Outline

- Light-cone OPE versus OPE in color dipoles.
- High-energy scattering and Wilson lines.
- Factorization in rapidity: Feynman diagrams in a shock-wave background.
- NLO Photon Impact Factor
- Leading order and NLO BK equation.
- Conclusions.
- Outlook.
Incoherent Interactions

Bjorken Limit

\[ Q^2 \to \infty, \ s \to \infty \]

\[ x_B = \frac{Q^2}{s} \text{ fixed} \]

\[ \text{resum} \ \alpha_s \ln \frac{Q^2}{\Lambda_{QCD}} \]
Incoherent-vs-Coherent

Bjorken Limit

\[ Q^2 \to \infty, \quad s \to \infty \]
\[ x_B = \frac{Q^2}{s} \text{ fixed} \]
\[ \text{resum } \alpha_s \ln \frac{Q^2}{\Lambda_{\text{QCD}}} \]

Regge Limit

\[ Q^2 \text{ fixed}, \quad s \to \infty \]
\[ x_B = \frac{Q^2}{s} \to 0 \]
\[ \text{resum } \alpha_s \ln \frac{1}{x_B} \]
Light-cone expansion and DGLAP evolution in the NLO
Light-cone expansion and DGLAP evolution in the NLO

\[ k_\perp^2 > \mu^2 \]

\[ k_\perp^2 < \mu^2 \]

$\mu^2$ - factorization scale (normalization point)

$k_\perp^2 > \mu^2$ - coefficient functions

$k_\perp^2 < \mu^2$ - matrix elements of light-ray operators (normalized at $\mu^2$)
Light-cone expansion and DGLAP evolution in the NLO

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OPE in light-ray operators

\((x-y)^2 \rightarrow 0\)

\[ T\{j_\mu(x)j_\nu(y)\} = \frac{(x-y)\xi}{2\pi^2(x-y)^4}\left[1 + \frac{\alpha_s}{\pi}(\ln(x-y)^2\mu^2 + C)\right]\bar{\psi}(x)\gamma_\mu\gamma_\xi\gamma_\nu[x, y]\psi(y) \]

\([x, y] \equiv \int_0^1 du (x-y)^\mu A_\mu(ux+(1-u)y)\] - gauge link
Light-cone expansion and DGLAP evolution in the NLO

\[ k_\perp^2 > \mu^2 \]

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\( \mu^2 \) - factorization scale (normalization point)

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\( k_\perp^2 < \mu^2 \) - matrix elements of light-ray operators (normalized at \( \mu^2 \))

Renorm-group equation for light-ray operators \( \Rightarrow \) DGLAP evolution of parton densities

\( (x - y)^2 = 0 \)

\[ \mu^2 \frac{d}{d\mu^2} \bar{\psi}(x)[x, y] \psi(y) = K_{\text{LO}} \bar{\psi}(x)[x, y] \psi(y) + \alpha_s K_{\text{NLO}} \bar{\psi}(x)[x, y] \psi(y) \]
High-energy expansion in color dipoles in the NLO
High-energy expansion in color dipoles in the NLO

\[ Y > \eta \]

\[ Y < \eta \]
High-energy expansion in color dipoles in the NLO

η > Y

- rapidity factorization scale

Rapidity Y > η - coefficient function ("impact factor")
Rapidity Y < η - matrix elements of (light-like) Wilson lines with rapidity divergence cut by η

\[ U_\eta^x = \text{Pexp}\left[ ig \int_{-\infty}^{\infty} dx^+ A_+^\eta(x_+, x_{\perp}) \right] \]

\[ A_\mu^\eta(x) = \int \frac{d^4 k}{(2\pi)^4} \theta(\eta - |\alpha_k|) e^{-ik\cdot x} A_\mu(k) \]
High-energy expansion in color dipoles in the NLO

\[
T\{j_\mu(x) j_\nu(y)\} = \int d^2z_1 d^2z_2 \, I^\text{LO}_{\mu\nu}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \\
+ \int d^2z_1 d^2z_2 d^2z_3 \, I^\text{NLO}_{\mu\nu}(z_1, z_2, z_3, x, y)[\text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\}\text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta}\} - N_c \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\}] 
\]

