Forced Linear Oscillators and the Dynamics of Euclidean Group Extensions

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Abstract. We study the generic dynamical behaviour of skew-product extensions generated by cocycles arising from equations of forced linear oscillators of special form. This work extends our earlier work on cocycles into compact Lie groups arising from differential equations of special form, (cf. [21]), to the case of non-compact fiber groups of Euclidean type. The earlier techniques do not work in the non-compact case. In the non-compact case one of the main obstacle is the lack of ‘recurrence’. Thus, our approach to studying Euclidean group extensions is : (i) first, to use a ‘twisted version’ of the so called ‘conjugation approximation method’ and then (ii) to use ‘geometric-control theoretic methods’ developed in our earlier work (cf. [20] and [21]). Even then, our arguments only work for base flows that admit a global Poincaré section, (e.g. for the irrational rotation flows on tori and for certain nil flows). We apply these results to study generic spectral behaviour of the forced quantum harmonic oscillator with time dependent stationary force restricted to satisfy given constraints.

1. Preliminaries. A : Introduction. Forced oscillations of time dependent linear systems are modelled by the equation

\[ y' = Ay + F(t), \]

where \( x \in \mathbb{R}^n \), \( F : \mathbb{R} \to \mathbb{R}^n \) and \( A : \mathbb{R} \to M(n, \mathbb{R}) \) are continuous functions. Here \( M(n, \mathbb{R}) \) denotes the set of \( n \times n \) real matrices. The function \( A \) models the unperturbed motion (i.e. the natural oscillations of the system) and the function \( F : \mathbb{R} \to \mathbb{R}^n \) is a perturbation modelling the external force. Typically the time dependence of \( A(t) \) and \( F(t) \) will be ‘recurrent’ or quasi-periodic. Such functions can be best modelled by thinking of them as evaluation of a given function along a trajectory of some compact recurrent flow \( (\Omega, \{T_t\}) \). Here, by a compact flow we mean a compact metric space \( \Omega \) with a one parameter family \( \{T_t\} \) of homeomorphisms of \( \Omega \) so that the map \( (t, \omega) \to T_t(\omega) \) is continuous on \( \mathbb{R} \times \Omega \). Given a single equation such as above, the space \( \Omega \) can be thought of as the ‘hull’ of the function \( t \to (F(t), A(t)) \), (i.e. as the closure of translates \( (F, A)_\tau(t) = (F(t+\tau), A(t+\tau)) \), \( (t, \tau \in \mathbb{R}) \), of the given function, where the closure is in a suitable topology on the space of functions). For example, quasi-periodic functions can be modelled this way by taking \( \Omega \) to be the torus with an irrational rotation flow. Thus, rather than studying a single equation such as above, we begin with a

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given flow \((\Omega, \{T_t\}_{t \in \mathbb{R}})\) on a compact metric space \(\Omega\) and given continuous functions \(A : \Omega \to M(n, \mathbb{R})\) and \(F : \Omega \to \mathbb{R}^n\) and consider a family of equations

\[ y'(t) = A(T_t \omega) y(t) + F(T_t \omega), \]

parametrized by a point \(\omega \in \Omega\). We observe that the above equation is generated by a right invariant vector field on the Lie group \(\Gamma = \mathbb{R}^n \ltimes GL(n, \mathbb{R})\) which is the semi-direct product of the vector group \(\mathbb{R}^n\) with \(GL(n, \mathbb{R})\). Note that the group multiplication on \(\Gamma\) is given by

\[ (v_1, g_1) \cdot (v_2, g_2) = (v_1 + U_{g_1} v_2, g_1 g_2), \quad (1.1) \]

where \(g \to U_g\) is a natural representation of \(G = GL(n, \mathbb{R})\) as linear transformations on \(\mathbb{R}^n\). For each \(\omega \in \Omega\) and \(t \in \mathbb{R}\), the given map \(\omega \to (F(\omega), A(\omega)) \in L(\Gamma) = \mathbb{R}^n \ltimes M(n, \mathbb{R})\)-the Lie algebra of \(\Gamma\), generates a time dependent right invariant vector field given by

\[ (y, x) \to (F(T_t \omega), A(T_t \omega)) \cdot (y, x) = (A(T_t \omega) y + F(T_t \omega), A(T_t \omega) x) : \Gamma \to T_{(y, x)} \Gamma, \]

where \(T_{(y, x)} \Gamma\)-the tangent space at \((y, x)\) to \(\Gamma\) is identified with \(L(\Gamma)\)-the Lie algebra of \(\Gamma\). Thus, the integral curve \(t \to (y(t), x(t)) \in \Gamma\) of this vector field satisfies

\[ (y'(t), x'(t)) = (A(T_t \omega) y(t) + F(T_t \omega), A(T_t \omega) x(t)). \]

Now we shall summarize as well as generalize the above set up:

**B : Notation and The Basic Set-Up.**

1. Let \((\Omega, \{T_t\}_{t \in \mathbb{R}}, \mu)\) be a flow, on a compact metric space \(\Omega\) and \(\mu\) is a flow invariant Borel probability measure on \(\Omega\).
2. Let \((V, \langle , \rangle)\) be a finite dimensional inner product space. Let || \cdot || denote the norm on \(V\) induced by this inner product and let \(\lambda\) denote a fixed Lebesgue measure on \(V\).
3. Let \(G\) be a compact, connected Lie group with Lie algebra \(L(G)\). Let \(\eta\) denote the normalized Harr measure on \(G\).
4. Let \(g \to U_g\) is a unitary (or orthogonal, if \(V\) is a real vector space) representation so that the action \((g, v) \to U_g v\) preserves \(\lambda\).
5. Let \(\Gamma = V \ltimes G\) be the semi-direct product of \(V\) and \(G\) with multiplication as in \((1.1)\).
6. Letter \(d\) will denote metric on all compact metric spaces. Given metric spaces \(X\) and \(Y\), \(C(X, Y)\) denotes the spaces of continuous functions from \(X\) to \(Y\) and letter \(D\) will denote the supremum metric on these function spaces when the domain space is compact.
7. Let \(S \subset L(G)\) and \(K \subset V\), thus \(K \times S \subset L(\Gamma)\) -the Lie algebra of \(\Gamma\), (here we identify the Lie algebra of \(V\) with \(V\) itself). Typically sets and \(S\) and \(K\) are ‘very thin’ sets in the respective Lie algebras.
8. Given any \(A \in C(\Omega, S)\) and \(F \in C(\Omega, K)\) consider the flow equation for the (non-autonomous) right invariant vector field on the Lie group \(\Gamma = V \ltimes G\), generated by the function \((F, A)\):

\[ x'(t) = A(T_t \omega) x(t), \quad (1.2) \]

\[ y'(t) = A(T_t \omega) y(t) + F(T_t \omega). \quad (1.3) \]

Here \(A(\omega)y\) is interpreted as \(A(\omega)y = \left. \frac{d}{dt} \Psi_A(\omega, t)(y) \right|_{t=0} \in V\), where or each \(\omega \in \Omega, \quad t \to \Psi_A(\omega, t) : \mathbb{R} \to G\) denote the ‘fundamental solution’ to the first of the above
equation, (i.e. the solution satisfying the initial condition $\Psi_A(\omega,0) = I$-the identity of $G$). Let

$$\Phi_{(F,A)}(\omega,t) = \int_0^t U_{\Phi_{A}(\omega,s)}^{-1} F(T_s \omega) ds \, .$$ (1.4)

Note that for each $\omega \in \Omega$, the map $t \to (U_{\Phi_{A}(\omega,t)} \Phi_{(F,A)}(\omega,t), \Psi_{A}(\omega,t)) : \mathbb{R} \to \Gamma$ is then the ‘fundamental solution’ for the non-homogenous system. Thus, for each $\omega \in \Omega$, $t \to y(t) = U_{\Phi_{A}(\omega,t)} \Phi_{(F,A)}(\omega,t) : \mathbb{R} \to V$ and $t \to x(t) = \Psi_{A}(\omega,t) : \mathbb{R} \to G$ satisfy the above equations with initial condition $(y(0), x(0)) = (0, I)$, where $I$ is the the identity of $G$. We want to view the map $(\omega,t) \to (U_{\Phi_{A}(\omega,t)} \Phi_{(F,A)}(\omega,t), \Psi_{A}(\omega,t)) : \mathbb{R} \to \Gamma$ as a $\Gamma$ valued cocycle. So we quickly recall the notion of a cocycle and that of a skew-product flow.

**C : Generalities on cocycles and skew-product flows.**

**Definition 1.1.** Given a flow $(\Omega,\{T_t\}_{t \in \mathbb{R}})$ and any arbitrary topological group $\hat{\Gamma}$, a continuous map $\rho : \Omega \times \mathbb{R} \to \hat{\Gamma}$, is a cocycle if it satisfies the following cocycle identity:

$$\rho(\omega,t+s) = \rho(T_t \omega,s) \rho(\omega,s) \, , \text{ for all } \omega \in \Omega \text{, and } t,s \in \mathbb{R} \, .$$

Given a cocycle, define two flows $(\hat{\Gamma} \times \Omega, \{T_t^\rho\}_{t \in \mathbb{R}})$ and $(\hat{\Gamma} \times \Omega, \{T_t^{\rho^R}\}_{t \in \mathbb{R}})$ by setting

$$T_t^{\rho^L}(\gamma, \omega) = (\rho(\omega,t) \gamma, T_t \omega) \text{ and } T_t^{\rho^R}(\gamma, \omega) = (\gamma \rho(\omega,t)^{-1}, T_t \omega) \, .$$

These flows will be referred to as the left and the right skew-product flows generated by $\rho$.

**Remark 1.2.** Let $H : \hat{\Gamma} \times \Omega \to \hat{\Gamma} \times \Omega$ be the map $H(\gamma, \omega) = (\gamma^{-1}, \omega)$. Note that

$$H \circ T_t^{\rho^L} = T_t^{\rho^R} \circ H \, , \text{ (} t \in \mathbb{R} \) \, .$$

Thus the left and right skew-product flows generated by $\rho$ are topologically isomorphic. Furthermore if $\mu$ is a flow invariant measure on $\Omega$, $\hat{\eta}$ is a left Harr measure on $\hat{\Gamma}$ and $\hat{\gamma}$ is unimodular, then the left and right skew-product flows preserve $\hat{\eta} \times \mu$ and are ergodic theoretically isomorphic. Since in this study, $\hat{\Gamma}$ is a Euclidean group, being unimodular, the dynamical and ergodic properties of both, left and the right skew-product flows will be the same.

**D : Cocycles into semi-direct product groups arising from non-homogenous linear systems.**

(1) Note that the map $\rho \equiv \rho_{(F,A)} : \Omega \times \mathbb{R} \to \Gamma \equiv V \times G$ defined by

$$\rho_{(F,A)}(\omega,t) \to (U_{\Phi_A(\omega,t)} \Phi_{(F,A)}(\omega,t), \Psi_A(\omega,t)) \, ,$$ (1.5)

is a cocycle, (i.e. satisfies the cocycle identity), into group $\Gamma$.

(2) Then $y(t)$-the ‘$v$-component’ of the orbit of $(v,g,\omega)$ under the flow $(V \times G \times \Omega, \{T_t^{\rho^L}\}_{t \in \mathbb{R}})$ is given by

$$y(t) = U_{\Phi_A(\omega,t)} \left( v + \Phi_{(F,A)}(\omega,t) \right) \, .$$
(3) The ‘left’ and ‘right’ skew product flows \( \{ T^p_{\rho,L} \}_{t \in \mathbb{R}}, \{ T^p_{\rho,R} \}_{t \in \mathbb{R}} \) on \( V \times G \times \Omega \) are given by

\[
T^p_{\rho,L}(v, g, \omega) = (\Psi_A(\omega, t)v + U\Psi_A(\omega, t)\Phi_{(F,A)}(\omega, t), (\Psi_A(\omega, t))g, T_{\rho} \omega),
\]

and

\[
T^p_{\rho,R}(v, g, \omega) = (v - Ug\Phi_{(F,A)}(\omega, t), g(\Psi_A(\omega, t))^{-1}, T_{\rho} \omega).
\]

Remark 1.3. In this paper, we investigate conditions on sets \( K \subseteq V \) and \( S \subseteq L(G) \) under which a generic \( K \times S \) valued map \( (F,A) \) on \( \Omega \) generates an ergodic cocycle \( \rho(F,A) \), (i.e. the corresponding skew product flow is ergodic).

2. Hypothesis and statements of the main theorems. A comment about the proof: One of the motivations for our efforts here is to see if our earlier ergodicity lifting results, (cf. [21]), can be extended to non-comact group extensions that arise from differential equations of special form. In general, certainly generic ergodicity lifting is not possible when the fiber group is non-compact. In order that such results hold in a class of cocycles, necessarily such a class must contain a dense set of ‘recurrent cocycles’. (Here ‘recurrent’ is in the sense of K. Schmidt, see [23]). When the fiber group is non-compact, (such as the Euclidean group or even \( \mathbb{R}^2 \)), there is no simple sufficiency condition implying recurrence of cocycles. There is no analogue of a well known recurrence criterion, (due to G. Atkinson and K. Schmidt), of ‘vanishing of the integral’ for \( \mathbb{R} \) valued cocycles. So apriori it is not clear which non-compact groups will have abundance of ergodic cocycles generated by solutions of non-autonomous systems of differential equations arising from a given set of its right invariant vector fields.

The problem of generic lifting of ergodicity was solved for compact connected Lie groups, ([21]) and in this case our arguments can be broadly split into two groups, ‘local arguments’ and ‘global arguments’. It is the global arguments that are heavily dependent on the compactness of the fiber group \( G \). These arguments utilize the so called ‘Property P’, which follows from the equicontinuity of the family of inner automorphisms - \( \{ x \to g^{-1}xg : G \to G \mid g \in G \} \) of \( G \), (see (cf.[20])). In general this property cannot hold for non-compact, non-abelian fiber groups. Thus our strategy of proof in this paper is completely different and is outlined below.

(a) We shall assume that our flow has a global section, (i.e. the flow is built from a discrete dynamical system and a ‘roof function’). We shall ‘discretize’ the ergodicity lifting problem and lift ergodicity generically to the Euclidean group extension of the Poincaré section of the flow. For this we shall develop a ‘twisted version’ of the ‘conjugation approximation technique’. We do get abundance of recurrent cocycles into Euclidean groups under the assumption that the representation \( g \to U_g \) does not have non-zero fixed points.

(b) Next, we develop an ‘open mapping theorem’ that will allow us to get ergodicity lifting results for the flow from the corresponding result for the Poincaré transformation on its Poincaré section. This will allow us to prove that, assuming ergodicity of a \( G \)-valued cocycle \( \Psi_A \) generated by a given \( A \in C(\Omega, S) \), a generic \( F \in C(\Omega, K) \) lifts ergodicity to the \( \Gamma = V \times G \) extension, (Theorem (2.8)). This is a perturbation argument where we need to suitably perturb the given function \( F \) keeping it \( K \) valued on ‘sufficiently long pieces’ of the orbits of points in the Poincaré section. This requirement results into a condition (called the ‘support condition’, described below), on the set \( K \), representation \( U_g \), the given \( S \) valued map \( A \), and the infimum of the roof function of the flow.
Finally we show that if $S$ has SAP and $(K - K) \times S$ has AP, then a generic map $(F, A) \in C(\Omega, K) \times C(\Omega, S)$ lifts ergodicity. Our earlier work shows that if $S$ has SAP, for a generic $A \in C(\Omega, S)$, the cocycle $\psi_A$ is ergodic. With the help of the other Lie algebraic condition we shall modify the proof of Theorem (2.8) and obtain this result, (Theorem 2.10). One may ask, how these two Lie algebraic conditions on sets $K$ and $S$ related to the condition that $K \times S$ has SAP? If $K$ contains $\{0\}$ they are equivalent, in general these conditions are very closely related.

(d) The other motivation for this work came from our another earlier work on spectral properties of the ‘forced quantum harmonic oscillator’ (see [19]) which is a topic of considerable interest in Mathematical Physics. In [19] we proved that under fairly general condition, a generic ‘ergodic forcing’ makes the discrete spectrum of the one dimensional quantum harmonic oscillator completely dissapear. Generic lifting of ergodicity to the 2-dimensional Euclidean group extensions plays a key role in this proof. The generic ergodicity result we had at the time [19] was written, was only in the class of all cocycles. This was the reason our results in [19] needed perturbations of both the ‘position and the momentum’ components of the Hamiltonian. From the Physics point of view one would like to have spectral results when only the position component is perturbed by a time dependent field. This motivation lead us to study lifting ergodicity with the constraint that the forcing function $F$ take values in a preassigned set. Our results in this paper yield new spectral results for higher dimensional quantum harmonic oscillators where the forcing functions are constrained to satisfy a variety of restrictions that are captured in terms of our Lie algebraic conditions on sets $S$ and $K$. In the last section of this paper we shall give an illustration of this.

B : A ‘global section condition’ on the flow.

