ABSTRACT

We compute flux-dependent corrections in the four-dimensional F-theory effective action using the M-theory dual description. In M-theory the 7-brane fluxes are encoded by four-form flux and modify the background geometry and Kaluza-Klein reduction ansatz. In particular, the flux sources a warp factor which also depends on the torus directions of the compactification fourfold. This dependence is crucial in the derivation of the four-dimensional action, although the torus fiber is auxiliary in F-theory. In M-theory the 7-branes are described by an infinite array of Taub-NUT spaces. We use the explicit metric on this geometry to derive the locally corrected warp factor and M-theory three-form as closed expressions. We focus on contributions to the 7-brane gauge coupling function from this M-theory back-reaction and show that terms quadratic in the internal seven-brane flux are induced. The real part of the gauge coupling function is modified by the M-theory warp factor while the imaginary part is corrected due to a modified M-theory three-form potential. The obtained contributions match the known weak string coupling result, but also yield additional terms suppressed at weak coupling. This shows that the completion of the M-theory reduction opens the way to compute various corrections in a genuine F-theory setting away from the weak string coupling limit.
1 Introduction

In Type IIB string theories non-Abelian gauge theories with gauge groups including ADE can arise on stacks of 7-branes. These eight-dimensional branes are special, since they back-react on the geometry also far away from the brane with a non-trivial deficit angle. F-theory provides a powerful non-perturbative description of 7-brane configurations including their back-reaction on the geometry and the non-trivial dilaton-axion profile. In an F-theory background the varying dilaton-axion is interpreted as the complex structure modulus of an auxiliary two-torus varying over the ten-dimensional space-time. This is the elliptic fibration of F-theory with 7-branes located at degeneration points of the torus fiber. In four-dimensional compactifications on a complex threefold $B_3$ the inclusion of a holomorphically varying dilaton-axion requires to consider an elliptically fibered Calabi-Yau fourfold $Y_4$ with a section. In this geometry the 7-branes wrap four-dimensional cycles in $B_3$ arising as the discriminant locus of the elliptic fibration of $Y_4$. The Calabi-Yau condition on $Y_4$ relates the class of the discriminant to the curvature of $B_3$. This ensures that F-theory on $Y_4$ is canceling 7-brane tadpoles and takes the 7-brane back-reaction into account.

To study the lower-dimensional effective actions of F-theory compactifications it is necessary to use the formulation of F-theory as an M-theory compactification in a special limit. This different point of view is inevitable due to the fact that F-theory has neither a low-energy effective action in twelve dimensions nor a fundamental microscopic formulation due to the strong coupling regions in a generic background. However, one argues that M-theory compactified on $Y_4$ to three dimensions is considered as four-dimensional F-theory on the same $Y_4$ compactified on an additional circle. This can be understood by applying the equivalence of M-theory on a two-torus via T-duality to Type IIB on a circle. In order to study any coupling of the four-dimensional effective action of F-theory it will be crucial to carefully follow it through this duality. Having a given question for F-theory in mind our strategy will thus be to first formulate this question in the dual M-theory and then perform the duality to F-theory.

The inclusion of background fluxes in F-theory compactifications is of crucial interest, for example, in moduli stabilization, or more recently in the generation of a chiral spectrum. Moreover, they are often essential to cancel D3-brane tadpoles. However, the understanding of fluxes would be incomplete when ignoring their back-reaction. In the dual M-theory picture the F-theory fluxes correspond to four-form flux $G_4$ and generalizations thereof. It is known that such fluxes source a non-trivial warp factor on the M-theory background. Thus also in F-theory the effect of warping has to be included in general. In addition to yielding the well-studied effect of the Type IIB warp factor from an M-theory perspective, it is in general unclear what the dependence of the warp factor $e^{3A/2}$ on the non-physical directions of the auxiliary two-torus of $Y_4$ should map to in F-theory. Recall that in F-theory the torus directions are not part of the physical spacetime but keep track of the dilaton-axion. Similarly, also a back-reaction of the flux $G_4$ altering the Kaluza-Klein reduction ansatz for the M-theory three-form
$C_3$ by including a non-closed three-form $\beta$ that is the Chern-Simons form of the flux $G_4 = d\beta$ has to be interpreted carefully in the F-theory dual. In this work, we follow both the dependence of the warp factor $e^{3A/2}$ on the unphysical torus direction and the non-closed three-form $\beta$ through the duality to F-theory and find that they map to corrections to 7-brane gauge coupling function for special choices of flux $G_4 = \omega_i \wedge F^i$. In the F-theory language $\omega_i$ are the $(1,1)$-forms from blow-ups of singularities of the elliptic fibration of $Y_4$ over a divisor $S_b$ in $B_3$ wrapped by a stack of 7-branes and $F^i$ denote 7-brane fluxes on $S_b$. The corresponding corrections of the 7-brane gauge coupling function are shown to precisely reproduce the known flux corrections to the D7-brane gauge-coupling in the weak coupling limit of F-theory [19]. These arise at weak coupling from the eight-dimensional couplings of the form $\text{Tr}(F^4)$ on the 7-branes when brane fluxes are included. However, the M-theory to F-theory lift provides a formalism to compute corrections valid even away from the weak coupling limit of F-theory. In our example we match for small $g_s$ the weak coupling contributions but also find further corrections suppressed by at least one power of $g_s$.

A deeper understanding of the corrections to the 7-brane gauge coupling function is of crucial importance both from a conceptual as well as phenomenological point of view. Since in four-dimensional $\mathcal{N} = 1$ theories the gauge coupling function is holomorphic in the chiral multiplets one expects that it is one of the $\mathcal{N} = 1$ data which should be computable in a controlled way. In the effective four-dimensional theory the leading gauge coupling of the 7-brane gauge theory is simply given by the volume of the cycle wrapped by the brane. Direct computations of the 7-brane flux corrections to the gauge couplings are possible in certain 7-brane configurations in F-theory on K3 with constant string coupling as demonstrated in [20, 21]. It was also shown these corrections can be related to purely geometric data of the K3 compactification space. There is an immediate phenomenological relevance of these corrections when building, for example, Grand Unified models in F-theory [6, 7, 22, 23, 24, 25, 26, 27, 28]. In fact, the flux-induced corrections to the gauge coupling functions can crucially alter their running spoiling unification.

While the geometric framework presented in this work is applicable more generally, our main focus will be on the computation of the corrections to the gauge coupling function in a local geometry. In this geometry we can use the local Calabi-Yau metric and show that the backreaction of the flux $G_4$ can be evaluated explicitly. More specifically, we consider a local model $Y_4$ that appropriately describes $Y_4$ in the vicinity of a stack of $k$ 7-branes on a complex surface $S_b$ in $B_3$. We construct this geometry explicitly by following 7-branes through the M-theory/F-theory duality, where the 7-branes become part of the M-theory geometry. As the first step to identify the corresponding M-theory geometry, we consider M-theory compactified on a Taub-NUT space with $k$ centers $TN_k$. A Taub-NUT space is a circle (Hopf) fibration over flat $\mathbb{R}^3$ apart from the $k$ centers where the circle shrinks to zero size. Its metric takes the form of the Gibbons-Hawking ansatz that is specified by $k$ functions $V_i$ that are formally the potentials of point charges in three dimensions [29]. For small asymptotic circle radii this setup yields Type IIA string theory with 6-branes located at the $k$ centers in $\mathbb{R}^3$ and filling the remaining seven dimensions. The asymptotic
Taub-NUT radius is identified with the radius of the A-circle \( r_A \), such that the Type IIA limit is \( r_A \to 0 \). One of the \( \mathbb{R}^3 \) directions we like to place on a further circle, called the B-circle, such that a T-duality along this circle yields a stack of \( k \) Type IIB 7-branes. Instead of putting the 6-branes on a circle, we can equally treat this as an image charge problem of infinitely many 6-branes along a line with a fixed spacing that defines the circle circumference \( r_B \). The advantage of this prescription is that one immediately infers the corresponding M-theory setup by extending the multi-Taub-NUT geometry \( TN_k \) to infinitely many centers \( TN_k^\infty \). Following this logic we argue that the M-theory metric still takes the Gibbons-Hawking form where \( V_I \) are the potentials of an infinite array of periodically repeating point charges along a line in \( \mathbb{R}^3 \). The metric is obtained by a Poisson resummation and involves defining functions already found in [30, 31] in another context. The resulting geometry describes a torus fibration over the normal space \( \mathbb{R}^2 \) to the line of monopoles with torus fiber pinching at the \( k \) centers in \( \mathbb{R}^2 \). In particular we read off the profile of the dilaton-axion from the metric data of \( TN_k^\infty \). Upon compactifying on the complex surface \( S_b \) we get a three-dimensional theory from M-theory on \( Y_4 = TN_k^\infty \times S_b \). We apply T-duality to dualize the 6-branes into 7-branes in F-theory compactified on a circle to three dimensions.

Having defined the M-theory or dual F-theory geometry \( Y_4 \), we can specify the M-theory flux \( G_4 \) explicitly. We switch on G-flux supported on the explicitly constructible normalizable two-forms of the Taub-NUT space \( TN_k^\infty \). This flux will descend to a 6-brane flux \( F^I \) with non-trivial instanton number on \( S_b \) that is mapped to a worldvolume flux on the dual 7-brane. We calculate explicitly the corrections of this flux to the effective three-dimensional 6-brane gauge coupling. In the M-theory picture this is induced by the back-reaction of the flux \( G_4 \) on the warp factor \( e^{3A/2} \) near the centers of the Taub-NUT spaces and on the Kaluza-Klein ansatz for the three-form \( C_3 \). In particular we are able to solve the equation for the warp factor using some core features of the metric on \( TN_k^\infty \). The corrections are sharply localized at the positions of the D6-branes in limit \( g_s \to 0 \).

At weak Type IIB string coupling we show explicitly that these new terms encode corrections to the four-dimensional D7-brane gauge coupling function which are linear in the dilaton-axion \( \tau \) and quadratic in the worldvolume flux \( F^2 \).

The paper is organized as follows. We start with a review of the gauge coupling on a stack of D7-branes in Type IIB in section 2. As a crucial ingredient we present the \( \mathcal{N} = 1 \) effective action in terms of linear multiplets in section 2.1 that we compactify on a circle to three dimensions in section 2.2. This is necessary to compare to three-dimensional M-theory obtained from the warped compactification on a fourfold with G-flux as reviewed in section 3.1. We embed \( k \) D6-branes as multi-center Taub-NUT into M-theory in section 3.2 and compactify on \( S^1 \) by constructing an infinite periodic array of multi-center Taub-NUT \( TN_k^\infty \) in section 3.3. In section 4 we turn to the calculation of the 7-brane gauge coupling. We start with a light review of F-theory as M-theory in section 4.1. Then we determine the leading gauge coupling on a general compact fourfold in section 4.2. We extend the M-theory reduction to include a back-reaction of the G-flux on the three-form reduction ansatz in section 4.3. With these preparations we derive the full flux-corrected gauge
coupling in section 4.4. We first obtain the real part from the back-reaction of the G-flux on the warp factor that we determine as a closed expression on $T N^\infty_k$ and then for the imaginary part by taking into account the altered reduction ansatz for the three-form that we also explicitly determine. We present our conclusions in section 5 and provide additional details, in particular on the linear multiplet formalism and the construction of $T N^\infty_k$, in four appendices A to D.

2 Motivation: D7-brane gauge coupling function

In this section we discuss the aspects of the four-dimensional effective theory on a stack of D7-branes in a weakly coupled orientifold compactification. We first recall in section 2.1 the expression of the D7-brane gauge coupling function as determined by a reduction of the D7-brane action. In section 2.2 we perform the reduction to three dimensions. This three-dimensional result will be later useful to compare to the F-theory gauge coupling function derived via M-theory.

2.1 D7-brane gauge couplings in 4d: Calabi-Yau orientifolds

We begin by recalling the basics from the computation of the four-dimensional D7-brane effective action in Type IIB $\mathcal{N} = 1$ compactifications on a Calabi-Yau orientifold $B_3 = Z_3/\sigma$ with O7-planes \cite{32,19,34}. Here $Z_3$ denotes the Calabi-Yau threefold covering space of the orientifold $B_3$ that is obtained by modding out an holomorphic involution $\sigma : Z_3 \to Z_3$. For appropriately chosen involution the fix point locus of $\sigma$ is a holomorphic divisor $D_{O7}$ that supports the O7-orientifold planes. The divisor $D_{O7}$ has to be homologous to $-8$ times the divisors $S$ in $Z_3$ wrapped by the D7-branes due to tadpole cancellation.

String theory on the orientifold is specified by the orientifold action $\mathcal{O} = \Omega(-1)^F_L \sigma$ acting on the fields, with $\Omega$ being the worldsheet parity operator, and $F_L$ the left-moving fermion number. The spectrum of orientifold invariant states, i.e. states transforming with an eigenvalue 1, determines the physical spectrum. To obtain the four-dimensional effective theory, all massless fields both of the bulk and the D7-brane have to be expanded in zero-modes that are counted by appropriate cohomology groups. Then these expansions are inserted into the ten-dimensional Type IIB supergravity and eight-dimensional D7-brane effective action, that are dimensionally reduced to four dimensions by integration over the internal directions and keeping only the orientifold invariant terms, to obtain the $\mathcal{N} = 1$ effective four-dimensional action.

Let us outline the gauge sector of this effective theory focusing on a stack of $k$ D7-branes with an eight-dimensional $U(k)$ gauge theory on $\mathbb{R}^{(3,1)} \times S_b$. If the divisor $S_b$ has a non-trivial topology, one can consider flux configurations $\mathcal{F}$ for the field strength $F_{D7}$ on the D7-brane. More precisely,

\footnote{See \cite{12,34,35} for a similar derivation of the dual D5- respectively D6-brane effective action.}
we split in the Kaluza-Klein ansatz the D7-brane field strength as

\[ F_{D7} = F + \mathcal{F} = (F^0 + \mathcal{F}^0)1 + (F^i + \mathcal{F}^i)T_i + (F^A + \mathcal{F}^A)\tilde{T}_A, \quad (2.1) \]

where \( F = dA + A \wedge A \) is the \( U(k) \) field strength in the four-dimensional effective theory, and \( \mathcal{F} \) is a background two-form flux on the D7-brane divisor \( S_b \). The field strength \( F_{D7} \) is a general element in the adjoint of \( U(k) \), that we have expanded in the generators \( \tilde{T}_A = (1, T_i, \tilde{T}_A) \) of the adjoint. Here \( T_I = (T_i, 1), i = 1, \ldots, k - 1, \) are the \( k \) Cartan generators of \( U(k) = U(1) \times SU(k) \), while \( \tilde{T}_A \) denote the generators associated to the roots of \( SU(k) \).

In the absence of flux \( \mathcal{F} \) it can be shown by a straightforward reduction of the D7-brane worldvolume action using the expansion (2.1) that the kinetic term for \( F \) takes the form

\[ S_{F_{D7}}^{(4)} = -2\pi \int_{M_4} \frac{1}{2} \text{Re} f_{AB} F^A \wedge \ast F^B + \frac{1}{2} \text{Im} f_{AB} F^A \wedge F^B \quad (2.2) \]

in the conventions \((A.3)\) of appendix A. Here \( f_{AB} \) denotes the gauge coupling function that is a holomorphic function of the chiral fields in the \( \mathcal{N} = 1 \) effective theory, and has adjoint indices \( A, B \).

The adjoint indices arise as we will show soon from the two traces

\[ C_{AB} = \text{Tr}(\tilde{T}_A \tilde{T}_B), \quad \tilde{C}_{\sigma(AB)} = \frac{1}{2} s\text{Tr}(\tilde{T}_A \tilde{T}_B \tilde{T}_C \tilde{T}_D) \int_{S_b} F^C \wedge F^D, \quad (2.3) \]

where \( F^C \) are the fluxes localized on the internal part \( S_b \) of the D7-brane and \( s\text{Tr}(,\cdots) \) denotes the symmetrized trace defined as the sum over all permutations \( \sigma \),

\[ s\text{Tr}(\tilde{T}_A \tilde{T}_B \tilde{T}_C \tilde{T}_D) = \frac{1}{4!} \sum_{\sigma} \text{Tr}(\tilde{T}_{\sigma(A)} \tilde{T}_{\sigma(B)} \tilde{T}_{\sigma(C)} \tilde{T}_{\sigma(D)}) \quad (2.4) \]

In the case at hand the chiral superfields are given by the the axiodilaton \( \tau = C_0 + i e^{-\phi} \), the combination \( G^a = \int_{\Sigma_a} C_2 - \tau B_2 \) and the Kähler moduli \([32]\)

\[ T_\alpha = \int_{D_\alpha} \frac{1}{2} (J \wedge J - e^{-\phi} B_2 \wedge B_2) + i(C_4 - C_2 \wedge B_2 + \frac{1}{2} C_6 B_2 \wedge B_2), \quad (2.5) \]

where \( \Sigma_a, D_\alpha \) denote a homology basis of odd curves respectively even divisors in \( Z_3 \) w.r.t. the involution \( \sigma \). The Kähler form on \( Z_3 \) is given by \( J \), while \( C_p \) denote the R-R \( p \)-forms, and \( B_2 \) is the NS-NS B-field. For simplicity we have frozen out the position and Wilson line moduli of the D7-brane and refer to \([19]\) for the open string corrected chiral coordinates.

In order to proceed we will need to recall some additional facts about the D7-brane theories following \([19, 36, 37]\). In particular, one finds that in the weak coupling description the gauge group is actually \( U(k) = SU(k) \times U(1) \). However, if the D7-brane and its orientifold image are not in the same cohomology class on \( Z_3 \), one finds that a geometric Stückelberg term is induced which renders the overall \( U(1) \) massive. More precisely, the moduli \( G^a \) are gauged due to the geometric Stückelberg coupling, and \( \text{Re} G^a \) is eaten by the overall \( U(1) \) which thus becomes massive. The mass of the
massive vector multiplet containing the \( \text{U}(1) \) and \( \text{Im}G^a \) is of the order of the Kaluza-Klein scale. In the following we will make the simplifying assumption, that for each stack of D7-branes there is exactly one \( G^a \) which becomes massive together with the overall \( \text{U}(1) \). While a detailed derivation of the effective action would require to actually integrate out this massive vector multiplets, we will in the following mostly drop it in our consideration. In other words, we will consider an \( \text{SU}(k) \) gauge theory and no \( G^a \) moduli.

Given these preliminaries we are now in the position to display the gauge coupling function \( f_{AB} \) for a stack of D7-branes. This generalizes the results given for a single D7-brane \cite{19}. Using the traces (2.3) one finds \footnote{We have set \( 2\pi\alpha' = 1 \) in the following.}

\[
f_{AB} = \frac{1}{4} (\delta_S^a T_a C_{AB} - i\tau \tilde{C}_{AB}) \equiv f^c(T) C_{AB} + f^\text{flux}_{AB} (\tau) .
\]

where \( \delta_S^a \) are the coefficients in the expansion of \( S = \delta_S^a D_a \) in a homology basis of orientifold-even divisors. Note that the general \( \mathcal{N} = 1 \) effective action (2.2) with the gauge coupling (2.6) is not a standard \( \mathcal{N} = 1 \) action due to the presence of the flux correction in (2.6). These fluxes actually break the gauge group in the eight-dimensional world volume theory of the D7-branes. To make this more explicit we display the action splitting into a flux-independent and a flux-dependent part as

\[
S^{(4)}_{F_D7} = -2\pi \int_{\mathcal{M}_4} \frac{1}{2} \text{Re} f^c \text{Tr}(F \wedge *F) + \frac{1}{2} \text{Im} f^c \text{Tr}(F \wedge F) \quad (2.7)
\]

\[
\quad + \frac{1}{2} \text{Re} f^\text{flux}_{AB} F^A \wedge *F^B + \frac{1}{2} \text{Im} f^\text{flux}_{AB} F^A \wedge F^B ,
\]

Clearly, a standard \( \mathcal{N} = 1 \) action can be found if the fluxes are zero and the gauge group is completely unbroken. A second possibility is to consider the breaking of the group, for example by moving the D7-branes apart on \( \mathbb{Z}_3 \). Then one finds a standard \( \mathcal{N} = 1 \) action for a gauge group \( \text{U}(1)^k \). For completeness we will summarize the result in this phase. Later on we will T-dualize the D7-branes to D6-branes which can then be moved apart in the T-dualized direction.