In the leading order the impact factor is Möbius invariant
In the NLO one should also expect conf. invariance since \( I^\text{NLO}_{\mu\nu} \) is given by tree diagrams
High-energy expansion in color dipoles in the NLO

\[ \frac{d}{d\eta} \text{tr}\{ U_x^\eta U_y^{\dagger\eta} \} = \frac{\alpha_s}{2\pi^2} \int d^2z \frac{(x-y)^2}{(x-z)^2(y-z)^2} \text{tr}\{ U_x^\eta U_y^{\dagger\eta} \} \text{tr}\{ U_x^\eta U_y^{\dagger\eta} \} - N_c \text{tr}\{ U_x^\eta U_y^{\dagger\eta} \} + \alpha_s K_{NLO} \text{tr}\{ U_x^\eta U_y^{\dagger\eta} \} + O(\alpha_s^2) \]

\( K_{NLO} = ? \) (Linear part of \( K_{NLO} = K_{NLO \ BFKL} \))

\( \eta \) - rapidity factorization scale

Evolution equation for color dipoles

\[ \text{Evolution equation for color dipoles} \]
Expansion of $F_2(x)$ in color dipoles in the next-to-leading order

\[ F_2(x_B) \simeq \int d^2 z_1 d^2 z_2 \, I^{LO}(z_1, z_2) \langle \text{tr}\{ U_{z_1}^\eta U_{z_2}^{\dagger \eta} \} \rangle \]

\[ + \frac{\alpha_s}{\pi} \int d^2 z_1 d^2 z_2 d^2 z_3 \, I^{NLO}(z_1, z_2, z_3) \langle \text{tr}\{ U_{z_1}^\eta U_{z_3}^{\dagger \eta} \} \text{tr}\{ U_{z_3} U_{z_2}^{\dagger \eta} \} \rangle \]

\[ \eta = \ln \frac{1}{x_B} \]

**plan**

- Calculate the NLO photon impact factor.
- Calculate the NLO evolution of color dipole.
Each path is weighted with the gauge factor \( P e^{i g} \int dx_\mu A^\mu \). Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction \( \Rightarrow \) we can replace the gauge factor along the actual path with the one along the straight-line path.

\[
U_z = [\infty p_1 + z_\perp, -\infty p_1 + z_\perp]
\]

\[
[x, y] = P e^{ig} \int_0^1 du (x-y)^\mu A_\mu (ux+(1-u)y) \quad p^\mu = \alpha p_1^\mu + \beta p_2^\mu + p_\perp^\mu
\]
Each path is weighted with the gauge factor $P e^{i g \int d\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
Each path is weighted with the gauge factor $P e^{ig \int dx_\mu A^\mu}$. Since the external field exists only within the infinitely thin wall, quarks and gluons do not have time to deviate in the transverse direction $\Rightarrow$ we can replace the gauge factor along the actual path with the one along the straight-line path.
\[ T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \, I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{Tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \]

\[ + \int d^2z_1 d^2z_2 d^2z_3 \, I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3, x, y) \left[ \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta}\} \text{tr}\{\hat{U}_{z_2}^\eta \hat{U}_{z_3}^{\dagger\eta}\} - N_c \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta}\} \right] \]
\[ T\{j_\mu(x)j_\nu(y)\} = \int d^2z_1 d^2z_2 I_{\mu\nu}^{LO}(z_1, z_2, x, y) \text{Tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger\eta}_{z_2}\} \]
\[ + \int d^2z_1 d^2z_2 d^2z_3 I_{\mu\nu}^{NLO}(z_1, z_2, z_3, x, y) [\text{tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger\eta}_{z_3}\} \text{tr}\{\hat{U}^\eta_{z_3} \hat{U}^{\dagger\eta}_{z_2}\} - N_c \text{tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger\eta}_{z_2}\}] \]
Conformal vectors:

\[ \kappa = \frac{\sqrt{s}}{2x^*} \left( \frac{p_1}{s} - x^2p_2 + x_{\perp} \right) - \frac{\sqrt{s}}{2y^*} \left( \frac{p_1}{s} - y^2p_2 + y_{\perp} \right) \]

\[ \zeta_1 = \sqrt{s} \left( \frac{p_1}{s} + z^2_{1\perp}p_2 + z_{1\perp} \right), \quad \zeta_2 = \sqrt{s} \left( \frac{p_1}{s} + z^2_{2\perp}p_2 + z_{2\perp} \right) \]

