Predicted electric field near small superconducting ellipsoids

J. E. Hirsch

Department of Physics, University of California, San Diego
La Jolla, CA 92093-0319
(Dated: September 22, 2003)

We predict the existence of large electric fields near the surface of superconducting bodies of ellipsoidal shape of dimensions comparable to the penetration depth. The electric field is quadrupolar in nature with significant corrections from higher order multipoles. Prolate (oblate) superconducting ellipsoids are predicted to exhibit fields consistent with negative (positive) quadrupole moments, reflecting the fundamental charge asymmetry of matter.

PACS numbers:

Electrostatic fields cannot exist inside normal metals because the mobile electrons will move along any existing electric field lines so as to minimize their potential energy, resulting in an equilibrium configuration with no electric field. Any kinetic energy gained by the electron in this process will be dissipated through inelastic scattering mechanisms.

Electrons in superconductors are also mobile (even more so than in normal metals), hence one is accustomed to think that no electrostatic field can exist inside superconductors either. However, we suggest that such preconception may actually be incorrect for the following reason: in a superconductor there are no inelastic scattering mechanisms that will dissipate the kinetic energy of a superfluid electron moving along an electric field line. As a consequence, as the electron moves along the field line and decreases its potential energy it will gain kinetic energy; when it reaches the minimum potential energy its kinetic energy will be maximum so it will 'overshoot' and move back to a region of higher potential energy, leaving the electric field unscreened. Thus we argue that at least in principle the existence of electrostatic fields inside superconductors is not ruled out by basic physics principles.

On the contrary, we have recently proposed that an electrostatic field indeed exists inside superconductors. This proposal is based on the theory of hole superconductivity, that predicts that negative charge is expelled from the interior of the superconductor towards the surface, resulting in a net positive charge density in the interior, a negative charge density near the surface, and an outward pointing electric field inside the superconductor.

In fact, the possibility of an electrostatic field in a superconductor follows directly from London’s equation for the supercurrent \( \vec{J} \) in the presence of a magnetic vector potential \( \vec{A} \):

\[
\vec{J} = -\frac{n_e e^2}{m_e c} \vec{A}
\]  

(1)

and Faraday’s law in the form that relates the electric field \( \vec{E} \) to the electric potential \( \phi \) and the magnetic vector potential:

\[
\vec{E} = -\nabla \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t}.
\]  

(2)

Taking the time derivative of Eq. (1) and using Eq. (2) yields

\[
\frac{\partial \vec{J}}{\partial t} = \frac{n_e e^2}{m_e} (\nabla \phi + \vec{E})
\]  

(3)

indicating that an electric field that derives from a potential will not lead to a time variation of the supercurrent, and in particular will not generate a supercurrent if one is not present initially.

The electric potential in the interior of the superconductor is assumed to obey the fundamental equation:

\[
\phi(\vec{r}) = -4\pi \lambda_L^2 \rho(\vec{r}) + \phi_0(\vec{r})
\]  

(4)

which is derived from the London equation under the assumption that the magnetic vector potential obeys the Lorenz gauge condition. We postulate Eq. (4) to describe the electric potential \( \phi(\vec{r}) \) and charge distribution \( \rho(\vec{r}) \) in the interior of all superconductors, with \( \lambda_L \) the London penetration depth and \( \phi_0(\vec{r}) \) the potential originating from a uniform positive charge density \( \rho_0 \):

\[
\phi_0(\vec{r}) = \int_V d^3r' \frac{\rho_0}{|\vec{r} - \vec{r}'|}
\]  

(5)

where the integral is over the volume of the superconducting body. The charge density \( \rho_0 \) is a function of parameters describing the superconducting material, of the dimensions of the body, and of temperature. It originates in the 'undressing' of carriers as the system goes superconducting, as described in the theory of hole superconductivity. In particular, it increases as the temperature is lowered below \( T_c \) and the superfluid density increases, and it is larger for superconductors with higher \( T_c \).

