Many-body symbolic dynamics of a classical oscillator chain

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Abstract. We study a certain type of the celebrated Fermi-Pasta-Ulam particle chain, namely the inverted FPU model, where the inter-particle potential has a form of a quartic double well. Numerical evidence is given in support of a simple symbolic description of dynamics (in the regime of sufficiently high potential barrier between the wells) in terms of an (approximate) Markov process. The corresponding transition matrix is formally identical to a ferromagnetic Heisenberg quantum spin-1/2 chain with long range coupling, whose diagonalization yields accurate estimates for a class of time correlation functions of the model.

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1. Introduction

In a historical numerical experiment Fermi Pasta and Ulam \[1\] studied the following non-linear model of a 1D crystal described by the Hamiltonian

\[ H = \sum_{i=1}^{N} \left( \frac{1}{2} m \ddot{x}_i^2 + V(x_i - x_{i-1} - l) \right), \quad V(x) = \frac{1}{2} m \omega^2 x^2 + \frac{1}{4} k x^4. \] (1)

Contrary to initial expectations of Fermi \emph{et al} the so-called FPU model (1) behaved in a strong contrast to the ergodic theorem of statistical mechanics, even in quite strongly non-linear regime when one would expect fast relaxation to statistical canonical equilibrium and equipartition of energy from an arbitrary initial state. Instead, FPU model triggered the discovery of non-linear normal modes, the so-called \textit{solitons}, and indirectly, the whole field of (computational) non-linear dynamics. Later, FPU chain has been studied in the spirit of KAM theory \[2\] and various estimates have been given for the critical strength of the dimensionless non-linearity parameter \( kl^2 / m \omega^2 \) required for the motion to become globally stochastic \[3, 4\].

Recently, FPU-type models have been studied in the context of energy transport and Fourier heat law in 1D chains of particles \[5, 6, 7\]. Quite unexpectedly, FPU-type models, namely Hamiltonians of the type (1) with non-linear inter-particle interaction \( V(x) \) (and no \textit{on-site} potential), turned out to be anomalous heat conductors, due to slow power law decay of transport (velocity-velocity or current-current) time correlation functions. These results, namely that correlations decay universally as \( C(t) \sim t^{-3/5} \), and correspondingly that Kubo transport coefficient diverges as \( \kappa(L) \sim L^{2/5} \) as a function of the chain length \( L \), have later been successfully explained in terms of hydrodynamic arguments and mode-mode coupling theory \[8\].

However, a perturbative-like approach such as mode-mode coupling theory should break down when the inter-particle potential has \textit{more than one} stable position. Indeed, Giardina \emph{et al} \[9\] have performed a series of numerical experiments on the particle chain with an oscillating inter-particle potential \( V(x) = 1 - \cos(x) \), as well as with the potential \( V(x) = -x^2 / 2 + x^4 / 4 \), and found normal heat conduction with a clean exponential decay of correlations. The key mechanism, which we believe produces non-KAM-like (almost) hyperbolic motion of such a high dimensional Hamiltonian system, are the hyperbolic saddles over which pairs of particles flip from one well to another. This motivated us to study in this paper some fundamental dynamical properties of the simplest version of such a hyperbolic chain with quartic double-well inter-particle potential \( V(x) \). We present here some intriguing numerical results which suggest existence of a simple (but only approximate) Markov partition with very simple many-body symbolic dynamics of this particle chain. In addition, we find that the transition matrix is formally generated by a Hamiltonian of a certain quantum spin-1/2 chain.

Section 2 introduces the model. In section 3 we define (approximate Markov) partition of phase space and obtain numerical and analytical estimates for the volumes of various cells. Expression for the average time between subsequent jumps among the cells is also obtained. In section 4 we introduce Markovian description of our model.
with the transition matrix and the flux matrix and numerically check the accuracy of the Markovian property in two independent ways. Section 5 represents the core of the present paper. We numerically calculate the transition matrix and make a formal correspondence between our model and a ferromagnetic Heisenberg quantum spin-1/2 chain. Using this correspondence we derive various estimates and bounds for the decay rates of the time correlation functions of a class of piece-wise constant functions.

2. Model: Inverted Fermi-Pasta-Ulam chain

With the choice for units of mass, time and length of \( m, 1/\omega \) and \( l \), respectively, and the new canonical coordinates \( q_i = x_i - il \) and \( p_i = m\dot{x}_i \), Hamiltonian (1) can be brought to a dimensionless form

\[
H = \sum_{i=1}^{N} \frac{p_i^2}{2} + V(q_i - q_{i-1}), \quad V(x) = -\alpha \frac{x^2}{2} + \frac{x^4}{4} + \frac{\alpha^2}{4},
\]

where \( \alpha = -m\omega^2/kl^2 \) is the only dimensionless parameter left. In order for the potential minima to be at level 0 (for \( \alpha > 0 \)) we have added a constant term \( \alpha^2/4 \) to the potential \( V(x) \). We can get all different orbits of the system described by the Hamilton function (2) just by varying one parameter \( \alpha \). At first it would seem that the energy \( E \) is also a free parameter. But by fixing all the three basic units we have also provided a natural unit for the energy density \( \varepsilon = E/N \). If we scale the dimensionless quantities by a factor \( \xi \), namely \((q_i, p_i, \alpha) \rightarrow (\xi q_i, \xi^2 p_i, \xi^2 \alpha)\), the Hamiltonian is simply multiplied by a factor \( \xi^4 \). Invariant parameter in this transformation is therefore \( \lambda = \varepsilon/\alpha^2 \). All different nonequivalent Hamiltonians can be obtained just by varying one parameter, namely \( \lambda \), or equivalently \( \alpha \) at fixed \( \varepsilon \). Energy density can therefore be fixed without any loss of generality. From now on \( \varepsilon = E/N = 1 \) is assumed, as well as periodic boundary conditions \((q_{N+1}, p_{N+1}) \equiv (q_1, p_1)\). Phase space point will be denoted by \( \mathbf{x} = (\mathbf{q}, \mathbf{p}) \).

For negative values of the parameter \( \alpha \) the system has a form of the standard \( \beta \)-FPU model (just in a different parametrization), but in this paper we focus on the positive values of the parameter \( \alpha \) in which case we call the system Inverted FPU model (IFPU). IFPU model is translationally invariant, therefore besides the energy \( \varepsilon \) there is also a second (trivial) constant of motion, namely the total momentum \( P = \sum_{i=1}^{N} p_i \). In all numerical calculations the total momentum has been fixed to \( P = 0 \).

