Service Chain and Virtual Network Embeddings: Approximations using Randomized Rounding

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Abstract—The SDN and NFV paradigms enable novel network services which can be realized and embedded in a flexible and rapid manner. For example, SDN can be used to flexibly steer traffic from a source to a destination through a sequence of virtualized middleboxes, in order to realize so-called service chains. The service chain embedding problem consists of three tasks: admission control, finding suitable locations to allocate the virtualized middleboxes and computing corresponding routing paths. This paper considers the offline batch embedding of multiple service chains. Concretely, we consider the objectives of maximizing the profit by embedding an optimal subset of requests or minimizing the costs when all requests need to be embedded. Interestingly, while the service chain embedding problem has recently received much attention, so far, only non-polynomial time algorithms (based on integer programming) as well as heuristics (which do not provide any formal guarantees) are known. This paper presents the first polynomial time service chain approximation algorithms both for the case with admission and without admission control. Our algorithm is based on a novel extension of the classic linear programming and randomized rounding technique, which may be of independent interest. In particular, we show that our approach can also be extended to more complex service graphs, containing cycles or sub-chains, hence also providing new insights into the classic virtual network embedding problem.

I. INTRODUCTION

Computer networks are currently undergoing a phase transition, and especially the Software-Defined Networking (SDN) and Network Function Virtualization (NFV) paradigms have the potential to overcome the ossification of computer networks and to introduce interesting new flexibilities and novel service abstractions such as service chaining.

In a nutshell, in a Software-Defined Network (SDN), the control over the forwarding switches in the data plane is outsourced and consolidated to a logically centralized software in the so-called control plane. This separation enables faster innovations, as the control plane can evolve independently from the data plane: software often trumps hardware in terms of supported innovation speed. Moreover, the logically centralized perspective introduced by SDN is natural and attractive, as many networking tasks (e.g., routing, spanning tree constructions) are inherently non-local. Indeed, a more flexible traffic engineering is considered one of the key benefits of SDN [10], [13]. Such routes are not necessarily shortest paths or destination-based, or not even loop-free [13]. In particular, OpenFlow [29], the standard SDN protocol today, allows to define routing paths which depend on Layer-2, Layer-3 and even Layer-4 header fields.

Network Function Virtualization (NFV) introduces flexibilities in terms of function and service deployments. Today’s computer networks rely on a large number of middleboxes (e.g., NATs, firewalls, WAN optimizers), typically realized using expensive hardware appliances which are cumbersome to manage. For example, it is known that the number of middleboxes in enterprise networks can be of the same order of magnitude as the number of routers [22]. The virtualization of these functions renders the network management more flexible, and allows to define and quickly deploy novel in-network services [7], [9], [21], [23], [27]. Virtualized network functions can easily be instantiated on the most suitable network nodes, e.g., running in a virtual machine on a commodity x86 server. The transition to NFV is discussed within standardization groups such as ETSI, and we currently also witness first deployments, e.g., TeraStream [39].

Service chaining [33], [34], [38] is a particularly interesting new service model, that combines the flexibilities from SDN and NFV. In a nutshell, a service chain describes a sequence of network functions which need to be traversed on the way from a given source s to a given destination t. For example, a service chain could define that traffic originating at the source is first steered through an intrusion detection system for security, next through a traffic optimizer, and only then is routed towards the destination. While NFV can be used to flexibly allocate network functions, SDN can be used to steer traffic through them.

A. The Scope and Problem

This paper studies the problem of how to algorithmically exploit the flexibilities introduced by the SDN+NFV paradigm. We attend the service chain embedding problem, which has recently received much attention. The problem generally consists of three tasks: (1) (if possible) admission control, i.e. selecting and serving only the most valuable requests, (2) the allocation of the virtualized middleboxes at the optimal locations and (3) the computation of routing paths via them. Assuming that one is allowed to exert admission control, the objective is to maximize the profit, i.e., the prizes collected for embedding service chains. We also study the problem variant, in which a given set of requests must be embedded, i.e. when admission control cannot be exerted. In this variant we
consider the natural objective of minimizing the cumulative allocation costs. The service chain embedding algorithms presented so far in the literature either have a non-polynomial runtime (e.g., are based on integer programming \cite{30, 34, 38}), do not provide any approximation guarantees \cite{31}, or ignore important aspects of the problem (such as link capacity constraints \cite{26}).

More generally, we also attend to the current trend towards more complex service chains, connecting network functions not only in a linear order but as arbitrary graphs, i.e. as a kind of virtual network.

B. Our Contributions

This paper makes the following contributions. We present the first polynomial time algorithms for the (offline) service chain embedding problem with and without admission control, which provide provable approximation guarantees.

We also initiate the study of approximation algorithms for more general service graphs (or “virtual networks”). In particular, we present polynomial time approximation algorithms for the embedding of service cactus graphs, which may contain branch sub-chains and even cycles. To this end, we develop a novel Integer Program formulation together with a novel decomposition algorithm, enabling the randomized rounding: we prove that known Integer Programming formulations are not applicable.

C. Technical Novelty

Our algorithms are based on the well-established randomized rounding approach \cite{35}: the algorithms use an exact Integer Program, for which however we only compute relaxed, i.e. linear, solutions, in polynomial time. Given the resulting fractional solution, an approximate integer solution is derived using randomized rounding, in the usual resource augmentation model.

However, while randomized rounding has been studied intensively and applied successfully in the context of path embeddings \cite{35}, to the best of our knowledge, besides our own work, the question of how to extend this approach to service chains (where paths need to traverse certain flexible waypoints) or even more complex graphs (such as virtual networks), has not been explored yet. Moreover, we are not aware of any extensions of the randomized rounding approach to problems allowing for admission control.

Indeed, the randomized rounding of more complex graph requests and the admission control pose some interesting new challenges. In particular, the more general setting requires both a novel Integer Programming formulation as well as a novel decomposition approach. Indeed, we show that solutions obtained using the standard formulation \cite{5, 37} may not be decomposable at all, as the relaxed embedding solutions are not a linear combination of elementary solutions. Besides the fact that the randomized rounding approach can therefore not be applied, we prove that the relaxation of our novel formulation is indeed provably stronger than the well-known formulation.

D. Organization

The remainder of this paper is organized as follows. Section \[I\] formally introduces our model. Section \[II\] presents the Integer Programs and our decomposition method. Section \[IV\] presents our randomized approximation algorithm for the service chain embedding problem with admission control and Section \[V\] extends the approximation for the case without admission control. In Section \[VI\] we derive a novel Integer Program and decomposition approach for approximating service graphs, and show why classic formulations are not sufficient. Section \[VII\] reviews related work and Section \[VIII\] concludes our work.

II. OFFLINE SERVICE CHAIN EMBEDDING PROBLEM

This paper studies the Service Chain Embedding Problem, short SCEP. Intuitively, a service chain consists of a set of Network Functions (NFs), such as a firewall or a NAT, and routes between these functions. We consider the offline setting, where batches of service chains have to be embedded simultaneously. Concretely, we study two problem variants: (1) SCEP-P where the task is to embed a subset of service chains to maximize the profit and (2) SCEP-C where all given service chains need to be embedded and the objective is to minimize the resource costs. Hence, service chain requests might be attributed with prizes and resources as e.g., link bandwidth or processing (e.g., of a firewall network function) may come at a certain cost.

A. Definitions & Formal Model

Given is a substrate network (the physical network representing the physical resources) which is modeled as directed network \(G_S = (V_S, E_S)\). We assume that the substrate network offers a finite set \(\mathcal{T}\) of different network function types (NFs) at nodes. The set of network function types may contain e.g., ‘FW’ (firewall), ‘DPI’ (deep packet inspection), etc. For each such type \(\tau \in \mathcal{N}\), we use the set \(V_S^\tau \subseteq V_S\) to denote the subset of substrate nodes that can host this type of network function. To simplify notation we introduce the set \(R_S^\tau = \{ (\tau, u) \mid \tau \in \mathcal{T}, u \in V_S^\tau \}\) to denote all node resources and denote by \(R_S = R_S^\tau \cup E_S\) the set of all substrate resources. Accordingly, the processing capabilities of substrate nodes and the available bandwidth on substrate edges are given by the function \(d_S : R_S \to \mathbb{R}_{\geq 0}\). Hence, for each type and substrate node we use a single numerical value to describe the node’s processing capability, e.g. given as the maximal throughput in Mbps. Additionally, we also allow to reference substrate node locations via types. To this end we introduce for each substrate node \(u \in V_S\) the abstract type \(\text{loc}_u \in \mathcal{T}\), such that \(V_S^{\text{loc}_u} = \{ u \}\) and \(d_S(\text{loc}_u, u) = \infty\) and \(d_S(\text{loc}_u, v) = 0\) for nodes \(v \in V_S \backslash \{ u \}\).

The set of service chain requests is denoted by \(\mathcal{R}\). A request \(r \in \mathcal{R}\) is a directed chain graph \(G_r = (V_r, E_r)\) with start node \(s_r \in V_r\) and end node \(t_r \in V_r\). Each of these virtual nodes corresponds to a specific network function type which is given via the function \(\tau_r : V_r \to \mathcal{T}\). We assume that the types of \(s_r\) and \(t_r\) denote specific nodes in the substrate. Edges of the service chain represent forwarding paths. Since the type for
Definition 1 (Valid Mapping). A valid mapping \( m_r \) of request \( r \in \mathcal{R} \) is a tuple \((m^V_r, m^E_r)\) of functions. The function \( m^V_r: \mathcal{V} \to \mathcal{V}' \) maps each virtual network onto paths in the substrate network, such that:

- All network functions \( i \in \mathcal{V}_r \) are mapped onto nodes that can host the particular function type. Formally, \( m^V_r(i) \in \mathcal{V}'_r \) for all \( i \in \mathcal{V}_r \).
- The edge mapping \( m^E_r \) connects the respective network functions using simple paths, i.e. given a virtual edge \((i, j) \in \mathcal{E}_r\), the embedding \( m^E_r(i, j) \) is an edge-path \((v_1, v_2, \ldots, v_{l-1}, v_{l})\) such that \( v_1, v_{l+1} \in \mathcal{E}_S \) for \( 1 \leq l \leq K \) and \( v_k = m^V_r(i) \) and \( v_k = m^V_r(j) \).

Next we define the notion of a feasible embedding for a set of requests, i.e. an embedding that obeys the network function and edge capacities.

Definition 2 (Feasible Embedding). A feasible embedding of a subset of requests \( \mathcal{R}' \subseteq \mathcal{R} \) is given by valid mappings \( m_\mathcal{R}' = (m^V_\mathcal{R}', m^E_\mathcal{R}') \) for \( r \in \mathcal{R}' \), such that network function and edge capacities are obeyed:

- For all types \( \tau \in \mathcal{T} \) and nodes \( u \in \mathcal{V}_S' \) holds:
  \[ \sum_{r \in \mathcal{R}'} \sum_{i \in \mathcal{V}_r} m^V_r(i) = u \] \[ d_{\tau}(i, u) \leq d_{\tau}(\tau, u) . \]
- For all edges \((u, v) \in \mathcal{E}_S\) holds:
  \[ \sum_{r \in \mathcal{R}'} \sum_{(i, j) \in \mathcal{E}_r(u, v), m^E_r(i, j)} d_{\tau}(i, j) \leq d_{\tau}(u, v) . \]

We first define the SCEP variant with admission control whose objective is to maximize the net profit (SCEP-P), i.e. the achieved profit for embedding a subset of requests.

Definition 3 (SCEP for Profit Maximization: SCEP-P).

Given: A substrate network \( \mathcal{G}_S = (\mathcal{V}_S, \mathcal{E}_S) \) and a set of requests \( \mathcal{R} \) as described above.

Task: Find a subset \( \mathcal{R}' \subseteq \mathcal{R} \) of requests to embed and a feasible embedding, given by a mapping \( m_\mathcal{R}' \), for each request \( r \in \mathcal{R}' \), maximizing the net profit \( \sum_{r \in \mathcal{R}'} b_r \).

In the variant without admission control, i.e. when all given requests must be embedded, we consider the natural objective of minimizing the cumulative cost of all embeddings. Concretely, the cost of the mapping \( m_r \) of request \( r \in \mathcal{R} \) is defined as the sum of costs for placing network functions plus the number of substrate links along which network bandwidth needs to be reserved, times the (processing or bandwidth) demand:

\[
\sum_{(i,j) \in \mathcal{E}_r} d_{\tau}(i, j) \sum_{a \in \mathcal{A}} \mathbf{m}_a(i,j) c_{\mathbf{S}}(u,v)
\]

The variant SCEP-C without admission control which asks for minimizing the costs is hence defined as follows.

Definition 4 (SCEP for Cost Minimization: SCEP-C).

Given: A substrate network \( \mathcal{G}_S = (\mathcal{V}_S, \mathcal{E}_S) \) and a set of requests \( \mathcal{R} \) as described above.

Task: Find a feasible embedding \( m_\mathcal{R} \) for all requests \( r \in \mathcal{R} \) of minimal cost \( \sum_{r \in \mathcal{R}} b_r \).

B. NP-Hardness

Both introduced SCEP variants are strongly NP-hard, i.e. they are hard independently of the parameters as e.g. the capacities. We prove the NP-hardness by establishing a connection to multi-commodity flow problems. Concretely, we present a polynomial time reduction from the Unsplittable Flow (USF) and the Edge-Disjoint Paths (EDP) problems [19] to the respective SCEP variants. Both USF and EDP are defined on a (directed) graph \( G = (\mathcal{V}, \mathcal{E}) \) with capacities \( \sum_{r \in \mathcal{R}} b_r \). The task is to route a set of \( K \) commodities \( (s_k, t_k) \) with demands \( d_k \) for \( 1 \leq k \leq K \) from \( s_k \in \mathcal{V} \) to \( t_k \in \mathcal{V} \) along simple paths inside \( G \). Concretely, EDP considers the decision problem in which both the edge capacities and the demands are 1 and the task is to find a feasible routing. The variant of EDP asking for the maximum number of routable commodities was one of Karp’s original 21 NP-complete problems and the decision variant was shown to be NP-complete even on series-parallel graphs [22]. In the USF problem, for each commodity an additional benefit \( b_k \) is given and the task is to find a selection of commodities to route, such that capacities are not violated and the sum of benefits of the selected commodities is maximized. Solving the USF problem is NP-hard and proven to be hard to approximate within a factor of \( |E|^{1/2-\varepsilon} \) for any \( \varepsilon > 0 \). [19]

We will argue in the following that EDP can be reduced to SCEP-C and USF can be reduced to SCEP-P. Both reductions use the same principal idea of expressing the given commodities as requests. Hence, we first describe this construction before discussing the respective reductions. For commodities \( (s_k, t_k) \) with \( 1 \leq k \leq K \) a request \( r_k \) consisting only of the two virtual nodes \( i_k \) and \( j_k \) and the edge \((i_k, j_k)\) is introduced. By setting \( s_{rk} = i_k \) and \( t_{rk} = j_k \) and \( \pi_r(i_k) = \text{loc}_{s_{rk}} \) and \( \pi_r(j_k) = \text{loc}_{t_{rk}} \), we can enforce that flow of request \( r_k \) originates at \( s_{rk} \in \mathcal{V} \) and terminates at \( t_{rk} \in \mathcal{V} \), hence modeling the original commodities. In both reductions presented below, we do not make use of network functions, i.e. \( \mathcal{T} = \{ \text{loc}_u | u \in \mathcal{V}_S \} \), and accordingly we do not need to specify network function capacities.
Regarding the polynomial time reduction from EDP to SCEP-C, we simply use unitary virtual demands and substrate capacities. As this yields an equivalent formulation of EDP, which is NP-hard, finding a feasible solution for SCEP-C is NP-hard. Hence, there cannot exist an approximation algorithm that (always) finds a feasible solution within polynomial time unless \( P = NP \). Therefore, we consider the more practical setting of capacities. As this yields an equivalent formulation of EDP, we simply use unitary virtual demands and substrate capacities. It is easy to see that any solution to this SCEP-P instance also induces a solution to the original USF instance. It follows that SCEP-P is strongly NP-hard.

### C. Further Notation

We generally denote directed graphs by \( G = (V, E) \). We use \( \delta^+_E(u) := \{(u, v) \in E\} \) to denote the outgoing edges of node \( u \) in \( E \) with respect to \( E \) and similarly define \( \delta^-_E(u) := \{(v, u) \in E\} \) to denote the incoming edges. If the set of edges \( E \) can be derived from the context, we often omit stating \( E \) explicitly. When considering functions on tuples, we often omit the (implicit) braces around a tuple and write e.g. \( f(x, y) \) instead of \( f((x, y)) \). Furthermore, when only some specific elements of a tuple are of importance, we write \( (x, \cdot) \in Z \) in favor of \( (x, y) \in Z \).

### III. Decomposing Linear Solutions

In this section, we lay the foundation for the approximation algorithms for both SCEP variants by introducing Integer Programming (IP) formulations to compute optimal embeddings (see Section III-A). Given the NP-hardness of the respective problems, solving any of the IPs to optimality is not possible within polynomial time (unless \( P = NP \)). Hence, we consider the linear relaxations of the respective formulations instead, as these naturally represent a conical or convex combination of valid mappings. We formally show that linear solutions can be decomposed into valid mappings in Section III-B. Given the ability to decompose solutions, we apply randomized rounding techniques in Sections IV and V to obtain tri-criteria approximation algorithms for the respective SCEP variants.

#### A. Integer Programming

To formulate the service chain embedding problems as Integer Programs we employ a flow formulation on a graph construction reminiscent of the one used by Merlin [38]. Concretely, we construct an extended and layered graph consisting of copies of the substrate network together with super sources and sinks. The underlying idea is to model the usage (and potentially the placement) of network functions by traversing inter-layer edges while intra-layer edges will be used for connecting the respective network functions. Figure 1 depicts a simple example of the used graph construction. The request \( r \) consists of the three nodes \( i, j, \text{and} l \). Recall that we assume that the start and the end node specify locations in the substrate network (cf. Section II). Hence, in the example the start node \( s_r = i \) and the end node \( t_r = l \) can only be mapped on the substrate nodes \( u \) and \( w \). Since for each connection of network functions a copy of the substrate network is introduced, the edges between these layers naturally represent the utilization of a network function.

### Definition 5 (Extended Graph). Let \( r \in \mathcal{R} \) be a request. The extended graph \( G^\text{ext} \) is defined as follows:

\[
V^\text{ext}_r = \{o^+_r, o^-_r\} \cup \{u^i,j|i, j \in E_r, u \in V_S\}
\]

\[
E^\text{ext}_r = \{(u^i,j, v^i,j)|i, j \in E_r, (u, v) \in E_S\} \cup \{(o^+_r, u^i,j)|i, j \in E_r, u \in V^\text{ext}_S\} \cup \{(o^-_r, o^-_r)|i, j \in E_r, u \in V^\text{ext}_S\} \cup \{(u^i,j, v^i,j)|i, j \in E_r, (i, j) \in \tau^-_S\} \cup \{(u^i,j, v^i,j)|i, j \in E_r, (i, j) \in \tau^+_S\}
\]

We denote by \( E^\text{ext}_{r, u, v} = \{(u^i,j, v^i,j)|i, j \in E_r\} \) all copies of the substrate edge \( (u, v) \in E_S \) together with the respective virtual edge \( (i, j) \in E_r \). Similarly, we denote by \( E^\text{ext}_{r, \tau^-_S} = \{(u^i,j, v^i,j)|i, j \in E_r\} \) the edges that indicate that node \( u \in V_S \) processes flow of network function \( j \in V_r \) having type \( \tau \) and \( \tau^- \).

