On Spaces of connected Graphs II

Relations in the algebra $\Lambda$

Jan A. Kneissler

Abstract

The graded algebra $\Lambda$ defined by Pierre Vogel is of general interest in the theory of finite-type invariants of knots and of 3-manifolds because it acts on spaces of connected graphs subject to relations called IHX and AS. We examine a subalgebra $\Lambda_0$ that is generated by certain elements called $t$ and $x_n$ with $n \geq 3$. Two families of relations in $\Lambda_0$ are derived and it is shown that the dimension of $\Lambda_0$ grows at most quadratically with respect to degree. Under the assumption that $t$ is not a zero divisor in $\Lambda_0$, a basis of $\Lambda_0$ and an isomorphism from $\Lambda_0$ to a sub-ring of $\mathbb{Z}[t, u, v]$ is given.

1 Main Results

The theory of Vassiliev invariants, initiated by V. A. Vassiliev in [5], leads to the study of modules of Chinese characters, i.e. embedded graphs with univalent and oriented trivalent vertices, subject to the following relations (the reader not familiar with these concepts is referred to [1], for instance).

$\quad \quad \quad = - \quad \quad \quad$ (AS)

$\quad \quad = 0$ (IHX)

Definition 1.1 Let $F(n)$ denote the $\mathbb{Q}$-vector space of connected tri-/univalent graphs having exactly $n$ univalent vertices that are labeled with the numbers 1 to $n$, modulo the relations (AS) and (IHX). If exchanging any two of the labels of $a \in F(n)$ turns $a$ into $-a$, then $a$ is called antisymmetric. The subspace of $F(3)$ spanned by antisymmetric elements is named $\Lambda$.

All our statements are valid over $\mathbb{Z}[\frac{1}{n}]$, but for convenience we work over $\mathbb{Q}$. First let us summarize some facts about $\Lambda$ (see [6] for details): It acts on modules of connected tri-/univalent graphs by insertion at a trivalent vertex; (AS) and (IHX) guarantee that the result does not depend on the choice of the trivalent vertex. So it also acts on itself, which turns $\Lambda$ into a graded commutative algebra. The following family of graphs named $t, x_3, x_4, x_5, \ldots$, represent elements of $\Lambda$ ($x_n$ is homogeneous of degree $n$ and $t$ has degree 1).
We set \( x_1 := 2t, x_2 := t^2 \) and let \( \Lambda_0 \) denote the subalgebra of \( \Lambda \) that is generated these elements \( x_n \) with \( n \geq 1 \). To our present knowledge there are no counterexamples to the following two conjectures:

**Conjecture 1.2** The algebra \( \Lambda \) is generated by the \( x_n \), i.e. \( \Lambda_0 = \Lambda \).

**Conjecture 1.3** The element \( t \) is not a zero divisor, i.e. the map \( \Lambda_0 \to \Lambda_0, \lambda \to t\lambda \) is injective.

Considering the results presented here, the verification of these conjectures might be the only remaining hurdle on the way to a complete understanding of \( \Lambda \).

**Remark 1.4** Computations that are based on results given in the subsequent paper [4] show that Conjecture 1.2 is true in degrees \( \leq 12 \). Due to Remark 1.8, Conjecture 1.3 is true up to degree 14.

To state the main result of this note, we define two families \( P_{i,j}, Q^k_{i,j} \) of polynomials in the variables \( t \) and \( x_n \):

\[
P_{i,j} := 3x_{i+2}x_{j+4} - 3tx_{i+1}x_{j+4} - 9tx_{i+2}x_{j+3} - 6t^2x_i x_{j+4} + 9t^2x_{i+1}x_{j+3} + 18t^3x_ix_{j+3} - 2(x_3 + 2t^3)x_{i+1} x_{j+2} + 4t(x_3 - 4t^3)x_i x_{j+2} + 8t^2(t^3 - x_3)x_i x_{j+1} + 3t^{i+2}x_{j+4} - 9t^{i+3}x_{j+3} + (7t^3 - x_3)t^{i+1}x_{j+2} + 3(x_3 - t^3)t^{i+2}x_{j+1} + 2(t^3 - x_3)t^{i+3}x_j
\]