For the numerical integration of equations of motion a fourth order symplectic algorithm of Ref. [10] has been used. This integrator has been checked to be the optimal choice (at least for the model studied here, and at relative accuracy \( \sim 10^{-5} - 10^{-8} \)) by making a careful comparison with a number of other symplectic and Runge-Kutta methods.

3. Phase space partition and statistical dynamics

If the barrier \( \Delta U = \alpha^2/4 \) between the potential minima is sufficiently high the differences \((q_{i+1}(t) - q_i(t))\) will spend most of the time either around the left or the right minimum,
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![Potential V(x) for the inverted FPU model with \( \alpha > 0 \).](image)

Figure 1. Potential \( V(x) \) for the inverted FPU model \( (2) \), \( \alpha > 0 \).

with quite infrequent jumps between the wells. As the motion around the minimum is approximately harmonic the more interesting part will be the transitions between the left and the right potential well, for each pair of neighboring particles, going over the hyperbolic saddle at \( q_{i+1} - q_i = 0 \). Understanding the motion in phase space of a high-dimensional system is generally very difficult. Some symbolic description of an orbit \( \mathbf{x}(t) \) is therefore highly desirable. In IFPU model with high potential barrier the choice of a partition of phase space is almost obvious:

For each instant of time \( t \) we will be interested only in a single binary bit of information \( a_i \in \{0,1\} \) for each pair of neighboring particles \( (i,i+1) \), namely if the neighbors are in the right or left well, \( (q_{i+1} - q_i) > 0 \) or \( < 0 \), we write \( a_i = 1 \) or \( a_i = 0 \), respectively. Binary positions for the whole chain can be compactly encoded in a binary integer (signature)

\[
S = (a_N \ldots a_2 a_1)_2 = \sum_{i=1}^{N} a_i 2^{i-1}.
\]

Thus we have defined a partition of \( 2^N \)-dimensional phase space cut by \( N \) hyper-planes \( q_{i+1} = q_i \) into \( 2^N \) cells labeled by signatures \( S \in \{0,1,\ldots,2^N-1\} \). By a slight abuse of language, we will use a term signature \( S \) also to refer to a phase space cell denoted by \( S \). If the transitions between various signatures are rare, which is obviously the case if the potential barrier \( \Delta U \) is high, it seems to be meaningful to concentrate on the (statistical) dynamics of transitions between the signatures. For the periodic boundary conditions the coordinate differences must fulfill the constraint \( \sum_{i=1}^{N} (q_{i+1} - q_i) \equiv 0 \) which in the limit of infinitely high potential barrier and even \( N \) translates into the condition \( \sum_{i=1}^{N} a_i = N/2 \). For odd \( N \) the number of pairs in the left and the right well should differ by 1. This presents only unnecessary technical complications and from now on we will assume \( N \) to be even. If the barrier is finite, signatures with different number of pairs in the left and right well are possible to visit. We will call signature \( S \)
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to be of order \( i \) if \( \sum_{i=1}^{N} a_i = N/2 \pm i \). The number of different signatures \( M_i \) of order \( i \) is
\[
M_i = \binom{N}{N/2 - i} = \frac{N!}{(N/2 - i)! (N/2 + i)!}
\]
(4)
For sufficiently high barrier the system will spend most of its time in signatures of order 0. We will specifically concentrate on signatures of order 0 and treat signatures of higher (mostly 1st) order only as 'tunnels' for transitions between different signatures of order 0. Figure 2 shows an example of a transition between two order 0 signatures via an intermediate short lived signature of order 1. First we have to know for which values of parameter \( \alpha \) and size \( N \) will the description by the signatures of order 0 only be adequate, i.e. will the relative time spent in higher order signatures be negligible.

Figure 2. Time dependence of differences \((q_{i+1}(t) - q_i(t))\) for \( N = 14, \alpha = 4.8 \). For time \( t < -20 \) the system is in the order 0 state \( S = 01010110101010_2 \), at \( t \approx -20 \) the system jumps to the temporary order 1 metastable state \( S'' = 0101010001010_2 \) and shortly thereafter at \( t = 0 \) jumps into the new stable order 0 state \( S' = 01110110001010_2 \). Such a transition from \( S \) to \( S' \) will be called a jump of length \( d = 6 \). Dashed lines denote the position of the saddles \( q_{i+1} = q_i \).

3.1. Estimating the fractional volumes of higher order signatures
We will now make a rough theoretical estimate for the time spent in signatures of different orders. For an ergodic system this time is proportional to the measure (volume) of the phase space covered with the signatures in question. Let us designate by \( t_i \) the time spent in signatures of order \( i \). We have
\[
t_i(E) \propto \int_{\text{order } i} \frac{dS}{|\nabla H|} = \frac{d\Gamma_i(E)}{dE},
\]
(5)
where the region of integration is the part of the energy surface \( H = E \) intersecting with the full set of signatures of order \( i \) and \( \Gamma_i(E) \) is the total volume of the phase space in the signatures of order \( i \) and with energy less than \( E \). First, let us make an estimate for \( \Gamma_0(E) \). Potential around both minima is to the lowest order harmonic, therefore we can roughly say \( \Gamma_0(E) \approx \Gamma(E)^N \), where \( \Gamma(E) \) is the volume for one harmonic oscillator, that is \( \Gamma(E) = 2\pi E/\sqrt{\alpha} \). For the signatures of order \( i \) the argument is very similar. In this case there are \( 2^i \) particles more in one potential well than in the other. Because the trivial condition \( \sum_{i=1}^{N} (q_{i+1} - q_i) \equiv 0 \) must still be satisfied, the equilibrium positions of pairs will not be at \( \pm \sqrt{\alpha} \) any more but will be shifted for \( \Delta q \) in order to keep the center of mass of all pairs at 0. This gives for the shift \( \Delta q = 2^i \sqrt{\alpha}/N \) and contributes \( \varepsilon_i = \alpha \Delta q^2 = 4i^2 \alpha^2/N^2 \) to the energy density, provided the cubic part of the potential is negligible, i.e. \( 2i/N \ll 1 \). Harmonic approximation for the volume \( \Gamma_i(E) \) can therefore still be used, but now at energy \( E - N\varepsilon_i \). Since we are interested only in the dependences for large \( N \) we can omit \('cancel\') energy derivatives coming from \( \varepsilon_i \), so that we have

\[
\frac{t_i}{t_0} \sim \frac{\Gamma(\varepsilon - \varepsilon_i)^N}{\Gamma(\varepsilon)^N} = \left(1 - \frac{\varepsilon_i}{\varepsilon}\right)^N \sim \exp\left(-4i^2 \frac{\alpha^2}{N\varepsilon}\right). \tag{6}
\]

