Raman scattering in a two-dimensional Fermi liquid with spin-orbit coupling

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We present a microscopic theory of Raman scattering in a two-dimensional Fermi liquid (FL) with Rashba and Dresselhaus types of spin-orbit coupling, and subject to an in-plane magnetic field ($\vec{B}$). In the long-wavelength limit, the Raman spectrum probes the collective modes of such a FL: the chiral spin waves. The characteristic features of these modes are a linear-in-$q$ term in the dispersion and the dependence of the mode frequency on the directions of both $\vec{q}$ and $\vec{B}$. All of these features have been observed in recent Raman experiments on Cd$_{1-x}$Mn$_x$Te quantum wells.

I. INTRODUCTION

Raman scattering is an inelastic light scattering process that allows to study dynamics of elementary excitations in solids both in space and time and to probe both single-particle and collective properties of electron systems. It helps to understand a variety of phenomena: from pairing mechanisms in high-temperature superconductors to properties of spin waves in spintronic devices. The latter requires probing dynamics of electron spins, which can be done if the incident and scattered light are polarized perpendicular to each other (cross-polarized geometry).

Non-trivial spin dynamics is encountered in systems with spontaneous magnetic order, or in an external magnetic field, or else in the presence of spin-orbit coupling (SOC). In this paper, we develop a microscopic theory of Raman scattering in a two-dimensional (2D) Fermi liquid (FL) in the presence of an in-plane magnetic field and both Rashba and Dresselhaus types of SOC. Breaking spin conservation leads to a substantial modification of the Raman spectrum already at the non-interacting level. Unlike in the $SU(2)$-invariant case, the cross-section of Raman scattering cannot be expressed solely in terms of charge and spin susceptibilities. As a result, it is not a priori clear how single-particle and collective spin-excitations in systems with SOC manifest themselves in a Raman scattering process. We show here that the scattering cross-section, in the cross-polarized geometry and in the long-wavelength limit, is parameterized by particular components of the spin susceptibility tensor. These components contain poles at frequencies that correspond to a new type of collective modes—chiral spin waves (CSW) arising from an interplay between electron-electron interaction ($eei$) and SOC.

In the absence of SOC, the dispersion of a collective mode in the spin channel must start with a $q^2$ term by analyticity. Recently, a dispersing peak was observed in resonant Raman scattering on magnetically doped CdTe quantum wells in the presence of an in-plane magnetic field. The unusual features observed in these experiments were a linear-in-$q$ term in the dispersion and a $\pi$-periodic modulation of the spectrum as the magnetic field is rotated in the plane of 2D electron gas (2DEG).

We argue that the observed peak is actually one of the long sought after CSW (in the presence of a field). The CSWs are collective oscillations of the magnetization ($\vec{M}$) that exist even in the absence of the magnetic field. In zero field, there are three such modes, which are massive, i.e., their frequencies are finite at $q \to 0$, and disperse with $q$ on a characteristic scale set by spin-orbit splitting. The modes are linearly polarized, with $\vec{M}$ being in the 2DEG plane for two of the modes and along the normal for the third one. If an in-plane magnetic field ($\vec{B}$) is applied, the mode with $\vec{M}_{||}\vec{B}$ remains linearly polarized while the other two modes with $\vec{M} \perp \vec{B}$ become elliptically polarized.

Figure 1a depicts the evolution of the excitation spectrum with $B$ at $q = 0$. As the field increases, two, out of the three, modes run into the continuum of spin-flip excitations (SFE). The third mode merges with the continuum at $B = B_{\text{c}}$, when the spin-split Fermi surfaces (FSs) become degenerate, and re-emerges to the right of this point. As the field is increased further, this mode transforms gradually into the spin wave [Silin-Leggett (SL) mode] of a partially polarized FL. In the experimental setup of Refs. 14–16, the effective Zeeman energy is larger than both Rashba and Dresselhaus splittings which, according to Fig. 1a, allows us to focus on the case of a single CSW adiabatically connected to the SL mode. We show that, at small $q$, the dispersion of this mode can be written as

$$\Omega(q, B) = \Omega(\theta_B) + w(\hat{\theta}_B, \theta_B)q + Aq^2,$$

where $\theta_B$ is the azimuthal angle of $\vec{B}$. We show, first by a symmetry argument and then by an explicit calculation, that the Rashba and Dresselhaus types of SOC contribute $\cos(\theta_B + \theta_B)$ terms to $w(\hat{\theta}_B, \theta_B)$, correspondingly, while the mass term, $\Omega(\theta_B)$ and the boundaries of the SFE continuum are $\pi$-periodic function of $\theta_B$. Our theory describes quantitatively the results of Refs. 14–16, where a single Raman peak was observed, and predicts that up to three peaks can be observed at lower magnetic fields in systems with higher mobility and/or stronger SOC.

The rest of the paper is organized as follows. In Sec. II, we discuss the general theory of Raman response in a Fermi-liquid with SOC. In Sec. III, we present symmetry arguments for the form of the CSW dispersion in the presence of a magnetic field. In Sec. IV, we present results of an explicit calculation for the CSW dispersion.
FIG. 1: Color online: a) Schematically: excitation spectrum (at \( q = 0 \)) of a 2D Fermi liquid with spin-orbit coupling and subject to an in-plane magnetic field \( B \). Shaded region: continuum of spin-flip excitations; lines: chiral spin waves. b) Continuum as a function of the angle \( \theta_B \) between \( B \) and the (100) direction. \( \Delta_R/\Delta_D = 0.5 \), \( \eta = \Delta_Z/\Delta_D \), where \( \Delta_D, \Delta_R, \Delta_Z \) is Dresselhaus/Rashba/Zeeman splitting. c) Continuum (shaded) and collective-mode frequency at \( q = 0 \) (line) as a function of \( \phi = \theta_B - \pi/2 \) for a Cd\(_{1-x}\)Mn\(_x\)Te quantum well.

In Sec. V we compare our theory to the experimental results of Refs. 14–16. In Sec. VI, we summarize our main results. Details of the calculations are given in Appendix.

II. GENERAL FORMALISM

In general, Raman scattering probes both charge and spin excitations. However, if the polarizations of the incident (\( \hat{e}_i \)) and scattered (\( \hat{e}_s \)) light are perpendicular to each other, the Raman vertex couples directly to the electron spin.\(^4\)\(^6\) The Feynman diagrams for the differential scattering cross-section in such a geometry are shown in Fig. 2. The first diagram on the right-hand side in the top row is a pure spin part, while the rest is a spin-charge part, which is non-zero due to SOC, and is represented by a sum of bubbles connected by the Coulomb interaction (dashed line).

