ALGEBRAS OF GENERALIZED SINGULAR INTEGRAL OPERATORS WITH CAUCHY KERNEL

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Abstract. For bounded Lebesgue measurable functions $f, g, \phi$ and $\psi$ on the unit circle, $P_+fP_+ + P_-gP_+ + P_+\phi P_- + P_-\psi P_-$ is called a generalized singular integral operator (GSIO) on $L^2(T)$, where $P_+$ is the Riesz projection, $P_- = I - P_+$. In this paper, we relate GSIOs to a number of operators, including Cauchy singular integral operator, (dual) truncated Toeplitz operator, Foguel-Hankel operator, multiplication operator, Toeplitz plus Hankel operator etc. We establish the short exact sequences associated of the $C^*$-algebras generated by GSIOs with bounded or quasi-continuous symbols. As a consequence we obtain the spectra of various classes of GSIOs, the spectral inclusion theorem and compute the Fredholm index of GSIOs. Moreover, we gave the necessary and sufficient conditions for invertibility (Fredholmness) of GSIOs via Wiener-Hopf factorization.

1. Introduction

Let $\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and $\mathbb{T} = \{\xi \in \mathbb{C} : |\xi| = 1\}$ be its boundary. Riemann-Hilbert boundary problem [25] on the unit circle can be reformulated as follows.

Given functions $\alpha, \beta, h$ on $\mathbb{T}$, find two analytic functions $f_+ \in \text{Hol}(\mathbb{D})$ and $f_- \in \text{Hol}(\mathbb{C} \setminus \mathbb{D})(f_-(\infty) = 0)$ such that

$$\alpha f_+ + \beta f_- = h$$

on $\mathbb{T}$.

$H^2$ denotes the classical Hardy space of the open unit disk $\mathbb{D}$, we let $L^2 = L^2(\mathbb{T}), L^\infty = L^\infty(\mathbb{T})$ denote the usual Lebesgue spaces on the unit circle [12]. $P_+$ is the orthogonal projection from $L^2(\mathbb{T})$ onto $H^2, P_- = I - P_+$. Suppose that $h \in L^2(\mathbb{T}), f_+ \in H^2$ and $f_- \in L^2(\mathbb{T}) \ominus H^2 = \mathbb{D} \overline{H^2}$. Put $f = f_+ + f_-$, the equation (1.1) becomes

$$S_{\alpha, \beta}f = h, \quad \text{where } S_{\alpha, \beta} = \alpha P_+ + \beta P_-.$$

$S_{\alpha, \beta}$ is called the singular integral operator with Cauchy kernel on $L^2(\mathbb{T})$, and

$$(S_{\alpha, \beta}f)(z) = \frac{\alpha(z) + \beta(z)}{2}f(z) + \frac{\alpha(z) - \beta(z)}{2} \frac{1}{\pi i} \int_T \frac{f(\xi)}{\xi - z} d\xi.$$

Riemann-Hilbert boundary problem is considered solved if one has found conditions for the operator $S_{\alpha, \beta}$ to be Fredholm or invertible. Most results about these operators can be found in [14, 15]. We are interested in the algebra of
singular integral operator, but the adjoint of \( R_{\alpha,\beta} \) is no longer a singular integral operator. Naturally, one can define the generalized singular integral operator.

Given a linear space \( X \), we denote by \( X_N \) the linear space of all \( N \)-dimensional vectors with components from \( X \) and let \( X_{N \times N} \) denote the linear space of \( N \times N \) matrices with entries from \( X \).

**Definition 1.1.** If \( H = \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} \in L^2_{2 \times 2} (\mathbb{T}) \), the generalized singular integral operator (GSIO) with symbol \( H \) is the operator \( R_H \) is defined by

\[
R_H x = P_+ f P_+ x + P_- g P_+ x + P_+ \phi P_- x + P_- \psi P_- x.
\]

for each \( x \in L^2 (\mathbb{T}) \).

The significance of GSIOs comes from the following special cases.

1. Multiplication operator on \( L^2 (\mathbb{T}) \) : if \( f = g = \phi = \psi \), then \( R_H \) is the multiplication operator on \( L^2 (\mathbb{T}) \).
2. Hilbert transform: if \( f = g = -\phi = -\psi = 1 \), then \( R_{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}} = 1 \otimes 1 = P_+ - P_- - 1 \otimes 1 \) is the Hilbert transform \( \mathbb{H} \) [12, Ch III].
3. Singular integral operator: if \( f = g = \alpha, \phi = \psi = \beta \), then \( R_{\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}} = S_{\alpha,\beta} \) is the singular integral operator. T. Nakazi and T. Yamamoto [21, 19, 20, 22, 23, 24] have study the boundedness and normality of \( S_{\alpha,\beta} \), and calculate its norm, C. Gu [16] have study the algebraic properties of \( S_{\alpha,\beta} \).
4. Toeplitz plus Hankel operators: if \( H = \begin{pmatrix} f & 0 \\ 0 & \phi \end{pmatrix} \), then \( (I \oplus J)R_H |_{H^2} = T_f + \Gamma_g \), where \( Jx(z) = \bar{x}(\bar{z}) \) for \( x \in L^2 (\mathbb{T}) \).
5. Foguel-Hankel operators: if \( \phi \in L^\infty \) and \( H = \begin{pmatrix} \bar{\phi} & 0 \\ 0 & \phi \end{pmatrix} \), then \( R_H \) and Foguel-Hankel operator \( \begin{pmatrix} T_f^* & X_f^* \\ 0 & T_f \end{pmatrix} \) are unitarily equivalent (see Section 2). Foguel-Hankel operators closely related to Halmos’ problem [17](whether or not any polynomially bounded operator on a Hilbert space \( H \) is similar to a contraction). J. Bourgain [4] has shown that \( R_H \) is similar to a contraction if \( \phi \in BMOA \), A. Aleksandrov and V. Peller [1] have shown that if \( R_H \) is polynomially bounded then \( \phi \in BMOA \). G. Pisier [27] and K. Davidson and V. Paulsen [7] give a negative answer to Halmos’ problem via vector-Foguel-Hankel operators.
6. (Dual) Truncated Toeplitz operators: let \( u \) is an inner function, suppose \( f \in L^\infty (\mathbb{T}) \) and \( H = \begin{pmatrix} f & uf \\ uf & \bar{f} \end{pmatrix} \), then \( R_H \) is unitary equivalent to the dual truncated Toeplitz operator \( D_f[8, 28, 29] \), furthermore, \( R_H \) is equivalent after extension to truncated Toeplitz operator for invertible symbol [6, Theorem 6.1].

Given a closed unital subalgebra \( A \subset L^\infty (\mathbb{T}) \), the \( C^* \)-algebra \( \mathfrak{A}_A \) is defined by

\[
\mathfrak{A}_A = \text{cl} \left\{ \sum_{i=1}^n \prod_{j=1}^m R_{H_{ij}} | H_{ij} \in A_{2 \times 2} \right\}.
\]

In fact, \( \mathfrak{A}_A \) equals the \( C^* \)-algebra generated by \( \{ R_{\alpha,\beta} | \alpha, \beta \in A \} \) and \( \{ R_{\phi,\psi} | \phi, \psi \in A \} \). In this paper, we explore the structure of the \( C^* \)-algebra \( \mathfrak{A}_{L^\infty (\mathbb{T})} \).
The earliest result on the $C^*-$algebra $\mathfrak{A}_{PC(\mathbb{T})}$ due to Gokhberg and Krupnik[13], where $PC(\mathbb{T})$ denote the algebra of all piecewise continuous and left continuous functions on $\mathbb{T}$. They proved that the sequence

$$0 \rightarrow \mathcal{C}(L^2(\mathbb{T})) \rightarrow \mathfrak{A}_{PC(\mathbb{T})} \rightarrow \mathcal{S} \rightarrow 0.$$ 

is exact. The algebra $\mathcal{S}$ consist of matrix-valued functions of second order $M(t, \mu) = (\alpha_{jk}(t, \mu))_{j,k}^{2,2}$ with the following properties:

- $\alpha_{11}(t, \mu), \alpha_{22}(t, 1 - \mu), \alpha_{12}(t, \mu), \alpha_{21}(t, \mu) \in C(\mathbb{T} \times [0, 1]),$
- $\alpha_{12}(t, 0) = \alpha_{21}(t, 0) = \alpha_{12}(t, 1) = \alpha_{21}(t, 1) = 0 \ \forall t \in \mathbb{T}.$

This paper is organized as follows. In section 2, we presents some pr eliminaries and basic properties of GISO. In section 3 and section 4, we establish the short exact sequences associated of the $C^*-$algebras generated by GISO with bounded symbols or quasicontinuous symbols, and obtain the essential spectrum of GISO and index forumla. In section 5, we establish vector we obtain the ne cessary and sufficient conditions for invertibility and Fredholmness of GSIO via equivalence after extension and Winer-Hopf factorization. In the last section, corresponding results apply for the spectrum of singular integral operators, Foguel-Hankel operators and dual truncated Toeplitz operators.

2. Preliminaries

The generalized singular integral operator $R(f \phi g \psi)$ can be expressed as an operator matrix with respect to the decomposition $L^2(\mathbb{T}) = H^2 \oplus z\overline{H^2}$, the result is of the form

$$(T_f \ H_g) \quad (H^*_g \ T_\psi), \quad (2.1)$$

where $T_f$ denote the Toeplitz operator on $H^2$ such that

$$T_f x = P_+(fx), \quad x \in H^2;$$

$H_g$ denote the Hankel operator on $H^2$ such that

$$H_g x = P_-(gx), \quad x \in H^2;$$

$H^*_g$ denote the adjoint of Hankel operator such that

$$H^*_g y = P_+(\phi y), \quad y \in z\overline{H^2};$$

$T_\psi$ denote the dual Toeplitz operator on $z\overline{H^2}$ such that

$$T_\psi y = P_-(\psi y), \quad y \in z\overline{H^2}.$$

Converse, if an operator $T$ on $L^2(\mathbb{T})$ has form (2.1), then $T$ is a GSIO. Moreover, the generalized singular integral operator $R(f \phi g \psi)$ is unitarily equivalent to an operator matrix on $H^2$. To illustrate this, we need to introduce two useful operators and their properties. For $x \in L^2$, define

$$V x(z) = \overline{x(z)};$$

$$J x(z) = \overline{x(z)}.$$
Note that $V$ is an anti-unitary operator and $U$ is an unitary operator, and they have the following properties:

1. $\langle Vx, Vy \rangle = \langle y, x \rangle, \quad \langle Ux, Uy \rangle = \langle x, y \rangle$;
2. $VM_fV = M_f, \quad UM_fU = M_f$, where $\tilde{f}(z) = f(\bar{z})$;
3. $VP_ = P_\bar{\Psi}$, $UP_ = P_\Psi$;
4. $VH^2 = \bar{z}H^2$, $UH^2 = \bar{z}H^2$;
5. $Uz^n = Vz^n = \bar{z}^{n+1}$.

