On the nature of the magnetic phase transition in a Weyl semimetal

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We investigate the nature of the magnetic phase transition induced by the short-ranged electron-electron interactions in a Weyl semimetal by using the perturbative renormalization-group method. We find that the critical point associated with the quantum phase transition is characterized by a Gaussian fixed point perturbed by a dangerously irrelevant operator. Although the low-energy and long-distance physics is governed by a free theory, the velocities of the fermionic quasiparticles and the magnetic excitations suffer from nontrivial renormalization effects. In particular, their ratio approaches one, which indicates an emergent Lorentz symmetry at low energies. We further investigate the stability of the fixed point in the presence of weak disorder. We show that while the fixed point is generally stable against weak disorder, among those disorders that are consistent with the emergent chiral symmetry of the clean system, a moderately strong random chemical potential and/or random vector potential may induce a quantum phase transition towards a disorder-dominated phase. We propose a global phase diagram of the Weyl semimetal in the presence of both electron-electron interactions and disorder based on our results.

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I. INTRODUCTION

Electronic materials whose low-energy excitations behave like Dirac fermions attract a lot of interests in condensed matter physics. These materials are characterized by nodal points in the Brillouin zone at which two distinct bands touch. Examples of these materials range from graphene to the surface state of a three-dimensional (3D) topological insulators. These studies not only help in developing a new generation of electronic devices, but also provide a theoretical link between condensed matter theory and high energy physics.

More recently, there has been growing interest in a close cousin of the above mentioned two-dimensional (2D) systems — the Weyl semimetals (WSM). Just like graphene, in these materials, two bands touch at certain point (the Weyl node) in the momentum space. As usual, band touching usually leads to Dirac fermions due to the presence of the time-reversal (T) and the space inversion (P) symmetry. To realize the Weyl fermion, which has only half a degree of freedom of a single Dirac fermion, the involved bands must be individually nondegenerate. This requires that either the T or the P symmetry is explicitly broken. The stability of these Weyl nodes are protected topologically. This is closely related to the fact that each Weyl node carries a quantized monopole charge \( Q = \pm 1 \) in the momentum space, which cannot be changed by a small perturbation. Moreover, since the net “magnetic charge” must be zero in a Brillouin zone, the Weyl nodes have to appear in pairs in a crystal.

Since a WSM is a gapless system, it is expected that the electron-electron interaction should have dramatic impact on its properties. Because of the vanishing density of states (DOS) at the Weyl nodes, the short-range interaction is perturbatively irrelevant while the long-range Coulomb interaction is marginally irrelevant in the renormalization-group (RG) sense. Therefore, to the leading order, it is reasonable to ignore the short-range interaction and the low-energy properties of the WSM are captured by the free Weyl fermions. While for the long-range Coulomb interaction, we only expect logarithmic corrections to the physical response functions at low frequencies and long distances.

On the other hand, in the presence of strong electron-electron interactions, we expect a quantum phase transition (QPT) towards some symmetry breaking phases. Previous studies based on the mean-field theory suggest that the possible states at strong repulsive interactions include the excitonic and the charge-density-wave (CDW) phases. Later an unbiased perturbative RG study on all possible four-fermion short-range repulsions that are consistent with the symmetries of the underlying lattice was performed. By extrapolating the one-loop RG equations to the strong-coupling regime, it was found that there is a unique direction in the asymptotic RG flow which points toward a spin-density-wave (SDW) ground state with the characteristic momentum of the SDW state corresponding to the momentum separation between the two Weyl nodes. Very recently, Ref. tackled this problem by mapping the lattice model with a strong on-site Hubbard interaction to a \( t-J \) type model. In terms of an appropriate mean-field treatment, it is concluded that the WSM with strong repulsions becomes magnetically ordered. Thus, both the perturbative RG and the strong-coupling analysis lead to a similar conclusion.

