Nearly Optimal Embeddings of Flat Tori
Technical Proofs

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Joint work with Ishan Agarwal and Oded Regev

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Notations & Definitions

- **lattice** $\mathcal{L} = B\mathbb{Z}^n$
  - **minimum distance** $\lambda_1(\mathcal{L}) := \min\{r > 0 : \text{rank}(\mathcal{L} \cap B_r) \geq 1\}$
  - **covering radius** $\mu(\mathcal{L}) := \max_{x \in \text{span}(\mathcal{L})} \text{dist}(x, \mathcal{L})$
  - **quotient lattice** $\mathcal{L}/\mathcal{L}' := \pi_{\text{span}(\mathcal{L}')} (\mathcal{L})$, for $\mathcal{L}' \subset \mathcal{L}$

- **(flat) torus** $\mathbb{R}^n/\mathcal{L}$
  - **torus metric**: $\text{dist}_{\mathbb{R}^n/\mathcal{L}}(x + \mathcal{L}, y + \mathcal{L}) = \text{dist}(x - y, \mathcal{L})$
    (write $\text{dist}_{\mathbb{R}^n/\mathcal{L}}(x, y)$ for simplicity)
  - **distortion** of (injective) embedding $f$: expansion/contraction;
    expansion: $\sup_{x,y} \frac{\text{dist}(f(x), f(y))}{\text{dist}_{\mathbb{R}^n/\mathcal{L}}(x, y)}$, contraction: $\inf \ldots$

- **Goal**: embed $\mathbb{R}^n/\mathcal{L}$ into $L_2$, with distortion $O(\sqrt{n \log n})$
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The HR10 Embedding

The HR10 Embedding $H_{L,k}(x)$ maps $x \in \mathbb{R}^n / L$ to a $k$-tuple (in $\ell_2$) of Gaussians centered at $x$ with certain variances and coefficients (determined by the “scale” $\lambda_1(L)$).

Wrapping the Gaussians:

- strictly speaking inputs to $H_{L,k}$ should be $x + L \in \mathbb{R}^n / L$
- consequently for $H_{L,k}$ to be well defined, the output Gaussians should be “wrapped around,” i.e., be the sum of all copies centered at $x + L$, and live in $L_2(\mathbb{R}^n / L)$ instead of $L_2(\mathbb{R}^n)$
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$H_{\mathcal{L},k}$ has distortion $O(\sqrt{nk})$:

- expansion: $\leq \sqrt{\pi k}$
- contraction: $\geq \sqrt{c_{H}/n}$, where $c_{H}$ is absolute constant
- caveat for contraction: saturation at $2^{k-1} \lambda_{1}(\mathcal{L})$, i.e., only have contraction w.r.t. $\min(\text{dist}_{\mathbb{R}^{n}/\mathcal{L}}(x, y), 2^{k-1} \lambda_{1}(\mathcal{L}))$

Choices of $k$ in HR10:

- $k = O(\log \frac{\mu(\mathcal{L})}{\lambda_{1}(\mathcal{L})})$: distortion $O\left(\sqrt{n \log \frac{\mu(\mathcal{L})}{\lambda_{1}(\mathcal{L})}}\right)$
- $k = O(\log n)$: distortion $O(\sqrt{n \log n})$, while requiring $\mu(\mathcal{L}) \leq \text{poly}(n) \cdot \lambda_{1}(\mathcal{L})$
**Distortion of The HR10 Embedding**

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The Partitioning Embedding

- given good filtration $\mathcal{F} : \{\vec{0}\} = \mathcal{L}_0 \subset \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_m = \mathcal{L}$
- define projections $\pi^{\leq j}_{\mathcal{F}} := \pi_{\text{span}(\mathcal{L}_j/\mathcal{L}_{j-1})}$ for $j \in [m]$ (and analogously $\pi^{> j}_{\mathcal{F}}, \pi^{> j}_{\mathcal{F}}, \pi^{< j}_{\mathcal{F}}, \pi^{< j}_{\mathcal{F}}$);
  this gives an orthogonal decomposition of the entire space
- define the compressed projections $E^{(j)}_{\mathcal{F},\alpha} := \sum_{i=j}^{m} \alpha^{i-j} \pi^{= i}_{\mathcal{F}}$ for $j \in [m]$, and the overall partitioning embedding $E_{\mathcal{F},\alpha}$ to be the tuple $(E^{(1)}_{\mathcal{F},\alpha}, \ldots, E^{(m)}_{\mathcal{F},\alpha})$ (in $\ell_2$)
- note that $E^{(j)}_{\mathcal{F},\alpha}(\mathcal{L})$ is not dense as long as $\alpha > 0$, and thus $E^{(j)}_{\mathcal{F},\alpha}(\mathbb{R}^n/\mathcal{L})$ gives a valid torus
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Where does Distortion Come from?

