Z}_k = \hat{Z}_k^{\text{vor}} = \text{Proj}_{\hat{\Gamma}_k}^{\text{vor}}(Z_k) \), \( k = 1, \ldots, n \), all are optimal quadratic Voronoi quantizations of \( Z_k \) at level \( N_k^2 \) \( = \text{card}(\Gamma_k^2) \). Assume \( X_0, Z_1, \ldots, Z_n \in L^{2+\eta}(\mathbb{P}) \). Then \( X_k \in L^{2+\eta}(\mathbb{P}) \) for every \( k \in \{0, \ldots, n\} \) and, for every \( k = 0, \ldots, n \),

\[
\| \hat{X}_k - X_k \|_2 \leq \left( (C_{d,\eta})^2 \prod_{i=1}^k (1 + q[\vartheta_{i-1}]_{\text{Lip}}^2) \frac{\sigma_{2+\eta}(X_0)}{N_0^{2/d}} \right) \frac{\sigma_{2+\eta}(Z_k)}{(N_k^2)^{2/q}} \frac{\sigma_{2+\eta}(\hat{X}_k)}{(N_k^2)^{2/d}} + \left( \tilde{C}_{d,\eta} \sigma_{2+\eta}(X_\ell) \right)^{1/2}.
\]
Remark 4.1 One has $\|\vartheta_{\ell-1}(X_{\ell-1})\|_2^2 \leq c(\vartheta_{\ell-1})(1 + \|X_{\ell-1}\|_2^2)$ with $\|X_{\ell-1}\|_2^2$ bounded from above according to (3.26). Moreover, if the $\tilde{Z}_k$ have diagonal covariance matrices, then, as emphasized in Remark 3.1, each factor $(1 + q[\vartheta_{i-1}]_{\text{Lip}})$ can be replaced by $(1 + [\vartheta_{i-1}]_{\text{Lip}}^2)$ which is smaller.

Proof. By Section 2.4.1 we know that both $(X_k)_{k=0:n}$ and $(\tilde{X}_k)_{k=0:n}$ are $\sigma(X_0, (Z_\ell, U_\ell)_{\ell=1:k})$-martingales so that, on the one hand, $\tilde{X}_k \preceq \tilde{X}_{k+1}$ for all $k = 0, \ldots, n-1$. On the other hand, their difference is also a martingale and one derives from the decomposition

$$\tilde{X}_{k+1} - X_{k+1} = \tilde{X}_k - X_k$$

$$+ \text{Proj}_{\text{G}^{\text{d}}}_{k+1} \left( \tilde{X}_{k+1}, U_{k+1} \right) - \tilde{X}_{k+1} + (\vartheta_k(\tilde{X}_k) - \vartheta_k(X_k))\tilde{Z}_{k+1} + \vartheta_k(X_k)(\tilde{Z}_{k+1} - Z_{k+1}).$$

that

$$\|\tilde{X}_{k+1} - X_{k+1}\|_2^2 = \|\tilde{X}_k - X_k\|_2^2$$

$$+ \|\text{Proj}_{\text{G}^{\text{d}}}_{k+1} \left( \tilde{X}_{k+1}, U_{k+1} \right) - \tilde{X}_{k+1} + (\vartheta_k(\tilde{X}_k) - \vartheta_k(X_k))\tilde{Z}_{k+1} + \vartheta_k(X_k)(\tilde{Z}_{k+1} - Z_{k+1})\|_2^2.$$

As r.v. $U_{k+1}$ is independent of $(\tilde{X}_{k+1}, \tilde{X}_k, X_k, Z_{k+1}, \tilde{Z}_{k+1})$, it follows from the dual stationarity property that

$$\|\tilde{X}_{k+1} - X_{k+1}\|_2^2 = \|\tilde{X}_k - X_k\|_2^2 + \|\text{Proj}_{\text{G}^{\text{d}}}_{k+1} \left( \tilde{X}_{k+1}, U_{k+1} \right) - \tilde{X}_{k+1}\|_2^2$$

$$+ \| (\vartheta_k(\tilde{X}_k) - \vartheta_k(X_k))\tilde{Z}_{k+1} + \vartheta_k(X_k)(\tilde{Z}_{k+1} - Z_{k+1})\|_2^2. \quad (4.38)$$

Moreover, by (3.21) and the independence of $(Z_{k+1}, \tilde{Z}_{k+1})$ and $(X_k, \tilde{X}_k)$, one has

$$\|(\vartheta_k(\tilde{X}_k) - \vartheta_k(X_k))\tilde{Z}_{k+1} + \vartheta_k(X_k)(\tilde{Z}_{k+1} - Z_{k+1})\|_2^2$$

$$= \|(\vartheta_k(\tilde{X}_k) - \vartheta_k(X_k))\tilde{Z}_{k+1}\|_2^2 + \|\vartheta_k(X_k)(\tilde{Z}_{k+1} - Z_{k+1})\|_2^2$$

$$\leq \|(\vartheta_k(\tilde{X}_k) - \vartheta_k(X_k))\|_2^2 \|\tilde{Z}_{k+1}\|_2^2 + \|\vartheta_k(X_k)\|_2^2 \|Z_{k+1} - \tilde{Z}_{k+1}\|_2^2. \quad (4.39)$$

