On the Work of Benjamin Olinde Rodrigues (1795-1851) — in particular, on “Expression of Spatial Motions” —

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Abstract. This is a translation of Proceedings of 22th Symposium on History of Mathematics, Tsuda University 2011, on the work of Benjamin Olinde Rodrigues and on his life. His chef-d’oeuvre is the work on Euclidean motion group in 1840. He invented Rodrigues expression of rotation and gave explicit calculation formula for product of two rotations, which might be considered as a discovery of quaternion product rule ahead of Hamilton. We follow a new proof of É. Cartan in his book on spineurs in 1938 for Rodrigues formula, which was called as Euler-Olind-Rodrigues formula mistakenly. We add as Appendix important parts of Lecture Note on applications of quaternion. There are given description of rotational movements in Rodrigues expression and an interesting compact formula for time derivative of rotation, applicable in many purposes.

1 Motivation for research report and others

Why I came to pay attention to Benjamin Olinde Rodrigues (1795/10/6 – 1851/12/17) is because, in fact, I wanted to know the historical process of the quaternion of Hamilton. Actually, in the workshop "Non-Commutative Harmonic Analysis" at Będlewo, Poland, August of 2007, I gave a talk on projective representations of complex reflection groups containing symmetric groups, and necessarily I discussed about the trilogy [Sch1, 1904] – [Sch3, 1911] of Schur’s theory on spin representations of finite groups and noticed that, in the third paper [Sch3], there appeared substantially a triplet of matrices which is called later as Pauli matrices, found independently by Physicist Pauli in [Paul, 1927]. About this historical comment, Prof. Marek Bożejko gave me a question “How about the case of quaternion of Hamilton?”, which means that in the theory of quaternion in 19th century, “Is there something like such triplet of matrices?” It might be possible, and mathematicians nowadays know generally that the unit ball $B$ consisting of quaternions with norm 1 gives a double covering group of the 3-dimensional rotation group $SO(3)$. However, at that time, I lacked totally such kind of historical point of view, and couldn’t answer his question. I felt deeply that I know nothing about Hamilton to answer this question and that I ought to study this sometime in future.

1 2010 Mathematics Subject Classification: Primary 20-03, 01A70; Secondary 20C99, 01A85.

Key Words and Phrases. Cartan’s spineur, quaternion of Hamilton, projective or spin representations of $SO(3)$, Pauli matrices, time derivative of Rodrigues formula of rotation

2 Translation (with small changes) of Proceedings of 22th Symposium on History of Mathematics in Reports of Institute for Mathematics and Computer Sciences, Tsuda University, 33 (2012), 59–79.
Note 1. Three matrices appeared in p.198 of Schur’s paper [Sch3, 1911] are
\[
F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
with \[
\begin{align*}
A^2 &= F, & B^2 &= -F, & C^2 &= F, & CBA &= F, \\
AB &= -BA &= -C, & BC &= -CB &= -A, & CA &= -AC &= B,
\end{align*}
\]
and they are used, as fundamental ingredients, to write down explicitly doubly-valued projective representations of symmetric groups \(\mathfrak{S}_n\) and alternating group \(\mathfrak{A}_n\).

On the other hand, three matrices appeared in Pauli’s paper [Paul, 1927] and later called as Pauli matrices, express rotation moment (spin) of electron, act on \((\mathbb{C}^2\text{-valued})\) wave function \(\psi\) as follows:
\[
\begin{align*}
s_x(\psi) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi, & s_y(\psi) &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \psi, & s_z(\psi) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \psi,
\end{align*}
\]
with commutation relation \([s_x, s_y] = 2is_z, [s_y, s_z] = 2is_x, [s_z, s_x] = 2is_y, i = \sqrt{-1}].

Further, Élie Cartan discussed in his paper [Car1, 1913], without introducing the terminology such as spineur (spinor in French), representations of \(SU(2)\) double covering of rotation group \(SO(3)\), actually containing spin representations together with linear representations, but did not appear there any triplet of matrices.

In the meantime, I was asked in 2008 from Prof. Satoshi Kawakami to give a Summer Intensive Course of one week for Mathematics Master Course of Nara Education University, and so I proposed as a subject “Quaternion, 3-dimensional Rotation Group and Introduction to Representation Theory of Groups” and begun to prepare a Lecture Note. In that occasion, when I searched documents about quaternion in website, I found a paper [Alt2] Hamilton, Rodrigues, and the Quaternion scandal. What does it mean scandal, in such an academic situation? I wondered and read it with a keen interest, then I noticed that the historical facts which I imagined myself until then, arround universal covering group of 3-dimensional rotation group, or parameter expressions of 3-dimensional rotations and so on, differ considerably from the true story. Thus I gradually went deeply into Rodrigues, the central figure of the present paper. As for Lecture Note above, I attach its important part with deep relation to this paper as an Appendix for more detailed commentary.

2 Chronology for Benjamin Olind Rodrigues

1789–1799: French Revolution,
1804–1814, 1815 (The One Hundred Days): Napoleon IĄC
1814–1815, 1815–1824: Bourbon Restoration, Louis XVIII,
1824–1830: Restoration, Charles X,
1830–1848: Louis-Philippe I.

3See also [Hir9].
1795/10/06, Benjamin Rodrigues was born.

1807, Jews living in France were required to modify their family name.

1808, Jews were required to add a name of a Christian Saint.

1811, Entrance examination for École Polytechnique and École Normale, Rodrigues was ranked first (15 years old).

Ranked first in the entrance examination, but he did not enter any of the above École. About this situation, I read several explanations: e.g., he was refused to enter because of Jews, or his oncre refused to pay his school expenses, or next year he is also ranked first or second in the entrance exam, etc. Anyhow, for usual French people, at the same time of entering in one of these Écoles, he is adopted by a government official, receive school expenses exemption, and a salary is paid, but it was not the case for Rodrigues. Finally he entered Université de Paris.

1815/06/23, Soutien pour le doctorat (19 years old),
à la Faculté des Sciences de Université de Paris, sous la présidence de M. Lacrois,
Doyen de la Faculté.

1815/06/28, Awarded a doctorate in Mathematics
from Faculté des Sciences, Université de Paris.

1815～, After the 1815 Restoration the Catholic hierarchy took control of educational and academic institutions, and Jewish people could not obtain any teaching position.

1816, The main part of the above thèse was published in
Mémoire (*) sur l’attraction des sphéroïdes, PREMIÈRE PARTIE. Formules générales pour l’attraction des corps quelconques, et application de ces formules à la sphère et aux ellipsoïdes. SECONDE PARTIE. Attraction des sphéroïdes infiniment peu différents d’une sphère, et développement général de la fonction V.
ibid., 3 (1814–1816): pp.361–385, 1816.
This paper contains Rodrigues formula for Legendre polynomials:

\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \]

—– between 1817–1837, no papers in Mathematics were published —–

1838, Three papers were published in Journal de Mathématiques pures et appliquées,
vol. 3 (1938), pp.547–548, pp.549–549, pp.550–551. (He was 42 years old)
The contents of 3 papers are the following:

[R3-1] The number of ways to decompose a convex polygon into triangles by diagonals
[R3-2] The number of ways to make product with n factors
[R3-3] Elementary and purely algebraic proof of development of binomial \((1 + a)^x\) into power series with negative or fractional powers

Length of each paper is short, but contents are important. The first two gave compact and clear proofs for main results of long papers published very recently in journals.

