Robustly Clustering a Mixture of Gaussians

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Abstract

We give an efficient algorithm for robustly clustering of a mixture of arbitrary Gaussians, a central open problem in the theory of computationally efficient robust estimation, assuming only that for each pair of component Gaussians, their means are well-separated or their covariances are well-separated.

1 Introduction

The Gaussian Mixture Model has been the quintessential generative statistical model for multi-dimensional data since its definition and application by Pearson [19] more than a century ago: A GMM is an unknown discrete distribution over \( k \) components, each a Gaussian with unknown mean and covariance. Remarkably, such a model is always uniquely identifiable. It has led to the development of important tools in statistics. Over the past two decades, the study of its computational aspects has been immensely fruitful. Since the seminal paper of Dasgupta [5], there has been much progress on efficiently clustering and learning Gaussian Mixture Models. One line of results assumes that the component Gaussians are spherical and their means are sufficiently separated [21, 12]. Another line avoids the spherical component assumption, but makes stronger assumptions on the mean separation [1, 15, 4]. A more general approach of estimating all parameters without requiring any separation was introduced by Kalai, Moitra and Valiant [14, 18] and Belkin and Sinha [2] and is polynomial for any fixed number of components and desired accuracy. We discuss these developments in more detail presently.

In spite of its mathematical appeal and wide usability, the Gaussian Mixture Model and approaches to estimating it have a serious vulnerability — noise in the data. Robust statistics, which seeks measures that are immune to noise, is itself a classical topic [13] and has led to the definition of robust statistical parameters such as the Tukey median [20] and the geometric median. While statistically sound, such classical parameters are either computationally intractable in high dimension — they are NP-hard and the dependence on the dimension of all known algorithms is exponential — or have error factors that grow polynomially with the dimension. This is the case even for the most basic problem of estimating the mean of a distribution, even of a single Gaussian. Over the past few years, there has been significant progress in computationally efficient robust estimation, starting with mean and covariance estimation for a large family of distributions [6, 17]. Such robust estimation has also been discovered for various generative models including mixtures of well-separated spherical Gaussians (early work by Brubaker [3]; and improved bounds more recently [6, 11, 16]), Independent Component Analysis [17, 16] and linear regression. In spite of impressive progress, the core motivating problem of robustly estimating a mixture of (even two) Gaussians has remained unsolved.

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In this paper, we give a polynomial-time algorithm for the robust estimation (and clustering) of a mixture of arbitrary Gaussians, assuming only that either their means are separated or their covariance matrices are separated. We measure the separation with respect to the full distribution being in isotropic position, i.e., if the mixture has covariance $\Sigma$, then we need that for every pair of Gaussians $i, j$, at least one of

$$
\|\Sigma^{-1/2}(\mu_i - \mu_j)\|_2, \quad \|\Sigma^{-1/2}(\Sigma_i - \Sigma_j)\Sigma^{-1/2}\|_F
$$

is large (the distance between the parameters when the overall covariance is the identity). We note that this is an affine-invariant measure of the separation between the component Gaussians.

Before we present our main result, we note that there are two results in the literature that are directly relevant. First, the clustering algorithm of Brubaker and Vempala [4] works assuming that the means of the Gaussians are well-separated in the above affine-invariant sense; they note that for two Gaussians, this corresponds to hyperplane separability, i.e., there is a hyperplane that separates most of one component Gaussian from most of the other. They also give a generalization for a mixture of $k$ Gaussians assuming each mean can be separated from the span of the other $k - 1$ by a hyperplane. They do not consider robustness or separation in the covariances, which becomes relevant when the means are very close (or coincide). The other directly relevant work is for mixtures of spherical Gaussians. Using the Sum-of-Squares convex programming hierarchy, it is possible to cluster a mixture of spherical Gaussians assuming pairwise mean separation that is close to the best possible in quasi-polynomial time; and in polytime for separation that is at least $k^\varepsilon$ standard deviations for any $\varepsilon > 0$ [16, 11], improving on the polynomial-time $k^{1/4}$-standard-deviations separation result of [21]. While the results of [21, 4] are based on spectral methods and are not robust, the SoS-based results for spherical Gaussian mixtures are robust to adversarial noise in addition to having a weaker separation requirement. We note that all of these results assume mean separation, and all but [4] are for spherical Gaussians.

Our main result is a robust algorithm to cluster a mixture of arbitrary Gaussians assuming only that for each pair of component Gaussians, either their means or their covariances are separated in an affine-invariant manner. The error in the estimation of the mixture parameters is the noise rate plus an additional term that can be controlled.

**Theorem 1.** Let $F$ be a mixture of $k$ unknown Gaussian components $N(\mu_i, \Sigma_i)$ in $\mathbb{R}^d$ with mixing weights $w_i$, and $w = \min w_i$. Let $\tilde{F}$ be a noisy mixture obtained from $F$ with noise fraction bounded by $\eta \leq w/k$. Assume that when $F$ is in isotropic position, and for every pair of distinct components $i, j \in [k]$, we have

$$
\|\mu_i - \mu_j\|^2 \geq \frac{\Delta}{w} \quad \text{(mean separation)}
$$

or

$$
\|\Sigma_i - \Sigma_j\|_F \geq \frac{\Delta}{w} \quad \text{(covariance separation)}
$$

Then, for $\Delta \geq C\log(k/\varepsilon)$ and

$$
t = \frac{5\log(k/\varepsilon)}{\log(\Delta) - \log \log(k/\varepsilon)} = O(\log(k/\varepsilon)),
$$

there is a randomized algorithm that given a sample of size $n = \text{poly}(d^{O(t)}, 1/w, \log(1/\phi))$ from $\tilde{F}$, with probability at least $1 - \phi$, finds a clustering that is correct for all but $\eta + O(\varepsilon)$ fraction of the sample, in time polynomial in $d^{O(t^2)}, 1/w, \log(1/\phi)$. Further, if $\Delta \geq Ck^{5/2}/\varepsilon$, then $t = 2$ suffices to get the same conclusion; and for any $\nu \in (0, 1)$ if $\Delta \geq C(k/\varepsilon)^\nu$ then $t = O(1/\nu)$ suffices.
A few remarks are in order. First, once we get an approximately correct clustering, the samples in each cluster can be used to estimate the component mixing weight, mean and covariance. Second, the exponent in the running time could possibly be reduced to $O(t)$ as done in [16] for spherical Gaussian mixtures. The complexity is $d^O(\log^2 k)$ with $\log(k/\epsilon)$ separation, and is polynomial with $\text{poly}(k, 1/\epsilon)$ separation (compare this to the $d^{O(k)}$ bound for learning a mixture with no separation requirements). Finally, we only need pairwise separation, and even when the means of components coincide, a separation in their covariances in Frobenius norm suffices for the algorithm. A further improvement is conceivable — efficient clustering when the components have pairwise large total variation distance.

1.1 Approach

Known efficient algorithms for Gaussian mixtures are typically either based on spectral considerations, or more general (and less efficient) convex programming. While the former methods work well in practice, and yield relatively small polynomial bounds, they generally appear to be vulnerable to noise. It is worth noting though, that the approach of [3, 17] as well as the filtering approach of [6] build on such spectral methods for the robust estimation of mean and covariance (which includes robust estimation of a single Gaussian).

Ideally, one would like an algorithm for Gaussian mixtures that is polynomial in all parameters. The general algorithm of Kalai, Moitra and Valiant [14, 18] has complexity $d^{O(k)}$, even without noise. Unfortunately, this appears unavoidable, at least for any Statistical Query (SQ) algorithm [7], a model that captures most existing algorithms for problems over distributions [8, 9].

On the other hand, the approach of [4] is polynomial in all parameters assuming a separation between the means of the components. The separation needed is considerably weaker than previous work for mixtures of arbitrary Gaussians (we will shortly draw inspiration from recent progress for the case of spherical Gaussians as well), in that the separation required depends only on the standard deviation in some direction, i.e., there is some hyperplane separating each Gaussian from the rest, and the separation needed is proportional to standard deviation along the normal to the hyperplane (not e.g., the largest standard deviation). This measure of separation is affine-invariant.

