The Cosmology of Ricci-Tensor-Squared Gravity in the Palatini Variational Approach

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We consider the cosmology of the Ricci-tensor-squared gravity in the Palatini variational approach. The gravitational action of standard general relativity is modified by adding a function $f(R^{ab}R_{ab})$ to the Einstein-Hilbert action, and the Palatini variation is used to derive the field equations. A general method of obtaining the background and first-order covariant and gauge-invariant perturbation equations is outlined. As an example, we consider the cosmological constraints on such theories arising from the supernova type Ia and cosmic microwave background observations. We find that the best fit to the data is a non-null leading-order correction to Einstein gravity, but the current data exhibit no significant preference over the concordance model. The growth of non-relativistic matter density perturbations at late times is also analyzed, and we find that a scale-dependent (positive or negative) sound-speed-squared term generally appears in the growth equation for small-scale density perturbations. We also estimate the observational bound imposed by the matter power spectrum for the model with $f(R^{ab}R_{ab}) = \alpha(R^{ab}R_{ab})^2$ to be roughly $|\alpha| \lesssim O(10^{-7})$ so long as the dark matter does not possess compensating anisotropic stresses.

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I. INTRODUCTION

The accumulating astronomical evidence for an accelerating cosmic expansion has stimulated many investigations into the nature of the dark energy, or other possible deviant gravitational effects, which might be responsible for this unexpected dynamics (for a review see, e.g. [1]). Besides proposing to add some new (and purely theoretical) matter species into the energy budget of the universe, many investigators have also focused their attentions on modifying general relativity (GR) on the largest scales, so as to introduce significant modifications in the behaviour of gravity at late times when it is comparatively weak. One example of the latter sort is provided by the family of $f(R)$ gravity models, which had also been considered before the discovery of cosmic acceleration (see for example Refs. [2, 3, 4]) with reference to alternative forms of inflation and the existence of singularities. In Refs. [2, 3], the authors discuss a specific model where the correction to GR is a polynomial function of the $R^2$, $R^{ab}R_{ab}$ and $R_{abcd}R^{abcd}$ quadratic curvature invariants (here, $R$, $R_{ab}$ and $R_{abcd}$ are respectively the Ricci scalar, Ricci tensor and Riemann tensor calculated in the standard way from the physical metric $g_{ab}$) and showed that there exist late-time accelerating attractors in Friedmann cosmological solutions to the theory. Barrow and Clifton established the general existence conditions for de Sitter, Einstein static, Gödel universes in theories where the Lagrangian is an arbitrary function of these three invariants [2].

When the Ricci scalar $R$ in the Einstein-Hilbert action is replaced by some general functions of $R$ and $R^{ab}R_{ab}$, it becomes necessary to distinguish between different variational approaches to deriving the field equations. In the metric approach, as in Refs. [2, 3], the metric components $g_{ab}$ are the only variational quantities and the field equations are generally of fourth-order, which makes the theories phenomenologically richer but more stringently constrained in most cases. Within the Palatini variational approach, on the other hand, we treat the metric $g_{ab}$ and the connection $\Gamma^\alpha_{\beta\gamma}$ as independent variables and extremize the action with respect to both of them; the resulting field equations are second order and easier to solve. The Palatini $f(R)$ gravity is also proposed as an alternative to dark energy in a series of works [8, 9, 10]. There has since been growing interest in these modified gravity theories: for the local tests of the Palatini and metric $f(R)$ gravity models see [11, 12]; for the cosmologies of these two classes of models see [13, 14, 15, 16, 17].

Both approaches to modifying gravity are far from problem-free. In the metric $f(R)$ gravity models, the theory is conformally related to standard GR plus a self-interacting scalar field [3], which generally introduces extra forces inconsistent with solar system tests [12]. The Palatini approach, on the other hand, generally leads to a large (or even negative) sound-speed-squared term in the growth equation of the non-relativistic matter perturbations on small scales. This induces effects on the cosmic microwave background (CMB) and the matter clustering power-spectra which deviate unacceptably from those which are observed [15, 16, 17]. Again, these examples highlight the difficulties encountered when trying to make modifications to standard GR which are compatible with
In this work we will focus on the Ricci-squared gravity models within the Palatini variational approach, which we also denote by the $f(R^{ab}R_{ab})$ gravity. It turns out that the Ricci tensor, $R_{ab}$, and Ricci scalar, $R$, appearing in the field equations in the Palatini approach are not the ones calculated from the physical metric, $g_{ab}$, (we consider the metric $g_{ab}$ as the physical one because it is this metric which the matter Lagrangian density depends on and the energy-momentum conservation law holds with respect to) as in GR, and we denote the GR equivalents by $R_{ab}$ and $R$ respectively to distinguish them from the Palatini quantities). Such a modification of gravity has indeed been considered in [18] and shown to give an accelerating cosmology. However, our work differs from [18] in that we replace the Ricci scalar in the gravitational action with $R + f(R^{ab}R_{ab})$ rather than simply $f(R^{ab}R_{ab})$, and we concentrate more on the cosmology at the first-order perturbation level, especially the late-time cold-dark-matter (CDM) density perturbation growth. We emphasize the similarity to the Palatini $f(R)$ gravity models also.

Our presentation is organized as follows. In Sec. II we briefly introduce the model and outline the methods used to derive the background and first-order covariant and gauge-invariant (CGI) perturbed field equations. In Sec. III we present the modified Friedmann equations and apply it to a specific family of theories with density perturbations at late cosmological times for this model. Since this analysis shares some similarities with the Christoffel symbol $\Gamma_{bc}^a$ calculated using the metric $g_{ab}$, $R_{ab}$ is assumed to be a symmetric tensor (if it contains an antisymmetric part then the field equation will be spoiled as discussed in [18]) and $g_{ab}$ could be used to raise or lower its indices. Varying the action Eq. (1) with respect to the metric $g_{ab}$ (note that $\nabla \Delta / \delta g = 0$ as they are independent) gives the modified Einstein equations:

\[ R_{ab} + 2F R^c_a R_{bc} - \frac{1}{2} g_{ab} \left[ R + f(R^{ab}R_{ab}) \right] = \kappa T^f_{ab} (3) \]

where $F = \partial f / \partial S$ with $S = R^{ab}R_{ab}$ and $T^f_{ab}$ is the energy-momentum tensor of the fluid matter (CDM and radiation).

On the other hand, varying the action with respect to the new variable $\Gamma_{bc}^a$ with the relation $\delta R_{ab} = D_c (\delta \Gamma_{ab}^c) - D_b (\delta \Gamma_{ca}^b)$, one arrives at another field equation

\[ D_c \left[ \sqrt{\det g} \left( g^{ac} + 2F g^{ac} R_{cd} g^{bd} \right) \right] = 0, \quad (4) \]

where $D_a$ represents the covariant derivative compatible to $\Gamma_{bc}^a$ (the covariant derivative compatible to $g_{ab}$ is denoted, as conventionally, by $\nabla_a$). Just like in the Palatini $f(R)$ models, this equation implies some relation between the physical metric $g_{ab}$ and the metric $g_{ab}$ whose Christoffel symbol is $\Gamma_{bc}^a$. However, because of the presence of the second term in the parentheses this relation is nontrivial and some further algebra will be needed to explicate it. Before doing that, we will present some preliminary definitions and expressions, one of which is the notation of $3 + 1$ decomposition.

