Chern-Simons Couplings and Inequivalent Vector-Tensor Multiplets

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Abstract

The off-shell vector-tensor multiplet is considered in an arbitrary background of $N = 2$ vector supermultiplets. We establish the existence of two inequivalent versions, characterized by different Chern-Simons couplings. In one version the vector field of the vector-tensor multiplet is contained quadratically in the Chern-Simons term, which implies nonlinear terms in the supersymmetry transformations and equations of motion. In the second version, which requires a background of at least two abelian vector supermultiplets, the supersymmetry transformations remain at most linear in the vector-tensor components. This version is of the type known to arise from reduction of tensor supermultiplets in six dimensions. Our work applies to any number of vector-tensor multiplets.

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1 Introduction

Supergravity actions are important tools in the study of string compactifications. Although physically relevant compactifications have a lower degree of symmetry, the more restrictive environment of $N = 2$ supersymmetry provides a rich testing ground for exploring perturbative and nonperturbative features of string theory, such as various kinds of dualities. An important role is played by the existence of different matter supermultiplets, whose mutual interactions are subject to stringent constraints. For vector and hyper multiplets these constraints form the basis for specific nonrenormalization theorems. Multiplets with two-form gauge potentials exhibit even more restrictive interactions, and presumably stronger nonrenormalization theorems. Such multiplets occur universally in string theory, as the world-sheet formulation naturally incorporates a target-space tensor gauge field. In four-dimensional effective field theories, this tensor field is commonly represented by a (pseudo)scalar “axion” field, which is obtained by performing a duality transformation on the tensor. The extra restrictions are then manifest in terms of an inevitable Peccei-Quinn invariance, which leads to specially constrained sigma-model geometries. In the presence of vector gauge fields, tensor gauge fields tend to couple to Chern-Simons forms.

In heterotic $N = 2$ four-dimensional supersymmetric string vacua, the axion/dilaton complex resides in a vector-tensor multiplet \cite{1, 2}. Off shell, this multiplet comprises a real scalar, a vector gauge field, a tensor gauge field, a real auxiliary scalar and a doublet of Majorana spinors. At the linearized level it can be obtained from reduction of a tensor supermultiplet in six spacetime dimensions. With respect to $N = 1$ supersymmetry, a vector-tensor multiplet decomposes into a tensor and a vector multiplet. The scalar and tensor components correspond to the dilaton and axion, which, after a duality conversion, are combined into a complex scalar dilaton field belonging to a vector supermultiplet. As the couplings of vector multiplets to supergravity have been extensively explored, it is convenient to work in such a dual formulation. However, it should be kept in mind that the couplings are much less restrictive in this context, and may not fully capture the relevant restrictions of the vector-tensor formulation. Thus, in this paper, we are motivated by the characterization of the heterotic axion/dilaton system in terms of an $N = 2$ effective action, within the more restrictive, and presumably more appropriate, context of the vector-tensor multiplet. Our results should be regarded as an extension of the work reported in \cite{3}. Rather surprisingly, but in line with the result of \cite{3}, we encounter couplings that are inconsistent with the typical couplings of the heterotic axion/dilaton complex. The couplings that we find appear in two varieties and instead play a role for the nonperturbative corrections to the effective Lagrangian for heterotic $N = 2$ supersymmetric string compactifications \cite{4}. One type of coupling could be of six-dimensional origin \cite{5}. For the relevance of nonperturbative phases with extra six-dimensional tensor multiplets, see \cite{4, 5} and references quoted therein.

An off-shell multiplet based on $8 + 8$ degrees of freedom, which includes one tensor and one vector gauge field, must have an (off-shell) central charge. Alternative off-shell formulations without a central charge require more, possibly infinite, degrees of freedom. Harmonic superspace would provide a natural setting for describing the latter case. In this paper we discuss the formulation with a central charge, which follows a similar pattern as was found for hypermultiplets \cite{6}. Specifically, the central charge acts as a translation operator, which links an infinite hierarchy of essentially identical multiplets. A system of
constraints then renders these additional multiplets dependent, in such a way that retains precisely $8 + 8$ off-shell degrees of freedom. The linearized version of this system has been discussed in a superspace context in [7].

Some comparison can be made between our results and previous results pertaining to hypermultiplets [8]. There exist quaternionic geometries known to hypermultiplet theories, which are inaccessible by the techniques which we employ. These restrictions have never been fully understood. In principle these could also occur for vector-tensor multiplets. As we will discuss in due course this seems indeed the case. Unlike in the hypermultiplet case, however, our approach yields the more complicated variants of the vector-tensor multiplet couplings, whereas some qualitatively simple ones do not fit into our framework. Here we have in mind certain theories that can be found (on-shell) by dimensional reduction of five-dimensional Einstein-Maxwell supergravity, after converting, in five dimensions, one of the vector fields into a tensor gauge field. On the other hand, some of our couplings are definitely not obtainable from higher-dimensional theories. Clearly the presentation in this paper constitutes only a first word on this issue and we feel that a further elucidation of these restrictions is highly desirable.

An important observation is that local supersymmetry requires a central charge to be realized locally. This necessitates a background of at least a single vector multiplet to provide the gauge field for the central charge transformation. In [3] the vector-tensor multiplet was considered in a background of a single vector multiplet with a local off-shell central charge. This resulted in the intricate hierarchical structure described above. Closure of the combined gauge and supersymmetry algebra then required a Chern-Simons coupling between the tensor field and the vector field of the vector-tensor multiplet itself. This in turn induced unavoidable nonlinearities into the supersymmetry transformation rules and into the action, in terms of the vector-tensor components.

In this paper we generalize the work of [3] by introducing a more general Chern-Simons term in an extended background of several vector multiplets. Crucial differences then occur. The most conspicuous is that in a background consisting of at least two vector multiplets, there exist two inequivalent classes of vector-tensor multiplets. In the first class the transformation rules are nonlinear. The theory considered in [3] belongs to this class. In the other class the transformation rules remain at most linear in the vector-tensor components so that the action is quadratic. The distinction between these cases is encoded in the particular Chern-Simons coupling chosen for the tensor field. In this way we are able to treat both the nonlinear and the linear versions in a common framework, and establish that, in truth, there are two inequivalent vector-tensor multiplets in four dimensions with $8 + 8$ off-shell degrees of freedom.

In our approach, we also exploit the presence of the vector multiplets for another purpose. We require all couplings to be invariant under constant scale and $U(1)$ transformations, so as to facilitate the coupling to supergravity via the superconformal multiplet calculus. Imposing these additional symmetries does not represent a significant restriction, as we can always freeze some of the background vector multiplets to a constant, a procedure that preserves supersymmetry and induces a breakdown of scale and chiral symmetry. The supergravity couplings will appear in a separate publication [8], together with a more detailed analysis of the couplings that seem to be outside the present framework.

In section 2 of this paper we exhibit the central-charge hierarchy and features of the Chern-Simons couplings. In section 3 we give the supersymmetry transformations for the two classes of vector-tensor multiplets. In section 4 we present the construction of
a linear multiplet from the vector-tensor multiplet, which enables the construction of a supersymmetric action, which is presented in section 5, where we also give the dual formulations of the corresponding theories in terms of vector supermultiplets.

2 Central charge, gauge structure and Chern-Simons couplings

The vector-tensor multiplet comprises a scalar field $\phi$, a vector gauge field $V_\mu$, a tensor gauge field $B_{\mu\nu}$, a doublet of spinors $\lambda_i$, and an auxiliary scalar field. As explained in the introduction, since this description involves $8 + 8$ off-shell degrees of freedom, it must incorporate an off-shell central charge. Infinitesimally, this charge acts as $\delta_z \phi = z\phi^{(z)}$.