Using the operator $U$, for $g \in L^2$, we can define the Hankel operator on $H^2$ by

$$\Gamma_g = UH_g.$$ 

The operator

$$\begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} : L^2 = H^2 \oplus \bar{z}H^2 \to H^2 \oplus H^2$$ 

is unitary. A simple computation gives

$$\begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} T_f & H^*_\phi \\ H_g & S_\psi \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} T_f & H^*_\phi \\ UH_g & US_\psi U \end{pmatrix} = \begin{pmatrix} T_f & \Gamma^*_\phi \\ \Gamma_g & UP_\bar{\Psi}P_\Psi \end{pmatrix} = \begin{pmatrix} T_f & \Gamma^*_\phi \\ \Gamma_g & P_+UM_\psi UP_+ \end{pmatrix} = \begin{pmatrix} T_f & \Gamma^*_\phi \\ \Gamma_g & P_+M_\bar{\Psi}P_+ \end{pmatrix} = \begin{pmatrix} T_f & \Gamma^*_\phi \\ \Gamma_g & T_\bar{\phi} \end{pmatrix}.$$ 

This shows that the operator $R(\begin{pmatrix} f & \phi \\ \psi & g \end{pmatrix}) : L^2 \to L^2$ is unitary equivalent to

$$\begin{pmatrix} T_f & \Gamma^*_\phi \\ \Gamma_g & T_\bar{\phi} \end{pmatrix} : H^2 \oplus H^2 \to H^2 \oplus H^2.$$ 

Therefore, $R(\bar{z}, 0, \phi, \bar{z})$ is unitary equivalent to the Foguel-Hankel operator[4]

$$\begin{pmatrix} T_{\bar{z}} & \Gamma_{\phi} \\ 0 & T_{\bar{z}} \end{pmatrix}.$$ 

**Example 2.1.** For $\alpha, \beta \in L^\infty$, the truncated singular integral operator

$$S^u_{\alpha, \beta}x = \alpha P_u x + \beta Q_u x, \quad x \in L^2.$$
It can be write as an operator matrix with respect to the decomposition $L^2(\mathbb{T}) = H^2 \oplus \mathbb{H}^2$,

$$
\begin{pmatrix}
T_\alpha + T_{(\beta - \alpha)u}T_{\bar{u}} & H^*_\beta \\
H_\alpha + H_{(\beta - \alpha)u}T_{\bar{u}} & S_\beta
\end{pmatrix},
$$

$$
= \begin{pmatrix}
T_\alpha & H^*_\beta \\
H_\alpha & S_\beta
\end{pmatrix} + \begin{pmatrix}
T_{(\beta - \alpha)u}T_{\bar{u}} & 0 \\
H_{(\beta - \alpha)u}T_{\bar{u}} & 0
\end{pmatrix}
$$

= \begin{pmatrix}
T_\alpha & H^*_\beta \\
H_\alpha & S_\beta
\end{pmatrix} + \begin{pmatrix}
T_{(\beta - \alpha)u} & 0 \\
H_{(\beta - \alpha)u} & 0
\end{pmatrix} \begin{pmatrix}
T_{\bar{u}} & 0 \\
0 & I
\end{pmatrix}
$$

**Example 2.2.** Asymmetric dual truncated Toeplitz operator $D^{\theta,\alpha}_\phi : (K^2_\theta)^\perp \to (K^2_\alpha)^\perp$ is unitarily equivalent to some general singular integral operator. Let $h, g \in H^2$, we have

$$
D^{\theta,\alpha}_\phi(\theta h + \bar{z}g) = (P_- + \alpha P_+ \bar{\alpha})\phi(\theta h + \bar{z}g) = \alpha P_+ \bar{\alpha}\phi \theta P_+ h + P_- \phi \theta P_+ h + \alpha P_+ \bar{\alpha}\phi \bar{z}g + P_- \phi \bar{z}g
$$

or

$$
D^{\theta,\alpha}_\phi \begin{pmatrix}
M_\theta & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix} h \\ \bar{z}g \end{pmatrix} = \begin{pmatrix}
M_\alpha & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}\phi} & H^*_\phi \\
H_{\phi\theta} & S_\phi
\end{pmatrix} \begin{pmatrix} M_\bar{\alpha} & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix} h \\ \bar{z}g \end{pmatrix}
$$

hence

$$
D^{\theta,\alpha}_\phi = \begin{pmatrix}
M_\theta & 0 \\
0 & I
\end{pmatrix} \begin{pmatrix} T_{\bar{\alpha}\phi} & H^*_\phi \\
H_{\phi\theta} & S_\phi
\end{pmatrix} \begin{pmatrix} M_\bar{\alpha} & 0 \\
0 & I
\end{pmatrix},
$$

where $\begin{pmatrix} M_\theta & 0 \\
0 & I \end{pmatrix} : (K^2_\theta)^\perp \to L^2$ and $\begin{pmatrix} M_\bar{\alpha} & 0 \\
0 & I \end{pmatrix} : L^2 \to (K^2_\alpha)^\perp$ are unitary.

We begin our study of GSIO by considering some elementary properties.

**Proposition 2.3.** Let $H = \begin{pmatrix} f \\ g \phi \psi \end{pmatrix} \in L^2_{2 \times 2}(\mathbb{T})$.

1. $R_H$ is bounded on $L^2(\mathbb{T})$ if and only if $f, \psi \in L^\infty$ and $g_-, (\bar{\phi})_- \in \text{BMO}(\mathbb{T})$. Where $\text{BMO}(\mathbb{T}) = L^\infty(\mathbb{T}) + \mathbb{H}L^\infty(\mathbb{T})$.
2. If $R_H$ is bounded, then $R_H$ is zero if and only if $f = \psi = 0$ and $g, \bar{\phi} \in H^2$.
3. If $R_H$ is bounded, then $R_H$ is compact if and only if $f = \psi = 0$ and $g, \bar{\phi} \in H^\infty + C(\mathbb{T})$.
4. If $R_H$ is bounded, then $R(f, g, \phi, \psi)$ is self-adjoint if and only if $f$ and $\psi$ are real valued, and $g - \bar{\phi} \in H^2$.
5. If $R_H$ is bounded and positive, then $f$ and $\psi$ are positive and $g - \bar{\phi} \in H^2$.
6. If $R_H$ is bounded, then $R_H$ is complex symmetric operator for $V$ if and only if $f = \psi$, where $V \bar{f}(z) = \bar{z}f(z)$.

**Proof.** (1)-(3) Clearly $R_H$ is bounded (resp. zero, compact) if and only if $T_f, H^*_\phi, H_\phi$ and $S_\psi$ are bounded (resp. zero, compact). Toeplitz operator $T_a$ is bounded [9, 7.8] (resp., zero, compact[5, p.94]) if and only if its symbol $a$ is bounded (resp., zero, zero). Hankel operator $H_a$ is bounded[26, Theorem 1.3](resp., zero, compact [26, Theorem 5.5]) if and only if $a_- \in \text{BMO}$ (resp., $a \in H^2$, $a \in H^\infty + C(\mathbb{T})$), the conclusion follows.
(4) By the matrix representation (2.1), we have $R_H$ is self-adjoint if and only if
\[ \begin{pmatrix} T_f & H^\phi \\ H_g & S_\psi \end{pmatrix} = \begin{pmatrix} T_f & H^\phi \\ H_\phi & S_\psi \end{pmatrix} \]
if and only if $T_f = T_f$, $H_g = H_\phi$, and $S_\psi = S_\psi$. $T_f = T_f$ is equivalent to $f$ is real, $H_g = H_\phi$ is equivalent to $g - \phi \in H^2$, $S_\psi = S_\psi$ is equivalent to $\psi$ is real.

(4) If $R_H$ is positive, then
\[ 0 \leq \langle R_H k_z, k_z \rangle = \langle (P_+ f P_+ + P_- g P_+ + P_+ \phi P_- + P_- \psi P_-) k_z, k_z \rangle \]
\[ = \langle (P_+ f P_+ + P_- g P_+ + P_+ \phi P_- + P_- \psi P_-) P_k z, P_k z \rangle \]
\[ = \langle P_+ f P_+ + P_- g P_+ + P_+ \phi P_- + P_- \psi P_- \rangle P_k z, P_k z \rangle \]
\[ = \langle f k_z, k_z \rangle \]
\[ = \int_0^{2\pi} f(e^{i\theta}) |k_z(e^{i\theta})|^2 \frac{d\theta}{2\pi}, \]
where $k_z(\omega) = \sqrt{\frac{1 - |\omega|^2}{1 - 2\omega e^{i\theta} + e^{2i\theta}}}$ is the normalized reproducing kernel of $H^2$. The last equality is the Poisson integral of $f$, so $f$ is positive almost everywhere on $\mathbb{T}$. Similarly,
\[ 0 \leq \langle R(f, g, \phi, \psi) z k_z, z k_z \rangle = \langle (P_+ f P_+ + P_- g P_+ + P_+ \phi P_- + P_- \psi P_-) z k_z, z k_z \rangle \]
\[ = \langle P_- \psi P_- z k_z, P_- z k_z \rangle \]
\[ = \langle \psi k_z, k_z \rangle \]
\[ = \int_0^{2\pi} \psi(e^{i\theta}) |k_z(e^{i\theta})|^2 \frac{d\theta}{2\pi}, \]
so $\psi$ is positive almost everywhere on $\mathbb{T}$. Since positive operator is self-adjoint and (4), $g - \phi \in H^2$.

