Topical Review

First exit times of harmonically trapped particles: a didactic review

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Abstract
We revise the classical problem of characterizing first exit times of a harmonically trapped particle whose motion is described by a one- or multidimensional Ornstein–Uhlenbeck process. We start by recalling the main derivation steps of a propagator using Langevin and Fokker–Planck equations. The mean exit time, the moment-generating function and the survival probability are then expressed through confluent hypergeometric functions and thoroughly analyzed. We also present a rapidly converging series representation of confluent hypergeometric functions that is particularly well suited for numerical computation of eigenvalues and eigenfunctions of the governing Fokker–Planck operator. We discuss several applications of first exit times, such as the detection of time intervals during which motor proteins exert a constant force onto a tracer in optical tweezers single-particle tracking experiments; adhesion bond dissociation under mechanical stress; characterization of active periods of trend-following and mean-reverting strategies in algorithmic trading on stock markets; relation to the distribution of first crossing times of a moving boundary by Brownian motion. Some extensions are described, including diffusion under quadratic double-well potential and anomalous diffusion.

Keywords: first exit time, harmonic potential, Ornstein–Uhlenbeck process, confluent hypergeometric function, Fokker–Planck equation, optical tweezers, survival probability
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(Some figures may appear in colour only in the online journal)
1. Introduction

First passage time (FPT) distributions have found numerous applications in applied mathematics, physics, biology and finance [1–3]. The FPT can characterize the time needed for an animal to find food; the time for an enzyme to localize a specific DNA sequence and to initiate a biochemical reaction; the time to exit from a confining domain (e.g., a maze); or the time to buy or sell an asset when its price deviation from the mean exceeds a prescribed threshold. The FPT distribution has been studied for a variety of diffusive processes, ranging from ordinary diffusion (Brownian motion) to continuous-time random walks (CTRWs) [4–11], fractional Brownian motion [12–15], Lévy flights [16–18], surface-mediated diffusion [19–22] and other intermittent processes [23, 24], diffusion in scale-invariant media [25, 26], trapped diffusion [27], thermally driven oscillators [28], Ornstein–Uhlenbeck process [29–38], and many others [1, 2, 39–48].

In this review, we revise the classical problem of characterizing the first exit time (FET) distribution of a multidimensional Ornstein–Uhlenbeck process from a ball [49–51]. The probability distribution can be found through the inverse Fourier (resp. Laplace) transform of the characteristic (resp. moment-generating) function for which explicit representations in terms of special functions are well known [30, 39]. Although the problem is formally solved, the solution involves confluent hypergeometric functions and thus requires subtle asymptotic methods and computational hints. The aim of the review is to provide a didactic self-consistent description of theoretical, numerical and practical aspects of this problem.

First, we recall the main derivation steps of the FET distribution, from the Langevin equation (section 2.1), through forward and backward Fokker–Planck (FP) equations (section 2.2 and 2.3), to spectral decompositions based on the eigenvalues and eigenfunctions of the FP operator (section 2.4). This general formalism is then applied to describe the first exit times of harmonically trapped particles in one dimension: the mean exit time (section 2.5), the survival probability (section 2.6) and the moment-generating function (section 2.7). In particular, we analyze the asymptotic behavior of the mean exit time and eigenvalues in different limits (e.g., strong trapping potential, large constant force, etc). Extensions to the radial Ornstein–Uhlenbeck process in higher-dimensional cases for both interior and exterior problems are presented in section 2.8 and 2.9, respectively. Although most of these results are classical, their systematic self-contained presentation and numerical illustrations are missing.

Section 3 starts from the summary of computational hints for computing confluent hypergeometric functions, while technical details are reported in appendix B. We then discuss three applications: (i) calibration of optical tweezers’ stiffness in single-particle tracking experiments and the detection of eventual constant forces exerted on a tracer by motor proteins (section 3.2), (ii) adhesion bond dissociation under mechanical stress (section 3.3), and (iii) distribution of triggering times of trend-following strategies in algorithmic trading on stock markets (section 3.4). We also illustrate a direct relation to the distribution of first-crossing times of a moving boundary by Brownian motion (section 3.5). Finally, we present several extensions of the spectral approach, including diffusion under quadratic double-well potential (section 3.6) and anomalous diffusion (section 3.7). Many technical details are summarized in the appendices.
2. First exit time distribution

We first recall the standard theoretical description of harmonically trapped particles by Langevin and Fokker–Planck equations [46, 52, 53]. We start with the one-dimensional Ornstein–Uhlenbeck process and then discuss straightforward extensions to higher dimensions.

2.1. Langevin equation

We consider a diffusing particle of mass \(m\) trapped by a harmonic potential of strength \(k\) and pulled by a constant force \(F_0\). The thermal bath surrounding the particle results in its stochastic trajectory, which can be described using a Langevin equation [52]

\[
m\ddot{X}(t) = -\gamma \dot{X}(t) + F(X(t)) + \xi(t),
\]

where \(-\gamma \dot{X}(t)\) is the viscous Stokes force (\(\gamma\) being the drag constant), \(F(X(t)) = -kX(t) + F_0\) includes the externally applied Hookean and constant forces, and \(\xi(t)\) is the thermal driving force with Gaussian distribution such that \(\langle \xi(t) \rangle = 0\) and \(\langle \xi(t) \xi(t') \rangle = 2k_b T \delta(t-t')\), with \(k_b \approx 1.38 \cdot 10^{-23} \text{J}/\text{K}\) being the Boltzmann constant, \(T\) the absolute temperature (in degrees Kelvin), \(\delta(t)\) the Dirac distribution, and \(\langle \ldots \rangle\) denoting the ensemble average or expectation. In the overdamped limit \((m = 0)\), one gets the first-order stochastic differential equation

\[
\dot{X}(t) = \frac{1}{\gamma} [F(X(t)) + \xi(t)] = \frac{k}{\gamma} (\hat{x} - X(t)) + \frac{\xi(t)}{\gamma}, \quad X(0) = x_0,
\]

where \(\hat{x} = F_0/k\) is the stationary position (mean value), and \(x_0\) is the starting position. The Langevin equation can also be written in a conventional (dimensionless) stochastic form [39, 40]

\[
d\dot{X}_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \quad X_0 = x_0,
\]

where \(W_t\) is the standard Wiener process (Brownian motion), \(\mu(x, t)\) and \(\sigma(x, t)\) are the drift and volatility, which in general can depend on \(x\) and \(t\). In our case, the volatility is constant, while the drift is a linear function of \(x\), \(\mu(x, t) = (\hat{x} - x)\theta\), i.e.

\[
d\dot{X}_t = \theta(\hat{x} - X_t)dt + \sigma dW_t, \quad X_0 = x_0,
\]

where \(\theta = k\delta/\gamma\), and \(\sigma = \sqrt{2D\delta}\), with \(\delta\) being a time scale, and \(D = k_b T/\gamma\) the diffusion coefficient. This stochastic differential equation defines an Ornstein–Uhlenbeck (OU) process, with mean \(\hat{x}\), variance \(\sigma^2\) and rate \(\theta\). An integral representation of equation (4) reads

\[
X_t = x_0 e^{-\theta t} + \hat{x}(1 - e^{-\theta t}) + \sigma \int_0^t e^{\theta(t-t')}dW_t.
\]

One can see that \(X_t\) is a Gaussian process with mean \(\langle X_t \rangle = x_0 e^{-\theta t} + \hat{x}(1 - e^{-\theta t})\) and covariance \(\langle X_t X_s \rangle = \langle X_t \rangle \langle X_s \rangle = \frac{\sigma^2}{2\theta} (e^{-\theta(t-s)} - e^{-\theta(t+s)})\). The discrete version of equation (4) with a fixed time step \(\delta\) is known as an auto-regressive model AR(1):

\[
X_n = (1 - k\delta/\gamma)X_{n-1} + F_0\delta/\gamma + \sqrt{2D\delta} \xi_n,
\]

where \(\xi_n\) are standard iid Gaussian variables with mean zero and unit variance. This discrete scheme can be used for numerical generation of stochastic trajectories. An extension of the above stochastic description to multidimensional processes is straightforward.
2.2. Forward Fokker–Planck equation

The Langevin equation (2) expresses the displacement \( \delta X(t) \) over a short time step \( \delta \) in terms of the current position \( X(t) \). In other words, the distribution of the next position is fully determined by the current position, the so-called Markov property. Such a Markov process can be characterized by a propagator or a transition density, i.e., the conditional probability density \( p(x, t \mid x_0, t_0) \) of finding the particle at \( x \) at time \( t \), given that it was at \( x_0 \) at earlier time \( t_0 \). The propagator can be seen as a ‘fraction’ of paths from \( x_0 \) to \( x \) among all paths started at \( x_0 \) (of duration \( t - t_0 \)), which formally writes as the average of the Dirac distribution \( \delta(X(t) - x) \) over all random paths started from \( x_0 \):

\[
p(x, t \mid x_0, t_0) = \int_{-\infty}^{\infty} \delta(x - x') p(x', t \mid x_0, t_0) \]

which expresses a simple fact that any continuous path from \( X(t_0) = x_0 \) to \( X(t) = x \) can be split at any intermediate time \( t' \) into two independent paths, from \( x_0 \) to \( x' \), and from \( x' \) to \( x \).

As a function of the arrival state \((x, t)\), the propagator satisfies the forward Fokker–Planck (FP) equation [52, 53]. We reproduce the derivation of this equation from [54], which relies on the evaluation of the integral

\[
I = \int_{-\infty}^{\infty} dx \, h(x) \left[ p(x, t + \delta \mid x_0, t_0) - p(x, t \mid x_0, t_0) \right]
\]

for any smooth function \( h(x) \) with compact support. One has

\[
I = \int_{-\infty}^{\infty} dx \, h(x) \int_{-\infty}^{\infty} dx' \, p(x, t + \delta \mid x', t) p(x', t \mid x_0, t_0)
- \int_{-\infty}^{\infty} dx' \, h(x') p(x', t \mid x_0, t_0) \int_{-\infty}^{\infty} dx \, p(x, t + \delta \mid x', t)
= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \, [p(x, t + \delta \mid x', t) p(x', t \mid x_0, t_0)] h(x) - h(x'),
\]

where the first term was represented using equation (7), while the normalization of the probability density \( p(x, t + \delta \mid x', t) \) allowed one to add the integral over \( x \) in the second term. Expanding \( h(x) \) into a Taylor series around \( x' \) and then exchanging the integration variables \( x \) and \( x' \), one gets

\[
I = \int_{-\infty}^{\infty} dx \, p(x, t \mid x_0, t_0) \sum_{n=1}^{\infty} \left( \frac{d^n}{dx^n} h(x) \right) \frac{1}{n!} \int_{-\infty}^{\infty} dx' \, p(x', t + \delta \mid x, t)(x' - x)^n.
\]

Finally, integrating each term by parts \( n \) times, dividing by \( \delta \) and taking the limit \( \delta \to 0 \) yield

\[
\int_{-\infty}^{\infty} dx \, h(x) \frac{\partial p(x, t \mid x_0, t_0)}{\partial t} = \int_{-\infty}^{\infty} dx \, h(x) \sum_{n=1}^{\infty} (-1)^n \frac{d^n}{dx^n} \left( D^{(n)}(x) p(x, t \mid x_0, t_0) \right),
\]

where the left-hand side is the limit of \( I/\delta \) as \( \delta \to 0 \), and

\[
D^{(n)}(x) = \frac{1}{n!} \lim_{\delta \to 0} \frac{1}{\delta} \int_{-\infty}^{\infty} dx' \, p(x', t + \delta \mid x, t)(x' - x)^n.
\]

Since the above integral relation is satisfied for the arbitrary function \( h(x) \), one deduces the so-called Kramers–Moyal expansion:
\[ \frac{\partial p(x, t \mid x_0, t_0)}{\partial t} = \sum_{n=1}^{\infty} (-1)^n \frac{\partial^n}{\partial x^n} \left( D^{(n)}(x)p(x, t \mid x_0, t_0) \right). \]  

(9)

Here, we assumed that the process is time homogeneous, i.e., \( p(x, t \mid x_0, t_0) = p(x, t + t' \mid x_0, t_0 + t') \), which implies the time-independence of \( D^{(n)}(x) \).

The density \( p(x', t + \delta \mid x, t) \) in equation (8) characterizes the displacement between \( X(t) = x \) and \( X(t + \delta) = x' \), which can be written as \( X(t + \delta) - X(t) \approx \delta [F(x) + \xi(t)] \) for small \( \delta \) according to the Langevin equation (2). After discretization in units of \( \delta \), the thermal force \( \xi(t) \) becomes a Gaussian variable with mean zero and variance \( \gamma \delta k_B T / \delta \). As a consequence, the displacement \( \delta x = x' - x \) is also a Gaussian variable with mean \( \langle \delta / \gamma \rangle F(x) \) and variance \( (\delta / \gamma)^2 2k_B T / \delta \), i.e.,

\[ p(x', t + \delta \mid x, t) = \frac{1}{\sqrt{4\pi \delta k_B T}} \exp \left( -\frac{(x' - x - F(x) \delta / \gamma)^2}{4\delta k_B T} \right) \]

for small \( \delta \). Substituting this density into equation (8) and evaluating Gaussian integrals, one gets \( D^{(1)}(x) = F(x)/\gamma \), \( D^{(2)} = D \), and \( D^{(n)} = 0 \) for \( n > 2 \) that yields the forward Fokker–Planck equation

\[ \frac{\partial p(x, t \mid x_0, t_0)}{\partial t} = \mathcal{L}_x p(x, t \mid x_0, t_0), \quad \mathcal{L}_x = -\frac{\partial}{\partial x} \frac{F(x)}{\gamma} + D \frac{\partial^2}{\partial x^2}, \]

(10)

where \( \mathcal{L}_x \) is the Fokker–Planck operator acting on the arrival point \( x \). This equation is completed by the initial condition \( p(x, t_0 \mid x_0, t_0) = \delta(x - x_0) \) at \( t = t_0 \), with a fixed starting point \( x_0 \). Note that the forward FP equation can be seen as the probability conservation law,

\[ \frac{\partial}{\partial t} p(x, t \mid x_0, t_0) = -\partial_x J(x, t \mid x_0, t_0), \]

where \( J(x, t \mid x_0, t_0) = \frac{F(x)}{\gamma} p(x, t \mid x_0, t_0) - D \partial_x p(x, t \mid x_0, t_0) \) is the probability flux. Setting \( J = 0 \), one can solves the first-order differential equation to retrieve the equilibrium solution \( R_{eq}(x) = Z w(x) \), where \( Z \) is the normalization factor, and

\[ w(x) = \exp \left( \int_0^x \frac{dx' F(x')}{k_B T} \right) = \exp \left( -\frac{V(x)}{k_B T} \right) = \exp \left( -\frac{kx^2}{2k_B T} + \frac{F_0 x}{k_B T} \right), \]

(11)

where \( V(x) = -\int_0^x dx' F(x') \) is the potential associated to the force \( F(x) \). This is the standard Boltzmann–Gibbs equilibrium distribution.

When the FP operator \( \mathcal{L}_x \) has a discrete spectrum, the probability density admits the spectral decomposition

\[ p(x, t \mid x_0, t_0) = \sum_{n=0}^{\infty} v_n(x) v_n(x_0) \tilde{w}(x_0) e^{-\lambda_n(t-t_0)} \]

(12)

over the eigenvalues \( \lambda_n \) and eigenfunctions \( v_n(x) \) of \( \mathcal{L}_x \):

\[ \mathcal{L}_x v_n(x) + \lambda_n v_n(x) = 0 \quad (n = 0, 1, 2, \ldots) \]

(13)

(eventually with appropriate boundary conditions; see below). The weight \( \tilde{w}(x) = 1/w(x) \) ensures the orthogonality of eigenfunctions:

\[ \int dx \tilde{w}(x) v_m(x) v_n(x) = \delta_{m,n}, \]

(14)
While the closure (or completeness) relation reads
\[ \sum_{n=0}^{\infty} \delta_n(x)v_n(x_0)\tilde{w}(x_0) = \delta(x - x_0). \] (15)

This relation implies the initial condition \( p(x, t_0 | x_0, t_0) = \delta(x - x_0) \). As for the Langevin equation, an extension to the multidimensional case is straightforward. In particular, the derivative \( \partial_t \) is replaced by the gradient operator, while \( \partial_t^2 \) becomes the Laplace operator \([53, 55]\).