Assuming that we can move the D7-branes apart on different internal cycles in the same class \([S_b]\). The gauge coupling function can be given for each individual brane labeled by \( I = 1, \ldots, k \). Fluxes are now only located on each separate D7-brane, which is reflected in the structure of adjoint indices. Indeed, in evaluating \( C_{IJ} \) and \( \tilde{C}_{IJ} \) from (2.3) we use the basis \( E_I = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0) \) with 1 at the \( I \)-the position that is related to the \( T_I = (T_i, 1) \) by a basis transformation. We then readily evaluate (2.3) as

\[
C_{IJ} = \delta_{IJ} , \quad \tilde{C}_{IJ} = \frac{1}{2} \delta_{IJ} \delta_{KL} \delta_{IK} \int_{S_b} \mathcal{F}^K \wedge \mathcal{F}^L = \frac{1}{2} \delta_{IJ} n^I , \quad (2.8)
\]

where we exploited that the \( E_I \) commute to evaluate the symmetrized trace \( s\text{Tr}(.) \). Here \( \mathcal{F}^I \) denotes the internal flux on the \( I \)th D7-brane. The numbers \( n^I \) characterize the topology of the
gauge configuration on the \( I \)-th brane. They are related to the integral instanton number \( k^I \) of the U(1) on the \( I \)-th brane as \( n^I = -8\pi^2 k^I \). Using these results the gauge coupling function on the \( I \)-th D7-brane is given by

\[
f^I = \frac{1}{4} (\delta^2 S T_n - i \frac{1}{2} \tau n^I) .
\] (2.9)

As we will see in section 4 for the comparison of the D7-brane action with the M-theory fourfold compactification it turns out to be convenient to dualize certain scalars into form fields. More precisely, we replace in four dimensions the chiral multiplet containing the complex scalars \( T^\alpha \) with a linear multiplet containing the bosonic fields \((L^\alpha, C^\alpha)\). Here \( L^\alpha \) are real scalars dual to the real part \( \text{Re} T^\alpha \) and the imaginary part \( \text{Im} T^\alpha \) is dual due to its shift symmetry to a two-form \( C^\alpha \). It will then be crucial to follow the terms involving \( f_{AB} \) through the dualization. As outlined in detail in appendix B this procedure dualizes the classical coupling \( \text{Im} f^c(T) \text{Tr}(F \wedge F) \) in (2.7) into a modification of the field strength \( H^3_3 \) of \( C^\alpha \) by the Chern-Simons form \( \omega_{\text{CS}} \) to \( \text{Tr}(F \wedge F) \),

\[
H^3_3 = d C^\alpha + \frac{1}{8} \delta^2_S \omega_{\text{CS}} , \quad \omega_{\text{CS}} = A \wedge dA + \frac{3}{8} A \wedge A \wedge A .
\] (2.10)

The complete dual action as given in (B.6) of appendix B then contains all terms in (2.7) except the term involving \( \text{Im} f^c(T) \) which is replaced, together with the kinetic term for \( \text{Im} T^\alpha \), by a kinetic term for \( H^\alpha \). Of course all other fields that do not couple to \( \text{Im} T^\alpha \) like \( \tau \) and \( \text{Re} T^\alpha \) or its dual \( L^\alpha \) are unaffected. For the later comparison to M-theory it is important to keep in mind the Kähler potential \( \tilde{K} \) for \( L^\alpha \) and \( \tau \) obtained by Legendre transformation of \( K(\tau|T) \) as

\[
\tilde{K}(\tau|L) = K + L^\alpha \text{Re} T^\alpha = \log(\frac{1}{6} L^\alpha L^\beta L^\gamma \mathcal{K}_{\alpha\beta\gamma}^\tau) - \log(\tau - \bar{\tau}) .
\] (2.11)

### 2.2 Dimensional reduction to three dimensions

In this subsection we discuss the circle reduction of the four-dimensional effective action of a D7-brane in an orientifold compactification to three dimensions. The final result will later be compared to the M-theory reduction on a Calabi-Yau fourfold when restricted to the weak coupling limit. It is important to stress that the M-theory reduction is performed on a smooth geometry at large volume. In the three-dimensional effective theory this yields a gauge theory on the Coulomb branch. In 4d F-theory compactified on an extra circle new terms in the effective theory are generated due to the necessity to integrate out massive vector multiplets containing the W-bosons and charged chiral matter multiplets [11, 38].

In the D7-brane picture we consider a reduction on a circle of circumference \( r \). Moving on the Coulomb branch is achieved by giving the scalars in the three-dimensional vector multiplets a vacuum expectation value. In order to make this more precise we make the following reduction ansatz for the four-dimensional fields,

\[
g^4_{\mu\nu} = \left( \begin{array}{ccc} g_{pq} + \frac{r^2}{r^2 A^0_q A^0_p} & \frac{r^2}{r^2} A^0_q \\ \frac{r^2}{r^2} A^0_p & \frac{1}{r^2} \end{array} \right) , \quad A = (A_3 - A^0_\zeta, \zeta) .
\] (2.12)
Here $A_3$ and $\zeta$ are a three-dimensional vector and a three-dimensional scalar both transforming in the adjoint of the gauge group $G$. The Coulomb branch is obtained by giving $\zeta$ a vev, and splitting

$$U(k) \rightarrow U(1)^k, \quad A \rightarrow A^I, \quad \zeta \rightarrow \zeta^I,$$

where $I = 1, \ldots, k$ runs only over the Cartan generators $T_I$ of $U(k)$. In this split one can now evaluate the traces (2.3). By the basis change to the $E_I = \text{diag}(0, \ldots, 1, \ldots, 0)$ the traces can be written, by the same calculations leading to (2.8), as

$$C_{IJ} = \delta_{IJ}, \quad \tilde{C}_{IJ} = \frac{1}{2}\delta_{IJ}n^I,$$

where we used the numbers $n^I$ introduced before. The couplings of the gauge-fields are thus encoded by

$$f_{IJ} = \frac{1}{4}(C_{IJ}^\alpha T_\alpha - i\frac{1}{2} \tau \delta_{IJ}n^I), \quad C_{IJ}^\alpha = \delta_{\alpha}^S C_{IJ}.$$

Note that this breaking has a natural interpretation in the T-dual picture, where the T-duality is performed along the reduction circle. In this duality the D7-branes become D6-branes localized on points of the reduction circle. The Coulomb branch corresponds to moving the D6-branes apart. The $\zeta^I$ can then be reinterpreted as positions on the circle.

Since we are reducing an $\mathcal{N} = 1$ supersymmetric action in four dimensions, we obtain an action with $\mathcal{N} = 2$ supersymmetry in three dimensions. It can be brought into the form

$$S^{(3)} = 2\pi \int_{\mathcal{M}_3} -\frac{1}{2} R_3 * 1 - \tilde{K}_{\alpha} dM^\alpha \wedge *d\tilde{M}^\beta + \frac{1}{4} \tilde{K}_{\Lambda\Sigma} d\xi^\Lambda \wedge *d\xi^\Sigma$$

$$+ \frac{1}{4} \tilde{K}_{\Lambda\Sigma} F^\Lambda \wedge *F^\Sigma + F^\Lambda \wedge \text{Im}(\tilde{K}_{\Lambda\alpha} dM^\alpha),$$

where one has to perform the Weyl-rescaling $g_{pq} \rightarrow r^2 g_{pq}$ to the Einstein-frame metric, and to make the following identifications,

$$R = r^{-2}, \quad \xi^\Lambda = (R, R\xi^I), \quad A^\Lambda = (A^0, A^I),$$

with $\Lambda = 0, 1, \ldots, k$. Here, the $M^\alpha$ collectively denote four-dimensional chiral multiplets. The three-dimensional kinetic potential $\tilde{K}$ depends on $M^\alpha, \tilde{M}^\beta$ as well as $\xi^\Lambda$ and reads

$$\tilde{K} = K(M, \tilde{M}) + \log R - \frac{1}{R} \text{Re} f_{IJ} (M) \xi^I \xi^J,$$

where $K(M, \tilde{M})$ is the four-dimensional Kähler potential evaluated for the three-dimensional fields. This kinetic potential contains also the gauge kinetic function since the third component of the four-dimensional vectors have become scalars $\xi^I$ in three dimensions as is obvious from (2.12).

Let us be more concrete by reducing the four-dimensional action (B.6) with linear multiplets. Since we are considering not only chiral and vector multiplets, but also linear multiplets containing
\( C_2 \), the form of the kinetic action in three dimensions we will obtain will be different from (2.16). Incorporating two-forms we specify the reduction ansatz for \( C_2 \) such that
\[
\mathcal{H}_3^\alpha \rightarrow (\tilde{F}_\alpha + \frac{1}{4} C_\alpha^{\alpha} \zeta^I F^J) \wedge dy ,
\]
(2.19)
where we introduced the field strength \( \tilde{F}_\alpha = dA_\alpha \) in three dimensions. Physically this means that the fields \( \tilde{T}_\alpha \), of which \( \text{Im}T_S = \delta_\alpha^S T_\alpha \) constitutes the leading part of the D7-brane gauge coupling, will occur after dualization into two-forms and dimensional reduction as vectors in three dimensions. Plugging that into the action (B.6) and performing a Weyl rescaling \( g_{\mu \nu} \rightarrow r^2 g_{\mu \nu} \) we integrate out the circle coordinate \( y \) to obtain
\[
S^{(3)}_{F_\alpha, F_\alpha} = 2\pi \int_{M_3} \tilde{K}_{\alpha \beta}(\tilde{F}_\alpha + \frac{1}{4} C_\alpha^{\alpha} \zeta^I F^J) \wedge *((\tilde{F}_\beta + \frac{1}{4} C_\beta^{\beta} \zeta^I F^J))
- \frac{1}{2\pi} \text{Ref}_{I,J}(F^I \wedge *F^J + d\zeta^I \wedge *d\xi^J) - \text{Im} f_{I,J}^{\text{flux}} \zeta^I \wedge F^J ,
\]
(2.20)
where we used the kinetic potential (2.18) with the four-dimensional Kähler potential (2.11). As one can easily check this matches the structure anticipated in (2.16) which is supplemented by additional terms involving the vectors \( \tilde{F}_\alpha \) contributed by the linear multiplets. The terms to determine \( f_{I,J} \) are:

1. the kinetic term \( F^I \wedge *F^J \) to determine the complete \( \text{Ref}_{I,J} \),
2. the mixed terms \( F^I \wedge *\tilde{F}_\alpha \) to determine the classical part of \( \text{Im} f_{I,J} \) proportional to \( \text{Im} T_\alpha \),
3. the term \( d\zeta^I \wedge F^J \) to obtain \( \text{Im} f_{I,J}^{\text{flux}} \).

Let us close this section by commenting on another choice of Cartan generators for \( U(k) \) which naturally appears in M-theory. This choice is associated to the split \( U(k) = SU(k) \times U(1) \) and yields the trace \( C_{I,J} \) in (2.3) as
\[
C_{ij} = C_{ij}, \quad C_{00} = k , \quad C_{i0} = 0 ,
\]
(2.21)
where \( i,j = 1, ..., k-1 \) label the Cartan generators \( T_i = E_i - E_{i+1} \) of \( SU(k) \) and \( C_{ij} \) is the Cartan matrix of \( SU(k) \). Decoupling the overall \( U(1) \) of \( \tilde{T}_0 = 1 \) in \( U(k) \) as in [36, 37], the classical part of the three-dimensional gauge coupling function (2.6) splits for the Cartan \( U(1) \)'s of \( SU(k) \) as
\[
f_{ij} = \frac{1}{4} C_{ij}^{\alpha} T_\alpha , \quad C_{ij}^{\alpha} = C_{ij} \delta_\alpha^S .
\]
(2.22)
It was this coupling which was found in [4] in a dimensional reduction of M-theory on a resolved Calabi-Yau fourfold. We will recall this reduction briefly in section 4.2.

---

\(^6\)However, we note that an action including linear multiplets can be brought in the standard form using duality of vectors and scalars in three dimensions.
3 M-theory compactifications and Taub-NUT geometries

In order to understand the gauge kinetic function of 7-branes in F-theory, we have to extend the Type IIB effective action discussed in the last section away from the weak coupling limit. This is achieved by considering F-theory as a limit of M-theory with G-fluxes. This section provides the necessary background material from the M-theory perspective to determine the full gauge kinetic coupling function of 7-branes from back-reaction effects as demonstrated in section 4.

In order to set the stage we first introduce the M-theory backgrounds with a non-trivial four-form flux $G_4$ in subsection 3.1. In particular, we stress that the $G_4$ background induces a non-trivial warp factor. Later on this back-reaction will be shown to correct the gauge coupling function. Since we will be interested in the gauge dynamics of one stack of 7-branes it will be necessary to introduce the dual local M-theory geometries. For a stack of $k$ D7-branes the form of this local M-theory geometry can be inferred via string duality. First we note that in compactifying Type IIB on a circle one can T-dualize the D7-branes into $k$ D6-branes. These D6-branes lift in M-theory to the geometry of Kaluza-Klein monopoles. Since the metric and cohomology of Kaluza-Klein monopoles in M-theory is just given by Taub-NUT space $TN_k$ with $k$ indicating the number of monopoles, we can explicitly analyze their local geometry in subsection 3.2.

Having introduced the multi-Taub-NUT spaces we discuss in subsection 3.3 a further compactification on a circle on which one can perform a T-duality to the F-theory setup. The resulting geometry will serve as a local model of the singular elliptic fibration of the M-/F-theory fourfold $Y_4$ with a 7-brane located on a divisor $S_b$ in the base. The compactification of the Taub-NUT geometry is achieved by considering an infinite chain of Kaluza-Klein monopoles with period $a$, denoted $TN_k^\infty$, and later considering the quotient. Technically, this process involves a resummation of certain divergent infinite sums in the corresponding metric.

3.1 M-theory on warped Calabi-Yau fourfolds

In this section we introduce vacuum solutions of M-theory on Calabi-Yau fourfolds with background fluxes following [12]. The eleven-dimensional low effective energy action of M-theory is given by

$$S_{M}^{(11)} = -2\pi \int_{\mathcal{M}_{11}} \frac{1}{2} R \ast 1 + \frac{1}{4} G_4 \wedge \ast G_4 + \frac{1}{12} C_3 \wedge G_4 \wedge G_4 - 2\pi \int_{\mathcal{M}_{11}} C_3 \wedge X_8 + \sum_k S_{M2}^k$$

(3.1)

where locally $G_4 = dC_3$ is the field strength of the M-theory three-form $C_3$, and $X_8$ is a forth order polynomial in the Riemann curvature of the eleven-dimensional space-time. The last term includes the coupling to M2-branes with action $S_{M2}^k$. The $G_4$ field strength, the curvature $X_8$, and

---

7 This name is due to Taub and Newman, Unti and Tamburino (NUT), but can also be traced back to nut at the origin which is the terminus for an isometrical fixed point introduced by Hawking.

8 We have set $\ell_M = 1$ in the conventions of [2].
the presence of M2-brane can serve as sources in the $C_3$ equations of motion

$$d \ast G_4 = \frac{1}{2} G_4 \wedge G_4 - X_8 + \sum_k \delta^{(8)}(\Sigma^k_3),$$  \hspace{1cm} (3.2)$$

where $\delta^{(8)}(\Sigma^k_3)$ is an eight-form current localizing on the world-volumes $\Sigma^k_3$ of the M2-branes.

Supersymmetric solutions can be analyzed by solving the equations of motion \[ (3.1) \] and its supersymmetry variations. A non-trivial background was found in [12] which allows for an internal Calabi-Yau geometry $Y_4$ times a flat space $\mathbb{R}^{(2,1)}$ and a background flux for the field strength $G_4$. The metric in the presence of such flux has to include a non-trivial warp factor $e^A$, and is given by

$$ds^2_{(11)} = e^{-A} \eta_{\mu \nu} dx^\mu dx^\nu + e^{A/2} g_{ab} dy^a d\bar{y}^b,$$

(3.3)

with $g_{ab}$ being the metric on the Calabi-Yau manifold $Y_4$. The warp factor only depends on the coordinates $y^a, \bar{y}^b$ of $Y_4$. The non-trivial field strength $G_4$ splits into a contribution with three flat indices $(G_4)_{\mu \nu \rho m}$ and an internal $G_4$-flux $G_4$ with indices only along $Y_4$. Supersymmetry implies the background component of $G_4$ with flat indices is determined by the warp factor

$$\quad (G_4)_{\mu \nu \rho m} = \epsilon_{\mu \nu \rho} \partial_m e^{3A/2},$$

(3.4)

where the derivative is taken with respect to the internal coordinates. The equations \[ (3.4) \] and \[ (3.2) \] require that the warp factor has to fulfill the Laplace equation

$$\Delta_{Y_4} (e^{3A/2}) = \ast_{Y_4} (\frac{1}{2} G_4 \wedge G_4 - X_8|_{Y_4} + \sum_k \delta^{(8)}(\Sigma^k_3)),$$

(3.5)

where $\Delta_{Y_4}, \ast_{Y_4}$ is the Laplacian and the Hodge-star evaluated in the Calabi-Yau metric $g_{ab}$. The last term in \[ (3.5) \] needs to be included if the background contains M2-branes which fill the non-compact space-time $\mathbb{R}^{(2,1)}$ and are pointlike in $Y_4$. There are further constraints by supersymmetry and the equations of motion on the background flux $G_4$. It can be shown that $G_4$ has to be selfdual and primitive,

$$\ast_{Y_4} G_4 = G_4, \quad J \wedge G_4 = 0,$$

(3.6)

where $J$ is the Kähler form on the fourfold $Y_4$. We will have to say more about the flux $G_4$ and its interpretation in the Type IIB picture in section 4.4.

Let us stress that for compact geometries the Laplace equation \[ (3.5) \] implies a non-trivial consistency condition when integrated over $Y_4$. This is the famous M2-brane tadpole condition

$$\frac{\chi(Y_4)}{24} = \frac{1}{2} \int_{Y_4} G_4 \wedge G_4 + N_{M2},$$

(3.7)

where $\chi(Y_4)$ is the Euler number of $Y_4$, and $N_{M2}$ is the number of space-time filling M2-branes. The condition \[ (3.7) \] together with \[ (3.0) \] implies that in a compact setting the corrections due to $X_8$ leading to $\chi(Y_4)$ in \[ (3.7) \] are crucial to find supersymmetric vacua with $G_4$ flux. However, in our local considerations we will focus mainly on the flux contribution in \[ (3.5) \], and leave the inclusion of the curvature corrections to future work.
3.2 Kaluza-Klein-monopoles: $\mathbb{T}_N^k$-spaces in M-theory

So far we have introduced the background geometry including a warped $Y_4$, and we recalled that the warp factor is sourced by internal fluxes $\mathcal{G}_4$. As a next step we like to identify local geometries in $Y_4$ which would correspond to D6-branes at weak coupling. Note that our geometries $Y_4$ will be elliptic fibrations in which such a weak coupling limit can be performed. The D6-branes are located at the points where the elliptic fibration pinches. In particular, a D6-brane will wrap the divisors $S_b$ in the base $B_3$ if the elliptic fiber pinches over this divisor. Clearly it is very hard to evaluate the warp factor equation (3.5) for the full geometry $Y_4$. To proceed we therefore will focus on a local model denoted as $\mathcal{Y}_4$ which arises in a patch of $Y_4$ near $S_b$.

Before considering the periodic case with an additional circle let us first recall some classical facts about the origin of D6-branes in M-theory. The D6-brane is realized in M-theory as a Kaluza-Klein monopole that is a solution to eleven-dimensional supergravity [39]. Roughly speaking, this monopole solution is an asymptotically locally flat circle fibration over $\mathbb{R}^3$ with degeneration loci at a point in $\mathbb{R}^3$. The asymptotic circumference of the circle fibration will be denoted by $r_A$, and corresponds to the Type IIA string coupling

$$g_{\text{IA}} = \frac{r_A}{2\pi}. \quad (3.8)$$

In the weak coupling limit $r_A \to 0$ the M-theory setup reduces to the Type IIA string with a D6-brane located at the point where the monopole circle pinches.

