Bordering for spectrally arbitrary sign patterns

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Abstract
We develop a matrix bordering technique that can be applied to an irreducible spectrally arbitrary sign pattern to construct a higher order spectrally arbitrary sign pattern. This technique generalizes a recently developed triangle extension method. We describe recursive constructions of spectrally arbitrary patterns using our bordering technique, and show that a slight variation of this technique can be used to construct inertially arbitrary sign patterns.

Keywords: nilpotent matrix, spectrally arbitrary pattern, nilpotent-Jacobian method, inertially arbitrary pattern.

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1. Introduction

A number of methods have been developed to check that a specific pattern is spectrally or inertially arbitrary, such as the analytic nilpotent-Jacobian method and the algebraic nilpotent-centralizer method (see e.g. [4, 7, 8, 9], and these have been applied to various classes of patterns (see e.g. [4, 7, 8, 9]). Recently in [11], a digraph method called triangle extension has been developed for constructing higher order spectrally or inertially arbitrary patterns from lower order patterns. In this paper, we generalize the triangle extension method by formulating it as a matrix bordering technique (see Remark 2.2). With this bordering technique, we construct higher order patterns (some of which cannot be obtained by triangle extension) that are spectrally or inertially arbitrary from lower order patterns. We give examples of new spectrally and inertially arbitrary sign patterns obtained by bordering.

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1.1. Definitions and the nilpotent Jacobian method.

Given an order $n$ matrix $A = [a_{ij}]$, denote the characteristic polynomial of $A$ by $p_A(z) = \det(zI - A)$. A sign pattern is a matrix $A = [\alpha_{ij}]$ of order $n$ with entries in $\{0, +, -, \}$. Let

$$Q(A) = \{ A \mid a_{ij} = 0 \text{ if } \alpha_{ij} = 0, a_{ij} > 0 \text{ if } \alpha_{ij} = + \text{ and } a_{ij} < 0 \text{ if } \alpha_{ij} = - \}.$$ 

If $A \in Q(A)$ for some pattern $\mathcal{A}$, then $A$ is a realization of $\mathcal{A}$ and we sometimes refer to $\mathcal{A}$ as $\text{sgn}(A)$. A pattern $\mathcal{A}$ is spectrally arbitrary if for every degree $n$ monic polynomial $p(z)$ over $\mathbb{R}$, there is some real matrix $A$ such that $A \in Q(A)$ and $p_A(z) = p(z)$. A pattern $B = [\beta_{ij}]$ is a superpattern of $\mathcal{A}$ if $\alpha_{ij} \neq 0$ implies $\beta_{ij} = \alpha_{ij}$, and $\mathcal{A}$ is a subpattern of $\mathcal{B}$.

Two patterns $\mathcal{A}$ and $\mathcal{B}$ are equivalent if $\mathcal{B}$ can be obtained from $\mathcal{A}$ via any combination of negation, transposition, permutation similarity and signature similarity.

A matrix $A$ is nilpotent if $A^k = 0$ for some positive integer $k$ and the smallest positive integer $k$ such that $A^k = 0$ is the index of $A$. An order $n$ nilpotent matrix $A$ has characteristic polynomial $p_A(z) = z^n$.

Suppose $\mathcal{A}$ is an order $n$ sign pattern with a nilpotent matrix $A \in Q(A)$ with $m \geq n$ nonzero entries $a_{i_{1} j_{1}}, a_{i_{2} j_{2}}, \ldots, a_{i_{m} j_{m}}$. Let $X = X_A(x_1, x_2, \ldots, x_m)$ denote the matrix obtained from $A$ by replacing $a_{i_{k} j_{k}}$ with the variable $x_k$ for $k = 1, \ldots, m$. Writing $p_X(z) = z^n + f_1 z^{n-1} + \cdots + f_n$ for some $f_i = f_i(x_1, x_2, \ldots, x_m)$, let $J = J_X$ be the $n \times m$ Jacobian matrix with $(i, j)$ entry equal to $\frac{\partial f_i}{\partial x_j}$ for $1 \leq i \leq n$, and $1 \leq j \leq m$. Let $J_{X=A}$ denote the Jacobian matrix evaluated at the nilpotent realization, that is $J_{X=A} = J_{(x_1, x_2, \ldots, x_m)=(a_{i_{1} j_{1}}, a_{i_{2} j_{2}}, \ldots, a_{i_{m} j_{m}})}$. A nilpotent matrix $A$ allows a full-rank Jacobian if the rank of $J_{X=A}$ is $n$. Finding a nilpotent matrix $A \in Q(A)$ that allows a full-rank Jacobian is known as the nilpotent-Jacobian method. As noted in part (c) of Theorem 1.1, this method guarantees that every superpattern of $\mathcal{A}$ is spectrally arbitrary.

A matrix $A$ (or pattern $\mathcal{A}$) is reducible if there is a permutation matrix $P$ such that $P A P^T$ (resp. $P A P^T$) is block triangular with more than one nonempty diagonal block. Otherwise it is irreducible. A matrix $A$ is nonderogatory if the dimension of the eigenspace of every eigenvalue is equal to one. The following theorem combines known results from [4] and [7].

**Theorem 1.1.** Let $A$ be a sign pattern of order $n$. If a nilpotent matrix $A \in Q(A)$ allows a full-rank Jacobian, then

(a) $A$ is irreducible,

(b) $A$ is nonderogatory, and

(c) every superpattern of $\mathcal{A}$ is spectrally arbitrary.