It follows from (4.38), (4.39), the Lipschitz property of the functions $\vartheta_k$ and the inequality $\|\tilde{Z}_{k+1}\|_2^2 \leq \|Z_{k+1}\|_2^2 = q$ deduced from Condition (3.21) (ii) that

$$\|\tilde{X}_{k+1} - X_{k+1}\|_2^2 \leq \|\tilde{X}_k - X_k\|_2^2 \left(1 + q[\vartheta_{i-1}]_{\text{Lip}}^2\right) + \|\vartheta_k(X_k)\|_2^2 \|Z_{k+1} - \tilde{Z}_{k+1}\|_2^2$$

$$+ \|\tilde{X}_k - \tilde{X}_{k+1}\|_2^2.$$

Discrete time Gronwall’s lemma yields for every $k = 0, \ldots, n$,

$$\|\tilde{X}_k - X_k\|_2^2 \leq \prod_{i=1}^k \left(1 + q[\vartheta_{i-1}]_{\text{Lip}}^2\right)\|\tilde{X}_0 - X_0\|_2^2 + \sum_{\ell=1}^k \prod_{i=\ell+1}^k \left(1 + q[\vartheta_{i-1}]_{\text{Lip}}^2\right)$$

$$\times \left(\|\vartheta_{\ell-1}(X_{\ell-1})\|_2^2 \|Z_{\ell} - \tilde{Z}_{\ell}\|_2^2 + \|\tilde{X}_\ell - \tilde{X}_{\ell}\|_2^2\right). \quad (4.40)$$

(b) Optimality of the quantizations $\tilde{Z}_k$ imply that the $\tilde{Z}_k$ are Voronoi stationary which makes $(Z_k, \tilde{Z}_k)$ a martingale coupling hence satisfying (3.21) for every $k = 1, \ldots, n$. As $X_0 \in L_{\mathbb{R}^d}^{2+\eta}(\mathbb{P})$ and
\( Z_k \in L_{R_2}^{2+\eta} (\mathbb{P}) \) for \( k = 1, \ldots, n \), the Voronoi (primal) non-asymptotic version of Zador’s Theorem (see Theorem \([A.1 \, b] \) in Appendix \([A.1] \)) implies that

\[
\| X_0 - \tilde{X}_0 \|_2 \leq C_{d,\eta}^{\text{vor}} (X_0) N_0^{-1/d} \quad \text{and} \quad \| Z_k - \tilde{Z}_k \|_2 \leq C_{d,\eta}^{\text{vor}} (Z_k) (N_k^2)^{-1/q}, \quad k = 1, \ldots, n,
\]

where \( C_{d,\eta}^{\text{vor}} \) is a positive real constant only depending on the dimension \( q \) and \( \eta > 0 \). Moreover, for every \( k = 1, \ldots, n \), the random variables \( \tilde{X}_k \) are compactly supported. Hence, owing to the dual form of Zador’s Theorem (see Appendix \([A.2 \, \text{Theorem } A.2(b)] \)), there exists a real constant \( C_{d,\eta}^{\text{del}} \in (0, +\infty) \) such that, for every \( k = 1, \ldots, n \),

\[
\| \tilde{X}_k - \tilde{X}_k \|_2 \leq C_{d,\eta}^{\text{del}} (\tilde{X}_k) N_k^{-1/d}.
\]

Plugging these bounds into (4.40) completes the proof. \( \square \)

**Application to the Euler scheme of a Brownian diffusion** The Euler scheme (2.19) is an ARCH with \( v_k(x) = \sqrt{\frac{T}{n}} \vartheta(t_k, x) \). If we assume that the diffusion function \( \vartheta(t, x) \) is Lipschitz in \( x \) uniformly in \( t \in [0, T] \) with constant \( [\vartheta]_{\text{Lip}} \), then \( \max_{0 \leq k \leq n-1} [\vartheta_k]_{\text{Lip}} \leq \sqrt{\frac{T}{n}} [\vartheta]_{\text{Lip}} \). As a consequence, under the assumptions of the above claim (b), for every \( k = 0, \ldots, n \),

\[
\| \tilde{X}_k - \tilde{X}_k \|_2 \leq \left( C_{d,\eta}^{\text{vor}} (\tilde{X}_0) \sigma_{d+\eta}^2 \frac{(\tilde{X}_0)^2}{N_0^2/d} \right) \left[ 1 + C_{d,\eta}^{\text{del}} (\tilde{X}_k) \right] \right)^{1/2}.
\]

Note that, setting \( c(\vartheta) = \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \frac{\| \vartheta(t, x) \|_{\text{Lip}}}{1 + |x|^2} \) \( c_{\text{Fr}} (\vartheta) = \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \frac{\| \vartheta(t, x) \|_{\text{Fr}}}{1 + |x|^2} \), we have \( \max_{0 \leq k \leq n-1} c_{\text{Fr}} (\vartheta_k) \leq \frac{T}{n} c_{\text{Fr}} (\vartheta) \) so that, according to (3.26),

\[
\| \vartheta(t_k, \tilde{X}_k) \|_2^2 \leq c(\vartheta) c_{\text{Fr}} (\vartheta) \| X_0 \|_2^2.
\]

**References**

[1] **Alfonsi, A. Corbetta J. and Jourdain, B.** (2017). Sampling of probability measures in the convex order by Wasserstein projection, arXiv:1709.05287v2.

[2] **Alfonsi, A. Corbetta J. and Jourdain, B.** (2018). Sampling of one-dimensional probability measures in the convex order and computation of robust option price bounds, accepted in **International Journal of Theoretical and Applied Finance**.

[3] **Backhoff-Veraguas J. and Pammer, G.** (2019). Stability of martingale optimal transport and weak optimal transport, arXiv:1904.04171.

[4] **Baker, D.** (2012). Martingales with specified marginals, PhD, Université Pierre et Marie Curie (Sorbonne-Université), Paris, France.

[5] **Beiglböck, M. Cox, A. and Huesmann, M.** (2017). Optimal transport and Skorokhod embedding. **Invent. Math.**, **208**(2):327–400.
[6] Beiglböck, M. Henry-Labordère, P. and Penkner, F. (2013). Model-independent bounds for option prices - a mass transport approach. Finance Stoch., 17(3):477–501.

[7] Beiglböck, M. and Juillet, N. (2016). On a problem of optimal transport under marginal martingale constraints. Ann. Probab., 44(1):42–106.