Our starting goal was to find a robust version of [4] that remains polynomial in all parameters. Their technique, isotropic PCA, an affine-invariant version of PCA, is not robust (showing this is a bit more involved than for most spectral algorithms). So we turn for inspiration to the special case of spherical Gaussians for which robust algorithms have been recently discovered, with near-optimal separation [16, 11]. The key idea there is to express the identifiability of a Gaussian component in terms of a polynomial system, solve this polynomial system using a sum-of-squares semi-definite programming relaxation, and round the fractional solution obtained to a nearly correct clustering. The requirement of the polynomial system for identification is that the means are sufficiently separated. The time complexity (level of the SoS hierarchy needed) grows as the mean separation decreases.

We combine and generalize the above approaches as follows: (1) we use the affine-invariant separation condition to formulate a polynomial program for identifiability, and show that the SoS-based approach can be extended to this setting. For this step, we crucially use the robust estimation algorithms for mean and covariance under bounded moment conditions (by showing that these conditions hold for noisy mixtures). This normalization of moments is used in our identifiability proofs. (2) Then we show that even if the means are too close to guarantee clusterability, as long as the covariances are sufficiently separated, again in an affine-invariant manner, we still get polynomial identifiability. Moreover, these requirements are pairwise and therefore considerably weaker than the previous affine-invariant requirements of [4], which needed separation between each component.
mean and the span of the rest. (3) Finally, we extend the rounding algorithms in a natural manner to peel off one cluster at a time in approximately decreasing order of mixing weight, while maintaining the robustness of the overall algorithm.

2 Background and Preliminaries

**Noise model.** We assume that the data is generated as follows. First, a sample is generated from a pure Gaussian mixture. Then an adversary replaces up to an $\eta$ fraction of the data with arbitrary points. We refer to the pure Gaussian mixture as $F$ with

$$F = w_1 F_1 + w_2 F_2 + \ldots + w_k F_k$$

and each $F_i$ is the Gaussian $N(\mu_i, \Sigma_i)$, and the nonnegative mixing weights $w_1 \geq w_2 \geq \ldots \geq w_k > 0$ sum to 1. We refer to the noisy mixture as $\bar{F}$.

**Gaussian moments.** As in previous work on Gaussian mixture learning, the structure of Gaussian moments will play an important role. In particular, we note that for $X \sim N(0, \sigma^2)$, the $2t$'th moment is $\mathbb{E}(X^{2t}) = (2t - 1)!! \sigma^{2t}$. And more generally, for $X \sim N(\mu, \Sigma)$, for any vector $v$ of the same dimension as $X$,

$$\mathbb{E} \left( (X - \mu, v)^{2t} \right) = (2t - 1)!! \left( \mathbb{E} \left( (X - \mu, v)^2 \right) \right)^t.$$

An early, influential result of Pearson is that moments of a mixture suffice to uniquely identify its component Gaussians and mixing weights. The work of Kalai et al. [14] on learning pure 2-mixtures shows that the first 6 moments suffice.

**Isotropic position.** We say that a distribution $D$ in $\mathbb{R}^n$ is in isotropic position if $X \sim D$ satisfies

$$\mathbb{E}(X) = 0 \quad \text{and} \quad \mathbb{E} \left( XX^T \right) = I.$$

Any distribution with a bounded, full-rank covariance matrix can be brought to isotropic position by an affine transformation. Namely, if $\mathbb{E}(X) = \mu$ and $\mathbb{E} \left( XX^T \right) = A$, then the distribution of the random variable $Y = A^{-1/2}(X - \mu)$ is in isotropic position. Isotropic position of a distribution can be computed to desired accuracy from a sample via the sample mean and covariance.

**SoS relaxations.** The Sum-of-Squares hierarchy is a sequence of semi-definite programs that provide increasingly tighter relaxations of solutions to polynomial inequalities over $\{0,1\}^n$. The basic idea is to use multilinear variables of degree up to $t$ for some $t$, and rewrite the constraints in terms of these variables. More specifically, given a problem with Boolean variables $x_1, \ldots, x_n$ and polynomial constraints $P_1(x), \ldots, P_m(x) \geq 0$, for the level-$t$ SoS relaxation, define variables $Y_S = \prod_{i \in S} x_i$ for each $S \subset [n]$ with $|S| \leq t$, and the matrix $\mathcal{M}(Y)_{I,J} = Y_{I \cup J}$ for all $I, J \subset [m]$ and $|I|, |J| \leq t$. For each polynomial $P_i(x) \geq 0$ define a matrix $\mathcal{M}(Y, P_i)_{I,J} = \sum_{K \subset [n]} (P_i)_K Y_{I \cup J \cup K}$ where $(P_i)_K$ is the coefficient of $\prod_{i \in K} x_i$ in $P_i$ and $I, J \subset [n]$ with $|I|, |J| \leq t - \deg(P_i)/2$. The level-$t$ SoS relaxation is

$$Y_0 = 1$$

$$\mathcal{M}(Y) \succeq 0$$

$$\mathcal{M}(Y, P_i) \geq 0 \quad \text{for all} \ i \in [m].$$

This SDP on $n^{O(t)}$ variables with any linear objective function in these variables can be solved in time polynomial in $n^{O(t)} \cdot m \cdot \log(1/\varepsilon)$ to any desired accuracy $\varepsilon > 0$. 

\[4\]
Pseudo-expectations and Pseudo-distributions. Any point in the convex hull of points from \(\{0,1\}^n\) can be viewed as a probability distribution (convex combination) of 0/1 extreme solutions and naturally defines an expectation. If \(f(x)\) is a function of interest, then the expectation corresponding to a fractional solution \(z \in \mathbb{R}^n\) with \(z = \sum_{i=1}^{n} \alpha_i z^i\) where \(z^i \in \{0,1\}^n\) is \(E(f) = \sum_{i=1}^{n} \alpha_i f(z^i)\). Any solution to a level-\(t\) SoS program above can be viewed as defining a pseudo-expectation \(\hat{E} : \mathbb{R}[x] \rightarrow \mathbb{R}\), where \(R[x]\) is the set of all multi-linear functions over \(x\), with the following properties:

\[
\begin{align*}
\hat{E}(1) &= 1 \\
\hat{E}(Q^2(x)) &\geq 0 \text{ for every polynomial } Q(x) \text{ with } \deg(Q) \leq t \\
\hat{E}(Q^2(x)P_i(x)) &\geq 0 \text{ for every polynomial constraint } P_i(x) \geq 0, \ i \in [m], \\
&\text{ and every polynomial } Q(x) \text{ with } \deg(Q) \leq t - \deg(P_i)/2.
\end{align*}
\]

The pseudo-expectation behaves like a true expectation for polynomials of degree up to \(t\), and the above constraints are implied by the SoS constraints. For more detailed background, see e.g., [10].

3 Identifiability

In this section, we describe a set of polynomial equations and inequalities that will lead to the SoS relaxation and imply the desired properties of a pseudo-expectation obtained by solving the relaxation.

Definition 2. Let \(A\) be the following system of polynomial equations and inequalities on the variable vectors \(p \in \mathbb{R}^n\), \(\mu \in \mathbb{R}^d\) and \(\Sigma \in \mathbb{R}^{d \times d}\), given data points \(X_1, \ldots, X_n:\)

(a) \(p_i^2 = p_i\) for all \(i \in [n]\),

(b) \(\sum_{i \in [n]} p_i = N\),

(c) \(\frac{1}{N} \sum_{i \in [n]} p_i X_i = \mu\),

(d) \(\frac{1}{N} \sum_{i \in [n]} p_i (X_i - \mu, v)^2 \leq 2t^t \left(\frac{1}{N} \sum_{i \in [n]} p_i (X_i - \mu, v)^2\right)^t\) for all \(v \in \mathbb{R}^d\),

(e) \(Y_i = (X_i - \mu)(X_i - \mu)^T\) for all \(i \in [n]\),

(f) \(\frac{1}{N} \sum_{i \in [n]} p_i Y_i = \Sigma\),

(g) \(\frac{1}{N} \sum_{i \in [n]} p_i (Y_i - \Sigma, M)^{2t} \leq (2t)^{2t} \|\Sigma\|_2^{2t} \|M\|_F^{2t}\) for all \(M \in \mathbb{R}^{d \times d}\).

