On Intrinsic Angular Momentum due to Edge Mass Current for Superfluid $^3$He A-Phase

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Abstract. In the superfluid $^3$He A-phase, the edge mass current which accompanies surface Andreev bound states flows due to topological features. We discuss the relation between the angular momentum by the edge mass current and the macroscopic intrinsic angular momentum $L = N\hbar$ by the Cooper pairs with aligned angular momenta, based on the quasi-classical theory. At the low temperature limit, half of the edge mass current from the bound state is canceled by that from the continuum state. Then, the magnitude of the total angular momentum is $L = N\hbar/2$ which corresponds to the macroscopic intrinsic angular momentum of $N$ particles. The net angular momentum decreases monotonously toward zero at the transition temperature. However, that is larger than the macroscopic intrinsic angular momentum expected from the same behavior of the superfluid density in middle temperatures.

1. Introduction
The superfluid $^3$He is a topological superfluid defined by a topological number. On a surface of the superfluid $^3$He faced to a topologically trivial vacuum, a superfluid gap is closed by a topological phase transition. Then, surface Andreev bound states appear. The topological features are different in the $^3$He A- and B-phases. The $^3$He A-phase is a chiral superfluid with edge mass current, and B-phase is a helical superfluid with edge spin current. In the A-phase, the magnitude of angular momentum by the edge mass current is $N\hbar/2$ at zero temperature [1], where $N$ is the total number of $^3$He atoms in slab geometry.

The value $N\hbar/2$ is also associated with the magnitude of intrinsic angular momentum by the Cooper pairs in the A-phase where the direction of angular momentum is aligned. For the magnitude of the intrinsic angular momentum, three classes have been proposed. They are summarized as $L = N\hbar/(\Delta/E_F)^n$ with $n = 0$ [2], $n = 1$ [3], and $n = 2$ [4], where $\Delta$ is a superfluid gap and $E_F$ is the Fermi energy. Since $\Delta/E_F \sim 10^{-3}$ for the superfluid $^3$He, the intrinsic angular momentum can be observed macroscopically only when $n = 0$. Thus, the magnitude of the angular momentum by the edge mass current and that of the macroscopic intrinsic angular momentum are the same. Here, we discuss whether the angular momentum by the edge mass current corresponds to the intrinsic angular momentum from microscopic features and the temperature dependence.

2. Quasi-classical theory
We consider disk geometry with a radius $R$ much larger than the coherence length and a small thickness along the $z$-direction to align $l$-vector toward the $z$-direction, where $d$-vector is also...
aligned toward the \( z \)-direction by the dipole interaction. In this geometry, the intrinsic angular momentum is expected as \( L_z = N h / 2 \) when \( n = 0 \). Since the \( d \)-vector is fixed, we consider the orbital part of the order parameter \( \Delta(x, k) = A_x(x)k_x + A_y(x)k_y \) in the one-dimension toward the radial direction \( x \), where \( k \) is the direction of the relative momentum of a Cooper pair and \( x \) is the center-of-mass coordinate of a Cooper pair. The coefficients \( A_x \) and \( A_y \) can be chosen as a real number without loss of generality. We assume that the surface at \( x = 0 \) is specular, where only the coefficient \( A_x \) is suppressed on the surface and \( |A_x| = |A_y| \) far from the surface.

Microscopic information of edge mass current is contained by the quasi-classical Green’s functions \( g(x, k, \omega_n) \), \( f(x, k, \omega_n) \), and \( \bar{f}(x, k, \omega_n) \) which satisfy the normalization condition \( g^2 - 1 - f \bar{f} \). The quasi-classical Green’s functions are calculated using Eilenberger equation [5]

\[
\left( \omega_n + \hbar v_F k_x \frac{\partial}{\partial x} \right) f = \Delta(x, k)g, \quad \left( \omega_n - \hbar v_F k_x \frac{\partial}{\partial x} \right) \bar{f} = \Delta(x, k)^*g, \tag{1}
\]

where \( v_F \) is the Fermi velocity and \( \omega_n = (2n + 1)\pi k_B T \) is the Matsubara frequency with \( n \in \mathbb{Z} \). The self-consistent condition for the pair potential is given as the gap equation

\[
\Delta(x, k) = N_0 \pi k_B T \sum_{0 \leq \omega_n \leq \omega_c} \langle V(k, k') \left[ f(x, k', \omega_n) + \bar{f}(x, k', \omega_n)^* \right] \rangle_{k'}, \tag{2}
\]

