PARABOLIC CURVES FOR DIFFEOMORPHISMS IN \((\mathbb{C}^2, 0)\)

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Abstract. We give a simple proof of the existence of parabolic curves for tangent to the identity diffeomorphisms in \((\mathbb{C}^2, 0)\) with isolated fixed point.

1. Introduction

Let \(F\) be a tangent to the identity diffeomorphism of \((\mathbb{C}^n, 0)\). A parabolic curve for \(F\) is an injective holomorphic map \(\varphi: \Omega \to \mathbb{C}^n\), where \(\Omega\) is a simply connected domain in \(\mathbb{C}\) with \(0 \in \partial \Omega\) such that

1. \(\varphi\) is continuous at the origin, and \(\varphi(0) = 0\).
2. \(F(\varphi(\Omega)) \subset \varphi(\Omega)\) and \(F^k(p)\) converges to 0 when \(k \to +\infty\), for \(p \in \varphi(\Omega)\).

We say that \(\varphi\) is tangent to \([v]\) if \([\varphi(\zeta)] \to [v]\) when \(\zeta \to 0\). Let us write

\[ F(z) = z + P_k(z) + P_{k+1}(z) + \cdots \]

where \(P_j\) is a \(n\)-dimensional vector of homogeneous polynomials of degree \(j\), and \(P_k(z) \not\equiv 0\). A characteristic direction for \(F\) is a point \([v]\) such that \(P_k(v) = \lambda v\), for some \(\lambda \in \mathbb{C}\); it is nondegenerate if \(\lambda \neq 0\). The integer \(\text{ord}(F) := k \geq 2\) is the tangency order of \(F\) at 0.

The following theorem is analogous to Briot and Bouquet’s theorem \([3]\) for diffeomorphisms of \((\mathbb{C}^n, 0)\).

Theorem 1.1 (Hakim \([6]\)). Let \(F\) be a tangent to the identity germ of diffeomorphism of \((\mathbb{C}^n, 0)\). For any nondegenerate characteristic direction \([v]\) there exist \(\text{ord}(F) - 1\) disjoint parabolic curves tangent to \([v]\) at the origin.

When \(n = 2\), Abate proved that the nondegeneracy condition can be dismissed.

Theorem 1.2 (Abate \([1, 2]\)). Let \(F\) be a tangent to the identity germ of diffeomorphism of \((\mathbb{C}^2, 0)\) such that 0 is an isolated fixed point. Then there exist \(\text{ord}(F) - 1\) disjoint parabolic curves for \(F\) at the origin.

This theorem is analogous to Camacho-Sad’s theorem \([4]\) of existence of invariant curves for holomorphic vector fields. We show in this note that the analogy is deeper enough to prove theorem \([1, 2]\) in a simple way starting with Hakim’s theorem.

2. Exponential Operator and Blow-up Transformation

Let \(\mathring{\mathcal{X}}_2(\mathbb{C}^2, 0)\) be the module of formal vector fields \(X = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}\) of order \(\geq 2\), i.e., \(\min\{\nu(a), \nu(b)\} \geq 2\). We denote by \(\mathring{\text{Diff}}_1(\mathbb{C}^2, 0)\) the group of tangent to the identity formal diffeomorphisms \(F(x, y) = (x + p(x, y), y + q(x, y))\) where
min{ν(p(x, y)), ν(q(x, y))} ≥ 2. Let us denote by \( \mathcal{X}_2(\mathbb{C}^2, 0) \) and by \( \text{Diff}_1(\mathbb{C}^2, 0) \) the convergent elements of \( \mathcal{X}_2(\mathbb{C}^2, 0) \) and \( \text{Diff}_1(\mathbb{C}^2, 0) \) respectively.

Let \( X \in \mathcal{X}_2(\mathbb{C}^2, 0) \). The exponential operator of \( X \) is the application \( \exp tX : \mathbb{C}[[x, y]] \rightarrow \mathbb{C}[[x, y, t]] \) defined by the formula

\[
\exp tX(g) = \sum_{j=0}^{\infty} \frac{t^j}{j!} X^j(g)
\]

where \( X^0(g) = g \) and \( X^{j+1}(g) = X(X^j(g)) \). Note that, since \( \nu(X^j(g)) \geq j + \nu(g) \), we can substitute \( t = 1 \) to get the element \( \exp X(g) \in \mathbb{C}[[x, y]] \). Moreover, \( \exp tX \) gives a homomorphism of \( \mathbb{C} \)-algebras, in particular, we have

\[
\exp tX(fg) = \exp tX(f) \exp tX(g)
\]

We get also

**Proposition 2.1.** The application

\[
\text{Exp} : \mathcal{X}_2(\mathbb{C}^2, 0) \rightarrow \text{Diff}_1(\mathbb{C}^2, 0)
\]

\[X \mapsto (\exp X(x), \exp X(y))\]

is a bijection.

