LECTURES ON MÖBIUS–LIE GEOMETRY
AND ITS EXTENSION

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Abstract. These lectures review the classical Möbius–Lie geometry and recent work on its extension. The latter considers ensembles of cycles (quadrics), which are interconnected through conformal-invariant geometric relations (e.g. “to be orthogonal”, “to be tangent”, etc.), as new objects in an extended Möbius–Lie geometry. It is shown on examples, that such ensembles of cycles naturally parameterise many other conformally-invariant families of objects, two examples—the Poincaré extension and continued fractions are considered in detail. Further examples, e.g. loxodromes, wave fronts and integrable systems, are published elsewhere.

The extended Möbius–Lie geometry is efficient due to a method, which reduces a collection of conformally invariant geometric relations to a system of linear equations, which may be accompanied by one fixed quadratic relation. The algorithmic nature of the method allows to implement it as a C++ library, which operates with numeric and symbolic data of cycles in spaces of arbitrary dimensionality and metrics with any signatures. Numeric calculations can be done in exact or approximate arithmetic. In the two- and three-dimensional cases illustrations and animations can be produced. An interactive Python wrapper of the library is provided as well.

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1. Introduction

Lie sphere geometry [14, Ch. 3; 18] in the simplest planar setup unifies circles, lines and points—all together called cycles in this setup. Symmetries of Lie spheres geometry include (but are not limited to) fractional linear transformations (FLT) of the form:

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} : x \mapsto \frac{ax + b}{cx + d}, \text{ where } \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.
\]

Following other sources, e.g. [72, § 9.2], we call (1) by FLT and reserve the name “Möbius maps” for the subgroup of FLT which fixes a particular cycle. For example, on the complex plane FLT are generated by elements of \(SL_2(\mathbb{C})\) and Möbius maps fixing the real line are produced by \(SL_2(\mathbb{R})\) [46, Ch. 1].

There is a natural set of FLT-invariant geometric relations between cycles (to be orthogonal, to be tangent, etc.) and the restriction of Lie sphere geometry to invariants of FLT is called Möbius–Lie geometry. Thus, an ensemble of cycles, structured by a set of such relations, will be mapped by FLT to another ensemble with the same structure.

It was shown recently that ensembles of cycles with certain FLT-invariant relations provide helpful parametrisations of new objects, e.g. points of the Poincaré extended space [52], loxodromes [54] or continued fractions [13, 51], see Example 5.1 below for further details. Thus, we propose to extend Möbius–Lie geometry and consider ensembles of cycles as its new objects, cf. formal Defn. 5.3. Naturally, “old” objects—cycles—are represented by simplest one-element ensembles without any relation. This paper provides conceptual foundations of such extension and demonstrates its practical implementation as a \texttt{C++} library \texttt{figure}.

Interestingly, the development of this library shaped the general approach, which leads to specific realisations in [51, 52, 54].

More specifically, the library \texttt{figure} manipulates ensembles of cycles (quadrics) interrelated by certain FLT-invariant geometric conditions. The code is build on top of the previous library \texttt{cycle} [40, 41, 46], which manipulates individual cycles within the \texttt{GiNaC} [7] computer algebra system. Thinking an ensemble as a graph, one can say that the library \texttt{cycle} deals with individual vertices (cycles), while \texttt{figure} considers edges (relations between pairs of cycles) and the whole graph. Intuitively, an interaction with the library \texttt{figure} reminds compass-and-straightedge constructions, where new lines or circles are added to a drawing one-by-one through relations to already presented objects (the line through two points, the intersection point or the circle with given centre and a point). See Example 5.4 of such interactive construction from the \texttt{Python} wrapper, which provides an analytic proof of a simple geometric statement.

\footnote{All described software is licensed under GNU GPLv3 [27].}
It is important that both libraries are capable to work in spaces of any dimensionality and metrics with an arbitrary signatures: Euclidean, Minkowski and even degenerate. Parameters of objects can be symbolic or numeric, the latter admit calculations with exact or approximate arithmetic. Drawing routines work with any (elliptic, parabolic or hyperbolic) metric in two dimensions and the euclidean metric in three dimensions.

The mathematical formalism employed in the library cycle is based on Clifford algebras, which are intimately connected to fundamental geometrical and physical objects [33, 34]. Thus, it is not surprising that Clifford algebras have been already used in various geometric algorithms for a long time, for example see [24, 35, 76] and further references there. Our package deals with cycles through Fillmore–Springer–Cnops construction (FSCc) which also has a long history, see [20, § 4.1; 25; 39, § 4.2; 44; 46, § 4.2; 71, § 1.1] and section 2.1 below. Compared to a plain analytical treatment [14, Ch. 3; 64, Ch. 2], FSCc is much more efficient and conceptually coherent in dealing with FLT-invariant properties of cycles. Correspondingly, the computer code based on FSCc is easy to write and maintain.

The paper outline is as follows. In Section 2 we sketch the mathematical theory (Möbius–Lie geometry) covered by the package of the previous library cycle [41] and the present library figure. We expose the subject with some references to its history since this can facilitate further development. Sec. 2.3 describes the principal mathematical tool used by the library figure. It allows to reduce a collection of various linear and quadratic equations (expressing geometrical relations like orthogonality and tangency) to a set of linear equations and at most one quadratic relation (8). Notably, the quadratic relation is the same in all cases, which greatly simplifies its handling. This approach is the cornerstone of the library effectiveness both in symbolic and numerical computations.

We consider two examples of ensembles of cycles in details. Section 3 presents the Poincaré extension. Given sphere preserving (Möbius) transformations in \( n \)-dimensional Euclidean space one can use the Poincaré extension to obtain sphere preserving transformations in a half-space of \( n + 1 \) dimensions. The Poincaré extension is usually provided either by an explicit formula or by some geometric construction. Here we present its algebraic background and describe all available options. The solution is given either in terms of one-parameter subgroups of Möbius transformations or ensembles of cycle representing equivalent triples of quadratic forms.

Another example, presented in Section 4, uses interrelations between continued fractions, Möbius transformations and representations of cycles by \( 2 \times 2 \) matrices. This leads us to several descriptions of continued fractions through ensembles of cycles consisting of chains of orthogonal or touching horocycles. One of these descriptions was proposed in a recent paper by A. Beardon and I. Short. The approach is extended to several dimensions in a way which is compatible to the early propositions of A. Beardon based on Clifford algebras.

In Sec. 5.1 we present several examples of ensembles, which were already used in mathematical theories [51, 52, 54], then we describe how ensembles are encoded in the present library figure through the functional programming framework.

Sec. 6 outlines several typical usages of the package. An example of a new statement discovered and demonstrated by the package is given in Thm. 6.1. All coding-related material is enclosed as appendices in the full documentation on the project page [41]. They contain:

1. Numerous examples of the library usage starting from the very simple ones.
2. A systematic list of callable methods.
3. Actual code of the library.
Sec. 2, Example 5.4 below or the above-mentioned first two appendices of the full documentation can serve as an entry point for a reader with respective preferences and background.

2. Möbius–Lie Geometry and the cycle Library

We briefly outline mathematical formalism of the extend Möbius–Lie geometry, which is implemented in the present package. We do not aim to present the complete theory here, instead we provide a minimal description with a sufficient amount of references to further sources. The hierarchical structure of the theory naturally splits the package into two components: the routines handling individual cycles (the library cycle briefly reviewed in this section), which were already introduced elsewhere [41], and the new component implemented in this work, which handles families of interrelated cycles (the library figure introduced in the next section).

2.1. Möbius–Lie geometry and FSC construction. Möbius–Lie geometry in $\mathbb{R}^n$ starts from an observation that points can be treated as spheres of zero radius and planes are the limiting case of spheres with radii diverging to infinity. Oriented spheres, planes and points are called together cycles. Then, the second crucial step is to treat cycles not as subsets of $\mathbb{R}^n$ but rather as points of some projective space of higher dimensionality, see [15, Ch. 3; 18; 64; 71].

To distinguish two spaces we will call $\mathbb{R}^n$ as the point space and the higher dimension space, where cycles are represented by points—the cycle space. Next important observation is that geometrical relations between cycles as subsets of the point space can be expressed in term of some indefinite metric on the cycle space. Therefore, if an indefinite metric shall be considered anyway, there is no reason to be limited to spheres in Euclidean space $\mathbb{R}^n$ only. The same approach shall be adopted for quadrics in spaces $\mathbb{R}^{pqr}$ of an arbitrary signature $p+q+r = n$, including $r$ nilpotent elements, cf. (2) below.

A useful addition to Möbius–Lie geometry is provided by the Fillmore–Springer–Cnops construction (FSCc) [20, § 4.1; 25; 39, § 4.2; 44; 46, § 4.2; 67, § 18; 71, § 1.1]. It is a correspondence between the cycles (as points of the cycle space) and certain $2 \times 2$-matrices defined in (4) below. The main advantages of FSCc are:

1. The correspondence between cycles and matrices respects the projective structure of the cycle space.
2. The correspondence is FLT covariant.
3. The indefinite metric on the cycle space can be expressed through natural operations on the respective matrices.

The last observation is that for restricted groups of Möbius transformations the metric of the cycle space may not be completely determined by the metric of the point space, see [40; 44; 46, § 4.2] for an example in two-dimensional space.

FSCc is useful in consideration of the Poincaré extension of Möbius maps [52], loxodromes [54] and continued fractions [51]. In theoretical physics FSCc nicely describes conformal compactifications of various space-time models [32; 42; 46, § 8.1]. Regretfully, FSCc have not yet propagated back to the most fundamental case of complex numbers, cf. [72, § 9.2] or somewhat cumbersome techniques used in [14, Ch. 3]. Interestingly, even the founding fathers were not always strict followers of their own techniques, see [26].

We turn now to the explicit definitions.

2.2. Clifford algebras, FLT transformations, and Cycles. We describe here the mathematics behind the first library called cycle, which implements fundamental geometrical relations between quadrics in the space $\mathbb{R}^{pqr}$ with the dimensionality $n = p+q+r$ and metric $x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$. A version simplified
for complex numbers only can be found in [51, 52, 54]. In two dimensions usage of dual and double numbers is preferable due to their commutativity [40, 46, 53, 77].

The Clifford algebra \( \mathcal{C}(p, q, r) \) is the associative unital algebra over \( \mathbb{R} \) generated by the elements \( e_1, \ldots, e_n \) satisfying the following relation:

\[
(2) \quad e_ie_j = -eje_i, \quad \text{and} \quad e_i^2 = \begin{cases} 
-1, & \text{if } 1 \leq i \leq p \\
1, & \text{if } p + 1 \leq i \leq p + q \\
0, & \text{if } p + q + 1 \leq i \leq p + q + r.
\end{cases}
\]

It is common [20, 22, 33, 34, 67] to consider mainly Clifford algebras \( \mathcal{C}(n) = \mathcal{C}(n, 0, 0) \) of the Euclidean space or the algebra \( \mathcal{C}(p, q) = \mathcal{C}(p, q, 0) \) of the pseudo-Euclidean (Minkowski) spaces. However, Clifford algebras \( \mathcal{C}(p, q, r), r > 0 \) with nilpotent generators \( e_i^2 = 0 \) correspond to interesting geometry [44, 46, 60, 77] and physics [28–30, 47, 48, 53] as well. Yet, the geometry with idempotent units in spaces with dimensionality \( n > 2 \) is still not sufficiently elaborated.

An element of \( \mathcal{C}(p, q, r) \) having the form \( x = x_1e_1 + \ldots + x_ne_n \) can be associated with the vector \( (x_1, \ldots, x_n) \in \mathbb{R}^{pq} \). The reversion \( a \mapsto a^* \) in \( \mathcal{C}(p, q, r) \) [20, (1.19(ii))] is defined on vectors by \( x^* = x \) and extended to other elements by the relation \( (ab)^* = b^*a^* \). Similarly the conjugation is defined on vectors by \( \bar{x} = -x \) and the relation \( \bar{ab} = b\bar{a} \). We also use the notation \( |a|^2 = a\bar{a} \) for any product \( a \) of vectors. An important observation is that any non-zero \( x \in \mathbb{R}^{pq} \) has a multiplicative inverse: \( x^{-1} = \frac{\bar{x}}{|x|^2} \). For a \( 2 \times 2 \)-matrix \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with Clifford entries we define, cf. [20, (4.7)]

\[
(3) \quad M = \begin{pmatrix} a^* & -b^* \\ -c^* & a^* \end{pmatrix} \quad \text{and} \quad M^* = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}.
\]

Then \( M\overline{M} = \delta I \) for the pseudodeterminant \( \delta := ad^* - bc^* \).

