A coherent categorification of the based ring of the lowest two-sided cell

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Abstract

We give a partial coherent categorification of $J_0$, the based ring of the lowest two-sided cell of an affine Weyl group, equipped with a monoidal functor from the category of coherent sheaves on the derived Steinberg variety. We show that our categorification acts on natural coherent categorifications of the Iwahori invariants of the Schwartz space of the basic affine space. In low rank cases, we construct complexes that lift the basis elements $t_w$ of $J_0$ and their structure constants.

Keywords— Asymptotic Hecke algebra, Iwahori-Hecke algebra, flag variety, Steinberg variety

1 Introduction

Let $\tilde{W}$ be an affine Weyl group. Its group algebra $C[\tilde{W}]$ is deformed by the affine Hecke algebra $H_{\text{aff}} = H(\tilde{W})$ of $\tilde{W}$. In turn, Lusztig defined the asymptotic Hecke algebra $J$, a based ring with basis $t_w$, $w \in \tilde{W}$ and structure constants determined from certain “leading terms” of the structure constants of $H_{\text{aff}}$. Further, he provided a morphism of algebras $\phi: H_{\text{aff}} \rightarrow J \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$ and showed it was an algebra after a mild completion. Thus $H_{\text{aff}}$ can be viewed as a subalgebra of $J$, and $J$ can be viewed as a subalgebra of a completion $\mathcal{H}_{\text{aff}}$ of $H_{\text{aff}}$. While the morphism $\phi$ is an essential part of Lusztig’s exploration of $J$, until recently there have been few compelling reasons to adopt the perspective of $J \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$ as a subalgebra of $H_{\text{aff}}$.

The algebra $H_{\text{aff}}$ appears in many areas of mathematics in many guises, but one of the most prominent relates to the representation theory of $p$-adic groups. Let $F$ be a local non-archimedean field and $q$ be the cardinality of the residue field of $F$. Let $G^\vee$ be a connected reductive algebraic group defined and split over $F$, with Langlands dual group $G$ taken over $\mathbb{C}$. For the purposes of harmonic analysis on $G^\vee(F)$, it is natural to consider $H_{\text{aff}}$, very much an algebraic object, as a subalgebra of the larger, analytically-characterized Harish-Chandra Schwartz algebra $\mathcal{C}(G^\vee(F))$. In [BK18], Braverman and Kazhdan gave an interpretation of $J$ in terms of harmonic analysis, casting $J$ as an algebraic version of $\mathcal{C}(G^\vee(F))$ by defining a map $J \rightarrow \mathcal{C}(G^\vee(F))$. In [Daw21], the author showed that this morphism was essentially the specialization of $\phi^{-1}$ for $q = q$, and in particular was injective. In [BKK23], Bezrukavnikov-Karpov-Krylov showed that this map was an isomorphism; the author did so via another approach for all but finitely-many cells of exceptional groups in [Daw23].

Lusztig gave a categorification of $J$ in [Lus97] in terms of perverse sheaves on the affine flag variety. In the spirit of the definition of $J$ as a ring, the underlying category is again $\text{Perv}(\mathcal{F} \ell)$, but with monoidal structure given by truncated convolution as opposed to convolution. In this paper we provide a categorification of a large direct summand $J_0$ of $J$ that is compatible with the perspective of [BK18]. Namely, we obtain a natural categorification of the action of $J_0$ on the unitary principal series, and produce a completely new category whose $K$-theory is $J_0$, as opposed a new monoidal structure.

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Theorem 1. Let $\mathcal{B}$ be the derived zero section of the Springer resolution $\tilde{N} \to N$. Then

1. the category 
$$\mathcal{J}_0 := D^b\text{Coh}(\mathcal{B}/G \times_{\text{pt}/G} \mathcal{B}/G),$$

is a triangulated subcategory of $\text{Coh}(\mathcal{B}/G \times_{\text{pt}/G} \mathcal{B}/G)$, has a monoidal structure given by convolution, and admits a natural monoidal functor
$$\text{Coh}(\text{St}/G \times G_m) \to \text{Coh}(\mathcal{B}/G \times G_m \times_{\text{pt}/G} \mathcal{B}/G \times G_m)!$$

such that

2. the induced morphism
$$H_{\text{aff}} \to K_0(\mathcal{J}_0) \to K_0(\mathcal{B}/G \times_{\text{pt}/G} \mathcal{B}/G)$$

is conjugate to $\phi_0$;

3. In the special case when $G$ has universal cover equal to $\text{SL}_2$ or $\text{SL}_3$, there exists a family of objects
$$\{\ell_w\}_{w \in c_{0}}$$
in $\mathcal{J}_0$, such that, if $t_w t_x = \sum z \gamma_{w,x,z}^{-1} t_z$ in $J_0$, then
$$\ell_w \star \ell_x = \bigoplus z \ell_z^\oplus \gamma_{w,x,z}^{-1}$$
in $\mathcal{J}_0$ and the image in $K_0(\mathcal{B}/G \times_{\text{pt}/G} \mathcal{B}/G)$ of the class $[\ell_w]$ under the above morphism is $[t_w]$.

4. The category $\mathcal{J}_0$ acts on $\text{Coh}(\mathcal{B}/T)$ and on $\text{Coh}(\mathcal{B}/G \times_{\text{pt}/G} \mathcal{B}/G)$;

Proof. The four propositions below each prove one statement of the theorem.

The algebra $J$ is very close to being a direct sum of matrix algebras. In type $A$, this is the main result of Xi’s monograph [Xi02], and Bezrukavnikov-Ostrik in [BO04] showed this up to central extensions in general. (It is however now known [BDD23] [QX22] that this partial result is sharp; the central extensions do in fact appear in general.)

Therefore the last two items are particularly relevant: $J$ is most interesting as a based algebra admitting a morphism from $H_{\text{aff}}$, and can be quite simple in isolation. Our categorification captures the failure of $\phi_0$ to be surjective, as explained in Remark 2 and the discussion preceding it. We hope to remove the very restrictive current hypothesis on item 3 in a future version of this paper; see remark 1 for an explanation of why it is currently necessary.

We would be remiss to not point out that, while the algebra $J_0$ is a quotient of $K_0(\mathcal{J}_0)$ by Proposition 2, the two are in fact not equal. After completing this paper, we learned of [Pro23], which succeeds in categorifying each summand of $J$, in particular $J_0$, in a way that also recovers $\phi_0$. The decategorification procedure of op. cit. is more sophisticated than simply taking Grothendieck groups.

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2 Functions and algebras

In this section we will recall the various algebras whose categorifications we will discuss in Section 3. There is no new material in this section, although we could not find a recollection of all the relationships below in one place in the existing literature.
2.1 The affine Hecke algebra

Let $G$ be a connected simply-connected reductive group defined over $\mathbb{C}$ with Borel subgroup $B$, maximal torus $T \subset B$, and classical flag variety $\mathcal{B}^G = G/B$. Let $X^*$ be the character lattice of $T$, and $\tilde{W} = W \ltimes X^*$, where $W$ is the finite Weyl group of $G$. Let $H = H_{aff}$ be the corresponding affine Hecke algebra over $\mathcal{A} = \mathbb{Z}[\frac{q^{1/2}}{q^{-1/2}}, q^{-1/2}]$ with standard basis $\{T_w\}_{w \in \tilde{W}}$. The multiplication in this, the Coxeter presentation, of $H$ is determined by $T_wT_{w'} = T_{ww'}$ when $\ell(ww') = \ell(w) + \ell(w')$ and the quadratic relation $(T_s + 1)(T_s - q) = 0$ for all $s \in S$, where $S \subset \tilde{W}$ is the set of simple reflections. The Coxeter presentation is well-suited to studying the action of $H$ on admissible representations and the constructible categorification of $H$.

There is a second presentation of $H$, due to Bernstein (and Bernstein-Zelevinskii in type $A$), which appears naturally in coherent descriptions of $H$, both on the level of $K$-theory and the level of categories.

**Definition 1.** The Bernstein presentation of $H$ is the presentation with basis $\{T_w\}_{w \in W, \lambda \in X^*}$, where

- For $w \in W$, $T_w$ is the same basis element as in the Coxeter presentation.
- If $\lambda \in \tilde{W}$ is an antidominant character, and hence corresponds to a geometrically dominant cocharacter in the sense of [CG97], then
  $$\theta_\lambda = q^{-\frac{\ell(\lambda)}{2}}T_\lambda.$$
- If $\lambda \in \tilde{W}$ is a dominant character, and hence corresponds to a geometrically antidominant cocharacter, then
  $$\theta_\lambda = q^{\frac{\ell(\lambda)}{2}}T_\lambda^{-1}.$$

The sets $\{\theta_\lambda, \theta_\lambda T_{s_0}\}_{\lambda \in X^*}$, and $\{\theta_\lambda, T_s\theta_\lambda\}_{\lambda \in X^*}$, are each $\mathcal{A}$-bases. The relations are as follows:

- The same quadratic relation for $T_{s_0}$;
- For any cocharacters $\lambda, \lambda'$, we have $\theta_\lambda \theta_{\lambda'} = \theta_{\lambda + \lambda'}$.
- The Bernstein relation
  $$\theta_{\frac{\lambda}{2}} T_{s_0\lambda} - T_{s_0\lambda} \theta_{-\frac{\lambda}{2}} = (q - 1) \frac{\theta_{\frac{\lambda}{2}} - \theta_{-\frac{\lambda}{2}}}{1 - \theta^{-\alpha}} = (q - 1) \theta_{\frac{\lambda}{2}}.$$

**Example 1.** Let $G^\vee = \text{PGL}_2$, so that $\tilde{W} = \mathfrak{S}_2 \ltimes X$, where $X$ is the character lattice of $G = \text{SL}_2$. Write $s_0$ for the finite simple reflection, and $s_1$ for the affine simple reflection in $\tilde{W}$. Then the generators of the Bernstein subalgebra are as follows:

1. If $\lambda = -n\alpha^\vee = (s_1s_0)^n \in \tilde{W}$, and hence corresponds to a geometrically dominant cocharacter in the sense of [CG97], then
   $$\theta_{(s_1s_0)^n} = \theta_{-n\alpha^\vee} = q^{-n}T_{(s_1s_0)^n}.$$

2. If $\lambda = n\alpha^\vee = (s_0s_1)^n \in \tilde{W}$, and hence corresponds to a geometrically antidominant cocharacter, then
   $$\theta_{(s_0s_1)^n} = \theta_{n\alpha^\vee} = q^nT_{(s_1s_0)^n}^{-1}.$$

In particular, under the geometric choice of dominance, we have $\rho = -1$.