Having defined the extended graph as above, we will first discuss our Integer Program 1 for SCEP-P. We use a single variable \( x_{r, (x, y)} \) per request \( r \) to indicate whether the request is to be embedded or not. If \( x_{r, (x, y)} = 1 \), then Constraint 5 induces a unit flow from \( o^+_r \) to \( o^-_r \) in the extended graph \( G^\text{ext}_r \) using the flow variables \( f_{r, e} \) for \( e \in E^\text{ext}_r \). Particularly, Constraint 6 states flow preservation at each node, except at the source and the sink.

Constraints 7 and 8 compute the effective load per request on the network functions and the substrate edges. Towards this end, variables \( l_{r, (x, y)} \) indicate the load induced by request \( r \in \mathcal{R} \) on resource \((x, y) \in R_S \). By the construction of the extended graph (see Definition 5), the sets \( E^\text{ext}_{r, u, v} \) and \( E^\text{ext}_{r, \tau^-_S} \) actually represent a partition of all edges in the extended graph for \( r \in \mathcal{R} \). Since each layer represents a virtual connection \( (i, j) \in E_r \) with a specific load \( d_{r, (i, j)} \) and each edge between layers \((i, j) \in E_r \) and \((j, k) \in E_r \) represents the usage of the network function \( j \) with demand \( d_{r, j} \), the unit flow is scaled by the respective demand. Constraint 9 ensures...
then the flow variables \( r \) versa). If a request to IP 1 induces a feasible solution to SCEP-P (and vice versa). The overall amount of used resources does not exceed the offered resources, therefore representing valid mappings. The flow enters and through which the flow leaves each layer. Together with the flow preservation this implies that the respective network functions are connected by the edges inside the layers, therefore representing valid mappings.

Considering SCEP-C, we adapt the IP 1 slightly to obtain the Integer Program 2 for the variant minimizing the costs: (i) all requests must be embedded by enforcing that \( x_r = 1 \) holds for all requests \( r \in R \) and (ii) the objective is changed to minimize the overall resource costs (cf. Equation 1). As the constraints safeguarding the feasibility of solutions are reused, the IP 2 indeed computes optimal solutions for SCEP-C.

While solving Integer Programs 1 and 2 with binary variables is computationally hard (cf. Section II-B), the respective linear relaxations can be computed in polynomial time. Concretely, the linear relaxation is obtained by simply replacing \([0,1]\) with \([0,1]\) in Constraints (10) and (11) respectively. We generally denote the set of feasible solutions to the linear relaxation by \( \mathcal{F}_L \) and the set of feasible solutions to the respective integer program by \( \mathcal{F}_I \). We omit the reference to any particular formulation here as it will be clear from the context. We recall the following well-known fact:

**Fact 6.** \( \mathcal{F}_I \subseteq \mathcal{F}_L \).

The above fact will e.g. imply that the profit of the optimal linear solution will be higher than the one of the optimal integer solution.

**B. Decomposition Algorithm for Linear Solutions**

As discussed above, any binary solution to the formulations 1 and 2 represents a feasible solution to the respective problem variant. However, as we will consider solutions to the respective linear relaxations instead, we shortly discuss how relaxed solutions can be decomposed into conical (SCE-P) or convex combinations (SCE-C) of valid mappings. Concretely, Algorithm 1 computes a set of triples \( D_r = \{ D_r^k = (f_r^k, m_r^k, l_r^k) \}_k \), where \( f_r^k \in [0,1] \) denotes the (fractional) embedding value of the \( k \)-th decomposition, and \( m_r^k \) and \( l_r^k \) represent the (valid) mapping and the induced loads on network functions and edges respectively. Importantly, the load function \( l_r^k : R_S \rightarrow \mathbb{R}_{\geq 0} \) represents the cumulative loads, when embedding the request \( r \in R \) fully according to the \( k \)-th decomposition.

The pseudocode for our decomposition scheme is given in Algorithm 1. For each request \( r \in R \), a path decomposition is performed from \( o_r^+ \) to \( o_r^- \) as long as the outgoing flow from the source \( o_r^+ \) is larger than 0. Note that the flow variables are input and originate from the linear program for SCEP-C or SCE-P respectively. We use \( G_{r,ext}^E \) to denote the graph in which an edge \( e \in \mathcal{E}_{ext}^r \) is contained, iff. the flow value along it is greater 0, i.e. for which \( f_{r,e} > 0 \) holds. Within this graph an arbitrary \( o_r^+ - o_r^- \) path \( P \) is chosen and the minimum available ‘capacity’ is stored in \( f_r^k \).

In Lines 8-15 the node mappings are set. For all virtual network functions \( i \in \mathcal{V} \) and all potential substrate nodes \( u \in V_r^{\mathcal{V}_{rea}(i)} \), we check whether \( u \) hosts \( i \) by considering the inter-layer connections contained in \( P \). Besides the trivial cases when \( i = o_r^+ \) or \( i = o_r^- \) holds, the network function \( i \) is mapped onto node \( u \) iff. edge \((u_r,i,u_{r,i+}) \) is contained in \( P \). As \( P \) is a directed path from \( o_r^+ \) to \( o_r^- \) and the extended graph does not contain any inter-layer cycles, this mapping is uniquely defined for each found path \( P \). For the start node \( s_r \) of the request \( r \in R \) and the end node \( t_r \), connections from \( o_r^+ \) or to \( o_r^- \) are checked respectively.

Concerning the mapping of the virtual edge \((i,j) \in \mathcal{E}_r \), the edges of \( P \) used in the substrate edge layer corresponding to the (virtual) connection \((i,j)\) are extracted in Line 18.
Algorithm 1: Decomposition Algorithm

Input: Substrate $G_S = (V_S, E_S)$, set of requests $\mathcal{R}$, solution $(\vec{x}, \vec{f}, \vec{l}) \in \mathcal{F}_{LP}$
Output: Fractional embeddings $\mathcal{D}_r = \{(f^k_r, m^k_r, k^r)\}_k$ for each $r \in \mathcal{R}$

\begin{algorithmic}[1]
\State for $r \in \mathcal{R}$ do
  \State set $\mathcal{D}_r \leftarrow \emptyset$ and $k \leftarrow 1$
  \While {true \And $\sum_{e \in \delta^+(u)} f_{r,e} > 0$ do
    \State \textbf{choose} $P = \langle o^+_r, \ldots, o^-_r \rangle \in G^e_{R_f}$
    \State $f^k_r \leftarrow \min_{e \in P} f_{r,e}$
    \State $m^k_r = (m^V_r, m^E_r) \leftarrow (\emptyset, \emptyset)$
    \State $l^k_r(x, y) \leftarrow 0$ for all $(x, y) \in R_S$
    \For {$i \in V_r$ do
      \For {$u \in V_{\tau(i)}$ do
        \If {$i = s_r$ \And $(o^+_r, u_{i\leftarrow})$ \in $P$ then
          \State set $m^V_r(i) \leftarrow u$
        \Else if $i = t_r$ \And $(u_{i\rightarrow}, o^-_r) \in P$ then
          \State set $m^V_r(i) \leftarrow u$
        \Else if $(u_{i\rightarrow}, u_{i\leftarrow}) \in P$ then
          \State set $m^V_r(i) \leftarrow u$
        \State set $m^V_r(i) \leftarrow u$
        \State $l^k_r(\tau_r(i), m^V_r(i)) \leftarrow l^k_r(\tau_r(i), m^V_r(i)) + d_r(i)$
      \EndFor
      \State $m^E_r(i, j) \leftarrow \langle (u, v) \in E_S\mid (u_{i\rightarrow}, v_{i\leftarrow}) \in P \rangle$
      \For {$u, v \in m^E_r(i, j)$ do
        \State set $l^k_r(u, v) \leftarrow l^k_r(u, v) + d_r(i)$
      \EndFor
    \EndFor
    \State $\mathcal{D}_r \leftarrow \mathcal{D}_r \cup \{D^k_r\}$ with $D^k_r = (f^k_r, m^k_r, k^r)$
    \State $f_{r,e} \leftarrow f_{r,e} - f^k_r$ for all $e \in P$ and $k \leftarrow k + 1$
  \EndWhile
  \State return $\{\mathcal{D}_r| r \in \mathcal{R}\}$
\EndFor
\end{algorithmic}

Note that $P$ is by construction a simple path and hence the constructed edge paths will be simple as well. In Lines 16 and 20 the cumulative load on all physical network functions and edges are computed, that would arise if request $r \in \mathcal{R}$ is fully embedded according to the $k$-th decomposition.

Lastly, the $k$-th decomposition $D^k_r$ is the sum (a triple) consisting of the fractional embedding value $f^k_r$, the mapping $m^k_r$, and the load $l^k_r$, is added to the set of potential embeddings $\mathcal{D}_r$ and the flow variables along $P$ are decreased by $f^k_r$. By decreasing the flow uniformly along $P$, flow preservation with respect to the adapted flow variables is preserved and the next iteration is started by incrementing $k$.

By construction, we obtain the following lemma:

**Lemma 7.** Each mapping $m^k_r$ constructed by Algorithm 1 in the $k$-th iteration is valid.

As initially the outgoing flow equals the embedding variable and as flow preservation is preserved after each iteration, the flow in the extended network is fully decomposed by Algorithm 1.

**Lemma 8.** The decomposition $\mathcal{D}_r$ computed in Algorithm 1 is complete, i.e. $\sum_{D^k_r \in \mathcal{D}_r} f^k \cdot x_r$ holds, for $r \in \mathcal{R}$.

Note that the above lemmas hold independently of whether the linear solutions are computed using IP 1 or IP 2. We give two lemmas relating the net profit (for SCEP-P) and the costs (for SCEP-C) of the decomposed mappings to the ones computed using the linear relaxations. We state the first without proof as it is a direct corollary of Lemma 8.

**Lemma 9.** Let $(\vec{x}, \vec{f}, \vec{l}) \in \mathcal{F}_{LP}$ denote a feasible solution to the linear relaxation of Integer Program 2 achieving a net profit of $\hat{B}$ and let $\mathcal{D}_r$ denote the respective decompositions of this linear solution for requests $r \in \mathcal{R}$ computed by Algorithm 2 then the following holds:

$$\sum_{r \in \mathcal{R}} \sum_{D^k_r \in \mathcal{D}_r} f^k \cdot h_r = \hat{B} \quad (15)$$

While the above shows that for SCEP-P the decomposition always achieves the same profit as the solution to the linear relaxation of IP 1 a similar statement holds for SCEP-C and IP 2.

**Lemma 10.** Let $(\vec{x}, \vec{f}, \vec{l}) \in \mathcal{F}_{LP}$ denote a feasible solution to the linear relaxation of Integer Program 2 having a cost of $\hat{C}$ and let $\mathcal{D}_r$ denote the respective decompositions of this linear solution computed by Algorithm 2 for requests $r \in \mathcal{R}$, then the following holds:

$$\sum_{r \in \mathcal{R}} \sum_{D^k_r \in \mathcal{D}_r} f^k \cdot c(m^k_r) \leq \hat{C} \quad (16)$$

Additionally, equality holds, if the solution $(\vec{x}, \vec{f}, \vec{l}) \in \mathcal{F}_{LP}$, respectively the objective $\hat{C}$, is optimal.

**Proof.** We consider a single request $r \in \mathcal{R}$ and show that $\sum_{D^k_r \in \mathcal{D}_r} f^k \cdot c(m^k_r) \leq \sum_{(x,y) \in R_S} c_S(x,y) \cdot l_{x,y}$ holds. The Integer Program 2 computes the loads on resources $(x, y) \in R_S$ in Constraints 7 and 8 based on the flow variables, which then drive the costs inside the objective (cf. Constraint 13). Within the decomposition algorithm, only paths $P \in G^e_{R_f}$ are selected such that $f_{r,e} > 0$ holds for all $e \in P$. Hence, the resulting mapping $m^r$, obtained by extracting the mapping information from $P$, uses only resources previously accounted for in the Integer Program 2. Since the computation of costs within the IP agrees with the definition of costs applied to the costs of a single mapping (cf. Equation 1), the reduction of flow variables along $P$ by $f^k_r$ (and the corresponding reduction of the loads) reduces the cost component of the objective by exactly $f^k_r \cdot c(m^r)$. Thus, the costs accumulated inside the decompositions $D^k_r \in \mathcal{D}_r$ are covered by the respective costs of the Integer Program 2. This proves the inequality. To prove equality, given an optimal solution, consider the following. According to the above argumentation, all costs accumulated within the resulting decomposition $\mathcal{D}_r$ are (at least) accounted for in the IP 2. Thus, the only possibility that the costs accounted for in the linear programming solution $(\vec{x}, \vec{f}, \vec{l})$ are greater than the costs accounted for in the decomposition $\mathcal{D}_r$ is that the linear programming solution still contains (cyclic) flows after having fully decomposed the request $r$. 


As these (cyclic) flows can be removed without violating any of the constraints while reducing the costs, the given solution cannot have been optimal.

By the above argumentation, it is easy to check that the (fractionally) accounted resources of the returned decompositions \( D_r \) are upper bounded by the resources allocations of the relaxations of Integer Programs \([1]\) and \([2]\) and hence are upper bounded by the respective capacities (cf. Constraint \([9]\)).

**Lemma 11.** The cumulative load induced by the fractional mappings obtained by Algorithm \([1]\) is less than the cumulative computed in the respective integer program and hence less than the offered capacity, i.e. for all resources \((x, y) \in R_S\)

\[
\sum_{r \in R} \sum_{D_r \in D_r^k} f_r^k \cdot \ell_r^k (x, y) \leq \sum_{r \in R} \ell_{r,x,y} \leq d_S(x, y), \tag{17}
\]

where \(\ell_{r,x,y}\) refers to the respective variables of the respective Integer Program.

**IV. APPROXIMATING SCEP-P**

This section presents our approximation algorithm for SCEP-P which is based on the randomized rounding of the decomposed fractional solutions of Integer Program \([1]\). In particular, our algorithm provides a tri-criteria approximation with high probability, that is, it computes approximate solutions with performance guarantees for the profit and for the maximal violation of capacities of both network functions and and edges with an arbitrarily high probability. We discuss the algorithm in Section \([IV-A]\) and then derive probabilistic bounds for the profit (see Section \([IV-B]\)) and the violation of capacities (see Section \([IV-C]\)). In Section \([IV-D]\) the results are condensed using a simple union-bound argument to prove our main theorem, namely that the presented algorithm is indeed a tri-criteria approximation for SCEP-P.

**A. Synopsis of the Approximation Algorithm**

The approximation scheme for SCEP-P is given as Algorithm \([2]\). Besides the problem specification, the approximation algorithm is handed four additional parameters: the parameters \(\alpha, \beta, \) and \(\gamma\) will bound the quality of the found solution with respect to the optimal solution in terms of profit achieved \((0 \leq \alpha \leq 1)\), the maximal violation of network function \((0 \leq \beta)\) and edge capacities \((0 \leq \gamma)\). As Algorithm \([2]\) is randomized and as we will only show that the algorithm has a constant success probability, the parameter \(Q\) controls the number of rounding tries to obtain a solution within the approximation factors \(\alpha, \beta, \) and \(\gamma\).

Algorithm \([2]\) first uses the relaxation of Integer Program \([1]\) to compute a fractional solution \((\bar{x}, \bar{f}, \bar{l})\). This solution is then decomposed according to Algorithm \([1]\) obtaining decompositions \(D_r\) for requests \(r \in R\). The while-loop (see Lines 4-19) attempts to construct a solution \(\{R', \{m_r | r \in R'\}\}\) for SCEP-P (cf. Definition \([3]\) according to the following scheme. Essentially, for every request \(r \in R\) a dice with \(|D_r| + 1\) many faces is cast, such that \(D_r^k \in D_r\) is chosen with probability \(f_r^k\) and none of the embeddings is selected with probability \(1 - \sum_{D_r \in D_r^k} f_r^k\). Within the algorithm, casting the dice is done by uniformly selecting a value \(p\) in the range of \([0, 1]\), such that the \(k\)-th decomposition \(D_r^k = (f_r^k, m_r^k, \ell_r^k) \in D_r\) is chosen iff. \(\sum_{l=1}^{k-1} f_r^l \leq p < \sum_{l=1}^{k} f_r^l\) holds. In case that a mapping was selected, the corresponding mapping and load informations are stored in the (globally visible) variables \(m_r\) and \(\ell_r\). In Lines 13-16 the request \(r \in R\) is added to the set of embedded requests \(R'\), the currently achieved objective \((B)\) and the cumulative loads on the physical network functions and edges are adapted accordingly.

After having iterated over all requests \(r \in R\), the obtained solution is returned only if the constructed solution achieves at least an \(\alpha\)-fraction of the objective of the linear program and violates node and edge capacities by factors less than \(1 + \beta\) and \(1 + \gamma\) respectively (see Lines 17 and 18). If after \(Q\) iterations no solution within the respective approximation bounds was found, the algorithm returns NULL.

**Algorithm 2: Approximation Algorithm for SCEP-P**

**Input:** Substrate \(G_S = (V_S, E_S)\), set of requests \(R\), approximation factors \(\alpha, \beta, \gamma \geq 0\), maximal number of rounding tries \(Q\)

**Output:** Approximate solution for SCEP-P

1. compute solution \((\bar{x}, \bar{f}, \bar{l})\) of Linear Program \([1]\)
2. compute \(\{D_r | r \in R\}\) using Algorithm \([1]\)
3. set \(q \leftarrow 1\)
4. while \(q \leq Q\) do
5. set \(R' \leftarrow \emptyset\) and \(\hat{m}_r \leftarrow \emptyset\) for all \(r \in R\)
6. set \(B \leftarrow 0\)
7. set \(L(x, y) = 0\) for all \((x, y) \in R_S\)
8. for \(r \in R\) do
9. choose \(p \in [0, 1]\) uniformly at random
10. if \(p \leq \sum_{D_r \in D_r^k} f_r^k\) then
11. set \(k \leftarrow \min\{k | k \in \{1, \ldots, |D_r|\} \land \sum_{l=1}^{k} f_r^l \geq p\}\)
12. set \(\hat{m}_r \leftarrow m_r^k\) and \(\hat{\ell}_r \leftarrow \ell_r^k\)
13. set \(R' \leftarrow R' \cup \{r\}\)
14. set \(B \leftarrow B + b_r\)
15. for \((x, y) \in R_S\) do
16. set \(L(x, y) \leftarrow L(x, y) + \hat{l}_r(x, y)\)
17. if \((B \geq \alpha \cdot \text{Opt}_{LP}\) and \(L(\tau, u) \leq (1 + \beta) \cdot d_S(\tau, u)\) for all \((\tau, u) \in R'_S\)) and \(L(u, v) \leq (1 + \gamma) \cdot d_S(u, v)\) for all \((u, v) \in E_S\) then
18. return \((R', \{\hat{m}_r | r \in R'\})\)
19. \(q \leftarrow q + 1\)
20. return NULL
be analyzed. Concretely, the analysis of the performance with respect to the objective is contained in Section IV-B while Section IV-C proves bounds for capacity violations and Section IV-D consolidates the results.

