Abstract

Sums of 1-dependent integer-valued random variables are approximated by compound Poisson, negative binomial and Binomial distributions and signed compound Poisson measures. Estimates are obtained for total variation and local metrics. The results are then applied to statistics of \(m\)-dependent \((k_1, k_2)\) events and 2-runs. Heinrich’s method and smoothing properties of convolutions are used for the proofs.

Key words: Compound Poisson distribution, signed compound Poisson measure, negative binomial, binomial, \(m\)-dependent variables, total variation norm, local norm.

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1 Introduction

In this paper, we consider sum \( S_n = X_1 + X_2 + \cdots + X_n \) of non-identically distributed 1-dependent random variables concentrated on nonnegative integers. Our aim is to estimate the closeness of \( S_n \) to compound Poisson, negative binomial and binomial distributions, under some analogue of Franken’s condition for factorial moments. For the proof of the main results, we use Heinrich’s \([16,17]\) version of the characteristic function method. Though this method does not allow to obtain small absolute constants, it is flexible enough for obtaining asymptotically sharp constants, as demonstrated for 2-runs statistic. Moreover, our approach allows for construction of asymptotic expansions.

We recall that the sequence of random variables \( \{X_k\}_{k \geq 1} \) is called \( m \)-dependent if, for \( 1 < s < t < \infty, t - s > m \), the sigma algebras generated by \( X_1, \ldots, X_s \) and \( X_t, X_{t+1} \ldots \) are independent. It is clear that, by grouping consecutive summands, we can reduce the sum of \( m \)-dependent variables to the sum of 1-dependent ones. Therefore, the results of this paper can be applied for some cases of \( m \)-dependent variables, as exemplified by binomial approximation to \((k_1, k_2)\) events.

Let us introduce necessary notations. Let \( \{Y_k\}_{k \geq 1} \) be a sequence of arbitrary real or complex-valued random variables. We assume that \( \hat{E}(Y_1) = E Y_1 \) and, for \( k \geq 2 \), define \( \hat{E}(Y_1, Y_2, \cdots Y_k) \) by

\[
\hat{E}(Y_1, Y_2, \cdots, Y_k) = E Y_1 Y_2 \cdots Y_k - \sum_{j=1}^{k-1} \hat{E}(Y_1, \cdots, Y_j) E Y_{j+1} \cdots Y_k.
\]

We define \( j \)-th factorial moment of \( X_k \) by \( \nu_j(k) = E X_k (X_k - 1) \cdots (X_k - j + 1), (k = 1, 2, \ldots, n, j = 1, 2, \ldots) \). Let

\[
\begin{align*}
\Gamma_1 &= E S_n = \sum_{k=1}^{n} \nu_1(k), \quad 
\Gamma_2 = \frac{1}{2} (\text{Var} S_n - E S_n) = \frac{1}{2} \sum_{k=1}^{n} (\nu_2(k) - \nu_1^2(k)) + \sum_{k=2}^{n} \hat{E}(X_{k-1}, X_k), \\
\Gamma_3 &= \frac{1}{6} \sum_{k=1}^{n} (\nu_3(k) - 3\nu_1(k)\nu_2(k) + 2\nu_1^3(k)) - \sum_{k=2}^{n} (\nu_1(k-1) + \nu_1(k)) \hat{E}(X_{k-1}, X_k) \\
&\quad + \frac{1}{2} \sum_{k=2}^{n} (\hat{E}(X_{k-1}(X_{k-1} - 1), X_k) + \hat{E}(X_{k-1}, X_k X_{k-1} - 1)) + \sum_{k=3}^{n} \hat{E}(X_{k-2}, X_{k-1}, X_k).
\end{align*}
\]

For the sake of convenience, we assume that \( X_k \equiv 0 \) and \( \nu_j(k) = 0 \) if \( k \leq 0 \) and \( \sum_{k=1}^{n} = 0 \) if \( k > n \). We denote the distribution and characteristic function of \( S_n \) by \( F_n \) and \( \hat{F}_n(t) \), respectively.
Below we show that $\Gamma_1, 2\Gamma_2$ and $6\Gamma_3$ are factorial cumulants of $F_n$, that is,

$$\hat{F}_n(t) = \exp\{\Gamma_1(e^{it} - 1) + \Gamma_2(e^{it} - 1)^2 + \Gamma_3(e^{it} - 1)^3 + \ldots \}$$

For approximation of $F_n$, it is natural to use measures or distributions which allow similar expressions.

Let $I_a$ denote the distribution concentrated at real $a$ and set $I = I_0$. Henceforth, the products and powers of measures are understood in the convolution sense. Further, for a measure $M$, we set $M^0 = I$ and

$$e^M := \exp\{M\} = \sum_{k=0}^{\infty} \frac{1}{k!} M^k.$$

Next we define Poisson and compound Poisson approximations of this paper. Let

$$\text{Pois}(\Gamma_1) = \exp\{\Gamma_1(I_1 - I)\}, \quad G = \exp\{\Gamma_1(I_1 - I) + \Gamma_2(I_1 - I)^2\},$$

$$\text{TP} = I_a\text{Pois}(\Gamma_1 + 2\Gamma_2 + \tilde{\delta}) = I_a\exp\{(\Gamma_1 + 2\Gamma_2 + \tilde{\delta})(I_1 - I)\}.$$  

Here $a = \lfloor -2\Gamma_2 \rfloor$ and $\tilde{\delta}$ are integer part and fractional part of $-2\Gamma_2$, respectively, that is $-2\Gamma_2 = a + \tilde{\delta}, \quad a \in \mathbb{Z}, \quad 0 \leq \tilde{\delta} < 1.$ It is easy to see, that Pois($\Gamma_1$) is Poisson distribution with parameter $\Gamma_1$. TP is called translated Poisson and was introduced in [19], see also [2], [3], [24], [25] and the references therein. In general, $G$ is signed measure, since $\Gamma_2$ can be negative. Signed compound Poisson measures similar to $G$ are used in numerous papers, see [2], [4], [10], [15], [29], and the references therein. In comparison to the Poisson distribution, the main benefit of $G$ and TP is matching of two moments, which then allows for the accuracy comparable to the one achieved by the normal approximation. This fact is illustrated in the next two sections. From a practical point of view, signed measures are not always convenient to use, since for calculation of their ‘probabilities’ one needs inverse Fourier transform or recursive algorithms. Therefore, we also prove estimates for such widely used distributions as binomial and negative binomial. We define the binomial distribution of this paper as

$$\text{Bi}(N, \bar{p}) = (I + \bar{p}(I_1 - I))^N, \quad N = \lfloor \bar{N} \rfloor, \quad \bar{N} = \frac{\Gamma_1^2}{2|\Gamma_2|}, \quad \bar{p} = \frac{\Gamma_1}{N}.$$
Here, we use $\lfloor \tilde{N} \rfloor$ to denote the integer part of $\tilde{N}$, that is, $\tilde{N} = N + \epsilon$, for some $0 \leq \epsilon < 1$. Also, we define negative binomial distribution and choose its parameters in the following way:

$$\text{NB}(r, \tilde{q}) \{ j \} = \frac{\Gamma(r + j)}{j! \Gamma(r)} \tilde{q}^j (1 - \tilde{q})^{r-j}, \quad (j \in \mathbb{Z}_+)$$

$$r \left( 1 - \frac{1 - \tilde{q}}{\tilde{q}} \right)^2 = 2 \Gamma_2. \quad (2)$$

Note that symbols $\tilde{q}$ and $\tilde{p}$ are not related and, in general, $\tilde{q} + \tilde{p} \neq 1$.

All estimates are obtained in the total variation and local norms. The total variation norm and the local norm of measure $M$ are denoted by

$$\|M\| = \sum_{k=-\infty}^{\infty} |M\{k\}|, \quad \|M\|_\infty = \sup_{k \in \mathbb{Z}} |M\{k\}|,$$

respectively. We use symbol $C$ to denote all (in general, different) positive absolute constants. Sometimes we supply $C$ with indices.

## 2 Known results

There are many results dealing with approximations to the sum of dependent integer-valued random variables. Note, however, that with very few exceptions: a) all papers are devoted to the sums of indicator variables only; b) results are not related to $k$-dependent variables. For example, indicators connected in a Markov chain are investigated in [10], [34]. The most general results, containing $k$-dependent variables as partial cases, are obtained for birth-death processes with some stochastic ordering, see [7], [11], [13] and the references therein.

Arguably the best explored case of sums of $k$-dependent integer-valued random variables is $k$-runs. Approximations of 2 or $k$-runs statistic by Poisson or centered Poisson, negative binomial distribution or signed compound Poisson measure are considered in [4], [7], [11], [24] and [33]. We formulate one of the most general results from [33].

Let $\eta_i \sim Be(p_i) \ (i=1,2,\ldots)$ be independent Bernoulli variables. Let us define $\xi_i = \prod_{j=i}^{i+k-1} \eta_j$, $S^* = \sum_{i=1}^{n} \xi_i$, where $\eta_{i+nm}$ is treated as $\eta_i$ for $1 \leq i \leq n$ and $m = \pm 1, \pm 2, \ldots$. Let

$$\tilde{r} = \frac{(ES^*)^2}{\text{Var} S^* - ES^*}, \quad \tilde{q} = \frac{ES^*}{\text{Var} S^*}.$$
If $\text{Var} S^* > ES^*$ and $n > 4k$, then

$$\|L(S^*) - NB(\bar{r}, \bar{q})\| \leq \frac{9(4k - 3)(2k - 1)}{ES^*} \sum_{i=1}^{n} m_i^2 E \xi_i.$$ 

Here $m_i = \max\{p_s : i - 2k + 2 \leq s \leq i + 2k - 2\}$,

$$\phi = 2 \wedge \frac{4.6}{\sqrt{\sum_{m=4k-1}^{n} v_m}}$$

and $v_m$ is the $m$th largest number of $(1 - p_{i-1})^2 p_i (1 - p_{i+1})p_{i+1} \ldots p_{1+k-1}$, $(i = 1, 2, \ldots, n)$.

The paper [33] also contains more detailed estimate with mixed moments of the form $E \xi_i \xi_{i+1}$. Note that in [33] total variation distance (which is half of the total variation norm) is used.

If $k = 2$ and $p_i \equiv p$, then $\bar{q} = (2p - 3p^2)/(1 + 2p - 3p^2)$, $(1 - \bar{q})/\bar{q} = np^2$, and more accurate result is proved in [7]. It states that, if $n \geq 2$ and $p < 2/3$, then

$$\|L(S^*) - NB(\bar{r}, \bar{q})\| \leq \frac{64.4p}{\sqrt{(n - 1)(1 - p)^3}}.$$  \hspace{1cm} (3)

The $k$-runs statistic has very explicit dependency of summands. Meanwhile, our aim is to obtain a general result which includes sums of independent random variables as a particular case. Except for examples, no specific assumptions about the structure of summands are made. For bounded and identically distributed random variables similar approach is taken in [22]. We give one example from [22] in the notation of previous Section. Let the $X_i$ be identically distributed, $|X_1| \leq C$, and, for $n \to \infty$,

$$\nu_1(1) = o(1), \quad \nu_2(1) = o(\nu_1(1)), \quad EX_1X_2 = o(\nu_1(1)), \quad n\nu_1(1) \to \infty.$$  \hspace{1cm} (4)

Then

$$\|F_n - G\| = O\left(\frac{\bar{R}}{\nu_1(1)\sqrt{n\nu_1(1)}}\right),$$

where

$$\bar{R} = \nu_3(1) + \nu_1(1)\nu_2(1) + \nu_1^3(1) + E(X_1(X_1 - 1)X_2 + X_1X_2(X_2 - 1)) + \nu_1(1)EX_1X_2 + EX_1X_2X_3.$$
Condition (4) implies that probabilities of $X_i$ depend on $n$. Thus, the classical case of a sequence of random variables, so typical for CLT, is completely excluded. Moreover, assumption $|X_1| \leq C$ seems rather strong. For example, then one can not consider Poisson or geometric random variables as possible summands.

Finally, we discuss Franken’s condition. In [14], Franken considers $S = X_1 + X_2 + \cdots + X_n$, when $X_i$ are independent nonnegative integer valued random variables, satisfying condition:

$$\nu_1(k) - \nu_1^2(k) - \nu_2(k) > 0, \text{ for } 1 \leq k \leq n.$$  \hspace{1cm} (5)

He proved that

$$\sup_x |\mathcal{L}(S)[(-\infty, x)] - \text{Pois}(\Gamma_1)[(-\infty, x)]| \leq C \sum_{k=1}^{n} \left[ \frac{\nu_1^2(k) + \nu_2(k)}{\nu_1(k) - \nu_1^2(k) - \nu_2(k)} \right].$$  \hspace{1cm} (6)

In (6), Kolmogorov’s uniform metric is used, which is weaker than the total variation norm. If we consider the sum of independent Bernoulli variables, (6) becomes the standard version of Poisson approximation to the Poisson-Binomial distribution. Franken’s result was extended and generalized, see, for example, [20], [32] or [10]. Particularly, it was shown that the signed compound Poisson approximation G significantly improves the accuracy of approximation. Condition (4) is stronger than the one in (5), albeit similar in assumption that the mean dominates other factorial moments.

In principle, Franken’s condition means that almost all probability mass of $F_n$ is concentrated at zero and unity. It is easy to check that any Bernoulli variable satisfies (5). However, it would be incorrect to assume that random variables satisfying Franken’s condition can be truncated to Bernoulli variables without significant loss of accuracy. For example, let us consider the sum of identically distributed random variables taking values 0 and 1 and 2 with probabilities 0.989; 0.010 and 0.001, respectively. The distribution of the sum differs from the binomial distribution with the same mean by some absolute constant. Thus, no improvement for large $n$. The proof of this fact is quite standard (for example, one can apply Lemma 4 from [32]) and we leave it out.
3 Results

All results are obtained under the following conditions:

\[
\nu_1(k) \leq 1/100, \quad \nu_2(k) \leq \nu_1(k), \quad \nu_4(k) < \infty, \quad (k = 1, 2, \ldots, n), \quad (7)
\]

\[
\lambda := \sum_{k=1}^{n} \nu_1(k) - 1.52 \sum_{k=1}^{n} \nu_2(k) - 12 \sum_{k=2}^{n} E X_{k-1} X_k > 0. \quad (8)
\]

The last condition is satisfied, if the following two assumptions hold

\[
\sum_{k=1}^{n} \nu_2(k) \leq \frac{\Gamma_1}{20}, \quad \sum_{k=2}^{n} |Cov(X_{k-1}, X_k)| \leq \frac{\Gamma_1}{20}. \quad (9)
\]

Moreover, if (7) and (9) hold, then \( \lambda > 0.2\Gamma_1 \). Indeed, then

\[
E X_{k-1} X_k \leq |Cov(X_{k-1}, X_k) + \nu_1(k - 1)\nu_1(k)| \leq |Cov(X_{k-1}, X_k)| + 0.01\nu_1(k).
\]

It is obvious that conditions in above are weaker than (4). For example, \( X_j \) are not necessarily bounded by some absolute constant. On the other hand, (7) and (8) are stronger than (5). Its a consequence of possible dependence of random variables and method of proof. Technically, it is possible to write a complete analogue of Franken’s condition for 1-dependent summands, which then reduces to (5) when all summands are independent. However, it contains various summands in the form \( \hat{E}(X_j, X_k, \ldots, X_l) \) and is hardly verifiable. It is quite probable, that by using different method of proof, one may succeed in weakening of (7) and (8) significantly.