[8] Beiglböck, M. Nutz, M. and Touzi, N. (2017). Complete duality for martingale optimal transport on the line. Ann. Probab., 44(1):3038-3074.

[9] Campi, L. Laachir, I. and Martini, C. (2017). Change of numeraire in the two-marginals martingale optimal transport problem. Finance Stoch., 21(2):471-486.

[10] De March, H. (2018). Entropic approximation for multi-dimensional martingale optimal transport, arXiv 1812.11104.

[11] De March, H. and Touzi, N. (2017). Irreducible convex paving for decomposition of multi-dimensional martingale transport plans, arXiv 1702.08298.

[12] Dolinsky, Y. and Soner, H.M. Robust hedging and martingale optimal transport in continuous time. Probab. Theory Relat. Fields, 160:391–427.

[13] Du, Q. Emelianenko, M. and Ju, L. (2006). Convergence of the Lloyd algorithm for computing centroidal Voronoi tessellations, SIAM Journal on Numerical Analysis, 44:102-119.

[14] Emelianenko, M. Ju, L. and Rand, A. (2008). Nondegeneracy and Weak Global Convergence of the Lloyd Algorithm in $\mathbb{R}^d$, SIAM Journal on Numerical Analysis, 46(3):1423-1441.

[15] Fadili, A. and Pagès, G. (2018). Functional convex order for stochastic differential equations and their approximation schemes. Technical report.

[16] Galichon, A. Henry-Labordère, P. and Touzi, N. (2014). A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. Ann. Appl. Probab., 24(1):312–336.

[17] Ghoussoub, N. Kim, Y.-H. and Lim, T. (2019). Structure of optimal martingale transport plans in general dimensions. Ann. Probab., 47(1):109-164.

[18] Graf, S. and Luschgy, H. (2000). Foundations of quantization for probability distributions, LNM 1730, Springer, Berlin, 230p.

[19] Guo, G. and Obloj, J. (2017). Computational Methods for Martingale Optimal Transport problems, arXiv:1710.07911.

[20] Henry-Labordère, P. (2019). (Martingale) optimal transport and anomaly detection with neural networks: a primal-dual algorithm, arXiv:1904.04340.

[21] Henry-Labordère, P. Tan X. and Touzi, N. (2016). An explicit martingale version of the one-dimensional Brenier’s theorem with full marginals constraint. Stochastic Process. Appl., 126(9):2800–2834.

[22] Henry-Labordère, P. and Touzi, N. (2016). An explicit martingale version of the one-dimensional Brenier theorem. Finance Stoch., 20(3):635–668.

[23] Hirsch, F Profeta, C Roynette, B. and Yor, M (2011). Peacocks and associated martingales, with explicit constructions. Bocconi & Springer Series, 3. Springer, Milan; Bocconi University Press, Milan, 2011. xxxii+384 pp.

[24] Hobson, D. (1998). Robust hedging of the lookback option. Finance Stoch., 2:329–347.

[25] Hobson, D. and Klimmek, M. (2015). Robust price bounds for the forward starting straddle. Finance Stoch., 19(1):189–214.

[26] Hobson, D. and Neuberger, A. (2012). Robust bounds for forward start options. Math. Finance, 22(1):31–56.

[27] Jourdain, B. and Margheriti, W. (2018). A new family of one-dimensional of martingale couplings, arXiv 1808.01390.
A Background on (optimal) primal and dual vector quantization

In what follows $\mathbb{R}^d$ is supposed to be equipped with the canonical Euclidean norm. For a more general presentation dealing with any norm, see [18] for Voronoi quantization and [37] for Delaunay quantization.

A.1 Optimal Voronoi quantization (primal)

Let $\Gamma = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ denote a finite subset of size $N$, that we will call grid. To such a grid we can associate Voronoi diagrams $(C_i(\Gamma))_{i=1}^N$ that are Borel partitions of $\mathbb{R}^d$ satisfying

$$\forall i \in \{1, \ldots, N\}, \quad C_i(\Gamma) \subset \{\xi \in \mathbb{R}^d : |\xi - x_i| \leq \min_{1 \leq j \leq N} |\xi - x_j|\}.$$
There is a one-to-one correspondence between Voronoi diagrams and Borel nearest neighbour projections, denoted $\text{Proj}_r$, defined as Borel mappings from $\mathbb{R}^d \to \Gamma$ such that

$$\forall \xi \in \mathbb{R}^d, \ |\xi - \text{Proj}_r(\xi)| = \text{dist}(\xi, \Gamma).$$

Indeed, if $\text{Proj}_r$ is a Borel nearest neighbour projection, then $(\{\text{Proj}_r = x_i\}_{i=1:N}$ is a Voronoi diagram and, conversely, for any Voronoi diagram $(C_i(\Gamma))_{i=1:N}$,

$$\text{Proj}_r = \sum_{i=1}^N x_i 1_{C_i(\Gamma)} \quad (A.41)$$

is a Borel nearest neighbour projection. The elements $(C_i(\Gamma))$ of a Voronoi diagram are called Voronoi cells.

We define a Voronoi or primal $\Gamma$-quantization of an $\mathbb{R}^d$-valued random vector $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ by

$$\tilde{X} = \tilde{X}^{\Gamma} := \text{Proj}_r(X) \quad (A.42)$$

whose distribution is given by $\tilde{\mu}^{\Gamma} = \mu \circ \text{Proj}^{-1}_r$ if $X$ is $\mu$-distributed.

If $\mu\left(\bigcup \partial C_i(\Gamma)\right) = 0$, then $\tilde{\mu}^{\Gamma}$ is unique and all $\Gamma$-quantizations are $\mathbb{P}$-a.s. equal. The mean $L^p$-quantization error induced by $\Gamma$ is defined by

$$e_p(\Gamma, \mu) = e_p(\Gamma, X) = \left\|\text{dist}(X, \Gamma)\right\|_p = \left\|X - \tilde{X}^{\Gamma}\right\|_p$$

for any Voronoi quantization of $X$ (still $\mu$-distributed).