We have fixed \( \varepsilon \equiv 1 \) so that the relative fraction of signatures of order 1, \( t_1/t_0 \), falls exponentially in \( \alpha^2/N \). We must stress that the exponential dependence is expected only when \( 4\alpha^2/N^2 \ll 1 \) and that the overall prefactor in front of an exponential function could still depend on \( N \). This is nicely confirmed by the numerical data in figure 3. Grouping the signatures by their order is meaningful since members of each group have the same phase space volume and the same minimal potential energy. The minimal potential energy of order 0 signatures is \( \varepsilon_0 = 0 \), of order 1 is \( \varepsilon_1 = 4\alpha^2/N^2 \) and of order \( i \) is \( \varepsilon_i \approx i^2 \varepsilon_1 \). The fraction of time spent in higher order signatures is just a power of relative time spent in order 1 signatures, \( (t_i/t_0) \approx (t_1/t_0)^i \) and this can be reduced by increasing \( \alpha \).

Of course, for the transitions to be possible at all, the barrier height must be smaller than the total energy \( E \). This yields the condition

\[
\alpha^2/4 < N \tag{7}
\]

which is just the opposite of the condition to keep the \( t_1/t_0 \) small. Nevertheless, we can still reduce the fraction of higher order states to around \( \exp(-16) \sim 10^{-6} \), which is small. But the condition \( \alpha \) can be problematic for small \( N \). Indeed, for the smallest nontrivial case of \( N = 4 \) we have numerically found that there are no transitions between signatures for \( \alpha \geq 2.8 \). For smaller \( \alpha \) we do find transitions, but there the system exhibits non-ergodic behaviour. Condition \( (7) \), although fulfilled, seems to be too weak and is still keeping the phase space cut into non-connected parts. The smallest system worth studying regarding transitions is therefore \( N = 6 \).

In all numerical work \( \alpha \) and \( N \) have been chosen so as to keep the fractional volume of higher order signatures small. We have therefore studied the dynamics on a set of \( M_0 \) signatures of order 0 exhibiting transitions thru short-lived order 1 signatures as shown
Figure 3. Fraction \( \eta = \left( \sum_{i=1}^{\infty} t_i \right)/t_0 \approx t_1/t_0 \) of time spent in signatures of higher order for \( N = 6, 8, 10, 12, 16, 20 \) and 30, from right to left, respectively. Lines connect points with the same value of \( N \). Inset shows by \( N \)-dependent prefactors rescaled values of \( \eta \) and exponential function (6) (chain straight line).

in figure 3. The most important physical scale that characterizes such a transport is the average time \( \tau \) between subsequent transitions.

### 3.2. Ergodicity and average transition time \( \tau \) between signatures of order 0

Though the main body of this paper is concerned with the transport and decay of time-correlations, i.e. with the system’s mixing property, we should first mention, that we have also checked the weaker dynamical property of ergodicity directly. We have employed a method of Robnik et al [11] of comparing the rate of visiting of different phase space cells (in our case, order 0 signatures) with the rate for a fully random dynamics. The results turned to be fully consistent with (uniformly) ergodic behaviour of our IFPU model in the high-barrier regime discussed above.

Now we define the time scale \( \tau \) to be an average time from the point when the orbit enters a certain order 0 signature \( S \) to the point where the same orbit enters the next order 0 signature \( S' \) \( (S' \neq S) \) concluding the transition from \( S \) to \( S' \). The average is taken over many different orbits with microcanonically distributed initial conditions, or over one very long orbit if we assume ergodicity. We obtain an approximate functional form for the dependence of \( \tau \) on \( N \) and \( \alpha \) by similar arguments as for the relative volume of higher order signatures. The probability for a jump will be estimated by the ratio between the phase space volume of the set of states just before a jump \( \Gamma_t \) and the volume of the set of equilibrium states \( \Gamma_{eq} \). The approximate volume \( \Gamma_{eq}(E) \) of the
phase space of equilibrium states has been derived in the previous subsection and is just \( \Gamma_{eq}(E) \approx \Gamma(E)^N = (2\pi/\sqrt{\alpha})^N E^N \). In the state just before the transitions there should be more than \( \Delta U \) of energy in one pair of particles. This energy will be denoted by \( a\alpha^2 \), where \( a \) is for now some unspecified constant. Certainly \( a \) must be greater than \( 1/4 \), but because our oscillators are coupled we allow for the possibility that the energy of a pair must actually be bigger than the barrier height. Phase volume \( \Gamma_t(E) \) for a state with one pair having the energy \( a\alpha^2 \) and the other \( N-1 \) pairs residing in the minima is

\[
\Gamma_t(E) = \int_0^{E-a\alpha^2} d\varepsilon \varepsilon^{N-1} \left( \frac{2\pi}{\sqrt{\alpha}} \right)^{N-1} (N-1)(\Gamma(E) - \Gamma(\varepsilon)) \\
\approx \left( \frac{2\pi}{\sqrt{\alpha}} \right)^N a\alpha^2 (E - a\alpha^2)^{N-1}.
\]

(8)

Probability for a jump is proportional to the quotient of derivatives \( d\Gamma_t(E)/d\Gamma_{eq}(E) \). Reciprocal average time between transitions \( 1/\tau(N,\alpha) \) is proportional to the above quotient and to the frequency of oscillation around potential minimum. Using also \( E = N \), we finally get

\[
\tau(N,\alpha) \approx \frac{A}{\sqrt{\alpha}} \frac{N}{a\alpha^2} \frac{1}{(1 - a\alpha^2/N)^N},
\]

(9)

where \( A \) is some overall numerical factor. We have determined numerical parameters \( a \) and \( A \) by fitting the numerical data for different lengths \( N \) in the range \( N = 6, \ldots, 30 \). Both parameters do depend on \( N \) for small chains, but are, of course, independent of \( \alpha \). The value of \( A \) is between 0.02 and 0.05, whether the \( a \) has dependence \( a = 1/4 + \mathcal{O}(1/N) \). In the limit of large chains \( a \) has the value \( 1/4 \) predicted by our simple physical picture. This picture can also be confirmed by looking at the local energy density just before the jump. Indeed, local energy of a pair is exactly \( a\alpha^2 \) for all chain sizes. So, the parameter \( a \) is not just some fitting parameter in equation (9) but it has a clear physical interpretation. In figure 4 we depict the estimate of \( \tau \) (9) together with the numerical data. Worth noticing is also an interesting scaling law, although only approximate. Global dependence of \( \tau(N,\alpha) \) is such that we can write approximately

\[
\tau(N,\alpha) \approx \frac{f(4N/\alpha^2)}{\sqrt{\alpha}} \exp(\alpha^2/4),
\]

(10)

where \( f(x) \) is some parameter-free function. The shape of this function can be seen in figure 4.

