Remarks on the equivalence of full additivity and monotonicity for the entanglement cost

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We analyse the relationship between the full additivity of the entanglement cost and its full monotonicity under local operations and classical communication. We show that the two properties are equivalent for the entanglement cost. The proof works for the regularization of any convex, subadditive, and asymptotically continuous entanglement monotone, and hence also applies to the asymptotic relative entropy of entanglement.

Introduction – Entanglement is the key resource in many quantum information processing protocols. In view of its central character it is of particular interest to be able to quantify entanglement. With this aim in mind, basic properties of so-called entanglement measures have been identified and studied in the literature in some detail (see [1–3] for some recent overviews).

The detailed character of entanglement and its quantification as a resource depends on the constraints that are being imposed on the set of available operations. In a communication setting where two spatially separated parties aim to manipulate a joint quantum state it is natural to restrict attention to local quantum operations and classical communication (LOCC). In this case separable states are freely available while non-separable states, which cannot be prepared by LOCC alone, attain the status of a resource that may achieve a task more efficiently than is possible by classical means [4]. The preparation of non-separable states by means of LOCC carries a cost in terms of pure singlet states. This gives rise to the concept of entanglement cost $E_C$. If we denote a general trace preserving LOCC operation by $\Psi$, and write $\Phi(K)$ for the density operator corresponding to the maximally entangled statevector in $K$ dimensions, i.e. $\Phi(K) = |\psi^+_K\rangle\langle\psi^+_K|$ where $|\psi^+_K\rangle = \sum_{j=1}^{K} |jj\rangle/\sqrt{K}$, then the entanglement cost may be defined formally as

$$E_C(\rho) = \inf \left\{ r : \lim_{n \to \infty} \frac{1}{n} \inf_{\Psi} D(\rho^\otimes n, \Psi(\Phi(2^n))) = 0 \right\}$$

where $D(\sigma, \eta)$ is a suitable measure of distance [1, 5, 6]. Clearly, the computation of this infimum is extraordinarily difficult and it is therefore fortunate that $E_C$ is closely related to the somewhat more easily computable entanglement of formation [7, 8]

$$E_F(\rho) = \min_{\sum_i p_i |\psi_i\rangle\langle\psi_i| = \rho} \sum_i p_i E_F(|\psi_i\rangle)$$

via the limit [9]

$$E_C(\rho) = \lim_{n \to \infty} \frac{E_F(\rho^\otimes n)}{n}.$$  

This equivalence is true for any distance measure $D(\sigma, \eta)$ that is equivalent to the trace norm with sufficiently weak dependence upon dimension [9]. Here we have defined $E_F$ for pure states as $E_F(|\psi_i\rangle) = S(tr_A(|\psi_i\rangle\langle\psi_i|))$, i.e. the entropy of entanglement [10] with the von Neumann entropy $S(\rho) = -\text{tr}[\rho \log_2 \rho]$.

Owing to its operational definition, the entanglement cost suggests itself as a good entanglement measure. A fundamental property which an entanglement measure should satisfy (apart from vanishing on separable states) is monotonicity under local operations and classical communication (LOCC):

$$E(\rho) \geq E(\Lambda(\rho))$$

for any LOCC operation $\Lambda$ which may include mappings to a larger Hilbert space. It is not hard to check that $E_C$ satisfies this condition. However, many measures satisfy the somewhat stronger condition, called full monotonicity, of the non-increase on average under LOCC, i.e.

$$E(\rho) \geq \sum_i p_i E(\rho_i)$$

where, in a LOCC protocol applied to state $\rho$, the state $\rho_i$ with label $i$ is obtained with probability $p_i$. While entanglement of formation satisfies this stronger monotonicity, it is not known whether the entanglement cost does.

Another nontrivial property, that may be shared by some entanglement measures is full additivity. A potential entanglement quantifier is additive if it satisfies $E(\rho \otimes \sigma) = 2E(\rho)$ for all $\rho$ and it is fully additive if it satisfies

$$E(\sigma \otimes \rho) = E(\sigma) + E(\rho)$$

for all $\sigma$ and $\rho$. Neither additivity nor full additivity are trivially satisfied for entanglement quantifiers. The entropy of entanglement, the so-called squashed entanglement [12], and the relative entropy of entanglement with reversed arguments [13] are fully additive. However the relative entropy of entanglement is not additive [14], and if NPPT-bound entangled states exist [15, 16] then the distillable entanglement would be super-additive [28]. It is an unsolved open question whether the entanglement of formation is additive or not [20, 22]. Note that the regularisation of an entanglement measure, i.e. a limit as in eq. [3] is automatically additive but not necessarily fully additive. Thus the entanglement cost and the regularized relative entropy of entanglement [23] are additive but it is unknown whether they are also fully additive.

In this paper we will demonstrate that the two properties of full monotonicity and full additivity are in fact equivalent for the entanglement cost. The proof is quite general, and applies to the regularization of any convex, subadditive, and asymptotically continuous entanglement monotone, and hence also applies to the asymptotic relative entropy of entanglement.
To begin with, we recall a useful characterization of the full monotonicity of convex functions that was established in [24].

