Renormalization Group Analysis of Boundary Conditions in Potential Scattering

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We analyze how a short distance boundary condition for the Schrödinger equation must change as a function of the boundary radius by imposing the physical requirement of phase shift independence on the boundary condition. The resulting equation can be interpreted as a variable phase equation of a complementary boundary value problem. We discuss the corresponding infrared fixed points and the perturbative expansion around them generating a short distance modified effective range theory. We also discuss ultraviolet fixed points, limit cycles and attractors with a given fractality which take place for singular attractive potentials at the origin. The scaling behaviour of scattering observables can analytically be determined and is studied with some emphasis on the low energy nucleon-nucleon interaction via singular pion exchange potentials. The generalization to coupled channels is also studied.

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I. INTRODUCTION

One of the most remarkable features of low energy scattering for a short range spherically symmetric potential is the onset of scale invariance and rotational invariance. For a short range potential, $U(r)$, which will be the subject of the present paper, with a typical size $a$ and in the long wavelength limit $ka \ll 1$, the Schrödinger’s equation for the reduced wave function in the $l=0$ channel becomes

$$u''(r) = 0.$$  

The solutions to this equation are straight lines

$$u(r) = a + br,$$  

where the coefficients $a$ and $b$ (or rather their ratio), are determined by matching to the solution in the region where the potential acts. The intersection of the asymptotic solution with the x-axis determines the scattering length $\alpha = -a/b$. Eq. (1) is obviously invariant under the scale transformation $r \to \lambda r$, but the general solution is not due to the presence of the potential $U(r)$. There are, however, two particular situations where the solution also transforms well under scaling, namely when either $a = 0$ or $b = 0$. These two cases correspond to $\alpha = 0$ (trivial scattering) and $\alpha = \pm \infty$ (zero energy bound state) respectively. The presence of a potential $U(r)$ at $r \sim a$ or finite energy, $ka \sim 1$, induces scaling violations which can be computed within perturbation theory, and obey scaling laws. As a matter of fact, it is interesting to see what kind of perturbation theory can be constructed around these two very simple scale invariant cases. To answer this question Wilsonian renormalization group (RG) methods seem the adequate tool $^1$.

Our interest and focus in the present paper is mainly motivated by encouraging developments in the last decade in nuclear physics, and more specifically on the Nucleon-Nucleon (NN) interaction problem in the framework of effective field theories triggered by Weinberg’s work $^2$ (for reviews see e.g. Refs $^3, 4, 5, 6, 7, 8$). Actually, much of the discussion has been unavoidably linked to the ability of designing adequate regularization schemes which in addition to preserve the symmetries can be removed beyond perturbation theory, allow to handle highly singular potentials and provide a one-valued renormalization group flow of low energy parameters.

In this paper we attack the problem by using the boundary condition (BC) regularization. It is based on the natural idea that all unknown information below some scale $R$ can always be parameterized in terms of a mixed boundary condition at the distance $R$ (see Sect. II). Strictly speaking, this is exactly true for a non-relativistic system which can be described through the Schrödinger equation, since it is a second order differential operator. The long distance physics is assumed to be determined in terms of a potential $U(r)$ above the boundary radius $R$. The BC method has been used extensively in the past for the treatment of NN scattering $^4$, and more recently for analyzing the commonly accepted analysis of phase-shifts $^7$ in terms of the chiral expansion $^11$ but always keeping the short distance cut-off finite. The equivalence between effective field theory (EFT) and boundary conditions at the origin for short range potentials has been established in Ref. $^5$. We as-
sume this equivalence to hold also in the presence of long distance interactions. Actually, we will see that it is possible to shrink the boundary to the origin with a smooth limit in the physical observables. From a Lagrangian viewpoint non-local momentum dependent terms in the interaction can, after suitable field redefinitions and using the equations of motion, be expressed as local and center-of-mass (CM) energy dependent potentials [12].

The boundary condition satisfies a renormalization group equation (see Sect. III) which infrared and ultraviolet critical points may be examined and characterized. Moreover, the boundary condition may be interpreted in terms of an outer truncated potential problem (see Sect. IV). The renormalization group equation for the boundary condition becomes a variable phase equation [13], which has a straightforward physical interpretation. Besides, one of the virtues of the variable phase approach is that it always deals with a on-shell problem, i.e., at any stage of the calculation the variable phase shift exactly corresponds to a physical phase shift of a certain on-shell problem. The price to pay is the non-linear character of the equation. Thus, the well known off-shell ambiguities characteristic of Lippmann-Schwinger equation (LSE), and which make the discussion on renormalization cumbersome, particularly when truncations are involved, never appear.

Although some of the aspects presented in this paper have been known or implicitly assumed for a long time in some way or another, we believe that the interpretation in terms of a short distance boundary condition becomes quite transparent and unifying, particularly when explicit long range, i.e. non-contact, interactions are considered [14, 15]. In the case of contact interactions, a general renormalization group analysis of elastic one channel potential scattering has been studied previously in Ref. [16] in momentum space within a Lippmann-Schwinger framework using a sharp momentum cut-off, which separates between the low and high energy region. The standard low momentum expansions for both the reaction matrix or the K-matrix (in fact the effective range expansion) arise as energy perturbations around the trivial and nontrivial scattering fixed points. The situation gets more involved when long distance forces are included and has been tackled in Ref. [17] providing the EFT understanding of the long distance modified effective range expansion [18]. This modified effective range expansion has been used to eliminate the one pion exchange (OPE) potential in the $^1S_0$ channel in Ref. [19]. The method of Ref. [17] has been applied more recently for the study of peripheral waves [20] and is based on delta shell regularization in coordinate space for the short range part of the interaction, but cutting off the large momentum components. This implies, in particular that the long range piece extends down to the origin. In the case of a singular potential at the origin this procedure becomes ill defined, because a sharp cut-off in momentum space does not suppress the short distance components entirely, unless states of high angular momentum with a short distance suppression cancelling the singularity are considered [20]. The relation of Wilson like renormalization and power counting has been treated in Ref. [21] in the particular case of the singular tensor component to the OPE potential (see also Ref. [22, 23, 24] and [25, 26] for the two pion exchange (TPE) extension). A momentum space treatment of Wilsonian renormalization ideas has also been proposed in Ref. [27] allowing for a determination of model independent low momentum potentials out of several realistic NN potentials [28, 29, 30]. The role of redundant operators in the absence of long range potentials has been discussed in Ref. [31].

A particular advantage of the BC has to do with the treatment of non-perturbative renormalization when long distance potentials are included. Although the momentum space treatment accommodates more general situations such as non-local potentials than those described here, we feel that for the most frequent case of long distance physics with local potentials the analysis in coordinate space becomes more transparent. Another important reason to prefer coordinate space in our analysis is that the Schrödinger equation defines a second order boundary value problem, and hence a sharp separation between long and short range physics becomes very natural. In particular, the short distance unknown physics may be handled in the spirit of old and modern works as a general boundary condition on the wave function at the origin. The question of how the origin should be approached is delicate, and depends on the postulated long range potential. We will discuss this issue along this paper in detail.

The paper is organized as follows. In Sect. II we elaborate on the boundary condition regularization, and its advantages as compared to other coordinate and momentum space regularizations. In Sect. III we make a comprehensive renormalization group analysis of boundary conditions. The corresponding infrared fixed points are determined and identified, and the renormalization group flow due to energy perturbations and potential perturbations is discussed. For power-like singular potentials at the origin we also establish the fixed points, limit cycles and attractors. In Sect. IV we establish the relation between the renormalization group flow of boundary conditions and the well known variable phase approach to potential scattering. This provides a nice interpretation of the boundary condition regularization, which suggests several working schemes. The generalization of higher partial waves and coupled inelastic channels is rather straightforward and is presented in a sketchy manner in Sect. V. Finally, in Sect. VI we draw our conclusions.

Since the present paper had its main motivation in the study of the NN interaction problem in effective field theories, most of the examples considered along this work

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1 Actually particle exchange implies in the non-relativistic limit a local and perhaps energy dependent potential
are taken from NN scattering in the $^1S_0$ singlet channel. According to Weinberg’s power counting, the long range reduced potential between the nucleons in this channel can be written as a low energy expansion which takes the form \[ M_NV_{NN} = U_{NN} = U_{LO} + U_{NLO} + U_{NNLO} + \ldots \] (3)

where LO refers to leading order, NLO to next to leading order, NNLO to next to next to leading order, and so on; $U_{LO}$ is the well-known one pion exchange (OPE) potential, while $U_{NLO}$ and $U_{NNLO}$ come mainly from two pion exchange (TPE), although they contain some minor contributions to the OPE potential. In this work, for convenience, we will mainly use the terms OPE and TPE for the potentials, which refer to

\[ U_{OPE} = U_{LO} \quad U_{TPE} = U_{LO} + U_{NLO} + U_{NNLO}. \] (4)

It should be noted that in the $^1S_0$ channel the OPE piece is just the well known Yukawa potential

\[ U_{OPE}(R) = -\frac{1}{aR^2}e^{-mR}, \] (5)

where $a = 16\pi f^2/(M_N m^2 g^2) = 0.7$ fm and $m = 0.699$ fm$^{-1}$ is the pion mass ($f = 0.468$ fm$^{-1}$ is the pion weak decay constant, $g = 1.29$ is the pion axial coupling and $M_N = 4.758$ fm$^{-1}$ is the nucleon mass). For the TPE potential, it is enough to know that for distances below the pion Compton wave length, $mR \ll 1$, it behaves as

\[ U_{TPE}(R) \rightarrow -\frac{a^4}{R^6}, \] (6)

where $a \simeq 1.64$ fm in the singlet channel, although it depends on the set of parameters one uses (for further details on the conventions used along this work for the TPE potential, see Refs. [23, 26]).

II. BOUNDARY CONDITION REGULARIZATION

For simplicity, let us consider the Schrödinger equation for $S$-wave scattering with reduced potential $U(r) = 2\mu V(r)$ and reduced wave function $u(r)$,

\[ -u''(r) + U(r)u_k(r) = k^2u_k, \] (7)

with the asymptotic behaviour at infinity

\[ u_k(r) \rightarrow \sin(kr + \delta(k)), \] (8)

and subject to the mixed boundary condition

\[ u_k(R) - L_k(R)u_k(R) = 0. \] (9)

The coefficient $L_k(R)$ encodes the physics below the scale $r \leq R$ down to the origin at the momentum $k$. For a given value of the inner boundary radius $R$, we get a solution $u(r, R)$ which depends both on the distance $r$ and on the inner boundary radius $R$. Obviously, the phase shift $\delta(k)$ inherits this $R$ dependence. The boundary condition represents our lack of explicit knowledge at some low scales, $r \leq R$, while we assume complete information on the potential, actually a local one, $U(r)$ for $r > R$. For a given $R$ we do not expect the scattering phase shift to depend strongly on this lack of explicit knowledge for wavelengths larger than $R$, i.e. $kR \ll 1$. We will show below how this statement can be made more quantitative.

Finally, as in any method we want ultimately to remove the regulator, i.e. to take the limit of the boundary radius to zero, $R \rightarrow 0$. This implicitly requires to extend the potential to short distances. In practice, we expect that for a radius much smaller than any other length scale in the problem a smooth limit should be obtained. As we will see in practical cases of interest in NN scattering some cautions must be taken, because although the limit in the bare parameters is not necessarily smooth, the physical results turn out to be indeed well behaved under certain circumstances.

The BC regularization method has several advantages over other methods. For clarity, we list them here, although their complete meaning will become obvious along the paper.

- It does not break any symmetry. In the present context this is a trivial statement, because it is applied to potential scattering once the CM motion and angular dependence are separated.

- It can incorporate any higher order derivative interactions. This is easily done by making an energy expansion of the boundary condition. This excludes the subtraction method of Ref. [33].

- It is a non-perturbative regularization. This is particularly interesting if one wants to discuss perturbative approximations, or power-counting schemes. A good test to those perturbative treatments is to see whether higher order corrections are indeed small by comparing with the full non-perturbative solution. In dimensional regularization (DR) there is no way to make such calculations, unless the regulators are analytically removed [34, 35] or the potential has a particularly simple separable structure [36].

- It can be applied to attractive or repulsive singular potentials at the origin. A crucial fea-
ture for these potentials is that they are non-perturbatively renormalizable but become perturbatively non-renormalizable \[37\]. Again, dimensional regularization both in the minimal subtraction (MS) or power divergence subtraction (PDS) scheme is unable to handle this problem, and to date there is no calculation dealing with singular potentials in DR. Also the delta shell regularization in coordinate space at short distances is excluded, because it assumes the singular potential to act at distances below the short distance regulator \[17\].

- It generates a one valued renormalization group flow because it involves one distance scale only. In contrast, regularization by a potential depends on at least two distance scales: the range of the potential as well as the strength \((1/\sqrt{U}\) has dimensions of length) of the potential. The square-well short distance regulators advocated in Ref. \[38, 39\] provide a multiple branched RG flow structure, which does influence the phase-shifts. It is not clear from that work which branch should one take, \textit{a priori}. In fact, these infinite branches reflect the well known non-uniqueness of the inverse scattering problem, rather than the short distance singularity of the long range potential.

- It allows a numerical elimination of the cut-off. In the calculations we will show below, we will always make sure that physical results are insensitive to the actual value of the short distance regulator. This improves e.g. on Ref. \[40\] where a finite cut-off was imposed.

- It provides a one-to-one mapping between the expansion of the BC and the physical scattering amplitude or inverse amplitude, both for the case of natural and unnatural scattering length. In other words, if the short distance wave function is truncated to a finite given order in a low energy expansion, the amplitude is also truncated at the same order. This property is shared with dimensional regularization in the MS scheme in the case of small scattering length.

- It is a uniquely defined regularization in the sense that the logarithmic derivative of the wave function computed from the asymptotic one can be uniquely determined from experiment by just integrating Schrödinger equation from infinity downwards to the origin.