Our results are valid for flows that have a global section. In other words we shall be dealing with flows which are ‘built’ over a discrete dynamical system by a continuous ‘roof function’. we begin by recalling the definition.

Definition 2.1. Let $(\Sigma, T)$ be a discrete dynamical system, i.e. a compact metric space $\Sigma$ with a homeomorphism $T : \Sigma \to \Sigma$. Let $r : \Sigma \to (0, \infty)$ be a continuous function. Let $\Omega = \tilde{\Omega}/\approx$ be the quotient set of the set $\tilde{\Omega} = \{ (\sigma, t) \mid 0 \leq t \leq r(\sigma), \sigma \in \Sigma \}$, where the equivalence relation $\approx$ identifies point $(\sigma, r(\sigma))$ with $(T\sigma, 0)$. Define a jointly continuous flow on $\Omega$ as follows: Let $\omega \in \Omega$ and $t > 0$, say $\omega = (\sigma, s), 0 \leq s \leq r(\sigma)$. Let $n \in \mathbb{N}$ and $s' \in R$ be uniquely defined by the conditions

$$\sum_{k=0}^{n-1} r(T^k \sigma) < s + t \leq \sum_{k=0}^{n} r(T^k \sigma) \quad \text{and} \quad \sum_{k=0}^{n-1} r(T^k \sigma) + s' = s + t,$$

then,

$$T_t \omega = T_t (\sigma, s) = (T^n \sigma, s').$$

This flow on $\Omega$ is called the flow built over $\Sigma$ by a roof function $r$ and will be denoted by $(\Omega_\Sigma, r, \{T_t\}_{t \in R})$. Note that the discrete dynamical system $(\Sigma, T)$ is the global Poincaré section for this flow.

Furthermore, if $\nu$ is a $T$ invariant Borel measure on $\Sigma$. Then the projection onto $\Omega$ of the restriction of the product measure $\nu \times \lambda$ to the set $\tilde{\Omega}$, is an invariant Borel measure for flow built on $\Omega$. We shall normalize this measure and denote this resulting flow invariant probability measure on $\Omega$ by $\mu$. 
Clearly irrational rotation flows on the $n$ torus are examples of such flows. Another class of examples of such flows is given by Heisenberg nil flows where the Poincaré map is the skew-shift on the 2-torus, (see [1]).

C : A ‘No fixed point condition’ on the representation $U_g$.

Now we introduce the condition the representation $g \rightarrow U_g$ needs to satisfy.

**Definition 2.2.** The representation $g \rightarrow U_g$ is said to satisfy a ‘no fixed point condition’ if the $G$ action on $V$ has no non-zero fixed points, i.e. if $U_g v = 0$ for all $g \in G$ then $v = 0$.

**Remark 2.3.** (1) In particular under this hypothesis if $G$ is trivial then $V$ is trivial. (2) The standard matrix representation of $G = SO(n)$ on $\mathbb{R}^n$ satisfies the ‘no fixed point condition’.

D : A ‘Support Condition’ on $K$, $A$ and $U_g$.

**Definition 2.4.** Consider the system as in the basic set up with a given flow on $\Omega$, a compact connected group $G$ with a unitary representation $U_g$ on a vector space $V$, let $A \in C(\Omega, S)$ and $K \subset V$. Let $V_{K-K}$ be the subspace generated by $K-K \equiv \{x-y \mid x, y \in K\}$. Let $T > 0$ and $\omega \in \Omega$. We shall say that $(K, A, U_g)$ has a $T$-support property at $\omega$ if for any $v \in V_{K-K}$, $\langle v, U_{\Psi_{K,A,\omega,t}} w \rangle = 0$ for all $t \in [0, T]$, then $v = 0$. We shall say that $(K, A, U_g)$ has a $T$-support property if this system has the $T$-support property at all $\omega \in \Omega$.

**Remark 2.5.** If $K$ has the $T$-support property, then $K$ cannot be a singleton set.

E : The AP and the SAP Conditions.

Next we describe the notion of ‘accessibility; property and ‘strong accessibility’ property of a subset of a Lie algebra. This notion has its roots in geometric control theory, (see [20] and [21] for more details).

**Definition 2.6.** Let $\Gamma$ be any Lie group with Lie algebra $L \equiv L(\Gamma)$. Let $D \subset L$. Let $L(D)$ denote the Lie subalgebra of $L$ generated by the set $D$ and let $L_0(D)$ denote the ideal of $L(D)$ generated by the set $D - D \equiv \{v_1 - v_2 \mid v_1, v_2 \in D\}$. Then,

(1) the set $\Lambda$ is said have property AP, (the accessibility property), if $L(D) = L$.

(2) Then set $\Lambda$ is said have property SAP, (the strong accessibility property), if $L_0(D) = L$.

**Remark 2.7.** Given a subset $D \subset L$, let

(i) $L^*(D)$ denote the linear span of Lie brackets of all elements of $\Lambda$;

(ii) $\Lambda(D)$ be the linear span of $D$ and

(iii) $\Lambda_0(D) = \{\sum_{i=1}^\ell \lambda_i d_i \mid \sum_{i=1}^\ell \lambda_i = 0, \ d_i \in D, \ \ell \in \mathbb{N}\}$.

Then it easy to show that $L_0(D) = L^*(D) + \Lambda_0(D)$ and $L(D) = L^*(D) + \Lambda(D)$, (see [20]). Thus $L_0(D)$ is already a ‘co-dimension 1’ subspace of $L(D)$.

**Theorem 2.8.** We assume the notation and standing assumptions of the Basic Set-Up. In addition, suppose that

(1) $(\Omega_{\Sigma,T}, \{T_t\}_{t \in \mathbb{R}}, \nu)$ be an ergodic flow built on its global Poincaré section $(\Sigma, T, \nu)$ by a continuous roof function $r : \Sigma \rightarrow (0, \infty)$. Suppose that the system $(\Sigma, T, \nu)$ is uniquely ergodic, aperiodic and measure $\nu$ is supported, (i.e. positive on non-empty open sets).
(2) The representation \( g \to U_g \) satisfies the no-fixed point condition.

(3) Let \( S \subset L(G) \) and \( A \in C(\Omega, S) \) be given so that the cocycle \( \Psi_A : \Omega \times \mathbb{R} \to G \) is ergodic, (i.e. the skew-product flow it generates on \( G \times \Omega \) is ergodic).

(4) Suppose that \( K \subset V \) is a closed convex set and \( (K, A, U_g) \) satisfy the T-support condition at \((\sigma,0)\) for every \( \sigma \in \Sigma \) and \( 0 < T \leq \inf \{ r(\sigma) \mid \sigma \in \Sigma \} \). Then the set

\[
\{ F \in C(\Omega, K) \mid (V \times G \times \Omega, \{ T^t_{(F,A)} \}_{t \in \mathbb{R}}, \lambda \times \eta \times \mu) \text{ is ergodic} \}
\]

is residual subset of \( C(\Omega, K) \).

**Remark 2.9.** We shall show that if the set \( K \) and \( S \) satify certain Lie algebraic conditions, then for a generic \((F, A) \in C(\Omega, K) \times C(\Omega, S)\) ‘certain properties’ hold. These ‘certain properties’ imply assumption (3) and a weaker version of assumption (4). So that a modification of the arguments in the proof of Theorem (2.8) leads to the following.

**Theorem 2.10.** We assume the notation and standing assumptions of the Basic Set-Up. Suppose that the assumptions (1) and (2) of (2.8) hold. In addition suppose that

(3’) the set \( S \) is a closed, convex set with the SAP property and

(4’) the set \( K \) is closed and convex and \((K - K) \times S \) has the property AP, (as a subset of the Lie algebra \( L(V \times G) \)). Then the set

\[
C_{\text{erg}}(\Omega, K \times S) = \{(F, A) \in C(\Omega, K) \times C(\Omega, S) \mid \text{the system } (V \times G \times \Omega, \{ T^t_{(F,A)} \}_{t \in \mathbb{R}}, \lambda \times \eta \times \mu) \text{ is ergodic} \}
\]

is a residual subset of \( C(\Omega, K \times S) \).

**Remark 2.11.** A perfect analog of our theorem for compact Lie group extension (cf. [21]), would be a statement that gives the same conclusion with the assumption that \( K \times S \) has SAP, (instead of \((K - K) \times S \) has AP and \( S \) has SAP). It will be interesting to see how these two Lie algebraic conditions are related. We can show that if \( 0 \in K \), then these two conditions are equivalent.

**Example 2.12.** (Non-autonomous forced harmonic oscillator) Consider the Hamiltonian for the \( n \)-particle Harmonic oscillator with time dependent frequencies that is forced by an external force,

\[
H(q_1, \ldots, q_n, p_1, \ldots, p_n) = \sum_{i=1}^n p_i^2 + \frac{\alpha_i^2 (T_i \omega_i) q_i^2}{2} + \sum_{i=1}^n f_i(T_i \omega_i) q_i,
\]

where the time dependence is modelled by a flow \((\Omega, \{ T_t \}_{t \in \mathbb{R}}) \) and \( \alpha_i : \Omega \to (0, \infty) \) and \( f_i : \Omega \to \mathbb{R} \) are continuous functions modelling the frequencies of the unperturbed oscillator and the external force respectively, for \( 1 \leq i \leq n \). The Hamilton’s equation of motion yields a non-autonomous, non-homogenous system

\[
\frac{d\mathbf{x}_k}{dt} = \begin{pmatrix} 0 & \alpha_k \\ -\alpha_k & 0 \end{pmatrix} \mathbf{x}_k + \begin{pmatrix} f_k(T \omega_k) \\ 0 \end{pmatrix},
\]

where \( \mathbf{x}_k = \begin{pmatrix} \bar{q}_k \\ \bar{p}_k \end{pmatrix} \) and \( \bar{q}_k = \alpha_k \bar{q}_k \). Here \( V = \mathbb{R}^{2n}, \ G = \mathbb{T}^n \) is the \( n \)-torus and the unitary representation of \( G \) on \( V \) is given by \( g \to U_g \) where \( U_g \mathbf{x} = U_{g^x}(\theta_1, \ldots, \theta_n) \mathbf{x} = \mathbf{x} \).
Let $R_0, x_1, \ldots, R_{n-1}$ and $R_\theta$ is the standard rotation matrix corresponding to angle $\theta$.

**Case (1).** Consider the case $n = 1$, thus $V = \mathbb{R}^2$, $K = \{(\frac{x}{\alpha}) : x \in C\} \subset \mathbb{R}^2$, where $C$ is some non-singleton closed convex subset of $\mathbb{R}$. Suppose the frequency $\alpha(t) = \alpha \notin \mathbb{Q}$ is a constant. Let $A = \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix}$. Then the cocycle $\psi_A$ is the constant cocycle $(\omega, t) \to e^{tA}$ and $G = SO(2) \equiv \mathbb{T}$-i.e. the circle group. Action of $G$ in $V - \mathbb{R}^2$ is the standard rotation action. For simplicity, let us just consider the flows built on a discrete dynamical system $(\Sigma, T, \nu)$ by a constant roof function $\sigma \to \tau$, where $\tau$ is a fixed positive real number. We shall assume the ‘resonance condition’-namely that $T$ is a rotation action. For simplicity, let us just consider the flows built on $C$ where $\alpha$ is some non-singleton closed convex subset of $\mathbb{R}$. Suppose the frequency $\alpha_1$ and $\alpha_2$ are rational and time independent and rationally independent. Then one can verify that the cocycle $(\omega, t) \to e^{tA}$ is uniquely ergodic. Thus, if $T$ is a rotation by an irrational number $\beta$, then if $T$ is weakly mixing, the resonance condition will hold. If $T$ is a rotation by an irrational number $\beta$, then resonance condition is equivalent to saying that $\beta$ and $\alpha_\tau$ are rationally independent. In this two cases hypothesis of Theorem (2.8) is satisfied. Thus we conclude that For a generic $F \in C(\Omega, K)$, the skew-product flow $(\mathbb{R}^2 \times \mathbb{T} \times \Omega, \{T_t^{F,A,R}\}_{t \in \mathbb{R}}, \lambda \times \eta \times \mu)$ is ergodic. We shall see a consequence of this result in the last section.

**Case (2).** Consider the case $n = 2$, with $K_1 = \{(x, 0, y, 0) \mid |x| < \varepsilon, |y| < \varepsilon\}$ and $K_2 = \{(x, 0, y, 0) \mid x + y = 1, x \geq 0, y \geq 0\}$. In either case if the frequencies $\alpha_1$ and $\alpha_2$ are time independent and rationally independent. Then one can verify that the triple $(K, A, \mathbb{R})$ satisfies the $T$-support property for any $T > 0$.

**Remark 2.13.** As mentioned above, our strategy of the proofs of these theorems is to first discretize the ergodicity lifting problem. This is not for just convenience, but is essential to our arguments. Our scheme is to first use the ‘conjugation approximation technique’ to lift ergodicity in the class of ‘ALL cocycles’ and then ‘approximate’ the ergodic cocycle by the one arising from the constrained system. This approximation technique works when our flow on $\Omega$ admits a nice ‘global section’. Thus, even though we can give the conjugation approximation argument for flows, we shall really need its discretized version.

3. **Discretization.** We recall that the flow $(\Omega, \{T_t\}_{t \in \mathbb{R}}, \mu)$ is the flow built, i.e. $\Omega \equiv \Omega_{\Sigma, T, \nu}$ and the flow is built over a discrete dynamical system $(\Sigma, T, \nu)$ by a continuous roof function $r$, $\nu$ being a $T$ invariant, supported Borel probability measure on $\Sigma$ and let $\mu$ be the flow invariant Borel measure on $\Omega$ generated by $\nu$. The ergodic properties of the flow on $\Omega$ and that of its Poincaré section $(\Sigma, T, \nu)$ are closely related. The following proposition states a result about this relation.

**Proposition 3.1.** Let $(\Omega_{\Sigma, T}, \{T_t\}_{t \in \mathbb{R}})$ be the flow built, with $\nu$ and $\mu$ as above.

1. The flow $(\Omega_{\Sigma, T}, \{T_t\}_{t \in \mathbb{R}}, \mu)$ is ergodic implies that the discrete dynamical system $(\Sigma, T, \nu)$ is ergodic.

Let $\hat{\Gamma}$ be a locally compact topological group and $\rho : \Omega \times \mathbb{R} \rightarrow \hat{\Gamma}$ be any continuous cocycle. Define $P(\rho) : \Sigma \rightarrow \Gamma$, by setting

$$P(\rho)(\sigma) = \rho((\sigma, 0), r(\sigma)) = e^{r(\sigma)}.$$  \hspace{1cm} (3.1)

Then $P(\rho)$ is a generator of a cocycle for the discrete system $(\Sigma, T, \nu)$. We shall call $P(\rho)$ the Poincaré section of the cocycle $\rho$. With this notation we have,

2. Let the flow be ergodic. Then $\rho$ is ergodic if and only if $P(\rho)$ is ergodic.
Recall that a cocycle is called ergodic if and only if the skew product it generates is ergodic, with respect to the product measure.

Given a pair of functions \( (F, A) : \Omega \to K \times S \subset V \times L(G) \), consider the cocycle \( \rho \equiv \rho^{(F, A)} = (\Psi_A \Phi(F, A), \Psi_A) : \Omega \to V \times G \). Then, the Poincaré section \( P(\rho^{(F, A)}) : \Sigma \to V \times G \) is given by

\[
P(\rho^{(F, A)})(\sigma) = (\varphi(F, A)(\sigma), \psi_A(\sigma)),
\]

where the generators of the component cocycles are denoted by \( \varphi(F, A) \) and \( \psi_A \) respectively and they are given by

\[
\psi_A(\sigma) = \Psi_A((\sigma, 0), r(\sigma)).
\]

and

\[
\varphi(F, A)(\sigma) = \Phi(F, A)((\sigma, 0), r(\sigma)).
\]

Let \( T_{(F, A)} \) and \( T_A \) denote the (right) skew product transformations on \( V \times G \times \Sigma \) and on \( G \times \Sigma \) generated by the cocycle \( P(\rho^{(F, A)}) \) and \( \psi_A \) respectively. Note that these are the Poincaré maps on the sections \( V \times G \times \Sigma \) and \( V \times G \) to the corresponding skew-product flows (i.e. \( \mathbb{R} \) actions), respectively. Note that,

\[
T^n_{(F, A)}(v, g, \sigma) = (v - U_g \varphi(F, A)(\sigma, n), g(\psi_A(\sigma, n))^{-1}, T^n\sigma),
\]

where

\[
\psi_A(\sigma, n) = \psi_A(T^{n-1}\sigma) \cdots \psi_A(\sigma) = \Psi_A(T^{k-1}\sigma, 0) r(T^{k-1}\sigma) \cdots \Psi_A((\sigma, 0), r(\sigma)) = \Psi_A((\sigma, 0), \sum_{j=0}^{k-1} r(T^j \sigma)) \tag{3.5}
\]

and

\[
\varphi(F, A)(\sigma, n) = \Phi(F, A)((\sigma, 0), \sum_{k=0}^{n-1} r(T^k \sigma)).
\]

Now let \( \omega = (\sigma, 0) \) and set \( r(\sigma, k) = \sum_{i=0}^{k-1} r(T^i \sigma), k \in \mathbb{N} \) and consider

\[
\varphi(F, A)(\sigma, n) = \Phi(F, A)(\omega, r(\sigma, n)) = \int_0^{r(\sigma, n)} U_{(\psi_A(\omega, s))^{-1} F(T_s \omega)} ds = \sum_{k=0}^{n-1} \int_0^{r(T^k \sigma)} U_{(\psi_A(\omega, s))^{-1} F(T_s \omega)} ds = \sum_{k=0}^{n-1} U_{(\psi_A(\omega, T_s \sigma))^{-1} F(T_s(T^k \sigma, 0))} ds = \sum_{k=0}^{n-1} U_{(\psi_A(\omega, T_s \sigma))^{-1} \xi_A(F)(T^k \sigma)} \tag{3.6},
\]

where

\[
\xi_A(F)(\sigma) = \int_0^{\rho(\sigma)} U_{(\psi_A(\omega, s))^{-1} F(T_s \omega)} ds.
\]
To relate the skew-product flow \((V \times G \times \Omega, \{T^\rho_{t,R}\}_{t \in \mathbb{R}})\) corresponding to the cocycle \(\rho^{(F,A)} = (\Psi_A \Phi_{(F,A)}, \Psi_A) : \Omega \to V \times G\) to the skew-product transformation on \(V \times G \times \Sigma\) generated by its Poincaré section \(P(\rho^{(F,A)})\), first we briefly recall/define these notions in the discrete setting.

