Boosted Cylindrical Magnetized Kaluza-Klein Wormhole

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Abstract

In this work, we consider a vacuum solution of Kaluza-Klein theory with cylindrical symmetry. We investigate the physical properties of the solution as viewed in four dimensional spacetime, which turns out to be a stationary, cylindrical wormhole supported by a scalar field and a magnetic field oriented along the wormhole. We then apply a boost to the five dimensional solution along the extra dimension, and perform the Kaluza-Klein reduction. As a result, we show that the new solution is still a wormhole with a radial electric field and a magnetic field stretched along the wormhole throat.

Keywords: Kaluza-Klein theory, magnetic wormhole, extra solutions

1 Introduction

Wormholes are topological structures like bridges or tunnels that connect different universes or different parts of the same universe. They were first suggested by Einstein and Rosen [1], who noted that the Schwarzschild black hole has two exterior asymptotic regions connected by a throat (or bridge). The Einstein-Rosen wormhole is spacelike and classical objects can not pass through it, however, it has been argued that it can perhaps connect quantum particles in order to produce quantum entanglement and also the Einstein-Pololsky-Rosen effect [2], [3]. Wormholes might thus provide a geometric description of elementary particles, perhaps, at the Planck scale [4]. On the other hand, wormholes can be used to describe initial data for the Einstein equations [5], [6] whose time evolution is equivalent to the black hole collisions that has been recently observed by LIGO [7]. Further, they can supply information for understanding the evaporation of black holes [8], and also the evolution of universe [8]-[10]. Additionally, the microscopic wormholes could present a mechanism for vanishing cosmological constant problem [11]-[13].

An interesting topic is the general theory of a traversable Lorentzian wormholes which were introduced by Morris and Thorne [14]. These wormholes are static and spherically symmetric bridges accessible for classical particles or light [15]. Moreover, the static wormholes can be the solutions of Einstein equations in the presence of exotic matter such as phantom fields with negative kinetic energy which violate the null energy condition [16, 17]. Henceforth, for solving this problem, wormholes can be studied in other alternative theories of gravity such as the Gauss-Bonnet theory [18], the theories of non-minimally coupled scalar fields [19], massive (bi)gravity

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and higher dimensional theories of gravity. Among possible extensions of higher dimensional models, in this paper, we will focus on the Kaluza-Klein theory, in which the existence of an extra dimension would give rise to a unified picture of general relativity and electromagnetism.

In the authors obtained an analytical solution of wormholes in Kaluza-Klein theory with the topology of spacetime being $R^1 \otimes S^3 \otimes M^d$, in which the dimension of spacetime equals $D = 1 + 3 + d$, with $d$ being the dimension of internal space. They supposed that the internal space is static and compact, with a time dependent four dimensional spacetime. Chodos and Detweiler, obtained a class of spherically symmetric and asymptotically flat solutions which described wormholes in Kaluza-Klein theory. Furthermore, their solution was expanded to axisymmetric multi-wormholes. In [25], [26], and [27], it can be seen that, by parametrizing the off-diagonal metric elements in five dimensional Kaluza-Klein theory, the new locally anisotropic wormholes, which are vacuum solutions of Einstein field equations will be defined. In addition, cylindrically symmetric Abelian wormholes in $(4 + n)$ dimensions in the context of Kaluza-Klein theory are obtained in [28].

In this paper, we consider the magnetized and cylindrical Kaluza-Klein wormhole, which is a vacuum solution of Einstein field equations in five dimensions. We apply a boost along the extra dimension, and by performing the Kaluza-Klein reduction we arrive at a four dimensional solution, which is still a wormhole. However, the boosted solution has both an electric field normal to the wormhole and a magnetic field stretched along the wormhole throat. Consequently, we investigate the behavior of the magnetic and electric flux parallel and through the wormhole. We then explore the gauge fields, which are due to the topology of the spatial compact manifold in a higher dimensional manifold.

The organization of the paper is as follows. We begin in section by reviewing and briefly discussing the cylindrical and magnetized wormhole obtained in [29]. We next present in section the boosted Kaluza-Klein magnetized wormhole, and perform a Kaluza-Klein reduction to obtain a new four dimensional wormhole, and investigate its properties. In section we will calculate the magnetic and electric flux through the wormhole. Section concludes with some comments.

## 2 The Kaluza-Klein Wormhole: a brief review

We begin by reviewing the original magnetized cylindrical wormhole which is a vacuum solution of Einstein field equations in five dimensions in the context of Kaluza-Klein theory (see [29] for derivations). The solution is

$$ds^2(5) = -\frac{c}{(ar)^{2/3}}dt^2 + dr^2 + dz^2 - r^{4/3} |c + d \ln r| dw^2 + 2(ar)^{4/3} dwd\theta,$$

where $r$ is cylindrical radial coordinate, $z \in (-\infty, +\infty)$ is the longitudinal coordinate, $\theta \in [0, 2\pi]$, and $w$ is the extra coordinate. $a$, $c$, and $d$ are parameters.