One of the problems which this regularization manages to deal with rather transparently is the disentanglement between short and long distance physics. Anticipating some of the results to come, we propose to solve \(^4\)

\[
-u''_k(r) + U(r)u_k(r) = k^2 u_k(r),
\]

subject to the boundary condition at the origin and normalization at infinity

\[
\frac{u'_k(0^+)}{u_k(0^+)} = k \cot \delta_S(k), \quad u_k(r) \to \frac{\sin(kr + \delta(k))}{\sin \delta(k)}.
\]

We use the notation \(u_k(0^+) = \lim_{R_{S} \to 0^+} u_k(R_{S})\), since we will see that in general \textit{a limit must be taken}. In the absence of long range potential \(U(r) = 0\) the phase shift is given by \(\delta_S(k)\). On the other hand, if we take \(\delta_S(k) = 0\) we get a standard problem with a regular boundary condition at the origin, \(u_k(0) = 0\). The actual problem is that \(\delta_S(k)\) is unknown. At low energies both the full phase-shift \(\delta(k)\) and the short distance phase-shift \(\delta_S(k)\) can be described by some low energy approximation, like e.g., an effective range expansion,

\[
k \cot \delta_S(k) = -\frac{1}{\alpha_{0,S}} + \frac{1}{2} r_{0,S} k^2 + \ldots
\]

\[
k \cot \delta(k) = -\frac{1}{\alpha} + \frac{1}{2} r_0 k^2 + \ldots
\]

where \(\alpha_{0,S}\) is the zero range scattering length, \(r_{0,S}\) the zero range effective range, and \(\alpha_0\) and \(r_0\) the full ones. If we also make an expansion at low energies of the reduced wave function

\[
u_k(r) = u_0(r) + k^2 u_2(r) + \ldots
\]

we get a recurrent hierarchy of equations, namely

\[
-u''_0(r) + U(r)u_0(r) = 0, \quad \alpha_S u_0'(0^+) + u_0(0^+) = 0,
\]

\[
u_0(r) \to 1 - \frac{r}{\alpha},
\]

at zeroth order and

\[
-u''_2(r) + U(r)u_2(r) = u_0(r), \quad \alpha_S u_2'(0^+) + u_2(0^+) = \frac{1}{2} r_S \alpha_S u_0(0^+),
\]

\[
u_2(r) \to \frac{r}{6\alpha} (r^2 - 3\alpha r + 3\alpha r_0),
\]

at second order and so on. These equations suggest a scheme to proceed in practice. If the short distance physics could be deduced entirely from the potential we would set \(\alpha_{0,S} = 0\), \(r_{0,S} = 0\), and so on. Then, the full phase shift \(\delta(k)\) and hence the full low energy threshold

\[^4\] For the time being we will assume a completely regular potential. Singular potentials will be discussed separately below.
parameters would be determined entirely from the solutions of the regular problem at the origin. On a leading order (LO) approximation, one can improve on that by treating $\alpha$ and $U(r)$ as independent variables and predict $\delta$ and the remaining parameters of the effective range expansion, i.e. $r_0, \nu_2$, and so on. In the next-to-leading order approximation (NLO) $\alpha_0, r_0$ and $U(r)$ are regarded as independent variables.

The standard way to proceed would be to integrate the equations, Eq. (10), (17) and so on, from the origin (or a sufficiently small radius $R_S$) and then to adjust the short distance parameters to get the proper threshold parameters. Instead, one can simply integrate from infinity downwards, with a known value of $\alpha_0$, and then one can use Eq. (10) together with Eq. (12) and Eq. (13) to compute $\delta(k)$ for any energy with a given truncated boundary condition. This procedure provides by definition the low energy parameters we started with and takes into account that the long range potential determines the form of the wave function at long distances. The only parameter in the procedure is the short distance radius $R_S$, which we expect to produce a smooth limit for the phase shift when we remove it by taking the limit $R_S \to 0$ (in practice $R_S$ should be smaller than any other length scale in the problem).

Unfortunately, numerical downwards integration of the Schrödinger equation is a rather unstable and delicate procedure because at short distances the irregular solution starts dominating and high precision may be required to determine the low energy parameters at short distances, in a way as to recover the long distance wave function. This can be accomplished by a sort of practical and spurious irreversibility triggered by the irregular solution; downwards and upwards integration may not necessarily be faithfully represented as inverse operations of each other at the numerical level (see e.g. [15] for further details).

### III. Renormalization Group Analysis of Boundary Conditions

#### A. RG Equation

We want to determine the evolution of the boundary condition, $L_k(R)$, on the boundary radius, $R$, by imposing the physical requirement of independence of phase shifts. This is very much in the spirit of the derivation of the Callan-Symanzik equation as applied to the renormalization of Green’s functions in Quantum Field Theory. The purpose is to establish contact with the methods of Ref. [16] where the analysis is carried out entirely in momentum space, and to show that the whole discussion can quite naturally be carried out in coordinate space. In Appendix E we show how the method can also be applied to the case where one uses a square well potential to regulate the short distance physics yielding a multivalued evolution. (See also the discussion in Ref. [41].)

In order to proceed further, we make the infinitesimal change of the boundary radius $R \to R + \Delta R$ and take into account the total derivative

$$\frac{\partial u(r, R)}{\partial R} = u_R(r, R).$$

Then, the derivative of the boundary condition with respect to the boundary radius is given by

$$u''(R, R) + u''_R(R, R) - L'_k(R)u(R, R) - L_k(R)(u'(R, R) + u(R, R)) = 0.$$  

Deriving also Schrödinger’s equation with respect to the inner boundary radius $R$ we get

$$-u''_R(r, R) + U(r)u_R(r, R) = k^2 u_R(r, R),$$

and the asymptotic wave function

$$u(r, R) \to \sin(\nu r + \delta(k)),$$

we get

$$u_R(r, R) \to \cos(\nu r + \delta(k)) \delta_R(k),$$

$$u'_R(r, R) \to -\sin(\nu r + \delta(k)) \delta_R(k).$$

Thus, using Lagrange’s identity we get

$$0 = -u_Ru'' - u''_Ru = (-u_Ru' + u'_Ru)'.$$  

Integrating between $R$ and $\infty$ and using the boundary condition, Eq. (9) and Eq. (20), we finally get

$$-k^2 \frac{d\delta}{dR} = \left[ k^2 - U(R) + L'_k(R) + L_k(R)^2 \right] u(R, R)^2.$$  

This equation tells us how the phase shift changes as the inner radius is changed. If we require the phase shift not to depend on the particular choice of $R$ (renormalization group invariance) we get

$$-L'_k(R) = k^2 - U(R) + L_k(R)^2.$$  

This equation governs the evolution of the boundary condition, i.e., the logarithmic derivative of the wave function at the boundary, which shows that in order to guarantee independence of the phase shifts with respect to $R$

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5 We remind that we are assuming a regular potential at the origin.

6 This defines RG invariance by a continuous change in the cut-off parameter. Note that there is another possible solution to the equation, namely the zeros of the wave function, $u_k(R_n(k), R) = 0$, at some discrete set of short distance cut-offs $R_n(k)$ which obviously depend on energy. This is equivalent a hard core potential, as opposed to the soft boundary condition. We analyze this possibility later on.
at all energies the boundary condition must also depend on energy, a not surprising result.

It is instructive to see that Eq. (25) can be almost trivially deduced, by applying Schrödinger's equation to the boundary condition at the point $R^+$, but the relation to renormalization and the approach of Ref. [16] is less obvious.

Eq. (24) has interesting consequences as regards the low energy limit of the boundary condition and provides in addition an error estimate of the phase shift for $k \to 0$. If we go to the zero energy limit, the phase shift behaves as $\delta \sim -a k$, with $a$ the s-wave scattering length and Eq. (24) becomes,

$$
\frac{dk}{dR} = \left[ -U(R) + L'_0(R) + L_0(R)^2 \right] \times \left[ \frac{u(R, R)}{k^2} \right].
$$

The previous estimate has, in addition a direct application in the renormalization of singular potentials (both attractive and repulsive) at the origin, of the form

$$
U(R) \to \pm \frac{1}{a^2} \left( \frac{a}{R} \right)^n (R \to 0),
$$

for $n \geq 2$. If we approach the origin $R \to 0$, we may neglect the energy term in Eq. (24). This means that the condition for the phase shift to be independent on the boundary condition becomes the condition of independence of the scattering length. This, in turn means that if the theory is renormalizable at zero energy it is renormalizable at any energy. We will see below an alternative and more appealing formulation of this fact. It has been known for a long time by a detailed study of the wave functions close to the origin using the WKB approximation [37], and it is very rewarding to provide such a simple derivation of this result within the present framework.

The BC parameter $L_k(R)$ has dimensions of inverse length, so it is natural to measure it in units of the boundary radius $R$,

$$
L_k(R) = \frac{\xi_k(R)}{R}.
$$

The equation satisfied by $\xi_k(R)$ is

$$
R \frac{d\xi_k}{dR} = \xi_k(1 - \xi_k) + \left[ U(R) - k^2 \right] R^2.
$$

This is a Ricatti type equation. By using the superposition principle for the wave function

$$
u_k(R) = u_{k,c}(R) + k \cot \delta_k u_{k,s}(R),
\text{with } u_{k,c} \to \cos(kR) \text{ and } u_{k,s} \to \sin(kR)/k \text{ we have that}
\xi_k(R) = R \frac{u'_k(R)}{u_k(R)} = R \left[ \frac{u'_{k,c}(R)}{u_{k,c}(R)} + \frac{k \cot \delta_k u'_{k,s}(R)}{u_{k,s}(R)} \right]
$$

whence the phase shift can be explicitly determined. Using the independence of the phase shift on the cut-off radius it can be easily shown that two solutions $\xi_k(R)$ and $\xi_k(R_0)$ are related by a Moebius bilinear transformation (see Ref. [42]),

$$
\xi_k(R) = \frac{A(R, R_0)\xi_k(R_0) + B(R, R_0)}{C(R, R_0)\xi_k(R_0) + D(R, R_0)},
$$

where $A, B, C$ and $D$ depend on the potential $U$ only. The former expression disentangles the boundary condition parameterizing the unknown short distance information from the known long range potential. Actually, the matrix

$$
M(R, R_0) = \begin{pmatrix} A(R, R_0) & B(R, R_0) \\ C(R, R_0) & D(R, R_0) \end{pmatrix}
$$

satisfies the group properties

$$
M(R, R')M(R', R'') = M(R, R''),
$$

which faithfully represents the dilatation group for the short distance cut-off $R \to \lambda R$. We will study below the stability structure of this group corresponding to the the infrared limit $R \to \infty$ and the ultraviolet limit $R \to 0$. For a study of the periodic case in log($R$) see e.g. Ref. [42].

### B. Long distance Fixed Points at zero energy

For a potential of typical size $a$ and at low energies $ka \ll 1$ we may look at the region $a \ll R \ll 1/k$ where both the potential and the energy can be neglected, yielding the equation

$$
R \frac{d\xi_0}{dR} = \xi_0(1 - \xi_0), \quad a \ll R \ll 1/k.
$$
Note that the equation is scale invariant under the change $R \rightarrow \lambda R$ within the interval $a \ll R \ll 1/k$. Obviously, having a finite interval breaks the scale invariance, since the interval boundaries also change. Thus, both the potential and the energy are scaling violating perturbations.

Fixed points in the dimensionless boundary condition $\xi$ are defined as those fulfilling the condition $\xi_0 = 0$. Hence, Eq. (37) has two fixed points at $\xi_0 = 0$ and $\xi_0 = 1$. The point $\xi_0 = 0$ is unstable since any small perturbation of its value at say $r \sim a$ results in increasingly large deviations from the fixed point, as can be seen directly by analyzing the differential equation, Eq. (37). On the contrary, the point $\xi_0 = 1$ is stable. Since both fixed points are associated with long distance behaviour they correspond to infrared (IR) fixed points. The physical interpretation of fixed points in the present context is clear; by varying the BC in the range $a \ll R \ll 1/k$ according to Eq. (37), we guarantee physics independence. Clearly, by analyzing the differential equation, Eq. (37). On the contrary, the point $\xi_0 = 1$ is stable. Since both fixed points are associated with long distance behaviour they correspond to infrared (IR) fixed points. The physical interpretation of fixed points in the present context is clear; by varying the BC in the range $a \ll R \ll 1/k$ according to Eq. (37), we guarantee physics independence. For this case it will be interesting to choose the scale $a \ll R \ll 1/k$ to describe the boundary condition function because we expect a weak dependence on the scale.

For the unstable point, $\xi_0 = 0$, the situation is such that an extreme fine tuning of the BC is required to have this scale invariance. On the contrary, the stable fixed point, $\xi_0 = 1$, does not require this accurate fine tuning.

To analyze the physical situation corresponding to these fixed points let us now take into account that $L_0(R) = u_0'(R)/u_0(R)$. In the region $a \ll R \ll 1/k$ we have the asymptotic wave function (we use the normalization condition $u_0(0) = 1$),

$$u_0(R) = 1 - \frac{R}{\alpha},$$

yielding

$$L_0(R) = \frac{\xi_0(R)}{R} = \frac{1}{R - \alpha}.$$  

Thus, the unstable fixed point $\xi_0 = 0$ corresponds to $\alpha \rightarrow \infty$ (zero energy bound state) and the stable fixed point $\xi = 1$ to $\alpha = 0$ (trivial scattering). These conclusions are in full agreement with the momentum space analysis of Ref. [16].

C. Positive Energy Perturbation

In the region where the potential does not act $r \gg a$, the equation satisfied by $\xi_k(R)$ is

$$R \frac{d \xi_k}{d R} = \xi_k (1 - \xi_k) - k^2 R^2,$$  

which solution is

$$\xi_k(R) = k R \cot [k R + \delta(k)].$$

We have fixed the arbitrary constant by imposing that the extrapolation of the logarithmic derivative to the origin can be related to the scattering phase shift, $\delta(k)$. Then we have

$$k \cot \delta(k) = k \frac{R + \xi_k(R) \cot(k R)}{k R \cot(k R) - \xi_k(R)},$$

Thus, an expansion of the scaled boundary condition in the form

$$\xi_k(R) = \xi_0(R) + (k R)^2 \xi_2(R) + (k R)^4 \xi_4(R) + \ldots,$$  

valid in the region $a \ll R \ll 1/k$ yields an expansion for the phase shift as given by Eq. (41).

$$k \cot \delta(k) = \frac{1}{R} \frac{\xi_0}{1 - \xi_0} + \frac{\xi_0^2 - 3 \xi_0 + 3 \xi_2 + 3}{3(\xi_0 - 1)^2} R k^2 + \ldots.$$  

Whereas written in this way the expansion for $k \cot \delta$ around the trivial fixed point $\xi_0 = 1$ induces increasingly large contributions for increasing orders in $k$, spoil- ing the convergence of the low energy expansion, in the case of the nontrivial fixed point $\xi_0 = 0$ this yields a perfectly well defined expansion. A similar situation occurs for the expansion in $\tan \delta/k$, although with opposite fixed points corresponding to the divergent and convergent case. Thus the perturbation theory around the non-trivial fixed point (large scattering length) corresponds to an effective range expansion of the form

$$k \cot \delta(k) = -\frac{1}{\alpha} + \frac{1}{2} r_0 k^2 + v_2 k^4 + \ldots,$$  

whereas for the trivial fixed point (small scattering length) the low energy expansion reads,

$$\tan \frac{\delta(k)}{k} = -\alpha - \beta k^2 - \gamma k^4 + \ldots.$$  

Matching both expansions, Eq. (45) and Eq. (46), we get the identifications

$$\beta = \frac{1}{2} r_0 \alpha^2,$$  

$$\gamma = \frac{1}{4} \alpha (\alpha r_0^2 + v_2).$$  

The convergence radius of these low energy expansions has to do with the longest distance singularities of the potential. For Yukawa like behaviour, $U \sim e^{-m R}/R$, one has $|k| < m/2$, due to the branch cut at $k = \pm im/2$.

Comparing the expansions (43) and (45), we have
\[ \xi_0(R) = \frac{R}{R - \alpha}, \]
\[ \xi_2(R) = \frac{6 \alpha R^2 - 2 R^3 + \alpha^2 (-6 R + 3 r_0)}{6 (\alpha - R)^2 R}, \]
\[ \xi_4(R) = -\frac{-24 \alpha R^2 + 4 R^6 - 15 \alpha^3 (4 R^3 - 6 R^2 r_0 + 3 R^2 r_0^2 - 12\nu_2) + 30 \alpha^2 (2 R^4 - R^3 r_0 - 6 R\nu_2)}{180 R^3 (-\alpha + R)^3}. \]

D. Zero energy Short distance critical points

1. Fixed points and Cycles.

We analyze now the short distance behaviour of Eq. [51] corresponding to the ultraviolet regime. Generally speaking, if the solution to the Schrödinger equation is written as a linear combination of a regular and irregular solution, \( u_0(r) = c_0 u_{\text{irreg}}(r) + c_1 u_{\text{reg}}(r) \), then
\[ \xi_0(R) = R \frac{u'_0(R)}{u_0(R)} = R \frac{c_0 u'_{\text{irreg}}(R)}{c_0 u_{\text{irreg}}(R) + c_1 u_{\text{reg}}(R)}, \]
where the ratio \( c_0/c_1 \) can be fixed by choosing \( \xi_0(R_0) \) with \( R_0 \) some reference scale. Obviously, regular solutions are unstable fixed points whereas irregular solutions correspond to stable fixed points since \( u_{\text{irreg}} \) always takes over when \( R \to 0 \).