Quadrics in \( \mathbb{R}^{pq} \)—which we continue to call cycles—can be associated to \( 2 \times 2 \) matrices through the FSC construction [20, (4.12); 25; 46, § 4.4]:

\[
(4) \quad k\bar{x}\bar{x} - \bar{l}\bar{x} - x\bar{l} + m = 0 \quad \leftrightarrow \quad C = \begin{pmatrix} l & m \\ k & \bar{l} \end{pmatrix},
\]

where \( k, m \in \mathbb{R} \) and \( l \in \mathbb{R}^{pq} \). For brevity we also encode a cycle by its coefficients \((k, l, m)\). A justification of (4) is provided by the identity:

\[
\begin{pmatrix} 1 & x \\ k & l \end{pmatrix} \begin{pmatrix} 1 & m \\ k & \bar{l} \end{pmatrix} = k\bar{x}\bar{x} - \bar{l}\bar{x} - x\bar{l} + m, \quad \text{since } \bar{x} = -x \text{ for } \bar{x} \in \mathbb{R}^{pq}.
\]

The identification is also FLT-covariant in the sense that the transformation (1) associated with the matrix \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) sends a cycle \( C \) to the cycle \( MCM^* \) [20, (4.16)]. We define the FLT-invariant inner product of cycles \( C_1 \) and \( C_2 \) by the identity

\[
(5) \quad \langle C_1, C_2 \rangle = \Re \text{tr}(C_1C_2)
\]

where \( \Re \) denotes the scalar part of a Clifford number. This definition in term of matrices immediately implies that the inner product is FLT-invariant. The explicit expression in terms of components of cycles \( C_1 = (k_1, l_1, m_1) \) and \( C_2 = (k_2, l_2, m_2) \) is also useful sometimes:

\[
(6) \quad \langle C_1, C_2 \rangle = l_1k_2 + \bar{l}_1\bar{k}_2 + m_1k_2 + m_2k_1.
\]

As usual, the relation \( \langle C_1, C_2 \rangle = 0 \) is called the orthogonality of cycles \( C_1 \) and \( C_2 \). In most cases it corresponds to orthogonality of quadrics in the point space. More generally, most of FLT-invariant relations between quadrics may be expressed in terms FLT-invariant inner product (5). For the full description of methods on
individual cycles, which are implemented in the library cycle, see the respective documentation [41].

**Remark 2.1.** Since cycles are elements of the projective space, the following normalized cycle product:

\[
[C_1, C_2] := \frac{\langle C_1, C_2 \rangle}{\sqrt{\langle C_1, C_1 \rangle \langle C_2, C_2 \rangle}}
\]

is more meaningful than the cycle product (5) itself. Note that, \([C_1, C_2]\) is defined only if neither \(C_1\) nor \(C_2\) is a zero-radius cycle (i.e. a point). Also, the normalised cycle product is \(\text{GL}_2(\mathbb{C})\)-invariant in comparison to \(\text{SL}_2(\mathbb{C})\)-invariance of (5).

We finish this brief review of the library cycle by pointing to its light version written in Asymptote language [31] and distributed together with the paper [54]. Although the light version mostly inherited API of the library cycle, there are some significant limitations caused by the absence of \(\text{GiNaC}\) support:

1. there is no symbolic computations of any sort,
2. the light version works in two dimensions only,
3. only elliptic metrics in the point and cycle spaces are supported.

On the other hand, being integrated with Asymptote the light version simplifies production of illustrations, which are its main target.

### 2.3. Connecting quadrics and cycles.

The library figure has an ability to store and resolve the system of geometric relations between cycles. We explain below some mathematical foundations, which greatly simplify this task.

We need a vocabulary, which translates geometric properties of quadrics on the point space to corresponding relations in the cycle space. The key ingredient is the cycle product (5)–(6), which is linear in each cycles’ parameters. However, certain conditions, e.g. tangency of cycles, involve polynomials of cycle products and thus are non-linear. For a successful algorithmic implementation, the following observation is important: all non-linear conditions below can be linearised if the additional quadratic condition of normalisation type is imposed:

\[
\langle C, C \rangle = \pm 1.
\]

This observation in the context of the Apollonius problem was already made in [26]. Conceptually the present work has a lot in common with the above mentioned paper of Fillmore and Springer, however a reader need to be warned that our implementation is totally different (and, interestingly, is more closer to another paper [25] of Fillmore and Springer).

**Remark 2.2.** Interestingly, the method of order reduction for algebraic equations is conceptually similar to the method of order reduction of differential equations used to build a geometric dynamics of quantum states in [4].

Here is the list of relations between cycles implemented in the current version of the library figure.

1. A quadric is flat (i.e. a hyperplane), that is, its equation is linear. Then, either of two equivalent conditions can be used:
   a. \(k\) component of the cycle vector is zero,
   b. the cycle is orthogonal \(\langle C_1, G_{\infty} \rangle = 0\) to the “zero-radius cycle at infinity” \(G_{\infty} = (0, 0, 1)\).
2. A quadric on the plane represents a line in Lobachevsky-type geometry if it is orthogonal \(\langle C_1, G_R \rangle = 0\) to the real line cycle \(C_R\). A similar condition is meaningful in higher dimensions as well.
(3) A quadric \( C \) represents a point, that is, it has zero radius at given metric of the point space. Then, the determinant of the corresponding FSC matrix is zero or, equivalently, the cycle is self-orthogonal (isotropic): \( \langle C, G \rangle = 0 \). Naturally, such a cycle cannot be normalised to the form (8).

(4) Two quadrics are orthogonal in the point space \( \mathbb{R}^q \). Then, the matrices representing cycles are orthogonal in the sense of the inner product (5).

(5) Two cycles \( C \) and \( S \) are tangent. Then we have the following quadratic condition:

\[
\langle C, S \rangle^2 = \langle C, C \rangle \langle S, S \rangle \quad \text{(or \( [C, S] = \pm 1 \))}
\]

With the assumption, that the cycle \( C \) is normalised by the condition (8), we may re-state this condition in the relation, which is linear to components of the cycle \( C \):

\[
\langle C, S \rangle = \pm \sqrt{\langle S, S \rangle}.
\]

Different signs here represent internal and outer touch.

(6) Inversive distance \( \theta \) of two (non-isotropic) cycles is defined by the formula:

\[
\langle C, S \rangle = \theta \sqrt{\langle C, C \rangle \langle S, S \rangle}
\]

In particular, the above discussed orthogonality corresponds to \( \theta = 0 \) and the tangency to \( \theta = \pm 1 \). For intersecting spheres \( \theta \) provides the cosine of the intersecting angle. For other metrics, the geometric interpretation of inversive distance shall be modified accordingly.

If we are looking for a cycle \( C \) with a given inversive distance \( \theta \) to a given cycle \( S \), then the normalisation (8) again turns the defining relation (11) into a linear with respect to parameters of the unknown cycle \( C \).

(7) A generalisation of Steiner power \( d \) of two cycles is defined as, cf. [26, § 1.1]:

\[
d = \langle C, S \rangle + \sqrt{\langle C, C \rangle \langle S, S \rangle}
\]

where both cycles \( C \) and \( S \) are \( k \)-normalised, that is the coefficient in front the quadratic term in (4) is 1. Geometrically, the generalised Steiner power for spheres provides the square of tangential distance. However, this relation is again non-linear for the cycle \( C \).

If we replace \( C \) by the cycle \( C_1 = \frac{1}{\sqrt{\langle C, C \rangle}} C \) satisfying (8), the identity (12) becomes:

\[
d \cdot k = \langle C_1, S \rangle + \sqrt{\langle S, S \rangle}
\]

where \( k = \frac{1}{\sqrt{\langle C, C \rangle}} \) is the coefficient in front of the quadratic term of \( C_1 \).

The last identity is linear in terms of the coefficients of \( C_1 \).

Summing up: if an unknown cycle is connected to already given cycles by any combination of the above relations, then all conditions can be expressed as a system of linear equations for coefficients of the unknown cycle and at most one quadratic equation (8).

3. Example: Poincaré Extension of Möbius Transformations

It is known, that Möbius transformations on \( \mathbb{R}^n \) can be expanded to the “upper” half-space in \( \mathbb{R}^{n+1} \) using the Poincaré extension [10, § 3.3; 61, § 5.2]. An explicit formula is usually presented without a discussion of its origin. In particular, one may get an impression that the solution is unique. Following [32] we consider various aspects of such extension and describe different possible realisations. Our consideration is restricted to the case of extension from the real line to the upper half-plane. However, we made an effort to present it in a way, which allows numerous further
generalisations. In section 5.1 we a partial realisation of Poincaré extension will be formulated through ensembles of cycles.

3.1. Geometric construction. We start from the geometric procedure in the standard situation. Recall, the group \( \text{SL}_2(\mathbb{R}) \) consists of real \( 2 \times 2 \) matrices with the unit determinant. \( \text{SL}_2(\mathbb{R}) \) acts on the real line by linear-fractional maps:

\[
\begin{pmatrix} a & b \\
 c & d \end{pmatrix} : x \mapsto \frac{ax + b}{cx + d}, \quad \text{where } x \in \mathbb{R} \text{ and } \begin{pmatrix} a & b \\
 c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}).
\]

A pair of (possibly equal) real numbers \( x \) and \( y \) uniquely determines a semicircle \( C_{xy} \) in the upper half-plane with the diameter \([x, y]\). For a linear-fractional transformation \( M \) (14), the images \( M(x) \) and \( M(y) \) define the semicircle with the diameter \([M(x), M(y)]\), thus, we can define the action of \( M \) on semicircles: \( M(C_{xy}) = C_{M(x)M(y)} \). Geometrically, the Poincaré extension is based on the following lemma, see Fig. 1(a) and more general Lem. 3.18 below:

**Lemma 3.1.** If a pencil of semicircles in the upper half-plane has a common point, then the images of these semicircles under a transformation (14) have a common point as well.

Elementary geometry of right triangles tells that a pair of intersecting intervals \([x, y], [x', y']\), where \( x < x' < y < y' \), defines the point

\[
\left( \frac{xy - x'y'}{x + y - x' - y'}, \frac{\sqrt{(x-y')(x-x')(x'-y)(y-y')}}{x + y - x' - y'} \right) \in \mathbb{R}^2_+.
\]

An alternative demonstration uses three observations:

1. the scaling \( x \mapsto ax, a > 0 \) on the real line produces the scaling \((u, v) \mapsto (au, av)\) on pairs (15),
2. the horizontal shift \( x \mapsto x + b \) on the real line produces the horizontal shift \( (u, v) \mapsto (u + b, v) \) on pairs (15),
3. for the special case \( y = -x^{-1} \) and \( y' = -x'^{-1} \) the pair (15) is \((0, 1)\).

Finally, expression (15), as well as (16)–(17) below, can be calculated by the specialised CAS for Möbius invariant geometries \([41, 49]\).

This standard approach can be widened as follows. The above semicircle can be equivalently described through the unique circle passing \( x \) and \( y \) and orthogonal to the real axis. Similarly, an interval \([x, y]\) uniquely defines a right-angle hyperbola in \( \mathbb{R}^2 \) orthogonal to the real line and passing (actually, having her vertices at) \((x, 0)\) and \((y, 0)\). An intersection with the second such hyperbola having vertices \( (x', 0) \) and \((y', 0)\) defines a point with coordinates, see Fig. 1(f):

\[
\left( \frac{xy - x'y'}{x + y - x' - y'}, \frac{\sqrt{(x-y')(x-x')(x'-y)(y-y')}}{x + y - x' - y'} \right),
\]

where \( x < y < x' < y' \). Note, the opposite sign of the product under the square roots in (15) and (16).

If we wish to consider the third type of conic sections—parabolas—we cannot use the unaltered procedure: there is no a non-degenerate parabola orthogonal to the real line and intersecting the real line in two points. We may recall, that a circle (or hyperbola) is orthogonal to the real line if its centre belongs to the real line. Analogously, a parabola is *focally orthogonal* (see \([46, \S 6.6]\)) for a general consideration) to the real line if its focus belongs to the real line. Then, an interval \([x, y]\) uniquely defines a downward-opened parabola with the real roots \( x \) and \( y \) and
focally orthogonal to the real line. Two such parabolas defined by intervals \([x, y]\) and \([x', y']\) have a common point, see Fig. 1(c):

\[
(17) \quad \left( \frac{xy' - yx'}{x - y - x' + y'}, \frac{(x' - x)(y' - y)(y - x + y' - x') + (x + y - x' - y')D}{(x - y - x' + y')^2} \right)
\]

where \(D = \pm \sqrt{(x - x')(y - y')(y - x)(y' - x')}\). For pencils of such hyperbolas and parabolas respective variants of Lem. 3.1 hold.

Focally orthogonal parabolas make the angle 45° with the real line. This suggests to replace orthogonal circles and hyperbolas by conic sections with a fixed angle to the real line, see Fig. 1(b)–(e). Of course, to be consistent this procedure requires a suitable modification of Lem. 3.1, we will obtain it as a byproduct of our study, see Lem. 3.18. However, the respective alterations of the above formulae (15)–(17) become more complicated in the general case.

The considered geometric construction is elementary and visually appealing. Now we turn to respective algebraic consideration.

3.2. Möbius transformations and Cycles. The group \(\text{SL}_2(\mathbb{R})\) acts on \(\mathbb{R}^2\) by matrix multiplication on column vectors:

\[
(18) \quad \mathcal{L}_g : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \text{where} \; g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}).
\]
A linear action respects the equivalence relation \( (x_1, x_2) \sim (\lambda x_1, \lambda x_2), \lambda \neq 0 \) on \( \mathbb{R}^2 \). The collection of all cosets for non-zero vectors in \( \mathbb{R}^2 \) is the projective line \( \mathbb{P}\mathbb{R}^1 \).

Explicitly, a non-zero vector \( \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \) corresponds to the point with homogeneous coordinates \([x_1 : x_2] \in \mathbb{P}\mathbb{R}^1\). If \( x_2 \neq 0 \) then this point is represented by \([\frac{x_1}{x_2} : 1]\) as well. The embedding \( \mathbb{R} \to \mathbb{P}\mathbb{R}^1 \) defined by \( x \mapsto [x : 1], x \in \mathbb{R} \) covers the all but one of points in \( \mathbb{P}\mathbb{R}^1 \). The exceptional point \([1 : 0]\) is naturally identified with the infinity.