Successive applications then generate a sequence of translations,

$$\phi \rightarrow \phi^{(z)} \rightarrow \phi^{(zz)} \rightarrow \text{etc},$$

and similarly on all other fields. It turns out that $\phi^{(z)}$ corresponds to the auxiliary field. All other objects in the hierarchy, $\phi^{(zz)}$, $V^{(z)}$, $V^{(zz)}$, etcetera, are dependent, and are given by particular combinations of the independent fields. This is enforced by a set of constraints, which we exhibit below. The central charge then acts so as to generate a sequence of multiplets, which are not independent; there are no new degrees of freedom beyond the $8 + 8$ described previously.

In addition to the central charge, the vector-tensor multiplet is subject to a pair of gauge transformations, consisting of a tensor transformation, with parameter $\Lambda_\mu$, under which $B_{\mu\nu} \rightarrow B_{\mu\nu} + \partial_\mu \Lambda_\nu$, and a vector transformation, with parameter $\theta^1$, under which $V_\mu \rightarrow V_\mu + \partial_\mu \theta^1$. (The reason for the superscript 1 will become clear shortly.) As will be described, in the interacting theory, the tensor field couples to certain Chern-Simons forms. Closure of the algebra then requires the vector transformation to act as well on the tensor field. The precise form of this transformation will be discussed below. It is necessary to first define the complete system of multiplets which we wish to describe, and to define some notational conventions.

We consider the vector-tensor multiplet in a background of $n$ vector multiplets. One of these provides the gauge field for the central charge, which we denote $W^0_\mu$. This must be an abelian gauge field. The remaining $n - 1$ vector multiplets supply additional background gauge fields $W^a_\mu$, which need not be abelian. The index $a$ is taken to run from 2 to $n$. We reserve the index 1 for the vector field $V_\mu$ of the vector-tensor multiplet. (The reason for this choice is based on the dual description of our theory, where the vector-tensor multiplet is replaced with a vector multiplet, so that the dual theory involves $n + 1$ vector multiplets.) Also, since $W^0_\mu$ is the gauge field for the central charge, the associated transformation parameter $\theta^0$ is identified with the central charge parameter $z$ introduced above, ie: $z \equiv \theta^0$. The vector gauge transformations act as follows on the background gauge fields,

$$\delta W^0_\mu = \partial_\mu z, \quad \delta W^a_\mu = \partial_\mu \theta^a + f_{bc}^a \theta^b W^c_\mu.$$

As mentioned above, in the interacting theory, the tensor field $B_{\mu\nu}$ necessarily couples to Chern-Simons forms. This coupling is evidenced by the transformation behavior of the

\footnote{A hierarchy such as (2.1) arises naturally when starting from a five-dimensional supersymmetric theory with one compactified coordinate, but this interpretation is not essential.}
tensor. To illustrate this, if we ignore the central charge (other than its contribution to $W_\mu^0$), then the vector field of the vector-tensor multiplet would transform as

$$\delta V_\mu = \partial_\mu \theta^1$$  \hspace{1cm} (2.3)$$

and the tensor field would transform as

$$\delta B_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} + \eta_{IJ} \theta^I \partial_{[\mu} W^J_{\nu]} ,$$  \hspace{1cm} (2.4)$$

where $\theta^I$ and $A_\mu$ are the parameters of the transformations gauged by $W^I_\mu$ and $B_{\mu\nu}$ respectively, and the index $I$ is summed from 0 to $n$. As mentioned above, in this context $W^1_\mu$ is identified with $V_\mu$. Closure of the combined vector and tensor gauge transformations requires that $\eta_{IJ}$ must be a constant tensor invariant under the gauge group. Furthermore, there is an ambiguity in the structure of $\eta_{IJ}$, which derives from the possibility of performing field redefinitions. For example, without loss of generality, $\eta_{IJ}$ can be chosen symmetric by absorbing a term $\eta_{[IJ]} W^I_\mu W^J_\nu$ into the definition of the tensor field $B_{\mu\nu}$. This point illustrates a feature that plays an important role in our results, namely that the presence of the vector multiplets allows for background-dependent field redefinitions. There is a similar issue for the other fields of the vector-tensor multiplet, which can also be redefined by terms depending on the background fields. We return to this issue later.

Using such redefinitions, we remove, without loss of generality, all components of $\eta_{IJ}$ except for $\eta_{11}$, $\eta_{1a}$ and $\eta_{ab}$, and also render $\eta_{ab}$ symmetric. Note that since $\eta_{1a}$ is invariant under the gauge group, it follows that $\eta_{1a} W^a_\mu$ is an abelian gauge field.

The situation is actually more complicated, since $V_\mu$ and $B_{\mu\nu}$ are also subject to the central charge transformation. As described above, under this transformation these fields transform into complicated dependent expressions, denoted $V^{(z)}_\mu$ and $B^{(z)}_{\mu\nu}$ respectively, which involve other fields of the theory. Accordingly, we deform the transformation rule (2.3) to

$$\delta V_\mu = \partial_\mu \theta^1 + z V^{(z)}_\mu ,$$  \hspace{1cm} (2.5)$$

and, at the same time, (2.4) to

$$\delta B_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} + \eta_{11} \theta^1 \partial_{[\mu} V^{(z)}_{\nu]} + \eta_{ab} \theta^a \partial_{[\mu} W^b_{\nu]} + \eta_{1a} \theta_{W^a_\mu} + \eta_{ab} \theta^a_{W^b_\mu} + z B^{(z)}_{\mu\nu} .$$  \hspace{1cm} (2.6)$$

All $\theta^0$-dependent terms, including any such Chern-Simons contributions, are now included in $V^{(z)}_\mu$ and $B^{(z)}_{\mu\nu}$, which are determined by closure of the full algebra, including supersymmetry. Note that the central charge acts trivially on the components of the vector multiplets, but not on the fields of the vector-tensor multiplet. The deformed transformation rules must still lead to a closed gauge algebra, which requires $B^{(z)}_{\mu\nu}$ to take the form

$$B^{(z)}_{\mu\nu} = -\eta_{11} V^{(z)}_{[\mu} V^{(z)}_{\nu]} + \hat{B}^{(z)}_{\mu\nu} ,$$  \hspace{1cm} (2.7)$$

where $\hat{B}^{(z)}_{\mu\nu}$ and $V^{(z)}_{\mu}$ transform covariantly under the central charge, but are invariant under all other gauge symmetries. The resulting gauge algebra now consists of the standard gauge algebra for the vector fields augmented by a tensor gauge transformation.

To complete the discussion of the geometrical features of the deformed gauge algebra, we list the fully covariant field strengths for the vector and tensor gauge fields. For the gauge fields of the background vector multiplets, we retain the standard expressions,

$$F^0_{\mu\nu} = 2\partial_{[\mu} W^0_{\nu]} , \quad F^a_{\mu\nu} = 2\partial_{[\mu} W^a_{\nu]} - f^a_{bc} W^b_{[\mu} W^c_{\nu]} ,$$  \hspace{1cm} (2.8)$$
where we have included possible nonabelian corrections in $F_{\mu\nu}^a$. As mentioned above, the gauge field associated with the central charge must be abelian. For the gauge fields of the vector-tensor multiplet the field strengths are

\[
F_{\mu\nu} = 2\partial_{[\mu} V_{\nu]} - 2W^0_{[\mu} V^{(z)}_{\nu]} ,
\]

\[
H^\mu = \frac{1}{2} i \varepsilon^{\mu\rho\sigma} \left[ \partial_\rho B_{\sigma} - \eta_{11} V_\rho V_\sigma - \eta_{1a} V_\rho \partial_\sigma W^a_\rho \right]
\]

\[-\eta_{ab} W^a_\nu \left( \partial_\sigma W^b_\rho - \frac{1}{3} \eta^{rb} W^c_\rho W^d_\sigma \right) - W^0_\nu \tilde{B}^{(a)}_{\rho\sigma} ,
\]

which are covariant under the combined gauge transformations, including those generated by the central charge. The Bianchi identities corresponding to the field strengths (2.8) and (2.9) are straightforward to determine, and are given by the following expressions,

\[
\partial_\mu \tilde{F}^{0\mu\nu} = 0 , \quad D_\mu \tilde{F}^{0\mu\nu} = 0 , \quad D_\mu \tilde{F}^{0\mu\nu} = -V^{(z)}_\mu \tilde{F}^{0\mu\nu} ,
\]

\[
D_\mu H^\mu = -\frac{1}{8} i \varepsilon^{\mu\rho\sigma} \left[ \eta_{11} F_{\mu\rho} F_{\sigma} + \eta_{1a} F_{\mu\rho} F^a_{\sigma} + \eta_{ab} F^a_{\mu\rho} F^b_{\sigma} + 2 \tilde{B}^{(a)}_\mu F^{0\mu\nu} \right] ,
\]

where the derivative on the left-hand side is covariant with respect to central-charge and nonabelian gauge transformations.