**Proof.** Suppose $A \in Q(A)$ is a nilpotent matrix of order $n$ that allows a full-rank Jacobian. Part (c) is [4, Theorem 3.1], which is a reframing of the nilpotent-Jacobian method introduced in [7]. Part (b) is [4, Corollary 4.5].

If $A$ is a reducible nilpotent matrix and $P A P^T$ is block triangular for some permutation matrix $P$, then the index of $A$ is at most the index of the largest order diagonal block of
Thus the index is bounded above by the order of the largest diagonal block. Since the index of $A$ is $n$, it follows that $A$ is irreducible, proving part (a).

Because of part (c) of Theorem 1.1, minimal spectrally arbitrary patterns (that is, spectrally arbitrary patterns for which no proper subpattern is spectrally arbitrary), are of special interest. For $n = 2$ and $n = 3$, the minimal spectrally arbitrary patterns are well-known (see, e.g., [1, 5]) and, up to equivalence, are:

$$T_2 = \begin{bmatrix} + & - \\ + & - \end{bmatrix}, \quad T_3 = \begin{bmatrix} + & 0 & 0 \\ 0 & - & - \end{bmatrix},$$

$$U_3 = \begin{bmatrix} + & + & 0 \\ + & 0 & - \\ 0 & + & 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} + & 0 & + \\ 0 & + & 0 \\ + & 0 & - \end{bmatrix}, \quad \text{and} \quad W_3 = \begin{bmatrix} + & 0 & + \\ + & 0 & + \\ 0 & + & 0 \end{bmatrix}.$$

1.2. Bordering

Let $A = [a_{ij}]$ be an order $n$ matrix, $x, z \in \mathbb{R}^n$, and let $B$ be the bordered matrix of order $n + 1$:

$$B = \begin{bmatrix} I_n & 0 \\ x^T & 1 \end{bmatrix} \begin{bmatrix} A & z \\ 0^T & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -x^T & 1 \end{bmatrix} = \begin{bmatrix} A - zx^T \\ x^T(A - zx^T) \end{bmatrix} \begin{bmatrix} z \\ x^T \end{bmatrix}. \quad (1)$$

Since this is a similarity transformation, it follows that $p_B(z) = z p_A(z)$, and thus $B$ is nilpotent if $A$ is nilpotent. Note that (1) is a special case of a construction introduced in [10, Theorem 3.1].

Let $e_i = [0, \ldots, 0, 1, 0, \ldots, 0]^T$ with a 1 in position $i$. In this paper, we focus on the special cases $z = e_j$ and $x^T = be_k$ for some $b \neq 0$, which we call standard unit bordering. In the next two sections we use bordering to construct higher order spectrally arbitrary patterns out of lower order patterns without having to recalculate a Jacobian matrix. In addition, at each stage, the construction provides an explicit nilpotent realization of the spectrally arbitrary pattern.

2. Standard unit bordering with equal indices

Let $A = [a_{ij}]$, and denote the $k$th row of $A$ by $r_k(A)$. Suppose $x = a_{kk}z = a_{kk}e_k$ for some $a_{kk} \neq 0$. Then $x^T A = a_{kk}r_k(A)$ and $A - zx^T = A - a_{kk}P_{kk}$ where $P_{kk}$ has a 1 in entry $(k,k)$ and zeros elsewhere. In this case, the matrix $B$ in (1) is

$$B = \frac{A - a_{kk}P_{kk}}{a_{kk}r_k(A - a_{kk}P_{kk})} e_k. \quad (2)$$

Let $A(u, v)$ denote the matrix obtained from $A$ by deleting row $u$ and column $v$.

**Theorem 2.1.** Let $A$ be a sign pattern of order $n$. Suppose $A = [a_{ij}] \in Q(A)$ is a nilpotent matrix and $A$ allows a full-rank Jacobian. Suppose $a_{kk} \neq 0$ and $a_{kv} \neq 0$ for some $v \neq k$. If $\det A(k,v) \neq 0$, then $B$ in (2) is a nilpotent matrix that allows a full-rank Jacobian and hence every superpattern of $B = \text{sgn}(B)$ is spectrally arbitrary.
Proof. Let $A \in \mathbb{Q}(A)$ be a nilpotent matrix and $X_A$ be a matrix with the nonzero pattern of $A$ having variable entries such that the Jacobian $J_{X_A=A}$ has rank $n$. For convenience, assume $k = n$ and the last row of $X_A$ is $[x_{n1}, x_{n2}, \ldots, x_{nn}]$, recognizing that some of these entries may be zero. Note that by assumption $x_{nv}$ and $x_{nn}$ are nonzero. Let $B$ be as in (2) and

$$X_B = \begin{bmatrix} X_A - x_{nn}P_{nn} & 0 \\ y & 0 \\ 0 & x_{nn} \end{bmatrix} \tag{3}$$

with $y = [y_1, y_2, \ldots, y_{n-1}]$ such that $y_i \neq 0$ if and only if $x_{ni} \neq 0$. (Note that, other than the placement of the variables in $y$, the nonzero entries of $y$ are independent of the variables in $X_A$.) Then $X_B$ has the nonzero pattern of $B$. Using cofactor expansion along the last row of $X_B$ gives

$$p_{X_B}(z) = \det(zI_n+1 - X_B)$$

$$= (z - x_{nn}) \det(zI_n - X_A + x_{nn}P_{nn}) + \sum_{\ell=1}^{n-1} (-1)^{n+\ell} y_{\ell} \det ([zI_n - X_A](n, \ell)).$$