### A. The $f(R^{ab}R_{ab})$ Gravity Model

We will start our discussion with the modified Einstein-Hilbert action in the present model,

\[ S = \int d^4x \sqrt{-g} \left[ \frac{R + f(R^{ab}R_{ab})}{2\kappa} + L_m \right], \quad (1) \]

in which $\kappa = 8\pi G_N$, with $G_N$ the Newtonian gravitational constant. Here, $R_{ab} = R_{ab}(\Gamma^a_b)$ is given by

\[ R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{cd} R_{ab} - \Gamma^c_{ad} R_{cb} \]

(2) and $R = g^{ab} R_{ab}$; note that $\Gamma_{bc}^a$ is a new and independent variable with respect to which we extremise the action, and is different from the Christoffel symbol $\Gamma_{bc}^a$ calculated using the metric $g_{ab}$. $R_{ab}$ is assumed to be a symmetric tensor (if it contains an antisymmetric part then the field equation will be spoiled as discussed in [18]) and $g_{ab}$ could be used to raise or lower its indices. Varying the action Eq. (1) with respect to the metric $g_{ab}$ (note that $\nabla \Delta / \delta g = 0$ as they are independent) gives the modified Einstein equations:

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### B. The $3 + 1$ Decomposition

The main idea of $3 + 1$ decomposition [20, 21, 22, 23] is to make spacetime splits of physical quantities with respect to the 4-velocity $u^a$ of an observer. The projection tensor $h_{ab}$ is defined as $h_{ab} = g_{ab} - u_a u_b$ and can be used to obtain covariant tensors perpendicular to $u^a$. For example, the covariant spatial derivative $\nabla_a$ of a tensor field $T^{ac}_{db} \cdots$ is defined as

\[ \nabla^a T^{bc} \cdots \equiv h^a_i h^b_j \cdots h^r_d \cdots h^k_e \nabla_i T^{r \cdots k}_{p \cdots q}. \]
The energy-momentum tensor and covariant derivative of the 4-velocity are decomposed respectively as

\[ T_{ab} = \pi_{ab} + 2q_a u_b + \rho u_u u_b - p h_{ab}, \]  

\[ \nabla_u u_b = \sigma_{ab} + \omega_{ab} + \frac{1}{3} \theta h_{ab} + u_a A_b. \]  

In the above, \( \pi_{ab} \) is the projected symmetric trace-free (PSTF) anisotropic stress, \( q_a \) the heat flux vector, \( p \) the isotropic pressure, \( \sigma_{ab} \) the PSTF shear tensor, \( \omega_{ab} = \nabla[u a b] \), is the vorticity, \( \theta = \nabla u_c \equiv 3a/a \) (\( a \) is defined here as the mean expansion scale factor) the volume expansion rate scalar, and \( A_b = \dot{u}_b \) is the fluid acceleration; the overdots denote time derivatives expressed as \( \dot{\phi} = u^a \nabla_a \phi \), brackets mean antisymmetrisation, and parentheses symmetrisation. The velocity normalization is chosen to be \( u^a u_a = 1 \). The quantities \( \pi_{ab}, q_a, \rho, p \) are referred to as dynamical quantities and \( \sigma_{ab}, \omega_{ab}, \theta, A_a \) as kinematical quantities. Note that the dynamical quantities can be obtained from the energy-momentum tensor \( T_{ab} \) through the relations

\[ \rho = T_{ab} u^a u^b, \]  

\[ p = -\frac{1}{3} h_{ab} T_{ab}, \]  

\[ q_a = h^d \dot{u}_c T_{cd}, \]  

\[ \pi_{ab} = h^d h^e T_{cd} + p h_{ab}. \]  

Decomposing the Riemann tensor and making use the Einstein equations, we could obtain, after linearization, the perturbed (constraint and propagation) equations [20, 21, 22, 23]. Here, we shall not list all of them because most are irrelevant for the following discussion; rather we will use the linearised Raychaudhuri equation

\[ \dot{\theta} + \frac{1}{3} \theta^2 - \nabla^a A_a + \frac{\kappa}{2} (\theta + 3p) = 0, \]  

the linearised conservation equations for the energy density:

\[ \dot{\rho} + (\rho + p) \theta + \nabla^a q_a = 0, \]  

and the linearised Friedmann equation

\[ \frac{1}{3} \theta^2 = \kappa \rho. \]  

The above equations are derived and presented for standard general relativity, and so the \( \rho, p, q_a \) variables describe imperfect fluid matter. For general modified gravity theories, such as those presented here, the modification to GR might be parameterized as an effective energy-momentum tensor. In this case the formalism of these equations is preserved and one just needs to replace \( \rho, p, q_a \) by the total effective quantities of the same sort: \( \rho^{\text{tot}}, p^{\text{tot}}, q_a^{\text{tot}}, \pi_{ab}^{\text{tot}} \) [24].

C. The Field Equations in \( f(R^{ab} R_{ab}) \) Gravity

In Eq. (4), we see that \( \sqrt{\det g} \left( g^{ab} + 2 F g^{ac} R_{cd} g^{bd} \right) \) is a symmetric \((2, 0)\) tensor density of weight 1, and so we can introduce a new metric \( g_{ab} \) by means of the following relation

\[ \sqrt{\det g} g^{ab} = \sqrt{\det g} \left( g^{ab} + 2 F g^{ac} R_{cd} g^{bd} \right), \]  

where the Levi-Civita connection of the metric \( g_{ab} \) is just \( \Gamma^c_{bc} \), as we referred to above.

To go further, we need to express \( R_{ab} \) explicitly. This is easy to do in principle, because Eq. (3) is just an algebraic equation for \( R_{ab} \). To see this, let us write the symmetric tensor \( R_{ab} \) in a general way as

\[ R_{ab} = \Delta u_a u_b + \Xi h_{ab} + 2 u(a \Upsilon_b) + \Sigma_{ab} \]  

where \( u_a \) is the 4-velocity of the observer referred to above. Substituting Eqs. (6) (13) into Eq. (3), we get

\[ \Delta u_a u_b + \Xi h_{ab} + 2 u(a \Upsilon_b) + \Sigma_{ab} + 2 F \left[ \Delta^2 u_a u_b + \Xi^2 h_{ab} + 2(\Delta + \Xi) u(a \Upsilon_b) + 2 \Xi \Sigma_{ab} \right] \]

\[ - \frac{1}{2} (\Delta + 3 \Xi + f) u_a u_b - \frac{1}{2} (\Delta + 3 \Xi + f) h_{ab} \]

