EULER NUMBERS OF HILBERT SCHEMES OF POINTS ON SIMPLE SURFACE SINGULARITIES AND QUANTUM DIMENSIONS OF STANDARD MODULES OF QUANTUM AFFINE ALGEBRAS

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Abstract. We prove the conjecture by Gyenge, Némethi and Szendrői [GNS17, GNS18] giving a formula of the generating function of Euler numbers of Hilbert schemes of points \( \text{Hilb}^n(\mathbb{C}^2/\Gamma) \) on a simple singularity \( \mathbb{C}^2/\Gamma \), where \( \Gamma \) is a finite subgroup of SL(2). We deduce it from the claim that quantum dimensions of standard modules for the quantum affine algebra associated with \( \Gamma \) at \( \zeta = \exp(\frac{2\pi i}{2(h^\vee+1)}) \) are always 1, which is a special case of a conjecture by Kuniba [Kun93]. Here \( h^\vee \) is the dual Coxeter number. We also prove the claim, which was not known for \( E_7, E_8 \) before.

Introduction

In this paper, we prove the conjecture by Gyenge, Némethi and Szendrői [GNS17, GNS18] giving a formula of the generating function of Euler numbers of Hilbert schemes of points \( \text{Hilb}^n(\mathbb{C}^2/\Gamma) \) on a simple singularity \( \mathbb{C}^2/\Gamma \), where \( \Gamma \) is a finite subgroup of SL(2). When \( \Gamma \) is of type \( A \), Euler numbers were computed by Dijkgraaf and Sulkowski [DS08], and Toda [Tod15]. The formula in [GNS17, GNS18] is given in a different form, and makes sense for arbitrary \( \Gamma \). The formula was proved for type \( D \), as well as type \( A \), in [GNS18]. Our proof of the formula in type \( E \) is new.

The conjectural formula in [GNS17], which we will prove, is the following:

\[
Z = \prod_{m=1}^{\infty} (1 - e^{-m\delta})^{-(n+1)} \sum_{\vec{m} \in \mathbb{Z}^n} e^{-\frac{\langle \vec{m}, C \vec{\alpha} \rangle}{2}} \delta \prod_{i=1}^{n} e^{-m_i \alpha_i},
\]

where \( \alpha_i \) is the \( i \)th simple root of \( \mathfrak{g}_{\text{fin}} \), the underlying finite dimensional complex simple Lie algebra, \( \delta \) is the positive primitive imaginary root, \( n \) is the rank of \( \mathfrak{g}_{\text{fin}} \), and \( C \) is the finite Cartan matrix for \( \mathfrak{g}_{\text{fin}} \). Here we omit the factor \( e^{\Lambda_0} \) for the 0th fundamental weight \( \Lambda_0 \). The well-known formula for the basic representation gives \((1 - e^{-m\delta})^{-n}\) instead of the power \(-(n+1)\). The additional \(-1\) comes from the Heisenberg algebra.

The conjectural formula in [GNS17], which we will prove, is the following:
Theorem 1. The generating function of Euler numbers of \( \text{Hilb}^n(C^2/\Gamma) \)

\[
\sum_{n=0}^{\infty} e^{-n\delta} \chi(\text{Hilb}^n(C^2/\Gamma))
\]

is obtained from \( Z \) by the substitution

\[
e^{-\alpha_i} = \exp \left( \frac{2\pi i}{h^\vee + 1} \right) \quad (i \neq 0),
\]

where \( h^\vee \) is the dual Coxeter number.

Let us sketch the strategy of our proof. In order to compute Euler numbers of \( \text{Hilb}^n(C^2/\Gamma) \), we use a recent result of Craw, Gammelgaard, Gyenge and Szendrői [CGGS19]: \( \text{Hilb}^n(C^2/\Gamma) \) with the reduced scheme structure is isomorphic to a quiver variety \( M_{\zeta^\bullet}^{n\delta, \Lambda_0} \), where \( \zeta^\bullet \) is a stability parameter

\[
\zeta^\bullet = 0 \quad \text{for all} \quad i \neq 0, \quad \zeta_0^\bullet < 0.
\]

This stability parameter is different from the standard one \( \zeta \)

\[
\zeta_i < 0 \quad \text{for all} \quad i \in I,
\]

which gives connected components \( M_{\zeta}^{n\delta, \Lambda_0} \) of \( \text{Hilb}^n(C^2/\Gamma) \). Here \( \zeta \) is an isomorphism class of \( \mathbb{C}[x,y]/I \) as a \( \Gamma \)-module for an ideal \( I \subset \mathbb{C}[x,y] \) of the coordinate ring \( \mathbb{C}[x,y] \) of \( C^2 \), invariant under the \( \Gamma \)-action.

As \( \zeta^\bullet \) lies in walls given by roots in the space of stability parameters, \( M_{\zeta^\bullet}^{n\delta, \Lambda_0} \) is singular in general. Nevertheless it has a representation theoretic meaning, as shown in [Nak09]: the quiver variety \( M_{\zeta^\bullet}^{n\delta, \Lambda_0} \) is responsible for the restriction of \( g_{\text{aff}} \) to \( g_{\text{fin}} \). The essential geometric ingredient for this relation is a projective morphism

\[
\pi_{\zeta^\bullet, \zeta}: M_{\zeta}^{n\delta, \Lambda_0} \to M_{\zeta^\bullet}^{n\delta, \Lambda_0} = \text{Hilb}^n(C^2/\Gamma), \quad n = v_0.
\]

By a local description of singularities of quiver varieties ([Nak09, §2.7], which went back to [Nak94, §6]), fibers of \( \pi_{\zeta^\bullet, \zeta} \) are lagrangian subvarieties in quiver varieties \( M_{\zeta}(v^*, w^*) \) associated with the finite \( ADE \) quiver for \( g_{\text{fin}} \), the finite dimensional complex simple Lie algebra underlying \( g_{\text{aff}} \). As a simple application of this result, we observe that

- Euler numbers of \( \text{Hilb}^n(C^2/\Gamma) \) can be written in terms of Euler numbers of \( M_{\zeta}(v, \Lambda_0) \) and quiver varieties \( M_{\zeta}(v^*, w^*) \) of the finite \( ADE \) type.

In fact, we get more data than we need: we have a stratification of \( \text{Hilb}^n(C^2/\Gamma) \) so that \( \pi_{\zeta^\bullet, \zeta} \) is a fiber bundle over each stratum. We can compute Euler numbers of all strata from Euler numbers of \( M_{\zeta}(v, \Lambda_0) \) and quiver varieties \( M_{\zeta}(v^*, w^*) \) of the finite \( ADE \) type. See (1.8) for the precise formula.

There are several algorithms\(^1\) to compute Euler numbers of quiver varieties. For example, [Nak04] and [Hau10]. They are complicated, and hard to use in practice. On the other

\(^1\)We make a distinction between an algorithm for a computation and an explicit computation as in [Nak10]. Namely a computation results finitely many \( \pm, \times, \) integers and variables. We do not require that the final expression is readable by a human. The expression like \( \sum_{i=1}^{2^{2^{100}}} a_i \) with explicit \( a_i \) is an algorithm.
hand, the conjectural formula in [GNS17] above does not contain such a complicated algorithm. It is just given by a simple substitution. It means that we should have a drastic simplification if we take a linear combination of complicated Euler numbers of quiver varieties of finite ADE type. We do not need to compute Euler numbers of individual strata, as we only need their sum.

