The applications of the general and reduced Yangian algebras

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The applications of the general and reduced Yangian $Y(sl(2))$ and $Y(su(3))$ algebras are discussed. By taking a special constraint, the representation of $Y(sl(2))$ and $Y(su(3))$ can be divided into two $2 \times 2$ and three $3 \times 3$ blocks diagonal respectively. The general and reduced Yangian $Y(sl(2))$ and $Y(su(3))$ are applied to the bi-qubit system and the mixed light pseudoscalar meson state, respectively. We can find that the general ones are not able to make the initial states disentangled by acting on the initial states, however the reduced ones are able to make the initial state disentangled. In addition, we show the effects of $Y(su(3))$ generators on the the decay channel.

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I. INTRODUCTION

Quantum entanglement, a nonlocal correlation, is of crucial importance in quantum computation [1], quantum teleportation [2], dense coding [3] and quantum key distribution [4]. However, in a real system, the deterioration of the coherence or even the decoherence and disentanglement due to the interaction with its surroundings, which is recognized as a main obstacle to realize quantum computing [5] and quantum information processing (QIP) [6], have to be taken into account in the researches in the field of quantum information. Earlier studies had indicated that entanglement decays exponentially [6,7] until T. Yu suggested that entanglement decays completely in finite time and called for concerted effort to research entanglement sudden death [10]. For example, different systems [11,13] and realizations in experiment [14,15] provide theoretical guidance to practical application of controlling entanglement.

In the last decades, Yangian algebras associated with simple Lie algebras has been studied systematically in both mathematics and physics, and have many applications through spin operators and quantum fields [16–18]. There have been some remarkable successes in studying the long-ranged interaction models by Yangian approaches [14,24] in which the Haldane-Shastry model was regarded as the representative of the spin chain $su(n)$ with long-range interaction [21]. Recently, Yangian $Y(sl(2))$ and $Y(su(3))$ have been studied for quantum entanglement [22,26]. In addition, Yangian $Y(su(3))$ algebra has been demonstrated to be able to realize the hadronic decay channels of light pseudoscalar mesons and predict a possible explanation of the unknown particle $X$ in the decay channel $K^0_L \rightarrow \pi^0\pi^0X$ [27].

In this work, we will study the effects of the generators of Yangian $Y(sl(2))$ on the entanglement degrees of two-qubit system and the mixed light pseudoscalar meson states are discussed in the cases of the general and reduced Yangian algebras respectively.

II. THE APPLICATIONS OF YANGIAN $Y(sl(2))$ ALGEBRA IN THE BI-QUBIT SYSTEM

To compare the effects of the general and the reduced $Y(sl(2))$ on the Bi-qubit system, we will firstly reduce the general $Y(sl(2))$ to the reduced one, then show the effects of them on the entanglement of the Bi-qubit system.

The Yangian $Y(sl(2))$ is generated by the genera-
tors \{I_a, J_a\} with the commutation relation \[29\]:

\[ [I_a, I_b] = i\epsilon_{\alpha\beta\gamma} I_{\gamma}, \quad [I_a, J_{\beta}] = i\epsilon_{\alpha\beta\gamma} J_{\gamma}, \quad (\alpha, \beta, \gamma = 1, 2, 3) \]

where the \{I_a\} form a simple Lie algebra \(\mathfrak{sl}(2)\) characterized by \(\epsilon_{\alpha\beta\gamma}\) and

\[ [J_{\pm}, [J_3, J_{\pm}]] = \frac{h^2}{4} I_{\pm}(J_{\pm} I_3 - I_3 J_{\pm}), \]

where \(h\) is the deformation parameter and the notations \(I_\pm = I_1 \pm i I_2\) and \(J_\pm = J_1 \pm J_2\).

Now let us consider a bi-spin system, the realization of generators of \(Y(\mathfrak{sl}(2))\) take the form of \[29\]

\[ I = S = S_1 + S_2, \quad \tag{1} \]

\[ J = \frac{\mu}{\mu + \nu} S_1 + \frac{\nu}{\mu + \nu} S_2 + \frac{i\lambda}{\mu + \nu} S_1 \times S_2, \quad \tag{2} \]

where \(S_1, S_2\) are the spin-\(\frac{1}{2}\) operators and \(\mu, \nu\) and \(\lambda\) are arbitrary parameters. \(I\) is the total spin operator satisfying \([I_i^+, I_j^-]\) = \(i\epsilon_{\alpha\beta\gamma} I_{\gamma}\), \(i, j = 1, 2\).