Then one defines, for $p > 0$ and an integer $N \geq 1$, the minimal mean $L^p$-quantization error at level $N$ by

$$e_{p,N}(\mu) = e_{p,N}(X) = \inf_{\Gamma:|\Gamma| \leq N} e_p(\Gamma, X).$$

If $\mu$ has a finite $p$th moment, then the above infimum is in fact a minimum and any optimal grid $\Gamma^{(N)}$ solution to the above minimization problem has a full size $N$ provided the support of $\mu$ has at least $N$ elements (see e.g. Theorem 4.12 in [18] or Theorem 5.1 in [31] among others). The random vector $\hat{X}^N = \tilde{X}^{\Gamma^{(N)}}$ is called an optimal $L^p$-quantization of $X$. Moreover, the optimal quantization $\hat{X}^N$ is $\mathbb{P}$-a.s. uniquely defined since one always has $\mu\left(\bigcup \partial C_i(\Gamma^{(N)})\right) = 0$ (see Theorem 4.2 in [18]).

Finally, in the quadratic case $p = 2$, any optimal quantization grid $\Gamma^{(N)}$ at level $N$ and its quantization $\hat{X}^N$ satisfy (see e.g. [18], [33] or [31], Proposition 5.1 among others) a stationarity (or self-consistency) equation reading

$$\hat{X}^N = \mathbb{E}\left(X \mid \hat{X}^N\right). \quad (A.43)$$

Quantization rates

**Theorem A.1 (Zador Theorem and Pierce Lemma for primal quantization) (a) ZADOR’S THEOREM FOR (PRIMAL) VORONOI QUANTIZATION: Let $X \in L_{\mathbb{R}^d}^{p+\eta}(\Omega, \mathcal{A}, \mathbb{P})$, $p, \eta > 0$, be a random vector with distribution $\mathbb{P}_X = \varphi \lambda_d + \nu_x$ where $\lambda_d$ denotes the Lebesgue measure and $\nu_x$ denotes the singular part of the distribution. Then

$$\lim_{N \to +\infty} N^{\frac{1}{2} + \frac{1}{2p}} e_{p,N}(X) = \tilde{J}^{\text{vor}}_{d,p} \left(\int_{\mathbb{R}^d} \varphi \frac{d}{d^d} d\lambda_d\right)^{\frac{1}{2} + \frac{1}{2p}}$$

where $\tilde{J}^{\text{vor}}_{d,p} = \inf_{N \geq 1} N^{\frac{1}{2} + \frac{1}{2p}} d_{p,N}(\mathcal{U}(\{0, 1\}^d))$. When $d = 1$, $\tilde{J}^{\text{vor}}_{1,p} = \frac{1}{(2p+1)^{\eta/\eta}}$.

(b) NON-ASYMPTOTIC BOUND (PIERCE LEMMA): Let $p, \eta > 0$. For every dimension $d \geq 1$, there exists a real constant $\tilde{C}^{\text{vor}}_{d,\eta,p} > 0$ such that, for every random vector $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$,

$$e_{p,N}(X) \leq \tilde{C}^{\text{vor}}_{d,\eta,p} N^{-\frac{1}{2} - \frac{\eta}{p}}$$

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where, for every $r > 0$, $\sigma_r(X) = \inf_{a \in \mathbb{R}^d} \|X - a\|_r \leq +\infty$.

**Remark.** Note that if we consider quadratic optimal product quantizations at levels $N \geq 1$, that is solutions – which exist – to the minimization problems

$$
e_{2,N}^{\text{prod}}(\mu) = \inf \{ e_2(\Gamma, X), \Gamma = \Gamma^1 \times \cdots \times \Gamma^d, |\Gamma| \leq N \}, \; N \geq 1,$$

then, such optimal product grids are still rate optimal and satisfy a universal non-asymptotic Pierce bound, see e.g. [31]

**Lloyd’s algorithm ($p = 2$)**

Let $\mu$ be a probability distribution supported by at least $N$ points of $\mathbb{R}^d$, $N \geq 1$. The Lloyd procedure at level $N$ provides a systematic way to make the quadratic primal quantization error decrease. Let $X \in L_2^2(\Omega, A, \mathbb{P})$ be $\mu$-distributed. Starting from a grid $\Gamma^{[0]} \subset \mathbb{R}^d$ with size $N$, we set for every $k \geq 0$,

$$\Gamma^{[k+1]} = \mathbb{E} (X | \hat{\Gamma}^{[k]})(\Omega) \quad \text{where} \quad \hat{\Gamma}^{[k]} = \text{Proj}_{\Gamma^{[k]}}(X).$$

One checks that $\Gamma^{[k]}$ has size $N$ for every $k \geq 0$ and that

$$\|X - \hat{\Gamma}^{[k+1]}\|_2 = \|\text{dist}(X, \Gamma^{[k+1]})\|_2 \leq \|X - \mathbb{E} (X | \hat{\Gamma}^{[k]})\|_2$$

$$= \left(\|X - \hat{\Gamma}^{[k]}\|_2^2 - \|\mathbb{E} (X | \hat{\Gamma}^{[k]}) - \hat{\Gamma}^{[k]}\|_2^2\right)^{1/2}$$

$$\leq \|X - \hat{\Gamma}^{[k]}\|_2.$$

This does not provide a proof that $\Gamma^{[k]}$ converges to an optimal grid $\Gamma_N$ as $k \to +\infty$. Some results in that direction have been obtained when $X$ has a compact support and the initial grid $\Gamma^{[0]}$ is chosen in an appropriate way (the so-called splitting method). For recent results on this topic, we refer to [13, 14] or [40] and the references therein.

