Cascades, Order and Ultrafilters

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Abstract

We investigate mutual behavior of cascades, contours of which are contained in a fixed ultrafilter. Using that relation we prove (ZFC) that the class of strict $J_{\omega}$-ultrafilters, introduced by J. E. Baumgartner in Ultrafilters on $\omega$, is empty. We translate the result to the language of $<\omega$-sequences under an ultrafilter, investigated by C. Laflamme in A few special ordinal ultrafilters, to show that if there is an arbitrary long finite $<\omega$-sequence under $u$ than $u$ is at least strict $J_{\omega^{+}}$-ultrafilter.

Keywords: ordinal ultrafilters, monotone sequential cascades

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1. Introduction

Baumgartner in the article Ultrafilters on $\omega$ (1) introduced a notion of $\mathcal{I}$-ultrafilters: Let $\mathcal{I}$ be an ideal on $X$, an ultrafilter (on $\omega$) is an $\mathcal{I}$-ultrafilter, if and only if, for every function $f : \omega \to X$ there is a set $U \in u$ such that $f[U] \in \mathcal{I}$. This kind of ultrafilters was studied by large group of mathematician. We shall mention only the most important papers in this subject from our point of view: J. Brendle [3], C. Laflamme [13], Shelah [14], [15], Blasscycz [2]. Among other types of ultrafilters J. E. Baumgartner introduced ordinal ultrafilters, precisely $\omega_1$ sequence of classes of ultrafilters. We say that $u$ is $J_{\alpha}$ ultrafilter (on $\omega$) if for each function $f : \omega \to \omega_1$ there is $U \in u$ such that $\text{ot}(f(U)) < \alpha$, where $\text{ot} (\cdot)$ denotes the order type. For additional information about ordinal ultrafilters a look at [1], [3], [18] is recommended. In [1] J. E. Baumgartner proved (in Theorems 4.2 and 4.6) that for each successor ordinal $\alpha < \omega_1$ the class of strict $J_{\omega^\alpha}$-ultrafilters (see below) is nonempty if P-points exist, he also pointed out that: “In general we do not know, whether, if $\alpha$ is limit, there is a $J_{\omega^\alpha}$-ultrafilter that is no $J_{\beta}$-ultrafilter, for some $\beta < \omega^\alpha$, even if CH or MA assumed”. Here, such ultrafilters we call strict $J_{\omega}$-ultrafilters, and we partially solve the problem, showing (ZFC) that the class of strict $J_{\omega}$-ultrafilters is empty.

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If $u$ is a filter(base) on $A \subset B$, then we identify $u$ with the filter on $B$ for which $u$ is a filter-base. Let $u, v$ be ultrafilters on $\omega$, recall that $v < \infty u$ if there is a function $f : \omega \to \omega$ such that $f(u) = v$ and $f$ is not finite-to-one or constant on any set $U \in u$. In [13] C. Laflamme proved (reformulation of Lemma 3.2) that if an ultrafilter $u$ has an infinite decreasing $<_{\infty}$-sequence below, then $u$ is at least strict $J_{\omega+1}$-ultrafilter. He also stated the following

[13] Open Problem 1) What about the corresponding influence of increasing $<_{\infty}$-chains below $u$? Given such an ultrafilter $u$ with an increasing infinite $<_{\infty}$-sequence $u >_{RK} \ldots >_{\infty} u_1 >_{\infty} u_0$ below, fix maps $g_i$ and $f_i$ witnessing $u >_{RK} u_i$ and $u_{i+1} >_{\infty} u_i$ respectively. The problem is really about the possible connections between $g_i$ and $f_i \circ g_{i+1}$ even relative to members of $u$.

[13] Open Problem 2) Can we have an ultrafilter $u$ with arbitrary long finite $<_{\infty}$-chains below $u$ without infinite one? This looks like the most promising way to build a strict $J_{\omega}$-ultrafilter.

We find affirmative answer to the first problem and negative answer to the second one.

2. Preliminaries

In [6] S. Dolecki and F. Mynard introduced monotone sequential cascades - special kind of trees - as a tool to describe topological sequential spaces. Cascades and their contours appeared to be also a useful tool to investigate certain types of ultrafilters on $\omega$, namely ordinal ultrafilters and P-hierarchy (see [18], [17]), here we focus on the first of them.

The cascade is a tree $V$, ordered by "$\subseteq$", without infinite branches and with a least element $\emptyset V$. A cascade is sequential if for each non-maximal element of $V$ ($v \in V \setminus \text{max} V$) the set $v + V$ of immediate successors of $v$ (in $V$) is countably infinite. We write $v^+$ instead of $v + W$ if it is known in which cascade the successors of $v$ are considered. If $v \in V \setminus \text{max} V$, then the set $v^+$ (if infinite) may be endowed with an order of the type $\omega$, and then by $(v_n)_{n<\omega}$ we denote the sequence of elements of $v^+$, and by $v_n W$ - the $n$-th element of $v + W$. We say that $v$ is a predecessor of $v'$ (we write $v = \text{pred}(v')$) if $v' \in v^+$.

The rank of $v \in V$ ($r_V(v)$ or $r(v)$) is defined inductively as follows: $r(v) = 0$ if $v \in \text{max} V$, and otherwise $r(v)$ is the least ordinal greater than the ranks of all immediate successors of $v$. The rank $r(V)$ of the cascade $V$ is, by definition, the rank of $\emptyset V$. If it is possible to order all sets $v^+$ (for $v \in V \setminus \text{max} V$) so that for each $v \in V \setminus \text{max} V$ the sequence $(r(v_n)_{n<\omega})$ is non-decreasing (other words if for each $v \in V \setminus \emptyset V$ the set $\{v' \in (\text{pred}(v))^+ : r(v') < \alpha\}$ is finite for each $\alpha < r(v)$), then the cascade $V$ is monotone, and we fix such an order on $V$ without indication. Thus we introduce lexicographic order $\leq_{\text{lex}}$ on $V$ in the following way: $v' \leq_{\text{lex}} v''$ if $v' \supseteq v''$ or if there exist $v, v' '$ such that $v' \in v^+$ and $v'' \in v'^+$ such that $v' \subseteq v'$ and $v'' \subseteq v''$ such that $v', v' ' \in v^+$ and $v'' = v_n$, $v' ' = v_m$ and $n < m$.

