Late Afterglow Bump/Plateau around the Jet Break: Signature of a Free-to-shocked Wind Environment in Gamma-Ray Burst

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Abstract

A number of gamma-ray bursts (GRBs) exhibit the simultaneous bumps in their optical and X-ray afterglows around the jet break. These bumps are similar to the afterglows of GRB 170817A, except preceded by a long shallow decay. Its origin is unclear. We suggest that these late simultaneous bumps may sound a transition of circumburst environment from a free-wind medium to a constant density medium, e.g., the shocked-wind medium. In this paper, we study the emission of an external-forward shock propagating in a free-to-shocked wind environment at different viewing angles. The late simultaneous bumps/plateaux followed by a steep decay are found in the optical and X-ray afterglows for high-viewing-angle observers. In addition, these theoretical bumps are preceded by a long plateau or shallow decay, which is formed during the external-forward shock propagating in the free-wind environment. For low-viewing-angle observers, the above bumps also appear but only in the situation where the structured jet has a low characteristic angle and the deceleration radius of the in-core jet flow is at around or beyond the free-wind boundary. We search GRBs for afterglows with the late simultaneous optical and X-ray bumps followed by a steep decay. GRBs 120326A, 100901A, 100814A, and 120404A are obtained. We find that an off-core (in-core) observed external-forward shock in a free-to-shocked wind environment can well explain the optical and X-ray afterglows in GRBs 120326A, 100901A, and 100814A (GRB 120404A).

Unified Astronomy Thesaurus concepts: Gamma-ray bursts (629); Interstellar medium (847); Interstellar medium wind (848)

1. Introduction

Gamma-ray bursts (GRBs) are the most luminous sources of electromagnetic radiation known in the universe. Observationally, GRBs generally appear as a brief and intense γ-rays followed by a long-lived afterglow emission. With Swift satellite observations, the diversity of light curves in X-ray afterglows have been revealed (Gehrels et al. 2004; Burrows et al. 2005b). However, a canonical light curve consisting of four power-law segments and a flaring component has been suggested, i.e., an initial steep decay with a typical slope $\sim-3$, a shallow decay with a typical slope $\sim-0.5$, a normal decay with a typical slope $\sim-1.2$, a late steep decay with a typical slope $\sim-2$, and one or several X-ray flares (Nousek et al. 2006; O’Brien et al. 2006; Zhang et al. 2006, 2007). The initial steep decay component is believed to be the tail of the prompt emission due to the curvature effect of the high-latitude emission (Barthelmy et al. 2005; Liang et al. 2006; O’Brien et al. 2006; Lin et al. 2017a, 2017b). Shallow, normal, and late steep decays are always attributed to the external shock (Nousek et al. 2006; Panaitescu et al. 2006; Zhang et al. 2006), which is developed when a relativistic jet propagates into the circumburst medium. If there is no energy injection into the external shock, a normal decay phase would appear in the afterglows. However, the decay may become shallow during the phase with continuous energy injection into the external shock. The late steep decay corresponds to the emission of the external shock after the jet core is fully visible. Correspondingly, the beginning of the transition into the late steep decay is referred to as jet break⁵ (e.g., Wang et al. 2018; Zhang 2018; Lamb et al. 2021). The X-ray flares generally show a sharp rise with a steep decay and thus could not be produced in the external-forward shock (e.g., Falcone et al. 2006; Romano et al. 2006). It is suggested that most X-ray flares have the same physical origin as prompt γ-rays (Burrows et al. 2005a; Dai et al. 2006; Falcone et al. 2006, 2007; Liang et al. 2006; Nousek et al. 2006; Zhang et al. 2006; Chincarini et al. 2007, 2010; Wang & Dai 2013; Wu et al. 2013; Hou et al. 2014; Yi et al. 2015, 2016; Mu et al. 2016a, 2016b; Lyu et al. 2020).

Multiwavelength observations of afterglows have also revealed some puzzling features, which are beyond expectations from the simple standard external shock scenario (Panaitescu et al. 2006; Panaitescu & Vestrand 2011; Li et al. 2012; Liang et al. 2013; Wang et al. 2015). For example, some GRBs exhibit late simultaneous bumps in their optical and X-ray afterglows around the jet break. In addition, their optical afterglows seem to feature a long plateau/shallow decay before the late bumps. An exemplar of these afterglows is GRB 120326A, which exhibits simultaneous bumps in its optical and X-ray afterglows at around 35 ks after the burst trigger (Melandri et al. 2014; Urata et al. 2014). We note that these late simultaneous bumps are all directly followed by a steep decay, which is very different from the decay of the phase preceding the late simultaneous bumps and thus is an intriguing phenomenon. Models explaining the late simultaneous bumps should also explain the steep decay following the bumps. Actually, this feature is strongly reminiscent of the afterglows in GRB 170817A, which is formed in an off-core observed external-forward shock propagating in an interstellar medium (ISM; Troja et al. 2018, 2019; Huang et al. 2019; Lamb et al. 2019; Ren et al. 2020), of which the mass density is constant. The observations of GRB 170817A reveal an achromatic bump followed by a steep decay, which is very similar to GRBs with the late bumps mentioned above. This may suggest that some of these bumps may be formed in an off-core observed external-forward shock in a homogeneous medium.

⁵ For the situation with a structured jet, especially for an off-core observer, the terminology of jet break becomes confusing. In this paper, jet break is used to represent the beginning of achromatic steepening in GRBs’ afterglows.
When a relativistic jet propagates in the circumburst medium of a GRB, an external shock develops and thus produces a long-term broadband afterglow emission (Sari et al. 1998; Mészáros & Rees 1999; Sari & Piran 1999a, 1999b). To describe the jet structure and the dynamics of the external-forward shock, we introduce a spherical coordinate \((R, \theta, \varphi)\) with \(R = 0\) locating at the burst’s central engine and \(\theta = 0\) along the jet axis. The jet flow moving at the direction of \((\theta, \varphi)\) is represented by \((\theta, \varphi)\) jet. We assume the observer location at the direction of \((\theta, \varphi)\) with \(\varphi = 0\) and \(\theta < \pi/2\).

In our calculations, the jet moving toward us\(^4\) is divided into \(I \times L\) small patches along the \(\theta\) and \(\varphi\) directions in their linear space, i.e., \([0, \theta_0]\), \([\theta_0, 2\theta_0]\), \([2\theta_0, 3\theta_0]\), \(\ldots\), \([l - 1\theta_0, l\theta_0]\) with \(\theta_0 = \theta/\Omega_1 + [0, \delta_\theta]\), \([\delta_\varphi, 2\delta_\varphi]\), \([2\delta_\varphi, 3\delta_\varphi]\), \(\ldots\), \([L - 1\delta_\varphi, L\delta_\varphi]\) with \(\delta_\varphi = \pi/L\). Here, \(\theta_0\) is the opening angle of the jet.

