ASPECTS OF THE DECOHERENT HISTORIES
APPROACH TO QUANTUM MECHANICS

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ABSTRACT
I give an informal overview of the decoherent histories approach to quantum mechanics, due to Griffiths, to Omnès, and to Gell-Mann and Hartle is given. Results on the connections between decoherence, records, correlation and entropy are described. The emphasis of the presentation is on understanding the broader meaning of the conditions of consistency and decoherence, and in particular, the extent to which they permit one to assign definite properties to the system. The quantum Brownian motion model is briefly discussed.

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1. Introduction

This is an informal account of the decoherent histories approach to quantum mechanics, loosely based on my talk at the workshop and the questions it generated. My aim is to give a brief summary of the approach, with some development of points not discussed elsewhere. Technical details will be kept to a minimum. They may be found in the excellent articles by the subject’s founders 1–6.

In a single sentence, the broad aim of the histories approach is this: we would like to be able to talk about the properties of a closed and isolated quantum system, without having to resort to notions of measurement or observation in an essential way. Here, by “talk about”, I mean make statements about the system pertaining to its physical properties, that may be related to each other by ordinary classical (i.e., Boolean) logic.

1.1 Copenhagen

Standard quantum mechanics is based on the Copenhagen interpretation. Formally, it may be founded on a (quite large) number of technical axioms (see, for example, Ref.[7]). These axioms place great emphasis on the notion of measurement of the system of interest by an external, classical observing apparatus. Indeed, the whole framework strictly applies only to a universe which has been divided into macroscopic classical systems and microscopic quantum systems.
Despite its great successes, it is inadequate on a number of counts. Many – if not all – of these inadequacies boil down to the fact that the Copenhagen interpretation does not supply a picture of what is actually happening, or at least, what one can meaningfully think of as actually happening, in a closed quantum system. It does not tell us for example, what is happening in a quantum system between measurements. More generally, it gives no indication as to what extent we can regard quantum systems as possessing definite properties, independently of whether or not they are being measured. These features of the Copenhagen interpretation make it difficult to extend it to the macroscopic domain, and in particular, to quantum cosmology.

1.2 The Histories Approach

The decoherent histories approach was designed to overcome these problems. In brief, the main features of the approach are as follows. It applies specifically to closed systems. It focuses on the histories of a closed system, rather than events at a fixed moment of time. It is a modest generalization of ordinary quantum mechanics, but relies on a far smaller list of axioms. These axioms are basically the statements that the closed system is described by the usual mathematical machinery of quantum mechanics, Hilbert space, unitary evolution of states, etc., together with a formula for the probabilities and a rule of interpretation. It makes no distinction between microscopic and macroscopic. A separate classical domain is therefore not assumed, but may be an emergent feature under calculable conditions. It makes no essential use of measurement, or collapse of the wave function, although these notions may be discussed within the framework of the approach. What replaces measurement is the more general notion of consistency (or the stronger notion of decoherence), determining which histories may be assigned probabilities. The approach also stresses classical logic, the conditions under which it may be applied, and thus, the conditions under which one can meaningfully talk about the properties of a physical system.

1.3 Why histories?

The basic building block in the decoherent histories approach is a history – a sequence of events at a succession of times. Why are these objects of particular interest?

(a) Histories are the most general class of situation one might be interested in. For example, a typical experimental situation might be of the form, a particle is emitted from a decaying nucleus at time $t_1$, then it passes through a magnetic field at time $t_2$, then it is absorbed by a detector at time $t_3$.

(b) We would like to understand how classical behaviour can emerge from the quantum mechanics of closed systems. It is therefore necessary to show, amongst other things, that successive positions in time of, say, a particle are approximately correlated according to classical laws. It is therefore necessary to study histories of position samplings.

(c) The basic practical aim of theoretical physics is to find patterns in presently
existing data. Why then should we not attempt to formulate our theories in the terms of the density matrix of the entire universe on a given spacelike surface? There are at least two reasons why not. Firstly, present records are stored in a wide variety of different ways – in computer memories, on photographic plates, on paper, in our own personal memories, in measuring apparatus. A theory explaining the correlation between present records would thus have to be a theory of measuring apparatus, photographic plates, etc. Surely a theory as fundamental as quantum mechanics should not become so embroiled in the details of measurement and storage. Secondly, the correlation between present records and past events can never be perfect. In order to discuss the approximate nature of correlations between past and present events it becomes necessary to talk about the histories of a system.