\( Q^k_{i,j} \) is defined recursively for \( i, j, k \geq 0 \) by

\[
Q^{k}_{-1,2} := t^k, \\
Q^{k}_{0,j} := x_{j+k}, \\
Q^{k}_{i+1,j} := tQ^{k}_{i,j+1} - \frac{1}{2}Q^{k}_{i,j+2} + \frac{1}{2}t^{j}Q^{k}_{i,2} + \frac{1}{2}(x_{j+2} - t^{j+2})Q^{k}_{i-1,2}.
\]

**Theorem 1** The equations \( P_{i,j} = P_{j,i} \) for \( i, j \geq 1 \), \( Q^{1}_{i,1} = Q^{2}_{i-1,2} \) for \( i \geq 0 \) are identities in \( \Lambda \).

**Remark 1.5** These relations in \( \Lambda \) are homogeneous of degree \( i + j + 6 \) and \( 2i + 2 \), respectively. The relation \( Q^{1}_{i,1} = Q^{2}_{i-1,2} \) allows us to express \( x_{2i+2} \) as a polynomial in \( t, x_3, x_4, \ldots, x_{2i+1} \). It has already been shown in [6] that \( \Lambda_0 \) is generated by the \( x_n \) with odd indices \( n \), so the existence of the \( Q \)-relations is no surprise.

The first (with respect to degree) \( P \)-relation that is not a consequence of the \( Q \)-relations is \( P_{1,3} = P_{3,1} \); if all \( x_{2i} \) in it are eliminated by \( Q \)-relations, one obtains exactly the relation \( P \) that has been established in [6] and proven in [2].

It is not known whether the relations of Theorem 1 together with \( x_1 = 2t \) generate the kernel of the map from the polynomial algebra \( \mathbb{Q}[t, x_1, x_2, \ldots] \) to \( \Lambda \), but it will be shown in section 4 that any further relation would be a multiple of \( t \), in contradiction to Conjecture 1.3.

**Corollary 1.6** The algebra \( \Lambda_0 \) is spanned (as \( \mathbb{Q} \)-vector space) by the following set of monomials

\[
M := \{ t^i, t^i x_{2n+1}^j, t^i x_{2n+1}^j x_{2n+3}^k \mid i \geq 0, j, k, n > 0 \}.
\]

If Conjecture 1.3 is true then \( M \) is a basis of \( \Lambda_0 \).

**Corollary 1.7** If \( \lambda_d \) denotes the dimension of the part in degree \( d \) of \( \Lambda_0 \), then

\[
\left| \frac{7d}{6} \right| - 3 \leq \lambda_d \leq \left| \frac{d^2}{12} + \frac{1}{2} \right| + 1.
\]

The upper bound is saturated if and only if Conjecture 1.3 is valid.
Remark 1.8

a) Using the eight characters of [6] and Corollary 1.6, we verified that Conjecture 1.3 is true for all $\lambda$ of degree $\leq 14$. This implies that the given upper bound is equal to $\lambda_d$ for $d \leq 15$.

b) Taking all characters into account, Pierre Vogel improved the lower bound: $\lambda_d \geq a_d$ with $\sum a_d x^d = \frac{1}{(1-x) (1-x^2)} + \frac{x^3-x^{16}}{(1-x) (1-x^2) (1-x^3)}$. This leads to $\lambda_d \geq \left\lceil \frac{5d}{4} \right\rceil - 2$ for $d \geq 21$.

Theorem 2 Conjecture 1.3 is equivalent to the following statement: There exists a unique “universal” character $\chi: \Lambda_0 \to \mathbb{Q}[t, u, v]$ satisfying $\chi(t) = t$, $\chi(x_0) = 0$, $\chi(x_1) = 2t$, $\chi(x_2) = t^2$ and for $n \geq 0$:

$$\chi(x_{n+3}) = t \chi(x_{n+2}) + u \left(2 \chi(x_{n+1}) - t^{n+1}\right) + v \left(2 \chi(x_n) - t^n - 2 (2t)^n\right).$$

If it exists, then $\chi$ is an isomorphism of graded algebras between $\Lambda_0$ and the sub-ring of $\mathbb{Q}[t, u, v]$ that consists of all elements of the form $a + (t^3 - tu + v)b$ with $a \in \mathbb{Q}[t]$ and $b \in \mathbb{Q}[t, u, v]$.