B. Probabilistic Guarantee for the Profit

To analyze the performance of Algorithm 2 with respect to the achieved profit, we recast the algorithm in terms of random variables. For bounding the profit achieved by the algorithm we introduce the discrete random variable $Y_r \in \{0, b_r\}$, which models the profit achieved by (potentially) embedding request $r \in \mathcal{R}$. According to Algorithm 2 request $r \in \mathcal{R}$ is embedded as long as the random variable $p$ in Line 9 was less than $\sum_{D \in D_r} f_r^k$. Hence, we have that $\Pr(Y_r = 0) = 1 - \sum_{D \in D_r} f_r^k$ holds, i.e. that the probability to achieve no profit for request $r \in \mathcal{R}$ is $1 - \sum_{D \in D_r} f_r^k$. On the other hand, the probability to embed request $r \in \mathcal{R}$ equals $\sum_{D \in D_r} f_r^k$ as in this case some decomposition will be chosen. Hence, we obtain $\Pr(Y_r = b_r) = \sum_{D \in D_r} f_r^k$. Given these random variables, we can model the achieved net profit of Algorithm 2 as:

$$B = \sum_{r \in \mathcal{R}} Y_r \ . \ (18)$$

The expectation of the random variable $B$ computes to

$$\sum_{r \in \mathcal{R}} \sum_{D \in D_r} f_r^k b_r \ = \mathbb{E}(B) \ ,$$

where $D_r$ denotes the decomposition of $(\tilde{x}, \tilde{f}, \tilde{l})$ obtained by Algorithm 7 for requests $r \in \mathcal{R}$.

To bound the probability of achieving a fraction of the profit of the optimal solution, we will make use of the following Chernoff-Bound over continuous random variables.

Theorem 13 (Chernoff-Bound [8]). Let $X = \sum_{i=1}^{n} X_i$ be a sum of $n$ independent random variables $X_i \in [0, 1]$, $1 \leq i \leq n$. Then the following holds for any $0 < \varepsilon < 1$:

$$\mathbb{P}(X \leq (1 - \varepsilon) \cdot \mathbb{E}(X)) \leq \exp(-\varepsilon^2 \cdot \mathbb{E}(X)/2) \quad (19)$$

Note that the above theorem only considers sums of random variables which are contained within the interval $[0, 1]$. In the following, we lay the foundation for appropriately rescaling the random variables $Y_r$ such that these are contained in the interval $[0, 1]$, while still allowing to bound the expected profit. To this end, we first show that we may assume that all requests can be fully fractionally embedded in the absence of other requests.

Lemma 14. We may assume without loss of generality that all requests can be fully fractionally embedded in the absence of other requests.

Proof. To show the claim we argue that we can extend the approximation scheme by a simple preprocessing step that filters out any request that cannot be fractionally embedded alone, i.e. having all the substrate’s available resources at its availability.

Practically, we can compute for each request $r \in \mathcal{R}$ the linear relaxation of Integer Program 1 in the absence of any other requests, i.e. for $\mathcal{R}' = \{r\}$. As the profit $b_r$ of request $r$ is positive, the variable $x_r$ is maximized effectively. If for the optimal (linear) solution $x_r < 1$ holds, then the substrate capacities are not sufficient to (fully) fractionally embed the request. Hence, in this case the request $r$ cannot be embedded (fully) under any circumstances in the original SCEP-P instance as any valid and feasible mapping $m_r$ for the original problem would induce a feasible solution with $x_r = 1$ when request $r$ is embedded alone. Note that this preprocessing step can be implemented in polynomial time, as the relaxation of IP 2 can be computed in polynomial time.

According to the above lemma we can assume that all requests can be fully fractionally embedded. Let $b_{\text{max}} = \max_{r \in \mathcal{R}} b_r$ denote the maximal profit of any single request. The following lemma shows that any fractional solution to the IP 1 will achieve at least a profit of $b_{\text{max}}$.

Lemma 15. $\mathbf{Opt}_{\text{LP}} \geq b_{\text{max}}$ holds, where $\mathbf{Opt}_{\text{LP}}$ denotes the optimal profit of the relaxation of IP 1.

Proof. Let $r' \in \mathcal{R}$ denote any of the requests having the maximal profit $b_{\text{max}}$ and let $(\tilde{x}_{r'}, \tilde{f}_{r'}, \tilde{l}_{r'})$ denote the linear solution obtained by embedding the request $r'$ according to IP 1 in the absence of other requests. Considering the set of original requests we now construct a linear solution $(\tilde{x}_r, \tilde{f}_r, \tilde{l}_r)$ over the variables corresponding to the original set of requests $\mathcal{R}$. Concretely, for $(\tilde{x}_r, \tilde{f}_r, \tilde{l}_r)$ we set all variables related to request $r'$ according to the solution $(\tilde{x}_{r'}, \tilde{f}_{r'}, \tilde{l}_{r'})$ and set all other variables to 0. This is a feasible solution, i.e. $(\tilde{x}_r, \tilde{f}_r, \tilde{l}_r) \in \mathcal{F}_{\text{LP}}$ holds, which achieves the same profit as the solution $(\tilde{x}_{r'}, \tilde{f}_{r'}, \tilde{l}_{r'})$, namely $b_{\text{max}}$ by Lemma 14. Hence, the profit of the optimal linear solution is lower bounded by this particular solution’s objective and the claim follows.

The above lemma is instrumental in proving the following probabilistic bound on the profit achieved by Algorithm 2.

Theorem 16. The probability of achieving less than $1/3$ of the profit of an optimal solution is upper bounded by $\exp(-2/9)$.

Proof. Instead of considering $B = \sum_{r \in \mathcal{R}} Y_r$, we consider the sum $B' = \sum_{r \in \mathcal{R}} Y'_r$ over the rescaled variables $Y'_r = Y_r/b_{\text{max}} \in [0, 1]$. Obviously $Y'_r \in [0, 1]$ holds. Choosing $\varepsilon = 2/3$ and applying Theorem 13 on $B'$ we obtain:

$$\mathbb{P}(B' \leq (1/3) \cdot \mathbb{E}(B')) \leq \exp(-2 \cdot \mathbb{E}(B')/9) \quad (20)$$
By Lemma 15 and as \( B' \cdot b_{\max} = B \) holds, we obtain that \( \mathbb{E}(B') = \mathbb{E}(B) / b_{\max} \geq 1 \) holds. Plugging in the minimal value of \( \mathbb{E}(B') \), i.e., 1, into the equation, we maximize the term \( \exp(-2 \cdot \mathbb{E}(B') / 9) \) and hence get

\[
\Pr(B' \leq (1/3) \cdot \mathbb{E}(B')) \leq \exp(-2/9).
\]

(21)

By using \( B' \cdot b_{\max} = B \), we obtain

\[
\Pr(B \leq (1/3) \cdot \mathbb{E}(B)) \leq \exp(-2/9).
\]

(22)

Denoting the optimal profit of Integer Program 1 by \( \text{Opt}_p \) and the optimal profit of the relaxation by \( \text{Opt}_{L_p} \), we note that \( \mathbb{E}(B) = \text{Opt}_{L_p} \) holds by Lemma 9. Furthermore, \( \text{Opt}_p \leq \text{Opt}_{L_p} \) follows from Fact 6. Thus,

\[
\frac{\text{Opt}_p}{3} \leq \frac{\text{Opt}_{L_p}}{3} = \frac{\mathbb{E}(B)}{3}
\]

(23)

holds, completing the proof together with Equation 22.

\( C. \) Probabilistic Guarantees for Capacity Violations

The input to Algorithm 2 encompasses the factors \( \beta, \gamma \geq 0 \), such that accepted solutions must satisfy \( L^\tau(u, v) \leq (1 + \delta) \cdot d_S(u, v) \) for \( (u, v) \in E_S \) and \( L^\tau(u, v) \leq (1 + \delta) \cdot d_S(u, v) \) for \( (u, v) \in E_S \). In the following, we will analyze which probability the above will hold. Our probabilistic bounds rely on Hoeffding’s inequality:

**Fact 17** (Hoeffding’s Inequality). Let \( \{X_i\} \) be independent random variables, such that \( X_i \in [a_i, b_i] \), then the following holds:

\[
\Pr \left( \sum_i X_i - \mathbb{E}(\sum_i X_i) \geq t \right) \leq \exp \left( -2t^2 / \left( \sum_i (b_i - a_i)^2 \right) \right)
\]

For our analysis to work, assumptions on the maximal allocation of a single request on substrate nodes and edges must be made. We define the maximal load as follows:

**Definition 18** (Maximal Load). We define the maximal load per network function and edge for each request as follows:

\[
\begin{align*}
\max_{i \in V^\tau, \tau(i) = \tau}^L d_r(i) & \quad \text{for } r \in \mathcal{R}, (\tau, u) \in R^\tau_S, \\
\max_{(i,j) \in E_r}^L d_r(i, j) & \quad \text{for } r \in \mathcal{R}, (u, v) \in E_S.
\end{align*}
\]

Additionally, we define the maximal load that a single request may induce on a network function type \( \tau \) on substrate node \( u \in V^\tau_S \) as the following sum:

\[
\max_{r, \tau, u}^L \sum_{i \in V_r, \tau(i) = \tau}^L d_r(i)
\]

(24)

(25)

(26)

The following lemma shows that we may assume \( \max_{r, \tau, u}^L \leq d_S(x, y) \) for all network resources \( (x, y) \in R_S \).

**Lemma 19.** We may assume the following without loss of generality.

\[
\max_{r, \tau, u}^L \leq d_S(x, y) \quad \forall (x, y) \in R_S
\]

(27)

Proof. Assume that \( \max_{r, \tau, u}^L > d_S(u) \) holds for some \( r \in \mathcal{R}, (\tau, u) \in R^\tau_S \). If this is the case, then there exists a virtual network function \( i \in V_r \) with \( \tau_r(i) = \tau \), such that \( d_r(i) > d_S(\tau, u) \) holds. However, if this is the case, then there cannot exist a feasible solution in which \( i \) is mapped onto \( u \), as this mapping would clearly exceed the capacity. Hence, we can remove the edges that indicate the mapping of the respective virtual function \( i \) on substrate node \( u \) in the extended graph construction a priori before computing the linear relaxation. The same argument holds true if \( \max_{r, \tau, u}^L > d_S(u, v) \) holds for some \( (u, v) \in E_S \).

We now model the load on the different network functions \( (\tau, u) \in R^\tau_S \) and the substrate edges \( (u, v) \in E_S \) induced by each of the requests \( r \in \mathcal{R} \) as random variables \( L_{r, \tau, u} \in [0, \max_{r, \tau, u}^L] \) and \( L_{r, u, v} \in [0, \max_{r, u, v}^L \cdot |E_r|] \) respectively. To this end, we note that Algorithm 2 chooses the \( k \)-th decomposition \( D_r \) with probability \( f_r^k \). Hence, with probability \( f_r^k \) the load \( L_r^\tau \) is induced for request \( r \in \mathcal{R} \). The respective variables can therefore be defined as \( P(L_{r, \tau, u} = i_r^k) = f_r^k \) (assuming pairwise different loads) and \( P(L_{r, u, v} = i_r^k) = 1 - \sum_{d_r \in D_r} f_r^d \) for \( (x, y) \in E_S \). Additionally, we denote by \( L_{r, x, y} = \sum_{r \in \mathcal{R}} L_{r, x, y} \) the overall load induced on function resource \( (x, y) \in R_S \). By definition, the expected load on the network nodes and the substrate edges \( (x, y) \in R^\tau_S \) compute to

\[
\mathbb{E}(L_{x, y}) = \sum_{r \in \mathcal{R}} \sum_{d_r \in D_r} f_r^d \cdot L_r^\tau(x, y).
\]

(28)

Together with Lemma 11 these equations yield:

\[
\mathbb{E}(L_{x, y}) \leq d_S(x, y) \quad \text{for } (x, y) \in R^\tau_S
\]

(29)

Using the above, we can apply Hoeffding’s Inequality:

**Lemma 20.** Let \( \Delta_T = \sum_{r \in \mathcal{R}} \max_{r, \tau, u}^L / \max_{r, \tau, u}^L \). The probability that the capacity of a single network function \( \tau \in \mathcal{T} \) on node \( u \in V^\tau_S \) is exceeded by more than a factor \( (1 + \sqrt{2 \cdot \log (|V^\tau_S| \cdot |\mathcal{T}|)} \cdot \Delta_T) \) is upper bounded by \( (|V^\tau_S| \cdot |\mathcal{T}|)^{-4} \).

Proof. Each variable \( L_{r, \tau, u} \) is clearly contained in the interval \([0, \max_{r, \tau, u}^L]\) and hence \( \sum_{r \in \mathcal{R}} (\max_{r, \tau, u}^L)^2 \) will be the denominator in Hoeffding’s Inequality. We choose \( t = \sqrt{2 \cdot \log (|V^\tau_S| \cdot |\mathcal{T}|)} \cdot \Delta_T \cdot d_S(\tau, u) \) and obtain:

\[
\Pr \left( L_{r, \tau, u} - \mathbb{E}(L_{r, \tau, u}) \geq t \right)
\]

\[
\leq \exp \left( \frac{-2 \cdot t^2}{\sum_{r \in \mathcal{R}} (\max_{r, \tau, u}^L)^2} \right)
\]

\[
\leq \exp \left( \frac{-4 \log (|V^\tau_S| \cdot |\mathcal{T}|) \cdot \Delta_T \cdot d_S(\tau, u)}{\sum_{r \in \mathcal{R}} (d_S(\tau, u) \cdot \max_{r, \tau, u}^L)^2} \right)
\]

(30)
We state the following corollaries without proof, showing that the above shown bounds work nicely if we assume more strict bounds on the maximal loads.

**Corollary 22.** Assume that \( \max_{r,u,v} L_{r,u,v} \leq \varepsilon \cdot d_S(r,u) \) holds for \( 0 < \varepsilon < 1 \) and all \((r,u) \in R_S^V \). With \( \Delta_E \) as defined in Lemma 20 we obtain: The probability that in Algorithm 2 the capacity of a single function network \( \tau \in T \) on node \( u \in V_S^\tau \) is exceeded by more than a factor \((1 + \varepsilon \cdot \sqrt{2 \cdot \log (|V_S| \cdot |T|)} \cdot \Delta_E)\) is upper bounded by \((|V_S| \cdot |T|)^{-4}\).

**Corollary 23.** Assume that \( \max_{r,u,v} L_{r,u,v} \leq \varepsilon \cdot d_S(u,v) \) holds for \( 0 < \varepsilon < 1 \) and all \((u,v) \in E_S \). With \( \Delta_E \) as defined in Lemma 27 we obtain: The probability that in Algorithm 2 the capacity of a single substrate edge \((u,v) \in E_S\) is exceeded by more than a factor \((1 + \varepsilon \cdot \sqrt{2 \cdot \log (|V_S| \cdot |T|)} \cdot \Delta_E)\) is upper bounded by \(|V_S|^{-4}\).

**D. Main Results**

We can now state the main tri-criteria approximation results obtained for SCEP-P. First, note that Algorithms 1 and 2 run in polynomial time. The runtime of Algorithm 1 is dominated by the search for paths and as in each iteration at least the flow of a single edge in the extended graph is set to 0, only \( O(\sum_{r \in R_S} |E_r| \cdot |E_S|) \) many graph searches are necessary. The runtime of the approximation itself is clearly dominated by the runtime to solve the Linear Program which has \( O(\sum_{r \in R_S} |E_r| \cdot |E_S|) \) many variables and constraints and can therefore be solved in polynomial time using e.g. the Ellipsoid algorithm 28.

The following lemma shows that Algorithm 2 can produce solutions of high quality with high probability.

**Lemma 24.** Let \( 0 < \varepsilon \leq 1 \) be chosen minimally, such that \( \max_{r=x,y} L_{r,x,y} \leq \varepsilon \cdot d(x,y) \) holds for all \((x,y) \in R_s\). Setting \( \alpha = 1/3, \beta = \varepsilon \cdot \sqrt{2 \cdot \log (|V_S| \cdot |T|)} \cdot \Delta_E \) and \( \gamma = \varepsilon \cdot \sqrt{2 \cdot \log (|V_S| \cdot |T|)} \cdot \Delta_E \) with \( \Delta_E \) as defined in Lemma 20 and 27 the probability that a solution is found within \( Q \in \mathbb{N} \) rounds is lower bounded by \( 1 - (19/20)^Q \) for \( |V_S| \geq 3 \).

**Proof.** We apply a union bound argument. By Lemma 20 the probability that for a single network function of type \( \tau \in T \) on node \( u \in V_S^\tau \) the allocations exceed the capacity by more than a factor \((1 + \beta)\) is less than \((|V_S| \cdot |T|)^{-4}\). Given that there are maximally \( |V_S| \cdot |T| \) many of network functions overall, the probability that any of these exceeds the capacity by a factor above \((1 + \beta)\) is less than \((|V_S| \cdot |T|)^{-3} \leq |V_S|^{-3}\). Similarly, by Lemma 21 the probability that the edge capacity of a single edge is violated by more than a factor \((1 + \gamma)\) is less than \(|V_S|^{-4}\). As there are at most \(|V_S|^2\) edges, the union bound gives us that the probability that the capacity of any of the edges is violated by a factor larger than \((1 + \gamma)\) is upper bounded by \(|V_S|^{-2}\). Lastly, by Theorem 16 the probability of not finding a solution having an \( \alpha \)-fraction of the optimal objective is less or equal to \( e^{\exp(-2/9)} \approx 0.8074\). The probability to not find a suitable solution, satisfying the objective and the capacity criteria, within a single round is

\[
\begin{align*}
\mathbb{P}(L_{r,u} \geq (1 + \sqrt{2 \cdot \log (|V_S| \cdot |T|)} \cdot \Delta_E) \cdot d_S(r,u)) \leq (|V_S| \cdot |T|)^{-4}.
\end{align*}
\]
therefore upper bounded by $\exp(-2/9) - 1/9 - 1/27 \leq 19/20$ if $|V_S| \geq 3$ holds. The probability find a suitable solution within $Q \in \mathbb{N}$ many rounds hence is $1 - (19/20)^Q$ for $|V_S| \geq 3$.