Observe that we have not yet specified the form of $\tilde{B}^{(a)}_{\mu\nu}$, which is determined by supersymmetry and will be discussed in the next section. As it turns out $\partial_\mu \tilde{F}^{0\mu\nu}$ is self proportional to the field strengths (at least, as far as the purely bosonic terms are concerned), but with field-dependent coefficients. Those contributions thus characterize additional Chern-Simons terms involving $W^0_\mu$ which also involve the scalar fields.

We also draw attention to the fact that the Bianchi identity for $H^\mu$ is not linear in the vector-tensor fields. On the right-hand side there are nonlinear terms that are either of second-order (the term proportional to $\eta_{11}$) or of zeroth-order (the term proportional to $\eta_{ab}$) in the vector-tensor fields. Furthermore the quantity $\tilde{B}^{(a)}_{\mu\nu}$ does not depend homogeneously on the vector-tensor fields either. Hence, generically the vector-tensor multiplet is realized in a nonlinear fashion. One may attempt, by restricting the parameters $\eta_{IJ}$ in a certain way, to find relatively simple representations. However, supersymmetry severely restricts the choices that one can make. In fact, as we intend to prove in the next section, there are just two inequivalent vector-tensor multiplets, associated with certain parameter choices.

### 3 The vector-tensor transformation rules

Transformation rules can be determined by imposing the supersymmetry algebra iteratively on multiplet component fields. The supersymmetry transformation rules for vector multiplets are fully known. Therefore, the algebra represented by the vector-tensor multiplet in the presence of a vector multiplet background is fixed up to gauge transformations which pertain exclusively to the vector-tensor multiplet. The most relevant commutator in this algebra involves two supersymmetry transformations, which reads

\[
[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta^{(cov)}(2\epsilon_2^i \eta^\mu \epsilon_1 \alpha + \text{h.c.}) + \delta_2(4\epsilon_2^i \bar{\epsilon}_2 \epsilon_1 \alpha \mu X^0 + \text{h.c.}) + \delta_0(\theta^1, \theta^\alpha, \Lambda_\mu) ,
\]

closing into a covariant translation $\delta^{(cov)}$, a central charge transformation $\delta_0$, and vector and a tensor gauge transformations (collectively denoted by $\delta_k$), each with field-dependent parameters. The field $X^0$ is the complex scalar of the vector multiplet associated with the central charge\(^2\). The gauge transformations in $\delta_k$ are found by imposing (3.1) on

\(^2\)Henceforth we will suppress the superscript on $X^0$ and define $X \equiv X^0$ to simplify the formulae.
the vector-tensor multiplet. For the final result one can verify that a supersymmetry transformation and a gauge or central-charge transformation close into a gauge or central-charge transformation. Gauge and central-charge transformations form a subalgebra, which can be evaluated on the basis of the results of the previous section.

We follow a procedure in which the vector multiplets play a double role. As emphasized above, one of the vector multiplets is required to realize the central charge in a local fashion. In addition we require all couplings to be invariant under constant scale and chiral $U(1)$ transformations. These transformations act on the vector multiplet components in a manner dictated by their behavior as superconformal multiplets. This leads to the scale and chiral weights for the vector multiplet components shown in table 1. By insisting on invariance under scale and chiral transformations we facilitate the coupling to supergravity in the context of the superconformal multiplet calculus. This topic will be discussed in a separate publication [8]. The requirement of scale and chiral invariance may seem overly restrictive. However, this is not so, since freezing some of the vector multiplets to a constant leaves supersymmetry unaffected while at the same time causing a breakdown of the scale and chiral symmetry.

For the vector-tensor multiplet, there remains some flexibility in the assignment of the scaling and chiral weights. This is because the scalar fields of the vector multiplets may serve as compensator fields. That is, we may arbitrarily adjust the weights for each of the vector-tensor components by suitably absorbing functions of $X^I$. In this way we choose the weights for the vector-tensor components to be as shown in table 1. The bosonic vector-tensor fields must all have chiral weight $c = 0$ since they are all real. In the context of supergravity, the scale and chiral transformations become local. To eventually permit a consistent coupling to supergravity, we must avoid a conflict between scale transformations and vector and tensor gauge transformations. For this reason we adjust $V_\mu$ and $B_{\mu\nu}$ to be also neutral under scale transformations. The scale weights for $\phi$ and $\lambda_i$, and the scale and chiral weight for $\lambda_i$ have been chosen somewhat arbitrarily.

There remains a freedom to absorb additional combinations of the background fields into the definition of $\phi$ and $\lambda_i$. Furthermore, the fields $V_\mu$ and $B_{\mu\nu}$ can be redefined by appropriate additive terms. The freedom to redefine the vector-tensor components by field redefinitions has already been mentioned in the previous section. It is important to separate relevant terms in the transformation rules from those that can be absorbed into such field redefinitions. In deriving our results this aspect has received proper attention. We will shortly demonstrate some of its consequences.

With diligence, solving the above prescription is straightforward. In the following we

| field       | vector multiplet | vector-tensor multiplet | parameter |
|-------------|------------------|-------------------------|-----------|
| $w$         | 1                | 0                       | $-\frac{1}{2}$ |
| $c$         | $-1$             | $0$                     | $-\frac{1}{2}$ |
| $\gamma_5$ | $+$              | $+$                     | $+$       |

Table 1: Scaling and chiral weights ($w$ and $c$, respectively) and fermion chirality ($\gamma_5$) of the vector and vector-tensor component fields and the supersymmetry parameter.
suppress nonabelian terms for the sake of clarity; they are not important for the rest of this paper. We are not aware of arguments that would prevent us from switching on the nonabelian interactions. Also for the sake of clarity, we introduce a couple of abbreviations, for factors which occur frequently in what follows,

\[ g = i\eta_{1a} X^a, \quad b = -\frac{1}{4}i\eta_{ab} \frac{X^a X^b}{X^2}. \]  