Let $W$ be a cascade, and let $\{V_w : w \in \text{max} W\}$ be a set of pairwise disjoint cascades such that $V_w \cap W = \emptyset$ for all $w \in \text{max} W$. Then, the confluence of cascades $V_w$ with respect to the cascade $W$ (we write $W \not\leftrightarrow V_w$) is defined as a
cascade constructed by the identification of \( w \in \max W \) with \( \emptyset \mathcal{V}_w \) and according to the following rules: \( \emptyset_W = \emptyset_{W + \mathcal{V}_W} \); if \( w \in W \setminus \max W \), then \( w + W + \mathcal{V}_w = w + W \); if \( w \in V_{u_0} \) (for a certain \( u_0 \in \max W \)), then \( w + W + \mathcal{V}_w = w + \mathcal{V}_{u_0} \); in each case we also assume that the order on the set of successors remains unchanged. By \((n) \mapsto V_n\) we denote \( W \mapsto V_w \) if \( W \) is a sequential cascade of rank 1.

Also we label elements of a cascade \( V \) by sequences of naturals of length \( r(V) \) or less, by the function which preserves the lexicographic order, \( v_l \) is a resulting name for an element of \( V \), where \( l \) is the mentioned sequence (i.e. \( v_{l-n} = (v_l)_n \in v^+ \)); by \( V_l \) we denote \( v_l^+ \) and by \( L_{\alpha,V} \) we understand \( \{ l \in \omega^{<\omega} : r_V(v_l) = \alpha \} \). Let \( v, v' \in V \), we say that \( v' \) is a predecessor of \( v \) (in \( V \)) if \( v \in v'^{+} \), we write \( v' = \text{pred}_V(v) \). For a finite sequence \( l = (n_0, \ldots, n_k) \) of natural numbers by \( l^- \) we denote a sequence \( l \) with the last element removed, i.e. \( l^- = (n_0, \ldots, n_{k-1}) \); by \( l^{+} \) we denote a set of all sequences \( l' \) such that \( l'^- = l \).

If \( \mathcal{U} = \{u_s : s \in S\} \) is a family of filters on \( X \) and if \( p \) is a filter on \( S \), then the contour of \( \{u_s\} \) along \( p \) is defined by

\[
\int_p \mathcal{U} = \int_p u_s = \bigcup_{p \in p} \bigcap u_s.
\]

Such a construction has been used by many authors (3, 9, 10) and is also known as a sum (or as a limit) of filters. On the sequential cascade, we consider the finest topology such that for all but the maximal elements \( v \) of \( V \), the co-finite filter on the set \( v^+ \) converges to \( v \). For the sequential cascade \( V \) we define the contour of \( V \) (we write \( \int V \)) as the trace on \( \max V \) of the neighborhood filter of \( \emptyset_V \) (the trace of a filter \( u \) on a set \( A \) is the family of intersections of elements of \( u \) with \( A \)). Similar filters were considered in 11, 12, 3. Let \( V \) be a monotone sequential cascade and let \( u = \int V \). Then the rank \( r(u) \) of \( u \) is, by definition, the rank of \( V \). It was shown in 7 that if \( \int V = \int W \), then \( r(V) = r(W) \).

Let \( S \) be a countable set. A family \( \{u_s\}_{s \in S} \) of filters is referred to as discrete if there exists a pairwise disjoint family \( \{U_s\}_{s \in S} \) of sets such that \( U_s \in u_s \) for each \( s \in S \). For \( v \in V \) we denote by \( v^+ \) a subcascade of \( V \) built by \( v \) and all successors of \( v \). If \( U \subseteq \max V \) and \( U \subseteq \int V \), then by \( U_{\int V} \) we denote the biggest (in the set-theoretical order) monotone sequential subcascade of \( V \).\( \mathcal{V} \) built of some \( v \in V \) such that \( U \cap \max v^+ \neq \emptyset \). We write \( v^+ \) and \( U^+ \) instead of \( v^{\int V} \) and \( U^{\int V} \) if we know in which cascade the subcascade is considered. The reader may find more information about monotone sequential cascades and their contours in 3, 6, 11, 16, 17, 18.

In the remainder of this paper each filter is considered to be on \( \omega \), unless indicated otherwise.

3. Existence of ordinal ultrafilters

For a monotone sequential cascade \( V \) by \( f_V \) we denote an lexicographic order respecting function \( \max V \rightarrow \omega_1 \), i.e., such a function that \( v' <_{\text{lex}} v'' \) iff \( f_V(v') < f_V(v'') \).
Let $V$ and $W$ be monotone sequential cascades such that $\max V \supset \max W$. We say that $W$ increases the order of $V$ (we write) $W \Rightarrow V$ if $\ot (f_W(U)) \geq \text{id}(\ot (f_V(U)))$ for each $U \subset \max W$, where $\text{id}(\alpha)$ is the biggest indecomposable ordinal less than, or equal to $\alpha$; by Cantor normal form theorem such a number exists and is defined uniquely. Clearly this relation is idempotent and transitive. Although relation of increasing of order says that one cascade is somehow bigger then another, this relation is quite independent with the containment of contours.

Example 3.1. $(\forall T \supset \forall V \not\Rightarrow T \Rightarrow V)$ Let $(V_n)_{n \in \omega}$ be a sequence of pairwise disjoint monotone sequential cascades of rank 2. For each $n < \omega$ choose $v_n$ - an arbitrary element of $\max V_n$. Let $V_n' = V_n \setminus \{v_n\}$ and let $V'_n$ be an arbitrary monotone sequential cascade of rank 2 such that $\max V'_n = \bigcup_{n < \omega} \{v_n\}$. Now put $T = (n) \not\Rightarrow_{n < \omega} V_n$ and $V = (n) \not\Rightarrow_{n < \omega} V'_n$.

Example 3.2. $(T \Rightarrow V \not\Rightarrow \forall T \supset \forall V)$ Let $(V_n)_{n \in \omega}$ be a sequence of pairwise disjoint monotone sequential cascades of rank 1. For each $n < \omega$ choose $v_n$ - an arbitrary element of $\max V_n$. Let $(B_n)_{n < \omega}$ be a partition of $\bigcup_{n < \omega} \{v_n\}$ into infinite sets. Let $V'_n$ be a monotone sequential cascade of rank 1, such that $\max V'_n = (\max V_n \setminus \{v_n\}) \cup B_n$. Put $T = (n) \not\Rightarrow V'_n$ and $V = (n) \not\Rightarrow V'_n$.

Let $V$ and $W$ be monotone sequential cascades, let $f : V \rightarrow W$ be a finite-to-one, $\subseteq_V$ order preserving surjection such that $F_{\max v \cup \emptyset v} = \text{id}_{\max v \cup \emptyset v}$, and $v \in f^{-1}(v)$ for each $V \in V$. Then, it is easy to see, that $V \Rightarrow f(V)$ and $f(V) \Rightarrow V$, we call this property locally finite partition property (LFPP)

Let $u, p$ be filters on $\omega$, then $u \cup p$ we define as $\{x \in \omega : \text{there exist } U \in u \text{ and } P \in p \text{ such that } U \cap P \subset X\}$.