The electric potential also obeys the usual Poisson equation

\[
\nabla^2 \phi(\vec{r}) = -4\pi \rho(\vec{r})
\]  

(6)
in the interior of the superconductor, and Laplace’s equation in the exterior. Furthermore we assume that \( \phi(\vec{r}) \) as well as its normal derivative are continuous across the surface of the superconductor, i.e. that no surface charge density exists.

Equation (4) predicts that \( \rho(\vec{r}) = \rho_0 \) deep in the interior of superconductors of dimensions much larger than the penetration depth, and an excess of negative charge within a layer of thickness \( \lambda_L \) of the surface. The solution of Eqs. (4)-(6) was discussed in Ref. [3] for the case of spherical superconductors. For that case, no electric field exists in the exterior of the superconductor and the proposed scenario cannot be detected. Fortunately, the situation is different for superconductors of non-spherical shape.

We consider here ellipsoids of revolution, defined by

\[
\frac{b^2}{a^2} + \frac{z^2}{b^2} = 1
\]

with \( \rho, z \) cylindrical coordinates. The dipole moment of any charge distribution with the symmetry of the ellipsoid is zero, but the quadrupole moment

\[
Q = \int d^3r \rho(\vec{r})[3z^2 - r^2]
\]

is not. For the uniformly charged ellipsoid with charge density \( \rho_0 \) the quadrupole moment is

\[
Q_0 = \frac{8}{15} \pi a^2 b(b^2 - a^2) \rho_0
\]

so that it is positive (negative) for prolate \((b > a)\) (oblate \((b < a)\)) ellipsoids for positive \( \rho_0 \).

We solve the differential equation resulting from Eqs. (4)-(6)

\[
\phi(\vec{r}) = \lambda_L^2 \nabla^2 \phi(\vec{r}) + \phi_0(\vec{r})
\]

numerically in the interior of the ellipsoid using the GENCOL algorithm. Initially we assume an arbitrary boundary condition consistent with overall charge neutrality, for example that the normal derivative \( \partial \phi / \partial n = 0 \) everywhere on the surface (Neumann boundary condition), which implies non-existence of electric fields in the exterior of the superconductor. However \( \partial \phi / \partial n = 0 \) on the surface implies \( \phi \) is constant on the surface. The first iteration of the procedure yields a \( \phi \) on the surface that is not constant (except for the case \( b = a \)), implying that the original assumption of no electric field in the exterior was incorrect.

To achieve self-consistency we obtain the electric potential in the exterior of the superconductor using

\[
\phi(\vec{r}) = \int_V d^3r' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|}
\]

where \( \rho(\vec{r}) \) is obtained from Eq. (4) using the solution obtained for \( \phi(\vec{r}) \) in the interior. For the next iteration step we can use as boundary condition for the interior problem the normal derivative \( \partial \phi / \partial n \) obtained from the exterior problem or the potential itself, obtained from Eq. (11) on the surface. After a few iterations the procedure converges to a unique solution \( \phi(\vec{r}) \) which is continuous and has continuous derivatives across the surface.

For a prolate ellipsoid with \( b/a = 1.5 \) and London penetration depth \( \lambda_L/a = 0.5 \) the quadrupole moment of the resulting charge distribution is \( Q = -0.485 \), in units so that \( \rho_0 = a = 1 \). For comparison the quadrupole moment of the bare positive ellipsoid for this case is \( Q_0 = \pi \).

Figure 1 shows the electric field configuration outside the ellipsoid obtained (the electric field inside is not shown). The magnitude of the electric field decays rapidly away from the surface because it is much larger). The electric field points out near the surface for \( \rho \sim a, z \sim 0 \) and points in near the surface for \( \rho \sim 0, z \sim b \). For an oblate ellipsoid the situation would be reversed. The tangential component of the field on the surface shown in Fig. 1 exerts a force on the electrons in the direction of smaller \( |z| \). As argued earlier, we interpret that electrons do not rearrange according to this force to screen the field because they will increase their kinetic energy when they move in the direction of the force and decrease it when they move opposite to it. The magnitude of the electric field decays rapidly away from the surface since it is of quadrupolar nature.

