The influence of the Kubo number on the transport of energetic particles

A Shalchi
Department of Physics and Astronomy, University of Manitoba, Winnipeg, Manitoba R3T 2N2, Canada
E-mail: andreasm4@yahoo.com

Keywords: turbulence, energetic particles, fusion plasmas, astrophysical plasmas, diffusion

Abstract
We discuss the interaction between charged energetic particles and magnetized plasmas by using analytical theory. Based on the unified nonlinear transport (UNLT) theory we compute the diffusion coefficient across a large scale magnetic field. To achieve analytical tractability we use a simple Gaussian approach to model the turbulent magnetic fields. We show that the perpendicular diffusion coefficient depends only on two parameters, namely the Kubo number and the parallel mean free path. We combine the aforementioned turbulence model with the UNLT theory and we solve the corresponding integral equation numerically to show how these two parameters control the perpendicular diffusion coefficient. Furthermore, we consider two extreme cases, namely the case of strong and suppressed pitch-angle scattering, respectively. For each case we consider small and large Kubo numbers to achieve a further simplification. All our analytical findings are compared with formulas which are known in diffusion theory.

1. Introduction
In physics one is interested in the interaction between electrically charged energetic particles and magnetized plasmas. In astrophysics and space science, for instance, cosmic rays interact with the plasma of the solar wind or the interstellar medium. It is subject of transport theory to study the aforementioned interactions analytically and numerically by using computer simulations [1–3].

To study the interaction between energetic particles and magnetized plasmas is also important for optimizing devices for controlled fusion such as tokamaks. In such devices so-called runaway electrons can cause damage to the walls of the containment device. In our work we explore the interaction of energetic particles and the plasma in a very fundamental way and, therefore, our results have impact on the physics of controlled fusion.

In the solar system, solar energetic particles and cosmic rays interact with the solar wind plasma. One of our aims is to reproduce different observations analytically and numerically. This helps us to test our understanding of particle transport but also to improve our knowledge of turbulence. The understanding of turbulence is one of the remaining puzzles of classical physics. However, studying energetic particles is also important for predicting radiation intensities of cosmic rays at and close to Earth. This will be important in order to protect astronauts during manned missions to Moon and Mars.

In astronomy and astrophysics we are interested in the motion of cosmic rays through our own and external galaxies. Therefore, a detailed knowledge of the transport processes is necessary in order to model their propagation. The mechanism of diffusive shock acceleration (DSA) is responsible for the creation of cosmic particles. In order to compute cosmic ray spectra in simulations of DSA, knowledge of the different particle diffusion coefficients is required.

In the scenarios described above, one finds a magnetic field configuration which can be understood as a superposition of a mean (or large scale) magnetic field $\mathbf{B}_0 = \langle \mathbf{B} \rangle = B_0 \mathbf{e}_z$ and a turbulent component $\delta \mathbf{B}$ due to the magnetized plasma. In the solar system, for instance, the mean field can be identified with the solar magnetic...
field and the turbulent fields are those of the solar wind plasma. A similar configuration can be found in interstellar space where the Galactic magnetic field is superposed by the turbulent fields of the interstellar medium. The mean magnetic field in any of those physical systems breaks the symmetry and, therefore, one has to distinguish between particle motion along and across that field. Due to the interaction with the turbulent magnetic field component, the particle motion is more a stochastic motion rather than a well-defined helical motion. Therefore, one has to employ methods of statistical physics.

Fundamental quantities in transport theory are the mean square displacements \((\langle \Delta x^2 \rangle)\) describing the spread of possible particle orbits. Here \(x\) corresponds to the considered direction in space. Often it is assumed that the transport is diffusive meaning that \((\langle \Delta x^2 \rangle) = 2\kappa_x t\) where we have used the diffusion coefficient in the \(x\)-direction \(\kappa_x\). Characteristic here is that the corresponding mean square displacement increases linearly with time. In the recent years, however, non-diffusive transport has been discussed in the literature [4–8]. In this case \((\langle \Delta x^2 \rangle) \sim t^\alpha\) and one needs to distinguish between subdiffusion \((\alpha < 1)\), Markovian diffusion \((\alpha = 1)\), superdiffusion \((\alpha > 1)\), and ballistic transport \((\alpha = 2)\).

If the transport is indeed diffusive and the scenario is axi-symmetric, the particle motion is described by a diffusion tensor of the form

\[
(\kappa_{ij}) = \begin{pmatrix}
\kappa_L & \kappa_A & 0 \\
-\kappa_A & \kappa_L & 0 \\
0 & 0 & \kappa_T
\end{pmatrix},
\]

(1)

Here we have used the parallel diffusion coefficient \(\kappa_L\), the perpendicular diffusion coefficient \(\kappa_A\), as well as the drift coefficient \(\kappa_T\). It has to be pointed out that in the case of the drift coefficient a definition of the diffusion coefficient via mean square displacements is no longer possible [9].

The knowledge of the diffusion tensor is required in different physical scenarios such as modeling the propagation of cosmic particles in the Galaxy [10, 11], studying solar modulation [12–16], and describing particles experiencing diffusive acceleration at interplanetary shocks [17–21] and interstellar shocks [22, 23].

In particular the analytical description of perpendicular diffusion is difficult [3]. In [24] the unified nonlinear transport (UNLT) theory was developed providing a nonlinear integral equation for \(\kappa_L\) (see section 2 of the current article). It was shown in several papers that the theory discussed here agrees with test-particle simulations:

- In [25] UNLT theory was combined with a two-component turbulence model. In the latter model the turbulence is approximated by a superposition of slab modes where \(\delta B(\vec{x}) = \delta B(x, y)\).
- In [26] UNLT theory was compared with simulations performed for the noisy reduced magnetohydrodynamic model proposed in [27]. The latter model can be understood as a broadened two-dimensional model.
- Wave propagation effects were included into UNLT theory in [28] and the results of analytical theory were compared with the corresponding simulations.
- In [29] results from UNLT theory were compared with simulations performed for a turbulence model based on Goldreich and Sridhar scaling [30].
- In [31] the UNLT theory was compared with simulations performed for a noisy slab model.

In all cases UNLT theory agrees well with simulations confirming its validity for perpendicular diffusion.

It is the purpose of the current article to discuss the UNLT theory and combine it with a Gaussian turbulence model to explore the influence of the Kubo number on the transport of energetic particles in a turbulent plasma. The paper is organized as follows. In section 2 we discuss the UNLT theory and in section 3 it is combined with the Gaussian model. In section 4 we consider the special case of strong pitch-angle scattering corresponding to the case where parallel diffusion influences perpendicular transport and both transport processes are coupled. In section 5 the case of suppressed pitch-angle scattering is discussed and in section 6 we summarize and conclude.

2. The UNLT theory

The UNLT theory was developed in [24]. In the following we briefly re-derive the theory and discuss the most important steps and assumptions.
2.1. Equations of motion

The fundamental equation describing the motion of energetic charged particles interacting with a turbulent magnetic field configuration is the Newton–Lorentz equation

\[
\frac{d\vec{p}}{dt} = q \left[ \frac{\vec{v}}{c} \times \vec{B} (\vec{x}, t) \right]
\]

where we have used cgs units. Furthermore we have used the electric charge of the particle \( q \), the speed of light \( c \), the particle momentum \( \vec{p} \) and particle velocity \( \vec{v} \). The magnetic field has the form \( \vec{B} (\vec{x}, t) = B_0 \vec{e}_z + \delta \vec{B} (\vec{x}, t) \).

In general one has to include electric fields [1]. However, electric fields are less important for spatial diffusion and, thus, those fields are neglected in the current paper.

Usually one is interested in the coordinates of the charged particle interacting with turbulence. In the following we refer to these coordinates as \( \text{particle coordinates} \) and we will use the symbols \( \vec{x} \) and \( \vec{v} \) for particle position and velocity, respectively. Alternatively, one can use the coordinates \( \vec{X} \) defined via [1, 32, 33]

\[
\vec{X} = \vec{x} + \frac{c}{q B_0} (\vec{p} \times \vec{e}_z) = \vec{x} + \frac{1}{\Omega} (\vec{v} \times \vec{e}_z),
\]

where \( \Omega = (q B_0) / (mc \gamma) \) is the unperturbed gyrofrequency. Here we have also used the rest mass \( m \) of the particle and the Lorentz factor \( \gamma \). For the velocity \( \vec{v} \) of the particle we use spherical coordinates

\[
\begin{align*}
\nu_x &= v \sqrt{1 - \mu^2 \cos (\Phi)}, \\
\nu_y &= v \sqrt{1 - \mu^2 \sin (\Phi)}, \\
\nu_z &= v \mu
\end{align*}
\]

with the particle speed \( v \), the pitch-angle cosine \( \mu \), and the gyrophase \( \Phi \).

