In this research we continue our previous investigation of wreath product normal structure \[1\].

The lattice of normal subgroups and their properties for finite iterated wreath products \(S_n \wr \ldots \wr S_n\), \(n, m \in \mathbb{N}\) are found. Special classes of normal subgroups and their orders and generators are found.

Further, the monolith of these wreath products is investigated by us.

Let \(k(\pi)\) be the number of cycles in decomposition of permutation \(\pi\) of degree \(n\). The number \(n - k(\pi)\) is denoted by \(\text{dec}(\pi)\), and is called a decrement \[2\] of permutation \(\pi\).

As well known \[2\] the minimal number of transpositions in factorization of a permutation \(\pi\) on transpositions is happen to be equal to \(\text{dec}(\pi)\). We set \(\text{dec}(\epsilon) = 0\). Therefore the decrement of \(n\)-cycle is \(n - 1\).

If \(\pi_1, \pi_2 \in S_n\), then the following formula holds:

\[
\text{dec}(\pi_1 \cdot \pi_2) = \text{dec}(\pi_1) + \text{dec}(\pi_2) - 2m, m \in \mathbb{N},
\]

where \(m\) is number of joint simplifying transpositions in \(\pi_1\) and \(\pi_2\).

The trivial subgroup of \(S_n\) we denote by \(E\).

**Definition 1.** The set of elements from \(S_n \wr S_n\), \(n \geq 5\) or \(n = 3\) of the tableaux form: \([e]_1, [a_1, a_2, \ldots, a_n]_2\), satisfying the following condition

\[
\sum_{i=1}^{n} \text{dec}[a_i]_2 = 2k, k \in \mathbb{N},
\]

we will call set of type \(\widetilde{A}^{(2)}\) and denote this set by \(E \wr \widetilde{A}_n\). For brevity of notation this subgroup be also denoted by \(\widetilde{A}_n^{(2)}\). It follows directly from the definition that the set of these elements supplemented by the operation of multiplication in the subdirect product, coincides with the group \(E \ltimes (S_n \boxtimes S_n \boxtimes S_n \ldots \boxtimes S_n)_n\), where subdirect product satisfies to condition \(2\).

We remind that the intersection of all non-trivial normal subgroups \(\text{Mon}(G)\) of \(G\) is called the monolith of a group \(G\).

**Proposition 2.** Elements of first type form the subgroup \(e \wr A_n\). This subgroup is the monolith of \(S_n \wr S_n\).

Now we can recursively construct easiest and elegant subgroup \(E \wr \widetilde{A}_n^{(2)}\) of \(S_n \wr S_n \wr S_n\).

**Definition 3.** The subgroup \(E \wr \widetilde{A}_n^{(2)}\) be denoted by \(\widetilde{A}_n^{(3)}\).

The order of \(E \wr \widetilde{A}_n^{(2)}\) is \((n!)^{3n} \cdot 2^3\). Furthermore we prove that \(E \wr \widetilde{A}_n^{(2)} \triangleleft S_n \wr S_n \wr S_n\).

Let the set of elements from \(S_n \wr S_n \wr S_n, n \geq 3\) of the form:

\[ [e]_1, [e, e, \ldots, e]_2, [a_1, a_2, \ldots, a_n]_3 \]

satisfying the following condition
\[ \sum_{i=1}^{n^2} \text{dec}(\{a_i\}^3) = 2k, k \in \mathbb{N}, \]  

be denoted by \( \tilde{A}_{n^2}^{(3)} \).

**Proposition 4.** The set of elements of type \( \tilde{A}_{n^2}^{(3)} \) forms a subgroup in \( S_n \wr S_n \wr S_n \). Moreover \( \tilde{A}_{n^2}^{(3)} < S_n \wr S_n \wr S_n \).

**Remark 5.** We note that \( \tilde{A}_{n^2}^{(3)} < A_{n^2}^{(3)} \). The order of \( A_{n^2}^{(3)} \) is \( (n!)^2 : 2 \). Furthermore \( \tilde{A}_{n^2}^{(3)} \triangleleft S_n \wr S_n \wr S_n \).

**Definition 6.** A subgroup in \( S_n \wr S_n \wr S_n \) is called \( \tilde{T}_n \) if it consists of:

1) elements of \( E \wr A_n \),
2) elements with the tableau \([3]\) presentation \([e]_1, [\pi_1, \ldots, \pi_n]_2\), that \( \pi_i \in S_n \setminus A_n \).

One easy can validates a correctness of this definition, i.e. that the set of such elements form a subgroup and its normality. This subgroup has structure \( \tilde{T}_n \cong (A_n \times A_n \times \cdots \times A_n) \times C_2 \cong S_n \bigsqcup S_n \cdots \bigsqcup S_n \), where the operation of a subdirect product \( \bigsqcup \) is subject to items 1) and 2).

**Definition 7.** A subgroup in \( S_n \wr S_n \wr S_n \) is of the type \( \tilde{T}_{n^2}^{(3)} \) if it consists of:

1) elements of the form \( E \wr E \wr A_n \),
2) elements with the tableau \([3]\) presentation \([e]_1, [e, \ldots, e]_2, [\pi_1, \ldots, \pi_n, \pi_{n+1}, \ldots, \pi_{n+2}]_3\), wherein \( \forall i = 1, \ldots, n: \pi_i \in S_n \setminus A_n \).

We define recursively the subgroup \( \tilde{T}_n^{(3)} \) having \( n \) different intervals of elements with the same parity permutations on \( X^2 \).

