RECOLOURING HOMOMORPHISMS TO TRIANGLE-FREE REFLEXIVE GRAPHS

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Abstract. For a graph $H$, the $H$-recolouring problem $\text{Recol}(H)$ asks, for two given homomorphisms from a given graph $G$ to $H$, if one can get between them by a sequence of homomorphisms of $G$ to $H$ in which consecutive homomorphisms differ on only one vertex. We show that, if $G$ and $H$ are reflexive and $H$ is triangle-free, then this problem can be solved in polynomial time. This shows, at the same time, that the closely related $H$-reconfiguration problem $\text{Recon}(H)$ of deciding whether two given homomorphisms from a given graph $G$ to $H$ are in the same component of the Hom-graph $\text{Hom}(G, H)$, can be solved in polynomial time for triangle-free reflexive graphs $H$.

1. Introduction

Reconfiguration, in various settings, is the common notion of moving between solutions of a combinatorial problem via small changes; see van den Heuvel [19] and Nishimura [17] for detailed surveys. For example, a reconfiguration, or recolouring, between two graph colourings is a sequence of colourings in which consecutive elements differ on one vertex. The $k$-recolouring problem is to decide, for two given $k$-colourings of $G$, whether or not there is a recolouring between them. Cereceda, van den Heuvel and Johnson [9] showed that 3-recolouring is polynomial time solvable, while Bonsma and Cereceda [1] showed that $k$-recolouring is PSPACE-complete for $k \geq 4$. One should compare this to the $k$-colouring problem which is well known to be polynomial time solvable for $k = 2$ but NP-complete for $k \geq 3$.

The $k$-colouring problem generalizes to the $H$-colouring problem $\text{Col}(H)$ for a graph $H$ and to the constraint satisfaction problem $\text{CSP}(\mathcal{H})$ for a relational structure $\mathcal{H}$. The computational complexity of $\text{Col}(H)$, and its myriad of generalizations and variations, have been well studied over the last 50 years, culminating in the recent CSP dichotomy theorem of Bulatov [7] and Zhuk [22], which says that $\text{CSP}(\mathcal{H})$ is either polynomial time solvable or NP-complete for every relational structure $\mathcal{H}$.

Analogously, the $k$-recolouring problem also generalizes to reconfiguration problems for $H$-colourings and CSP, both of which are well studied; see, e.g., [4–6, 13, 20] and [2, 8, 10, 11, 14–16, 18], respectively. In particular, Gopalan et al. [10] proved a dichotomy theorem for the reconfiguration variation of $\text{CSP}(\mathcal{H})$ for structures $\mathcal{H}$ with two vertices. In this paper, we focus on the complexity of $H$-colouring reconfiguration in the setting of graphs which are reflexive in the sense that they have a

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While we will state our case that there is a recolouring between them.

Several authors (e.g. [3]) have noted that, for undirected graphs, the $H$-recolouring problem is essentially the same as asking whether there is a path between a given pair of homomorphisms in $\text{Hom}(G,H)$; we call this the $H$-reconfiguration problem and denote it by $\text{Recon}(H)$. In one direction, any path from $\phi$ to $\psi$ in $\text{Hom}(G,H)$ can be converted into a path in which any two consecutive $H$-colourings differ on one vertex by simply changing the colours “one at a time.” In the other direction, if $(G,\phi,\psi)$ is a YES instance for $\text{Recol}(H)$, then the sequence of colourings is a path from $\phi$ to $\psi$ in $\text{Hom}(G,H)$ unless there exists an isolated vertex $v$ of $G$ with a loop which is mapped to different components of $H$ by $\phi$ and $\psi$. We will state our results in terms of $\text{Recol}(H)$ in the introductory sections, in the later sections it is convenient to speak in terms of paths in the graph $\text{Hom}(G,H)$, and when we do so, we talk of $\text{Recon}(H)$ and reconfigurations rather than $\text{Recol}(H)$ and recolourings.

The diameter of $\text{Hom}(G,H)$ can be superpolynomial in $|V(G)|$ (see, e.g., [1]) and so, a priori, a YES instance of $\text{Recol}(H)$ may not have a certificate of polynomial size. Thus, the natural complexity class for $\text{Recol}(H)$ is PSPACE, rather than NP.

Some of the known results for irreflexive graphs are summarized as follows. The case that $H$ is a clique follows from the results of [1, 9] for $k$-recolouring discussed earlier. In [4], the dichotomy for cliques was generalized to ‘circular cliques’ $G_{p/q}$; specifically, $\text{Recol}(G_{p/q})$ is polynomial time solvable if $p/q < 4$ and PSPACE-complete if $p/q \geq 4$. The polynomial time algorithm for 3-recolouring in [9] had a topological flavor which Wrochna distilled and extended in [20] to prove the very general result that $\text{Recol}(H)$ is polynomial time solvable for any graph $H$ not containing a 4-cycle. In [13], we proved that, if $H$ is a $K_{2,3}$-free quadrangulation of the sphere different from the 4-cycle, then $\text{Recol}(H)$ is PSPACE-complete.

In the current paper, we use several ideas of these earlier papers, and adapt Wrochna’s language and many of his ideas, to prove an analogue of his result for reflexive graphs $G$ and $H$. Reflexive graphs are actually a smoother setting for some of these ideas; e.g. the fact that two consecutive vertices in a path in $G$ can be

\footnote{In contrast, it is interesting to remark that the problems $\text{Recol}(H)$ and $\text{Recon}(H)$ are not so closely related in the digraph case. For given $H$ they may have different sets of connected YES instances; though we do not have examples yet where they have different complexity [5].}
map to the same vertex of $H$ eliminates a parity obstruction which turns up in \cite{20}. We prove the following.

**Theorem 1.1.** If $H$ is a reflexive triangle-free graph, then $\text{Recol}(H)$ can be solved in polynomial time for reflexive instances.

In contrast, we recently proved in \cite{13} that $\text{Recol}(H)$ is PSPACE-complete when $H$ is any $K_4$-free reflexive triangulation of the sphere, other than a triangle itself. We refer to this paper for a deeper suggestion of the connection of topology to the complexity of the reconfiguration problem. We remark that the argument of Wrochna \cite{20} directly generalizes to get that $\text{Recol}(H)$ is solvable in polynomial time for general instances if $H$ is a graph which may have loops on some vertices which does not contain a 4-cycle, a triangle with a loop on one vertex, or two adjacent vertices with loops; see the footnote on p. 330 of \cite{21}.

Before we move on to the proof of the main theorem, we observe that it implies the following result for a less general class of graphs $H$, but more general instances. Recall that the *girth* of a graph is the length of its shortest cycle.

**Corollary 1.2.** If $H$ is a reflexive symmetric graph with girth at least 5, then $\text{Recol}(H)$ can be solved in polynomial time for all instances.

**Proof.** Let $(G, \phi, \psi)$ be an instance of $\text{Recol}(H)$. If $G$ has isolated vertices, then we can simply change the colour of every such vertex $v$ from $\phi(v)$ to $\psi(v)$ at the beginning without any issues. So, we assume that $G$ has no isolated vertices.

