Gauge Theory Formulations for Continuous and Higher Spin Fields

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ABSTRACT: We consider a gauge theory action for continuous spin particles formulated in a spacetime enlarged by an extra coordinate recently proposed by Schuster and Toro. It requires one scalar gauge field and has two gauge symmetries. We show that the gauge symmetry is reducible in the sense that the gauge parameters also have a gauge symmetry. Using reducibility we get an action which has two scalar gauge fields but just one irreducible gauge symmetry. We then show how the equations of motion are related to previously proposed equations for continuous spin particles. There is a limit where the continuous spin Schuster-Toro action reduces to a higher spin action containing all integer helicities. We then show how the action with two fields and its field equations are related to other known formulations of higher spin field theories. We also discuss briefly the physical content of each formulation.
1 Introduction

The less well known particle type allowed by special relativity and quantum mechanics is
the continuous spin particle (CSP). Along with massive and massless particles of integer
and half-integer spins they constitute the irreducible representations of the Poincaré group
in flat spacetime [1]. CSPs are massless states labelled by a real parameter ρ, its continuous
spin, and comprises infinitely many helicity states which mix with each other under Lorentz
transformations. There is a bosonic representation where all helicities are integer and a
fermionic one with all helicities being half-integer. When ρ = 0 the helicity states decouple
of each other and reduce to a set of ordinary massless states in which each helicity appears
once giving rise to a higher spin (HS) theory having all integer (or half-integer) helicities
present. CSPs are largely ignored not only because they are not found in Nature but
also because even its free quantum formulation is beset with problems [2–6]. However, it
was found recently that CSPs have covariant soft emission amplitudes which approach the
amplitudes for ordinary low helicity (0, ±1 and ±2) particles at energies large compared
with ρ or in the non-relativistic regime [7, 8]. This led naturally to a search for an action
principle, at least for the free case, and soon an action was proposed for a bosonic CSP\(^1\)
[10]. As for other theories of massless particles it is a gauge theory. It can be coupled to
currents which are consistent with no-go theorems for lower spins and with the covariant
soft factors of [7, 8]. The equations of motion describe degrees of freedom with the expected
polarization content of a single CSP. When ρ vanishes the equations of motion reduce to
the well known higher spin Fronsdal equations [11] for all helicities.

The action is formulated in an enlarged spacetime with the usual spacetime coordinates
x^μ and an additional 4-vector coordinate η^μ. The gauge field Ψ(x, η) is a scalar field and

\(^1\)A previously proposed action [9] described not a single CSP but a continuum of CSPs, with every value
of ρ, making the coupling to a conserved current in the ρ → 0 limit problematic.
it is assumed to be analytic in $\eta^\mu$. The action is given by [10]

$$S = \frac{1}{2} \int d^4x \, d^4\eta \left[ \delta'(\eta^2 + 1)(\partial_\eta \Psi(\eta, x))^2 + \frac{1}{2} \delta(\eta^2 + 1) ((\partial_\eta \cdot \partial_x + \rho)\Psi(\eta, x))^2 \right], \quad (1.1)$$

where $\delta'$ is the derivative of the delta function with respect to its argument. The spacetime metric is mostly minus. There are no terms with two derivatives of $\eta^0$ so that the dynamics is restricted to the usual spacetime. The action is invariant under Lorentz transformations and translations in $x^\mu$ but not translations in $\eta^\mu$. It is also invariant under two gauge transformations

$$\delta \Psi(\eta, x) = [\eta \cdot \partial_x - \frac{1}{2}(\eta^2 + 1)(\partial_\eta \cdot \partial_x + \rho)]\epsilon(\eta, x) + \frac{1}{4}(\eta^2 + 1)^2 \chi(\eta, x), \quad (1.2)$$

where $\epsilon(\eta, x)$ and $\chi(\eta, x)$ are the gauge parameters. As remarked in [10] this represents a huge gauge freedom which was used to show that the action (1.1) propagates only one CSP degree of freedom. Furthermore, when $\rho$ vanishes it was shown that the action describes HS fields for all integer helicities.

We wish to remark that the gauge transformations (1.2) are in fact reducible since

$$\delta \epsilon = \frac{1}{2}(\eta^2 + 1)\Lambda(\eta, x), \quad (1.3)$$

$$\delta \chi = (\partial_\eta \cdot \partial_x + \rho)\Lambda(\eta, x), \quad (1.4)$$

with $\Lambda(\eta, x)$ arbitrary, leaves (1.2) invariant. This is one of the main results of this paper and we will explore its consequences in Section 2. We will make heavy use of this reducibility in the rest of the paper.