2.2 The asymptotic Hecke algebra

**Definition 2.** Lusztig’s $a$-function $a: \tilde{W} \rightarrow \mathbb{N}$ is defined such that $a(w)$ is the minimal value such that

$$q^{a(w)} h_{x,y,w} \in \mathcal{A}^+$$

for all $x, y \in \tilde{W}$.
It is known that $a$ is constant on two-sided cells of $\tilde{W}$ and that
\[
a(c) = \dim B_u
\]
where $u$ is the unipotent conjugacy class in $G$ corresponding to $c$ under Lusztig’s bijection. It is also known that $a(w) \leq \ell(w)$ for all $w \in \tilde{W}$.

In [Lus87] Lusztig defined an associative algebra $J$ over $\mathbb{Z}$ equipped with an injection $\phi: H \rightarrow J \otimes_{\mathbb{Z}} \mathfrak{a}$ which becomes an isomorphism after taking a certain completion of both sides. As an abelian group, $J$ has a basis $\{t_w\}_{w \in \tilde{W}}$. Recalling the Kazhdan-Lusztig basis elements
\[
C_w = \sum_{y \leq w} (-1)^{\ell(w)-\ell(y)} q^{\frac{\ell(w)-\ell(y)}{2}} P_{y,w}(q^{-1}) T_y,
\]
the structure constants of $J$ are obtained from those in $H$ written in the $\{C_w\}_{w \in W}$-basis under the following procedure. Using the structure constants $C_x C_y = \sum_{z \in W} h_{x,y,z} C_z$
for $h_{x,y,z} \in \mathfrak{a}$, Lusztig then defines the integer $\gamma_{x,y,z}$ by the condition
\[
q^{\gamma_{x,y,z}} h_{x,y,z} - \gamma_{x,y,z} \in q\mathfrak{a}^+.
\]
The product in $J$ is then defined as
\[
t_x t_y = \sum_{z} \gamma_{x,y,z} t_z^{-1}.
\]
One then defines
\[
\phi(C_w) = \sum_{z \in W, d \in D, a(z) = a(d)} h_{x,d,z} t_z,
\]
where $D \subset \tilde{W}$ is the set of distinguished involutions. The elements $t_d$ for distinguished involutions $d$ are orthogonal idempotents. Moreover, $J = \bigoplus_c J_c$ is a direct sum of two-sided ideals indexed by two-sided cells $c \subset \tilde{W}$. The unit element in each summand is $\sum_{d \in D} t_d$, and the unit element of $J$ is $\sum_{d \in D} t_d$.

2.2.1 The lowest two-sided cell

Let $c_0$ be the lowest two-sided cell, also called the “big cell.” It can be characterized as the two-sided cell containing the longest element of $W$, and we have $a(c_0) = \ell(w_0)$. The summand $J_0 := J_{c_0}$ is particularly well-understood, and has historically been the first summand for which any structure-theoretic result has been achieved (consider, for example, the progression [Xi90], [Xi02], [BO04]). Write $\phi_0$ for the composition of $\phi$ composed by the projection $J \otimes_{\mathbb{Z}} \mathfrak{a} \rightarrow J_0 \otimes_{\mathbb{Z}} \mathfrak{a}$.

By [Xi90] and [Nie11], we have the following description of $c_0 \subset \tilde{W}$. Let $c_0$ be the lowest cell of the affine Weyl group of the universal covering group $\tilde{G}$ of $G$ with maximal torus $\tilde{T}$. Then $c_0 = c_0 \cap \tilde{W}$, and
\[
c_0 = \left\{ f^{-1} w_0 \chi g \, \bigg| \, f, g \in \Sigma, \chi \in X^*(\tilde{T})^+ \right\},
\]
where $\Sigma = \{wx \mid x \in W\} \subset \tilde{W}(\tilde{G})$, where
\[
x_w = w^{-1} \left( \prod_{\alpha \in \Delta, \langle w^{-1}(\alpha), \alpha \rangle < 0} \omega_\alpha \right) \in X^*(\tilde{T}),
\]
where \( \varpi_\alpha \) is the fundamental dominant weight corresponding to \( \alpha \). We have

\[
t_f^{-1}w_0\lambda t(f')^{-1}w_0\nu g' = 0
\]

if \( g \neq f' \), and

\[
t_f^{-1}w_0\lambda t g^{-1}w_0\nu g' = \sum \mu m^\mu_{\lambda,\nu} t_f^{-1}w_0\mu g'
\]

where \( m^\mu_{\lambda,\nu} \) is the multiplicity of \( V(\mu) \) in \( V(\lambda) \otimes V(\nu) \).

### 2.2.2 On a theorem of Steinberg

Write \( pt = \text{Spec} \mathbb{C} \).

Steinberg showed in [Ste75] that \( \tilde{K}_T(pt) \) is a free \( \tilde{K}_G(pt) \) module with basis \( \{x_w\}_{w \in W} \). Under the isomorphism \( K_0(pt/T) \simeq K_0(\mathcal{B}^\otimes/\tilde{G}) \), the \( x_w \) define an \( K_0(\tilde{G}/\tilde{G}) \)-basis \( \{\mathcal{F}_w\}_w \) of the latter ring, where \( \mathcal{F}_w = \mathcal{O}(x_w) \), and [KL87] show that the natural pairing

\[
(\cdot, \cdot) : K_0(\mathcal{B}^\otimes/\tilde{G}) \otimes K_0(\tilde{G}/\tilde{G}) \to K_0(pt/\tilde{G})
\]

is nondegenerate. While the dual basis is often employed in the literature starting from loc. cit., we are not aware of an explicit description of it. We provide one here in very low rank cases in type \( A \). The lack of a description in other cases of the dual basis elements as classes in \( K \)-theory of some natural objects of \( \text{Coh}(\mathcal{B}^\otimes/\tilde{G}) \) is the only obstruction to proving Proposition 3 in greater generality.

**Lemma 1.** Let \( \tilde{G} = \text{SL}_2 \) or \( \text{SL}_3 \). The collection \( \mathcal{G}_w = \mathcal{O}(y_w)[\ell(w)] \), where

\[
y_w = \left( w^{-1} \prod_{\alpha \in \Delta} \varpi_\alpha \right) ^{-\rho^{-1}}
\]

defines the basis dual to Steinberg’s basis of \( K_0(\mathcal{B}^\otimes/\tilde{G}) \) under the above pairing.

**Example 2.** In type \( A_1 \) and additive notation, we have \( x_1 = 0 \) and \( x_s = s_\alpha(\varpi_\alpha) = 1 - 2 = -1 \). In this case the Steinberg basis is self-dual, with \( y_1 = \varpi_\alpha - \rho = 1 - 1 = 0 \) and \( y_s = s_\alpha(0) - \rho = -1 \).

The lemma can be proved by direct computation.

**Remark 1.** The classes \( [\mathcal{G}_w] \) cease to pair correctly with the Steinberg basis classes starting for \( G = \text{SL}_4 \), in a way apparently governed by singularities of Schubert cells. For example, for \( \text{SL}_4 \), one has

\[
([\mathcal{F}_w], [\mathcal{F}_1]) = \text{triv}_{\text{SL}_4}
\]

where \( w = 1 \), or when \( w = \sigma \) is the product of the two permutations in \( \mathcal{S}_4 \) that index singular Schubert varieties. In this case, the element dual to \( [\mathcal{F}_1] \) is \( [\mathcal{G}_1] + [\mathcal{G}_\sigma] \). We hope to produce natural complexes in a future version of this paper that will lift these sums and pair correctly.

### 3 Sheaves and categories

All categories, functors, and schemes are derived unless indicated otherwise. We emphasize especially that all fibre products are derived (although frequently this consideration will have no effect). Sections 3.1 and 3.3 recall the necessary material to define the category \( \mathcal{J}_0 \), and contain no new material.
3.1 Derived stacks

Unless otherwise indicated, to “apply base-change” means to apply Proposition 2.2.2 (b) of [GR17].

If $X$ is classical, then $\text{Coh}(X)$ and $\text{Coh}(X/G)$ are the usual bounded derived categories. We write $\text{Rep}(G) := \text{Coh}(pt/G)$. We will often use silently the fact that if $f : X \to Y$ is a morphism of smooth locally-Noetherian schemes, then the pullback functor $f^*$ preserves coherence. The classical schemes we work with will of course be exclusively locally-Noetherian, and the flag variety and bundles over it are smooth.

3.2 The scheme of singularities and singular support

Given a coherent sheaf $\mathcal{F}$ on a stack $X$, Arinkin and Gaitsgory in [AG15] define a classical stack $\text{SingSupp}(\mathcal{F})$, the singular support of $\mathcal{F}$. The singular support serves in particular to measure the extent to which an object of $\text{Coh}(X)$ fails to lie in $\text{Perf}(X)$. We will require only very special cases of the theory of singular support.

Let $X$ first be a quasi-smooth derived scheme. The classical scheme $\text{Sing}(X)$ measures how far from being smooth $X$ is. Let $T^*(X)$ be the cotangent complex of $X$ and $\mathcal{T}(X)$ its dual. Then one defines

$$\text{Sing}(X) := \text{Spec} \left( \text{Sym}_{\mathcal{O}_X} H^1(T(X)) \right) \to X^\circ.$$

The scheme of singularities is affine over $X^\circ$, but is not in general a vector bundle. The singular support will be a conical subset of $\text{Sing}(X)$. In general, if a morphism $f : X \to Y$ exhibits $f^{-1}(\text{pt})$ as quasi-smooth, then given $x \in X$,

$$\text{Sing}(X)_x = \text{coker}(df_x)^*.$$

Note that if $f : V \to W$ is a linear map between vector spaces, then the dg-algebra of functions on the derived scheme $f^{-1}(0)$ is

$$\mathcal{O}_{\ker f} \otimes \text{Sym}(\text{coker}(f)^*[1]).$$

The case of quotient stacks is analogous; one must only keep track of equivariance. See [AG15, Sections 8,9].

