Long-distance interactions of branes: correspondence between supergravity and super Yang-Mills descriptions

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Abstract

We address the issue of correspondence between classical supergravity and quantum super Yang-Mills (or Matrix theory) expressions for the long-distance, low-velocity interaction potentials between 0-branes and bound states of branes. The leading-order potentials are reproduced by the \(F^4\) terms in the 1-loop SYM effective action. Using self-consistency considerations, we determine a universal combination of \(F^6\) terms in the 2-loop SYM effective action that corresponds to the subleading terms in the supergravity potentials in many cases, including 0-brane scattering off 1/8 supersymmetric \(4\|1\|^0\) and \(4\|4\\perp4\|^0\) bound states representing extremal \(D=5\) and \(D=4\) black holes. We give explicit descriptions of these configurations in terms of 1/4 supersymmetric SYM backgrounds on dual tori. Under a proper choice of the gauge field backgrounds, the 2-loop \(F^6\) SYM action reproduces the full expression for the subleading term in the supergravity potentials, including its subtle \(v^2\) part.
1 Introduction

One of the remarkable consequences of the open string theory description of D-branes \[ 1 \] is the existence of a close correspondence between the supergravity and super Yang-Mills theory results for certain interactions of D-branes and their bound states \[ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \]. For configurations of branes with enough amount of underlying supersymmetry, the long-distance and short-distance limits of the string-theory potential given by the annulus diagram are the same, implying that the leading-order (long-distance) interaction potential determined by the classical supergravity limit of the closed string theory is the same as the (short-distance) one-loop potential produced by the massless open string theory modes, i.e. by the super Yang-Mills theory \[ 3 \]. This was demonstrated explicitly for the leading-order terms in the potentials of interactions of 0-branes with 1/2 supersymmetric (non-marginal) bound states \[ 7, 8, 9 \] and with 1/4 supersymmetric marginal bound states \[ 13, 14, 15 \]. Similar conclusions were reached for the leading-order interaction of D-brane probes with 1/8 supersymmetric bound states representing \( D = 5 \) black holes \[ 16, 17 \].

On the SYM side, all of the leading-order potentials (including also the cases of interaction with non-supersymmetric bound states of branes like \( 6 + 0 \) \[ 4, 18, 19, 20, 21 \], \( 8 + 0 \) \[ 22 \] and configurations corresponding to near-extremal \( D = 5 \) black holes \[ 17 \]) may be obtained by plugging the corresponding SYM backgrounds into the leading universal \( F^4 \) terms in the IR part of the one-loop SYM effective action in \( D < 10 \). The general form of these \( F^4 \) terms was discussed in detail in \[ 23, 14, 17 \].

The aim of this paper will be to attempt to understand if the supergravity-SYM correspondence extends also to the level of subleading terms in the interaction potentials. The first step in this direction was made in \[ 24, 25 \] where the 0-brane - 0-brane interaction was considered. It was shown that the 2-loop effective action in the \( D = 1 + 0 \) dimensional reduction of SYM theory computed for the relevant (velocity \( v \), distance \( r \)) background has \( v^6/r^{14} \) as the leading IR term (i.e. does not contain a \( v^4 \)-term \[ 24 \]) and its coefficient is in precise agreement \[ 25 \] with the first subleading term in the supergravity potential (computed using large \( N \) limit or the ‘null reduction’ prescription \[ 23 \] implementing the suggestion of \[ 20 \]).

We expect that as in the case of the leading \( v^4/r^7 \) potential \[ 3 \], this coincidence should have a weak-coupling open string theory explanation and thus should be more universal, i.e. should apply also to other appropriate configurations of branes of different dimensions and with various amounts of supersymmetry (in particular, to a Dp-brane bound to other branes interacting with a Dp-brane bound to the same or different combination of branes, and to T-dual configurations). Moreover, similar relations may be true also between all higher order terms in the classical supergravity potentials and the leading infra-red (low-energy) contributions to the higher-loop terms in the SYM effective action in \( D < 10 \) dimensions.
Like the $v^4/r^7$ potential originates from the leading IR term $F^4/M^7$ in the 1-loop SYM effective action (with an IR cutoff $M \sim r$), the $v^6/r^{14}$ potential may be related to the leading IR term $F^6/M^{14}$ in the 2-loop SYM effective action. We shall conjecture that in general

(i) the leading IR part (which has also an appropriate scaling with $N$) of the $L$-loop term in the $U(N)$ SYM effective action in $D = 1 + p$ dimensions has a universal $F^{2L+2}/M^{(7-p)L}$ structure;

(ii) computed for a SYM background representing a configuration of interacting branes, the $F^{2L+2}/M^{(7-p)L}$ term should reproduce the $1/r^{(7-p)L}$ term in the corresponding long-distance classical supergravity potential.

The first part of this conjecture is known to be true for $L = 1$, and we interpret the result of [24] about the vanishing of the $v^4$ term in the 2-loop $D = 1$ SYM effective action as suggesting that it is also true in general for $L = 2$. The assumption that the leading part of the 2-loop term is the $F^6$ one is also implied by the $F^4$ term non-renormalisation theorem of [27]. While we formulated the above conjecture for general $L$, most of considerations in this paper will be restricted to the $L = 2$ case.

Given that direct computation of the $L > 1$ terms in the $D > 1$ SYM effective action is hard, our strategy in trying to test this conjecture will be to make a plausible assumption about the structure of the relevant part of the SYM effective action and then check if our ansatz can match known supergravity potentials for various special configurations of branes. Since different brane systems with different amounts of supersymmetry have very different SYM descriptions, the conjecture that all of the corresponding interaction potentials originate from a single universal $F^{2L+2}$-type SYM expression provides highly non-trivial constraints on the latter.

We shall study the first non-trivial case of $L = 2$ and demonstrate that indeed the interaction potentials for various examples of interactions of 0-branes with type IIA BPS bound state of branes with $1/2, 1/4$ or $1/8$ of supersymmetry and non-trivial 0-brane content can be described by a certain universal combination of $F^6$ terms in the leading IR part of the 2-loop SYM effective action in $D = 1 + p$, thus suggesting a non-trivial generalisation of the 0-brane – 0-brane result of [25].

The type IIA brane systems we shall deal with will be of the following special type: a 0-brane probe (a cluster of $n_0$ 0-branes) interacting with a BPS (marginal or non-marginal, $1/2^n$ supersymmetric) bound state of branes having a non-zero 0-brane charge ($N_0$) component and wrapped over a torus $T^p$. The probe will have a velocity along a direction transversal to the ‘internal’ torus. T-duality along all of the directions of $T^p$

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1While our discussion may have obvious implications for the Matrix theory proposal [4], we shall be assuming the weak string coupling limit and consider only perturbative SYM contributions.

2Though ‘universality’ or ‘BPS saturation’ of terms with higher than 6 powers of $F$ may seem less plausible, there are, in fact, string-theory examples of higher-order terms that receive contributions only from one particular loop order, to all orders in loop expansion [28].
relates this system to a system of Dp-brane probe (with charge $n_0$) parallel to a bound state of $N_0$ Dp-branes bound to D$q$-branes ($q < p$) and wrapped over the dual torus $\tilde{T}^p$. Such system should thus have a $U(n_0 + N_0)$ SYM description \cite{29} with a non-trivial SYM background reflecting the presence of other branes in the bound state \cite{30} and the velocity of the probe \cite{3, 31}.

On the supergravity side, the action for a 0-brane probe interacting with a background produced by a marginal bound state of branes \((1||0, 4||0, 4\perp1||0\text{ or } 4\perp4||0)\) which is essentially the same as the action for a $D = 11$ graviton scattering off an M-brane configuration \((2+\text{wave}, 5+\text{wave}, 2\perp5+\text{wave} \text{ or } 5\perp5+\text{wave})\) has the following general structure \cite{32, 33, 16}

\[
I_0 = -T_0 \int dt \ H_0^{-1} \left[ \sqrt{1 - H_0 H_1...H_k v^2} - 1 \right] \equiv \int dt \left[ \frac{1}{2} T_0 v^2 - V(v, r) \right]. \tag{1.1}
\]

Here $H_0$ and $H_1, ..., H_k$ \((k = 1 \text{ for } 1/4 \text{ supersymmetric bound states and } k = 2 \text{ or } k = 3 \text{ for } 1/8 \text{ supersymmetric bound states})\) are the harmonic functions $H_i = 1 + Q_i/r^{7-p}$ corresponding to the constituent charges of the bound state. Since for D-branes $Q_i \sim g_s N_i$ and $T_0 \sim n_0 g_s^{-1}$ where $g_s$ is the string coupling constant, the long-distance expansion of the classical supergravity interaction potential $V$ has the following form\footnote{For simplicity, we are assuming here that the bound state has only RR charges; cases with non-vanishing fundamental string charge or momentum \((\tilde{Q}_1 \sim g_s^2)\) can be treated in a similar way, see \cite{14, 15} and section 3.1 below.}

\[
V = \sum_{L=1}^{\infty} V^{(L)} = \frac{n_0}{g_s} \sum_{L=1}^{\infty} \left( \frac{g_s}{r^{7-p}} \right)^L k_L(v, N_i), \tag{1.2}
\]

so that the \((1/r^{7-p})^L\) term has the same $g_s$ dependence as in the $L$-loop term in SYM theory with coupling $g_{YM}^2 \sim g_s$.

The detailed form of the coefficients $k_L$ in the potential in (1.2) reflects the two important features of the action (1.1): (1) the special role played by the 0-brane function $H_0$, and (2) the appearance of the product of the ‘constituent’ harmonic functions. The second property is the direct consequence of the ‘harmonic function rule’ structure \cite{34} of the supergravity backgrounds representing marginal BPS bound states of branes. It implies that all the constituent charges, except the 0-brane one, enter $V$ in a completely symmetric way.

The asymmetry between $H_0$ and $H_1, ..., H_k$ is strengthened by further important assumption that in expanding (1.1) in powers of $1/r^{7-p}$ one should take $H_0$ without the usual asymptotic term 1, i.e. as $H_0 = Q_0/r^{7-p}$, $Q_0 \sim g_s N_0$. This prescription which is crucial for the precise correspondence with the SYM theory already at the leading (1-loop) level may be interpreted at least in two possible ways. One may assume (as was done in \cite{3, 8, 13}) that $N_0$ is large for fixed $r$ and $g_s$ (in particular, $N_0 \gg N_1, ..., N_k$), so that $H_0 = 1 + Q_0/r^{7-p} \approx Q_0/r^{7-p}$. Alternatively, one may keep $N_0$ finite but consider the $D = 10$ brane system as resulting from an M-theory configuration with $x^- = x^{11} - t$.
direction being compact $[20]$. As was pointed out in $[25]$, the dimensional reduction of $D = 11$ gravitational wave combined with M-brane configurations along $x^-$ results in supergravity backgrounds with $H_0 \rightarrow H_0 - 1 = Q_0/r^{7-p}$. In what follows we shall always set $H_0 = Q_0/r^{7-p}$ without making any assumptions about magnitude of $N_0$.

To reproduce the detailed form of the subleading terms in (1.2) from the SYM theory it should be possible to encode the structure of the supergravity expression (1.1) (in particular, the cross-terms coming from the product $H_1 H_2 \ldots H_k$ of the harmonic functions and reflecting effective interactions of constituent branes in the bound state) in the explicit form of (a) universal $F^{2L+2}$ terms in the SYM effective action, and (b) specific SYM background representing the bound state of branes on the SYM side.

A $U(N_0)$ SYM background $F_{ab}$ on the dual torus $\tilde{T}^p$ which is a candidate for a description of a marginal BPS D-brane bound state with $1/2^n$ of $\mathcal{N} = 2$, $D = 10$ supersymmetry should satisfy the following conditions: (1) $F_{ab}$ should preserve $1/2^{n-1}$ of $\mathcal{N} = 1$, $D = 10$ supersymmetry of SYM theory; (2) considered as a gauge field background on a D$p$-brane world volume, $F_{ab}$ should induce only the charges $\sim \int \text{tr}(F \wedge \ldots \wedge F)$ of constituent branes; (3) the classical D-brane (Born-Infeld) action computed in this background

$$I = T_p \int d^p \tilde{x} \text{Str} \sqrt{-\det(\eta_{ab}I + F_{ab})}, \quad (1.3)$$

should reproduce the mass of the corresponding marginal BPS bound state which is in agreement with the supergravity background describing the bound state, i.e. is proportional to the sum of charges of the constituent branes. In general, these conditions do not fix the required SYM background uniquely. One of the lessons of our discussion below will be the crucial role of an appropriate choice of the SYM representation of the brane bound states for supergravity – SYM correspondence at the subleading level.

To fix the form of the relevant 2-loop $F^6$ term in the SYM effective action $\Gamma$ we shall proceed in steps, first considering the interaction of a 0-brane probe with $1/2$ supersymmetric non-marginal bound state $(p\ldots+0)$ and then turning to more complex cases of interaction with $1/4$ and $1/8$ supersymmetric bound states. Demanding the agreement with the supergravity potential $V$ in the simplest cases we will be able to extract the information about the required structure of $\Gamma$ which will then be checked and sharpened (by including terms that were vanishing on the previous less complicated backgrounds) on more complex examples of 0-brane scattering off bound states with reduced amount of supersymmetry. This procedure will turn out to be by far less arbitrary as it may seem first and the emerging consistent picture will provide support for the above conjectures.

In section 2 we shall describe an ansatz for the leading IR part $\Gamma$ of the full SYM effective action $\Gamma$ which is expected to reproduce the supergravity potential $V$ (1.2). The proposal for the higher-loop terms in $\Gamma$, and, in particular, for the 2-loop $F^6$ term, which we shall make will be motivated and tested by comparing with subleading terms in $V$ in various special cases in the following sections.

In section 3 we shall consider the interaction between a 0-brane and a non-marginal
1/2 supersymmetric bound state \((p + ... + 0)\) of type IIA D-branes. Since the action for the latter treated as a probe can be described by switching on a constant abelian background on the \(Dp\)-brane world sheet, it will be straightforward to demonstrate that the corresponding supergravity potential \(V\) admits a SYM interpretation, suggesting as a result the required pattern of the Lorentz-index (BI polynomials) and internal index (single trace in adjoint representation \(\text{Tr}\)) contractions in \(\Gamma\). As a particular example, we shall consider the 0-brane - 0-brane scattering, interpreting the 2-loop result of \([25]\) as a special case of the general \(\text{Tr}F^6\) SYM expression and suggesting its extension to all loop orders.

The 0-brane interactions with 1/4 supersymmetric marginal bound states (of a fundamental string and a 0-brane 1\(\parallel\)0 and of a 4-brane and a 0-brane 4\(\parallel\)0) will be discussed in section 4. We shall find that exact all-order expression for the classical 0 \(- (1\parallel0)\) supergravity potential is reproduced by the same ansatz for the SYM effective action that was giving the full 0 \(- (p + ... + 0)\) potential in section 3. The situation in the 0 \(- (4\parallel0)\) case turns out to be more subtle as the corresponding gauge field background is described by two different (though still commuting) \(su(N)\) matrices. We shall determine an extra correction term in the 2-loop part of \(\Gamma\) which was vanishing in the previous 0 \(- (p + ... + 0)\) and 0 \(- (1\parallel0)\) cases but is necessary for the exact agreement between the subleading term \(V^{(2)}\) in \((1,2)\) and the \(O(F^6)\) 2-loop SYM term in the 0 \(- (4\parallel0)\) case.

The consistency of the emerging expression for the 2-loop SYM effective action will be tested further in section 5 where we shall consider the 0-brane interactions with 1/8 supersymmetric marginal bound states 4\(\perp\)1\(\parallel\)0 (or 5\(\perp\)2\(\perp\)wave in \(D = 11\)) and 4\(\perp\)4\(\perp\)4\(\parallel\)0 (or 5\(\perp\)5\(\perp\)5\(\perp\)wave in \(D = 11\)), which (when wrapped over \(T^5\) and \(T^6\)) correspond to \(D = 5\) and \(D = 4\) extremal black holes with regular horizons. These bound state configurations are represented by curved type IIA \(D = 10\) supergravity backgrounds but also admit simple descriptions in terms of 1/4 supersymmetric SYM gauge field backgrounds on the dual tori, which, as we shall see, are not unique. We shall find that in the 4\(\perp\)1\(\parallel\)0 \((D = 5\) black hole) case there exists a natural choice of a SYM background on \(\tilde{T}^5\) which when substituted into the 2-loop SYM action determined in the previous sections reproduces the complete expression for the subleading term \(V^{(2)}\) in the supergravity potential, including the \(v^2\) term (cf. \([16]\)). The 4\(\perp\)4\(\perp\)4\(\parallel\)0 \((D = 4\) black hole) configuration will be represented either by commuting \(([F,F] = 0)\) or non-commuting (‘three instanton’) 1/4 supersymmetric gauge field backgrounds on \(\tilde{T}^6\). We shall find that to reproduce the full expression for the subleading supergravity potential \(V^{(2)}\) it is necessary to use the non-commuting SYM background and to include in the 2-loop effective action terms with commutators of \(F\). Such commutator terms should, in general, be present in \(\Gamma\) but were not contributing in our previous examples which were described by commuting SYM backgrounds.

Some important remaining questions will be mentioned in section 6. In Appendix we shall describe another ‘commuting’ representation for the \(D = 4\) black hole configuration.
2 Interactions between branes and Super Yang-Mills effective action

Our aim in this section will be to describe possible structure of the leading IR part $\Gamma$ of the full SYM effective action $\Gamma$ which is relevant for the discussion of interaction potentials between a 0-brane probe and a bound state of branes wrapped over a p-torus. We shall consider the case of weak string coupling and assume that such configurations can be represented by appropriate backgrounds in SYM theory in $D = p + 1$ dimensions. The proposal for the higher-loop terms in $\Gamma$, and, in particular, for the 2-loop $F^6$ term, we shall make below will be motivated and tested by comparing with the subleading terms in the supergravity interaction potentials in various special cases considered in the following sections. Our ansatz for $\Gamma$ will be a natural generalisation of the one-loop $F^4$-term in $\Gamma$ which governs the leading-order interaction potentials between different combinations of branes [14] and in the special 0-brane scattering case it will also agree with the direct 2-loop $D = 1$ SYM calculations in [24, 25, 35].