However, applying cofactor expansion along the last row of the first summand gives

$$\det(zI_n - X_A + x_{nn}P_{nn}) = z \det ([zI_n - X_A](n, n))$$

$$+ \sum_{\ell=1}^{n-1} (-1)^{n+\ell} x_{n\ell} \det ([zI_n - X_A](n, \ell)).$$

Thus

$$p_{X_B}(z) = z \det(zI_n - X_A + x_{nn}P_{nn}) - x_{nn}z \det([zI_n - X_A](n, n))$$

$$+ \sum_{\ell=1}^{n-1} (-1)^{n+\ell} y_{\ell} - x_{nn}x_{n\ell} \det ([zI_n - X_A](n, \ell)).$$

Since the determinant is linear in the rows (or using a rank 1 perturbation of a determinant), it follows that

$$p_{X_B}(z) = z p_{X_A}(z) + \sum_{\ell=1}^{n-1} (-1)^{n+\ell} (y_{\ell} - x_{nn}x_{n\ell}) \det ([zI_n - X_A](n, \ell)). \tag{4}$$

Focusing on the coefficients of $p_{X_B}(z)$, the second summand can be rewritten as

$$\sum_{r=3}^{n+1} \left[ \sum_{\ell=1}^{n-1} S_{r,\ell}(y_{\ell} - x_{nn}x_{n\ell}) \right] z^{n-r+1}$$

for some polynomials $S_{r,\ell}$ of the variable entries in $X_A$. To consider the Jacobian of $X_B$, we assume the last columns of $J_{X_B}$ are indexed by the nonzeros of $x_{n1}, \ldots, x_{nn}, y_1, \ldots, y_{n-1}$. 


Let \( m \) be the number of nonzero entries of \( y \) and \( w \) be the number of variables in \( X_A \). Then the \((n + 1) \times (w + m)\) Jacobian matrix \( J_{X_B} \) is

\[
J_{X_B} = \begin{bmatrix}
J_{X_A} & O \\
O^T & 0^T
\end{bmatrix} + \sum_{\ell=1}^{n-1} (y_\ell - x_{nn}x_{n\ell})M_\ell + \begin{bmatrix} O & N \end{bmatrix}
\]

for some matrices \( M_\ell \) and \((n + 1) \times (2m + 1)\) matrix \( N \) with columns indexed by the nonzeros of \( x_{n1}, \ldots, x_{nn}, y_1, \ldots, y_{n-1} \). Note that by (1) and (3), \( y_\ell = a_{nn}a_{n\ell} = x_{nn}x_{n\ell} \) in the nilpotent realization, so that we can ignore each matrix \( M_\ell \) in (5), since its coefficient vanishes at the nilpotent realization. Further the column of \( N \) corresponding to \( y_\ell \) is \( \tilde{N}_{y_\ell} = [0, 0, S_{3,\ell}, S_{4,\ell}, \ldots, S_{n+1,\ell}]^T \) for \( 1 \leq \ell \leq n \), and in addition, the column corresponding to \( x_{n\ell} \) is \( \tilde{N}_{x_{n\ell}} = -x_{nn}\tilde{N}_{y_\ell} \) for \( 1 \leq \ell \leq n-1 \) and \( \tilde{N}_{x_{nn}} = \sum_{\ell=1}^{n-1} -x_{n\ell}\tilde{N}_{y_\ell} \). It follows that \( N \) is column equivalent to \([ O \mid \tilde{N}_{y_1} \vert \tilde{N}_{y_2} \vert \cdots \vert \tilde{N}_{y_{n-1}}] \). From (4), with \( z = 0 \),

\[
S_{n+1,\ell} = (-1)^{\ell-1} \det(X_A(n, \ell)),
\]

giving

\[
S_{n+1,\ell}|_{X_B=B} = (-1)^{\ell-1} \det(A(n, \ell)).
\]

Thus, the condition that there exists an index \( v \neq n \) such that \( a_{nv} \neq 0 \) and \( \det(A(n, v)) \neq 0 \) implies that \( S_{n+1,\ell}|_{X_B=B} \neq 0 \) for some \( \ell, 1 \leq \ell \leq n-1 \). It follows that \( J_{X_B=B} \) is equivalent to

\[
\begin{bmatrix}
J_{X_A=A} & * \\
O^T & s^T
\end{bmatrix}
\]

for some \( s \neq 0 \). Hence \( B \) allows a full-rank Jacobian. Thus by Theorem 1.1, every super-pattern of \( B \) is spectrally arbitrary.

**Remark 2.2.** A method in [11] called triangle extension on arc \((u, v)\) (in the digraph associated with \( A \)) is equivalent to a special case of applying Theorem 2.1 to row \( u \) of \( A \) and entry \((u, v)\), namely in the situation that \( a_{uu} \) and \( a_{uv} \) are the only nonzero entries in row \( u \) of \( A \).