### 2.3. Backward Fokker–Planck equation

The forward FP equation describes the evolution of the probability density \( p(x, t | x_0, t_0) \) from a given initial state (here, the starting point \( x_0 \) at time \( t_0 \)). Alternatively, if the particle is found at the arrival point \( x \) at time \( t \) (or, more generally, in a prescribed subset of states), one can interpret \( p(x, t | x_0, t_0) \) as the conditional probability density that the particle is started from \( x_0 \) at time \( t_0 \) knowing that it arrived at \( x \) at later time \( t \). As a function of \( x_0 \) and \( t_0 \), this probability density satisfies the backward Fokker–Planck (or Kolmogorov) equation \([53]\):

\[
\frac{\partial}{\partial t} p(x, t | x_0, t_0) = \mathcal{L}^*_{x_0} p(x, t | x_0, t_0),
\] (16)

where the backward FP operator \( \mathcal{L}^* \) is adjoint to the forward FP operator \( \mathcal{L} \) (i.e., \( \langle \mathcal{L} f, g \rangle = \langle f, \mathcal{L}^* g \rangle \) for any two functions \( f \) and \( g \) from an appropriate functional space). Equation (10) implies

\[
\mathcal{L}^*_{x_0} = \frac{F(x_0)}{\gamma} \partial_{x_0} + D\partial_{x_0}^2 = \frac{k}{\gamma} (\hat{x} - x_0)\partial_{x_0} + D\partial_{x_0}^2.
\] (17)

Note that this operator acts on the starting point \( x_0 \) while the minus sign in front of the time derivative reflects the backward time direction. Equation (16) is easily obtained by differentiating the Chapman–Kolmogorov equation (7) with respect to the intermediate time \( t' \).

The eigenvalues of both forward and backward FP operators are identical, while the eigenfunctions \( u_n(x) \) of the backward FP operator \( \mathcal{L}^* \) are simply \( u_n(x) = v_n(x)/w(x) \). As a consequence, one can rewrite the spectral decomposition (12) as

\[
p(x, t | x_0, t_0) = \sum_{n=0}^{\infty} u_n(x_0)u_n(x)w(x)e^{-\frac{k}{\gamma}(t-t_0)},
\] (18)

with the weight \( w(x) \) from equation (11). The eigenfunctions \( u_n(x) \) are as well orthogonal:

\[
\int dx w(x)u_m(x)u_n(x) = \delta_{m,n},
\] (19)

while the closure (or completeness) relation reads

\[
\sum_{n=0}^{\infty} u_n(x_0)u_n(x)w(x) = \delta(x - x_0).
\] (20)

This relation implies the terminal condition \( p(x, t | x_0, t) = \delta(x - x_0) \) at \( t_0 = t \). In contrast to equation (12), the weight \( w(x) \) in the spectral representation (18) depends on the fixed arrival point \( x \), while the backward FP operator \( \mathcal{L}^*_{x_0} \) acts on eigenfunctions \( u_n(x_0) \).
When there is no force term, the operator $\mathcal{L}$ is self-adjoint, $\mathcal{L} = \mathcal{L}^*$, and the probability density is invariant under time reversal: $p(x, t | x_0, t_0) = p(x_0, t_0 | x, t)$. This property does not hold in the presence of force.

Finally, the backward FP equation is closely related to the Feynman–Kac formula for determining distributions of various Wiener functionals [56–60]. For instance, we already mentioned that the probability density $p(x, t | x_0, t_0)$ can be understood as the conditional expectation:

$$u(x_0, t_0) = \left\{ \exp \left( - \int_{t_0}^t dt' U(X(t'), t') \right) \psi(X(t)) + \int_{t_0}^t dt' f(X(t'), t') \exp \left( - \int_{t_0}^{t'} dt'' U(X(t''), t'') \right) \right\}_{X(t_0) = x_0}$$

satisfies the backward FP equation

$$- \frac{\partial}{\partial t_0} u(x_0, t_0) = \mathcal{L}_{x_0}^* u(x_0, t_0) = U(x_0, t_0) u(x_0, t_0) + f(x_0, t_0),$$

subject to the terminal condition $u(x_0, t) = \psi(x_0)$ at a later time $t > t_0$.

### 2.4. First exit times

In this review, we study the random variable $\tau = \inf \{ t > 0 : |X(t)| > L \}$, i.e., the first exit time of the process $X(t)$ from an interval $[-L, L]$ when started from $x_0$ at $t_0 = 0$. The cumulative distribution function of $\tau$ is related to the survival probability $S(x_0, t) = P\{ \tau > t \}$ up to time $t$ of a particle that started from $x_0$. The notion of survival is associated with the disappearing of the particle that hit either endpoint, due to chemical reaction, permeation, adsorption, relaxation, annihilation, transformation or any other ‘killing’ mechanism. The survival probability $S(x_0, t)$ can be expressed through the probability density $p(x, t | x_0, 0)$ of moving from $x_0$ to $x$ in time $t$ without visiting the endpoints $\pm L$ during this motion. Alternatively, $p(x, t | x_0, 0)$ can be seen as the conditional probability density of starting from point $x_0$ at time $t_0 = 0$ under condition to be at $x$ at time $t$. This condition includes the survival up to time $t$, i.e., not visiting the endpoints $\pm L$. The probability density $p(x, t | x_0, 0)$ satisfies the backward FP equation with Dirichlet boundary condition at $x = \pm L$: $p(x, t, \pm L, 0) = 0$. This condition simply states that a particle started from either endpoint has immediately hit this endpoint, i.e. not survived. Note that this condition is preserved during all intermediate times $t'$ due to the Chapman–Kolmogorov equation (7).

Since the survival probability $S(x_0, t)$ ignores the actual position $x$ at time $t$, one just needs to average the density $p(x, t | x_0, 0)$ over $x$:

$$S(x_0, t) = \int_{-L}^{L} dx p(x, t | x_0, 0) = \sum_{n=0}^{\infty} u_n(x_0) e^{-\lambda_n t} \int_{-L}^{L} dx u_n(x) w(x),$$

where the spectral decomposition (18) was used. The eigenfunctions $u_n(x)$ of the backward FP operator should satisfy the Dirichlet boundary condition at $x = \pm L$: $u_n(\pm L) = 0$. Equation (23) also implies the backward FP equation.
\[ \frac{\partial S(x_0, t)}{\partial t} = \mathcal{L}^*_x S(x_0, t), \]  

(24)

which is completed by the initial condition \( S(x_0, 0) = 1 \) (the particle exists at the beginning) and Dirichlet boundary condition \( S(\pm L, t) = 0 \) (the process is stopped upon the first arrival at either endpoint of the confining interval \([ -L, L ]\)). Since the process is homogeneous in time, \( p(x, t \mid x_0, t_0) \) depends on \( t - t_0 \) and thus \( \partial p / \partial t_0 = -\partial p / \partial t \) that allows one to write the left-hand side of the backward FP equation (24) with the plus sign. Note that the characterization of first passage times through the backward FP equation goes back to the seminal work in 1933 by Pontryagin et al [61]. Similar equations emerge in quantum mechanics when one searches for eigenstates of a particle trapped by a short-range harmonic potential [62] (see also appendix D for the quantum harmonic oscillator).

The FET probability density is \( q(x_0, t) = \frac{\partial S(x_0, t)}{\partial x} \), while the moment-generating function is given by its Laplace transform:

\[ \langle e^{-st} \rangle = \int_0^\infty dt \ e^{-st} q(x_0, t) \equiv \tilde{q}(x_0, s), \]  

(25)

with tilde denoting Laplace-transformed quantities. The Laplace transform of equation (24) yields the equation \((\mathcal{L}^*_x - s) \tilde{S}(x_0, s) = -1 \) with Dirichlet boundary conditions. Since \( \tilde{q}(x_0, s) = 1 - s \tilde{S}(x_0, s) \), one gets

\[ (\mathcal{L}^*_x - s) \tilde{q}(x_0, s) = 0, \]  

(26)

with Dirichlet boundary condition \( \tilde{q}(\pm L, s) = 1 \).

Finally, the moments \( \langle \tau^m \rangle_{x_0} \) can be found in one of these standard ways:

(i) from the moment-generating function,

\[ \langle \tau^m \rangle_{x_0} = (-1)^m \lim_{s \to 0} \frac{\partial^m}{\partial s^m} \tilde{q}(x_0, s); \]  

(27)

(ii) from the spectral representation of the survival probability:

\[ \langle \tau^m \rangle_{x_0} = m! \sum_{n=0}^\infty u_n(x_0) \lambda_n^{-m} \int_{-L}^{L} dx \ u_n(x) w(x); \]  

(28)

(iii) from recurrence partial differential equations (PDEs)

\[ \mathcal{L}^*_x \{ \tau^m \rangle_{x_0} = -m \{ \tau^{m-1} \rangle_{x_0}, \]  

(29)

with Dirichlet boundary conditions [30].

In what follows, we focus on the mean exit time \( \langle \tau \rangle_{x_0} \), the moment-generating function \( \tilde{q}(x_0, s) \), and the survival probability \( S(x_0, t) \) for harmonically trapped particles.

2.5. Mean exit time

The mean exit times of diffusive processes were studied particularly well because of their practical importance and simpler mathematical analysis (see [1, 2, 63–65] and references therein). In fact, the mean exit time,
\[ \langle \tau \rangle_{x_0} = \int_0^\infty dt \ t q(x_0, t) = \int_0^\infty dt \ S(x_0, t), \]  
(30)
satisfies the simpler equation than the time-dependent PDE (16):
\[ \mathcal{L}^*_{x_0}(\tau)_{x_0} = \mathbf{-1}, \]  
(31)
with Dirichlet boundary conditions at \( x_0 = \pm L \). The double integration and imposed boundary conditions yield \([66]1\)
\[ \langle \tau \rangle_{x_0} = \frac{1}{D} \left\{ \left[ \int_{-L}^L \frac{dx}{w(x)} \right]^{-1} \int_{-L}^L \frac{dx}{w(x)} \int_0^L dx' w(x') \right\} \times \int_{-L}^{x_0} \frac{dx}{w(x)} - \int_{-L}^{x_0} \frac{dx}{w(x)} \int_0^x dx' w(x') \right\}. \]  
(32)
Substituting \( w(x) \) from equation (11), one gets
\[ \langle \tau \rangle_{x_0} = \frac{L^2 \sqrt{\pi}}{D} \kappa \left\{ \text{erf} \left( i \sqrt{\kappa} (x_0/L - \varphi) \right) \right\} \times \left\{ \int_{\sqrt{\kappa} (1-\varphi)}^{\sqrt{\kappa} (-1-\varphi)} dz \ e^{-z^2} \text{erf} (z) \right\} - \int_{\sqrt{\kappa} (-1-\varphi)}^{\sqrt{\kappa} (x_0/L-\varphi)} dz \ e^{-z^2} \text{erf} (z) \right\}, \]  
(33)
where \( \text{erf} \) is the error function, and \( \kappa \) and \( \varphi \) are two dimensionless parameters characterizing the trapping harmonic potential and the pulling constant force, respectively
\[ \kappa \equiv \frac{kL^2}{2k_B T}, \quad \varphi \equiv \frac{\hat{\kappa}}{L} = \frac{F_0}{kL}. \]  
(34)
Throughout the paper, we consider \( \varphi \geq 0 \), while all the results for \( \varphi < 0 \) can be obtained by replacing \( \varphi \rightarrow -\varphi \) and \( x_0 \rightarrow -x_0 \). For large \( \kappa \) or \( \varphi \), one can use an equivalent representation (A.1) provided in appendix A.1.
Several limiting cases are of interest:
- When \( \varphi = 0 \) (i.e., \( F_0 = 0 \)), equation (33) is reduced to
\[ \langle \tau \rangle_{x_0} = \frac{L^2 \sqrt{\pi}}{D} \kappa \int_{\sqrt{\kappa} (x_0/L)}^{\sqrt{\kappa} L} dz \ e^{-z^2} \text{erf} (z). \]  
(35)
- In the limit \( k \rightarrow 0 \), one gets a simpler expression
\[ \langle \tau \rangle_{x_0} = \frac{L^2}{D \eta} \left( 1 - x_0/L - 2 \frac{e^{-\eta x_0/L} - e^{-\eta}}{e^\eta - e^{-\eta}} \right), \]  
(36)
where \( \eta = F_0 L/(k_B T) \) is another dimensionless parameter. If \( F_0 = 0 \), one retrieves the classical result for Brownian motion:
\[ \langle \tau \rangle_{x_0} = \frac{L^2}{2D} \left( 1 - \left( \frac{x_0}{L} \right)^2 \right). \]  
(37)
\[^1\text{ In [66], the minus sign in front of } U(z) \text{ in the second integral in the numerator of the first term in equation (7.7) is missing.} \]
For small $\kappa$, the Taylor expansion of equation (33) yields

$$
\tau_\kappa \varphi_\kappa \approx - + - + \left( L x D x L x 0 0 2 1 12 3 \right). \quad (38)
$$

We emphasize that the limits $\kappa \to 0$ and $k \to 0$ are not equivalent because in the latter case, $\varphi \to \infty$ according to equation (34).

In the opposite limit of large $\kappa$, four cases can be distinguished (see appendix A.1):

- For small $\kappa$, the Taylor expansion of equation (33) yields

$$
\langle \tau \rangle_{x_0} \approx \frac{L^2 - x_0^2}{2D} \left( 1 + \kappa \frac{1 - 2\varphi(x_0/L) + (x_0/L)^2}{3} + O(\kappa^2) \right) \quad (38)
$$

and the exponential growth in the first two relations is valid for any $x_0$ not too close to $\pm L$.

Note that the limit of the second asymptotic relation (for $0 < \varphi < 1$) as $\varphi \to 0$ is different from the case $\varphi = 0$ by a factor of 2. In fact, when $\varphi > 0$, it is much more probable to reach the right endpoint than the left one, and $\langle \tau \rangle_{x_0}$ characterizes mainly the exit through the right endpoint at large $\kappa$. In turn, when $\varphi = 0$, both endpoints are equivalent, which doubles the chances to exit and thus reduces by a factor of 2 the mean exit time. Note that the first two relations (up to a numerical prefactor) can be obtained by the Kramers theory of escape [66, 67]. The last relation in equations (39) can be retrieved from the last line of equation (7.9) of [66].

The behavior of the mean exit time $\langle \tau \rangle_{x_0}$ as a function of the starting point $x_0$ is illustrated in figure 1. The increase of $\kappa$ at fixed $\varphi = 0$ transforms the spatial profile of the mean exit time from the parabolic shape (35) at $\kappa = 0$ to a $\Pi$-shape at large $\kappa$ (figure 1(a)). In other words, the dependence on the starting point becomes weak at large $\kappa$. At the same time, the
height of the profile rapidly grows with $\kappa$ according to equation (39). On the other hand, the spatial profile becomes more skewed and sensitive to the starting point as $\varphi$ increases at fixed $\kappa = 1$, while the height is decreasing (figure 1(b)). As expected, the constant force breaks the initial symmetry of the harmonic potential and facilitates the escape from the trap.

Figure 2 shows how the mean exit time $\langle \tau \rangle_0$ from the center varies with $\kappa$ and $\varphi$. When there is no constant force ($\varphi = 0$), one observes a rapid (exponential) growth at large $\kappa$, in agreement with equation (39) (shown by circles). The presence of a moderate constant force (with $0 < \varphi < 1$) slows down the increase of the mean exit time. For instance, at $\varphi = 0.5$, $\langle \tau \rangle_0$ exhibits a broad minimum at intermediate values of $\kappa$, but it resumes growing at larger values of $\kappa$. In turn, for $\varphi \geq 1$, there is no exponential growth with $\kappa$, and the mean exit time slowly decreases, as expected from equation (39). Since the constant force shifts the minimum of the harmonic potential from $0$ to $\hat{x} = F_0/k$, the border value $\varphi = 1$ corresponds to the minimum $\hat{x}$ at the exit position ($\hat{x} = L$). For $\varphi < 1$, the harmonic potential keeps the particle away from the exit and thus greatly increases the mean exit time. In turn, for $\varphi > 1$, the harmonic potential attracts the particle to $\hat{x}$, which is outside the interval $[-L, L]$ and thus speeds up the escape.