Indeed, as presented, the Lloyd procedure appears as a pseudo-algorithm since computing a conditional expectation is a non-trivial exercise, especially in higher dimension. In its original form, the field of application of Lloyd’s algorithm is mainly the one dimensional framework

**One dimensional setting ($d = 1$).** Assume that the c.d.f $F(x) = \mu(-\infty, x] = \mathbb{P}(X \leq x)$ and the partial first moment $K(\xi) = \int_{-\infty}^{\xi} \mu(dx) = \mathbb{E} X 1_{(X \leq \xi)}$ both have closed form expressions(such is the case for the normal or the exponential distributions for example). In a one-dimensional setting the Voronoi cells of a grid $\Gamma = \{x_1, \ldots, x_N\} \subset \mathbb{R}$ with size $N \geq 1$ are defined by

$$C_i(\Gamma) = (x_{i-1/2}, x_{i+1/2}], \; i = 1 : N$$

where $x_{1/2} = -\infty, x_{N+1/2} = +\infty$ and $x_{i+1/2} = \frac{x_{i+1/2} + x_{i-1/2}}{2}, \; i = 1 : N - 1$.

Then, if we denote by $\Gamma^{[\ell]} = \{x_{1}^{[\ell]}, \ldots, x_{N}^{[\ell]}\}$ the elements of the grid $\Gamma^{[\ell]}$ labelled in an increasing order (i.e. so that $x_{1}^{[\ell]} < \cdots < x_{N}^{[\ell]}$), the procedure reads

$$x_{i}^{[\ell+1]} = \frac{K(x_{i+1/2}^{[\ell]}) - K(x_{i-1/2}^{[\ell]})}{F(x_{i+1/2}^{[\ell]}) - F(x_{i-1/2}^{[\ell]}), \; i = 1 : N.}$$

(A.44)

If the distribution $\mu$ has a non-piecewise affine log-concave density, then it is proved in [29] that $x^{[\ell]}$ converges toward $x^{[\infty]}$, unique stationary $N$-quantizer of $\mu$, at an exponential rate. Then, one computes the weights of this quantizer by

$$p_{i}^{[\infty]} := \mathbb{P}(X \in C_i(\Gamma^{[\infty]})) = F(x_{i+1/2}^{[\infty]}) - F(x_{i-1/2}^{[\infty]}), \; i = 1 : N.$$
Higher dimensional setting: the \textit{k-means algorithm}. In higher dimensions no closed forms are available for the Lloyd algorithm and a randomized version of the procedure is required for an easy implementation (in low dimension $d = 2$ or $3$ the algorithm can be implemented by computing all the integrals by cubature formulas using the QHull library \cite{www.qhull.org} (see also \cite{30}). This randomized (approximate) avatar of the original procedure is also known in datascience as the \textit{k-means algorithm}. One simulates a large sample of the distribution of $X$ and replaces the distribution $\mu = \mathbb{P}$ of $X$ by the induced empirical measure $\hat{\mu} = \frac{1}{M} \sum_{m=1}^{M} \delta_{X_m}$. Then, the above recursion (A.44) reads

$$x_i^{[\ell+1]} = \frac{\sum_{1 \leq m \leq M} X_m 1_{(X_m \in C_i(\Gamma^{[\ell]}))}}{\text{card}\{1 \leq m \leq M : X_m \in C_i(\Gamma^{[\ell]})\}}, \quad i = 1 : N, \ell \geq 1,$$

and the weights are given by $p_i^{[\ell]} = \frac{\text{card}\{1 \leq m \leq M : X_m \in C_i(\Gamma^{[\ell]})\}}{M}$, $i = 1 : N$ (can be computed at the end of the procedure). This approach based on a Monte Carlo simulation is much more time consuming to compute optimal quantization grids.

### A.2 Optimal Delaunay (dual) quantization

Let $X : (\Omega, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d$ be a random vector lying in $L^\infty(\mathbb{P})$. We will assume for convenience in what follows that the support of its distribution $\mu = \mathbb{P}_X$ spans $\mathbb{R}^d$ as an affine space. Otherwise one may always consider the affine space $A_\mu$ spanned by $\text{supp}(\mu)$ and reduce the problem to the former framework by combining a translation with a change of coordinates into an orthonormal basis of the vector space associated with $A_\mu$. Optimal dual (or Delaunay) quantization relies on the best approximation which can be achieved by a discrete random vector $\hat{X}$ that satisfies a certain stationarity assumption on the extended probability space $(\Omega \times \Omega_0, \mathcal{A} \otimes A_0, \mathbb{P} \otimes \mathbb{P}_0)$ with $(\Omega_0, A_0, \mathbb{P}_0)$ supporting a random variable uniformly distributed on $[0, 1]$. That is why we define, for $p \in [1, +\infty)$:

$$\forall N \geq d + 1, \quad d_{p,N}(X) = \inf_X \left\{ \|X - \hat{X}\|_p : \hat{X} : (\Omega \times \Omega_0, \mathcal{A} \otimes A_0, \mathbb{P} \otimes \mathbb{P}_0) \to \mathbb{R}^d, \text{card}\hat{X}(\Omega \times \Omega_0) \leq N \text{ and } \mathbb{E}(\hat{X}|X) = X \right\}.$$

One checks that $d_{p,N}(X)$ only depends on the distribution $\mu$ of $X$ and can subsequently be denoted $d_{p,N}(\mu)$. One shows (see \cite{37}) that, for a given distribution $\mu$ on $(\mathbb{R}^d, \text{Bor}(\mathbb{R}^d))$,

$$d_{p,N}(\mu) = \inf \left\{ \|\Xi - \xi\|_p, (\Xi, \xi) : (\Omega_\Xi, \mathcal{A}, \mathbb{P}) \to \mathbb{R}^d \times \mathbb{R}^d, \Xi \sim \mu, \mathbb{E}(\xi | \Xi) = \Xi, \text{card}(Y(\Omega)) \leq N \right\}. \quad (A.46)$$