- want to embed $\text{dist}_{\mathbb{R}^n/L}(x, y) = \text{dist}(x - y, L)$
  (for simplicity suppose $y = \vec{0}$)
- let $v \in L$ be a closest lattice vector (CV) to $x$; then $\text{dist}(x, L) = \|x - v\|
- want $\text{dist}(E_{F, \alpha}^{(j)}(x), E_{F, \alpha}^{(j)}(L)) = \|E_{F, \alpha}^{(j)}(x - v)\|$ so that they add up to $\Theta(1) \cdot \|x - v\|$ and there is constant distortion
- however $E_{F, \alpha}^{(j)}(v)$ is not necessarily CV to $E_{F, \alpha}^{(j)}(x)$ due to:
  1. projection (left figure: project onto $y$-direction)
  2. compression (right figure: compress $y$-direction by $\alpha = 1/2$)
both distorting the geometry

\[
\begin{align*}
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- let $\mathbf{v} \in \mathcal{L}$ be a closest lattice vector (CV) to $\mathbf{x}$; then $\text{dist}(\mathbf{x}, \mathcal{L}) = \|\mathbf{x} - \mathbf{v}\|$
- want $\text{dist}(E_{\mathcal{F}, \alpha}^{(j)}(\mathbf{x}), E_{\mathcal{F}, \alpha}^{(j)}(\mathcal{L})) = \|E_{\mathcal{F}, \alpha}^{(j)}(\mathbf{x} - \mathbf{v})\|$ so that they add up to $\Theta(1) \cdot \|\mathbf{x} - \mathbf{v}\|$ and there is constant distortion
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Although CV could change in each compressed projection, this only leads to shorter embedded distance and does not harm expansion.

The expansion is easily \( \leq \sqrt{\frac{1}{1-\alpha^2}} \) thanks to the geometric series (and square root due to using \( \ell_2 \) tuple).
Contraction of The Partitioning Embedding: Act 1

want to prove constant contraction

let $j_1$ be the last index where CV changes

we know the part $\|\pi_{>j_1}^F (x - v)\|$ is “captured” by $E_{F,\alpha}^{(>j_1)}$

if this part is already a constant fraction of $\|x - v\|$ then we get constant contraction

so from now on suppose, say, $\|\pi_{\leq j_1}^F (x - v)\|^2 > \frac{1}{2} \|x - v\|^2$

we also know $\|E_{F,\alpha}^{(j_1)} (x - v)\| \geq \frac{1}{2} \lambda_1 (E_{F,\alpha}^{(j_1)} (L))$, due to change of CV (by triangle ineq., $\|E_{F,\alpha}^{(j_1)} (v - v^{(j_1)})\| \leq 2 \|E_{F,\alpha}^{(j_1)} (x - v)\|$, where $E_{F,\alpha}^{(j_0)} (v^{(j)})$ is CV to $E_{F,\alpha}^{(j)} (x)$)
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- we know the part $\|\pi_{\mathcal{F}^*}^{j_1}(x - v)\|$ is “captured” by $E_{\mathcal{F},\alpha}^{(\geq j_1)}$
- if this part is already a constant fraction of $\|x - v\|$ then we get constant contraction
- so from now on suppose, say, $\|\pi_{\mathcal{F}^*}^{< j_1}(x - v)\|^2 > \frac{1}{2}\|x - v\|^2$
- we also know $\|E_{\mathcal{F},\alpha}^{(j_1)}(x - v)\| \geq \frac{1}{2} \lambda_1(E_{\mathcal{F},\alpha}^{(j_1)}(\mathcal{L}))$, due to change of CV (by triangle ineq., $\|E_{\mathcal{F},\alpha}^{(j_1)}(v - v^{(j_1)})\| \leq 2\|E_{\mathcal{F},\alpha}^{(j_1)}(x - v)\|$, where $E_{\mathcal{F},\alpha}^{(j_0)}(v^{(j)})$ is CV to $E_{\mathcal{F},\alpha}^{(j)}(x)$)
suffice to find $j_0 \leq j_1$ s.t. $\|E_{F,\alpha}^{(j_0)}(x - v')\|$ captures $\|\pi_{\leq j_1}^F(x - v)\|$, where $v' = v^{(j_0)} (E_{F,\alpha}^{(j_0)}(v')$ is CV to $E_{F,\alpha}^{(j_0)}(x)$ (w.l.o.g. $\|\pi_{\leq j_0}^F(x - v')\| \leq \mu(L_{j_0-1})$)