The fields of the maximally supersymmetric $D$-dimensional SYM theory (obtained by reduction from $D = 10$ SYM) are the vectors $A_a$ ($a = 0, ..., D - 1$) and the scalars $X_i$ ($i = D, ..., 9$). In general, both may have non-trivial background values. The system we will be interested in consists of a 0-brane probe (a marginal bound state of $n_0$ 0-branes) interacting with a BPS bound state of branes containing, in particular, $N_0$ 0-branes, and wrapped over $T^p$. Under T-duality along all of the directions of the torus it becomes a D$p$-brane probe with charge $n_0$ parallel to a D$p$-brane source with charge $N_0$ bound to some other branes of lower dimensions. Assuming that the probe and the source are separated by a distance $r$ in the direction 8 and that probe has velocity $v$ along the transverse direction 9, this configuration may be described by the following $u(N)$, $N = n_0 + N_0$, SYM background on the dual torus $\tilde{T}^p$

$$\hat{A}_a = \left( \begin{array}{cc} 0_{n_0 \times n_0} & 0 \\ 0 & A_a \end{array} \right), \quad \hat{X}_i = \left( \begin{array}{cc} 0_{n_0 \times n_0} & 0 \\ 0 & X_i \end{array} \right), \quad i \neq 8, 9,$$

$$\hat{X}_8 = \left( \begin{array}{cc} r I_{n_0 \times n_0} & 0 \\ 0 & 0_{N_0 \times N_0} \end{array} \right), \quad \hat{X}_9 = \left( \begin{array}{cc} vt I_{n_0 \times n_0} & 0 \\ 0 & 0_{N_0 \times N_0} \end{array} \right),$$

where $A_a$ and $X_i$ are $N_0 \times N_0$ matrices in the fundamental representation of $u(N_0)$ which describe the source bound state. For example, $A_m$ may be an instanton field representing a 0-brane charge on a 4-brane (i.e. $4||0$ bound state) [30, 36] while $X_i$ may be a wave representing a momentum flow along some direction of $T^p$, or, after T-duality along that direction, a fundamental string charge (for $p = 1$ this corresponds to the $1||0$ bound state) [37, 38, 39].

In general, the SYM effective action $\Gamma$ (computed using the background field gauge) is a gauge-invariant functional of the background fields $\Gamma(A, X) = \Gamma(F, X, DF, DX, ...)$.

Assuming that the source brane configuration is a supersymmetric (BPS) one, $\Gamma$ should
vanish for \( v = 0 \) as well as for \( r \to \infty \). The long-distance interaction potential will be given by the low-energy expansion of \( \Gamma \) in powers of \( F \) multiplied by powers of \( 1/X_8 \) or \( 1/r \). The background value of \( X_8 \) plays the role of an effective IR cutoff (in the open string theory picture it is related to the mass of the open string states stretched between the probe and source branes). \(^4\)

The dependence on derivatives of the scalar fields \( X_i \) (and thus, in particular, on \( v \)) \(^2\) \([10, 14]\) may be formally determined from the dependence of \( \Gamma \) on the gauge field in a higher-dimensional background representing T-dual \( (X_i \to A_i) \) configuration. Indeed, from the point of view of the open string theory description of D-branes \([1]\) \( \Gamma \) should be related to the short-distance limit of the open string loop diagrams and thus should be ‘covariant’ under the T-duality \([1, 2]\) \( \Gamma \) (the string partition function is given by the path integral with the source term \( \int dt[\partial_t x^a A_a(x) + \partial_a x^i X_i(x)] \) so that \( A_s \leftrightarrow X_s \) under T-duality). The dependence on \( \partial_n X_i \) may thus be determined from \( \Gamma(F) \) in a higher-dimensional pure gauge field background with \( F_{mi} = D_m X_i \).

The problem is then formally reduced to finding the SYM effective action in the case of a purely gauge field background and with an effective IR cutoff \( M = r \) provided by the scalar field background (we set the string tension \( T = (2\pi \alpha')^{-1} = 1 \)) \(^3\) The corresponding SYM theory on \( \bar{T}^p \) has the action

\[
S = -\frac{1}{4g_{YM}^2} \int d^{p+1}\bar{x} \; \text{tr} F^2 + ... = -\frac{1}{8g_{YM}^2 N} \int d^{p+1}\bar{x} \; \text{Tr} F^2 + ... , \quad d^{p+1}\bar{x} = dt d^{p} \bar{x} , \tag{2.3}
\]

\[
g_{YM}^2 = (2\pi)^{-1/2} g_s \bar{V}_p , \tag{2.4}
\]

where \( g_s \) is the string coupling constant, \( \bar{V}_p = \int d^{p} \bar{x} \) is the volume of \( \bar{T}^p \) \( (V_p \bar{V}_p = (2\pi/T)^p) \) and \( \text{tr} \) and \( \text{Tr} \) are the traces in the fundamental and adjoint representations of \( su(N) \). The value of the SYM coupling constant is dictated by T-duality considerations for a system of 0-branes on a torus \([1] [10, 11, 12]\) The action for a collection of Dp-branes wrapped over \( \bar{T}^p \) is \( S = -T_p \int d^{p+1}\bar{x} \; \text{tr} \sqrt{-\det(\eta_{ab} + \partial_a \bar{X}^i \partial_b \bar{X}^i + T^{-1} F_{ab})} + ... \), where \( T_p \) is the Dp-brane tension \([1]\). Viewed as the leading term in this action, eq. \((2.3)\) should thus have the coefficient \( g_{YM}^2 = T^2/T_p = (2\pi)^{(p-1)/2} T^{(3-p)/2} g_s \), where \( \bar{g}_s \) is the string coupling constant of the T-dual theory satisfying the standard relation \( V_p/g_{YM}^2 = \bar{V}_p/\bar{g}_s^2 \). For \( T = 1 \) this gives \((2.4)\).

In \( D = p+1 \) dimensions \( g_{YM}^2 \) has mass dimension \( 3-p \) so that on dimensional grounds, the relevant part of \( \Gamma \) expanded in powers of \( F \) should have the following structure

\[
\Gamma = \sum_{L=1}^{\infty} (g_{YM}^2 N)^{L-1} \int d^{p+1}\bar{x} \sum_n C_n L \frac{1}{M^{2n-(p-3)L-4}} F^n , \tag{2.5}
\]

\(^4\)Though not all of the SYM excitations are getting explicit mass terms, the remaining IR divergences must cancel out as in \([24]\) and should not contribute to the leading IR (‘interaction potential’ \( \Gamma \)) part of \( \Gamma \).

\(^5\)We shall be interested only in the low-energy limit of the SYM theory, i.e. will not consider the UV cutoff dependent parts in the corresponding effective actions (assuming the existence of an explicit UV cutoff effectively provided in the weak-coupling case by the string theory).
where \( L \) is the number of loops and \( F^n \) stand for all possible contractions of \( n \) factors of the field strength matrix (we shall not explicitly consider terms with covariant derivatives of \( F \) assuming that they can be ignored for the relevant backgrounds).

In what follows we shall be interested only in a special subset of terms in (2.5) (generalising the ‘diagonal terms’ in [25]) which have the right coupling \( g_s \), 0-brane charge \( N_0 \) and distance \( r = M \) dependence to be in correspondence with the terms in the long-distance expansion (1.2) of the classical supergravity interaction potential \( V \) between a Dp-brane probe (with tension \( \sim n_0/g_s \)) and a Dp-brane source (with ‘charge’ parameter \( \sim g_s N_0 \)). As it is clear from the string-theory description of interaction between two Dp-branes, these terms should come out of planar diagrams of SYM theory. As was already mentioned in the Introduction, our conjecture is that due to maximal underlying supersymmetry of the SYM theory, the terms \( F^{2L+2}/M^{(7-p)L} \) represent, in fact, the leading IR contribution to \( \Gamma \) at \( L \)-th loop order. This is true for \( L = 1 \) [43, 23] and, in view of the results of [24, 27], should certainly be the case also for \( L = 2 \).

The sum of these leading IR terms at each loop level will be denoted by \( \Gamma \)

\[
\Gamma = \sum_{L=1}^{\infty} \Gamma^{(L)} = \frac{1}{2} \sum_{L=1}^{\infty} \int d^{D+1}x \left( \frac{a_p}{M^{7-p}} \right)^L (g_{YM} N)^{L-1} \hat{C}_{2L+2}(F) ,
\]

(2.6)

The coefficients \( a_p \) here must be universal, i.e. they cannot depend on \( N \). As we shall find from comparison with the supergravity potential \( 2\pi \alpha' = 1 \)

\[
a_p = 2^{2-p} \pi^{-(p+1)/2} \Gamma \left( \frac{5-p}{2} \right) .
\]

(2.7)

The structure of the coefficients \( \hat{C}_{2L+2}(F) \) should be such that when computed for the relevant gauge field backgrounds they should contain an extra factor of \( n_0 N_0 \) so that the order \( n_0 N_0^L \) term in (2.6) could match the corresponding term in the supergravity expression \( \int dt V \) (1.2).

The central question which we shall be addressing below is the following: which Lorentz and internal index structure of \( \hat{C}_{2L+2} \) or the ‘diagonal’ \( F^{2L+2} \) terms in (2.6) is required in order for the resulting \( \Gamma \) to agree with the supergravity potential (1.2). Since there are several inequivalent configurations of branes (involving BPS bound states of branes with different amounts of supersymmetry), the assumption that the interaction potentials for all of them should be described by the same universal SYM expression \( \Gamma \) (2.6) imposes non-trivial constraints on the latter.

Let us first assume that the background field strength \( F_{ab} \) belongs to the Cartan part of \( su(N) \), i.e. that all of its components commute, \([F_{ab}, F_{cd}] = 0 \). This will be the case for most of the examples discussed below. Our main assumption (motivated by the form of the supergravity potential in the case of 0-brane interaction with 1/2 supersymmetric non-marginal bound states of D-branes discussed in section 3) will be that at least for commuting backgrounds, i.e. up to the ‘commutator terms’ involving \([F, F] \), the structure
of the Lorentz-index contraction in $\hat{C}_{2n} \sim F^{2n}$ in (2.6) is the same as in the polynomials $C_{2n} = O(F^{2n})$ appearing in the expansion of the abelian Born-Infeld action,

$$\sqrt{-\det(\eta_{ab} + F_{ab})} = \sum_{n=0}^{\infty} C_{2n}(F),$$

(2.8)

$$C_0 = 1, \quad C_2 = -\frac{1}{4} F^2, \quad C_4 = -\frac{1}{8} \left[ F^4 - \frac{1}{4} (F^2)^2 \right],$$

(2.9)

$$C_6 = -\frac{1}{12} \left[ F^6 - \frac{3}{8} F^4 F^2 + \frac{1}{32} (F^2)^3 \right], \quad \ldots,$$

(2.10)

where $F^k$ is the trace of the matrix product in Lorentz indices, i.e.

$$F^2 = F_{ab} F_{ba}, \quad F^k = F_{a_1 a_2} F_{a_2 a_3} \ldots F_{a_k a_1}.$$

This is indeed what happens (for generic $F_{ab}$) in the explicitly known 1-loop expression for $\Gamma$ \cite{23, 14}

$$\Gamma^{(1)} = -\frac{\Gamma^{(7-p)}}{2(4\pi)^{(p+1)/2} M^{7-p}} \int d^{p+1}\hat{x} \ b_8 + O(\frac{1}{M^{9-p}}) = \Gamma^{(1)} + O(\frac{1}{M^{9-p}}),$$

(2.11)

i.e. (cf. [2.6])

$$\Gamma^{(1)} = \frac{\alpha_p}{2M^{7-p}} \int d^{p+1}\hat{x} \ \hat{C}_4(F),$$

(2.12)

$$\hat{C}_4 = \text{STr} \ C_4 = -\frac{1}{8} b_8 = -\frac{1}{8} \text{STr} \left[ F^4 - \frac{1}{4} (F^2)^2 \right]$$

$$= -\frac{1}{12} \text{Tr} \left( F_{ab} F_{bc} F_{cd} F_{dc} + \frac{1}{2} F_{ab} F_{bc} F_{dc} F_{ad} - \frac{1}{4} F_{ab} F_{cd} F_{ad} F_{cd} - \frac{1}{8} F_{ab} F_{cd} F_{cd} F_{cd} \right).$$

(2.13)

Here $\text{STr}$ is the symmetrised trace in the adjoint representation of $su(N)$ which may be expressed in terms of traces $\text{tr}$ in the fundamental representation $[14]$.

In general, only the traceless $su(N)$ part of the $u(N)$ background field matrix $F$ couples to the quantum fields and thus enters the effective action. Symbolically, if $F = \left( \begin{array}{cc} F_1 & 0 \\ 0 & F_2 \end{array} \right)$, where $F_i$ belong to $su(N_i)$, $N = N_1 + N_2$, then $\text{Tr} F^4 = \text{Tr}_1 F_1^4 + \text{Tr}_2 F_2^4 + f(F_1, F_2) \cdot f(F_1, F_2) = 2[N_1 \text{tr} F_1^2 + N_2 \text{tr} F_2^2 + 6(\text{tr}_1 F_1^2)(\text{tr}_2 F_2^2)]$ (see eq. (2.16) below) and it is the latter part $f(F_1, F_2)$ that describes interaction (see also [17] \footnote{We are grateful to J. Maldacena for a discussion of this point.} between two clusters of branes). The ‘self-energy’ terms $\text{Tr}_1 F_1^4$ and $\text{Tr}_2 F_2^4$ vanish in the case when $F_1$ and $F_2$ are supersymmetric SYM backgrounds representing BPS states of branes. Since this is the case we will be discussing below, we will not discriminate between $\Gamma$ and interaction potential between branes.

Our next assumption will be that for commuting $F_{ab}$ backgrounds, the pattern of contraction over the internal indices in $\hat{C}_{2n}(F)$ should be similar to that in (2.13), i.e. to
the single (symmetrised) trace in the adjoint representation. We shall see that the simple ansatz
\[
\hat{C}_{2n}(F) = \text{STr } C_{2n}(F),
\]
where STr is applied to the polynomial \( C_{2n}(F) \) appearing in the expansion of the BI action \((2.8)\) with each \( F_{ab} \) now promoted to an \( su(N) \) matrix,\(^8\) leads to \( \Gamma \) \((2.6)\) which reproduces the full supergravity potential \( V \) in the case of a 0-brane interacting with a general 1/2 BPS bound state \( p + \ldots + 0 \) (e.g., with another 0-brane or \( 2 + 0 \) bound state) as well as in the case of a 0-brane interacting with the 1/4 supersymmetric bound state \( 1 \parallel 0 \). The Tr structure of \( \hat{C}_{2n} \) provides, in particular, the required \( n_0 N_0 \) factor.

We shall find, however, that STr in \((2.14)\) should be modified by certain correction terms (which vanish in the above special cases where the background \( F_{ab} \) is essentially proportional to a single \( su(N) \) matrix) but whose presence is needed for the SYM – supergravity correspondence in the case of more complicated backgrounds involving self-dual gauge field strengths \((4 \parallel 0 \text{ and its generalisations})\).

The difference between the 1-loop and higher loop terms in \( \Gamma \) in what concerns the internal index structure is clear from the form of the \( SU(N) \) YM Lagrangian expanded \((2.5)\) near a background \( A_{a}^{s} \). In the background field gauge (\( s = 1, \ldots, N^{2} - 1 \))
\[
L = \frac{1}{g_{\text{YM}}^{2} N} \left[ \frac{1}{2} B_{a}^{s} \Delta_{ab} B_{b}^{s} + f_{rst} B_{a}^{s} B_{b}^{t} D_{a} B_{b}^{r} + f_{rst} B_{a}^{s} B_{b}^{t} f_{st} B_{a}^{r} B_{b}^{t} \right],
\]
where \( \Delta_{ab} = -\eta_{ab} D^{2} - 2 F_{ab}, \ D = D(A) \). The 1-loop effective action (obtained after \( \Delta_{ab} \rightarrow \Delta_{ab} + M^{2} \)) has, indeed, the form of a sum of \( 1/M^{n} \) terms multiplied by a single trace in the adjoint representation of a polynomial in \( F_{ab} \). This is no longer so in general for higher-loop corrections to \( \Gamma \) (there are many different contractions from products of the structure constants \( f_{rst} \)). It appears, however, that the contributions to the ‘diagonal’ or leading IR part \( \Gamma \) \((2.6)\) of \( \Gamma \) \((2.5)\) do have a Tr-type structure, up to the ‘commutator terms’ (i.e. up to the terms that vanish when evaluated on simple ‘commuting’ backgrounds). One may attempt to understand this using large \( N \) limit considerations.

While the 1-loop coefficient \( \hat{C}_{4} \) \((2.13)\) is equal to \( \text{STr} C_{4} \) for generic \( F_{ab} \), we shall use the following ansatz for \( \hat{C}_{2n} \) with higher \( n > 2 \)
\[
\hat{C}_{2n}(F) = \hat{\text{STr}} \ C_{2n}(F) = \text{STr } C_{2n}(F) + \text{STr'} C_{2n}(F).
\]
Here \( \hat{\text{STr}} \) will be defined in terms of somewhat different as compared to STr combination of symmetrised traces in the fundamental representation (see below). We shall explicitly determine \( \hat{\text{STr}} \) or the ‘correction term’ \( \text{STr'} C_{2n} \) in \((2.13)\) from comparison with supergravity potential for \( n = 3 \), i.e. in the case of the 2-loop coefficient \( \hat{C}_{6} \).