Earlier attempts to a formulation of CSPs involved the handling of the Bargmann-Wigner equations [6, 12, 13], the proposal of covariant equations [3, 5] and also attempts to derive them from massive higher spin equations [14]. No action was ever proposed before preventing the use of perturbation theory and coupling to ordinary fields to better understand the CSPs properties. Even so the CSP representations can be extended to higher dimensions and to the supersymmetric case [15] forming supermultiplets of the super-Poincaré algebra. Perturbative string theory does not allow CSPs providing one of the few model independent properties of low energy string theory [16]. Tensionless strings, however, do propagate CSPs [17, 18]. The proposal of the gauge invariant action (1.1) for free CSPs opens new doors to the understanding of this class of particles. It is also relevant in $2 + 1$ dimensions providing a massless generalization of anyons [19]. No self interactions or matter interaction are known presently but mass terms, for instance, are excluded [10]. Another important consequence of (1.1) is that for $\rho = 0$ it reduces to a sum of Fronsdal actions for massless higher spins for all helicities [10] providing an alternative formulation for massless higher spin particles.

The use of extra coordinates as a bookkeeping device has already been employed in some formulations of CSPs and HS fields. Starting with the Wigner conditions the authors of [14] show that a limit of massive HS field equations results in the CSP equations of motion. They obtain a gauge invariant equation for a single CSP in terms of a constrained
field. To remove the constraint a compensator is introduced ending with a formulation with two gauge fields. No action giving these equations of motion was found. The field used in [14] is composed of a totally symmetric tensor with all of its indices contracted with an auxiliary vector. It is natural to relate this extra vector with the coordinate \( \eta \) of [10]. We will show in Section 3 how to derive both formulations of [14] starting with the action (1.1) and a suitable choice for the \( \eta \) dependence of \( \Psi(\eta, x) \). They also consider the case of a massless HS field with a constrained field and we will show how to derive their equations for \( \rho = 0 \) in Section 4. Another formulation for HS fields considers two gauge fields and also makes use of extra coordinates [20]. Starting from the Fronsdal equations in flat and AdS spaces an action is build up describing all massless integer helicities. We will show in Section 4 how this action can be derived from (1.1). There is also a HS field formulation which uses an oscillator basis [21] instead of extra vectors. The indices of a totally symmetric HS field are contracted with creation operators forming a HS ket. An action can then be written in flat or AdS spaces. For the flat case the action reproduces Fronsdal equations. We will show in Section 5 how to relate the oscillator formalism action with the Schuster-Toro action. Finally in the last section we make some comments and discuss the physical content for each formulation.

2 Reducibility of the Gauge Transformations

In order to explore the reducibility of the gauge transformations (1.2) and to make contact with the results of [14] we will choose an specific form for the \( \eta \) dependence of \( \Psi(\eta, x) \). We must first notice that the delta functions in (1.1) are essentially restricting the dynamics to a hyperboloid in \( \eta \) space so its natural to assume the expansion

\[
\Psi(\eta, x) = \sum_{n=0}^{\infty} \frac{1}{n!} (\eta^2 + 1)^n \psi_n(\eta, x),
\]

where \( \psi_n(\eta, x) \) are also scalar fields which are analytic in \( \eta^\mu \). Taking into account that the fields in [14] depend on an extra vector contracted with HS fields we also assume that

\[
\psi_n(\eta, x) = \sum_{s=0}^{\infty} \frac{1}{s!} \eta^{\mu_1} \ldots \eta^{\mu_s} \psi_{(ns)}^{\mu_1 \ldots \mu_s}(x),
\]

where \( \psi_{(ns)}^{\mu_1 \ldots \mu_s}(x) \) is a completely symmetric and unconstrained tensor field in spacetime. Apparently this does not seem to be a good idea since we are replacing the original field \( \Psi \) by an infinite number of other fields \( \psi_n \) but shortly we will see its relevance when we take into account the reducibility of the gauge transformations. The parameters of the gauge transformation (1.2-1.4), \( \epsilon, \chi \) and \( \Lambda \), can also be expanded like (2.1) and (2.2). We then find that (1.2) reduces to

\[
\delta \psi_n = (1 - n) \eta \cdot \partial_x \epsilon_n - \frac{1}{2} n (\partial_\eta \cdot \partial_x + \rho) \epsilon_{n-1} + \frac{1}{4} n (n - 1) \chi_{n-2},
\]
while (1.3) and (1.4) become
\[\delta \epsilon_n = \frac{1}{2} n \Lambda_{n-1},\]  
(2.4)
\[\delta \chi_n = 2 \eta \cdot \partial_x \Lambda_{n+1} + (\partial_\eta \cdot \partial_x + \rho) \Lambda_n.\]  
(2.5)