The dynamics and the emission of the external-forward shock is estimated independently in each patch. In addition, the following assumptions are adopted in estimating the dynamics and radiation of the external-forward shock: (1) The sideways expansion are ignored and the dynamics at each patch are assumed to be independent of other patches. The amount of sideways expansion is a debated topic, with numerical simulations suggesting a limited amount of spreading (e.g., Granot et al. 2001; Cannizzo et al. 2004; Zhang & MacFadyen 2009; van Eerten et al. 2010; van Eerten & MacFadyen 2012). In addition, it is difficult to consider the sideways expansion in the context of a structured jet and the related afterglow fittings. (2) The microphysical parameters for synchrotron emission, e.g., \(e_c\) and \(e_B\), are set as constants, where \(e_c\) and \(e_B\) are the fractions of the shock energy used to accelerate electrons and contributing to the magnetic energy, respectively. (3) The energy injection in the external shock is not considered in our work.

\(^4\) The central engine of a GRB usually launches a pair of outflows, i.e., a near jet moving toward us and a counter jet moving away from us. In general, the emission from the counter jet is negligible compared with that from the near jet. Therefore, we only consider the emission from the near jet.
\[ \frac{d\Gamma(\theta, R)}{dR} = -\frac{\Gamma^2 - 1}{M_0(\theta) + \epsilon m + 2(1 - \epsilon) \Gamma m} \frac{dm(R)}{dR}, \]

where \( M_0(\theta) = E_{51} \cos(\theta) / (4\pi \Gamma \rho \rho_0^2) \) with \( c \) the light speed, and \( m \) is the swept-up mass of the external shock from \( R_0 = 10^{13} \text{ cm} \) to \( R \) per solid angle and can be estimated with

\[ \frac{dm(R)}{dR} = \rho R^2. \]

In Equation (3), \( \epsilon(\theta, R) = \epsilon_0 \cdot \min[1, (\gamma'_{c,\text{max}}/\gamma'_{c,\text{in}})^{p-2}] \) is the radiation efficiency of the external-forward shock, where \( \gamma'_{c,\text{in}} = \epsilon_0 (p - 2) m_e \Gamma / (1 - p) m_e \) is the minimum Lorentz factor of the electrons accelerated in the shock, \( \gamma'_{c,\text{max}} = 6\pi m_e c / (\sigma_T \Gamma^2 B_0^2 \rho_0) \) is the efficient cooling Lorentz factor of electrons (Sari et al. 1998). Besides, \( B' \theta(\theta, R) = (2\pi \rho_0 B_0 \Gamma^2 \Gamma c \) is the magnetic field behind the shock, \( t_{\text{obs}}(\theta, R) = \int_{R_0}^{R} (c - v) dr / c \) with \( v(\theta, R) = c\beta \) and \( \beta(\theta, R) = \sqrt{1 - 1/\Gamma^2} \), and \( m_e \), \( p \), and \( \sigma_T \) are the electron mass, the spectral index of accelerated electrons in the shock, and the Thomson cross section, respectively.

With Equations (1)–(4), one can estimate the value of \( \Gamma \) at different \( R \). Then, the observed time for a photon from the \( (\theta, \varphi) \) jet can be estimated with

\[ t_{\text{obs}}(\theta, \varphi, \theta_0, R) = (1 + z) \times \left[ \frac{R_0}{2\Gamma' c^2} + \frac{R_{\text{last}}(1 - \cos \Theta)}{c} \right], \]

where \( \cos \Theta = \cos(\theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \cdot \|(\theta_0, \varphi_0) \), \( \cos \theta_0 = \sin \theta \cos \varphi \sin \theta_0 \), and \( \Theta \) is the angle between the direction of \( (\theta, \varphi) \) and the line of sight, i.e., \( (\theta_0, \varphi_0) \) with \( \varphi_0 = 0 \). In Equation (5), the sum of the first two terms on the right-hand side of the square bracket is the arrival time of photons for an observer in the direction of \( (\theta_0, \varphi_0) \). Then, if the observer is in the direction of \( (\theta_0, \varphi_0) \) with \( \varphi_0 = 0 \), the last term should be added. For a given observer time \( t_{\text{obs}} \), one can obtain the corresponding value of \( R = R_{\text{obs}}(\theta, \varphi, \theta_0) \) based on Equation (5). The value of \( R_{\text{obs}} \) is the location of the external-forward shock for the \( (\theta, \varphi) \) jet observed at \( t_{\text{obs}} \) and is used to calculate the observed flux from the \( (\theta, \varphi) \) jet.

**Evolution of the electron energy spectrum:** In the X-ray and optical bands, the main radiation mechanism of the external-forward shock in GRBs is the synchrotron radiation of the swept-up electrons (Sari et al. 1998; Sari & Piran 1999a). We denote the instantaneous electron spectrum for the swept-up electrons per solid angle in the \( (\theta, \varphi) \) jet as \( n'_e(\gamma'_{c,\text{in}} \theta, R) \), where \( n'_e \) is the number of electrons in \( [\gamma'_{c,\text{in}}, \gamma'_{c,\text{max}} + d\gamma'_{c,\text{in}}] \) with \( \gamma'_{c,\text{in}} \) the Lorentz factor of the electrons. The evolution of \( n'_e \) can be described as

\[ \frac{dn'_e}{dt'} + \frac{\partial}{\partial \gamma'_{c,\text{in}}} (\gamma'_{c,\text{in}} n'_e) = Q', \]

where \( \gamma'_{c,\text{in}} \) is the cooling rate of the electrons and \( Q' \) is the injection rate of the electrons from the shock, i.e.,

\[ \gamma'_{c,\text{in}}(\gamma'_{c,\text{in}} \theta, R) \equiv \frac{d\gamma'_{c,\text{in}}}{dt'} = -\frac{\sigma_T \gamma'_{c,\text{in}} B'^2}{6\pi m_e c} - \frac{2 \gamma'_{c,\text{in}}}{3 R} \frac{dR}{dt'}. \]

\[ Q'(\gamma'_{c,\text{in}} \theta, R) = \begin{cases} Q_0 \gamma'_{c,\text{in}}^{p-2}, & \gamma'_{c,\text{in}} \leq \gamma'_{c,\text{in}} \leq \gamma'_{c,\text{in},\text{max}}. \\ 0, & \text{others}. \end{cases} \]

Here, \( p > 2 \) is the power-law index, \( \gamma'_{c,\text{in},\text{max}} = \sqrt{9m_e c^2 / (8B_0^2 \rho_0)} \) with \( q_c \) the electron charge (e.g., Kumar et al. 2012). \( Q_0 \) is obtained by solving \( \int_{\gamma'_{c,\text{in},\text{min}}}^{\gamma'_{c,\text{in},\text{max}}} Q'd\gamma'_{c,\text{in}} = \int_{R_0}^{R} (\rho R / m_0) dR / dt' \) with \( dR / dt' = c \Gamma \), and \( t'(\theta, R) \) is the time elapsed in the comoving frame of the blast wave. The first and second terms in the right-hand side of Equation (7) are respectively the synchrotron radiative cooling and adiabatic cooling of the electrons, and the inverse-Compton cooling is not considered here. We solve Equation (6) and \( dR / dt' = c \Gamma \) for \( n'_e(\gamma'_{c,\text{in}} \theta, R) \) at different \( R \). In our calculations, the fourth-order Runge-Kutta method is used. In addition, an appropriate time step \( \Delta t' < \min \{ \Delta \gamma'_e / \gamma'_{c,\text{in}} \} \) is adopted in our calculations, where \( \Delta \gamma'_e \) is the width of our adopted electron energy grids (e.g., see appendix A of Geng et al. 2018).