try to bound $\|E_{F,\alpha}^{(j_0)}(x - v')\|^2 = \sum_{i=j_0}^m \alpha^{2(i-j_0)} \|\pi_{\leq i}^F(x - v')\|^2$

truncate the sum at some $j_2 \geq j_1$ to handle the exponential factor: $\|E_{F,\alpha}^{(j_0)}(x - v')\|^2 \geq \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi_{\leq i}^F(x - v')\|^2$

note that $\sum_{i=j_0}^{j_2} \|\pi_{\leq i}^F(\cdot)\|^2 = \|\cdot\|^2 - \|\pi_{\leq j_0}^F(\cdot)\|^2 - \|\pi_{j_2}^F(\cdot)\|^2$

1. $\|x - v'\|^2 \geq \|x - v\|^2$ as $v$ is CV
2. $\|\pi_{\leq j_0}^F(x - v')\| \leq \mu(L_{j_0-1})$ for free;
   want $\mu(L_{j_0-1}) \leq \frac{1}{4} \lambda_1(E_{F,\alpha}^{(j_1)}(L))$;
   then $\|\pi_{\leq j_0}^F(x - v')\| \leq \frac{1}{2} \|x - v\|$
3. hopefully $\|\pi_{> j_2}^F(x - v')\| = \|\pi_{> j_2}^F(x - v)\| (< \frac{1}{\sqrt{2}} \|x - v\|$)
suffice to find $j_0 \leq j_1$ s.t. $\|E_{F,\alpha}^{(j_0)}(x - v')\|$ captures $\|\pi_{F}^{\leq j_1}(x - v)\|$, where $v' = v^{(j_0)}$ ($E_{F,\alpha}^{(j_0)}(v')$ is CV to $E_{F,\alpha}^{(j_0)}(x)$) (w.l.o.g. $\|\pi_{F}^{\leq j_0}(x - v')\| \leq \mu(L_{j_0-1})$)

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Contraction of The Partitioning Embedding: Act 2

- suffice to find $j_0 \leq j_1$ s.t. $\|E_{F,\alpha}^{(j_0)}(x - v')\|$ captures $\|\pi_{<j_1}^F(x - v)\|$, where $v' = v^{(j_0)} (E_{F,\alpha}^{(j_0)}(v')$ is CV to $E_{F,\alpha}^{(j_0)}(x))$
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- truncate the sum at some $j_2 \geq j_1$ to handle the exponential factor: $\|E_{F,\alpha}^{(j_0)}(x - v')\|^2 \geq \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi_{=i}^F(x - v')\|^2$

- note that $\sum_{i=j_0}^{j_2} \|\pi_{=i}^F(\cdot)\|^2 = \|\cdot\|^2 - \|\pi_{<j_0}^F(\cdot)\|^2 - \|\pi_{>j_2}^F(\cdot)\|^2$

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abla \pi_{F}^{\leq j_1}(x - v)$, where $v' = v^{(j_0)}$ ($E_{F,\alpha}^{(j_0)}(v')$ is CV to $E_{F,\alpha}^{(j_0)}(x)$) (w.l.o.g. $\|\pi_{F}^{< j_0}(x - v')\| \leq \mu(L_{j_0-1})$)

try to bound $\|E_{F,\alpha}^{(j_0)}(x - v')\|^2 = \sum_{i=j_0}^{m} \alpha^{2(i-j_0)} \|\pi_{F}^{= i}(x - v')\|^2$

truncate the sum at some $j_2 \geq j_1$ to handle the exponential factor: $\|E_{F,\alpha}^{(j_0)}(x - v')\|^2 \geq \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi_{F}^{= i}(x - v')\|^2$

note that $\sum_{i=j_0}^{j_2} \|\pi_{F}^{= i}(\cdot)\|^2 = \|\cdot\|^2 - \|\pi_{F}^{< j_0}(\cdot)\|^2 - \|\pi_{F}^{> j_2}(\cdot)\|^2$