In the unperturbed case, where we have by definition \( \delta \vec{B} = 0 \), the vector \( \vec{X} \) corresponds to the position of the particle’s guiding center. Therefore, we call \( \vec{X} \) the \( \text{guiding center coordinates} \). To obtain the velocity of the guiding center \( \vec{X} \), we consider the time derivative of equation (3). This gives us

\[
\vec{V} = \frac{d\vec{X}}{dt} = \vec{v} + \frac{c}{q B_0} \left( \frac{d\vec{p}}{dt} \times \vec{e}_z \right)
\]

\[
= \vec{v} + \frac{1}{B_0} [\vec{e} \times \vec{B}] = \vec{V} = \nu \vec{B} + \nu \delta B_z \vec{B}_0.
\]

where we have employed equation (2) and used the \( \text{Grássmann Identity} \)

\[
\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} (\vec{a} \cdot \vec{c}) - \vec{c} (\vec{a} \cdot \vec{b}).
\]

Therefore, we can express the components \( V_x \) of the guiding center velocity vector by the components of the particle velocity vector \( \nu \) and the magnetic field components. Very often in diffusion theory, turbulence models with \( \delta B_z = 0 \) are considered. In this particular case equation (5) simplifies to

\[
\begin{align*}
V_x &= \nu \times \vec{B}_0 \delta B_z, \quad V_y &= \nu \times \vec{B}_0 \delta B_y, \quad \text{and} \quad V_z = \nu \delta B_z.
\end{align*}
\]

From a more practical point of view, these equations are valid as long as the condition \( \delta B_z \ll B_0 \) is satisfied. The latter condition is, for instance, fulfilled in the solar wind [34].

Equation (7) is often directly used as a starting point. This is justified by stating that the diffusion coefficient of the particles corresponds to the diffusion coefficient of their guiding centers [32, 33]. In [35], however, the following equation has been used as starting point to compute the perpendicular diffusion coefficient

\[
\nu \delta B_0 [\vec{X} (t), t]
\]

where we have written down explicitly how the different quantities depend upon time \( t \) and particle position \( \vec{x} (t) \). The latter equation is similar compared to equation (7) but a \( \text{correction factor ‘a’} \) was included allowing to balance out different approximations used in diffusion theory (see below for more details). An examples would be finite gyroradius effects which lead to an \( a \) smaller than one [32, 33].

2.2. The UNLT theory

To compute the perpendicular diffusion coefficient \( \nu \), one can employ the \( \text{Taylor–Green–Kubo (TGK) formulation} \) [36–38]
\[ \kappa_L = \int_0^\infty dt \langle v_x(t) v_x(0) \rangle , \tag{9} \]

where we have assumed axis-symmetric turbulence with \( \kappa_L = \kappa_{xx} = \kappa_{yy} \). According to equation (9), the diffusion coefficient is given as time integral over the corresponding velocity correlation function. By combining equations (8) and (9) we find the emergence of 4th order correlations

\[ \kappa_L = \frac{a^2}{B_0^2} \int_0^\infty dt \langle v_x(t) v_x(0) \delta B_x (\vec{x}(t), t) \delta B_x [\vec{x}(0), 0] \rangle . \tag{10} \]

To proceed we replace the turbulent magnetic fields therein by the Fourier representation

\[ \delta B_x (\vec{x}, t) = \int d^3 k \delta B_x (\vec{k}, t) e^{i \vec{k} \cdot \vec{x}} . \tag{11} \]

and we assume homogeneous turbulence to derive

\[ \kappa_L = \frac{a^2}{B_0^2} \Re \int_0^\infty dt \int d^3 k \langle v_x(t) v_x(0) \delta B_x (\vec{k}, t) \delta B_x^* (\vec{k}, 0) e^{i \vec{k} \cdot \vec{x}} \rangle . \tag{12} \]

Here \( \vec{x} \equiv \vec{x} (t) \) and we have set \( \vec{x} (t = 0) = 0 \).

Equation (12) contains the ensemble average operator acting on a quantity depending on magnetic fields and particle orbits. A powerful tool in order to deal with such configurations is provided by the so-called random phase approximation. The latter approximation is also known as the Corrsin approximation \cite{39} and is frequently used in diffusion theory \cite{24, 35, 40}. More details on the validity of the Corrsin approximation can be found in \cite{41}. In our case, the random phase approximation can be written as

\[ \langle v_x(t) v_x(0) \delta B_x (\vec{k}, t) \delta B_x^* (\vec{k}, 0) e^{i \vec{k} \cdot \vec{x}} \rangle 
\approx \langle \delta B_x (\vec{k}, t) \delta B_x^* (\vec{k}, 0) \rangle \langle v_x(t) v_x(0) e^{i \vec{k} \cdot \vec{x}} \rangle . \tag{13} \]

By using this type of approximation and by employing the magnetic correlation tensor in the Fourier or wave vector space

\[ P_\gamma (\vec{k}, t) = \langle \delta B_x (\vec{k}, t) \delta B_x^* (\vec{k}, 0) \rangle , \tag{14} \]

we can write equation (12) as

\[ \kappa_L = \frac{a^2}{B_0^2} \Re \int_0^\infty dt \int d^3 k P_\gamma (\vec{k}, t) \langle v_x(t) v_x(0) e^{i \vec{k} \cdot \vec{x}} \rangle . \tag{15} \]

In the following we use the magnetostatic approximation meaning that we assume

\[ P_\gamma (\vec{k}, t) = P_\gamma (\vec{k}) . \tag{16} \]

The UNLT theory was extended to allow for dynamical turbulence in \cite{42}. If we consider particles moving much faster than the Alfvén speed, the static approximation should be accurate and equation (15) becomes in this case

\[ \kappa_L = \frac{a^2 \gamma^2}{B_0^2} \int d^3 k P_\gamma (\vec{k}) \Re [T (\vec{k})] \tag{17} \]

with

\[ T (\vec{k}) = \frac{1}{\nu^2} \int_0^\infty dt \langle v_x(t) v_x(0) e^{i \vec{k} \cdot \vec{x}} \rangle . \tag{18} \]

Above we have used the ensemble average operator \( \langle \ldots \rangle \). The latter operator can be expressed as

\[ \langle A \rangle = \frac{1}{4} \int_{-1}^{+1} d \mu \int_{-1}^{+1} d \mu \int_{-\infty}^{+\infty} dz \ A (\mu_0, \mu, z, t) \]

meaning that we average over all particle properties. The parameter \( \mu_0 \) denotes the initial pitch-angle cosine. Therewith, the 4th order correlations in equation (18) can be written as

\[ 1 \text{ in the literature one often finds that the terms Corrsin’s hypothesis and random phase approximation are used interchangeably to describe the same physical approximation. It was pointed out by one of the referees that this is not really correct. Random phases for the magnetic field is a special case of Corrsin in the original context in which the Lagrangian correlation function is evaluated. The correct statement, somewhat broader, would be that the joint distribution function of displacements and magnetic fluctuations can be written as the product of two distribution functions. That is the displacements and magnetic fluctuations are independent random variables. Here there is also a third random variable, the velocity, so the stated equivalence of Corrsin and random phase approximation becomes even more misleading. In the current paper we use the terms Corrsin’s hypothesis and random phase approximation in order to refer to equation (13) as it is often done in the literature.}
\[
\langle v_z(t)v_z(0)e^{ikx} \rangle = \frac{v^2}{4} \int_{-1}^{1} d\mu_0 \int_{-1}^{1} d\mu \mu_0 \mu \Gamma(\vec{k}, \mu, t)
\]
with the pitch-angle dependent characteristic function
\[
\Gamma(\vec{k}, \mu, t) = \int d^4x \ e^{ikx} \langle \vec{x}, \mu, \mu_0, t \rangle.
\]

Here we have employed again the guiding center approximation meaning that we did not distinguish between guiding center and particle coordinates. Finite gyroradius corrections are discussed in the context of UNLT theory in \cite{32, 33}.