The linear action (18) induces a morphism of the projective line \( \mathbb{P}\mathbb{R}^1 \), which is called a Möbius transformation. Considered on the real line within \( \mathbb{P}\mathbb{R}^1 \), Möbius transformations takes fraction-linear form:

\[
g : [x : 1] \mapsto \begin{pmatrix} ax + b \\ cx + d \end{pmatrix} : 1, \quad \text{where } g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}) \text{ and } cx + d \neq 0.
\]

This \( \text{SL}_2(\mathbb{R}) \)-action on \( \mathbb{P}\mathbb{R}^1 \) is denoted as \( g : x \mapsto g \cdot x \). We note that the correspondence of column vectors and row vectors \( i : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto (x_2, -x_1) \) intertwines the left multiplication \( L_g \) (18) and the right multiplication \( R_{g^{-1}} \) by the inverse matrix:

\[
(19) \quad R_{g^{-1}} : (x_2, -x_1) \mapsto (cx_1 + dx_2, -ax_1 - bx_2) = (x_2, -x_1) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.
\]

We extended the map \( i \) to \( 2 \times 2 \)-matrices by the rule:

\[
(20) \quad i : \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_2 \\ -y_1 \\ x_2 \\ -x_1 \end{pmatrix}.
\]

Two columns \( \begin{pmatrix} x \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} y \\ 1 \end{pmatrix} \) form the \( 2 \times 2 \) matrix \( M_{xy} = \begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix} \). For geometrical reasons appearing in Cor. 3.3, we call a cycle the \( 2 \times 2 \)-matrix \( C_{xy} \) defined by

\[
(21) \quad C_{xy} = \frac{1}{2} M_{xy} \cdot i(M_{xy}) = \frac{1}{2} M_{yx} \cdot i(M_{yx}) = \begin{pmatrix} \frac{x+y}{2} & \frac{-x+y}{2} \\ \frac{-x+y}{2} & \frac{x+y}{2} \end{pmatrix}.
\]

We note that \( \det C_{xy} = -(x-y)^2/4 \), thus \( \det C_{xy} = 0 \) if and only if \( x = y \). Also, we can consider the Möbius transformation produced by the \( 2 \times 2 \)-matrix \( C_{xy} \) and calculate:

\[
(22) \quad C_{xy} \begin{pmatrix} x \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} x \\ 1 \end{pmatrix} \quad \text{and} \quad C_{xy} \begin{pmatrix} y \\ 1 \end{pmatrix} = -\lambda \begin{pmatrix} y \\ 1 \end{pmatrix} \quad \text{where } \lambda = \frac{x-y}{2}.
\]

Thus, points \([x : 1], [y : 1] \in \mathbb{P}\mathbb{R}^1 \) are fixed by \( C_{xy} \). Also, \( C_{xy} \) swaps the interval \([x, y] \) and its complement.

Due to their structure, matrices \( C_{xy} \) can be parametrised by points of \( \mathbb{R}^3 \). Furthermore, the map from \( \mathbb{R}^2 \to \mathbb{R}^3 \) given by \( (x, y) \mapsto C_{xy} \) naturally induces the projective map \( (\mathbb{P}\mathbb{R}^1)^2 \to \mathbb{P}\mathbb{R}^2 \) due to the identity:

\[
\frac{1}{2} \begin{pmatrix} \lambda x & \mu y \\ \mu & -\lambda x \end{pmatrix} \begin{pmatrix} \mu & -\mu y \\ \mu & \mu \lambda \end{pmatrix} = \lambda \mu \begin{pmatrix} \frac{x+y}{2} & \frac{-x+y}{2} \\ \frac{-x+y}{2} & \frac{x+y}{2} \end{pmatrix} = \lambda \mu C_{xy}.
\]

Conversely, a zero-trace matrix \( \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \) with a non-positive determinant is projectively equivalent to a product \( C_{xy} \) (21) with \( x, y = \frac{a + \sqrt{a^2 + 4bc}}{2} \). In particular, we can embed a point \([x : 1] \in \mathbb{P}\mathbb{R}^1 \) to \( 2 \times 2 \)-matrix \( C_{xy} \) with zero determinant.

The combination of (18)–(21) implies that the correspondence \( (x, y) \mapsto C_{xy} \) is \( \text{SL}_2(\mathbb{R}) \)-covariant in the following sense:

\[
(23) \quad g C_{xy} g^{-1} = C_{x'y'}, \quad \text{where } x' = g \cdot x \text{ and } y' = g \cdot y.
\]
To achieve a geometric interpretation of all matrices, we consider the bilinear form $Q : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ generated by a $2 \times 2$-matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \):

\begin{equation}
Q(x, y) = (x_1 \ x_2) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \text{ where } x = (x_1, x_2), \ y = (y_1, y_2).
\end{equation}

Due to linearity of $Q$, the null set
\begin{equation}
\{ (x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid Q(x, y) = 0 \}
\end{equation}
factors to a subset of $P\mathbb{R}^1 \times P\mathbb{R}^1$. Furthermore, for the matrices $C_{xy}$ (21), a direct calculation shows that:

**Lemma 3.2.** The following identity holds:
\begin{equation}
C_{xy}(i(x'), y') = \text{tr}(C_{xy}C_{x' y'}) = \frac{1}{2}(x + y)(x' + y') - (xy + x'y').
\end{equation}

In particular, the above expression is a symmetric function of the pairs $(x, y)$ and $(x', y')$.

The map $i$ appearance in (26) is justified once more by the following result.

**Corollary 3.3.** The null set of the quadratic form $C_{xy}(x') = C_{xy}(i(x')$, $x'$) consists of two points $x$ and $y$.

Alternatively, the identities $C_{xy}(x) = C_{xy}(y) = 0$ follows from (22) and the fact that $i(z)$ is orthogonal to $z$ for all $z \in \mathbb{R}^2$. Also, we note that:
\begin{equation}
i \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (x_1 \ x_2) \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\end{equation}

Motivated by Lem. 3.2, we call $\langle C_{xy}, C_{x' y'} \rangle := -\text{tr}(C_{xy}C_{x' y'})$ the pairing of two cycles. It shall be noted that the pairing is not positively defined, this follows from the explicit expression (26). The sign is chosen in such way, that
\begin{equation}
\langle C_{xy}, C_{xy} \rangle = -2\det(C_{xy}) = \frac{1}{2}(x - y)^2 \geq 0.
\end{equation}

Also, an immediate consequence of Lem. 3.2 or identity (24) is

**Corollary 3.4.** The pairing of cycles is invariant under the action (23) of $\text{SL}_2(\mathbb{R})$:
\begin{equation}
\langle g \cdot C_{x' y'}, g^{-1} \rangle = \langle C_{xy}, C_{x' y'} \rangle.
\end{equation}

From (26), the null set (25) of the form $Q = C_{xy}$ can be associated to the family of cycles \( \{ C_{x' y'} \mid \langle C_{xy}, C_{x' y'} \rangle = 0, (x', y') \in \mathbb{R}^2 \times \mathbb{R}^2 \} \) which we will call orthogonal to $C_{xy}$.

### 3.3 Extending cycles.

Since bilinear forms with matrices $C_{xy}$ have numerous geometric connections with $P\mathbb{R}^1$, we are looking for a similar interpretation of the generic matrices. The previous discussion identified the key ingredient of the recipe: $\text{SL}_2(\mathbb{R})$-invariant pairing (26) of two forms. Keeping in mind the structure of $C_{xy}$, we will parameterise \( a generic 2 \times 2 \) matrix as \( \begin{pmatrix} l + n & -m \\ k & -l + n \end{pmatrix} \) and consider the corresponding four dimensional vector $(n, l, k, m)$. Then, the similarity with
\begin{equation}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}):
\end{equation}

\begin{equation}
\begin{pmatrix} l' + n' & -m' \\ k' & -l' + n' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} l + n & -m \\ k & -l + n \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}
\end{equation}

\[ \text{Further justification of this parametrisation will be obtained from the equation of a quadratic curve (31).} \]
corresponds to the linear transformation of \( \mathbb{R}^4 \), cf. [46, Ex. 4.15]:

\[
\begin{pmatrix}
  n' \\
  l' \\
  k' \\
  m'
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & cb + ad & bd & ca \\
  0 & 2cd & d^2 & c^2 \\
  0 & 2ab & b^2 & a^2
\end{pmatrix}
\begin{pmatrix}
  n \\
  l \\
  k \\
  m
\end{pmatrix}.
\]

In particular, this action commutes with the scaling of the first component:

\[
\lambda : (n, l, k, m) \mapsto (\lambda n, l, k, m).
\]

This expression is helpful in proving the following statement.

**Lemma 3.5.** Any \( \text{SL}_2(\mathbb{R}) \)-invariant (in the sense of the action (28)) pairing in \( \mathbb{R}^4 \) is isomorphic to

\[
2\tau n' - 2ll' + km' + mk' = (n' \ l' \ k' \ m')
\begin{pmatrix}
  2\tau & 0 & 0 & 0 \\
  0 & -2 & 0 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
  n \\
  l \\
  k \\
  m
\end{pmatrix}
\]

where \( \tau = -1, 0 \) or 1 and \( (n, l, k, m), (n', l', k', m') \in \mathbb{R}^4 \).

**Proof.** Let \( T \) be a 4 \times 4 matrix from (28), if a \( \text{SL}_2(\mathbb{R}) \)-invariant pairing is defined by a 4 \times 4 matrix \( J = (j_{ij}) \), then \( T'JT = J \), where \( T' \) is transpose of \( T \). The equivalent identity \( T'^*J = JT^{-1} \) produces a system of homogeneous linear equations which has the generic solution:

\[
\begin{align*}
  j_{12} &= j_{13} = j_{14} = j_{21} = j_{31} = j_{41} = 0, \\
  j_{22} &= \frac{(d-a)j_{42} - 2bj_{44}}{c} - 2j_{43}, \\
  j_{23} &= -\frac{b}{c}j_{42}, \\
  j_{24} &= -\frac{cj_{42} + 2(a-d)j_{44}}{c}, \\
  j_{34} &= \frac{c(a-d)j_{42} + (a-d)^2j_{44}}{c^2} + j_{43}, \\
  j_{33} &= \frac{b^2}{c^2}j_{44}, \\
  j_{32} &= \frac{b}{c}j_{42} + 2(a-d)j_{44}
\end{align*}
\]

with four free variables \( j_{11}, j_{42}, j_{43} \) and \( j_{44} \). Since a solution shall not depend on \( a, b, c, d \), we have to put \( j_{42} = j_{44} = 0 \). Then by the homogeneity of the identity \( T'^*J = JT^{-1} \), we can scale \( j_{43} \) to 1. Thereafter, an independent (sign-preserving) scaling (29) of \( n \) leaves only three non-isomorphic values \(-1, 0, 1\) of \( j_{11} \). \( \square \)

The appearance of the three essential different cases \( \tau = -1, 0 \) or 1 in Lem. 3.5 is a manifestation of the common division of mathematical objects into elliptic, parabolic and hyperbolic cases [40; 46, Ch. 1]. Thus, we will use letters “e”, “p”, “h” to encode the corresponding three values of \( \tau \).

Now we may describe all \( \text{SL}_2(\mathbb{R}) \)-invariant pairings of bilinear forms.

**Corollary 3.6.** Any \( \text{SL}_2(\mathbb{R}) \)-invariant (in the sense of the similarity (27)) pairing between two bilinear forms \( Q = \begin{pmatrix} l + n & -m \\
  k & -l + n \end{pmatrix} \) and \( Q' = \begin{pmatrix} l' + n' & -m' \\
  k' & -l' + n' \end{pmatrix} \) is isomorphic to:

\[
\langle Q, Q' \rangle_\tau = -\text{tr}(Q_\tau Q')
\]

\[
= 2\tau n'n - 2l'l + k'm + m'k,
\]

where \( Q_\tau = \begin{pmatrix} l - \tau n & -m \\
  k & -l - \tau n \end{pmatrix} \) and \( \tau = -1, 0 \) or 1.

Note that we can explicitly write \( Q_\tau \) for \( Q = \begin{pmatrix} a & b \\
  c & d \end{pmatrix} \) as follows:

\[
Q_e = \begin{pmatrix} a & b \\
  c & d \end{pmatrix}, \quad Q_p = \begin{pmatrix} \frac{1}{2}(a - d) & b \\
  c & -\frac{1}{2}(a - d) \end{pmatrix}, \quad Q_h = \begin{pmatrix} -d & b \\
  c & -a \end{pmatrix}.
\]
In particular, \( Q_h = -Q^{-1} \) and \( Q_\rho = \frac{1}{2}(Q_e + Q_h) \). Furthermore, \( Q_\rho \) has the same structure as \( C_{xy} \). Now, we are ready to extend the projective line \( \mathbb{P}\mathbb{R}^1 \) to two dimensions using the analogy with properties of cycles \( C_{xy} \).

**Definition 3.7.**  
(1) Two bilinear forms \( Q \) and \( Q' \) are \( \tau \)-orthogonal if \( \langle Q, Q' \rangle_\tau = 0 \).

(2) A form is \( \tau \)-isotropic if it is \( \tau \)-orthogonal to itself.