With a special constraint relation \(\mu\nu = -\frac{1}{2}\lambda^2 \tag{28}\), we can get \(J^2 = \frac{1}{4}, [J_a, J_b] = i\epsilon_{abc} J_c\). Similarity transformations of the generators can be made by the use of the matrix \(\tau\) which takes the form of

\[ \tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \nu & -\frac{1}{2}\lambda & 0 \\ 0 & -\frac{1}{2}\lambda & \nu & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad \tag{3} \]

After the similarity transformations, the generators become

\[ Y^+ = \tau^{-1} J^+ \tau = \begin{pmatrix} \xi \sigma^x & 0 \\ 0 & \frac{1}{2} \xi^{-1} \sigma^x \end{pmatrix}, \quad \tag{4} \]

\[ Y^- = \tau^{-1} J^- \tau = \begin{pmatrix} \frac{1}{2} \xi^{-1} \sigma^x & 0 \\ 0 & -\xi \sigma^x \end{pmatrix}, \quad \tag{4} \]

\[ Y^3 = \tau^{-1} J^3 \tau = \frac{1}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \quad \tag{4} \]

where \(\xi = \nu - \frac{1}{2}\lambda\) and \(\sigma\) are pauli matrices. \(\{Y^a, a = \pm, 3\}\) reduce to two \(2 \times 2\) blocks diagonal and \(4 \times 4\) matrix is essentially the 4-dimension representation of \(\mathfrak{sl}(2)\) algebra, so it is marked as the reduced \(Y(\mathfrak{sl}(2))\) algebra in this case.

We will study the effects of the \(Y(\mathfrak{sl}(2))\) algebra generators on the entanglement degree of Bi-qubit system as follows.

For an arbitrary two-qubit pure state \(|\Phi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle\), where \(a_{00}, a_{01}, a_{10}, a_{11}\) are the normalized complex amplitudes, the concurrence (the measurement of entanglement) \(C\) is given by \[30\]

\[ C = 2|a_{00}a_{11} - a_{01}a_{10}| \quad \text{and} \quad 0 \leq C \leq 1. \quad \tag{5} \]

with the maximally entangled state (MES) \(C = 1\) and the separable state \(C = 0\). Of course, we can construct another general state as the initial state

\[ |\phi\rangle = \frac{1}{\sqrt{2}}[\alpha(|00\rangle + |11\rangle) + \beta(|01\rangle + |10\rangle)] \quad \tag{6} \]

where \(|\alpha|^2 + |\beta|^2 = 1\).

The concurrence \(C\) of the initial state is

\[ C = |\alpha^2 - \beta^2|. \quad \tag{7} \]

By making use of the transition characteristic of Yangian operators \(J_i\) and \(Y^i\) \((i = +, -, 3)\) \[29, 31\], we can construct transition operators \(P\), which can make the initial state transit another state

\[ |\phi_i\rangle = P|\phi\rangle. \quad \tag{8} \]

To illustrate the relation of every transition operator versus the concurrence, we will discuss two cases as follows.

(i) The general \(Y(\mathfrak{sl}(2))\)

Due to the transition effect of Yangian generators, transition operators \(P\) can be constructed as compositions of Yangian \(Y(\mathfrak{sl}(2))\) generators.

(a) Let us first take the transition operator

\[ P_1 = J^+, \quad \tag{9} \]

which is composed of the general Yangian \(Y(\mathfrak{sl}(2))\) generators. By acting \(P_1\) to the initial state, we can get the final state \(|\phi_1\rangle = P_1|\phi\rangle = \frac{1}{\sqrt{2}}(\frac{\nu + \frac{1}{2}\lambda}{\mu + \nu}\alpha|01\rangle + \frac{\mu - \frac{1}{2}\lambda}{\mu + \nu}\beta|11\rangle)\) with the normalization condition

\[ |\phi_1\rangle = \frac{\nu + \frac{1}{2}\lambda}{\mu + \nu} |\alpha|^2 + |\frac{\mu - \frac{1}{2}\lambda}{\mu + \nu} |\beta|^2 = 2. \quad \tag{10} \]