Next we define remainder terms. Let

\[
R_0 = \sum_{k=1}^{n} \left\{ \nu_2(k) + \nu_1^2(k) + E X_{k-1} X_k \right\},
\]

\[
R_1 = \sum_{k=1}^{n} \left\{ \nu_1^3(k) + \nu_1(k)\nu_2(k) + \nu_3(k) + [\nu_1(k - 2) + \nu_1(k - 1) + \nu_1(k)]E X_{k-1} X_k \\
+ \hat{E}_2^+(X_{k-1}, X_k) + \hat{E}_2^+(X_{k-2}, X_{k-1}, X_k) \right\},
\]
\[ R_2 = \sum_{k=1}^{n} \left\{ \nu_1^3(k) + \nu_2^2(k) + \nu_4(k) + [\nu_1(k-1) + \nu_1(k)][\nu_3(k) + \hat{E}_2^+(X_{k-1}, X_k)] \right\} \]
\[ + \left( EX_{k-1}X_k \right)^2 + \sum_{l=0}^{3} \nu_1(k-l)\hat{E}_2^+(X_{k-2}, X_{k-1}, X_k) + \hat{E}_2^+(X_{k-2}, X_{k-1}, X_k) \]
\[ + \hat{E}_3^+(X_{k-1}, X_k) + \hat{E}_2^+(X_{k-3}, X_{k-2}, X_{k-2}, X_k) \} \].

Here
\[
\hat{E}^+(X_1) = EX_1, \quad \hat{E}^+(X_1, X_2) = EX_1X_2 + EX_1EX_2, \\
\hat{E}^+(X_1, \ldots, X_k) = EX_1 \ldots X_k + \sum_{j=1}^{k-1} \hat{E}^+(X_1, X_2, \ldots, X_j)EX_{j+1}X_{j+2} \cdots X_k, \\
\hat{E}_2^+(X_{k-1}, X_k) = \hat{E}^+(X_{k-1}(X_{k-1}-1), X_k) + \hat{E}^+(X_{k-1}, X_k(X_k-1)), \\
\hat{E}_2^+(X_{k-2}, X_{k-1}, X_k) = \hat{E}^+(X_{k-2}(X_{k-2}-1), X_{k-1}, X_k) + \hat{E}^+(X_{k-2}, X_{k-1}(X_{k-1}-1), X_k) \\
+ \hat{E}^+(X_{k-2}, X_{k-1}, X_k(X_k-1)), \\
\hat{E}_3^+(X_{k-1}, X_k) = \hat{E}^+(X_{k-1}(X_{k-1}-1)(X_{k-1}-2), X_k) + \hat{E}^+(X_{k-1}(X_{k-1}-1), X_k(X_k-1)) \\
+ \hat{E}^+(X_{k-1}, X_k(X_k-1)(X_k-2)).
\]

For better understanding of the order of remainder terms, let us consider the case of Bernoulli variables \( P(X_i = 1) = 1 - P(X_i = 0) = p_i \). If all \( X_i \) are independent, then \( R_0 = C \sum_{i=1}^{n} p_i^2 \) and \( R_1 = C \sum_{i=1}^{n} p_i^3 \). If \( X_i \) are 1-dependent, then at least \( R_0 \leq C \sum_{i=1}^{n} p_i \) and \( R_1 \leq C \sum_{i=1}^{n} p_i^{3/2} \). If some additional information about \( X_i \) is available (for example, that they form 2-runs), then the estimates are somewhat in between.

Our aim is investigation of approximations with at least two parameters. However, for the completeness, we begin from the Poisson approximation. Note that Poisson approximation (for indicator variables) is considered in \([1, 5]\) under much more general conditions than assumed in this paper.
Theorem 3.1 Let conditions (7) and (8) be satisfied. Then, for all \( n \),

\[
\|F_n - \text{Pois}(\Gamma_1)\| \leq C_1 R_0 \{1 + \Gamma_1 \min(1, \lambda^{-1})\} \min(1, \lambda^{-1}),
\]

(10)

\[
\|F_n - \text{Pois}(\Gamma_1)(I + \Gamma_2(I_1 - I)^2)\| \leq C_2 \{1 + \Gamma_1 \min(1, \lambda^{-1})\} \left( R_0^2 \min(1, \lambda^{-2}) + R_1 \min(1, \lambda^{-3/2}) \right),
\]

(11)

\[
\|F_n - \text{Pois}(\Gamma_1)\|_{\infty} \leq C_3 R_0 \min(1, \lambda^{-3/2}),
\]

(12)

\[
\|F_n - \text{Pois}(\Gamma_1)(I + \Gamma_2(I_1 - I)^2)\|_{\infty} \leq C_4 \{R_0^2 \min(1, \lambda^{-5/2}) + R_1 \min(1, \lambda^{-1})\}.
\]

(13)

If all \( X_i \sim \text{Be}(1, p_i) \) are independent, then the order of accuracy in (10) is correct (see, for example [5]) and is equal to \( C_1 \sum\limits_{i=1}^n p_i^2 (1 \lor \sum\limits_{i=1}^n p_i)^{-1} \). Similarly, in (11) the order of accuracy is \( C(\max p_i)^2 \). As one can expect, the accuracy of approximation is trivial, if all \( p_i \) are uniformly bounded from zero, i.e., \( p_i > C \).

The accuracy of approximation is much better for \( G \).

Theorem 3.2 Let conditions (7) and (8) be satisfied. Then, for all \( n \),

\[
\|F_n - G\| \leq C_5 R_1 \{1 + \Gamma_1 \min(1, \lambda^{-1})\} \min(1, \lambda^{-3/2}),
\]

(14)

\[
\|F_n - G(I + \Gamma_3(I_1 - I)^3)\| \leq C_6 \{1 + \Gamma_1 \min(1, \lambda^{-1})\} \left( R_1^2 \min(1, \lambda^{-3}) + R_2 \min(1, \lambda^{-2}) \right),
\]

(15)

\[
\|F_n - G\|_{\infty} \leq C_7 R_1 \min(1, \lambda^{-2}),
\]

(16)

\[
\|F_n - G(I + \Gamma_3(I_1 - I)^3)\|_{\infty} \leq C_8 \{R_1^2 \min(1, \lambda^{-7/2}) + R_2 \min(1, \lambda^{-5/2})\}.
\]

(17)

If, instead of (8), we assume (9), then \( 1 + \Gamma_1 \min(1, \lambda^{-1}) \leq C \) and \( \lambda \geq C \Gamma_1 \). If, in addition, all \( X_i \) do not depend on \( n \) and are bounded, then estimates in (14) and (15) are of orders \( O(n^{-1/2}) \) and \( O(n^{-1}) \), respectively. Thus, the order of accuracy is comparable to CLT and Edgeworth’s expansion. If all \( X_i \sim \text{Be}(1, p_i) \) are independent, then the order of accuracy in (14) is the right one (see [19]) and is equal to \( C \sum\limits_{i=1}^n p_i^2 (1 \lor \sum\limits_{i=1}^n p_i)^{-3/2} \).

Approximation \( G \) has two parameters, but: a) is not always a distribution, b) its ”probabilities” are not easily calculable. Some authors argue (see, for example, [7]) that, therefore, probabilistic approximations are more preferable. We start from translated Poisson distribution. Its probabilities
are the same as probabilities of the Poisson law (albeit shifted). Unlike Poisson approximation it has two parameters, which are chosen to (almost) match two moments of \( S_n \). The necessity to take shift by integer number is stipulated by the total variation norm. Both discrete distributions must have the same support, otherwise the total variation of their difference is equal to 2. The integer shift also means that the matching of variances is incomplete.

**Theorem 3.3** Let \( \Gamma_1 \geq 1 \) and conditions \([7]\) and \([9]\) be satisfied. Then, for all \( n \),

\[
\|F_n - TP\| \leq C_9 \left( \frac{R_1 + |\Gamma_2|}{\Gamma_1 \sqrt{\Gamma_1}} + \frac{\delta}{\Gamma_1} \right),
\]

(18)

\[
\|F_n - TP\|_\infty \leq C_{10} \left( \frac{R_1 + |\Gamma_2|}{\sqrt{\Gamma_1}} + \frac{\delta}{\Gamma_1 \sqrt{\Gamma_1}} \right).
\]

(19)

If all \( X_i \sim Be(1, p) \), \( p < 0.01 \) are independent, then the order of accuracy in \([18]\) is the right one (see \([19]\), \([25]\)) and is equal to \( O(\sqrt{p/n} + (np)^{-1}) \).

The negative binomial approximation is meaningful only if \( \text{Var}S_n > E S_n \).

**Theorem 3.4** Let conditions \([7]\) and \([9]\) be satisfied and let \( \Gamma_2 > 0 \). Then, for all \( n \),

\[
\|F_n - \text{NB}(r, \bar{q})\| \leq C_{11} \min(1, \Gamma_1^{-3/2})(R_1 + \Gamma_2^2 \Gamma_1^{-1}),
\]

(20)

\[
\|F_n - \text{NB}(r, \bar{q})(I + |\Gamma_3 - 4\Gamma_2^2(3\Gamma_1)^{-1}|(I_1 - I)^3)| \leq C_{12} \left\{ R_1^2 \min(1, \Gamma_1^{-3}) + R_2 \min(1, \Gamma_1^{-1}) + \Gamma_2^2 \Gamma_1^{-1} \left| \Gamma_3 - 4\Gamma_2^2(3\Gamma_1)^{-1} \right| \min(1, \Gamma_1^{-1}) + \Gamma_2^2 \Gamma_1^{-1} \min(1, \Gamma_1^{-1}) \right\},
\]

(21)

\[
\|F_n - \text{NB}(r, \bar{q})\|_\infty \leq C_{13} \min(1, \Gamma_1^{-2})(R_1 + \Gamma_2^2 \Gamma_1^{-1}),
\]

(22)

\[
\|F_n - \text{NB}(r, \bar{q})(I + |\Gamma_3 - 4\Gamma_2^2(3\Gamma_1)^{-1}|(I_1 - I)^3)\|_\infty \leq C_{14} \left\{ R_1^2 \min(1, \Gamma_1^{-1/2}) + R_2 \min(1, \Gamma_1^{-5/2}) + \Gamma_2^2 \Gamma_1^{-1} \left| \Gamma_3 - 4\Gamma_2^2(3\Gamma_1)^{-1} \right| \min(1, \Gamma_1^{-1/2}) + \Gamma_2^2 \Gamma_1^{-1} \min(1, \Gamma_1^{-5/2}) \right\}.
\]

(23)

It seems that asymptotic expansion for the negative binomial approximation was so far never considered in the context of 1-dependent summands. If all \( X_i \) do not depend on \( n \) and are bounded, the accuracies of approximation in \([20]\) and \([21]\) are \( O(n^{-1/2}) \) and \( O(n^{-1}) \), respectively.

If \( \text{Var}S_n < E S_n \), it is more natural to use the binomial approximation.
Theorem 3.5 Let conditions \((7)\) and \((9)\) be satisfied, \(\Gamma_1 \geq 1\) and \(\Gamma_2 < 0\). Then, for all \(n\),

\[
\| F_n - \text{Bi}(N, \bar{p}) \| \leq C_{15}(\Gamma_2^2 \Gamma_1^{-5/2} + R_1 \Gamma_1^{-3/2}), \\
\| F_n - \text{Bi}(N, \bar{p})(I + [\Gamma_3 - N \bar{p}^3/3](I_1 - I)^3) \| \leq C_{16}\{ R_1^2 \Gamma_1^{-3} + R_2 \Gamma_2^{-2} + |\Gamma_2|^3 \Gamma_1^{-1} \}
\]

\[
+ c \Gamma_2^2 \Gamma_1^{-3} + \Gamma_2^2 \Gamma_3 \Gamma_1^{-1}, \\
\| F_n - \text{Bi}(N, \bar{p}) \|_\infty \leq C_{17}(\Gamma_2^2 \Gamma_1^{-3} + R_1 \Gamma_1^{-2}), \\
\| F_n - \text{Bi}(N, \bar{p})(I + [\Gamma_3 - N \bar{p}^3/3](I_1 - I)^3) \|_\infty \leq C_{18}\{ R_1^2 \Gamma_1^{-7/2} + R_2 \Gamma_2^{-5/2} + |\Gamma_2|^3 \Gamma_1^{-9/2} \}
\]

\[
+ c \Gamma_2^2 \Gamma_1^{-7/2} + \Gamma_2^2 \Gamma_3 \Gamma_1^{-9/2}. \\
\]

If all the \(X_i\) do not depend on \(n\) and are bounded, the accuracies of approximation in \((24)\) and \((25)\) are \(O(n^{-1/2})\) and \(O(n^{-1})\), respectively.