\[ = \kappa \left( \rho^f u_a u_b - p^f h_{ab} + 2 u(a q_b^f) + \pi_{ab}^f \right), \]

which leads to the following four equations:

\[ \Delta + 2 F \Delta^2 - \frac{1}{2} (\Delta + 3 \Xi + f) = \kappa \rho^f, \]

\[ \Xi + 2 F \Xi^2 - \frac{1}{2} (\Delta + 3 \Xi + f) = -\kappa p^f, \]

\[ [1 + 2 F(\Delta + \Xi)] \Upsilon_a = \kappa q_a^f, \]

\[ (1 + 4 F) \Sigma_{ab} = \kappa \pi_{ab}^f, \]

where \( f, F \) are functions of \( R^{ab} R_{ab} = \Delta^2 + 3 \Xi^2 \). Thus given the specified form of \( f, F \), the quantities \( \Delta, \Xi \) can be obtained from Eqs. (11) (15), at least numerically. Then, \( \Upsilon_a \) and \( \Sigma_{ab} \) can also be calculated from Eqs. (16) (17) provided the values of \( q^f_a \) and \( \pi^f_{ab} \) are given. Note that \( \Upsilon_a \) and \( \Sigma_{ab} \) are nonzero only at first order in perturbation. Taking the time derivatives of Eqs. (14) (15), and using the background values of \( \Delta, \Xi \), we could easily obtain \( \Delta, \Xi \) by solving the two linear algebraic equations. Similarly, \( \nabla_a \Delta \) and \( \nabla_a \Xi \) could be worked out (here, \( \nabla \) is the spatial derivative). In what follows, we shall assume that \( \Delta, \Xi \) and their derivatives have been calculated.

The next step is to find out the relation between \( g_{ab} \) and \( g_{ab} \). We could rewrite Eq. (12) as

\[ \sqrt{\det g} g^{ab} = \sqrt{\det g} g^{ac} (g_c^b + 2 F R_c^b). \]

Taking the determinants of both sides and equating we get

\[ \det g = \det g \cdot \det P, \]
with
\[ P^a_b = \delta^a_b + 2FR^a_b. \] (20)

Thus, we conclude from Eq. (18) that
\[ g^{ab} = \frac{\sqrt{\det g}}{\sqrt{\det g}} \alpha^c \rho^a_c \]
\[ = \frac{1}{\sqrt{\det g}} \left( g^{ab} + 2FR^{ab} \right); \] (21)
\[ g_{ab} = \frac{\sqrt{\det g}}{\sqrt{\det g}} \alpha^c (P^{-1})^c_b \]
\[ = \sqrt{\det P} (P^{-1})_{ab}. \] (22)

Obviously, \( \det P \) and \( P^{-1} \) need to be evaluated respectively. For \( \det P \), we have
\[ \det P = (1 + 2F\Delta)(1 + 2F\Xi)b^0 \]
\[ + 4F^u(\alpha \gamma_b) + 2F\Sigma^b_0. \]

To calculate this, let us write \( g_{00} = a^2(1 + 2 \Psi) \), \( g_{0a} = g_{ab} = a^2 \alpha^2 \), \( g_{ab} = -a^2 B_a \), \( g_{\alpha\beta} = a^2 \gamma_{\alpha\beta} + 2H\tau_{\alpha\beta} \) and \( u_0 = a(1 + \Psi), u_a = -a(v_a - B_a) \) where \( \alpha, \beta \) run over 1, 2, 3, \( \Psi, B_a, H \) and \( H\tau_{\alpha\beta} \) are first order metric variables of which \( H\tau_{\alpha\beta} \) is traceless, \( v_a \) is the spatial component of \( u_a \), and \( \gamma_{\alpha\beta} \) is the metric of 3 dimensional flat space. As a result, \( g_{00} = a^{-2}(1 - 2 \Psi) \), \( g_{0a} = g_{ab} = a^{-2}B^a \), \( g_{ab} = (1 - 2H) \gamma_{\alpha\beta} - 2H\tau_{\alpha\beta} \) and \( u_0 = a^{-1}(1 - \Psi) \), \( u_a = -a^{-1}v_a \). From these expressions the components of \( h^b_0 \) can also be obtained and one can substitute all these quantities into the above equation to get \( \det P \). Since \( \gamma_a \) and \( \Sigma_a \) are only of first order and because \( \Sigma_a \) is traceless, it is then not difficult to see that up to first order (note that the facts \( u^a \gamma_a = u^a \gamma^a \gamma_a = 0 \) and \( u^a \Sigma_a = 0 \) indicate that \( \gamma^a = \gamma_0 = 0 \) and \( \Sigma^a_0 = \Sigma^a_0 = 0 \))
\[ \det P = (1 + 2F\Delta)(1 + 2F\Xi)^3. \] (23)

For \( (P^{-1})^b_a \), we know that it is symmetric as the inverse matrix of a symmetric matrix, and so could be written as
\[ (P^{-1})^b_a = A u^a u_b + B h^a_b + 2u(\alpha C_b) + D^b_a. \] (24)

Using
\[ P^b_c = (1 + 2F\Delta)u^b u_c + (1 + 2F\Xi)h^b_c + 4F^u(b \gamma_c) + 2F\Sigma^b_c \]
and
\[ (P^{-1})^a_b P^b_c = \delta^a_c \]
it is then easy to obtain, to first order, that
\[ A = \frac{1}{1 + 2F\Delta}; \]
\[ B = \frac{1}{1 + 2F\Xi}; \]
\[ C_a = \frac{2F}{(1 + 2F\Delta)(1 + 2F\Xi)} \Xi_a; \]
\[ D_{ab} = \frac{2F}{(1 + 2F\Xi)^2} \Sigma_{ab}. \] (25)

As a result, we have now the relations between the two metrics \( g_{ab} \) and \( g_{ab} \) and their inverses as
\[ g_{ab} = \lambda g_{ab} + \xi_{ab}, \]
\[ g^{ab} = \frac{1}{\lambda} g^{ab} + \zeta_{ab}, \] (27)
where
\[ \lambda = \sqrt{(1 + 2F\Delta)(1 + 2F\Xi)}, \]
\[ \omega = \frac{1 + 2F\Xi}{1 + 2F\Delta}, \]
\[ \xi_{ab} = \lambda(\lambda - 1)u_a u_b - 4\sqrt{\omega} F u_a \gamma_b - \frac{2F}{\sqrt{\omega}} \Sigma_{ab}, \]
\[ \zeta_{ab} = \frac{1}{\lambda} \left( \frac{1}{\omega} - 1 \right) u^a u^b + \frac{2F}{\lambda^2} \frac{2u^a \gamma^b + \Sigma_{ab}}{31} \]