Although we considered the FET from a symmetric interval $[-L, L]$ for convenience, shifting the coordinate by $\hat{x}$ allows one to map the original problem to the FET from a nonsymmetric interval $[-a, b]$ with $a = L(1 + \varphi)$ and $b = L(1 - \varphi)$, with the starting point $x_0$ being shifted by $L_0$ to vary from $-a$ to $b$. As a consequence, the choice of the symmetric interval $[-L, L]$ is not restrictive, and all the results can be recast for a general interval $[-a, b]$ by shifts.

2.6. Survival probability

The survival probability is fully determined by the eigenvalues and eigenfunctions of the backward FP operator. The eigenvalue equation (13) reads

$$Du'' - (k/\gamma)(x - \hat{x})u' + \lambda u = 0.$$  

(40)
A general solution to this equation is well known [68]

\[ u(z) = c_1 M\left(-\frac{\alpha^2}{4\kappa}, \frac{1}{2}, \kappa (z - \varphi)^2\right) + c_2 (z - \varphi) M\left(-\frac{\alpha^2}{4\kappa} + \frac{1}{2}, \frac{3}{2}, \kappa (z - \varphi)^2\right), \]  
(41)

where \( z = x/L \) is the dimensionless coordinate, \( \lambda = D\alpha^2/L^2 \), \( c_1 \) and \( c_2 \) are arbitrary constants, and

\[ M(a, b, z) = \sum_{n=0}^{\infty} \frac{a^{(n)} z^n}{b^{(n)} n!} \]  
(42)

is the confluent hypergeometric function of the first kind (also known as the Kummer function), with \( a^{(0)} = 1 \) and \( a^{(n)} = \alpha (\alpha + 1) \cdots (\alpha + n - 1) = \Gamma(\alpha + n)/\Gamma(\alpha) \), where \( \Gamma(z) \) is the gamma function. The first and second terms in equation (41) are, respectively, the symmetric and antisymmetric functions with respect to \( \varphi \).

To shorten notations, we set

\[ m^{(1)}_{\alpha,\kappa}(z) \equiv \alpha M\left(-\frac{\alpha^2}{4\kappa}, \frac{1}{2}, \kappa z^2\right), \]  
(43)

\[ m^{(2)}_{\alpha,\kappa}(z) \equiv \varphi M\left(-\frac{\alpha^2}{4\kappa} + \frac{1}{2}, \frac{3}{2}, \kappa z^2\right), \]  
(44)

so that

\[ u(z) = c_1 m^{(1)}_{\alpha,\kappa}(z - \varphi) + c_2 m^{(2)}_{\alpha,\kappa}(z - \varphi). \]  
(45)

The Dirichlet boundary conditions read

\[ c_1 m^{(1)}_{\alpha,\kappa}(-1 - \varphi) + c_2 m^{(2)}_{\alpha,\kappa}(-1 - \varphi) = 0 \quad \text{(at } x_0 = -L), \]

\[ c_1 m^{(1)}_{\alpha,\kappa}(1 - \varphi) + c_2 m^{(2)}_{\alpha,\kappa}(1 - \varphi) = 0 \quad \text{(at } x_0 = L). \]

In the special case \( \varphi = 1 \), one gets \( c_1 = 0 \), and the eigenvalues are determined from the equation \( m^{(2)}_{\alpha,\kappa}(2) = 0 \). In general, for \( \varphi \neq 1 \), one considers the determinant of the underlying 2 × 2 matrix:

\[ D_{\alpha,\kappa,\varphi} = m^{(1)}_{\alpha,\kappa}(-1 - \varphi)m^{(2)}_{\alpha,\kappa}(1 - \varphi) - m^{(2)}_{\alpha,\kappa}(-1 - \varphi)m^{(1)}_{\alpha,\kappa}(1 - \varphi). \]  
(46)

Setting this determinant to 0 yields the equation on \( \alpha \):

\[ D_{\alpha,\kappa,\varphi} = 0, \]  
(47)

where \( \alpha_n \) (\( n = 0, 1, 2, \ldots \)) denote all positive solutions of this equation (for fixed \( \kappa \) and \( \varphi \)). The eigenfunctions then read

\[ u_n(z) = \frac{\beta_n}{\sqrt{L}} \left[ c^{(1)}_n m^{(1)}_{\alpha_n,\kappa}(z - \varphi) - c^{(2)}_n m^{(2)}_{\alpha_n,\kappa}(z - \varphi) \right]. \]  
(48)
where
\[ c_n^{(1)} = m_{a_n,k}^{(2)} (1 - \varphi), \quad c_n^{(2)} = m_{a_n,k}^{(1)} (1 - \varphi), \quad (49) \]
and the normalization constant is
\[ \beta_n^{-2} = \int_{-1}^{1} \varphi(x) e^{-\kappa \zeta^2} \left[ c_n^{(1)} m_{a_n,k}^{(1)} (z) - c_n^{(2)} m_{a_n,k}^{(2)} (z) \right]^2 \, dz. \quad (50) \]

Multiplying equation (40) by \( w(x) \) and integrating from \( a \) to \( b \), one obtains
\[ \int_a^b dx \, u(x) w(x) = \frac{D}{\lambda} [u'(a) w(a) - u'(b) w(b)]. \quad (51) \]
The derivative of the Kummer function can be expressed through Kummer functions, in particular,
\[ \partial_z m_{a_n,k}^{(1)} (z) = \frac{\alpha^2}{2 \kappa} \left( m_{a_n,k}^{(1)} (z) - \frac{m_{a_n,k}^{(1)} (z)}{\sqrt{\omega^2 - 4 \kappa \zeta^2}} \right), \quad (52) \]
\[ \partial_z m_{a_n,k}^{(2)} (z) = \left( 2 \kappa \zeta^2 - 1 \right) \frac{\alpha^2}{2 \kappa} m_{a_n,k}^{(2)} (z) + \left( 2 + \frac{\alpha^2}{2 \kappa} \right) m_{a_n,k}^{(3)} \left( \frac{\alpha}{2 \kappa} \zeta^2 \right), \quad (53) \]
from which one gets explicit formulas for \( u'_n(a) \) and \( u'_n(b) \) and thus for the integral in equation (51). We get therefore,
\[ S(x_0, t) = \sum_{n=0}^{\infty} w_n e^{-D \alpha_n^2 / L^2} \left[ c_n^{(1)} m_{a_n,k}^{(1)} (x_0 \varphi - \varphi) - c_n^{(2)} m_{a_n,k}^{(2)} (x_0 \varphi - \varphi) \right], \quad (54) \]
where
\[ w_n = \frac{\beta_n^{-2} e^{-\kappa}}{\alpha_n^2} [\nu(-1) - \nu(1)], \quad (55) \]
with
\[ \nu(z) = e^{2 \varphi \kappa \zeta} \left( c_n^{(1)} \partial_z m_{a_n,k}^{(1)} (\varphi - \varphi) - c_n^{(2)} \partial_z m_{a_n,k}^{(2)} (\varphi - \varphi) \right). \quad (56) \]

Taking the derivative with respect to time, one obtains the FET probability density
\[ q(x_0, t) = \frac{D}{L^2} \sum_{n=0}^{\infty} w_n \alpha_n^2 e^{-D \alpha_n^2 / L^2} \left[ c_n^{(1)} m_{a_n,k}^{(1)} (x_0 L - \varphi) - c_n^{(2)} m_{a_n,k}^{(2)} (x_0 L - \varphi) \right]. \quad (57) \]

In the limit \( \kappa \to 0 \), functions \( m_{a_n,k}^{(1)} (z) \) and \( m_{a_n,k}^{(2)} (z) \) approach \( \cos (\alpha z) \) and \( \sin (\alpha z) \), respectively, so that eigenfunctions from equation (48) become \( u_n(z) = \frac{\beta_n}{\sqrt{L}} \sin (\alpha (1 - z)) \), while the determinant in equation (46) is reduced to \( \sin (2 \alpha) \), from which \( \alpha_n = \pi (n + 1)/2 \).

In this limit, the dependence on \( \varphi \) vanishes, and one retrieves the classical result for Brownian motion
\[ S(x_0, t) = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n e^{-D \alpha_n^2 (n+1/2)^2 / L^2} \cos \left( \pi (n + 1/2) x_0 / L \right). \quad (58) \]

Only symmetric eigenfunctions with \( \alpha_n = \pi (n + 1/2) \) contribute to this expression.
For centered harmonic potential ($\varphi = 0$), equation (47) is reduced to
\[ m^{(1)}(1) m^{(2)}(1) = 0, \]
which determines two sequences of zeros: $\alpha_{n,1}$ from $m^{(1)}(1) = 0$, and $\alpha_{n,2}$ from $m^{(2)}(1) = 0$. Consequently, one can consider separately two sequences of symmetric and antisymmetric eigenfunctions: $m^{(1)}(z)$ and $m^{(2)}(z)$. According to equation (23), integration over arrival points removes all the terms containing antisymmetric eigenfunctions. This simpler situation is considered as a particular case in section 2.8.

Figure 3 illustrates the behavior of the probability density $q(x_0, t)$. For fixed $\varphi = 0$, an increase of $\kappa$ increases the mean exit time and makes the distribution wider. Note that the most probable FET remains almost constant. The opposite trend appears for variable $\varphi$ at fixed $\kappa = 1$: an increase of $\varphi$ diminishes the mean exit time and makes the distribution narrower. This is expected because a strong constant force would drive the particle to one exit and dominate over the stochastic part.

Figure 4 shows the dependence of the survival probability $S(x_0, t)$ on the starting point $x_0$. At short times, the survival probability is close to 1 independently of $x_0$, except for the close vicinity of the endpoints. As time increases, $S(x_0, t)$ is progressively attenuated. The spatial profile is symmetric for centered harmonic potential ($\varphi = 0$), and skewed to the left in

\[ \text{Figure 3. FET probability density } q(0, t) \text{ for several } \kappa \text{ at fixed } \varphi = 0 \text{ (a) and for several } \varphi \text{ at fixed } \kappa = 1 \text{ (b). The timescale } L^2/D \text{ is set to 1. The spectral decomposition (57) is truncated after 30 terms.} \]

\[ \text{Figure 4. Survival probability } S(x_0, t) \text{ as a function of the starting point } z_0 \text{ with } \kappa = 1, \text{ and } \varphi = 0 \text{ (a) and } \varphi = 0.9 \text{ (b). The timescale } L^2/D \text{ is set to 1. The spectral decomposition (54) is truncated after 30 terms.} \]
the presence of a positive constant force ($\varphi = 0.9$); reaching the right endpoint is more probable due to the drift caused by a constant force.

2.7. Moment-generating function

Since any linear combination of functions in equation (45) satisfies equation (26), with $s = -D\alpha^2/L^2$, one can easily find the moment-generating function $\hat{q}(x_0, s)$ by imposing the boundary condition $\hat{q}(\pm L, s) = 1$:

$$
\hat{q}(x_0, s) = \frac{A_{a, s, \varphi}^{(1)}}{D_{a, s, \varphi}} m_{a, x}^{(1)}(x_0/L - \varphi) + \frac{A_{a, s, \varphi}^{(2)}}{D_{a, s, \varphi}} m_{a, x}^{(2)}(x_0/L - \varphi),
$$

(60)

where

$$
A_{a, s, \varphi}^{(1)} = m_{a, x}^{(2)} (1 - \varphi) - m_{a, x}^{(2)} (-1 - \varphi),
$$

$$
A_{a, s, \varphi}^{(2)} = m_{a, x}^{(1)} (-1 - \varphi) - m_{a, x}^{(1)} (1 - \varphi),
$$

and $D_{a, s, \varphi}$ is defined by equation (46). Setting $a = L(1 + \varphi)$ and $b = L(1 - \varphi)$, one retrieves the moment-generating function of the FET of an Ornstein–Uhlenbeck process from an interval $[-a, b]$ reported in [39] (p. 548, 3.0.1), in which equation (60) is written more compactly in terms of the two-parametric family $S(\alpha, a, b)$ of parabolic cylinder functions (see appendix B.1). For the symmetric interval $[-a, a]$, a similar expression for the moment-generating function was provided in [30].

It is worth noting that the probability density $q(x_0, t)$ could be alternatively found by the inverse Laplace transform of equation (60). For this purpose, one determines the poles $s_n$ of $\hat{q}(x_0, s)$ in the complex plane which are given by zeros $\alpha_n$ of $D_{a, s, \varphi}$ according to equation (47). In other words, one has $s_n = -D\alpha_n^2/L^2$, and the residue theorem yields

$$
q(x_0, t) = \frac{4\pi D}{L^2} \sum_{n=0}^{\infty} e^{-D\alpha_n^2/2L^2} \left[ \frac{A_{a, s, \varphi}^{(1)}}{D'_{a, s, \varphi}} m_{a, x}^{(1)}(x_0/L - \varphi) + \frac{A_{a, s, \varphi}^{(2)}}{D'_{a, s, \varphi}} m_{a, x}^{(2)}(x_0/L - \varphi) \right],
$$

(61)

where $D'_{a, s, \varphi}$ denotes the derivative of $D_{a, s, \varphi}$ with respect to $s = -\alpha^2/(4\kappa)$. Comparing the above formula to equation (57), one gets another representation for coefficients $w_n$

$$
w_n = \frac{4\kappa}{\alpha_n^2 D'_{a, s, \varphi}},
$$

(62)

where we used the identity $c_{n}^{(1)} A_{a, s, \varphi}^{(2)} = -c_{n}^{(2)} A_{a, s, \varphi}^{(1)}$, with $c_{n}^{(1,2)}$ from equation (49). Two alternative representations (55) and (62) allow one to compute the normalization coefficients $\beta_n$ without numerical integration in equation (50).

2.8. Higher-dimensional case

In higher dimensions, we consider the FET of a multidimensional Ornstein–Uhlenbeck process from a ball of radius $L$. For centered harmonic potential (i.e., $F_0 = 0$), the derivation follows the same steps used earlier. In fact, the integration of the probability density $p(x, \delta x_0, 0)$ over the arrival point $x$ in the multidimensional version of equation (23) removes the angular dependence of the survival probability so that the eigenvalue equation is reduced to the radial part.
where \( d \) is the space dimension. In other words, we consider the FPT of the radial Ornstein–Uhlenbeck process to the level \( L \). In turn, the analysis for noncentered harmonic potential with \( F_0 \neq 0 \) is much more involved in higher dimensions due to angular dependence, and is beyond the scope of this review.

**Survival probability.** A solution of equation (63) is given by the Kummer function, which is regular at \( r = 0 \)

\[
u_n(r) = \frac{\beta_n}{L^{d/2}} M\left(-\frac{\alpha_n^2}{4\kappa}, \frac{d}{2}, \kappa \frac{r}{L}\right) \quad (n = 0, 1, 2, \ldots),
\]

where \( \beta_n \) is the normalization factor:

\[
\beta_n = l^n \int_0^1 dz z^{d-1} e^{-\kappa z^2} M\left(-\frac{\alpha_n^2}{4\kappa}, \frac{d}{2}, \kappa z^2\right)^2.
\]

The eigenvalues \( \lambda_n = D\alpha_n^2 L^2 \) are determined by the positive zeros \( \alpha_n \) of the equation

\[
M\left(-\frac{\alpha_n^2}{4\kappa}, \frac{d}{2}, \kappa\right) = 0.
\]

Repeating the same steps as in section 2.6 yields the spectral representation of the survival probability

\[
S(r_0, t) = \sum_{n=0}^{\infty} w_n e^{-D\alpha_n^2 t/2} M\left(-\frac{\alpha_n^2}{4\kappa}, \frac{d}{2}, \kappa \frac{r_0}{L}\right),
\]

where

\[
w_n = \frac{\beta_n^2 e^{-\kappa}}{2\kappa} M\left(-\frac{\alpha_n^2}{4\kappa} + 1, \frac{d}{2}, \kappa\right).
\]

and we used the identity

\[
\int_0^1 dz z^{d-1} e^{-\kappa z^2} M\left(-\frac{\alpha_n^2}{4\kappa}, \frac{d}{2}, \kappa z^2\right) = \frac{e^{-\kappa}}{2\kappa} M\left(-\frac{\alpha_n^2}{4\kappa} + 1, \frac{d}{2}, \kappa\right).
\]

The FET probability density is then

\[
q(r_0, t) = \frac{D}{L^2} \sum_{n=0}^{\infty} w_n \alpha_n^2 e^{-D\alpha_n^2 t/2} M\left(-\frac{\alpha_n^2}{4\kappa}, \frac{d}{2}, \kappa \frac{r_0}{L}\right).
\]

In the limit \( \kappa \to 0 \), one can use the identity (see appendix B.1)

\[
\lim_{\kappa \to 0} \frac{M\left(-\frac{\alpha_n^2}{4\kappa}, \frac{d}{2}, \kappa z^2\right)}{\kappa} = \frac{\Gamma(d/2) J_{d/2-1}(\alpha z)}{J_d(\alpha)} =
\begin{cases}
\frac{\cos(\alpha z)}{\alpha z} & (d = 1) \\
\frac{J_0(\alpha z)}{(\alpha z)^{d/2-1}} & (d = 2) \\
\frac{\sin(\alpha z)}{\alpha z} & (d = 3)
\end{cases}
\]

to retrieve the classical results for Brownian motion (here \( J_d(z) \) is the Bessel function of the first kind). In particular, one retrieves \( \alpha_n = \pi (n + 1/2) \) in one dimension and \( \alpha_n = \pi (n + 1) \) in two dimensions.
in three dimensions (with \( n = 0, 1, 2, \ldots \)). For the one-dimensional case, we retrieved only the zeros of symmetric eigenfunctions that contribute to the survival probability (cf discussion in section 2.6).