A solution \(p\) is an indicator vector of a subset \(S\) of the given points so that the subset approximately satisfies the moment conditions of a Gaussian up to the 2\(t\)th moment (d), and the points \(Y_i = (X_i - \mu(S))(X_i - \mu(S))^T\) also satisfy moment conditions up to the 2\(t\)th moment (g). Our goal is to ensure that the subset \(S\) identified is essentially one of the components of the mixture. We note that when the mixture is isotropic, then each component Gaussian has covariance bounded as \(\Sigma_i \preceq (1/w_i) I\), so the RHS of (d) is bounded by \(2(t/w)^t \|v\|^{2t}\), and the RHS of (g) is bounded by \((2t/w)^{2t} \|M\|_F^{2t}\).

In the next section, we describe the corresponding SoS relaxation \(\hat{A}\) and prove that its solution is a pseudo-distribution that satisfies \(A\) (Lemma 7). Here we analyze the constraint system and its implications.
Definition 3 (Well-separated Isotropic Sample). We say that \( X = \{X_1, \ldots, X_n\} \) is a \((\tau, \Delta)\)-separated isotropic sample with true clusters \( S_j, j \in [k] \) if for \( \mu_j = (1/|S_j|) \sum_{i \in S_j} X_i \) and \( \Sigma_j = (1/|S_j|) \sum_{i \in S_j} (X_i - \mu_j)(X_i - \mu_j)^T \), we have

1. for every pair \( i, j \in [k] \), either \( \|\mu_i - \mu_j\|_2^2 \geq \Delta/w \) or \( \|\Sigma_i - \Sigma_j\|_F \geq \Delta/w \),
2. for every \( j \in [k] \), \( |S_j| \geq w_j (1 - \eta)(1 - \tau)n \),
3. for every \( j \in [k] \), \( S_j \) satisfies \( A \) with \( N = |S_j| \).

Lemma 4 (Completeness). Suppose \( X = \{X_1, \ldots, X_n\} \) where each \( X_i \) is an i.i.d. sample generated from an isotropic noisy Gaussian mixture \( \hat{F} \) such that for every pair of Gaussian components \( i, j \in [k] \), either \( \|\mu_i - \mu_j\|_2^2 \geq (\Delta + \xi)/w \) or \( \|\Sigma_i - \Sigma_j\|_F \geq (\Delta + \xi)/w \). Then, with \( n = \text{poly}(d^{O(t)}, 1/\xi, \log(1/\phi)) \), \( X \) is a \((\tau, \Delta)\)-separated sample with probability at least \( 1 - \phi \).

The next lemma will be useful for the proof. We postpone its proof to Section 8.

Lemma 5. Suppose \( C_{4t} = (4t - 1)!! \). If \( X \sim N(0, I) \) and \( Y = XX^T \), then for any \( M \) with \( \|M\|_F = 1 \),

\[
\mathbb{E}((Y - I) \cdot M)^{2t} \leq C_{4t}.
\]

Let \( a_j \) be the indicator of \( S_j \), \( j \in [k] \). The main lemma of the analysis is stated below.

Lemma 6. Suppose that \( X = \{X_1, \ldots, X_n\} \) is a \((\tau, \Delta)\)-separated sample. Let \( \hat{E} \) be a degree-\( 2t \) pseudo-expectation which satisfies \( A \). Then for every pair \( i, j \in [k] \),

1. if \( \|\mu_i - \mu_j\|_2^2 \geq \Delta/w \), then \( \mathbb{E}((a_i, p)(a_j, p)) \leq 2 \left( \frac{16t}{\Delta} \right)^t N_i N_j \),
2. if \( \|\Sigma_i - \Sigma_j\|_F \geq \Delta/w \), then \( \mathbb{E}((a_i, p)(a_j, p)) \leq \left( \frac{\Delta}{\chi} \right)^{2t} N_i N_j \).

We note that these proofs themselves (in Section 8) will be based on degree-\( 2t \) SoS proofs, so that they can be preserved by the solution to an SoS relaxation of \( A \).

4 SoS Relaxation

This section is devoted to the following lemma.

Lemma 7. There is a system \( \hat{A} \) of polynomial equations on at most \( d^{O(t)} + n \) variables and containing at most \( d^{O(t)} \) constraints, such that, there is a degree-\( 2t \)-SoS proof that \( \hat{A} \) implies \( A \). Moreover, any feasible solution of the level-\( 2t \) SoS relaxation SoS\(_{2t}(\hat{A}) \) of \( \hat{A} \) is a pseudo-expectation \( \hat{E} \) satisfying \( A \).

We start by defining polynomial equations \( \hat{A} \). First, we need to deal with the fact that constraints (d) and (g) in \( A \) are for every unit vector \( v \). For this we use the quantifier elimination idea for SoS programs. We introduce extra variables \( Q_1 \) and \( Q_2 \) that are order-\( t \) tensors. By adding the nonnegative terms \( \|Q_1 v^{\otimes t}\|^2 \) and \( \|Q_2 M^{\otimes t}\|^2 \) to the LHS of the inequalities (d) and (g) in \( A \), we can eliminate \( v \) and \( M \); then we have \( d^{O(t)} \) equality constraints. Thus, \( \hat{A} \) will be a system of constraints on \( d^{O(t)} + O(n) \) variables and \( d^{O(t)} + \text{poly}(n) \) constraints as we desired.

Definition 8. Let \( \hat{A} \) be the following system of polynomial equations on the variable vectors \( p, \mu, \hat{\Sigma}, Q_1 \) and \( Q_2 \), given data points \( X_1, \ldots, X_n \):

(a') \( p_i^2 = p_i \) for all \( i \in [n] \).
(b') $\sum_{i \in [n]} p_i = N,$
(c') $\frac{1}{N} \sum_{i \in [n]} p_i X_i = \hat{\mu},$
(d') $\frac{1}{N} \sum_{i \in [n]} p_i (\langle X_i, X - \hat{\mu}, v \rangle)^2 + \|Q_1 v^{\otimes t}\|^2 = 2t \left( \frac{1}{N} \sum_{i \in [n]} p_i (\langle X_i, X - \hat{\mu}, v \rangle)^2 \right)^t$ for all $v \in \mathbb{R}^d,$
(e') $Y_i = (X_i - \hat{\mu})^T (X_i - \hat{\mu})$ for all $i \in [n],$ 
(f') $\frac{1}{N} \sum_{i \in [n]} p_i Y_i = \tilde{\Sigma},$
(g') $\frac{1}{N} \sum_{i \in [n]} p_i (Y_i - \tilde{\Sigma}, M)^2 + \|Q_2 M^{\otimes t}\|^2 = (2t)^2 \|\tilde{\Sigma}\|_2 \|M\|_F^2$ for all $M \in \mathbb{R}^{d \times d}.$

Next we describe the level-$\ell$ SoS-relaxation of $\hat{A},$ denoted by SoS$_{\ell}(\hat{A}).$

**Definition 9** (Sum-of-Squares Relaxation). Let $\ell \geq \deg(\hat{A})/2.$ Suppose that $\hat{A}$ is on $m$ variables $x_1, \ldots, x_m$ as in Definition 8. Define variables $Y_S = \prod_{i \in S} x_i$ where $S$ is any subset of $[m]$ with $|S| \leq \ell.$ Define matrix

$$M(Y)_{I,J} = Y_{I \cup J}$$

where $I$ and $J$ are subsets of $[m]$ of size at most $\ell.$ For each polynomial equations $(A_i = 0) \in \hat{A},$ define matrix

$$M(Y, A_i)_{I,J} = A_i Y_{I \cup J} = \sum_K (A_i)_K Y_{I \cup J \cup K}$$

where $I$ and $J$ are subsets of $[m]$ of size at most $\ell - \deg(A_i)/2,$ and $(A_i)_K$ is the coefficient of the term $\prod_{i \in K} x_i$ in $A_i.$ The resulting SoS relaxation SoS$_{\ell}(\hat{A})$ is defined by the set of constraints:

$$Y_{\emptyset} = 1,$$

$$M(Y) \succeq 0,$$

$$M(Y, A_i) = 0 \text{ for all } A_i \in \hat{A}.$$ 

The resulting system is defined on $(m_{\leq \ell}) = d^{O(\ell t)}$ variables $Y_S$ with $|S| \leq \ell.$ Using the ellipsoid method, this SDP can be solved up to an additive $\epsilon$-error in time proportional to $d^{O(\ell t)}.$ We will use $\ell = 2t.$ The following theorem (see [10]) shows that the solution of the SDP is actually a pseudo-expectation for $\hat{A}.$

