Heavy Quark Expansion for the Inclusive Decay $\bar{B} \to \tau \bar{\nu} X$

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Abstract:

We calculate the differential decay rate for inclusive $\bar{B} \to \tau \bar{\nu} X$ transitions to order $1/m_b^2$ in the heavy quark expansion, for both polarized and unpolarized tau leptons. We show that using a systematic $1/m_b$ expansion significantly reduces the theoretical uncertainties in the calculation. We obtain for the total branching ratio $\text{BR}(\bar{B} \to \tau \bar{\nu} X) = 2.30 \pm 0.25\%$, and for the tau polarization $A_{pol} = -0.706 \pm 0.006$. From the experimental measurement of the branching ratio at LEP, we derive the upper bound $\lambda_1 \leq 0.8 \text{ GeV}^2$ for one of the parameters of the heavy quark effective theory.

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1. Introduction

Recently, it has been observed that inclusive semileptonic decays of hadrons containing a single heavy quark allow for a systematic, QCD-based expansion in powers of $1/m_Q$ [4]. In particular, it has been shown that the inclusive decay rates computed in the heavy quark limit $m_Q \to \infty$ coincide with those obtained in the free quark decay model, while corrections of order $1/m_Q$ vanish. The leading nonperturbative corrections are of order $1/m_Q^2$ and depend on only two hadronic parameters, which parameterize certain forward matrix elements of local dimension-five operators. These corrections have been computed for a number of processes [2]–[7]. These new theoretical developments not only provide a theoretical justification for the parton model, but also allow a model independent calculation of the nonperturbative corrections to a high level of accuracy.

Semileptonic $B$ meson decays into a tau lepton are potentially very interesting. First, they are sensitive to certain form factors which are unmeasurable with a massless lepton in the final state. As a consequence, the ratio of the decay rates for the tau channel and the light lepton channels is sensitive to the nonperturbative corrections of order $1/m_Q^2$, while independent of the Cabbibo–Kobayashi–Maskawa matrix elements $V_{cb}$ and $V_{ub}$. Second, the possibility to study the tau polarization offers a greater variety in the experimental and theoretical analysis and an independent determination of parameters. Third, the inclusive $\bar{B} \to \tau \bar{\nu} X$ decay rate is useful for constraining or probing certain extensions of the standard model, such as models with many scalar fields.

In this paper, we study the decay $\bar{B} \to \tau \bar{\nu} X$ using a combination of the operator product expansion and heavy quark effective theory. The main points which we address are the following:

a. We calculate analytically, to order $1/m_b^2$, the total decay rate and the lepton spectrum for the cases of unpolarized and polarized tau leptons. Effects of the finite tau lepton mass are treated exactly.

b. We study in detail the numerical predictions for the total decay rate and the tau polarization. In particular, we stress that by using a systematic $1/m_Q$ expansion the theoretical predictions become more accurate than those of the free quark decay model, not only because $1/m_b^2$ corrections are included but, more importantly, because the masses
of the charm and bottom quarks are correlated in a specific way.

c. We use the experimental value of the inclusive branching ratio $\text{BR}(\bar{B} \to \tau \bar{\nu} X)$ to derive a bound on the hadronic parameter $\lambda_1$, which is related to the kinetic energy of the $b$-quark inside the $B$ meson.

2. Analytic Expressions

We begin by presenting the analytic results necessary for our numerical analysis. The techniques by which they are obtained are described in detail elsewhere [2]–[5], so we will present only a brief summary, followed by the results of the computation.

The inclusive differential decay distribution is determined by the imaginary part of the time-ordered product of two flavor-changing currents,

$$ T^{\mu \nu} = -i \int d^4 x e^{-i q \cdot x} \langle B | T \{ J^\mu \dagger (x), J^\nu (0) \} | B \rangle, \quad (2.1) $$

where $J^\mu = \bar{c} \gamma^\mu (1 - \gamma_5) b$ or $J^\mu = \bar{u} \gamma^\mu (1 - \gamma_5) b$. Since over most of the Dalitz plot the energy release is large (of order $m_b$), the time-ordered product can be written as an operator product expansion, in which higher-dimension operators are suppressed by powers of $\Lambda/m_b$, where $\Lambda$ is a typical low energy scale of the strong interactions. To this end, however, it is necessary to separate the large part of the $b$-quark momentum by writing $p_b = m_b v + k$, where $v$ is the velocity of the decaying $B$ meson. The aim is to construct an expansion in powers of $k/m_b$, where the residual momentum $k$ is of order $\Lambda_{\text{QCD}}$. This separation is most conveniently performed by using the formalism of the heavy quark effective theory [8]. One can then evaluate the matrix elements of the resulting tower of nonrenormalizable operators with the help of the heavy quark symmetries.