**Theorem 25.** Assuming that $|V_S| \geq 3$ holds, and that $\max_{r,x,y} s_{r,x,y} = \epsilon d_S(x, y)$ holds for all resources $(x, y) \in R_S$ with $0 < \epsilon \leq 1$ and by setting $\beta = \epsilon \cdot \sqrt{2 \cdot \log (|V_S| - 2)} \cdot \Delta V$ and $\gamma = \epsilon \cdot \sqrt{2 \cdot \log |V_S|} \cdot \Delta E$ with $\Delta V, \Delta E$ as defined in Lemmas 20 and 21, Algorithm 2 is a $(\alpha, 1 + \beta, 1 + \gamma)$ tri-criteria approximation algorithm for SCEP-P, such that it finds a solution with high probability, that achieves at least an $\alpha = 1/3$ fraction of the optimal profit and violates network function and edge capacities only within the factors $1 + \beta$ and $1 + \gamma$ respectively.

**V. APPROXIMATING SCEP-C**

In the previous section we have derived a tri-criteria approximation for the SCEP-P variant that maximizes the profit of embedding requests while only exceeding capacities within certain bounds. We show in this section that the approximation scheme for SCEP-P can be adapted for the cost minimization variant SCEP-C by introducing an additional preprocessing step.

Recall that the cost variant SCEP-C (see Definition 4) asks for finding a feasible embedding $m_r$ for all given requests $r \in R$, such that the sum of costs $\sum_{r \in R} c(m_r)$ is minimized. We propose Algorithm 3 to approximate SCEP-C. After shortly discussing the adaptations necessitated with respect to Algorithm 2 we proceed to prove the respective probabilistic guarantees analogously to the previous section with the main results contained in Section V.D.

**A. Synopsis of the Approximation Algorithm 3**

The approximation for SCEP-C given in Algorithm 3 is based on Algorithm 2. Algorithm 3 first computes a solution to the linear relaxation of Integer Program 2, which only differs from the previously used Integer Program 1 by requiring to embed all requests and adopting the objective to minimize the overall induced costs (cf. Section III-A). While for the relaxation of Integer Program 1 a feasible solution always exists – namely, not embedding any requests – this is not the case for the relaxation of Integer Program 2 i.e. $\mathcal{F}_{LP} = \emptyset$ might hold. Hence, if solving the formulation was determined to be infeasible, the algorithm returns that no solution exists. This is valid, as $\mathcal{F}_{IP} \subseteq \mathcal{F}_{LP}$ holds (cf. Fact 6) and, if $\mathcal{F}_{LP} = \emptyset$ holds, $\mathcal{F}_{IP} = \emptyset$ must follow.

Having found a linear programming solution $\bar{\mathcal{F}} = (\bar{x}, \bar{f}, \bar{l}) \in \mathcal{F}_{LP}$, we apply the decomposition algorithm presented in Section III-B to obtain the set of decomposed embeddings $D_r = \{(f^k_r, m^k_r, l^k_r)\}_k$ for all requests $r \in R$. As the Integer Program 2 enforces that $x_r = 1$ holds for all requests $r \in R$, we derive the following corollary from Lemma 8 stating that the sum of fractional embedding values is one for all requests.

**Corollary 26.** $\sum_{D^k_r \in D_r} f^k_r = 1$ holds for all requests.

Lines 5-14 is the core addition of Algorithm 3 compared to Algorithm 2. This preprocessing step effectively removes fractional mappings that are too costly from the set of decompositions $D_r$ by setting their fractional embedding values to zero and rescaling the remaining ones. Concretely, given a request $r \in R$, first the weighted (averaged) cost $W_{C_r} = \sum_{D^k_r \in D_r} f^k_r \cdot c(m^k_r)$ is computed. In the next step, the fractional embedding values of those decompositions costing less than two times the weighted cost $W_{C_r}$ is computed and assigned to $\lambda_r$. Then, for each decomposition $D^k_r = (f^k_r, m^k_r, l^k_r) \in D_r$ a new fractional embedding value $\hat{f}^k_r$ is defined: either $\hat{f}^k_r$ is set to zero or is set to $f^k_r$ rescaled by dividing by $\lambda_r$. The rescaling guarantees, that also for $D_r$ the sum of newly defined fractional embedding values $\hat{f}^k_r$ equals one, i.e. $\sum_{k=1}^{\vert D_r \vert} \hat{f}^k_r = 1$ holds. As there will exist at least a single decomposition $m^k_r$ such that $c(m^k_r) \leq 2 \cdot W_{C_r}$ holds, the set of decompositions will not be empty and the rest of the algorithm is well-defined. We formally prove this in Lemma 28 in the next section.

After having preprocessed the decomposed solution of the linear relaxation of Integer Program 2, the randomized rounding scheme already presented in Section IV is employed. For each request one of the decompositions $D^k_r \in D_r$ is chosen according to the fractional embedding values $\hat{f}^k_r$ (see Lines 20-24). Note that as $\sum_{D^k_r \in D_r} \hat{f}^k_r = 1$ holds, the index $k$ will always be well-defined and hence for each request $r \in R$ exactly one mapping $\hat{m}_r$ will be selected. Analogously to Algorithm 2, the variables $L(x, y)$ store the induced loads by the current solution on substrate resource $(x, y) \in R_S$ and a solution is only returned if neither any of the node or any of the edge capacities are violated by more than a factor of $(2 + \beta)$ and $(2 + \gamma)$ respectively, where $\beta, \gamma > 0$ are again the respective approximation guarantee parameters (cf. Section IV), which are an input to the algorithm. In the following sections, we prove that Algorithm 3 yields solutions having at most two times the optimal costs and which exceed node and edge capacities by no more than a factor of $(2 + \beta)$ and $(2 + \gamma)$ respectively with high probability.

**B. Deterministic Guarantee for the Cost**

In the following we show that Algorithm 2 will only produce solutions whose costs are upper bounded by two times the optimal costs. The result follows from restricting the set of potential embeddings $D_r$ to only contain decompositions having less than two times the weighted cost $W_{C_r}$ for each request $r \in R$. To show this, we first reformulate Lemma 10 in terms of the weighted cost as follows.

**Corollary 27.** Let $\bar{\mathcal{F}} = (\bar{x}, \bar{f}, \bar{l}) \in \mathcal{F}_{LP}$ denote an optimal solution to the linear relaxation of Integer Program 2 having a cost of $W_{Opt_{LP}}$ and let $D_r$ denote the decomposition of this linear solution computed by Algorithm 2 then the following holds:

$$\sum_{r \in R} W_{C_r} = W_{Opt_{LP}} \cdot (37)$$

The above corollary follows directly from Lemma 10 and the definition of $W_{C_r}$, as computed in Line 6. As a next
Lemma 28. The sum of fractional embedding values of decompositions whose cost is upper bounded by two times WC$_r$ is at least 1/2. Formally, $\lambda_r \geq 1/2$ holds for all requests $r \in \mathcal{R}$.

Proof. For the sake of contradiction, assume that $\lambda_r < 1/2$ holds for any request $r \in \mathcal{R}$. By the definition of WC$_r$ and the assumption on $\lambda_r$, we obtain the following contradiction:

$$\text{WC}_r = \sum_{D_r \in \mathcal{D}_r} f^k_r \cdot c(m^k_r) \tag{38}$$
$$\leq \sum_{D_r \in \mathcal{D}_r, c(m^k_r) \geq 2 \cdot \text{WC}_r} f^k_r \cdot c(m^k_r) \tag{39}$$
$$\leq \sum_{D_r \in \mathcal{D}_r, c(m^k_r) \geq 2 \cdot \text{WC}_r} f^k_r \cdot 2 \cdot \text{WC}_r \tag{40}$$
$$\leq (1 - \lambda_r) \cdot 2 \cdot \text{WC}_r \tag{41}$$
$$< \text{WC}_r \tag{42}$$

For Equation 38 the value of WC$_r$ as computed in Algorithm 3 was used. Equation 39 holds as only a subset of decompositions, namely the ones with mappings of costs higher than two times WC$_r$, are considered and $f^k_r \geq 0$ holds by definition. The validity of Equation 40 follows as all the considered decompositions have a cost of at least two times WC$_r$ and Equation 41 follows as $(1 - \lambda_r) > 1/2$ holds by assumption. Lastly, Equation 42 yields the contradiction, showing that indeed $\lambda_r \geq 1/2$ holds for all requests $r \in \mathcal{R}$. \qed

By the above lemma, the set $\mathcal{D}_r$ will indeed not be empty. As per rescaling $\sum_{D_r \in \mathcal{D}_r} f^k_r = 1$ holds, for each request exactly one decomposition will be selected. Finally, we derive the following lemma.

Lemma 29. The cost $\sum_{r \in \mathcal{R}} c(\hat{m}_r)$ of any solution returned by Algorithm 3 is upper bounded by two times the optimal cost.

Proof. Let Opt$_{LP}$ denote the cost of the optimal linear solution to the linear relaxation of Integer Program 2 and let Opt$_{IP}$ denote the cost of the respective optimal integer solution. By allowing to select only decompositions $m_r$ for which $c(m_r) \leq 2 \cdot \text{WC}_r$ holds in Algorithm 3 and as for each request a single mapping is chosen, we have $c(\hat{m}_r) \leq 2 \cdot \text{WC}_r$, where $\hat{m}_r$ refers to the actually selected mapping. Hence $\sum_{r \in \mathcal{R}} c(\hat{m}_r) \leq 2 \cdot \sum_{r \in \mathcal{R}} \text{WC}_r$ holds. By Corollary 27 and the fact that Opt$_{LP} \leq$ Opt$_{IP}$ holds (cf. Fact 6), we can conclude that $\sum_{r \in \mathcal{R}} c(\hat{m}_r) \leq 2 \cdot \text{Opt}_{IP}$ holds, proving the lemma. \qed

Note that the above upper bound on the cost of any produced solution is deterministic and does not depend on the random choices made in Algorithm 3.

C. Probabilistic Guarantees for Capacity Violations

Algorithm 3 employs the same rounding procedure as Algorithm 2 presented in Section V for computing approximations for SCEPT-P. Indeed, the only major change with respect to Algorithm 2 is the preprocessing that replaces the fractional embedding values $f^k_r$ with $\hat{f}^k_r$. As the changed values $\hat{f}^k_r$ are used for probabilistically selecting the mapping for each of the requests, the analysis of the capacity violations needs to reflect these changes. To this end, Lemma 28 will be instrumental as it shows that each of the decompositions is scaled by at most a factor of two. As the following lemma shows, this implies that the decompositions contained in $\mathcal{D}_r$ are using at most two times the original capacities of nodes and functions.
Lemma 30. The fractional decompositions of all requests violate node function and edge capacities by at most a factor of two, i.e.,
\[ \sum_{r \in \mathcal{R}} \sum_{D \in \mathcal{D}_r} f^k_r : l^k_r(x, y) \leq 2 \cdot d_S(x, y) \] (43)
holds for all resources \((x, y) \in R_S^V\).

Proof. By Lemma 11, the fractional allocations of the original decompositions \(\mathcal{D}_r\) do not violate capacities. Hence, by multiplying Equation 17 by two, we obtain that
\[ 2 \cdot \sum_{r \in \mathcal{R}} \sum_{D \in \mathcal{D}_r} f^k_r \leq 2 \cdot d_S(x, y) \] (44)
holds for all resources \((x, y) \in R_S^V\). By Lemma 28, \(\lambda_p \geq 1/2\) holds for all requests \(r \in \mathcal{R}\). Hence, the probabilities of the decompositions having costs less than \(\mathbb{W}_c\) are scaled by at most a factor of two, i.e., \(f^k_r \leq 2 \cdot f^k_r\) holds for all requests \(r \in \mathcal{R}\) and each decomposition. As the mappings and the respective loads of \(\mathcal{D}_r\) and \(\mathcal{D}_r\) are the same, we obtain that
\[ \sum_{r \in \mathcal{R}} \sum_{D \in \mathcal{D}_r} f^k_r : l^k_r(x, y) \leq 2 \cdot \sum_{r \in \mathcal{R}} \sum_{D \in \mathcal{D}_r} f^k_r : l^k_r(x, y) \]
holds for all resources \((x, y) \in R_S^V\). Together with Equation 44, this proves the lemma.

Given the above lemma, we restate most of the lemmas already contained in Section 4C with only minor changes. We note that the definition of the maximal loads (cf. Definition 18) are independent of the decompositions found and that the corresponding assumptions made in Lemma 19 are still valid when using the relaxation of Integer Program 2 and are independent of the scaling. Concretely, forbidding mappings of network functions or edges onto elements that can never support them is still feasible. Furthermore note that while we effectively scale the probabilities of choosing specific decomposition, this does not change the corresponding loads of the (identical) mappings.

We again model the load on the different network functions \((\tau, u) \in R_S^V\) and the substrate edges \((u, v) \in E_S^V\) induced by each of the requests \(r \in \mathcal{R}\) as random variables \(L_{r, \tau, u} \in [0, \max_{r, \tau, u} L]\) and \(L_{r, u, v} \in [0, \max_{r, u, v} |E_r|]\) respectively. To this end, we note that Algorithm 3 chooses the \(k\)-th decomposition \(D^k_r = (f^k_r, l^k_r, m^k_r)\) with probability \(f^k_r\). Hence, with probability \(f^k_r\) the load \(l^k_r\) is induced for request \(r \in \mathcal{R}\). The respective variables can therefore be defined as \(P(L_{r, \tau, u} = l^k_r(x, y)) = f^k_r\) (assuming pairwise different loads) and \(P(L_{r, u, v} = 0) = 1 - \sum_{D^k_r \in \mathcal{D}_r} f^k_r\) for \((x, y) \in R_S^V\). Again, we denote by \(L_{x, y} = \sum_{r \in \mathcal{R}} L_{r, x, y}\) the overall load induced on resource \((x, y) \in R_S^V\).

By definition of the expectation, the expected load on the network resource \((x, y) \in R_S^V\) computes to
\[ \mathbb{E}(L_{x, y}) = \sum_{r \in \mathcal{R}} \sum_{D^k_r \in \mathcal{D}_r} f^k_r : l^k_r(x, y) \] (45)
Together with Lemma 30, we obtain:
\[ \mathbb{E}(L_{x, y}) \leq 2 \cdot d_S(x, y) \quad \text{for } (x, y) \in R_S^V \] (46)
Using the above, we again apply Hoeffding’s Inequality, while slightly adapting the probabilistic bounds compared with Lemma 20.

Lemma 31. Let \(\Delta_T = \sum_{r \in \mathcal{R}} \frac{L_{r, \tau, u}}{\max_{r, \tau, u}}.\) The probability that the capacity of a single network function \(r \in \mathcal{T}\) on node \(u \in V_S^V\) is exceeded by more than a factor \((2 + \sqrt{\log(|V_S^V| \cdot |\mathcal{T}|)} \cdot \Delta_T)\) is upper bounded by \(|V_S^V| \cdot |\mathcal{T}|^{-2}\).

Proof. Each variable \(L_{r, \tau, u}\) is clearly contained in the interval \([0, \max_{r, \tau, u} L]\). We choose \(t = \sqrt{\log(|V_S^V| \cdot |\mathcal{T}|)} \cdot \Delta_T \cdot d_S(r, u)\) and obtain:
\[ \mathbb{P}(L_{r, \tau, u} - \mathbb{E}(L_{r, \tau, u}) \geq t) \]
\[ \leq \exp \left( -\frac{2t^2}{\sum_{r \in \mathcal{R}} (\max_{r, \tau, u} L)^2} \right) \]
\[ \leq \exp \left( -\frac{2t^2}{|V_S^V| \cdot |\mathcal{T}| \cdot \Delta_T \cdot d_S(r, u)^2} \right) \]
\[ = \exp \left( -\frac{2t^2}{|V_S^V| \cdot |\mathcal{T}| \cdot \max_{r, \tau, u} L_{r, \tau, u}^2} \right) \]
\[ = |V_S^V| \cdot |\mathcal{T}|^{-2} \]
Analogously to Lemma 20, we have used again
\[ \max_{r, \tau, u} L_{r, \tau, u} \leq d_S(r, u) \cdot \max_{r, \tau, u, v} \frac{L_{r, v}}{\max_{r, \tau, u, v} |E_r|} \] (48)
in Equation 47. Analogously to the proof of Lemma 20 we have \(\mathbb{E}(L_{r, \tau, u}) \leq 2 \cdot d_S(r, u)\) by Equation 46 and the lemma follows.

We restate Lemma 21 without proof as the only change is replacing the factor of \(\sqrt{2}\) with the factor \(\sqrt{3}/2\) and observing that by using Equation 46 we obtain a \(2 + \gamma\) approximation for the load instead of a \(1 + \gamma\) one.

Lemma 32. Let \(\Delta_E = \sum_{r \in \mathcal{R}} |E_r|^2.\) The probability that the capacity of a single substrate edge \((u, v) \in E_S^V\) is exceeded by more than a factor \((2 + \sqrt{3}/2 \cdot \log(|V_S^V|) \cdot \Delta_E)\) is bounded by \(|V_S^V|^{-2}\).

The following corollaries show that the above bounds play out nicely if we assume more strict bounds on the maximal loads.

Corollary 33. Assume that \(\max_{r, \tau, u} L_{r, \tau, u} \leq \varepsilon \cdot d_S(r, u)\) holds for \(0 < \varepsilon < 1\) and all \(r \in \mathcal{T}, u \in V_S^V\). With \(\Delta_T\) as defined in Lemma 20 we obtain: The probability that in Algorithm the capacity of a single network function \(r \in \mathcal{T}\) on node \(u \in V_S^V\) is exceeded by more than a factor \((2 + \varepsilon \cdot \sqrt{\log(|V_S^V| \cdot |\mathcal{T}|)} \cdot \Delta_T)\) is upper bounded by \(|V_S^V| \cdot |\mathcal{T}|^{-2}\).
Corollary 34. Assume that $\max_{x,y} d_{S}(x,y) \leq \varepsilon \cdot d_{S}(u,v)$ holds for $0 < \varepsilon < 1$ and all $(u,v) \in E_{S}$. With $\Delta_{E}$ as defined in Lemma 27, we obtain: The probability that in Algorithm 2 the capacity of a single substrate node $(u,v) \in E_{S}$ is exceeded by more than a factor $(2 + \varepsilon \cdot \sqrt{3/2} \cdot \log |V_{S}| \cdot \Delta_{E})$ is upper bounded by $|V_{S}|^{-2}$.

D. Main Results

We can now proceed to state the main tri-criteria approximation results obtained for SCEP-C. The argument for Algorithm 3 having a polynomial runtime is the same as for Algorithm 2 since the preprocessing can be implemented in polynomial time.

The following lemma shows that Algorithm 2 can produce solutions of high quality with high probability.

Lemma 35. Let $0 < \varepsilon < 1$ be chosen minimally, such that $\max_{x,y} d_{S}(x,y) \leq \varepsilon \cdot d_{S}(u,v)$ holds for all network resources $(x,y) \in R_{S}$. Setting $\beta = \varepsilon \cdot \sqrt{\log |V_{S}| \cdot |T|} \cdot \Delta_{V}$ and $\gamma = \varepsilon \cdot \sqrt{3/2} \cdot \log |V_{S}| \cdot \Delta_{E}$ as defined in Lemmas 31 and 32, the probability that a solution is found within $Q \in \mathbb{N}$ rounds is lower bounded by $1 - (2/3)^{Q}$ if $|V_{S}| \geq 3$ holds.