A discussion of how the two Ricci tensors \( R_{ab} \) and \( R_{ab} \) are related to one another is given in the appendix, with the help of which the Einstein equation Eq. (3) can be rewritten as
\[ R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab} + \kappa T_{ab}^{eff} \] (32)
where
\[ \kappa T_{ab}^{eff} = \frac{1}{2} g_{ab}(f + \delta R) - \delta R_{ab} - 2FR^a_c R_{cb}. \] (33)

and (see the appendix for a definition of the tensor \( \gamma_{bc} \))
\[ \delta R_{ab} = \nabla_c \gamma_{ac} - \nabla_b \gamma_{ac} + \gamma_{ab} \gamma_{cd} - \gamma_{ac} \gamma_{db}, \]
\[ \delta R = g^{ab} \delta R_{ab}. \] (35)

With the aid of Eqs. (3) we can identify \( \rho^{eff}, p^{eff}, g^{eff} \) and \( \pi^{eff} \) and express them in terms of \( \omega, \lambda, F, \Delta \) and \( \Xi \), which are functions of \( \rho^f, p^f \) (c.f. Eqs. (14, 15, 28, 29)), and \( \gamma_a, \Sigma_a, \) which are also functions of \( g_{ab} \) (c.f. Eqs. (16, 17)). However, from Eqs. (26 - 31, 69, 70) one can see that this process will involve a lot of calculation. In the present work we will not perform detailed numerical calculations of the perturbation equations of the \( f(R_{ab} R_{ab}) \) model; instead, in the next two sections of the paper we will:
1. Study the background evolutions of general Ricci-squared gravity models. As an example, we will consider a specific family of theories with \( f(R_{ab} R_{ab}) = \)
\(\alpha(R^{ab}R_{ab})^\beta\), and constrain the allowed \((\alpha, \beta)\) parameter space using data sets supernovae (SNe) luminosity distances and the CMB shift parameter.

2. Present a simple argument to show that this class of modified gravity theory, like those arising in the Palatini \(f(\mathbf{R})\) theory, generally possesses a scale-dependent effective sound-speed-squared term which affects the growth of CDM density perturbations and thus influences the matter power spectrum \([13, 16, 17]\) on small scales.

III. THE COSMOLOGICAL BACKGROUND EVOLUTION

In order to analyse the cosmological background evolution we can neglect the \(q^f_t\) and \(\pi^f_{ab}\) terms, and hence the quantities \(\Upsilon_\alpha, \Sigma_{ab}\). As a result, the equations are greatly simplified.

We are interested in the modified Friedmann equation in the present model. From Eqs. (38, 39, 33) we have

\[
3H^2 = \kappa \rho^\text{tot},
\]

in which \(H \equiv \frac{\dot{a}}{a}\) is the Hubble expansion rate and \(\rho^\text{tot}\) is expressed as

\[
\kappa \rho^\text{tot} = \kappa \rho^f + \frac{1}{2} f - 2F \Delta^2 - \frac{2}{2} \delta R - \delta R_{ab} u^a u^b ,
\]

with \(\delta R\) and \(\delta R_{ab}\) given in Eqs. (34, 35). After a lengthy calculation, and using Eq. (28) to eliminate the term \(\ddot{\lambda} + 3\dot{\lambda}/2\) which appears in Eq. (37), we obtain the following simple result

\[
\left( H + \frac{\dot{\lambda}}{2\lambda} \right)^2 = \frac{1}{6} (\Delta - 3\omega \Xi). \tag{38}
\]

There are two interesting points regarding Eq. (38). Firstly, we see that only \(\omega\), and not its time derivatives \(\dot{\omega}\) or \(\ddot{\omega}\), enter the equation. Secondly, the second-order derivative of \(\lambda\) does not appear either; to see the consequence of this, note that since \(\lambda = \frac{\partial \lambda}{\partial \rho^f}/\partial \rho^f \dot{\rho}^f = -\zeta \partial \lambda(\rho^f)/\partial \rho^f \dot{\rho}^f H\) (with \(\zeta = 3\) for matter and \(\zeta = 4\) for radiation), the dependence of \(\lambda\) on \(\rho^f\) could be expressed in terms of \(\rho^f\), e.g., in radiation-dominated era \(\rho^f = \rho^f/3\) and in matter-dominated era \(\rho^f = 0\), and so we have \(\lambda \propto H\). Consequently, Eq. (38) has the form

\[
H^2 = \Theta(\rho^f, \rho^f), \tag{39}
\]

where \(\Theta\) is a complicated function of \(\rho^f\) and \(\dot{\rho}^f\) (at late times \(\rho^f \equiv 0\) and it becomes a function of \(\rho^f\) alone).

As we discussed above, knowing \(\rho^f\) and \(\dot{\rho}^f\) means that we know \(\Delta, \Xi, \lambda\) and \(\omega\). Thus, given a specific form for \(f(R^{ab}R_{ab})\), Eq. (35) completely determines the background cosmological evolution of the model. As a particular example let us consider the case of

\[
f(R^{ab}R_{ab}) = \alpha(R^{ab}R_{ab})^\beta \tag{40}
\]

where \(\alpha\) and \(\beta\) are the model parameters. Note that here \(R^{ab}R_{ab} = \Delta^2 + 3\Xi^2\) is always non-negative, and \(\beta = 0\) corresponds to a picking the standard \(\Lambda\)CDM cosmology of GR.

For convenience, we shall define the following dimensionless quantities

\[
\tilde{f} \equiv \frac{f}{H_0^2}, \quad \tilde{\Delta} \equiv \frac{\Delta}{H_0^2}, \quad \tilde{\Xi} \equiv \frac{\Xi}{H_0^2}, \quad \tilde{\rho} \equiv \frac{\rho}{H_0^2}, \quad \Omega_m \equiv \frac{\kappa \rho_m}{3H_0^2}, \quad \Omega_r \equiv \frac{\kappa \rho_r}{3H_0^2}, \quad \tilde{\omega} \equiv \alpha H_0^{\beta - 2}, \tag{41}
\]

then Eqs. (14, 15) could be rewritten as

\[
\tilde{\Delta} + 2\tilde{\rho}\tilde{\Delta}^2 - \frac{1}{2} \left( \tilde{\Delta} + 3\tilde{\Xi} + \tilde{f} \right) = 3\Omega_m + 3\Omega_r, \tag{42}
\]

\[
\tilde{\Xi} + 2\tilde{\rho}\tilde{\Xi}^2 - \frac{1}{2} \left( \tilde{\Delta} + 3\tilde{\Xi} + \tilde{f} \right) = -\Omega_r. \tag{43}
\]
where

\[ f(R^{ab}R_{ab}) = \dot{\alpha} \left( \Delta^2 + 3 \Xi^2 \right)^\beta, \]  
\[ F(R^{ab}R_{ab}) = \ddot{\alpha} \beta \left( \Delta^2 + 3 \Xi^2 \right)^{-1}; \]

and then Eq. (38) reduces to

\[ \left[ 1 - \frac{\zeta \lambda_{0} \rho^f}{2 \lambda} \right]^2 \frac{H^2}{H_0^2} = \frac{1}{6} \left( \Delta - 3 \omega \Xi \right) \]  
(46)

where \( \lambda_{0} \rho^f \equiv \partial \lambda / \partial \rho^f \).