This simplification has a representation theoretic origin. Let us explain it.

The above specialized character is called the quantum dimension:

$$\dim_q V = \left. \text{ch} V \right|_{e^{-\alpha_i} = \exp(\frac{2\pi i}{h^\vee + 1})}.$$ 

Here $V$ is a finite dimensional representation $V$ of $\mathfrak{g}_{\text{fin}}$. It was introduced by Andersen [And92, Def. 3.1] (see also Parshall-Wang [PW93]). We choose a specific root of unity $\exp(\frac{2\pi i}{2(h^\vee + 1)})$, where these papers study more general roots of unity $\zeta$.

Therefore the conjectural formula in [GNS17] states that the generating function of Euler numbers of Hilb$^n(\mathbb{C}^2/\Gamma)$ is given by the quantum dimension of the Fock space at $\exp(\frac{2\pi i}{2(h^\vee + 1)})$, restricted from $\mathfrak{g}_{\text{aff}}$ to $\mathfrak{g}_{\text{fin}}$, just keeping track $e^{-\delta}$.

A direct sum of homology groups (which are isomorphic to complexified $K$-group) of Lagrangian subvarieties of quiver varieties of the finite ADE type carries a structure of a finite dimensional representation of the quantum loop algebra $U_q(\mathfrak{L}_{\text{fin}})$. See [Nak01]. It is called a standard module of $U_q(\mathfrak{L}_{\text{fin}})$.

It turns out that the above simplification is a consequence of the following representation theoretic result of an independent interest:

**Theorem 2.** The quantum dimension of arbitrary standard module of $U_q(\mathfrak{L}_{\text{fin}})$ of type ADE at $\zeta = \exp(\frac{2\pi i}{2(h^\vee + 1)})$ is equal to 1.

Since standard modules are tensor products of $l$-fundamental modules [VV02], it is enough to prove this result for $l$-fundamental modules. The author is told by Naoi that this result is a special case of more general conjecture posed by Kuniba [Kun93, Conj. 2 (A.6a)] (see also Kuniba-Nakanishi-Suzuki [KNS11, Conjecture 14.2]) formulated for Kirillov-Reshetikhin modules. (Recall $l$-fundamental modules are the simplest examples of Kirillov-Reshetikhin modules.) It is not difficult to check the general conjecture for type $A$. Type $D$ case was shown in [Lee13], while type $E_6$ case was shown in [Gle14].

Although we only need the simplest special case of more general conjecture, we could not find a proof of the relevant result for type $E_7$, $E_8$ in the literature. Therefore we give its proof. Fortunately the necessary explicit computation for type $E$ from the algorithm in [Nak04] was already done in [Nak10] by using a supercomputer. Alternatively we could quote the computation by Kleber [Kle97], which assumed fermionic formula conjectured at that time. The fermionic formula was proved later by Di Francesco and Kedem [FK08].

We also give a proof of the known cases $A$, $D$, $E_6$ for the completeness. We encounter a new feature in $E_7$, $E_8$, which did not arise in other cases. Hence our check in the simplest case is yet nontrivial.

Because it is about Kirillov-Reshetikhin modules, it suggests that a general conjecture should be studied in the framework of cluster algebras, as in [FK08]. Note also that Euler
numbers of (graded) quiver varieties are understood in the context of cluster algebras in a recent work of Bittmann [Bit19]. Thus the suggestion is compatible with the approach in this paper, though cluster algebras play no role in this paper.

The paper is organized as follows. In §1 we deduce Theorem 1 from Theorem 2 after recalling results from [Nak09]. In §2 we prove Theorem 2. It is proved by the case by case analysis. In §3 we discuss an additional topic, the rationally smoothness of Hilb^n(\mathbb{C}^2/\Gamma). It is rationally smooth if n = 2, but not so in general.

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1. Euler numbers of quiver varieties

1(i). Quiver varieties. Let \( \Gamma \) be a nontrivial finite subgroup of \( \text{SL}(2) \). We define the affine Dynkin diagram via the McKay correspondence (see e.g., [Kir16, Ch. 8] for detail): let \( \{ \rho_i \}_{i \in I} \) be the set of isomorphism classes of irreducible representations of \( \Gamma \) with the trivial representation \( \rho_0 \). We identify \( \rho_i \) with a vertex of a graph. We draw \( n_{ij} \) edges between \( \rho_i \) and \( \rho_j \) where \( n_{ij} = \dim \text{Hom}_\Gamma(\rho_i, \rho \otimes \rho_j) = \dim \text{Hom}_\Gamma(\rho_j, \rho \otimes \rho_i) \), where \( \rho \) is the 2-dimensional representation of \( \Gamma \) given by the inclusion \( \Gamma \subset \text{SL}(2) \). Then the graph is an affine Dynkin diagram of type \( ADE \). Let \( g_{\text{aff}} \) denote the corresponding affine Lie algebra, and \( g_{\text{fin}} \) the underlying finite dimensional complex simple Lie algebra corresponding to the Dynkin diagram obtained from the affine one by removing \( \rho_0 \). Let \( n \) be the rank of \( g_{\text{fin}} \), which is the number of vertices in \( I \) minus 1.

We use the convention of the root system for \( g_{\text{aff}} \) as in [Kac90, Ch. 6 and §12.4]. Let \( \alpha_i \) be the \( i \)th simple root of \( g_{\text{aff}} \) corresponding to \( \rho_i \), \( \delta \) be the primitive positive imaginary root of \( g_{\text{aff}} \). We have \( \delta = \sum a_i \alpha_i \), and \( a_i \) is equal to the dimension of \( \rho_i \). Let \( \alpha_i^\vee \) be the \( i \)th simple coroot. We take the scaling element \( d \) satisfying

\[ \langle \alpha_i, d \rangle = \delta_0. \]

Then \( \{ \alpha_i^\vee, d \}_{i \in I} \) is a base of \( h_{\text{aff}}^* \), the Cartan subalgebra of \( g_{\text{aff}} \). We define the fundamental weights \( \Lambda_i \ (i \in I) \) by

\[ \langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij}, \quad \langle \Lambda_i, d \rangle = 0. \]

Then \( \{ \alpha_i, \Lambda_0 \}_{i \in I} \) forms a base of \( h_{\text{aff}}^* \).

We choose an orientation of edges in the affine Dynkin diagram and consider the corresponding affine quiver \( Q = (I, \Omega) \). We take dimension vectors \( w = (w_i) \), and \( v = (v_i) \in \mathbb{Z}_{\geq 0}^I \), and consider quiver varieties \( \mathcal{M}_{\zeta}(v, w) \), \( \mathcal{M}_{\zeta^*}(v, w) \), where \( \zeta, \zeta^* \) are stability parameters such that

\[ \zeta_i < 0 \quad \text{for all } i \in I, \quad \zeta_i^* = 0 \quad \text{for all } i \neq 0, \quad \zeta_0^* < 0. \]
See [Nak09, §2] for the definition of quiver varieties for these stability conditions. Since $\zeta^\bullet$ lives in the boundary of a chamber containing $\zeta$, we have a projective morphism

$$\pi_{\zeta^\bullet, \zeta} : M_{\zeta}(v, w) \to M_{\zeta^\bullet}(v, w).$$

See [Nak09, §2]. In fact, this is an example studied in [Nak09, §2.8] associated with a division $I = I^0 \sqcup I^+$ with $I^0 = I \setminus \{0\}$, $I^+ = \{0\}$.