As derived schemes or stacks will appear below with approximately the same frequency as their truncations, there are no notational savings to be had by adopting either the convention that all schemes are derived unless otherwise indicated, or the opposite convention. To match our convention about functors, we declare that in any case where a derived stack and its classical truncation appears, the derived stack will be without decoration, as will all classical stacks that appear without any derived enhancement.

3.3 Categorification of $H_{\text{aff}}$, Bezrukavnikov’s equivalence

Let $\tilde{\mathcal{N}} = T^*(\mathcal{B}^\circ)$, and denote the Steinberg variety by $\text{St} = \tilde{\mathcal{N}} \times_{\mathfrak{g}} \tilde{\mathcal{N}}$. It is naturally a global complete intersection derived scheme, fitting into the pullback diagram

$$\begin{array}{ccc}
\text{St} & \longrightarrow & \tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \\
\downarrow & & \downarrow f_{\text{oi}} \\
\text{pt} & \longrightarrow & \mathfrak{g}
\end{array}$$

where the vertical morphism is induced by the composite

$$\tilde{\mathcal{N}} \times \tilde{\mathcal{N}} \to \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}$$

sending $(x, b, y, b') \mapsto x - y$. Therefore $\text{Sing}(\text{St})$ is defined. Its fibre over a point $(X, b_1, b_2)$ of $\text{St}^\circ$ is computed in [CD23, Lemma 3.3.5] to be

$$\text{Sing}(\text{St})_{(X,b_1,b_2)} = \{ Y \in \mathfrak{g} \mid Y \in b_1 \cap b_2, \kappa(Y, [X,-]) = 0 \} \subseteq (\mathfrak{g}/(n_1 \oplus n_2))^* = b_1 \cap b_2, \quad (2)$$
where $\kappa$ is the Killing form.

The category $\text{Coh}(\text{St}/G \times G_m)$ is monoidal under convolution of sheaves. It categorifies $H$, and its unmixed $\text{Coh}(\text{St}/G)$ version is one side of Bezrukavnikov’s celebrated equivalence $[\text{Bez16}]$ upgrading the $K$-theoretic results we recall in the sequel.

**Theorem 2** (Bezrukavnikov, [Bez16]). Let $G^\vee$ be split over $F = \overline{F}_q$ and dual to $G$. Let $I^\vee \subset \mathcal{LG}^\vee$ be an Iwahori subgroup of the loop group $\mathcal{LG}^\vee$. Then there is an equivalence of categories

$$D_{I^\vee/I^\vee} := D^b(I^\vee/\mathcal{LG}^\vee/I^\vee) \to \text{Coh}(\text{St}/G).$$

The equivalence intertwines the automorphism of the left hand side induced by the Frobenius automorphism with pullback by the automorphism of $\text{St}/G$ induced by $(X, b_1, b_2) \mapsto (qX, b_1, b_2)$.

### 3.4 Categorification of $J_0$

#### 3.4.1 Derived enhancement of the flag variety

Let $|\mathfrak{B}|$ be the total space of the quotient

$$0 \to \tilde{N} \to B^\circ \times \mathfrak{g} \to |\mathfrak{B}| \to 0$$

and define

$$\mathfrak{B} = \text{Spec}(\text{Sym}_{B^\circ} |\mathfrak{B}|) = \text{Spec} \left( \text{Sym} \left( B^\circ \times \mathfrak{g}/\tilde{N} \right)^* \right).$$

(3)

Then $\mathfrak{B}$ is naturally a derived scheme with classical truncation $B^\circ$ equipped with a morphism $i: \mathfrak{B} \to \tilde{N}$, and hence with a morphism $i_{\text{der}}: \mathfrak{B} \times \mathfrak{B} \to \text{St}$.

By construction, we have a pullback diagram

$$\begin{array}{ccc}
\mathfrak{B} & \to & \tilde{N} \\
\downarrow & & \downarrow \\
\{0\} & \to & B^\circ \times \mathfrak{g},
\end{array}$$

(4)

where $B^\circ \times \mathfrak{g} \to B^\circ$ is the trivial bundle with fibre $\mathfrak{g}$ and $\{0\}$ is its zero-section. Therefore $\mathfrak{B}$ is a quasi-smooth DG-scheme in the sense of [AG15]. The description of $\mathfrak{B}$ as a fibre product yields a similar description of $\mathfrak{B} \times \mathfrak{B}$. Indeed, the diagram

$$\begin{array}{ccc}
\tilde{N} & \to & \mathfrak{g} \\
\downarrow & & \downarrow \text{id} \\
B^\circ \times \mathfrak{g} & \to & \mathfrak{g} & \to & B^\circ \times \mathfrak{g} \\
\uparrow & & \uparrow & & \uparrow \\
\{0\} & \to & \text{pt} & \to & \{0\}
\end{array}$$

(5)

gives immediately the description

$$\begin{array}{ccc}
\mathfrak{B} \times \mathfrak{B} & \xrightarrow{i_{\text{der}}} & \text{St} \\
\downarrow & & \downarrow \text{pr}_{\text{St}} \\
\{0\} & \xrightarrow{i_{\{0\}}} & B^\circ \times B^\circ \times \mathfrak{g}.
\end{array}$$

(6)
where \( \{0\} \) now means the zero-section of the trivial bundle \( \mathcal{B} \times \mathcal{B} \times \mathfrak{g} \). We get the same description for the quotients by \( G \) or \( G \times \mathbb{G}_m \). Note that (6) says that \( i_{\text{der}} \) is a quasi-smooth closed immersion by [AG15, Prop. 2.1.10].

The category \( \text{Coh}(\mathcal{B}/G \times \text{pt}/G/\mathcal{B}/G) \) is a module category over the monoidal category \( \text{Coh}(\text{pt}/G) \), via
\[
V \cdot \mathcal{F} = \pi^* V \otimes_{\mathcal{O}_{\text{pt}},\pi} \mathcal{F},
\]
for \( V \in \text{Coh}(\text{pt}/G) \), where \( \pi: \mathcal{B}/G \times \text{pt}/G/\mathcal{B}/G \to \text{pt}/G \). The same procedure makes \( J_0 \) into a module category over \( \text{Coh}(\text{pt}/G) \).

**Lemma 2.** If \( \mathcal{F} \in \text{Coh}_{G \times \mathbb{G}_m}(\text{St}) \), then
\[
\text{SingSupp}(i_{\text{der}}^* \mathcal{F}) \subseteq \Delta \tilde{\mathfrak{g}} \subset \tilde{\mathfrak{g}} \times \tilde{\mathfrak{g}}.
\]

**Proof.** We seek to apply Proposition 7.1.3 of [AG15] (c.f. Section 7.4.5 and Lemma 8.4.2 of op. cit.) to the quasi-smooth closed embedding \( i_{\text{der}} \). We have
\[
\text{Spec Sym}_{\mathcal{O}_{\text{pt}} \otimes \mathcal{O}_{\mathcal{B}}}(i_{\text{der}}^* T(\text{St})[1]) \rightarrow \text{Sing}(\mathcal{B} \times \mathcal{B}) \rightarrow \text{Sing}(\mathcal{B} \times \mathcal{B})
\]
Fibrewise, the pairing condition of (2) becomes vacuous, and the singular codifferential is the linear map
\[
\text{Sing}(\text{St})(0, b_1, b_2) = (\mathfrak{g}/(n_1 \oplus n_2))^* = b_1 \cap b_2 \rightarrow (\mathfrak{g}/n_1)^* \oplus (\mathfrak{g}/n_2)^* = b_1 \oplus b_2 = \text{Sing}(\mathcal{B} \times \mathcal{B})(b_1, b_2)
\]
induced by the direct sum of projections
\[
\mathfrak{g}/n_1 \oplus \mathfrak{g}/n_2 \rightarrow \mathfrak{g}/(n_1 \oplus n_2).
\]
This implies that (7) is the simply the diagonal embedding, and the lemma follows. \( \Box \)

**3.4.2 Definition of the category \( J_0 \)**

We now define the category \( J_0 \), the main point being the condition we impose on the singular supports of its objects. In our case
\[
\text{Sing}(\mathcal{B}) = \text{Spec} \left( \text{Sym}_{\mathcal{O}_{\mathcal{B}}} \mathfrak{g}[2] \right) \rightarrow \mathcal{B}^{\nabla}.
\]
Koszul duality gives an equivalence
\[
\text{KD}: \text{Coh}(\mathcal{B}) \rightarrow \text{Sym}_{\mathcal{O}_{\mathcal{B}}} \mathfrak{g}[2] \rightarrow \text{Mod}^{\mathfrak{g}} -
\]
and following [AG15], we set
\[
\text{SingSupp}(\mathcal{F}) = \text{supp}(\text{KD}(\mathcal{F})) \subset [\mathfrak{g}].
\]
Thus for \( \mathcal{F} = \mathcal{F}_1 \boxtimes \mathcal{F}_2 \in \text{Coh}(\mathcal{B} \times \mathcal{B}) \), \( \text{SingSupp}(\mathcal{F}) \) is just the usual support of some other sheaf on the total space of the bundle \( \text{Sing}(\mathcal{B}) \times \text{Sing}(\mathcal{B}) \). We note that \( \text{Sing}(\mathcal{B}) \) is none other than the bundle \( \tilde{\mathfrak{g}} \). Indeed, the fibres of \( \mathcal{B} \) are
\[
\text{Spec Sym}_C((\mathfrak{g}/n)^*[1]) = \text{Spec Sym}_C(b[1])
\]
and Koszul duality identifies
\[
\text{Sym}_C(b[1]) - \text{Mod} \simeq \text{Sym}_C(b^*[2]) - \text{Mod},
\]
and it makes sense to take the support of a module on the right-hand side on the scheme \( b \), by defining the support to be the support of the cohomology over the classical ring \( \text{Sym}_C b^* \).
We define $\mathcal{J}_0$ to be the full subcategory of $\text{Coh}_G(\mathcal{B} \times \mathcal{B})$ with objects $\mathcal{F}$ such that the projection $\text{SingSupp}(\mathcal{F}) \to \text{Sing}(\mathcal{B})$ onto the first factor is a proper morphism. We write

$$\mathcal{J}_0 := \text{Coh}(\mathcal{B}/G \times \text{pt}/G/G!);$$

and

$$\mathcal{J}_{0!} := \text{Coh}(\mathcal{B}/G \times \mathbb{G}_m \times \text{pt}/G \times \mathbb{G}_m; \mathcal{B} \times \mathbb{G}_m);,$$

where $\mathbb{G}_m$ acts trivially on $\mathcal{B}$.