Proof. We again apply a union bound argument. By Lemma 31, the probability that for a single network function of type $\tau \in T$ on node $u \in V_{S}$ the allocations exceed $(2 + \beta) \cdot d_{S}(\tau,u)$ is less than $(|V_{S}| \cdot |T|)^{-1}$. Given that there are maximally $|V_{S}| \cdot |T|$ many network functions overall, the probability that on any network function more than $(2 + \beta) \cdot d_{S}(\tau,u)$ resources will be used is less than $(|V_{S}| \cdot |T|)^{-1} \leq |V_{S}|^{-1}$. Similarly, by Lemma 32 the probability that the allocation on a specific edge is larger than $(2 + \beta) \cdot d_{S}(u,v)$ is less than $|V_{S}|^{-3}$. As there are at most $|V_{S}|^{2}$ edges, the union bound gives us that the probability that any of these edges’ allocations will be higher than $(2 + \gamma) \cdot d_{S}(u,v)$ is less than $|V_{S}|^{-1}$. As the cost of any solution found is deterministically always smaller than two times the optimal cost (cf. Lemma 29), the probability of not finding an appropriate solution within a single round is upper bounded by $2/|V_{S}|$. For $|V_{S}| \geq 3$ the probability of finding a feasible solution within $Q \in \mathbb{N}$ rounds is therefore $1 - (2/3)^{Q}$.

Finally, we can state the main theorem showing that Algorithm 3 is indeed a tri-criteria approximation algorithm.

Theorem 36. Assuming that $|V_{S}| \geq 3$ holds and that $\max_{x,y} d_{S}(x,y) \leq \varepsilon \cdot d_{S}(u,v)$ holds for all network resources $(x,y) \in R_{S}$ with $0 < \varepsilon < 1$ and by setting $\beta = \varepsilon \cdot \sqrt{\log |V_{S}| \cdot |T|} \cdot \Delta_{V}$ and $\gamma = \varepsilon \cdot \sqrt{3/2} \cdot \log |V_{S}| \cdot \Delta_{E}$ as defined in Lemmas 27 and 28, Algorithm 2 is a $(\alpha, 2 + \beta, 2 + \gamma)$ tri-criteria approximation algorithm for SCEP-C, such that it finds a solution with high probability, with costs less than $\alpha = 2$ times higher than the optimal cost and violates network function and edge capacities only within the factors $2 + \beta$ and $2 + \gamma$ respectively.

VI. APPROXIMATE CACTUS GRAPH EMBEDDINGS

Having discussed approximations for linear service chains, we now turn towards more general service graphs (essentially a virtual network), i.e. service specifications that may contain cycles or branch separate sub-chains. Concretely, we propose a novel linear programming formulation in conjunction with a novel decomposition algorithm for service graphs whose undirected interpretation is a cactus graph. Given the ability to decompose fractional solutions, we show that we can still apply the results of Sections V and VI for this case. Our results show that our approximation scheme can be applied as long as linear solutions can be appropriately decomposed. We highlight the advantage of our novel formulation by showing that the standard multi-commodity flow approach applied in the Virtual Network Embedding Problem (VNEP) literature cannot be decomposed and hence cannot be used in our approximation framework.

This section is structured as follows. In Section VI-A we motivate why considering more complex service graphs is of importance. Section VI-B introduces the notion of service cactus graphs and introduces the respective generalizations of SCEP-P and SCEP-C. Section VI-C shows how these particular service graphs can be decomposed into subgraphs. Building on this a priori decomposition of the service graphs, we introduce extended graphs for cactus graphs and the respective integer programming formulation in Sections VI-D and VI-E. In Section VI-F we show how the linear solutions can be decomposed into a set of fractional embeddings analogously to the decompositions computed in Section III. Section VI-G shows that the approximation results for SCEP-P and SCEP-C still hold for this general graph class. Lastly, Section VI-H shows that the standard approach of using multi-commodity flow formulations yields non-decomposable solutions, i.e. our approximation framework cannot be applied when using the standard approach. This also sheds light on the question why no approximations are known for the Virtual Network Embedding Problem, which considers the embedding of general graphs.

A. Motivation & Use Cases

While service chains were originally understood as linear sequences of service functions (cf. [38]), we witness a trend toward more complex chaining models, where a single network function may spawn multiple flows towards other functions, or merge multiple incoming flows. We will discuss one of these use cases in detail and refer the reader to [11], [17], [20], [25], [30] for an overview on more complex chaining models.

Let us give an example which includes functionality for load balancing, flow splitting and merging. It also makes the case for “cyclic” service chains. The use case is situated in the context of LTE networks and Quality-of-Experience (QoE) for mobile users. The service chain is depicted in Figure 2 and is discussed in-depth in the IETF draft [20]. Depending on whether the incoming traffic from the packet gateway (P-GW) at the first load balancer LB, is destined for port 80 and has type TCP, it is forwarded through a performance
P-GW → LB1 → PEP → LB2 → FW → NAT → Internet

[TCP, port:80]

[otherwise]

G

graphs like the one in Figure 2. While for service chains request chain definition of Section II to more complex service

Figure 2 (depicted as dashed or solid), but can be extended to the outbound or the inbound connections of the example in Our approach presented henceforth allows to embed either “cycle” LB

statefulness of the network functions, all connections between state on incoming and outgoing flows. Note that the PEP

1 needs to be passed through the PEP (e.g., for caching) or needs to know whether the traffic received from the firewall
deadged rectangles

a higher quality of experience for the end-user and reduces transcoding if the content is a video. This allows to offer both – depending on user agent information – additionally perform

may redirect the traffic towards one of the caches. Receiving Figure 2, cf. [20]) and whether the content is cached, the PEP

outgoing traffic destined to the Internet. Depending on the ratio on the bandwidth requirements. Furthermore, e.g., video streams received by the PEP from one of the caches may be transcoded on-the-fly, and hence the outgoing bandwidth of the PEP towards LB1 might be less than the traffic received.

enhancement proxy (PEP). Otherwise, LB1 forwards the flow directly to a second load balancer LB2 which merges all outgoing traffic destined to the Internet. Depending on the deep-packet inspection performed by the PEP (not depicted in Figure 2) cf. [20]) and whether the content is cached, the PEP may redirect the traffic towards one of the caches. Receiving the content either from the caches or the Internet, the PEP may – depending on user agent information – additionally perform transcoding if the content is a video. This allows to offer both a higher quality of experience for the end-user and reduces network utilization.

It must be noted that all depicted network functions (depicted as rectangles) are stateful. The load balancer LB2 e.g., needs to know whether the traffic received from the firewall needs to be passed through the PEP (e.g., for caching) or whether it can be forwarded directly towards LB1. Similarly, the firewall and the network-address translation need to keep state on incoming and outgoing flows. Note that the PEP spawns sub-chains towards the different caches and that the example also contains multiple types of cycles. Based on the statefulness of the network functions, all connections between network functions are bi-directed. Furthermore, there exists a “cycle” LB1, PEP, LB2, LB1 (in the undirected interpretation). Our approach presented henceforth allows to embed either the outbound or the inbound connections of the example in Figure 2 (depicted as dashed or solid), but can be extended to also take bi-directed connections into account.

B. Service Cactus Graphs Embedding Problems

Given the above motivation, we will now extend the service chain definition of Section II to more complex service graphs, like the one in Figure 2. While for service chains request graphs \( G_r = (V_r, E_r) \) were constrained to be lines (cf. Section II), we relax this constraint in the following definition.

Definition 37 (Service Cactus Graphs). Let \( G_r = (V_r, E_r) \) denote the service graph of request \( r \in R \). We call \( G_r \) a service cactus graph if the following two conditions hold:

1) \( E_r \) does not contain opposite edges, i.e., for all \((v, u) \in E_r\) the opposite edge \((u, v)\) is not contained in \( E_r \).
2) The undirected interpretation \( G_r = (\bar{V}_r, \bar{E}_r) \), with \( \bar{V}_r = V_r \) and \( \bar{E}_r = \{(u, v) | (u, v) \in E_r \} \), is a cactus graph, i.e. any two simple cycles share at most a single node.

Moreover, we do not assume that the service cactus graphs have unique start or end nodes. Specifically, all virtual nodes \( i \in V_r \) can have arbitrary types. The definitions of the capacity function \( d_r : V_r \cup \bar{E}_r \rightarrow \mathbb{R}_{\geq 0} \) and the cost functions \( c_S : R_S \rightarrow \mathbb{R}_{\geq 0} \) are not changed. As the definitions of valid and feasible embeddings as well as the definition of SCEP-P and SCEP-C (see Definitions 1, 2, 3, and 4 respectively) do not depend on the underlying graph model, these are still valid, when considering service cactus graphs as input. To avoid ambiguities, we refer to the respective embedding problems on cactus graphs as SCGEP-P and SCGEP-C.

C. Decomposing Service Cactus Graphs

As the Integer Programming formulation (see IP 3 in Section VI-F) for service cactus graphs is much more complex, we first introduce a graph decomposition for cactus graphs to enable the concise descriptions of the subsequent algorithms.

Concretely, we apply a breadth-first search to re-orient edges as follows. For a service cactus graph \( G_r = (\bar{V}_r, \bar{E}_r) \) and whether the content is cached, the PEP may choose any node as the root node and denote it by \( r_r \). Now, we perform a simple breadth-first search in the undirected service graph \( G_r \) (see Definition 37) and denote by \( \pi_r : \bar{V}_r \rightarrow \bar{V}_r \cup \{\text{NULL}\} \) the predecessor function, such that if \( \pi_r(i) = j \) then virtual node \( i \in V_r \) was explored first from virtual node \( j \in V_r \). Let \((V_r^{\text{dfs}}, E_r^{\text{dfs}})\) denote the graph \( G_r^{\text{dfs}} \) with \( V_r^{\text{dfs}} = V_r \) and \( E_r^{\text{dfs}} = \{(i, j) | j \in V_r, \pi_r(j) = i \land \pi_r(j) \neq \text{NULL}\} \). Clearly, \( G_r^{\text{dfs}} \) is a directed acyclic graph. We note the following based on the cactus graph nature of \( G_r \):

Lemma 38.

1) The in-degree of any node is bounded by 2, i.e. \( |\delta_{E_r^{\text{dfs}}}(i)| \leq 2 \) holds for all \( i \in V_r^{\text{dfs}} \).
2) Furthermore, if \( |\delta_{E_r^{\text{dfs}}}(i)| = 2 \) holds for some \( i \in V_r^{\text{dfs}} \), then there exists a node \( j \in V_r^{\text{dfs}} \), such that there exist exactly two paths \( P_r, P_l \) in \( V_r^{\text{dfs}} \) from \( j \) to \( i \) which only coincide in \( i \) and \( j \).

Proof. With respect to Statement 1) note that node \( i \) must be reached from the (arbitrarily) chosen root \( r_r \in V_r^{\text{dfs}} \). For the sake of contradiction, if \( |\delta_{E_r^{\text{dfs}}}(i)| > 2 \) then there exist at least three (pair-wise different) paths \( P_1, P_2, P_3 \) from \( r_r \) to \( i \). Hence, in the undirected representation \( G_r \) there must exist at least two cycles overlapping in more than one node, namely in \( i \) as well as some common predecessor of \( i \). This is not allowed by the cactus nature of \( G_r \) (cf. Definition 37) and hence \( |\delta_{E_r^{\text{dfs}}}(i)| \leq 2 \) must hold for any \( i \in V_r \).

Statement 2) holds due to the following observations. As all nodes are reachable from the root \( r_r \), we are guaranteed
to find a common predecessor \( j \in V_{\text{bfs}} \) while backtracking along the reverse directions of edges in \( E_{\text{bfs}} \). If this was not the case, then there would exist two different sources, which is – by construction – not possible.

Based on the above lemma, we will now present a specific graph decomposition for service cactus graphs. Concretely, we show that service cactus graphs can be decomposed into a set of cyclic subgraphs together with a set of line subgraphs with unique source and sink nodes. For each of these subgraphs, we will define an extended graph construction that will be used in our Integer Programming formulation. Concretely, the graph decomposition given below, will enable a structured induction of flow in the extended graphs: the flow reaching a sink in one of the subgraphs will induce flow in all subgraphs having this node as source. This will also enable the efficient decomposition of the computed flows (cf. Section VI-F).

**Definition 39 (Service Cactus Graph Decomposition).** Any graph \( G_{\text{bfs}} \) of a service cactus graph \( G_r \) can be decomposed into a set of cycles \( C_r = \{C_1, C_2, \ldots\} \) and a set of simple paths \( P_r = \{P_1, P_2, \ldots\} \) with corresponding subgraphs \( G^C_r = (V^C_r, E^C_r) \) and \( G^P_r = (V^P_r, E^P_r) \) for \( C_k \in C_r \) and \( P_k \in P_r \), such that:

1) The graphs \( G^C_r \) and \( G^P_r \) are connected subgraphs of \( G_{\text{bfs}} \) for \( C_k \in C_r \) and \( P_k \in P_r \) respectively.

2) Sources \( s^{C_k}_{\text{brs}} \), \( t^{C_k}_{\text{brs}} \in V^C_r \) and sinks \( s^{P_k}_{\text{brs}}, t^{P_k}_{\text{brs}} \in V^P_r \) are given for \( C_k \in C_r \) and \( P_k \in P_r \) respectively, such that \( \delta^+_{E^C_r}(s^{C_k}_{\text{brs}}) = \delta^+_{E^C_r}(t^{C_k}_{\text{brs}}) = \delta^+_{E^P_r}(s^{P_k}_{\text{brs}}) = \delta^+_{E^P_r}(t^{P_k}_{\text{brs}}) = \emptyset \) holds within \( G^C_r \) and \( G^P_r \) respectively.

3) \( \{E^C_r|C_k \in C_r\} \cup \{E^P_r|P_k \in P_r\} \) is a (pair-wise disjoint) partition of \( E_{\text{bfs}} \).

4) Each \( C_k \in C_r \) consists of exactly two branches \( B^C_{r_1}, B^C_{r_2} \subset E^C_r \) such that both these branches start at \( s^{C_k}_{\text{brs}} \) and terminate at \( t^{C_k}_{\text{brs}} \) and cover all edges of \( G^C_r \).

5) For \( P_k \in P_r \) the graph \( G^P_r \) is a simple path from \( s^{P_k}_{\text{brs}} \) to \( t^{P_k}_{\text{brs}} \).

6) Paths may only overlap at sources and sinks: \( \forall P_k, P'_k \in P_r, P_k \neq P'_k : V^{P_k} \cap V^{P'_k} \subseteq \{s^{P_k}, t^{P_k}, s^{P'_k}, t^{P'_k}\} \).

We introduce the following sets:

\[ V^{C_k} = \{s^{C_k}_{|C_k \in C_r}\} \quad V^{P_k} = \{s^{P_k}_{|P_k \in P_r}\} \]
\[ V^{C_k} = \{s^{C_k}_{|C_k \in C_r}\} \quad V^{P_k} = \{s^{P_k}_{|P_k \in P_r}\} \]
\[ V^{C_k} = \{s^{C_k}_{|C_k \in C_r}\} \quad V^{P_k} = \{s^{P_k}_{|P_k \in P_r}\} \]
\[ V^{P_k} = \{t^{P_k}_{|P_k \in P_r}\} \]
\[ V^{C_k} = \{s^{C_k}, t^{C_k}_{|C_k \in C_r}\} \quad V^{P_k} = \{s^{P_k}, t^{P_k}_{|P_k \in P_r}\} \]
\[ V^{P_k} = \{t^{P_k}_{|P_k \in P_r}\} \]
\[ E^C_r = \{e \in E^C_r|C_k \in C_r\} \quad E^P_r = \{e \in E^P_r|P_k \in P_r\} \]
\[ E^P_r = E^P_r \cup \{t^{P_k}_1\} \quad E^P_r = E^P_r \cup \{t^{P_k}_1\} \]
\[ E^C_r = E^C_r \cup \{t^{C_k}_1\} \quad E^P_r = E^P_r \cup \{t^{P_k}_1\} \]
\[ E^C_r = E^C_r \cup \{t^{C_k}_1\} \quad E^P_r = E^P_r \cup \{t^{P_k}_1\} \]
\[ E^C_r = E^C_r \cup \{t^{C_k}_1\} \quad E^P_r = E^P_r \cup \{t^{P_k}_1\} \]
\[ E^C_r = E^C_r \cup \{t^{C_k}_1\} \quad E^P_r = E^P_r \cup \{t^{P_k}_1\} \]
\[ E^C_r = E^C_r \cup \{t^{C_k}_1\} \quad E^P_r = E^P_r \cup \{t^{P_k}_1\} \]

\[ B^{C_k} = \{i \in V^{C_k}_r|\delta^+_{E^C_r}(i) > 1, i \notin V^{C_k}\} \]

Note that the node sets \( V^{C_k}, V^{P_k}, V^{C_k}, V^{P_k}, V^{P_k}, \) contain virtual nodes or cycles that are either the source, or the target or any of both. Similarly, \( E^C_r \) and \( E^P_r \) contain all virtual edges that are covered by any of the paths or any of the cycles. With respect to \( E^C_r \) and \( E^P_r \), we note that these edges’ orientation agrees with the original specification \( E_r \) and the edges in \( E^C_r \) and \( E^P_r \) are reversed. \( E_{\text{bfs}} \) contains all edges whose orientation was reversed in \( G_{\text{bfs}} \). \( B^{C_k} \) denotes the set of branching nodes of cycle \( C_k \in C_r \), i.e. nodes which have an out-degree of larger than one in the graph \( G_{\text{bfs}} \) and which are not a source or a target of any of the cycles.

Lastly, we abbreviate \( V^P_r(t^{C_k}) \) by \( V^P_{\text{brs}} \), i.e. the substrate nodes onto which the target \( t^{C_k} \) of cycle \( C_k \) can be mapped.

The constructive existence of the above decomposition follows from Lemma 38 as proven in the following lemma.

**Lemma 40.** Given a service cactus graph \( G_r = (V_r, E_r) \), the graph \( G_{\text{bfs}} = (V_{\text{bfs}}, E_{\text{bfs}}) \) and its decomposition can be constructed in polynomial time.

**Proof.** Note that the graph \( G_{\text{bfs}} \) is constructed in polynomial time by choosing an arbitrary node as root and then performing a breadth-first search. Having computed the graph \( G_{\text{bfs}} \), the decomposition of \( G_{\text{bfs}} \) (according to Definition 39) can also be computed in polynomial time. First identify all ‘cycles’ in \( G_{\text{bfs}} \) by performing backtracking from nodes having an in-degree of 2. Afterwards, no cycles exist anymore and for each remaining edge \( e = (i, j) \in E_{\text{bfs}} \setminus \{E^C_r|C_k \in C_r\} \) a path \( P_k e = (V^P_e, E^P_e) \) with \( V^P_k = \{i, j\} \) and \( E^P_k = \{(i, j)\} \) can be introduced.

An example of a decomposition is shown in Figure 3.