By performing a Kaluza-Klein reduction along the coordinate $w$, the following scalar, and gauge fields can be obtained

$$\phi^2 = |r^{2/3} (c + d \ln r)|, \quad A_\theta = \frac{a^4}{\kappa|c + d \ln r|}.$$  

One can immediately find the electromagnetic tensor field as

$$F_{\theta r} = -F_{r \theta} = -\frac{da^4}{\kappa r (c + d \ln r)^2},$$  

where $\kappa = \frac{1}{\sqrt{8\pi G}}$. The magnetic and electric flux parallel and through the wormhole are calculated in [29].

The electric flux is

$$\Phi_E = \int_{\Sigma} F_{\theta r} dw = \int_{\Sigma} \frac{da^4}{\kappa r (c + d \ln r)^2} dw,$$

and the magnetic flux is

$$\Phi_M = \int_{\Sigma} F_{r \theta} dw = \int_{\Sigma} -\frac{da^4}{\kappa r (c + d \ln r)^2} dw.$$  

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which corresponds to a magnetic field along the $z$ coordinate ($B_z$). Furthermore, the four-dimensional spacetime can be calculated via the reduction from the five dimensional metric (1), given by

$$d s^2 = - \frac{c}{(ar)^{2/3}} dt^2 + dr^2 + \frac{a r^{4/3}}{|c + d \ln r|} d\theta^2 + dz^2,$$

which is a static, cylindrical wormhole supported by a scalar field and a magnetic field oriented along the wormhole ($z$-axis).

To conclude this section, we may remark that the magnetic flux on either side of the throat converge to a finite value. For understanding the gravitational and electromagnetic effects of the wormhole, the equations of motion for a neutral and a charged test particle were calculated, and we found the repulsive character of the wormhole gravitational field. Finally, we showed that the null energy condition is violated for this solution [29].

### 3 The Boosted Kaluza-Klein Wormhole

In this section, we perform a boost to the Kaluza-Klein magnetized cylindrical wormhole (1). The applied boost is along the extra coordinate $w$ with the boost parameter $\alpha$. We can make the following coordinate boost by renaming the metric coordinates (1) as $(t', r', z', \theta', w')$ and representing the boosted coordinate as $(t, r, z, \theta, w)$

$$t' = t \cosh \alpha - w \sinh \alpha,$$

$$w' = w \cosh \alpha - t \sinh \alpha,$$

with these transformations, the metric (1) will become

$$d s^2 = - \left( \frac{c}{(ar)^{2/3}} \cosh^2 \alpha + r^{4/3} |(c + d \ln r)| \sinh^2 \alpha \right) dt^2 + dr^2 + dz^2$$

$$- \left( \frac{c}{(ar)^{2/3}} \sinh^2 \alpha + r^{4/3} |(c + d \ln r)| \cosh^2 \alpha \right) dw^2$$

$$+ \left( \frac{c}{(ar)^{2/3}} + r^{4/3} |(c + d \ln r)| \right) \sinh 2\alpha dt dw$$

$$- 2 (ar)^{4/3} \sinh \alpha d\theta dt$$

$$+ 2 (ar)^{4/3} \cosh \alpha d\theta dw,$$

which again is a stationary (but not static) vacuum solution of Einstein field equations with vanishing Ricci scalar and Ricci tensor.

For our new metric (7), we can perform the Kaluza-Klein reduction on the extra coordinate $w$, and using the following ansatz for the metric which lies in the premise that the compact dimension of a $N$ dimensional differential manifold is orthogonal to the manifold [30]-[32]

$$(\hat{g}_{AB}) = \begin{pmatrix} g_{\mu\nu} + \kappa^2 \psi A_\mu A_\nu & \kappa \psi A_\mu \\ \kappa \psi A_\nu & \psi \end{pmatrix},$$

where $\hat{g}_{AB}$ is the five dimensional metric, and the four dimensional spacetime is given by $g_{\mu\nu}$, $A_\mu$, and $\psi$ are the vector (gauge) and scalar fields as new gravitational degrees of freedom, respectively.
Therefore, by using (8) we will obtain the four dimensional metric with the following components

\[ g_{tt} = -c \cosh^2 \alpha \frac{(ar)^{2/3}}{(ar)^{4/3}} - r^{4/3} \sinh^2 \alpha [(c + d \ln r) + c \sinh^2 \alpha] \]