We first list the transformation rules for arbitrary Chern-Simons terms. Subsequently we will show that, in fact, they incorporate two distinct versions of the vector-tensor multiplet. The transformation rules for the vector-tensor multiplet are given as follows,

\[
\begin{align*}
\delta \phi &= \xi^i \lambda_i + \xi_i \lambda^i, \\
\delta V_{\mu} &= i\varepsilon^{ij} \xi_i \gamma_\mu \left(2 X \lambda_j + \phi \Omega^0_j\right) - i W^0_i \xi^i \lambda_i + \text{h.c.}, \\
\delta B_{\mu\nu} &= -2[X]^2 \xi^i \sigma_{\mu\nu} \left(4 \eta_{11} \phi - g - \hat{g}\right) \lambda_i \\
&\quad -2 \xi^i \sigma_{\mu\nu} \left(2 \eta_{11} \phi^2 \hat{X} \Omega_i^0 + \phi \hat{X} \partial_i \hat{g} \Omega_i^1 - 4i \hat{X} \Re[\partial_j (Xb)] \Omega_i^1 \right) \\
&\quad + \varepsilon^{ij} \xi_i \gamma_\mu \left(2i \eta_{11} X \lambda_j + \eta_{11} \phi \Omega_j^0 + \eta_{1a} \Omega_j^a\right) \\
&\quad + \varepsilon^{ij} \xi_i \gamma_\mu \left(2 X (2 \eta_{11} \phi - g) \lambda_j + \eta_{11} \phi^2 \Omega_j^0 - i \eta_{1a} \phi \Omega_j^a - 4i \partial_j (Xb) \Omega_j^0\right) \\
&\quad - i \eta_{11} W^0_{\mu\nu} \xi^i \lambda_i + \eta_{ab} \varepsilon^{ij} \xi_i \gamma_\mu \Omega_j^a + \text{h.c.}, \\
\delta \lambda_i &= \left(\bar{D} \phi - i \Psi(\phi)\right) \xi_i - \frac{i}{2X} \varepsilon_{ij} \xi^j \eta_i \left(\frac{1}{X} \bar{D} \lambda^0_j \Omega_i^0 \right) - \frac{1}{X} \bar{D} \left(\bar{D} \Omega^0_j \right) \lambda_i \\
&\quad - \frac{1}{4 \eta_{11} \phi - g - \hat{g}} \frac{1}{X} \left\{\left(2 \eta_{11} \phi^2 Y^0_{ij} + \phi \hat{X} \partial_j \hat{g} Y^1_{ij} - 4i \hat{X} \Re[\partial_j (Xb)] Y^0_{ij}\right) \xi^j \\
&\quad - 2 \eta_{11} e^j \left(X \lambda_i \lambda_j - \hat{X} \varepsilon_{ik} \varepsilon_{jl} \lambda^k \lambda^l \right) \\
&\quad + X e^j \left(X \partial_j \hat{g} \Omega_i^1 \lambda_k - \hat{X} \varepsilon_{ik} \varepsilon_{jm} \partial_j \hat{g} \Omega_i^1 \lambda^m \right) \\
&\quad + i e^j (\partial_j \partial_j (Xb) \Omega_j^1 \Omega_j^1 + \varepsilon_{ik} e^j (\partial_j \partial_j (Xb) \Omega_j^1 \Omega_j^1 \right) \right\}. \tag{3.3}
\end{align*}
\]

Imposing closure of the supersymmetry algebra also leads to the constraints on the higher elements of the central charge hierarchy, as described in the previous section. This reflects the inability of the basic representation to constitute an off-shell multiplet without central charge. The following constraints are obtained in this manner,

\[
V_{\mu}^{(x)} = \frac{-1}{4 \eta_{11} \phi - g - \hat{g}} \frac{1}{X^2} \left\{H_{\mu} - [i(2 \eta_{11} \delta^0 \phi^2 + \phi X \partial_j \hat{g} - 4 \Im \partial_j (Xb)] X \partial_j \hat{X}^1 + \text{h.c.}] \right\} + \text{fermion terms,}
\]

\[
\hat{B}_{\mu\nu}^{(x)} = \frac{1}{4} i (g - \hat{g}) F_{\mu\nu} + \frac{1}{4} i (4 \eta_{11} \phi - g - \hat{g}) \hat{F}_{\mu\nu} - \frac{1}{4} \phi (2 \eta_{11} \phi - g - \hat{g}) F_{\mu\nu} \\
+ \frac{1}{4} i \phi \Im (X \partial_j \hat{g}) \hat{F}_{\mu\nu} + 2 \Im [\partial_j (Xb)] (F_{\mu\nu}^I - \hat{F}_{\mu\nu}^I) + \text{fermion terms}, \tag{3.4}
\]

as well as similar relations for \(\lambda^{(x)}\) and \(\phi^{(x)}\), which are of less direct relevance. The specific constraints shown in (3.4) are crucial for determining supersymmetric couplings, as we discuss in subsequent sections. By acting on the above constraints with central-charge transformations, one recovers an infinite hierarchy of constraints. These relate the components of the higher multiplets \((\phi^{(x)}, V_{\mu}^{(x)}, B_{\mu\nu}^{(x)}, \lambda^{(x)}_{(x)}, \phi^{(x)}_{(x)})\), etcetera, in such a way as to
retain precisely $8 + 8$ independent degrees of freedom. The transformation rules (3.4) are completely general, in the sense that no modifications are possible which cannot be removed by field redefinitions. These, then, constitute a unique representation of the vector-tensor multiplet. As exhibited in [3] the transformation rules of the higher multiplets take almost the same form. Their characteristic feature is that the transformations involve objects both at the next and at the preceding level. The transformations given above involve only the next level as there is no lower level. The consistency of this is ensured by the gauge transformations of the fields $V_\mu$ and $B_{\mu\nu}$.

At this point we come to a crucial feature of our results. We will now demonstrate that, depending on whether $\eta_{11}$ vanishes or not, i.e. depending on whether or not we include a $V \wedge dV$ Chern-Simons form in the tensor couplings, we describe two inequivalent representations of the vector-tensor multiplet.

**The nonlinear vector-tensor multiplet:**

If $\eta_{11} \neq 0$, signifying the presence of a $V \wedge dV$ Chern-Simons form in the tensor couplings, then we can redefine fields in such a way that $\eta_1$ disappears completely from the formulation described so far, signifying the absence of $V \wedge dW^a$ Chern-Simons forms in the tensor couplings. Specifically, if we begin with the transformation rules (3.4), and if $\eta_{11} \neq 0$, then we may perform the following redefinition,

$$
\phi \rightarrow \phi - \frac{i}{4} \eta_a \left( X^a - \bar{X}^a \right),
$$

$$
V_\mu \rightarrow V_\mu - \frac{i}{4} \eta_a \left( \frac{X^a}{X} + \frac{\bar{X}^a}{\bar{X}} \right) W^0_\mu + \frac{1}{2} \eta_a W^a_\mu,
$$

$$
B_{\mu\nu} \rightarrow B_{\mu\nu} + \frac{1}{4} \eta_a \left( \frac{X^a}{X} + \frac{\bar{X}^a}{\bar{X}} \right) V_{[\mu} W^0_{\nu]} + \frac{1}{2} \eta_a V_{[\mu} W^a_{\nu]} - \frac{1}{16} \eta_a \eta_b \left( \frac{X^b}{X} - \frac{\bar{X}^b}{\bar{X}} \right) W^0_{[\mu} W^a_{\nu]}.
$$

(3.5)

In terms of the shifted fields, we then obtain precisely the rules (3.4), but without the $\eta_1$ terms. This version of the vector-tensor multiplet is a straightforward extension of the result presented in [3], but with the background extended to several vector multiplets. A characteristic feature of this version is that the transformation rules are nonlinear in the vector-tensor components, as a result of the Chern-Simons coupling between $V_\mu$ and $B_{\mu\nu}$. Observe that this version contains at least two abelian vector gauge fields, $W^0_\mu$ and $V_\mu$.

The linear vector-tensor multiplet:

If $\eta_{11} = 0$, signifying the absence of a $V \wedge dV$ Chern-Simons coupling, thereby avoiding nonlinearities (in terms of vector-tensor fields) in the Bianchi identities (2.11), we arrive at a distinct formulation. This case has at least three abelian vector fields, $W^0_\mu$, $\eta_1 W^a_\mu$, and $V_\mu$. As one can check from (3.4), all the transformation rules now become at most linear in the fields of the vector-tensor multiplet.

### 4 Linear Multiplet Construction

It is possible to form products of vector tensor-multiplets, using the background vector multiplets judiciously, so as to form $N = 2$ linear multiplets. This enables the construction
of supersymmetric actions using known results in multiplet calculus. In this section we
describe the construction of such linear multiplets. In the following section we present
the associated supersymmetric actions, and give the dual descriptions in terms of vector
multiplets alone.

