Some aspects of analysis related to
\( p \)-adic numbers

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Preface

A field with an absolute value function is a basic type of metric space, which includes the real and complex numbers with their standard metrics, and ultrametrics on fields like the $p$-adic numbers. Here we try to give some perspectives of analysis in situations like these.
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Chapter 1

Preliminaries

1.1 Metrics and ultrametrics

As usual, a metric space is a set $M$ with a nonnegative real-valued function $d(x, y)$ defined for $x, y \in M$ that satisfies the following three conditions: first,

\begin{equation}
    d(x, y) = 0 \quad \text{if and only if} \quad x = y;
\end{equation}

second, $d(x, y)$ is symmetric in $x$ and $y$, so that

\begin{equation}
    d(x, y) = d(y, x)
\end{equation}

for every $x, y \in M$; and third,

\begin{equation}
    d(x, z) \leq d(x, y) + d(y, z)
\end{equation}

for every $x, y, z \in M$, which is known as the triangle inequality. If

\begin{equation}
    d(x, z) \leq \max(d(x, y), d(y, z))
\end{equation}

for every $x, y, z \in M$, then the metric $d(x, y)$ is said to be an ultrametric on $M$. Of course, (1.4) implies (1.3). The discrete metric on a set $M$ is defined by putting $d(x, y) = 1$ when $x \neq y$, and $d(x, y) = 0$ when $x = y$, and is an ultrametric on $M$.

Let $a$ be a real number such that $0 < a \leq 1$, and let $r, t$ be nonnegative real numbers. Observe that

\begin{equation}
    \max(r, t) \leq (r^a + t^a)^{1/a},
\end{equation}

and hence that

\begin{equation}
    r + t \leq \max(r, t)^{1-a} (r^a + t^a) \leq (r^a + t^a)^{(1-a)/a} + 1 = (r^a + t^a)^{1/a}.
\end{equation}

Equivalently,

\begin{equation}
    (r + t)^a \leq r^a + t^a.
\end{equation}
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If $d(x,y)$ is a metric on a set $M$ and $0 < a \leq 1$, then it follows that $d(x,y)^a$ also defines a metric on $M$, which determines the same topology on $M$ as $d(x,y)$. Similarly, if $d(x,y)$ is an ultrametric on $M$, then $d(x,y)^a$ is an ultrametric on $M$ for every $a > 0$, which defines the same topology on $M$ as $d(x,y)$.

Remember that the absolute value $|x|$ of a real number $x$ is defined to be equal to $x$ when $x \geq 0$ and to $-x$ when $x \leq 0$. Of course,

\begin{equation}
|x + y| \leq |x| + |y|
\end{equation}

and

\begin{equation}
|xy| = |x||y|
\end{equation}

for all real numbers $x, y$. The standard Euclidean metric on the set $\mathbb{R}$ of real numbers is defined by

\begin{equation}
d(x,y) = |x - y|,
\end{equation}

and is not an ultrametric. Note that $|x|^a$ satisfies the analogue of (1.8) when $0 < a \leq 1$, by (1.7), and that it satisfies the analogue of (1.9) for every $a > 0$.

Let $(M, d(x,y))$ be a metric space. The open ball centered at a point $x \in M$ and with radius $r > 0$ is defined by

\begin{equation}
B(x,r) = \{ z \in M : d(x,z) < r \}.
\end{equation}

If $y \in B(x,r)$, then $t = r - d(x,y) > 0$, and one can check that

\begin{equation}
B(y,t) \subseteq B(x,r),
\end{equation}

using the triangle inequality. If $d(\cdot, \cdot)$ is an ultrametric on $M$, then it is easy to see that

\begin{equation}
B(y,r) \subseteq B(x,r)
\end{equation}

under these conditions. More precisely,

\begin{equation}
B(x,r) = B(y,r)
\end{equation}

when $d(x,y) < r$, since the opposite inclusion may be obtained by reversing the roles of $x$ and $y$.

Similarly, the closed ball centered at $x$ with radius $r \geq 0$ is defined by

\begin{equation}
\overline{B}(x,r) = \{ z \in M : d(x,z) \leq r \}.
\end{equation}

If $d(\cdot, \cdot)$ is an ultrametric on $M$, and $d(x,y) \leq r$, then

\begin{equation}
\overline{B}(y,r) \subseteq \overline{B}(x,r),
\end{equation}

and hence

\begin{equation}
\overline{B}(x,r) = \overline{B}(y,r),
\end{equation}

as before. In particular, this implies that closed balls in $M$ of positive radius are open sets when $d(\cdot, \cdot)$ is an ultrametric. One can also check that open balls in $M$ are closed sets when $d(\cdot, \cdot)$ is an ultrametric, using (1.13). Equivalently, the
complement of an open ball in $M$ is an open set when $d(\cdot, \cdot)$ is an ultrametric, which can also be derived from the remarks in the next paragraph.

Let us continue to suppose that $d(\cdot, \cdot)$ is an ultrametric on $M$. If $x, y, z \in M$ satisfy $d(y, z) \leq d(x, y)$, then

$$d(x, z) \leq \max(d(x, y), d(y, z)) = d(x, y),$$

by the ultrametric version of the triangle inequality. If $d(y, z) < d(x, y)$, then

$$d(x, y) \leq \max(d(x, z), d(y, z))$$

implies that $d(x, y) \leq d(x, z)$. Combining this with (1.18), we get that

$$d(x, y) = d(x, z)$$

when $d(y, z) < d(x, y)$.

Let $(M, d(x, y))$ be an arbitrary metric space again. As usual, a sequence \( \{x_j\}_{j=1}^{\infty} \) of elements of $M$ is said to be a Cauchy sequence if for each $\epsilon > 0$ there is a positive integer $L(\epsilon)$ such that

$$d(x_j, x_l) < \epsilon$$

for every $j, l \geq L(\epsilon)$. Convergent sequences are Cauchy sequences, and $M$ is said to be complete if every Cauchy sequence in $M$ converges to an element of $M$. If \( \{x_j\}_{j=1}^{\infty} \) is a Cauchy sequence in $M$, then it follows that

$$\lim_{j \to \infty} d(x_j, x_{j+1}) = 0,$$

by taking $l = j + 1$ in (1.21). Conversely, if $d(\cdot, \cdot)$ is an ultrametric, and if \( \{x_j\}_{j=1}^{\infty} \) is a sequence of elements of $M$ that satisfies (1.22), then one can check that \( \{x_j\}_{j=1}^{\infty} \) is a Cauchy sequence in $M$.

## 1.2 Completions

Let $(M, d(x, y))$ be a metric space, and let \( \{x_j\}_{j=1}^{\infty} \) and \( \{y_j\}_{j=1}^{\infty} \) be Cauchy sequences of elements of $M$. If

$$\lim_{j \to \infty} d(x_j, y_j) = 0,$$

then \( \{x_j\}_{j=1}^{\infty} \) and \( \{y_j\}_{j=1}^{\infty} \) are said to be equivalent as Cauchy sequences in $M$. It is easy to see that this defines an equivalence relation on the collection of Cauchy sequences in $M$. Of course, convergent sequences in $M$ are Cauchy sequences, and two convergent sequences in $M$ are equivalent as Cauchy sequences if and only if they converge to the same element of $M$. Similarly, if a Cauchy sequence in $M$ is equivalent to a convergent sequence in $M$, then that Cauchy sequence converges to the same element of $M$. In particular, constant sequences in $M$ are convergent, and a Cauchy sequence in $M$ converges to an element of $M$ if and
only if it is equivalent to the corresponding constant sequence. The completion of $M$ is defined to be the set of equivalence classes of Cauchy sequences in $M$. There is a natural embedding of $M$ into its completion, which associates to each $x \in M$ the equivalence class of Cauchy sequences that contains the constant sequence $\{x_j\}_{j=1}^\infty$ with $x_j = x$ for each $j$.

Suppose for the moment that $M$ is the set $\mathbb{Q}$ of rational numbers, equipped with the standard metric (1.10). It is well known that the completion of $\mathbb{Q}$ with respect to the standard metric can be identified with the real line. More precisely, if one uses this to construct the real numbers, then one should not use the definition of a metric space in the previous section, which assumes that the real numbers have already been defined. However, the standard metric on $\mathbb{Q}$ still makes sense, and takes values in $\mathbb{Q}$. One can also define what it means for a sequence of rational numbers to converge to a rational number with respect to the standard metric in the usual way, and what it means for a sequence of rational numbers to be a Cauchy sequence. Thus the completion of $\mathbb{Q}$ can be defined as in the previous paragraph, with the same properties as before. We shall return to this in a moment.

Let $(M, d(x, y))$ be an arbitrary metric space again. If $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ are Cauchy sequences in $M$, then it is well known that $\{d(x_j, y_j)\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$, with respect to the standard metric on $\mathbb{R}$. It follows that $\{d(x_j, y_j)\}_{j=1}^\infty$ converges in $\mathbb{R}$, by the completeness of $\mathbb{R}$. If $\{x_j'\}_{j=1}^\infty$ and $\{y_j'\}_{j=1}^\infty$ are Cauchy sequences in $M$ that are equivalent to $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$, respectively, then it is easy to see that

$$\lim_{j \to \infty} (d(x_j', y_j') - d(x_j, y_j)) = 0,$$

so that $\{d(x_j', y_j')\}_{j=1}^\infty$ and $\{d(x_j, y_j)\}_{j=1}^\infty$ have the same limit in $\mathbb{R}$. Thus the limit of $\{d(x_j, y_j)\}_{j=1}^\infty$ leads to a well-defined distance function on the completion of $M$, and one can check that this distance function is a metric on the completion of $M$. It is well known that the completion of $M$ is complete with respect to this metric. The natural embedding of $M$ into its completion preserves distances, and maps $M$ onto a dense subset of its completion.

Let us go back to the case where $M = \mathbb{Q}$, with the standard metric. If $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbb{Q}$, then it is easy to see that $\{|x_j|\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbb{Q}$. If $\{x_j'\}_{j=1}^\infty$ is another Cauchy sequence in $\mathbb{Q}$ that is equivalent to $\{x_j\}_{j=1}^\infty$, then one can check that $\{|x_j'|\}_{j=1}^\infty$ and $\{|x_j|\}_{j=1}^\infty$ are equivalent as Cauchy sequences in $\mathbb{Q}$. This permits one to extend the absolute value function to a mapping from the completion of $\mathbb{Q}$ into itself. Similarly, if $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ are Cauchy sequences in $\mathbb{Q}$, then $\{|x_j - y_j|\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbb{Q}$, as in the previous paragraph. If $\{x_j'\}_{j=1}^\infty$ and $\{y_j'\}_{j=1}^\infty$ are Cauchy sequences in $\mathbb{Q}$ that are equivalent to $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$, then $\{|x_j' - y_j'|\}_{j=1}^\infty$ and $\{|x_j - y_j|\}_{j=1}^\infty$ are equivalent as Cauchy sequences in $\mathbb{Q}$, as before. Thus the standard metric on $\mathbb{Q}$ extends to a function defined on pairs of elements of the completion of $\mathbb{Q}$, and with values in the completion of $\mathbb{Q}$.

If $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ are Cauchy sequences in $\mathbb{Q}$, then one can check that $\{x_j + y_j\}_{j=1}^\infty$ and $\{x_j y_j\}_{j=1}^\infty$ are Cauchy sequences too, by standard arguments.
In the case of products, this uses the fact that Cauchy sequences are bounded. If \( \{x'_j\}_{j=1}^{\infty} \) and \( \{y'_j\}_{j=1}^{\infty} \) are Cauchy sequences in \( \mathbb{Q} \) that are equivalent to \( \{x_j\}_{j=1}^{\infty} \) and \( \{y_j\}_{j=1}^{\infty} \), respectively, then \( \{x'_j + y'_j\}_{j=1}^{\infty} \) and \( \{x'_j y'_j\}_{j=1}^{\infty} \) are equivalent as Cauchy sequences in \( \mathbb{Q} \) to \( \{x_j + y_j\}_{j=1}^{\infty} \) and \( \{x_j y_j\}_{j=1}^{\infty} \), respectively. Using this, one can extend addition and multiplication to the completion of \( \mathbb{Q} \), so that the completion of \( \mathbb{Q} \) becomes a commutative ring. Note that the extension of the standard metric on \( \mathbb{Q} \) to its completion mentioned in the preceding paragraph is the same as the extension of the absolute value to the completion of \( \mathbb{Q} \) applied to the difference of two elements of the completion of \( \mathbb{Q} \).

If \( \{x_j\}_{j=1}^{\infty} \) is a sequence of rational numbers that does not converge to 0, then there is an \( r \in \mathbb{Q} \) such that \( r > 0 \) and

\[
(1.25) \quad |x_j| \geq 2r \quad \text{for infinitely many } j.
\]

If \( \{x_j\}_{j=1}^{\infty} \) is also a Cauchy sequence of elements of \( \mathbb{Q} \), then it follows that

\[
(1.26) \quad |x_j| \geq r \quad \text{for all but finitely many } j,
\]

and in particular \( x_j \neq 0 \) for all but finitely many \( j \). If \( x_j \neq 0 \) for every \( j \), then one can use this to show that \( \{1/x_j\}_{j=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{Q} \). If \( \{x'_j\}_{j=1}^{\infty} \) is another Cauchy sequence of nonzero elements of \( \mathbb{Q} \) that is equivalent to \( \{x_j\}_{j=1}^{\infty} \), then \( \{1/x'_j\}_{j=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{Q} \) that is equivalent to \( \{1/x_j\}_{j=1}^{\infty} \). Using these remarks, one can extend the mapping \( x \mapsto 1/x \) to the nonzero elements of the completion of \( \mathbb{Q} \), so that the completion of \( \mathbb{Q} \) becomes a field.

Let \( \{x_j\}_{j=1}^{\infty} \) be a sequence of elements of \( \mathbb{Q} \) that does not converge to 0 again, so that (1.25) holds for some \( r \in \mathbb{Q} \) with \( r > 0 \). This implies that either

\[
(1.27) \quad x_j \geq 2r \quad \text{for infinitely many } j,
\]

or that

\[
(1.28) \quad x_j \leq -2r \quad \text{for infinitely many } j.
\]

If \( \{x_j\}_{j=1}^{\infty} \) is a Cauchy sequence in \( \mathbb{Q} \), then it follows that either

\[
(1.29) \quad x_j \geq r \quad \text{for all but finitely many } j,
\]

or that

\[
(1.30) \quad x_j \leq -r \quad \text{for all but finitely many } j.
\]

Let us say that \( \{x_j\}_{j=1}^{\infty} \) is positive in the first case, and negative in the second case. If \( \{x'_j\}_{j=1}^{\infty} \) is another Cauchy sequence in \( \mathbb{Q} \) that is equivalent to \( \{x_j\}_{j=1}^{\infty} \), then it is easy to see that \( \{x'_j\}_{j=1}^{\infty} \) is positive when \( \{x_j\}_{j=1}^{\infty} \) is positive, and that \( \{x'_j\}_{j=1}^{\infty} \) is negative when \( \{x_j\}_{j=1}^{\infty} \) is negative. This permits one to extend the standard ordering on \( \mathbb{Q} \) to its completion, with the usual properties with respect to addition and multiplication. Once the ordering on the completion of \( \mathbb{Q} \) is defined, it is easy to see that the extensions of the absolute value and distance functions to the completion of \( \mathbb{Q} \) satisfy the corresponding triangle inequalities.
Thus the completion of $\mathbb{Q}$ basically becomes a metric space, but where the metric also takes values in the completion of $\mathbb{Q}$. One can define convergence of sequences in the completion of $\mathbb{Q}$ in the usual way, as well as Cauchy sequences, and show that the completion of $\mathbb{Q}$ is complete, as before.

Suppose now that $d(x, y)$ is an ultrametric on a set $M$. If $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ are Cauchy sequences of elements of $M$ that are not equivalent, then there is an $r > 0$ such that

$$d(x_j, y_j) \geq r$$

for infinitely many $j$, and in fact for all but finitely many $j$. This implies that $d(x_j, y_j)$ is eventually constant in this case, as in (1.20). Similarly, if $\{x'_j\}_{j=1}^\infty$ and $\{y'_j\}_{j=1}^\infty$ are Cauchy sequences in $M$ that are equivalent to $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$, respectively, then

$$d(x'_j, y'_j) = d(x_j, y_j)$$

for all but finitely many $j$ under these conditions. It is easy to see that the extension of $d(\cdot, \cdot)$ to the completion of $M$ is also an ultrametric in this situation.

### 1.3 Continuous extensions

Let $(M_1, d_1(x, y))$ and $(M_2, d_2(u, v))$ be metric spaces, and suppose that $f$ is a uniformly continuous mapping from $M_1$ into $M_2$. Thus for each $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that

$$d_2(f(x), f(y)) < \epsilon$$

for every $x, y \in M_1$ with $d_1(x, y) < \delta$. If $\{x_j\}_{j=1}^\infty$ and $\{x'_j\}_{j=1}^\infty$ are sequences of elements of $M_1$ such that

$$\lim_{j \to \infty} d_1(x_j, x'_j) = 0,$$

then it is easy to see that

$$\lim_{j \to \infty} d_2(f(x_j), f(x'_j)) = 0.$$

Conversely, if $f$ is not uniformly continuous, then there are sequences $\{x_j\}_{j=1}^\infty$ and $\{x'_j\}_{j=1}^\infty$ in $M_1$ that satisfy (1.34) and not (1.35). If $f$ is uniformly continuous and $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence of elements of $M_1$, then $\{f(x_j)\}_{j=1}^\infty$ is a Cauchy sequence of elements of $M_2$, which converges to an element of $M_2$ when $M_2$ is complete.

Now let $E$ be a dense subset of $M_1$, and suppose that $f$ is a uniformly continuous mapping from $E$ into $M_2$, with respect to the restriction of $d_1(x, y)$ to $x, y \in E$. If $M_2$ is complete, then it is well known that there is a unique extension of $f$ to a uniformly continuous mapping from $M_1$ into $M_2$. More precisely, uniqueness only requires continuity instead of uniform continuity, and completeness of $M_2$ is not needed. To get the existence of such an extension, let $x \in M_1$ be given, and let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of $E$ that converges
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to \(x\) in \(M_1\). Thus \(\{x_j\}_{j=1}^\infty\) is a Cauchy sequence in \(E\), so that \(\{f(x_j)\}_{j=1}^\infty\) converges in \(M_2\), as before. If \(\{x'_j\}_{j=1}^\infty\) is another sequence of elements of \(E\) that converges to \(x\) in \(M_1\), then \(\{x_j\}_{j=1}^\infty\) and \(\{x'_j\}_{j=1}^\infty\) satisfy (1.34), and hence (1.35). This implies that \(\{f(x_j)\}_{j=1}^\infty\) and \(\{f(x'_j)\}_{j=1}^\infty\) converge to the same element of \(M_2\). If we put \(f(x)\) equal to the limit of \(\{f(x_j)\}_{j=1}^\infty\) under these conditions, then this agrees with the original definition of \(f(x)\) when \(x \in E\), and it does not depend on the choice of sequence \(\{x_j\}_{j=1}^\infty\).

To show that this extension is uniformly continuous on \(M_1\), let \(\epsilon > 0\) be given, and let \(\delta = \delta(\epsilon)\) be a positive real number such that (1.33) holds for every \(x, y \in E\) with \(d_1(x, y) < \delta\). Let \(x, y \in M_1\) be given, with \(d_1(x, y) < \delta\), and let \(\{x_j\}_{j=1}^\infty\) and \(\{y_j\}_{j=1}^\infty\) be sequences of elements of \(E\) that converge to \(x\) and \(y\) in \(M_1\), respectively. Thus \(d_1(x_j, y_j) < \delta\) for all sufficiently large \(j\), so that

\[
d_2(f(x_j), f(y_j)) < \epsilon
\]

for all sufficiently large \(j\). This implies that

\[
d_2(f(x), f(y)) \leq \epsilon,
\]

since \(\{f(x_j)\}_{j=1}^\infty\) and \(\{f(y_j)\}_{j=1}^\infty\) converge to \(f(x)\) and \(f(y)\) in \(M_2\), respectively, by construction. Similarly, if \(f : E \to M_2\) is an isometric embedding, in the sense that

\[
d_2(f(x), f(y)) = d_1(x, y)
\]

for every \(x, y \in E\), then \(f\) is obviously uniformly continuous, and this extension of \(f\) to \(M_1\) satisfies (1.38) for every \(x, y \in M_1\).

Let \((M, d(x, y))\) be a metric space, and suppose that \(\phi_1\) and \(\phi_2\) are isometric embeddings of \(M\) into \(M_1\) and \(M_2\), respectively, so that

\[
d_1(\phi_1(x), \phi_1(y)) = d_2(\phi_2(x), \phi_2(y)) = d(x, y)
\]

for every \(x, y \in M\). Suppose also that \(\phi_j(M)\) is dense in \(M_j\) for \(j = 1, 2\), which can always be arranged by replacing \(M_j\) with the closure of \(\phi_j(M_j)\). Put

\[
f = \phi_2 \circ \phi_1^{-1}
\]

on \(\phi_1(M)\), which is an isometric embedding of \(\phi_1(M)\) into \(M_2\), with respect to the restriction of \(d_1(\cdot, \cdot)\) to \(\phi_1(M)\). If \(M_2\) is complete, then \(f\) has a unique extension to an isometric embedding of \(M_1\) into \(M_2\), as before. If \(M_1\) is complete, then \(f(M_1)\) is complete with respect to the restriction of \(d_2(\cdot, \cdot)\) to \(f(M_1)\). This implies that \(f(M_1)\) is a closed subset of \(M_2\), because any sequence of elements of \(f(M_1)\) that converges to an element of \(M_2\) is a Cauchy sequence in \(f(M_1)\), and hence converges to an element of \(f(M_1)\), by completeness. It follows that

\[
f(M_1) = M_2
\]

under these conditions, because \(f(M_1) = \phi_2(M)\) is dense in \(M_2\), by hypothesis.
1.4 Quasimetrics

Let $M$ be a set, and let $d(x,y)$ be a nonnegative real-valued function defined for $x, y \in M$ such that

\begin{equation}
    d(x, y) = 0 \quad \text{if and only if} \quad x = y,
\end{equation}

and

\begin{equation}
    d(x, y) = d(y, x)
\end{equation}

for every $x, y \in M$. We say that $d(x, y)$ is a quasimetric on $M$ if

\begin{equation}
    d(x, z) \leq C \left( d(x, y) + d(y, z) \right)
\end{equation}

for some $C \geq 1$ and every $x, y, z \in M$. This is equivalent to asking that

\begin{equation}
    d(x, z) \leq C' \max(d(x, y), d(y, z))
\end{equation}

for some $C' \geq 1$ and every $x, y, z \in M$. More precisely, (1.45) implies (1.44) with $C = C'$, and (1.44) implies (1.45) with $C' = 2C$. Of course, (1.44) reduces to the ordinary triangle inequality (1.3) when $C = 1$, and (1.45) reduces to the ultrametric version of the triangle inequality (1.4) when $C' = 1$.

If $d(\cdot, \cdot)$ satisfies (1.45) and $a$ is a positive real number, then

\begin{equation}
    d(x, z)^a \leq C' \max(d(x, y)^a, d(y, z)^a)
\end{equation}

for every $x, y, z \in M$, so that $d(\cdot, \cdot)^a$ is also a quasimetric on $M$. Similarly, if $d(\cdot, \cdot)$ satisfies (1.44) and $0 < a \leq 1$, then

\begin{equation}
    d(x, z)^a \leq C'' \left( d(x, y)^a + d(y, z)^a \right)
\end{equation}

for every $x, y, z \in M$, using (1.7) in the second step. If $a \geq 1$, then $f(r) = r^a$ is a convex function on $[0, \infty)$, and hence

\begin{equation}
    (r + t)^a = 2^a \frac{r}{2} + 2^a \frac{t}{2} \leq 2^a (r^a/2 + t^a/2) = 2^{a-1} (r^a + t^a)
\end{equation}

for every $r, t \geq 0$. This implies that

\begin{equation}
    d(x, z)^a \leq C'' \left( d(x, y)^a + d(y, z)^a \right) \leq 2^{a-1} C'' \left( d(x, y)^a + d(y, z)^a \right)
\end{equation}

for every $x, y, z \in M$ when $d(\cdot, \cdot)$ satisfies (1.44) and $a \geq 1$. Note that $|x - y|^a$ is not a metric on $\mathbb{R}$ when $a > 1$.

Let $d(\cdot, \cdot)$ be a quasimetric on a set $M$ that satisfies (1.44) for some $C \geq 1$. Also let $n$ be a nonnegative integer, and let us check that

\begin{equation}
    d(x_0, x_{2^n}) \leq C^n \sum_{j=1}^{2^n} d(x_{j-1}, x_j)
\end{equation}

for any finite sequence $x_0, x_1, \ldots, x_{2^n}$ of $2^n + 1$ elements of $M$. This is trivial when $n = 0$, and this is the same as (1.44) when $n = 1$. Suppose now that (1.50)
holds for some \( n \geq 0 \), and let us verify that the analogous statement holds for \( n+1 \) as well. If \( x_0, x_1, \ldots, x_{2^n+1} \) is a finite sequence of \( 2^n+1 + 1 \) elements of \( M \), then we can apply (1.50) to the first \( 2^n + 1 \) terms \( x_0, x_1, \ldots, x_{2^n} \) of this sequence. Similarly, we can apply the induction hypothesis to the sequence \( x_{2^n}, x_{2^n+1}, \ldots, x_{2^n+1} \) of \( 2^n + 1 \) elements of \( M \), to get that

\[
(1.51) \quad d(x_{2^n}, x_{2^n+1}) \leq C^n \sum_{j=1}^{2^n} d(x_{2^n+j-1}, x_{2^n+j}).
\]

It follows that

\[
(1.52) \quad d(x_0, x_{2^n+1}) \leq C (d(x_0, x_{2^n}) + d(x_{2^n}, x_{2^n+1})) \leq C^n \sum_{j=1}^{2^n} d(x_{j-1}, x_j) + C^{n+1} \sum_{j=1}^{2^n} d(x_{2^n+j-1}, x_{2^n+j}) = C^{n+1} \sum_{j=1}^{2^n+1} d(x_{j-1}, x_j)
\]

using (1.44) in the first step.

If instead \( d(\cdot, \cdot) \) satisfies (1.45) for some \( C' \geq 1 \), then

\[
(1.53) \quad d(x_0, x_{2^n}) \leq (C')^n \max\{d(x_{j-1}, x_j) : j = 1, \ldots, 2^n\}
\]

for any sequence finite \( x_0, x_1, \ldots, x_{2^n} \) of \( 2^n + 1 \) elements of \( M \). As before, this is trivial when \( n = 0 \), and this is the same as (1.45) when \( n = 1 \). If (1.53) holds for some \( n \geq 0 \) and \( x_0, x_1, \ldots, x_{2^n+1} \) is a finite sequence of \( 2^n+1 + 1 \) elements of \( M \), then we can apply (1.53) to the first \( 2^n + 1 \) terms \( x_0, x_1, \ldots, x_{2^n} \) of this sequence. We can also apply this induction hypothesis to the sequence \( x_{2^n}, x_{2^n+1}, \ldots, x_{2^n+1} \) of \( 2^n + 1 \) elements of \( M \), to get that

\[
(1.54) \quad d(x_{2^n}, x_{2^n+1}) \leq (C')^n \max\{d(x_{2^n+j-1}, x_{2^n+j}) : j = 1, \ldots, 2^n\}.
\]

It follows that

\[
(1.55) \quad d(x_0, x_{2^n+1}) \leq C' \max(d(x_0, x_{2^n}), d(x_{2^n}, x_{2^n+1})) \leq (C')^{n+1} \max\{d(x_{j-1}, x_j) : j = 1, \ldots, 2^{n+1}\},
\]

using (1.45) in the first step.

### 1.5 Quasimetrics, 2

Let \( d(x, y) \) be a quasimetric on a set \( M \). Thus the open ball \( B(x, r) \) in \( M \) centered at a point \( x \in M \) and with radius \( r > 0 \) can be defined with respect to \( d(x, y) \) as in (1.11). Let us say that a set \( U \subseteq M \) is an open set if for each \( x \in M \) there is an \( r > 0 \) such that

\[
(1.56) \quad B(x, r) \subseteq U,
\]
as usual. It is easy to see that this defines a topology on $M$, in the same way as for metric spaces. If $d(x, y)$ is a metric on $M$, then open balls in $M$ are open sets, as in (1.12). This uses the ordinary version of the triangle inequality in a significant way, and does not work for quasimetrics, without additional hypotheses. However, there are some substitutes for this, as follows.

Let us begin with some variants of (1.12). Suppose that $d(x, y)$ satisfies (1.44) for some $C \geq 1$, and let $x \in M$ and $r > 0$ be given. If $y \in M$ and $d(x, y) < (2C)^{-1}r$, then one can check that

\[(1.57) \quad B(y, (2C)^{-1}r) \subseteq B(x, r).\]

Similarly, if $d(\cdot, \cdot)$ satisfies (1.45) for some $C' \geq 1$, then

\[(1.58) \quad B(y, (C')^{-1}r) \subseteq B(x, r)\]

for every $y \in M$ with $d(x, y) < (C')^{-1}r$.

Let $E$ be any subset of $M$, and put

\[(1.59) \quad U = \{x \in M : B(x, r) \subseteq E \text{ for some } r > 0\}.

Let $x \in U$ be given, and let $r$ be a positive real number such that $B(x, r) \subseteq E$. If $d(\cdot, \cdot)$ satisfies (1.44), then

\[(1.60) \quad B(y, (2C)^{-1}r) \subseteq B(x, r) \subseteq E\]

for every $y \in M$ with $d(x, y) < (2C)^{-1}r$, by (1.57). Similarly, if $d(\cdot, \cdot)$ satisfies (1.58), then

\[(1.61) \quad B(y, (C')^{-1}r) \subseteq B(x, r) \subseteq E\]

for every $y \in M$ with $d(x, y) < (C')^{-1}r$. In both cases, it follows that $y \in U$, which means that $U$ contains an open ball centered at $x$ with positive radius. This implies that $U$ is an open set in $M$. Of course, any open subset of $M$ that is contained in $E$ is also contained in $U$, so that $U$ is equal to the interior of $E$ in $M$ with respect to this topology.

Let $w \in M$ and $t > 0$ be given, and let us apply the previous remarks to $E = B(w, t)$. This leads to a set $U = U(w, t)$ as in (1.59), which is the interior of $B(w, t)$. Of course, $w \in U(w, t)$, by construction. More precisely, if $d(\cdot, \cdot)$ satisfies (1.44), then we get that

\[(1.62) \quad B(w, (2C)^{-1}t) \subseteq U(w, t),\]

by (1.57). Similarly, if $d(\cdot, \cdot)$ satisfies (1.45), then

\[(1.63) \quad B(w, (C')^{-1}t) \subseteq U(w, t),\]

by (1.58).

Now let $w, z \in M$ be given, with $w \neq z$, so that $d(w, z) > 0$. If $d(\cdot, \cdot)$ satisfies (1.44), then it follows that

\[(1.64) \quad B(w, (2C)^{-1}d(w, z)) \cap B(z, (2C)^{-1}d(w, z)) = \emptyset\]
1.5. QUASIMETRICS, 2

Similarly, if \( d(\cdot, \cdot) \) satisfies (1.45), then

\[
(1.65) \quad B(w, (C')^{-1} d(w, z)) \cap B(z, (C')^{-1} d(w, z)) = \emptyset.
\]

This implies that \( M \) is Hausdorff, since \( w \) and \( z \) are in the interiors of the corresponding open balls, by the remarks in the preceding paragraph.

Let \( \{x_j\}_{j=1}^{\infty} \) be a sequence of elements of \( M \), and let \( x \) be an element of \( M \). It is natural to say that \( \{x_j\}_{j=1}^{\infty} \) converges to \( x \) in \( M \) with respect to \( d(\cdot, \cdot) \) when

\[
(1.66) \quad \lim_{j \to \infty} d(x_j, x) = 0.
\]

Alternatively, one might use the definition of convergence of sequences in a topological space, so that \( \{x_j\}_{j=1}^{\infty} \) converges to \( x \in M \) if for every open set \( U \subseteq M \) with \( x \in U \) there is an \( L \geq 1 \) such that

\[
(1.67) \quad x_j \in U
\]

for each \( j \geq L \). The first condition obviously implies the second condition, by the definition of open sets in \( M \). Conversely, the second condition implies the first condition, because every open ball centered at \( x \) contains an open set in \( M \) that contains \( x \) as an element, as before.

Similarly, one might like to say that a point \( x \in M \) is a limit point of a set \( E \subseteq M \) with respect to the quasimetric \( d(\cdot, \cdot) \) if for each \( r > 0 \) there is a \( y \in E \) such that \( x \neq y \) and \( d(x, y) < r \). The usual topological definition says that \( x \in M \) is a limit point of \( E \subseteq M \) if for each open set \( U \subseteq M \) with \( x \in U \), there is a point \( y \in E \) such that \( x \neq y \) and \( y \in U \). It is easy to see that the first definition implies the second definition in this situation, by the definition of open sets in \( M \). Conversely, the second definition implies the first definition, because every open ball in \( M \) centered at \( x \) contains an open set that contains \( x \) as an element. As usual, the closure of \( E \subseteq M \) is the set \( \overline{E} \) of \( x \in M \) such that \( x \in E \) or \( x \) is a limit point of \( E \), and is a closed subset of \( E \), and the topological characterization implies that \( \overline{E} \) is always a closed set in \( M \).

Let \( x \in M \) and a positive integer \( j \) be given, and let \( U_j(x) \) be the interior of \( B(x, 1/j) \). Thus \( U_j(x) \) is an open set that contains \( x \) and is contained in \( B(x, 1/j) \), as before. If \( U \) is any open set in \( M \) that contains \( x \), then \( U_j(x) \subseteq U \) for all sufficiently large \( j \), by the definition of an open set in \( M \). This shows that there is a local base for the topology of \( M \) at \( x \) with only finitely or countably many elements, as in the case of metric spaces. In particular, this implies that sequences can be used for many standard topological arguments involving \( M \), concerning limit points and continuity, for instance.

Of course, one can define the closed ball \( \overline{B}(x, r) \) centered at a point \( x \in M \) and with radius \( r \geq 0 \) as in (1.15). In a metric space, closed balls are closed sets, but this does not work in quasimetric spaces without additional hypotheses. However, if \( d(\cdot, \cdot) \) satisfies (1.44), then it is easy to see that the closure of \( \overline{B}(x, r) \) is contained in \( \overline{B}(x, C r) \). This uses the characterization of limit points of subsets of \( M \) in terms of \( d(\cdot, \cdot) \) mentioned earlier. Similarly, if \( d(\cdot, \cdot) \) satisfies (1.45), then the closure of \( \overline{B}(x, r) \) is contained in \( \overline{B}(x, C' r) \).
One can also define a uniform structure on $M$ corresponding to the quasimetric $d(\cdot, \cdot)$ in essentially the same way as for metric spaces, as in [21]. The topology on $M$ determined by $d(\cdot, \cdot)$ described earlier is the same as the topology associated to this uniform structure as in [21]. Note that the characterization of the interior of a set $E \subseteq M$ as the set $U$ in (1.59) is the same as Theorem 4 on p178 of [21] in this context. Cauchy sequences and uniform continuity can be defined for quasimetrics in the same way as for metrics, and are determined by the corresponding uniform structure as well.

The metrization theorem for uniform spaces discussed in [21] implies that there is a metric on $M$ that determines the same uniform structure as the one associated to $d(x, y)$, and hence the same topology. Of course, many related properties of $M$ can be shown more directly, as before. Remember that $d(x, y)^a$ is a quasimetric on $M$ for every positive real number $a$, as in the previous section. It is easy to see that $d(x, y)^a$ determines the same uniform structure on $M$ as $d(x, y)$ for each $a > 0$, and the same topology on $M$ in particular. In [25], it is shown that there is a metric $d_0(x, y)$ on $M$ and a positive real number $a_0$ such that $d(x, y)$ and $d_0(x, y)^{a_0}$ are each bounded by constant multiples of the other.

1.6 Lipschitz mappings

Let $(M_1, d_1(x, y))$ and $(M_2, d_2(u, v))$ be quasimetric spaces, so that $M_1$ and $M_2$ are sets, and $d_1(x, y)$ and $d_2(u, v)$ are quasimetrics on them, respectively. A mapping $f : M_1 \rightarrow M_2$ is said to be Lipschitz of order $a > 0$ if there is a nonnegative real number $C$ such that

\begin{equation}
(1.68) \quad d_2(f(x), f(y)) \leq C \cdot d_1(x, y)^a
\end{equation}

for every $x, y \in M_1$. If $a = 1$, then one may simply say that $f$ is Lipschitz. Note that a Lipschitz mapping of any order is uniformly continuous. Of course, $f$ satisfies (1.68) with $C = 0$ if and only if $f$ is constant.

Remember that $d_1(x, y)^a$ is also a quasimetric on $M_1$ for every $a > 0$, as in Section 1.4. Thus $f$ is Lipschitz of order $a$ as a mapping from $(M_1, d_1(x, y))$ into $(M_2, d_2(u, v))$ if and only if $f$ is Lipschitz of order 1 as a mapping from $(M_1, d_1(x, y)^a)$ into $(M_2, d_2(u, v))$, with the same constant $C$. Similarly, $f$ is Lipschitz of order $a$ with constant $C$ as a mapping from $(M_1, d_1(x, y))$ into $(M_2, d_2(u, v))$ if and only if $f$ is Lipschitz of order 1 with constant $C^{1/a}$ as a mapping from $(M_1, d_1(x, y))$ into $(M_2, d_2(u, v)^{1/a})$. If $d_1(x, y)$ and $d_2(u, v)$ are metrics on $M_1$ and $M_2$, respectively, then $d_1(x, y)^a$ is a metric on $M_1$ when $0 < a \leq 1$, and $d_2(u, v)^{1/a}$ is a metric on $M_2$ when $a \geq 1$, as in Section 1.1.

Let us now restrict our attention to the case where $M_2$ is the real line, equipped with the standard metric. If $f$ is a real-valued function on $M_1$ that satisfies

\begin{equation}
(1.69) \quad f(x) \leq f(y) + C \cdot d_1(x, y)
\end{equation}
for some $C \geq 0$ and every $x, y \in M_1$, then we also have that

(1.70) \[ f(y) \leq f(x) + C d_1(x, y) \]

for every $x, y \in M_1$, by interchanging the roles of $x$ and $y$. This implies that

(1.71) \[ |f(x) - f(y)| = \max(f(x) - f(y), f(y) - f(x)) \leq C d_1(x, y) \]

for every $x, y \in M_1$, so that $f$ is Lipschitz of order 1 with constant $C$. In particular,

(1.72) \[ f_p(x) = d_1(x, p) \]

has this property with $C = 1$ for every $p \in M_1$ when $d_1(x, y)$ is a metric on $M_1$, by the triangle inequality. In this case, $d_1(x, y)^a$ also defines a metric on $M_1$ when $0 < a \leq 1$, as in Section 1.1. It follows that

(1.73) \[ f_{p,a}(x) = d_1(x, p)^a \]

defines a Lipschitz mapping from $(M_1, d_1(x, y)^a)$ into $\mathbb{R}$ of order 1 with constant $C = 1$ for each $p \in M_1$, by the same argument. Equivalently, this means that (1.73) is a Lipschitz mapping from $(M_1, d_1(x, y))$ into $\mathbb{R}$ of order $a$ with constant $C = 1$ for each $p \in M_1$ when $0 < a \leq 1$ and $d_1(\cdot, \cdot)$ is a metric on $M_1$.

Suppose for the moment that $M_1 = \mathbb{R}$, and that $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz of order $a > 1$ with respect to the standard metric on $\mathbb{R}$ on the domain and range. It is easy to see that $f$ is constant on $\mathbb{R}$ under these conditions, because $f'(x) = 0$ for every $x \in \mathbb{R}$. Equivalently, if $M_1 = \mathbb{R}$ equipped with the quasimetric

(1.74) \[ d_1(x, y) = |x - y|^a \]

for some $a > 1$, and if $f$ is Lipschitz of order 1 as a mapping from $(M_1, d_1(x, y))$ into $\mathbb{R}$ with the standard metric, then $f$ is constant. However, if $d_1(x, y)$ is any quasimetric on a set $M_1$, then one can use metrics on $M_1$ as in [25] to get real-valued Lipschitz functions of positive order on $M_1$ with respect to $d_1(x, y)$.

1.7 Haar measure

Let $A$ be a commutative group, with the group operations expressed additively. Suppose that $A$ is also equipped with a topology, such that the group operations on $A$ are continuous. More precisely, this means that addition on $A$ is continuous as a mapping from $A \times A$ into $A$, with respect to the product topology on $A \times A$ associated to the given topology on $A$. The mapping

(1.75) \[ x \mapsto -x \]

should be continuous on $A$ too, where $-x$ is the additive inverse of $x \in A$. In order for $A$ to be a topological group, it is customary to ask that $\{0\}$ be a closed set in $A$. It is well known that this implies that $A$ is Hausdorff, and in fact regular as a topological space. Note that the translation mapping

(1.76) \[ x \mapsto a + x \]
is continuous on $A$ for every $a \in A$, because of continuity of addition on $A$. This implies that (1.76) is a homeomorphism from $A$ onto itself for each $a \in A$, since the inverse mapping corresponds to translation by $-a$. Similarly, (1.75) is a homeomorphism on $A$, because it is its own inverse mapping.

Put

\begin{equation}
-E = \{-x : x \in E\} \tag{1.77}
\end{equation}

for each $E \subseteq A$, and

\begin{equation}
a + E = \{a + x : x \in E\} \tag{1.78}
\end{equation}

for each $a \in A$ and $E \subseteq A$. If $E$ is an open set in $A$, then (1.77) is an open set in $A$ too, and (1.78) is an open set in $A$ for every $a \in A$, because (1.75) and (1.76) are homeomorphisms on $A$. There are analogous statements for closed sets, compact sets, and Borel sets. In particular, if there is a nonempty open subset of $A$ that is contained in a compact set, then it follows that $A$ is locally compact as a topological space.

If $A$ is locally compact, then a famous theorem states that there is a well-behaved nonnegative translation-invariant Borel measure $H$ on $A$, known as \textit{Haar measure}. To say that $H$ is invariant under translations on $A$ means that

\begin{equation}
H(a + E) = H(E) \tag{1.79}
\end{equation}

for every Borel set $E \subseteq A$ and $a \in A$. Haar measure is also supposed to satisfy $H(U) > 0$ for every nonempty open set $U \subseteq A$, $H(K) < \infty$ for every compact set $K \subseteq A$, and some additional regularity properties. It is well known that $H$ is uniquely determined up to multiplication by a positive real number under these conditions. Using this, one can show that

\begin{equation}
H(-E) = H(E) \tag{1.80}
\end{equation}

for every Borel set $E \subseteq A$. Of course, any commutative group $A$ is a locally compact topological group with respect to the discrete topology, in which case counting measure on $A$ satisfies the requirements of Haar measure. The real line is a locally compact commutative topological group with respect to addition and the standard topology, and Lebesgue measure on $\mathbb{R}$ satisfies the requirements of Haar measure.

Let $A$ be a locally compact commutative topological group again, and let $C_{\text{com}}(A)$ be the space of continuous real or complex-valued functions on $A$ with compact support. If $H$ satisfies the requirements of Haar measure on $A$, then

\begin{equation}
L(f) = \int_A f \, dH \tag{1.81}
\end{equation}

defines a nonnegative linear functional on $C_{\text{com}}(A)$. More precisely, if $f$ is a continuous nonnegative real-valued function with compact support on $A$ such that $f(x) > 0$ for some $x \in A$, then it is easy to see that $L(f)$ is a positive real number. If $f \in C_{\text{com}}(A)$ and $a \in A$, then

\begin{equation}
f_a(x) = f(x + a) \tag{1.82}
\end{equation}
defines an element of $C_{com}(A)$, and

\begin{equation}
L(f_a) = L(f),
\end{equation}

because of the translation-invariance of $H$. A linear functional on $C_{com}(A)$ with these properties is known as a Haar integral on $A$. Haar measure on $A$ can also be obtained from a Haar integral, using the Riesz representation theorem. A Haar integral can be defined on the real line using the Riemann integral, for instance.
Chapter 2

Absolute value functions

2.1 Definitions and examples

Let \( k \) be a field. A nonnegative real-valued function \(| \cdot |\) on \( k \) is said to be an absolute value function on \( k \) if it satisfies the following three conditions: first,

\[
|x| = 0 \text{ if and only if } x = 0;
\]

(2.1)

second,

\[
|x y| = |x| |y|
\]

(2.2)

for every \( x, y \in k \); and third,

\[
|x + y| \leq |x| + |y|
\]

(2.3)

for every \( x, y \in k \). Of course, the standard absolute value function on \( \mathbb{R} \) satisfies these conditions, as in Section 1.1. Similarly, it is well known that the standard norm or modulus on the field \( \mathbb{C} \) of complex numbers satisfies these conditions. If \( k \) is any field, then the trivial absolute value function on \( k \) is defined by putting \(|0| = 0\) and \(|x| = 1\) for every \( x \in k \) with \( x \neq 0 \), and is easily seen to satisfy these conditions as well.