+ \frac{\sinh^2 2\alpha (r^{4/3}(ar)^{2/3}[(c + d \ln r) + c] - 4r^{4/3}(ar)^{2/3} \cosh \alpha (c + d \ln r) + c \sinh^2 \alpha)}{(4r^{4/3}(ar)^{2/3} \cosh \alpha (c + d \ln r) + c \sinh^2 \alpha)^2}, \]  

(9)

\[ g_{rr} = g_{zz} = 1, \]  

(10)

\[ g_{\theta \theta} = \frac{a^4 r^4 \cosh^2 \alpha}{(ar)^{2/3} (r^{4/3}(ar)^{2/3} \cosh^2 \alpha [(c + d \ln r) + c \sinh^2 \alpha])}, \]  

(11)

and

\[ g_{\theta \phi} = \frac{a^2 r^2 \sinh 2\alpha \cosh \alpha (r^{4/3}(ar)^{2/3}[(c + d \ln r) + c] - (ar)^{4/3} \sinh \alpha)}{2(r^{4/3}(ar)^{2/3} \cosh^2 \alpha [(c + d \ln r) + c \sinh^2 \alpha])}. \]  

(12)

It is well known that a boost along the extra dimension, when applied to a five dimensional Kaluza Klein solution, leads to electrically charged solutions in the projected (3 + 1) dimensional solution. For example, if we start with a neutral, Schwarzschild-like solution in five dimensions, the boosted solution becomes something like the Reissner-Nordström charged black hole [33].

The boost along the fifth dimension, in the wormhole case we are considering, has led to a charged and magnetic solution. It is seen that a cross term \( dt d\theta \) appears in the transformed metric. Although this term looks like a rotation term in familiar spacetimes like the Kerr black hole, but in this case, it is fictitious and lacks a physical basis.

In order to see whether the new solution is still a wormhole or not, we calculate the circumference of a circle with \( r, t, z = \text{const} \), that is

\[ C(r) = \int_0^{2\pi} g_{\theta \theta} d\theta = 2\pi \frac{(ar)^{10/3} \cosh^2 \alpha}{r^{4/3}(ar)^{2/3} \cosh^2 \alpha [(c + d \ln r) + c \sinh^2 \alpha]} \]

(13)

In Fig. 1 we present the behavior of \( C(r) \) for representative values of \( a, c, d, \) and \( \alpha \). We can see that \( C(r) \) has a minimum, which is identified as the radius of the wormhole’s throat.

![Figure 1](image_url)  

Figure 1: Plot shows \( C(r) \). The radius of the throat is obtained after setting \( a = 0.6, c = 0.6, d = 13, \) and \( \alpha = 1.2 \), which corresponds to the value \( r_{th} = r_{min} = 1.96 \).
In the procedure of Kaluza-Klein reduction, the gauge fields $A_\mu$, and the scalar field $\psi$ can also be obtained by using (8), which are given by

$$A_t = -\frac{\sinh 2\alpha (r^{4/3}(ar)^{2/3}|(c + d \ln r)| + c)}{2\kappa (r^{4/3}(ar)^{2/3}\cosh^2 \alpha|(c + d \ln r)| + c \sinh^2 \alpha)}, \quad (14)$$

$$A_\theta = -\frac{a^2 r^2 \cosh \alpha}{\kappa (r^{4/3}(ar)^{2/3}\cosh^2 \alpha|(c + d \ln r)| + c \sinh^2 \alpha)}, \quad (15)$$

and

$$\psi = -\left(\frac{c}{(ar)^{2/3}} \sinh^2 \alpha + r^{4/3} \cosh^2 \alpha|(c + d \ln r)|\right), \quad (16)$$

respectively.

Substituting the gauge fields $A_\mu$ into the equation $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, it will give rise to the following electromagnetic field tensors

$$F_{rt} = \frac{c \sqrt[T]{(ar)^{2/3}} \sinh 2\alpha (2|(c + d \ln r)| + d)}{2\kappa (r^{4/3}(ar)^{2/3}\cosh^2 \alpha|(c + d \ln r)| + c \sinh^2 \alpha)^2}, \quad (17)$$

and

$$F_{r\theta} = \frac{a^2 r \cosh \alpha (a^{2/3} dr^2 \cosh^2 \alpha - 2c \sinh^2 \alpha)}{\kappa (r^{4/3}(ar)^{2/3}\cosh^2 \alpha|(c + d \ln r)| + c \sinh^2 \alpha)^2}. \quad (18)$$

It is worth noting that the electromagnetic field tensor $F_{r\theta}$ corresponds to a magnetic field along the $z$-axis, and $F_{rt}$ shows a transverse (radial) electric field, which appears after boost. Consequently, we have

$$F^{\pi \theta} = \frac{B_z}{r}, \quad (19)$$

$$F^{rt} = -E_r \rightarrow F_{rt} g^{tt} g^{rr} = -E_r. \quad (20)$$

In Fig. 2 an schematic of the wormhole after boosting, together with the directions of electric and magnetic fields is shown. We also present electric field $E(r)$ and magnetic field $B_z(r)$ in Figs. 3, 4, which both are a decreasing function of radial coordinate $r$. On the other hand, in Fig. 5 we present a visualization of the wormhole.
Figure 2: Schematic diagram of the wormhole after boosting with a radial electric field and a magnetic field along the $z$-axis.