Remark 1.9

a) The recursion already appeared in [6], where it is shown that for certain values of $t, u, v$ it leads to well-defined homomorphisms of $\Lambda_0$, corresponding to the characters coming from Lie (super)algebras. Thus all these characters may be regarded as specializations of the (yet hypothetical) universal character.

b) We have been informed by Pierre Vogel that $\chi$ might be derived from a speculative object called “the universal Lie algebra” (see [7]); this may open an approach to Conjecture 1.3.

Acknowledgments

After the computer calculations described in [2] had revealed that $\Lambda_0$ is not a polynomial algebra, it was natural to ask if the identity found in degree 10 is member of a whole family of relations in $\Lambda$. In this sense, the main stimulus had been provided by Pierre Vogel; he noticed that an identity of values of characters – used in the proof of Lemma 7.5 in his paper [6] – can be modified to become an element of $F(4)$. If it is trivial, then one gets a one-parameter family of relations in $\Lambda_0$. Trying to prove this $F(4)$-relation, we realized that there should exist a $F(6)$-generalization of it, which produces a two-parameter family of relations in $\Lambda_0$. This idea finally lead to the investigations that are presented in [3].

I am also thankful to Jens Lieberum for advises concerning the manuscript, and to the Studienstiftung for financial support. A special thank is addressed to all participants of the “Knot Theory Week” in Bonn in July ’97 for creating an animating atmosphere which promoted the arising of the ideas that led to these results.

2 The $P$-relations

All $P$-Relations are consequences of a single relation in $F(6)$ that has been proven in [3]. To describe it, let $\alpha_{rs}$ denote elements of $F(6)$ that are given by the following picture:

\[ \begin{array}{c}
\cdots \\
\hline
r \\
\hline
\cdots \\
\end{array} \quad - \quad \begin{array}{c}
\cdots \\
\hline
s \\
\hline
\cdots \\
\end{array} \]
Thus $\alpha_{rs}$ is a linear combination of 16 graphs, each of them has $2(r + s)$ trivalent vertices. The univalent vertices are numbered clockwise, starting with the one on the lower left side. Then $\alpha_{r+1,s+1}$ is the same linear combination of graphs as the elements $[y]^n_r$ introduced in [3]. So we may rewrite equation (39) of [3] in the new terminology:

$$3\alpha_{35} = 9\alpha_{34} + 3t\alpha_{25} - 9t^2\alpha_{24} + 6t^2\alpha_{15} + (4t^3 + 2x_3)\alpha_{23} - 18t^3\alpha_{14} + 4t(4t^3 - x_3)\alpha_{13} + 8t^2(x_3 - t^3)\alpha_{12}$$

(1)

Now we consider the following operation for graphs of $F(6)$: glue $i$ edges between univalent vertices 1 and 2 and $j$ edges between 5 and 6; then identify the univalent vertices 2, 3 and 5 to form a new trivalent vertex:

```
    i

    j

    \Gamma_{i,j}(\alpha_{rs}) = 2(2x_{i+r} - t^{i+r})(t^{j+s} - 2x_{j+s}) - 2(2x_{i+s} - t^{i+s})(t^{j+r} - 2x_{j+r}).
```

Finally, by applying $\Gamma_{i,j}$ to (1), we obtain the desired relation $P_{i,j} = P_{j,i}$ in $\Lambda_0$.

3 The $Q$-relations

We use the calculus that has been established in section 5 of [6]. Elements in $\Lambda_0$ are presented by words of length $\geq 2$ in three letters 1, 2, 3. To the reader's convenience, we will repeat the relations of [6] that will be used:

$$\langle 12 \rangle = -t, \quad \langle 12^n 1 \rangle = x_n$$

$\forall \sigma \in S_3 : \langle w \rangle = \langle \sigma(w) \rangle$

$$\langle 1v \rangle + \langle 2v \rangle + \langle 3v \rangle = \langle v1 \rangle + \langle v2 \rangle + \langle v3 \rangle = 0$$

$$\langle u1v \rangle + \langle u2v \rangle + \langle u3v \rangle = 2t \langle uv \rangle$$

$$\langle \gamma v \rangle = t \langle \gamma \rangle, \quad \langle v\gamma^2 \rangle = t \langle \gamma \rangle$$

$$\langle u\gamma v \rangle + \langle u\gamma\sigma_2(v) \rangle = \langle u\gamma \rangle \langle \gamma v \rangle$$