In this paper we will set \( \Omega_m = 8.5 \times 10^{-5} \), so today we have \( H^2 / H_0^2 = 1 \) and there are 3 equations (Eqs. 42, 43, 46) for the 5 parameters \( \Delta, \Xi, \dot{\alpha}, \beta \) and \( \Omega_m \). Therefore, we are able to express all the other quantities in terms of \( \beta \) and \( \Omega_m \), which can therefore be treated as the two independent degrees of freedom of our model. Note that \( \dot{\alpha} \) is a constant, and once evaluated at the present day, it could be used all through the cosmic history, which helps determine \( \Delta, \Xi \) at arbitrary times.

In Figure 1 we have plotted the effective equation of state, defined by \( w_{eff} \equiv -1 - \frac{\dot{H}}{H^2} = -1 - \frac{\dot{H}}{H^2} \) (where a star-superscript denotes the derivative with respect to \( \log(a) \)), as a function of the redshift. The values of \( \beta \) are indicated beside the curves. At early times, when the \( f(R^{ab}R_{ab}) \) corrections are negligible, the models all mimic the \( \Lambda \)CDM evolution of \( w_{eff} \), and the same thing happens in the future. This is because during this era the matter (relativistic and non-relativistic) is greatly diluted so that the right-hand sides of Eqs. 12, 13 both vanish; consequently, we can solve them to show that \( \Delta = \Xi = \text{const.} \) and so \( f(R^{ab}R_{ab}) \) is also constant. The deviation from \( \Lambda \)CDM occurs mainly at intermediate times, that is, in the recent past and future.

We now use the observational data on the background cosmology to constrain the parameter space (in the \( \beta - \Omega_m \) plane) of the present model. For this we jointly use the 157 measurements on SNe luminosity distance in the Gold data sets of Riess et al. [28] and the CMB shift (R) parameter. The SNe luminosity distance is expressed as

\[ d_L(z) = (1 + z) \int_0^z \frac{du}{H(u)} = \frac{1 + z}{H_0} \int_0^z \frac{du}{E(u)} \]  
(47)

where \( E(z) = H(z)/H_0 \). The measurements supply the extinction-corrected distance modulus \( \mu_0 = 5 \log d_L + 25 \) (with \( d_L \) in units of Mega-parsecs) and its uncertainty, \( \sigma \), for individual SNe, so that the standard \( \chi^2 \) minimization, defined by

\[ \chi^2 = \sum_{i=1}^{157} \left[ \mu_{p,i}(z; H_0, \Omega_m, \beta) - \mu_{0,i} \right]^2 / \sigma_i^2 \]  
(48)

is easy to implement, where \( \mu_p \) is the theoretically predicted distance modulus. As \( H_0 \) appears only as it does in Eq. (47), we could marginalize over it by integrating the probability density \( p(\chi) \propto \exp(-\chi^2/2) \) for all values of \( H_0 \). For the CMB R-parameter, defined as

\[ R = \sqrt{\Omega_m H_0} \int_0^{z_{dec}} \frac{dz}{H(z)} \]  
(49)

we adopt the observational value \( R^{obs} = 1.70 \pm 0.03 \) at \( z_{dec} = 1089 \) from [29]. Note that this does not depend on the specified value of \( H_0 \).

Our constraining result is shown in Figure 2 where we have shown the 68% and 95% confidence regions respectively. The constrained intervals are roughly \( 0.20 \lesssim \Omega_m \lesssim 0.36 \) and \( -0.13 \lesssim \beta \lesssim 0.29 \) at the 95% confidence level, with the best fitting values being \( (\beta, \Omega_m) \approx (0.07, 0.265) \) with \( \chi^2/dof \approx 1.126 \). Also note that the concordance \( \Lambda \)CDM model (the white star) lies within the 68% confidence region of our constraints.

Thus, we see that the background cosmological data is able to constrain \( |\beta| \) to be of order 0.1. In the next section we will briefly investigate the possible constraint from the growth of dark-matter density perturbations, and show that this may provide a potentially more stringent restriction on \( \beta \). However, considering that this latter limit depends on the properties of the dark matter, our background constraints given in this section are less model dependent.

FIG. 2: The constraints on the parameter space of \( \Omega_m \) and \( \beta \) in the present model from joint SNe and CMB shift parameter data sets. The grey and light grey regions represent the 68.3% and 95.4% confidence contours respectively. The white circle \( (\Omega_m = 0.265, \beta = 0.07) \) is the best-fitting parameter of our model, and the star is the concordance \( \Lambda \)CDM model.
IV. EFFECTS ON LATE-TIME CDM DENSITY PERTURBATION GROWTH

In this section we study the effects of the $f(R^{ab}R_{ab})$ corrections to GR on the CDM density perturbation growth. We start by recalling the case of $f(R)$ gravity because it shares some similarities with the $f(R^{ab}R_{ab})$ one, while being technically simpler than the latter, and because a similar analysis for the former is still missing from Refs. \[16, 17\] (see however \[15\] for a slightly different treatment).

A. The Case of Palatini $f(R)$ Gravity

Recall that in our simplified model the universe is filled with CDM and radiation, and at later times the radiation energy density is negligible, so $\rho^{f} \approx \rho_{\mathrm{CDM}}$.

Taking the spatial derivative of the Raychaudhuri equation Eq. (31) (with the $\rho, p$ there being replaced by $\rho^{tot} \approx \rho^{f} + \rho^{f^{eff}}, p^{tot} \approx p^{f} + p^{f^{eff}}$), and working in the CDM frame (where the observer is comoving with CDM particles and thus $A = 0$) \[25\], we have

$$\Delta_{\mathrm{CDM}}^{\lambda} + H\Delta_{\mathrm{CDM}}^{\prime} - \frac{\kappa}{2}(\lambda^{2\mathrm{tot}} + 3\lambda^{p^{tot}})a^{2} = 0, \quad (50)$$

where $\Delta_{\mathrm{CDM}}$ is the CDM density perturbation contrast that is defined through $\nabla_{a}\rho_{\mathrm{CDM}} = \rho_{\mathrm{CDM}} \sum_{k} \frac{k}{a} \Delta Q_{a}^{k}$, and $H = \frac{\dot{a}}{a}$ is the Hubble expansion rate with respect to conformal time (note that a prime denotes the conformal time derivative, and a dot the cosmic comoving proper-time derivative); $\lambda^{\prime}$ and $\lambda^{p^{\prime \mathrm{eff}}}$ are respectively the harmonic expansion coefficients for $\nabla_{a}\rho$ and $\nabla_{a}p$ (defined via $\nabla_{a}\rho = \sum_{k} \frac{k}{a} \lambda^{\prime} Q_{a}^{k}$ and $\nabla_{a}p = \sum_{k} \frac{k}{a} \lambda^{p^{\prime \mathrm{eff}}}$ \[20\]). Clearly we need to know about $\lambda^{\prime \prime}$ and $\lambda^{p^{\prime \prime \mathrm{eff}}}$ which arise from the $f(R^{ab}R_{ab})$ modifications to GR (c.f. Eq. (33)).