Let $u$ be an ultrafilter and let $V, W$ be monotone sequential cascades such that $\int V \subset u$ and $\int W \subset u$. Then we say that rank $\alpha$ in cascade $V$ agree with rank $\beta$ in cascade $W$ with respect to the ultrafilter $u$ if for any choice of $\hat{V}_p, s \in f V_p$ and $\hat{W}_p, s \in f W_s$ there is: $\bigcup_{(p, s) \in L_{\alpha, V} \times L_{\beta, W}} (\hat{W}_p, s \cap \hat{V}_p, s) \in u$; this relation is denoted by $\alpha V E_u \beta W$.

Proposition 3.3. Let $u$ be an ultrafilter and let $V, W$ be monotone sequential cascades such that $\int V \subset u$ and $\int W \subset u$. Then $1_V E_u 1_W$ and $r(V) V E_u r(W) W$.

Proof. First suppose that $r(V) = 1$ or $r(W) = 1$, say $r(V) = 1$. Clearly card $(L_{1, V}) = 1$. For each $(p, s) \in L_{1, V} \times L_{1, W}$ take any $\hat{V}_p, s \in f V_p$ and any $\hat{W}_p, s \in f W_s$. Notice that since $\int V$ is a co-finite filter on $\max V$ thus for each $(p, s) \in L_{1, V} \times L_{1, W}$ the set $(\hat{W}_p, s \cap \max V) \setminus (\hat{W}_p, s \cap \hat{V}_p, s)$ is finite. Therefore there exist $\hat{W}_p, s \subset \hat{W}_p, s, \hat{W}_p, s \in f W_s$ such that $\hat{W}_p, s \cap \hat{V}_p, s = \hat{W}_p, s \cap \max V$. Thus $\bigcup_{(p, s) \in L_{1, V} \times L_{1, W}} (\hat{W}_p, s \cap \hat{V}_p, s) \supset \bigcup_{(p, s) \in L_{1, V} \times L_{1, W}} (\hat{W}_p, s \cap \hat{V}_p, s) = \bigcup_{(p, s) \in L_{1, V} \times L_{1, W}} (\hat{W}_p, s) \cap \max V$. Since $\bigcup_{(p, s) \in L_{1, V} \times L_{1, W}} (\hat{W}_p, s) \in f W \subset u$ and $\max V \in f V \subset u$ thus $\bigcup_{(p, s) \in L_{1, V} \times L_{1, W}} (\hat{W}_p, s \cap \hat{V}_p, s) \in u$.

Before we deal with case $r(V) \geq 2$ and $r(W) \geq 2$ we state the following claim: in assumption of this Proposition, if a set $U$ is such that for each $(p, s) \in L_{1, V} \times \mathbb{N}$
The intersection $U \cap \max V_p \cap \max W_s$ is finite, then $U \not\subset u$. Let $f : \omega \rightarrow L_{1,V}, h : \omega \rightarrow L_{1,W}$ be bijections, and let $G(i, j) = U \cap \max V_{f(i)} \cap \max W_{h(j)}$ for $i, j \in \omega$. Let $\Delta^\geq = \{(i, j) : i \geq j\}, \Delta^\leq = \{(i, j) : i \leq j\}$. Since $u$ is an ultrafilter thus either $G(\Delta^\geq) \in u$ or $G(\Delta^\leq) \in u$. But $G(\Delta^\geq)$ is finite on each $\max V_p$ for $p \in L_{1,V}$ and so $(G(\Delta^\geq))^c \in \int V$ therefore $G(\Delta^\geq) \not\subset u$. Also $G(\Delta^\leq)$ is finite on each $\max W_s$ for $s \in L_{1,W}$ and so $(G(\Delta^\leq))^c \in \int W$ therefore $G(\Delta^\leq) \not\subset u$.

Now let $r(V) \geq 2$ and $r(W) \geq 2$ and suppose on the contrary that $K = \bigcup_{(p,s) \in L_{1,V} \times L_{1,W}} (\max V_p \cap \max W_s) \not\subset u$ for some choice of $\max V_p \in \int V$ and $\max W_s \in \int W$. Thus $K^c \subset u$. Put $R^\# = \{(p, s) \in L_{1,V} \times L_{1,W} : \int V_p^\# \not\subset \int W_s\}$ and $R^\# = \{(p, s) \in L_{1,V} \times L_{1,W} : \not\exists p, s \in \int V_p \cap \max W_s\}$. Define $K_1 = \bigcup_{(p,s) \in R^\#} (\max V_p \cap \max W_s) \not\subset u$. Since traces of $\max W_s$ and of $\int V_p$ on max $V_p \cap \max W_s$ are co-finite filters (on max $V_p \cap \max W_s$) thus $K_1 \cap \max V_p \cap \max W_s$ is finite on each $(p, s) \in R^\#$ and empty on each $(p, s) \in R^\#$. By similar reasoning $K_2$ is finite on each $(p, s) \in L_{1,V} \times L_{1,W}$. Therefore by claim above $K_1 \not\subset u$ and $K_2 \not\subset u$ - contradiction. Second statement of Proposition 3.3 is clear.

In the above Proposition 3.3 the inverse of the implication does not hold.

**Example 3.4.** Let $(A_\alpha)_{\alpha \leq 2\omega}$ be a partition of $\omega$ into infinite sets. Let $V_\alpha$ be a monotone sequential cascade of rank 1 such that $\max V_\alpha = A_\alpha$. Let $W = (\alpha) \rightarrow V_\alpha, V = (\alpha) \rightarrow V_\alpha \cup W_\alpha$, and let $u$ be any free ultrafilter containing $A_\omega$. Clearly $\int V \not\subset u$, $\int W \not\subset u$ but $1_V E_{1,W}$.

Although the following Theorem 3.5 is stated using the "$\Rightarrow$" relation, it is worth to look at the proof of it as on the description of possible relations of cascades whose contours are contained in the same ultrafilter, and as a description of operation which leads from such cascades to others whose contours are also contained in the same ultrafilter.

**Theorem 3.5.** Let $u$ be an ultrafilter and let $V, W$ be monotone sequential cascades of finite ranks such that $\int V \subset u$ and $\int W \subset u$. Then $n_{V} E_{u} n_{W}$ implies the existence of a monotone sequential cascade $T$ of rank $\max \{r(V), r(W)\} \leq r(T) \leq r(V) + r(W)$ and such that $\int T \subset u, T \Rightarrow V, T \Rightarrow W, \int V \subset \int T$ and $\int W \subset \int T$.