**Moment-generating function.** The moment-generating function, obeying equation (63) with \(-s\) instead of \(\lambda\), is

\[
\tilde{q}(0, s) = \frac{M\left(\frac{dL^2}{4\pi D}, \frac{d}{2}, \kappa \frac{a^2}{L^2}\right)}{M\left(\frac{dL^2}{4\pi D}, \frac{d}{2}, \kappa\right)},
\]

in agreement with [39] (p. 581, 2.0.1). This function satisfies the boundary condition \(\tilde{q}(L, s) = 1\) and is regular at \(n = 0\). The Laplace inversion of this expression yields another representation of the probability density

\[
\tilde{q}(n, t) = \frac{4\kappa D}{L^2} \sum_{n=0}^{\infty} e^{-Dn^2/L^2} \frac{M\left(-\frac{a^2}{4\kappa}, \frac{d}{2}, \kappa \frac{a^2}{L^2}\right)}{M\left(-\frac{a^2}{4\kappa}, \frac{d}{2}, \kappa\right)},
\]

where \( M'(a, b, z) \) denotes the derivative of \( M(a, b, z) \) with respect to \( a \). Comparing this relation to equation (70), the coefficients \(w_n\) from equation (68) can also be identified as

\[
w_n = \frac{4\kappa}{\alpha^n M'(a, b, z)}.
\]

As mentioned above, two expressions (68, 74) for \(w_n\) can be used to compute the normalization constants \(\beta_n\) without numerical integration in equation (65).

**Mean exit time.** Following the same steps as in section 2.5, one gets the mean exit time for the higher-dimensional case

\[
\langle \tau \rangle_0 = \frac{L^2}{D} \frac{1}{\kappa} \int_{\tau r_0/L}^{\infty} d\eta \int_0^{\eta} d\rho \int_0^{\rho} e^\frac{\rho}{\kappa} e^{-\frac{\rho^2}{2}},
\]

where we imposed the Dirichlet boundary condition at \(n = L\) and the regularity condition at \(n = 0\). In the limit \(\kappa \to 0\), one retrieves the classical result \(\langle \tau \rangle_0 = (L^2 - n_0^2)/(2dD)\). In the opposite limit \(\kappa \gg 1\), one gets

\[
\langle \tau \rangle_0 \approx \frac{L^2}{D} \frac{\Gamma(\frac{d}{2}) e^\frac{d}{\kappa}}{4\kappa^{\frac{1+d}{2}}} (\kappa \gg 1),
\]

which is applicable for any \(r_0\) not too close to \(L\). The behavior of the mean exit time for general spherically symmetric potentials is discussed in [66].

In appendix A.2, the asymptotic behavior of the smallest eigenvalue \(\lambda_0 = D\alpha^2/L^2\) is obtained:

\[
\lambda_0 \approx \frac{D}{L^2} \frac{4\kappa^{\frac{1+d/2}}}{\Gamma(\frac{d}{2})} e^{-\kappa} (\kappa \gg 1),
\]

which is just the inverse of the above asymptotic relation for the mean exit time. While the first eigenvalue exponentially decays with \(\kappa\), the other eigenvalues linearly grow with \(\kappa\) (see appendix A.2):
As a consequence, the gap between the lowest eigenvalue \( \lambda_0 \) and the next eigenvalue \( \lambda_1 \) grows linearly with \( \kappa \). For \( \lambda \gg \kappa \), the contribution of all excited eigenmodes becomes negligible as compared to the lowest mode, and the first exit time follows approximately an exponential law, \( \mathbb{P} \{ \tau > t \} \approx \exp(-t/\langle \tau \rangle) \), with the mean \( \langle \tau \rangle \) from equation (76).

We illustrate this behavior for the 3D case in figure 5(a), which presents the first three eigenvalues \( \lambda_n \) as functions of \( \kappa \). Note that the correction term to the asymptotic line for \( \lambda_3 \) is significant even for \( \kappa = 10 \) (see appendix A.2 for details). For comparison, figure 5(b) shows the first three eigenvalues for the exterior problem discussed in the next subsection.

2.9. Exterior problem

For the exterior problem, when the process is started outside the interval \([-L, L]\) (or outside the ball of radius \( L \) in higher dimensions), the FET is also referred to as the first passage time to the boundary of this domain: \( \tau = \inf \{ t > 0 : |X(t)| < L \} \). While the mean exit time and the probability distribution can be found in a very similar way (see below), their properties are very different from the earlier considered interior problem. For the sake of simplicity, we only consider the centered harmonic potential (i.e., \( F_0 = 0 \)), although the noncentered case in one dimension can be treated similarly.

In one dimension, the domain \((-\infty, -L) \cup (L, \infty)\) is split into two disjointed subdomains so that \( \tau \) is in fact the first passage time to a single barrier, either at \( x = L \) (if started from \( x_0 > L \)), or at \( x = -L \) (if started from \( x_0 < -L \)). This situation is described in appendix C.

Mean exit time. Following the steps of section 2.5, one obtains the mean exit time

\[
\langle \tau \rangle_0 = \frac{L^2}{D} \kappa \int_{\mathcal{D}} \eta_{n/L} \int_0^{\infty} dr_1 e^{r_1^2} \int_0^{\infty} dr_2 e^{-r_2^2},
\]

where we imposed the Dirichlet boundary condition at \( r_0 = L \) and the regularity condition at infinity.
For even dimensions $d$, the change of integration variables yields the explicit formula

$$
\langle \tau \rangle_n = \frac{L^2}{4D_\kappa} \left[ 2 \ln \left( \frac{n_0}{L} \right) + \sum_{j=1}^{d-1} \frac{\Gamma \left( \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - j \right)} \left( 1 - \left( \frac{n_0}{L} \right)^{-2j} \right) \right] \tag{80}
$$

(we use the convention that $\sum_{j=1}^{n} a_j$ is zero if $n < 1$). For instance, the mean exit time in two dimensions is particularly simple:

$$
\langle \tau \rangle_0 = \frac{L^2}{2D_\kappa} \ln \left( \frac{n_0}{L} \right) \quad (d = 2). \tag{81}
$$

For odd $d$, repeated integration by parts yields

$$
\langle \tau \rangle_0 = \frac{L^2}{4D_\kappa} \left[ 2 \sqrt{\pi} \int_{\pi z_0}^{\pi x} \, dz \, e^{z^2} \text{erfc} (z) + \sum_{j=1}^{d-1} \frac{\Gamma \left( \frac{d}{2} \right)}{\Gamma \left( \frac{d}{2} - j \right)} \frac{\Gamma \left( j + \frac{1}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} \left( 1 - \left( \frac{n_0}{L} \right)^{-2j} \right) \right] \tag{82}
$$

where $z_0 = n_0/L$, and $\text{erfc} (z) = 1 - \text{erf} (z)$. Note that for $d = 1$, all terms vanish except the integral.

For large $r_0$ or large $\kappa$, the leading asymptotic term is $\frac{L^2}{2D_\kappa} \ln \left( \frac{n_0}{L} \right)$ for all dimensions (for odd dimensions, this term comes from the integral). When $n_0/L$ approaches 1, the mean exit time vanishes as $(1 - \left( \frac{n_0}{L} \right))$, where the prefactor $c$ depends on $\kappa$ and $d$.

When $\kappa \to 0$, the mean exit time diverges:

$$
\langle \tau \rangle_n \approx \frac{L^2}{D} \frac{\Gamma \left( \frac{d}{2} \right)}{2(d-2)} \left( 1 - \left( \frac{n_0}{L} \right)^{2-d} \right)^{-d/2} \quad (d \neq 2) \tag{83}
$$

(for $d = 2$, see equation (81)). This divergence is expected because, for the exterior problem, the mean exit time for Brownian motion is infinite in all dimensions, irrespectively of its recurrent or transient character.

Finally, the mean exit time for non-centered harmonic potential (i.e., $F_0 \neq 0$) in one dimension reads for $x_0 > L$ as

$$
\langle \tau \rangle_{x_0} = \frac{L^2}{D} \frac{\sqrt{\pi}}{2\kappa} \int_{\pi (l-x_0)}^{\pi (x_0-L)} \, dz \, e^{z^2} \text{erfc} (z). \tag{84}
$$

In the limit of large $\kappa$, two asymptotic regimes are distinguished:

(i) when $\varphi < 1$, the upper and lower limits of integration go to infinity so that the mean exit time behaves as

$$
\langle \tau \rangle_{x_0} \approx \frac{L^2}{D} \frac{1}{2\kappa} \ln \left( \frac{x_0/L - \varphi}{1 - \varphi} \right); \tag{85}
$$
(ii) when \( \varphi > 1 \), the lower limit of integration goes to \(-\infty\), and the mean exit time exponentially diverges as
\[
\langle t \rangle_{\infty} \approx \frac{L^2 \sqrt{\pi} e^{(\varphi - 1)^2}}{D 2^{3/2} (\varphi - 1)}.
\] (86)

Both regimes are similar to that of the interior problem considered in section 2.5.

**Probability distribution.** The moment-generating function \( \tilde{q}(r_0, s) \) for the exterior problem satisfies the same equation (63), with \(-s\) instead of \( \lambda_n \), as \( \tilde{q}(r_0, s) \) from equation (72) for the interior problem. In order to ensure the regularity condition at infinity (as \( r_0 \to \infty \)), one replaces \( M(a, b, z) \) by the confluent hypergeometric function of the second kind (also known as Tricomi function):
\[
U(a, b, z) = \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} M(a, b, z) + \frac{\Gamma(b - 1)}{\Gamma(a)} z^{1-b} M(a - b + 1, 2 - b, z)
\] (87)
(for integer \( b \), this relation is undefined but can be extended by continuity; see appendix B.2). For \( a > 0 \), the function \( U(a, b, z) \) vanishes as \( z \to \infty \), in contrast to an exponential growth of \( M(a, b, z) \) according to equations (B.7) and (B.8). In turn, \( U(a, b, z) \) exhibits non-analytic behavior near \( z = 0 \), \( U(a, b, z) \approx \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} + \frac{\Gamma(b - 1)}{\Gamma(a)} z^{1-b} + \ldots \), that limited its use for the interior problem.

The moment-generating function for the exterior problem is then
\[
\tilde{q}(r_0, s) = \frac{U\left(\frac{sl^2}{4\kappa D + \frac{d}{2} \kappa}, \frac{r_0^2}{L^2}\right)}{U\left(\frac{sl^2}{4\kappa D + \frac{d}{2} \kappa}, \kappa\right)} \quad (r_0 \geq L),
\] (88)
in agreement with [39] (p. 581, 2.0.1).

Denoting by \( \alpha_n \) the positive zeros of the equation
\[
U\left(-\frac{\alpha_n^2}{4\kappa}, \frac{d}{2}, \kappa\right) = 0,
\] (89)
the inverse Laplace transform yields the FET probability density:
\[
q(r_0, t) = \frac{4\kappa D}{L^2} \sum_{n=0}^{\infty} e^{-\alpha_n^2 L^2} U\left(-\frac{\alpha_n^2}{4\kappa}, \frac{d}{2}, \kappa\right),
\] (90)
where \( U'(a, b, z) \) is the derivative with respect to \( a \). Its integral over time is the survival probability:
\[
S(r_0, t) = \sum_{n=0}^{\infty} \alpha_n e^{-\alpha_n^2 L^2} U\left(-\frac{\alpha_n^2}{4\kappa}, \frac{d}{2}, \kappa\right),
\] (91)
with
\[
\omega_n = \frac{4\kappa}{\alpha^2_n U \left( -\frac{\alpha^2_n}{4\kappa}, \frac{d}{2}, \kappa \right)}.
\]  

(92)

Alternatively, one can use the eigenvalues \( \lambda_n = D\alpha^2_n/L^2 \) and the corresponding eigenfunctions
\[
u_n(r) = \beta_n \frac{L^{d/2}}{L} U \left( -\frac{\alpha^2_n}{4\kappa}, \frac{d}{2}, \kappa (r/L)^2 \right),
\]

(93)

where \( \beta_n \) is the normalization factor:
\[
\beta_n = \frac{1}{\kappa} \int_1^\infty \left( 1 + e^{-\kappa z^2} \right) \left( e^{-\kappa z^2} \right)^2 \, dz.
\]

(94)

The normalization factors \( \beta_n \) diverge as \( \kappa \to 0 \).

Repeating the same steps as earlier, one retrieves the spectral representation (91) of the survival probability with
\[
\omega_n = \frac{\beta_n^2 e^{-\kappa}}{2\kappa} U \left( -\frac{\alpha^2_n}{4\kappa} + 1, \frac{d}{2}, \kappa \right).
\]

(95)

The asymptotic behavior of eigenvalues as \( \kappa \to 0 \) is discussed in appendix A.3.

2.10. Similarities and distinctions

In spite of the apparent similarities between the interior and the exterior problems, there is a significant difference in the spectral properties of the two problems. This difference becomes particularly clear in the limit \( \kappa \to 0 \) when the harmonic potential is switched off (see figure 5). For the interior problem, the spectrum remains discrete and continuously approaches to the spectrum of the radial Laplacian. In this limit, one retrieves the classical results for Brownian motion (e.g., \( \alpha_n \to \pi (n + 1)/2 \) as \( \kappa \to 0 \) in one dimension). In turn, the Laplace operator for the exterior problem has a continuum spectrum so that the continuous transition from discrete to continuum spectrum as \( \kappa \to 0 \) is prohibited. In particular, all eigenvalues \( \lambda_n \) vanish as \( \kappa \to 0 \) (see appendix A.3). In other words, the spectral properties for infinitely small \( \kappa > 0 \) and \( \kappa = 0 \) are drastically different. One can see that the asymptotic behavior of the eigenvalues \( \lambda_n \) is quite different for the interior and the exterior problems.

3. Discussion

In this section, we discuss computational hints for confluent hypergeometric functions (section 3.1), three applications in biophysics and finance (section 3.2, 3.3 and 3.4), relation to the distribution of first crossing times of a moving boundary by Brownian motion (section 3.5), diffusion under quadratic double-well potential (section 3.6), and further extensions (section 3.7).

3.1. Computational hints

The probability distribution of first exit times involves confluent hypergeometric functions \( M(a, b, z) \) (for interior problem) and \( U(a, b, z) \) (for exterior problem). For instance, the
The eigenvalues of the Fokker–Planck operator are obtained through zeros \( \alpha_n \) of the equation

\[
\kappa - \alpha_n \kappa M(a, b, z) = 0
\]

or similar. As a consequence, one needs to compute these functions for large \( |a|/\kappa \). Although the series in the definition (42) of \( M(a, b, z) \) converges for all \( z \), numerical summation becomes inaccurate for large \( |a| \), and other representations of confluent hypergeometric functions are needed. In appendix B.2, we discuss an efficient numerical scheme for rapid and accurate computation of \( M(a, b, z) \) for large \( |a| \) and moderate \( z \) that relies on the expansion (B.14). Moreover, we show that this scheme is appropriate for computing the derivative of \( M(a, b, z) \) with respect to \( a \) that appears in equation (73) or similar after the inverse Laplace transform.