Let $G'$ be the graph obtained from $G$ by adding all possible loops. We claim that the instance $(G', \phi, \psi)$ of $\text{Recol}(H)$ is a YES instance if and only if $(G, \phi, \psi)$ is a YES instance. In one direction, it is clear that if $(G', \phi, \psi)$ is a YES instance, then so is $(G, \phi, \psi)$.

Now, suppose that $(G, \phi, \psi)$ is a YES instance. Then there is a path from $\phi$ to $\psi$ in $\text{Hom}(G, H)$ in which consecutive $H$-colourings differ on one vertex. It suffices to consider the case that $\phi$ and $\psi$ differ on one vertex $x$, from which the general case follows. If $\phi_1\psi$ is an edge of $\text{Hom}(G', H)$, then we are done; so, we assume that it is not. Since $\phi$ and $\psi$ only differ on $x$, the only way that $\phi_1\psi$ can be an edge of $\text{Hom}(G, H)$ and not $\text{Hom}(G', H)$ is if $x$ had no loop in $G$ and $\phi(x) \sim \psi(x)$. Observe that $\phi(x)\phi(y)\psi(x)$ is walk in $H$ for any neighbour $y$ of $x$ in $G'$. If there are neighbours $y$ and $z$ of $x$ such that $\phi(y) \neq \phi(z)$, then $\phi(x)\phi(y)\psi(x)\phi(z)$ forms a 4-cycle in $H$. Therefore, $\phi$ maps all neighbours of $x$ to the same vertex of $H$. Let $y$ be any neighbour of $x$ in $G$ (which exists, as $x$ is not isolated). Let $\phi'$ be the map obtained from $\phi$ by changing the colour of $x$ to $\phi(y)$. Then we have $\phi \sim \phi' \sim \psi$ in $\text{Hom}(G', H)$. Thus, there is a recolouring between $\phi$ and $\psi$. \hfill $\square$

**2. Outline of proof**

We assume, throughout the rest of the paper, that $H$ is a triangle-free reflexive graph and that $G$ is a reflexive graph. We also assume that $G$ and $H$ are connected; otherwise, we could simply deal with each component separately. Let $\phi$ and $\psi$ be two $H$-colourings of $G$. Our goal is to present an algorithm to determine, in polynomial time, whether there is a recolouring between $\phi$ and $\psi$ or, equivalently, whether there is a $(\phi, \psi)$-reconfiguration—a walk from $\phi$ to $\psi$ in $\text{Hom}(G, H)$.

The main obstructions to the existence of a $(\phi, \psi)$-reconfiguration in $\text{Hom}(G, H)$ are related to the ways in which $\phi$ and $\psi$ map the cycles (or, more generally, closed
walks) of $G$ into $H$. Of course, for any cycle $C$ of $G$, a $(\phi, \psi)$-reconfiguration in $\text{Hom}(G, H)$ yields a $(\phi_C, \psi_C)$-reconfiguration $W_C$ in $\text{Hom}(C, H)$ between the restrictions $\phi_C$ and $\psi_C$ of $\phi$ and $\psi$ to $C$. If the length of $C$ is equal to $mk$ for some $m \geq 1$ and $k \geq 4$ and $\phi$ maps $C$ exactly $m$ times around an induced cycle of length $k$ in $H$, then, since $H$ is triangle-free, $\phi_C$ is an isolated vertex of $\text{Hom}(C, H)$; such a cycle $C$ is said to be a tight cycle of $\phi$. All homomorphisms in the same component of $\text{Hom}(G, H)$ as $\phi$ must agree on the vertices of every tight cycle (or, more generally, tight closed walk). If $\psi$ does not agree with $\phi$ on some such vertex, then we can conclude that the desired reconfiguration does not exist.

A more subtle obstruction comes from cycles in $G$ which “wind around” a given induced cycle of $H$ in a non-tight way. As a specific example, for $k \geq 4$, imagine that a cycle $C$ of length $3k + 100$ is mapped by $\phi$ three times around an induced cycle $C'$ of length $k$ in $H$ in such a way that the final 100 vertices are all mapped to the same vertex as the $(3k)$th vertex of $C$; note that this is a homomorphism of $C$ to $H$ because $H$ is reflexive. In this case, the cycle $C$ is not tight under $\phi$ and so $\phi_C$ is not isolated in $\text{Hom}(C, H)$. In particular, the “slack” introduced by the 100 extra vertices allows $\phi_C$ to be reconfigured to any homomorphism which wraps $C$ three times around $C'$ in the same direction as $\phi_C$ does. Moreover, unlike in the tight case, the image of $C$ is not constrained to stay within $C'$. However, since $H$ is triangle-free, the ways in which the image of a vertex $v \in V(C)$ can “leave” the set $V(C')$ under a walk in $\text{Hom}(C, H)$ starting with $\phi_C$ are heavily restricted. Any homomorphism in the same component of $\text{Hom}(C, H)$ as $\phi_C$ essentially wraps $C$ three times around the cycle $C'$ in the same direction as $\phi_C$ does, with the exception of a few “excursions” which leave $V(C')$ for a few steps and then return to $C'$ by doubling back along essentially the same route; see Figure 1 for an illustration and Lemma 3.5 for a more formal statement and proof of this fact.

![Figure 1](attachment:image.png)

**Figure 1.** Two homomorphisms from a long cycle $C$ to a graph $H$. The bold edges are the non-loop edges in the image of each homomorphism. The first homomorphism “wraps around” a cycle $C'$ of $H$ a bounded number of times. If $C$ is sufficiently long, then the first homomorphism can be reconfigured to the second. Triangle-freeness of $H$ makes it impossible for the image of $C$ to completely leave cycle $C'$.

We refer to the latter obstruction as a topological obstruction. To properly describe such obstructions we define an analogue, for triangle-free graphs, of homotopy theory and the fundamental group of a space. The statement that the images $\phi(C)$ and $\psi(C)$ of a cycle $C$ “wrap around the same cycles of $H$ the same number
of times" essentially translates to the closed walks that they trace out being homotopic in the fundamental group of $H$. The goal is to show that, if no frozen vertex or topological obstructions exist, then there is a $(\phi, \psi)$-reconfiguration. As it turns out, dealing with obstructions for cycles only is not enough. It is easy to come up with examples in which there are no tight cycles, and for all cycles $C$ of $G$ the closed walks $\phi(C)$ and $\psi(C)$ are homotopic, but there is no $(\phi, \psi)$-reconfiguration, even when $H$ is triangle-free; see Figure 2. Indeed, there is a stronger topological obstruction. If two cycles $C$ and $C'$ of $G$ share a vertex $r$, then a $(\phi, \psi)$-reconfiguration $W$ induces cycle reconfigurations $W_C$ and $W_{C'}$ that agree on $r$. That is; where the trace of a vertex $r$ under a reconfiguration $W = \phi_1 \sim \cdots \sim \phi_d$ in Hom$(G, H)$ is the walk $W_r = \phi_1(r) \sim \cdots \sim \phi_d(r)$, $W_C$ and $W_{C'}$ have the same trace $W_r$. For a $(\phi, \psi)$-reconfiguration to exist, we need to be able to reconfigure the homomorphisms of all cycles of $G$ simultaneously in a consistent manner.