Fix any discrete dynamical system \((\Sigma, T)\) and let \(\Gamma = V \times G\) be the semi-direct product group as above. Then given a pair of function \((f, \psi)\), where \(f \in C(\Sigma, V)\) and \(\psi \in C(\Sigma, G)\), one defines a (right) skew-product transformation,

\[
T_{(f, \psi)}(v, g, \sigma) = (v - U_g f(\sigma), g(\psi(\sigma))^{-1}, T\sigma).
\]

Note that

\[
T^n_{(f, \psi)}(v, g, \sigma) = (v - \varphi_{(f, \psi)} \cdot g(\psi(\sigma))^{-1}, T^n\sigma),
\]

where

\[
\psi(\sigma, n) = \psi(T^n\sigma) \cdots \psi(T\sigma) \psi(\sigma),
\]

and

\[
\varphi_{(f, \psi)}(\sigma, n) = \sum_{k=0}^{n-1} U_{\psi(\sigma, k)}^{-1} f(T^k\sigma).
\]

With this notation, the above computation shows that: given a pair of functions \((F, A) : \Omega \to K \times S \subset V \times L(G)\), the Poincaré section to the skew product flow \((V \times G \times \Omega, \{T^\rho_{t,R}\}_{t \in \mathbb{R}})\), (where \(\rho\) is as above), is the discrete skew product system \((V \times G \times \Sigma, T_{(f, \psi)})\) where

\[
\psi = \psi_A, \quad \text{and} \quad f = \xi_A(F),
\]

where \(\psi_A\) and \(\xi_A(f)\) have been defined before in (3.5) and (3.6). This observation will help us relate ergodicity of these skew-product systems. Having done this, we now focus completely on the discrete set-up.

**Twisted-Cohomology.**

**Definition 3.2.** Let \((\Sigma, T)\) be any discrete dynamical system.

1. Two continuous cocycles \((f_1, \psi), (f_2, \psi) : \Sigma \to V \times G\) are cohomologous via a continuous transfer function \(\beta \in C(\Sigma, V)\) if

\[
H_\beta \circ T_{(f_2, \psi)} = T_{(f_1, \psi)} \circ H_\beta,
\]

where the homeomorphism \(H_\beta : V \times G \times \Sigma \to V \times G \times \Sigma\) is given by

\[
H_\beta(v, g, \omega) = (v - U_g \beta(\sigma), g, \sigma).
\]

In this case we shall also call \(f_1\) and \(f_2\) ‘\(\psi\)-cohomologous’ via a transfer function \(\beta\) and this is equivalent to the following condition,

\[
f_2(\sigma) = f_1(\sigma) + \beta(\sigma) - U_{\psi(\sigma)}^{-1} \beta(T\sigma).
\]

2. Given a map \(\psi \in C(\Sigma, G)\) and \(\beta \in C(\Sigma, V)\), define the map \(1^\beta_\psi : \Sigma \to V\) by setting

\[
1^\beta_\psi(\sigma) = \beta(\sigma) - U_{\psi(\sigma)}^{-1} \beta(T\sigma).
\]

Note that the map \(1^\beta_\psi\) is \(\psi\)-cohomologous to the trivial map \(f = 0\) and is called a \(\psi\)-coboundary. Given a fixed representation \(g \to U_g\) and a map \(\psi \in C(\Sigma, G)\), denote the set of all \(\psi\)-coboundaries by \(B_\psi(\Sigma, V)\), i.e.

\[
B_\psi(\Sigma, V) = \{1^\beta_\psi \mid \beta \in C(\Sigma, V)\}.
\]
Lemma 3.3. Let $(\Sigma, T)$ be a discrete dynamical system and $\psi \in C(\Sigma, G)$. Let the representation $g \to U_g$ be uniformly bounded, (in particular unitary). Then, the following statements are equivalent.

1. $f \in B_\psi(\Sigma, V)$,
2. $\frac{\varphi(f,\psi)(\sigma,n)}{n} \to 0$ as $n \to \infty$, uniformly in $\sigma$.

Proof. (1) implies (2): Let $f = 1_\psi^{1/\beta}$, then

$$\frac{\varphi(f,\psi)(\sigma,n)}{n} = \frac{\varphi(1_\psi^{1/\beta},\psi)(\sigma,n)}{n} = \frac{1}{n}(\beta(\sigma) - U_{\psi(\sigma,n)}\beta(T^n\sigma)) \to 0,$$

uniformly, (since $g \to U_g$ is uniformly bounded). Hence the same holds for any $f \in B_\psi(\Sigma, V)$.

(2) implies (1): For any $f \in C(\Sigma, V)$ and $n \in \mathbb{N}$, set

$$G_n(\sigma) = \frac{1}{n} \sum_{k=0}^{n-1} (n-k)U_{\psi(\sigma,k)}^{-1}f(T^k\sigma).$$

Note that

$$G_n(\sigma) - U_{\psi(\sigma)}^{-1}G_n(T\sigma) = f(x) - \frac{1}{n}U_{\psi(\sigma,n)}f(T^n\sigma) - \frac{1}{n}(\varphi(f,\psi)(\sigma,n) - f(\sigma)).$$

By the hypothesis $1_\psi^G \to f$ uniformly as $n \to \infty$. Hence (1) follows.

Remark 3.4. Suppose that $\psi \in C(\Sigma, G)$ is ergodic, i.e. the system $(G \times \Sigma, T_\psi, \eta \times \nu)$ is ergodic and that the representation $g \to U_g$ satisfies the ‘no fixed point condition’. Then

$$\frac{\varphi(f,\psi)(\sigma,n)}{n} = \frac{1}{n} \sum_{k=0}^{n-1} U_{\psi(\sigma,k)}^{-1}f(T^k\sigma) \to 0,$$

as $n \to \infty$, a.e $(g, \sigma)$.

To see this, observe that the above average is the ergodic average of the function $(g, \sigma) \to U_g f(\sigma)$. Hence by the ergodicity of $\psi$, the limit is $\int_{G \times \Sigma} U_g f(\sigma)d(\eta \times \nu)(g, \sigma)$. This limit is a $G$ invariant vector and hence by the ‘no fixed point’ hypothesis, it must be zero.

Corollary 3.5. With the notation as above, suppose

1. that the ‘no fixed point condition’ holds,
2. $(\Sigma, T)$ is uniquely ergodic and
3. $\psi$ is ergodic.

Then, $C(\Sigma, V) = B_\psi(\Sigma, V)$.

Proof. Since $\psi$ is ergodic and ergodic compact group extensions of uniquely systems are uniquely ergodic (cf. ([12])), it follows that $\psi$ is uniquely ergodic, i.e. the skew product transformation $(G \times \Sigma, T_\psi, \eta \times \nu)$ is uniquely ergodic. Hence by the above remark $\frac{\varphi(\psi,f)(\sigma,n)}{n} \to 0$ uniformly on $G \times \Sigma$. Now Lemma (3.3) completes the proof.
4. A twisted version of the conjugation approximation technique. Now we show that if we fix an ergodic $\psi \in C(\Sigma, G)$ then a generic choice of $f \in B_{\psi}(\Sigma, V)$ lifts ergodicity to the discrete system $(V \times G \times \Sigma, T_{(f, \psi)}, \lambda \times \eta \times \nu)$. This is proved by employing a ‘twisted version’ of the so called conjugation approximation technique. This technique originated in the works of A. Katok and D. Anosov, (see [2]) in the construction of ergodic diffeomorphisms on certain manifolds and refined by M. Herman (see [10]) in the context of group extensions. S. Glasner and B. Weiss abstracted it to lift minimality to compact connected group extensions in the class of closures of continuous coboundaries, (see [9]). This technique was extended to lift ergodicity to non-compact group extensions in the category of closures of smooth coboundaries in [17]. In this section we show that this technique can be suitably modified to lift ergodicity to Euclidean group extensions, (where the cocycle into the ‘$G$-component’ is fixed and ergodic). Now we state the main theorem.

**Theorem 4.1.** Suppose

(1) $(\Sigma, T, \nu)$ is aperiodic and ergodic, and $\nu$ supported, (i.e. is positive on non-empty open sets).

(2) $G$ is a compact, connected Lie group.

(3) let $g \rightarrow U_g$ be a finite dimensional unitary representation of $G$ on a vector space $V$ so that the ‘no fixed point condition’ holds.

(4) $\psi \in C(\Sigma, G)$ be ergodic, i.e. the skew-product system $(G \times \Sigma, T_{\psi}, \eta \times \nu)$ is ergodic.

Let $\Gamma = V \rtimes G$ be the semi-direct product of $G$ and $V$. Then the set

$$\{ f \in B_{\psi}(\Sigma, V) \mid (V \times G \times \Sigma, T_{(f, \psi)}, \lambda \times \eta \times \nu) \text{ is ergodic} \}$$

is a residual subset of $B_{\psi}(\Sigma, V)$.

Then, the following is a consequence of Corollary (3.5).

**Corollary 4.2.** In the above theorem if ergodicity of $(\Sigma, T, \nu)$ is replaced by unique ergodicity, then we conclude that the set

$$\{ f \in C(\Sigma, V) \mid (V \times G \times \Sigma, T_{(f, \psi)}, \lambda \times \eta \times \nu) \text{ is ergodic} \}$$

is a residual subset of $C(\Sigma, V)$.

**Remark 4.3.** (1) We remark that in the discrete setting some authors have studied the problem of lifting point transitivity in Euclidean group extensions when the base transformation is hyperbolic, (cf. [16], [22]). These techniques are completely different than ours, they are only for hyperbolic base, (and hence use ‘hyperbolic techniques’ i.e. A. Livsic’s cohomology theorem etc.).

(2) Again, in discrete case, some results on lifting point transitivity or ergodicity can be found for special examples in ([3] [4]) and ([8]). The results in this paper are much more general and answer some of the questions raised in these works.

**Proof of Theorem (4.1).** Let $\mathcal{H} = L^1(V \times G \times \Sigma)$ and $\mathcal{H}_0 = \{ h \in \mathcal{H} \mid \int h d(\lambda \times \eta \times \nu) = 0 \}$. Let $||| \cdot |||$ denote the $L^1$ norm on $\mathcal{H}$, (recall that $||| \cdot ||$ is the norm on $V$). Given $h \in \mathcal{H}_0$, $\varepsilon > 0$ and $M \in \mathbb{N}$ set

$$U(h, \varepsilon, M) = \left\{ f \in B_{\psi}(\Sigma, V) \mid \text{there exists } N \in \mathbb{N}, M < N \text{ such that } \frac{1}{N} \sum_{j=0}^{N-1} h \circ T_{(f, \psi)}^j \right\}.$$
Note that if \( h \in \bigcap_{i,n,M \in \mathbb{N}} U(h_i, \frac{1}{n}, M) \), where \( \{h_i \mid i \in \mathbb{N}\} \) is a countable dense set in \( \mathcal{H}_0 \), then \( T_{(f, \psi)} \) is ergodic. We show that each \( U(h, \varepsilon, M) \) is open and dense in \( B_{\psi}(\Sigma, V) \), then the Baire category theorem completes the proof. We also remark that without loss of generality \( h \) can be taken to be continuous with compact support, as the space of such maps contains a countable dense subset of \( \mathcal{H}_0 \). Openness of \( U(h, \varepsilon, M) \) is straightforward, so we proceed to prove the density.

**Density of \( U(h, \varepsilon, M) \).**

Given \( f \in C(\Sigma, V) \), set \( H_f(v, g, \sigma) = (v + U_g f(\sigma), g, \sigma) \).

Then, if \( f, u \in C(\Sigma, V) \) we have

\[
T_{(u+1_{i,0}, \psi)} \circ H_f = H_f \circ T_{(u, \psi)}.
\]

Since \( H_f \) is measure preserving, taking \( u = 0 \) it follows that \( 1_{i,0}^f \in \overline{U(h, \varepsilon, M)} \) if and only if \( 0 \equiv 1_{i,0}^u \in \overline{U(h \circ H_f, \varepsilon, M)} \). Thus density of \( U(h, \varepsilon, M) \) follows if we prove that \( 0 \in \overline{U(h, \varepsilon, M)} \) for any \( h \in \mathcal{H}_0 \), (notice that \( h \in \mathcal{H}_0 \) implies \( h \circ H_f \in \mathcal{H}_0 \)).

Next, note that since \( H_f \) is measure preserving,

\[
\| \frac{1}{N} \sum_{k=0}^{N-1} h \circ T^k_{(1_{i,0}, \psi)} \|_1 = \| \frac{1}{N} \sum_{k=0}^{N-1} h \circ (H_f \circ T^k_{(0, \psi)})H_f^{-1} \|_1 = \| \frac{1}{N} \sum_{k=0}^{N-1} h \circ H_f(T^k_{(0, \psi)}) \|_1.
\]

By the ergodicity of \((G \times \Sigma, T, \eta \times \nu)\),

\[
\frac{1}{N} \sum_{k=0}^{N-1} h \circ T^k_{(1_{i,0}, \psi)} \to \int_{G \times \Sigma} h \circ H_f d(\eta \times \nu) = 0,
\]

in the \( \| \cdot \|_1 \) norm, as \( N \to \infty \). Thus, to prove the density it is enough to prove that given any \( h \in \mathcal{H}_0 \), \( \varepsilon > 0 \), there exists a map \( f \in C(\Sigma, V) \) such that

1. \( D(1_{i,0}^f, 1) \equiv \sup_{\sigma \in \Sigma} \| f(\sigma) - U_{(\psi(\sigma))^{-1}} f(T\sigma) \| < \varepsilon \) and
2. \( \| \int_{G \times \Sigma} h \circ H_f d(\eta \times \nu) \|_1 < \varepsilon \).

Next, we examine requirement (2) above and modify it a bit. First note that

\[
H_f = R \circ \tilde{H}_f \circ R^{-1},
\]

where

\[
R(v, g, \sigma) = (U_g v, g, \sigma) \quad \text{and} \quad \tilde{H}_f(v, g, \sigma) = (v + f(\sigma), g, \sigma).
\]

Since \( R \) is measure preserving, \( h \in \mathcal{H}_0 \) if and only if \( h \circ R \in \mathcal{H}_0 \). Hence replacing \( h \) by \( h \circ R \), condition (2) is equivalent to the following

\[
\| \int_{G \times \Sigma} h \circ (\tilde{H}_f \circ R^{-1}) d(\eta \times \nu) \|_1 < \varepsilon.
\]

Since \( \tilde{H}_f \circ R^{-1}(v, g, \sigma) = (f(\sigma) + U_{g^{-1}} v, g, \sigma) \) we have reduced the proof of Theorem (4.1) to proving the following lemma.

**Lemma 4.4.** Given \( h \in \mathcal{H}_0 \) and \( \varepsilon > 0 \) there exists a \( f \in C(\Sigma, V) \) such that

1. \( D(1_{i,0}^f, 1) = \sup_{\sigma \in \Sigma} \| f(\sigma) - U_{(\psi(\sigma))^{-1}} f(T\sigma) \| < \varepsilon \) and

(2) \(|\int_{G \times \Sigma} h(U_{g^{-1}}v + f(\sigma), g, \sigma) d\eta(g) \nu(\sigma)\|_1 < \varepsilon.\)

\[\Box\]

The Special Case. \(h \equiv h(v), \text{i.e. } h \text{ depends only on the V component.}\) First we claim that in this special case it is enough to prove that

\[\| \int_{\Sigma} h(v + f(\sigma)) d\nu(\sigma) \|_{L^1(V)} < \varepsilon,\]

where \(\| \|_{L^1(V)}\) denotes the \(L^1\) norm on \(L^1(V)\). To see this, note that the map \((v, g, \sigma) \mapsto \int_{G \times \Sigma} h(U_{g^{-1}}v + f(\sigma), g, \sigma) d\eta(g) \nu(\sigma)\) depends only on \(v\), thus its \(\| \|_1\) norm is same as its \(\| \|_{L^1(V)}\) norm and this norm is preserved by the map \(v \mapsto U_g v\).