1839, A paper [R4-1] was published in ibid., 4(1839), 236–240. Its contents are
[R4-1] *Number of inversions in the order of products of permutations*

**Contents.** The generating function for this number was given, and this result continued to have important influences until nowadays. For \( \sigma \in \mathfrak{S}_n \), count the number of pairs \( i < j \) such that the inversion \( \sigma(i) > \sigma(j) \) occurs, and let \( N_n(k) \) be the number of \( \sigma \) for which the number of inversions is just equal to \( k \). Define the generating function of \( N_n(k) \) as

\[
R_n(q) := \sum_k N_n(k) q^k.
\]

Rodrigues gave a method of computing \( R_n(q) \) inductively.

**Note.** For this problem, later there were studies of Netto and MacMahon. But, only in 1970, Leonard Carlitz discovered this paper [R4-1].

1840, A paper [R5-1] was published in *ibid.* vol. 5, pp.380–440. Its subjects are

[R5-1] *Eulidean Motions in the Space, in particular Rotations*

This long paper gives very important results and can be said as his Chef d’Oeuvre. The contents, containing substantial discovery of Quaternion, will be explained in §3.

1843, Two papers were published in *ibid.* vol. 8,

[R8-1] pp.217–224, [R8-2] pp.225–234.

Cf. 1843/10/16, Discovery of *Quaternion* by William Rowan Hamilton (1805–1865): by giving the fundamental formula \( i^2 = j^2 = k^2 = ijk = -1 \).

1843/10/17, Letter from Sir W. R. Hamilton to John T. Graves, Esq. *on Quaternions* (which is handwritten) [Later, in printed form, 1 line about 75 letters and total 167 lines, 4 pages, containing footnotes, in Collected Works.]

**Note.** In these two days, he wrote, with a quill and ink, a letter of this volume and its fair copy for himself. This is a great concentration of a man of genius.

1851/12/17, Benjamin Olinde Rodrigues 56 years old died.

3 Chef-d’Oeuvre Paper [R5-1] in 1840

Paper [R5-1] by Olinde Rodrigues: *Des lois géométriques qui régissent les déplacements d’un système solide dans l’espace, et la variation des coordonnées provenant de ses déplacements considérés indépendamment des causes qui peuvent les produire.*

Translation of Title: *Geometric rules which govern the movements of a system of solid bodies in the space, and changes of coordinates coming from its displacements considered independently of the reason which produces them.*

The Style of Writing of this paper. There is no independent Introduction. The paper is separated into parts with numbers from 1 to 33, which we call here as *parts* (maybe *numeros* in French). Each *part* has only its number and no title. As general style, there are 18 Titles in italic under each of which a group of *parts* are gathered. But there exist several exceptions, for instance in some *parts*, there are one or two italic titles (something like as
subsections in a section). Theorems are not separated from ground sentences as in modern style where theorems are numbered and their assertions are written in italic. In the middle of part 4 (or n° 4), there is a big italic title such as

*Théorème fondamental.*

But this might be a title of a subnumero (like subsubsection in a subsection) and I found several of such italic titles. There are neither Propositions, Lemmas nor Diagrams.

**Contents.** He discussed very generally on displacements of solid bodies, that is, *Euclidean motion group* in modern language. Naturally there are two kind of motions, rotations and parallel translations. He treated the latter as a kind of *infinitesimally small rotations* with rotation axis situated in the perpendicular direction at the infinite long distances. This is the general idea (Idée générale) throughout of this paper. Accordingly, he asserted (in n° 1 and n° 2) that “the properties of parallel translations are contained in the properties of rotations”. In his original expression,

Ainsi donc, toute translation d’un système peut rigoureusement être considéré comme une rotation d’une amplitude infiniment petite autour d’un axe fixe infiniment éloigné et normal à la direction de cette translation.

On ne sera donc pas surpris de trouver ultérieurement toutes les propriétés des translations comprises dans celles des rotations, ·····

Thus, as methods of discussions, repeatedly he used calculations using infinitely small pararell displacements $\Delta x, \Delta y, \Delta z$, of the directions of $x$-axis and so on, and limit transitions from spherical triangles to planar triangles.

**List of Titles and the corresponding numbers of parts:**

*Idée générale de la translation et de la rotation d’un système solide.* n° 1 ∼ 2

*Du déplacement d’un système d’un point fixe.* n° 3

(Displacement fixing a point, or a rotation arround a fixing point.)

*Du déplacement quelconque d’un système solide dans l’espace.* n° 4 ∼ 7

*De la composition des rotations successives d’un solide autour de deux axes convergents.* n° 8

(Product of rotations arround two axis that intersect each other, or product of two rotations arround the intersecting point.)

This is one of highlight points of the paper where the composition of two rotations is treated. Here jumping over the usual product structure in the rotation group $SO(3)$, there appears the product structure in its universal covering (double covering) group $Spin(3)$. Thit is the *product formula in Rodrigues expression* of rotations. I quote his original sentences at this important critical point:

Telle est la différence caractéristique à signaler entre la composition des rotations et celle des translations successives. Il y a d’ailleurs entre des deux sortes de composition l’analogie qui existe entre les propriétés du triangle rectiligne et celles
Denote by $R$ a synthetic calculation method, and one can say that the calculation rule of known formula in spherical trigonometry (cf. Note A3.1) appeared substantially here (cf. [Agn], [Alt1]). It seems that French mathematicians at the same center have rather considerable background in spherical trigonometry.

(Explanation as I understand) The first sentence, in response to the previous sentences, refers to “difference between rotation composition and translation composition”. The following sentence explains how to calculate the compositions of two rotations around the same center $O$. But it is difficult for me to translate this and relating portions of this n°8 well into English correctly, so I will explain it in my own words.

On a unit sphere with center $O$, take two point $A$ and $B$ and put $n_A = \overrightarrow{OA}$, $n_B = \overrightarrow{OB}$. Denote by $R(\phi_A n_A)$ with $\phi_A \in R$ the rotation around the unit vector $n_A$ (as rotation axis) with the angle $\phi_A$ to the direction of right-handed screw. Then the assertion is

**Assertion 3.1** The product $R(\phi_B n_B)R(\phi_A n_A)$ of two rotations is expressed as $R(\phi_C n_C)$ with rotation axis $n_C = \overrightarrow{OC}$ and rotation angle $\phi_C$ given as follows: rotate the plane $OAB$ around $n_A$ by angle $-\phi_A/2$, then we get a line (= big circle) on the sphere. Similarly rotate the plane $OAB$ around $n_B$ by angle $\phi_B/2$, then we get another line on the sphere. Two lines intersect at a point $C$ (the nearest intersecting point), and we put inner angle at $C$ as $\pi - \phi_C/2$ (or outer angle $\phi_C/2$).

**Proof.** We draw several auxiliary lines (cf. Figure 3 in [Alt1]). Rotate the plane $OAB$ around $n_A$ by angle $\phi_A/2$, then we get a line on the other side of $\triangle ABC$, and rotate $OAB$ around $n_B$ by angle $-\phi_B/2$, we get another line, and they cross each other at a point $C'$.