A linear multiplet has \(8 + 8\) independent off-shell degrees of freedom. It comprises a
triplet of real scalars, \(L^{ij}\), which satisfy \(L^{ij} = \varepsilon_{ik}\varepsilon_{jl}L^{kl}\), a fermion doublet \(\varphi_i\), a complex
scalar \(G\), and a real vector field \(E_\mu\). It can support a nontrivial central charge, which
would then generate an infinite hierarchy of multiplets, supplemented by constraints, in
a manner completely analogous to the vector-tensor multiplet, discussed above. The
component fields transform under supersymmetry as follows \([6]\),

\[
\begin{align*}
\delta L^{ij} &= 2\varepsilon^{(ij}\varphi^{kl)}) + 2\varepsilon_{ik}\varepsilon_{jl}\varepsilon^{(k}\varphi^{l)}, \\
\delta \varphi^i &= \partial L^{ij}\varepsilon^i_j + \bar{E}\varepsilon^{ij}\varepsilon^i_j - G\varepsilon^i + 2\bar{X}L^{(z)ij}\varepsilon^i_k\varepsilon^j_k, \\
\delta G &= -2\varepsilon_i\partial L^{ji} + 2\bar{X}(\varepsilon^{ij}\varepsilon^{(z)^j - h.c.)} - 2\varepsilon_i\Omega^{(z)}ij\varepsilon^i_j, \\
\delta E_\mu &= 2\varepsilon^{ij}\varepsilon^i_j\varphi^{(z)} + 2\bar{X}\varepsilon^{ij}\gamma_\mu\varphi^{(z)} + \bar{\varepsilon}^{ij}\gamma_\mu\Omega^{(z)}ij + h.c.,
\end{align*}
\]

(4.1)

where \(L^{(z)}ij\) and \(\varphi^{(z)}i\) are the image of \(L^{ij}\) and \(\varphi_i\) under the central charge. We stress that the
fields \(X\) and \(\Omega_i\) which appear in (4.1) are the scalar and fermion doublet, respectively, of
the vector multiplet which contains the gauge field for the central charge transformation,
referred to as \(X^0\) and \(\Omega_i^0\) above. As usual, we suppress the superscript zeros for the sake
of clarity. As mentioned, objects higher in the central-charge hierarchy, like \(L^{(z)}ij\) and \(\varphi^{(z)}i\)
are dependent. Their exact form is not particularly relevant because we construct a linear
multiplet as wholly dependent on vector-tensor and vector multiplet components.

We determine the linear multiplet by requiring the lowest component \(L^{ij}\) to have
weights \(w = 2\) and \(c = 0\) and to have the suitable transformation property. We wish to
construct \(L^{ij}\) as a function of the vector-tensor fields and the background vector multiplet
fields. To discover the form of this construction, we notice that the \(L^{ij}\) must transform
nontrivially under chiral \(SU(2)\) transformations. The only vector-tensor component which
transforms under \(SU(2)\) is the fermion \(\lambda_i\). For the vector multiplets, only the fermions
\(\Omega^I_i\) and the auxiliary fields \(Y^I_{ij}\) transform nontrivially under \(SU(2)\). Therefore, the most
general possible linear multiplet must include an \(L^{ij}\) of the following form

\[
L^{ij} = X^A \bar{\lambda}^i \lambda_j + \bar{X}^A \varepsilon_{ik} \varepsilon_{jl} \bar{\lambda}^k \lambda^l + \bar{X}^B_i \bar{\lambda}_j + \bar{X}^B_i \varepsilon_{ik} \varepsilon_{jl} \bar{\lambda}^k \Omega^{(z)}j + \bar{C}_I j \Omega^{I}_j + \bar{C}_I j \varepsilon_{ik} \varepsilon_{jl} \bar{\Omega}^{(z)}I \Omega^{I}j + \bar{D}_I Y^I_{ij},
\]

(4.2)
where $A, B, C_{IJ}$ and $D_I$ are functions of $\phi, X^I$ and $X^I$, with scaling and chiral weights as shown in table 2. Equation (4.2) is the natural ansatz to begin a systematic analysis.

Requiring that $L_{ij}$ transform into a spinor doublet as indicated in (4.1) puts stringent requirements on each of the functions $A(\phi, X^I, X^I)$, $B_I(\phi, X^I, X^I)$, $C_{IJ}(\phi, X^I, X^I)$ and $D_I(\phi, X^I, X^I)$. These are encapsulated by a system of coupled first-order, linear differential equations, which are determined as follows. Upon varying (4.2) with respect to $L$, one finds that the resulting 3-fermi terms and terms involving $Y_{ij}$ take the required form if and only if the following conditions are satisfied,

\[
\begin{align*}
\mathcal{E}\partial_\phi A &= -4\eta_{11}\bar{A}, & \mathcal{E}\partial_1 B_I &= \bar{B}_I\partial_J g, \\
\mathcal{E}\partial_1 A &= (A + \bar{A})\partial_J g + 2\eta_{11}B_I, & \mathcal{E}\partial_1 C_{IJ} &= 2i\bar{A}\partial_J(Xb), \\
\mathcal{E}\partial_1 A &= -2\eta_{11}\bar{B}_I, & \mathcal{E}\partial_2 C_{IJ} &= i\bar{B}_I\partial_J(Xb), \\
\mathcal{E}\partial_\phi B_I &= 2\bar{A}\partial_J g, & \partial_\phi D_I &= -Xb_I - 2AP_I, \\
\partial_1 (X^2\mathcal{E}B_I) &= 4i(A + \bar{A})X\partial_J(Xb), & \partial_1 D_J &= -2C_{IJ} - B_IP_J, \\
\end{align*}
\] (4.3)

where $g$ and $b$ were defined in (3.2) and

\[
\begin{align*}
\mathcal{E} &= -4\eta_{11}\phi + g + \bar{g}, & P_I &= -\phi\delta_I^0 I - i\mathcal{E}^{-1}\text{Im}(\phi X\partial_I g + 4i\partial_I(Xb)). \\
\end{align*}
\] (4.4)

Furthermore, the reality condition on $L_{ij}$ requires that $D_I$ be real. It is satisfying that the system of equations (4.3) turns out to be integrable, despite its complexity. After some work, one can prove that the general solution decomposes as a linear combination of three distinct solutions, each with an independent physical interpretation. The most interesting of these is given as follows,

\[
\begin{align*}
[A]_1 &= \eta_{11}(\phi + i\zeta) - \frac{1}{2}g, \\
[B]_1 &= -\frac{1}{2}(\phi + i\zeta)\partial_J g - 2i\partial_1 b, \\
[C_{IJ}]_1 &= -\frac{1}{2}i(\phi + i\zeta)\partial_I\partial_J(Xb), \\
[D]_1 &= \text{Re}\left\{\left[\frac{1}{2}\eta_{11}(\phi + i\zeta)^3 - \frac{1}{2}i\zeta(\phi + i\zeta)g\right]\delta_I^0 + \frac{1}{2}(\phi + i\zeta)X\partial_I(g\phi + 4ib)\right\}, \\
\end{align*}
\] (4.5)

where

\[
\zeta = \frac{\text{Im}(\phi g + 4ib)}{2\eta_{11}\phi - \text{Re} g}. \\
\] (4.6)

In terms of the action, which is discussed in the next section, this solution provides the couplings which involve the vector-tensor fields. The remaining two solutions, which we discuss presently, give rise either to a total divergence or to interactions which involve only the background fields. The latter of these correspond to previously known results. For this reason, the remaining solutions are secondary to that presented in (4.3). We discuss them to be complete, but also to demonstrate the pervasive nature of the techniques which we employ.