Thus in the above condition (2), we can replace \(v\) by \(U_g v\) and then integration over \(G\) drops off and the claim is proved.

Now we construct the map \(f\) in a series of steps. This construction involves first picking a ‘tall enough Rokhlin tower’. Then defining \(f\) on its base ‘with the aim of achieving condition (2) of the lemma’, (see the following Steps 1 to 5). Then one extends \(f\) to the entire tower by a careful averaging procedure, (Steps 6 to 8), so as to achieve Condition (1) of the lemma. Finally one has to verify that this extension also satisfies Condition (2). The details follow.

**Step 1.** Note that the family of functions \(\{h_g \mid g \in G\}\) is a precompact subset of \(L^1(V)\), where \(h_g(v) = h(U_g v)\). Furthermore integral (with respect to \(\eta\)) of each \(h_g\) is zero. Hence for each \(g \in G\), there is a convex combination (which is independent of \(g \in G\), of translates of \(h_g\) that is close to 0 in the supremum (and hence in the \(L^1\) norm). More precisely (see [17]), there exists real numbers \(c_i\) and vectors \(v_i \in V\), \((1 \leq i \leq s)\) independent of \(g \in G\) such that

\[c_i \in [0, 1] \quad \text{and} \quad \sum_{i=1}^s c_i = 1 \quad \text{and} \quad \| \sum_{i=1}^s c_i (h_g)_{v_i} \|_{L^1(V)} < \frac{\varepsilon}{4}, \quad \text{for all } g \in G, \quad (4.2)\]

where \((h_g)_{v_i}(w) = h_g(v_i + w) = h(U_g(v_i + w))\). Note that this is just a consequence of the ergodic theorem applied to the translation action of the additive group \(V\) on itself. Since each \(U_{g^{-1}}\) is \(\lambda\) preserving and

\[(h_g)_{v_i} \circ U_{g^{-1}}(w) = h(U_g(v_i + U_{g^{-1}}w)) = h(w + U_g v_i), \quad (4.3)\]

equation (4.2) implies that

\[\| \sum_{i=1}^s c_i h^\#_{(g, i)} \|_{L^1(V)} < \frac{\varepsilon}{4}, \quad \text{for all } g \in G, \quad (4.4)\]

where \(h_{(g, i)}(w) = h(w + U_g v_i)\).

**Step 2.** Divide the interval \([0, 1]\) into subintervals \(\Lambda_i\) \((1 \leq i \leq s)\) of length \(c_i\) and define a continuous map \(\kappa : [0, 1] \to V\) by setting it equal to \(v_i\) on ‘most of the interval \(\Lambda_i\)’, using (4.4) make sure that

\[\| \int_0^1 h(w + U_g(\kappa(t))) dt \|_1 < \frac{\varepsilon}{8}, \quad \text{for all } g \in G. \quad (4.5)\]

**Step 3.** Pick \(N \in \mathbb{N}\) such that

\[\operatorname{Max}(M, \frac{2C}{\varepsilon}, \frac{8C^*}{\varepsilon}) < N, \quad (4.6)\]
Note that

\[ C = \sup\{|U_y(\kappa(t))| : t \in [0,1], g \in G\} \quad \text{and} \quad C^* = \sup\{|h(w)| : w \in V\}. \]

**Step 4.** Using Rokhlin’s lemma pick a Borel set \( E \subset \Sigma \) such that \( E, TE, \ldots, T^{N^2-1}E \) are mutually disjoint and

\[ \nu\left( \bigcup_{i=0}^{N^2-1} T^iE \right) > 1 - \frac{\varepsilon}{8C^*}. \]  \hspace{1cm} (4.7)

**Step 5.** Partition \( E \) into disjoint compact sets \( \{E_i \mid 1 \leq i \leq p\} \) and fix points \( \sigma_i \in E_i \) such that

\[ |h(v + U_{\psi(\sigma,j)}w) - h(v + U_{\psi(\sigma,j)}w)| < \frac{\varepsilon}{16}, \]

for all \( j \in [0, N^2 - 1], \sigma \in E_i, 1 \leq i \leq p \) and \( w \in V \). \hspace{1cm} (4.8)

Since \( h \) has compact support, this choice is possible by the continuity of \( \psi \) and regularity of \( \nu \).

**Step 6.** Define a Borel map \( f^* : \Sigma \to V \) as follows. First let \( \theta_i : E_i \to [0,1], (1 \leq i \leq p) \) be a Borel isomorphism such that

\[ a^*_i(\nu|_{E_i}) = \ell, \]  \hspace{1cm} (4.9)

where \( \nu|_{E_i} \) and \( \ell \) are respectively the normalized restriction of \( \nu \) to \( E_i \) and the usual Lebesgue measure on \([0,1]\). Set \( f^* = \kappa \circ \theta_i \) on \( E_i \) and extend the domain of definition definition of \( f^* \) by setting

\[ f^*(T^j\sigma) = U_{\psi(\sigma,j)}f^*(\sigma) \text{ if } \sigma \in E \text{ and } j \in [0, N^2 - 1], \]

\[ 0 \text{ otherwise.} \]  \hspace{1cm} (4.10)

**Step 7.** Using Lusin’s approximation theorem select a continuous map \( f^\# : \Sigma \to V \) such that

\[ \nu(\Sigma_0) < \frac{\varepsilon}{32C^*N^2}, \quad \text{and} \quad ||f^\#(\sigma)|| \leq C, \text{ for all } \sigma \in \Sigma, \]  \hspace{1cm} (4.11)

where \( \Sigma_0 = \{\sigma : f^*(\sigma) \neq f^\#(\sigma)\} \). Let \( \Sigma_1 = \{\sigma \in \Sigma : T^j\sigma \in \Sigma_0 \text{ for some } j \in [0, N^2 - 1]\} \). Then

\[ \nu(\Sigma_1) < \frac{\varepsilon}{32C^*}. \]  \hspace{1cm} (4.12)

**Step 8.** Set

\[ f(\sigma) = \frac{1}{N} \sum_{k=0}^{N-1} U_{(\psi(\sigma,k))^{-1}}f^\#(T^k\sigma), \quad (\sigma \in \Sigma). \]  \hspace{1cm} (4.13)

We shall show that \( f \) is the required map. Clearly \( f : \Sigma \to V \) is continuous. Note that

\[ ||U_{(\psi(\sigma))^{-1}}f(T\sigma) - f(\sigma)|| \]

\[ = \frac{1}{N}|| \sum_{k=0}^{N-1} U_{(\psi(\sigma))^{-1}} U_{(\psi(T\sigma,k))^{-1}}f^\#(T^{k+1}\sigma) - \sum_{j=0}^{N-1} U_{(\psi(\sigma,j))^{-1}}f^\#(T^j\sigma)|| \]

\[ = \frac{1}{N}|| \sum_{k=0}^{N-1} U_{(\psi(\sigma,k+1))^{-1}}f^*(T^{k+1}\sigma) - \sum_{j=0}^{N-1} U_{(\psi(\sigma,j))^{-1}}f^\#(T^j\sigma)|| \]
This proves the claim.

Next, we claim that
\[ f(T^k y) = U_{\psi(y,k)} f^*(y), \quad \text{if } y \in T^j(E) \setminus \Sigma_1 \text{ and } j, k, j + k \in [0, N^2 - N - 1]. \] (4.14)

To prove this claim, let \( y = T^j(\sigma) \), where \( \sigma \in E \) and consider,

\[ U_{\psi(y,k))}^{-1} f(T^k y) \] (4.15)

\[ = \frac{1}{N} \sum_{i=0}^{N-1} U_{\psi(y,k))}^{-1} U_{\psi(T^k y,i)}^{-1} f^*(T^{k+j+i} \sigma) \]

\[ = \frac{1}{N} \sum_{i=0}^{N-1} U_{\psi(y,k))}^{-1} U_{\psi(T^k y,i)}^{-1} f^*(T^{k+j+i} \sigma), \quad \text{(since } y \notin \Sigma_1), \]

\[ = \frac{1}{N} \sum_{i=0}^{N-1} U_{\psi(y,k))}^{-1} U_{\psi(T^k y,i)}^{-1} U_{\psi(\sigma,k+j+i)} f^*(\sigma), \quad \text{(by (4.10))}, \]

\[ = \frac{1}{N} \sum_{i=0}^{N-1} U_{\psi(y,k))}^{-1} U_{\psi(\sigma,k+j)} f^*(\sigma) \]

\[ = U_{\psi(\sigma,j)} f^*(\sigma), \quad \text{(by the cocycle identity)}, \]

\[ = f^*(T^j \sigma), \quad \text{(by (4.10))}, \]

\[ = f^*(y). \] (4.16)

This proves the claim.

Now for any Borel subset \( B \subset \Sigma \), let \( \nu|_B \) the normalized restriction of \( \nu \) to \( B \). Then for \( 1 \leq i \leq p \) and \( j \in [0, N^2 - N - 1] \), we have

\[ \int_{T^j E_i} h(v + f(\sigma))d\nu(\sigma) \] (4.17)

\[ = \nu(T^j E) \int_{T^j E_i} h(v + f(\sigma))d\nu|_{T^j E_i}(\sigma) \]

\[ = \nu(T^j E) \int_{E_i} h(v + f(T^j \sigma))d\nu|_{E_i}(\sigma) \]

\[ \leq \nu(E_i) \int_{E_i} h(v + U_{\psi(\sigma,j)} f^*(\sigma))d\nu|_{E_i}(\sigma) + 2\nu(E_i)C^* \nu(\Sigma_1), \quad \text{(by (4.14))}, \]

\[ \leq \frac{\varepsilon}{16} \nu(T^j E_i) + \nu(E_i) \int_{E_i} h(v + U_{\psi(\sigma,j)} f^*(\sigma))d\nu|_{E_i}(\sigma), \quad \text{(by (4.12))}, \]

\[ \leq \frac{\varepsilon}{8} \nu(T^j E_i) + \nu(E_i) \int_{E_i} h(v + U_{\psi(\sigma,j)} f^*(\sigma))d\nu|_{E_i}(\sigma), \quad \text{(by (4.8))}, \]

\[ \leq \frac{\varepsilon}{8} \nu(T^j E_i) + \nu(E_i) \int_{0}^{\kappa(t)} h(v + U_{\psi(\sigma,j)} f^*(\kappa(t)))dt, \quad \text{(by (4.9))}, \]

\[ \leq \frac{\varepsilon}{4} \nu(T^j E_i), \quad \text{(by (4.5))}. \] (4.18)
Thus for any $v \in V$ we have,
\[ \left| \int_{\Sigma} h(v + f(\sigma))d\nu(\sigma) \right| \]
\[ \leq \left| \sum_{i=1}^{p} \sum_{j=0}^{N^2-N-1} \int_{T^jE_i} h(v + f(\sigma))d\nu(\sigma) + C^*\nu(\Sigma \setminus \bigcup_{k=0}^{N^2-1} T^k E) + C^*N\nu(E) \right| \]
\[ \leq \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \left| \sum_{i=1}^{p} \sum_{j=0}^{N^2-N-1} \nu(T^jE_i) \int_{T^jE_i} h(v + f(\sigma))d\nu(\sigma) \right|, \text{(by (4.6) and (4.7))} \]
\[ \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} \sum_{i=1}^{p} \sum_{j=0}^{N^2-N-1} \nu(T^jE_i), \text{(by 4.17),} \]
\[ \leq \varepsilon \]

Thus, the above estimate implies that
\[ \left| \int_{\Sigma} h(v + f(y))d\nu(y) \right|_1 \leq \varepsilon. \]

Thus, the proof is complete in the special case.

**Extension of the proof to the general case.**

Let $\bar{h} \in \mathcal{H}_0$ be a given continuous function with compact support. Set
\[ \bar{h}(v) = \int_{G \times \Sigma} h(U_g v, g, \sigma)d(\eta \times \nu). \]

Then given $\varepsilon' > 0$, by the previous construction we get a map $f \in C(\Sigma, V)$ such that (1) of Lemma (4.4) holds and
\[ \left| \int_{G \times \Sigma} \bar{h}(U_g^{-1}v + f(\sigma))d\nu(\sigma) \right|_1 < \varepsilon'. \quad (4.19) \]

In constructing this $f$ we have to be more careful in choosing $N \in \mathbb{N}$. Here, in addition to the conditions in Step 3, we need to select $N$ to make sure that
\[ \left| \bar{h} - h_1 \right|_1 < \varepsilon', \quad (4.20) \]
where
\[ h_1(v, g, \sigma) = \frac{1}{N} \sum_{k=0}^{N-1} h(U_{\psi(\sigma, -k)}v, g\psi(\sigma, -k)^{-1}, T^{-k}\sigma). \]

The ergodicity of $(G \times \Sigma, T, \eta \times \nu)$ allows one to pick sufficiently large $N$ so that (4.20) holds. Now we claim that
\[ \left| \int_{G \times \Sigma} h_1(U_g^{-1}v + f(\sigma), g, \sigma)d(\eta \times \nu) - \int_{G \times \Sigma} h(U_g^{-1}v + f(\sigma), g, \sigma)d(\eta \times \nu) \right|_1 < \varepsilon'. \quad (4.21) \]

Thus (4.19, 4.20) and (4.21) will imply condition (2) of Lemma (4.4) if $3\varepsilon' < \varepsilon$ and the proof will be complete.
So we turn to the proof of (4.21). Note that
\[
\int_{G \times \Sigma} h_1(U_g^{-1}v + f(\sigma), g, \sigma) d\eta \times \nu
\]
\[
= \frac{1}{N} \sum_{k=0}^{N-1} \int_{G \times \Sigma} h(U_{\psi(\sigma,-k)}(U_g^{-1}v + f(\sigma)), g\psi(\sigma,-k)^{-1}, T^{-k}\sigma) d(\eta \times \nu)
\]
\[
= \frac{1}{N} \sum_{k=0}^{N-1} \int_{G \times \Sigma} h(U_{\psi(\sigma,-k)}g^{-1}v - U_{\psi(T^{-k}\sigma,k)}^{-1}f(T^{-k}(T_k\sigma)), g\psi(x,-k)^{-1}, T^{-k}\sigma) d(\eta \times \nu)
\]
\[
= \frac{1}{N} \sum_{k=0}^{N-1} \int_{G \times \Sigma} h(U_g^{-1}v - U_{\psi(\sigma,k)}^{-1}f(T_k\sigma), g, \sigma) d(\eta \times \nu).
\]
The last step follows because the map \((g, \sigma) \to (g\psi(\sigma,-k)^{-1}, T^{-k}\sigma)\) is measure preserving for each \(k \in \mathbb{Z}\). Now by (4.14), \(U_{\psi(\sigma,k)^{-1}}f(T_k\sigma) = f(\sigma)\) for \(0 \leq k \leq N-1\) if \(\sigma \in \bigcup_{j \in [0,N^2-N-1]} T^j E\). Thus, by (4.6) and (4.7) we have
\[
\left\| \int_{G \times \Sigma} h_1(U_g^{-1}v - f(v, g, \sigma)d(\eta \times \nu) - \int_{G \times \Sigma} h(U_g^{-1}v - f(\sigma), g, \sigma)d(\eta \times \nu) \right\|_1 < \frac{\varepsilon}{4}.
\]
Thus, if we take \(\varepsilon' = \frac{\varepsilon}{4}\), the proof of Lemma (4.4) as well as that of Theorem (4.1) is complete.

5. Proof of Theorem (2.8).

Proof. Using (2) of Proposition (3.1), it is enough to show that the set
\[
C_{\text{erg}} = \{ F \in C(\Omega, K) \mid (V \times G \times \Omega, T_{(\xi_A(F),\psi_A)}, \lambda \times \eta \times \mu) \text{ is ergodic} \},
\]
is residual in \(C(\Omega, K)\), where recall that \(f_{(F,A)}(\sigma) = \xi_A(F)(\sigma) = \int_{0}^{r(\sigma)} U_{\psi_A(\omega,s)^{-1}} F(T^\omega)ds\). As before, given \(h \in \mathcal{H}_0 = \{ f \in L^1(V \times G \times \Sigma) \mid \int f d(\lambda \times \eta \times \nu) = 0 \}, \varepsilon > 0\) and \(M \in \mathbb{N}\) set
\[
W(h, \varepsilon, M) = \{ F \in C(\Omega, K) \mid \text{there exists } N \in \mathbb{N}, 
M < N \text{ such that } \left\| \frac{1}{N} \sum_{j=0}^{N-1} h \circ T^j_{(\xi_A(F),\psi_A)} \right\|_1 < \varepsilon \}.
\]
Without loss of generality we shall assume that \(h\) has compact support. Again, as a consequence of the Baire category theorem, the proof will essentially boil down to proving the density of each \(W(h, \varepsilon, M)\) in \(C(\Omega, K)\).