For noninteger \( b \), the Tricomi function \( U(a, b, z) \) is expressed through \( M(a, b, z) \) by equation (87) that allows one to apply the same numerical scheme for the exterior problem in odd dimensions \( d \). Although the Tricomi function for integer \( b \) can be obtained by continuation, the derivation of its rapidly converging representation is more subtle. In practice, one can compute \( U(a, b, z) \) for an integer \( b \) by the extrapolation of a sequence \( U(a, h, z) \) computed for noninteger \( h \) approaching \( b \) as \( \epsilon \to 0 \).

In the case of large \( z \) and moderate \( |a|/\kappa \), one can use integral representations of \( M(a, b, z) \) and \( U(a, b, z) \) (see appendix B.2). The same algorithms can also be applied to compute parabolic cylinder function \( D(z) \) and Whittaker functions (see appendix B.1). The MATLAB code for computing both Kummer and Tricomi functions is available.

3.2. Single-particle tracking

The Langevin equation (2) can describe the thermal motion of a small tracer in a viscous medium. The Hookean force \(-kX(t)\) incorporates the harmonic potential of an optical tweezer which is used to trap the tracer in a specific region of the medium [69]. Optical trapping strongly diminishes the region accessible to the tracer and thus enables reduction of the field of view and increase of the acquisition rate up to few MHz [70–74]. At the same time, trapping affects the intrinsic dynamics of the tracer and may screen or fully remove its features at long times. The choice of the stiffness \( k \) is therefore a compromise between the risk of losing the tracer from the field of view (too small \( k \)) and risk of suppressing important dynamical features (too large \( k \)). The FET statistics can then be used for estimating the appropriate stiffness due to a quantitative characterization of escape events. For instance, one can choose the stiffness to ensure that the mean exit time strongly exceeds the duration of experiment, or that the escape probability is below a prescribed threshold.

Another interesting option consists of detecting events in which a constant force is applied to the tracer. In living cells, such events can mimic the action of motor proteins that attach to the tracer and pull it in one direction [75–79]. The presence of a constant force facilitates the escape from the optical trap, while higher fraction of escape events (as compared to the case without constant force) can be an indicator of such active transport mechanisms.

Originally, the idea of fast escape in the case of comparable Hookean and external forces was used to estimate the force generated by a single protein motor [75]. A ‘trap and escape’ experiment consisted in trapping a single organelle moving along microtubules at strong stiffness and then gradually reducing it until the organelle escapes the trap. Repeating such measurements, one can estimate the ‘escape power’ \( kL \) as a measure of the driving force \( F_0 \) when \( \phi = F_0/(kL) \sim 1 \), where \( L \) is the size of the trap. In this way, the driving force generated by a single (presumably dynein-like) motor was estimated to be 2.6 pN [75].

\[\text{See http://pmc.polytechnique.fr/pagesperso/dg/confluent/confluent.html}\]
approximate but direct way of force measuring relies on the drastic change in the mean exit time behavior at $\phi = 1$ according to equation (39).

Interestingly, these mechanisms can even be detected from a single trajectory. When there is no constant force, the mean-square displacement (MSD) of a trapped tracer, $\langle \Delta X^2 \rangle$, is known to approach the constant level $k B T$. In other words, the square root of the long-time asymptotic MSD determines the typical size $\tau_0 = \ell k T / k$, i.e., the dimensionless parameter $\kappa$ can be interpreted as the squared ratio between the exit distance $L$ and the characteristic size of the trap $\ell_k$. The above analysis showed that a tracer can rapidly reach levels below or slightly above $\ell_k$. However, significantly longer explorations are extremely improbable. In fact, according to equation (76), the mean exit time for $\kappa \gg 1$ is

$$
\langle \tau \rangle_0 \approx \tau_0 \frac{\Gamma (d/2)e^x}{2x^{d/2}} \quad (\kappa \gg 1),
$$

where we set $L = q\ell_k = \sqrt{k D\ell_k}$. For large enough $t$ (i.e., $t \gg \tau_0$), the contributions of all excited eigenstates vanish, and the survival probability exhibits a mono-exponential decay: $S(0, t) \approx \exp (-t/\langle \tau \rangle_0)$, where we replaced the smallest eigenvalue $\lambda_0$ by $1/\langle \tau \rangle_0$ for $\kappa \gg 1$ according to equation (77), while $w_0 \approx 1$ as shown in appendix A.2. In the intermediate regime $\tau_0 \ll t \ll \langle \tau \rangle_0$, the survival probability therefore remains close to 1.

A constant force $F_0$ pulling the tracer from the optical trap strongly affects the mean exit time and the survival probability. The dimensionless parameter $\phi$ from equation (34) is the ratio between the new stationary position $\hat{x}$ of the trajectory and the exit level $\kappa = L / \ell_k$:

$$
\phi = \frac{F_0}{kL} = \frac{\hat{x}}{\ell_k} = \frac{F_0\sqrt{2D\tau_k}}{2k_B T \sqrt{k}}.
$$

For large $\phi$, the mean exit time can be approximated according to equation (39) as $\langle \tau \rangle_0 \approx \tau_k \ln \frac{1}{1 - \phi / \phi_0} \approx \tau_k / \phi$, i.e., it becomes smaller than $\tau_k$, and much smaller than the mean exit time from equation (96) without force. As expected, exit events would be observed much more often in the presence of strong constant force.

For a long acquired trajectory, one can characterize how often different levels are reached. Strong deviations from the expected statistics (given by the survival probability)
would suggest the presence of a constant force. To illustrate this idea, we simulate the thermal motion of a spherical tracer of radius \( a = 1 \mu m \) submerged in water and trapped by an optical tweezer with a typical stiffness constant \( k = 10^{-6} \) N m\(^{-1}\) [73, 74]. The Stokes relation implies \( \gamma = 6\pi a \eta_0 \approx 1.88 \cdot 10^{-8} \) kg s\(^{-1}\), from which the diffusion coefficient is \( D = k_B T / \gamma \approx 2.20 \cdot 10^{-13} \) m\(^2\) s\(^{-1}\) at \( T = 300 \) K (with \( \eta_0 \approx 10^{-3} \) kg m\(^{-1}\) s\(^{-1}\) being the water viscosity). The characteristic trapping time is \( \tau_I = \gamma / k \approx 18.8 \) ms, while the confinement length is \( \ell_I = \sqrt{2k_B T / k} \approx 91 \) nm. Figure 6(a) shows one simulated trajectory of the tracer. According to equation (33), the mean exit times from the intervals \((-\ell_I, \ell_I)\) and \((-2\ell_I, 2\ell_I)\) are 27.2 ms and 517 ms, respectively. For a generated sample of duration 1 s, one observes multiple crossings of levels \( \pm \ell_I \) and only a few crossings of levels \( \pm 2\ell_I \). For comparison, we generated another trajectory for which a constant force \( F_0 = 0.2 \) pN (yielding \( \varphi = 2.20 / \sqrt{k} \)) is applied between 0.3 s and 0.5 s (figure 6(b)). Since motor proteins exert forces that are typically tenfold higher [75, 76], their effect is expected to be much stronger and thus easier to detect. The constant force reduces the mean exit times to 9.8 ms and 28 ms, i.e., by factors 2.8 and 18, respectively. As expected, once the constant force is applied, the tracer tends to reach the new stationary level \( \hat{x} = 200 \) nm so that the trajectory crosses the level \( \ell_I \) and remains above this level for whole duration of the forced period. Once the force is switched off, the trajectory returns to its initial regime with zero mean. One can see that the use of FET statistics presents a promising perspective for the design and analysis of single-particle tracking experiments, while Bayesian techniques can be further applied to get more reliable results [82, 83]. Note that the FPT statistics have also been suggested as robust estimators of diffusion characteristics [84] (see also [26]). Another method relying on the time evolution of the tracer probability distribution was proposed for simultaneously extracting the restoring-force constant and diffusion coefficient [85].

At the same time, we emphasize that this perspective needs further analysis. First, we focused on normal diffusion in a harmonic potential while numerous single-particle tracking experiments evidenced anomalous diffusion in living cells and polymer solutions [73, 86–89]. Several theoretical models have been developed to describe anomalous processes such as continuous-time random walks (CTRW), fractional Brownian motion (fBm), and generalized Langevin equation [5, 6, 78, 79]. While an extension of the presented results is rather straightforward for CTRW (section 3.7), the FPT problems for non-Markovian fBm or generalized Langevin equation are challenging due to the lack of an equivalent Fokker–Planck formulation. Second, the quadratic profile is an accurate approximation for optical trapping potential only for moderate deviations from the center of the laser beam [69], while the spatial profile can be more complicated for strong deviations. In other words, an accurate description of the tracer escape may require more sophisticated analysis. Finally, the inference of constant forces from a single trajectory may present some statistical challenges because different escape events can be correlated.

### 3.3. Adhesion bond dissociation under mechanical stress

We briefly mention another biophysical example of bond dissociation. Adhesion between cells or of cells to surfaces is mediated by weak noncovalent interactions. While a reversible bond between two molecules can break spontaneously (due to thermal fluctuations), an external force is needed to rupture the multiple bonds linking two cells together [90]. The dynamics of bond rupture can be seen as the first exit time problem in which exit or escape occurs when the intermolecular distance exceeds an effective interaction radius. Bell suggested application of the kinetic theory of the strength of solids to describe the lifetime of a bond (i.e., the mean exit time) as
where \( E_b \) is the bond energy, \( r_b \) is the range of the minimum of the binding free energy, \( F_0 \) is the applied external force per bond, and \( t_0 \) is the lifetime at the critical force \( E_b/r_b \) at which the minimum of the free energy vanishes \([90]\). This relation became a canonical description of adhesion bond dissociation under force.

If the binding potential can be approximated as quadratic, then the lifetime of a bond is precisely the mean exit time \( \langle \tau \rangle \) of a harmonically trapped particle. In that case, the second asymptotic relation in equation (39) implies the quadratic dependence on the force,

\[
\langle \tau \rangle \sim e^{(1-\varphi^2)},
\]

where \( \varphi = F_0/(k_b) \), and \( \kappa = E_b/(k_B T) = kr_b^2/(2k_B T) \). In other words, equation (98) is retrieved only for weak forces when the quadratic term \( \varphi^2 \) can be neglected. However, in the regime where the bond is most likely to break, the applied force is large, and the mean exit time may have completely different asymptotics (see, e.g., the last line of equations (39) for \( \varphi > 1 \)). This discrepancy was already outlined in [91], in which the cases of a harmonic potential and an inverse power law attraction were discussed, and in [92], which presented a molecular dynamics study of unbinding and the related analysis of first exit times. Other effects such as the dependence of the bond strength and survival time on the loading rate, were investigated both theoretically and experimentally (see [91–96] and references therein).

### 3.4. Algorithmic trading

Algorithmic trading is another field for applications of FETs. In algorithmic trading, a set of trading rules is developed in order to anticipate the next price variation of an asset from its earlier (historical) prices \([97]\). Although the next price is random (and thus unpredictable), one aims to catch some global or local trends which can be induced by collective behavior of multiple traders or macroeconomic tendencies \([98–100]\). Many trading strategies rely on the exponential moving average \( \bar{p}_n \) of the earlier prices \( p_k \) \([101–104]\)

\[
\bar{p}_n = \lambda \sum_{k=0}^{\infty} (1 - \lambda)^k p_{n-k},
\]

where \( 0 < \lambda \leq 1 \) characterizes how fast the exponential weights of more distant prices decay. The difference between the current price \( p_n \) and the ‘anticipated’ average price \( \bar{p}_n \),

\[
\delta_n \equiv p_n - \bar{p}_n = (1 - \lambda) \sum_{k=0}^{\infty} (1 - \lambda)^k r_{n-k}, \quad (r_n = p_n - p_{n-1}),
\]

can be seen as an indicator of a new trend. For independent Gaussian price variations \( r_n \), writing \( \delta_{n+1} = (1 - \lambda)\delta_n + (1 - \lambda)r_{n+1}, \) one retrieves equation (5) for a discrete version of an Ornstein–Uhlenbeck process, where \( \lambda = \mu (1 - \lambda)/\lambda \) is related to the mean price variation \( \mu, \theta = -\ln (1 - \lambda), \) and \( \sigma = \sigma_0 \sqrt{\frac{(1 - \lambda)}{\sqrt{1 - (1 - \lambda)^2}}} \) is proportional to the standard deviation (volatility) \( \sigma_0 \) of price variations.

The indicator \( \delta_n \) can be used in both mean-reverting and trend-following strategies. In the mean-reverting frame, if \( \delta_n \) exceeds a prescribed threshold \( L \), this is a trigger to sell the asset at its actual (high) price, in anticipation of its return to the expected (lower) level \( \bar{p}_n \) in the near future. Similarly, the event \( \delta_n < -L \) triggers buying the asset. In the opposite trend-following frame, the condition \( \delta_n > L \) is interpreted as the beginning of a strong trend, and is thus the signal to buy the asset at its actual price, in anticipation of its further growth (similarly for \( \delta_n < -L \)). In other words, the same condition \( \delta_n > L \) (or \( \delta_n < -L \)) can be interpreted
differently depending on the empirical knowledge on the asset behavior. Whatever the strategy is used, the statistics of crossings of the prescribed levels $\pm L$ is precisely the FET problem. Theoretical results in section 2 can help to characterize durations between buying and selling moments. In particular, the choice of the threshold $L$ is a compromise between execution of too frequent buying/selling transactions (i.e., higher transaction costs) at small $L$ and missing intermediate trends (i.e. smaller profits) at large $L$. We also note that Ornstein–Uhlenbeck processes often appear in finance to model, e.g., interest rates (Vasicek model) and currency exchange rates [105, 106]. A general frame of using eigenfunctions for pricing options is discussed in [107].

### 3.5. First crossing of a moving boundary by Brownian motion

The first exit time problem can be extended to time-evolving domains [108–112]. For instance, one can investigate the first passage time of Brownian motion to a time-dependent barrier $L(t)$, $\tau = \inf \{ t > 0 : X(t) = L(t) \}$, or the first exit time from a symmetric ‘envelope’ $[-L(t), L(t)]$, $\tau = \inf \{ t > 0 : |X(t)| = L(t) \}$. Although the survival probability $S(x_0, t)$ satisfies the standard diffusion equation with the Dirichlet boundary condition, the boundary $L(t)$ evolves with time. For a smooth $L(t)$, setting $S(x_0, t) = v(z, t)$ with a new space variable $z = x_0/L(t)$ yields

$$\frac{\partial v(z, t)}{\partial t} = \frac{D(L(t)^2)}{L(t)^2} \frac{d^2 v(z, t)}{d z^2} - \frac{L'(t)}{L(t)} z d v(z, t),$$

with Dirichlet boundary condition $v(\pm 1, t) = 0$ at two fixed endpoints (here we focus on the exit time). Setting a new time variable $T = \ln (L(t)/L(0))$, the above equation can also be written as the backward Fokker–Planck equation with time-dependent diffusion coefficient $D(T) = D.L(t)^{-1} = D e^{t}$ and a centered harmonic potential:

$$\frac{\partial v(z, T)}{\partial T} = D(T) \frac{d^2 v(z, T)}{d z^2} - z d v(z, T).$$

In higher dimensions, the second derivative $d^2$ is simply replaced by the radial Laplace operator $\frac{\partial^2}{\partial r^2} + \frac{d-1}{r} \frac{\partial}{\partial r}$.