The concurrence of the final state \(|\phi_1\rangle\) is obtained as

\[ C_1 = |\frac{(\mu - \frac{1}{2}\lambda)(\nu + \frac{1}{2}\lambda)}{(\mu + \nu)^2}\alpha^2|, \quad \tag{11} \]

Taking the transition operator as \(P_2 = J^-\), we can obtain \(|\phi_2\rangle = P_2|\phi\rangle = \frac{1}{\sqrt{2}}(\beta|00\rangle + \frac{\nu + \frac{1}{2}\lambda}{\mu + \nu}\alpha|01\rangle + \frac{\mu + \frac{1}{2}\lambda}{\mu + \nu}\alpha|10\rangle)\). The concurrence of \(|\phi_2\rangle\) is same with \(|\phi_1\rangle\), namely, \(C_1 = C_2\) with range from 0 to 1.

(b) Choose the transition operator

\[ P_3 = J^3, \quad \tag{12} \]

we can obtain the final state \(|\phi_3\rangle = \frac{1}{\sqrt{2}}[\alpha(-|00\rangle + |11\rangle) - \frac{\nu + \frac{1}{2}\lambda}{\mu + \nu}\beta(|01\rangle - |10\rangle)]\). Utilizing the normalization condition, the concurrence is gotten as

\[ C_3 = C = |\alpha^2 - \beta^2|, \quad \tag{13} \]
which is the same with the concurrence of the initial state in Eq. (7). That is to say, the transition operator $J^3$ can not change the entanglement degree of the initial state.

(ii) The reduced $Y(sl(2))$

(a) When we take the transition operator

$$P_4 = Y^+,$$  

which is composed of the reduced Yangian $Y(sl(2))$ generators. By acting $P_4$ to the initial state, we will obtain the final state $|\phi_4\rangle = \frac{1}{\sqrt{2}}(|\xi^{-1}\alpha|00\rangle + |\xi\beta|11\rangle)$. The corresponding normalization condition leads to

$$|\xi^{-1}\alpha|^2 + |\xi\beta|^2 = 2.$$  

The concurrence of the final state $|\phi_4\rangle$ is

$$C_4 = 0,$$  

When $P_5 = Y^-$, the final state yields $|\phi_5\rangle = \frac{1}{\sqrt{2}}(|\alpha|-|00\rangle + |\beta|01\rangle + |\xi\beta|10\rangle)$. We can obtain the same concurrence of the final state $|\phi_5\rangle$ with $C_4$. So we can obtain that the reduced Yangian $Y^\pm$ can make the initial state disentangled.

(b) If we choose the transition operator

$$P_6 = Y^3$$  

corresponding to the final state $|\phi_6\rangle = \frac{1}{\sqrt{2}}(|\alpha|00\rangle + |\beta|11\rangle)$. Utilizing the normalization condition, the concurrence is gotten as

$$C_4 = C = |\alpha^2 - \beta^2|,$$  

which is the same with the concurrence of the initial state in Eq. (7). That is to say, the transition operator $Y^3$ can not change the entanglement degree of the initial state too. By comparing the effects of the transition operators of the general and reduced Yangian on entanglement, we find that the reduced Yangian operators ($Y^+$ and $Y^-$) make the initial state disentangled directly, but the general Yangian operators can not. In addition, through carefully observing, we find that $\alpha$ is the parameter of $|00\rangle$ and $|11\rangle$ at first, but become the parameter of $|01\rangle$ and $|10\rangle$ after action of $P_1$, $P_2$, $P_4$ and $P_5$, namely, these four operators make the transformation happen between states. However, there is not so transformation for $P_3$ and $P_6$. $J^+$ and $Y^+$ make $\beta$ transition to $|11\rangle$, not $|00\rangle$. However, the same case, $J^-$ and $Y^-$ make $\beta$ transition to $|00\rangle$, not $|11\rangle$. By above different transition cases, we can fully understand the transition effect of Yangian $Y(sl(2))$ operators.