**Proof.** Before we start the proof, let us make the following remarks: in this theorem we claim (in place of $\int T \subset u$) that (under the same assumption and notation) $\int T \subset (\int V \cup \int W)$, and this formulation also will be used in the proof; cascade $T$ build in this proof has ranks not less then max $\{n, m\} + \max \{r(V) - n, r(W) - m\}$ and not greater then $r(V) + r(W) - 1$ and this is inductively used in the proof. Without loss of generality, we may assume that each branch in $V$ has length $r(V)$ and each branch in $W$ has length $r(W)$, and that $r(V) \leq r(W)$.

We proceed by induction by $r(W)$, and for each $r(W)$ by sub-induction by $r(V)$. First step of induction and of sub-inductions is $r(V) = 1$ and then we
take $T = W^{\max V}$ which clearly fulfills the claim. Assume that the claim is proved for all cascades $V$, $W$ which behave like in assumptions and such that $r(\tilde{V}) \leq r(\tilde{W})$ and either $r(\tilde{W}) < r(W)$ or else $(r(W) = r(\tilde{W})$ and $r(\tilde{V}) < r(V)$).

We consider 3 cases

1) $n < r(V)$ and $m < r(W)$;

2) $(n = r(V)$ and $m < r(W))$ or $(n < r(V)$ and $m = r(W))$;

3) $n = r(V)$ and $m = r(W)$.

Case 1) Let $R^\# = \{(l, s) \in L_{n,v} \times L_{m,w} : \int V^\# \int W_s\}$. Notice that exactly one of the following 3 subcases holds:

1.1) There is $K_{F-F} \subsetneq R^\#$ such that $\text{card} (K_{F-F}(l)) < \omega$ and $\text{card} (K_{F-F}(l)) < \omega$ for each $l \in \text{dom} K_{F-F}$, $s \in \text{rng} K_{F-F}$ and $\bigcup_{(l,s) \in K_{F-F}} (\tilde{V}_{l,s} \cap \tilde{W}_{l,s}) \in u$ for each choice of $\tilde{V}_{l,s} \in \int V_{l,s}$ and of $\tilde{W}_{l,s} \in \int W_{l,s}$;

1.2) $\sim 1.1$ and there is $K_{\infty-F} \subsetneq R^\#$ such that $\text{card} (K_{\infty-F}(s)) < \omega$ for each $s \in \text{rng} K_{\infty-n}$ and $\bigcup_{(l,s) \in K_{\infty-F}} (\tilde{V}_{l,s} \cap \tilde{W}_{l,s}) \in u$ for each choice of $\tilde{V}_{l,s} \in \int V_{l,s}$ and of $\tilde{W}_{l,s} \in \int W_{l,s}$;

1.3) $\sim 1.1$ and there is $K_{\infty-F} \subsetneq R^\#$ such that $\text{card} (K_{\infty-F}(l)) < \omega$ for each $l \in \text{dom} K_{\infty-F}$ and $\bigcup_{(l,s) \in K_{\infty-F}} (\tilde{V}_{l,s} \cap \tilde{W}_{l,s}) \in u$ for each choice of $\tilde{V}_{l,s} \in \int V_{l,s}$ and of $\tilde{W}_{l,s} \in \int W_{l,s}$.

Let $\Delta^\leq = \{(i,j) : i \geq j; i,j \in \omega\}$ and $\Delta^\geq = \{(i,j) : i \leq j; i,j \in \omega\}$. Let $p: \omega \to L_{n,v}$ and $q: \omega \to L_{m,w}$ be bijections. For $X \subseteq \omega \times \omega$ define $G(X) = \bigcup_{(i,j) \in X} (\omega \times \omega \times \omega$ and $\Delta^\leq \cup G(\Delta^\leq) \in u$ thus either $G(\Delta^\leq) \in u$ (case 1.1 or 1.2) or $G(\Delta^\geq) \in u$ (case 1.1 or 1.3).

Since $u$ is an ultrafilter thus without loss of generality (by LFPP, for case 1.1 used twice, its property of increasing order and transitivity of "$\Rightarrow$" relation) exactly one of the following subcases holds.

1.1') There is $K_{-1-1} \subsetneq R^\#$ such that $\text{card} (K_{-1-1}(l)) = 1$ and $\text{card} (K_{-1-1}(s)) = 1$ for each $(l, s) \in K_{-1-1} \cap \bigcup_{(l,s) \in K_{-1-1}} (\tilde{V}_{l,s} \cap \tilde{W}_{l,s}) \in u$ for each choice of $\tilde{V}_{l,s} \in \int V_{l,s}$ and of $\tilde{W}_{l,s} \in \int W_{l,s}$;

1.2') $\sim 1.1$ and there is $K_{-1-1} \subsetneq R^\#$ such that $\text{card} (K_{-1-1}(s)) = 1$ and $\text{card} (K_{-1-1}(l)) = \omega$ for each $(l, s) \in K_{-1-1} \cap \bigcup_{(l,s) \in K_{-1-1}} (\tilde{V}_{l,s} \cap \tilde{W}_{l,s}) \in u$ for each choice of $\tilde{V}_{l,s} \in \int V_{l,s}$ and of $\tilde{W}_{l,s} \in \int W_{l,s}$;

1.3') $\sim 1.1$ and there is $K_{-1-1} \subsetneq R^\#$ such that $\text{card} (K_{-1-1}(l)) = 1$ and $\text{card} (K_{-1-1}(s)) = \omega$ for each $(l, s) \in K_{-1-1} \cap \bigcup_{(l,s) \in K_{-1-1}} (\tilde{V}_{l,s} \cap \tilde{W}_{l,s}) \in u$ for each choice of $\tilde{V}_{l,s} \in \int V_{l,s}$ and of $\tilde{W}_{l,s} \in \int W_{l,s}$.

Subcase 1.1') Without loss of generality, we may assume that $\max V = \max W = H(K_{-1-1})$. Define a series of sets: $R = L_{n,v} \times L_{m,w}$, $R(A) = \{(l, s) \in R : \text{card} (\max V_l \cap \max W_s \cap A) = \omega\}$ for $A \subseteq \omega$, $\tilde{u} = \{R(U) : \bar{U} \in \bar{u}\}$, $\tilde{V} = \{v \in V : r(v) > \max V\}$, $\tilde{W} = \{w \in W : r(w) > \max W\}$; On $\tilde{V}$ we define order $\subseteq_{\tilde{V}}$ by: if $v_1, v_2 \in \tilde{V}$, $(l, s) \in R$ then: $v_1 \subseteq_{\tilde{V}} v_2$ iff $v_1 \subseteq_{\tilde{V}} v_2$; $v_1 \subseteq_{\tilde{V}} (l, s)$ iff $\text{card} (\max V_l \cap \max W_s \cap \max V_l \cap V) = \omega$. On $\tilde{W}$ we introduce order in the analogous way. Notice that $\tilde{u}$ is an ultrafilter.
on $R$ and that $\tilde{V}$ and $\tilde{W}$ are monotone sequential cascades (on $R$) and that $r(\tilde{V}) = r(V) - n$, $r(\tilde{W}) = r(W) - m$ and that $\int \tilde{V} \subset \hat{u}$ and $\int \tilde{W} \subset \hat{u}$.