We identify $v, w$ with weights of $g_{aff}$ by

$$v = \sum_i v_i \alpha_i, \quad w = \sum_i w_i \Lambda_i.$$

It is convenient to use a different convention for the dimension vector $v$:

$$(1.1) \quad v = m\delta + \sum_{i \in I^0} m_i \alpha_i, \quad \text{i.e.,} \quad m = v_0, \quad m_i = v_i - v_0 a_i.$$

Let us take $w = \Lambda_0$. It is well-known that $M_{\zeta}(v, \Lambda_0)$ is the $\Gamma$-fixed point component of Hilbert schemes $I$ of points in the affine plane $\mathbb{C}^2$ such that $\mathbb{C}[x, y]/I$ is isomorphic to $\bigoplus \rho^{\mu}_{\zeta \alpha_i}$ as a $\Gamma$-module. Euler number for $M_{\zeta}(v, \Lambda_0)$ is also known. It was given in [Nak02]. Since it was stated without an explanation, let us explain how it is derived. We change the stability condition $\zeta$ to see that $M_{\zeta}(v, w)$ is diffeomorphic to a moduli space of framed rank 1 torsion free sheaves on the minimal resolution of $\mathbb{C}^2/\Gamma$. Then rank 1 torsion free sheaves are ideal sheaves twisted by line bundles, hence Euler numbers are given by Göttsche formula for Hilbert schemes of points [Göt90]. Moreover this latter picture gives the Frenkel-Kac construction of the basic representation of $g_{aff}$ (see e.g., [Kac90, §14.8]), hence we get the formula of $Z$ in Introduction. We have used the convention in (1.1).

We can define a structure of a representation of the affine Lie algebra $g_{aff}$ on the direct sum of homology groups $\bigoplus_{\zeta} H_* (M_{\zeta}(v, \Lambda_0))$. See [Nak02] and references therein. We can also construct a structure of a representation of the Heisenberg algebra commuting with $g_{aff}$ by [Nak97, Nak99].

On the other hand, Craw, Gammelgaard, Gyenge and Szendrői [CGGS19] recently proved that the Hilbert scheme $\text{Hilb}^n(\mathbb{C}^2/\Gamma)$ of $n$ points in $\mathbb{C}^2/\Gamma$ with the reduced scheme structure is isomorphic to $M_{\zeta^\bullet}(v, \Lambda_0)$ with $v = n\delta$.

1(ii). Stratification. As is explained in [Nak09, §2], $M_{\zeta^\bullet}(v, w)$ parametrizes $S$-equivalence classes of $\zeta^\bullet$-semistable framed representations of the preprojective algebra. Therefore its points are represented by direct sum of $\zeta^\bullet$-stable representations. Under the above choice of $\zeta^\bullet$, we have one distinguished summand, giving a point in $M_{\zeta^\bullet}(v', w)$ for some $v' \leq v$ (component-wise) such that $v_0 = v_0$, and other summands are simple representations $S_i$ with $i \in I^0$. See [Nak09, §2.6]. Since multiplicities of $S_i$ can be read off from the difference $v - v'$, we can regard $M_{\zeta^\bullet}(v', w)$ as a subset of $M_{\zeta^\bullet}(v, w)$. By [Nak09, Prop. 2.30] $M_{\zeta^\bullet}(v', w) \neq \emptyset$ if and only if $w - v'$ is an $I^0$-dominant weight appearing in the basic representation $V(\Lambda_0)$ of the affine Lie algebra $g$ associated with $\Gamma$. Here $I^0$-dominant means that $\langle w - v', \alpha_i \rangle \geq 0$ for $i \in I^0$. We thus have the stratification

$$M_{\zeta^\bullet}(v, w) = \bigsqcup_{v' \text{ as above}} M_{\zeta^\bullet}(v', w).$$
Moreover the transversal slice to the stratum $\mathcal{M}_{\bullet}(v', w)$ is locally isomorphic to a quiver variety $\mathcal{M}_0(v^s, w^s)$ around 0, associated with the finite ADE quiver $Q \setminus \{\rho_0\}$ such that dimension vectors are given by
\[ v^s = v - v', \quad w_i^s = \langle w - v', \alpha'_i \rangle \quad i \in I^0. \]

Note that $v - v'$ has no 0th component as $v_0^s = v_0$. Note also that $w_i^s \geq 0$, as $w - v'$ is $I^0$-dominant. It is also known that the inverse image of the slice in $\mathcal{M}_c(v, w)$ under $\pi_{c, \xi}$ is locally isomorphic to $\mathcal{M}_c(v^s, w^s)$ around $\mathcal{L}(v^s, w^s)$, the inverse image of the origin 0 of $\mathcal{M}_0(v^s, w^s)$ under the projective morphism $\mathcal{M}_c(v^s, w^s) \to \mathcal{M}_0(v^s, w^s)$. See [Nak09, §2.7] and the references therein for these claims on transversal slices. There is also an algebraic approach in [CB03].

For $w = \Lambda_0$, we have
\[
w_i^s = -\sum_{j \in I^0} m'_{ij} \langle \alpha_j, \alpha'_i \rangle, \quad \text{where} \quad v' = m\delta + \sum_{j \in I^0} m'_{ij} \alpha_j \text{ in the convention (1.1)}. 
\]

Note that $v = m\delta + \sum m_i \alpha_i$ and $v'$ share the same $m$ for the coefficient of $\delta$, as $v_0 = v_0'$. In particular,
\[
\sum_{i \in I^0} w_i^s \Lambda_i - v_i^s \alpha_i = -\sum_{i \in I^0} m_i \alpha_i. 
\]

We also note
\[
m'_{ij} \leq 0 \quad (i \in I^0), \tag{1.3}
\]

as $\langle m'_{ij}, i \in I^0 \rangle = -(w^s)_{i \in I^0} C^{-1}$, and the inverse of the Cartan matrix $C$ has positive entries. This was proved in [CGGS19, Prop. A.1] by a different method.

1(iii). Examples.

**Example 1.4.** Let $\mathcal{M}_{\bullet}(m\delta, \Lambda_0) \cong \text{Hilb}^n(\mathbb{C}^2/\Gamma)$. Consider a stratum containing $(n - 1)$-distinct points in $\mathbb{C}^2 \setminus \{0\}/\Gamma$ together with trivial representations. We have $v' = (n - 1)\delta + \alpha_0 = n\delta - \sum_{i \in I^0} a_i \alpha_i$, and hence
\[ v^s = \sum_{i \in I^0} a_i \alpha_i, \quad w_i^s = \sum_{j \in I^0} C_{ij} a_j. \]

Note that $w^s$ has entries 1 at vertices in the finite quiver $I^0$ which are connected to the 0-vertex in the affine quiver. In particular, $\mathcal{M}_0(v^s, w^s)$ is $\mathbb{C}^2/\Gamma$, $\mathcal{M}_c(v^s, w^s)$ is its minimal resolution, the very first example of a quiver variety considered by Kronheimer [Kro89], before the definition of quiver varieties was introduced.

This is obvious, as the transversal slice is $\mathcal{M}_{\bullet}(\delta, \Lambda_0) \cong \mathbb{C}^2/\Gamma$, as we can ignore $(n - 1)$-distinct points.

