GRAPH-WREATH PRODUCTS AND FINITENESS CONDITIONS

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ABSTRACT. A notion of graph-wreath product is introduced. We obtain sufficient conditions for these products to satisfy the topologically inspired finiteness condition type $F_n$. Under various additional assumptions we show that these conditions are necessary. Our results generalise results of Cornulier about wreath products in case $n = 2$. Graph-wreath products include classical permutational wreath products and semidirect products of right-angled Artin groups by groups of automorphisms amongst others.

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1. INTRODUCTION AND BACKGROUND MATERIAL

In this paper we introduce a notion of graph-wreath product of groups and we explore finiteness conditions of groups constructed in this way. Cornulier established necessary and sufficient conditions for permutational wreath products to be finitely presented. We generalise Cornulier’s results in two ways, establishing results for a wider class than wreath products and extending the results to the higher finiteness conditions: type $F_n$. As we write this paper, Bartholdi, Cornulier and Kochloukova have announced a similar generalisation for the finiteness condition $F_{P_n}$: see [1]. Their results are for wreath products only, but are stronger than ours in that special case.

The graph-wreath product unites the concepts of permutational wreath product of two groups and graph product of a family of groups over a graph. Constructions of this kind have been considered by other authors: see [11] for an example. Our methods build on ideas of Davis, [8].

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1.1. Finiteness conditions. We are concerned with the finiteness condition type $F_n$ of a group $G$, meaning that there is an Eilenberg–Mac Lane space with finite $n$-skeleton. This is a property enjoyed by all finite groups but is topologically inspired and so may be called a homotopical finiteness condition. We shall also need the finiteness conditions type $FP_n$ for groups and modules. The following are some elementary facts about these definitions. We refer the reader to one of the standards texts [3], [4] for this and other background material.

1.2. Lemma. A group $G$ is of type $FP_n$ if and only if the trivial module $\mathbb{Z}$ is of type $FP_n$ as a $\mathbb{Z}G$-module. Every group of type $F_n$ is of type $FP_n$. If $H$ is a subgroup of $G$ then $H$ is of type $FP_n$ as a group if and only if $\mathbb{Z} \otimes \mathbb{Z}H \mathbb{Z}$ is of type $FP_n$ as a $\mathbb{Z}G$-module.

1.3. Wreath products. The restricted wreath product $A \wr H$ of two groups $A$ and $H$ has base $B$ the set of functions from $H$ to $A$ with finite support and head $H$. It is the semidirect product $B \ltimes H$. More generally, if $H$ has a permutation representation through an action on a set $\Omega$ then the restricted permutational wreath product $A \wr \Omega H$ is constructed in the same way with base the set of functions from $\Omega$ to $A$ with finite support. We do not have anything to say about the unrestricted wreath product.

1.4. Simple graphs, cliques, and flag complexes. By a simple graph we mean a 1-dimensional simplicial complex. A clique in such a graph consists of a non-empty finite set of vertices each pair of which are joined by an edge. For $p \geq 1$ a $p$-clique is a clique with exactly $p$ distinct vertices. The flag complex generated by a simple graph $\Gamma$ is the simplicial complex whose 1-skeleton coincides with $\Gamma$ and in which every $p$-clique supports a $(p-1)$-simplex whenever $p \geq 3$. If $X$ is a simplicial complex then $X$ is called a flag complex if and only if it is the flag complex generated by its 1-skeleton. In general, by a non-face of $X$ we mean a set of $p$ distinct vertices with $p \geq 3$ which do not support a $(p-1)$-simplex. A non-edge is a pair of distinct vertices which are not joined by an edge. Another way of defining a flag complex is to say that it is a simplicial complex in which each non-face has a non-edge.

1.5. Group actions on graphs. Let $H$ be a group. By a simple $H$-graph we mean a simple graph on which $H$ acts by graph automorphisms: we allow elements of $H$ to invert edges. An action of $H$ on a simple graph $\Gamma$ induces an action on the flag complex generated by $\Gamma$: note that such an action need not be admissible in the sense of Brown [5], but becomes admissible on barycentric subdivision.

The following lemma is fundamental. It translates between the language of graph theory and the language of simplicial complexes.

1.6. Lemma. Let $H$ be a group and let $\Gamma$ be a non-empty simple $H$-graph. Let $X$ be the flag complex generated by $H$. Let $\Delta_m$ denote the set of $m$-simplices of $X$ and let $X^m$ denote the $m$-skeleton of $X$, for $m \geq 0$. Then the action of $H$ extends naturally to $X$ and the following are equivalent:

(i) $\mathbb{Z}\Delta_m$ is of type $FP_n$ as a $\mathbb{Z}H$-module.

(ii) $H$ has finitely many orbits of $(m+1)$-cliques and the stabilizer of each $(m+1)$-clique has type $FP_n$.

Proof. Assume that (i) holds. Since $n \geq 0$ we infer from (i) that $\mathbb{Z}\Delta_m$ is finitely generated as a $\mathbb{Z}H$-module. The decomposition of $\Delta_m$ into $H$-orbits gives rise to a
direct sum decomposition of $\mathbb{Z}\Delta_m$ as $\mathbb{Z}H$-module. Hence $\Delta_m$ falls into finitely many orbits. An $m$-simplex in the flag complex has $m + 1$ vertices which form a clique. So having finitely many orbits of $m$-simplices is the same condition as having finitely many orbits of $(m + 1)$-cliques. Choose a set $\Delta_0$ of orbit representatives in $\Delta$. Then $\Delta_0$ is finite and $\mathbb{Z}\Delta$ is isomorphic to $\bigoplus_{\delta \in \Delta_0} \mathbb{Z} \otimes \mathbb{Z}H$. Now $\mathbb{Z}\Delta$ is of type $\text{FP}_n$ over $\mathbb{Z}H$ if and only if each $\mathbb{Z} \otimes \mathbb{Z}H$ is of type $\text{FP}_n$ over $\mathbb{Z}H$. This in turn is equivalent to each stabilizer $H_\delta$ being of type $\text{FP}_n$, as a group by Lemma 1.2. The converse is equally straightforward and we leave the details.

1.7. **Graph products.** If $\Gamma$ is a simple graph and $A = (A_v)_{v \in V}$ is a family of groups indexed by the vertices of $\Gamma$ then the graph product $A^\Gamma$ is defined to be the quotient of the free product formed by imposing commutator relations that force elements over distinct vertices to commute when those vertices are joined by an edge. For example:

- If $\Gamma$ is the graph consisting of two vertices and one edge then $(A_1, A_2)^\Gamma$ is the direct product $A_1 \times A_2$.
- If $\Gamma$ is the graph consisting of two vertices and no edges then $(A_1, A_2)^\Gamma$ is the free product $A_1 \ast A_2$.
- If $\Gamma$ is a complete graph then $A^\Gamma$ is the direct sum $\bigoplus_{v \in V} A_v$.
- If $\Gamma$ is a discrete graph (with no edges) then $A^\Gamma$ is the free product $\ast_{v \in V} A_v$.

If all the vertex groups $A_v$ are equal to the same group $A$ then we write $A^\Gamma$ instead of $A^\Gamma$. These further examples belong to this case:

- For any graph, $\mathbb{Z}^\Gamma$ is the right-angled Artin group determined by $\Gamma$ and $(\mathbb{Z}/2\mathbb{Z})^\Gamma$ the right-angled Coxeter group.