There are two obvious ways that the projection onto the first factor can be proper: either $\text{KD}(\mathcal{F})$ is of form $\Delta^* \mathcal{F}'$, where $\Delta$ is the diagonal, or $\text{KD}(\mathcal{F}) = \mathcal{F}_1 \boxtimes \mathcal{F}_2$ with $\text{supp}(\mathcal{F}_2)$ contained in the zero section. This latter case arises precisely from sheaves $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \in \mathcal{J}_0$ such that $\mathcal{F}_2$ is perfect. These are essentially the only examples that we will encounter. The image of $i^*_\text{der}$ consists of sheaves of the first type (this is especially easy to see for those sheaves whose images in $K$-theory are contained in $Z(J_0) = \phi_0(\text{Z}(\text{H}_\text{aff}))$), and the sheaves $\mathcal{I}_w$ that we define in Section 3.4.3 are all examples of the second kind.

**Remark 2.** This fact, together with the second statement of the main theorem, can be viewed as a categorification of the fact that $\phi_0$ is injective but not surjective.

Consider the pairing operation defined by

$$\langle \mathcal{F}, \mathcal{G} \rangle = \pi^*_\text{der}(\mathcal{F} \otimes \mathcal{G})$$

where $\pi : \mathcal{B}/G \to \text{pt}/G$. In general, this operation does not define a functor

$$\langle -, - \rangle : \text{Coh}(\mathcal{B}/G) \otimes \text{Coh}(\mathcal{B}/G) \to \text{pt}/G,$$

but will do so when it comes to convolution of objects of $\mathcal{J}_0$.

**Proposition 1.** The category $\mathcal{J}_0$ is a monoidal category under convolution of sheaves, and the pullback $i^*_\text{der}$ defines a monoidal functor

$$i^*_{\text{der}} : \text{Coh}(\text{St}/G \times \mathbb{G}_m) \to \mathcal{J}_{0!}$$

such that

$$\text{SingSupp}(i^*_{\text{der}} \mathcal{F}) \subset \Delta \tilde{\mathcal{E}}$$

for all $\mathcal{F}$. The category $\mathcal{J}_0$ is a triangulated subcategory of $\text{Coh}_G(\mathcal{B} \times \mathcal{B})$.

Additionally,

1. If $\mathcal{F}_1 \boxtimes \mathcal{G}$ and $\mathcal{G}' \boxtimes \mathcal{F}_2$ are in $\mathcal{J}_0$, then $\langle \mathcal{G}, \mathcal{G}' \rangle \in \text{Coh}(\text{pt}/G)$ and

$$\mathcal{F}_1 \boxtimes \mathcal{G} \ast \mathcal{G}' \boxtimes \mathcal{F}_2 = \langle \mathcal{G}, \mathcal{G}' \rangle \mathcal{F}_1 \boxtimes \mathcal{F}_2;$$

2. If $\mathcal{V}_1, \mathcal{V}_2 \in \text{Coh}(\text{pt}/G)$, then

$$\mathcal{V}_1 \ast \mathcal{F} \ast \mathcal{V}_2 = (\mathcal{V}_1 \otimes \mathcal{V}_2) \ast \mathcal{F}$$

for $\mathcal{F}, \mathcal{G} \in \mathcal{J}_0$.

**Remark 3.** The category $\text{Coh}(\mathcal{B}/G \times \text{pt}/G/G!)$ is not monoidal.

**Proof.** Let

$$\mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to$$

be a distinguished triangle in $\text{Coh}(\mathcal{B}/G \times \text{pt}/G/G!)$ such that $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{J}_0$. Applying Koszul duality, we get

$$\text{KD}(\mathcal{F}_1) \to \text{KD}(\mathcal{F}_2) \to \text{KD}(\mathcal{F}_3) \to$$
in \( \text{Coh}(\tilde{\mathfrak{g}}/G \times_{pt/G} \tilde{\mathfrak{g}}/G) \). Then
\[
\text{supp} K\text{D}(\mathcal{F}_3) \subset \text{supp} K\text{D}(\mathcal{F}_1) \cup \text{supp} K\text{D}(\mathcal{F}_2),
\]
and we see that projection \( \text{supp} K\text{D}(\mathcal{F}_3) \to \tilde{\mathfrak{g}}/G \) onto the first factor is proper.

Let \( \mathcal{F} \in \mathcal{J}_0 \) and let \( \mathcal{G} \in \text{Coh}(\mathcal{B}/G \times_{pt/G} \mathcal{B}/G) \). Then their convolution will be coherent if
\[
(\mathcal{F}_{12} \boxtimes \mathcal{O}_{\mathcal{B}/G}^{\mathcal{B}/G} \mathcal{G}_{23})_{ij} \text{ is coherent, where the subscripts } i,j \text{ indicate which factors inside } \mathcal{B}/G \times_{pt/G} \mathcal{B}/G \times_{pt/G} \mathcal{B}/G \text{ a given sheaf sits on. Noting that } \text{SingSupp}(\mathcal{O}_{\mathcal{B}/G}^{\mathcal{B}/G}) = \{0\}, \text{ we have}
\]
\[
\text{SingSupp}(\mathcal{F}_{12} \boxtimes \mathcal{O}_{\mathcal{B}/G}^{\mathcal{B}/G}) \cap \text{SingSupp}(\mathcal{O}_{\mathcal{B}/G}^{\mathcal{B}/G} \mathcal{G}_{23}) \subset \{0\}/G \times_{pt/G} \tilde{\mathfrak{g}}/G \times_{pt/G} \{0\}/G.
\]
We claim that this intersection is in fact contained in the zero section of \( \tilde{\mathfrak{g}}/G \times_{pt/G} \tilde{\mathfrak{g}}/G \times_{pt/G} \tilde{\mathfrak{g}}/G \). First, projection from \( \text{SingSupp}(\mathcal{F}_{12}) \times \{0\} \) to the first coordinate is a proper morphism, and so the same is true for projection from the intersection: \( \{0\} \times \text{SingSupp}(\mathcal{G}_{23}) \) is closed. As \( \text{SingSupp}(\mathcal{F}_{12}) \) is a conical subset, it now follows that the intersection is contained in \( \{0\} \times \{0\} \times \{0\} \). The claim now follows from [AG15] Corollary 8.4.8.

Now suppose that projection to the first factor from \( \text{SingSupp}(\mathcal{G}_{23}) \) is also proper. We have
\[
\text{Sing}(p_{13}): \tilde{\mathfrak{g}}/G \times_{pt/G} \tilde{\mathfrak{g}}/G \rightarrow \tilde{\mathfrak{g}}/G \times_{pt/G} \tilde{\mathfrak{g}}/G \times_{pt/G} \tilde{\mathfrak{g}}/G
\]
is the inclusion of the zero section into the second coordinate, and is the identity on the other coordinates.
Define
\[
Y_2 = \{(x_1, x_3) \mid (x_1, z) \in \text{SingSupp}(\mathcal{F}_{12}), \ (z, x_3) \in \text{SingSupp}(\mathcal{G}_{23}) \text{ for some } z \in \{0\}\}.
\]
Then
\[
\text{Sing}(p_{13})^{-1}\{(x_1, x_2, x_3) \mid (x_1, x_2) \in \text{SingSupp}(\mathcal{F}_{12}), \ (x_2, x_3) \in \text{SingSupp}(\mathcal{G}_{23})\}
\]
\[
= \{(x_1, z, x_3) \mid (x_1, z) \in \text{SingSupp}(\mathcal{F}_{12}), \ (x_2, z) \in \text{SingSupp}(\mathcal{G}_{23}), \ z \in \{0\}\}
\]
\[
= Y_2 \times_{\mathcal{B}/G \times_{pt/G} \mathcal{B}/G} \mathcal{B}/G \times_{pt/G} \mathcal{B}/G \times_{pt/G} \mathcal{B}/G.
\]
It follows from [AG15], Lemma 8.4.5 that \( \text{SingSupp}(\mathcal{F} \otimes \mathcal{G}) \subset Y_2 \). It therefore suffices to show that the projection \( Y_2 \to \tilde{\mathfrak{g}}/G \) is proper. Indeed, we have
\[
p_1^{-1}(K) \subset K \times p_2, \left( p_1^{-1}(\{0\}) \right),
\]
for any \( K \), where \( p_i, x \) is the projection \( \text{SingSupp}(\mathcal{F}) \to \tilde{\mathfrak{g}}/G \) onto the \( i \)-th factor.
Therefore \( \mathcal{J}_0 \) is a monoidal category. The same of course goes for \( \mathcal{J}_{0, \mathfrak{g}} \).

Now we show the first formula. It is easy to see that if \( \mathcal{F} \boxtimes \mathcal{G} \in \mathcal{J}_0 \), then \( \text{SingSupp}(\mathcal{G}) \) must be contained in the zero section, i.e. that \( \mathcal{G} \) must be perfect. Then, \( \langle \mathcal{F}, \mathcal{G} \rangle \) is coherent because \( \mathcal{G} \boxtimes \mathcal{G} \) is and the map to pt is proper. Its pullback to \( \mathcal{B} \times \mathcal{B} \) is then perfect. The remainder of the formula is obtained by carrying out the calculations in Lemma 5.2.28 of [CG97]. The required projection formula and base-change are provided by Lemma 3.2.4 and Proposition 2.2.2 (b) of [GR17], respectively.