**Example 1.5** (Yamagishi [Yam17]). Consider $\mathcal{M}_{\bullet}(2\delta, \Lambda_0) = \text{Hilb}^2(\mathbb{C}^2/\Gamma)$. The formal neighborhoods of fibers of $\pi_{c, \xi}$ over 0-dimensional strata in $\mathcal{M}_c(2\delta, \Lambda_0)$ were determined by Yamagishi [Yam17]. He identified the formal neighborhood with that of the intersection of the nilpotent cone for the complex simple Lie algebra $\mathfrak{g}_{\text{fin}}$ and the Slodowy slice to a
‘sub-subregular’ orbit. There is only one 0-dimensional stratum except type $A_1$, $A_2$, $D_n$, while there are none for $A_1$, $A_2$, two for $D_n$ ($n > 4$) and three for $D_4$. We can give corresponding vector $v'$ as follows:

- If $Q$ is of type $A_n^{(1)}$ or $A_2^{(1)}$, there is no such $v'$.
- If $Q$ is of type $A_n^{(1)}$ with $n > 2$, $v' = 2\alpha_0 + \alpha_1 + \alpha_n$.
- If $Q$ is of type $D_n^{(1)}$ with $n > 4$, $v' = 2\alpha_0 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ or $v' = 2\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3$.
- If $Q$ is of type $D_4^{(1)}$, we have three possibilities: in addition to the second example in $D_n^{(1)}$ above, we have $(\alpha_1, \alpha_3) \mapsto (\alpha_1, \alpha_4), (\alpha_3, \alpha_4)$, as its cyclic permutation.
- If $Q$ is of type $E_8^{(1)}$, we have $v' = \begin{array}{ccccccc}
1 & 2 & 2 & 2 & 2 & 1 \\
1 & 0 & 0 & 0 & 0 & 0
\end{array}$. The cases $E_6^{(1)}, E_7^{(1)}$ are similar.

Here 0th vertex is $\circ$, and other vertices are $\bullet$.

Transversal slices are

- If $Q$ is of type $A_n^{(1)}$ with $n > 2$, $v^s = \begin{array}{cccccccc}
1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}$, $w^s = \begin{array}{cccccccc}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}$.

- If $Q$ is of type $D_n^{(1)}$, $v^s = \begin{array}{ccccccc}
1 & 2 & 2 & 2 & 2 & 1 & 0 \\
1 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}$, $w^s = \begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$ in the first case, and $v^s = \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 4 & 4 & 4 & 2 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}$, $w^s = \begin{array}{cccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$ in the second case.

- If $Q$ is of type $D_4^{(1)}$, $v^s = \begin{array}{cccccc}
1 & 2 & 1 & 0 & 0 & 2 \\
1 & 2 & 2 & 2 & 2 & 2
\end{array}$, $w^s = \begin{array}{cccccc}
0 & 0 & 2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}$, and its cyclic permutation.

- If $Q$ is of type $E_8^{(1)}$, $v^s = \begin{array}{ccccccccc}
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}$, $w^s = \begin{array}{ccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$.

In particular, it means that the quiver variety $\mathcal{M}_0(v^s, w^s)$ is isomorphic to the intersection of Slodowy slice and the nilpotent cone in the formal neighborhood of the origin.

This result was known before for type $A_n$ and the second case of $D_n$. The type $A_n$ case was proved in [Nak94, §8]. See also [Maf05]. For the second case of type $D_n$, it was shown in [HL14]. In these cases, $\mathcal{M}_0(v^s, w^s)$ itself is isomorphic to the intersection of Slodowy slice and the nilpotent cone, not only in the formal neighborhood. The exceptional cases are conjectured, but not shown as far as the author knows.
Remark 1.6. In cases for the above Example, it is known that corresponding Coulomb branches of the quiver gauge theories (which are affine Grassmannian slices by [BFN19]) are ‘next-to-minimal’ nilpotent orbits closures in $\mathfrak{g}_{\text{fin}}$ by [AH13].

Example 1.7 (Yamagishi [Yam17]). Let us consider $v = \delta + \alpha_0$, $w = \Lambda_0$, hence $\mathcal{M}_{v} (\delta + \alpha_0, \Lambda_0)$. Note that this is different from $\mathcal{M}_{v} (\delta, \Lambda_0) = \text{Hilb}^1 (\mathbb{C}^2 / \Gamma) = \mathbb{C}^2 / \Gamma$. Let us show that $\mathcal{M}_{v} (\delta + \alpha_0, \Lambda_0)$ is not isomorphic to $\mathbb{C}^2 / \Gamma$.

This $\mathcal{M}_{v} (\delta + \alpha_0, \Lambda_0)$ appears as the union of strata in $\mathcal{M}_{v} (2\delta, \Lambda_0) = \text{Hilb}^2 (\mathbb{C}^2 / \Gamma)$. Thus all possible strata are determined already as above: besides the open stratum for $v' = v$, we have a single 0-dimensional stratum except for $A_1$, $A_2$, $D_n$, none for $A_1$, $A_2$, three for $D_4$, two for $D_n$ ($n > 4$). The transversal slice $\mathcal{M}_{v} (v^s, w^s)$ is given by the same $w^s$ as above, $v^s$ is subtracting $\sum_{i \in I_0} a_i \alpha_i$ from the above example. Concretely it is

- For type $A_n^{(1)}$ with $n > 2$, $v^s = \begin{array}{cccccccc} 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ \end{array}$, $w^s = \begin{array}{cccccccc} 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ \end{array}$.

- For type $D_n^{(1)}$ with $n > 4$, $v^s = \begin{array}{cccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{array}$, $w^s = \begin{array}{cccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \end{array}$ in the first case, and $v^s = \begin{array}{cccccccc} 0 & 1 & 2 & 2 & 2 & 2 & 0 & 0 \\ 1 & 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ \end{array}$, $w^s = \begin{array}{cccccccc} 0 & 1 & 2 & 2 & 2 & 2 & 0 & 0 \\ 1 & 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ \end{array}$ in the second case.

- For type $D_4^{(1)}$, $v^s = \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 2 & 2 \\ \end{array}$, $w^s = \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 2 & 2 \\ \end{array}$, and its cyclic permutation.

- For type $E_8^{(1)}$, $v^s = \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ \end{array}$, $w^s = \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ 0 & 1 & 2 & 3 & 4 & 3 & 2 & 1 \\ \end{array}$.

Let us consider $\pi_{\cdot, \cdot} : \mathcal{M}_{v} (v, w) \rightarrow \mathcal{M}_{v} (v, w)$. The domain $\mathcal{M}_{v} (v, w)$ is the minimal resolution of $\mathbb{C}^2 / \Gamma$. The above description of the transversal slice immediately conclude the following:

- $\mathcal{M}_{v} (v, w)$ is obtained from the minimal resolution of $\mathbb{C}^2 / \Gamma$ by collapsing $\mathbb{P}^1$’s whose vertices are not connected to the 0-vertex in the affine Dynkin diagram.