**Observed flux calculation:** With the obtained \( n'_e(\gamma'_{c,\text{in}} \theta, R) \), the spectral power of synchrotron radiation at a given frequency \( \nu' \) can be described as

\[ P'(\nu', \theta, R) = \frac{\sqrt{3} q_c^3 B'^2}{m_e c^2} \int_0^{\gamma'_{c,\text{in}}} \nu'^{5/3} \left[ \frac{1 + \nu dR}{\nu dR} \right] F(\nu') \nu'^{5/3} \nu' d\gamma'_{c,\text{in}}. \]

where \( F(\nu) = x \int_{\nu_{\text{in}}}^{\nu_{\text{out}}} K_{5/3}(k) dk \) with \( K_{5/3}(k) \) the modified Bessel function of 5/3 order and \( \nu'^{5/3} \approx 3q_c B'(\theta, R) \gamma'^{2/3} / (4\pi m_e c) \). Then, the observed flux density \( f_o(\theta, R) \) from the per solid angle of the jet flow in the direction of \( (\theta, \varphi) \) is

\[ f_o(\theta, \varphi, R) = (1 + z) P'(\nu, \theta, R) \left[ \frac{1 + \nu dR}{\nu dR} \right] D^3 \frac{1}{4\pi d^2}, \]

where \( \nu \) is the observed photon frequency, \( d_L \) is the luminosity distance at the cosmological redshift \( z \), and \( D(\theta, \varphi, R, R) = 1 / [\Gamma(1 - \cos \Theta)] \) is the Doppler factor of the \( (\theta, \varphi) \) jet relative to the observer. Then, the observed total flux density \( f_o \) at \( t_{\text{obs}} \) is

\[ f_o(t_{\text{obs}}) = \frac{1}{L} \sum_{i=1}^{L} f_o(\theta_i, \varphi_i, R_{\text{obs}}) \sin \theta_i d\theta_i d\varphi, \]

where \( \theta_i = (i - 0.5) \Delta \theta \) and \( \varphi_i = (i - 0.5) \Delta \varphi \). The observed flux \( F_{\text{XRT}} \) in the X-ray band (0.3–10 keV) of the X-ray Telescope (XRT) on board the Swift satellite is calculated with \( F_{\text{XRT}} = \int_{h_{\text{in}}}^{h_{\text{out}}} f_o(t_{\text{obs}}) dh \), where \( h_{\text{in}} = 10 \text{ keV} \) and \( h_{\text{out}} = 0.3 \text{ keV} \).

**3. Theoretical Afterglows and Analysis**

### 3.1. Numerical Results

We mainly focused on the afterglows of GRBs in a free-to-shocked wind circumburst environment, especially for those with late simultaneous bumps followed by a steep decay and preceded by a shallow decay.
In Figure 1, we show the light curves (solid lines) of the radiation from the external-forward shock propagating in a free-to-shocked wind circumburst environment, where $E_{k,\text{iso, on}} = 10^{52}$ erg, $\Gamma_0 = 250$, $\theta_c = 5^\circ$, $\theta_{\text{jet}} = 8\theta_c$, $p = 2.2$, $\epsilon_e = 0.1$, $\epsilon_B = 10^{-3}$, $A_\star = 0.01$, $R_\text{tr} = 10^{17}$ cm, and $z = 1$ are adopted to calculate the dynamics and emission of the external-forward shock. It can easily be found that the late simultaneous bumps/plateaux followed by a steep decay appear in the optical and X-ray afterglows for high-viewing-angle observers (e.g., $\theta_v \lesssim 4\theta_c$). In addition, these simultaneous bumps are preceded by a shallow decay, which is very similar to the late simultaneous bumps found in some bursts, e.g., GRB 120326A. In Figure 2, we also show the optical afterglows (solid lines) of the external-forward shock for a jet with $\theta_v = \theta_c$, where the circumburst environment and the values of other parameters are the same as those in Figure 1. By comparing the light curves shown in Figures 1 and 2, one can conclude that the opening angle of the jet may affect the light curves before the late bump/plateau. The late simultaneous bumps/plateaux, which are both directly followed by a steep decay and preceded by a shallow decay, presented in the afterglows for high-viewing-angle observers are the key finding of this work.

For the jet adopted in Figures 1 and 2, the deceleration radius of the in-core jet flow is around $10^{14}$ cm in the free-wind medium. One can conclude that there are no broad late bumps in the afterglows for in-core observers. We would like to point out that if the deceleration radius of the in-core jet flow lies...
The jet may affect the light curves before the late simultaneous bumps afterglows driven by a top-hat jet. One can conclude that the opening angle of emission of the external-forward shock are the same as those in Figure 1 but θ preceded by a long plateau or shallow decay can appear in the Optical light curves set beyond the free-wind boundary in Figure 3, where a forward shock in a free-to-shocked/free-to-shocked wind environment. In Figure 3, late broad bumps preceded by a long plateau or shallow decay indeed appear in the afterglows for low-viewing-angle observers. In addition, the late broad bump is followed by a normal/steep decay if a structured jet with high/low characteristic angle θc (e.g., θc = 5°/θc = 0.5°) is adopted.

Then, we can conclude that the late simultaneous broad bumps directly followed by a steep decay and preceded by a long plateau/shallow decay may sound the external-forward shock propagating in a free-to-shocked wind environment. In addition, the appearance of such feature requires that the observer have a high-viewing angle with respect to the jet axis, or, the structured jet should have a low characteristic angle with the deceleration radius of the in-core jet flow at around or beyond the free-wind boundary.

3.2. Understanding the Appearance of the Late Bump/Plateau

In Figures 1 and 2, we also plot the afterglows formed in a pure free-wind medium with ρ = 5 × 10^3 A_ν R^{-k} g cm^{-3} (dotted lines) and those formed in a pure shocked-wind medium with ρ = n_0 m_p cm^{-3} (dashed lines) for some viewing angles. Hereafter, the afterglow formed in a free/shocked/free-to-shocked wind environment is represented by free/shocked/free-to-shocked-wind afterglow, and the symbols and lines with the same color in the same figure correspond to the situation with the same viewing angle. One can conclude that the afterglow formed in a free-to-shocked wind environment can be approximately estimated by superposition of the free-wind afterglow and the shocked-wind afterglow from the same viewing angle. Then, the appearance of a free-to-shocked-wind afterglow can be understood based on the appearance of the corresponding free-wind afterglow and the corresponding shocked-wind afterglow.

First, we offer analytical insight into the appearance of the free/shocked-wind afterglows shown in Figure 2. Here, a Gaussian structured jet with θ_{jet} = θc can be approximated to a top-hat jet with the same opening angle. Table 2 in Appendix presents the relations between the analytical temporal index α and the spectral index β with f_{\nu}(t_{\text{obs}}) \propto t_{\text{obs}}^{-\alpha} \nu^{-\beta} for free/shocked-wind afterglows driven by a top-hat jet. According to Table 2, we show the analytical temporal index α and the corresponding light curves (short dashed lines) for the free/shocked-wind afterglows with θ_{c} = 4θc in Figure 2. Here, the light curves are decomposed into different segment based on the situation (i.e., I, II, III, or IV listed in Table 2) and spectral regime, which are shown below the corresponding segment. One can conclude that the analytical light curves generally describe the corresponding numerical results.