Shading of inner parts of the bubbles in Fig. 2 denotes renormalizations due to the exchange part of \( eei \). If this renormalization is neglected, the differential (per unit energy and per unit solid angle) scattering cross-section is reduced, upon projection onto the conduction band, to the result of Ref. 6:

\[
\frac{d^2A}{d\Omega d\hat{q}} \propto \sum_{\mu \mu'} |\gamma_{\mu \mu'}|^2 \text{Im} L^0_{\mu \mu'} - \frac{2\pi e^2}{q} \text{Im} \frac{Z Z}{\epsilon(q, \Omega)}, \tag{2a}
\]

\[
\gamma_{\mu \mu'} = i \langle \hat{\mu} \cdot \hat{\sigma} e^{i\hat{q} \cdot \hat{r}} \rangle_{\mu \mu'}, \tag{2b}
\]

\[
L^0_{\mu \mu'} = \frac{\int (\varepsilon_\mu - \varepsilon_{\mu'}) - f(\varepsilon_{\mu'}) - f(\varepsilon_\mu)}{\Omega + \varepsilon_\mu - \varepsilon_{\mu'} + i\gamma}, \tag{2c}
\]

\[
Z = \sum_{\mu \mu'} \gamma_{\mu \mu'} (e^{-i\hat{q} \cdot \hat{r}})_{\mu \mu'} L^0_{\mu \mu'}, \tag{2d}
\]

\[
\tilde{Z} = \sum_{\mu \mu'} \gamma_{\mu \mu'}^* (e^{i\hat{q} \cdot \hat{r}})_{\mu \mu'} L^0_{\mu \mu'}, \tag{2e}
\]

\[
\epsilon(q, \Omega) = \epsilon + \frac{2\pi e^2}{q} \sum_{\mu \mu'} |\langle e^{i\hat{q} \cdot \hat{r}} \rangle_{\mu \mu'}|^2 L^0_{\mu \mu'}. \tag{2f}
\]

Here, \( \epsilon \) is the background dielectric constant, \( \mu \) and \( \mu' \) refer to the quantum numbers of electrons which include the momentum and spin/chirality: \( \mu = \{ \xi, \nu \} \), \( \mu' = \{ \xi', \nu' \} \) with \( \nu, \nu' = \pm 1 \), \( \varepsilon_\mu \) is the energy of a state with quantum number \( \mu \), \( f(\varepsilon) \) is the Fermi function, \( \langle X \rangle_{\mu \mu'} \equiv \int d\hat{q} \psi^{\dagger}_\mu(\hat{r}) X(\hat{r}) \psi_{\mu'}(\hat{r}) \) is the matrix element of \( X(\hat{r}) \) between states \( \mu \) and \( \mu' \), and \( \hat{n} = \hat{e}_i \times \hat{e}_S \). The matrix elements are computed with respect to the eigenvectors of the non-interacting Hamiltonian for a (001) quantum well with Rashba and Dresselhaus SOC and in the presence of an in-plane magnetic field. In what follows, we will be assuming that the energy scales associated with the magnetic field and SOC are much smaller than the Fermi energy. In this case, it suffices to use a low-energy version of the Hamiltonian

\[
\hat{H}_L = v_F(k - k_F)\hat{\sigma}_0 + \sum_{i=1,2} s_i \hat{\sigma}_i, \tag{3}
\]

where \( \hat{\sigma}_0 \) is the identity matrix, \( v_F \) and \( k_F \) are the Fermi velocity and momentum in the absence of SOC and magnetic field, correspondingly, and vector \( \vec{s} \) parametrizes...
the effects of SOC and magnetic field:

\[ s_1 = \frac{1}{2} (\Delta_R \sin \theta_k + \Delta_D \cos \theta_k + \Delta_Z \cos \theta_B), \]

\[ s_2 = -\frac{1}{2} (\Delta_R \cos \theta_k + \Delta_D \sin \theta_k - \Delta_Z \sin \theta_B), \]

where \( \Delta_R, \Delta_D, \) and \( \Delta_Z \) are the Rashba, Dresselhaus, and Zeeman energy splittings, correspondingly. The \( x_1 \) axis is chosen along the \((100)\), and \( x_2, \) and \( x_3 \) axes are chosen along the \((010)\) and \((001)\) directions, correspondingly. The Green’s function of Eq. (3) is given by

\[ \hat{G}_K^0 = \frac{g_+ + g_-}{2} \hat{\sigma}_0 + \frac{g_+ - g_-}{2|s|} \sum_{i=1,2} s_i \hat{\sigma}_i, \]

\[ g_\pm = \frac{1}{\omega_n - \varepsilon_{K,\pm}}, \quad \varepsilon_{K,\pm} = v_F (k - k_F) \pm |s|, \]

where \( |s| = \sqrt{s_1^2 + s_2^2} \).

Using the eigenvectors of Hamiltonian (3)

\[ \psi_\mu (\vec{r}) = e^{i\vec{k}.\vec{r}} \left( i e^{-i \phi_\mu} 1 \right), \]

where \( \tan \phi_\mu = -s_1/s_2 \), we obtain for the matrix elements of Raman scattering

\[ \gamma_{\mu \mu'} = i \delta_{\tilde{k} - \tilde{k}' - \vec{q}} \left[ n_1 \left\{ i \nu e^{-i \phi_{\mu} + q} - i \nu e^{i \phi_{\mu'}} \right\} \right. \]

\[ \left. + n_2 \left\{ -\nu e^{-i \phi_{\mu} - q} - \nu e^{i \phi_{\mu'}} \right\} \right. \]

\[ \left. + n_3 \left\{ \nu \nu e^{-i (\phi_{\mu} - q) - \phi_{\mu'}} - 1 \right\} \right], \]

\[ \langle e^{i \vec{q}.\vec{r}} \rangle_{\mu \mu'} = \delta_{\tilde{k} - \tilde{k}' - \vec{q}} \left[ n_1 \nu e^{-i (\phi_{\mu} + q) - \phi_{\mu'}} + 1 \right]. \]

We are interested in the small-\( q \) limit, when the \( q \)-dependent terms in the dispersions of CSWs corrections to the frequencies of these waves at \( q = 0 \), which implies that \( v_F q \ll \Omega \). At the same time, both the magnetic field and SOC are assumed to be weak compared to the Fermi energy (in appropriate units), i.e., \( \Omega \ll v_F k_F \). Combining the two inequalities above, we obtain

\[ q \ll \Omega/v_F \ll k_F. \]

The first part of this inequality ensures that the diagonal elements of \( L^0_{\mu \mu'}, \) in Eq. (2a) are small by charge conservation: \( L^0_{\mu \mu'} \propto q \). Therefore, the main contribution to the cross-section in this limit comes from off-diagonal terms with \( \nu \nu' = -1 \), which correspond to processes that flip spin/chirality. The second part of Eq. (8) implies that \( \phi_{\mu + q} \) can be approximated by \( \phi_{\mu} \), upon which the off-diagonal matrix elements are simplified to

\[ \gamma_{\mu \mu'} \approx -2i \delta_{\tilde{k} - \tilde{k}' - \vec{q}} \times \left[ i n_1 \nu \cos \phi_{\mu} + i n_2 \nu \sin \phi_{\mu} + n_3 \right] \]

\[ \langle e^{i \vec{q}.\vec{r}} \rangle_{\mu \mu'} \approx \nu \nu' + 1 = 0. \]

By the same argument, the diagonal terms in \( Z \) and \( \bar{Z} \) in Eq. (2a) are small as \( q \), while the off-diagonal terms are small because they contain off-diagonal matrix elements \( \langle e^{i \vec{q}.\vec{r}} \rangle_{\mu \mu'} \), which are small by Eq. (9c). Therefore, the second term in Eq. (2a) can be neglected compared to the first one, and the cross-section is reduced to

\[ \frac{d^2 A}{d\Omega d\Omega} \propto \sum_{\mu \mu'} |\gamma_{\mu \mu'}|^2 \text{Im} L^0_{\mu \mu'} = \int_k \left[ n_1^2 \cos^2 \phi_k + n_2^2 \sin^2 \phi_k + n_1 n_2 \sin 2 \phi_k + n_3^2 \right] \int_\omega (g_+ \tilde{g}_- + g_+ \tilde{g}_-), \]

where \( \int_k \equiv \int d^2 k / (2\pi)^2; \int_\omega \equiv \int dw / 2\pi; \quad g_\nu = 1/(i\omega - \varepsilon_{\tilde{k},\nu}) \) and \( \tilde{g}_\nu = 1/(i\omega + \Omega - \varepsilon_{\tilde{k} + \vec{q},\nu'}) \). The various terms in the equation above can be expressed via the components of the spin-spin correlation function in the chiral basis:

\[ \chi_{11}(Q) = - \int_k \int_\omega (g_+ \tilde{g}_- + g_+ \tilde{g}_-) \cos^2 \phi_k, \]

\[ \chi_{22}(Q) = - \int_k \int_\omega (g_+ \tilde{g}_- + g_+ \tilde{g}_-) \sin^2 \phi_k, \]

\[ \chi_{12}(Q) = - \int_k \int_\omega (g_+ \tilde{g}_- + g_+ \tilde{g}_-) \sin \phi_k \cos \phi_k, \]

\[ \chi_{21}(Q) = \chi_{12}(Q), \]

\[ \chi_{33}(Q) = - \int_k \int_\omega (g_+ \tilde{g}_- + g_+ \tilde{g}_-) \left( \chi_{33} \right). \]