There are no such vertices for $A_1$, $A_2$. For type $A_n$ ($n > 2$), we collapse $(n - 2)$ $\mathbb{P}^1$’s corresponding to vertices except the leftmost and rightmost. For type $D_n$, we collapse $(n - 1)$ $\mathbb{P}^1$’s except the second one from the left, and producing singularities of type $A_1$ and $D_{n-2}$, where we understand $D_2 = A_1 \times A_1$, $D_3 = A_3$. For type $E_8$, the 7 $\mathbb{P}^1$’s except the rightmost one are collapsed.

1(iv). Euler numbers. By §1(iii) we can relate Euler numbers of $\mathcal{M}_{v} (v, w)$, $\mathcal{M}_{v} (v', w)$ and $\mathcal{M}_{v} (v^s, w^s)$. Let $\chi (\cdot)$ denote the Euler number of a space. We have

$$\chi (\mathcal{M}_{v} (v, w)) = \sum_{v'} \chi (\mathcal{M}_{v'} (v', w)) \chi (\mathcal{L}_{v^s} (v^s, w^s))$$

$$= \sum_{v'} \chi (\mathcal{M}_{v'} (v', w)) \chi (\mathcal{M}_{v} (v^s, w^s)).$$
It is known [Nak94, Cor. 5.5] that the central fiber $\mathcal{L}_\zeta(v^s, w^s)$ of $\mathcal{M}_\zeta(v^s, w^s) \rightarrow \mathcal{M}_0(v^s, w^s)$ is homotopic to $\mathcal{M}_\zeta(v^s, w^s)$, hence the second equality follows.

From $\chi(\mathcal{M}_\zeta(v, w))$, $\chi(\mathcal{M}_\zeta(v^s, w^s))$, we compute $\chi(\mathcal{M}_\zeta(v^s, w^s))$ recursively as

$$\chi(\mathcal{M}_\zeta(v^s, w^s)) = \chi(\mathcal{M}_\zeta(v, w)) - \sum_{v' \neq v} \chi(\mathcal{M}_\zeta(v^s, w^s))\chi(\mathcal{M}_\zeta(v^s, w^s)). \quad (1.8)$$

Here we use $v^s = 0$, hence $\mathcal{M}_\zeta(v^s, w^s)$ is a point for $v' = v$.

We take the generating function of Euler numbers as

$$\sum_v \chi(\mathcal{M}_\zeta(v, w))e^{-v} = \sum_{v'} \chi(\mathcal{M}_\zeta(v', w))e^{-v'} \sum_{v^s} \chi(\mathcal{M}_\zeta(v^s, w - v'|j))e^{-v^s}, \quad (1.9)$$

where $w - v'|j = w^s = \sum_{i \in j} (w - v', \alpha^s_i)\Lambda_i$.

We claim

$$\sum_{v^s} \chi(\mathcal{M}_\zeta(v^s, w^s)) \prod_{i \in j} e^{w_i^s\Lambda_i - v^s_i\alpha_i} \bigg|_{e^{-\alpha_i} = \exp\left(\frac{2\pi i}{h^\times +1}\right)} = 1. \quad (1.10)$$

We take $w = \Lambda_0$ and switch to the convention (1.1). Then the left hand side of (1.9) is $Z$ in Introduction. By (1.2), (1.10) implies that

$$Z|_{e^{-\alpha_i} = \exp\left(\frac{2\pi i}{h^\times +1}\right)} = \sum_{m, \tilde{m}} \chi(\mathcal{M}_\zeta(m\delta + \sum_{i \in j} m_i\alpha_i, \Lambda_0))e^{-m\delta}. \quad (1.11)$$

By (1.3) we consider $\mathcal{M}_\zeta^s(m\delta + \sum_{i \in j} m_i\alpha_i, \Lambda_0)$ as a stratum of $\mathcal{M}_\zeta^s(m\delta, \Lambda_0) = \text{Hilb}^m(\mathbb{C}^2/\Gamma)$. Therefore the right hand side of the above is the generating function of Euler numbers of Hilb$^m(\mathbb{C}^2/\Gamma)$. Thus we have proved Theorem 1.

Recall that $\bigoplus_{v^s} H_*(\mathcal{L}_\zeta(v^s, w^s)) \cong \bigoplus_{v^s} K(\mathcal{L}_\zeta(v^s, w^s)) \otimes_{\mathbb{C}} \mathbb{C}$ is the so-called standard module of the quantum loop algebra $U_q(\mathfrak{gl}_\text{fin})$, specialized at $q = 1$ [Nak01]. The above (1.10) means that its character (as a $\mathfrak{gl}_\text{fin}$-module), specialized at $e^{-\alpha_i} = \exp\left(\frac{2\pi i}{h^\times +1}\right)$ is equal to 1. As we mentioned in Introduction, this specialization is the quantum dimension. Thus (1.10) follows from Theorem 2.

In order to prove (1.10), we may assume $w^s$ is a fundamental weight: there is a torus action on framing vector spaces, and the induced action on $\mathcal{M}_\zeta(v^s, w^s)$. Let us suppose $w^s = \Lambda_i + \Lambda_j$ for $i, j \in I^0$ for simplicity. Then the torus fixed point set is

$$\bigsqcup_{v^1 + v^2 = v^s} \mathcal{M}_\zeta(v^1, \Lambda_i) \times \mathcal{M}_\zeta(v^2, \Lambda_j).$$

As Euler number is equal to the sum of Euler numbers of fixed points with respect to a torus action, (1.10) for $w^s$ follows from (1.10) for $\Lambda_i$ and $\Lambda_j$. This result is compatible with what we mentioned in Introduction: standard modules are tensor products of $l$-fundamental modules [VV02], as $l$-fundamental modules correspond to the case $w^s$ is a fundamental weight.

A standard module depends also on spectral parameter, which is specialization homomorphism $K_{\prod_{i \in j^0} \text{GL}(w_i^s)}(pt) \rightarrow \mathbb{C}$. But the restriction of a standard module to $U_q(\mathfrak{g}_\text{fin})$ is
independent of the spectral parameter. Hence the spectral parameter is not relevant for Theorem 2.

Remark 1.11. In order to prove Theorem 1, we need to check (1.10) only when \( w^* \) is contained in the root lattice. But \( \Lambda_i \) does not satisfy this condition in general, hence the above reduction cannot be performed among \( w^* \) in the root lattice.

2. Quantum dimensions of standard modules

2(i). Quantum dimension. Let \( V \) be a finite dimensional representation of \( g_{\text{fin}} \). The specialized character

\[
\left. \text{ch} \right|_{e^{-\alpha_i = \exp(\frac{2\pi i}{h^\vee + 1})}} V
\]

is called the quantum dimension of \( V \), denoted by \( \dim_q V \), and was introduced by Andersen [And92, Def. 3.1] (see also Parshall-Wang [PW93]), where \( \zeta = \exp\left(\frac{2\pi i}{2(h^\vee + 1)}\right) \). Note \( 2\rho \) in \( K_{2\rho} = q^{2\rho} \) in [And92] should be understood as an element in the dual of the weight lattice by \( \langle 2\rho, \lambda \rangle = (2\rho, \lambda)/(\alpha_0, \alpha_0) \) for a weight \( \lambda \). Here \( \alpha_0 \) is a short root. See [PW93, Lemma 1.1]. It was assumed that \( \zeta \) is a primitive \( \ell \)-th root of unity with odd \( \ell \) in [And92], the definition still makes sense for our choice. By the Weyl character formula, we have (see [And92, (3.2)], [PW93, Th. 1.3])