Since the WSM phase is stable at weak coupling and an ordered phase occurs at strong coupling, the two phases, the WSM and the ordered phase, must be separated by a
quantum critical point (QCP). Up to now, it is still unclear what is the nature of this QCP. We try to answer it in this paper. We focus on the transition to SDW phase. We start with the minimal model of WSM which consists of two Weyl nodes with opposite monopole charges. We propose that the critical theory is described by a single Dirac fermion (consisting of two copies of Weyl fermions) and a complex bosonic order parameter characterizing the SDW fluctuations. By a one-loop RG analysis, we find that the Gaussian fixed is stable at low energies so that the correlation length exponent \( \nu = 1/2 \). Although the QPT is characterized by a trivial fixed point in the sense that the boson-fermion coupling is marginally irrelevant, we find that the velocities of the fermionic quasiparticles and the magnetic excitations suffer from non-trivial renormalization effects. In fact, their ratio approaches one in the low-energy limit. This indicates that the QCP has an emergent Lorentz symmetry, which results in the dynamical critical exponent \( z = 1 \).

We also investigate the stability of the above fixed point in the presence of weak disorder. The latter is inevitable in condensed matter systems. The proposed critical theory has a chiral symmetry which is absent in the underlying lattice model. To simplify our analysis, we consider all sorts of disorder that are consistent with the chiral symmetry. In three dimensions, there are four types of disorder satisfying this condition: the random chemical potential, the random vector potential, the random chiral chemical potential, and the random chiral vector potential. We find that while the critical properties of the clean system is stable against weak disorder for all types of the disorder mentioned above, a strong random chemical potential or a strong random vector potential may induce a quantum phase transition toward a disorder-dominated phase.

The disorder physics for a single non-interacting Dirac fermion is well-studied for the random chemical potential. Already in the 1980’s, Fradkin predicted the existence of a disorder-driven QPT from a semimetallic to a diffusive metallic (DM) phase with increasing disorder strength. With the renewed interest in the WSM, further theoretical studies have been performed, including the global phase diagram of lattice models, the transport properties, the calculation of critical exponents, and the single-particle Green function. Recently, the role of the random vector potential in a non-interacting WSM has been examined, and a similar disorder-driven QPT from a semimetallic to a DM phase with increasing disorder strength is predicted.

Since our analysis of the disorder effect is based on an effective theory of the interacting WSM, our work complements these previous works in the sense that our results are about the disorder effect on a WSM with intermediate strength of electron-electron interactions. Combined the knowledge on the disorder effects of a non-interacting WSM, we propose a global phase diagram of an interacting WSM in the presence of disorder at zero temperature as shown in Fig. I. At weak disorder strength and weak interacting strength, the WSM phase is stable. By increasing the interacting strength, the WSM becomes unstable and an ordered phase is developed. In our case, this strong-coupling phase exhibits the SDW ordering. On the other hand, for a weakly interacting WSM, a disorder-driven QPT occurs by increasing the disorder strength and the system turns into the DM phase at strong disorder. When both the disorder strength and the interaction strength are strong, there may be two possibilities: an unknown phase lying between the DM and the SDW phase (Fig. I(a)) or a direct transition between the DM and the SDW phase (Fig. I(b)). In any case, there will be a multicritical point (point C) at which the WSM, the DM, and the SDW phase meet with each other. The study on the nature of this multicritical point is beyond the scope of the present work.

The rest of the paper is organized as follows. In Sec. II we describe the model to fix our notation and discuss the structure of the effective theory for the QCP. The one-loop RG analysis of the effective theory for the clean system and the effects of quenched disorder are presented in Sec. III and IV, respectively. The last section is devoted to a conclusive discussion.

II. THE MODEL

We start with a minimal lattice model describing the non-interacting WSM whose Bloch Hamiltonian is given by:

\[
\mathcal{H}(k) = d(k) \cdot \sigma ,
\]
where $\sigma_i$ with $i = 1, 2, 3$ are Pauli matrices in the spin space and

$$
d_3(k) = t_3(2 + \gamma - \cos k_1 - \cos k_2 - \cos k_3), \\
d_1(k) = t_1 \sin k_1 , \quad d_2(k) = t_2 \sin k_2 . \tag{2}
$$

Without loss of generality, we take $t_1, t_2, t_3 > 0$. $\gamma$ is a real number. One may show that when $|\gamma| < 1$, there is a pair of Weyl nodes at $k = \pm K$, where $K = (0, 0, k_0)$ and $k_0 = \cos^{-1} \gamma$. We assume that the system preserves the $P$ symmetry but breaks the $T$ symmetry so that the two Weyl nodes have the same energy.