Thus, we fix a given \(h \in \mathcal{H}_0\), \(\varepsilon > 0\) and \(M \in \mathbb{N}\). Recall that \(A \in C(\Omega, S)\) is given so that the cocycle \(\Psi_A\) is ergodic, (and hence so is \(\psi_A\)). Furthermore sets \(K, S\) and the unitary representation \(U_g\) are given so that the hypothesis of Theorem (2.8) holds. Let \(F_0 \in C(\Omega, K)\) and \(\delta > 0\) be given and we need to find \(F \in C(\Omega, K)\) such that
\[
(1) \, \|F(\omega) - F_0(\omega)\| < \delta \text{ for all } \omega \in \Omega \text{ and} \\
(2) \, F \in W(h, \varepsilon, M).
\]
The construction of the map \(F\) will be carried out in the following steps.

Step 1. Let \(V_K\) and \(V_{K-K}\) be the subspace of \(V\) spanned by elements of \(K\) and \(K-K\) respectively. Our assumption implies that \(\text{dim}(V_{K-K}) \geq 1\). Since \(K\) is convex, \(K\) has a non-empty relative interior \(K = \text{int}(K)\) as a subset of \(V_K\).
Without loss of generality, we shall assume that the given map \( F_0 \in C(\Omega, K) \) takes values in \( K \) and hence there exists a \( \delta_1 > 0 \) such that
\[
F_0(\omega) + B_{\delta_1}(0, V_{K^-}) \subseteq K, \quad \text{for all } \omega \in \Omega.
\]
(5.1)
where \( B_{\delta_1}(0, V_{K^-}) \) is the open ball centered at 0 with radius \( \delta_1 \) in the vector space \( V_{K^-} \).

**The strategy of the proof.** Let \( f_0 = \xi_A(F_0) \in C(\Sigma, V) \). Applying Theorem (4.1) we shall get a map \( f \in C(\Sigma, V) \) so that
(a) the cocycle \((f, \psi_A) \) based on the discrete Poincaré map \( T(f, \psi_A) \) is ergodic, (i.e., generates an ergodic skew-product transformation) and
(b) \( ||f(\sigma) - f_0(\sigma)|| < \delta_2 \), for all \( \sigma \in \Sigma \), where \( \delta_2 > 0 \) is suitably chosen.

Our required map \( F \) will be of the form \( F = F_0 + F^* \) and we want to make sure that \( \xi_A(F) = f \). Thus, \( F^* \) should be such that
\[
\xi_A(F^*) = \xi_A(F - F_0) = \xi_A(F) - \xi_A(F_0) = f - f_0.
\]
Thus we need to select \( \delta_2 > 0 \) so small that \( ||f(\sigma) - f_0(\sigma)|| < \delta_2 \), (\( \sigma \in \Sigma \)), will guarantee that the map \( f - f_0 \) can be ‘lifted’ i.e can be written as \( \xi_A(F^*) \) for a suitable map \( F^* \). In fact, if we make \( F^* \) take values in \( B_{\delta_1}(0, V_{K^-}) \), then by (5.1) the map \( F = F_0 + F^* \) will be \( K \) valued as desired. So before choosing \( \delta_2 \), first we need to show that such a ‘lift’ is possible and for this we are going to appeal to the T-support property. We begin with the following lemma.

**Lemma 5.1.** Let \( A \in C(\Omega, S) \) be given. Assume the \((K, A, U_g) \) has the T-support property. For each \( \omega \in \Omega \), define a linear operator \( L_A^\omega : C^{00}([0, T], V_{K^-}) \to V \) by setting,
\[
L_A^\omega F = \int_0^T U(\psi_A(\omega, s))^{-1} F(s) ds,
\]
where \( F \in C^{00}([0, T], V_{K^-}) \) is \( \{ F \in C([0, T], V_{K^-}) \mid F(0) = F(T) = 0 \} \). Then each \( L_A^\omega \) is onto, (uniformly in \( \omega \)). More precisely, given any \( \varepsilon > 0 \) ther exists some \( \hat{\delta} > 0 \) such that \( L_A^\omega \) maps the \( \varepsilon \) ball centered at 0 onto the \( \hat{\delta} \) ball centered at 0 of vector space \( V \), for each \( \omega \in \Omega \).

**Proof.** Fix any \( \omega \in \Omega \). Suppose \( L_A^\omega \) is not onto, then there exists a \( v \in V \setminus \{0\} \) such that
\[
\langle \int_0^T U(\psi_A(\omega, s))^{-1} F(s) ds, v \rangle = 0, \quad \text{for all } F \in C^{00}([0, T], V_{K^-}).
\]
This implies that for any \( \tau \in [0, T] \) and any \( F \in C^{00}([0, \tau], V_{K^-}) \),
\[
\int_0^\tau \langle U(\psi_A(\omega, s))^{-1} F(s), v \rangle ds = \langle \int_0^\tau U(\psi_A(\omega, s))^{-1} F(s) ds, v \rangle = 0.
\]
Thus,
\[
0 = \langle U(\psi_A(\omega, s))^{-1} F(s), v \rangle = \langle F(s), U(\psi_A(\omega, s))v \rangle,
\]
for all \( s \in [0, T] \) and \( F \in C^{00}([0, T], V_{K^-}) \).
This implies that \( \langle w, U(\psi_A(\omega, s))v \rangle = 0 \) for all \( w \in V_{K^-} \) and \( s \in [0, T] \). Thus \( v = 0 \)-a contradiction.

Now given a \( \varepsilon > 0 \), the uniformity (in \( \omega \)), of the \( \hat{\delta} \) ball in the image of the \( \varepsilon \) ball under \( L_A^\omega \) comes from the fact that norm \( ||L_A^\omega|| \), (a continuous function of \( \omega \)), is uniformly bounded away from 0 as \( \omega \) varies over the compact set \( \Omega \). \( \square \)
Step 2. Now in the above lemma letting $\bar{\varepsilon} = \varepsilon, (-the given \varepsilon),$ we get a a $\delta_2 \equiv \tilde{\delta} > 0$ satisfying the conclusion of the above lemma. Let

$$\delta = \min\{\delta_1, \delta_2\}.$$ 

Step 3. For this $\delta$, applying Theorem (4.1), choose a $f \in C(\Sigma, V)$ such that

(a) the skew-product map $T_{(f, \psi_A)}$ is ergodic and

(b) $|| f(\sigma) - f_0(\sigma) || < \delta$, for all $\sigma \in \Sigma$.

Step 4. Since $(f, \psi_A)$ is ergodic, select $N \in \mathbb{N}$ such that $M < N$ and

$$\left|\left| \frac{1}{N} \sum_{k=0}^{N-1} h \circ T^k_{(f, \psi_A)} \right|\right|_1 < \frac{\varepsilon}{8} \quad \text{and} \quad \frac{16C^*}{\varepsilon} < N,$$  

(5.2)

where $C^* = \sup\{|h(v, g, \sigma)| \mid (v, g, \sigma) \in V \times G \times \Sigma\}$.

Step 5. Now for each $\sigma \in \Sigma$ one may define $F^*_\sigma : [0, r(\sigma)] \to V$ by ‘lifting’ the map $t \to (f - f_0)(T_t \omega)$, that is, making sure that $L^*_{\omega}(F^*_\sigma)(t) = (f - f_0)(T_t \omega)$, where $\omega = (\sigma, 0)$. Then setting $F^*(\sigma, t) = F^*_\sigma(t), 0 \leq t \leq r(\sigma)$, should give us the required map $F^*$ on $\hat{\Omega}$. The problem with this construction is the following: Eventhough for each $\sigma$ the map $t \to F^*(\sigma, t)$ is continuous, it may not be so as a function of $\sigma$. To avoid this ‘lack of regularity in the lift’, we shall define $F^*$ on selected ‘strips of orbits’ to be locally constant and then continuously extend it to all of $\hat{\Omega}$. In order to do this, first we shall need a Rokhlin tower.

Chose a Rokhlin tower of height $N$, i.e. Pick a Borel set $E \in \Sigma$ such that $T^i E$ are mutually disjoint for $i \in [0, N^2 - 1]$ such that

$$\nu(\Sigma \setminus \bigcup_{i=0}^{N^2-1} T^i(E)) < \frac{\varepsilon}{8C^*}.$$  

(5.3)

Step 6. Since $h$ is continuous with compact support, choose $m \in \mathbb{N}$ so that

$$\left|\left| \frac{1}{N} \sum_{k=0}^{N-1} h(T^k_{(f, \psi_A)}(v, g, \sigma')) - \frac{1}{N} \sum_{k=0}^{N-1} h(T^k_{(f, \psi_A)}(v, g, \sigma)) \right|\right| < \frac{\varepsilon}{4},$$  

\text{if} \quad d(\sigma, \sigma') < \frac{1}{m} \quad \text{and for all} \quad v, g. \quad (5.4)

Step 7. Next, select disjoint compact sets $E_k \subset E$, $1 \leq k \leq q$, so that

$$\nu(E \setminus \bigcup_{k=1}^{q} E_k) < \frac{\varepsilon}{16C^*N^2},$$  

(5.5)

and

$$\text{diam}(T^iE_k) < \frac{1}{m} \quad \text{for all} \quad 0 \leq i \leq N^2 - 1 \quad \text{and} \quad 1 \leq k \leq q. \quad (5.6)$$

Now set

$$\Sigma = \bigcup_{j=0}^{N^2-N-1} \bigcup_{k=1}^{q} T^j(E_k)$$

and note that

$$\nu(\Sigma) > \nu(\bigcup_{j=0}^{N^2-N-1} T^jE) - \frac{\varepsilon}{16C^*} \geq \nu\left(\bigcup_{j=0}^{N^2-1} T^jE\right) - \frac{\varepsilon}{16C^*} - \frac{\varepsilon}{16C^*} \geq 1 - \frac{\varepsilon}{4C^*}. \quad (5.7)$$

Next, fix a point $\sigma_k \in E_k$ for each $1 \leq k \leq q$. 


Step 8. For each $\sigma \in \Sigma$, using Lemma (5.1) find $F^*_\sigma \in C^0([0, T], V_{K-\mathcal{K}})$ such that
\[ \xi_A(F^*_\sigma)(\sigma) = f(\sigma) - f_0(\sigma), \quad \text{for all } \sigma \in \Sigma. \]

Step 9. Now let
\[ \tilde{\Omega} = \{(\sigma, t) \mid \sigma \in \tilde{\Sigma}, \ t \in [0, T]\}. \]
and define $F^* : \tilde{\Omega} \to B_\delta(0, V_{K-\mathcal{K}})$ by setting
\[ F^*(T^j\sigma, t) = F^*_{T^j(\sigma_k)}(t), \quad \text{if } t \in [0, T] \text{ and } \sigma \in T^j(E_k), \]
for $0 \leq j \leq N^2 - 1$ and $1 \leq k \leq q$. Note that $F^*$ is continuous on its compact domain of definition $\tilde{\Omega}$. Now using Tietze’s extension theorem, extend $F^*$ continuously on all of $\tilde{\Omega}$ so that (i) the extension takes values in $B_\delta(0, V_{K-\mathcal{K}})$ and (ii) takes value 0 on some open neighbourhood of the set $\bigcup_{\sigma \in \Sigma}\{\sigma, 0\}, (\sigma, r(\sigma))$. This last condition uses the hypothesis $T < \inf\{r(\sigma) \mid \sigma \in \Sigma\}$ and guarantees that $F^*$ defines a continuous function on $\Omega = \tilde{\Omega}/\approx$ with values in $B_\delta(0, V_{K-\mathcal{K}})$. We now claim that $F = F_0 + F^*$ is the required map. Our construction already shows that $||F(\sigma) - F_0(\sigma)|| < \delta$ for all $\sigma \in \Sigma$. We now show that $F \in W(h, \varepsilon, M)$.

Claim. $F \in W(h, \varepsilon, M) :$ Define $f^# : \tilde{\Sigma} \to V$ by setting $f^#(\sigma) = f(T^j(\sigma_k))$ if $\sigma \in T^j(E_k), 1 \leq k \leq q$ and $0 \leq j \leq N^2 - 1$ and equal to 0 outside $\tilde{\Sigma}$. Now note that,

(i) Now estimates (5.8), (5.4) and (5.7) imply
\[ \left\| \left( \frac{1}{M} \right) \sum_{k=0}^{N-1} h \circ T^k_{(f, \psi_A)} \right\|_1 - \left\| \frac{1}{M} \sum_{k=0}^{N-1} h \circ T^k_{(f^#, \psi_A)} \right\|_1 < \frac{\varepsilon}{2}. \tag{5.8} \]

(ii) Next, by our choice of $F^*_\sigma$ in Step 8, functions $\xi_A(F) = \xi_A(F_0 + F^*)$ and $f^#$ agree on the set $\tilde{\Sigma}$, which has $\nu$ measure greater than or equal to $1 - \frac{\varepsilon}{4M\nu}$. Thus,
\[ \left\| \left( \frac{1}{M} \right) \sum_{k=0}^{N-1} h \circ T^k_{(f^#, \psi_A)} \right\|_1 - \left\| \frac{1}{M} \sum_{k=0}^{N-1} h \circ T^k_{(\xi_A(F), \psi_A)} \right\|_1 < \frac{\varepsilon}{2}. \tag{5.9} \]

Now equations (5.2), (5.8) and (5.9) implies that $\left\| \frac{1}{N} \sum_{j=0}^{N-1} h \circ T^j_{(\xi_A(F), \psi_A)} \right\|_1 < \varepsilon$. This proves that $F \in W(h, \varepsilon, M)$.

Remark 5.2. In the above proof the support property plays a key role. It is a kind of ‘small time null controllability’ condition. It demands that for any $\sigma \in \Sigma$ and any unit vector in $V$, the trajectory $t \to U_{\psi_A(\omega, t)^y}$ should leave orthocomplement of $V_{K-\mathcal{K}}$ before time $T$, (note that $T$ can be very small). Suppose the representation $U_g$ is irreducible, the cocycle $\psi_A$ is ergodic and $\text{dim}(V_{K-\mathcal{K}}) \geq 1$. Then one can see that the above mentioned trajectory will leave orthocomplement of $V_{K-\mathcal{K}}$ at some $T > 0$, a.e. $\omega$. However this $T$ may not be small, (small enough to be less than the infimum of the roof function, so that our proof will work).

Now suppose that the map $A$ is constant and ‘everything is real analytic’. In addition, suppose the representation is irreducible, the cocycle $\psi_A$ is ergodic and $\text{dim}(V_{K-\mathcal{K}}) \geq 1$. Then the above argument shows that the support property must hold for arbitrarily small $T > 0$. Note that the standard one-dimensional harmonic oscillator with constant frequency is an example of this case.

In the next theorem we show that the Lie algebraic conditions in the hypothesis of Theorem (2.10) will imply this ‘small time null controllability condition’ for the cocycle generated by a generic $K \times S$ valued function.
6. Proof of Theorem (2.10). This theorem uses two results. The first one is about generic lifting of ergodicity by cocycles generated by linear differential equations with coefficients in the class $C(\Omega, S)$, when $G$ is a compact connected Lie group. This is the main result of (21).

**Theorem 6.1.** With the notation and the basic set up as above. Let $(\Omega, \{T_t\}_{t \in \mathbb{R}, \mu})$ be an ergodic aperiodic flow, where the ergodic invariant measure is supported. Let $S \subset L(G)$ be a closed, convex subset with SAP. Then the set

$$C_{erg} = \{ A \in C(\Omega, S) \mid \text{the flow } (G \times \Omega, \{T_t^{\psi A}\}_{t \in \mathbb{R}, \eta \times \mu}) \text{ is ergodic} \},$$

is a residual subset of $C(\Omega, S)$.

The second result is the following.

**Proposition 6.2.** Let the notation and the basic set-up be as above, where $(\Omega, \{T_t\}_{t \in \mathbb{R}, \mu})$ is the flow built over $\Sigma$ with roof function $r$. Suppose the set $S \subset L(G)$ has the SAP property as a subset of $L(G)$ and the set $(K-K)\times S$ has the AP property as a subset of $L(V \times G)$. Let $T > 0$ and $\varepsilon > 0$. Then given any $A \in C(\Omega, S)$ and $\varepsilon > 0$, there exists $A^* \in C(\Omega, S)$ and $\delta > 0$, $\delta$ depends on $\varepsilon$ and $A$, such that

(a) $||A(\omega) - A^*(\omega)|| < \varepsilon$, for all $\omega \in \Omega$ and
(b) $\nu(\Sigma) \leq B_\delta(0, V) \subset L^*_A(B_1) \leq 1 - \varepsilon$, where $B_1$ denote the unit ball in $C^{00}(0, T, V_{K-K})$, (here $L^*_A \equiv L^*_{A^*}$ is the operator in Lemma (5.1)).