The request graph \( G_r \) is first reoriented by a breadth-first search to obtain \( G_{\text{bfs}} \). As the virtual network function \( l \) is the only node with an in-degree of 2, first the “cycle” along \( j, k, l \) is decomposed to obtain cycle \( C_1 \). Afterwards, all remaining edges are decomposed into single paths \( P_r = \{P_1, P_2, P_3, P_4\} \), consisting only of one edge. With respect to the notation introduced in Definition 39 we have e.g. \( s^{P_4}_k = j, s^{C_1} = j, t^{P_1}_k = i, \) and \( t^{C_1} = l. \) Furthermore, according to the original edge orientation we have \( \bar{E}^{C_1}_r = \emptyset, \bar{E}^{P_1}_r = \{(i, j)\}, \bar{E}^{C_1}_r = \{(j, k), (k, l)\}, \) and \( \bar{E}^{P_1}_r = \{(j, m), (m, l)\} \). We note that node \( m \) is a branching node, i.e. \( B^{C_1} = \{m\}, \) as it is contained in the subgraph of \( C_1 \) and has a degree larger than 1 and is neither a source nor the target of a cycle.

**D. Extended Graphs for Service Cacti Graphs**

Based on the above decomposition scheme for service cactus graphs, we now introduce extended graphs for each path and cycle respectively. Effectively, the extended graphs will be used in the Integer Programming formulation for SCGEP as well as for the decomposition algorithm. In contrast to the
Definition 41 (Extended Graph for Paths). The extended graph $G^p_{r,ext}$ for path $P_k$ is defined as follows: $V^p_{r,ext} = V^r_{+} \cup V^p_{r,S} \cup V^p_{r,F}$, with $V^p_{r,k} = \{u_{r,i}^k | i = s_k, u \in V^r_{S}(i)\}$, $V^p_{r,k} = \{u_{r,j}^k | j = t_k, u \in V^r_{S}(j)\}$, and $V^p_{r,k} = \{u_{r,i}^k | (i, j) \in E^r_{P}, u \in V_S\}$.

We set $E^p_{r,s} = E^p_{r,+} \cup E^p_{r,S} \cup E^p_{r,F} \cup E^p_{r,S}$, with:

$$E^p_{r,S} = \{(u^r_{j}, u^r_{j}) | (i, j) \in E^{r}_{P}, (u, v) \in V_{S}\} \cup$$

$$E^p_{r,k} = \{(u^r_{i}, u^r_{j}) | i = s_k, (i, j) \in E^{r}_{P}, u \in V^r_{S}(i)\}$$

$$E^p_{r,k} = \{(u^r_{i}, u^r_{j}) | j = t_k, (i, j) \in E^{r}_{P}, u \in V^r_{S}(j)\}$$

$$E^p_{r,F} = \{(u^r_{j}, u^r_{j}) | (i, j) \in E^{r}_{P}, u \in V^r_{S}(j)\}$$

For cycles we employ a more complex graph construction (cf. Definition 42), that is exemplarily depicted in Figure 3. As noted in Definition 39, the edges $V^C_{k}$ of cycle $C_k \subset C_r$ are partitioned into two sets, namely the branches $B^C_{r,1}$ and $B^C_{r,2}$. Within the extended graph construction, these branches are transformed into a set of parallel paths, namely one for each potential substrate node that can host the function of target function $t^C_k$. Concretely, in Figure 5, two parallel path constructions are employed for both realizing the branches $B^C_{r,1}$ and $B^C_{r,2}$ such that for the left construction the network function $l \in V_r$ will be mapped to $v \in V_S$ and in the right construction the function $l \in V_r$ will be hosted on the substrate node $w \in V_S$. This construction will effectively allow the decomposition of flows, as the amount of flow that will be sent into the left path construction of branch $B^C_{r,1}$ will be required to be equal to the amount of flow that is sent into the left path construction of branch $B^C_{r,2}$. Furthermore, for each substrate node $u \in V^r_{S}(s^C_k)$ that may host the source network function $s^C_k$ of cycle $C_k \subset C_r$, there exists a single super source $u_{r, s^C_k}$. Together with the parallel paths construction, the flow along the edges from the super sources to the respective first layers will effectively determine how much flow is forwarded from each substrate node $u \in V^r_{S}(s^C_k)$ hosting the source functionality towards each substrate node $v \in V^r_{S}(t^C_k)$ hosting the target functionality $t^C_k$. The amount of flow along edge $(u^r_{m}, u^r_{n})$ in Figure 5 will e.g. indicate to which extent the mapping of virtual node $j \in V_r$ to substrate node $v \in V_S$ will coincide with the mapping of virtual function $l \in V_r$ to substrate node $w \in V_S$. In fact, to be able to decompose the linear solutions later on, we will enforce the equality of flow along edges $(u^r_{i,j}, u^r_{j,k})$ and $(u^r_{i,j}, u^r_{j,m})$. We lastly note that the flow variables $f^r_{+}$ depicted in Figure 5 will be used to induce flows inside the respective constructions or will be used to propagate flows respectively.

Definition 42 (Extended Graph for Cycles). The extended graph $G^{C}_{r,ext} = (V^{C}_{ext}, E^{C}_{ext})$ for cycle $C_k \subset C_r$ is defined as follows:

Fig. 3. Service cactus graph as well as the decomposition of it according to Definition 39. We visualize the network types by annotating the substrate nodes with the virtual nodes that can be mapped to them and we have e.g. $V^r_{S}(i) = \{v, w\}$. The decomposition graph $G^b_{r,bs}$ is rooted at the arbitrarily chosen substrate node $j = r \in V^b_{bs}$. The graph $G^b_{r,bs}$ is decomposed into four paths and a single cycle. Note that some of the edges in $G^b_{r,bs}$ are reversed with respect to the original graph $G_{r}$, and that these re-orientations also reflect in the edge orientations within the decompositions.

Fig. 4. Extended graphs $G^{P}_{r,ext}$ of the paths of service cactus request $r$ of Figure 3. Note that the orientation of edges is reversed, if the virtual edge was reversed in the breadth-first search. This is e.g. the case for the edge $(i, j) \in G_{r}$ respectively the path $P_k$. The flow values $f^{r,+}$ depicted here are used in the Integer Program 3 to induce flows in the extended graphs.

Definition 5 in Section III-A, these extended graphs will not be directly connected by edges, but thoughtfully stitched using additional variables (see Section VI-E).

The definition of the extended graphs for paths $P_k \in P_r$ generally follows the Definition 5 for linear service chains: for each path $P_k \in P_r$ and each virtual edge $(i, j) \in E^{r}_{P}$ a copy of the substrate graph is generated (cf. Figure 3). If an edge’s orientation was reversed when constructing $G^b_{r,bs}$, the edges in the substrate copy are reversed. Additionally, each extended graph $G^{P}_{r,ext}$ contains a set of source and sink nodes with respect to the types of $s_k$ and $t^C_k$. Concretely, the extended graph $G^{P}_{r,ext}$ contains two super sources, as the virtual node $j$ can be mapped onto the substrate nodes $v$ and $w$. The concise definition of the extended graph construction for the extended graphs of paths is given below.
and substrate node extended graphs (paths or cycles) for virtual node \( i \) and SCGEP-C respectively. The IPs are based on the individual flow will be sent along them.

\[
V_{r,ext} = V_{r,C_k} \cup V_{r,-} \cup V_{r,S}, \text{ with }
\]

\[
V_{r,+} = \{ \upsilon r, u | i = s r, u \in V_{r,T}^+(i) \}
\]

\[
V_{r,-} = \{ \upsilon r, j | j = t r, u \in V_{r,T}^-(j) \}
\]

\[
V_{r,S} = \{ \upsilon r, w | (i,j) \in E_{r,C_k}, u \in V_{r,T}^+(i), w \in V_{r,S,t} \}
\]

We set \( E_{r,ext} = E_{r,+} \cup E_{r,-} \cup E_{r,S} \cup E_{r,P} \), with:

\[
E_{r,S} = \{ (\upsilon r, u, \upsilon r, j) | (i,j) \in E_{r,C_k}, (u,v) \in E_{S,C_k} \} \cup \{ (\upsilon r, u, \upsilon r, w) | (i,j) \in E_{r,C_k}, (u,v) \in E_{S,C_k} \}
\]

\[
E_{r,+} = \{ (\upsilon r, u, \upsilon r, w) | (i,j) \in E_{r,C_k}, (u,v) \in V_{T}^+(i), w \in V_{S,t} \}
\]

\[
E_{r,-} = \{ (\upsilon r, u, \upsilon r, w) | (i,j) \in E_{r,C_k}, (u,v) \in V_{T}^-(i), w \in V_{S,t} \}
\]

\[
E_{r,S} = \{ (\upsilon r, u, \upsilon r, j) | (i,j) \in E_{r,C_k}, i = s r, u \in V_{T}^+(i), j = t r, w \in V_{S,t} \}
\]

Note that we employ the abbreviation \( V_{S,t} \) to denote the substrate nodes that may host the cycle’s target network function, i.e. \( V^T_{S,t}(C_k) \) (cf. Definition 42).

E. Linear Programming Formulation

We propose Integer Programs [3] and [4] to solve SCGEP-P and SCGEP-C respectively. The IPs are based on the individual service cactus graph decompositions and the corresponding extended graph constructions discussed above. As IP[4] reuses all of the core Constraints, we restrict our presentation to IP[3].

We use variables \( f_{r,i,u} \in \{0,1\} \) to indicate the amount of flow that will be induced at the super sources of the different extended graphs (paths or cycles) for virtual node \( i \in V_{r,ext}^+ \) and substrate node \( u \in V_{r,T}^+(i) \). Note that these flow variables are included in the Figures 4 and 5. Additionally flow variables \( f_{r,e} \in \{0,1\} \) are defined for all edges contained in any extended graph construction, i.e. for \( r \in R \) and \( e \in E_{r,ext} \), with \( E_{r,ext} = (\cup_{r \in R_P} E_{r,ext}^P) \cup (\cup_{r \in R_C} V^C_{r,ext}) \). Furthermore, the already introduced variables (see Section III-A: \( x_r \in \{0,1\} \) and \( l_{r,x,y} \geq 0 \) are used to model the embedding decision of request \( r \in R \) and the allocated loads on resource \((x,y) \in R_S \) for request \( r \in R \) respectively.

Embeddings are realized as follows. If \( x_r = 1 \) holds, by Constraint [50] exactly one of the flow values \( f_{r,i,u} \) is set to 1 for \((u,v) \in V^S \) and hence a flow will be induced in all paths and cycles having the arbitrarily chosen root \( r \in V_r \) as source. Specifically, for all paths \( P_k \in \mathcal{P}_r \), Constraint [53] sets the flow values of edges incident to the respective super source nodes in the extended graphs \( G^P_{r,k} \) according to the values to \( f_{r,i,u} \). For cycles, Constraint [51] induces a flow analogously, albeit deciding which target substrate node \( w \in V_{r,S,t}^+ \) will host the function \( f_{r,v,k} \). Importantly, Constraint [51] only induces flow in the branch \( B_{r,C_k} \) of cycle \( C_k \in C_r \). Indeed, Constraint [52] forces the flow in the branch \( B_{r,C_k} \) to reflect the flow decisions of the branch \( B_{r,C_k} \). Having induced flow in the extended networks for paths \( P_k \in \mathcal{P}_r \) and cycles \( C_k \in C_r \), Constraint [54] enforces flow preservation in all of the extended networks. Here we define \( V_{r,ext} = (\cup_{r \in R_P} V^P_{r,S}) \cup (\cup_{r \in R_C} V^C_{r,S}) \) to be all nodes that are neither a super source nor a super sink in the respective extended graphs. As flow preservation is enforced within all extended graphs, the flow of path and cycle graphs must eventually terminate in the respective super sinks. By Constraints [55] and [56], the variables \( f_{r,i,u} \) of the target of the cycle \( C_k \in C_r \) — i.e. \( j = t_{r,k}^+ \) or the target of the path \( P_k \in \mathcal{P}_r \) — i.e. \( j = t_{r,k}^P \) are set according to the amount of flow that reaches the respective sink in the respective extended graph. This effectively induces flows in all the extended graphs of cycle \( C_k \in C_r \) or path \( P_k \in \mathcal{P}_r \) for which \( s_{r,k}^+ \in V^S \) or \( s_{r,k}^P \in V^P \) holds, respectively. paths \( P_k \in \mathcal{P}_r \) or cycles \( C_k \in C_r \) may be spawned also at nodes lying inside another cycle \( C_k \). This is e.g. the case for path \( P_2 \) in Figure 3 as the virtual node \( m \in V_r \) is an inner node of the branch \( B_{r,C_2}^P \) of cycle \( C_1 \). To nevertheless, induce the right amount of flow at the different super sources in the graph \( G^P_{r,ext} \), the sum of flows along processing edges inside the parallel path constructions have to be computed, as visualized in Figure 5.}

In the flow variable \( f_{r,i,u,v} \) needs to equal the sum of the flow along edges \((\upsilon r, u, \upsilon r, v) \) and \((\upsilon r, u, \upsilon r, w) \) and the flow variable \( f_{r,m,u} \) needs to equal the sum of the flow along edges \((\upsilon r, u, \upsilon r, m) \) and \((\upsilon r, u, \upsilon r, w) \) as the respective edges denote the processing of function \( m \in V_r \) on node \( v \in V_S \) and \( u \in V_S \). Constraints [57] realizes this functionality, i.e. the inter-layer edges that denote processing are summed up to set the respective flow inducing variables \( f_{r,i,u} \). As paths may only overlap at sources and sinks (cf. Definition 59), Constraints [53], [55], and [57] propagate flow induction across extended graphs.

Constraints [58] and [59] set the load variables \( l_{r,x,y} \) induced by the embedding of request \( r \in R \) substrate resources \((x,y) \in R_S \). Within the Integer Program [3] we again make
Graphs representing the processing at node \(E\) of use of specific index sets to simplify notation (cf. Definition \(E\)) in Section III-A and Integer Program 1). Concretely, in Constraint 58 for all nodes \(i \in V_i^\pm \cup \bigcup_{C_k \in C_S} B_{r,i}^{C_k}\), the respective flow induction variables \(f_{r,i,u}^{+}\) are added.

With respect to accounting for the load of substrate edges \((u,v) \in E_S\), we note that the only difference to the construction in Section III-A is the re-orientation of edge directions and the consideration of both paths and cycles. Lastly, the objective 49 as well as the Constraint 50 have not changed with respect to (Integer) Linear Program 1.

Summarizing the workings of Integer Program 3 we note the following:

- For each request \(r \in R\), a breadth-first search from an arbitrarily root \(r_r \in V_r\) is performed to obtain graph \(G_r^{ bfs}\). This graph is decomposed according to Definition 39.
- Within IP 1 for each request \(r \in R\) flow is induced in the respective extended graphs of cycles and paths via the variables \(f_{r,i,u}^{+}\) for all nodes \(i \in V_i^\pm\) and substrate nodes \(u \in V_S^{r,\tau(i)}\) (see Constraints 51, 52, and 53).

\[
\begin{align*}
\text{Integer Program 3: SCGEP-P} & \quad \text{(49)} \\
\max & \quad \sum_{r \in R} b_r \cdot x_r \\
\text{s.t.} & \quad \sum_{u \in V_{S,\tau(i)}^{r}} f_{r,i,u}^{+} = x_r \quad \forall r \in R \quad \text{(50)} \\
& \quad \sum_{u \in V_{S,\tau(i)}^{C_k}} f_{r,i,u}^{+} = f_{r,i,u}^{+} \quad \forall r \in R, C_k \in C_S, (i,j) \in B_{r,i}^{C_k}, i = s_{r,i}^{C_k}, u \in V_{S,\tau(i)}^{r} \quad \text{(51)} \\
& \quad f_{r,i}^{+} = f_{r,i,u}^{+} \quad \forall r \in R, C_k \in C_S, (i,j) \in B_{r,i}^{C_k}, (i,j') \in B_{r,i}^{C_k}, i = s_{r,i}^{C_k}, w \in V_{S,\tau(i)}^{r} \quad \text{(52)} \\
& \quad f_{r,i}^{+} = f_{r,i,u}^{+} \quad \forall r \in R, P_k \in \mathcal{P}_r, (i,j) \in E_{P_k}^{r}, i = s_{r,i}^{P_k}, u \in V_{S,\tau(i)}^{r} \quad \text{(53)} \\
& \quad f_{r,i}^{+} = f_{r,i,u}^{+} \quad \forall \tau \in \mathcal{R}, u \in V^{ext,SCG} \quad \text{(54)} \\
& \quad f_{r,i}^{+} = f_{r,i,u}^{+} \quad \forall \tau \in \mathcal{R}, C_k \in C_S, (i,j) \in B_{r,i}^{C_k}, j = t_{r,i}^{C_k}, w \in V_{S,\tau(i)}^{r} \quad \text{(55)} \\
& \quad f_{r,i}^{+} = f_{r,i,u}^{+} \quad \forall \tau \in \mathcal{R}, P_k \in \mathcal{P}_r, (i,j) \in E_{P_k}^{r}, j = t_{r,i}^{P_k}, u \in V_{S,\tau(i)}^{r} \quad \text{(56)} \\
& \quad \sum_{\tau \in \tau^{+}(u)} f_{r,i}^{+} = 0 \quad \forall \tau \in \mathcal{R}, \tau \in \tau^{+}(u) \quad \text{(57)} \\
& \quad \sum_{d_{r}(i) \cdot f_{r,i}^{+} + d_{r}(i) \cdot f_{r,i}^{+} = l_{r,i,u}^{+}} \quad \forall \tau \in \mathcal{R}, (\tau, u) \in R_{S}^{\tau} \quad \text{(58)} \\
& \quad \sum_{d_{r}(i,j) \cdot f_{r,i}^{+} = l_{r,i,u}^{+}} \quad \forall \tau \in \mathcal{R}, (u,v) \in E_{S} \quad \text{(59)} \\
& \quad \sum_{r \in R} l_{r,x,y} \leq d_{S}(x,y) \quad \forall (x,y) \in R_{S} \quad \text{(60)} \\
& \quad x_r \in \{0,1\} \quad \forall r \in R \quad \text{(61)} \\
& \quad f_{r,i,u}^{+} \in \{0,1\} \quad \forall i \in V_i^{\pm} \quad \text{(62)} \\
& \quad f_{r,i}^{+} \in \{0,1\} \quad \forall \tau \in \mathcal{R}, i \in E_{r,flow}^{ext,SCG} \quad \text{(63)} \\
& \quad l_{r,x,y} \geq 0 \quad \forall \tau \in \mathcal{R}, (x,y) \in R_{S} \quad \text{(64)}
\end{align*}
\]
F. Decomposition Algorithm / Correctness

In the following we present Algorithm 4 to decompose a given solution to the (relaxation of) Integer Program 3 as in the decomposition algorithm for linear service chains (cf. Algorithm 1), the output of the algorithm is a set $D_r$ for each request $r \in \mathcal{R}$ consisting of triples $(f_r^+, m_r^+, t_r^+)$, where $f_r^+ \in [0,1]$ is the fractional embedding value of the (fractional) embedding defined by the mapping $m_r^+$. We again note that the load function $l_r^+ : R_S \rightarrow \mathbb{R}_{\geq 0}$ indicates the load on substrate resource $(x,y) \in R_S$, if the mapping $m_r^+$ is fully embedded.