\(^8\)For commuting \( F_{ab} \) the symmetrised trace STr is of course equal simply to the trace Tr. The use of symmetrisation here is to isolate the terms that are non-vanishing on commuting \( F_{ab} \) from the remaining ‘commutator terms’ (cf. \[15\]). The symmetrisation is also helpful in order to express Tr in terms of traces in the fundamental representation (Tr of a product of several different matrices takes simpler form if one symmetrises the factors in the product).
For a single $su(N)$ matrix $Y$ the traces in the adjoint and fundamental representations
are related as follows (see, e.g., [16]):

\[
\text{Tr}Y^2 = 2N\text{tr}Y^2 , \quad \text{Tr}Y^4 = 2N\text{tr}Y^4 + 6\text{tr}Y^2\text{tr}Y^2 , \quad (2.16)
\]

\[
\text{Tr}Y^6 = 2N\text{tr}Y^6 + 30\text{tr}Y^4\text{tr}Y^2 - 20\text{tr}Y^3\text{tr}Y^3 . \quad (2.17)
\]

Similar relations apply to symmetrised products of $su(N)$ generators $Y_a$, e.g.,

\[
\text{STr}(Y_{s_1}\ldots Y_{s_N}) = \text{Tr}[Y_{s_1}\ldots Y_{s_N}]
\]

\[
= 2N\text{tr}[Y_{s_1}\ldots Y_{s_N}] + 30\text{tr}[Y_{s_1}\ldots Y_{s_4}\text{tr}[Y_{s_5}Y_{s_6}]] - 20\text{tr}[Y_{s_1}\ldots Y_{s_3}\text{tr}[Y_{s_4}\ldots Y_{s_6}]] . \quad (2.18)
\]

We assume that $\hat{C}_6$ in (2.6) is given by

\[
\hat{C}_6(F) = \text{STr}(Y_{s_1}\ldots Y_{s_6}) C^{s_1\ldots s_6}(F) , \quad (2.19)
\]

\[
C^{s_1\ldots s_6}(F) \equiv -\frac{1}{12} \left[ (F^{s_1}F^{s_6}) - \frac{3}{8} (F^{s_1}F^{s_4})(F^{s_5}F^{s_6}) + \frac{1}{32} (F^{s_1}F^{s_2})(F^{s_3}F^{s_4})(F^{s_5}F^{s_6}) \right] , \quad (2.20)
\]

where $(F\ldots F)$ indicates traces of matrix products over suppressed Lorentz indices, and

\[
\text{STr}(Y_{s_1}\ldots Y_{s_6}) \equiv 2N\text{tr}[Y_{s_1}\ldots Y_{s_N}] + \alpha_1\text{tr}[Y_{s_1}\ldots Y_{s_4}\text{tr}[Y_{s_5}Y_{s_6}]] + \alpha_2\text{tr}[Y_{s_1}\ldots Y_{s_3}\text{tr}[Y_{s_4}\ldots Y_{s_6}]]
\]

\[
+ \alpha_3 N^{-1}\text{tr}[Y_{s_1}Y_{s_2}\text{tr}[Y_{s_3}Y_{s_4}\text{tr}[Y_{s_5}Y_{s_6}]] . \quad (2.21)
\]

$\text{STr}(Y_{s_1}\ldots Y_{s_6})$ is equal to $\text{STr}(Y_{s_1}\ldots Y_{s_N})$, when $\alpha_1 = 30, \alpha_2 = -20, \alpha_3 = 0$ (cf. (2.18)).

The condition (implied by the SYM–supergravity correspondence in the $0 - (p + \ldots + 0)$ and $0 - (1||0)$ cases as mentioned above) that $\hat{C}_6$ should coincide with $\text{STr}C_6$ (2.14) for the simplest commuting backgrounds with all $F_{ab}$ components proportional to the same $su(N)$ matrix, gives the constraints

\[
\alpha_1 = 30 - \alpha_3 , \quad \alpha_2 = \alpha_3 - 20 . \quad (2.22)
\]

Then (2.21) becomes (cf. (2.15),(2.19))

\[
\text{STr}(Y_{s_1}\ldots Y_{s_6}) = \text{STr}(Y_{s_1}\ldots Y_{s_6})
\]

\[
+ \alpha_3 \left( -\text{tr}[Y_{s_1}\ldots Y_{s_4}\text{tr}[Y_{s_5}Y_{s_6}]] + \text{tr}[Y_{s_1}\ldots Y_{s_3}\text{tr}[Y_{s_4}\ldots Y_{s_6}]]
\]

\[
+ N^{-1}\text{tr}[Y_{s_1}Y_{s_2}\text{tr}[Y_{s_3}Y_{s_4}\text{tr}[Y_{s_5}Y_{s_6}]] \right) . \quad (2.23)
\]

We will show that demanding also the agreement between the subleading term $\mathcal{V}^{(2)}$ in the supergravity potential (1.2) for the $0 - (4||0)$ system and the 2-loop term in $\Gamma$ for the corresponding (instanton) gauge field background implies that

\[
\alpha_3 = -30 . \quad (2.24)
\]
To summarise, the expression (2.13) for \( \hat{C}_{2n} \) and the definition (2.8) of the polynomials \( C_{2n}(F) \) lead to the following all-loop BI-type ansatz for the relevant part \( \Gamma \) (2.8) of the SYM effective action
\[
\Gamma = \frac{1}{2Ng_{YM}^2} \int d^{p+1}\hat{x} \hat{STr} \left[ H_p^{-1}(\sqrt{-\det(\eta_{ab}I + H_p^{1/2}F_{ab}) - 1}) + \frac{1}{4}F^2 \right] + ... ,
\] (2.25)
\[
H_p \equiv \frac{a_pNg_{YM}^2}{M^{7-p}} .
\] (2.26)

The dots in (2.25) stand for possible commutator terms which vanish for commuting \( F_{ab} \). \( I \) is the unit \( N \times N \) matrix and the square root of the determinant is understood as in the expansion (2.8) with each \( F_{ab} \) now replaced by an \( su(N) \) matrix.

Let us finally turn to the general case of non-commuting \( F_{ab} \) backgrounds for which additional commutator terms which, in general, may be present in \( \hat{C}_{2n} \) with \( n > 2 \) may be non-vanishing. An example of a background with \( [F_{ab}, F_{cd}] \neq 0 \) will be used in section 5.2 to describe the 1/8 supersymmetric bound state \( 4 \| 4 \| 4 \| 0 \). We shall assume that the 2-loop coefficient \( \hat{C}_6 \) is given by the totally symmetric expression \( \hat{STr}C_6(F) \) (2.19) plus an order \( F^6 \) commutator term \( \mathcal{C}_6 \)
\[
\hat{C}_6(F) = \hat{STr} C_6(F) + \mathcal{C}_6(F) ,
\] (2.27)
\[
\mathcal{C}_6 \sim \text{Tr}(FF[F,F]FF) + ... .
\] (2.28)

In general, \( \mathcal{C}_6 \) may have different Lorentz index structure than \( C_6 \) (2.10) and different internal index structure than a single \( \text{Tr} \). Though we will not be able to determine the detailed form of \( \mathcal{C}_6 \), we will see that the presence of such commutator terms is necessary for the complete correspondence between the supergravity and SYM expressions for the subleading term in the interaction potential of a 0-brane with the \( 4 \| 4 \| 4 \| 0 \) bound state. The single \( \text{Tr} \) form of the representative term in \( \mathcal{C}_6 \) in (2.28) will be important for getting the correct \( n_0N_0^2 \) scaling of the whole 2-loop correction.

### 3 0-brane interaction with 1/2 supersymmetric D-brane bound states

Below we shall consider the interaction potential between a 0-brane and a non-marginal 1/2 supersymmetric bound state \( p + (p - 2) + ... + 0 \) of type IIA D-branes \((p = 0, 2, 4, 6)\). Special cases are the 0-brane – 0-brane and 0-brane – (2 + 0)-brane interactions. From

\footnote{Note that in contrast to the classical non-abelian BI Lagrangian [1.3] [13], i.e. \( L \sim g_s^{-1} \text{Str} \sqrt{-\det(\eta_{ab}I + F_{ab})} \) which represents a part of the tree-level open string effective action and is defined in terms of the symmetrised trace in the fundamental representation, \( \Gamma \), which is a sum of certain quantum loop corrections, is defined in terms of the modified symmetrised trace \( \hat{STr} \). Expressed in terms of the traces in the fundamental representation \( \hat{STr}Y^n \) starts with the \( 2N\text{Str}Y^n \) term but contains also terms with multiple traces \( \text{tr} \) (see (2.17)).}
M-theory point of view this corresponds to scattering of a $D = 11$ graviton (with fixed finite $p_-$ or fixed large $p_{11}$) off a transverse 'p-brane'.

We shall demonstrate that the classical supergravity expression for the potential $V$ (1.2) admits a SYM interpretation (to all orders in string coupling) which is equivalent to a special case of (2.6),(2.25). In particular, the $v^6/r^{14}$ term in the 0-brane – 0-brane interaction [23] and the corresponding terms in the more general $0 - (p + ... + 0)$ case originate from the same 2-loop $F^6$-term in (2.6). This generalises the previously known relation between the leading-order term in $V$ and the $F^4$ term in the 1-loop SYM effective action (2.11).

In the case of the scattering off 1/2 supersymmetric bound state s of D-branes the correspondence with the SYM theory is rather explicit because of the simple BI structure of the action of the $(p + ... + 0)$ brane treated as a probe. This correspondence will no longer be so transparent for the 0-brane interactions with 1/4 and 1/8 supersymmetric bound states discussed in sections 4 and 5; these cases will provide non-trivial checks of the consistency of the ansatz (2.6),(2.15),(2.25).

3.1 Supergravity expression for the $0 - (p + ... + 0)$ potential

To describe the interaction of a 0-brane and a type IIA $(p + ... + 0)$ brane we shall follow [13, 14] and consider the action of $(p + ... + 0)$ brane (i.e. a Dp-brane action with a constant abelian background $F_{mn}$) as a probe moving in the background produced by a 0-brane as a source. One can check that the same expression (to all orders in velocity and charges) is found by considering a 0-brane as a probe moving in a background produced by the $(p + ... + 0)$ brane as a source. The $(p + ... + 0)$ probe action is (see [13, 14])

$$I_p = -T_p V_p \int dt \left[ H_0^{-1} \sqrt{1 - H_0 v^2} \det(H_0^{1/2} \delta_{mn} + F_{mn}) - H_0^{-1} \sqrt{\det F_{mn}} \right]. \quad (3.1)$$

We have assumed that the coordinates $X_i$ transverse to the $p$-brane depend only on the world-volume time $t$ with $v$ being the velocity in a transverse direction. $V_p$ is the volume of a torus around which $p$-brane is wrapped. The Dp-brane tension $T_p$ is [1]

$$T_p \equiv n_p \tilde{T}_p = n_p g_s^{-1} (2\pi)^{(1-p)/2} T^{(p+1)/2}, \quad T \equiv (2\pi\alpha')^{-1}, \quad (3.2)$$

where the integer $n_p$ is the number of $p$-branes (in what follows $T = 1$). The 0-brane charge of $(p + ... + 0)$ brane is

$$n_0 = n_p (2\pi)^{-p/2} V_p \sqrt{\det F_{mn}}. \quad (3.3)$$

$H_0$ is the harmonic function corresponding to the 0-brane source

$$H_0 = \frac{Q_0^{(p)}}{r^{7-p}}, \quad (3.4)$$
where $Q_q^{(n)}$ denotes the ‘charge’ of a D$q$-brane background smeared in $n$ dimensions,

$$Q_q^{(n)} = N_q g_s (2\pi)^{(5-q)/2} T^{(q-7)/2} (V_n \omega_{6-q-n})^{-1}, \quad \omega_{k-1} = \frac{2\pi^{k/2}}{\Gamma(\frac{k}{2})}. \quad (3.5)$$

As already noted in the Introduction, we shall drop out the usual asymptotic value 1 in $H_0$ (assuming either the ‘null reduction’ or ‘fixed $p_-$’ prescription [23] or that the 0-brane charge $Q_0^{(p)} \sim V_p^{-1} g_s N_0$ is large, so that $Q_0^{(p)} \gg 1$). It is under this prescription that the expression for the resulting interaction potential will have a simple SYM interpretation.

The action (3.1) may be rewritten as

$$I_p = -T_0 \int dt \ H_0^{-1} \left[ \sqrt{1 - H_0 v^2} \sqrt{\det(\delta_{mn} + H_0^{1/2} \tilde{F}_{mn})} - 1 \right], \quad (3.6)$$

where $T_0 = n_0 g_s^{-1} (2\pi)^{1/2}$ and

$$\tilde{F}_{mn} \equiv (F_{mn})^{-1}. \quad (3.7)$$

This corresponds to a T-dual configuration obtained by applying T-duality along all of the directions of the $p$-torus, i.e. (3.6) describes the interaction of a $p$-brane source (with charge $N_0$) with parallel $(0 + \ldots + p)$-brane probe (with $p$-brane charge $n_0$) moving in a relative transverse direction.

As suggested by (3.6), a gauge theory description should be based on a SYM theory in $p + 1$ dimensions. It is natural, however, to go one dimension higher to be able treat the transverse velocity on the same formal footing with the gauge field components $\tilde{F}_{mn}$. Let us assume that the direction of motion is $X_9$. T-duality along this direction transforms the configuration in question into a static system of a D-string parallel to $(p + 1) + \ldots + 1$ brane with velocity becoming an electric field background. Indeed, (3.6) is equivalent to

$$I_p = -T_0 \int dt \ H_0^{-1} \left[ \sqrt{-\det(\delta_{ab} + H_0^{1/2} \tilde{F}_{ab})} - 1 \right], \quad (3.8)$$

where $a, b = 0, 1, \ldots, p, 9$ and

$$\tilde{F}_{09} = v.$$  

In this form the action (3.6) is the same as for the type IIB T-dual configuration of a D-string interacting with a $(p + 1) + \ldots + 1$ brane or for a D-instanton interacting with a $(p - 1) + \ldots + (-1)$ brane (the relation between the 0-brane and instanton cases involves duality in the euclidean time direction). The action of the composite brane probe in the instanton background is [14] (we are ignoring the dependence of the brane coordinates $X_i$ on the world-volume coordinates)

$$I_{p-1} = -T_{p-1} V_p H_0^{-1} \left[ \sqrt{\det(H_0^{1/2} \delta_{ab} + F_{ab})} - \sqrt{\det F_{ab}} \right]$$

$$= -T_{p-1} V_p \sqrt{\det F_{ab}} H_0^{-1} \left[ \sqrt{\det(\delta_{ab} + H_0^{1/2} F_{ab}^{-1})} - 1 \right]. \quad (3.9)$$
Here $H_0$ has the interpretation of the harmonic function of the instanton background (smeared in the 9-th direction). This expression is indeed equivalent to (3.6) for constant $v$ and $F_{mn}$.

The interaction potential $V$ in (3.8) written as
\[
I = \int dt \left( \frac{1}{2} T_0 v^2 - V \right),
\]
is thus
\[
V = T_0 \sum_{L=1}^{\infty} C_{2L+2}(\tilde{F}) \, H_0^L,
\]
where $C_{2n}(\tilde{F})$ are the polynomials of degree $2n$ in $\tilde{F}_{ab}$ which appear in the expansion of the BI action (2.8). The explicit form of the potential is thus found to be (see (3.2),(3.5))
\[
V = \sum_{L=1}^{\infty} V^{(L)} = n_0 N_0 \tilde{V}_p \sum_{L=1}^{\infty} \left( \frac{a_p}{r^{7-p}} \right)^L (g_{YM}^2 N_0)^{L-1} C_{2L+2}(\tilde{F}),
\]
where $a_p = 2^{2-p-\pi^2(p+1)/2}(\frac{7-p}{7-p})$ and $\tilde{V}_p$ is the volume of the dual torus, $V_p \tilde{V}_p = (2\pi)^p$. We have defined $g_{YM} = [\left(2\pi\right)^{-1/2} g_s \tilde{V}_p]^{1/2}$ as the effective coupling of the corresponding SYM theory (2.4).

### 3.2 Relation to SYM effective action

The dependence of $V$ (3.12) on the string coupling $g_s$ or on $g_{YM}^2$ suggests that the $1/r^{(7-p)L}$ term in it should originate from the $L$-th loop contribution in the relevant part $\Gamma$ of the SYM effective action on the dual torus. As we shall demonstrate, (3.12) is indeed reproduced by our ansatz (2.6),(2.25) for $\Gamma$ to all orders in the long-distance expansion. For the simple gauge field background which describes the $0-(p+...+0)$ configuration $\hat{\text{STr}} \, C_{2n} = \text{STr} \, C_{2n} = \text{Tr} \, C_{2n}$, i.e. the correction term in (2.13) vanishes and $\hat{C}_{2n}(F)$ in (2.6) is given by (2.14). The correspondence between $\Gamma$ and $V$ was previously checked [8, 9, 13, 14] at the 1-loop level where $\Gamma$ is given by (2.12).