The leading term in the expansion reproduces the result of the free quark decay model [1], while giving a unambiguous meaning to the heavy quark mass [9]. The leading nonperturbative corrections are of relative order $1/m_b^2$ and may be written in terms of two parameters, $\lambda_1$ and $\lambda_2$, which are related to the kinetic energy $K_b$ of the $b$-quark inside the $B$ meson, and to the mass splitting between $B$ and $B^*$ mesons [10]:

$$ K_b = -\frac{\lambda_1}{2m_b}, \quad m_{B^*}^2 - m_B^2 = 4\lambda_2. \quad (2.2) $$
The operator product expansion for semileptonic decays to massless leptons has been performed in refs. [2]–[4]. The inclusion of the tau mass in the process $\bar{B} \to \tau \bar{\nu} X$ is a straightforward, but cumbersome, generalization. For the sake of brevity, we shall present only the final expressions. The tau lepton can have spin up ($s = +$) or spin down ($s = -$) relative to the direction of its momentum, and it is convenient to decompose the corresponding decay rates as

$$\Gamma (\bar{B} \to \tau(s = \pm) \bar{\nu} X) = \frac{1}{2} \Gamma \pm \tilde{\Gamma}. \quad (2.3)$$

The total rate, summed over the tau polarizations, is given by $\Gamma$, while the tau polarization is $A_{\text{pol}} = 2\tilde{\Gamma} / \Gamma$. The differential decay rate depends on the kinematic variables $q^2$, $E_\tau$, and $E_\nu$, where $q^2$ is the invariant mass of the lepton pair, and $E_\tau$ and $E_\nu$ denote the tau and neutrino energies in the parent rest frame. Let us introduce a set of related dimensionless variables by

$$\hat{q}^2 = \frac{q^2}{m_b^2}, \quad y = \frac{2E_\tau}{m_b}, \quad x = \frac{2E_\nu}{m_b}, \quad (2.4)$$

and define the mass ratios

$$\rho_j = \frac{m_j^2}{m_b^2}, \quad \rho_\tau = \frac{m_\tau^2}{m_b^2}, \quad (2.5)$$

where $j = c$ or $u$ is a flavor label. The triple differential decay rate may be written in terms of five invariant form factors $\hat{W}_i$, for which we adopt the conventions of ref. [4]. We obtain

$$\frac{1}{\Gamma_b} \frac{d\Gamma}{dq^2 dy dx} = 24 \Theta \left( x - \frac{2(q^2 - \rho_\tau)}{y + \sqrt{y^2 - 4\rho_\tau}} \right) \Theta \left( \frac{2(q^2 - \rho_\tau)}{y - \sqrt{y^2 - 4\rho_\tau}} - x \right) \times \left\{ (\hat{q}^2 - \rho_\tau)\hat{W}_1 + \frac{1}{2}(y x - \hat{q}^2 + \rho_\tau)\hat{W}_2 + \frac{1}{2}[\hat{q}^2(y - x) - \rho_\tau(y + x)]\hat{W}_3 \right. \right.$$  

$$\left. + \frac{1}{2}\rho_\tau(\hat{q}^2 - \rho_\tau)\hat{W}_4 + \rho_\tau x\hat{W}_5 \right\}, \quad (2.6)$$

and

$$\frac{1}{\Gamma_b} \frac{d\tilde{\Gamma}}{dq^2 dy dx} = 6 \Theta \left( x - \frac{2(q^2 - \rho_\tau)}{y + \sqrt{y^2 - 4\rho_\tau}} \right) \Theta \left( \frac{2(q^2 - \rho_\tau)}{y - \sqrt{y^2 - 4\rho_\tau}} - x \right) \frac{1}{\sqrt{y^2 - 4\rho_\tau}} \times \left\{ [(\hat{q}^2 - \rho_\tau)y - 2\rho_\tau x](-2\hat{W}_1 + \hat{W}_2 + x\hat{W}_3 + \rho_\tau\hat{W}_4 + y\hat{W}_5) \right. \right.$$  

$$\left. - (y^2 - 4\rho_\tau)[x\hat{W}_2 + (\hat{q}^2 - \rho_\tau)(\hat{W}_3 + \hat{W}_5)] \right\}, \quad (2.7)$$

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where
\[
\Gamma_b = \frac{|V_{tb}|^2 G_F^2 m_b^5}{192\pi^3}; \quad j = c \text{ or } u. \tag{2.8}
\]

