ON THE SATURATION OF COROTATION RESONANCES: A NUMERICAL STUDY

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ABSTRACT

The torque exerted by an external potential on a two-dimensional gaseous disk at non-coorbital corotation resonances is studied by means of numerical simulations. The degree of saturation of these resonances is important in determining whether an eccentric giant planet embedded in a protoplanetary disk experiences an eccentricity excitation or damping. Previous analytic studies of the saturation properties of these resonances suffered two important restrictions, since they neglected (1) the possible overlap between neighboring first-order corotation resonances and (2) the fact that first-order corotation resonances share their location with principal Lindblad resonances. We perform calculations restricted to one or two resonances to investigate the properties of two neighboring corotation resonances, as well as the properties of a corotation resonance that overlaps a Lindblad resonance. We find that these properties hardly differ from the case of an isolated corotation resonance, which is found in a first step to agree with the analytical theory. In particular, although the torque of two neighboring corotation resonances may differ from the sum of the torques of the corresponding resonances when considered as isolated, it never exceeds the sum of the fully unsaturated isolated corotation resonances, and it saturates in a fashion similar to an isolated resonance. Similarly, the presence of an underlying Lindblad resonance hardly affects the corotation torque, even if that resonance implies a torque strong enough to significantly redistribute the azimuthally averaged surface density profile, in which case the corotation torque scales with the resulting vortensity gradient. This set of numerical experiments thus essentially validates previous analytic studies. As a side result, we show that corotation libration islands misrepresented by a mesh of too low resolution can lead to a strongly overestimated corotation torque. This may be an explanation of why the eccentricity of embedded Jupiter-sized planets was never observed to undergo an excitation in the numerical simulations performed so far.

Subject headings: accretion, accretion disks — hydrodynamics — planetary systems: protoplanetary disks — planets and satellites: general

1. INTRODUCTION

Among the puzzles raised by the statistics of the recently discovered extrasolar giant planets (EGPs) is the origin of their eccentricities. The population of EGPs with periods larger than \( \sim 10 \) days displays an important scatter in eccentricity, which reaches for some systems values as extreme as 0.9, while most of the eccentricities of longer period planets are almost uniformly distributed between 0 and 0.6–0.7. Several explanations have been envisaged to date in order to account for these eccentricities, such as planet-planet interactions (Rasio & Ford 1996; Ford et al. 2001), external perturbers such as distant binary companions or passing stars (Holman et al. 1997; Mazeh et al. 1997), or disk-planet interactions (Goldreich & Sari 2003; Ogilvie & Lubow 2003 and references therein). In this last case, these interactions endow the giant protoplanets with nonvanishing eccentricities that remain after the dispersal of the disk. Here we examine some aspects of this tidal excitation of the eccentricity of a giant protoplanet, by means of restricted numerical simulations. In order for these simulations to describe as closely as possible the resonances excited by an eccentric giant planet, it is useful to bear in mind their location and their action on the eccentricity. In Table 1 we indicate the ratio \( \Omega_1/\Omega_p \) at the location of principal and first-order resonances and the impact of these resonances on the planet’s eccentricity, where we denote by \( \Omega \) the orbital angular velocity of fluid elements in the disk and by \( \Omega_p \) the planet’s orbital frequency. The qualification fast refers to first-order terms of the potential that have the pattern frequency \( \Omega_p + \kappa_p/m \), where \( \kappa_p \) is the planet’s epicyclic frequency and \( m \) the azimuthal wavenumber, while the slow denomination refers to first-order terms that have a pattern frequency \( \Omega_p - \kappa_p/m \). In Table 1 we neglect the slight sub-Keplerianity of the disk due to the radial pressure gradient; we also neglect the slight offset between the nominal position of a Lindblad resonance (LR) and its effective position (corresponding to the turning point in the WKB dispersion relation of density waves in the disk), and we exploit the Keplerian degeneracy \( \Omega_p = \kappa_p \) and \( \Omega = \kappa \).

In the case of an embedded giant protoplanet that clears a gap around its orbit, none of the coorbital resonances (the principal corotation resonances [CRs], the fast first-order outer LRs [OLRs], and the slow first-order inner LRs [ILRs]) contributes to the exchange of energy and angular momentum with the planet, and the eccentricity budget of the planet is therefore determined by the extreme LRs (the fast ILR and the slow OLR), which excite the eccentricity, and by the first-order CRs, which damp it. It has long been known that the damping provided by the CRs slightly exceeds the excitation due to the LRs. Goldreich & Tremaine (1980), using the large-\( m \) Bessel-function approximation for the torques and treating \( m \) as a continuous variable (an approximation that might not be accurate for gap-opening giant planets, which involve low
m-values), have shown that the CRs dominate over the LRs by a 5% amount. The CRs can, however, easily saturate, unlike the LRs. If they reach a 5% saturation, the net effect on the eccentricity is an excitation. Ogilvie & Lubow (2003) and Goldreich & Sari (2003) have studied the saturation properties of an isolated CR. The saturation of a CR results from the competition between libration (which tends to flatten the vortensity profile across the libration islands of the resonance) and viscous diffusion (which tends to establish the large-scale gradient of vortensity across the width of the libration islands). The saturation of a resonance therefore requires a sufficiently small viscosity for a given resonance width. For a given mass and eccentricity of the planet, there exists a critical viscosity below which the planet undergoes an eccentricity excitation, at least according to the balance between the resonant interactions. Ogilvie & Lubow (2003) show that for a planet-to-primary mass ratio \( q = 0.001 \) and a disk with aspect ratio \( H/r = 0.05 \), a starting eccentricity on the order of \( e = 0.01 \) is sufficient to trigger an eccentricity growth. Their analysis, however, neglected the fact that non–co-orbital CRs are not isolated. Successive CRs can indeed overlap, and these resonances share their location with the principal LRs, as can be seen in Table 1.