If a form \( Q = \begin{pmatrix} l + n & -m \\ k & -l + n \end{pmatrix} \) has \( k \neq 0 \) then we can scale it to obtain \( k = 1 \), this form of \( Q \) is called normalised. A normalised \( \tau \)-isotropic form is completely determined by its diagonal values: \( \begin{pmatrix} u + v & -u^2 + \tau v^2 \\ 1 & -u + v \end{pmatrix} \). Thus, the set of such forms is in a one-to-one correspondence with points of \( \mathbb{R}^2 \). Finally, a form \( Q = \begin{pmatrix} l + n & -m \\ k & -l + n \end{pmatrix} \) is e-orthogonal to the \( \tau \)-isotropic form \( \begin{pmatrix} u + v & -u^2 + \tau v^2 \\ 1 & -u + v \end{pmatrix} \) if:

\[
(31) \quad k(u^2 - \tau v^2) - 2lu - 2nv + m = 0
\]
that is the point \((u, v) \in \mathbb{R}^2\) belongs to the quadratic curve with coefficients \((k, l, n, m)\).

### 3.4. Homogeneous spaces of cycles

Obviously, the group \( \text{SL}_2(\mathbb{R}) \) acts on \( \mathbb{P}\mathbb{R}^1 \) transitively, in fact it is even 3-transitive in the following sense. We say that a triple \( \{x_1, x_2, x_3\} \subset \mathbb{P}\mathbb{R}^1 \) of distinct points is **positively oriented** if

either \quad x_1 < x_2 < x_3, \quad \text{or} \quad x_3 < x_1 < x_2

where we agree that the ideal point \( \infty \in \mathbb{P}\mathbb{R}^1 \) is greater than any \( x \in \mathbb{R} \). Equivalently, a triple \( \{x_1, x_2, x_3\} \) of reals is positively oriented if:

\[
(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) > 0.
\]

Also, a triple of distinct points, which is not positively oriented, is **negatively oriented**. A simple calculation based on the resolvent-type identity:

\[
\frac{ax + b}{cx + d} = \frac{ay + b}{cy + d} = \frac{(x - y)(ad - bc)}{(cx + b)(cy + d)}
\]

shows that both the positive and negative orientations of triples are \( \text{SL}_2(\mathbb{R}) \)-invariant. On the other hand, the reflection \( x \mapsto -x \) swaps orientations of triples. Note, that the reflection is a Möbius transformation associated to the cycle

\[
(32) \quad C_{0\infty} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \text{with } \det C_{0\infty} = -1.
\]

A significant amount of information about Möbius transformations follows from the fact, that any continuous one-parametric subgroup of \( \text{SL}_2(\mathbb{R}) \) is conjugated to one of the three following subgroups\(^3\):

\[
(33) \quad A = \left\{ \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix} \right\}, \quad N = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right\}, \quad K = \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \right\}
\]

where \( t \in \mathbb{R} \). Also, it is useful to introduce subgroups \( A' \) and \( N' \) conjugated to \( A \) and \( N \) respectively:

\[
(34) \quad A' = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \mid t \in \mathbb{R} \right\}, \quad N' = \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}.
\]

---

\(^3\)A reader may know that \( A, N \) and \( K \) are factors in the Iwasawa decomposition \( \text{SL}_2(\mathbb{R}) = ANK \) (cf. Cor. 3.10), however this important result does not play any rôle in our consideration.
Thereafter, all three one-parameter subgroups $A'$, $N'$ and $K$ consist of all matrices with the universal structure

\[
\begin{pmatrix}
a & \tau b \\
b & a
\end{pmatrix}
\]

where $\tau = 1, 0, -1$ for $A'$, $N'$ and $K$ respectively.

We use the notation $H_\tau$ for these subgroups. Again, any continuous one-dimensional subgroup of $\text{SL}_2(\mathbb{R})$ is conjugated to $H_\tau$ for an appropriate $\tau$.

We note, that matrices from $A$, $N$ and $K$ with $t \neq 0$ have two, one and none different real eigenvalues respectively. Eigenvectors in $\mathbb{R}^2$ correspond to fixed points of Möbius transformations on $\mathbb{RP}^1$. Clearly, the number of eigenvectors (and thus fixed points) is limited by the dimensionality of the space, that is two. For this reason, if $g_1$ and $g_2$ take equal values on three different points of $\mathbb{RP}^1$, then $g_1 = g_2$.

Also, eigenvectors provide an effective classification tool: $g \in \text{SL}_2(\mathbb{R})$ belongs to a one-dimensional continuous subgroup conjugated to $A$, $N$ or $K$ if and only if the characteristic polynomial $\det(g - \lambda I)$ has two, one and none different real root(s) respectively. We will illustrate an application of fixed points techniques through the following well-known result, which will be used later.

**Lemma 3.8.** Let \( \{x_1, x_2, x_3\} \) and \( \{y_1, y_2, y_3\} \) be positively oriented triples of points in $\mathbb{R}$. Then, there is a unique (computable!) Möbius map $\phi \in \text{SL}_2(\mathbb{R})$ with $\phi(x_j) = y_j$ for $j = 1, 2, 3$.

**Proof.** Often, the statement is quickly demonstrated through an explicit expression for $\phi$, cf. [12, Thm. 13.2.1]. We will use properties of the subgroups $A$, $N$ and $K$ to describe an algorithm to find such a map. First, we notice that it is sufficient to show the Lemma for the particular case $y_1 = 0$, $y_2 = 1$, $y_3 = \infty$. The general case can be obtained from composition of two such maps. Another useful observation is that the fixed point for $N$, that is $\infty$, is also a fixed point of $A$.

Now, we will use subgroups $K$, $N$ and $A$ in order of increasing number of their fixed points. First, for any $x_3$ the matrix $g' = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \in K$ such that $\cot t = x_3$ maps $x_3$ to $y_1 = \infty$. Let $x'_1 = g'x_1$ and $x'_2 = g'x_2$. Then the matrix $g'' = \begin{pmatrix} 1 & -x'_1 \\ 0 & 1 \end{pmatrix} \in N$, fixes $\infty = g'x_3$ and sends $x'_1$ to $y_1 = 0$. Let $x''_2 = g''x'_2$, from positive orientation of triples we have $0 < x''_2 < \infty$. Next, the matrix $g''' = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \in A$ with $a = \sqrt{x''_2}$ sends $x''_2$ to $1$ and fixes both $\infty = g''g'x_3$ and $0 = g''g'x_1$. Thus, $g = g'''g''g'$ makes the required transformation $(x_1, x_2, x_3) \mapsto (0, 1, \infty)$. \qed

**Corollary 3.9.** Let \( \{x_1, x_2, x_3\} \) and \( \{y_1, y_2, y_3\} \) be two triples with the opposite orientations. Then, there is a unique Möbius map $\phi \in \text{SL}_2(\mathbb{R})$ with $\phi \circ C_{0\infty}(x_j) = y_j$ for $j = 1, 2, 3$.

We will denote by $\phi_{XY}$ the unique map from Lem. 3.8 defined by triples $X = \{x_1, x_2, x_3\}$ and $Y = \{y_1, y_2, y_3\}$.

Although we are not going to use it in this paper, we note that the following important result [58, § III.1] is an immediate consequence of our proof of Lem. 3.8.

**Corollary 3.10** (Iwasawa decomposition). Any element of $g \in \text{SL}_2(\mathbb{R})$ is a product $g = g_AG_Ng_K$, where $g_A$, $g_N$ and $g_K$ belong to subgroups $A$, $N$, $K$ respectively and those factors are uniquely defined.

In particular, we note that it is not a coincidence that the subgroups appear in the Iwasawa decomposition $\text{SL}_2(\mathbb{R}) = ANK$ in order of decreasing number of their fixed points.
3.5. Triples of intervals. We change our point of view and instead of two ordered triples of points consider three ordered pairs, that is—three intervals or, equivalently, three cycles. This is done in line with our extension of Lie–Möbius geometry.

For such triples of intervals we will need the following definition.

**Definition 3.11.** We say that a triple of intervals \( \{[x_1, y_1], [x_2, y_2], [x_3, y_3]\} \) is **aligned** if the triples \( X = \{x_1, x_2, x_3\} \) and \( Y = \{y_1, y_2, y_3\} \) of their endpoints have the same orientation.

Aligned triples determine certain one-parameter subgroups of Möbius transformations as follows:

**Proposition 3.12.** Let \( \{[x_1, y_1], [x_2, y_2], [x_3, y_3]\} \) be an aligned triple of intervals.

1. If \( \phi_{XY} \) has at most one fixed point, then there is a unique (up to a parametrisation) one-parameter semigroup of Möbius map \( \psi(t) \subset \text{SL}_2(\mathbb{R}) \), which maps \([x_1, y_1]\) to \([x_2, y_2]\) and \([x_3, y_3]\):

\[
\psi(t_j)(x_j) = x_j, \quad \psi(t_j)(y_j) = y_j, \quad \text{for some } t_j \in \mathbb{R} \text{ and } j = 2, 3.
\]

2. Let \( \phi_{XY} \) have two fixed points \( x < y \) and \( C_{xy} \) be the orientation inverting Möbius transformation with the matrix (21).

\[
\begin{align*}
x_j' &= x_j, \quad y_j' = y_j, \quad x_j'' = C_{xy}x_j, \quad y_j'' = C_{xy}y_j \quad \text{if } x < x_j < y, \\
&\quad x_j' = C_{xy}x_j, \quad y_j' = C_{xy}y_j, \quad x_j'' = x_j, \quad y_j'' = y_j, \quad \text{otherwise}.
\end{align*}
\]

Then, there is a one-parameter semigroup of Möbius map \( \psi(t) \subset \text{SL}_2(\mathbb{R}) \), and \( t_2, t_3 \in \mathbb{R} \) such that:

\[
\psi(t_j)(x_j') = x_j'', \quad \psi(t_j)(x_j'') = x_j''', \quad \psi(t_j)(y_j') = y_j'', \quad \psi(t_j)(y_j'') = y_j'''
\]

where \( j = 2, 3 \).

**Proof.** Consider the one-parameter subgroup of \( \psi(t) \subset \text{SL}_2(\mathbb{R}) \) such that \( \phi_{XY} = \psi(1) \). Note, that \( \psi(t) \) and \( \phi_{XY} \) have the same fixed points (if any) and no point \( x_j \) is fixed since \( x_j \neq y_j \). If the number of fixed points is less than 2, then \( \psi(t)x_1, t \in \mathbb{R} \) produces the entire real line except a possible single fixed point. Therefore, there are \( t_2 \) and \( t_3 \) such that \( \psi(t_2)x_1 = x_2 \) and \( \psi(t_3)x_1 = x_3 \). Since \( \psi(t) \) and \( \phi_{XY} \) commute for all \( t \) we also have:

\[
\psi(t_j)y_j = \psi(t_j)\phi_{XY}x_1 = \phi_{XY}\psi(t_j)x_1 = \phi_{XY}x_j = y_j, \quad \text{for } j = 2, 3.
\]

If there are two fixed points \( x < y \), then the open interval \( (x, y) \) is an orbit for the subgroup \( \psi(t) \). Since all \( x_1', x_2' \) and \( x_3' \) belong to this orbit and \( C_{xy} \) commutes with \( \phi_{XY} \), we may repeat the above reasoning for the dashed intervals \([x_j', y_j']\). Finally, \( x_j'' = C_{xy}x_j' \) and \( y_j'' = C_{xy}y_j' \), where \( C_{xy} \) commutes with \( \phi = \psi(t_j), j = 2, 3 \). Uniqueness of the subgroup follows from Lemma 3.13.

The group \( \text{SL}_2(\mathbb{R}) \) acts transitively on collection of all cycles \( C_{xy} \), thus this is an \( \text{SL}_2(\mathbb{R}) \)-homogeneous space. It is easy to see that the fix-group of the cycle \( C_{-1,1} \) is \( A' \) (34). Thus the homogeneous space of cycles is isomorphic to \( \text{SL}_2(\mathbb{R})/A' \).

**Lemma 3.13.** Let \( H \) be a one-parameter continuous subgroup of \( \text{SL}_2(\mathbb{R}) \) and \( X = \text{SL}_2(\mathbb{R})/H \) be the corresponding homogeneous space. If two orbits of one-parameter continuous subgroups on \( X \) have at least three common points then these orbits coincide.

**Proof.** Since \( H \) is conjugated either to \( A' \), \( N' \) or \( K \), the homogeneous space \( X = \text{SL}_2(\mathbb{R})/H \) is isomorphic to the upper half-plane in double, dual or complex numbers [46, § 3.3.4]. Orbits of one-parameter continuous subgroups in \( X \) are conic sections, which are circles, parabolas (with vertical axis) or equilateral hyperbolas.
Alternatively, we can reformulate Prop. 3.12 as follows: three different cycles \( C_{x,y}, C_{x',y}, C_{x,y}, \) define a one-parameter subgroup, which generate either one orbit or two related orbits passing the three cycles.

We have seen that the number of fixed points is the key characteristics for the map \( \phi_{XY} \). The next result gives an explicit expression for it.

**Proposition 3.14.** The map \( \phi_{XY} \) has zero, one or two fixed points if the expression

\[
\text{(36) } \det \begin{pmatrix} 1 & x_1 y_1 & -x_1 y_1 & 1 \\ 1 & x_2 y_2 & -x_2 y_2 & 1 \\ 1 & x_3 y_3 & -x_3 y_3 & 1 \end{pmatrix} - 4 \det \begin{pmatrix} x_1 & 1 & y_1 & 1 \\ x_2 & 1 & y_2 & 1 \\ x_3 & 1 & y_3 & 1 \end{pmatrix} \cdot \det \begin{pmatrix} x_1 & -x_1 y_1 & y_1 & 1 \\ x_2 & -x_2 y_2 & y_2 & 1 \\ x_3 & -x_3 y_3 & y_3 & 1 \end{pmatrix}
\]

is negative, zero or positive respectively.