(5) By the definition of complex symmetric operator [11], we have $R_H$ is complex symmetric with the conjugation $V$ if and only if $VR_H V = R_H^*$. Using the properties of $V$ yields
\[ VR_H V = \begin{pmatrix} T_\phi & H^* \\ H_\phi & T_f \end{pmatrix} \]
(2.3)
On the other hand, \( R_H^* = \begin{pmatrix} T_f & \phi \\ \hat{H}_g & S_\psi \end{pmatrix} \). It follows that \( VRHV = R_H^* \) holds if and only if \( T_f = T_\psi \) and \( S_f = S_\psi \) hold if and only if \( f = \psi \). \( \square \)

3. \( C^*\)-ALGEBRAS \( \mathfrak{R}_{L^\infty} \)

Recall the \( C^*\)-algebra \( \mathfrak{R}_{L^\infty} \) is defined by

\[
\mathfrak{R}_{L^\infty} = \overline{\text{clos} \left\{ \sum_{i=1}^{n} \prod_{j=1}^{m} R_{H_{ij}} \bigg| H_{ij} \in L_{2^{\times 2}}(\mathbb{T}) \right\} } .
\]

Let \( \mathfrak{S}\mathfrak{R}_{L^\infty} \) be the closed ideal of \( \mathfrak{R}_{L^\infty} \) generated by operators of the form

\[
R( f_1, \phi_1 ) R( f_2, \phi_2 ) - R( f_1 f_2, \phi_1, \phi_2 )
\]

where \( f_i, g_i, \phi_i, \psi_i, g, \phi \) are in \( L^\infty(\mathbb{T})(i = 1, 2) \). Furthermore, the \( C^*\)-algebra \( \mathfrak{R}_{L^\infty} \) equals the algebra generated by Riesz projection and all multiplication operators with \( L^\infty(\mathbb{T}) \) symbols, i.e.

\[
\mathfrak{R}_{L^\infty} = \overline{\text{clos} \{ P, M_\phi | \phi \in L^\infty(\mathbb{T}) \} } .
\]

Next, we will establish the symbol map of \( \mathfrak{R}_{L^\infty} \) with the normalized reproducing kernel of \( H^2 \).

**Lemma 3.1.** Let \( H_i = ( f_i, \phi_i ) \in \cap_{p \geq 1} L^p_{2^{\times 2}}(\mathbb{T}), i \in \mathbb{Z}_+ \).

1. The radial limit

\[
\lim_{r \to 1^-} \langle R_{H_1} \cdots R_{H_m}, k_{r\xi}, k_{r\zeta} \rangle = f_1(\xi) \cdots f_m(\xi) \quad \text{a.e. on } \mathbb{T}.
\]

2. The radial limit

\[
\lim_{r \to 1^-} \langle R_{H_1} \cdots R_{H_m}, \bar{z}k_{r\xi}, \bar{z}k_{r\zeta} \rangle = \psi_1(\xi) \cdots \psi_m(\xi) \quad \text{a.e. on } \mathbb{T}.
\]

3. If \( g, \phi \in \cap_{p \geq 1} L^p(\mathbb{T}) \), then \( \prod_{i=1}^{n} R_{H_i} - R( \prod_{i=1}^{n} f_i, \phi ) \in \mathfrak{S}\mathfrak{R}_{L^\infty} \).

4. If \( T \in \mathfrak{S}\mathfrak{R}_{L^\infty} \), then

\[
\lim_{r \to 1^-} \langle Tk_{r\xi}, k_{r\zeta} \rangle = 0, \\
\lim_{r \to 1^-} \langle T\bar{z}k_{r\xi}, \bar{z}k_{r\zeta} \rangle = 0.
\]

5. The uniform limit of GSIO is also a GSIO.

**Proof.** (1) We will prove this lemma by induction on \( m \). For \( m = 1 \), applying (2.2), we obtain

\[
\langle R_{H_1}, k_{r\xi}, k_{r\zeta} \rangle = \int_0^{2\pi} f_1(e^{i\theta})|k_{r\xi}(e^{i\theta})|^2 \frac{d\theta}{2\pi}
\]

where \( |k_{r\xi}|^2 \) is the Poisson kernel for \( r\xi \in \mathbb{D} \). By Fatou’s theorem,

\[
\lim_{r \to 1^-} \langle R_{H_1}, k_{r\xi}, k_{r\zeta} \rangle = f_1(\xi)
\]

for almost all \( \xi \in \mathbb{T} \).
Let $m \geq 2$, assume the result true up to $n - 1$. A simple computation gives
\[
\langle RH_1 \cdots RH_n k_{r\xi}, k_{r\xi} \rangle = \langle RH_2 \cdots RH_n k_{r\xi}, k_{r\xi} \rangle
\]
where $f_{1+} = P_+ f_{1+}, f_{1-} = P_- f_{1-}$. Note that
\[
\langle RH_2 \cdots RH_n k_{r\xi}, P f_{1-} \rangle = \langle f_{1-} P_+ RH_2 \cdots RH_n k_{r\xi}, k_{r\xi} \rangle
\]
and
\[
\|P_- \phi_{1+} k_{r\xi}\| = \|P_-(\phi_{1+} + \phi_{1-})k_{r\xi}\|
\]
By induction hypothesis, the result holds.

(2) Using the properties of $V$, we have
\[
\langle RH_1 \cdots RH_n z k_{r\xi}, z k_{r\xi} \rangle = \langle RH_1 \cdots RH_n V k_{r\xi}, V k_{r\xi} \rangle
\]
By (2.3), we have
\[
VRH_i V = R \left( \frac{\bar{\phi}_i \bar{f}_i}{\phi_i f_i} \right) \quad 1 \leq i \leq m.
\]
Hence Lemma 3.1 (1) implies the result.

(3) For \( k = 2 \), by the definition 4.1, we have

\[
R\left(f_1 \phi_1 \atop g_1 \psi_1\right)R\left(f_2 \phi_2 \atop g_2 \psi_2\right) - R\left(f_1 f_2 \phi \atop g \psi_1 \psi_2\right) \in \mathfrak{A}_{L^\infty}.
\]

Assume the result true up to \( n - 1 \). Observe that

\[
\prod_{i=1}^{n} R\left(f_i \phi_i \atop g_i \psi_i\right) - R\left(\prod_{i=1}^{n} f_i \phi_i \atop g \prod_{i=1}^{n} \psi_i\right) = \prod_{i=1}^{n} R\left(f_i \phi_i \atop g_i \psi_i\right) - R\left(\prod_{i=2}^{n} f_i \phi_i \atop g \prod_{i=2}^{n} \psi_i\right) - R\left(\prod_{i=1}^{n} f_i \phi_i \atop g \prod_{i=1}^{n} \psi_i\right).
\]

By induction hypothesis, the result holds.

(4) Suppose \( g, \phi \in L^\infty \). Linear combinations of operators of the form

\[
R_{H_1}R_{H_2} \cdots R_{H_{n-1}} \left(R_{H_n}R_{H_{n+1}} - R\left(f_n f_{n+1} \phi \atop g \psi_n \psi_{n+1}\right)\right)R_{H_{n+2}} \cdots R_{H_{n+k}}
\]

form a dense subset of \( \mathfrak{A}_{L^\infty} \). Lemma 3.1 (1)(2) gives the result.

(5) If \( R \) is a bounded operator on \( L^2 \) and \( \lim_{n \to \infty} ||R_{H_n} - R|| = 0 \), then

\[
\lim_{n \to \infty} ||P_+(R_{H_n} - R)P_+|| \leq \lim_{n \to \infty} ||R_{H_n} - R|| = 0,
\]

\[
\lim_{n \to \infty} ||P_-(R_{H_n} - R)P_+|| \leq \lim_{n \to \infty} ||R_{H_n} - R|| = 0,
\]

\[
\lim_{n \to \infty} ||P_-(R_{H_n} - R)P_-|| \leq \lim_{n \to \infty} ||R_{H_n} - R|| = 0,
\]

\[
\lim_{n \to \infty} ||P_-(R_{H_n} - R)P_-|| \leq \lim_{n \to \infty} ||R_{H_n} - R|| = 0.
\]

Since

\[
P_+ R_{H_n} P_+ |_{H^2} = T_{\phi_n},
\]

\[
P_- R_{H_n} P_+ |_{H^2} = H_{g_n},
\]

\[
P_+ R_{H_n} P_- |_{\overline{H}^2} = H_{\psi_n},
\]

\[
P_- R_{H_n} P_- |_{\overline{H}^2} = \tilde{T}_{\psi_n},
\]
and
\[
\|T_z P_+ R P_+ T_z - P_+ R P_+\|
\]
\[
= \|T_z P_+ R P_+ T_z - T_z T_f n T_z + T_f n - P_+ R P_+\|
\]
\[
\leq \|T_z P_+ R P_+ T_z - T_z T_f n T_z\| + \|T_f n - P_+ R P_+\|
\]
\[
\leq \|T_z (P_+ R P_+ - T_f n) T_z\| + \|T_f n - P_+ R P_+\|
\]
\[
\leq \|T_z\| \|P_+ R P_+ - T_f n\| \|T_z\| + \|T_f n - P_+ R P_+\| \to 0 \quad (n \to \infty).
\]
it follows that $T_z P_+ R P_+ T_z = P_+ R P_+$. We have $P_+ R P_+|_{H^2}$ is a Toeplitz operator, because an operator $T$ is a Toeplitz operator if and only if $T z T z = T$ [5, Theorem 6]. Moreover,
\[
\|P_+ R P_+ T_z - S_z P_+ R P_+\|
\]
\[
= \|P_+ R P_+ T_z - P_- R H a P_+ T_z + S_z P_- R H a P_+ - S_z P_+ R P_+\|
\]
\[
\leq \|P_+ R P_+ T_z - P_- R H a P_+ T_z\| + \|S_z P_- R H a P_+ - S_z P_+ R P_+\|
\]
\[
\leq \|P_+ R P_+ - P_- R H a P_+\| \|T_z\| + \|S_z\| \|P_- R H a P_+ - P_+ R P_+\|
\]
shows that $P_+ R P_+ T_z = S_z P_+ R P_+$. Since an operator $H$ is a Hankel operator if and only if $H T z = T z H$ [26, Theorem 1.8], we have $P_- R H a P_+|_{H^2}$ is a Hankel operator. Similarly, $P_+ R P_+|_{H^2}$ is the adjoint of a Hankel operator. By $V T \psi V = T \psi$, then $P_- R H a P_-|_{H^2}$ is a dual Toeplitz operator. Hence $\tilde{R}$ is a GSIO. \hfill \Box

**Theorem 3.2.** The sequence
\[
0 \longrightarrow \mathfrak{SA}_{L^\infty} \longrightarrow \mathfrak{R}_{L^\infty} \longrightarrow L^\infty_2 (\mathbb{T}) \longrightarrow 0
\]
is a short exact sequence; that is, the quotient algebra $\mathfrak{R}_{L^\infty} / \mathfrak{SA}_{L^\infty}$ is *-isometrically isomorphic to $L^\infty \oplus L^\infty$.

