A Gas of Bags: Weirdness and Criticality\textsuperscript{1}

L. G. Moretto, J. B. Elliott\textsuperscript{2}, P. T. Lake, L. Phair

Nuclear Science Division,
Lawrence Berkeley National Laboratory
Berkeley, CA, USA 94720
E-mail: lgmoretto@lbl.gov

Abstract. A gas of MIT bags has peculiar properties, one of which is that of not existing. We shall discuss its instability against coalescence. The absence of surface energy makes the bags already critical at their unique temperature $T_H$. The addition of a surface energy does not seem to cure the problem.

1. Introduction

It was shown elsewhere that both Hagedorn’s model \cite{1, 2} and the MIT bag model \cite{3} have an exponential spectrum of the form

$$
g(m) = \exp\left(\frac{m}{T_H}\right).
$$

This is a spectrum of a thermostat with temperature $T_H$. It will impart its unique, immutable temperature to any other “normal” system in contact with it.

It was also shown that the evaporation of the bag into “normal” particles produces a saturated vapor with a concentration \cite{4, 5}

$$
\frac{N(m)}{V} = g(m) \left(\frac{mT_H}{2\pi}\right)^{3/2} \exp\left(-\frac{m}{T_H}\right).
$$

2. A Gas of Bags

The question arises as to whether particles with the same exponential spectrum can give rise to a stable gas akin to a physical gas of resonances. It is clear that, apart from translational degrees of freedom, bags are indifferent towards coalescence or fragmentation as shown in figure 1. Now let us include the translational modes and consider no interactions other than coalescence or fragmentation.

\textsuperscript{1} Dedicated to W. Swiatecki in fond memory.

\textsuperscript{2} Current Address:
Lawrence Livermore National Laboratory,
94551 Livermore, CA, USA
Let us consider the equilibrium between a bag $A$ and the $n$ equal bags $B$ resulting from its fragmentation. The equilibrium constant is

$$K = \frac{c_B^n}{c_A} = \frac{q_B^n}{q_A} = \frac{q_A^{n-1}}{n^{2n}}$$

where: $q_B = \left(\frac{2\pi m_A T}{h^2 n}\right)^{3/2} = \frac{1}{n^{3/2}} q_A$.

c$_A$, c$_B$ are the concentrations of the two species and q$_A$, q$_B$ are the corresponding reduced partition functions.

It follows that

$$\lim_{n \to \infty} c_b = \lim_{n \to \infty} \frac{c_A^{1/n} q_A^{n-1}}{n^{3/2}} = 0$$

and that the percent dissociation, $\alpha$, is

$$\alpha = \frac{1}{n^{3/2}} c_A^{1/n} q_A^{n-1}$$

which also goes to 0 as $n \to \infty$.

Thus the translational degrees of freedom strongly bias the system towards coalescence.

From equation 2 we obtain the mass distribution

$$\frac{N(m)}{V} = \left(\frac{mT_H}{2\pi}\right)^{3/2},$$

again strongly biased towards the largest possible bag or, grand canonically, towards an infinite bag.

3. Surface Energy and Criticality

Normal van der Waals fluids exhibit a first order coexistence line between liquid and vapor which terminates at the critical temperature. Above the critical temperature the two phases are not distinguishable any longer but a scar remains in the phase diagram which is visualized with, for instance, density fluctuations or heat capacities.

The steam tables give a good example [6]. Figure 2 shows the isobaric heat capacity overlaid on the phase diagram of water. At and below the critical temperature, there is a singularity in the heat capacity in going from one phase to another. Above the critical point, the heat capacity stays finite, but there is a maximum value given a constant pressure. This figure should be kept in mind when thinking about criticality in QCD.

But what about MIT bags? Let us recall how, in Fisher’s cluster picture, criticality comes about.
Figure 2. The isobaric heat capacity of water shows how even though the heat capacity stays finite everywhere above the critical temperature, a scar remains in that there is a maximum value for any given pressure.  