![Figure 2](image-url)

**Figure 2.** Two homomorphisms $\phi$ and $\psi$ from a graph $G$ to a triangle-free graph $H$. The bold edges indicate the non-loop edges in the images of the homomorphisms. There are no tight closed walks and the restrictions of these homomorphisms to any given cycle of $G$ can be reconfigured, but $\phi$ cannot be reconfigured to $\psi$.

To deal with this we define, in Section 3, an analogue $\pi(H, r)$ of the fundamental group of a topological space. The definition is similar to, but simpler than, a more sophisticated definition of homotopy theory for digraphs found in [12]. We start with a graph $\Pi(H, r)$ whose vertices are closed walks of $H$ starting and ending at the basepoint $r$, and in which two are adjacent if they are adjacent in Hom$(C, H)$ (where $C$ is a cycle), or if you get one from the other by subdividing an edge and mapping the new vertex to the same vertex as one of its neighbours. The elements of $\pi(H, r)$ are the components of $\Pi(H, r)$. The group operation is concatenation: from elements $[C]$ and $[C']$ of $\phi(H, r)$, the element $[C] \cdot [C']$ is the class of the closed walk $C \cdot C'$ with basepoint $r$ that you get by traversing $C$ and then $C'$. We show that when $H$ is triangle-free, classes of $\pi(H, r)$ have a canonical reduced form that can be computed in polynomial time.

Any cycle $C$ of $H$ with a basepoint $r'$ different from $r$ can be viewed, via a 'basepoint change' by a any $(r, r')$-walk $W$, as a cycle $W \cdot C \cdot W^{-1}$ in $\pi(H, r)$, where
$W^{-1}$ is $W$ traversed in reverse. The trace $W_r$ of $r$ under a $(\phi, \psi)$-reconfiguration in $\text{Hom}(G, H)$ yields a basepoint change that induces equality $[\phi(C)] = [W_r \cdot \psi(C) \cdot W_r^{-1}]$ for all cycles $C$ in $\Pi(H, r)$ at the same time. Following Wrochna, in [21], a walk $W_r$ in $H$ that induces this common basepoint change of all cycles is called 'topologically valid'. Given a topologically valid walk, the only remaining obstructions are tight closed walks. In Section 4, following the approach refined by Wrochna, give several useful equivalent conditions for the existence of a walk that is topologically valid for $\phi$ and $\psi$. This allows us to give a polynomial time algorithm to determine if a given $(\phi(r), \psi(r))$-walk is topologically valid.

Determining whether a given $(\phi(r), \psi(r))$-walk is topologically valid, and determining whether there exists a topologically valid $(\phi(r), \psi(r))$-walk are different problems. For the later problem, we consider a basepoint-free version of homotopy theory in Section 5. Using this basepoint-free homotopy theory we give a polynomial time algorithm to determine whether or not there is a topologically valid walk $W_r$ for given $\phi$ and $\psi$. This is the main technical part of the paper.

In Section 6 we define tight closed walks, and show that given a topologically valid walk $W_r$ one can determine in polynomial time if there are any tight closed walks that obstruct a $(\phi, \psi)$-reconfiguration that uses $W_r$.

All these ideas are brought together in Section 7 to give the formal proof of Theorem 7.1. This is more descriptive version of Theorem 1.1 and yields it as an immediate corollary.

3. Discrete analogue of the fundamental group

For this section, and all following sections, $H$ is a triangle-free reflexive graph. All algorithms are about a given instance $(G, \phi, \psi)$ of $\text{Recon}(H)$, and running times are given in terms of $|V(G)|$.

3.1. The fundamental group of a reflexive graph. For any integer $\ell \geq 1$ let $P_\ell$ be the reflexive path on $\ell + 1$ vertices. We refer to elements of $\text{Hom}(P_\ell, H)$ as walks of $H$, and represent them as $(x_0 x_1 \ldots x_\ell)$. If $x_0 = a$ and $x_\ell = b$ the walk is an $(a, b)$-walk.

**Definition 3.1.** Let $\Pi(H; a, b)$ be the graph whose vertices are $(a, b)$-walks in $H$, and in which a walk $Y$ is adjacent to a walk $X = (x_0, \ldots, x_\ell)$ if either of the following are true:

- (P1) $Y = (x_0, x_1, \ldots, x_i, x_i, x_i, x_i, x_{i+1}, \ldots, x_\ell)$ for any $i$, or
- (P2) $Y = (x_0, x_1, \ldots, x_{i-1}, x_i', x_i, x_{i+1}, \ldots, x_\ell)$ for some $i \notin \{0, \ell\}$, where $x_i' \sim x_i$.

Of course, $Y$ is also adjacent to $X$ if $X$ is adjacent to $Y$ via this definition. For an $(a, b)$-walk $X$, let $[X]$ be the component of $\Pi(H; a, b)$ containing $X$.

**Remark 3.2.** As suggested in the previous section, we will build from this a version of $\text{Hom}(C, H)$ in which edges of $C$ can be subdivided. Property (P1) allows subdivision. Property (P2) mimics adjacency in $\text{Hom}(C, H)$: here we use the fact mentioned in the introduction that even in $\text{Hom}(C, H)$ one need only change one colour at a time.

The concatenation operation builds a $(a, c)$-walk $X_1 \cdot X_2$ from an $(a, b)$-walk $X_1$ and a $(b, c)$-walk $X_2$ by identifying the last vertex of $X_1$ with the first vertex of $X_2$. The reversal operation changes an $(a, b)$-walk $X_1$ to a $(b, a)$-walk $X_1^{-1}$ by reversing the order of the vertices. Though concatenation is only a partial operation on the
Lemma 3.5. \( \phi, \psi \) is a group isomorphism. We observed that if \( r \) is a basepoint, then a \((\phi, \psi)\)-path \(0_a \) (or just \(0 \) when \(a\) is understood) at \(a\), which acts as an identity with respect to concatenation. Thus restricting to closed walks with a fixed first vertex, or basepoint, \(r\), it is easy to see then that concatenation is a group operation on the components of \(\Pi(H; r, r)\). We thus define the following ‘fundamental group’ of a graph \(H\).

**Definition 3.3.** The fundamental group of a reflexive graph \(H\), with basepoint \(r\), is the set \(\pi(H; r)\) of components of \(\Pi(H; r, r)\) under the concatenation operation.

### 3.2. The fundamental group of a triangle-free graph.

The above construction and definition agrees with a more general construction in [12], where the authors go on to show that \(\pi(H; a)\) is isomorphic to the fundamental group of the clique complex of \(H\). Without formalizing this, we simply view \(\pi(H, r)\) as an analogue of the homotopy group of a space. In light of this analogy, if \([C] = [C']\) we say that \(C\) and \(C'\) are homotopic, and if \([C] = 0\) we call it contractible. Under this analogy, a triangle-free graph \(H\) corresponds to a purely one-dimensional space—a 1-manifold, so several things become simpler. In fact, the simplification begins with the observation that in property (P2) of Definition 3.1, such \(x_i'\) can exist in a triangle-free graph \(H\) only if \(x_{i-1} = x_{i+1}\) and this is either \(x_i\) or \(x_i'\). It is not hard to see then that the following version of Definition 3.1 yields an \(\Pi(H; a, b)\) with the same components, and so yields the same \(\pi(H; r)\).