First we show how Theorem (2.10) can be proved by modifying the arguments in the proof of Theorem (2.8), using the above two results.

**Prof of Theorem (2.10).** Given $h \in H_0$, $\varepsilon > 0$ and $M \in \mathbb{N}$ define

$$W(h, \varepsilon, M) = \{(F, A) \in C(\Omega, K \times S) \mid$$

$$\exists N \in \mathbb{N}, M < N \text{ such that } \left| \frac{1}{N} \sum_{j=0}^{N-1} h \circ T^j_{\xi(\xi_0)} \right|_1 < \varepsilon \}.$$

As before, the crucial step is the density of $W(h, \varepsilon, M)$. Let $(F_0, A_0) \in C(\Omega, K \times S)$ and $\delta > 0$ be given. In this proof our set of ‘nice points’ in $\Sigma$, (i.e. analog of set $\Sigma$ in the proof of Theorem (2.8)), will have to be chosen more carefully. We proceed as follows.

**Step 1.** For the given function $F$, we choose $\delta_1 > 0$ as in Step 1 of Theorem (2.8).

**Step 2.** Applying Proposition (6.2) with $A = A_0$ and $\varepsilon$ replaced by $\frac{\varepsilon}{16C^*}$, we get a $\delta_2 > 0$ and $A^* \in C(\Omega, S)$ such that $||A_0(\omega) - A^*(\omega)|| < \frac{\varepsilon}{16C^*}$ for all $\omega \in \Omega$ and

$$\nu(\Sigma) < \frac{\varepsilon}{16C^*},$$

where $\Sigma_{\delta_2} = \{ \sigma \in \Sigma \mid B_{\delta_2}(0, V) \subset L^*_A(B_1) \}$. (6.1)

where $C^*$ is as before, the supremum norm of $h$.

**Step 3.** Since $A \in C(\Omega, S)$, $A \rightarrow ||L^*_A||$ is a continuous map, (uniformly in $\sigma \in \Sigma$) and $\nu$ is a regular Borel measure. Thus, property of $A^*$ listed in equation (6.1) does not change if $A^*$ is replaced by another map $A$ close enough to $A^*$ in the supremum metric in $C(\Omega, S)$. Now since $S$ has SAP, by Theorem (6.1), we can choose a map $A$ close enough to $A^*$ so that $\Psi_A$ is an ergodic cocycle. Thus, if necessary replacing $A^*$ by a close enough map, without loss of generality, we can assume that in addition to property (6.1), $\Psi_{A^*}$ is ergodic.

**Step 4.** Let $\delta = \min\{\delta_1, \delta_2\}$. 
Step 5. Since \((\Sigma, T, \nu)\) is ergodic, by applying the \(L^2\) ergodic theorem to the characteristic function of set \(\Sigma^1\), given \(\varepsilon > 0\) as above, we can find a \(N_1\) such that if \(N_1 < n\) then
\[
\nu(\Sigma^2_n) < \frac{\varepsilon}{16C^*} \quad \text{where}
\]
\[
\Sigma^2_n = \{ \sigma \in \Sigma \mid \left\lfloor \frac{0 \leq j \leq n - 1}{n} \middle| T^j \sigma \in \Sigma^1 \right\rfloor > 1 - \frac{\varepsilon}{8C^*} \}.
\]

Step 6. Now we continue exactly as in the proof of Theorem (2.8), for \(\delta\) in Step 4, applying Theorem (4.1), choose a \(f \in C(\Sigma, V)\) such that
(a) the skew-product map \(T^*_{(f, \psi^{A*})}\) is ergodic and
(b) \(\|f(\sigma) - f_0(\sigma)\| < \delta\), for all \(\sigma \in \Sigma\), where \(f_0 = \xi^{A*}(F_0)\).

Step 7. Since \((f, \psi^A)\) is ergodic, select \(N \in \mathbb{N}\) such that \(\max\{M, N_1\} < N\), and
\[
\left\lfloor \frac{1}{N} \sum_{k=0}^{N-1} h \circ T^k_{(f, \psi^{A*})} \right\rfloor 1 < \frac{\varepsilon}{8} \quad \text{and} \quad \frac{16C^*}{\varepsilon} < N.
\]

Step 8. Chose a Rokhlin tower of height \(N\), i.e. Pick a Borel set \(E \in \Sigma\) such that \(T^i E\) are mutually disjoint for \(i \in [0, N^2 - 1]\) such that
\[
\nu(\Sigma \setminus \bigcup_{i=0}^{N^2-1} T^i(E)) < \frac{\varepsilon}{16C^*}.
\]

Step 9. Since \(h\) is continuous with compact support, choose \(m \in \mathbb{N}\) so that for all \((v, g) \in V \times G\),
\[
\text{if } d(\sigma, \sigma') < \frac{1}{m},
\]
then
\[
\left\| \frac{1}{N} \sum_{k=0}^{N-1} h(T^k_{(f, \psi^{A*})}(v, g, \sigma')) - \frac{1}{N} \sum_{k=0}^{N-1} h(T^k_{(f, \psi^{A*})}(v, g, \sigma)) \right\| < \frac{\varepsilon}{8}.
\]

Step 10. Next, select compact sets \(E_i \subset E, 1 \leq i \leq q\), so that
\[
\nu(E \setminus \bigcup_{i=1}^{q} E_i) < \frac{\varepsilon}{16C^*N^2}, \quad \text{and}
\]
\[
\text{diam}(T^kE_i) < \frac{1}{m}, \text{ for all } 0 \leq k \leq N^2 - 1 \text{ and for all } 1 \leq i \leq q.
\]

Step 11. The following set \(J\) of indices will be called the set of admissible indices,
\[
J = \{(i, k) \mid 1 \leq i \leq q, \ 0 \leq k \leq N^2 - N - 1 \text{ such that } \nu\{T^kE_i \setminus (\Sigma^1 \cup \Sigma^2) > 0\}\},
\]
where \(\Sigma^2 = \Sigma^2_N\). For each \((i, k) \in J\) fix a point \(\sigma_{(i,k)} \in T^kE_i\).

Step 12. First, for each \((i, k) \in J\), using Lemma (5.1) find \(F^*_{\sigma_{(i,k)}} \in C^{00}([0, T], V_{K-K})\) such that
\[
\xi^{A*}(F^*_{\sigma_{(i,k)}}) = f(\sigma_{(i,k)}) - f_0(\sigma_{(i,k)}).
\]
Note that this is possible, since $\sigma_{(i, k)} \notin \Sigma^1$. Now set $\hat{\Sigma} = \bigcup_{(i, k) \in J} T^k(E_i)$ and note that
\[\nu(\Sigma \setminus \hat{\Sigma}) < \nu(\Sigma \setminus \bigcup_{j=0}^{N^2-N-1} T^j E) + \nu(\Sigma^1 \cup \Sigma^2) + \nu(\bigcup_{k=0}^{N^2-N-1} \bigcup_{i=1}^q T^k(E \setminus E_i)) \]
\[< \frac{\varepsilon}{8C^*} + \frac{\varepsilon}{8C^*} + \frac{\varepsilon}{16C^*} = \frac{5\varepsilon}{16C^*}. \quad (6.8)\]

**Step 13.** Now let $\tilde{\Omega} = \{(\sigma, t) \mid \sigma \in \hat{\Sigma}, \ t \in [0, T]\}$ and define $F^* : \tilde{\Omega} \to B_8(0, V_{K-\kappa})$ by setting
\[F^* (T^k \sigma, t) = F^*_{(\sigma, (i, k), t)} \text{, if } t \in [0, T] \text{ and } \sigma \in T^k(E_i) \text{ where } (i, k) \in J,\]
Note that $F$ is continuous on its compact domain of definition $\tilde{\Omega}$. Now exactly as before, using Tietze’s extension theorem extend $F$ continuously on all of $\Omega$ so that $F$ defines a continuous function on $\Omega = \tilde{\Omega} / \approx$ with values in $B_8(0, V_{K-\kappa})$. Now we show that $F = F_0 + F^*$ is the required map. Our construction already shows that $\|F(\sigma) - F_0(\sigma)\| < \delta$ for all $\sigma \in \Sigma$. We now show that $F \in W(h, \varepsilon, M)$.

**Claim.** $F \in W(h, \varepsilon, M)$. To show this, we make the following three observations.

(i) Note that $F$ was defined (in Step (13)), so that $\xi_{A^*}(F)$ is (locally) constant on each $T^k(E_i)$, if $(i, j) \in J$. Since $\nu(\Sigma) > 1 - \frac{5\varepsilon}{16C^*}$, this implies
\[\left\| \frac{1}{N} \sum_{j=0}^{N^2-N-1} h \circ T^k_{(\xi_{A^*}(F), \psi_A)} \right\|_1 \]
\[< \sum_{(i, k) \in J} \nu(T^k(E_i)) \int_{V \times G} \left| \frac{1}{N} \sum_{j=0}^{N^2-N-1} h \circ T^j_{(\xi_{A^*}(F), \psi_A)}(v, g, T^k \sigma_i) \right| d(\lambda \times \nu) < \frac{5\varepsilon}{16}.\]

(ii) The choices in Step (9) and (10) make sure that the average
\[\left\| \frac{1}{N} \sum_{k=0}^{N-1} h \circ T^k_{(f, \psi_A)} \right\|_1 \]
\[< \sum_{(i, k) \in J} \nu(T^k(E_i)) \int_{V \times G} \left| \frac{1}{N} \sum_{j=0}^{N^2-N-1} h \circ T^j_{(f, \psi_A)}(v, g, T^k \sigma_i) \right| d(\lambda \times \nu) < \frac{5\varepsilon}{16}.\]

(iii) Finally observe that $F$ was constructed so that
\[\xi_{A^*}(F)(T^k \sigma_i) = f(T^k \sigma_i), \text{ for all } (i, k) \in J.\]

Now fix any $(i, k) \in J$, define a set $L_N(i, k)$ by setting
\[L_N(i, k) = \{j \in \{0, 1, \cdots, N-1\} \mid T^j(T^k \sigma_i) \notin \Sigma^1\}.\]
Since $T^k \sigma_i \notin \Sigma^2$, $\frac{|L_N(i, k)|}{N} > 1 - \frac{\varepsilon}{8C^*}$, (by (6.2)). Thus, for any $(v, g) \in V \times G$ and $(i, k) \in J$, we hav
\[\left| \frac{1}{N} \sum_{j=0}^{N^2-N-1} h \circ T^j_{(f, \psi_A)}(v, g, T^k \sigma_i) - \frac{1}{N} \sum_{j=0}^{N^2-N-1} h \circ T^j_{(\xi_{A^*}(F), \psi_A)}(v, g, T^k \sigma_i) \right| < \frac{\varepsilon}{4}. \quad (6.9)\]
Thus,

\[ \sum_{(i,k) \in J} \nu(T^k(E_i)) \int_{V \times G} \frac{1}{N} \sum_{j=0}^{N-1} h \circ T^j_{(f \psi A^*)} (v, g, T^k \sigma_i) - h \circ T^j_{(\xi \ast \psi A^*)} (v, g, T^k \sigma_i) \, d(\lambda \times \nu) < \frac{\varepsilon}{4} \]

Thus these estimates lead to

\[ \left\| \frac{1}{M} \sum_{k=0}^{N-1} h \circ T^k_{(f \psi A^*)} \right\|_1 - \left\| \frac{1}{M} \sum_{k=0}^{N-1} h \circ T^k_{(\xi \ast \psi A^*)} \right\|_1 < \frac{5\varepsilon}{16} + \frac{5\varepsilon}{16} + \frac{4\varepsilon}{16} = \frac{7\varepsilon}{8}. \quad (6.10) \]

This equation along with equation (6.4) leads to \[ \left\| \frac{1}{M} \sum_{k=0}^{N-1} h \circ T^k_{(\xi \ast \psi A^*)} \right\|_1 < \varepsilon, \]
which proves that \( F \in W(h, \varepsilon, M) \). \( \Box \)

7. Proof of Proposition (6.2) and other technicalities. Finally we prove Proposition (6.2). First we begin with a geometric lemma that generalizes Proposition (2.1) in [20]. For this we need to introduce some notation, (one may compare it with that used in [20]).

Fix a \( T, 0 < T < \inf \{ r(\sigma) \mid \sigma \in \sigma \} \). Let \( C^p([0, T], (K - K) \times S) \) denote the set of all piecewise continuous, \((K - K) \times S\) valued functions on \([0, T]\), the space \( C^p([0, T], S) \) is defined similarly. Given a map \( A \in C^p([0, T], S) \) and \( \delta > 0 \) define a set

\[ N(\Delta, \varepsilon, T) = \{ (F, B) \in C^p([0, T], (K - K) \times S) \mid \sup_{t \in [0, T]} \| A(t) - B(t) \| < \varepsilon \text{ and } \sup_{t \in [0, T]} \| F(t) \| < \varepsilon \} \text{.} \]

For any \( (F, B) \in C^p([0, T], (K - K) \times S) \), let \( t \to \gamma_{(F, B)}(t) = (y(t), x(t)) : [0, T] \to V \times G \) be the solution of the system

\[ \begin{align*}
 y'(t) &= B(T_t \omega) y(t) + F(T_t \omega) \\
 x'(t) &= B(T_t \omega) x(t),
\end{align*} \]

with initial condition \((y(0), x(0)) = (0, I)\), \((I)\) is the identity of \( G \).

Now given \((F, B) \in C(\Omega, (K - K) \times S)\), set

\[ (F^\sigma(t), B^\sigma(t)) = (F(\sigma, t), B(\sigma, t)), \quad t \in [0, T], \quad \sigma \in \Sigma. \]

Then \((F^\sigma, B^\sigma) \in C^p([0, T], (K - K) \times S)\). Note that, in our previous notation, for each \( \sigma \in \Sigma \),

\[ \gamma_{(F^\sigma, B^\sigma)}(t) = (U_{\Psi_B(\omega, t)} \Phi_{(F, B)}(\omega, t), g \Psi_B(\omega, t)^{-1}) \in V \times G, \quad \text{where} \ \omega = (\sigma, 0). \]

With this notation, we have the following

Proposition 7.1. Fix a \( T \) as above. Let \( A \in C(\Omega, S) \) and \( \varepsilon > 0 \) be given. Suppose \((K - K) \times S\) has \( A \) as a subset of the Lie algebra \( V \times L(G) \). Then there exists a \( \delta > 0 \) such that, for each \( \sigma \in \Sigma \), the set \( \{ \gamma_{(F, B)}(T) \mid (F, B) \in N(A^\sigma, \varepsilon, T) \} \) contains a right translate of \( B_{\delta}(0, I) \)-closed \( \delta \) ball centered at \((0, I) \in V \times G \).

This means that for each \( \sigma \in \Sigma \), there exists some \((F^\sigma_0, B^\sigma_0) \in N(A, \varepsilon, T)\) such that

\[ B_{\delta}(0, I) \gamma_{(F^\sigma_0, B^\sigma_0)}(T) \subseteq \{ \gamma_{(F, B)}(T) \mid (F, B) \in N(A^\sigma, \varepsilon, T) \}. \]
In particular, for each $\sigma \in \Sigma$, given a point $(v^\sigma, g^\sigma) \in B_\delta(0, I)$, there exists a $(F^\sigma, B^\sigma) \in N(A^\sigma, \varepsilon, T)$ such that

$$\gamma_{(F,B)}(T) = (v^\sigma, g^\sigma) \cdot \gamma_{(F_0^\sigma,B_0^\sigma)}(T).$$

We remark that in the above statement $\delta$ is independent of $\sigma \in \Sigma$ is the most crucial point.

Remark 7.2. (1) Above Proposition is a (non-trivial) generalization of Proposition (2.1) of ([20]). The reader may compare both proofs.

(2) Note that the conclusion of this Proposition does not guarantee a ball centered at $\gamma_{(0, A^\sigma_0)}(T)$ to be inside $N(A, \varepsilon, T)$. This makes our proof a bit more technical.

(3) Note that the function $(F^\sigma, B^\sigma) \in N(A^\sigma, \varepsilon, T)$ obtained in the conclusion of the above Proposition only belong to $C([0, T], (K - K) \times S)$. They are not necessarily restrictions of a map $(F, B) \in C([\Omega, (K - K) \times S])$ to each $\{T_t(\sigma) \mid t \in [0, T]\}$. Thus the map $\sigma \rightarrow (F^\sigma(t), B^\sigma(t))$ need not be continuous as a function of $\sigma$. However, for our purpose this result is sufficient, because our argument needs continuity only on a set of $\sigma$’s with large measure and this can be achieved by approximation.

To prove this proposition, first we need two observations which are formulated in the following two lemmas. The first one is Lemma (2.3) of ([20]), we refer the reader to this reference for its proof.