This work aims at studying the properties of a nonisolated CR by means of numerical simulations. Our analysis is threefold: we first check that numerical simulations of an isolated CR provide results in agreement with the analytical theory; then we investigate the properties of a pair of CRs of similar strength with wavenumbers that differ by 1, from the well-separated case to the strongly overlapping case; and finally we investigate the behavior of a first-order CR that lies at the location of a principal LR. Disentangling the torque from each resonance in this last case is made possible by the fact that the azimuthal wavenumber of the LR and that of the CR differ by 1, as can be seen in Table 1.

2. NUMERICAL ISSUES AND SETUP

2.1. Numerical Scheme

The scheme is one commonly used in the context of the tidal interaction of a protoplanetary disk and an embedded planet (Nelson et al. 2000; d’Angelo et al. 2003; Bate et al. 2003). It is an upwind scheme on a staggered polar mesh, in which the slopes are computed using a second-order estimate (van Leer 1977; Stone & Norman 1992). This kind of scheme is known to exhibit a spurious postshock oscillatory behavior that is regularized by the use of a second-order artificial viscosity. In the present work, however, in which we hardly ever had to deal with shocks (which appear only in the case of a strong overlapping LR), this artificial viscosity plays no role. We work in a frame rotating with angular speed \( \Omega_f \), which corotates either with one of the resonances or with the midradius of our annular domain. We use the angular momentum conservative scheme described by Kley (1998) for a rotating frame, rather than considering the Coriolis force as a source term. A significant speed up is obtained using the fast azimuthal advection method described by Masset (2000). In a standard manner, we consider a locally isothermal two-dimensional gaseous disk with sound speed \( c_s(r) \) (\( r \) being the distance to the mesh center), which we characterize by its aspect ratio \( H/r = c_s(r)/[\sqrt{\Gamma}(r)] \), assumed to be uniform over the computational domain. The disk is torqued by a potential, \( \Phi(r, \phi, t) \), that corresponds either to one or two rotating components:

\[
\Phi(r, \phi, t) = T(t/\tau) \sum_{i=1}^{1 \text{ or } 2} \psi_i(r) \cos[m_i(\phi - \Omega_i t)],
\]

where \( \psi_i(r) \) is the radial profile of the \( i \)th component of the potential, \( m_i \) its azimuthal wavenumber, \( \Omega_i \) its pattern speed, and where

\[
T(x) = \begin{cases} 
\sin^2 \left( \frac{\pi x}{2\sigma} \right) & \text{if } x < 1 \\
1 & \text{otherwise}
\end{cases}
\]

is a temporal tapering that turns on the potential on the timescale \( \tau \).

2.2. Annular Domain Width

In order for the torque arising from each component of equation (1) to correspond only to a specific resonance, some care must be taken about the radial width over which the disk is torqued. One can either adopt a profile, \( \psi(r) \), that is nonvanishing on a radial range smaller than that of the computational domain, or one can adopt a constant \( \psi(r) \) profile and a sufficiently narrow computational domain, so as to remain everywhere sufficiently far from the other resonances associated to that potential component (e.g., from the LRs in the case in which we want to consider a CR). We have adopted this second solution, and we work on radially narrow meshes. Namely, we consider low-\( m \) resonances (\( m = 2 \) or 3). For the less favorable case \( m = 3 \), the LR to CR ratio radii are \( r_{\text{OLR}}/r_{\text{CR}} = 1.21 \) and \( r_{\text{ILR}}/r_{\text{CR}} = 0.76 \). We want the computational domain to avoid the LR locations, at least by typically 2 or 3 driving lengths (Meyer-Vernet & Sicardy 1987):

\[
\lambda_D \approx \left( \frac{H^2 r}{3m} \right)^{1/3}.
\]

This implies that we have a sufficiently thin disk (the “standard” choice \( H/r = 0.05 \), for which the driving length is \( \lambda_D \approx 0.06r \), would give marginally too large LR widths, since the CR, which is at least as large as \( H \), would slightly overlap the perturbation inside of the OLR). Instead, we use a much thinner disk (\( H/r = 0.01 \)), for which the wave driving length at the LR is approximately 3 times shorter. With this choice, we can safely consider two distinct CRs that are well separated from the LRs.