1. $\|x - v'\|^2 \geq \|x - v\|^2$ as $v$ is CV
2. $\|\pi_{F}^{< j_0}(x - v')\| \leq \mu(L_{j_0-1})$ for free;
   want $\mu(L_{j_0-1}) \leq \frac{1}{4} \lambda_1(E_{F,\alpha}^{(j_1)}(L))$;
   then $\|\pi_{F}^{< j_0}(x - v')\| \leq \frac{1}{2} \|x - v\|$.
3. hopefully $\|\pi_{F}^{> j_2}(x - v')\| = \|\pi_{F}^{> j_2}(x - v)\| (\leq \frac{1}{\sqrt{2}} \|x - v\|)$
Contraction of The Partitioning Embedding: Act 2

- suffice to find $j_0 \leq j_1$ s.t. $\|E_{\mathcal{F},\alpha}^{(j_0)}(x - v')\|$ captures $\|\pi_{\mathcal{F}}^{\leq j_1}(x - v)\|$, where $v' = v^{(j_0)}$ ($E_{\mathcal{F},\alpha}^{(j_0)}(v')$ is CV to $E_{\mathcal{F},\alpha}^{(j_0)}(x)$) (w.l.o.g. $\|\pi_{\mathcal{F}}^{\leq j_0}(x - v')\| \leq \mu(L_{j_0-1})$

- try to bound $\|E_{\mathcal{F},\alpha}^{(j_0)}(x - v')\|^2 = \sum_{i=j_0}^{m} \alpha^{2(i-j_0)} \|\pi_{\mathcal{F}} = i(x - v')\|^2$

- truncate the sum at some $j_2 \geq j_1$ to handle the exponential factor: $\|E_{\mathcal{F},\alpha}^{(j_0)}(x - v')\|^2 \geq \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi_{\mathcal{F}} = i(x - v')\|^2$

- note that $\sum_{i=j_0}^{j_2} \|\pi_{\mathcal{F}} = i(\cdot)\|^2 = \|\cdot\|^2 - \|\pi_{\mathcal{F}}^{< j_0}(\cdot)\|^2 - \|\pi_{\mathcal{F}}^{> j_2}(\cdot)\|^2$
  1. $\|x - v'\|^2 \geq \|x - v\|^2$ as $v$ is CV
  2. $\|\pi_{\mathcal{F}}^{< j_0}(x - v')\| \leq \mu(L_{j_0-1})$ for free; want $\mu(L_{j_0-1}) \leq \frac{1}{4} \lambda_1(E_{\mathcal{F},\alpha}(\mathcal{L}))$; then $\|\pi_{\mathcal{F}}^{< j_0}(x - v')\| \leq \frac{1}{2} \|x - v\|$
  3. hopefully $\|\pi_{\mathcal{F}}^{> j_2}(x - v')\| = \|\pi_{\mathcal{F}}^{> j_2}(x - v)\| (\leq \frac{1}{\sqrt{2}} \|x - v\|)$
suffice to find $j_0 \leq j_1$ s.t. $\|E_{\mathcal{F},\alpha}^{(j_0)}(x - v')\|$ captures $\|\pi_{\leq j_1}^<(x - v)\|$, where $v' = v^{(j_0)}$ ($E_{\mathcal{F},\alpha}^{(j_0)}(v')$ is CV to $E_{\mathcal{F},\alpha}^{(j_0)}(x)$) (w.l.o.g. $\|\pi_{\leq j_0}^<(x - v')\| \leq \mu(\mathcal{L}_{j_0-1})$)

try to bound $\|E_{\mathcal{F},\alpha}^{(j_0)}(x - v')\|^2 = \sum_{i = j_0}^{m} \alpha^{2(i-j_0)} \|\pi_{= i}^= (x - v')\|^2$

truncate the sum at some $j_2 \geq j_1$ to handle the exponential factor: $\|E_{\mathcal{F},\alpha}^{(j_0)}(x - v')\|^2 \geq \alpha^{2(j_2-j_0)} \cdot \sum_{i = j_0}^{j_2} \|\pi_{= i}^= (x - v')\|^2$

note that $\sum_{i = j_0}^{j_2} \|\pi_{= i}^= (\cdot)\|^2 = \|\cdot\|^2 - \|\pi_{\leq j_0}^<(\cdot)\|^2 - \|\pi_{> j_2}^>(\cdot)\|^2$