To compare (3.12) to (2.13) we need to identify the corresponding $u(n_0 + N_0)$ SYM background which should be substituted into $\Gamma$ (2.6). It is given by (2.1),(2.2) with $A_m$ having constant field strength given by $\hat{F}_{mn}$ times the unit matrix $I_{N_0 \times N_0}$. As was noted in section 2, $\Gamma$ and thus $\hat{C}_{2n}$ should, in general, depend also on the scalar field background $\hat{X}_i$. To determine the dependence of $\Gamma$ on the velocity $\partial_0 \hat{X}_9$ one may use

\[10\]Note that the structure of the $\hat{C}_6$ (2-loop) term in (2.6) is different from the $O(F^6)$ term in the 1-loop SYM effective action (even though the latter also has a single $\text{Tr}$ structure); in particular, as can be seen from the general expression for the 1-loop effective action in a constant abelian background in [14], or from the results of [3, 8], the 1-loop $O(F^6)$ term vanishes for the abelian $D = 2$ background (and thus does not contribute to the 0-brane scattering) while the 2-loop $F^6$ term does not. The vanishing of the coefficient of the 1-loop $v^6/r^{10}$ term in the SYM expression for the 0-brane interaction potential was noted in [25].
the fact the lower-dimensional SYM theory is a dimensional reduction of \( D = 10 \) SYM theory which contains only the gauge field. Since \( \tilde{C}_{2n}(F) \) are universal functions of \( F \) which do not explicitly depend on a space-time dimension, one may expect that the dependence on \( D_9X_i \) should be the same as on \( F_{ai} \) in the coefficient \( \tilde{C}_{2n}(F) \) in a higher-dimensional SYM theory. Though this is not true in general for the full effective action in SYM theory (cf. \[10\]), this should be true for the part of the effective action \( \Gamma \) \[2.9\] we are interested in, since it should originate from the open string loop diagrams with boundaries on the two different D-branes and thus should be covariant under the T-duality \[1, 2\] interchanging the abelian \( A_k \) and \( X_k \) components. One then needs to compute \( C_{2n} \) in a higher-dimensional SYM with \( F_{ai} \rightarrow D_aX_i \). Note, however, that does not mean that the full \( \Gamma \) is computed in a higher-dimensional space: the structure of \( \Gamma \) \[2.9\], i.e. the power of \( M \) and the definition of \( g_{YM} \) are the same as in the original \( D = p + 1 \) dimensional SYM theory with a scalar field background \[10\].

The basic example is the interaction between two parallel p-branes one of which has a velocity in the direction 9 transverse to the world volume of a p-brane described by the scalar field background \( \hat{X}_9 \) \[2.2\] in \( D = p + 1 \) dimensional SYM theory. To determine the dependence of \( \Gamma \) on \( \partial_9X_9 \) one is thus to compute \( \tilde{C}_{2n}(F) \) in \( D' = D + 1 \) dimensions in the electric background \( F_{09} = \partial_9X_9 \sim v \) and substitute the result into the original expression \( \Gamma \) \[2.6\] for \( \Gamma \) in \( D = p + 1 \) dimensions. Similar considerations apply in the case of oscillating \( X_i \) field representing a wave carrying momentum in some direction along the brane (for example, \( X_1 = X_1(x_5 + t) \) is related to a gauge field wave \( A_1 = A_1(x_5 + t) \) in one higher dimension, see section 4.1 below).

The pure gauge field background in the ‘auxiliary’ \( D' = p + 2 \) dimensional SYM theory which corresponds to the scattering of a 0-brane off a \((p + \ldots + 0)\) bound state is thus represented by the following gauge field matrices in the fundamental representation of \( u(N) \) \((N = n_0 + N_0)\)

\[
\hat{F}_{09} = \begin{pmatrix} vI_{n_0 \times n_0} & 0 \\ 0 & 0_{N_0 \times N_0} \end{pmatrix}, \quad \hat{F}_{mn} = \begin{pmatrix} 0_{n_0 \times n_0} & 0 \\ 0 & -\hat{F}_{mn}I_{N_0 \times N_0} \end{pmatrix}.
\]

(3.13)

It is useful to subtract the traces and to describe the background by the \( su(N) \) matrices \( F_{ab} \) which are proportional to the same matrix \( J_0 \)

\[
F_{09} = \hat{F}_{09}J_0 = vJ_0, \quad F_{mn} = \hat{F}_{mn}J_0,
\]

(3.14)

\[\text{Note, however, that does not mean that the full } \Gamma \text{ is computed in a higher-dimensional space: the structure of } \Gamma \text{, i.e. the power of } M \text{ and the definition of } g_{YM} \text{ are the same as in the original } D = p + 1 \text{ dimensional SYM theory with a scalar field background.} \]

A somewhat different description of the 1-loop result which is closely related to the discussion of the D-instanton – Dp-brane interactions in \[10\] is based on adding one extra 9-th dimension, choosing time to be euclidean and compact and assuming that the 2-space \((\hat{x}_0, \hat{x}_9)\) has volume \( V_2 \) related to the velocity as \( f_0V_2 = 2\pi, \ f_0 = iv \). As in \[10\] one can show that for the background \( (3.14) \) the \( F^4 \)-term in \( (2.13) \) becomes \( b_8 = 2a_0[N_0f_0^2 - f_0^2\text{tr}(F_{mn}F_{mn})] \), so that the 1-loop effective action is \( (D = p + 1 \rightarrow D + 1) \)

\[
\Gamma = i\frac{a_0}{2(4\pi)^{n_0/2}}\int_0^\infty dss\frac{s^{\frac{6-D}{2}}}{e^{-s^2}}\int d^p x \left[ v^3N_0 + v\text{ tr}(F_{mn}F_{mn}) \right] = -i\int_{-\infty}^\infty d\tau \mathcal{V}^{(1)}(r \rightarrow \sqrt{r^2 + v^2\tau^2}),
\]

where \( \mathcal{V}^{(1)}(r) = -\frac{a_0}{(4\pi)^{n_0/2}}\frac{1}{r^7}\frac{\Gamma(\frac{9-D}{2})}{\Gamma(\frac{7-D}{2})}\int d^p x \left[ v^3N_0 + v^2\text{ tr}(F_{mn}F_{mn}) \right].\)
\[ J_0 \equiv \frac{1}{n_0 + N_0} \begin{pmatrix} N_0 I_{n_0 \times n_0} & 0 \\ 0 & -n_0 I_{N_0 \times N_0} \end{pmatrix}, \quad \text{tr} J_0 = 0. \] (3.15)

Since all of the components of \( F_{\alpha \beta} \) commute and \( \text{Tr} J_0^{2n} = 2n_0 N_0 \) one finds that\(^{12}\)

\[ \text{STr}[C_{2L+2}(F)] = \text{Tr}[C_{2L+2}(\tilde{F})] = 2n_0 N_0 C_{2L+2}(\tilde{F}) \text{.} \] (3.16)

To reproduce (3.12) one should take only the \( n_0 N_0^L v^{2L+2} \) part of the \( L \)-loop term in (2.3),(2.23) since the supergravity calculation based on ‘probe-source’ picture is asymmetric in \( n_0, N_0 \), i.e. gives only the terms which are linear in the probe charge \( n_0 \).\(^{13}\) As a result, \( \mathcal{V} \) (3.12) is in precise agreement with \( \Gamma \) (2.4),(2.14).

It is important to stress that the presence of only a single \( \text{Tr} \) in the ansatz (2.3),(2.14) is crucial for the correspondence between \( \mathcal{V} \) and \( \Gamma \): this trace produces the \( n_0 N_0 \) factor, while the additional powers of \( N = N_0 + n_0 \) are correlated with the power of the gauge field coupling as implied by (2.3),(2.5).

Demanding that \( \hat{C}_{2n} = \hat{\text{STr}} C_{2n} = 2n_0 N_0 C_{2L+2}(\tilde{F}) \) or \( \hat{\text{STr}} J_0^{2n} = 2n_0 N_0 \) implies that the correction terms in (2.15) must vanish. In particular, starting with the most general 2-loop combination (2.21) and requiring that \( \hat{\text{STr}} J_0^6 = 2n_0 N_0 \) fixes the two coefficients \( \alpha_1 \) and \( \alpha_2 \) according to (2.22), i.e. the correction term in (2.23) vanishes for \( Y_s = J_0 \).

### 3.3 Special cases: 0 − 0, 0 − (2 + 0) and 0 − (4 + 2 + 0)

The simplest special case is that of the 0-brane − 0-brane scattering. Here \( p = 0 \) and \( \partial_0 X_9 \rightarrow F_{09} = v J_0 \) so that the 1 + 0 dimensional SYM effective action (2.6) in its closed BI-type form (2.25) becomes

\[ \Gamma = \frac{n_0 N_0}{N g_{\text{YM}}^2} \int dt \left[ H_0^{-1}(\sqrt{1 - H_0 v^2} - 1) + \frac{1}{2} v^2 \right]. \] (3.17)

The factor \( 2n_0 N_0 \) came from \( \text{Tr} J_0^{2k} \) and \( (a_0 = \frac{15}{2}, M = r) \)

\[ H_0 = \frac{Q_0}{r^7}, \quad Q_0 = \frac{15}{2(2\pi)^{1/2} N_0 g_s}. \] (3.18)

This agrees with the exact expression for the supergravity potential following from the 0-brane probe action \( I_0 = -T_0 \int dt H_0^{-1} \sqrt{1 - H_0 v^2} \) after we use that \( T_0 = n g_s^{-1}(2\pi)^{1/2} \) (see (3.2),(3.5)) and replace \( N = n_0 + N_0 \) in (3.17) by \( N_0 \), i.e. separate the \( n_0 N_0^L v^{2L+2} \) part\(^ {25} \) of the \( L \)-loop term in (3.17). This is the extension of the previous one-loop \((v^4/r^7)\)\(^ {2, 4, 5, 10} \) and two-loop \((v^6/r^{14})\)\(^ {25} \) results to all orders in \( 1/r^7 \) or loop expansion.

---

\(^{12}\)Given a diagonal matrix in the fundamental representation of \( u(N) \) with entries \( a_i \) the corresponding matrix in the adjoint representation has entries \( a_i - a_j \). That implies that \( J_0 \) has \( 2n_0 N_0 \) non-vanishing diagonal elements equal to \( \pm 1 \).

\(^{13}\)Equivalently, this corresponds to assuming that the source is much heavier than the probe, i.e. \( N_0 \gg n_0 \), so that one may replace \( N = n_0 + N_0 \) by \( N_0 \).
Similar conclusion is reached also in the case when some $p$ of the spatial dimensions are compactified, i.e. the $0–0$ system is described by a $p + 1$ dimensional SYM theory.

Our discussion also clarifies the SYM structure of the 2-loop result of \[25\]. The $v^6/r^{14}$ term in $\mathcal{V}$ considered in \[25\] corresponds to the 2-loop $\tilde{C}_6 \sim \text{Tr} F^6$ term in the SYM description.\[4\] The general $L = 2 F^6$ term in \[2.6\] generalises the $1 + 0$ dimensional SYM result of \[25\] to a higher-dimensional case. The explicit 2-loop computation of $\Gamma(v, r)$ in \[24, 25\] provides the overall normalisation of the 2-loop term chosen in our ansatz for $\Gamma$ \[2.6\], \[2.7\]. Thus the checks of consistency of the detailed structure of the 2-loop part of $\Gamma$ discussed below will be concerned only with the relative normalisations of different terms in $\tilde{C}_6$.

The probe action for the $0–(2 + 0)$ interaction \[4, 8, 11, 13\] is a special case of (3.1)\[3\]

\[
I_2 = -T_0 \int dt \ H_0^{-1} \left[ \sqrt{(1 - H_0v^2)(1 + H_0f_1^2)} - 1 \right],
\]

(3.19)

where $f_1 = (\mathcal{F}_{12})^{-1} = (2\pi)^{-1}V_2n_2/n_0$. Since in the general case of a block-diagonal euclidean matrix $F_{ab}$ with non-zero entries $f_k$ \(f_0 = iv, \ f_1 = F_{12}, \ f_2 = F_{34}, \ldots\) the polynomials $C_{2n}$ in \[2.9\], \[2.10\] are

\[
C_4 = -\frac{1}{8} \left[ 2 \sum_k f_k^4 - (\sum_k f_k^2)^2 \right],
\]

(3.20)

\[
C_6 = \frac{1}{12} \left[ 2 \sum_k f_k^6 - \frac{3}{2} \sum_k f_k^4 \sum_n f_n^2 + \frac{1}{4} (\sum_k f_k^2)^3 \right],
\]

(3.21)

we conclude that the leading term in the resulting potential $\mathcal{V}^{(1)} \sim \frac{1}{v^2}(v^2 + f_1^2)^2$ originates from the one-loop term $\sim \text{Tr}[F^4 - \frac{1}{4}(F^2)^2]$ in \[2.6\] while the first subleading term $\mathcal{V}^{(2)} \sim \frac{1}{v^2}(v^2 - f_1^2)(v^2 + f_1^2)^2$ is reproduced by the 2-loop SYM term $\sim \text{Tr}[F^6 - \frac{2}{8} F^4 F^2 + \frac{1}{32}(F^2)^3]$ with $F_{09} = vJ_0$, $F_{12} = f_1J_0$ as in \[3.14\]. Note that since we have subtracted 1 in $H_0$ we do not need to consider the limit of large field $\mathcal{F}_{mn}$ (i.e. of large $n_0 \gg n_2$) in order to establish the precise agreement between the supergravity and SYM expressions (cf. \[25\] and \[8, 11, 13\]).

In the $0–(4 + 2 + 0)$ case we get one extra factor of $(1 + H_0f_2^4)$ under the square root in \[3.19\] \((f_2 = (\mathcal{F}_{34})^{-1})\), so that the leading and subleading terms in the interaction

\footnote{Let us stress again that the single Tr form of $\tilde{C}_6$ is crucial for getting the correct $n_0\sqrt{n_0}$ factor in this first subleading term in the 0-brane interaction potential (this would not be so for a general combination of single tr and double tr terms in \[2.21\]).}

\footnote{The action of the same structure is found by considering a 0-brane probe moving in the background corresponding to the non-marginal $2 + 0$ bound state. Using the explicit form of the $2 + 0$ solution \[14\] one finds for the 0-brane action: $I_0 = -T_0' \int dt[K^{1/2} \sqrt{1 - K - v^2} - \cos \theta(K^{-1} - 1)]$, where $K = 1 + Q_2/\sqrt{5}$, $\tilde{K} = \cos^2 \theta + K \sin^2 \theta$ and $\tan \theta = f_1 = n_0/n_2$ (we consider the self-dual torus with $V_2 = 2\pi$). This action becomes equivalent to the action in \[3.19\] provided we make identifications $H_0 = \tilde{K}$ and $T_0 = T_0' \cos \theta$. The condition $H_0 = K$ or $Q_2' = Q_2\sqrt{1 + n_0^2/n_2^2} = Q_0^{(2)}$ is satisfied in the limit of large $n_0$ (the relation between tensions is also satisfied in this limit).}
potential in (3.12) are
\[ V^{(1)} = -\frac{1}{8r^3} T_0 Q_0^{(4)} [(f_1^2 - f_2^2)^2 + 2v^2(f_1^2 + f_2^2) + v^4] , \]
\[ V^{(2)} = -\frac{1}{16r^6} T_0 (Q_0^{(4)})^2 (f_1^2 + f_2^2 + v^2) [-(f_1^2 - f_2^2)^2 + v^4] , \]
and are reproduced by the 1-loop and 2-loop terms in \( \Gamma \) (2.6),(3.20),(3.21). Note that in case of the self-dual background \( f_1 = f_2 = f \) the static terms in the potential cancel out (for \( F_{mn} = F^*_{mn} \) the \( D = 4 \) BI action becomes quadratic in the field strength) so that
\[ V^{(1)} = -\frac{1}{8r^3} T_0 Q_0^{(4)} (4v^2 f_1^4 + v^4) , \]
\[ V^{(2)} = -\frac{1}{16r^6} T_0 (Q_0^{(4)})^2 (2v_4 f_1^2 + v^6) . \]

4 0-brane interaction with 1/4 supersymmetric bound states

In this section we shall study the subleading terms in the interaction potentials of a 0-brane probe with 1/4 supersymmetric marginal bound states of branes: a bound state of a fundamental string and a 0-brane \( 1\parallel 0 \) and a bound state of a 4-brane and a 0-brane \( 4\parallel 0 \). The same potentials describe the scattering of \( D = 11 \) gravitons (with fixed large \( p_{11} \) or fixed finite \( p_- \) ) off the bound states of M2-brane with wave and M5-brane with wave (‘longitudinal M2-brane’ and ‘longitudinal M5-brane’). In contrast to the scattering off 1/2 supersymmetric bound states discussed in the previous section, here the supergravity expression for the potential (again obtained from a Born-Infeld-Nambu type action in curved space) is no longer immediately interpretable as a SYM expression because of a more complicated structure of the background fields (containing products of different harmonic functions).

This structure is not seen at the leading order in large-distance expansion since the leading term \( V^{(1)} \) in the potential depends on the constituent charges in an ‘additive’ way. Indeed, \( V^{(1)} \) is reproduced by the one-loop \( (\text{STr} C_4 \sim \text{Tr} F^4 + ...) \) term in the SYM effective action (3.15). The correspondence between the supergravity and SYM (or matrix theory) expressions for \( V^{(1)} \) was previously established in the \( 0 - (4\parallel 0) \) case in \([13]\) (with \( F^4 \) interpretation given in \([14]\) and in the \( 0 - (1\parallel 0) \) case in \([13]\).

We will find that in the \( 0 - (1\parallel 0) \) case the first subleading term \( V^{(2)} \) in the potential is again reproduced by the 2-loop term in \( \Gamma \) (2.6),(2.14), i.e. it has the same structure (and coefficient) as \( \hat{C}_6 \sim \text{Tr} F^6 \). Remarkably, this correspondence extends also to all higher-order terms in \( \mathcal{V} \) and \( \Gamma \) (2.6),(2.25), just as in the \( 0 - (p + ...) + 0 \) case of the previous section. This is a consequence of the fact that the corresponding YM background is again proportional to a single \( su(N) \) matrix.

The situation in the \( 0 - (4\parallel 0) \) case is more complicated: while the supergravity potential has the same form as in the \( 1\parallel 0 \) case, the relevant YM background is less trivial.
as it now depends two different (but still commuting) $su(N)$ matrices. As a result, $V^{(2)}$ is reproduced by (2.6) with $\hat{C}_6$ given by (2.13), i.e. with $F^6$ terms having the same Lorentz-index structure but a modified prescription $\hat{\text{Str}}$ (2.23) for taking traces over the internal indices. $\hat{\text{Str}}C_6$ differs from $\text{Str}C_6$ by the $\alpha_3$ term in (2.23) which was vanishing in all previous cases (i.e. $0 - (p + \ldots + 0)$ and $0 - (1||0)$) but turns out to be non-zero in the $0 - (4||0)$ case.