**Proof.** If a Möbius transformation \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) maps \( x_1 \mapsto y_1, x_2 \mapsto y_2, x_3 \mapsto y_3 \) and \( s \mapsto s \), then we have a homogeneous linear system, cf. [12, Ex. 13.2.4]:

\[
\begin{pmatrix} x_1 & 1 & -x_1 y_1 & -y_1 \\ x_2 & 1 & -x_2 y_2 & -y_2 \\ x_3 & 1 & -x_3 y_3 & -y_3 \\ s & 1 & -s^2 & -s \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]

A non-zero solution exists if the determinant of the \( 4 \times 4 \) matrix is zero. Expanding it over the last row and rearranging terms we obtain the quadratic equation for the fixed point \( s \):

\[
s^2 \det \begin{pmatrix} x_1 & 1 & y_1 \\ x_2 & 1 & y_2 \\ x_3 & 1 & y_3 \end{pmatrix} + s \det \begin{pmatrix} 1 & x_1 y_1 & y_1 - x_1 \\ 1 & x_2 y_2 & y_2 - x_2 \\ 1 & x_3 y_3 & y_3 - x_3 \end{pmatrix} + \det \begin{pmatrix} x_1 & -x_1 y_1 & y_1 \\ x_2 & -x_2 y_2 & y_2 \\ x_3 & -x_3 y_3 & y_3 \end{pmatrix} = 0.
\]

The value (36) is the discriminant of this equation. □

**Remark 3.15.** It is interesting to note, that the relation \( ax + b - cxy - dy = 0 \) used in (37) can be stated as \( e \)-orthogonality of the cycle \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and the isotropic bilinear form \( \begin{pmatrix} x & -xy \\ 1 & -y \end{pmatrix} \).

If \( y = g_0 \cdot x \) for some \( g_0 \in H_\tau \), then for any \( g \in H_\tau \) we also have \( y_g = g_0 \cdot x_g \), where \( x_g = g \cdot x \) and \( y_g = g \cdot y \). Thus, we demonstrated the first part of the following result.

**Lemma 3.16.** Let \( \tau = 1, 0 \) or \(-1 \) and a real constant \( t \neq 0 \) be such that \( 1 - \tau t^2 > 0 \).

1. The collections of intervals:

\[
I_{\tau,t} = \left\{ [x, \frac{x + \tau t}{1 - \tau t}] \mid x \in \mathbb{R} \right\}
\]

is preserved by the actions of subgroup \( H_\tau \). Any three different intervals from \( I_{\tau,t} \) define the subgroup \( H_\tau \) in the sense of Prop. 3.12.

2. All \( H_\tau \)-invariant bilinear forms compose the family \( P_{\tau,t} = \left\{ \begin{pmatrix} a & \tau b \\ b & a \end{pmatrix} \right\} \).

The family \( P_{\tau,t} \) consists of the eigenvectors of the \( 4 \times 4 \) matrix from (28) with suitably substituted entries. There is (up to a factor) exactly one \( \tau \)-isotropic form in \( P_{\tau,t} \), namely \( \begin{pmatrix} 1 & \tau \\ 1 & 1 \end{pmatrix} \). We denote this form by \( \iota \). It corresponds to the point
the real line, the square of the cosine of this angle is:

\[
\frac{\langle Q,R \rangle}{\sqrt{|\langle Q,Q \rangle|}} = \frac{\tau}{\sqrt{\tau^2 - \tau}}.
\]

Here, the real line is represented by the bilinear form \( R = \begin{pmatrix} 2^{-1/2} & 0 \\ 0 & 2^{-1/2} \end{pmatrix} \) normalised such that \( \langle R,R \rangle = \pm 1 \).

Proof. The first statement is verified by a short calculation. A form \( Q = \begin{pmatrix} l+n & -m \\ k & -l+n \end{pmatrix} \) in the second statement may be calculated from the homogeneous system:

\[
\begin{pmatrix}
0 & -2x & x^2 - 1 \\
0 & -2y & y^2 - 1 \\
-2 & 0 & -\tau - 1
\end{pmatrix}
\begin{pmatrix}
l \\
k \\
m
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

which has the rank 3 if \( x \neq y \). The third statement can be checked by a calculation as well. Finally, the last item is a particular case of the more general statement as indicated. Yet, we can derive it here from the implicit derivative \( \frac{dv}{du} = \frac{ku-\tau}{\tau} \) of the function \( k(u^2 + \tau v^2) - 2lu - 2nv + m = 0 \) (31) at the point \((u,0)\). Note that this value is independent from \( \tau \). Since this is the tangent of the intersection angle with the real line, the square of the cosine of this angle is:

\[
\frac{1}{1 + (\frac{dv}{du})^2} = \frac{n^2}{l^2 + k^2 u^2 + n^2 - 2ku} = \frac{\langle Q,R \rangle^2}{\langle Q,Q \rangle}.
\]

if \( ku^2 - 2lu + m = 0 \). 

Also, we note that, the independence of the left-hand side of (39) from \( x \) can be shown from basic principles. Indeed, for a fixed \( t \) the subgroup \( H_\tau \) acts transitively on the family of triples \( \{x, \frac{x+\tau}{1+\tau}, t\} \), thus \( H_\tau \) acts transitively on all bilinear forms orthogonal to such triples. However, the left-hand side of (39) is \( \text{SL}_2(\mathbb{R}) \)-invariant, thus may not depend on \( x \). This simple reasoning cannot provide the
exact expression in the right-hand side of (39), which is essential for the geometric interpretation of the Poincaré extension.

To restore a cycle from its intersection points with the real line we need also to know its cycle product with the real line. If this product is non-zero then the sign of the parameter $n$ is additionally required. At the cycles’ language, a common point of cycles $C$ and $S$ is encoded by a cycle $\hat{C}$ such that:

$$\langle G, C \rangle_e = \langle G, S \rangle_e = \langle G, G \rangle_{\tau} = 0. \quad (40)$$

For a given value of $\tau$, this produces two linear and one quadratic equation for parameters of $\hat{C}$. Thus, a pair of cycles may not have a common point or have up to two such points. Furthermore, Möbius-invariance of the above conditions (39) and (40) supports the geometrical construction of Poincaré extension, cf. Lem. 3.1:

**Lemma 3.18.** Let a family consist of cycles, which are $e$-orthogonal to a given $\tau$-isotropic cycle $G$ and have the fixed value of the fraction in the left-hand side of (39). Then, for a given Möbius transformation $g$ and any cycle $C$ from the family, $gC$ is $e$-orthogonal to the $\tau$-isotropic cycle $gG$ and has the same fixed value of the fraction in the left-hand side of (39) as $C$.

Summarising the geometrical construction, the Poincaré extension based on two intervals and the additional data produces two situations:

1. If the cycles $C$ and $C'$ are orthogonal to the real line, then a pair of overlapping cycles produces a point of the elliptic upper half-plane, a pair of disjoint cycles defines a point of the hyperbolic. However, there is no orthogonal cycles uniquely defining a parabolic extension.

2. If we admit cycles, which are not orthogonal to the real line, then the same pair of cycles may define any of the three different types (EHP) of extension. These peculiarities make the extension based on three intervals, described above, a bit more preferable.

### 3.7. Summary of the construction and generalisations.

Based on the consideration in Sections 3.2–3.6 we describe the following steps to carry out the generalised extension procedure:

1. Points of the extended space are equivalence classes of aligned triples of cycles in $\mathbb{P} \mathbb{R}^1$, see Defn. 3.11. The equivalence relation between triples will emerge at step 3.

2. A triple $T$ of different cycles defines the unique one-parameter continues subgroup $S(t)$ of Möbius transformations as defined in Prop. 3.12.

3. Two triples of cycles are equivalent if and only if the subgroups defined in step 2 coincide (up to a parametrisation).

4. The geometry of the extended space, defined by the equivalence class of a triple $T$, is elliptic, parabolic or hyperbolic depending on the subgroup $S(t)$ being similar $S(t) = qH_{\tau}(t)g^{-1}$, $g \in \text{SL}_2(\mathbb{R})$ (up to parametrisation) to $H_{\tau}$ (35) with $\tau = -1, 0$ or 1 respectively. The value of $\tau$ may be identified from the triple using Prop. 3.14.

5. For the above $\tau$ and $g \in \text{SL}_2(\mathbb{R})$, the point of the extended space, defined by the equivalence class of a triple $T$, is represented by $\tau$-isotropic (see Defn. 3.7(2)) bilinear form $g^{-1}\begin{pmatrix} 1 & \tau \\ 1 & 1 \end{pmatrix}g$, which is $S$-invariant, see the end of Section 3.5.

Obviously, the above procedure is more complicated that the geometric construction from Section 3.1. There are reasons for this, as discussed in Section 3.6: our procedure is uniform and we are avoiding consideration of numerous subcases created by an incompatible selection of parameters. Furthermore, our presentation
is aimed for generalisations to Möbius transformations of moduli over other rings. This can be considered as an analog of Cayley–Klein geometries [65; 77, Apps. A–B]. It shall be rather straightforward to adopt the extension for $\mathbb{R}^n$. Möbius transformations in $\mathbb{R}^n$ are naturally expressed as linear-fractional transformations in Clifford algebras, cf. [20] and Sect. 4.5. There is a similar classification of subgroups based on fixed points [1, 78] in multidimensional case. The Möbius invariant matrix presentation of cycles $\mathbb{R}^n$ is already known [20, (4.12); 25; 51, § 5]. Of course, it is necessary to enlarge the number of defining cycles from 3 to, say, $n + 2$. This shall have a connection with Cauchy–Kovalevskaya extension considered in Clifford analysis [68, 74]. Naturally, a consideration of other moduli and rings may require some more serious adjustments in our scheme.

Our construction is based on the matrix presentations of cycles. This technique is effective in many different cases [46, 51]. Thus, it is not surprising that such ideas (with some technical variation) appeared independently on many occasions [20, (4.12); 25; 39, § 4.2; 71, § 1.1]. The interesting feature of the present work is the complete absence of any (hy)complex numbers. It deemed to be unavoidable [46, § 3.3.4] to employ complex, dual and double numbers to represent three different types of Möbius transformations extended from the real line to a plane. Also (hy)complex numbers were essential in [40, 46] to define three possible types of cycle product (30), and now we managed without them.

Apart from having real entries, our matrices for cycles share the structure of matrices from [20, (4.12); 25; 40; 46]. To obtain another variant, one replaces the map $i$ (20) by

$$ t : \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 & y_2 \\ x_1 & x_2 \end{pmatrix}. $$

Then, we may define symmetric matrices in a manner similar to (21):

$$ C_{xy}^t = \frac{1}{2} M_{xy} \cdot t(M_{xy}) = \begin{pmatrix} xy & x + y \\ x + y & 2 \end{pmatrix}. $$

This is the form of matrices for cycles similar to [39, § 4.2; 71, § 1.1]. The property (23) with matrix similarity shall be replaced by the respective one with matrix congruence: $g \cdot C_{xy}^t \cdot g^t = C_{x'y'}^t$. The rest of our construction may be adjusted for these matrices accordingly.

4. Example: Continued Fractions

Continued fractions remain an important and attractive topic of current research [17; 36; 38; 39, § E.3]. A fruitful and geometrically appealing method considers a continued fraction as an (infinite) product of linear-fractional transformations from the Möbius group, see Sec. 4.2 below for a brief overview, works [9; 62; 66; 70; 71, Ex. 10.2] and in particular [11; 73, § 7.5] contain further references and historical notes. Partial products of linear-fractional maps form a sequence in the Möbius group, the corresponding sequence of transformations can be viewed as a discrete dynamical system [11, 59]. Many important questions on continued fractions, e.g. their convergence, can be stated in terms of asymptotic behaviour of the associated dynamical system. Geometrical generalisations of continued fractions to many dimensions were introduced recently as well [8, 36].

4.1. Continued fractions and Möbius–Lie geometry. A comprehensive consideration of the Möbius group involves cycles—the Möbius invariant set of all circles and straight lines. Furthermore, an efficient treatment cycles and Möbius transformations is realised through certain $2 \times 2$ matrices, which we will review in Sec. 4.3, see also [20, § 4.1; 25; 39, § 4.2; 40; 46; 71]. Linking the above relations we present continued fractions within the extend Möbius–Lie geometry:
Claim 4.1 (Continued fractions and cycles). Properties of continued fractions may be illustrated and demonstrated using related cycles, in particular, in the form of respective $2 \times 2$ matrices.

One may expect that such an observation has been made a while ago, e.g. in the book [71], where both topics were studied. However, this seems did not happen for some reasons. It is only the recent paper [13], which pioneered a connection between continued fractions and cycles. We hope that the explicit statement of the claim will stimulate its further fruitful implementations. In particular, Sec. 4.4 reveals all three possible cycle arrangements similar to one used in [13]. Secs. 4.5–4.6 shows that relations between continued fractions and cycles can be used in the multidimensional case as well.

As an illustration, we draw on Fig. 2 chains of tangent horocycles (circles tangent to the real line, see [13] and Sec. 4.4) for two classical simple continued fractions:

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \ldots}}}$$

$$\pi = 3 + \frac{1}{7 + \frac{1}{15 + \frac{1}{292 + \ldots}}}$$

One can immediately see, that the convergence for $\pi$ is much faster: already the third horocycle is too small to be visible even if it is drawn. This is related to larger coefficients in the continued fraction for $\pi$.