We will directly consider the case of multiple Kaluza-Klein monopoles since we will need to consider periodic arrays later on. The solution with $k$ Kaluza-Klein monopoles will be denoted by $T\mathbb{N}_k$. The metric of $T\mathbb{N}_k$ is given by

$$ds_{T\mathbb{N}_k}^2 = \frac{1}{V}(dt + U)^2 + V d\vec{r}^2, \quad (3.9)$$

where $t \sim t + r_A$ is a periodic coordinate on a circle $S^1$ of circumference $r_A = 4\pi m$ with $m$ being the mass of the Taub-NUT solution. The flat part of $T\mathbb{N}_k$ is $\mathbb{R}^3$ with coordinates $\vec{r} = (x, y, z)$. The one-form $U$ on $\mathbb{R}^3$ is the $S^1$ connection. In this metric one has the functions

$$V = 1 + \sum_{i=1}^{k} V_I, \quad U = \sum_{I=1}^{k} U_I, \quad V_I = \frac{m}{|\vec{r} - \vec{r}_I|}, \quad *_3 dU_I = -dV_I, \quad (3.10)$$

where $\vec{r}_I$ denote the positions of the $k$ monopoles, and $*_3$ is the Hodge star in $\mathbb{R}^3$. We denote this space as $T\mathbb{N}_k$. We see that the circle fibration degenerates at the $k$ points $\vec{r}_I$ in $\mathbb{R}^3$. Note that one has to use two patches around each monopole in order to obtain a globally well-defined connection $U_I$. Furthermore, one has to have the same mass $m$ for all monopoles in order to get a smooth solution. The multi-center solution $T\mathbb{N}_k$ admits $k$ anti-selfdual two-forms locally defined by

$$\Omega_I = d\eta_I = \frac{1}{4\pi m} d\left(\frac{V_I}{V} (dt + U) - U_I\right), \quad I = 1, \ldots, k. \quad (3.11)$$

---

9The geometry approaches an $S^1$-bundle over $S^2 \times \mathbb{R}$ at infinity in $\mathbb{R}^3$. 

12
It is straightforward although technically involved to check that

\[ \int_{TN_k} \Omega_I \wedge \Omega_J = -\delta_{IJ}, \tag{3.12} \]

as was noted in [40] and is shown in detail in appendix C.

Let us comment on the topology of \( TN_k \). One can introduce the following real two-dimensional subvarieties of \( TN_k \) defined as

\[ S_i = \{ (t, \vec{r}) \mid \exists p \in [0, 1] \ \text{s. t.} \ \vec{r} = (1-p) \vec{r}_i + p \vec{r}_{i+1} \}, \quad i = 1, \ldots, k-1. \tag{3.13} \]

These subvarieties are indeed closed two-cycles by noting the degeneration of the \( S^1 \)-fiber at the position of the monopoles which gives them the topology of a sphere \( S^2 = \mathbb{P}^1 \). The generators \( S_1, \ldots, S_{k-1} \) span the second homology of \( TN_k \) that is thus given by \( \mathbb{Z}^{k-1} \). Furthermore these surfaces intersect each other as the negative Cartan matrix \( C_{ij} \) of \( A_{k-1} \) which matches the fact that these geometries give \( SU(k) \) gauge theories [41]. To see this one notices that \( S_i \) and \( S_j, i \neq j \), intersect each other exactly once if and only if \( i = j-1 \) but with reversed orientation. To find the self-intersection of \( S_i \), deform the base curve generically, which intersects the old one precisely at \( \vec{r}_i \) and at \( \vec{r}_{i+1} \) this time with the same orientation resulting in the self-intersection two. If we add the cycle \( S_0 \) connecting \( \vec{r}_1 \) and \( \vec{r}_k \), that is minus the sum of the \( S_i \), we obtain the Cartan matrix of affine \( A_{k-1} \). This is consistent with the fact, that \( TN_k \) is for generic moduli the resolution of an \( A_{k-1} \)-singularity. In summary \( H_2(TN_k, \mathbb{Z}) \) is isomorphic to the weight lattice of \( A_{k-1} \).

The Poincaré dual of \( H_2(TN_k, \mathbb{Z}) \) is given by \( H^2_{cpt}(TN_k, \mathbb{Z}) \), the second cohomology with compact support. Hence it is isomorphic to \( \mathbb{Z}^{k-1} \) and its generators are given by [42]

\[ \hat{\omega}_i = \Omega_i - \Omega_{i+1}. \tag{3.14} \]

These fulfill the following conditions, see appendix C,

\[ \int_{TN_k} \hat{\omega}_i \wedge \hat{\omega}_j = -C_{ij}, \quad \int_{S_i} \hat{\omega}_j = -C_{ij}. \tag{3.15} \]

This concludes our discussion of the space \( TN_k \). It will be crucial in a next step to generalize these geometries to have infinitely many centers in order to describe periodic configurations.

### 3.3 \( S^1 \)-compactification of \( TN_k \): \( TN^\infty_k \)-space in M-theory

Our goal is to to eventually describe 7-branes F-theory and to derive their gauge coupling function. At weak Type IIB string coupling the corresponding D7-branes T-dualize to D6-branes localized on

\[ \text{indeed if all monopoles approach each other the area of the } S_i \text{ vanishes and the space develops a } \mathbb{Z}_{k-1} \text{-singularity. To see this one expands the metric along the lines as was done for the case of the single monopole. The configuration which arises by squeezing all monopoles together corresponds to a monopole of charge } nm \text{ which equips } \psi \text{ with a periodicity of } \frac{2\pi}{k} \text{ what shows the desired deficit angle.} \]
a circle, which we termed the B-circle. In order to describe this situation in M-theory we consider an infinite array of Kaluza-Klein monopoles separated by a distance \( r_B \) in the \( z \)-direction of \( \mathbb{R}^3 \) introduced in (3.9). To effectively compactify this \( z \)-direction on a circle we mod out the relation \( z \sim z + r_B \). This is analogous to the geometries considered in [30, 43, 44].

We first introduce the metric structure on the infinite array denoted by \( TN_k^\infty \) in the following. This space is obtained as follows. We first consider the special situation of \( TN_k \) with centers located in the \((x, y)\)-origin but separated along the \( z \)-coordinate in (3.9) by a distance \( z_I \). This implies that we take the vectors \( \vec{r}_I \) in (3.10) of the form
\[
\vec{r}_I = (0, 0, z_I), \quad 0 \leq z_I < r_B.
\]
Next we periodically extend this space to \( TN_k^\infty \) in the \( z \)-direction with period \( r_B \). The metric for such a configuration still takes the form
\[
\text{ds}^2_{TN_k^\infty} = \frac{1}{V} (dt + U)^2 + V d\vec{r}^2, \quad (3.17)
\]
where \( V \) is a harmonic function on \( \mathbb{R}^3 \) except at the points \( \vec{r}_I \) and \( U \) a connection one-form,
\[
V = 1 + \sum_{I=1}^k V_I, \quad U = \sum_{I=1}^k U_I. \quad (3.18)
\]
Since we consider an infinite array \( V_I, U_I \) are of the form
\[
V_I = \frac{r_A}{4\pi} \sum_{\ell \in \mathbb{Z}} \frac{1}{\sqrt{\rho^2 + (z + \ell r_B - z_I)^2}} - \frac{r_A}{4\pi} \sum_{\ell \in \mathbb{Z}} \frac{1}{r_B|\ell|}, \quad *_3 dU_I = -dV_I. \quad (3.19)
\]
where \( r_A = 4\pi m, \mathbb{Z}^* = \mathbb{Z}\backslash\{0\} \), and \( \rho = \sqrt{x^2 + y^2} \). The first term in \( V_I \) is just the potential of a periodic configuration of monopoles along the \( z \)-axis with spacing \( r_B \). The second term in \( V_I \) is a regulator which ensures convergence and can be modified by any finite constant. This metric is also called the Ooguri-Vafa metric, that was initially constructed in the analysis of the hypermultiplet moduli space of Type II string theory [30]. To see that the metric (3.17) defined with \( V \) and \( U \) in (3.18) is smooth for finite and different \( z_I \neq z_J \) for \( I \neq J \) one notices that locally near the singularities of \( V \) the space looks like that of one single Kaluza-Klein monopole which is known to be smooth. For our later discussion it will be crucial to introduce the rescaled coordinates
\[
\hat{t} = \frac{t}{r_A}, \quad \hat{z} = \frac{z}{r_B}, \quad \hat{z}^I = \frac{z^I}{r_B}, \quad \hat{\rho} = \frac{\rho}{r_B}. \quad (3.20)
\]
Note that in these coordinates one has the periodic identifications
\[
\hat{t} = \hat{t} + 1, \quad \hat{z} = \hat{z} + 1, \quad \hat{z}^I = \hat{z}^I + 1. \quad (3.21)
\]
To obtain a better understanding of the regularity and the physical meaning of the solution one has to perform a Poisson resummation of \( V \) and \( U \) [30, 31]. The details of the calculations are
relegated to appendix D. Finally we may then write
\[ V_I = -\frac{r_A}{2\pi r_B} \left( \log \left( \frac{\hat{\rho}}{\Lambda} \right) - \sum_{\ell \in \mathbb{Z}^+} K_0 \left( \frac{2\pi \hat{\rho} \ell}{r_B} \right) e^{2\pi i \ell (\hat{z} - \hat{z}_I)} / r_B \right) \]
\[ = -\frac{r_A}{2\pi r_B} \left( \log \left( \frac{\hat{\rho}}{\Lambda} \right) - 2 \sum_{\ell > 0} K_0(2\pi \hat{\rho} \ell) \cos(2\pi \ell (\hat{z} - \hat{z}_I)) \right), \quad (3.22) \]

where \( \Lambda \) is a constant which can be chosen arbitrarily in the regularization of \( (3.19) \). The function \( K_0(x) \) is the zeroth Bessel function of second kind. Let us note that \( V_I \) satisfies the Poisson equation
\[ \Delta_3 V_I = -\frac{r_A}{r_B \hat{\rho}} \delta(\hat{z} - \hat{z}_I) \delta(\hat{\rho} \delta(\varphi)), \quad (3.23) \]

where \( \Delta_3 = \frac{\partial^2}{\hat{\rho}^2} + \frac{1}{\hat{\rho}} \frac{\partial}{\hat{\rho}} + \frac{1}{\hat{\rho}^2} \frac{\partial^2}{\varphi^2} + \frac{\partial^2}{z^2} \) is the Laplacian in cylinder coordinates. One can also perform a Poisson resummation for \( U \), as we do in appendix D, finding up to an ambiguity of an exact form
\[ U_I = \frac{r_A}{4\pi} \left( -1 - 2(\hat{z} - \hat{z}_I) + 2i\hat{\rho} \sum_{\ell \in \mathbb{Z}^+} \text{sign}(\ell) K_1 \left( 2\pi \hat{\rho} |\ell| \right) e^{2\pi i \ell (\hat{z} - \hat{z}_I)} \right) d\varphi \]
\[ = -\frac{r_A}{4\pi} \left( 1 + 2(\hat{z} - \hat{z}_I) + 4\hat{\rho} \sum_{\ell > 0} K_1(2\pi \hat{\rho} \ell) \sin(2\pi \ell (\hat{z} - \hat{z}_I)) \right) d\varphi, \quad (3.24) \]

for \( \hat{z}_I \leq \hat{z} < \hat{z}_I + 1 \), where \( \varphi = \arctan(y/x) \), and \( K_1 \) is the first Bessel function of second kind. In the first term in this expression we have included an integration constant \( \hat{z}_I \) which arises when solving \( (3.19) \). Note that this form is gauge equivalent to \( U_I \) with leading term given by \( U_I = \frac{r_A}{2\pi}(\varphi_0 + \varphi) d\hat{z} + \ldots \) by the gauge transformation by \( d(\hat{z} \varphi) \). It will turn out below that it is important for the F-theory interpretation to define the full circle connection \( U \) in this gauge reading
\[ U = \frac{k}{2\pi} r_A(\varphi + \varphi_0) d\hat{z} - \frac{r_A}{2\pi} \left( 2\hat{\rho} \sum_{\ell > 0} K_1(2\pi \hat{\rho} \ell) \sin(2\pi \ell (\hat{z} - \hat{z}_I)) \right) d\varphi. \quad (3.25) \]

Here we introduced an integration constant \( \varphi_0 \). As we will show next this choice of integration constant is required when matching the local geometry with an asymptotic elliptic fibration required in F-theory and equivalently for the identification of the three-dimensional RR-form \( C_0 \equiv k \varphi_0 \).

For completeness we note that also the definition of the two-forms \( \Omega_i \) can be extended to \( TN_k^\infty \). They are given by
\[ \Omega_i^\infty = dh_i = \frac{1}{r_A} d \left( \frac{V_i}{V}(dt + U) - U_I \right), \quad (3.26) \]

As demonstrated in appendix D these forms still satisfy
\[ \int_{TN_k^\infty} \Omega_i^\infty \wedge \Omega_j^\infty = -\delta_{ij}, \quad *_4 \Omega_i^\infty = -\Omega_i^\infty, \quad (3.27) \]

where the Hodge-star \( *_4 \) is in the \( TN_k^\infty \) metric \( (3.17) \). In addition, we introduce the generalization of the forms introduced in \( (3.14) \) to the geometry \( TN_k^\infty \),
\[ \omega_i^\infty = \Omega_i^\infty - \Omega_i^{\infty} \quad (3.28) \]

\[ ^{11} \text{In appendix D we have fixed } 1/\Lambda = \pi e^{2\gamma} \text{ with } \gamma \approx 0.577 \text{ denoting the Euler-Mascheroni constant.} \]
As on the Taub-NUT space $\text{TN}_k$ we expect them to generate the second cohomology with compact support $H^2_{\text{cpc}}(\text{TN}_k, \mathbb{Z})$ and to be dual to the connecting $\mathbb{P}^1$ between $z_i$ and $z_{i+1}$ of the resolved $A_{k-1}$ singularity. In particular, the intersections are given by the Cartan matrix $C_{ij}$ as in (3.15).

To close this section, let us now discuss the limit of large $\hat{\rho}$, which means that we are moving away from the centers of the monopoles. In this limit one can expand

$$K_0(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}, \quad x \gg 1,$$

(3.29)

so that the terms involving the Bessel functions in (3.22) and (3.24) are exponentially suppressed as $e^{-2\pi|\hat{\rho}|} \to 0$ for large $\hat{\rho}$. Since the $z_I$ are the positions in the $z$-direction with period $r_B$ this is equivalent to smearing one Kaluza-Klein monopole along the $z$-direction in the base $\mathbb{R}^3$ to obtain a new isometrical direction $^{12}$ One can then use this isometry to gauge away two components of the connection $U$ keeping only the component $U_3$ in the $z$-direction. We therefore obtain the approximate potential and gauge connection

$$V = 1 - \frac{k}{2\pi} \frac{r_A}{r_B} \log \left( \frac{\hat{\rho}}{\Lambda} \right), \quad U = \frac{k}{2\pi} r_A (\varphi + \varphi_0) d\hat{z},$$

(3.30)

up to leading order in $r_B$. Clearly, this means simply that we have dropped the exponentials in (3.22) and (3.25). In the limit (3.30) we can rewrite the metric (3.17) as

$$ds^2_{\text{TN}_k} \approx \frac{1}{V} \left( (dt + U_z dz)^2 + V^2 dz^2 \right) + V (d\rho^2 + \rho d\varphi^2),$$

(3.31)

where the coordinates have periods $(t, z) = (t + r_A, z + r_B)$. In the next step we show that this is simply a two-torus bundle over the $(\hat{\rho}, \varphi)$-plane

$$ds^2 = \frac{v_0}{\text{Im}\tau} \left( (d\hat{t} + \text{Re}\tau d\hat{z})^2 + (\text{Im}\tau)^2 d\hat{z}^2 \right) + ds^2_{\text{base}},$$

(3.32)

where $v_0$ is the volume of the two-torus fiber. The rescaled coordinates $\hat{t}$ and $\hat{z}$ with integral periods were introduced already in (3.21). Note that this torus structure is present due to the careful choice of boundary conditions, involving the constant $\varphi_0$ only, in the determination of (3.30). Comparing (3.31) and (3.32) volume of the torus fiber is given by

$$v_0 = r_A r_B.$$

(3.33)

The complex structure of the torus-fiber at a fixed point $u = \rho e^{i\varphi}$ in the $(\hat{\rho}, \varphi)$-plane, is given by

$$\tau(u) = \frac{k}{2\pi} (\varphi_0 + \varphi) + i \left( \frac{r_B}{r_A} - \frac{k}{2\pi} \log \left( \frac{\hat{\rho}}{\Lambda} \right) \right) = \tau + \frac{k}{2\pi i} \log \left( \frac{u}{\Lambda} \right).$$

(3.34)

Furthermore, the condition $dV = -*dU$ ensures that $\tau$ is a holomorphic function in $u$. Anticipating the discussion of F-theory in section 4.1 we thus obtain precisely the expected monodromy of the

$^{12}$In the picture of point particles in $\mathbb{R}^3$ this corresponds to a charged wire extended along the $z$-axis.
the axio-dilaton in an F-theory with $k$ D7-branes at $u = 0$. We identify the background value

$$\tau = C_0 + i g_s^{-1}$$

as

$$C_0 = \frac{1}{2\pi} k \varphi_0 , \quad g_s = \frac{r_A}{r_B} . \quad (3.35)$$

We also introduce the notation

$$\tau_I(u) = \frac{1}{2\pi} (\varphi + \varphi_0) - i \frac{1}{2\pi} \log \left( \frac{\hat{\rho}}{\Lambda} \right) . \quad (3.36)$$

That the right-hand side of the equation carries no index is explained by the fact that we have neglected the subleading corrections.

For completeness and later reference we list the leading parts of the anti-selfdual two-form $\Omega$. Inserting (3.30) and (3.34) into (3.26) we obtain

$$\eta^\infty_I = \frac{\text{Im} \tau_I}{\text{Im} \tau} (d\hat{t} + \text{Re} \tau d\hat{z}) - \text{Re} \tau_I d\hat{z} , \quad \Omega^\infty_I = d\eta^\infty_I . \quad (3.37)$$

For the case of just one monopole we reproduce (in cohomology) the model discussed in [2] to describe a local 7-brane geometry.

$$\eta^\infty = \frac{r_B}{r_A} \frac{1}{\text{Im} \tau} (d\hat{t} + \text{Re} \tau d\hat{z}) , \quad \Omega^\infty = d\eta^\infty . \quad (3.38)$$

As a next step one would have to construct the forms $\omega^\infty_i = \Omega^\infty_{i+1} - \Omega^\infty_i$ as in (3.28). However, having neglected the subleading corrections depending on the $z$-coordinate the forms $\omega^\infty_i$ would vanish identically for the forms (3.37). In other words, if we want to localize fluxes or gauge fields along the forms $\omega^\infty_i$ it will be crucial to include the non-trivial $z$-dependence in (3.22).

We conclude by interpreting the geometric meaning of the subleading exponential sums in (3.22) and (3.24). Approaching $\rho = 0$ where the fiber torus degenerates, we note that the leading term of $V$ does not “know” about the position of the degeneration of the fibration of the A-circle on the $z$-direction. The corresponding degenerated torus that arises from the leading term only, i.e. the metric (3.31), merely looks like a very thin tire. However, the degenerated torus that arises from M-/F-theory should look like a torus that pinches at a point only, so that the pinched torus forms a $\mathbb{P}^1$. These two different pictures of the degeneration of the torus are called the “differential geometric” and the “algebraic geometric” degeneration in reference [46]. Including now the exponential corrections in $V$ and $U$, however, localizes the A-cycle degeneration and thus the torus degeneration at the point $z = 0$ on the B-cycle, which reconciles the differential and algebraic geometric pictures.

4 7-brane gauge coupling functions in warped F-theory

In this section we turn to the computation of the gauge-coupling function of a stack of 7-branes in F-theory by using the dual M-theory. In order to do that we first recall some basics about F-theory
on singular elliptically fibered Calabi-Yau fourfolds with an $A_{k-1}$ singularity along a divisor $S_b$ in section 4.1. This setup leads to an SU$(k)$ gauge theory in the effective four-dimensional theory, and has a weak coupling limit introduced in section 2. In section 4.1 we also recall how F-theory can be viewed as a limit of M-theory. In section 4.2 we use this map of F-theory to a dual three-dimensional M-theory compactification on a resolved Calabi-Yau fourfold to compute the leading gauge coupling function as in [4]. In order to include the corrections due to brane fluxes we perform a refined but local reduction in section 4.4, and include a non-trivial warp factor and a back-reacted M-theory three-form as introduced in section 4.3. The resulting correction to the D7-brane gauge coupling can be matched with the weak coupling result of section 2.