Theoretically, the off-core observed afterglow shown in Figure 2 is related to Situations I, III, and IV in Table 2, which lead to the appearance of a late bump both followed by a steep decay and preceded by a shallow rise/decay. The reasons are as follows: (1) Situation IV describes the behavior of the afterglow in its late phase (post-core phase) in which the jet core is fully visible to the observer. Based on Table 2, the post-core phase generally appears as a steep decay. (2) Except for four cases in the free-wind medium (shaded in pink in Table 2), the temporal indexes α for the phase before the post-core phase are less than 0, leading to a rising pattern of the corresponding light-curve segment. (3) In Situation I or III, the absolute value of α in the free-wind medium is less than that in the shocked-wind medium for the same spectral regime. The combination of the behaviors of (1)–(3) results in that for high-viewing-angle observers, i.e., the free-to-shocked-wind afterglow appears as a shallow rise/decay in its early phase, followed by a fast rise, and then joined by a steep decay. Besides, Situation I shows that if the in-core jet flow is decelerated at around or beyond the free-wind boundary, a bump preceded by a shallow rise/decay also appears in the afterglows for low-viewing-angle observers, e.g., afterglows with θ_{c} \leq 3θc as shown in Figure 3.

Second, we discuss the appearance of the free-to-shocked-wind afterglows shown in Figure 1. Intuitively, the afterglows driven by a structured jet can be decomposed into the emission from the jet core and envelopes. In Figure 4, the light curves of the emission from the jet core (i.e., θ \leq θc) and the envelopes with θ ∈ (θc, 2θc), θ ∈ (2θc, 3θc), and θ ∈ (3θc, 4θc) are shown, where the free-to-shocked-wind afterglow observed at θ_{c} = 4θc is taken as an example. One can generally find the following two behaviors:5 the emission from both the jet core and the envelope with θ ∈ (θc, 2θc) dominates the afterglow in the post-core phase, and the emission of the jet flow close to the line of sight dominates the early phase. These two behaviors may lead to the appearance of a late bump in the afterglows for high-viewing-angle observers if the shocked-wind medium is involved. The reasons are as follows: For off-core observed afterglows driven by the jet core, Situations I and III in Table 2

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5 These two behaviors are associated with the following three facts: (1) the external-forward shock is relativistic in its early phase, (2) almost all of the jet is visible for the observer in the post-core phase, and (3) the total kinetic energy of the jet is dominated by that of both the jet core and the envelope with θ ∈ (θc, 2θc).
reveal that the light curve of the shocked-wind afterglow may appear as a narrow bump with a fast rise ($\alpha \approx -4$), and the light curve of the free-wind afterglow may appear as an extremely wide bump with a shallow rise ($\alpha \gtrsim -2$). This is also applied to the afterglows driven by the envelope with $\theta_c(\theta_{c,2}, \theta_c)$. With the contribution of the emission from the jet flow ($E_{k,iso} \ll E_{k,iso, on}$) close to the line of sight, the shallow rise in the free-wind afterglows driven by the jet flow with $\theta \in (0, 2\theta_c)$ may disappear but the fast rise in the shocked-wind afterglows may stand, e.g., the late bump found in the shocked-wind afterglows with $\theta_c = 4\theta_c$ as shown in Figure 1. In addition, the transition from a free-wind medium to a shocked-wind medium may result in the shocked-wind afterglow dominating the late phase of the free-to-shocked-wind afterglows. Then a bump followed by a steep decay may appear in the free-to-shocked-wind afterglow if the shocked-wind medium is involved.

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**Figure 3.** X-ray (solid lines) and optical (dashed lines) light curves of the external-forward shock in a free-to-shocked wind environment for different viewing angle $\theta_v$, where a structured jet with $E_{k,iso, on} = 10^{55}$ erg, $\Gamma_0 = 100$, and $\theta_c = 5^\circ$ (top panel) or $\theta_c = 0.5^\circ$ (bottom panel) are adopted and the values of other parameters are the same as those in Figure 1. For the given structured jet, the deceleration radius of the in-core jet flow is beyond the free-wind boundary. It is found that late broad bumps preceded by a long plateau or shallow decay indeed appear in the afterglows for low-viewing-angle observers. In addition, the late broad bump is followed by a normal/steep decay if a structured jet with high/low characteristic angle $\theta_c$ (e.g., $\theta_c = 5^\circ/\theta_c = 0.5^\circ$) is adopted.

3.3. Characteristic Transition Time

The free-to-shocked-wind afterglows have two characteristic transitions: the transition from the free-wind phase to the shocked-wind phase, referred to as the first transition; the transition from the shocked-wind phase to the post-core phase, referred to as the second transition. Here, the light curves of free-to-shocked-wind afterglows are decomposed into three phases: the free-wind phase, shocked-wind phase, and post-core phase, where the free-wind phase (shocked-wind phase) is formed during the external-forward shock propagating in the free-wind (shocked-wind) medium.

The first transition may be related to the observed time for the jet flow crossing $R_u$. This can be found by comparing the light curves shown in Figure 4 for the situations with $\xi = 4$ (solid lines), $\xi = 16$ (dashed lines), and $\xi = 64$ (dotted lines), where $R_u$ is changed to obtain different $\xi = \rho_{shocked-wind}/\rho_{free-wind}(R_u)$ by keeping the same $A_u$ and $n_0$ in different situations. Figure 4 shows that when a higher value of $R_u$ is adopted, the first transition appears later. In Figure 1, we plot the observation...
times of the \((\theta_v, \varphi_v)\) jet and \((0, 0)\) jet\(^7\) crossing \(R_{\text{L}}\) with \(\circ\) and \(\bullet\) symbols, respectively. One can conclude that for low-viewing-angle observers, the first transition occurs during the \((\theta_v, \varphi_v)\) jet crossing \(R_{\text{L}}\). For high-viewing-angle observers, however, the time for the \((\theta_v, \varphi_v)\) jet crossing \(R_{\text{L}}\) is significantly larger than the first transition time. This implies that the free-wind phase is shaped by the emission of the jet flow at around \((\theta, \varphi) = (\theta_v, \varphi_v)\) only in the situations with a low-viewing angle. Figure 1 also shows that the observed time of the \((0, 0)\) jet crossing \(R_{\text{L}}\) (i.e., the \(\circ\) symbols) is generally larger than the first transition time. Then, we can find that the first transition is neither associated with the dynamics of the \((\theta_v, \varphi_v)\) jet nor the \((0, 0)\) jet for the high-viewing-angle situations. Figure 4 shows that the first transition occurs later by decreasing \(\theta_{\text{jet}}\). Then, one can find that the first transition should be related to the dynamics of the jet flow being between the \((\theta_v, \varphi_v)\) and \((0, 0)\) directions for a high-viewing-angle observer. We have found that the observed times of the \((2\theta_v, \varphi_v)\) jet and \((3\theta_v, \varphi_v)\) jet crossing \(R_{\text{L}}\) are at around the first transition for the situations with \(\theta_v = 4\theta_c\) and \(\theta_v = 8\theta_c\), respectively.