2. THE GRAPH-WREATH PRODUCT AND STATEMENT OF RESULTS

2.1. **Definition.** Suppose that $H$ is a group acting on a graph $\Gamma$. Given a group $A$ we can form the graph product of the family in which the same group $A$ is placed over each vertex of $\Gamma$. Then the action of $H$ on $\Gamma$ induces an action of $H$ on the graph product $A^\Gamma$. We define the graph-wreath product to be the semidirect product $A^\Gamma \rtimes H$. We introduce the notation $A \wr_{\text{w}} H$ for this construction. When $H$ is trivial this is just the graph product. When $\Gamma$ is the complete graph on its vertex set then this is the wreath product. If $\Omega$ denotes the complete graph on the $H$-set $\Omega$ then $A \wr_\Omega H = A \rtimes_\Omega H$. If $\Gamma$ has two vertices and one edge and $C_2$ is the group of order two that inverts the edge then $A \wr_\Gamma C_2$ is the wreath product $A \rtimes C_2$. If on the other hand, $\Gamma$ has two vertices and no edge then $A \wr_\Gamma C_2$ is isomorphic to the free product $A \ast C_2$. The notation $\wr$ is intended to distinguish our definition from the wreath product while at the same time indicating a close resemblance and relationship.

Our results fall naturally into two categories: sufficient conditions which are easier to establish and may have wider immediate applicability; and the necessity of those sufficient conditions under various additional hypotheses. In the end it remains open to find a clean set of necessary and sufficient conditions.

2.2. **Main results.** Henceforth, $H$ is a group, $\Gamma$ is a non-empty simple $H$-graph and $A$ is a non-trivial group. For the remainder of this section, $G = A \wr_\Gamma H$ is a graph wreath product and $X$ is the flag complex associated to $\Gamma$. Let $\Delta_m$ denote the set of $m$-simplices of $X$. 
We state our results in terms of flag complexes. The stylistic difference from the works [1], [9] that results from this choice is nothing more than translation using Lemma 1.6.

2.3. **Theorem A.** The following conditions are sufficient for \( G \) to be of type \( \text{F}_n \).

(i) \( H \) is of type \( \text{F}_n \);

(ii) \( A \) is of type \( \text{F}_n \);

(iii) \( \mathbb{Z}\Delta_p(X) \) is of type \( \text{FP}_{n-1-p} \) over \( \mathbb{Z}H \) for \( 0 \leq p \leq n-1 \).

It may well be possible to establish the natural version of this result for the property \( \text{FP}_n \): we conjecture that the result continues to hold when the three instances of \( \text{F}_n \) are replaced by \( \text{FP}_n \) in the above statement. Indeed, this is the story presented in [1] in the case of wreath products. One can speculate that in the generality of graph-wreath products, such a generalisation continues to hold, but since it would involve intricate bookkeeping with signs in definitions of boundary maps in chain complexes we prefer to leave that case for the present.

In the case of type \( \text{F}_2 \) the sufficient conditions are also necessary.

2.4. **Theorem.** \( G \) is finitely presented if and only if \( A \) and \( H \) are finitely presented, \( \Gamma \) has finitely many orbits of edges, and each vertex of \( \Gamma \) has finitely generated stabiliser.

This is easy to prove by the same methods as Cornulier uses for wreath products and we leave the details to the reader. For completeness we state the very simple result regarding finite generation.

2.5. **Lemma.** \( G \) is finitely generated if and only if both \( A \) and \( H \) are finitely generated and \( \Gamma \) has finitely many orbits of vertices.

**Proof.** Cornulier’s proof for wreath products can be employed with essentially no modification, (see [9], Proposition 2.1).

We now see that the following is an alternative route to proving Theorem 2.3.

2.6. **Conjecture.** The following conditions are sufficient for \( G \) to be of type \( \text{FP}_n \).

(i) \( H \) is of type \( \text{FP}_n \);

(ii) \( A \) is of type \( \text{FP}_n \);

(iii) \( \mathbb{Z}\Delta_p(X) \) is of type \( \text{FP}_{n-1-p} \) over \( \mathbb{Z}H \) for \( 0 \leq p \leq n-1 \).

**Deduction of Theorem 2.3 from this Conjecture.** This is immediate from Theorems 2.3 and 2.4 because, for \( n \geq 3 \), a group is of type \( \text{F}_n \) if and only if it is both of type \( \text{F}_2 \) and of type \( \text{FP}_n \).

As a step towards finding necessary conditions for the graph wreath product to have homotopical finiteness we offer the following three results.

2.7. **Theorem B.** Suppose that \( G \) is of type \( \text{F}_n \) with \( n \geq 3 \). Assume that the stabilisers of \( p \)-cells of \( X \) are of type \( \text{FP}_{n-1-p} \) over \( \mathbb{Z}H \) for \( 0 \leq p \leq n-2 \). Then \( A \) and \( H \) are of type \( \text{F}_n \) and \( H \) acts cocompactly on the \((n-1)\)-skeleton of \( X \).

We have improved results in certain special cases and the following two cases are perhaps worthy of special mention.
2.8. **Theorem C.** If $A$ has infinite abelianisation then the following conditions are necessary and sufficient for $G$ to be of type $F_n$.

(i) $H$ is of type $F_n$
(ii) $A$ is of type $F_n$
(iii) $\mathbb{Z} \Delta_p(X)$ is of type $FP_{n-1-p}$ as a $\mathbb{Z}H$-module for $0 \leq p \leq n - 1$.

The assumption here that $A$ has infinite abelianisation was also used in [1].

2.9. **Theorem D.** If $H$ is polycyclic-by-finite then $G$ is of type $F_n$ if and only if the following conditions hold:

(i) $A$ is of type $F_n$.
(ii) $H$ acts cocompactly on the $(n-1)$-skeleton of $X$.

2.10. **Methods.** Our main tool is the polyhedral product of spaces described in detail by Davis [8]. If $X$ is a simplicial complex and $Y$ is a family of pointed pairs of spaces indexed by the vertices of $X$ then the polyhedral product $Y^X$ is a space constructed from products of finite subfamilies of $Y$ corresponding to simplices of $X$ which are then glued together in accordance with the adjacency of faces in $X$. In our situation, the graph $\Gamma$ that appears in the statements of our results is the 1-skeleton of the simplicial complex $X$. In particular, we are interested in the case when $X$ is an infinite complex. We take care to clarify the polyhedral product construction for this generality, but the essentials are in Davis’ work [8].

In the special case when $Y$ is a family of Eilenberg–Mac Lane spaces and $X$ is a flag complex then $Y$ is an Eilenberg–Mac Lane space for the graph product of the fundamental groups of $Y$.

We shall apply this by taking the family $Y$ to consist of a single pointed space copies of which are placed at each vertex of $X$. If $H$ is a group acting on $X$ by simplicial automorphisms then $H$ has an induced action on the polyhedral product $Y^X$ and the graph wreath product $\pi_1(Y) \ltimes \Gamma H$ acts on the universal cover $Y^X$. This construction can be varied by using subcomplexes of $X$ and subcomplexes of $Y$.

Building on ideas in [1] we show that more can be said for graph-wreath products in which $A = \mathbb{Z}$ is infinite cyclic. In these, the base $B$ is a right-angled Artin group and the polyhedral product of copies of $S^1 = K(\mathbb{Z}, 1)$ is the Salvetti complex. The cell structure of the Salvetti complex makes is particularly simple and this is why more can be said in case $A = \mathbb{Z}$ or more generally in case $A$ has $\mathbb{Z}$ as a homomorphic image. The observation that a group with $\mathbb{Z}$ as a quotient therefore has $\mathbb{Z}$ as a retract was used to advantage in [1] and can be used in the same way here.

3. **Some homological algebra**

We collect some homological algebra for later use. The reader may find it convenient to skip this section on first reading. We begin with some well known facts that apply to exact sequences of modules over an arbitrary associative ring $R$. Recall that a module $M$ is said to be of type $FP_n$ if and only if there is a projective resolution $P_* \rightarrow M$ in which the first $n + 1$ projective modules, that is $P_j$ with $0 \leq j \leq n$, are finitely generated. $M$ is of type $FP_0$ if and only if it is finitely generated. $M$ is of type $FP_1$ if and only if it is finitely presented. $M$ is of type $FP_\infty$ if $P_n$ can be chosen with all $P_j$ finitely generated: this condition is equivalent to being of type $FP_n$ for all $n$. For convenience we interpret $FP_\infty$ as a vacuous condition when $n < 0$.