$\triangle ABC'$ is a mirror image of $\triangle ABC$. Rotate $OAB$ around $n_A$ by angle $-\phi_A$, then we get a line on the other side of $\triangle ABC$, and rotate $OAB$ around $n_B$ by angle $\phi_B$, we get another line, and they cross each other at a point $A'$.

1) $C$ is invariant under $R(\phi_B n_B)R(\phi_A n_A)$.

In fact, under the first rotation $R(\phi_A n_A)$, $C$ is mapped to $C'$, and under the second $R(\phi_B n_B)$, $C'$ is mapped back to $C$.

2) $A$ is mapped to $A'$. In fact, under the first rotation, $A$ is mapped to $A$, and under the second, $A$ is mapped to $A'$.

3) $\angle ACA' = \phi_C$.

In fact, denote by $D$ the crossing point of $BC$ and $AA'$. Then $\triangle ACD$ and $\triangle DCA'$ are mutually mirror images of the other. So, $\angle ACA' = 2\angle ACD = 2(\phi_C/2) = \phi_C$. $\square$

Thus Assertion 3.1 is proved and so $\sin(\phi_C/2)$ and $\cos(\phi_C/2)$ can be calculated using known formula in spherical trigonometry (cf. Note A3.1 in Appendix below). This gives a synthetic calculation method, and one can say that the calculation rule of quaternion has appeared substantially here (cf. [Agn], [Alt1]). It seems that French mathematicians at the time had rather considerable background in spherical trigonometry.

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4 The title Assertion 3.1 is temporarily given here by me for convenience of quotation.

5 Note that if we rotate the plane $OAB$ by angle $\pi$ (a half of $2\pi$), then it comes back to itself.
Composition des rotations infiniment petites. n° 9
Here the translations are treated as infinitesimally small rotations arround an axis at infinity, and discuss their compositions (= products).

De la composition des rotations autour de deux axes parallèles. n° 10 ~ 11
Composition of rotations arround two axis parallel to each other.

De la composition des rotations autour d’axes fixes non convergents en nombre quelconques.
Composition of rotations arround non-intersecting two axis, very complicated.

Examen du cas particulier des axes non convergents.
Examination in the special case of two axis non-intersecting.

De la composition des déplacements successifs d’un système combinés de rotations et de translations.
On the composition of rotations and translations.

Équation de l’axe central.
Equations which determines the axis of the composed rotation.

Examen du cas des variations infiniment petites.
Examination of the case of infinitely small displacements.

De la composition analytique des rotations autour d’axes non convergents.
Calculation formula for composition of rotations around two axis non-intersecting.

Composition des rotations successives autour de trois axes rectangulaires.
Composition of rotations arround three orthogonal axis (this corresponds so-called Euler product of rotation).

De la composition des déplacements infiniment petits successifs d’un système solide.

Conditions d’équilibre de plusieurs déplacements successifs infiniment petits. n° 24 (25 missing)
Discussions on the realization of the state of Equilibrium, that is, the condition for that, after successive displacements, the solid body comes back to the original position.

Analogie de ces lois de composition et d’équilibre avec celles de la composition et de l’équilibre des forces appliquées à un système invariable. n° 26
Discussions on the striking analogy (l’analogie frappante) between the above state of Equilibrium and Equilibrium state when force is applied.

De la détermination des variations des coordonnées d’un système solide dues à un déplacement quelconque de ce système, analytiquement déduites des conditions

*By the way, Euler discussed the existence of axis for a rotation, but he didn’t discuss Euler expression of a rotation, as I understand.*
4 É. Cartan’s Proof of Rodrigues formula for rotations

Élie Cartan quoted in no 59. Représentation d’une rotation, p.57, in his book [Car2], 1938, one of main results of Rodrigues as follows and gave a proof of his own. I quote the central part of the proof of Cartan:

La formule (3) permet de retrouver les formules d’Euler-Olinde-Rodrigues. Soit \( L \) le vecteur unitaire porté sur l’axe de rotation et \( \theta \) l’angle de rotation; les deux vecteurs unitaires \( A, B \) ont pour produit scalaire \( \cos \frac{\theta}{2} \) et leur produit vectoriel \( \frac{1}{2}(AB - BA) \) est égal à \( iL \sin \frac{\theta}{2} \). On en déduit

\[
BA = \cos \frac{\theta}{2} - iL \sin \frac{\theta}{2}, \quad AB = \cos \frac{\theta}{2} + iL \sin \frac{\theta}{2},
\]

d’où

\[
(5) \quad X' = \left( \cos \frac{\theta}{2} - iL \sin \frac{\theta}{2} \right) X \left( \cos \frac{\theta}{2} + iL \sin \frac{\theta}{2} \right).
\]

Si l’on désigne par \( l_1, l_2, l_3 \) les cosinus directeurs de \( L \), les paramètres d’Euler-Olinde-Rodrigues sont les quatre quantités

\[
\rho = \cos \frac{\theta}{2}, \quad \lambda = l_1 \sin \frac{\theta}{2}, \quad \mu = l_2 \sin \frac{\theta}{2}, \quad \nu = l_3 \sin \frac{\theta}{2},
\]

dont la somme des carrés est égal à 1.

Here a matrix \( X \) is associated to a vector \( \vec{x} \) (as defined in no 55) as

\[
X = \begin{pmatrix}
x_3 \\
x_1 + ix_2 \\
x_1 - ix_2 \\
\end{pmatrix} \quad \longleftrightarrow \quad \vec{x} = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{pmatrix} \in \mathbb{R}^3,
\]

and \( \lambda E_2 = \text{diag}(\lambda, \lambda) \) is identified with a scalar \( \lambda \), where \( E_2 \) is the identity matrix of order 2. Take a rotation \( R(\theta I) \) around a unit rotation axis \( l = l(l_1, l_2, l_3) \) and of rotation angle \( \theta \). the matrix \( L = \begin{pmatrix}
l_3 \\
l_1 + il_2 \\
l_1 - il_2 \\
\end{pmatrix} \) is associated to the axis \( l \), and \( X' \) is the image of \( X \) under \( R(\theta I) \). The matrices \( A \) and \( B \) are associated to unit vectors \( \vec{a} \) and \( \vec{b} \) respectively, chosen in such a way that

\[
\frac{1}{2}(AB + BA) = \langle \vec{a}, \vec{b} \rangle E_2 = \cos \frac{\theta}{2} E_2, \quad \frac{1}{2}(AB - BA) = iL \sin \frac{\theta}{2}.
\]

Thus the formula (5) gives correctly Rodrigues expression of the rotation \( R(\theta I) \) given in the paper [R5-1].

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7 One of the main results of the paper [R5-1], 1840, of Benjamin Olinde Rodrigues. The author’s name of this paper is written as Olinde Rodrigues.