2.3. Boundary Conditions

Far from an isolated CR, but still inside of the band forbidden for the propagation of acoustic perturbations around corotation,
the perturbation corresponds to an evanescent wave, with a purely imaginary radial wavenumber given by

\[ k_r^2 = \frac{1}{c_s^2} \left[ m_i^2 (\Omega - \Omega_i)^2 - \Omega^2 \right] - \frac{m_i^2}{r^2}, \]

obtained from the dispersion relation of a density wave in a Keplerian disk. In the case in which we consider only CRs, we evaluate this wavenumber for each resonance, \( i \), at the mesh boundaries, and we apply the following procedure:

1. We evaluate the azimuthal Fourier transform of the hydrodynamic variables (surface density and radial and azimuthal velocities) with wavenumber \( m_i \) in the second ring at the inner boundary and in the penultimate ring at the outer boundary:

\[
A_{m_i} = \frac{2}{N_\phi} \sum_{k=0}^{N_\phi-1} \xi_k \cos \left( \frac{2\pi}{N_\phi} m_i k \right),
\]

\[
B_{m_i} = \frac{2}{N_\phi} \sum_{k=0}^{N_\phi-1} \xi_k \sin \left( \frac{2\pi}{N_\phi} m_i k \right),
\]

where \((\xi_k)_{0 \leq k < N_\phi}\) is any of the aforementioned quantities on the corresponding ring and where \(N_\phi\) is the number of zones of the mesh in azimuth.

2. We then overwrite the content of the first ring (at the inner boundary) and of the last ring (at the outer boundary) with

\[
\xi_k = \xi^0 + \sum_i \left[ A_{m_i} \cos \left( \frac{2\pi}{N_\phi} m_i k \right) + B_{m_i} \sin \left( \frac{2\pi}{N_\phi} m_i k \right) \right] \exp (-|k_r| \Delta r),
\]

where \(\Delta r\) is the radial resolution of the mesh and where \(\xi^0\) stands for the unperturbed hydrodynamic variable.

This simple prescription ensures a correct description at the mesh boundaries of the evanescent waves for all the CRs. In the case in which one considers both a CR and an LR, a similar procedure does not produce satisfactory results, and the computational domain is rapidly invaded by spurious propagating waves. We then turn to a procedure proposed by W. Kley et al. (2003) on the Web site Comparison of Hydrocode Performance in Disk-Planet Interaction, supported by the EU Research and Training Network The Origin of Planetary Systems, for a comparison test problem,\(^2\) in which the perturbation is artificially damped on a short range of radius near either boundary. Although this prescription mangles the response of the disk to a CR near its boundaries, it is very effective at avoiding the reflection of the wave launched at the LR.

### 2.4. Centrifugal Balance

The mechanism we want to investigate relies on a minute motion inside libration islands, which involves very small radial velocities. The idealized picture of these libration islands can easily be mangled by an inaccurate initial centrifugal balance, which gives rise to axisymmetric waves with an associated radial velocity that can be larger than the one corresponding to libration. It is therefore critical to impose a strict centrifugal balance at \( t = 0 \) that takes into account the discretization of the problem on the mesh. Since the mesh is staggered, there is an infinite number of sequences \( (v_\phi^j)_{0 \leq j < N_r} \) (where \(N_r\) is the radial number of zones of the mesh) that lead to a strict centrifugal balance. We choose the one that minimizes the quantity \( R \), defined by

\[
R = \sum_{j=1}^{N_r-2} \left| v_\phi^{j+1} - v_\phi^{j-1} - 2v_\phi^j \right|.
\]

#### 2.5. Torque Expression

The total torque exerted by the potential on the disk is evaluated as

\[
\Gamma = \sum_{i=0}^{N_\phi-1} \sum_{j=0}^{N_\phi-1} \Phi_{ij} - \Phi_{ij} S_{ij} \sin \left( \frac{m \Delta \phi}{\Delta \phi} \right),
\]

where \(S_{ij}\) is the surface area of zone \((i,j)\), \(\Sigma_{ij}\) and \(\Phi_{ij}\) are the surface density and gravitational potential pertaining to the center of this zone, respectively, and \(\Delta \phi = 2\pi/N_\phi\) is the resolution in azimuth.

We also use the torque exerted on the disk by the specific potential component with azimuthal wavenumber \(m\). Its expression is given by

\[
\Gamma_m = \sum_{i=0}^{N_\phi-1} \sum_{j=0}^{N_\phi-1} S_{ij} \sin \left( \frac{m \Delta \phi}{\Delta \phi} \right) \times \left[ A^m_j \sin \left( m \Delta \phi \right) - B^m_j \cos \left( m \Delta \phi \right) \right],
\]

where

\[
A^m_j = \frac{2}{N_\phi} \sum_{j=0}^{N_\phi-1} \Phi_{ij} \cos \left( m \Delta \phi \right),
\]

\[
B^m_j = \frac{2}{N_\phi} \sum_{j=0}^{N_\phi-1} \Phi_{ij} \sin \left( m \Delta \phi \right).
\]

The value of \(\Gamma_m\) coincides exactly with the value of \(\Gamma\) whenever only one potential component is present, regardless of the resolution.

#### 2.6. Units

As in a large number of publications on the disk-planet tidal interaction problem, the mass unit is the mass of the central object, the distance unit is arbitrary (in what follows it either corresponds to the corotation radius of an isolated resonance or to the mean of the corotation radii of two resonances), and the time unit is \(\Omega^{-1}\), where \(\Omega\) is the Keplerian angular speed at \(r = 1\). The surface density is initially uniform and equal to 1.