In general, the above equation does not admit explicit solutions. A notable exception is the case of square-root boundaries, which have been thoroughly investigated [113–118]. In fact, when $L(t) = t + t_0$ (with $b > 0$ and $t_0 > 0$), one has $L'(t)L(t) = b$ so that $D(T)$ is independent of $T$ (or $t$). In other words, one retrieves the backward Fokker–Planck problem (17, 24) with $\tilde{\zeta} = 0, \tilde{\gamma} = 1, L = 1$, and $D$ replaced by $b/D$. Its exact solution is given by equation (73) for the $d$-dimensional case:

$$q(z_0, T) = 2 \sum_{n=0}^{\infty} e^{-2 n_t} \frac{M\left(-\nu_n, \frac{d}{2}, \frac{b z_0^2}{2D} \right)}{M'\left(-\nu_n, \frac{d}{2}, \frac{b}{2D} \right)},$$

where $z_0 = n_0/L(0) = n_0/\sqrt{2b t_0}$ denotes the rescaled starting point $r_0$, and $\nu_n = \alpha_n^2/(4\kappa)$ are zeros of $M\left(-\nu, \frac{d}{2}, \frac{b}{2D} \right) = 0$. Changing back $T$ to $t$, one gets

$$p(z_0, t) = \frac{1}{t_0} \sum_{n=0}^{\infty} \left( 1 + t/t_0 \right)^{-n_t-1} \frac{M\left(-\nu_n, \frac{d}{2}, \frac{b z_0^2}{2D} \right)}{M'\left(-\nu_n, \frac{d}{2}, \frac{b}{2D} \right)}.$$
This expression in a slightly different form was provided for \( d = 1 \) in [118]. Note also that the probability \( \mathbb{P}\{\sup_{t \leq T}\{W_t/t\} < c\} \) admits a similar expansion [119].

In addition, equation (72) yields

\[
t_0^{-\nu} \langle (\tau + t_0)^\nu \rangle = \langle e^{2\nu T} \rangle = \tilde{q}(z_0, -2\nu) = \frac{M(-\nu, \frac{d}{2}, \frac{b}{2\nu})}{M(-\nu, \frac{d}{2}, \frac{b}{2\nu})},
\]

from which one retrieves

\[
\langle (\tau + t_0)^\nu \rangle = \frac{t_0^\nu}{M(-\nu, \frac{d}{2}, \frac{b}{2\nu})} \quad \text{(at } z_0 = 0)\),
\]

which was reported for \( d = 1 \) in [114]. Note that the \( \nu \)th moment exists under the condition \( \Re\{\nu\} < \nu_0 \), as is clearly seen from equation (104). In the special case \( b = D \), the square-root boundary \( \sqrt{\langle W_t^2 \rangle} = \sqrt{2Dt} \) grows in the same way as the root mean square of Brownian motion \( \langle W_t^2 \rangle = \langle WD t \rangle \). Since \( \nu_0 = 1 \) at \( b = D \), the mean exit time is infinite. More generally, the mean exit time is infinite for broader envelopes (\( b \geq D \)) and finite for narrower envelopes (\( b < D \)), as expected. The shift \( t_0 \) plays a minor role of a time scale.

3.6. Quadratic double-well potential

The above spectral approach can be extended to more complicated trapping potentials. As an example, we briefly describe diffusion under double-well (or bistable) piecewise quadratic potential:

\[
V(x) = \begin{cases} 
\frac{1}{2} k_1 (x + x_1)^2, & x \leq 0, \\
\frac{1}{2} k_2 (x - x_2)^2 + v_0, & x \geq 0,
\end{cases}
\]

where two minima are located at \(-x_1\) and \(x_2\) (with \(x_1 > 0\) and \(x_2 > 0\)), \(k_1\) and \(k_2\) are two spring constants, and \(v_0 = \frac{1}{2}(k_1 x_1^2 - k_2 x_2^2)\) is a constant ensuring the continuity of the potential at \(x = 0\). The resulting Langevin equation remains linear, in contrast to other bistable potentials such as a quartic potential (e.g., \(V(x) = ax^4 + bx^2 + cx\)). The diffusive dynamics under double-well potentials were thoroughly investigated by using general theoretical tools (e.g. Kramers’ theory [66, 67] or WKB approximation [120–122]) and exactly solvable models (see [123–129] and references therein).

For each semi-axis, an eigenfunction satisfies equation (40) with the proper \( k_i \). However, neither the Kummer, nor the Tricomi functions are appropriate to represent the solution in this case. In fact, the Kummer function \( M(a, 1/2, z^2) \) rapidly grows at infinity, while the Tricomi function \( U(a, 1/2, z^2) \) behaves as \( \frac{\sqrt{\pi}}{\Gamma(a + 1/2)} - \frac{2\sqrt{\pi}}{\Gamma(a)\Gamma(1/2)}|z| + \ldots \) for small \( z \), i.e., its derivative is discontinuous at 0. A convenient representation can still be obtained as a linear combination of two Kummer functions, in which the rapid growth of these functions is compensated. This is precisely the case of parabolic cylinder functions \( D_{\nu}(z) \) and \( D_\nu(-z) \) which vanish as \( z \to \infty \) (resp., \( z \to -\infty \)) but rapidly grow as \( z \to -\infty \) (resp. \( z \to \infty \)) unless \( \nu \) is a non-negative integer [see equations (B.3), (D.2) and (D.3)]. An eigenfunction can therefore be written as
where \( \kappa_i = \frac{k_i x_i^2}{2k_B T} \), \( \nu_i = \frac{\lambda x_i^2}{2(\kappa_i D)} \), and \( \lambda, \, c_1, \, c_2 \) are determined by normalization and two interface conditions at \( x = 0 \). The continuity of the eigenfunction at \( x = 0 \) can be satisfied by choosing

\[
\begin{align*}
    c_1 &= \beta e^{\kappa_2} D_{x_2} \left(-\sqrt{2\kappa_2}\right), \\
    c_2 &= \beta e^{\kappa_1} D_{x_1} \left(-\sqrt{2\kappa_1}\right),
\end{align*}
\]

where \( \beta \) is a normalization constant.

The second interface condition is deduced from the orthogonality of eigenfunctions with two weights \( \nu_{1,2} \) from equation (11) for positive and negative semi-axes, and

\[
   w_i(x) = \exp \left( \kappa_i \left[ 1 - \left(\frac{x}{x_i} \pm 1\right)^2 \right] \right),
\]

where plus (resp., minus) corresponds to \( i = 1 \) (resp., \( i = 2 \)). The orthogonality imposes the interface condition

\[
   D w_1(0) u^- (0^-) - D w_2(0) u^+ (0^+) = 0,
\]

where the same diffusion coefficient \( D \) is assumed on both sides. Since \( w_i(0) = 1 \), one retrieves the standard flux continuity equation, \( u^-(0^-) = u^+(0^+) \), yielding an equation determining the eigenvalues \( \lambda \):

\[
   \begin{align*}
   & x_1 D_{x_1} \left(-\sqrt{2\kappa_1}\right) \left[ 2\kappa_1 D_{x_1} \left(-\sqrt{2\kappa_1}\right) + \sqrt{2}\kappa_2 D_{x_1} \left(-\sqrt{2\kappa_2}\right) \right] \\
   & + x_2 D_{x_2} \left(-\sqrt{2\kappa_2}\right) \left[ 2\kappa_2 D_{x_2} \left(-\sqrt{2\kappa_2}\right) + \sqrt{2}\kappa_1 D_{x_2} \left(-\sqrt{2\kappa_1}\right) \right] = 0,
\end{align*}
\]

where we used the identity \( \frac{\partial}{\partial z} D_k(z) = \frac{z}{2} D_k(z) - D_k(z) \), and \( \lambda \) appears in \( \nu_i = \lambda x_i^2 / (2\kappa_i D) \). The smallest eigenvalue \( \lambda = 0 \) corresponds to the steady state.

The normalization constant \( \beta \) is found according to

\[
   \beta^{-2} = e^{\kappa_1 + \kappa_2} \left[ x_1 D_{x_1} \left(-\sqrt{2\kappa_1}\right) \int_{-\sqrt{2\kappa_1}}^{\sqrt{2\kappa_1}} dz \, D_{x_1}^2(z) \right]
   + \frac{x_2 D_{x_2} \left(-\sqrt{2\kappa_2}\right)}{\sqrt{2\kappa_2}} \int_{-\sqrt{2\kappa_2}}^{\sqrt{2\kappa_2}} dz \, D_{x_2}^2(z),
\]

in which both integrals can be partly computed by using the identity [130]

\[
   \int_0^\infty dz \, D_k^2(z) = \sqrt{\pi} \frac{\psi \left(1 - \frac{k^2}{2}\right) - \psi \left(\frac{k^2}{2}\right)}{\Gamma (-\nu)},
\]

where \( \psi(z) = \Gamma'(z)/\Gamma(z) \) is the digamma function. The lowest eigenfunction corresponding to \( \lambda_0 = 0 \), is constant, \( u_0(x) = \beta_0 \), with

\[
   \beta_0^{-2} = \frac{\sqrt{\pi}}{2} \left[ x_1 e^{\kappa_1} \left(1 + \text{erf} \left(\frac{\sqrt{\kappa_1}}{\sqrt{\kappa_1}}\right)\right) + x_2 e^{\kappa_2} \left(1 + \text{erf} \left(\frac{\sqrt{\kappa_2}}{\sqrt{\kappa_2}}\right)\right) \right].
\]
As a consequence, one retrieves the equilibrium Boltzmann–Gibbs distribution,
\[ p_{eq}(x) = x p(x, \alpha|0) = u_0(x_0) u_0(x) w(x) = \beta^2 w(x). \]

Figure 7 illustrates the evolution of the probability density \( p(x, t|x_0, t_0) \) for diffusion under quadratic double-well potential with two minima at ±1 (i.e., \( x_1 = x_2 = 1 \), and \( \kappa_1 = 2, \kappa_2 = 1 \). Dashed vertical line indicates the starting point at \( t_0 = 0; x_0 = 2 \) (a) and \( x_0 = -2 \) (b). Symbols represent normalized histograms of arrival positions obtained by Monte Carlo simulations of an adapted version of equation (5) (with time step \( \delta = 10^{-3} \) and \( 10^5 \) sample trajectories), while lines show the spectral decomposition (18) with 50 terms. We set \( D = 1 \).

Figure 7. Evolution of the probability density \( p(x, t|x_0, t_0) \) for diffusion under quadratic double-well potential (sketched by black dotted line) with two minima at ±1 (i.e., \( x_1 = x_2 = 1 \)), and \( \kappa_1 = 2 \) and \( \kappa_2 = 1 \). Dashed vertical line indicates the starting point at \( t_0 = 0; x_0 = 2 \) (a) and \( x_0 = -2 \) (b). Symbols represent normalized histograms of arrival positions obtained by Monte Carlo simulations of an adapted version of equation (5) (with time step \( \delta = 10^{-3} \) and \( 10^5 \) sample trajectories), while lines show the spectral decomposition (18) with 50 terms. We set \( D = 1 \).

3.7. Further extensions

The spectral approach is a general tool for computing FETs and other first-passage quantities. We briefly mention four straightforward extensions.

(i) In one dimension, one can easily derive the splitting probability \( H(x_0) \), i.e., the probability to exit from one endpoint (e.g., \( x_0 = L \)) before the other (\( x_0 = -L \)). The splitting probability is governed by the stationary equation \( \mathcal{L}_0 H(x_0) = 0 \) so that \( H(x_0) \) is given by a general solution in equation (45) with \( \alpha = 0 \). Two constants \( c_1 \) and \( c_2 \) are set by boundary conditions \( H(L) = 1 \) and \( H(-L) = 0 \), from which

\[
H(x_0) = \frac{\text{erf}(i \sqrt{\kappa}(x_0/L - \varphi)) + \text{erf}(i \sqrt{\kappa}(1 + \varphi))}{\text{erf}(i \sqrt{\kappa}(1 - \varphi)) + \text{erf}(i \sqrt{\kappa}(1 + \varphi))},
\]

where we used \( M(0, b, z) = 1 \) and \( M(1/2, 3/2, z) = \frac{\sqrt{\pi} \text{erf}(i \sqrt{\kappa})}{2j \sqrt{\kappa}} \). Note that this expression can be recognized in equation (33) for the mean exit time.

(ii) The Dirichlet boundary conditions, \( q(\pm L, t) = 0 \), were imposed on the FET probability density at both endpoints in order to stop the process whenever it exits from the interval. One can consider other boundary value problems, e.g., with one reflecting endpoint or one/two semi-reflecting points. In this case, the Dirichlet boundary condition at one or both endpoints is replaced by Neumann or Robin boundary conditions [46]. For instance,
the condition \( \frac{\partial}{\partial x_0} q(x_0, t) = 0 \) at \( x_0 = -L \) describes the reflecting barrier at \( -L \). The Robin boundary condition, \( \frac{\partial}{\partial x_0} q(x_0, t) + h q(x_0, t) = 0 \), allows one to consider partial absorptions/reflections for modeling various transport mechanisms on the boundary and to switch continuously between Neumann (pure reflections) and Dirichlet (pure absorptions) cases by varying \( h \) from 0 to infinity [131–137]. The solution can be obtained in the same way.

(iii) The first passage time to a single barrier can be deduced from the first exit time from an interval by sending one endpoint to infinity (see appendix C).

(iv) A straightforward extension of the spectral approach allows one to deduce FETs of continuous-time random walks (CTRW) [5, 6]. In this model, long stalling periods between moves result in anomalous subdiffusion, when the mean-square displacement evolves sublinearly with time: \( \langle (X(t) - X(0))^2 \rangle \approx 2D_h t^\alpha \), with the exponent \( 0 < \alpha < 1 \) and the generalized diffusion coefficient \( D_h \). The same derivations can be formally repeated for the fractional Fokker–Planck equation governing the survival probability of CTRWs. In practice, it is sufficient to replace \( s/D \) by \( s^\alpha/D \) in the Laplace domain that in the time domain yields the replacement of exponential functions \( \exp (-\lambda t) \) by Mittag–Leffler functions \( E_{\alpha}(\lambda D_h t^\alpha/D) \) in spectral decompositions such as equation (23) or similar. As expected for CTRWs, the mean exit time diverges due to long stalling periods while the survival probability exhibits a power law decay \( t^{-\alpha} \) at long times instead of the exponential decay for normal diffusion.

Conclusion

We revised the classical problem of finding first exit times for harmonically trapped particles. Although the explicit formulas for the moment-generating function \( \langle e^{-s\tau} \rangle \) can be found in standard textbooks (e.g. [39]), the computation of the probability density and the survival probability through the inverse Laplace transform requires substantial analysis of confluent hypergeometric functions. For didactic purposes, we reproduced the main derivation steps and resulting spectral decompositions that involve the eigenvalues and eigenfunctions of the governing Fokker–Planck operator. We also provided explicit formulas for the mean exit time and discussed its asymptotic behavior in different limits. We considered the general case of noncentered harmonic potential in one dimension (Ornstein–Uhlenbeck process with nonzero mean) and the centered harmonic potential in higher dimensions (radial Ornstein–Uhlenbeck process). Both interior and exterior problems were analyzed.

After revising this classical problem, we discussed some practical issues. First, we described a rapidly converging series representation of confluent hypergeometric functions which is particularly well suited for rapid numerical computation of eigenvalues and eigenfunctions of the governing Fokker–Planck operator. Second, we showed how the mean exit time and the survival probability can be used for the analysis of single-particle tracking experiments with optically trapped tracers. The derived formulas allow one to choose the appropriate value of the optical tweezers’ stiffness and to detect in acquired trajectories the active periods with nonzero force exerted by motor proteins. Third, we mentioned the relation of the first exit time problem to the dynamics of bond dissociation under mechanical stress, which plays an important role in cell adhesion and motility. Fourth, we considered an application of FETs for algorithmic trading in stock markets in which buying or selling signals are triggered when the difference between the current and anticipated prices exceeds a prescribed threshold. In a first approximation, these events correspond to exits of an Ornstein–
Uhlenbeck process from an interval so that the FET statistics can be used to estimate strategy-holding periods and to choose the appropriate threshold that ensures the desired transaction rate. Fifth, we mentioned the relation to the distribution of first crossing times of a moving boundary by Brownian motion. Finally, we discussed several extensions of the spectral approach, including diffusion under quadratic double-well potential and anomalous diffusion.