**Proof:** Let \( G(x, y) = \left(x + \sum_{n=2}^{\infty} p_n(x, y), y + \sum_{n=2}^{\infty} q_n(x, y)\right) \) and \( X = \sum_{n=2}^{\infty} \left(a_n(x, y) \frac{\partial}{\partial x} + b_n(x, y) \frac{\partial}{\partial y}\right) \). The identity \( \text{Exp}(X) = G \) is equivalent to

\[
p_{m+1} = a_{m+1} + HT_{m+1}\left(\sum_{j=2}^{m} \frac{1}{j!} X^j_m(x)\right)
\]

\[
q_{m+1} = b_{m+1} + HT_{m+1}\left(\sum_{j=2}^{m} \frac{1}{j!} X^j_m(y)\right),
\]

where \( X_m = \sum_{n=2}^{m} \left(a_n(x, y) \frac{\partial}{\partial x} + b_n(x, y) \frac{\partial}{\partial y}\right) \), and \( HT_{m+1}(h) \) is the homogeneous term of \( h \) of order \( m + 1 \). These equations determine univocally \( X \) if \( G \) is given. □

In general, \( X \) may be not convergent for certain convergent \( G \). The formal vector field \( X \) such that \( G = \text{Exp}(X) \) is called the **infinitesimal generator** of \( G \). Note that \( \text{ord}(G) = \nu(X) \). If \( k = \nu(X) \), then \( a_k = p_k \) and \( b_k = q_k \), thus the characteristic directions of \( F \) correspond to the points of the tangent cone of \( X \). Moreover, if \( X = fX' \) with \( X' \in \mathcal{X}(\mathbb{C}^2, 0) \) and \( f \in \mathbb{C}[[x, y]] \) then \( \text{Exp}(X)(x, y) = (x + f(x, y)p(x, y), y + f(x, y)q(x, y)) \). The converse statement follows by a process similar to the proof of proposition 2.1. In particular, 0 is an isolated singular point of \( X \) if and only if 0 is an isolated fixed point of \( F \). In the case \( f(x, y) = x^k \), and \( S = (x = 0) \) invariant by \( X' \), Camacho-Sad’s index of \( X \) at 0 along \( S \) is exactly Abate’s residual index of \( F \) at 0 along \( S \).

Now, let \( \pi : (M, D) \rightarrow (\mathbb{C}^2, 0) \) be the blow up of \( \mathbb{C}^2 \) at the origin, where \( D = \pi^{-1}(0) = \mathbb{P}^1 \), thus each characteristic direction determines a point of \( D \).

**Proposition 2.2.** Let \( F \in \text{Diff}_1(\mathbb{C}^2, 0) \). There exists a unique germ of diffeomorphism \( \tilde{F} \) in \((M, D)\) such that \( \pi \circ \tilde{F} = F \circ \pi \) and \( \tilde{F}|_D = \text{id} |_D \). Moreover, the germ \( \tilde{F}^p \) has order \( \geq \text{ord}(F) \) for any characteristic direction \( p \in D \) and hence \( \tilde{F}^p \in \text{Diff}_1(M, p) \).
Proof: Let $F(x, y) = (x + p_k(x, y) + \cdots + y + q_k(x, y) + \cdots)$ where $k = \text{ord}(F) \geq 2$. We have two charts of $M = U_1 \cup U_2$ such that $\pi|_{U_i} : U_i \to \mathbb{C}^2$, is defined by $\pi(x, v) = (x, xv)$ and $\pi|_{U_2} : U_2 \to \mathbb{C}^2$, is defined by $\pi(u, y) = (uy, y)$. We define $\tilde{F}$ in the first chart as

$$
\tilde{F}(x, v) = \frac{1}{F(x)} \circ F \circ \pi(x, v) = \left( x + \frac{p_k(x, xv) + \cdots}{x} + \frac{q_k(x, xv) + \cdots}{v} \right) = (x + x^k(p_k(1, v) + x(\cdots)), v + x^{k-1}(q_k(1, v) - vp_k(1, v) + x(\cdots)))
$$

Observe that $\tilde{F}(0, v) = (0, v)$, thus any point of the divisor is fixed. Moreover, if $q_k(1, v_0) - v_0p_k(1, v_0) = 0$ we have $d\tilde{F}(0, v_0) = I$, and thus for any characteristic direction $p = (0, v_0) \in D$, $\text{ord}(\tilde{F}_p) \geq \text{ord}(F)$.

Proposition 2.3. Let $X \in \mathfrak{X}_2(\mathbb{C}^2, 0)$. Let $\tilde{X}$ be the formal vector field in $(M, D)$ such that $D\pi \cdot \tilde{X} = X \circ \pi$. If $p$ is a point of the tangent cone of $X$ then $\tilde{X}_p \in \mathfrak{X}_2(M, p)$.