### 3. The isolated corotation resonance

In a first step, we have checked what resolution is needed for the code to properly capture the torque properties (unsaturated torque value and saturation properties) in the case of an isolated CR. For this purpose, we have made a number of test calculations with different resolutions and viscosities. Our first setup consists of an isolated \(m = 3\) resonance at frequency \(\Omega = 1\). The inner mesh boundary is located at \(R_{\text{min}} = 0.93\) and

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\(^2\) For more information on the EU Networks comparison of hydrocode performance in disk-planet interaction, see http://www.astro.su.se/~pawel/planets/test.hydro.html.
the outer mesh boundary is located at $R_{\text{max}} = 1.07$. The potential is turned on on a timescale $\tau = 150 \Omega^{-1}$. The number of zones in $\phi$ is set to 90, which means that each libration island spans 30 zones azimuthally. The width of the libration islands is given, if one neglects pressure effects, by

$$W = (32)^{1/2} \delta_{\phi} = \frac{8}{\Omega} \left(\frac{\psi}{3}\right)^{1/2},$$

where

$$\delta_{\phi} = \left(\frac{2\psi}{3\Omega^2}\right)^{1/2}$$

is the characteristic length scale for libration in a Keplerian disk (Ogilvie & Lubow 2003). The forcing potential amplitude is $\psi = 10^{-5}$, so the width of the libration islands, neglecting pressure effects, is $W \approx 0.014$. This typically corresponds to a Jupiter-sized planet orbiting a solar-type star on an eccentric orbit with $e \sim 4 \times 10^{-3}$. We have tried to vary the number of zones radially, from $N_r = 30$ to $N_r = 240$. This corresponds to a radial sampling of the libration islands that varies from 3 to 25 zones. In this last case, the number of zones over which a libration island is described is approximately the same in azimuth and radius. The kinematic viscosity is either set to 0, in which case diffusion is only of numerical origin, or to the constant value $10^{-6}$, which corresponds to a small ($\sim 7\%$) saturation of the resonance. This can be evaluated by using the saturation parameter introduced by Ogilvie & Lubow (2003), the expression of which, in a Keplerian disk, is

$$p = \left(\frac{2\psi}{3\Omega^2}\right) \left(\frac{3m\Omega}{2\nu}\right)^{2/3},$$

which leads in our case to the value $p = 0.182$. The $7\%$ saturation is then deduced using equation (66) of Ogilvie & Lubow (2003), which holds for slightly saturated resonances.

The results are presented in Figure 1. One can notice an oscillatory behavior of the torque value in the inviscid case, with a period of the order of the libration time,

$$\tau_{\text{lib}} \approx \frac{8r}{3\Omega_\phi m},$$

where

$$\Omega_\phi = \left(\frac{2\psi}{3\Omega^2}\right)^{1/2}$$

is the characteristic timescale for libration in a Keplerian disk, which leads in our case to the value $\tau_{\text{lib}} \approx 100$. The expected oscillatory behavior of the torque value in the inviscid case, increases as the resolution increases.

The torque value at large viscosity is in correct agreement with the estimate that can be deduced from the Goldreich & Tremaine (1980) formula, which leads to

$$\Gamma_{\text{unsaturated}} = -5.92 \times 10^{-9}.$$

The expected torque value is therefore $7\%$ smaller in absolute value; i.e., $\Gamma = -5.51 \times 10^{-9}$. The error in the torque given by our calculation is therefore typically $5\%$. One can also notice that the unsaturated torque value is very well described even at low radial resolution (bottom curves). This had already been noticed by Masset (2002) in the context of the horseshoe region torque. On the other hand, the curves of different resolutions noticeably differ in the inviscid case. The expected behavior is that the torque will tend toward zero (saturates) as time goes to infinity. The three highest resolutions seem to be in reasonable agreement with this expectation, within a small offset. The case of lowest resolution, however, does have a significant residual torque at large $t$, and does not exhibit any oscillatory behavior. It is likely that at these low resolutions, the numerical diffusion itself acts to unsaturate the torque. One can also note that the oscillatory behavior of the torque, in the inviscid case, increases as the resolution increases.

In Figure 2 we plot the dimensionless residual torque value at large time as a function of the resolution, expressed as the number of zones spanned by the libration islands. We see that a resolution $w/\Delta r$ of 12–25 is sufficient to ensure a residual unsaturation of $\sim 1\%$ after approximately 1000 orbits.

In Figure 3 we plot the streamlines (from the calculation with highest radial resolution) in the inviscid case and in the viscous case. As we consider a thin disk, for which the pressure length scale is on the order of the libration scale, the radial width of the libration islands is satisfactorily accounted for by the ballistic expression of equation (2).

We have performed additional calculations for the isolated CR case in order to investigate whether the viscosity prescription ($\nu = \text{constant}$ or $\nu \propto \Sigma^{-2/3}$, as used in the analysis of Ogilvie & Lubow 2003) and the boundary condition prescription, could significantly alter the value and saturation properties of the corotation torque.