We will first consider (Sect. III D 2) the case of regular potentials, for which the following condition is fulfilled
\[ \lim_{R \to 0} R^2 U(R) = 0 \]
and afterwards the case of singular potentials which behave as \( 1/R^n \) at short distances. In this latter case we are going to make a further distinction between those power-law potentials with \( n > 2 \) (Sect. III D 3) and the inverse square potential, \( U(R) = g/R^2 \) (Sect. III D 4). For a short review of the short-distance solutions of singular power-law potentials see Appendix A.

2. Regular potentials.

For a regular potential the wave function behaves linearly at short distances, \( u(R) = c_0 + c_1 R \), yielding
\[ \xi_0(R) = \frac{c_1 R}{c_0 + c_1 R}, \]
which has the fixed points \( \xi_0 = 0, 1 \). The first one, \( \xi_0 = 0 \) is stable and corresponds to irregular solution of the Schrödinger equation. The convergence towards this fixed point is linear in \( R \)
\[ \xi_0(R) \to \frac{c_1}{c_0} R + \mathcal{O}(R^2). \]
The other fixed point, \( \xi_0 = 1 \) is unstable and corresponds to the regular solution \( u(R) \sim R \).

A special case is that of the Yukawa potential

\[
U(R) = -\frac{1}{aR} e^{-mR},
\]

which has a \( 1/R \) singularity for short distances. For this potential the behaviour of the wave function at short distances is given by

\[
u\left(\frac{R}{a}\right) = \sum_{n=0}^{\infty} \left[ \frac{\nu}{n^2 - 1} \right],
\]

for which the same fixed points than discussed before for a common regular potential are reproduced (namely \( \xi_0 = 0, 1 \)), but with logarithmic convergence towards the stable fixed point \( \xi_0 = 0 \)

\[
\xi_0(R) = \frac{c_1}{c_0} R + m R - 5 R \frac{R}{a} \log (\frac{R}{a}) + O(R^2),
\]

from which an \( R \log R \) convergence trend is deduced (which, incidentally is independent of the actual \( c_1/c_0 \) ratio).

### 3. Power-law singular potentials.

For a power-law potential which behaves as

\[
U(R) = \pm \frac{1}{a^2} \left( \frac{a}{R} \right)^n, \quad n > 2,
\]

as \( R \to 0 \) we have one scale only, so we can define the variable \( R = \alpha R \), in such a way that the coupling constant becomes one. The solution to the Schrödinger equation for the repulsive case reads

\[
u(\alpha R) = c_0 \sqrt{\alpha} I_{-n/2} \left( \frac{x^{1-n/2}}{\alpha^{n/2 - 1}} \right) + c_1 \sqrt{\alpha} J_{-n/2} \left( \frac{x^{1-n/2}}{\alpha^{n/2 - 1}} \right),
\]

where \( c_{0,1} \) are integration constants and \( K_{-n/2}(z) \) and \( I_{-n/2}(z) \) are regular (exponentially increasing) and irregular (exponentially decreasing) modified Bessel functions respectively. The fixed points are \( \xi_0 = \pm \infty \) corresponding to take \( c_0 = 0 \) and \( c_1 = 0 \) respectively; \( \xi_0 = \pm \infty \) is unstable, while \( \xi_0 = \pm \infty \) is stable.

For the attractive \( n > 2 \) case, the solution to the Schrödinger equation for the attractive case reads

\[
u(\alpha R) = c_0 \sqrt{\alpha} J_{n/2} \left( \frac{x^{1-n/2}}{\alpha^{n/2 - 1}} \right) + c_1 \sqrt{\alpha} J_{n/2} \left( \frac{x^{1-n/2}}{\alpha^{n/2 - 1}} \right),
\]

where \( J_{-n} \) are oscillating spherical Bessel functions. From here one can obtain the dimensionless logarithmic derivative \( \xi_0(R) \), while the ratio \( c_0/c_1 \) may be determined by choosing \( \xi_0(R_0) \). Close to the origin we can use the asymptotic expansions

\[
u(z) \to \sqrt{2/\pi z} \cos \left( (\nu/2 + 1/4)\pi - z \right),
\]

and hence we have

\[
x_0(R) = \frac{x_0(R_0) \cot \Phi - 1}{x_0(R_0) + \cot \Phi},
\]

with

\[
\Phi = 1 - \frac{1}{n-2} \left[ \left( \frac{R}{R_0} \right)^{1-n/2} - 1 \right],
\]

where \( R_0 \) is some short distance reference scale. So we see that for \( n \geq 2 \) we have an attractor \( (R, 0) \) with a fractal asymptotic \( (R \to 0) \) dimension \( d = 2 - 2/n \). The fractal dimension can be deduced from the scaling properties of the zeros of \( \xi_0(R) \), \( R_0/2^n \sim 1/N \), being \( R_0 \) the \( N \)-th zero.

### 4. Inverse square potential

The inverse square potential, \( U = g/R^2 \), requires a separate study. Note that in that case the RG equation, Eq. \( 31 \), is formally invariant under the continuous scaling transformation, \( R \to \mu R \). This symmetry is, however, explicitly broken by an initial condition \( \xi_0(R_0) \). Using Eq. \( 31 \) we get the zero energy fixed points at

\[
x_0 = \pm \sqrt{1 + 4g}
\]

for \( g > -1/4 \). The positive and negative roots correspond to stable and unstable fixed points respectively. For \( g < -1/4 \) we have purely imaginary solutions. This is the signal for a limit cycle. Actually, the solutions are given by

\[
u_0(R) = c_0 R^{\lambda_-} + c_1 R^{\lambda_+}
\]

\[
\lambda_\pm = (1 \pm \sqrt{1 + 4g})/2 \quad \text{for} \quad g > -1/4,
\]

\[
u_0(R) = c_0 \sqrt{R} \cos (\lambda \log R) + c_1 \sqrt{\lambda} \sin (\lambda \log R)
\]

\[
\lambda = \sqrt{-1 + 4g} \quad \text{for} \quad g < -1/4.
\]

So, in the first case we obtain

\[
x_0(R) = \frac{c_0 \lambda_- R^\lambda_- + c_1 \lambda_+ R^\lambda_+}{c_0 R^\lambda_- + c_1 R^\lambda_+} \quad \text{for} \quad g > -1/4,
\]

which has an attractive fixed point at \( \xi_0 = \lambda_- \), and a repulsive one at \( \xi_0 = \lambda_+ \). In the latter case, \( g < -1/4 \), we obtain

\[
x_0(R) = \lambda \cot \left[ \tan^{-1} \frac{2 \lambda}{2x_0(R_0) - 1} + \lambda \log \frac{R}{R_0} \right] + \frac{1}{2},
\]

(70)
FIG. 2: Running of $\xi_0(R)$ for regular and singular potentials. In the left panel we show $\xi_0(R)$ for the case with no potential in which $\xi_0(R) = R/(R - \alpha_0)$, for a square well potential and for a Yukawa potential (the One Pion Exchange potential in the $^1S_0$ channel). In the right channel we show $\xi_0(R)$ for the case of the Chiral Two Pion Exchange potential which for short distances behaves as $-a^4/R^6$; the lines representing the renormalization flow of $\xi_0(R)$ becomes more dense in its way to the origin, giving rise to a nontrivial fractal dimension $d = 2 - 2/n = 5/3$ in the $R \to 0$ limit. These examples are taken from neutron-proton scattering in the singlet s-wave $^1S_0$ channel, in which the scattering length is $\alpha_0 = -23.74$ fm, and $R$ is expressed in fm.

FIG. 3: Running of $\xi_0$ vs $\xi_2$ for regular and singular potentials (note the log scale). In the left panel we show the case of a Yukawa potential (the One Pion Exchange potential in the $^1S_0$ channel). In the middle channel we show the case of the Chiral Two Pion Exchange potential. In the left panel we compare the case with no potential to that of a OPE and TPE potential. These examples are taken from neutron-proton scattering in the singlet s-wave $^1S_0$ channel, in which the scattering length is $\alpha_0 = -23.74$ fm, and $r_0 = 2.77$ fm. These plots should be compared with Fig. (1).

In this case the discrete scaling property $\xi_0(Re^{N\pi/\lambda}) = \xi_0(R)$ with $N = 0, \pm1, \pm2, \ldots$ typical of the limit cycles [44] is reproduced. For instance, if we consider $\xi_0(R_N) = 0$, then the sequence of the zeros of $\xi_0$ is given by $R_{N+1} = e^{i/\lambda}R_N$. This corresponds to the Russian doll renormalization (see e.g. Ref. [45]). Coordinate and momentum space analyses have been treated in [46] and [47] respectively.

5. Overview of the ultraviolet limit

On the light of the previous discussion we obtain

$\xi_0(R)$, the following behaviour

- For a regular potential we have two fixed points, an attractive one, corresponding to the irregular solution of the Schrödinger equation, and a repulsive one, corresponding to the regular solution. This means that all solutions go to the irregular solution at the origin.
- For a singular potential with $n = 2$ and coupling $g > -1/4$ we have two fixed points. For $g < -1/4$ there are limit cycles.
- For a repulsive singular potential with $n > 2$ we have two fixed points. The attractive one corre-
sponds to the irregular solution.

- For an attractive singular potential with \( n > 2 \) we have an attractor with asymptotic fractal dimension \( d = 2 - 2/n \).

In Fig. 2 we present three of the possible situations. The examples are taken from neutron-proton scattering in the singlet s-wave \(^1S_0\) channel, for which the scattering length is \( \alpha_0 = -23.74 \) fm. In the regular potentials example we use for the discussion the One Pion Exchange (OPE) potential, which, for this channel, takes the form of an usual Yukawa potential as the one given in Eq. (37), for which \( m = 0.699 \) fm\(^{-1} (= 138 \) MeV) and \( a = 0.7 \) fm, and a Square Well potential with range \( R_0 = 1/m = 1.43 \) fm and depth \( U_0 = -0.1 \) fm\(^2\). In the singular potential example we use the chiral Two Pion Exchange (TPE) potential, which behaves as \( -\xi^2/R^6 \) at distances below the pion Compton wavelength. For completeness we draw also in Fig. 3 the behaviour of \( \xi_0 \) vs. \( \xi_2 \) very much in spirit of our study of Sect. III C on positive energy perturbations (see Fig. 1).

E. Low energy expansion of the BC with a potential

If we relax the condition \( R \gg a \), we can undertake a low energy expansion to get the set of differential equations for the low energy scaled BC coefficients,

\[
R \frac{d \xi_0}{d R} = \xi_0(1 - \xi_0) + U(R)R^2, \tag{71}
\]

\[
R \frac{d \xi_2}{d R} = -\xi_2(1 + 2\xi_0) - 1, \tag{72}
\]

\[
R \frac{d \xi_4}{d R} = -\xi_4(3 + 2\xi_0) - \xi_2^2, \tag{73}
\]

valid for any \( R \). The r.h.s. are the corresponding beta functions of the renormalization group flow. Note that unlike the standard RG equations a manifest scale dependence shows up due to the potential \( U(R) \) which describes the long range physics. The fulfillment of these equations guarantees cut-off independence of low energy parameters, \( \alpha, r_0, v_2 \) and so on for any value of the cut-off \( R_S \). Moreover, these equations exhibit a natural hierarchy: the solution of a given coefficient \( \xi_{2n} \) depends only on the previous ones \( \xi_{2n-2}, \ldots, \xi_0 \). Furthermore, they are non-perturbative in the potential and do not require any off-shell information, as in the momentum space treatments \[17\]. Actually, the solution for the first equation in the absence of a potential reads,

\[
\xi_0(R) = \frac{R}{R - \alpha}, \quad R \gg a, \tag{74}
\]

where \( \alpha \) is the integration constant which can be identified with the physical scattering length. In the presence of the potential \( U(R) \), this suggests a solution of the form

\[
\xi_0(R) = \frac{R}{R - \alpha_0(R)}, \tag{75}
\]

where \( \alpha_0(R) \) is an undetermined coefficient, satisfying Eq. (122) of Sect. IV. We will show in the next section that \( \alpha_0(R) \) is the scattering length corresponding to the truncated potential \( U(r) \) for \( r < R \) and zero otherwise. Actually, making use of the analogy we can solve the equations, (71), (72) and (73) in a more efficient manner, see Eqs. (124), (125) and (126).

The set of equations (71), (72) and (73) have to be solved with some initial conditions. For asymptotically large distances we must have \( \xi_0' = 0 \), and hence

\[
\xi_0(\infty) = 1 \quad (\alpha \neq \infty), \tag{76}
\]

\[
\xi_0(\infty) = 0 \quad (\alpha = \infty). \tag{77}
\]

Thus, unless \( \alpha = \infty \), all solutions go asymptotically to the stable fixed point, according to the general theory. If \( \alpha \gg a \) then there is a region for \( r \sim a \) where \( \xi_0 \ll 1 \) and \( \xi_0 \) remains almost constant. Likewise for \( \alpha \ll a \) we have \( \xi_0 \sim 1 \) for \( r \geq a \). Finally, if \( \alpha \sim a \) we have \( \xi_0 \gg 1 \) only in the region \( r \sim a \).

If we solve around the fixed point \( \xi_0 = 1 \) we get

\[
\xi_0(R) = 1 + R \int_R^\infty U(r)dr. \tag{78}
\]

Going to \( R \to \infty \) since already at \( R \sim a \) we may have

\[
\xi_0(a) = \frac{a}{a - \alpha}. \tag{79}
\]

Thus, if \( a \gg \alpha \) we have \( \xi_0(a) \sim 1 \) and if \( a \ll \alpha \) then \( \xi_0(a) \sim 0 \), even if \( \xi_0(\infty) = 1 \) (unless \( \alpha = \infty \), in which case we do have \( \xi_0(\infty) = 0 \)). Thus, taking the scale \( R \sim a \) seems like a good choice to be as close as possible to the fixed point situation \( \xi' \sim 0 \).

F. Error estimates

1. General Considerations

Exact renormalization group invariance requires the knowledge of the complete phase shifts at all energies for

fixed by \( \alpha \) and the effective range. A way out is to consider energy dependent perturbations instead of momentum dependent ones \[16\,17\] and the hierarchy that we find is recovered.
the fulfillment of Eq. (23), but practical computations demand the use of a limited amount of physical information as input in order to have predictive power. One example is a theory in which we know the value of the scattering length, $a_0$, and consequently we want to fix its renormalization group flow, which can be obtained from the low energy limit of Eq. (24), yielding

$$\frac{d a}{d R} = \lim_{k \to 0} \left( \frac{u_k(R,R)}{k} \right)^2 \times \left[ L_0'(R) + L_0(R)^2 - U(R) \right] = 0. \quad (80)$$

Thus, $a_0(R) = a_0(R_0)$ implies the fulfillment of Eq. (71), but not of Eq. (72) or higher order ones. This theory can be shown equivalent to truncating the boundary condition (see Appendix B) in the $R \to 0$ limit

$$\lim_{R \to 0} L_k(R) = \lim_{R \to 0} L_0(R), \quad (81)$$

with $L_0(R)$ fulfilling the RG equation

$$L_0'(R) + L_0(R)^2 = U(R) \quad (82)$$

subjected to the asymptotic boundary condition $L_0(R) \to 1/(R - a_0)$ at $R \to \infty$.