Suppose for the moment that \(| \cdot |\) is a nonnegative real-valued function on \( k \) that satisfies (2.1) and (2.2). Thus \(|1| > 0\), since \( 1 \neq 0 \) in \( k \), by definition of a field. Here we use 0 and 1 to denote both the additive and multiplicative identity elements in \( k \) and their counterparts in \( \mathbb{R} \), and it should always be clear from the context which is being considered in any given instance. We also have that \(|1|^2 = |1|^2 = |1|\), by (2.2), which implies that

\[
|1| = 1.
\]

(2.4)

Similarly, if \( x \in k \) satisfies \( x^n = 1 \) for some positive integer \( n \), then

\[
|x|^n = |x^n| = |1| = 1,
\]

(2.5)
and hence $|x| = 1$. In particular, $(-1)^2 = 1$ in $k$, so that

$$| - 1| = 1.$$ (2.6)

If $x \in k$ and $x \neq 0$, then $x$ has a multiplicative inverse $x^{-1}$ in $k$, which satisfies

$$|x| |x^{-1}| = |1| = 1,$$ (2.7)

and hence

$$|x^{-1}| = |x|^{-1}.$$ (2.8)

If $| \cdot |$ is an absolute value function on $k$, then it follows from (2.6) that

$$d(x, y) = |x - y|$$ (2.9)

is symmetric in $x$ and $y$. Thus (2.9) defines a metric on $k$. Let us say that $| \cdot |$ is an ultrametric absolute value function on $k$ if

$$|x + y| \leq \max(|x|, |y|)$$ (2.10)

for every $x, y \in k$. This implies that the associated metric (2.9) is an ultrametric on $k$. The trivial absolute value function on any field $k$ is an ultrametric absolute value function, for which the associated metric (2.9) is the same as the discrete metric on $k$.

The $p$-adic absolute value $| \cdot |_p$ is defined on $\mathbb{Q}$ for each prime number $p$ as follows. Let $x \in \mathbb{Q}$ be given, and put $|x|_p = 0$ when $x = 0$. Otherwise, if $x \neq 0$, then $x$ can be expressed as $p^j a/b$ for some integers $a, b$, and $j$ such that neither $a$ nor $b$ is an integer multiple of $p$, including 0. In this case, we put

$$|x|_p = p^{-j},$$ (2.11)

which does not depend on the particular choices of $a$ and $b$. One can check that $| \cdot |_p$ is an ultrametric absolute value function on $\mathbb{Q}$, and the corresponding ultrametric

$$d_p(x, y) = |x - y|_p,$$ (2.12)

is known as the $p$-adic metric on $\mathbb{Q}$.

Let $k$ be an arbitrary field again, and let $| \cdot |$ be a nonnegative real-valued function on $k$ that satisfies (2.1) and (2.2). This implies that (2.4) and (2.6) still hold, for the same reasons as before. Let us say that $| \cdot |$ is a quasimetric absolute value function if there is a real number $C \geq 1$ such that

$$|x + y| \leq C (|x| + |y|)$$ (2.13)

for every $x, y \in k$. This means that (2.9) satisfies (1.44), and hence is a quasimetric on $k$. Equivalently, $| \cdot |$ is a quasimetric absolute value function on $k$ if there is a real number $C' \geq 1$ such that

$$|x + y| \leq C' \max(|x|, |y|)$$ (2.14)
CHAPTER 2. ABSOLUTE VALUE FUNCTIONS

for every \(x, y \in k\), in which case (2.9) satisfies (1.45) on \(k\). As before, (2.14) implies (2.13) with \(C = C'\), and (2.13) implies (2.14) with \(C' = 2C\). Of course, (2.13) reduces to (2.3) when \(C = 1\), and (2.14) reduces to (2.10) when \(C' = 1\).

If \(|·|\) satisfies (2.13) and \(0 < a \leq 1\), then

\[
|x + y|^a \leq C^a(|x| + |y|)^a \leq C^a(|x|^a + |y|^a)
\]

(2.15)

for every \(x, y \in k\), by (1.7). In particular, if \(|x|\) is an absolute value function on \(k\), then \(|x|^a\) is also an absolute value function on \(k\) when \(0 < a \leq 1\). If \(|·|\) satisfies (2.13) and \(a \geq 1\), then

\[
|x + y|^a \leq C^a(|x| + |y|)^a \leq 2^{a-1}C^a(|x|^a + |y|^a)
\]

(2.16)

for every \(x, y \in k\), by (1.48). Similarly, if \(|·|\) satisfies (2.14) and \(a > 0\), then

\[
|x + y|^a \leq (C')^a \max(|x|^a, |y|^a)
\]

(2.17)

for every \(x, y \in k\). It follows that \(|x|^a\) is an ultrametric absolute value function on \(k\) for every \(a > 0\) when \(|x|\) is an ultrametric absolute value function on \(k\), and that \(|x|^a\) is a quasimetric absolute value function on \(k\) for every \(a > 0\) when \(|x|\) is a quasimetric absolute value function on \(k\).

Let \(|·|\) be a nonnegative real-valued function on \(k\) that satisfies (2.1) and (2.2) again, and hence (2.4) and (2.6). If \(|·|\) also satisfies (2.14) for some \(C' \geq 1\), then

\[
|1 + z| \leq C' \quad \text{for every } z \in k \text{ with } |z| \leq 1,
\]

(2.18)

by (2.4). Conversely, suppose that \(|·|\) satisfies (2.18) for some \(C' \geq 1\), and let us check that (2.14) holds for every \(x, y \in k\). We may as well restrict our attention to the case where \(|y| \leq |x|\), since otherwise we can interchange the roles of \(x\) and \(y\). If \(x = 0\), then (2.14) is trivial, and so we can suppose that \(x \neq 0\) too. Thus we can put \(z = y/x\), so that \(|z| = |y|/|x| \leq 1\), by (2.8). This permits us to use (2.18) to get that

\[
|x + y| = |1 + z||x| \leq C' \max(|x|, |y|),
\]

(2.19)

as desired.

The definition of a quasimetric absolute value function in terms of (2.18) corresponds to Definition 1.1 on p12 of [4], but with different terminology. The definition of an ordinary absolute value function corresponds to Definition 2.1.1 on p21-2 of [14]. The relationship between ordinary absolute value functions and quasimetric absolute value functions will be clarified in the next section, as in Lemma 1.2 on p13-4 of [4]. Ultrametric absolute value functions are also called non-archimedian, and we shall return to this in Section 2.3.
2.2 Some refinements

Let $|\cdot|$ be a quasimetric absolute value function on a field $k$, that satisfies (2.14) for some $C' \geq 1$. Also let $n$ be a nonnegative integer, and let us check that

$$\sum_{j=1}^{2^n} |z_j| \leq (C')^n \max \{|z_j| : j = 1, \ldots, 2^n\}$$

(2.20)

for any finite sequence $z_1, \ldots, z_{2^n}$ of $2^n$ elements of $k$. This is trivial when $n = 0$, and this is the same as (2.14) when $n = 1$. Suppose now that (2.20) holds for some $n \geq 1$, and let us verify that the analogous statement holds for $n + 1$. Let $z_1, \ldots, z_{2^{n+1}}$ be a finite sequence of $2^{n+1}$ elements of $k$, so that (2.20) can be applied to the first $2^n$ terms $z_1, \ldots, z_{2^n}$ of this sequence. Similarly, we can apply the induction hypothesis to the last $2^n$ terms $z_{2^n+1}, \ldots, z_{2^{n+1}}$ of this sequence, to get that

$$\left| \sum_{j=1}^{2^n} z_{2^n+j} \right| \leq (C')^n \max \{|z_{2^n+j}| : j = 1, \ldots, 2^n\}.$$ 

(2.21)

It follows that

$$\left| \sum_{j=1}^{2^{n+1}} z_j \right| \leq C' \max \left\{ \left| \sum_{j=1}^{2^n} z_j \right|, \left| \sum_{j=1}^{2^n} z_{2^n+j} \right| \right\} \leq (C')^{n+1} \max \{|z_j| : j = 1, \ldots, 2^{n+1}\},$$

(2.22)

using (2.14) in the first step. Note that (2.20) is equivalent to (1.55) in this setting, where $d(\cdot, \cdot)$ is the quasimetric (2.9) corresponding to $|\cdot|$. More precisely, (2.20) follows from (1.53) with $x_0 = 0$ and $x_l = \sum_{j=1}^{l-1} z_j$ when $l \geq 1$. Conversely, (1.53) follows from (2.20) applied to $z_j = x_j - x_{j-1}$ in this situation.

Suppose that $|\cdot|$ is a quasimetric absolute value function on $k$ that satisfies (2.14) with $C' = 2$. Let $N$ be a positive integer, and let $n$ be the smallest nonnegative integer such that

$$N \leq 2^n,$$

(2.23)

so that $2^{n-1} < N$. Also let $z_1, \ldots, z_N$ be a finite sequence of $N$ elements of $k$, and put $z_j = 0$ when $N < j \leq 2^n$. Applying (2.20) with $C' = 2$, we get that

$$\left| \sum_{j=1}^{N} z_j \right| \leq 2^n \max \{|z_j| : j = 1, \ldots, N\} \leq 2N \max \{|z_j| : j = 1, \ldots, N\}.$$ 

(2.24)

Let $z \in k$ be given, and let $N \cdot z$ be the sum of $N$ $z$’s in $k$. Observe that

$$|N \cdot z| \leq 2N |z|,$$

(2.25)
by applying (2.24) with \( z_j = z \) for each \( j = 1, \ldots, N \). In particular,
\[(2.26) \quad |N \cdot 1| \leq 2N,\]
by (2.4).

Let \( x, y \in k \) be given, and let \( r \) be a positive integer. The binomial theorem implies that
\[(2.27) \quad (x + y)^r = \sum_{j=0}^{r} \binom{r}{j} \cdot x^j y^{r-j},\]
where \( \binom{r}{j} \) is the usual binomial coefficient, which is a positive integer, and \( x^0, y^0 \) are both interpreted as being equal to 1 in \( k \). If \(| \cdot |\) is a quasimetric absolute value function on \( k \) that satisfies (2.14) with \( C' = 2 \), then it follows that
\[(2.28) \quad |(x + y)^r| \leq 2 \left( r + 1 \right) \max \left\{ \left| \binom{r}{j} \cdot x^j y^{r-j} \right| : j = 0, \ldots, r \right\},\]
by (2.24) with \( N = r + 1 \). This implies that
\[(2.29) \quad |(x + y)^r| \leq 2 \left( r + 1 \right) \max \left\{ 2 \binom{r}{j} |x|^j |y|^{r-j} : j = 0, \ldots, r \right\},\]
using (2.25) with \( N = \binom{r}{j} \). Of course,
\[(2.30) \quad \max \left\{ \binom{r}{j} |x|^j |y|^{r-j} : j = 0, \ldots, r \right\} \leq \sum_{j=0}^{\infty} \binom{r}{j} |x|^j |y|^{r-j} = (|x| + |y|)^r,\]
using the binomial theorem again in the second step. Combining this with (2.29), we get that
\[(2.31) \quad |x + y|^r = |(x + y)^r| \leq 4 \left( r + 1 \right) (|x| + |y|)^r \]
for each positive integer \( r \) under these conditions. Thus
\[(2.32) \quad |x + y| \leq (4 \left( r + 1 \right))^{1/r} (|x| + |y|) \]
for every \( x, y \in k \) and \( r \geq 1 \), which implies that
\[(2.33) \quad |x + y| \leq |x| + |y|,\]
by taking the limit as \( r \to \infty \).

This shows that a quasimetric absolute value function \(| \cdot |\) on \( k \) that satisfies (2.14) with \( C' = 2 \) is actually an absolute value function on \( k \). Suppose that \(| \cdot |\) is any quasimetric absolute value function on \( k \), so that \(| \cdot |\) satisfies (2.14) for some \( C' \geq 1 \). As in the previous section, \(| \cdot |^a\) is also a quasimetric absolute value function on \( k \) for each \( a > 0 \), which satisfies (2.17). If \( a \) is sufficiently small, then
\[(2.34) \quad (C')^a \leq 2,\]
which implies that \(| \cdot |^a\) is an absolute value function on \( k \), by the preceding argument.
2.3 Some more refinements

Let $|\cdot|$ be an absolute value function on a field $k$, and suppose that there is a real number $A \geq 1$ such that
\begin{equation}
|N \cdot 1| \leq A
\end{equation}
for every positive integer $N$. Observe that
\begin{equation}
N \cdot z = (N \cdot 1) z
\end{equation}
for every $z \in k$ and positive integer $N$, so that
\begin{equation}
|N \cdot z| = |N \cdot 1| |z| \leq A |z|
\end{equation}
under these conditions. Let $x, y \in k$ and a positive integer $r$ be given, so that
\begin{equation}
(x + y)^r \text{ can be expressed as in (2.27), using the binomial theorem. This implies that}
\end{equation}
\begin{equation}
|(x + y)^r| \leq \sum_{j=0}^{r} \binom{r}{j} x^j y^{r-j}
\end{equation}
\begin{equation}
\leq A \sum_{j=0}^{r} |x|^j |y|^{r-j} \leq A (r + 1) \max(|x|, |y|)^r,
\end{equation}
using (2.37) in the second step. Equivalently,
\begin{equation}
|x + y|^r = |(x + y)^r| \leq A (r + 1) \max(|x|, |y|)^r
\end{equation}
for each $r \geq 1$, and hence
\begin{equation}
|x + y| \leq (A (r + 1))^{1/r} \max(|x|, |y|).
\end{equation}
Taking the limit as $r \to \infty$, we get that
\begin{equation}
|x + y| \leq \max(|x|, |y|)
\end{equation}
for every $x, y \in k$, so that $|\cdot|$ is an ultrametric absolute value function on $k$. If $|\cdot|$ is a quasimetric absolute value function on $k$ that satisfies (2.35) for some $A \geq 1$ and every positive integer $N$, then
\begin{equation}
|N \cdot 1|^a \leq A^a
\end{equation}
for every positive real number $a$ and positive integer $N$, and we know from the previous section that $|\cdot|^a$ is an absolute value function on $k$ when $a$ is sufficiently small. It follows from the preceding argument that $|\cdot|^a$ is an ultrametric absolute value function on $k$ when $a$ is sufficiently small, which implies that $|\cdot|$ is an ultrametric absolute value function on $k$, as in Section 2.1. Alternatively, one could extend the preceding argument directly to quasimetric absolute value functions, using (2.20).
Of course, if $|·|$ is an ultrametric absolute value function on $k$, then

\begin{equation}
|N \cdot 1| \leq 1
\end{equation}

for every positive integer $N$, by (2.4). Note that

\begin{equation}
(N_1 N_2) \cdot 1 = N_1 \cdot (N_2 \cdot 1) = (N_1 \cdot 1) (N_2 \cdot 1)
\end{equation}

for any two positive integers $N_1$, $N_2$, and hence that

\begin{equation}
|N_j \cdot 1| = |(N \cdot 1)^j| = |N \cdot 1|^j
\end{equation}

for all positive integers $N$ and $j$. If $|N \cdot 1| > 1$ for some positive integer $N$, then it follows that $|N^j \cdot 1|$ is unbounded, so that (2.43) can also be derived directly from (2.35). A quasimetric absolute value function $|·|$ on $k$ is said to be archimedian if $|N \cdot 1|$ has no finite upper bound for all positive integers $N$, and otherwise $|·|$ is said to be non-archimedian. Thus $|·|$ is non-archimedean if and only if it is an ultrametric absolute value function.

Suppose that $|·|$ is an ultrametric absolute value function on $k$. If $x, y \in k$ satisfy $|y| \leq |x|$, then

\begin{equation}
|x + y| \leq \max(|x|, |y|) = |x|.
\end{equation}

We also have that

\begin{equation}
|x| = |(x + y) - y| \leq \max(|x + y|, |y|),
\end{equation}

by (2.6), which implies that $|x| \leq |x + y|$ when $|y| < |x|$. It follows that

\begin{equation}
x + y = |x|
\end{equation}

when $|y| < |x|$, which also corresponds to (1.20) in Section 1.1.

If $|·|$ is a nontrivial quasimetric absolute value function on $Q$, then a famous theorem of Ostrowiak states that $|·|$ is either a positive power of the standard absolute value function on $Q$, or a positive power of the $p$-adic absolute value function on $Q$ for some prime number $p$. More precisely, if $|·|$ is archimedian, then $|N \cdot 1| > 1$ for some positive integer $N$, and one can show that $|·|$ is a positive power of the standard absolute value function on $Q$. Otherwise, if $|·|$ is non-achimedian, then $|N \cdot 1| \leq 1$ for every positive integer $N$, and $|N \cdot 1| < 1$ for some positive integer $N$, because $|·|$ is nontrivial. If $p$ is the smallest positive integer such that $|p \cdot 1| < 1$, then one can show that $p$ is a prime number, and that $|·|$ is a positive power of the $p$-adic absolute value on $Q$. See Theorem 2.1 on p16 of [4] or Theorem 3.1.3 on p44 of [14] for more details.

### 2.4 Some topological properties

Let $k$ be a field, and let $|·|$ be a quasimetric absolute value function on $k$, with the associated quasimetric $d(·, ·)$ on $k$, as in (2.9). By construction, $d(·, ·)$ is invariant under translations on $k$, in the sense that

\begin{equation}
d(x + z, y + z) = d(x, y)
\end{equation}
for every \( x, y, z \in k \). If \( a \) is a positive real number, then \( |\cdot|^a \) is also a quasimetric absolute value function on \( k \), as in Section 2.1, for which the corresponding quasimetric on \( k \) is equal to
\[
d(x, y)^a = |x - y|^a.
\]

(2.50)

Each of these quasimetrics determines the same topology on \( k \), and in fact the same uniform structure.

Remember that \( |\cdot|^a \) is an absolute value function on \( k \) when \( a \) is sufficiently small, as in Section 2.2, in which case (2.50) is a metric on \( k \). This implies that open balls in \( k \) with respect to (2.50) are open sets with respect to the corresponding topology when \( a \) is sufficiently small, and that closed balls in \( k \) with respect to (2.50) are closed sets. It is easy to see that an open or closed ball in \( k \) with respect to \( d(\cdot, \cdot) \) centered at a point \( x \in k \) and with radius \( r \) is the same as the open or closed ball in \( k \) with respect to (2.50) centered at \( x \) with radius \( r^a \), for each \( a > 0 \). It follows that open balls in \( k \) with respect to \( d(\cdot, \cdot) \) are open sets, and that closed balls in \( k \) with respect to \( d(\cdot, \cdot) \) are closed sets, even when \( |\cdot|^a \) is an absolute value function on \( k \).

If \( a > 0 \) is sufficiently small so that \( |\cdot|^a \) is an absolute value function on \( k \), then \( |x|^a \) is continuous with respect to the corresponding metric (2.50) on \( k \).

More precisely, this means that \( |x|^a \) is continuous as a mapping from \( k \) into \( \mathbb{R} \), with respect to the standard topology on \( \mathbb{R} \). In fact, \( |x|^a \) is Lipschitz of order \( 1 \) with constant \( C = 1 \) with respect to the metric (2.50) on \( k \) and the standard metric on \( \mathbb{R} \), as in Section 1.6. Equivalently, this means that \( |x|^a \) is Lipschitz of order \( a \) with constant \( C = 1 \) with respect to the associated quasimetric (2.9) on \( k \) and the standard metric on \( \mathbb{R} \). In particular, \( |x|^a \) is continuous with respect to the topology determined by the associated quasimetric (2.9) on \( k \), so that \( |x|^a \) is continuous with respect to this topology as well.

Of course, addition and multiplication on \( k \) correspond to mappings from the Cartesian product \( k \times k \) of \( k \) with itself into \( k \). Using the topology on \( k \) determined by the quasimetric \( d(\cdot, \cdot) \) associated to the quasimetric \( |\cdot| \), one can define the corresponding product topology on \( k \times k \). With respect to this topology, addition and multiplication on \( k \) correspond to continuous mappings from \( k \times k \) into \( k \). This can be verified in essentially the same way as for real or complex numbers. If \( x, y \in k \) and \( x, y \neq 0 \), then
\[
x^{-1} - y^{-1} = (y - x) x^{-1} y^{-1},
\]

(2.51)

and hence
\[
|x^{-1} - y^{-1}| = |y - x| |x^{-1}| |y^{-1}| = |y - x| |x|^{-1} |y|^{-1},
\]

(2.52)

using (2.8) in the last step. If \( x, y \in k \), \( x \neq 0 \), and \( y \) is sufficiently close to \( x \), then one can also check that there is a uniform positive lower bound for \( |y| \) in terms of \( |x| \), using the quasimetric version of the triangle inequality. This permits one to show that \( x \mapsto x^{-1} \) is continuous as a mapping from \( k \setminus \{0\} \) into itself, in essentially the same way as for real or complex numbers. It follows that \( k \) is a topological field with respect to the topology determined by \( d(\cdot, \cdot) \).
2.5 Completions, 2

Let $k$ be a field, and let $| \cdot |$ be an absolute value function on $k$, with the corresponding metric $d(\cdot, \cdot)$ as in (2.9). It is convenient to restrict our attention here to absolute value functions instead of quasimetric absolute value functions, in order to follow the usual discussion for metric spaces, as in Section 1.2. One could also start with a quasimetric absolute value function on $k$, and then reduce to the case of ordinary absolute value functions, as in Section 2.2. In the context of arbitrary quasimetric spaces, one has to be a bit careful about continuity properties of the quasimetric, or work with a metric that determines the same uniform structure. In this situation, quasimetric absolute value functions and their associated quasimetrics already have nice continuity properties, because they are related to ordinary absolute value functions and their associated metrics as in Section 2.2, and one may as well work directly with the latter.

Before considering the completion of $k$, let us look at some properties of Cauchy sequences of elements of $k$. If $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence of elements of $k$, then it is easy to see that $\{|x_j|\}_{j=1}^\infty$ is a Cauchy sequence in $\mathbb{R}$ with respect to the standard metric, which thus converges to a nonnegative real number. This uses the triangle inequality, and may be considered as a special case of an analogous statement about distances between Cauchy sequences in Section 1.2, because $|x_j|$ is the same as the distance between $x_j$ and 0. If $|\cdot|$ is an ultrametric absolute value function on $k$, and if $\{x_j\}_{j=1}^\infty$ does not converge to 0, then $\{|x_j|\}_{j=1}^\infty$ is eventually constant. This can be derived from (2.48), and may also be considered as a special case of an analogous statement for distances in ultrametric spaces, as in Section 1.2.

Suppose that $\{x_j\}_{j=1}^\infty$, $\{x'_j\}_{j=1}^\infty$ are equivalent Cauchy sequences of elements of $M$, in the sense discussed in Section 1.2. In this case, one can check that

\[
\lim_{j \to \infty} \left( |x_j| - |x'_j| \right) = 0
\]

in $\mathbb{R}$, so that $\{|x_j|\}_{j=1}^\infty$ and $\{|x'_j|\}_{j=1}^\infty$ have the same limit in $\mathbb{R}$. This permits one to extend the absolute value function on $k$ to a nonnegative real-valued function on the completion of $k$. If $|\cdot|$ is an ultrametric absolute value function on $k$, and $\{x_j\}_{j=1}^\infty$ or $\{x'_j\}_{j=1}^\infty$ does not converge to 0 in $k$, then $|x_j| = |x'_j|$ for all sufficiently large $j$. As before, these statements may be considered as special cases of analogous statements for distances, as in Section 1.2.

If $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$ are Cauchy sequences of elements of $k$, then one can check that $\{x_j + y_j\}_{j=1}^\infty$ and $\{x_j y_j\}_{j=1}^\infty$ are Cauchy sequences of elements of $k$ as well. In the case of products, this also uses the fact that Cauchy sequences in $k$ are bounded. If $\{x'_j\}_{j=1}^\infty$, and $\{y'_j\}_{j=1}^\infty$ are Cauchy sequences of elements of $k$ that are equivalent to $\{x_j\}_{j=1}^\infty$ and $\{y_j\}_{j=1}^\infty$, respectively, then $\{x'_j + y'_j\}_{j=1}^\infty$ and $\{x'_j y'_j\}_{j=1}^\infty$ are equivalent as Cauchy sequences in $k$ to $\{x_j + y_j\}_{j=1}^\infty$ and $\{x_j y_j\}_{j=1}^\infty$, respectively. This permits one to extend addition and multiplication on $k$ to the completion of $k$, so that the completion of $k$ becomes a commutative ring. The extension of the absolute value function to the completion of $k$ satisfies the same type of properties as on $k$, and the extension of the associated metric
on $k$ to the completion of $k$ as in Section 1.2 corresponds to the extension of the absolute value function to the completion of $k$ as in (2.9).

If $\{x_j\}_{j=1}^{\infty}$ is a sequence of elements of $k$ that does not converge to 0, then there is an $r > 0$ such that

$$|x_j| \geq 2r \quad \text{for infinitely many } j. \tag{2.54}$$

If $\{x_j\}_{j=1}^{\infty}$ is also a Cauchy sequence of elements of $k$, then it follows that

$$|x_j| \geq r \quad \text{for all but finitely many } j, \tag{2.55}$$

and in particular $x_j \neq 0$ for all but finitely many $j$. Of course, it is not necessary to switch between $2r$ in (2.54) and $r$ in (2.55) when $|\cdot|$ is an ultrametric absolute value function. If $\{x_j\}_{j=1}^{\infty}$ is a Cauchy sequence of nonzero elements of $k$ that does not converge to 0, then one can check that $\{1/x_j\}_{j=1}^{\infty}$ is a Cauchy sequence in $k$ too, using (2.52). Similarly, if $\{x'_j\}_{j=1}^{\infty}$ is another Cauchy sequence of nonzero elements of $k$ that does not converge to 0 and which is equivalent to $\{x_j\}_{j=1}^{\infty}$, then one can verify that $\{1/x'_j\}_{j=1}^{\infty}$ is equivalent to $\{1/x_j\}_{j=1}^{\infty}$, using (2.52) again. This permits one to extend the mapping $x \mapsto 1/x$ to the nonzero elements of the completion of $k$, so that the completion of $k$ becomes a field. Note that the extension of the absolute value function to the completion of $k$ still satisfies (2.8).

The extension of the absolute value function to the completion of $k$ is an absolute value function on the completion of $k$. If $|\cdot|$ is an ultrametric absolute value function on $k$, then its extension to the completion of $k$ is an ultrametric absolute value function too. As in Section 1.2, there is a natural embedding of $k$ into its completion, which associates to each $x \in k$ the equivalence class of Cauchy sequences that contains the constant sequence $\{x_j\}_{j=1}^{\infty}$ with $x_j = x$ for each $j$. By construction, this embedding is a field isomorphism from $k$ onto its image in the completion of $k$, which preserves absolute values and hence distance. It is customary to identify $k$ with its image in the completion under this embedding, which is a dense subset of the completion.

Alternatively, suppose that we start with a completion of $k$ as a metric space, which is to say an isometric embedding of $k$ onto a dense subset of a complete metric space. Note that addition on $k$ may be considered as a uniformly continuous mapping from $k \times k$ into $k$, with respect to a suitable product metric on $k \times k$ corresponding to the metric on $k$ associated to the absolute value function. One can then extend addition on $k$ to the completion of $k$ as in Section 1.3. Similarly, multiplication on $k$ corresponds to a mapping from $k \times k$ into $k$ that is uniformly continuous on bounded subsets of $k \times k$, which is sufficient for this type of extension argument. If $r$ is any positive real number, then $x \mapsto 1/x$ is uniformly continuous as a mapping from

$$\{x \in k : |x| \geq r\} \tag{2.56}$$

into $k$, which is again sufficient for this type of extension argument. This gives another way to look at the extension of the field operations to the completion of
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The absolute value function can also be considered as a uniformly continuous mapping from \( k \) into \( \mathbb{R} \), but its extension to the completion of \( k \) is already implicitly included in the metric, since it is the same as the distance to 0 in \( k \).

Suppose now that \( k_1 \) and \( k_2 \) are fields equipped with absolute value functions \( |\cdot|_1 \) and \( |\cdot|_2 \), respectively, and let \( d_1(\cdot, \cdot) \) and \( d_2(\cdot, \cdot) \) be the corresponding metrics on \( k_1 \) and \( k_2 \), as in (2.9). Also let \( \phi_1 \) and \( \phi_2 \) be field isomorphisms from \( k \) onto subsets of \( k_1 \) and \( k_2 \) that are dense with respect to the corresponding metrics. If \( \phi_1 \) and \( \phi_2 \) preserve the appropriate absolute value functions, in the sense that

\[
|\phi_1(x)|_1 = |\phi_2(x)|_2 = |x|
\]

for every \( x \in k \), then \( \phi_1 \) and \( \phi_2 \) are isometric embeddings of \( k \) into \( k_1 \) and \( k_2 \), respectively. If \( k_1 \) and \( k_2 \) are also complete with respect to their corresponding metrics, then it follows that there is an isometry \( f \) from \( k_1 \) onto \( k_2 \) such that

\[
f \circ \phi_1 = \phi_2,
\]

by the remarks at the end of Section 1.3. Under these conditions, it is easy to see that \( f \) is a field isomorphism, and that

\[
|f(z)|_2 = |z|_1
\]

for every \( z \in k_1 \).

Let \( k \) be a field, and let \( |\cdot| \) be an ultrametric absolute value function on \( k \). As before, if \( \{x_j\}_{j=1}^\infty \) is a Cauchy sequence of elements of \( k \) that does not converge to 0, then \( \{|x_j|\}_{j=1}^\infty \) is eventually constant. In particular, this implies that the natural extension of \( |\cdot| \) to the completion of \( k \) takes values in the same set of nonnegative real numbers as \( |\cdot| \) does on \( k \).

Now let \( k = \mathbb{Q} \), let \( p \) be a prime number, and let \( |\cdot|_p \) be the \( p \)-adic absolute value function on \( \mathbb{Q} \), as in Section 2.1. The completion of \( \mathbb{Q} \) with respect to \( |\cdot|_p \) is known as the field \( \mathbb{Q}_p \) of \( p \)-adic numbers. The natural extension of \( |\cdot|_p \) to \( \mathbb{Q}_p \) is known as the \( p \)-adic absolute value function on \( \mathbb{Q}_p \), and is also denoted \( |\cdot|_p \). Similarly, the natural extension of the \( p \)-adic metric \( d_p(\cdot, \cdot) \) on \( \mathbb{Q} \) to \( \mathbb{Q}_p \) is known as the \( p \)-adic metric on \( \mathbb{Q}_p \), and is denoted \( d_p(\cdot, \cdot) \) as well. The remark in the preceding paragraph implies that \( |x|_p \) is either equal to 0 or to an integer power of \( p \) for every \( x \in \mathbb{Q}_p \), and \( d_p(\cdot, \cdot) \) has the same property on \( \mathbb{Q}_p \).

### 2.6 Infinite series

Let \( k \) be a field, let \( |\cdot| \) be a quasimetric absolute value function on \( k \), and let \( d(\cdot, \cdot) \) be the associated quasimetric on \( k \), as in (2.9). Also let \( \sum_{j=1}^\infty a_j \) be an infinite series with \( a_j \in k \) for each \( j \). As usual, this series \( \sum_{j=1}^\infty a_j \) is said to converge in \( k \) if the corresponding sequence of partial sums

\[
s_n = \sum_{j=1}^n a_j
\]
2.6. INFINITE SERIES

converges to an element of \( k \) with respect to \( d(\cdot, \cdot) \), in which case the value of the sum \( \sum_{j=1}^{\infty} a_j \) is equal to the limit of the sequence \( \{s_n\}_{n=1}^{\infty} \) of partial sums. If \( \sum_{j=1}^{\infty} a_j \) converges in \( k \) and \( c \in k \), then it is easy to see that \( \sum_{j=1}^{\infty} c a_j \) converges in \( k \) too, with

\[
\sum_{j=1}^{\infty} c a_j = c \sum_{j=1}^{\infty} a_j.
\]

Similarly, if \( \sum_{j=1}^{\infty} a_j \) and \( \sum_{j=1}^{\infty} b_j \) are convergent series with terms in \( k \), then \( \sum_{j=1}^{\infty} (a_j + b_j) \) converges in \( k \) too, with

\[
\sum_{j=1}^{\infty} (a_j + b_j) = \sum_{j=1}^{\infty} a_j + \sum_{j=1}^{\infty} b_j.
\]

Note that \( \{s_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( k \) with respect to \( d(\cdot, \cdot) \) if and only if for each \( \epsilon > 0 \) there is an \( L \geq 1 \) such that

\[
|s_n - s_l| = \left| \sum_{j=l+1}^{n} a_j \right| < \epsilon
\]

for every \( n > l \geq L \). In particular, this implies that \( \{a_j\}_{j=1}^{\infty} \) converges to 0 in \( k \), by taking \( n = l + 1 \). Of course, the convergence of \( \sum_{j=1}^{\infty} a_j \) implies that \( \{s_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( k \), and the converse holds when \( k \) is complete with respect to \( d(\cdot, \cdot) \).

If \( | \cdot | \) is an absolute value function on \( k \), then we say that \( \sum_{j=1}^{\infty} a_j \) converges absolutely when

\[
\sum_{j=1}^{\infty} |a_j|
\]

converges as an infinite series of nonnegative real numbers. This implies that \( \{s_n\}_{n=1}^{\infty} \) is a Cauchy sequence in \( k \), because

\[
\left| \sum_{j=l+1}^{n} a_j \right| \leq \sum_{j=l+1}^{n} |a_j|
\]

for every \( n > l \geq 1 \). It follows that \( \sum_{j=1}^{\infty} a_j \) converges in \( k \) when \( k \) is complete, in which case we also have that

\[
\left| \sum_{j=1}^{\infty} a_j \right| \leq \sum_{j=1}^{\infty} |a_j|.
\]

If \( | \cdot | \) is an ultrametric absolute value function on \( k \), then

\[
\left| \sum_{j=l+1}^{n} a_j \right| \leq \max_{l+1 \leq j \leq n} |a_j|
\]

\[
\left| \sum_{j=1}^{\infty} a_j \right| \leq \sum_{j=1}^{\infty} |a_j|.
\]
for every $n > l \geq 1$. This implies that $\{s_n\}_{n=1}^\infty$ is a Cauchy sequence in $k$ when $\{a_j\}_{j=1}^\infty$ converges to 0. If $k$ is complete, then $\sum_{j=1}^\infty a_j$ converges in $k$, and satisfies

$$\left| \sum_{j=1}^\infty a_j \right| \leq \max_{j \geq 1} |a_j|.$$  (2.68)

More precisely, it is easy to see that the maximum on the right side of (2.68) exists under these conditions, because $\{|a_j|\}_{j=1}^\infty$ is a sequence of nonnegative real numbers that converges to 0.

Of course, we can also consider series that start with $j = 0$. If $x \in k$ and $n$ is a nonnegative integer, then it is well known that

$$\sum_{j=0}^n x^j = 1 - x^{n+1},$$  (2.69)

where $x^0$ is interpreted as being equal to 1, as usual. Thus

$$\sum_{j=0}^n x^j = \frac{1 - x^{n+1}}{1 - x}$$  (2.70)

for every $n \geq 0$ when $x \neq 1$, so that

$$\lim_{n \to \infty} \sum_{j=0}^n x^j = \frac{1}{1 - x}$$  (2.71)

when $|x| < 1$, because $|x^{n+1}| = |x|^{n+1} \to 0$ as $n \to \infty$.

Let $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ be infinite series with terms in $k$. The Cauchy product of these two series is the infinite series $\sum_{n=0}^\infty c_n$, where

$$c_n = \sum_{j=0}^n a_j b_{n-j}$$  (2.72)

for each nonnegative integer $n$. It is easy to see that

$$\sum_{n=0}^\infty c_n = \left( \sum_{j=0}^\infty a_j \right) \left( \sum_{l=0}^\infty b_l \right)$$  (2.73)

formally. In particular, if $a_j = 0$ for all but finitely many $j$, and if $b_l = 0$ for all but finitely many $l$, then $c_n = 0$ for all but finitely many $n$, and (2.73) holds.

If $|\cdot|$ is an absolute value function on $k$, then

$$|c_n| \leq \sum_{j=0}^n |a_j| |b_{n-j}|$$  (2.74)
for each \( n \geq 0 \), where the right side corresponds to the Cauchy product of
\[
\sum_{j=0}^{\infty} |a_j| \quad \text{and} \quad \sum_{l=0}^{\infty} |b_l|.
\]
If \( \sum_{j=0}^{\infty} a_j \) and \( \sum_{l=0}^{\infty} b_l \) converge absolutely, then one
can check that \( \sum_{n=0}^{\infty} c_n \) converges absolutely, with
\[
(2.75) \quad \sum_{n=0}^{\infty} |c_n| \leq \left( \sum_{j=0}^{\infty} |a_j| \right) \left( \sum_{l=0}^{\infty} |b_l| \right).
\]
If \( k \) is complete, then it follows that these series converge in \( k \), and one can
check that \( (2.73) \) holds. The main point is to approximate the infinite series by
finite sums, using \( (2.75) \) to estimate the errors.

Similarly, if \( | \cdot | \) is an ultrametric absolute value function on \( k \), then
\[
(2.76) \quad |c_n| \leq \max_{0 \leq j \leq n} (|a_j| |b_{n-j}|)
\]
for each \( n \geq 0 \). If \( \{a_j\}_{j=0}^{\infty} \) and \( \{b_l\}_{l=0}^{\infty} \) both converge to 0, then it is easy to
see that \( \{c_n\}_{n=0}^{\infty} \) converges to 0 as well. This implies that the series \( \sum_{j=0}^{\infty} a_j \),
\( \sum_{l=0}^{\infty} b_l \), and \( \sum_{n=0}^{\infty} c_n \) converge in \( k \) when \( k \) is complete, in which case one can
check that \( (2.73) \) holds again. As before, the main point is to approximate
the infinite series by finite sums, but now using the ultrametric version of the
triangle inequality to estimate the errors.

### 2.7 Topological equivalence

Let \( k \) be a field, and let \( | \cdot | \) be a quasimetric absolute value function on \( k \). As
before, this leads to a quasimetric \( d_1(\cdot, \cdot) \) on \( k \) as in \( (2.9) \), and thus a topology
on \( k \). If \( x \in k \) and \( |x| < 1 \), then
\[
(2.77) \quad |x^j|_1 = |x|_1^j \to 0 \quad \text{as} \quad j \to \infty
\]
as a sequence of nonnegative real numbers, which implies that \( \{x^j\}_{j=1}^{\infty} \) converges
to 0 in \( k \) with respect to the topology corresponding to \( | \cdot |_1 \). Conversely, if
\( \{x^j\}_{j=1}^{\infty} \) converges to 0 in \( k \) with respect to the topology corresponding to \( | \cdot |_1 \),
then
\[
(2.78) \quad |x|^1_1 = |x|^j_1 < 1
\]
for all but finitely many \( j \), and hence \( |x|_1 < 1 \). This shows that the open unit
ball in \( k \) with respect to \( | \cdot |_1 \) is uniquely determined by the topology on \( k \) that
Corresponds to \( | \cdot |_1 \). Note that \( x \in k \) satisfies \( |x|_1 > 1 \) if and only if \( x \neq 0 \)
and \( |x^{-1}|_1 = |x|_1^{-1} < 1 \), so that the set of \( x \in k \) with \( |x|_1 > 1 \) is also uniquely
determined by the topology on \( k \) that corresponds to \( | \cdot |_1 \). It follows that the
set of \( x \in k \) with \( |x|_1 = 1 \) is uniquely determined by the topology on \( k \) that
Corresponds to \( | \cdot |_1 \) as well.

Now let \( | \cdot |_1 \) and \( | \cdot |_2 \) be quasimetric absolute value functions on \( k \), and let
\( d_1(\cdot, \cdot) \) and \( d_2(\cdot, \cdot) \) be the associated quasimetrics on \( k \), as in \( (2.9) \). If
\[
(2.79) \quad |x|_2 = |x|^q_1
\]
for some $a > 0$ and every $x \in k$, then
\begin{equation}
(2.80) \quad d_2(x, y) = d_1(x, y)^a
\end{equation}
for every $x, y \in k$, and the corresponding topologies on $k$ are the same, as in Section 2.4. Conversely, suppose that the topologies on $k$ corresponding to $|\cdot|_1$ and $|\cdot|_2$ are the same, and let us show that there is an $a > 0$ such that (2.79) holds for every $x \in k$. The topological equivalence of $|\cdot|_1$ and $|\cdot|_2$ implies that
\begin{equation}
(2.81) \quad |x|_1 < 1 \text{ if and only if } |x|_2 < 1
\end{equation}
for every $x \in k$, by the remarks in the preceding paragraph. Similarly,
\begin{equation}
(2.82) \quad |x|_1 = 1 \text{ if and only if } |x|_2 = 1
\end{equation}
and
\begin{equation}
(2.83) \quad |x|_1 > 1 \text{ if and only if } |x|_2 > 1
\end{equation}
for every $x \in k$. Of course, (2.79) is trivial when $x = 0$, and when $|x|_1 = |x|_2 = 1$. We would like to show that there is an $a > 0$ such that (2.79) holds for every $x \in k$ with $x \neq 0$ and $|x|_1, |x|_2 < 1$. This would imply that (2.79) also holds for every $x \in k$ with $|x|_1, |x|_2 > 1$, by applying the previous statement to $1/x$.

Let $y, z \in k$ be given, with $y, z \neq 0$ and $|y|_1, |z|_1 < 1$, which implies that $|y|_2, |z|_2 < 1$. If $m$ and $n$ are positive integers, then we can apply (2.81) to $x = y^m/z^n$, to get that
\begin{equation}
(2.84) \quad |y|^m_1 < |z|^n_1 \text{ if and only if } |y|^n_2 < |z|^m_2.
\end{equation}
Equivalently, this means that
\begin{equation}
(2.85) \quad m \log |y|_1 < n \log |z|_1 \text{ if and only if } m \log |y|_2 < n \log |z|_2,
\end{equation}
and hence that
\begin{equation}
(2.86) \quad \frac{n}{m} < \frac{\log |y|_1}{\log |z|_1} \text{ if and only if } \frac{n}{m} < \frac{\log |y|_2}{\log |z|_2},
\end{equation}
since the logarithms are negative in this situation. It follows that
\begin{equation}
(2.87) \quad \frac{\log |y|_1}{\log |z|_1} = \frac{\log |y|_2}{\log |z|_2},
\end{equation}
so that
\begin{equation}
(2.88) \quad \frac{\log |y|_1}{\log |y|_2} = \frac{\log |z|_1}{\log |z|_2}.
\end{equation}
This implies that there is an $a > 0$ such that (2.79) holds for every $x \in k$ with $x \neq 0$ and $|x|_1, |x|_2 < 1$, as desired.

This follows the discussion that begins on p.42 of [14], and we shall consider a refinement in the next section, corresponding to Lemma 3.1 on p.18 of [4]. Of course, if $|\cdot|_1$ is the trivial absolute value function on $k$, as in Section 2.1, then the corresponding topology on $k$ is the discrete topology. Conversely, if the topology on $k$ corresponding to a quasimetric absolute value function $|\cdot|_1$ on $k$ is the discrete topology, then one can check that $|\cdot|_1$ is the trivial absolute value function on $k$, using the remarks at the beginning of the section.
2.8. TOPOLOGICAL EQUIVALENCE, 2

Let $|\cdot|_1$ and $|\cdot|_2$ be quasimetric absolute value functions on a field $k$ again, and suppose that the topology on $k$ corresponding to $|\cdot|_1$ is at least as strong as the topology on $k$ corresponding to $|\cdot|_2$. More precisely, this means that every open set in $k$ with respect to the topology corresponding to $|\cdot|_2$ is also an open set in $k$ with respect to the topology corresponding to $|\cdot|_1$. In particular, if $x \in k$ and $\{x^j\}_{j=1}^\infty$ converges to 0 with respect to the topology on $k$ corresponding to $|\cdot|_1$, then it follows that $\{x^j\}_{j=1}^\infty$ also converges to 0 with respect to the topology on $k$ corresponding to $|\cdot|_2$. Thus

\begin{equation}
|x|_1 < 1 \implies |x|_2 < 1
\end{equation}

for every $x \in k$ under these conditions, by the remarks at the beginning of the previous section. If $x \neq 0$, then we can apply (2.89) to $1/x$, to get that

\begin{equation}
|x|_1 > 1 \implies |x|_2 > 1.
\end{equation}

Equivalently,

\begin{equation}
|x|_2 \leq 1 \implies |x|_1 \leq 1
\end{equation}

for every $x \in k$, which is the contrapositive of (2.90). If $u, v \in k$ and $v \neq 0$, then we can apply (2.91) to $x = u/v$, to get that

\begin{equation}
|u|_2 \leq |v|_2 \implies |u|_1 \leq |v|_1.
\end{equation}

Of course, all of these statements are trivial when $|\cdot|_1$ is the trivial absolute value function on $k$. Suppose now that $|\cdot|_1$ is not the trivial absolute value function on $k$, in addition to the condition on the associated topologies in the preceding paragraph. This means that there is a $v_0 \in k$ such that $v_0 \neq 0$ and $|v_0|_1 \neq 1$, and we may as well ask that $|v_0|_1 < 1$, since otherwise we could replace $v_0$ with $1/v_0$. Note that $|v_0|_2 < 1$ as well, by (2.89). If we take $v = v_0$ in (2.92), then we get that

\begin{equation}
|u|_2 \leq |v_0|_2 \implies |u|_1 \leq |v_0|_1 < 1
\end{equation}

for every $u \in k$. If $x \in k$ satisfies $|x|_2 < 1$, then

\begin{equation}
|x^n|_2 = |x^n|_1 \leq |v_0|_2
\end{equation}

for all sufficiently large positive integers $n$, which implies that $|x^n|_1 = |x^n|_1 < 1$, by taking $u = x^n$ in (2.93). It follows that $|x|_1 < 1$, so that

\begin{equation}
|x|_2 < 1 \implies |x|_1 < 1
\end{equation}

for every $x \in k$ under these conditions. As before, one can apply this to $1/x$ when $x \neq 0$, to get that

\begin{equation}
|x|_2 > 1 \implies |x|_1 > 1.
\end{equation}
At this stage, we are in essentially the same situation as in the previous section. More precisely, the combination of (2.89) and (2.95) corresponds exactly to (2.81), and similarly the combination of (2.90) and (2.96) corresponds exactly to (2.83). The third condition (2.82) follows automatically from (2.81) and (2.83), and one can use this to show that there is an \( a > 0 \) such that (2.79) holds for every \( x \in k \), as before. In [4], one uses (2.89) and the nontriviality of \( | \cdot |_1 \) to get that
\[
|x|_1 = 1 \implies |x|_2 = 1
\]
for every \( x \in k \). The combination of (2.89), (2.90) and (2.97) imply (2.81), (2.82), and (2.83) again, so that one can continue as before. In particular, if the topology on \( k \) corresponding to \( | \cdot |_1 \) is at least as strong as the topology on corresponding to \( | \cdot |_2 \), and if \( | \cdot |_1 \) is nontrivial, then it follows that the topologies on \( k \) corresponding to \( | \cdot |_1 \) and \( | \cdot |_2 \) are the same. One can show that the topology on \( k \) corresponding to \( | \cdot |_2 \) is at least as strong as the topology corresponding to \( | \cdot |_1 \) more directly under these conditions, using (2.92). This also uses the fact that there is a \( v_0 \in k \) such that \( v_0 \neq 0 \) and \( |v_0|_1 < 1 \), as before, so that \( |v_0|_1 = |v_0|_1^j \to 0 \) as \( j \to \infty \).