Proof: Let $X = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$ with $a(x, y) = a_k(x, y) + \cdots, b(x, y) = b_k(x, y) + \cdots$ and $k \geq 2$. Let $U_1$ and $U_2$ be two charts of $M = U_1 \cup U_2$ as in the proposition above. Then $\tilde{X}$ is given in the chart $U_1$ by

$$
\tilde{X}(x, v) = a(x, xv)\frac{\partial}{\partial x} + \frac{b(x, xv) - va(x, xv)}{v} \frac{\partial}{\partial v} = x^k(a_k(1, v) + x(\cdots))\frac{\partial}{\partial x} + x^{k-1}((b_k(1, v) - va_k(1, v)) + x(\cdots))\frac{\partial}{\partial y}
$$

Now, if $p = (0, v_0) \in D$ is such that $b_k(1, v_0) - v_0a_k(1, v_0) = 0$, then $\nu_p(a(x, xv)) \geq k$ and $\nu_p\left(\frac{b(x, xv) - va(x, xv)}{x}\right) \geq k$ so $\tilde{X}_p \in \mathfrak{X}_2(M, p)$. We say that $X$ is strictly non singular if $X = f X'$, where $X'$ is a non singular formal vector field. Otherwise, we say that $X$ is strictly singular. Note that in the above statement any strictly singular point of $\tilde{X}$ is in the tangent cone of $X$. Let us also recall that Seidenberg’s reduction of singularities is done by blowing-up at strictly singular points.

Lemma 2.4. Let $F \in \text{Diff}_1(\mathbb{C}^2, 0)$ and $X \in \mathfrak{X}_2(\mathbb{C}^2, 0)$ such that $F = \text{Exp}(X)$. Let $\tilde{X}$ be as in the proposition above. Then for any characteristic direction $p \in D$ $\tilde{F}_p = \text{Exp}(\tilde{X}_p)$.

Proof: Let $U \simeq \mathbb{C}^2$ be a chart of $M$ such that $\pi|_U : U \to \mathbb{C}^2$ is defined by $\pi(x, v) = (x, xv)$ and $p \in U \cap D = \{(0, v) \in U\}$ be a point on the divisor. Without lost of generality, applying a linear change of coordinates, we can suppose that $p = (0, 0) \in U$. Since

$$
F(x, y) = \text{Exp}(X) = (\exp X(x), \exp X(y)),
$$

using the definition of $\tilde{F}$, we have

$$
\tilde{F}(x, v) = \left( \frac{\exp X(x)}{\exp X(x)}, \frac{\exp X(xv)}{\exp X(x)} \right) = \left( \frac{\exp X(x)}{\exp X(x)}, \frac{\exp X(x) \exp X(v)}{\exp X(x)} \right) = (\exp X(x), \exp X(v)) = (\exp \tilde{X}(x), \exp \tilde{X}(v)) = \text{Exp}(\tilde{X}_p)(x, v).
$$

This ends the proof.
3. Existence of parabolic curves

We need the following formal version of Camacho-Sad’s theorem [4] whose proof goes exactly as the original one (see also [5]).

**Theorem 3.1** (Camacho and Sad). Take $X \in \hat{X}_1(C^2,0)$ with an isolated singularity at the origin. There is a desingularization morphism $\sigma : (\tilde{M}, \tilde{D}) \to (C^2,0)$ composition of a finite sequence of blow-ups with centers at strictly singular points and a point $p \in \tilde{D}$ satisfying the following property: There are local coordinates $(u, v)$ at $p$ such that $\tilde{D}_p = (u = 0)$ and the transform $X^*$ of $X$ at $p$ is of the form:

$$X^*(u, v) = u^k\left(\lambda u + u^2(\cdots)\frac{\partial}{\partial u} + (\mu v + u(\cdots))\frac{\partial}{\partial v}\right)$$

where $\lambda \neq 0$ and $\frac{\mu}{\lambda} \notin \mathbb{Q}_{>0}$.

Let us prove theorem 1.2. Take $X$ the infinitesimal generator of $F$, and consider $X^*$ and $p$ as in Camacho-Sad’s Theorem. By lemma 2.4 we have

$$F_p^*(u, v) := \text{Exp}(X^*_p) = (u + \lambda u^{k+1} + O(u^{k+2}), v + \mu u^k v + O(u^{k+1}))$$

so $F_p^*$ is a diffeomorphism tangent to the identity, with $(1,0)$ as a nondegenerate characteristic direction. By Hakim’s Theorem, there exist $\text{ord}(F) - 1$ disjoint parabolic curves $\varphi_j : \Omega_j \to M$ for $F_p^*$ tangent to the direction $(1,0)$ at $p$. Since this direction is transversal to the divisor, it follows that $\varphi_j(\Omega_j) \cap \tilde{D} = \{p\}$ and thereby $\pi \circ \varphi_j$ is also a parabolic curve for $F$.

Furthermore, according to J. Cano’s proof [5] of Camacho-Sad’s theorem, to find the points $p \in \tilde{D}$ that satisfy Camacho-Sad’s theorem, it is enough to follow after the first blow up, the singularities with Camacho-Sad’s index not in $\mathbb{Q}_{>0}$. Thus, there exist parabolic curves for any characteristic direction of $F$ that gives at the divisor Abate’s residual index not in $\mathbb{Q}_{>0}$ (see corollary 3.1. in [1]).

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