1. $\|x - v'\|^2 \geq \|x - v\|^2$ as $v$ is CV
2. $\|\pi_{\leq j_0}^<(x - v')\| \leq \mu(\mathcal{L}_{j_0-1})$ for free; want $\mu(\mathcal{L}_{j_0-1}) \leq \frac{1}{4} \lambda_1(\mathcal{E}_{\mathcal{F},\alpha}^{(j_1)}(\mathcal{L}))$; then $\|\pi_{\leq j_0}^<(x - v')\| \leq \frac{1}{2}\|x - v\|$.
3. hopefully $\|\pi_{> j_2}^>(x - v')\| = \|\pi_{> j_2}^>(x - v)\| (\leq \frac{1}{\sqrt{2}}\|x - v\|)$.
Contraction of The Partitioning Embedding: Act 2

- suffice to find $j_0 \leq j_1$ s.t. $\|E_{F, \alpha}^{(j_0)}(x - v')\|$ captures $\|\pi_{j_1}^* (x - v)\|$, where $v' = v^{(j_0)}$ ($E_{F, \alpha}^{(j_0)}(v')$ is CV to $E_{F, \alpha}^{(j_0)}(x)$) (w.l.o.g. $\|\pi_{j_0}^* (x - v')\| \leq \mu(L_{j_0-1})$)

- try to bound $\|E_{F, \alpha}^{(j_0)}(x - v')\|^2 = \sum_{i=j_0}^{m} \alpha^{2(i-j_0)} \|\pi_i^* (x - v')\|^2$

- truncate the sum at some $j_2 \geq j_1$ to handle the exponential factor: $\|E_{F, \alpha}^{(j_0)}(x - v')\|^2 \geq \alpha^{2(j_2-j_0)} \cdot \sum_{i=j_0}^{j_2} \|\pi_i^* (x - v')\|^2$

- note that $\sum_{i=j_0}^{j_2} \|\pi_i^* (\cdot)\|^2 = \|\cdot\|^2 - \|\pi_{j_0}^{<} (\cdot)\|^2 - \|\pi_{j_2}^{>} (\cdot)\|^2$

  1. $\|x - v'\|^2 \geq \|x - v\|^2$ as $v$ is CV
  2. $\|\pi_{j_0}^{<} (x - v')\| \leq \mu(L_{j_0-1})$ for free; want $\mu(L_{j_0-1}) \leq \frac{1}{4} \lambda_1(E_{F, \alpha}^{(j_1)}(L))$; then $\|\pi_{j_0}^{<} (x - v')\| \leq \frac{1}{2} \|x - v\|$
  3. hopefully $\|\pi_{j_2}^{>} (x - v')\| = \|\pi_{j_2}^{>} (x - v)\| \left( < \frac{1}{\sqrt{2}} \|x - v\| \right)$
“hopefully $\|\pi^{>j_2}(x - v')\| = \|\pi^{>j_2}(x - v)\|$”

suffice to show $\pi^{>j_2}(v') = \pi^{>j_2}(v)$, or $E^{(j_2+1)}(v') = E^{(j_2+1)}(v)$

if not, they are distant: $\|E^{(j_2+1)}(v - v')\| \geq \lambda_1(E^{(j_2+1)}(\mathcal{L}))$

note that by algebra, $\|E^{(j_0)}(\cdot)\| \geq \alpha^{j_2+1-j_0} \|E^{(j_2+1)}(\cdot)\|$ 

hence

$$\|\pi^{\leq j_1}(x - v)\| > \frac{1}{\sqrt{2}} \|x - v\| \geq \frac{1}{\sqrt{2}} \|E^{(j_0)}(x - v)\|$$