**Theorem 10.** Let $\hat{A}$ be a set of polynomial constraints on $x$ and $Y \in \mathbb{R}^{m_{\leq \ell}}$ be any feasible point in SoS$_{\ell}(\hat{A}).$ Define the multilinearizing map $\overline{E}_Y : \mathbb{R}[x] \rightarrow \mathbb{R}$ as, where $\mathbb{R}[x]$ is the set of all multi-linear functions over $x,$

$$\overline{E}_Y \left( \prod_{i \in S} x_i \right) := Y_S,$$

for every $S \in [m],$ and extend $\overline{E}_Y$ linearly. Then $\overline{E}_Y$ is a degree $\ell$ pseudo-expectation for $\hat{A}.$

From (1), we can compute the pseudo-expectation of each multi-linear functions of degree at most $\ell$ over variable $p.$ In the rounding algorithm, we will only use $\overline{E}_Y$ up to the degree 2.
5 Robust Isotropic Position

We will need the following robust estimate of mean and covariance of the full Gaussian mixture. The theorem state below follows by combining the algorithm of [6] with the moment condition of [17], and proving the corresponding moment bounds.

**Theorem 11.** Let \( \tilde{F} \) be a noisy Gaussian mixture with unknown noise fraction \( \eta \) and Gaussian mixture of unknown mean \( \mu \) and covariance \( \Sigma \). There is a polynomial time algorithm which given \( n \) samples from \( \tilde{F} \) with \( n \geq \tilde{\Omega} \left( \frac{d^3 \log(1/\tau)}{\eta^2} \right) \), computes \( \hat{\mu} \) and \( \hat{\Sigma} \) with probability \( 1 - \tau \) within error

\[
\|\hat{\mu} - \mu\| = O(\eta \log(1/\eta)),
\]

\[
\left\| \hat{\Sigma} - \Sigma \right\|_F = O(\eta \log(1/\eta)).
\]

**Lemma 12.** Let \( F \) be a mixture of \( k \) Gaussians with mean \( \mu \). Then for \( X \sim F \), \( X \) satisfies the following bounded moment condition (2) for \( t = 2 \), \( C(2) = 3/w \) and \( t = 4 \), \( C(4) = 35/(3w) \):

\[
E \left( (X - \mu)^T v \right)^2 \leq C(t) \left( E \left( (X - \mu)^T v \right)^2 \right)^t \text{ for every unit vector } v,
\]

where \( w \) is the lower bound on the minimum mixing weight.

6 Algorithm

We can finally state the main algorithm for robust clustering.

**Algorithm 1 GMM Clustering**

Input: a \((\tau, \Delta)\)-separated sample \( X = \{X_1, \ldots, X_n\} \), error parameter \( \epsilon > 0 \) and SoS-degree \( t \in \mathbb{N}^+ \).

1. Use robust estimation (Theorem 11) to approximate the mean and covariance of the mixture; denote the results by \( \tilde{\mu} \) and \( \tilde{\Sigma} \). Apply the affine transformation \( \tilde{\Sigma}^{-1/2}(x - \tilde{\mu}) \) to make the mixture nearly isotropic.

2. Fix \( \alpha_0 = 1 \). For \( \ell = 1 \) to \( k \):

   (a) Find \( \alpha_\ell \) as the maximum \( \alpha \in (\eta, \alpha_{\ell-1}] \) s.t. the SoS SDP at level \( 2t \) with the polynomial constraint system \( \mathcal{A} \) and \( N = \alpha(1 - \eta)n \) is feasible. Let the solution be a pseudo-expectation \( \tilde{E} \) of degree \( 2t \).

   (b) Run the following rounding algorithm on \( \tilde{E} \) and \( \alpha_\ell \) to obtain a subset \( S \) of \([n]\):

      i. Let \( M = \tilde{E}pp^T \). Choose a uniformly random row \( i \) of \( M \) such that \( M_{ii} \geq \alpha_\ell/2 \). Let \( S \) be the largest \( N = \alpha_\ell(1 - \eta)n \) entries in the \( i \)’th row of \( M \).

      ii. Output \( S \). Set \( X = X \setminus S \).

7 Rounding Analysis

Suppose \( \tilde{E} \) is a pseudo-expectation of degree-\( 4t \) on \( p \) satisfying \( \mathcal{A} \) and \( \tilde{E}(a_i, p)\langle a_j, p \rangle \leq \delta N_i N_j \). The following main lemma of the rounding analysis gives the error guarantee of the rounding algorithm.
Lemma 13. The rounding algorithm outputs clusters $\hat{S}_j, j \in [k]$ such that with probability at least $\frac{1}{2} - \frac{2k\eta}{w}$, for all $j \in [k],$

$$|\hat{S}_j \cap S_j| \geq w_j(1 - \eta)n - \frac{2\sqrt{8k}\delta}{w_j} n - \eta_j n,$$

where $\sum_{j=1}^k \eta_j \leq \eta$. Moreover, $||S_j| - \alpha(1 - \eta)n| \leq (2\sqrt{8k}\delta/\alpha + \eta_j)n$.

Let $M = \tilde{E}pp^T$. To prove the main lemma, we use Lemma 14 and Lemma 15 to analyze each entry and each row of $M$.

Lemma 14. Let $\tilde{E}$ be a pseudo-expectation of degree-4$t$ on $p$ satisfying $A$. If a sample point $i \in [n]$ is generated from a Gaussian components, let $S(i)$ be the true cluster of $i$. Then

1. For any $i, j \leq n$, $0 \leq M_{ij} \leq M_{ii} \leq 1$,
2. For any $i \leq n$, $\sum_{j \leq n} M_{ij} \geq M_{ii} \alpha(1 - \eta)n$, and,
3. $\sum_{i,j : S(i) \neq S(j)} M_{ij} \leq \delta n^2$.

Proof. We first bound each entry of $M$

$$M_{ij} = \tilde{E}p_ip_j \leq \tilde{E}p_i = \tilde{E}p_i^2 = M_{ii}.$$  

Then for each row we have

$$\sum_{j \leq n} M_{ij} = \sum_{j \leq n} \tilde{E}p_ip_j = \tilde{E} \left( \sum_{j \leq n} p_j \right) = \tilde{E}p_i N \geq M_{ii} \alpha(1 - \eta)n.$$  

Finally, from the assumption at the beginning of this section, we have for every pair of components $l, m \in [k]$

$$\sum_{i \in S_l} \sum_{j \in S_m} M_{ij} = \tilde{E} \left( \sum_{i \in S_l} \sum_{j \in S_m} p_ip_j \right) = \tilde{E}\langle a_i, p \rangle \langle a_m, p \rangle \leq \delta N_i N_j.$$  

Then

$$\sum_{i, j : S(i) \neq S(j)} M_{ij} = \sum_{S_l, S_m \neq S(i)} \sum_{i \in S_l} \sum_{j \in S_m} M_{ij} \leq \delta n^2.$$  