We can now use the \( \Lambda \) symmetry to choose the gauge \( \Lambda_{n-1} = -(2/n) \epsilon_n \) for \( n \neq 0 \) so that all \( \epsilon_n \) with \( n > 0 \) vanish while \( \epsilon_0 \) remains free. We can now fix the \( \chi \) symmetry by choosing \( \chi_{n-2} = -\frac{4}{n(n-1)} \psi_n \) for \( n \geq 2 \) so that \( \psi_n = 0 \) for \( n \geq 2 \). Then only \( \psi_0, \psi_1 \) and \( \epsilon_0 \) are non-vanishing. We then find that (2.3) and (2.4) become
\[\delta \psi_0 = \eta \cdot \partial_x \epsilon_0,\]  
(2.6)
\[\delta \psi_1 = -\frac{1}{2} (\partial_\eta \cdot \partial_x + \rho) \epsilon_0,\]  
(2.7)
\[\delta \epsilon_0 = 0.\]  
(2.8)

Then the reducibility allowed us to eliminate all terms of \( \Psi \) except \( \psi_1 \) and \( \psi_2 \) which have a much more simple gauge transformation with an unconstrained parameter \( \epsilon_0 \).

When the expansion (2.1) is used in the action (1.1) we get
\[S = \frac{1}{2} \int d^4x \, d^4\eta \left[ \delta'(\eta^2 + 1)(\partial_x \psi_0)^2 + \frac{1}{2} \delta(\eta^2 + 1) \left[ (\partial_\eta \cdot \partial_x + \rho) \psi_0 + 2 \eta \cdot \partial_x \psi_1 \right]^2 - 4 \partial_x \psi_0 \cdot \partial_x \psi_1 \right].\]  
(2.9)

Notice that all \( \psi_n \) with \( n \geq 2 \) have dropped out of the action because of the delta functions. So even without fixing the \( \Lambda \) symmetry the action knows about the reducibility of the original gauge transformations and only \( \psi_0 \) and \( \psi_1 \) remain at the end. We can easily verify that the action (2.9) is invariant under the gauge transformations (2.6) and (2.7). The equation of motion obtained by varying \( \psi_0 \) is\(^\text{2}\)
\[\delta'(\eta^2 + 1) \left[ \square_x \psi_0 - \eta \cdot \partial_x (\partial_\eta \cdot \partial_x + \rho) \psi_0 - 2(\eta \cdot \partial_x)^2 \psi_1 \right] - 2 \delta(\eta^2 + 1) \left[ \square_x \psi_1 + \frac{1}{2} \eta \cdot \partial_x (\partial_\eta \cdot \partial_x + \rho) \psi_1 + \frac{1}{4} (\partial_\eta \cdot \partial_x + \rho)^2 \psi_0 \right] = 0,\]  
(2.10)
while varying \( \psi_1 \) yields
\[\delta(\eta^2 + 1) \left[ \square_x \psi_0 - \eta \cdot \partial_x (\partial_\eta \cdot \partial_x + \rho) \psi_0 - 2(\eta \cdot \partial_x)^2 \psi_1 \right] = 0.\]  
(2.11)

They are not independent since multiplying (2.10) by \( \eta^2 + 1 \) we get (2.11). The delta function in (2.11) implies that the quantity inside the square brackets vanishes on the hyperboloid \( \eta^2 = -1 \). We can now use the expansion (2.2) for \( \psi_0 \) and \( \psi_1 \) so that the quantity inside the square brackets is also a power series in \( \eta \). Then the delta function restriction reduces the square bracket power series in \( \eta \) to a power series in \( \eta^\mu = \eta^\mu/|\eta| \).