The expressions for the form factors \( \hat{W}_i \), at tree level and to order \( 1/m_b^2 \) in the operator product expansion, are

\[
\hat{W}_1 = \delta(\hat{z}) \left\{ \frac{1}{4} (2 - y - x) - \frac{\lambda_1 + 3\lambda_2}{12m_b^2} \right\} \\
+ \delta'(\hat{z}) \left\{ \frac{\lambda_1}{24m_b^2} [8\hat{q}^2 + 6(y + x) - 5(y + x)^2] + \frac{\lambda_2}{8m_b^2} [8\hat{q}^2 + 14(y + x) - 5(y + x)^2 - 16] \right\} \\
+ \delta''(\hat{z}) \frac{\lambda_1}{24m_b} (2 - y - x)[4\hat{q}^2 - (y + x)^2],
\]

\[
\hat{W}_2 = \delta(\hat{z}) \left\{ 1 - \frac{5(\lambda_1 + 3\lambda_2)}{6m_b^2} \right\} + \delta'(\hat{z}) \left\{ \frac{7\lambda_1}{6m_b^2} (y + x) + \frac{\lambda_2}{2m_b^5} [5(y + x) - 4] \right\} \\
+ \delta''(\hat{z}) \frac{\lambda_1}{6m_b^2} [4\hat{q}^2 - (y + x)^2],
\]

\[
\hat{W}_3 = \frac{\hat{W}_1}{2} + \delta'(\hat{z}) \left\{ \frac{5\lambda_1}{12m_b^5} (y + x) + \frac{\lambda_2}{4m_b^5} [5(y + x) - 12] \right\} + \delta''(\hat{z}) \frac{\lambda_1}{12m_b^2} [4\hat{q}^2 - (y + x)^2],
\]

\[
\hat{W}_4 = \delta'(\hat{z}) \frac{2(\lambda_1 + 3\lambda_2)}{3m_b^2},
\]

\[
\hat{W}_5 = -\frac{\hat{W}_1}{2} - \delta'(\hat{z}) \left\{ \frac{\lambda_1}{12m_b^5} [5(y + x) + 8] + \frac{5\lambda_2}{4m_b^5} (y + x) \right\} - \delta''(\hat{z}) \frac{\lambda_1}{12m_b^2} [4\hat{q}^2 - (y + x)^2], \tag{2.9}
\]

where
\[
\hat{z} = 1 + \hat{q}^2 - \rho_j - y - x + i\epsilon. \tag{2.10}
\]

Our results for \( \hat{W}_1, \hat{W}_2 \) and \( \hat{W}_3 \) coincide with those obtained in ref. [4]. The two form factors \( \hat{W}_4 \) and \( \hat{W}_5 \) do not contribute when the final lepton is massless, but are important for the tau channel.

We now integrate over the kinematic variables \( \hat{q}^2 \) and \( x \) to obtain the differential decay rates with respect to the rescaled lepton energy \( y \). Notice that a subtlety specific for the case of non-vanishing tau lepton mass is the appearance of the second step function in (2.4) and (2.7), which gives an upper limit for the variable \( x \). For the case of a massless
lepton, this limit becomes trivial \((x < \infty)\). Integrating (2.6), we find

\[
\frac{1}{\Gamma_b} \frac{d\Gamma}{dy} = 2 \sqrt{y^2 - 4\rho_\tau} \left\{ x_0^3 \left[ y^2 - 3y(1 + \rho_\tau) + 8\rho_\tau \right] + x_0^2 \left[ -3y^2 + 6y(1 + \rho_\tau) - 12\rho_\tau \right] \\
- \frac{\lambda_2 x_0}{m_b^2 (1 + \rho_\tau - y)} \left[ 5x_0^2 \left( y^3 - 4y^2(1 + \rho_\tau) + 2y(3 + 7\rho_\tau) + 4\rho_\tau(\rho_\tau - 5) \right) \\
+ 3x_0 \left( -5y^3 + y^2(17 + 15\rho_\tau) - y(24 + 46\rho_\tau) - \rho_\tau(18\rho_\tau - 70) \right) \\
+ 3 \left( 5y^3 - 10y^2(1 + \rho_\tau) + 4y(3 + 4\rho_\tau) + 16\rho_\tau(\rho_\tau - 2) \right) \right]\right\
\]