Here $w$ represents a word of length $\geq 2$, $u, v$ words of length $\geq 1$, $\gamma$ a word of length 1 and $\sigma_2$ is the transposition of $S_3$ that keeps $\gamma$ fixed; $\gamma^n$ denotes a word consisting of $n$ copies of the letter $\gamma$.

First let us deduce some simple relations:

$$\langle u212 \rangle = -(\langle u211 \rangle - \langle u213 \rangle) = -(\langle u211 \rangle - 2t \langle u23 \rangle + \langle u223 \rangle + \langle u233 \rangle) = -t \langle u21 \rangle - t \langle u23 \rangle - \langle u222 \rangle - \langle u221 \rangle = -(\langle u221 \rangle)$$

(2)

$$\langle 212^n 1 \rangle = -(112^n 1) - (312^n 1) = -t \langle 12^n 1 \rangle - 2t \langle 32^n 1 \rangle + \langle 32^{n+1} 1 \rangle + \langle 332^n 1 \rangle = -t \left( \langle 32^n 1 \rangle + \langle 12^n 1 \rangle + \langle 22^n 1 \rangle \right) - (12^n + 11) = -x_{n+2}$$

(3)

$$\langle 232^n 1 \rangle = -(332^n 1) - (132^n 1) = -t \langle 32^n 1 \rangle - 2t \langle 12^n 1 \rangle + \langle 112^n 1 \rangle + \langle 12^n + 1 \rangle = t \langle 22^n 1 \rangle + (12^n + 1) = x_{n+2} - t^{n+2}$$

(4)

$$\langle u212^n 1 \rangle = 2t \langle u2^n 1 \rangle - \langle u2^{n+2} 1 \rangle - \langle u232^n 1 \rangle$$

(5)

$$\langle u232^n 1 \rangle = \langle u2 \rangle \langle 232^n 1 \rangle - \langle u212^n 3 \rangle = \langle u2 \rangle \langle 232^n 1 \rangle + \langle u212^n 1 \rangle + \langle u212^n + 1 \rangle$$

(6)
The difference of the last two equations (5) - (6) leads to

\[ (u212^n1) = t (u2^{n+1}1) - \frac{1}{2} (u2^{n+2}1) - \frac{1}{2} t^n (u212) - \frac{1}{2} (u2) (232^n1) \]  

(7)

For \( i, j \geq 0, k \in \{1, 2\} \) let \( q_{i,j}^k := (2^{k-1}(12)^i12/1) \), then (7), (2), (4) imply for \( i \geq 2, j \geq 1 \):

\[
q_{i,j}^k = t q_{i-1,j+1}^k - \frac{1}{2} q_{i-1,j+2}^k - \frac{1}{2} t^i (2^{k-1}(12)^i1) - \frac{1}{2} (2^{k-1}(12)^i) (232^j1) \\
\quad = t q_{i-1,j+1}^k - \frac{1}{2} q_{i-1,j+2}^k + \frac{1}{2} t^j q_{i-1,2}^k + \frac{1}{2} q_{i-2,j}^k (x_j + t^{j+2})
\]

This is exactly the recursive formula in the definition of \( Q_{i,j}^k \). We have \( q_{0,j}^1 = \langle 12/1 \rangle = x_{j+1} = Q_{0,j}^1 \) and (3) \( q_{0,j}^2 = \langle 212/1 \rangle = -x_{j+2} = -Q_{0,j}^2 \) for \( j \geq 1 \). A little calculation shows that \( q_{1,j}^k = Q_{1,j}^k \) and \( q_{2,j}^k = -Q_{1,j}^k \) as well. Thus the equality \( Q_{i,j}^k = (-1)^{k+1} q_{i,j}^k \) holds in \( \Lambda_0 \) for all \( i \geq 0, j \geq 1 \) and \( k \in \{0,1\} \).