$$\geq \frac{1}{2\sqrt{2}} \|E^{(j_0)}(v - v')\|$$

$$\geq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}} \lambda_1(E^{(j_2+1)}(\mathcal{L}))$$

on the other hand $\|\pi^{\leq j_1}(x - v)\| \leq \mu(\mathcal{L}_{j_1})$;

so want $\mu(\mathcal{L}_{j_1}) \leq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}} \lambda_1(E^{(j_2+1)}(\mathcal{L}))$ for contradiction
“hopefully \( \| \pi_{\mathcal{F}}^\geq j_2 (x - v') \| = \| \pi_{\mathcal{F}}^\geq j_2 (x - v) \| \)”

suffice to show \( \pi_{\mathcal{F}}^\geq j_2 (v') = \pi_{\mathcal{F}}^\geq j_2 (v) \), or \( E_{\mathcal{F}, \alpha}^{(j_2 + 1)} (v') = E_{\mathcal{F}, \alpha}^{(j_2 + 1)} (v) \)

if not, they are distant: \( \| E_{\mathcal{F}, \alpha}^{(j_2 + 1)} (v - v') \| \geq \lambda_1 (E_{\mathcal{F}, \alpha}^{(j_2 + 1)} (\mathcal{L})) \)

note that by algebra, \( \| E_{\mathcal{F}, \alpha}^{(j_0)} (\cdot) \| \geq \alpha^{j_2 + 1 - j_0} \| E_{\mathcal{F}, \alpha}^{(j_2 + 1)} (\cdot) \| \)

hence

\[
\| \pi_{\mathcal{F}}^{\leq j_1} (x - v) \| > \frac{1}{\sqrt{2}} \| x - v \| \geq \frac{1}{\sqrt{2}} \| E_{\mathcal{F}, \alpha}^{(j_0)} (x - v) \|
\]

\[
\geq \frac{1}{2 \sqrt{2}} \| E_{\mathcal{F}, \alpha}^{(j_0)} (v - v') \|
\]

\[
\geq \frac{\alpha^{j_2 + 1 - j_0}}{2 \sqrt{2}} \lambda_1 (E_{\mathcal{F}, \alpha}^{(j_2 + 1)} (\mathcal{L}))
\]

on the other hand \( \| \pi_{\mathcal{F}}^{\leq j_1} (x - v) \| \leq \mu (\mathcal{L}_{j_1}) \);

so want \( \mu (\mathcal{L}_{j_1}) \leq \frac{\alpha^{j_2 + 1 - j_0}}{2 \sqrt{2}} \lambda_1 (E_{\mathcal{F}, \alpha}^{(j_2 + 1)} (\mathcal{L})) \) for contradiction
“hopefully $\|\pi_{\mathcal{F}}^{> j_2} (x - v')\| = \|\pi_{\mathcal{F}}^{> j_2} (x - v)\|”$

suffice to show $\pi_{\mathcal{F}}^{> j_2} (v') = \pi_{\mathcal{F}}^{> j_2} (v)$, or $E_{\mathcal{F}, \alpha}^{(j_2+1)} (v') = E_{\mathcal{F}, \alpha}^{(j_2+1)} (v)$

if not, they are distant: $\|E_{\mathcal{F}, \alpha}^{(j_2+1)} (v - v')\| \geq \lambda_1 (E_{\mathcal{F}, \alpha}^{(j_2+1)} (L))$

note that by algebra, $\|E_{\mathcal{F}, \alpha}^{(j_0)} (\cdot)\| \geq \alpha^{j_2 + 1 - j_0} \|E_{\mathcal{F}, \alpha}^{(j_2+1)} (\cdot)\|$

hence

$$\|\pi_{\mathcal{F}}^{\leq j_1} (x - v)\| > \frac{1}{\sqrt{2}} \|x - v\| \geq \frac{1}{\sqrt{2}} \|E_{\mathcal{F}, \alpha}^{(j_0)} (x - v)\|$$

$$\geq \frac{1}{2\sqrt{2}} \|E_{\mathcal{F}, \alpha}^{(j_0)} (v - v')\|$$

$$\geq \frac{\alpha^{j_2 + 1 - j_0}}{2\sqrt{2}} \lambda_1 (E_{\mathcal{F}, \alpha}^{(j_2+1)} (L))$$