We next claim that if \( \mathcal{F} \in \text{DG Coh}_{G \times \mathbb{G}_m}(\text{St}) \), then \( i_{\text{der}}^* \mathcal{F} \) is coherent, and the projection
\[
p: \text{SingSupp}(i_{\text{der}}^* \mathcal{F}) \subset \tilde{\mathfrak{g}}/G \times_{pt/G} \tilde{\mathfrak{g}}/G \rightarrow \tilde{\mathfrak{g}}/G
\]
ono onto the first factor is proper. Coherence follows again from (5). Indeed, we need only show that the pushforward of \( i_{\text{der}}^* \mathcal{F} \) to \( \mathcal{B} \otimes \mathcal{B} \) is coherent, and base-change says that this equals \( p_{\text{St}}^{-1} \mathcal{F} \). By hypothesis \( p_{\text{St}} \mathcal{F} \) is coherent, and hence by smoothness of \( \mathcal{B} \otimes \mathcal{B} \times \mathfrak{g} \) the pullback is also coherent. We must check that \( i_{\text{der}}^* \mathcal{F} \in \mathcal{J}_0 \). Indeed, this follows immediately from Lemma 2, which says that \( \text{SingSupp}(i_{\text{der}}^* \mathcal{F}) \subset \Delta \tilde{\mathfrak{g}} \).

We now check that \( i_{\text{der}}^* \) is monoidal. Diagrams (4) and
We used base-change for the diagram (9) with implicit categories in the special case of (9) for \(ij\) or \(SL_2\) or \(SL_3\), where it can be carried out almost verbatim. As remarked above, we hope to move beyond these two special cases in future work.

Recalling the equivalence

\[ \text{Ind}^G_B : \text{Coh}(pt/B) \to \text{Coh}(\mathcal{B}/G^\triangleright), \]

we define

\[ \mathcal{F}_w = \text{Ind}^G_B \text{Ind}^B_H(x_w), \]

implies that

\[ \tilde{N}/G \times \mathbb{G}_m \to g/G \times \mathbb{G}_m \to g/G \times \mathbb{G}_m \]

\[ \mathcal{B}/G \times \mathbb{G}_m \to pt/G \times \mathbb{G}_m \to g/G \times \mathbb{G}_m \]

\[ \mathcal{B}/G \times \mathbb{G}_m \to pt/G \times \mathbb{G}_m \to pt/G \times \mathbb{G}_m \]

Thus

\[ \mathcal{B}/G \times \mathbb{G}_m \times pt/G \times \mathbb{G}_m \to \mathcal{B}/G \times \mathbb{G}_m \times pt/G \times \mathbb{G}_m \to \mathcal{B}/G \times \mathbb{G}_m \times pt/G \times \mathbb{G}_m \]

\[ \tilde{N}/G \times \mathbb{G}_m \to \tilde{N}/G \times \mathbb{G}_m \]

and we can apply base-change to the pullback diagram

\[ \mathcal{B}/G \times \mathbb{G}_m \times pt/G \times \mathbb{G}_m \to \mathcal{B}/G \times \mathbb{G}_m \times pt/G \times \mathbb{G}_m \to \mathcal{B}/G \times \mathbb{G}_m \]

\[ \tilde{N}/G \times \mathbb{G}_m \to \tilde{N}/G \times \mathbb{G}_m \]

\[ \text{St}/G \times \mathbb{G}_m \]

where \(\pi_{ij}\) and \(p_{ij}\) are the projections.

With diagram (9) in hand, the remainder is entirely formal. Indeed, according to the definition of convolution on \(\text{St}\), we compute as follows: Let \(\mathcal{F}, \mathcal{G} \in \text{Coh}(\text{St}/G \times \mathbb{G}_m)\). Then

\[ i^*_\text{der}(\mathcal{F} \ast \mathcal{G}) = i^*_\text{der}p_{13*}(p^*_{12}\mathcal{F} \otimes p^*_{23}\mathcal{G}) \]

\[ \simeq \pi_{13!}(i \times i)^*(p^*_{12}\mathcal{F} \otimes p^*_{23}\mathcal{G}) \]

\[ \simeq \pi_{13!}(\pi^*_{12}\mathcal{F} \otimes \pi^*_{23}\mathcal{G}) \]

\[ = i^*_\text{der} \mathcal{F} \ast i^*_\text{der} \mathcal{G}. \]

We used base-change for the diagram (9) with \(ij = 13\) between lines (10) and (11), and just commutativity of (9) for \(ij = 12\) and \(ij = 23\) on line (12).

Recall that \(a(c_0) = \dim \mathcal{B}\), any quasicoherent sheaf on \(\mathcal{B}^\triangleright\) has cohomology only in degrees at most \(a(c_0)\). On the level of \(K\)-theory, this reflects the influence of the \(a\)-function on the multiplication in \(J\).

### 3.4.3 The sheaves \(\ell_w\)

Xi, in [Xi90] for \(G\) simply-connected, and Nie in [Nie11] in general gave a description in \(K\)-theory of the elements \(\ell_w\) for \(w \in c_0\). We recall this construction below in Section 2.2.1; here we follow it on the level of categories in the special case \(G = SL_2\) or \(SL_3\), where it can be carried out almost verbatim. As remarked above, we hope to move beyond these two special cases in future work.

Recalling the equivalence

\[ \text{Ind}^G_H : \text{Coh}(pt/B) \to \text{Coh}(\mathcal{B}/G^\triangleright), \]

we define

\[ \mathcal{F}_w = \text{Ind}^G_H \text{Ind}^B_H(x_w), \]
where \(x_w \in \text{Coh}_T(\text{pt})\) is as in (1) and
\[
\text{Infl}_T^B : \text{Coh}(\text{pt}/T) \to \text{Coh}(\text{pt}/B)
\]
is inflation. Likewise, define
\[
\mathcal{G}_w = \text{Ind}_G^B \text{Infl}_T^B(y_w)
\]
where \(y_w\) is the dual basis from Lemma 1.

Now if \(w = f w_0 g^{-1}\), define
\[
\ell_w = \mathcal{F}_f \boxtimes p^* \mathcal{G}_g,
\]
where \(p : B \to B^\circ\), and if \(w = f w_0 \chi g^{-1}\), define
\[
\ell_w = V(\chi) \ell_{f w_0 g^{-1}},
\]
where we view \(\mathcal{F}_f\) as pushed forward under the inclusion of the zero section of \(B\). Clearly \(\ell_w \in \text{Coh}(B^\circ/\text{pt})\).

Moreover, as \(B^\circ\) is smooth, \(\mathcal{G}_g\) is perfect, and hence \(p^* \mathcal{G}_g\) is perfect. Therefore \(\text{SingSupp}(\mathcal{G})\) is contained in the zero section of \(\text{Sing}(B)\), and \(\ell_w \in j_0\).

By Proposition 1 (or using Corollary 8.4.8 of [AG15] directly), \(\langle \mathcal{G}_g, \mathcal{F}_f \rangle\) is defined for all \(g, f\) and takes values in \(\text{Coh}(\text{pt}/G)\). In fact, it agrees with the pairing \(\langle -, - \rangle^\circ\) on the classical truncation given by the same procedure:
\[
\langle p^* \mathcal{G}_g, \mathcal{F}_f \rangle = \pi_* (p^* \mathcal{G}_g \otimes \mathcal{F}_f) = \pi^\circ_* p_* (p^* \mathcal{G}_g \otimes \mathcal{F}_f) = \pi^\circ_* (\mathcal{G}_g \otimes p_* \mathcal{F}_f) = \langle \mathcal{G}_g, \mathcal{F}_f \rangle^\circ
\]
where
\[
\begin{array}{ccc}
\mathcal{B}/G & \xrightarrow{p} & \mathcal{B}/G^\circ \\
\pi & \xrightarrow{\pi^\circ} & \text{pt}/G.
\end{array}
\]

Therefore by the Borel-Weil-Bott theorem, we have
\[
\mathcal{F}_f \boxtimes p^* \mathcal{G}_g \star \mathcal{F}_{f'} \boxtimes p^* \mathcal{G}_{g'} = \begin{cases} 
\mathcal{F}_f \boxtimes \mathcal{G}_{g'} & \text{if } g = f' \\
0 & \text{otherwise}
\end{cases}.
\]

Moreover, as \(G\) is reductive, we have again by Proposition 1 that
\[
(V(\lambda) \mathcal{F}_f \boxtimes p^* \mathcal{G}_g) \star (V(\nu) \mathcal{F}_g \boxtimes p^* \mathcal{G}_{g'}) = (V(\lambda) \otimes V(\nu)) \mathcal{F}_f \boxtimes p^* \mathcal{G}_{g'} = \bigoplus_{\mu} V(\mu)(\mathcal{F}_f \boxtimes p^* \mathcal{G}_g) \otimes m_{\lambda, \nu}^{\mu},
\]
where \(m_{\lambda, \nu}^{\mu}\) is the multiplicity of \(V(\mu)\) in \(V(\lambda) \otimes V(\nu)\).

4 \(K\)-theory

In this section we show that the functor \(\pi_{\text{der}}^*\) categorifies Lusztig’s homomorphism \(\phi_0\). By \(K\)-theory we shall always mean simply the Grothendieck group. We write \(R(G) = K_G(\text{pt}) = K_0(\text{Rep}(G))\).

4.1 \(K\)-theory of classical schemes and Lusztig’s homomorphism

We first relate Lusztig’s homomorphism to a construction in \(K\)-theory of classical schemes given in [CG97]. There is no new material in this section; when \(G\) is simply-connected the relationship is given by [Xi16], and the analogous result in general follows from [BO04]. In order to perform calculations, though, we must devote significant space to fixing conventions.
Recall that the $K_{G \times \mathbb{G}_m}(\text{St}) \simeq H$ as $\mathcal{A}$-algebras. We will use the explicit isomorphism given by Chriss and Ginzburg in [CG97], Theorem 7.2.5.