4 Applications

1. Asymptotically sharp constant for the negative binomial approximation to 2-runs.

As already mentioned in above, the 2-runs statistic is one of the best investigated cases of sums of 1-dependent discrete random variables. It is easy to check that the rate of accuracy in \((3)\) is \(O(pn^{-1/2})\). However, the constant 64.4 does not look particularly small. Here, we shall show, that, on the other hand, asymptotically sharp constant is small. We shall consider 2-runs with edge effects, which we think to be more realistic case than \(S^*\). Let \(S_\xi = \xi_1 + \xi_2 + \cdots + \xi_n\), where \(\xi_i = \eta_i \eta_{i+1}\) and \(\eta_i \sim \text{Be}(p)\), \((i = 1, 2, \ldots, n + 1)\) are independent Bernoulli variables. The sum \(S^*\) differs from \(S_\xi\) by the last summand only, which is equal to \(\eta_n \eta_1\). For \(S_\xi\) we have

\[
\Gamma_1 = np^2, \quad \Gamma_2 = \frac{np^3(2 - 3p) - 2p^3(1 - p)}{2}, \quad \Gamma_3 = \frac{np^4(3 - 12p + 10p^2) - 6p^4(1 - p)(1 - 2p)}{3},
\]

see [21]. Let \(\text{NB}(r, \bar{q})\) be defined as in \((2)\) and

\[
\hat{C}_{TV} = \frac{1}{3} \sqrt{\frac{2}{\pi}}(1 + 4e^{-3/2}) = 0.5033...; \quad \hat{C}_L = \frac{1}{\sqrt{3\pi}} \exp\left\{\sqrt{\frac{3}{2}} - \frac{3}{2}\right\} \sqrt{3 - \sqrt{6}} = 0.1835... .
\]
Theorem 4.1 Let $p \leq 1/20$, $np^2 \geq 1$. Then

$$
\left\| \mathcal{L}(S_\xi) - \text{NB}(r, \bar{q}) \right\| - \tilde{C}_{TV} \frac{p}{\sqrt{n}} \right\| \leq C_{19} \left( \frac{p^2}{\sqrt{n}} + \frac{1}{n} \right),
\left\| \mathcal{L}(S_\xi) - \text{NB}(r, \bar{q}) \right\| - \tilde{C}_L \frac{n}{\sqrt{np^2}} \right\| \leq C_{20} \left( \frac{p}{n} + \frac{1}{n\sqrt{np^2}} \right).
$$

We now get the following corollary.

Corollary 4.1 Let $p \to 0$ and $np^2 \to \infty$, as $n \to \infty$. Then

$$
\lim_{n \to \infty} \frac{\left\| \mathcal{L}(S_\xi) - \text{NB}(r, \bar{q}) \right\| \sqrt{n}}{p} = \tilde{C}_{TV}, \quad \lim_{n \to \infty} n \left\| \mathcal{L}(S_\xi) - \text{NB}(r, \bar{q}) \right\| = \tilde{C}_L.
$$

2. Binomial approximation to $N(k_1, k_2)$ events. Let $\eta_i \sim \text{Be}(p), (0 < p < 1)$ be independent Bernoulli variables and let $Y_j = (1 - \eta_{j-m+1}) \cdots (1 - \eta_{j-k_2})\eta_{j-k_2+1} \cdots \eta_{j-1}\eta_j, j = m, m+1, \ldots, n, k_1 + k_2 = m$. Further, we assume that $k_1 > 0$ and $k_2 > 0$. Let $N(n; k_1, k_2) = Y_m + Y_{m+1} + \cdots + Y_n$. We denote the distribution of $N(n; k_1, k_2)$ by $H$. Let $a(p) = (1 - p)^{k_1}p^{k_2}$. It is well known that $N(n; k_1, k_2)$ has limiting Poisson distribution and the accuracy of Poisson approximation is $O(a(p))$, see [18] and [31], respectively. However, Poisson approximation has just one parameter. Consequently, the closeness of $p$ to zero is crucial. We can expect any two-parametric approximation to be more universal. It is known that

$$
\begin{align*}
\text{EN}(n; k_1, k_2) &= (n - m + 1)a(p), \\
\text{Var}N(n; k_1, k_2) &= (n - m + 1)a(p) + (1 - 4m + 3m^2 - n(2m - 1))a^2(p),
\end{align*}
$$

see [30]. Under quite mild assumptions $\text{Var}N(n; k_1, k_2) < \text{EN}(n; k_1, k_2)$. Consequently, the natural probabilistic approximation is Binomial one. The Binomial approximation to $N(n; k_1, k_1)$ was already considered in [30]. Regrettably, the estimate in [30] contains expression which is of the constant order when $a(p) \to 0$. 

12
Note that \(Y_1, Y_2, \ldots\) are \(m\)-dependent. Consequently, results of the previous Section can not be applied directly. However, one can group summands in the following natural way:

\[
N(n; k_1, k_2) = (Y_m + Y_{m+1} + \cdots + Y_{2m-1}) + (Y_{2m} + Y_{2m+1} + \cdots + Y_{3m-1}) + \cdots = X_1 + X_2 + \ldots
\]

Each \(X_j\), with probable exception of the last one, contains \(m\) summands. It is not difficult to check that \(X_1, X_2, \ldots\) are 1-dependent Bernoulli variables. All parameters can be written explicitly. Set \(N = \lfloor \tilde{N} \rfloor\) be the integer part of \(\tilde{N}\),

\[
\tilde{N} = \frac{(n - m + 1)^2}{(n - m + 1)(2m - 1) - m(m - 1)}, \quad \tilde{N} = N + \epsilon, \quad 0 \leq \epsilon < 1, \quad \bar{p} = \frac{(n - m + 1)a(p)}{N}.
\]

For the asymptotic expansion we need the following notation

\[
A = \frac{a^3(p)}{6} (n - m + 1)m(m - 1).
\]

The two-parametric binomial approximation is more natural, when \(EN(n; k_1, k_2) \geq 1\), which means that we deal with large values of \(n\) only.

**Theorem 4.2** Let \((n - m + 1)a(p) \geq 1\) and \(ma(p) \leq 0.01\). Then

\[
\|H - \text{Bi}(N, \bar{p})\| \leq C_{21} \frac{a^{3/2}(p)m^2}{\sqrt{n - m + 1}}, \quad (28)
\]

\[
\|H - \text{Bi}(N, \bar{p})(I + A(I_1 - I)^3)\| \leq C_{22} \frac{a(p)m^2(a(p)m + \epsilon)}{n - m + 1}, \quad (29)
\]

\[
\|H - \text{Bi}(N, \bar{p})\|_{\infty} \leq C_{23} \frac{a(p)m^2}{n - m + 1}, \quad (30)
\]

\[
\|H - \text{Bi}(N, \bar{p})(I + A(I_1 - I)^3)\|_{\infty} \leq C_{24} \frac{\sqrt{a(p)m^2(a(p)m + \epsilon)}}{(n - m + 1)^{3/2}}. \quad (31)
\]

Note that the assumption \(ma(p) \leq 0.01\) in Theorem 4.2 is not very restrictive on \(p\) when \(k_1, k_2 > 1\). For example, it is satisfied for \(p \leq 1/4\) and \(N(n; 4, 4)\).
\textbf{Theorem 4.3} Let \((n - m + 1)a(p) \geq 1\) and \(ma(p) \leq 0.01\). Then

\[
\left\| H - Bi(N, \bar{p}) \right\| - \tilde{C}_{TV} \frac{a^{3/2}(p)m(m-1)}{2\sqrt{n - m + 1}} \leq C_{25}(m) \frac{a^{3/2}(p)m(m-1)}{\sqrt{n - m + 1}} \left( \frac{1}{\sqrt{(n - m + 1)a(p)}} + \frac{\epsilon}{N - \epsilon} + a(p) \right),
\]

\[
\left\| H - Bi(N, \bar{p}) \right\|_{\infty} - \tilde{C}_{L} \frac{a(p)m(m-1)}{2(n - m + 1)} \leq C_{26}(m) \frac{a(p)m(m-1)}{(n - m + 1)} \left( \frac{1}{\sqrt{(n - m + 1)a(p)}} + \frac{\epsilon}{N - \epsilon} + a(p) \right).
\]

Constants \(C_{25}(m)\) and \(C_{26}(m)\) depend on \(m\).

\textbf{Corollary 4.2} Let \(m\) be fixed, \(a(p) \to 0\), \((n - m + 1)a(p) \to \infty\), as \(n \to \infty\). Then

\[
\lim_{n \to \infty} \frac{\left\| H - Bi(N, \bar{p}) \right\|\sqrt{n - m + 1}}{a^{3/2}(p)m(m-1)} = \frac{\tilde{C}_{TV}}{2},
\]

\[
\lim_{n \to \infty} \frac{\left\| H - Bi(N, \bar{p}) \right\|_{\infty}(n - m + 1)}{a(p)m(m-1)} = \frac{\tilde{C}_{L}}{2}.
\]

\section{5 Bergström expansions}

The results of this section can be of independent interest. On one hand, we construct long asymptotic expansions. On the other hand, all these expansions contain measures from specific representation of \(F\) as convolution of measures. Therefore, probably one should treat Bergström expansions as auxiliary measures which can be used for derivation of more advanced expansions, as we have demonstrated in this paper. Bergström expansion was introduced in [6].

First, we need representation of the characteristic function \(\hat{F}(t)\) as product of functions.

\textbf{Lemma 5.1} Let conditions (7) and (8) be satisfied. Then

\[
\hat{F}(t) = \varphi_1(t)\varphi_2(t) \ldots \varphi_n(t),
\]

where \(\varphi_1(t) = Ee^{itX_1}\) and, for \(k = 2, \ldots, n\),

\[
\varphi_k(t) = 1 + E(e^{itX_k} - 1) + \sum_{j=1}^{k-1} \frac{E((e^{itX_j} - 1), (e^{itX_{j+1}} - 1), \ldots, (e^{itX_k} - 1))}{\varphi_j(t)\varphi_{j+1}(t) \ldots \varphi_{k-1}(t)}.
\]
Lemma 5.1 follows from more general Lemma 3.1 in [16]. Representation holds for all $t$, since

$$
\sqrt{E|e^{itX_k} - 1|^2} \leq \sqrt{2E|e^{itX_k} - 1|} \leq \sqrt{2\nu_1(k)} \leq \sqrt{0.02} < 1/6
$$

is satisfied for all $t$.

Let us define by $\sum_{n,m}^{ls}$ the sum by all possible collections of $m_i$ such that $m_1 + \ldots + m_n = l$, $m_i \in \{0, 1\}$, that is,

$$
\sum_{n,m}^{ls} = \sum \{m_1 + \ldots + m_n = l; m_i \in \{0, 1\}; i = 1, \ldots n\}.
$$

Let, for $j = 1, \ldots, n$ and $l = 2, \ldots, n$,

$$
\psi_j(t) = \exp\left\{\nu_1(j)(e^{it} - 1)\right\},
$$
$$
g_j(t) = \exp\left\{\nu_1(j)(e^{it} - 1) + \left(\frac{\nu_2(j) - \nu_1^2(j)}{2} + \hat{E}(X_{j-1}, X_j)\right)(e^{it} - 1)^2\right\}, \quad (32)
$$

$$
\hat{B}_l(Pois)(t) := \sum_{n,m}^{ls} \prod_{j=1}^{n} \psi_j(t)^{(1-m_j)}(\varphi_j(t) - \psi_j(t))^{m_j},
$$
$$
\hat{B}_l(G)(t) := \sum_{n,m}^{ls} \prod_{j=1}^{n} g_j(t)^{(1-m_j)}(\varphi_j(t) - g_j(t))^{m_j}.
$$

We define measures $B_l(Pois)$ and $B_l(G)$ by their inverse Fourier transform:

$$
B_l(Pois)\{k\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \hat{B}_l(Pois)(t) dt,
$$
$$
B_l(G)\{k\} = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikt} \hat{B}_l(G)(t) dt.
$$
Theorem 5.1 Let conditions (7) and (8) be satisfied. Then for, any \( s = 0, 1, 2, \ldots, n \) the following estimates hold

\[
\|F - \text{Pois}(\Gamma_1) - \sum_{l=1}^{s} B_l(\text{Pois})\| \leq C_{27}(s) \left\{ 1 + \Gamma_1 \min(1, \lambda^{-1}) \right\} R_0^{s+1} \min(1, \lambda^{-s-1}), \tag{33}
\]

\[
\|F - \text{Pois}(\Gamma_1) - \sum_{l=1}^{s} B_l(\text{Pois})\|_\infty \leq C_{28}(s) \left\{ 1 + \Gamma_1 \min(1, \lambda^{-1}) \right\} R_0^{s+1} \min(1, \lambda^{-s-3/2}), \tag{34}
\]

\[
\|F - G - \sum_{l=1}^{s} B_l(G)\| \leq C_{29}(s) \left\{ 1 + \Gamma_1 \min(1, \lambda^{-1}) \right\} R_1^{s+1} \min(1, \lambda^{-(3s+3)/2}), \tag{35}
\]

\[
\|F - G - \sum_{l=1}^{s} B_l(G)\|_\infty \leq C_{30}(s) \left\{ 1 + \Gamma_1 \min(1, \lambda^{-1}) \right\} R_1^{s+1} \min(1, \lambda^{-(3s+4)/2}). \tag{36}
\]

Since we have assumed that \( \sum_{l=1}^{0} = 0 \), the case \( s = 0 \) corresponds to (10), (12), (16) and (14). If instead of (8) we assume (9) and if all the \( X_i \) do not depend on \( n \) and \( s \) is fixed, then the orders of accuracy for the last two estimates are \( O(n^{-(s+1)/2}) \) and \( O(n^{-(s+2)/2}) \), respectively.

6 Auxiliary results

In this section, some auxiliary results from other papers are collected. For the sake of brevity, we will use notation \( U = I_1 - I \).

Lemma 6.1 Let \( t \in (0, \infty) \), \( 0 < p < 1 \) and \( n, j = 1, 2, \ldots \). We then have

\[
\|U^2 e^{tU}\| \leq \frac{3}{4te}, \quad \|U^j e^{tU}\| \leq \left( \frac{2j}{te} \right)^{j/2}, \quad \|U^j e^{tU}\|_\infty \leq \frac{C(j)}{t^{(j+1)/2}},
\]

\[
\|U^j(I + pU)^n\| \leq \left( \frac{n+j}{j} \right)^{-1/2} (p(1-p))^{-j/2},
\]

\[
\|U^j(I + pU)^n\|_\infty \leq \frac{\sqrt{n}}{2} \left( 1 + \sqrt{\frac{\pi}{2j}} \right) \frac{n}{n+j+1} \frac{(n+j+1)/2}{np(1-p)}^{(j+1)/2}.
\]

The first inequality was proved in [28] (formula (29)). The second bound follows from formula (3.8) in [12] and the properties of the total variation norm. The third relation follows from the formula of inversion. For the proof of other estimates, see Lemma 4 and formula (35) from [27].

For our asymptotically sharp norm estimates, we need the following lemmas.
Lemma 6.2 Let $t > 0$ and $p \in (0,1)$. Then
\[
\left\| U^3 e^{itU} \right\| - \frac{3\tilde{C}_{TV}}{t^{3/2}} \leq C \frac{t^2}{t^{3/2}}, \quad \left\| U^3 e^{itU} \right\|_\infty - \frac{3\tilde{C}_L}{t^2} \leq C \frac{t^{3/2}}{t^{3/2+1}}, \\
\left\| U^3 (I + pU)^n \right\| - \frac{3\tilde{C}_{TV}}{(np(1-p))^{3/2}} \leq C \frac{1}{(np(1-p))^2},
\]
\[
\left\| U^3 (I + pU)^n \right\|_\infty - \frac{3\tilde{C}_L}{(np(1-p))^2} \leq C \frac{1}{(np(1-p))^{3/2+1}}.
\]

Lemma’s statement follows from a more general Proposition 4 in [26] and from [9].