Lemma 7.3. Let $\Gamma$ be a $\rho$-dimensional Lie group and $R \subset L(\Gamma)$-the Lie algebra of $\Gamma$ Suppose $R$ has the SAP property. Then there exists an $\rho$ tuple $X = (X_1, X_2, \cdots, X_p) \in R^\rho$ such that for every $\varepsilon > 0$, the set $\{e^{sX} \equiv e^{s_1X_1}e^{s_2X_2}\cdots e^{s_pX_p} \mid s \equiv (s_1, s_2, \cdots, s_p) \in P_\varepsilon \}$ has non-empty interior, where $P_\varepsilon = \{s \equiv (s_1, s_2, \cdots, s_p) \mid 0 < s_i, \ 1 \leq i \leq \rho, \ \text{and} \ \sum_{i=1}^\rho s_i = \varepsilon\}.$

We shall apply this lemma to $\Gamma = V \ltimes G$ with $R = (K - K) \times S$. Our second lemma shows that the hypothesis of the above lemma is satisfied as a consequence of the hypothesis of Theorem (2.10).

Lemma 7.4. With the notation as above, if $(K - K) \times S$ has AP and $S$ has SAP, then $(K - K) \times S$ has SAP.

Proof of Proposition (7.1). Note that $F = \{A^\sigma \mid \sigma \in \Sigma\} \subset C^\infty([0, T], (K - K) \times S)$ is a compact set and $C = \{(0, A^\sigma)(t) \mid t \in [0, T], \ \sigma \in \Sigma\}$ is also compact. Let $\{U_j \mid 1 \leq j \leq \ell\}$ be a cover of $C$ by relatively open subsets of $(K - K) \times S$ with diameter less than $\varepsilon$. Using the Lebesgue number for this cover and the equicontinuity of the family $F$, we can find a $p$ such that $0 < p < T$ with the property that for each $\sigma \in \Sigma$, there exists a $j(\sigma)$ such that $\{A^\sigma(t) \mid 0 \leq t \leq p\} \subseteq U_j$, $1 \leq j \leq \ell$. Since $(K - K) \times S$ is convex and has AP, (and hence SAP), its every non-empty relatively open subset has SAP. Then applying Lemma (7.3) with $R = U_j$, we get, for every $j \in \{1, \cdots, \ell\}$ an $\rho$ tuple, $(\rho = \dim(V \ltimes G))$,

$$X^j = (X^j_1, X^j_2, \cdots, X^j_\rho),$$

where $X^j_i = (k^j_{i,1} - k^j_{i,2}, s^j_i)$, where $k^j_{i,1}, k^j_{i,2} \in K, \ s^j_i \in S, \ 1 \leq i \leq \rho,$

such that $Q_j = \{e^{sX^j} \mid s \in P_\varepsilon\}$ contains a translate $W_jh_j$ of some neighbourhood $W_j$ of the identity element $(0, I) \in \Gamma \equiv V \ltimes G$. Set,

$$\tilde{W} = \bigcap_{1 \leq j \leq \ell} W_j, \quad \text{and} \quad W = \bigcap_{g \in G}(0, g)\tilde{W}(0, g)^{-1}.$$
Note that here the proof differs from that of Proposition (2.1) of ([20]) because we are taking $W$ to be the intersection over conjugates of $W$ by elements of $G$ and not by all elements of $\Gamma = G \times V$. As we shall see, we will be able to carry out the remaining argument just as in the compact case. Note that $W$ is a neighbourhood of the identity $(0, I)$, since $G$ is compact. Moreover $W(0, g) = (0, g)W$ for all $g \in G$. Clearly, $Wh_j \subseteq Q_j$ for every $j$. Also note that $W$ does not depend on $\sigma \in \Sigma$.

Now for any $s \in P_e$ and $1 \leq i \leq \ell$, define $H_{s,j} : [0, p] \to (K - K) \times S$ to be the piecewise constant map whose value on the interval $[s_1 + \cdots + s_{k-1}, s_1 + \cdots + s_k]$ is $X_k^j$. For a given $\sigma \in \Sigma$ and $s \in P_e$, define $(F^\sigma_s, B^\sigma_s) : [0, T] \to (K - K) \times S$ by setting

$$(F^\sigma_s(t), B^\sigma_s(t)) = H_{s,j}(\sigma)(t) \quad \text{if} \quad 0 \leq t < p,$$

$$= (0, A^\sigma(t)) \quad \text{if} \quad p \leq t \leq T.$$

Then for any $\sigma \in \Sigma$, $B^\sigma_s \in N(A^\sigma, \varepsilon, T)$ for every $s \in P_e$. Moreover,

$$Wh_j(\sigma) \subseteq Q_j(\sigma) = \{\gamma_{H_{s,j}}(p) \mid s \in P_e\},$$

and then

$$W\tilde{h}_\sigma \subseteq \{\gamma(F^\sigma_s, B^\sigma_s)(p) \mid s \in P_e\} \subseteq \{\gamma(F, B)(p) \mid (F, B) \in N(A^\sigma, \varepsilon, T)\},$$

where $\tilde{h}_\sigma = \gamma_{(0, A^\sigma)}(T)\gamma_{(0, A^\sigma)}(p)^{-1}h_j(\sigma)$. Here we have used the fact that

$$\gamma_{(0, A^\sigma)}(T)\gamma_{(0, A^\sigma)}(p)^{-1}Wh_j(\sigma) = W\gamma_{(0, A^\sigma)}(T)\gamma_{(0, A^\sigma)}(p)^{-1}\tilde{h}_\sigma.$$  

This fact is a consequence of

$$\gamma_{(0, A^\sigma)}(T)\gamma_{(0, A^\sigma)}(p)^{-1} \in G,$$

which in turns, is a consequence of the fact that because the first component, (i.e. the ‘$V$-component’,) of the above point of $\Gamma = V \times G$ is the zero vector. Being an element of $G$ it ‘commutes with $W$’. Thus, because we are perturbing functions of the form $(0, A^\sigma)$, where the first component is the identically zero function, we can extend the argument in the proof of Proposition (2.3) of ([20]) beyond compact case to this very special case. Thus, we have produced the desired neighbourhood of the identity. Finally, take $\delta > 0$ small enough so that the $\delta$ ball centered at $(0, I)$ is contained in this neighbourhood and the proof is complete. Note that $\delta$ is independent of $\sigma \in \Sigma$ which is the crucial, desired fact.

Proof of Proposition (6.2). Recall that $m = \dim(V)$. We are going to apply Proposition (7.1) where $T$ will be replaced by $\frac{T}{m}$ and the compact section $\Sigma$ will be replaced by ‘$m$-layers’ of $\Sigma$. More precisely, $\Sigma$ is replaced by the set $\Sigma(m) = \bigcup_{k=0}^{m-1} \{T_k(\sigma, 0) \mid \sigma \in \Sigma\}$, where $T_k = i \frac{T}{m}$. Note that Proposition (7.1) still applies, because the only property of $\Sigma$ that was used is its compactness and the fact that $\Sigma$ and its translates $T_k(\Sigma \times \{0\})$ are disjoing for $0 \leq t \leq T$.

For $\sigma \in \Sigma$ and $i \in \{0, 1, \cdots, m - 1\}$, let $I^{(\sigma,i)} = \{T_k(\sigma, 0) \mid t_i \leq t \leq t_{i+1}\} \subseteq \Omega$. Given $(F, B) \in C(\Omega, (K - K) \times S)$, $\sigma \in \Sigma$ and $i \in \{0, \cdots, m - 1\}$, set

$$(F^{(\sigma,i)}, B^{(\sigma,i)})(t) = (F(T_i(\sigma, 0), B(T_i\sigma, 0)) = (F(\sigma, t), B(\sigma, t)), \quad \text{for} \quad t \in [t_i, t_{i+1}].$$

Then $(F^{(\sigma,i)}, B^{(\sigma,i)}) \in C^p((0, \frac{T}{m}], (K - K) \times S)$. Note that, this extends our previous notation, (observe that, $(F^\sigma, B^\sigma) \equiv (F^{(\sigma,0)}, B^{(\sigma,0)}))$.

Now recall that we are given $A \in C(\Omega, S)$, $\varepsilon > 0$ and $T > 0$. Applying Proposition (7.1) with this $A$, $\varepsilon$ and replacing $T$ by $\frac{T}{m}$ and $\Sigma$ by $\Sigma(m)$, we get
a $\delta > 0$ and for each $\sigma \in \Sigma$ and each $i \in \{0, \cdots, m - 1\}$, a piecewise continuous map

$$(F_0^{(\sigma,i)}, B_0^{(\sigma,i)}) : [0, \frac{T}{m}] \to (K - K) \times S$$ such that

$$(F_0^{(\sigma,i)}, B_0^{(\sigma,i)}) \in N(A^{(\sigma,i)}, \varepsilon, \frac{T}{m}) \quad \text{and} \quad (7.1)$$

$$B_0(0, I)\gamma(F_0^{(\sigma,i)}, B_0^{(\sigma,i)})\left(\frac{T}{m}\right) \subseteq \{\gamma(F, B) \mid (F, B) \in N(A^{(\sigma,i)}, \varepsilon, \frac{T}{m})\}. \quad (7.2)$$

It will be convenient to set

$$\gamma(F_0^{(\sigma,i)}, B_0^{(\sigma,i)})\left(\frac{T}{m}\right) = (k_0^{(\sigma,i)}, g_0^{(\sigma,i)}).$$

For each $\sigma \in \Sigma$ and $i \in \{0, \cdots, m - 1\}$, we shall pick a map $(F_1^{(\sigma,i)}, B_1^{(\sigma,i)}) : [0, \frac{T}{m}] : (K - K) \times S$ such that

(1) $(F_1^{(\sigma,i)}, B_1^{(\sigma,i)}) \in N(A^{(\sigma,i)}, \varepsilon, \frac{T}{m})$ and

(2) $\gamma(F_1^{(\sigma,i)}, B_1^{(\sigma,i)})\left(\frac{T}{m}\right) \equiv (k_1^{(\sigma,i)}, g_1^{(\sigma,i)})$, where we select $g_1^{(\sigma,i)} = g_0^{(\sigma,i)}$ and the vector $k_1^{(\sigma,i)}$ is picked from the $\delta$ ball centered at $k_0^{(\sigma,i)}$ so that the set of $m$ vectors

$$\{g_0^{(\sigma,m-1)} \cdots g_0^{(\sigma,0)} k_1^{(\sigma,0)}, g_0^{(\sigma,m-1)} \cdots g_0^{(\sigma,1)} k_1^{(\sigma,1)},$$

$$g_0^{(\sigma,m-1)} \cdots g_0^{(\sigma,2)} k_1^{(\sigma,2)}, \cdots, g_0^{(\sigma,m-1)} k_1^{(\sigma,m-1)}\},$$

form a basis of $V$ and each of these vectors have norm bigger than $\frac{\delta}{2}$. Note that such a choice is possible as a consequence of a general fact: Given $m$ open balls of radius $\delta$ in a $m$ dimensional vector space $V$, one can choose one vector from each ball so that the vectors so chosen form a basis of $V$ and each has norm larger than $\frac{\delta}{2}$. Apply this fact to the set of $m$ translates of $B_3(I)$-the $\delta$ ball in $G$ centered at the identity $I$, given by $\{g_0^{(\sigma,m-1)} \cdots g_0^{(\sigma,\ell)} B_3(I) \mid \ell \in \{0, \cdots, m - 1\}\}$. Since the representation $U_g$ is unitary, all of these balls have radii $\delta$.

Then for each $\sigma \in \Sigma$ and $i \in \{0, \cdots, m - 1\}$, identifying the interval $[0, \frac{T}{m}]$ with the piece of the orbit $I^{(\sigma,i)} \subset \Omega$, define a piecewise continuous, $S$ valued map $B^{(\sigma,*)}$ by setting

$$B^{(\sigma,*)}(T_i \sigma) = B_1^{(\sigma,i)}(t - t_i), \quad \text{if} \ t_i \leq t \leq t_{i+1}.$$ Now observe that the operator $L_{B^{(\sigma,*)}}^\sigma = L_{B^{(\sigma,0)}}^\sigma$ is still well defined operator on $C^0([0,T], V_{K-K})$ eventhough $B^{(\sigma,*)}$ is only piecewise continuous and furthermore it is well defined on the space of piecewise continuous functions as well. Now fix any $i \in \{0, \cdots, m - 1\}$ and define a piecewise continuous, $K - K$ valued map $F^{(\sigma,i,*)}$ by setting

$$F^{(\sigma,i,*)}(T_i \sigma) = F_1^{(\sigma,i)}(t - t_i), \quad \text{if} \ t_i \leq t \leq t_{i+1},$$

$$= 0 \quad \text{if} \ t \in [0,T]\backslash [t_i, t_{i+1}].$$ Then our choice of $(k_1^{(\sigma,i)}, g_1^{(\sigma,i)}) \in (K - K) \times S$ shows that

$$L_{B^{(\sigma,*)}}^\sigma(F^{(\sigma,i,*)}) = g_0^{(\sigma,m-1)} \cdots g_0^{(\sigma,1)} k_1^{(\sigma,1)}, \quad \text{for each} \ i \in \{0, \cdots, m - 1\}.$$ In particular, the operator $L_{B^{(\sigma,*)}}^\sigma$ is onto $V$ and its norm is no less than $\frac{\delta}{2\varepsilon}$, (which is independent of $\sigma \in \Sigma$).

Now for each $\sigma \in \Sigma$, we approximate the piecewise continuous function $B^{(\sigma,*)}$ by a continuous function $B^* : \{(\sigma, t) \mid t \in [0,r(\sigma)]\} \to S$, so that

(i) $\|A^*(T_i \sigma) - B^*(T_i \sigma)\| < \varepsilon$ for all $t \in [0, T]$,
(ii) the operators $L_{B^*}^{\sigma}$ remains onto and with norms bounded away from 0, uniformly in $\sigma$, for all $\sigma$ and

(iii) $B^*(T_1\sigma)$ is $0$ for values of $t$ very close to $0$ and $r(\sigma)$.

For the precise construction of this approximation we refer the reader to page 319, step 11 in the proof of Theorem (1.1) of reference [20]. The basic idea in the construction is to approximate $B^{(\sigma,*)}$ by a sequence of continuous $S$ valued maps converging to $B^{(\sigma,*)}$ in the $L^1$ norm. Then Gronwall’s inequality allows us to achieve (ii), uniformly in $\sigma$. In fact, for each $\sigma \in \Sigma$, the operator $L_{B^*}^{\sigma}$ still maps the set of vectors $\{ \{ F^{\sigma,i,*} | 0 \leq i \leq m - 1 \} \}$ onto a basis of $V$ and the norm of the operator $L_{B^*}^{\sigma}$ on the space $C^{\infty}([0,T], S)$ is bounded away from 0 uniformly in $\sigma$.

Now to see that the norms of $L_{B^*}^{\sigma}$ as an operators on $C^{\infty}([0,T], V_{K-K})$ are bounded away from 0, (uniformly in $\sigma$), again we ‘approximate’ functions $F^{\sigma,i,*}$ by continuous functions $F^{\sigma,i,*}_{\sigma}: \{ (\sigma, t) | t \in [0,r(\sigma)] \} \rightarrow V_{K-K}$ for each $i \in \{0, \cdots, m - 1\}$, such that (i) $F^{\sigma,i,*} \in C^{\infty}([0,T], V_{K-K})$ and (ii) the operators $F^{\sigma,i,*}_{\sigma}$ remain onto with norms bounded away from 0, uniformly in $\sigma$. The construction of these approximations is exactly as before (see [20] for details).

Thus for each $\sigma \in \Sigma$ we have constructed a continuous map $B^\sigma$ within $\varepsilon$ of $A^\sigma$ such that the image of a unit ball in $C^{\infty}([0,T], V_{K-K})$ contains a ball centered at $0$ in $V$ with some fixed radius radius that is independent of $\sigma \in \Sigma$. Now since the map $A \in C(\Omega, S)$, by continuity, each $\sigma$ has a closed neighbourhood $V_\sigma \in \Sigma$ such that if $\sigma' \in V_\sigma$, then $||B^\sigma(T_1\sigma) - A^\sigma(T_1\sigma')|| < \varepsilon$ for all $t \in [0,r(\sigma)]$. Now pick a finite number of points $\sigma_i | 1 \leq i \leq \ell$ such $V_{\sigma_i}$’s are disjoint and their union has $\nu$ measure greater than $1-\varepsilon$. Finally define $B: \Omega \rightarrow S$, by setting $B(T_1\sigma) = B^\sigma(T_1\sigma)$ if $t \in [0,r(\sigma))$ and $\sigma \in V_{\sigma_i}$. Extend $B$ continuously to all of $\Omega$ so that it defines a continuous map on $\Omega$. It follows that the map $B$ is the required map $A^*$.

Proof of Lemma (7.4). Recall that $K \times S \subset V \times L(G)$ has AP and $S$ has SAP. We ask the reader to recall the notation introduced in part (E) of the first section on Preliminaries. First we prove the following claim.