**Proof.** Linear combinations of operators of the form $\prod_{j=1}^m R(\hat{f_i} \hat{\phi_i})$ span a dense subset of $\mathfrak{R}_{L^\infty}$, compute
\[
\prod_{j=1}^m R(\hat{f_i} \hat{\phi_i}) = R\left(\frac{\Pi_{i=1}^n f_i}{0} \frac{0}{\Pi_{i=1}^n \psi_i}\right) + \prod_{j=1}^m R(\hat{f_i} \hat{\phi_i}) - R\left(\frac{\Pi_{i=1}^n f_i}{0} \frac{0}{\Pi_{i=1}^n \psi_i}\right) \in \mathfrak{SA}_{L^\infty} \quad (\text{By Lemma 3.1}(3))
\]
This shows that operators of the form
\[
T = R\left(\begin{array}{c} f \\ 0 \end{array}\right) + E_0, \quad f, \psi \in L^\infty, E_0 \in \mathfrak{SA}_{L^\infty}
\]
form a dense subset of $\mathfrak{R}_{L^\infty}$. Therefore, for every operator $T$ in $\mathfrak{R}_{L^\infty}$, there exists a sequence of operators
\[
T_n = R\left(\begin{array}{c} f_n \\ 0 \end{array}\right) + E_n, \quad E_n \in \mathfrak{SA}_{L^\infty}
\]
such that $\lim_{n \to \infty} \|T_n - T\| = 0$. By Lemma 3.1(1) and (4), we have
\[
f_n (\xi) = \lim_{r \to 1^-} \langle T_n k_{r \xi}, k_{r \xi}\rangle.
\]
and
\[ |f_n(\xi) - f_m(\xi)| \leq \|T_n - T_m\|. \] (3.2)
So \( \{f_n(\xi)\} \) is a Cauchy sequence. Define
\[ f(\xi) \triangleq \lim_{n \to \infty} f_n(\xi). \]
we then have
\[ \left| \lim_{r \to 1-} \langle Tk^r \xi, k^r \xi \rangle - f(\xi) \right| \]
\[ = \left| \lim_{r \to 1-} \langle Tk^r \xi, k^r \xi \rangle - \lim_{r \to 1-} \langle Tn k^r \xi, k^r \xi \rangle + \lim_{r \to 1-} \langle Tn k^r \xi, k^r \xi \rangle - f_n(\xi) + f_n(\xi) - f(\xi) \right| \]
\[ \leq \left| \lim_{r \to 1-} \langle Tk^r \xi, k^r \xi \rangle - \lim_{r \to 1-} \langle Tn k^r \xi, k^r \xi \rangle \right| + | f_n(\xi) - f(\xi) | \]
\[ \leq \|T - T_n\| + | f_n(\xi) - f(\xi) | \]
and it follows that
\[ \lim_{r \to 1-} \langle Tk^r \xi, k^r \xi \rangle = f(\xi). \]
Similarly, define
\[ \psi(\xi) \triangleq \lim_{n \to \infty} \psi_n(\xi), \]
we have
\[ \lim_{r \to 1-} \langle T\tilde{k}^r \xi, \tilde{k}^r \xi \rangle = \psi(\xi). \]
Using (3.2), \( \lim_{n \to \infty} \|f_n - f\|_{\infty} = 0. \) Similarly, \( \lim_{n \to \infty} \|\psi_n - \psi\|_{\infty} = 0. \) Thus
\[ \|R(f_n f - 0_{\psi}) \| \leq \|f_n - f\| + \|\psi_n - \psi\| \to 0 \quad (n \to \infty). \]
Let \( E = T - R(f_0 0_{\psi}), \) we have \( \lim_{n \to \infty} \|E_n - E\| = 0, \) since \( \mathfrak{S}R_{L^\infty} \) is closed, \( E \in \mathfrak{S}R_{L^\infty}. \) It follows that \( T \) have the following form
\[ T = R(f 0_{\psi}) + E, \quad f, \psi \in L^\infty(\mathbb{T}), E \in \mathfrak{S}R_{L^\infty}. \]
Define the map \( \rho : \mathfrak{R}_{L^\infty} \to L_2^\infty(\mathbb{T}) \) by
\[ \rho(T)(\xi) = \left( \lim_{r \to 1-} \langle Tk^r \xi, k^r \xi \rangle, \lim_{r \to 1-} \langle T\tilde{k}^r \xi, \tilde{k}^r \xi \rangle \right). \] (3.3)
Recall the norm of \( L_2^\infty(\mathbb{T}), \| (a, b) \| = \max \{ \|a\|_\infty, \|b\|_\infty \}. \) Clearly, \( \|\rho(T)\| \leq \|T\|. \) The map \( \rho \) is linear, contractive, and preserves conjugation. Moreover,
\[ \rho(T) = (f, \psi). \]
If \( A_1, A_2 \in \mathfrak{R}_{L^\infty} \), and
\[ A_1 = R(f_1 0_{\psi_1}) + E_1, \quad A_2 = R(f_2 0_{\psi_2}) + E_2, \quad E_1, E_2 \in \mathfrak{S}R_{L^\infty}, \]
then
\[ A_1 A_2 = R(f_1 0_{\psi_1}) R(f_2 0_{\psi_2}) + \underbrace{R(f_1 0_{\psi_1}) E_2 + E_1 R(f_2 0_{\psi_2}) + E_1 E_2}_{\in \mathfrak{S}R_{L^\infty}}. \]
Using Lemma 3.1(1) and (4), we have
\[
\lim_{r \to 1^{-}} \langle A_1 A_2 \tilde{k}_r \xi, k_r \xi \rangle = \lim_{r \to 1^{-}} \langle R \left( f_1, 0 \psi \right) R \left( f_2, 0 \psi \right) k_r \xi, k_r \xi \rangle
\]
\[
= \lim_{r \to 1^{-}} \langle R \left( f_1, 0 \psi \right) R \left( f_2, 0 \psi \right) k_r \xi, k_r \xi \rangle
\]
\[
= f_1(\xi) \cdot f_2(\xi)
\]
\[
= \lim_{r \to 1^{-}} \langle R \left( f_1, 0 \psi \right) k_r \xi, k_r \xi \rangle \cdot \lim_{r \to 1^{-}} \langle R \left( f_2, 0 \psi \right) k_r \xi, k_r \xi \rangle
\]
\[
= \lim_{r \to 1^{-}} \langle A_1 k_r \xi, k_r \xi \rangle \cdot \lim_{r \to 1^{-}} \langle A_2 k_r \xi, k_r \xi \rangle \quad \text{a.e. on } \mathbb{T}.
\]
Similarly,
\[
\lim_{r \to 1^{-}} \langle A_1 A_2 \tilde{z} \tilde{k}_r \xi, \tilde{z} k_r \xi \rangle = \lim_{r \to 1^{-}} \langle A_1 \tilde{z} \tilde{k}_r \xi, \tilde{z} \tilde{k}_r \xi \rangle \cdot \lim_{r \to 1^{-}} \langle A_2 \tilde{z} \tilde{k}_r \xi, \tilde{z} \tilde{k}_r \xi \rangle \quad \text{a.e. on } \mathbb{T}.
\]
Since the algebraic operations of \( L^\infty_2(\mathbb{T}) \) are all performed coordinated-wise, we have \( \tilde{\rho} \) is multiplicative.

By Lemma 3.1(4), we have \( \mathcal{SR}_{L^\infty} \subseteq \ker \rho \). For every \( T = R(f, 0, 0, \psi) + E \in \ker \rho \), thus, \( f = \psi = 0 \). Hence \( \mathcal{SR}_{L^\infty} = \ker \rho \).

We define the map
\[
\tilde{\rho} : \mathcal{R}_{L^\infty} / \mathcal{SR}_{L^\infty} \longrightarrow L^\infty_2(\mathbb{T}),
\]
\[
R \left( f, 0 \psi \right) + \mathcal{SR}_{L^\infty} \longmapsto (f, \psi).
\]
Hence, \( \tilde{\rho} \) is a \( C^* \)-isomorphism. \[ \square \]

**Corollary 3.3.** If \( T \in \mathcal{R}_{L^\infty} \), then \( \rho(T^* T - TT^*) = (0, 0) \).

**Example 3.4.** In fact, \( \mathcal{R}_{L^\infty} \) is a proper subalgebra of \( B(L^2(\mathbb{T})) \). We make some modification to [10, Example 4]. Let \( T \) be the operator defined by
\[
T z^n = z^{2n+1}, \quad n \in \mathbb{Z}.
\]
Note that
\[
T^* z^n = \begin{cases} 
\frac{z^{n-1}}{2}, & \text{if } n \text{ is odd;} \\
0, & \text{if } n \text{ is even.}
\end{cases}
\]
and
\[
(T^* T - TT^*) z^n = \begin{cases} 
0, & \text{if } n \text{ is odd;} \\
z^n, & \text{if } n \text{ is even.}
\end{cases}
\]
Hence \( T^* T - TT^* \) is the orthogonal projectiton onto \( \text{span}\{z^{2n}\}_{n \in \mathbb{Z}} \).

\[
\langle (T^* T - TT^*) k_r \xi, k_r \xi \rangle = (1 - r^2) \langle (T^* T - TT^*) \sum_{i=0}^{\infty} (r \xi)^i z^i, \sum_{j=0}^{\infty} (r \xi)^j z^j \rangle
\]
\[
= (1 - r^2) \langle \sum_{n=0}^{\infty} (r \xi)^{2n} z^{2n}, \sum_{m=0}^{\infty} (r \xi)^{2m} z^{2m} \rangle
\]
\[
= \langle k(r \xi)^2, k(r \xi)^2 \rangle
\]
\[
= \frac{1 - r^2}{1 + r^2} \longrightarrow \frac{1}{2} (r \to 1^-).
\]
By Corollary 3.3, we have \( T \notin \mathcal{R}_{L^\infty} \).