### III. THE REDUCED $Y(su(3))$ ALGEBRA

The subalgebra of $Y(su(3))$ is Lie algebra $su(3)$, which we have been familiar with the $su(3)$ symmetry of elementary particles $\{\mathbf{32}, \mathbf{32}'\}$. $su(3)$ generators are defined by $[F^a, F^b] = \frac{i}{2}f_{abc}F^c$ (a, b, c = 1, 2, · · · , 8), where the structure constants $f_{abc}$ are antisymmetric $f_{123} = 1, f_{458} = f_{678} = \frac{2\sqrt{3}}{3}, f_{147} = f_{256} = f_{257} = f_{345} = -f_{346} = \frac{1}{2}$. The 3-dimensional representation of $su(3)$ is formed by the well-known Gell-Mann matrices, i.e.,

$$\Lambda^a = 2F^a, \quad \{\Lambda^a, a = 1, 2, \cdots 8\}$$

$$[\Lambda^a, \Lambda^b] = 2if_{abc}\Lambda^c.$$  

Now we can introduce the shift operators

$$I^\pm = F^1 \pm iF^2, \quad U^\pm = F^6 \pm iF^7, \quad V^\pm = F^4 \pm iF^5,$$

$$I^3 = F^3, \quad Y = \frac{2}{\sqrt{3}}F^8 = I^8, \quad (Y - \text{hypercharge})$$

and the notations

$$U^3 = \frac{1}{2}I^3 + \frac{3}{4}I^8, \quad V^3 = \frac{1}{2}I^3 - \frac{3}{4}I^8,$$

$$Q = I^3 + \frac{1}{2}Y.$$  

where $U^i (i = \pm 3), V^i (V^i, i = \pm 3)$ and $Q$ are called the $U$-spin, $V$-spin and the charge operator respectively. And it is easy to check that $[U^i, Q] = 0$, but there is no the same property between $V$-spin and $I$-spin.

The fundamental representation of local $su(3)$ is given by

$$I^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$V^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad I^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Y = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$  

and $I^-(I^+)^\dagger, U^-=(U^+)^\dagger, V^-=(V^+)^\dagger$.

For the system of two particles, we can define the operators of $Y(su(3))$ as follows:

$$I^a = \sum_i F_i^a,$$

$$Y^a = \frac{\mu + \nu}{\mu + \nu}I_i^8 + \frac{\nu}{\mu + \nu}I_i^a + \frac{i\lambda}{2(\mu + \nu)}f_{abc} \sum_{i \neq j} \omega_{ij} I_i^b I_j^c$$  

(i, j = 1, 2).  

Here $I^a$ form a $SU(3)$ algebra characterized by $f_{abc}$, $\{F_i^a, a = 1, 2, \cdots 8\}$ form a local $su(3)$ on the $i$ site and they obey the commutation relation

$$[F_i^a, F_j^b] = if_{abc}\delta_{ij}F_i^c.$$  

(24)
\[\omega_{ij} = \begin{cases} 
1, & \text{if } i > j \\
-1, & \text{if } i < j \\
0, & \text{if } i = j 
\end{cases}\]

Eq. (23) plays an important role in explaining the physical meaning of the representation theory of Chari-Pressley [34] through more calculation [35]. If we take the condition \(\mu \nu = -\frac{\lambda^2}{4}\), in terms of the notations

\[
\tilde{I}^\pm = Y^1 \pm i Y^2, \quad \tilde{U}^\pm = Y^6 \pm i Y^7, \quad \tilde{V}^\pm = Y^4 \pm i Y^5, \\
\tilde{I}^3 = Y^3, \quad \tilde{I}^8 = \frac{2}{\sqrt{3}} Y^8,
\]

we can directly calculate and obtain

\[
(Y)^2 = \frac{1}{3}, \quad (Y)^3 = 0.
\]