The second transition from the shocked-wind phase to the post-core phase may be related to the following three parts of the jet observed times:\(^8\) (i) the near core edge, i.e., the \((\theta_L, \varphi_L)\) jet begins to be visible to the observer, i.e., \(\Gamma(\theta_L, R)\sin(\theta_L) \approx 1\); (ii) the jet axis, i.e., \((0, 0)\) jet, begins to be visible to the observer, i.e., \(\Gamma(0, R)\sin(\theta_L) \approx 1\); (iii) the far core edge, i.e., \((\theta_v, \varphi_v + \pi)\) jet, begins to be visible to the observer, i.e., \(\Gamma(\theta_v, R)\sin(\theta_L + \theta_L) \approx 1\). It is worth pointing out that when the far core edge becomes visible, the afterglow fully enters the post-core phase and appears as a steep decay. As shown in Figures 1 and 2, the observation times of (i), (ii), and (iii) are indicated with \(\triangleright\), \(\triangleleft\), and \(\circ\) symbols, respectively. It can be found that the second transition begins at around \(\triangleright\) and ends at \(\triangleleft\) for an observer with a low-viewing angle. In addition, the \(\triangleright\) symbols are generally around the end of the plateau or the peak of the bump if the late bump/plateau appears in the afterglows. For high-viewing-angle observers, the shocked-wind phase generally appears as a plateau or fast rise. Then, the second transition begins at the end of the plateau or fast rise, i.e., the observation time of the \(\triangleright\) symbols for high-viewing-angle observers. In summary, the second transition begins at around \(\triangleright\) and ends at \(\triangleleft\).

4. Case Study

With Swift satellite observations and the optical data reported in the literature, we search GRBs for afterglows with the late simultaneous X-ray and optical bumps directly followed by a steep decay. Only four bursts, i.e., GRBs 120326A, 100901A, 100814A, and 120404A, are obtained. Then, we perform the fitting on the X-ray and optical afterglows of these bursts. The light curves from prompt emission to the late afterglow of these bursts are shown in Figure 5, where the late bumps directly followed by a steep decay and preceded by a plateau/shallow decay can be easily found in these light curves.

1. GRB 120326A is a long GRB at redshift \(z = 1.798\) with unusual X-ray and optical afterglows. The late simultaneous bumps in the optical and X-ray afterglows are at around \(t_{\text{obs}} \sim 4 \times 10^5\) s (Melandri et al. 2014; Uragawa et al. 2014). Some authors have proposed that the energy injection model may be responsible for this late bump (e.g., Hou et al. 2014; Melandri et al. 2014; Laskar et al. 2015). Since the observed bumps are very similar to those shown in Figures 1–3, we would like to model the late simultaneous bumps based on an external-forward shock in a free-to-shocked wind environment. The XRT data at \(t_{\text{obs}} \gtrsim 3 \times 10^5\) s and the optical data at \(t_{\text{obs}} \gtrsim 10^6\) s are used in our fitting.

2. GRB 100901A, with redshift \(z = 1.408\), is a long GRB with a significant rebrightening in the optical and X-ray bands at \(t_{\text{obs}} \sim 3 \times 10^3\) s (Gorbovskoy et al. 2012; Laskar et al. 2015). Laskar et al. (2015) proposed that the energy injection model may be responsible for this late bump. Gorbovskoy et al. (2012) proposed that the bump may be caused by a two-step collapse in the long engine activity. We note that these late bumps are all directly followed by a steep decay, which is very similar to those shown in Figures 1–3. Then, we would like to model these two late simultaneous bumps based on an external-forward shock in the free-to-shocked wind environment. In our fitting, the XRT and the optical data at \(t_{\text{obs}} > 10^3\) s are used.

3. GRB 100814A is a long GRB with redshift \(z = 1.44\). The optical rebrightening appears at \(10^3\) s. De Pasquale et al. (2015) presented broadband observations of GRB 100814A and they attributed the late optical rebrightening to a long-lived external-reverse shock and external-forward shock.

---

\(^7\) The \((0, 0)\) jet represents the jet flow along the jet axis. For a high-viewing-angle observer, the observed time of the \((0, 0)\) jet crossing \(R_{\text{L}}\) can be analytically estimated with Equation \(t_{\text{obs}}(0, 0, \theta_v, R_{\text{L}}) = (1 + z_0(1 - \cos \theta_v))/R_{\text{L}}/c_2\), which is consistent with that obtained based on the numerical method (i.e., \(\circ\) symbols in Figure 1).

\(^8\) If \(\Gamma_0 \gg 1/\sin(\theta_L - \theta_L)\) is satisfied, the observation times of (i), (ii), and (iii) can be analytically estimated with \([\sin(\theta_L - \theta_L)]^{-3/2} \Gamma_0^{-3/2} \Phi_{\text{dec}}\), \([\sin(\theta_L + \theta_L)]^{-3/2} \Gamma_0^{-3/2} \Phi_{\text{dec}}\), and \([\sin(\theta_L + \theta_L)]^{-3/2} \Gamma_0^{-3/2} \Phi_{\text{dec}}\), respectively. The analytical results of the above three observation times are consistent with those estimated based on the numerical method, i.e., the values shown with \(\triangleright\), \(\triangleleft\), and \(\circ\) symbols in Figures 1 and 2.
Nardini et al. (2014) attributed the late optical rebrightening to the late-time activity of the central engine in the observed afterglow emission. Yu et al. (2015) invoked a magnetar with spin evolution to explain the afterglow emission. Besides, Geng et al. (2016) used an ultrarelativistic $e^+e^-$ wind injection model to explain the late rebrightening. It should be noted that the optical/X-ray afterglows all turn into a steep decay after the peak time of the optical bump. In addition, there seems to be a dip in the X-ray afterglow at $t_{\text{obs}} \sim 6 \times 10^4$ s, which is shown in the inset of the bottom left panel of Figure 5. Then, we find that there may be simultaneous late bumps in the optical and X-ray afterglows and the rise of the X-ray bump is covered by the late central engine activities. The XRT data at $t_{\text{obs}} \gtrsim 9 \times 10^4$ s and the optical data at $t_{\text{obs}} \gtrsim 10^5$ s are taken in our fitting.

4. GRB 120404A, with redshift $z = 2.876$, is a long GRB with a significant rebrightening in the optical and near-infrared bands at $t_{\text{obs}} \sim 2 \times 10^3$ s (Guidorzi et al. 2014; Laskar et al. 2015). The X-ray observations around this time also show a bump, although the data are sparse owing to the orbital gap of Swift (Laskar et al. 2015). They proposed that the energy injection model may be responsible for this late bump. Interestingly, the bumps are all directly followed by a steep decay and very similar to those shown in Figures 1–3. Then, the XRT data at $t_{\text{obs}} > 550$ s and the optical data at $t_{\text{obs}} \gtrsim 200$ s are also modeled as the emission from an external-forward shock in a free-to-shocked wind environment.