**Definition 3.4.** Let \(\Pi(H; a, b)\) be the graph whose vertices are \((a, b)\)-walks in \(H\), and in which a walk \(Y\) is adjacent to a walk \(X = (x_0, \ldots, x_\ell)\) if either of the following are true:

1. \((P1)\) \(Y = (x_0, x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_\ell)\) for any \(i\), or
2. \((P2')\) \(Y = (x_0, x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_\ell)\) for some \(i \not\in \{0, \ell\}\), where \(x_{i-1} = x_i' = x_{i+1}\).

For an \((a, b)\)-walk \(X\), let \([X]\) be the component of \(\Pi(H; a, b)\) containing \(X\).

A closed walk with a basepoint other then \(r\) can be viewed as a closed walk with basepoint \(r\) via what is known as a basepoint change. For an \((r, b)\)-walk \(X\) and a closed walk \(C\) in based at \(b\) define the closed walk \(\beta_X(C) := X \cdot C \cdot X^{-1}\). It is standard, and easily shown, that the map \(\pi(H; b) \to \pi(H; r) : [C] \mapsto [\beta_X(C)]\) is a group isomorphism. We observed that if \(C\) is a closed walk with basepoint \(r\), then a \((\phi, \psi)\)-path \(W\) in \(\text{Hom}(C, H)\) induces a \((\phi(r), \psi(r))\)-walk \(W_r\) in \(\text{Hom}(r, H)\). In fact, it induces a homotopy between \(\phi(C)\) and the cycle \(\psi(C)\) with basepoint changed by the trace \(W_r\) of \(r\). This gives us our basic topological obstruction to a \((\phi, \psi)\)-reconfiguration. The following would hold for non triangle-free \(H\) as well, but the proof would be a bit longer.

**Lemma 3.5.** For triangle-free \(H\), if there is a \((\phi, \psi)\)-path \(W\) in \(\text{Hom}(C, H)\) for some cycle \(C\) with basepoint \(r\), then \([\psi(C)] = [\beta_{W_r}(\phi(C))]\) in \(\pi(H; \phi(r))\).

**Proof.** Recall that if there is a path from \(\phi\) to \(\psi\) in \(\text{Hom}(C, H)\), then there is a path in which any two consecutive elements differ on one vertex. By induction on the length of this path, it suffices to consider the case that \(\phi\) and \(\psi\) differ on only one vertex.
If they differ on any vertex but their basepoint, then there is nothing to prove, as then \( \psi(C) = \phi(C) \) by (P2'), and so we are done with empty \( W_r \). So we may assume that they differ on the first vertex. That is, we have

\[
\psi(C) = (x_0, x_1, \ldots, x_\ell, x_0) \quad \text{and} \quad \phi(C) = (x_0', x_1, \ldots, x_\ell, x_0').
\]

In this case take \( W_r = (x_0, x_0') \), we will show that

\[
[\psi(C)] = [(x_0, x_1, \ldots, x_\ell, x_0)] = [(x_0, x_0', x_1, \ldots, x_\ell, x_0)] = [\beta_{W_r}(\phi(C))].
\]

Well, \( x_0, x_0' \) are adjacent (as \( C \) is reflexive) and are both adjacent to \( x_1 \) and \( x_{\ell-1} \). So, as \( H \) is triangle-free, we have that \( x_0' = x_1 \) or \( x_0 = x_1 \). In the first case we have

\[
[(x_0, x_1, \ldots, x_\ell, x_0)] = [(x_0, x_0'x_1, \ldots, x_\ell, x_0)]
\]

by (P1) and in the second case we have it by (P2'). Similarly \( x_0' = x_\ell \) or \( x_0 = x_\ell \), and we get the second equality that we need to finish the lemma by transitivity. \( \square \)

### 3.3. Computation of reduced cycles

We say that a walk \( X \) in \( \Pi(H; r) \) is reduced if it is a shortest walk in \( [X] \).

**Lemma 3.6.** Let \( H \) be a triangle-free graph. Any class \([X]\) in \( \pi(H; r) \) has a unique reduced walk, and it can be found in linear time.

**Proof.** We prove the result more generally for walks in \( \Pi(H; a, b) \) using the topological notion of a covering space—the idea is to define a graph \( U \) for which the result is trivial, and to show that walks in \( H \) ‘lift’ uniquely to walks in \( U \) in such a way that components of \( \Pi \) are preserved.

For a graph \( H \) and a vertex \( r \), the universal cover \( U \) is the infinite graph whose vertices are walks in \( H \) starting at \( r \), and having no occurrences of the same vertex at distance 1 or 2. Two walks \( X \) and \( Y \) are adjacent in \( U \) if we can get one from the other by dropping the last vertex. See Figure 3 for an example.

![Figure 3. The Petersen graph and a finite portion of its universal cover.](image)

Because \( U \) is a tree, it is easy to see that \( \Pi(U; (r), (r, x_1, x_2, \ldots, x_\ell)) \) is connected for any vertex \((r, x_1, x_2, \ldots, x_\ell)\) and that its unique shortest walk is

\[
(r)(r, x_1)(r, x_1, x_2) \ldots (r, x_1, x_2, \ldots, x_\ell).
\]
Now, the map
\[ \phi : U \rightarrow H : (r, x_1, \ldots, x_\ell) \mapsto x_\ell \]
is a locally bijective homomorphism; i.e., induces a bijection between closed
neighbourhoods \( \{x\} \cup N(x) \) of vertices in \( U \) and vertices in \( H \). It follows that the map
\[ \Phi : (r, x_1, \ldots, x_\ell) \mapsto (\phi(r), \phi(r, x_1), \ldots, \phi(r, x_1, \ldots, x_\ell)) \]
is a bijective homomorphism between the set \( \Pi(U; r) \) of walks in \( U \) starting at \( r \) and the set \( \Pi(H; r) \) of walks in \( H \) starting at \( r \). Indeed the lift \( \Phi^{-1}(X) \) of a walk \( X = (r x_1 \ldots x_n) \) of \( H \) from \( \Pi(H; r) \) to \( \Pi(U; r) \) can be given explicitly: its first vertex is \( \Phi^{-1}(X)_0 := (r) \) and its \( i \)th vertex is
\[ \Phi^{-1}(X)_i := \Phi^{-1}(X)_{i-1} \cdot \phi^{-1}_{x_{i-1}}(x_i) \]
where \( \phi^{-1}_x \) is the bijective restriction of \( \phi^{-1} \) to the closed neighbourhood of \( x \).

Observe that for \( x_{i-1} \sim x_i \), we have \( \phi^{-1}_{x_{i-1}}(x_i) = \phi^{-1}_{x_{i-1}}(x_i) \).