Making use of the definitions, we re-write Eq. (10) as

\[ \frac{d^2 A}{d\Omega d\Omega} \propto n_1^2 \chi_{11} + n_2^2 \chi_{22} + n_1 n_2 (\chi_{12} + \chi_{21}) + n_3^2 \chi_{33}. \]

Accounting for the exchange part of \( cee \) amounts simply to replacing the free-electron spin-spin correlation function by the renormalized ones, i.e., to replacing the bare bubbles in Fig. 2 by the shaded ones:

\[ \frac{d^2 A}{d\Omega d\Omega} \propto n_1^2 \chi_{11} + n_2^2 \chi_{22} + n_1 n_2 (\chi_{12} + \chi_{21}) + n_3^2 \chi_{33}. \]

The poles of the renormalized components of the spin susceptibility tensor correspond to CSWs. This is the main point of departure from the theory of Ref. 6, which
accounts for plasmons but not for CSWs. On a technical level, renormalized $\chi_{ij}$ can be found either from the equations of motion of the FL theory\textsuperscript{7,8,13} or from Random Phase Approximation (RPA) in the spin channel.\textsuperscript{10} Equation (13) is valid at finite $q$ provided that $q \ll k_F$.

Suppose for a moment that SOC is absent while the magnetic field is applied along the $x_1$-axis. In this case, the $q = 0$ susceptibility components satisfy $\chi_{22} = \chi_{33}$ and $\chi_{11} = \chi_{12} = 0$, due to conservation of spin component along the direction of the field.\textsuperscript{12} [At small but finite $q$, $\chi_{11}$ and $\chi_{12}$ are also finite but small (in proportion to $q$) and can be neglected.] The Raman cross-section is then proportional to $(n_d^2 + n_d^2)\chi_{22}$. Because of spin-rotation-symmetry in the absence of the field, $\chi_{22}$ is proportional to the transverse susceptibility $\chi_{\perp} = \chi_{+-} + \chi_{-+}$\textsuperscript{20,21} The pole of $\chi_{\perp}$ corresponds to the SL mode.\textsuperscript{17–19} The direction of $\vec{n} = \vec{e}_1 \times \vec{e}_2$ affects only the magnitude of the Raman signal but not its profile.

In the presence of SOC, however, the result for the Raman cross-section does not reduce to $\chi_{\perp}$. Instead, the partial components of $\chi_{lm}$ contribute to the cross-section with weights determined by the direction of $\vec{n}$. The poles of $\chi_{lm}$ now correspond to CSWs shown in Fig. 1a, one of them being adiabatically connected to the SL mode. In general, equations of motion for the three components of magnetization are coupled to each other, and hence all the components of $\chi$ have poles at the same frequencies but with different residues. The profile of the spectrum is determined by the residues as well as by the relative weights of the geometrical factors $n_i n_j$ as given in Eq. (13).

### III. Symmetry Arguments for Dispersion of Chiral Spin Waves

In this section, we show that the general form of the CSW dispersion can be determined solely by symmetry and dimensional analyses. The role of the microscopic theory, presented in Sec. IV, amounts then to fixing the dimensionless functions of the exchange interaction parameter(s), which enter this form.

As SOC and magnetic field are weak, the interaction of two quasiparticles with momenta $\vec{k}$ and $\vec{k}'$ can be described by an $SU(2)$-invariant Landau function

$$F(\vartheta) = \alpha_0 \delta_0' F_0(\vartheta) + \tilde{\sigma} \cdot \tilde{\sigma}' F_0(\vartheta),$$

where $\vartheta = \theta_0 - \theta_0'$. Within this approximation, the charge and spin sectors of the theory are decoupled, and we focus on the exchange interaction parameterized by $F_0(\vartheta)$.

To set the stage, we discuss the SL mode in the absence of SOC. Kohn’s theorem protects the $q = 0$ term in the dispersion from renormalization by $\varepsilon e i \vec{k} \cdot \vec{B}$; therefore, $\Omega_0(\theta_0) = \Delta_z$ in Eq. (1). Now we consider the dispersion at small but finite $q$: $q \ll \Delta_z / v_F$. In the absence of SOC, rotations in the orbital and spin spaces are independent, which excludes the dot product $\vec{q} \cdot \vec{B}$ and any power of it thereof. Therefore, the dispersion starts with a $q^2$ term. Combining the symmetry arguments with dimensional analysis, we find

$$\Omega(q, B) = \Delta_z + a_2 \{F_0\} \frac{(v_F q)^2}{\Delta_z},$$

where $a_2$ is a dimensionless function which depends on the angular harmonics of $F_0(\vartheta)$: $a_2 \{F_0\} = a_2(F_0, F_0', F_0'', \ldots )$. A precise form of $a_2$ is determined by the microscopic theory.\textsuperscript{17,24}

We now apply the same reasoning to CSWs. If only Rashba SOC is present, there is no preferred in-plane direction, hence a linear-in-$q$ term is absent while the quadratic term is isotropic. There are three CSWs in this case, whose dispersions for $q \ll \Delta_R / v_F$ can be written as

$$\Omega_{\alpha}(q, B) = \tilde{a}_0 \{\{F_0\}\} |\Delta_R| + \tilde{a}_2 \{\{F_0\}\} \frac{(v_F q)^2}{|\Delta_R|},$$

where $\alpha = 1 \ldots 3$ and $\tilde{a}_{0,2}$ are some other dimensionless functions of the FL parameters. Since Kohn’s theorem does not apply in the presence of SOC, $\tilde{a}_0 \neq 1$. Explicit forms of the functions $\tilde{a}_{0,2}$ were found in Refs. 8–10. If only Dresselhaus SOC is present, the Hamiltonian can be transformed to the Rashba form by replacing $\theta_0 \rightarrow \pi / 2 - \theta_0$. Therefore, the CSWs are the same in both cases although the Hamiltonians have different symmetries.

If both Rashba and Dresselhaus types of SOC are present, the $q = 0$ term in the dispersion remains isotropic because in the absence of the magnetic field, the limit of $q = 0$ can be reached along any direction in the plane), while a linear-in-$q$ term is still not allowed by symmetry. Indeed, since the dispersion must be analytic in $q$, a term of the type $|\vec{q}|$ is not allowed, and the linear term could only be of the form $c_{1,2} = \text{const}$. However, such a form does not obey the symmetries of the $D_{2d}$ group (rotation by $\pi$ and reflection about the diagonal plane). The quadratic term can have an anisotropic part of the form $q_1 q_2 \propto \sin 2\theta$, but we already know that such a term is absent if only Dresselhaus SOC is present. Therefore, such a term can only arise due to interplay between Rashba and Dresselhaus types of SOC, and its should be proportional to the product $\Delta_R \Delta_D$. Hence the anisotropic part is small compared to the isotropic, $q^2$ part.