4. \( C^* \)-algebras \( \mathcal{R}_{C(T)} \) and \( \mathcal{R}_{QC} \)

Let \( C(T) \) denote the set of continuous complex-valued functions on \( T \), and \( C(T) \) is a closed subalgebra of \( L^\infty \). The set of all compact operators on \( L^2(T) \) is denoted by \( \mathcal{K}(L^2(T)) \).

**Lemma 4.1.** The \( C^* \)-algebra \( \mathcal{R}_{C(T)} \) is irreducible. Furthermore, \( LC(L^2(T)) \subset \mathcal{R}_{C(T)} \).

**Proof.** If \( \mathcal{R}_{C(T)} \) is reducible, then there exists a nontrivial orthogonal projection \( Q \) which commutes with each element of \( \mathcal{R}_{C(T)} \). In particular, \( Q R(z \bar{z}) = R(z \bar{z}) R(z \bar{z}) \) and \( R(z \bar{z}) \) is the bilateral shift. Since the commutant of the bilateral shift is the set of all multiplications\([18, 146]\), it follows that \( Q = M_{\chi_\Delta} \), where \( \chi_\Delta \) is a characteristic function. Note that

\[
R(z \bar{z}) = QR(z \bar{z}),
\]

\[
\begin{pmatrix}
T_z & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
T_{\chi_\Delta} & H^*_\Delta \\
H_{\chi_\Delta} & T_{\chi_\Delta}
\end{pmatrix} = \begin{pmatrix}
T_z & 0 \\
0 & 0
\end{pmatrix} \begin{pmatrix}
T_{\chi_\Delta} & H^*_\Delta \\
H_{\chi_\Delta} & T_{\chi_\Delta}
\end{pmatrix},
\]

implies

\[
T_z T_{\chi_\Delta} = T_{\chi_\Delta} T_z.
\]

Since the commutant of \( T_z \) is the set of all analytic Toeplitz operators on \( H^2 [18, 147] \), it follows that \( \chi_\Delta \) is 0 or 1, and \( Q = I \) or \( Q = 0 \). This contradicts our assumption. Therefore \( \mathcal{R}_{C(T)} \) is irreducible.

Applying the formula \( I - T_z T_z = 1 \otimes 1 \) yields

\[
R(\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}) - R(\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}) = 1 \otimes 1,
\]

where \( 1 \otimes 1 \) is an operator of rank 1, thus \( LC(L^2(T)) \cap \mathcal{R}_{C(T)} \neq \{0\} \). By \([9, 539]\), we have \( LC(L^2(T)) \subset \mathcal{R}_{C(T)} \). \( \square \)

The algebra \( QC \triangleq (H^\infty + C(T)) \cap (\overline{H^\infty + C(T)}) \) is a closed subalgebra of \( L^\infty(T) \) which properly contains \( C(T) \). Let \( \mathfrak{S}R_{QC} \) (resp. \( \mathfrak{S}R_{C(T)} \)) be the closed ideal of \( \mathcal{R}_{QC} \) (resp. \( \mathcal{R}_{C(T)} \)) generated by operators of the form

\[
R(\begin{pmatrix}
f_1 & \psi_1 \\
g_1 & \phi_1
\end{pmatrix}) R(\begin{pmatrix}
f_2 & \psi_2 \\
g_2 & \phi_2
\end{pmatrix}) - R(\begin{pmatrix}
f_1 f_2 & \phi_1 \phi_2 \\
g_1 g_2 & \psi_1 \psi_2
\end{pmatrix})\]

where \( f_i, g_i, \phi_i, \psi_i, g, \phi \) are in \( QC \) (resp. \( C(T) \)) \((i = 1, 2)\).

**Lemma 4.2.** \( \mathfrak{S}R_{C(T)} = \mathcal{K}(L^2(T)) \), and \( \mathfrak{S}R_{QC} = \mathcal{K}(L^2(T)) \).
Proof. If \( f_i, g_i, \phi_i, \psi_i, g, \phi \in C(\mathbb{T}) \) (resp. \( QC \)) (\( i=1,2 \)), an easy computation shows that
\[
R_{(f_1 \phi_{1})_{g_1 \psi_{1}}} R_{(f_2 \phi_{2})_{g_2 \psi_{2}}} - R_{(f_1 f_2 \phi)_{g \psi_{1} \psi_{2}}}
= (T_{f_1 f_2} + H_{\phi_1}^* H_{g_2} - T_{f_1 f_2} T_{f_1} H_{\phi_2}^* + H_{\phi_1}^* \tilde{T}_{\psi_2} - H_{\phi}^*)
= (H_{\phi_1}^* H_{g_2} - H_{\phi_1}^* H_{f_2} T_{f_1} H_{\phi_2}^* + H_{\phi_1}^* \tilde{T}_{\psi_2} - H_{\phi}^*)
\]

The second equality follows form the formulas \( T_{ab} - T_a T_b = H_a^* H_b \) and \( \tilde{T}_{ab} - \tilde{T}_a \tilde{T}_b = H_a H_b^* \). Since the Hankel operator \( H_\phi \) is compact if and only if \( \phi \in H^\infty + C(\mathbb{T}) \) by [26, p.27], it follows that
\[
R_{(f_1 \phi_{1})_{g_1 \psi_{1}}} R_{(f_2 \phi_{2})_{g_2 \psi_{2}}} - R_{(f_1 f_2 \phi)_{g \psi_{1} \psi_{2}}}
\]
is compact, and \( \mathfrak{SR}_{C(\mathbb{T})} \subset \mathcal{K}(L^2(\mathbb{T})) \) (resp. \( \mathfrak{SR}_{QC} \subset \mathcal{K}(L^2(\mathbb{T})) \)). On the other hand, \( LC(L^2(\mathbb{T})) \) contains no proper closed ideal. Hence, \( \mathfrak{SR}_{C(\mathbb{T})} = LC(L^2(\mathbb{T})) \) (resp. \( \mathfrak{SR}_{QC} = \mathcal{K}(L^2(\mathbb{T})) \)). \( \square \)

**Corollary 4.3.** For every \( T \in \mathfrak{R}_{L^\infty} \), we have
\[
\|\rho(T)\| \leq \|T\|_e.
\]
In particular, if \( H = \left(\begin{array}{c} f \\ \phi \\ g \end{array}\right) \), then
\[
\max\{\|f\|_\infty, \|\psi\|_\infty\} \leq \|R_H\|_e.
\]

**Proof.** If \( T \in \mathfrak{R}_{L^\infty} \), by Theorem 3.2, we have
\[
\inf_{A \in \mathfrak{SR}_{L^\infty}} \|T + A\| = \|\rho(T)\|.
\]
On the other hand, \( \mathcal{K} \subset \mathfrak{SR}_{L^\infty} \) by Lemma 4.2. Therefore,
\[
\inf_{A \in \mathfrak{SR}_{L^\infty}} \|T + A\| \leq \inf_{K \in \mathcal{K}(L^2(\mathbb{T}))} \|T + K\| = \|T\|_e.
\]
Use Theorem 3.2 again,
\[
\|\rho(R_H)\| = \max\{\|f\|_\infty, \|\psi\|_\infty\}.
\]
\( \square \)

If \( T \) is a bounded linear operator on Hilbert space \( H \), \( \sigma_\varepsilon(T) \) denotes the essential spectrum of \( T \). For \( \phi \in L^\infty \), \( Ran_{ess}\phi \) denotes the essential range of \( \phi \). If \( E \) is a subset of complex plane \( \mathbb{C} \), the convex hull of \( E \) will be denoted by \( coE \). Combining Theorem 3.2 and Lemma 4.2, we get the following result.

**Corollary 4.4.** There exists a *-homomorphism \( \zeta \) from the quotient algebra \( \mathfrak{R}_{L^\infty}/\mathcal{K} \) onto \( L^\infty_2(\mathbb{T}) \) such that the diagram
commutes. Moreover,

1. For every $T \in \mathcal{R}_L$, if $T$ is Fredholm, then $\rho(T)$ is invertible in $L^\infty_2(\mathbb{T})$;
2. $\text{Ran}_{\text{ess}} f \cup \text{Ran}_{\text{ess}} \psi \subset \sigma_e(R_H)$.

Recall the spectral inclusion theorem of Toeplitz operator\[9],

\[ \text{Ran}_{\text{ess}} f \subset \sigma_e(T_f) \subset \sigma(T_f) \subset \text{coRan}_{\text{ess}} f. \tag{4.2} \]

Corollary 4.4 give the first inclusion similar to (4.2), the next theorem will show the third inclusion similar to (4.2).