Lemma 15. Choose a uniformly random row $i$ such that $M_{ii} \geq \alpha/2$. Then with probability at least $1 - \frac{2\eta}{\alpha} - \frac{1}{2k}$, $i$ is a “good row”, i.e, $i \in S(i)$, and denoting all points in clusters other than $S(i)$ as $\bar{S}(i)$, we have

$$\sum_{j \in \bar{S}(i)} M_{ij} \leq \frac{8k\delta}{\alpha^2} M_{ii} n.$$  

Proof. Since $\sum_{i \leq n} M_{ii} \geq N = \alpha(1 - \eta)n$, we have the number of rows such that $M_{ii} \geq \alpha/2$ is at least $\frac{\alpha - 2\eta}{2 - \alpha} n$. In these rows with $M_{ii} \geq \alpha/2$, there are at most $\eta n$ rows $i$ for which $X_i$ is from noise. By Part(3) of Lemma 14 and Markov’s inequality, the number of rows such that $\sum_{j \in \bar{S}(i)} M_{ij} \geq (8k\delta/\alpha^2)n M_{ii}$ is at most

$$\frac{\sum_{i,j : S(i) \neq S(j)} M_{ij}}{(8k\delta/\alpha^2)n M_{ii}} \leq \frac{\delta n^2}{(8k\delta/\alpha^2)n M_{ii}} \leq \frac{\alpha n}{4k}.$$
Thus, there are at least \( \left( \frac{\alpha(1-\eta)}{2-\alpha} - \eta - \frac{\alpha}{2k} \right) n \) good rows and the probability to pick a good row is at least
\[
\frac{\alpha-2\eta}{2-\alpha} - \eta - \frac{\alpha}{2k} \geq 1 - \frac{2\eta}{\alpha} - \frac{1}{2k}.
\]

Proof of Lemma 13. Suppose \( i \) is a good row as defined in Lemma 15. Call an entry of \( i \)'th row of \( M \) “large” if it exceeds \( \sqrt{\frac{8k\delta}{\alpha}} M_{ii} \). Using part (1) and (2) of Lemma 14, we obtain that the number of entries in the \( i \)'th row that exceed \( \sqrt{\frac{8k\delta}{\alpha}} M_{ii} \) is at least \( \alpha(1-\eta) n - \sqrt{\frac{8k\delta}{\alpha}} n \).

On the other hand, using Lemma 15 along with Markov’s inequality, we obtain that the number of large entries in \( \bar{S}(i) \) is at most
\[
\sum_{j \in \bar{S}(i)} M_{ij} \leq \frac{\sqrt{\frac{8k\delta}{\alpha}} n}{\sqrt{\frac{8k\delta}{\alpha}} M_{ii}} = \frac{\sqrt{8k\delta}}{\alpha} n.
\]

Suppose there are \( \eta_{S(i)} \) large entries corresponding to noise in the \( i \)'th row. Then there are at least \( \alpha(1-\eta) n - 2\sqrt{\frac{8k\delta}{\alpha}} n - \eta_{S(i)} n \) large entries with \( j \in S(i) \). When picking the largest \( \alpha(1-\eta) n \) entries of \( i \)'th row, we will pick all the large entries. So we have, using the fact that \( \alpha \geq w_{S(i)} \),
\[
|\hat{S}(i) \cap S(i)| \geq w_{S(i)}(1-\eta) n - 2\sqrt{\frac{8k\delta}{w_{S(i)}}} n - \eta_{S(i)} n.
\]

Since every sample point is in at most one cluster, we have \( \sum_{j=1}^{k} \eta_j \leq \eta \). And the last conclusion also follows from the above lower bound on the size of \( S(i) \).

8 Proofs

8.1 Moments of Mixtures

We first prove the bounded moments condition of a mixture of \( k \) Gaussians to apply the robust estimation algorithm on the mixture.

Proof of Lemma 12. Without loss of generality, we can assume that \( \mu = 0 \). It suffices to prove the inequality for one dimension case because the projection of mixture of \( k \) Gaussians into any direction is still a mixture of \( k \) Gaussians.

Suppose \( E_i \) is the expectation over \( i \)'th Gaussian component, and \( \mu_i \) and \( \sigma_i^2 \) are the mean and the variance of \( i \)'th Gaussian component. Let \( C_{2k} \) be the \( 2k \)’th moment constant for a Gaussian. Then \( C_{2k} = (2k - 1)!! \). For one component, we have
\[
E_i(X - \mu)^4 = E_i((X - \mu_i) + \mu_i)^4 = \mu_i^4 + 6\mu_i^2E_i(X - \mu_i)^2 + E_i(X - \mu_i)^4
\]
\[
= \mu_i^4 + 6\mu_i^2\sigma_i^2 + C_4 \sigma_i^4
\]

and
\[
E_i(X - \mu)^2 = E_i((X - \mu_i) + \mu_i)^2 = \mu_i^2 + 2\mu_iE_i(X - \mu_i) + E_i(X - \mu_i)^2
\]
\[
= \mu_i^2 + \sigma_i^2.
\]
Thus
\[ E_i(X - \mu)^4 \leq 3 E_i ((X - \mu)^2)^2. \]

Then
\[
E(X - \mu)^4 = \sum_{i=1}^{k} w_i E_i(X - \mu)^4 \\
\leq 3 \sum_{i=1}^{k} w_i E_i ((X - \mu)^2)^2 \\
\leq \frac{3}{w} \sum_{i=1}^{k} w_i^2 E_i ((X - \mu)^2)^2 \\
\leq \frac{3}{w} \left( \sum_{i=1}^{k} w_i E_i(X - \mu)^2 \right)^2 = \frac{3}{w} (E(X - \mu)^2)^2.
\]

Similarly, for one component we have
\[
E_i(X - \mu)^8 = E_i \left( ((X - \mu_i) + \mu_i)^8 \right) \\
= \mu_i^8 + 28 \mu_i^6 \sigma_i^2 + 46 C_4 \mu_i^4 \sigma_i^4 + 28 C_6 \mu_i^2 \sigma_i^6 + C_8 \sigma_i^8.
\]

Thus
\[
E_i(X - \mu)^8 \leq \frac{35}{3} E_i ((X - \mu)^2)^4.
\]

Then
\[
E(X - \mu)^8 = \sum_{i=1}^{k} w_i E_i(X - \mu)^8 \\
\leq \frac{35}{3} \sum_{i=1}^{k} w_i E_i ((X - \mu)^2)^4 \\
\leq \frac{35}{3w} \sum_{i=1}^{k} w_i^2 E_i ((X - \mu)^2)^4 \\
\leq \frac{35}{3w} \left( \sum_{i=1}^{k} w_i E_i(X - \mu)^2 \right)^4 = \frac{35}{3w} (E(X - \mu)^2)^4.
\]

Next we bound the moments of covariance estimates, used in the proof of identifiability.

**Proof of Lemma 5.** Note that \( \mathbb{E}Y = I \).

\[
\begin{align*}
\mathbb{E} \left( (Y - I) \cdot M \right)^{2t} &= \mathbb{E} \left( \sum_{i,j} (Y_{ij} - I_{ij}) M_{ij} \right) \\
&= \sum_{i_1,\ldots,i_{2t},j_1,\ldots,j_{2t}} \mathbb{E} ((Y_{i_1,j_1} - I_{i_1,j_1}) \cdots (Y_{i_{2t},j_{2t}} - I_{i_{2t},j_{2t}})) M_{i_1,j_1} \cdots M_{i_{2t},j_{2t}} \\
&= \sum_{\alpha,\beta} M(\alpha, \beta) \prod_{i \leq n} \mathbb{E} X_i^{\alpha(i)} (Y_{ii} - I_{ii})^{\beta(i)}
\end{align*}
\]
where \( \alpha(i) \) is even and \( \sum_{i \leq n} \alpha(i) + 2\beta(i) = 4t \). We will first prove that \( \mathbb{E} \left( X_i^{\alpha(i)} (Y_{ii} - 1)^{\beta(i)} \right) \leq \mathbb{E} \left( X_i^{\alpha(i)+2\beta(i)} \right) \).

\[
\mathbb{E} \left( X_i^{\alpha(i)+2\beta(i)} \right) - \mathbb{E} \left( X_i^{\alpha(i)} (Y_{ii} - 1)^{\beta(i)} \right) = \mathbb{E} \left( X_i^{\alpha(i)} \left( Y_{ii}^{\beta(i)} - (Y_{ii} - 1)^{\beta(i)} \right) \right)
\]

\[
= \int_{Y_{ii} \leq 1/2} + \int_{1/2 < Y_{ii} \leq 1} + \int_{Y_{ii} > 1}
\]

\[
\geq \int_{Y_{ii} \leq 1/2} + \int_{Y_{ii} > 1}.
\]