\[
\left. \text{ch} \right|_{e^{-\alpha_i = \exp(\frac{2\pi i}{h^\vee + 1})}} V = \prod_{\alpha \in \Delta_{\text{fin}}^+} \frac{\zeta^{d_\alpha(\lambda + \rho, \alpha^\vee)} - \zeta^{-d_\alpha(\lambda + \rho, \alpha^\vee)}}{\zeta^{d_\alpha(\rho, \alpha^\vee)} - \zeta^{-d_\alpha(\rho, \alpha^\vee)}} ,
\]

where \( \lambda \) is the highest weight of an irreducible representation \( V = V(\lambda) \), \( \Delta_{\text{fin}}^+ \) is the set of positive roots of \( g_{\text{fin}} \), \( d_\alpha \in \{1, 2, 3\} \) is the square length of \( \alpha \) divided by the length of a short root, and \( \rho \) is the half sum of positive roots. (Since we are only considering type ADE, we have \( d_\alpha = 1 \) for any \( \alpha \).) We have \( d_\alpha \langle \rho, \alpha^\vee \rangle \leq \langle \rho, \theta \rangle = h^\vee \), hence the denominator does not vanish.

Let us introduce \( \zeta \)-integers by

\[
[n]_\zeta \overset{\text{def}}{=} \frac{\zeta^n - \zeta^{-n}}{\zeta - \zeta^{-1}},
\]

so that

\[
(2.1) \quad \left. \text{ch} \right|_{e^{-\alpha_i = \exp(\frac{2\pi i}{h^\vee + 1})}} V = \prod_{\alpha \in \Delta^+} \frac{[d_\alpha(\lambda + \rho, \alpha^\vee)]_\zeta}{[d_\alpha(\rho, \alpha^\vee)]_\zeta}.
\]

We have

\[
[2(h^\vee + 1) + k]_\zeta = [k]_\zeta
\]

as \( \zeta^{2(h^\vee + 1)} = 1 \), as well as

\[
(2.2) \quad [h^\vee + 1 - k]_\zeta = [k]_\zeta
\]

as \( \zeta^{h^\vee + 1 - k} \zeta^k = \zeta^{h^\vee + 1} = -1 \). In particular, we have \([h^\vee + 1]_\zeta = 0 \). The former is usual analogy between roots of unity and characteristic \( 2(h^\vee + 1) \). The latter is a new feature at an even root of unity.
2(ii). **Type A.** Consider type $A_{n-1}$, i.e., $\mathfrak{g}_{\text{fin}} = \mathfrak{sl}(n, \mathbb{C})$. We have $h^\vee = n$.

In type $A_{n-1}$, it is known that the $k$th $l$-fundamental module is the $k$th fundamental representation of $\text{SL}(n)$.

The quantum dimension of the $k$th fundamental representation $V(\Lambda_k)$ is

$$\dim_q V(\Lambda_k) = \frac{[n][n-1]\cdots[n-k+1]}{[k][k-1]\cdots[1]}.$$  

By (2.2), this is equal to 1, as $[n-i]$ cancels with $[i+1]$.

Hence Theorem 2 is proved for type $A$.

2(iii). **Type D.** Consider type $D_n$, i.e., $\mathfrak{g}_{\text{fin}} = \mathfrak{so}(2n, \mathbb{C})$. We have $h^\vee = 2n - 2$.

It is known that the $k$th $l$-fundamental module for $1 \leq k \leq n - 2$ is isomorphic as an $\mathfrak{so}(2n, \mathbb{C})$-module to

$$\bigwedge^k (\mathbb{C}^{2n}) \oplus \bigwedge^{k-2} (\mathbb{C}^{2n}) \oplus \cdots,$$

where $\cdots$ ends as $\bigwedge^1 (\mathbb{C}^{2n}) = \mathbb{C}^{2n}$ if $k$ is odd, and $\bigwedge^0 (\mathbb{C}^{2n}) = \mathbb{C}$ if $k$ is even. See [Nak03, Remark 5.9]. For $k = n - 1, n$, the $l$-fundamental module is isomorphic to the $k$th fundamental representation of $\mathfrak{so}(2n, \mathbb{C})$.

As in [FH91], positive roots are

$$\{L_i + L_j \mid i < j\} \cup \{L_i - L_j \mid i < j\},$$

and simple roots are

$$\alpha_i = L_i - L_{i+1} \quad (i = 1, \ldots, n-1), \quad \alpha_n = L_{n-1} + L_n$$

with the standard inner product $(L_i, L_j) = \delta_{ij}$. Fundamental weights are

$$\Lambda_i = L_1 + L_2 + \cdots + L_i \quad (i = 1, \ldots, n-2),$$

$$\Lambda_{n-1} = \frac{1}{2}(L_1 + L_2 + \cdots + L_{n-2} + L_{n-1} - L_n),$$

$$\Lambda_n = \frac{1}{2}(L_1 + L_2 + \cdots + L_{n-2} + L_{n-1} + L_n).$$

Weyl vector $\rho$ is

$$\rho = \sum_{i=1}^{n} (n-i)L_i.$$  

We have $(\Lambda_k, L_1 + L_2) = 2$ unless $k = 1, n-1, n$. If this holds, we have

$$[(\Lambda_k + \rho, L_1 + L_2)] = [2n-1] = 0.$$

Thus

$$\dim_q V(\Lambda_k) = 0 \quad \text{if} \quad k \neq 1, n-1, n.$$

We have

$$\dim_q V(\Lambda_1) = \frac{[n][2n-2]}{[1][n-1]} = 1,$$

as $[2n-2] = [1], [n] = [n-1]$ by (2.2).
Next we consider the case $\lambda = \Lambda_{n-1}$. We have
\[
(\Lambda_{n-1}, L_i - L_j) = \begin{cases}
0 & \text{if } j \leq n - 1, \\
1 & \text{if } j = n,
\end{cases}
\]
\[
(\Lambda_{n-1}, L_i + L_j) = \begin{cases}
1 & \text{if } j \leq n - 1, \\
0 & \text{if } j = n.
\end{cases}
\]
These imply
\[
\dim_q V(\Lambda_{n-1}) = \begin{cases}
\frac{[2n-2]_\zeta [2n-4]_\zeta \ldots [4]_\zeta}{[n-1]_\zeta [n-2]_\zeta \ldots [3]_\zeta} & \text{if } n \text{ is even}, \\
\frac{[2n-2]_\zeta [2n-4]_\zeta \ldots [n]_\zeta}{[n-1]_\zeta [n-2]_\zeta [n-3]_\zeta \ldots [3]_\zeta} & \text{if } n \text{ is odd}.
\end{cases}
\]
This is 1 by (2.2). Note $[2n-2]_\zeta = [1]_\zeta = 1$. We also have $\dim_q V(\Lambda_n) = 1$ as we have a diagram automorphism $n \leftrightarrow n - 1$, $i \leftrightarrow i$ ($i \neq n - 1, n$).

In summary,
\[
\dim_q V(\Lambda_k) = \begin{cases}
1 & \text{if } k = 1, n - 1, n, \\
0 & \text{otherwise}.
\end{cases}
\]
Hence Theorem 2 is proved.