### 2.9 Another refinement

Let \( k \) be a field, and let \( | \cdot | \) be a quasimetric absolute value function on \( k \). Thus \( | \cdot | \) satisfies (2.14) in Section 2.1 for some \( C' \geq 1 \), which implies that \( | \cdot | \) satisfies (2.20) in Section 2.2 for each nonnegative integer \( n \). Let \( r \) be a nonnegative integer, and let \( n \) be the smallest nonnegative integer such that \( r + 1 \leq 2^n \). The minimality of \( n \) implies that \( 2^n - 1 < r + 1 \), or equivalently
\[
2^n < 2 (r + 1).
\]
If \( z_0, z_1, \ldots, z_r \) are \( r + 1 \) elements of \( k \), then
\[
\left| \sum_{j=0}^r z_j \right| \leq (C')^n \max\{|z_j| : j = 0, 1, \ldots, r\},
\]
by (2.20) and the choice of \( n \). Let us choose \( b \geq 0 \) so that \( 2^b = C' \), and hence
\[
(C')^n = (2^b)^n = (2^n)^b = (2 (r + 1))^b = 2^b (r + 1)^b,
\]
by (2.98). Combining this with (2.99), we get that
\[
\left| \sum_{j=0}^r z_j \right| \leq 2^b (r + 1)^b \max\{|z_j| : j = 0, 1, \ldots, r\}
\]
for all \( z_0, z_1, \ldots, z_r \in k \).

Suppose that there is a real number \( B \geq 1 \) such that
\[
|N \cdot 1| \leq B N
\]
for every positive integer $N$. Of course, this implies that
\[(2.103) \quad |N \cdot z| \leq B |z|\]
for every $z \in k$ and positive integer $N$, using (2.36) at the beginning of Section 2.3. Let $x, y \in k$ and a positive integer $r$ be given, so that $(x + y)^r$ can be expressed as in (2.27) in Section 2.2, by the binomial theorem. It follows that
\[(2.104) \quad |(x + y)^r| \leq 2^b (r + 1)^b \max \left\{ \left| \binom{r}{j} \cdot x^j y^{r-j} \right| : j = 0, 1, \ldots, r \right\},\]
by (2.101). This reduces to
\[(2.105) \quad |(x + y)^r| \leq 2^b (r + 1)^b B \max \left\{ \left| x^j \right| \left| y^{r-j} \right| : j = 0, 1, \ldots, r \right\},\]
using (2.103) for each term in the maximum on the right side. Combining this with (2.30) in Section 2.2, we get that
\[(2.106) \quad |x + y|^r = |(x + y)^r| \leq 2^b (r + 1)^b B (|x| + |y|)^r.\]
Equivalently,
\[(2.107) \quad |x + y| \leq 2^{b/r} (r + 1)^{1/r} B^{1/r} (|x| + |y|)\]
for every $x, y \in k$ and positive integer $r$, which implies that
\[(2.108) \quad |x + y| \leq |x| + |y|\]
for every $x, y \in k$, by taking the limit as $r \to \infty$. This shows that $| \cdot |$ is an absolute value function on $k$ under these conditions.

Let $| \cdot |$ be any quasimetric absolute value function on a field $k$ again. If $N_0 \cdot 1 = 0$ in $k$ for some positive integer $N_0$, then there are only finitely many elements of $k$ of the form $N \cdot 1$, where $N$ is a positive integer. In particular, this implies that there is a real number $A \geq 1$ such that $|N \cdot 1| \leq A$ for every positive integer $N$, which is the same as (2.35) in Section 2.3. It follows that $| \cdot |$ is an ultrametric absolute value function on $k$ under these conditions, as before. Otherwise, if $N \cdot 1 \neq 0$ in $k$ for every positive integer $N$, then $k$ has characteristic 0, and one can define $r \cdot 1$ in $k$ for every rational number $r$ in the usual way. This leads to an embedding of $\mathbb{Q}$ into $k$, so that
\[(2.109) \quad |r \cdot 1|\]
defines a quasimetric absolute value function on $\mathbb{Q}$. As mentioned at the end of Section 2.3, a theorem of Ostrowski implies that (2.109) is either a positive power of the standard absolute value function on $\mathbb{Q}$, or a positive power of the $p$-adic absolute value function on $\mathbb{Q}$ for some prime number $p$, or the trivial absolute value function on $\mathbb{Q}$. In the second and third cases, $|N \cdot 1| \leq 1$ for every positive integer $N$, so that the discussion in Section 2.3 implies that $| \cdot |$ is an ultrametric absolute value function on $k$. In the first case, let us suppose that (2.109) is equal to the standard absolute value function on $\mathbb{Q}$, which can be arranged by replacing $| \cdot |$ with the appropriate positive power of itself. This implies that $| \cdot |$ satisfies (2.102) with $B = 1$, and hence that $| \cdot |$ is an absolute value function on $k$. 


2.10 Complex numbers

Let \(|z|_0\) denote the standard absolute value function on the complex numbers \(\mathbb{C}\), so that
\[
|z|_0 = (x^2 + y^2)^{1/2}
\]
for each \(z \in \mathbb{C}\), where \(z = x + iy\) and \(x, y \in \mathbb{R}\). Suppose that \(|z|\) is another absolute value function on \(\mathbb{C}\) such that
\[
|z| = |z|_0
\]
for every \(z \in \mathbb{R}\). Note that
\[
|i| = 1,
\]
as in (2.5), because \(i^4 = 1\). If \(z = x + iy\) with \(x, y \in \mathbb{R}\) again, then it follows that
\[
|z| \leq |x| + |y| = |x|_0 + |y|_0 \leq 2|z|_0,
\]
where the 2 in the last step could be replaced by a \(\sqrt{2}\), using the triangle inequality. This implies that the standard topology on \(\mathbb{C}\), which corresponds to \(|z|_0\), is at least as strong as the topology corresponding to \(|z|\).

Of course, \(|z|_0\) is not the trivial absolute value function on \(\mathbb{C}\). Thus the discussion in Section 2.8 implies that the topology on \(\mathbb{C}\) corresponding to \(|z|\) is the same as the standard topology. Alternatively, one could get the same conclusion by considering \(\mathbb{C}\) as a two-dimensional vector space over \(\mathbb{R}\), as in Section 3.7. At any rate, the argument in Section 2.7 implies that \(|z|\) is equal to a positive power of \(|z|_0\). It is easy to see that this power has to be 1, because of (2.111), so that (2.111) holds for every \(z \in \mathbb{C}\) under these conditions.

Now let \(k\) be a field with a quasimetric absolute value function \(|\cdot|\), and suppose that \(|\cdot|\) is archimedian. This implies that \(k\) has characteristic 0, so that there is a natural embedding of \(\mathbb{Q}\) into \(k\). The restriction of \(|\cdot|\) to the image of \(\mathbb{Q}\) in \(k\) is also archimedian, and hence the restriction of \(|\cdot|\) to the image of \(\mathbb{Q}\) is equal to a positive power of the standard absolute value function on \(\mathbb{Q}\). By replacing \(|\cdot|\) with a suitable positive power of itself, if necessary, we may as well ask that the restriction of \(|\cdot|\) to the image of \(\mathbb{Q}\) in \(k\) be equal to the standard absolute value function on \(\mathbb{Q}\). The argument in the previous section then implies that \(|\cdot|\) is an absolute value function on \(k\).

If \(k\) is complete with respect to the metric corresponding to \(|\cdot|\), then the natural embedding of \(\mathbb{Q}\) into \(k\) can be extended to an embedding of \(\mathbb{R}\) into \(k\), such that the restriction of \(|\cdot|\) to the image of \(\mathbb{R}\) in \(k\) is equal to the standard absolute value function on \(\mathbb{R}\). Suppose that there is an element \(i\) of \(k\) such that \(i^2 = -1\). This implies that the embedding of \(\mathbb{R}\) into \(k\) can be extended to an embedding of \(\mathbb{C}\) into \(k\) in the obvious way. The restriction of \(|\cdot|\) to the image of \(\mathbb{C}\) in \(k\) is equal to the standard absolute value function on \(\mathbb{C}\), by the argument at the beginning of the section.

In particular, this permits one to consider \(k\) as a complex Banach algebra. A famous theorem implies that the image of \(\mathbb{C}\) in \(k\) is equal to \(k\) under these
conditions, because $k$ is a field. More precisely, suppose that $\zeta \in k$ is not in the image of $C$ in $k$, and put
\[ f(z) = (\zeta - z)^{-1} \]  
for each $z \in C$. Here we identify $z$ with its image in $k$ on the right side of (2.114), so that $\zeta - z$ is considered as an element of $k$ for each $z \in C$. By hypothesis, $\zeta - z \neq 0$ in $k$ for each $z \in C$, so that the right side of (2.114) makes sense as an element of $k$.

The theory of holomorphic functions can be extended to functions that take values in a complex Banach space, such as $k$. One can show that (2.114) is holomorphic as a mapping from $C$ into $k$ in this sense, and in fact that (2.114) has a local power series expansion at each point. We also have that
\[ |\zeta - z| \to \infty \text{ as } z \to \infty \text{ in } C, \]  
by the triangle inequality, which implies that
\[ (\zeta - z)^{-1} \to 0 \text{ as } z \to \infty \text{ in } C, \]  
by the multiplicative property of absolute value functions. Although the second step does not work in the context of Banach algebras, where the norm is sub-multiplicative and not necessarily multiplicative, one can still get (2.116) using another argument. Of course, (2.114) is a continuous function on $C$, which implies that it is bounded on compact subsets of $C$. Thus (2.114) is bounded on $C$, by (2.116). An appropriate version of Liouville’s theorem implies that (2.114) should be constant on $C$, and hence identically 0, by (2.116). This is a contradiction, because (2.114) is not supposed to be 0 for any $z \in C$. It follows that $\zeta \in k$ is in the image of $C$, as desired.

A more elementary approach in the setting of absolute value functions on fields can be found on p38-9 of [4]. Let $\zeta \in k$ be given, and put
\[ g(z) = |\zeta - z| \]  
for each $z \in C$, where $z$ is identified with an element of $k$, as before. Thus $g(z)$ defines a continuous real-valued function on $C$ that satisfies (2.115), which implies that the minimum of $g(z)$ is attained on $C$, since closed and bounded subsets of $C$ are compact. If $\zeta$ is not in the image of $C$ in $k$, then the minimum of $g(z)$ on $k$ is positive, and it is shown in [4] that $g(z)$ is constant on $C$, contradicting (2.115). Note that minimizing $g(z)$ on $C$ corresponds to maximizing $|f(z)|$ on $C$, where $f(z)$ is as in (2.114). That the maximum of $|f(z)|$ is attained on $C$ can be derived from (2.116), which also works in the context of Banach algebras. One can show that $|f(z)|$ is subharmonic on $C$, because $f(z)$ is holomorphic on $C$. It follows from well-known results about subharmonic functions that $|f(z)|$ is constant on $C$, since it attains its maximum. This implies that $|f(z)|$ is identically 0 on $C$, by (2.116), which is a contradiction again. Of course, this argument is very closely related to the one using Liouville’s theorem discussed in the preceding paragraph.
2.11 Local compactness

Remember that a topological space $X$ is said to be \textit{locally compact} if for each $x \in X$ there is an open set $U \subseteq X$ such that $x \in U$ and $U$ is contained in compact subset $K$ of $X$. If $X$ is Hausdorff, then $K$ is also a closed set in $X$, and hence the closure $\overline{U}$ of $U$ in $X$ is contained in $K$. This implies that $\overline{U}$ is compact, since closed subsets of compact sets are compact. Now let $k$ be a field, and let $|\cdot|$ be a quasimetric absolute value function on $k$. Thus $k$ is a Hausdorff topological space with respect to the topology determined by the corresponding quasimetric $(2.9)$, as usual. If $k$ is locally compact, then there is an open set $U$ in $k$ such that $0 \in U$ and the closure $\overline{U}$ of $U$ in $k$ is compact. This implies that there is an $r > 0$ such that the closed ball $B(0, r)$ centered at $0$ with radius $r$ in $k$ is compact. It follows that every closed ball in $k$ with radius $r$ is compact, by continuity of translations.

If $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence of elements of $k$, then for each $\epsilon > 0$ there is an $L(\epsilon) \geq 1$ such that
\begin{equation}
|x_j - x_l| < \epsilon
\end{equation}
for every $j, l \geq L(\epsilon)$. If $k$ is locally compact, then there is an $r > 0$ such that every closed ball in $k$ of radius $r$ is compact, as in the previous paragraph. This implies that all but finitely many terms of the sequence $\{x_j\}_{j=1}^\infty$ are contained in a compact set, so that there is a subsequence of $\{x_j\}_{j=1}^\infty$ that converges to an element of this compact set. It follows that the whole sequence $\{x_j\}_{j=1}^\infty$ converges to the same element of $k$, because $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence. This shows that $k$ is complete as a quasimetric space with respect to the quasimetric associated to $|\cdot|$ when $k$ is locally compact.

Suppose that $|\cdot|$ is not the trivial absolute value function on $k$, so that there is an $w_0 \in k$ such that $w_0 \neq 0$ and $|w_0| \neq 1$. More precisely, either $0 < |w_0| < 1$ or $|w_0| > 1$, and we may as well ask that $|w_0| > 1$, since otherwise we could replace $w_0$ with $1/w_0$. If $k$ is locally compact, then there is an $r > 0$ such that $B(0, r)$ is compact, as before. Of course,
\begin{equation}
x \mapsto x w_0^n
\end{equation}
is a continuous mapping on $k$ for each positive integer $n$, which sends $B(0, r)$ onto $B(0, r|w_0|^n)$. Thus $B(0, r|w_0|^n)$ is a compact set in $k$ for each positive integer $n$ under these conditions. Every bounded subset of $k$ is contained in $B(0, r|w_0|^n)$ when $n$ is sufficiently large, because $|w_0|^n \to \infty$ as $n \to \infty$, since $|w_0| > 1$. It follows that closed and bounded subsets of $k$ are compact in this situation, because closed subsets of compact sets are compact.

If $|\cdot|$ is the trivial absolute value function on $k$, then the corresponding topology on $k$ is discrete, and hence locally compact. However, the closed unit ball in $k$ is not compact in this case, unless $k$ has only finitely many elements. Note that every Cauchy sequence in $k$ is eventually constant when $|\cdot|$ is trivial, so that $k$ is automatically complete. If $|\cdot|$ is nontrivial on $k$ and $k$ is locally compact, then every Cauchy sequence of elements of $k$ is contained in a compact subset of $k$, because Cauchy sequences are bounded, and closed and bounded
subsets of $k$ are compact. This gives a slightly different way to look at the completeness of $k$ when $k$ is locally compact, by combining these two cases.

2.12 An auxiliary fact

Let $k$ be a field, and suppose that

$$x^2 \neq -1$$

in $k$ for every $x \in k$. Note that this implies that $k$ does not have characteristic 2, because $1 = -1$ in $k$ when $k$ has characteristic 2. Also let $|\cdot|$ be an absolute value function on $k$, and suppose that $k$ is complete with respect to the metric corresponding to $|\cdot|$. Under these conditions, there is a positive real number $c$ such that

$$|x^2 + 1| \geq c$$

for every $x \in k$. In fact, one can take

$$c = |4 \cdot 1| (1 + |4 \cdot 1|)^{-1},$$

as in the proof of Lemma 2.2 starting on p35 of [4]. More precisely, if there is an $x_1 \in k$ such that $|x_1^2 + 1|$ is less than (2.122), then it is shown in [4] that there is a Cauchy sequence $\{x_j\}_{j=1}^{\infty}$ of elements of $k$ such that $\{x_j^2\}_{j=1}^{\infty}$ converges to $-1$. If $k$ is complete, then it follows that $\{x_j\}_{j=1}^{\infty}$ converges to an element $x$ of $k$ such that $x^2 = -1$, contradicting (2.120).

Of course, if $x \in k$ satisfies $|x| \geq \sqrt{2}$, then $|x^2| \geq 2$, and hence

$$|x^2 + 1| \geq 2 - 1 = 1,$$

by the triangle inequality. Thus it suffices to consider $x \in k$ with $|x| \leq \sqrt{2}$ to get (2.121). If the set of $x \in k$ with $|x| \leq \sqrt{2}$ is compact, then the existence of $c > 0$ as in (2.121) can be derived from the extreme value theorem. If $k$ is locally compact, and if $|\cdot|$ is not the trivial absolute value function on $k$, then every closed ball in $k$ is compact, as in the previous section. If $|\cdot|$ is the trivial absolute value function on $k$, then (2.121) holds with $c = 1$ whenever $x^2 \neq -1$.

Here is another argument that works when $k$ is complete, and not necessarily locally compact. Suppose that $x, y \in k$ satisfy

$$|x^2 + 1|, |y^2 + 1| \leq \eta$$

for some $\eta > 0$, so that

$$1 - \eta \leq |x^2|, |y^2| \leq 1 + \eta,$$

by the triangle inequality. Observe that

$$|x^2 - y^2| = |(x^2 + 1) - (y^2 + 1)| \leq |x^2 + 1| + |y^2 + 1| \leq 2 \eta,$$
and hence

\[(2.127) \quad |x - y| |x + y| = |(x - y) (x + y)| = |x^2 - y^2| \leq 2 \eta.\]

It follows that

\[(2.128) \quad |x - y| \leq \sqrt{2 \eta} \quad \text{or} \quad |x + y| \leq \sqrt{2 \eta}.
\]

We also have that

\[(2.129) \quad |x + x| = |(1 + 1)x| = |2 \cdot 1| |x| \geq |2 \cdot 1| \frac{(1 - \eta)}{2},\]

by (2.125), and thus

\[(2.130) \quad |2 \cdot 1| \frac{(1 - \eta)}{2} \leq |x + x| \leq |x - y| + |x + y|.
\]

Suppose that \(\{x_j\}_{j=1}^\infty\) is a sequence of elements of \(k\) such that

\[(2.131) \quad |x_j^2 + 1| \leq \eta\]

for each \(j\). Of course, \(-x_j\) satisfies the same condition, and so we may suppose that

\[(2.132) \quad |x_1 - x_j| \leq \sqrt{2 \eta}\]

for each \(j\), by (2.128), and replacing \(x_j\) with \(-x_j\) whenever necessary. This implies that

\[(2.133) \quad |x_j - x_l| \leq |x_j - x_1| + |x_1 - x_l| \leq 2 \sqrt{2 \eta}\]

for every \(j, l \geq 1\), by the triangle inequality. Combining this with (2.130), we get that

\[(2.134) \quad |2 \cdot 1| \frac{(1 - \eta)}{2} \leq |x_j - x_l| + |x_j + x_l| \leq 2 \sqrt{2 \eta} + |x_j + x_l|\]

for every \(j, l \geq 1\). If \(\eta\) is sufficiently small, then this gives a positive lower bound for \(|x_j + x_l|\) for every \(j, l \geq 1\).

Let us now fix \(\eta > 0\) so that the preceding statement holds. If there is no \(c > 0\) such that (2.121) holds for every \(x \in k\), then there is a sequence \(\{x_j\}_{j=1}^\infty\) of elements of \(k\) such that \(\{x_j^2\}_{j=1}^\infty\) converges to \(-1\). In particular, we may ask that \(x_j\) satisfy (2.131) for every \(j\), and also (2.132), by replacing \(x_j\) with \(-x_j\), as before. Because \(\{x_j^2\}_{j=1}^\infty\) converges to \(-1\), we have that

\[(2.135) \quad |x_j^2 - x_l^2| \to 0 \quad \text{as} \ j, l \to \infty.
\]

This implies that either \(|x_j - x_l|\) or \(|x_j + x_l|\) is as small as we like when \(j\) and \(l\) are sufficiently large, as in (2.127) and (2.128). Using the positive lower bound for \(|x_j + x_l|\) mentioned in the previous paragraph, we get that \(|x_j - x_l|\) is as small as we like when \(j\) and \(l\) are sufficiently small. Thus \(\{x_j\}_{j=1}^\infty\) is a Cauchy sequence in \(k\) under these conditions, which converges to an element \(x\) of \(k\) such that \(x^2 = 1\) when \(k\) is complete.

This argument would also work for a quasimetric absolute value function \(|\cdot|\) on \(k\), with some minor adjustments. Alternatively, one could reduce to the
case of an absolute value function, by replacing $| \cdot |$ with a suitable positive power of itself. If $| \cdot |$ is an ultrametric absolute value function on $k$, then this argument could be simplified a bit. Otherwise, in the archimedian case, $k$ has characteristic 0, and one can choose $| \cdot |$ so that it agrees with the usual absolute value function on the natural image of $\mathbb{Q}$ in $k$, as in Section 2.9. In this case, $|n \cdot 1| = n$ for each positive integer $n$, which leads to some other simplifications.

If (2.121) holds for every $x \in k$, then it is easy to see that

$$|x^2 + y^2| \geq c \max(|x|^2, |y|^2)$$

(2.136)

for every $x, y \in k$. More precisely, this holds with $c = 1$ when $x = 0$ or $y = 0$, and so we may as well suppose that $x, y \neq 0$. If $y \neq 0$, then

$$|x^2 + y^2| = |(x/y)^2 + 1||y|^2 \geq c|y|^2$$

(2.137)

for every $x \in k$, by (2.121). Using the analogous estimate when $x \neq 0$, we get that (2.136) holds for every $x, y \in k$, as desired.

### 2.13 Complex numbers, 2

Let $k$ be a field, and suppose that $x^2 \neq -1$ for every $x \in k$. Consider the field $k(i)$ obtained by adjoining to $k$ an additional element $i$ such that $i^2 = -1$. More precisely, one can think of $k(i)$ initially as a two-dimensional vector space over $k$, in which every element can be expressed as $x + iy$, where $x, y \in k$ and $i \notin k$.

It is easy to define multiplication on $k(i)$ in such a way that $k(i)$ becomes a commutative ring, using multiplication on $k$, and by defining $i^2$ to be $-1$ in $k$.

One can also define complex conjugation on $k(i)$ in the usual way, so that

$$
(x + iy) = x - iy
$$

(2.138)

for every $x, y \in k$. This determines a ring automorphism on $k$, which satisfies

$$
(x + iy)(x + iy) = (x + iy)(x - iy) = x^2 + y^2
$$

(2.139)

for every $x, y \in k$. If $y \neq 0$, then

$$
x^2 + y^2 = ((x/y)^2 + 1) y^2 \neq 0
$$

(2.140)

for every $x \in k$, because $(x/y)^2 \neq -1$ by hypothesis. Similarly, $x^2 + y^2 \neq 0$ when $x \neq 0$. If $x + iy \neq 0$ in $k(i)$, then $x \neq 0$ or $y \neq 0$, which implies that $x^2 + y^2 \neq 0$, and hence that $x^2 + y^2$ has a multiplicative inverse in $k$, because $x^2 + y^2 \in k$ and $k$ is a field. It follows that $x + iy$ has a multiplicative inverse in $k(i)$ under these conditions, given by

$$
(x + iy)^{-1} = (x - iy)(x^2 + y^2)^{-1},
$$

(2.141)

so that $k(i)$ is a field too.
Let \( |\cdot| \) be a quasimetric absolute value function on \( k \). It is natural to extend \( |\cdot| \) to \( k(i) \) by putting
\[
|x + iy| = |x^2 + y^2|^{1/2}
\]
for every \( x, y \in k \), where the right side of (2.142) is defined using the absolute value of \( x^2 + y^2 \) as an element of \( k \). If \( x + iy \neq 0 \) in \( k(i) \), then \( x^2 + y^2 \neq 0 \) in \( k \), as in the preceding paragraph. This implies that the right side of (2.142) is positive when \( x + iy \neq 0 \) in \( k(i) \). Observe also that
\[
|z| = |\overline{z}|
\]
for every \( z \in k(i) \), by construction.

Equivalently,
\[
|z|^2 = |z \overline{z}|
\]
for every \( z \in k(i) \), by (2.139). Thus
\[
|zw|^2 = |z(w \overline{w})| = |z \overline{w}w| = |z \overline{z}w|w|
\]
for every \( z, w \in k(i) \), using the fact that complex conjugation is a field automorphism in the second step, and commutativity of multiplication on \( k(i) \) in the third step. Because \( z \overline{z}, w \overline{w} \in k \), the multiplicative property (2.2) of \( |\cdot| \) on \( k \) implies that
\[
|zw|^2 = |z| |w \overline{w}| = |z|^2 |w|^2
\]
for every \( z, w \in k(i) \), using (2.144) in the second step. This shows that this extension of \( |\cdot| \) to \( k(i) \) also satisfies the multiplicative property (2.2).

Note that
\[
|x + iy| = |x^2 + y^2|^{1/2} \leq \sqrt{C'} \max(|x|, |y|)
\]
for every \( x, y \in k \), where \( C' \geq 1 \) is as in (2.14), and where (2.14) is applied to the absolute value of \( x^2 + y^2 \) as an element of \( k \) in the second step. Suppose now that there is a \( c > 0 \) such that (2.136) holds for every \( x, y \in k \). This implies that
\[
|x + iy| = |x^2 + y^2|^{1/2} \geq \sqrt{c} \max(|x|, |y|)
\]
for every \( x, y \in k \). Under these conditions, it is easy to see that this extension of \( |\cdot| \) to \( k(i) \) is a quasimetric absolute value function, using (2.147) and (2.148).

Alternatively, if \( z, w \in k(i) \), then
\[
(z + w)(z + w) = (z + w)(\overline{z} + \overline{w}) = z \overline{z} + z \overline{w} + w \overline{z} + w \overline{w},
\]
using the fact that complex conjugation is a field automorphism in the first step. It follows that
\[
|z + w|^2 = |(z + w)(z + w)| = |z \overline{z} + (z \overline{w} + w \overline{z}) + w \overline{w}|
\]
for every \( z, w \in k(i) \). Of course, \( z \overline{z}, w \overline{w} \), and (2.149) are elements of \( k \), which implies that
\[
z \overline{w} + w \overline{z}
\]
is an element of $k$ too. Because $z \overline{z}$, $w \overline{w}$, and (2.151) are elements of $k$, (2.150) can be estimated in terms of $|z \overline{z}| = |z|^2$, $|w \overline{w}| = |w|^2$, and the absolute value of (2.151), using the quasimetric version of the triangle inequality for $|\cdot|$ on $k$.

More precisely, $2 \cdot z \overline{w}$ is the sum of (2.151) and $z \overline{w} - w \overline{z}$, (2.152)

where (2.151) is in $k$, and (2.152) is $i$ times an element of $k$. Thus (2.148) implies that

(2.153) $\sqrt{c} |z \overline{w} + \overline{z} w| \leq |2 \cdot z \overline{w}|$.

We also have that

(2.154) $|2 \cdot z \overline{w}| = |2 \cdot 1| |z| |\overline{w}| = |2 \cdot 1| |z| |w|$, using the multiplicative property of $|\cdot|$ on $k(i)$ in the first step, and (2.143) in the second step. This permits one to estimate (2.150) in terms of $|z|^2$, $|w|^2$, and $|z| |w|$, to get that this extension of $|\cdot|$ to $k(i)$ is a quasimetric absolute value function too. If $|\cdot|$ is an absolute value function on $k$ and (2.148) holds with $c = 1$, then one could use this argument to show that this extension of $|\cdot|$ to $k(i)$ is an absolute value function as well.

Of course, $k(i)$ can be identified with $k \times k$ in an obvious way. This leads to a topology on $k(i)$ that corresponds to the product topology on $k \times k$, using the topology on $k$ determined by the quasimetric associated to $|\cdot|$. There is also a topology on $k(i)$ determined by the quasimetric associated to the extension of $|\cdot|$ to $k(i)$ just defined. It is easy to see that these two topologies on $k(i)$ are the same, using (2.147) and (2.148). Similarly, if $k$ is complete with respect to the quasimetric associated to $|\cdot|$, then $k(i)$ is complete with respect to the quasimetric associated to the extension of $|\cdot|$ to $k(i)$ just defined.

Let us now consider the case where $|\cdot|$ is an archimedian absolute value function on $k$. As before, this implies that $k$ has characteristic 0, and we may as well suppose that the restriction of $|\cdot|$ to the natural image of $\mathbb{Q}$ in $k$ is equal to the standard absolute value on $\mathbb{Q}$. Let us also suppose that $k$ is complete with respect to the metric associated to $|\cdot|$. This implies that the natural embedding of $\mathbb{Q}$ into $k$ extends to an embedding of $\mathbb{R}$ into $k$, and that the restriction of $|\cdot|$ to the image of $\mathbb{R}$ in $k$ is equal to the standard absolute value on $\mathbb{R}$. If $x^2 \neq -1$ for every $x \in k$, then there is a $c > 0$ such that (2.136) holds for every $x, y \in k$, as in the previous section. Thus the extension of $|\cdot|$ to $k(i)$ is a quasimetric absolute value function, and in fact this extension is an absolute value function on $k(i)$, as in Section 2.9. Remember that $k(i)$ is complete with respect to the metric associated to the extension of $|\cdot|$ to $k(i)$, as in the preceding paragraph. The natural embedding of $\mathbb{R}$ into $k$ leads to an embedding of $\mathbb{C}$ into $k(i)$. The image of $\mathbb{C}$ in $k(i)$ is actually equal to $k(i)$ under these conditions, as in Section 2.10. This implies that the image of $\mathbb{R}$ in $k$ is equal to $k$. The characterization of complete fields with archimedian absolute value functions described in Section 2.10 and this section is another famous theorem of Ostrowski, as on p33 of [4].
3.1 \( p \)-Adic integers

Let \( p \) be a prime number, and let \( | \cdot |_p \) be the \( p \)-adic absolute value function on \( \mathbb{Q} \), as in Section 2.1. Also let \( \mathbb{Z} \) be the set of integers, and observe that

\[
|x|_p \leq 1 \tag{3.1}
\]

for every \( x \in \mathbb{Z} \). Suppose now that \( y \in \mathbb{Q} \) satisfies \( |y|_p \leq 1 \), and let us check that \( y \) can be approximated by integers with respect to the \( p \)-adic metric. By hypothesis, \( y \) can be expressed as \( a/b \), where \( a \) and \( b \) are integers, \( b \neq 0 \), and \( b \) is not an integer multiple of \( p \). Thus \( b \neq 0 \) modulo \( p \), which implies that there is an integer \( c \) such that \( b c \equiv 1 \) modulo \( p \), because \( p \) is prime. Equivalently, there is a \( w \in \mathbb{Z} \) such that \( b c = 1 - p w \), and hence

\[
y = \frac{a}{b} = \frac{ac}{bc} = \frac{ac}{1 - pw} = ac \lim_{n \to \infty} \sum_{j=0}^{n} (pw)^j. \tag{3.2}
\]

This uses (2.71) applied to \( pw \), which is permissible because

\[
|pw|_p = |p|_p |w|_p \leq 1/p < 1 \tag{3.3}
\]

for every \( w \in \mathbb{Z} \). It follows from (3.2) that \( y \) can be approximated by integers with respect to the \( p \)-adic metric, as desired.

The set \( \mathbb{Z}_p \) of \( p \)-adic integers is defined by

\[
\mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \}, \tag{3.4}
\]

which is the closed unit ball \( \overline{B}(0,1) \) in \( \mathbb{Q}_p \) with respect to the \( p \)-adic metric. Let us check that

\[
\mathbb{Z}_p = \overline{\mathbb{Z}}, \tag{3.5}
\]

where \( \overline{\mathbb{Z}} \) is the closure of \( \mathbb{Z} \) in \( \mathbb{Q}_p \), with respect to the \( p \)-adic metric. Of course, \( \mathbb{Z} \subseteq \mathbb{Z}_p \), by (3.1), which implies that \( \overline{\mathbb{Z}} \subseteq \mathbb{Z}_p \), because \( \mathbb{Z}_p \) is a closed set in
3.1. \textit{P-ADIC INTEGERS}\n
\textbf{Q}_p. \textit{by construction. Now let }x \in \textbf{Z}_p \textit{be given, and let us show that }x \textit{can be approximated by ordinary integers with respect to the }p\textit{-adic metric. Remember that }\textbf{Q} \textit{is dense in }\textbf{Q}_p, \textit{so that }x \textit{can be approximated by rational numbers with respect to the }p\textit{-adic metric. If }y \in \textbf{Q} \textit{and }|x - y|_p \leq 1, \textit{then}

\begin{equation}
|y|_p \leq \max(|y - x|_p, |x|_p) \leq 1
\end{equation}

\textit{by the ultrametric version of the triangle inequality, so that }y \in \textbf{Q} \cap \textbf{Z}_p. \textit{Thus }x \textit{can actually be approximated by elements of }\textbf{Q} \cap \textbf{Z}_p \textit{with respect to the }p\textit{-adic metric. The argument in the preceding paragraph says exactly that elements of }\textbf{Q} \cap \textbf{Z}_p \textit{can be approximated by ordinary integers with respect to the }p\textit{-adic metric, which implies that }x \textit{can be approximated by ordinary integers with respect to the }p\textit{-adic metric, as desired.}

\textit{Note that }\textbf{Z}_p \textit{is a subgroup of }\textbf{Q}_p \textit{with respect to addition, and in fact a subring of }\textbf{Q}_p \textit{with respect to addition and multiplication. Put}

\begin{equation}
p^j \textbf{Z} = \{p^j x : x \in \textbf{Z}\}
\end{equation}

\begin{equation}
p^j \textbf{Z}_p = \{p^j x : x \in \textbf{Z}_p\} = \{y \in \textbf{Q}_p : |y|_p \leq p^{-j}\}
\end{equation}

\textit{for each }j \in \textbf{Z}, \textit{so that }p^j \textbf{Z}_p \textit{is the same as the closure of }p^j \textbf{Z} \textit{in }\textbf{Q}_p \textit{with respect to the }p\textit{-adic metric, by (3.5). Of course, }p^j \textbf{Z} \textit{is a subgroup of }\textbf{Q} \textit{with respect to addition for every }j \in \textbf{Z}, \textit{and }p^j \textbf{Z} \textit{is an ideal in }\textbf{Z} \textit{when }j \geq 0. \textit{Similarly, }p^j \textbf{Z}_p \textit{is a subgroup of }\textbf{Q}_p \textit{with respect to addition for every }j \in \textbf{Z}, \textit{and }p^j \textbf{Z}_p \textit{is an ideal in }\textbf{Z}_p \textit{when }j \geq 0. \textit{Thus the quotients}

\begin{equation}
\textbf{Z}/p^j \textbf{Z}
\end{equation}

\begin{equation}
\textbf{Z}_p/p^j \textbf{Z}_p
\end{equation}

\textit{are defined as commutative rings when }j \geq 0. \textit{The obvious inclusion of }\textbf{Z} \textit{into }\textbf{Z}_p \textit{leads to a ring homomorphism from }\textbf{Z} \textit{into (3.10), whose kernel is}

\begin{equation}
\textbf{Z} \cap (p^j \textbf{Z}_p).
\end{equation}

\textit{This is the same as }p^j \textbf{Z}, \textit{because }x \in \textbf{Z} \textit{satisfies }|x|_p \leq p^{-j} \textit{if and only if }x \in p^j \textbf{Z}. \textit{Every element of }\textbf{Z}_p \textit{can be expressed as a sum of elements of }\textbf{Z} \textit{and }p^j \textbf{Z}_p, \textit{because }\textbf{Z}_p \textit{is the closure of }\textbf{Z} \textit{in }\textbf{Q}_p. \textit{It follows that the homomorphism from }\textbf{Z} \textit{into (3.10) mentioned earlier is surjective, and leads to a ring isomorphism from (3.9) onto (3.10). In particular, (3.10) has exactly }p^j \textit{elements for each }j \geq 0.

\textit{This implies that }\textbf{Z}_p \textit{is totally bounded with respect to the }p\textit{-adic metric, in the sense that }\textbf{Z}_p \textit{can be covered by finitely many balls of arbitrarily small radius. It follows that }\textbf{Z}_p \textit{is compact with respect to the topology determined by the }p\textit{-adic metric, because }\textbf{Q}_p \textit{is complete, and }\textbf{Z}_p \textit{is a closed subset of }\textbf{Q}_p. \textit{Similarly, }p^j \textbf{Z}_p \textit{is compact for every }j \in \textbf{Z}, \textit{which can either be derived by an analogous argument, or using continuity of multiplication on }\textbf{Q}_p. \textit{If }E \textit{is a bounded subset of }\textbf{Q}_p \textit{with respect to the }p\textit{-adic metric, then }E \subseteq p^j \textbf{Z}_p \textit{when }-j \textit{is sufficiently large. If }E \textit{is closed and bounded, then }E \textit{is compact, because closed subsets of compact sets are compact.}
3.2 Formal series

Let $k_0$ be a field, and let $T$ be an indeterminate. In this section, we shall be interested in formal series of the form

$$f(T) = \sum_{j=n}^{\infty} f_j T^j,$$

where $n \in \mathbb{Z}$ and $f_j \in k_0$ for each $j \geq n$. More precisely, one can ask that $f_j \in k_0$ be defined for every $j \in \mathbb{Z}$, with the condition that $f_j = 0$ for all but finitely many negative integers $j$. This permits the space $k_0((T))$ of all such formal series in $T$ with coefficients in $k_0$ to be identified with the collection of functions from $\mathbb{Z}$ into $k_0$ that are equal to 0 at all but at most finitely many negative integers. As on p27 of [4], it is convenient to use the notation

$$f(T) = \sum_{j \gg -\infty} f_j T^j,$$

instead of (3.12), to indicate that $f_j = 0$ for all but finitely many negative integers $j$, without specifying $n \in \mathbb{Z}$ as in (3.12).

Of course, the space of all functions from $\mathbb{Z}$ into $k_0$ is a vector space over $k_0$ with respect to pointwise addition and scalar multiplication. The space of such functions that are equal to 0 at all but finitely many negative integers is a linear subspace of this vector space, and hence a vector space over $k_0$ too. Thus $k_0((T))$ is a vector space over $k_0$ in a natural way, where the vector space operations correspond to termwise addition and scalar multiplication of formal series as in (3.12).

It is easy to define multiplication of formal series as in (3.12), where

$$T^j T^l = T^{j+l}$$

for all $j, l \in \mathbb{Z}$. To be more precise, suppose that $f(T)$ is as in (3.13), and that

$$g(T) = \sum_{l \gg -\infty} g_l T^l$$

is another element of $k_0((T))$. Under these conditions, their product $f(T) g(T)$ is given by

$$f(T) g(T) = \sum_{r \gg -\infty} (f g)_r T^r,$$

where

$$(f g)_r = \sum_{j+l=r} f_j g_l$$

for each $r \in \mathbb{Z}$. The sum in (3.17) is taken over all $j, l \in \mathbb{Z}$ with $j + l = r$, and it is easy to see that all but at most finitely many terms in this sum are equal to 0, because $f_j = 0$ and $g_l = 0$ for all but finitely many negative integers $j$,
3.2. FORMAL SERIES

\[ l, \ \text{respectively. Similarly, } (3.17) \text{ is equal to } 0 \text{ for all but finitely many negative integers } r, \text{ so that } (3.16) \text{ defines an element of } k_0((T)). \text{ One can check that } k_0((T)) \text{ is a commutative ring with respect to this definition of multiplication, and in fact an algebra over } k_0. \text{ One can also identify } k_0 \text{ with the subalgebra of } k_0((T)) \text{ consisting of series for which only the constant term may be nonzero. In particular, one can identify the multiplicative identity element } 1 \text{ of } k_0 \text{ with } T^0, \text{ which is the multiplicative identity element of } k_0((T)). \]

\[ \text{Let } k_0[[T]] \text{ be the subset of } k_0((T)) \text{ consisting of series of the form} \]

\[ f(T) = \sum_{j=0}^{\infty} f_j T^j, \]

so that \( f_j = 0 \text{ when } j < 0. \text{ This is a subalgebra of } k_0((T)) \text{ that contains } k_0, \text{ which is the usual algebra of formal power series with coefficients in } k_0. \text{ The elements of } k_0((T)) \text{ may be described as formal Laurent series with coefficients in } k_0 \text{ and a pole of finite order. Note that every element of } k_0((T)) \text{ can be expressed as } T^n f(T) \text{ for some } n \in \mathbb{Z} \text{ and } f(T) \in k_0[[T]]. \text{ The algebra } k_0[T] \text{ of formal polynomials in } T \text{ with coefficients in } k_0 \text{ may be identified with the subalgebra of } k_0[[T]] \text{ consisting of formal power series } f(T) \text{ such that } f_j = 0 \text{ for all but finitely many } j. \]

If \( a(T) \) is any element of \( k_0[[T]] \), then \( a(T)^n \in k_0[[T]] \) for each positive integer \( n \), and we interpret \( a(T)^0 \) as being the series with only the constant term 1. This permits one to define

\[ \sum_{n=0}^{\infty} T^n a(T)^n \]

as an element of \( k_0[[T]] \), since for each nonnegative integer \( j \), the coefficient of \( T^j \) in \( T^n a(T)^n \) is equal to 0 when \( j < n \). Observe that

\[ (1 - T a(T)) \sum_{n=0}^{\infty} T^n a(T)^n = \sum_{n=0}^{\infty} T^n a(T)^n - T a(T) \sum_{n=0}^{\infty} T^n a(T)^n \]

\[ = \sum_{n=0}^{\infty} T^n a(T)^n - \sum_{n=1}^{\infty} T^n a(T)^n = 1. \]

This shows that \( 1 - T a(T) \) has a multiplicative inverse in \( k_0[[T]] \) under these conditions, which is given by (3.19).

If \( f(T) \) is a nonzero element of \( k_0((T)) \), then \( f(T) \) can be expressed as

\[ f(T) = c T^n (1 - T a(T)) \]

for some \( c \in k_0 \text{ with } c \neq 0, n \in \mathbb{Z} \text{, and } a(T) \in k_0[[T]]. \) It follows that \( f(T) \) has a multiplicative inverse in \( k_0((T)), \text{ which is given by} \]

\[ f(T)^{-1} = c^{-1} T^{-n} (1 - T a(T))^{-1}, \]
where \((1 - Ta(T))^{-1}\) is as in the previous paragraph. Thus \(k_0((T))\) is a field, which contains \(k_0\) as a subfield.

Let \(r\) be a positive real number strictly less than 1, and let us use \(r\) to define an absolute value function on \(k_0((T))\). Let \(f(T) \in k_0((T))\) be given, and put 
\[
|f(T)| = 0 \quad \text{when} \quad f(T) = 0.
\]
Otherwise, if \(f(T) \neq 0\), then there is a unique integer \(n = n(f)\) such that
\[
(3.23) \quad f_n \neq 0 \quad \text{and} \quad f_j = 0 \quad \text{for every} \quad j < n,
\]
in which case we put
\[
(3.24) \quad |f(T)| = r^n.
\]
It is easy to see that
\[
(3.25) \quad |f(T)g(T)| = |f(T)||g(T)|
\]
for every \(f(T), g(T) \in k_0((T))\), because \(n(f(T)g(T)) = n(f(T)) + n(g(T))\) when \(f(T), g(T) \neq 0\). Similarly,
\[
(3.26) \quad |f(T) + g(T)| \leq \max(|f(T)|, |g(T)|)
\]
for every \(f(T), g(T) \in k_0((T))\), because \(n(f(T) + g(T)) \geq \min(n(f(T)), n(g(T)))\) when \(f(T), g(T) \neq 0\).

Thus \(|f(T)|\) defines an ultrametric absolute value function on \(k_0((T))\) for each \(r \in (0, 1)\). If \(a\) is a positive real number, then
\[
(3.27) \quad |f(T)|^a
\]
is the same as the ultrametric absolute value function associated in this way to \(r^a\) instead of \(r\). In particular, the topology on \(k_0((T))\) determined by the ultrametric corresponding to \(|f(T)|\) as in (2.9) does not depend on the choice of \(r \in (0, 1)\). If one were to take \(r = 1\) in the definition of \(|f(T)|\), then one would get the trivial absolute value function on \(k_0((T))\). Note that the restriction of \(|f(T)|\) to \(k_0\) is the trivial absolute value function on \(k_0\) for every \(r \in (0, 1)\).