Lemma 6.3 Let $\lambda > 0$ and $k = 0, 1, 2, \ldots$. Then
\[
|\sin(t/2)|^k e^{-\lambda \sin^2(t/2)} \leq C(k) \frac{\lambda^{k/2}}{\lambda^{k/2}}, \\
\int_{-\pi}^{\pi} |\sin(t/2)|^k e^{-\lambda \sin^2(t/2)} dt \leq \frac{C(k)}{\max(1, \lambda^{(k+1)/2})}.
\]

Both estimates are trivial and very rough. Note that, for $|t| \leq \pi$, we have $|\sin(t/2)| \geq |t|/\pi$.

Lemma 6.4 Let $M$ be finite variation measure concentrated on integers. For $v \in \mathbb{R}$ and $u > 0$, we have
\[
\|M\| \leq \left(1 + u\pi\right)^{1/2} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{M}(t)|^2 + \frac{1}{u^2} \left|\left(e^{-itv}\hat{M}(t)\right)'\right|^2 dt\right)^{1/2}, \quad (37)
\]
and
\[
\|M\|_\infty \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{M}(t)| dt. \quad (38)
\]

The estimate (37) is well-known; see, for example, [23]. The estimate (38) follows from the formula of inversion.

Lemma 6.5 ([6]) For all numbers $A, B > 0$, $s = 0, 1, 2, \ldots, n$, the following identity holds:
\[
A^n = \sum_{m=0}^{s} \binom{n}{m} B^{n-m} (A - B)^m + \sum_{m=s+1}^{n} \binom{m-1}{s} A^{n-m} (A - B)^{s+1} B^{m-s-1}. \quad (39)
\]

Identity (39) holds also for convolutions of measures. The next lemma is its generalization.
Lemma 6.6 \((8)\) Let \(F_1, F_2, \ldots, F_n, G_1, \ldots, G_n\) be finite measures, \(s \in \{0, 1, 2, \ldots\}\). Then the following identity holds:

\[
\prod_{j=1}^{n} F_j = \prod_{j=1}^{n} G_j + \sum_{l=1}^{s} \sum_{n,m} \prod_{j=1}^{n} G_j^{1-m_j} (F_j - G_j)^{m_j} + R_n(s + 1),
\]

where

\[
R_n(s + 1) = \sum_{l=s+1}^{n} (F_l - G_l) \prod_{j=l+1}^{n} F_j \sum_{l-1,m}^{s} \prod_{j=1}^{l-1} G_j^{1-m_j} (F_j - G_j)^{m_j}.
\]

Lemma 6.7 Let \(s = 1, 2, 3\). For all \(t \in \mathbb{R}\),

\[
E \exp \{itX_k\} = 1 + \sum_{l=1}^{s} \nu_l(k) \frac{(e^{it} - 1)^l}{l!} + \theta \nu_{s+1}(k) \frac{|e^{it} - 1|^{s+1}}{s!},
\]

\[
E(\exp \{itX_k\})' = \sum_{l=1}^{s} \nu_l(k) \frac{ie^{it}(e^{it} - 1)^{l-1}}{l!} + \theta \nu_{s+1}(k) \frac{|e^{it} - 1|^s}{(s-1)!}.
\]

Lemma 6.7 is a particular case of Lemma 3 from \(32\).

Lemma 6.8 \((16)\) Let \(Z_1, Z_2, \ldots, Z_k\) be 1-dependent complex-valued random variables with \(E|Z_m|^2 < \infty\) \(1 \leq m \leq k\). Then

\[
|\tilde{E}(Z_1, Z_2, \ldots, Z_k)| \leq 2^{k-1} \prod_{m=1}^{k} (E|Z_m|^2)^{1/2}.
\]

7 Preliminary results

We use notation \(U = I_1 - I\) and symbols \(\theta\) and \(\Theta\) to denote all real or complex quantities satisfying \(|\theta| \leq 1\) and all measures of finite variation satisfying \(||\Theta|| = 1\) respectively. Moreover, let \(z = e^{it} - 1\) and \(Z_j = \exp \{itX_j\} - 1\). As before we assume that \(\nu_j(k) = 0\) and \(X_k = 0\) for \(k \leq 0\). Also, we omit the argument \(t\), wherever possible and, for example, write \(\varphi_k\) instead of \(\varphi_k(t)\).

The next lemma can easily be proved by induction.

Lemma 7.1 For all \(t \in \mathbb{R}\) and \(k \geq 2\), the following estimate holds:

\[
\tilde{E}^+(|Z_1|, \ldots, |Z_k|) \leq 4\tilde{E}^+(|Z_1|, \ldots, |Z_{k-1}|).
\]
Lemma 7.2 Let \(\max_k \nu_1(k) \leq 0.01\). Then, for \(k = 1, 2, \ldots, n\),

\[
|\varphi_k - 1| \leq \frac{1}{10}, \quad \frac{1}{|\varphi_k|} \leq \frac{10}{9}, \quad (41)
\]

\[
|\varphi_k - 1| \leq |z|[(0.66)\nu_1(k - 1) + (4.13)\nu_1(k)], \quad (42)
\]

\[
|\varphi_k - 1 - EZ_k| \leq \sin^2(t/2)[(0.374)\nu_1(k) + (0.288)\nu_1(k - 1)
+ (15.58)EX_{k-1}X_k + (0.1)EX_{k-2}X_{k-1}], \quad (43)
\]

Proof. We repeatedly apply below the following trivial inequalities:

\[
|z| \leq 2, \quad |Z_k| \leq 2, \quad |Z_k| \leq X_k|z|. \quad (44)
\]

The second estimate in (41) follows from the first estimate:

\[
|\varphi_k| \geq |1 - |\varphi_k - 1|| \geq 1 - (1/10) = 9/10.
\]

The first estimate in (41) follows from (42) and (44) and by the assumption of the lemma. It remains to prove the (42) and (43). Both proofs are very similar. From Lemma 5.1 and equation (41), we get

\[
|\varphi_k - 1 - EZ_k| \leq \frac{1}{|\varphi_k-1|}^{\frac{10}{9}} |\tilde{E}(Z_{k-1}, Z_k)| + \frac{1}{|\varphi_k-3\varphi_{k-1}|} |\tilde{E}(Z_{k-2}, Z_{k-1}, Z_k)| + \frac{1}{|\varphi_k-3\cdots\varphi_{k-1}|} |\tilde{E}(Z_{k-3}, \ldots, Z_k)|
+ \frac{1}{|\varphi_k-4\cdots\varphi_{k-1}|} |\tilde{E}(Z_{k-4}, \ldots, Z_k)| + \frac{1}{|\varphi_k-5\cdots\varphi_{k-1}|} |\tilde{E}(Z_{k-5}, \ldots, Z_k)| + \sum_{j=1}^{k-6} \frac{1}{|\varphi_j\varphi_{j+1}\cdots\varphi_{k-1}|} |\tilde{E}(Z_j, \ldots, Z_k)|
\leq \left(\frac{10}{9}\right)^2 |\tilde{E}(Z_{k-1}, Z_k)| + \left(\frac{10}{9}\right)^2 |\tilde{E}(Z_{k-2}, Z_{k-1}, Z_k)| + \left(\frac{10}{9}\right) |\tilde{E}(Z_{k-3}, \ldots, Z_k)|
+ \left(\frac{10}{9}\right)^4 |\tilde{E}(Z_{k-4}, \ldots, Z_k)| + \left(\frac{10}{9}\right)^5 |\tilde{E}(Z_{k-5}, \ldots, Z_k)|
+ \sum_{j=1}^{k-6} \left(\frac{10}{9}\right)^{k-j} |\tilde{E}(Z_j, \ldots, Z_k)|. \quad (45)
\]

By (44) and Lemma 6.8 we obtain

\[
E|Z_j| \leq \nu_1(j)|z| \leq 0.02|\sin(t/2)|, \quad E|Z_j|^2 \leq 2E|Z_j| \leq 2\nu_1(j)|z| = 4\nu_1(j)|\sin(t/2)| \quad (46)
\]
and

\[
|\hat{E}(Z_j, \ldots Z_k)| \leq 2^{k-j} 2^{(k-j+1)/2} |z|^{(k-j+1)/2} \prod_{l=j}^{k} \sqrt{\nu_1(l)} \leq 2^{2(k-j)-1} |z|^2 \sqrt{\nu_1(k)\nu_1(k-1)0.1^{k-j-1}} \leq 4^{k-j} \sin^2 \frac{t}{2} [\nu_1(k) + \nu_1(k-1)]0.1^{k-j-1} = 10 \sin^2 \frac{t}{2} [\nu_1(k) + \nu_1(k-1)](0.4)^{k-j}. \quad (47)
\]

Consequently,

\[
\sum_{j=1}^{k-6} \left( \frac{10}{9} \right)^{k-j} |\hat{E}(Z_j, \ldots Z_k)| \leq 10 \sin^2 \frac{t}{2} [\nu_1(k) + \nu_1(k-1)] \sum_{j=1}^{k-6} \left( \frac{4}{9} \right)^{k-j} \leq 2 \sin^2 \frac{t}{2} [\nu_1(k) + \nu_1(k-1)](0.0694). \quad (48)
\]

By 1-dependence, (44) and Hölder’s inequality (see also [16]), we have for \( j \geq 3, \)

\[
|E Z_{k-1} \cdots Z_k| \leq \prod_{i=k-j}^{k} \sqrt{E|Z_i|^2} \leq \prod_{i=k-j}^{k} \sqrt{2\nu_1(i)|z|^2} \leq 2^{(j+1)/2} |z|(j+1)/2 \sqrt{\nu_1(k-1)\nu_1(k)(0.1)^j-j-1} \leq 2^{j-1} |z|^2 \nu_1(k-1) + \nu_1(k) (0.1)^j-j-1 = 2^j \sin^2 (t/2) [\nu_1(k-1) + \nu_1(k)] (0.1)^j-j. \quad (49)
\]

Moreover, for any \( j, \)

\[
|E Z_{j-1} Z_j| \leq 2E|Z_j| \leq 4 \sin (t/2)|\nu_1(j)|, \quad |E Z_{j-1} Z_j| \leq |z|^2 E X_{j-1} X_j = 4 \sin^2 (t/2) E X_{j-1} X_j \quad (50)
\]

and

\[
|E Z_{j-2} Z_{j-1} Z_j| \leq 2E|Z_{j-1} Z_j| \leq 8 \sin^2 (t/2) E X_{j-1} X_j. \quad (51)
\]

Therefore, from (46), we have

\[
|\hat{E}(Z_{j-1}, Z_j)| \leq E|Z_{j-1} Z_j| + \nu_1(j-1)\nu_1(j)|z|^2 \leq 2.02|z|\nu_1(j) \leq 0.0404 \sin (t/2). \quad (52)
\]
Similarly, applying (49), (50), (51) and (54), we obtain the following rough estimates:

\[
|\hat{E}(Z_{j-2}, Z_{j-1}, Z_j)| \leq |z||\sin(t/2)|\{0.2\nu_1(j-1) + 0.2804\nu_1(j)\} \leq 0.01\sin^2(t/2),
\]

\[
|\hat{E}(Z_{j-3}, \ldots, Z_j)| \leq |z||\sin(t/2)|\{0.044\nu_1(j-1) + 0.1348\nu_1(j)\} \leq 0.0036\sin^2(t/2),
\]

\[
|\hat{E}(Z_{j-4}, \ldots, Z_j)| \leq |z||\sin(t/2)|\{0.0169\nu_1(j-1) + 0.0405\nu_1(j)\} \leq 0.00115\sin^2(t/2).
\]

It is easy to get estimate

\[
|\hat{E}(Z_{k-1}, Z_k)| \leq E|Z_{k-1}Z_k| + \nu_1(k-1)\nu_1(k)|z|^2 \leq \sin^2(t/2)\{4EX_{k-1}X_k + 0.04\nu_1(k)\}.
\]

Similarly, taking into account (49)–(53), we get

\[
|\hat{E}(Z_{k-2}, Z_{k-1}, Z_k)| \leq \sin^2(t/2)\{8.08EX_{k-1}X_k + 0.08EX_{k-2}X_{k-1} + 0.0008\nu_1(k-1)\},
\]

\[
|\hat{E}(Z_{k-3}, \ldots, Z_k)| \leq \sin^2(t/2)\{0.3216EX_{k-1}X_k + 0.08\nu_1(k-1) + 0.1\nu_1(k)\},
\]

\[
|\hat{E}(Z_{k-4}, \ldots, Z_k)| \leq \sin^2(t/2)\{0.3632EX_{k-1}X_k + 0.0176\nu_1(k-1) + 0.0248\nu_1(k)\},
\]

\[
|\hat{E}(Z_{k-5}, \ldots, Z_k)| \leq \sin^2(t/2)\{0.0944EX_{k-1}X_k + 0.0068\nu_1(k-1) + 0.0091\nu_1(k)\}.
\]

Combining (48), (51)–(55) with (45) we prove (43).

For the proof of (42), we apply mathematical induction. Let us assume that (41) holds for first \(k-1\) functions and let \(k \geq 6\). Then the proof is almost identical to the proof of (43). We expand \(\varphi_k\) just like in (45):

\[
|\varphi_k - 1| \leq E|Z_k| + \left(\frac{10}{9}\right)|\hat{E}(Z_{k-1}, Z_k)| + \cdots + \sum_{j=1}^{k-4} \left(\frac{10}{9}\right)^{k-j} |\hat{E}(Z_j, \ldots, Z_k)|.
\]

Applying (49), (46) and (52)–(53), we easily complete the proof of (42). The proof for \(k < 6\) is analogous. □

**Lemma 7.3** Let \(\nu_1(k) \leq 0.01, \nu_2(k) < \infty, \text{ for } 1 \leq k \leq n\). Then, for all \(t \in \mathbb{R}\),

\[
|\varphi_k| \leq 1 - \lambda_k\sin^2(t/2) \leq \exp\{-\lambda_k\sin^2(t/2)\}
\]
\[
|\tilde{F}(t)| = \prod_{k=1}^{n} |\varphi_k| \leq \exp\{-1.3\lambda \sin^2(t/2)\}.
\]

Here \(\lambda_k = 1.606\nu_1(k) - 0.288\nu_1(k-1) - 2\nu_2(k) - 0.1EX_{k-2}X_{k-1} - 15.58EX_{k-1}X_k\) and \(\lambda\) is defined by (3).