In the following we get the commutation relations

\[
\begin{aligned}
[\tilde{I}^+, \tilde{I}^-] &= 2\tilde{I}^3, & [\tilde{I}^3, \tilde{I}^\pm] &= [\tilde{I}^\pm, \tilde{I}^\pm] = 0
\end{aligned}
\]

\[
[\tilde{I}^3, \tilde{I}^\pm] = \pm \tilde{I}^\pm, & \quad [\tilde{I}^3, \tilde{U}^\pm] = \mp \frac{1}{2} \tilde{U}^\pm \\
[\tilde{I}^8, \tilde{U}^\pm] = \pm \tilde{U}^\pm, & \quad [\tilde{I}^3, \tilde{V}^\pm] = \mp \frac{1}{2} \tilde{V}^\pm \\
[\tilde{I}^8, \tilde{V}^\pm] = \mp \tilde{V}^\pm, & \quad [\tilde{U}^+, \tilde{U}^-] = 2\tilde{U}^3 \\
[\tilde{I}^3, \tilde{V}^\pm] = \pm \tilde{V}^\pm, & \quad [\tilde{V}^+, \tilde{V}^-] = 2\tilde{V}^3 \\
[\tilde{I}^\pm, \tilde{U}^\pm] = \pm \tilde{U}^\pm, & \quad [\tilde{U}^\pm, \tilde{V}^\pm] = \pm \tilde{I}^\pm \\
[\tilde{I}^\pm, \tilde{U}^\pm] = \pm \tilde{U}^\pm, & \quad [\tilde{V}^\pm, \tilde{I}^\pm] = [\tilde{U}^\pm, \tilde{V}^\pm] = 0
\end{aligned}
\]

where \(\tilde{U}^3 = -\frac{1}{4} \tilde{I}^3 + \frac{3}{4} \tilde{I}^8\) and \(\tilde{V}^3 = -\frac{1}{4} \tilde{I}^3 - \frac{3}{4} \tilde{I}^8\). They are similar with the commutation relations of the \(I\)-spin, \(U\)-spin, and \(V\)-spin.

Setting \(x\) the eigenvalue of \(\tilde{I}^3\), then \(|x E - \tilde{I}^3| = 0\) with \(E\) a unite matrix and its solutions are

\[
\begin{aligned}
x_1 &= 0, & x_{2,3} &= \pm \frac{1}{2}, \\
x_{4,5} &= \pm \frac{1}{2} \sqrt{\mu^2 - 2\mu \nu + \nu^2 - \lambda^2}, \\
x_{6,7} &= \pm \frac{1}{2} \sqrt{\mu^2 - 2\mu \nu + \nu^2 + \lambda^2}, \\
x_{8,9} &= \pm \frac{1}{2} \sqrt{\mu^2 - 2\mu \nu + \nu^2 + \lambda^2}.
\end{aligned}
\]

By taking \(\mu \nu = -\frac{\lambda^2}{4}\), we can get

\[
x_{1,2,3} = 0, \quad x_{4,5,6} = \frac{1}{2}, \quad x_{7,8,9} = -\frac{1}{2}.
\]

If we take the similar matrix as

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \nu & 0 & -\frac{\lambda}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \nu & 0 & 0 & 0 & -\frac{\lambda}{2} & 0 & 0 \\
0 & -\frac{\lambda}{2} & 0 & \nu & 0 & 0 & 0 & -\frac{\lambda}{2} & 0 \\
0 & 0 & 0 & 0 & \nu & 0 & 0 & \nu & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{\lambda}{2} & 0 & \nu & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{\lambda}{2} & 0 & \nu \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{\lambda}{2} & 0
\end{pmatrix}
\]

and its inverse matrix \(A^{-1}\) can be obtained easily, thus we can obtain

\[
\begin{aligned}
\tilde{I}^3 &= A^{-1} \tilde{I}^3 A = \begin{pmatrix} I^3 & 0 \\ 0 & I^3 \end{pmatrix}, \\
\tilde{I}^8 &= A^{-1} \tilde{I}^8 A = \begin{pmatrix} I^8 & 0 \\ 0 & I^8 \end{pmatrix}
\end{aligned}
\]

\[
\begin{aligned}
\tilde{I}^+ &= A^{-1} \tilde{I}^+ A = \begin{pmatrix} \alpha \tilde{I}^+ & 0 \\ 0 & -\alpha^{-1} \tilde{I}^+ \end{pmatrix}, \\
\tilde{I}^- &= A^{-1} \tilde{I}^- A = \begin{pmatrix} -\alpha^{-1} \tilde{I}^- & 0 \\ 0 & \alpha \tilde{I}^- \end{pmatrix}
\end{aligned}
\]