\[
+ \frac{\lambda_1}{3m_b^2 (1 + \rho_\tau - y)^2} \left[ 3 \left( y^4 - 2y^3(1 + \rho_\tau) + 8y\rho_\tau(1 + \rho_\tau) - 16\rho_\tau^2 \right) \\
+ 2x_0^3 \left( y^4 - 5y^3(1 + \rho_\tau) + 2y^2(5 + 11\rho_\tau + 5\rho_\tau^2) - 40y\rho_\tau(1 + \rho_\tau) - 2\rho_\tau(5 - 38\rho_\tau + 5\rho_\tau^2) \right) \\
+ 3x_0^2 \left( -2y^4 + 8y^3(1 + \rho_\tau) - y^2(15 + 28\rho_\tau + 15\rho_\tau^2) + 52y\rho_\tau(1 + \rho_\tau) + 18\rho_\tau(1 - 6\rho_\tau + \rho_\tau^2) \right) \\
+ 6x_0 \left( y^4 - 3y^3(1 + \rho_\tau) + y^2(5 + 6\rho_\tau + 5\rho_\tau^2) - 12y\rho_\tau(1 + \rho_\tau) - 8\rho_\tau(1 - 4\rho_\tau + \rho_\tau^2) \right) \right\},
\]

while (2.7) yields

\[
\frac{1}{\Gamma_b} \frac{d\tilde{\Gamma}}{dy} = (y^2 - 4\rho_\tau) \left\{ x_0^3 \left[ 3 - y - \rho_\tau \right] + 3x_0^2 \left[ y - 2 \right] \\
+ \frac{\lambda_2 x_0}{m_b^2 (1 + \rho_\tau - y)} \left[ 5x_0^2 \left( y^2 - 2y(2 - \rho_\tau) + 6(1 - \rho_\tau) \right) \\
+ 3x_0 \left( -5y^2 + y(17 - 5\rho_\tau) - 4(6 - 5\rho_\tau) \right) + 3 \left( 5y^2 - 10y + 12(1 - \rho_\tau) \right) \right]\right\
\]

\[
+ \frac{\lambda_1}{3m_b^2 (1 + \rho_\tau - y)^2} \left[ 3 \left( -y^3 + 2y^2 + 4y\rho_\tau - 8\rho_\tau \right) \\
+ 2x_0^3 \left( -y^3 + y^2(5 + 2\rho_\tau) + y(-10 - 11\rho_\tau + 5\rho_\tau^2) + 6\rho_\tau(5 - 3\rho_\tau) \right) \\
+ 3x_0^2 \left( 2y^3 - 2y^2(4 + 3\rho_\tau) + y(15 + 22\rho_\tau - 5\rho_\tau^2) - 24\rho_\tau(2 - \rho_\tau) \right) \\
+ 6x_0 \left( -y^3 + y^2(3 + 4\rho_\tau) - y(5 + 11\rho_\tau) + 6\rho_\tau(3 - \rho_\tau) \right) \right\}.
\]

Here

\[
x_0 = 1 - \frac{\rho_j}{1 + \rho_\tau - y}.
\]

At this point, we like to mention that there is an elegant alternative way to obtain the spectra \(d\Gamma/dy\) and \(d\tilde{\Gamma}/dy\). Instead of starting from a triple differential decay rate, one can construct the operator product expansion for the decay \(\bar{B} \to \tau + X\), where the neutrino is now part of the final state \(X\). The leading contribution in the expansion is given by
a diagram containing a charm quark–neutrino loop. The discontinuity of this diagram gives the lepton spectrum. In this approach, the variable $x_0$ appears in a natural way as the upper limit in the integration over a Feynman parameter. We have checked that this approach leads to the same results as given above.

Finally, we perform the $y$-integration over the kinematic region

$$2\sqrt{\rho_\tau} \leq y \leq 1 + \rho_\tau - \rho_j$$

(2.14)

to obtain the total decay rates. For $\Gamma$, we find

$$\frac{\Gamma}{\Gamma_b} = \sqrt{\lambda} \left\{ (1 + \frac{\lambda_1}{2m_b^2}) \left[ 1 - 7(\rho_j + \rho_\tau) - 7(\rho_j^2 + \rho_\tau^2) + \rho_j^3 + \rho_\tau^3 + \rho_j \rho_\tau (12 - 7(\rho_\tau + \rho_j)) \right] \\
+ \frac{3\lambda_2}{2m_b^2} \left[ -3 + 5(\rho_j + \rho_\tau) - 19(\rho_j^2 + \rho_\tau^2) + 5(\rho_j^3 + \rho_\tau^3) + 7\rho_j \rho_\tau (4 - 5(\rho_\tau + \rho_j)) \right] \right\} \\
+ 12 \left( 1 + \frac{\lambda_1 + 3\lambda_2}{2m_b^2} \right) \left[ \rho_j^2 \ln \frac{(1 + \rho_j - \rho_\tau + \sqrt{\lambda})^2}{4\rho_j} + \rho_\tau^2 \ln \frac{(1 + \rho_\tau - \rho_j + \sqrt{\lambda})^2}{4\rho_\tau} \right] \\
- 12 \left( 1 + \frac{\lambda_1 + 15\lambda_2}{2m_b^2} \right) \rho_j^2 \rho_\tau^2 \ln \frac{(1 - \rho_\tau - \rho_j + \sqrt{\lambda})^2}{4\rho_\tau \rho_j} ,$$