8 By definition, of the length 1.

9 It is very much regrettable that Cartan did not give an exact reference to this paper, and he misunderstood the name of its author Olinde Rodrigues as names of two persons called Olinde and Rodrigues respectively (cf. Comment 5.1 below).
The matrix \(X\) is represented by \(\det X = -(x_1^2 + x_2^2 + x_3^2)E_2 = -\|\vec{x}\|^2E_2\), where \(\|\vec{x}\|\) denotes Euclidian norm of \(\vec{x}\). Moreover \(X^2 = \|\vec{x}\|^2E_2\), and \(A^2 = E_2\), \(A^{-1} = A\) for unit vector \(\vec{x}\). Let \(Y\) be associated to \(\vec{y}\), then

\[
\frac{1}{2}(XY + YX) = \langle \vec{x}, \vec{y} \rangle, \quad \frac{1}{2}(XY - YX) = \vec{x} \wedge \vec{y},
\]

where \(\vec{x} \wedge \vec{y}\) denotes the vector product of \(\vec{x}\) and \(\vec{y}\). Two vectors \(\vec{x}\) and \(\vec{y}\) are perpendicular to each other if and only if \(XY = -YX\), and in such a case \(\vec{x} \wedge \vec{y}\) is called \textit{bivecteur} (by Cartan) and is represented by \(-iXY = -i\frac{1}{2}(XY - YX)\).

In \(\text{n}^\circ 55\), a triplet \(H_1, H_2, H_3\) of \(2 \times 2\) matrices is introduced as

\[
H_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and in \(\text{n}^\circ 57\), another triplet \(I_j := -iH_j\) \((j = 1, 2, 3)\) is introduced. Each of them has the following relations respectively

\[
H_j^2 = 1\ (j = 1, 2, 3), \quad H_jH_k = -H_kH_j\ (j \neq k), \quad H_1H_2H_3 = i,
\]

\[
I_j^2 = 1\ (j = 1, 2, 3), \quad I_jI_k = -I_kI_j\ (j \neq k), \quad I_1I_2I_3 = -1.
\]

The matrix \(X\) associated to \(\vec{x}\) is \(X = x_1H_1 + x_2H_2 + x_3H_3\), and \(\{I_1, I_2, I_3\}\) is a triplet satisfying the fundamental formula of \textit{Quaternions} (this is remarked in \(\text{n}^\circ 57\), \textit{Relation avec la théorie des quaternions}). Moreover \(\{1, H_1, H_2, H_3, i, I_1, I_2, I_3\}\) gives a basis over \(R\) of \(M(2, \mathbb{C})\) of full matrix algebra of order 2 over \(\mathbb{C}\).

Now we come to explain the meaning of the above quotation. Consider a unit vector \(\vec{x}\) and the reflection (\textit{symétrie} in [Car2]) \(\text{Ref}(\vec{a})\) with respect to the hyperplain orthogonal to it, which is given as

\[
\vec{x}' = \vec{x} - 2\langle \vec{x}, \vec{a}\rangle.
\]

Translating this into the matrix form, we have

(4.1) \[
\text{Ref}(\vec{a})X = X' = -AXA
\]

In fact, \(X' = X - 2A\frac{1}{2}(XA + AX) = X - AXA - A^2X = -AXA\quad (\because A^2 = 1)\).

Take another unit vector \(\vec{b}\), then \(\text{Ref}(\vec{b})\text{Ref}(\vec{a})X = BAXAB\). Also take a vector \(\vec{y}\) perpendicular to \(\vec{x}\) and take \textit{bivecteur} \(\vec{a} = \vec{x} \wedge \vec{y}\) which is represented by \(U := -iXY\).

Then, under \(\text{Ref}(\vec{a})\), \(XY\) is transformed to \(X'Y' = AXYA = AXYA^{-1}\), and so \(\text{Ref}(\vec{b})\text{Ref}(\vec{x})U = (BA)U(AB), \ AB = (BA)^{-1}\).

Thus stated, we should come back to the fundamental principle of Cartan’s idea for the proof of so-called \textit{les paramètres d’Euler-Olinde-Rodrigues} above. In \(\text{n}^\circ 10\), \textit{Décomposition d’une rotation en un produit de symétries} of [Car2], it is proved that, in the space of dimension \(n\) over \(R\) or \(\mathbb{C}\),

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\(^{10}\text{The formula below is exactly ‘La formule (3)’ at the top of the above quotation from [Car2], p.57.}\)
Toute rotation est le produit d’un nomble pair \( \leq n \) de symétries.

Hence, in the case of \( n = 3 \) over \( \mathbb{R} \), every rotation is a product of two reflections (symétries), that is, for any non-trivial rotation \( R \), there exists two unit vectors \( \vec{a} \) and \( \vec{b} \), with the angle from \( \vec{a} \) to \( \vec{b} \) smaller than \( \pi \), such that \( R = \text{Ref}(\vec{b})\text{Ref}(\vec{a}) \).

The rotation axis \( \vec{l} \) (= \( l \) in our notation) of \( R \) is a positive multiple of \( \vec{a} \wedge \vec{b} \) and let the angle from \( \vec{a} \) to \( \vec{b} \) be \( \frac{\theta}{2} \). Then \( \langle \vec{a}, \vec{b} \rangle = \cos \frac{\theta}{2} \) or \( \frac{1}{2}(AB + BA) = \cos \frac{\theta}{2} \), and \( \frac{1}{2}(AB - BA) = i\vec{a} \wedge \vec{b} = \sin \frac{\theta}{2} \cdot iL \).

If we check the movement of \( \text{Ref}(\vec{b})\text{Ref}(\vec{a}) \) on the 2-dimensional plane spanned by \( \vec{d} \) and \( \vec{b} \), then it is exactly the movement of \( R(\theta l) \), as we can see easily.

Thus the assertion in the quotation above is newly proved by Cartan.

## 5 Comments and Literatures

**Comment 5.1.** For the middle name Olinde of Rodrigues, I have checked the lists of Christian Saints and the lists of traditional French boys’ names, downloaded from website. Curiously enough, I couldn’t find Olinde in these lists. I understood that this name is added by his father under the order of Christian Church around 1808, but in some literature it is explained that this name Olinde, along with the second names of his brother and sisters, was taken by his father from literary works and the like. Anyhow he signed to his mathematical papers as Olinde Rodrigues, not using Benjamin. The reason why, I cannot imagine, but this seems to work against him. For instance, Élie Cartan misunderstood it as two person’s names, and in his book [Car2] used the terminology as les formules (and les paramètres) d’Euler-Olinde-Rodrigues. My friend Prof. emeritus Michel Duflo helped me very much to search Rodrigues’ papers, difficult to take copies. He wrote me that he didn’t know there exists French name Olinde in that time.

**Comment 5.2.** About confusions on the first names of mathematicians, also there is the case of J. Schur (1875–1941) and I. Schur (Issai Schur). About 90 years later of the case of Benjamin Olind Rodrigues, it was the times when hidden (?) Jewish misanthropy turned into persecution by Nazi who held power. I cannot but shed tears in the latest years when Schur was becoming very unhappy because of the persecution (Cf. [Hir6]).

**Comment 5.3.** Some years ago, at about 2008, when I was checking systematically website files under the questioning title ‘mathematician Benjamin Rodreagues’ or something similar, the considerable date and time were necessary while throwing away useless files to look for possibly valuable documents. In addition, there were many files which put wrong informations. For example, there appeared a file saying ‘the mathematician who wrote only one article during life’ (it means [R5-1]), and after many files passing, ‘the mathematician who wrote only two articles during life’, and so on. Also about his birth place and nationality, there are estimations ‘Spain or Portgal from the spelling of family name’, and someone reported as ‘I found a family documents in a Portuguese ancient document house, so it’s done’ and so on. Finally I found a file reporting a third paper of Rodrigues, and I felt something really curious
and so decided to study seriously the situation, in particular, search how many papers are there of him etc. and wanted to collect copies of all of them.