In addition to (4.3), a second solution to (4.3) is given as follows,

\[
\begin{align*}
[A]_2 &= i\eta_{11}\zeta' - i\alpha, \\
[B]_2 &= -\frac{1}{2}i\zeta'\partial_I g - 2i\partial_1 \gamma, \\
[C_{IJ}]_2 &= \frac{1}{2}\zeta'\partial_I\partial_J(Xb), \\
[D]_2 &= \text{Re}\left\{2iX\phi\partial_I\gamma + \frac{i}{2}\zeta'X\phi\partial_I g - 2\zeta'\partial_I(Xb)\right\}, \\
\end{align*}
\] (4.7)
where $\gamma = \frac{1}{4}i\alpha_a X^a/X$, which is a holomorphic, homogeneous function of the background scalars $X^a$ and $X^0$,

$$\zeta' = \frac{2\alpha \phi + 4 \text{Re } \gamma}{2\eta_1 \phi - \text{Re } g},$$  

(4.8)

and where $\alpha$ and $\alpha_a$ are arbitrary real parameters. Note that this solution could be concisely included with the first solution by redefining $g \rightarrow g + 2i\alpha$ and $b \rightarrow b + \gamma$. In fact, this second solution indicates that the functions $g$ and $b$ are actually defined modulo the modifications indicated by these shifts. In terms of the action, this ambiguity is analogous to the shift of the theta angle in an ordinary Yang-Mills theory. The third and final solution to (4.3) is given by

\begin{align*}
[A]_3 &= 0, \\
[B]_3 &= 0, \\
[C]_{IJ} &= -\frac{1}{8}i\partial_{IJ}(X^{-1}f), \\
[D]_I &= -\frac{1}{2}\text{Im}\partial_I(X^{-1}f),
\end{align*}

(4.9)

where $f$ is an arbitrary holomorphic function of $X^0$ and $X^a$, of degree two. In terms of the action, this solution corresponds to interactions amongst the background vector multiplets alone. Since the possible vector multiplet self-couplings have been fully classified, this solution does not provide us with new information. At the same time, however, it is reassuring and satisfying that the previously established solutions have been found anew in the present context. The function $f$ provides the well-known holomorphic prepotential for describing the background self-interactions.

Now that we have determined the scalar triplet $L_{ij}$, in terms of the specific functions $A(\phi, X^I, \bar{X}^I)$, $B_I(\phi, X^I, \bar{X}^I)$, $C_{IJ}(\phi, X^I, \bar{X}^I)$, and $D_I(\phi, X^I, \bar{X}^I)$ given above, it is straightforward to generate the remaining components of the linear multiplet, $\varphi_i$, $G$, and $E_{\mu}$ by varying (4.1) with respect to supersymmetry. The precise functional form of these higher components, in terms of the vector-tensor and the background fields, is not so illuminating, so we will not present them here. Given the complete linear multiplet, it is straightforward to determine a supersymmetric action. This is discussed in the following section.

5 Vector-tensor Lagrangians and their dual versions

As mentioned above, there are known results which describe supersymmetric densities as multiplet products. Particularly useful is a product between an $N = 2$ linear multiplet and an $N = 2$ vector multiplet which yields such a density,

$$\mathcal{L} = -W_{\mu}E_{\mu} - \frac{1}{2}Y_{ij}L_{ij} + \{(XG + \bar{\Omega}_i\varphi^i) + \text{h.c.}\},$$  

(5.1)

where $L_{ij}$, $\varphi_i$, $G$, and $E_{\mu}$ are the components of a linear multiplet, and where $X$, $W_{\mu}$, $\Omega_i$ and $Y_{ij}$ are the components of the vector multiplet used to gauge the central charge, which appear explicitly in the linear multiplet transformation rules (4.1). Again, we suppress the superscript zero in the interest of clarity. Equation (5.1) represents a supersymmetric Lagrangian for a generic linear multiplet. We choose the linear multiplet in (5.1) to be the one discussed in the previous section. In this way we are able to construct an interacting supersymmetric Lagrangian involving the vector-tensor component fields, which also
respects the central charge as a local symmetry. This explains the utility of constructing the linear multiplet in the previous section.

In this paper we restrict attention to the bosonic terms in the action, since these are of the most interest. The terms involving fermions are straightforward to generate, however. Looking at equation (1.2), we see that \( L_{ij} \) consists of terms quadratic in fermions, with one additional \( Y_{ij}^I \) term. From this we deduce the following. From the second term of (5.1) only the \( Y_{ij}^I \) term in (1.2) contributes to the bosonic Lagrangian, since the balance of \( L_{ij} \) contributes only fermion bilinears. Therefore, the second term of (5.1) supplies only a term \(-\frac{1}{2} D_I Y_{ij}^0 Y_{ij}^I \) to the bosonic Lagrangian. The bulk of the bosonic Lagrangian then comes from the \( G \) and \( E_\mu \) components of the linear multiplet. From (1.1) we see that \( G \) and \( E_\mu \) are generated by varying \( L_{ij} \) twice with respect to supersymmetry. Therefore, to obtain the purely bosonic action we need only consider the variation of the fermion fields in each of the fermion bilinear terms of \( L_{ij} \). Any additional contributions will necessarily involve fermions, since each of the two variations of \( L_{ij} \) needed to generate \( G \) and \( E_\mu \) must remove one of the two fermions from the respective term of \( L_{ij} \). The contributions which follow from the \( Y_{ij} \) term in (1.2) needs to be treated differently; there, one has to consider separately the second variation of \( D_I \) and also that of \( Y_{ij} \) in order to generate the purely bosonic contributions to \( G \) and \( E_\mu \). Finally, the \( \Omega (\phi^I) \) term in (5.1) obviously does not contribute to the bosonic action at all, so we ignore it.

Given the complexity of the transformation rule for \( \lambda_i \) found in (3.4), it is clear that a fair amount of work is involved in carrying out this process. Nevertheless, it is straightforward to vary (1.2) to generate \( \varphi_i, G, \) and \( E_\mu \), which can then be read off from (1.1), then to insert these expressions into the action formula (5.1), in precisely the manner described above. Carrying out this process, one finds the following form for the bosonic action,

\[
\mathcal{L} = |X|^2 A (\partial_\mu \phi - iV^{(\mu)}_\phi)^2 + 2|X|^2 B_I \partial^\mu X^I (\partial_\mu \phi - iV^{(\mu)}_\phi) \\
+ 4XC_{IJ} \partial^\mu X^I \partial_\mu X^J - 2D_I X \Box X^I \\
+ A (F^{-\mu\nu} - i\phi F^{-0\mu\nu})\left(\frac{1}{2}(F^{\mu\nu}_{\phi} - i\phi F^{-0\mu\nu} + iW^{0}_{\mu}(\partial_\nu \phi - iV^{(\nu)}_\phi))\right) \\
+ iXB_I F^{-I\mu\nu} \left(\frac{1}{2}(F^{\mu\nu}_{\phi} - i\phi F^{-0\mu\nu} + iW^{0}_{\mu}(\partial_\nu \phi - iV^{(\nu)}_\phi))\right) \\
+ iB_I (F^{-\mu\nu} - i\phi F^{-0\mu\nu})W^{0\mu}_{\nu}\partial_\nu X^I \\
- C_{IJ} F^{-\mu\nu} (X F_{\mu\nu}^J + 4W^{0\mu}_{\nu}\partial_\nu X^J) + D_I W^{0\mu\nu} F^{-\mu\nu} \\
+ |X|^2 A (W^{\mu}_{\nu} + 4|X|^2)(\phi^{(\mu)})^2 \\
- \frac{1}{4}(X \partial_1 D_I + P(t \partial_2 D_I)) Y_{ij} Y^{Iji} - \frac{1}{4} D_I Y_{ij} Y^{Iij} \\
+ \text{h.c.} \\
\tag{5.2}
\]

This describes the bosonic coupling of a vector-tensor multiplet to \( n \) vector multiplets. Note that each term involves a factor of one of the functions \( A(\phi, X^I, \bar{X}^I) \), \( B_I(\phi, X^I, \bar{X}^I) \), \( C_{IJ}(\phi, X^I, \bar{X}^I) D_I(\phi, X^I, \bar{X}^I) \), or \( P_I(\phi, X^I, \bar{X}^I) \), which were given explicitly in the previous section. It would be desirable to utilize the precise forms of these functions, as well as the properties (4.3) to cast this Lagrangian in a more elegant fashion, but this is not essential for the purposes of this paper. A special case of (5.2) was presented previously in [8], where the background consisted of a single vector multiplet. In that case \( \eta_{1a} \) and \( \eta_{ab} \) were necessarily zero, so that the functions \( g \) and \( b \) were vanishing.