(d) As stated above, the minimal pragmatic aim of theoretical physics is to explain the data. Yet many feel that our theories should explain more than just the numbers: it should supply us with a picture of the world the way it really is. Histories arguably supply us with that picture.

2. The Formalism of Decoherent Histories

I now briefly outline the mathematical formalism of the decoherent histories approach. Further details may be found in Refs.[1-6,8].

2.1 Probabilities for Histories

In quantum mechanics, propositions about the attributes of a system at a fixed moment of time are represented by sets of projections operators. The projection operators $P_\alpha$ effect a partition of the possible alternatives $\alpha$ a system may exhibit at each moment of time. They are exhaustive and exclusive,

$$\sum_\alpha P_\alpha = 1, \quad P_\alpha P_\beta = \delta_{\alpha\beta} P_\alpha \quad (2.1)$$

A quantum-mechanical history is characterized by a string of time-dependent projections, $P^n_\alpha(t_n) \cdots P^1_\alpha(t_1)$, together with an initial state $\rho$. The time-dependent projections are related to the time-independent ones by

$$P^k_{\alpha_k}(t_k) = e^{iH(t_k-t_0)} P^k_{\alpha_k} e^{-iH(t_k-t_0)} \quad (2.2)$$

where $H$ is the Hamiltonian. The candidate probability for such histories is

$$p(\alpha_1,\alpha_2,\cdots,\alpha_n) = Tr \left( P^n_{\alpha_n}(t_n) \cdots P^1_{\alpha_1}(t_1) \rho P^1_{\alpha_1}(t_1) \cdots P^n_{\alpha_n}(t_n) \right) \quad (2.3)$$

It is straightforward to show that (2.3) is both non-negative and normalized to unity when summed over $\alpha_1,\cdots,\alpha_n$. However, (2.3) does not satisfy all the axioms of probability theory, and for that reason it is referred to as a candidate probability. It does not satisfy the requirement of additivity on disjoint regions of sample space. More precisely, for each set of histories, one may construct coarser-grained histories
by grouping the histories together. This may be achieved, for example, by summing over the projections at each moment of time,

$$\bar{P}_\alpha = \sum_{\alpha \in \bar{\alpha}} P_\alpha$$

(2.4)

although this is not the most general type of coarse graining. The additivity requirement is then that the probabilities for each coarser-grained history should be the sum of the probabilities of the finer-grained histories of which it is comprised. Quantum-mechanical interference generally prevents this requirement from being satisfied; thus histories of closed quantum systems cannot in general be assigned probabilities.

The standard illustrative example is the double slit experiment. The histories consist of projections at two moments of time: projections determining which slit the particle went through at time $t_1$, and projections determining the point at which the particle hit the screen at time $t_2$. As is well-known, the probability distribution for the interference pattern on the screen cannot be written as a sum of the probabilities for going through each slit; hence the candidate probabilities do not satisfy the additivity requirement.

There are, however, certain types of histories for which interference is negligible, and the candidate probabilities for histories do satisfy the sum rules. These histories may be found using the decoherence functional:

$$D(\alpha, \alpha') = \text{Tr} \left( P_{\alpha_n}(t_n) \cdots P_{\alpha_1}(t_1) \rho P_{\alpha'_1}(t_1) \cdots P_{\alpha'_n}(t_n) \right)$$

(2.5)

Here $\alpha$ denotes the string $\alpha_1, \alpha_2, \cdots, \alpha_n$. If the real part of decoherence functional vanishes for all distinct pairs of histories $\alpha, \alpha'$, then it may be shown that all probability sum rules are satisfied. Sets of such histories are said to be consistent, or weakly decoherent.

A stronger condition is that both real and imaginary parts of the decoherence functional vanish,

$$D(\alpha, \alpha') = 0, \quad \text{for} \quad \alpha \neq \alpha'$$

(2.6)

This I shall refer to quite simply as decoherence (although it is sometimes further classified as medium or strong decoherence). We will discuss this condition extensively in the next section.