In the Palatini $f(R)$ model, in which the Ricci scalar $R$ in the gravitational action is replaced with $R + f(R)$, Eqs. \[26, 27\] still hold, but with (see for example \[10\])

$$\lambda = 1 + \frac{\partial f}{\partial R},$$
$$\xi_{ab} = 0,$$
$$\zeta_{ab} = 0. \quad (51)$$

Then, with the help of the calculations in the appendix, it is straightforward to show that

$$R_{ab} = R_{ab} + \delta R_{ab}$$
$$= R_{ab} + \frac{3}{2\lambda^{2}} \nabla_{a}\nabla_{b}a - \frac{1}{\lambda} \nabla_{a} \nabla_{b} \lambda - \frac{1}{2\lambda} g_{ab} \nabla^{2} \lambda \quad (52)$$

where $\nabla^{2} = \Box$, and the modified Einstein equation \[10\],

$$\lambda R_{ab} - \frac{1}{2} g_{ab} (R + f) = \kappa T^{f}_{ab},$$

can be rewritten as

$$R_{ab} - \frac{1}{2} g_{ab} R = \kappa T^{\mathrm{tot}}_{ab},$$

in which the effective total energy-momentum tensor is given by

$$\kappa T^{\mathrm{tot}}_{ab} = \frac{1}{\lambda^{2}} \kappa T^{f}_{ab} + \frac{1}{2\lambda} g_{ab} (R + f) - \frac{1}{2} g_{ab} (R - \delta R) - \delta R_{ab}. \quad (53)$$

Using Eq. (8), we can now identify

$$\kappa \rho^{\prime} = \frac{1}{\lambda^{2}} \kappa \rho^{f} + \frac{1}{2\lambda} (R + f)$$
$$- \frac{1}{2} \left[ R + \frac{3}{\lambda} \Box \lambda - \frac{3}{2\lambda^{2}} \nabla^{a} \lambda \nabla_{a} \lambda \right]$$
$$- \frac{3\lambda^{2}}{2\lambda^{2}} + \frac{\dot{\rho}}{\rho} + \frac{1}{2} \Box \lambda, \quad (54)$$

$$\kappa p^{\prime} = \frac{1}{\lambda^{2}} \kappa p^{f} - \frac{1}{2\lambda} (R + f)$$
$$+ \frac{1}{2} \left[ R + \frac{3}{\lambda} \Box \lambda - \frac{3}{2\lambda^{2}} \nabla^{a} \lambda \nabla_{a} \lambda \right]$$
$$- \frac{1}{3\lambda} (\dot{\lambda} + \nabla^{2} \lambda) - \frac{1}{2\lambda} \Box \lambda. \quad (55)$$

Thus

$$\kappa (\rho^{\prime} + 3p^{\prime}) = \frac{3}{\lambda} \dot{\lambda} + \frac{1}{\lambda^{2}} \nabla^{2} \lambda + \cdots \quad (56)$$

in which $\cdots$ represent the terms not involving second order (time and spatial) derivatives.

The reason why we keep only two second derivative terms explicitly on the right-hand side of Eq. (56) is that, after taking the spatial covariant derivative, the first term contributes a $\Delta_{\mathrm{CDM}}^{\prime}$ piece to Eq. (50) while the second term contributes a $\kappa^{2} \Delta_{\mathrm{CDM}}^{\prime}$ piece. None of the remaining terms in $\cdots$ contribute these two pieces to Eq. (50). To be more explicit, recall that $\lambda = \lambda(\rho_{\mathrm{CDM}})$ in the model, so

$$\nabla_{a} \lambda = \frac{\partial \lambda(\rho_{\mathrm{CDM}})}{\partial \rho_{\mathrm{CDM}}} \nabla_{a} \rho_{\mathrm{CDM}}$$
$$= \frac{\dot{\lambda}}{\rho_{\mathrm{CDM}}} + \frac{\lambda}{3H} \nabla_{a} \rho_{\mathrm{CDM}}$$
$$= -\frac{\dot{\lambda}}{3H} \nabla_{a} \rho_{\mathrm{CDM}}$$
$$= -\frac{\dot{\lambda}}{3H} \sum_{k} \frac{k}{a} \Delta_{\mathrm{CDM}}^{k} Q_{a}^{k}, \quad (57)$$

where we have used Eq. (10) to background order. As a result,

$$\kappa (\lambda^{\prime \prime} + \lambda^{p^{\prime \prime \mathrm{eff}}}) a^{2}$$
$$= -\frac{\dot{\lambda}}{\lambda H} \Delta_{\mathrm{CDM}}^{\prime \prime} - \frac{\dot{\lambda}}{3\lambda H} k^{2} \Delta_{\mathrm{CDM}}^{\prime \prime} + \cdots,$$
and Eq. (50) can be recast into the form
\[
\left[1 + \frac{\lambda}{2\lambda H}\right] \Delta''_{\text{CDM}} + [\cdots]\Delta'_{\text{CDM}}
+ \left[\cdots + \frac{\lambda}{3(2\lambda H + \lambda)} k^2\right] \Delta_{\text{CDM}} = 0
\]
which, after rearrangement, gives
\[
\Delta''_{\text{CDM}} + [\cdots]\Delta'_{\text{CDM}}
+ \left[\cdots + \frac{\lambda}{3(2\lambda H + \lambda)} k^2\right] \Delta_{\text{CDM}} = 0. \tag{58}
\]
Here, \(\cdots\) denotes complicated terms that are determined completely by the background evolutions of the model, and are unimportant for our analysis here. What is essential in Eq. (58) is that it tells us that, as long as the quantity \(\lambda\) does not vanish, in general there will appear an effective sound-speed-squared term for the growth of matter density perturbations. Depending on the sign of \(\lambda\), this sound-speed-squared term could be either positive or negative, in both cases the small-scale density perturbation growth becomes extremely scale-dependent, altering the shape of the matter power spectra significantly [15, 16, 17]. Notice that the terms in \(\cdots\) could also modify the evolution of density contrasts differently as compared with the prediction in standard general relativity, but in a scale-independent manner, and at small scales their effects are subdominant.

One more comment on the modified Friedmann equation in the Palatini gravity models is appropriate. Using Eq. (11) with \(\rho\) replaced by the \(\rho_{\text{tot}}\) given in Eq. (53), we can see that only the \(\lambda\theta, \lambda^2\) terms are involved and the \(\lambda\) terms cancel (we only consider terms to background order here). Since \(\lambda = \partial\lambda(\rho^f)/\partial \rho^f\partial \rho^f H\) we see that \(\lambda\theta, \lambda^2 \propto H^2\) and could be moved to the left-hand side of Eq. (11); the remaining terms on the right-hand side are also functions of \(\rho^f\) only and so Eq. (59) is also realized. To be more explicit, the modified Friedmann equation in Palatini \(f(R)\) gravity is [16]
\[
\left[H + \frac{\lambda}{2\lambda}\right]^2 = \frac{1}{6\lambda} \left[\kappa(\rho^f + 3p^f) - (R + f)\right]
\]
which can be shown to be just Eq. (38) if \(\omega = 1\) there, as expected, because the metrics then take the same form (of course, the definitions of \(\lambda\) are different in the two cases).