The following lemma records information which can be found in Bieri’s notes, ([8] Proposition 1.4).
3.1. **Lemma.** Let $j$ be an integer and let $L \rightarrow M \rightarrow N$ be a short exact sequence of $R$-modules.

(i) If $M$ has type $FP_j$ then $N$ is of type $FP_j$ if and only if $L$ has type $FP_{j-1}$.

(ii) If $N$ has type $FP_{j+1}$ and $M$ has type $FP_j$ then $L$ has type $FP_j$.

(iii) If $L$ and $N$ have type $FP_j$ then $M$ has type $FP_j$.

3.2. **Lemma.** Let $k$ be a non-negative integer. Suppose that

$$\cdots \rightarrow C_j \rightarrow C_{j-1} \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow C \rightarrow 0$$

is an exact sequence of modules and that $k$ is a natural number such that $C_i$ is of type $FP_{k-1}$ for each $i$. Then $C$ is of type $FP_k$.

**Proof.** Write $C_{-1} := C$ and let $Z_j$ denote the kernel of the map $C_j \rightarrow C_{j-1}$ for $j \geq 0$. Set $Z_{-1} := C_{-1}$. Exactness of $C_*$ yields short exact sequences

$$Z_j \rightarrow C_j \rightarrow Z_{j-1}$$

Then $C_k$ is finitely generated and so $Z_{k-1}$ is also finitely generated. Applying Lemma 3.1 to the sequence $Z_{k-1} \rightarrow C_{k-1} \rightarrow Z_{k-2}$ we deduce that $Z_{k-2}$ is of type $FP_1$. Continuing in this way inductively we deduce that $Z_{k-j}$ is of type $FP_{j-1}$. The result now follows by setting $j = k + 1$. □

3.3. **Lemma.** Fix $k \geq 0$. Suppose that

$$0 \rightarrow M \rightarrow M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{k-1} \rightarrow M_k \rightarrow 0$$

is an exact sequence of modules in which $M_i$ is of type $FP_i$ ($0 \leq i \leq k$). Then $M$ is finitely generated.

**Proof.** This can be proved by induction on $k$. When $k = 0$ the result follows because type $FP_0$ for modules is equivalent to finite generation. If $k > 0$ then let $L$ denote $\text{Ker}(M_{k-1} \rightarrow M_k)$. From the short exact sequence

$$L \rightarrow M_{k-1} \rightarrow M_k$$

we deduce that $L$ has type $FP_{k-1}$ by Lemma 3.1. Now we can apply the inductive hypothesis to the exact sequence

$$0 \rightarrow M \rightarrow M_0 \rightarrow M_1 \rightarrow \cdots \rightarrow M_{k-2} \rightarrow L \rightarrow 0$$

which has length one less than the original. □

3.4. **Lemma.** Suppose that $\cdots \rightarrow Q_m \rightarrow Q_{m-1} \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow 0$ is a chain complex of finitely generated projective $R$-modules such that the $i$th homology group $H_i(Q_*)$ is of type $FP_{m-i}$ for $1 \leq i \leq m - 1$. Then $H_0(Q_*)$ is of type $FP_m$.

**Proof.** For $1 \leq i \leq m$ set $Z_i := \text{Ker}(Q_i \rightarrow Q_{i-1})$ and $B_i := \text{Im}(Q_{i+1} \rightarrow Q_i)$. Then the $i$th homology group $H_i := H_i(Q_*)$ fits into the short exact sequence $B_i \rightarrow Z_i \rightarrow H_i$ for $0 \leq i \leq m - 1$. Since $Q_m$ is finitely generated so is the quotient $B_{m-1}$. Since $B_{m-1}$ and $H_{m-1}$ are both finitely generated it follows that $Z_m$ is finitely generated. Now the short exact sequence $Z_{m-1} \rightarrow Q_{m-1} \rightarrow B_{m-2}$ in which $Q_{m-1}$ is finitely generated and projective is a finite presentation of $B_{m-2}$ and so $B_{m-2}$ is of type $FP_1$. Combining this with the information that $H_{m-2}$ is of type $FP_1$ we deduced that $Z_{m-2}$ is of type $FP_1$. Continuing in this way inductively, we deduce that $Z_{m-i}$ is of type $FP_{i-1}$ for each $i$ and in particular $Z_0$ is of type $FP_{m-1}$. From the short exact sequence $Z_0 \rightarrow Q_0 \rightarrow H_0$ we conclude that $H_0$ is of type $FP_m$. □
The next results concern a group \( G \) which is a semidirect product of a normal subgroup \( B \) by a subgroup \( H \). Following [1], we write \( G = B \rtimes H \) or \( G = H \rtimes B \) to emphasise this structure. The group algebra \( \mathbb{Z}B \) admits an action of \( G \) defined by \( \xi \cdot h b := \xi h b \) for \( \xi \in \mathbb{Z}B \), \( h \in H \) and \( b \in B \). We write \( b \) for the augmentation ideal of \( \mathbb{Z}B \); this is a \( \mathbb{Z}G \)-submodule.

3.5. **Lemma.** If \( G = B \rtimes H \) is of type \( F_n \) (resp. \( F_{n+1} \)) then so is \( H \).

**Proof.** This is because \( H \) is a retract of \( G \).

3.6. **Lemma.** ([1], Lemma 6). \( G = B \rtimes H \) is of type \( F_n \) if and only if \( H \) is of type \( F_p \) and \( b \) is of type \( FP_{n-1} \) as a \( \mathbb{Z}G \)-module.

For the proof of Theorem C we shall need two variations on two results of [1]. The first is a generalisation of ([1], Lemma 17).

3.7. **Lemma.** Suppose that \( G = B \rtimes H \) is a group of type \( F_n \) and that \( H_i(B, Z) \) is of type \( FP_{n-i-1} \) as a \( \mathbb{Z}H \)-module for \( 1 \leq i \leq n-1 \). Then \( H_i(B, Z) \) is of type \( FP_{n-i-1} \) as a \( \mathbb{Z}H \)-module.

**Proof.** Choose a projective resolution \( P_* \to b \) of \( b \) over \( \mathbb{Z}G \) in which \( P_i \) is finitely generated for \( 0 \leq i \leq n-1 \):

\[
\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to b \to 0.
\]

Apply the functor \(- \otimes_{\mathbb{Z}B} \mathbb{Z}\) to obtain the chain complex

\[
\cdots \to Q_n \to Q_{n-1} \to \cdots \to Q_1 \to Q_0 \to b/b^2 \to 0
\]

of \( \mathbb{Z}H \)-modules. Here the \( Q_i \) are projective modules over \( \mathbb{Z}H \) and they are finitely generated for \( i \leq n-1 \). Note that the chain complex is exact at \( Q_0 \) and at \( b/b^2 \). For each \( i \geq 1 \), we have homology \( H_i(Q_*) \cong \text{Tor}_i^{\mathbb{Z}H}(b, \mathbb{Z}) \cong H_{i+1}(B, \mathbb{Z}) \). For each \( i \geq 1 \), let \( B_{i-1} \) be the module of boundaries and let \( Z_i \) be the module of cycles. Thus we have short exact sequences

\[
B_i \to Z_i \to H_i
\]

and

\[
Z_i \to Q_i \to B_{i-1}
\]

for all \( i \geq 1 \). For \( i \leq n-1 \), \( Q_i \) is finitely generated as a \( \mathbb{Z}H \)-module and therefore \( B_i \) is also finitely generated. We also have the hypothesis that \( H_i \) is of type \( FP_{n-i-2} \) for each \( i \). Taking \( i = n-2 \) we deduce that \( Z_{n-2} \) is finitely generated as a \( \mathbb{Z}H \)-module, being the extension of \( B_{n-2} \) by \( H_{n-2} \) both of which are finitely generated. Since the \( Q_i \) are finitely generated and projective for \( i \leq n-1 \) we deduce from the exact sequence \( Z_{n-2} \to Q_{n-2} \to B_{n-3} \) that \( B_{n-3} \) is of type \( FP_1 \) as a \( \mathbb{Z}H \)-module. We also have that \( H_{n-3} \) is of type \( FP_1 \) and hence the extension \( Z_{n-3} \) is of type \( FP_1 \) from which we can deduce that the cokernel \( B_{n-4} \) is of type \( FP_2 \). Continuing inductively in this way, we deduce that \( B_0 \) is of type \( FP_{n-2} \) and therefore \( H_1(B, Z) = b/b^2 = \text{Coker}(B_0 \to Q_0) \) is of type \( FP_{n-1} \).