When the chemical potential coincides with the energy of the Weyl nodes, the system at low energies can be described by the continuum Hamiltonian $H = H_0 + V$, where

$$
H_0 = \sum_{a=1,2,3} \int d^3x \psi^\dagger \alpha_a \psi , 
$$

where $v_1 = t_1$, $v_2 = t_2$, $v_3 = t_3 \sin k_0$, and the Dirac matrices $\alpha_a$ are given by

$$
\alpha_{1,2} = \tau_0 \otimes \sigma_1, \quad \alpha_3 = \tau_3 \otimes \sigma_3 .
$$

Here the Pauli matrices $\tau_a$ with $a = 1, 2, 3$ and the $2 \times 2$ unit matrix $\tau_0$ describe the node degrees of freedom. The Dirac field $\psi = [\chi^+, \chi^-]^t$, where $\chi^+$ and $\chi^-$ describe the Weyl fermions at node $K$ and $-K$, respectively. $V$ consists of the short-ranged four-fermion interactions whose actual forms can be found in Ref. [14].

In terms of the low-energy degrees of freedom, one may expand the electron operator $c(r)$ as

$$
c(r) \sim e^{iK \cdot r} \chi^+_r(r) + \chi^+_r(r) \cdot \chi^-_r(r) + \cdots , \tag{4}
$$

where $\cdots$ represents the operators with scaling dimensions higher than $\chi^+$. Hence, the spin density operator $S = \frac{i}{2} \psi^\dagger \sigma c$ can be written as

$$
S_a(r) \sim \psi^\dagger \tau_0 \otimes \sigma_a \psi + \left(2iK \cdot \chi^+_r \sigma_a \chi^+ + \text{H.c.} \right) + \cdots . \tag{5}
$$

One may identify the operator $\chi^+_r \sigma_a \chi^+$ as the order parameter for the SDW ordering since a nonvanishing value of its expectation value results in $\langle S_a \rangle = 2|N_0| \cos(2K \cdot r + \theta_0)$, where $N_0 = \langle \chi^+_r \sigma_3 \chi_+ \rangle$ and $\theta_0$ is the phase of $N_0$.

Based on the above observation, we propose that near the critical point lying between the WSM and the SDW phase, the system is described by the effective theory whose Lagrangian density in the imaginary-time formulation is of the form: $L = L_\psi + L_\phi + L_{\text{int}}$, where

$$
L_\psi = \bar{\psi} \left(\gamma_0 \partial_\tau + v \gamma_5 \partial_\theta \right) \psi , \tag{6}
$$

$$
L_\phi = \left| \partial_\tau \phi \right|^2 + \frac{v_3}{2} \left( \nabla \phi \right)^2 + r |\phi|^2 + \lambda |\phi|^4 , \tag{7}
$$

$$
L_{\text{int}} = g \left( \phi_1 \bar{\psi} \psi + i \phi_2 \bar{\gamma}_5 \psi \right) , \tag{8}
$$

with $\lambda, v_3 > 0$. The Dirac field $\psi$ and the complex bosonic field $\phi = \frac{\phi_1 + i \phi_2}{\sqrt{2}}$ describe the gapless fermionic quasiparticles and the SDW fluctuations, respectively. To simplify our analysis, we take $v_1 = v_2 = v_3 = v$. In general, $v_k \neq v$ and $L$ does not respect the Lorentz symmetry, as we would expect in condensed matter physics. The representation of the $\gamma$-matrices is chosen to be

$$
\gamma_0 = \tau_1 \otimes \sigma_3 , \quad \gamma_1 = \tau_1 \otimes \sigma_2 , \\
\gamma_2 = -\tau_1 \otimes \sigma_1 , \quad \gamma_3 = -\tau_2 \otimes \sigma_0 . \tag{9}
$$

where $\sigma_0$ is the $2 \times 2$ unit matrix in the spin space. One may verify that they are Hermitian and obey the Clifford algebra

$$
\{ \gamma_\mu, \gamma_\nu \} = 2 \delta_{\mu\nu} . \tag{10}
$$

The matrix $\gamma_5$ is defined as $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \tau_3 \otimes \sigma_0$.