The results are shown in Figure 4. The curves with diamonds show the results with a constant viscosity, while the curves with squares show the results with the viscosity prescription $\nu \propto \Sigma^{-2/3}$. All these calculations have $W/\Delta r = 21.5$ or 22.
The solid curves show calculations with an inner boundary at $R_{\text{min}} = 0.88$ and an outer boundary at $R_{\text{max}} = 1.12$, with damping boundary conditions on the 7% inner and outer parts of the mesh. The dotted curves show calculations with the same boundary conditions but with an inner boundary at $R_{\text{min}} = 0.83$ and an outer boundary at $R_{\text{max}} = 1.17$. Finally, the dashed curves show calculations with boundary conditions adapted to the evanescent waves. The dashed and dotted curves strictly have all same parameters, and differ only in their boundary conditions. One can make the following comments:

1. There is a systematic offset between a uniform viscosity prescription and the $\nu \propto \Sigma^{-2/3}$ prescription, which amounts to a bit less than 10%, the latter prescription leading to a more saturated resonance.

2. The calculation that one would expect to best fit the analytical results of Ogilvie & Lubow (2003) is the one with adapted boundary conditions and a $\nu \propto \Sigma^{-2/3}$ prescription. This corresponds to the dashed curve with squares. Although it satisfactorily agrees with the theoretical expectations for $p \leq 0.3$, it leads to a resonance that is less saturated than expected for larger values of $p$. For $p = 2.0$, the measured dimensionless torque is offset of 0.1 from its analytical estimate.

Such an offset is unlikely to be due to numerical diffusion, as can be seen in Figure 2.

3. Damping boundary conditions have a small but measurable impact on the fully unsaturated torque value. This might be linked to the fact that with the radially extended mesh that we consider, the boundaries lie within $\sim 2$ driving lengths from the LRs. In any case, whenever we study a situation with two resonances (either two CRs or one CR and one LR), we compare our results to the isolated CR case by performing separately an isolated resonance test calculation, with exactly the same boundary location and prescription.

4. SATURATION OF A PAIR OF COROTATION RESONANCES

We have performed a first set of calculations in which we consider the $m = 3$ CR presented at length in the previous section ($\psi = 10^{-5}$, $\Omega = 1$) and a second resonance, with potential amplitude $\psi' = 1.3 \times 10^{-5}$, azimuthal wavenumber $m' = 2$, and frequency $f'$. Such that the corotation radius is located at $r = 1 + s \times 10^{-4}$, where $s$ takes the values 25, 50, 100, 200, 300, and 500. With the largest separation ($s = 500$), the resonances are well apart and their distance in radius amounts to slightly more than 3 times the sum of their half-widths, while for the smallest separation the distance between the corotation radii amounts to only 16% of the sum of their half-widths, so the resonances strongly overlap. For each two-resonance calculation, two other separate calculations with exactly the same parameters are performed, in which one only of the resonances is present. We have performed all these calculations with three different values for the viscosity (chosen here as uniform): $\nu = 10^{-6}$, $10^{-7}$, and 0. The first choice corresponds to a low value of the $p$ parameter, and to resonances that are therefore almost unsaturated, while the second choice corresponds to values of $p$ that are close to unity ($p = 0.844$ for the $m = 3$ resonance, and $p = O(1)$ for the $m' = 2$ resonance as well, with an exact value that depends on $s$). The mesh parameters were $R_{\text{min}} = 0.87$, $R_{\text{max}} = 1.18$, $N_r = 270$, and $N_{\phi} = 90$.

4.1. High Viscosity

In Figure 5 we show results for four values of the separation ($s = 300, 100, 50$, and 25) of the highly viscous case. On these plots the variable $\delta$ represents the distance between the
corotation radii of the two resonances, \( w_1 \) is the \( m = 3 \) resonance width, and \( w_2 \) is the \( m' = 2 \) resonance width, estimated with the ballistic approximation of equation (2). The resonances should overlap when the parameter \( \delta = 2\bar{\delta}/(w_1 + w_2) \) becomes smaller than unity, i.e., when the distance between the corotation radii is smaller than the sum of the resonances’ half-widths.

The variable \( \tau_{\text{beat}} \) features in Figure 5. It represents the amount of time that separates identical configurations of potential components:

\[
\tau_{\text{beat}} = 2\pi \left| \frac{m^{-1} - m'^{-1}}{\Omega - \Omega'} \right|.
\]  

(4)

When the two resonances overlap, a passive scalar advected by the flow displays a highly time-dependent behavior and remarkably, in this highly viscous case, this has no impact on the torque value. The two-resonance torque value is almost insensitive to the resonance overlap, and exhibits a value that is nearly constant in time, whatever the value of \( \delta \). A tentative explanation for this property could be the following: Ogilvie & Lubow (2003) have emphasized that the nonlinear effect on the corotation torque of an isolated resonance can be understood as the feedback of the perturbation on the vortensity gradient of the disk, and this is likely generalizable to a set of CRs. For a viscosity large enough that the viscous timescale across the libration islands be shorter than any other timescale (libration timescale or beating timescale), the vortensity gradient across the resonances is hardly modified with respect to its unperturbed value; hence, nonlinear effects should be negligible and the torque value should be the sum of the linear estimates for each resonance.