(2.15)

with

$$\lambda = 1 - 2(\rho_\tau + \rho_j) + (\rho_\tau - \rho_j)^2 .$$

(2.16)
For $\tilde{\Gamma}$, we obtain the lengthy expression
\[
\frac{\tilde{\Gamma}}{\Gamma_b} = \frac{1}{6} \left[ (1 - \hat{m}_\tau)^2 - \rho_j \right] \left[ - (1 - \hat{m}_\tau)^3 (3 + 15\hat{m}_\tau + 5\hat{m}_\tau^2 + \hat{m}_\tau^3) \\
+ \rho_j (1 - \hat{m}_\tau)(21 + 57\hat{m}_\tau + 31\hat{m}_\tau^2 + 11\hat{m}_\tau^3) \\
+ \rho_j^2 (21 - 15\hat{m}_\tau - 5\hat{m}_\tau^2 + 47\hat{m}_\tau^3)/(1 - \hat{m}_\tau) - 3\rho_j^3 \right] \\
- 2\rho_j^2 (3 - 3\hat{m}_\tau^4 - 2\hat{m}_\tau^2 \rho_j) \ln \left( \frac{1 - \hat{m}_\tau)^2}{\rho_j} \right)
\]
\[+ \frac{\lambda_1}{12m_b^2} \left\{ (1 - \hat{m}_\tau)^2 - \rho_j \right\} \left[ (1 - \hat{m}_\tau)^3 (3 +\hat{m}_\tau^2) \\
+ \rho_j (1 + \hat{m}_\tau)(21 - 6\hat{m}_\tau - 8\hat{m}_\tau^2 - 2\hat{m}_\tau^3 + 11\hat{m}_\tau^4)/(1 - \hat{m}_\tau) \\
+ \rho_j^2 (21 - 57\hat{m}_\tau + 14\hat{m}_\tau^2 - 42\hat{m}_\tau^3 - 99\hat{m}_\tau^4 + 47\hat{m}_\tau^5)/(1 - \hat{m}_\tau)^3 - 3\rho_j^3 \right] \\
- 12\rho_j^2 (3 - 3\hat{m}_\tau^4 - 2\hat{m}_\tau^2 \rho_j) \ln \left( \frac{1 - \hat{m}_\tau)^2}{\rho_j} \right) \right\}
\]
\[+ \frac{\lambda_2}{4m_b^2} \left\{ [(1 - \hat{m}_\tau)^2 - \rho_j] \left[ (1 - \hat{m}_\tau)(9 + 27\hat{m}_\tau + 70\hat{m}_\tau^2 + 10\hat{m}_\tau^3 - 15\hat{m}_\tau^4 - 5\hat{m}_\tau^5) \\
- \rho_j (15 - 3\hat{m}_\tau + 62\hat{m}_\tau^2 - 70\hat{m}_\tau^3 - 45\hat{m}_\tau^4 - 55\hat{m}_\tau^5)/(1 - \hat{m}_\tau) \\
+ \rho_j^2 (57 - 84\hat{m}_\tau + 82\hat{m}_\tau^2 + 260\hat{m}_\tau^3 - 235\hat{m}_\tau^4)/(1 - \hat{m}_\tau)^2 - 15\rho_j^3 \right] \\
- 12\rho_j (8\hat{m}_\tau^2 - 8\hat{m}_\tau^4 + 3\rho_j + 4\hat{m}_\tau^2 \rho_j - 15\hat{m}_\tau^4 \rho_j - 10\hat{m}_\tau^2 \rho_j^2) \ln \left( \frac{1 - \hat{m}_\tau)^2}{\rho_j} \right) \right\},
\]
where
\[\hat{m}_\tau = m_\tau/m_b = \sqrt{\rho_\tau}.
\]

Our results for the differential and total unpolarized decay rates confirm a very recent calculation of Koyrakh [11]. In the limit $\rho_\tau \to 0$, these results reduce to the expressions given in [2]–[4]. In the limit $\lambda_1, \lambda_2 \to 0$, corresponding to the free quark decay model, our results agree with ref. [12]. In the same limit, our expressions for $d\tilde{\Gamma}/dy$ and $\tilde{\Gamma}$ agree with those of ref. [13]. Finally, for $\hat{m}_\tau = 0$ we find $-2d\tilde{\Gamma}/dy = d\Gamma/dy$ and $-2\tilde{\Gamma} = \Gamma$, as required.