Figure 3: The radial electric field $E_r$ as a function of $r$. In the special case $\alpha = 0$, the electric field vanishes, and we get back to the un-boosted solution.
Figure 4: The plot represents the magnetic field $B_z(r)$ in two different cases. (a): $a = 1.4$, $c = 2.4$, $d = 7$, and $\alpha = 2.8$. (b): $a = 0.2$, $c = 1.85$, $d = 2.24$, and $\alpha = 1.82$.

Figure 5: Visualization of the magnetic and electric wormhole. This plot is obtained by the requirement that the circumference of horizontal circles equals $C(r)$ and the vertical axis is $Z \equiv r$ with $r \geq r_{th}$.

4 Magnetic and Electric Flux

In this section, we derive the magnetic and electric flux through the wormhole. We will obtain the magnetic flux across the two dimensional hypersurface $t, z = constant$. The general expression for the magnetic flux is given by the following Gaussian integral [30]

$$\Phi_B = \int_{\theta_{min}}^{\theta} \frac{1}{2} F_{\mu\nu} ds_{\mu\nu} = \int_{r_{min}}^{r} \frac{1}{2} F_{r\theta} |g^{(2)}| g^{\mu\nu} g^{rr} g^\theta\theta dr d\theta, \quad (21)$$

where $ds_{\mu\nu}$ is an element of two dimensional surface area normal to the $z$ direction, and we take the magnetic flux integral from the location of the throat ($r_{min}$) to a distance $r$. The integral can not be calculated analytically, but according to our numerical calculations, for some ranges of the constants $a, c, d$, and $\alpha$, it converges to a constant value.
Now, the electric flux per unit length (see Fig. 2) can be computed via \[34\]

\[ \frac{\Phi_E}{z} = \int g^{rr} g^{tt} F_{rt} \sqrt{|g^{(2)}|} \mathrm{d}\theta, \tag{22} \]

which gives the following result

\[ \frac{\Phi_E}{z} = \frac{4\pi a^3 r^{10/3} \sinh 2\alpha \cosh^3 \alpha (d + 2|c + d \ln r|)}{(r^{4/3}(ar)^{2/3} \cosh^2 \alpha |c + d \ln r| + c \sinh^2 \alpha)^{3/2} (4r^{4/3}(ar)^{2/3} \cosh^2 \alpha |c + d \ln r| + c \sinh^2 \alpha)^{1/2}} \]  \[ \tag{23} \]

In order to understand the behavior of the electric flux, we present in Fig. 6 the quantity $\Phi_E/z$ as a function of $r$. It can be seen from the figure that the electric charge is not localized, since otherwise we would have obtained a constant electric flux. Moreover, the electric lines of force should pass through the throat, since the electric flux does not vanish at the throat (i.e., $\Phi_E(r_{\text{min}}) \neq 0$)

![Figure 6](image.png)

Figure 6: $\Phi_E/z$ in the case with $a = 1.28$, $c = 1.8$, $d = 0.7$, and $\alpha = 1.8$.

5 Conclusions

In [29], we presented a cylindrical five dimensional Ricci flat Kaluza-Klein solution. By using the Kaluza-Klein reduction, we derived at a four dimensional static metric, which described a wormhole. The wormhole geometry was cylindrically symmetric supported by a scalar field and a magnetic field. We further studied the magnetic flux for $r \geq r_{\text{th}}$, which converged to a constant value. We also showed that the null energy condition was violated.

The major focus of this work is on the boosted Kaluza-Klein cylindrical and magnetized wormhole. We first apply a boost along the extra dimension, and arrive to a five dimensional Kaluza-Klein vacuum solution. By performing the Kaluza-Klein reduction, we obtain a four dimensional spacetime supported by scalar, electric, and magnetic fields. In order to understand the structure of the new four dimensional solution, we minimize the proper circumference of concentric circles along the Killing field $\partial_\theta$, and show that there is a minimum, which corresponds to a wormhole throat. Consequently, after the boost, and as the most interesting outcome, the new solution is still a wormhole with a radial electric field, and a magnetic field oriented along the wormhole throat.
Next, we find the magnetic flux on one side of the wormhole, and show that it converges to a constant value for some ranges of $a$, $c$, $d$, and $\alpha$. Moreover, the electric flux per unit length is calculated, which shows that there is an extended charge distribution.

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