The second is a version of ([1], Proposition 19) designed to handle right-angled Artin groups.
3.8. **Proposition.** Let \( G = B \times H \) be a group of type \( \text{FP}_n \). Suppose that there is a \( \mathbb{Z}B \)-projective \( \mathbb{Z}G \)-resolution \( P_* \rightarrow \mathbb{Z} \) of the trivial module \( \mathbb{Z} \) with the property that for each \( i \geq 1 \), the induced map \( P_i \otimes_{\mathbb{Z}B} \mathbb{Z} \rightarrow P_{i-1} \otimes_{\mathbb{Z}B} \mathbb{Z} \) is zero. Assume also that for each \( i \), \( P_i \) is isomorphic to \( H_1(B, \mathbb{Z}) \otimes_{\mathbb{Z}H} \mathbb{Z}G \). Then \( H_j(B, \mathbb{Z}) \) is of type \( \text{FP}_{n-j} \) as a \( \mathbb{Z}H \)-module for \( 1 \leq j \leq n \).

Here, by a resolution of \( \mathbb{Z} \) we mean an exact sequence terminating at the right with \( \rightarrow \mathbb{Z} \rightarrow 0 \). The Proposition creates flexibility by considering resolutions using \( \mathbb{Z}G \)-modules which are not necessarily projective as \( \mathbb{Z}G \)-modules, only projective as modules for the subgroup \( B \). Before commencing the proof it may be helpful to draw attention to the following ingredient because there is a subtlety.

3.9. **Lemma.** Let \( G \) be a group with a normal subgroup \( B \). Let \( P_* \rightarrow \mathbb{Z} \) be as in the statement of Lemma \([5,3]\). Then the homology groups of the chain complex \( P_* \otimes_{\mathbb{Z}B} \mathbb{Z} \) inherit a \( G \)-action and as such they are isomorphic to the standard homology groups \( H_i(B, \mathbb{Z}) \) with their inherited \( G/B \)-action.

**Proof.** Let \( Q_* \rightarrow \mathbb{Z} \) be a projective resolution \( \mathbb{Z} \) over \( \mathbb{Z}G \). Then both chain complexes \( Q_* \otimes_{\mathbb{Z}B} \mathbb{Z} \) and \( P_* \otimes_{\mathbb{Z}B} \mathbb{Z} \) compute the homology of \( B \). Using the projectivity of the \( Q_i \) over \( G \) we deduce the existence of a \( G \)-chain map \( \phi : Q_* \rightarrow P_* \) extending the identity on \( \mathbb{Z} \). Since both resolutions are projective resolutions over \( \mathbb{Z}B \) this chain map is a \( \mathbb{Z}B \)-chain homotopy equivalence and so induces isomorphisms \( \phi_i : H_i(Q_* \otimes_{\mathbb{Z}B} \mathbb{Z}) \rightarrow H_i(P_* \otimes_{\mathbb{Z}B} \mathbb{Z}) \) for all \( i \). But clearly the maps \( \phi_* \) are \( G \)-maps. Therefore we have computed not only the same homology groups \( H_i(B, \mathbb{Z}) \) but also the same \( G \)-structure. [The subtlety derives from the fact that there may not exist a \( G \)-chain map from \( P_* \) to \( Q_* \) that is a \( G \)-chain homotopy inverse to \( \phi \).]

Keeping this in mind, the proof becomes straightforward.

**Proof of Proposition \([3,3]\)** We work by induction on \( n \). Thus we may assume that \( H_j(B, \mathbb{Z}) \) is of type \( \text{FP}_{n-j} \) for \( 1 \leq j \leq n-1 \). Lemma \([3,7]\) shows that \( H_1(B, \mathbb{Z}) \) is of type \( \text{FP}_{n-1} \). This deals with the case \( j = 1 \). With \( n \) fixed, we work by induction on \( j \). Suppose that \( H_j(B, \mathbb{Z}) \) is known to be of type \( \text{FP}_{n-j} \) for \( i \leq j-1 \) and that \( j > 1 \). We want to show that \( H_j(B, \mathbb{Z}) \) is of type \( \text{FP}_{n-j} \). The hypotheses ensure that \( H_j(B, \mathbb{Z}) \) is equal to \( P_j \otimes_{\mathbb{Z}B} \mathbb{Z} \). Let \( Z_i := \text{Ker}(P_i \rightarrow P_{i-1}) \) for each \( i \geq 1 \).

Let

\[
\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_{j+1} \rightarrow P_j \rightarrow P_{j-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow \mathbb{Z} \rightarrow 0
\]

be the a resolution as in the statement. Since \( G \) is acting on \( B \) in a way that respects the Artin group structure, this is an exact sequence of \( \mathbb{Z}G \)-modules which are free as \( \mathbb{Z}B \)-modules. In fact for each \( i \), \( P_i \) is isomorphic as a \( \mathbb{Z}G \)-module to the induced module \( H_i(B, \mathbb{Z}) \otimes_{\mathbb{Z}H} \mathbb{Z}G \). From the inductive hypothesis on \( j \), \( P_i \) is of type \( \text{FP}_{n-i} \) over \( \mathbb{Z}G \) for \( 0 \leq i < j-1 \). From the overarching inductive hypothesis on \( n \), \( P_i \) is of type \( \text{FP}_{n-i} \) for \( j \leq i \leq n \). In particular, the kernel \( Z \) of the map \( P_{j-1} \rightarrow P_{j-2} \) is of type \( \text{FP}_{n-j} \) over \( \mathbb{Z}G \) by Lemma \([3,3]\.)

Let

\[
\cdots \rightarrow P'_m \rightarrow P'_{m-1} \rightarrow \cdots \rightarrow P'_0 \rightarrow Z \rightarrow 0
\]

be a projective resolution of \( Z \) over \( \mathbb{Z}G \) that is witness to the type \( \text{FP}_{n-j} \). Take \( m = n - j \). Then \( P'_i \) is finitely generated for \( 0 \leq i \leq m \). Applying the functor \( - \otimes_{\mathbb{Z}B} \mathbb{Z} \) to this resolution we obtain a chain complex

\[
Q_m \rightarrow Q_{m-1} \rightarrow \cdots \rightarrow Q_0 \rightarrow 0
\]
in which \( Q_i = P_i' \otimes_{\mathbb{Z}H} \mathbb{Z} \) of finitely generated projective \( \mathbb{Z}H \)-modules. Using Lemma \( \ref{lemma} \) we deduce that the zeroth homology of this chain complex is of type \( \text{FP}_m \) and this is the desired conclusion. \( \square \)