**Proof.** We have

\[
|\varphi_k| \leq |1 + EZ_k| + |\varphi_k - 1 - EZ_k| \leq |1 + \nu_1(k)z| + \frac{\nu_2(k)}{2}|z|^2 + |\varphi_k - 1 - EZ_k|.
\]

Applying the definition of the square of the absolute value for complex number we get

\[
|1 + \nu_1(k)z|^2 = (1 - \nu_1(k)\cos t)^2 + (\nu_1(k)\sin t)^2 = 1 - 4\nu_1(k)(1 - \nu_1(k))\sin^2(t/2).
\]

Consequently,

\[
|1 + \nu_1(k)z| \leq 1 - \nu_1(k)(1 - \nu_1(k))\sin^2(t/2) \leq 1 - 2\nu_1(k)(1 - \nu_1(k))\sin^2(t/2).
\]

Combining the last estimate with (13) and and using Lemma 6.7, we get the first estimate of the lemma. The second estimate follows immediately. \(\square\)

For expansions of \(\varphi_k\) in powers of \(z\), we use the following notation:

\[
\begin{align*}
\gamma_2(k) &= \frac{\nu_2(k)}{2} + \tilde{E}(X_{k-1}, X_k), \\
\gamma_3(k) &= \frac{\nu_3(k)}{6} + \frac{\tilde{E}_2(X_{k-1}, X_k)}{2} + \tilde{E}(X_{k-2}, X_{k-1}, X_k) - \nu_1(k-1)\tilde{E}(X_{k-1}, X_k), \\
r_0(k) &= \nu_2(k) + \sum_{l=0}^{3} \nu_1^2(k-l) + EX_{k-1}X_k, \\
r_1(k) &= \nu_3(k) + \sum_{l=0}^{5} \nu_1^3(k-l) + \nu_1(k-1)EX_{k-1}X_k + \tilde{E}_2^+(X_{k-1}, X_k) + \tilde{E}^+(X_{k-2}, X_{k-1}, X_k),
\end{align*}
\]
\[ r_2(k) = \nu_4(k) + \sum_{l=0}^{7} \nu_1^1(k-l) + \nu_2^2(k) + (EX_{k-1}X_k)^2 + (EX_{k-2}X_{k-1})^2 \]
\[ + \nu_1(k-1)\tilde{E}_2^+(X_{k-1}, X_k) + \sum_{l=0}^{3} \nu_1(k-l)\tilde{E}_2^+(X_{k-2}, X_{k-1}, X_k) \]
\[ + \tilde{E}_2^+(X_{k-2}, X_{k-1}, X_k) + \tilde{E}_3^+(X_{k-1}, X_{k-2}) + \tilde{E}_3^+(X_{k-3}, X_{k-2}, X_{k-1}, X_k). \]

Lemma 7.4  Let condition (7) be satisfied, \( k = 1, \ldots, n \). Then, for all \( t \in \mathbb{R} \),
\[
\varphi_k = 1 + \nu_1(k)z + \theta C|z|^2r_0(k), \quad (56)
\]
\[
\varphi_k = 1 + \nu_1(k)z + \gamma_2(k)z^2 + \theta C|z|^3r_1(k), \quad (57)
\]
\[
\varphi_k = 1 + \nu_1(k)z + \gamma_2(k)z^2 + \gamma_3(k)z^3 + \theta C|z|^4r_2(k), \quad (58)
\]
\[
\frac{1}{\varphi_{k-1}} = 1 + C\theta|z|[\nu_1(k-2) + \nu_1(k-1)], \quad (59)
\]
\[
\frac{1}{\varphi_{k-1}} = 1 - \nu_1(k-1)z + C\theta|z|^2[\nu_2(k-1) + \sum_{l=1}^{4} \nu_1^2(k-l) + EX_{k-2}X_{k-1}], \quad (60)
\]
\[
\frac{1}{\varphi_{k-1}} = 1 - \nu_1(k-1)z - \left( \frac{\nu_2(k)}{2} - \nu_1^2(k) + \tilde{E}(X_{k-1}, X_k) \right)z^2
\]
\[ + C\theta|z|^3\left\{ \nu_3(k-1) + \sum_{l=1}^{5} \nu_1^3(k-l) + \tilde{E}_2^+(X_{k-1}, X_k) \right\}
\]
\[ + \tilde{E}_2^+(X_{k-2}, X_{k-1}, X_k) + \nu_1(k-1)\tilde{E}_2^+(X_{k-1}, X_k) \]
\[ + (\nu_1(k-1) + \nu_1(k))[\nu_2(k) + \sum_{l=0}^{3} + \tilde{E}_3^+(X_{k-1}, X_k)] \right\}, \quad (61)
\]
\[
\frac{1}{\varphi_{k-1}\varphi_{k-2}} = 1 + C\theta|z|[\nu_1(k-3) + \nu_1(k-2) + \nu_1(k-1)], \quad (62)
\]
\[
(\varphi_k - 1)^2 = \nu - 1^2(k)z^2 + C\theta|z|[\nu_1(k-1)
\]
\[ + \nu_1(k)[\nu_2(k) + \sum_{l=0}^{3} \nu_1^2(k-l) + \tilde{E}(X_{k-1}, X_k)] \right\}, \quad (63)
\]
\[
(\varphi_k - 1)^3 = \nu_1^3(k)z^3
\]
\[ + C\theta|z|[\nu_2^2(k) + \sum_{l=0}^{3} \nu_1^4(k-l) + (EX_{k-1}X_k)^2 + \tilde{E}^+(X_{k-1}, X_k)] \right\}. \quad (64)
\]

Proof. Further on we assume that \( k \geq 7 \). For smaller values of \( k \), all proofs just become shorter. The lemma is proved in four steps. First, we prove (56), (57), (59) and (62). Second, we obtain (60) and (62). Then we prove (61) and (63). Final step is the proof of (68). At each step, we employ results from the previous step. Since all proofs are very similar, we give just some of them.
Due to (41), we have
\[
\frac{1}{\varphi_{k-1}} = \frac{1}{1 - (1 - \varphi_{k-1})} = 1 + \theta \sum_{j=1}^{\infty} |1 - \varphi_k|^j = 1 + (1 - \varphi_{k-1}) + C\theta |1 - \varphi_{k-1}|.
\]
Therefore, (59) and (62) follow from (42).

From Lemmas 5.1, 6.7, 7.1, equation (41) and second estimate in (47), we get
\[
|\varphi_k| = 1 + EZ_k + \frac{|\bar{E}(Z_{k-1}, Z_k)|}{|\varphi_{k-1}|} + \frac{|\bar{E}(Z_{k-2}, Z_{k-1}, Z_k)|}{|\varphi_{k-3}\varphi_{k-1}|} + \sum_{j=1}^{k-3} \frac{|\bar{E}(Z_j, \ldots, Z_k)|}{|\varphi_j\varphi_{j+1} \cdots \varphi_{k-1}|}
\]
\[\leq 1 + \nu_1(k)z + C\theta z|z|^2 \nu_2(k) + C\theta \bar{E}^+(|Z_{k-1}|, |Z_k|)
\]
\[+ C\theta z^3 \sqrt{\nu_1(k-3)\nu_1(k-2)\nu_1(k-1)\nu_1(k)}
\]
\[= 1 + \nu_1(k)z + C\theta z^2 \left\{ \nu_2(k) + EX_{k-1}X_k + \nu_1(k-1)\nu_1(k) + \sum_{i=0}^{3} \nu_1^2(k-i) \right\}
\]
\[= 1 + \nu_1(k)z + C\theta z^2 \tau_0(k),
\]
which proves (56).

Proof of (57) is almost identical. We take longer expansion in Lemma 5.1 and note that due to (39)
\[
Z_k = X_kz + \theta X_k(X_k - 1)\frac{|z|^2}{2}.
\]
Therefore,
\[
\bar{E}(Z_{k-1}, Z_k) = \bar{E}(X_{k-1}z + \theta |z|^2 X_{k-1}(X_{k-1} - 1), Z_k) = z\bar{E}X_{k-1}Z_k
\]
\[+ C\theta |z|^3 \bar{E}^+(X_{k-1}(X_{k-1} - 1), X_k) = z^2\bar{E}(X_{k-1}, X_k) + C\theta |z|^3 \bar{E}^+_2(X_{k-1}, X_k).
\]
Other proofs are simple repetition of the given ones with the only exception that results from previous steps are used. For example, for the proof of (58), we apply Lemma 5.1 and get
\[
|\varphi_k| = 1 + EZ_k + \sum_{j=1}^k \frac{|\bar{E}(Z_j, \ldots, Z_k)|}{|\varphi_j\varphi_{j+1} \cdots \varphi_{k-1}|} = 1 + EZ_k + \sum_{j=k-2}^{k-3} + \sum_{j=k-6}^{k-7}.
\]
By (47),

$$\sum_{j=1}^{7} \frac{|\hat{E}(Z_j, \ldots, Z_k)|}{|\varphi_j\varphi_{j+1} \cdots \varphi_{k-1}|} \leq C|z|^4 \sqrt{\nu_1(k-1) \cdots \nu_1(k)} \leq C|z|^4 \sum_{l=0}^{7} \nu_1^4(k-l)$$

and by (40)

$$\left| \sum_{j=k-6}^{k-3} \right| \leq C \sum_{j=k-6}^{k-3} \hat{E}^+(|Z_j|, \ldots, |Z_k|) \leq C \hat{E}^+(|Z_{k-3}|, \ldots, |Z_k|) \leq C|z|^4 \hat{E}^+(X_{k-3}, \ldots, X_k).$$

For other summands, we apply Lemma 6.7 and use the previous estimates.

□

Lemma 7.5 Let condition (7) hold. Then, for all $t \in \mathbb{R}$,

$$\left( \hat{E}(Z_j, \ldots, Z_k) \right)^\prime = \sum_{i=j}^{k} \hat{E}(Z_j, \ldots, Z_i', \ldots, Z_k),$$

$$|\hat{E}(Z_j, \ldots, Z_i', \ldots, Z_k)| \leq 2^{3(k-j)+1}/2 |z|^{(k-j)/2} \prod_{l=j}^{k} \sqrt{\nu_1(l)}.$$

The first identity was proved in [16]. Applying (49) we obtain

$$|\hat{E}(Z_j, \ldots, Z_i', \ldots, Z_k)| \leq 2^{k-j} \sqrt{E|Z_i'|^2} \prod_{l \neq i}^{k} \sqrt{E|Z_l|^2}.$$

Due to assumptions

$$E|Z_i'|^2 = E|e^{itX_i}X_i|^2 = EX_i^2 = EX_i(X_i - 1 + 1) = \nu_2(l) + \nu_1(l) \leq 2\nu_1(l).$$

Combining the last estimate with $E|Z_i|^2 \leq 2E|Z_i| \leq 2|z|\nu_1(l)$, the proof follows. □

Lemma 7.6 Let condition (7) be satisfied, $k = 1, \ldots, n$ and $\varphi_k$ be defined as in Lemma 5.1. Then, for all $t \in \mathbb{R}$,

$$\varphi_k' = 33\theta[\nu_1(k) + \nu_1(k-1)], \quad (65)$$

$$\varphi_k' = \nu_1(k)z' + \theta C|z|(r_0(k) + \hat{E}^+(X_{k-2}, X_{k-1})), \quad (66)$$
\[
\varphi_k' = 1 + \nu_1(k)z' + \gamma_2(k)(z^2)' + \theta C|z|^2(\nu_1(k) + [\nu_1(k - 2) + \nu_1(k)]\text{Ex}_{k-1}X_k
\]
\[
+ \hat{E}^+(X_{k-4}, X_{k-3}, X_{k-2}) + \hat{E}^+(X_{k-3}, X_{k-2}, X_{k-1})
\],
(67)
\[
\varphi_k' = 1 + \nu_1(k)z' + \gamma_2(k)(z^2)' + \gamma_3(k)(z^3)' + \theta C|z|^3(\nu_1(k) + \hat{E}^+(X_{k-4}, \ldots, X_{k-1})
\]
\[
+ \hat{E}^+(X_{k-5}, \ldots, X_{k-2}).
\]
(68)

**Proof.** Note that
\[
\left( \hat{E}(Z_j, \ldots, Z_k) \right)' \left( \varphi_j \cdots \varphi_{k-1} \right) = \frac{\hat{E}(Z_j, \ldots, Z_k)'}{\varphi_j \cdots \varphi_k} - \frac{\hat{E}(Z_j, \ldots, Z_k)}{\varphi_j \cdots \varphi_k} \sum_{m=j}^{k-1} \varphi_m' \varphi_m.
\]

Now the proof is just repetition of the proof of Lemma 7.4. For example, (65) is easily verifiable for \( k = 0, 1 \). Let us assume that it holds for \( 1, 2, \ldots, k - 1 \). From Lemmas 5.1 and 6.7 and equation (47), we get
\[
|\varphi_k'| \leq \nu_1(k) + \sum_{j=1}^{k-1} \left| \hat{E}(Z_j, \ldots, Z_k) \right| + \sum_{j=1}^{k-1} \left| \hat{E}(Z_j, \ldots, Z_k) \right| \sum_{m=j}^{k-1} \left| \varphi_m' \right| \left| \varphi_m \right|
\]
\[
\leq \nu_1(k) + \sum_{j=1}^{k-1} \left( \frac{10}{9} \right)^{k-j} \sum_{i=j}^{k} \left| \hat{E}(Z_j, \ldots, Z_i, \ldots, Z_k) \right|
\]
\[
+ \sum_{j=1}^{k-1} \left( \frac{10}{9} \right)^{k-j} \left| \hat{E}(Z_j, \ldots, Z_k) \right|(k - j)33 \cdot 0.02 \cdot \left( \frac{10}{9} \right).
\]

By Lemma 7.5
\[
\left| \hat{E}(Z_j, \ldots, Z_i, \ldots, Z_k) \right| \leq |\nu_1(k - 1) + \nu_1(k)|(0.04)^{k-j} \frac{10}{\sqrt{2}}.
\]
Combining the last two estimates and (47), the proof of (65) is completed.