By inductive assumption there is $T$ monotone sequential cascade (on $R$) of rank $\max \{r(\tilde{V}) - n, r(\tilde{W}) - m\} \leq r(T) \leq r(\tilde{V}) + r(\tilde{W}) - n - m - 1$ and such that $\int T \subset \hat{u}$ and $T \supset \tilde{V}$, and $T \supset \tilde{W}$, also by inductive assumption, for each $(l, s) \in R^g$ there is a monotone sequential cascade $T_{l,s}$ of rank $\max \{n, m\} \leq r(T_{l,s}) \leq n + m - 1$ such that $T_{l,s} \supset \tilde{V}_l$, $T_{l,s} \supset \tilde{W}_s$, $\int T_{l,s} \subset \int V_l \lor \int W_s$. Define $\hat{T} = \hat{T} + \{\{(l, s) : \max V_i (\max W_s : \max \hat{T}) \} T_{l,s}$. Take any $A \in \int \hat{T}$, thus there exist $\hat{A} \in \int \hat{T}$ and $A_{l,s} \in \int T_{l,s}$ such that $A = \bigcup_{(l, s) \in A} A_{l,s}$. Since $A_{l,s} \supset \tilde{V}_l \lor \tilde{W}_s$ for some $\tilde{V}_l \in \int V_l$ and $\tilde{W}_s \in \int W_s$ so $A \supset \bigcup_{(l, s) \in \hat{A}} (\tilde{V}_l : \land \tilde{W}_s) = \bigcup_{(l, s) \in R} (\tilde{V}_l : \land \tilde{W}_s) \cap \bigcup_{(l, s) \in A} (\max V_i : \land \max W_s)$. Since $\bigcup_{(l, s) \in A} (\max V_i : \land \max W_s) \in u$ thus $\bigcup_{(l, s) \in \hat{A}} (\tilde{V}_l : \land \tilde{W}_s) \cap \bigcup_{(l, s) \in A} (\max V_i : \land \max W_s) \in u$ and so $A \in u$ and so $\int \hat{T} \subset u$. Consider sets $U_i = \bigcup_{(l, s) \in R^a T_{l,s} \land \int \tilde{T}} \max V_i \land \max W_s \land P = \omega^b$, thus $\max V_i \land \max W_s \land P = \omega^b$, thus $\omega^{b-a}$.

Therefore $\max V_i \land \max W_s \land P = \omega^{b-a}$ and so $\omega^{b-a}$ and so $\int \hat{T} \subset u$. Proof that $\int \hat{T} \subset u$ is analogous.

Subcase 1.2') Without loss of generality, we may assume that $\max V = \max W = H(K_{\infty-1})$. Consider cascade $V'$ - such a modification of cascade $V$ that in the place of the cascade $V_l$, for each $l \in dom K_{\infty-1}$ we put a following cascade: $(s) \rightarrow_{\{(s) : (l, s) \in K_{\infty-1}\}} V_l = \max W_{s} \cap (\bigcup_{l \neq l_{(s)}} \max V_{l}) \land k$. Notice that $H(K_{\infty}) \in \int V$ and so $H(K_{\infty}) \in u$ so without loss of generality we may assume that $K_{\infty} = K_{\infty-1}$ and so $V' = V \lor H(K_{\infty})$. Notice that $V'$ is a monotone sequential cascade of rank $r(V') = r(V) + 1$ and that $\int V' \in u$. Calculation of the rank is straightforward, so take $P \in \int V'$ for each $l \in dom K_{\infty-1}$ label elements of the set $\{s : (l, s) \in K_{\infty-1}\}$ by natural numbers by preserving lexicographic order bijection, $s_n$ is a resulting name. If $P \in \int V'$ then there exists $\tilde{P} \in \int \{V : V \lor r(V) \leq \max\}$, also for each $l : v_l \in \tilde{P}$ there exists a cofinite subset $A_l$ of $\omega$ that for each $(l, s_n)$, such that $v_l \in \tilde{P}, n \in A_l$, there is a set $P_{l,s_n} \subset \int V_l = \max W_{s_n} \cap \bigcup_{l \neq l_{(s)}} \max V_{l} \land k$, such that $P = \bigcup_{l,v_l \in \tilde{P} \lor \max A_l \lor \max V_{l}} P_{l,s_n}$. Since for each pair $(l, s_n)$ there exist sets $\tilde{V}_{l,s_n} \subset \int V_l$ and $\tilde{W}_{l,s_n} \subset \int W_{s_n}$ such that $P_{l,s_n} \supset \tilde{V}_{l,s_n} \land \tilde{W}_{l,s_n} : \supset \bigcup_{l,v_l \in \tilde{P} \lor \max A_l \lor \max V_{l}} P_{l,s_n} \supset u$ so $\bigcup_{l \in dom K_{\infty-1} \lor v_l \in \tilde{P} \lor \max V_{l}} P_{l,s_n} \supset u$. On the other hand (by assumption $\sim 1.1$) $\bigcup_{l \in dom K_{\infty-1} \lor v_l \in \tilde{P} \lor \max A_l \lor \max V_{l}} P_{l,s_n} \neq u$, where $\tilde{W}_{l,s_n} = \max W_{s_n}$, $\tilde{V}_{l,s_n} = \max V_l$ and $A_l = \omega$ for $(l, s_n) \in K_{\infty-1}$ such that $v_l \notin \tilde{P}$. Thus $\bigcup_{l \in dom K_{\infty-1} \lor v_l \in \tilde{P} \lor \max A_l \lor \max V_{l}} P_{l,s_n} \in u$, therefore since
\[ P \supset \bigcup_{l \in \text{dom } K_{\omega_1}} \max V_l \cap \bigcup_{l \in \text{dom } K_{\omega_1-1}} \bigcup_{t, s : n \in A_i} (\hat{V}_{l, s} \cap \hat{W}_{l, s}) \text{ we have } P \in u \text{ and so } \int V' \in u. \]