4.1 0-brane – (1||0)-brane interaction

To find the interaction potential we shall consider a 0-brane probe moving in the background produced by 1||0 as a source. The type IIA supergravity solution representing the 1/4 supersymmetric marginal bound state of a fundamental string and a 0-brane is a dimensional reduction of the $D = 11$ ‘2-brane + wave’ solution [34, 33] and is given by

$$ds^2_{10} = H_0^{-1/2}\tilde{H}_1^{-1}[-dt^2 + H_0 dx_5^2 + H_0 H_1 dx_i dx_i], \quad (4.1)$$

$$e^{2\phi} = \tilde{H}_1^{-1/2}H_0^{3/2}, \quad A = H_0^{-1}dt,$$

$$H_0 = \frac{Q_0^{(1)}}{r^6}, \quad \tilde{H}_1 = 1 + \frac{\tilde{Q}_1}{r^6}, \quad \tilde{Q}_1 = g_s Q_1 = (2\pi)^{-1/2}g_s L_5 \frac{N_1}{N_0} Q_0^{(1)}, \quad (4.2)$$

where $\tilde{Q}_1$ is the fundamental string charge ($N_1$ is the winding number) and $Q_1$ and $Q_0^{(1)}$ are defined in (3.5). $L_5$ is the length of the circular direction (chosen to be the 5-th one) along which the fundamental string is wound.

The action of a 0-brane probe with a transverse velocity $v$ is then

$$I = -T_0 \int dt \left[H_0^{-1} \left(\sqrt{1 - H_0 \tilde{H}_1 v^2} - 1\right)\right], \quad (4.3)$$

so that the first two terms in the interaction potential $V$ (1.1), (1.2) are thus

$$V^{(1)} = -\frac{1}{8r^6} T_0 (4\tilde{Q}_1 v^2 + Q_0^{(1)} v^4), \quad V^{(2)} = -\frac{1}{16r^{12}} T_0 Q_0^{(1)} (4\tilde{Q}_1 v^4 + Q_0^{(1)} v^6). \quad (4.4)$$

It is the product of the two harmonic functions under the square root in (4.3) compared to a single factor of $H_0$ in (3.6), (3.19) that makes comparison with a SYM action non-trivial. Similar complex structure of the probe action containing the product of several harmonic functions (1.1) is characteristic to all cases of scattering off 1/4 and 1/8 supersymmetric BPS configurations discussed below. This structure is a consequence of the ‘harmonic function rule’ form of the supergravity backgrounds representing marginal BPS bound states of different branes [34], and it makes establishing a connection between a curved space 0-brane action and a flat space SYM effective action quite non-trivial.

Since $T_0 \sim g_s^{-1} n_0$, $\tilde{Q}_1 \sim g_s^2 N_1$, $Q_0^{(1)} \sim g_s N_0$ it may seem that $V^{(1)}$ and $V^{(2)}$ contain terms of different orders in the string coupling. Still, $V^{(1)}$ is reproduced by the 1-loop correction in the SYM effective action, and $V^{(2)}$ – by the 2-loop one, as for the pure
D-brane configurations discussed in the previous section (and for the 4\parallel 0 case considered below). The reason is that after T-duality the fundamental string winding number \(N_1\) will have the interpretation of a momentum carried by the classical SYM wave, i.e. the corresponding gauge field background will explicitly depend on \(g_s\), \(F_{ab} \sim g_s^{1/2}\). That will bring in an extra power of \(g_{YM}^2 \sim g_s\) on the SYM side.

The 1\parallel 0 state is, indeed, T-dual to a bound state of a D-string and a wave: the action \((4.3)\) is the same as for a D-string probe in the D-string + wave background (with the string probe oriented parallel to the string source and moving in the orthogonal direction). This relation suggests that the 0\parallel 0 state is, indeed, T-dual to a bound state of a D-string and a wave: the action \((4.3)\) is the same as for a D-string probe in the D-string + wave background (with the string probe oriented parallel to the string source and moving in the orthogonal direction). This relation suggests that the 0\parallel 0 configuration should have a description in terms of the \(D = 2\) SYM theory with \(\tilde{Q}_1\) having the interpretation of a SYM momentum \([37, 47, 38, 39, 15]\). The latter is represented by a periodic scalar field background \(X_1 = X_1(\tilde{x}_5 + t)\) with \(\tilde{x}_5\) being the direction of the momentum flow.

To find the dependence of the coefficients \(\hat{C}_{2n}\) in the SYM effective action \(\Gamma (2.0)\) on derivatives of \(X_1\) we may formally trade the \(X_1\) wave for a gauge field wave by going to a \(D = 3\) SYM theory and considering the T-dual background \(A_1 = A_1(x_5 + t)\). This is similar to the trick used above to find the dependence on \(\partial \bar{X}_5 \sim \nu\): one considers the SYM theory in one dimension higher with an electric gauge field background \(F_{09}\).

Altogether this corresponds to performing T-duality along the direction of the transverse D-string momentum-carrying oscillations \((x_1)\) and the direction of motion of its center of mass \((x_9)\). We end up with a configuration of two parallel 3-branes described by a a plane wave (in the direction 5) on one 3-brane and a constant electric field (in the direction 9) on another 3-brane. It is represented by the the following stationary abelian \(u(N)\) gauge field background in the \(D = 4\) SYM theory on the dual torus (cf. \((3.14)\))

\[
\hat{F}_{09} = \left(\begin{array}{cc} v_{I_{0} \times N_0} & 0 \\ 0 & 0_{N_0 \times N_0} \end{array} \right), \quad \hat{F}_{51} = \hat{F}_{01} = \left(\begin{array}{cc} 0_{N_0 \times N_0} & 0 \\ 0 & -h(\tilde{x}_5 + t) I_{N_0 \times N_0} \end{array} \right),
\]

or, equivalently, by the following \(su(N)\) background

\[
F_{09} = v J_0, \quad F_{51} = F_{01} = h(\tilde{x}_5 + t) J_0,
\]

where the \(su(N)\) matrix \(J_0\) was defined in \((3.13)\). The function \(h\) which is the derivative of \(A_1 = A_1(\tilde{x}_5 + t)\) may be chosen, e.g., as \(h \sim \sqrt{g_s} \sin \left[\frac{2\pi}{L_5}(\tilde{x}_5 + t)\right]\) \([13]\) and is normalised so that

\[
<h^2> = \frac{1}{L_5} \int d\tilde{x}_5 h^2 = g_s (2\pi)^{1/2} \frac{N_1}{N_0 L_5} = \frac{\tilde{Q}_1}{Q_0^{(1)}}.
\]

\(N_1\) is thus the momentum carried by the gauge field wave along the 5-th direction,

\[
\frac{1}{(g_{YM}^2)_3} \int d\tilde{x}_1 d\tilde{x}_5 \text{tr}(\hat{F}_{01} \hat{F}_{51}) = \frac{2\pi N_1}{L_5}, \quad (g_{YM}^2)_3 = (2\pi)^{-1/2} g_s \tilde{L}_1 \tilde{L}_5.
\]

\(^{16}\)Let us note that for the aim of reproducing the expressions for interaction potentials here and in all examples discussed below one, in principle, does not need to know the explicit form of the gauge field backgrounds representing a plane wave or instanton or their superpositions – all what is needed are the basic properties like constraints on the field strengths (i.e. \(F_{1+} = 0\) or \(F_{mn} = F_{mn}^*\)) and normalisation conditions for the integrals of the squares of the field strength components.
Here the dual torus dimension $\tilde{x}_1$ (with length $\tilde{L}_1$) is the auxiliary dimension of the plane wave and $\tilde{L}_5$ is the length of the dimension $\tilde{x}_5$ dual to the one along the fundamental string ($\tilde{L}_5 L_5 = 2\pi$).

It should be stressed again that the passage to the $2 + 1$ or $3 + 1$ dimensional SYM theory serves only to determine the dependence on the derivatives of the scalar fields $X_1$ and $X_9$ in the original $D = 1 + 1$ dimensional SYM theory: to find the SYM effective action $\Gamma$ which correspond to the supergravity potential $V$ we should set $p = 1$, $D = 2$, $g_{YM}^2 = (g_{YM}^2)_2 = (2\pi)^{-1/2} g_s \tilde{L}_5$ in (2.6).

Since all of the components of $F_{ab}$ in (4.6) are, as in (3.14), proportional to the same $su(N)$ matrix $J_0$, the expression for $\hat{C}_{2n}(F)$ in (2.6) is again given by (2.14) and, as in (3.16), $\hat{C}_{2n} = 2 n_0 N_0 C_{2n}(F)$. Thus we only need to compute the coefficients $C_4 \sim F^4$ and $C_6 \sim F^6$ in (2.9),(2.10) for the abelian background $F_{03} = v$, $F_{51} = F_{01} = h$,

$$C_4 = -\frac{1}{8} (4h^2 v^2 + v^4) , \quad C_6 = -\frac{1}{16} (4h^2 v^4 + v^6) . \quad (4.9)$$

Using (4.7) we find the precise agreement between the two leading terms in $V$ (4.4) and the 1-loop and 2-loop terms in the SYM effective action (2.6). At the leading-order level this correspondence was also checked by direct 1-loop $D = 2$ SYM calculation in $X_1$ background in (13)\(^{19}\).

As in the $0-(p+\ldots+0)$ case, the relation between $V$ in (4.3) and $\Gamma$ in (2.6),(2.14),(2.25) holds not only for the first two leading terms, but also for the complete expressions, i.e. for all terms in the expansion in $1/r^6$. Indeed, $\Gamma$ in (2.23) with $p = 1$ and the $D = 4$ BI determinant $\det(n_{ab} I + F_{ab})$ computed on the abelian background (4.6) (which looks the same as a D3-brane action in the gauge field background or a D-string action in the scalar field background) is found to be (cf. (3.17))

$$\Gamma = \frac{n_0 N_0}{N g_{YM}^2} \int dt d\tilde{x}_5 \left[ H_1^{-1} \left( 1 - (1 + H_1 h^2) H_1 v^2 - 1 \right) + \frac{1}{2} v^2 \right] . \quad (4.10)$$

Here $H_1$ given by (2.26) is equivalent to $H_0 = Q_0^{(1)} / r^6$ in (4.2). This expression (its part linear in $n_0$) takes exactly the same form as $V$ in (4.3),(4.2) after we replace $h^2$ by $< h^2 >$ in (4.7) so that $1 + H_1 h^2$ becomes $\tilde{H}_1$ in (4.2). This establishes the agreement between all higher-order terms in the expansion of the interaction potential in (4.3) and the SYM effective action (2.6),(2.14).

\(^{17}\)The extra power of $g_{YM}^2 \sim g_s$ in the classical momentum (4.8) explains the correspondence with the supergravity expressions in (4.4) (see also (13)\(^{17}\)).

\(^{18}\)As was already mentioned above, here and in the examples discussed in the following sections, it is sufficient to check only the agreement between the supergravity and the SYM expressions for the relative coefficients between the highest power of $v$ and its lower powers at each order of $1/r^{7-p}$ expansion since the agreement of the coefficients of the $v^4$ and $v^6$ terms (same as in the 0-brane–0-brane scattering in the case of compactification on a torus) was already established by the one-loop (3)\(^{10}\) and two-loop (24)\(^{25}\) computations.

\(^{19}\)The potential vanishes in the static limit reflecting the BPS nature of the plane wave background (which preserves 1/2 of supersymmetry in SYM theory). Let us note in passing that the vanishing of the 1-loop YM effective action in the non-abelian plane wave background was discussed in (18).
4.2 0-brane – (4||0)-brane interaction

The action of a 0-brane probe moving in the background produced by the bound state 4||0 as a source is [32, 13]

\[ I_0 = -T_0 \int dt \, H_0^{-1}(\sqrt{1 - H_0 H_4 v^2} - 1), \]  
\[ H_0 = \frac{Q_0^{(4)}}{r^3}, \quad H_4 = 1 + \frac{Q_4^{(4)}}{r^3}, \]  
were \( Q_4 \) and \( Q_0^{(4)} \) are given by \((3.3)\). This action is formally the same as in the \( 0 - (1||0) \) case \((4.3)\) with \( H_4 \) replacing the fundamental string function \( \tilde{H}_1 \). The two leading terms in the classical potential thus have the same form as in \((4.4)\) with \( Q_0^{(1)} \to Q_0^{(4)}, \, \tilde{Q}_1 \to Q_4 \).

The 4||0 brane wrapped over a 4-torus \( T^4 \) and having even \( N_0 \) may be described by the following self-dual \( u(N_0) \) background on the dual 4-torus \( T^4 \) [30, 36, 13, 19]

\[ F_{12} = F_{34} = q \, \sigma_3 \otimes I_{N_0 \times N_0}, \]  
with all other components of \( F_{mn} \) being zero. Here \( \sigma_3 = \text{diag}(1, -1) \) and

\[ q^2 = (2\pi)^2 \tilde{V}_4^{-1} \frac{N_4}{N_0} = \frac{Q_4}{Q_0^{(4)}}, \quad \text{i.e.} \quad \frac{1}{16\pi^2} \int_{T^4} d^4x \, \text{tr}(F_{mn}F_{mn}) = N_4. \]  

The leading and subleading terms of the supergravity potential, expressed in terms of the YM background, read

\[ \mathcal{V}^{(1)} = -\frac{n_0}{16r^3} \left[ 4v^2 N_4 + (2\pi)^{-2} \tilde{V}_4 v^4 N_0 \right] \]

\[ = -\frac{n_0 N_0}{64\pi^2 r^3} \tilde{V}_4 \left( 4v^2 q^2 + v^4 \right) = -\frac{n_0 N_0 g_s}{64(2\pi)^{5/2} r^6} \tilde{V}_4 \left[ v^2 \text{tr}(F_{mn}F_{mn}) + v^4 N_0 \right], \]  
\[ \mathcal{V}^{(2)} = -\frac{n_0 N_0 g_{YM}}{64(2\pi)^{5/2} r^6} \tilde{V}_4 \left( 4v^4 N_4 + (2\pi)^{-2} \tilde{V}_4 v^6 N_0 \right) \]

\[ = -\frac{n_0 N_0 g_{YM}^2}{64(2\pi)^{5} r^6} \tilde{V}_4 \left( 4v^4 q^2 + v^6 \right) = -\frac{n_0 N_0 g_{YM}^2}{64(2\pi)^{5} r^6} \tilde{V}_4 \left[ v^4 \text{tr}(F_{mn}F_{mn}) + v^6 N_0 \right]. \]

While the leading-order potential \( \mathcal{V}^{(1)} \) is the same as in the special case of \( 0 - (4 + 2 + 0) \) interaction in \((3.23)\) with the \((4 + 2 + 0)\) brane described by the self-dual abelian background \( \tilde{F}_{12} = \tilde{F}_{34} = q \) (so that \((3.15)\) can be found also by using such \((4 + 2 + 0)\) configuration to represent \( 4||0 \) as a probe and treating the 0-brane as a source \((3.3)\) this is no longer so for the subleading term in the potential \( \mathcal{V}^{(2)} \): as follows from \((3.21)\) (with \( f_0 = iv, \, f_1 = f_2 = q \)) in the \( 0 - (4 + 2 + 0) \) case the coefficient of the 'mixed' \( v^4 q^2 \) term in \((3.24)\) is factor of 2 smaller than in the supergravity expression \((3.16)\). This extra factor of 2 in \( \mathcal{V}^{(2)} \) is a direct consequence of the different structure of the action \((1.11)\) compared to \((3.6)\): it originates from the coefficient 2 in \( (H_4)^2 = 1 + 2Q_4/r^3 + ... \) which appears in the expansion of \((4.11)\).
The two configurations, i.e. $0 - (4 + 2 + 0)$ and $0 - (4||0)$, having different subleading terms in the supergravity potentials may still be described by the same universal 2-loop SYM action $\Gamma^{(2)}$ because the corresponding $su(N)$ gauge field backgrounds are different. While in (2.14) all the components of the field strength are proportional to the same $su(N)$ matrix $J_0$, here the electric (velocity) component $F_{09}$ and the instanton (4-brane) components $F_{12} = F_{34}$ are expressed in terms of two different matrices $J_0$ and $J_1$ from the Cartan subalgebra of $su(N)$,

$$F_{09} = v J_0, \quad F_{12} = F_{34} = q J_1,$$  \hspace{1cm} (4.17)

$$J_1 \equiv \begin{pmatrix} 0_{n_0 \times n_0} & 0 \\ 0 & \sigma_3 \otimes I_{N_2 \times N_2} \end{pmatrix} = \begin{pmatrix} 0_{n_0 \times n_0} & 0 \\ 0 & I_{N_2 \times N_2} \\ 0 & -I_{N_2 \times N_2} \end{pmatrix}, \quad \text{tr} J_1 = 0. \hspace{1cm} (4.18)$$

As a result, this background is more sensitive to the non-abelian structure of the coefficients $\hat{C}_{2n}$ (2.15) in the SYM effective action (2.3).

Let us first recall how the 1-loop term in (2.6) reproduces the leading term in the potential (1.13). Substituting the background (4.17) into $\hat{C}_4 = \text{STr} C_4(F)$ and using (2.9), (3.20) we find that

$$\hat{C}_4 = \text{STr} C_4(F) = -\frac{1}{8} \text{STr}(4v^2 q^2 J_0^2 J_0^2 + v^4 J_0^4) = -\frac{1}{4} n_0 N_0 (4v^2 q^2 + v^4),$$

and, as a result, complete agreement between (2.4) and (4.17).