With the above choice of the $\gamma$-matrices, the SDW order parameter in the $z$ direction is given by

$$
\chi^+_r \sigma_3 \chi^+ + \text{H.c.} = \bar{\psi} \psi .
$$

$\langle \bar{\psi} \psi \rangle \neq 0$ also implies dynamical chiral symmetry breaking. Hence, the SDW ordering in the lattice model appears in the guise of chiral symmetry breaking at long distances.

By the chiral transformation

$$
\psi \rightarrow e^{i\theta \gamma_5 / 2} \psi , \quad \bar{\psi} \rightarrow \bar{\psi} e^{-i\theta \gamma_5 / 2} , \tag{11}
$$

where $\theta$ is a real constant, a term involving $\bar{\psi} \gamma_5 \psi$ will be generated. Hence, $L_{\text{int}}$ contains two terms whose structure is fixed by the chiral symmetry. That is, $L_{\text{int}}$ is invariant against the chiral transformation. To see this, we notice that the operators $\bar{\psi} \psi$ and $\bar{\psi} \gamma_5 \psi$ transform as

$$
\left( \begin{array}{c} \bar{\psi} \psi \\ i \bar{\psi} \gamma_5 \psi \end{array} \right) \rightarrow \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} \bar{\psi} \psi \\ i \bar{\psi} \gamma_5 \psi \end{array} \right) , \tag{12}
$$

under the chiral transformation [Eq. [11]]. We further write $L_{\text{int}}$ as

$$
L_{\text{int}} = g(\phi_1, \phi_2) \left( \begin{array}{c} \bar{\psi} \psi \\ i \bar{\psi} \gamma_5 \psi \end{array} \right) ,
$$

Then, the invariance of $L_{\text{int}}$ under the chiral transformation, Eq. [11], requires that $\phi_1$ and $\phi_2$ transform as

$$
\left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) \rightarrow \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right) \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) . \tag{13}
$$

Equation [13] implies that $\phi$ transforms as

$$
\phi \rightarrow e^{-i\theta} \phi . \tag{14}
$$

That is, the $U(1)$ transformation of the $\phi$ field corresponds to the chiral transformation, not the charge $U(1)$ transformation. This chiral symmetry also determines the structure of $L_{\phi}$. We have to emphasize that this chiral symmetry is an emergent one since it is not the symmetry of the microscopic lattice model.
The phase diagram of $\mathcal{L}$ can be seen as follows. For $r > 0$, the $\phi$ field is gapped and we may integrate it out. The resulting theory is a single species of Dirac fermions with short-range spin-dependent four-fermion interactions. Since the four-fermion interactions are irrelevant at weak coupling, this corresponds to the WSM phase. On the other hand, for $r < 0$, $\langle \phi \rangle$ is pinned at some nonzero value and the chiral symmetry is broken. This gives a mass to the Dirac fermion, and the resulting phase exhibits the SDW ordering. Hence, $r = 0$ corresponds to the QCP. A RG analysis is warranted to study its critical properties.

III. ONE-LOOP RG ANALYSIS

In the analysis of quantum phase transitions involving gapless fermions, people usually employ the Hertz-Millis-Moriya theory, where the fermions are integrated out to obtain an effective action of the order parameter, with the assumption that each term in the resulting action can be written as a local functional of the order parameter. Such an approach, however, is shown to be incomplete due to either the breakdown of Fermi-liquid theory, or an infinite number of local marginal operators being generated. Moreover, a recent large-scale quantum Monte-Carlo study reported results not consistent with the Hertz-Millis-Moriya theory. Here we shall treat the fermionic and bosonic fields on equal footing.