Here \( x^2 = -x_{\perp}^2 \), (similarly for \( y \))

\[
I^{LO} \propto \frac{2}{\pi^6} \int d^2z_{1\perp}d^2z_{2\perp} \text{tr}\{U_{z_{1\perp}} U_{z_{2\perp}^\dagger}\} \frac{z_{12\perp}^2}{x^*_2y^*_2 (\kappa \cdot \zeta_1)^3 (\kappa \cdot \zeta_2)^3} \\
\times \frac{\partial^2}{\partial x^\mu \partial y^\nu} \left[ -2(\kappa \cdot \zeta_1)(\kappa \cdot \zeta_2) + \kappa^2(\zeta_1 \cdot \zeta_2) \right]
\]
The NLO impact factor is not Möbius invariant $\Rightarrow$ the color dipole with the cutoff $\eta = \ln \sigma$ is not invariant.
The NLO impact factor is not Möbius invariant \( \Rightarrow \) the color dipole with the cutoff \( \eta = \ln \sigma \) is not invariant.

However, if we define a composite operator \( (a \text{- analog of } \mu^{-2} \text{ for usual OPE}) \)

\[
\left[ \text{Tr}\{ \hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_2} \} \right]^{\text{conf}} = \text{Tr}\{ \hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_2} \} \\
+ \frac{\alpha_s}{4\pi} \int d^2 z_3 \frac{z_{12}^2}{z_1^2 z_{23}^2} \left[ \frac{1}{N_c} \text{tr}\{ \hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_3} \} \text{tr}\{ \hat{U}^\eta_{z_3} \hat{U}^\dagger_{z_2} \} - \text{Tr}\{ \hat{U}^\eta_{z_1} \hat{U}^\dagger_{z_2} \} \right] \ln \frac{a z_{12}^2}{z_{13}^2 z_{23}^2} + O(\alpha_s^2)
\]

the impact factor becomes conformal in the NLO.
Operator expansion in conformal dipoles

\[ T\{j_\mu(x)j_\nu(y)\} = \int d^2z_1 d^2z_2 \ I_{\mu\nu}^{\text{LO}}(z_1, z_2, x, y) \text{tr}\{\hat{U}_z \hat{U}_z^\dagger\}\text{conf} \]

\[ + \int d^2z_1 d^2z_2 d^2z_3 \ I_{\mu\nu}^{\text{NLO}}(z_1, z_2, z_3, x, y) \left[ \frac{1}{N_c} \text{tr}\{\hat{U}_z \hat{U}_z^\dagger\} \text{tr}\{\hat{U}_{z_3} \hat{U}_{z_3}^\dagger\} - \text{tr}\{\hat{U}_{z_1} \hat{U}_{z_2}^\dagger\} \right] \]
Operator expansion in conformal dipoles

\[ T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2z_1 d^2z_2 \ I^{\text{LO}}_{\mu\nu}(z_1, z_2, x, y) \text{tr}[\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger\eta}_{z_2}\}]^{\text{conf}} \]

\[ + \int d^2z_1 d^2z_2 d^2z_3 \ I^{\text{NLO}}_{\mu\nu}(z_1, z_2, z_3, x, y) \left[ \frac{1}{N_c} \text{tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger\eta}_{z_3}\} \text{tr}\{\hat{U}^\eta_{z_3} \hat{U}^{\dagger\eta}_{z_2}\} - \text{tr}\{\hat{U}^\eta_{z_1} \hat{U}^{\dagger\eta}_{z_2}\} \right] \]

\[ I^{\text{NLO}}_{\mu\nu} = - I^{\text{LO}}_{\mu\nu} \frac{\alpha_s N_c}{4\pi} \int d\tilde{z}_3 \frac{\tilde{z}_{12}^2}{\tilde{z}_{13}^2 \tilde{z}_{23}^2} \ln \frac{\tilde{z}_{12}^2 e^{2\eta} \alpha_s^2}{\tilde{z}_{13}^2 \tilde{z}_{23}^2} Z_3^2 + \text{conf}. \]