To reproduce the subleading term (4.16) in $V$ we shall use $\hat{C}_6 = \hat{C}_6 = \text{STr} C_6(F)$ in (2.14) with $\text{STr}$ defined according to (2.23). Remarkably, the coefficient $\alpha_3$ in (2.23) is fixed uniquely (2.24) once we demand the correspondence between $V^{(2)}$ and the 2-loop SYM effective action $\Gamma^{(2)}$ in (2.6). The extra $\alpha_3$-term in (2.23) is responsible for correcting the coefficient 2 of $v^4$ term in $V^{(2)}$ in (3.14) into 4 in (4.16). Indeed, the expression for $\hat{C}_6$ for the background (1.17) is found to be (cf. (4.17), (4.19))

$$\hat{C}_6 = -\frac{1}{16} \text{STr}(2v^4 q^2 J_0^2 J_1^2 + v^6 J_1^6).$$

Applying the definition (2.23), (2.24) of $\text{STr}$ one finishes with

$$\hat{C}_6 = -\frac{1}{8N} n_0 N_0 \left[ (n_0 + N_0)(2v^2 q^2 + v^6) + 2N_0 v^4 q^2 \right].$$

\footnote{Since the $F_{ab}$ background here is commuting, $\text{STr} C_{2n} = \text{Tr} C_{2n}$. It is useful, however, to keep symmetrisation in order to express Tr-expressions in terms of tr ones. Note that the symmetris product of $\text{tr}(J_0^2 J_1^2 J_0^2)$ where $J = J_0$ or $J_1$ is $\text{Sym}(\text{tr}(J_0^2 J_1^2)) \to \frac{1}{4} \text{tr}[J_0^2 J_1^2 + 2 \text{tr}(J_0 J_1) J_0 J_1]$. Thus $\text{STr}(J_0^2 J_1^2) = 2N_0 [2 \text{tr}(J_0^2 J_1^2) + 2 \text{tr}[J_0^2 J_1^2 + 2N_0 N_0 N_0]$, where we have used (2.16) and $\text{tr}(J_0 J_1) = 0$.}

\footnote{Note again that since we have subtracted 1 from $H_0$ in (1.13) we do not need to assume (as was done in [13]) that $N_0 \gg N_4$ in the supergravity expression for the potential.}

\footnote{Useful symmetrised products of traces of $J_0^2 J_1^2 J_0^2$ are $\text{Sym}(\text{tr}(J_0^2 J_1^2)) \to \frac{1}{4} \text{tr}[J_0^2 J_1^2 + 8 \text{tr}(J_0^2 J_1^2) J_0 J_1 + 6 \text{tr}(J_0^2 J_1^2) J_0 J_1 J_0 J_1]$, $\text{Sym}(\text{tr}(J_0^2 J_1^2)) \to \frac{1}{8} \text{tr}[2 \text{tr}(J_0 J_1) J_0 J_1 + 3 \text{tr}(J_0^2 J_1^2 J_0 J_1)]$, $\text{Sym}(\text{tr}(J_0^2 J_1^2) J_0 J_1) \to \frac{1}{16} \text{tr}[J_0^2 J_1^2 J_0 J_1 + 4 \text{tr}(J_0^2 J_1^2 J_0 J_1)]$, which can be simplified using that $\text{tr} J_0 = \text{tr} J_1 = \text{tr}(J_0 J_1) = 0$. Let us note also that since, in general, $\text{tr}(Y_1 Y_2) \text{tr}(Y_3 Y_4) \text{tr}(Y_5 Y_6) = \frac{1}{4} \text{tr} \left[Y_1 Y_2 \text{tr}(Y_3 Y_4) \text{tr}(Y_5 Y_6) + 2 \text{terms} + 4 \text{terms} \right]$, we find that $\text{Sym}(\text{tr} X^2 Y^2 Z^2) \to \frac{1}{16} \text{tr} X^2 \text{tr} Y^2 \text{tr} Z^2 + 2 \text{tr}(X Y) \text{tr}(Y Z) + 2 \text{tr}(XY)[...] + 2 \text{tr}(XZ)[...].$}
Isolating the part linear in $n_0$ in the 2-loop term in (2.6) which is proportional to $N\hat{C}_6$ we find the agreement with (4.16). Note that while in the case of the 0-brane–0-brane scattering in [25] the 2-loop SYM expression for the potential was probe–source ($n_0 \leftrightarrow N_0$) symmetric, this is no longer true in the $0 - (4||0)$ case: the symmetry is broken by the non-vanishing correction term in $\hat{\text{Str}}C_6$ producing the second $v^4$ term in (4.21).

5 0-brane interaction with 1/8 supersymmetric bound states of branes

By demanding the precise agreement between the supergravity potentials for the $0 - (p + ... + 0), 0 - (1||0)$ and $0 - (4||0)$ cases we have so far fixed the structure of the 2-loop term $\sim \hat{C}_6$ (2.21)–(2.24) in the leading IR part $\Gamma$ of the SYM effective action, at least up to commutator terms which vanish on commuting $F_{ab}$ backgrounds. A further test of consistency of our ansatz for $\Gamma$ and thus of correspondence between the supergravity and SYM descriptions is provided by non-trivial examples of 0-brane interactions with 1/8 supersymmetric bound states of branes. These bound states are represented by intersections of three and four branes in $D = 10$ (or $D = 11$) and, when wrapped over 5-torus and 6-torus, are related to $D = 5$ and $D = 4$ extremal black holes with regular horizons.

We will show that these bound state configurations admit simple SYM descriptions (which, however, are not unique). They are essentially superpositions of plane wave and/or instanton backgrounds which represent 1/4 supersymmetric BPS states of SYM theory.23 Plugging these backgrounds into the $F^4$-term in the 1-loop SYM effective action demonstrates the agreement between the supergravity and SYM expressions for the leading-order interaction potentials (in the $D = 5$ black hole case this was previously shown in [16, 17]).

In the $D = 5$ black hole case the issue of correspondence between the supergravity and SYM descriptions at the first subleading order was previously addressed in [16] where the $v^2$ term present in the supergravity potential $\mathcal{V}^{(2)}$ was not explicitly determined from a 2-loop SYM expression. We shall demonstrate that making a certain natural choice of underlying SYM background and substituting it into the 2-loop SYM term $\Gamma^{(2)}$ implicitly determined by considerations of the previous sections reproduces the complete expression for the subleading term $\mathcal{V}^{(2)}$ in the supergravity potential.

To be able to reproduce the $v^2$ term in $\mathcal{V}^{(2)}$ in the case of a 0-brane interacting with a bound state representing the $D = 4$ black hole we shall need to choose a non-abelian ([$F$, $F$] $\neq 0$) representation for the corresponding SYM background and to include in the the 2-loop SYM effective action $\Gamma^{(2)}$ additional ‘commutator terms’ which were vanishing in all previous examples described by commuting gauge field backgrounds.

\footnote{Previous discussions of SYM/Matrix theory description of $D = 5$ black holes appeared in [16, 50, 51]; some comments on $D = 4$ black holes were also made in [50].}
5.1 0-brane interaction with $D = 5$ black hole ($0 - (4\perp 1\parallel 0)$)

One particular representation for the three-charge $D = 5$ black holes with regular horizons \[53\] is given by the $D = 11$ configuration of intersecting \[54\] longitudinal M5-brane and M2-brane with a wave along the common string, i.e. by $5\perp 2 + \text{wave}$ \[34\] \[24\] Upon dimensional reduction along the common 11-th dimension this becomes the $4\perp 1 \parallel 0$ type IIA configuration with the 4-brane and the fundamental string wrapped over a 4-torus and a circle which are orthogonal cycles of the 5-torus. Special cases of this 1/8 supersymmetric marginal BPS bound state are the 1/4 supersymmetric $1 \parallel 0$ and $4 \parallel 0$ states discussed in section 4.

As in the previous cases, to find the supergravity interaction potential we shall consider the action of a 0-brane probe moving in the background produced by this composite source. The explicit form of the $D = 10$ type IIA string-frame metric, dilaton and vector field representing the $4 \perp 1 \parallel 0$ configuration may be obtained, e.g., by dimensional reduction of the $D = 11$ \[5\] solution in \[34\] (cf. (4.1))

\[
d_{10}^2 = H_0^{-1/2}H_4^{-1/2}\tilde{H}_1^{-1}\left[ -dt^2 + H_0\tilde{H}_1(dx_1^2 + ... + dx_4^2) \right. \\
+ H_0H_4dx_5^2 + H_0\tilde{H}_1H_4(dx_6^2 + ... + dx_9^2) \left. \right] ,
\]

\[
e^{2\phi} = \tilde{H}_1^{-1}H_0^{3/2}H_4^{-1/2} , \quad A = H_0^{-1}dt ,
\]

\[
\tilde{H}_1 = 1 + \frac{\tilde{Q}_1^{(4)}}{r^2} , \quad H_4 = 1 + \frac{Q_4^{(1)}}{r^2} , \quad H_0 = \frac{Q_0^{(5)}}{r^2} ,
\]

\[
\tilde{Q}_1^{(4)} = g_sQ_1^{(4)} = (2\pi)^{-1/2}g_sL_5\frac{N_1}{N_0}Q_0^{(5)} ,
\]

where the directions $x_1, ..., x_4$ are parallel to the 4-brane and $x_5$ - to the string. In the $D = 11$ interpretation $H_0$ is the harmonic function of the wave (we again drop 1 in $H_0$ assuming 'null reduction' \[23\]). The action of a 0-brane (moving transversely to the internal 5-torus $x_1, ..., x_5$) is a direct generalisation of (4.3) and (4.11)

\[
I_0 = -T_0\int dt \; H_0^{-1}\left( \sqrt{1 - H_0\tilde{H}_1H_4v^2} - 1 \right) ,
\]

so that the two leading terms in the interaction potential are (cf. (4.4), (4.15), (4.16))

\[
\mathcal{V}^{(1)} = -\frac{1}{8r^2}T_0\left( 4v^2(\tilde{Q}_1^{(4)} + Q_4^{(1)}) + v^4Q_0^{(5)} \right) ,
\]

\[
\mathcal{V}^{(2)} = -\frac{1}{16r^4}T_0\left( 8v^2\tilde{Q}_1^{(4)}Q_4^{(1)} + 4v^4(\tilde{Q}_1^{(4)} + Q_4^{(1)})Q_0^{(5)} + v^6(Q_0^{(5)})^2 \right) .
\]

To find a description of this configuration in terms of a SYM background let us note that performing T-duality along all of the directions of $T^5$ transforms $0\parallel 4\perp 1$ into $5\parallel 1\perp \text{wave}$,
i.e. a bound state of a D5-brane, D-string and a wave. A D-string charge on D5-brane may be represented by an 4d instanton in $D = 5 + 1$ SYM theory \[30, 36\] while a momentum wave – by a plane wave configuration of SYM fields \[13, 17, 39\].

The explicit representation for the momentum wave in $D = 6$ SYM theory is not unique. Since the SYM stress tensor contains the two contributions – of the gauge fields $A_a$ ($a = 0, 1, ..., 5$) and of the scalars $X_i$ (6, 7, 8, 9), oscillations of both of the fields may, in general, carry parts of the total momentum. This is what we shall assume below, choosing the following specific background

$$A_1 = A_1(\tilde{x}_5 + t) , \quad X_6 = X_6(\tilde{x}_5 + t) . \quad (5.6)$$

The wave of $X_6$ is natural to include as it represents the momentum in the special case of the D-string+wave system, or the fundamental string winding number in the T-dual $1||0$ state considered in section 4.1.\[25\]

Adding a 0-brane probe to the $4||1||0$ state corresponds in the T-dual picture to adding a D5-brane probe parallel to the composite ‘D5-brane’ source. As discussed in the previous sections, to determine the dependence on the scalar field background $X_6$ and $X_9 \sim vt$ representing the velocity of the probe we may formally perform further T-duality transformations along the 6-th and 9-th transverse directions. We then finish with a stationary pure gauge field $D = 8$ configuration describing a 7-brane probe with a constant electric flux $F_{09} \sim v$, and a 7-brane source with a 3-brane charge (represented by an instanton background in 1234 subspace) and a YM wave ($A_1 = A_1(\tilde{x}_5+t)$, $A_6 = A_6(\tilde{x}_5+t)$) carrying momentum along $\tilde{x}_5$.

Let us first ignore the $X_6$ or $A_6$ background. Then the $0 - (4||0\perp1)$ configuration is described by the superposition of the (anti)self-dual (4.13),(4.17) and $A_1$ wave (4.6) backgrounds in the SYM theory on the dual 5-torus, i.e. by the following $su(n_0 + N_0)$ gauge field strength (cf. (4.6),(4.17),(4.18))

$$F_{09} = vJ_0 , \quad F_{12} = F_{34} = qJ_1 , \quad F_{51} = F_{01} = hJ_0 , \quad (5.7)$$

where $J_0$ and $J_1$ were defined in (3.15) and (4.18), and $q$ is the same as in (4.17) while $h$ is the same as in (4.6),(4.7), i.e.

$$q^2 = \frac{Q_4}{Q_0^{(4)}} = \frac{Q_4^{(1)}}{Q_0^{(5)}} , \quad <h^2> = \frac{1}{L_5} \int d\tilde{x}_5 h^2 = \frac{\hat{Q}_1}{Q_0^{(1)}} = \frac{\hat{Q}_4^{(4)}}{Q_0^{(5)}} . \quad (5.8)$$

Including the $A_6$ background we get instead of (5.7)

$$F_{09} = vJ_0 , \quad F_{12} = F_{34} = qJ_1 , \quad (5.9)$$

$$F_{51} = F_{01} = hJ_0 , \quad F_{56} = F_{06} = wJ_0 , \quad (5.10)$$

\[25\]It is also natural to expect that in another special case of D5-brane+wave state the momentum should be carried by the transverse $(X_i)$ oscillations of the 5-brane.
where the ‘vector’ and ‘scalar’ wave functions \( h = h(\tilde{x}_5 + t) \) and \( w = w(\tilde{x}_5 + t) \) satisfy a generalisation of the second condition in (5.8)

\[
< h^2 + w^2 > = \frac{1}{L_5} \int d\tilde{x}_5 \left( h^2 + w^2 \right) = \frac{\hat{Q}_1}{Q^{(1)}_0} = \frac{\hat{Q}_1^{(4)}}{\tilde{Q}_0^{(5)}} .
\]

(5.11)

Thus \( < h^2 + w^2 > \) is proportional to the momentum of the wave in the 5\( \parallel \)1+wave configuration.

The ‘instanton+wave’ field background \((5.9),(5.10)\) with \( v = 0 \), which should be representing the marginal BPS bound state 5\( \parallel \)1+wave configuration invariant under 1/8 of \( \mathcal{N} = 2, D = 10 \) type IIB supersymmetry, is indeed preserving the 1/4 of the \( \mathcal{N} = 1, D = 10 \) supersymmetry of the SYM theory. The standard condition of the vanishing of the gaugino variation is

\[
F_{rs} \gamma^{ab} = 0 ,
\]

(5.12)

where \( r, s \) are the internal \( u(N) \) indices and \( \gamma_{ab} = \gamma[a\gamma b] \), \( \gamma(a\gamma b) = \eta_{ab} = \text{diag}(-1, 1, ..., 1) \). As the diagonal matrices \( J_0 \) and \( J_1 \) are not proportional, choosing different values of \( r, s \) and combining the resulting equations we get two separate (‘instanton’ and ‘wave’) conditions:

\[
(\gamma_{12} + \gamma_{34})\epsilon = 0, \quad \text{i.e.} \quad P_{1234}\epsilon = \epsilon , \quad P_{mnkl} \equiv \frac{1}{2} (1 + \gamma_{mnkl}) ,
\]

(5.13)

and \( (h \gamma_1 + w \gamma_6)(\gamma_0 + \gamma_5)\epsilon = 0 \), or

\[
(\gamma_0 + \gamma_5)\epsilon = 0, \quad \text{i.e.} \quad P_{05}\epsilon = \epsilon , \quad P_{05} \equiv \frac{1}{2} (1 - \gamma_{05}) .
\]

(5.14)

Since the projectors \( P_{1234} \) and \( P_{05} \) commute, we conclude that the amount of unbroken supersymmetry is reduced to 1/4 of the original one.

Computing the classical BI Lagrangian for this commuting \( u(N) \) background \((5.9),(5.10)\) one finds

\[
L = \text{tr} \sqrt{-\text{det}(\eta_{ab} I + \hat{F}_{ab})} = N(1 + q^2) \sqrt{1 - \frac{h^2 + (1 + q^2)(1 + w^2)}{1 + q^2}} v^2 .
\]

(5.15)

This provides a test of the marginal BPS property of the \( v = 0 \) background: higher-order terms vanish for \( v = 0 \) and the coefficient of the \( v^2 \) term \( \sim h^2 + (1 + q^2)(1 + w^2) \) has its leading-order part being proportional to the mass \( (M \sim \hat{Q}_1^{(4)} + Q^{(1)}_4 + Q^{(5)}_0 \sim 1 + q^2 + < h^2 + w^2 >) \) of the 5\( \parallel \)1+wave bound state (see also [17]). The higher-order correction term \( \sim w^2 q^2 v^2 \) will be discussed below.

Since the background \((5.9),(5.10)\) involves only two commuting diagonal \( u(N) \) matrices, the computation of the 1-loop \( \hat{C}_4 \) \((2.13)\) and 2-loop \( \hat{C}_6 \) \((2.19)\) terms in the SYM effective action is essentially the same as in section 4. Both \( \hat{C}_4 \) and \( \hat{C}_6 \) vanish for \( v = 0 \) as expected for a supersymmetric configuration. \( \hat{C}_4 \) has the form (cf. \((1.9),(1.19)\))

\[
\hat{C}_4 = \text{STr} \ C_4(F) = -\frac{1}{8} \text{STr} \left[ 4v^2 q^2 J_0^2 J_1^2 + 4v^2 (h^2 + w^2) J_0^4 + v^4 J_0^4 \right]
\]
Using (5.8), (5.11) we find the agreement between the supergravity (5.4) and the SYM (2.6), (2.12) expressions for the leading-order term in the potential. Note that the leading-order potential $\sim \hat{C}_4$ depends only on the sum $<h^2 + w^2>$ of the gauge field and scalar field contributions to the momentum, i.e. it is not sensitive to how the momentum is distributed between the two terms.\footnote{The agreement between the leading-order terms in $\mathcal{V}$ and $\Gamma$ was previously checked in [16, 17] where the scalar wave contribution was not included, i.e. $w$ was equal to zero.} This will no longer be so for $\hat{C}_6$.