Now, not all \((r, b)\)-walks of \( H \) must lift to walks of \( U \) with the same endpoint, but
those in a component of \( \Pi(H; r, b) \) should; in fact, we show now that for triangle-free
\( H \), a component of \( \Pi(H; r, b) \) lifts isomorphically to \( \Pi(U; r, B) \) for some walk \( B \),
by showing \( \Phi^{-1} \) is a homomorphism. Indeed let \( X \) and \( X' \) be adjacent in \( \Pi(H; r, b) \).
If \( X \) and \( X' \) are related by \((P1)\), then we may assume \( X = (r, v_1, \ldots, v_n) \) while
\( X' = (r, v_1, \ldots, v_{n-1}, v_i, v_{i+1}, \ldots, v_n) \). These lift to \( W \cdot U \) and \( W \cdot \phi^{-1}_{v_i}(v_i) \cdot U \)
respectively where \( W = \Phi^{-1}(r, v_1, \ldots, v_i) \) as \( \phi^{-1}_{v_i}(v_i) \) is equal to the last vertex
\( \phi^{-1}_{v_{n-1}}(v_i) \) of \( W \). But as these vertices are equal \( W \cdot U \) and \( W \cdot \phi^{-1}_{v_i}(v_i) \cdot U \) are
adjacent in \( \Pi(H; r, B) \) by \((P1)\). Similar arguments about \((P2)'\) complete the proof
that adjacent walks in \( H \) lift to adjacent walks in \( U \). Thus \( \Phi^{-1} \) maps a component of \( \Pi(H; r, b) \) isomorphically to \( \Pi(U; r, B) \) for some \( B \). Clearly \( \Phi \) takes
the shortest walk in \( \Pi(U; r, B) \) to a unique shortest walk in \( \Pi(H; r, b) \), so \( \Pi(H; r, b) \)
has a unique shortest walk.

One can lift to \( U \) to easily find the unique shortest walk, but the fact that it is
unique means that one can simply apply \((P1)\) or \((P2)'\) to greedily shorten a walk,
and so find its reduced form in linear time.

\[ \square \]

3.4. Computational Tools. We finish this section by observing some simple
computational tools that are familiar from homotopy theory which we will use several
times. It is easy to show, for \((a, b)\)-walks \( X_1 \) and \( X_2 \), that
\[ |X_1| = |X_2| \iff |X_1 \cdot X_2^{-1}| = 0. \tag{1} \]

Taking \( I \) as some initial segment of a closed walk \( C \), it is not hard to see that
the cyclic shift \( \sigma_I(C) \) of \( C \) by \(|I|\) vertices is in \([\beta_I^{-1}(C)]\), and so
\[ |C| = |0| \iff |\sigma_I(C)| = |0|. \tag{2} \]
Thus \(|C| \) is contractible if and only if all cyclic shifts of it are contractible. Breaking
a cycle up as the concatenation of walks, this yield identities such as the following:
\[ |X_1 \cdot X_2 \cdot X_3 \cdot X_4| = |0| \iff |X_2 \cdot X_3 \cdot X_4 \cdot X_1| = |0| \tag{3} \]
or using \((1)\):
\[ |X_i^{-1}| = |X_2 \cdot X_3 \cdot X_4| \iff |X_2^{-1}| = |X_4 \cdot X_3 \cdot X_1|. \tag{4} \]

There is one more simple observation which allows us to apply these notions
succinctly.
Figure 4. The cycle $C = C_1 \cdot X C_2$ where $C_1 = U_1 \cdot X \cdot V_1$ and $C_2 = U_2 \cdot X^{-1} \cdot V_2$.

**Definition 3.7.** When $C_1 = U_1 \cdot X \cdot V_1$ and $C_2 = U_2 \cdot X^{-1} \cdot V_2$ are closed walks as in Figure 4, let $C_1 \cdot X C_2 = U_1 \cdot V_2 \cdot U_2 \cdot V_1$.

With this definition, we have the following.

**Lemma 3.8.** Let $C = C_1 \cdot X C_2$. If two of $[C_1], [C_2]$ and $[C]$ are contractible, then they all are.

**Proof.** Taking shifts $C'_1 = V_1 \cdot U_1 \cdot X$ of $C_1$ and $C'_2 = X^{-1} \cdot V_2 \cdot U_1$ we have that $[C'_1 \cdot C'_2] = [C']$ where $C'$ is the shift $V_1 \cdot U_1 \cdot V_2 \cdot U_1$ of $C = C_1 \cdot X C_2$. It is easy to see therefore that two of $[C'_1], [C'_2]$ and $[C']$ are contractible if and only if the third of them is. The result follows by (2). \[\square\]

4. **Topologically valid system of walks**

From Lemma 3.5 we see a reconfiguration of $\phi$ to $\psi$ in $\text{Hom}(G, H)$ induces for each cycle $C$ of $G$ fixed-basepoint homotopy from $\phi(C)$ to $\beta_{W_r}(\psi(C))$ with a common basepoint change $W_r$. Not only is such a walk $W_r$ necessary for a reconfiguration it is almost sufficient. If it exists, the only obstructions a reconfiguration will be non-topological (tight closed walks). So recognizing the existence of such a basepoint change is key. This task is simplified by the fact that choice of the basepoint $r$ is unimportant. We address this in this section.

**Definition 4.1.** A system of walks for $\phi$ and $\psi$ is a vector $(W_v) := (W_v)_{v \in V(G)}$ of walks in $H$ such that $W_v$ is a $(\phi(v), \psi(v))$-walk. A walk $W_v$ is topologically valid if for every closed walk $C$ of $G$ with basepoint $v$ we have 

$$[\phi(C)] = [\beta_{W_v}(\psi(C))].$$

The system of walks is topologically valid if all walks in it are topologically valid.

As suggested above, we get a topologically valid sytem of walks from a reconfiguration.

**Fact 4.2.** Let $W$ be a reconfiguration from $\phi$ to $\psi$ in $\text{Hom}(G, H)$. Where for every vertex $v$ of $G$, $W_v$ is the trace of $v$, the system $(W_v)$ of walks is topologically valid for $\phi$ and $\psi$.

**Proof.** In Section 2 we observed that $W$ induces a $(\phi(C), \psi(C))$-walk in $\text{Hom}(C, H)$ for any closed walk $C$ of $G$. In Lemma 3.5 we observed that where $v$ is the basepoint
of $C$ this implies $[\psi(C)] = [\beta_{W_r}(\phi(C))]$ in $\pi(H; \phi(v))$. Thus any reconfiguration $W$ from $\phi$ to $\psi$ yields a system $(W_v)$ of walks that is topologically valid for $\phi$ and $\psi$. \hfill \Box

Generalizing the condition for closed walks, a $uv$-walk $U$ in $G$ is (topologically) preserved by a system $(W_v)$ of walks for $\phi$ and $\psi$ if

$$[\phi(U)] = [W_u \cdot \psi(U) \cdot W_{v}^{-1}].$$

(6)

By (1) and (2) we can write in the following form:

$$[\phi(U) \cdot W_v \cdot \psi(U)^{-1} \cdot W_{v}^{-1}] = 0 \quad (6')$$

Now, for a $uv$-walk $U$ and a $uv$-walk $V$ we have

$$\phi(U \cdot V) \cdot W_u \cdot \psi(U \cdot V)^{-1} \cdot W_{v}^{-1} = (\phi(U) \cdot W_u \cdot \psi(U)^{-1} \cdot W_{u}^{-1}) \cdot W_v \cdot (\phi(V) \cdot W_v \cdot \psi(V)^{-1} \cdot W_{v}^{-1}),$$

so by Lemma 3.8 and form $(6')$, we get that if two of $U, V$ and $U \cdot V$ are preserved by $(W_v)$, then so is the third of them. This yields the following alternate definition.