Since the particle distribution function \( f(\vec{x}, \mu, \mu_0, t) \) used in equation (21) involves velocities and the particle position, we need a transport equation for describing diffusion in the phase space. Such an equation does exist and is known as the \textit{Fokker–Planck equation}
\[
\frac{\partial f}{\partial t} + v_\mu \frac{\partial f}{\partial z} = \frac{\partial}{\partial \mu} \left[ D_{\mu\mu} \frac{\partial f}{\partial \mu} \right] + D_\perp \left[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right].
\]

Here we have assumed again axis-symmetry and we have used the \textit{Fokker–Planck} coefficients of pitch-angle diffusion \( D_{\mu\mu}(\mu) \) and perpendicular diffusion \( D_\perp(\mu) \), respectively. A more complete version of the cosmic ray Fokker–Planck equation can be found in \cite{1}.

It has to be pointed out that the \textit{Fokker–Planck equation} can be averaged over the parameter \( \mu \) and a late time limit can then be considered. This leads exactly to the standard cosmic ray diffusion equation (see again \cite{1}). Therefore, using the \textit{Fokker–Planck equation} (22) in diffusion theory is more accurate because the \textit{Fokker–Planck equation} does not only provide a pitch-angle dependent description of the transport, it is also accurate at earlier times for which the pitch-angle isotropization is still incomplete. Therefore, the \textit{Fokker–Planck equation} can be understood as a pitch-angle dependent diffusion equation which works for early as well as late times. This is the reason why UNLT theory contains even the quasi-linear diffusion coefficient if the appropriate limit is considered (see section 5.3). The \textit{Fokker–Planck equation} describes the distribution of particles at early times where the parallel motion is still unperturbed as well as the late times where the parallel motion is diffusive. The usual particle diffusion equation on the other hand, only describes particle transport in the late time limit and does not contain the early ballistic motion. The idea that early times have to be taken into account is not new since quasi-linear theory, for instance, is entirely based on the assumption that unperturbed orbits are used (see \cite{43}).

To proceed we multiply the \textit{Fokker–Planck equation} (22) by \( \exp(ik \cdot \vec{x}) \) and thereafter we integrate over space to obtain
\[
\frac{\partial \Gamma}{\partial t} = ivkz_0 \Gamma + \frac{\partial}{\partial \mu} \left[ D_{\mu\mu} \frac{\partial \Gamma}{\partial \mu} \right] - D_\perp k_z^2 \Gamma
\]
with \( \Gamma(\vec{k}, \mu, t) \) defined in equation (21). To proceed it is useful to define
\[
S(\mu, \vec{k}) = \frac{1}{2} \int_{-1}^{1} d\mu_0 \mu_0 \int_{0}^{\infty} dt \Gamma(\vec{k}, \mu, t)
\]
and with equations (18) and (20) we obtain
\[
T(\vec{k}) = \frac{1}{2} \int_{-1}^{1} d\mu \mu S(\mu, \vec{k}).
\]

By using integration by parts, one can rewrite the latter formula as
\[
T(\vec{k}) = \frac{1}{4} \int_{-1}^{1} d\mu (1 - \mu^2) \frac{\partial S(\mu, \vec{k})}{\partial \mu}.
\]

In order to derive an ordinary differential equation for the function \( S(\mu, \vec{k}) \), we multiply equation (23) by \( \mu_0 \), integrate over time, and average over the initial pitch-angle cosine \( \mu_0 \) to find
\[
\frac{1}{2} \int_{-1}^{1} d\mu_0 \mu_0 \left[ \Gamma(t = \infty) - \Gamma(t = 0) \right] = ivkz_0 S + \frac{\partial}{\partial \mu} \left[ D_{\mu\mu} \frac{\partial S}{\partial \mu} \right] - k_z^2 D_\perp S.
\]

In order to continue, we have to know \( \Gamma(t = 0) \) and \( \Gamma(t = \infty) \). The first function is given by \( \Gamma(t = 0) = 2\delta(\mu - \mu_0) \) for every possible value of \( \vec{k} \). It is well-known that particles experience pitch-angle isotropization due to pitch-angle scattering described by the parameter \( D_{\mu\mu} \). Therefore, the particle distribution function \( f \) and therewith the function \( \Gamma \) become \( \mu_0 \)- and \( \mu \)-independent for late times, i.e.,
\[
\Gamma(\vec{k}, \mu_0, \mu, t \to \infty) = \Gamma(\vec{k}, t \to \infty).
\]
If we multiply the pitch-angle independent function \( \Gamma(\vec{k}, t \to \infty) \) by
\( \mu \), and integrate over \( \mu \), we find zero. Therewith, equation (27) becomes

\[
-\mu = \text{ivk}_\parallel S_v + \frac{\partial}{\partial \mu} \left[ D_{\mu \mu} \frac{\partial S_v}{\partial \mu} \right] - k^2 D_{\perp} S_v.
\] (28)

It is usual to assume that \( D_{\mu \mu} \) and \( D_{\perp} \) are even functions in \( \mu \). To proceed we split \( S_v \) in an even contribution \( S_+ \) and an odd contribution \( S_- \). Then we can derive from equation (28) a system of two ordinary differential equations for \( S_+ \) and \( S_- \)

\[
0 = \text{ivk}_\parallel S_+ + \frac{\partial}{\partial \mu} \left[ D_{\mu \mu} \frac{\partial S_+}{\partial \mu} \right] - k^2 D_{\perp} S_+ \quad \text{(29)}
\]

which is even in \( \mu \) and

\[
-\mu = \text{ivk}_\parallel S_+ + \frac{\partial}{\partial \mu} \left[ D_{\mu \mu} \frac{\partial S_-}{\partial \mu} \right] - k^2 D_{\perp} S_- \quad \text{(30)}
\]

which is odd in \( \mu \). Equation (29) can be averaged over the pitch-angle cosine. By using \( D_{\mu \mu} (\mu = \pm 1) = 0 \) and equation (25), we can easily derive

\[
T \equiv \frac{1}{2} \int_{-1}^{-1} d\mu \mu S_+ (\mu, k) = \frac{k^2}{2\text{ivk}_\parallel} \int_{-1}^{-1} d\mu D_{\perp} S_+.
\] (31)

To proceed we assume

\[
D_{\perp} = 2|\mu|\kappa_\perp
\] (32)

which is a simple model based on the assumption that \( D_{\perp} (\mu) \) is symmetric in \( \mu \) and increases with \( |\mu| \). The assumption that \( D_{\perp} \sim |\mu| \) is supported by test-particle simulations [44]. The factor 2 in equation (32) has been chosen so, that [1]

\[
\kappa_\perp = \frac{1}{2} \int_{-1}^{-1} d\mu D_{\perp} (\mu).
\] (33)

With this form equation (31) becomes

\[
T = \int_{0}^{1} d\mu \mu S_+ = \frac{2\kappa_\perp k^2}{\text{ivk}_\parallel} \int_{0}^{1} d\mu \mu S_+.
\] (34)

Now we integrate equation (30) over the pitch-angle cosine from 0 to 1 and we use equation (34) to obtain

\[
-\frac{1}{2} = \text{ivk}_\parallel \int_{0}^{1} d\mu \mu S_+ - D S'_\parallel (\mu = 0) - k^2 \int_{0}^{1} d\mu D_{\perp} S_-.
\] (35)

Using equations (32) and (34) yields

\[
\frac{1}{2} = D S'_\parallel (\mu = 0) + \left[ 2\kappa_\parallel k^2 + \frac{(\text{ivk}_\parallel)^2}{2\kappa_\parallel k^2} \right] T
\] (36)

with \( S'_\parallel \equiv \partial S_\parallel / \partial \mu \). Furthermore, we have used \( D \equiv D_{\mu \mu} (\mu = 0) \) which is the pitch-angle Fokker–Planck coefficient at 90°. Next we consider equation (26). Due to the factor \( 1 - \mu^2 \) therein, we can approximate the derivative of \( S_\parallel (k, \mu) \) by its value at \( \mu = 0 \); i.e., \( S'_\parallel (k, \mu) \approx S'_\parallel (k, \mu = 0) \). We find

\[
T (k) \approx S'_\parallel (k, \mu = 0) / 3
\] (37)

and equation (36) becomes

\[
T (k) = \frac{1}{3} 2D + (4/3) \kappa_\parallel k^2 + (\text{ivk}_\parallel)^2 (3\kappa_\parallel k^2).
\] (38)

This result is valid for arbitrary but finite \( D \equiv D_{\mu \mu} (\mu = 0) \). A further simplification can be achieved by assuming \( D_{\mu \mu} = (1 - \mu^2) D \) corresponding to isotropic pitch-angle scattering. This form has been derived in [45] in the strong turbulence limit \( \delta B \gg B_0 \) and should also be valid for intermediate strong turbulence \( \delta B \approx B_0 \) which can be found in the interplanetary space or the interstellar medium. In this case we have \( 2D_{\mu \mu} (0) = 2D = v/\lambda_\parallel \) [46]. With this result equation (17) becomes

\[
\kappa_\parallel = \frac{a^2 v^2}{3B_0^2} \int d^3k \frac{P_{\perp} (k)}{F (k_\parallel, k_\perp) + (4/3) \kappa_\parallel k^2 + v/\lambda_\parallel}
\] (39)

with \( F (k_\parallel, k_\perp) = (v^2 k^2) / (3\kappa_\parallel k^2) \). Equation (39) is valid for axis-symmetric static turbulence, \( \delta B_\parallel \ll B_0 \), and a constant guide field.
To derive equation (39) we explicitly assumed that pitch-angle scattering is isotropic. However, equation (39) was derived elsewhere without this assumptions. Based on the work presented in [42, 47, 48] we conclude that equation (39) is also valid if pitch-angle scattering is not isotropic.