We will show that also \( V' \Rightarrow V \) holds. Take any \( A \subseteq \max V' \) and notice that it suffices to prove \( (f_{V'}(A \cap \max V)) \geq \text{ind} \) (ot \( (f_V(A \cap \max V)) \)) for such \( V \) that \( r_{V'}(v) = n \). So we fix such \( l \) and consider \( A \cap \max V' \) assuming, without loss of generality, that \( (f_V(A \cap \max V)) = \omega^c \) for some \( c \leq n \). Consider a following sequence of sets \( (\max V_{l-k} \cap A)_{k<\omega} \), there is \( k_0 \) that \( (f_V(A \cap V_{l-k_0})) = \omega^c \) or there is \( O \) - infinite subset of \( \omega \) such that \( (f_V(A \cap V_{l-k})) = \omega^{c-1} \) for each \( k \in O \). Notice that each \( V_{l-k} \) is split, during the construction of \( V' \), into finitely many pieces by sets \( \max W_s \cap \bigcup_{k>\omega-1(s)} \max V_{l-k} \). So there is \( s_0 \) such that \( (f_V(A \cap \max V_{l-k})) = (f_V(A \cap \max V_{l-k} \cap \max W_{s_0})) \). Therefore either \( (f_V(A \cap \max V_{l-k})) = \omega^c \) for some \( k < \omega \), or \( (f_V(A \cap \max V_{l-k})) \geq \omega^{c-1} \) for infinite number of \( k \)'s. Thus \( ot \) \( (f_V(A \cap \max V)) \) \( \geq \omega^c \) and so \( V' \Rightarrow V \).

We notice that for cascades \( V' \) and \( W \) conditions described as 1.1 hold. Now we proceed like in subcase 1.1'. Define a series of sets \( R' = L_{1,V'} \times L_{1,W}, R'^# = \{(l, s) \in R' : \int V'_l \int \max W \leq\} \{l, s) \in R : \text{card} (\max V'_l \cap \max W \cap A) = \omega \} \). for \( A \subseteq \omega, \hat{u} = \{R(U) : U \in u\}, \int V' = \{v \in V : r(v) > 1 \} \cup R(\max W), W = \{w \in W : r(w) > 1 \} \cup R(\max W). \) Observe that \( (l, s) \in R \iff (l''n, s) \in R' \) and \( (l, s) \in R'^# \iff (l''n, s) \in R'^# \). On \( V' \) we define order \( \leq_{V'} \) by: if \( v_1, v_2 \in V' \cap V', (l, s) \in R' \) then \( v_1 \leq_{V'} v_2 \) if \( v_1 \in V', v_2 \in V' \). (l, s) if \( \text{card} (\max V'_l \cap \max W \cap V_{l''n}) = \omega \). On \( W \) we introduce order in the analogical way. Notice that \( \hat{u} \) is an ultrafilter on \( R' \) and that \( \hat{V'} \) and \( \hat{W} \) are monotone sequential cascades (on \( R' \)) and that \( r(\hat{V'}) = r(V') - 1 = r(V), r(\hat{W}) = r(W) - m \) and that \( \int \hat{V'} \subseteq \hat{u} \) and \( \int \hat{W} \subseteq \hat{u} \).

By inductive (or sub-inductive) assumption (for \( V', W \) and \( u \)) there is \( \hat{T} \) a monotone sequential cascade on \( R' \) of rank \( r \{r(V), r(W) - 1 \} \leq r(\hat{T}) \leq r(V) + r(W) - 2 \) and such that \( \int \hat{T} \cap \hat{u} \) and \( \hat{T} \Rightarrow V' \) and \( \hat{T} \Rightarrow W, \) also by inductive assumption, for each \( (l, s) \in R'^# \) there is a monotone sequential cascade \( T_{l,s} \), of rank \( r(\hat{T}_{l,s}) = 1 \) such that \( T_{l,s} \Rightarrow V'_l, T_{l,s} \Rightarrow W_s, \int T_{l,s} \subseteq \int V'_l \lor \int W_s \). Define \( \hat{T} = \hat{T} \lor_{\text{max}(V'_l \cap \max W_s, \max V'_l \cap \max W_s)} T_{l,s} \). Take any \( A \in \int \hat{T} \) such that \( \exists A \in \int \hat{T} \) such that \( A = \bigcup_{(l,s) \in \hat{A}} A_{(l,s)} \). Since \( A_{l,s} \supseteq V'_l \cap W_s \) for some \( \hat{V'}_l \in \int V'_l \) and \( \hat{W}_s \in \int W_s \), also \( \hat{A} = \bigcup_{(l,s) \in \hat{A}} (\hat{V'}_l \cap \hat{W}_s) = \bigcup_{(l,s) \in \hat{A}} (\max V'_l \cap \max W_s) \). Since \( \int \hat{T} \subseteq \hat{u} \) and \( \hat{u} \subseteq \hat{u} \) thus \( \int \hat{T} \subseteq \hat{u} \) and \( \hat{u} \subseteq \hat{u} \) and so \( \hat{T} \subseteq \hat{u} \). Consider sets \( U_{i} = \bigcup_{(l,s) \in R^#(T_{l,s}) = i} \max T_{l,s} \). By inductive assumption - upper limiations of ranks, only finite number of these sets are nonempty, and since \( \bigcup_{i<\omega} U_i \subseteq u \) and \( U_{i_0} \subseteq u \) for some \( i_0 \). Let \( T = \bigcap_{i<\omega} U_i \). Clearly \( \int T \subseteq u \). Calculation of the rank of \( T \) follows easily. Take any \( P \in \max T \), without loss of generality, we may assume that \( ot (f_{V'}(P)) = \omega^b \) for some \( b \leq r(V) \). Split \( R' \) into following sets \( R'_a = \{(l, s) \in R' : \omega^{a-1} \leq \omega \leq \omega^a \} \leq ot (f_{V'}(P \cap \max V'_l \cap \max W_s)) < \omega^a \) for \( a \in \{1, \ldots, b+1\} \). For some \( a \), say \( a \), we have ot \( (f_{V'}(\bigcup_{l,s} \max V'_l \cap \max W_s)) < \omega^a \)
max $W_n \cap P = \omega^b$, thus $\text{ot}(f_{V'}(R_{\alpha_0}')) \geq \omega^{b-a}$. Therefore $\text{ot}(f_T(R_{\alpha_0})) \geq \omega^{b-a}$ and so $\text{ot}(f_T(P)) \geq \omega^b$, and so $T \Rightarrow V'$ and $T \Rightarrow V$ by transitivity of $\Rightarrow$ relation. Proof that $T \Rightarrow W$ is analo
gical.

Subcase 1.3') Proof is analo
gical to 1.2'.