**Property 1** A convex function $E$ is full monotone if and only if it satisfies the following conditions:

- **LUI** It is invariant under local unitary operations.
- **FLAGS** It satisfies

$$ E\left(\sum_i p_i \rho_i \otimes |i\rangle \langle i|\right) = \sum_i p_i E(\rho_i) \tag{7} $$

where $|i\rangle$ are local orthogonal flags, $\sum_i p_i = 1$, $\rho_i$ are arbitrary states.

As the entanglement cost is trivially invariant under local unitary operations, and is known to be convex [25] it will therefore be our aim in the following to look for the equivalence of full additivity with the FLAGS condition eq. (7).

We will establish that the FLAGS condition is true if and only if $E_C$ is fully additive, thereby implying that full monotonicity is true if and only if $E_C$ is fully additive:

**Proposition 1** $E_C$ satisfies

$$ E_C(pp \otimes P_0 + (1-p)\sigma \otimes P_1) = p E_C(\rho) + (1-p) E_C(\sigma) \tag{8} $$

for any $\rho$ and $\sigma$ if and only if $E_C$ is additive. Here $P_i = |i\rangle \langle i|$ are local orthogonal flags.

**Proof** Denoting by $S$ the symmetrised tensor product, e.g. $S(\rho \otimes \sigma) = \rho \otimes \sigma + \sigma \otimes \rho$ then we find

$$ E_C(pp \otimes P_0 + (1-p)\sigma \otimes P_1) = \lim \frac{1}{n} \frac{1}{n} E_F\left(\sum_{i=1}^n (p \otimes P_0) \otimes (\sigma \otimes P_1)\right) = \lim \frac{1}{n} \frac{1}{n} E_F\left((1-\epsilon)\hat{\rho}_{typ} + \epsilon \rho_{atyp}\right) \tag{9} $$

where we define the ‘typical’ $\hat{\rho}_{typ}$ as the normalized state proportional to $\sum_{k=1}^{nm} S((p \otimes P_0) \otimes (\sigma \otimes P_1))$, and $\epsilon$ is defined by eqn. (9). Note that we assert that the weight of the typical part is exactly $1 - \epsilon$, which we are free to do as any excess weight can be moved into the definition of $\rho_{atyp}$.

In $\rho_{atyp}$ the local flags give us information about exactly which of the $n$ copies of our basic two-party system are in state $\rho$, and which are in state $\sigma$. The typicality condition ensures that $\rho$ will each occur at least $[n\rho(1-\delta)]$ times, and $\sigma$ will occur at least $[n(1-p)(1-\delta)]$ times. Hence we are free to perform local unitaries to transform $\hat{\rho}_{typ}$ to a sorted normalized state, $\rho^{\sigma}_{typ}$, that is proportional to:

$$ \rho^{\sigma}_{typ} = (\rho \otimes P_0) \otimes [np(1-\delta)] \otimes (\sigma \otimes P_1) \otimes [n(1-p)(1-\delta)] \otimes \tilde{\omega}_{rem} $$

i.e. we ensure that in $\rho_{typ}$ the first $np(1-\delta)$ copies are in state $\rho$, and the next $n(1-p)(1-\delta)$ copies are in the states $\sigma$, with the remaining statistical fluctuations shifted to the last $(n\delta + 1) \pm 1$ copies represented by the remainder state $\tilde{\omega}_{rem}$. Note that in this process one can ensure that all local information in the flags will be preserved, so that in particular one retains the ability to determine precisely whether one is in the typical or atypical part of the entire state.

Now we use the fact that $E_F$ satisfies the FLAGS condition to find

$$ E_C(pp \otimes P_0 + (1-p)\sigma \otimes P_1) = E_F\left([pp \otimes P_0 + (1-p)\sigma \otimes P_1]\otimes^n\right) - (1-\epsilon)E_F(\rho^{\sigma}_{typ}) \tag{10} $$

where $\rho^{\sigma}_{typ}$ is the state obtained when we remove all flags from $\rho_{typ}$ (in general the removal of a single flag will always signify the removal of any flags). Now we may use the asymptotic continuity [23, 27] of the entanglement of formation to conclude that

$$ \frac{1}{n} E_F\left([pp \otimes P_0 + (1-p)\sigma \otimes P_1]\otimes^n\right) - (1-\epsilon)E_F(\rho^{\sigma}_{typ}) \leq 2\epsilon C \log d $$

where $C$ is a constant, $d$ is the dimension of a single system (half of a pair). Now we may use the monotonicity and subadditivity of the entanglement of formation

$$ E_F(X) \leq E_F(X \otimes Y) \leq E_F(X) + E_F(Y) \tag{11} $$

to conclude that

$$ E_F(\rho^{\otimes [np(1-\delta)]} \otimes \sigma^{\otimes [n(1-p)(1-\delta)]}) \leq E_F(\rho^{\sigma}_{typ}) $$

and hence the triangular inequality gives

$$ \frac{1}{n} E_F\left([pp \otimes P_0 + (1-p)\sigma \otimes P_1]\otimes^n\right) - (1-\epsilon)E_F(\rho^{\otimes [np(1-\delta)]} \otimes \sigma^{\otimes [n(1-p)(1-\delta)]}) \leq 2\epsilon C \log d + 4\delta \log d $$