As described in the previous section, the functions \( A(\phi, X^I, \bar{X}^I) \), \( B_I(\phi, X^I, \bar{X}^I) \), \( C_{IJ}(\phi, X^I, \bar{X}^I) \) and \( D_I(\phi, X^I, \bar{X}^I) \), which define the Lagrangian are linear superpositions...
of three distinct terms, one of which describes the local couplings of the vector-tensor multiplet components, another which is a total derivative, and one which codifies the self-interactions of the background. As a result of this, the Lagrangian \((5.2)\) can be written as a sum of three analogous pieces: a vector-tensor piece, a total-derivative piece, and a piece that exclusively depends on the background.

It would be interesting to use our Lagrangian \((5.2)\) to address physical questions, such as the quantum mechanical properties of \(N = 2\) supersymmetric theories involving tensor fields. As discussed in the introduction, it is expected that the restrictions on the couplings inferred by tensor gauge fields presumably implies special renormalization theorems, perhaps even finiteness. Having Lagrangians like \((5.2)\) (including the fermionic additions which have been suppressed in this paper, but which can be obtained straightforwardly using methods presented above) is imperative for addressing such issues.

In addition to the quantum aspects just mentioned, there exist more humble but nevertheless relevant questions which may be addressed with relative ease. For instance, a vector-tensor multiplet is classically equivalent to a vector multiplet. The theory which we have presented, involving one vector-tensor multiplet and \(n\) vector multiplets is classically equivalent to a theory involving \(n + 1\) vector multiplets. Since these latter theories are well understood, it is of interest to determine what subset of vector multiplet theories are classically equivalent to vector-tensor theories. Furthermore, heterotic low-energy string Lagrangians with \(N = 2\) supersymmetry are usually based on vector multiplets rather than on vector-tensor multiplets, despite the fact that the dilaton in a heterotic string theory resides in a vector-tensor multiplet. This is justified by the classical equivalence just mentioned.

A significant restriction along these lines has to do with the Kähler spaces on which the scalar fields of the theory may live. In the case of \(N = 2\) vector multiplets these consist of so-called “special Kähler” spaces, and the attendant geometry is known as “special geometry”. For the case of effective Lagrangians corresponding to heterotic \(N = 2\) supersymmetric string compactifications, a well-known theorem \([3]\) indicates that this space must involve, at least at the string tree level, an independent \(SU(1,1)/U(1)\) coset factor to accommodate the axion/dilaton complex. It would be interesting to discover such a restriction from supersymmetry in a field theoretic context, such as having it follow from the requirement of a dual relationship with a vector-tensor theory. Such questions had lead us to suppose that the vector-tensor theories had dual formulations which uniformly possessed such a factorization. As we discuss below, this has proved not to be the case. Irrespective of this, it is useful to cast our Lagrangian in terms of its dual form, since there the couplings are concisely encoded in a single holomorphic function.

One goes about constructing the dual formulation, in the usual manner, by introducing a Lagrange multiplier field \(a\), which, upon integration, would enforce the Bianchi identity on the field strength \(H_\mu\). The relevant term to add to the Lagrangian is therefore

\[
\mathcal{L}(a) = a(D_\mu H^\mu + \frac{i}{4} \eta_{11} F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{i}{4} \eta_{1a} F_{\mu\nu} \tilde{F}^{a\mu\nu} + \frac{i}{4} \eta_{ab} F_{\mu\nu} \tilde{F}^{b\mu\nu} + \frac{1}{2} i \hat{B}^{(a)}_{\mu\nu} \tilde{F}^{0\mu\nu}).
\]

Including this term, we treat \(H_\mu\) as unconstrained and integrate it out of the action, thereby trading the single on-shell degree of freedom represented by \(B_{\mu\nu}\) for the real scalar \(a\). Doing this, and also eliminating the auxiliary fields, \(\phi^{(z)}\) and \(Y_{ij}^{\dagger}\), we obtain a dual theory involving only vector multiplets. To perform these operations, it is instructive to note that all occurrences of \(H_\mu\) in \((5.2)\) and \((5.3)\) are most conveniently written in terms...
of \(V^{(z)}_{\mu}\), which can be done using (3.4). All such terms can be then be collected, and written as follows,

\[
\mathcal{L}(V^{(z)}_{\mu}) = \frac{1}{4} \mathcal{E} \left( W^\mu W^\nu - (W^2 + 4|X|^2)\delta^{\mu\nu} \right) \left( -\frac{1}{2} V^{(z)}_{\mu} V^{(z)}_{\nu} + V^{(z)}_{\mu} \partial_\nu (a - \zeta) \right),
\]

where \(\mathcal{E}\) was defined in (4.4) and \(\zeta\) was defined in (4.6). It is interesting how the terms involving \(V^{(z)}_{\mu}\) factorize into the form given in (5.4). The equation of motion for \(H_{\mu}\) is conveniently written in terms of \(V^{(z)}_{\mu}\). From (5.4), this follows immediately, and is given by the following simple expression,

\[
V^{(z)}_{\mu} = \partial_\mu (a - \zeta).
\]

The equations of motion for the auxiliary fields are \(\phi^{(z)} = 0\), and \(Y^I_{ij} = 0\). After substituting these solutions, we then manipulate the result into the familiar form for the bosonic Lagrangian involving vector multiplets,

\[
\mathcal{L} = \frac{i}{2} \left( \partial_\mu F^I - \partial_\mu X^I \partial^\mu \bar{F}^I \right) - \frac{1}{8} i \left( \bar{F}^I_{IJ} F^{+IJ} - F^{IJ}_{\mu\nu} F^{-IJ}_{\mu\nu} \right),
\]

characterized by a holomorphic function \(F(X^0, X^1, X^a)\). In (5.6), a subscript \(I\) denotes differentiation with respect to \(X^I\). The natural bosons in the dual theory are found to be

\[
X^1 = X^0 (a - \zeta + i\phi),
\]

\[
W^I_{\mu} = V_{\mu} + (a - \zeta) W^0_{\mu}.
\]

One can check that these transform as components of a common vector multiplet. For the general case, the dual theory obtained in this manner is described by the following holomorphic prepotential,

\[
F = -\frac{1}{X^0} \left( \frac{1}{3} \eta_{11} X^1 X^1 X^1 + \frac{1}{2} \eta_{1a} X^1 X^1 X^a + \eta_{ab} X^1 X^a X^b \right) - \alpha X^1 X^1 + \alpha_a X^1 X^a + f(X^0, X^a).
\]

The quadratic terms proportional to \(\alpha\) and \(\alpha_a\) give rise to total derivatives since their coefficients are real. The term involving the function \(f(X^0, X^a)\) represents the self-interactions of the background vector multiplets. The first three terms in (5.8) encode the couplings of the erstwhile vector-tensor fields, \(\phi\) and \(a\), and it is these which we are most interested in. As mentioned above, it is natural to question whether the Kähler space described by this prepotential function conforms to the theorem of [9]. The fact that it does not derive from the fact that the would-be complex dilaton \(X^1/X^0\) appears more than linearly in the prepotential. Nor can the quadratic or cubic terms be removed by a clever field redefinition. As discussed earlier in this paper, the best one can do it to remove either \(\eta_{11}\) or \(\eta_{1a}\). There exists an obstruction to removing both of these. We remark that these parameters are related to the Chern-Simons couplings to the tensor field in the dual formulation. The obstruction to removing the unwanted terms in the prepotential derive from the inability to formulate an interacting vector-tensor theory without any such Chern-Simons couplings.