We have also numerically computed statistical distribution of transition times \( \tau_n \), i.e. the time intervals between entering different subsequent signatures of order 0 such that \( \tau = \langle \tau_n \rangle \), for a fixed \( N \) and \( \alpha \). This distribution is shown in figure 5 and is clearly seen to be exponential. There is a discrepancy only for extremely small times. This is a consequence of the duration of intermediate unstable order 1 state that occurs between order 0 states (see figure 2). The fraction of order 1 states was in this case \( t_1/t_0 = 0.0013 \) which is the same as the time scale of the discrepancy in the figure. Our distribution should be a convolution of distributions of \( t_0 \) and of \( t_1 \). But the precise explanation of
Figure 4. In the top figure experimental values of $\tau$ for $N = 6, 8, 10, 16, 20$ and $30$ (symbols from left to right) and semi-theoretical estimate (with continuous curves) for $\tau$ are drawn. Errors in numerical $\tau$ are of the same order as the symbol sizes. In the bottom figure we plot the dependence of $\tau$ on $\eta = (\sum_{i=1}^{\infty} t_i)/t_0$. The data and the symbols used are the same as in the top figure, while the straight line segments are here merely connecting the points with the same value of $N$.

this discrepancy is the following: we obtain too many of “short” jumps because of the way we measure the time when the system has arrived into a new order 0 signature. We mark the system as arriving into a new signature exactly at the top of the barrier, when
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**Figure 5.** Numerical data for $\tau(N, \alpha)$ divided by $\exp(\alpha^2/4)/\sqrt{\alpha}$ versus $4N/\alpha^2$. The solid curve is only drawn to guide an eye, perhaps suggesting some universal scaling function.

**Figure 6.** Probability distribution of times $t = \tau_n$ (in a relative scale $t/\tau$) between two different consecutive order 0 signatures. On the right we plot enlarged distribution for small times. All is for the chain of length $N = 20$ and $\alpha = 5.4$ corresponding to $\tau = 3500$. Dashed line is an exponential function.

the difference $(q_{i+1} - q_i)$ changes the sign and not when it relaxes to the bottom of the well. This allows for some ‘fake transitions’ in the regions of phase space where two or more 0 order signatures almost touch with a saddle-like region in between (for instance near coordinate origin $q_i \equiv 0$). There it is possible that the system, which although it gets over the barrier, subsequently relaxes into some nearby well because it does not have the ‘right’ momentum vector.
4. Markov process and its transition matrix

We decided to describe dynamics with a sequence of binary signatures instead of a full orbit $\mathbf{x}(t)$ with the hope of some simplification. We are hence looking for some simple statistical description of the transitions between signatures. Stochastic system is fully described by giving conditional transition probabilities $P(S, S'; S'', \ldots ; t, t', \ldots)$. The simplest kind of a stochastic process is called Markov process. For a Markov process the probabilities $P(S, S'; t)$ contain entire information about the system. We will shortly write a matrix $P(t)$ instead of $P(S, S'; t)$. For now let us define $P(S, S'; t)$ on the full space of $2^N$ signatures although we will later be interested only in transition probabilities among $M_0$ signatures of order 0. The transition matrix $P$ must satisfy the condition

$$P(t + t') = P(t)P(t').$$

(11)

The matrix $P(t)$ contains therefore some redundant information. We can write instead

$$P(t) = \exp (Ft),$$

(12)

where for a Markov process the matrix $F$ is time independent. The matrix element $F(S, S')$ is a probability flux from the signature $S$ to the signature $S'$, and can be defined by the limit

$$F = \lim_{t \to 0} \frac{P(t) - I}{t}. \quad (13)$$

The conservation of probability imposes the condition $\sum_{S'} F(S, S') = 0$ for the matrix elements.

By a statistical description, using $P(t)$ instead of $\mathbf{x}(t)$, we lose information about the details within a single cell of a partition. In other words, we study the dynamics on the space of functions that are constant over one phase space cell. This can be formalized by introducing characteristic functions $B_S(\mathbf{x})$ over signatures $S$

$$B_S(\mathbf{x}) = \begin{cases} 1 & \mathbf{x} \in S \\ 0 & \text{otherwise} \end{cases}. \quad (14)$$

Characteristic functions $B_S$ span the space of all observables which are piece-wise constant over the signatures. Every piece-wise constant observable $W(\mathbf{x})$ can be expanded over the base functions $B_S$ as $W(\mathbf{x}) = \sum_S w SB_S(\mathbf{x})$ with the expansion coefficients $w_S$ given by

$$w_S(t) = \frac{\langle B_S W(t) \rangle_E}{\langle B_S \rangle_E}, \quad (15)$$

where the brackets $\langle \rangle_E$ denote a microcanonical phase space average over the energy surface $H = E$. The matrix elements of a transition matrix $P$ are nothing but the correlation functions between the characteristic functions

$$P(S, S'; t) = \frac{\langle B_{S'}(t)B_S \rangle_E}{\langle B_S \rangle_E}. \quad (16)$$

As the transition matrix propagates the probabilities between the signatures we can write the time dependent vector $\mathbf{w}(t) = (w_S(t), w_{S'}(t), \ldots)$ as

$$\mathbf{w}(t) = P(t)\mathbf{w}(0) = \exp (Ft)\mathbf{w}(0). \quad (17)$$
Similarly, the autocorrelation function of the observable $W$ is
\[ \langle W(t)W(0) \rangle_E = w^T \exp(Ft)w. \] (18)

We can see that the behaviour of correlation functions is determined by the spectrum of the (Markov) propagator $\exp(Ft)$.