### 4.1 F-theory as a limit of M-theory

To get started, let us recall some basic facts about a four-dimensional F-theory compactification on an elliptically fibered Calabi-Yau fourfold $Y_4$. In general the elliptic fibration over a base $B_3$ is described by the Weierstrass form

\[ y^2 = x^3 + f(\vec{w})x^4 + g(\vec{w}) , \]  

where $f(\vec{w})$ and $g(\vec{w})$ are sections of $K_{B_3}^{-4}$ respectively $K_{B_3}^{-6}$, and hence depend on the coordinates $\vec{w}$ of $B_3$. The modular parameter $\tau$ of the elliptic fiber of $Y_4$ is only defined up to PSL$(2, \mathbb{Z})$ transformation and thus most invariantly specified by

\[ j(\tau(\vec{w})) = \frac{4 \cdot (24f)^3}{\Delta} , \quad \Delta = 27g^2 + 4f^3 . \]  

where the $j$-function provides away from the singularities $\Delta = 0$ a biholomorphic map from the fundamental region to the complex plane. The fibration in particular implies, that $\tau$ is a section $\tau(\vec{w})$ on the base $B_3$ and describes a varying coupling $\tau = C_0 + ie^{-\phi}$ of Type IIB. Clearly, near the singularities $\Delta = 0$ this simple picture breaks down and has to be replaced by a refined local treatment as we discussed above in section 3.3.

The special subloci on $B_3$ where the discriminant $\Delta$ vanishes indicate the presence of objects charged under $\tau$. These loci geometrically describe divisors in $B_3$ over which the elliptic fiber becomes singular. In Type IIB string theory these divisors are wrapped by $(p,q)7$-branes. The particular type of fiber degeneration leads to different monodromies of $\tau$ around the singular divisors that encode the type of $(p,q)7$-branes and the gauge groups on these branes. As an example we consider a singular $Y_4$ with an $A_{k-1}$ singularity in the elliptic fiber over a divisor $S_b \subset B_3$ which describes a stack of $k$ D7-branes on $S_b$. In other words we consider the split of the class $[\Delta]$ of the discriminant as

\[ [\Delta] = k[S_b] + [\Delta'] , \]  

where $[\Delta']$ is the residual part of $\Delta$ wrapped by a single complicated 7-brane. While $\Delta'$ might intersect $S_b$ the new physics at these intersections will not be of crucial importance to the discussion.
of this work. We will mainly focus on a local model near \( S_b \) and concentrate on the back-reaction of the flux on the geometry. In this local model we introduce a local complex coordinate \( u \) such that \( S_b \) is given by \( u = 0 \). In the vicinity of \( S_b \) we have the local behaviour

\[
  j(\tau(\vec{w})) = a \frac{1}{u^k} + b \quad \Rightarrow \quad \tau(\vec{w}) = \begin{cases} 
  j^{-1}(b) & \text{far way from the D7-branes} \\
  -i \frac{k}{2\pi} \log(u) & \text{near the D7-branes}
\end{cases}
\]

where we have used that \( j(\tau) \sim e^{-2\pi i \tau} \) for large \( \text{Im}(\tau) \). This is precisely the naively expected dilaton in the neighborhood of a D7-brane in perturbative Type IIB theory.

Before turning to the discussion of the formulation of F-theory via M-theory, let us make contact with the presentation of section 2. In this section we have considered the weakly coupled limit of F-theory \[47, 48\]. In this very special case the axio-dilaton \( \tau(u) \) is constant almost everywhere on \( B_3 \) and chosen to have \( \text{Im} \tau \gg 1 \) corresponding to a small Type IIB string coupling \( g_{\text{IIB}} \). The fundamental objects are in this limit D7-branes and O7-planes.

To study F-theory compactifications away from the weak coupling limit is in general a hard task. The complication arises due to the fact that there is no fundamental twelve-dimensional effective action for F-theory which could be used at low energies. To nevertheless investigate general F-theory configurations one has to take a detour via M-theory and a three-dimensional compactification. One starts with a compactification of M-theory on the singular elliptically fibered Calabi-Yau fourfold \( Y_4 \). Due to the singularities of \( Y_4 \) there are massless M2-brane states which are massless and generate a non-trivial gauge theory and spectrum in the effective three-dimensional theory. For our setups with an \( A_{k-1} \) singularity over \( S_b \) one finds a three-dimensional non-Abelian gauge theory with gauge group \( G = SU(k) \). The F-theory limit is performed by shrinking the volume of the elliptic fiber of \( Y_4 \). Since the A-circle shrinks, this yields a Type IIA compactification on a small B-circle over \( B_3 \). After T-duality along the B-circle this will yield a Type IIB string compactification on \( B_3 \) times the T-dual B-circle. In the F-theory limit this growing extra circle yields an additional non-compact direction and hence the effective theory will be four-dimensional. In our discussion it will be crucial to include the warp factor in the general M-theory solution \[3.3\] when performing this duality.

To actually derive the couplings of this theory one can resolve \( Y_4 \) to obtain a smooth geometry \( \hat{Y}_4 \), on which one can Kaluza-Klein reduce eleven-dimensional supergravity \[3.1\]. Geometrically this yields \( k - 1 \) new exceptional divisors \( D_i \) in \( \hat{Y}_4 \) resolving the \( A_{k-1} \) singularity over \( S_b \). We denote the Poincaré dual two-forms to \( D_i \) by \( \omega_i \). The Kaluza-Klein reduction of M-theory to three dimensions requires to expand the Kähler form \( J \) of \( Y_4 \), as well as the M-theory three-form potential \( C_3 \) into harmonic modes. Explicitly, one has\[13\]

\[
  \begin{align*}
    \frac{J}{V} &= R \omega_0 + L^\alpha \omega_\alpha + \xi^i \omega_i \\
    C_3 &= A^0 \wedge \omega_0 + A^\alpha \wedge \omega_\alpha + A^i \wedge \omega_i
  \end{align*}
\]

\[\text{Note that we restrict to Calabi-Yau fourfolds with } h^{2,1}(\hat{Y}_4) = 0, \text{ such that no extra scalars arise from } C_3.\]
where $V$ is the volume of the Calabi-Yau fourfold $\hat{Y}_4$. Here we have included the two-form $\omega_0$ Poincaré dual to the base $B_3$, and the two-forms $\omega_\alpha$ Poincaré dual to divisors $D_\alpha = \pi^{-1}(D^b_\alpha)$ inherited from divisors $D^b_\alpha$ of the base. The coefficients $(R, L_\alpha, \xi^i)$, and $(A^A) = (A^0, A^\alpha, A^i)$, with $A \in \{\alpha, 0, i\}$, are real scalars and vectors in the three-dimensional effective theory. In the F-theory limit to four dimensions, the vector multiplet with bosonic components $(R, A^0)$ becomes part of the four-dimensional metric, and one identifies

$$R = r_B^2,$$

where $r_B$ is circumference of the circle on which the T-duality to Type IIB is performed, and $V$ is the volume of $\hat{Y}_4$. The vector $A^0$ is the Kaluza-Klein vector in the four-dimensional metric as in (2.12). The vector multiplets with bosonic components $(L_\alpha, A^\alpha)$ lift to complex scalars $T_\alpha$ in the F-theory limit, just as in appendix B. Finally, the vector multiplets with bosonic components $(\xi^i, A^i)$ lift to four-dimensional U(1) vector multiplets gauging the Cartan generators $T_i$ of the four-dimensional SU($k$) gauge group as in section 2.1.

In order to proceed further in the discussion, let us recall the behavior of the fields in the F-theory lift. The latter is given by the vanishing of the fiber volume and the blow-down map from $\hat{Y}_4$ to $Y_4$. To make this more precise we introduce the following $\epsilon$-scaling [4]

$$r_B \mapsto \epsilon r_B, \quad R \mapsto \epsilon^2 R, \quad \zeta^i \equiv \frac{\xi^i}{R} \mapsto \epsilon^{2/3} \zeta^i, \quad L^\alpha = 2L^\alpha_{\text{IIB}},$$

where the scalars $L^\alpha$ do not scale with $\epsilon$ but are identified with a factor two with the Type IIB variables $L^\alpha_{\text{IIB}}$ used in appendix B. Note that the Type IIB string coupling is given by $g_{\text{IIB}} = r_A / r_B$. As can be inferred by using the T-duality rules applied to the Type IIA coupling (3.8). Since $g_{\text{IIB}}$ should not scale in the F-theory limit, we find that also $r_A \mapsto \epsilon r_A$. We note in addition that this identification of the string coupling perfectly agrees with (3.35) from M-theory on $TN_k^\infty$.

We can thus give a diagrammatic summary of the limit we will consider. Recalling all identifications from M-theory on $TN_k^\infty$ in section 3,

$$v^0 = r_A r_B, \quad g_s^{\text{IA}} = \frac{r_A}{2\pi}, \quad g_s^{\text{IIB}} = \frac{r_A}{r_B},$$

we consider the following limits:

$$\begin{array}{ccc}
\text{M-theory on } TN_k^\infty & \text{F-limit} & \text{10d F-theory} \\
r_A, r_B \text{ finite} & v^0 \to 0 & g_s^{\text{IIB}} \text{ finite} \\
\frac{g_s^{\text{IA}}}{r_A} \to 0 & \text{Type IIA in 9d} & g_s^{\text{IIB}} \to 0 \\
r_A \text{ finite, } g_s^A \sim 0 & \text{F-limit} & \text{weakly coupled 10d IIB} \\
v^0 \to 0 & \text{10d F-theory} & g_s^{\text{IIB}} \sim 0.
\end{array}$$

Understanding the geometry and the physics of the four corners of this diagram is essential for the calculations of the corrections to the gauge kinetic function in section 4.4.
It is important for us to also follow the space $TN_k^\infty$ through the M-theory to F-theory lift. In fact, since the space $TN_k$ corresponds in Type IIA to $k$ parallel D6-branes, the space $TN_k^\infty$ yields an infinite array of periodically repeating parallel D6-branes. The periodic coordinate in section 3.3 was $z = z + r_B$, which we normalized to have integer periods by setting $\hat{z} = z/r_B$. In the $z$-direction the monopoles are separated by distances $z_{i+1} - z_i$, where $z_I$ are the locations of the $k$ monopoles. Without loss of generality we will take in the following $z_1 = 0$, setting the location of the first monopole to be the origin. We identify the blow-up modes $\xi^i$ in (4.5) with the normalized differences as we will later justify in section 4.4.2 as

$$\xi^i = r_B(z_{i+1} - z_i) . \quad (4.10)$$

In the F-theory limit $\epsilon \to 0$ the vanishing of the $\xi^i$ requires to also moving the centers on top of each other by sending $z_{i+1} \to z_i$, i.e. one has to send $z_I \to 0$.

### 4.2 Leading 7-brane gauge coupling functions

In this section we recall how the classical volume parts of the 7-brane gauge coupling function can be derived in F-theory via M-theory. This derivation only involves topological methods and can therefore be treated in a rigorous global picture of a compact Calabi-Yau fourfold $Y_4$. We return to a local analysis when deriving the corrections to the gauge-coupling function in section 4.4.

Let us note that the field strength of $C_3$ given in (4.5) is given by

$$G_4 = F^A \wedge \omega_A = F^0 \wedge \omega_0 + F^\alpha \wedge \omega_\alpha + F^i \wedge \omega_i . \quad (4.11)$$

In this expression $F^A = dA^A$ are the field strengths of the three-dimensional $U(1)$ gauge fields. The three-dimensional effective action is computed by inserting the expansion (4.11) into the eleven-dimensional supergravity action (3.1). Since we are interested in the leading flux-independent gauge coupling function we assume here that the metric is not warped by demanding that the warp factor $e^{3A/2}$ in (3.3) is constant, and we set the background flux $G_4 = 0$. Here we are interested in the reduction of the kinetic term of $G_4$, and derive

$$S_{\text{kin}}^{(11)} = \frac{\pi}{2} \int G_4 \wedge *G_4 \cong 2\pi \int_{\mathbb{R}^{(2,1)}} G_{AB} F^A \wedge *F^B \quad (4.12)$$

where in the second equality we have performed a Weyl rescaling of the three-dimensional metric $g^{(3)} \to V^2 g^{(3)}$ in order to bring the action into the Einstein frame, and introduced the metric

$$G_{AB} = \frac{V}{4} \int_{Y_4} \omega_A \wedge *\omega_B , \quad \omega_A = (\omega_0, \omega_i, \omega_\alpha) \quad (4.13)$$

In the following we compute the metric $G_{AB}$ explicitly and discuss the matching with (2.20) in order to read off the gauge-coupling function.
In order to compute the metric $G_{AB}$ explicitly we need some information about the intersections of the various forms $\omega_A$. We define $K_{ABCD} = \int \hat{Y} \omega_A \wedge \omega_B \wedge \omega_C \wedge \omega_D$. Due to the elliptic fibration structure one has $K_{\alpha\beta\gamma\delta} = 0$. In addition we have $\omega_i \wedge \omega_0 = 0$ in cohomology. We will need the following non-vanishing intersections:

\begin{align}
K_{0\alpha\beta\gamma} &\equiv \frac{1}{2} K_{\alpha\beta\gamma}, \\
K_{ij\alpha\beta} &\equiv -\frac{1}{2} C^\gamma_{ij} K_{\alpha\beta\gamma}, \\
C^\gamma_{ij} &\equiv C_{ij} C^{\gamma}.
\end{align}

(4.14)

where $C_{ij}$ denotes the Cartan matrix of $G$ as above in section 2. We recall that in the M-theory reduction the complex coordinates are given by

\begin{align}
T_\alpha &\equiv \frac{1}{6} \int_{D_6} J \wedge J \wedge J + i \int_{D_6} C_6 \\
&\equiv \frac{1}{4} \sqrt{3} \omega_0 (L^\beta L^\gamma R - C^\gamma_{ij} L^\beta \xi^i \xi^j) + i \rho_\alpha + \ldots,
\end{align}

(4.15)

where we have used (4.5). Using the intersections (4.14) one evaluates

\begin{align}
G_{ij} &= \frac{C^\alpha_{ij}}{4R} \text{Re} T_\alpha + \frac{C^\alpha_{ij} C^\beta_{kl} \xi^l \xi^k}{R^2} + \ldots, \\
G_{i\alpha} &= - \frac{G_{\alpha\beta}}{R} C^\beta_{ij} \xi^j + \ldots
\end{align}

(4.16)

where the dots indicate terms which are of higher power in $R$. Inserting these expressions into (4.12) we find the action

\begin{align}
S_{\text{kin}}^{(3)} &= -2\pi \int G_{\alpha\beta} (F^\alpha - R^{-1} \xi^i C^\alpha_{ij} F^j) \wedge \ast(\tilde{F}^\beta - R^{-1} \xi^i C^\beta_{ij} F^j) + \frac{1}{16 \pi} C^\alpha_{ij} \text{Re} T_\alpha F^i \wedge \ast F^j.
\end{align}

(4.17)

Comparing this action with (2.20) we infer that the leading gauge coupling function is simply given by

\begin{align}
f_{ij} &= \frac{1}{2} C^\alpha_{ij} T_\alpha = \frac{1}{4} C^\alpha_{ij} T_\alpha^{0\text{IBB}},
\end{align}

(4.18)

where we recall from (4.17) that we have to identify $T_\alpha = \frac{1}{2} T_\alpha^{0\text{IBB}}$. Note that this expression agrees with the weak coupling result (2.22) if we drop the correction term $Q_\alpha$ containing the flux. It will be the task of the final subsection to also reproduce this correction.

Let us conclude this section by noting that the expression (4.20) can also be directly inferred from an M-theory kinetic potential $\tilde{K}$. It was shown in [37] that for an elliptic fibration it takes the form

\begin{align}
\tilde{K}^M &= \log \left( \frac{1}{12} RL^\alpha L^\beta L^\gamma K_{\alpha\beta\gamma} - \frac{1}{8} \xi^i \xi^j C^\alpha_{ij} L^\beta L^\gamma K_{\alpha\beta\gamma} + \ldots \right).
\end{align}

(4.21)
and can be obtained from a Kähler potential given by $K^M = -3 \log V$ via a Legendre transform. If one Taylor expands (4.21) around the F-theory point in moduli space with small $\xi_i$ one finds

$$\tilde{K}^M = \log \left( \frac{1}{12} L^\alpha L^\beta L^\gamma K_{\alpha\beta\gamma} \right) + \log(R) - \frac{C_{ij}^0 K_{\alpha\beta\gamma} L^\beta L^\gamma}{g K_{\alpha\beta\gamma} R L^\alpha L^\beta L^\gamma} \xi_i \xi_j.$$ (4.22)

with $K_{\alpha\beta\gamma}$ the intersection numbers (4.14). Comparing this form with the general expression (2.18) of a three-dimensional kinetic potential one confirms the identification (4.20) of the classical gauge coupling function.

4.3 On dimensional reduction with fluxes and warp factor

In this subsection we discuss the dimensional reduction of M-theory with a warp factor and background four-form fluxes $G_4$. Our main focus will be on the modifications arising in the reduction of the M-theory three-form. Our results will extend the discussion in [3].

Let us now perform the reduction including the warp factor. For simplicity we will not include higher curvature corrections and mobile M2-branes in the supergravity action (3.1). We will focus on the terms involving $G_4$ only, i.e. the kinetic terms and the Chern-Simons term. For the M-theory three-form $C_3$ itself we make the reduction Ansatz

$$C_3 = A^A \wedge \tilde{\omega}_A + \beta(M^\Sigma),$$ (4.23)

where $\tilde{\omega}_A$ are two-forms and $\beta$ is a three-form on $\hat{Y}_4$. The fluctuations are parameterized by three-dimensional vectors $A^A$ and scalars $M^\Sigma$, which change the geometry of $\hat{Y}_4$. To restrict to the case of massless vectors $A^A$ we demand in the following

$$d\tilde{\omega}_A = 0.$$ (4.24)

We introduce the three-forms

$$\beta_\Sigma = \frac{\partial \beta}{\partial M^\Sigma}.$$ (4.25)

The three-form $\beta$ is only patchwise defined, since we demand that in cohomology $d_8 \beta$ encodes the topologically non-trivial background flux $G_4$. This yields the field strength

$$G_4 = F^A \wedge \tilde{\omega}_A + dM^\Sigma \wedge \beta_\Sigma + G_4.$$ (4.26)

On next inserts the expressions (4.23) and (4.26) in the 11d supergravity action

$$S^{(11)}_{G_4} = 2\pi \int \frac{1}{4} G_4 \wedge * G_4 + \frac{1}{12} C_3 \wedge G_4 \wedge G_4 ,$$ (4.27)

using the warped metric (3.3). In order to bring the Einstein-Hilbert term into the standard 3d from one has to perform a Weyl rescaling with the warped volume

$$V_w = \int_{\hat{Y}_4} e^{3A/2} J \wedge J \wedge J \wedge J.$$ (4.28)

23
As a result one finds the 3d action\footnote{Note that the reduction of the Chern-Simons term is complicated by the fact that the M-theory potential $C_3$ appears without derivatives. We suppress terms of the form $\beta \wedge \partial_M \beta$, which are manifestly not gauge invariant. Terms of this type appear in Chern-Simons couplings for D-branes and it would be interesting to interpret them. These terms can be computed explicitly in our example and vanish for the derivatives w.r.t. the $M^2$ we study.}

\[
S_{G_4}^{(3)} = 2\pi \int_{M_3} G_{AB} F^A \wedge * F^B + \frac{d}{dM} dM^\Sigma \wedge * dM^\Lambda + V_w * 1 \\
+ \frac{1}{4} \Theta_{AB} A^A \wedge F^B + d_{A\Sigma\Lambda} (M^2 dM^\Lambda \wedge F^A) .
\]

We discuss the various terms appearing in this action in turn. Firstly, there is the kinetic term for the vectors $A^A$ with coupling

\[
G_{AB}^w = \frac{V_w}{4} \int_{Y_4} e^{3A/2} \tilde{\omega}_A \wedge \tilde{\omega}_B .
\]

Note that in contrast to (3.13) a warp factor appears in the integral. By solving the warp factor equation (3.5) we will later show that this induces a flux correction to the gauge coupling function.

The terms in the second line of (4.29) arise from the reduction of the 11d Chern-Simons coupling. The term proportional to $\Theta_{AB}$ is a three-dimensional Chern-Simons term with constant coefficient $\Theta_{AB} = \int_{Y_4} G_4 \wedge \tilde{\omega}_A \wedge \tilde{\omega}_B$. Depending on the index structure this Chern-Simons term either induces a gauging for non-trivial $\Theta_{i\alpha}$ in the dual 4d F-theory compactification \cite{4, 37}, or for $\Theta_{ij}$ generated at one loop by the four-dimensional chiral matter \cite{11}. Finally, the last term in (4.29) contains the coupling

\[
d_{A\Sigma\Lambda} = -\frac{1}{2} \int_{Y_4} \tilde{\omega}_A \wedge \beta_\Sigma \wedge \beta_\Lambda .
\]

We will later show that coupling induces a flux correction to the imaginary part of the F-theory gauge coupling function.