**Proposition 4.5.** Let $H = (f_1 \begin{pmatrix} \phi \\ g \end{pmatrix}, f_2) \in L^\infty_2(\mathbb{T})$. If we define

\[ G_i = \text{coRan}_{\text{ess}} f_i \cup \{ \lambda \notin \text{Ran}_{\text{ess}} f_i : d_i(\lambda) \leq \delta\| (f_i - \lambda)^{-1} \|_\infty \} \]

where

\[ d_i(\lambda) = (1 - \text{dist}((f_i - \lambda)/|f_i - \lambda|, H^\infty)^2)^{1/2}, \quad \delta = \min\{ \text{dist}(\phi, H^\infty), \text{dist}(g, H^\infty) \} \]

for $n = 1, 2$, then

\[ \sigma(R_H) \subset G_1 \cup G_2. \]

**Proof.** Suppose $\lambda \in \rho(T_{f_1}) \cap \rho(T_{f_2})$, we have

\[
R_H - \lambda I_{L^2} = \begin{pmatrix} T_{f_1 - \lambda} & H^*_\phi \\ H_g & \tilde{T}_{f_2 - \lambda} \end{pmatrix}
\]

\[
= \begin{pmatrix} T_{f_1 - \lambda} & 0 \\ 0 & \tilde{T}_{f_2 - \lambda} \end{pmatrix} + \begin{pmatrix} 0 & H^*_\phi \\ H_g & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} T_{f_1 - \lambda} & 0 \\ 0 & \tilde{T}_{f_2 - \lambda} \end{pmatrix} \begin{pmatrix} I_{L^2} & \begin{pmatrix} 0 & T_{f_1 - \lambda} H^*_\phi \\ T_{f_2 - \lambda} H_g & 0 \end{pmatrix} \end{pmatrix}
\]

If $\|H_\phi\| < \|T_{f_1 - \lambda}^{-1}\|$ and $\|H_g\| < \|\tilde{T}_{f_2 - \lambda}^{-1}\|^{-1}$, then

\[
\left\| \begin{pmatrix} 0 & T_{f_1 - \lambda} H^*_\phi \\ \tilde{T}_{f_2 - \lambda} H_g & 0 \end{pmatrix} \right\| = \max\{\|T_{f_1 - \lambda} H^*_\phi\|, \|\tilde{T}_{f_2 - \lambda} H_g\|\} < 1,
\]

and $\lambda \in \rho(R_H)$. This mean that

\[ \{ \lambda \in \rho(T_{f_1}) : \|H_\phi\| < \|T_{f_1 - \lambda}^{-1}\|^{-1} \} \cap \{ \lambda \in \rho(T_{f_2}) : \|H_g\| < \|\tilde{T}_{f_2 - \lambda}^{-1}\|^{-1} \} \subset \rho(R_H) \]

or

\[ \sigma(R_H) \subset \sigma(T_{f_1}) \cup \{ \lambda \in \rho(T_{f_1}) : \|T_{f_1 - \lambda}^{-1}\|^{-1} \leq \|H_\phi\| \}
\]

\[ \cup \sigma(T_{f_2}) \cup \{ \lambda \in \rho(T_{f_2}) : \|\tilde{T}_{f_2 - \lambda}^{-1}\|^{-1} \leq \|H_g\| \}. \tag{4.3} \]
Repeat the above reasoning for $R^*_H$, we have
\[\sigma(R^*_H) \subset \sigma(T_{f_1}) \cup \{\lambda \in \rho(T_{f_1}) : \|T_{f_1-\lambda}^{-1}\| \leq \|H_g\|\}\]
\[\cup \sigma(\tilde{T}_{f_2}) \cup \{\lambda \in \rho(\tilde{T}_{f_2}) : \|\tilde{T}_{f_2-\lambda}^{-1}\| \leq \|H_{\tilde{\phi}}\}\}.
\]
Taking conjugates, we get
\[\sigma(R_H) \subset \sigma(T_{\tilde{f}_1}) \cup \{\bar{\lambda} \in \rho(T_{\tilde{f}_1}) : \|T_{\tilde{f}_1-\bar{\lambda}}^{-1}\| \leq \|H_{\tilde{g}}\|\}\]
\[\cup \sigma(\tilde{T}_{f_2}) \cup \{\bar{\lambda} \in \rho(\tilde{T}_{f_2}) : \|\tilde{T}_{f_2-\bar{\lambda}}^{-1}\| \leq \|H_{\tilde{\phi}}\}\}.
\]
Since $\|T_{f_1-\bar{\lambda}}^{-1}\| = \|(T_{f_1-\bar{\lambda}})^{-1}\|$ and $\|	ilde{T}_{f_2-\bar{\lambda}}^{-1}\| = \|(\tilde{T}_{f_2-\bar{\lambda}})^{-1}\|$, it follows that
\[\sigma(R_H) \subset \sigma(T_{f_1}) \cup \{\lambda \in \rho(T_{f_1}) : \|T_{f_1-\lambda}^{-1}\| \leq \|H_g\|\}\]
\[\cup \sigma(\tilde{T}_{f_2}) \cup \{\lambda \in \rho(\tilde{T}_{f_2}) : \|\tilde{T}_{f_2-\lambda}^{-1}\| \leq \|H_{\tilde{\phi}}\}\}. \tag{4.4}
\]
According to the norm of Hankel operator ([26, Theorem 1.4]), we have $\|H_{\tilde{\phi}}\| = dist(\tilde{\phi}, H^\infty)$ and $\|H_g\| = dist(g, H^\infty)$. Let $\delta = \min\{dist(\tilde{\phi}, H^\infty), dist(g, H^\infty)\}$. We combine (4.3) and (4.4). Thus
\[\sigma(R_H) \subset \sigma(T_{f_1}) \cup \{\lambda \in \rho(T_{f_1}) : \|T_{f_1-\lambda}^{-1}\| \leq \delta\}
\[\cap \sigma(\tilde{T}_{f_2}) \cap \{\lambda \in \rho(\tilde{T}_{f_2}) : \|\tilde{T}_{f_2-\lambda}^{-1}\| \leq \delta\}\}.
\]
Since $\tilde{T}_{f_2}$ and $T_{\tilde{f}_2}$ are anti-unitary, $\sigma(\tilde{T}_{f_2}) = \sigma(T_{f_2})$ and $\|\tilde{T}_{f_2-\lambda}\| = \|T_{f_2-\lambda}\|. Using the (4.2) and norm estimation of the inverse of Toeplitz operator[25, page.125.]
\[\frac{(1 - \text{dist}(\varphi/|\varphi|, H^\infty)^2)^{1/2}}{|\varphi^{-1}|} \leq \|T_{\varphi^{-1}}^{-1}\|^{-1},\]
we have
\[\{\lambda \in \rho(T_{f_1}) : \|T_{f_1-\lambda}^{-1}\| \leq \delta\} \subset \{\lambda \notin \text{Ran}_{ess} f_1 : d_1(\lambda) \leq \delta\|(f_1 - \lambda)^{-1}\|_{\infty}\},\]
\[\{\lambda \in \rho(T_{f_2}) : \|T_{f_2-\lambda}^{-1}\| \leq \delta\} \subset \{\lambda \notin \text{Ran}_{ess} f_2 : d_2(\lambda) \leq \delta\|(f_2 - \lambda)^{-1}\|_{\infty}\},\]
\[\sigma(T_{f_1}) \subset \text{coRan}_{ess} f_1,\]
\[\text{and } \sigma(T_{f_2}) \subset \text{coRan}_{ess} f_2,\]
where $d_i(\lambda) = (1 - \text{dist}((f_i - \lambda)/|f_i - \lambda|, H^\infty)^2)^{1/2}, i = 1, 2. \square$

**Theorem 4.6.** The sequence
\[0 \to \mathcal{K}(L^2(T)) \to \mathcal{R}_{C(T)} \to C_2(T) \to 0\]
is a short exact sequence; that is, the quotient algebra $\mathcal{R}_{C(T)}/\mathcal{K}$ is $*$-isometrically isomorphic to $C_2(T)$.

**Proof.** Using the proof of Theorem 3.2 and Lemma 4.2, for every operator $T \in \mathcal{R}_{C(T)}$ have the following form
\[T = R(f \circ \psi) + K, \quad f, \psi \in C(T), K \in \mathcal{K}. \tag{4.5}\]
The map $\bar{\rho}$ defined in (3.4) is $*$-isometrically isomorphic from $\mathcal{R}_{C(T)}/\mathcal{K}(L^2(T))$ to $C_2(T)$. \square
Remark 4.7. In fact, the previous theorem can be extend to the algebra $QC$. The sequence

$$0 \to \mathcal{K}(L^2(\mathbb{T})) \to R_{QC} \to QC_2 \to 0$$

is a short exact sequence. The proof is similar in spirit to Theorem 4.6.

**Corollary 4.8.** For every $T \in R_{QC}$, we have

$$\|\rho(T)\| = \|T\|_e.$$ 

In particular, if $H = \begin{pmatrix} f & \varphi \\ g & f \end{pmatrix} \in QC_{2 \times 2}$, then

$$\max\{\|f\|_\infty, \|\varphi\|_\infty\} = \|R_H\|_e.$$ 

**Corollary 4.9.** If $H = \begin{pmatrix} f & \varphi \\ g & f \end{pmatrix} \in QC_{2 \times 2}$, then $\sigma_e(R_H) = \text{Ran}_{\text{ess}} f \cup \text{Ran}_{\text{ess}} \psi$. Moreover, $R_H$ is Fredholm if and only if $f$ and $\psi$ are invertible in $QC$.

**Remark 4.10.** If $H = \begin{pmatrix} f & \varphi \\ g & f \end{pmatrix} \in C(T)_{2 \times 2}$, then $\sigma_e(R_H) = f(T) \cup \psi(T)$.

**Definition 4.11.** Let $f$ is an invertible function in $C(T)$, the winding number of $f$ about the origin is defined by

$$\sharp(f) = \frac{1}{2\pi i} \int_{f(T)} \frac{dz}{z}.$$ 

**Definition 4.12.** Let $T$ be a bounded linear operator on Hilbert space $H$, a bounded linear operator $B$ on $H$ is called the regularizer of $T$ if $BT - I$ and $TB - I$ are compact. If $T$ is Fredholm, the difference $\text{ind}(T) = \dim \ker T - \dim \ker T^*$ is call the index of $T$.

**Corollary 4.13.** If $T$ is Fredholm operator in $R_{C(T)}$, then

1. $R\left(\begin{pmatrix} f_0^{-1} & 0 \\ 0 & \psi_0^{-1} \end{pmatrix}\right)$ is a regularizer of $T$;
2. $\text{ind}(T) = \sharp(\psi_0) - \sharp(f_0)$, where $f_0(\xi) = \lim_{r \to 1^-} \langle Tk_r \xi, k_r \xi \rangle$, $\psi_0(\xi) = \lim_{r \to 1^-} \langle T \bar{k}_r \xi, \bar{k}_r \xi \rangle$.