3. Numerical Analysis

3.1. Input parameters

When we neglect the tiny contribution from $b \to u$ transitions, the input parameters
entering our calculations are the mass of the tau lepton \[14\],
\[
m_{\tau} = 1.777 \text{ GeV},
\] (3.1)
the heavy quark masses \(m_b\) and \(m_c\), the hadronic parameters \(\lambda_1\) and \(\lambda_2\), and the quark mixing parameter \(|V_{cb}|\). We stress, however, that not all of these parameters are independent. In fact, the heavy quark effective theory can be used to construct a systematic \(1/m_Q\) expansion of the masses of hadrons containing a heavy quark, in which the parameters \(\lambda_1\) and \(\lambda_2\) appear at second order. The relevant relations are \[10\]
\[
m_B = m_b + \Lambda - \frac{\lambda_1 + 3\lambda_2}{2m_b} + \ldots,
\]
\[
m_D = m_c + \Lambda - \frac{\lambda_1 + 3\lambda_2}{2m_c} + \ldots,
\] (3.2)
where we neglect higher-order power corrections. The parameter \(\Lambda\) can be associated with the effective mass of the light degrees of freedom in the heavy mesons \[4\]. For each set of values for \(\Lambda, \lambda_1,\) and \(\lambda_2\), we solve (3.2) to find the heavy quark masses that should be used in the results of sect. 2. Hence, instead of the four parameters \(m_b, m_c, \lambda_1, \lambda_2\) we are left with three parameters \(\Lambda, \lambda_1, \lambda_2\). It will turn out that the correlation between the heavy quark masses, which is imposed by (3.2), reduces the theoretical uncertainties in the results in a significant way.

In our analysis, we use the value of \(\lambda_2\) that is obtained from the known value of the \(B - B^*\) mass splitting [cf. (2.2)]
\[
\lambda_2 = \frac{m_{B^*}^2 - m_B^2}{4} = 0.12 \text{ GeV}^2.
\] (3.3)
We expect this value to be accurate up to power corrections of order \(\Lambda/m_b \sim 10\%\). At this point, the parameters \(\Lambda\) and \(\lambda_1\) must still be obtained from nonperturbative calculations, introducing a certain amount of model dependence. From a QCD sum rule analysis, one finds that \(\Lambda\) lies in the range \[15\] \[16\]
\[
0.45 < \Lambda < 0.60 \text{ GeV}.
\] (3.4)
QCD sum rules have also been used to compute \(\lambda_1\) \[15\] \[17\] \[18\], but these calculations suffer from large uncertainties. There are theoretical arguments, however, that \(\lambda_1\) should
be negative (as one expects, since \(-\lambda_1\) is proportional to the kinetic energy of the heavy quark) \([7]\), and that its magnitude cannot be too large \([14]\). Here we shall use the range

\[0 < -\lambda_1 < 0.3 \text{ GeV}^2.\]

(3.5)

3.2. \(\text{BR}(\bar{B} \to \tau \bar{\nu} X)\)

Let us then present our numerical results. It is convenient to normalize the total branching ratio \(\text{BR}(\bar{B} \to \tau \bar{\nu} X)\) to the measured branching fraction into final states with an electron \([20]\),

\[\text{BR}(\bar{B} \to e \bar{\nu} X) = 10.7 \pm 0.5\% .\]

(3.6)

This eliminates the otherwise significant uncertainties from the values of \(|V_{cb}|^2\) and \(m_b^5\). We incorporate the one-loop QCD corrections, which may be extracted from \([21]\). Using \(\alpha_s(m_b) \approx 0.22\), one finds that the total rates are corrected by multiplicative factors \(\eta_{\tau} = 0.90\) and \(\eta_e = 0.88\), respectively. What is relevant for us is the ratio, \(\eta_{\tau}/\eta_e = 1.02\), for which the error due to the choice of scale in the running coupling constant is very small. From (2.15), we then obtain the main result of this section:

\[\text{BR}(\bar{B} \to \tau \bar{\nu} X) = 2.30 \pm 0.25\% .\]

(3.7)

There are a number of points to be made regarding this result:

\(a\). The \(1/m_b^2\) corrections to the free quark decay model reduce the prediction for the rate by approximately 4%. For example, if we take the values \(\Lambda = 0.5\) GeV and \(\lambda_1 = -0.25\) GeV, corresponding to the heavy quark masses \(m_b = 4.8\) GeV and \(m_c = 1.45\) GeV, the spectator model result of 2.37% is lowered to 2.28%. The fractional decreases in the individual rates are 7.3% for the tau channel and 3.8% for the electron channel.