Our fitting is performed based on the Markov Chain Monte Carlo (MCMC) method to produce posterior predictions for the model parameters. The MCMC method is widely used in finding the best set of parameters for a specified model, e.g., GRB 080413B (Geng et al. 2016); GRBs 100418A, 100901A, 100814A, 120326A, and 120404A (Laskar et al. 2015). In our work, fitting the afterglows of the four bursts with the MCMC method is to test whether or not the late simultaneous bumps followed by a steep decay can be explained with an external-forward shock in a free-to-shocked wind environment. The posterior probability density functions for the physical parameters, i.e., $E_{\text{iso,cm}}, \Gamma_0, \theta_0, \theta_p, \theta_\gamma, p, \epsilon_r, \epsilon_b, A_{\text{iso}},$ and $R_{\text{tr}}$, are presented in Figure 6, where only the fitting result of GRB 120326A is shown as an example. The optimal result from MCMC fitting is shown in Figure 5 with the blue line (XRT) and red line (optical), and the obtained parameters at the $1\sigma$ confidence level are reported in Table 1, where the values of the transition radius $R_{\text{tr}}$ (i.e., $1.05 \times 10^{17}$, $1.91 \times 10^{17}$, $3.98 \times 10^{17}$, and $6.31 \times 10^{16}$ cm for GRBs 120326A, 100901A, 100814A, and 120404A, respectively) are consistent with those found in other bursts (e.g., Ramirez-Ruiz et al. 2001; Kong et al. 2010; Feng & Dai 2011; Li et al. 2020). It can be found that both the X-ray afterglow and the optical afterglow of these bumps can be well modeled with an external-forward shock in a free-to-shocked wind environment.

1. The theoretical light curves do not well fit the transition behavior from the free-wind phase to the shocked-wind phase in the afterglows of GRB 120326A. This may imply that the density jump factor $\xi$ from the free-wind medium to the shocked-wind medium may be less than 4,
i.e., $\xi < 4$. Figure 1 reveals that the external-forward shock propagating in a shocked-wind environment can yield a plateau before the late simultaneous bumps in the case of $\theta_v = 4\theta_c$. Then, we try to fit the afterglows of GRB 120326A with the emission of the external-forward shock propagating in a homogeneous medium, where the priors of $\log_{10}(E_{k,\text{iso, on}}\text{ erg}^{-1})$, $\log_{10} \Gamma_0$, $\theta_v/\theta_c$, $\theta_v/\theta_c$, $p$, $\log_{10} \epsilon_e$, $\log_{10} \epsilon_B$, and $\log_{10} \Gamma_0$ are set as a uniform distribution in the range of (52, 55), (1.5, 3.0), (0.3, 8.0), (2.5, 4.5), (0.0, 8.0), (2.1, 2.9), (−3.2, −0.5), (−6.5, −2.0), and (0.0, 1.7) in our MCMC fittings, respectively. However, no well-fitting result is found. The fitting result with minimum reduced $\chi^2$ is shown with dashed lines in the top left panel of Figure 5. It can be found that the early plateau of GRB 120326A could not be well fitted based on the emission of the external-forward shock in a homogeneous medium.

2. There seems to be a plateau at $t_{\text{obs}} \sim 600$ s in the optical afterglow of GRB 100814A, which may indicate an energy injection into the external shock. If so, our model could not well describe the afterglows of GRB 100814A in its early phase. This may be the reason for the deviation in our theoretical results relative to the observations in the early phase. In addition, the X-ray
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Table 1

| Parameters | GRB 120326A | GRB 100901A | GRB 100814A | GRB 120404A |
|------------|-------------|-------------|-------------|-------------|
| $\theta_e$ (°) | 0.69$^{+0.12}_{-0.12}$ | 0.51$^{+0.09}_{-0.09}$ | 0.89$^{+0.12}_{-0.05}$ | 0.58$^{+0.14}_{-0.11}$ |
| $\log_{10}(E_{\text{iso, core, erg}})$ | 54.69$^{+0.15}_{-0.14}$ | 54.44$^{+0.14}_{-0.13}$ | 54.89$^{+0.07}_{-0.13}$ | 53.78$^{+0.16}_{-0.14}$ |
| $\log_{10} T_0$ | 1.83$^{+0.27}_{-0.19}$ | 1.74$^{+0.04}_{-0.03}$ | 2.28$^{+0.11}_{-0.09}$ | 1.86$^{+0.03}_{-0.04}$ |
| $\log_{10} \epsilon_e$ | $-0.87^{+0.19}_{-0.16}$ | $-0.67^{+0.05}_{-0.05}$ | $-0.51^{+0.10}_{-0.05}$ | $-1.01^{+0.12}_{-0.08}$ |
| $\log_{10} \epsilon_b$ | $-3.95^{+0.18}_{-0.31}$ | $-3.51^{+0.13}_{-0.24}$ | $-4.69^{+0.10}_{-0.17}$ | $-3.21^{+0.16}_{-0.18}$ |
| $\rho$ | 2.27$^{+0.08}_{-0.07}$ | 2.30$^{+0.09}_{-0.06}$ | 2.60$^{+0.04}_{-0.05}$ | 2.29$^{+0.07}_{-0.05}$ |
| $\log_{10}(R_0, \text{cm})$ | 17.02$^{+0.12}_{-0.14}$ | 17.28$^{+0.09}_{-0.13}$ | 17.60$^{+0.07}_{-0.13}$ | 16.80$^{+0.06}_{-0.08}$ |
| $\theta_e/\theta_c$ | 4.14$^{+0.28}_{-0.19}$ | 4.50$^{+0.37}_{-0.25}$ | 3.66$^{+0.09}_{-0.05}$ | 0.68$^{+0.25}_{-0.28}$ |
| $\log_{10} A_S$ | $-1.01^{+0.19}_{-0.15}$ | $-0.78^{+0.19}_{-0.13}$ | $-0.66^{+0.11}_{-0.13}$ | $-1.46^{+0.11}_{-0.13}$ |
| $\theta_{\text{obs}} / \theta_e$ | 3.54$^{+0.51}_{-0.48}$ | 3.40$^{+0.44}_{-0.39}$ | 4.03$^{+0.00}_{-0.44}$ | 1.17$^{+0.10}_{-0.10}$ |

$E_k$ (erg) | $2.23 \times 10^{51}$ | $6.83 \times 10^{50}$ | $5.88 \times 10^{51}$ | $9.63 \times 10^{59}$

Note. $E_k$ is the total kinetic energy of the structured jet.

and optical afterglows of GRB 100814A show achromatic behavior, which could not be explained only by our scenario (i.e., an off-core observed structured jet propagating into a free-to-shocked wind environment) since the appearance of the late simultaneous bump/plateau in our scenario is achromatic. Other emission rather than that of the external-forward shock may contribute to the long X-ray shallow decay. In addition, our model could not well describe the X-ray emission at $t_{\text{obs}} \gtrsim 3 \times 10^3$ s in GRB 100901A, which seems to be another component in this phase.

3. The fitting result of GRB 120404A shows that this burst is observed in the core. Based on the fitting result of this burst, the deceleration radius of the (0, 0) jet can be estimated and is $2.43 \times 10^{16}$ cm, which is at around the transition radius $R_0 = 6.31 \times 10^{16}$ cm of this burst.