2.3. Alternative theories for perpendicular diffusion

Above the UNLT theory has been discussed and in the following sections of the paper we will explore solutions of integral equation (39). It has to be emphasized that alternative theories for perpendicular transport can be found in the literature ranging from the quasi-linear approach (see [43]) to nonlinear theories similar compared to UNLT theory. Nonlinear theories for perpendicular diffusion have been proposed a few years after quasi-linear theory has been presented (see, e.g., [49]). A certain breakthrough was achieved due to the development of the nonlinear guiding center (NLGC) theory presented in [35]. As shown in the latter paper, NLGC theory agrees well with certain test-particle simulations performed for the two-component turbulence model. However, NLGC theory fails for pure slab turbulence where one expects compound diffusion due to the fact that particles are tied to magnetic field lines while they experience parallel diffusion (see section 4.5 for more details).

Compared to UNLT theory, NLGC theory is based on a different treatment of the higher order correlations occurring in equation (10). In [35] it was suggested to approximate

\[ (v_x(t)v_x(0)\delta B_x[\vec{x}(t), t]\delta B_y[\vec{x}(0), 0]) \approx (v_x(t)v_x(0)) \langle \delta B_x[\vec{x}(t), t]\delta B_y[\vec{x}(0), 0] \rangle. \]

In [50], however, it was shown that the latter approximation fails completely for slab turbulence explaining why NLGC theory does not provide the correct subdiffusive behavior for this type of turbulence. Therefore, alternative theories for perpendicular diffusion were developed in the past few years. One extension of NLGC theory incorporates a modified treatment of the particle trajectories, called random ballistic decorrelation (see [51]). UNLT theory uses an approach based on the Fokker–Planck equation in order to compute directly the correlations between particle position and particle velocity. UNLT and NLGC theories provide the same integral equation in the limit of large Kubo numbers. For small Kubo numbers, however, completely different results can be obtained. In this small Kubo number regime, test-particle simulations have been performed in order to test different nonlinear transport theories and the results they provide (see [26, 31]). It was shown that only UNLT theory works in this regime whereas NLGC theory and its extensions presented in the literature completely fail.

3. The Gaussian decorrelation model

In order to compute the perpendicular diffusion coefficient based on equation (39), we need to specify the tensor component \( P_{xx}(\vec{k}) \). In the current paper we employ the Gaussian decorrelation model used in [52]. In this case the combined correlation function

\[ P(\vec{k}) = P_{xx}(\vec{k}) + P_{yy}(\vec{k}) \]

has the form

\[ P_{xx}(\vec{k}) = \frac{l_x l_x \delta B_x^2}{(2\pi)^3/2} k_x^2 e^{-\frac{1}{2}(\vec{k})^2 - \frac{1}{2}(\vec{k}, \vec{k})^2}. \]

Here we have used the turbulent magnetic field component \( \delta B_x \) as well as the two length scales \( l_x \) and \( l_{xl} \) along and across the mean magnetic field, respectively. The latter two length scales are characteristic scales over which the turbulent magnetic field decorrelates. The spectrum used here is normalized to that one always obtains

\[ \delta B_x^2 = \delta B_y^2 = \int d\vec{k} P(\vec{k}). \]

A problem of the model used here is the rapid decorrelation if the wave numbers are larger than the inverse scales. Such small scales are usually referred to as inertial range. It was shown in the famous paper [53] that turbulence in the inertial range can be described as power law with \( k^{-5/3} \). The same power law can be observed for magnetic turbulence in the solar wind (see, e.g., [54]). The small scales of the inertial range are essential if it comes to pitch-angle scattering and therewith parallel diffusion [3, 56]. However, small scales are not important for perpendicular diffusion. Perpendicular transport of particles as well as the random walk of magnetic fields lines are entirely controlled by the large scales of the energy range of the spectrum. Therefore, the question whether spectrum (42) is realistic or not does not matter for the study presented in the current paper as long as the spectrum has the correct behavior at large scales. Furthermore, it was shown in [53] that one always obtains the same asymptotic limits from UNLT theory regardless what the spectrum really is. In the current paper we, therefore, employ model (42) to warrant analytical tractability.
Table 1. Shown are the original variables occurring in transport theory and the new variables used in \cite{53} and in the current paper. Here we have used the parallel and perpendicular scales of the turbulence \( l_\parallel \) and \( l_\perp \). Furthermore, \( B_0 \) and \( \delta B_0 \) are the mean magnetic field and the turbulent magnetic field components, respectively.

| Original variables                  | New variables                  |
|-------------------------------------|--------------------------------|
| Parallel wavenumber                 | \( k_\parallel \)             |
| Perpendicular wavenumber            | \( k_\perp \)                 |
| Mean square displacement in \( x \)-direction | \( \langle (\Delta x)^2 \rangle \) | \( \langle (\Delta x)^2 \rangle / l_\parallel^2 \) |
| Mean square displacement in \( y \)-direction | \( \langle (\Delta y)^2 \rangle \) | \( \langle (\Delta y)^2 \rangle / l_\parallel^2 \) |
| Mean square displacement in \( z \)-direction | \( \langle (\Delta z)^2 \rangle \) | \( \langle (\Delta z)^2 \rangle / l_\parallel^2 \) |
| Parallel mean free path             | \( \lambda_\parallel \)       |
| Perpendicular mean free path        | \( \lambda_\perp \)           |
| Diffusion ratio                     | \( \lambda_\parallel / \lambda_\perp \) | \( \lambda_\parallel \delta B_0 / B_0 \) |
| Magnetic field ratio/Kubo number    | \( \delta B_0 / B_0 \)        |
| Field line diffusion coefficient   | \( n_{QL} \)                 |

If UNLT theory represented by equation (39) is combined with the Gaussian model (42), and by using the integral transformations \( x = l_\parallel k_\parallel \) and \( y = l_\perp k_\perp \) one finds after some straightforward algebra

\[
D = \frac{a^2}{\sqrt{2\pi}} K^2 \int_0^\infty dx \int_0^\infty dy \frac{y^2 e^{-\frac{x^2}{4y^2} - \frac{2y^2}{4y^2}}}{x^2 + 4\lambda_\parallel \lambda_\perp y^2 + 1},
\]

where we have used the diffusion ratio

\[
D := \frac{K \lambda_\parallel \lambda_\perp}{l_\parallel l_\perp^2} = \frac{\lambda_\parallel \lambda_\perp}{\lambda_\parallel l_\perp^2}
\]

and the Kubo number

\[
K = \frac{l_\parallel}{l_\perp} \frac{\delta B_0}{B_0}.
\]

The idea here is that all particle displacements are normalized with respect to the corresponding length scale of turbulence. This means, for instance, that we use \( \langle (\Delta x)^2 \rangle / l_\parallel^2 \) instead of \( \langle (\Delta x)^2 \rangle \). In table 1 we summarize the new variables and compare them with standard parameters usually used in diffusion theory.

By additionally using the parameter

\[
\tilde{\lambda}_\parallel = \lambda_\parallel / l_\parallel,
\]

we can write equation (44) as

\[
D = \frac{a^2}{\sqrt{2\pi}} K^2 \int_0^\infty dx \int_0^\infty dy \frac{Dy^2 e^{-\frac{x^2}{4y^2} - \frac{2y^2}{4y^2}}}{x^2 + 4D^2 \tilde{\lambda}_\parallel y^4 / 9 + Dy^2}.
\]

Very clearly we can see that the diffusion ratio \( D \) depends only on the Kubo number \( K \) and the parallel mean free path normalized with respect to the parallel scale \( \lambda_\parallel / l_\parallel \).