For $\hat{C}_6 = \hat{\text{STr}}C_6$ we find (cf. (4.9), (4.20))

$$\hat{C}_6 = -\frac{1}{16}\hat{\text{STr}}\left[8v^2w^2q^2J^4_0J^2_1 + 2v^4q^2J^4_0J^2_1 + 4v^4(h^2 + w^2)J^6_0 + v^6J^6_0\right].$$  

(5.17)

Computing the modified trace $\hat{\text{STr}}$ as in (4.21) we get for the 2-loop coefficient in $\Gamma$ (2.6) ($N = n_0 + N_0$)

$$N\hat{C}_6 = -\frac{1}{8}n_0N_0\left[(n_0 + N_0)[8v^2w^2q^2 + 2v^4q^2 + 4v^4(h^2 + w^2) + v^6] + N_0(8v^2w^2q^2 + 2v^4q^2)\right].$$  

(5.18)

The second $N_0$-term in the brackets in (5.18) originates from the correction term in $\hat{\text{STr}}$ in (2.23). This effectively doubles the coefficients of the $J^4_0J^2_1$ terms in (5.17). The $n_0N^2_0$ part of (5.18) which should be compared with the supergravity potential becomes

$$N\hat{C}_6 = -\frac{1}{8}n_0N_0\left[16v^2w^2q^2 + 4v^4(q^2 + h^2 + w^2) + v^6\right] + O(n^4_0).$$  

(5.19)

Using (5.8), (5.11) we conclude that the $v^4$ and $v^6$ terms in the 2-loop term in $\Gamma$ (2.6) are indeed in agreement with the classical supergravity potential (5.3).

As for the $v^2$-term in (5.19) which is present only if $w \neq 0$, we find that it reproduces the $v^2$ term in (5.5) provided the momentum is distributed equally between the gauge field and the scalar field oscillations, i.e. if (cf. (5.11))

$$<w^2> = <h^2> = \frac{1}{2} <w^2 + h^2> = \frac{1}{2} \tilde{Q}^{(4)}_{(1)}\tilde{Q}^{(5)}_{(0)}.$$  

(5.20)

It would be interesting to find an independent reason for imposing this condition.

### 5.2 0-brane interaction with $D = 4$ black hole ($0 - (4 \perp 4 \perp 4 \| 0)$)

The 1/8 supersymmetric marginal bound state configurations of $D = 11$ theory corresponding to $D = 4$ extremal black holes with regular horizons [55] may be represented as M-brane intersections $2 \perp 2 \perp 5 \perp 5$ or $5 \perp 5 \perp 5 + \text{wave wrapped over } T^6 \times S^1 [56]$. Assuming...
that the intersection direction is the 11-th one, the second configuration (which is thus a combination of the three longitudinal 5-branes) admits a simple SYM description in terms of three ‘overlapping’ instantons on the dual 6-torus.

Dimensional reduction of the $D = 11$ background $5 \perp 5 \perp 5$-wave to $D = 10$ gives the $4 \perp 4 \perp 4 \perp 0$ configuration which becomes $2 \perp 2 \perp 2 \perp 6$ after T-duality along the directions of 6-torus. As we shall demonstrate below, it can be described in terms of certain 1/4 supersymmetric gauge field background in the $D = 6 + 1$ SYM theory on $\tilde{T}^6$ which may be interpreted as a superposition of the three ‘overlapping’ instantons (with the instanton numbers being the charges of the three orthogonal 2-branes). The choice of such gauge field background is not unique, and we shall present both ‘commuting’ (see also Appendix) and ‘non-commuting’ representations for it.

In contrast to the $D = 5$ black hole case considered above which does not admit a purely D-brane description (one of the charge is always from the NS-NS sector), in the $4 \perp 4 \perp 4 \perp 0$ case all the charges are from the R-R sector and so the correspondence between the supergravity and SYM expressions for the interaction potential should, in principle, be more straightforward to establish. This is indeed so for the leading-order term in the $1/r$ expansion, irrespective of a particular choice of the gauge field theory representation for $4 \perp 4 \perp 4 \perp 0$. We shall find, however, that to be able to reproduce the subleading term in the supergravity potential (in particular, its $v^2$ part) one must use the non-commuting version of the corresponding SYM background.

### 5.2.1 Supergravity background and interaction potential

The $5 \perp 5 \perp 5$-wave configuration (with 5-brane coordinates \{1, 2, 3, 4, 11\}, \{1, 2, 5, 6, 11\}, and \{3, 4, 5, 6, 11\}) reduced along $x_{11}$ gives the type IIA $4 \perp 4 \perp 4 \perp 0$ background with the following string-frame metric, dilaton and R-R vector field [56] (cf. (5.1))

\[
\begin{align*}
\text{d}s^2_{10} &= (H_0 H_{4(1)} H_{4(2)} H_{4(3)})^{-1/2} \left[ -\text{d}t^2 + H_0 H_{4(3)} (\text{d}x_1^2 + \text{d}x_2^2) + H_0 H_{4(2)} (\text{d}x_3^2 + \text{d}x_4^2) \\
&\quad + H_0 H_{4(1)} (\text{d}x_5^2 + \text{d}x_6^2) + H_0 H_{4(1)} H_{4(2)} H_{4(3)} (\text{d}x_7^2 + \text{d}x_8^2 + \text{d}x_9^2) \right], \\
\text{e}^{2\phi} &= (H_{4(1)} H_{4(2)} H_{4(3)})^{-1/2} H_0^{3/2}, \\
A &= H_0^{-1} \text{d}t.
\end{align*}
\]

\(^{27}\)There are other possible SYM (or matrix theory) embeddings of $D = 4$ black holes involving finite boosts which represent extra parameters of the corresponding non-marginal generalisations of the marginal 1/8 BPS bound states (we need not discuss them here since we keep 0-brane number $N_0$ finite as in [27]). As an example, one may consider $2 \perp 2 \perp 5 \perp 5$ with the brane directions being \{5, 6\}, \{4, 7\}, \{1, 2, 3, 4, 5\}, \{1, 2, 3, 6, 7\} and add a boost along direction 1. Reducing down to $D = 10$ along the boost direction, we get a non-marginal bound state $0 + (4 \perp 4 \perp 2 \perp 2)$ parametrised by 5 charges, or, after T-duality along all of the directions of 6-torus, $6 + (2 \perp 2 \perp 4 \perp 4)$ (this configuration is related to the 5-charge $D = 4$ black hole in [53]).

\(^{28}\)Other U-dual $D = 10$ configurations like $3 \perp 3 \perp 3 \perp 3$ [48, 71, 55] do not have as simple SYM description as the one existing for $2 \perp 2 \perp 2 \perp 6$ we describe below. The $2 \perp 2 \perp 2 \perp 6$ configuration was also mentioned as a possible matrix theory representation for $D = 4$ black holes in [50].
Here the charges are proportional to the numbers of branes according to (3.5), i.e. $Q_0^{(6)} \sim N_0$, $Q_{4(k)}^{(2)} \sim N_{4(k)}$. The action of a 0-brane probe in this background is again of the same form as (1.1), (1.11), (5.3)\footnote{Essentially the same action is found for the T-dual configuration of a 3-brane probe in the 3±3±3±3 background\footnote{Note that these leading terms in the potential simplify when the tori are self-dual, i.e. when $V_p = (2 \pi)^{p/2}$: all factors of $2 \pi$ inside the brackets cancel out and one is left with combinations of integer numbers of branes multiplying powers of velocity.}}.

\[
I_0 = -T_0 \int dt \, H_0^{-1} \left( \sqrt{1 - H_0 H_{4(1)} H_{4(2)} H_{4(3)} v^2} - 1 \right),
\]  

so that the two leading terms in the interaction potential (1.2) are (cf. (4.15), (4.16) and (5.23))

\[
\mathcal{V}^{(1)} = -\frac{1}{8r} T_0 \left[ 4r^2 \left( \frac{Q_{4(1)}^{(2)} + Q_{4(2)}^{(2)} + Q_{4(3)}^{(2)}}{V_{2(1)}} + v^2 Q_0^{(6)} \right) \right]
\]

\[
= -\frac{\pi n_0}{8r} \left[ 4r^2 \left( \frac{N_{4(1)} + N_{4(2)}}{V_{2(1)}} + \frac{N_{4(3)}}{V_{2(3)}} + v^6 N_0 (2 \pi R)^2 \right) \right],
\]

\[
\mathcal{V}^{(2)} = -\frac{1}{16r^2} T_0 \left[ \frac{8r^2}{V_0} \left( \frac{Q_{4(1)}^{(2)} Q_{4(2)}^{(2)} + Q_{4(2)}^{(2)} Q_{4(3)}^{(2)} + Q_{4(1)}^{(2)} Q_{4(3)}^{(2)}}{V_{2(1)}} + v^6 Q_0^{(6)} \right) \right]
\]

\[
+ 4v^4 \left( \frac{Q_{4(1)}^{(2)} + Q_{4(2)} + Q_{4(3)}^{(2)}}{V_{2(1)}} + v^6 Q_0^{(6)} \right) \right],
\]

\[
= -\frac{n_0 g_s}{64(2\pi)^{1/2} r^2} \left\{ \frac{1}{2} (2 \pi)^2 v^2 \left( \frac{N_{4(1)} N_{4(2)}}{V_{4(3)}} + \frac{N_{4(2)} N_{4(3)}}{V_{4(1)}} + \frac{N_{4(1)} N_{4(3)}}{V_{4(2)}} \right) \right.
\]

\[
+ 4v^4 N_0 \left( \frac{2 \pi R}{V_0} \left( \frac{N_{4(1)}}{V_{2(1)}} + \frac{N_{4(2)}}{V_{2(2)}} + \frac{N_{4(3)}}{V_{2(3)}} \right) + v^6 N_0 (2 \pi R)_0^2 \right) \right\},
\]

where $V_0 = (2 \pi)^6 R_1 \ldots R_6$ is the volume of $T^6$, $V_{2(1)} = (2 \pi)^2 R_5 R_6$, $V_{2(2)} = (2 \pi)^2 R_3 R_4$, $V_{2(3)} = (2 \pi)^2 R_1 R_2$ and $V_{4(k)} = V_0 / V_{2(k)}$\footnote{Essentially the same action is found for the T-dual configuration of a 3-brane probe in the 3±3±3±3 background\footnote{Note that these leading terms in the potential simplify when the tori are self-dual, i.e. when $V_p = (2 \pi)^{p/2}$: all factors of $2 \pi$ inside the brackets cancel out and one is left with combinations of integer numbers of branes multiplying powers of velocity.}}. Our aim will be to reproduce these expressions by substituting appropriate SYM background into the 1-loop and 2-loop terms in the SYM effective action (2.6). Note that the remaining part ($\sim g_s^2 N_{4(1)} N_{4(2)} N_{4(3)}$) of the $v^2$ term in the potential in (5.24) which is contained in $\mathcal{V}^{(3)}$ should come from a 3-loop term in the SYM effective action.

### 5.2.2 SYM backgrounds representing $4\perp 4\perp 4\perp 0$

To find a description of $4\perp 4\perp 4\perp 0$ wrapped over $T^6$ or of its T-dual configuration $6\perp 2\perp 2\perp 2$ wrapped over $\tilde{T}^6$ in terms of a gauge field background $F_{mn}$ ($m, n = 1, \ldots, 6$) in the SYM theory on $\tilde{T}^6$ one needs to satisfy the following conditions:

1. $F_{mn}$ should preserve $1/4$ of $\mathcal{N} = 1, D = 10$ supersymmetry, i.e. there should exist the corresponding $\epsilon \neq 0$ solution of (5.12);
(2) Substituted into the (non-abelian, $U(N_0)$) D6-brane action, $F_{mn}$ should induce only the required charges of the three 2-branes: it should satisfy \( \text{tr} F = 0 \), $\int \text{tr}(F \wedge F \wedge F) = 0$ with $\text{tr}(F \wedge F) \neq 0$ such that $\int C_3 \wedge \text{tr}(F \wedge F)$ gives the coupling of 3-form field to the charges $N_{4(1)}, N_{4(2)}$ and $N_{4(3)}$ of 2-branes wrapped over the three orthogonal cycles of 6-torus;

(3) The classical BI Lagrangian \( L = \text{Str} \sqrt{\text{det}(\delta_{mn}I + F_{mn})} \) should reproduce the mass of the marginal BPS bound state $6 \parallel 2 \perp 2 \perp 2$, i.e. all higher-order terms in $L_6$ should vanish, $L = \text{tr}(I + \frac{1}{4}F_{mn}F_{mn})$.

The 2-brane charges on a collection of $N_0$ 6-branes may be represented by a 4d (anti)self-dual $SU(N_0)$ gauge field backgrounds. There exists several $D = 6 + 1$ SYM backgrounds which may be interpreted as ‘superpositions’ of the three instantons and which satisfy the above conditions and thus are candidates for a description of $6 \parallel 2 \perp 2 \perp 2$.

While all of them, when substituted into the SYM effective action \( (2.6) \), reproduce the leading term \( (5.24) \) and the \( v^4 \) and \( v^6 \) parts of the subleading term \( (5.25) \) in the supergravity potential, it turns out to be impossible to reproduce the \( v^2 \) term in \( V^{(2)} \). We shall show that there exists a non-commuting background which produces that needed \( v^2 \) term under a natural assumption that the 2-loop term in the SYM effective action \( (2.6) \) should, in general, contain terms with commutators of $F$.

We shall consider the following $su(N_0)$ constant gauge field strength background on $\tilde{T}^6$ which may be viewed as a generalisation of the three (anti)self-dual 4d backgrounds in three different 4-spaces which intersect over 2-spaces

\[
F_{14} = -F_{23} = p_1 h_1, \quad F_{45} = -F_{36} = p_2 h_2, \quad F_{15} = F_{26} = p_3 h_3, \quad (5.27)
\]

with all other components being zero. Here $p_k$ are constants which we shall fix as

\[
p_k^2 = V^{-1} \frac{N_{4(k)}}{N_0} = (2\pi)^2 \tilde{V}^{-1} \frac{N_{4(k)}}{N_0}, \quad (5.28)
\]

while $h_k$ are independent $su(N_0)$ matrices. We shall consider the following two different choices for $h_k$. The first one will be

\[
h_k = \mu_k \otimes I_{\frac{N_0 \times N_0}{4}}, \quad (5.29)
\]

where $\mu_k$ are diagonal $4 \times 4$ matrices from the Cartan subalgebra of $su(4)$ (used in \( [13] \) to describe a YM background representing a non-supersymmetric $6 + 0$ configuration)

\[
\mu_1 = \text{diag}(1,1,-1,-1), \quad \mu_2 = \text{diag}(1,-1,-1,1), \quad \mu_3 = \text{diag}(1,-1,1,-1), \quad (5.30)
\]

\[
\text{tr} \mu_k = 0, \quad \text{tr}(\mu_k \mu_l) = 4\delta_{kl}, \quad \mu_k \mu_l = \epsilon_{klm} \mu_m, \quad [\mu_k, \mu_l] = 0 .
\]
Our second choice will be

\[ h_k = \sigma_k \otimes I_{N_0 \times N_0}, \quad \sigma_k \sigma_l = \delta_{kl} + i \epsilon_{klm} \sigma_m, \]

where \( \sigma_k \) are the \( SU(2) \) Pauli matrices.

These two choices of \( h_k \) define a commuting and a non-commuting \( F_{mn} \) backgrounds \( (5.27) \) which will have the same basic properties.\(^{31}\) Since the matrices \( h_k \) are linearly independent, the condition of preservation of supersymmetry \((5.12)\) leads simply to

\[ (\gamma_{14} - \gamma_{23}) \epsilon = 0, \quad (\gamma_{45} - \gamma_{36}) \epsilon = 0, \quad (\gamma_{15} + \gamma_{26}) \epsilon = 0. \] \( (5.32) \)

The third condition here is a consequence of the first two, so that \((5.32)\) may be expressed in terms of the two commuting projectors \( (cf. (5.13)) \) \( P_{1234} \epsilon = 0, \quad P_{3456} \epsilon = 0. \) As a result, there exists a solution for \( \epsilon \) representing the remaining 1/4 of the \( \mathcal{N} = 1, D = 10 \) \( (\text{corresponding to } 1/8 \text{ of } \mathcal{N} = 2, D = 10) \) supersymmetry.\(^{32}\)

It is also easy to check that because of the properties of the matrices \( \mu_k \) or \( \sigma_k \) the background \( (5.27) \) induces only the required 2-brane charges on the 6-brane, with the three 2-branes oriented along the 12, 34 and 56 cycles of the 6-torus.