Under the scaling transformation $x_\alpha \to s^{-1} x_\alpha$ ($\alpha = 1, 2, 3$) and $\tau \to s^{-\frac{r}{2}} \tau$, the scaling dimensions of various fields and parameters at the tree level are given by $[\psi] = \frac{3}{2}$, $[\phi] = \frac{3}{2}$, $[v] = z - 1 = [v_6]$, $[r] = 2z$, $[\lambda] = 3(z - 1)$, and $[g] = \frac{3}{2}(z - 1)$. If we choose $v$ to be RG invariant, then $z = 1$. Thus, we get $[\psi] = \frac{3}{2}$, $[\phi] = 1$, $[r] = 2$, and $[\lambda] = 0 = [g]$. We see that $r$ is a relevant perturbation around the Gaussian fixed point (characterized by $(r, \lambda, g) = (0, 0, 0)$), whereas both $\lambda$ and $g$ are marginal perturbations at the tree level.

To determine the fate of the boson-fermion coupling $g$, we calculate the RG equations to the one-loop order. To proceed, we assume that there are $N$ species of Dirac fermions, i.e., $\psi \to \psi_\alpha$, with $\alpha = 1, 2, \ldots, N$, and rescale the coupling constant $g$ by $g \to g/\sqrt{N}$. To calculate the RG equations, we separate the fields $\psi_\alpha$ and $\phi_i$ ($i = 1, 2$) into the slow and fast modes: $\psi_\alpha = \phi_\alpha^< + \phi_\alpha^>$ and $\phi_i = \phi_i^< + \phi_i^>$, where the fast modes $\phi_\alpha^>$ and $\phi_i^>$ consist of the Fourier components with $e^{-\Lambda} < |k| < \Lambda$, while the slow modes $\phi_\alpha^<$ and $\phi_i^<$ consist of the Fourier components with $|k| < e^{-\Lambda}$. Here $\Lambda$ is the UV cutoff in momenta and the scaling parameter $l > 0$.

By integrating out the fast modes to the one-loop order and rescaling the the space, time, and fields by $x_\alpha \to e^l x_\alpha$, $\tau \to e^{\tau l}$, $\psi_\alpha \to Z_\alpha^{-1/2} \psi_\alpha$, and $\phi_i \to Z_i^{-1/2} \phi_i$, we obtain the one-loop RG equations:

$$
\frac{d\ln v}{dl} = z - 1 - \frac{8v^2(v-v_6)\alpha}{3Nv_6(v+v_6)^2},
$$

$$
\frac{d\ln v_6}{dl} = z - 1 + \frac{\nu^2 - 1}{v_6^2} \alpha,
$$

$$
\frac{d\beta}{dl} = 3(z-1)\beta - 4\beta\alpha - \frac{5v^2\beta^2}{v_6^2} + \frac{8\alpha^2}{N},
$$

$$
\frac{d\alpha}{dl} = 3(z-1)\alpha - 2 \left[ 1 + \frac{4v^2}{Nv_6(v+v_6)^2} \right] \alpha^2,
$$

$$
\frac{d\tilde{r}}{dl} = 2(z-\alpha)\tilde{r} + \frac{4\beta}{v^2 + v^2\tilde{r}} - 8\alpha,
$$

where the dimensionless couplings are defined as $\alpha = \frac{g^2}{8\pi^2}, \beta = \frac{\lambda}{4\pi^2 v^2}$, and $\tilde{r} = \frac{r}{v^2\Lambda^2}$. The wavefunction renormalization constants $Z_\phi$ and $Z_\psi$ are chosen such that the terms $\psi_\alpha \partial_\tau \psi$ and $|\partial_\tau \phi|^2$ in $\mathcal{L}$ are RG invariant. If we take $v$ to be a RG invariant, then Eq. (15) gives $z = 1 + \frac{8(1-\eta)\alpha}{3N\eta(1+\eta)^2}$, and the other RG equations become