Finally, let both types of SOC and the magnetic field be present but, in agreement with the experimental conditions of Refs. 14–16, we focus on the case when the Zeeman energy is largest scale in the problem, i.e., $\Delta_z \gg \Delta_R, \Delta_D$. The $q = 0$ term in the dispersion then depends on the orientation of $\vec{B}$ in the plane; the $D_{2d}$-symmetry forces this dependence to be of the $\sin 2\theta$ form. Since the anisotropic part of $\Omega_{\alpha}(\theta_0)$ is non-zero only if both types of SOC are present, the coefficient of the $\sin 2\theta$ term must be on the order of $\Delta_R / \Delta_D$. In addition to the anisotropic part, there are also isotropic corrections of order $\Delta_R^2 / \Delta_D \Delta_z$ and $\Delta_D^2 / \Delta_z$. 

To lowest order in SOC, the form of the linear term is determined by the symmetries of the Rashba (group $C_{\infty v}$) and Dresselhaus (group $D_{2d}$) types of SOC. In both cases, we need to form a scalar ($\Omega$) out of a polar vector ($\vec{q}$) and a pseudovector ($\vec{B}$). In the $C_{\infty v}$ group, this is only possible by forming the Rashba invariant $B_1 q_2 - B_2 q_1 \propto \sin(\theta_\vec{q} - \theta_\vec{B})$, which is the same term as in the original Rashba Hamiltonian with $\vec{\sigma} \rightarrow \vec{B}$. Likewise, the only possible scalar in the $D_{2d}$ group is the Dresselhaus invariant $B_1 q_1 - B_2 q_2 \propto \cos(\theta_\vec{q} + \theta_\vec{B})$. The quadratic term in the high-field limit can be taken the same as the quadratic term in the SL mode, Eq. (15).\textsuperscript{25}

Combining together all the arguments given above, we arrive at the following form of the coefficients in Eq. (1)

$$\begin{align*}
\Omega_0(\theta_\vec{B}) & = \Delta_Z + a_0(\{F^a\}) \frac{\Delta_R^2 + \Delta_D^2}{\Delta_Z} \\
& + \tilde{a}_0(\{F^a\}) \frac{\Delta_R \Delta_D}{\Delta_Z} \sin 2\theta_\vec{B},

w(\theta_\vec{q}, \theta_\vec{B}) & = v_F a_1(\{F^a\}) \left[ \frac{\Delta_R}{\Delta_Z} \sin(\theta_\vec{q} - \theta_\vec{B}) \\
& + \frac{\Delta_D}{\Delta_Z} \cos(\theta_\vec{q} + \theta_\vec{B}) \right],

A & = a_2(\{F^a\}) \frac{v_F^2}{\Delta_Z},
\end{align*}$$

(17)

where $a_0$, $\tilde{a}_0$, $a_1$, and $a_2$ are dimensionless functions. We will now perform a microscopic calculation to confirm the above form for the dispersion of collective modes. The linear-in-$q$ term of the dispersion is the most interesting one, as it is the only term that breaks the symmetry between $\Delta_R$ and $\Delta_D$. This allows one to extract separately the Rashba and Dresselhaus components of SOC from the spectrum of the collective modes. Note that this information cannot be deduced from the $q = 0$ term alone.

### IV. MICROSCOPIC CALCULATION OF CHIRAL SPIN WAVE DISPERSION

In this section, we confirm Eq. (17) by an explicit calculation within the $s$-wave approximation, in which the Landau function contains only the zeroth harmonic: $F^a(\theta) = F_0^a$. In this case, the FL theory is equivalent to the Random Phase Approximation (RPA) in the spin channel.\textsuperscript{10,12} and we adopt the RPA method for convenience. In this scheme, the shaded bubbles in Fig. 2 are approximated by the ladder series of diagrams.\textsuperscript{10,12} Summing up this series, we obtain for the tensor of the spin susceptibility

$$\chi(Q) = \frac{(g \mu_B)^2}{4} \hat{\chi}^+(Q) \left[ \hat{I} + \frac{F_0^a}{2N_F} \hat{\chi}^+(Q) \right]^{-1},$$

(18)

where $Q = (\vec{q}, i\Omega_n)$, $g$ is the effective Landé factor, $\mu_B$ is the Bohr magneton, $N_F$ is the density of states at the Fermi surface, $\hat{I}$ is the $3 \times 3$ identity matrix, $\chi^i_j(Q) = -\int K \text{ Tr} \left[ \hat{\sigma}_i \hat{G}_K^a \hat{G}_j K + Q \right]$, $K \equiv (\vec{k}, i\omega_n)$ and $G^a_K$ differs from $G^a_0$ in Eq. (5) only in that the bare Zeeman energy is replaced by the renormalized one, $\Delta^*_Z = \Delta_Z/(1 + F_0^a)$, while $\Delta_R$ and $\Delta_D$ are not renormalized in the $s$-wave approximation.\textsuperscript{7,26,27} The collective modes correspond to the poles of $\chi$ and can be found as roots of the equation

$$\text{Det} \left[ \hat{I} + \frac{F_0^a}{2N_F} \hat{\chi}^*(\vec{q}, i\Omega_n \rightarrow \Omega + i0^+) \right] = 0.$$  

(19)

The details of solving this equation are purely mathematical in nature and are presented in Appendices A1, A2, and A3. Here, we only state that the results indeed coincide with those in Eq. (17) and give explicit expressions for the dimensionless functions, which read

$$\begin{align*}
\tilde{a}_0(F_0^a) & = -\frac{(1 + F_0^a)(2 + F_0^a)}{2F_0^a}, \\
a_0(F_0^a) & = \frac{(1 + F_0^a)(2 + 3F_0^a)}{4F_0^a}, \\
a_1(F_0^a) & = -\frac{(1 + F_0^a)^2 \left[ 4 + (F_0^a)(1 + F_0^a) + (F_0^a)^2 \right]}{F_0^a(2 + F_0^a)^2}, \\
a_2(F_0^a) & = \frac{(1 + F_0^a)^2}{2F_0^a}.
\end{align*}$$

(20)

[The forms of $a_0$ and $a_2$ coincide with the results of Refs. 12 and 24, respectively, in the $s$-wave approximation.] For repulsive $ee$ $F_0^a < 0$ and the quadratic term in the dispersion is always negative, while the sign of the interaction-dependent prefactor of the linear term $a_1(F_0^a)$ is positive. However, the overall sign of the linear term depends on the signs of the Rashba and Dresselhaus couplings, as well as on the orientations of $\vec{q}$ and $\vec{B}$.

The continuum of SFE corresponds to interband transitions and is given by a set of $\Omega$ that satisfy $\Omega = |\varepsilon_{\vec{k}, \pm}^\pm - \varepsilon_{\vec{k}, \pm}^-|$ for all states on the FS. Because $\varepsilon_{\vec{k}, \pm}$ vary around the FS, $\Omega$ varies between the minimum ($\Omega_{\min}$) and maximum ($\Omega_{\max}$) values, which determine a finite width of the continuum even at $q = 0$ (see Fig. 1a). In the presence of $\vec{B}$, both $\Omega_{\min}$ and $\Omega_{\max}$ depend on $\theta_\vec{B}$. The anisotropic part of $\varepsilon_{\vec{k}, \pm}$ is given by $\Delta_R \Delta_D \sin 2\theta_\vec{B} + \Delta_R \Delta_Z \sin(\theta_\vec{k} - \theta_\vec{B}) + \Delta_D \Delta_Z \cos(\theta_\vec{k} + \theta_\vec{B})$. As one can see, changing $\theta_\vec{B} \rightarrow \theta_\vec{B} + \pi$ can be compensated by $\theta_\vec{k} \rightarrow \theta_\vec{k} + \pi$, and thus $\Omega_{\min,\max}(\theta_\vec{B})$ have a period of $\pi$. Figure 1b shows $\Omega_{\min,\max}(\theta_\vec{B})$ for a range of the magnetic fields.