It is important to realize that these results are a concise description of two very different situations. As described in detail in section 3, depending on whether the parameter \(\eta_{11}\)
is vanishing or not, indicating the absence or presence, respectively, of a $V \wedge dV$ Chern-Simons coupling to the tensor field, the theory takes on very distinct characters. It is instructive then, to summarize our results independently for each of these two cases.

The nonlinear vector-tensor multiplet:

As described above, when the parameter $\eta_{11}$ does not vanish, the tensor field involves a coupling to the Chern-Simons form $V \wedge dV$, which is quadratic in terms of vector-tensor fields. Consequently, the corresponding transformation rules contain significant nonlinearities. As also discussed above, in this case it is possible to remove the parameter $\eta_{1a}$, and therefore the $V \wedge dW^a$ Chern-Simons couplings, without loss of generality, by the field redefinition given in (3.3). Without loss of generality, we then define $\eta_{11} = 1$ and $\eta_{1a} = 0$. In this case the functions $A(\phi, X^I, \bar{X}^I)$, $B_I(\phi, X^I, \bar{X}^I)$, $C_{IJ}(\phi, X^I, \bar{X}^I)$, and $D_I(\phi, X^I, \bar{X}^I)$ which define the linear multiplet and, more importantly, the vector-tensor Lagrangian (5.2) are given by the following expressions

\begin{align*}
A &= \phi + i \frac{b + \bar{b}}{\phi}, \\
B_I &= -2i \partial_I b, \\
C_{IJ} &= -\frac{1}{2} i(\phi + \frac{b + \bar{b}}{\phi}) \partial_I \partial_J(Xb) - \frac{1}{2} i \partial_I \partial_J(X^{-1}f), \\
D_I &= \text{Re}\left(\frac{1}{3} \phi^3 \delta^0_i + 2i \phi \partial_I b - 2 \frac{b + \bar{b}}{\phi} \partial_I(Xb)\right) - \frac{1}{2} \text{Im} \partial_I \left(X^{-1}f\right). \quad (5.9)
\end{align*}

For the sake of clarity, we have absorbed the parameters $\alpha$ and $\alpha_a$ into the functions $b$ and $g$ in the manner described immediately after equation (4.8). Substituting these functions into the Lagrangian (5.2), one may perform the duality transformation. This is laborious, but completely straightforward. In this way we obtain a dual description involving only vector multiplets, which is characterized by the following holomorphic prepotential,

\begin{align*}
F &= -\frac{1}{X^0} \left(\frac{1}{3} X^1 X^1 X^1 + \eta_{ab} X^1 X^a X^b\right) - \alpha X^1 X^1 + \alpha_a X^1 X^a + f(X^0, X^a). \quad (5.10)
\end{align*}

As discussed above, the quadratic terms proportional to $\alpha$ and $\alpha_a$ represent total derivatives, and the final term involves the background self-interactions. Notice that in this case the prepotential is cubic in $X^1$. No higher-dimensional tensor theory is known that could possibly give rise to this coupling.

The linear vector-tensor multiplet:

As described previously, if $\eta_{11} = 0$, implying the absence of the $V \wedge dV$ Chern-Simons coupling, we obtain a vector-tensor multiplet which is distinct from the nonlinear case just discussed. In this case, it is not possible to perform a field redefinition to remove all of the $\eta_{1a}$ parameters. Formally, this is indicated by the presence of $\eta_{11}$ factors in denominators in (3.3). In this case, the supersymmetric transformation rules do not contain any terms more than linear in terms of the vector-tensor component fields. The functions $A(\phi, X^I, \bar{X}^I)$, $B_I(\phi, X^I, \bar{X}^I)$, $C_{IJ}(\phi, X^I, \bar{X}^I)$, and $D_I(\phi, X^I, \bar{X}^I)$ which define the linear multiplet and, more importantly, the vector-tensor Lagrangian (5.3) are now
given by the following expressions

\[ A = -\frac{1}{2}g, \]
\[ B_I = -\frac{1}{g + \bar{g}}(\phi \bar{g} \partial_I g - 2i(b + \bar{b})\partial_I g) - 2i\partial_I b, \]
\[ C_{IJ} = -\frac{1}{g + \bar{g}}(i\phi \bar{g} + 2(b + \bar{b}))\partial_I \partial_J (Xb) - \frac{1}{8}i\partial_I \partial_J (X^{-1} f), \]
\[ D_I = \frac{1}{g + \bar{g}} \text{Re}\left\{ \phi \bar{g} X \partial_I (\phi g + 4ib) - 2i(b + \bar{b})\partial_I [X(\phi g + 4ib)] \right\}. \quad (5.11) \]

As above, for the sake of clarity we have absorbed the parameters \( \alpha \) and \( \alpha_a \) into the functions \( b \) and \( g \) in the manner described immediately after equation (4.8). Substituting these functions into the Lagrangian (5.2), and performing the duality transformation, one obtains a dual description involving vector multiplets alone, which is characterized by the following prepotential,

\[ F = -\frac{1}{X^0} \left( \frac{1}{2} \eta_{ab} X^1 X^a X^b + \eta_{ab} X^1 X^a X^b \right) - \alpha X^1 X^1 + \alpha_a X^1 X^a + f(X^0, X^a). \]

Again, as discussed above, the quadratic terms involving \( \alpha \) and \( \alpha_a \) represent total derivatives, while the last term involves the background self-interactions. Notice that in this case the prepotential has a term quadratic in \( X^1 \), which cannot be suppressed. Such a term also arises from the reduction of six-dimensional tensor multiplets to four dimensions. In that case, the presence of the quadratic term is inevitable, because it originates from the kinetic term of the tensor field. Although there is no consistent action in six dimensions, because of the self-duality of the tensors, this term can still be defined via the various field equations [3]. Observe that we have at least three abelian vector fields coupling to the vector-tensor multiplet, namely \( W_0^\mu, W_1^\mu \) and \( \eta_{1a} W_1^\mu \).

The work presented in this paper represents an exhaustive analysis of the \( N = 2 \) vector-tensor multiplet with 8 + 8 off-shell degrees of freedom, for the case of global supersymmetry and local central charge, in the presence of \( n \) background vector multiplets, one of which gauges the central charge. We have presented the complete and general transformation rules in this context, and have shown that these actually include two distinct cases, one of which is nonlinear in the vector-tensor components, and the other of which is linear. Furthermore we have constructed a supersymmetric action for this system, and exhibited the bosonic part of this. The dual descriptions in terms of vector multiplets have been obtained, and the respective prepotential functions exhibited. Neither in the nonlinear case nor in the linear case do the associated Kähler spaces exhibit the factorization property expected for the case of the heterotic string, at least in the classical limit. As we discussed in the introduction, the inability to construct certain couplings in off-shell formulations with central charges, is not a new phenomenon and is, at present, not understood. For the couplings constructed in this paper, it is therefore not appropriate to associate the vector-tensor multiplet with the heterotic axion/dilaton complex. However, as stressed in [4], there can be additional vector-tensor multiplets which are of nonperturbative origin. Their contribution to the prepotential cannot be evaluated in heterotic perturbation theory. They contribute in precisely two distinct ways to the prepotential,
corresponding to the two different couplings found above. The linear vector-tensor multiplet coupling, with terms quadratic and linear in the vector-tensor field $X^1$ is of the type that one obtains from six-dimensional supergravity \[5\]. Both vector-tensor couplings can be reproduced from the dual type-II description, in addition to the perturbative heterotic couplings, as was demonstrated in \[4\]. Note that our results are applicable to any number of vector-tensor supermultiplets.

For the results presented above, the fact that the central charge is realized locally enables the inclusion of supergravity couplings in a rather straightforward manner. That program has also been carried out, and the complete supergravity couplings will appear in a separate publication \[8\].

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