\subsection{4.4 Calculation of corrections to the gauge kinetic function}

Finally we are well equipped in order to derive the correction to the gauge kinetic function induced by a non-trivial background flux $G_4$. We will show that these corrections match in the weak coupling limit the well-known corrections to the gauge kinetic function due to D7-brane flux.
The basic idea to compute the corrections to the real part of the gauge coupling function \( (4.20) \) is to derive the gravitational back-reaction of the fluxes on the warp factor in M-theory via \( (3.5) \). This computation requires an explicit knowledge of the metric on the M-/F-theory fourfold \( \hat{Y}_4 \).

We describe the elliptic fourfold \( \hat{Y}_4 \rightarrow Y_4 \) with a resolved SU\((k)\) singularity in the elliptic fibration locally in the vicinity of the resolved singularity by the local geometry constructed in section 3.3.

\[
\mathcal{Y}_4 = S_b \times T N^\infty_k .
\] (4.33)

Here \( S_b \) is that divisor in the base \( B_3 \) of the elliptic fibration \( Y_4 \) with the SU\((k)\)-fibre singularity. \( T N^\infty_k \) is the periodic chain of multi-center Taub-NUT spaces with metric \( (3.17) \), that locally describes the normal space in \( \hat{Y}_4 \) to the resolved singularity over \( S_b \). As discussed in section 3.3, the metric on \( T N^\infty_k \) is known and governed by the function \( V = 1 + \sum_{I=1}^{k} V_I \) and the gauge connection \( U \) of \( (3.22) \) respectively \( (3.24) \).

In a brane picture in Type IIA and IIB or F-theory, the compactification of M-theory on \( Y_4 \) describes the Coulomb branch with U(1)\(^k\) gauge symmetry of the 3-dimensional gauge theory from \( k \) parallel spacetime-filling 6-branes or T-dual, fluxed \( k \) 7-branes wrapping \( S_b \times M_3 \) respectively \( S_b \times S^1 \times M_3 \) where \( S^1 \) denotes the circle in the basis of Taub-NUT \( T N^\infty_k \), \( \mathbb{R}^2 \times S^1 \). In this picture we also introduce the localized \( G_4 \)-flux in M-theory. This flux is identified with two-form flux \( \hat{F}^I \) of the \( I \)-th 6-brane on \( S_b \) or its T-dual 7-brane that is valued in the U(1) gauge group of the corresponding brane. It can be embedded into the Cartan subalgebra of the enhanced gauge group U(1)\(\times\)SU\((k)\) by defining new fluxes \( F^0 \) and \( F^i \), \( i = 1, \ldots, k - 1 \), as

\[
\hat{F}^m = F^0 + F^m - F^{m-1}, \quad \hat{F}^1 = F^0 + F^1, \quad \hat{F}^k = F^0 - F^{k-1},
\] (4.34)

where \( m = 2, \ldots, k - 1 \). The flux on \( \mathcal{Y}_4 \) is thus of the form

\[
G_4 = \hat{F}^I \wedge \Omega^\infty_I = F^i \wedge \omega^\infty_i + F^0 \wedge \sum_j \Omega^\infty_j,
\] (4.35)

where the second equality can be checked easily using \((4.34)\) and where \( I = 1, \ldots, k \) and \( i = 1, \ldots, k - 1 \). Recall that \( \omega^\infty_i = \Omega^\infty_i - \Omega^\infty_{i+1} \) are two-forms on \( T N^\infty_k \) which have been introduced already in \((3.28)\), and satisfy \( \int_{T N^\infty_k} \omega^\infty_i \wedge \omega^\infty_j = -C_{ij} \). Note that these forms should be identified with the blow-up forms \( \omega_i \) appearing in \((4.5)\) in the global embedding. Note that the two-form in the expansion with \( F^0 \) is trivial in cohomology in \( T N^\infty_k \), which matches the fact that the corresponding diagonal U(1) in the enhancement to gauge group U\((k)\) is massive and integrated out in the effective theory.

\(^{17}\) We focus here on SU\((k)\)-singularities only in co-dimension 1 in \( B_3 \), i.e. \( S_b \) is the full internal world-volume of the wrapped branes in a D-brane picture.

\(^{18}\) Note that the flux on the 7-brane is T-dual to the separation of 6-branes on \( S^1 \), i.e. has one leg on \( M_3 \) and one leg on \( S^1 \). It breaks U\((k)\) \(\rightarrow\) U\((1)^k\) and is not to be confused with the fluxes \( F^i \) introduced next in \((4.35)\).
4.4.1 Corrections to the real part of the gauge coupling function

We first calculate the correction to the real part of the gauge coupling function from the back-reaction of the \( G_4 \)-flux (4.35) on the warp factor. We find this corrected warp factor analytically for the full metric (3.17) on the local geometry \( Y_4 \) with fluxes \( G_4 \). Qualitatively, the corrected warp factor then modifies all integrals over the internal space \( \hat{Y}_4 \), in particular (4.12), and thus corrects the gauge kinetic function.

The warp factor equation (3.5) on \( Y_4 \) is given by

\[
\Delta_{Y_4} e^{3A/2} = *_{Y_4} (1/2 G_4 \wedge G_4),
\]

where on the right hand side we have only included the background flux \( G_4 \) and dropped the remaining terms in (3.5). In general the precise expression of the the two-forms \( \hat{F}^I \) on \( S_b \) will induce a non-trivial behaviour of the warp factor on \( S_b \). However, for simplicity we will neglect the non-trivial profile of \( \hat{F}^I \) on \( S_b \) by averaging over \( S_b \) as

\[
\langle \hat{F}^I \wedge \hat{F}^J \rangle_{S_b} = \delta^{IJ} \frac{1}{V_{S_b}} \int_{S_b} \hat{F}^I \wedge \hat{F}^I = \delta^{IJ} \frac{n^I}{V_{S_b}},
\]

where \( V_{S_b} = \frac{1}{2} \int_{S_b} J \wedge J \) for \( J \) denoting the Kähler form on \( S_b \). Note that we additionally assumed that the off-diagonal elements \( I \neq J \) vanish identically. In the brane picture the numbers \( n^I \) are then related to the instanton numbers on \( S_b \) in the \( U(1) \) of the \( I \)-th brane, respectively, as discussed below (2.8). Similarly we average over the dependence of the warp factor \( e^{3A/2} \) on \( S_b \) by integrating the right hand side of the warp factor equation (4.36) over the \( S_b \). Then we obtain an equation between four-forms on \( T N^\infty_k \) reading

\[
d \star_4 d e^{3A/2} = \frac{n^I}{2V_{S_b}} \Omega^\infty_I \wedge \Omega^\infty_I,
\]

where \( d \) and \( \star_4 \) denote the exterior derivative respectively the Hodge star on \( T N^\infty_k \). In order to solve the warp factor equation (4.36) we first evaluate

\[
\Omega^\infty_I \wedge \Omega^\infty_J = \frac{2}{r_A^2} V d\left( \frac{V_I}{V} \right) \wedge (dt + U) \wedge \star_3 d\left( \frac{V_I}{V} \right) = - \frac{2}{r_A^2} V d\left( \frac{V_I}{V} \right) \wedge \star_4 d\left( \frac{V_I}{V} \right),
\]

where we used the relation \( \star_3 dU_I = -dV_I \) and \( \star_4 dV_I = -(dt + U) \wedge \star_3 dV_I \) where the latter follows from (10.24) and the orientation on \( T N^\infty_k \) specified there. Then it is straightforward to show that (4.38) is solved by

\[
e^{3A/2} = 1 - \frac{n^I}{2r_A^2 V_{S_b}} \left( \frac{V_I^2}{V} - V_I \right),
\]

where we made use of \( \Delta_3 V_I \sim \delta(z - z_I) \) on the three-dimensional base of the Taub- NUT geometry \( T N^\infty_k \) as well as

\[
\frac{V_I}{V}(\hat{z} = \hat{z}_I, \hat{\rho} = 0) = \delta_{IJ}.
\]
The integration constant in (4.40) is chosen to be 1 to reproduce the unwarped case. With this convention the boundary behavior of the warp factor is analyzed as follows. First we introduce a cutoff $M$ “at infinity” in the $\hat{\rho}$-direction so that

$$V_1|_{\hat{\rho}=M} = 0, \quad e^{3A/2}|_{\hat{\rho}=M} = 1.$$  \quad (4.42)

Indeed, this behavior at large $\hat{\rho}$ is necessary to glue the local model $\mathcal{Y}_4$ into a compact Calabi-Yau fourfold $\hat{Y}_4$. Then we evaluate the warp factor on the locus $\hat{Z}_J := (\hat{\rho} = 0, \hat{z} = \hat{z}_J)$ of one monopole in $TN_k^\infty$. We obtain the warp factor

$$(e^{3A/2} - 1)|_{\hat{Z}_J} = \frac{n^I}{2r_A^2 V_{S_b}} \frac{V_I}{V} \left(1 + \sum_{K \neq I} V_K \right)|_{\hat{Z}_J} = \frac{n^I}{2r_A^2 V_{S_b}} \left[\delta_{IJ}(1 + \sum_{K \neq I} V_K) + V_I \sum_{K \neq I} \delta_{KJ} \right]|_{\hat{Z}_J},$$  \quad (4.43)

which is finite since the potentials $V_K$ are regular at $\hat{Z}_J$ for $K \neq J$. This result is expected since in the one-monopole case the warp factor at the position of the 6-brane should only see the localized flux $n^J$ on that brane and fall off to 1 at distances $\hat{\rho}$ far away from the brane. However, we see from (4.43) that in the case of $k$ monopoles, besides this back-reaction of the localized flux $n^J$ on the same 6-brane at $\hat{Z}_J$, the gravitational back-reaction of the localized fluxes $n^K$ from different branes, $K \neq J$, also affects the warp factor at $\hat{Z}_J$ with a suppression factor $V_K|_{\hat{Z}_J}$.

Now we are able to calculate the gauge coupling function. This is carried out by considering the kinetic term (4.12) corrected by the warp factor in the general metric ansatz (3.3). Following the same logic as for the flux $\mathcal{G}_4$ in (4.35) we include a three-dimensional field strength in the expansion of $G_4$ as

$$G_4 = \hat{F}^I \wedge \Omega^\infty_I + \mathcal{G}_4 = F^i \wedge \omega^\infty_i + F^0 \wedge \sum J \Omega^\infty_J + \mathcal{G}_4,$$  \quad (4.44)

where $I = 1, \ldots, k$ and $i = 1, \ldots, k - 1$. Then, the three-dimensional gauge fields are embedded into $U(k)$ as

$$\hat{F}^m = F^0 + F^m - F^{m-1}, \quad \hat{F}^1 = F^0 + F^1, \quad \hat{F}^k = F^0 - F^{k-1},$$  \quad (4.45)

for $m = 2, \ldots, k - 1$, which is completely analogous to (4.35). The three-dimensional kinetic term for $\hat{F}^I$ is evaluated in the warped background as in section 4.3 and contains the warped metric (4.30). Focusing on the warped metric in the local fourfold $\mathcal{Y}_4$ we obtain

$$G^w_{I,J} = \frac{V_{w}}{4} \int_{\mathcal{Y}_4} e^{3A/2} \Omega^\infty_I \wedge \star_{\mathcal{Y}_4} \Omega^\infty_J = -\frac{V_{w} V_{S_b}}{4} \int_{TN_k^\infty} e^{3A/2} \Omega^\infty_I \wedge \Omega^\infty_J$$  \quad (4.46)

which is the corrected version of (4.13). Here we used that the Hodge star on $\mathcal{Y}_4$ acts as $\frac{1}{2} \mathcal{J}^2 \star_4$ and in addition the anti-selfduality (3.27) of $\Omega^\infty_I$. Noting that the forms $\Omega^\infty_I$ are constant over $S_b$ we

\[\text{In general the precise linear combination of the two solutions to the homogeneous equation } d \star_4 d g = 0 \text{ we have to add has to be determined by global boundary conditions on } e^{3A/2}.\]
readily integrate out the Kähler form to obtain a volume factor $\mathcal{V}_S$. Then we read off the gauge coupling function $R f_{I,J}$ simply as the coefficient of the kinetic term $\hat{F}^I \wedge \ast \hat{F}^J$ in (2.20) from which we see that we have to take into account an additional factor of $-2R = -2v^0/\mathcal{V}_w$. In addition we note that the Type IIB volume $\text{Re} T_S = 2\mathcal{V}_S$. Since the warp factor only appears linearly in (4.46) we insert the solution (4.40) for $e^{3A/2}$ to obtain

$$\text{Re} f_{I,J} \equiv -2\frac{v_0^0}{\mathcal{V}_w} G^w_{I,J} = 1 - \frac{v_0^0}{4r_A^2} n^K \int_{T_N^\infty} \left( \frac{V_K^2}{V} - V_K \right) \Omega_I^\infty \wedge \Omega_J^\infty,$$  \hspace{1cm} (4.47)$$

where we used the property (3.27) of the $\Omega_I$ on the first term to obtain the proportionality to $\delta_{IJ}$.

We immediately recognize the first term in (4.47) as the leading part of the gauge coupling function (2.15) on the Coulomb branch of the three-dimensional gauge theory. The second term in (4.47) already resembles the real part of the flux induced contribution $\text{Re} f_{I,J}^{\text{flux}}$ to the gauge coupling (2.6) respectively (2.15). We obtain the final expression for the gauge coupling function by evaluating the integral in (4.47) over the local geometry $TN^\infty_k$. However, instead of evaluating this in general, which is hard due to complicated integrand, we focus on the weak coupling result $g_s \sim 0$. For small $g_s$, as discussed rigorously in appendix C we can use the localization property

$$\Omega_I^\infty \wedge \Omega_J^\infty \rightarrow -\frac{1}{2\pi} \delta_{IJ} \delta(\hat{\rho}) \delta(\hat{z} - \hat{z}_I) d\hat{t} \wedge d\hat{\rho} \wedge d\varphi \wedge d\hat{z}$$  \hspace{1cm} (4.48)$$
in local coordinates $\hat{z}$ on the quotient $\mathbb{R}/\mathbb{Z} = S^1$. Then we evaluate the integral in (4.47) as

$$\text{Re} f_{I,J}^{\text{flux}} = -\frac{v_0^0}{4r_A^2} n^K \int_{T_N^\infty} \left( \frac{V_K^2}{V} - V_K \right) \Omega_I^\infty \wedge \Omega_J^\infty = \frac{1}{2} \delta_{IJ} v_0 \mathcal{V}_S (e^{3A/2} - 1) \bigg|_{\hat{z}_I}$$

$$= \frac{1}{8} g_s^{-1} \delta_{IJ} \left[ n^I_{IIB} + \sum_{K \neq I} (n^I_{IIB} + n^K_{IIB}) V_K \right](\hat{z}_I),$$  \hspace{1cm} (4.49)$$

where we used the evaluation of the warp factor (4.43) in the last equality and the basic relation $\frac{v_0^0}{r_A^2} = g_s^{-1}$ following from (4.8). Moreover the remaining integrals over $\hat{t}$ and $\varphi$ yield a factor 1 respectively $2\pi$. In addition we identified the flux number $n^I_{IIB} = 2n^K$ due to the orientifolding as noted already in (4.7).

We note, that in the result (2.15) for $\text{Re} f_{I,J}^{\text{flux}}$ that we obtained by dimensional reduction of the D7-brane effective action to three dimensions we only see the first term in (4.49) proportional to $n^I$. However, this is perfectly consistent recalling that $V_K \sim g_s$, cf. (3.22), which reveals the corrections proportional to $V_K$ in (4.49) as one loop corrections to the gauge coupling $f_{I,J}$. These are not visible in the string-tree-level D7-brane effective action obtained in section 2. More precisely the corrections are suppressed by $g_s$ and the separation $|\hat{z}_I - \hat{z}_K|$ between the branes as

$$V_K |\hat{z}_I = \frac{g_s}{4\pi} \left( \frac{1}{|\hat{z}_I - \hat{z}_K|} - 2\gamma - \psi(1 - |\hat{z}_I - \hat{z}_K|) - \psi(1 + |\hat{z}_I - \hat{z}_K|) \right),$$  \hspace{1cm} (4.50)$$

where we used (3.19) and introduced Euler’s constant $\gamma = 0.577216\ldots$ as well as $\psi(x)$ denoting the digamma function. The function $\psi(x)$ is well-defined except at $x \in \{0, -1, -2, \ldots\}$ and since
0 < |\hat{z}_I - \hat{z}_K| < 1, the composition \psi(1 - |\hat{z}_I - \hat{z}_K|) is finite. We note, however, that this implies that the corrections in (4.49) diverge as \frac{1}{|\hat{z}_I - \hat{z}_K|} in the case that the branes move on top of each other \hat{z}_K = \hat{z}_I. Intuitively this is clear since the integral (4.38) calculates formally the self-energy \int \phi dV of charges in three dimensions by identifying e^{3A/2} with the electric potential \phi and \Omega_I^\infty \wedge \Omega_J^\infty with the charge density \rho. Thus, by using the approximation (4.48) we formally calculate the self-energy of a point charge, that is infinite. However, the self-energy i.e. the integral (4.38) is regularized in M-theory by the smooth forms \Omega_I^\infty that smear out the charge density \rho.

4.4.2 Corrections to the imaginary part of the gauge coupling function

In this final section we calculate the flux-induced corrections to the imaginary part of the gauge coupling function. These corrections originate from the 11-dimensional Chern-Simons term \mathcal{C}_3 \wedge G_4 \wedge G_4 with an altered reduction ansatz (4.23) in the presence of a non-trivial flux \mathcal{G}_4. Following the logic of section 4.3 the dependence of the new three-form \beta(M^2) on the moduli M^2 of the compactification geometry is crucial to obtain the coupling \text{d} A_{\Sigma L} in (4.29). It is a Chern-Simons term in three dimensions and is identified with the reduction of the topological term Tr(\text{F} \wedge \text{F}) of the four-dimensional gauge theory to three dimensions in (2.20). We demonstrate this identification and the reproduction of the right flux correction to the imaginary part of the gauge coupling and obtain a perfect match in the weak coupling limit where we reproduce the flux correction \sim n^I in (2.15) to the D7-brane gauge coupling.

First we have to identify the appropriate form for the three-form \beta that we defined in (4.26) as the Chern-Simons form of the flux \mathcal{G}_4 = \delta \beta. From the expansion (4.35) and recalling \Omega_I^\infty = \delta A_I we make the ansatz

\[ \beta = \mathcal{F}^I \wedge \eta_I(\varphi_0, \hat{z}), \]

where we indicated the moduli dependence of \beta on the angle \varphi_0 and the position of the k periodic monopoles \hat{z} = (\hat{z}_I) through the one-forms \eta_I. From this it follows that the relevant terms in the three-dimensional action (4.29) take the form

\[ S_{G_4}^{(3)} \supset 2\pi \int_{M_3} (d_{ICbK} C_0 d\hat{z}_K \wedge \hat{F}^I + d_{IKC_0} \hat{z}_K dC_0 \wedge \hat{F}^I), \]

where we identified the RR-axion \frac{k}{2\pi} \varphi_0 = C_0 as before in the definition of the axio-dilaton (3.35) and set \tilde{\omega}_I = \Omega_I as in (4.44). Then the coupling \text{d} ICbK is given by

\[ \text{d} ICbK = -\frac{1}{4} \int_{\hat{Y}_4} \Omega_I^\infty \wedge \frac{\partial \beta}{\partial C_0} \wedge \frac{\partial \beta}{\partial \hat{z}_K} = -\frac{1}{4} n^I \delta^{JL} \int_{TN_K^\infty} \Omega_I^\infty \wedge \frac{\partial \Omega_J^\infty}{\partial C_0} \wedge \frac{\partial \eta_L}{\partial \hat{z}_K}. \]

Here we replaced the compact fourfold \hat{Y}_4 by our local geometry \hat{Y}_4 that by its direct product structure \hat{Y}_4 = S_b \times TN_K^\infty allowed us to pull out the integral of the flux over \hat{S}_b. We note that the

\[ \text{In contrast the Poisson re-summed } V_K \text{ in (3.22) diverges at } \rho = 0, \text{ though, since Poisson re-summation breaks down for } \rho = 0 \text{ and (3.22) is not valid at } \rho = 0. \]
two terms in (4.52) are equal, up to a term proportional to \(d(dI_{C_0})C_0 \hat{z}^K \hat{F}^I\), by partial integration and by virtue of the antisymmetry of \(dI_{C_0}\) in the last two indices. In general this can yield further subleading correction to \(\text{Im}\, f_{IJ}\) that we ignore in the following.