where \( N_0 \) is the density of states in the normal state, \( \omega_c \) is a cutoff energy set to be \( \omega_c = 40 \pi k_B T \) with the transition temperature \( T_c \), and \( \langle \cdot \cdot \cdot \rangle_k \) indicates the Fermi surface average. The pairing interaction is given as \( V(k, k') = 3g_1 k \cdot k' \) for Cooper pairs with an orbital angular momentum \( l = 1 \), where \( g_1 \) is a coupling constant. In our calculation, we use the relation \( (g_1 N_0)^{-1} = \ln(T/T_c) + 2 \pi k_B T \sum_{0 \leq \omega_n \leq \omega_c} \omega^{-1} \) and solve eq. (1) by the Riccati method [6].

By using the quasi-classical Green’s function, the mass current is calculated by

\[
j(x) = m v_F N_0 \pi k_B T \sum_{-\omega_c \leq \omega_n \leq \omega_c} \langle kg(x, k, \omega_n) \rangle_k, \tag{3}
\]

where \( m \) is the mass of the \( ^3 \text{He} \) atom. The mass current and local density of states (LDOS) for an energy \( E \) are given by

\[
j(x, E) = \langle j(x, k, E) \rangle_k = m v_F N_0 \langle k \text{Re} \left[ g(x, k, \omega_n) |_{\omega_n = E + i\eta} \right] \rangle_k, \tag{4}
\]

\[
N(x, E) = \langle N(x, k, E) \rangle_k = N_0 \langle \text{Re} \left[ g(x, k, \omega_n) |_{\omega_n = E + i\eta} \right] \rangle_k, \tag{5}
\]

respectively, where \( \eta \) is a positive infinitesimal constant.

### 3. Bound and continuum states

First, we show the microscopic features of the edge mass current by analytical calculation at the low temperature limit. Since the edge mass current accompanies surface Andreev bound states, we decompose the edge mass current into the bound state below a superfluid gap and the continuum state above the gap.

We solve Eilenberger equation (1) under the pair potential with \( A_x = \Delta_0 \tanh(x/\xi) \) and \( A_y = \Delta_0 \), where \( \Delta_0 \) is a superfluid gap in a bulk and \( \xi = \hbar v_F / \Delta_0 \) is the coherence length. The quasi-classical Green’s function

\[
g(x, k, \omega_n) = \frac{1}{\sqrt{\omega_n^2 + \Delta_0^2 \sin^2 \theta}} \left[ \omega_n + \frac{\Delta_0^2 \sin^2 \theta \cos^2 \phi}{2(\omega_n + i \Delta_0 \sin \theta \sin \phi)} \text{sech}^2 \left( \frac{x}{\xi} \right) \right], \tag{6}
\]

satisfies eq. (1), where \( k_x = \sin \theta \cos \phi \) and \( k_y = \sin \theta \sin \phi \). By the quasi-classical Green’s
function in eq. (6), we can calculate θ-angle-resolved LDOS by eq. (5) as

\[ N(x, \sin \theta, E) = \frac{N_0}{2} \sech^2 \left( \frac{x}{\xi} \right), \]  

for the bound state \( |E| < \Delta_0 \sin \theta \) and

\[ N(x, \sin \theta, E) = N_0 \left[ \frac{|E|}{\sqrt{E^2 - \Delta_0^2 \sin^2 \theta}} - \frac{1}{2} \left( \frac{|E|}{\sqrt{E^2 - \Delta_0^2 \sin^2 \theta}} - 1 \right) \sech^2 \left( \frac{x}{\xi} \right) \right], \]  

for the continuum state \( |E| > \Delta_0 \sin \theta \). Thus, quasiparticles feel the pair potential \( \Delta_0 \sin \theta \), where \( \theta \) is an angle from a point node. On the surface, the zero energy density of states has a finite value \( N_0 \). Since the quasiparticles fill the energy state up to the Fermi energy, namely \( E = 0 \) at the low temperature limit, the mass current along the edge from the bound state is

\[ j_y^{\text{bound}}(x) = \int_{-\Delta_0 \sin \theta}^{0} dE j_y(x, k, E) \bigg|_k = -\frac{m v_F N_0 \Delta_0}{6} \sech^2 \left( \frac{x}{\xi} \right). \]  

and from the continuum state is

\[ j_y^{\text{cont}}(x) = \int_{-\infty}^{-\Delta_0 \sin \theta} dE j_y(x, k, E) \bigg|_k = \frac{m v_F N_0 \Delta_0}{12} \sech^2 \left( \frac{x}{\xi} \right). \]  