Recall that by [BO04], $J_0 \simeq \mathcal{K}_G(Y \times Y)$ for a centrally-extended set $Y$ of cardinality $\#W$. Moreover, by 5.5 (a) of loc. cit., the stabilizer of every $y \in Y$ is $G$. When $G$ is simply-connected, it has no nontrivial central extensions, and hence in this case $J_0 \simeq \text{Mat}_{\#W}(R(G))$ is a matrix ring, as first shown in [Xi90]. Combining these results, we obtain an injection $\varphi_1 : J_0 \hookrightarrow \text{Mat}_{\#W}(R(\tilde{G}))$, where $\tilde{G} \to G$ is a simply-connected. Write $H(\tilde{W}(\tilde{G}))$ for the affine Hecke algebra of $\tilde{G}$.

In parallel, when $G$ is simply-connected, the external tensor product gives an isomorphism $K_0(\mathcal{B}/G \times pt/G \otimes (\mathcal{B}/G)) \simeq K(\mathcal{B}/G) \otimes R(G) K_0(\mathcal{B}/G) \simeq \text{Mat}_{\#W}(R(G))$, by [CG97], Theorem 6.2.4. This theorem does not hold when $G$ is not simply-connected. Indeed, the one-dimensional tempered representations of $H$ that are not discrete series give of $J_0$-modules for $G^\vee = SL_2$. On the other hand, the external tensor product still gives an inclusion

$$\psi_1 : K_0(\mathcal{B}/G \times pt/G \otimes (\mathcal{B}/G)) \hookrightarrow K_0(\mathcal{B}/\tilde{G} \times pt/\tilde{G} \otimes (\mathcal{B}/\tilde{G})) \simeq \text{Mat}_{\#W}(R(\tilde{G})).$$

(Note that $G$ and $\tilde{G}$ have canonically isomorphic flag varieties and Weyl groups.)

By [Nie11], we have an isomorphism $\sigma : J_0 \to K_0(\mathcal{B}/G^\vee \times pt/G \otimes (\mathcal{B}/G^\vee))$ regardless of whether $G$ is simply-connected or not.

**Lemma 3** ([Xi16]). The following diagram of $\mathcal{A}$-algebras

$$
\begin{array}{ccc}
K_0(\text{St}/G \times \mathbb{G}_m) & \xrightarrow{\phi_0} & J_0 \otimes \mathbb{Z} \mathcal{A} \\
\sim & & \varphi_1 \\
H & \xrightarrow{\phi_0} & \text{Mat}_{\#W}(\text{Mat}_{\#W}(R(\tilde{G} \times \mathbb{G}_m)))
\end{array}
$$

commutes, where $A$ is the change-of-basis matrix from the the $Z(H(\tilde{W}(\tilde{G})))$-basis $\{\theta_{e_w} C \mid w \in W\}$ of $H(\tilde{W}(\tilde{G}))$ to the $Z(H(\tilde{W}(\tilde{G})))$-basis $\{C_{d_w, w_0} \mid w \in W\}$, where $e_w$ and $d_w$ are as in [Xi16].

### 4.1.1 $K$-theory of derived schemes and Lusztig’s homomorphism

Koszul duality identifies $\text{Coh}(\mathcal{B}/G \times \mathbb{G}_m)$ with $\text{Coh}(\mathcal{C}_{[-2]}^* G \times \mathbb{G}_m)$.

We now establish the relationship in $K$-theory between the monoidal functor $i_*^*$, from Section 3.4.1 and the morphism $\phi_0$. Write

$$K_0(\text{St}/G \times \mathbb{G}_m) := K_0(\text{Coh}(\text{St}/G \times \mathbb{G}_m))$$

and write $K(X^\vee) := K_0(\text{Coh}(X^\vee))$ whenever $X^\vee$ is a classical scheme, and similarly for stacks.

If $X$ is a derived stack with classical truncation $X^\vee$, we may define a morphism

$$K(X) \to K(X^\vee)$$

by

$$[\mathcal{F}] \mapsto \sum_i (-1)^i [\pi_i(\mathcal{F})],$$

where $\pi_i(F)$ is viewed as a $\pi_0(\mathcal{O}_X)$-module.

Recalling that the derived structure on $X$ is to be thought of as “higher nilpotents,” this morphism is identical in spirit to identifying $K_0(\text{Coh}(\text{Spec} A))$ and $K_0(\text{Coh}(\text{Spec} A_{\text{red}}))$, where $A$ is Noetherian and $A_{\text{red}}$ is reduced. Indeed, the map (13) is also an isomorphism of abelian groups, and both isomorphisms are consequences of dévissage; see e.g. [Toë14].
**Lemma 4.** Pushforward by bundle projection $p: \text{St}/G \times \mathbb{G}_m \to \text{St}/G \times \mathbb{G}_m^\triangleright$ induces the map (13) on $K$-theory. This map is an isomorphism of rings.

**Proof.** By the remarks preceding the lemma, it suffices to show that $p_*$ respects convolution in $K$-theory. By definition, we have

$$p_*([\mathcal{F}]) * p_*([\mathcal{G}]) = \sum_{i,j} (-1)^{i+j} \pi_i(\mathcal{F}) * \pi_j(\mathcal{G}),$$

(14)

whereas

$$p_*([\mathcal{F}] * [\mathcal{G}]) = p_*p_{13*}(p_1^{12}\mathcal{F} \otimes q_2^*\mathcal{G}) = p_1^{\triangleright}p_{13*}(p_1^{12}\mathcal{F} \otimes q_2^*\mathcal{G})$$

by commutativity of the diagram

$$\begin{array}{ccc}
\left(\left(\hat{N} \times \hat{N}\right) \times \hat{N} \times \hat{N}\right) / G \times \mathbb{G}_m & \xrightarrow{p_{13}} & \text{St}/G \times \mathbb{G}_m \\
\downarrow & & \downarrow \\
\hat{N} / G \times \mathbb{G}_m \times \hat{N} / G \times \mathbb{G}_m & \xrightarrow{p_{13}^{\triangleright}} & \text{St}/G \times \mathbb{G}_m^\triangleright.
\end{array}$$

We have

$$p_{13*}p_{13*}(p_1^{12}\mathcal{F} \otimes q_2^*\mathcal{G}) = p_1^{\triangleright}p_{13*}(p_1^{12}\mathcal{F} \otimes q_2^*\mathcal{G}),$$

(15)

and so for (14) to equal (15), we need

$$\pi_n(p_1^{12}\mathcal{F} \otimes q_2^*\mathcal{G}) = \sum_{i+j=n} p_1^{\vartrianglelefteq}(\mathcal{F}) \otimes \hat{N} \times \hat{N} p_1^{\vartriangleleft}(\mathcal{G}),$$

which follows from the Künneth formula.

In the case of $K_0(\mathcal{F}_0)$, we can define another map to the $K$-theory of the truncation. Recall $p: \mathcal{B} / G \times \mathbb{G}_m \to \mathcal{B} / G \times \mathbb{G}_m^\triangleright$ is the bundle projection morphism, and let $i: \mathcal{B} / G \times \mathbb{G}_m^\triangleright \to \mathcal{B} / G \times \mathbb{G}_m$ be the inclusion of the zero section. Then define $\Phi$ to be the composite

$$\Phi: K(\mathcal{F}_0)^{\text{id} \times i^*} \to K_0(\mathcal{B} / G \times \mathbb{G}_m \times pt / G \times \mathbb{G}_m) \xrightarrow{p_{\times \text{id}}} K_0(\mathcal{B} / G \times \mathbb{G}_m \times pt / G \times \mathbb{G}_m) \to \mathcal{B} / G \times \mathbb{G}_m^\triangleright.$$ 

That is, if $\mathcal{F} \boxtimes \mathcal{G} \in \mathcal{F}_0$, then

$$\Phi([\mathcal{F} \boxtimes \mathcal{G}]) = [p_*\mathcal{F}] \boxtimes [i^*\mathcal{G}].$$

**Remark 4.** It is necessary that the source of $(\text{id} \times i)^*$ (which we will show makes sense as a functor) is $\mathcal{F}_0$ and not all of $\text{Coh}_G(\mathcal{B} \times \mathcal{B})$; the functor

$$i^*: \text{QCoh}(\mathcal{B} / G \times \mathbb{G}_m) \to \text{QCoh}(\mathcal{B} / G \times \mathbb{G}_m^\triangleright)$$

does not preserve coherence in general.

**Lemma 5.** The morphism $\Phi$ is well-defined and is a surjective morphism of $R(\mathbb{G}_m \times \mathbb{G}_m^\triangleright)$-algebras.

**Proof.** By the Künneth formula, we have

$$\text{Sing}(\mathcal{B} \times \mathcal{B}^\triangleright) \simeq \mathcal{B}^\triangleright / G \times \mathbb{G}_m \times \mathcal{B} / G \times \mathbb{G}_m.$$ 

Then

$$\text{Sing}(\text{id} \times i): \mathcal{B}^\triangleright / G \times \mathbb{G}_m \times pt / G \times \mathbb{G}_m \to \mathcal{B}^\triangleright / G \times \mathbb{G}_m \times pt / G \times \mathbb{G}_m$$

is given by the identity in the first coordinate, and then the zero map in the second coordinate.

Therefore $\text{Sing}(\text{id} \times i)^{-1}(\{0\}) = \{0\} \times \mathcal{B}^\triangleright / G \times \mathbb{G}_m$. Now let $\mathcal{F} \in \mathcal{F}_0$. The argument is essentially the same as in the proof of Proposition 1. The set-theoretic intersection

$$\left(\text{SingSupp}(\mathcal{F}) \times (\mathcal{B} \times \mathcal{B}) / G \times \mathbb{G}_m \times \mathcal{B} / G \times \mathbb{G}_m^\triangleright \times \mathcal{B} / G \times \mathbb{G}_m\right) \cap \text{Sing}(\text{id} \times i)^{-1}(\{0\})$$

$$= \text{SingSupp}(\mathcal{F}) \cap (\{0\} \times \mathcal{B}^\triangleright) / G \times \mathbb{G}_m$$
is contained in the zero-section of $\text{Sing}(\mathcal{B}/G \times \mathbb{G}_m \times_{\text{pt}/G \times \mathbb{G}_m} \mathcal{B}/G \times \mathbb{G}_m)$. Indeed, $\mathcal{F} \in J_0$ and $\mathcal{B}^{\triangleright}$ is compact, so the above intersection must be compact. As $\text{SingSupp}(\mathcal{F})$ is conical, we see the intersection must be contained in the zero-section. We conclude by [AG15], Corollary 8.4.8 that $(\text{id} \times i)^*$ is well-defined. Obviously $p_\ast \times \text{id}$ is well-defined, and hence $\Phi$ is well-defined. As each of $(\text{id} \times i)^*$ and $p_\ast \times \text{id}$ are $R(G)$-linear (the latter by the projection formula), so is $\Phi$.