By construction,
\[
(3.28) \quad k_0[[T]] = \{f(T) \in k_0((T)) : |f(T)| \leq 1\}
\]
for every \(r \in (0, 1)\), which is the closed unit ball in \(k_0((T))\) with respect to \(|f(T)|\). Of course, \(k_0[[T]]\) can be identified with a Cartesian product of a sequence of copies of \(k_0\), indexed by the nonnegative integers. One can check that the topology on \(k_0[[T]]\) determined by the ultrametric (2.9) corresponding to \(|f(T)|\) is the same as the product topology associated to the discrete topology on each copy of \(k_0\) in the Cartesian product. In particular, this topology does not depend on the choice of \(r \in (0, 1)\), as before. It is easy to see that \(k_0[[T]]\) is dense in \(k_0[[T]]\) with respect to this topology.

Similarly,
\[
(3.29) \quad T^n k_0[[T]] = \{T^n f(T) : f(T) \in k_0[[T]]\}
\]
\[
= \{f(T) \in k_0((T)) : |f(T)| \leq r^n\}
\]
3.3. DISCRETE ABSOLUTE VALUE FUNCTIONS

for each \(n \in \mathbb{Z}\). Thus every bounded subset of \(k_0((T))\) with respect to \(|f(T)|\) is contained in (3.29) for some \(n \in \mathbb{Z}\). In particular, every Cauchy sequence of elements of \(k_0((T))\) with respect to the ultrametric (2.9) corresponding to \(|f(T)|\) is contained in (3.29) for some \(n \in \mathbb{Z}\). One can check that the coefficients of \(T^j\) of the terms of such a Cauchy sequence are eventually constant for each \(j\), and hence that the Cauchy sequence converges in \(k_0((T))\). This implies that \(k_0((T))\) is complete with respect to the ultrametric (2.9) corresponding to \(|f(T)|\), for each \(r \in (0, 1)\).

If \(k_0\) has only finitely many elements, then \(k_0[[T]]\) is compact with respect to the topology determined by the ultrametric (2.9) corresponding to \(|f(T)|\) for each \(r \in (0, 1)\), because \(k_0[[T]]\) is topologically equivalent to a product of finite sets. In this case, (3.29) is also compact for each \(n \in \mathbb{Z}\), either by an analogous argument, or using continuity of multiplication on \(k_0((T))\). If \(E \subseteq k_0((T))\) is bounded with respect to \(|f(T)|\), then \(E\) is contained in (3.29) for some \(n \in \mathbb{Z}\), as in the preceding paragraph. If \(E\) is closed and bounded, then it follows that \(E\) is compact, since closed subsets of compact sets are compact.

3.3 Discrete absolute value functions

Let \(k\) be a field, and let \(\cdot\) be a quasimetric absolute value function on \(k\). Thus

\[
(3.30) \quad \{|x| : x \in k, x \neq 0\}
\]

is a subgroup of the multiplicative group \(\mathbb{R}^+\) of positive real numbers. Suppose that there is a positive real number \(\rho < 1\) such that for each \(x \in k\) with \(|x| < 1\), we have that

\[
(3.31) \quad |x| \leq \rho.
\]

This implies that for each \(x \in k\) with \(|x| > 1\), we have that

\[
(3.32) \quad |x| \geq 1/\rho,
\]

by applying the previous statement to \(1/x\). If \(y, z \in k\) satisfy \(|y| < |z|\), then we get that

\[
(3.33) \quad |y| \leq \rho |z|,
\]

by applying (3.30) to \(x = y/z\).

It follows that 1 is not a limit point of (3.30) with respect to the standard metric on \(\mathbb{R}\) under these conditions. Conversely, if 1 is not a limit point of (3.30) with respect to the standard metric on \(\mathbb{R}\), then there is a \(\rho \in (0, 1)\) with the property described in the preceding paragraph. In this case, one can check that (3.30) is a discrete subgroup of \(\mathbb{R}^+_+\), in the sense that (3.30) has no limit points in \(\mathbb{R}^+_+\). More precisely, if \(t \in \mathbb{R}^+_+\) is a limit point of (3.30) in \(\mathbb{R}^+_+\), then there exist \(y, z \in k\) such that \(|y| < |z|\) and \(|y|, |z|\) are arbitrarily close to \(t\) with respect to the standard metric on \(\mathbb{R}\). This implies that \(|y|/|z|\) is arbitrarily close to 1, which would contradict (3.33). Of course, 0 is always a limit point of (3.30) in \(\mathbb{R}\) when \(\cdot\) is nontrivial on \(k\). If \(\cdot\) satisfies the condition described in the previous paragraph, then \(\cdot\) is said to be discrete on \(k\).
The trivial absolute value function is discrete on any field $k$. In this case, one can take $\rho = 0$ in (3.31), and (3.32) is vacuous. If $p$ is a prime number, then the $p$-adic absolute value function on $k = \mathbb{Q}$ is discrete, and one can take $\rho = 1/p$ in (3.31). However, the standard absolute value function on $k = \mathbb{Q}$ is not discrete. If $k_0$ is a field, $T$ is an indeterminate, $k$ is the field $k_0((T))$ of formal series in $T$ with coefficients in $k_0$ discussed in the previous section, and $|f(T)|$ is the absolute value function on $k_0((T))$ associated to some $r \in (0, 1)$ as in (3.24), then $|f(T)|$ is discrete on $k_0((T))$, and one can take $\rho = r$ in (3.31).

If a quasimetric absolute value function $| \cdot |$ on a field $k$ is discrete, then $| \cdot |$ is discrete on $k$ for every positive real number $a$. More precisely, if $| \cdot |$ satisfies (3.31) for some $\rho \in (0, 1)$, then $| \cdot |^a$ satisfies the analogous condition with $\rho^a$ instead of $\rho$. If $| \cdot |$ is discrete on $k$, then the natural extension of $| \cdot |$ to the corresponding completion of $k$ is also discrete, and one can use the same value of $\rho$ in (3.31) on the completion of $k$.

Suppose for the moment that $k$ has characteristic 0, so that there is a natural embedding of $\mathbb{Q}$ into $k$. Thus a quasimetric absolute value function $| \cdot |$ on $k$ leads to a quasimetric absolute value function on $\mathbb{Q}$, as in (2.109) in Section 2.9. If this quasimetric absolute value function on $\mathbb{Q}$ is archimedian, then it is equal to a positive power of the standard absolute value function on $\mathbb{Q}$, as mentioned in Section 2.3. This implies that $| \cdot |$ is not discrete on $k$, because any positive power of the standard absolute value function on $\mathbb{Q}$ is not discrete.

If $| \cdot |$ is a discrete quasimetric absolute value function on any field $k$, then $| \cdot |$ is an ultrametric absolute value function on $k$. More precisely, if $k$ does not have characteristic 0, then every quasimetric absolute value function on $k$ is an ultrametric absolute value function. This follows from the discussion in Section 2.3, as mentioned at the end of Section 2.9. If $k$ has characteristic 0, then there is a natural embedding of $\mathbb{Q}$ into $k$, which leads to a quasimetric absolute value function on $\mathbb{Q}$, as before. If this quasimetric absolute value function on $\mathbb{Q}$ is non-archimedian, then $| \cdot |$ is non-archimedian on $k$. This implies that $| \cdot |$ is an ultrametric absolute value function on $k$ again, as in Section 2.3. The remaining possibility is that $k$ has characteristic 0, and that the corresponding absolute value function on $\mathbb{Q}$ is archimedian. In this case, $| \cdot |$ is not discrete on $k$, as in the preceding paragraph.

Let $| \cdot |$ be a quasimetric absolute value function on a field $k$ again, and put

$$
(3.34) \quad \rho_1 = \sup \{ |x| : x \in k, |x| < 1 \},
$$

so that $0 \leq \rho_1 \leq 1$. Thus $| \cdot |$ is discrete on $k$ if and only if $\rho_1 < 1$. It is easy to see that $\rho_1 = 0$ if and only if $| \cdot |$ is the trivial absolute value function on $k$. Let us suppose from now on in this section that $| \cdot |$ is nontrivial and discrete on $k$, so that $0 < \rho_1 < 1$. Under these conditions, one can check that the supremum in (3.34) is attained, which is to say that there is an $x_1 \in k$ such that

$$
(3.35) \quad |x_1| = \rho_1.
$$

This uses the fact that (3.30) is a discrete subgroup of $\mathbb{R}_+$. It follows that

$$
(3.36) \quad |x_1^2| = |x_1|^2 = \rho_1^2.
$$
for each $j \in \mathbb{Z}$, so that $\rho_1^j$ is an element of (3.30) for each $j \in \mathbb{Z}$.

Conversely, let $w \in k$ with $w \neq 0$ be given, and let us show that

$$|w| = \rho^j$$

for some $j \in \mathbb{Z}$. Of course, there is a $j \in \mathbb{Z}$ such that

$$\rho_1^{j+1} < |w| \leq \rho_1^j,$$

because $|w| > 0$. Suppose for the sake of a contradiction that $|w| < \rho_1^j$, and put $u = w/x_1^j$, so that

$$|u| = |w|/|x_1|^j = |w|/\rho_1^j < 1.$$  

The definition (3.34) of $\rho_1$ implies that $|u| \leq \rho_1$, and hence that

$$|w| = |ux_1^j| \leq \rho_1^{j+1},$$

contradicting (3.38). Thus we get (3.37), which means that (3.30) consists of exactly the integer powers of $\rho_1$ under these conditions.

### 3.4 The ultrametric case

Let $k$ be a field, and let $| \cdot |$ be an ultrametric absolute value function on $k$. It is easy to see that the closed unit ball

$$(3.41) \quad \mathcal{B}(0, 1) = \{ x \in k : |x| \leq 1 \}$$

in $k$ with respect to $| \cdot |$ is a subring of $k$ under these conditions. Note that $\mathcal{B}(0, 1)$ contains the multiplicative identity element 1 in $k$, by (2.4). If $x \in k$ satisfies $|x| = 1$, then $x$ and $x^{-1}$ are in $\mathcal{B}(0, 1)$, so that $x$ is invertible in $\mathcal{B}(0, 1)$ as a ring. Conversely, if $x$ is an element of $\mathcal{B}(0, 1)$ that is invertible in $\mathcal{B}(0, 1)$, then $|x| \leq 1, x \neq 0$, and $|x^{-1}| \leq 1$, which implies that $|x| = 1$.

Similarly, the open unit ball

$$(3.42) \quad B(0, 1) = \{ x \in k : |x| < 1 \}$$

in $k$ with respect to $| \cdot |$ is an ideal in $\mathcal{B}(0, 1)$. More precisely, $B(0, 1)$ is a maximal ideal in $\mathcal{B}(0, 1)$, by the remarks in the previous paragraph. This implies that the quotient ring

$$(3.43) \quad \mathcal{B}(0, 1)/B(0, 1)$$

is a field, which also follows more directly from the characterization of invertible elements in $\mathcal{B}(0, 1)$. Of course, the multiplicative identity element 1 in $k$ is in $\mathcal{B}(0, 1)$ and not in $B(0, 1)$, by (2.4), so that its image in the quotient (3.43) is nonzero. The quotient (3.43) is known as the residue field associated to $| \cdot |$ on $k$.

If $| \cdot |$ is the trivial absolute value function on any field $k$, then $\mathcal{B}(0, 1)$ is equal to $k$, $B(0, 1)$ is the trivial ideal $\{0\}$, and hence (3.43) is isomorphic to
If \( k = \mathbb{Q} \) equipped with the \( p \)-adic absolute value function for some prime number \( p \), then \( \overline{B}(0, 1) \) is the same as the ring \( \mathbb{Z}_p \) of \( p \)-adic integers, and \( B(0, 1) \) reduces to \( p\mathbb{Z}_p \). It follows that the residue field (3.43) is isomorphic to \( \mathbb{Z}/p\mathbb{Z} \) in this case, as in Section 3.1. If \( | \cdot | \) is an ultrametric absolute value function on a field \( k \) and \( a \) is a positive real number, then \( | \cdot |^a \) is also an ultrametric absolute value function on \( k \), as in Section 2.1. Clearly \( \overline{B}(0, 1) \) and \( B(0, 1) \) are the same for \( | \cdot |^a \) as for \( | \cdot | \), so that the residue field (3.43) is the same for \( | \cdot |^a \) as for \( | \cdot | \) too.

Suppose that \( k \) is a field of characteristic \( p \) for some prime number \( p \), so that \( p \cdot 1 = 0 \) in \( k \). If \( | \cdot | \) is an ultrametric absolute value function on \( k \), then \( n \cdot 1 \) is an element of \( \overline{B}(0, 1) \) for every positive integer \( n \), whose image in (3.43) is the same as \( n \cdot 1 \) in (3.43). It follows that \( p \cdot 1 = 0 \) in (3.43) as well, so that the residue field also has characteristic \( p \) under these conditions.

Now let \( k_0 \) be a field, let \( T \) be an indeterminate, and let \( k \) be the corresponding field \( k_0((T)) \) of formal series in \( T \) with coefficients in \( k_0 \) discussed in Section 3.2. Also let \( |f(T)| \) be the absolute value function on \( k_0((T)) \) associated to some \( r \in (0, 1) \) as in (3.24). In this situation, \( \overline{B}(0, 1) \) is the ring \( k_0[[T]] \) of formal power series in \( T \) with coefficients in \( k_0 \), and \( B(0, 1) \) is equal to \( T k_0[[T]] \). Note that there is a natural homomorphism from \( k_0[[T]] \) onto \( k_0 \), which sends a formal power series to its constant term. The kernel of this homomorphism is equal to \( T k_0[[T]] \), which leads to an isomorphism from the residue field (3.43) onto \( k_0 \).

Let \( | \cdot | \) be an ultrametric absolute value function on a field \( k \) again, and let \( k_1 \) be a subfield of \( k \). Thus the restriction of \(| \cdot |\) to \( k_1 \) is an ultrametric absolute value function on \( k_1 \), and the corresponding open and closed unit balls in \( k_1 \) are the same as the intersections of their counterparts in \( k \) with \( k_1 \). This leads to a natural injective homomorphism from the residue field associated to \( k_1 \) into the residue field associated to \( k \). If \( k_1 \) is dense in \( k \) with respect to the ultrametric corresponding to \( | \cdot | \), then one can check that this homomorphism is surjective. In particular, the residue field associated to the completion of \( k \) with respect to \( | \cdot | \) is isomorphic to the residue field associated to \( k \).

Note that \( \overline{B}(0, 1) \) can be expressed as the union of a family of pairwise-disjoint open balls in \( k \) of radius 1, which are the cosets of \( B(0, 1) \) in \( \overline{B}(0, 1) \). If the residue field (3.43) has only finitely many elements, then \( \overline{B}(0, 1) \) can be expressed as the union of finitely many pairwise-disjoint open balls of radius 1. Conversely, if \( \overline{B}(0, 1) \) can be covered by finitely many open balls of radius 1, then one can check that the residue field (3.43) has only finitely many elements. Of course, any open ball of radius 1 in \( k \) that intersects \( \overline{B}(0, 1) \) is contained in \( \overline{B}(0, 1) \), because of the ultrametric version of the triangle inequality. Thus any open ball in \( k \) of radius 1 that intersects \( \overline{B}(0, 1) \) is actually a coset of \( B(0, 1) \) in \( \overline{B}(0, 1) \).

Suppose that \( | \cdot | \) is discrete on \( k \), as in the previous section. Let \( \rho_1 \) be as in (3.34), so that \( 0 \leq \rho_1 < 1 \), and \( B(0, 1) \) is the same as

\[
B(0, \rho_1) = \{ x \in k : |x| \leq \rho_1 \}.
\]

Suppose also that \( | \cdot | \) is nontrivial on \( k \), so that \( \rho_1 > 0 \). Remember that the
nonzero values of $|\cdot|$ on $k$ are the same as the integer powers of $\rho_1$ under these conditions, as in the previous section. If the residue field (3.43) has only finitely many elements, then the closed unit ball in $k$ can be covered by finitely many closed balls of radius $\rho_1$. This implies that every closed ball in $k$ of radius $\rho_1^j$ for some $j \in \mathbb{Z}$ can be covered by finitely many closed balls of radius $\rho_1^{j+1}$, using translations and dilations on $k$. Repeating the process, it follows that every closed ball in $k$ can be covered by finite many closed balls of arbitrarily small radius. Thus closed balls in $k$ are totally bounded under these conditions, with respect to the ultrametric on $k$ corresponding to $|\cdot|$. If $k$ is also complete, then it follows that closed balls in $k$ are compact. In this case, closed and bounded subsets of $k$ are compact, because closed subsets of compact sets are compact.

Conversely, let $|\cdot|$ be an ultrametric absolute value function on a field $k$, and suppose that $B(0, 1)$ is totally bounded with respect to the corresponding ultrametric on $k$. In particular, this implies that $B(0, 1)$ can be covered by finitely many open balls of radius 1, so that the residue field (3.43) has only finitely many elements. We also get that $B(0, 1)$ can be covered by finitely many closed balls of radius $1/2$, say, so that there are finitely many elements $x_1, \ldots, x_n$ of $k$ such that

$$B(0, 1) \subseteq \bigcup_{j=1}^n B(x_j, 1/2). \tag{3.45}$$

We may as well suppose that

$$B(x_j, 1/2) \cap B(0, 1) \neq \emptyset \tag{3.46}$$

for each $j = 1, \ldots, n$, since otherwise $B(x_j, 1/2)$ is not needed in the covering of $B(0, 1)$. It follows that $x_j \in B(0, 1)$ for each $j$, by the ultrametric version of the triangle inequality, which is to say that $|x_j| < 1$ for each $j$. If $x \in k$ and $|x| < 1$, so that $x \in B(0, 1)$, then $x \in B(x_j, 1/2)$ for some $j$, by (3.45). Thus

$$|x| \leq \max(|x_1|, \ldots, |x_n|, 1/2) < 1, \tag{3.47}$$

by the ultrametric version of the triangle inequality again, which shows that $|\cdot|$ is discrete on $k$ under these conditions.

### 3.5 Haar measure, 2

If $p$ is a prime number, then the $p$-adic numbers $\mathbb{Q}_p$ form a locally compact commutative topological group with respect to addition. If $H$ is a choice of Haar measure on $\mathbb{Q}_p$, then $H(\mathbb{Z}_p)$ should be positive and finite, because $\mathbb{Z}_p$ is nonempty, open, and compact. It is convenient to normalize Haar measure on $\mathbb{Q}_p$ so that

$$H(\mathbb{Z}_p) = 1, \tag{3.48}$$

in which case $H$ is unique. Let us check that

$$H(p^j \mathbb{Z}_p) = p^{-j} \tag{3.49}$$
for each integer $j$, using (3.48) and the translation-invariance of $H$. If $j \geq 0$, then this follows from the fact that $\mathbb{Z}_p$ can be expressed as the union of $p^j$ pairwise-disjoint translates of $p^j \mathbb{Z}_p$, because (3.10) has exactly $p^j$ elements. If $j < 0$, then $p^{-j} \mathbb{Z}_p$ can be expressed as the union of $p^{-j}$ pairwise-disjoint translates of $\mathbb{Z}_p$, since one can multiply by $p^j$ to reduce to the previous situation. Of course, (3.49) implies that the Haar measure of every translate of $p^j \mathbb{Z}_p$ should be equal to $p^{-j}$ too. One way to get the existence of Haar measure on $\mathbb{Q}_p$ is to first define a Haar integral on $C_{com}(\mathbb{Q}_p)$, as a Riemann integral.

Let $k$ be a field, and let $| \cdot |$ be a quasimetric absolute value function on $k$. In particular, $k$ is a commutative topological group with respect to addition, and using the topology on $k$ determined by the quasimetric corresponding to $| \cdot |$. If $| \cdot |$ is the trivial absolute value function on $k$, then the corresponding topology on $k$ is discrete, and counting measure on $k$ satisfies the requirements of Haar measure, as before. If $k$ is locally compact with respect to the topology determined by the quasimetric associated to $| \cdot |$, then we have seen that $k$ is complete, as in Section 2.11.

If $k$ is complete and archimedian, then $k$ is isomorphic to the real or complex numbers, and $| \cdot |$ corresponds to a positive power of the standard absolute value function on $\mathbb{R}$ or $\mathbb{C}$, as in Sections 2.10 and 2.13. Thus $k$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ as a topological group with respect to addition, and where $\mathbb{R}$ and $\mathbb{C}$ are equipped with their standard topologies. As in Section 1.7, Lebesgue measure on $\mathbb{R}$ satisfies the requirements of Haar measure. Similarly, $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$ as a topological group with respect to addition, and two-dimensional Lebesgue measure satisfies the requirements of Haar measure on $\mathbb{C}$.

Let us suppose from now on in this section that $| \cdot |$ is a nontrivial ultrametric absolute value function on $k$. If $k$ is locally compact, then the closed unit ball in $k$ is compact and hence totally bounded, as in Section 2.11. This implies that the residue field (3.43) is finite, and that $| \cdot |$ is discrete on $k$, by the remarks at the end of the previous section. Conversely, if the residue field is finite, and if $| \cdot |$ is discrete on $k$, then we have seen that closed balls in $k$ are totally bounded, as in the previous section. If $k$ is also complete with respect to the ultrametric corresponding to $| \cdot |$, then it follows that closed balls in $k$ are compact, so that $k$ is locally compact.

Continuing with these hypotheses, let $N$ be the number of elements of the residue field (3.43), and let $\rho_1$ be as in (3.34). Thus $\rho_1 < 1$, because $| \cdot |$ is discrete on $k$, and $\rho_1 > 0$, because $| \cdot |$ is nontrivial on $k$. Remember that the nonzero values of $| \cdot |$ on $k$ are the same as the integer powers of $\rho_1$, as in Section 3.3. We have also seen that $\overline{B}(0,1)$ can be expressed as the union of $N$ pairwise-disjoint open balls of radius 1, as in the previous section. Equivalently, $\overline{B}(0,1)$ can be expressed as the union of $N$ pairwise-disjoint closed balls of radius $\rho_1$, by the definition of $\rho_1$. This implies that every closed ball in $k$ of radius $\rho_1^j$ for some $j \in \mathbb{Z}$ can be expressed as the union of $N$ pairwise-disjoint closed balls of radius $\rho_1^{j+1}$. It follows that the Haar measure of a closed ball in $k$ of radius $\rho_1^j$ is equal to $N$ times the Haar measure of a closed ball of radius $\rho_1^{j+1}$, for any choice of Haar measure on $k$, using invariance under translations. If Haar measure on $k$
is normalized so that
\[ H(B(0, 1)) = 1, \]
then we get that
\[ H(B(x, \rho^n_j)) = N^{-j} \]
for every \( x \in k \) and \( j \in \mathbb{Z} \). As before, one can first define a Haar integral on \( C_{\text{com}}(k) \) as a Riemann integral, and then get Haar measure on \( k \) using the Riesz representation theorem.

Let \( k_0 \) be a field, let \( T \) be an indeterminate, and consider the corresponding field \( k_0([T]) \) of formal series, as in Section 3.2. Also let \( |f(T)| \) be the absolute value function on \( k_0((T)) \) associated to some \( r \in (0, 1) \) as in (3.24). Thus \( |f(T)| \) is nontrivial and discrete on \( k \), and we have seen that \( k_0((T)) \) is complete with respect to \( |f(T)| \). The corresponding residue field is isomorphic to \( k_0 \), as in the previous section. If \( k_0 \) has only finitely many elements, then we have already mentioned in Section 3.2 that \( k_0[[T]] \) is a compact subset of \( k_0((T)) \), and hence that closed balls in \( k_0((T)) \) are compact.

In particular, \( k_0((T)) \) is locally compact when \( k_0 \) has only finitely many elements, in which case Haar measure on \( k_0((T)) \) can be analyzed as before. Alternatively, \( k_0[[T]] \) is a compact commutative topological group with respect to addition when \( k_0 \) has only finitely many elements, using the topology induced by the one on \( k_0((T)) \). As a topological group with respect to addition, \( k_0[[T]] \) is isomorphic to a product of a sequence of copies of \( k_0 \), where \( k_0 \) is considered as a topological group with respect to addition and the discrete topology. It is convenient to take Haar measure on \( k_0 \) to be counting measure divided by the total number of elements of \( k_0 \), so that the Haar measure of \( k_0 \) is equal to 1. With this normalization, Haar measure on \( k_0[[T]] \) corresponds to a product measure on a product of a sequence of copies of \( k_0 \), and one can get Haar measure on \( k_0((T)) \) from Haar measure on \( k_0[[T]] \).

### 3.6 Norms and ultranorms

Let \( k \) be a field, let \( |\cdot| \) be an absolute value function on \( k \), and let \( V \) be a vector space over \( k \). A nonnegative real-valued function \( N \) on \( V \) is said to be a norm if it satisfies the following three conditions: first, \( N(v) = 0 \) if and only if \( v = 0 \); second,
\[ N(tv) = |t|N(v) \]
for every \( t \in k \) and \( v \in V \); and third,
\[ N(v + w) \leq N(v) + N(w) \]
for every \( v, w \in V \). Under these conditions,
\[ d(v, w) = N(v - w) \]
defines a metric on \( V \). More precisely, the fact that (3.54) is symmetric in \( v \) and \( w \) follows from (3.52) with \( t = -1 \), since \(|-1| = 1\), as in (2.6).
A norm $N$ on $V$ is said to be an ultranorm if
\begin{equation}
N(v + w) \leq \max(N(v), N(w))
\end{equation}
for every $v, w \in V$, which implies (3.53). In this case, (3.54) is an ultrametric on $V$. If $V \neq \{0\}$ and $N$ is an ultranorm on $V$, then it is easy to see that $| \cdot |$ is an ultrametric absolute value function on $k$, because of (3.52). If $| \cdot |$ is the trivial absolute value function on $k$, then one can get an ultranorm $N$ on $V$ by putting $N(v) = 1$ for every $v \in V$ with $v \neq 0$, and $N(0) = 0$. Let us call this the trivial ultranorm on $V$, for which the corresponding metric on $V$ is the discrete metric.

If $N$ is any norm on $V$, then one can check that
\begin{equation}
\max(N(v) - N(w), N(w) - N(v)) \leq N(v - w)
\end{equation}
for every $v, w \in V$, using (3.53). This is a special case of (1.71) in Section 1.6. It follows that $N$ is continuous as a real-valued function on $V$, with respect to the topology on $V$ determined by the metric (3.54) corresponding to $N$, and with respect to the standard topology on $R$. If $N$ is an ultrametric on $V$, then
\begin{equation}
N(v) = N(w)
\end{equation}
for every $v, w \in V$ such that $N(v - w) < N(v)$. This is a special case of (1.20) in Section 1.1.

Let $n$ be a positive integer, and let $k^n$ be the set of $n$-tuples of elements of $k$. This is a vector space over $k$ with respect to coordinatewise addition and scalar multiplication. Put
\begin{equation}
N_0(v) = \max(|v_1|, \ldots, |v_n|)
\end{equation}
for each $v = (v_1, \ldots, v_n) \in k^n$, which defines a norm on $k^n$. If $| \cdot |$ is an ultrametric absolute value function on $k$, then $N_0$ is an ultranorm on $k^n$. The topology on $k^n$ determined by the corresponding metric as in (3.54) is the same as the product topology associated to the topology on $k$ determined by the metric (2.9) corresponding to $| \cdot |$.

Let $e(1), \ldots, e(n)$ be the standard basis vectors in $k^n$, so that the $l$th coordinate of $e(j)$ is equal to 1 when $j = l$, and to 0 otherwise. Thus each $v \in k^n$ can be expressed as
\begin{equation}
v = \sum_{j=1}^{n} v_j e(j).
\end{equation}
If $N$ is any norm on $k^n$, then it follows that
\begin{equation}
N(v) \leq \sum_{j=1}^{n} N(v_j e(j)) = \sum_{j=1}^{n} |v_j| N(e(j)) \leq \left( \sum_{j=1}^{n} N(e(j)) \right) N_0(v)
\end{equation}
for every $v \in k^n$. If $N$ is an ultranorm on $k^n$, then we get that
\begin{align}
N(v) &\leq \max_{1 \leq j \leq n} N(v_j e(j)) = \max_{1 \leq j \leq n} (|v_j| N(e(j))) \\
&\leq \left( \max_{1 \leq j \leq n} N(e(j)) \right) N_0(v)
\end{align}
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for every \( v \in k^n \).

Let \( V \) be any vector space over \( k \), and let \( N_1, N_2 \) be norms on \( V \). Suppose that there is a positive real number \( C \) such that

\[
N_1(v) \leq C N_2(v)
\]

for every \( v \in V \). This implies that the corresponding metrics as in (3.54) satisfy the analogous condition. It follows that every open set in \( V \) with respect to the topology determined by the metric associated to \( N_1 \) is also an open set with respect to the topology determined by the metric associated to \( N_2 \).

Conversely, suppose that every open set in \( V \) with respect to the topology determined by the metric associated to \( N_1 \) is also an open set with respect to the topology determined by the metric associated to \( N_2 \). Let \( r_1 > 0 \) be given, and note that the open ball

\[
\{ v \in V : N_1(v) < r_1 \}
\]

with respect to \( N_1 \) centered at 0 with radius \( r_1 \) is an open set with respect to the topology determined by the metric associated to \( N_1 \). The hypothesis that this also be an open set in \( V \) with respect to the topology determined by the metric associated to \( N_2 \) implies that there is an \( r_2 > 0 \) such that

\[
\{ v \in V : N_2(v) < r_2 \}
\]

is contained in (3.63). If \( |\cdot| \) is not the trivial absolute value function on \( k \), then one can use this with \( r_1 = 1 \) and the homogeneity property (3.52) of norms to get that there is a \( C > 0 \) such that (3.62) holds for every \( v \in V \).

Let \( V \) be a vector space over \( k \) again, and let \( N \) be a norm on \( V \). If \( V \) is not complete as a metric space with respect to the corresponding metric (3.54), then one can define its completion in the usual way. The vector space operations and norm \( N \) can be extended to the completion, in such a way that the completion becomes a vector space over \( k \), the extension of the norm to the completion is also a norm, and \( V \) is a dense linear subspace of the completion. If \( V \) is complete and \( k \) is not complete with respect to the metric associated to \( |\cdot| \), then one can extend scalar multiplication on \( V \) to the completion of \( k \), so that \( V \) becomes a vector space over the completion of \( k \). One can also check that \( N \) will still be a norm on \( V \) as a vector over the completion of \( k \), which is to say that (3.52) holds for \( t \) in the completion of \( k \).

3.7 Finite-dimensional vector spaces

Let \( k \) be a field, and let \( |\cdot| \) be an absolute value function on \( k \). Also let \( n \) be a positive integer, and let \( N \) be a norm on \( k^n \). If \( N_0 \) is the norm (3.58) on \( k^n \), then we have seen that

\[
N(v) \leq C_1 N_0(v)
\]
for some $C_1 > 0$ and every $v \in k^n$, as in (3.60) and (3.61). Under certain conditions, we would like to show that there is a $C_2 > 0$ such that

\begin{equation}
N_0(v) \leq C_2 N(v)
\end{equation}

for every $v \in k^n$. This would imply that the topologies on $k^n$ determined by the metrics associated to $N$ and $N_0$ as in (3.54) are the same, as in the previous section.

Suppose for the moment that $k$ is locally compact with respect to the metric associated to $| \cdot |$, and that $| \cdot |$ is not the trivial absolute value function on $k$. This implies that every closed ball in $k$ is compact, as in Section 2.11. Note that a closed ball in $k^n$ with respect to $N_0$ with radius $r > 0$ is the same a Cartesian product of $n$ closed balls in $k$ with respect to $| \cdot |$ with radius $r$, by the definition of $N_0$. Thus closed balls in $k^n$ with respect to $N_0$ are also compact with respect to the product topology on $k^n$, because products of compact sets are compact.

Remember that $| \cdot |$ is continuous as a real-valued function on $k$, with respect to the topology on $k$ determined by the metric corresponding to $| \cdot |$, and the standard topology on $\mathbb{R}$. This implies that $N_0$ is continuous as a real-valued function on $k^n$, with respect to the product topology on $k^n$, and the standard topology on $\mathbb{R}$. This could also be derived from the analogue of (3.56) for $N_0$. It follows that

\begin{equation}
\{ v \in k^n : N_0(v) = 1 \}
\end{equation}

is a closed set in $k^n$ with respect to the product topology. This implies that (3.67) is a compact subset of $k^n$ with respect to the product topology, because closed subsets of compact sets are compact.

It is easy to see that $N$ is also continuous as a real-valued function on $k^n$, with respect to the product topology on $k^n$ and the standard topology on $\mathbb{R}$, using (3.56) and (3.65). This implies that $N$ attains its minimum on (3.67), because (3.67) is nonempty and compact. Let $c$ be the minimum value of $N$ on (3.67), so that $c > 0$, because $N(v) > 0$ when $v \neq 0$. We would like to check that

\begin{equation}
c N_0(v) \leq N(v)
\end{equation}

for every $v \in k^n$ under these conditions. More precisely, (3.68) is trivial when $v = 0$, and (3.68) holds by definition of $c$ when $N_0(v) = 1$. If $v \in k^n$ and $v \neq 0$, then there is a $t \in k$ such that $|t| = N_0(v) > 0$, by the definition (3.58) of $N_0(v)$. Thus $N_0(t^{-1} v) = 1$, which implies that $N(t^{-1} v) \geq c$, and hence (3.68), by (3.52). Of course, (3.66) is the same as (3.68), with $C_2 = 1/c$.

Now suppose that $k$ is complete, and let us show that there is a $C_2 > 0$ such that (3.66) holds for every $v \in k^n$. To do this, we use induction on $n$. The $n = 1$ case is very easy, using the homogeneity property (3.52) of norms. Thus we let an integer $n \geq 2$ be given such that the statement holds for $n - 1$, and we would like to prove the analogous statement for $n$.

Let $N$ be a norm on $k^n$, as before. Put

\begin{equation}
L = \{ v \in k^n : v_n = 0 \},
\end{equation}
which is an \((n - 1)\)-dimensional linear subspace of \(k^n\) that can be identified with \(k^{n-1}\) in an obvious way. Note that the restriction of \(N_0\) to \(L\) corresponds exactly to the analogue of \(N_0\) on \(k^{n-1}\). Thus the induction hypothesis can be applied to the restriction of \(N\) to \(L\), to get that there is a \(C'_2 > 0\) such that

\[(3.70)\quad N_0(v) \leq C'_2 N(v)\]

for every \(v \in L\).

It is easy to see that \(k^n\) is complete with respect to the metric associated to \(N_0\) when \(k\) is complete. Similarly, \(L\) is complete with respect to the metric associated to the restriction of \(N_0\) to \(L\). One can check that \(L\) is also complete with respect to the metric associated to the restriction of \(N\) to \(L\), using (3.65) and (3.70). It follows from this that \(L\) is a closed set in \(k^n\) with respect to the metric associated to \(N\). More precisely, if a sequence \(\{v(l)\}_{l=1}^{\infty}\) of elements of \(L\) converges to an element \(v\) of \(k^n\) with respect to this metric, then \(\{v(l)\}_{l=1}^{\infty}\) is a Cauchy sequence in \(L\). The completeness of \(L\) implies that \(\{v(l)\}_{l=1}^{\infty}\) already converges to an element \(v'\) of \(L\), so that \(v = v'\), because the limit of a convergent sequence in a metric space is unique. In particular, \(v \in L\), as desired.

Let \(e(n)\) be the element of \(k^n\) whose \(j\)th coordinate is equal to 0 when \(j < n\), and whose \(n\)th coordinate is equal to 1, as in the previous section. Thus \(e(n) \notin L\), and hence there is a positive real number \(c\) such that

\[(3.71)\quad N(e(n) - w) \geq c\]

for every \(w \in L\), because \(L\) is a closed set in \(k^n\) with respect to the metric associated to \(N\). Equivalently, \(N(v) \geq c\) for every \(v \in k^n\) such that \(v_n = 1\). This implies that

\[(3.72)\quad N(v) \geq c |v_n|\]

for every \(v \in k^n\) with \(v_n \neq 0\), by the homogeneity property (3.52) of norms. Of course, (3.72) is trivial when \(v_n = 0\).

Let \(v \in k^n\) be given, and observe that

\[(3.73)\quad N(v - v_n e(n)) \leq N(v) + |v_n| N(e(n)) \leq (1 + c^{-1} N(e(n))) N(v),\]

by (3.72). By construction, \(v - v_n e(n) \in L\), so that

\[(3.74)\quad N_0(v - v_n e(n)) \leq C'_2 N(v - v_n e(n)) \leq C'_2 (1 + c^{-1} N(e(n))) N(v),\]

by (3.70). Note that

\[(3.75)\quad N_0(v - v_n e(n)) = \max(|v_1|, \ldots, |v_{n-1}|),\]

which implies that \(N_0(v) = \max(N_0(v - v_n e(n)), |v_n|)\). It follows that

\[(3.76)\quad N_0(v) \leq \max(C'_2 (1 + c^{-1} N(e(n))), |v_n|) N(v)\]

for every \(v \in k^n\), as desired, by (3.72) and (3.74).
3.8 Some related facts

Let $k$ be a field, and let $| \cdot |$ be an absolute value function on $k$. Also let $V$ be a vector space over $k$, and let $N$ be a norm on $V$. If $k$ is complete with respect to the metric associated to $| \cdot |$, and if $V$ has finite dimension as a vector space over $k$, then $V$ is also complete with respect to the metric associated to $N$. Of course, this is trivial when $V = \{0\}$, and otherwise one can reduce to the case where $V = k^n$ for some positive integer $n$. It is easy to see that $k^n$ is complete with respect to the metric associated to the norm $N_0$ in (3.58), as mentioned in the previous section, and completeness with respect to the metric associated to any other norm $N$ can be derived from (3.65) and (3.66).

Suppose that $| \cdot |$ is nontrivial on $k$, and that $k$ is locally compact. If $V$ is finite-dimensional, then $V$ is locally compact with respect to the topology determined by the metric associated to $N$, and in fact closed balls in $V$ with respect to $N$ are compact with respect to the corresponding topology. To see this, it suffices to consider the case where $V = k^n$, as before. We have already seen that closed balls in $k^n$ with respect to $N_0$ are compact, as in the previous section. This implies that closed balls with respect to $N$ are compact, by (3.65) and (3.66), and because closed subsets of compact sets are compact.

If $| \cdot |$ is the trivial absolute value function on $k$, then $N_0$ is the trivial ultranorm on $k^n$ for each positive integer $n$. If $N$ is any other norm on $k^n$, then we still have (3.65) and (3.66), because $k$ is complete. Of course, the proof in the previous section could be simplified in this case. It follows that the topology on $k^n$ determined by the metric associated to $N$ is still the discrete topology. There is an analogous statement for arbitrary finite-dimensional vector spaces over $k$, as before.

Let $| \cdot |$ be an absolute value function on a field $k$ again, and let $N$ be a norm on a vector space $V$ over $k$, which is not necessarily finite-dimensional. If $k$ is complete and $W$ is a finite-dimensional linear subspace of $V$, then $W$ is complete with respect to the restriction to $W$ of the metric associated to $N$, as before. This implies that $W$ is a closed set in $V$ with respect to the topology determined by the metric associated to $N$. This follows from a standard argument that was also used in the previous section.

Let $W$ be a linear subspace of $V$ again. Suppose that there is a positive real number $c < 1$ such that for each $v \in V$ there is a $w \in W$ such that

\[
N(v - w) \leq c N(v).
\]

(3.77)

Applying this property to $v - w$ and repeating the process, one can check that $W$ is dense in $V$ with respect to the metric associated to $N$. If $W$ is a closed set in $V$ with respect to the topology determined by this metric, then it follows that $V = W$. In particular, this holds when $k$ is complete and $W$ is finite-dimensional, as in the preceding paragraph.

Let us now restrict our attention to the case where $| \cdot |$ is nontrivial on $k$. Thus there is a $t_0 \in k$ such that $t_0 \neq 0$ and $|t_0| \neq 1$. As usual, we may as well ask that $|t_0| > 1$, since otherwise we could replace $t_0$ with $1/t_0$. 

3.9. HAAR MEASURE, 3

Suppose that $V$ is locally compact with respect to the topology determined by the metric associated to $N$. This implies that every closed ball in $V$ with respect to $N$ is compact, because $|\cdot|$ is nontrivial on $k$, as in Section 2.11. In particular, the closed unit ball in $V$ is compact, and hence totally bounded. Thus for each $\epsilon > 0$ the closed unit ball in $V$ with respect to $N$ can be covered by finitely many closed balls of radius $\epsilon$. Let us apply this with $\epsilon = 1/(2|t_0|)$, to get finitely many vectors $w_1,\ldots,w_n$ in $V$ such that for each $u \in V$ with $N(u) \leq 1$ there is a $j \in \{1,\ldots,n\}$ that satisfies

$$N(u - w_j) \leq 1/(2|t_0|).$$

(3.78)

Let $W$ be the linear span of $w_1,\ldots,w_n$ in $V$, and let $v \in V$ be given. We would like to show that there is a $w \in W$ that satisfies (3.77) with $c = 1/2$. Of course, this is trivial when $v = 0$, and so we may suppose that $v \neq 0$. Let $l$ be the unique integer such that $|t_0|^{l-1} < N(v) \leq |t_0|^l$. If $u = t_0^{-l}v$, then $N(u) \leq 1$, and hence there is a $j \in \{1,\ldots,n\}$ that satisfies (3.78). It follows that

$$N(v - t_0^l w_j) = |t_0|^l N(u - w_j) \leq |t_0|^{l-1}/2 < N(v)/2.$$

(3.79)

This is exactly what we wanted, since $t_0^l w_j \in W$.

This implies that $W$ is dense in $V$, as before. If $k$ is complete, then $V = W$, because $W$ is finite-dimensional, by construction. This shows that $V$ is finite-dimensional when $V$ is locally compact, $|\cdot|$ is nontrivial on $k$, and $k$ is complete.

3.9 Haar measure, 3

Let $X$ be a locally compact Hausdorff topological space, and let $W$ be an open subset of $X$. It is well known that $X$ is regular as a topological space under these conditions, so that for each $p \in W$ there is an open subset $V(p)$ of $X$ such that $p \in V(p)$ and the closure $\overline{V(p)}$ of $V(p)$ in $X$ is contained in $W$. Because $X$ is locally compact, one can also choose $V(p)$ so that $\overline{V(p)}$ is compact for each $p \in W$. If there is also a base for the topology of $X$ with only finitely or countably many elements, then Lindelöf’s theorem implies that $W$ can be expressed as the union of finitely or countably many of the sets $V(p)$. In this case, it follows that $W$ is $\sigma$-compact, which means that $W$ can be expressed as the union of finitely or countably many compact sets, because $W$ is the union of finitely or countably many of the sets $\overline{V(p)}$.

Now let $n$ be a positive integer, and let $A_1, A_2,\ldots,A_n$ be $n$ commutative topological groups. The Cartesian product

$$A = \prod_{j=1}^n A_j$$

(3.80)

is also a commutative group, where the group operations on $A$ are defined coordinatewise. If $A$ is equipped with the product topology corresponding to the given topologies on the $A_j$’s, then it is easy to see that $A$ is a topological
group as well. Suppose from now on that \( A_j \) is locally compact for each \( j \), which implies that \( A \) is locally compact too. This uses the fact that Cartesian products of compact subsets of the \( A_j \)'s are compact subsets of \( A \), with respect to the product topology.

Under these conditions, there is a Haar measure \( H_j \) on \( A_j \) for each \( j \). To get a Haar measure \( H \) on \( A \), one can basically use a suitable product measure construction, but there are some details involved with this. Suppose first that there is a base \( B_j \) for the topology of \( A_j \) with only finitely or countably many elements for each \( j \). Let \( B \) be the collection of sets \( U \subseteq A \) of the form

\[
U = \prod_{j=1}^{n} U_j,
\]

(3.81)

where \( U_j \in B_j \) for each \( j \). It is easy to see that \( B \) is a base for the product topology on \( A \), and that \( B \) has only finitely or countably many elements.

In this situation, \( A_j \) is \( \sigma \)-compact for each \( j \), by the remarks at the beginning of the section. This implies that \( A_j \) is \( \sigma \)-finite with respect to \( H_j \) for each \( j \), because compact sets have finite Haar measure. This permits one to apply the standard construction of product measures to the \( H_j \)'s, as Borel measures on the \( A_j \)'s. This leads to a product measure \( H \), defined on a suitable \( \sigma \)-algebra of subsets of \( A \). In particular, if \( E \subseteq A \) is a product of Borel subsets of the \( A_j \)'s, then \( E \) is measurable with respect to the product measure construction.

Of course, open subsets of the \( A_j \)'s are Borel sets, by definition, so that products of open subsets of the \( A_j \)'s are measurable with respect to the product measure construction. Thus the elements of the base \( B \) for the topology of \( A \) mentioned earlier are all measurable with respect to the product measure construction. Remember that every open set in \( A \) can be expressed as a union of elements of \( B \), and that \( B \) has only finitely or countably many elements. It follows that every open set in \( A \) is measurable with respect to the product measure construction, since it can be expressed as a union of finitely or countably many measurable sets. This implies that Borel subsets of \( A \) are measurable with respect to the product measure construction under these conditions.

Thus the product measure \( H \) may be considered as a Borel measure on \( A \). One can also check that \( H \) is invariant under translations on \( A \), because \( H_j \) is invariant under translations on \( A_j \) for each \( j \). If \( V \) is a nonempty open subset of \( A \), then \( V \) contains a product of nonempty open subsets of the \( A_j \)'s, by definition of the product topology. This implies that \( H(V) > 0 \), because of the corresponding property of the \( H_j \)'s, which is one of the requirements of a Haar measure. If \( K \) is a compact subset of \( A \), then the projection of \( K \) in \( A_j \) is compact for each \( j \), because the projection mappings are continuous. Of course, \( K \) is contained in the Cartesian product of its projections in the \( A_j \)'s. This implies that \( H(K) \) is finite, because of the corresponding property of the \( H_j \)'s. Note that open subsets of \( A \) are \( \sigma \)-compact in this situation, by the remarks at the beginning of the section. Because of this, it is well known that \( H \) satisfies the regularity conditions required of a Haar measure.