Because of equation (2) we have

\[ q_{i-1,2}^k = \langle 2(12)^{i-1}1221 \rangle = -\langle 2(12)^i1-1212 \rangle = -\langle 2(21)^i212 \rangle = -\langle (12)^i121 \rangle = -q_{i,1}^1. \]

This proves that for \( i \geq 1 \) the relation \( Q_{i,1}^1 = Q_{i-1,2}^2 \) is valid in \( \Lambda_0 \). It remains to check the case \( i = 0 \) which is just the identity \( x_2 = t^2 \).

4 Implications

**Lemma 4.1** There exists a degree-preserving bijection \( \beta \) between the following sets of monomials

\[ \{ x_{2n+1}^i x_{2n+3}^j \mid i, n > 0, j \geq 0 \} \text{ and } \{ u^k v^l \mid k \geq 0, l > 0 \}, \]

where the degree of \( x_m, u, v \) are assumed to be \( m, 2, 3 \), respectively.

**Proof** The map given by \( \beta(x_{2n+1}) := u^{n-1}v \) and \( \beta(pq) := \beta(p) \beta(q) \) obviously relates monomials of the same degree. To show that \( \beta \) is a bijection, one has to verify that for any integers \( k \geq 0 \) and \( l > 0 \) the Diophantine system

\[
(n - 1) i + n j = k \\
i + j = l \\
n > 0 \\
i > 0 \\
j \geq 0
\]

has exactly one solution \( (n, i, j) = (\lfloor \frac{k}{l} \rfloor + 1, \lfloor \frac{k}{l} \rfloor l - k + l, k - \lfloor \frac{k}{l} \rfloor l) \).

If \( \chi(x_m) \in \mathbb{Z}[u, v, w] \) is calculated via the recursion of Theorem 2 and if the monomials \( t^i u^j v^k \) are ordered by the lexicographical order of the pairs \( (-i, -k) \), we make an important observation: for all \( n \geq 1 \) the leading term of \( \chi(x_{2n+1}) \) is \( -3 \cdot 2^n v \cdot \beta(x_{2n+1}) = -3 \cdot (2u)^{n-1}v \). This assertion is true for \( \chi(x_3) = -3v + 3u + t^3 \) and we obtain for \( n \geq 1 \) per induction:

\[
\chi(x_{2n+3}) = 2u \chi(x_{2n+1}) + 2v \chi(x_{2n}) + t \cdot (\ldots) = -3 \cdot (2u)^{n-1}v + v^2 \cdot (\ldots) + t \cdot (\ldots)
\]

The ordering of monomials is multiplicative, so (up to a non-zero scalar) the leading term of \( \chi(x_{2n+1}^i x_{2n+3}^j) \) is \( \beta(x_{2n+1}^i x_{2n+3}^j) \).

**Lemma 4.2** The monomials \( \{ 1, t^i, t^i x_{2n+1}, t^i x_3 x_{2n+1}, x_{2n+1}^j, x_{2n+1}^j x_{2n+3}^j \mid i, j, n > 0 \} \) are linearly independent in \( \Lambda \).
Proof Let $\chi_D$ denote the character to $Q[\sigma_2, \sigma_3]$ corresponding to the Lie super-algebra $D(2,1,\alpha)$. Due to results of [6], one can calculate $\chi_D(x_n)$ with the recursion of Theorem 2 by setting $t = 0$, $u = -2\sigma_2$ and $v = 4\sigma_3$. The upper observation and Lemma 4.1 make clear that $\chi_D$ maps the monomials of type $x_{2n+1}^d$ and $x_{2n+1,2n+3}^d$ to linearly independent elements of $Q[\sigma_2, \sigma_3]$.

All monomials that contain $t$ are mapped to 0 by $\chi_D$, so it only remains to show the linear independence of the monomials of type $t^s t^t x_{2n+1}, t^s x_3 x_{2n+1}$ of a given degree $d$. We will use the $t = 1$ specialization of the character $\chi_{osp}$ of [6] which we will call $\chi_{osp}$ as well. The target of $\chi_{osp}$ is $Q[\alpha]$ and $\chi_{osp}(x_n)$ is given by setting $t = 1, u = -\alpha + 6\alpha^2$ and $v = -4\alpha^2 + 8\alpha^3$ in $\chi(x_n)$. It is an immediate consequence of the recursion formula that the degree of the polynomial $\chi_{osp}(x_n)$ is $n$ for $n = 3$ or $n \geq 5$. So the $\chi_{osp}$-images of $t^d, t^{d-2n-1} x_{2n+1}, t^{d-2n-4} x_3 x_{2n+1}$ all have different degrees; this completes the proof.