We can try to improve the description of the phase shifts by making RG independent not only the scattering length $a_0$, but also the effective range $r_0$, or even higher order parameters of the effective range expansion. In case we fix $a_0$, $r_0$, $v_2$, ..., $v_n$, we can write the boundary condition as

$$L_k(R) = L_0 + k^2 L_2 + \cdots + k^{2n} L_{2n} + \cdots \quad (83)$$

where $L_0$, $L_2$, ..., $L_{2n}$ obey their respective renormalization group equations, while the higher order terms, represented by the dots, do not.

### 2. Zeroth Order Truncation

If we truncate the BC to zeroth order in the energy, i.e. Eq. (71) is fulfilled while Eq. (72) and higher order ones are not, then the scattering length is independent on the short distance cut-off $R$. The truncation is made in order to fix the scattering length of the system, while the effective range, the shape parameter and higher order terms of the expansion of $k \cot \delta$ are determined by both the scattering length and the long range potential.

This situation is equivalent to take $L_k(R) = L_0(R)$, from which the cut-off dependence of the phase shift can be deduced by making the substitution $L_k(R) \to L_0(R)$ in Eq. (24), yielding

$$\frac{d \delta}{d R} = -k^3 \left( \frac{u_k(R,R)}{k} \right)^2. \quad (84)$$

For a regular potential, for which we have the behaviour $u_k(R,R)/k \sim c_0 + c_1 R$ for small cut-offs, the phase shift shows a linear dependence in the cut-off radius which slope increases with the momentum $k$, so for small cut-off radii we have

$$\frac{d \delta}{d R} \sim -k^3 \frac{r_0^2}{c_0} \quad (85)$$

and consequently we find a linear dependence of the phase shift with respect to the cut-off radius, $\Delta \delta \sim R$.

For an attractive singular potential, which behaves as $U(R) \sim -1/R^n$, the situation changes, since the wave function is very much suppressed at short distances, $u_k(R,R)/k \sim R^{n/4}$ as compared to the regular potential case. Then we obtain, for small cut-offs, the behaviour

$$\frac{d \delta}{d R} \sim -k^3 R^{n/2}, \quad (86)$$

from which one can deduce a cut-off error $\Delta \delta \sim R^{n/2+1}$, and we recover renormalizability, i.e. cut-off independence, in the small cut-off limit

$$\lim_{R \to 0} \frac{d \delta}{d R} = 0. \quad (87)$$

This cut-off dependence is depicted in Fig. (4), in which we show the nucleon-nucleon phase shift in the $3^1 S_0$ channel as a function of the cut-off $R$ for various center-of-mass momenta. The phase shift $\delta(k,R)$ is computed using an energy independent boundary condition which turns out to be equivalent to fixing the scattering length to some given value, as for example the experimental one $a_0 = -23.74$ fm, for both the one and two pion exchange potentials. The OPE potential is just an usual Yukawa potential, while the TPE potential behaves as $-a^4/R^4$ for distances smaller than the pion Compton wavelength. As we can see in the figures, in the OPE case the phase shift shows a linear dependence in $R$ near the origin, while in the TPE case the phase shift becomes insensitive to the cut-off much earlier.

Finally, for a repulsive singular potential behaving as $U(R) \sim 1/R^n$, the phase shifts shows an exponential behaviour as the cut-off is removed. This can be understood from the behaviour of the wave function at short

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8 The relation between the truncation of the boundary condition and the fixation of low energy scattering observables, i.e. the different terms of the effective range expansion, can be made clear by rewriting Eq. (24) as

$$\frac{d}{d R} k \cot \delta = \left[ L_k'(R) + L_k(R)^2 + k^2 - U(R) \right] \times u(R,R)^2,$$

which is obtained in just the same way as Eq. (23), but changing the asymptotic normalization of $u(r,R)$ to

$$u(r,R) \to \cos k r + k \cot \delta(k,R) \frac{\sin k r}{k},$$

from this it is obvious that fixing $a_0$ is equivalent to the fulfillment of Eq. (21), fixing $a_0$ and $r_0$ to fulfilling Eq. (21) and Eq. (22), and so on.
distances (see Appendix A)
\[
\frac{u_k(R, R)}{k} \sim R^{n/2} e^{-(a/R)^{n/2-1}/(n/2-1)},
\]
(88)
where we have taken the irregular solution since for an arbitrary scattering length this is the UV fixed point and the regular solution may be discarded. Then, for small cut-off
\[
\frac{d \delta}{dR} \sim -k^3 R^{n/2} e^{-2(a/R)^{n/2-1}/(n/2-1)},
\]
(89)
from which one can see that the phase shift develops a very strong cut-off dependence when one tries to fix the scattering length to an arbitrary value, unlike the previous cases.

3. Relation to Orthogonality Constraints

The truncation of the boundary condition at zeroth order is equivalent to the fulfillment of orthogonality relations between different energy solutions, which was used in Refs. 22–25 to renormalize and obtain model-independent predictions for the $^1S_0$ singlet and $^3S_1 - ^3D_1$ triplet (deuteron) channels. Orthogonality constraints imply that different energy wave functions fulfill the integral relation
\[
\int_0^\infty u_k(r)u_{k'}(r) \, dr = \delta(k - k'),
\]
(90)
If we use the corresponding Lagrange identity between $u_k$ and $u'_{k'}$, we can rewrite the previous orthogonality constraint as
\[
(k^2 - k'^2) \int_0^\infty u_k(r)u_{k'}(r) \, dr = u'_{k}u_{k'} - u_ku'_{k'},
\]
(91)
so for $k' \neq k$, and including a short distance cut-off $R$, we obtain
\[
\frac{u'_{k}(R)}{u_k(R)} = \frac{u'_{k'}(R)}{u_{k'}(R)},
\]
(92)
Finally, taking $k' = 0$ as a reference state
\[
\frac{u'_{k}(R)}{u_k(R)} = \frac{u'_{0}(R)}{u_{0}(R)},
\]
(93)
or, equivalently, $L_k(R) = L_0(R)$, i.e., imposing orthogonality between different energy solutions is equivalent to truncate the boundary condition at zeroth order.

4. Higher Order Truncations and Inconsistencies

Naively one can think that a truncated energy expansion of the boundary condition would lead to a systematically more accurate description of the data. In this section we will analyze this problem and show cases where this naive expectation is not fulfilled.

To begin with we can try to truncate the boundary condition, for example, to second order in the energy
\[
L_k(R) = L_0(R) + k^2 L_2(R),
\]
(94)
so Eqs. (71) and (72) are fulfilled while Eq. (73) and higher order ones are not. In such a case both the scattering length and the effective range are cut-off independent, while the phase shift is not. In a similar fashion to the previous case, the cut-off dependence of the phase shift can be estimated, yielding
\[
\frac{d \delta}{dR} = -k^5 L_2(R)^2 \left( \frac{u_k(R, R)}{k} \right)^2,
\]
(95)
FIG. 4: Cut-off dependence of the phase shifts for NN scattering in the $^1S_0$ channel with the OPE and TPE potentials, which behave as $1/R$ and $1/R^6$ respectively. For computing the phase shift we use a energy independent boundary condition fulfilling the equation $L_0(R) + L_2(R) - U(R) = 0$, which in turns mean that the scattering length is independent on the cut-off (we take $\alpha_0 = -23.74 \text{ fm}$). We show the phase shifts for center of mass momenta of $k_{cm} = 0.25, 0.5, 1.0 \text{ fm}^{-1}$.
which can be easily computed for regular potentials. For such potentials we can take into account that (see Sect. XIV)

$$\lim_{R \to 0} L_k(R) = k \cot \delta_S(k)$$

$$= -\frac{1}{\alpha_S} + \frac{1}{2} r_s k^2 + \ldots$$

(96)

being $\alpha_S$ and $r_S$ the short-range scattering length and effective range. Inserting the previous expression on Eq. (95), we arrive to the following result

$$\lim_{R \to 0} \frac{d \delta}{dR} = -k^5 \left( \frac{r_S}{2} \right)^2 \left( \frac{u_k(0,0)}{k} \right)^2 \neq 0.$$  

(97)

This means that by adding more terms in the energy expansion of the boundary condition, i.e. fixing higher order parameters in the effective range expansion, we will find the same linear dependence as in the zeroth order case of the phase shift on the cut-off $R$ in the ultraviolet limit. With each new order in the energy expansion, the slope of $\delta(k,R)$ in the $R \to 0$ limit is progressively suppressed in energy with respect to the zeroth order case, e.g. by a factor of $(k r_S)^2$ at second order, $k^4 r_S v_S$ at fourth order, etc.

One example of this energy suppression is given by the OPE potential in the $S_0$ channel, for which $r_S = 4.46$ fm (see Sect. XIV). In this case, the cut-off dependence is in principle smoothened for $k < 0.45$ fm$^{-1}$ ($\sim 88$ MeV), indeed a bit more due to the higher order terms $k^5, k^3, \ldots$ which have been ignored here. The cut-off dependence for different energies is depicted in Fig. (5), both for the case where $\alpha$ and both $\alpha$ and $r_0$ parameters are fixed. As we can see, the effective cut-off dependence is still a bit smoother for two parameters at $k = 0.5$ fm$^{-1}$, and at $k = 1.0$ fm$^{-1}$ is only slightly worse than with one parameter.

In the case of attractive singular potentials the renormalization group analysis becomes more complex. The first step is the evaluation of $L_2$, which is computed in Appendix [13] yielding for $R \to 0$

$$L_2(R) \to \left( \frac{\alpha_0}{u_0(R,R)} \right)^2 \frac{\Delta r_0}{2},$$

(98)

where $\Delta r_0$ is the short distance contribution to the effective range$^9$, and the zero energy wave function is defined by

$$u_0(R,R) = \lim_{k \to 0} \frac{u_k(R,R)}{k}.$$  

(99)

The similarity with the $L_2(R) \to r_s/2$ result for regular potentials is striking. In fact the formula above can be used to obtain $r_s$ for a regular potential (for which $u_0(R,R)$ goes to a constant value at short distances).

Now we can evaluate the convergence of the phase shift with respect to the cut-off for $L_k = L_0 + k^2 L_2$, which is given by

$$\lim_{R \to 0} \frac{d \delta}{dR} = -k^5 \left( \frac{\Delta r_0}{2} \right)^2 \left( \frac{\alpha_0^2}{u_0(R,R)} \right)^2 \neq 0.$$  

(100)

This behaviour is depicted in Fig. (6) and at first sight looks weird. With every zero of the wave function there is a $\pi$ jump in the phase shifts, meaning that the system is explicitly sensitive to the appearance of deeply bound states, something that should not have practical consequences within the domain of applicability of a correctly formulated effective theory. There is no obvious and unique $R \to 0$ limit.

The previous result seems to suggest that one cannot fix both $\alpha_0$ and $r_0$ and obtain renormalized phase shifts. However, a further analysis below will show that it is indeed possible and the results are well defined and unique. $^{10}$

Instead of a naive truncation of the boundary condition as we have done, we suggest to reorder the expansion as follows

$$L_k(R) = L_0(R) + k^2 L_2(R) + k^4 \Delta_4(R) + \ldots$$

(101)

where $L_0$ and $L_2$ fulfill exactly the RG equations, but higher order terms do not (and hence the notation $\Delta_4, \Delta_6, \ldots$ for these terms). In such a case we can see that the scale dependence of the phase shift is given by

$$\frac{d \delta}{dR} = -k^5 \left[ L_2(R)^2 + 2 L_0 \Delta_4 + \Delta_4'(R) \right] \times \left( \frac{u_k(R,R)}{k} \right)^2.$$  

(102)

$^9$ Thus, the total effective range of the system is

$$r_0 = \Delta r_0 + r_0(\alpha_0)$$

where $r_0(\alpha_0)$ is the effective range due to the zero energy boundary condition used to fix the scattering length of the system. It can be computed with the well known integral formula

$$r_0(\alpha_0) = 2 \int_0^\infty dr \left[ \hat{v}_0^2(r) - \check{u}_0^2(r) \right],$$

where $\hat{v}_0(r) = 1 - r/\alpha_0$ and $\check{u}_0(r)$ is the solution of the zero energy reduced Schrödinger equation with the asymptotic normalization $\check{u}_0(r) \to \hat{v}_0(r)$ for large distances.

$^{10}$ In fact, we can consider for example the shape parameter $v_2$, which has the following integral definition

$$v_2 = \int_0^\infty dr \left[ \check{v}_2 - \check{u}_2 \right],$$

where $\check{v}_2(r) = 1 - r/\alpha_0$ and $\check{u}_2(r) = (r^2 - 3 \alpha_0 r^2 / 2 \alpha_0) r/6 \alpha_0$, while $\check{u}_0$ and $\check{u}_2$ are solutions of Eqs. (10) and (11) respectively, subjected to the asymptotic boundary conditions $\check{u}_0 \to 0$ and $\check{u}_2 \to \hat{v}_2$. Since the solutions for attractive singular potentials behave at short distances as $r^{n/4}$ times some trigonometric function, $v_2$ is convergent. A similar argument can be applied to $v_3, v_4$, and so on, and even to the phase shifts, thus giving a well defined limit when the cut-off is removed.
Then, analyzing the behaviour of $L_0$, $L_2$ and $\Delta_4$ (see Appendix 13), we find that
\[
\frac{d\delta}{dR} = -k^5 u_0(R, R) u_2(R, R) + O(k^7)
\sim -k^5 R^{n/2},
\] (103)
which means that the convergence pattern that emerges when we fix both $\alpha$ and $r_0$ is similar to that found when we fixed $\alpha_0$ only, i.e. there is no improvement on the short distance scaling suppression. The same arguments, but considering $\Delta_6$, $\Delta_8$, etc, can be applied to higher orders in the momentum expansion of the phase shifts yielding identical short distance scaling suppression.

The nontrivial fact is that fixing both the scattering length and the effective range is not exactly equivalent to any truncation of the boundary condition $L_k(R)$. All the terms in the energy expansion are equally singular, and therefore equally relevant. The correct parameterization of the truncated boundary condition corresponds to a unique Padé approximant looking representation of the boundary condition
\[
L_k(R) = \frac{u_0 + k^2 u_2}{u_0 + k^2 u_2} + k^4 \Delta(R),
\] (104)
where $\Delta(R)$ is a remainder. Indeed, for short distances, the Padé behaves as $1/R^{n/2}$, while the remainder $\Delta(R) \sim R$, and hence can be safely ignored when
the cut-off is removed. This expression for \( L_k(R) \) gives an \( R^{n/2+1} \) UV scaling for the phase shift, regardless on how many low energy parameters \( \alpha, r_0, v_2 \) and so on are fixed. The form of this Padé, which could be visualized as a renormalization group improvement is driven by the Moebius bilinear transformation, Eq. \((34)\) which embodies the dilatation group properties of the short distance cut-off given in Eq. \((35)\). A different Padé approximant will lead to spurious cut-off dependences, which will jeopardize the \( R \to 0 \) scaling behaviour.