**Fact 4.3.** A system of walks $(W_v)$ is topologically valid for $\phi$ and $\psi$ if and only if every edge in $G$ is topologically preserved by $(W_v)$ if and only if every walk in $G$ is topologically preserved by $(W_v)$.

The topologically valid system of walks $(W_v)$ we get from a reconfiguration $W$ is not generally unique, indeed each walk in general may have many repeated vertices, but it is unique up to homotopy classes.

**Lemma 4.4.** Let $W_r$ be a reduced topologically valid $(\phi(r), \psi(r))$-walk in $H$. There is a unique topologically valid system $(W_v)$ of reduced walks for $\phi$ and $\psi$ that contains $W_r$.

**Proof.** First we show that there is a topologically valid system $(W_v)$ of walks containing $W_r$. Choose a spanning tree $T$ of $G$ with root $r$. Inductively build $(W_v)$ by setting $W_u$ to be the reduction of $\phi(u) \cdot W_u \cdot \psi(uv)$, where the $(\phi(u), \psi(u))$-walk $W_u$ is already defined for the precursor $u$ of $v$ in $T$.

By construction any edge in $T$, so any path in $T$ is preserved by $(W_v)$. Thus the system $(W_v)$ is topologically valid if and only if every edge $e$ in $G \setminus T$ is also preserved by $(W_v)$.

If there is some edge $e = uv$ that is not preserved by $(W_v)$. Letting $T_v$ denote the unique path in $T$ from $r$ to $v$, we clearly have then that the closed walk $C = T_u \cdot e \cdot T_v^{-1}$ is not preserved by $(W_v)$, so $[\phi(C)] \neq [\beta_{W_r}(\psi(C))]$, contradicting the fact that $W_r$ was topologically valid. \hfill \Box

In light of this fact, we make the following definition.

**Definition 4.5.** A topologically valid $(\phi(r), \psi(r))$-walk $W_r$ contained in a topologically valid system $(W_v) = (W_v)_{v \in V(G)}$ of walks, generates the system $(W_v)$.

Actually our proof of Lemma 4.4 did more. Defining $(W_v)$ as we did in the proof can clearly be done in polynomial time. Once we have done so, checking that at most quadratic number of edges not in $T$ are preserved can be done in polynomial time. Thus we showed the following.
Corollary 4.6. Let $W_r$ be reduced walk from $\phi(r)$ to $\psi(r)$. We can determine in polynomial time if $W_r$ is topologically valid. If it is, we produce the system $(W_r)$ it generates, if not, we provide a closed walk $C$ of $G$ such that $[\phi(C)] \neq [\beta_{W_r}(\psi(C))]$.

5. Basepoint-free homotopy

Fact 4.2 tells us that there is a $(\phi, \psi)$-reconfiguration in $\text{Hom}(G, H)$ only if there is a topologically valid walk $W_r$ between the basepoints $\phi(r)$ and $\psi(r)$, and Corollary 4.6 tells us how to decide if a given $W_r$ is topologically valid. Our main result in this section, Lemma 5.7, tells us how to decide if there is any walk $W_r$ that is topologically valid for $\phi$ and $\psi$. If a randomly chose $(\phi(r), \psi(r))$ walk $W_r$ is not valid, then Corollary 4.6 gives us a non-contractible closed walk $C$ such that $[\phi(C)] \neq [\beta_{W_r}(\psi(C))]$. This $C$ is our starting point in this section, as it greatly limits the possibilities for a valid $W_r$. To characterise the possibilities, we use a version of $\Pi(H, r)$ in which we let the basepoint move freely.

Definition 5.1. Let $\Pi(H)$ be the graph we get from the disjoint union, over all $r \in V(H)$, of $\Pi(H; r)$ by adding an edge between walks $y$ and $x = (x_0, x_1, \ldots, x_\ell, x_0)$ if

(P3) $y = (x_1, \ldots, x_\ell)$ and $x_1 = x_{\ell-1}$.

Let $\pi(H)$ be the set of components of $\Pi(H)$, and let $[C]$ be the component of $\Pi(H)$ containing $C$. Clearly we have that

$$[C] = [C'] \text{ implies } [C'] = [C']. \quad (7)$$

The converse is not true in general, indeed not even heuristically, as $C$ and $C'$ may satisfy $[C] = [C']$ and have different basepoints. However, if we have a walk $W$ from $C$ to $C'$ in $\Pi(H)$, then it is not hard to see that $[C] = [\beta_{W_r}(C')]$ where $W_r$ is the trace of the basepoint $r$ of $C$.

If $C$ is a closed walk with basepoint $r$, then it is trivial to see that $[C] = [\beta_{W_r}(C)]$ for any walk $W$ ending at $r$, and so from Lemma 3.5 we get that if there is a $(\phi, \psi)$-path in $\text{Hom}(C, H)$ then $[\phi(C)] = [\psi(C)]$. A closed walk $C$ is free-reduced if it is a shortest closed walk in $[C]$. Though any class in $\pi(H; r)$ contains a unique reduced closed walk, this does not hold for free-reduced closed walks in classes of $\pi(H)$. Indeed, if $C$ is free-reduced, any cyclic shift of it is also free reduced. However, any non-contractible reduced closed walk $C$, ‘contains’ a unique free-reduced closed walk.

Fact 5.2. If $C$ is a non-contractible reduced closed walk with basepoint $r$, then $C$ decomposes uniquely as $C = \beta_T(C_f)$ for some free reduced closed walk $C_f$, and some reduced walk $T$ from $r$ to the first vertex of $C_f$.

Proof. Indeed, as $C = (x_0, x_1, \ldots, x_\ell, x_0)$ is reduced with respect to operations (P1) and (P2) the only possible reduction is operation (P3), meaning $x_{\ell-1} = x_1$ and it reduces to $C_1 = (x_1, x_2, \ldots, x_{\ell-1})$. Where $T_1 = (x_0, x_1)$ we have $C_1 = \beta_{T_1}(C_1)$. By induction we get the fact. □

As the reduced form of $C$ is unique and can be found in linear time, so can its decomposition into $\beta_T(C_f)$. We call this its free decomposition and call $T$ its tail.

This free decomposition does not change by much under basepoint change. To show this we start with a somewhat technical calculation. The intuition for this lemma is perhaps aided by first reading the corollaries that follow it.
A walk $I$ is \textit{initial} in a walk $X$ if $X = I \cdot J$ for some $J$. A walk is \textit{terminal} in $X$ if it is initial in $X^{-1}$.