4. Fisher cluster model

To describe cluster formation in a simple fluid, Fisher proposed the degeneracy of clusters of a given size $A$ to be of the form \[ g(A) = q_0 A^{-\tau} \exp \left[ \frac{c_0}{T_c} A^\sigma \right]. \] (7)

This is in contrast to the MIT bag picture, unless $\sigma = 1$. In the case Fisher considered, $c_0 A^\sigma$ is the surface energy, and $\sigma = 2/3$. What gives rise to this discrepancy?

A simple model to study clusters is the Ising model. Being confined to a lattice, a cluster is just a collection of connected lattice points with the same spin. The Ising model also exhibits both a first order and a second order phase transition, making it an invaluable tool in studying phase transitions in terms of clusters.

The study of the degeneracies of clusters on a grid, also known as lattice animals, is an active field of research. There is no closed form to enumerate all the lattice animals of a given size and surface. Much work has gone in finding these values, both exactly and approximately.

4.1. Lattice animal degeneracy

One method of approximating lattice animal degeneracies is by studying pure percolation on a grid. \[ P(X; x_1, y_1) = p^A (1 - p)^s. \] (8)
If periodic boundary conditions are used, there are $V$ unique locations where animal $X$ can exist. Furthermore, since $P(X; x_1, y_1)$ is only a function of $A$ and $s_s$, any other animal with the same $A$ and $s_s$ will have the same probability. Putting this together, the probability of finding any animal of size $A$ and surface $s_s$ anywhere on the grid is

$$P(A, s_s) = V p^A (1 - p)^{s_s} g(A, s_s)$$

(9)

Notice that the degeneracy of the clusters, $g(A, s_s)$, enters the previous equation.

With high statistics of a percolating grid, we can measure $P(A, s_s)$ and solve for $g(A, s_s)$:

$$g(A, s_s) = \frac{P(A, s_s)}{V p^A (1 - p)^{s_s}}.$$  

(10)

One further note needs to be made before these values can be applied to the Ising model. The parameter $s_s$ is seen to be a site surface, which is different from the surface used in the Ising model. The Ising model considers the bond surface, $s_b$, of the cluster. Instead of the number of sites surrounding the animal, the number of bonds between the cluster and the outside determine the energy. Figure 3 gives an example of an animal with different values for its site and bond surfaces.

Further classification of the animals as a function of $A$, $s_s$, and $s_b$ yields

$$g(A, s_s, s_b) = \frac{P(A, s_s, s_b)}{V p^A (1 - p)^{s_s}}.$$  

(11)

To find the physically significant degeneracy for the Ising model, $g(A, s_b)$, the degeneracies are summed over the different $s_s$:

$$g(A, s_b) = \sum_{s_s} g(A, s_s, s_b).$$  

(12)

4.2. Lattice animals in the Ising model

Taking the degeneracies of these lattice animals, first consider the total degeneracy of animals as a function of size, as seen in figure 4. It can be seen that the data fits to the case $\sigma = 1$, a volume effect.

This method does not model the Ising system properly. In the case of the Ising model, the appearance of a lattice animal is affected by the amount of surface the animal possesses. The
The total degeneracy of lattice animals follows the same trend as the Hagedorn spectrum.

The shape of the clusters changes as a function of temperature, controlled by the Boltzmann factor for the surface energy.

Concentration of a cluster as a function of size and (bond) surface, \( n(A, s) \), is thus weighted by a Boltzmann factor

\[
\begin{align*}
  n(A, s) &= g(A, s) \exp \left[ \frac{-E(A, s)}{T} \right] \\
             &= g(A, s) \exp \left[ -\frac{2s}{T} \right] 
\end{align*}
\]

(13)

As an example, figure 5 shows the degeneracy of clusters as a function of surface and the effects of applying a Boltzmann factor at various temperatures.

A result of this procedure is that the average surface of the clusters changes as a function of temperature. In fact, for a given temperature, plotting the average surface as a function of cluster size yields the relation

\[
\langle s \rangle \propto A^{\sigma(T)}
\]

(14)

where the power \( \sigma \) becomes a function of the temperature.