Paper [13] also presents examples of proofs based on chains of horocycles. This intriguing construction was introduced in [13] ad hoc. Guided by the above claim we reveal sources of this and other similar arrangements of horocycles. Also, we can produce multi-dimensional variant of the framework.

4.2. Preliminaries on Continued Fractions. We use the following compact notation for a continued fraction:

$$K(a_n | b_n) = \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ldots}}} = \frac{a_1}{a_2} \frac{a_2}{a_3} \frac{a_3}{a_4} \cdots$$

Without loss of generality we can assume $a_j \neq 0$ for all $j$. The important particular case of simple continued fractions, with $a_n = 1$ for all $n$, is denoted by $K(b_n) = K(1 | b_n)$. Any continued fraction can be transformed to an equivalent simple one.
It is easy to see, that continued fractions are related to the following linear-fractional (Möbius) transformation, cf. [9,62,66,70]:

\[(42) \quad S_n = s_1 \circ s_2 \circ \ldots \circ s_n, \quad \text{where} \quad s_j(z) = \frac{a_j}{b_j + z}.\]

These Möbius transformation are considered as bijective maps of the Riemann sphere \(\hat{C} = \mathbb{C} \cup \{\infty\}\) onto itself. If we associate the matrix \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) to a linear-fractional transformation \(z \mapsto \frac{az + b}{cz + d}\), then the composition of two such transformations corresponds to multiplication of the respective matrices. Thus, relation (42) has the matrix form:

\[(43) \quad \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} = \begin{pmatrix} 0 & a_1 \\ 1 & b_1 \end{pmatrix} \begin{pmatrix} 0 & a_2 \\ 1 & b_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix}.\]

The last identity can be fold into the recursive formula:

\[(44) \quad \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} = \begin{pmatrix} P_{n-2} & P_{n-1} \\ Q_{n-2} & Q_{n-1} \end{pmatrix} \begin{pmatrix} 0 & a_n \\ 1 & b_n \end{pmatrix}.\]

This is equivalent to the main recurrence relation:

\[(45) \quad P_n = b_n P_{n-1} + a_n P_{n-2}, \quad Q_n = b_n Q_{n-1} + a_n Q_{n-2}, \quad n = 1, 2, 3, \ldots, \quad \text{with} \quad P_1 = a_1, \quad P_0 = 0, \quad Q_1 = b_1, \quad Q_0 = 1.\]

The meaning of entries \(P_n\) and \(Q_n\) from the matrix (43) is revealed as follows. Möbius transformation (42)–(43) maps 0 and \(\infty\) to

\[(46) \quad \frac{P_n}{Q_n} = S_n(0), \quad \frac{P_{n-1}}{Q_{n-1}} = S_n(\infty).\]

It is easy to see that \(S_n(0)\) is the partial quotient of (41):

\[(47) \quad \frac{P_n}{Q_n} = \frac{a_1}{b_1} + \frac{a_2}{b_2} + \ldots + \frac{a_n}{b_n}.\]

Properties of the sequence of partial quotients \(\{\frac{P_n}{Q_n}\}\) in terms of sequences \(\{a_n\}\) and \(\{b_n\}\) are the core of the continued fraction theory. Equation (46) links partial quotients with the Möbius map (42). Circles form an invariant family under Möbius transformations, thus their appearance for continued fractions is natural. Surprisingly, this happened only recently in [13].

4.3. Möbius Transformations and Cycles. If \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) is a matrix with real entries then for the purpose of the associated Möbius transformations \(M : \bar{z} \mapsto \frac{az + b}{cz + d}\) we may assume that \(\det M = \pm 1\). The collection of all such matrices form a group. Möbius maps commute with the complex conjugation \(z \mapsto \bar{z}\). If \(\det M > 0\) then both the upper and the lower half-planes are preserved. If \(\det M < 0\) then the two half-planes are swapped. Thus, we can treat \(M\) as the map of equivalence classes \(z \sim \bar{z}\), which are labelled by respective points of the closed upper half-plane. Under this identification we consider any map produced by \(M\) with \(\det M = \pm 1\) as the map of the closed upper-half plane to itself.

The characteristic property of Möbius maps is that circles and lines are transformed to circles and lines. We use the word cycles for elements of this Möbius-invariant family [40,46,77]. We abbreviate a cycle given by the equation

\[(48) \quad k(u^2 + v^2) - 2lu - 2nu + m = 0\]

to the point \((k, l, n, m)\) of the three dimensional projective space \(PR^3\). The equivalence relation \(z \sim \bar{z}\) is lifted to the equivalence relation

\[(49) \quad (k, l, n, m) \sim (k, l, -n, m)\]
in the space of cycles, which again is compatible with the Möbius transformations acting on cycles.

The most efficient connection between cycles and Möbius transformations is realised through the construction, which may be traced back to [71] and was subsequently rediscovered by various authors [20, § 4.1; 25; 39, § 4.2], see also [40, 46]. The construction associates a cycle \((k, l, n, m)\) with the \(2 \times 2\) matrix \(C = \begin{pmatrix} l + in & -m \\ k & -l + in \end{pmatrix}\), see discussion in [46, § 4.4] for a justification. This identification is Möbius covariant: the Möbius transformation defined by \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) maps a cycle with matrix \(C\) to the cycle with matrix \(MCM^{-1}\). Therefore, any Möbius-invariant relation between cycles can be expressed in terms of corresponding matrices. The central role is played by the Möbius-invariant inner product [46, § 5.3]:

\[
\langle C, G \rangle = \Re \text{tr}(CS) \tag{50}
\]

which is a cousin of the product used in GNS construction of \(C^*\)-algebras. Notably, the relation:

\[
\langle C, S \rangle = 0 \quad \text{or} \quad km' + mk' - 2nn' - 2ll' = 0 \tag{51}
\]

describes two cycles \(C = (k, l, m, n)\) and \(S = (k', l', m', n')\) orthogonal in Euclidean geometry. Also, the inner product (50) expresses the Descartes–Kirillov condition [39, Lem. 4.1(c); 46, Ex. 5.26] of \(C\) and \(S\) to be externally tangent:

\[
\langle C + S, C + S \rangle = 0 \quad \text{or} \quad (l + l')^2 + (n + n')^2 - (m + m')(k + k') = 0 \tag{52}
\]

where the representing vectors \(C = (k, l, m, n)\) and \(S = (k', l', m', n')\) from \(PR^3\) need to be normalised by the conditions \(\langle C, C \rangle = 1\) and \(\langle S, S \rangle = 1\).

### 4.4. Continued Fractions and Cycles.

Now we are following [51]. Let \(M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\) be a matrix with real entries and the determinant \(\det M\) equal to \(\pm 1\), we denote this by \(\delta = \det M\). As mentioned in the previous section, to calculate the image of a cycle \(C\) under Möbius transformations \(M\) we can use matrix similarity \(MCM^{-1}\). If \(M = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix}\) is the matrix (43) associated to a continued fraction and we are interested in the partial fractions \(\frac{P_n}{Q_n}\), it is natural to ask:

**Which cycles \(C\) have transformations \(MCM^{-1}\) depending on the first (or on the second) columns of \(M\) only?**

It is a straightforward calculation with matrices\(^4\) to check the following statements:

#### Lemma 4.2.
The cycles \((0, 0, 1, m)\) (the horizontal lines \(v = m\)) are the only cycles, such that their images under the Möbius transformation \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) are independent from the column \(\begin{pmatrix} b \\ d \end{pmatrix}\). The image associated to the column \(\begin{pmatrix} a \\ c \end{pmatrix}\) is the horocycle \((c^2m, acm, \delta, ca^2m)\), which touches the real line at \(\frac{a}{c}\) and has the radius \(\frac{1}{mc^2}\).

In particular, for the matrix (44) the horocycle is touching the real line at the point \(\frac{P_n}{Q_{n-1}} = S_n(\infty) \tag{46}\).
Lemma 4.3. The cycles \((k, 0, 1, 0)\) (with the equation \(k(u^2 + v^2) - 2v = 0\)) are the only cycles, such that their images under the Möbius transformation \((a b \\ c d)\) are independent from the column \((a c)\). The image associated to the column \((b d)\) is the horocycle \((d^2 k, bdk, \delta, b^2 k)\), which touches the real line at \(b\) and has the radius \(\frac{1}{kd}\).

In particular, for the matrix \((44)\) the horocycle is touching the real line at the point \(\frac{P_n}{Q_n} = S_n(0)\) \((46)\). In view of these partial quotients the following cycles joining them are of interest.

Lemma 4.4. A cycle \((0, 1, n, 0)\) (any non-horizontal line passing \(0\)) is transformed by \((42)-(43)\) to the cycle \((2Q_nQ_{n-1}, P_nQ_{n-1} + Q_nP_{n-1}, \delta n, 2P_nP_{n-1})\), which passes points \(\frac{P_n}{Q_n} = S_n(0)\) and \(\frac{P_{n-1}}{Q_{n-1}} = S(\infty)\) on the real line.

The above three families contain cycles with specific relations to partial quotients through Möbius transformations. There is one degree of freedom in each family: \(m, k\) and \(n\), respectively. We can use the parameters to create an ensemble of three cycles (one from each family) with some Möbius-invariant interconnections. Three most natural arrangements are illustrated by Fig. 3. The first row presents the initial selection of cycles, the second row—their images after a Möbius transformation (colours are preserved). The arrangements are as follows:

![Figure 3](image-url)

Figure 3. Various arrangements for three cycles. The first row shows the initial position, the second row—after a Möbius transformation (colours are preserved).

The left column shows the arrangement used in the paper [13]: two horocycles touching, the connecting cycle is passing their common point and is orthogonal to the real line.

The central column presents two orthogonal horocycles and the connecting cycle orthogonal to them.

The horocycles in the right column are again orthogonal but the connecting cycle passes one of their intersection points and makes 45° with the real axis.

(1) The left column shows the arrangement used in the paper [13]: two horocycles are tangent, the third cycle, which we call connecting, passes three
points of pair-wise contact between horocycles and the real line. The connecting cycle is also orthogonal to horocycles and the real line. The arrangement corresponds to the following values \( m = 2, k = 2, n = 0 \). These parameters are uniquely defined by the above tangent and orthogonality conditions together with the requirement that the horocycles’ radii agreeably depend from the consecutive partial quotients’ denominators: \( \frac{1}{Q_{n-1}} \) and \( \frac{1}{\sqrt{Q_n}} \) respectively. This follows from the explicit formulae of image cycles calculated in Lemmas 4.2 and 4.3.

(2) The central column of Fig. 3 presents two orthogonal horocycles and the connecting cycle orthogonal to them. The initial cycles have parameters \( m = \sqrt{2}, k = \sqrt{2}, n = 0 \). Again, these values follow from the geometric conditions and the alike dependence of radii from the partial quotients’ denominators: \( \frac{\sqrt{2}}{\sqrt{Q_{n-1}}} \) and \( \frac{\sqrt{2}}{\sqrt{Q_n}} \).

(3) Finally, the right column have the same two orthogonal horocycles, but the connecting cycle passes one of two horocycles’ intersection points. Its mirror reflection in the real axis satisfying (49) (drawn in the dashed style) passes the second intersection point. This corresponds to the values \( m = \sqrt{2}, k = \sqrt{2}, n = \pm 1 \). The connecting cycle makes the angle 45° at the points of intersection with the real axis. It also has the radius \( \frac{\sqrt{2}}{\sqrt{Q_n}} \left| \frac{P_n}{Q_n} - \frac{P_{n-1}}{Q_{n-1}} \right| = \frac{\sqrt{2}}{\sqrt{Q_n}} \left| P_n Q_{n-1} - P_{n-1} Q_n \right| \) — the geometric mean of radii of two other cycles. This again repeats the relation between cycles’ radii in the first case.

Three configurations have fixed ratio \( \sqrt{2} \) between respective horocycles’ radii. Thus, they are equally suitable for the proofs based on the size of horocycles, e.g. [13, Thm. 4.1].

On the other hand, there is a tiny computational advantage in the case of orthogonal horocycles. Let we have the sequence \( p_j \) of partial fractions \( p_j = \frac{P_j}{Q_j} \) and want to rebuild the corresponding chain of horocycles. A horocycle with the point of contact \( p_j \) has components \( (1, p_j, n_j, P_j^2) \), thus only the value of \( n_j \) need to be calculated at every step. If we use the condition “to be tangent to the previous horocycle”, then the quadratic relation (52) shall be solved. Meanwhile, the orthogonality relation (51) is linear in \( n_j \).

4.5. Multi-dimensional Möbius maps and cycles. It is natural to look for multidimensional generalisations of continued fractions. A geometric approach based on Möbius transformation and Clifford algebras was proposed in [8]. Let us restrict the consideration from Sect. 2.2 to Clifford algebra \( \mathcal{O}(n) \) of the Euclidean space \( \mathbb{R}^n \). It is the associative unital algebra over \( \mathbb{R} \) generated by the elements \( e_1, \ldots, e_n \) satisfying the following relation:

\[
e_i e_j + e_j e_i = -2\delta_{ij}
\]

where \( \delta_{ij} \) is the Kronecker delta. An element of \( \mathcal{O}(n) \) having the form \( x = x_1 e_1 + \ldots + x_n e_n \) can be associated with the vector \( (x_1, \ldots, x_n) \in \mathbb{R}^n \). The **reversion** \( a \rightarrow a^* \) in \( \mathcal{O}(n) \) [20, (1.19(ii))] is defined on vectors by \( x^* = x \) and extended to other elements by the relation \((ab)^* = b^*a^* \). Similarly, the **conjugation** is defined on vectors by \( \bar{x} = -x \) and the relation \( \bar{ab} = b\bar{a} \). We also use the notation \( |a|^2 = aa^* \geq 0 \) for any product of vectors. An important observation is that any non-zero vectors \( x \) has a multiplicative inverse: \( x^{-1} = \frac{x}{|x|^2} \).