on the other hand $\|\pi_{\mathcal{F}}^{\leq j_1} (x - v)\| \leq \mu (L_{j_1})$;
so want $\mu (L_{j_1}) \leq \frac{\alpha^{j_2 + 1 - j_0}}{2\sqrt{2}} \lambda_1 (E_{\mathcal{F}, \alpha}^{(j_2+1)} (L))$ for contradiction
“hopefully $\|\pi_\mathcal{F}^{>j_2}(x - v')\| = \|\pi_\mathcal{F}^{>j_2}(x - v)\|$$

suffice to show $\pi_\mathcal{F}^{>j_2}(v') = \pi_\mathcal{F}^{>j_2}(v)$, or $E_{\mathcal{F},\alpha}^{(j_2+1)}(v') = E_{\mathcal{F},\alpha}^{(j_2+1)}(v)$

if not, they are distant: $\|E_{\mathcal{F},\alpha}^{(j_2+1)}(v - v')\| \geq \lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))$

note that by algebra, $\|E_{\mathcal{F},\alpha}^{(j_0)}(\cdot)\| \geq \alpha^{j_2+1-j_0} \|E_{\mathcal{F},\alpha}^{(j_2+1)}(\cdot)\|$

hence

$$\|\pi_{\mathcal{F}}^{\leq j_1}(x - v)\| > \frac{1}{\sqrt{2}} \|x - v\| \geq \frac{1}{\sqrt{2}} \|E_{\mathcal{F},\alpha}^{(j_0)}(x - v)\|$$

$$\geq \frac{1}{2\sqrt{2}} \|E_{\mathcal{F},\alpha}^{(j_0)}(v - v')\|$$

$$\geq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}} \lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))$$

on the other hand $\|\pi_{\mathcal{F}}^{\leq j_1}(x - v)\| \leq \mu(\mathcal{L}_{j_1})$;
so want $\mu(\mathcal{L}_{j_1}) \leq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}} \lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L}))$ for contradiction
already manage to capture $\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x} - \mathbf{v})\|$, even the entire $\|\mathbf{x} - \mathbf{v}\|$, by $\|E^{(j_0)}_{\mathcal{F},\alpha}(\mathbf{x} - \mathbf{v}^{'})\|$?

need to consider saturation of HR10, i.e., can only use 
$\min(\|E_{\mathcal{F},\alpha}^{(j)}(\mathbf{x} - \mathbf{v}^{(j)})\|, \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}^{(j)}(\mathcal{L})))$ for each $j$

for $E_{\mathcal{F},\alpha}^{(> j_1)}$, they still capture $\|\pi_{\mathcal{F}}^{> j_1}(\mathbf{x} - \mathbf{v})\|$ as long as 
$\mu(\mathcal{L}_{j}) \leq \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}^{(j)}(\mathcal{L}))$

for $E_{\mathcal{F},\alpha}^{(j_0)}$ (to capture $\|\pi_{\mathcal{F}}^{\leq j_1}(\mathbf{x} - \mathbf{v})\|$), want 
$\mu(\mathcal{L}_{j_1}) \leq \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}^{(j_0)}(\mathcal{L}))$

Finally we have $\Theta(1)$ contraction (considering saturation of HR10),
and thus $\Theta(1)$ distortion of the partitioning embedding,
and thus $O(\sqrt{n \log n})$ overall distortion after composing with the HR10 embedding.
already manage to capture $\|\pi_{\mathcal{F}}(x - v)\|$, even the entire $\|x - v\|$, by $\|E_{\mathcal{F},\alpha}(x - v')\|$?

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$$\min(\|E_{\mathcal{F},\alpha}(x - v^{(j)})\|, \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}(\mathcal{L})))$$

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for $E_{\mathcal{F},\alpha}(x - v^{(j)})$, they still capture $\|\pi_{\mathcal{F}}(x - v)\|$ as long as
$$\mu(L_j) \leq \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}(\mathcal{L}))$$

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for $E_{\mathcal{F},\alpha}^{(j_0)}$ (to capture $\|\pi^{\leq j_1}(x - v)\|$), want $\mu(\mathcal{L}_{j_1}) \leq \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}^{(j_0)}(\mathcal{L}))$