4. Polyhedral products

4.1. Polyhedral products of groups and spaces. Given a simplicial complex \( X \) with vertex set \( V \), the 1-skeleton of \( X \) is a simple graph and so we write \( A^X \) for the graph product \( A^X \) over the 1-skeleton of \( X \): we may say that the polyhedral product \( A^X \) of groups is, by definition, the graph product \( A^X \). Now suppose that \( Y = (Y_v, Z_v, \ast_v)_{v \in V} \) is a family of pointed pairs of spaces indexed by the vertex set \( V \). For each \( v \) we suppose given a space \( Y_v \), a subspace \( Z_v \subseteq Y_v \) and a basepoint \( \ast_v \in Z_v \). Then we write \( Y^X \) for the polyhedral product which is defined as follows. As a set, \( Y^X \) consists of those tuples \( (y_v)_{v \in V} \) of elements of \( \prod_{v \in V} Y_v \) that satisfy the following conditions

\[
(\alpha) \ y_v = \ast_v \text{ for all but finitely many } v;
\]

\[
(\beta) \ \{v \in V; \ y_v \notin Z_v\} \text{ is the vertex set of a face of } X.
\]

The empty set of vertices is considered to be a face of \( X \) for the purposes of this definition. Thus if \( X = \emptyset \) then \( Y^X \) is a one point space. If \( X \) is a finite complex, condition \((\alpha)\) makes no impact and it is then clear how the space \( Y^X \) inherits a topology: namely we endow it with the compactly generated topology following from the product topology. In general, condition \((\alpha)\) allows us to view \( Y^X \) as the colimit (i.e. directed union) of the \( Y^{X_i} \) as the \( X_3 \) run through the finite subcomplexes of \( X \). The presence of basepoints \( \ast \) have the role of specifying the canonical inclusion maps \( Y^{X_i} \to Y^{X_j} \) when \( X_3 \subseteq X_j \) are finite subcomplexes of \( X \). Now we see that the natural topology is the colimit topology.

4.2. Cell structure. If the \( Y_v \) are cell complexes and the \( Z_v \) are subcomplexes, then there is a natural cell complex structure on the polyhedral product \( Y^X \).

In the special case when all the \( (Y_v, Z_v, \ast_v) \) are equal to the same basepointed pair \( (Y, Z, \ast) \) we write \( (Y, Z)^X \) for the polyhedral product. A cell in \( Y^X \) of dimension \( d \) arises is a product of cells \( \sigma_i \) in \( Y_v \) over some finite subset of vertices where \( \sum_i \dim \sigma_i = d \). Thus the \( (n-1) \)-skeleton of \( Y^X \) depends only on the \( (n-1) \)-skeleton of \( Y \) and the \( (n-1) \)-skeleton of \( X \).

If in addition, \( Z = \{\ast\} \) then we further abbreviate the notation by writing \( Y^X \). If \( Y \) is a connected cell complex with basepoint \( \ast \) then Davis has shown that \( Y^X = (Y, \{\ast\})^X \) is an Eilenberg–Mac Lane space for \( \pi_1(Y)^X \) if and only if \( Y \) is aspherical and \( X \) is a flag complex, (\cite{Davis}, Theorem 2.22 and Corollary 2.23). We describe these observations and results of Davis \cite{Davis} in a little more detail as follows.

4.3. Proposition. Let \( V = X^0 \) be the vertex set of a polyhedron \( X \). Let \( Y = (Y_v, \ast_v)_{v \in V} \) be a family of pointed Eilenberg–Mac Lane spaces with chosen basepoints. Let \( p_v : \bar{Y}_v \to Y_v \) be the projection map from the universal covering space and let \( \bullet \) be a basepoint in \( \bar{Y}_v \) lying over \( \ast_v \). Let \( \bar{Y} \) be the family of pointed pairs \( (\bar{Y}_v, p_v^{-1}(\ast_v), \bullet)_{v \in V} \) where \( p_v : \bar{Y}_v \to Y_v \) is the projection map from the universal covering space. Let \( A_v = \pi_1(Y_v, \ast_v) \) and let \( A \) denote the family \( (A_v)_{v \in V} \). Let \( K \) denote the kernel of the natural homomorphism from the graph product \( A^X \) to the direct sum \( \bigoplus_{v \in V} A_v \). Then
(i) If $X$ is a flag complex then $\tilde{\Y}^X$ is contractible and $\Y^X$ is an Eilenberg–Mac Lane space for the graph product $\Lambda^X$.
(ii) The universal covering map $\tilde{\Y}^X \to \Y^X$ factors through the polyhedral product $\Y^X$ and the intermediate covering has Galois correspondent $K$.

If $X$ is an $H$-simplicial complex and $Y$ is a pointed cell complex then the polyhedral product $\Y^X$ admits a natural action of the permutational graph-wreath product $\pi_1(Y)^{X_1} \wr_\chi H$.

4.4. Proposition. Let $A$ be a group. Assume that $X$ is a flag complex. If $Y$ is an Eilenberg–Mac Lane space of type $K(A, 1)$ then $\Y^X$ is an Eilenberg–Mac Lane space of type $K(A^X, 1)$.

4.5. Some Lemmas. Let $Y$ be a connected pointed cell complex with universal cover $\tilde{Y}$, and let $X$ be a simplicial complex. Write $\ast$ for the base point of $Y$ and write $p: \tilde{Y} \to Y$ for the universal covering map. We shall fix a basepoint $\ast$ over $\ast$ in $\tilde{Y}$ and the symbol $\tilde{Y}$ should be interpreted as meaning the triple $(\tilde{Y}, p^{-1}(\ast), \ast)$ when this is required by the context. Let $k$ be a natural number.

4.6. Lemma. If $X$ is non-empty and $\tilde{\Y}^X$ is $k$-connected then $\tilde{Y}$ is $k$-connected.

Proof. Fix any $v \in X^0$. Projection onto the $v$-coordinate defines a retraction $Y^X \to Y$. Therefore $\pi_j(Y)$ is isomorphic to a direct summand of $\pi_j(Y^X)$ for each $j$. The result follows.

4.7. Lemma. If $X$ is a flag complex and $\tilde{Y}$ is $k$-connected then $\tilde{\Y}^X$ is $k$-connected.

Proof. Writing $X$ as the directed union of the family $(X_\lambda)$ of finite full subcomplexes we have $Y^X = \lim \Y_{X_\lambda}$ and hence $\pi_j(Y^X) = \lim \pi_j(Y^X_{X_\lambda})$ for any $j$. Therefore it suffices to prove that $\Y^X_{X_\lambda}$ is $k$-connected for all $\lambda$. Since every full subcomplex of a flag complex is again a flag complex we have that every $X_\lambda$ is a flag complex and thus we may assume that $X$ is finite.

Since $Y$ is $k$-connected and $k \geq 1$ we can add cells of dimension $k + 1$ and greater to $Y$ in order to embed $Y$ into an aspherical space $U$ with the same $k + 1$-skeleton as $Y$. By ([8], Theorem 2.22), $U^X$ is aspherical and $U^X$ is contractible. Therefore the $(k + 1)$-skeleton of $\tilde{U}^X$ is $k$-connected. Since $Y^X$ and $U^X$ have the same $(k + 1)$-skeleton, so also do their universal covers, and so it follows that $\Y^X$ has a $k$-connected $(k + 1)$-skeleton and therefore $\tilde{\Y}^X$ is $k$-connected.

4.8. Lemma. Fix $k \geq 2$. Let $Y$ be a cell complex with a single 0-cell that is aspherical and has non-trivial fundamental group. Let $\sigma$ be a standard $k$-simplex and let $\partial \sigma$ be its boundary. Then $\tilde{\Y}^{\partial \sigma}$ is a homotopy equivalent to a bouquet of $k$-spheres and $H_k(\tilde{\Y}^{\partial \sigma})$ is non-zero.

Proof. This is proved as in the proof of ([8], Theorem 2.22). If $E$ denotes the preimage of the basepoint of $Y$ in $\tilde{Y}$ then $(\tilde{Y}, E)^{\partial \sigma}$ is homotopy equivalent to $(\text{Cone}(E), E)^{\partial \sigma}$ and is the spherical realization of a product building and so is homotopy equivalent to a bouquet of spheres as required. The non-vanished of homology is non-zero because this bouquet of spheres is not contractible.