For each request $r \in \mathcal{R}$, the algorithm decomposes the flows in the extended graphs iteratively as long as $x_r > 0$ holds. This is done by placing initially, only the root $r_r \in \mathcal{V}_r$ is placed in the queue $Q$ and one of the substrate nodes $u \in V_{S_r^+}$ with $f_{r,r,u}^+ > 0$ is chosen as the node to host $r_r$. Such a node must always exist by Constraint 50 of Integer Program 3 and the node mapping is set accordingly. The queue $Q$ will only contain nodes that are sources of paths or cycles in the decomposition, i.e. which are contained in $V_{r}^\pm$. Furthermore, by construction, each extracted node from the queue will already be mapped to some specific substrate node via the function $m_r^+$. The set $\mathcal{V}$ is used to keep track of all variables of the Integer Linear Program whose value used in the decomposition process.

For each node in the queue $i \in Q$, all cycles $C_k \in \mathcal{C}_r$ and paths $P_k \in \mathcal{P}_r$ starting at $i$, i.e. that $s_i^{C_k} = i$ or $s_i^{P_k} = i$ holds, are handled one after another using the function ProcessPath. We start by discussing how paths $P_k$ are handled (see Lines 17-20). Note that by construction, the source

---

**Integer Program 4: SCGEP-C**

\[
\begin{align*}
\min \sum_{r \in \mathcal{R}} \sum_{(x,y) \in R_S} c_s(x,y) \cdot l_{r,x,y} \\
\text{s.t.} \quad 50.1, \ldots, 62.64 \quad x_r = 1 \quad \forall r \in \mathcal{R}
\end{align*}
\]

---

**Algorithm 4: Decomposition Algorithm Service Cacti**

**Input**: Substrate $G_S = (V_S, E_S)$, set of requests $\mathcal{R}$, solution $\bar{\bar{\bar{x}}} \in \mathcal{X}$

**Output**: Set of fractional mappings $D_r = \{ (f^+_r, m^+_r, t^+_r) \}_k$ for each request $r \in \mathcal{R}$

1. for $r \in \mathcal{R}$ do
   2. set $D_r := \emptyset$ and $k \leftarrow 1$
   3. while $x_r > 0$ do
      4. set $m^+_k : (m^+_r, m^+_r) \rightarrow (\emptyset, \emptyset)$
      5. set $l^+_k(x,y) \leftarrow 0 \quad \forall (x,y) \in R_S$
      6. set $Q = \{ r_r \}$
      7. choose $u \in V_{S_r^+}^+ (r)$ with $f_{r,r,u}^+ > 0$
      8. set $m^+_r(r_r) \leftarrow u$
      9. set $\mathcal{V} \leftarrow \{ r_r, f^+_r, m^+_r, t^+_r \}$
     10. while $|Q| > 0$ do
         11. choose $i \in Q$ and set $Q \leftarrow Q \setminus \{ i \}$
         12. foreach $C_k \in \mathcal{C}_r$ with $s_i^{C_k} = i$ do
             13. choose $P_l = \{ s_{i,l} \} \in G_{S,ext,f}$
                with $s = (m^+_r(s_{i,l}))_{r \in C_k}^+$ and $t \in V_{S_r}^+$
             14. choose $P_l = \{ s_{i,l} \} \in G_{S,ext,f}$
                ProcessPath($C_k, P_l^+, P_m, m^+_r, m^+_r, \mathcal{V}, Q$)
             15. ProcessPath($C_k, P_l^+, P_m, m^+_r, m^+_r, \mathcal{V}, Q$)
         16. foreach $P_m \in \mathcal{P}_r$ with $s_i^{P_m} = i$ do
             17. choose $P = \{ s_{i,j} \} \in G_{S,ext,f}$
                with $s = (m^+_r(s_{i,l}))_{r \in C_k}^+$ and $t \in V_{S_r}^+$
             18. ProcessPath($P_k, P_m^+, P_m, m^+_r, m^+_r, \mathcal{V}, Q$)
         19. for $i \in V_r$ do
             20. set $l^+_k(s_{i,l}(i), m^+_r(i)) \leftarrow l^+_k(r_r, m^+_r(i)) + d_r(i)$
         21. for $(i,j) \in E_r$ do
             22. set $f^+_k(i,j) \leftarrow \min \mathcal{V}$
             23. set $v \leftarrow v - f^+_k$ for all $v \in \mathcal{V}$
             24. set $l^+_k(i,j, m^+_r(i)) \leftarrow l^+_k(i,j, m^+_r(i)) + d_r(i,j)$
         25. set $f^+_k \leftarrow \min \mathcal{V}$
         26. set $v \leftarrow v - f^+_k$ for all $v \in \mathcal{V}$
         27. set $l^+_k(i,j, m^+_r(i)) \leftarrow l^+_k(i,j, m^+_r(i)) + d_r(i,j)$
     28. set $k \leftarrow k + 1$
     29. return $\{ D_r | r \in \mathcal{R} \}$
30. ProcessPath($K, P_r, P_{ext}, m^+_r, m^+_r, \mathcal{V}, Q$)
31. foreach $(i,j) \in P_r$ do
    32. set $P_{ext} \leftarrow (\{ u, v \}, [u_{i,j}^+, u_{i,j}^+] \in P_{ext}, u \neq v)$
    33. set $P_{ext} \leftarrow (\{ u, v \}, [u_{i,j}^+, u_{i,j}^+] \in P_{ext}, u \neq v)$
    34. if $(i,j) \in E_{ext}$ then
        35. set $m^+_r(j,i) \leftarrow (u, u_{i,j}^+, u_{i,j}^+) \quad \forall (u,u_{i,j}^+, u_{i,j}^+) \in P_{ext}$
    36. else
        37. set $m^+_r(j,i) \leftarrow (u, u_{i,j}^+, u_{i,j}^+) \quad \forall (u,u_{i,j}^+, u_{i,j}^+) \in P_{ext}$
    38. if $j \neq t^+_i$ and $(u_{i,j}^+, u_{i,j}^+) \in P_{ext}$ then
        39. set $m^+_r(j) \leftarrow u$
    40. else
        41. set $m^+_r(j) \leftarrow u$
    42. if $j \in V_{r}^+$ then
        43. set $\mathcal{V} \leftarrow \mathcal{V} \cup \{ f^+_j, m^+_r(j) \}$
    44. set $Q \leftarrow Q \cup \{ j \}$
    45. set $\mathcal{V} \leftarrow \mathcal{V} \cup \{ f^+_j, m^+_r(j) \}$
    46. set $Q \leftarrow Q \cup \{ j \}$
    47. set $\mathcal{V} \leftarrow \mathcal{V} \cup \{ f^+_j, m^+_r(j) \}$
    48. set $Q \leftarrow Q \cup \{ j \}$

of \( P_k \) is already mapped to the substrate node \( m^V_r(i) \). Within the graph \( G^{ exempt}_r f \) \( = (V^{ exempt}_r f \ , E^{ exempt}_r f) \) with \( V^{ exempt}_r f = V^{ exempt}_r \) and \( E^{ exempt}_r f = \{ e \in E^{ exempt}_r [f, e > 0] \} \) a path \( P \) is chosen from the respective super source \( (m^V_r(i))^+_r \) \( \in V^{ exempt}_r \) towards any of the sinks \( V^{ exempt}_r \). By definition of \( E^{ exempt}_r f \) the flow along \( P \) is greater than 0. The function \( \text{Path} \) is handed the path identifier \( K = P_k \), the virtual path \( V_r = E^{ exempt}_r \subseteq E_r \), the path in the extended network \( P^{ exempt}_r = P \subseteq E^{ exempt}_r \), as well as the mappings and the list of variables \( V \) and the queue \( Q \).

The function \( \text{Path} \) proceeds as follows. For all virtual edges \( (i, j) \in E_r \) the edge mapping is set by projecting the edges used in the extended graph to the substrate path \( P_S \in E_S \) in the substrate network (see Line 34). Then, depending on whether the virtual edge’s direction was reversed, the edge mapping \( m^E_r \) is set by either reversing the order of path \( P_S \) or keeping it the same. Note that by Definition 1 the function indeed maps all edges \( i, j \) held in the virtual edge \( V_r \). The edge variables used along the path \( P^{ exempt}_r \) are added to the queue \( Q \). By construction, the start node of \( P^{ exempt}_r \) is added to the queue of nodes \( Q \) to pass. Lastly, also all virtual nodes \( r \) are reachable from \( r \). The cycles and paths cover all edges (and hence all nodes) and sources and targets are added by the function \( \text{Path} \) to the path \( Q \). Eventually all virtual edges and hence all virtual nodes are reachable and thus all virtual nodes are mapped to the substrate (cf. Definition 1). Without further proof, we state the analogue to Lemma 2.

Lemma 43. Each mapping \( m^k_r = (m^V_r, m^E_r) \) found by Algorithm 4 for request \( r \in R \) in the \( k \)-th iteration is valid.

Proof sketch. The proof works by induction and we consider only a single request \( r \in R \). We argue that as long as the constraints specified in Integer Program 3 hold, always a valid mapping can be constructed. It clear, that initially the Constraints of Integer Program 3 are all satisfied.

In the above synopsis of Algorithm 4 we have already argued that the “choose”-operations in Lines 7, 13, 14, and 18 are well-defined, given that all constraints hold. Similarly, we have argued that the \( \text{ProcessPath} \) function correctly maps nodes and edges. Within this proof we therefore only show (i) that indeed all nodes and edges are mapped and that (ii) all constraints of Integer Program 3 hold after decreasing the respective variables.

With respect to (i), we note that by the definition of \( G^{ exempt}_r \) all virtual nodes are reachable from \( r \). As the cycles and paths cover all edges (and hence all nodes) and sources and targets are added by the function \( \text{ProcessPath} \) to the path \( Q \), eventually all virtual edges and hence all virtual nodes will be processed. As the mapping of \( r \) is fixed initially and the targets of edges in \( E^{ exempt}_r \) are always mapped, the node mapping will be complete and valid by construction. Furthermore, by definition of the extended graphs, the path assignment of Lines 34-39 connects the respective substrate node locations and therefore is also valid.

Considering (ii), i.e. the preservation of the integer programs constraints validity, we note that all variables denoting the usage of resources are collected in the set \( V \). Concretely, the flow variables \( f_r \) along each path or cycle branch in the extended graph are added (cf. Line 46) together with the flow induction variables \( f_r^{ exempt} \), (cf. Lines 9 and 45) and the embedding variable \( x_r \) (cf. Line 9). Then in Line 27 the values of the respective variables are reduced exactly by the amount \( f_r^{ exempt} > 0 \). Furthermore, the load variables are reduced in Line 28. We claim that again all constraints of Integer Program 3 hold. We omit the proof that the Constraints 58 and 59, computing the loads \( l_{r,x,y} \) of substrate resource \( (x,y) \in R_S \) for request \( r \in R \), still hold as the “choose”-operations only depend on the other Constraints. Validating the correctness of the other constraints after having reduced the values of the variables in Line 27 is quite easy. The function \( \text{ProcessPath} \) includes all variables along the edges of \( P^{ exempt}_r \) in the set \( V \). Hence, flow preservation will hold inside each extended graph. Furthermore, starting with the initial flow induction variable \( f_r^{ exempt} \), all intermediate flow induction variables are added to \( V \). Concretely, if a node is placelem:validity-of-kth-decomposition-scgepd into \( Q \), then
its respective flow induction variable will have already been placed in $V$ (cf. 45). Also, the embedding variable $x_{r}$ is added, such that Constraints 50-57 are still valid. All flow inducing variables $f_{r}^{k}$, together with flows in the respective extended networks are reduced together with the variable $x_{r}$.

Hence, as after reducing the Integer Program’s variables, all constraints are still valid and the later on found mappings are also valid by induction.

G. Approximating Service Cacti Embeddings

Given Algorithm 4 to decompose (relaxed) solutions to Integer Programs 3 and 4, we can readily apply the approximation framework presented in Sections IV and V to obtain approximations for SCGEP-P and SCGEP-C. For completeness, we reiterate the important lemmas that link the Integer Programs and its decompositions to the respective approximation Algorithms 2 and 3.

The analogue of Lemma 8 follows from Lemma 43 as in each iteration the variable $x_{r}$ is decremented by $f_{r}^{k}$ while adding a decomposition of value $f_{r}^{k}$:

**Lemma 44.** The decomposition $D_{r}$ computed in Algorithm 4 is complete, i.e. $\sum_{D^{k} \in D_{r}} f_{r}^{k} = x_{r}$, holds, for $r \in R$.

We next prove the analogue of Lemma 11, stating that the cumulative load induced by the decompositions does not exceed the cumulative load in the respective Integer Programs 3 and 4.

**Lemma 45.** The cumulative load induced by the fractional mappings obtained by Algorithm 4 is less than the load computed in the respective integer program and hence less than the offered capacity, i.e. for all resources $(x, y) \in R_{S}$ holds

$$\sum_{r \in R} \sum_{D^{k} \in D_{r}} f_{r}^{k} \cdot l_{r}^{k}(x, y) \leq \sum_{r \in R} l_{r, x, y} \leq d_{S}(x, y),$$

where $l_{r, x, y}$ refers to the respective variables of the respective Integer Program.

**Proof.** It suffices to consider a single iteration of Algorithm 4. By Lemma 43, each virtual node $i \in V_{r}$ and each virtual edge $(i, j) \in E_{r}$ is mapped to a substrate node and a substrate path respectively. By construction, each of these mappings was accounted for by the flow variables and the flow induction variables contained in $V$. The load variables for resource $(x, y) \in R_{S}$ are computed in Integer Programs 3 and 4 via the flow values in Constraints 58 and 59. As the variables in $V$ are uniformly decremented by the value $f_{r}^{k}$, and each node $i \in V_{r}$ or edge $(i, j) \in E_{r}$ induces a load of $d_{r}(i)$ or $d_{r}(i, j)$ respectively and the computation of loads in Algorithm 4 agrees with the computation in Constraints 58 and 59, the respective constraints pertaining to the loads, will be valid again after Line 28 of Algorithm 4. Hence, in each iteration the load accounted for in the decomposition $D^{k}_{r}$ is upper bounded by the decrease in the load variables in the respective iteration.

Based on the above Lemmas, it is easy to obtain the analogues of Lemmas 9 and 10.

**Lemma 46.** Let $(x, f^{k}, f_{r}^{k}, l^{k}) \in F_{LP}$ denote a feasible solution to the linear relaxation of Integer Program 3 achieving a net profit of $\hat{B}$ and let $D_{r}$ denote the respective decompositions of this linear solution for requests $r \in R$ computed by Algorithm 4, then the following holds:

$$\sum_{r \in R} \sum_{D^{k} \in D_{r}} f_{r}^{k} \cdot b_{r} = \hat{B}. \tag{68}$$

**Lemma 47.** Let $(x, f^{k}, f_{r}^{k}, l^{k}) \in F_{LP}$ denote a feasible solution to the linear relaxation of Integer Program 4 having a cost of $\hat{C}$ and let $D_{r}$ denote the respective decompositions of this linear solution computed by Algorithm 4 for requests $r \in R$, then the following holds:

$$\sum_{r \in R} \sum_{D^{k} \in D_{r}} f_{r}^{k} \cdot c(\bar{m}_{r}^{k}) \leq \hat{C}. \tag{69}$$

Additionally, equality holds, if the solution $(x, f^{k}, f_{r}^{k}, l^{k}) \in F_{LP}$, respectively the objective $\hat{C}$, is optimal.

The proofs and lemmas contained in Sections IV and V to obtain the approximations to SCGEP-P and SCGEP-C are purely based on Lemmas 7 - 11. The above presented analogues (Lemmas 43 - 47) hence give all the prerequisites to employ the approximation framework developed in Sections IV and V.

The only changes to the respective approximations (cf. Algorithms 2 and 3) are to employ the novel Integer Programs and its decompositions to the respective graph sizes. Concretely, the number of variables and constraints of Integer Programs 3 and 4 is still polynomial in the respective graph sizes. Concretely, the number of variables and constraints is bounded by $O(|E_{r}| \cdot |V_{S}| \cdot |E_{S}|)$, as for each edge $(i, j) \in E_{r}$, there lies on a cycle $C_{k} \in \mathcal{C}_{r}$, exactly $|V_{S, C_{k}}^{r}| \leq |V_{S}|$ many parallel path constructions with $O(|E_{S}|)$ many edges is used. Hence, the runtime to compute the linear relaxations of the respective integer programs is still polynomial and the runtime of the novel decomposition algorithm increases at most by a factor $1 + \gamma$.

Without further proofs, we state the following two theorems.

**Theorem 48.** We adapt Algorithm 2 by replacing Integer Program 2 by Integer Program 3 and Algorithm 7 by Algorithm 4. Assuming that $|V_{S}| \geq 3$ holds, and that $\max_{x, y} \leq \varepsilon \cdot d_{S}(x, y)$ holds for all resources $(x, y) \in R_{S}$ with $0 < \varepsilon \leq 1$ and by setting $\beta = \varepsilon \cdot \sqrt{2 \cdot \log (|V_{S}| \cdot |F|) \cdot \Delta V}$ and $\gamma = \varepsilon \cdot \sqrt{2 \cdot \log |V_{S}| \cdot \Delta E}$, we obtain a $(\alpha, 1 + \beta, 1 + \gamma)$ tri-criteria approximation algorithm for SCGEP-P, such that it finds a solution with high probability, that achieves at least an $\alpha = 1/3$ fraction of the optimal profit and violates network function and edge capacities only within the factors $1 + \beta$ and $1 + \gamma$ respectively.

**Theorem 49.** We adapt Algorithm 3 by replacing Integer Program 2 by Integer Program 4 and Algorithm 7 by Algorithm 4. Assuming that $|V_{S}| \geq 3$ holds, and that $\max_{x, y} \leq \varepsilon \cdot d_{S}(x, y)$ holds for all resources $(x, y) \in R_{S}$ with $0 < \varepsilon \leq 1$ and by setting $\beta = \varepsilon \cdot \sqrt{2 \cdot \log (|V_{S}| \cdot |F|) \cdot \Delta V}$ and $\gamma = \varepsilon \cdot \sqrt{2 \cdot \log |V_{S}| \cdot \Delta E}$, we obtain a $(\alpha, 1 + \beta, 1 + \gamma)$ tri-criteria approximation algorithm for SCGEP-P, such that it finds a solution with high probability, that achieves at least an $\alpha = 1/3$ fraction of the optimal profit and violates network function and edge capacities only within the factors $1 + \beta$ and $1 + \gamma$ respectively.
Assuming that \(|V_S| \geq 3\) holds and that \(\max_{r \in R} \leq \varepsilon \cdot d_S(r, u)\) holds for all network resources \((x, y) \in R_S\) with \(0 < \varepsilon \leq 1\) and by setting \(\beta = \varepsilon \cdot \sqrt{\log |V_S|/|T|} \cdot \Delta_V\) and \(\gamma = \varepsilon \cdot 3.2 \cdot \log |V_S| \cdot \Delta_E\) with \(\Delta_V, \Delta_E\) as defined in Lemmas 20 and 21, we obtain a \((\alpha, 2 + \beta, 2 + \gamma)\) tri-criteria approximation algorithm for SCGEP-C, such that it finds a solution with high probability, with costs less than \(\alpha = 2\) times higher than the optimal cost and violates network function and edge capacities only within the factors \(2 + \beta\) and \(2 + \gamma\) respectively.