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Appendix A. Limit of large $\kappa$

A.1. Mean exit time

For large $\kappa$ or $\phi$, equation (33) is not appropriate for numerical computation of the mean exit time because integrals and error functions are exponentially large. Re-arranging these terms, one can rewrite equation (33) as

$$
\langle \tau \rangle_{z_0} = \frac{L^2}{D} \frac{\sqrt{\pi}}{2\kappa} \left\{ 1 + e^{-2\phi(1+z_0^2)-\kappa\{1-z_0^2\}} \cdot \frac{D\left(\sqrt{\kappa}(z_0 - \phi)\right)}{D\left(\sqrt{\kappa}(1 + \phi)\right)} \right. 
\left. \times \int \frac{\tau^2}{\tau^2(\phi - 1)} \, \text{erf}(z) \right\},
$$

where $z_0 = x_0/L$, and $D(x)$ is the Dawson function:

$$
D(x) = e^{-x^2} \int_0^\infty dt \, e^{tx^2},
$$

which is related to the error function of an imaginary argument as

$$
\text{erf}(ix) = \frac{2i}{\sqrt{\pi}} e^{x^2} D(x).
$$

For large $x$, the Dawson function decays as

$$
D(x) \simeq \frac{1}{2x} + \frac{1}{4x^3} + \frac{3}{8x^5} + \ldots.
$$

The relation (A.1) allows one to compute the mean exit time in the limit of large $\kappa$ and/or $\phi$. In fact, since the Dawson function vanishes for large argument, the ratio in front of the first integral in equation (A.1) becomes exponentially close to 1 so that

$$
\langle \tau \rangle_{z_0} \simeq \frac{L^2}{D} \frac{\sqrt{\pi}}{2\kappa} \int \frac{\tau^2}{\tau^2(\phi - 1)} \, \text{erf}(z).
$$

Three situations can be considered separately.
(i) If $\varphi > 1$, the upper and lower limits of the above integral are positive and large, so that

$$
\langle \tau \rangle z_0 \simeq \frac{L^2}{D} \frac{1}{2\kappa} \ln \frac{\varphi - z_0}{\varphi - 1} \quad (\kappa \gg 1),
$$

(A.6)

where we used the asymptotic relation

$$
\int_a^b dz \ e^{z^2} \ erfc (z) \simeq \ln \left( \frac{b}{a} \right) \sqrt{\pi} \quad (a, b \gg 1).
$$

(A.7)

Note that equation (A.6) is accurate already for $\varphi \gtrsim 2$ (and $\kappa \geq 1$).

(ii) If $0 < \varphi < 1$ but $\kappa \to \infty$, the lower limit goes to $-\infty$, and the integral exponentially diverges:

$$
\langle \tau \rangle 0 \simeq \frac{L^2}{D} \sqrt{\kappa} e^{\kappa(1-\varphi)^2} \quad (\kappa \gg 1)
$$

(A.8)

(here, the starting point is set to 0, but the result holds for all $z_0$ not too close to 1). This relation is valid for any $0 < \varphi < 1$. Setting formally $\varphi = 0$, one gets the relation that is twice larger than the asymptotic equation (39) derived for $\varphi = 0$. The missing factor 2 can be retrieved from the ratio in front of the first integral in equation (A.1). The difference between the cases $\varphi = 0$ and $\varphi > 0$ (small but strictly positive) can also be explained by the following argument. For the nonsymmetric case ($\varphi > 0$), the right endpoint $x_0 = L$ is closer to the minimum position $\hat{x}$ than the left endpoint $x_0 = -L$. When $\kappa$ is large, the probability of large deviations from $\hat{x}$ rapidly decays with distance, so that the probability of exiting through the left endpoint is exponentially smaller than that from the right endpoint. In other words, the above relation essentially describes the mean exit time from the right endpoint. In turn, when $\varphi = 0$ (and thus $\hat{x} = 0$), both endpoints are equivalent, which doubles the chances to exit and thus twice reduces the mean exit time.

(iii) In the marginal case $\varphi = 1$, the integral in equation (A.5) grows logarithmically with $\kappa$. One can split the integral by an intermediate point $\xi \gg 1$ so that

$$
\int_0^z dz \ e^{z^2} \ erfc (z) + \int_z^\infty dz \ e^{z^2} \ erfc (z) \simeq \frac{1}{\sqrt{\kappa}} \ln \frac{\sqrt{\kappa}(1 - z_0)}{c (\xi)},
$$

where

$$
c (\xi) \equiv \xi \exp \left( -\sqrt{\kappa} \int_0^\xi dz \ e^{z^2} \ erfc (z) \right) \to 0.375 \ldots \quad (\xi \to \infty).
$$

We get therefore

$$
\langle \tau \rangle z_0 \simeq \frac{L^2}{D} \frac{1}{2\kappa} \ln \frac{\sqrt{\kappa}(1 - z_0)}{0.375 \ldots} \quad (\kappa \gg 1).
$$

(A.9)

This asymptotic relation is accurate starting from $\sqrt{\kappa}(1 - z_0) \gtrsim 2$.

A.2. Eigenvalues (interior problem)

For large $\kappa$, we search for positive solutions $\alpha_n$ of the equation $M \left( -\frac{a^2}{4\kappa}, b, \kappa \right) = 0$ in the form: $a_n^2/(4\kappa) = n - \epsilon$, where $\epsilon$ is a small parameter, and $n = 0, 1, 2, \ldots$ One gets then
\[ 0 = M(-n + \varepsilon; b, \kappa) \approx \sum_{j=0}^{b} (-n)(-n + 1) \ldots (-n + j - 1)k^j / b^{(j+1)!} + \varepsilon (-1)^{n} \sum_{j=n+1}^{\infty} (-n + j - 1)k^j / b^{(j+1)!} + O(\varepsilon^2), \]

from which the small parameter \( \varepsilon \) can be determined as

\[ \varepsilon \approx -\frac{S_1}{(-1)^n n! S_2}, \tag{A.10} \]

where \( S_1 \) and \( S_2 \) denote two above sums. The second sum can be written as

\[ S_2 = \sum_{j=n+1}^{\infty} (-n + j - 1)k^j / b^{(j+1)!} = \Gamma(b) \sum_{j=0}^{\infty} \frac{k^{j+n+1}}{\Gamma(b + j + 1 + n)(j + 1) \ldots (j + 1 + n)}. \]

This expression can be obtained by integrating \( n + 1 \) times the Mittag–Leffler function \( E_{1,b+n+1}(1) \) which asymptotically behaves as \( E_{1,b+n+1}(1) \approx \kappa^{-b-n}e^{(1 + O(1/\kappa))} \) as \( \kappa \gg 1 \).

Since the integration does not change the leading term, one concludes that

\[ S_2 \approx \Gamma(b)\kappa^{-b-n}e^{(1 + O(1/\kappa))} \quad (\kappa \gg 1). \]

Keeping the highest-order term in the first sum, \( S_1 \approx (-1)^n k^n / b(n) \), one gets

\[ \varepsilon \approx -\frac{\kappa^{b+2n}}{n! \Gamma(b + n)} e^{-\kappa}, \]

from which we obtain the asymptotic behavior of the positive solution \( \alpha_n \) as \( \kappa \gg 1 \):

\[ \alpha_n \approx 4\kappa \left[ n + \frac{\kappa^{b+2n}e^{-\kappa}}{n! \Gamma(b + n)} \right] \quad (n = 0, 1, 2, \ldots). \tag{A.11} \]

In particular, the smallest solution \( \alpha_0 \) exponentially decays with \( \kappa \),

\[ \alpha_0^2 \approx \frac{4\kappa^{1+b}}{\Gamma(b)} e^{-\kappa} \quad (\kappa \gg 1), \tag{A.12} \]

while the other eigenvalues grow linearly with \( \kappa \):

\[ \alpha_n^2 \approx 4\kappa n \quad (\kappa \gg 1, \ n = 1, 2, \ldots), \tag{A.13} \]

and the first-order correction \( \varepsilon \) decays exponentially fast. This asymptotic behavior can be related to equidistant energy levels of a quantum harmonic oscillator (see appendix D).

Since \( \alpha_0 \) rapidly vanishes, the first eigenfunction approaches the unity:

\[ M\left( \frac{-\alpha_0^2}{4\kappa^2}, d, z^2 \right) \to 1. \]

As a consequence, the normalization constant is simply

\[ \beta_0^2 \approx 2d^{d/2} / \Gamma(d/2) \]

so that \( w_0 \approx 1 \), because \( M(1, b, z) = \Gamma(b)E_{1,b}(z) \).

A.3. Eigenvalues (exterior problem)

For the exterior problem, we consider the asymptotic behavior of solutions of

\[ U\left( \frac{-\alpha_i^2}{4\kappa_z^2}, b, \kappa \right) = 0 \quad \text{as} \ \kappa \to 0. \]

For noninteger \( b \), one can use equation (87) to write in the lowest order in \( \kappa \)
For \( b < 1 \), \( \kappa^{1-b} \) is a small parameter so that the first term has to be small. Setting \( 1 - b - \frac{a^2}{4\kappa} = -n + \varepsilon \) (with \( n = 0, 1, 2, \ldots \)) one gets
\[
\varepsilon = (-1)^{n-1} \frac{\Gamma(b-1)}{n! \Gamma(b-1-n) \Gamma(1-b)} \kappa^{1-b},
\]
from which
\[
\alpha_n^2 \approx 4\kappa \left(1 - b + n + \frac{(-1)^n \Gamma(b-1)}{n! \Gamma(b-1-n) \Gamma(1-b)} \kappa^{1-b} + \ldots \right).
\]

In turn, if \( b > 1 \), \( \kappa^{1-b} \) is a large parameter so that the second term has to be small. Setting \(-\frac{a^2}{4\kappa} = -n + \varepsilon\), one gets
\[
\varepsilon = (-1)^{n-1} \frac{\Gamma(1-b)}{n! \Gamma(1-b-n) \Gamma(b-1)} \kappa^{b-1},
\]
from which
\[
\alpha_n^2 \approx 4\kappa \left(n + \frac{(-1)^n \Gamma(1-b)}{n! \Gamma(1-b-n) \Gamma(b-1)} \kappa^{b-1} + \ldots \right).
\]

For integer \( b \), the analysis is more subtle and is beyond the scope of this paper. We just checked numerically that \( \alpha_n^2 \propto \kappa^b \) as \( \kappa \to 0 \) for \( b = 1 \) and \( b = 2 \) that corresponds to dimensions \( d = 2 \) and \( d = 4 \).

**Appendix B. Confluent hypergeometric functions**

For the sake of completeness, we summarize selected relations between special functions often used to describe first passage times of Ornstein–Uhlenbeck processes (see [68] for details). After that, we describe a rapidly converging representation of confluent hypergeometric functions.

**B.1. Relations**

The Kummer confluent hypergeometric function \( M(a, b, z) = \mathcal{F}_1(a; b; z) \) defined in equation (42) satisfies the Kummer equation:
\[
zy'' + (b - z)y' - ay = 0.
\]
For \( b = 1/2 \), this equation is also related to Weber’s equation
\[
y'' - \left(z^2/4 + c\right)y = 0,
\]
which has two independent solutions: \( e^{-z^2/4}M(c/2 + 1/4, 1/2, z^2/2) \) (even) and \( ze^{-z^2/4}M(c/2 + 3/4, 3/2, z^2/2) \) (odd). These solutions are often expressed through the parabolic cylinder function \( D_l(z) \), which satisfies equation (B.2) with \( v = -c - 1/2 \).
\[ D_v(z) = \frac{\cos\left(\frac{\pi v}{2}\right) \Gamma\left(\frac{1 + v}{2}\right)}{\sqrt{\pi} 2^{-v/2}} e^{-z^2/4} M\left(-\frac{v}{2}, \frac{1}{2}, \frac{z^2}{2}\right) \]

\[ + \frac{\sin\left(\frac{\pi v}{2}\right) \Gamma\left(\frac{2 + v}{2}\right)}{\sqrt{\pi} 2^{-(v+1)/2}} e^{-z^2/4} zM\left(-\frac{v}{2} + 1, \frac{3}{2}, \frac{z^2}{2}\right) \]  

(B.3)

\[ = 2^{v/2} e^{-z^2/4} U\left(-\frac{v}{2}, \frac{1}{2}, \frac{z^2}{2}\right) \]  

(B.4)

(the last relation is only valid for \( \Re\{z\} \geq 0 \)).

The confluent hypergeometric functions \( M(a, b, z) \) and \( U(a, b, z) \) are also related to the Whittaker functions \( M_{a,b}(z) \) and \( W_{a,b}(z) \) \[68\]

\[ M_{a,b}(z) = e^{-z^2/2} z^{b+1/2} M(1/2 + b - a, 1 + 2b, z) , \]

\[ W_{a,b}(z) = e^{-z^2/2} z^{b+1/2} U(1/2 + b - a, 1 + 2b, z) . \]

The following relations help to analyze the Brownian motion limit \[39\]

\[ \lim_{\kappa \to 0} M\left(\frac{a}{4\kappa}, b + 1, \kappa x\right) = 2^b \Gamma\left(b + 1\right) \frac{I_b\left(\sqrt{\kappa a}\right)}{(\kappa a)^{b/2}} , \]  

(B.5)

\[ \lim_{\kappa \to 0} \kappa^b \Gamma\left(\frac{a}{4\kappa}\right) U\left(\frac{a}{4\kappa}, b + 1, \kappa x\right) = 2^{1-b} K_b\left(\sqrt{\kappa a}\right) (\kappa a)^{b/2} , \]  

(B.6)

where \( I_b(z) \) and \( K_b(z) \) are the modified Bessel functions of the first and second kind, respectively. The asymptotic expansions for large \( |z| \) (and fixed \( a \) and \( b \)) are \[68\] (section 13.5):

\[ M(a, b, z) \approx \frac{e^{z^2/2} \Gamma(b)}{\Gamma(a)} \left( \sum_{n=0}^{N_1} \frac{(b-a)^{(n)}(1 - a)^{(n)}}{n! z^n} + O\left(|z|^{-N_1}\right) \right) \]

\[ + \frac{e^{z^2/2} \Gamma(b)}{\Gamma(a) (b - a)} \left( \sum_{n=0}^{N_2} \frac{(a)^{(n)}(1 + a - b)^{(n)}}{n! (-z)^n} + O\left(|z|^{-N_2}\right) \right), \]  

(B.7)

\[ U(a, b, z) \approx \frac{z^{-a}}{\Gamma(a)} \left( \sum_{n=0}^{N_1} \frac{(a)^{(n)}(1 + a - b)^{(n)}}{n! (z)^n} + O\left(|z|^{-N_1}\right) \right) \]  

(B.8)

where the upper (resp., lower) sign in the second line is taken if \(-\pi/2 < \arg(z) < 3\pi/2\) (resp., \(-3\pi/2 < \arg(z) \leq -\pi/2\)), and \( N_1, N_1, \) and \( N_2 \) are truncation orders.

B.2. Computation

Series representations. The computation of the Kummer function \( M(a, b, z) \) by direct series summation in equation (42) is not convenient for large \( |z| \). For this case, two equivalent representations were proposed:
(i) \[ M(a, b, z) = \Gamma(b) e^{z/2} 2^{b-1} \sum_{n=0}^{\infty} A_n z^n \left( \frac{\sqrt{z(2b-4a)}}{\sqrt{z(2b-4a)}} \right)^{b-1+n}, \]  
where the coefficients \( A_n \) are defined by
\[ A_0 = 1, \quad A_1 = 0, \quad A_2 = b/2, \quad nA_n = (n-2+b)A_{n-2} + (2a-b)A_{n-3} \]
(see [68], section 13.3.7, and [138], section 4.8). Note that the coefficients \( A_n \) depend on \( a \) and grow with \( |a| \).

(ii) \[ M(a, b, z) = \Gamma(b) e^{z/2} 2^{b-1} \sum_{n=0}^{\infty} p_n(b, z) \left( \frac{\sqrt{z(2b-4a)}}{\sqrt{z(2b-4a)}} \right)^{b-1+n}, \]  
where \( p_n(b, z) \) are the Buchholz polynomials in \( b \) and \( z \) (see [139], section 7.4). These polynomials are less explicit than the coefficients \( A_n \), but they are independent of \( a \). Consequently, this representation is particularly convenient for large \( |a| \).