One can easily solve equation (48) numerically. In figure 1 we show the parameter \( \tilde{\lambda}_\parallel = \lambda_\parallel / l_\parallel \) versus \( \tilde{\lambda}_\parallel \) and in figure 2 we visualize the diffusion ratio \( D \) versus \( \tilde{\lambda}_\parallel \) for different values of the Kubo number \( K \). For our calculations we have set \( a^2 = 1 \) in equation (48) as an example. Usually it is assumed that \( a^2 \) is an order one constant but its exact value if still not clear (see section 4 for a more detailed discussion of this matter). One can easily see by considering figures 1 and 2 that for all values of the Kubo number, the perpendicular mean free path is directly proportional to the parallel mean free path as long as \( \lambda_\parallel \ll l_\parallel \). For \( \lambda_\parallel \gg l_\parallel \), however, the perpendicular mean free path becomes constant. Furthermore, the Kubo number has a strong influence on magnitude of the perpendicular diffusion coefficient. Below our numerical findings are explained analytically.

4. Strong pitch-angle scattering

In most astrophysical scenarios parallel diffusion is the dominant transport process \cite{1}. Parallel diffusion is caused by pitch-angle scattering. The latter process can influence the transport across the mean magnetic field. In the current section we, therefore, consider the case of strong pitch-angle scattering.
4.1. Pitch-angle scattering and parallel diffusion

Pitch-angle scattering is causing parallel diffusion described by the parallel diffusion coefficient \( \kappa_\parallel \) or, alternatively, the parallel mean free path \( \lambda_\parallel = \lambda_\parallel / l \). Pitch-angle scattering is described by the pitch-angle Fokker–Planck coefficient \( \mathcal{D}_{\mu
u} \) and is related to the parallel mean free path via the well-known formula \([57]\)

\[
\lambda_\parallel = \frac{3v}{8} \int_{-1}^{+1} d\mu \frac{(1 - \mu^2)^2}{\mathcal{D}_{\mu
u}(\mu)}. \tag{49}
\]

For strong pitch-angle scattering we find a short parallel mean free path. This behavior can easily be understood. If the turbulent magnetic field is weak, we find a small scattering parameter \( \mathcal{D}_{\mu
u} \). Pitch-angle scattering prevents the particle from following the unperturbed orbit and is, therefore, reducing the parallel mean free path. Therefore, strong pitch-angle scattering leads to a short parallel mean free path and, therewith, a small parallel diffusion coefficient.

If pitch-angle scattering is strong, it has a back-reaction on the perpendicular motion of the charged particles. Therefore, in the limit discussed here, we have to take into account parallel diffusion in the theoretical description of perpendicular transport. The UNLT theory represented by equation (39), takes into account this effect since the parallel mean free path \( \lambda_\parallel \) enters the latter equation.
4.2. UNLT theory for strong pitch-angle scattering

For short parallel mean free paths, we also expect that the perpendicular mean free path is short [31]. Therefore, in the limit of strong pitch-angle scattering, we can omit the term $\kappa_p k^2_p$ in the denominator of equation (39). Thus, the UNLT theory provides the following nonlinear integral equation in the limit of strong pitch-angle scattering

$$\kappa_l = \frac{a^2 v^2}{3B_0^2} \int d^3k \frac{P_{\perp}(\hat{k})}{(v^2 k^2_l)/(3\kappa_l k^2_l) + v/\lambda_l}.$$  

(50)

For axis-symmetric turbulence we can write

$$\kappa_l = 2\pi \frac{a^2 v^2}{3B_0^2} \int_0^\infty dk_|| \int_0^\infty dk_\perp \frac{k_\perp P(k_||, k_\perp)}{(v^2 k^2_l)/(3\kappa_l k^2_l) + v/\lambda_l},$$

(51)

where we have used the combined correlation function defined in equation (41). To continue we employ the Gaussian decorrelation model [42]. Alternatively, we can use equation (48) as starting point. In the limit considered here we have $\lambda_|| \to 0$ and the second term in the denominator in the latter equation can be neglected so that equation (48) simplifies to

$$D = \frac{a^2}{\sqrt{2\pi}} K^2 D \int_0^\infty dy y^4 e^{-y^2} \int_0^\infty dx \frac{e^{-x^2}}{x^2 + Dy^2}.$$  

(52)

The $x$-integral can easily be solved by [58]

$$\int_0^\infty dx \frac{e^{-x^2}}{x^2 + c^2} = \frac{\pi}{2c} \frac{e^{c^2/2}}{\text{Erfc}\left(\frac{c}{\sqrt{2}}\right)},$$

(53)

where we have used the complementary error function $\text{Erfc}(z)$. Therewith, we derive

$$\sqrt{D} = a^2 \frac{2\pi}{\sqrt{8}} K^2 D \int_0^\infty dy y^4 e^{(D-y^2)/2} \text{Erfc}\left(\frac{\sqrt{D}}{\sqrt{2}} y\right).$$

(54)

By using [58]

$$\int_0^\infty dy y^4 e^{(D-1)y^2} \text{Erfc}(\sqrt{D} y) = \frac{3\text{ArcTan}\left(\sqrt{1 - D} / D\right) - \left(5 - 2D\right) \sqrt{D - D^2}}{4\sqrt{\pi}(1 - D)^{5/2}},$$

(55)

this becomes

$$\sqrt{D} = a^2 K^2 \frac{3\text{ArcTan}\left(\sqrt{1 - D} / D\right) - \left(5 - 2D\right) \sqrt{D - D^2}}{2(1 - D)^{5/2}}.$$  

(56)

and we can easily rewrite equation (56) so that $K = f(D)$. In figure 3 we show the latter function giving us the Kubo number dependence of the diffusion ratio $D$. As demonstrated, for strong pitch-angle scattering, the diffusion ratio $D$ depends only on one single parameter and that is the Kubo number $K$. In figure 4 we show the ratio $\kappa_l/\kappa_l$ versus the scale ratio $l_\perp/l_\perp$ for the case $B_\perp/B_0 = 0.5$. The magnetic field ratio employed here should be realistic in the solar wind [59]. Therefore, figure 4 shows how the remaining parameter $l_\perp/l_\perp$ controls the ratio of the two diffusion coefficients for the case of strong pitch-angle scattering corresponding to low energy particles.

In the following we consider two asymptotic limits in order to relate our general formula (56) to known results. These two limits correspond to small and large Kubo numbers, respectively. We expect that if the Kubo number $K$ is small, the diffusion ratio $D$ is small as well (see, e.g., figure 3). Therefore, we can simplify equation (56) by considering small and large values of the parameter $D$. This is done in the following two paragraphs.

4.3. The CLRR scaling

In the limit $D \to 0$, and by using $\text{ArcTan}(x \to \infty) \to \pi/2$, equation (56) becomes

$$\sqrt{D} = \frac{3\pi}{4} a^2 K^2.$$  

(57)

This result agrees perfectly with the one derived in [55]. By using equations (45) and (46) to replace $D$ and $K$, equation (57) can be written as
Equation (58) was derived for the Gaussian turbulence model represented by equation (42). In [55] the following formula was derived for arbitrary turbulence

\[ \sqrt{\frac{\kappa_{\perp}}{\kappa_{||}}} = \frac{\pi a^2}{B_0} \int k \delta(k) \delta(k). \]  

Equation (58) is very similar compared to the result obtained in the famous work of Rechester and Rosenbluth [60]. Therefore, we call this limit the collisionless Rechester and Rosenbluth (CLRR) scaling.

4.4. The Zybin and Istomin scaling

In the limit \( D \to \infty \), and by using complex numbers, equation (56) becomes

\[ D = a^2 K^2. \]
By using equations (45) and (46) to replace $D$ and $K$, our finding can be written as

$$\kappa_\parallel = a^2 \kappa_|| \delta B^2 \over B_0^2.$$  \hfill (61)

The latter formula agrees with the limit already derived in [55] and was originally discussed in [61]. In this case the ratio $\kappa_\parallel / \kappa_\parallel$ depends only on the magnetic field ratio $\delta B_x / B_0$ and no other turbulence properties.

### 4.5. Slab turbulence and compound diffusion

A very extreme turbulence model is the so-called slab model. Within this model it is assumed that the turbulent magnetic field depends only on the coordinate parallel with respect to the mean magnetic field. From our more general equations, the slab result can be obtained by considering the limiting process $K \to 0$ because slab turbulence can be understood as zero Kubo number turbulence. In this case we need to consider the aforementioned limiting process in equation (57). Easily we can see that for $K \to 0$ we also obtain $\kappa_\parallel \to 0$ which is usually understood as subdiffusion. A more detailed investigation of the pure slab model shows [62]

$$\frac{1}{2} \frac{d}{dt} \langle (\Delta x)^2 \rangle = k_{\text{FL}} \sqrt{\kappa/\langle \pi \rangle}$$  \hfill (62)

with the field line diffusion coefficient $k_{\text{FL}}$. In the literature this type of transport is usually called compound diffusion [5, 63, 64]. Since we deal with slab turbulence, the field line diffusion coefficient is in this case given by equation (74).