4.2. Moderate Viscosity

In Figure 6 we show results for six values of the separation \( s = 500, 300, 200, 100, 50, \) and 25) of the moderately viscous case. On these curves, one can see that the two-resonance torque is nearly equal to the sum of the one-resonance torques when the resonances are separated \( (\delta > 1) \), although there exists a small but noticeable difference between these two values that increases as \( \delta \) gets close to unity. When both resonances overlap \( (\delta \approx 1) \), the time-averaged two-resonance torque value can be either larger (in absolute value, for \( \delta = 0.63 \)) or smaller \( (\delta = 0.32 \text{ or } 0.16) \) than the sum of the one-resonance torque values. One can also notice a torque modulation with period \( \tau_{\text{beat}} \). The findings of the highly viscous case therefore do not hold, even for the time-averaged values. Note that \( \tau_{\text{beat}} \) is only the shortest period over which one can expect the torque modulation to occur, while the longest is \( 2\pi/|\text{GCD}(m, m')/\Omega - \Omega'| \), where \( \text{GCD}(m, m') \) is the greatest common divisor of \( m \) and \( m' \), which is 1 in the case in which \( |m' - m| = 1 \).

4.3. Inviscid Case

In Figure 7 we show results for six values of the separation \( s = 500, 300, 200, 100, 50, \) and 25) of the inviscid case.
Again, the solid and dash-dotted curves are almost indistinguishable by eye when the resonances do not overlap ($\delta > 1$, top panels), while they are markedly different when they overlap ($\delta < 1$, bottom panels). All curves slowly decay in absolute value. This is normal for the one-resonance torques and their sum, since the corotation torque in that case tends to saturate, and it is reasonable to expect it also to saturate, on the average, in the two-resonance case. A quasi-periodic torque modulation is apparent for the closest case (bottom right panel), with a pseudoperiod that corresponds to $2\tau_{\text{beat}}$.

Fig. 6.—Corotation torque of a pair of CRs in the moderate-viscosity case as a function of time. The curve styles are the same as in the previous figure.

Fig. 7.—Corotation torque of a pair of CRs in the inviscid case as a function of time. The curve styles are the same as in Fig. 5.
4.4. Systematic Exploration

We have performed calculations similar to the illustrative cases of two CRs of the previous paragraphs, with a more systematic exploration of the (separation, saturation) parameter space. Our aim is to check the following trends:

1. The two-resonance torque value is the sum of the one-resonance torques, whenever the viscosity is high or the resonances do not overlap.

2. The two-resonance torque value of overlapping resonances saturates at low viscosities, in a fashion similar to the saturation of a one-resonance torque.

3. The time-averaged torque value of two overlapping resonances for a small or moderate viscosity may sensibly differ from the sum of the isolated resonances torques, but it never exceeds the sum of the fully unsaturated isolated resonance torques.

We therefore set up the following series of calculations:

1. We consider two fixed CRs, with frequencies and azimuthal wavenumbers of \( \Omega = 1.015 \) and \( m = 3 \) and \( \Omega' = 0.985 \) and \( m' = 2 \), respectively. These resonances have the same forcing potential amplitude, \( \psi \), which we vary. The disk still has \( H/r = 0.01 \); the mesh has \( N_r = 240 \) by \( N_\phi = 90 \) zones, the inner mesh boundary is at \( R_{\min} = 0.84 \), and the outer mesh boundary is at \( R_{\max} = 1.16 \). The potential is turned on on the timescale \( \tau = 20 \, \Omega^{-1} \). We use damping boundary conditions.

2. The amplitude of the forcing potential is chosen such that the parameter \( \delta^{-1} = (w_1 + w_2)/2\Omega \) takes the values 0.3, 0.45, 0.6, 0.8, 1.0, 1.15, 1.3, 1.45, 1.8, 2.1, 2.5, 2.75, 3.0, 4.0, 5.0, and 6.0; i.e., from well-separated resonances to strongly overlapping resonances. In the above expression, \( w_1 \) and \( w_2 \) are evaluated as if the resonance frequencies were both \( \Omega = 1 \); therefore, \( w_1 = w_2 \), since the libration island width, in the ballistic approximation, does not depend on \( m \).

3. The viscosity is varied such that the saturation parameter, given by equation (3), of the “mean” fictitious resonance that would have \( \Omega = 1 \), the same forcing potential as the two resonances considered, and \( m = 2.5 \), would take the values \( p = 0.2, 0.4, 0.6, \ldots, 2.0 \).

4. Each calculation is performed over a time that is larger than the libration time of each resonance, the beating time between them, and the viscous time across each libration zone.

The torque is then measured and averaged over the last beating time.

5. The same procedure is repeated for two other calculations in which only one of the resonances is present, and which is performed over exactly the same time, and again the torque is measured and time averaged over the same temporal window (although the beating time is then meaningless). The whole calculation therefore amounts in total to \( 3 \times 16 \times 10 = 480 \) such runs.

In Figure 8, we plot the ratio of the time-averaged two-resonance torque to the sum of the one-resonance torques as a function of viscosity and resonance separation. As expected, the result is very close to unity near the top and left axes, which corresponds to the well-separated case and the highly viscous case, respectively. This confirms expectation (1) above.

The fact that the ratio plotted in Figure 8 does not differ from unity by more than roughly 40% confirms expectation (2) above; i.e., the saturation curve of total torque versus \( p \), taken at any given value for the overlap parameter, does not significantly differ from the sum of the saturation curves of the corresponding resonances, considered as isolated.

We plot in Figure 9 the ratio of the two-resonance torque value to the sum of the fully unsaturated torque values of both resonances as a function of viscosity and resonance separation. This ratio can be seen to never exceed unity, which confirms expectation (3) above.