We omit the proofs of remaining expansions and note only that
\[
(e^{itX} - 1)' = iXe^{itX} = ie^{itX}X = z'X\left( 1 + (X - 1)z + \theta \frac{(X - 1)(X - 2)}{2} |z|^2 \right),
\]
due to Bergström’s identity. \( \square \)

We next need expansions for \( g_k \) defined by (32).
Lemma 7.7 Let conditions in (7) be satisfied, \(k = 1, 2, \ldots, n\). Then, for all \(t \in \mathbb{R}\),

\[
\begin{align*}
g_k &= 1 + C\theta |z| \left[ \nu_1(k - 1) + \nu_1(k) \right], \\
g_k' &= C\theta [\nu_1(k - 1) + \nu_1(k)], \\
g_k &= 1 + \nu_1(k)z + \gamma_2(k)z^2 + C\theta |z|^3 \left\{ \nu_1^2(k - 1) + \nu_1^3(k) + \nu_1(k)\nu_2(k) \\
&\quad + [\nu_1(k - 1) + \nu_1(k)]E_{X_{k-1}X_k} \right\}, \\
g_k' &= \nu_1(k)z' + \gamma_2(k)(z^2)' + C\theta |z|^3 \left\{ \nu_1^4(k - 1) + \nu_1^3(k) + \nu_1(k)\nu_2(k) \\
&\quad + [\nu_1(k - 1) + \nu_1(k)]E_{X_{k-1}X_k} \right\}, \\
g_k &= 1 + \nu_1(k)z + \gamma_2(k)z^2 + 3 \gamma_3(k)z^3 \\
&\quad + C\theta |z|^4 \left\{ \nu_1^5(k - 1) + \nu_1^4(k) + \nu_1^2(k) \right\}, \\
g_k' &= \nu_1(k)z' + \gamma_2(k)(z^2)' + 3 \gamma_3(k)(z^3)' \\
&\quad + C\theta |z|^3 \left\{ \nu_1^6(k - 1) + \nu_1^5(k) + \nu_1^3(k) \right\}, \\
|g_k| &\leq \exp \{-\lambda_k \sin^2(t/2)\}.
\end{align*}
\]

Here \(\lambda_k\) is as in Lemma 7.3 and

\[
\hat{\gamma}_3(k) = \frac{\nu_1(k)\nu_2(k) - \nu_1^3(k)}{2} + \nu_1(k)\hat{E}(X_{k-1}, X_k) + \frac{\nu_1^3(k)}{6}.
\]

Proof. For any complex number \(b\), we have

\[
e^b = 1 + b + \frac{b^2}{2} + \cdots + \frac{b^s}{s!} + \theta \frac{|b|^{s+1}}{(s + 1)!} e^{|b|}.
\]

Due to assumptions

\[
\begin{align*}
E_{X_{j-1}X_j} &\leq \sqrt{E_{X_{j-1}X_j}^2} \leq \sqrt{[\nu_2(j - 1) + \nu_1(j - 1)] [\nu_2(j) + \nu_1(j)]} \\
&\leq 2\sqrt{\nu_1(j - 1)\nu(j)} \leq 2[\nu_1(j - 1) + \nu_1(j)].
\end{align*}
\]

Therefore, the exponent of \(g_k\) is bounded by some absolute constant \(C\) and (69) and (70) easily
follow. We have

\[ g_k = 1 + \nu_1(k)z + \gamma_2(k)z^2 + C\theta\{\nu_1^3(k) + \nu_2^2(k) + \nu_1(k)\nu_2(k) + \nu_1(k)\nu_2(k) + \nu_1(k)\nu_2(k) + (\tilde{E}^+(X_{k-1}, X_k))^2\}. \]

Moreover,

\[ \nu_2^2(k) \leq \nu_1(k)\nu_2(k), \quad \nu_1(k-1)\nu_1^2(k) \leq \nu_1^3(k-1) + \nu_1^3(k) \]

and

\[ (\tilde{E}^+(X_{k-1}, X_k) \leq 2(EX_{k-1}X_k)^2 + 2\nu_2^2(k-1)\nu_1^2(k) \leq 2[\nu_1(k-1) + \nu_1(k)]EX_{k-1}X_k + 2\nu_1^3(k-1) + 2\nu_1^3(k). \]

Thus, (71) easily follows. The estimates (72) – (74) are proved similarly.

For the proof of (75), note that

\[ \tilde{E}^+(X_{k-1}, X_k) \leq EX_{k-1}X_k + 0.01\nu_1(k), \quad \nu_1^2(k) \leq 0.01\nu_1(k) \]

and

\[ |g_k| \leq \exp\left\{-2\nu_1(k)\sin^2(t/2) + 2[\nu_2(k) + \nu_1^2(k) + 2\tilde{E}^+(X_{k-1}, X_k)]\sin^2(t/2)\right\} \]

\[ \leq \exp\left\{-1.92\nu_1(k)\sin^2(t/2) + 2\nu_2(k)\sin^2(t/2) + 4EX_{k-1}X_k\sin^2(t/2)\right\}, \]

which completes the proof. \( \square \)

For asymptotic expansions, we need a few smoothing estimates.

**Lemma 7.8** Let conditions (7) and (8) be satisfied, 0 \( \leq \alpha \leq 1 \), and \( M \) be any finite (signed) measure. Then

\[ \|M \exp\{\Gamma_1U + \alpha\Gamma_2U^2\}\| \leq C\|M \exp\{0.9\lambda U\}\|, \quad \|M \exp\{\Gamma_1U + \alpha\Gamma_2U^2\}\|_\infty \leq C\|M \exp\{0.9\lambda U\}\|_\infty. \]
Proof. Due to (7) and (8), we have

$$\Gamma_1 - 3.1|\Gamma_2| \geq \Gamma_1 - 1.55 \sum_{k=1}^{n} \nu_2(k) - 0.0155\Gamma_1 - 3.1 \sum_{k=1}^{n} \text{EX}_{k-1}X_k - 0.031\Gamma_1 \geq 0.9\lambda.$$ 

Thus,

$$\|M \exp\{\Gamma_1 U + \alpha\Gamma_2 U^2\}\| \leq \|M \exp\{(\Gamma_1 - 3.1|\Gamma_2|)U\}\|\|\exp\{3.1|\Gamma_2|U + \alpha\Gamma_2 U^2\}\| \leq \|M \exp\{0.9\lambda U\}\|\|\exp\{3.1|\Gamma_2|U + \alpha\Gamma_2 U^2\}\|.$$ 

Analogous estimate holds for local norm. It remains to prove that the second exponent measure is bounded by some absolute constant. Note that the total variation of any distribution equals unity. Therefore, by Lemma 6.1

$$\|\exp\{3.1|\Gamma_2|U + \alpha\Gamma_2 U^2\}\| = \|\exp\{3.1|\Gamma_2|U\}\left(I + \sum_{m=1}^{\infty} \frac{(\alpha\Gamma_2 U^2)^m}{m!}\right)\| \leq 1 + \sum_{m=1}^{\infty} \frac{|\Gamma_2|^m}{m!} U^2 \exp\{3.1|\Gamma_2|U/m\}\| \leq 1 + \sum_{m=1}^{\infty} \frac{|\Gamma_2|^m}{m! e^{-m} \sqrt{2\pi m}} \left(\frac{3m}{3.1|\Gamma_2|e}\right)^m \leq C.$$ 

Combining both inequalities given above, we complete the proof of the lemma. □

Lemma 7.9 Let conditions (7) and (9) hold and M be any finite (signed) measure. Then

$$\text{NB}(r, \theta) = \exp\left\{\Gamma_1 U + \Gamma_2 U^2 + \frac{4\Gamma_2^2 U^3}{3\Gamma_1} + \frac{2\Gamma_2^4 U^4 \Theta}{\Gamma_1^4} \frac{1}{0.7}\right\} = \exp\left\{\Gamma_1 U + \Gamma_2 U^2 + \frac{4\Gamma_2^2 U^3 \Theta}{3\Gamma_1} \frac{3}{28}\right\} = \exp\left\{0.5\Gamma_1 U\right\}\Theta C. \quad (77)$$

Proof. Due to (9),

$$\Gamma_2 = \frac{1}{2} \sum_{k=1}^{n} (\nu_2^2(k) - \nu_1^2(k)) + \sum_{k=1}^{n} \text{Cov}(X_{k-1}, X_k) \leq \frac{1}{2} \sum_{k=1}^{n} \nu_2(k) + \sum_{k=1}^{n} |\text{Cov}(X_{k-1}, X_k)| \leq \frac{3}{40} \Gamma_1.$$ 

Therefore,

$$\frac{1 - \theta}{q} = \frac{2\Gamma_2}{\Gamma_1} \leq 0.15, \quad \left(\frac{1 - \theta}{q}\right)\|U\| \leq 0.15(\|I_1\| + \|I\|) \leq 0.3.$$
Consequently, from (2),

\[
\text{NB}(r, \overline{q}) = \exp \left\{ \sum_{j=1}^{\infty} \frac{r}{j} \left( \frac{1-\overline{q}}{\overline{q}} \right)^j U^j \right\}
\]

\[
= \exp \left\{ \Gamma_1 U + r \left( \frac{1-\overline{q}}{\overline{q}} \right)^2 U^2 + \frac{r}{3} \left( \frac{1-\overline{q}}{\overline{q}} \right)^3 U^3 + \frac{r}{4} \left( \frac{1-\overline{q}}{\overline{q}} \right)^4 U^4 \right\}
\]

\[
= \exp \left\{ \Gamma_1 U + r \left( \frac{1-\overline{q}}{\overline{q}} \right)^2 U^2 + \frac{r}{3} \left( \frac{1-\overline{q}}{\overline{q}} \right)^3 U^3 \right\}
\]

Recalling that \( r(1-\overline{q})/\overline{q} = \Gamma_1 \), we obtain all equalities except the last one. The last equality is equivalent to

\[
\left\| \exp \left\{ 0.5\Gamma_1 U + \Gamma_1 U^2 \Theta \frac{3}{28} \right\} \right\| \leq C
\]

which is proved similarly to Lemma 7.8.

**Lemma 7.10** Let \( M \) be any finite (signed) measure. Then under conditions (7) and (9),

\[
\text{Bi}(N, \overline{p}) = \exp \left\{ -N \sum_{j=1}^{\infty} \frac{(-\overline{p}U)^j}{j} \right\}
\]

\[
= \exp \left\{ \Gamma_1 U + 2\Gamma_2 U^2 + U^2 \Theta \frac{50\Gamma_2^2}{21\Gamma_1} + \frac{N\overline{p}U^3}{3} + \frac{N\overline{p}U^4}{4} \Theta \frac{5}{3} \right\}
\]

\[
= \exp \left\{ \Gamma_1 U + \Gamma_2 U^2 + U^2 \Theta \frac{50\Gamma_2^2}{21\Gamma_1} + \frac{N\overline{p}U^3}{3} \right\}
\]

\[
= \exp \left\{ \Gamma_1 U + \frac{N\overline{p}U^2}{2} \Theta \frac{5}{3} \right\} = \exp \left\{ \Gamma_1 U + \Gamma_1 U^2 \Theta \frac{1}{6} \right\} = \exp \left\{ 0.5\Gamma_1 U \right\} \Theta C.
\]

**Proof.** Due to (9),

\[
|\Gamma_2| \leq \frac{1}{2} \sum_{k=1}^{n} (\nu_2^2(k) + 0.01\nu_1(k)) + \sum_{k=1}^{n} |\text{Cov}(X_{k-1}, X_k)| \leq \Gamma_1 (0.025 + 0.005 + 0.05) = 0.08\Gamma_1.
\]

Therefore,

\[
\overline{p} = \frac{\Gamma_1}{N - \epsilon} \leq \frac{\Gamma_1}{N - 1} \leq \frac{2|\Gamma_2|}{\Gamma_1 - 2|\Gamma_2|} \leq \frac{50|\Gamma_2|}{21\Gamma_1} < \frac{1}{5},
\]

and

\[
\frac{\epsilon}{N} \leq \frac{2|\Gamma_2|}{\Gamma_1} \leq \frac{2|\Gamma_2|}{\Gamma_1} \leq 0.16.
\]
Consequently,
\[ N\bar{p}^2 = 2|\Gamma_2|\frac{\bar{N}}{N} = 2|\Gamma_2|\frac{1}{1 - \epsilon/N} = 2|\Gamma_2|\left(1 + \frac{\epsilon}{N}\theta_{100}\right) \]
and
\[ -\frac{N\bar{p}^2}{2} = \Gamma_2 + \theta_{50}\Gamma_2^2\frac{\epsilon}{21\Gamma_2^2} \tag{79} \]

Taking into account (78), we prove
\[
\text{Bi}(N, \bar{p}) = \exp \{-N\sum_{j=1}^{s} \frac{(-\bar{p}U)^j}{j}\} = \exp \left\{\Gamma_1 U - \frac{N(\bar{p}U)^2}{2} + \frac{N(\bar{p}U)^3}{3} + \frac{N\bar{p}^4U^4}{4}\Theta_{5}^3\right\} = \exp \left\{\Gamma_1 U + \frac{N(\bar{p}U)^2}{2}\Theta_{5}^3\right\} = \exp \left\{\Gamma_1 U + \Gamma_1 U^2\Theta_{5}^3\right\} = \exp \left\{\Gamma_1 U + \frac{\Gamma_1 U^2}{6}\right\}.
\]

Combining (79) with the last expansions, we obtain all equalities except the last one whose proof is similar to that of Lemma 7.8.

8 Proofs

Proof of Theorem 5.1. For any real numbers \(b_1, \ldots, b_n\), the following inequality holds:
\[ \sum_{l=s+1}^{n} b_j \sum_{l-m=1}^{s} \prod_{j=1}^{l-m} b_j \leq \left(\sum_{j=1}^{n} b_j\right)^{s+1} \tag{80} \]

The exponent of \(g_k\) is bounded by some absolute constant (see the proof of Lemma 7.7). Therefore,
\[ 1 \leq \exp\left\{-N(\bar{p}U)^2 + \frac{N(\bar{p}U)^3}{3} + \frac{N\bar{p}^4U^4}{4}\Theta_{5}^3\right\} = \exp\left\{\Gamma_1 U + \frac{\Gamma_1 U^2}{6}\right\}. \]

and a similar estimate holds for any other collection of \(\lambda_{m_1}, \ldots, \lambda_{m_s}\). Consequently, from Lemmas 6.6, 7.8 and equation (75), we obtain
\[ \left|\prod_{k=1}^{n} \varphi_k - \sum_{l=0}^{s} \hat{B}_l(G)\right| \leq C(s) \exp\{-1.3\lambda\sin^2(t/2)\} \left(\sum_{k=1}^{n} |\varphi_k - g_k|\right)^{s+1} \tag{81} \]

Applying Lemma 6.3 and using equations (57), (71) and (75), the result in (36) follows. Since,

\[ |\varphi_k - \exp\{\nu_1(k)z\}| \leq C|z|^3 \left( \sum_{l=0}^{3} \nu_1^2(l) + \nu_2(k) + EX_{k-1}X_k \right) \]

the result (34) also can be proved quite similarly.