The decoherence functional obeys a simple inequality which turns out to be rather useful. It is,

$$|D(\alpha, \alpha')|^2 \leq D(\alpha, \alpha) D(\alpha', \alpha')$$

(2.7)

Intuitively, this result says that there can be no interference with a history which has candidate probability zero. It is useful mathematically because if connects the off-diagonal components of the decoherence functional to the candidate probabilities – more familiar objects about which more is known. We will exploit this fact in the next section. Eq.(2.7) also suggests a possible measure of approximate decoherence: we say that a system decoheres to order $\epsilon$ if the decoherence functional satisfies (2.7)
with a factor of $\epsilon^2$ on the right-hand side. As shown in Ref.[8], such a condition implies that most (but not all) probability sum rules will then be satisfied to order $\epsilon$.

The focus of the decoherent histories approach is on sets of histories satisfying the decoherence condition (2.6) (or the weaker condition of consistency). Whether or not the decoherence condition is satisfied will depend on the initial state, the Hamiltonian, and the projections at each moment of time. Changing any one of these in a decoherent set of histories will generally not preserve decoherence. One typically expects, for a given system, to be supplied with the initial state and the Hamiltonian. It is then matter for investigation to determine which histories, i.e., which strings of projections, will lead to the decoherence condition being satisfied.

2.2 Consistency and Classical Logic

Now I discuss why sets of consistent histories are of interest. As stated, propositions about the attributes of a quantum system may be represented by projection operators. The set of all projections have the mathematical structure of a lattice. This lattice is non-distributive, and this means that the corresponding propositions may not be submitted to Boolean logic. Similar remarks hold for the more complex propositions expressed by general sets of quantum-mechanical histories.

The reason why consistent sets of histories are of interest is that they can be submitted to Boolean logic. Indeed, a theorem of Omnès states that a set of histories forms a consistent representation of Boolean logic if and only if it is a consistent set$^6$. That is, in a consistent set of histories, each history corresponds to a proposition about the properties of a physical system and we can meaningfully manipulate these propositions without contradiction using ordinary classical logic. It is in this sense that the decoherent histories approach allows one to “talk about” the properties of a system in a meaningful way.

Based on these considerations, Omnès introduced the following rule of interpretation of quantum mechanics alluded to in the Introduction: 

\textit{Any description of a physical system should consist of propositions belonging to a common consistent quantum logic and any reasoning about it should consist of valid implications}.$^6$

As an example, consider the case of retrodiction of the past, given present data. Suppose we have a consistent set of histories. We would say that the alternative $\alpha_n$ (present data) implies the alternatives $\alpha_{n-1} \cdots \alpha_1$ (past events) if

$$p(\alpha_1, \cdots, \alpha_{n-1}|\alpha_n) = \frac{p(\alpha_1, \cdots, \alpha_n)}{p(\alpha_n)} = 1$$

(2.8)

In this way, we can in quantum mechanics build a picture of \textit{what actually happened} in the past, given the present data, using only logic and the consistency of the histories. It is not necessary for a measuring device to actually be there in the past. Similarly, one can in certain circumstances use logic and consistency to deduce what is happening in quantum systems between measurements.

There is however, a caveat. Some situations in quantum mechanics admit multiple representations of logic that are not equivalent. They are described by two or
more inequivalent sets of consistent histories the union of which is not a consistent set. There then exist statements about the system that are logically implied in some sets of histories but not in all. Such statements are referred to as “reliable” rather than “true”. If, for example, the retrodiction process outlined above suffered from this type of ambiguity, one would not say that the past alternatives “actually happened”. See Ref.[9] for a discussion of this very subtle issue.

3. Decoherence, Correlation and Records

Physically, decoherence characterized by Eq.(2.6) is intimately related to the storage of information about a system of interest somewhere in the universe. It is in this sense that decoherence replaces and generalizes the notion of measurement in ordinary quantum mechanics. Systems decohere, and hence acquire definite properties, not necessarily through measurement, but through their interactions and correlations with other systems. In this section I discuss these issues.