By Ahlfors [2] (see also [8, § 5; 20, Thm. 4.10]) a matrix \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with Clifford entries defines a linear-fractional transformation of \( \mathbb{R}^n \) if the following conditions are satisfied:
(1) $a$, $b$, $c$ and $d$ are products of vectors in $\mathbb{R}^n$.
(2) $ab^*$, $cd^*$, $c^*a$ and $d^*b$ are vectors in $\mathbb{R}^n$.
(3) the pseudodeterminant $\delta := ad^* - bc^*$ is a non-zero real number.

Clearly we can scale the matrix to have the pseudodeterminant $\delta = \pm 1$ without an effect on the related linear-fractional transformation. Recall notations (3), cf. [20, (4.7)]

$$
\bar{M} = \begin{pmatrix}
  a^* & -b^* \\
  -c^* & a^*
\end{pmatrix}
\quad \text{and} \quad
M^* = \begin{pmatrix}
  d & b \\
  c & a
\end{pmatrix}.
$$

Then $M \bar{M} = \delta I$ and $\bar{M} = \kappa M^*$, where $\kappa = 1$ or $-1$ depending either $d$ is a product of even or odd number of vectors.

To adopt the discussion from Section 4.3 to several dimensions we use vector rather than paravector formalism, see [20, (1.42)] for a discussion. Namely, we consider vectors $x \in \mathbb{R}^{n+1}$ as elements $x = x_1 e_1 + \ldots + x_n e_n + x_{n+1} e_{n+1}$ in $\mathcal{O}(n+1)$. Therefore we can extend the Möbius transformation defined by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathcal{O}(n)$ to act on $\mathbb{R}^{n+1}$. Again, such transformations commute with the reflection $R$ in the hyperplane $x_{n+1} = 0$:

$$
R: \quad x_1 e_1 + \ldots + x_n e_n + x_{n+1} e_{n+1} \mapsto x_1 e_1 + \ldots + x_n e_n - x_{n+1} e_{n+1}.
$$

Thus we can consider the Möbius maps acting on the equivalence classes $x \sim R(x)$.

Spheres and hyperplanes in $\mathbb{R}^{n+1}$—which we continue to call cycles—can be associated to $2 \times 2$ matrices by (4), cf. [20, (4.12), 25]:

$$
k\bar{x}x - l\bar{x} - x\bar{l} + m = 0 \iff C = \begin{pmatrix} l & m \\
 k & l
\end{pmatrix}
$$

where $k, m \in \mathbb{R}$ and $l \in \mathbb{R}^{n+1}$. For brevity we also encode a cycle by its coefficients $(k, l, m)$. The identification is also Möbius-covariant in the sense that the transformation associated with the Ahlfors matrix $M$ sends a cycle $C$ to the cycle $MCM^*$ [20, (4.16)]. The equivalence $x \sim R(x)$ is extended to spheres:

$$
\begin{pmatrix} l & m \\
 k & l
\end{pmatrix} \sim \begin{pmatrix} R(l) & m \\
 k & R(l)
\end{pmatrix}
$$

since it is preserved by the Möbius transformations with coefficients from $\mathcal{O}(n)$.

Similarly to (50) we define the Möbius-invariant inner product of cycles by the identity $\langle C, S \rangle = \Re \text{tr}(CS)$, where $\Re$ denotes the scalar part of a Clifford number. The orthogonality condition $\langle C, S \rangle = 0$ means that the respective cycle are geometrically orthogonal in $\mathbb{R}^{n+1}$.

4.6. Continued fractions from Clifford algebras and horocycles. There is an association between the triangular matrices and the elementary Möbius maps of $\mathbb{R}^n$, cf. (42):

$$
\begin{pmatrix} 1 & b \\
 0 & 1
\end{pmatrix} : \quad x \mapsto (x + b)^{-1}, \quad \text{where} \quad x = x_1 e_1 + \ldots + x_n e_n, \quad b = b_1 e_1 + \ldots + b_n e_n.
$$

Similar to the real line case in Section 4.2, Beardon proposed [8] to consider the composition of a series of such transformations as multidimensional continued fraction, cf. (42). It can be again represented as the the product (43) of the respective $2 \times 2$ matrices. Another construction of multidimensional continued fractions based on horocycles was hinted in [13]. We wish to clarify the connection between them. The bridge is provided by the respective modifications of Lem. 4.2–4.4.

**Lemma 4.5.** The cycles $(0, e_{n+1}, m)$ (the “horizontal” hyperplane $x_{n+1} = m$) are the only cycles, such that their images under the Möbius transformation $\begin{pmatrix} a & b \\
 c & d
\end{pmatrix}$ are...
connecting cycles

\[\text{connecting cycle is the image under the corresponding Möbius transformation } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with Clifford entries.} \]

valid in multidimensional situation as well.

end of Section 4.4 on computational advantage of orthogonal horocycles remains presented by the chain of tangent/orthogonal horocycles. The observation made at the

\[\text{Lemma 4.6. The cycles } (k,e_{n+1},0) \text{ (with the equation } k(u^2 + v^2) - 2v = 0 \text{) are the only cycles, such that their images under the Möbius transformation } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ are independent from the column } \begin{pmatrix} a \\ c \end{pmatrix}. \]

\text{The image associated to the column } \begin{pmatrix} b \\ d \end{pmatrix} \text{ is the horocycle } (k|d|^2, kb\bar{c} + \delta e_{n+1}, -k\bar{b}), \text{ which touches the hyperplane } x_{n+1} = 0 \text{ at } \frac{bd}{|d|^2} \text{ and has the radius } \frac{1}{k|d|^2}. \]

The proof of the above lemmas are reduced to multiplications of respective matrices with Clifford entries.

\[\text{Lemma 4.7. A cycle } C = (0, l, 0), \text{ where } l = x + re_{n+1} \text{ and } 0 \neq x \in \mathbb{R}^n, r \in \mathbb{R}, \text{ that is any non-horizontal hyperplane passing the origin, is transformed into } MCM^* = (cx\bar{d} + dx\bar{a}, ax\bar{d} + bx\bar{c} + \delta re_{n+1}, ax\bar{b} + b\bar{x}\bar{a}). \text{ This cycle passes points } \frac{ac}{|c|^2} \text{ and } \frac{bd}{|d|^2} \text{ and orthogonal to the hyperplane } x_{n+1} = 0. \]

\[\text{Proof. We note that } e_{n+1}x = -xe_{n+1} \text{ for all } x \in \mathbb{R}^n. \text{ Thus, for a product of vectors } d \in \mathcal{C}(n) \text{ we have } e_{n+1}d = d^*e_{n+1}. \text{ Then} \]

\[ce_{n+1}\bar{d} + de_{n+1}\bar{c} = (cd^* - dc^*)e_{n+1} = (cd^* - (cd^*)^*)e_{n+1} = 0 \]

due to the Ahlfors condition 2. Similarly, \[ace_{n+1}\bar{d} + be_{n+1}\bar{a} = 0 \text{ and } ace_{n+1}\bar{d} + be_{n+1}\bar{c} = (ad^* - bc^*)e_{n+1} = \delta e_{n+1}. \]

\[\text{The image } MCM^* \text{ of the cycle } C = (0, l, 0) \text{ is } (cl\bar{d} + dl\bar{c}, al\bar{d} + bl\bar{c}, a\bar{d} + b\bar{c}). \text{ From the above calculations for } l = x + re_{n+1} \text{ it becomes } (cx\bar{d} + dx\bar{a}, ax\bar{d} + bx\bar{c} + \delta re_{n+1}, ax\bar{b} + b\bar{x}\bar{a}). \]

Thus, we have exactly the same freedom to choose representing horocycles as in Section 4.4: make two consecutive horocycles either tangent or orthogonal. To visualise this, we may use the two-dimensional plane \(V\) passing the points of contacts of two consecutive horocycles and orthogonal to \(x_{n+1} = 0\). It is natural to choose the connecting cycle (drawn in blue on Fig. 3) with the centre belonging to \(V\), this eliminates excessive degrees of freedom. The corresponding parameters are described in the second part of Lem. 4.7. Then, the intersection of horocycles with \(V\) are the same as on Fig. 3.

Thus, the continued fraction with the partial quotients \[\frac{P_nQ_n}{Q_{n+1}} \in \mathbb{R}^n \text{ can be represented by the chain of tangent/orthogonal horocycles. The observation made at the end of Section 4.4 on computational advantage of orthogonal horocycles remains valid in multidimensional situation as well.} \]

As a further alternative we may shift the focus from horocycles to the connecting cycle (drawn in blue on Fig. 3). The part of the space \(\mathbb{R}^n\) encloses inside the connecting cycle is the image under the corresponding Möbius transformation of the half-space of \(\mathbb{R}^n\) cut by the hyperplane \((0, l, 0)\) from Lem. 4.7. Assume a sequence of connecting cycles \(C_j\) satisfies the following two conditions, e.g. in Seidel–Stern-type theorem [13, Thm 4.1]:
(1) for any \(j\), the cycle \(C_j\) is enclosed within the cycle \(C_{j-1}\).
(2) the sequence of radii of \(C_j\) tends to zero.

Under the above assumption the sequence of partial fractions converges. Furthermore, if we use the connecting cycles in the third arrangement, that is generated by the cycle \((0, x + e_{n+1}, 0)\), where \(\|x\| = 1, x \in \mathbb{R}^n\), then the above second condition can be replaced by following

(2') the sequence of \(x^{(j)}_{n+1}\) of \((n + 1)\)-th coordinates of the centres of the connecting cycles \(C_j\) tends to zero.

Thus, the sequence of connecting cycles is a useful tool to describe a continued fraction even without a relation to horocycles.

Summing up, we started from multidimensional continued fractions defined by the composition of Möbius transformations in Clifford algebras and associated to it the respective chain of horocycles. This establishes the equivalence of two approaches proposed in [8] and [13] respectively.

5. Extension of Möbius–Lie Geometry by Ensembles of Interrelated Cycles

Previous consideration suggests a far-reaching generalisation Möbius–Lie geometry which we are presenting in this section.

5.1. Figures as ensembles of cycles. We start from some examples of ensembles of cycles, which conveniently describe FLT-invariant families of objects.

Example 5.1. (1) The Poincaré extension of Möbius transformations from the real line to the upper half-plane of complex numbers is described by a pair of distinct intersecting cycles \(\{C_1, C_2\}\), that is \(\langle C_1, C_2 \rangle^2 \leq \langle C_1, C_1 \rangle \langle C_2, C_2 \rangle\).

Two such pairs \(\{C_1, C_2\}\) and \(\{S_1, S_2\}\) are equivalent if any cycle \(S_i\) from the second pair is a linear combination of (i.e. belongs to the pencil spanned by) the cycles from the first pair.

A modification from Sect. 3.5 with ensembles of three cycles describes an extension from the real line to the upper half-plane of complex, dual or double numbers. The construction can be generalised to arbitrary dimensions, cf. Sect. 3.7.

(2) A parametrisation of loxodromes is provided by a triple of cycles \(\{C_1, C_2, C_3\}\) such that, cf. [54] and Fig. 4:
(a) \(C_1\) is orthogonal to \(C_2\) and \(C_3\),
(b) \(\langle C_2, C_3 \rangle^2 \geq \langle C_2, C_2 \rangle \langle C_3, C_3 \rangle\).

Then, main invariant properties of Möbius–Lie geometry, e.g. tangency of loxodromes, can be expressed in terms of this parametrisation [54].

(3) A continued fraction is described by an infinite ensemble of cycles \(\{C_k\}\) such that, cf. Sect. 4.4:
(a) All \(C_k\) are touching the real line (i.e. are horocycles),
(b) \(\{C_1\}\) is a horizontal line passing through \((0, 1)\),
(c) \(C_{k+1}\) is tangent to \(C_k\) for all \(k > 1\).

Several similar arrangements are described in Sect. 4.4. The key analytic properties of continued fractions—their convergence—can be linked to asymptotic behaviour of such an infinite ensemble [13].

(4) A remarkable relation exists between discrete integrable systems and Möbius geometry of finite configurations of cycles [16, 55–57, 69]. It comes from “reciprocal force diagrams” used in 19th-century statics, starting with J.C. Maxwell. It is demonstrated in that the geometric compatibility of reciprocal
figures corresponds to the algebraic compatibility of linear systems defining these configurations. On the other hand, the algebraic compatibility of linear systems lies in the basis of integrable systems. In particular [55,56], important integrability conditions encapsulate nothing but a fundamental theorem of ancient Greek geometry.

(5) An important example of an infinite ensemble is provided by the representation of an arbitrary wave as the envelope of a continuous family of spherical waves. A finite subset of spheres can be used as an approximation to the infinite family. Then, discrete snapshots of time evolution of sphere wave packets represent a FLT-covariant ensemble of cycles [6]. Further physical applications of FLT-invariant ensembles may be looked at [37].

One can easily note that the above parametrisations of some objects by ensembles of cycles are not necessary unique. Naturally, two ensembles parametrising the same object are again connected by FLT-invariant conditions. We presented only one example here, cf. [54].