2(iv). Type $E_6$. We have $h^\vee = 12$. The numbering of vertices is $1 \rightarrow 3 \rightarrow 5 \rightarrow 6$. We need to compute quantum dimensions of $V(\Lambda_k)$ as well as of $V(\Lambda_1 + \Lambda_6)$ since it appears in standard representations when $w^*$ is the 4th fundamental weight.

Positive roots are
\[
\alpha_1, \ldots, \alpha_6, \alpha_1 + \alpha_3, \alpha_3 + \alpha_4, \alpha_4 + \alpha_5, \alpha_5 + \alpha_6, \alpha_2 + \alpha_4,
\]
\[
\alpha_1 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_4 + \alpha_5, \alpha_3 + \alpha_4 + \alpha_5, \alpha_4 + \alpha_5 + \alpha_6,
\]
\[
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5,
\]
\[
\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,
\]
\[
\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6,
\]
\[
\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6,
\]
\[
\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6,
\]
\[
\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6,
\]
\[
\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6,
\]
\[
\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6.
\]

Hence the denominator of (2.1) is
\[
[1]_\zeta^6 [2]_\zeta^5 [3]_\zeta^5 [4]_\zeta^5 [5]_\zeta^4 [6]_\zeta^3 [7]_\zeta^2 [8]_\zeta^2 [9]_\zeta^2 [10]_\zeta^2 [11]_\zeta.
\]

We compute the quantum dimension of the first fundamental representation as
\[
\dim_q V(\Lambda_1) = \frac{[2]_\zeta [3]_\zeta [4]_\zeta [5]_\zeta [6]_\zeta [7]_\zeta [8]_\zeta [9]_\zeta [10]_\zeta [11]_\zeta}{[1]_\zeta [2]_\zeta [3]_\zeta [4]_\zeta [5]_\zeta [6]_\zeta [7]_\zeta [8]_\zeta [9]_\zeta [10]_\zeta [11]_\zeta} = \frac{[9]_\zeta [12]_\zeta}{[1]_\zeta [14]_\zeta}.
\]
This is 1 by (2.2). We also have \( \dim_q V(\Lambda_6) = 1 \) by the diagram automorphism. We have
\[
\begin{align*}
(\Lambda_k + \rho, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6) &= 13 \quad \text{for } k = 2, 3, 5, \\
(\Lambda_1 + \Lambda_6 + \rho, \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6) &= 13.
\end{align*}
\]
Therefore \( \dim_q V(\Lambda_k) = 0 \) for \( k = 2, 3, 5 \), \( \dim_q V(\Lambda_1 + \Lambda_6) = 0 \). We also have
\[
(\Lambda_4 + \rho, \alpha_1 + \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6) = 13,
\]
hence \( \dim_q V(\Lambda_4) = 0 \).

It is known that the \( k \)th fundamental module is isomorphic as a \( g_{\text{fin}} \)-module to
\[
\begin{align*}
V(\Lambda_k) & \quad \text{if } k = 1, 6, \\
V(\Lambda_2) \oplus V(0) & \quad \text{if } k = 2, \\
V(\Lambda_3) \oplus V(\Lambda_6) & \quad \text{if } k = 3, \\
V(\Lambda_5) \oplus V(\Lambda_1) & \quad \text{if } k = 5, \\
V(\Lambda_4) \oplus V(\Lambda_2) \oplus V(\Lambda_1 + \Lambda_6) \oplus V(0) & \quad \text{if } k = 4.
\end{align*}
\]
This can be given by using the algorithm [Nak04], as we did for \( E_8 \) in [Nak10] with much smaller efforts. Instead, the list can be found in [Kle97], which assumed fermionic formula conjectured at that time, and proved later in [FK08].

We substitute the above computation of quantum dimensions of various modules to the above combination. We find that the combination has the quantum dimension always 1. Hence Theorem 2 is proved.

2(v). type \( E_7 \). We have \( h^v = 18 \). The numbering of vertices is

\[
\begin{array}{cccccc}
& & & & 2 & \\
1 & 3 & 4 & 5 & 6 & 7
\end{array}
\]

We calculate as above, using Sage:
\[
\dim_q V(\Lambda_7) = \frac{[10]_\zeta[14]_\zeta[18]_\zeta}{[1]_\zeta[5]_\zeta[9]_\zeta} = 1
\]
and
\[
\dim_q V(\Lambda_k) = 0 \quad \text{if } k \neq 7.
\]

We have a new pattern:
\[
\dim_q V(2\Lambda_1) = \frac{[12]_\zeta[13]_\zeta[14]_\zeta[15]_\zeta[18]_\zeta[21]_\zeta}{[1]_\zeta[2]_\zeta[4]_\zeta[5]_\zeta[6]_\zeta[7]_\zeta} = \frac{[21]_\zeta}{[2]_\zeta}.
\]
We use (2.2) for \( k = -2 \) this time: \( [21]_\zeta = [-2]_\zeta \). This is \( -[2]_\zeta \). Hence
\[
\dim_q V(2\Lambda_1) = -1.
\]
This is the first example of a representation whose quantum dimension is \(-1\). Similarly we have
\[
\dim_q V(\Lambda_1 + \Lambda_7) = \frac{[12]_\zeta[14]_\zeta[15]_\zeta[18]_\zeta[20]_\zeta}{[1]_\zeta[2]_\zeta[4]_\zeta[5]_\zeta[7]_\zeta} = -1.
\]
On the other hand, we get
\[ \dim_q V(2\Lambda_7) = \dim_q V(\Lambda_1 + \Lambda_6) = \dim_q V(\Lambda_2 + \Lambda_7) = 0. \]

It is known that the \( k \)th \( l \)-fundamental module is isomorphic as a \( g_\text{fin} \)-module to
\[
\begin{cases} 
V(\Lambda_k) & \text{if } k = 7, \\
V(\Lambda_1) \oplus V(0) & \text{if } k = 1, \\
V(\Lambda_2) \oplus V(\Lambda_7) & \text{if } k = 2, \\
V(\Lambda_6) \oplus V(\Lambda_1) \oplus V(0) & \text{if } k = 6, \\
V(\Lambda_3) \oplus V(\Lambda_6) \oplus V(\Lambda_1)^\oplus \oplus V(0) & \text{if } k = 3, \\
V(\Lambda_5) \oplus V(\Lambda_2)^\oplus \oplus V(\Lambda_1 + \Lambda_7) \oplus V(\Lambda_7)^\oplus & \text{if } k = 5, \\
V(\Lambda_4) \oplus V(\Lambda_3)^\oplus \oplus V(\Lambda_2 + \Lambda_7)^\oplus \oplus V(\Lambda_1 + \Lambda_6) \oplus V(2\Lambda_1) & \text{if } k = 4. \\
\end{cases}
\]

Substituting the above computation, we find that these combinations have its quantum dimension always 1: for example, we have \( \dim_q V(2\Lambda_1) + 2 \dim_q V(0) = (-1) + 2 = 1 \) in the last case. Hence Theorem 2 is proved.