**Example 2.3.** If

\[
A = \begin{bmatrix}
0 + 0 0 \\
0 - + 0 \\
+ 0 0 + \\
+ 0 - +
\end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix}
0 1 0 0 \\
0 -1 1 0 \\
1 0 0 1 \\
1 0 -1 1
\end{bmatrix},
\]

then \( A \) is nilpotent, \( A \in Q(A) \), and \( A \) allows a full-rank Jacobian. Hence \( A \) is spectrally arbitrary \((A \) is equivalent to the second matrix in Appendix A of [6]). Further, \( a_{44} \neq 0 \),

\[
5
\]
\(a_{41} \neq 0\) and \(\det(A(4,1)) \neq 0\). Applying Theorem 2.1 to row 4 and entry \((4,1)\) gives a spectrally arbitrary pattern \(B_5\) with nilpotent matrix \(B \in \mathbb{Q}(B_5)\) for

\[
B_5 = \begin{bmatrix}
0 & + & 0 & 0 & 0 \\
0 & - & + & 0 & 0 \\
+ & 0 & 0 & + & 0 \\
+ & 0 & - & 0 & + \\
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 & 1 \\
\end{bmatrix}.
\]

Note that since row 4 of \(A\) has more than one off-diagonal entry, triangle extension as described [11] is not possible on the arc \((4,1)\) in the digraph associated with \(A\), demonstrating that Theorem 2.1 provides a more general technique than triangle extension in [11].

**Remark 2.4.** Theorem 2.1 can be applied recursively. In particular, suppose \(\det(A(n,v)) \neq 0\) and Theorem 2.1 was applied to row \(n\) and entry \((n,v)\) of \(A\) to obtain \(B\). It follows that \(\det(B(n+1,v)) = (-1)^n \det(A(n,v)) \neq 0\) since there is only one nonzero entry in the last column of \(B(n+1,v)\), namely 1 in the last row. Thus Theorem 2.1 can now be applied to row \(n+1\) and entry \((n+1,v)\) of \(B\).

**Example 2.5.** The sign pattern \(B_5\) in Example 2.3 can be recursively bordered using Theorem 2.1, starting with row 5 and entry \((5,1)\), to obtain a spectrally arbitrary pattern of order \(n \geq 6\), with \(3n - 4\) nonzero entries, of the form

\[
B_n = \begin{bmatrix}
0 & + & & & & \cdots & O \\
0 & - & + & & & & \cdots \cdot \cdot \cdot \\
+ & 0 & 0 & + & & & \cdots \\
+ & 0 & - & 0 & + & & \cdots \\
& & & & & \cdots & 0 & + \\
& & & & & \cdots & 0 & + \\
\end{bmatrix}.
\]

Note that each nonzero entry of the nilpotent realization of \(B_n\) has magnitude 1. It can be shown that \(B_5\) and \(B_4 = A\) in Example 2.3 are minimally spectrally arbitrary.

### 3. Standard unit bordering with unequal indices

Referring to (1), suppose \(x = be_k\) for some \(b \neq 0\) and \(z = e_j\) for \(j \neq k\); thus \(x^Tz = 0\). With the \(k\)th row of \(A\) denoted by \(r_k(A)\), \(x^T A = br_k(A)\) and \(A - zx^T = A - bP_{jk}\) where \(P_{jk}\) has a 1 in entry \((j,k)\) and zeros elsewhere. In this case, the matrix \(B\) in (1) is

\[
B = \begin{bmatrix}
A - bP_{jk} \\
br_k(A - bP_{jk})
\end{bmatrix}
\begin{bmatrix}
e_j \\
0
\end{bmatrix}.
\]

Recall that \(X_A\) is obtained from \(A\) by replacing some of the nonzero entries with variables. In the case that \(\text{rank } J_{X=A} = n\), we call a nonzero entry of \(A \text{ Jacobian in } X_A\) if
it is replaced by a variable in $X_A$, otherwise the entry is non-Jacobian in $X_A$. Note that a non-Jacobian entry may be zero. To simplify the next proof, for $U, V \subseteq \{1, 2, \ldots, n\}$, let $A(U, V)$ denote the matrix obtained from $A$ by deleting the rows in $U$ and the columns in $V$.

**Theorem 3.1.** Let $A$ be a sign pattern of order $n$. Suppose $A = [a_{ij}] \in Q(A)$ is a nilpotent matrix and $A$ allows a full-rank Jacobian. Suppose $a_{jk}, j \neq k$, is non-Jacobian for some choice of $X_A$. If $a_{kv} \neq 0$ and $\det(A(j, v)) \neq 0$, for some $v$, then $B$ in (6) is a nilpotent matrix that allows a full-rank Jacobian and hence every superpattern of $B = \text{sgn}(B)$ is spectrally arbitrary.

**Proof.** Let $A \in Q(A)$ be a nilpotent matrix and $X_A$ be a matrix with the nonzero pattern of $A$ having variable entries such that the Jacobian $J_{X_A} = A$ has rank $n$ with no variable placed in position $(j, k)$. For convenience, assume that $j = 1, k = n$, and the last row of $X_A$ is $[x_{n1}, x_{n2}, \ldots, x_{nn}]$, recognizing that some of these entries may be zero. Let $B$ be as in (6) and

$$X_B = \begin{bmatrix} X_A - bP_{1n} & 1 \\ y & 0 \\ 0 & 0 \end{bmatrix}$$

with $y = [y_1, y_2, \ldots, y_n]$ such that $y_i \neq 0$ if and only if $x_{ni} \neq 0$. (Note that, other than the placement of the variables in $y$, the nonzero entries of $y$ are independent of the variables in $X_A$.) Then $X_B$ has the nonzero pattern of $B$. Using cofactor expansion along the last row of $X_B$ gives

$$p_{X_B}(z) = \det(zI_{n+1} - X_B) = z \det(zI_n - X_A + bP_{1n}) + \sum_{\ell=1}^{n} (-1)^{\ell + 1} y_\ell \det([zI_n - X_A](1, \ell))$$

$$= zp_{X_A}(z) + (-1)^n z b \det([zI_n - X_A](1, n)) + \sum_{\ell=1}^{n} (-1)^{\ell + 1} y_\ell \det([zI_n - X_A](1, \ell))$$