In order to show that (4.52) reproduces the flux correction to the imaginary part of the gauge coupling we have to evaluate (4.53). This is a lengthy but straightforward calculation. Omitting the details we obtain up to exact forms the result

\[
\Omega_I \wedge \partial_{C_0} \eta_L \wedge \partial_{\hat{z}_K} \eta_J = \frac{1}{2} \left[ -\frac{V_I}{V} \left( \frac{V_J}{V} - \delta_{KJ} \right) \Omega_L \wedge \Omega_J - \frac{V_J}{V} \frac{V_I}{V} \Omega_I \wedge \Omega_J + 2 \frac{V_K}{V} \left( \frac{V_I}{V} - \delta_{KJ} \right) \sum_S \Omega_I \wedge \Omega_S \right]
\]

\[+ \frac{r_B}{r_A} \frac{V_L}{V} \left( \frac{V_K}{V} - \delta_{KJ} \right) \left( \Delta V_I - \frac{V_I}{V} \Delta V \right) dt \wedge \hat{\rho} d\hat{\rho} \wedge d\varphi \wedge d\hat{z}, \quad (4.54)\]

where we omitted the superscript \(^\infty\) for brevity. In the derivation we first recall from (3.25) that \(U = r_A C_0 d\hat{z}\) and evaluate \(\partial_{C_0} \eta_L = \frac{V_L}{V} d\hat{z}\) that follows from (3.26). Thus we can drop all terms in \(\Omega_I^\infty \wedge \partial_{\hat{z}_K} \eta_I\) which are proportional to \(d\hat{z}\). Next we plug in the definitions for these forms and formally calculate the derivatives in local coordinates. We note that due to the dependence of \(V_I\) and \(U_I\) in (3.22), (3.24) on only the combination \((\hat{z} - \hat{z}_I)\) we can write

\[
\partial_{\hat{z}_K} V_I = -\delta_{IK} \partial_{\hat{z}} V_I, \quad \partial_{\hat{z}_K} U_I = -\delta_{IK} \partial_{\hat{z}} U_I. \quad (4.55)
\]

Next we write the relation \(*_3 dU_I = -dV_I\) in local coordinates for the \(\varphi\)-component \(U_I^\varphi\) of \(U_I\) as

\[
r_B \hat{\rho} \partial_{\hat{\rho}} V_I = \partial_{\hat{z}} U_I^\varphi, \quad r_B \hat{\rho} \partial_{\hat{\rho}} V_I = -\partial_{\hat{\varphi}} U_I^\varphi, \quad (4.56)
\]

which is of course in perfect agreement with (3.22), (3.24), to recast every term in (4.54) as a function of derivatives of \(V_I\) and \(V\) multiplying the top-form \(dt \wedge \hat{\rho} \hat{\rho} \wedge d\varphi \wedge d\hat{z}\). Then we perform partial integrations, ignoring boundary terms, until every single partial derivative acts only on fractions \(V\). Comparing to (4.39) in local coordinates,

\[
\Omega_I^\infty \wedge \Omega_J^\infty = -\frac{2r_B V}{r_A} \left[ \partial_{\hat{\rho}} \left( \frac{V_I}{V} \right) \partial_{\hat{\rho}} \left( \frac{V_J}{V} \right) + \partial_{\hat{z}} \left( \frac{V_I}{V} \right) \partial_{\hat{z}} \left( \frac{V_J}{V} \right) \right] dt \wedge \hat{\rho} \hat{\rho} \wedge d\varphi \wedge d\hat{z} \quad (4.57)
\]

where we used \(*_4 d\hat{\rho} = -\hat{\rho} (dt + U) \wedge d\varphi \wedge d\hat{z}\) and \(*_4 d\hat{z} = -\hat{\rho} (dt + U) \wedge d\hat{\rho} \wedge d\varphi\) exploiting the vierbein formalism (D.24), allows us to obtain the first two terms in (4.54). However, partial integration in addition produces a term

\[
\frac{r_B}{r_A} \frac{V_L}{V} L \left( \frac{V_K}{V} - \delta_{KJ} \right) \Delta_3 \left( \frac{V_I}{V} \right) dt \wedge \hat{\rho} \hat{\rho} \wedge d\varphi \wedge d\hat{z}. \quad (4.58)
\]

Applying \(\Delta_3 = *_3 d *_3 d\) we obtain the last two terms in (4.54) and the mixed terms with derivatives acting on different terms can be rewritten using

\[
\sum_S \Omega_I^\infty \wedge \Omega_S^\infty = \frac{2}{r_A^2} dV \wedge (dt + U) \wedge *_3 d \left( \frac{V_I}{V} \right) = -\frac{2}{r_A^2} dV \wedge *_4 d \left( \frac{V_I}{V} \right) \quad (4.59)
\]

and \(*_4 dV_I = -(dt + U) \wedge *_3 dV_I\) yielding the third term in (4.54).
With the result (4.54) we can now evaluate the coupling \(d_{IC_0K}\) in (4.53). According to (2.15), (2.20) and (4.52) it is related to the imaginary part of the flux correction to the the gauge coupling function in the Coulomb branch of \(U(k)\) as \(\frac{1}{2} \text{Im} f_{IJ}^{\text{flux}}\) if we identify

\[
\tilde{z}_I = \zeta^I = \frac{\xi^I}{r_B}.
\]  

(4.60)

Again we focus on the extraction of the weak coupling behavior \(g_s \sim 0\) where the integral (4.53) for \(d_{IC_0K}\) can be evaluated explicitly. We recall first the limit (4.48) and note that the potentials \(V, V_I\) obey the Poisson equation (3.23). As in the evaluation of the real part (4.49) we then replace all four-forms in (4.54) by delta-functions and by integration we just have to evaluate the different pre-factors at points. Then only the second and third term in (4.54) contribute yielding

\[
n^J \delta_{JL} \left( \frac{V_J}{V} - \delta_{KK} \right) \sum S \Omega_I \wedge \Omega_S \rightarrow \delta_{IK} \left[ n^I + n^I \sum S \not= I V_S |Z_I| + (\delta_{IK} - 1) n^I V_K |Z_I| \right],
\]  

(4.61)

where it is important for the latter formula to separately consider the cases \(I = K\) and \(I \neq K\) and to split the sum over \(J\) into \(J \neq I\) and \(J = I\). Thus we obtain the imaginary part of the flux correction to \(f_{IJ}\) as

\[
\text{Im} f_{IJ}^{\text{flux}} \simeq 2d_{IC_0K} = -\frac{1}{8} C_0 \left[ \delta_{IJ} \left( 3n_{IIB}^I + 2 \sum K \not= I n_{IIB}^I V_K |Z_I| \right) + 2n_{IIB}^I V_K |Z_I| \delta_{IJ} - 1 \right],
\]  

(4.62)

where we used as before the definition \(\tilde{Z}_I := (\hat{\rho} = 0, \hat{z} = \hat{z}_I)\) and the relation \(n_{IIB}^K = 2n^K\). Note that the structure is similar to the real part however a precise matching requires to keep all terms, most importantly those related to the terms in (4.52) by partial integration\(^{21}\) in the reduced action (4.29). We emphasize that we not only obtain the expected flux correction to the imaginary part of \(f_{IJ}\) in (2.15) but also subleading corrections proportional to \(g_s\) via \(V_J \sim g_s\). These corrections are analogous to the to those of the real part in (4.49) and are accordingly identified as one-loop corrections that are absent in the strict weak coupling limit and in particular in the the tree-level result (2.15) of the D7-brane gauge coupling. Using the finite expression (4.50) for \(V_K |\tilde{Z}_I|\) we can predict some of this leading loop correction.

5 Conclusion

In this work we have studied corrections to the four-dimensional F-theory effective action induced by 7-brane fluxes. We have argued that this can be done via a three-dimensional M-theory compactification by comparing the result to a circle reduction of a genuine four-dimensional \(\mathcal{N} = 1\)  

\(^{21}\)In particular the factor 3 in (4.62) arises precisely by partial integration and should be cancelled by the omitted terms in (4.29) that are also obtained by partial integration.
supergravity theory. The 7-brane fluxes are lifted to M-theory four-form flux $G_4$. The crucial observation was that the $G_4$ flux then backreacts and requires a more general Kaluza-Klein Ansatz including a non-trivial warp factor and a modified three-form potential. The vacuum solutions are the warped Calabi-Yau fourfold backgrounds found in [12]. Thus, the determination of the effective action requires to perform a warped Kaluza-Klein reduction and results in new terms depending on the background fluxes. The warping and modified M-theory potential crucially depend on the circle direction decompactified in the M-theory to F-theory limit. This dependence induces additional terms interpreted as 7-brane flux corrections in four-dimensions.

To explicitly derive these corrections to the effective action it was necessary to solve the warp factor equation and give the explicit representatives of the harmonic forms on the internal geometry. Clearly, this requires a knowledge of the metric on the internal Calabi-Yau space and is very hard in general. Therefore, we have focused on local M-theory geometries, which yield D6-branes at weak coupling. More precisely, we considered M-theory on Taub-NUT spaces and made use of the explicit form of the Taub-NUT metric and its harmonic forms and potentials. For a stack of $k$ D6-branes the M-theory background is in fact the multi-centered Taub-NUT space $TN_k$. In order to compare with the F-theory action one has to effectively perform a T-duality along a circle to move from Type IIA to Type IIB string theory. Hence, it was necessary to introduce an infinite array $TN_k^\infty$ of multi-centered Taub-NUT spaces with period given by the circle radius. The metric and harmonic forms have been determined by using a Poisson resummation.

To determine the warp factor and M-theory three-form potential we have considered the local fourfold geometry $S_b \times TN_k^\infty$, where $S_b$ is a complex surface over which we averaged the solutions. Remarkably, it was sufficient to use some of the key properties of the $TN_k^\infty$ geometry to solve the warp factor equation and give a closed expression of the warp factor in terms of the fluxes and the Taub-NUT potentials determining its metric. Using this solution in the warped Kaluza-Klein reduction we were able to show that the real part of the gauge-coupling function receives a correction quadratic in the flux. In F-theory this corresponds to a flux correction to the real part of the gauge coupling function of a space-time filling 7-brane wrapped on $S_b$. At weak string coupling this additional term precisely yields the flux square correction linear in the inverse Type IIB string coupling $\text{Im} \tau = 1/g_s$, which can be derived from the Dirac-Born-Infeld action of a D7-brane. To derive the correction to the imaginary part of the gauge coupling function we had to dimensionally reduce the M-theory Chern-Simons term taking into account a back-reacted three-form potential. On our local geometry it was given by $F^I \wedge \eta_I$, where $F^I$ is a two-from flux on $S_b$ and $\eta_I$ are the $k$ fundamental one-forms on $TN_k^\infty$. While the internal derivative of this correction gives the background flux, its external derivatives induce the flux-square correction to the 7-brane gauge coupling function. Let us note that the classical part of the imaginary part of the 7-brane gauge coupling arises not from the Chern-Simons term of M-theory, but rather from the kinetic term of $G_4$ as a kinetic mixing of U(1) field strengths. This is due to the fact, that the imaginary part of the Kähler modulus $T_\alpha$ is in fact arising as a three-dimensional vector in the M-theory reduction.
to three dimensions.

In addition to matching the known weak coupling result from the D7-brane action we have shown that there extra terms going with a factor $g_s$. It would be desirable to analyze these in more detail. This would involve solving the integrals over the warp factor corrected gauge coupling without employing a sharp localization of $\Omega_I \wedge \Omega_J$. While this is beyond the scope of this work, one expects that this can be also done in a closed form using techniques known from the study of one-loop integrals.\footnote{See, e.g. \cite{50,51}, for some recent progress in this direction.} A further extension is to include the higher curvature terms in the equation determining the warp factor. These should yield the missing higher curvature terms on the D7-brane action and further complete the four-dimensional effective action of F-theory. This is likely doable in our local geometry where the solution to the warp factor equations can be determined using the explicit metric.

Let us stress that the basic idea presented in this work is much wider applicable. We argued that the M-theory to F-theory lift is subtle, since non-trivial profiles of the fields in the growing extra dimension have to be incorporated with care. We have shown an example computation for the local geometry near a stack of branes. To perform the evaluation in more general backgrounds appears significantly harder. However, let us point out that in most of our computations only some basic properties of the defining functions were necessary. One expects that one can develop a formalism which does not make use of the metric but rather employs appropriate $\mathcal{N} = 1$ periods of the Calabi-Yau fourfold depending on the complex structure moduli. In the most optimistic scenario, one can use these in a complete warped Kaluza-Klein reduction inducing geometric corrections to all $\mathcal{N} = 1$ characteristic functions determining the four-dimensional effective action.

Acknowledgements

We would like to thank Yi-Zen Chu, Mirjam Cvetič, Monica Guica, Babak Haghighat, Jim Halverson, Albrecht Klemm, Hans-Peter Nilles, Eran Palti, Daniel Park, Raffaele Savelli, Stephan Stieberger, Wati Taylor, and Timo Weigand for interesting discussions. TG likes to thank the Bethe Center Bonn, UPenn, KITPC, and MIT for hospitality and support during the preparation of this work. DK acknowledges hospitality of the Bethe Center Bonn. DK and MP are also grateful for the hospitality of the MPI Munich. The research of TG was supported by a research grant of the Max Planck Society. The research of DK was supported by the ‘Deutsche Telekom Stiftung’ and DOE under grant DE- FG02-95ER40893-A0. The work of MP was supported by the graduate school BCGS, the German National Academic Foundation, and the ‘Deutsche Telekom Stiftung’.
A Conventions of $\mathcal{N} = 1$ actions and dimensionful constants

For reference in the main text, let us briefly introduce our conventions for the four-dimensional $\mathcal{N} = 1$ effective action used in this work. The action takes the general form

$$S_{\mathcal{N}=1}^{(4)} = \frac{1}{\kappa_4^2} \int_{\mathbb{R}^{(3,1)}} \left( -\frac{1}{2} R + 1 - K_{MN} \nabla M^M \wedge \nabla \bar{M}^N - \frac{1}{2} \text{Re} f_{AB} F^A \wedge * F^B - \frac{1}{2} \text{Im} f_{AB} F^A \wedge F^B - * V \right).$$  \hspace{1cm} (A.1)

Here we introduced the four-dimensional gravitational constant, the four-dimensional Ricci-scalar $R$, a number of chiral superfields with scalar components $M^N$ that are the coordinates of the Kähler manifold of scalar fields with Kähler metric $K_{MN} = \frac{\partial^2 K}{\partial M^M \partial \bar{M}^N}$ and a number of vectormultiplets with field strengths $F^A$ with gauge kinetic function $f_{AB}$ of the chiral multiplets $M^M$. By $*$ we denote the four-dimensional Hodge star operator and $V$ is the scalar potential that consists of the F-term and D-term scalar potential, $V = V_F + V_D$ for

$$V_F = e^K (K^{MN} D_M W D_N \bar{W} - 3|W|^2), \hspace{1cm} V_D = \frac{1}{2} \text{Re} f^{-1} AB D_A D_B. \hspace{1cm} (A.2)$$

We introduced the superpotential $W$ that is a holomorphic function of the chiral superfields $M^M$ as well as the $\mathcal{N} = 1$ covariant derivative $D_M = \partial_M + K_M$

In the course of deriving this action from String/M-/F-theory it is furthermore useful to introduce our conventions for the String, ten- and eleven-dimensional Planck scale as well as their relation to the D7-brane tension and the four- and three-dimensional Planck scale. These conventions were originally used in $[37]$

$$\kappa_{11}^{-2} = \kappa_{10}^{-2} = \kappa_4^{-2} = \kappa_3^{-2} = 2\pi = \mu_7 = T_7. \hspace{1cm} (A.3)$$

B Linear multiplets and gauge couplings

Let us begin with the dualization of the chiral multiplets with complex scalars $T_\alpha$ into linear multiplets. More precisely, if $\text{Im} T_\alpha$ has a shift symmetry it can be dualized into a two-form $C_2^\alpha$, which together with $\text{Re} T_\alpha$ forms the bosonic components of a linear multiplet $[52]$. To actually perform the dualization we collect all terms involving $\text{Im} T_\alpha$. First we turn to the kinetic terms for the $T_\alpha$. These are determined by the four-dimensional Kähler potential $[32]$

$$K = - \log(\tau - \bar{\tau}) - 2 \log(\mathcal{V}(T + \bar{T})), \hspace{1cm} (B.1)$$

where $\mathcal{V}$ is the volume of the Calabi-Yau threefold $Z_3$, which considered as a function of $T_\alpha$ is independent of $\tau, \bar{\tau}$. The metric for all complex scalars $M_I = (\tau, T_\alpha)$ is given by $K_{I,J} = \frac{\partial^2}{\partial M_I \partial M_J} K$. We note that the structure of $K$ at this order implies that there are no kinetic mixing terms between $T_\alpha$ and $\tau$. 

34
Next we note that in \(2.7\) the imaginary part \(\text{Im} \, T_\alpha\) also appears in front of the theta-angle term \(\text{Tr}(F \wedge F)\) in the non-Abelian gauge theory. In this case we perform a partial integration and write

\[
S_{\text{gauge, im}}^{(4)} = -\frac{2\pi}{8} \int_{\mathcal{M}_4} \delta^2_S \text{Im} \, T_\alpha \, \text{Tr}(F \wedge F) = \frac{2\pi}{8} \int_{\mathcal{M}_4} \delta^2_S d\text{Im} \, T_\alpha \wedge \omega_{\text{CS}} ,
\]

which holds up to a total derivative, and we have defined

\[
\omega_{\text{CS}} = A \wedge dA + \frac{2}{3} A \wedge A \wedge A .
\]

One can now eliminate \(G_\alpha = d\text{Im} \, T_\alpha\) in favor of its dual \(dC_2^\alpha\). We formally achieve this by adding the Lagrange multiplier

\[
S_{\text{Lag}}^{(4)} = \frac{2\pi}{8} \int_{\mathcal{M}_4} G_\alpha \wedge dC_2^\alpha .
\]

and eliminate \(G_\alpha\) by its equations of motion. First we evaluate the equations of motion yielding

\[
G_\beta = -\frac{1}{2} \tilde{K}^{\tau \gamma} T_\beta \wedge *H_3^{\alpha} , \quad H_3^{\alpha} = dC_2^\alpha + \frac{1}{8} \delta^2_S \omega_{\text{CS}} ,
\]

where we have introduced the modified field strength \(H_3^{\alpha}\). Then we rewrite the relevant effective action including \(2.2\), \(B.2\) and \(B.4\) in terms of \(G_\alpha = d\text{Im} \, T_\alpha\) and eliminate \(G_\alpha\) by using \(B.5\). Inserting this into the above action we obtain

\[
S_{C_2,F}^{(4)} = 2\pi \int_{\mathcal{M}_4} \tilde{K}^{\alpha \beta} H_3^{\alpha} \wedge *H_3^{\beta} + \frac{1}{4} \tilde{K}^{\alpha \beta} d\text{Re} T_\alpha \wedge *d\text{Re} T_\beta - \tilde{K}_{\tau \bar{\tau}} d\tau \wedge *d\bar{\tau} - \frac{1}{2} \text{Im} f^{\text{flux}}_{AB} F^A \wedge F^B - \frac{1}{2} \text{Ref} f_{AB} F^A \wedge *F^B .
\]

In order to bring the kinetic term for \(C_2^\alpha\) in the canonical form we have in addition used the Legendre-transformed dual Kähler potential of \(B.1\) given by \(B.2\)

\[
\tilde{K}(\tau | L) = K + L^\alpha \text{Re} T_\alpha = \log(\frac{1}{6} L^\alpha L^\beta L^\gamma K_{\alpha \beta \gamma}) - \log(\tau - \bar{\tau})
\]

for the Legendre-transformed dual variables

\[
L^\alpha = -\frac{\partial K}{\partial \text{Re} T_\alpha} = \frac{v^\alpha}{V} , \quad \text{Re} T_\alpha = \frac{\partial \tilde{K}}{\partial L^\alpha}
\]

that was defined in \(B.2\) to dualize the real part \(\text{Re} T_\alpha\) of the Kähler moduli to the scalar component of different linear multiplets.\(^{23}\) Essentially we exploited here the basic relation \(K_{T_\alpha T_\beta} = -\frac{1}{4} \tilde{K}^{\alpha \beta}\), which is an immediate consequence of the general relations of Legendre transformations \(B.8\).