5. The Theorems A, B, C, D

We fix the notation exactly as in §2.2.
5.1. Proof of Theorem A. Before commencing the proof we make some remarks about the action of \( H \) and the nature of the cellular chain complex of \( X \). Let \( X^m \) denote the \( m \)-skeleton of \( X \). The \( i \)th cellular chain group \( C_i(X) \) of \( X \) is, at a technical level, the \( i \)th singular homology group of the pair \( H(X^i, X^{i-1}) \). This is additively isomorphic to the free abelian group on the set of \( i \)-simplices in \( X \). The action of \( H \) may permute the vertices of a cell and so as a \( \mathbb{Z}H \)-module, \( C_i(X) \) is a direct sum of signed permutation modules \( \bigoplus_{i=1}^{\dim \sigma \in \Sigma} \mathbb{Z}_{\sigma} \) where the sum is taken over orbit representatives of cells and the subgroups \( H_\sigma \) are the setwise stabilizers of each cell and \( \mathbb{Z}_{\sigma} \) denotes the orientation module for \( \sigma \): it is additively isomorphic to \( \mathbb{Z} \) on which orientation reversing elements of \( H_\sigma \) act as \(-1\).

We shall conclude the proof by finally appealing to Brown’s elementary criterion for a group, \( G \), to be of type FP\(_n\), [5], which we recall here for convenience. Suppose that \( G \) acts on a CW-complex \( X \), which is acyclic in dimensions less than \( n \), and such that the stabiliser of any \( p \)-cell is of type FP\(_{n-p}\), for \( 0 \leq p \leq n \). If then \( X \) has a finite \( n \)-skeleton mod \( G \), then \( G \) will be of type FP\(_n\). [1.1 Proposition, [5]].

We now assume the conditions of Theorem A, that both \( H \) and \( A \) are of type FP\(_n\) and that \( \mathbb{Z}\Delta_n(X) \) is of type FP\(_{n-1-p}\) for \( 0 \leq p \leq n-1 \).

**Proof that \( G \) is of type FP\(_n\).** Choose an Eilenberg–Mac Lane space \( Y \) for \( A \) with finite \( n \)-skeleton and with a single 0-cell. Since \( X \) is a flag complex and \( Y \) is an Eilenberg–Mac Lane space for the polyhedral product \( Y^X \) is an Eilenberg–Mac Lane space for the graph product \( A^X \) by Proposition 4.4. There is an action of \( H \) on both \( A^X \) and \( Y^X \) and this leads to an action of the graph-wreath product \( G \) on the universal cover \( \tilde{Y}^X \). The \( i \)th cellular chain group \( C_i(\tilde{Y}^X) \) is isomorphic to the induced module \( C_i(\tilde{Y}^X) \otimes_{\mathbb{Z}H} \mathbb{Z}G \). An \( i \)-cell \( \sigma \) of \( Y^X \) is a product of a family \( (\sigma_i)_{i \in V} \) of cells of \( Y \) indexed by the set \( V \) vertices of \( X \) and the dimensions of these cells must sum to \( i \). All but finitely many of the \( \sigma_i \) are equal to the base point of \( Y^X \) and this is fixed by \( H \). Thus the stabilizer of a typical cell is equal to the pointwise stabilizer of the vertices in the support. An \( i \)-dimensional cell is thus supported on a set of at most \( i+1 \) vertices. The setwise stabilizer of this cell is commensurate with the pointwise stabilizer of the supporting vertices. The number of orbits of \( i \)-dimensional cells is bounded by a function of the number of orbits of cells in the \( i \)-skeleton of \( X \) and the number of cells in the in the \( i \)-skeleton of \( Y \). This gives a finite bound on the number of orbits. Each \( i \)-cell stabilizer is commensurate with an \((i-1)\)-cell stabilizer of \( H \) on \( X \) and so we deduce from the assumption that \( H \) acts cocompactly on \( \mathbb{Z}X^i \) that \( C_i(Y^X) \) is of type FP\(_{n-i}\) as a \( \mathbb{Z}H \)-module for each \( i \). Now it follows at once that \( C_i(\tilde{Y}^X) \) is of type FP\(_{n-i}\) as a \( \mathbb{Z}G \)-module for each \( i \). It now follows from Brown’s elementary criterion, via Lemma 1.6 and Theorem 2.4 that \( G \) is of type FP\(_n\) and FP\(_2\) and hence of type FP\(_n\).

\( \square \)

5.2. Proof of Theorem B. We now assume that \( G \) has type FP\(_n\). The aim is to investigate what can be said about \( H \), \( A \) and \( X \). We may assume that \( n \geq 3 \) since the cases \( n \leq 2 \) are understood (see Theorem 2.4 and Lemma 2.5). It is immediate that \( H \) must have type FP\(_2\) because \( H \) is a retract of \( G \).

We work by induction on \( n \) and so we may assume that \( A \) is of type FP\(_{n-1}\), and that \( H \) acts cocompactly on the \((n-2)\)-skeleton, \( X^{n-2} \). Fix an Eilenberg–Mac Lane space \( Y \) for \( A \) with finite \((n-1)\)-skeleton and a single 0-cell. Recall from §4 that \( Y \) is now shorthand for \((Y, \ast)\) or \((Y, \{\ast\}, \ast)\) where \( \ast \) denotes the base point.
Applying Lemma 1.6, the cocompactness of the $H$-action on the $X^{n-2}$-skeleton and to the assumption in our statement of Theorem B we have the following condition.

$(\dagger)$ $\mathbb{Z}\Delta_p(X)$ is of type $\text{FP}_{n-1-p}$ as a $\mathbb{Z}H$-module for $0 \leq p \leq n - 2$.

To prove that $A$ is of type $F_n$ and that there are finitely many orbits of $(n - 1)$-cells in the flag complex $X$ we shall use constructions involving subcomplexes of $X$ and $Y$.

**Proof that $A$ is of type $F_n$.** Let $(Y_\alpha)$ be the family of finite subcomplexes of the Eilenberg–Mac Lane space $Y$ which contain the $(n - 1)$-skeleton. These are ordered by inclusion and have $Y$ as their filtered colimit. Note that since $n \geq 3$ these subspaces all have the same fundamental group $\alpha$.

Let $W_\alpha$ be the universal cover of the polyhedral product $Y_\alpha^X$. The family of $W_\alpha$ also form a filtered colimit system with colimit $W := \tilde{Y}^X$. The $W_\alpha$ all have the same $(n - 1)$-skeleton as each other and as $W$. Recall that $W$ is an Eilenberg–Mac Lane space for $A^X$. The $W_\alpha$ are therefore $(n - 2)$-connected and the structure of their cellular chain complex up to dimension $n - 1$ gives the exact sequence

$$0 \rightarrow Z_{n-1}(W) \rightarrow C_{n-1}(W) \rightarrow \cdots \rightarrow C_1(W) \rightarrow C_0(W) \rightarrow \mathbb{Z} \rightarrow 0$$

in which $Z_{n-1}(W)$ denotes the group of $(n - 1)$-cycles of $W$. As in the proof of Theorem A the conditions we have through induction and through $(\dagger)$ imply that $C_j(W)$ is of type $\text{FP}_{n-j-1}$ as a $\mathbb{Z}G$-module for $0 \leq j \leq n - 1$. The trivial module at the right hand end of this sequence is of type $\text{FP}_n$ over $\mathbb{Z}G$ because $G$ is of type $\text{FP}_n$. We deduce from Lemma 3.3 that $Z_{n-1}(W)$ is finitely generated. For each $\alpha$ we have short exact sequences

$$0 \rightarrow Z_{n-1}(W_\alpha) \rightarrow Z_{n-1}(W) \rightarrow H_{n-1}(W_\alpha).$$

Since $\lim\limits_{\alpha} H_{n-1}(W_\alpha) = H_{n-1}(W) = 0$ and $Z_{n-1}(W)$ is finitely generated we conclude that there is a choice $\alpha_0$ for which $H_{n-1}(W_{\alpha_0}) = 0$. This tells us that $Y_{\alpha_0}$ has trivial $(n - 1)$st homology group and since $n \geq 3$, the Hurewicz isomorphism holds and $Y_{\alpha_0}$ is $(n - 1)$-connected. Then an Eilenberg–Mac Lane space for $A$ can be chosen by adding cells of dimension $n + 1$ and greater to $Y_{\alpha_0}$. Therefore $A$ is of type $F_n$.