Then (see \cite{37}), one may show that such a definition is equivalent to

$$d_{p,N}(X) = \inf \left\{ \|\Delta_p(X; \Gamma)\|_p : \text{conv}(\text{supp}(\mu)) \subset \Gamma \subset \mathbb{R}^d, \text{card}(\Gamma) \leq N \right\}$$

where the \textit{local dual quantization functional} $\Delta_p$ reads on a given grid $\Gamma$ which contains an affine basis of $\mathbb{R}^d$ (or, equivalently, whose convex hull has a non-empty interior):

$$\Delta_p(\xi; \Gamma) = \inf_X \left\{ \left( \sum_{i=1}^{N} \lambda_i |\xi - x_i|^p \right)^{1/p} : (\lambda_i)_{i=1:N} \in [0, 1]^N \text{ and } \sum_{i=1}^{N} \lambda_i x_i = \xi, \sum_{i=1}^{N} \lambda_i = 1 \right\}.$$

When $p = 2$ (quadratic case), one has the following result about the zones where the infimum $\Delta_p(\xi; \Gamma)$ is attained: if the grid $\Gamma \subset \mathbb{R}^d$ contains an affine basis with its points are in \textit{general position} – none of its subset of size $d + 1$ lies on the same sphere – then it admits a unique Delaunay triangulation in the following sense (see \cite{41} or, for our setting, Proposition 6 and Theorem 4 in \cite{37}):

\footnote{4. www.qhull.org}
1. For every $\xi \in \text{conv}(\Gamma)$, there exist a unique $I = I(\xi) \subset \{1, \ldots, N\}$ of cardinality $d + 1$ such that

   (a) $(x_i)_{i \in I}$ is an affine basis,
   
   (b) $\text{conv}\{x_i, i \in I\} \cap \{x_j, j \in I^c\} = \emptyset$ (so-called Delaunay property),
   
   (c) $\Delta_p(\xi; \Gamma)$ is attained as a minimum at an $N$-tuple $\lambda_1, \ldots, \lambda_N$ satisfying the constraints with $\lambda_i = 0$ if $i \notin I$.

2. If $I = I(\xi)$ as above for some $\xi \in \text{conv}(\Gamma)$, then for every $\xi' \in \text{conv}(x_i, i \in I(\xi))$, $I(\xi') = I(\xi)$.

A collection of simplexes $(x_i)_{i \in I}$ where $I$ is admissible for some $\xi \in \text{conv}(\Gamma)$ is called a triangulation of $\Gamma$. When the points of $\Gamma$ are not in general position, several subsets $I$ of $\{1, \ldots, N\}$ can satisfy condition 1. However, if such is the case, $I$ remains admissible for all points $\xi$ in $\text{conv}(x_i, i \in I)$. Thus, several triangulations may exist, each one giving raise to its own splitting operator (see (A.47) below). A typical example is a rectangle split by one of its two diagonals which yields two triangulations, one for each diagonal.

It was proved in [37] that for such grids, we can construct a dual quantization projection (or splitting operator) which is the counterpart of the nearest neighbour projection for Voronoi quantization. This operator maps the random variable $X$ randomly to the vertices of the Delaunay “hyper-triangle” (in fact a $d$-simplex) in which $X$ falls (see Figure 1 further on), where the probability of mapping/projecting $X$ to a given vertex $t_i$ is determined by the $i$-th barycentric coordinate of $X$ in the (non-degenerated) “hyper-triangle” (or $d$-simplex) $\text{conv}\{t_j : j = 1, \ldots, d + 1\}$. When $p \neq 2$, an extension of the notion of Delaunay “triangulation”can still be defined although slightly more involved (similarly, the Voronoi cells are no longer convex when $p \neq 2$). We refer again to [37] for details.

Mathematically speaking, let $(D_k(\Gamma))_{1 \leq k \leq m}$ be a Delaunay partition of the convex hull $\text{conv}(\Gamma)$ of $\Gamma$. Let us denote by $\lambda_i^k(\xi)$ the barycentric coordinates of $\xi$ in the triangle $D_k(\Gamma)$, with the convention $\lambda_i^k(\xi) = 0$ if $x_i \notin D_k(\Gamma)$. We define the dual (or Delaunay) projection operator – also called splitting operator – by

$$
\text{Proj}^{\text{del}}_{\Gamma}(\xi, u) = \sum_{k=1}^{m} \left[ \sum_{i=1}^{N} x_i \cdot \mathbf{1}_{\left\{ \sum_{j=1}^{i-1} \lambda_j^k(\xi) \leq u < \sum_{j=1}^{i} \lambda_j^k(\xi) \right\}} \right] \mathbf{1}_{D_k(\Gamma)}(\xi).
$$

(A.47)

Figure 1 – Voronoi (left) and Delaunay (right) projections for the realization $X(\omega) = \bullet$.