Alternatively, there is another product measure construction for nonnegative
3.10. SUMMABLE FUNCTIONS

Let $X$ be a nonempty set, and let $f(x)$ be a nonnegative real-valued function on $X$. Thus

\[ \sum_{x \in A} f(x) \]  

is defined for every nonempty finite subset $A$ of $X$, and the sum

\[ \sum_{x \in X} f(x) \]

can be defined as the supremum of (3.82) over all nonempty finite subsets $A$ of $X$. More precisely, the supremum is finite when there is a finite upper bound for the subsums (3.82), and otherwise (3.83) is equal to $+\infty$. If $g$ is another
nonnegative real-valued function on \( X \), then one can check that
\[
\sum_{x \in X} (f(x) + g(x)) = \sum_{x \in X} f(x) + \sum_{x \in X} g(x),
\]
where the right side of (3.84) is interpreted as being \(+\infty\) when either of the individual sums is equal to \(+\infty\). Similarly, if \( a \) is a nonnegative real number, then
\[
\sum_{x \in X} a f(x) = a \sum_{x \in X} f(x),
\]
where the right side of (3.85) is interpreted as being \(+\infty\) when (3.83) is \(+\infty\) and \( a > 0 \), and equal to 0 when \( a = 0 \), even when (3.83) is infinite.

A nonnegative real-valued function \( f(x) \) on \( X \) is said to be summable on \( X \) if (3.83) is finite. Let \( \epsilon > 0 \) be given, and put
\[
E(f, \epsilon) = \{ x \in X : f(x) \geq \epsilon \}.
\]
If \( A \) is a nonempty finite subset of \( E(f, \epsilon) \), then
\[
\epsilon (#A) \leq \sum_{x \in A} f(x) \leq \sum_{x \in X} f(x),
\]
where \(#A\) denotes the number of elements of \( A \). If \( f \) is summable on \( X \), then it follows that \( E(f, \epsilon) \) has only finitely many elements, and that
\[
#E(f, \epsilon) \leq \epsilon^{-1} \sum_{x \in X} f(x)
\]
for each \( \epsilon > 0 \). In particular, if \( f \) is summable on \( X \), then the set of \( x \in X \) such that \( f(x) \neq 0 \) has only finitely or countably many elements, since it can be expressed as the union of \( E(f, 1/n) \) over all positive integers \( n \).

Now let \( k \) be a field, and let \( V \) be a vector space over \( k \). Also let \( | \cdot | \) be an absolute value function on \( k \), and let \( N \) be a norm on \( V \) with respect to \( | \cdot | \), as in Section 3.6. Under these conditions, a \( V \)-valued function \( f(x) \) on \( X \) is said to be summable if \( N(f(x)) \) is summable on \( X \) as a nonnegative real-valued function on \( X \), as in the preceding paragraph. Of course, one can consider \( V = k \) as a one-dimensional vector space over \( k \), and \( | \cdot | \) as a norm on this vector space, so that this definition applies to \( k \)-valued functions on \( X \) in particular.

Let \( \ell^1(X, V) \) be the space of summable \( V \)-valued functions on \( X \), in the sense just described, and put
\[
\|f\|_1 = \|f\|_{\ell^1(X, V)} = \sum_{x \in X} N(f(x))
\]
for every \( f \in \ell^1(X, V) \). If \( f, g \in \ell^1(X, V) \), then it is easy to see that \( f + g \) is summable on \( X \) too, and that
\[
\|f + g\|_1 = \sum_{x \in X} N(f(x) + g(x)) \leq \sum_{x \in X} (N(f(x)) + N(g(x))) = \|f\|_1 + \|g\|_1.
\]
Similarly, if $f \in \ell^1(X, V)$ and $t \in k$, then $t f(x)$ is summable on $X$ as well, and

$$
(3.91) \quad \|t f\|_1 = \sum_{x \in X} N(t f(x)) = \sum_{x \in X} |t| N(f(x)) = |t| \|f\|_1.
$$

This shows that $\ell^1(X, V)$ is a vector space over $k$ with respect to pointwise addition and scalar multiplication, and that $\|f\|_1$ is a norm on $\ell^1(X, V)$.

If $f$ is any $V$-valued function on $X$, then the support of $f$ is defined by

$$
(3.92) \quad \text{supp } f = \{ x \in X : f(x) \neq 0 \}.
$$

Let $c_{00}(X, V)$ be the collection of $V$-valued functions $f$ on $X$ such that supp $f$ has only finitely many elements. This is a vector space over $k$ with respect to pointwise addition and scalar multiplication, and a linear subspace of $\ell^1(X, V)$. If $f$ is summable on $X$, then we have seen that the support of $N(f(x))$ has only finitely or countably many elements, which means that supp $f$ has the same property. If $| \cdot |$ is the trivial absolute value function on $k$, and $N$ is the trivial ultranorm on $V$, then $f$ is summable on $V$ if and only if supp $f$ has only finitely many elements.

Suppose that $f$ is a summable $V$-valued function on $X$, and let $\epsilon > 0$ be given. Thus $N(f(x))$ is summable as a nonnegative real-valued function on $X$, and hence there is a finite set $A(\epsilon) \subseteq X$ such that

$$
(3.93) \quad \sum_{x \in X} N(f(x)) < \sum_{x \in A(\epsilon)} N(f(x)) + \epsilon,
$$

by the definition of the sum on the left side of (3.93). If we put $f_\epsilon(x) = f(x)$ when $x \in A(\epsilon)$, and $f_\epsilon(x) = 0$ otherwise, then $f_\epsilon \in c_{00}(X, V)$, and

$$
(3.94) \quad \|f - f_\epsilon\|_1 = \sum_{x \in X \setminus A(\epsilon)} N(f(x)) < \epsilon.
$$

This uses (3.93) in the second step, and it follows that $c_{00}(X, V)$ is dense in $\ell^1(X, V)$ with respect to the metric associated to the $\ell^1$ norm.

If $f \in c_{00}(X, V)$, then

$$
(3.95) \quad \sum_{x \in X} f(x)
$$

can be defined as an element of $V$ in the obvious way, and satisfies

$$
(3.96) \quad N\left( \sum_{x \in X} f(x) \right) \leq \sum_{x \in X} N(f(x)) = \|f\|_1.
$$

Of course,

$$
(3.97) \quad f \mapsto \sum_{x \in X} f(x)
$$

defines a linear mapping from $c_{00}(X, V)$ into $V$, so that

$$
(3.98) \quad N\left( \sum_{x \in X} f(x) - \sum_{x \in X} g(x) \right) = N\left( \sum_{x \in X} (f(x) - g(x)) \right) \leq \|f - g\|_1
$$
for every \( f, g \in c_{00}(X, V) \), as in (3.96). This implies that (3.97) is uniformly continuous with respect to the metric associated to the \( \ell^1 \) norm on \( c_{00}(X, V) \), and the metric associated to \( N \) on \( V \).

If \( V \) is complete with respect to the metric associated to \( N \), then there is a unique extension of (3.97) to a uniformly continuous mapping from \( \ell^1(X, V) \) into \( V \), with respect to the metric associated to the \( \ell^1 \) norm on \( \ell^1(X, V) \), and the metric associated to \( N \) on \( V \). This follows from the discussion in Section 1.3, and uses the fact that \( c_{00}(X, V) \) is dense in \( \ell^1(X, V) \). In this situation, it is easy to see that this extension of (3.97) to \( \ell^1(X, V) \) is also linear, and satisfies (3.96). At any rate, one can use this to define (3.95) as an element of \( V \) when \( f \in \ell^1(X, V) \) and \( V \) is complete.

Alternatively, if \( f \in \ell^1(X, V) \), then we have seen that \( \text{supp } f \) has only finitely or countably many elements. If \( \text{supp } f \) has only finitely many elements, then we already know how to define (3.95), and so we may as well suppose that \( \text{supp } f \) is countably infinite. Thus the elements of \( \text{supp } f \) may be enumerated by a sequence, and (3.95) may be identified with an infinite series with terms in \( V \). The summability of \( f \) implies that this series is absolutely convergent, and hence that the partial sums form a Cauchy sequence in \( V \), as in Section 2.6. If \( V \) is complete, then it follows that the sequence of partial sums converges in \( V \). Any other enumeration of the elements of \( \text{supp } f \) would lead to a rearrangement of the same series, and one can check that the corresponding sums would be the same, using absolute convergence. Of course, this approach to the definition of (3.95) for \( f \in \ell^1(X, V) \) is equivalent to the one described in the preceding paragraph.

Let us continue to suppose that \( V \) is complete, and verify that \( \ell^1(X, V) \) is complete with respect to the \( \ell^1 \) metric. Let \( \{f_j\}_{j=1}^\infty \) be a Cauchy sequence in \( \ell^1(X, V) \), so that for each \( \epsilon > 0 \) there is an \( L(\epsilon) \geq 1 \) such that

\[
(3.99) \quad \|f_j - f_l\|_1 < \epsilon
\]

for every \( j, l \geq L(\epsilon) \). In particular,

\[
(3.100) \quad N(f_j(x) - f_l(x)) \leq \|f_j - f_l\|_1 < \epsilon
\]

for every \( x \in X \) and \( j, l \geq L(\epsilon) \), which implies that \( \{f_j(x)\}_{j=1}^\infty \) is a Cauchy sequence in \( V \) for every \( x \in X \). If \( V \) is complete, then it follows that \( \{f_j(x)\}_{j=1}^\infty \) converges to an element of \( V \) for every \( x \in X \), and we let \( f(x) \) denote the limit of this sequence. We would like to show that \( f \in \ell^1(X, V) \), and that \( \{f_j\}_{j=1}^\infty \) converges to \( f \) with respect to the \( \ell^1 \) norm.

If \( A \) is any nonempty finite subset of \( X \), then

\[
(3.101) \quad \sum_{x \in A} N(f_j(x) - f(x)) = \lim_{l \to \infty} \sum_{x \in A} N(f_j(x) - f_l(x))
\]

for every \( j \), because \( N \) is continuous on \( V \) with respect to the metric associated to itself. Combining this with (3.99), we get that

\[
(3.102) \quad \sum_{x \in A} N(f_j(x) - f(x)) \leq \epsilon
\]
for every $j \geq L(\epsilon)$. This implies that

\[(3.103) \quad \sum_{x \in X} N(f_j(x) - f(x)) \leq \epsilon\]

for every $j \geq L(\epsilon)$, by taking the supremum over all finite subsets $A$ of $X$ in (3.102). In particular, one can use this to get that $f$ is summable on $X$, because $f_j$ is summable on $X$ for every $j$, by hypothesis. It follows easily that $\{f_j\}_{j=1}^\infty$ converges to $f$ with respect to the $\ell^1$ norm, as desired, using (3.103) again.

### 3.11 Vanishing at infinity

As in the previous section, we let $X$ be a nonempty set, $k$ be a field, and $V$ be a vector space over $k$. We also let $| \cdot |$ be an absolute value function on $k$, and $N$ be a norm on $V$ with respect to $| \cdot |$. Under these conditions, a $V$-valued function $f$ on $X$ is said to be bounded if $N(f(x))$ is bounded on $X$, as a nonnegative real-valued function on $X$. It is easy to see that the space $\ell^\infty(X, V)$ of bounded $V$-valued functions on $X$ is a vector space over $k$ with respect to pointwise addition and scalar multiplication. If $f \in \ell^\infty(X, V)$, then we put

\[(3.104) \quad \|f\|_\infty = \|f\|_{\ell^\infty(X, V)} = \sup_{x \in X} N(f(x)),\]

which is easily seen to be a norm on $\ell^\infty(X, V)$.

A $V$-valued function $f$ on $X$ is said to vanish at infinity on $X$ if for each $\epsilon > 0$,

\[(3.105) \quad N(f(x)) < \epsilon\]

for all but finitely many $x \in X$. Let $c_0(X, V)$ be the collection of $V$-valued functions on $X$ that vanish at infinity. It is easy to see that $c_0(X, V)$ is a linear subspace of $\ell^\infty(X, V)$, and that the space $c_00(X, V)$ of $V$-valued functions on $X$ with finite support is a linear subspace of $c_0(X, V)$. One can also check that $c_0(X, V)$ is a closed set in $\ell^\infty(X, V)$ with respect to the metric associated to the $\ell^\infty$ norm. If $| \cdot |$ is the trivial absolute value function on $k$, and $N$ is the trivial ultranorm on $V$, then a $V$-valued function $f$ on $X$ vanishes at infinity if and only if $\text{supp } f$ has only finitely many elements.

Let $f$ be a $V$-valued function on $X$ that vanishes at infinity, and let $\epsilon > 0$ be given. Put $f_\epsilon(x) = f(x)$ when $N(f(x)) \geq \epsilon$ and $f_\epsilon(x) = 0$ otherwise, so that $\text{supp } f_\epsilon$ has only finitely many elements, by hypothesis. Thus $f_\epsilon \in c_00(X, V)$, and it is easy to see that

\[(3.106) \quad \|f - f_\epsilon\|_\infty \leq \epsilon,\]

by construction. This shows that $c_00(X, V)$ is dense in $c_0(X, V)$, with respect to the metric associated to the $\ell^\infty$ metric.

If $V$ is complete with respect to the metric associated to $N$, then $\ell^\infty(X, V)$ is complete with respect to the metric corresponding to the $\ell^\infty$ norm. In fact, it is well known that the space of bounded mappings from $X$ into any complete metric space is itself a complete metric space with respect to the supremum
CHAPTER 3. ADDITIONAL STRUCTURE

It follows that $c_0(X, V)$ is complete with respect to the $\ell^\infty$ norm when $V$ is complete, because a closed subset of a complete metric space is also complete, with respect to the restriction of the metric to the subset.

Let us suppose from now on in this section that $|\cdot|$ is an ultrametric absolute value function on $k$, and that $N$ is an ultranorm on $V$ with respect to $|\cdot|$. It is easy to see that the $\ell^\infty$ norm is an ultranorm on $\ell^\infty(X, V)$ under these conditions. If $f \in c_0(X, V)$, then

$$\sum_{x \in X} f(x) \tag{3.107}$$

can be defined in the usual way, and satisfies

$$N\left( \sum_{x \in X} f(x) \right) \leq \max_{x \in X} N(f(x)) = \|f\|_\infty, \tag{3.108}$$

by the ultrametric version of the triangle inequality. As before,

$$f \mapsto \sum_{x \in X} f(x) \tag{3.109}$$

is a linear mapping from $c_0(X, V)$ into $V$, and in this case we have that

$$N\left( \sum_{x \in X} f(x) - \sum_{x \in X} g(x) \right) = N\left( \sum_{x \in X} (f(x) - g(x)) \right) \leq \|f - g\|_\infty \tag{3.110}$$

for every $f, g \in c_0(X, V)$, as in (3.108). It follows that (3.109) is uniformly continuous with respect to the metric associated to the $\ell^\infty$ norm on $c_0(X, V)$, and the metric associated to $N$ on $V$.

If $V$ is complete with respect to the metric associated to $N$, then there is a unique extension of (3.109) to a uniformly continuous mapping from $c_0(X, V)$ into $V$, with respect to the metric on $c_0(X, V)$ associated to the $\ell^\infty$ norm, and the metric on $V$ associated to $N$. This follows from the discussion in Section 1.3, and uses the fact that $c_0(X, V)$ is dense in $c_0(X, V)$. As in the previous section, it is easy to see that this extension of (3.109) to $c_0(X, V)$ is linear, and that it also satisfies (3.108). As before, this can be used to define (3.107) when $f \in c_0(X, V)$ and $V$ is complete.

If $f \in c_0(X, V)$, then for each positive integer $n$, $N(f(x)) \geq 1/n$ for at most finitely many $x \in X$. This implies that $f(x) \neq 0$ for at most finitely or countably many $x \in X$, by taking the union over $n \geq 1$. If $f(x) \neq 0$ for only finitely many $x \in X$, then (3.107) can be defined in the usual way. Otherwise, the support of $f$ is countably infinite, so that its elements may be enumerated by an infinite sequence. Thus (3.107) can be identified with an infinite series, as in the previous section. In this case, the terms of the series converge to 0, which implies that the corresponding sequence of partial sums is a Cauchy sequence in $V$, because of the ultrametric version of the triangle inequality. This is similar to the ultrametric case in Section 2.6, and it follows that the sequence of partial sums converges in $V$ when $V$ is complete. One can also check that
rearrangements of the series have the same sum under these conditions, so that the value of the sum does not depend on the particular enumeration of the elements of the support of $f$. As before, this approach to the definition of (3.107) for $f \in c_0(X,V)$ is equivalent to the one described in the preceding paragraph.

### 3.12 Double sums

Let $X_1$, $X_2$ be nonempty sets, and let $X = X_1 \times X_2$ be their Cartesian product. Also let $f(x_1, x_2)$ be a nonnegative real-valued function on $X$, so that the sum

$$(3.111) \quad \sum_{(x_1, x_2) \in X} f(x_1, x_2)$$

can be defined as in Section 3.10. Similarly,

$$(3.112) \quad \sum_{x_2 \in X_2} f(x_1, x_2)$$

is defined as a nonnegative extended real number for each $x_1 \in X_1$, and

$$(3.113) \quad \sum_{x_1 \in X_1} f(x_1, x_2)$$

is defined as a nonnegative extended real number for each $x_2 \in X_2$. This permits the iterated sums

$$(3.114) \quad \sum_{x_1 \in X_1} \left( \sum_{x_2 \in X_2} f(x_1, x_2) \right)$$

and

$$(3.115) \quad \sum_{x_2 \in X_2} \left( \sum_{x_1 \in X_1} f(x_1, x_2) \right)$$

to be defined in essentially the same way as before. More precisely, if (3.112) is equal to $+\infty$ for any $x_1 \in X_1$, then (3.114) is interpreted as being $+\infty$, and otherwise (3.114) may be defined as in Section 3.10. The sum (3.115) is defined analogously, and one can show that (3.111), (3.114), and (3.115) are equal to each other under these conditions. One can start by checking that (3.111) and (3.114) are each less than or equal to the other, by considering approximations to these sums by finite subsums. This implies that (3.111) and (3.114) are equal to each other, and the equality of (3.111) and (3.115) is similar.

Now let $k$ be a field, and let $V$ be a vector space over $k$. If $f(x_1, x_2)$ is a $V$-valued function on $X$ with finite support, then all of the sums mentioned in the preceding paragraph can be defined as elements of $V$. More precisely, (3.112) is equal to 0 for all but finitely many $x_1 \in X_1$, and (3.113) is equal to 0 for all but finitely many $x_2 \in X_2$, so that (3.114) and (3.115) are defined. Of course, (3.111), (3.114), and (3.115) are equal under these conditions. This is basically the same as the case where $X_1$ and $X_2$ are both finite sets.
Suppose that $| \cdot |$ is an absolute value function on $k$, $N$ is a norm on $V$ with respect to $| \cdot |$, and that $V$ is complete with respect to the metric corresponding to $N$. Let $f(x_1, x_2)$ be a summable $V$-valued function on $X$, so that $N(f(x_1, x_2))$ is summable as a nonnegative real-valued function on $X$. Thus (3.111) is defined as an element of $V$, as in Section 3.10. Note that $f(x_1, x_2)$ is also summable as a $V$-valued function of $x_2 \in X_2$ for every $x_1 \in X_1$, and as a $V$-valued function of $x_1 \in X_1$ for every $x_2 \in X_2$. This implies that (3.112) is defined as an element of $V$ for every $x_1 \in X_1$, and that (3.113) is defined as an element of $V$ for every $x_2 \in X_2$. We also have that

\[
(3.116) \quad N\left( \sum_{x_2 \in X_2} f(x_1, x_2) \right) \leq \sum_{x_2 \in X_2} N(f(x_1, x_2))
\]

for every $x_1 \in X_1$, and that

\[
(3.117) \quad N\left( \sum_{x_1 \in X_1} f(x_1, x_2) \right) \leq \sum_{x_1 \in X_1} N(f(x_1, x_2))
\]

for every $x_2 \in X_2$, as in (3.96). It follows that

\[
(3.118) \quad \sum_{x_1 \in X_1} N\left( \sum_{x_2 \in X_2} f(x_1, x_2) \right) \leq \sum_{x_1 \in X_1} \left( \sum_{x_2 \in X_2} N(f(x_1, x_2)) \right),
\]

and that

\[
(3.119) \quad \sum_{x_2 \in X_2} N\left( \sum_{x_1 \in X_1} f(x_1, x_2) \right) \leq \sum_{x_2 \in X_2} \left( \sum_{x_1 \in X_1} N(f(x_1, x_2)) \right).
\]

Thus (3.112) is summable as a $V$-valued function of $x_1 \in X_1$, and (3.113) is summable as a $V$-valued function of $x_2 \in X_2$, by the remarks about nonnegative real-valued functions on $X$ at the beginning of the section. This implies that (3.114) and (3.115) are defined as elements of $V$, as in Section 3.10. One can check that (3.111), (3.114), and (3.115) are equal to each other under these conditions. This follows from the remarks in the preceding paragraph when $f$ has finite support in $X$, and otherwise one can approximate $f$ by functions with finite support in $X$ with respect to the $\ell^1$ norm, as in Section 3.10.

Let us now restrict our attention to the case where $| \cdot |$ is an ultrametric absolute value function on $k$, and $N$ is an ultranorm on $V$. We continue to ask that $V$ be complete with respect to the ultrametric corresponding to $N$. If $f(x_1, x_2)$ is a $V$-valued function on $X$ that vanishes at infinity, then (3.111) can be defined as an element of $V$, as in the previous section. It is easy to see that $f(x_1, x_2)$ also vanishes at infinity as a $V$-valued function of $x_2 \in X_2$ for every $x_1 \in X_1$, and as a $V$-valued function of $x_1 \in X_1$ for every $x_2 \in X_2$. This implies that (3.112) is defined as an element of $V$ for every $x_1 \in X_1$, and that (3.113) is defined as an element of $V$ for every $x_2 \in X_2$, as in the previous section. We also have that

\[
(3.120) \quad N\left( \sum_{x_2 \in X_2} f(x_1, x_2) \right) \leq \max_{x_2 \in X_2} N(f(x_1, x_2))
\]
3.13. **GENERALIZED CONVERGENCE**

for every $x_1 \in X_1$, and that

\[(3.121) \quad N \left( \sum_{x_1 \in X_1} f(x_1, x_2) \right) \leq \max_{x_1 \in X_1} N(f(x_1, x_2)) \]

for every $x_2 \in X_2$, by (3.108). Using this, one can check that (3.112) vanishes at infinity as a $V$-valued function of $x_1 \in X_1$, and that (3.113) vanishes at infinity as a function of $x_2 \in X_2$, because $f(x_1, x_2)$ vanishes at infinity on $X$. This implies that (3.114) and (3.115) are defined as elements of $V$, as in the previous section. Moreover,

\[(3.122) \quad N \left( \sum_{x_1 \in X_1} \left( \sum_{x_2 \in X_2} f(x_1, x_2) \right) \right) \leq \max_{x_1 \in X_1} N \left( \sum_{x_2 \in X_2} f(x_1, x_2) \right) \leq \max_{(x_1, x_2) \in X} N(f(x_1, x_2)) \]

and

\[(3.123) \quad N \left( \sum_{x_2 \in X_2} \left( \sum_{x_1 \in X_1} f(x_1, x_2) \right) \right) \leq \max_{x_2 \in X_2} N \left( \sum_{x_1 \in X_1} f(x_1, x_2) \right) \leq \max_{(x_1, x_2) \in X} N(f(x_1, x_2)) \]

using (3.108) in the first steps in (3.122) and (3.123), and (3.120) and (3.121) in the second steps, respectively. As before, one can show that (3.111), (3.114), and (3.115) are the same under these conditions, by approximating $f$ by functions with finite support in $X$ with respect to the $\ell^\infty$ norm.

### 3.13 Generalized convergence

Let $X$ be a nonempty set, let $k$ be a field, and let $V$ be a vector space over $k$. Also let $| \cdot |$ be an absolute value function on $k$, and let $N$ be a norm on $V$ with respect to $| \cdot |$. If $f$ is a $V$-valued function on $X$ and $A$ is a finite subset of $X$, then the sum

\[(3.124) \quad \sum_{x \in A} f(x) \]

can be defined as an element of $V$ in the usual way, which is interpreted to be 0 when $A = \emptyset$. The family of these finite sums may be considered as a net of elements of $V$, indexed by the collection of finite subsets of $X$. More precisely, the collection of finite subsets of $X$ is partially ordered by inclusion. If $A_1$, $A_2$ are finite subsets of $X$, then their union $A_1 \cup A_2$ is a finite subset of $X$ that contains $A_1$ and $A_2$, which implies that the collection of finite subsets of $X$ is a directed system. The convergence of the sum

\[(3.125) \quad \sum_{x \in X} f(x) \]
in \( V \) can be defined in terms of the convergence of the corresponding net of finite subsums (3.124) in \( V \). This means that there is a \( v \in V \) with the property that for every \( \epsilon > 0 \) there is a finite set \( A(\epsilon) \subseteq X \) such that

\[
N \left( \sum_{x \in A} f(x) - v \right) < \epsilon
\]

for every finite set \( A \subseteq X \) with \( A(\epsilon) \subseteq A \). It is easy to see that the limit \( v \) of this net is unique when it exists, in which case the value of the sum (3.125) is defined to be \( v \).

Similarly, let us say that the sum (3.125) satisfies the generalized Cauchy criterion if for each \( \epsilon > 0 \) there is a finite set \( A_0(\epsilon) \subseteq X \) such that

\[
N \left( \sum_{x \in B} f(x) \right) < \epsilon
\]

for every finite set \( B \subseteq X \) with \( A_0(\epsilon) \cap B = \emptyset \). If the sum (3.125) converges in the sense described in the preceding paragraph, then one can check that it satisfies the generalized Cauchy criterion, in essentially the same way as for ordinary infinite series. More precisely, let \( A(\epsilon) \) be a finite subset of \( X \) for which (3.126) holds, and let \( B \subseteq X \) be a finite set that is disjoint from \( A(\epsilon) \). Thus (3.126) can be applied to \( A = A(\epsilon) \) and \( A = A(\epsilon) \cup B \), to get that

\[
N \left( \sum_{x \in B} f(x) \right) = N \left( \sum_{x \in A(\epsilon) \cup B} f(x) - \sum_{x \in A(\epsilon)} f(x) \right) \\
\leq N \left( \sum_{x \in A(\epsilon) \cup B} f(x) - v \right) + N \left( \sum_{x \in A(\epsilon)} f(x) - v \right) \\
< \epsilon + \epsilon = 2 \epsilon.
\]

This shows that (3.127) holds with \( A_0(\epsilon) = A(\epsilon/2) \), and one can take \( A_0(\epsilon) = A(\epsilon) \) when \( N \) is an ultranorm on \( V \).

Suppose that (3.125) satisfies the generalized Cauchy criterion, and let \( A_0(\epsilon) \) be as in the preceding paragraph. If \( x \in X \setminus A_0(\epsilon) \), then we can take \( B = \{x\} \) in (3.127), to get that

\[
N(f(x)) < \epsilon.
\]

This implies that \( f \) vanishes at infinity on \( X \). Conversely, if \( N \) is an ultranorm on \( V \), and if \( f \) vanishes at infinity on \( X \), then it is easy to see that (3.125) satisfies the generalized Cauchy criterion.

Let \( N \) be any norm on \( V \) again, and suppose that \( f \) is a summable \( V \)-valued function on \( X \), as in Section 3.10. Thus for each \( \epsilon > 0 \) there is a finite set \( A_0(\epsilon) \subseteq X \) such that

\[
N \left( \sum_{x \in X \setminus A_0(\epsilon)} f(x) \right) < \epsilon,
\]

as in (3.93). If \( B \subseteq X \) is a finite set that is disjoint from \( A_0(\epsilon) \), then we get that

\[
N \left( \sum_{x \in B} f(x) \right) \leq \sum_{x \in B} N(f(x)) \leq \sum_{x \in X \setminus A_0(\epsilon)} N(f(x)) < \epsilon.
\]
This shows that (3.125) satisfies the generalized Cauchy criterion under these conditions.

If (3.125) satisfies the generalized Cauchy criterion, then the norms of the finite subsums (3.124) are uniformly bounded, over all finite sets \( A \subseteq X \). More precisely, let \( A_0(1) \) be a finite subset of \( X \) such that (3.127) holds with \( \epsilon = 1 \) for all finite sets \( B \subseteq X \setminus A_0(1) \). If \( A \) is any finite subset of \( X \), then

\[
(3.132) \quad N\left( \sum_{x \in A} f(x) \right) \leq N\left( \sum_{x \in A \setminus A_0(1)} f(x) \right) + N\left( \sum_{x \in A \cap A_0(1)} f(x) \right) \\
\leq 1 + \sum_{x \in A_0(1)} N(f(x)),
\]

using (3.127) applied to \( B = A \setminus A_0(1) \) in the second step.

Suppose that \( k = \mathbb{R} \) with the standard absolute value function, and that \( V = \mathbb{R} \) with the standard absolute value function as the norm. If \( f \) is a real-valued function on \( X \) such that (3.125) satisfies the generalized Cauchy criterion, then the finite subsums (3.124) are uniformly bounded, as in the previous paragraph. In particular, this can be applied to finite sets \( A \subseteq X \) such that \( f(x) \geq 0 \) for every \( x \in X \), or such that \( f(x) \leq 0 \) for every \( x \in X \). Using this, one can check that \( f \) is actually summable on \( X \) in this case. The analogous statement also holds when \( k = V = \mathbb{C} \), with the standard absolute value function, by applying the previous argument to the real and imaginary parts of a complex-valued function on \( X \).

Let \( k, V \) be arbitrary again, and suppose that (3.125) satisfies the generalized Cauchy criterion. Thus \( f \) vanishes at infinity on \( X \), as before, which implies that the support of \( f \) has only finitely or countably many elements. If the support of \( f \) is finite, then (3.125) obviously converges. Otherwise, suppose that the support of \( f \) is countably infinite, and let \( \{x_j\}_{j=1}^{\infty} \) be a sequence of distinct elements of \( X \) that includes every element in the support of \( f \). It is easy to see that the partial sums

\[
(3.133) \quad \sum_{j=1}^{n} f(x_j)
\]

of the infinite series

\[
(3.134) \quad \sum_{j=1}^{\infty} f(x_j)
\]

form a Cauchy sequence in \( V \) under these conditions, because (3.125) satisfies the generalized Cauchy criterion. If \( V \) is complete with respect to the metric associated to \( N \), then it follows that this sequence of partial sums converges in \( V \), which is to say that (3.134) converges in \( V \) in the usual sense. In this case, it is easy to see that (3.125) also converges in \( V \) in the sense described at the beginning of the section, with the same value of the sum. This uses the generalized Cauchy criterion again, to get that finite sums of the form (3.124) are close to partial sums of the form (3.133) under suitable conditions.
Chapter 4

Power series

4.1 Complex coefficients

Let \(a_0, a_1, a_2, a_3, \ldots\) be a sequence of complex numbers, and consider the power series

\[
\sum_{j=0}^{\infty} a_j z^j, \tag{4.1}
\]

where \(z \in \mathbb{C}\). As usual, \(z^j\) is interpreted as being equal to 1 for every \(z \in \mathbb{C}\) when \(j = 0\). If (4.1) converges for some \(z \in \mathbb{C}\), then \(\{a_j z^j\}_{j=0}^{\infty}\) converges to 0 as a sequence in \(\mathbb{C}\), which implies in particular that \(\{a_j z^j\}_{j=0}^{\infty}\) is bounded.

Using this, one can check that \(\sum_{j=0}^{\infty} a_j w^j\) converges absolutely for every \(w \in \mathbb{C}\) such that \(|w| < |z|\), by comparison with a convergent geometric series.

The radius of convergence \(\rho\) of (4.1) may be defined by

\[
\rho = \sup \left\{ |z| : z \in \mathbb{C}, \sum_{j=0}^{\infty} a_j z^j \text{ converges} \right\}. \tag{4.2}
\]

Of course, (4.1) converges trivially when \(z = 0\), so that (4.2) is the supremum of a nonempty set. If (4.1) converges for \(z \in \mathbb{C}\) with arbitrarily large modulus, then (4.2) is interpreted as being \(+\infty\), as usual. If \(w \in \mathbb{C}\) satisfies \(|w| < \rho\), then it is easy to see that \(\sum_{j=0}^{\infty} a_j w^j\) converges absolutely, using the remarks in the previous paragraph. If \(w \in \mathbb{C}\) satisfies \(|w| > \rho\), then \(\sum_{j=0}^{\infty} a_j w^j\) does not converge, by the definition (4.2) of \(\rho\). Note that \(\rho\) is uniquely determined by these two properties. It is well known that

\[
\rho = \left( \limsup_{j \to \infty} |a_j|^{1/j} \right)^{-1}, \tag{4.3}
\]

by the root test, with the standard conventions that \(1/0 = +\infty\) and \(1/\infty = 0\).

Suppose that

\[
\sum_{j=0}^{\infty} |a_j| r^j \tag{4.4}
\]
4.1. COMPLEX COEFFICIENTS

converges for some nonnegative real number \( r \). This implies that (4.1) converges absolutely for every \( z \in \mathbb{C} \) with \( |z| \leq r \), by the comparison test. Moreover,

\[
\left| \sum_{j=0}^{\infty} a_j z^j - \sum_{j=0}^{n} a_j z^j \right| = \left| \sum_{j=n+1}^{\infty} a_j z^j \right| \leq \sum_{j=n+1}^{\infty} |a_j| |z|^j \leq \sum_{j=n+1}^{\infty} |a_j| r^j
\]

(4.5)

for each \( z \in \mathbb{C} \) with \( |z| \leq r \) and nonnegative integer \( n \). It follows that the partial sums

\[
\sum_{j=0}^{n} a_j z^j
\]

(4.6)

converge to (4.1) uniformly on the closed disk

\[
\{ z \in \mathbb{C} : |z| \leq r \}
\]

(4.7)

under these conditions. This shows that (4.1) defines a continuous function on (4.7), since the partial sums (4.6) are continuous.

Similarly, if the radius of convergence \( \rho \) of (4.1) is positive, then (4.1) defines a continuous function on the open disk

\[
\{ z \in \mathbb{C} : |z| < \rho \}.
\]

(4.8)

More precisely, let \( z_0 \in \mathbb{C} \) with \( |z_0| < \rho \) be given, and let \( r \) be a positive real number such that

\[
|z_0| < r < \rho.
\]

(4.9)

Thus (4.4) converges, because \( r < \rho \), so that (4.1) defines a continuous function on (4.7), as in the previous paragraph. In particular, (4.1) is continuous at \( z_0 \) as a function on (4.7), which implies that (4.1) is continuous at \( z_0 \) as a function on (4.8), because \( |z_0| < r \). It follows that (4.1) is continuous as a function on (4.8) at every point in (4.8), as desired.

Consider the power series

\[
\sum_{j=1}^{\infty} j a_j z^{j-1},
\]

(4.10)

obtained by differentiating (4.1) term by term. It is well known that the radius of convergence of (4.10) is equal to the radius of convergence of (4.1), which can be derived from (4.3), for instance. Alternatively, if (4.10) converges absolutely for some \( z \in \mathbb{C} \), then (4.1) converges absolutely as well, by the comparison test. Conversely, if \( z \in \mathbb{C} \) and \( |z| < \rho \), then (4.1) converges by comparison with a convergent geometric series, in which case (4.10) converges absolutely too. It is also well known that (4.1) is a holomorphic function on (4.8), whose complex derivative is given by (4.10).

The complex exponential function may be defined for \( z \in \mathbb{C} \) by

\[
E(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}.
\]

(4.11)
where \( j! \) is \( j \) factorial, the product of the positive integers from 1 to \( j \), which is interpreted as being equal to 1 when \( j = 0 \). It is easy to see that this series converges absolutely for every \( z \in \mathbb{C} \), using the ratio test, for instance. If \( w, z \in \mathbb{C} \), then

\[
E(w + z) = \sum_{l=0}^{\infty} \frac{(w + z)^l}{l!} = \sum_{l=0}^{\infty} \sum_{j=0}^{l} \frac{w^j z^{l-j}}{j!(l-j)!},
\]

using the binomial theorem in the second step. This implies that

\[
E(w + z) = E(w) E(z)
\]

for every \( w, z \in \mathbb{C} \), because the right side of (4.12) is the same as the Cauchy product of the series representing \( E(w) \) and \( E(z) \). More precisely, this uses the absolute convergence of the series representing \( E(w) \) and \( E(z) \), to ensure that the right side of (4.12) is equal to the right side of (4.13).

If we take \( w = -z \) in (4.13), then we get that

\[
E(z) E(-z) = E(0) = 1
\]

for every \( z \in \mathbb{C} \). Equivalently, this means that \( E(z) \neq 0 \) for every \( z \in \mathbb{C} \), and that \( E(-z) = 1/E(z) \). Observe that

\[
\overline{E(z)} = E(\overline{z})
\]

for every \( z \in \mathbb{C} \), where \( \overline{z} \) is the complex conjugate of \( z \), since one can take the complex-conjugate of (4.11) term by term. This implies that

\[
E(2 \text{Re } z) = E(z + \overline{z}) = E(z) E(\overline{z}) = E(z) \overline{E(z)} = |E(z)|^2
\]

for every \( z \in \mathbb{C} \), where \( \text{Re } z \) denotes the real part of \( z \). In particular,

\[
|E(i y)| = 1
\]

for every \( y \in \mathbb{R} \).

Note that \( E(x) \in \mathbb{R} \) for every \( x \in \mathbb{R} \), by (4.11). More precisely, \( E(x) \geq 1 \) when \( x \geq 0 \), which implies that \( 0 < E(x) \leq 1 \) when \( x \leq 0 \), because \( E(x) = 1/E(-x) \). Of course, \( E(x) \) is the same as the usual real exponential function on \( \mathbb{R} \) when \( x \in \mathbb{R} \). It is easy to see that \( E(x) \) is strictly increasing on \([0, +\infty)\), and that \( E(x) \to +\infty \) as \( x \to +\infty \), directly from (4.11). This implies that \( E(x) \) is also strictly increasing on \((-\infty, 0]\), and hence on \( \mathbb{R} \), and that \( E(x) \to 0 \) as \( x \to -\infty \), using \( E(x) = 1/E(-x) \) again.

If the coefficients \( a_j \) of (4.1) are real numbers, then one can think of (4.1) as a power series on \( \mathbb{R} \), with many of the same properties as before. In particular, the radius of convergence of (4.1) as a power series on \( \mathbb{R} \) is the same as the radius of convergence of (4.1) as a power series on \( \mathbb{C} \). If the radius of convergence \( \rho \) is positive, then (4.1) defines a continuous real-valued function on the open interval \((-\rho, \rho)\) in \( \mathbb{R} \). This function is also differentiable on \((-\rho, \rho)\), with the derivative given by the power series (4.10).
4.2 Ultrametric absolute value functions

Let \( k \) be a field, and let \( a_0, a_1, a_2, a_3, \ldots \) be a sequence of elements of \( k \). Consider the corresponding formal power series

\[
f(X) = \sum_{j=0}^{\infty} a_j X^j,
\]

where \( X \) is an indeterminate. As in [4, 14], we shall use upper-case letters like \( X \) for indeterminates, and lower-case letters like \( x \) for elements of \( k \) or other fields. Let \( | \cdot | \) be an ultrametric absolute value function on \( k \), and suppose that \( k \) is complete with respect to the ultrametric that corresponds to \( | \cdot | \), as in (2.9).

If \( x \in k \), then

\[
\sum_{j=0}^{\infty} a_j x^j,
\]

converges in \( k \) exactly when \( \{a_j x^j \}_{j=0}^{\infty} \) converges to 0 in \( k \), as in Section 2.6. Equivalently, this means that

\[
|a_j x^j| = |a_j| |x|^j \to 0 \quad \text{as } j \to \infty
\]

as a sequence of nonnegative real numbers. In this case, the value of (4.19) may be denoted \( f(x) \).

The radius of convergence \( \rho \) of this power series may be defined as the supremum of the set of nonnegative real numbers \( r \) such that

\[
\lim_{j \to \infty} |a_j| r^j = 0.
\]

Of course, \( r = 0 \) automatically has this property, so that this is the supremum of a nonempty set. If (4.21) holds for some arbitrarily large real numbers \( r \), then the supremum is interpreted as being equal to \( +\infty \), as usual. If \( r, t \) are nonnegative real numbers such that \( t \leq r \) and \( r \) satisfies (4.21), then

\[
\lim_{j \to \infty} |a_j| t^j = 0
\]

as well. This implies that (4.22) holds when \( t < \rho \), by the definition of \( \rho \). If \( t > \rho \), then (4.22) does not hold, again by the definition of \( \rho \). It is easy to see that \( \rho \) is uniquely determined by these two properties. One can also check that

\[
\rho = \left( \lim_{j \to \infty} |a_j|^{1/j} \right)^{-1},
\]

using standard properties of the limsup, and with the usual conventions for \( 1/0 \) and \( 1/\infty \). It follows from these properties of \( \rho \) that (4.19) converges in \( k \) when \( x \in k \) satisfies \( |x| < \rho \), and not when \( |x| > \rho \). However, this may not determine \( \rho \) uniquely, depending on the possible values of \( | \cdot | \) on \( k \).
If (4.19) converges for some \( x \in k \), then

\[
\left| \sum_{j=0}^{\infty} a_j x^j \right| \leq \max_{j \geq 0} |a_j x^j| = \max_{j \geq 0} (|a_j| |x|^j),
\]

as in (2.68) in Section 2.6. Suppose now that \( r \) is a positive real number that satisfies (4.21), which implies that (4.20) holds for every \( x \in k \) with \( |x| \leq r \). Thus (4.19) converges when \( |x| \leq r \), and

\[
\left| \sum_{j=0}^{\infty} a_j x^j - \sum_{j=0}^{n} a_j x^j \right| = \left| \sum_{j=n+1}^{\infty} a_j x^j \right| \\
\leq \max_{j \geq n+1} (|a_j| |x|^j) \leq \max_{j \geq n+1} (|a_j| r^j)
\]

for every nonnegative integer \( n \), by (4.24). It follows that the partial sums

\[
\sum_{j=0}^{n} a_j x^j
\]

converge to (4.19) uniformly on the set

\[
\{ x \in k : |x| \leq r \},
\]

because the right side of (4.25) tends to 0 as \( n \to \infty \), by (4.21). This implies that (4.19) defines a continuous function on (4.27) under these conditions, because the partial sums (4.26) are continuous.

Using this, one can check that (4.19) defines a continuous function on

\[
\{ x \in k : |x| < \rho \}
\]

when \( \rho > 0 \). Of course, if \( \rho < +\infty \) and

\[
\lim_{j \to \infty} |a_j| \rho^j = 0,
\]

then one might as well apply the previous discussion to \( r = \rho \), to get that (4.19) is continuous on

\[
\{ x \in k : |x| \leq \rho \}.
\]

Otherwise, one can apply the previous discussion to each \( r > 0 \) such that \( r < \rho \). More precisely, if \( x_0 \in k \) satisfies \( |x_0| < \rho \), then one can choose \( r > 0 \) such that \( |x_0| \leq r < \rho \). One can also choose \( r \) so that \( r > |x_0| \) when \( x_0 \neq 0 \), but this is not really necessary here, because (4.27) is an open set in \( k \) when \( r > 0 \) and \(| \cdot |\) is an ultrametric absolute value function.
4.3 Differentiation and Lipschitz conditions

Let $k$ be a field, and let $|\cdot|$ be an ultrametric absolute value function on $k$. Also let $r$ be a positive real number, and suppose that $x, y \in k$ satisfy $|x|, |y| \leq r$, so that $|x - y| \leq r$ too. If $j$ is a positive integer, then

$$x^j - y^j = ((x - y) + y)^j - y^j = \sum_{l=1}^{j} \binom{j}{l} \cdot (x - y)^l y^{j-l},$$

(4.31)

by the binomial theorem. This implies that

$$|x^j - y^j| \leq |x - y| r^{j-1},$$

(4.32)

because of the ultrametric version of the triangle inequality, which implies in particular that $|n \cdot 1| \leq 1$ for every positive integer $n$. Similarly, if $j \geq 2$, then

$$x^j - y^j - j y^{j-1} (x - y) = ((x - y) + y)^j - y^j - j y^{j-1} (x - y)$$

$$= \sum_{l=2}^{j} \binom{j}{l} \cdot (x - y)^l y^{j-l},$$

(4.33)

and hence

$$|x^j - y^j - j y^{j-1} (x - y)| \leq |x - y|^2 r^{j-2}.$$

(4.34)

Suppose that $k$ is complete with respect to the ultrametric that corresponds to $|\cdot|$. Let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of elements of $k$ that satisfies

$$\lim_{j \to \infty} |a_j| r^j = 0,$$

(4.35)

and put

$$f(x) = \sum_{j=0}^{\infty} a_j x^j$$

(4.36)

for each $x \in k$ with $|x| \leq r$. As before, the convergence of this series follows from the completeness of $k$, although this is not needed when $a_j = 0$ for all but finitely many $j$, in which case $f(x)$ is a polynomial function. Note that

$$|f(x)| \leq \max_{j \geq 0} (|a_j| r^j)$$

(4.37)

for every $x \in k$ with $|x| \leq r$, as in (4.24).