Finally, we show that $\Phi$ is a morphism of rings. Using linearity, we compute

\[
\Phi([\mathcal{F}_1] \boxtimes [\mathcal{F}]) \ast \Phi([\mathcal{G}'] \boxtimes [\mathcal{F}_2]) = p_\ast [\mathcal{F}_1] \boxtimes i_{\text{der}}^\ast [\mathcal{G}] \ast p_\ast [\mathcal{G}'] \boxtimes i_{\text{der}}^\ast [\mathcal{F}_2] = \langle i_{\text{der}}^\ast [\mathcal{G}], p_\ast [\mathcal{G}'] \rangle_{\mathcal{B}^{\triangleright}} \cdot p_\ast [\mathcal{F}_1] \boxtimes i_{\text{der}}^\ast [\mathcal{F}_2],
\]

whereas

\[
\Phi([\mathcal{F}_1] \boxtimes [\mathcal{G}] \ast [\mathcal{G}'] \boxtimes [\mathcal{F}_2]) = \langle [\mathcal{G}], [\mathcal{G}'] \rangle_{\mathcal{B}^{\triangleright}} \cdot p_\ast [\mathcal{F}_1] \boxtimes i_{\text{der}}^\ast [\mathcal{F}_2].
\]

Write $\pi : \mathcal{B}/G \times \mathbb{G}_m^{\triangleright} \to \text{pt}/G \times \mathbb{G}_m$. Then

\[
(i_{\text{der}}^\ast [\mathcal{G}], p_\ast [\mathcal{G}'])_{\mathcal{B}^{\triangleright}} = \pi_\ast (i_{\text{der}}^\ast [\mathcal{G}] \otimes_{\mathcal{B}^{\triangleright}} p_\ast [\mathcal{G}']) = \pi_\ast p_\ast (p_\ast i_{\text{der}}^\ast [\mathcal{G}] \otimes_{\mathcal{B}^{\triangleright}} [\mathcal{G}']) = \langle [\mathcal{G}], [\mathcal{G}'] \rangle_{\mathcal{B}^{\triangleright}}
\]

by the formulation of the projection formula in [GR17], Lemma 3.2.4. Surjectivity follows as $\Phi$ has a section defined by $\mathcal{F} \boxtimes [\mathcal{G}] \to i_\ast [\mathcal{F}] \boxtimes p_\ast [\mathcal{G}]$. This completes the proof.

**Lemma 6.** We have $\Phi(\Theta_{\Delta,\mathcal{A}}(\lambda)) = \Theta_{\Delta,\mathcal{B}^{\triangleright}}(\lambda)$.

**Proof.** This is a local computation that amounts to the map

\[
\mathbb{C}[x,\epsilon] \otimes \mathbb{C}[y,\delta]/(x - y, \epsilon - \delta) \to \mathbb{C}[x, y]
\]

quotienting by $\delta$ and leaving the first factor untouched, where $|x| = |y| = 0$ and $|\epsilon| = |\delta| = -1$. One sees that quotienting by $\delta$ also kills $\epsilon$.

**Proposition 2.** The following diagram of $R(\mathbb{G}_m)$-algebras

\[
\begin{array}{c}
K_0(\text{St}/G \times \mathbb{G}_m) \ar[d]^{p_{\text{pt},\ast}} \ar[r]^-{i_{\text{der}}^\ast} & K_0(J_0) \ar[d]^-{\Phi} \\
K_0(\text{St}^{\triangleright}/G \times \mathbb{G}_m) \ar[d]^\sim & K_0(\tilde{N}/G \times \mathbb{G}_m \times_{\text{pt}/G \times \mathbb{G}_m} \tilde{N}/G \times \mathbb{G}_m) \ar[r]^-{i_{\text{der}}^\ast \circ p_\ast} & K_0(\mathcal{B}^{\triangleright} \times \mathcal{B}^{\triangleright})/G \times \mathbb{G}_m) \\
& \ar[ul]^-{\phi_0} \ar[r]^-{\phi_1 \otimes \text{id}_G} & J_0 \otimes \mathbb{Z} \mathcal{A}\\
\end{array}
\]

commutes.

We will first describe the middle morphism on the $K$-theory of the classical schemes. Consider the diagrams

\[
\begin{array}{ccc}
\tilde{N}/G \times \mathbb{G}_m \ar[d]^-{\Delta} & \ar[l]^-{\pi_\Delta} & \mathcal{B}/G \times \mathbb{G}_m^{\triangleright} \ar[d]^-{\Delta} \\
\text{St}/G \times \mathbb{G}_m^{\triangleright} & \ar[l]^-{\tilde{\rho}} \ar[d]^-{\Delta} & \mathcal{B}/G \times \mathbb{G}_m \times_{\text{pt}/G \times \mathbb{G}_m} \mathcal{B}/G \times \mathbb{G}_m, \tag{16}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{B}/G \times \mathbb{G}_m^{\triangleright} \times_{\text{pt}/G \times \mathbb{G}_m} \tilde{N}/G \times \mathbb{G}_m \ar[d] & \ar[r]^-{\text{id} \times m} & \mathcal{B}/G \times \mathbb{G}_m \times_{\text{pt}/G \times \mathbb{G}_m} \mathcal{B}/G \times \mathbb{G}_m, \\
\end{array}
\]
which is Cartesian, and the diagram

\[
\begin{array}{c}
\left(\mathbb{P}^1 \times \mathbb{P}^1\right)/G \times \mathbb{G}_m \\
\downarrow \quad \downarrow
\end{array}
\begin{array}{c}
\left(T^* \mathbb{P}^1 \times \mathbb{P}^1\right)/G \times \mathbb{G}_m \\
\downarrow \quad \downarrow
\end{array}
\begin{array}{c}
\left(T^* \mathbb{P}^1 \times S^\vee T^* \mathbb{P}^1\right)/G \times \mathbb{G}_m \\
\downarrow \quad \downarrow
\end{array}
\begin{array}{c}
\left(\mathbb{P}^1 \times \mathbb{P}^1\right)/G \times \mathbb{G}_m
\end{array}
\]

(17)

which relates to the case \(G = \text{SL}_2\). Put

\[\mathcal{O}_\lambda := [\iota_\Delta^* \pi^*_\Delta \mathcal{O}_{\mathcal{G}_\mathcal{R}}(\lambda)].\]

**Lemma 7.** We have

\[\iota^* \circ \bar{\mu}_*(\mathcal{O}_\lambda) = \Delta_* \mathcal{O}_{\mathcal{G}_\mathcal{R}}(\lambda),\]

and in the case when \(G = \text{SL}_2\), we have

\[\iota^* \circ \bar{\mu}_*(-q^{1/2} j_* \mathcal{O}(0, -2)) = -q^{\frac{1}{2}} \mathcal{O}(0, -2) + q^{-\frac{1}{2}} \mathcal{O}(0, 0).\]

**Proof.** By diagram (17)

\[\iota^* \circ \bar{\mu}_*(-q^{1/2} j_* \mathcal{O}(0, -2)) = -q^{\frac{1}{2}} \iota^* \iota_* \mathcal{O}(0, -2) = -q^{\frac{1}{2}} \lambda \otimes \mathcal{O}(0, -2)\]

by [CG97], Lemma 5.4.9, where \(\lambda = [\text{Sym}(\mathbb{P}^1 \times T^* \mathbb{P}^1[1])] = [\mathcal{O}(0, 0)] - q^{-1}[\mathcal{O}(0, 2)].\) Here we have multiplied \(\mathcal{O}(0, 2)\) by the character \(q^{-1}\), giving its fibres the trivial \(\mathbb{C}^\times\)-action, which restores equivariance of the complex defining the class \(\lambda\). (When confronted with a linear map \(V \to W\) where \(W\) has trivial \(\mathbb{C}^\times\)-action and \(V\) is scaled by a character, one restores equivariance by tensoring \(V\) with the inverse character.) Thus we have

\[\iota^* \circ \bar{\mu}_*(-q^{1/2} j_* \mathcal{O}(0, -2)) = -q^{\frac{1}{2}} \mathcal{O}(0, 0) - q^{-1} \mathcal{O}(0, 2)) \otimes \mathcal{O}(0, -2) = -q^{\frac{1}{2}} \mathcal{O}(0, -2) + q^{-\frac{1}{2}} \mathcal{O}(0, 0).\]

To prove the second formula, apply base-change diagram (16) and use that, according to the Thom isomorphism theorem, \(((\text{id} \times \pi)^*)^{-1} = \iota^*\). Then we have

\[\iota^* \circ \bar{\mu}_*(\mathcal{O}_\lambda) = \iota^*(\bar{\mu} \circ \iota_\Delta)_* \pi^*_\Delta \mathcal{O}(\lambda).\]

By base-change, we have \((\text{id} \times \pi)^* \Delta_* = (\bar{\mu} \circ \iota_\Delta)_* \pi^*_\Delta\), hence \(\Delta_* = ((\text{id} \times \pi)^*)^{-1}(\bar{\mu} \circ \iota_\Delta)_* \pi^*_\Delta\). \(\square\)

**Proof of Proposition 2.** That the bottom square commutes is the combination of the main results of [Xi16] and [Nie11].

By Proposition 1, the morphism \(i^*\text{der}\) induces a morphism on \(K\)-theory as above. Hence by Lemma 5 and the above discussion, all the morphisms in the diagram are well-defined morphisms of algebras.