The new NLO impact factor is conformally invariant.
Operator expansion in conformal dipoles

\[ T\{\hat{j}_\mu(x)\hat{j}_\nu(y)\} = \int d^2 z_1 d^2 z_2 \, I^\text{LO}_{\mu\nu}(z_1, z_2, x, y) \text{tr}\{\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta}\}\}^{\text{conf}} \]

\[ + \int d^2 z_1 d^2 z_2 d^2 z_3 \, I^\text{NLO}_{\mu\nu}(z_1, z_2, z_3, x, y) \left[ \frac{1}{N_c} \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger \eta}\} \text{tr}\{\hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger \eta}\} - \text{tr}\{\hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger \eta}\}\right] \]

\[ I^\text{NLO}_{\mu\nu} = - I^\text{LO}_{\mu\nu} \frac{\alpha_s N_c}{4\pi} \int dz_3 \, \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{12}^2 e^{2\eta} a_s^2}{z_{13}^2 z_{23}^2} Z_3^2 + \text{conf}. \]

The new NLO impact factor is conformally invariant.

In conformal \( \mathcal{N} = 4 \) SYM theory (where the \( \beta \)-function vanishes) one can construct the composite conformal dipole operator order by order in perturbation theory.
Operator expansion in conformal dipoles

\[ T\{ \hat{j}_\mu(x)\hat{j}_\nu(y) \} = \int d^2z_1d^2z_2 \ I^{\text{LO}}_{\mu\nu}(z_1, z_2, x, y) \text{tr}[\{ \hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta} \}]^{\text{conf}} \]

\[ + \int d^2z_1d^2z_2d^2z_3 \ I^{\text{NLO}}_{\mu\nu}(z_1, z_2, z_3, x, y) \left[ \frac{1}{N_c} \text{tr}\{ \hat{U}_{z_1}^\eta \hat{U}_{z_3}^{\dagger\eta} \} \text{tr}\{ \hat{U}_{z_3}^\eta \hat{U}_{z_2}^{\dagger\eta} \} - \text{tr}\{ \hat{U}_{z_1}^\eta \hat{U}_{z_2}^{\dagger\eta} \} \right] \]

\[ I^{\text{NLO}}_{\mu\nu} = - I^{\text{LO}}_{\mu\nu} \frac{\alpha_s N_c}{4\pi} \int dz_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \ln \frac{z_{12}^2 e^{2\eta} a_s^2}{z_{13}^2 z_{23}^2} z_3^2 + \text{conf.} \]

The new NLO impact factor is conformally invariant.

In conformal $\mathcal{N} = 4$ SYM theory (where the $\beta$-function vanishes) one can construct the composite conformal dipole operator order by order in perturbation theory.