\textbf{Lemma 5.3.} Let $A$ and $B$ be non-contractible reduced closed walks with respective free decompositions $\beta_T(A_f)$ and $\beta_S(B_f)$. If $[A] = [\beta_W(B)]$ for a walk $W$ then there is an integer $d$ and initial $I$ of $A_f^d$ such that $B_f = \sigma_f(A_f)$ is a cyclic shift of $A_f$, and $[W] = [T \cdot I \cdot S^{-1}]$. This is reduced unless $T$ and $S$ end in a common walk.

\textbf{Proof.} As $[A] = [\beta_W(B)]$ we have $[A^d] = [\beta_W(B^d)]$ so $[A^dW] = [WB^d]$ for any integer $d$. While $A^dW$ and $WB^d$ are not generally reduced, $A^d$ and $B^d$ reduce to $\beta_T(A_f^d)$ and $\beta_S(B_f^d)$, and so by taking $d$ large enough, we can say $A^dW = WB^d$ reduces to

$$I_A \cdot T_W = I_W \cdot T_B \quad (*)$$

for some initial $I_A$ of $\beta_T(A_f^d)$, some terminal $T_W$ of $W$, some initial $I_W$ of $W$ and some terminal $T_B$ of $\beta_S(B_f^d)$. As $(*)$ holds for all large enough $d$, and $I_A$ and $T_B$ grow with $d$, we have that $[A_f] = [B_f]$, and that by taking $d$ large enough, $I_A$ and $T_B$ can be assured to have an arbitrarily long overlap, so $B_f$ is some shift of $A_f$, as needed. We now just have to verify the decomposition of $W$.

Again from $(*)$ for large enough $d$ we get that the end $T_W$ of $W$ is terminal in $T_B$ so in $A_f^d \cdot S^{-1}$, and as $\beta_T(A_f^d) \cdot W$ reduced to $I_A \cdot T_W$, we get that $W = W' \cdot T_W$ for some $W'$ which cancels with the end of $A_f^d \cdot T^{-1}$. Thus $W'$ is initial in $T \cdot A_f^{-d}$.

Now, $W'$ ends where $T_W$ starts. If $T$ and $S$ end in a common walk, then it is possible that $W' \cdot T_W$ is the reduction of $T \cdot S$. Otherwise $W'$ contains $T$ and $T_W$ contains $S$. In this case $W'$ begins with $T$ and then contains some initial walk of $A_f^{-d}$. So $W' \cdot T_W$ is $T \cdot I \cdot S^{-1}$ for some initial $I$ of $A_f^d$ or $A_f^{-d}$, as needed. That $B_f = \beta_I(A_f)$ now follows from the fact that $B_f$ is a shift of $A$, and that $I$ is an initial walk in $A_f$ or its inverse that has the right endpoints. \hfill $\Box$

From this we get a couple of immediate corollaries.

\textbf{Corollary 5.4.} If $A_f$ and $B_f$ are free reduced non-contractible closed walks with $[A_f] = [B_f]$ then $B_f$ is a cyclic shift of $A_f$.

\textbf{Proof.} As $[A_f] = [B_f]$ we have $B_f \in [\beta_W(A_f)]$ where $W_r$ is the trace of the basepoint $r$ under the walk $W$ from $A_f$ to $B_f$ in $\Pi(H)$. By the lemma $B_f$ is a shift of $A_f$. \hfill $\Box$

For a reduced closed walk $C$, let $\sqrt{C}$ be the shortest initial walk of $C$ such that $\sqrt{C^d} = C$ for some positive $d$.

\textbf{Corollary 5.5.} If $A$ and $B$ are non-contractible reduced closed walks with free decompositions $A = \beta_T(A_f)$ and $B = \beta_S(B_f)$, then $[A] = [\beta_W(B)]$ if and only if $W$ is of the form

$$W_d = \sqrt{A_f^d} \cdot T \cdot I \cdot S^{-1}$$

for some integer $d$, where $I$ is an initial walk of $A$.

\textbf{Proof.} The ‘if’ part is trivial as $[W_d] = T \cdot \sqrt{A_f^d} \cdot I \cdot S^{-1}$ and this is the trace of the path from $A$ to $B$ that we get by composing the reduction of $A$ to $A_f$ with the cyclic shift of $A_f$ by $A_f^d \cdot I$ to $B_f$, and then composing this with the inverse of the reduction of $B$ to this shift of $B_f$. The ‘only if’ is from Lemma 5.3 the lemma. \hfill $\Box$
Corollary 5.6. Let $A$ and $B$ be reduced closed walks in $G$ with basepoint $v$ having respective free decompositions $\beta_T(A)$ and $\beta_S(B)$, and such that $\sqrt{A} \neq \pm \sqrt{B}$. If there exists a walk $W$ such that $[\phi(A)] = [\beta_W, \psi(A)]$ and $[\phi(B)] = [\beta_W, \psi(B)]$, then there is a unique reduced such walk and it is the reduction of either $T \cdot I \cdot S^{-1}$ for some initial $I$ of $A_f$ or $T' \cdot I' \cdot S'^{-1}$ for some initial $I'$ of $B_f$.

Proof. Let $A$ and $B$ and $W$ be as in the premise of the corollary. By Corollary 5.5 we have then

$$[\sqrt{A}^d \cdot T \cdot I \cdot S^{-1}] = [\sqrt{B}^{d'} \cdot T' \cdot I' \cdot S'^{-1}]$$

for some $d$ and $d'$ where $I$ and $I'$ (and all other components) are fixed. Reducing these, we get

$$[T \cdot \sqrt{A}^d \cdot I \cdot S^{-1}] = [T' \cdot \sqrt{B}^{d'} \cdot I' \cdot S'^{-1}]$$

where $A_f$ is the free reduction of $A$ and $B_f$ is the free reduction of $B$. As $\sqrt{A} \neq \sqrt{B}$ we have either $T \neq T'$ or $\sqrt{A_f} \neq \sqrt{B_f}$. Either way, the only possible solution has $d = 0$ or $d' = 0$. These cannot yield different solutions by considerations of length. $\square$

We are now ready to prove the main result of the section.

Lemma 5.7. Let $r$ be a vertex of $G$. We can determine in polynomial time if there is a $(\phi(r), \psi(r))$-walk $W_r$ that is topologically valid for $\phi, \psi$ in Hom$(G, H)$. If there is, one with length at most $2n$ is provided.

Proof. Let $W_r$ be any shortest $(\phi(r), \psi(r))$-walk in $H$. By Corollary 4.6 we can determine in polynomial time if $W_r$ is topologically valid. If it is, we are done, so we may assume that it is not, and the algorithm returns a closed walk $C$ of $G$ such that $[\phi(C)] \neq [\beta_W, \psi(C)]$, so necessarily $\phi(C)$ is non-contractible.