Note that in [37] this projection is denoted $J^u_\Gamma$ (this change is motivated by notational consistency). It is clear that, by construction,

$$
\forall \xi \in \text{conv}(\Gamma), \quad \int_0^1 \text{Proj}^{\text{del}}_{\Gamma}(\xi, u)du = \xi
$$
Moreover, it follows from (A.47), that

$$\Delta_p(\xi; \Gamma) = \left( \mathbb{E}_{\tilde{\mathbb{P}}_0}[\xi - \text{Proj}_{\Gamma}^{\text{dual}}(\xi, U)]^p \right)^{1/p},$$

where $U$ is defined on $(\Omega_0, \mathcal{A}_0, \tilde{\mathbb{P}}_0)$ with a $\mathcal{U}([0,1])$-distributed (so that the operator $\text{Proj}_{\Gamma}(\xi, u)$ is defined on this exogenous space). Then we define (on the product probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{\mathbb{P}})$) the dual (or Delaunay) quantization

$$\tilde{X}_{\Gamma, \text{dual}} := \text{Proj}_{\Gamma}^{\text{dual}}(X, U)$$

so that

$$\|\Delta_p(X; \Gamma)\|_p = \|X - \tilde{X}_{\Gamma, \text{dual}}\|_p \quad \text{and} \quad \mathbb{E}(\tilde{X}_{\Gamma, \text{dual}} \mid X) = X.$$

**Remark.** $L^p$-Dual quantization can be extended in a canonical way to $L^p(\tilde{\mathbb{P}})$-integrable random vectors by defining in a proper way the splitting operator outside the convex hull of the grid $\Gamma$. Unfortunately, as expected the dual stationarity property is not preserved by this extension.

**Optimal $L^p$-dual quantizers (existence).** It is shown in [37] that, for every integer $N \geq d + 1$, there exists at least one optimal dual quantizer $\Gamma^{(N), \text{dual}}$ at level $N \geq d + 1$ which achieves the infimum $d_{p,N}(X)$ and any such optimal dual quantizer has cardinality $N$. Furthermore, $d_{p,N}(X) \to 0$ as $N \to +\infty$. We recall below the main result on convergence rate of dual quantization for bounded random vectors established in [39].

**Theorem A.2 (Zador Theorem and Pierce Lemma for dual quantization)** (a) **Zador’s Theorem for Dual Quantization:** Let $X \in L^\infty_{\tilde{\mathbb{P}}_x}(\Omega, \mathcal{A}, \tilde{\mathbb{P}})$ be a bounded random vector with distribution $\tilde{\mathbb{P}}_x = \varphi.\lambda_d + \nu_x$ where $\lambda_d$ denotes the Lebesgue measure and $\nu_x$ denotes its singular component. Then, for every $p \in (0, +\infty)$,

$$\lim_{N \to +\infty} N^{\frac{1}{2}} d_{p,N}(X) = J_{d,p}^{\text{dual}} \left( \int_{\mathbb{R}^d} \varphi \frac{d\nu_x}{d\lambda_d} \right)^{\frac{1}{2} + \frac{1}{p}} \quad \text{where} \quad J_{d,p}^{\text{dual}} = \inf_{N \geq 1} N^{\frac{1}{2}} d_{p,N}(\mathcal{U}([0,1]^d)) \geq \tilde{J}_{d,p}^{\text{vor}}.$$  

When $d = 1$, $J_{1,p}^{\text{dual}} = \left( \frac{2^{p+1}}{p+2} \right)^{1/p}$ as $p \uparrow +\infty$.

(b) **Non-asymptotic bound (Pierce Lemma):** Let $p, \eta > 0$. For every dimension $d \geq 1$, there exists a real constant $C_{d,p}^{\text{dual}} > 0$ such that, for every random vector $X : (\Omega, \mathcal{A}, \tilde{\mathbb{P}}) \to \mathbb{R}^d$, $L^\infty(\tilde{\mathbb{P}})$-bounded,

$$d_{p,N}(X) \leq \tilde{C}_{d,p}^{\text{dual}} N^{-\frac{1}{2}} \sigma_{p+\eta}(X) \quad \text{(A.48)}$$

where, for every $r > 0$, $\sigma_r(X) = \inf_{a \in \mathbb{R}^d} \|X - a\|_r < +\infty$.

**Remark.** Note that claim (b) remains true if the support of $\tilde{\mathbb{P}}_x$ does not span $\mathbb{R}^d$ as an affine space, but $A_\mu$ with dimension $d'$. However, if such is the case (A.48) holds with $N^{-1/d'}$ so that $N^{-1/d}$ is suboptimal.

**Voronoi versus Delaunay quantization.** To illustrate the difference between Voronoi and Delaunay quantization (in the case $d = p = 2$), we compare in Figure 4 below the nearest neighbor projection and the dual quantization operator.

For a given grid $\Gamma \subset \mathbb{R}^d$, the nearest neighbor projection $\text{Proj}_{\Gamma}^{\text{vor}}$ maps $X(\omega)$ entirely to the generator of the Voronoi cell $C_\gamma(\Gamma)$ in which $X(\omega)$ falls. By contrast, the Delaunay random splitting operator $\text{Proj}_{\Gamma}^{\text{dual}}$ splits up the “weight” 1 of $X(\omega)$ across the vertices of the Delaunay triangle in which $X(\omega)$ falls. Since each vertex receives here a proportion according to the barycentric coordinate of the point $X(\omega)$ in that specific

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Delaunay triangle, this splitting operator fulfills a backward interpolation property, i.e. \( X(\omega) \) is given by a convex combination of the vertices of the Delaunay triangle.

Finally, this property also implies the intrinsic dual stationarity condition

\[
\mathbb{E}(\hat{X}^{\Gamma, \text{dual}}|X) = X.
\]

Note that, by contrast with regular Voronoi quantization where (A.43) holds for optimal quadratic grids, this dual stationarity equation is satisfied by any dual quantization grid.

**Remark.** For a comparison in one dimension, we give the example of optimal quantizations for \( U([0,1]) \). Following [37], Section 5.1, we derive for an optimal dual quantizer of \( U([0,1]) \) with size \( N \)

\[
\Gamma^{(N), \text{del}} = \left\{ \frac{i-1}{N} : i = 1, \ldots, N \right\}.
\]

On the other hand, it holds in the case of optimal Voronoi quantization

\[
\Gamma^{(N), \text{vor}} = \left\{ \frac{2i-1}{2N} : i = 1, \ldots, N \right\}
\]

so that an optimal Voronoi quantizer of size \( N \) is made up by the midpoints of an optimal Delaunay of size \( N + 1 \). Such a property does not hold for general distributions in arbitrary dimensions.