**Analogy:**

When the UV cutoff does not respect the symmetry of a local operator, the composite local renormalized operator must be corrected by finite counter-terms order by order in perturbation theory.
\[
\Delta \equiv (x - y), \quad x_* = x^+ \sqrt{s/2}, \quad y_* = x^- \sqrt{s/2}, \quad R \equiv -\frac{\Delta^2 z_{12}}{x_* y_* Z_1 Z_2}
\]
\[
Z_1 = -\frac{(x-z_1)^2}{x_*} + \frac{(y-z_1)^2}{y_*}, \quad Z_2 = -\frac{(x-z_2)^2}{x_*} + \frac{(y-z_2)^2}{y_*}
\]
\[
I_{\mu \nu}^{NLO}(x, y) = -\frac{\alpha_s N_c^2}{8 \pi^2 x_* y_*} \int d^2 z_1 d^2 z_2 U_{\text{conf}}(z_1, z_2) \left\{ \frac{1}{Z_1^2 Z_2^2} \partial^x \partial^y \ln \frac{\Delta^2}{x_* y_*} + 2 \left( \frac{\partial^x Z_1}{Z_1^3 Z_2} + \frac{\partial^y Z_1}{Z_1^3 Z_2} \right) \left[ \ln \frac{1}{R} + \frac{1}{2R} - 2 \right] + \frac{2 (\partial^x Z_1) (\partial^y Z_1)}{Z_1^4 Z_2^2} \left[ \ln \frac{1}{R} - \frac{1}{2R} \right] - \frac{1}{2} \left[ \frac{\partial^x Z_1}{Z_1^3 Z_2} \partial^y \ln \frac{\Delta^2}{x_* y_*} + \frac{\partial^y Z_1}{Z_1^3 Z_2} \partial^x \ln \frac{\Delta^2}{x_* y_*} \right] \right\} \]
\[
\equiv \frac{1}{Z_1 Z_2} \left\{ \frac{1}{Z_1 Z_2} \partial^x \partial^y \ln \frac{\Delta^2}{x_* y_*} + \frac{2 (\partial^x Z_1) (\partial^y Z_1)}{Z_1^4 Z_2^2} \left[ \ln \frac{1}{R} - \frac{1}{2R} \right] \right\} \]

\[\Delta = (x - y), \quad x_* = x^+ \sqrt{s/2}, \quad y_* = x^- \sqrt{s/2}, \quad R = -\frac{\Delta^2 z_{12}}{x_* y_* Z_1 Z_2}\]
Different tensor structures appearing at NLO

\[
\frac{z_{12}^2}{Z_1 Z_2} \frac{x_\mu y_\nu}{x_\ast y_\ast} \frac{\Delta^2}{x_\ast y_\ast} \\
\frac{\partial^x Z_1}{Z_1} \left( \frac{\partial^y \Delta^2}{x_\ast y_\ast} \right) + \frac{\partial^y Z_1}{Z_1} \left( \frac{\partial^x \Delta^2}{x_\ast y_\ast} \right) + (z_1 \leftrightarrow z_2) \\
\frac{\partial^x Z_1}{Z^2_1} \left( \frac{\partial^y Z_1}{Z_1} \right) + \frac{\partial^x Z_2}{Z^2_2} \left( \frac{\partial^y Z_2}{Z_2} \right) \]

\[
\frac{1}{Z_1 Z_2} \left[ \left( \frac{\partial^x Z_1}{Z_1} \right) \left( \frac{\partial^y Z_2}{Z_2} \right) + \left( \frac{\partial^x Z_2}{Z_2} \right) \left( \frac{\partial^y Z_1}{Z_1} \right) \right]
\]
Conformal vectors

\[
\kappa_{\mu} = \frac{\sqrt{s}}{2x_\star} \left( \frac{p_1^\mu}{s} - x^2 p_2^\mu + x_{\perp}^\mu \right) - \frac{\sqrt{s}}{2y_\star} \left( \frac{p_1^\mu}{s} - y^2 p_2^\mu + y_{\perp}^\mu \right)
\]

\[
\zeta_{1\mu} = \left( \frac{p_1^\mu}{s} + z_{1\perp}^\mu p_2^\mu + z_{1\perp}^\mu \right), \quad \zeta_{2\mu} = \left( \frac{p_1^\mu}{s} + z_{2\perp}^\mu p_2^\mu + z_{2\perp}^\mu \right)
\]
Conformal vectors

\[ \kappa^\mu = \frac{\sqrt{s}}{2x^*_s} \left( \frac{p_1^\mu}{s} - x^2p_2^\mu + x^\mu \right) - \frac{\sqrt{s}}{2y^*_s} \left( \frac{p_1^\mu}{s} - y^2p_2^\mu + y^\mu \right) \]

\[ \zeta_{1}^\mu = \left( \frac{p_1^\mu}{s} + z_1^2 p_2^\mu + z_1^\mu \right), \quad \zeta_{2}^\mu = \left( \frac{p_1^\mu}{s} + z_2^2 p_2^\mu + z_2^\mu \right) \]