The recurrence relations for the Buchholz polynomials were derived in [140]:
\[ p_k(b, z) = \left( \frac{i}{2} \right)^n \sum_{k=0}^{[n/2]} \left( \begin{array}{c} n \\ 2k \end{array} \right) f_k(b) g_{n-2k}(z), \]  
where the polynomials \( f_k(b) \) and \( g_k(z) \) are defined recursively by
\[ f_k(b) = -\left( \frac{b}{2} - 1 \right) \sum_{j=0}^{k-1} \left( \begin{array}{c} 2k-1 \\ 2j \end{array} \right) \frac{B_{2j+k}}{k-j} f_j(b), \quad f_0(b) = 1, \]  
\[ g_k(z) = -\frac{i}{4} \sum_{j=0}^{(k-1)/2} \left( \begin{array}{c} k-1 \\ 2j \end{array} \right) \frac{B_{2j+k+1}}{j+1} g_{k-2j-1}(z), \quad g_0(z) = 1, \]
and \( B_j \) are the Bernoulli numbers. Using the recurrence relations between Bessel functions, \( \frac{dx}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x) \), one can express
\[ J_{b-1+n}(x) = P_n(1/x) J_{b-1}(x) + Q_n(1/x) J_b(x), \]
where the polynomials \( P_n(z) \) and \( Q_n(z) \) are defined recursively
\[ P_0(z) = 1, \quad P_1(z) = 0, \quad P_{n+1}(z) = 2(b-1+n)zP_n(z) - P_{n-1}(z), \]  
\[ Q_0(z) = 0, \quad Q_1(z) = 1, \quad Q_{n+1}(z) = 2(b-1+n)zQ_n(z) - Q_{n-1}(z). \]
We get therefore the following expansion that rapidly converges for large \( x \) and moderate \( z \):
\[ M(a, b, z) = e^{z/2} \sum_{n=0}^{\infty} p_n(b, z) \left( \frac{F_n(1/x)}{x^n} + G_n(x) \frac{Q_n(1/x)}{x^{n-1}} \right), \]
where \( x = \sqrt{z(2b-4a)} \), and
\[ F_b(x) = \Gamma(b) 2^{b-1} x^{1-b} J_{b-1}(x), \quad G_b(x) = \Gamma(b) 2^{b-1} x^{1-b} J_b(x). \]
In particular, for $b = d/2$, one has

\[
\begin{align*}
&d \quad F_d(x) \\
&1 \quad \cos (x) \\
&2 \quad J_0(x) \\
&3 \quad \sin (x)/x \quad (\sin (x) - x \cos (x))/x^3
\end{align*}
\]

(B.16)

The above recursive relations allow one to compute rapidly the polynomials $p_b(x, z)$, $P_b(1/x)$ and $Q_b(1/x)$. The series (B.14) can be truncated after 5–10 terms when $|az|$ is large enough and $z$ is not too large (see [140] for several examples).

According to equation (87), one can apply this method to compute the Tricomi confluent hypergeometric function $U(a, b, z)$ for noninteger $b$. Other series expansions for $U(a, b, z)$ are discussed in [141, 142]. For integer $b$, one can substitute $\varepsilon = b + \varepsilon$ into equation (87) and then take the limit $\varepsilon \to 0$. This extension by continuity yields [68]

\[
\begin{align*}
&\sum_{k=0}^{b-2} \frac{(a - b + 1)^{(k)}}{k!} x^{k-b+1} \\
&+ \sum_{k=0}^{b-1} \frac{\psi(b+k)}{k!} \left( \psi(a+k) - \psi(1+k) - \psi(b+k) \right), \quad b = 1, 2, \ldots,
\end{align*}
\]

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function, and the intermediate sum is omitted for $b=1$. In practice, one can apply the above numerical scheme to rapidly compute $M(a, b, z)$ with several noninteger $b$ approaching the integer $b$, and then to extrapolate them in the limit $b \to b$.

Taking the derivative of equation (B.10) with respect to $a$ and using the relation

\[
\frac{\partial}{\partial a} J_a(x) = \frac{1}{2} J_a(x) - J_{a+1}(x),
\]

one obtains

\[
\frac{\partial}{\partial a} M(a, b, z) = \Gamma(b)e^{i\pi/2}2\sqrt{z} \sum_{n=0}^{\infty} p_b(n, z) \frac{J_{b+n} \left( \sqrt{z}(2b - 4a) \right)}{\left( \sqrt{z}(2b - 4a) \right)^{b+n}}.
\]

(B.17)

or, equivalently,

\[
\frac{\partial}{\partial a} M(a, b, z) = 2\sqrt{z} \sum_{n=0}^{\infty} p_b(n, z) \left[ F_b(x) \frac{P_{b+n}(1/x)}{x^{n+1}} + G_b(x) \frac{Q_{b+n}(1/x)}{x^n} \right],
\]

(B.18)

with $x = \sqrt{z}(2b - 4a)$. This expression allows one to rapidly compute the coefficients $w_n$ in the spectral representation of the survival probability. Similar relations can be derived for $\frac{\partial}{\partial a} U(a, b, z)$ using equation (87) for noninteger $b$. Finally, one can also apply these formulas for computing the parabolic cylinder function $D_0(z)$ and its derivative $\frac{\partial}{\partial x} D_0(z)$, which are used to characterize the first passage time to a single barrier (appendix C).

**Integral representations.** The above scheme is convenient for large $|a|$ and moderate $|z|$. However, if $|a|$ is moderate while $|z|$ is large, the numerical convergence of the above series is slowed down due to a rapid growth of Buchholz polynomials with $z$. In addition, the computation of the Tricomi function $U(a, b, z)$ as a linear combination (87) of two large Kummer functions can result in significant round-off errors at large $z$. In this case, one can
apply a different technique that relies on integral representations of confluent hypergeometric functions.

For the Kummer function $M(a, b, z)$, one can use the following integral representation for $\Re\{b - a\} > 0$

$$M(a, b, z) = \frac{e^{z}z^{b-\frac{1}{2}}I(b)}{I(b-a)} \int_{0}^{\infty} dt e^{-t}t^{\frac{a+1}{2}-a}J_{b-1}(2\sqrt{zt}). \quad (B.19)$$

This representation is convenient for computing eigenvalues and eigenfunctions because $a = -a^2/(4\kappa) < 0$ and $b = d/2 > 0$.

The Tricomi function $U(a, b, z)$ has an integral representation for positive $a$ [68]

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_{0}^{\infty} dt e^{-t}t^{a-1}(1 + t)^{b-a-1} \quad (\Re\{a\} > 0, \Re\{z\} > 0). \quad (B.20)$$

When $a$ is negative, one can use the recurrence relation to increase $a$:

$$U(a - 1, b, z) + (b - 2a - z)U(a, b, z) + a(a + 1 - b)U(a + 1, b, z) = 0. \quad (B.21)$$

Applying this relation repeatedly, one gets

$$U(a, b, z) = p_{n}(a, b, z)U(a + n, b, z) + q_{n}(a, b, z)U(a + n + 1, b, z), \quad (B.22)$$

where the polynomials $p_{n}(a, b, z)$ and $q_{n}(a, b, z)$ can be rapidly computed through recurrence relations:

$$p_{n}(a, b, z) = q_{n-1}(a, b, z) - (b - 2(a + n) - z)p_{n-1}(a, b, z), \quad p_{0} = 1,$$

$$q_{n}(a, b, z) = -(a + n)(a + n + 1 - b)p_{n-1}(a, b, z), \quad q_{0} = 0.$$  

Choosing $n$ such that $a + n > 0$, one can express $U(a, b, z)$ in terms of $U(a + n, b, z)$ and $U(a + n + 1, b, z)$ which are found by numerical integration of equation (B.20). If $z$ is too large, it is convenient to divide each recurrence relation by $z$ and to consider them as polynomials of $1/z$. The resulting value can be compared with the asymptotic expansion (B.8).

Appendix C. First passage time to a single barrier

The first passage times (one-barrier problem) for harmonically trapped particles have attracted more attention than the first exit times (two-barrier problem) [29, 32–36]. In general, the first passage time $\tau_{\ell}$ to a single barrier at $\ell > 0$ in one dimension can be found following the steps from section 2.6. In practice, these results can be deduced from the FET statistics. If the starting point $x_{0}$ lies on the right to $\ell$ (i.e., $x_{0} > \ell$), this problem is equivalent to the exterior problem to reach the interval $[-\ell, \ell]$ from outside (see section 2.9). In turn, if $0 < x_{0} < \ell$, the FPT to a single barrier $\ell$ can be deduced from the FET from the interval $[-a, \ell]$ in the limit $a \to \infty$.

In order to illustrate this point, we focus on the moment-generating function $\tilde{q}(x_{0}, s)$ given by equation (60), with $\ell = L(1 - \varrho)$ and $a = L(1 + \varrho)$. Setting $1 - \varrho = \epsilon$, we consider the limit $\epsilon \to 0$, for which $L = \ell/\epsilon \to \infty$ and $1 - \varrho = \epsilon \to 0$, so that $a \to \infty$ while $\ell$ is kept fixed. The asymptotic behavior of functions $n_{a,\ell}^{(1,2)}$ from equation (43) as $\epsilon \to 0$ can be easily found:

See http://dlmf.nist.gov/13.16.E3
where we replaced \( \alpha \) and \( \kappa \) by \(-D_\ell /L_2^2\) and \( kL_2^2/(2D_\gamma)\), introduced short notations \( a = s_\ell/(2k) \) and \( y = k\ell^2/(2D_\gamma)\), and used the asymptotic relation (B.7) for the last two functions. Substituting the above expressions into equation (60), one deduces in the limit \( \varepsilon \to 0 \)

\[
\hat{q}(x_0, s) = M\left(a, \frac{1}{2}, y_0\right) + 2\left(\frac{a}{y}\right)\sqrt{\pi} M\left(a + 1/2, \frac{3}{2}, y_0\right),
\]

(C.1)

where \( y_0 = k\ell_0^2/(2D_\gamma) \). Using equation (B.3), one can alternatively write the moment-generating function as

\[
\hat{q}(x_0, s) = \exp\left(\frac{k\left(x_0^2 - \ell^2\right)}{4D_\gamma}\right) D_{-s\gamma/k}\left(\frac{x_0}{\sqrt{D_\gamma}}\right) D_{-s\gamma/k}\left(\ell \frac{1}{\sqrt{D_\gamma}}\right) \quad (0 \leq x_0 \leq \ell).\]

(C.2)

Note also that equation (88) for the exterior case \( x_0 > \ell \) can also be written in terms of the parabolic cylinder function \( D_\ell(z) \) according to equation (B.4):

\[
\hat{q}(x_0, s) = \exp\left(\frac{k\left(x_0^2 - \ell^2\right)}{4D_\gamma}\right) D_{-s\gamma/k}\left(0, \frac{x_0}{\sqrt{D_\gamma}}\right) D_{-s\gamma/k}\left(\ell \frac{1}{\sqrt{D_\gamma}}\right) \quad (x_0 > \ell),
\]

(C.3)

in agreement with [39] (see also [32, 42]).

The inverse Laplace transform yields the probability density \( p_{x,a}(t) \) [35, 42]

\[
q(x_0, t) = -\frac{k}{\gamma} \exp\left(\frac{k\left(x_0^2 - \ell^2\right)}{4D_\gamma}\right) \sum_{n=1}^{\infty} \frac{1}{D_{s\ell}'} \left\{ \pm x_0 \frac{1}{\sqrt{D_\gamma}} \right\} e^{-\nu_n k t / \gamma},
\]

(C.4)

where \( 0 < \nu_1 < \ldots < \nu_6 < \ldots \) are the zeros of the function \( D_\ell(\pm \ell, k/(D_\gamma)) \), and \( D_\ell'(z) \) is the derivative of \( D_\ell(z) \) with respect to \( \nu \), evaluated at point \( \nu = \nu_6 \) [42] (p.154). The signs plus and minus correspond to \( x_0 > \ell \) and \( x_0 < \ell \), respectively. Both \( D_\ell(z) \) and \( D_\ell'(z) \) can be rapidly evaluated by the numerical scheme presented in appendix B.2.

In the special case \( \ell = 0 \), the FET probability density gets a simple explicit form:

\[
q(x_0, t) = \frac{x_0}{\sqrt{4\pi D}} \left(\frac{k/\gamma}{\sinh (k t/\gamma)}\right)^{3/2} \exp\left(-\frac{k x_0^2}{4D_\gamma} e^{-k t / \gamma} + \frac{k t}{2\gamma}\right).
\]

(C.5)

In the limit \( k \to 0 \), one retrieves the classical formula for the FPT of Brownian motion at the origin.
Appendix D. Quantum harmonic oscillator

The eigenvalue problem (40) with \( b = 1/2 \) is closely related to eigenstates of a quantum harmonic oscillator of mass \( m \) and frequency \( \omega \) \cite{120}. In fact, the eigenstates \( \psi_n \) and energies \( E_n \) of the Hamiltonian \( H = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 x^2}{2} \) satisfy the time-independent Schrödinger equation

\[
\left[ -\frac{\hbar^2}{2m} \partial_x^2 + \frac{m\omega^2 x^2}{2} \right] \psi(x) = E\psi(x),
\]

where \( \hat{p} = -i\hbar \partial_x \) is the momentum operator, and \( \hbar \) is the (reduced) Planck constant. In terms of the dimensionless coordinate \( \eta = \frac{x}{\sqrt{2m\omega}} \), the above Schrödinger equation is reduced to Weber’s equation (B.2), with \( \epsilon = -\frac{E}{\hbar\omega} \). Setting \( \psi(z) = e^{-z^2/4}\tilde{u}(z) \) yields \( \tilde{u}' - z\tilde{u}' - (c + 1/2)\tilde{u} = 0 \), from which the rescaling \( u(x) = \tilde{u}\left(\sqrt{\hbar/(\Delta \xi)}(x - \bar{x})\right) \) implies equation (40), with \( \lambda = -\frac{\hbar}{\sqrt{2m\omega}}(c + 1/2) \). Consequently, the energies of the quantum oscillator and the eigenvalues of the FP operator are simply related as: \( \lambda = -\frac{\hbar}{\sqrt{2m\omega}}(c + 1/2) \).

If no boundary condition is imposed, the non-normalized eigenstate is simply \( \psi(x) = D_n(x\sqrt{2m\omega}/\hbar) \), where \( D_n(z) \) is the parabolic cylinder function (see appendix B.1), and \( \nu = -c - 1/2 \). One can check that

\[
D_n(z) \approx \frac{1}{\sqrt{2\pi n}} e^{-z^2/4} \left[ 1 - \frac{\nu(\nu - 1)}{2z^2} + O\left(z^{-4}\right) \right] \quad (z \gg 1),
\]

\[
D_n(z) \approx e^{-z^2/4} \left[ 1 - \frac{\nu(\nu - 1)}{2z^2} + O\left(z^{-4}\right) \right]
- \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{\nu^2/4} e^{-z^2/4} \left[ 1 + \frac{(\nu + 1)(\nu + 2)}{2z^2} + O\left(z^{-4}\right) \right] \quad (z \ll -1).
\]

In order to eliminate the unphysical rapid growth of the eigenstate as \( z \to -\infty \), one needs to impose \( \nu = n \) with \( n = 0, 1, 2, \ldots \) to remove the last term, from which one retrieves the quantized energies of the quantum harmonic oscillator: \( E_n = \hbar\omega (n + 1/2) \), while the eigenfunctions become expressed through the Hermite polynomials \( H_n(z) \)

\[
\psi_n(x) = \frac{1}{\sqrt{2^n n!}} \frac{m\omega}{\pi\hbar}^{1/4} e^{-\frac{m\omega x^2}{2\hbar}} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right),
\]

where the usual normalization prefactor is included, and we used \( D_n(z) = 2^{-n/2} e^{-z^2/4} H_n(z/\sqrt{2}) \). Since imposing no boundary condition corresponds to barriers at distance \( L \to \infty \), we retrieve the asymptotic behavior \( \lambda_n \approx \frac{k}{\sqrt{\ell^2}} \) or, equivalently, \( a_\text{eff}^2 \approx \frac{\ell^2}{2\lambda_n} \approx 2\kappa \) as \( \kappa \to \infty \). Note that the prefactor \( 2\kappa \) is twice smaller than that of equation (A.13) because the latter relation accounts only for symmetric eigenfunctions that contribute to the survival probability.

Imposing the Dirichlet boundary condition at \( x = \pm L \) corresponds to setting infinite potential outside the interval \([-L, L]\) (and keeping the harmonic potential inside). The
eigenvalue problem for a quantum oscillator in such potential is equivalent to the analysis of the first exit time distribution in section 2.6.

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