Figure 2 shows the obtained charge density \( \rho(\vec{r}) \) in the interior along the axis \( z = 0 \) and \( \rho = 0 \) and the electric field in the interior along these axis for the parameters of Fig. 1. Because \( \lambda_L \) is comparable to the size of the body,
FIG. 2: For the parameters of Fig. 1: (a) Charge density in the interior of the superconductor along the horizontal axis plotted versus $\rho/a$ (curve labeled $\rho/a$) and along the vertical axis plotted versus $z/b$ (curve labeled $z/b$) and on the boundary plotted versus $\theta/\pi$), with $\theta = \tan^{-1}(z/b)/(\rho/a)$; note that the negative charge density near the surface is larger in magnitude along the $z$ direction. (b) Electric fields in the interior along the $\rho/a$ (dashed) and $z/b$ (dot-dashed) directions. Note that the electric field along the $z$ direction changes sign near the surface, and that the electric fields are finite at the surface.

$\rho(r)$ does not reach the value $\rho_0 (=1)$ deep in the interior. Contrary to the case of the sphere discussed in ref. 3, here the charge density and electric fields are different in the different directions. Furthermore the negative charge near the surface cannot fully screen the electric field as in the case of the sphere, so that the field is finite at the surface of the ellipsoid and leaks out. Note that the obtained negative charge density near the surface is larger in the direction of the larger axis of the ellipsoid. This is found to be the case generally for both prolate and oblate ellipsoids, which implies that the quadrupole moment will always be negative (positive) for prolate (oblate) ellipsoids. This would even be the case if the charge density near the surface was equal in the directions of large and small axis, but this feature enhances the effect even more. For smaller values of the penetration depth the charge density is larger in the interior and more negative near the surface as in the case of the sphere[3], and a finite electric field always occurs at the surface except for $a = b$.

Figure 3 shows the electric potential as well as the normal and tangential components of the electric field on the surface of a prolate ellipsoid for the parameters of Fig. 1.

The agreement of the results from the inside and outside solutions indicates that the numerical iteration procedure has converged. The electric potential is larger near the equator ($\theta = 0$) than near the poles ($\theta = \pi/2$), consistent with a tangential electric field in the polar direction and a normal field that points out near the equator and in near the poles. The results differ substantially from what would be obtained from a pure quadrupole with rotational symmetry along the $z$ axis, where the potential is given by

$$\phi_Q = \frac{Q}{4} \frac{3z^2 - r^2}{r^5}$$

These results are shown in Fig. 3 as dashed lines for comparison. This implies that there is substantial contribution from higher order multipoles. Moving away from the surface we find that the pure quadrupolar behavior predicted by Eq. (12) rapidly becomes dominant. It is easy
to obtain the contribution from the different higher order multipoles by either comparing the potential outside for different distances or by computing integrals of the obtained charge distribution inside. A precise experimental study of the field distribution in the neighborhood of superconducting ellipsoids should give detailed information on the charge distribution inside the superconductor.

As $\lambda_T$ decreases compared with the dimensions of the body the electric fields increase, and the electric field in the interior approaches the one of a uniformly charged ellipsoid of charge density $\rho_0$, given by $E_\rho = C_\rho \rho_0 \rho$, $E_z = C_z \rho_0 z$, with $C_\rho, C_z$ scale-independent geometrical constants. The electric fields at the surface are a fraction (whose value depends on $b/a$) of the fields at the surface of the uniformly charged ellipsoid. For example for $b/a = 1.5$ the limiting values for the fields at the surface are approximately $E_\rho^{\text{max}} \sim 0.13 E_\rho^{\text{max}}$, $E_z^{\text{max}} \sim -0.3 E_z^{\text{max}}$, with $E_\rho^{\text{max}}, E_z^{\text{max}}$ the maximum values of the fields inside, i.e. $C_\rho \rho_0$ and $C_z \rho_0 b$ respectively. The difference in potential at the surface between the poles and the equator reaches a limiting value which for the case $b/a = 1.5$ is approximately $\Delta \phi \sim 1.2 E_\rho^{\text{max}} a$.