\(^{2}\)We could also derive the equations of motion directly from (1.1) before making the expansion (2.1) and get \( \delta'(\eta^2 + 1) \square_x \Psi - \frac{1}{2} (\partial_\eta \cdot \partial_x + \rho) (\delta(\eta^2 + 1)(\partial_\eta \cdot \partial_x + \rho) \Psi = 0. \) Using now the expansion (2.1) we get only (2.10).
since it is just fixing $|\eta|$. Then the vanishing of the square bracket power series on the hyperboloid implies the vanishing of each of its coefficients so that the square bracket itself vanishes. Using a similar reasoning for the second term of (2.10) we get

$$\Box x \psi_0 - \eta \cdot \partial_x (\partial_\eta \cdot \partial_x + \rho) \psi_0 - 2(\eta \cdot \partial_x)^2 \psi_1 = 0, \quad (2.12)$$

$$\Box x \psi_1 + \frac{1}{2} \eta \cdot \partial_x (\partial_\eta \cdot \partial_x + \rho) \psi_1 + \frac{1}{4} (\partial_\eta \cdot \partial_x + \rho)^2 \psi_0 = 0. \quad (2.13)$$

### 3 Continuous Spin

To make contact with the results of [14] we perform a Fourier transformation in $x^\mu$ and $\eta^\mu$

$$\psi_0(\eta, x) = \int d^4 \omega \int d^4 p e^{i \eta \cdot \omega + i p \cdot x} \tilde{\psi}_0(\omega, p), \quad (3.1)$$

and similarly for $\psi_1$ and for $\epsilon_0$. We then find that (2.12) and (2.13) reduce to

$$p^2 \tilde{\psi}_0 + \frac{1}{2} (p \cdot \omega - \rho) p \cdot \partial_\omega \tilde{\psi}_0 + (p \cdot \partial_\omega)^2 \tilde{\psi}_1 = 0, \quad (3.2)$$

$$p^2 \tilde{\psi}_1 - (p \cdot \omega - \rho) p \cdot \partial_\omega \tilde{\psi}_1 - \frac{1}{2} (p \cdot \omega - \rho)^2 \tilde{\psi}_0 = 0, \quad (3.3)$$

and for the gauge transformations (2.6) and (2.7) we find

$$\delta \tilde{\psi}_0 = - p \cdot \partial_\omega \tilde{\epsilon}_0, \quad (3.4)$$

$$\delta \tilde{\psi}_1 = \frac{1}{2} (p \cdot \omega - \rho) \tilde{\epsilon}_0. \quad (3.5)$$

To reproduce the results of [14] we now impose the constraint

$$\tilde{\psi}_0 = -(\Box_\omega - 1) \tilde{\psi}_1, \quad (3.6)$$

so that (3.3) becomes

$$p^2 \tilde{\psi}_1 - (p \cdot \omega - \rho) p \cdot \partial_\omega \tilde{\psi}_1 + \frac{1}{2} (p \cdot \omega - \rho)^2 (\Box_\omega - 1) \tilde{\psi}_1 = 0, \quad (3.7)$$

which is precisely equation (5.2) found in [14]. They also find that the field is constrained and we find the constraint after using (3.6) in the field equation (3.2)

$$(p \cdot \omega - \rho)^2 (\Box_\omega - 1)^2 \tilde{\psi}_1 = 0. \quad (3.8)$$

Since we do not want to impose any condition on the momenta then $(\Box_\omega - 1)^2 \tilde{\psi}_1 = 0$ given the trace condition (5.3) of [14]. Finally we find that the gauge transformation (3.5) coincides with (5.5) of [14], while the consistency of the constraint (3.6) with (3.4) yields

$$(p \cdot \omega - \rho)(\Box_\omega - 1) \tilde{\epsilon}_0 = 0. \quad (3.9)$$

Not constraining the momenta means that $(\Box_\omega - 1) \tilde{\epsilon}_0 = 0$, the same condition found in equation (5.6) of [14]. In this way we have reproduced the constrained formulation found in [14] starting from the action (2.9). Notice that the constraint on $\tilde{\psi}_1$, $(\Box_\omega - 1)^2 \tilde{\psi}_1 = 0$,
is in fact one of the equations of motion while our formulation with two fields $\psi_0$ and $\psi_1$ is completely unconstrained.

The trace condition on $\tilde{\psi}_1$ was removed in [14] using a compensator field $\chi$. This corresponds to our initial formulation with two fields $\psi_1$ and $\psi_2$. The field $\chi$ of [14] can be introduced by a combination of $\tilde{\psi}_0$ and $\tilde{\psi}_1$ as

$$ (p \cdot \omega - \rho) \chi = \tilde{\psi}_0 + (\Box - 1) \tilde{\psi}_1. $$

We then get from (3.2) and (3.3) that

$$ p^2 \tilde{\psi}_1 - (p \cdot \omega - \rho) p \cdot \partial_\omega \tilde{\psi}_1 + \frac{1}{2} (p \cdot \omega - \rho)^2 (\Box - 1) \tilde{\psi}_1 - \frac{1}{2} (p \cdot \omega - \rho)^3 \chi = 0, $$

$$ (\Box - 1)^2 \tilde{\psi}_1 - (p \cdot \omega - \rho) (\Box - 1) \chi - 4p \cdot \partial_\omega \chi = 0, $$

while the gauge transformation of $\chi$ becomes

$$ \delta \chi = (\Box - 1) \tilde{\epsilon}_0. $$

These correspond to equations (5.13-5.15) of [14]. By choosing $\chi = 0$ we recover from (3.10) the constraint (3.6), the trace condition of $\tilde{\psi}_1$ and $\tilde{\epsilon}_0$ besides (3.7). Then we have shown that the equations proposed for CSPs in [14] can be obtained from the action (2.9).