Now let $x, y \in k$ be given, with $|x|, |y| \leq r$, so that the series expansions for both $f(x)$ and $f(y)$ converge. Thus

$$f(x) - f(y) = \sum_{j=1}^{\infty} a_j (x^j - y^j),$$

(4.38)

and hence

$$|f(x) - f(y)| \leq \max_{j \geq 1} (|a_j| |x^j - y^j|),$$

(4.39)
As in (2.68) in Section 2.6. Combining this with (4.32), we get that
\begin{equation}
|f(x) - f(y)| \leq \max_{j \geq 1}(|a_j|r^{j-1}) |x - y|.
\end{equation}

If \( f(X) \) is the formal power series associated to this sequence of coefficients, as in (4.18), then the formal derivative of \( f(X) \) is the formal power series
\begin{equation}
f'(X) = \sum_{j=1}^{\infty} j \cdot a_j X^{j-1}.
\end{equation}

Of course,
\begin{equation}
|j \cdot a_j x^{j-1}| \leq |a_j| |x|^{j-1}
\end{equation}
for every \( x \in k \) and positive integer \( j \), because \( |j \cdot 1| \leq 1 \). If \( |x| \leq r \), then
\begin{equation}
\lim_{j \to \infty} j \cdot a_j x^{j-1} = 0
\end{equation}
in \( k \), by (4.35), so that
\begin{equation}
\sum_{j=1}^{\infty} j \cdot a_j x^{j-1}.
\end{equation}
converges in \( k \). Let \( f'(x) \) denote the value of the sum (4.44), so that
\begin{equation}
|f'(x)| \leq \max_{j \geq 1}(|j \cdot a_j|r^{j-1}) \leq \max_{j \geq 1}(|a_j|r^{j-1})
\end{equation}
for every \( x \in k \) with \( |x| \leq r \), which is the analogue of (4.37) for \( f'(x) \).

If \( x, y \in k \) satisfy \( |x|, |y| \leq r \), then
\begin{equation}
f(x) - f(y) - f'(y)(x - y)
\end{equation}

\begin{equation}
= \sum_{j=1}^{\infty} a_j (x^j - y^j) - \sum_{j=1}^{\infty} j \cdot a_j y^{j-1} (x - y)
\end{equation}

\begin{equation}
= \sum_{j=2}^{\infty} a_j (x^j - y^j - j \cdot y^{j-1} (x - y)).
\end{equation}

This implies that
\begin{equation}
|f(x) - f(y) - f'(y)(x - y)| \leq \max_{j \geq 2}(|a_j||x^j - y^j - j \cdot y^{j-1} (x - y)|),
\end{equation}
as in (2.68) in Section 2.6. It follows that
\begin{equation}
|f(x) - f(y) - f'(y)(x - y)| \leq \max_{j \geq 2}(|a_j|r^{j-2}) |x - y|^2,
\end{equation}
by (4.34). In particular, this shows that \( f'(y) \) is the derivative of \( f(y) \) in the usual sense when \( | \cdot | \) is not the trivial absolute value function on \( k \).
4.4. HENSEL’S LEMMA

Observe that

\[ |f(x) - f(y)| \leq \max \left( |f'(x)||x - y|, \max_{j \geq 2}(|a_j| r^{j-2}) |x - y|^2 \right) \]
\[ \quad = \max \left( |f'(y)|, \max_{j \geq 2}(|a_j| r^{j-2}) |x - y| \right) |x - y| \]

for every \( x, y \in k \) with \( |x|, |y| \leq r \), by (4.47) and the ultrametric version of the triangle inequality. Similarly,

\[ |f'(y)||x - y| \leq \max \left( |f(x) - f(y)|, \max_{j \geq 2}(|a_j| r^{j-2}) |x - y|^2 \right) \]

for every \( x, y \in k \) with \( |x|, |y| \leq r \). If we also have that

\[ \max_{j \geq 2}(|a_j| r^{j-2}) |x - y| < |f'(y)|, \]

then it follows that

\[ |f(x) - f(y)| = |f'(y)||x - y|. \]

More precisely, this is trivial when \( x = y \), and otherwise one can multiply both sides of (4.51) by \( |x - y| \), and still have a strict inequality.

If \( x, y \in k \) satisfy \( |x|, |y| \leq r \) again, then

\[ |f'(x) - f'(y)| \leq \max_{j \geq 2}(j \cdot |a_j| r^{j-2}) |x - y| \leq \max_{j \geq 2}(|a_j| r^{j-2}) |x - y|, \]

which is the analogue of (4.40) for \( f'(x) \) in place of \( f(x) \). This implies that

\[ |f'(x)| = |f'(y)| \]

when (4.51) holds, by the ultrametric version of the triangle inequality.

As mentioned earlier, the completeness of \( k \) is only needed in this section to ensure the convergence of the various infinite series. If \( a_j = 0 \) for all but finitely many \( j \), then completeness of \( k \) is not needed, and the various estimates for \( f(x) \) and \( f'(x) \) still hold.

4.4 Hensel’s lemma

Let \( k \) be a field, let \(| \cdot |\) be an ultrametric absolute value function on \( k \), and suppose that \( k \) is complete with respect to the corresponding ultrametric. Also let \( r \) be a positive real number, and let \( a_0, a_1, a_2, a_3, \ldots \) be a sequence of elements of \( k \) that satisfies (4.35). Thus \( f(x) \) may be defined for \( x \in k \) with \( |x| \leq r \) as in (4.36). Let \( x_0 \in k \) be given, with \( |x_0| \leq r \) and \( f'(x_0) \neq 0 \). If \( z \in k \) is sufficiently close to \( f(x_0) \), then we would like to find an \( x \in k \) close to \( x_0 \) that satisfies

\[ f(x) = z. \]

In particular, if \( |x - x_0| \leq r \), then \( |x| \leq r \), so that \( f(x) \) is defined. In order to find \( x \), we consider an appropriate sequence of approximations.
Suppose that the $l$th approximation $x_l \in k$ has been chosen for some non-negative integer $l$, in such a way that $|x_l| \leq r$ and $f'(x_l) \neq 0$, where $f'$ is as defined in the previous section. Let us choose $x_{l+1} \in k$ so that

\begin{equation}
(4.56) \quad f(x_l) + f'(x_l)(x_{l+1} - x_l) = z,
\end{equation}

which is to say that

\begin{equation}
(4.57) \quad x_{l+1} = x_l + f'(x_l)^{-1}(z - f(x_l)).
\end{equation}

Thus

\begin{equation}
(4.58) \quad |x_{l+1} - x_l| = |f'(x_l)|^{-1}|z - f(x_l)|.
\end{equation}

Suppose also that

\begin{equation}
(4.59) \quad |x_{l+1} - x_l| \leq r,
\end{equation}

which implies that $|x_{l+1}| \leq r$, so that $f(x_{l+1})$ is defined. In this case,

\begin{equation}
(4.60) \quad f(x_{l+1}) - z = f(x_{l+1}) - f(x_l) - f'(x_l)(x_{l+1} - x_l),
\end{equation}

by (4.56), and hence

\begin{equation}
(4.61) \quad |f(x_{l+1}) - z| = |f(x_{l+1}) - f(x_l) - f'(x_l)(x_{l+1} - x_l)|
\leq \max_{j \geq 2}(|a_j| r^{j-2}) |x_{l+1} - x_l|^2,
\end{equation}

using (4.48) in the second step. Plugging (4.58) into the right side of (4.61), we get that

\begin{equation}
(4.62) \quad |f(x_{l+1}) - z| \leq \max_{j \geq 2}(|a_j| r^{j-2}) |f'(x_l)|^{-2} |f(x_l) - z|^2.
\end{equation}

Similarly, if (4.59) holds, and thus $|x_{l+1}| \leq r$, then

\begin{equation}
(4.63) \quad |f'(x_{l+1}) - f'(x_l)| \leq \max_{j \geq 2}(|a_j| r^{j-2}) |x_{l+1} - x_l|,
\end{equation}

by (4.53). As before, we can combine this with (4.58), to get that

\begin{equation}
(4.64) \quad |f'(x_{l+1}) - f'(x_l)| \leq \max_{j \geq 2}(|a_j| r^{j-2}) |f'(x_l)|^{-1} |f(x_l) - z|.
\end{equation}

If

\begin{equation}
(4.65) \quad \max_{j \geq 2}(|a_j| r^{j-2}) |f'(x_l)|^{-1} |f(x_l) - z| < |f'(x_l)|,
\end{equation}

then (4.64) implies that

\begin{equation}
(4.66) \quad |f'(x_{l+1}) - f'(x_l)| < |f'(x_l)|,
\end{equation}

and hence

\begin{equation}
(4.67) \quad |f'(x_{l+1})| = |f'(x_l)|,
\end{equation}

by the ultrametric version of the triangle inequality. Of course, we would like (4.67) to hold for each $l \geq 0$, so that $|f'(x_l)| = |f'(x_0)|$ for every $l$. 

4.4. HENSEL’S LEMMA

Put
\[ b_l = \max_{j \geq 2} (|a_j| r^{j-2} |f'(x_l)|^{-2} |f(x_l) - z|), \]
so that (4.65) is the same as saying that \( b_l < 1 \). Using this notation, (4.62) can be reexpressed as
\[ |f(x_{l+1}) - z| \leq b_l |f(x_l) - z|. \]
If (4.59) holds and \( b_l < 1 \), then
\[
\begin{align*}
(4.68) & \quad b_{l+1} = \max_{j \geq 2} (|a_j| r^{j-2} |f'(x_l)|^{-2} |f(x_l) - z|) \\
& \quad = \max_{j \geq 2} (|a_j| r^{j-2} |f'(x_l)|^{-2} |f(x_{l+1}) - z|),
\end{align*}
\]
by (4.67). Combining this with (4.69), we get that
\[
\begin{align*}
(4.70) & \quad b_{l+1} \leq b_l \max_{j \geq 2} (|a_j| r^{j-2} |f'(x_l)|^{-2} |f(x_l) - z|) = b_l^2,
\end{align*}
\]
using the definition (4.68) of \( b_l \) in the second step. In particular, this implies that \( b_{l+1} < 1 \) under these conditions.

If (4.59) holds and \( b_l < 1 \), then \( f'(x_{l+1}) \neq 0 \), by (4.67), so that we can repeat the process. If \( x_{l+2} \) is obtained from \( x_{l+1} \) as in (4.57), then we get that
\[
\begin{align*}
(4.72) & \quad |x_{l+2} - x_{l+1}| = |f'(x_{l+1})|^{-1} |z - f(x_{l+1})|,
\end{align*}
\]
as in (4.58). This implies that
\[
\begin{align*}
(4.73) & \quad |x_{l+2} - x_{l+1}| \leq b_l |f'(x_l)|^{-1} |z - f(x_l)|,
\end{align*}
\]
by (4.67) and (4.69). Equivalently,
\[
\begin{align*}
(4.74) & \quad |x_{l+2} - x_{l+1}| \leq b_l |x_{l+1} - x_l|
\end{align*}
\]
under these conditions, by (4.58).

In order for all of this to work, we need \( z \) to be sufficiently close to \( f(x_0) \), as mentioned at the beginning of the section. More precisely, let us suppose that
\[
\begin{align*}
(4.75) & \quad |z - f(x_0)| \leq |f'(x_0)| r
\end{align*}
\]
and
\[
\begin{align*}
(4.76) & \quad \max_{j \geq 2} (|a_j| r^{j-2} |f(x_0) - z|) < |f'(x_0)|^2.
\end{align*}
\]
Note that (4.75) is the same as (4.59) with \( l = 0 \), by (4.58). Similarly, (4.76) is the same as (4.65) with \( l = 0 \), which is the same as saying that \( b_0 < 1 \). Under these conditions, we can repeat the process described above to get a sequence \( \{x_l\}_{l=0}^{\infty} \) of elements of \( k \) defined recursively by (4.57), and which satisfies (4.59) and \( b_l < 1 \) for every \( l \).

It is easy to see that
\[
\begin{align*}
(4.77) & \quad \lim_{l \to \infty} b_l = 0,
\end{align*}
\]
by (4.71) and the hypothesis that \( b_0 < 1 \). This implies that
\[
\lim_{l \to \infty} |x_{l+1} - x_l| = 0,
\]
(4.78)

because of (4.74). Thus \( \{x_l\}_{l=0}^\infty \) is a Cauchy sequence of elements of \( k \), by the ultrametric version of the triangle inequality. It follows that \( \{x_l\}_{l=0}^\infty \) converges to an element \( x \) of \( k \), since \( k \) is supposed to be complete. If \( a_j = 0 \) for all but finitely many \( j \), then the completeness of \( k \) is not needed to define \( f(x) \) as in (4.36), but it is still needed here.

Of course, \( |x_{l+1} - x_l| \) decreases monotonically, by (4.74) and the fact that \( b_l < 1 \) for each \( l \). This implies that
\[
|\lim_{l \to \infty} x_l - x_0| \leq |x_1 - x_0|,
\]
(4.79)

for each \( l \geq 0 \), by the ultrametric version of the triangle inequality. Equivalently,
\[
|x_l - x_0| \leq |f'(x_0)|^{-1}|z - f(x_0)| \leq r
\]
(4.80)

for every \( l \geq 0 \), using (4.58) with \( l = 0 \) in the first step, and (4.75) in the second step. It follows that
\[
|x - x_0| \leq |f'(x_0)|^{-1}|z - f(x_0)| \leq r,
\]
(4.81)

where \( x \) is the limit of \( \{x_l\}_{l=1}^\infty \), as in the preceding paragraph. In particular,
\[
|x| \leq \max(|x - x_0|, |x_0|) \leq r,
\]
(4.82)

since \( |x_0| \leq r \) by hypothesis, so that \( f(x) \) is defined. It is easy to see that
\[
\lim_{l \to \infty} |f(x_l) - z| = 0,
\]
(4.83)

by (4.69) and (4.77). Thus \( f(x) = z \), as desired, because \( f \) is continuous on the closed ball in \( k \) centered at 0 with radius \( r \), as in Section 4.2.

Remember that (4.67) holds for each \( l \geq 0 \) in this situation, which implies that
\[
|f'(x_l)| = |f'(x_0)|
\]
(4.84)

for each \( l \geq 0 \). Taking the limit as \( l \to \infty \), we get that
\[
|f'(x)| = |f'(x_0)|,
\]
(4.85)

because \( f' \) is also continuous on the closed ball in \( k \) centered at 0 with radius \( r \). Alternatively, (4.85) could be derived from (4.76) and the first inequality in (4.81), as in (4.54). More precisely, let \( w \) be any element of \( k \) that satisfies
\[
|w - x_0| \leq |f'(x_0)|^{-1}|z - f(x_0)|.
\]
(4.86)

Thus \( |w - x_0| \leq r \), by (4.75), which implies that
\[
|w| \leq \max(|w - x_0|, |x_0|) \leq r.
\]
(4.87)
as before. Under these conditions, we have that

\[ |f'(w)| = |f'(x_0)|, \]

as in (4.54), where \( x \) and \( y \) in (4.54) correspond to \( w \) and \( x_0 \) here, respectively. This uses (4.76) and (4.86) to get the hypothesis (4.51) for (4.54).

If \( w \) is any element of \( k \) that satisfies (4.86) and hence (4.87), then

\[ |x - w| \leq \max(|x - x_0|, |x_0 - w|) \leq |f'(x_0)|^{-1} |z - f(x_0)|, \]

by (4.81). In this case, we get that

\[ |f(x) - f(w)| = |f'(x)| |x - w| = |f'(x_0)| |x - w|, \]

as in (4.52), where \( x \) and \( y \) in (4.52) correspond to \( w \) and \( x_0 \) here, respectively. The hypothesis (4.51) for (4.52) follows from (4.76), (4.85), and (4.89) here, and (4.89) is also used in the second step in (4.90). This implies that \( w = x \) when \( f(w) = f(x) \), since \( f'(x_0) \neq 0 \) by hypothesis. This shows that \( x \) is the unique element of \( k \) that satisfies (4.81) and \( f(x) = z \) under these conditions.

4.5 Some additional remarks

Let us continue with the notation in the previous section. Note that

\[ |f'(x_0)| \leq \max_{j \geq 1}(|a_j| r^{j-1}) = r \max_{j \geq 1}(|a_j| r^{j-2}), \]

by (4.45) in Section 4.3 applied to \( x_0 \). Suppose that

\[ \max_{j \geq 1}(|a_j| r^{j-2}) |f(x_0) - z| < |f'(x_0)|^2. \]

This automatically implies (4.76), since the maximum is taken over all \( j \geq 1 \) instead of \( j \geq 2 \). Combining (4.91) and (4.92), we also get that

\[ |f'(x_0)| |f(x_0) - z| \leq r \max_{j \geq 1}(|a_j| r^{j-2}) |f(x_0) - z| < r |f'(x_0)|^2, \]

which implies that

\[ |z - f(x_0)| < |f'(x_0)| r. \]

This is a bit stronger than (4.75), so that (4.92) implies both (4.75) and (4.76). Thus it suffices to ask that (4.92) hold, in order to get a unique point \( x \in k \) that satisfies (4.81) and \( f(x) = z \), as in the previous section.

Suppose for the moment that

\[ r \max_{j \geq 2}(|a_j| r^{j-2}) < |f'(x_0)|. \]

If (4.75) also holds, then we get that

\[ r \max_{j \geq 2}(|a_j| r^{j-2}) |f(x_0) - z| < |f'(x_0)|^2 r, \]
by multiplying the left and right sides of (4.75) and (4.95) together. Of course, (4.96) is equivalent to (4.76), by dividing by \( r \), so that (4.75) automatically implies (4.76) when (4.95) holds. Note that

\[
|f'(x_0) - a_1| = \sum_{j=2}^{\infty} j \cdot a_j x_0^j \leq \max_{j \geq 2} (|j \cdot a_j| r^{j-1}) \\
\leq \max_{j \geq 2} (|a_j| r^{j-1}) = r \max_{j \geq 2} (|a_j| r^{j-2}),
\]

using the definition of \( f'(x_0) \) in the first step, and (2.68) in Section 2.6 in the second step. This implies that

\[
|f'(x_0)| = |a_1|
\]

when (4.95) holds, by the ultrametric version of the triangle inequality.

Similarly, if

\[
r \max_{j \geq 2} (|a_j| r^{j-2}) < |a_1| = |f'(0)|,
\]

then (4.98) holds again, by (4.97) and the ultrametric version of the triangle inequality. In this case, we also have that

\[
|f(x_0) - f(0)| = \sum_{j=1}^{\infty} a_j x_0^j \leq \max_{j \geq 1} (|a_j| |x_0|^j) \leq \max_{j \geq 1} (|a_j| r^j) \leq |a_1| r = |f'(0)| r,
\]

since \(|x_0| \leq r\). Consider the inequality

\[
|z - f(0)| \leq |a_1| r = |f'(0)| r,
\]

which is the analogue of (4.75) with \( x_0 \) replaced by 0. If (4.99) holds, then (4.75) is equivalent to (4.101), by (4.98), (4.100), and the ultrametric version of the triangle inequality. If (4.99) and (4.101) both hold, then the analogue of (4.76) with \( x_0 \) replaced by 0 holds too, as in the preceding paragraph.

Now suppose that

\[
|f'(x_0)| \leq r \max_{j \geq 2} (|a_j| r^{j-2}),
\]

which is the opposite of (4.95). If (4.76) also holds, then we get that

\[
|f'(x_0)| |f(x_0) - z| \leq r \max_{j \geq 2} (|a_j| r^{j-2}) |f(x_0) - z| < r |f'(x_0)|^2,
\]

as in (4.93). This implies (4.94), as before, so that (4.76) automatically implies (4.75) when (4.102) holds. Observe that (4.102) holds when

\[
|a_1| \leq r \max_{j \geq 2} (|a_j| r^{j-2}),
\]

by (4.91). Of course, (4.104) is the opposite of (4.99).
4.6 Another look at regularity

Let $k$ be a field, and let $|\cdot|$ be an ultrametric absolute value function on $k$. If $x, y \in k$ and $j$ is a positive integer, then

$$\begin{align*}
(x - y) \sum_{l=0}^{j-1} x^l y^{j-l-1} &= x \sum_{l=0}^{j-1} x^l y^{j-l-1} - y \sum_{l=0}^{j-1} x^l y^{j-l-1} \\
&= \sum_{l=0}^{j-1} x^{l+1} y^{j-l-1} - \sum_{l=0}^{j-1} x^l y^{j-l} \\
&= \sum_{l=1}^{j} x^l y^{j-l-1} - \sum_{l=0}^{j-1} x^l y^{j-l} = x^j - y^j.
\end{align*}$$

This implies that

$$\begin{align*}
|x^j - y^j| &= |x - y| \left| \sum_{l=0}^{j-1} x^l y^{j-l-1} \right| \\
&\leq |x - y| \left( \max_{0 \leq l \leq j-1} |x^l y^{j-l-1}| \right) \\
&\leq |x - y| \left( \max(|x|, |y|) \right)^{j-1},
\end{align*}$$

by the ultrametric version of the triangle inequality. This gives another way to look at (4.32) in Section 4.3.

Suppose that $k$ is complete with respect to the ultrametric associated to $|\cdot|$, and let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of elements of $k$ that satisfies

$$\lim_{j \to \infty} |a_j| r^j = 0$$

for some positive real number $r$. Thus the corresponding power series

$$f(x) = \sum_{j=0}^{\infty} a_j x^j$$

is defined for each $x \in k$ with $|x| \leq r$, as usual. Note that the Lipschitz condition (4.40) in Section 4.3 could be derived from (4.39) using (4.106) instead of (4.32).

Let $x, x_0 \in k$ be given, with $|x|, |x_0| \leq r$, and observe that

$$f(x) - f(x_0) = \sum_{j=1}^{\infty} a_j (x^j - x_0^j) = \sum_{j=1}^{\infty} a_j (x - x_0) \left( \sum_{l=0}^{j-1} x^l x_0^{j-l-1} \right),$$

using (4.105) with $y = x_0$ in the second step. Put

$$b_l = \sum_{j=l+1}^{\infty} a_j x_0^{j-l-1}$$
for each nonnegative integer \( l \), where the convergence of the series follows from (4.107). We also have that

\[
|b_l| \leq \max_{j \geq l+1} |a_j x_0^{j-l-1}| \leq \max_{j \geq l+1} (|a_j| r^{j-l-1})
\]

for each \( l \geq 0 \), using (2.68) in Section 2.6 in the first step. Thus

\[
|b_l| r^l \leq \max_{j \geq l+1} (|a_j| r^{j-1})
\]

for each \( l \geq 0 \), which tends to 0 as \( l \to \infty \), by (4.107). This implies that the power series

\[
g_0(x) = \sum_{l=0}^{\infty} b_l x^l
\]

also converges for every \( x \in k \) with \(|x| \leq r\). Of course,

\[
|a_j| |x|^l |x_0|^{j-l-1} \leq |a_j| r^{j-l-1} \to 0 \quad \text{as} \quad j \to \infty,
\]

by (4.107), which permits us to interchange the order of summation in (4.109). It follows that

\[
f(x) - f(x_0) = (x - x_0) g_0(x)
\]

for every \( x \in k \) with \(|x| \leq r\).

By construction,

\[
g_0(x) = \sum_{l=0}^{\infty} \left( \sum_{j=l+1}^{\infty} a_j x_0^{j-l-1} \right) x^l = \sum_{j=1}^{\infty} \left( \sum_{l=0}^{j-1} a_j x_0^{j-l-1} \right) x^l
\]

\[
= \sum_{j=1}^{\infty} j a_j x_0^{j-1} = f'(x_0).
\]

Here the order of summation has been interchanged in the second step, using (4.114) with \( x = x_0 \). We also have that

\[
|g_0(x) - g_0(y)| \leq \max_{l \geq 1} (|b_l| r^{l-1}) |x - y|
\]

for every \( x, y \in k \) with \(|x|, |y| \leq r\), by (4.40) in Section 4.3 applied to \( g_0 \). Combining this with (4.111), we get that

\[
|g_0(x) - g_0(y)| \leq \max_{j \geq 2} (|a_j| r^{j-2}) |x - y|
\]

for every \( x, y \in k \) with \(|x|, |y| \leq r\). Note that

\[
f(x) - f(x_0) - f'(x_0) (x - x_0) = (x - x_0) (g_0(x) - g_0(x_0))
\]

for every \( x \in k \) with \(|x| \leq r\), by (4.115) and (4.116). Thus

\[
|f(x) - f(x_0) - f'(x_0) (x - x_0)| = |x - x_0| |g_0(x) - g_0(x_0)|
\]

\[
\leq \max_{j \geq 2} (|a_j| r^{j-2}) |x - x_0|^2
\]

for every \( x \in k \) with \(|x| \leq r\), using (4.118) with \( y = x_0 \) in the second step. This gives another way to look at (4.48) in Section 4.3.
Let $k$ be a field, and let

(4.121) \[ f(X) = \sum_{j=0}^{\infty} a_j X^j \]

be a formal power series with coefficients in $k$. If $X$ and $Y$ are commuting indeterminates, then we have that

(4.122) \[ f(X + Y) = \sum_{l=0}^{\infty} a_l (X + Y)^l = \sum_{l=0}^{\infty} a_l \left( \sum_{j=0}^{l} \binom{l}{j} \cdot X^j Y^{l-j} \right) \]

as a formal power series in $X$ and $Y$. Equivalently,

(4.123) \[ f(X + Y) = \sum_{l=0}^{\infty} \left( \sum_{j=0}^{\infty} \binom{l}{j} \cdot a_l Y^{l-j} \right) X^j, \]

by interchanging the order of summation in (4.122).

Suppose that $k = \mathbb{C}$, with the standard absolute value function, and that

(4.124) \[ \sum_{j=0}^{\infty} |a_j| r^j \]

converges for some $r > 0$. This implies that

(4.125) \[ \sum_{j=0}^{\infty} a_j z^j \]

converges absolutely for every $z \in \mathbb{C}$ with $|z| \leq r$, and we denote the value of the sum by $f(z)$. Let $z_0 \in \mathbb{C}$ be given, with $|z_0| < r$, and put

(4.126) \[ r_0 = r - |z_0| > 0. \]

Also let $w \in \mathbb{C}$ be given, with $|w| \leq r_0$, so that

(4.127) \[ |w + z_0| \leq |w| + |z_0| \leq r_0 + |z_0| = r. \]

Thus $f(w + z_0)$ is defined, and can be expressed as

(4.128) \[ f(w + z_0) = \sum_{l=0}^{\infty} a_l (w + z_0)^l = \sum_{l=0}^{\infty} a_l \left( \sum_{j=0}^{l} \binom{l}{j} w^j z_0^{l-j} \right). \]

Note that

(4.129) \[ \sum_{l=0}^{\infty} |a_l| \left( \sum_{j=0}^{\infty} \binom{l}{j} |w|^j |z_0|^{l-j} \right) = \sum_{l=0}^{\infty} |a_l| (|z_0| + |w|)^l \leq \sum_{l=0}^{\infty} |a_l| r^l, \]
which is finite, by hypothesis. This permits us to interchange the order of summation in (4.128), to get that

\[ f(w + z_0) = \sum_{j=0}^{\infty} \left( \sum_{l=j}^{\infty} \binom{l}{j} a_l z_0^{l-j} \right) w^j. \]  

(4.130)

More precisely,

\[ \sum_{j=0}^{\infty} \left( \sum_{l=j}^{\infty} \binom{l}{j} |a_l| |z_0|^{l-j} \right) |w|^j = \sum_{l=0}^{\infty} \left( \sum_{j=0}^{l} \binom{l}{j} |a_l| |z_0|^{l-j} |w|^j \right) \]

(4.131)

is the same as (4.129), and hence is finite, by hypothesis. This implies that the sum in \( l \) on the right side of (4.130) converges absolutely for each \( j \geq 0 \), and that the resulting sum in \( j \) converges absolutely too.

Let \( k \) be an arbitrary field again, and let |·| be an ultrametric absolute value function on \( k \). Suppose that \( k \) is complete with respect to the metric corresponding to |·|, and that the coefficients \( a_j \) of (4.121) satisfy

\[ \lim_{j \to \infty} |a_j| r^j = 0 \]  

(4.132)

for some \( r > 0 \). This implies that

\[ f(x) = \sum_{j=0}^{\infty} a_j x^j \]

(4.133)

is defined for \( x \in k \) with \( |x| \leq r \), as before. If \( w, x_0 \in k \) satisfy \( |w|, |x_0| \leq r \), then

\[ |w + x_0| \leq \max(|w|, |x_0|) \leq r \]

(4.134)

by the ultrametric version of the triangle inequality. Thus \( f(w + x_0) \) is defined, and can be expressed as

\[ f(w + x_0) = \sum_{l=0}^{\infty} a_l \left( w + x_0 \right)^l = \sum_{l=0}^{\infty} \left( \sum_{j=0}^{l} \binom{l}{j} \cdot w^j x_0^{l-j} \right). \]

(4.135)

Remember that \( |N \cdot 1| \leq 1 \) for every positive integer \( N \), because |·| is an ultrametric absolute value function on \( k \). This implies that

\[ |a_l| \left| \binom{l}{j} \cdot w^j x_0^{l-j} \right| \leq |a_l| |w|^j |x_0|^{l-j} \leq |a_l| r^j \]

(4.136)

for all \( l \geq j \geq 0 \), which tends to 0 as \( l \to \infty \), by (4.132). This permits us to interchange the order of summation in (4.135), as follows.

Put

\[ \tilde{a}_j = \sum_{l=j}^{\infty} \binom{l}{j} \cdot a_l x_0^{l-j} \]

(4.137)
for each nonnegative integer \( j \), where the convergence of the series in \( k \) follows from (4.132) and the hypothesis that \(|x_0| \leq r\). We also have that

\[
|\tilde{a}_j| \leq \max_{l \geq j} \left( \frac{l}{j} \right) \cdot a_l x_0^{l-j} \leq \max_{l \geq j}(|a_l| |x_0|^{l-j}) \leq \max_{l \geq j}(|a_l| r^{l-j})
\]

for each \( j \geq 0 \), using (2.68) in Section 2.6 in the first step. Thus

\[
|\tilde{a}_j| r^j \leq \max_{l \geq j}(|a_l| r^l)
\]

for each \( j \geq 0 \), which tends to 0 as \( j \to \infty \), by (4.132). If \( w \in k \) satisfies \(|w| \leq r\), as before, then we get that

\[
f(w + x_0) = \sum_{j=0}^{\infty} \tilde{a}_j w^j,
\]

by interchanging the order of summation in (4.135). Note that the series on the right side of (4.140) converges when \(|w| \leq r\), because (4.139) tends to 0 as \( j \to \infty \).

### 4.8 Contraction mappings

Let \( k \) be a field, and let \(| \cdot |\) be an ultrametric absolute value function on \( k \). Also let \( x_0 \in k \) and \( t > 0 \) be given, and consider the closed ball

\[
B(x_0, t) = \{ x \in k : |x - x_0| \leq t \}
\]

centered at \( x_0 \) with radius \( t \) in \( k \). Suppose that \( g \) is a \( k \)-valued function on \( B(x_0, t) \) which is Lipschitz with constant \( c \geq 0 \) with respect to the ultrametric on \( k \) associated to \(| \cdot |\), so that

\[
|g(x) - g(y)| \leq c |x - y|
\]

for every \( x, y \in B(x_0, t) \). Let \( \alpha \in k \) be given, and put

\[
f(x) = \alpha x + g(x)
\]

for each \( x \in B(x_0, t) \). Thus

\[
|f(x) - f(y)| \leq \max(|\alpha| |x - y|, |g(x) - g(y)|) \leq \max(|\alpha|, c) |x - y|
\]

for every \( x, y \in B(x_0, t) \), by the ultrametric version of the triangle inequality. Similarly,

\[
|\alpha| |x - y| \leq \max(|f(x) - f(y)|, |g(x) - g(y)|)
\]
for each \( x, y \in \overline{B}(x_0, t) \). If
\[
|g(x) - g(y)| < |\alpha| |x - y| \tag{4.146}
\]
for each \( x, y \in \overline{B}(x_0, t) \) with \( x \neq y \), then we get that
\[
|\alpha| |x - y| \leq |f(x) - f(y)| \tag{4.147}
\]
for every \( x, y \in \overline{B}(x_0, t) \), which is trivial when \( x = y \). It follows that
\[
|f(x) - f(y)| = |\alpha| |x - y| \tag{4.148}
\]
for every \( x, y \in \overline{B}(x_0, t) \) under these conditions, since (4.144) holds with \( c = |\alpha| \).

Let \( z \in k \) be given, with
\[
|f(x_0) - z| \leq |\alpha| t. \tag{4.149}
\]
Suppose that \( \alpha \neq 0 \), and put
\[
h(x) = \alpha^{-1} (z - g(x)) \tag{4.150}
\]
for every \( x \in \overline{B}(x_0, t) \). Thus
\[
\alpha (x - h(x)) = f(x) - z \tag{4.151}
\]
for every \( x \in \overline{B}(x_0, t) \), so that (4.149) is the same as saying that
\[
|h(x_0) - x_0| \leq t. \tag{4.152}
\]
Note that \( h \) is Lipschitz with constant \( c/|\alpha| \), because \( g \) is Lipschitz with constant \( c \). In particular, if \( c \leq |\alpha| \), then \( h \) is Lipschitz with constant 1, and (4.152) implies that
\[
h(\overline{B}(x_0, t)) \subseteq \overline{B}(x_0, t), \tag{4.153}
\]
by the ultrametric version of the triangle inequality.

If \( k \) is complete with respect to the ultrametric associated to \( | \cdot | \), then every closed set \( E \subseteq k \) is also complete with respect to the restriction of this ultrametric to \( E \), including \( E = \overline{B}(x_0, r) \). Suppose that
\[
c < |\alpha|, \tag{4.154}
\]
so that \( h \) is Lipschitz with constant \( c/|\alpha| < 1 \). Under these conditions, the contraction mapping theorem implies that there is a unique \( x \in \overline{B}(x_0, t) \) such that
\[
h(x) = x. \tag{4.155}
\]
Equivalently, (4.155) means that
\[
f(x) = z, \tag{4.156}
\]
by (4.151).
4.9 Contraction mappings, 2

Let $k$ be a field with an ultrametric absolute value function $| \cdot |$ again, and suppose that $k$ is complete with respect to the associated ultrametric. Also let

$a_0, a_1, a_2, a_3, \ldots$ be a sequence of elements of $k$ such that

\[
\lim_{j \to \infty} |a_j| r^j = 0
\]

for some $r > 0$, so that the corresponding power series

\[
f(x) = \sum_{j=0}^{\infty} a_j x^j
\]

converges for every $x \in k$ with $|x| \leq r$. If $x_0 \in k$ and $t > 0$ satisfy $|x_0| \leq r$ and

\[
t \leq r,
\]

then

\[
B(x_0, t) \subseteq B(x_0, r) \subseteq B(0, r),
\]

by the ultrametric version of the triangle inequality. In particular, this implies that $f$ is defined on $B(x_0, t)$.

Put

\[
g(x) = f(x) - f'(x_0) x
\]

for each $x \in B(0, r)$. Thus

\[
g(x) - g(y) = f(x) - f(y) - f'(x_0) (x - y)
\]

for every $x, y \in B(0, r)$, and hence

\[
|g(x) - g(y)| \leq \max_{j \geq 2} (|a_j| r^{j-2}) \max(|x - y|, |y - x_0|) |x - y|.
\]

The right side of (4.163) can be estimated using (4.48) and (4.53) in Section 4.3, to get that

\[
|g(x) - g(y)| \leq \left( \max_{j \geq 2} (|a_j| r^{j-2}) \right) \max(|x - y|, |y - x_0|) |x - y|
\]

for every $x, y \in B(0, r)$. If $x, y \in B(x_0, t)$, then $|x - y| \leq t$, by the ultrametric version of the triangle inequality, and so

\[
|g(x) - g(y)| \leq \left( \max_{j \geq 2} (|a_j| r^{j-2}) \right) t |x - y|.
\]

Suppose that $f'(x_0) \neq 0$, and let us take

\[
\alpha = f'(x_0),
\]
so that (4.143) is the same as (4.161). The restriction of $g$ to $\overline{B}(x_0, t)$ is Lipschitz with constant
\[
c = t \max_{j \geq 2}(|a_j| r^{j-2}),
\]
(4.167) by (4.165). Thus (4.154) holds when
\[
t \max_{j \geq 2}(|a_j| r^{j-2}) < |f'(x_0)|.
\]
(4.168) The hypotheses (4.75) and (4.76) in Section 4.4 basically correspond to (4.149), (4.159), and (4.168) here. More precisely, if we take
\[
t = |f'(x_0)|^{-1} |f(x_0) - z|,
\]
(4.169) then (4.149) becomes an equality, by (4.166). With this choice of $t$, (4.75) is equivalent to (4.159), and (4.76) is equivalent to (4.168). This shows that the conclusions of Section 4.4 can also be derived from the discussion in the previous section.

Note that
\[
|f'(x)| = |f'(x_0)|
\]
(4.170) for every $x \in \overline{B}(x_0, t)$ when $f'(x_0) \neq 0$ and $t > 0$ satisfies (4.159) and (4.168). This is basically the same as (4.54) in Section 4.3.

## 4.10 Strassmann’s theorem

Let $k$ be a field, and let $| \cdot |$ be an ultrametric absolute value function on $k$. Suppose that $k$ is complete with respect to the ultrametric associated to $| \cdot |$, and let $a_0, a_1, a_2, a_3, \ldots$ be a sequence of elements of $k$ that satisfies
\[
\lim_{j \to \infty} |a_j| r^j = 0
\]
(4.171) for some positive real number $r$. Thus the power series
\[
f(x) = \sum_{j=0}^{\infty} a_j x^j
\]
(4.172) converges for every $x \in k$ with $|x| \leq r$, as usual. Suppose also that $a_j \neq 0$ for some $j$, and let $N$ be a nonnegative integer such that
\[
|a_j| r^j < |a_N| r^N \text{ for every } j > N.
\]
(4.173) More precisely, one can take $N$ to be the largest nonnegative integer such that
\[
|a_N| r^N = \max_{j \geq 0}(|a_j| r^j),
\]
(4.174) and this will be the smallest $N$ that satisfies (4.173).
4.10. **STRASSMANN’S THEOREM**

Under these conditions, *Strassmann’s theorem* implies that $f$ can have at most $N$ zeros in the closed ball $B(0, r)$ in $k$ centered at 0 and with radius $r$. We follow the very nice proof by induction on $N$ in [4], which is also discussed in [14]. To deal with the base case $N = 0$, observe that

$$ |f(x) - a_0| = \left| \sum_{j=1}^{\infty} a_j x^j \right| \leq \max_{j \geq 1} (|a_j| |x|^j) \leq \max_{j \geq 1} (|a_j| r^j) $$

for every $x \in k$ with $|x| \leq r$, using (2.68) in Section 2.6 in the second step. If (4.173) holds with $N = 0$, then we get that

$$ |f(x) - a_0| < |a_0| $$

for every $x \in k$ with $|x| \leq r$, which implies that

$$ |f(x)| = |a_0| $$

by the ultrametric version of the triangle inequality. In particular, this means that $f(x) \neq 0$ for every $x \in k$ with $|x| \leq r$, because $a_0 \neq 0$ in this situation.

Now suppose that $N \geq 1$, and that Strassmann’s theorem holds for $N - 1$. If $f(x) \neq 0$ for every $x \in k$ with $|x| \leq r$, then there is nothing to do, and so we suppose also that there is an $x_0 \in k$ such that $|x_0| \leq r$ and $f(x_0) = 0$. Let

$$ g_0(x) = \sum_{l=0}^{\infty} b_l x^l $$

be as in (4.113) in Section 4.6, where

$$ b_l = \sum_{j=l+1}^{\infty} a_j x_0^{j-l-1} $$

for each nonnegative integer $l$, as in (4.110). Thus

$$ |b_l| r^l \leq \max_{j \geq l+1} (|a_j| r^{j-1}) $$

for each $l \geq 0$, as in (4.112). By construction,

$$ |b_l - a_{l+1}| = \left| \sum_{j=l+2}^{\infty} a_j x_0^{j-l-1} \right| \leq \max_{j \geq l+2} (|a_j| |x_0|^{j-l-1}) \leq \max_{j \geq l+2} (|a_j| r^{j-l-1}) $$

for each $l \geq 0$, using (2.68) in Section 2.6 in the second step. If $N$ satisfies (4.173), then it is easy to see that

$$ |b_{N-1} - a_N| < |a_N|, $$
and hence
\[ |b_{N-1}| = |a_N|, \]  
by the ultrametric version of the triangle inequality. It follows that
\[ |b_l| r^l < |a_N| r^{N-1} = |b_{N-1}| r^{N-1} \]
for every \( l > N - 1 \), by (4.173) and (4.180). This means that \( g_0 \) satisfies the analogous condition with \( N - 1 \) instead of \( N \), so that \( g_0 \) has at most \( N - 1 \) zeros in \( \mathcal{B}(0, r) \), by the induction hypothesis. We also have that
\[ f(x) = (x - x_0) g_0(x) \]
for every \( x \in k \) with \( |x| \leq r \), by (4.115) in Section 4.6, and the condition that \( f(x_0) = 0 \). This implies that \( f \) can have at most \( N \) zeros in \( \mathcal{B}(0, r) \), as desired.

As an application of Strassmann’s theorem, let \( N \) be a positive integer that satisfies (4.174). As before, one can take \( N \) to be the largest positive integer such that
\[ |a_N| r^N = \max_{j \geq 1} (|a_j| r^j). \]
If \( z \) is any element of \( k \), then \( f(x) - z \) can be expressed by a power series in \( x \) on \( \mathcal{B}(0, r) \), where the constant term is equal to \( a_0 - z \), and the other coefficients are the same as the coefficients \( a_j \) of \( f \), with \( j \geq 1 \). This means that \( N \) also satisfies the analogue of (4.173) for \( f(x) - z \) instead of \( f(x) \), because \( N \geq 1 \) by hypothesis. Thus Strassmann’s theorem implies that \( f(x) - z \) can have at most \( N \) zeros in \( \mathcal{B}(0, r) \), which is to say that there are at most \( N \) points \( x \in \mathcal{B}(0, r) \) at which \( f(x) = z \).

### 4.11 The exponential function

Let \( k \) be a field of characteristic 0, and let
\[ E(X) = \sum_{j=0}^{\infty} \frac{X^j}{j!} \]
be the exponential function on \( k \), which is initially considered as a formal power series in the indeterminate \( X \). Of course, this uses the natural embedding of \( \mathbb{Q} \) into \( k \), that results from \( k \) having characteristic 0. If \( X \) and \( Y \) are commuting indeterminates, then it is easy to see that
\[ E(X + Y) = E(X) E(Y) \]
as formal power series in \( X \) and \( Y \), as in (4.13) in Section 4.1. Note that the formal derivative of \( E(X) \) is equal to \( E(X) \), as usual.

Let \( |\cdot| \) be an absolute value function on \( k \), and let us consider the convergence properties of
\[ E(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!} \]
for $x \in k$. To do this, we also suppose from now on in this section that $k$ is complete with respect to the metric that corresponds to $| \cdot |$. Remember that $| \cdot |$ induces an absolute value function on $\mathbb{Q}$, using the natural embedding of $\mathbb{Q}$ into $k$. The behavior of this absolute value function on $\mathbb{Q}$ is characterized by Ostrowski’s theorem, as discussed at the end of Section 2.3. This leads to various cases for $k$, as follows.

Suppose first that the induced absolute value function on $\mathbb{Q}$ is archimedean, and hence that $| \cdot |$ is archimedean on $k$. In this case, Ostrowski’s theorem implies that the induced absolute value function on $\mathbb{Q}$ is a positive power of the standard absolute value function. We may as well suppose that the induced absolute value function on $\mathbb{Q}$ is equal to the standard absolute value function, by replacing $| \cdot |$ on $k$ by a suitable power of itself, as in Section 2.9. Under these conditions, $k$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, and $| \cdot |$ corresponds to the standard absolute value function, as in Sections 2.10 and 2.13. Similarly, (4.189) corresponds to the usual exponential function on $\mathbb{R}$ or $\mathbb{C}$, as in Section 4.1.

Otherwise, if the induced absolute value function on $\mathbb{Q}$ is non-archimedean, then $| \cdot |$ is non-archimedean on $k$ too. This implies that $| \cdot |$ is an ultrametric absolute value function on $k$, as in Section 2.3. In this situation, the right side of (4.189) converges in $k$ exactly when
\begin{equation}
x^j / j! \to 0 \quad \text{as } j \to \infty,
\end{equation}
as in Section 2.6. If $x, y \in k$ both have this property, then the Cauchy product of the series used to define $E(x)$ and $E(y)$ also converges in $k$, and has sum equal to $E(x) E(y)$, as in Section 2.6 again. This means that the series used to define $E(x + y)$ converges in $k$, and that
\begin{equation}
E(x + y) = E(x) E(y),
\end{equation}
since the series defining $E(x + y)$ is the same as the Cauchy product of the series defining $E(x)$ and $E(y)$, as before.

In particular, the induced absolute value function on $\mathbb{Q}$ is non-archimedean when it is trivial, in which case (4.190) holds exactly when $|x| < 1$. Suppose now that the induced absolute value function on $\mathbb{Q}$ is non-archimedean and not trivial. By Ostrowski’s theorem, there is a prime number $p$ such that the induced absolute value function on $\mathbb{Q}$ is equal to a positive power of the $p$-adic absolute value. As before, we may as well suppose that the induced absolute value function on $\mathbb{Q}$ is equal to the $p$-adic absolute value, by replacing $| \cdot |$ with a suitable positive power of itself on $k$, if necessary. We shall restrict our attention to this case for the rest of the section.