3.1 Records Imply Decoherence

Consider a closed system $S$ which consists of two interacting subsystems $A$ and $B$. The Hilbert space $\mathcal{H}$ of $S$ is therefore of the form $\mathcal{H}_A \otimes \mathcal{H}_B$. Suppose we are interested in the histories characterized solely by properties of system $A$, thus $B$ is regarded as the environment. The system is analyzed using the decoherence functional (2.5), where we take the $P_\alpha$ to denote a projection on $\mathcal{H}_A$ (strictly one should therefore write $P_\alpha \otimes I_B$, where $I_B$ denotes the identity on $\mathcal{H}_B$, but for convenience I largely neglect this notation). I also introduce projections $R_\beta$ on the Hilbert space $\mathcal{H}_B$.

I shall show that histories of $A$ decohere if the sequences of alternatives the histories consist of exhibit exact and persistent correlations with sequences of alternatives of $B$. More precisely, I imagine that the alternatives of $A$ characterized by $P_\alpha^n$ at each moment of time $t_k$ are perfectly recorded in $B$ as a result of their interaction. I also imagine that this record in $B$ is perfectly persistent (i.e., permanent). This means that at any time $t_f$ after the time $t_n$ of the last projection on $A$ there exist a sequence of alternatives of $B$, $\beta_1, \ldots, \beta_n$, that are in perfect correlation with the alternatives of $A$, $\alpha_1 \cdot \cdot \cdot \alpha_n$ at times $t_1 \cdot \cdot \cdot t_n$.

Correlations between subsystems are generally analyzed using the joint probability distribution

$$p(\alpha, \beta) = \text{Tr}(P_\alpha \otimes R_\beta \rho) \quad (3.1)$$

This is a bona fide probability because it involves projections at a single moment of time and, by the cyclic property of the trace, histories consisting of alternatives at a single moment of time automatically decohere. Then the alternatives of $A$ and $B$ characterized by the projections $P_\alpha$ and $R_\beta$ are said to be exactly correlated if

$$p(\alpha, \beta) = \delta_{\alpha \beta} p(\alpha) \quad (3.2)$$

The decoherence functional (2.6) may be written,

$$D(\alpha, \alpha') = \sum_{\beta_1 \cdot \cdot \cdot \beta_n} \text{Tr} \left( R_{\beta_1} \cdot \cdot \cdot R_{\beta_n} P_\alpha^n(t_n) \cdot \cdot \cdot P_{\alpha_1}(t_1) \rho P_{\alpha_1}(t_1) \cdot \cdot \cdot P_{\alpha_n}(t_n) \right) \quad (3.3)$$
using the exhaustivity of the projections $R^k_{\beta_k}$. Now using the inequality (2.7), one finds,

$$|D(\alpha', \beta')| \leq \sum_{\beta_1 \cdots \beta_n} [p(\alpha, \beta)p(\alpha', \beta')]^{1/2}$$

(3.4)

where $p(\alpha, \beta)$ denotes the diagonal elements of the summand in the right-hand side of (3.3). Now the object is to show that the strings of alternatives $\alpha$ and $\beta$ are perfectly correlated, and hence the right-hand side of (3.4) vanishes unless $\alpha = \alpha'$. Since $p(\alpha, \beta)$ are only candidate probabilities, we cannot use them to discuss these correlations. However, they are non-negative, and we may use this to derive the following simple inequality:

$$p(\alpha, \beta) \leq \sum_{\beta_1 \cdots \beta_{k-1}} \sum_{\alpha_{k+1} \cdots \alpha_n} \sum p(\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n)$$

$$= Tr \left( R^k_{\beta_k} \otimes P_{\alpha_k}(t_k) \rho_{eff}(\alpha_1 \cdots \alpha_{k-1}) \right) p(\alpha_1 \cdots \alpha_{k-1})$$

(3.5)

Here

$$\rho_{eff}(\alpha_1 \cdots \alpha_{k-1}) = \frac{P^{k-1}_{\alpha_k}(t_{k-1}) \cdots P^1_{\alpha_1}(t_1) \rho P^1_{\alpha_1}(t_1) \cdots P^{k-1}_{\alpha_k}(t_{k-1})}{p(\alpha_1 \cdots \alpha_{k-1})}$$