If \( \beta(i) \) is odd, both terms are positive and the statement is proved. Then we may assume \( \beta(i) \) is even. Since \( Y_{ii} = X_i^2 \) is always non-negative, we can see that if \( 0 \leq Y_{ii} \leq 1/2 \) or \( Y_{ii} > 1 \),

\[
X_i^{\alpha(i)} \left( Y_{ii}^{\beta(i)} - (Y_{ii} - 1)^{\beta(i)} \right) \geq Y_{ii} - 1.
\]

Then

\[
\left( \int_{Y_{ii} \leq 1/2} + \int_{Y_{ii} > 1} \right) X_i^{\alpha(i)} \left( Y_{ii}^{\beta(i)} - (Y_{ii} - 1)^{\beta(i)} \right) \geq \left( \int_{Y_{ii} \leq 1/2} + \int_{Y_{ii} > 1} \right) (Y_{ii} - 1).
\]

Since \( \mathbb{E}Y_{ii} = 1 \),

\[
\left( \int_{Y_{ii} \leq 1/2} + \int_{1/2 < Y_{ii} \leq 1} + \int_{Y_{ii} > 1} \right) (Y_{ii} - 1) = \mathbb{E}Y_{ii} - 1 = 0.
\]

So we get

\[
\left( \int_{Y_{ii} \leq 1/2} + \int_{Y_{ii} > 1} \right) (Y_{ii} - 1) \geq 0.
\]

We will show \( \sum_{\alpha, \beta} M(\alpha, \beta) \leq 1 \) next. Without loss of generality, we may assume \( M \) is symmetric since \( Y \) and \( I \) are symmetric. We note that \( M(\alpha, \beta) \) is a sum of degree-2t polynomials of \( M \), which have the form \( \prod M_{i_1 i_2} M_{i_2 i_3} \ldots M_{i_k i_1} \). Thus \( M(\alpha, \beta) \) are part of diagonal entries of \( M^{2t} \).

\[
\sum_{\alpha, \beta} M(\alpha, \beta) = \text{tr} \left( M^{2t} \right) = \sum \lambda \left( M^{2t} \right) \leq \left( \sum \lambda \left( M^2 \right) \right)^t = \|M\|_{F}^{2t} = 1.
\]

\[
\sum_{\alpha, \beta} M(\alpha, \beta) \prod_{i \leq n} \mathbb{E} \left( X_i^{\alpha(i)} (Y_{ii} - I_{ii})^{\beta(i)} \right) \leq \max_{\alpha, \beta} \prod_{i \leq n} \mathbb{E} \left( X_i^{\alpha(i)} (Y_{ii} - I_{ii})^{\beta(i)} \right)
\]

\[
\leq \max_{\alpha, \beta} \prod_{i \leq n} \mathbb{E} X_i^{\alpha(i)+2\beta(i)}
\]

\[
\leq \max_{i} \mathbb{E} X_i^{4t} \leq C_{4t}.
\]

8.2 Identifiability

In the proof of identifiability, we will use only inequalities that can themselves be proved using low-degree SoS proofs.
We first prove Lemma 6 Item 1 and Lemma 20 Equation (3), for any pair of components, assuming that their means are separated. We then prove the second part of both lemmas, assuming covariance separation.

The proof will use the following well-known facts. The notation $\vdash_t$ below indicates that the proof is a sum-of-squares proof of degree at most $t$. We omit the notation $\vdash_t$ indicating this explicitly when clear from context.

**Fact 16** (SoS Triangle Inequality). Let $a, b$ be indeterminates. Let $t$ be a power of 2.

$$\vdash_t (a + b)^t \leq 2^{t-1}(a^t + b^t).$$

The next two facts apply to pseudo-expectations.

**Fact 17** (SoS Hölder’s). Let $p_1, \ldots, p_n$ and $x_1, \ldots, x_n$ be indeterminates. Let $t$ be a power of 2.

$$\{p_i^2 = p_i \forall i \leq n\} \vdash_{O(t)} \left(\sum_{i \leq n} p_i x_i\right)^t \leq \left(\sum_{i \leq n} p_i\right)^t \left(\sum_{i \leq n} x_i^t\right).$$

and

$$\{p_i^2 = p_i \forall i \leq n\} \vdash_{O(t)} \left(\sum_{i \leq n} p_i x_i\right)^t \leq \left(\sum_{i \leq n} p_i\right)^t \left(\sum_{i \leq n} p_i^t x_i\right).$$

**Fact 18** (Pseudo-expectation Hölder’s). Let $p$ be a degree-$O(1)$ polynomial. Let $E$ be a degree-$4t$ pseudo-expectation on indeterminates $x$.

$$\tilde{E}p(x)^{2t-2} \leq \left(\tilde{E}p(x)^{2t}\right)^{\frac{t-1}{t}}.$$

**Fact 19** (Pseudo-expectation Cauchy-Schwarz). Let $E$ be a degree-$t$ pseudo-expectation on indeterminates $x$. Let $p, q$ be polynomials of degree at most $t/2$.

$$\tilde{E}p(x)q(x) \leq \left(\tilde{E}p(x)^2\right)^{1/2} \left(\tilde{E}q(x)^2\right)^{1/2}.$$

We will use the following lemma to prove Lemma 6.

**Lemma 20.** For every $j \in [k]$, there exists degree-$2t$ sum-of-squares proofs that $A$ implies

1. $\langle a_j, p \rangle^{2t} \|\hat{\mu} - \mu_j\|^{4t} \leq 2 \left(\frac{4t}{w}\right)^t n \|\hat{\mu} - \mu_j\|^{2t} \langle a_j, p \rangle^{2t-1}$ (3)

2. $\langle a_j, p \rangle^{2t} \|\hat{\Sigma} - \Sigma_j\|^{4t}_{F} \leq 2 \left(\frac{2t}{w}\right)^t n \|\hat{\Sigma} - \Sigma_j\|^{2t}_{F} \langle a_j, p \rangle^{2t-1}$. (4)

**Proof of Lemma 6 Item 1.** By Lemma 20 Equation (3) and Fact 19,

$$\tilde{E}\langle a_i, p \rangle^{2t} \langle a_j, p \rangle^{2t} \|\hat{\mu} - \mu_i\|^{4t} \leq 2^{2t+1} n \left(\frac{t}{w}\right)^t \tilde{E}\langle a_i, p \rangle^{2t-1} \langle a_j, p \rangle^{2t} \|\hat{\mu} - \mu_i\|^{2t}$$

$$\leq 2^{2t+1} n \left(\frac{t}{w}\right)^t \left(\tilde{E}\langle a_i, p \rangle^{2t-2} \langle a_j, p \rangle^{2t}\right)^{1/2} \left(\tilde{E}\langle a_i, p \rangle^{2t} \langle a_j, p \rangle^{2t} \|\hat{\mu} - \mu_i\|^{4t}\right)^{1/2}.$$
Rearranging gives

\[
\mathbb{E}(a_i, p)^{2t}(a_j, p)^{2t} \| \hat{\mu} - \mu_i \|^4 \leq 2^{4t+2}n^2 \left( \frac{t}{w} \right)^{2t} \mathbb{E}(a_i, p)^{2t-2}(a_j, p)^{2t}.
\]

This inequality also holds by symmetry with \( i \) and \( j \) exchanged. By the SoS triangle inequality,

\[
\| \hat{\mu} - \mu_i \|^4 + \| \hat{\mu} - \mu_j \|^4 \geq 2^{-4t+1} \| \mu_i - \mu_j \|^4 \geq 2^{-4t+1} \left( \frac{\Delta}{w} \right)^{2t}.
\]

Then

\[
\mathbb{E}(a_i, p)^{2t}(a_j, p)^{2t} \leq 2^{4t-1} \left( \frac{\Delta}{w} \right)^{-2t} \mathbb{E}(a_i, p)^{2t}(a_j, p)^{2t} \left( \| \mu_i - \hat{\mu} \|^4 + \| \mu_j - \hat{\mu} \|^4 \right)
\leq 2^{8t+1} t^{2t} \Delta^{-2t} n^2 \mathbb{E} \left( \langle a_i, p \rangle^{2t-2}(a_j, p)^{2t} + \langle a_i, p \rangle^{2t}(a_j, p)^{2t-2} \right)
\leq 2^{8t+1} t^{2t} \Delta^{-2t} n^4 \mathbb{E}(a_i, p)^{2t-2}(a_j, p)^{2t-2}
\leq 2^{8t+1} t^{2t} \Delta^{-2t} n^4 \left( \mathbb{E}(a_i, p)^{2t}(a_j, p)^{2t} \right)^{\frac{t-1}{t}}.
\]