**Proof that $H$ acts cocompactly on the $(n - 1)$-skeleton of $X$.** For a contradiction, suppose that there are infinitely many orbits of $(n - 1)$-simplices in $X$. These orbits are countable in number so we may write the $(n - 1)$-skeleton of $X$ as a strictly ascending union of a chain of $H$-finite $H$-complexes beginning with the $(n - 2)$-skeleton:

$$X^{n-2} = X_0 \subset X_1 \subset X_2 \subset \cdots \subset X^{n-1},$$

$$X^{n-1} = \bigcup_{j \in \mathbb{N}} X_j.$$

Now the spaces $Y_{X_j}$, for $j \geq 0$ and $Y^{X^{n-1}}$ all have the same fundamental group $A^X$ and $Y^{X^{n-1}}$ is the union of the $Y_{X_j}$. On passing to universal covers we obtain a chain of $(n - 2)$-connected spaces, the $\tilde{Y}^X_j$, with union $\tilde{Y}^{X^{n-1}}$. We may assume that each $X_j$ has been chosen with exactly one orbit of missing $(n - 1)$-cells that are included in the next, $X_{j+1}$. We now show that the induced maps in homology:

$$H_{n-1}(\tilde{Y}^X_j) \rightarrow H_{n-1}(\tilde{Y}^{X_{j+1}})$$
have non-trivial kernel for all $j$. Let $\sigma$ be a representative of the orbit of missing $(n-1)$-cells in $X_j$ that are present in $X_{j+1}$. Then $\partial \sigma$ is a subcomplex of $X_j$ and $X_j \cup \sigma$ is a subcomplex of $X_{j+1}$. There is an inclusion and retraction of pairs

$$(Y^\sigma, Y^{\partial \sigma}) \rightarrow (Y^{X_j \cup \sigma}, Y^X_j) \rightarrow (Y^\sigma, Y^{\partial \sigma}).$$

This in Thus it suffices to show that the kernel of the map

$$H_{n-1}(\tilde{Y}^{\partial \sigma}) \rightarrow H_{n-1}(\tilde{Y}^\sigma)$$

is non-zero because we have a commutative square

$$\begin{array}{ccc}
H_{n-1}(\tilde{Y}^{\partial \sigma}) & \longrightarrow & H_{n-1}(\tilde{Y}^\sigma) \\
\bigg\downarrow & & \bigg\downarrow \\
H_{n-1}(\tilde{Y}^{X_j}) & \longrightarrow & H_{n-1}(\tilde{Y}^{X_{j+1}})
\end{array}$$

in which the vertical maps are injective. Note that $Y^{\partial \sigma}$ and $Y^\sigma$ have the same $(n-1)$-skeleton: $\sigma$ is the flag complex on the complete graph on its vertices, so by Proposition 4.3, $\tilde{Y}^\sigma = \tilde{Y}^{\partial \sigma}$ and since $n \geq 3$, there is an inclusion of $Y^{\partial \sigma}$ into $\tilde{Y}^\sigma$ because $Y^{\partial \sigma}$ and $Y^\sigma$ both have the same fundamental group. Moreover, $\tilde{Y}^{\partial \sigma}$ and $\tilde{Y}^\sigma$ are the same since they are both simply connected. So it suffices to prove that the map

$$H_{n-1}(\tilde{Y}^{\partial \sigma}) \rightarrow H_{n-1}(\tilde{Y}^\sigma)$$

has non-trivial kernel. This can be deduced from Lemma 5.8. Just as in the proof that $A$ is of type $F_n$, the $(n-1)$-cycles of the complexes $Y^X_j$ are independent of $j$ and form a finitely generated $\mathbb{Z}H$-module. We reach a contradiction by essentially the same reasoning as in the proof that $A$ is of type $F_n$. □

5.3. **Proof of Theorem C.** We refer the reader to Charney’s survey [7] for the following fact, which can be deduced from the observation that every cell in the Salvetti complex is a cycle. The reason for this is that every cell is a torus, which is a closed manifold.

5.4. **Lemma.** If $B$ is a right-angled Artin group and $S$ is its Salvetti complex then the boundary maps $d_i : C_i(S) \rightarrow C_{i-1}(S)$ are zero for $i \geq 1$ and $C_i(S) \cong H_i(S) \cong H_i(B, \mathbb{Z})$.

*Proof.* Details and further information can be found in [7, §2.7]. □

5.5. **Proposition.** Suppose that $A$ is infinite cyclic and that $G$ is of type $FP_n$. Then $H_j(B, \mathbb{Z})$ is of type $FP_{n-j}$ as a $\mathbb{Z}H$-module for $1 \leq j \leq n$.

*Proof.* Since $A$ is infinite cyclic, $B$ is a right-angled Artin group. By combining Proposition 5.8 with Lemma 5.4, the result follows. □

*Proof of Theorem C.* Suppose that the graph-wreath product $G$ is of type $FP_n$ with $n \geq 3$ and that $A$ has infinite abelianisation. Since $G$ is finitely generated so is $A$, by Lemma 2.5 Therefore $A$ has an infinite cyclic quotient $A \rightarrow \mathbb{Z}$ and clearly $\mathbb{Z}$ is a retract of $A$. Moreover the polyhedral product $A \rightarrow A^X$ is functorial in $A$ so it follows that $\mathbb{Z}^X$ is a retract of $A^X$, and $\mathbb{Z} \otimes_\Gamma H$ is a retract of $A \otimes_{\Gamma} H$. We infer that $\mathbb{Z} \otimes_\Gamma H$ is of type $FP_n$. The result now follows from Proposition 5.5 because
the homology groups \( H_i(B, \mathbb{Z}) \) of a right angled Artin group coincide with the permutation modules on the cells of the Salvetti complex (Lemma 5.4).

5.6. **Proof of Theorem D.**

*Proof.* We have the same setup as for Theorems A and B but here \( H \) is polycyclic and so \( \mathbb{Z}H \) is a Noetherian ring. Therefore \( \mathbb{Z}H \)-modules are of type \( \text{FP}_\infty \) whenever they are finitely generated. We prove Theorem D by induction on \( n \). We may assume that, as an inductive hypothesis, \( n \geq 3 \), \( H \) is of type \( \text{FP}_{n-1} \), \( A \) is of type \( \text{FP}_n \) and that \( H \) acts cocompactly on the \((n-2)\)skeleton of \( X \). The Noetherian property ensures that the chain groups of \( X \) up to dimension \( n-2 \) are of type \( \text{FP}_\infty \). Therefore we may argue in exactly the same way as for Theorem B that \( A \) is of type \( \text{F}_{n-1} \) and that \( X \) has only finitely many orbits of \((n-1)\)-cells.

□

6. **Examples**

We briefly describe some corollaries of our main theorems.

6.1. **Houghton groups.** The \( n \)th Houghton group \( H_n \) is the group of permutations of the set \( R_n := \{1, \ldots, n\} \times \mathbb{N} \) of \( n \) copies of \( \mathbb{N} \) (called rays) that act as translation far along each ray. There is a homomorphism \( H_n \to \mathbb{Z}^n \) which sends each permutation to its eventual translation vector. The image of this homomorphism is the group of zero-sum vectors and the kernel is the group of finitary permutations. In particular \( H_n \) is elementary amenable, being \((\text{locally finite})\)-by-(free abelian of rank \( n-1 \)). It is shown in [5] that \( H_n \) is of type \( \text{F}_{n-1} \) but not of type \( \text{F}_n \) for each \( n \geq 1 \). This makes Houghton’s group ideally suited to our theorems.