5. Remarks on Regular Solutions

The RG behaviour of regular solutions represents a very special case. In it we fix the scattering observables to the values corresponding to the regular solution at the origin. By doing this the convergence with respect to the cut-off is improved noticeably.

The first case we are going to consider is the trivial one, that of a regular potential in which we do not fix anything, but rather enforce the regular solution \( u(R, R) = c_0 R \) at a small cut-off radius \( R \). For such a case we have the cut-off dependence

\[
\frac{d \delta}{d R} = k [U(R) - k^2] \left( \frac{u_k(R, R)}{k} \right)^2,
\]

since \( L_k(R) = 1/R \). In it \( R \) can be interpreted as the starting integration point in any usual integration procedure for the Schrödinger equation. For a potential which goes to a constant value at the origin, \( U(R) \to U_0 \), we have \( \Delta\delta \sim R^3 \), while for a Coulomb-like potential at short distances, like the Yukawa potential, we have \( \Delta\delta \sim R^2 \).

When one fixes the scattering length to the regular value, the convergence rate is given by Eq. \((84)\). In such a case, there would be a small improvement for a Yukawa potential, which will now converge as \( \Delta\delta \sim R^3 \). These results should be compared with the convergence when fixing the scattering length to an arbitrary value, which is \( \Delta\delta \sim R \). When one additionally fixes the effective range, Eq. \((103)\) describes the convergence rate. In case the effective range is taken to be the regular value, we have that the short distance effective range is zero, \( r_S = 0 \); \( L_2(R) \) can be estimated, for example, by the Green function methods used in Appendix \( A \) giving \( L_2(R) \sim R \). Then we have \( \Delta\delta \sim R^3 \), in accordance with naive expectations that increasing the number of counterterms will smoothen the cut-off dependence. Further suppressions will take place at higher orders.

The second case is when the potential is repulsive sing-
the same as setting counterterms. Counterterms are not observables, but they represent the unknown short distance potential as a low energy expansion in terms of the delta function and its derivatives. On the other hand, setting physical observables is the visible effect for the need of counterterms.  

Although there are no clear rules in the literature, we will take the point of view that (i) the cut-off should be removed, (ii) it is the cut-off dependence what drives the construction of an acceptable power counting, and (iii) the long range potential is going to be treated non-perturbatively at any order (for a singular potential this is absolutely necessary). These assumptions have been adopted in our previous works \[22, 25, 26\] but it is fair to mention that they are not universally agreed upon. (i) is for example accepted in Ref. \[22\], while rejected in Refs. \[48, 49, 50\], (ii) is explicitly used in the RG analysis of Birse \[16, 17, 21\], and in Refs. \[22, 33\] for promoting certain counterterms, and (iii) is accepted in Refs. \[48, 49, 50\] (the same which rejected (i)), but not in Ref. \[23\], in which it is advocated the perturbative treatment of the potential beyond LO, while LO remains non-perturbative (although no actual computation is done for NLO or NNLO). The amazing aspect of all these disagreements is that there is no operational definition of what would be a valid criterion and discrepancies are utterly based on the favourite prejudices of different authors (including ourselves). Here we are not going to discuss the validity or convenience of these assumptions (this has been partially done in our previous work \[22, 25, 26\]) but looking instead for their consequences.

The first consequence of our previous analysis is that the long range potential and the counterterms (or short range potential) are not independent: the singularity structure of the potential determines whether counterterms should be included or not. Only in the very special case of long range regular potentials are the counterterms independent (see also \[22, 25, 26\]). This disagrees with power counting schemes based on naive dimensional analysis, such as Weinberg's, in which all the counterterms with appropriate dimensions are included in the computation, regardless on the structure of the long range potential. But if the cut-off is to be removed, one must always include a counterterm if the long range potential is singular attractive (in agreement with the previous findings of Ref. \[39\] and the results of Ref. \[23\]) and one cannot include any counterterm at all in case it is singular repulsive.

From a naive viewpoint, it might seem counterintuitive that for a repulsive singular potential no counterterm can be included, but as we discuss now it is indeed quite natural. If we consider a singular repulsive potential with a characteristic long distance scale \(a\), then the physics associated with a short distance scale \(a^2\), such that \(a^2 \ll a\), would not affect at all the long distance physics, since the long distance potential itself would act as a potential barrier which destroys any effect coming from scales \(a^2\) smaller than \(a\). Then, there must be a strong short distance insensitivity which manifests itself as the dominance of the regular solution and thus a lack of counterterms when the cut-off is removed.

The issue of repulsive singular potentials raises an unexpected consequence for the power counting of the long distance potential: if the potential is singular attractive at a given order, it should remain singular attractive at higher orders. If the long distance potential between two particles is known to be attractive, the effective theory should reproduce this feature to all orders in the expansion of the potential (if this is going to be used in any non-perturbative computation). Of course the full interaction between two particles cannot truly be singular attractive at all distances; if this were to be the case the system would collapse. However, in the effective theory it is the unknown short distance physics which are repulsive, and these interactions are not explicitly modelled in effective theories, but implicitly via counterterms.

Another important issue which arises from the previous RG analysis regards the number of counterterms and/or renormalization conditions one should include when one has an attractive singular potential. As we have seen, the qualitatively power law cut-off scaling behaviour does not depend at all on the number of counterterms included in the computation, but only on the power law divergence of the potential near the origin. Although this seems to suggest that there is no reason for adding counterterms beyond the first one, when we look at the quantitative cut-off dependence we can see that the situation may change. One clear example is given by the TPE potential at NNLO in the \(1S_0\) singlet channel. If we look at how much the phase shifts change between the cut-offs \(R = 0.15\text{ fm}\) and \(R = 1.5\text{ fm}\) at a center of mass momentum \(k = 1.0\text{ fm}^{-1}\), we see

\[
|\Delta \delta(k)|_{1C} \simeq 25.21^o \\
|\Delta \delta(k)|_{2C} \simeq 15.11^o
\]

where the subscripts \(1C\) and \(2C\) refer to the number of counterterms used. As one can appreciate, there is a noticeable improvement of the convergence when a second counterterm is added, and which can justify its inclusion \[12\]. It should be noted that this improvement is not

\[11\] Actually, in momentum space one may find redundant counterterms appearing in the short distance potential at fourth order in momentum \(V_4(k',k) = C_0 + C_2(k^2 + k'^2) + C_4(k^4 + k'^4) + D_4 k^2 k'^2 + \ldots \). Obviously, \(C_4\) and \(D_4\) are redundant or else do not correspond to the same order. This actually shows that there are appear more counterterms than renormalization conditions.

\[12\] As a matter of fact, it is interesting to notice that a third counterterm (to fix \(\epsilon_2 = -0.48\text{ fm}\)) does not improve at all the convergence, but worsens it \((|\Delta \delta(k)|_{3C} \simeq 56.50^o)\).
This remark applies equally when going from a pure effective range prediction when one fixes the scattering length, which is \( r_0(\theta_0) = 2.86 \text{ fm} \), is very close to the experimental value, \( r_0 = 2.77 \text{ fm} \), to which we fix the second counterterm. Taking into account that the convergence when fixing two counterterms depends on the \( \Delta r_0 \) needed to correctly fix the effective range (if it were zero we would have an \( R^2 \) convergence pattern instead \( R^1 \)) it is not a surprise that there is dramatic improvement.

On the other hand, there are also good reasons not to include these extra counterterms. As argued in Refs. 22, 25, 26, the inclusion of more counterterms than the minimum required for finiteness break orthogonality constraints between different energy solutions, and in some cases, as the singlet at NLO 13, also the Wigner causality bound 21, although the unphysical consequences of breaking this bound have not been studied. Another good reason is the lack of an obvious improvement in the NNLO results in the singlet channel, as can be seen in Fig. 17. While NLO phases noticeably improve when fixing the effective range, the improvement for NNLO phases is very small. Thus it may be a better option to preserve orthogonality. Actually, Fig. 17 suggests that it would be more instructive and perhaps profitable to improve on the long distance potential than adding more and more counterterms 14. Work along these lines is on the way 52.

It is also important to notice that the renormalizability of singular potentials depends on a very specific representation of the short range physics, a Padé approximant if we write them in terms of an energy dependent boundary condition. This casts doubts on the renormalizability of momentum space treatments in which the contact interactions are parameterized as an ad-hoc expansion of deltas and their derivatives, since only a very precise short distance interaction will be able to renormalize the Lippmann-Schwinger equation.

### IV. BOUNDARY CONDITION AND VARIABLE PHASE EQUATION

#### A. Equivalence between BC and Variable Phase

The boundary condition, Eq. (9), corresponds to the inner radius \( R \) of a boundary value problem defined in the region \( R \leq r < \infty \). We can give a physically appealing and computationally convenient interpretation of this BC in terms of a complementary outer boundary value problem in the region \( 0 < r \leq R \). If we consider the family of potentials \( U(r, R) = U(r)\theta(R - r) \), which corresponds to a set of truncated potentials, \( U(r) \), at distances below a certain radius, \( r < R \), acting only from the origin to the boundary radius \( R \) we would have at the boundary, \( r = R \), the asymptotic wave function

\[
u(r) = \sin(kr + \delta(k, R)) \quad r > R,
\]

where we keep explicitly the dependence on \( R \) of the phase shift. The logarithmic derivative at the boundary \( r = R \) from the left is therefore

\[
L_k(R) = k \cot(kR + \delta(k, R)).
\]

If we identify this expression with that of Eq. (9) we get the following equation for \( \delta(k, R) \)

\[
\frac{d\delta(k, R)}{dR} = -\frac{1}{k} U(R) \sin^2(kR + \delta(k, R)).
\]

This is a variable phase equation of the type analyzed in Ref. 13, describing the evolution of the phase shift, \( \delta(k, R) \), corresponding to the truncated potential, \( U(r, R) = U(r)\theta(R - r) \). The standard derivation for a regular wave function at the origin, \( u(0) = 0 \), is well known. For the case of general solutions including also energy dependence see e.g. 14.

Thus, the renormalization group equation for the boundary condition of an inner truncated potential \( U(r, R) = U(r)\theta(R - r) \) is solved by the logarithmic derivative of an outer cut-off potential \( \bar{U}(r, R) = U(r)\theta(R - r) \) through the variable phase equation Eq. (111). This result builds a one to one relation between the evolution of effective boundary conditions of the outer problem and that of a variable phases of the inner problem, which is illustrated in fig. 18. Obviously, the physical phase shift can be obtained from the variable phase as an asymptotic limit

\[
\delta(k) = \delta(k, \infty).
\]

On the other hand, the boundary condition extrapolated to the origin is given by

\[
L_k(0^+) = k \cot \delta(k, 0^+).
\]

In the standard variable phase approach 13 one assumes \( \delta(k, 0) = 0 \) corresponding to the absence of zero range interaction, and hence \( L_k(0) = \infty \), i.e. a regularity condition at the origin, \( u(0) = 0 \). In the present context it makes sense to define the short range phase-shifts as the variable phase extrapolated to the origin, once the potential has been completely switched off

\[
\delta_S(k) = \delta(k, 0^+) \equiv \lim_{R \to 0^+} \delta(k, R).
\]

In fact, it turns out that the way the former limit must be taken is a bit subtle, particularly in the case of singular

\[\text{13} \quad \text{The NLO prediction for the effective range is } 2.29 \text{ fm, smaller than the experimental one.}
\]

\[\text{14} \quad \text{This remark applies equally when going from a pure effective range expansion, which describes the low energy data with arbitrary accuracy but randomly agrees to the intermediate energy region sensitive to the explicit pion exchange potential.} \]
FIG. 7: Phase shifts for the $^1S_0$ single channel for the NLO TPE potential (left panel) and the NNLO TPE potential (right panel). We show the renormalized phase shifts computed when fixing one (1C curve) or two (2C curve) parameters, i.e. the scattering length and the effective range ($a_0 = -23.74$ fm and $r_0 = 2.77$ fm).

potentials (see also below). In the absence of a potential one gets a constant variable phase, and hence we would simply get $\delta(k) = \delta_S(k)$.

On these grounds, in the presence of a long range potential, the total phase shift $\delta(k)$ can be understood as a long distance distortion of the short range phase shift $\delta_S(k)$

$$\delta_S(k) \rightarrow \text{(Long range distortion)} \rightarrow \delta(k) \quad (115)$$

In this interpretation the boundary condition regularization can be used to disentangle the short and long range physics from $\delta(k)$ to obtain $\delta_S(k)$

$$\delta(k) \rightarrow \text{(Remove the distortion)} \rightarrow \delta_S(k) \quad (116)$$

Another aspect of the variable-phase approach is the fact that we always deal with a given on-shell problem, that corresponding to the truncated potential. So, in the whole process there is no need to invoke any smooth off-shell behaviour, although we are changing the Hilbert space when the boundary radius is moved.

For later purposes it is convenient to introduce the variable effective range $\hat{M}$-matrix and its inverse the reaction $\hat{V}$-matrix,

$$\hat{M}(k, R) = k \cot \delta(k, R), \quad (117)$$

$$\hat{V}(k, R) = \frac{\tan \delta(k, R)}{k}, \quad (118)$$

respectively.

B. Renormalization of Low energy Threshold Parameters

Using the equivalence discussed above between the variable phase and boundary condition problems, it is
instructive to do a low energy expansion. At zero energy we have,

\[ \alpha_0(R) \equiv -\lim_{k \to 0} \frac{\delta(k)}{k}, \quad (121) \]

so one can obtain

\[ \frac{d\alpha_0}{dR} = U(R) (R - \alpha_0)^2. \quad (122) \]

For increasing \( R \) this equation describes how the scattering length evolves as the potential \( U(r) \) is switched on. The physical scattering length is given by the value of \( \alpha_0(R) \) at \( R = \infty \), that is, \( \alpha_0 = \alpha_0(\infty) \), provided we specify an initial condition at any point, say the origin. In the standard approach, one takes \( \alpha(0) = 0 \). However, this is not the only possibility. We may use \( R = \infty \) as the initial condition, and compute \( \alpha(0) \) from there by integrating Eq. (122) for decreasing \( R \). This can be interpreted as the evolution of the scattering length as the potential is switched off for \( r > R \). If we use an arbitrary value for \( \alpha \) at \( R = \infty \) we obtain in general \( \alpha(0) \equiv \alpha_S \neq 0 \). Thus, we may interpret the value of \( \alpha_S \) as the scattering length corresponding to switching off the potential entirely.

Going beyond the zero energy limit \( k = 0 \) is not uniquely defined, because there are many equivalent ways of parameterizing the phase shifts by a low energy expansion. The coefficients of the expansion are, however, well defined. If for definiteness we use the effective range expansion for the running \( \hat{M} \)-matrix

\[ \hat{M}(k, R) = k \cot \delta(k, R) \]

\[ = -\frac{1}{\alpha_0(R)} + \frac{1}{2} r_0(R)k^2 + v_2(R)k^4 + \cdots \]

(123)

one has the set of equations \([14, 15]\)

\[ \frac{d\alpha_0}{dR} = U(R) (R - \alpha_0)^2, \quad (124) \]

\[ \frac{dr_0}{dR} = 2U(R)R^2 \left( 1 - \frac{R}{\alpha_0} \right) \left( \frac{r_0}{R} + \frac{R}{3\alpha_0} - 1 \right), \quad (125) \]

\[ \frac{dv_2}{dR} = R^4U(R) \left\{ \frac{1}{4} \left( \frac{r_0}{R} + \frac{R}{3\alpha_0} - 1 \right)^2 \right. \]

\[ + 2 \left( 1 - \frac{R}{\alpha_0} \right) \left( -\frac{1}{12} \frac{r_0}{R} + \frac{v_2}{R^3} - \frac{1}{120} \frac{R}{\alpha_0} + \frac{1}{24} \right) \} \].