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Good Filtration

During the contraction proof we assumed:

\[ \mu(\mathcal{L}_j) \leq \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}(\mathcal{L})), \quad \mu(\mathcal{L}_{j_0-1}) \leq \frac{1}{4} \lambda_1(E_{\mathcal{F},\alpha}(\mathcal{L})), \]
\[ \mu(\mathcal{L}_{j_1}) \leq \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}(\mathcal{L})), \quad \mu(\mathcal{L}_{j_1}) \leq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}} \lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L})). \]

These reduce to \((\beta, \gamma)\)-filtration:

- \[ \mu(\mathcal{L}_j/\mathcal{L}_{j-1}) \leq \beta \cdot \lambda_1(\mathcal{L}_j/\mathcal{L}_{j-1}) \]
- \[ \lambda_1(\mathcal{L}_{j+1}/\mathcal{L}_j) \geq \gamma \cdot \lambda_1(\mathcal{L}_j/\mathcal{L}_{j-1}) \]
- i.e., separated scales!

(\(\alpha \geq 1/\gamma\))

(\(\gamma\sqrt{n}, \gamma\))-filtration can be achieved using Korkine–Zolotarev basis. The idea is intuitive: to group shortest bases into one sublattice until reaching a next scale that is \(\gamma\) times larger.
During the contraction proof we assumed:
\begin{align*}
\mu(L_j) &\leq \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}^{(j)}(\mathcal{L})), \\
\mu(L_{j_0-1}) &\leq \frac{1}{4} \lambda_1(E_{\mathcal{F},\alpha}^{(j_1)}(\mathcal{L})), \\
\mu(L_{j_1}) &\leq \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}^{(j_0)}(\mathcal{L})), \\
\mu(L_{j_1}) &\leq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}} \lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(\mathcal{L})).
\end{align*}

These reduce to \((\beta, \gamma)-\text{filtration}:
\begin{align*}
\triangleright \quad \mu(L_j/L_{j-1}) &\leq \beta \cdot \lambda_1(L_j/L_{j-1}) \\
\triangleright \quad \lambda_1(L_{j+1}/L_j) &\geq \gamma \cdot \lambda_1(L_j/L_{j-1}) \\
\triangleright \quad \text{i.e., separated scales!}
\end{align*}

(along with mild enough compression \(\alpha \geq 1/\gamma\))

\((\gamma \sqrt{n}, \gamma)-\text{filtration can be achieved using Korkine–Zolotarev basis.}
The idea is intuitive: to group shortest bases into one sublattice until reaching a next scale that is \(\gamma\) times larger.
During the contraction proof we assumed:
\[ \mu(L_j) \leq \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}(L)), \quad \mu(L_{j_0-1}) \leq \frac{1}{4} \lambda_1(E_{\mathcal{F},\alpha}(L)), \]
\[ \mu(L_{j_1}) \leq \text{poly}(n) \cdot \lambda_1(E_{\mathcal{F},\alpha}(L)), \quad \mu(L_{j_1}) \leq \frac{\alpha^{j_2+1-j_0}}{2\sqrt{2}} \lambda_1(E_{\mathcal{F},\alpha}^{(j_2+1)}(L)). \]

These reduce to \((\beta, \gamma)-\text{filtration}:\)

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The idea is intuitive: to group shortest bases into one sublattice until reaching a next scale that is \(\gamma\) times larger.
Open Questions, Extended

- Finite dimensional embeddings
  - via discretization?
  - trade-off between distortion and dimensionality

- Efficiently computable embeddings
  - Korkine–Zolotarev basis is hard to compute
  - necessary for algorithmic applications, if any:

- Lattice-specific distortion upper bound
  - e.g. $\Theta(1)$ distortion for $\mathbb{Z}^n$
  - potentially involving $\lambda_1(\mathcal{L})$ and $\mu(\mathcal{L})$
  - lower bound: $\Omega(\lambda_1(\mathcal{L}^*) \cdot \mu(\mathcal{L}) / \sqrt{n}) \geq \Omega(\frac{\lambda_1(\mathcal{L}^*)}{\mu(\mathcal{L}^*)} \cdot \sqrt{n})$
  - by having “adaptive” compression factor $\alpha$?

- Embedding into $L_p$ instead of $L_2$
  - for $L_1$, same lower bound
  - for finite dimensionality we can embed from $\ell_2$ into $\ell_p$
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