\(b\). The 11% uncertainty in our prediction (3.7) takes into account the experimental uncertainty in the electronic branching ratio (3.6), the theoretical uncertainties in (3.4) and (3.3), as well as an estimate of higher-order power corrections which we have not included. The most important of these are the \(1/m_c^2\) corrections to the relation (3.2) between \(m_D\) and \(m_c\). We estimate the uncertainty related to this effect to be about 2%. We emphasize that the largest error of all of these is the \textit{experimental} one, so any improvement in the determination of \(\text{BR}(\bar{B} \to e \bar{\nu} X)\) will improve the accuracy of our prediction.
c. Numerically, the main improvement of our calculation over the spectator model calculation is not in the incorporation of the nonperturbative $1/m_b^2$ corrections, but in using $m_c$ and $m_b$ as determined by (3.2) rather than treating them as uncorrelated input parameters. To demonstrate this, let us take for a moment the central value $\text{BR}(\bar{B} \to e\bar{\nu}X) = 10.7\%$. Then, if we allow $m_c$ and $m_b$ to vary independently within the ranges $1.4 < m_c < 1.5 \text{ GeV}$ and $4.6 < m_b < 5.0 \text{ GeV}$, we find $\text{BR}(\bar{B} \to \tau\bar{\nu}X) = 2.27 \pm 0.55\%$. However, if we use $\Lambda$ and $\lambda_1$ as our input parameters and calculate $m_c$ and $m_b$ from (3.2), we obtain $\text{BR}(\bar{B} \to \tau\bar{\nu}X) = 2.30 \pm 0.09\%$. Accounting for the correlation between $m_c$ and $m_b$ reduces the error from 24% to 4%. Even if we allow the very conservative range $-0.5 < \lambda_1 < 0.5 \text{ GeV}$ and $0.4 < \Lambda < 0.7 \text{ GeV}$ (which covers a range for $m_c$ and $m_b$ larger than commonly accepted), we get $\text{BR}(\bar{B} \to \tau\bar{\nu}X) = 2.23 \pm 0.33\%$. This is still more accurate than if we allow independent variation of $m_c$ and $m_b$.

d. We have not included the final charmless modes $X_u$ in our numerical analysis. Since $|V_{ub}/V_{cb}| \sim 0.1$ $^{20}$, their effect on the inclusive rate for each of the electron and tau channel is of order 1%. The effect on the ratio of rates is even smaller and certainly well below our error in (3.7).

### 3.3. The tau polarization

The polarization of the tau lepton, $A_{\text{pol}} = 2\tilde{\Gamma}/\Gamma$, being a ratio of decay rates, is by itself subject to much smaller uncertainties than are the rates themselves. We find that the numerical value for $A_{\text{pol}}$ is rather insensitive to variations of $\Lambda$ and $\lambda_1$. Allowing these parameters to vary within the ranges (3.4) and (3.5), we find

$$A_{\text{pol}} = -0.706 \pm 0.006 \, .$$

A few points are in order regarding this result:

a. The $1/m_b^2$ corrections to the free quark decay model reduce the prediction for the tau polarization by approximately 4%.

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1 It seems to us that previous results from spectator model calculations ignore the uncertainty in quark masses and thus significantly underestimate the theoretical errors.
b. The 0.9% uncertainty in $A_{\text{pol}}$ is due almost entirely to our estimate of higher-order nonperturbative corrections. The uncertainty from the theoretical errors on $\Lambda$ and $\lambda_1$ is surprisingly small, about 0.1%.

c. Numerically, the main improvement is again in using $m_c$ and $m_b$ as determined by (3.2). As before, if we allow $m_c$ and $m_b$ to vary independently, we find $A_{\text{pol}} = -0.70 \pm 0.04$, while if we calculate them from $\Lambda$ and $\lambda_1$ and allow those inputs to vary, we find $A_{\text{pol}} = -0.7045 \pm 0.0005$. Thus, the correlation between $m_c$ and $m_b$ reduces the error on $A_{\text{pol}}$ from 6% to 0.1%.

d. The effect of final charmless hadronic states $X_u$ is included in the result (3.8). The effect, however, is rather small: it shifts the central value of $A_{\text{pol}}$ from $-0.705$ to $-0.706$, assuming $|V_{ub}/V_{cb}| = 0.1$.

e. We have also numerically calculated the tau polarization as a function of the tau energy, $A_{\text{pol}}(y) = -2(d\Gamma/dy)/(d\Gamma/dy)$. The $1/m_b^2$ corrections to the spectator results are small except for a region close to the parton model endpoint $y_{\text{max}} = 1 - \rho_c + \rho_\tau$. However, in this endpoint region (corresponding to $|\vec{p}_\tau| \sim 1.6$ GeV) our results cannot be trusted, since the operator product expansion becomes singular, and a resummation is necessary [22, 23].