### 5. Suppressed pitch-angle scattering

In the current section we consider the case that pitch-angle scattering is very weak corresponding to a very long parallel mean free path. In astrophysical scenarios this corresponds to the case of very high particle energies.

#### 5.1. UNLT theory for suppressed pitch-angle scattering

Here we assume that pitch-angle scattering is suppressed, corresponding to $\nu / \lambda \parallel = 0$. In this particular case equation (39) becomes

$$\kappa_\parallel = \frac{v^2}{2B_0^2} \int d^3k \ P_x \tilde{k} \frac{2 \kappa_\parallel k_x^2}{(vk_x)^2 + (2 \kappa_\parallel k_x^2)^2},$$  \hfill (63)

where we have set again $a^2 = 1$ as an example. As already pointed out in [24], the integral equation (63) can be solved by

$$\kappa_\parallel = \frac{v}{2} k_{\text{FL}}$$  \hfill (64)

with the field line diffusion coefficient $k_{\text{FL}}$ given by

$$k_{\text{FL}} = \frac{1}{B_0^2} \int d^3k \ P_x \tilde{k} \frac{\kappa_{\text{FL}} k_x^2}{(\kappa_{\text{FL}} k_x^2)^2 + k_x^2}.$$  \hfill (65)

In this limit perpendicular diffusion is only controlled by the random walk of magnetic field lines. Therefore, equation (64) is known as field line random walk (FLRW) limit. The nonlinear integral equation (65) was originally derived in [65] and describes the diffusion of magnetic field lines.

Based on equation (64) we find for the perpendicular mean free path

$$\lambda_\perp = \frac{3}{v} \kappa_\parallel = \frac{3}{2} k_{\text{FL}}.$$  \hfill (66)

Clearly we find that the perpendicular mean free path does not depend on any particle properties. In particular we find that $\lambda_\perp$ does not depend on particle energy. Furthermore, the perpendicular mean free path of the particle is basically equal to the field line diffusion coefficient except the numerical factor $3/2$. In terms of our new variables equation (64) can be written as (see table 1)

$$\tilde{\lambda}_\perp = \frac{\lambda_\perp l_\parallel}{l_\perp^2} = \frac{3}{2} D_{\text{FL}}.$$  \hfill (67)

Obviously, understanding of perpendicular diffusion in the limit considered here requires knowledge of the field line diffusion coefficient $k_{\text{FL}}$. Therefore, we discuss FLRW in the next paragraph.
5.2. Field line diffusion

In the following we discuss magnetic field line diffusion based on equation (65). If we combine the latter equation with the Gaussian model (42) and employ the integral transformations \( x = k_l \parallel l \) as well as \( y = k_l l_\perp \), we derive:

\[
D_{FL} = \frac{K^2}{\sqrt{2\pi}} \int_0^\infty dx \int_0^\infty dy \frac{D_{FL} y^5}{(D_{FL} y^2)^2 + x^2 e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2}},
\]

where we have used again the Kubo number \( K \) as well as the dimensionless field line diffusion coefficient

\[
D_{FL} = \frac{k_\parallel}{l_\perp} \kappa_{FL}.
\]

By considering equation (68) we can easily see that the parameter \( D_{FL} \) depends only on one quantity and this is the Kubo number \( K \). Therefore, based on analytically theory, we conclude that the Kubo number is the critical parameter if one describes the random walk of magnetic field lines.

We can easily evaluate equation (68) numerically and visualize the field line diffusion parameter \( D_{FL} \) versus the Kubo number \( K \). This is done in figure 5. We can clearly see that the field line diffusion coefficient \( D_{FL} \) increases with the Kubo number \( K \). How \( D_{FL} \) depends on \( K \) and what the different transport regimes are, can be understood by comparing our numerical findings with the analytical results. In order to explore equation (68) analytically, we have to consider asymptotic limits. In the following we, therefore, explore the cases of small and large Kubo numbers, respectively.

5.3. The quasi-linear regime

For small Kubo numbers we expect \( \kappa_{FL} \) to be small. By employing [66]

\[
\lim_{a \to 0} \frac{a}{a^2 + x^2} \to \pi \delta (x)
\]

we derive from equation (65) in the limit \( \kappa_{FL} \to 0 \)

\[
\kappa_{FL} = \frac{\pi}{2B_0^2} \int d^3k \, P(\vec{k}) \delta (k_\parallel).
\]

For the Gaussian model (42) this becomes

\[
\kappa_{FL} = \frac{\pi^2}{B_0^2} \frac{l_\parallel^4 \delta B_0^2}{(2\pi)^{3/2}} \int_0^\infty dk_\perp k_\perp^3 e^{-\frac{1}{2}(l, k_\perp)^2}.
\]

Using the integral transformation \( y = l_\parallel k_\perp \) leads to

\[
\kappa_{FL} = \frac{\pi^2}{(2\pi)^{3/2} B_0^2} \int_0^\infty dy \, y^3 e^{-\frac{1}{2}y^2}.
\]
The $y$-integral yields 2 and, thus, we obtain for the field line diffusion coefficient

$$\kappa_{FL} = \frac{\sqrt{\pi}}{2} \int \frac{\delta B_x^2}{B_0^2}. \quad (74)$$

The latter formula corresponds to the well-known quasi-linear scaling [67]. By using equation (69) this becomes

$$D_{FL} = \frac{\sqrt{\pi}}{2} K^2. \quad (75)$$

Characteristic for the quasi-linear regime is that the field line diffusion coefficient is directly proportional to $K$. Formula (75) is also visualized in figure 5. Together with equation (64) we obtain

$$\kappa_L = \frac{\nu}{2} \sqrt{\frac{\pi}{2}} \frac{\delta B_x^2}{B_0^2} \quad (76)$$

and for the perpendicular mean free path we get

$$\lambda_L = \frac{3}{2} \sqrt{\frac{\pi}{2}} \frac{\delta B_x^2}{B_0^2}. \quad (77)$$

A characteristic property of this limit is that the perpendicular mean free path does not depend on the particle velocity and scales like $\lambda_L \sim l_\perp \delta B_x^2/B_0^2$.

Equation (71) represents the quasi-linear formula for the general case. Obviously quasi-linear theory is valid for perpendicular diffusion if pitch-angle scattering is suppressed and if the Kubo number is small.

### 5.4. The Kadomtsev and Pogutse scaling

For large Kubo numbers we expect $\kappa_{FL}$ to be large. In the limit $\kappa_{FL} \to \infty$, equation (65) becomes

$$\kappa_{FL}^2 = \frac{1}{2B_0^2} \int dk P(k) k_{FL}^{-2}. \quad (78)$$

For the Gaussian model (42) this becomes

$$\kappa_{FL}^2 = \frac{l_\perp^4}{2(2\pi)^{3/2}} \frac{\delta B_x^2}{B_0^2} \int dk e^{-\frac{1}{2}(k/k_\perp)^2} P(k). \quad (79)$$

The integral transformations $x = k/k_\perp$ and $y = k_\perp l_\perp$ lead to

$$\kappa_{FL}^2 = \frac{l_\perp^4}{2\sqrt{\pi}} \frac{\delta B_x^2}{B_0^2} \int_0^\infty dx e^{-x^2} \int_0^\infty dy e^{-\frac{1}{2}y^2}. \quad (80)$$

The first integral yields $\sqrt{\pi}/2$ and the second 1. Therefore, we find for the field line diffusion coefficient

$$\kappa_{FL} = \frac{1}{\sqrt{2}} l_\perp \frac{\delta B_x}{B_0}. \quad (81)$$

The latter limit is sometimes called the nonlinear regime or the Bohm limit of field line diffusion and was originally obtained in [68]. By using equation (69) this becomes

$$D_{FL} = \frac{1}{\sqrt{2}} K. \quad (82)$$

The latter formula is also shown in figure 5. Characteristic for this regime is that $D_{FL} \sim K$. Together with equation (64) we obtain for the diffusion coefficient of the charged particle

$$\kappa_L = \frac{\nu}{2} l_\perp \frac{\delta B_x}{B_0}. \quad (83)$$

and for the perpendicular mean free path we get

$$\lambda_L = \frac{3}{\sqrt{8}} l_\perp \frac{\delta B_x}{B_0}. \quad (84)$$

Again we find a perpendicular mean free path which does not depend on particle velocity. In this limit the scaling behavior is $\lambda_L \sim l_\perp \delta B_x^2/B_0$.