We first show the diagram commutes on the Bernstein subalgebra. Recalling from Section 3.3 that the diagonal component of the Steinberg variety is a classical rather than a derived scheme, and so \(\mathcal{O}_\lambda\) is naturally an element of \(K_0(\text{St}/G \times \mathbb{G}_m)\) for which \(p_{\text{St}}_* \mathcal{O}_\lambda = \mathcal{O}_\lambda\) with the right-hand side regarded as an object of \(K_0(\text{St}^\vee/G \times \mathbb{G}_m)\). Then by Lemma 7, it suffices to show that \(\Phi([i^* \mathcal{O}_\lambda]) = [\Delta_* \mathcal{O}_{\mathcal{G}_\mathcal{R}}(\lambda)]\). Indeed, though, we have

\[i^* (\mathcal{O}_\lambda) = (\mathcal{O}_{\Delta \mathcal{G}}(\lambda)),\]

as the structure sheaf of the diagonal pulls back to the structure sheaf of the diagonal. By Lemma 6 we have \(\Phi([\mathcal{O}_{\Delta \mathcal{G}}]) = [\mathcal{O}_{\Delta \mathcal{G}^\circ}],\) and likewise for the twists. Thus the diagram commutes for the Bernstein subalgebra.
Consider the diagram
\[
\begin{array}{c}
(\mathcal{B} \times \mathcal{B} \times \text{Spec}(\text{Sym}(\mathfrak{g}[1]))) / G \times \mathbb{G}_m \\
\downarrow \phi \\
(\mathcal{B} \times \mathcal{B}) / G \times \mathbb{G}_m \\
\downarrow \pi_{\text{der}} \\
\text{pt}/G \times \mathbb{G}_m \\
\downarrow \\
g / G \times \mathbb{G}_m.
\end{array}
\]
(18)

Note that, by extending (6), the bottom square is Cartesian. Moreover, we have
\[
(\mathcal{B} \times \mathcal{B}) / G \times \mathbb{G}_m \times_{g/G \times \mathbb{G}_m} \text{pt}/G \times \mathbb{G}_m = (\mathcal{B} \times \mathcal{B}) / G \times \mathbb{G}_m \times_{\text{pt}/G \times \mathbb{G}_m} \text{pt}/G \times \mathbb{G}_m \times_{g/G \times \mathbb{G}_m} \text{pt}/G \times \mathbb{G}_m
\]
where we identify \(g\) with \(g^\ast\) via the Killing form. Therefore the large square is also Cartesian, and so the upper square is also Cartesian.

We will use this to explain how to compute \(i_{\text{der}}^! j_* \mathcal{F}\) for any coherent sheaf \(\mathcal{F}\) on \(\mathcal{B} \times \mathcal{B}\). Pullback by the top map sends \(\mathcal{F} \mapsto \mathcal{F} \otimes \text{Sym}(\mathfrak{g}[1])\). By (3), the map \(p\) is given by the identity on classical truncations, together with the map of cdgas which is pointwise the obvious map \(\text{Sym}(\mathfrak{b}[1]) \to \text{Sym}(\mathfrak{g}[1])\) given by including a Borel subalgebra \(\mathfrak{b} \hookrightarrow \mathfrak{g}\). Pushforward by \(r\) corresponds pointwise to equipping the module structure given by
\[
\text{Sym}(\mathfrak{b}[1]) \otimes \text{Sym}(\mathfrak{b}_2[1]) \to \text{Sym}(\mathfrak{g}[1]).
\]

Applying the functor \((p \times \text{id})_*\) forgets the \(\text{Sym}(\mathfrak{b}[1])\)-action, and then applying \((\text{id} \times i)^\ast\) quotients by the remaining \(\text{Sym}(\mathfrak{b}_2[1])\)-action. That is, fibrewise we have
\[
\mathbb{C} \otimes_{\text{Sym}(\mathfrak{b}_2[1])} \text{Sym}(\mathfrak{g}[1]) \otimes \mathcal{F}_{\mathfrak{b}_1, \mathfrak{b}_2} = \text{Sym}(\mathfrak{g}/\mathfrak{b}_2[1]) \otimes \mathcal{F} = \text{Sym}(\mathfrak{n}_2[1]) \otimes \mathcal{F}_{\mathfrak{b}_1, \mathfrak{b}_2},
\]
where \(\mathfrak{n}_2\) is the radical of \(\mathfrak{b}_2\). Comparing with Lemma 7 and the first sentence of its proof, this says precisely that the diagram commutes when \(G = \text{SL}_2\).

Returning to general \(G\), let \(s\) be a simple reflection. Let \(Y_s \subset \mathcal{B} \times \mathcal{B}\) be the closure of the \(G\)-orbit labelled by \(s\), and let
\[
\pi_s : T^\ast_{Y_s} (\mathcal{B} \times \mathcal{B}) \to Y_s
\]
be the conormal bundle. Put \(\mathcal{Q}_s = \pi_s^\ast \Omega^1_{Y_s/\mathcal{B} \times \mathcal{B}}\). Then by equation 7.6.34 in [CG97], it suffices to show that
\[
\Phi (i^\ast s_{\text{St}}^! \mathcal{Q}_s) = \mathcal{Q} \otimes \mathcal{Q} \otimes i_{X_s^\ast} (\mathfrak{g}/\mathfrak{g}[2]) - \mathcal{Q},
\]
where \(i_{X_s}\) is the inclusion of the Schubert variety \(X_s \cong \mathbb{P}^1\) into \(\mathcal{B} \times \mathcal{B}\). That is, the image of \(\mathcal{Q}_s\) is just the pushforward of the answer in the \(\text{SL}_2\) case. But it is clear that this is indeed the case.

We have now nearly proved

**Proposition 3.** For \(G = \text{SL}_2\) or \(\text{SL}_3\), there exists a family of objects \(\{t_w\}_{w \in \mathfrak{e}_0}\) in \(\mathfrak{j}_0\), such that that if \(t_w t_x = \sum z \gamma_{w\cdot x, z} t_z\) in \(J_0\), then
\[
t_w \ast t_x = \bigoplus_z t_z^{\gamma_{w, x, z} - 1}
\]
in \(J_0\) and such that \(\Phi([t_w]) = [t_w]\).

**Proof.** The discussion in Section 3.4.3 proves all but the last statement of the proposition. Finally, by Proposition 2, if \(w = f w_0 g^{-1}\) we have
\[
\Phi ([t_w]) = \Phi ([\mathcal{F}_f] \boxtimes [p^\ast \mathcal{Q}_f]) = [\mathcal{F}_f] \boxtimes i_{\text{St}}^* p^\ast [\mathcal{Q}_f] = [\mathcal{F}_f] \boxtimes [\mathcal{Q}_f].
\]
The general claim follows by linearity over \(K_0(\text{pt}/G)\) and the parametrization in Section 3.4.3. □
4.2 The Schwartz space of the basic affine space

Let $F$ be a non-archimedean local field and $G^\vee$ be a reductive group defined over $F$ dual to $G = G$. The Schwartz space of the basic affine space $\mathcal{S}$ was defined by Braverman-Kazhdan in [BK99] to organize the principal series representations of $G^\vee(F)$ in a way insensitive to the poles of intertwining operators. In [BK18], Braverman and Kazhdan gave the following description of $J_0$ in terms of the Iwahori-invariants $\mathcal{S}^{I}$ of $\mathcal{S}$. In loc. cit. it was shown that $\mathcal{S}^{I}$ is isomorphic to $K_0(\mathcal{B}/T \times \mathcal{G}_m^\vee)$ as an $\mathcal{H} \otimes \mathbb{C}[\hat{W}]$-module, and in [BK18], it was proven that $J_0 \simeq \text{End}_{\mathcal{H}}(\mathcal{S}^{I})$, where the action of $\hat{W}$ is as defined in loc. cit.

Example 3. Let $G^\vee = \text{SL}_2$, with $\hat{W} = W \ltimes X = \langle s_0, s_1 \mid s_0^2 = s_1^2 = 1 \rangle$, where $X_\ast$ is the cocharacter lattice of $G^\vee$, and $s_0$ is the finite simple reflection. Then we have that $K_0(\mathcal{B}/T \times \mathcal{G}_m)$ has basis $\{[\mathcal{O}_{\mathcal{P}_1}], [\mathcal{O}_{\mathcal{P}_2}(-1)]\}$, and we have $t_{s_0} = [\mathcal{O}_{\mathcal{P}_1}] \boxtimes [\mathcal{O}_{\mathcal{P}_2}]$ and $t_{s_1} = [\mathcal{O}(-1)] \boxtimes [\mathcal{O}(-1)][1]$ under the identification in Lemma 3. The basis elements corresponding to the two distinguished involutions in $\mathfrak{c}_0$ act by projectors, with $t_{s_0}$ preserving $\mathcal{O}_{\mathcal{P}_1}$ and killing $\mathcal{O}_{\mathcal{P}_2}(-1)$, and vice-versa for $t_{s_1}$.

Recalling that $K_0(\mathcal{B}/G \times \mathcal{G}_m^\vee \times pt/G \times \mathcal{G}_m \mathcal{B}/G \times \mathcal{G}_m^\vee) \simeq K_0(\mathcal{B}/T \times \mathcal{G}_m^\vee)$, we see that we have two natural coherent categorifications of $\mathcal{S}^{I}$, and that $J_0$ acts on both of them:

**Proposition 4.** The category $\mathcal{J}_0$ acts on $\text{Coh}(\mathcal{B}/G \times \mathcal{G}_m^\vee \times pt/G \times \mathcal{G}_m \mathcal{B}/G \times \mathcal{G}_m^\vee)$ and on $\text{Coh}(\mathcal{B}/T \times \mathcal{G}_m^\vee)$.

**Proof.** This is a port of Proposition 1; the proof that $\mathcal{F} \star \mathcal{S}$ is coherent if $\mathcal{F}, \mathcal{S} \in \mathcal{J}_0$ used nothing about SingSupp($\mathcal{F}$). The proof for $\text{Coh}(\mathcal{B}/T \times \mathcal{G}_m^\vee)$ is entirely similar. \qed

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