### V. COMPARISON WITH EXPERIMENT

We are now in a position to apply our results to recent Raman data on a Cd$_{1-x}$Mn$_x$Te quantum well.\textsuperscript{14-16} In these experiments, $\vec{q}$ and $\vec{B}$ are chosen to be perpendicular to each other, i.e., $\theta_\vec{B} - \theta_\vec{q} = \pm \pi/2$. Accordingly,
the dispersion in Eqs. (1) and (17) is simplified to
\[
\Omega(q, B) = \Delta_Z \left[ 1 + a_0 (r^2 + d^2) + \tilde{a}_0 r \sin 2\theta_B \right. \\
\left. \pm a_1 (r - d \sin 2\theta_B) \frac{v_F q}{\Delta_Z} + a_2 (v_F q)^2 \right].
\]
(21)
where \(r = \Delta_R/\Delta_Z\) and \(d = \Delta_D/\Delta_Z\). Up to the angular dependence of the mode mass, this form was conjectured in Refs. 15,16 on phenomenological grounds. The measured frequency of the mode at \(q = 0\) gives \(\Delta_Z = 0.4\) meV at \(B = 2\) T. For the number density of \(n = 2.7 \times 10^{11}\) cm\(^{-2}\) and effective mass of \(m^* = 0.1 m_e\) (\(m_e\) is the bare electron mass), \(k_F = 1.3 \times 10^{-2}\) Å\(^{-1}\) and \(v_F = 1.0\) eVÅ. The range of \(v_F q\) is from 0 to 0.6 meV.

The theoretical results for the mass term, \(\Omega_0(\theta_B)\), and spin-wave velocity, \(w(\pm \pi/2 + \theta_B, \theta_B)\) [cf. Eq. (1)] are plotted in Fig. 3. Using the parameters specified above, the best agreement with the data is achieved by choosing \(F^0_0 = -0.41\), \(\Delta_R \approx 0.05\) meV, and \(\Delta_D \approx 0.1\) meV, which corresponds to the Rashba and Dresselhaus coupling constants of \(\alpha = 1.9\) meVÅ and \(\beta = 3.8\) meVÅ. The values of \(\alpha\) and \(\beta\) are very close to those found in Ref. 16. An estimate for \(F^0_R\), obtained using a screened Coulomb interaction with a dielectric constant \(\varepsilon = 10.0\) for a CdTe quantum well, \(28\) yields \(F^0_R \approx -0.35\), which is pretty close to the number that reproduces the experimental data. The calculated \(q\)-dependence of the mode (Fig. 3b) also reproduces the experimentally observed one very well.

Based on this agreement between the theory and experiment, we argue that the collective mode observed in Refs. 14–16 is, in fact, one of the sought-after CSWs,7–10 probed in the regime of a strong magnetic field. The same type of experiments performed at lower fields on systems with stronger SOC and/or higher mobilities, e.g., on GaAs/GaAlAs or InGaAs/InAlAs, should reveal the whole spectrum shown in Fig. 1a.

FIG. 3: a) Variation of the CSW velocity \(w\), as defined in Eq. (1), with the angle \(\phi = \theta_B - \pi/2\). Inset: variation of the frequency at \(q = 0\) with \(\phi\). b) Dispersion of a CSW in the CdMnTe quantum well in the magnetic field of 2 T for \(\phi = \pi/4\). To be consistent with experimental geometry, we fixed \(B \perp \vec{q}\). The negative sign of the field implies flipping its direction.

Comparison with other theoretical approaches is now in order. The phenomenological model of Refs. 15 and 16 describes the data assuming very strong (up to a factor of 6.5) renormalization of SOC by many-body effects, which is not consistent with the moderate (< 2) values of parameter \(r_s\) in Cd\(_{1-x}\)Mn\(_x\)Te. Our microscopic theory explains the data without such an assumption. In addition, it is argued in Ref. 15 and 16 the entire \(q\)-dependence of a collective-mode spectrum in the presence of SOC can be obtained by a linear shift of the quadratic term in the SL dispersion: \(A q^2 \rightarrow A [\vec{q} + \vec{q}_0]^2\). We see from Eq. (20) that this would require \(|a_1| = 2 |a_2|\). This is reproduced only in the weak coupling limit \(|F^0_0| \ll 1\).

**Note.** When this manuscript was almost completed, a subset of authors of Refs. 15 and 16 announced a first-principles study in Ref. 29, in which they also identified the \(\pi\)-periodic modulation of the mode frequency at \(q = 0\) as a second-order effect in SOC.

**VI. CONCLUSIONS**

In conclusion, we developed a microscopic theory of Raman scattering in a two-dimensional Fermi liquid in the presence of both Rashba and Dresselhaus spin-orbit couplings, and subject to an in-plane magnetic field. Interplay between an exchange part of the electron-electron interaction and spin-orbit coupling leads to resonance peaks at frequencies corresponding to novel collective modes–chiral spin waves. We derived the polarization dependence of the Raman signal and showed that the Raman spectrum can be used to uniquely determine different components of spin-orbit coupling by measuring a characteristic linear-in-\(q\) term in the dispersion of chiral spin waves. Our theory describes quantitatively all the features of the Raman signal observed recently on a CdTe quantum well.14–16 The formalism developed here can be readily extended to other two-dimensional systems with broken inversion symmetry, such as graphene on transition-metal-dichalcogenide substrates and surface states of topological insulators/superconductors.

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**Appendix A: Computational details**

The general form of the dispersion a CSW is established in Eqs. (1) and (17) using symmetry and dimensional analysis. Here, we confirm this form by an explicit calculation which yields the forms of functions \(a_0, a_0, a_1\), and \(a_2\). We start from Eq. (19) and find the expansions of the spin-spin correlation functions, \(\chi^{+\alpha}_{\beta}(Q)\), to order \(q^2\). Unless stated otherwise, all frequencies in the Appendix are the Matsubara ones. Whenever it does not lead to a
confusion, the Matsubara index \( n \) will be suppressed and the frequencies will be denoted simply as \( \Omega \). Analytic continuation to real frequencies will be done at the final step. We follow the general scheme developed in Refs. 10 and 12 to find the dispersions of the collective modes. For brevity, we will switch to notations where \( F_0^a \equiv -u \) and \( \chi_{ab}^* = -\Pi_{ab} \).

1. Spin-charge correlation functions in the presence of magnetic field and spin-orbit coupling

In this Appendix, we consider the general properties of the spin-charge correlation function defined as

\[
\Pi_{ab}(Q) = \int \frac{d\omega}{2\pi} \int \frac{d\xi}{4\pi^2} \text{Tr} \left[ \hat{\sigma}_a \hat{G}_K^* + i\hat{q} \right], \quad (A1)
\]

where summation over repeated indices \( i, j \in \{1, 2\} \) is implied and a tilde above a quantity means that its \((2 + 1)\) momentum is shifted by \( Q \) with respect to the momentun of the corresponding quantity without a tilde, i.e., \( \tilde{g}_\pm = g_\pm(K + Q) \), etc. Since \( s_l \) \((l = 1, 2)\) depend only on the spatial momentum, it is understood that in this case \( s_l \equiv s_l(K + \mathbf{q}) \). Because the magnetic field and SOC are assumed to be weak, integration over the actual (anisotropic) FS can be replaced by that over a circular FS of radius \( k_F \). Accordingly,

\[
\int \cdots = N_F \int \frac{d\theta_k}{2\pi} \int d\xi_k \cdots = N_F \int \frac{d\theta_k}{2\pi} \int \frac{d\xi_k}{4\pi^2} \cdots \quad (A3)
\]

where \( N_F = k_F/2\pi\nu_F \) is the 2D density of states and \( \xi_k = v_F(k - k_F) \). In the same approximation, \( \varepsilon_{\mathbf{q},\nu;\mathbf{q}',\nu'} = \xi_k + \nu|\mathbf{s}| \), where \( |\mathbf{s}| = \sqrt{s_1^2 + s_2^2} \) is evaluated at \( \xi = 0 \) but does depend on the azimuthal angle \( \theta_k \) of \( \bar{k} \). Furthermore, we will be neglecting the difference between \( s_l \) (evaluated at \( \mathbf{k} + \mathbf{q} \)) and \( s_l \) (evaluated at \( \mathbf{k} \)); keeping such terms would amount to higher order corrections. Accordingly, we approximate \( \varepsilon_{\mathbf{q},\nu;\mathbf{q}',\nu'} = \xi_k + \nu|\mathbf{s}| \equiv \xi_k + \nu \bar{F} \cdot \bar{q} + \nu|\mathbf{s}| \), where we also neglected the \( \mathcal{O}(q^2) \) term as being higher order in \( q/k_F \).