### 5.5. The percolative regime

In the previous two paragraphs we have shown how the quasi-linear and the nonlinear or bohmian limit can be derived for small and large Kubo numbers. However, as pointed out in [69] there can be another regime which is known as percolative regime. Characteristic for the quasi-linear regime is the scaling $D_{FL} \sim K^2$ (see, e.g., equation (75) of the current paper) and for the Bohm regime $D_{FL} \sim K$ (see equation (82)). The percolative
regime described in [69] predicts the scaling \( D_{\text{FL}} \sim K^{0.7} \) for large Kubo numbers. It is usually assumed that the percolative regime is a consequence of the breakdown of the Corrsin approximation and, therefore, analytical theories based on that approximation don’t contain percolation and therewith the scaling \( D_{\text{FL}} \sim K^{0.7} \). However, UNLT theory and other theories such as the field line diffusion theory presented in [65] agree remarkably well with computer simulations performed for large Kubo number turbulence (see, e.g., [26, 70]). Therefore, it is unclear how important the percolative regime is in real physical scenarios. From an analytical point of view one would have to develop a theory for field line and particle diffusion which is not based on Corrsin’s approximation in order to obtain this type of transport and scaling.

6. Summary and conclusion

The purpose of this article is to review the analytical description of perpendicular diffusion of energetic particles interacting with a magnetized plasma and to explore the influence of the Kubo number on the transport. To do this we have combined the UNLT theory with a Gaussian model for the magnetic correlations. By relating all occurring quantities in the latter theory to the parallel and perpendicular length scales of the turbulence \( l_{\parallel} \) and \( l_{\perp} \) (see table 1) we have shown that the perpendicular mean free path depends only on two parameters namely on the parallel mean free path divided by the parallel scale \( \lambda_{\parallel}/l_{\parallel} \) and the Kubo number \( K = (l_{\parallel} B_{\parallel})/(l_{\perp} B_{\parallel}) \).

In figure 1 the perpendicular mean free path versus the parallel mean free path is shown for different values of the Kubo number. Figure 2 shows the ratio of the two diffusion coefficients versus the parallel mean free path for different Kubo numbers. In all considered cases we find that the perpendicular mean free path is directly proportional to the parallel mean free path as long as the latter parameter satisfies \( \lambda_{\parallel} \ll l_{\parallel} \). For \( \lambda_{\parallel} \gg l_{\parallel} \) however, the perpendicular mean free path becomes constant meaning that it does no longer depend on the parallel mean free path or particle energy. The Kubo number controls the magnitude of the perpendicular diffusion coefficient in all cases.

We have also investigated UNLT theory for the Gaussian model analytically. We have explored the case of strong pitch–angle scattering corresponding to short parallel mean free paths. We show that in this case the perpendicular mean free path is directly proportional to the parallel mean free path as already found numerically. Furthermore, we found two asymptotic limits namely one for small Kubo numbers and one for large Kubo numbers. The former limit can be called the CLRR scaling [55] and the latter one the Zybin and Istomin scaling. The findings obtained for strong pitch–angle scattering are visualized in figures 3 and 4.

For suppressed pitch–angle scattering, corresponding to long parallel mean free path, UNLT theory provides the FLRW limit for which the perpendicular diffusion coefficient of the particle satisfies \( \kappa_{\perp} = \nu s_{\perp} / 2 \) where we have used the field line diffusion coefficient \( \kappa_{\perp} \). For the latter parameter we have derived again two asymptotic limits namely one for small Kubo numbers and one for large Kubo numbers. For small Kubo numbers the perpendicular particle diffusion coefficient is given by the quasi-linear formula and for large Kubo numbers by the Kadomtsev and Pogutse limit. Figure 5 shows all obtained results for the field line diffusion coefficient.

The formulas obtained in the current paper for field line and perpendicular diffusion coefficients were obtained for a Gaussian turbulence model. However, in previous work the theory was evaluated for different spectra and turbulence geometries. In [71], for instance, a two-dimensional turbulence model was employed in combination with a power-law spectrum. Very similar results were obtained compared to the results presented in the current paper for the case of large Kubo numbers. In [55] the UNLT theory was explore for very general turbulence and it was shown that the asymptotic limits the theory provides only depend on the parallel diffusion coefficient of the particle and the Kubo number. The general formulas listed in [55] agree perfectly with the results obtained in the current article.

Although UNLT theory seems to describe perpendicular diffusion accurately, there are still some unanswered questions in the theory of perpendicular transport. The theory still contains the parameter \( a^2 \) as originally introduced in [35]. The physical meaning and the exactly numerical value of this parameter remain unclear. However, a possible explanation was given in [32, 33] where finite gyroradius effects have been taken into account. It has to be emphasized that such additional effects alter the diffusion coefficients presented in the current paper. Replacing Corrsin’s hypothesis by a more general approximation could also provide different formulas for the diffusion coefficients. Future work will hopefully resolve these last puzzles in the analytical theory of perpendicular diffusion.

Acknowledgments

A Shalchi acknowledges support by the Natural Sciences and Engineering Research Council (NSERC) of Canada.
References