5. Discussion and Conclusions

In this paper, the emission of the external-forward shock in a free-to-shocked wind circumburst environment is studied. We mainly focus on the light curves of the afterglows in the late phase. Late simultaneous bumps/plateaux both directly followed by a steep decay and preceded by a plateau/shallow decay in the afterglows may sound the transition of the circumburst environment from a free-wind to a homogeneous shocked-wind medium.

In this work, a Gaussian structured jet is adopted. However, we would like to point out that our conclusion about the circumburst environment (i.e., the free-to-shocked wind environment) in these bursts may be robust. The reasons are shown as follows. First, the late simultaneous bumps found in these bursts are very similar to those in GRB 170817A, which is formed in an off-core observed external-forward shock propagating in an ISM (Troja et al. 2018, 2019; Huang et al. 2019; Lamb et al. 2019; Ren et al. 2020). This implies that the late bumps in these bursts may be formed in an off-core observed external-forward shock in a homogeneous medium. Interestingly, there seems to be no plateau/shallow decay in the early phase of GRB 170817A’s afterglows (see Figure 1 of Ren et al. 2020, especially the X-ray afterglow in $t_{\text{obs}} \sim 2–100$ day). This behavior may also appear in the afterglows of GRBs 120326A, 100901A, 100814A if their late simultaneous bumps appear with the same mechanism as that in GRB 170817A. In this paradigm, the plateau/shallow decay preceding the bumps in these three bursts would require another explanation and the external-forward shock in the free-wind environment may be a natural candidate, especially for long GRBs. Second, the external-forward shock with other kinds of structured jet are also studied. For example, Huang et al. (2019) studied the jet structure owing to the jet precession and the corresponding afterglows in a homogeneous environment. Very complex structured jets have been found. Granot et al. (2018) studied the afterglows from 2D relativistic hydrodynamic simulations of a GRB jet propagating in a homogeneous medium. It seems to be the case that the late simultaneous bumps generally appear without being preceded by a plateau/shallow decay in the afterglows for a structured jet propagating in a homogeneous medium (e.g., Figures 3 and 5 of Huang et al. 2019, and Figures 6 and 7 of Granot et al. 2018). However, our work could not rule out the scenario that other kinds of structured jet propagation only in a homogeneous medium could explain the observed late simultaneous bumps both directly followed by a steep decay and preceded by a plateau/shallow decay in these bursts. It is interesting to point out that a power-law structured jet propagating in a free-to-shocked wind is also studied in Lu et al. (2020). It also leads to
a bump/plateau in the light curve for large viewing angles. However, the bump/plateau appeared in small viewing angles is quite mild compared with this work and observations.

We note that the energy injection into the external shock is always used to explain the bump/plateau in the afterglows (e.g., Hou et al. 2014; Melandri et al. 2014; Nardini et al. 2014; Laskar et al. 2015). The energy injection scenario and our off-core observed scenario (i.e., the late simultaneous bumps/plateaux appear in the free-to-shocked-wind afterglows for high-viewing-angle observers) may not be fundamentally different in shaping the simultaneous bumps/plateaux because both scenarios try to account for light-curve morphology by varying the kinetic energy of the visible jet flow. In the energy injection scenario, the kinetic energy varies in the lab and observer time owing to the energy injection into the external shock. In our off-core observed scenario, the kinetic energy of the visible jet flow implicitly varies with observer time since the observer sees an increasing region of the structured jet owing to the deceleration of the jet.

Besides, the fitting results reported in Table 1 reveal that the in-core observed isotropic kinetic energy \(E_{k, \text{iso,obs}}\) of these bursts (i.e., \(\sim 5 \times 10^{54}, 3 \times 10^{54}, 8 \times 10^{54},\) and \(6 \times 10^{53}\) erg for GRBs 120326A, 100901A, 100814A, and 120404A, respectively) are significantly high but consistent with those found in other bursts (see Figure 10 in Racusin et al. 2011 and Table 1 in Yi et al. 2017). Compared with the beaming corrected kinetic energy reported in Table 2 in Yi et al. (2017), the total kinetic energy of our obtained structured jet is also consistent with those found in other bursts. However, it should be noted that the in-core observed fluence of GRBs 120326A, 100901A, and 100814A would be significantly high since the value of \(\theta_{v} \sim 4\theta_{c}\) is obtained from our MCMC fittings. According to the structure of the jet, the in-core observed fluence would be around \(1.58 \times 10^{-5}, 4.99 \times 10^{-5},\) and \(1.62 \times 10^{-2}\) erg cm\(^{-2}\) for GRBs 120326A, 100901A, and 100814A, respectively. Here, the observed fluences of GRBs 120326A, 100901A, and 100814A are respectively \(3 \times 10^{-5}\) erg cm\(^{-2}\) (10–1000 keV), \(2 \times 10^{-6}\) erg cm\(^{-2}\) (15–150 keV), and \(2 \times 10^{-5}\) erg cm\(^{-2}\) (10–1000 keV) based on the observations of the Fermi or Swift satellites. The in-core fluences estimated from these bursts are 10 times larger than the largest measured fluence of 2000 GRBs in the Burst And Transient Source Experiment (BATSE) catalog.

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### Appendix

#### Analytical Afterglows for a Top-hat Jet

Theoretically, the analytical light curves of a free/shocked-wind afterglow driven by a top-hat jet are useful in understanding the appearance of the late simultaneous bumps/plateaux shown in Figure 2. Same as Sari et al. (1998), the observed flux \(f_{\nu}(t_{\text{obs}})\) from the external-forward shock developed by a top-hat jet can be read as

\[
\begin{align*}
fn\left(t_{\text{obs}}\right) &= \begin{cases} 
(\nu/\nu_{c})^{1/2}F_{v,\text{max}}, & \nu < \nu_{c}, \\
(\nu/\nu_{m})^{-1/2}F_{v,\text{max}}, & \nu_{c} < \nu < \nu_{m}, \\
(\nu_{m}/\nu_{c})^{-1/2}(\nu/\nu_{m})^{-p/2}F_{v,\text{max}}, & \nu > \nu_{m},
\end{cases}
\end{align*}
\]

(A1)

for the fast cooling case, and

\[
\begin{align*}
fn\left(t_{\text{obs}}\right) &= \begin{cases} 
(\nu/\nu_{m})^{1/2}F_{v,\text{max}}, & \nu < \nu_{m}, \\
(\nu/\nu_{m})^{(p-1)/2}F_{v,\text{max}}, & \nu_{m} < \nu < \nu_{c}, \\
(\nu_{c}/\nu_{m})^{(p-1)/2}(\nu/\nu_{c})^{-p/2}F_{v,\text{max}}, & \nu > \nu_{c},
\end{cases}
\end{align*}
\]