The chiral Green’s functions at momentum \( \mathbf{k} + \mathbf{q} \) need to be expanded to second order in \( \mathbf{q} \). Within the same approximations as specified above,

\[
\tilde{g}_{\nu} = g_{\nu}(\mathbf{k} + \mathbf{q}, i\omega + i\Omega)
\]

\[
\approx g_{\nu} + \partial_l g_{\nu} q_l + \frac{1}{2} \partial_l \partial_l' g_{\nu} q_l q_l' + \bar{F} \cdot \bar{q} + \bar{q}^2 (\bar{F} \cdot \bar{q})^2, \quad (A4)
\]

where \( g_{\nu} \equiv g_{\nu}(\mathbf{k}, i\omega + i\Omega) \) and \( \partial_l \equiv \partial_l/\partial k_l \).

In what follows, we will need the following integrals

\[
\int_{\xi_k} \int_{\omega} g_{\nu} g_{\nu'} = -\frac{(\nu - \nu')|\mathbf{s}|}{i\Omega + (\nu - \nu')|\mathbf{s}|},
\]

\[
\int_{\xi_k} \int_{\omega} g_{\nu} g_{\nu} = \frac{i\Omega}{(i\Omega + (\nu - \nu')|\mathbf{s}|)^2},
\]

\[
\int_{\xi_k} \int_{\omega} g_{\nu} g_{\nu} = \frac{i\Omega}{(i\Omega + (\nu - \nu')|\mathbf{s}|)^3}, \quad (A5)
\]

where \( \xi_{\mathbf{q}} \equiv \int d\xi_k \). Using these integrals, we find

\[
\int_{\xi_k} \int_{\omega} g_{\nu} g_{\nu} = \frac{\bar{F} \cdot \bar{q}}{i\Omega} + \frac{(\bar{F} \cdot \bar{q})^2}{(i\Omega)^2}, \quad (A6a)
\]

\[
\int_{\xi_k} \int_{\omega} g_{\nu} g_{\nu} = \int_{\xi_k} \int_{\omega} g_{\nu} + g_{\nu}, \quad (A6b)
\]

\[
\int_{\xi_k} \int_{\omega} g_{\nu} g_{\nu} = \frac{2 |\mathbf{s}|}{i\Omega \pm 2 |\mathbf{s}|} + \frac{i\Omega}{(i\Omega \pm 2 |\mathbf{s}|)^2} \bar{F} \cdot \bar{q}
\]

\[
+ \frac{i\Omega}{(i\Omega \pm 2 |\mathbf{s}|)^3} (\bar{F} \cdot \bar{q})^2. \quad (A6c)
\]

where, as before, \( K = (\bar{k}, i\omega_m) \), etc., \( a, b \in \{0, 1, 2\} \) with the 0th component corresponding to the charge and the 1st...3rd components corresponding to three spin projections. The Green’s function \( G^\sigma_{\nu} \) is obtained from \( G^0_{\nu} \) in Eq. (5) by replacing \( \Delta_Z \rightarrow \Delta^*_{Z} = \Delta_Z/(1 - u) \). Substituting Eq. (5) into Eq. (A1), we obtain
Since we put \( \hat{s}_t = s_t \) in Eq. (A2), the spin-charge components of the polarization operator, \( \Pi_{ab}(Q) \) with \( a \neq 0 \), are reduced to a combination \( \Pi_{ab} \propto \int_{\xi} \int_{\omega} (g_+ \hat{g}_+ - g_- \hat{g}_-) s_a \), which is equal to zero by Eq. (A6b). Thus the spin sector is decoupled from the charge sector in the limit of \( q/k_F \to 0 \). Restricting to \( a, b \in \{1, 2, 3\} \), we evaluate the traces occurring in Eq. (A2) as

\[
\begin{align*}
\text{Tr}[\hat{s}_a \hat{s}_b] s_j &= i \lambda_{ab} s_j \\
\text{Tr}[\hat{s}_a \hat{s}_b \hat{s}_c] s_j s_j &= -2 \delta_{ab} s^2 + 4 s_a s_b,
\end{align*}
\] (A7)

where \( \lambda_{abc} \) is the Levi-Civita tensor and it is understood that \( s_3 = 0 \). We thus obtain a compact form of the spin-spin correlation function with \( a, b \in \{1, 2, 3\} \):

\[
\Pi_{ab}(Q) = N_F \int_{\theta_k} \int_{\xi} \int_{\omega} \left[ (g_+ \hat{g}_+ - g_- \hat{g}_-) \frac{s_a s_b}{|s|^2} + (g_+ \hat{g}_+ - g_- \hat{g}_+) \left( \delta_{ab} - \frac{s_a s_b}{|s|^2} \right) - i (g_+ \hat{g}_+ - g_- \hat{g}_+) \lambda_{abc} \frac{s_c}{|s|^2} \right],
\] (A8)

where the integrals over \( \omega \) and \( \xi \) are to be substituted from Eqs. (A6a-A6b). For further convenience, we also list explicit formulas for \( s_a \) and relevant quantities:

\[
\begin{align*}
s_1 &= \frac{1}{2} \left( \Delta_R \sin \theta_k + \Delta_D \cos \theta_k - \Delta_Z^* \cos \theta_B \right), \\
s_2 &= \frac{1}{2} \left( -\Delta_R \cos \theta_k - \Delta_D \sin \theta_k - \Delta_Z^* \sin \theta_B \right), \\
4s_1^2 &= \Delta_R^2 \sin^2 \theta_k + \Delta_D^2 \cos^2 \theta_k + (\Delta_Z^*)^2 \cos^2 \theta_B + \Delta_R \Delta_D \sin \theta_k \cos \theta_B - 2 \Delta_D \Delta_Z^* \cos \theta_k \cos \theta_B, \\
4s_2^2 &= \Delta_R^2 \cos^2 \theta_k + \Delta_D^2 \sin^2 \theta_k + (\Delta_Z^*)^2 \sin^2 \theta_B + \Delta_R \Delta_D \sin \theta_k \sin \theta_B + 2 \Delta_D \Delta_Z^* \sin \theta_k \sin \theta_B, \\
4s_1 s_2 &= \Delta_R^2 + \Delta_D^2 + (\Delta_Z^*)^2 + 2 \Delta_R \Delta_D \sin \theta_k \sin \theta_B + 2 \Delta_D \Delta_Z^* \sin \theta_k \sin \theta_B, \\
4s_1 s_2 &= -(\Delta_R^2 + \Delta_D^2) \sin \theta_k \cos \theta_B + (\Delta_Z^*)^2 \sin \theta_B \cos \theta_B - \Delta_R \Delta_D \sin \theta_k \cos \theta_B + \Delta_D \Delta_Z^* \cos \theta_k \cos \theta_B, \\
\end{align*}
\] (A9)

where \( \theta_B \) is the angle between \( \hat{B} \) and the \( x_1 \) axis.