**Definition 8.** The subgroup of \( S_n \wr S_n \wr S_n \) having structure \( E \wr \tilde{T}_n \) is denoted by \( \tilde{T}_{n^2}^{(3)} \). The following isomorphism \( \tilde{T}_{n^2}^{(3)} \cong S_n \bigsqcup \cdots \bigsqcup S_n \times S_n \bigsqcup \cdots \bigsqcup S_n \times \cdots \times S_n \bigsqcup \cdots \bigsqcup S_n \), where a tuple \( S_n \bigsqcup S_n \cdots \bigsqcup S_n \) repeats \( n \) times, holds. The operation of a subdirect product \( \bigsqcup \) is determined by Definition 6.

The operation \( \bigsqcup \) accords with the properties described in item 1 and 2 of Definition 7, also \( \bigsqcup \) is determined by automorphism in \( \tilde{T}_n \cong (A_n \times A_n \times \cdots \times A_n) \times C_2 \) in this case.

**Remark 9.** Note that in \( \tilde{T}_n^{(3)} \) vertex permutation of tableau third part satisfy the condition: elements with the tableau presentation \([e]_1, [e, \ldots, e]_2, [\pi_1, \ldots, \pi_n; \pi_{n+1}, \ldots, \pi_{n+2}]_3\), that either all \( \pi_i \in S_n \setminus A_n \) or all \([\pi_i]_3 \in A_n \) for \( 1 < i \leq n, n+1 < i < 2n, \ldots, n^2 - n < i \leq n^2 \).

Here are the names of (almost all) predefined theorem-like environments.

**Proposition 10.** The subgroup \( E \wr A_n \) is the monolith of \( S_n \wr S_n \).

We call level of \( AutX^* \) as active if it has at least one non-trivial permutation. Denote by \( Aut_f X^* \) the group of all finite automorphism of spherically homogeneous rooted tree.

**Proposition 11.** Let \( H < Aut_f X^* \) with depth \( k \) then \( H \) contains \( k \)-th level subgroup \( P \) having all even vertex permutations \( p_k \in A_n \) on \( X_k \) and trivial permutations in vertices of rest of levels. Furthermore \( P \) is normal in \( W \) provided \( k \) is last active level of \( Aut_f X^* \).
Theorem 12. Proper normal subgroups in $S_n \wr S_m$ (action of group is left), where $n, m \geq 3$ with $n, m \neq 4$ are of the following types:

1) the subgroups of the first level stabilizer $[4, 5]$ are
   $$E \wr A_m, \overline{T}_m, E \wr S_m, E \wr A_n,$$

2) the subgroups that act on both levels are $A_n \wr A_m, S_n \wr A_m, A_n \wr S_m,$

wherein the subgroup $S_n \wr A_m \simeq S_n \wr (S_m \times S_m \times \ldots \times S_m)$ endowed with the subdirect product $[5]$

satisfying to condition $[3]$, moreover $S_n \wr A_m$ has two isomorphic copies, embedded into $S_n \wr S_m$ in different ways.

In total there are 8 proper normal subgroups in $S_n \wr S_m$.

Proposition 13. All normal subgroups of $S_n \wr (S_m \times S_k)$ can be partitioned in 2 types:

1) $E \wr (N_i \times N_j)$, where $N_i \not\subset \prod_{k=1}^{n} S_{m}^{(k)}$ and $N_j \not\subset \prod_{l=1}^{n} S_{l}^{(l)}$.

2) $\mathcal{A}_i \wr (N_i \times N_j)$, where $\mathcal{A}_i \triangleleft S_n$, $N_i$ and $N_j$ are subgroups from item 1) possessing an extension by $\mathcal{A}_i$ in a correspondent groups $S_n \wr S_m$ and in $S_n \wr S_k$. The full list of them: $S_n \wr (S_m \times \mathcal{A}_k), S_n \wr (\mathcal{A}_m \times S_k), S_n \wr (\mathcal{A}_m \times S_k)$,

We denote the set of normal subgroup of $S_n \wr S_n$ by $N(S_n \wr S_n)$. Subgroup with number $i$ from $N(S_n \wr S_n)$ is denoted by $N_i(S_n \wr S_n)$.

Theorem 14. The full list of normal subgroups of $S_n \wr S_n \wr S_n$ consists of 50 normal subgroups. These subgroups are the following:

1) **Type** $T_{023}$ contains: $E \wr \mathcal{A}_n \wr H$, $\overline{T}_n \wr H$, where $H \in \{\mathcal{A}_n, \mathcal{A}_n^2, S_n\}$. There are 6 subgroups.

2) **The second type of subgroups is subclass in** $T_{023}$ with new base of wreath product subgroup $\mathcal{A}_n^2$: $E \wr S_n \wr \mathcal{A}_n^2$, $E \wr \mathcal{A}_n \wr \mathcal{A}_n^2$, $E \wr N_i(S_n \wr S_n)$. Therefore this class has 12 new subgroups.

Thus, the total number of normal subgroups in **Type** $T_{023}$ is 18.

3) **Type** $T_{003}$: $A_{00}^{(3)}(n^2) = E \wr E \wr \mathcal{A}_n^2, \overline{T}_n \wr \mathcal{A}_n^2, \overline{T}_n \wr (3)$. Hence, here are 3 new subgroups.

4) **Type** $T_{123}$: $N_i(S_n \wr S_n) \wr S_n$, $N_i(S_n \wr S_n) \wr \mathcal{A}_n$ and $N_i(S_n \wr S_n) \wr \mathcal{A}_n^2$. Thus, there are 29 new normal subgroups in $T_{123}$, taking into account repetition $[5]$.

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