$$
\frac{d\eta}{dl} = \frac{1 - \eta^2}{\eta^2} \left[ 1 + \frac{8\eta}{3N(1+\eta)^3} \right] \alpha,
$$

$$
\frac{d\beta}{dl} = 4 \left[ \frac{2(1-\eta)}{N\eta(1+\eta)^2} - 1 \right] \alpha \beta - \frac{5\beta^2}{\eta^3} + \frac{8\alpha^2}{N},
$$

$$
\frac{d\alpha}{dl} = -2 \left[ 1 + \frac{4}{N(1+\eta)^2} \right] \alpha^2,
$$

$$
\frac{d\tilde{r}}{dl} = 2 \left[ 1 - \left( 1 - \frac{8}{3N\eta(1+\eta)^2} \right) \alpha \right] \tilde{r} + \frac{4\beta}{\eta^2 + \tilde{r}} - 8\alpha.
$$
where \( \eta = v_2/v_1 \).

Equations (20) – (23) have only one fixed point, the Gaussian fixed point, characterized by \((\bar{r}, \alpha, \beta, \eta) = (0, 0, 0, 1)\). The RG flows of \( \alpha, \beta \) and \( \eta \) are shown in Fig. 2. A few comments on the critical properties are in order. (i) First of all, the boson-fermion coupling \( \alpha \) is marginally irrelevant around this fixed point. Hence, the mean-field critical exponents are exact. In particular, the correlation length exponent \( \nu \) is related to the scaling dimension of \( r \), leading to \( \nu = 1/2 \). (ii) We notice that the velocity ratio \( \eta \) will flow to one. This indicates that the Lorentz symmetry is recovered at low energies. Since \( v_1 \) and \( v_2 \) are the velocities of the fermionic quasiparticles and the magnetic excitations in the SDW phase, this fact can be examined by experiments. (iii) Due to the emergent Lorentz symmetry at low energies, the dynamical critical exponent \( z = 1 \) for this QCP.

IV. THE EFFECTS OF QUENCHED DISORDER

A. The random potential

Now we would like to study the effects of quenched disorder on this QCP. In order to describe the disorder effects, the Dirac fields are coupled to a random field \( A(r) \) through the Hamiltonian

\[
H_{\text{dis}} = -v_1 \int d^3 x \bar{\psi} \Gamma \psi A(r) ,
\]

where \( v_1 \) measures the strength of the single-impurity potential. Since the chiral symmetry plays an important role on the critical properties of the clean system, we would like to respect this symmetry. In three dimensions (3D), there are 16 linearly independent choices of \( \Gamma \): \( I, \gamma_0, i\gamma_0 \gamma_5, \) and \( \gamma_5 \), where \( I \) is the 4 \times 4 unit matrix, \( \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu] \), and \( \mu, \nu = 0, 1, 2, 3 \). Among these vertices, only half of them, i.e., \( \Gamma = I, \sigma_{\mu\nu}, \gamma_5 \), preserve the chiral symmetry. We will focus on these types of random potentials. We will see later that these random potentials are divided into four classes: \( \Gamma = I, \gamma_5, \sigma_{ab}, \) and \( \Gamma = i\gamma_0 \gamma \). The random field \( A(r) \) is nonuniform and random in space, but constant in time. Thus, it mixes up the momenta but not the frequencies. We further assume that it is a quenched, Gaussian white-noise field with the correlation functions:

\[
\langle A(r) \rangle = 0 , \quad \langle A(r_1)A(r_2) \rangle = \Delta \delta(r_1 - r_2) ,
\]

for \( \Gamma = I, \gamma_5, \sigma_{ab} \), and

\[
\langle A_a(r) \rangle = 0 , \quad \langle A_a(r_1)A_b(r_2) \rangle = \frac{1}{3} \Delta \delta_{ab} \delta(r_1 - r_2) ,
\]

for \( \Gamma = i\gamma_0 \gamma \), where \( a, b = 1, 2, 3 \) and the variance \( \Delta \) is chosen to be dimensionless. Let us understand the significance of these random potentials in terms of the original lattice fermions.