**One dimensional setting (quadratic case)**

**Dual weights attached to a fixed grid.** Let \( \mu \) be probability distribution such that \( \text{conv}(\text{supp}(\mu)) = [a, b], a, b \in \mathbb{R}, a < b \) and let \( \Gamma = \{x_1, \ldots, x_N\} \) be a grid of size \( N \) with \( x_1 = a \) and \( x_N = b \). We denote by \( F \) and \( K \) respectively, the c.d.f. and the first partial moment functions of \( \mu \).

Starting from the fact that, for every \( \xi \in [x_i, x_{i+1}] \), \( \xi = \frac{x_{i+1}-x_i}{x_{i+1}-x_i} x_i + \frac{x_i-x_{i+1}}{x_{i+1}-x_i} x_{i+1} \), we derive that, for every \( i = 1 : N \),

\[
p_i(\Gamma) = \int_{(x_{i-1}, x_i]} \frac{x_{i+1}-\xi}{x_{i+1}-x_i} \mu(d\xi) + \int_{(x_i, x_{i+1}]} \frac{\xi-x_i}{x_{i+1}-x_i} \mu(d\xi) = \frac{K(x_i) - K(x_{i-1}) - x_{i-1}(F(x_i) - F(x_{i-1}))}{x_i - x_{i-1}} + \frac{(F(x_{i+1}) - F(x_i))x_{i+1} - (K(x_{i+1}) - K(x_i))}{x_{i+1} - x_i}.
\]

**Optimizing a dual grid** One computes likewise \( d_{2,N}^2(\Gamma, \mu) \):

\[
d_{2,N}(\Gamma)^2 = \int_{[a,b]} \mu(d\xi) \int_0^1 du[|\xi - \text{Proj}^{\text{del}}(\xi, u)|]^2
\]

\[
= \sum_{i=1}^{N-1} \int_{(x_i, x_{i+1}]} \mu(d\xi) \left[ \frac{x_{i+1}-\xi}{x_{i+1}-x_i} (\xi - x_i)^2 + \frac{\xi-x_i}{x_{i+1}-x_i} (x_{i+1} - \xi)^2 \right] = \sum_{i=1}^{N-1} \int_{(x_i, x_{i+1}]} \mu(d\xi)(x_{i+1} - \xi)(\xi - x_i)
\]

\[
= \sum_{i=1}^{N-1} \left( (x_i + x_{i+1})(K(x_{i+1}) - K(x_i)) - x_i x_{i+1}(F(x_{i+1}) - F(x_i)) \right) - \int \xi^2 \mu(d\xi).
\]

Then, one shows that, viewed as a function of the \( N \)-tuple \( x = (x_1, \ldots, x_N) \), the mapping \( x \mapsto d_{2,N}^2(x, \mu) \) is continuously differentiable when \( F \) is continuous and differentiable on the set of vectors \( x \) with all coordinates outside the at most countable set of discontinuities of \( F \) otherwise with,

\[
\frac{\partial d_{2,N}(x, \mu)^2}{\partial x_i} = K(x_{i+1}) - K(x_{i-1}) - \left[ x_{i+1}(F(x_{i+1}) - F(x_i)) + x_{i-1}(F(x_i) - F(x_{i-1})) \right]
\]
with the convention $F(x_0) = F(x_1)$ and $F(x_{N+1}) = F(x_N)$.

As any optimal $N$-tuple satisfies $\nabla d_{2,N}^2(x,\mu) = 0$, elementary computations show that this equation reads

$$x = T(x) = (T_1(x), \ldots, T_N(x))$$

where the mapping $T$, defined from the simplex $S_{a,b} = \{a = x_1 < x_2 < \cdots < x_N\}$ onto it, is given by

$$T_i(x) = \frac{K(x_{i+1}) - K(x_i) - (x_{i+1} - x_i)(F(x_{i+1}) - F(x_i))}{F(x_{i+1}) - F(x_{i-1})}$$

$$+ \frac{K(x_i) - K(x_{i-1}) + (x_i - x_{i-1})(F(x_i) - F(x_{i-1}))}{F(x_{i+1}) - F(x_{i-1})}, \ i = 1 : N,$$

(A.50)

still with the above convention.

From this fixed point equality, one can devise an iterative fixed point procedure which can be seen as the counterpart of Lloyd I procedure for dual quantization:

$$x^{[\ell + 1]} = T(x^{[\ell]}), \ \ell \geq 0, \ x^{[0]} \in S_{a,b}.$$  

(A.52)

Although it turns out to be quite efficient with (truncated) usual distributions like normal, exponential, $\gamma$ distributions, no theoretical result is available yet to prove its convergence (except for the uniform distribution on the unit interval which is of no practical interest). In particular we have not yet a counterpart of Kieffer’s theorem (see [23]) which proves the exponentially fast convergence of the one dimensional regular “Voronoi” Lloyd procedure for non-piecewise affine log-concave distributions.

**Algorithmic aspects in higher dimensions (quadratic setting)**

For higher dimensional numerical aspects, we refer to [38] where two stochastic algorithms have been devised to compute optimal dual quantization grids in the spirit of the randomized avatar of Lloyd I (fixed point method) and CLVQ algorithms (stochastic gradient descent) respectively. Figure 2 displays three examples of dual quantization of $2D$-random vectors.

![Dual Quantizations](image)

Figure 2 – Dual quantizations ($d = 2$). Left: $U([0,1]^2)$, $N = 16$. Middle: truncated $N(0;\mathcal{I}_2)$, $N = 250$. Right: truncated law of $(W_1, \sup_{t \in [0,1]} W_t)$, $W$ standard Brownian motion, $N = 250$ (with B. Wilbertz).