\[
\begin{aligned}
\tilde{U}^+ &= A^{-1} \tilde{U}^+ A = \begin{pmatrix} U^+ & 0 \\ 0 & \alpha U^+ \end{pmatrix}, \\
\tilde{U}^- &= A^{-1} \tilde{U}^- A = \begin{pmatrix} U^- & 0 \\ 0 & \alpha^{-1} U^- \end{pmatrix}
\end{aligned}
\]

\[
\begin{aligned}
\tilde{V}^+ &= A^{-1} \tilde{V}^+ A = \begin{pmatrix} \alpha \tilde{V}^+ & 0 \\ 0 & \alpha^{-1} \tilde{V}^+ \end{pmatrix}, \\
\tilde{V}^- &= A^{-1} \tilde{V}^- A = \begin{pmatrix} V^- & 0 \\ 0 & \alpha^{-1} V^- \end{pmatrix}
\end{aligned}
\]

where \(\alpha = \nu - \frac{\lambda}{2}\). In virtue of a similar transformation, we can reduce the eight 9 \(\times\) 9 matrix to the three 3 \(\times\) 3 block diagonal, so it is marked as the reduced \(Y(su(3))\) algebra in this case. Taking the correspondence

\[
\begin{aligned}
I^+ \rightarrow \alpha \tilde{I}^+, & \quad I^- \rightarrow -\alpha^{-1} \tilde{I}^-, & \quad I^3 \rightarrow I^3, & \quad I^8 \rightarrow I^8, \\
U^+ \rightarrow \alpha \tilde{U}^+, & \quad U^- \rightarrow -\alpha^{-1} \tilde{U}^-, \\
V^+ \rightarrow \alpha \tilde{V}^+, & \quad V^- \rightarrow -\alpha^{-1} \tilde{V}^-.
\end{aligned}
\]

Eq. (27) will get the same result, that is, the Yangian algebra we discussed hides a \(u(1)\) algebra.
TABLE I: The general Yangian in decay channels.

| $|\eta\rangle_{ini}$ | $|\eta\rangle_{fin}$ | $C_{fin}$ | decay |
|-----------------|-----------------|----------|-------|
| $\eta = \alpha_1 |\eta^0\rangle + \alpha_2 |\eta^0\rangle'$ | $\eta' = \eta \rightarrow \pi^+ \pi^-$ | $C_1$ | $\eta \rightarrow \pi^+ \pi^-$ |
| $I^+$ | $\eta'$ | $C_2$ | $\eta \rightarrow \pi^+ \pi^-$ |
| $I^-$ | $\eta'$ | $C_2$ | $\eta \rightarrow \pi^+ \pi^-$ |
| $\tilde{U}^+$ | $\eta'$ | $C_2$ | $\eta \rightarrow \pi^+ \pi^-$ |
| $\tilde{U}^-$ | $\eta'$ | $C_2$ | $\eta \rightarrow \pi^+ \pi^-$ |
| $\tilde{V}^+$ | $\eta'$ | $C_2$ | $\eta \rightarrow \pi^+ \pi^-$ |
| $\tilde{V}^-$ | $\eta'$ | $C_2$ | $\eta \rightarrow \pi^+ \pi^-$ |
| $\tilde{I}^3$ | $\eta'$ | $\text{Log}_2$ | $\eta \rightarrow \pi^+ \pi^-$ |
| $\tilde{I}^8$ | $\eta'$ | $\text{Log}_2$ | $\eta \rightarrow \pi^+ \pi^-$ |

IV. THE APPLICATIONS OF Y(su(3)) ALGEBRA IN THE MIXED LIGHT PSEUDOSCALAR MESON STATE $\eta$

The $\eta$ and $\eta'$ mesons play the important role in low energy QCD. $\eta - \eta'$ mixing system is one of the most attractive problems all along.\cite{27, 40, 38}. Hence we choose the superposition of singlet and octet of $su(3)$ as the initial state

$$|\eta\rangle = \alpha_1 |\eta^0\rangle + \alpha_2 |\eta^0\rangle', \quad (32)$$

where $\alpha_1$ and $\alpha_2$ are the normalized real amplitudes and satisfy $\alpha_1^2 + \alpha_2^2 = 1$. $|\eta^0\rangle = |\xi^0\rangle = \frac{1}{\sqrt{3}}(|u\bar{u}) + |d\bar{d}) + |s\bar{s})\rangle$, $|\eta^0\rangle' = \frac{1}{\sqrt{6}}(-|u\bar{u}) - |d\bar{d}) + 2|s\bar{s})\rangle$.