We conclude the discussion of the four-dimensional effective action by noting that \(C_2^\alpha\) has to also transform under a non-Abelian gauge transformations \(A \rightarrow A + d\Lambda\) of the vector fields as \(C_2^\alpha \rightarrow C_2^\alpha - \frac{1}{8} \delta^2_S \text{Tr}(\Lambda F)\) to ensure invariance of \(H_3^{\alpha}\) introduced in \(B.3\). Furthermore, the field strength \(H_2^{\alpha}\) obeys the Bianchi identity \(dH_2^{\alpha} = \frac{1}{8} \delta^2_S \text{Tr}(F \wedge F)\).

\(^{23}\)This is not to be mixed up with the dualization of the imaginary part \(\text{Im} T_\alpha\) performed in this section. In particular, the two-forms \(D_2^{\alpha}\) forming the linear multiplet together with the \(L^\alpha\) are different from the two-forms \(C_2^{\alpha}\) defined in \(B.4\).
C Details of $TN_k$

In this appendix we review some details of the geometry of multi-center Taub-NUT space, $TN_k$. We start with the discussion of one monopole, $TN_1$. The metric is given as

$$ds^2_{TN} = \frac{1}{V} (dt + U)^2 + V d\vec{r}^2,$$  \hspace{1cm} (C.1)

where $t$ denotes a periodic coordinate on an $S^1$ and $\vec{r}=(x,y,z)$ three-dimensional Cartesian coordinates on $\mathbb{R}^3$. The circle is non-trivially fibred over $\mathbb{R}^3$. The function $V$ and the $S^1$-connection $U$ are related by

$$*_3 dU = \pm dV,$$  \hspace{1cm} (C.2)

where $*_3$ denotes the Hodge star operator on the base $\mathbb{R}^3$ with standard orientation. The $\pm$-sign will lead to a self-dual respectively anti-selfdual two-form $\Omega$ as introduced below in (C.6). Note that the closedness of $dU$ requires $V = 1 + V_1$ to be harmonic. (C.2) is solved by

$$V_1 = \frac{r_A}{4\pi|\vec{r}|}, \quad U = \pm \frac{r_A}{4\pi} \left( -1 + \frac{z}{|\vec{r}|} \right) \frac{x dy - y dx}{x^2 + y^2} = \pm \frac{r_A}{4\pi} \left( -1 + \frac{z}{|\vec{r}|} \right) d\varphi,$$  \hspace{1cm} (C.3)

where we have also introduced cylindrical coordinates with $|\vec{r}| = \sqrt{\rho^2 + z^2}$ for $\rho \in \mathbb{R}^+, \varphi \in [0, 2\pi]$, $z \in \mathbb{R}^+$. $r_A$ can be thought of as the charge of the monopole, but more importantly in our context is its interpretation as the circumference of the $S^1$-fibre at infinity, as discussed below. We note that the term $\mp d\varphi$ in $U$ is an integration constant, that is not fixed by the condition $*_3 dU = \pm dV_1$ but by the condition of smoothness of $U$, i.e. the absence of a Dirac string. Indeed, the one-form $U$ in (C.3) is only a local one-form representing the global connection of the Dirac monopole in a coordinate patch. To see this note the presence of the Dirac string, that is the locus where the local expression $U$ is not well-defined. In cylindrical coordinates since $d\varphi$ is not well-defined at $\rho = 0$, in order to have a well-defined one-form containing $d\varphi$ its pre-factor has to vanish on the locus $\rho = 0$. However the pre-factor of $d\varphi$ in the one-form $U$ in (C.3) vanishes only on the positive $z$-axis for the choice of integration constant $-1$ and $U$ is ill-defined on the negative $z$-axis. This is precisely the Dirac string. Thus, $U$ is only a local one-form well-defined only on the positive $z$-axis and one has to introduce at least one further patch and with another local one-form that is required to be well-defined on the negative $z$-axis. Thus, one introduces the two patches $U_\pm$ and the corresponding connections denoted $U_\pm$ reading

$$U_+ = \left\{ (r, \varphi, z) \mid 0 \leq z \right\} : \quad U^+ = \pm \frac{r_A}{4\pi} \left( -1 + \frac{z}{\sqrt{\rho^2 + z^2}} \right) d\varphi,$$

$$U_- = \left\{ (r, \varphi, z) \mid z \leq 0 \right\} : \quad U^- = \pm \frac{r_A}{4\pi} \left( 1 + \frac{z}{\sqrt{\rho^2 + z^2}} \right) d\varphi,$$  \hspace{1cm} (C.4)

---

24 Actually we consider fundamental solutions $V$ with $\Delta V_1 = \delta^3(\vec{r})$ in the distributional sense.

25 We note the spherical symmetry of the one-monopole configuration with $U = \pm \frac{r_A}{4\pi} (-1 + \cos \theta) d\varphi$ in spherical coordinates. We use cylinder coordinates to prepare for the discussion of appendix D

---

36
that differ only by the integration constant in the \( d\phi \)-component. They are related by the gauge transformation \( U^+ = U^+ + \frac{r_A}{2\pi} d\phi \). In particular we note that \( U^\pm \) vanishes precisely on the positive (negative) \( z \)-axis and thus can be glued together to form a smooth global gauge connection.

Consequently, in order for the metric to be gauge invariant, i.e. the term \( dt + U \) to be globally defined, the coordinate \( t \) has to compensate this gauge transformation and cannot be globally defined either. Thus we have to introduce two coordinates \( t^\pm \) on \( U^\pm \) that are related by the gauge transformation

\[
t^+ = t^- \pm \frac{r_A}{2\pi} \varphi \quad \Rightarrow \quad t = t + r_A,
\]

where the sign \( \pm \) again refers to the choice in (C.2) and we inferred the periodicity of \( \varphi \) as \( \varphi \) is identified modulo \( 2\pi \). Thus we see that the parameter \( r_A \) sets the circumference of the \( S^1 \) at infinity, \( |\vec{r}| \to \infty \), as the potential \( V \to 1 \) in the metric (C.1). It is important to emphasize that only with this circumference we have a globally well-defined \( S^1 \)-fibre radius.

Next we comment on the smoothness of \( TN_1 \), where we assume \( *dU = +dV_1 \) for this paragraph to avoid confusion. In fact the singularity of \( V \) at the origin is just a coordinate singularity. In spherical coordinates one can expand the metric (C.1) around the origin using \( V \sim V_1 \) and the coordinate transformation \( q_2 = |\vec{r}|, t^\pm = -\frac{r_A}{4\pi} (\psi \pm \varphi) \) to identify it near the origin as the flat metric \( \mathbb{R}^4 \) iff \( \psi \) has period \( 4\pi \). This metric is obviously smooth. We note that in the case of multiple monopoles \( TN_k \) discussed next, the space is still smooth for generic positions of the \( k \) monopoles, however, develops a deficit angle \( 2\pi/k \), i.e. locally becomes \( \mathbb{R}^4/Z_k \), for \( k \) coincident monopoles.

We conclude the analysis of the \( TN_1 \) geometry by analyzing its (co)homology. Depending on the sign in (C.2) \( TN_1 \) admits a selfdual (sign +1) respectively anti-selfdual (sign -1) two-form that is locally given by

\[
\Omega = d\eta = \frac{1}{r_A} d\left( \frac{V_1}{V} (dt + U) - U \right) \tag{C.6}
\]

As the one-form \( U \) is not globally defined as pointed out in (C.4), the one-form \( \eta \) in turn is not a global form and thus \( \Omega \) is not a globally exact form. On the two patches \( U^\pm \) the two local one-forms denoted \( \eta^\pm \) are given by inserting \( U^\pm \) defined in (C.4) into (C.6) yielding

\[
\eta^\pm = \frac{1}{r_A} \left( \frac{V_1}{V} (dt + U) - U^\pm \right) \tag{C.7}
\]

where we used that the term \( dt + U \) is a global one-form by virtue of (C.5). It further holds the normalization\(^{26}\)

\[
\int \Omega \wedge \Omega = \pm 1. \tag{C.8}
\]

Furthermore we note the limit\(^{27}\)

\[
\Omega \wedge \Omega \to \pm \frac{1}{2\pi} \delta(\rho) \delta(z) d\vec{r} \wedge d\rho \wedge d\varphi \wedge dz, \quad \text{for} \quad r_A \to 0,
\]

\(^{26}\)The sign of the \( \Omega^2 \) can be obtained for any (anti-)selfdual \( \Omega \) since \( \Omega \wedge \Omega = \pm \Omega \wedge *\Omega \), but \( \int \Omega \wedge *\Omega \) positive (negative). We can also switch between a self-dual and anti-selfdual form by changing the orientation on \( \mathbb{R}^3 \).

\(^{27}\)To prove that we use the following mathematical statement. Let \((f_j)_{j\in\mathbb{N}}\) be a sequence of positive functions defined on \( \mathbb{R}^n \), s.t. \( \int_{B(a)} f_j(x)dx = 1 \forall j \). Furthermore \( f_j \) converges uniformly to zero on any set \( 0 < a < |x| < 1/a \),
where we have introduced a new coordinate \( \tilde{t} \) by \( \tilde{t} = t/r_A \). That identifies \( \Omega \wedge \Omega \) as the dual of the origin in \( \mathbb{R}^3 \) for \( r_A \to 0 \).

The results of the one-monopole geometry carry easily over to the multi-center case, denoted \( TN_k \). For this one makes the multi-center ansatz

\[
V = 1 + \sum_{l=1}^{k} V_l, \quad U = \sum_{l=1}^{k} U_l, \quad V_l = \frac{r_A}{4\pi |\vec{r} - \vec{r}_l|}, \quad *3dU_l = dV_l, \tag{C.11}
\]

where \( \vec{r}_l \) denote the positions of the \( k \) monopoles. The connection \( U \) is defined as the sum of gauge connections \( U_l \) constructed for each monopole \( I \) along the lines of (C.4). To write down an expression for the connection \( U \) in local coordinates is a bit subtle due to the dependence of the integration constant in \( dV_l = *_{\mathbb{R}^3}dU_l \) on the choice of coordinate patches covering \( TN_k \). As in (C.4) we have use two patches around each of the \( k \) monopoles with corresponding local one-forms \( U_I^\pm \) in order to avoid a corresponding Dirac string. Placing the \( I \)-th monopole at the origin, we identify \( U_I^\pm = U^\pm \) as defined in (C.4). Then, in writing down \( U = \sum_I U_I \) at a given point on \( TN_k \) we have to decide for each connection \( U_I \) separately to either use the local one-form \( U_I^+ \) with integration constant \( -1 \cdot d\varphi \) or \( U_I^- \) with \( 1 \cdot d\varphi \). Thus, adding up the respective integration constants of the \( U_I \) the integration constant in the local expression for \( U \) can take any value between \( -k \cdot d\varphi \) and \( k \cdot d\varphi \) depending on the point on \( TN_k \).\(^{28}\)

In contrast, the combination \( dt + U \) is again unique since it is globally well defined by virtue of the condition (C.5) around each individual monopole. This then implies that in order to get a smooth solution all monopoles have to have the same charge \( r_A \).

The multi-center solution \( TN_k \) admits \( k \) two-forms locally defined by

\[
\Omega_I = d\eta_I = \frac{1}{r_A} d\left( \frac{V_I}{V} (dt + U) - U_I \right), \tag{C.12}
\]

where the two different signs in \(*3dU_I = \pm dV_I \) yield (anti-)selfduality. They obey the relation

\[
\int_{\mathbb{R}^3 \times S^1} \Omega_I \wedge \Omega_J = \pm \delta_{IJ}. \tag{C.13}
\]

Indeed, we can choose coordinates such that the \( I \)-th monopole is centred at the origin and that the two-plane \( z = 0 \) does not contain a different, \( K \)-th monopole, \( K \neq I \).\(^{29}\) This allows us to identify \( V_I \) for any \( a > 0 \), then \( f_j \to \delta \) in the distributional sense. This can be seen by recalling that uniform convergence means convergence in the maximum norm and it is easy to see that

\[
\max_{|\vec{r}| \in [a, \infty)} \frac{\frac{2a}{r} \vec{r}}{(r + \frac{a}{\vec{r}})^3} \to \frac{\frac{2a}{r} \vec{r}}{(a + \frac{a}{\vec{r}})^3} r \to 0, \tag{C.9}
\]

which establishes the desired result.

\(^{28}\) To illustrate this further, let us define a patching of \( TN_k \) by drawing \( k \) two-dimensional planes in \( \mathbb{R}^3 \) through each of the \( k \) monopoles so that no other monopole is contained in the same plane. For each monopole this defines a partial order by what we call “above” and “below” the corresponding plane in \( \mathbb{R}^3 \) and we accordingly assign \( U_I^\pm \cong U^\pm \). Then for every point in \( TN_k \) we know whether it lies above or below the \( I \)-th plane and can thus write down the local expression for \( U \) by adding up the integration constants \( \mp1 \) of the individual \( U_I^\pm \).

\(^{29}\) We demand that the plane \( z = 0 \) contains no other monopole although both \( \Omega_I \) and \( \eta_I \) are well-defined at \( \vec{r} = \vec{r}_K \).
and $U_I$ with the one-monopole connection of $TN_1$ in (C.3). Then we introduce spherical coordinates and the coordinate patches of (C.4) and identify $U_I^\pm \equiv U^\pm$. Since the coordinate patches $U_\pm$ are just the upper and lower halfspaces of $\mathbb{R}^3$, $z \leq 0$ respectively $z \geq 0$, they share, though with opposite orientation, the common boundary $H$ given by

$$H = \{(r, \varphi, z = 0)\}.$$

(C.14)

By virtue of Stokes’ theorem we may pull the integral of any exact form to this boundary $H$. Then we evaluate (C.13) taking into account the opposite orientation of $H$,

$$\int_{S^1} \Omega_I \wedge \Omega_J = \int_{S^1} \int_0^{\infty} \frac{1}{2\pi} d\varphi \wedge \Omega_J = \pm \int_{S^1} \int_0^{\infty} \frac{1}{r_A} \frac{dV_I}{V} \wedge dt = \pm \frac{V_I}{V} \bigg|_{\rho=0}^{\infty} = \pm \delta_{IJ},$$

(C.15)

where we first used (C.7) with (C.4) and then integrated $dt$ over $S^1$. In the last step we exploited that $V_J/V$ vanishes at $\rho = \infty$, as $V \to 1$ while $V_I \to 0$, and vanishes at $\rho = 0$ as well except when $V_J = V_I$ yielding $V_I/V = 1$, since the pole $V_I \to \infty$ cancels precisely the pole $V \to \infty$.

We note that the area of the two-cycles $S_i$ spanning $H_2(TN_k, \mathbb{Z})$ introduced in (3.13) reads

$$\int_{S_i} \text{vol}_{S_i} = \int_{S^1} \int_{\tilde{r}_i}^{\tilde{r}_i+1} V^\frac{1}{2} V^{-\frac{1}{2}} = r_A |\tilde{r}_i - \tilde{r}_i+1|.$$

(C.16)

The forms $\tilde{\omega}_i = \Omega_i - \Omega_{i+1}$, $i = 1, \ldots, k-1$, spanning its Poincare dual fulfill the following conditions

$$\int_{S_i} \tilde{\omega}_i \wedge \tilde{\omega}_j = \pm C_{ij}, \quad \int_{S_j} \omega_j = \pm C_{ij},$$

(C.17)

again depending on (anti-)selfduality of $\Omega_I$. The first statement is clear due to (C.13). For the second one we calculate

$$\int_{S_i} \omega_j = \int_{\partial S_i} \eta_j - \eta_{j+1} = \pm \left( \frac{V_j}{V} \bigg|_{\tilde{r}_i}^{\tilde{r}_i+1} - \frac{V_{j+1}}{V} \bigg|_{\tilde{r}_i}^{\tilde{r}_i+1} \right).$$

(C.18)

### D Details of $TN^\infty_k$

The metric of infinitely many Kaluza-Klein monopoles placed with equal spacing $r_B$ along a straight line in $\mathbb{R}^3$ is again of the from (C.1). Moreover due to the cylinder symmetry of the set-up it is convenient to introduce cylindrical coordinates $\rho = \sqrt{x^2 + y^2}$, $\varphi = \arctan(y/x)$ and $z$ being a coordinate on the axis along which the monopoles are aligned. After forming the quotient $z \sim z + r_B$ we denote this space by $TN^\infty_1$. The potential $V$ reads

$$V = 1 + \frac{r_A}{4\pi} \left( \sum_{\ell \in \mathbb{Z}} \frac{1}{\sqrt{\rho^2 + (z + \ell r_B)^2}} - \sum_{\ell \in \mathbb{Z}} \frac{1}{r_B|\ell|} \right).$$

(D.1)
We note that $V$ is now a harmonic function on $\mathbb{R}^2 \times S^1$ due to the periodicity along the $z$-axis. Thus we can view the geometry of $TN_1^\infty$ as a single Kaluza-Klein monopole on $\mathbb{R}^2 \times S^1$, treated as an image charge problem on $\mathbb{R}^3$. The last term in (D.1) is a regulator that assures the convergence of the sum. Note that the precise form of the regulator can be modified by any finite constant. The corresponding connection $U = \sum_I U_I$ with $\ast_3 dU_I = dV_I$ is given on the patch $z \in [0, r_B]$ as

$$U = \frac{r_A}{4\pi} \left( -1 + \sum_{\ell \in \mathbb{Z}} \frac{z - \ell r_B}{\sqrt{\rho^2 + (z - \ell r_B)^2}} \right) d\varphi,$$

(D.2)

where $-1 \cdot d\varphi$ is a choice of integration constant so that $U$ is regular on $[0, r_B]$. In fact, treating $TN_1^\infty$ as an image charge problem there is a Dirac string for every monopole at $\vec{r}_I = (0, 0, \ell r_B)$ as in appendix C. Again $d\varphi$ is ill-defined for $\rho = 0$ and so is $U$ unless the coefficient of $d\varphi$ vanishes. Evaluating $U$ in (D.2) at $\rho = 0$ we have chosen our regularization such that for $z \in [0, r_B]$,

$$\sum_{\ell} \frac{z - \ell r_B}{\sqrt{\rho^2 + (z - \ell r_B)^2}} \bigg|_{\rho=0} = \sum_{\ell} \text{sign}(z - \ell r_B) = 1$$

(D.3)

and $U = 0$, i.e. well-defined. However, when considering for instance $z \in [r_B, 2r_B]$ we evaluate, in the same regularization $\sum_{\ell} \text{sign}(z - \ell r_B) = 3$ and the one-form $U$ in (D.2) is ill-defined. Thus, we introduce patches $U_n$, $n$ integer, that cover the $z$-axis in increments of $r_B$ and local one-forms $U^n$,

$$U_n = \{(\rho, \varphi, z) | nr_B \leq z < (n + 1)r_B\} : \quad U^n = \frac{r_A}{4\pi} \left( -1 - 2n + \sum_{\ell \in \mathbb{Z}} \frac{z - \ell r_B}{\sqrt{\rho^2 + (z - \ell r_B)^2}} \right) d\varphi.$$ 

(D.4)

The $U^n$ are well-defined on $U_n$ and related by the gauge transformation $U^{n+1} = U^n - \frac{r_A}{2\pi} d\varphi$. In other words, when crossing the lines $z = nr_B$ from below (above) we have to change the integration constant in the local one-form by $-2d\varphi$ ($+2d\varphi$). It is important to note, that $U$ in (D.2) descends to a one-form which is well-defined along the whole $S^1$ of the compactified $z$-direction, $z \sim z + r_B$.