Proof of Corollary 1.6 Let us introduce an ordering on the monomials $t^c x_3^e \ldots x_{2n+1}^e$ by the lexicographical order on the tuples $(-\sum_{i=0}^n e_i, e_1, e_2, \ldots, e_n)$. The $P$- and $Q$-relations are essentially of the form

$$3 x_{i+2} x_{j+4} = 3 x_{i+4} x_{j+2} + \text{terms with exponent sum} > 2$$

$$3 \cdot (-2)^{-i} x_{2i+2} = \text{terms with exponent sum} > 1 \quad \text{(for} \; i \geq 1\text{).}$$

Let $m$ denote a monomial that contains $x_n$ and $x_m$ with $3 \leq n \leq m - 3$. Via the relation $P_{n-2,m-4} = P_{m-4,n-2}$ one can express $m$ as sum of $m/(x_n \cdot x_m) \cdot x_{n+2} \cdot x_{m-2}$ and terms having a bigger exponent sum. All monomials in this sum are smaller than $m$.

The $Q$-relations enable us to write any monomial containing $x_{2i}$ in terms of smaller monomials. Doing both substitutions repeatedly, we obtain (after a finite number of steps because the degree is preserved) a linear combination of monomials that have only odd indices, which differ at most by 2 inside each monomial. This shows that $M$ spans $\Lambda_0$.

To prove the second statement, let us assume that there is a non-empty linear combination $c$ of elements of $M$ that is trivial in $\Lambda_0$. Without loss of generality we may further require that $c$ is homogeneous of degree $d$ with the smallest possible $d$. Write $c = t \cdot c + r$ where $r$ is a combination of elements of $M$ that do not contain $t$. Because of $\chi_D(c) = 0$ and $\chi_D(t) = 0$ we have $\chi_D(r) = 0$ and Lemma 4.2 implies that $r$ is the empty linear combination. So $c$ is a non-empty linear combination of degree $d - 1$ and since we assumed Conjecture 1.3, $\check{c}$ must be trivial in $\Lambda_0$ contradicting the minimality of $d$.

Proof of Corollary 1.7 For the lower bound we have to estimate the number of monomials of degree $d$ that appear in Lemma 4.2. The number of monomials of the form $t^d, t^{d-2n-1} x_{2n+1}$ or $t^{d-2n-4} x_3 x_{2n+1}$ is at least $1 + \left\lfloor \frac{d-3}{2} \right\rfloor$ + $\left\lfloor \frac{d-3}{2} \right\rfloor = d - 3$ for $d \geq 0$ (for $d \geq 5$ this is even the exact value). To complete the proof of the lower bound, in view of Lemma 4.1 it remains to show

$$\# \{ (i, j) \mid 2i + 3j = d, \; i \geq 0, \; j > 0 \} \geq \left\lfloor \frac{d}{6} \right\rfloor.$$  

For any $d$ one may write $d = 6 \cdot \left\lfloor \frac{d}{6} \right\rfloor + 2p+3q$ with $(p, q) \in \{(0, 0), (2, -1), (1, 0), (0, 1), (2, 0), (1, 1)\}$. Then for $1 \leq k \leq \left\lfloor \frac{d}{6} \right\rfloor$ the pair $(p + 3 \left(\left\lfloor \frac{d}{6} \right\rfloor - k), q + 2k)$ is an element of this set.