(8)

Let $W_\ell = z \det([zI_n - X_A](\{1, n\}, \{\ell, n\}))$. Applying cofactor expansion on the determinant in the second summand of (8) gives

$$z \det([zI - X_A](1, n)) = \sum_{\ell=1}^{n-1} (-1)^{n+\ell + 1} x_{n\ell} W_\ell.$$ 

(9)

Using the fact that the last row of $zI_n - X_A$ is $[0, \cdots, 0, z] - [x_{n1}, x_{n2}, \ldots, x_{nn}]$, and that a determinant is linear in the last row,

$$\sum_{\ell=1}^{n} (-1)^{\ell + 1} y_\ell \det([zI_n - X_A](1, \ell)) = \sum_{\ell=1}^{n-1} (-1)^{\ell + 1} y_\ell W_\ell + \sum_{\ell=1}^{n} (-1)^{\ell + 1} y_\ell U_\ell$$

(10)
for  
\[ U_\ell = \det \left( [zI_n - X_A](1, \ell) - z \begin{bmatrix} O & 0 \\ 0^T & 1 \end{bmatrix} \right), \]  
with  \( 1 \leq \ell \leq n - 1 \),

and  \( U_n = \det([zI_n - X_A](1, n)) \). Using (9) and (10) in (3) gives

\[ p_{X_B}(z) = zp_{X_A}(z) + \sum_{\ell=1}^{n-1} (-1)^{\ell+1} b_{x_n,\ell} W_\ell + (-1)^\ell y_\ell W_\ell + \sum_{\ell=1}^{n} (-1)^\ell y_\ell U_\ell. \]

However, using cofactor expansion along the last row of the matrix in \( U_\ell \) gives

\[ U_\ell = \sum_{i=1}^{\ell-1} (-1)^{n+i} x_{ni} \det ([zI_n - x_A]([1, n], \{i, \ell\})) \]

\[ + \sum_{i=\ell+1}^{n} (-1)^{n+i-1} x_{ni} \det ([zI_n - x_A]([1, n], \{\ell, i\})). \]

Thus

\[ p_{X_B} = zp_{X_A} + \sum_{\ell=1}^{n-1} (-1)^\ell (y_\ell - b_{x_n,\ell}) W_\ell \]

\[ + \sum_{1 \leq i < \ell \leq n} (y_\ell x_{ni} - y_i x_{n\ell})(-1)^{n+i+\ell} \det ([zI_n - x_A]([1, n], \{\ell, i\})). \]

Focusing on the coefficients of \( p_{X_B}(z) \), we can rewrite \( p_{X_B}(z) \) as

\[ zp_{X_A}(z) + \sum_{r=3}^{n+1} \sum_{\ell=1}^{n-1} S_{r,\ell}(y_\ell - b_{x_n,\ell}) z^{n-r+1} + \sum_{r=5}^{n+1} \sum_{1 \leq i < \ell \leq n} T_{r,\ell}(y_\ell x_{ni} - y_i x_{n\ell}) z^{n-r+1} \]  \( (11) \)

for some polynomials \( S_{r,\ell} \) and \( T_{r,\ell} \) in the variable entries of \( X_A \). Note that the variables in the last row of \( X_A \) do not appear in \( S_{r,\ell} \) or \( T_{r,\ell} \). To consider the Jacobian of \( X_B \), we assume the last columns of \( J_{X_B} \) are indexed by the nonzeros of \( x_{n1}, \ldots, x_{nn}, \) and \( y_1, \ldots, y_n \).

Let \( m \) be the number of nonzero entries of \( y \) and \( w \) be the number of variables in \( X_A \). Since \( x_{n1} \) is non-Jacobian in \( X_A \), the \((n + 1) \times (w + m)\) Jacobian matrix \( J_{X_B} \) is

\[ J_{X_B} = \begin{bmatrix} J_{X_A} & O \\ 0^T & 0^T \end{bmatrix} + \sum_{\ell=1}^{n-1} (y_\ell - b_{x_n,\ell}) M_\ell + \sum_{1 \leq i < \ell \leq n} (y_\ell x_{ni} - y_i x_{n\ell}) H_{i,\ell} \]  \( (12) \)

for some matrices \( M_\ell, H_{i,\ell} \) and \((n + 1) \times (2m)\) matrix \( N \) with columns indexed by the nonzeros of \( x_{n1}, \ldots, x_{nn}, y_1, \ldots, y_n \). Note that by (6) and (7), \( y_\ell = b_{a,\ell} = b_{x_n,\ell} \) in the nilpotent realization, so that we can ignore each matrix \( M_\ell \) and \( H_{i,\ell} \) in (12), since their coefficients vanish at the nilpotent realization. Let \( s_\ell = [0, 0, S_{3,\ell}, S_{4,\ell}, \ldots, S_{n+1,\ell}]^T \) and \( t_\ell = [0, 0, 0, T_{3,\ell}, T_{4,\ell}, \ldots, T_{n+1,\ell}]^T \). By (11), the column of \( N \) corresponding to \( y_\ell \) is

\[ \vec{N}_{y_\ell} = s_\ell + \sum_{i=1}^{\ell-1} x_{ni} t_{i,\ell} - \sum_{i=\ell+1}^{n} x_{ni} t_{i,\ell}, \]
and the column corresponding to $x_{n\ell}$ is

$$\hat{N}_{x_{n\ell}} = -bs_{\ell} - \sum_{i=1}^{\ell-1} y_i t_{i\ell} + \sum_{i=\ell+1}^{n} y_i t_{i\ell}$$

for $1 \leq \ell \leq n - 1$. Thus, evaluated at the nilpotent realization with $y_i = ba_{ni} = bx_{ni}$, $\hat{N}_{x_{n\ell}} = -b\hat{N}_{y_{\ell}}$ for $1 \leq \ell \leq n - 1$. Further $\hat{N}_{y_{n}} = \sum_{i=1}^{n-1} x_{ni} t_{in}$ with $\hat{N}_{x_{nn}} = \sum_{i=1}^{n-1} (-y_i) t_{in}$.