5. SATURATION OF A COROTATION RESONANCE OVERLAPPING A LINDBLAD RESONANCE

As the first-order CRs lie on top of principal LRs, we have addressed the restricted problem of the saturation of one CR that exactly overlaps a LR. We have examined the consequences of this overlap in two different situations:

1. The Lindblad torque is small enough so that it does not significantly affect the azimuthally averaged radial profile of surface density. This allows us to compare directly the corotation torque value with its value in the case in which no LR is present.

2. The Lindblad torque is large and significantly redistributes the disk material and thus it considerably alters the...
slope of vortensity across the resonance. In this case, a direct comparison with a single corotation calculation is not possible. Rather, we measure the slope of vortensity across the resonance and compare the value of the unsaturated resonance to the Goldreich & Tremaine (1979) estimate.

5.1. Small Lindblad Torque

We consider a CR excited by a potential with wavenumber \( m = 3 \), frequency \( \Omega = 1 \), and amplitude \( \psi = 2 \times 10^{-6} \), which has exactly the same location as an LR excited by a potential with wavenumber \( m' = 2 \), frequency \( \Omega' = 3/2 \), and amplitude \( \psi' = 5 \times 10^{-5} \). For the case of an eccentric planet, this corresponds to a slow first-order CR overlapping a principal outer LR. The reason why we consider such a low-amplitude forcing potential for the CR is that we wish the onset of possible nonlinear effects between the corotation and LRs to be as low as possible.

Our mesh extends from \( R_{\text{min}} = 0.84 \) to \( R_{\text{max}} = 1.16 \) and its resolution is \( N_r = 1000 \) by \( N_\phi = 90 \). The radial resolution is large in order to have a satisfactory resolution of the narrow libration islands of the CR. The potentials are turned on over the timescale \( \tau = 150 \, \Omega^{-1} \).

In Figure 10 we plot the torque value of the CR as a function of time for different values of the viscosity, corresponding to an arithmetic sequence of the saturation parameter, \( p \), for a mixed corotation/Lindblad resonance and for the case of an isolated CR with the same characteristics.

The two series of curves exhibit very similar trends. The torque of the isolated resonance tends to oscillate more than that overlapping a LR at low viscosity. One can also notice the very long libration time, which is due to the narrowness of the libration islands.

In Figure 11 we plot the dimensionless torque as a function of the saturation parameter, obtained from the time average of the torque values presented in Figure 10, over the interval \( 800 \, \Omega^{-1} < \tau < 1600 \, \Omega^{-1} \).

The time-averaged torque estimate at low viscosity, especially for the isolated resonance, may be slightly biased by the fact that it has not had enough time to reach a steady saturation level over the full duration of the calculation. The impact of this bias is, however, small, and the conclusion that can be drawn from the examination of Figure 11 is that the torque saturation properties are hardly affected by the presence of the underlying LR. The tiny difference between both curves can be attributed either to the relatively small number of libration times over which the time average is performed (at low viscosity) or to the slight radial redistribution of the disk material under the action of the Lindblad torque. The tiny change of the vortensity slope, defined as \( -d \log (\Sigma/B)/d \log r \) (where \( B \) is the second Oort’s constant), can indeed be estimated to be on the order of a few percent, compatible with the discrepancy of the two saturation curves.

5.2. Large Lindblad Torque

We have also performed a test calculation in which the forcing potential amplitude of the LR is much larger, so that it modifies the surface density profile in a reduced version of what occurs at a gap edge. We cannot vary the viscosity in this kind of calculation, since we want the surface density profile to converge toward the redistributed profile in a reasonable amount of time. At viscosities low enough to significantly saturate the corotation torque, the viscous timescale across the excavated region becomes prohibitively large. We therefore just compare the corotation torque value to the Goldreich & Tremaine (1979) estimate, obtained by measuring the slope of \( \Sigma/B \) across the resonance width at any instant in time. In Figure 12, the left panel shows the comparison between this estimate and the actual corotation torque value. They agree to within a few percent, and the actual torque value is slightly smaller than the Goldreich-Tremaine estimate. The parameters that we adopted for this calculation are:

1. \( \psi = 10^{-5}, m = 3 \), and \( \Omega = 1 \) for the CR.
2. \( \psi' = 10^{-4}, m' = 2 \), and \( \Omega' = 1/2 \) for the LR. This frequency implies that it is an ILR. If we wanted to strictly reproduce the overlap between a first-order CR and a principal ILR, we should have \( m_{\text{ILR}} = m_{\text{CR}} + 1 \). We have chosen instead \( m_{\text{ILR}} = m_{\text{CR}} - 1 \), which allows us to better resolve the wave launched at the LR for a given value of \( N_\phi \). We anticipate that our conclusions are not affected by this choice.