We can now apply the general result, Eq. (A8) to particular situations. In what follows, we will make \( v_f q \) and \( \Omega \) dimensionless by rescaling to \( \Delta_Z^* = \Delta_Z/(1 - u) \) and define, for the use in Appendices only, \( r = \Delta_R/\Delta_Z^* \) and \( d = \Delta_D/\Delta_Z^* \). Note that these definitions differ from those in the main text, where \( \Delta_R \) and \( \Delta_D \) are rescaled to \( \Delta_Z \).

### 2. Silin-Leggett mode

To test our general formula, we apply it first to a simple case of the Silin-Leggett mode, the dispersion of which is known.\(^{24}\) The Silin-Leggett mode is the collective mode in the absence of SOC \((r = d = 0)\) and in the presence of \( \hat{B} \). Although the coordinate system in this case is in fact defined by the direction of \( \hat{B} \), we choose \( \hat{B} \) to point in an arbitrary direction with respect to a fixed coordinate system. This will useful for a more general case, when the spin-rotational symmetry is broken by SOC. In this case, \( s = \Delta_Z^* \) is independent of angle \( \theta_B \). Using Eq. (A8), we find:

\[
\begin{align*}
\Pi_{11}(Q) &= \frac{(v_f q)^2}{2 \Omega^2} + \frac{\sin^2 \theta_B}{\Omega^2 + 1} \left( 1 - \frac{6 \Omega^4 + 3 \Omega^2 + 1 (v_f q)^2}{\Omega^2 (\Omega^2 + 1)^2} \right), \\
\Pi_{12}(Q) &= \frac{(v_f q)^2}{2 \Omega^2} + \frac{\cos \theta_B}{\Omega^2 + 1} \left( 1 - \frac{6 \Omega^4 + 3 \Omega^2 + 1 (v_f q)^2}{\Omega^2 (\Omega^2 + 1)^2} \right), \\
\Pi_{13}(Q) &= \frac{\Omega \sin \theta_B}{\Omega^2 + 1} \left( 1 - \frac{3 \Omega^2 - 1 (v_f q)^2}{(\Omega^2 + 1)^2} \right), \\
\Pi_{22}(Q) &= \frac{(v_f q)^2}{2 \Omega^2} + \frac{\cos^2 \theta_B}{\Omega^2 + 1} \left( 1 - \frac{6 \Omega^4 + 3 \Omega^2 + 1 (v_f q)^2}{\Omega^2 (\Omega^2 + 1)^2} \right), \\
\Pi_{23}(Q) &= \frac{\Omega \cos \theta_B}{\Omega^2 + 1} \left( 1 - \frac{3 \Omega^2 + 1 (v_f q)^2}{(\Omega^2 + 1)^2} \right), \\
\Pi_{33}(Q) &= \frac{\Omega \cos \theta_B}{\Omega^2 + 1} \left( 1 \right),
\end{align*}
\] (A10)
\[
\Pi_{ab}(Q) = \left( \begin{array}{ccc}
\kappa_a + (B + \kappa_b) \sin^2 \theta_B & \kappa_f - (B + \kappa_b) \sin \theta_B \cos \theta_B & (C + \kappa_c) \sin \theta_B + \kappa_d \cos \theta_B \\
-(B + \kappa_b) \sin \theta_B \cos \theta_B & \kappa_a + (B + \kappa_b) \cos^2 \theta_B & -(C + \kappa_c) \cos \theta_B - \kappa_d \sin \theta_B \\
-(C + \kappa_c) \sin \theta_B - \kappa_d \cos \theta_B & (C + \kappa_c) \cos \theta_B + \kappa_d \sin \theta_B & 2\kappa_a + (B + \kappa_b)
\end{array} \right),
\]

(A11)

where

\[
\mathcal{B} = \frac{1}{\Omega^2 + 1}, \quad C = -\frac{\Omega}{\Omega^2 + 1}.
\]

(A12)

The other definitions are apparent from a term-by-term comparison entries in Eq. (A10) and the corresponding entries in Eq. (A11). All the \(\kappa\)'s are corrections that are small in \(v_{Fq}\). At \(v_{Fq} = 0\), the eigenmode equation \(\text{Det}(1 + u\Pi/2N_F) = 0\) has a solution \(\Omega^2 + 1 = s_0 = 2u - u^2\) such that \(\Omega^2 = -(1 - u)^2\).

Restoring the dimensional frequency and continuing analytically to real frequencies, we find that the frequency at the \(q = 0\) is simply given by \(\Omega = \Delta_Z\), in agreement with Kohn’s theorem. To find corrections to this result at small but finite \(v_{Fq}\), we look for a solution of the form \(\Omega^2 + 1 = s_0 + \kappa_0\). Expanding the eigenmode equation to linear order in all the \(\kappa\)'s, we obtain

\[
1 - 2Bu + (B^2 + C^2)u^2 = u [(1 - Bu)(\kappa_a + \kappa_b + \kappa_g) - 2u\kappa_c].
\]

(A13)

Solving for \(\kappa_0\), we obtain

\[
\kappa_0 = \frac{2(1 - u)^2 (v_{Fq})^2}{u}.
\]

(A14)

Restoring the units and continuing analytically, we obtain the frequency of the mode at finite \(q\) as

\[
\Omega = \Delta_Z - \frac{(1 - u)^2 (v_{Fq})^2}{2u \Delta_Z}.
\]

(A15)

Relabeling \(u \rightarrow -F_0\), we obtain the coefficient \(a_2\), as given by Eq. (20).

3. Chiral spin wave in the high-field limit

\subsection*{a. Frequency of a chiral spin wave at \(q = 0\)}

In this section, we derive the frequency of the chiral spin wave at \(q = 0\) and in the high-field limit, when \(\Delta_R, \Delta_D \ll \Delta_Z\) or \(r, d \ll 1\) for dimensionless quantities. Unless \(u\) is very close 1, i.e., the system is close to a ferromagnetic transition, there is no real difference between conditions \(\Delta_R, \Delta_D \ll \Delta_Z\) and \(\Delta_R, \Delta_D \ll \Delta_Z\). With \(\Pi_{ab}(i\Omega) \equiv \Pi_{ab}(i\Omega, \vec{q} = 0)\), we obtain from Eq. (A8)

In a matrix form,

\[
\Pi(i\Omega) = \left( \begin{array}{ccc}
\kappa_f - (B + \kappa_b) \sin \theta_B \cos \theta_B & \kappa_a + (B + \kappa_b) \sin^2 \theta_B & (C + \kappa_c) \sin \theta_B + \kappa_d \cos \theta_B \\
(B + \kappa_b) \sin \theta_B \cos \theta_B & \kappa_a + (B + \kappa_b) \cos^2 \theta_B & -(C + \kappa_c) \cos \theta_B - \kappa_d \sin \theta_B \\
-(C + \kappa_c) \sin \theta_B - \kappa_d \cos \theta_B & (C + \kappa_c) \cos \theta_B + \kappa_d \sin \theta_B & 2\kappa_a + (B + \kappa_b)
\end{array} \right),
\]

(A17)
where \( B \) and \( C \) are the same as in Eq. (A12) and the \( \kappa \)'s, which are small in \( r \) and \( d \), are again defined by a term-by-term comparison of Eq. (A16) and the entries in Eq. (A17). We seek a solution of the form \( \Omega^2 + 1 = s_0 + \kappa_0 \), where \( \kappa_0 \) is also small in \( r \) and \( d \). To linear order in \( \kappa \)'s, the eigenmode equation reads

\[
1 - 2Bu + (B^2 + C^2)u^2 = u \left[ (1 - Bu)(2\kappa_0 + 3\kappa_a - \kappa_f \sin 2\theta_B) - 2uC(\kappa_c + \kappa_d \sin 2\theta_B) \right].
\]  

(A18)

Solving for \( \kappa_0 \) we get:

\[
\kappa_0 = (r^2 + d^2) \left\{ \frac{(1 - u)(2 - 3u)}{2u} \right\} - rd \sin 2\theta_B \left\{ \frac{(1 - u)(2 - u)}{u} \right\}.
\]  

(A19)

Restoring the units and continuing analytically to real frequencies, we obtain the coefficients \( a_0 \) and \( \tilde{a}_0 \) in Eq. (20).