From now on we assume that the fraction of higher order signatures is very small $\eta \approx t_1/t_0 \ll 1$ and we restrict the matrices $P(t)$ and $F$ on the $M_0$-dimensional subspace of signatures of order 0, which essentially contain all non-vanishing matrix elements. Numerically we calculated the transition matrix $P(t)$ from one very long orbit as an average
\[ P(S, S'; t) = \frac{\langle \delta_{S',S(t+\tau)} \delta_{S,S(t)} \rangle_{\tau}}{\langle \delta_{S,S(t)} \rangle_{\tau}}, \] (19)

where the brackets $\langle \rangle_{\tau}$ denote a time average over the orbit and $\delta_{S,S(t)}$ has value 1 if the orbit is in a signature $S$ at time $t$ and 0 otherwise. For small $t$ the average can be performed, and by using expression (13), yields for $F(S, S')$, ($S \neq S'$)
\[ F(S, S') = \frac{n(S, S')}{t_S}, \] (20)

where $n(S, S')$ is the number of direct transitions between the order 0 signatures $S$ and $S'$ and $t_S$ is the total time spent in the signature $S$. Here a small comment regarding higher order signatures is in order. We saw in figure 2 that between each order 0 states there is a short intermediate order 1 state of duration $\sim \tau t_1/t_0$. So there are no direct transitions between order 0 states and all fluxes $F(S, S')$ among $M_0$ order 0 signatures are strictly zero. As we have decided to study only (indirect) transitions between order 0 states (for the parameter values where the fraction of higher order states is negligible) we must somehow circumvent this difficulty. One solution is to calculate the derivative of the transition matrix $P(t)$ in equation (13) not at time $t = 0$ but at the time $t$ with $\tau t_1/t_0 \ll t \ll \tau$. In this case the number $n(S, S')$ in (20) is the number of jumps between order 0 states $S$ and $S'$ with one short intermediate order 1 state. Still more simple and convenient solution is to replace the orbit $x(t)$ by a sequence of times $t_i$ when the orbit hits the boundary of a cell of order 0 coming from outside. Orbit $x(t)$ is therefore replaced by a sequence of pairs $\ldots, (S_i, t_i), (S_{i+1}, t_{i+1}), \ldots$ which tell you that at time $t_i$ orbit $x(t)$ came to the order 0 signature $S_i$, then until time $t_{i+1}$ was in signature $S_i$ or any higher order signature and at time $t_{i+1}$ arrived into the next order 0 signature $S_{i+1}$ ($S_{i+1} \neq S_i$), etc. Then we define $n(S, S')$ as the number of subsequent pairs $(S, S')$ in the sequence $\{S_i\}$, and $t_S = \sum_i^{S_{i+1} = S} (t_{i+1} - t_i)$, hence the flux matrix is calculated as
\[ F(S, S') = \frac{\sum_{i}^{(S, S') = (S_i, S_{i+1})} 1}{\sum_{i}^{S_{i+1} = S} (t_{i+1} - t_i)}. \] (21)

This definition is used in the numerical calculation of this paper.
4.1. Tests for the Markov process

Before describing the process of transitions as Markovian we must check if this is at all permissible. Checking the rigorous conditions for the partition to signatures to be Markovian, see e.g. [12], is a considerable mathematical problem, at least to our judgement. On the other hand, heuristic arguments and numerical results support the idea, that for sufficiently large transition times $\tau$ the process will indeed be Markovian. The system has positive Lyapunov exponents. Vaguely this means that the system quickly “forgets” its history. If the average time between the jumps $\tau$ is longer than the Lyapunov time, we can expect transitions to be Markovian. However, the most straightforward test is to check explicitly the composition formula (11).

Condition $P(2t) = P^2(t)$ is trivially fulfilled for small and for large times $t$, $t \ll \tau$ and $t \gg \tau$, respectively. We checked it for time $t = \tau$ by calculating the quantity

$$\sigma = \frac{||P(2\tau) - P^2(\tau)||_E}{||P(2\tau)||_E},$$

where $||A||_E = (\sum_{i,j} |A_{i,j}|^2)^{1/2}$ is the Euclidean norm of a matrix $A$ (all results have also been re-checked by using the spectral norm giving almost identical results). As the matrix $P(t)$ will be calculated numerically from one very long but finite orbit it will have statistical error $\sigma_T$ due to the finite number of simulated transitions. In addition, $\sigma$ will also have a contribution $\sigma_M$ from a systematic error because the process may not be precisely Markovian. We have assumed $\sigma^2 = \sigma^2_M + \sigma^2_T$ and made an estimate for the statistical error $\sigma_T$ as a square root of a number of average transitions per matrix element $n$. It is

$$\sigma_T = c \frac{1}{\sqrt{n}} = c \frac{N_0}{2} \sqrt{M_0 \over N_0},$$

where $N_0$ is the total number of all transitions between the signatures of order 0 and $c$ is some unspecified numerical constant which has been determined from the numerical data. Its value turned out to be $c \approx 0.7$. Results for the dependence of a systematic error $\sigma_M$ are shown in figure 7. The error $\sigma_M(\tau)$ goes to zero with increasing $\tau$, as expected. Even more interesting is the fact that the error also seems to decrease with increasing $N$ at constant $\tau$. This is interesting as it means that with increasing $N$ we do not have to keep the fraction of higher order states small in order for the process to stay Markovian.

Whether a given system can be described as a Markov process can be checked also by calculating the transition matrix $P(t)$ and comparing it with the exponential formula (12) as a function of time. $P(t)$ can be calculated numerically directly from definition, for instance by using the formula (19). This definition would have a meaning also if the transitions were not Markovian. We then compare the correlation functions between the transitions between signatures of various orders as a Markov process on $2^N$ dimensional space.

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1 We have numerically checked that the distribution of Lyapunov exponents is approximately linear, as is typically the case for sufficiently chaotic systems [13].
2 In such a case one could bring higher order signatures back and describe a complete hierarchy of transitions between signatures of various orders as a Markov process on $2^N$ dimensional space.
characteristic functions (16) and the matrix elements of the propagator \( \langle S | \exp \left( F t \right) | S' \rangle \). These calculations should agree only if the flux matrix \( F(t) = \log P(t)/t \) were time independent and this would signal that the transitions are described by a Markov process. Numerical results for this test are shown in figure 8. Correlation functions calculated in both ways have been computed for the characteristic function \( B_S \equiv | S \rangle \) on the signature \( S = 54 \) and for the macroscopic observable with the vector \( w_S = S \mod 2 \), i.e. characteristic function on a union of half of all signatures with the first bit \( a_1 = 1 \), denoted by \( | s_1 \rangle \). It can be seen from the figure that the autocorrelation function of the observable \( | s_1 \rangle \) begins to fall as \( \exp \left( -\lambda_1 t \right) \), where \( \lambda_1 \) is the biggest nontrivial eigenvalue of \( F \), very early. This is a consequence of relatively large overlap between the macroscopic state \( | s_1 \rangle \) and the eigenstate \( | v_1 \rangle \) corresponding to the eigenvalue \( \lambda_1 \). We will show in subsection 5.2 that \( | \langle s_1 | v_1 \rangle |^2 \sim 1/N \) while \( | \langle S | v_1 \rangle |^2 \sim 1/M_0 \), so \( | \langle s_1 | v_1 \rangle |^2 \gg | \langle S | v_1 \rangle |^2 \) (for \( N \gg 1 \)).