Let \( \tilde{\varphi}_k = \varphi_k \exp\{-i\nu_1(k)t\}, \tilde{g}_k = g_k \exp\{-i\nu_1(k)t\} \) and denote by \( \tilde{R}_n(s+1) \) the remainder term in Bergström expansion for the difference of \( \tilde{F}(t) \) and \( \sum_0^s \tilde{B}_l(G) \). We then obtain

\[
\left( e^{-it\Gamma_1} \tilde{R}_n(s + 1) \right)' = \sum_{l=s+1}^{n} (\tilde{\varphi}_l - \tilde{g}_l)' \prod_{j=l+1}^{n} \tilde{\varphi}_j \sum_{j-1,m}^{s} \prod_{j=1}^{l-1} \tilde{g}_j^{l-m_j}(\tilde{\varphi}_j - \tilde{g}_j)^{m_j} \\
+ \sum_{l=s+1}^{n} (\tilde{\varphi}_l - \tilde{g}_l) \left( \prod_{j=l+1}^{n} \tilde{\varphi}_j \right)' \sum_{j-1,m}^{s} \prod_{j=1}^{l-1} \tilde{g}_j^{l-m_j}(\tilde{\varphi}_j - \tilde{g}_j)^{m_j} \\
+ \sum_{l=s+1}^{n} (\tilde{\varphi}_l - \tilde{g}_l) \prod_{j=l+1}^{n} \tilde{\varphi}_j \sum_{j-1,m}^{s} \prod_{j=1}^{l-1} \tilde{g}_j^{l-m_j}(\tilde{\varphi}_j - \tilde{g}_j)^{m_j} \\
+ \sum_{l=s+1}^{n} (\tilde{\varphi}_l - \tilde{g}_l) \prod_{j=l+1}^{n} \tilde{\varphi}_j \sum_{j-1,m}^{s} \prod_{j=1}^{l-1} \tilde{g}_j^{l-m_j}(\tilde{\varphi}_j - \tilde{g}_j)^{m_j}' \\
=: J_{11} + J_{12} + J_{13} + J_{14} \text{ (say).}
\]

We have

\[ |\tilde{\varphi}_l - \tilde{g}_l| \leq C(|\varphi_l' - g_l' + \nu_1(k)|\varphi_l - g_l|). \]

Combining the last estimate with (80), (57), (71), (75) and using Lemma 7.3, we obtain

\[ J_{11} \leq C(s)|z|^{3s+2}R_0^{s+1} \exp\{-C\lambda \sin^2(t/2)\}. \] (82)

Taking into account (66), (67) and (76), we get

\[ |\tilde{\varphi}_l' | \leq |\varphi_l' - \nu_1(l)z' | + \nu_1(l)e^{it} - \varphi_l | \leq |\varphi_l' - \nu_1(l)z' | + \nu_1(l)|z| + \nu_1(l)|1 - \varphi_l| \leq C|z| \sum_{j=0}^{3} \nu_1(l - j). \]

Similar estimate holds for \( |\tilde{g}_l' | \). Therefore,

\[
\left| \left( \prod_{j=l+1}^{n} \tilde{\varphi}_j \right) \right| \leq \sum_{j=l+1}^{n} |\tilde{\varphi}_j| \prod_{k=l+1,k \neq j}^{n} |\tilde{\varphi}_k| \leq C(s) \exp\{-C\lambda \sin^2(t/2)\}|z|\Gamma_1. \]

32
Similarly, \((\tilde{g}_j^{1-m_j})' = (1 - m_j)\tilde{g}_j'\) and
\[
\left| \left( \prod_{j=1}^{l-1} \tilde{g}_j^{1-m_j} \right) \right|' \leq C(s) \sum_{j=1}^{l-1} \sum_{i=0}^{3} \nu_1(j - i) \prod_{k=1, k \neq j}^{l-1} |g_k|^{1-m_k} \leq C(s)\Gamma_1 |z| \prod_{j=1}^{l} \exp\{-\lambda_j \sin^2(t/2)\}.
\]
Consequently,
\[
J_{12}, J_{13} \leq C(s)|z|^{3s+4}\Gamma_1 R_1^{s+1} \exp\{-C\lambda \sin^2(t/2)\}.
\]
Let
\[
\tilde{r}_1(k) = r_1(k) + \nu_1(k)\nu_2(k) + [\nu_1(k-2) + \nu_1(k-1)]EX_{k-1}X_k + \tilde{E}^+(X_{k-4}, X_{k-3}, X_{k-2}) + \tilde{E}^+(X_{k-3}, X_{k-2}, X_{k-1}).
\]
Quite similar arguments as in above lead to
\[
\left| \left( \prod_{k=1}^{l} (\varphi_k - \tilde{g}_k)^{m_k} \right) \right|' \leq \sum_{k=1}^{l-1} m_k|\varphi_k' - \tilde{g}_k'| \prod_{j=1, j \neq k}^{l-1} |\varphi_j' - \tilde{g}_j'|^{m_j}
\leq C(s) \sum_{k=1}^{l-1} m_k|z|^2 \tilde{r}_1(l) \prod_{j=1, j \neq k}^{l-1} (|z|^2 \tilde{r}_1(j))^{m_j} \leq C(s)|z|^{3s-1} \prod_{j=1}^{l-1} \tilde{r}_1(j)^{m_j}.
\]
Consequently,
\[
J_{14} \leq C(s)|z|^{3s+4}\Gamma_1 R_1^{s+1} \exp\{-C\lambda \sin^2(t/2)\}. \tag{83}
\]
Using (82) and (83), we obtain
\[
\left| \left( \exp\{-it\Gamma_1\} \left( \prod_{k=1}^{n} \varphi_k - \sum_{l=0}^{s} \hat{B}_l(G) \right) \right) \right|' \leq C(s)|z|^{3s+2} R_1^{s+1} (1 + |z|^2 \Gamma_1). \tag{84}
\]
Applying (37) with \(u = \max(1, \sqrt{\Gamma_1})\) and \(v = \Gamma_1\) and using (84) and (81), we prove (35). The proof of (33) is similar and is omitted. \(\Box\)

**Proof of Theorem 3.2** As we already noted above, (14) and (16) follow from Theorem 5.1 when
\[ s = 0. \text{ Now, we have} \]
\[
\left| \prod_{j=1}^{n} g_j (1 + \Gamma_3 z^3) - \prod_{j=1}^{n} g_j - \sum_{m=1}^{n} (\varphi_m - g_m) \prod_{j \neq m} g_j \right|
\leq \left| \prod_{j=1}^{n} g_j (1 + \Gamma_3 z^3) - \prod_{j=1}^{n} g_j - \prod_{j=1}^{n} g_j \sum_{m=1}^{n} (\varphi_m - g_m) \prod_{j \neq m} g_j \right| + \left| \sum_{m=1}^{n} (\varphi_m - g_m) \left( \prod_{j=1}^{n} g_j - \prod_{j \neq m} g_j \right) \right|
\leq \left| \prod_{j=1}^{n} g_j \left( \sum_{m=1}^{n} (\varphi_m - g_m - \Gamma_3 z^3) \right) \right| + \left| \sum_{m=1}^{n} (\varphi_m - g_m) \prod_{j \neq m} g_j (g_m - 1) \right| =: J_{21} + J_{22}.
\]

Taking into account (68), (69), (73) and (75), we get
\[
J_{21} + J_{22} \leq CR_2 |z|^4 \exp \{-C \lambda \sin^2(t/2)\}. \tag{85}
\]

Applying to the last estimate (85) and Lemma 6.3 and combining the resulting estimate with (56), for \( s = 1 \), we obtain (17). For the proof of (15), we use (85), the triangle inequality and (57) with \( v = \Gamma_1 \) and \( u = \max(1, \Gamma_1) \). We need estimates for derivative
\[
\left| \left( e^{-it \Gamma_1} \left[ \prod_{j=1}^{n} g_j (1 + \Gamma_3 z^3) - \prod_{j=1}^{n} g_j - \sum_{m=1}^{n} (\varphi_m - g_m) \prod_{j \neq m} g_j \right] \right) \right|
\leq \left| \left( e^{-it \Gamma_1} \prod_{j=1}^{n} g_j \sum_{m=1}^{n} (\varphi_m - g_m - \gamma_3(m) z^3) \right) \right| + \left| \left( e^{-it \Gamma_1} \prod_{m=1}^{n} (\varphi_m - g_m) \prod_{j \neq m} g_j (g_m - 1) \right) \right|
\leq \left| \left( \prod_{j=1}^{n} \tilde{g}_j \right) \sum_{m=1}^{n} (\varphi_m - g_m - \gamma_3(m) z^3) \right| + \left| \prod_{j=1}^{n} \tilde{g}_j \sum_{m=1}^{n} (\varphi'_m - g'_m - \gamma_3(m) z^3) \right|
\]

\[
+ \sum_{m=1}^{n} (\varphi'_m - g'_m) \prod_{j \neq m} \tilde{g}_j (\tilde{g}_m - e^{-it \nu_1(m)}) \left( \prod_{j \neq m} \tilde{g}_j \right) \right| + \sum_{m=1}^{n} |\varphi_m - g_m| |g_m - 1| \left( \prod_{j \neq m} \tilde{g}_j \right) \right| 
+ \sum_{\varphi'_m - g'_m} \left| \prod_{k \neq m} \tilde{g}_j \right| \left| g'_m \right| + \nu_1(m). \right|
\]

Applying Lemmas 7.6, 7.7 and 6.3 it is not difficult to prove that the derivative given above is less than \( C |z|^5 T_1 R_2 \exp \{-C \lambda \sin^2(t/2)\} \). Combining this estimate with (85) and applying Lemma 6.4, we obtain (15). \( \square \)

**Proof of Theorem 3.1** As seen above, (10) and (12) follow from Theorem 5.1 when \( s = 0. \)
Applying Lemma 7.8 and using the following identity

\[ e^b - 1 - b = b^2 \int_0^1 (1 - \tau)e^{\tau b}d\tau, \quad (86) \]

we get

\[
\| G - \text{Pois}(\Gamma_1)(I + \Gamma_2 U^2) \| = \| \exp\{\Gamma_1 U\} \int_0^1 (1 - \tau)(\gamma_2 U^2)^2 \exp\{\tau \Gamma_2 U^2\}d\tau \| \\
\leq \int_0^1 \| \Gamma_2^2 U^4 \exp\{\Gamma_1 U + \tau \Gamma_2 U^2\}\|d\tau \leq C|\Gamma_2| \| U^4 \exp\{0.9\lambda U\}\| \leq CR_0^2 \min(1, \lambda^{-2}).
\]

Combining this estimate with Bergström expansion \((s = 1)\) for \(G\), we prove (11). The proof of (13) is practically the same and is therefore omitted. □

**Proof of Theorem 3.3.** Let \( A = \Gamma_1 z + \Gamma_2 z^2, B = \Gamma_1 z + (2\Gamma_2 + \tilde{\delta})(z - it) \). Due to (9), \(|\Gamma_2| \leq C\Gamma_1\) and \(\max(|e^A|, |e^B|) \leq \exp\{-0.2\Gamma_1 \sin^2(t/2)\}\). If the real part of a complex number \(\text{Re}Z \leq 0\), then

\[
|e^Z - 1| \leq \left| \int_0^1 Ze^{\tau Z}d\tau \right| \leq |Z| \int_0^1 \exp\{\text{Re}Z\}d\tau \leq |Z|.
\]

Consequently,

\[
|e^A - e^B| \leq \exp\{-0.2\Gamma_1 \sin^2(t/2)\}|A - B| \leq \exp\{-0.2\sin^2(t/2)\}C(|\Gamma_2| |t| + \tilde{\delta}t^2).
\]

Similarly, for \(|t| \leq \pi\),

\[
\left| \left( e^{A - it\Gamma_1} - e^{B - it\Gamma_1} \right) \right| \leq |A' - i\Gamma_1| e^{-0.2\Gamma_1 \sin^2(t/2)} C(|\Gamma_2| |t| + \tilde{\delta}t^2) + |A' - B'| e^{-0.2\Gamma_1 \sin^2(t/2)}
\]

\[
\leq Ce^{-0.1\Gamma_1 \sin^2(t/2)} (|\Gamma_2| |t|^2 + \tilde{\delta}|t|)(1 + \Gamma_1 t^2) e^{-0.2\Gamma_1 t^2/\pi}
\]

\[
\leq Ce^{-0.1\Gamma_1 \sin^2(t/2)} (|\Gamma_2| |t|^2 + \tilde{\delta}|t|).
\]

It remains to apply Lemma 6.4 with \(u = \sqrt{\Gamma_1}\) and \(v = \Gamma_1\), Theorem 3.2 and the triangle inequality.
Proof of Theorem 3.4. Applying (77) and Lemma 6.1 we obtain
\[
\|G - NB(r, \bar{q})\| = \|G - G \exp \left\{ \frac{4\Gamma_2^2 U^3 \Theta}{\Gamma_1} \frac{1}{0.7} \right\}\| = \|G \int_0^1 \left( \exp \left\{ \frac{4\Gamma_2^2 U^3 \Theta}{\Gamma_1} \frac{1}{0.7} \right\} \right)' d\tau\|
\lesssim C \| \frac{\Gamma_2^2}{\Gamma_1} \| \| U^3 \exp \{\Gamma_1 U\} \| \lesssim C \frac{\Gamma_2^2}{\Gamma_1} \min(1, \Gamma_1^{-3/2}).
\]

Combining the last estimate with (14), we prove (20). The estimate (22) is proved analogously.