As is known, the entanglement degree of the genuine N-particle qudit pure state \cite{39} is measured by its mean entropy \cite{40}

$$C_P^{(N)} = \left\{\begin{array}{ll}
\frac{1}{N} \sum_{i=1}^{N} S_i & \text{if } S_i \neq 0 \forall i \\
0 & \text{otherwise}
\end{array}\right., \quad (33)$$

where $S_i = -Tr((\rho_k)_{\lambda} \text{Log}_3(\rho_k)_{\lambda})$ is the reduced partial Von Neumann entropy for the $i$th particle only, with the other $N-1$ particles traced out, and $(\rho_k)_{\lambda}$ is the corresponding reduced density matrix. The system we discuss here is bipartite qudit, $N = 2$. Thus the corresponding reduced density matrix is

$$C_{ini} = -2 \frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}} \text{Log}_3 \left( \frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}} \right)^2 - \frac{\alpha_1}{\sqrt{3}} + \frac{2\alpha_2}{\sqrt{6}} \text{Log}_3 \left( \frac{\alpha_1}{\sqrt{3}} + \frac{2\alpha_2}{\sqrt{6}} \right)^2. \quad (34)$$

The value of $C_{ini}$ has been discussed in detail with range from 0 to 1.\cite{27}.

To illustrate the effects of every generator of Yangian $Y(su(3))$ on the initial state, we will take the following operators as the transition operators in the general and reduced Yangian cases respectively.

(i) The general $Y(su(3))$

Every general transitions operator of $Y(su(3))$ is applied to the initial state, then we can obtain the following results.

$$|\eta_1\rangle = \tilde{I}^+ |\eta\rangle = \xi^{-1} \left( \frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}} \right) |d\bar{u}\rangle,$$

where $\xi$ is the normalization condition, $\nu$ is not an independent parameter. The behavior of $C_1$ depends on $\mu$ for the case of $\lambda = 2$, as shown in Fig. 1. It is worth noting that the maximum value of $C_1$ is $\text{Log}_32$. In addition, we can find that the entanglement degrees of the final states are same and the maximum value of $C_1$ after the actions of the operators $\tilde{I}^3$ and $\tilde{I}^8$.

(ii) The reduced $Y(su(3))$

Every reduced transitions operator of $Y(su(3))$ is applied to the initial state by the same way, then we can obtain the following results.

$$|\eta_1\rangle = \tilde{I}^+ |\eta\rangle = \xi^{-1} \left( \frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}} \right) |d\bar{u}\rangle.$$
TABLE II: The reduced Yangian in decay channels.

| $|\eta\rangle_{\text{ini}}$ | $P$ | $|\eta\rangle_{\text{fin}}$ | $C_{\text{fin}}$ | decay |
|-----------------|-----|-----------------|----------------|-------|
| $|\eta\rangle = \alpha_1|\eta^\prime\rangle + \alpha_2|\eta^\prime\prime\rangle$ | $I^+$ | $|\eta\rangle$ | 0 | $\eta \rightarrow \pi^-$ |
| | $I^-$ | $|\eta\rangle$ | 0 | $\eta \rightarrow \pi^+$ |
| | $U^+|\eta\rangle$ | 0 | $\eta \rightarrow K_0^+$ |
| | $U^-|\eta\rangle$ | 0 | $\eta \rightarrow K_0^+$ |
| | $V^+|\eta\rangle$ | 0 | $\eta \rightarrow K^+$ |
| | $V^-|\eta\rangle$ | 0 | $\eta \rightarrow K^-$ |
| | $F^3$ | $12\log_2\gamma$ | $\eta \rightarrow \pi^0$ |
| | $F^8$ | $12\log_2\gamma$ | $\eta \rightarrow \eta^0\eta^{'0}\pi^0$ |

$|\eta_2\rangle = I^-|\eta\rangle = \xi^{-1}(\frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}})|u\bar{d}\rangle$