Claim. $\Lambda((K-K) \times S) \subset \Lambda_0((K-K) \times S) + (0, L^*(S))$. Let $(k^*, s^*) \in \Lambda((K-K) \times S)$, then

$$(k^*, s^*) = \left( \sum_{i=1}^\ell \lambda_i(k_i - k_i'), \sum_{i=1}^\ell s_i \right), \text{ where } k_i, k_i' \in K, \text{ and } s_i \in S,$$

$$= \left( \sum_{i=1}^\ell \lambda_i(k_i - k) - \sum_{i=1}^\ell \lambda_i(k_i' - k), \sum_{i=1}^\ell s_i - s \right) + (0, \sum_{i=1}^\ell \lambda_i s),$$

where $k \in K$ and $s \in S$ are (arbitrarily chosen), fixed elements. Now note that the first term in the above expression belongs to $\Lambda_0((K-K) \times S)$, because it can be expressed as $\sum_{i=1}^{2\ell} \mu_i(\tilde{k}_i - \tilde{k}_i'), \tilde{s}_i)$ for suitably choose $\mu_i \in \mathbb{R}, \tilde{k}_i, \tilde{k}_i' \in K$ and $\tilde{s}_i \in S$, where $\sum_{i=1}^{2\ell} \mu_i = 0$. Thus,

$$(k^*, s^*) \in \Lambda_0((K-K) \times S) + (0, L(S))$$

$$\subset \Lambda_0((K-K) \times S) + (0, L^*(S) + \Lambda_0(S)), \text{ (since } S \text{ has SAP}),$$

$$\subset \Lambda_0((K-K) \times S) + (0, L^*(S)), \text{ (since } (0, \Lambda_0(S)) \subset \Lambda_0((K-K) \times S),$$
This proves the claim. Thus,

\[ L(V \ltimes G) = L^*((K - K) \times S) \]
\[ \subseteq L^*((K - K) \times S) + A_0((K - K) \times S) + (0, L^*(S)) \]
\[ \subseteq L^*((K - K) \times S) + A_0((K - K) \times S) = L_0((K - K) \times S). \]

Note that in the above chain of equalities/inclusions, the first one is because \((K - K) \times S\) has AP, the last but one comes from the observation \((0, L^*(S)) \subseteq L^*((K - K) \times S)\). This shows that \((K - K) \times S\) has SAP.

\[ \square \]

8. **Application to spectral properties of forced quantum harmonic oscillator.** We shall quickly outline the general set-up and then focus on the example of a forced quantum harmonic oscillator.

(1) Let \((\Omega, \{T_t\}_{t \in \mathbb{R}}, \mu)\) be a continuous, ergodic flow on a compact metric space \(\Omega\).

(2) Let \(H\) is a separable Hilbert space.

(3) Let \(H_0\) be a given (unperturbed) time independent Hamiltonian (i.e. an essentially self adjoint operator) on \(H\).

(4) Let \(\omega \to H_\omega\) be a continuous map from \(\Omega\) to the set of essentially self adjoint operators on \(H\). Thus, for each \(\omega \in \Omega\) \(t \to H_0 + H_1(T_t \omega)\) yields a time dependent family of Hamiltonian operators on \(H\).

The evolution of the quantum system with (time dependent) Hamiltonian \(H = H_0 + H_1(T_t \omega)\) is given by the Schrödinger equation

\[ i \frac{\partial \psi}{\partial t} = \left( H_0 + H_1(T_t \omega) \right) \psi. \quad (8.1) \]

Let \((\omega, t) \to U_H(\omega, t)\) be the unitary evolution operator (propagator) determined by this equation (at this point we shall assume its existence). Thus, for each \(\omega \in \Omega\) the curve \(t \to U_H(\omega, t)\) satisfies the above equation with the initial condition \(U_H(\omega, 0) = I\) - the identity operator. In fact the map \((\omega, t) \to U_H(\omega, t) : \Omega \times \mathbb{R} \to \mathcal{U}\) is a cocycle into \(\mathcal{U}\) the unitary group of the Hilbert space \(H\).

In the non-autonomous set-up, the appropriate Hilbert space in which the unitary evolution takes place is \(L^2(\Omega, H, \mu) \equiv H \otimes L^2(\Omega, \mu)\) - the space of \(H\) valued square integrable functions on \(\Omega\). The evolution is given by the unitary one parameter group \(\{V_t^H\}_{t \in \mathbb{R}}\) given by

\[ V_t^H f(\omega) = U(\omega, t)^{-1} f(T_t(\omega)), \quad f \in L^2(\Omega, H, \mu). \]

In Physics literature the infinitesimal generator of this representation is called the ‘quasi-energy operator’ and the stability properties of the quantum system are studied in terms of the spectral properties of \(\{V_t^H\}_{t \in \mathbb{R}}\). A quantum system is considered to be stable if the spectrum of this representation is discrete pure point and unstable otherwise. Here we shall focus only on the following special example of quantum harmonic oscillator.

**Example 8.1.** Consider the Case (1) of the example of harmonic oscillator discussed before. The quantum version of example is where \(H = L^2(\mathbb{R})\) - the space of square integrable functions with respect to the usual Lebesgue measure on \(\mathbb{R}\),

\[ H_0 = \frac{p^2}{2} + \frac{\alpha^2 q^2}{2}, \]
Thus in this case \( \alpha \) as in Case (1) of example (2.12), we shall assume that the frequency \( \tau \) function is the natural frequency. We shall take the flow to be the flow built by a constant roof function \( p \), where \( \alpha \) and \( \alpha \tau \) are rationally independent. Then the set \( C \) such that the unitary operator \( V \) has only singular continuous spectrum \( F \) is the constant frequency of the unperturbed oscillator. Suppose \( \tau,\alpha \rightarrow \infty \) \( k \) the identity mapping, uniformly.)

The unitary propagator (i.e. the cocycle) \( U^F \) into the unitary group of Hilbert space \( \mathcal{H} \) can be explicitly written as, (see [20]),

\[
U_F(\omega, t) = e^{-itH_0}\text{exp}(-i\langle \Phi_{(F,A)}(\omega, t), \left( \begin{array}{c} q \\ p \end{array} \right) \rangle),
\]

where the inner product in the above formula is the ‘formal inner product’ given by \( \langle \left( \begin{array}{c} q \\ p \end{array} \right), \left( \begin{array}{c} q' \\ p' \end{array} \right) \rangle = \beta q + \gamma p \), for \( \left( \begin{array}{c} q \\ p \end{array} \right) \in \mathbb{R}^2 \). We shall study the quantum dynamics via the spectral properties of the ‘Poincaré section \( P(U_F) \) of this cocycle and the unitary representation \( P(V_F) \) it generates. Note that \( P(U_F) : \Sigma \rightarrow \mathcal{U}(\mathcal{H}) \) is a (i.e. a generator of a) cocycle into the unitary group of Hilbert space \( \mathcal{H} \) and \( P(V_F) \) is a unitary operator on \( L^2(\Sigma, \mathcal{H}, \nu) \), which is the ‘quantum analog’ of the Poincaré section for the quantum evolution. More explicitly,

\[
P(U_F)(\sigma) = U_F((\sigma, 0), \tau), \quad \text{and}
\]

\[
V_F(\xi)(\sigma) = (P(U_F)(\sigma))^{-1}\xi(T\sigma), \quad \xi \in L^2(\Sigma, \mathcal{H}, \nu). \tag{8.2}
\]

With this notation we have the following theorem.

**Theorem 8.2.** Consider the example of the forced quantum oscillator described above. We summarize the assumptions:

(1) The flow is built by a constant roof function \( \tau \) over an aperiodic, uniquely ergodic system \( (\Sigma, T, \nu) \).

(2) Recall that \( \alpha \) is the constant frequency of the unperturbed oscillator. Suppose the ‘resonance condition’ holds, i.e. the transformation \( T_{(\tau,\alpha)}(\theta, \sigma) \rightarrow (\theta + \tau\alpha, T\sigma) \) on \( \mathbb{T} \times \Sigma \) is ergodic. (In particular \( (\Sigma, T\nu) \) be weakly mixing or be a circle rotation by an irrational \( \beta \) where \( \beta \) and \( \tau\alpha \) are rationally independent).

(3) Let \( C \subset \mathbb{R} \) be non-singleton, compact and convex. Then,

(I) Then the set

\[
\{ F \in C(\Omega, C) \mid \text{such that the unitary operator } V_F \text{ has only continuous spectrum} \},
\]

is a residua subset of \( C(\Omega, C) \).

(II) Furthermore, Suppose the following ‘strong resonance condition holds’: The transformation \( T_{(\tau,\alpha)} \) is uniformly rigid, (in particular, when \( T \) is a circle rotation by an irrational \( \beta \) and \( \beta \) and \( \tau\alpha \) are rationally independent). Then the set

\[
\{ F \in C(\Omega, C) \mid \text{such that the unitary operator } V_F \text{ has only singular continuous spectrum} \},
\]

is a residua subset of \( C(\Omega, C) \). (Transformation \( T_{(\tau,\alpha)} \) is uniformly rigid if there is a sequence \( n_k \rightarrow \infty \) of integers such that \( T_{(\tau,\alpha)}^{n_k} \rightarrow \text{Id-the identity mapping, uniformly.} \)
Remark 8.3. (1) The proof of part (I) is essentially the one given in ([19]) even though it is in a rather special case when $F$ depends only on $\sigma$. In the following, we sketch the argument for reader’s convenience.

(2) Once one goes through the following sketch of arguments, with our Theorems (2.8) and (2.10), one can prove various statements of above type for a more general $d$ dimensional quantum harmonic oscillator where the external field $F$ is restricted to take values in a preassigned set $K \subset \mathbb{R}^d$. One has to only make sure that the ‘frequency vector’ of the unperturbed Hamiltonian and the set $K$ satisfy the Lie algebraic conditions in our theorems. We shall leave this task to the reader.

A brief sketch of the proof. (I) First note that the Hamiltonian operator $H_0$ acting on $\mathcal{H}$, has only discrete pure point spectrum with eigenvalues $\lambda_n = (n + \frac{1}{2})\alpha$, $n \in \mathbb{N}$ and the corresponding eigen functions $e_n(x) = e^{-(x^2)/2}H_n(x)$, where $H_n(x)$ are the Hermite polynomials. Note that the set $\{e_n\}_{n \in \mathbb{N}}$ forms a complete orthonormal set in $\mathcal{H}$. Let $\hat{e}_n \in L^2(\Sigma, \mathcal{H}, \nu)$ denote the constant function $\sigma \mapsto e_n$. Note that the linear span of functions of the form $\sigma \mapsto a(\sigma)\hat{e}_n$ is dense in $L^2(\Sigma, \mathcal{H}, \nu)$, where $a$ varies over bounded, real valued, measurable functions on $\Sigma$ and $n \in \mathbb{N}$. We shall show that each such vector is a ‘weakly mixing vector’ for the unitary operator $V_F$, (for a generic choice of $F$). This will show that $V_F$ has only continuous spectrum for a generic choice of $F$. The following formula is a key to showing this, (see [7] for the formula),

$$\langle (V_F^*\hat{e}_m)(\sigma), \hat{e}_m(\sigma) \rangle_{\mathcal{H}} = e^{i\pi\alpha(m + \frac{1}{2})}\exp(-\frac{\|\varphi(\xi_\Lambda, \psi_\Lambda)(\sigma, n)\|^2}{4\alpha})L^0_m\left(\frac{\|\varphi(\xi_\Lambda, \psi_\Lambda)(\sigma, n)\|^2}{2\alpha}\right), \quad (8.3)$$

where $\langle \ , \ \rangle_{\mathcal{H}}$ is the standard inner product on the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ and $L^0_m$ are the Laguerre polynomials. Now consider the map

$$\rho(v) = e^{-(\|v\|^2/4\alpha)}L^0_m\left(\frac{\|v\|^2}{2\alpha}\right), \quad v \in \mathbb{R}^2.$$

We view $\rho$ as a map in $L^1(\mathbb{R}^2 \times \mathbb{T} \times \Sigma)$, then the ergodicity of the skew-product system $(\mathbb{R}^2 \times \mathbb{T} \times \Sigma, T_{(\xi_\Lambda, \psi_\Lambda)}, \lambda \times \eta \times \nu)$ implies that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \rho(T^n_{(\xi_\Lambda, \psi_\Lambda)}(v, \theta, \sigma)) = 0, \quad \text{a.e.} (v, \theta, \sigma).$$

Now we refer to ([19]) for the details of the argument that shows that the above equation implies

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \langle V_F^*\hat{e}_m, \hat{e}_m \rangle_{\mathcal{H}} = 0, \quad (m \in \mathbb{N}),$$

where $\langle \ , \ \rangle_{\mathcal{H}}$ is the inner product on $L^2(\Sigma, \mathcal{H}, \nu)$. This means each $\hat{e}_n$ is a weakly mixing vector for $V_F$, which in turn imply that all vectors in $L^2(\Sigma, \mathcal{H}, \nu)$ are weakly mixing vectors. Hence for a generic $F$, $V_F$ has only continuous spectrum.

(II) Fix a countable set of functions of the form $\sigma \mapsto a_k\hat{e}_m \in L^2(\Sigma, \mathcal{H}, \nu)$, $(k, m \in \mathbb{N})$, whose linear span is dense in $L^2(\Sigma, \mathcal{H}, \nu)$, where $a_k$ is a real valued function on $\Sigma$ and
\( \hat{e}_m \) is as above. Given \( m, k, M \in \mathbb{N} \) define,

\[
W(M, m, k) = \{ F \in C(\Omega, C) \mid \text{there exists a } N \in \mathbb{N} \text{ such that } M < N \text{ and } |\langle V_F^{*} (a_k \hat{e}_m), a_k \hat{e}_m \rangle_* | > \frac{1}{2} \}.
\]

Note that if \( F \in \bigcap_{m, k, M} W(M, m, k) \) then the spectral measure corresponding to the vector \( a_k \hat{e}_m \), (for operator \( V_F \), cannot be absolutely continuous. This is so because \( \langle V_F^{*} (a_k \hat{e}_m), a_k \hat{e}_m \rangle_* \) does not converge to 0 as \( n \to \infty \). Assuming that each \( W(M, m, k) \) is open and dense, it is easy to see that for a generic choice of \( F \), the spectral measure of \( V_F \) is singular continuous. To see this pick a generic \( F \) for which the spectral measure of \( V_F \) is continuous and such that \( F \) belongs to each \( W(M, m, k) \).

Now we show that each \( W(M, m, k) \) is dense in \( C(\Sigma, C) \). Fix \( M, m, n \). The density of twisted coboundaries in \( C(\Sigma, \mathbb{R}^2) \), the uniform rigidity and formula (8.3) plays a role here. First we show that the set

\[
\{ f \in C(\Sigma, \mathbb{R}^2) \mid \lim_{n \to \infty} \inf \left| \varphi_{(f, \psi, A)}(\sigma, n) \right| = 0 \text{ uniformly in } \sigma \},
\]

is residual and hence dense in \( C(\Sigma, \mathbb{R}^2) \). The reason for this is under our hypothesis, (i) the set of twisted coboundaries are dense in \( C(\Sigma, \mathbb{R}^2) \) and (ii) the ‘strong rigidity condition’ holds. We shall leave the details for the reader.

Next, observe that our hypothesis implies that the hypothesis of Theorem (2.8) holds. Now given \( F \in C(\Omega, C) \), and \( \varepsilon > 0 \), we proceed exactly as in the proof of Theorem (2.8) and in the Step 3 of the proof, we pick the \( f \) so that \( T_{(f, \psi, A)} \) is ergodic as well as it belongs to the above set. Then ‘the lifting argument’ argument will work as before and using the ‘approximation’ on a set of large measure one can construct a function \( F_1 \in C(\Omega, C) \) that is \( \varepsilon \) close to \( F \) and for which

\[
\nu(\sigma \mid \lim_{n \to \infty} \inf \left| \varphi(\xi_A(F_1), \sigma, n) \right| = 0) > 1 - \varepsilon.
\]

Finally, observe that the key formula (8.3) will then imply that \( F_1 \in W(M, m, n) \), proving the density of \( W(M, m, n) \).

Unfortunately, due to limitations on space, time and patience and to avoid repetative construction, I have only expressed the basic idea in the above proof. The reader who has gone over the earlier proof should be able to fill in all the details.

**Remark 8.4.** Finally, we would like to pose some questions regarding the spectral properties of these oscillators.

1. Under what assumptions on the dynamics of \( (\Sigma, T, \nu) \) can one produce forcing functions so that the spectrum of \( V_F \) is only absolutely continuous? One would like to believe that if \( T \) is ‘rapidly mixing’ this might be true.

2. One can also ask for a ‘smooth version’ of these results which I think will need more sophisticated technical machinery.

3. For what other quantum oscillators these results can be extended? I believe that for quantum oscillators given by other quadratic time dependent Hamiltonians this should be possible and these questions will also lead to exploring dynamics of group extensions which are semi-direct products of nilpotent and compact or semi-simple groups.

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