Let
\[
M_1 := \frac{4\Gamma_2^2}{3\Gamma_1} U^3, \quad M_2 := \frac{2\Gamma_3^2 U^4 \Theta}{\Gamma_1} \frac{1}{0.7}, \quad M_3 := \Gamma_3 U^3 - M_1.
\]

Then by Lemmas 7.9 and 6.1 and using equation (86),
\[
NB(r, \bar{q}) = G \exp \{M_1 + M_2\}
= G \left( I + M_1 + M_2^2 \int_0^1 (1 - \tau) \exp \{\tau M_1\} d\tau \right) \left( I + M_2 \int_0^1 \exp \{x M_2\} dx \right)
= G(I + M_1) + M_1^2 \int_0^1 (1 - \tau) G \exp \{\tau M_1\} d\tau
+ \int_0^1 \int_0^1 M_2(I + M_1 + M_1^2(1 - \tau)) G \exp \{\tau M_1 + x M_2\} d\tau dx
= G(I + M_1) + \exp \{0.5\Gamma_1 U\} \{M_2^2 \Theta C + [M_2 + M_1 M_2] \Theta C + M_2^2 M_2 \Theta C\}
= G(I + M_1) + \exp \{0.25\Gamma_1 U\} \Gamma_2^2 \Gamma_1^{-2} U^4 \Theta C.
\]

By the triangle inequality,
\[
\|F - NB(r, \bar{q})(I + M_3)\|
\lesssim \|F - G(I + \Gamma_3 U^3)\| + \|G(I + \Gamma_3 U^3) - G(I + M_1)(I + M_3)\|
+ C \| \exp \{0.25\Gamma_1 U\} \Gamma_2^2 \Gamma_1^{-1} (I + M_3)\| =: J_{31} + J_{32} + J_{33}.
\]

By Lemmas 7.8 and 6.1
\[
J_{32} \lesssim C \| \exp \{0.9\lambda U\} \Gamma_2^2 \Gamma_1^{-1} (\Gamma_3 - 4\Gamma_2^2 (3\Gamma_1)^{-1}) U^6\| \lesssim \Gamma_2^2 \Gamma_1^{-1} |\Gamma_3 - 4\Gamma_2^2 (3\Gamma_1)^{-1}| \min(1, \Gamma_1^{-3}).
\]

36
Similarly refsmoothing and using Lemma 6.1,

\[ J_{33} \leq C \| \exp \{0.25 \Gamma_1 U \} \Gamma_2 \Gamma_1^{-2} U^4 \| + C \| \exp \{0.25 \Gamma_1 U \} \Gamma_2 \Gamma_1^{-2} (\Gamma_3 - 4 \Gamma_2^2 (3 \Gamma_1)^{-1}) U^7 \| \]
\[ \leq C \Gamma_2 \Gamma_1^{-1} \min(1, \Gamma_1^{-3}) + C \Gamma_2 \Gamma_1^{-2} |\Gamma_3 - 4 \Gamma_2^2 (3 \Gamma_1)^{-1}| \min(1, \Gamma_1^{-7/2}). \]

Combining the last two estimates and applying (15) for \( J_{31} \), we prove (21). The estimate in (23) is proved quite similarly. □

**Proof of Theorem 3.5.** Let

\[ \tilde{M}_1 := \frac{N \Gamma_1^2 U^3}{3}, \quad \tilde{M}_2 := \frac{N \Gamma_1^4 U^4}{4} \Theta \frac{5}{3} + U^2 \theta \frac{50 \Gamma_1^2 \epsilon}{21 \Gamma_1^2}, \quad \tilde{M}_3 := \Gamma_3 U^3 - M_1. \quad (87) \]

Since the proof is almost identical to that of Theorem 3.4, it is omitted. □

**Proof of Theorem 4.1.** Let \( M_3 \) be defined as (87). It is easy to check that

\[ \nu_1(k) = p^2, \quad \nu_2(k) = \nu_3(k) = 0, \quad \text{EX}_{k-1} X_k \leq C p^3, \quad \text{EX}_{k-2} X_{k-1} X_k \leq C p^4, \]
\[ \text{EX}_{k-3} \cdots X_k \leq C p^5, \quad \Gamma_2 \leq C n p^3, \quad \Gamma_3 \leq C n p^4, \quad R_1 \leq C n p^4, \quad R_2 \leq C n p^5. \]

and

\[ M_3 = -\frac{np^4}{3} U^3 + U^3 \theta C n p^5. \]

From Lemmas 7.9 and 6.1, we have

\[ \left\| (\text{NB}(\tau, \theta) - \exp\{np^2 U\}) U^3 \right\| \leq \left\| \exp\{np^2 U\} \int_0^1 (\Gamma_2 U^2 \Theta / 0.7) \exp\{\tau (\Gamma_2 U^2 \Theta / 0.7)\} d\tau U^3 \right\| \]
\[ \leq C n p^3 \| \exp\{0.5 np^2 U\} U^5 \| \leq \frac{C}{p^2 n \sqrt{n}}. \quad (88) \]
Applying (21), (88) and Lemmas 7.9 and 7.8, we obtain

\[ \|F - NB(r, \bar{q})\| - \frac{\tilde{C}_{TV}p}{\sqrt{n}} \leq \|F - NB(r, \bar{q})(I + M_3)\| + \|NB(r, \bar{q})M_3\| - \frac{\tilde{C}_{TV}p}{\sqrt{n}} \]

\[ \leq \frac{Cp}{n} + \|NB(r, \bar{q})(M_3 + np^4U^3/3)\| + \left| \frac{np^4}{3} \right| \|NB(r, \bar{q})U^3\| - \frac{\tilde{C}_{TV}p}{\sqrt{n}} \]

\[ \leq \frac{Cp^2}{\sqrt{n}} + \frac{np^4}{3} \|NB(r, \bar{q}) - \exp\{np^2U\}\| + \left| \frac{np^4}{3} \right| \|\exp\{np^2U\}U^3\| - \frac{\tilde{C}_{TV}p}{\sqrt{n}} \]

\[ \leq \frac{Cp^2}{\sqrt{n}} + \frac{np^4}{3} \left| \exp\{np^2U\}U^3\| - \frac{3\tilde{C}_{TV}p}{(np^2)^{3/2}} \right| \leq \frac{Cp^2}{\sqrt{n}} + \frac{C}{n}. \]

For the local estimates, one should use the local metric. □

**Proof of Theorem 4.2** The direct consequence of conditions assumed are the following estimates

\[(n - m + 1) \geq 100m, \quad \tilde{N} = \frac{(n - m + 1)}{2m - 1 - m(m - 1)/(n - m + 1)} \geq \frac{100m}{2m} = 50.\]

We have

\[
\bar{p} = \frac{(n - m + 1)a(p)}{\tilde{N}} + \frac{(n - m + 1)a(p)}{\tilde{N}} \left( \frac{\tilde{N}}{N} - 1 \right) = \frac{(n - m + 1)a(p)}{\tilde{N}} \left( 1 + \frac{\epsilon}{N - \epsilon} \right)
\]

\[
= a(p) \left( 2m - 1 - \frac{m(m - 1)}{n - m + 1} \right) \left( 1 + \frac{\epsilon}{N - \epsilon} \right) \leq a(p) \left( 2m + \frac{m}{100} \right) \left( 1 + \frac{1}{49} \right)
\]

\[
\leq 2.05a(p)m \leq 0.03. \quad (89)
\]

The sum \( \tilde{S} \) has \( n - m + 1 \) summands. After grouping, we get \( K \) 1-dependent random variables containing \( m \) initial summands each, and (possibly) one additional variable, equal to the sum of \( \delta m \) initial summands. Here, \( K \) and \( \delta \) are the integer and fractional parts of \( (n - m + 1)/m \), respectively. That is,

\[
K = \left\lfloor \frac{n - m + 1}{m} \right\rfloor, \quad \frac{n - m + 1}{m} = K + \delta, \quad 0 \leq \delta < 1. \quad (90)
\]

The analysis of the structure of new variables \( X_j \) shows that, for \( j = 1, \ldots, K \)

\[
X_j = \begin{cases} 
1, & \text{with probability } ma(p), \\
0, & \text{with probability } 1 - ma(p), 
\end{cases}
\quad X_{K+1} = \begin{cases} 
1, & \text{with probability } \delta ma(p), \\
0, & \text{with probability } 1 - \delta ma(p). 
\end{cases}
\]

38
Consequently, \( \nu_2(j) = \nu_3(j) = \nu_4(j) = \hat{E}_2^+(X_1, X_2) = \hat{E}_2^+(X_1, X_2, X_3) = \hat{E}_3(X_1, X_2) = 0 \). For calculation of \( E(X_1 X_2) \), note that there are the following non-zero product events: a) the first summand of \( X_1 \) equals 1 and any of the summands of \( X_2 \) equals 1 (\( m \) variants); b) the second summand of \( X_1 \) equals 1 and any of the summands of \( X_2 \), beginning from the second one, equals 1 (\( m - 1 \) variant) and etc. Each event has the probability of occurrence \( a^2(p) \). Therefore,

\[
E(X_1 X_2) = a^2(p)(m + (m - 1) + (m - 2) + \cdots + 1) = \frac{a(p)^2 m(m + 1)}{2}.
\]

Similarly, arguing we obtain the following relations for \( j = 1, \ldots, K \), \( j = 2, \ldots, K \) and \( j = 3, \ldots, K \) if more variables are involved) and \( X_{K+1} \) (if \( \delta > 0 \)):

\[
\begin{align*}
E_X(j) &= ma(p), \quad E_{X_{j-1} X_j} = \frac{m(m + 1)a^2(p)}{2}, \quad \hat{E}(X_{j-1}, X_j) = -\frac{m(m - 1)a^2(p)}{2}, \\
E_{X_{j-2} X_{j-1} X_j} &= \frac{m(m + 1)(m + 2)a^3(p)}{6}, \quad \hat{E}(X_{j-2}, X_{j-1}, X_j) = \frac{a^3(p)m(m - 1)(m - 2)}{6}, \\
E_{X_{K+1}} &= \delta ma(p), \quad E_{X_K X_{K+1}} = \frac{\delta m(\delta m + 1)a^2(p)}{2}, \\
\hat{E}(X_K, X_{K+1}) &= \frac{a^2(p)\delta m(\delta m + 1 - 2m)}{2}, \quad E_{X_{K-1} X_K X_{K+1}} = \frac{\delta m(\delta m + 1)(\delta m + 2)a^3(p)}{6}, \\
\hat{E}_{X_{K-1} X_K X_{K+1}} &= \frac{a^3(p)\delta m(9m^2 - 9m + 2)}{6}.
\end{align*}
\]

(91)

It is obvious, that \( \Gamma_1 = (n - m + 1)a(p) \). Taking into account (90) and (91) we can calculate \( \Gamma_2 \):

\[
\begin{align*}
\Gamma_2 &= -\frac{1}{2}[Km^2 a^2(p) + \delta^2 m^2 a^2(p)] - \frac{(K - 1)m(m - 1)a^2(p)}{2} + \frac{\delta ma^2(p)\delta m + 1 - 2m}{2} \\
&= -\frac{a^2(p)m}{2}[2m(K + \delta) - (K + \delta) - (m - 1)] \\
&= -\frac{a^2(p)}{2}[(n - m + 1)(2m - 1) - m(m - 1)].
\end{align*}
\]

(92)

Similarly,

\[
\Gamma_3 = \frac{a^3(p)}{6}[(n - m + 1)(3m - 1)(3m - 2) - 4m(2m - 1)(m - 1)].
\]

Making use of all the formulas given above and noting that \( m \geq 2 \), it is easy to get the estimate

\[
R_1 \leq \quad K(ma(p))^3 + (\delta ma(p))^3 + 3ma(p)[(K - 2)m(m + 1)a^2(p)]/2 + \delta m(\delta m + 1)a^2(p)/2 \\
+ C(K + \delta)m^3 a^3(p) \leq Cm^3 a^3(p)(K + \delta) \leq C(n - m + 1)m^2 a^3(p).
\]

39
Similarly, 
\[ \hat{E}(X_1, X_2, X_3, X_4) \leq Cm^4a^4(p), \quad R_2 \leq C(n - m + 1)m^3a^4(p). \]

Using (92) and (89), we get
\[
\frac{N\overline{p}^3}{3} = \frac{\Gamma_1 \overline{p}^2}{3} = \frac{\Gamma_1}{3} \frac{4\Gamma_4}{3} \frac{1 + \frac{\epsilon}{N - \epsilon}}{N} = \frac{4\Gamma_4}{3\Gamma_1} \frac{\epsilon}{N} \left( 2 + \frac{\epsilon}{N - \epsilon} \right) \\
= \frac{a^3(p)}{3} (n - m + 1)(2m - 1)^2 + \theta Cm^3a^3(p).
\]

Similarly,
\[ \Gamma_3 = \frac{a^3(p)}{6}(n - m + 1)(3m - 1)(3m - 2) + \theta Cm^3a^3(p). \]

Therefore,
\[ \Gamma_3 - \frac{N\overline{p}^3}{3} = A + C\theta m^3a^3(p). \]

By Lemma 6.1
\[ m^3a^3(p)\|U^3\text{Bi}(N,\overline{p})\| \leq C \frac{m^3a^3(p)}{(n - m + 1)a(p)\sqrt{(n - m + 1)a(p)}} \leq C \frac{m^3a^2(p)}{n - m + 1}. \tag{93} \]

Next, we check the conditions in (9). Indeed, we already noted that \( \nu_2(j) = 0. \) Now
\[
(K - 1)|\hat{E}(X_1, X_2)| + |\hat{E}(X_{K-1}, X_K)| \leq \frac{Km(m - 1)a^2(p)}{2} + \frac{\delta m^2a^2}{2} \leq (K + \delta)2m^2a^2(p) \\
\leq \frac{2ma^2}{n - m + 1} = 2ma \Gamma_1 \leq 0.02 \Gamma_1.
\]

It remains to apply Theorem 3.5 and (93). The local estimate is proved similarly. \( \square \)

**Proof of Theorem 4.3** We have
\[
\left| \|H - \text{Bi}(N,\overline{p}) - \hat{C}_{TV}a^{3/2}(p)m(m - 1)\| \over 2\sqrt{n - m + 1} \right| \leq \|H - \text{Bi}(N,\overline{p})(I + AU^3)\| \\
+ \|\text{Bi}(N,\overline{p})U^3\left( A - \frac{a^3(p)}{6}(n - m + 1)m(m - 1) \right)\| \\
+ \frac{a^3(p)}{6}(n - m + 1)m(m - 1) \|\text{Bi}(N,\overline{p})U^3\| - \frac{3\hat{C}_{TV}}{(N\overline{p}(1 - \overline{p}))^{3/2}} \|N\overline{p}^{3/2} \left( \frac{1}{(1 - \overline{p})^{3/2}} - 1 \right). \]
We easily check that

\[
\frac{1}{(1 - \theta)^{3/2}} - 1 = \frac{1 - (1 - \theta)^3}{(1 - \theta)^{3/2}[1 + (1 - \theta)^{3/2}]} = \frac{\theta[1 + (1 - \theta) + (1 - \theta)^2]}{(1 - \theta)^{3/2}[1 + (1 - \theta)^{3/2}]} = a(p)C(m)\theta.
\]

All that now remains is to apply (31) and use Lemmas [6.1] and [6.2]. The local estimate is proved similarly. □

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