### b. Linear-in-\( q \) term in the dispersion

Now we are interested in all terms to linear order in \( r \), \( d \), and \( v_Fq \). Accordingly, we need to expand the quantities in Eq. (A9) to linear order in these variables:

\[
2s_1 = \Delta^*_B (-\cos \theta_B + r \sin \theta_B + d \cos \theta_B),
\]

\[
2s_2 = \Delta^*_B (-\sin \theta_B - r \cos \theta_B - d \sin \theta_B),
\]

\[
4s_1^2 = (\Delta^*_B)^2 (\cos^2 \theta_B + 2r \sin \theta_B \cos \theta_B - 2d \cos \theta_B \cos \theta_B),
\]

\[
4s_2^2 = (\Delta^*_B)^2 (\sin^2 \theta_B + 2r \cos \theta_B \sin \theta_B + 2d \sin \theta_B \sin \theta_B),
\]

\[
4s_1s_2 = (\Delta^*_B)^2 [\sin \theta_B \cos \theta_B + r \cos (\theta_B + \theta_B) + d \sin (\theta_B - \theta_B)].
\]  

(A20)

Furthermore,

\[
\tilde{v}_F \cdot \tilde{q} = v_F q \cos (\theta_B - \theta_B)
\]

\[
\int_{\xi_F} \int_{\omega} (\tilde{g}_+ \tilde{g}_+ + \tilde{g}_- \tilde{g}_-) = \frac{2v_Fq}{\Omega^2 + 1} \cos (\theta_B - \theta_B),
\]

\[
\int_{\xi_F} \int_{\omega} (\tilde{g}_+ \tilde{g}_- + \tilde{g}_- \tilde{g}_+) = -\frac{2}{\Omega^2 + 1} \left[ 1 - \tilde{r}_\theta_B \frac{\Omega^2}{\Omega^2 + 1} \right] - \frac{2\Omega v_F q}{(\Omega^2 + 1)^2} \left[ \Omega^2 - 1 + \tilde{r}_\theta_B (3\Omega^2 - 1) \right] \cos (\theta_B - \theta_B),
\]

\[
\int_{\xi_F} \int_{\omega} (\tilde{g}_+ \tilde{g}_- - \tilde{g}_- \tilde{g}_+) = 2 \frac{\Omega}{\Omega^2 + 1} \left[ 1 \right] + \frac{4\Omega^2 v_F q}{(\Omega^2 + 1)^2} \left[ 1 - \tilde{r}_\theta_B (\Omega^2 - 3) \right] \cos (\theta_B - \theta_B),
\]  

(A21)

where \( \tilde{r}_\theta = r \sin (\theta - \theta_B) + d \cos (\theta + \theta_B). \) Let’s further introduce

\[
r_{1,\theta} \equiv r \sin \theta + d \cos \theta,
\]

\[
r_{2,\theta} \equiv r \cos \theta + d \sin \theta,
\]

\[
r_{3,\theta} \equiv r_{1,\theta} \sin \theta_B + r_{2,\theta} \cos \theta_B
\]  

(A22)
such that \( r_{1,\theta} \cos \theta_B - r_{2,\theta} \sin \theta_B = \tilde{r}_\theta \). Note that the pairs \((\tilde{r}_\theta, r_{3,\theta})\) and \((r_{1,\theta}, r_{2,\theta})\) are related by a \( \theta_B \) rotation. This leads to:

\[
\begin{align}
\Pi_{11}(Q) &= \frac{\sin^2 \theta_B}{2 N_F} \left[ 1 + \frac{i \Omega(3 \Omega^2 - 1)}{(\Omega^2 + 1)^2} v_F q \tilde{r}_\theta q \right] + \frac{v_F q}{\Omega} \frac{\Omega^2(3 \Omega^2 + 1)}{2(\Omega^2 + 1)^2} r_{3,\theta} q \sin^2 \theta_B, \\
\Pi_{22}(Q) &= \frac{\cos^2 \theta_B}{2 N_F} \left[ 1 + \frac{i \Omega(3 \Omega^2 - 1)}{(\Omega^2 + 1)^2} v_F q \tilde{r}_\theta q \right] - \frac{v_F q}{\Omega} \frac{\Omega^2(3 \Omega^2 + 1)}{2(\Omega^2 + 1)^2} r_{3,\theta} q \sin^2 \theta_B, \\
\Pi_{12}(Q) &= \frac{1}{2 N_F} \left[ 1 + \frac{i \Omega(3 \Omega^2 - 1)}{(\Omega^2 + 1)^2} v_F q \tilde{r}_\theta q \right], \\
\Pi_{13}(Q) &= -\frac{\Omega \sin \theta_B}{2 N_F} \left[ 1 + \frac{i \Omega(3 \Omega^2 - 1)}{(\Omega^2 + 1)^2} v_F q \tilde{r}_\theta q \right], \\
\Pi_{23}(Q) &= \frac{\Omega \cos \theta_B}{2 N_F} \left[ 1 + \frac{i \Omega(3 \Omega^2 - 1)}{(\Omega^2 + 1)^2} v_F q \tilde{r}_\theta q \right], \\
\end{align}
\]

In a matrix form,

\[
\hat{\Pi}(Q) = \begin{pmatrix}
\kappa_a \sin 2 \theta_B + (B + \kappa_b) \sin^2 \theta_B & -\kappa_a \cos 2 \theta_B - (B + \kappa_b) \sin \theta_B \cos \theta_B & -\kappa_a \sin \theta_B + (B + \kappa_b) \cos^2 \theta_B - (C + \kappa_c) \sin \theta_B - \kappa_d \cos \theta_B \\
-\kappa_a \cos 2 \theta_B - (B + \kappa_b) \sin \theta_B \cos \theta_B & \kappa_a \cos 2 \theta_B - (B + \kappa_b) \sin \theta_B \cos \theta_B & (C + \kappa_c) \cos \theta_B - \kappa_d \sin \theta_B \\
(C + \kappa_c) \sin \theta_B + \kappa_d \cos \theta_B & -(C + \kappa_c) \cos \theta_B + \kappa_d \sin \theta_B & (B + \kappa_b)
\end{pmatrix}.
\]

Once again \( B \) and \( C \) are the same as before and the \( \kappa ' \)'s are obtained by a term-term comparison of Eqs. (A23-A28) and Eq. (A29).

We look for a solution of the form \( \Omega^2 + 1 = s_0 + \kappa_0 \). To linear order in \( \kappa ' \)'s, the eigenmode equation is now of the following form

\[
1 - 2 Bu + (B^2 + C^2) u^2 = 2u \{(1 - Bu)\kappa_b - uC\kappa_c\}
\]

Solving for \( \kappa_0 \), we get

\[
\begin{align}
\kappa_0 &= -\frac{2(1-u)^2[(4-u)(1-u) + u^2]}{u(2-u)^2} v_F q \tilde{r}_\theta q \\
&= -\frac{2(1-u)^2[(4-u)(1-u) + u^2]}{u(2-u)^2} v_F q \left(r \sin(\theta_q - \theta_B) + d \cos(\theta_q + \theta_B)\right).
\end{align}
\]

From here, one can read the coefficient \( a_1 \) as given by Eq. (20).

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