These equations have to be supplemented with the initial conditions \([14, 15]\)

\[ \alpha(0^+) = \alpha_S \quad \alpha(\infty) = \alpha, \quad (127) \]

\[ r_0(0^+) = r_0, \quad r_0(\infty) = r_0, \quad (128) \]

\[ v_2(0^+) = v_0, \quad v_2(\infty) = v_2. \quad (129) \]

It should be noted here that \( \alpha_S, r_0, v_2 \) and \( \alpha\) can only be defined for regular potentials \([15]\). The set of Eqs. (124), (125) and (126) for the running low energy threshold parameters solve the set of equations for the parameters \( \alpha_0(R), \alpha_2(R) \) and \( \xi_4(R) \), Eqs. (71), (72) and (73) if in the relations (49), (50) and (51) one substitutes the asymptotic low energy threshold parameters for the running ones.

The previous reasoning may also be applied to a low energy expansion of the variable reaction \( \hat{V} \)-matrix

\[ \hat{V}(k, R) = \frac{\tan \delta(k, R)}{k} \]

\[ = -\alpha_0(R) - \frac{1}{2} \beta_0(R)k^2 + \cdots \quad (131) \]

which generates the following equations for \( \alpha_0 \) and \( \beta_0 \)

\[ \frac{d\alpha_0}{dR} = U(R) (\alpha_0 - R)^2, \quad (132) \]

\[ \frac{d\beta_0}{dR} = -U(R) (R - \alpha_0) (R^3 - 3\alpha_0 R^2 + 3\beta_0) \quad (133) \]

Obviously, we have the relation

\[ \beta_0(R) = r_0(R)\alpha_0(R)^2. \quad (134) \]

It is straightforward to check that the set of Eqs. (124), and (125) and Eqs. (132) and (133) are mutually compatible, as it should be.

All low energy expansions share a common hierarchy for the low energy parameters; the evolution of a given low energy parameter contributing to a given order depends only on the evolution of lower order low energy parameters. The set of equations express the evolution of the low energy parameters at zero range when the long distance contribution is switched on. Conversely, they offer a possibility to determine the zero range low energy parameters from the total ones by downwards evolution in the cut-off variable \( R \).

1. Application to NN Scattering

The previous equations, Eqs. (124), (125) and (126), can be used to study the renormalization behaviour of the low energy parameters for neutron-proton scattering in the \( ^1S_0 \) channel with the OPE potential. Although the OPE potential is regular for the \( ^1S_0 \) channel, it behaves as \( 1/r \) for distances below the pion Compton wave length. The consequence of this mild singularity is that the short distance scattering length, \( \alpha_S = \alpha_0(0^+) = 0 \), is zero, regardless the fact that we are not taking the regular solution of the OPE potential. This can be understood

\[ \text{Note}: \quad (130) \]

\[ \text{Note}: \quad (131) \]

\[ \text{Note}: \quad (132) \]
by studying the behaviour of $\alpha_0(R)$ for the ultraviolet limit, $R \to 0$ (see Ref. [14]). We can solve Eq. (124) for short distances and large scattering lengths with the Yukawa potential, given by Eq. (57), yielding

$$\frac{1}{\alpha_0(R)} \simeq \frac{1}{a} \log R + C,$$  \hspace{1cm} (135)

where $C$ is an integration constant, and which is valid for small distances ($mR \ll 1$) and large scattering lengths ($R \ll \alpha_0(R)$). This solution of Eq. (124) is equivalent to take the irregular solution of the Schrödinger equation or the stable ultraviolet fixed point studied in Sect. III D 2. On the other hand, if we assume the scattering length to be small compared to the cut-off scale, $\alpha_0(R) \ll R$, we obtain

$$\alpha_0(R) \simeq -\frac{R^2}{2a},$$  \hspace{1cm} (136)

which turns out to be equivalent to take the regular solution of the Schrödinger equation or the unstable ultraviolet fixed point for the Yukawa potential. So the difference between the regular and irregular solution lies in the ultraviolet behaviour of $\alpha_0(R)$: although they both approach the same ultraviolet limit, $\alpha_0(0^+) = 0$, the trend is dissimilar.

We can integrate Eqs. (124), (125) and (126) for the other low energy parameters of NN scattering in the $1S_0$ channel with OPE, by taking as initial conditions for $R = \infty$ the experimental scattering length and effective range, while for the shape parameter $v_2$ we use the value obtained from the Nijmegen II potential, yielding

$$\alpha(0^+) = 0 \quad \alpha(\infty) = -23.74 \text{ fm},$$  \hspace{1cm} (137)

$$r_0(0^+) = 4.46 \text{ fm} \quad r_0(\infty) = 2.77 \text{ fm},$$  \hspace{1cm} (138)

$$v_2(0^+) = 1.24 \text{ fm}^3 \quad v_2(\infty) = -0.48 \text{ fm}^3.$$  \hspace{1cm} (139)

In Ref. [14] we can find a more detailed discussion although different results are obtained due to the low infrared cut used ($R_\infty = 10 \text{ fm}$) $^{16}$. In Ref. [53] the result $r_{0,S} = 4.0 \text{ fm}$ is obtained by a renormalization analysis in momentum space, and in Ref. [19] they obtain $r_{0,S} = 3.10 \text{ fm}$ although an ultraviolet cut-off of half the rho mass $\Lambda = m_\rho/2$ is employed to perform the calculation; if we use $R = \pi/2\Lambda = \pi/m_\rho \sim 0.8 \text{ fm}$, equivalent to the previous cut-off, we obtain $r_{0,S} = 2.86 \text{ fm}$.

C. Low energy expansion of the short distance interaction

So far, all we have done has to do with relating long and short range physics along the trajectory defined by the long distance potential. If it was for that nothing would be achieved. We propose to make a low energy expansion of the short range physics. According to our previous discussion for a truncated potential it makes sense to expand either the $\tilde{M}$-matrix or the $\tilde{V}$-matrix in powers of $k$,

$$\tilde{M}_S = k \cot \delta_S = \frac{u_k'(0^+)}{u_k(0^+)}$$

$$= -\frac{1}{\alpha_S} + \frac{1}{2}r_S k^2 + v_S k^4 + \ldots$$  \hspace{1cm} (140)

$$\tilde{V}_S = \frac{\tan \delta_S}{k} = \frac{u_k(0^+)}{u_k'(0^+)}$$

$$= -\alpha_S - \frac{1}{2} \beta_S k^2 - \gamma_S k^4 + \ldots$$  \hspace{1cm} (141)

respectively. To achieve consistency with the low energy expansions, Eqs. (135) and (140), up to some order of the full $\tilde{M}$-matrix and $\tilde{V}$-matrix we have to compute the short distance low energy parameters by integrating downwards the set of Eqs. (124), (125) and (126) or Eqs. (124) and (138). This way we exactly reproduce the low energy expansion up to a desired order, and generate all higher orders in energy according to the long distance part of the potential. In particular, we also generate at any level of truncation the remaining higher order parameters. Thus, in the LO approximation we consider $\alpha$ and $U(r)$ as independent parameters, corresponding to keep one term in Eq. (140), and hence setting $r_0(0^+) = v_2(0^+) = \ldots = 0$. One then obtains $r_0, v_2, \text{etc.}$, from $\alpha$ and $U(r)$, and as a consequence the phase shift $\delta(k)$. In the NLO approximation one considers $\alpha$ and $r_0$ as independent variables and predicts $v_2, v_3, \text{etc.}$, and hence the phase shift $\delta(k)$ from the knowledge of $\alpha$, $r_0$ and $U(r)$.

Note that if the physical low energy parameters, say $\alpha$, $r_0$ and so on, and the potential $U$ are known, the result is unique $^{17}$. Another important point is that if we have a singular potential we cannot start directly at the origin, but at a given small radius $R$. An advantage of our method that will become clear below is that we can make $R$ much smaller than any other scale in the problem. Even if we do this numerically, this is an effective way of eliminating the regularization. Moreover, if we take the limit of large $R$, much larger than the range of the potential $\alpha$, we are effectively having a constant variable phase given by the short distance theory. It thus makes sense to compare the full result including the potential with that of the low energy expansion.

$^{16}$ Here $R_\infty = 20 \text{ fm}$ is used instead.

$^{17}$ In practice this situation may be too optimistic, since low energy parameters may be directly deduced from the data, which are analyzed with a given model. In Ref. [54] we provide a thorough determination of the low energy parameters for NN interaction in all partial waves for the high quality potentials of Ref. [53].
V. HIGHER PARTIAL WAVES AND COUPLED CHANNELS

The generalization of the present ideas to coupled channels is in principle straightforward and runs parallel to what was done in Sect. [III] with some modification which we outline in the following. The coupled channel Schrödinger equation for the relative motion reads

\[-u''(r) + U(r) + \frac{l^2}{r^2} u(r) = k^2 u(r),\]

where \(U(r)\) is the coupled channel matrix potential, \(u(r)\) is the reduced matrix wave function in the initial and final state and

\[l^2 = \text{diag}(l_1(l_1 + 1), \ldots, l_N(l_N + 1)),\]

\[k^2 = \text{diag}(2\mu_1(E - E_1), \ldots, 2\mu_N(E - E_N)),\]

are the angular momentum and the CM momentum in the coupled channel space respectively. \(E_i\) is the threshold energy and \(\mu_i\) the reduced mass in the i-th channel. We assume for \(u(r)\) the boundary condition,

\[u'(R) - L_k(R)u(R) = 0,\]

where \(L_k(R)\) is a real hermitean matrix in coupled channel space, which in our framework encodes the unknown physics at distances \(r\) below the boundary radius \(R\). In addition, we assume the asymptotic normalization condition for scattering states

\[u(r) \rightarrow u_{\text{in}}(r) - u_{\text{out}}(r)S,\]

with \(S\) the standard coupled channel unitary S-matrix, \(S\dagger S = 1\), from which the scattering amplitude \(f = 1 + 2ik^{1/2}S\dagger k^{1/2}\) can be obtained. The out-going and in-going wave functions can be defined as

\[u_{\text{in}}(r) = \hat{h}^-(r), \quad u_{\text{out}}(r) = \hat{h}^+(r),\]

where \(h^+(r)\) and \(h^-(r)\) are given by

\[h^+(r) = \text{diag}(\hat{h}^+_{i_1}(k_1 r), \ldots, \hat{h}^+_{i_N}(k_N r)), \quad h^-(r) = \text{diag}(\hat{h}^-_{i_1}(k_1 r), \ldots, \hat{h}^-_{i_N}(k_N r)),\]

with \(\hat{h}^\pm_i(x)\) the reduced Haekkel functions of order \(i\),

\[\hat{h}^\pm_i(x) = xH^\pm_{i+1/2}(x) (\hat{h}^0_0 = e^{\pm ix}),\]

which satisfy the Schrödinger’s equation for a free particle.

To determine how should the matrix boundary condition depend on the boundary radius in order to achieve the same S-matrix, we proceed similarly as in Sect. [III]. Making an infinitesimal displacement of the radius, \(R \rightarrow R + \Delta R\), and taking into account the total derivative of the wave function with respect to the boundary radius

\[\frac{\partial u(r, R)}{\partial R} = u_R(r, R),\]

then, the derivative of the boundary condition is given by

\[u''(R, R) + u'_R(R, R) - L'_k(R)u(R, R) - L_k(R)u'(R, R) - L_k(R)u_R(R, R) = 0,\]

Deriving also Schrödinger’s equation with respect to the inner radius \(R\)

\[-u''(r, R) + U(r)u(r, R) = k^2 u(r, R),\]

and the asymptotic wave function, Eq. (149)

\[u_R(r, R) \rightarrow -u_{\text{out}}(r)\frac{dS}{dR},\]

\[u'(r, R) \rightarrow \left( u_{\text{in}}(r) - u_{\text{out}}'(r)S \right),\]

\[u'_R(r, R) \rightarrow u_{\text{out}}'(r)\frac{dS}{dR}.\]

Thus, using Lagrange’s identity we get

\[0 = u(r, R)u_R(R, R) - u'(r, R)u_R(R, R),\]

\[= \left( u(r, R)u_R(R, R) - u'(r, R)u_R(R, R) \right)' \cdot\]

Integrating between \(R\) and \(\infty\) and using the boundary condition, Eq. (155) and Eq. (156) and the identity \(u_{\text{in}}u_{\text{out}}\) as well as the Wronskian \(u_{\text{in}}u'_{\text{out}} - u_{\text{in}}'u_{\text{out}} = 2ik\) we finally get

\[2iS\dagger k\frac{dS}{dR} = u(R, R)\dagger \left[ k^2 - U(R) - \frac{l^2}{r^2} \right. + L'(R) + L(R)^2 \left] u(R, R) \right.,\]

This equation tells us how the S-matrix changes as the inner boundary radius is changed. If we require the S-matrix not to be dependent on the particular choice of \(R\) we get

\[L(R)^2 + L'(R) = U(R) + \frac{l^2}{r^2} - k^2,\]

which is the coupled channel generalization of Eq. (29), which likewise accounts for the coupled channel momentum \(k\) dependent evolution of the boundary condition. The evolution of the low energy parameters can be translated into the corresponding evolution of the short distance boundary condition as a function of the boundary radius. Defining the dimensionless quantity

\[\Xi_k(R) = R L_k(R) = R u_k^\dagger(R)u_k(R)^{-1},\]

and using Eq. (159) we get

\[R \frac{d\Xi_k(R)}{dR} = \Xi_k(1 - \Xi_k) + U(R)R^2 + \frac{l^2}{r^2} - k^2 R^2,\]
a result already derived in our previous work \cite{14} by different means. Assuming for simplicity the degenerate case, \( k_i = k \) and expanding into powers of the common momentum \( k \) one gets,

\[
\Xi_k(R) = \Xi_0(R) + (kR)^2 \Xi_2(R) + \ldots
\]  

(162)

so the RG flow at zero energy becomes

\[
R \frac{d\Xi_0}{dR} = \Xi_0(1 - \Xi_0) + U(R)R^2 + 1^2.
\]

(163)

A thorough study for the general multichannel case is beyond the scope of the present work. Nevertheless, for some special cases some general conclusions may be drawn. The most appealing regime has to do with the possible appearance of chaos. If that would be so, the whole renormalization program would not be implementable in practice, since an absolute knowledge of the short distance conditions would be needed. It is well known, that the minimal order for a dynamical autonomous system to develop chaotic solutions is three since for two independent variables the Poincaré-Bendixson theorem \cite{49} guarantees integrability. Actually, if we introduce the variable \( t = -\log(R/a) \), as a new variable with \( R \) as a dependent variable, the simplest case for potential scattering would correspond to two-coupled channels. In Ref. \cite{15} we have discussed the case corresponding to OPE NN potential in the \( ^3S_1 - ^3D_1 \) coupled channel, and infrared fixed points have been determined.