(3.6)

is the effective density matrix at time $t_k$. The first factor in the right-hand side of (3.5) is now of the form (3.1), and the degree of correlation between the alternatives $\alpha_k$ and $\beta_k$ may be discussed. Note that the relevant density operator is not $\rho$ but the effective density operator (3.6). (A further sum over $\alpha_1 \cdots \alpha_{k-1}$ could be performed on the right-hand side of (3.5), preserving the inequality, in which case it would be the averaged effective density operator that would enter). In the case of perfect correlation assumed here, the right-hand side of (3.5) is zero, and thus $p(\alpha, \beta) = 0$, unless $\alpha_k = \beta_k$. From (3.4), the decoherence functional is therefore diagonal in $\alpha_k$. Carrying out the same for all values of $k$, one finds that the decoherence functional is diagonal in all the $\alpha_k$’s. This shows that, as advertized, a perfect and persistent correlation of alternatives of $A$ with those of $B$ leads to exact decoherence of the histories of $A$.

This argument was inspired by an argument given by Hartle\textsuperscript{3} in his discussion of the recovery of the Copenhagen interpretation from the decoherent histories approach. It is, however, an improvement of his derivation, since it utilizes a proper definition of correlation, (3.1), (3.2), which may be extended to the case of approximate correlations discussed below.

3.2 Approximate Correlations, Entropy, Fluctuations.

More generally, one would expect the correlation between $A$ and $B$ to be only approximate, and the consequent decoherence will then also be approximate. Measures of approximate correlation are therefore required. A possible measure of the approximate correlation in (3.1) is the mutual information,

$$I(A; B) = \sum_{\alpha, \beta} p(\alpha, \beta) \ln \left( \frac{p(\alpha, \beta)}{p(\alpha)p(\beta)} \right)$$

(3.7)
The mutual information vanishes in the case of no correlation, \( p(\alpha, \beta) = p(\alpha)p(\beta) \), and is positive otherwise. A non-trivial theorem due to Kholevo and others\(^\text{10}\) shows that the mutual information of the joint probability distribution (3.1) is bounded above by the von Neumann entropy of the subsystems,

\[
I(A; B) \leq S[\rho_A], \quad I(A; B) \leq S[\rho_B]
\]

Here \( S[\rho] = -\text{Tr}(\rho \ln \rho) \) is the von Neumann entropy, \( \rho_A = \text{Tr}_B \rho \) is the reduced density matrix of subsystem \( A \), and similarly for \( B \). This means that the von Neumann entropy supplies an upper limit to the degree of correlation between subsystems. From the discussion above, it therefore also supplies an upper limit to the degree of decoherence. (More precisely, the degree of diagonality in \( \alpha_k \) for each \( k \) is controlled by the von Neumann entropy of the partially traced effective density operator at time \( t_k \)). This conclusion is intuitively satisfying: decoherence is, as stated, related to the storage of information somewhere in the universe. The degree of decoherence is therefore limited by the physical capacity of the “communication channel” transmitting the information about the distinguished system to its environment, and by the capacity of the environment to store information.

The von Neumann entropy frequently appears in discussions of decoherence of density matrices, where large entropy for the distinguished subsystem is held to be a signal of destruction of interference\(^\text{11}\). The above is, I believe, the first indication of a formal connection between decoherence of histories and von Neumann entropy. It would be of interest to explore the connection between these notions of approximate correlation with an environment and the approximate decoherence condition based on Eq.(2.7).

The von Neumann entropy also appears in a related context. The interaction with the environment typically needed for decoherence of histories also induces fluctuations in the evolution of the distinguished system. There is therefore a certain degree of tension between the demands of decoherence and approximate classical predictability: decoherence requires interaction with an environment, which inevitably produces fluctuations, but classical predictability requires that these fluctuations be small\(^2\). We will see an example of this in the quantum Brownian motion model of the next section. In Ref.[12], an information-theoretic measure of the size of these fluctuations was proposed – the Shannon information of the Husimi distribution (a certain type of smeared Wigner function) of the density matrix \( \rho \) of the distinguished system. This measure of uncertainty is in fact bounded from below by the von Neumann entropy of \( \rho \) (Ref.[12]). This suggests that the von Neumann entropy is the key to understanding the connection between decoherence and fluctuations: it limits the amount of decoherence from above but bounds the size of the fluctuations from below. Large entropy therefore permits good decoherence, but leads to large fluctuations; on the other hand, small entropy allows small fluctuations, but the amount of decoherence is also small.