5. The form and the spectral properties of the flux matrix

Something can be said immediately about the spectrum of \( F \). Conservation of probability implies all eigenvalues \( \lambda_i \) to be smaller or equal to zero. Further, because the system is believed to be ergodic, there should be a non-degenerate constant eigenvector with the corresponding eigenvalue \( \lambda_0 = 0 \). This is all the consequence of the Perron-Frobenius form of the matrix \( P \). For finite \( N \) the spectrum of the flux matrix is discrete and the correlation functions will fall asymptotically as \( \exp \left( -\lambda_1 t \right) \) where \( \lambda_1 \) is the largest nonzero eigenvalue. An interesting question is what happens in the thermodynamic limit \( N \to \infty \). If there is a spectral gap we will have an exponential decay, otherwise some anomalous behaviour can occur.
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Figure 8. Comparison of the correlation functions calculated from the propagator \( \exp(\mathbf{F}t) \) (dashed curves) and from directly determined transition matrix \( P(S, S'; t) \) (full curves). In figure (a) are autocorrelation functions \( S = S' \) and in figure (b) cross-correlation functions \( S \neq S' \). Bottom pair of curves in figure (a) is for the signature \( S = 54 \), the middle pair for the one particle state \( |s1\rangle \) and the top curve is a simple exponential decay with the biggest nonzero eigenvalue of the matrix \( \mathbf{F} \). From all correlation functions we have subtracted equilibrium value which is \( P(S, S; t \to \infty) = 1/M_0 \) and \( 1/2 \) for the state \( |s1\rangle \). In figure (b) are correlation functions between the signature \( S = 54 \) and the signature \( S' = 58, S' = 116, \) and \( S' = 135 \), from top to bottom, respectively. The chain length and the parameter were \( N = 8, \alpha = 3.83, \tau \sim 1000 \).

5.1. Numerical calculation of the flux matrix

Non-vanishing matrix elements of \( \mathbf{F} \) are expected only between the signatures that differ just by two bits provided that \( t_1/t_0 \ll 1 \). In other words, two bits \( a_i = 1 \) and \( a_j = 0 \) only switch their position. Such a transition will be called a (bit) jump of length \( d \), where
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Table 1. Dependence of $c_d$ on distance $d$ for different $\tau$ and $N$.

| $N=6$ | $N=12$ |
|-------|--------|
| $\tau$ | $\eta[10^{-2}]$ | $c_2$ | $c_3$ | $\tau$ | $\eta[10^{-2}]$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ | $c_6$ |
| 50 | 4.7 | 0.71 | 0.64 | 37 | 10 | 0.79 | 0.68 | 0.63 | 0.62 | 0.63 |
| 360 | 1.4 | 0.72 | 0.66 | 130 | 4.0 | 0.81 | 0.73 | 0.71 | 0.76 | 0.79 |
| 1100 | 0.8 | 0.73 | 0.66 | 366 | 1.8 | 0.83 | 0.76 | 0.76 | 0.85 | 0.92 |

| $N=8$ | $N=10$ |
|-------|--------|
| $\tau$ | $\eta[10^{-2}]$ | $c_2$ | $c_3$ | $c_4$ | $\tau$ | $\eta[10^{-2}]$ | $c_2$ | $c_3$ | $c_4$ | $c_5$ |
| 17 | 12 | 0.72 | 0.61 | 0.57 | 11 | 20 | 0.74 | 0.61 | 0.54 | 0.52 |
| 180 | 2.2 | 0.76 | 0.67 | 0.65 | 250 | 2.0 | 0.80 | 0.74 | 0.76 | 0.78 |
| 1030 | 0.6 | 0.76 | 0.71 | 0.71 | 621 | 1.0 | 0.82 | 0.76 | 0.82 | 0.86 |

$d = |j - i|$ is the distance between the two bits involved. Example of a such transition has been shown in figure 2. We have numerically calculated complete matrices $F$ for sizes $N = 6, 8, 10$ and 12 and different $\tau$ (or $\alpha$), in order to check the above hypothesis and to obtain some knowledge about the sizes and the structure of matrix elements. The only matrix elements which are really non-vanishing are those of the bit jumps, and what is more, the value of the matrix element depends only on the distance $d$ of the jump and not on the particular signatures involved. This practically means that in addition to the diagonal elements we have only $N/2$ different matrix elements in the flux matrix $F$. This number must be compared to the total number of matrix elements $M^2_0$ which grows exponentially with $N$. We have checked this also for the matrices $F$ of sizes up to $N = 30$ but in this case we have compared only the jumps of length $d$ with different number of bits 1 on the sites between the two jump sites. We confirmed that the probability fluxes of jumps depend only on the length $d$. It is therefore meaningful to introduce dimensionless coefficients $c_d$ which are ratios between the matrix elements of a jump of length $d$ and a jump of length $d = 1$. Therefore $c_1 \equiv 1$, by definition, and the rest of $N/2 - 1$ coefficients $c_d$ together with the average transition time $\tau$ is all that we need in order to specify the flux matrix $F$ completely. We can write

$$F = \frac{1}{\tau N^2(c_{N/2}/2 + \sum_{d=1}^{N/2-1} c_d)} C,$$

where the matrix $C$ is just a matrix involving the coefficients $c_d$ only. Numerical values of $c_d$ for different $N$ and $\tau$ are listed in table 1. It can be seen that for not too big $\tau$ (e.g. $\tau = 11$ and $N = 10$) the coefficients $c_d$ decrease monotonically with the distance $d$. For bigger $\tau$ (e.g. $\tau = 621$ and $N = 10$), this dependence ceases to be monotonic but becomes slightly well-shaped with a minimum at around $d \approx N/4$. There also seems to be a general trend that with increasing $N$, at a constant fraction $\eta \approx t_1/t_0$, the coefficients $c_d$ all approach 1. This can be explained by the following heuristic argument. To keep the fraction $t_1/t_0$ of higher order states constant with increasing $N$ we must increase the barrier height $\Delta U$. As a consequence, the transition time $\tau$ increases and
correspondingly also the average time $\tau t_1/t_0$ spent in the non-equilibrium intermediate state of order 1 in between the states of order 0. At a constant $t_1/t_0$ the transition time $\tau$ depends approximately exponentially on the chain length $N$ (see figure 4). On the other hand, the relaxation time for the non-equilibrium energy distribution at the moment of transition grows perhaps only linearly with $N$ (inversely proportional to the sound speed), but certainly slower than exponentially. For sufficiently large $N$ the relaxation time will therefore be smaller than the time spent in the intermediate order 1 cell. Initial locally non-equilibrium distribution at the beginning of a jump will therefore relax before the transition to the new order 0 state, which will therefore all be equally probable. Because of this we can assume that the coefficients $c_d$ will not depend on $d$ in the thermodynamic limit. This hypothesis still needs further verification as the limit of high $N$ could not be tested due to the fast growth of the dimension $M_0$ with $N$. Despite of that, the numerical results are consistent and point in the right direction (see table 1). We will show now, that our flux matrix has a specially appealing form which can be described using a formal connection to a certain Hamiltonian of a quantum spin-1/2 chain. Based on this we will also deduce the essential spectral properties of the flux matrix.