Once I asked Prof. M. Duflo to find out the above 3rd paper and so on in Correspondence sur l'École Impériale Polytechnique, 3. Then, together with copies of all papers of his in J. Math. Pures Appl., I could make a report on Rodrigues and sent him my draft. After that I again asked him to find another document [R4*] in Bulletin Scientifique de la Société Philomatique de Paris, as shown below in a part-copy of his e-mail:

Cher Takeshi,

Merci beaucoup pour ton intéressant et amusant texte sur Olinde. Je suis content d'avoir pu t’aider à obtenir de la documentation.

> Je te demandrai cette fois-ci encore de trouver l’article
> \bibitem[R04]{Rodr04} Olinde Rodrigues,
> Sur quelques propri\'et\'es des int\'egrales doubles et des rayons
> de courbure des surfaces,
> Bulletin Scientifique de la Soci\'et\'e Philomatique de Paris,
> pp.34-36, 1815. [Signed \lq P.' by Poisson.]

Cela a été difficile ! Un excellent exercice d’internet !

Je ne l’ai pas trouvé dans les bibliothèques parisiennes, y compris la bibliothèque nationale, jussieu, ihp, ens... J’ai peut-être mal cherché, je l’ai découvert après, il s’appelle souvent Rodrigue (sans s) ce qui rend les recherches difficiles.

Je ne l’ai pas trouvé sur le site web de la société philomatique de Paris. En fait c’est amusant de consulter ce site web; je ne connaissais pas l’existence de cette société; ·······

Puis j’ai pensé a books.google.com. C’est genial : j’ai mis "philomatique paris 1815" dans la boîte de recherches, et j’ai pu lire le livre — et donc tu peux aussi le faire. J’ignore dans quelle bibliothèque Google a scanné ce volume, probablement une bibliothèque d’une université americaine. ·······

Comment 5.4. In this occasion (at the beginning of 2012), when I tried again to look around website putting ‘mathematician Benjamin Olinde Rodrigues’ in the box of research, I was quite surprised that the situation has been completely changed from that of several years ago so that many files containing wrong data disappeared and the files, which should be appeared even then, such as [AlOr] etc. appear ranked high enough.

5.5. Literatures.
Quoted from review [Dav] on the book [AlOr], AMS-LMS, 2005.

Rodrigues produced only 17 mathematical papers but wrote extensively about social, economic, and political matters, on banking and on alleviating problems of labor. From 1816 to 1837, Rodrigues produced no mathematical papers. Between 1838 and 1845, he wrote eight, including one on transformation groups that some consider his chef-d’oeuvre. Taken at face value, this is a remarkable achievement.
How many of us could get back into mathematical shape after doing something else (writing reviews or becoming a provost, say) for two decades? Perhaps Rodrigues was theorematizing all along but didn’t have the time to write up his findings properly. He left no personal papers, so we can’t tell. We can safely conjecture, though, that he kept abreast of the contents of the mathematical journals of the day.

Quoted from [URL1] on Rodrigues.

······

Rather in 1807 Jews living in France were required to modify their family names and in the following year they were required to add a name of French origin. At this point Olinde was added to Rodrigues name. ·····

Appendix.

Quoted from [URL1] on Rodrigues.

······

Appendix. 11

Quaternion, 3-dimensional Rotation Group and Introduction to Theory of Representations of Groups

Section 1 is omitted. Section 2, Introduction and §2.1, are omitted. Below begins with §2.2. We add the character A to the top of section numbers and subsection numbers etc.

A2. Represent Complex Numbers and Quaternions by means of $2 \times 2$ Matrices

A2.2. Complex $2 \times 2$ matrices representing quaternions

The total of quaternion number $q = \alpha + \beta i + \gamma j + \delta k$ ($\alpha, \beta, \gamma, \delta \in \mathbb{R}$) gives a non-commutative number field, and we denote it by $H$. Quaternion $q$ with $\alpha = 0$ is called pure quaternion and we denote by $H_{\text{p}}$ their totality. Here $i, j, k$ are imaginary units invented by W.R. Hamilton (1805–’65) satisfying

$$i^2 = j^2 = k^2 = ijk = -1,$$

and, together with 1, they form a basis of $H$ over $\mathbb{R}$. Hamilton discovered on the way of morning walk on the 16th Oct. 1843 that, with three imaginary units $i, j, k$ satisfying the so-called fundamental formula (2.1), in the totality of $q = \alpha + \beta i + \gamma j + \delta k$, four arithmetic operations are possible. His paper [Ham1] appeared in 1843. Besides that, he wrote in the next day a detailed report about this discovery to his friend J.T. Graves as a long letter, written with a quill and ink, which is reprinted in 1844. In the printed form of Collected Works of Hamilton, 4 pages, total 167 lines [Ham2].

Quaternion $H$ is a linear algebra over $\mathbb{R}$ and can be immersed into $M(2, \mathbb{C})$. The immersion $\Psi$ is given as an linear extension of the correspondence, with $i = \sqrt{-1} \in \mathbb{C}$,

$$i \to I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j \to J = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad k \to K = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

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so that for $\alpha, \alpha' \in \mathbb{R}$ and $q, q' \in \mathbb{H}$,

\[
\begin{align*}
\Psi(\alpha q + \alpha' q') &= \alpha \Psi(q) + \alpha' \Psi(q'), \\
\Psi(q q') &= \Psi(q) \Psi(q').
\end{align*}
\]

(2.3)

The conjugate of $q = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$ is defined as $\bar{q} = \alpha - \beta i - \gamma j - \delta k$ and the norm of $q$ by

\[
\|q\| = \sqrt{q \bar{q}} = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)^{1/2}.
\]

Then we have $q^{-1} = \|q\|^{-2} \bar{q}$.

**Problem 2.2.1.** Prove that the system of relations (2.2) is equivalent with the system of relations (2.1).

**Problem 2.2.2.** Prove that the triplet of matrices $\{I, J, K\}$ satisfies the similar relations as the triplet $\{i, j, k\}$.

**Problem 2.2.3.** Prove the formula (2.3). Also prove $\det \Psi(q) = \|q\|^2$, and $qq' = q' \bar{q}$ (the order of the product is inverted).

**Note A2.2.4.** When Hamilton discovered quaternion on the way of morning walk, he was near to a bridge, and at that time he curved the so-called *fundamental formula* on a stone of the bridge. It is the formula (2.1). I read that even today some peoples of Department of Mathematics are used to take a morning walk to the bridge on the same date as the 16th October, the date of Great Discovery.

**A3. Quaternion and rotation group $SO(3)$, Hamilton’s discrepancies**

**A3.1. Expression of a 3-dimensional rotation**

After the discovery, Hamilton was pursuing applications of quaternion. One of the themes was the problem of describing a rotation in 3D Euclidean space $E^3$. As is explained in §A2.1 (omitted), a rotation in 2D Euclidean plane $E^2 \cong \mathbb{C}$ can be expressed in a simple way by multiplication of complex number with modulus 1, and so Hamilton was aiming for something similar to that with respect to quaternion.