It follows that $N|_{X_B=B}$ is column equivalent to $[O | \hat{N}_{y_{1}} | \hat{N}_{y_{2}} | \cdots | \hat{N}_{y_{n}}]$.

From (8), with $z = 0$, the $(n + 1)$ entry of $\hat{N}_{y_{\ell}}$ is

$$(-1)^{\ell} \det X_A(1, \ell),$$

which evaluated at $X_B = B$ is

$$(-1)^{\ell} \det A(1, \ell).$$

Thus, the hypothesis that there exists an index $v$ such that $a_{kv} \neq 0$ and $\det A(j, v) \neq 0$ implies that the $(n + 1)$ entry of $\hat{N}_{y_{\ell}}$ is nonzero. It follows that $J_{X_B=B}$ is equivalent to

$$\begin{bmatrix} J_{X_A=A} & * \\ 0^T & r^T \end{bmatrix}$$

for some $r \neq 0$. Hence $B$ allows a full-rank Jacobian. Thus by Theorem 1.1 every superpattern of $B$ is spectrally arbitrary.

**Example 3.2.** Starting with $\mathcal{T}_2$, the unique spectrally arbitrary pattern of order 2 up to equivalence [6], the bordering technique of Theorem 3.1 gives spectrally arbitrary patterns of order 3. In particular, consider the nilpotent matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \in Q(\mathcal{T}_2) \quad \text{and let} \quad X_A = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}. $$

Then $J_{X=A}$ has full rank and entry $a_{12}$ is non-Jacobian in $X_A$. Thus, the bordering technique of Theorem 3.1 gives the matrix

$$\begin{bmatrix} 1 & -1 - b & 1 \\ 1 & -1 & 0 \\ b & -b & 0 \end{bmatrix},$$

providing different spectrally arbitrary patterns depending on the chosen value of $b \neq 0$. Taking $b = \frac{1}{2}$ gives a sign pattern equivalent to $\mathcal{W}_3$ (see [1]) with a full-rank Jacobian. Taking $b = -\frac{1}{2}$ gives a pattern equivalent to a superpattern of $\mathcal{V}_3$ (see [1]) with a full-rank Jacobian. Taking $b = -1$ gives a pattern equivalent to $\mathcal{V}_3$ with a full-rank Jacobian. This last option, using $b = a_{12}$ maintains sparsity (i.e, it gives a minimal spectrally arbitrary pattern.)
Remark 3.3. With a well-chosen example, Theorem 3.1 can be applied recursively. In particular, note that in (13), the \( n \) variables of \( X_A \) are used to show that \( B \) allows a full-rank Jacobian. Thus, at most one of the nonzero entries in row \( n + 1 \) of \( B \) needs to be Jacobian in \( X_B \). Further, an entry in row \( n + 1 \) that is Jacobian in \( X_B \) can be chosen to be any nonzero position \( (n+1,v) \) for which \( \det A(j,v) \) is nonzero. Note that, since the last column of \( B \) has only one nonzero entry, \( \det B(n+1,v) = (-1)^{j+n} \det A(j,v) \). Thus, if there is more than one \( v \) with \( a_{kv} \neq 0 \), and \( \det A(j,v) \neq 0 \), then bordering can be repeated recursively, applying it to \( (j,k) = (n+1,k) \) in \( B \).

Example 3.4. Consider the nilpotent realization

\[
\begin{bmatrix}
1 & -1 & 0 \\
1 & 0 & -1 \\
1 & 0 & -1
\end{bmatrix}
\]

of the spectrally arbitrary pattern \( V_3 \). By applying Theorem 3.1 with \( b = k = 1, j = 3 \) and \( v = 2 \), and repeating recursively, increasing \( j \) but keeping \( b = k = 1 \) and \( v = 2 \), a spectrally arbitrary pattern \( K_n \) is obtained for \( n \geq 4 \), with

\[
K_n = \begin{bmatrix}
+ & - & O \\
+ & 0 & - & + \\
0 & 0 & - & + \\
0 & - & 0 & 0 & + \\
\vdots & \vdots & \vdots & \ddots & \ddots \\
0 & - & 0 & 0 & \cdots & 0 & + \\
+ & - & 0 & 0 & \cdots & 0 & 0
\end{bmatrix}.
\]

The nonzero entries in a nilpotent realization of \( K_n \) have magnitude 1.

As far as we know, the spectrally arbitrary sign patterns \( B_n \) in Example 2.5 and \( K_n \) in Example 3.4 have not previously appeared in the literature.

4. General bordering for \( n = 3 \)

In Theorems 2.1 and 3.1, we restricted to bordering with standard unit vectors in the place of \( x \) and \( z \) in (1). We next illustrate the more general bordering (1) with a couple of examples.