Now we let $p_i$ be a rational probability that approximates $p$ closely, i.e. the difference $\epsilon_i := p - p_i$ is a small real number. Then there exists an integer $m_i$ such that both $m_i(p_i)$ and $m_i(1-p_i)$ are integers. Inserting these definitions into the previous inequality gives:

$$ \frac{1}{nm_i} E_F\left([pp \otimes P_0 + (1-p)\sigma \otimes P_1]\otimes^{nm_i}\right) - (1-\epsilon)E_F(\rho^{\otimes [nm_i(p_i(1-\delta)]+nm_i\epsilon_i(1-\delta)]} \otimes \sigma^{\otimes [nm_i(1-p_i)(1-\delta)]-nm_i\epsilon_i(1-\delta)]}) \leq 2\epsilon C \log d + 4\delta \log d $$

The state in the middle lines of this inequality can be written as (in somewhat loose terms in order not to obscure the argument):

$$ \rho^{\otimes [nm_i(p_i(1-\delta)]+nm_i\epsilon_i(1-\delta)]} \otimes \sigma^{\otimes [nm_i(1-p_i)(1-\delta)]-nm_i\epsilon_i(1-\delta)]} \approx (\rho^{\otimes m_i(p_i)} \otimes \sigma^{\otimes m_i(1-p_i)}) \otimes (\tilde{\omega}_{rem(1-\delta)]} \pm 1 \otimes \sigma^{\otimes -nm_i\epsilon_i(1-\delta)]} \pm 1} $$
Hence a second application of inequality (11) together with the triangular inequality gives:

\[
\frac{1}{nm_i}|E_F\left((pp \otimes P_0 + (1 - p)\sigma \otimes P_1) \otimes \delta_i\right) - 2\epsilon_i\delta_i + 2 \log(d)/nm_i.
\]

Now sending \(\epsilon \to 0, \delta \to 0\), and \(n \to \infty\) we obtain

\[
|E_C\left((pp \otimes P_0 + (1 - p)\sigma \otimes P_1)\right) - \frac{1}{nm_i}E_C(\rho\otimes m_i, p_i - (1 - p_i)E_C(\sigma)) \leq 2\epsilon_i,
\]

(12)

This inequality is the key to obtaining the proposition. In this equation if we pick \(m_i = 2, p = p_i = 1/2\), then we find that \(\epsilon_i = 0\) and recover the equality:

\[
E_C\left(\frac{1}{2}\rho \otimes |0\rangle\langle 0| + \frac{1}{2}\sigma \otimes |1\rangle\langle 1|\right) = \frac{1}{2}E_C(\rho \otimes \sigma)
\]

(13)

By comparing this equality to the FLAGS condition we see that if FLAGS is true for \(E_C\), then \(E_C\) is fully additive. On the other hand, if \(E_C\) is fully additive, then equation (12) becomes:

\[
|E_C\left((pp \otimes P_0 + (1 - p)\sigma \otimes P_1)\right) - p_iE_C(\rho) - (1 - p_i)E_C(\sigma)| \leq 2\epsilon_i,
\]

in which case sending \(p_i \to p\) and hence \(\epsilon_i \to 0\) establishes the FLAGS condition for all \(p\). Hence fully additivity of \(E_C\) also implies full monotonicity.

Finally let us note that we have used the following properties of \(E_F\) and \(E_C\): 1) full monotonicity of \(E_F\), 2) asymptotic continuity of \(E_F\), 3) sub-additivity of \(E_F\), 4) convexity of \(E_C\). The asymptotic continuity follows from convexity and sub-additivity of \(E_F\). We thus obtain the following proposition:

**Proposition** Suppose that a function \(E\) that is convex, sub-additive, fully monotone under LOCC and asymptotically continuous. Then full monotonicity and full additivity of \(E^\infty\) are equivalent.

In particular this demonstrates that the asymptotic relative entropy of entanglement is fully additive iff it is fully monotone.

**Summary and Conclusions** – We have considered the relationship between the full additivity and full monotonicity of the entanglement cost. We have established that the two properties are equivalent for the regularization of any convex, sub-additive, and asymptotically continuous entanglement monotone, including the entanglement cost.

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[27] M.A. Nielsen, Phys. Rev. A 61, 064301 (2000).
[28] This was first noted in Ref. [19], where it was shown that if a particular NPPT Werner state is undistillable, then the distillable entanglement would be super-additive. In Refs. [16,17], in turn, the implication was fully established, as it was shown that any NPPT state becomes distillable when assisted by a PPT entangled state.