(A2)

for the slow cooling case. For the external-forward shock developed by a top-hat jet, the values of \(\nu_{c}, \nu_{m}, F_{v,\text{max}}\) can be approximately associated with the Lorentz factor \(\bar{\Gamma}_{\text{obs}}\) and the location \(R_{\text{obs}}\) of the external-forward shock developed by the jet flow closest to the light of sight, i.e.,

\[
\begin{align*}
&\begin{cases}
\text{for } \theta_{v} < \theta_{\text{jet}}: \quad \nu_{c} \propto \bar{\Gamma}_{\text{obs}}^{8}[\rho(\bar{R}_{\text{obs}})]^{1/2}\bar{R}_{\text{obs}}^{-2}, \quad \nu_{m} \propto \bar{\Gamma}_{\text{obs}}^{1/2}[\rho(\bar{R}_{\text{obs}})]^{1/2}, \quad F_{v,\text{max}} \propto \kappa[\bar{\Gamma}_{\text{obs}}^{2}[\rho(\bar{R}_{\text{obs}})]^{1/2}\bar{R}_{\text{obs}}^{3}, \\
\text{for } \theta_{v} > \theta_{\text{jet}} + 1/\bar{\Gamma}: \quad \nu_{c} \propto \bar{\Gamma}_{\text{obs}}^{6}[\rho(\bar{R}_{\text{obs}})]^{3/2}\bar{R}_{\text{obs}}^{-2}, \quad \nu_{m} \propto \bar{\Gamma}_{\text{obs}}^{2}[\rho(\bar{R}_{\text{obs}})]^{3/2}, \quad F_{v,\text{max}} \propto \kappa[\bar{\Gamma}_{\text{obs}}^{2}[\rho(\bar{R}_{\text{obs}})]^{3/2}\bar{R}_{\text{obs}}^{3},
\end{cases}
\end{align*}
\]

(A3)

where \(\kappa = \{1 - 1/[1 + 2\bar{\Gamma}^{2}[1 - \cos(\theta_{\text{jet}} + \theta_{v})]\}^{2}\). In addition, one can have

\[
\begin{align*}
&\begin{cases}
\text{for } \bar{R}_{\text{obs}} < R_{\text{dec}}: \quad \bar{\Gamma}_{\text{obs}} = \text{const}, \quad \bar{R}_{\text{obs}} \propto t_{\text{obs}}; \\
\text{for } \bar{R}_{\text{obs}} > R_{\text{dec}}, \quad s = 2, \quad \theta_{v} < \theta_{\text{jet}}: \quad \bar{\Gamma}_{\text{obs}} \propto \bar{R}_{\text{obs}}^{1/2}, \quad \bar{R}_{\text{obs}} \propto t_{\text{obs}}^{1/2}; \\
\text{for } \bar{R}_{\text{obs}} > R_{\text{dec}}, \quad s = 2, \quad \theta_{v} > \theta_{\text{jet}} + 1/\bar{\Gamma}: \quad \bar{\Gamma}_{\text{obs}} \propto \bar{R}_{\text{obs}}^{1/2}, \quad \bar{R}_{\text{obs}} \propto t_{\text{obs}}; \\
\text{for } \bar{R}_{\text{obs}} > R_{\text{dec}}, \quad s = 0, \quad \theta_{v} < \theta_{\text{jet}}: \quad \bar{\Gamma}_{\text{obs}} \propto \bar{R}_{\text{obs}}^{3/8}, \quad \bar{R}_{\text{obs}} \propto t_{\text{obs}}^{1/4}; \\
\text{for } \bar{R}_{\text{obs}} > R_{\text{dec}}, \quad s = 0, \quad \theta_{v} > \theta_{\text{jet}} + 1/\bar{\Gamma}: \quad \bar{\Gamma}_{\text{obs}} \propto \bar{R}_{\text{obs}}^{3/2}, \quad \bar{R}_{\text{obs}} \propto t_{\text{obs}}.
\end{cases}
\end{align*}
\]

(A4)

In general, the observed spectral flux is expressed as \(f_{\nu}(t_{\text{obs}}) \propto t_{\text{obs}}^{-\beta}\) and the so-called closure relations, i.e., the relationship between the temporal index \(\alpha\) and the spectral index \(\beta\), are discussed (e.g., Zhang & Mészáros 2004; Zhang et al. 2006). With Equations (A1)–(A4), the values of \(\alpha\) and \(\beta\) for different situations are reported in Table 2.
### Table 2
The Values of \( \alpha \) and \( \beta \) in Different Situations

| Situations\(^a\) | Spectral Regime\(^b\) | Free-wind medium | Shocked-wind medium |
|------------------|----------------------|------------------|---------------------|
|                  | \( f \) | \( \nu < \nu_\ell (\beta = -\frac{1}{2}) \) | \(- \beta \) | \( \frac{1}{3} \) |
|                  | \( \nu_\ell < \nu < \nu_m (\beta = -\frac{1}{2}) \) | \(- \frac{3}{2} \) | \( 2 \beta - 3 \) | \(- \frac{1}{2} \) |
| I. \( R < R_{\text{dec}} \) & s | \( \nu < \nu_m (\beta = \frac{1}{2}) \) | \( \beta \) | \(- \frac{1}{3} \) |
|                  | \( \nu_m < \nu < \nu_\ell (\beta = \frac{3}{2}) \) | \(- \frac{1}{2} \) | \(~0.6 \) |
| f & s | \( \nu > \max \{\nu_\ell, \nu_m\} (\beta = \frac{5}{2}) \) | \(- \frac{2}{3} \) | \(~0.1 \) | \(- 2 \) |
| II. In core (i.e., \( \theta_\psi < \theta_{\text{jet}} \)) & s | \( \nu < \nu_m (\beta = \frac{1}{2}) \) | \( \frac{3\beta + 1}{2} \) | \( 0 \) |
|                  | \( \nu_m < \nu < \nu_\ell (\beta = \frac{3}{2}) \) | \(- \frac{1}{3} \) | \(~1.4 \) |
| f & s | \( \nu > \max \{\nu_\ell, \nu_m\} (\beta = \frac{5}{2}) \) | \(- \frac{2}{3} \) | \(~1.15 \) | \(- \frac{2}{3} \) |
| III. Off core (i.e., \( \theta_\psi > \theta_{\text{jet}} \)) & s | \( \nu < \nu_m (\beta = \frac{1}{2}) \) | \( \frac{3\beta + 1}{2} \) | \( 0 \) |
|                  | \( \nu_m < \nu < \nu_\ell (\beta = \frac{3}{2}) \) | \(- \frac{1}{3} \) | \(~1.4 \) |
| f & s | \( \nu > \max \{\nu_\ell, \nu_m\} (\beta = \frac{5}{2}) \) | \(- \frac{2}{3} \) | \(~1.15 \) | \(- \frac{2}{3} \) |
| IV. & s | \( \nu < \nu_m (\beta = \frac{1}{2}) \) | \( \frac{3\beta + 1}{2} \) | \( 0 \) |
|                  | \( \nu_m < \nu < \nu_\ell (\beta = \frac{3}{2}) \) | \(- \frac{1}{3} \) | \(~1.4 \) |
| f & s | \( \nu > \max \{\nu_\ell, \nu_m\} (\beta = \frac{5}{2}) \) | \(- \frac{2}{3} \) | \(~1.15 \) | \(- \frac{2}{3} \) |

Notes.

\( ^a \) The closure relations in Situation I or IV are the same for in-core and off-core observers. Here, Situation I describes the pre-deceleration behavior of afterglows, where the Lorentz factor of the external-forward shock remains constant; and Situation IV describes the behavior of afterglows in the post-core phase, where the jet is fully visible for the observer.

\( ^b \) “f" and “s" represent the fast cooling and slow cooling, respectively.

\( ^c \) \( p = 2.2 \) is adopted.

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