For the upper bound, we have to count the monomials of degree $d$ in the set $M$ of Corollary 1.6. Lemma 4.1 implies that this number is $1 + N(d)$, where

$$N(d) := \# \{ u^i v^j \mid 2i + 3j \leq d, \; i \geq 0, \; j > 0 \}.$$  

With $t := \left\lfloor \frac{d}{3} \right\rfloor$ and $\varepsilon(n) := \frac{(-1)^{n-1}}{2}$ we get:

$$N(d) = \sum_{j=1}^{t} \# \{ i \mid 0 \leq 2i \leq d - 3j \} = \sum_{j=1}^{t} \left\lfloor \frac{d-3j}{2} \right\rfloor + 1 = \sum_{j=1}^{t} \frac{d-3j+2 + \varepsilon(d+j)}{2}$$
\[
\begin{align*}
\frac{1}{2}t d - \frac{3}{4}(t + 1) + t + \frac{1}{2} \sum_{j=1}^{t} \varepsilon(d + j) &= \frac{1}{2} t d - \frac{3}{4} t^2 + \frac{1}{4} \mu(d) \\
\end{align*}
\]

In the last equality we made the abbreviation \(\mu(d) := \left[ \frac{d}{2} \right] + 2 \cdot \sum_{j=1}^{d} \varepsilon(d + j)\).

It turns out that \(\mu(d) = 0, 0, 0, 1, -1, 1\) when \(d \equiv 0, 1, 2, 3, 4, 5\) modulo 6. If we set \(r := d - 3 t\), we finally obtain \(\frac{d}{2} t + \frac{t}{2} - N(d) = \frac{t^2}{2} - \frac{\mu(d)}{4} + \frac{t}{2}\), which is a value between 0 and 1 for all \(d\). This shows that \(N(d) = \left[ \frac{d^2}{2} + \frac{t}{2} \right] \).

\[\square\]

**Proof of Theorem 2** The \(\leftarrow\)-direction is easy; we have already seen that if \(\chi\) with demanded properties exists, it maps the elements of the spanning set \(M\) of type \(t^i x_{2n+1}^i x_{2n+3}^k\) to polynomials with leading terms \(t^i \beta(x_{2n+1}^i x_{2n+3}^k)\), which are linearly independent. Thus \(\chi\) is injective and \(\Lambda_0\) has no zero divisors because \(Q[t, u, v]\) is an integral domain.

Let \(\hat{x}_n\) denote the polynomial in \(t, u, v\) that is assigned to \(x_n\) by the recursive rule in Theorem 2. To prove the \(\Rightarrow\)-direction of the statement, we have to show that the \(\hat{x}_n\) satisfy the \(P\)- and \(Q\)-relations (i.e. replacing all \(x_n\) by \(\hat{x}_n\) turns \(P\)- and \(Q\)-relations into identities in \(Q[t, u, v]\)).

The \(P\)-relations this can be shown in a straightforward way: Let \(R_{i,j} := P_{i,j} - P_{j,i}\) and \(S_i := t x_{i-1} + u (x_{i-2} - t^{n+1}) + v (2 x_n - t^n - 2 (2t)^n)\). With a little patience, one verifies that replacing every \(x_{j+p}\) in \(R_{i,j}\) by \(S_{j+p}\) (i.e. for \(p = 0, 1, 2, 3, 4\)) leads to a combination of \(R_{i,k}\)-s with \(k < j\):

\[
R_{i,j} | x_j = S_j, \ldots, x_{j+i} = S_{j+i} = t R_{i,j-1} + 2 u R_{i,j-2} + 2 v R_{i,j-3}
\]

So if the \(\hat{x}_n\) satisfy the relation \(P_{i,j} - P_{j,i}\) for \(j \leq n, n \geq 4\), then \(P_{i,n+1} - P_{n+1,i}\) is also satisfied. To complete the inductive argument, only the cases \((i, j) = (1, 2), (1, 3), (2, 3)\) have to be checked by explicit calculation.

The \(Q\)-relations are more complicated to handle. Let \(q_{i,j}^k\) denote the polynomial of \(Q[t, u, v]\) that is obtained when in \(Q_{i,j}^k\) every \(x_n\) is replaced by \(\hat{x}_n\). By definition we have for \(i \geq 1, j \geq 0\)

\[
q_{i,j}^k = t q_{i-1,j+1}^k + \frac{1}{2} q_{i-1,j+2} + \frac{1}{2} t q_{i-1,j+2}^k + \frac{1}{2} (\hat{x}_{j+2} - t^{j+2}) q_{i-2,j+2}^k.
\]