5.2. Correspondence between the Markov process and the quantum spin chain

We can make the announced correspondence by connecting the eigenvalue problem for the flux matrix $F$ with the eigenvalue problem for the quantum Hamilton operator $\hat{H}$ over $M_0$-dimensional Hilbert space with the same matrix elements as $F$. This correspondence would generally be of no particular use since the Hamiltonian $H$ does not represent any simple quantum system. In our example of IFPU chain this is not the case, since $F$ has a particularly nice and simple form, with non-vanishing elements only between the signatures connected by a bit jump where signatures are sequences of binary bits, 0 and 1. This should immediately remind us of Heisenberg chains of quantum spin-1/2 particles. Let us write the Hamilton function for one-dimensional ferromagnetic Heisenberg spin chain of length $N$, with Pauli variables $\vec{\sigma}_j = (\sigma^x_j, \sigma^y_j, \sigma^z_j)$ and coupling constants $J(d)$

$$
\hat{H} = -\frac{1}{2} \sum_{j,d=1}^{N} J(d) \vec{\sigma}_j \cdot \vec{\sigma}_{j+d} = -\frac{1}{2} \sum_{j,d=1}^{N} J(d) \left\{ \sigma^x_j \sigma^x_{j+d} + \sigma^y_j \sigma^y_{j+d} + \sigma^z_j \sigma^z_{j+d} \right\},
$$

(25)

where periodic boundary condition is assumed $\sigma_{N+j} \equiv \sigma_j$ and $J(d) \equiv J(N-d), J(0) \equiv 0$. Operators $\sigma^\pm_j = \sigma^x_j \pm i \sigma^y_j$ are the standard raising and lowering operators. Quantum state of the spin chain, an eigenstate of $\sigma^z_j$, will be denoted by the signature $|S\rangle$, with the obvious interpretation. Bit $a_j = 1$ or 0 denotes spin $j$ up or down, respectively. Now we can calculate the matrix elements of the Hamilton operator (25). For the off diagonal elements we get

$$
\langle S|\hat{H}|S'\rangle = \begin{cases} 
-2J(d) & \text{if there is a jump of length } d \text{ connecting } S \text{ and } S' \\
0 & \text{otherwise}
\end{cases}.
$$

(26)
Diagonal elements are
\[
\langle S|\hat{H}|S\rangle = -J(N/2)\left[N/2 - 2s_{N/2}(S)\right] - \sum_{d=1}^{N/2-1} J(d)\left[N - 2s_d(S)\right],
\]
where \(s_d(S)\) is a number of different signatures that can be reached from the signature \(S\) with a jump of length \(d\). For any signature \(S\) we can write an identity
\[
\sum_{d=1}^{N/2} s_d(S) = \left(\frac{N}{2}\right)^2.
\]
By denoting \(s = M_0N^2/(2N - 2)\), the following equality is also valid
\[
\sum_{i=1}^{M_0} s_d(S_i) = \begin{cases} 
  s & d = 1, \ldots, N/2 - 1 \\
  s/2 & d = N/2
\end{cases}.
\]

The ground state for the ferromagnetic Heisenberg Hamiltonian \[23\] can be found immediately for any \(J(d) > 0\). We have to keep in mind that the Hilbert space is in our case spanned just by \(M_0\) signatures of order 0 and not by all possible signature states as is usual for the quantum spin chains. Order \(i\) of the signature is simply an eigenvalue of the \(S_z\) component of the total spin, namely \(S_z = \sum_j \sigma_j^z = i\). But the operator \(S_z\) commutes with the Hamilton function and the Hilbert space is therefore a direct sum of subspaces labelled by eigenvalues of \(S_z\). We choose \(M_0\) dimensional Hilbert subspace with \(S_z(= i) = 0\). Ground state denoted by \(|0\rangle\) is
\[
|0\rangle = \frac{1}{\sqrt{M_0}} \sum_{i=1}^{M_0} |S_i\rangle,
\]
with the energy
\[
E_0 \equiv \langle 0|\hat{H}|0\rangle = -N\left(\sum_{d=1}^{N/2-1} J(d) + \frac{1}{2}J(N/2)\right).
\]

Now that we have the matrix elements of \(H\) and the ground state, we can see that if we prescribe \(J(d)\) to equal \(c_d\) multiplied by the constant prefactor in front of the matrix \(C\) in equation for \(F\) \[24\], we can formally write our flux matrix in terms of the Heisenberg spin Hamiltonian \[25\]
\[
\mathbf{F} = -\mathbf{H} + E_0\mathbf{I}.
\]
Transition probabilities \(P(S,S';t)\) can now be written as
\[
P(t) = \exp (\mathbf{F}t) = A(t) \exp (-\beta\mathbf{H}),
\]
where we wrote \(\beta = t\) and \(A(t) = \exp (E_0t)\). Expression for \(P\) has the same form as the density operator for the quantum canonical distribution at temperature \(1/t\). Time dependence of \(P\) is therefore the same as the dependence of the canonical distribution on cooling. Understanding time dynamics of the IFPU model is equivalent to the cooling of the ferromagnetic Heisenberg spin chain or its imaginary time dynamics. In the limit \(t \to \infty\) the matrix elements of \(P\) go towards the constant value \(1/M_0\) and in the corresponding Heisenberg spin chain any non-equilibrium distribution relaxes to the ground state \(|0\rangle\).
For higher lying states we numerically solved the eigenvalue problem for the flux matrix at various $\tau$ and sizes up to $N = 12$. One such example of a numerical spectrum is shown in figure 9. We have said that with increasing $N$, at constant $t_1/t_0$, we expect coefficients $c_d$ to approach a constant value $c_d = 1$ independent of $d$. This case corresponds to the simplest Heisenberg spin chain with uniform coupling. Apart from the constant factor the Hamilton function (25) is in this case $H = -2S^2 + 3N/2$ and the matrix $F$ reads