DIS photon impact factor is a linear combination of the following tensor basis

\[ \mathcal{I}_{1}^{\mu\nu} = g^{\mu\nu} \quad \mathcal{I}_{2}^{\mu\nu} = \frac{\kappa^{\mu} \kappa^{\nu}}{\kappa^2} \]

\[ \mathcal{I}_{3}^{\mu\nu} = \frac{\kappa^{\mu} \zeta_{1}^{\nu} + \kappa^{\nu} \zeta_{1}^{\mu}}{\kappa \cdot \zeta_{1}} + \frac{\kappa^{\mu} \zeta_{2}^{\nu} + \kappa^{\nu} \zeta_{2}^{\mu}}{\kappa \cdot \zeta_{2}} \]

\[ \mathcal{I}_{4}^{\mu\nu} = \frac{\kappa_{1}^{2} \zeta_{1}^{\mu} \zeta_{1}^{\nu}}{(\kappa \cdot \zeta_{1})^2} + \frac{\kappa_{2}^{2} \zeta_{2}^{\mu} \zeta_{2}^{\nu}}{(\kappa \cdot \zeta_{2})^2} \quad \mathcal{I}_{5}^{\mu\nu} = \frac{\zeta_{1}^{\mu} \zeta_{2}^{\nu} + \zeta_{2}^{\mu} \zeta_{1}^{\nu}}{\zeta_{1} \cdot \zeta_{2}} \]

Cornalba, Costa, Penedones (2010)
Regularization of the rapidity divergence

For light-like Wilson lines loop integrals are divergent in the longitudinal direction

\[
\int_0^\infty \frac{d\alpha}{\alpha} = \int_{-\infty}^\infty d\eta = \infty
\]

Regularization by: slope

\[
U_\eta^n(x_\perp) = \text{Pexp}\left\{ig \int_{-\infty}^\infty du \, n_\mu \, A_\mu(u_n + x_\perp)\right\} \quad n_\mu = p_1^\mu + e^{-2\eta} p_2^\mu
\]

Regularization by: Rigid cut-off (used in NLO)

\[
U_\eta^n_x = \text{Pexp}\left[ig \int_{-\infty}^\infty du \, p_1^\mu A_\mu^n(u_p + x_\perp)\right]
\]

\[
A_\mu^n(x) = \int \frac{d^4k}{(2\pi)^4} \theta(\eta - |\alpha_k|) e^{-i k \cdot x} A_\mu(k)
\]

\[
k_\mu = \alpha_k p_1^\mu + \beta_k p_2^\mu + k_\perp^\mu
\]
To get the evolution equation, consider the dipole with the rapidies up to \( \eta_1 \) and integrate over the gluons with rapidity \( \eta_1 > \eta > \eta_2 \). This integral gives the kernel of the evolution equation (multiplied by the dipole(s) with rapidity up to \( \eta_2 \)).

In the frame || to \( \eta_1 \) the gluons with \( \eta < \eta_1 \) are seen as pancake.

Particles with different rapidity perceive each other as Wilson lines.
Leading order: BK equation

\[
\frac{d}{d\eta} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} = K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} + \ldots \quad \Rightarrow
\]

\[
\frac{d}{d\eta} \langle \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}} = \langle K_{\text{LO}} \text{Tr}\{\hat{U}_x \hat{U}_y^\dagger\} \rangle_{\text{shockwave}}
\]
Non linear evolution equation: BK equation

\[ U_{z}^{ab} = \text{Tr}\{t^{a}U_{z}t^{b}U_{z}^{\dagger}\} \Rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{1}} \rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{2}} + \alpha_{s}(\eta_{1} - \eta_{2})(U_{x}U_{z}^{\dagger}U_{z}U_{y}^{\dagger})^{n_{2}} \]
Non linear evolution equation: BK equation

\[
U^{ab}_z = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2}
\]

\[
\hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_\perp) \hat{U}^\dagger(y_\perp)\}
\]