4 Higher Spin Fields

It was shown in [10] that taking $\rho = 0$ in (1.1) reduces the action to a sum of Fronsdal actions for all integer helicities. Just setting $\rho = 0$ in the equation for $\tilde{\psi}_0$ and $\tilde{\psi}_1$ of the last section does not lead us in an obvious way to any known formulation for HS theories. So let us go back to (2.9) and set $\rho = 0$. Then the equations of motion (2.10) and (2.11) reduce to

$$ \delta'(\eta^2 + 1) \left[ \Box x \psi_0 - \eta \cdot \partial_x \partial_\eta \cdot \partial_x \psi_0 - 2(\eta \cdot \partial_x)^2 \psi_1 \right] $$

$$ -2\delta(\eta^2 + 1) \left[ \Box x \psi_1 + \frac{1}{2} \eta \cdot \partial_x \partial_\eta \cdot \partial_x \psi_1 + \frac{1}{4} (\partial_\eta \cdot \partial_x)^2 \psi_0 \right] = 0, $$

and

$$ \delta(\eta^2 + 1) \left[ \Box x \psi_0 - \eta \cdot \partial_x \partial_\eta \cdot \partial_x \psi_0 - 2(\eta \cdot \partial_x)^2 \psi_1 \right] = 0, $$

respectively. Using the same reasoning which lead to (2.12) and (2.13) we find

$$ \Box x \psi_0 - \eta \cdot \partial_x \partial_\eta \cdot \partial_x \psi_0 - 2(\eta \cdot \partial_x)^2 \psi_1 = 0, $$

$$ \Box x \psi_1 + \frac{1}{2} \eta \cdot \partial_x \partial_\eta \cdot \partial_x \psi_1 + \frac{1}{4} (\partial_\eta \cdot \partial_x)^2 \psi_0 = 0. $$

The gauge transformations (2.6) and (2.7) are now

$$ \delta \psi_0 = \eta \cdot \partial_x \epsilon_0, $$

$$ \delta \psi_1 = -\frac{1}{2} \partial_\eta \cdot \partial_x \epsilon_0. $$
To get the results of [14] for the HS case we use the constraint
\[ \psi_1 = -\frac{1}{4} \Box_\eta \psi_0, \tag{4.7} \]
so that (4.3) and (4.4) become
\[ \Box_x \psi_0 - \eta \cdot \partial_x \eta \cdot \partial_x \psi_0 + \frac{1}{2} (\eta \cdot \partial_x)^2 \Box_\eta \psi_0 = 0, \tag{4.8} \]
\[ \Box_x \Box_\eta \psi_0 + \frac{1}{2} \eta \cdot \partial_x \eta \cdot \partial_x \Box_\eta \psi_0 - (\partial_\eta \cdot \partial_x)^2 \psi_0 = 0. \tag{4.9} \]
Applying \( \Box_\eta \) in (4.8) and comparing with (4.9) gives
\[ (\eta \cdot \partial_x)^2 \Box_\eta^2 \psi_0 = 0. \tag{4.10} \]
This condition is too restrictive for \( \psi_0 \) unless we choose \( \Box_\eta^2 \psi_0 = 0 \). The gauge condition for \( \psi_1 \) (4.6) yields
\[ \eta \cdot \partial_x \Box_\eta \epsilon_0 = 0, \tag{4.11} \]
and again it is too restrictive on \( \epsilon_0 \) unless we take \( \Box_\eta \epsilon_0 = 0 \).