Note that
\begin{equation}
|x^j / j!| = |x|^j / |j!|_p
\end{equation}
for every $x \in k$ and nonnegative integer $j$, where $|j!|_p$ is the $p$-adic absolute value of $j!$. This follows from the hypothesis that the absolute value function induced on $\mathbb{Q}$ by $| \cdot |$ on $k$ is equal to $| \cdot |_p$. In order to find out when $x \in k$ satisfies (4.190), we would like to estimate $|j!|_p$, which is determined by the total
number of factors of $p$ in $j!$. Let $\lfloor r \rfloor$ denote the integer part of a nonnegative real number $r$, which is the largest nonnegative integer less than or equal to $r$. It is well known that the total number of factors of $p$ in $j!$ is equal to

\begin{equation}
\sum_{l=1}^{\infty} \lfloor j/p^l \rfloor
\end{equation}

for each nonnegative integer $j$. More precisely, $\lfloor j/p^l \rfloor$ is the number of positive integers less than or equal to $j$ that are divisible by $p^l$. Thus the first term in the series (4.193) counts a factor of $p$ in $j!$ for each positive integer less than or equal to $j$ that is divisible by $p$. The second term in (4.193) counts another factor of $p$ for each positive integer less than or equal to $j$ that is divisible by $p^2$, and so on.

Of course, the infinite series (4.193) is really a finite sum, since $\lfloor j/p^l \rfloor = 0$ when $j < p^l$. We also have that

\begin{equation}
\sum_{l=1}^{\infty} \lfloor j/p^l \rfloor < \sum_{l=1}^{\infty} j/p^l = j \frac{(1/p)}{1 - (1/p)} = \frac{j}{p - 1}
\end{equation}

for each positive integer $j$. It follows that

\begin{equation}
1/|j!|_p \leq p^{j/(p-1)}
\end{equation}

for every nonnegative integer $j$, so that

\begin{equation}
|x^{j}/j!| = |x|^j/|j!|_p \leq (|x| p^{1/(p-1)})^j
\end{equation}

for every $x \in k$. If $x \in k$ satisfies

\begin{equation}
|x| < p^{-1/(p-1)},
\end{equation}

then $|x| p^{j/(p-1)} < 1$, and hence $(|x| p^{j/(p-1)})^j \to 0$ as $j \to \infty$. Combining this with (4.196), we get that (4.190) holds when $x$ satisfies (4.197).

If $j = p^n$ for some positive integer $n$, then we have that

\begin{equation}
\sum_{l=1}^{\infty} \lfloor j/p^l \rfloor = \sum_{l=1}^{n} p^{-l} = \sum_{l=0}^{n-1} p^l = \frac{p^n - 1}{p - 1} = \frac{j - 1}{p - 1}.
\end{equation}

This means that

\begin{equation}
1/|j!|_p = p^{(j-1)/(p-1)}
\end{equation}

when $j$ is a positive power of $p$, so that

\begin{equation}
|x^{j}/j!| = |x|^j/|j!|_p = (|x| p^{1/(p-1)})^j p^{-1/(p-1)}
\end{equation}

for every $x \in k$ in this case. If $x \in k$ satisfies

\begin{equation}
|x| \geq p^{-1/(p-1)},
\end{equation}

then $|x| p^{j/(p-1)} \geq 1$, and the right side of (4.200) is greater than or equal to $p^{-1/(p-1)}$. This implies that $x^{j}/j!$ does not converge to 0 in $k$ as $j \to \infty$ when $x$ satisfies (4.201), since (4.200) holds for infinitely many $j$. It follows that (4.190) holds if and only if $x \in k$ satisfies (4.197) under these conditions.
4.12 Some additional properties

Let \( k \) be a field with characteristic 0 again, and let \(|\cdot|\) be an ultrametric absolute value function on \( k \). Suppose from now on in this section that \( k \) is complete with respect to the ultrametric associated to \(|\cdot|\). Let us also suppose for the first part of the section that the induced absolute value function on \( \mathbb{Q} \) is the \( p \)-adic absolute value for some prime number \( p \). Thus the domain of the exponential function (4.189) is given by

\[
D = B(0, p^{-1/(p-1)}) = \{ x \in k : |x| < p^{-1/(p-1)} \},
\]

as in the previous section.

Note that

\[
|E(x) - 1| \leq \max_{j \geq 1} |x^j/j!|
\]

for every \( x \in D \), as in (2.68) in Section 2.6. This implies that

\[
|E(x) - 1| < 1
\]

for every \( x \in D \), by (4.196) and the definition (4.202) of \( D \). It follows that

\[
|E(x)| = 1
\]

for every \( x \in D \), by the ultrametric version of the triangle inequality.

Let \( j \) be a positive integer, and remember that

\[
(p - 1) \sum_{l=1}^{\infty} [j/p^l] < j,
\]

by (4.194). The left side of (4.206) is an integer, and so we get that

\[
(p - 1) \sum_{l=1}^{\infty} [j/p^l] \leq j - 1.
\]

Equivalently,

\[
\sum_{l=1}^{\infty} [j/p^l] \leq \frac{j - 1}{p - 1},
\]

which implies that

\[
1/|j!|_p \leq p^{(j-1)/(p-1)}
\]

for every positive integer \( j \). A more precise analysis of (4.193) and hence \(|j!|_p\) is given in Problem 165 on p113 of [14]. Note that the sum in the statement of Lemma 4.5.4 on p112 of [14] should start at \( i = 1 \), as well as the sum in Problem 164 on p112 of [14].

Using (4.209), we get that

\[
|x^{j-1}/j!| = |x|^{j-1}/|j!|_p \leq \left(|x| p^{(j-1)/(p-1)}\right)^{j-1}
\]
for every \( x \in k \) and positive integer \( j \). Equivalently,

\[
|x^j/j!| \leq (|x|^{p^{1/(p-1)}})^{j-1} |x|
\]

for every \( x \in k \) and positive integer \( j \), which is a refinement of (4.196). If \( x \) satisfies (4.197), then we get that

\[
|x^j/j!| \leq |x|
\]

for every positive integer \( j \), which is a refinement of (4.196). If \( x \) satisfies (4.197), then we get that

\[
|E(x) - 1| \leq |x|
\]

for every \( x \in D \), by (4.203). In particular,

\[
|E(x) - 1| < p^{-1/(p-1)}
\]

for every \( x \in D \), by (4.213) and the definition (4.202) of \( D \).

More precisely, (4.211) implies that

\[
|E(x) - 1 - x| \leq \max_{j \geq 2} |x^j/j!|
\]

for every \( x \in D \) with \( x \neq 0 \), and every integer \( j \geq 2 \). We also have that

\[
|E(x) - 1| = |x|
\]

for every \( x \in D \) with \( x \neq 0 \), by the ultrametric version of the triangle inequality. Of course, (4.218) also holds when \( x = 0 \), so that (4.218) holds for every \( x \in D \).

Let \( z \in k \) be given, with

\[
|z - 1| < p^{-1/(p-1)}.
\]

We would like to find \( x \in D \) such that \( E(x) = z \). If \( z = 1 \), then we can take \( x = 0 \), and so we suppose from now on that \( z \neq 1 \). Put \( r = |z - 1| > 0 \) and \( x_0 = 0 \), and let us apply the discussion in Section 4.4 to \( f = E \). Thus \( f(x_0) = E(0) = 1 \) and \( f'(x_0) = E'(0) = 1 \) in this case. It is easy to see that (4.75) holds under these conditions, by definition of \( r \). One can also check that (4.76) holds in this situation, because of (4.209). The discussion in Section 4.4 leads to a point \( x \in k \) such that \( |x| \leq r \) and \( E(x) = f(x) = z \), as desired.

Let us now consider the case where the induced absolute value function on \( Q \) is trivial. Thus the domain of the exponential function (4.189) is

\[
D = B(0, 1) = \{x \in k : |x| < 1\},
\]
as in the previous section. It is easy to see that (4.204) still holds for every $x \in D$ in this situation, using (4.203) and the fact that

\begin{equation}
|x^j/j!| = |x|^j
\end{equation}

for each $x \in k$ and positive integer $j$. This implies that (4.205) also holds for every $x \in D$, as before. Similarly, (4.213) holds for every $x \in D$, by (4.203), (4.221), and the definition (4.220) of $D$. We also have (4.218) for every $x \in D$, for essentially the same reasons as before. If $z \in k$ satisfies

\begin{equation}
|z - 1| < 1,
\end{equation}

then one can find an $x \in D$ such that $E(x) = z$, using the discussion in Section 4.4 again.
Chapter 5

Geometry and measure

5.1 Diameters of sets

Let \((M, d(x, y))\) be a metric space. As usual, a subset \(A\) of \(M\) is said to be \textit{bounded} if \(A\) is contained in a ball of finite radius in \(M\). If \(A \subseteq M\) is bounded and nonempty, then the \textit{diameter} of \(A\) is defined by

\[
\text{diam } A = \sup \{d(x, y) : x, y \in A\}.
\]

(5.1)

It will be convenient to put \(\text{diam } A = +\infty\) when \(A\) is unbounded, and \(\text{diam } A = 0\) when \(A = \emptyset\). Note that

\[
\text{diam } \overline{A} = \text{diam } A
\]

(5.2)

for every \(A \subseteq M\), where \(\overline{A}\) denotes the closure of \(A\) in \(M\).

Let \(A \subseteq M\) and a positive real number \(r\) be given, and put

\[
A_r = \bigcup_{x \in A} B(x, r) = \{y \in M : d(x, y) < r \text{ for some } x \in A\}.
\]

(5.3)

Thus \(A \subseteq A_r\), and \(A_r\) is an open set in \(M\), because \(A_r\) is a union of open sets. It is easy to see that

\[
\text{diam } A_r \leq \text{diam } A + 2r,
\]

(5.4)

by the triangle inequality. If \(d(\cdot, \cdot)\) is an ultrametric on \(M\), then we have that

\[
\text{diam } A_r \leq \max(\text{diam } A, r).
\]

(5.5)

If \(M\) is any metric space, \(A \subseteq M\) is bounded, and \(p \in A\), then

\[
A \subseteq \overline{B}(p, \text{diam } A).
\]

(5.6)

Of course,

\[
\text{diam } \overline{B}(x, r) \leq 2r
\]

(5.7)
5.2. **Hausdorff Content**

for every \( x \in M \) and \( r \geq 0 \), by the triangle inequality. If \( d(\cdot, \cdot) \) is an ultrametric
on \( M \), then

\[
\text{diam} B(x, r) \leq r
\]

for every \( x \in M \) and \( r \geq 0 \). If \( M \) is the real line with the standard metric, and
\( A \subseteq \mathbb{R} \) is bounded, then there is a closed interval \( I \subseteq \mathbb{R} \) such that

\[
\text{diam} A = \text{diam} I.
\]

In this case, \( I \) is the same as the closed ball centered at its midpoint, with radius
equal to one-half the diameter of \( I \), which is the same as the length of \( I \).

Let \( n \) be a positive integer, and suppose that \( M = \mathbb{R}^n \), equipped with the
metric

\[
d(x, y) = \max_{1 \leq j \leq n} |x_j - y_j|.
\]

Of course, the topology on \( \mathbb{R}^n \) determined by this metric is the same as the
standard topology on \( \mathbb{R}^n \), which is the product topology on \( \mathbb{R} \). Note that open and closed balls in \( \mathbb{R}^n \) with respect
to (5.10) are open and closed cubes with sides parallel to the axes, respectively.
If \( A \subseteq \mathbb{R}^n \) is a bounded set with diameter less than or equal to \( r \) for some \( r \geq 0 \),
then the \( n \) coordinate projections of \( A \) into \( \mathbb{R} \) each have diameter less than or
equal to \( r \) as well. This implies that \( A \) is contained in a closed cube in \( \mathbb{R}^n \) with
sides parallel to the axes and with side length \( r \), which is the same as a closed
ball in \( \mathbb{R}^n \) with radius \( r/2 \) with respect to the metric (5.10).

**5.2 Hausdorff content**

Let \((M, d(x, y))\) be a metric space, and let \( \alpha \) be a nonnegative real number. If
\( A \subseteq M \) is bounded and nonempty, then \( \text{diam} A \) is defined as in the previous
section, and so

\[
(\text{diam} A)^\alpha
\]

is defined for each \( \alpha > 0 \). If \( \alpha = 0 \), then (5.11) is interpreted as being equal
to 1 when \( A \) is bounded and nonempty, even when \( A \) has only one element, so
that \( \text{diam} A = 0 \). It will be convenient to interpret (5.11) as being equal to \( +\infty \)
for every \( \alpha \geq 0 \) when \( A \) is unbounded, including \( \alpha = 0 \). Similarly, we interpret
(5.11) as being equal to 0 for every \( \alpha \geq 0 \) when \( A = \emptyset \), including \( \alpha = 0 \).

The \( \alpha \)-dimensional Hausdorff content \( H_\alpha^\alpha(E) \) of a set \( E \subseteq M \) is defined to
be the infimum of

\[
\sum_j (\text{diam} A_j)^\alpha
\]

over all collections \( \{A_j\}_j \) of finitely or countably many subsets of \( M \) such that
\( E \subseteq \bigcup_j A_j \). If \( A_j \) is unbounded for any \( j \), then \( (\text{diam} A_j)^\alpha = +\infty \), and (5.12)
is infinite. If there are infinitely many \( A_j \)'s, then (5.12) is interpreted as the
supremum over all finite subsums, as in Section 3.10, which may be infinite even
when \( A_j \) is bounded for each \( j \).
By construction,
\[ H^\alpha_{\text{con}}(E) \leq (\text{diam } E)^\alpha \]
for every \( E \subseteq M \) and \( \alpha \geq 0 \), since we can cover \( E \) by itself. In particular,
\[ H^\alpha_{\text{con}}(\emptyset) = 0. \]

Alternatively, one might consider the empty set to be covered by the empty family of subsets of \( M \), for which the corresponding empty sum (5.12) is equal to 0. If \( E \subseteq \tilde{E} \subseteq M \), then it is easy to see that
\[ H^\alpha_{\text{con}}(E) \leq H^\alpha_{\text{con}}(\tilde{E}) \]
for every \( \alpha \geq 0 \), because every covering of \( \tilde{E} \) in \( M \) is a covering of \( E \) too. If \( E_1, E_2, E_3, \ldots \) is any sequence of subsets of \( M \), then one can check that
\[ H^\alpha_{\text{con}} \left( \bigcup_{l=1}^{\infty} E_l \right) \leq \sum_{l=1}^{\infty} H^\alpha_{\text{con}}(E_l), \]
for every \( \alpha \geq 0 \), by combining coverings of the \( E_l \)'s to get coverings of their union. This is a bit simpler in the case of finite unions, and of course (5.16) is trivial when \( H^\alpha_{\text{con}}(E_l) = +\infty \) for any \( l \). Thus \( H^\alpha_{\text{con}} \) defines an outer measure on \( M \) for every \( \alpha \geq 0 \).

Because of (5.2), one might as well restrict one’s attention to coverings of \( E \subseteq M \) by closed subsets of \( M \) in the definition of \( H^\alpha_{\text{con}}(E) \). Alternatively, one can restrict one’s attention to coverings of \( E \) by open subsets of \( M \) and get the same result for \( H^\alpha_{\text{con}}(E) \), because of (5.4). This implies that one can restrict one’s attention to coverings of \( E \) by finitely many open subsets of \( M \) when \( E \) is compact. If \( M \) is the real line with the standard metric, then one can restrict one’s attention to coverings of \( E \) by intervals, using (5.9). Similarly, if \( d(\cdot, \cdot) \) is an ultrametric on \( M \), then one can restrict one’s attention to coverings of \( E \subseteq M \) by closed balls in the definition of \( H^\alpha_{\text{con}}(E) \), by the remarks in the previous section.

### 5.3 Restricting the diameters

Let \((M, d(x, y))\) be a metric space again, and let \( \alpha \geq 0 \) and \( 0 < \delta \leq \infty \) be given. If \( E \subseteq M \), then \( H^\alpha_{\delta}(E) \) is defined to be the infimum of
\[ \sum_j (\text{diam } A_j)^\alpha \]
over all collections \( \{A_j\}_j \) of finitely or countably many subsets of \( M \) such that \( E \subseteq \bigcup_j A_j \) and
\[ \text{diam } A_j < \delta \]
for each \( j \), if such a covering exists, and otherwise we put \( H^\alpha_{\delta}(E) = +\infty \). If \( M \) is separable, then \( M \) can be covered by finitely or countably many balls of
radius $r$ for each $r > 0$, which implies that these coverings always exist. If $0 < \delta \leq \eta \leq \infty$, then it is easy to see that

$$H^\alpha_{\eta}(E) \leq H^\alpha_{\delta}(E),$$

(5.19)

because the coverings used to define $H^\alpha_{\delta}(E)$ can also be used in the definition of $H^\alpha_{\eta}(E)$ in this case.

Note that $H^\alpha_{\infty}(E)$ is the infimum of (5.17) over all collections $\{A_j\}_j$ of finitely or countably many bounded subsets of $M$ such that $E \subseteq \bigcup_j A_j$, and that these coverings always exist. The only difference between this and the definition of $H^\alpha_{\text{con}}(E)$ is the restriction to coverings of $E$ by bounded subsets of $M$. This does not affect the infimum, because (5.17) is infinite as soon as $A_j$ is unbounded for any $j$. It follows that

$$H^\alpha_{\infty}(E) = H^\alpha_{\text{con}}(E),$$

(5.20)

for every $E \subseteq M$.

It is easy to see that $H^\alpha_{\delta}$ is an outer measure on $M$ for each $\delta > 0$, for essentially the same reasons as for $H^\alpha_{\text{con}}$. As before, one can restrict one’s attention to coverings of $E$ by open subsets of $M$ or closed subsets of $M$ in the definition of $H^\alpha_{\delta}(E)$, and to finite coverings of $E$ when $E$ is compact. If $M = \mathbb{R}$ with the standard metric, then one can restrict one’s attention to coverings of $E \subseteq \mathbb{R}$ by intervals. If $d(x, y)$ is an ultrametric on a set $M$, then one can restrict one’s attention to coverings of $E \subseteq M$ by closed balls.

Alternatively, let $\bar{H}^\alpha_{\delta}(E)$ be the infimum of (5.17) over all collections $\{A_j\}_j$ of finitely or countably many subsets of $M$ such that $E \subseteq \bigcup_j A_j$ and

$$\text{diam } A_j \leq \delta$$

(5.21)

for each $j$, if such a covering exists, and otherwise put $\bar{H}^\alpha_{\delta}(E) = +\infty$. The only difference between this and $H^\alpha_{\delta}(E)$ is that we replace (5.18) with (5.21). Of course, (5.21) is vacuous when $\delta = \infty$, so that $\bar{H}^\alpha_{\infty}(E) = H^\alpha_{\text{con}}(E)$ by definition. If $M$ is separable, then these coverings exist for every $\delta > 0$, as before.

If $0 < \delta < \eta \leq \infty$, then

$$\bar{H}^\alpha_{\delta}(E) \leq \bar{H}^\alpha_{\eta}(E),$$

(5.22)

for the same reasons as in (5.19). Similarly, it is easy to see that

$$\bar{H}^\alpha_{\eta}(E) \leq \bar{H}^\alpha_{\delta}(E),$$

(5.23)

because any of the coverings of $E$ that can be used in the definition of $H^\alpha_{\delta}(E)$ can also be used in the definition of $\bar{H}^\alpha_{\delta}(E)$. If $0 < \delta < \eta \leq \infty$, then

$$H^\alpha_{\eta}(E) \leq \bar{H}^\alpha_{\delta}(E),$$

(5.24)

because the coverings of $E$ used in the definition of $\bar{H}^\alpha_{\delta}(E)$ can also be used in the definition of $H^\alpha_{\eta}(E)$. 

As before, one can check that $\tilde{H}_\alpha^\delta$ is an outer measure on $M$ for each $\delta > 0$. One can also restrict one’s attention to coverings of $E$ by closed subsets of $M$ in the definition of $\tilde{H}_\alpha^\delta(E)$, because of (5.2). If $M = \mathbb{R}$ with the standard metric, then one can restrict one’s attention to coverings of $E$ by closed intervals in the definition of $\tilde{H}_\alpha^\delta(E)$, by (5.9). If $d(x,y)$ is an ultrametric on $M$, then one can restrict one’s attention to coverings of $E$ by closed balls in the definition of $\tilde{H}_\alpha^\delta(E)$, using the remarks in Section 5.1.

The condition (5.21) seems to be used more commonly, but there are some advantages to (5.18). In particular, the argument for restricting to coverings of $E$ by open subsets of $M$ does not work for the condition (5.21). However, the two conditions are essentially equivalent in the limit as $\delta \to 0$, by (5.23) and (5.24).

### 5.4 Hausdorff measure

Let $(M,d(x,y))$ be a metric space again, and let $\alpha \geq 0$ be given. The $\alpha$-dimensional Hausdorff measure of $E \subseteq M$ is defined by

\[
H^\alpha(E) = \sup_{\delta > 0} H^\alpha_\delta(E).
\]

This can also be considered as a limit of $H^\alpha_\delta(E)$ as $\delta \to 0$, because of (5.19). It is easy to see that $H^\alpha$ is an outer measure on $M$, because of the corresponding property of $H^\alpha_\delta$ for each $\delta > 0$. If $\alpha = 0$, then one can check that $H^\alpha$ reduces to counting measure on $M$.

Suppose for the moment that $M = \mathbb{R}$ with the standard metric. In this case, we have that

\[
H^1_\delta(E) = H^1_{\text{con}}(E)
\]

for each $\delta > 0$ and $E \subseteq \mathbb{R}$. This is because every interval in $\mathbb{R}$ can be expressed as a union of finitely many subintervals with arbitrarily small diameter, where the sum of the lengths of the smaller intervals is equal to the length of the initial interval. It follows that

\[
H^1(E) = H^1_{\text{con}}(E)
\]

for every $E \subseteq \mathbb{R}$, by taking the supremum over $\delta > 0$ of (5.26). Note that $H^1_{\text{con}}$ is basically the same as Lebesgue outer measure on $\mathbb{R}$. In particular,

\[
H^1_{\text{con}}([a,b]) = b - a
\]

for every $a, b \in \mathbb{R}$ with $a \leq b$, where $[a,b]$ is the usual closed interval in $\mathbb{R}$ from $a$ to $b$, consisting of $x \in \mathbb{R}$ with $a \leq x \leq b$. More precisely,

\[
H^1_{\text{con}}([a,b]) \leq \text{diam}[a,b] = b - a
\]

by (5.13) in Section 5.2. In order to get the opposite inequality, it suffices to show that

\[
b - a \leq \sum_j \text{diam} A_j
\]
5.5. HAUSDORFF DIMENSION

for any collection \( \{ A_j \}_j \) of finitely or countably many subsets of \( \mathbb{R} \) such that \([a, b] \subseteq \bigcup_j A_j\). As in Section 5.2, one can reduce to the case where the \( A_j \)'s are open intervals in \( \mathbb{R} \), and it is enough to consider coverings of \([a, b]\) by finitely many open intervals, because \([a, b]\) is compact.

Let \((M, d(x, y))\) be an arbitrary metric space again, and suppose that \(E_1, E_2\) are subsets of \(M\) such that

\[
d(x, y) \geq \eta
\]

for some \(\eta > 0\) and every \(x \in E_1, y \in E_2\). This implies that any subset of \(M\) with diameter less than \(\delta \leq \eta\) cannot intersect both \(E_1\) and \(E_2\). Thus coverings of \(E_1 \cup E_2\) by such subsets of \(M\) can be split into coverings of \(E_1\) and \(E_2\) separately. It follows that

\[
H_\alpha^\delta(E_1 \cup E_2) \geq H_\alpha^\delta(E_1) + H_\alpha^\delta(E_2)
\]

for \(0 < \delta \leq \eta\), by splitting the corresponding sums (5.31). Taking the limit as \(\delta \to 0\), we get that

\[
H^\alpha(E_1 \cup E_2) \geq H^\alpha(E_1) + H^\alpha(E_2)
\]

under these conditions. Because \(H^\alpha\) is an outer measure on \(M\), there is a standard way to define a collection of measurable subsets of \(M\) with respect to \(H^\alpha\). More precisely, this collection of measurable sets is a \(\sigma\)-algebra of subsets of \(M\), and \(H^\alpha\) is countably additive on this \(\sigma\)-algebra. It is also well known that the Borel subsets of \(M\) are measurable with respect to \(H^\alpha\), because of (5.33).

Of course,

\[
H^\alpha_{\text{con}}(E) \leq H^\alpha(E)
\]

for every \(E \subseteq M\), by (5.20). If \(H^\alpha_{\text{con}}(E) = 0\), then the coverings of \(E\) for which the corresponding sums (5.12) are small automatically involve subsets of \(M\) with small diameter. This implies that \(H^\alpha_\delta(E) = 0\) for every \(\delta > 0\), and hence that \(H^\alpha(E) = 0\). It follows that \(H^\alpha(E) = 0\) if and only if \(H^\alpha_{\text{con}}(E) = 0\), using (5.34) for the “only if” part.

5.5 Hausdorff dimension

Let \((M, d(x, y))\) be a metric space, and let \(E \subseteq M\), \(0 \leq \alpha < \beta < \infty\), and \(0 < \delta < \infty\) be given. If \(\{A_j\}_j\) is a collection of finitely or countably many subsets of \(M\) such that \(E \subseteq \bigcup_j A_j\) and \(\text{diam } A_j < \delta\) for each \(j\), then

\[
H_\delta^\beta(E) \leq \sum_j (\text{diam } A_j)^\beta \leq \delta^{\beta - \alpha} \sum_j (\text{diam } A_j)^\alpha,
\]

using the definition of \(H_\delta^\beta(E)\) in the first step. Taking the infimum over all such collections \(\{A_j\}_j\), we get that

\[
H_\delta^\beta(E) \leq \delta^{\beta - \alpha} H_\delta^\alpha(E) \leq \delta^{\beta - \alpha} H^\alpha(E),
\]
using the definition (5.25) of $H^\alpha(E)$ in the second step. If $H^\alpha(E) < \infty$, then it follows that

$$H^\beta(E) = 0$$

for every $\beta > \alpha$, by taking the limit as $\delta \to 0$ in (5.36).

The Hausdorff dimension of $E$ may be defined as

$$\dim_H E = \inf \{ \alpha \geq 0 : H^\alpha(E) < \infty \},$$

which is interpreted as being $+\infty$ when $H^\alpha(E) = \infty$ for every $\alpha \geq 0$. If $\beta \geq 0$ satisfies $\beta > \dim_H E$, then there is an $\alpha \geq 0$ such that $\alpha < \beta$ and $H^\alpha(E) < \infty$. This implies that $H^\beta(E) = 0$, as in (5.37). Thus $\dim_H E$ can also be given by

$$\dim_H E = \inf \{ \beta \geq 0 : H^\beta(E) = 0 \},$$

where the infimum is interpreted as being $+\infty$ when $H^\beta(E) > 0$ for every $\beta \geq 0$. Combining this with the remarks at the end of the preceding section, we get that $\dim_H E$ may be defined equivalently by

$$\dim_H(E) = \inf \{ \beta \geq 0 : H^\beta_{\text{con}}(E) = 0 \},$$

which is also interpreted as being $+\infty$ when $H^\beta_{\text{con}}(E) > 0$ for every $\beta \geq 0$.

If $E \subseteq \tilde{E} \subseteq M$, then it is easy to see that

$$\dim_H E \leq \dim_H \tilde{E},$$

because of the analogous property of the Hausdorff measures. Let $\alpha \geq 0$ be given, and suppose that $E_1, E_2, E_3, \ldots$ is a sequence of subsets of $M$ such that

$$\dim_H E_l \leq \alpha$$

for each $l \geq 1$. This implies that

$$H^\beta(E_l) = 0$$

for every $\beta > \alpha$ and $l \geq 1$, as in (5.37). It follows that

$$H^\beta \left( \bigcup_{l=1}^{\infty} E_l \right) = 0$$

for every $\beta > \alpha$, and hence that

$$\dim_H \left( \bigcup_{l=1}^{\infty} E_l \right) \leq \alpha.$$
for every $a > 0$ when $d(x, y)$ is an ultrametric on $M$. If $A \subseteq M$, then it is easy to see that the diameter of $A$ with respect to $d(x, y)^a$ is the same as the diameter of $A$ with respect to $d(x, y)$ to the power $a$. Using this, one can check that the $\alpha$-dimensional Hausdorff content of $E \subseteq M$ with respect to $d(x, y)^a$ is the same as the $\alpha a$-dimensional content of $E$ with respect to $d(x, y)$. Similarly, the $\alpha$-dimensional Hausdorff measure of $E \subseteq M$ with respect to $d(x, y)^a$ is equal to the $\alpha a$-dimensional Hausdorff measure of $E$ with respect to $d(x, y)$. There is an analogous statement for $H_\alpha$ for each $\delta > 0$, where the $\delta$ used for $d(x, y)$ corresponds to $\delta^a$ for $d(x, y)^a$. It follows that the Hausdorff dimension of $E \subseteq M$ with respect to $d(x, y)^a$ is equal to the Hausdorff dimension of $E$ with respect to $d(x, y)$ divided by $a$.

5.6 Some regularity properties

Let $(M, d(x,y))$ be a metric space, and let $E$ be a subset of $M$. Suppose that the $\alpha$-dimensional Hausdorff measure of $E$ is finite for some $\alpha \geq 0$. In particular, this means that $H_\alpha^\delta(E) < \infty$ for each $\delta > 0$, where $H_\alpha^\delta$ is as in Section 5.3. Let $l$ be a positive integer, and consider $H_\alpha^\delta(E)$ with $\delta = 1/l$. By definition of $H_\alpha^\delta(E)$, there is a collection $\{A_{j,l}\}_j$ of finitely or countable many subsets of $M$ such that $E \subseteq \bigcup_j A_{j,l}$,}

\begin{equation}
\text{diam } A_{j,l} < 1/l
\end{equation}

for each $j$, and

\begin{equation}
\sum_j (\text{diam } A_{j,l})^\alpha < H_1^\alpha(E) + 1/l \leq H^\alpha(E) + 1/l.
\end{equation}

As in Section 5.3, we may as well ask that $A_{j,l}$ be an open set in $M$ for each $j$ as well. This implies that

\begin{equation}
U_l = \bigcup_j A_{j,n}
\end{equation}

is an open set in $M$ for each $l$, so that

\begin{equation}
\tilde{E} = \bigcap_{l=1}^\infty U_l
\end{equation}

is a Borel set in $M$, and more precisely a $G_\delta$ set in $M$. Of course, $E \subseteq U_l$ for each $l$, by construction, and hence $E \subseteq \tilde{E}$. Similarly, if $A_{j,l}$ is a Borel set for each $j$ and $l$, then $U_l$ is a Borel set for each $l$, and hence $\tilde{E}$ is a Borel set. In particular, if we had used the outer measures $H_\alpha^\delta$ in Section 5.3 instead of $H_\alpha^\delta$, then we could take $A_{j,l}$ to be a closed set for each $j$ and $l$, if not an open set.

Let us check that

\begin{equation}
H^\alpha(\tilde{E}) = H^\alpha(E).
\end{equation}

To do this, it suffices to show that

\begin{equation}
H^\alpha(\tilde{E}) \leq H^\alpha(E),
\end{equation}

\begin{equation}
H^\alpha(E) \leq H^\alpha(\tilde{E}).
\end{equation}
because the opposite inequality follows from the fact that $E \subseteq \tilde{E}$. Observe that

$$H_{1/l}^\alpha(\tilde{E}) \leq \sum_j (\text{diam } A_{j,l})^\alpha \tag{5.52}$$

for each $l$, because $\{A_{j,l}\}_j$ can also be used as a covering of $\tilde{E}$ in the definition of $H_{1/l}^\alpha(\tilde{E})$ for each $l$. Combining this with (5.47), we get that

$$H_{1/l}^\alpha(\tilde{E}) < H^\alpha(E) + 1/l \tag{5.53}$$

for each $l$. If we take the limit as $l \to \infty$ in (5.53), then we get (5.51), as desired.

It would be nice if for every $\epsilon > 0$ there is an open set $U \subseteq M$ such that $E \subseteq U$ and

$$H^\alpha(U) < H^\alpha(E) + \epsilon, \tag{5.54}$$

but this does not always work. If $M$ is the real line with the standard metric and $\alpha < 1$, for instance, then $H^\alpha(U) = +\infty$ for every nonempty open set $U \subseteq \mathbb{R}$. It is well known that this type of approximation does work when $E$ is contained in a countable union of open sets with finite measure. More precisely, if $E$ is a Borel set, then this follows from Theorem 2.2.2 on p60 of [11], or Theorem 1.10 on p11 of [27]. If $E$ is not a Borel set, then one can reduce to this case by replacing $E$ with the intersection of $\tilde{E}$ from (5.49) with the countable union of open sets just mentioned.

Alternatively, put

$$V_n = \bigcap_{l=1}^n U_l \tag{5.55}$$

for each positive integer $n$, so that $V_n$ is an open set, $V_{n+1} \subseteq V_n$, and

$$\bigcap_{n=1}^\infty V_n = \bigcap_{l=1}^\infty U_l = \tilde{E}. \tag{5.56}$$

If $H^\alpha(V_n) < \infty$ for any $n$, then

$$\lim_{n \to \infty} H^\alpha(V_n) = H^\alpha(\tilde{E}) = H^\alpha(E), \tag{5.57}$$

by standard results from measure theory, since $H^\alpha$ is countably additive on Borel sets. This gives another way to look at (5.54), using the definition of $H^\alpha$. Similarly, if $E$ is contained in an open set $W \subseteq M$ with $H^\alpha(W) < \infty$, then one can apply the same argument to $V_n \cap W$. A slightly more complicated version of this argument can also be used when $E$ is contained in a countable union of open sets with finite $H^\alpha$ measure.

In some situations, we may have that

$$H^\alpha(A) \leq (\text{diam } A)^\alpha \tag{5.58}$$
for every \( A \subseteq M \), and for a fixed \( \alpha \geq 0 \). This holds when \( A = \mathbb{R} \) with the standard metric and \( \alpha = 1 \), for instance. In this case, if \( U_l \) is as in (5.48), then
\[
H^\alpha(U_l) \leq \sum_j H^\alpha(A_{j,l}) \leq \sum_j (\text{diam } A_{j,l})^\alpha \tag{5.59}
\]
for each \( l \). Combining this with (5.47), we get that
\[
H^\alpha(U_l) < H^\alpha(E) + 1/l \tag{5.60}
\]
for each \( l \).

Suppose instead that
\[
H^\alpha(A) \leq C (\text{diam } A)^\alpha \tag{5.61}
\]
for some \( C \geq 1 \) and every \( A \subseteq M \). This implies that
\[
H^\alpha(U_l) \leq C \sum_j (\text{diam } A_{j,l})^\alpha \tag{5.62}
\]
for each \( l \), as in (5.59). It follows that \( H^\alpha(U_l) \) is finite for each \( l \), by (5.47) and the hypothesis that \( H^\alpha(E) < \infty \). In particular, \( H^\alpha(V_n) \) is finite for each \( n \), where \( V_n \) is as in (5.55), so that (5.57) holds, as before.

### 5.7 Lipschitz mappings, 2

Let \((M_1, d_1(x, y))\) and \((M_2, d_2(u, v))\) be metric spaces, and suppose that \( f \) is a Lipschitz mapping from \( M_1 \) into \( M_2 \) of order \( a > 0 \) with constant \( C \geq 0 \), as in Section 1.6. If \( A \) is a bounded subset of \( M_1 \), then \( f(A) \) is a bounded subset of \( M_2 \), and
\[
\text{diam } f(A) \leq C (\text{diam } A)^a. \tag{5.63}
\]
More precisely, \( \text{diam } A \) refers to the diameter of \( A \) as a subset of \( M_1 \), with respect to \( d_1(x, y) \), and \( \text{diam } f(A) \) refers to the diameter of \( f(A) \) in \( M_2 \), with respect to \( d_2(u, v) \). This also works when \( A \) is unbounded, in which case we interpret \( C (\text{diam } A)^a \) to be \( +\infty \) when \( C > 0 \), and to be 0 when \( C = 0 \).

Let \( E \subseteq M_1 \) and a nonnegative real number \( \alpha \) be given. If \( \{A_j\}_j \) is any collection of finitely or countably many subsets of \( M_1 \) such that \( E \subseteq \bigcup_j A_j \), then \( \{f(A_j)\}_j \) is a collection of finitely or countably many subsets of \( M_2 \) such that \( f(E) \subseteq \bigcup_j f(A_j) \). Using this and (5.63), it is easy to see that
\[
H^\alpha_{\text{con}}(f(E)) \leq C^\alpha H^\alpha_{\text{con}}(E). \tag{5.64}
\]
Similarly, if \( \delta, \eta > 0 \) satisfy \( C \delta^a \leq \eta \), then we get that
\[
H^\alpha_{\eta}(f(E)) \leq C^\alpha H^\alpha_{\delta^a}(E) \tag{5.65}
\]
It follows that
\[
H^\alpha_{\eta}(f(E)) \leq C^\alpha H^\alpha_{\text{a}}(E) \tag{5.66}
\]
for every \( \eta > 0 \), and hence that

\[ H^\alpha(f(E)) \leq C^\alpha H^\alpha(E). \tag{5.67} \]

As before, the various measures of \( E \) are defined using \( d_1(x, y) \) on \( M_1 \), and the various measures of \( f(E) \) are defined using \( d_2(u, v) \) on \( M_2 \). If \( \alpha = 0 \), then \( C^\alpha \) is interpreted as being equal to 1 for every \( C \geq 0 \). If \( \alpha > 0 \) and \( C = 0 \), then the right sides of these inequalities may be interpreted as being equal to 0, even when the corresponding measure of \( E \) is infinite. In particular,

\[ \dim_H f(E) \leq a^{-1} \dim_H E \tag{5.68} \]

for every \( E \subseteq M_1 \) under these conditions.

A mapping \( f : M_1 \to M_2 \) is said to be bilipschitz with constant \( C \geq 1 \) if

\[ C^{-1} d_1(x, y) \leq d_2(f(x), f(y)) \leq C d_1(x, y) \tag{5.69} \]

for every \( x, y \in M_1 \). In this case, we have that

\[ C^{-\alpha} H^\alpha(E) \leq H^\alpha(f(E)) \leq C^\alpha H^\alpha(E) \tag{5.70} \]

for every \( \alpha \geq 0 \) and \( E \subseteq M_1 \). More precisely, the second inequality in (5.70) follows from (5.67), with \( a = 1 \). The first inequality in (5.70) is essentially the same as the second inequality, applied to the inverse of \( f \). This also uses the fact that we can restrict our attention to coverings of \( f(E) \) by subsets of \( f(E) \) in the definition of \( H^\alpha(f(E)) \). As before, it follows that

\[ \dim_H f(E) = \dim_H E \tag{5.71} \]

for every \( E \subseteq M_1 \) under these conditions. Note that \( f \) is bilipschitz with constant \( C = 1 \) if and only if \( f \) is an isometric embedding.

Let us now take \( M_2 = \mathbb{R} \), equipped with the standard metric. Remember that

\[ f_p(x) = d_1(x, p) \tag{5.72} \]

is a Lipschitz mapping of order 1 from \( M_1 \) into \( \mathbb{R} \) with constant \( C = 1 \) for every \( p \in M_1 \), as in Section 1.6. If \( p, q \in E \subseteq M_1 \), then

\[ f_p(p) = d_1(p, p) = 0 \quad \text{and} \quad f_p(q) = d_1(p, q) \tag{5.73} \]

are elements of \( f_p(E) \), so that

\[ d_1(p, q) \leq \text{diam } f_p(E). \tag{5.74} \]

If \( E \) is connected, then \( f_p(E) \) is a connected subset of \( \mathbb{R} \), because \( f \) is continuous. Of course, \( H^1_{\text{con}} \) is the same as Lebesgue outer measure on \( \mathbb{R} \), as in Section 5.4, and the Lebesgue measure of a connected subset of \( \mathbb{R} \) is equal to its diameter. Thus

\[ \text{diam } f_p(E) = H^1_{\text{con}}(f_p(E)) \tag{5.75} \]
when $E$ is connected. We also have that

$$H^1_{con}(f_p(E)) \leq H^1_{con}(E),$$

by (5.64). It follows that

$$d_1(p, q) \leq H^1_{con}(E)$$

for every $p, q \in E$ when $E$ is connected, and hence that

$$\text{diam } E \leq H^1_{con}(E) \leq H^1(E).$$

### 5.8 Spherical measure

Let $(M, d(x, y))$ be a metric space, and let $\alpha$ be a nonnegative real number. Suppose that in the definition of $\alpha$-dimensional Hausdorff content of $E \subseteq M$, we restrict our attention to collections $\{A_j\}_j$ of finitely or countably many closed balls in $M$, instead of arbitrary subsets of $M$. Let us call the infimum of (5.12) in Section 5.2 over all such coverings of $E$ the $\alpha$-dimensional spherical content of $E$. Thus $H^\alpha_{con}(E)$ is automatically less than or equal to the $\alpha$-dimensional spherical content of $E$, since all of the coverings of $E$ used in the definition of the $\alpha$-dimensional spherical content of $E$ can also be used in the definition of $H^\alpha_{con}(E)$. In the other direction, the $\alpha$-dimensional spherical content of $E$ is less than or equal to $2^\alpha$ times $H^\alpha_{con}(E)$. This follows from the fact that every nonempty bounded set $A \subseteq M$ is contained in a closed ball with radius $\text{diam } A$ and diameter less than or equal to $2 \text{ diam } A$, as in Section 5.1. If $A$ is unbounded, then one might interpret $M$ as a ball of infinite radius, but this case does not really matter for estimating the $\alpha$-dimensional spherical content of $E$ in terms of $H^\alpha_{con}(E)$.

Similarly, we can restrict our attention to coverings of $E$ by collections $\{A_j\}_j$ of finitely or countably many closed balls $A_j$ in $M$ with diameter less than $\delta$ in Section 5.3, to get spherical versions of the outer measures $H^\alpha_\delta$. As before, $H^\alpha_\delta(E)$ is automatically less than or equal to its spherical version, because every covering of $E$ used for the spherical version can also be used for $H^\alpha_\delta(E)$. One can also check that the spherical version of $H^\alpha_\delta(E)$ is less than or equal to $2^\alpha$ times $H^\alpha_{2\delta}(E)$, using the same fact about nonempty bounded subsets $A$ of $M$ being contained in closed balls as in the preceding paragraph. Note that $\delta$ is replaced by $2\delta$ in $H^\alpha_{2\delta}(E)$, because of the extra factor of $2$ in the estimate for the diameter of the closed ball that contains $A$.

The supremum of the spherical version of $H^\alpha_\delta(E)$ over $\delta > 0$ is known as the $\alpha$-dimensional spherical measure of $E$. Thus $H^\alpha(E)$ is less than or equal to the $\alpha$-dimensional spherical measure of $E$, because of the analogous inequality for $H^\alpha_\delta(E)$. The $\alpha$-dimensional spherical measure of $E$ is less than or equal to $2^\alpha$ times $H^\alpha(E)_\delta$, because of the corresponding estimate for the spherical version of $H^\alpha_\delta(E)$. Note that the analogue of Hausdorff dimension for spherical measures is the same as the ordinary Hausdorff dimension.

If $M$ is the real line with the standard metric, then spherical measures are the same as Hausdorff measures. This is because Hausdorff measures on $\mathbb{R}$ can
already be defined in terms of coverings by closed intervals, which are the same as closed balls in this case. Similarly, if \(d(x, y)\) is an ultrametric on any set \(M\), then we have seen that the corresponding Hausdorff measures can be defined in terms of coverings by closed balls, so that spherical measures are the same as Hausdorff measures in this situation as well.

Let \((M, d(x, y))\) be an arbitrary metric space again, and let \(Y\) be a subset of \(M\). Thus \(Y\) can also be considered as a metric space, using the restriction of \(d(x, y)\) to \(Y\). If \(E \subseteq Y\), then the \(\alpha\)-dimensional Hausdorff measure of \(E\) as a subset of \(M\) is equal to the \(\alpha\)-dimensional Hausdorff measure of \(E\) as a subset of \(Y\). In the other direction, if \(\{A_j\}_j\) is any collection of subsets of \(M\) such that \(E \subseteq \bigcup_j A_j\), then \(\{A_j \cap Y\}_j\) is a collection of subsets of \(Y\) such that \(E \subseteq \bigcup_j A_j \cap Y\), and

\[
\text{diam}(A_j \cap Y) \leq \text{diam} A_j
\]

for each \(j\). This fact was implicitly used in the first inequality in (5.70). More precisely, this is essentially the same as (5.70), with \(C = 1\).

However, this type of argument does not work for spherical measures. One problem is that the intersection of a ball in \(M\) with \(Y\) may not be a ball in \(Y\). Another problem is that although every ball in \(Y\) can be expressed as the intersection of \(Y\) with a ball in \(M\) with the same radius, the ball in \(M\) may have larger diameter. One way to try to avoid the first problem is to restrict one’s attention further to coverings of a set \(E\) by balls centered on \(E\). This leads to another problem, which is that such a covering of \(E\) might not be admissible as a covering of a subset of \(E\).