H. Non-Decomposability of the Standard IP

The a priori graph decompositions, the extended graph constructions, the respective integer program, and the corresponding decomposition algorithm to obtain approximations for SCGEP are very technical (cf. Sections 3.1 to 3.3). In the following we argue that standard multi-commodity flow integer programming formulations typically used in the virtual network embedding literature, see e.g. [5], [30], [37], cannot be employed for our randomized rounding approach. This result also suggests that the hope expressed by Chowdhury et al. to obtain approximations using this standard formulation [5], is unlikely to be true in general.

In particular, we consider the solutions of the relaxation of the archetypical Integer Program 5 for SCEP-C, which uses multi-commodity flows to connect virtual network functions in the substrate (cf. 5, 50, 37), and show that this formulation allows for solutions which cannot be decomposed into valid embeddings. As we will show, this has ramifications beyond not being able to apply the randomized rounding approach, as the relaxations obtained from Integer Program 5 are provably weaker than the ones obtained by Integer Program 5.

Integer Program 5 employs three classes of variables. The variable \(x_r \in \{0, 1\}\) indicates whether request \(r \in R\) shall be embedded. Variables \(y_{r,i,u} \in \{0, 1\}\) and \(z_{r,i,j,u,v}\) denote the embedding of virtual node \(i \in V_r\) on substrate node \(u \in V_S\) and the embedding of virtual edge \((i, j) \in E_r\) on substrate edge \((u, v) \in E_S\) respectively. By Constraint 71 the virtual node \(i \in V_r\) of request \(r \in R\) must be placed on any of the appropriate substrate nodes in \(V_S^{r,t}\) iff. \(x_r = 1\) holds. Constraint 72 induces a unit flow for each virtual edge \((i, j) \in E_r\): if virtual node \(i \in V_r\) is mapped onto substrate node \(u \in V_S\) and virtual node \(j \in V_r\) is mapped onto \(v \in V_S\), then the flow balance at node \(u\) is 1, and the flow balance at substrate node \(v\) is -1 and flow preservation holds elsewhere. Note also that in the case that both the source and the target are mapped onto the same node no flow is induced. Lastly, Constraints 73 to 74 compute the effective node and edge allocations and bound these by the respective capacities.

By the above explanation, it is easy to check that any integral solution to Integer Program 5 defines a valid mapping. Indeed, for each request \(r \in R\) with \(x_r = 1\) the node \(i \in V_r\) is mapped onto substrate node \(u\), i.e. \(m_r^u(i) = u\) holds, iff. \(y_{r,i,u} = 1\) holds and the edge mapping of \((i, j) \in E_r\) can be recovered from the flow variables \(y_{r,i,j,u,v}\): by performing a breadth-first search from \(m_r^u(i) \in V_S\) to \(m_r^u(j) \in V_S\) where an edge \((u, v) \in E_S\) is only considered if \(y_{r,i,j,u,v} = 1\) holds.

We denote the set of feasible integral solutions of Integer Program 5 by \(F_{IP}^{\text{IP}}\) and the set of linear solutions by \(F_{IP}^{\text{LP}}\).

Reversely, each solution to SCEP-P, i.e. each subset \(R' \subseteq R\) of requests with mappings \((m_r^V, m_r^E)\) for \(r \in R'\), induces a specific solution to the Integer Program 5. We denote by \(\varphi\) the function that given a set \(R'\) and corresponding mappings \(\{m_r\}_{r \in R'}\) yields the respective integer programming solution \((\vec{x}, \vec{y}, \vec{z}) \in \mathbb{F}_{IP}^{\text{IP}}\) defined as follows:

- \(x_r = 1\) iff. \(r \in R'\) for all requests \(r \in R\),
- \(y_{r,i,u} = 1\) iff. \(r \in R'\) and \(m_r^u(i) = u\) for all requests \(r \in R\), virtual nodes \(i \in V_r\), and substrate nodes \(u \in V_S^{r,t}\).
- \(z_{r,i,j,u,v} = 1\) iff. \(r \in R'\) and \((u, v) \in E_r\) for all requests \(r \in R\), virtual edges \((i, j) \in E_r\) and substrate edges \((u, v) \in E_S\).

By the above argumentation, we observe that Integer Program 5 indeed is a valid formulation for SCEP-P. We will now investigate the linear relaxations of Integer Program 5 and show that the formulation may produce solutions, which cannot be embedded. As an important precursor to this result, we first define the set of all linear programming solutions that may originate from convex combinations of singular embeddings as follows.

Definition 50 (Decomposable LP Solution Spaces), Let \(M_r\) denote the set of all valid mappings for the request \(r \in R\). We denote by \(F_{LP,r}\) the set of all convex combinations of projections of valid mappings into the solutions space \(\mathbb{F}_{IP}^{\text{IP}}\) of Integer Program 5 i.e. \(\mathbb{F}_{LP,r}\) equals

\[
\{ \sum_{k=1}^{n} \lambda_k \cdot \varphi(r, M_k) \mid n \in \mathbb{N}, M_k \in M_r, \lambda_k \geq 0, \sum_{k=1}^{n} \lambda_k \leq 1 \}.
\]

Note that the above also allows for the possibility of the non-embedding, as \(\sum_{k=1}^{n} \lambda_k = 0\) is allowed. We denote the space of all decomposable linear programming solutions as

\[
\mathbb{F}_{LP}^{LP} = \bigcap_{r \in R} \mathbb{F}_{LP,r} = \{ (\vec{x}, \vec{y}, \vec{z}) \in \mathbb{F}_{LP}^{\text{IP}} \mid (\vec{x}, \vec{y}, \vec{z}) \text{ satisfies Constraints } 71 \text{ to } 74 \}.
\]

As stated before, we denote by \(\mathbb{F}_{IP}^{LP} = \{ (\vec{x}, \vec{y}, \vec{z}) \mid (\vec{x}, \vec{y}, \vec{z}) \text{ satisfies Constraints } 71 \text{ to } 74 \}\) denote the set of linear programming solutions feasible according to the constraints of Integer Program 5. Using the definitions of \(\mathbb{F}_{LP}\) and \(\mathbb{F}_{IP}^{LP}\) we can now formally prove, that the linear relaxation of Integer Program 5 does contain solutions, which cannot be decomposed.

Lemma 51. The set of feasible LP solutions of Integer Program 5 may contain non decomposable solutions, i.e. \(\mathbb{F}_{IP}^{LP} \neq \mathbb{F}_{LP}\).

Proof. We show \(\mathbb{F}_{LP}^{LP} \cap \mathbb{F}_{LP} = \emptyset\) by constructing an example in Figure 6 using a single request and a 8-node substrate.

We assume that \(V_r^{\tau(i)} = \{u_1, u_3\}\), \(V_r^{\tau(j)} = \{u_2, u_6\}\), \(V_r^{\tau(k)} = \{u_4, u_8\}\), \(V_r^{\tau(l)} = \{u_3, u_7\}\) holds. The depicted request shall be fully embedded, i.e. \(x_r = 1\) holds. The fractional embedding is represented indicating the node mapping variables, \(\frac{1}{2}\) means that the corresponding mapping value \(y_{r,i,u_1} = 1/2\). Edge mappings are represented according to the
Integer Program 5: Classic Multi-Commodity Flow Formulation for SCGEP-P

\[
\max \sum_{r \in \mathcal{R}} b_r x_r
\]
\[
\sum_{u \in V_{S_{r(i)}}} y_{r,i,u} = x_r \quad \forall r \in \mathcal{R}, i \in V_r
\]
\[
\sum_{(u,v) \in \delta^+(u)} \delta_{r,i,j,u,v} - \sum_{(v,u) \in \delta^-(u)} \delta_{r,i,j,u,v} = y_{r,i,u} - y_{r,j,u} \quad \forall r \in \mathcal{R}, (i,j) \in E_r
\]
\[
\sum_{r \in \mathcal{R}} \sum_{(i,j) \in E_r} d_r(i) \cdot \delta_{r,i,j,u,v} \leq d_S(\tau, u) \quad \forall (\tau,u) \in R_S^V
\]
\[
x_r \in \{0, 1\} \quad \forall r \in \mathcal{R}
\]
\[
y_{r,i,u} \in \{0, 1\} \quad \forall r \in \mathcal{R}, i \in V_r, u \in V_{S_{r(i)}}
\]
\[
y_{r,i,u} = 0 \quad \forall r \in \mathcal{R}, i \in V_r, u \notin V_{S_{r(i)}}
\]
\[
z_{r,i,j,u,v} = 0 \quad \forall r \in \mathcal{R}, (i,j) \in E_r, (u,v) \in E_S
\]

Assume for the sake of contradiction that the depicted embedding is a linear combination of elementary solutions, i.e. there exist mappings \(\lambda_k\) for \(k \in K\) such that the depicted solution is a linear combination \(\sum_{k \in K} \lambda_k \cdot \phi(r)\) with \(\lambda_k \geq 0\) and \(\sum_{k \in K} \lambda_k \leq 1\). As virtual node \(i\) is mapped onto substrate node \(u_1\), and \(u_2\) and \(u_8\) are the only neighboring nodes that host \(j\) and \(k\) respectively there must exist a mapping \((m^V_{r(i)}, m^E_{r(i)})\) within \(M_k\) with \(m^V_{r(i)}(i) = u_1\), \(m^V_{r(i)}(j) = u_2\), \(m^V_{r(i)}(k) = u_8\) and \(m^E_{r(i)}(i,k) = (u_1, u_2)\) and \(m^E_{r(i)}(i,k) = (u_1, u_8)\). Similarly, as the flow of virtual edge \((j,l)\) at \(u_2\) only leads to \(u_3\) and the flow of virtual edge \((j,k)\) at \(u_8\) only leads to \(u_6\), the virtual node \(l\) must be embedded both on \(u_3\) and \(u_7\). As the virtual node \(l\) must be mapped onto exactly one substrate node, this partial decomposition cannot possible be extended to a valid embedding. Hence, the depicted solution cannot be decomposed into elementary mappings and \(\mathcal{F}_{LP}^{\text{new}} \not\subseteq \mathcal{F}_{LP}^{\text{old}}\) holds (in general).

We lastly prove the following theorem, showing that the relaxations of the novel Integer Program 5 are provably stronger than the ones of the standard IP 5. To compare the strength of formulations over different variable spaces, we project relaxed solutions of Integer Program 3 to the solution space \(\mathcal{F}_{LP}^{\text{new}}\) of Integer Program 1 (see 1 for an introduction on comparing formulations using projections).

**Theorem 52.** The linear relaxations of Integer Program 3 are provably stronger than the linear relaxation of Integer Program 5.

**Proof.** We denote by \(\mathcal{F}_{LP}^{\text{new}}\) the solution space of the linear relaxations of Integer Program 5. Let \(\pi : \mathcal{F}_{LP}^{\text{new}} \rightarrow \mathcal{F}_{LP}^{\text{old}}\) denote the projection of the novel solution space to the solution space of the standard formulation. Concretely, given a solution \((\bar{x}, \bar{f}^+), \bar{f}^+, \bar{f}^-) \in \mathcal{F}_{LP}^{\text{new}}\) the function \(\pi\) first computes the according decomposition \(D_r\), according to Algorithm 4, and then returns the following solution contained in \(\mathcal{F}_{LP}^{\text{old}}\).

- \(x_r = \sum_{D_r \in D_r} f_r^k\) for all \(r \in \mathcal{R}\).
- \(y_{r,i,u} = \sum_{D_r \in D_r, m^V_{r(i)}(i)=u} f_r^k\) for all requests \(r \in \mathcal{R}\), virtual nodes \(i \in V_r\) and substrate nodes \(u \in V_{S_{r(i)}}\).
As the Formulation 3 and the decomposition Algorithm 4 are valid, it is easy to check that the above projection is valid, i.e., \( \pi(F_{LP}^{new}) \subseteq F_{LP}^{mcf} \) holds. Furthermore, given any solution \( (\vec{x}, \vec{f}, \vec{l}) \in F_{LP}^{mcf} \), it is obvious that the projected solution \( \pi(\vec{x}, \vec{f}, \vec{l}) \) is decomposable, as the respective mappings were computed explicitly in the decomposition algorithm. Thus, \( \pi(F_{LP}^{new}) \) holds and under this projection all solutions are decomposable. Together with Lemma 51 we obtain \( \pi(F_{LP}^{new}) \subseteq F_{LP}^{mcf} \subseteq F_{LP}^{new} \). Lastly, as \( F_{LP}^{mcf}/F_{LP} \neq \emptyset \) holds, the relaxations of Integer Program 3 are indeed provably stronger than the ones of Integer Program 5.

The above theorem is based on the structural deficits of the Integer Program 5 and does neither depend on the particular way we have formulated IP or nor does it depend on whether we consider SCGEP-P or SCGEP-C. Furthermore, we note that the above theorem can also have practical implications, when trying to obtain good bounds via linear relaxations, as the difference in the benefit (or cost) can be unbounded.

**Lemma 53.** The objectives of the linear relaxations of Integer Programs 3 and 5 can diverge arbitrarily. Concretely, considering the embedding of a particular set of requests \( R \) on substrate \( G_S \) and denoting the benefit obtained by Integer Program 3 as \( \text{Opt}_{LP}^{new} \) and the benefit obtained by Integer Program 5 as \( \text{Opt}_{LP}^{mcf} \), the absolute difference \( \text{Opt}_{LP}^{mcf} - \text{Opt}_{LP}^{new} \) as well as the relative difference \( (\text{Opt}_{LP}^{mcf} - \text{Opt}_{LP}^{new}) / \text{Opt}_{LP}^{new} \) may be unbounded.

**Proof.** We first note that by Theorem 52 we have \( \text{Opt}_{LP}^{new} \leq \text{Opt}_{LP}^{mcf} \leq \text{Opt}_{LP}^{new} \) holds. We reuse the example depicted in Figure 6 and we assume that the depicted request is the only one. As discussed, the depicted fractional solution is a feasible relaxation of Integer Program 5. As \( x_r = 1 \) holds, we have \( \text{Opt}_{LP}^{new} = b_r \).

Considering the relaxations of Integer Program 3 we claim that only \( x_r = 0 \) is feasible. This is easy to check as there does not exist a potential substrate location for virtual node \( l \), such that the branches \( i \rightarrow j \rightarrow l \) and \( i \rightarrow k \rightarrow l \) can end in the same substrate location while also emerging at the same location. Hence, the benefit obtained by the relaxation of Integer Program 5 is \( \text{Opt}_{LP}^{new} = 0 \).

Hence, the absolute difference is \( \text{Opt}_{LP}^{mcf} - \text{Opt}_{LP}^{new} = b_r \) and \( -b_r \) as \( b_r \) can be set arbitrarily – the absolute as well as the relative difference are unbounded.

**VII. Related Work**

Service chaining has recently received much attention by both researchers and practitioners [31, 34, 35]. Soule et al. [38] present Merlin, a flexible framework which allows to define and embed service chain policies. The authors also present an integer program to embed service chains, however, solving this program requires an exponential runtime. Also Hartet et al. [34] have studied the service chain embedding problem, and proposed a constraint optimization approach. However also this approach requires exponential runtime in the worst case. Besides the optimal but exponential solutions presented in [34], [38], there also exist heuristic solutions, e.g., [31, 33]: while heuristic approaches may be attractive for their low runtime, they do not provide any worst-case quality guarantees. We are the first to present polynomial time algorithms for service chain embeddings which comes with formal approximation guarantees. Concretely, our results based on randomized rounding techniques are structured into two papers: In [12], we only consider the admission control variant and derive a constant approximation under assumptions on the relationship between demands and capacities as well as on the optimal benefit. In contrast, we have considered in this paper a more general setting, in which the decomposition approach based on random walks [12] is not feasible, as we allow for arbitrary service cactus graphs. Importantly, while our approach does not require specific assumptions on loads and benefits, we obtain worse approximation ratios.

From an algorithmic perspective, the service chain embedding problem can be seen as a variant of a graph embedding problem, see [6] for a survey. Minimal Linear Arrangement (MLA) is the archetypical embedding footprint minimization problem, where the substrate network has a most simple topology: a linear chain. It is known that MLA can be \( O(\sqrt{\log n \log \log n}) \) approximated in polynomial time [4]. However, the approximation algorithms for VLSI layout problems [6, 10], cannot be adopted for embedding service chains in general substrate graphs.

Very general graph embedding problems have recently also been studied by the networking community in the context of virtual network embeddings (sometimes also known as testbed mapping problems). Due to the inherent computational hardness of the underlying problems, the problem has mainly been approached using mixed integer programming [5] and heuristics [40]. For a survey on the more practical literature, we refer the reader to [15]. Algorithmically interesting and computationally tractable embedding problems arise in more specific contexts, e.g., in fat-tree like datacenter networks [2]. In their interesting work, Bansal et al. [3] give an \( n^{O(d)} \) time \( O(d^2 \log (nd)) \)-approximation algorithm for minimizing the load of embeddings in tree-like datacenter networks, based on a strong LP relaxation inspired by the Sherali-Adams hierarchy. The problem of embedding star graphs has recently also been explored on various substrate topologies (see also [36]), but to the best of our knowledge, no approximation algorithm is known for embedding chain- and cactus-like virtual networks on arbitrary topologies.

Finally, our work is closely related to unsplittable path problems, for which various approximation algorithms exist, also for admission control variants [24]. In particular, our work leverages techniques introduced by Raghavan and Thompson [35]: in their seminal work, the authors develop provably good algorithms based on relaxed, polynomial time versions of 0-1 integer programs. However, our more general setting...
not only requires a novel and more complex decomposition approach, but also a novel and advanced formulation of the mixed integer program itself: as we have shown, standard formulations cannot be decomposed at all. Moreover, we are not aware of any extensions of the randomized rounding approach to problems allowing for admission control and the objective of maximizing the profit.

VIII. Conclusion

This paper initiated the study of polynomial time approximation algorithms for the service chains embedding problem, and beyond. In particular, we have presented novel approximation algorithms, which apply randomized rounding on the decomposition of linear programming solutions, and which also support admission control. We have shown that using this approach, a constant approximation of the objective is possible in multi-criteria models with non-trivial augmentations, both for service chains as well as for more complex virtual networks, particularly service cactus graphs. Besides our result and decomposition technique, we believe that also our new integer program formulation may be of independent interest.

Our paper opens several interesting directions for future research. In particular, we believe that our algorithmic approach is of interest beyond the service chain and service cactus graph embedding problems considered in this paper, and can be used more generally. In particular, we have shown that our approach can be employed as long as the relaxations can be decomposed. Hence, we strongly believe that our approach can be applied to further graph classes as well. Moreover, it will also be interesting to study the tightness of the novel bounds obtained in this work.

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