**BK equation:** Ian Balitsky (1996), Yu. Kovchegov (1999)

\[
\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2 z}{(x - z)^2(y - z)^2}\left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z) \hat{U}(z, y) \right\}
\]

Alternative approach: JIMWLK (1997-2000)
Non linear evolution equation: BK equation

\[ U_{ab}^z = \text{Tr}\{t^a U_z t^b U_z^\dagger\} \Rightarrow (U_x U_y^\dagger)^{\eta_1} \rightarrow (U_x U_y^\dagger)^{\eta_2} + \alpha_s(\eta_1 - \eta_2)(U_x U_z^\dagger U_z U_y^\dagger)^{\eta_2} \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_c} \text{Tr}\{\hat{U}(x_\perp)\hat{U}^\dagger(y_\perp)\} \]

BK equation: Ian Balitsky (1996), Yu. Kovchegov (1999)

\[ \frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_s N_c}{2\pi^2} \int \frac{d^2z (x - y)^2}{(x - z)^2(y - z)^2} \left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\} \]

Alternative approach: JIMWLK (1997-2000)

LLA for DIS in pQCD \Rightarrow BFKL \quad \text{(LLA: } \alpha_s \ll 1, \alpha_s \eta \sim 1) \]
Non linear evolution equation: BK equation

\[ U_{z}^{ab} = \text{Tr}\{t^{a}U_{z}t^{b}U_{z}^{\dagger}\} \Rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{1}} \rightarrow (U_{x}U_{y}^{\dagger})^{\eta_{2}} + \alpha_{s}(\eta_{1} - \eta_{2})(U_{x}U_{z}^{\dagger}U_{z}U_{y}^{\dagger})^{\eta_{2}} \]

\[ \hat{U}(x, y) \equiv 1 - \frac{1}{N_{c}}\text{Tr}\{\hat{U}(x_{\perp})\hat{U}^{\dagger}(y_{\perp})\} \]

**BK equation:** Ian Balitsky (1996), Yu. Kovchegov (1999)

\[
\frac{d}{d\eta} \hat{U}(x, y) = \frac{\alpha_{s}N_{c}}{2\pi^{2}}\int \frac{d^{2}z (x - y)^{2}}{(x - z)^{2}(y - z)^{2}}\left\{ \hat{U}(x, z) + \hat{U}(z, y) - \hat{U}(x, y) - \hat{U}(x, z)\hat{U}(z, y) \right\}
\]

Alternative approach: JIMWLK (1997-2000)

- LLA for DIS in pQCD $\Rightarrow$ BFKL  
  \( \text{(LLA: } \alpha_{s} \ll 1, \alpha_{s}\eta \sim 1) \)

- LLA for DIS in sQCD $\Rightarrow$ BK eqn  
  \( \text{(LLA: } \alpha_{s} \ll 1, \alpha_{s}\eta \sim 1, \alpha_{s}^{2}A^{1/3} \sim 1) \)

(s for semi-classical)
Non linear evolution equation at NLO

\[
\frac{d}{d\eta} \text{Tr}\{U_x U_y^\dagger\} = \\
\int \frac{d^2 z}{2\pi^2} \left( \alpha_s \frac{(x - y)^2}{(x - z)^2(z - y)^2} + \alpha_s^2 K_{NLO}(x, y, z) \right) \left[ \text{Tr}\{U_x U_z^\dagger\} \text{Tr}\{U_z U_y^\dagger\} - N_c \text{Tr}\{U_x U_y^\dagger\} \right] + \\
\alpha_s^2 \int d^2 z d^2 z' \left( K_4(x, y, z, z') \{U_x, U_z^\dagger, U_z, U_y^\dagger\} + K_6(x, y, z, z') \{U_x, U_z^\dagger, U_z', U_z, U_z^\dagger, U_y^\dagger\} \right)
\]