We now expand \( \psi_0 \) as in (2.2) to get from (4.8) that
\[ \sum_{n=0}^{\infty} \frac{1}{n!} \eta^{\mu_1} \ldots \eta^{\mu_n} \left[ \Box_x \psi_0^{(0,n)} - n \partial_{\mu_1} \partial_x \psi_0^{(0,n)} + \frac{1}{2} n(n-1) \partial_{\mu_1} \partial_{\mu_2} \psi_0^{(0,n)} \right] = 0, \tag{4.12} \]
where a prime denotes contraction of two indices. These are the Fronsdal equations for all integer higher spins. Then from (4.10) we get the double trace condition on \( \psi_0^{(0,n)}(x) \) while from (4.11) we get the trace condition on the gauge parameters \( \epsilon_0 \) which appears as a consistency condition for the constraint (4.7). These equations were also obtained in [14] where they were derived from Fronsdal equations but no action was provided. It should be remarked that [14] considered only one helicity in \( \psi_0 \) but as we have shown here it can be extended to any number of helicities.

To single out just one helicity \( s \) we have to impose one further condition
\[ (\eta \cdot \partial_\eta - s) \psi_0 = 0. \tag{4.13} \]
This selects the term \( \psi_0^{(0,s)} \) in \( \psi_0 \). Gauge invariance now requires that \( (\eta \cdot \partial_\eta - s + 1) \epsilon_0 = 0 \) so that \( \epsilon_0 \) has just one component of rank \( s - 1 \) as expected.

Alternatively, we could go back to the original action (1.1) and set \( \rho = 0 \). We can still use the expansion of \( \Psi \) (2.1) so that all \( \psi_n \) with \( n > 1 \) vanish and we get
\[ S = \frac{1}{2} \int d^4x \ d^4\eta \left[ \delta' (\eta^2 + 1) (\partial_x \psi_0)^2 + \frac{1}{2} \delta (\eta^2 + 1) \left( (\partial_\eta \cdot \partial_x \psi_0 + 2 \eta \cdot \partial_x \psi_1)^2 - 4 \partial_x \psi_0 \cdot \partial_x \psi_1 \right) \right]. \tag{4.14} \]
It is gauge invariant under (4.5) and (4.6) with no constraint on the gauge parameter. The equations of motion from the variation of $\psi_0$ gives (4.1) while the variation of $\psi_1$ gives (4.2). There are no constraints on the fields. This action is identical to the action (52) of [20] for the flat case if we identify their fields $h_1$ and $h_2$ as $h_1 = \psi_0 / \sqrt{2}$ and $h_2 = -2\psi_1 / \sqrt{2}$, respectively, when $\mu^2 = 1$.

We now implement the constraint (4.7) in the action to get

$$ S = \frac{1}{2} \int d^4x \ d^4\eta \left[ \delta'(\eta^2 + 1)(\partial_x \psi_0)^2 + \frac{1}{2} \delta(\eta^2 + 1) \left( (\partial_\eta \cdot \partial_x \psi_0 - \frac{1}{2} \eta \cdot \partial_x \Box_\eta \psi_0)^2 + \partial_x \psi_0 \cdot \partial_x \Box_\eta \psi_0 \right) \right]. \quad (4.15) $$

Now the action is no longer gauge invariant. The variation of the action is proportional to a term depending on $\psi_0$ (and its derivatives) multiplied by $\eta \cdot \partial_x \Box_\eta \psi_0$ so that we regain gauge invariance if the gauge parameter is traceless. Because of the presence of $\Box_\eta^2$ in the action the equations of motion will have terms up to the second derivative of the delta function. This term will give rise to (4.8) while the term with one derivative of the delta function will identically vanish after using (4.8). The term with the delta function without derivatives reduces to $(\eta \cdot \partial_x)^2 \Box_\eta^2 \psi_0$ after the use of (4.8) giving rise to the double trace condition on $\psi_0$. At the end we get again the Fronsdal action for all spins.

A third way to proceed is to use the expansion (2.2) for $\psi_0$ directly in the action (4.15). The $\eta$ integration is divergent since it has to be performed on the hyperboloid enforced by the delta functions. We can perform a Wick rotation $\eta^0 \rightarrow i\eta^0$ so that the integration is now done on the sphere. Alternatively we could have started with the Euclidean version of (1.1) so that the delta functions enforce an integration over a sphere. Anyway, since the action is quadratic in $\psi^{(0,0)}(x)$ there will be contributions involving fields of different ranks. If the sum (or difference) of the ranks is odd there will appear an odd number of $\eta$’s so that the integral vanishes. When the sum (or difference) of the ranks is even there appears two equal terms with opposite signs so that they cancel out. Then the action reduces to a sum of quadratic terms with fields of the same rank. The integration on $\eta$ can be performed and we get

$$ S = -\pi^2 \sum_{s=0}^{\infty} \frac{1}{2^s (s!)^2} S_s^{(F)}, \quad (4.16) $$

where $S_s^{(F)}$ is the Fronsdal action for helicity $s$.