Much remains to be done to make these ideas precise. They will be developed
3.3 Decoherence Implies Generalized Records

The decoherence achieved through persistent correlation with another system is stronger than consistency, since both real and imaginary parts of the decoherence functional vanish, i.e., Eq.(2.6) holds. There is in fact a converse to this result, namely that Eq.(2.6) is related to the existence of records. This subsection is a modest elaboration on the results of Gell-Mann and Hartle\textsuperscript{2}.

Consider the decoherence functional (2.5), for any system (not just the special one discussed above). Introduce the convenient notation

$$C_{\alpha} = P_{\alpha_{n}}(t_{n}) \cdots P_{\alpha_{1}}(t_{1})$$

Let the initial state be pure, $\rho = |\Psi\rangle\langle\Psi|$. In this case, the decoherence condition (2.6) is referred to as medium decoherence. It implies that the states $C_{\alpha}|\Psi\rangle$ are an orthogonal (but in general incomplete) set. There therefore exists a set of projection operators $R_{\beta}$ (not in general unique) of which these states are eigenstates,

$$R_{\beta} C_{\alpha}|\Psi\rangle = \delta_{\alpha\beta} C_{\alpha}|\Psi\rangle$$

Note that the $C_{\alpha}$'s are not themselves projectors in general. One may then consider histories consisting of the string of projections (3.9), adjoined by the projections $R_{\beta}$ at any time after the final time. The decoherence functional for such histories is of the form of that in the summand in Eq.(3.3):

$$D(\alpha, \beta|\alpha', \beta') = Tr \left( R_{\beta} C_{\alpha}|\Psi\rangle\langle\Psi| C_{\alpha}^\dagger R_{\beta}' \right)$$

These histories decohere exactly by virtue of (3.10) and (2.6), and thus the diagonal elements of (3.11), which we denote $p(\alpha, \beta)$, are true probabilities. The correlations contained in these probabilities may therefore be discussed. Indeed, Eq.(3.10) implies that $p(\alpha, \beta) = \delta_{\alpha\beta} p(\alpha)$, and thus $\alpha$ and $\beta$ are perfectly correlated.

Medium decoherence therefore implies the existence of a string of alternatives $\beta_{1} \cdots \beta_{n}$ perfectly correlated with the string $\alpha_{1} \cdots \alpha_{n}$. For this reason the projection operators $R_{\alpha}$ are referred to as generalized records: information about the histories characterized by alternatives $\alpha_{1} \cdots \alpha_{n}$ is recorded somewhere. It is, however, not possible to say that the information resides in a particular subsystem, since we have not specified the form of the system $S$; indeed, it may not even be possible to divide it into subsystems. It would be of interest to study the special case in which $S$ consists of two interacting subsystems, with the projections in (3.9) onto one of the subsystems. One could then ask whether medium decoherence of the distinguished subsystem alternatives implies the existence of physical records in the other subsystem.

One can give a similar discussion of the decoherence condition (2.6) when the initial state is mixed, although the connection with the existence of generalized records does not appear to come out as cleanly\textsuperscript{2}.

3.4 Reiteration: Consistency v. Decoherence

It is worth reiterating the discussions of Subsections 2.2, 3.1 and 3.3 on decoherence and consistency, and stressing their meaning. Consistency – i.e., $ReD(\alpha, \alpha') = 0$
for $\alpha \neq \alpha'$ – permits classical logic to be applied to the description of the system. It permits one to “talk about” the system – to make statements about it which may be related to each other by classical logic. It does not invite one to think of the system’s potential properties as actual, but it allows conditional statements to be made: given that a certain property is actualized one may consistently make logical deductions about which other properties then also hold. Decoherence – i.e. $D(\alpha, \alpha') = 0$ for $\alpha \neq \alpha'$ – is stronger than consistency. Since it implies consistency, it also permits classical logic to be applied. But it is in addition related to the existence of records. A set of histories which are decoherent, rather than merely consistent, may be thought of as possessing definite properties, since information about those properties is stored somewhere in the universe.