In particular, if $H = \begin{pmatrix} f & \varphi \\ g & f \end{pmatrix} \in C(T)_{2 \times 2}$ and $R_H$ is a Fredholm operator, then

$$\text{ind}(R_H) = \sharp(\psi) - \sharp(f).$$ 

**Proof.** If $T \in R_{C(T)}$, by the formula (4.5), we have

$$T = R\left(\begin{pmatrix} f_0 & 0 \\ 0 & \psi_0 \end{pmatrix}\right) + K, \quad f_0, \psi_0 \in C(T), \quad K \in \mathcal{K}(L^2(\mathbb{T})).$$
where \( f_0(\xi) = \lim_{r \to 1^-} \langle Tk_r \xi, k_r \xi \rangle, \psi_0(\xi) = \lim_{r \to 1^-} \langle T \bar{k}_r \xi, \bar{k}_r \xi \rangle \), and hence \( f \) and \( \psi \) are invertible in \( C(\mathbb{T}) \) by the remark 4.10. A calculation shows that

\[
R\left( f_{0^{-1}} \circ \psi_0^{-1} \right) T = R\left( f_{0^{-1}} \circ \psi_0^{-1} \right) R\left( f_0 \circ \psi_0 \right) + R\left( f_0 \circ \psi_0 \right) K
\]

\[
= \left( \begin{array}{cc} T_{f_0^{-1}} & 0 \\ 0 & \tilde{T}_{\psi_0^{-1}} \end{array} \right) \left( \begin{array}{cc} T_{f_0} & 0 \\ 0 & \tilde{T}_{\psi_0} \end{array} \right) + R\left( f_0 \circ \psi_0 \right) K
\]

\[
= \left( I - H^2 - H_{f_0^{-1}Hf_0} \right) \left( \begin{array}{cc} 0 & 0 \\ 0 & \tilde{T}_{\psi_0^{-1}}-H_{\psi_0^{-1}H^*_{\psi_0}} \end{array} \right) + R\left( f_0 \circ \psi_0 \right) K
\]

\[
= I + \left( -H_{f_0^{-1}Hf_0} \right) \left( \begin{array}{cc} 0 & 0 \\ 0 & H_{\psi_0^{-1}H^*_{\psi_0}} \end{array} \right) + R\left( f_0 \circ \psi_0 \right) K.
\]

Since the Hankel operator \( H_\phi \) is compact if and only if \( \phi \in H^\infty + C(\mathbb{T}) \) by [26, p.27], we have \( H_{f_0^{-1}Hf_0} \) and \( H_{\psi_0^{-1}H^*_{\psi_0}} \) are compact, so \( R\left( f_{0^{-1}} \circ \psi_0^{-1} \right) T - I \) is compact, similarly, \( TR\left( f_{0^{-1}} \circ \psi_0^{-1} \right) - I \) is compact.

Since the Fredholm index is stable under compact operator perturbations [2, p.98], it follows that

\[
\text{ind}(T) = \text{ind}(R\left( f_0 \circ \psi_0 \right) + K)
\]

\[
= \text{ind}R\left( f_0 \circ \psi_0 \right)
\]

\[
= \text{ind}\left( T_{f_0} \circ \psi_0 \right)
\]

\[
= \text{ind} T_{f_0} + \text{ind} \tilde{T}_{\psi_0}.
\]

Note that

\[
\text{ind} \tilde{T}_{\psi_0} = \dim \ker(\tilde{T}_{\psi_0}) - \dim \ker(\tilde{T}_{\psi_0}^*)
\]

\[
= \dim \ker(VT^*_{\psi_0} V) - \dim \ker(VT_{\psi_0} V)
\]

\[
= \dim \ker(T^*_{\psi_0}) - \dim \ker(T_{\psi_0})
\]

\[
= -\text{ind} T_{\psi_0}.
\]

By the theorem [9, 7, 26], we have \( \text{ind} T_{f_0} = -\sharp(f_0) \) and \( \text{ind} \tilde{T}_{\psi_0} = \sharp(\psi_0) \). Therefore, \( \text{ind}(T) = \sharp(\psi) - \sharp(f_0) \).

**Corollary 4.14.** If \( H = \left( \begin{array}{cc} f & \phi \\ g & \psi \end{array} \right) \in C(\mathbb{T})_{2 \times 2} \), then \( R_H \) is invertible if and only if the following conditions hold:

(1) \( f \) and \( \phi \) are invertible,

(2) \( \sharp(\psi) = \sharp(f) \), and
(3) either \( \ker(R_H) = \{0\} \) or \( \ker(R_H^*) = \{0\} \).

**Proof.** By Corollary 4.4, we have \( \text{Ran}_{\text{ess}} f \cup \text{Ran}_{\text{ess}} \psi \subset \sigma(R_H) \). Suppose that \( R_H \) is invertible, then

(a) \( f \) and \( \phi \) are invertible;

(b) \( \ker(R_H) = \{0\} \) and \( \ker(R_H^*) = \{0\} \).

It follows that \( \text{ind}(R_H) = 0 \). By Corollary 4.13, we have \( \sharp(\psi) = \sharp(f) \);

On the other hand, if \( R_H \) is Fredholm, then \( R_H \) is invertible if and only if

(i) \( \text{ind}(R_H) = 0 \);

(ii) either \( \ker(R_H) = \{0\} \) or \( \ker(R_H^*) = \{0\} \).

By Remark 4.10, \( R_H \) is Fredholm if and only if \( f \) and \( \psi \) are invertible, hence the result follows. \( \square \)

**Remark 4.15.** There exist some examples showing that both of \( \ker(R_H) \) and \( \ker(R_H^*) \) are nontrivial. For example, if \( u \) and \( \theta \) are nonconstant inner functions, then

\[
\overline{z}(H^2 \ominus \theta H^2) \subseteq \ker R_{\begin{pmatrix} u & 0 \\ 0 & \theta \end{pmatrix}}
\]

and

\[
H^2 \ominus u H^2 \subseteq \ker R_{\begin{pmatrix} u & 0 \\ 0 & \theta \end{pmatrix}}^*.
\]

Let \( \Delta \) is a proper subset of \( \mathbb{T} \) and has positive measure, \( \chi_\Delta \) is the characteristic function of \( \Delta \), we have \( R_{\begin{pmatrix} \chi_\Delta & \chi_\Delta \\ \chi_\Delta & \chi_\Delta \end{pmatrix}} = M_{\chi_\Delta} \) and \( \dim \ker M_{\chi_\Delta} = \dim \ker M_{\chi_\Delta}^* = \infty. \)

## 5. Invertible and Fredholm of GISO

In this section, we found that GSIOs and singular integral operators with \( 2 \times 2 \) matrix symbol are equivalent after extension.

**Definition 5.1.** [3] Let \( T \) and \( S \) are bounded operator on Hilbert space \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) respectively. The operators \( T \) and \( S \) are called equivalent after extension, written \( T \sim S \), if there exist Hilbert spaces \( Z \) and \( W \) such that \( T \oplus I_Z \) and \( S \oplus I_W \) are equivalent operators. This means that there exist invertible bounded linear operators \( E \) and \( F \) such that

\[
\begin{pmatrix} T & 0 \\ 0 & I_Z \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_W \end{pmatrix} F.
\]

The relation \( \sim \) is reflexive, symmetric and transitive.

**Theorem 5.2.** [3] If \( T \sim S \), then \( T \) is invertible(Fredholm) if and only if \( S \) is invertible (Fredholm).

Let

\[
A = \begin{pmatrix} f & 0 \\ g & -1 \end{pmatrix}, B = \begin{pmatrix} \varphi & -1 \\ \psi & 0 \end{pmatrix}.
\]
where $f, g, \phi, \psi \in L^\infty(\mathbb{T})$. Write the Cauchy singular integral operators with $2 \times 2$ matrix symbol

$$A\mathbb{P}_+ + B\mathbb{P}_- = \begin{pmatrix} f \\ g \end{pmatrix} \begin{pmatrix} P_+ & 0 \\ 0 & P_+ \end{pmatrix} + \begin{pmatrix} \phi & 0 \\ \psi & 0 \end{pmatrix} \begin{pmatrix} P_- & 0 \\ 0 & P_- \end{pmatrix} : L^2_2(\mathbb{T}) \to L^2_2(\mathbb{T}). \quad (5.1)$$

**Theorem 5.3.** Let $H = \begin{pmatrix} f \\ g \end{pmatrix} \in L^\infty_{2 \times 2}(\mathbb{T})$, $R_H \sim A\mathbb{P}_+ + B\mathbb{P}_-$. 

**Proof.** Let $H_1 = \begin{pmatrix} g \\ \psi \end{pmatrix}$, an easy computation shows that

$$\begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix} (A\mathbb{P}_+ + B\mathbb{P}_-) \begin{pmatrix} I_{L^2_2} \\ R_{H_1} \end{pmatrix} = \begin{pmatrix} fP_+ + \varphi P_- \\ gP_+ + \psi P_- \end{pmatrix} \begin{pmatrix} I_{L^2_2} \\ R_{H_1} \end{pmatrix} = \begin{pmatrix} I_{L^2_2} \\ R_{H_1} \end{pmatrix} \begin{pmatrix} R_H \\ R_{H_1} \end{pmatrix} = \begin{pmatrix} I_{L^2_2} \\ R_{H_1} \end{pmatrix}.$$ 

The operators $\begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix}$ and $\begin{pmatrix} I_{L^2_2} \\ R_{H_1} \end{pmatrix}$ are invertible, and

$$\begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix}^{-1} = \begin{pmatrix} P_+ & P_- \\ P_- & P_+ \end{pmatrix},$$

$$\begin{pmatrix} I_{L^2_2} \\ R_{H_1} \end{pmatrix}^{-1} = \begin{pmatrix} I_{L^2_2} \\ R_{H_1} \end{pmatrix}.$$ 

Hence the operators $R_H$ and $A\mathbb{P}_+ + B\mathbb{P}_-$ are equivalent after extension. \hfill \Box

If $f$ and $\psi$ are invertible, then $A$ and $B$ are invertible and

$$A^{-1} = \begin{pmatrix} f^{-1} & 0 \\ f^{-1}g & -1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 0 & \psi^{-1} \\ -1 & \phi\psi^{-1} \end{pmatrix},$$