Again, in order to get a description for just one helicity $s$ in (4.16) we can impose (4.13) to pick up the helicity $s$ component. We could try to implement this condition directly into the action through a Lagrange multiplier but this seems very difficult to be accomplished since this condition is not gauge invariant.

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*Other regularizations are also discussed in [10]. Since we get Fronsdal field equations for all spins from (4.15) the regularization of the action must not be a fundamental problem.*
5 Oscillator Basis

Another approach to handle the tensor indices in HS field theory is through the use of oscillators instead of extra coordinates. The version developed in [21] for flat space makes use of creation and annihilation operators, $\alpha^\mu, \overline{\alpha}^\mu$ satisfying the usual commutation relations $[\overline{\alpha}^\mu, \alpha^\nu] = \eta^\mu\nu$ with $\overline{\alpha}^\mu = (\alpha^\mu)\dagger$. A totally symmetric field of helicity $s$, $\phi_{\mu_1\ldots\mu_s}(x)$, is saturated with the creation operators to form the ket

$$|\phi> = \frac{1}{s!} \alpha^{\mu_1} \ldots \alpha^{\mu_s} \phi_{\mu_1\ldots\mu_s}(x)|0>.$$  \hspace{1cm} (5.1)

An action, which reduces to the Fronsdal action, is then written as

$$S^{(M)} = -\frac{1}{2} \int d^4x \left[ <\partial^\mu \phi|(1 - \frac{1}{4} \alpha^2)|\partial_\mu \phi> - <(\overline{\alpha} \cdot \partial_x - \frac{1}{2} \alpha \cdot \partial_x \overline{\alpha}^2)\phi|(\overline{\alpha} \cdot \partial_x - \frac{1}{2} \alpha \cdot \partial_x \overline{\alpha}^2)\phi> \right],$$ \hspace{1cm} (5.2)

where $<\phi| = (|\phi>)\dagger$. The double traceless condition is now $(\overline{\alpha}^2)|\phi> = 0$, while the gauge symmetry has the form

$$\delta |\phi> = \alpha \cdot \partial_x |\epsilon>, \hspace{1cm} (5.3)$$

$$|\epsilon> = \frac{1}{(s-1)!} \alpha^{\mu_1} \ldots \alpha^{\mu_{s-1}} \epsilon_{\mu_1\ldots\mu_{s-1}}(x)|0>,$$ \hspace{1cm} (5.4)

with the traceless condition $\overline{\alpha}^2|\epsilon> = 0$. We can find explicitly the relation among the terms in the action (5.2) and the action (4.15) with $\psi_0$ satisfying (4.13) to have just one helicity. The result is

$$<\partial^\mu \phi|\partial_\mu \phi> = \frac{2^s s!}{\pi^2} \int d^4 \eta \left[ \delta(\eta^2 + 1)(\partial_x \psi_0)^2 + \frac{1}{4} \delta(\eta^2 + 1)\partial_x \psi_0 \cdot \square \eta \partial_x \psi_0 \right],$$ \hspace{1cm} (5.5)

$$<\overline{\alpha}^2 \partial^\mu \phi|\overline{\alpha}^2 \partial_\mu \phi> = -\frac{2^s s!}{\pi^2} \int d^4 \eta \delta(\eta^2 + 1)\partial_x \psi_0 \cdot \square \eta \partial_x \psi_0,$$ \hspace{1cm} (5.6)

$$<\overline{\alpha} \cdot \partial_x \phi|\overline{\alpha} \cdot \partial_x \phi> = \frac{2^s s!}{\pi^2} \int d^4 \eta \delta(\eta^2 + 1) \left[ -\frac{1}{2} (\partial_\eta \cdot \partial_x \psi_0)^2 + \frac{1}{16} (\eta \cdot \partial_x \square \eta \psi_0)^2 - \frac{1}{8} \partial_x \psi_0 \cdot \square \eta \partial_x \psi_0 \right],$$ \hspace{1cm} (5.7)

$$<\overline{\alpha} \cdot \partial_x \phi|\alpha \cdot \partial_x \overline{\alpha}^2 \phi> = \frac{2^s s!}{\pi^2} \int d^4 \eta \delta(\eta^2 + 1) \left[ -\frac{1}{2} \partial_\eta \cdot \partial_x \psi_0 \cdot \eta \cdot \partial_x \square \eta \psi_0 + \frac{1}{8} (\eta \cdot \partial_x \square \eta \psi_0)^2 - \frac{1}{4} \partial_x \psi_0 \cdot \square \eta \partial_x \psi_0 \right],$$ \hspace{1cm} (5.8)

$$<\alpha \cdot \partial_x \overline{\alpha}^2 \phi|\alpha \cdot \partial_x \overline{\alpha}^2 \phi> = \frac{2^s s!}{\pi^2} \int d^4 \eta \delta(\eta^2 + 1) \left[ -\frac{1}{2} \partial_x \psi_0 \cdot \square \eta \partial_x \psi_0 - \frac{1}{4} (\eta \cdot \partial_x \square \eta \psi_0)^2 \right].$$ \hspace{1cm} (5.9)