4. An Experimental Bound on $\lambda_1$

The only quantity concerning inclusive $\bar{B} \to \tau \bar{\nu} X$ decays which has so far been measured is the total branching ratio. Adding the statistical and systematic errors on the recent ALEPH measurement [24] in quadrature, we will use the value

$$\text{BR}(\bar{B} \to \tau \bar{\nu} X) = 2.76 \pm 0.63 \%.$$  \hspace{1cm} (4.1)

Among the parameters which go into the prediction (3.7), the one that is subject to the largest theoretical uncertainties is $\lambda_1$. Using experimental data on inclusive semileptonic $D$ decays (in the electron channel), one can derive the experimental lower bound [25]

$$\lambda_1 \geq -0.5 \text{ GeV}^2.$$  \hspace{1cm} (4.2)
We find that the semileptonic $B$ decay in the tau channel provides an upper bound on this quantity. Taking into account the $1\sigma$ lower bound on the branching ratio \((4.1)\), the $1\sigma$ range for the branching ratio to electrons \((3.6)\), our range \((3.4)\) for \(\Lambda\), and the uncertainty from higher-order corrections, we obtain

$$\lambda_1 \leq 0.8 \text{ GeV}^2. \quad (4.3)$$

A lower bound on \(\lambda_1\) may also be derived from the same data, but it is weaker than \((4.2)\).

Let us clarify the role of the various input data in deriving the bound \((4.3)\):

\(a.\) The upper bound on \(\lambda_1\) is sensitive to the lower bound on the total branching ratio in \((4.1)\). If, for example, \(\text{BR}(\bar{B} \to \tau \bar{\nu} X) = 2.76\%\) (the central value of the ALEPH measurement), then \(\lambda_1 \leq -0.7 \text{ GeV}^2\) is required, which is inconsistent with \((4.2)\). Consistency of our theoretical analysis (which is based on the assumption of lepton universality in the standard model) then suggests that the branching ratio should be at the lower end of the range quoted by the ALEPH collaboration. This is also clear if one simply compares our result \((3.7)\) with the experimental result \((4.1)\).

\(b.\) Our bound is also sensitive to the experimental value of \(\text{BR}(\bar{B} \to e \bar{\nu} X)\). Thus, an improvement in the experimental data on this mode could lead to very strong constraints on \(\lambda_1\).

\(c.\) The bound is weakly dependent on the precise value of \(\Lambda\). This is in contrast to the bound derived in \([25]\), which is very sensitive to \(\Lambda\) through its dependence on \(m_c^5\).

\(d.\) We emphasize that in deriving the bound \((4.3)\), we have allowed each of the input parameters independently to take their extreme $1\sigma$ values, so the result we quote is rather conservative.

5. Summary

Using the operator product expansion and heavy quark effective theory, we have calculated the inclusive rate for $B$ meson decays into a tau lepton plus anything, including the effects of the non-vanishing tau mass and the leading nonperturbative corrections of order $1/m_\tau^2$. For the total branching ratio, we find \(\text{BR}(\bar{B} \to \tau \bar{\nu} X) = 2.30 \pm 0.25\%\). For
the polarization of the tau lepton, we obtain $A_{\text{pol}} = -0.706 \pm 0.006$. The effect of the $1/m_b^2$ corrections on both observables is a small shift of about 4% as compared to the free quark decay model. However, we have emphasized that, from the numerical point of view, the most significant improvement of our analysis is to implement the tight correlation between the heavy quark masses $m_b$ and $m_c$, which is imposed by the structure of the heavy quark expansion. This correlation reduces the theoretical uncertainties in a very significant way. The experimental value for the branching ratio allows us to derive the upper bound $\lambda_1 \leq 0.8 \text{ GeV}^2$ on one of the fundamental parameters of the heavy quark effective theory.

While this paper was in writing, we received a preprint by Balk et al. [26] with results similar to ours and to ref. [11]. Our study of the tau polarization in sect. 2, and the numerical analysis of sects. 3 and 4, have no overlap with either of these papers.

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