Example 4.1. Starting with the nilpotent realization \( A \) of \( T_2 \) given in Example 3.2, a nilpotent realization of \( T_3 \) can be obtained as follows:

\[
B = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\frac{1}{2} & 1 & 1
\end{bmatrix}
\begin{bmatrix}
1 & -1 & 0 \\
1 & -1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\frac{1}{2} & -1 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & -1 & 0 \\
\frac{1}{2} & 0 & -1 \\
0 & \frac{1}{2} & -1
\end{bmatrix}
\in Q(T_3).
\]

Matrix \( B \) allows a full-rank Jacobian and hence (as is well-known) every superpattern of \( T_3 \) is spectrally arbitrary by Theorem 1.1.
Example 4.2. Starting with the nilpotent realization $A$ of $T_2$ given in Example 3.2, a nilpotent realization of $U_3$ (see [1]) can be obtained as follows:

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & \frac{1}{2} \\ 1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -2 & \frac{1}{2} \\ 1 & -1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$  

In particular, matrix $B$ is a nilpotent realization of $U_3$ that allows a full-rank Jacobian and hence every superpattern of $U_3$ is spectrally arbitrary by Theorem 1.1.

As demonstrated in [1], every spectrally arbitrary sign pattern of order 3 is a superpattern of one of the four patterns $T_3$, $U_3$, $V_3$ and $W_3$. From Example 3.2, every superpattern of $V_3$ and $W_3$ is spectrally arbitrary by Theorem 3.1 using a standard unit bordering of $T_2$. The other two order 3 patterns can be obtained by using a general bordering of $T_2$ as demonstrated in Examples 4.1 and 4.2.

Corollary 4.3. Every spectrally arbitrary sign pattern of order 3 is a superpattern of a pattern obtained from $T_2$ by bordering as in (1).

5. Inertially arbitrary borderings

We conclude by extending the main results in Sections 2 and 3 to obtain inertially arbitrary sign patterns. The inertia of a matrix $A$ is the ordered triple $i(A) = (a, b, c)$ for which $a$ is the number of eigenvalues of $A$ with positive real parts, $b$ is the number with negative real parts, and $c$ is the number of eigenvalues with real parts zero. The refined inertia of a matrix $A$ is the ordered 4-tuple $ri(A) = (a, b, c_1, c_2)$ for which $c_1$ is the algebraic multiplicity of zero as an eigenvalue for $A$ and $c_1 + c_2 = c$. Then $c_2$ is the number of nonzero imaginary eigenvalues of $A$. A sign pattern $A$ of order $n$ is inertially arbitrary if, for every non-negative integer choice of $(a, b, c)$ with $a + b + c = n$, there is some matrix $A \in Q(A)$ with $i(A) = (a, b, c)$. As with nilpotent matrices, a matrix $A$ of order $n$ with refined inertia $(0, 0, c_1, c_2)$ allows a full-rank Jacobian if the Jacobian matrix $J_{X=A}$ has rank $n$.

The next theorem combines [3, Theorem 2.13] and [1, Corollary 4.5].

Theorem 5.1. Let $A$ be a sign pattern and $A \in Q(A)$ be a matrix with $ri(A) = (0, 0, c_1, c_2)$ for some $c_1 \geq 2$. If $A$ allows a full-rank Jacobian, then

(a) $A$ is nonderogatory, and

(b) every superpattern of $A$ is inertially arbitrary.

Note that, unlike the context of Theorem 1.1, $A$ is not necessarily irreducible if $A$ allows a full-rank Jacobian in Theorem 5.1.

Example 5.2. Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad X_A = \begin{bmatrix} 1 & x_1 & 0 & 0 \\ 1 & x_2 & 0 & 0 \\ 0 & 0 & 1 & x_3 \\ 0 & 0 & 1 & x_4 \end{bmatrix}.$$
This matrix $A \in Q(\mathcal{T}_2 \oplus \mathcal{T}_2)$ is a nonderogatory reducible matrix with refined inertia $(0, 0, 2, 2)$ and $J_{X_A=A}$ has rank 4. Therefore, by Theorem 5.1 every superpattern of

$$A = \begin{bmatrix} + & - & 0 & 0 \\ + & - & 0 & 0 \\ 0 & 0 & + & - \\ 0 & 0 & + & - \end{bmatrix}$$

is inertially arbitrary. Note that while it is known [2] that $\mathcal{T}_2 \oplus \mathcal{T}_2$ is spectrally arbitrary (and hence inertially arbitrary), it is not yet known if every superpattern of $\mathcal{T}_2 \oplus \mathcal{T}_2$ is spectrally arbitrary.

The proof of the next theorem is the same as that for Theorem 2.1 except it uses Theorem 5.1 instead of Theorem 1.1.

**Theorem 5.3.** Let $A$ be a sign pattern. Suppose $A = [a_{ij}] \in Q(A)$ is a matrix having refined inertia $(0, 0, c_1, c_2)$ with $c_1 \geq 2$, and $A$ allows a full-rank Jacobian. Suppose $a_{kk} \neq 0$ and $a_{kv} \neq 0$ for some $v \neq k$. If $\det A(k, v) \neq 0$, then $B$ in (2) has refined inertia $(0, 0, c_1 + 1, c_2)$, $B$ allows a full-rank Jacobian and every superpattern of $B = \text{sgn}(B)$ is inertially arbitrary.

**Example 5.4.** Let $A$ be the matrix in Example 5.2. With $k = 2$ and $v = 1$, Theorem 5.3 implies that every superpattern of

$$B = \begin{bmatrix} + & - & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + \\ 0 & 0 & + & - & 0 \\ - & 0 & 0 & 0 & - \end{bmatrix}$$

is inertially arbitrary. Note that $B$ is spectrally arbitrary since $B$ is equivalent to $\mathcal{T}_2 \oplus V_3$, but it is not known if every superpattern of $B$ is spectrally arbitrary.