In this case

$$A\mathbb{P}_+ + B\mathbb{P}_- = B(I - A\mathbb{P}_+ + \mathbb{P}_-),$$

$$= B(P_+B^{-1}A\mathbb{P}_+ + \mathbb{P}_+ - I + P_-B^{-1}A\mathbb{P}_+ + \mathbb{P}_- B^{-1}A\mathbb{P}_+ + \mathbb{P}_-)$$

$$= B(P_+B^{-1}A\mathbb{P}_+(I + P_-B^{-1}A\mathbb{P}_+) + \mathbb{P}_-(P_-B^{-1}A\mathbb{P}_+ + I))$$

$$= B(P_+B^{-1}A\mathbb{P}_+ + \mathbb{P}_-)(P_-B^{-1}A\mathbb{P}_+ + I)$$

where $I + P_-B^{-1}A\mathbb{P}_+$ is invertible on , the inverse is $I - P_-B^{-1}A\mathbb{P}_+$. This implies

$$A\mathbb{P}_+ + B\mathbb{P}_- \sim P_+B^{-1}A\mathbb{P}_+ + \mathbb{P}_-. \quad (5.2)$$
Moreover, under the decomposition $L^2_2(\mathbb{T}) = H^2_2(\mathbb{T}) \oplus (H^2(\mathbb{T}))_{\frac{1}{2}}$, we have
\[
P_+ B^{-1} A P_+ + P_- = \begin{pmatrix} T_{B^{-1}A} & 0 \\ 0 & I_{(H^2(\mathbb{T}))_{\frac{1}{2}}} \end{pmatrix},
\]
where $T_{B^{-1}A}$ is a block Toeplitz operator on $H^2_2(\mathbb{T})$ and
\[
B^{-1}A = \begin{pmatrix} g\psi^{-1} & -\psi^{-1} \\ g\phi\psi^{-1} - f & -\phi\psi^{-1} \end{pmatrix},
\]
det $B^{-1}A = -f\psi^{-1}$. Hence,
\[
P_+ B^{-1} A P_+ + P_- \sim \sim T_{B^{-1}A}.
\]
Similarly,
\[
AP_+ + B P_- = A(P_+ + P_- A^{-1} B P_-(P_+ A^{-1} B P_- + I)
\]
This implies
\[
AP_+ + B P_- \sim P_+ + P_- A^{-1} B P_-.
\]
and
\[
P_+ + P_- A^{-1} B P_- \sim \sim \mathbb{J}T_{(A^{-1}B)^*} \mathbb{J}
\]
where $\mathbb{J}(f, f)^T = (J f, Jf)^T = (\bar{f}, \bar{f})^T$ for $f \in L^2(\mathbb{T})$.

Recall the invertibility and Fredholm of Toeplitz operators with matrix-symbols via Wiener-Hopf factorization.

**Definition 5.4.** A representation of the form $F = F_- DF_+$ is called Wiener-Hopf factorization of the invertible matrix function $F \in L^\infty_{\mathbb{N} \times \mathbb{N}}(\mathbb{T})$ if $D = \text{diag}(z^{\kappa_j})_{j=1}^N$ with $\kappa_j \in \mathbb{Z}$, and if $F_-$ and $F_+$ satisfy the following conditions:

1. $F_+, F_-^{-1} \in H^2_{\mathbb{N} \times \mathbb{N}}(\mathbb{T})$, $F_-, F_1^{-1} \in H^2_{\mathbb{N} \times \mathbb{N}}(\mathbb{T})$,
2. The operator $F_1^{-1}P_+ F_+$ is defined on the linear space of all $\mathbb{C}^N$-valued trigonometric polynomials, can be extended to a bounded operator on $H^2_{\mathbb{N}}(\mathbb{T})$.

**Theorem 5.5.** [30] Let $F \in L^\infty_{\mathbb{N} \times \mathbb{N}}(\mathbb{T})$. Then $T_F$ is invertible (resp. Fredholm) if and only if $F$ admits a Wiener-Hopf factorization $F = F_- DF_+$ (resp. $F = F_- DF_+$).

If $T_a$ is Fredholm, then
\[
\dim \ker T_a = -\sum_{\kappa_j < 0} \kappa_j, \quad \dim \text{coker} T_a = \sum_{\kappa_j > 0} \kappa_j.
\]

**Theorem 5.6.** If $H = \begin{pmatrix} f & \phi \\ g & \psi \end{pmatrix} \in L^\infty_{2 \times 2}(\mathbb{T})$, then $R_H$ is invertible (resp. Fredholm) if and only if $f$ and $\psi$ are invertible in $L^\infty(\mathbb{T})$ and $\begin{pmatrix} g\psi^{-1} & -\psi^{-1} \\ g\phi\psi^{-1} - f & -\phi\psi^{-1} \end{pmatrix}$ admit a Winer-Hopf factorization $F_- F_+$ (resp. $F_- DF_+$).

If $R_H$ is Fredholm, then
\[
\dim \ker R_H = -\sum_{k_j < 0} k_j, \quad \dim \ker R_H^* = \sum_{k_j > 0} k_j.
\]
Proof. If $R_H$ is invertible or Fredholm, by Corollary 4.4, we have $f$ and $\psi$ are invertible in $L^\infty(\mathbb{T})$. Since the relation $\sim$ is transitive, combining Theorem 5.3, (5.2) and (5.4), it follows that $R_H \sim T_{B^{-1}A}$. Using Theorem 5.2 and Theorem 5.5, we get the result. \qed

6. Applications

6.1. The Spectral Inclusion Theorem. In the theory of Toeplitz operator, the spectrum of $T_\phi$ always includes the essential range of $\phi$. Corollary 4.4 shows that

$$Ran_{ess} f \cup Ran_{ess} \psi \subset \sigma(R(f,g,\phi,\psi)).$$

Hence, for the bounded singular integral operator $R_{\alpha,\beta}$, we have

$$Ran_{ess} \alpha \cup Ran_{ess} \beta \subset \sigma(R_{\alpha,\beta}),$$

for the bounded dual truncated Toeplitz operator $D_\phi$, we have

$$Ran_{ess} \phi \subset \sigma(D_\phi);$$

for the bounded Foguel-Hankel operator \( \begin{pmatrix} T^*_z & X \\ 0 & T_z \end{pmatrix} \), we have

$$T \subset \sigma \left( \begin{pmatrix} T^*_z & X \\ 0 & S \end{pmatrix} \right).$$

Moreover, for every constant $\lambda$, we have

$$\lambda I - \begin{pmatrix} T^*_z & X \\ 0 & T_z \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & \lambda I - T_z \end{pmatrix} \begin{pmatrix} I & -X \\ 0 & I \end{pmatrix} \begin{pmatrix} \lambda I - T_z & 0 \\ 0 & I \end{pmatrix}.$$

Note that

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}$$

is always invertible and

$$\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}.$$ 

If both of $\lambda I - T_z$ and $\lambda I - T_z^*$ are invertible, then $\lambda I - \begin{pmatrix} T^*_z & X \\ 0 & T_z \end{pmatrix}$ is invertible. Therefore,

$$\sigma \left( \begin{pmatrix} T^*_z & X \\ 0 & T_z \end{pmatrix} \right) \subset \sigma(T_z) = \overline{\mathbb{D}}.$$
6.2. Essential spectrum. The essential spectrum of Toeplitz operator with continuous symbol equals the essential range of the symbol. Corollary 4.9 shows that if \( f, g, \phi, \psi \in C(\mathbb{T}) \), then

\[
\sigma_e(R(f, g, \phi, \psi)) = f(\mathbb{T}) \cup \psi(\mathbb{T}).
\]

Hence, for bounded singular integral operator, if \( \alpha, \beta \in C(\mathbb{T}) \), then

\[
\sigma_e(R_{\alpha, \beta}) = \alpha(\mathbb{T}) \cup \beta(\mathbb{T}).
\]

For bounded dual truncated Toeplitz operator, if \( \varphi \in C(\mathbb{T}) \), then

\[
\sigma_e(D_{\varphi}) = \varphi(\mathbb{T}).
\]

For bounded Foguel-Hankel operator, if \( X = \Gamma_\phi \) and \( \phi \in H^\infty + C(\mathbb{T}) \), then

\[
\sigma_e(T^*_z X_0 T_z) = \mathbb{T}.
\]

6.3. Special cases. We consider one of operators \( T_f \) and \( S_\psi \) is invertible. In particular, suppose \( S_\psi = I \), Suppose that \( \lambda \notin \text{Ran}_{\text{ess}} f \cup \{1\} \). Now

\[
\begin{pmatrix}
 T_{f-\lambda} & H^*_\phi \\
 H_g & I - \lambda
\end{pmatrix} = \begin{pmatrix}
 I & H^*_\phi \\
 0 & \frac{1}{1-\lambda} I
\end{pmatrix} \begin{pmatrix}
 T_{f-\lambda} - H^*_\phi H_g & 0 \\
 0 & I
\end{pmatrix} \begin{pmatrix}
 I & 0 \\
 (1-\lambda) H_g & I
\end{pmatrix}.
\]

Since \( \begin{pmatrix}
 I & H^*_\phi \\
 0 & \frac{1}{1-\lambda} I
\end{pmatrix} \) and \( \begin{pmatrix}
 I & 0 \\
 (1-\lambda) H_g & I
\end{pmatrix} \) are always invertible, or \( \left( \begin{pmatrix}
 I & H^*_\phi \\
 0 & \frac{1}{1-\lambda} I
\end{pmatrix} \right)^{-1} = \left( \begin{pmatrix}
 I & 0 \\
 (1-\lambda) H_g & I
\end{pmatrix} \right)^{-1} = \left( \begin{pmatrix}
 I & 0 \\
 -(1-\lambda) H_g & I
\end{pmatrix} \right) \), it follows that

\[
\begin{pmatrix}
 T_{f-\lambda} & H^*_\phi \\
 H_g & I - \lambda
\end{pmatrix}
\]

is invertible if and only if

\[
\begin{pmatrix}
 T_{f-\lambda} & 0 \\
 0 & I
\end{pmatrix}
\]

is invertible. Therefore, we have

\[
\sigma(R(f, g, \phi, 1)) = \sigma(T_f - H^*_\phi H_g) \cup \text{Ran}_{\text{ess}} f \cup \{1\}.
\]

Since \( T_f - H^*_\phi H_g = T_f - T_{\phi g} + T_{\phi} T_g \), we have

\[
\lim_{r \to 1-} \langle T_f - H^*_\phi H_g k_r \xi, k_r \xi \rangle = f(\xi) \quad \text{a.e. on } \mathbb{T}.
\]

By Corollary 4.4, we have \( \text{Ran}_{\text{ess}} f \subseteq \sigma(T_f - H^*_\phi H_g) \). Hence,

\[
\sigma(R(f, g, \phi, 1)) = \sigma(T_f - H^*_\phi H_g) \cup \{1\}.
\]

Similarly,

\[
\sigma(R(1, g, \phi, \psi)) = \sigma(S_\psi - H_g H^*_\phi) \cup \{1\}.
\]
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