$$F = \frac{2}{N^2 \tau} \left( 2S^2 - \frac{N(N+2)}{2} I \right),$$

(34)

where $S^2 = 1/4(\vec{\sigma}_1 + \ldots + \vec{\sigma}_N)^2$. Eigenvalues of $S^2$ are $S(S+1)$, where $S$ is the quantum number of the total spin with values from $S = 0, \ldots, N/2$. The Hilbert space is composed of order 0 states with $S_z = 0$. If we denote by $\lambda_j$ the eigenvalues of operator (34) and by $n(j)$ the corresponding multiplices we have

$$\lambda_j = -\frac{4j(N+1-j)}{N^2 \tau},$$

$$n(j) = \frac{(N+1-2j)N!}{(N+1-j)!j!} \quad j = 0, \ldots, N/2.$$

(35)

Particularly interesting are the first two eigenvalues $\lambda_0$ and $\lambda_1$. The biggest, non degenerate eigenvalue $\lambda_0 = 0$ belongs to the ground state $|0\rangle$. The largest nontrivial eigenvalue is $\lambda_1 = -4/N\tau$ with the multiplicity $n(1) = N - 1$. In the thermodynamic

**Figure 9.** Spectrum of $F$ for $N = 8$ at three different $\tau = 17, 180$ and 1030 with stars, pluses and crosses, respectively. With boxes is plotted the spectrum in the case $c_d \equiv 1$ (35) for comparison.
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limit and keeping $\tau$ constant, it goes to zero as $1/N$. The correlation functions therefore decay slower than exponential in the thermodynamic limit. From the figures 9 and 10 it can also be deduced that with increasing $\tau$ the spectrum is indeed approaching the spectrum for the $c_d \equiv 1$ (35), as predicted. What is more, the eigenvalue $\lambda_1$ seems to be monotonically approaching the limiting case ($c_d \equiv 1$) from above. We can therefore conclude by observation that the correlation functions of observables which can be spanned by $|S\rangle$ decay slower than $\exp(-4t/N\tau)$.

![Figure 10](image)

Figure 10. Enlarged first multiplet for $N = 10$ and $\tau = 11,40,250$ and 620 with pluses, squares, full squares and crosses, respectively. Referential degenerate set of first $N - 1$ eigenvalues at $c_d \equiv 1$ is plotted with stars.

Now that we expect the spectrum to be close to the one for $c_d \equiv 1$ we can explain the decay of the correlation functions for one particle state $|s1\rangle$ and the one signature state $|S\rangle$ in figure 8. State vector for $|S\rangle$ has only one component different from 0. Eigenvector $|v1\rangle$ corresponding to the eigenvalue $\lambda_1$ has on the other hand all components approximately of the same order, i.e. $1/\sqrt{M_0}$. Square of the scalar product is therefore $|\langle S|v_1\rangle|^2 \sim 1/M_0$. One particle state $|s1\rangle$ is proportional to the $|s1\rangle \sim \sigma_1^+\sigma_1^-|0\rangle$. Because we act on the state $|0\rangle$ with the eigenvalue $S = N/2$ with the operator $\sigma_1^+\sigma_1^-$, we can decompose the state $|s1\rangle$ as a linear combination of states with the eigenvalue $S = N/2$ and $S = N/2 - 1$. State $|s1\rangle$ can therefore be written as a sum of $N$ states, one of which is also $|v1\rangle$. If we assume all the expansion coefficients to be of the same order we immediately obtain $|\langle s1|v_1\rangle|^2 \sim 1/N$. 
6. Discussion and conclusion

We divide the phase space of a $N$-body Hamiltonian, namely the inverted FPU model, into $2^N$ cells that are uniquely tagged by a binary number, called a signature $\mathcal{S}$. Signatures are ordered according to the absolute difference in the number of particle pairs/bits in the left/0 and right/1 potential well. For a sufficiently high potential barrier an approximate Markovian description on space spanned by order 0 signatures is possible. The accuracy of the Markovian property has been checked numerically to be increasing with increasing chain length $N$ at a constant fraction of higher order states. In the flux matrix $\mathbf{F}$ we have non-vanishing matrix elements only between the signatures connected by an exchange of a pair of bits. This enables for the formal correspondence between the IFPU and ferromagnetic Heisenberg quantum spin-$1/2$ chains. Understanding time dependence of a Markov transition matrix is equivalent to cooling or imaginary time dynamics of a quantum spin chain. If we increase the chain length $N$ at a fixed fraction of higher order states, the transition rates are expected to become independent of a jump length $d$. Such trend is confirmed by a numerically calculated matrices $\mathbf{F}$. In this limit exact eigenvalues of a flux matrix can be obtained explicitly. The largest nontrivial eigenvalue in this case is $\lambda_1 = -4/N\tau$ and goes to 0 in the thermodynamic limit. There is also a numerical evidence that the largest nontrivial eigenvalue in a general case (for a non-constant transition rate) is strictly larger than $-4/N\tau$. Therefore, the correlation functions of the observables which can be spanned by functions $|\mathcal{S}\rangle$ decay asymptotically slower than $\exp(-4t/N\tau)$.

Unfortunately these results do not imply directly the behaviour of more general correlation functions, such as current-current correlation which is needed in order to understand the transport properties. We have made some numerical calculations of current-current time correlation functions of IFPU chain for different sizes $N$ and different parameters. As expected, the decay of correlations for observables that vary on time scales smaller than $\tau$ (e.g. current) is faster than the decay of correlation functions spanned by the piece-wise constant basis $|\mathcal{S}\rangle$. Transition from the algebraic (anomalous conductivity) to the exponential (normal conductivity) decay of current autocorrelation function occurs rather abruptly at $\alpha \approx 3$ (for $N \geq 20$).

Finally, though we suggested several interesting properties of the above model, we must admit that most of our conclusions are based on numerical evidence. Therefore, we believe that it should be a challenging and not impossible future task to try to provide more rigorous justifications (and perhaps proofs) of our results. Establishing rigorous asymptotic Markovian property would enable one to systematically code and enumerate all the many-body (unstable, hyperbolic) periodic orbits and to use them explicitly in a classical or semi-classical trace formulae, for example to calculate the transport coefficients directly. We feel that IFPU chain may become a useful toy model of a chaotic field theory [14].
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