The proof of the next theorem is the same as that for Theorem 3.1 except it uses Theorem 5.1 instead of Theorem 1.1.

**Theorem 5.5.** Let $A$ be a sign pattern. Suppose $A = [a_{ij}] \in Q(A)$ is a matrix with refined inertia $(0, 0, c_1, c_2)$ for some $c_1 \geq 2$ and $A$ allows a full-rank Jacobian. Suppose $a_{jk} \neq 0$ and $a_{kv} \neq 0$ for some $v \neq k$, is non-Jacobian for some choice of $X_A$. If $a_{kv} \neq 0$ and $\det A(j, v) \neq 0$, for some $v$, then $B$ in (6) has refined inertia $(0, 0, c_1 + 1, c_2)$, $B$ allows a full-rank Jacobian, and every superpattern of $B = \text{sgn}(B)$ is inertially arbitrary.

**Example 5.6.** Consider the matrix $A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$, and $X_A = \begin{bmatrix} -1 & x_1 & -1 & 0 & 0 \\ 2 & x_2 & x_3 & 0 & 0 \\ 0 & 0 & 0 & -1 & x_4 \\ 0 & x_5 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix}$. 

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Matrix $A$ has sign pattern $G_5$ from [12] (see also Section 5.3 of [4]), $A$ has refined inertia $(0,0,3,2)$, $J_{X=A}$ has full rank and entry $(1,3)$ is non-Jacobian. Thus with $j = 1$, $k = 3$, $v = 4$, and $b = -1$ in Theorem [5.5] we obtain the inertially arbitrary pattern

$$B = \begin{bmatrix} - & - & 0 & 0 & 0 & + \\ + & + & + & 0 & 0 & 0 \\ 0 & 0 & 0 & - & - & 0 \\ 0 & - & 0 & 0 & - & 0 \\ - & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & + & 0 \end{bmatrix}.$$  

Since $G_5$ has no nilpotent realization, it follows that $B$ has no nilpotent realization; thus $B$ is not spectrally arbitrary. Note that, for $n \geq 2$, using the sign pattern $G_{2n+1}$ with matrix $\tilde{A}_{2n+1}$ as listed in Section 5.3 of [4], then $\tilde{A}_{2n+1}$ has refined inertia $(0,0,2n-1,2)$, $\det(\tilde{A}(1,4)) \neq 0$, and entry $(1,3)$ is non-Jacobian. Thus, using Theorem [5.5] with $j = 1$, $k = 3$, $v = 4$, and $b = -1$ applied to $\tilde{A}_{2n+1}$, we can construct an even order inertially arbitrary sign pattern with no nilpotent realization for each even order $2n+2 \geq 6$. In [12], only odd order sign patterns were provided with these conditions.

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References

[1] T. Britz, J.J. McDonald, D.D. Olesky, and P. van den Driessche. Minimal spectrally arbitrary sign patterns. *SIAM J. Matrix Anal. Appl.* 26 (2004) 257–271.

[2] M.S. Cavers. On reducible matrix patterns. *Linear and Multilinear Algebra* 58.2 (2010) 257–267.

[3] M.S. Cavers and S.M. Fallat. Allow problems concerning spectral properties of patterns. *Electron. J. Linear Algebra* 23:1 (2012) 731–754.

[4] M. Cavers, C. Garnett, I.-J. Kim, D.D. Olesky, P. van den Driessche, and K. Vander Meulen. Techniques for identifying inertially arbitrary patterns. *Electron. J. Linear Algebra* 26 (2013) 71–89.

[5] M.S. Cavers and K.N. Vander Meulen. Spectrally and inertially arbitrary patterns. *Linear Algebra Appl.* 394 (2005) 53-72.
[6] L. Corpuz and J.J. McDonald. Spectrally arbitrary nonzero patterns of order 4. *Linear and Multilinear Algebra* 55 (2007) 249–273.

[7] J.H. Drew, C.R. Johnson, D.D. Olesky and P. van den Driessche. Spectrally arbitrary patterns. *Linear Algebra Appl.* 308 (2000) 121–137.

[8] C. Garnett, B.L. Shader. A proof of the $T_n$ conjecture: centralizers, Jacobians and spectrally arbitrary sign patterns. *Linear Algebra Appl.* 436 (2012) 4451–4458.

[9] C. Garnett, B.L. Shader. The nilpotent-centralizer method for spectrally arbitrary patterns. *Linear Algebra Appl.* 438 (2013) 3836–3850.

[10] I.-J. Kim, D.D. Olesky, B.L. Shader, P. van den Driessche, H. van der Holst and K.N. Vander Meulen. Generating potentially nilpotent full sign patterns. *Electron. J. Linear Algebra* 18 (2009) 162–175.

[11] I.-J. Kim, B. Shader, K.N. Vander Meulen and M. West. Spectrally arbitrary pattern extensions. *Linear Algebra Appl.* 517 (2017) 120–128.

[12] I.-J. Kim, D.D. Olesky and P. van den Driessche. Inertially arbitrary sign patterns with no nilpotent realization. *Linear Algebra Appl.* 421 (2007) 264–283.

[13] R.J. Pereira. Nilpotent matrices and spectrally arbitrary sign patterns. *Electron. J. Linear Algebra* 16 (2007) 232–236.