In order to get a better understanding of $V$ and $U$ we perform a Poisson resummation of these two quantities. Recall that a Poisson resummation relates a function $f$ of period one and its Fourier-transform $\hat{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ikx} dx$ via

$$\sum_{k \in \mathbb{Z}} \hat{f}(k)e^{2\pi ikx} = \sum_{k \in \mathbb{Z}} f(x + k).$$

(D.5)

The Fourier-transform of $f(z) = \frac{1}{\sqrt{\rho^2 + z^2}}$ is $\hat{f}(k) = 2K_0(2\pi k)$, which is the zeroth modified Bessel function of second kind and shows the following asymptotic behaviour near zero,

$$K_0(x) = -\log \frac{x}{2} - \gamma, \quad x \to 0, \quad \gamma = \lim_{N \to \infty} \sum_{k=1}^{N} \frac{1}{k} - \log N,$$

(D.6)

\[30\] Again we have $\Delta_3 V = \delta^3(\vec{r})$ in the distributional sense.
where $\gamma$ is the Euler-Mascheroni constant. We now plug $f(z) = \frac{r_A}{4\pi r_B} \frac{1}{|\rho^2 + \zeta^2|}$ with $\hat{\rho} = \frac{\rho}{r_B}$ and $\hat{z} = \frac{z}{r_B}$ as well as $\hat{f}(k) = \frac{r_A}{2\pi} K_0(2\pi|\hat{k}|)$ into (D.5) and obtain

$$V = 1 + \frac{r_A}{4\pi r_B} \left( \sum_{\ell \in \mathbb{Z}} \frac{1}{\sqrt{\rho^2 + (\zeta + \ell)^2}} - \sum_{\ell \in \mathbb{Z}^*} \frac{1}{|\ell|} \right) = 1 + \frac{r_A}{2\pi r_B} \left( \sum_{\ell \in \mathbb{Z}^*} K_0(2\pi|\hat{\rho}|e^{2\pi i \ell} \hat{\zeta} + K_0(0) - \sum_{\ell > 0} \frac{1}{|\ell|} \right).$$

(D.7)

The right hand side contains two divergent terms, $K_0(0)$ and $\sum_{\ell > 0} \frac{1}{|\ell|}$. We therefore have to take a suitable limit to get a finite result by considering and calculating, using (D.6),

$$\frac{r_A}{2\pi r_B} \lim_{N \to \infty} \left( K_0(\frac{2\pi \hat{\rho}}{N}) - \sum_{\ell = 1}^N \frac{1}{\ell} \right) = \frac{r_A}{2\pi r_B} \lim_{N \to \infty} \left( - \log \frac{\pi \hat{\rho}}{N} - \gamma - \sum_{\ell = 1}^N \frac{1}{\ell} \right) = -\frac{r_A}{2\pi r_B} \log(\hat{\rho}/\Lambda),$$

(D.8)

where $\Lambda$ comprises all constants including an eventually shift in the regulator term. For the concrete regulator in (3.18) we have $\Lambda = 1/(\pi e^{2\gamma})$. Finally we obtain

$$V = 1 + \frac{r_A}{4\pi r_B} \left( \sum_{\ell \in \mathbb{Z}} \frac{1}{\sqrt{\rho^2 + (\zeta + \ell)^2}} - \sum_{\ell \in \mathbb{Z}^*} \frac{1}{|\ell|} \right) = 1 - \frac{r_A}{2\pi r_B} \left( \log \frac{\hat{\rho}}{\Lambda} - 2 \sum_{\ell > 0} K_0(2\pi |\hat{\rho}|\ell) \cos(2\pi \ell \hat{\zeta}) \right).$$

(D.9)

Similarly one can also perform a Poisson resummation for the connection $U$, which is given by

$$U = \frac{r_A}{4\pi} \left( -1 + \sum_{\ell \in \mathbb{Z}} \frac{(\hat{\zeta} - \ell)}{\sqrt{\rho^2 + (\hat{\zeta} - \ell)^2}} \right) d\phi,$$

(D.10)

for $0 \leq \hat{\zeta} < 1$. Using that the Fourier transform of $f(\hat{z}) = \frac{\hat{z}}{\sqrt{\rho^2 + \hat{z}^2}}$ reads $\hat{f}(k) = 2i\hat{\rho}\text{sign}(k)K_1(2\pi|\hat{k}|)$ we can perform a Poisson resummation for the connection as well, finding naively

$$U = \frac{r_A}{4\pi} \left( -1 + 2i\hat{\rho} \sum_{\ell \in \mathbb{Z}} \text{sign}(\ell) K_1(2\pi|\hat{\rho}|\ell) e^{2\pi i \ell \hat{\zeta}} \right) d\phi.$$

(D.11)

Note that the contribution $\ell = 0$ is again ill defined. We recall that

$$K_1(x) \sim \frac{1}{x}, \quad x \ll 1.$$

(D.12)

This enables us to regularize the $\ell = 0$ contribution, i.e. sign($\ell$), as

$$\lim_{\ell \to 0} \frac{1}{2} \left( i\hat{\rho} \frac{1}{2\pi |\hat{k}|} (1 + 2\pi i \ell \hat{\zeta}) - i\hat{\rho} \frac{1}{2\pi |\hat{k}|} (1 - 2\pi i \ell \hat{\zeta}) \right) = -\hat{\zeta}.$$

(D.13)

We finally obtain

$$U = -\frac{r_A}{4\pi} \left( 1 + 2\hat{\zeta} + 4\hat{\rho} \sum_{\ell > 0} K_1(2\pi |\hat{\rho}|\ell) \sin(2\pi \ell \hat{\zeta}) \right) d\phi.$$

(D.14)

Note that this is cohomologically equivalent by adding term proportional to $d(\hat{\zeta} \phi)$ and $d\phi$ yielding

$$U = \frac{r_A}{2\pi} \hat{\rho} d\hat{\zeta} + \frac{r_A}{2\pi} \left( -2\hat{\rho} \sum_{\ell > 0} K_1(2\pi |\hat{\rho}|\ell) \sin(2\pi \ell \hat{\zeta}) \right) d\phi.$$

(D.15)
As in the non-periodic case, one can easily generalize to the multi-center case \( TN_k^\infty \). We restrict ourselves to the case that all monopoles are located at \( (\dot{\rho} = 0, \dot{z} = \dot{z}_I)_{I=1,\ldots,k} \), i.e. we consider \( k \) periodic chains of monopoles that are shifted among each other. The corresponding re-summed potentials and connections are given for \( I = 1,\ldots,k \) and \( \hat{z} \in [\hat{z}_I, \hat{z}_I + 1] \) by

\[
V_I = -\frac{r_A}{2\pi r_B} \left( \log \frac{\dot{\rho}}{\Lambda} - 2 \sum_{\ell>0} K_0(2\pi \ell \dot{\rho}) \cos(2\pi \ell (\hat{z} - \hat{z}_I)) \right) , \tag{D.16}
\]

\[
U_I = -\frac{r_A}{4\pi} \left( 1 + 2(\hat{z} - \hat{z}_I) + 4\dot{\rho} \sum_{\ell>0} K_1(2\pi \ell \dot{\rho}) \sin(2\pi \ell (\hat{z} - \hat{z}_I)) \right) d\varphi , \tag{D.17}
\]

that obey \(*_3 dU_I = -dV_I\). Generalizing the patches of (C.4) to \( k \) monopoles as

\[
\mathcal{U}_n(I) = \{ (\hat{\rho}, \varphi, \hat{z}) \mid n + \hat{z}_I \leq \hat{z} < \hat{z}_I + n + 1 \} , \tag{D.18}
\]

we can construct local one-forms \( U^n_I \) for other values of \( \hat{z} \) by changing the integration constant by \( \pm 2 \). In direct analogy with (C.4) they read on \( \mathcal{U}_n(I) \) as

\[
U^n_I = -\frac{r_A}{4\pi} \left( 1 + 2n + 2(\hat{z} - \hat{z}_I) + 4\dot{\rho} \sum_{\ell>0} K_1(2\pi \ell \dot{\rho}) \sin(2\pi \ell (\hat{z} - \hat{z}_I)) \right) d\varphi . \tag{D.19}
\]

Analogously to (C.12) the space \( TN_k^\infty \) also exhibits \( k \) anti-self-dual two-forms given by

\[
\Omega^n_I = d\eta_I = \frac{1}{r_A} dt \left( \frac{V_I}{V}(dt + U) - U_I \right) . \tag{D.20}
\]

The expression for the local one-forms \( \eta_I \) depends on the coordinate patches \( \mathcal{U}_n(I) \), i.e. the value of \( \hat{z} \), through the dependence of the \( U^n_I \) in (D.19) on the coordinate patch. The local one-forms are denoted \( \eta^n_I \). The combination \((dt + U)\) for \( U = \sum_I U_I \) is again globally defined by appropriately defining local coordinates \( t \).

We would like to check that the relation

\[
\int \Omega^n_I \wedge \Omega^n_J = -\delta_{IJ} \tag{D.21}
\]

still holds in the periodic case. First we center the \( I \)-th monopole at the origin \((\hat{\rho}, \varphi, \hat{z}) = 0\). Then we use as in the one monopole case (C.15) the exactness of \( \Omega^n_I \) on the patches \( \mathcal{U}_n(I) \) of (D.18). Since we eventually work on the quotient \( \hat{z} \sim \hat{z} + 1 \) we integrate over the interval \( \hat{z} \in [0,1] \), but have to keep in mind that the integration constant in \( U_I \) jumps by \(-2d\varphi\) when \( \hat{z} \to 1 \) from below. As mentioned earlier the boundaries of \( \hat{z} \in [0,1] \) representing \( S^1 \) are simply

\[
H = \{ (\hat{\rho}, \varphi, z = 0) \} , \tag{D.22}
\]

with opposite orientation, respectively. We readily perform the pullback of the integral by Stokes theorem as

\[
\int \Omega_I \wedge \Omega_J = \int_{S^1_H} (\eta^1_I - \eta^0_I) \wedge \Omega_I = \int_{S^1_H} \frac{1}{2\pi} d\varphi \wedge \Omega_I = \int_{S^1_H} \int_0^\infty \frac{1}{r_A} \left. \frac{V_J}{V} \right| \dot{\rho} = 0 \wedge dt = \left. \frac{V_J}{V} \right|_{\dot{\rho} = 0} = -\delta_{IJ} . \tag{D.23}
\]
Here we used the local expression \[(D.19)\] and \[(D.20)\] to evaluate $\eta_1^1 - \eta_0^0 \sim 2d\phi$ in the second equality and exploited the behaviour of $V_J/V$ at $\hat{\rho} = 0, \infty$ as for $TN_1$ to obtain the last equality.

We conclude by representing any metric of the form \[(C.1)\] in terms of Vierbeins $\epsilon_i$ [53]
\[
\epsilon^0 = \frac{1}{\sqrt{V}}(dt + U), \quad \epsilon^i = \sqrt{V} dx^i, \quad i = 1, 2, 3. \tag{D.24}
\]
Vierbeins make it particularly easy to evaluate the Hodge star $\ast_4$ on Taub-NUT with any number of monopoles by specifying the orientation by the volume form as $\epsilon^0 \wedge \epsilon^1 \wedge \epsilon^2 \wedge \epsilon^3$. Then it is straightforward to check for instance the (anti-)selfduality of $\Omega_I$ respectively $\Omega_I^{\infty}$ noting that
\[
\Omega_I = (1 \pm \ast_4) \left( \frac{V_I}{V} dU - dU_I \right). \tag{D.25}
\]

References

[1] C. Vafa, “Evidence for F theory,” *Nucl.Phys.* **B469** (1996) 403–418, arXiv:hep-th/9602022 [hep-th].

[2] F. Denef, “Les Houches Lectures on Constructing String Vacua,” arXiv:0803.1194 [hep-th].

[3] K. Dasgupta, G. Rajesh and S. Sethi, “M theory, orientifolds and G - flux,” *JHEP* **9908** (1999) 023, hep-th/9908088 [hep-th].

[4] T. W. Grimm, “The N=1 effective action of F-theory compactifications,” *Nucl. Phys.* **B845** (2011) 48, [arXiv:1008.4133 [hep-th]]

[5] M. R. Douglas and S. Kachru, “Flux compactification,” *Rev. Mod. Phys.* **79** 733 (2007), arXiv:hep-th/0610102.

[6] R. Donagi and M. Wijnholt, “Model Building with F-Theory,” arXiv:0802.2969 [hep-th].

[7] C. Beasley, J. J. Heckman, and C. Vafa, “GUTs and Exceptional Branes in F-theory - I,” *JHEP* **0901** (2009) 058, arXiv:0802.3391 [hep-th].

[8] A. P. Braun, A. Collinucci and R. Valandro, “G-flux in F-theory and algebraic cycles,” *Nucl. Phys.* **B856** (2012) 129, arXiv:1107.5337 [hep-th].

[9] J. Marsano and S. Schafer-Nameki, “Yukawas, G-flux, and Spectral Covers from Resolved Calabi-Yau’s,” *JHEP* **1111** (2011) 098, arXiv:1108.1794 [hep-th].

[10] S. Krause, C. Mayrhofer and T. Weigand, “$G_4$ flux, chiral matter and singularity resolution in F-theory compactifications,” *Nucl. Phys.* **B858** (2012) 1, arXiv:1109.3454 [hep-th].
[11] T. W. Grimm and H. Hayashi, “F-theory fluxes, Chirality and Chern-Simons theories,” arXiv:1111.1232 [hep-th]

[12] K. Becker and M. Becker, “M-Theory on Eight-Manifolds,” Nucl. Phys. B477 (1996) 155–167. arXiv:hep-th/9605053

[13] S. B. Giddings, S. Kachru, and J. Polchinski, “Hierarchies from fluxes in string compactifications,” Phys. Rev. D66 (2002) 106006. arXiv:hep-th/0105097 [hep-th]

[14] S. B. Giddings and A. Maharana, “Dynamics of warped compactifications and the shape of the warped landscape,” Phys. Rev. D73 (2006) 126003. arXiv:hep-th/0507158 [hep-th]

[15] C. P. Burgess, P. G. Camara, S. P. de Alwis, S. B. Giddings, A. Maharana, F. Quevedo and K. Suruliz, “Warped Supersymmetry Breaking,” JHEP 0804 (2008) 053. arXiv:hep-th/0610255 [hep-th]

[16] G. Shiu, G. Torroba, B. Underwood and M. R. Douglas, “Dynamics of Warped Flux Compactifications,” JHEP 0806 (2008) 024. arXiv:0803.3068 [hep-th]

[17] M. R. Douglas and G. Torroba, “Kinetic terms in warped compactifications,” JHEP 0905 (2009) 013. arXiv:0805.3700 [hep-th]

[18] L. Martucci, “On moduli and effective theory of N=1 warped flux compactifications,” JHEP 0905 (2009) 027. arXiv:0902.4031 [hep-th]

[19] H. Jockers and J. Louis, “The effective action of D7-branes in N = 1 Calabi-Yau orientifolds,” Nucl. Phys. B705 (2005) 167–211. arXiv:hep-th/0409098

[20] W. Lerche and S. Stieberger, “Prepotential, mirror map and F theory on K3,” Adv. Theor. Math. Phys. 2 (1998) 1105, [Erratum-ibid. 3 (1999) 1199]. arXiv:hep-th/9804176 [hep-th].

[21] W. Lerche, S. Stieberger and N. P. Warner, “Quartic gauge couplings from K3 geometry,” Adv. Theor. Math. Phys. 3 (1999) 1575. arXiv:hep-th/9811228 [hep-th]

[22] J. Marsano, N. Saulina and S. Schafer-Nameki, “F-theory Compactifications for Supersymmetric GUTs,” JHEP 0908 (2009) 030. arXiv:0904.3932 [hep-th]

[23] R. Blumenhagen, T. W. Grimm, B. Jurke and T. Weigand, “Global F-theory GUTs,” Nucl. Phys. B829, (2010) 325. arXiv:0908.1784 [hep-th]

[24] C. -M. Chen, J. Knapp, M. Kreuzer and C. Mayrhofer, “Global SO(10) F-theory GUTs,” JHEP 1010 (2010) 057. arXiv:1005.5735 [hep-th]

[25] M. Cvetic, I. Garcia-Etxebarria and J. Halverson, “Global F-theory Models: Instantons and Gauge Dynamics,” JHEP 1101 (2011) 073. arXiv:1003.5337 [hep-th]
[26] P. G. Camara, E. Dudas and E. Palti, “Massive wavefunctions, proton decay and FCNCs in local F-theory GUTs,” JHEP 1112 (2011) 112, arXiv:1110.2206 [hep-th].

[27] R. Donagi and M. Wijnholt, “Breaking GUT Groups in F-Theory,” arXiv:0808.2223 [hep-th].

[28] R. Blumenhagen, “Gauge Coupling Unification in F-Theory Grand Unified Theories,” Phys. Rev. Lett. 102 (2009) 071601 arXiv:0812.0248 [hep-th].

[29] G. W. Gibbons and S. W. Hawking, “Gravitational Multi-Instantons,” Phys. Lett. B78 (1978) 430.

[30] H. Ooguri and C. Vafa, “Summing up D-instantons,” Phys. Rev. Lett. 77 (1996) 3296–3298, arXiv:hep-th/9608079 [hep-th].

[31] D. Gaiotto, G. W. Moore and A. Neitzke, “Four-dimensional wall-crossing via three-dimensional field theory,” Commun. Math. Phys. 299 (2010) 163 arXiv:0807.4723 [hep-th].

[32] T. W. Grimm and J. Louis, “The effective action of N = 1 Calabi-Yau orientifolds,” Nucl. Phys. B699 (2004) 387–426, arXiv:hep-th/0403067

[33] T. W. Grimm, T.-W. Ha, A. Klemm, and D. Klevers, “The D5-brane effective action and superpotential in N=1 compactifications,” Nucl. Phys. B816 (2009) 139–184, arXiv:0811.2996 [hep-th].

[34] T. W. Grimm and D. V. Lopes, “The N=1 effective actions of D-branes in Type IIA and IIB orientifolds,” Nucl. Phys. B855, (2012) 639, arXiv:1104.2328 [hep-th].

[35] M. Kerstan and T. Weigand, “The Effective action of D6-branes in N=1 type IIA orientifolds,” JHEP 1106 (2011) 105 arXiv:1104.2329 [hep-th].

[36] T. W. Grimm and T. Weigand, “On Abelian Gauge Symmetries and Proton Decay in Global F-theory GUTs,” Phys. Rev. D82 (2010) 086009 arXiv:1006.0226 [hep-th].

[37] T. W. Grimm, M. Kerstan, E. Palti and T. Weigand, “Massive Abelian Gauge Symmetries and Fluxes in F-theory,” JHEP 1112 (2011) 004 arXiv:1107.3842 [hep-th].

[38] F. Bonetti and T. W. Grimm, “Six-dimensional (1,0) effective action of F-theory via M-theory on Calabi-Yau threefolds,” arXiv:1112.1082 [hep-th].

[39] P. Townsend, “The eleven-dimensional supermembrane revisited,” Phys. Lett. B350 (1995) 184–187, arXiv:hep-th/9501068 [hep-th].

[40] P. Ruback, “The motion of Kaluza-Klein monopoles,” Commun. Math. Phys. 107 (1986) 93–102.
[41] A. Sen, “A Note on enhanced gauge symmetries in M and string theory,” JHEP 9709 (1997) 001, arXiv:hep-th/9707123 [hep-th]

[42] M. Bianchi, F. Fucito, G. Rossi, and M. Martellini, “Explicit construction of Yang-Mills instantons on ALE spaces,” Nucl.Phys. B473 (1996) 367–404, arXiv:hep-th/9601162 [hep-th].

[43] M. Blau and M. O’Loughlin, “Aspects of U duality in matrix theory,” Nucl.Phys. B525 (1998) 182–214, arXiv:hep-th/9712047 [hep-th].

[44] E. Eyras and Y. Lozano, “Exotic branes and nonperturbative seven-branes,” Nucl.Phys. B573 (2000) 735–767, arXiv:hep-th/9908094 [hep-th].

[45] L. Grafakos, Classical Fourier Analysis. Springer Verlag, 2008.

[46] P. S. Aspinwall, “M theory versus F theory pictures of the heterotic string,” Adv.Theor.Math.Phys. 1 (1998) 127–147, arXiv:hep-th/9707014 [hep-th].

[47] A. Sen, “F theory and orientifolds,” Nucl.Phys. B475 (1996) 562–578 arXiv:hep-th/9605150 [hep-th].

[48] A. Sen, “Orientifold limit of F theory vacua,” Phys.Rev. D55 (1997) 7345–7349, arXiv:hep-th/9702165 [hep-th].

[49] T. W. Grimm and R. Savelli, “Gravitational Instantons and Fluxes from M/F-theory on Calabi-Yau fourfolds,” Phys.Rev. D85 (2012) 026003 arXiv:1109.3191 [hep-th].

[50] C. Angelantonj, I. Florakis and B. Pioline, “A new look at one-loop integrals in string theory,” arXiv:1110.5318 [hep-th].

[51] M. B. Green, S. D. Miller and P. Vanhove, “Small representations, string instantons, and Fourier modes of Eisenstein series (with an appendix by D. Ciubotaru and P. Trapa),” arXiv:1111.2983 [hep-th].

[52] G. Girardi and R. Grimm, “The Superspace geometry of gravitational Chern-Simons forms and their couplings to linear multiplets: A Review,” Annals Phys. 272 (1999) 49–129, arXiv:hep-th/9801201 [hep-th].

[53] C. W. Misner, “The Flatter regions of Newman, Unti and Tamburino’s generalized Schwarzschild space,” J.Math.Phys. 4 (1963) 924–938.