As discussed in ref. [3], energy considerations show that $\rho_0$ decreases inversely with the linear dimension of the sample. Consequently the electric fields at the surface first increase as the sample size increases and then reach their limiting values once the sample is much larger than the penetration depth. The potential difference $\Delta \phi$ at the surface increases linearly with the linear size of the sample when the fields have reached their limiting values. We estimated in ref. [3] the maximum electric field in the interior of both high $T_c$ cuprates and $Nb$ to be of order $10^8 V/cm$, which according to the above discussion then yields for ellipsoids with $b/a \sim 1.5$ electric fields near the surface of order $10^8 V/cm$ for samples larger than the penetration depth, and potential difference between polar and equatorial points on the surface $\Delta \phi \sim 10^5 V \times a(cm)$, with $a$ the linear dimension of the body.

Such large electric fields are not seen near the surface of macroscopic superconductors. We propose that this is so because when the potential difference $\Delta \phi$ on the surface becomes larger than the work function $W$, electrons will ‘pop out’ of the superconductor near the region of low potential and migrate to the region of high potential on the surface, and the resulting electronic layer outside the superconductor will screen the electric field so that it becomes unobservable. For the example discussed above with a work function $W \sim 5 eV$ this implies that the predicted large electric fields will only be observed for samples with linear dimensions smaller than $a \sim 5000 A$. We emphasize however that this estimate is very rough and depends on intrinsic parameters of the superconducting material [2] and on the shape parameters. The maximum sample size for which the fields are unscreened will increase as the temperature increases (since $\rho_0$ will decrease) and the magnitude of the fields will decrease, such that the potential difference on the surface never exceeds the work function.

Note also that an attractive or repulsive electric force will exist between small superconducting ellipsoids depending on their relative orientation. A set of small superconducting ellipsoids in close proximity will lower their energy by adopting the proper orientation and coming close to each other, thus bunching into spherical aggregates as shown in Fig. 4. We suggest that this may be related to the remarkable experimental finding of formation of superconducting balls reported by Tao and coworkers [3].

Unlike the conventional description that is non- Lorentz invariant, Eq. (1) with $\hat{A}$ in the Lorenz gauge leads to a relativistically covariant description of the electrodynamics of superconductors [4, 10, 11], which we do not believe violates any basic physical law nor contradicts any known experimental fact. The resulting electric field in the interior of superconductors does not lead to a time dependent supercurrent because Eq. (3) replaces the conventional relation between current and electric field for ‘perfect conductors’ usually assumed to be valid for superconductors. Further experimental consequences and the relation with microscopic theory will be the subject of future work.

Acknowledgments

The author is grateful to E.N. Houstis, W.F. Mitchell, and J.R. Rice for making their GENCOL software available for public use, and to J. Kuti for help with implementation of the software.

[1] J.E. Hirsch, Phys.Lett.A 281, 44 (2001).
[2] J.E. Hirsch, Phys.Lett. A 309, 457 (2003).
[3] J.E. Hirsch, Phys.Rev. B 68, 184502 (2003).
[4] J.E. Hirsch and F. Marsiglio, Phys. Rev. B 39, 11515 ; J.E. Hirsch, Physica C 158, 326 (1989); Physica C 364-365, 37 (2001) and references therein.
[5] M. Tinkham, 'Introduction to Superconductivity', McGraw-Hill, New York, 1996.
[6] F. London and H. London, Physica 2, 341 (1935).
[7] J.E. Hirsch, Phys.Rev.B 62, 14487 (2000); Phys.Rev.B 62, 14498 (2000).
[8] Netlib Repository, [http://www.netlib.org/toms/637](http://www.netlib.org/toms/637)
[9] R. Tao, X. Xu, Y.C. Lan, Y. Shiroyanagi, Physica C 377, 357 (2002).
[10] J.E. Hirsch, [cond-mat/0312619](http://arxiv.org/abs/cond-mat/0312619) (2003).
[11] J. Govaerts, D. Bertrand and G. Stenuit, Supercond.Sci. Technol. 14, 463 (2001).