Computing the determinant in the classical BI action \((1.3)\) (defined with the symmetrised trace, i.e. ignoring possible commutator terms in non-commuting case \((15)\)) one finds

\[ L = \text{Str} \sqrt{\det(\delta_{mn}I + F_{mn})} = \sum_{n=0}^{\infty} \text{Str} C_{2n}(F) \]

\[ = \text{Str} \sqrt{(I + p_1^2 h_1^2 + p_2^2 h_2^2 + p_3^2 h_3^2)^2} = N_0(1 + p_1^2 + p_2^2 + p_3^2) = \text{tr}(I + \frac{1}{4} F_{mn} F_{mn}). \]

\( (5.33) \)

Note, in particular, that

\[ C_{2n}(F) = 0, \quad n = 2, 3, \ldots. \] \( (5.34) \)

As a result, the energy of the gauge field configuration \((5.27)\) is, indeed, equal to the mass of the 1/8 supersymmetric marginal bound state 414141||0 or 6||212121.\(^{33}\)

\[ M = T_6 \int d^6 \bar{x} \text{Str} \sqrt{\det(\delta_{mn}I + F_{mn})} = (2\pi)^{1/2} g_s^{-1} \left( N_0 + \frac{N_4(1)}{V_{2(1)}} + \frac{N_4(2)}{V_{2(2)}} + \frac{N_4(3)}{V_{2(3)}} \right). \]

\( (5.35) \)

In the commuting \( h_k \sim \mu_k \) case \((1.33)\) follows from the fact that the squares of \( h_k \) are equal to the unit matrix \( I = I_{N_0 \times N_0} \). In the non-commuting \( h_k \sim \sigma_k \) case one is to note that \( \text{Str} \sqrt{\det(\delta_{mn}I + F_{mn})} \) is computed by first expanding the square root in powers of \( F \) \((2.8)\) and then applying \( \text{Str} \). Since under the symmetrised trace the factors of \( F_{mn} \) in \( C_{2n}(F) \) may be treated as commuting, the vanishing of all \( C_{2n}, \quad n > 1 \), which follows from the structure of the abelian version of \((5.27)\), implies also that \( \text{Str} C_{2n}(F) = 0, \quad n = 2, 3, \ldots. \)

\(^{31}\)The commuting background is obviously a solution of the \( D = 6 \) YM equations. The same should be true also for the non-commuting case, i.e. there should exist a potential \( A_m \) which solves the classical \( SU(2) \) YM equations and has \((5.27), (5.31)\) as its field strength.

\(^{32}\)Related discussions of supersymmetry-preserving conditions appeared in \([58, 60, 61, 62]\). Supersymmetric SYM solutions on 8-torus were discussed in \([53]\).

\(^{33}\)See also \([64, 13, 19]\) for a similar discussion of energies of 4||0 and other gauge field configurations.
5.2.3 Supergravity - SYM correspondence

Let us now demonstrate that while both commuting (5.27),(5.29) (and (A.4),(A.3)) and non-commuting (5.27),(5.31) SYM backgrounds (supplemented by the velocity component and substituted into the SYM effective action (2.6)) reproduce \( V^{(1)} \) (5.25) and the \( v^4 \) and \( v^6 \) parts of \( V^{(2)} \) (5.26), it is only the non-commuting background (5.27),(5.31) that may generate the subtle \( v^2 \) term in (5.26) provided also that the 2-loop coefficient \( \hat{C}_6(F) \) (2.27) contains commutator terms like (2.28).

The supergravity potential in (5.25),(5.26) expressed in terms of the SYM background (5.27) takes the following form (cf. (4.15),(4.16))

\[
V^{(1)} = -\frac{n_0}{16r} \tilde{V}_6 \left[ v^2 \text{tr}(F_{mn}F_{mn}) + v^4 N_0 \right],
\]

\[
V^{(2)} = -\frac{n_0 g_{YM}^2}{(4\pi)^{d/2}} \tilde{V}_6 \left[ \frac{1}{2} v^2 (|F|^2) + v^4 N_0 \text{tr}(F_{mn}F_{mn}) + v^6 N_0^2 \right],
\]

where \( g_{YM}^2 = (2\pi)^{-1/2} g_s \tilde{V}_6 \) and \(|F|^2\) is a notation for

\[
(|F,F|^2) = 16 N_0^2 (p_{1}^2 + p_{2}^2 + p_{3}^2).
\]

The \( su(N_0 + n_0) \) SYM background \( F_{ab} = (F_{09}, F_{mn}) \) describing the interaction between a 0-brane with velocity \( v \) and the 4 \( \perp 4 \perp 0 \) bound state is given by the \( su(N_0) \) field strength (5.27) embedded into \( su(N_0 + n_0) \) as \( F_{mn} \rightarrow \text{diag}(0_{n_0 \times n_0}, F_{mn}) \) and the ‘velocity’ component \( F_{09} \) (cf. (4.17),(4.18) and (5.9),(5.10))

\[
F_{09} = v J_0 , \quad [F_{09}, F_{mn}] = 0 , \quad m,n = 1,...,6 .
\]

Since the indices of the commuting ‘velocity’ and ‘instanton’ parts of \( F_{ab} \) do not overlap (in contrast to what was in the case of \( D = 5 \) black hole background (5.9),(5.10)), one finds that the 1-loop and 2-loop coefficients in (2.6), (2.13),(2.19) take the following form (cf. (4.19),(4.20))

\[
\hat{C}_4(F_{ab}) = \hat{C}_4(F_{mn}) - \frac{1}{4} \text{STr} \left( v^2 J_0^2 F_{mn} F_{mn} + v^4 J_0^4 \right),
\]

\[
\hat{C}_6(F_{ab}) = \hat{C}_6(F_{mn}) - \frac{1}{16} \text{STr} \left( -v^2 J_0^2 [F_{mn}F_{nk}F_{kl}F_{lm} - \frac{1}{4} (F_{mn}^2)^2] + \frac{1}{2} v^4 J_0^4 F_{mn} F_{mn} + v^6 J_0^6 \right).
\]

As a result, the background (5.27) (or (A.4)) substituted into the effective action (2.6) exactly reproduces (5.36) and the \( v^4 \) and \( v^6 \) terms in (5.37). This is, of course, not too surprising since the coefficients of the \( v^2 \) term in (5.36) and the \( v^4 \) term in (5.37) are additive in the constituent instanton (4-brane) charges, so that the agreement is essentially the consequence of the one in the \( 4 \perp 0 \) case (4.19),(4.21).

At the same time, if we assume that \( \hat{C}_6 \) is totally symmetric in the six \( F \)-factors as in (2.13),(2.19), then the coefficient of the \( v^2 \) term in (5.41) vanishes and thus does match
the one in $\mathcal{V}^{(2)}$ (5.37). Indeed, this coefficient is proportional to $C_4(F_{mn})$ (2.3) which vanishes identically for the background (5.27) (see (5.34)).

The non-trivial composite structure of the 4.4.4.4||0 bound state which is responsible for the appearance of the product of the harmonic functions in (5.24) and thus for the fact that the $v^2$ term in (5.26),(5.37) is proportional to a combination of products of charges of the constituent 4-branes, should, in fact, be reflected in a non-abelian nature of the corresponding background (5.27),(5.31).

On general grounds, the 2-loop term $\hat{C}_0$ in (2.6) may contain various $F^6$ commutator terms (2.28). To illustrate that they may, indeed, produce the required $v^2$ term let us we shall consider a particular commutator term with simple Lorentz index and internal index contractions (cf. (2.28))

$$C_0 = i\beta_1 \text{Tr}(F_{ab}F_{cd}F_{ef}F_{cd}F_{ef}) ,$$

(5.42)

where $\beta_1$ is a universal numerical coefficient. Using (5.38),(5.27),(5.31) and noting that

$$\text{Tr}(F_{ab}F_{cd}F_{ef}F_{cd}F_{ef}) \to v^2 \text{Tr}(J_0^3[F_{mn},F_{kl}^2]) + ...$$

it is easy to see that this term (multiplied by $N$ according to (2.6)) contains indeed the same $v^2$ contribution as in (5.37), i.e. is proportional to $n_0 N^2 v^2(p_1^2 p_2^2 + p_2^2 p_3^2 + p_1^2 p_3^2)$.

6 Concluding remarks

The approach we have used in this paper – first extracting an ansatz for the leading IR part $\Gamma^{(2)} \sim \int \hat{C}_0$ of the 2-loop SYM effective action from comparison with supergravity potentials for 0-brane interactions with simple bound states of branes and then checking its consistency against more complicated examples with less supersymmetry may be extended to various other cases. One may consider non-marginal generalisations of, e.g., 4||0 bound state (described by a combination of the instanton and constant magnetic backgrounds [49]) as well as interactions between two different bound states of branes (e.g., 2 + 0 and 4||0 with the former treated as a probe as in [13]). One may also study subleading terms in the interaction potentials for non-supersymmetric configurations involving 6 + 0 states [19] or near-extremal $D = 5$ and $D = 4$ black holes. This may lead to further non-trivial

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34The same conclusion is reached in the case of another commuting background (5.37). Since $J_0$ is proportional to the unit matrix on the subspace where $F_{mn}$ is non-vanishing, $\text{STr}[J_0^3C_4(F_{mn})] \sim \text{STr}C_4(F_{mn})$. Using that the explicit form of $F_{mn}$ in this case is $F_{12} = q_1 J_{1}^{(1)} + q_2 J_{2}^{(2)}$, $F_{34} = q_1 J_{1}^{(1)} + q_3 J_{1}^{(3)}$, $F_{56} = q_2 J_{1}^{(2)} + q_3 J_{1}^{(3)}$, where $J_{1}^{(k)}$ are defined as in (4.18) with zeroes in the complementary blocks, we find that $C_4(F_{mn}) = -16q_1 q_2 q_3 J_{1}^{(1)} J_{1}^{(2)} J_{1}^{(2)} (q_1 J_{1}^{(1)} + q_2 J_{1}^{(2)} + q_3 J_{1}^{(3)})$ which does not contain the required structure in (5.38) ($\sim q_1^2 q_2^2 + ...$) even before one takes the traces.

35Dots stand for $v$-independent terms that should cancel in an appropriate combination of $F^6$ commutator terms (the $v = 0$ configuration is a BPS one).

36Since, e.g., $[\sigma_1, \sigma_2] = 2i\sigma_3$, the same relation is true for the corresponding components of $F_{mn}$ in (5.27) embedded into $su(n_0 + N_0)$, so that we conclude (cf. (5.27) and (4.17),(4.18)) that the $v^2 p_1^2 p_2^2$ term in (5.42) has coefficient $\text{Tr}(J_0^3 J_0^2)$ as in (4.13) which is equal to $2n_0 N_0$.  

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checks of the expression for $\hat{C}_6$ we have suggested above and may, in particular, allow to fix the form of the commutator terms in it.

It would be important, of course, to compute the relevant $F^6$ terms in the SYM effective action directly, extending the $D = 1 + 0$ result of [24, 25] to higher dimensions and more general SYM backgrounds and thus verifying our conjectures about the structure of $\hat{C}_6$. This may be feasible using a combination of techniques in [25, 26, 27].

Furthermore, it would be most interesting to perform a string-theory computation of the subleading (2-loop) terms in the interaction potential, checking that the $r \to 0$ and $r \to \infty$ limits of the string result continue to agree (for relevant supersymmetric configurations of branes) beyond the leading 1-loop level considered in [2, 3, 8]. This would provide an explanation for the supergravity-SYM correspondence at the subleading level demonstrated in [25] and in the present paper.

Finally, there remain also questions about the role of 0-branes (and thus of Matrix theory relation) in this correspondence. Does it hold also for other appropriate configurations of branes with no 0-brane content (or, in T-dual picture, for configurations other than a ‘Dp-brane+...’ parallel to a ‘Dp-brane + ...’)? Related question is about the role of the large $N_0$ (0-brane number) limit or the ‘null reduction’ ansatz of dropping 1 in the 0-brane harmonic function [25]. If there is indeed a weak-coupling string-theory explanation for the supergravity-SYM correspondence at the subleading level, it may presumably apply also to some other (nearly) supersymmetric configurations of (large number of) branes.

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Appendix A

Here we shall describe another representation for the $4\perp 4\perp 4\parallel 0$ bound state in terms of a commuting SYM background on the dual 6-torus which is different from the one given in section 5.2.2.

Each of the three longitudinal M5-branes in $5\perp 5\perp 5$+wave configuration may be described [11, 28] by a SYM instanton on the dual torus, with the instanton charge being the wrapping number of the five-brane. One way to combine them together is to split the total number of 0-branes $N_0$ in $4\perp 4\perp 4\parallel 0$ (equal to the number of 6-branes in the T-dual $6\parallel 2\perp 2\perp 2$ configuration) into the three parts, $N_0 = N_0(1) + N_0(2) + N_0(3)$, and to consider the three instantons embedded, respectively, into the $su(N_0(1))$, $su(N_0(2))$ and $su(N_0(3))$ subalgebras of $su(N_0)$. The components of the SYM gauge field potential on $\tilde{T}^6$ may
be chosen as $A_1 = \text{diag} \left( A_1^{(1)}(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4), A_1^{(2)}(\bar{x}_1, \bar{x}_2, \bar{x}_5, \bar{x}_6), 0_{N_0(3) \times N_0(3)} \right)$, ..., $A_6 = \text{diag} \left( 0_{N_0(1) \times N_0(1)}, \ A_6^{(2)}(\bar{x}_1, \bar{x}_2, \bar{x}_5, \bar{x}_6), \ A_6^{(3)}(\bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6) \right)$, where $A_m^{(k)} \in su(N_0(k))$ have (anti)self-dual field strengths,

$$F^{(k)}_{mn} = \pm (F^{(k)}_{mn})^*, \quad \frac{1}{16\pi^2} \int_{\mathcal{F}^{(k)}} d^4 \bar{x} \text{ tr } (F^{(k)}_{mn} F^{(k)}_{mn}) = N_{4(k)}.$$ (A.1)

Assuming that $N_0(k)$ are even, $F^{(k)}_{mn}$ may be taken in the same explicit (anti)self-dual form as in \([1.13]\),\([1.14]\)

$$F^{(1)}_{12} = -F^{(1)}_{34} = q_1 \sigma_3 \otimes I_{N_0(1) \times N_0(1)/2}, \quad F^{(2)}_{12} = -F^{(2)}_{56} = q_2 \sigma_3 \otimes I_{N_0(2) \times N_0(2)/2}, \quad F^{(3)}_{34} = -F^{(3)}_{56} = q_3 \sigma_3 \otimes I_{N_0(3) \times N_0(3)/2},$$ (A.2)

where (cf. \([5.28]\))

$$q_k^2 = (2\pi)^2 \bar{V}^{-1}_{4(k)} N_{4(k)} / N_{0(k)}.$$ (A.3)

The choice of the signs here is important for preservation of supersymmetry (see below).

The non-vanishing components of $F_{mn} = \text{diag}(F^{(1)}_{mn}, F^{(2)}_{mn}, F^{(3)}_{mn})$ are then (cf. \([5.27]\))

$$F_{12} = f_1, \quad F_{34} = f_2, \quad F_{56} = f_3,$$ (A.4)

where $f_k$ are the following commuting block-diagonal $su(N_0)$ matrices

$$f_1 = \text{diag} \left( q_1 \sigma_3 \otimes I_{N_0(1) \times N_0(1)/2}, q_2 \sigma_3 \otimes I_{N_0(2) \times N_0(2)/2}, 0_{N_0(3) \times N_0(3)} \right),$$

$$f_2 = \text{diag} \left( -q_1 \sigma_3 \otimes I_{N_0(1) \times N_0(1)/2}, 0_{N_0(2) \times N_0(2)}, q_3 \sigma_3 \otimes I_{N_0(3) \times N_0(3)/2} \right),$$

$$f_3 = \text{diag} \left( 0_{N_0(1) \times N_0(1)}, -q_2 \sigma_3 \otimes I_{N_0(2) \times N_0(2)/2}, -q_3 \sigma_3 \otimes I_{N_0(3) \times N_0(3)/2} \right).$$ (A.5)

To see if this SYM background preserves some amount of supersymmetry, we write down the condition \([5.12]\) for each of the three diagonal blocks of the $su(N_0)$ matrix $F_{mn}$. As a result, we get three copies of the ‘single-instanton’ condition \([5.13]\)

$$(\gamma_{12} - \gamma_{34})\epsilon = 0, \quad (\gamma_{12} - \gamma_{56})\epsilon = 0, \quad (\gamma_{34} - \gamma_{56})\epsilon = 0,$$ (A.6)

with the third one being a consequence of the first two.\(^\text{37}\) The conditions \((A.6)\) may thus be expressed in terms of two commuting projectors

$$P_{1234}\epsilon = \epsilon, \quad P_{1256}\epsilon = \epsilon,$$ (A.7)

\(^{37}\)The choice of all conditions in \([A.2]\) as self-duality ones would thus lead to a contradiction unless $\epsilon = 0$ and thus to the complete breakdown of supersymmetry.

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implying that $1/4$ of the $\mathcal{N} = 1, D = 10$ supersymmetry is preserved.

Since $\text{tr}F_{mn} = 0$ and $F_{12}F_{34}F_{56} = 0$ (as follows from (A.4), (A.5)) the only non-vanishing charges induced by this background on the 6-brane world volume are the 2-brane charges ($\sim \int \text{tr}(F_{mn}F_{kl})$) in the 12, 34 and 56 cycles of the 6-torus. They are equal to $N_{4(1)}, N_{4(2)}$ and $N_{4(3)}$.

The energy of this gauge field configuration (A.4) obtained from the classical BI action is, indeed, equal to the mass of the $1/8$ supersymmetric marginal bound state $4 \perp 4 \perp 4 \parallel 0$. Since all components of $F_{mn}$ here are commuting, the symmetrised trace in (1.3) is equivalent simply to the trace in the fundamental representation and we find the result similar to that in (5.33)

$$M = T_6 \int d^6\tilde{x} \, \text{tr} \sqrt{\text{det}(\delta_{mn}I + F_{mn})} = T_6 \tilde{V}_6 \left[ N_0 + N_{0(1)}q_1^2 + N_{0(2)}q_2^2 + N_{0(3)}q_3^2 \right]$$

$$= T_6 \int d^6\tilde{x} \, \text{tr}(I + \frac{1}{4}F_{mn}F_{mn}) = (2\pi)^{1/2}g_s^{-1} \left( N_0 + \frac{N_{4(1)}}{V_{2(1)}} + \frac{N_{4(2)}}{V_{2(2)}} + \frac{N_{4(3)}}{V_{2(3)}} \right). \quad (A.8)$$
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