$= -\xi^{-1}(\frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}})|\bar{s}d\rangle$

$|\eta_3\rangle = U^+|\eta\rangle = \xi^{-1}(\frac{\alpha_1}{\sqrt{3}} + \frac{2\alpha_2}{\sqrt{6}})|s\bar{d}\rangle$

$= \xi^{-1}(\frac{\alpha_1}{\sqrt{3}} + \frac{2\alpha_2}{\sqrt{6}})|K_0^\prime\rangle$

$|\eta_4\rangle = U^-|\eta\rangle = \xi^{-1}(\frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}})|d\bar{s}\rangle$

$= \xi^{-1}(\frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}})|K_0\rangle$

$|\eta_5\rangle = V^+|\eta\rangle = \xi^{-1}(\frac{\alpha_1}{\sqrt{3}} + \frac{2\alpha_2}{\sqrt{6}})|u\bar{s}\rangle$

$= \xi^{-1}(\frac{\alpha_1}{\sqrt{3}} + \frac{2\alpha_2}{\sqrt{6}})|K^\prime\rangle$

$|\eta_6\rangle = V^-|\eta\rangle = \xi^{-1}(\frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}})|\bar{s}d\rangle$

$= \xi^{-1}(\frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}})|K^-\rangle$

$|\eta_7\rangle = F^3|\eta\rangle = (\frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}})(\frac{1}{2}|u\bar{u}\rangle - |d\bar{d}\rangle)$

$= (\frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}})\frac{1}{\sqrt{2}}|\pi^0\rangle$

$|\eta_8\rangle = F^8|\eta\rangle = (\frac{\alpha_1}{\sqrt{3}} - \frac{\alpha_2}{\sqrt{6}})(\frac{1}{3}|u\bar{u}\rangle - \frac{1}{2}|d\bar{d}\rangle)$

$- \frac{2}{3}\frac{\alpha_1}{\sqrt{3}} + \frac{2\alpha_2}{\sqrt{6}}|s\bar{s}\rangle$

$= (-\frac{5\alpha_1}{18} - \frac{7\alpha_2}{18\sqrt{2}})|\eta^{'0}\rangle - (\frac{7\alpha_1}{18\sqrt{2}} + \frac{17\alpha_2}{36})|\eta^0\rangle$

$+ (\frac{5\alpha_1}{6\sqrt{6}} - \frac{5\alpha_2}{12\sqrt{3}})|\pi^0\rangle$

(V. CONCLUSION)

In this paper, we have studied the applications of Yangian $Y(sl(2))$ and $Y(su(3))$ for quantum entanglement in the general and reduced Yangian cases. For $Y(sl(2))$, we only studied the effects Yangian algebra on the entanglement. By calculating, we can find that the general transition operators Yangian $Y(sl(2))$ can not make the initial state disentangled, and the entanglement degree of the final state is from 0 to 1. However, the reduced ones can make the initial state disentangled except $Y^3$. It is worth noting that the general transition operators, $J^3$ and $Y^3$ can not change the entanglement of the initial state.

For $Y(su(3))$, we have studied not only the change of entanglement degree, but also the decay channel in the mixed light pseudoscalar meson states. Our results show that the general $Y(su(3))$ generators can not make the initial state disentangled, but the reduced ones can make the initial state disentangled except $F^3$ and $F^8$. In addition, we can obtain that $F^3$, $F^8$, $F^3$ and $F^8$ can not change the entanglement of the initial state, and the entanglement degree of the initial and final states are $\log_2\gamma$ with the normalization condition. Moreover, some hadronic decay channels of pseudo-scalar mesons can be reformulated under the framework of Yangian by acting transition operators consisting of generators of Yangian $Y(su(3))$ on the initial state.

Now a question put forward: can we use this method to $su(n)$? To our knowledge, this problem has not been discussed in this thesis. But we can surmise that for $Y(su(n))$ the matrices of the generators...
Y can be written as $n$ pieces of $n \times n$ matrices, furthermore each pieces is formed by the consequently generators of $su(n)$. Consequently, it is of interesting area how to generalize the idea in $Y(su(n))$, which makes the system contact with physical application.

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