The last inequality follows from Hölder’s inequality for pseudo-expectations (Fact 18). Rearranging gives

\[
\left( \mathbb{E}(a_i, p)^{2t}(a_j, p)^{2t} \right)^{1/t} \leq 2^{8t+1} t^{2t} \Delta^{-2t} n^4.
\]

Applying Cauchy-Schwarz again,

\[
\mathbb{E}(a_i, p) \langle a_j, p \rangle \leq \left( \mathbb{E}(a_i, p)^{2t}(a_j, p)^{2t} \right)^{1/2t} \leq 2^{4t+1} t^t \Delta^{-t} n^2.
\]

**Proof of Lemma 20 Equation (3).**

\[
\langle a_j, p \rangle \| \hat{\mu} - \mu_j \|^2 = \sum_{i \in S_j} p_i \langle \hat{\mu} - \mu_j, \hat{\mu} - \mu_j \rangle
= \sum_{i \in S_j} p_i \langle \hat{\mu} - X_i, \hat{\mu} - \mu_j \rangle + \sum_{i \in S_j} p_i \langle X_i - \mu_j, \hat{\mu} - \mu_j \rangle.
\]

Using SoS triangle inequality, we get

\[
\langle a_j, p \rangle^{2t} \| \hat{\mu} - \mu_j \|^{4t} \leq 2^{2t-1} \left( \sum_{i \in S_j} p_i \langle \hat{\mu} - X_i, \hat{\mu} - \mu_j \rangle \right)^{2t} + 2^{2t-1} \left( \sum_{i \in S_j} p_i \langle X_i - \mu_j, \hat{\mu} - \mu_j \rangle \right)^{2t}.
\]

By Hölder’s inequality, \( \{p_i^2 = p_i \ \forall i \leq n \} \) implies

\[
\left( \sum_{i \in S_j} p_i \langle \hat{\mu} - X_i, \hat{\mu} - \mu_j \rangle \right)^{2t} \leq \left( \sum_{i \in S_j} p_i \right)^{2t-1} \left( \sum_{i \in S_j} p_i \langle \hat{\mu} - X_i, \hat{\mu} - \mu_j \rangle^{2t} \right).
\]
Then using the inequality in $\mathcal{A}$,

$$\sum_{i \leq n} p_i(\hat{\mu} - X_i, \hat{\mu} - \mu_j)^{2t} \leq 2N_j t^t \left( \frac{1}{N} \sum_{i \leq n} p_i(\hat{\mu} - X_i, \hat{\mu} - \mu_j)^{2} \right)^t$$

$$\leq 2N_j t^t \left( \frac{1}{N} \sum_{i \leq n} p_i(X_i, \hat{\mu} - \mu_j)^{2} \right)^t$$

$$\leq 2N_j \left( \frac{t}{w} \right)^t \|\hat{\mu} - \mu\|^{2t}.$$  

The last inequality follows that the mixture is near isotropic. Thus

$$\left( \sum_{i \in S_j} p_i(\hat{\mu} - X_i, \hat{\mu} - \mu_j) \right)^{2t} \leq 2N_j \left( \frac{t}{w} \right)^t \|\hat{\mu} - \mu\|^{2t}. \tag{6}$$

Using Hölder’s inequality again,

$$\left( \sum_{i \in S_j} p_i(X_i - \mu_j, \hat{\mu} - \mu_j) \right)^{2t} \leq \left( \sum_{i \in S_j} p_i \right)^{2t-1} \left( \sum_{i \in S_j} \langle X_i - \mu_j, \hat{\mu} - \mu_j \rangle^{2t} \right).$$

Then condition 3 in Definition 3 gives

$$\sum_{i \in S_j} \langle X_i - \mu_j, \hat{\mu} - \mu_j \rangle^{2t} \leq 2N_j t^t \left( \frac{1}{N} \sum_{i \in S_j} \langle X_i - \mu_j, \hat{\mu} - \mu_j \rangle^{2} \right)^t$$

$$\leq 2N_j t^t \left( \frac{1}{N} \sum_{i \in S_j} \langle X_i, \hat{\mu} - \mu_j \rangle^{2} \right)^t$$

$$\leq 2N_j \left( \frac{t}{w} \right)^t \|\hat{\mu} - \mu\|^{2t}.$$  

Thus

$$\left( \sum_{i \in S_j} p_i(X_i - \mu_j, \hat{\mu} - \mu_j) \right)^{2t} \leq 2N_j \left( \frac{t}{w} \right)^t \langle a_j, p \rangle^{2t-1} \|\hat{\mu} - \mu_j\|^{2t}. \tag{7}$$

Combining the equations (5), (6), (7), we get

$$\langle a_j, p \rangle^{2t} \|\hat{\mu} - \mu_j\|^{4t} \leq 2^{2t+1} N_j \left( \frac{t}{w} \right)^t \|\hat{\mu} - \mu\|^{2t} \langle a_j, p \rangle^{2t-1}.$$  

The proof of identifiability when the pair of components is covariance-separated is similar as the mean-separated case.
Proof of Lemma 6 Item 2. By Lemma 20 Equation (4) and Fact 19,

\[ \mathbb{E}(a_i, p)^{2t} \langle a_j, p \rangle^{2t} \| \hat{\Sigma} - \Sigma_i \|_{F}^{4t} \leq \left( \frac{4t}{w} \right)^{2t} n \mathbb{E}(a_i, p)^{2t-1} \langle a_j, p \rangle^{2t} \| \hat{\Sigma} - \Sigma_j \|_{F}^{2t} \]

\[ \leq \left( \frac{4t}{w} \right)^{2t} n \left( \mathbb{E}(a_i, p)^{2t-2} \langle a_j, p \rangle^{2t} \right)^{1/2} \left( \mathbb{E}(a_i, p)^{2t} \langle a_j, p \rangle^{2t} \| \hat{\Sigma} - \Sigma_i \|_{F}^{4t} \right)^{1/2}. \]

Rearranging gives

\[ \mathbb{E}(a_i, p)^{2t} \langle a_j, p \rangle^{2t} \| \hat{\Sigma} - \Sigma_i \|_{F}^{4t} \leq \left( \frac{4t}{w} \right)^{4t} n^2 \mathbb{E}(a_i, p)^{2t-2} \langle a_j, p \rangle^{2t}. \]

This inequality also holds by symmetry with 1 and 2 exchanged. By SoS triangle inequality,

\[ \| \hat{\Sigma} - \Sigma_i \|_{F}^{4t} + \| \hat{\Sigma} - \Sigma_j \|_{F}^{4t} \geq 2^{-4t+1} \| \Sigma_i - \Sigma_j \|_{F}^{4t} \geq 2 \left( \frac{\Delta}{2w} \right)^{4t}. \]

Then

\[ \mathbb{E}(a_i, p)^{2t} \langle a_j, p \rangle^{2t} \leq \frac{1}{2} \left( \frac{\Delta}{2w} \right)^{-4t} \mathbb{E}(a_i, p)^{2t} \langle a_j, p \rangle^{2t} \left( \| \hat{\Sigma} - \Sigma_i \|_{F}^{4t} + \| \hat{\Sigma} - \Sigma_j \|_{F}^{4t} \right) \]

\[ \leq \frac{1}{2} \left( \frac{8t}{\Delta} \right)^{4t} n^2 \mathbb{E}(a_i, p)^{2t-2} \langle a_j, p \rangle^{2t} + \langle a_i, p \rangle^{2t} \langle a_j, p \rangle^{2t-2} \]

\[ \leq \left( \frac{8t}{\Delta} \right)^{4t} n^4 \mathbb{E}(a_i, p)^{2t-2} \langle a_j, p \rangle^{2t-2} \]

\[ \leq \left( \frac{8t}{\Delta} \right)^{4t} n^4 \left( \mathbb{E}(a_i, p)^{2t} \langle a_j, p \rangle^{2t} \right)^{1/4}. \]