4. Quantum Brownian Motion Model

I now very briefly consider a particular model, namely the quantum Brownian motion model. This model has been extensively studied in the literature so only the briefest of accounts will be given here\textsuperscript{2,8,13}. The model consists of a particle of mass $M$ in a potential $V(x)$ linearly coupled to an environment consisting of a large bath of harmonic oscillators in a thermal state at temperature $T$. I consider histories of position samplings of the distinguished system. The samplings are continuous in time and Gaussian sampling functions are used (corresponding to approximate projection operators). The decoherence functional for the model is most conveniently given in path-integral form:

$$D[\bar{x}(t), \bar{y}(t)] = \int \mathcal{D}x \mathcal{D}y \delta(x_f - y_f) \rho(x_0, y_0)$$

$$\times \exp \left( \frac{i}{\hbar} S[x(t)] - \frac{i}{\hbar} S[y(t)] + \frac{i}{\hbar} W[x, y] \right)$$

$$\times \exp \left( - \int dt \frac{(x(t) - \bar{x}(t))^2}{2\sigma^2} - \int dt \frac{(y(t) - \bar{y}(t))^2}{2\sigma^2} \right)$$

(4.1)

Here, $S$ is the action for a particle in a potential $V(x)$, $\bar{x}(t)$, $\bar{y}(t)$ are the sampled positions and $x_f$ and $x_0$ denote the final and initial values respectively. The effects of the environment are summarized entirely by the Feynman-Vernon influence functional phase, $W[x, y]$, given by,

$$W[x(t), y(t)] = -\int_0^t ds \int_0^s ds' [x(s) - y(s)] \eta(s - s') \left[ x(s') + y(s') \right]$$

$$+ i \int_0^t ds \int_0^s ds' [x(s) - y(s)] \nu(s - s') \left[ x(s') - y(s') \right]$$

(4.2)

The explicit forms of the non-local kernels $\eta$ and $\nu$ may be found in Refs.[13]. Here it is assumed, as is typical in these models, that the initial density matrix of the total system is simply a product of the initial system and environment density matrices, and the initial environment density matrix is a thermal state at temperature $T$. 
Considerable simplifications occur in a purely ohmic environment in the FokkerPlanck limit (a particular form of the high temperature limit), in which one has

\begin{align}
\eta(s-s') &= M\gamma \delta'(s-s') \\
\nu(s-s') &= \frac{2M\gamma kT}{\hbar} \delta(s-s')
\end{align}

(4.3)

where \(\gamma\) is the dissipation. For convenience I will work in this limit.

One can see almost immediately that the imaginary part of \(W\), together with the Gaussian samplings in (4.1), will have the effect of suppressing widely differing paths \(\bar{x}(t), \bar{y}(t)\). Indeed, the suppression factor will be of order

\[\exp\left(-\frac{2M\gamma kT\sigma^2}{\hbar^2}\right)\]  

(4.5)

In cgs units \(\hbar \sim 10^{-27}\) and \(k \sim 10^{-16}\), so \(kT/\hbar^2 \sim 10^{40}\) if \(T\) is room temperature. Values of order 1 for \(M, \gamma\) and \(\sigma\) therefore lead to an astoundingly small suppression factor.

Decoherence through interaction with a thermal environment is thus a very effective process indeed.

More precisely, one can approximately evaluate the functional integral (4.1). Let \(X = (x+y)/2, \xi = x-y\), and use the smallness of the suppression factor to expand about \(\xi = 0\). Then the \(\xi\) functional integral may be carried out with the result,

\[D[\bar{x}(t), \bar{y}(t)] = \int DX W(M\dot{X}_0, X_0) \exp\left(-\int dt \frac{(X-\bar{x}+\bar{y})^2}{\sigma^2}\right)\]

\[\times \exp\left(-\int dt \frac{F[X]^2}{2(\Delta F)^2} - \int dt \frac{(\bar{x}-\bar{y})^2}{2\ell^2} - i\hbar \int dt \frac{(\bar{x}-\bar{y})F[X]}{4\sigma^2(\Delta F)^2}\right)\]  

(4.6)

where

\[F[X] = M\dddot{X} + M\gamma \ddot{X} + V'(X)\]  

(4.7)

are the classical field equations with dissipation, and

\[\Delta F^2 = \frac{\hbar^2}{\sigma^2} + 4M\gamma kT\]  

(4.8)

\[\ell^2 = 2\sigma^2 + \frac{\hbar^2}{4M\gamma kT}\]  

(4.9)

\(W(M\dot{X}_0, X_0)\) is the Wigner transform of the initial density operator.