6.2. **Corollary.** Fix \( n \). Let \( A \) be a group. Then \( A \wr \Omega H_n \) is of type \( \text{F}_{n-1} \) if and only if \( A \) is of type \( \text{F}_{n-1} \).

*Proof.* Consider the action of \( H_n \) on its ray system. The stabilizer of any finite set of points in the ray system is isomorphic to a copy of the group \( H_n \) and is therefore always of type \( \text{F}_{n-1} \). Moreover, Houghton’s group acts \( k \)-transitively for all \( k \). Therefore the result follows from Theorems A and B.

□

Thompson’s groups, also discussed in [5] where the \( \text{FP}_\infty \) property is established, also have useful permutation actions. For example the classical Thompson group \( F \) acts as a group of order preserving permutations of the set \( D \) of dyadic rationals in the interval \([0, 1]\). In just the same way as above we may state:

6.3. **Corollary.** For \( 0 \leq n \leq \infty \), \( A \wr D F \) is of type \( \text{F}_n \) if and only if \( A \) is of type \( \text{F}_n \).

7. **Iterated Wreath Products**

We shall be concerned only with the classically defined wreath product. Our interest is to explore which graph-wreath products contain iterated wreath products of finitely generated abelian groups as subgroups. For background information about these we refer the reader to [6].

If \( \Omega \) is a right \( H \)-set we write \( A \wr_\Omega H \) or \( A \wr (H, \Omega) \) for the generalised graph-wreath product \( A \wr_\Omega H \) where \( \Omega \) denotes the complete graph on \( \Omega \) with the induced action of \( H \).

A permutation group \((H, \Omega)\) consists of a group \( H \) and a right \( H \)-set \( \Omega \). For our purposes it is important to consider pointed permutation representations. Henceforth, our permutation representations will be triples \((H, \Omega, \ast \Omega)\) consisting of a
group $H$, a left $H$-set $\Omega$ and a chosen base point $*$ $\in \Omega$. Subsequently, if we use the notation $(H, \Omega)$ for a permutation representation it is to be understood that there is a choice of basepoint which we have chosen to suppress in the notation by way of shorthand. If $(K, \Delta, *_{\Delta})$ is a second pointed permutation representation then the wreath product $(K, \Delta, *_{\Delta}) (H, \Omega, *_{\Omega})$ is defined as follows. The base $B$ of the wreath product is the set of functions from $\Omega$ to $K$ with finite support. $B$ inherits a group structure pointwise from $K$. The group $H$ acts on $B$ by $b^\delta (\omega) = b(\omega h^{-1})$ for $b \in B$, $h \in H$ and $\omega \in \Omega$. The head of the wreath product is the group $H$. The wreath product itself is written $(K, \Delta) \wr (H, \Omega)$ and this is the semi-direct product $H \ltimes B$ of $B$ by $H$. It is the group of consisting of ordered pairs $(h, b)$ with multiplication defined by

$$(h, b)(h', b') = (hh', b^h(b')).$$  

There are two ways to make the wreath product into a permutation representation. These are often referred to as the imprimitive action and the product action. The value of making the wreath product into a permutation group is that one can then consider iterated wreath products.

7.1. **The wreath product with the imprimitive action.** In this case we define a right action of $(K, \Delta) \wr (H, \Omega)$ on $\Omega \times \Delta$. A typical element $(h, b)$ of the wreath product is an ordered pair with $h \in H$ and with $b$ being a function with finite support from $\Omega$ to $K$. The action is defined by

$$(\omega, \delta) \cdot (h, b) = (\omega \cdot h, \delta \cdot b(\omega h)).$$  

It is natural to choose $(*_{\Omega}, *_{\Delta})$ for the basepoint in $\Omega \times \Delta$ when such a choice is required.

The imprimitive wreath product satisfies an associative law:

7.2. **Lemma.** Let $(H, \Omega)$, $(K, \Delta)$ and $(L, \Xi)$ be three permutation representations. Then the iterated wreath product with imprimitive action,

$$((L, \Xi) \wr (K, \Omega)) \wr (H, \Omega), \Omega \times \Delta \times \Xi,$$

is independent of bracketing, up to isomorphism. However it is most convenient to think of this as bracketed to the left and that is how we refer to it.

7.3. **The wreath product with product action.** In this case we define a right action of $(K, \Delta) \wr (H, \Omega)$ on the set $\Delta^\Omega$ of functions from $\Omega$ to $\Delta$ with finite support. Recall that it is understood that $\Delta$ has a chosen basepoint $*_{\Delta}$ and the support of a function $b: \Omega \to \Delta$ is $\{\omega; b(\omega) \neq *_{\Delta}\}$. The constant function $*$ with value $*_{\Delta}$ serves as the basepoint of $\Delta^\Omega$. The action of $(K, \Delta) \wr (H, \Omega)$ is given by

$$(f \cdot (h, b))(\omega) = f(\omega \cdot h^{-1})b(\omega).$$  

We can use the imprimitive action to show that an iterated wreath product of copies of $\mathbb{Z}$ can be embedded in an iterated permutational wreath product of copies of Houghton’s group. It is easy to finite a regular orbit of an infinite cyclic subgroup of Houghton’s group on the ray system and this process can be iterated. In this way we can embed $\bigotimes_{m} \mathbb{Z} \cdots \mathbb{Z}$ into

$$H_n \wr_{R_n \cdots R_n} \cdots \wr_{R_n \cdots R_n} H_n.$$
where the bracketing is to the left and the iterated wreath product of Houghton groups can be shown to be of type $\text{FP}_{n-1}$ using Theorem A. More generally we may use

7.4. Lemma. Let $A$ be a finitely generated abelian group of Hirsch length $d$. Then there is an isomorphic copy of $A$ in the Houghton group $H_n$ which has a regular orbit on the ray system $R_n$, for any $n \geq 2d$.

Proof. Write $A$ as the direct sum $F \oplus \mathbb{Z}^d$ with $F$ finite and then define an action of $A$ on $F \sqcup \mathbb{Z}^d$ where elements of $F$ act with the regular action on $F$ and trivially on $\mathbb{Z}^d$ and elements of $\mathbb{Z}^d$ act by translation on $\mathbb{Z}^d$ and trivially on $F$. This realizes $A$ as a group of permutations in an isomorphic copy of Houghton’s $H_{2d}$. \qed

7.5. Corollary. Any finitely iterated wreath product of finitely generated abelian groups, bracketed to the left, can be embedded in an elementary amenable group of type $\text{F}_n$ for arbitrarily large $n$.

Note that by increasing $n$ not only do we have the opportunity to embed more wreath products but the finiteness property $\text{F}_n$ is strengthened.

This is already surprising and reminds us of some tantalizing questions. Can other bracketings of wreath products of copies of $\mathbb{Z}$ be embedded in finitely presented groups: this would be interesting because other bracketings yields faster growing Følner functions, (see [10] for further information). We do not know how to do this. The wreath product with the product action seems to be inadequate for the purpose because we lose the cocompactness of action. The wreath product with imprimitive action seems to be inadequate because we lose the opportunity to find regular permutation representations of iterated wreath products of copies of $\mathbb{Z}$.

In addition to this there remains the fundamental question of how to improve the Theorems A, B, C, D so that there is a clear statement of necessary and sufficient conditions. It is also of interest to work through the results of this paper for type $\text{FP}_n$ and this has already been largely completed in the recent work [11] and also for the type $\text{FH}_n$, the finiteness condition introduced by Bestvina and Brady in [2]. Our difficulty with the property $\text{FP}_n$ is simply the immense task of keeping track of signs conventions in resolutions and it would be of interest to find an elegant solution to this issue. Our difficulty with managing $\text{FH}_n$ is that we are building complexes and if the building block is not simply connected at the outset then the Hurewicz isomorphism is unavailable: other authors may know a solution to this.

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