However it is not apparent how the two set of variables are connected. If there exists a connection between the extra coordinates $\eta$ and the oscillators $\alpha, \overline{\alpha}$ it is not a simple one.
Also it is not clear how to lift the trace constraints using the oscillator formalism so that a connection with our formulation with two fields is not apparent.

A similar situation happens with the Francia-Sagnotti formulation in [22]. The trace constraint on the completely symmetric tensor for a given helicity is lifted with two completely symmetric gauge fields of rank $s-3$ and $s-4$. Their action contains higher spacetime derivatives while ours has only two spacetime derivatives so it seems that there is no clear connection between the two formalisms as well.

6 Conclusions

We have shown that the gauge transformations that leave the Schuster-Toro action invariant are reducible. After the expansion of $\Psi(\eta, x)$ in powers of $\eta^2 + 1$ the reducibility is responsible for the elimination of all terms except the first two. The reduced action depends on two fields $\psi_0$ and $\psi_1$ and is invariant under an irreducible gauge transformation. As shown here it reproduces the equations found by [14] and the action in [20]. Therefore (2.9) gives rise to an alternative formulation for CSPs and HS field theory.

It should also be remarked that going from the formulation with two fields to the one field formulation a constraint had to be imposed. For the HS case (4.7) is not a gauge fixing condition. Usually the gauge fixing condition restricts the gauge parameter through an equation involving spacetime derivatives. Even though we get an equation like (4.11), with one spacetime derivative, we find that the components of $\epsilon_0$ must satisfy $p(p^\mu p_1^{\mu_1}...p_n^{\mu_n}) = 0$ (where a prime means the contraction of two indices) in momentum space. The only non trivial solution is the vanishing of the gauge parameter trace, $\Box_\eta \epsilon_0 = 0$. Then the gauge parameter equation effectively has no spacetime derivatives so that (4.7) is in fact a constraint. This can be checked by handling (4.3) and (4.4) at lowest orders in $\eta$. Notice that these equations relate $\psi_0^{(s)}$ and $\psi_1^{(s-2)}$ so that they decouple. For $s = 0$ and $s = 1$, $\psi_1$ does not contribute and we find that $\psi_0$ propagates $s = 0$ and $s = 1$ helicities, respectively. For $s = 2$ we find that $\psi_0$ propagates $s = 2$ helicity while $\psi_1$ propagates $s = 0$. This scalar is not the trace of the $s = 2$ field, so up to this point we have two $s = 0$ fields. As we go to higher $s$, $\psi_1$ starts to propagate higher helicities adding up to the helicities generated by $\psi_0$. Since the double traces of $\psi_0$ and $\psi_1$ are non vanishing, they also propagate lower $s$ helicities. At the end we find that (4.3) and (4.4) propagate all integer helicities and each one appears an infinity number of times. On the other side, after using the constraint (4.7) and reducing the theory to one field $\psi_0$ we get all integer helicities appearing just once. This can be understood since (4.7) is just stating that $\psi_1$ is essentially the trace of $\psi_0$. This will be discussed in details in a forthcoming work [25]. The situation for the CSP case is much more involved and it is not clear at this stage whether (3.6) is a constraint. Work in this direction is also in progress.

The extension of our results to dimensions other than four is straightforward. The most important extension right now is the addition of possible deformations of the gauge symmetry to include self-interactions. The coupling to gravity is also possible and it may provide an alternative way to look for CSPs and HS interactions in non-flat backgrounds.
in particular in AdS. It is not clear which form of the action is more suitable for these purposes.

It is known that in the context of $AdS_4/CFT_3$ some critical models in the boundary are dual to higher spin fields in $AdS_4$ [23]. Such higher spin theories have been extensively studied (for a recent review see [24]) and only recently an action has been proposed [26, 27]. The new actions presented here may be an alternative to the study of this duality. Another situation where the HS results found here can be applied is in the small tension limit of string theory [17, 18].

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