**B. The Case of Palatini \(f(R^{ab}R_{ab})\) Gravity**

Now consider the \(f(R^{ab}R_{ab})\) gravity model. As discussed in the last section, the detailed forms of \(\rho^{fI}, p^{fI}\) are very complicated, but fortunately we need not evaluate the full formulae explicitly. Our experience of the simpler theory described in the last subsection shows that what is most relevant for our analysis are the second order (time and space) derivative terms (note that the term \(\dot{\theta}\), if exists, can also contribute to \(\dot{\Delta}_{\text{CDM}}\) because \(\dot{\nabla}_a \theta\) contains \(\dot{Z}\) where \(Z\) is the Harmonic expansion coefficient of \(\dot{\nabla}_a \theta\) via \(\dot{\nabla}_a \theta = \frac{\lambda}{\lambda H} 2 Q_{ab}^f\), and because \(\xi\Delta_{\text{CDM}} = -k\dot{Z}\); however it turns out no \(\theta, \dot{\theta}\) terms appear in \(\kappa(\rho^{\text{tot}} + 3p^{\text{tot}})\), which are straightforward to identify.

We shall formally repeat the procedure of the last subsection. Note that the quantities \(T_a, \Sigma_{ab}\) are determined by \(p^{fI}_{ab}, \pi^{fI}_{ab}\) which in our case are due to the radiation matter species. At late times the radiation energy density is negligible so that, to a good approximation, \(p^{fI}_{ab}, \pi^{fI}_{ab}\) and thus \(T_a, \Sigma_{ab}\), vanish. As a result, Eqs. (26, 27) become
\[
\begin{align*}
g_{ab} &= \lambda g_{ab} + \lambda(\omega - 1)u_a u_b, \tag{59} \\
g^{ab} &= \frac{1}{\lambda} g^{ab} + \frac{1}{\lambda^2} \left(\frac{1}{\omega} - 1\right) u^a u^b, \tag{60}
\end{align*}
\]
with \(\lambda, \omega\) defined in Eqs. (28, 29). Meanwhile, since \(\rho^f \approx \rho_{\text{CDM}}, \rho^f \equiv 0\) at late times; \(F, \Delta, \Xi, \) and hence \(\lambda\) and \(\omega\), become functions of \(\rho_{\text{CDM}}\) only, i.e., \(\lambda = \lambda(\rho_{\text{CDM}}), \omega = \omega(\rho_{\text{CDM}})\). The analysis in Eq. (57) then also applies to \(\lambda\) and \(\omega\) here, so
\[
\begin{align*}
\dot{\nabla}_a \lambda &= -\frac{\lambda}{3H} \sum_k k^2 \Delta_{\text{CDM}} Q^k_{a}, \\
\dot{\nabla}_a (\lambda \omega) &= \frac{(\lambda \omega)}{3H} \sum_k k^2 \Delta_{\text{CDM}} Q^k_{a}.
\end{align*}
\]
Now, from Eq. (62) we obtain
\[
\kappa(\rho^{\text{tot}} + 3p^{\text{tot}}) = 2\kappa(\rho^{\text{tot}} - 3p^{\text{tot}}) = \kappa(\rho^f + 3p^f) - 2F(\Delta^2 - 3\Xi^2) - f - 2\delta R_{ab} u^a u^b. \tag{61}
\]
where the relation \(R_{ab} = \Delta^2 u_a u_b + 3\Xi h_{ab}\) and Eq. (8) are used; \(\delta R_{ab}\) is given in Eq. (51). Note that \(\delta R\) does not appear in this formula, and what we need to evaluate is just the collection of second-order derivative terms in \(\delta R_{ab} u^a u^b\). After some manipulation we obtain a similar expression to Eq. (56) for the Palatini \(f(R)\) model:
\[
\kappa(\rho^{\text{tot}} + 3p^{\text{tot}}) = \frac{3\lambda}{\lambda} + \frac{1}{\lambda} \dot{\nabla}^2 (\lambda \omega) + \cdots \tag{62}
\]
The following analysis then completely parallels that for the Palatini \(f(R)\) model and the CDM density perturbation growth equation can be shown to be (like Eq. (55))
\[
\Delta''_{\text{CDM}} + [\cdots]\Delta'_{\text{CDM}}
+ \left[\cdots + \frac{(\lambda \omega)}{3(2\lambda H + \lambda)} k^2\right] \Delta_{\text{CDM}} = 0. \tag{63}
\]
Again the \([\cdots]\) in Eq. (63) denotes the terms which are completely determined by the background evolution and
are not of interests to us here. Thus we see that, similar to the case of Palatini $f(R)$ gravity model, a scale-dependent sound-speed-squared term also appears in the Palatini $f(R^a b R^{a b})$ gravity model, whose sign depends on $(\lambda \omega)$. Note that in the case of $\omega = 1$ the metric $g_{a b}$ (Eq. (53)) has the same form as that in the Palatini $f(R)$ model, and Eq. (63) reduces to Eq. (58) as expected.

In the general $f(R^a b R^{a b})$ gravity models it is possible that $(\lambda \omega)' \neq 0$, thus the scale-dependent effective sound-speed-squared term influences the matter perturbation growth and could alter the shape of the predicted matter power spectrum. Our previous knowledge derived from refs. [12,13,17] suggests that this effect might allow observational data to place stringent constraints on the parameter space of these theories. In fact, we can give a rough estimate of how stringent the constraint can be. Consider first Eq. (62) for the Palatini $f(R)$ model: since on small scales the $k^2$ term dominates the other terms in front of $\Delta_{C_D M}$ (the ‘$\cdots$’ terms), it is obvious that the magnitude of the quantity $\lambda/3(2\Lambda H + \lambda)$ determines the deviation from $\Lambda C D M$ results. The observational constraint on the model parameter $\beta$ (recall that $f(R) = \alpha(-R)^{\beta}$, as shown in Refs. [12,13], is $|\beta| < O(10^{-6} \sim 10^{-7})$; we take $|\beta| \sim 10^{-5}$ and $\Omega_m = 0.3$ for illustrative purposes, and find that $|\lambda/3(2\Lambda H + \lambda)| \sim O(10^{-7} - 10^{-6})$ for the relevant redshift range of $0 \leq z \leq 10$. In the analogous case of Palatini $f(R^a b R^{a b})$ gravity, Eq. (63), we have $(\lambda \omega)'/3(2\Lambda H + \lambda)$ as the dominant term instead of $\lambda/3(2\Lambda H + \lambda)$, and so the constraint should be $|\lambda(\omega)'/3(2\Lambda H + \lambda)| < O(10^{-7} - 10^{-6})$ in the same redshift range. Again, taking $\Omega_m = 0.3$, for the theories with $f(R^a b R^{a b}) = \alpha(-R^a b R^{a b})^{\beta}$ we find that to satisfy the above constraint $|\beta|$ must also be limited to be $O(10^{-5})$ or even smaller.