\(K_{NLO}\) is the next-to-leading order correction to the dipole kernel and \(K_4\) and \(K_6\) are the coefficients in front of the (tree) four- and six-Wilson line operators with arbitrary white arrangements of color indices.
Sample of diagrams of the NLO gluon contribution

(I)  

(II) 

(III) 

(IV) 

(V)  

(VI) 

(VII) 

(VIII)
NLO evolution of composite “conformal” dipoles in QCD

\[
\frac{d}{d\eta} \left[ \text{tr} \{ \hat{U}_{z_1} U^\dagger_{z_2} \} \right]^{\text{conf}} = \frac{\alpha_s}{2\pi^2} \int d^2z_3 \left( \text{tr} \{ \hat{U}_{z_1} \hat{U}^\dagger_{z_3} \} \text{tr} \{ \hat{U}_{z_3} \hat{U}^\dagger_{z_2} \} - N_c \text{tr} \{ \hat{U}_{z_1} \hat{U}^\dagger_{z_2} \} \right)^{\text{conf}}
\]

\[
\times \left[ 1 + \frac{\alpha_s N_c}{4\pi} \left( b \ln \frac{z_{12}^2}{\mu^2} + b \frac{z_{12}^2 - 2}{z_{13}^2 z_{23}^2} \ln \frac{z_{13}^2}{z_{23}^2} + \frac{67}{9} - \frac{\pi^2}{3} \right) \right]
\]

\[
+ \frac{\alpha_s}{4\pi^2} \int \frac{d^2z_4}{z_{34}^4} \left\{ -2 + \frac{z_{14}^2 z_{23}^2 + z_{24}^2 z_{13}^2 - 4z_{12}^2 z_{34}^2}{2(z_{14}^2 z_{23}^2 - z_{24}^2 z_{13}^2) \ln \frac{z_{14}^2 z_{23}^2}{z_{24}^2 z_{13}^2}} \ln \frac{z_{13}^2 z_{24}^2}{z_{23}^2 z_{14}^2} + \left( 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right) \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right\}
\]

\[
\times \left[ \text{tr} \{ \hat{U}_{z_1} \hat{U}^\dagger_{z_3} \} \text{tr} \{ \hat{U}_{z_3} \hat{U}^\dagger_{z_4} \} \text{tr} \{ \hat{U}_{z_4} \hat{U}^\dagger_{z_2} \} - \text{tr} \{ \hat{U}_{z_1} \hat{U}^\dagger_{z_4} \hat{U}_{z_3} \hat{U}^\dagger_{z_2} \hat{U}_{z_3} \hat{U}^\dagger_{z_2} \hat{U}_{z_4} \hat{U}^\dagger_{z_4} \} \right] \right) \}
\]

\[
b = \frac{11}{3} N_c - \frac{2}{3} n_f
\]

I. Balitsky and G.A.C

\[K_{\text{NLO BK}} = \text{Running coupling part} + \text{Conformal "non-analytic" (in } j) \text{ part} + \text{Conformal analytic } (N = 4) \text{ part}\]

Linearized \(K_{\text{NLO BK}}\) reproduces the known result for the forward NLO BFKL kernel.
Factorization in rapidity

- NLO amplitude in $\mathcal{N}=4$ SYM: I. Balitsky and G.A.C. (2009)
- NLO amplitude in QCD: (in preparation)
NLO Amplitude in $\mathcal{N}=4$ SYM theory and in QCD

Factorization in rapidity

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Conclusions

- High-energy operator expansion in color dipoles works at the NLO level.
- NLO photon impact factor in coordinate space has been calculated and presented: the result is conformal.
- The NLO BK kernel in QCD and $\mathcal{N} = 4$ SYM agrees with NLO BFKL eigenvalues.
- The NLO BK kernel in QCD is a sum of the running-coupling part and conformal part.
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Outlook

- Fourier transform of the NLO Photon Impact Factor.
- NLO amplitude of $\gamma^* \gamma^*$ scattering (QCD).