Case 2) In both subcases proof is an easier version of proof in case 1 (subcases 1.2 and 1.3).

Case 3) a) Case $r(V) = r(W) = 1$ was done at the beginning of the proof;
b) If $\min \{r(V), r(W)\} = 1$ and $\max \{r(V), r(W)\} > 1$ then $1_V E_{u_1} V_1$ by Proposition 3.3, and by already proved part 2 of the proof, the required cascade $T$ exists;
c) If $\min \{r(V), r(W)\} > 1$ then $1_V E_{u_1} V_1$ by Proposition 3.3, and by already proved case 1, the required cascade $T$ exists.

Inclusions of contours is straightforward by monotonicity of contour opera
tion with respect to the confu
cence. ■

By Proposition 3.3 and Theorem 3.5 we have

**Corollary 3.6.** Let $u$ be an ultrafilter and let $V$, $W$ be monotone sequential cascades of finite ranks. If $\int V \subset u$ and $\int W \subset u$ then there is a monotone sequential cascade $T$ of finite rank not less then $\max \{r(V), r(W)\}$ and such that $\int T \subset u$ and $T \Rightarrow V$.

**Proposition 3.7.** [18. Proposition 3.3 redefined in virtue of it’s proof] Let $V$ be a monotone sequential cascade of rank $\alpha$. If $u$ is such an ultrafilter that $\int V \subset u$ then $\text{ot}(f_V(U)) \geq \omega^\alpha$ for all $U \in u$.

**Proposition 3.8.** [18. Proposition 3.6] Let $\alpha$ be a countable indecomposable ordinal, let $n < \omega$ and let $u$ be an ultrafilter. If there is a function $f : \omega \to \omega_1$ such that $\text{ot}(f(U)) \geq \omega^{\alpha+n}$ for each $U \in u$ and for each $g : \omega \to \omega_1$ there is $U_g \in u$ such that $\text{ot}(g(U_g)) < \omega^{\alpha+\omega}$, then there exists a monotone sequential cascade $V$ of rank $n$ such that $\int V \subset u$.

**Theorem 3.9.** (ZFC) The class of strict $J_{\omega^\omega}$-ultrafilters is empty.

**Proof.** Suppose that $u$ is a strict $J_{\omega^\omega}$-ultrafilter, thus by definition of this class, for each $n < \omega$ there exists a function $f_n : \omega \to \omega_1$ such that $\text{ot}(f_n(U)) \geq \omega^n$ for each $U \in u$ and there is no function $f_{\infty} : \omega \to \omega_1$ that $\text{ot}(f_{\infty}(U)) \geq \omega^\omega$ for each $U \in u$. Let $K$ be a set of all such $n < \omega$ that there is $f_n : \omega \to \omega_1$ such that $\text{ot}(f_n(U)) \geq \omega^n$ and that there is $U_n \in u$ such that $\text{ot}(f_n(U_n)) = \omega^n$.

By Proposition 3.7 there is a sequence $W_n$ of monotone sequential cascades such that $r(W_{n+1}) > r(W_n)$ and $\int W_n \subset u$.

We will build a sequence $(T_n)$ of monotone sequential cascades such that
1) $(r(T_n))$ is an increasing sequence
2) $\int T_n \subset u$

\(^{01)}\) In fact $K = \omega$, but since we do not need this in the theorem, we omit a short proof of this fact.
3) For each $n < \omega$ there exist sets $A_n^0$ and $A_n^1$ that $A_n^i \not\supseteq T_n$ for $i \in \{0, 1\}$ and there is such $i_n \in \{0, 1\}$ that $T_{n+1} \supseteq T_n^{A_n^{i_n}}$.

4) $\bigcup_{i \in \{1, \ldots, n\}, j \in \{1, \ldots, i\}} \max T_{i,j} \cap \max T_{n+1} = \emptyset$, for $T_i = (j) \leftrightarrow T_{i,j}$;

5) $\bigcap_{n < \omega} \max T_n = \emptyset$.

Define $T_1$ as the monotone sequential cascade of rank 1 with $\max T_1 = \omega$, clearly $\int T_1 \in u$. Suppose that cascades $T_n$ are already defined for $n \leq m$. For a cascade $T_m = (k) \leftrightarrow T_{m,k}$ consider sets $B_{m,k}^i = \bigcup_{k=j+i} \max T_{m,k}$ for $i \in \{0, 1\}$. Clearly $B_{m,0}^0 \not\supseteq T_m$ and $B_{m,1}^1 \not\supseteq T_m$, and since $B_{m,0}^1 \cup B_{m,1}^0 = \max T_m$ one of these sets belongs to $u$, call $i_m$ the $i$ for which it happens. Let $C_m = \emptyset$ for $m = 2$ and $C_m = \bigcup_{i \in \{1, \ldots, n\}, j \in \{1, \ldots, i\}} \max T_{i,j}$ for $m \geq 3$. Clearly $B_{m,1}^m \setminus C_m \in u$.

So $\int T_{m+1}^{B_{m,1}^m} \setminus C_m \in u$. By Corollary 3.6 applied to $T_m^{B_{m,1}^m}$, $W_m + 1$ and to $u$, there is a monotone sequential cascade $T_{m+1}$ of finite rank not less then $m + 1$ that $\int T_{m+1} \in u$ and $T_{m+1} \supseteq (T_{m+1}^{B_{m,1}^m})$. We define $A_{m+1}^i = \max T_{m+1}$ and $A_m^b = \omega \setminus A_m^i$ for $b \in \{0, 1\}$, $b \neq i_m$. Clearly $T_{m+1}$ with $A_{m+1}^i$ and $A_m^b$ fulfill the claim for $n = m + 1$. To see that $\bigcap_{n < \omega} \max T_n = \emptyset$, it suffices to notice that $\bigcup_{k \in \{1, \ldots, m\}} \max T_{k} \cap \max T_{m+1} = \emptyset$.

Define $T = \bigcup_{n < \omega} T_n^{(\omega \setminus A_n^{i_n+1})}$ ordered by:

1) If $t_1 \in T_n$ and $t_2 \in T_m$ $(n \neq m)$ then $t_1 \supseteq t_2 \iff n < m$
2) If $T_1, T_2 \in T_n$ then $t_1 \subseteq t_2 \iff t_1 \subseteq V_n, t_2$, where "$\subseteq V_n$" is an order on $V_n$.

Let $f_\infty : \omega \to \omega_1$ be a preserving $\subseteq_T$ order function. Take any $U \in u$ and $n \in \omega$. Since $\int T_{n+1} \in u$ thus $U \not\supseteq \int T_{n+1}$ and so $U \not\supseteq \int T_{n+1,k}$ for infinitely many $k$, take $k_0$ from this set.