6. LOW-RESOLUTION ISSUES

The above sections essentially validate the previous work of Ogilvie & Lubow (2003) on the saturation of an isolated CR in the context of the first-order CRs of an eccentric giant planet, which are not isolated. The numerical simulations performed
thus far on the problem of the eccentricity excitation of a giant planet did not exhibit any eccentricity growth for Jupiter-sized giant planets (Papaloizou et al. 2001). While there may be a number of reasons for this, we show in this section how the corotation torque estimate is mangled at very low resolution; i.e., when the width of the libration islands is on the order of the mesh radial size. The width of a first-order CR is given by

\[ W_r / C^2 \approx 4.1(C_m^\pm \text{meq})^{1/2}, \]  

(5)

where \( q \) is the mass ratio and \( C_m^{\pm} \) is a factor on the order of unity given in Tables 1 and 2 of Ogilvie & Lubow (2003). Assuming one has a Jupiter-sized planet orbiting a solar mass star (\( q = 10^{-3} \)) with eccentricity \( e = 0.01 \), the resonance width is therefore \( W / r \approx 0.013(C_m^\pm \text{meq})^{1/2} \). As a point of comparison, in run N1 in Papaloizou et al. (2001), which corresponds to a Jupiter-sized giant planet, the radial resolution is \( \Delta r / r = 0.043 \).

Figure 13 shows the results (as before, diamonds are for a uniform viscosity prescription, while squares are for a \( \nu \propto \Sigma^{-2/3} \) viscosity prescription). The solid curves are for a resolution \( W / \Delta r = 1.0 \), the dotted curve is for \( W / \Delta r = 0.5 \), and the dashed curve is for \( W / \Delta r = 2.0 \). This shows again that the value of the fully unsaturated corotation torque is relatively well reproduced even at very low resolution, but the saturated resonant torque is completely mangled at low resolution. The case \( W = \Delta r \) is particularly striking, since in this case one gets a torque value larger than the Goldreich & Tremaine (1979) estimate. Therefore, in a low-resolution simulation of an embedded giant planet, as soon as the eccentricity gets to values that give the libration islands a size comparable to the radial resolution, these effects could lead to a strong spurious damping of the eccentricity, which would therefore be bounded by small values (\( O(0.01) \), depending on the resolution).

7. CONCLUSIONS

By means of reduced two-dimensional numerical simulations, involving two resonances at the same time and each resonance separately in order to compare the isolated case to the two-resonance case, we reach the following conclusions:

1. The corotation torque of two overlapping resonances coincides with the sum of the corotation torques of each resonance, considered as isolated, either when the resonances do not overlap (their separation is larger than the sum of the half-widths of their libration islands) or when the viscosity is large (fully unsaturated resonances).
2. When the resonances overlap and are partially saturated, the two-resonance torque may differ at most by a few tens of percent from the sum of the isolated torques, evaluated for the same viscosity. This implies that a given degree of saturation of a nonisolated resonance is reached for a value of the viscosity of the same order of magnitude as the viscosity needed to achieve the same degree of saturation for the isolated resonance.
3. The time-averaged two-resonance torque never exceeds the sum of the Goldreich & Tremaine (1979) estimates of each resonance.
4. In the case in which a CR overlaps a weak LR (which does not significantly redistribute the vortensity profile), the
time-averaged torque value differs from the isolated case by at most a few percent, whatever the value of the viscosity.

5. In the case in which a CR overlaps a strong LR, for which our calculations are limited to high-viscosity situations, the time-averaged torque value corresponds to the Goldreich & Tremaine (1979) estimate; i.e., to the torque of the unsaturated CR, considered as isolated.

6. The only case in which we have observed a corotation torque value that is larger than the Goldreich & Tremaine (1979) estimate (which therefore jeopardizes an eccentricity excitation for the set of parameters quoted in the introduction) is when the mesh resolution is too low. This is, therefore, a numerical artifact, and we conclude that a high resolution is needed in the region of first-order CR (i.e., close to the gap edges) in order to observe an eccentricity excitation, with libration islands spanning at least ~10 zones radially.

Finally, we emphasize that the saturation of the first-order CRs is only one of the factors determining whether the eccentricity of a giant protoplanet grows or decays through its interaction with the disk. An aspect of this problem that tends to be overlooked is the secular exchange of eccentricity between the planet and the disk, which may occur on a timescale of thousands of orbital periods. The mechanism responsible for this exchange is sometimes referred to as the apsidal resonance (Goldreich & Sari 2003), although this resonance is so broad in a nearly Keplerian disk that we prefer to avoid this denomination. When a planet is placed on an elliptical orbit in a circular disk, its eccentricity should initially decrease as eccentricity is imparted to the disk. This exchange can, in principle, occur in a way that conserves the angular momentum deficit of the coupled disk-planet system, in which case the eccentricity of the planet will be restored later in the cycle of secular exchange. Therefore, a transient decrease of the planet’s eccentricity in such circumstances should not be confused with eccentricity damping. In the analogous problem of the secular inclination dynamics of coupled disk-planet systems, Lubow & Ogilvie (2001) have defined a positive definite measure of the bending disturbance of the system, equivalent to an angular momentum deficit, which is conserved in the secular exchange and evolves only through the effects of mean motion resonances and viscous damping. Papaloizou (2002) has also treated aspects of the eccentricity dynamics of coupled disk-planet systems, while Ogilvie (2001) has drawn attention to the subtle issue of whether eccentricity is damped or excited by viscous or turbulent stresses in the disk, and at what rate.

A fully self-consistent calculation therefore involves many additional complications, such as the treatment of the force arising from the material inside of the Roche lobe, the role of self-gravity, and the eccentric coupling between the disk and the planet through the apsidal waves. It will be presented in a forthcoming work.

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