1. Schlickeiser R 2002 Cosmic Ray Astrophysics (Berlin: Springer)
2. Spatschek K H 2008 Aspects of stochastic transport in laboratory and astrophysical plasmas Plasma Phys. Control. Fusion 50 124027
3. Shalchi A 2009 Nonlinear Cosmic Ray Diffusion Theories (Astrophysics and Space Science Library vol 362) (Berlin: Springer)
4. Perri S and Zimbardo G 2007 Evidence of superdiffusive transport of electrons accelerated at interplanetary shocks Astrophys. J. 671 L377
5. Shalchi A and Kourakis I 2007 A new theory for perpendicular transport of cosmic rays Astron. Astrophys. 470 405
6. Perri S and Zimbardo G 2009 Ion and electron superdiffusive transport in the interplanetary space Adv. Space Res. 44 465
7. Zimbardo G, Perri S, Pompiois P and Velti P 2012 Anomalous particle transport in the heliosphere Adv. Space Res. 49 1633
8. Perri S, Zimbardo G, Effenberger F and Fichtner H 2015 Parameter estimation of superdiffusive motion of energetic particles upstream of heliospheric shocks Astron. Astrophys. 578 10
9. Shalchi A 2011 Applicability of the Taylor–Green–Kubo formula in particle diffusion theory Phys. Rev. E 83 046402
10. Busching J and Potgieter M S 2008 The variability of the proton cosmic ray flux on the Suns way around the galactic center Adv. Space Res. 42 504
11. Thornbury A and Drury L 2014 Power requirements for cosmic ray propagation models involving re-acceleration and a comment on second-order Fermi acceleration theory Mon. Not. R. Astron. Soc. 442 3010
12. Ferreira S E S, Scherker K and Potgieter M S 2008 Cosmic rays in the dynamic heliosheath Adv. Space Res. 41 351
13. Engelbrecht N E and Burger R A 2010 Effects of various dissipation range onset models on the 26-day variations of low-energy galactic cosmic-ray electrons Adv. Space Res. 45 1015
14. Hitge M and Burger R A 2010 Cosmic ray modulation with a Fisk-type heliospheric magnetic field and a latitude-dependent solar wind speed Adv. Space Res. 45 18
15. Alania M V, Modrzejewski R and Wawrzynca A 2014 Peculiarities of cosmic ray modulation in the solar minimum 23/24 J. Geophys. Res.: Space Phys. 119 4164
16. Engelbrecht N E and Burger R A 2015 A comparison of turbulence-reduced drift coefficients of importance for the modulation of galactic cosmic-ray protons in the supersonic solar wind Adv. Space Res. 55 390
17. Zank G P, Li G, Florinski V, Hu Q, Lario D and Smith C W 2006 Particle acceleration at perpendicular shock waves: model and observations J. Geophys. Res. 111 A06108
18. Sandroos A and Vainio R 2009 Diffusive shock acceleration to relativistic energies in the solar corona Astron. Astrophys. 507 L21
19. Li G, Shalchi A, Ao X, Zank G and Verkhoglyadova O P 2012 Particle acceleration and transport at an oblique CME-driven shock Adv. Space Res. 49 1067
20. Battarbee M, Vainio R, Laitinen T and Hietala H 2013 Injection of thermal and suprathermal seed particles into coronal shocks of varying obliquity Astron. Astrophys. 558 13
21. Kocharov L, Laitinen T, Vainio R, Afanasiev A, Mursula K and Ryan J M 2015 Solar interacting protons versus interplanetary protons in the core plus halo model of diffusive shock acceleration and stochastic re-acceleration Astrophys. J. 806 11
22. Ferrand G, Danos R J, Shalchi A, Safi-Harb S, Edmon P and Mendoza P 2014 Cosmic ray acceleration at perpendicular shocks in supernova remnants Astrophys. J. 792 13
23. Takamoto M and Kirk J G 2015 Rapid cosmic-ray acceleration at perpendicular shocks in supernova remnants Astrophys. J. 809 11
24. Shalchi A 2010 A unified particle diffusion theory for cross-field scattering: subdiffusion, recovery of diffusion, and diffusion in 3D turbulence Astrophys. J. 720 L127
25. Taut R C and Shalchi A 2011 Numerical test of improved nonlinear guiding center theories Astrophys. J. 735 92
26. Shalchi A and Hussein M 2014 Perpendicular diffusion of energetic particles in noisy reduced magnetohydrodynamic turbulence Astrophys. J. 794 56
27. Ruffolo D and Matthaeus W H 2013 Theory of magnetic field line random walk in noisy reduced magnetohydrodynamic turbulence Phys. Plasmas 20 012308
28. Hussein M and Shalchi A 2014 Parallel and perpendicular diffusion coefficients of energetic particles interacting with shear Alfvén waves Mon. Not. R. Astron. Soc. 444 2676
29. Shalchi A and Hussein M 2015 Erratum to: benchmarking the unified nonlinear transport theory for Goldreich–Sridhar turbulence Astrophys. Space Sci. 353 243
30. Goldreich P and Sridhar S 1995 Toward a theory of interstellar turbulence: II. Strong alfvenic turbulence Astrophys. J. 438 763
31. Hussein M, Taut R C and Shalchi A 2013 The influence of different turbulence models on the diffusion coefficients of energetic particles J. Geophys. Res.: Space Phys. 112 A09312
32. Shalchi A 2015 Finite gyroradius corrections in the theory of perpendicular diffusion: I. Suppressed velocity diffusion Adv. Space Res. 56 1264
33. Shalchi A 2015 Finite gyroradius corrections in the theory of perpendicular diffusion: II. Strong velocity diffusion Adv. Space Res. 57 431–42
34. Belcher J W and Davis L Jr 1971 Large-amplitude Alfvén waves in the interplanetary medium J. Geophys. Res. 76 3534
35. Matthaeus W H, Qin G, Bieber J W and Zank G P 2003 Nonlinear collisionless perpendicular diffusion of charged particles Astrophys. J. 590 L53
36. Taylor G I 1922 Diffusion by continuous movement Proc. London Math. Soc. 20 196
37. Green M S 1951 Brownian motion in a gas of noninteracting molecules J. Chem. Phys. 19 1036
38. Kubo R 1957 Statistical-mechanical theory of irreversible processes: I. General theory and simple applications to magnetic and conduction problems J. Phys. Soc. Japan 12 570
39. Corssin S 1999 Progress report on some turbulent diffusion research Atmospheric Diffusion and Air Pollution (Advances in Geophysics vol 6) ed F Frenkkel and P Sheppard (New York: Academic)
40. Lerche I 1973 Enhanced diffusion in strongly magnetized astrophysical plasmas Astrophys. Space Sci. 23 359
41. Taut R C and Shalchi A 2010 On the widespread use of the Corssin hypothesis in diffusion theories Phys. Plasmas 17 122313
42. Shalchi A 2011b Charged-particle transport in space plasmas: an improved theory for cross-field scattering Plasma Phys. Control. Fusion 53 074010
43. Jokipii J R 1966 Cosmic-ray propagation: I. Charged particles in a random magnetic field Astrophys. J. 146 480
44. Qin G and Shalchi A 2014 Pitch-angle dependent perpendicular diffusion of energetic particles interacting with magnetic turbulence Appl. Phys. Res. 6 1

16
[45] Shalchi A, Skoda T, Tautz R C and Schlücker R 2009 Analytical description of nonlinear cosmic ray scattering: isotropic and quasilinear regimes of pitch-angle diffusion Astron. Astrophys. 507 589
[46] Shalchi A 2006 Analytical investigation of the two-dimensional cosmic ray Fokker–Planck equation Astron. Astrophys. 448 809
[47] Shalchi A 2011 A heuristic derivation of an improved analytical theory for perpendicular diffusion of charged particles Adv. Space Res. 48 1499
[48] Lerche I and Tautz R C 2011 Cosmic ray diffusion: detailed investigation of a recent model Phys. Plasmas 18 082305
[49] Owens A J 1974 The effects of nonlinear terms in cosmic-ray diffusion theory Astrophys. J. 191 235
[50] Shalchi A 2006 Extended nonlinear guiding center theory of perpendicular diffusion Astron. Astrophys. 453 L43
[51] Ruffolo D, Pianapani T, Matthaeus W H and Chuychai P 2012 Random ballistic interpretation of nonlinear guiding center theory Astrophys. J. Lett. 747 L34
[52] Neuer M and Petschek H K 2006 Diffusion of test particles in stochastic magnetic fields for small Kubo numbers Phys. Rev. E 73 26404
[53] Kolmogorov A N 1941 The local structure of turbulence in incompressible viscous fluid for very large Reynolds’ numbers Dokl. Akad. Nauk. SSSR 30 301
[54] Denskat K U and Neuhauser F M 1982 Observations of hydrodynamik turbulence in the solar wind Solar Wind Five (NASA Conf. Publication) (Woodstock, Vermont, 1–5 November 1982) vol 2280 ed M Neugebauer (Washington, DC: NASA) pp 81–91
[55] Shalchi A 2015 Perpendicular diffusion of energetic particles in collisionless plasmas Phys. Plasmas 22 010704
[56] Bieber J W, Matthaeus W H, Smith C W, Warner W, Kallenrode M B and Wibberenz G 1994 Proton and electron mean free paths: the Palmer consensus revisited Astrophys. J. 420 L294
[57] Earl J A 1974 The diffusive idealization of charged-particle transport in random magnetic fields Astrophys. J. 193 231
[58] Gradshteyn I S and Ryzhik I M 2000 Table of Integrals, Series, and Products (New York: Academic)
[59] Ruffolo D, Pianapani T, Matthaeus W H and Chuychai P 2012 Random ballistic interpretation of nonlinear guiding center theory Astrophys. J. Lett. 747 L34
[60] Rechester A B and Rosenbluth M N 1978 Electron heat transport in a Tokamak with destroyed magnetic surfaces Phys. Rev. Lett. 40 38
[61] Zybkin K P and Istanto Y N 1985 Diffusion of charged particles in a random magnetic field Z. Eksp. Teor. Fiz. 89 836
[62] Shalchi A, Tautz R C and Rempe T J 2011 Test-particle transport: higher-order correlations and time-dependent diffusion Plasma Phys. Control. Fusion 53 105016
[63] Kota J and Jokipii J R 2000 Velocity correlation and the spatial diffusion coefficients of cosmic rays: compound diffusion Astrophys. J. 531 1067
[64] Webb G M, Zank G P, Kaghavshivi E K and le Roux J A 2006 Compound and perpendicular diffusion of cosmic rays and random walk of the field lines: I. Parallel particle transport models Astrophys. J. 651 211
[65] Matthaeus W H, Gray P C, Pontius D H Jr and Bieber J W 1995 Spatial structure and field-line diffusion in transverse magnetic turbulence Phys. Rev. Lett. 75 2136
[66] Zwilling R 2007 Standard Mathematical Tables and Formulas (Boca Raton, FL: CRC)
[67] Jokipii J R and Parker E N 1969 Stochastic aspects of magnetic lines of force with application to cosmic-ray propagation Astrophys. J. 155 777
[68] Kadomtsev B B and Pogutse O P 1978 Plasma electron thermal conductivity across a braided magnetic field Plasma Phys. Control. Nucl. Fusion Res. 40 38
[69] Isichenko M B 1992 Percolation, statistical topography, and transport in random media Rev. Mod. Phys. 64 961
[70] Shalchi A and Qin G 2010 Random walk of magnetic field lines: analytical theory versus simulations Astrophys. Space Sci. 330 279
[71] Shalchi A 2013 Simple analytical forms of the perpendicular diffusion coefficient for two-component turbulence: I. Magnetostatic turbulence Astrophys. J. 774 7