Example 5.2. Two non-degenerate triples \( \{C_1, C_2, C_3\} \) and \( \{S_1, S_2, S_3\} \) parameterise the same loxodrome as in Ex. 5.12 if and only if all the following conditions are satisfied:

1. Pairs \( \{C_2, C_3\} \) and \( \{S_2, S_3\} \) span the same hyperbolic pencil. That is cycles \( S_2 \) and \( S_3 \) are linear combinations of \( C_2 \) and \( C_3 \) and vice versa.
(2) Pairs \( \{C_2, C_3\} \) and \( \{S_2, S_3\} \) have the same normalised cycle product (7):

\[
[C_2, C_3] = [S_2, S_3].
\]

(3) The elliptic-hyperbolic identity holds:

\[
\text{arccosh} [C_j, S_j] \equiv \frac{1}{2\pi} \text{arccos} [C_1, S_1] \pmod{1}
\]

where \( j \) is either 2 or 3.

Various triples of cycles parametrising the same loxodrome are animated on Fig. 4.

The respective equivalence relation for parametrisation of elliptic, parabolic and hyperbolic Poincaré extensions, cf. Ex. 5.1(1), by triple of cycles can be deduced from Prop. 3.12.

5.2. Extension of Möbius–Lie geometry and its implementation through functional approach. The above examples suggest that one can expand the subject and applicability of Möbius–Lie geometry through the following formal definition.

**Definition 5.3.** Let \( X \) be a set, \( R \subset X \times X \) be an oriented graph on \( X \) and \( f \) be a function on \( R \) with values in FLT-invariant relations from § 2.3. Then \( (R, f) \)-ensemble is a collection of cycles \( \{C_j\}_{j \in X} \) such that

\[
C_i \text{ and } C_j \text{ are in the relation } f(i, j) \text{ for all } (i, j) \in R.
\]

For a fixed FLT-invariant equivalence relations \( \sim \) on the set \( E \) of all \( (R, f) \)-ensembles, the extended Möbius–Lie geometry studies properties of cosets \( E/\sim \).

This definition can be suitably modified for

1. ensembles with relations of more then two cycles, and/or
2. ensembles parametrised by continuous sets \( X \), cf. wave envelopes in Ex. 5.15.

The above extension was developed along with the realisation the library figure within the functional programming framework. More specifically, an object from the class figure stores defining relations, which link new cycles to the previously introduced ones. This also may be treated as classical geometric compass-and-straightedge constructions, where new lines or circles are drawn through already existing elements. If requested, an explicit evaluation of cycles’ parameters from this data may be attempted.

To avoid “chicken or the egg” dilemma all cycles are stored in a hierarchical structure of generations, numbered by integers. The basic principles are:

1. Any explicitly defined cycle (i.e., a cycle which is not related to any previously known cycle) is placed into generation-0,
2. Any new cycle defined by relations to previous cycles from generations \( k_1, k_2, \ldots, k_n \) is placed to the generation \( k \) calculated as:

\[
k = \max(k_1, k_2, \ldots, k_n) + 1.
\]

This rule does not forbid a cycle to have a relation to itself, e.g. isotropy (self-orthogonality) condition \( \langle C, C \rangle = 0 \), which specifies point-like cycles, cf. relation 3 in § 2.3. In fact, this is the only allowed type of relations to cycles in the same (not even speaking about younger) generations.

There are the following alterations of the above rules:

1. From the beginning, every figure has two pre-defined cycles: the real line (hyperplane) \( C_R \), and the zero radius cycle at infinity \( C_\infty = (0, 0, 1) \). These cycles are required for relations 1 and 2 from the previous subsection. As predefined cycles, \( C_R \) and \( C_\infty \) are placed in negative-numbered generations defined by the macros \( \text{REAL\_LINE\_GEN} \) and \( \text{INFINITY\_GEN} \).
(2) If a point is added to generation-0 of a figure, then it is represented by a zero-radius cycle with its centre at the given point. Particular parameter of such cycle dependent on the used metric, thus this cycle is not considered as explicitly defined. Thereafter, the cycle shall have some parents at a negative-numbered generation defined by the macro `GHOST GEN`.

A figure can be in two different modes: freeze or unfreeze, the second is default. In the unfreeze mode an addition of a new cycle by its relation prompts an evaluation of its parameters. If the evaluation was successful then the obtained parameters are stored and will be used in further calculations for all children of the cycle. Since many relations (see the previous Subsection) are connected to quadratic equation (8), the solutions may come in pairs. Furthermore, if the number or nature of conditions is not sufficient to define the cycle uniquely (up to natural quadratic multiplicity), then the cycle will depend on a number of free (symbolic) variable.

There is a macro-like tool, which is called `subfigure`. Such a `subfigure` is a figure itself, such that its inner hierarchy of generations and relations is not visible from the current figure. Instead, some cycles (of any generations) of the current figure are used as predefined cycles of generation-0 of `subfigure`. Then only one dependent cycle of `subfigure`, which is known as result, is returned back to the current figure. The generation of the result is calculated from generations of input cycles by the same formula (56).

There is a possibility to test certain conditions (“are two cycles orthogonal?”) or measure certain quantities (“what is their intersection angle?”) for already defined cycles. In particular, such methods can be used to prove geometrical statements according to the Cartesian programme, that is replacing the synthetic geometry by purely algebraic manipulations.

**Example 5.4.** As an elementary demonstration, let us prove that if a cycle `r` is orthogonal to a circle `a` at the point `C` of its contact with a tangent line `l`, then `r` is also orthogonal to the line `l`. To simplify setup we assume that `a` is the unit circle. Here is the Python code:

```python
F=figure()
a=F.add_cycle(cycle2D(1,[0,0],-1),"a")
l=symbol("l")
C=symbol("C")
F.add_cycle_rel([[is_tangent_i(a),
                 is_orthogonal(F.get_infinity()),only_reals(1)],l])
F.add_cycle_rel([[is_orthogonal(C),is_orthogonal(a),
                 is_orthogonal(1),only_reals(C)],C])
r=F.add_cycle_rel([[is_orthogonal(C),
                  is_orthogonal(a)],"r")
Res=F.check_rel(1,r,"cycle_orthogonal")
for i in range(len(Res)):
    print("Tangent and radius are orthogonal: \%s\%\"
        %bool(Res[i].subs(pow(cos(wild(0)),2))
        ==-1-pow(sin(wild(0)),2)).normal())
```

The first line creates an empty figure `F` with the default euclidean metric. The next line explicitly uses parameters (1,0,0,−1) of `a` to add it to `F`. Lines 3–4 define symbols `l` and `C`, which are needed because cycles with these labels are defined in lines 5–8 through some relations to themselves and the cycle `a`. In both cases we want to have cycles with real coefficients only and `C` is additionally self-orthogonal (i.e. is a zero-radius). Also, `l` is orthogonal to infinity (i.e. is a line) and `C` is...
orthogonal to \( a \) and \( l \) (i.e. is their common point). The tangency condition for \( l \) and self-orthogonality of \( C \) are both quadratic relations. The former has two solutions each depending on one real parameter, thus line \( l \) has two instances. Correspondingly, the point of contact \( C \) and the orthogonal cycle \( r \) through \( C \) (defined in line 7) each have two instances as well. Finally, lines 11–15 verify that every instance of \( l \) is orthogonal to the respective instance of \( r \), this is assisted by the trigonometric substitution \( \cos^2(*) = 1 - \sin^2(*) \) used for parameters of \( l \) in line 15. The output predictably is:

Tangent and circle \( r \) are orthogonal: True
Tangent and circle \( r \) are orthogonal: True

An original statement proved by the library \texttt{figure} for the first time will be considered in the next Section.

6. Mathematical Usage of Libraries \texttt{cycle} and \texttt{figure}

The developed libraries \texttt{cycle} and \texttt{figure} has several different uses:

- It is easy to produce high-quality illustrations, which are fully-accurate in mathematical sense. The user is not responsible for evaluation of cycles' parameters, all computations are done by the library as soon as the figure is defined in terms of few geometrical relations. This is especially helpful for complicated images which may contain thousands of interrelated cycles. See Escher-like Fig. 5 which shows images of two circles under the modular group action [75, § 14.4].

![Figure 5. Action of the modular group on the upper half-plane.](image)

- The package can be used for computer experiments in Möbius–Lie geometry. There is a possibility to create an arrangement of cycles depending on one or several parameters. Then, for particular values of those parameters certain conditions, e.g. concurrency of cycles, may be numerically tested or graphically visualised. It is possible to create animations with gradual change of the parameters, which are especially convenient for illustrations, see Fig. 8 and [50].

- Since the library is based on the \texttt{GiNaC} system, which provides a symbolic computation engine, there is a possibility to make fully automatic proofs of various statements in Möbius–Lie geometry. Usage of computer-supported proofs in geometry is already an established practice [46, 63] and it is naturally to expect its further rapid growth.
Last but not least, the combination of classical beauty of Lie sphere geometry and modern computer technologies is a useful pedagogical tool to widen interest in mathematics through visual and hands-on experience.

Computer experiments are especially valuable for Lie geometry of indefinite or nilpotent metrics since our intuition is not elaborated there in contrast to the Euclidean space [40, 43, 44]. Some advances in the two-dimensional space were achieved recently [46, 60], however further developments in higher dimensions are still awaiting their researchers.

As a non-trivial example of automated proof accomplished by the figure library for the first time, we present a FLT-invariant version of the classical nine-point theorem [21, §1.8; 64, §1.1], cf. Fig. 7(a):

**Theorem 6.1 (Nine-point cycle).** Let $ABC$ be an arbitrary triangle with the orthocenter (the points of intersection of three altitudes) $H$, then the following nine points belongs to the same cycle, which may be a circle or a hyperbola:

1. Feet of three altitudes, that is points of pair-wise intersections $AB$ and $CH$, $AC$ and $BH$, $BC$ and $AH$.
2. Midpoints of sides $AB$, $BC$ and $CA$.
3. Midpoints of intervals $AH$, $BH$ and $CH$.

There are many further interesting properties, e.g. nine-point circle is externally tangent to that triangle three excircles and internally tangent to its incircle as it seen from Fig. 7(a).

To adopt the statement for cycles geometry we need to find a FLT-invariant meaning of the midpoint $A_m$ of an interval $BC$, because the equality of distances $BA_m$ and $A_mC$ is not FLT-invariant. The definition in cycles geometry can be done by either of the following equivalent relations:
The midpoint $A_m$ of an interval $BC$ is defined by the cross-ratio $\frac{BA_m}{CA_m} = 1$, where $I$ is the point at infinity.

We construct the midpoint $A_m$ of an interval $BC$ as the intersection of the interval and the line orthogonal to $BC$ and to the cycle, which uses $BC$ as its diameter. The latter condition means that the cycle passes both points $B$ and $C$ and is orthogonal to the line $BC$.

Both procedures are meaningful if we replace the point at infinity $I$ by an arbitrary fixed point $N$ of the plane. In the second case all lines will be replaced by cycles passing through $N$, for example the line through $B$ and $C$ shall be replaced by a cycle through $B$, $C$ and $N$. If we similarly replace “lines” by “cycles passing through $N” in Thm. 6.1 it turns into a valid FLT-invariant version, cf. Fig. 7(b). Some additional properties, e.g. the tangency of the nine-points circle to the ex-/in-circles, are preserved in the new version as well. Furthermore, we can illustrate the connection between two versions of the theorem by an animation, where the infinity is transformed to a finite point $N$ by a continuous one-parameter group of FLT, see. Fig. 8 and further examples at [50].

It is natural to test the nine-point theorem in the hyperbolic and the parabolic spaces. Fortunately, it is very easy under the given implementation: we only need to change the defining metric of the point space, cf. [3], this can be done for an already defined figure. The corresponding figures Fig. 7(c) and (d) suggest that the hyperbolic version of the theorem is still true in the plain and even FLT-invariant forms. We shall clarify that the hyperbolic version of the Thm. 6.1 specialises the nine-point conic of a complete quadrilateral [19, 23]: in addition to the existence of this conic, our theorem specifies its type for this particular arrangement as equilateral hyperbola with the vertical axis of symmetry.

The computational power of the package is sufficient not only to hint that the new theorem is true but also to make a complete proof. To this end we define an ensemble of cycles with exactly same interrelations, but populate the generation-0 with points $A$, $B$ and $C$ with symbolic coordinates, that is, objects of the GiNaC class realsymbol. Thus, the entire figure defined from them will be completely general. Then, we may define the hyperbola passing through three bases of altitudes and check by the symbolic computations that this hyperbola passes another six “midpoints” as well.

In the parabolic space the nine-point Thm. 6.1 is not preserved in this manner. It is already observed [5, 43–46, 48, 52, 60], that the degeneracy of parabolic metric in the point space requires certain revision of traditional definitions. The parabolic variation of nine-point theorem may prompt some further considerations as well. An expanded discussion of various aspects of the nine-point construction shall be the subject of a separate paper.

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Figure 7. The illustration of the conformal nine-points theorem. The left column is the statement for a triangle with straight sides (the point $N$ is at infinity), the right column is its conformal version (the point $N$ is at the finite part). The first row shows the elliptic point space, the second row—the hyperbolic point space. Thus, the top-left picture shows the traditional theorem, three other pictures—its different modifications.

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Figure 8. Animated transition between the classical and conformal versions of the nine-point theorem. Use control buttons to activate it. You may need Adobe Acrobat Reader for this feature.

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