For \(i \geq 0, j \geq 3\) and \(k \in \{1, 2\}\) there is another helpful relation, namely

\[
q_{i,j}^k = t q_{i,j-1}^k + 2 u q_{i,j-2} + 2 v q_{i,j-3} - (u t^{-2} + v t^{-3}) q_{i-1,j-2}^k - v (2t)^{-1} q_{i-2,j-2}^k.
\]

To make it correct for \(i = 0\), we set \(q_{1,2}^k := 2 (2t)^{-k-2}\). With \(q_{1,2}^k = t^k\) and \(q_{0,1} = \hat{x}_{j+k}\) this relation translates into the recursive definition of the \(\hat{x}_n\) for \(i = 0\). To prove the relation for \(i \geq 0\), one uses (8) to reduce it to terms with first index \(< i\) and concludes by induction; we leave the detailed calculation as an exercise.

The equation (8) is kindly simple for \(j = 0\): \(q_{i,0}^k = t q_{i-1,1}^k\). Together with (9) we can thus write any \(q_{i,j}^k\) in terms of \(q_{i,a}^b\)-s with \(a \leq i\) and \(b \in \{1, 2\}\). When we do this to the right side of equation (8) for the cases \(j = 1\) and \(j = 2\), we obtain (for \(i \geq 1\)):

\[
\begin{align*}
q_{i,1}^k &= t q_{i-1,1}^k - 2 u q_{i-1,1}^k + 2 v q_{i-1,2}^k + 2 t v q_{i-2,3}^k \quad (10) \\
q_{i,2}^k &= (t^2 - u) q_{i-1,2}^k + (t u - v) q_{i-1,1}^k + 2 t (t u - v) q_{i-2,2}^k - 2 t v q_{i-2,1}^k + 2 t^2 v q_{i-3,2}^k \quad (11)
\end{align*}
\]

Now we are able to prove that the following two identities

\[
q_{i,1}^2 = q_{i,2}^1, \quad \text{and} \quad q_{i,1}^2 = q_{i,1}^1,
\]

hold for \(i \geq 0\). It is easy to check them for \(i = 0, 1, 2\). For \(n \geq 3\), one has to assume that both equations hold for \(i < n\) and deduce the case \(i = n\) by skillful use of (10) and (11). Once more, this is a lengthy calculation that we will leave to the reader’s ambition. Up to some nasty computation, we have finally shown that the \(\hat{x}_n\) satisfy the relations \(P_{i,j} = P_{j,i}\) and \(Q_{1,1}^1 = Q_{1-1,2}^2\).
Under the premise that $t$ is injective, we have already seen that these are the only relations in terms of $x_n$’s in $\Lambda_0$, so $\chi$ is well defined.

As we have seen above, $\chi$ is then injective, and the recursive formula for $\chi$ allows us to verify that $\chi(x_n)$ is always of the form $a + (t^3 - tu + v)b$ with $a \in \mathbb{Q}[t]$ and $b \in \mathbb{Q}[t,u,v]$. So finally, Corollary 1.6 and Lemma 4.1 imply the last statement of Theorem 2, which completes the proof.

□

References

[1] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology 34 (1995), 423-472.

[2] J. A. Kneissler, *The number of primitive Vassiliev invariants up to degree twelve*, e-print archive (http://xxx.lanl.gov), q-alg/9706022.

[3] J. A. Kneissler, *On spaces of connected graphs I: Properties of ladders*, Proc. Internat. Conf. "Knots in Hellas '98", Series on Knots and Everything, vol. 24 (2000), 252-273.

[4] J. A. Kneissler, *On spaces of connected graphs III: The ladder filtration*, Jour. of Knot Theory and its Ramif. Vol. 10, No. 5 (2001), 675-686.

[5] V. A. Vassiliev, *Cohomology of knot spaces*, Theory of Singularities and its Applications (ed. V. I. Arnold), Advances in Soviet Math., 1 (1990), 23-69.

[6] Pierre Vogel, *Algebraic structures on modules of diagrams*, Université Paris VII preprint, July 1995 (revised 1997).

[7] Pierre Vogel, *The universal Lie algebra*, Université Paris VII preprint, June 1999.

e-mail:jan@kneissler.info
http://www.kneissler.info