First of all, the matrix \( \Gamma = i\gamma_0 \gamma \) corresponds to the random vector potential, which describes the randomness in the phases of the hopping amplitudes for the lattice fermions in the presence of a random magnetic field. Its coupling is uniquely fixed by the gauge principle. This term breaks the T symmetry in a fixed sample, but not on the average. This is not a problem since we have assumed the breaking of the T symmetry in the underlying lattice model.

Next, consider the case with \( \Gamma = I \), which corresponds to the random chemical potential. Since this term couples to the two Weyl fermions equally, it describes a smooth non-staggered potential that varies very little over each unit cell of the original lattice. This term preserves the T symmetry, but violates the particle-hole symmetry in a fixed sample.

Let us turn into the case with \( \Gamma = \gamma_5 \). Clearly, this term couples to the two Weyl fermions with opposite signs, which is thus dubbed as the random chiral chemical potential. It arises from a staggered component of a potential, and results in chirality imbalance in a fixed sample, but not on average.

Finally, we consider the case with \( \Gamma = \sigma_{ab} \). We notice that \( \sigma_{ab} = -\epsilon_{abc}(i\gamma_0 \gamma_5) \gamma_5 \), which form the three components of a chiral vector potential. Thus, this term is dubbed as the random chiral vector potential. This term also breaks the T symmetry in a fixed sample, but not on average. However, the three components have different physical origins. In terms of the original lattice fermions, the term \( \Gamma = I = \gamma_0 \otimes \delta \) describes a smooth non-staggered Zeemann coupling that varies very little over each unit cell of the original lattice, which arises from a random magnetic field in the z direction. On the other hand, the rest two components, \( \sigma_3 = \gamma_3 \otimes \gamma_1 \) and \( \sigma_1 = \gamma_3 \otimes \gamma_2 \), involve chirality imbalance and Zeeman coupling simultaneously. The latter arises from a random magnetic field in the xy-plane. Due to this reason, we will study the effects of the three components separately. In fact, it suffices to consider the term \( \Gamma = \sigma_{12} \).

B. The one-loop RG analysis

Following the method employed in Ref. 41–43, the RG equations in the presence of weak disorder can be obtained by integrating out the fast modes for the fields \( \psi, \phi \) and then performing the rescaling for space, time, and fields: \( x_a \rightarrow e^l x_a, \tau \rightarrow e^{3l/2} \tau, \psi \rightarrow Z_\psi^{-1/2} \psi, \phi \rightarrow Z_\phi^{-1/2} \phi, \) and \( A \rightarrow e^{-3l/2} A \) to bring the terms \( \bar{\psi} \gamma_0 \partial_\tau \psi \) and \( |\partial_\tau \phi|^2 \) back to the original forms and to take \( \Delta \) to be a RG invariant. To the one-loop order, we find that

\[
\frac{d \ln v}{dl} = z - 1 - \frac{8\nu^2(v - v_b)\alpha}{3N\lambda_4(v + v_b)^2} - \gamma ,
\]

\[
\frac{d \ln v_b}{dl} = z - 1 + \left( \frac{v^2}{v_b^2} - 1 \right) \alpha ,
\]

where \( \eta = v_b/v_1 \).

Equations (20) – (23) have only one fixed point, the Gaussian fixed point, characterized by \((\bar{r}, \alpha, \beta, \eta) = (0, 0, 0, 1)\). The RG flows of \( \alpha, \beta \) and \( \eta \) are shown in Fig. 2.
where the definitions of $\alpha$, $\beta$, and $\tilde{r}$ are the same as before, $\gamma = \Delta \tilde{v}_1^2$, and