2(vi). type \( E_8 \). We have \( h \vee = 30 \). The numbering of vertices is
\[ 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8. \]

We have similar patterns:
\[ \dim_q V(\Lambda_k) = 0 \quad \text{for any } k = 1, \ldots, 8, \]
\[ \dim_q V(2\Lambda_8) = \frac{[20]\zeta[21]\zeta[24]\zeta[25]\zeta[30]\zeta[33]\zeta}{[1]\zeta[2]\zeta[6]\zeta[7]\zeta[10]\zeta[11]\zeta} = -1, \]
\[ \dim_q V(\Lambda_7 + \Lambda_8) = \frac{[14]\zeta[18]\zeta[20]\zeta[22]\zeta[24]\zeta[25]\zeta[26]\zeta[30]\zeta[32]\zeta[34]\zeta}{[1]\zeta[3]\zeta[5]\zeta[6]\zeta[7]\zeta[9]\zeta[11]\zeta[13]\zeta[17]\zeta} = 1, \]
\[ \dim_q V(\Lambda_6 + \Lambda_8) = \frac{[15]\zeta[18]\zeta[20]\zeta[21]\zeta[24]\zeta[25]\zeta[26]\zeta[27]\zeta[30]\zeta[32]\zeta[33]\zeta[35]\zeta}{[1]\zeta[2]\zeta[4]\zeta[5]\zeta[6]\zeta[7]\zeta[10]\zeta[11]\zeta[13]\zeta[16]\zeta} = -1, \]
and
\[ \dim_q V(\Lambda_1 + \Lambda_8) = \dim_q V(2\Lambda_1) = \dim_q V(\Lambda_2 + \Lambda_8) = \dim_q V(\Lambda_1 + \Lambda_7) = \dim_q V(\Lambda_1 + 2\Lambda_8) = \dim_q V(2\Lambda_7) = \dim_q V(3\Lambda_8) = \dim_q V(2\Lambda_1 + \Lambda_8) = \dim_q V(\Lambda_3 + \Lambda_8) = \dim_q V(\Lambda_2 + \Lambda_7) = \dim_q V(\Lambda_1 + \Lambda_6) = \dim_q V(\Lambda_1 + \Lambda_2) = 0. \]

The \( g_\text{fin} \)-module structure of the \( k \)th \( l \)-fundamental module is known. Let us omit negligible modules (i.e., those with \( \dim_q = 0 \)) for brevity. We have
\[
\begin{cases} 
V(0) & \text{if } k = 1, 2, 7, 8, \\
V(2\Lambda_8) \oplus V(0)^\oplus & \text{if } k = 3, 6, \\
V(2\Lambda_8)^\oplus \oplus V(\Lambda_7 + \Lambda_8)^\oplus \oplus V(0)^\oplus & \text{if } k = 5, \\
V(\Lambda_6 + \Lambda_8)^\oplus \oplus V(\Lambda_7 + \Lambda_8)^\oplus \oplus V(2\Lambda_8)^\oplus \oplus V(0)^\oplus & \text{if } k = 4. \\
\end{cases}
\]
Rather miraculously we find that all have quantum dimensions 1. For example, we calculate the result as $(-4) + 18 + (-23) + 10 = 1$ in the last case. Hence Theorem 2 is proved.

3. Rationally smoothness

As mentioned in Introduction, the original motivation of study of $\mathcal{M}_{\zeta}(v, w)$ in [Nak09] was its relation between the restriction from $\mathfrak{g}_{\text{aff}}$ to $\mathfrak{g}_{\text{fin}}$. However the Euler numbers of $\mathcal{M}_{\zeta}(v, w)$ did not play any role in [Nak09]. Intersection cohomology groups of $\mathcal{M}_{\zeta}(v, w)$ appeared instead.

Therefore we ask whether $\mathcal{M}_{\zeta}(v, w)$ is a rational homology manifold, i.e., its intersection cohomology complex is quasi-isomorphic to the constant sheaf. See [BM83, 1.4] for the original definition of a rational homology manifold, and its equivalence to the present one. This sounds optimistic, but Borho-MacPherson [BM83, §2.3] showed that the nilpotent variety is a rational homology manifold. Also $\mathcal{M}_0(v, \Lambda)$ is a symmetric power of $\mathbb{C}^2/\Gamma$, which has only finite quotient singularity, hence is a rational homology manifold.

3(i). Example: Hilbert schemes of two points. Consider $\mathcal{M}_{\zeta}(2\delta, \Lambda_0) = \text{Hilb}^2(\mathbb{C}^2/\Gamma)$. As we already mentioned in Example 1.5, Yamagishi showed that $\mathcal{M}_{\zeta}(2\delta, \Lambda_0)$ is locally isomorphic to the intersection of the nilpotent cone of $\mathfrak{g}_{\text{fin}}$ and the Slodowy slice to a sub-subregular orbit around 0-dimensional strata. The transversal slice to the bigger stratum is $\mathbb{C}^2/\Gamma$. It is also a rational homology manifold. Combined with the rational smoothness of the nilpotent variety mentioned above, we obtain the following.

**Corollary 3.1.** $\mathcal{M}_{\zeta}(2\delta, \Lambda_0) = \text{Hilb}^2(\mathbb{C}^2/\Gamma)$ is a rational homology manifold.

This result can be also proved from [Nak09], together with [Nak01] and some Euler number computation. See the argument below in a simpler situation.

3(ii). Counter-example. Consider $A^{(1)}_3$ with $n = 4$, i.e., $\text{Hilb}^4(\mathbb{C}^2/(\mathbb{Z}/2)) = \mathcal{M}_{\zeta}(4\delta, \Lambda_0)$. We take $v' = \begin{array}{c} 4 \\ 2 \end{array}$. Then $v^* = 2$, $w^* = 4$. Hence the transversal slice is $\mathcal{M}_0(v^*, w^*)$, which is the nilpotent orbit in $\mathfrak{sl}(4)$ of type $(2^2)$. It is well known to be a non-rational homology manifold. For example, we can argue as follows. By [Nak01, Th. 15.1.1] $\dim i_0^! \text{IC}(\mathcal{M}_0(v^*, w^*))$ gives the multiplicities of the trivial representation in the standard module for $w^*$. The latter is 4th tensor power of the natural 2-dimensional representation of $\mathfrak{sl}(2)$. The multiplicity of the trivial representation is 2. Hence $\mathcal{M}_0(v^*, w^*)$ is not a rational homology manifold, and $\mathcal{M}_{\zeta}(4\delta, \Lambda_0)$ neither.

Alternatively we argue as follows. We realize this variety as the Coulomb branch of a quiver gauge theory of type $A_3$ with $v'' = \begin{array}{c} 1 \\ 2 \\ 1 \end{array}$, $w'' = \begin{array}{c} 0 \\ 2 \\ 0 \end{array}$. It is the closure of a $\text{SL}(4)[[z]]$-orbit in the affine Grassmannian for $\text{SL}(4)$, and hence the intersection cohomology is known by geometric Satake correspondence. We have

$$\dim i_0^! \text{IC}(\mathcal{M}_C(v'', w'')) = \dim V_0(2\Lambda_2),$$

where $i_0$ is the embedding of the identity element to the affine Grassmannian. The right hand side is the 0-weight space in the representation of $\text{SL}(4)$ with the highest weight $2\Lambda_2$. This weight space is 2-dimensional.
In general, if $\mathfrak{M}_0(v^*, w^*)$ is happen to be the closure of a $G[[z]]$-orbit in an affine Grassmannian $\text{Gr}_G = G((z))/G[[z]]$ (e.g., it is true in type $A$ by [MV03] or the combination of [NT17] and [BFN19]), it is a rational homology manifold if and only if weight spaces of the corresponding representation are all 1-dimensional.\footnote{The author thanks Dinakar Muthiah for this remark.}

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