Let us take a bijective correspondence between the space $H_-$ of pure quaternions and 3D Euclidean space $E^3$ given as

\[
H_- \ni x = x_1 i + x_2 j + x_3 k \mapsto x = (x_1, x_2, x_3) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in E^3.
\]

(3.1)

Here $x$ is a vertical vector but we express it by a transposed horizontal vector to save space. The length of $x$ is given by

\[
\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2},
\]

and so we put $\|x\| := \|x\|$. We can express the length preserving isomorphism (3.1) by a symbol as $H_- \cong E^3$ (this modern symbol expression is powerful but didn’t exist in the times of Hamilton!).
Recall that a rotation in $E^3$ fixing the origin is expressed by an orthogonal matrix $U = (u_{ij})_{i,j=1}^3 \in SO(3)$ as $x \to x' = Ux$, where, with $E_3$ the unit matrix of order 3,

\begin{equation}
SO(3) = \{ U \in M(3, \mathbb{R}) ; \quad U^tU = UU^t = E_3, \quad \det U = 1 \}.
\end{equation}

On the other hand, denote by $B$ the unit ball of $H$ given as $B := \{ a \in H ; \|a\| = 1 \}$, then it is a group under multiplication. From the beginning, Hamilton estimated that, as in the case of 2D rotation group $SO(2)$ and the torus group $T^1 := \{ z \in C ; |z| = 1 \}$,

The group $B$ should be isomorphic to the group $SO(3)$.

He took as above the space of pure quaternions $H_-$ as Euclidean space $E^3$ and look for ways of action of $B$ on it. As we see, the simplest way of action is the left multiplication as $L(a) : H_- \ni x \mapsto ax \in H_-$ ($a \in B$). Alas! for Hamilton. The image $L(a)x = ax$ belongs to $H_-$ only when $a$ is orthogonal to $x$, that is, $\langle a, x \rangle = \frac{1}{2}(ax + xa) = 0$ (Cf. Problem 3.1 below). So the big and difficult problem for Hamilton was

What kind of action of $a \in B$ on $H_- \cong E^3$ gives an isomorphism from $B$ onto $SO(3)$?

However, judging from the result, the above estimation of Hamilton was a misleading wrong estimate or a wrong button, from the beginning. Before 40 years old, He discovered quaternion and after that he wrote several huge books e.g. [Ham3] and [Ham4], and tried to spread the theory of quaternion in the world, but it seems that the wrong buttons were stuck for his whole later life. In the essay [Alt2], Altmann wrote this situation in detail using some emotional terms such as The sad truth or entirely unacceptable or Optical illusion or causing endless damage etc. Still more expressed as

\begin{itemize}
  \item \ldots, and that Hamilton committed a serious error of judgement in basing his parametrization on the special case of the rectangular transformation.
\end{itemize}

(this is the transformation appeared in Problem 3.1 below).

**Problem 3.1.** Put $B_- := B \cap H_-$ and express $a \in B$ as

\[ a = \cos \theta + \sin \theta w \quad \text{with} \quad w \in B_-, \quad \theta \in \mathbb{R}. \]

Prove that, in case $\sin \theta \neq 0$, we have $ax \in H_-$ for an $x \in H_- \cong E^3$ if and only if $\langle x, w \rangle = 0$, that is, $x \perp w$ (perpendicular to each other).

Also prove that, in that case, the left multiplication $L(a) = L(\cos \theta + \sin \theta w)$ induces on the hyperplane $w^\perp := \{ x \in H_- ; x \perp w \}$ a rotation of angle $\theta$ around the origin.

Well now, what is the correct expression of the rotation group by means of quaternion? An answer has been given substantially in the paper [R5-1] of Rodrigues in 1840, but it is ignored historically until very recently, except an early comment by É. Cartan in [Car2].

**Lemma A3.2.** For $a \in B$, we have $a^{-1} = \overline{a}$, and the group $B$ acts on $H_- \cong E^3$ through

\begin{equation}
T(a) : \quad H_- \ni x \mapsto x' = axa^{-1} = ax\bar{a} \in H_-,
\end{equation}
that is, \( T(a)T(b) = T(ab) \) \((a, b \in B)\).

**Proof.** Since \( \overline{qq'} = \overline{q} \overline{q'} \), we have \( \overline{x'} = \overline{a} \overline{x} \overline{a} = a(-x)a = -x' \), whence \( x' \in H_- \). Moreover \( T(a)T(b)x = a(bx) \overline{a} = (ab)x(ab) = T(ab)x \).

\( \square \)

**Lemma A3.3.** For a \( w \in B_- = H_- \cap B \), put \( g_w(\theta) = \cos \theta + \sin \theta w \) \((\theta \in \mathbb{R})\). Then \( \theta \mapsto g_w(\theta) \) is a one-parameter subgroup of \( B \), and \( \frac{dg_w(\theta)}{d\theta} \big|_{\theta=0} = w \).

**Proof.** For \( \theta, \theta' \in \mathbb{R} \), we have, from \( w^2 = -1 \),

\[
g_w(\theta)g_w(\theta') = (\cos(\theta)\cos(\theta') - \sin(\theta)\sin(\theta')) + (\sin(\theta)\cos(\theta') + \cos(\theta)\sin(\theta')) w = g_w(\theta + \theta').
\]

\( \square \)

**Problem 3.4.** For a \( w \in B_- \), take \( u, v \in B_- \) in such a way that \( \{u, v, w\} \); \{w\} gives a right-handed orthonormal coordinate system. Then \( uv = w \), \( vw = u \), \( wu = v \), and for \( g_w(\theta) \), we have

\[
\begin{align*}
T(g_w(\theta))u &= \cos(2\theta)u + \sin(2\theta)v, \\
T(g_w(\theta))v &= -\sin(2\theta)u + \cos(2\theta)v, \\
T(g_w(\theta))w &= w.
\end{align*}
\]

The matrix expressin of \( T(g_w(\theta)) \) with respect to the basis \( \{u, v, w\} \) is

\[
\begin{pmatrix}
\cos(2\theta) & -\sin(2\theta) & 0 \\
\sin(2\theta) & \cos(2\theta) & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

**Theorem A3.5.** The group \( B \) is a double covering and universal covering of rotation group \( SO(3) \) and a covering map is given by \( T(a) \) \((a \in B)\).

**Proof.**  1) The map \( T \) is surjective. In fact, it is known that any rotation \( g \in SO(3) \) has a non-zero invariant vector \( w \in E^3 \) and so it is a rotation of of some angle \( \phi \) arround \( w \). Take the vector \( w \in H_- \cong E^3 \) corresponding to it and put \( \theta = \phi/2 \). Then, as seen from (3.5) in Problem 3.4, we have \( T(\cos \theta + \sin \theta w) = g \).

2) The kernel of \( T \) is \( \{ \pm 1 \} \subset B \). In fact, as seen from (3.4), \( T(g_w(\theta)) = E_3 \) (unit matrix) if and only if \( 2\theta \equiv 0 \pmod{2\pi} \), whence \( \theta \equiv 0 \pmod{\pi} \) and so \( g_w(\theta) = \pm 1 \).

3) The unit ball \( B \) is topologically homeomorphic to 3-dimensional sphere \( S^3 \), and is simply connected. \( \square \)

**Problem 3.6.** Prove the following. For \( q \in H \), define exp \( q \) by an absolutely convergent infinite series as

\[
(3.6) \quad \exp q = \sum_{n=0}^{\infty} \frac{q^n}{n!} = 1 + q + \frac{q^2}{2!} + \cdots.
\]
Then, for \( w \in B_− \), \( \exp(\theta w) = \cos \theta + \sin \theta \, w = g_w(\theta) \) \((\theta \in R)\).