Since $U \not\supseteq \int T_{n+1,1,k_0}$ thus $\ot (f_{T_{n+1}}(U \cap \max T_{n+1,1,k_0}) = r(T_{n+1,1,k_0}) \geq \omega^n$.) By condition 4) $U \cap \max T_{n+1,1,k_0} \cap \max T_i \neq \emptyset$ only for a finite number of $i > n + 1$. So $\{\max T_{n+1,1,k_0} \cap U \cap \max T_{i} \setminus \bigcup_{j > i_0} \max T_{j} : i \geq n + 1\}$ is a finite partition of $U \cap \max T_{n+1,1,k_0}$. Thus there is $i_0 \geq n + 1$ such that $\ot (f_{T_{n+1}}(\max T_{n+1,1,k_0} \cap U \cap (\max T_{i} \setminus \bigcup_{j > i_0} \max T_{j}))) = 0$

and $\ot (f_{T_{n+1}}(\max T_{n+1,1,k_0} \cap U) \geq \omega^n$, and since $T_{i_0} \supseteq T_{n+1}$ thus $\ot (f_{T_{i_0}}(\max T_{n+1,1,k_0} \cap U \cap (\max T_{i} \setminus \bigcup_{j > i_0} \max T_{j}))) \geq 0$

indeed $\ot (f_{T_{n+1}}(\max T_{n+1,1,k_0} \cap U \cap (\max T_{i} \setminus \bigcup_{j > i_0} \max T_{j})))$, and $f_{\infty} \supseteq \max T_{i_0} \setminus \max T_j = f_{T_{i_0}}(\max T_{i_0} \setminus \bigcup_{j > i_0} \max T_{j})$ thus $\ot (f_{\infty}(\max T_{n+1,1,k_0} \cap U \cap (\max T_{i_0} \setminus \bigcup_{j > i_0} \max T_{j}))) \geq \omega^n$.

Therefore $\ot (f_{\infty}(U)) \geq \omega^n$. ■

There is a straight correspondence between cascades and $<_\omega$-sequences. Let $u$ be an ultrafilter, take sequence $u = u_0 > u_1 > u_2 > \ldots > u_n$ and functions $f_n : \omega \to \omega$ - witnesses that $u_{m-1} > u_m$.

We will build a monotone sequential cascade $V$ which correspond to the sequence above with respect to some $U \in u$. In this aim we build a sequence of cascades $(W_i)_{i < n}$. Take any monotone sequential cascade $W_1$ of rank 1 and label elements of $\max W_1$ by natural numbers by any bijections. Clearly $W \in$
$u_n$ for each $W \in \int W_1$. Take $W_2 = W_1 \cup \bigcup_{i \in \max W_1} (f_n^{-1}(n)) = \infty f_n^{-1}(i)$ \footnote{Since formally levels in cascade can not intersect we may assume that domain of $f_1$ and ranges of $f_m$ are subsets of a pairwise disjoint copies of $\omega$.} ordered by, extended by transitivity, the following preorder: If $w_1, w_2 \in W_1$ then $w_1 \leq_{W_2} w_2$ iff $w_1 \leq_{W_1} w_2$; if $w_1 \in \max W_1$ and $w_2 \in \max W_2$ then $w_1 \leq_{W_2} w_2$ if $f_n^{-1}(w_1) = w_2$. Clearly $W \in u_{n-1}$ for each $w \in \int W_2$. We continue this procedure to get $W_n$ and define $V_n = W_n$.

Now take any monotone sequential cascade $V$ of finite rank, with $\int W \subset u$, without loss of generality we may assume that all branches of $V$ have the same length $n$. For each $v \in V$ let $\hat{v}$ be an arbitrary element of $\max v$. Consider functions $f_i : \omega \to \omega$ such that $f_1(v_1) = \hat{v}$ for each $v_1 \in \max v$ where $r(v) = i$. Thus $u >_{\infty} f_i(u) >_{\infty} f_2 \circ f_1(u) >_{\infty} \ldots >_{\infty} f_n \circ f_{n-1} \circ \ldots \circ f_2 \circ f_1(u)$, (for details see \cite{17}).

This cascades - $<_{\infty}$-sequences correspondence allows us to look at the Proposition 3.3 and Theorems 3.5 and 3.9 (in virtue of its proofs) in the following way:

Proposition 3.3 and Theorem 3.5 describes mutual behavior of the functions - witnesses of $<_{\infty}$-sequences. Clearly existence of infinite increasing $<_{\infty}$-sequences under some ultrafilter implies existence of an arbitrary long finite $<_{\infty}$-sequences under this ultrafilter. Theorem 3.9 shows that if an ultrafilter has an arbitrary long finite $<_{\infty}$-sequences then is at least a strict $J_{\omega+1}$-ultrafilter.

We’d like to drew attention, not only to benefits, but also to limitations of the construction presented in the paper. Probably Theorem 3.5 can be proved in a stronger, i.e. infinite version, but still there is rather no hope to extend our construction to other limit ordinals. The problem lays in the relations between order ultrafilters and monotone sequential contours, contained in an ultrafilter, described in Proposition 3.8, with a special emphasis on of the upper limitation of the order-type of images. This limitation is non-removable, what was shown in \cite{18} Theorem 3.9 by proving (under MA $\sigma$-cetr) that there is a strict $J_{\omega+1}$-ultrafilter that does not contain any monotone sequential contour of rank 3. Thus we restate Baumgartner question in virtue of our result.

**Problem 3.10.** What about other limit classes? Is there a model with non-empty class of the strict $J_{\omega\alpha}$ ultrafilters for some (all) limit $\omega < \alpha < \omega_1$?

Opposite side of this problem is a Shelah question

**Problem 3.11.** \cite{14} Question 3.12] Prove the consistency "there is no $J_\alpha$-ultrafilter on $\omega$".

Under following three theorems of Baumgartner and remembering Shelah model with no P-points, the above question essentially asks about classes of limit index and classes whose index is a successor of a limit ordinal.

**Theorem 3.12.** \cite{14} Theorem 4.1] The strict-$J_{\omega^2}$ ultrafilters are P-point ultrafilters.
Theorem 3.13. \[1, \text{Theorem 4.2}\] If there is a P-point then there are strict $J_{\omega^\alpha+1}$ ultrafilters for all $\alpha < \omega_1$.

Theorem 3.14. \[1, \text{Theorem 4.6}\] Let $\alpha < \omega_1$ and assume $u$ is a strict $J_{\omega^\alpha+2}$ ultrafilter. Then there is a P-point $v$ such that $v \leq_{\text{RK}} u$.

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