Hausdorff and spherical measures do have a number of properties in common, and indeed both are examples of a well-known construction of Carathéodory. In particular, \(\alpha\)-dimensional spherical measure defines an outer measure on \(M\) for each \(\alpha \geq 0\). This leads to a corresponding \(\sigma\)-algebra of measurable subsets of \(M\) for each \(\alpha \geq 0\), on which \(\alpha\)-dimensional spherical measure is countably additive. If \(E_1, E_2 \subseteq M\) satisfy (5.31) in Section 5.4 for some \(\eta > 0\), then it is easy to see that the analogues of (5.32) and (5.33) for spherical measures also hold. This implies that Borel subsets of \(M\) are measurable with respect to \(\alpha\)-dimensional spherical measure for each \(\alpha \geq 0\). If \(E \subseteq M\) has finite \(\alpha\)-dimensional spherical measure for some \(\alpha \geq 0\), then there is a Borel set \(\tilde{E} \subseteq M\) that contains \(E\) and has the same \(\alpha\)-dimensional spherical measure as \(E\). This can be derived from the same type of argument as in Section 5.6, although in this case the sets \(U_i\) in (5.48) are \(F_\sigma\) sets.

One could also consider coverings by open balls instead of closed balls, and get similar conclusions. In some situations, the analogues of spherical measures using coverings by open balls are the same as for coverings by closed balls. This happens when the diameter of any open or closed ball of radius \(r\) in \(M\) is equal to \(2r\), for instance. One can check that this happens when \(d(x, y)\) is an ultrametric on \(M\) too, using the fact that the diameter of a ball of radius \(r\) is less than or equal to \(r\). Of course, in some examples of ultrametric spaces, every closed ball of positive radius can be expressed as an open ball, and vice-versa.
Suppose that \( f \) is a Lipschitz mapping of order \( a > 0 \) from a metric space \( M_1 \) into another metric space \( M_2 \), with constant \( C \geq 0 \). If \( B \) is an open or closed ball in \( M_1 \) centered at a point \( x \) and with radius \( r \), then \( f(B) \) is contained in the open or closed ball in \( M_2 \) centered at \( f(x) \) with radius \( C r^a \), respectively. In order to estimate the effect on spherical measures, one should look at the diameter of a ball in \( M_2 \) that contains \( f(B) \), such as the one just mentioned. In some situations, it may be possible to represent \( B \) as a ball in \( M_1 \) centered at \( x \) with more than one radius \( r \). In this case, it is better to take \( r \) to be as small as possible when working with closed balls, and at least approximately minimal when working with open balls.

If \( f \) is a bilipschitz mapping from \( M_1 \) onto \( M_2 \), then \( f \) is a Lipschitz mapping of order 1 from \( M_1 \) into \( M_2 \), and the inverse mapping \( f^{-1} \) defines a Lipschitz mapping from \( M_2 \) into \( M_1 \). The behavior of these Lipschitz mappings can be analyzed as in the previous paragraph. However, if \( f \) is not surjective, then the inverse mapping \( f^{-1} \) is only defined as a Lipschitz mapping from \( f(E) \) into \( M_1 \), and we are back to some of the problems mentioned earlier in the section.

### 5.9 Euclidean spaces

Let \((M, d(x, y))\) be a metric space, and let \(\mathcal{A}\) be a \(\sigma\)-algebra of subsets of \(M\) that contains the Borel subsets of \(M\). Also let \(\mu\) be an outer measure defined on \(\mathcal{A}\), and suppose that

\[
\mu(A) \leq C (\text{diam } A)^\alpha
\]

for some \(\alpha, C \geq 0\) and every \(A \in \mathcal{A}\). If \(E \in \mathcal{A}\), and if \(\{A_j\}_j\) is a collection of finitely or countably many elements of \(\mathcal{A}\) such that \(E \subseteq \bigcup_j A_j\), then

\[
\mu(E) \leq \sum_j \mu(A_j) \leq \sum_j C (\text{diam } A_j)^\alpha .
\]

This implies that

\[
\mu(E) \leq C H^\alpha_{con}(E),
\]

by taking the infimum over all such collections \(\{A_j\}_j\) in (5.81). More precisely, this also uses the fact that one can restrict one’s attention to coverings of \(E\) by Borel subsets of \(M\) in the definition of \(H^\alpha_{con}(E)\), as Section 5.2.

Similarly, suppose that \(\mu\) satisfies (5.80) for some \(\alpha, C \geq 0\) and all closed balls \(A\) in \(M\). If \(E \in \mathcal{A}\), then the same type of argument as in the preceding paragraph implies that \(\mu(E)\) is less than or equal to \(C\) times the spherical version of \(H^\alpha_{con}(E)\). Of course, one may be able to use a smaller constant \(C\) in this case, since one is considering a smaller collection of sets \(A\) in (5.80).

Let \(n\) be a positive integer, and suppose that \(M = \mathbb{R}^n\), with the standard metric. If \(\mu\) is Lebesgue measure on \(\mathbb{R}^n\), then it is easy to see that (5.80) holds with \(\alpha = n\) for some \(C \geq 0 \) and all Borel sets \(A \subseteq \mathbb{R}^n\). If \(A\) is a ball in \(\mathbb{R}^n\), then (5.80) holds with \(\alpha = n\) and the same constant \(C\) as for the unit ball. It is well known that (5.80) actually holds with \(\alpha = n\) for all Borel sets \(A \subseteq \mathbb{R}^n\),
and with the same constant $C$ as for balls. This is known as the isodiametric inequality. In fact, $n$-dimensional Hausdorff measure is equal to a constant multiple of Lebesgue measure on $\mathbb{R}^n$, where the constant corresponds to the one in the isodiametric inequality. It can also be shown that $n$-dimensional Hausdorff measure is equal to $n$-dimensional spherical measure on $\mathbb{R}^n$. These measures are often defined with additional constant factors included, so that they agree with Lebesgue measure on $\mathbb{R}^n$.

It is much easier to check directly that

$$H^n(A) \leq C'(\text{diam } A)^n$$

for some $C' \geq 0$ and every $A \subseteq \mathbb{R}^n$, where $C'$ depends only on $n$. It suffices to consider the case where $A$ is a cube, which can be covered by smaller cubes in the usual way. This implies that $H^n(U)$ is bounded by a constant multiple of the Lebesgue measure of $U$ for every open set $U \subseteq \mathbb{R}^n$, by expressing $U$ as a union of cubes with disjoint interiors. The same inequality holds for all Lebesgue measurable subsets of $\mathbb{R}^n$, by approximation by open sets.

Remember that $\mathbb{R}^n$ is a locally compact commutative topological group with respect to addition and the standard topology. Of course, Lebesgue measure satisfies the requirements of Haar measure on $\mathbb{R}^n$, and one can check that $H^n$ satisfies the requirements of Haar measure on $\mathbb{R}^n$ too. This implies that $H^n$ is a constant multiple of Lebesgue measure on the Borel subsets of $\mathbb{R}^n$, by the uniqueness of Haar measure. More precisely, Hausdorff measure of any dimension is invariant under translations on $\mathbb{R}^n$, because the standard Euclidean metric on $\mathbb{R}^n$ is invariant under translations. The $n$-dimensional Hausdorff measure of a bounded subset of $\mathbb{R}^n$ is finite, as in the preceding paragraph. If $U$ is a nonempty open subset of $\mathbb{R}^n$, then the Lebesgue measure of $U$ is positive, and hence $H^n(U) > 0$, since we have already seen that the Lebesgue measure of $U$ is bounded by a constant times $H^n(U)$. One can also verify that $H^n(U) > 0$ more directly.

Let $Q$ be a cube in $\mathbb{R}^n$ with sides parallel to the axes, with sidelength equal to 1. Thus $H^n(Q)$ and the Lebesgue measure are both positive and finite, and $H^n(Q)$ is a constant multiple of the Lebesgue measure of $Q$, by invariance under translations. One can extend this to cubes with sidelength equal to $2^l$ for some $l \in \mathbb{Z}$, using invariance under translations again. This implies that $H^n(U)$ is equal to the same constant multiple of the Lebesgue measure of $U$ when $U \subseteq \mathbb{R}^n$ is an open set, by expressing $U$ as a union of such cubes with disjoint interiors. It follows that $H^n(E)$ is equal to the same constant multiple of the Lebesgue measure of $E$ for every Borel set $E \subseteq \mathbb{R}^n$, by the outer regularity properties of both measures.

If $N$ is any norm on $\mathbb{R}^n$, then there is a translation-invariant metric on $\mathbb{R}^n$ associated to $N$, as in (3.54) in Section 3.6. This norm is equivalent to the standard Euclidean norm on $\mathbb{R}^n$, in the sense that each is bounded by a constant multiple of the other, as in Section 3.7. Of course, this implies that the corresponding metrics satisfy the same property, and in particular that they determine the same topology on $\mathbb{R}^n$. Similarly, it is easy to see that $n$-dimensional Hausdorff measure on $\mathbb{R}^n$ with respect to the metric associated to $N$
5.10 A SIMPLE COVERING ARGUMENT

is comparable to \(n\)-dimensional Hausdorff measure with respect to the standard metric on \(\mathbb{R}^n\). As before, one can check that \(n\)-dimensional Hausdorff measure on \(\mathbb{R}^n\) with respect to the metric associated to \(N\) satisfies the requirements of Haar measure, and hence is equal to a constant multiple of Lebesgue measure on the Borel subsets of \(\mathbb{R}^n\). This is all much simpler when \(N(v)\) is the maximum of the absolute values of the coordinates of \(v \in \mathbb{R}^n\), as in (3.58). In this case, the corresponding metric on \(\mathbb{R}^n\) is the same as (5.10) in Section 5.1.

5.10 A simple covering argument

Let \((M, d(x, y))\) be an ultrametric space, and let \(B\) be an open or closed ball in \(M\) with radius \(r > 0\). If \(x \in B\), then \(B\) may be considered as the open or closed ball in \(M\) centered at \(x\) with radius \(r\), as appropriate. This follows from (1.14) in Section 1.1 in the case of open balls, and (1.17) in the case of closed balls.

Let \(B, B'\) be open or closed balls in \(M\) with radii \(r, r' > 0\), respectively, such that

\[
B \cap B' \neq \emptyset.
\]

If \(x \in B \cap B'\), then \(B\) and \(B'\) can both be considered as balls centered at \(x\) in \(M\), as in the preceding paragraph. Using this, one can check that either

\[
B \subseteq B' \quad \text{(5.85)}
\]

or

\[
B' \subseteq B. \quad \text{(5.86)}
\]

More precisely, (5.85) holds when \(r < r'\), and (5.86) holds when \(r' < r\). If \(r = r'\), then (5.85) holds when \(B = B(x, r)\), and (5.86) holds when \(B' = B(x, r')\). If \(r = r'\) and \(B, B'\) are both open or both closed, then \(B = B'\). Of course, \(B\) and \(B'\) may be the same as subsets of \(M\), even if they are initially defined as balls of different radii, or one is initially defined as an open ball and the other is initially defined as a closed ball. This also works when \(B\) or \(B'\) is a closed ball of radius 0 in \(M\), which is to say a subset of \(M\) with exactly one element.

Now let \(B\) be a collection of open or closed balls in \(M\). An element \(B\) of \(B\) is said to be maximal in \(B\) if it is maximal with respect to inclusion, which is to say that for each \(B' \in B\) with \(B \subseteq B'\) we have that \(B = B'\). It is easy to see that the maximal elements of \(B\) are pairwise disjoint, by the remarks in the preceding paragraph.

Let \(B_0\) be the collection of maximal elements of \(B\), so that

\[
\bigcup \{B : B \in B_0\} \subseteq \bigcup \{B : B \in B\}, \quad \text{(5.87)}
\]

since \(B_0 \subseteq B\). Let us say that \(B\) is a nice collection of balls in \(M\) if every element of \(B\) is contained in a maximal element of \(B\). In this case, we have that

\[
\bigcup \{B : B \in B\} \subseteq \bigcup \{B : B \in B_0\}, \quad \text{(5.88)}
\]
and hence
\[ \bigcup \{B \in \mathcal{B}_0\} = \bigcup \{B \in \mathcal{B}\}. \tag{5.89} \]

Actually, in order to get (5.88), it suffices to know that for each \( B \in \mathcal{B} \) and \( x \in B \) there is a \( B_0 \in \mathcal{B}_0 \) such that \( x \in B_0 \). However, this would imply that \( B \subseteq B_0 \) or \( B_0 \subseteq B \), since \( B \cap B_0 \neq \emptyset \). If \( B_0 \subseteq B \), then \( B = B_0 \), because \( B_0 \) is supposed to be maximal in \( \mathcal{B} \). Thus \( B \subseteq B_0 \) under these conditions. In particular, for each \( B \in \mathcal{B} \), this could be applied to any \( x \in B \), to get that \( B \) is contained in some \( B_0 \in \mathcal{B}_0 \). It follows that (5.88) holds if and only if \( \mathcal{B} \) is nice.

Let \( B_1 \in \mathcal{B} \) be given, and put
\[ \mathcal{C}(B_1) = \{B \in \mathcal{B} : B_1 \subseteq B\}. \tag{5.90} \]

Note that \( B_1 \) is automatically an element of \( \mathcal{C}(B_1) \). If \( x_1 \) is any element of \( B_1 \), then the elements of \( \mathcal{C}(B_1) \) may be considered as balls centered at \( x_1 \), by the remarks at the beginning of the section. In particular, the elements of \( \mathcal{C}(B_1) \) are linearly ordered by inclusion. If \( B \in \mathcal{C}(B_1) \) and \( B \) is a maximal element of \( \mathcal{B} \), then \( B \) is a maximal element of \( \mathcal{C}(B_1) \). Conversely, if \( B \in \mathcal{C}(B_1) \) is a maximal element of \( \mathcal{C}(B_1) \), then it is easy to see that \( B \) is a maximal element of \( \mathcal{B} \) too. Of course, if \( \mathcal{C}(B_1) \) has only finitely many elements, then it has a maximal element. If \( \mathcal{B} \) has only finitely many elements, then \( \mathcal{C}(B_1) \) has only finitely many elements for each \( B_1 \in \mathcal{B} \), and \( \mathcal{B} \) is nice.

Put
\[ U(B_1) = \bigcup \{B : B \in \mathcal{C}(B_1)\}, \tag{5.91} \]

so that \( B_1 \subseteq U(B_1) \), by construction. Let \( x_1 \) be an element of \( B_1 \) again, so that the elements of \( \mathcal{C}(B_1) \) may be considered as balls centered at \( x_1 \), as before. If \( \mathcal{C}(B_1) \) has a maximal element, then it is the same as \( U(B_1) \) as a subset of \( M \). If \( \mathcal{C}(B_1) \) contains balls of arbitrarily large radius, then \( U(B_1) = M \). Of course, if \( M \) is bounded, then \( M \) may be considered as a ball centered at \( x_1 \) with sufficiently large radius. Otherwise, suppose that \( \mathcal{C}(B_1) \) does not have a maximal element, and that the elements of \( \mathcal{C}(B_1) \) can be expressed as open or closed balls with bounded radius. In this case, \( U(B_1) \) is an open ball centered at \( x_1 \) with positive finite radius, which is the supremum of the radii of the elements of \( \mathcal{C}(B_1) \).

Observe that
\[ \bigcup \{U(B_1) : B_1 \in \mathcal{B}\} = \bigcup \{B : B \in \mathcal{B}\}. \tag{5.92} \]

More precisely, the left side of (5.92) is contained in the right side of (5.92), because \( U(B_1) \) is a union of elements of \( \mathcal{B} \) for every \( B_1 \in \mathcal{B} \), by construction. Similarly, the right side of (5.92) is contained in the left side of (5.92), because \( B_1 \subseteq U(B_1) \) for every \( B_1 \in \mathcal{B} \).

Suppose that \( B_1, B_2 \in \mathcal{B} \) satisfy \( B_1 \subseteq B_2 \), so that \( B_2 \in \mathcal{C}(B_1) \), and in fact \( \mathcal{C}(B_2) \subseteq \mathcal{C}(B_1) \). Let us check that
\[ U(B_1) = U(B_2). \tag{5.93} \]
5.10. A SIMPLE COVERING ARGUMENT

Of course, \( U(B_2) \subseteq U(B_1) \), because \( C(B_2) \subseteq C(B_1) \). To get the opposite inclusion, let \( B \in C(B_1) \) be given, and let us verify that \( B \subseteq U(B_2) \). Note that either either \( B \subseteq B_2 \) or \( B_2 \subseteq B \), because \( B_2 \in C(B_1) \) and the elements of \( C(B_1) \) are linearly ordered by inclusion. If \( B \subseteq B_2 \), then \( B \subseteq U(B_2) \), since \( B_2 \subseteq U(B_2) \), by construction. Otherwise, if \( B_2 \subseteq B \), then \( B \in C(B_2) \), and hence \( B \subseteq U(B_2) \), as desired.

If \( B_1, B'_1 \in \mathcal{B} \) are both contained in \( B_2 \in \mathcal{B} \), then it follows that

\[
(5.94) \quad U(B_1) = U(B'_1).
\]

Conversely, suppose that \( B_1, B'_1 \in \mathcal{B} \) have the property that

\[
(5.95) \quad U(B_1) \cap U(B'_1) \neq \emptyset.
\]

This implies that there are \( B \in C(B_1), B' \in C(B'_1) \) such that \( B \cap B' \neq \emptyset \), by definition of \( U(B_1), U(B'_1) \). In this case, \( B \subseteq B' \) or \( B' \subseteq B \), as mentioned earlier in the section. Put \( B_2 = B' \) when \( B \subseteq B' \), and \( B_2 = B \) when \( B' \subseteq B \). Thus \( B_2 \in \mathcal{B} \), because \( B, B' \in \mathcal{B} \), by definition of \( C(B_1), C(B'_1) \). We also have that \( B_1 \subseteq B \) and \( B'_1 \subseteq B' \), by definition of \( C(B_1), C(B'_1) \), and hence that \( B_1, B'_1 \subseteq B_2 \). This brings us back to the situation described at the beginning of the paragraph, so that (5.94) holds under these conditions. It follows that for each \( B_1, B'_1 \in \mathcal{B} \), either

\[
(5.96) \quad U(B_1) \cap U(B'_1) = \emptyset,
\]

or (5.94) holds.

As a basic class of examples, let \( V \) be a nonempty open subset of \( M \), and let \( \mathcal{B} \) be the collection of open or closed balls in \( M \) with positive radius that are contained in \( V \). Then

\[
(5.97) \quad \bigcup \{ B : B \in \mathcal{B} \} = V
\]

in this case. If \( V = M \), then \( \mathcal{B} \) consists of all open or closed balls in \( M \) with positive radius, and \( U(B_1) = M \) for every \( B_1 \in \mathcal{B} \). Otherwise, if \( V \neq M \), then for each \( B_1 \in \mathcal{B} \), the elements of \( C(B_1) \) have bounded radius. This implies that \( U(B_1) \) is an open or closed ball of finite radius for each \( B_1 \in \mathcal{B} \). By construction,

\[
(5.98) \quad U(B_1) \subseteq V
\]

for every \( B_1 \in \mathcal{B} \), because \( U(B_1) \) is a union of elements of \( \mathcal{B} \), each of which is contained in \( V \). It follows that \( U(B_1) \in \mathcal{B} \) for every \( B_1 \in \mathcal{B} \), and these are the maximal elements of \( \mathcal{B} \).

Now let \( \alpha \) be a nonnegative real number, and let \( \mathcal{B} \) be a nonempty collection of open or closed balls in \( M \) such that

\[
(5.99) \quad \sum_{B \in \mathcal{B}} (\text{diam } B)^\alpha < \infty.
\]

If \( B_1 \in \mathcal{B} \) and \( \text{diam } B_1 > 0 \), then it is easy to see that \( C(B_1) \) has only finitely many elements, because of (5.99). In particular, this implies that \( C(B_1) \) has a
maximal element, as before. Otherwise, if \( \text{diam} B_1 = 0 \), then \( B_1 \) consists of a single point. This happens for closed balls of radius 0, but it can also happen for open or closed balls of positive radius around an isolated point in \( M \). If \( B_1 \) is not maximal, then there is a \( B_2 \in \mathcal{B} \) such that \( B_1 \subseteq B_2 \) and \( B_1 \neq B_2 \). This implies that \( \text{diam} B_2 > 0 \), and hence that \( B_2 \) is contained in a maximal element of \( \mathcal{B} \). This shows that \( \mathcal{B} \) is nice under these conditions.

### 5.11 Haar and Hausdorff measures

Let \( k \) be a field, and let \( | \cdot | \) be an ultrametric absolute value function on \( k \). In this section, we suppose that \( | \cdot | \) is nontrivial on \( k \), and that \( k \) is locally compact with respect to the corresponding ultrametric. This implies that closed balls in \( k \) are compact, as in Section 2.11. It follows that \( | \cdot | \) is a discrete absolute value function on \( k \), in the sense described in Section 3.3, and as discussed at the end of Section 3.4. In this situation, we also have that the corresponding residue field (3.43) is finite, as mentioned near the end of Section 3.4.

Let \( p_1 \) be as in (3.34) in Section 3.3, so that \( 0 < p_1 < 1 \), because \( | \cdot | \) is discrete and nontrivial. Thus the positive values of \( |x| \) on \( k \) are integer powers of \( p_1 \), as in Section 3.3. In particular,

\[
(5.100) \quad \text{diam} B(x, p_1^j) = p_1^j
\]

for each \( x \in k \) and \( j \in \mathbb{Z} \), where \( B(x, r) \) is the closed ball in \( k \) centered at \( x \in k \) with radius \( r > 0 \) with respect to the ultrametric associated to \( | \cdot | \). Remember that \( B(0, 1) \) is a subring of \( k \), \( B(0, 1) \) is an ideal in \( B(0, 1) \), and that the quotient ring

\[
(5.101) \quad B(0, 1)/B(0, 1)
\]

is the residue field corresponding to \( | \cdot | \) on \( k \), as in Section 3.4. Let \( \alpha \) be the positive real number that satisfies

\[
(5.102) \quad p_1^{-\alpha} = \#(B(0, 1)/B(0, 1)),
\]

where \( \#(B(0, 1)/B(0, 1)) \) is the number of elements of the residue field (5.101).

Let \( H \) be Haar measure on \( k \), normalized so that

\[
(5.103) \quad H(B(0, 1)) = 1.
\]

Observe that

\[
(5.104) \quad H(B(x, p_1^j)) = p_1^{\alpha j}
\]

for every \( x \in k \) and \( j \in \mathbb{Z} \), by (3.51) in Section 3.5. Equivalently,

\[
(5.105) \quad H(B(x, p_1^j)) = (\text{diam} B(x, p_1^j))^\alpha
\]

for every \( x \in k \) and \( j \in \mathbb{Z} \), by (5.100). It follows that

\[
(5.106) \quad H(A) \leq (\text{diam} A)^\alpha
\]
for every bounded Borel set \( A \subseteq k \). More precisely, if \( \text{diam} \, A = 0 \), then \( A \) contains at most one element, and (5.106) says that \( H(A) = 0 \). Otherwise, if the diameter of \( A \) is positive, then it is equal to \( \rho^j_1 \) for some \( j \in \mathbb{Z} \). In this case, \( A \) is contained in a closed ball of radius \( \rho^j_1 \), so that (5.106) follows from (5.104).

Using (5.106), we get that
\[
H(E) \leq H_{\text{con}}^\alpha(E)
\]
for every Borel set \( E \subseteq k \), where \( H_{\text{con}}^\alpha(E) \) is the \( \alpha \)-dimensional Hausdorff content of \( E \) with respect to the ultrametric associated to \( | \cdot | \) on \( k \). This is the same as (5.82) in Section 5.9, with \( \mu = H \) and \( C = 1 \).

Remember that \( B(0, 1) \) can be expressed as the union of finitely many pairwise-disjoint open balls of radius 1, where the number of these open balls of radius 1 is equal to (5.102). Equivalently, \( B(0, 1) \) can be expressed as the union of the same number of pairwise-disjoint closed balls of radius \( \rho_1 \), because an open ball in \( k \) of radius 1 is the same as a closed ball in \( k \) centered at the same point with radius \( \rho_1 \) in this situation. It follows that for each \( x \in k \) and \( j \in \mathbb{Z} \), \( B(x, \rho^j_1) \) can be expressed as the union of the same number of pairwise-disjoint closed balls of radius \( \rho^{j+1}_1 \). If \( l \) is any positive integer, then one can repeat the process to get that \( B(x, \rho^j_1) \) can be expressed as the union of
\[
\rho^\alpha_1 \rho^{j+l}_1
\]

pairwise-disjoint closed balls of radius \( \rho^{j+1}_1 \).

Using this and (5.100), one can check directly that
\[
H_{\delta}^\alpha(B(x, \rho^j_1)) \leq \rho^{(j+\delta)}_1
\]
for every \( x \in k \), \( j \in \mathbb{Z} \), and \( \delta > 0 \). This implies that
\[
H^\alpha(B(x, \rho^j_1)) \leq \rho^{(j+\delta)}_1
\]
for every \( x \in k \) and \( j \in \mathbb{Z} \), by taking the supremum of the left side of (5.109) over \( \delta > 0 \). It follows that
\[
H^\alpha(A) \leq (\text{diam} \, A)^\alpha
\]
for every \( A \subseteq k \), as before. We also get that \( H^\alpha(E) \leq H_{\text{con}}^\alpha(E) \) for every \( E \subseteq k \), and hence that
\[
H^\alpha(E) = H_{\text{con}}^\alpha(E),
\]
since \( H_{\text{con}}^\alpha(E) \leq H^\alpha(E) \) automatically.

Observe that
\[
H_{\text{con}}^\alpha(B(x, \rho^j_1)) \geq \rho^{(j+\delta)}_1
\]
for every \( x \in k \) and \( j \in \mathbb{Z} \), by (5.104) and (5.107). This can also be verified more directly, as follows. In order to estimate \( H_{\text{con}}^\alpha(E) \), it suffices to consider coverings of \( B(x, \rho^j_1) \) by finitely or countably many closed balls of positive radius in \( k \), as in Section 5.2. Of course, this uses the fact that the metric associated to \( | \cdot | \) on \( k \) is an ultrametric. More precisely, it suffices to consider coverings of
by finitely many closed balls of positive radius, because closed balls of positive radius are open sets in \( k \), and \( \overline{B}(x, \rho_1^j) \) is compact. In this situation, one can reduce further to coverings of \( \overline{B}(x, \rho_1^j) \) by finitely many closed balls of equal radius, using the earlier arguments about expressing closed balls as unions of balls of smaller radius. If \( l \) is a nonnegative integer, then (5.108) is the minimum number of closed balls of radius \( \rho_1^{j+l} \) needed to cover \( \overline{B}(x, \rho_1^j) \). This implies (5.113), as desired.

Combining (5.110) and (5.113), we get that

\[
H^\alpha(\overline{B}(x, \rho_1^j)) = H^\alpha_{\text{con}}(\overline{B}(x, \rho_1^j)) = \rho_1^j
\]

for every \( x \in k \) and \( j \in \mathbb{Z} \). It follows that

\[
H^\alpha(E) = H(E)
\]

when \( E \) is a ball in \( k \), by (5.104) and (5.114). If \( E \) is a nonempty proper open subset of \( k \), then \( E \) can be expressed as a union of pairwise-disjoint balls of positive radius in \( k \), as in the previous section. Note that \( k \) is separable, because closed balls in \( k \) are separable. This implies that any collection of pairwise-disjoint balls in \( k \) of positive radius can have only finitely or countably many elements. In particular, if \( E \) is a nonempty proper open subset of \( k \), then \( E \) can be expressed as the union of finitely or countably many pairwise-disjoint balls in \( k \). This permits (5.115) to be derived from the corresponding statement for balls in \( k \) in this case. One can check that (5.115) holds for Borel sets \( E \subseteq k \), using this and outer regularity properties of \( H \) and \( H^\alpha \).

Of course, the metric on \( k \) associated to \( | \cdot | \) is invariant under translations, by construction. This implies that the corresponding Hausdorff measure of any dimension is invariant under translations as well. One can check that \( H^\alpha \) satisfies the other requirements of Haar measure on \( k \), and indeed this may be considered as a way to construct Haar measure on \( k \). Similarly, the argument in the preceding paragraph may be considered as a way to deal with uniqueness of Haar measure in this situation.

Let \( p \) be a prime number, and suppose that \( k = \mathbb{Q}_p \), equipped with the \( p \)-adic metric. This satisfies the conditions mentioned at the beginning of the section, with \( \rho_1 = 1/p \). In this case, (5.101) has exactly \( p \) elements, and (5.102) corresponds to taking \( \alpha = 1 \). Thus 1-dimensional Hausdorff measure is the same as Haar measure on \( \mathbb{Q}_p \).

Let \( k \) be any field with an ultrametric absolute value function \( | \cdot | \) that satisfies the conditions mentioned at the beginning of the section again. Also let \( n \) be a positive integer, so that \( k^n \) may be considered as an \( n \)-dimensional vector space over \( k \). Similarly, \( k^n \) may be considered as a locally compact commutative group with respect to addition, using the product topology associated to the topology on \( k \) determined by the ultrametric corresponding to \( | \cdot | \). Remember that Haar measure on \( k^n \) basically corresponds to a product of \( n \) copies of Haar measure on \( k \), as in Section 3.9.

Let \( N_0 \) be the ultranorm on \( k^n \) defined in (3.58) in Section 3.6, so that closed balls in \( k^n \) with respect to \( N_0 \) are the same as products of \( n \) closed balls
5.12. SIMILARITIES

in \( k \) with respect to \(| \cdot |\), with the same radius. If \( \alpha \) is as in (5.102), then \((\alpha n)\)-dimensional Hausdorff measure on \( k^n \) with respect to the ultrametric associated to \( N_0 \) satisfies the requirements of Haar measure, and has other properties like those in the \( n = 1 \) case. Any other norm on \( k^n \) is equivalent to \( N_0 \), as in Section 3.7, which implies that the corresponding Hausdorff measures are bounded by constant multiples of each other too. Using this, one can check that \((\alpha n)\)-dimensional Hausdorff measure on \( k^n \) with respect to the metric associated to \( N \) also satisfies the requirements of Haar measure.

5.12 Similarities

Let \( k \) be a field, let \(| \cdot |\) be an ultrametric absolute value function on \( k \), and suppose that \( k \) is complete with respect to the ultrametric corresponding to \(| \cdot |\). Also let \( a_0, a_1, a_2, a_3, \ldots \) be a sequence of elements of \( k \), and consider the corresponding power series

\[
(5.116) \quad f(x) = \sum_{j=0}^{\infty} a_j x^j.
\]

If

\[
(5.117) \quad \lim_{j \to \infty} |a_j| r^j = 0
\]

for some \( r > 0 \), then \( f(x) \) may be considered as a \( k \)-valued function on

\[
(5.118) \quad D = \overline{B}(0, r).
\]

Alternatively, if \( \rho > 0 \), and (5.117) holds for every \( r \in (0, \rho) \), then \( f(x) \) may be considered as a \( k \)-valued function on

\[
(5.119) \quad D = B(0, \rho).
\]

If (5.117) holds for every \( r > 0 \), then we can take \( \rho = +\infty \) and \( D = k \). Note that the series expansion (4.44) in Section 4.3 for \( f'(x) \) converges for every \( x \in D \). This uses the fact that \(|j \cdot a_j| \leq |a_j|\) for each \( j \), by the ultrametric version of the triangle inequality.

Let \( B \) be an open or closed ball centered at a point \( x \in k \) with radius \( t > 0 \) such that \( B \subseteq D \). Let us say that \( B \) is “admissible” if it satisfies the following four conditions. First, \( f'(x) \neq 0 \). Second,

\[
(5.120) \quad |f'(y)| = |f'(x)|
\]

for every \( y \in B \). Third,

\[
(5.121) \quad |f(y) - f(w)| = |f'(x)||y - w|
\]

for every \( y, w \in B \). To state the fourth condition, let \( B' \) be the ball in \( k \) centered at \( f(x) \), with radius equal to \( t|f'(x)| \), and which is an open ball when \( B \) is an
open ball, and a closed ball when \( B \) is a closed ball. Thus (5.121) implies that
\[ f(B) \subseteq B', \]
and the fourth condition is that
\[ f(B) = B'. \]

Of course, \( B \) may be considered as being centered at any of its elements, as mentioned at the beginning of Section 5.10. However, it is easy to see that the admissibility of \( B \) does not depend on the choice \( x \) of center of \( B \). This also uses the fact that \( B' \) may be considered as being centered at any of its elements, which include the elements of \( f(B) \). If \( x \in D \) and \( f'(x) \neq 0 \), then the discussion in Sections 4.8 and 4.9 gives a criterion for the admissibility of balls centered at \( x \). In particular, this criterion implies that balls centered at \( x \) with sufficiently small radius are admissible.

If \( B \) is the collection of these admissible balls, then
\[ \bigcup \{ B : B \in B \} = \{ x \in D : f'(x) \neq 0 \}. \]

More precisely, the left side of (5.123) is contained in the right side of (5.123), because of the first and second conditions in the definition of admissibility. Similarly, the right side of (5.123) is contained in the left side of (5.123), because every \( x \in D \) with \( f'(x) \neq 0 \) is contained in an admissible ball, as mentioned at the end of the preceding paragraph. If \( D \) is as in (5.118) and \( j \cdot a_j \neq 0 \) for some positive integer \( j \), then Strassmann’s theorem implies that \( f'(x) = 0 \) for only finitely many \( x \in D \). Otherwise, if \( D \) is as in (5.119) or \( D = k \), and if \( j \cdot a_j \neq 0 \) for some \( j \geq 1 \), then one can use Strassmann’s theorem to get that \( f'(x) = 0 \) for at most finitely or countably many \( x \in D \).

Let \( B_1 \) be an admissible ball, let \( C(B_1) \) be as in (5.90), and let \( U(B_1) \) be as in (5.91). If the elements of \( C(B_1) \) have bounded radii, then we have seen that \( U(B_1) \) is an open or closed ball with finite radius. In this situation, one can also check that \( U(B_1) \) is automatically admissible, because it is the union of a collection of admissible balls centered at the same point. This implies that \( U(B_1) \) is an element of \( C(B_1) \), which is automatically maximal, and hence that \( U(B_1) \) is a maximal element of \( B \). Note that \( C(B_1) \) only has finitely many elements when the elements of \( C(B_1) \) have bounded radii and \( | \cdot | \) is a discrete absolute value function on \( k \), as in Section 3.3.

Let us suppose from now on in this section that \( | \cdot | \) is nontrivial on \( k \), and that \( k \) is locally compact. As in the previous section, we let \( H \) be Haar measure on \( k \), normalized as in (5.103). We also let \( \rho_1 \) be as in (3.34) in Section 3.3, and we let \( \alpha > 0 \) be as in (5.102). Put
\[ x E = \{ xy : y \in E \} \]
for each \( x \in k \) and \( E \subseteq k \). Of course, (5.124) reduces to \( \{0\} \) when \( x = 0 \) and \( E \neq \emptyset \), and (5.124) is the empty set for every \( x \in k \) when \( E = \emptyset \). If \( x \neq 0 \), then multiplication by \( x \) defines a homeomorphism on \( k \), and the mapping from \( E \subseteq k \) to (5.124) preserves open sets, closed sets, compact sets, and Borel sets. Let us check that
\[ H(x E) = |x|^{\alpha} H(E) \]
for every \( x \in k \) and Borel set \( E \subseteq k \). This is trivial when \( x = 0 \), and so we may as well restrict our attention to \( x \neq 0 \). In this case, it is easy to see that \( H(xE) \) also satisfies the requirements of Haar measure on \( k \). The uniqueness of Haar measure implies that \( H(xE) \) is equal to a positive real number times \( H(E) \), where this positive real number depends on \( x \) but not \( E \). In order to determine this positive real number, it suffices to take \( E = B(0, 1) \), for which we have that
\[
H(xB(0, 1)) = H(B(0, |x|)) = |x|^\alpha,
\]
by (5.104). This implies (5.125), because of the normalization (5.103) for \( H \).

Let \( x_0 \in k \) and \( j_0 \in \mathbb{Z} \) be given, and let \( g \) be a mapping from \( B(x_0, \rho_{1}^{j_0}) \) into \( k \). Let \( l_0 \) be another integer, and suppose that
\[
|g(x) - g(y)| = \rho_{1}^{l_0} |x - y|
\]
for every \( x, y \in B(x_0, \rho_{1}^{j_0}) \). This implies that \( g(B(x_0, \rho_{1}^{j_0})) \subseteq B(g(x_0), \rho_{1}^{j_0+l_0}) \), and in fact one can show that
\[
g(B(x_0, \rho_{1}^{j_0})) = B(g(x_0), \rho_{1}^{j_0+l_0})
\]
in this situation. Let us simply take this as an additional condition here, for the sake of convenience. If \( x \in B(x_0, \rho_{1}^{j_0}) \), \( j \in \mathbb{Z} \), and \( j \geq j_0 \), then \( \rho_{1}^{j} \leq \rho_{1}^{j_0} \), and hence
\[
B(x, \rho_{1}^{j}) \subseteq B(x_0, \rho_{1}^{j_0}),
\]
by the ultrametric version of the triangle inequality. In this case, it is easy to see that
\[
g(B(x, \rho_{1}^{j})) = B(g(x), \rho_{1}^{j+l_0}),
\]
using (5.127) and (5.128). This can also be shown directly, as for (5.128).

If \( E \subseteq B(x_0, \rho_{1}^{j_0}) \) is a Borel set, then one can check that
\[
H(g(E)) = \rho_{1}^{a_{1}l_0} H(E).
\]
If \( H \) is identified with \( \alpha \)-dimensional Hausdorff measure on \( k \), as in the previous section, then (5.131) can be obtained directly from (5.127). Otherwise, one can first check that (5.131) holds when \( E \) is a ball contained in \( B(x_0, \rho_{1}^{j_0}) \), using (5.104) and (5.130). If \( E \) is an open set contained in \( B(x_0, \rho_{1}^{j_0}) \), then \( E \) can be expressed as the union of the maximal balls contained in \( E \), as in Section 5.10. The maximal balls contained in \( E \) are pairwise disjoint, as before, and there can be only finitely or countably many of these maximal balls contained in \( E \), because \( B(x_0, \rho_{1}^{j_0}) \) is compact and hence separable. This permits (5.131) to be derived from the analogous statement for balls contained in \( B(x_0, \rho_{1}^{j_0}) \), by countable additivity of Haar measure. Once one has (5.131) for open subsets of \( B(x_0, \rho_{1}^{j_0}) \), the analogous statement for Borel sets can be derived from the outer regularity of Haar measure.

5.12. SIMILARITIES

for every \( x \in k \) and Borel set \( E \subseteq k \). This is trivial when \( x = 0 \), and so we may as well restrict our attention to \( x \neq 0 \). In this case, it is easy to see that \( H(xE) \) also satisfies the requirements of Haar measure on \( k \). The uniqueness of Haar measure implies that \( H(xE) \) is equal to a positive real number times \( H(E) \), where this positive real number depends on \( x \) but not \( E \). In order to determine this positive real number, it suffices to take \( E = B(0, 1) \), for which we have that
\[
H(xB(0, 1)) = H(B(0, |x|)) = |x|^\alpha,
\]
by (5.104). This implies (5.125), because of the normalization (5.103) for \( H \).

Let \( x_0 \in k \) and \( j_0 \in \mathbb{Z} \) be given, and let \( g \) be a mapping from \( B(x_0, \rho_{1}^{j_0}) \) into \( k \). Let \( l_0 \) be another integer, and suppose that
\[
|g(x) - g(y)| = \rho_{1}^{l_0} |x - y|
\]
for every \( x, y \in B(x_0, \rho_{1}^{j_0}) \). This implies that \( g(B(x_0, \rho_{1}^{j_0})) \subseteq B(g(x_0), \rho_{1}^{j_0+l_0}) \), and in fact one can show that
\[
g(B(x_0, \rho_{1}^{j_0})) = B(g(x_0), \rho_{1}^{j_0+l_0})
\]
in this situation. Let us simply take this as an additional condition here, for the sake of convenience. If \( x \in B(x_0, \rho_{1}^{j_0}) \), \( j \in \mathbb{Z} \), and \( j \geq j_0 \), then \( \rho_{1}^{j} \leq \rho_{1}^{j_0} \), and hence
\[
B(x, \rho_{1}^{j}) \subseteq B(x_0, \rho_{1}^{j_0}),
\]
by the ultrametric version of the triangle inequality. In this case, it is easy to see that
\[
g(B(x, \rho_{1}^{j})) = B(g(x), \rho_{1}^{j+l_0}),
\]
using (5.127) and (5.128). This can also be shown directly, as for (5.128).

If \( E \subseteq B(x_0, \rho_{1}^{j_0}) \) is a Borel set, then one can check that
\[
H(g(E)) = \rho_{1}^{a_{1}l_0} H(E).
\]
If \( H \) is identified with \( \alpha \)-dimensional Hausdorff measure on \( k \), as in the previous section, then (5.131) can be obtained directly from (5.127). Otherwise, one can first check that (5.131) holds when \( E \) is a ball contained in \( B(x_0, \rho_{1}^{j_0}) \), using (5.104) and (5.130). If \( E \) is an open set contained in \( B(x_0, \rho_{1}^{j_0}) \), then \( E \) can be expressed as the union of the maximal balls contained in \( E \), as in Section 5.10. The maximal balls contained in \( E \) are pairwise disjoint, as before, and there can be only finitely or countably many of these maximal balls contained in \( E \), because \( B(x_0, \rho_{1}^{j_0}) \) is compact and hence separable. This permits (5.131) to be derived from the analogous statement for balls contained in \( B(x_0, \rho_{1}^{j_0}) \), by countable additivity of Haar measure. Once one has (5.131) for open subsets of \( B(x_0, \rho_{1}^{j_0}) \), the analogous statement for Borel sets can be derived from the outer regularity of Haar measure.
Bibliography

[1] J. Benedetto and W. Czaja, Integration and Modern Analysis, Birkhäuser, 2009.

[2] G. Birkhoff and S. Mac Lane, A Survey of Modern Algebra, 4th edition, Macmillan, 1977.

[3] H. Cartan, Elementary Theory of Analytic Functions of One or Several Complex Variables, translated from the French, Dover, 1995.

[4] J. Cassels, Local Fields, Cambridge University Press, 1986.

[5] R. Coifman and G. Weiss, Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes, Lecture Notes in Mathematics 242, Springer-Verlag, 1971.

[6] R. Coifman and G. Weiss, Extensions of Hardy spaces and their use in analysis, Bulletin of the American Mathematical Society 83 (1977), 569–645.

[7] G. David and S. Semmes, Fractured Fractals and Broken Dreams: Self-Similar Geometry through Metric and Measure, Oxford University Press, 1997.

[8] L. Evans and R. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992.

[9] K. Falconer, The Geometry of Fractal Sets, Cambridge University Press, 1986.

[10] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications, 2nd edition, Wiley, 2003.

[11] H. Federer, Geometric Measure Theory, Springer-Verlag, 1969.

[12] G. Folland, A Course in Abstract Harmonic Analysis, CRC Press, 1995.

[13] G. Folland, Real Analysis, 2nd edition, Wiley, 1999.

[14] F. Gouvea, p-Adic Numbers: An Introduction, 2nd edition, Springer-Verlag, 1997.
[15] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, 2001.

[16] E. Hewitt and K. Ross, *Abstract Harmonic Analysis*, Volumes I and II, 1970, 1979.

[17] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer-Verlag, 1975.

[18] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton University Press, 1948.

[19] F. Jones, *Lebesgue Integration on Euclidean Space*, Jones and Bartlett, 1993.

[20] I. Kaplansky, *Set Theory and Metric Spaces*, Chelsea, 1977.

[21] J. Kelley, *General Topology*, Springer-Verlag, 1975.

[22] J. Kelley and T. Srinivasan, *Measure and Integral*, Springer-Verlag, 1988.

[23] S. Krantz, *A Panorama of Harmonic Analysis*, Mathematical Association of America, 1999.

[24] S. Krantz and H. Parks, *The Geometry of Domains in Space*, Birkhäuser, 1999.

[25] R. Macías and C. Segovia, *Lipschitz functions on spaces of homogeneous type*, Advances in Mathematics 33 (1979), 257–270.

[26] S. Mac Lane and G. Birkhoff, *Algebra*, 3rd edition, Chelsea, 1988.

[27] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, 1995.

[28] L. Nachbin, *The Haar Integral*, translated from the Portuguese by L. Bechtolsheim, Krieger, 1976.

[29] H. Royden, *Real Analysis*, 3rd edition, Macmillan, 1988.

[30] W. Rudin, *Principles of Mathematical Analysis*, 3rd edition, McGraw-Hill, 1976.

[31] W. Rudin, *Real and Complex Analysis*, 3rd edition, McGraw-Hill, 1976.

[32] W. Rudin, *Fourier Analysis on Groups*, Wiley, 1990.

[33] W. Rudin, *Functional Analysis*, 2nd edition, McGraw-Hill, 1991.

[34] H. Schaefer and M. Wolff, *Topological Vector Spaces*, Springer-Verlag, 1999.

[35] J.-P. Serre, *Lie Algebras and Lie Groups*, 2nd edition, Lecture Notes in Mathematics 1500, Springer-Verlag, 2006.
[36] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[37] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, with the assistance of T. Murphy, Princeton University Press, 1993.

[38] E. Stein and R. Shakarchi, *Fourier Analysis: An Introduction*, Princeton University Press, 2003.

[39] E. Stein and R. Shakarchi, *Complex Analysis*, Princeton University Press, 2003.

[40] E. Stein and R. Shakarchi, *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*, Princeton University Press, 2005.

[41] E. Stein and R. Shakarchi, *Functional Analysis: Introduction to Further Topics in Analysis*, Princeton University Press, 2011.

[42] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[43] M. Taibleson, *Fourier Analysis on Local Fields*, Princeton University Press, 1975.

[44] A. Thompson, *Minkowski Geometry*, Cambridge University Press, 1996.