The last inequality follows from pseudo-expectation Hölder’s. Rearranging gives

\[ \left( \mathbb{E}(a_i, p)^{2t} \langle a_j, p \rangle^{2t} \right)^{1/t} \leq \left( \frac{8t}{\Delta} \right)^{4t} n^4. \]

Applying Cauchy-Schwarz again,

\[ \mathbb{E}(a_i, p) \langle a_j, p \rangle \leq \left( \mathbb{E}(a_i, p)^{2t} \langle a_j, p \rangle^{2t} \right)^{1/2t} \leq \left( \frac{8t}{\Delta} \right)^{2t} n^2. \]

Proof of Lemma 20 Equation (4).

\[ \langle a_j, p \rangle \| \hat{\Sigma} - \Sigma_j \|_{F}^{2} = \sum_{i \in S_j} p_i (\hat{\Sigma} - \Sigma_j, \hat{\Sigma} - \Sigma_j) \]

\[ = \sum_{i \in S_j} p_i (\hat{\Sigma} - Y_i, \hat{\Sigma} - \Sigma_j) + \sum_{i \in S_j} p_i (Y_i - \Sigma_j, \hat{\Sigma} - \Sigma_j). \]

Using SoS triangle inequality, we get

\[ \langle a_j, p \rangle^{2t} \| \hat{\Sigma} - \Sigma_j \|_{F}^{4t} \leq 2^{2t-1} \left( \sum_{i \in S_j} p_i (\hat{\Sigma} - Y_i, \hat{\Sigma} - \Sigma_j) \right)^{2t} + 2^{2t-1} \left( \sum_{i \in S_j} p_i (Y_i - \Sigma_j, \hat{\Sigma} - \Sigma_j) \right)^{2t}. \]

(8)
By Hölder’s inequality, \( \{p_i^2 = p_i \; \forall i \leq n\} \) implies
\[
\left( \sum_{i \in S_j} p_i(\hat{\Sigma} - Y_i, \hat{\Sigma} - \Sigma_j) \right)^{2t} \leq \left( \sum_{i \in S_j} p_i \right)^{2t-1} \left( \sum_{i \in S_j} p_i(\hat{\Sigma} - Y_i, \hat{\Sigma} - \Sigma_j)^{2t} \right).
\]
Then using the inequality in \( A \),
\[
\sum_{i \leq n} p_i(\hat{\Sigma} - Y_i, \hat{\Sigma} - \Sigma_j)^{2t} \leq N(2t)^{2t} \left\| \Sigma_j \right\|^{2t}_2 \left\| \hat{\Sigma} - \Sigma_j \right\|_F^{2t} \leq N \left( \frac{2t}{w} \right)^{2t} \left\| \hat{\Sigma} - \Sigma_j \right\|_F^{2t}.
\]
Thus
\[
\left( \sum_{i \in S_j} p_i(\hat{\Sigma} - Y_i, \hat{\Sigma} - \Sigma_j) \right)^{2t} \leq N \left( \frac{2t}{w} \right)^{2t} (a_j, p)^{2t-1} \left\| \hat{\Sigma} - \Sigma_j \right\|_F^{2t}.
\]
(9)

Using Hölder’s inequality again, \( \{p_i^2 = p_i \; \forall i \leq n\} \) implies
\[
\left( \sum_{i \in S_j} p_i(Y_i - \Sigma_j, \hat{\Sigma} - \Sigma_j) \right)^{2t} \leq (a_j, p)^{2t-1} \left( \sum_{i \in S_j} (Y_i - \Sigma_j, \hat{\Sigma} - \Sigma_j)^{2t} \right).
\]

Then condition 3 in Definition 3 gives
\[
\sum_{i \in S_j} (Y_i - \Sigma_j, \hat{\Sigma} - \Sigma_j)^{2t} \leq N_j(2t)^{2t} \left\| \Sigma_j \right\|^{2t}_2 \left\| \hat{\Sigma} - \Sigma_j \right\|_F^{2t} \leq N_j \left( \frac{2t}{w} \right)^{2t} \left\| \hat{\Sigma} - \Sigma_j \right\|_F^{2t}.
\]
Thus
\[
\left( \sum_{i \in S_j} p_i(Y_i - \Sigma_j, \hat{\Sigma} - \Sigma_j) \right)^{2t} \leq N_j \left( \frac{2t}{w} \right)^{2t} (a_j, p)^{2t-1} \left\| \hat{\Sigma} - \Sigma_j \right\|_F^{2t}.
\]
(10)

Combining the equations (8), (9), (10) and \( N, N_j \leq n \), we get
\[
(a_j, p)^{2t} \left\| \hat{\Sigma} - \Sigma_j \right\|^{2t}_F \leq \left( \frac{4t}{w} \right)^{2t} n \left\| \hat{\Sigma} - \Sigma_j \right\|^{2t}_F (a_j, p)^{2t-1}.
\]

8.3 Proof of Main Theorem

Proof of Theorem 1. By Theorem 11, we can robustly estimate the mean of and the covariance matrix of the mixture in polynomial time. Then we can assume the sample is nearly isotropic.

By the separation assumption and Lemma 4, we have the set of noisy samples \( X \) is a \((\tau, \Delta - \xi)\)-separated sample with probability \( 1 - \tau \). If \( \|\mu_i - \mu_j\|^2 \geq (\Delta - \xi)/w \), by Lemma 6 Item 1, we have
\[
\mathbb{E}(a_i, p) (a_j, p) \leq 2 \left( \frac{16t}{\Delta - \xi} \right)^{t} N_i N_j.
\]
(11)
If \(\|\Sigma_i - \Sigma_j\|_F \geq (\Delta - \xi)/w\), by Lemma 6, we have
\[
\tilde{E}\langle a_i, p \rangle \langle a_j, p \rangle \leq \left(\frac{8t}{\Delta - \xi}\right)^{2t} N_i N_j.
\] (12)

Suppose \(\delta > 0\) satisfies \(\tilde{E}\langle a_i, p \rangle \langle a_j, p \rangle \leq \delta N_i N_j\). Then from (11), we can take any \(\delta \geq 2 \left(\frac{16t}{\Delta - \xi}\right)^t\) in the mean-separated case, and from (12), we can take any \(\delta \geq \left(\frac{8t}{\Delta - \xi}\right)^{2t}\) in the covariance-separated case. If we choose \(t\) so that
\[
2 \left(\frac{16t}{\Delta - \xi}\right)^t = O(\epsilon^2 k^{-5})
\] (13)
and
\[
\left(\frac{8t}{\Delta - \xi}\right)^{2t} = O(\epsilon^2 k^{-5}),
\] (14)
then we can take \(\delta = O(\epsilon^2 k^{-5})\).

Then Lemma 13 shows that the clustering is correct for all but \(\frac{2\sqrt{8k\delta}}{w} + \eta_j\) fraction of sample in cluster \(S_j\). So with probability \(\geq 1/2 - \frac{2k\eta}{w}\), the fraction of the sample wrongly clustered is at most
\[
\sum_{j=1}^{k} \frac{2\sqrt{8k\delta}}{w_j} + \eta_j \leq 2\sqrt{8k\delta} \sum_{j=1}^{k} (k - j + 1) + \eta \leq O(\epsilon) + \eta.
\]
The remaining part is to choose the minimal \(t\) satisfying (13) and (14). If \(\Delta \geq C \log(k/\epsilon)\), then we can let
\[
t = \frac{5 \log(k/\epsilon)}{\log(\Delta) - \log \log(k/\epsilon)}.
\]
If \(\Delta \geq Ck^{5/2}/\epsilon\), then \(t = 2\) suffices. If \(\Delta \geq C(k/\epsilon)^\nu\), then \(t = O(1/\nu)\) suffices.

If \(\eta < \frac{w}{6k}\), then the failure probability of the rounding algorithm is \(\frac{2k\eta}{w} + 1/2 < 5/6\). If we run the rounding algorithm \(O(\log(1/\phi))\) times, the failure probability is less than \(\phi\).

\[\blacksquare\]

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