Thus, we conclude that the Palatini $f(R^a b R^{a b})$ models may be constrained by the observational data on the matter power spectrum just as stringently as are the Palatini $f(R)$ models. Yet, we should note that a more exact quantitative constraint can only be obtained by exploiting a full parameter-space search as done in [17], and that the above conclusion depends on the assumption that the CDM particles have vanishing anisotropic stress. If, in contrast, the dark matter particles admit an anisotropic stress, in a manner similar to that prescribed in [27], then the effective sound-speed-squared terms might be canceled and leave no significant traces.

V. DISCUSSION AND CONCLUSIONS

We considered a general class of modified gravity models where the Ricci scalar in the gravitational action of GR is replaced by a function $R + f(R^a b R^{a b})$ and the field equations are derived using the Palatini variational approach, i.e., treating the metric $g_{a b}$ and connection $\Gamma^c_{a b}$ as independent variables so that the action is varied with respect to both of them. The strategy for deriving the cosmological equations at both the background (zero-order) and the first-order perturbation levels is outlined. The main step in this process is to determine the metric, $g_{a b}$, whose Levi-Civita connection is $\Gamma^c_{a b}$, relate it to the physical metric $g_{a b}$ and thereby fix the relation between $R_{a b}$ and $R_{a b}$. Then, the correction to GR is treated as a new effective energy-momentum tensor while the field equations take the same form as in GR. The formulae laid down here might be useful for the numerical implementations of such modified gravity models.

We also investigated in detail a power-law correction to the usual Einstein-Hilbert action given by $f(R^a b R^{a b}) = \alpha(R^a b R^{a b})^{\beta}$. We used the SNIa luminosity distance and CMB shift parameter data to constrain its (independent) two-parameter space ($\beta, \Omega_m$). We found that at 95% confidence level $\beta \in [-0.13, 0.29]$ and $\Omega_m \in [0.20, 0.36]$. A slightly positive value of $\beta$ ($\beta \approx 0.07$) is preferred by the data used. However, the standard $\Lambda C D M$ model (equivalent to $\beta = 0$) with $\Omega_m = 0.27$ is still within the 68% confidence contour. Hence, although the best fit to the data is a null leading-order correction to Einstein gravity, the current data exhibits no significant preference over the concordance $\Lambda C D M$ model of GR.

The late-time growth of matter density perturbations in general Palatini $f(R^a b R^{a b})$ gravity models was also studied. It was shown that the equations governing this class of models look very similar to that in the Palatini $f(R)$ models. In particular, there exists a scale-dependent effective sound-speed-squared term in the perturbation growth equation which may be either positive or negative, depending on the background evolution in both models. In the $f(R)$ case it is well known that these terms can lead to strong scale-dependence of the matter power spectrum, which is highly constrained by observational data [13,16], and we expect a similar feature to exist in the $f(R^a b R^{a b})$ case. We estimate that this will produce a strong observational bound of $|\beta| \lesssim O(10^{-5})$ unless some exotic properties are added to the dark matter candidate [27].

As the final remark, we give a brief comment about the static and spherically-symmetric solutions of the present model. The analogue for the Palatini $f(R)$ model was considered in [30] and the authors found that the exterior spherically-symmetric vacuum solutions are unique. Here we just want to point out the $f(R^a b R^{a b})$ model also shares this feature. In fact, in the vacuum where $p^a = p^b = 0$ it is easy to show that for our model $\Delta = \Xi = 0$ is const. are uniquely determined by Eqs. (14,15) and so is $R_{a b}$. The full consideration of a static system also needs the interior solution and its matching the exterior, which is beyond the scope of this paper and will be left for further investigation.
Appendix

In this Appendix we present the relation between $R_{ab}(\Gamma_{bc}^a)$ and $R_{ab}(g_{ab})$ if the two metrics $g_{ab}$ and $g_{ab}$ satisfy the following relations

$$ g_{ab} = \lambda g_{ab} + \xi_{ab}, \quad (64) $$

$$ g^{ab} = \lambda^{-1} g^{ab} + \xi^{ab}, \quad (65) $$

where $\lambda$ is a scalar function and $\xi_{ab}$, $\zeta_{ab}$ symmetric tensors.

Firstly, the requirement

$$ g^{ac} g_{cb} = g^{ac} g_{cb} = \delta_b^c \quad (66) $$

implies that

$$ \lambda \xi_{ab} + \lambda^{-1} \xi^{ab} + \zeta_{ab} = 0. \quad (67) $$

Then, with some algebra, and using Eq. (67), we can easily show that (here a comma denotes the ordinary derivative)

$$ \Gamma_{bc}^a(g_{ab}) = \frac{1}{2} \left( \frac{1}{\lambda} g^{ad} + \zeta^{ad} \right) \times $$

$$ \left[ \lambda g_{bd} + \xi_{bd}, c \right] + \left( \lambda g_{cd} + \zeta_{cd} \right), b + \left( \lambda g_{bc} + \xi_{bc} \right), d \right] $$

$$ = \Gamma_{bc}^a(g_{ab}) + \gamma_{bc}^a, \quad (68) $$

where the difference between $\Gamma_{bc}^a$ and $\Gamma_{bc}^a$, denoted $\gamma_{bc}^a$, is defined by

$$ \gamma_{bc}^a = \frac{1}{2 \lambda} \left[ \delta_b^c \nabla_c \lambda + \delta_c^b \nabla_b \lambda - g_{bc} \nabla^a \lambda \right] + \frac{1}{2} \left[ \zeta_b^c \nabla_c \lambda + \zeta_c^b \nabla_b \lambda - g_{bc} \zeta^{ad} \nabla_d \lambda \right] + \frac{1}{2} \left[ \nabla_b \xi_c^a + \nabla_c \xi_b^a - \nabla^a \xi_{bc} \right] + \frac{1}{2} \zeta^{ad} \nabla_c \delta_{bd} + \nabla_b \delta_{cd} - \nabla_d \delta_{bc} \right] \quad (69) $$

and is a true tensor, as expected. From the definition in Eq. (2) we can thus derive that

$$ R_{ab} = R_{ab} + \nabla_c \gamma_{ab}^c - \nabla_b \gamma_{ab}^c + \gamma_{ab}^d \gamma_{cd} - \gamma_{ac}^d \gamma_{bd} \quad (70) $$

Note that $R_{ab}$ differs from $R_{ab}$ by a rank-2 symmetric tensor.

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[Here $Q^k_k = \frac{k}{2} \nabla^2 k^2$ and $Q^k$ is the zero order eigenfunction of the comoving Laplacian $a^2 \nabla^2 k^2$ with eigenvalue $k^2$, $k$ representing a specified scale.]

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