To analyze the short distance behaviour, let us assume the potential to behave as an inverse singular power

\[
U(R)R^2 = \frac{a^n}{R^n},
\]

(164)

where \( a \) is a matrix with length dimension. We can diagonalize the potential by a \textit{global} transformation, say \( G \), so that if we have a set of diagonal \( GC_0G^{-1} \) at some initial value the solution will always be diagonal. This means that the system is integrable and hence chaos is precluded.

Similarly to the findings of Sect. \textbf{IV} (see also Ref. \cite{15}) the coupled channel boundary condition, Eq. \textbf{(152)}, for the \textit{outer} boundary values problem, Eq. \textbf{(142)} and Eq. \textbf{(143)}, can be interpreted in simple physical terms of a complementary \textit{inner} problem where the potential \( U(r) \) acts in the interval \( R \leq r < \infty \). If we switch off the potential above a given boundary radius \( R \) we have, at the boundary

\[
L_k(R) = u'(R)u^{-1}(R) = \left[ u'_n(R) - u'_n(out)(R)S(R) \right] \times \left[ u_n(R) - u_{out}(R)S(R) \right]^{-1},
\]

(165)

where \( S(R) \) is the S-matrix associated to the potential \( U(r) \) acting in the region \( 0 < r \leq R \), which inherits the dependence on the chosen boundary radius \( R \). It is straightforward to obtain the equation for the variable S-matrix,

\[
2ik \frac{dS(R)}{dR} = \left[ S(R)\hat{h}^{(+)}(R) - \hat{h}^{(-)}(R) \right] U(R) \times \left[ \hat{h}^{(-)}(R) - \hat{h}^{(+)}(R)S(R) \right].
\]

(166)

Further consequences of this equation, in particular its low energy limit can be looked up in Ref. \cite{54}.

\section{VI. SUMMARY AND CONCLUSIONS}

In the present work we have analyzed the role of boundary conditions in potential two-body scattering from a renormalization point of view. We remind that in quantum mechanical problems the most general form of interactions can be accommodated not only in terms of a given potential, but also as a nontrivial mixed boundary condition. The discussion has been carried out in configuration space, because there the disentanglement of the short and long distance physics is naturally formulated using both a suitable boundary condition at the origin and a local potential respectively. The suitability of the boundary condition is governed by a simple renormalization group equation, and depends explicitly on the choice of the potential, the scattering energy as well as some renormalization conditions. In potential scattering the most appropriate conditions at low energies are the threshold parameters, such as the scattering length, the effective range and so on. The generalization of such an equation for the case of coupled channels is straightforward and has also been analyzed in some detail.

The resulting equations can be applied to many cases of interest, like the study of both infrared and ultraviolet fixed points. We find stable and unstable infrared fixed points corresponding to the limit of small and large scattering lengths respectively, in agreement with previous authors \cite{10, 17}. The same kind of fixed points is also found for regular potentials and repulsive singular potentials in the ultraviolet regime. In contrast, for the case of attractive singular power law potentials, we also describe ultraviolet limit cycles and attractors with a computable fractal dimension. Actually, these exotic renormalization group solutions are genuinely non-perturbative short distance phenomena, and have a close relationship with the cut-off dependence of observables and hence with power counting. Once the RG trajectory of the boundary condition enters the attractor, renormalized observables show a very smooth dependence with respect to the cut-off, as can be seen in Refs. \cite{19, 20, 21}. Moreover, these results imply fine tuning short-distance conditions as already found in our previous works \cite{12, 15}. Obviously, the appearance of an RG trajectory attractor and fractality immediately suggests the search for possible short distance chaotic behaviour of the renormalization group equations. Let us note that the presence of chaos at short distances would demand not only fine tuning but abso-
lute tuning of renormalization constants in the ultraviolet limit. Although our equations look very similar to those appearing in dynamical systems with chaotic behaviour, for the obvious candidate of coupled channel zero energy s-wave scattering with a singular power law potential we have not seen any traces of a chaotic pattern.

Furthermore, the ultraviolet renormalization group behaviour of singular potentials provides interesting insights into the power counting rules for non-perturbative effective field theories. Boundary conditions allow a clear estimate of the cut-off behaviour of scattering observables when certain renormalization conditions are imposed. Since one expects that an effective theory leads to a low energy description which becomes increasingly insensitive to detailed short distance dynamics, then the cut-off dependence of the observables can be used to construct a power counting for the unknown short range physics, i.e. the counterterms. One consequence is that counterterms are no longer completely independent of the long range potential, as happened in the original Weinberg’s counting. Thus, a counterterm must be included in channels with an attractive singular potential, in agreement with previous works [22, 24, 39]. The inclusion of extra counterterms in these channels do not change the power law suppression in the cut-off of short range physics. However, the proportionality coefficient does depend on the renormalization conditions.

On the contrary in channels with a repulsive singular potential no counterterm can be added if the cut-off is to be removed [25]. This result, which seems very counterintuitive from a naive point of view, is indeed very natural when one takes into account that a long range repulsive potential acts as a potential barrier for the unknown short distance physics, thus completely screening their contribution to low energy physics.

A very appealing perspective of the renormalization group method is provided by the definition of a complementary scattering problem, which can be mapped into a variable phase equation [13] with nontrivial initial (short-distance) conditions and used extensively in our previous works [14, 15, 27]. Actually, using the long distance fixed points we find a kind of short-distance modified effective range expansion both for large and small scattering lengths, which in some cases is amenable to a perturbative discussion. Moreover, our method allows to handle the case of singular potentials at the origin. This complements the long distance modified effective range expansion proposed long-ago [18] and also deduced more recently by renormalization group arguments in Ref. [17], where by construction genuinely singular potentials at the origin where excluded.

One of the most rewarding aspects of the previous and present investigations has to do with the practical elimination of short distance cut-offs in the NN scattering problem which long range pion exchange contributions become singular at the origin. Since Effective Field Theory ideas were proposed to study the NN problem, a lot a progress has been made, but the existence of finite cut-offs has clouded the key renormalization issues. The renormalization program becomes necessary to make truly model independent calculations based on both the long distance physics explicitly governed by pion exchanges and hence sensitive to chiral symmetry, and the unknown short distance dynamics. The present approach suggests a way from a renormalization group viewpoint to effectively remove these cut-offs, and suggests that the non-perturbative counterterms to achieve scale independence can indeed have a quite unexpected behaviour, as we have discussed and illustrated in detail for the $^1S_0$ channel with the singular TPE Potential. Actually, our results are more general and can be applied in other contexts where the unknown short distance physics plays a significant role.

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APPENDIX A: SOLUTIONS TO THE SCHRODINGER EQUATION FOR SINGULAR POTENTIALS

Singular Potentials are commonplace on Effective Theories. Since the effective interaction is developed as a long range (or low energy) expansion, each new order gives rise to potentials which are more singular at short range scales. This can be seen in the NN chiral potential, which at $Q^0$ order displays a $1/r^{3+\nu}$ singularity for distances below the pion Compton wave length $^{18}$, Consequently the study of the solutions of the Schrödinger equation for singular potentials is of main interest for this work, in which we treat the renormalization of the effective interactions in coordinate space.

We will consider the case of a potential which shows a power law divergence near the origin, i.e. $U(r) \rightarrow$
and so on. Once a certain solution for $u_0(r)$ is given, the higher order terms can be solved via Green functions

$$u_2(r) = \int_0^\infty dr' G_0(r,r') u_0(r'), \quad \text{and so on},$$

where the Green function can be defined in terms of two linear independent solutions $u_{0,a}$ and $u_{0,b}$ of the zero energy Schrödinger equation

$$G_0(r,r') = u_{0,a}(r) u_{0,b}(r') \theta(r'-r) + u_{0,a}(r') u_{0,b}(r) \theta(r-r'),$$

where $u_{0,a}$ and $u_{0,b}$ fulfill the relation

$$u_{0,a}(r) u_{0,b}(r) - u_{0,a}(r) u_{0,b}(r) = 1,$$

corresponding to unity Wronskian normalization. From this representation of $u_2$, $u_4$, and so on, in terms of Green function it is very easy to obtain their power law behaviour near the origin. Since any solution of the zero energy Schrödinger equation for a singular potential behaves as $r^{n/4}$, times some trigonometric or exponential function depending whether the potential is attractive or repulsive, we see that $u_2$ behaves as $r^{n/2+1} \times r^{n/4}$, $u_4$ as $r^{n/2+2} \times r^{n/4}$, and so on, each power of $k^2$ adding an $r^{n/2+1}$ to the wave function. This means that there is a strong short distance suppression in the expansion of the wave function in powers of momenta.

**APPENDIX B: EVALUATION OF THE TRUNCATION OF THE BOUNDARY CONDITION FOR ATTRACTIVE SINGULAR POTENTIALS**

In this appendix we want to discuss in further detail the error estimates when truncations are made in the boundary condition, Sect. IIII thus breaking exact RG invariance. The idea behind these truncations is to fix some physical observables of the system, and then make predictions for the other observables. This violates exact RG invariance, which would imply a complete knowledge of the phase shifts of the system at all energies, something which is not possible in practical computations, and which would also lack predictive power. Nevertheless, as we saw in Sect. IIII, although exact RG invariance is broken for an arbitrary cut-off $R$, under some circumstances we recover it in the $R \to 0$ limit.

In all these cases, higher order RG equations are not fulfilled. Thus, if we only fix $\alpha_0$ for example, we can rewrite the expansion of the boundary condition as

$$L_k(R) = L_0(R) + k^2 \Delta_2(R) + \ldots$$

meaning by this notation that $L_0$ fulfills its RG equation, while $\Delta_2$ and higher order terms don’t. If we fix both $\alpha_0$ and $r_0$, we could write accordingly

$$L_k(R) = L_0(R) + k^2 L_2(R) + k^4 \Delta_4(R) + \ldots$$

in which now both $L_0$ and $L_2$ fulfill their respective RG equations, while the higher orders don’t. In many occasions, the higher order terms can be neglected, and then the boundary condition reduces to

$$L_k(R) = L_0(R),$$

$$L_k(R) = L_0(R) + k^2 L_2(R),$$
which in turn implies that $\Delta_2/L_0 \to 0$ or $\Delta_4/L_2 \to 0$ when the cut-off $R$ goes to zero, thus simplifying the computations.

For computing $L_0$, $L_2$, and higher order terms in the expansion of the boundary condition it is useful to define the next momentum expansion of the wave function

$$\frac{u_k(r, R)}{k} = u_0(r, R) + k^2 u_2(r, R) + k^4 u_4(r, R) + \ldots$$  \hspace{1cm} (B5)

in which we have normalized the wave functions according to the asymptotic normalization

$$u_k(r, R) \to \sin(k r + \delta(k, R)),$$ \hspace{1cm} (B6)

$$u_0(r, R) \to r - \alpha_0,$$ \hspace{1cm} (B7)

$$u_2(r, R) \to \frac{r}{6} (3 \alpha_0 (r - r_0) - r^2),$$ \hspace{1cm} (B8)

$$u_4(r, R) \to \frac{r}{120} (r^4 - 5 \alpha_0 r_0^2 + 10 \alpha_0 r_0 r^2 - 120 \alpha_0 v_2),$$ \hspace{1cm} (B9)

and so on. Note that here $\delta(k, R)$ is the phase shift dependence on the short distance cut-off (not to be confused with the variable phase). This normalization for the wave functions is consistent with the one used in the derivation of Eq. \{(24)\}. The terms $u_0$, $u_2$, etc, obey the following equations

$$-u''_0 + U(r) u_0 = 0,$$ \hspace{1cm} (B10)

$$-u''_2 + U(r) u_2 = u_0,$$ \hspace{1cm} (B11)

$$-u''_4 + U(r) u_4 = u_2,$$ \hspace{1cm} (B12)

which can be trivially deduced by applying the reduced Schrödinger equation to the expansion in powers of momenta of the wave function, Eq. \{(25)\}.

Now let’s study the case in which we fix the scattering length of the system, thus fulfilling Eq. \{(71)\}, while Eq. \{(72)\} and higher order ones are not. In such a case we can write the expansion of the boundary condition as

$$L_k(R) = L_0(R) + k^2 \Delta_2(R) + \ldots$$ \hspace{1cm} (B13)

in which the relation of the terms in this expansion and the ones of the expansion of the wave function is given by

$$L_0(R) = \frac{u'_0}{u_0}$$ \hspace{1cm} (B14)

$$\Delta_2(R) = \frac{u'_2 u_0 - u_2 u'_0}{u_0^2}.$$ \hspace{1cm} (B15)

From the behaviour of the zero energy wave function for power-law singular potential, it is trivial to check that for small cut-offs

$$L_0(R) \sim \frac{1}{R^{n/2} \cot \left[ \frac{2}{n-2} \left( \frac{a}{R} \right)^{n/2-1} + \phi_0 \right]}.$$ \hspace{1cm} (B16)

On the other side, the behaviour of $\Delta_2$ can be easily evaluated if we take into account the following Lagrange identity

$$\left( u'_2 u_0 - u_2 u'_0 \right) = u_0(r)^2,$$ \hspace{1cm} (B17)

where we have dropped the dependence of $u_0$ on the cut-off, since this wave function is cut-off independent. From this we find that

$$\Delta_2(R) = \frac{1}{u_0(R)^2} \int_0^R dr u_0(r)^2,$$ \hspace{1cm} (B18)

which for small cut-offs approximately behaves as

$$\Delta_2(R) \sim R f \left[ \frac{2}{n-2} \left( \frac{a}{R} \right)^{n/2-1} + \phi_0 \right],$$ \hspace{1cm} (B19)

with $f(x)$ some unspecified trigonometric function. It can be easily shown that $\Delta_2$ obeys the differential equation

$$\Delta'_2(R) + 2 L_0(R) \Delta_2(R) - 1 = 0,$$ \hspace{1cm} (B20)

in contrast with the renormalization group equation

$$L'_k(R) + 2 L_0(R) L_2(R) + 1 = 0.$$ \hspace{1cm} (B21)

As we can see, $k^2 \Delta_2 \ll L_0$, which means that we can drop the $\Delta_2$ term, without jeopardizing the smooth cut-off dependence of the phase shifts for small cut-offs. In fact, it is curious to see that dropping $\Delta_2$ improves the convergence. When we take $L_k(R) = L_0(R)$, we arrive at the following cut-off dependence for the phase shift

$$\frac{d \delta}{dR} = -k^3 \left( \frac{u_k(R, R)}{k} \right)^2 \sim k^3 R^{n/2},$$ \hspace{1cm} (B22)

while when considering the contribution for the $\Delta_2$ term, i.e. we take $L_k(R) = L_0(R) + k^2 \Delta_2(R)$, we arrive at the following result

$$\frac{d \delta}{dR} = -2 k^3 \left( \frac{u_k(R, R)}{k} \right)^2 \sim k^3 R^{n/2},$$ \hspace{1cm} (B23)

which is qualitatively the same dependence, although the cut-off dependence is doubled. In short, what we can see is that, when fixing the scattering length, $\Delta_2(R)$ is negligible in comparison with $L_0(R)$, and can therefore be removed.