In the disk with \( R \gg \xi \), the angular momentum by the edge mass current from each state is

\[ L_z^{\text{bound}} = N \hbar, \quad L_z^{\text{cont}} = -\frac{N \hbar}{2}, \]  

where angular momentum from the bound state corresponds to that in the two-dimensional Fermi surface calculated by Furusaki et al. [7]. Therefore, the total angular momentum is

\[ L_z = L_z^{\text{bound}} + L_z^{\text{cont}} = \frac{N \hbar}{2}, \]  

which corresponds to the intrinsic angular momentum when \( n = 0 \). In regard to microscopic features, half of the angular momentum from the bound state is canceled by that from the continuum state.

**Figure 1.** Profiles of mass current along the edge at \( T = 0.5 T_c \) under self-consistent pair potential (red line) and uniform pair potential (blue line). The units are \( \xi_0 = \hbar v_F / 2 \pi k_B T_c \) and \( j_0 = m v_F N_0 \pi k_B T_c \). Inset: A profile of the self-consistent pair potential.

**Figure 2.** Left axis: Temperature dependence of angular momentum under self-consistent pair potential (solid circle) and uniform pair potential (open circle). Right axis: Temperature dependence of superfluid density \( \rho_s^\parallel \) (solid line) and \( \rho_s^\perp \) (dotted line).
4. Temperature dependence
Next, we show the temperature dependence of the angular momentum by numerical calculation, compared with the superfluid density in the two-fluid model. The intrinsic angular momentum associated with Cooper pairs is expected to decrease as the superfluid density.

We solve Eilenberger equation (1) and gap equation (2) self-consistently and compare to the solution of Eilenberger equation under uniform pair potential. In Fig. 1, profiles of the edge mass current at \( T = 0.5T_c \) under self-consistent pair potential (red line) and uniform pair potential (blue line) are shown. The profile of the edge mass current under self-consistent pair potential is varied gradually over the region within \( -10 \xi_0 \) owing to the variation of the pair potential (the inset of Fig. 1). In contrast, the edge mass current under uniform pair potential is localized near the surface. A profile of the self-consistent pair potential is also shown in the inset of Fig. 1. Because of the specular boundary condition, the \( k_x \)-component becomes zero at the surface. In contrast, the \( k_y \)-component is enhanced by compensating for the loss of the \( k_x \)-component on the surface, where the polar state is realized. In other words, the inverse chiral state \( -k_x + ik_y \) is mixed with the chiral state \( k_x + ik_y \) near the surface. The inverse chiral state must have the 4\( \pi \) phase winding to minimize the free energy in the axisymmetric disk [8].

The temperature dependence of the angular momentum by the edge mass current under these two forms of pair potential is shown in Fig. 2 with the component of the superfluid density tensor parallel (perpendicular) to the \( l \)-vector \( \rho_{s\|}^0 (\rho_{s\perp}^0) \) [4]. In both cases, the angular momentum heads toward \( Nh/2 \) at the low temperature limit and toward zero at the transition temperature. Under the uniform pair potential, the angular momentum has the same temperature dependence as \( \rho_{s\|}^0 \), which corresponds to the previous work by Kita [9]. However, the angular momentum under the self-consistent pair potential is larger than that under the uniform pair potential in middle temperatures. This increment of the angular momentum is due to the additional inverse chiral state with the 4\( \pi \) phase winding yielding the extra current.

5. Summary
We have discussed the relation between the angular momentum by the edge mass current which accompanies surface Andreev bound states and the intrinsic angular momentum by Cooper pairs, based on the quasi-classical theory. At the low temperature limit, half of the edge mass current from the bound state is canceled by that from the continuum state. Then, the magnitude of the total angular momentum is \( L = Nh/2 \) which corresponds to the macroscopic intrinsic angular momentum. However, the angular momentum by the edge mass current is larger than that expected from Cooper pairs in the temperature region between \( T = 0 \) and \( T_c \).

Acknowledgments
We thank K. Nagai, S. Higashitani, and Y. Nagato for helpful discussions. Y.T. acknowledges the support of the Research Fellowships of the Japan Society for the Promotion of Science for Young Scientists. This work was supported by the MEXT KAKENHI (No. 22103002).

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