The most important point of above discussions is that, under the correspondence
\[
H_− \ni \theta \, w \rightarrow \exp(\theta w) \in B \rightarrow T(\exp(\theta w)) \in SO(3)
\]
with \( \theta \in R, w \in B_− \), the angle of rotation is doubled as \( \theta \rightarrow 2\theta \) as is shown in (3.5). This means that \( a = \exp(\theta w) \) and \(-a = \exp((-\theta + \pi)w)\) have the same image \( T(a) = T(-a)\), and the map \( T : B \rightarrow SO(3) \) is a 2:1 correspondence. Hamilton seems to have been insisting particularly, with the great pioneer’s stubbornness, to obtain 1:1 correspondence. According to some biographies, Hamilton became eventually to suffer from excessive alcohol intake [Bell].

Altmann points out, as one of the reasons, a serious psychological distress in this rotation expression problem. Looking at the cause of his worries, from the present age of mathematical standard, I can suspect that, in this problem there are two different objects such as
1) object which operate on something (operators),
2) object to be operated (operands),
however they both are the same quaternion and might be confused mutually or might not be clearly distinguished.

Dear readers! You may not feel much sympathy to Hamilton’s serious anxiety, when reading this explanation. However it is because firstly you have been taught already under a modern mathematical basic training, and secondly here the author (Hirai) have chosen adequate notation in such a way that
- for the object 1), the characters such as \( a, b, g_w \) etc.,
- for the object 2), the characters such as \( x, y \) etc.
Thus, because of the hints drawn carefully for the reader, your understanding is unconsciously guided in the right direction.

I have already mentioned the misunderstandings that Hamilton had. For his life and also about quaternion, somewhat ironic story telling can be found in website [http://members.fortunecity.com/jonhays/clifhistory.htm](http://members.fortunecity.com/jonhays/clifhistory.htm) and the author of this website MR. jonhays noted that at age 17 he read about Hamilton in Men of Mathematics by Eric Temple Bell (1883–1960)\(^{12}\).

**Problem 3.7 (formula for calculation).** For \( u, v \in H_− \), the corresponding elements in \( E^3 \) are denoted by \( u = t(u_1, u_2, u_3), v = t(v_1, v_2, v_3) \), and their inner product is defined as \( \langle u, v \rangle := u_1v_1 + u_2v_2 + u_3v_3 \) and denoted by \( u \cdot v = \langle u, v \rangle \). Then, prove the following formula:
\[
(3.8) \quad uv = -u \cdot v + u \times v,
\]
where \( u \times v \in H_− \), \( u \times v := \begin{vmatrix} i & = & u_1 & v_1 \\ j & = & u_2 & v_2 \\ k & = & u_3 & v_3 \end{vmatrix} \).

**Problem 3.8.** Prove the following formula:
\[
(3.9) \quad u \times v = -v \times u, \quad (u \times v) \perp u, \quad (u \times v) \perp v.
\]

\(^{12}\)Chapter 19, An Irish Tragedy, pp.340–361.
A3.2. Rodrigues expression of rotation and product formula

A3.2.1. Rodrigues parameter $\theta w \in H_-$ ($\theta \in R, w \in B_-$)

It was proved by Euler that any rotation $\rho$ in 3D Euclidean space $E^3$ around the origin has necessarily a rotation axis. Let $w \in E^3, \|w\| = 1$, be the axis and $\theta$ the angle of $\rho$ around the axis $w$ in right-handed screw rotation. Then, as is shown above, $\rho$ is expressed in (3.7) as

$$
\rho = R(\theta w) := T((\exp(\frac{1}{2}\theta w)) \in SO(3),
$$

where $w \in H_\sim \cong E^3$ corresponds to $w$. We call this expression as Rodrigues expression, and $(\theta, w) \in R \times B_-$ or $\theta w \in H_-$ as Rodrigues parameter of rotation $\rho$.

This expression is, unlike the expression by means of Euler angles (cf. §A3.3 below), the parameters are seamless, and locally univalent but globally multivalent. If one wishes to make completely univalent and put some restriction on $\theta w$, there appears inevitably some breaks in the parameter. So that, as parameter space, it is natural to take the whole space $H_-$ and enjoy the advantage of capability of describing smoothly multi-rotations of machines or airplanes etc.

A3.2.2. Product formula for two rotations

The main contribution of Rodrigues is the description of the product of two rotations $R(\theta w)R(\theta' w') = R(\theta'' w'')$. To describe Rodrigues parameters $\theta'' w''$ from $\theta w$ and $\theta' w'$, we calculate it according to the quaternion product rule in $H$ as

$$
(\cos(\frac{1}{2}\theta) + \sin(\frac{1}{2}\theta)w)(\cos(\frac{1}{2}\theta') + \sin(\frac{1}{2}\theta')w') = \cos(\frac{1}{2}\theta'') + \sin(\frac{1}{2}\theta'')w''.
$$

It gives us the so-called Rodrigues formula in [R5-1] in our notations as

$$
\begin{align*}
\cos(\frac{1}{2}\theta'') &= \cos(\frac{1}{2}\theta) \cos(\frac{1}{2}\theta') - \sin(\frac{1}{2}\theta) \sin(\frac{1}{2}\theta') w \cdot w', \\
\sin(\frac{1}{2}\theta'')w'' &= \cos(\frac{1}{2}\theta) \sin(\frac{1}{2}\theta') w' + \sin(\frac{1}{2}\theta) \cos(\frac{1}{2}\theta') w +
+ \sin(\frac{1}{2}\theta) \sin(\frac{1}{2}\theta') w \times w'.
\end{align*}
$$

Originally he induced his product formula (equivalent to the above) from some formulas for spherical trigonometric functions, and thus we may say that Rodrigues substantially gave the product rule for quaternion, in advance of Hamilton.

**Note A3.9.** When $\theta, \theta'$ are both small, we can obtain the first approximation from the second equation above as $\theta'' w'' \doteq \theta' w' + \theta w$. Furthermore if $w, w'$ are near to each other, then the first approximation is

$$
\theta'' \doteq \theta + \theta', \quad w'' \doteq \frac{1}{2}(w + w').
$$

Note that, in the case of Euler angle expression, there does not exist such an approximation.

**Note A3.10.** Basic formulas for spherical trigonometry are given as follows. For a spherical triangle $ABC$, let the interior angles be $\alpha, \beta, \gamma$, the opposite side lengths be $a, b, c$, the area be $S$, and the radius of the sphere be $\rho$. Then a formula Spherical Excess is

$$
\alpha + \beta + \gamma - \pi = S/\rho^2 > 0,
$$

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we encounter immediately some difficulties, for instances: Euler angle components again.