Unfortunately this changes when considering the theory in which one fixes both the scattering length and the effective range. In it, the boundary condition can be expanded as

$$L_k(R) = L_0(R) + k^2 L_2(R) + k^4 \Delta_4(R) + \ldots$$ \hspace{1cm} (B24)

and, as we will show below, the $\Delta_4$ contribution cannot be ignored. The behaviour of $L_0(R)$ is the one given by Eq. \{(B10)\}, i.e. exactly the same as in the previous case. On the contrary, the behaviour of $L_2(R)$ differs
significantly from that of $\Delta_2(R)$, given by Eq. (B19). The reason lies in the fact that we are fixing the effective range. This can be deduced from the behaviour of the $u_2$ wave function, which obeys the equation

$$-u_2'' + U(r) u_2 = u_0 .$$

(B25)

The solutions of the above equation can be separated into an homogeneous and inhomogeneous piece

$$u_2 = u_{2,H} + u_{2,I},$$

which in turn means that $L_2(R)$ can be written as the contribution of these two pieces

$$L_2 = L_{2,H} + L_{2,I},$$

(B27)

where

$$L_{2,H} = \frac{u_{2,H} u_0 - u_{2,H} u_0'}{u_0^2} ,$$

(B28)

$$L_{2,I} = \frac{u_{2,I} u_0 - u_{2,I} u_0'}{u_0^2} .$$

(B29)

The inhomogeneous piece of $u_2$ comes from the nontrivial contribution stemming from the $u_0$ wave function and can be computed via Green’s functions. Its short distance behaviour is much smoother than that of $u_0$, $u_{2,I}$ scales as $r^{3n/4 + 1}$ times an oscillating function $f(x)$ with $x = \frac{2}{n-2} (\frac{\alpha_0}{R})^{n/2 - 1}$, while $u_0$ just scales as $r^{n/4}$. From these behaviours it is trivial to see that

$$L_{2,I}(R) = \Delta_2(R) \sim R ,$$

(B30)

in which we wanted to make clear the identification between the inhomogeneous piece of $L_2$ and what we previously called $\Delta_2$.

The problem with the inhomogeneous piece of $u_2$ is that it contributes to the effective range is fixed, and is given by the following formula

$$r_{0,I} = \frac{2}{\alpha_0} \int_0^\infty dr \left[ (r - \alpha_0)^2 - u_0^2(r) \right] ,$$

(B31)

as can be trivially deduced from Eq. (B17), and in which we have taken the cut-off to zero for simplicity. For modifying the total effective range we need to add one contribution from the asymptotic behaviour at large distances of the homogeneous wave function in such a way that

$$u_{2,H} \rightarrow -\frac{1}{2} \alpha_0 r_{0,H} r.$$  

(B32)

Then, the complete effective range of the wave function $u_2 = u_{2,H} + u_{2,I}$ will be $r_0 = r_{0,H} + r_{0,I}$, thus fixing the effective range to the desired value. But the homogeneous contribution to $u_2$ is much more singular than the inhomogeneous one. The short distance behaviour of $u_{2,H}$ is given by

$$u_{2,H}(R) \rightarrow R^{n/4} \sin \left[ \frac{2}{n-2} \left( \frac{\alpha_0}{R} \right)^{n/2 - 1} + \varphi_2 \right],$$

(B33)

with $\varphi_2$ a semiclassical phase. From the previous behaviour for $u_{2,H}$, Eq. (B33), one can see that $L_{2,H}$ (and therefore $L_2$) will behave in a very similar way to $L_0$, i.e. a $1/R^{n/2}$ singularity times some oscillating function. But one can get a much better evaluation of $L_{2,H}$ by considering the following Lagrange identity for $u_0$ and $u_{2,H}$

$$(u_{2,H} u_0 - u_{2,H} u_0')' = 0 ,$$

(B34)

in contrast with $u_{2,I}$ which obeys the Lagrange identity Eq. (B17). This means that $u_{2,H} u_0 - u_{2,H} u_0'$, which appears in the numerator of $L_{2,H}$, is a constant value. Evaluating at large distances, we obtain

$$u_{2,H} u_0 - u_{2,H} u_0' = \frac{r_0 H}{2} \alpha_0^2,$$

(B35)

from which trivially follows

$$L_{2,H}(R) = \left( \frac{\alpha_0}{u_0(R)} \right)^2 \frac{r_0 H}{2} .$$

(B36)

It is curious to see the equations which both contributions to $L_2$ follow

$$L_{2,H} + 2 L_0(R) L_{2,H}(R) + 2 = 0 ,$$

(B37)

$$L_{2,I} + 2 L_0(R) L_{2,I}(R) - 1 = 0 ,$$

(B38)

from which taking into account that $L_2 = L_{2,H} + L_{2,I}$ one recovers the renormalization group equation.

Finally, to evaluate $\Delta_4$ we need its expression in terms of wave functions

$$\Delta_4(R) = \frac{u_4 u_0 - u_4 u_0'}{u_0^2} + \frac{u_2}{u_0} L_2(R) ,$$

(B39)

where the $u_4$ contribution stems solely from the inhomogeneous piece (since we are not fixing $u_2$). By considering the appropriate Lagrange’s identity

$$(u_4 u_0 - u_4 u_0')' = u_0 u_2,$$

(B40)

it is trivial to deduce the entire evaluation of $\Delta_4(R)$

$$\Delta_4(R) = \frac{1}{u_0^2} \int_0^R dr u_0 u_2 + \frac{u_2}{u_0} L_2(R) .$$

(B41)

Due to the $u_2 L_2/u_0$ contribution, we see that $\Delta_4$ is as singular as $L_2$, and therefore as $L_0$, so its presence cannot be neglected in the momentum expansion of the boundary condition. The same happens with $\Delta_6$, $\Delta_8$, and so on, but for the purposes of this appendix is enough to consider only $\Delta_4$. With all this we are prepared to evaluate the cut-off dependence of the phase shifts, which is given by

$$\frac{d\delta}{dR} = -k^3 \left[ L_2^2 + \Delta_4^2 + 2 L_0 \Delta_4 \right] u_0^2(R) + O(k^7).$$

(B42)
After evaluating $\Delta_4$ inside the brackets, we are left with the much simpler expression
\begin{equation}
\frac{d \delta}{dR} = -k^5 u_2(R) u_0(R) + \mathcal{O}(k^7) \\
\sim -k^5 R^{n/2}, \tag{B43}
\end{equation}
which is just the same cut-off dependence as in the previous case, but somewhat diminished by the appearance of an extra $k^2$ factor.

Of course this is only the first step in showing that the phase shift is well behaved. The $\mathcal{O}(k^7)$ term needs the explicit consideration of $\Delta_6$, the $\mathcal{O}(k^9)$ term of $\Delta_8$, and so on. Although it is straightforward to show that including the appropriate expansion of the boundary condition, the phase shift has a convergent behaviour for the appropriate expansion of the boundary condition, the on. Although it is straightforward to show that including the much simpler expression
\begin{equation}
\alpha \Delta u'(0) + u(0) = 0, \tag{C2}
\end{equation}
\[ \text{where } \Delta \text{ contains the inhomogeneous higher order pieces.} \]

This amplitude does not have any pole in the complex plane. Thus, one starts from $\alpha = \infty$ to $\alpha = 0^+$ using Eq. (C3) which corresponds to a repulsive core and then goes on for $\alpha = 0^-$ with Eq. (C3).

For the purely short distance theory, the generalization to coupled channel scattering is almost trivial. If we assume $s$-wave scattering, and using the boundary condition at the origin
\begin{equation}
\alpha u'(0) + u(0) = 0, \tag{C5}
\end{equation}
with $\alpha$ the $s$-wave scattering length matrix, we get for the coupled channel amplitude
\begin{equation} f^{-1} = -\alpha^{-1} + i k, \tag{C6} \end{equation}
with $k$ given by Eq. (144). The simplicity of the derivation contrasts with the cumbersome treatment of the Lippmann-Schwinger equation with a sharp three-momentum cut-off presented in Ref. [59].

**APPENDIX D: WHY A BOUNDARY CONDITION INSTEAD OF A POTENTIAL AS A REGULATOR IN COORDINATE SPACE?**

In this appendix we elaborate further on the suitability of the BC as compared to the use of square well or delta shell regulators. Our main concern has to do with choosing a regularization method where a truncated low energy expansion of the amplitude corresponds exactly with a truncated low energy expansion of the regulator. One of the advantages of a boundary condition as a short

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19 There are situations where this pole is absent. For instance a square well repulsive potential
distance regulator, as opposed to a short range potential, is related to the non appearance of spurious high order effects in a low energy expansion of the amplitude. As we will see here this requires infinitely many counter-terms in an energy dependent potential. Of course, any of these methods when considered to all orders ought to yield the same result; the difference has to do with truncating the regularized potential to a given order. As we will show the BC is the only method of the three analyzed where truncation is consistent order by order. To understand the situation let us consider the three regularizations in coordinate space: Square well regularization [14], delta shell regularization [17] and boundary condition regularization [14].

- Square well regulator. In this case we get

$$U_k(r) = (U_0 + k^2 U_2 + \ldots) \theta(a-r).$$

The phase-shift reads

$$k \cot \delta = \frac{\sqrt{k^2 + U_k} \cot(ka) \cot(\sqrt{k^2 + U_k} a) + k}{k \cot(ka) - \sqrt{k^2 + U_k} \cot(\sqrt{k^2 + U_k} a)}. \quad (D2)$$

Thus, the scattering length is given by

$$\alpha = a \left(1 - \frac{\tan(\sqrt{U_0} \alpha)}{\sqrt{U_0} \alpha}\right). \quad (D3)$$

and the effective range

$$r_0 = 2a \left(1 - \frac{1}{\alpha U_0} - \frac{a^2}{3\alpha^2}\right) + U_2 \frac{a - \alpha}{a^2 U_0} \left(1 - \sqrt{U_0} \alpha + \alpha a U_0 - U_0^2 a^2\right). \quad (D4)$$

and so on for $r_2$, etc. Obviously, we can always fit any set of low energy parameters by adding sufficient energy dependent terms to the potential $U_0$, $U_2$, etc. any value of the low energy constants we want. The mapping, however, is not one to one; we have infinitely many solutions for $U_0$ Eq. (D3) for a given value of $\alpha$ at a given scale $a$. This multivaluation propagates to higher order low energy parameters. As a consequence, if we only specify a set of low energy parameters, the remaining higher order ones are multivalued, and hence the phase-shift is not uniquely predicted. To make the ambiguity more explicit let us take the limit $a \to 0$ and consider the LO truncated potential, $U_0$. In this limit we obtain from inversion of Eq. (D3),

$$\sqrt{U_0} a = \left(n + \frac{1}{2}\right) \pi \left(1 - \frac{a}{\alpha}\right) + \ldots. \quad (D5)$$

with $n$ an arbitrary integer number. Using this asymptotic solution we get for the effective range,

$$r_0 = 2a \left(1 - \frac{a}{\alpha} - \frac{1}{(n + 1/2)^2 \pi^2} + \ldots\right), \quad (D6)$$

which depends manifestly on the arbitrary value of $n$. The $n$ dependence might be cancelled by including a NLO term, $U_2$. More generally, for any finite $a$ one should include infinitely many terms in the energy expansion of the square well potential, Eq. (D11) to get rid of these multi-valuation in the pure short range theory.

- Delta Shell regulator. In this case we get

$$U(r) = (U_0 + k^2 U_2 + \ldots) \delta(a-r). \quad (D7)$$

Then one gets for the scattering length

$$\alpha = \frac{a^3 U_0}{1 + U_0 a^2}, \quad (D8)$$

whereas the effective range reads,

$$r_0 = \frac{4a}{3} \left(1 - \frac{a}{2\alpha}\right) + 2a U_2 \left(1 - \frac{a}{\alpha}\right)^2. \quad (D9)$$

In this case there are no multiple solutions, but similarly to the square well regularization we get spurious terms at higher orders.

- Boundary condition regulator. In this case we have

$$\frac{u'_k(a)}{u_k(a)} = k \cot(ka + \delta) = \xi_0 + (ka)^2 \xi_2 + \ldots. \quad (E1)$$

In this case the translation between the boundary condition and an effective range expansion is most straightforward; each energy contribution to the BC provides a new term in the effective range. If we take the limit $a \to 0$ we get a one-to-one mapping. Thus, the BC method provides a compatible hierarchy of equations in a low momentum expansion.

### APPENDIX E: RENORMALIZATION GROUP EQUATION FOR POTENTIAL REGULARIZATION

The standard way of visualizing renormalization is by means of potentials. In this section we use our renormalization group ideas similar to those presented in Sect. 311 to determine the RG evolution of these short distance regulators.

#### 1. Square well Potential Regularization

In Refs. 39 a short range energy dependent square well has been employed to regulate the short distance behaviour. That means taking the family of potentials

$$U_k(r, R) = U_k(R) \theta(R-r) + U(r) \theta(r-R). \quad (E1)$$
The regular solution for \( r < R \) is (assuming \( U_k(R) < 0 \) for definiteness)
\[
u(r) = A \sin \left( \sqrt{k^2 - U_k(R)} r \right) \quad r < R, \tag{E2}\]

\[
- \frac{1}{k} \frac{d \delta}{d \ln k} = \begin{cases} 
\sin^2 \left( \sqrt{k^2 - U_k(R)} R \right) (U_k(R) - U(R)) + U_k''(R) \left( \frac{R}{A^2} \right)^2 
\end{cases}, \tag{E3}
\]

If we demand the phase shift to be independent on the short-distance regulator we get the renormalization group equation,
\[
R U_k'(R) = (U_k(R) - U(R)) \frac{2 \sin^2(\phi)}{1 - \phi \cos \phi \sin \phi}, \tag{E4}
\]

where we have defined the dimensionless combination,
\[
\phi = \sqrt{k^2 - U_k(R)} R. \tag{E5}
\]

A similar equation has also been found in Ref. \[41\] by different means. The low energy fixed points are given by
\[
\sin(\phi) = 0 \quad 1 = \phi \sin(\phi) \cos(\phi). \tag{E6}
\]

The first equation has the analytical solution \( \phi_n = n\pi \). The other equation has also infinitely many solutions \( \phi_m \). All these fixed points are stable, so we have infinitely many branches, in agreement with the observation of Ref. \[41\].

### 2. Delta shell Potential

In Ref. \[17\] a delta shell potential regularization for the short range potential has been introduced, and a RG equation in momentum space has been obtained by cutting off the high energy components. According to our

where \( A \) has to be determined by matching to the long range piece of the wave function. Then

\[
U(r) = U_k(R) \delta(r - R) + U(r) \theta(r - R). \tag{E7}
\]

Let us denote by \( u_L \) and \( v_L(r) \) the wave functions regular and singular at the origin respectively associated to the long range potential alone, \( U(r) \), and fulfilling
\[
u_L \rightarrow \sin(kr + \delta_L), \tag{E8}
v_L \rightarrow \cos(kr + \delta_L). \tag{E9}
\]

If we take the \( u(r) = Au_L(r) + Bv_L(r) \) for \( r > R \), then we have
\[
u'(R) = -k \cot(kR) = U_S(R). \tag{E10}
\]

To obtain the RG equation let us compute the change of the phase shift,
\[
\Delta \delta = -\frac{1}{k} \Delta \left[ U_S(R) u(R)^2 \right] + \frac{1}{k} U_L(R) \Delta u(R)^2. \tag{E11}
\]

Using the condition \[E10\] we get
\[
U_S'(R) = U_L(R) - 2 \left[ U_S(R) + k \cot(kR) \right] U_S(R). \tag{E12}
\]

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