The decoherence width (4.9) does not, in fact, immediately indicate the expected suppression of interference, because the temperature-dependent term will typically be utterly negligible compared to the \(\sigma^2\) term. The point, however, is that more precise notions of decoherence need to be employed. One should check some of the probability sum rules, or use the approximate decoherence condition based on Eq.(2.7), in which the sizes of the off and on-diagonal terms are compared. This has not been carried out for the general expression (4.6), and in fact seems to be rather hard. Satisfaction of the approximate decoherence condition was checked for
some special cases in Ref.[8]. Still, one expects the standard to which decoherence is attained to be of the order of the suppression factor (4.5), i.e., very good indeed.

Now consider the diagonal elements of the decoherence function, representing the probabilities for histories.

\[ p[\bar{x}(t)] = \int \mathcal{D}X \ W(MX_0, X_0) \]
\[ \times \ \exp \left( -\int dt \ \frac{(X - \bar{x})^2}{\sigma^2} - \int dt \ \frac{F[X]^2}{2(\Delta F)^2} \right) \]  

(4.10)

The distribution is peaked about configurations \( \bar{x}(t) \) satisfying the classical field equations with dissipation; thus approximate classical predictability is exhibited. The width of the peak is given by (4.8). Loosely speaking, a given classical history occurs with a weight given by the Wigner function of its initial data. This cannot be strictly correct, because the Wigner function is not positive in general, although it is if coarse-grained over an \( \bar{h} \)-sized region of phase space. For a more precise discussion of the interpretation of (4.10), see Ref.[14].

The width (2.8) has clearly identifiable contributions from quantum and thermal fluctuations. The thermal fluctuations dominate the quantum ones when \( 8M\gamma kT\sigma^2 >> \bar{h}^2 \), which, from (4.5), is precisely the condition required for decoherence, as previously noted\(^{12,14,15} \). Environmentally-induced fluctuations are therefore inescapable if one is to have decoherence.

As mentioned in the previous section, there is a tension between the demands of decoherence and classical predictability, both of which are necessary (although generally not sufficient) for the emergence of a quasiclassical domain\(^2 \). This tension is due to the fact that the degree of decoherence (4.5) improves with increasing environment temperature, but predictability deteriorates, because the fluctuations (4.8) grow. However, the smallness of Boltzmann’s constant ensures that the fluctuations (4.8) will be small compared to \( F[X] \) for a wide range of temperatures if \( M \) is sufficiently large. Moreover, the efficiency of decoherence as evidenced through (4.5) is largely due to the smallness of \( \bar{h} \), and will hold for a wide range of temperatures. So although there is some tension, there is a broad compromise regime in which decoherence and classical predictability can each hold extremely well.

5. Concluding Remarks

In this contribution, I have given an account of the decoherent histories approach to quantum mechanics. I have tried to cover some aspects of the subject that are not fully described elsewhere.

For me, one of the most interesting aspects of the meeting was to learn about the stochastic Schrödinger equation approaches of various workers, including Diósi, Ghirardi, Gisin, Karolyhazy, Pearle, Percival, Rimini and Spiller. Although very different from the decoherent histories approach, it was gratifying to discover that some of the deepest and most difficult questions are common to both of these approaches to quantum mechanics. An example is the general question of the most
natural most way to divide a sufficiently large complex system into distinguished system and environment. I feel that workers in these fields have much learn from each other. Indeed, as a consequence of the meeting, a project was commenced with the aim of exploring the seemingly close connections between the quantum state diffusion approach of Gisin and Percival\(^\text{16}\) and the decoherent histories approach\(^\text{17}\).

For further literature on the decoherent histories approach, see Refs.[18-28].

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