First calculate the product of 6 matrices of Euler angles, then decompose the product into 3 radians. For example, the length $\rho$ in (3.14)–(3.17), the radius $\rho$ comes from the 3rd formula in (3.16). In fact, the computational load is heavy. In fact, to calculate the product of two rotations, we apply Euler angle expression to calculations such as product of two rotations, $g(x_1e_1+x_2e_2+x_3e_3)\leftrightarrow x = t(x_1, x_2, x_3)$. Denote by $g_1(\theta)$ a rotation around $e_1$ at right-handed screw angle $\theta$, and similarly $g_2(\theta), g_3(\theta)$ for $e_2, e_3$ respectively, then

$$
\begin{align*}
g_1(\theta) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix},
g_2(\theta) &= \begin{pmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{pmatrix},
g_3(\theta) &= \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\end{align*}
$$

A rotation $\rho$ of $E^3$ or $\rho \in SO(3)$ can be expressed as

$$
\rho = g_3(\varphi)g_2(\theta)g_3(\psi) = \begin{pmatrix} \cos \varphi - \sin \varphi & 0 & \sin \varphi \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \psi & 0 & \sin \psi \\ 0 & 1 & 0 \\ -\sin \psi & 0 & \cos \psi \end{pmatrix} = \begin{pmatrix} \cos \varphi \cos \psi & \sin \varphi \cos \psi & \sin \varphi \sin \psi \\ \sin \varphi \cos \psi & \cos \varphi \cos \psi & \cos \varphi \sin \psi \\ -\sin \varphi \sin \psi & \cos \varphi \sin \psi & \cos \varphi \cos \psi \end{pmatrix}.
$$

When we apply Euler angle expression to calculations such as product of two rotations, we encounter immediately some difficulties, for instances:

1) The computational load is heavy. In fact, to calculate the product of two rotations, first calculate the product of 6 matrices of Euler angles, then decompose the product into 3 Euler angle components again.
2) It is not possible to evaluate calculation errors. Even if the change of rotations are small enough, the deviations between their Euler angles can be very big, sometimes there would be jumps.

For more details, you can read Altmann’s text book [Alt1].

A4. Applications of Rodrigues expression, Time derivative

Recently, in many directions, Rodrigues expression of rotation is applied adequately. Its parameter is given by $	heta w \in \mathbb{H}_-(\theta \in \mathbb{R}, w \in \mathbb{B}_-)$. and the rotation $R(\theta w) = T(\exp(\frac{1}{2}\theta w))$ acts on $x \in \mathbb{H}_- \cong E^3$ as

$$H_- \ni x \rightarrow \exp(\frac{1}{2}\theta w) x \exp(\frac{1}{2}\theta w)^{-1} = \left( \cos\left(\frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta\right)w \right) x \left( \cos\left(\frac{1}{2}\theta\right) - \sin\left(\frac{1}{2}\theta\right)w \right) \in \mathbb{H}_- \cong E^3,$$

A4.1. Examples of application in various fields

In geophysics, it is important to describe rotations for problems such as:

- In plate tectonics, describe plates movement according to geological time, by means of a rotation which leaves the earth center invariant.
- In geodesy, describe the relationship of the inertial coordinate system of the universe, which is the basis of Newton’s equation of motion, with the Earth coordinate system. This is used for satellite orbit calculation.
- In seismology, it is necessary to quantify the two rotations “difference” to quantify how much the fault plane deviates from the reference plane.

For more purposes such as

- Computer graphics,
- Aircraft design, spacecraft attitude control, dynamics such as aviation.

For more detailed comments, see e.g. [Agn], 2006.

A4.2. Time derivative of rotation in Rodrigues expression

When we study rotations in a dynamical system, such as in aeronautical engineering, the rotation that depends on the time $t$ is treated. We give here an interesting compact formula for time derivative of rotation, which can be easily applied in many purposes.

In case $\theta$ and $w$ depend on the time $t$, put $Q(t) := R(\theta w)$, and let us study its time derivative $\dot{Q}(t) = \frac{dQ(t)}{dt}$.

Theorem A4.1. For $w \in \mathbb{B}_- (\subset \mathbb{H}_-)$ depending on $t$, we have $\dot{w}w = -\dot{w}w$ and so $\dot{w} \perp w$ in $\mathbb{H}_-$ and $\dot{w}w = \dot{w} \times w$, $\perp \dot{w}$, $\perp w$ (mutually orthogonal). The time derivative $\dot{Q}(t)$ of rotation $Q(t)$ is given as

$$\dot{Q}(t) x = Q(t) \left( [\dot{\theta} w + \sin \theta \dot{w} + (1 - \cos \theta) \dot{w} w] \times x \right),$$
or \[ Q(t)^{-1} \dot{Q}(t) x = \left[ \dot{\theta} w + \sin \theta \, \dot{w} + (1 - \cos \theta) \, \dot{w} \cdot w \right] \times x , \]

where \( x \in H_1 (\cong E^3) \) is a fixed vector, and for \( a, b \in H_1 \), \( a \times b \) denotes the vector product in the 3-dimensional vector space \( H_1 \).

**Proof.** Differentiate the both side of \( w^2 = w \times w = -1 \) with respect to \( t \), then

\[
\dot{w} \cdot w + \dot{w} \cdot w = 0 \quad \therefore \quad \dot{w} \left( \frac{1}{2} \right) \cdot w \quad \text{(in } H_1) \quad \therefore \quad \dot{w} \cdot w = \dot{w} \times w .
\]

\[
\frac{d}{dt} \exp \left( \frac{1}{2} \theta w \right) = \frac{d}{dt} \left( \cos \left( \frac{1}{2} \theta \right) \right) + \sin \left( \frac{1}{2} \theta \right) w =
\]

\[
= \left( -\sin \left( \frac{1}{2} \theta \right) \right) \cos \left( \frac{1}{2} \theta \right) w + \sin \left( \frac{1}{2} \theta \right) \dot{w} = \exp \left( \frac{1}{2} \theta w \right) \frac{1}{2} \dot{\theta} w + \sin \left( \frac{1}{2} \theta \right) \dot{w} .
\]

Therefore

\[
\frac{d}{dt} Q(t)x = \frac{d}{dt} \left\{ \exp \left( \frac{1}{2} \theta w \right) x \exp \left( -\frac{1}{2} \theta w \right) \right\} =
\]

\[
= \frac{d}{dt} \left( \exp \left( \frac{1}{2} \theta w \right) \right) x \exp \left( -\frac{1}{2} \theta w \right) + \exp \left( \frac{1}{2} \theta w \right) \left\{ \left( \cos \left( \frac{1}{2} \theta \right) - \sin \left( \frac{1}{2} \theta \right) \right) \sin \left( \frac{1}{2} \theta \right) \dot{w} \cdot x +
\]

\[
+ x \left( -\sin \left( \frac{1}{2} \theta \right) \right) + \cos \left( \frac{1}{2} \theta \right) \sin \left( \frac{1}{2} \theta \right) \right) \exp \left( -\frac{1}{2} \theta w \right)\}
\]

\[
= \exp \left( \frac{1}{2} \theta w \right) \left\{ \dot{\theta} w \times x + \cos \left( \frac{1}{2} \theta \right) \sin \left( \frac{1}{2} \theta \right) \left( \dot{w} \cdot x - x \cdot \dot{w} \right) - \sin \left( \frac{1}{2} \theta \right) \left( \dot{w} \cdot x + x \cdot \dot{w} \right) \right\} \exp \left( -\frac{1}{2} \theta w \right) \quad \text{(use (1.31))}
\]

\[
= Q(t) \left\{ \left[ \dot{\theta} w + \sin \theta \, \dot{w} + (1 - \cos \theta) \, \dot{w} \right] \times x \right\} .
\]

(Proof completed)

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