Figure 6 shows the values of \( \sigma \) over a large temperature range. At low temperatures \( \sigma = 2/3 \), as originally stated by Fisher. There is then a transition as the temperature increases where \( \sigma \to 1 \), which corresponds to the second order transition in the Ising model. The presence of the surface energy causes the clusters to be compact at low temperatures. Then as the temperature increases, the degeneracy of the clusters favors the dendritic, fractal form. This is analogous to the second order phase transition in that there is a limiting temperature where the surface free energy goes to zero.

Herein lies the contradiction. Since there is no surface energy present in the MIT bag model, the natural shape of the bags should be fractal. This implies that the natural temperature of
Figure 6. Surface dimensionality \( (S \propto A^\sigma) \) of 3-dimensional lattice clusters with surface as a function of temperature.

the system, \( T_H \), is above the critical temperature. Yet, the formation of a bag is described well as a first order phase transition, which do not exist above the critical temperature.

5. Effects of surface energy

How does the picture change if surface energy is added to the MIT bags? First of all, it would make the situation of a gas of bags even more precarious, since there would be an ever greater tendency to maximize the drop size and to minimize the surface. Even more interesting is the effect of the surface on the bag temperature. The surface translates into an additional pressure on the bag. The enthalpy of the bag becomes:

\[
EV = H = [f(T) + B]V + c_s V^{2/3}.
\]  
(15)

The pressure is:

\[
p = \frac{1}{3} f(T) - \left(B + \frac{2}{3} c_s V^{-1/3}\right) = 0 \text{ at equilibrium},
\]  
(16)

for which:

\[
T = f^{-1} \left[3 \left(B + \frac{2}{3} c_s V^{-1/3}\right)\right].
\]  
(17)

In figure 7 we show the dependence of the bag size on the temperature, energy, and heat capacity. Notice that the temperature of a bag increases and tends to infinity with decreasing bag size.

The introduction of a temperature dependent surface energy is also easily implemented. If:

\[
c_s = c_s^0 \left(1 - \frac{T}{T_c}\right),
\]  
(18)

Figure 7. Effects of bags having a surface energy on various thermodynamic properties.
Figure 8. Effects of the surface energy varying with temperature. Notice how the temperature stays finite with this change when previously it diverges.

Figure 9. Schematic picture of the effects of a bag having surface energy.

We have for the stability condition:

$$\sigma T^4 = 3 \left[ B + \frac{2}{3} c_0^0 \left( 1 - \frac{T}{T_c} \right) V^{-2/3} \right]. \quad (19)$$

Compared to the previous situation, the bag temperature goes to $T_c$ rather than infinity as the bag becomes small, as seen in figure 8. It also preserves the trend that the large bag sizes go to a temperature of $T_H$.

This temperature dependence of the bag leads to two interesting conclusions:

1) A gas of bags of different sizes, and thus of different temperatures, is out of equilibrium. The system will tend to make one single bag of maximum size and minimum surface.

2) If a bag decays, its temperature, as manifested by the decay products such as pions, will progressively increase as the bag evaporates.

These conclusions are shown in figure 9.

Acknowledgments

This work was supported by the US Department of Energy under Contract No. DE-AC02-05CH11231.

References

[1] Hagedorn R 1965 Suppl. Nuovo Cimento 3 147
[2] Hagedorn R and Ranft J 1968 Suppl. Nuovo Cimento 6 169
[3] Chodos A, Jaffe R L, Johnson K, Thorn C B and Weissskopf V F 1974 Phys. Rev. D 9 3471
[4] Moretto L G, Bugaev K A, Elliot J B and Phair L 2006 Europhys. Let. 76 402
[5] Moretto L G, Bugaev K A, Elliot J B and Phair L 2006 Proc. Sci. CPOD2006 37
[6] Wagner W and Kretzschmar H 2008 *International Steam Tables - Properties of Water and Steam based on the Industrial Formulation IAPWS-IF97* (Berlin, Heidelberg: Springer-Verlag Berlin Heidelberg) p 360

[7] Fisher M E 1967 *Rep. Prog. Phys.* **30** 615

[8] Fisher M E 1967 *Physics (N. Y.)* **3** 255

[9] Stauffer D 1985 *Introduction to Percolation Theory* (London, Philadelphia: Taylor & Francis)