By Lemma 3.6, (and Fact 5.2) we can find the free decompositions $\phi(C) = \beta_T(C_f)$ and $\psi(C) = \beta_S(C_f)$ in polynomial time. We can check in polynomial time if $C_f'$ is a shift of $C_f$. If it is not, then $[\phi(C)] \neq [\psi(C)]$ by Corollary 5.4, and so by (7), we have

$$[\phi(C)] \neq [\beta_W, \psi(C)]$$

for any $W$. Thus there can be no topologically valid $W_r$, and we are done.

We may assume, therefore, that $C_f'$ is the shift $\sigma_f(C_f)$ for some initial walk $I$ of $C_f$. By Corollary 5.5 we thus have that the only reduced walks $W$ that preserve $C$ are reductions of

$$W_d := \sqrt{\phi(C)}^d \cdot T \cdot I \cdot S^{-1}$$

for integers $d$.

By Corollary 4.6 we can decide if $W_0$ is topologically valid. If it is we return it and are done, so we may assume that it is not, and Corollary 4.6 yields another closed walk $C'$ with $[\phi(C')] \neq [\beta_W, \psi(C')]$. By Corollary 5.5 we get that $\sqrt{C'} \neq \pm \sqrt{C}$ and so by Corollary 5.6 the only walks $W$ that can preserve both $C$ and $C'$ are the reductions of $T \cdot I \cdot S^{-1}$ and $T' \cdot I' \cdot S'^{-1}$ where $\beta_T(C_f)$ is the reduction of $\phi(C')$ and $\beta_S(\sigma_f(C_f))$ is the reduction of $\psi(C')$. Both of these have length at most $2n$ so we can check them for topological validity in polynomial time by Corollary 4.6, and return the shortest one that is valid if any are. $\square$
6. Tight closed walks

Though having a topologically valid $(\phi(r), \psi(r))$-walk in $H$ guarantees there are no topological obstructions to a $(\phi, \psi)$-reconfiguration, there may still be non-topological obstructions.

**Definition 6.1.** A system of walks $(W_v)$ is realizable if (by possibly adding repeated vertices into walks) we get a system of walks that is induced by a reconfiguration.

**Definition 6.2.** A closed walk $C$ of $G$ is tight under $\phi$ if $\phi(C)$ is free reduced in $\pi(H)$. Any vertex of a tight closed walk is fixed under $\phi$.

It is not hard to see that if $C$ is tight under $\phi$, then any neighbour of $\phi$ in $\text{Hom}(G, H)$ must agree with $\phi$ on $C$. Thus a fixed vertex is one that cannot change under a reconfiguration. A walk is constant if all vertices are the same.

**Fact 6.3.** If the system of walks $(W_v)$ is realizable for $\phi$ and $\psi$ in $\text{Hom}(G, H)$ and $c$ is fixed under $\phi$, then $W_v$ is constant.

**Lemma 6.4.** For a given system $(W_v)$ of paths that is topologically valid for $\phi, \psi \in \text{Hom}(G, H)$, we can decide in time linear in the sum of the lengths of the paths if $(W_v)$ is realizable. If it is not realizable, we give a closed walk $C$ on which $\phi$ is tight.

**Proof.** Let homomorphisms $\phi, \psi : G \to H$ and a system of walks $(W_v)_{v \in G}$ that is topologically valid for $\phi$ and $\psi$ be given. Starting with $\phi$ we attempt to construct a reconfiguration $W$ from $\phi$ to $\psi$, by changing one vertex $v$ at a time from its current position $W_v$ to its next position. If we succeed, this witnesses the realizability of $(W_v)$. As any edge $e = uv$ of $G$ is topologically valid under the system of walks,

$$[W_u] = [\phi(e) \cdot W_v \cdot \phi(e)^{-1}],$$

and as by Lemma 3.6 there is a unique reduced walk in $[W_u]$, we have, where a superscript of $i$ on the walk designates that $i^{th}$ vertex of the walk, either $W_u^i = W_v^{i-1}$ for all $i$ except maybe $i = |V(W_u)|$, or $W_v^i = W_u^{i-1}$ for all $i$ except maybe $i = |V(W_v)|$.

Construct an auxiliary digraph $A$ on $V(G)$ by setting $u \to v$ if

$$W_v^1 = W_u^0 \text{ and } W_v^0 \sim W_u^1.$$  \hspace{1cm} (8)

(Interpret this as 'u wants to move to $W_v^0$ but v has to move first'.) Observe that if $uv$ is an edge, and $|W_u| = 0$, then $|W_v| \leq 1$, and $W_v^1 \sim W_u^0$, so $u$ has no arcs in $A$.

If $A$ has any sink $v$, then on moving $v$ to $W_v^1$, we still have a homomorphism. Append this homomorphism to $W$, and relabelling $W_v$ by removing its first vertex. As each relabelling reduces the length of a walk in $(W_v)$ by one, we either complete in time linear in the sum of the lengths of the paths, (and we are done) or at some step $A$ has no sinks, so contains a directed cycle.

Assume that $A$ has a directed cycle $C = a_1 \to a_2 \to \cdots \to a_t$. By construction, every arc $a_i \to a_{i+1}$ of $C$ is an edge $a_i \sim a_{i+1}$ of $G$, and (8) ensures that $\phi(a_i) \notin \{\phi(a_{i-1}), \phi(a_{i-2})\}$, (indices modulo $t$) for all $i$. Thus is $C$ is a tight closed walk under $\phi$. 

$\square$
7. Full statement and proof of the main theorem

We are now ready to state and prove the main technical theorem, from which Theorem 1.1 follows immediately.

**Theorem 7.1.** Let $H$ be a reflexive symmetric triangle-free graph, and $(G, \phi, \psi)$ be an instance of $\text{Recol}(H)$. There is a reconfiguration from $\phi$ to $\psi$ if and only if there is a system of walks $(W_v)_{v \in V(G)}$ that is topologically valid for $\phi$ and $\psi$, such that $W_v$ is a constant walk for any vertex $c$ in closed walk $C$ that is tight under $\phi$.

Moreover, these validity of this conditions can be determined by an algorithm that runs in time that is polynomial in $|V(G)|$.

**Proof.** The ‘only if’ part of the first statement of the theorem is proved in Facts 4.2 and 6.3. The ‘sufficiency’ is proved in Lemma 6.4. The validity algorithm comes from Lemma 5.7 and Lemma 6.4.

Indeed, choosing a vertex $r$ of $G$, by Lemma 5.7 we determine if there is a topologically valid $(\phi(r), \psi(r))$-path. If there is not, we are done, and if there is, we get one with $W_r$ of length at most $2n$. In fact we get the system $(W_v)$ of walks that it generates, and no path in this system can have length greater than $4n$ so the sum of the lengths of these paths are polynomial in $n$. By Lemma 6.4 we determine if this system of walks is realizable in polynomial time. If it is we are done, and if it is not, then the lemma returns a tight closed walk. Choosing new $r$ in $C$, any realizable system of paths is constant on $r$. If $\phi(r) \neq \psi(r)$ there is no such system, and we are done. If $\phi(r) = \psi(r)$, set $W_r$ to be constant, and apply Corollary 4.6 to determine if it is topologically valid. If it is apply Lemma 6.4 we determine if it is realizable.

□

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