\[
\eta_I = \begin{cases} 
1 & \Gamma = I, \sigma_{12} \\
-1 & \Gamma = i\gamma_0, \gamma_5 \\
\frac{v^3}{v_b(v + v_b)^2} & \Gamma = I \\
-\frac{v^3}{v_b(v + v_b)^2} & \Gamma = \gamma_5 \\
\frac{v^3(v + 2v_b)}{2v_b(v + v_b)^2} & \Gamma = i\gamma_0 \\
-\frac{v^3(v + 2v_b)}{2v_b(v + v_b)^2} & \Gamma = \sigma_{12} \\
\end{cases}
\]

\[
\xi_{1r} = \begin{cases} 
1 & \Gamma = I, \gamma_5 \\
1/9 & \Gamma = i\gamma_0 \gamma \\
-1/3 & \Gamma = \sigma_{12} \\
\end{cases}
\]

If we take $v$ to be a RG invariant, then we have $z = 1 + \frac{8\alpha(1-\eta)}{3N\eta(1+\eta)^2} + \gamma$ and the one-loop RG equations become

\[
\frac{\text{d} \ln \eta}{\text{d} l} = \left[ 1 - \frac{\eta^2}{\eta^2} + \frac{8(1-\eta)}{3N\eta(1+\eta)^2} \right] \alpha + \gamma, \\
\frac{\text{d} \alpha}{\text{d} l} = -2\left[ 1 + \frac{4}{N(1+\eta)^2} \right] \alpha^2 + (1 - 2\eta_I) \alpha \gamma, \\
\frac{\text{d} \gamma}{\text{d} l} = -2\gamma + \frac{8\alpha \gamma}{N} \left[ \xi_{1r} - \frac{1 + 2\eta_I}{3\eta(1+\eta)^2} \right] + 2\xi_{2r} \gamma^2, \\
\frac{\text{d} \beta}{\text{d} l} = 2 \left[ \frac{2(1-\eta)}{N\eta(1+\eta)^2} - 1 \right] \alpha \beta + \frac{8\alpha^2}{N} - \frac{5\beta^2}{\eta^3} + 3 \beta \gamma, \\
\frac{\text{d} \tilde{r}}{\text{d} l} = 2\tilde{r} + 2 \left[ \frac{8(1-\eta)}{3N\eta(1+\eta)^2} - 1 \right] \alpha \tilde{r} + 2\gamma \tilde{r} + \frac{4\beta}{\sqrt{\eta^2 + \tilde{r}^2}} - 8\alpha .
\]

Equations (33) - (37) have a fixed point characterized by $(\tilde{r}, \alpha, \beta, \gamma) = (0, 0, 0, 0)$, the Gaussian fixed point. As we have discussed before, in the absence of disorder, this fixed point is IR stable and the Lorentz symmetry is recovered at low energy. In the presence of weak disorder, the stability of the Gaussian fixed point remains intact since $\gamma$ is an irrelevant coupling around the Gaussian fixed point. Moreover, $\eta$ still flows to 1 at low energy, as shown in Fig. 3. Hence, the Lorentz symmetry emerges at low energy even in the presence of weak disorder. This results in $z = 1$. The RG flows of the boson-fermion coupling $\alpha$ and the disorder strength $\gamma$ for different types of random potentials with $N = 1$ and $\eta = 1$. The solid (red) circles denote the fixed points. (a) $\Gamma = I$, (b) $\Gamma = \gamma_5$, (c) $\Gamma = i\gamma_0 \gamma$, and (d) $\Gamma = \sigma_{12}$.

FIG. 4: The RG flow of the boson-fermion coupling $\alpha$ and the disorder strength $\gamma$ for different types of random potentials with $N = 1$. The solid (red) circles denote the fixed points. (a) $\Gamma = I$, (b) $\Gamma = \gamma_5$, (c) $\Gamma = i\gamma_0 \gamma$, and (d) $\Gamma = \sigma_{12}$. For clarity, we have set $\eta = 1$. We discuss their behaviors in the following.

For $\Gamma = I$, the boson-fermion $\alpha$ is a marginally irrelevant coupling around the Gaussian fixed point and will flow to zero at low energy. On the other hand, the RG equation for the disorder strength $\gamma$ has two fixed points: $\gamma = 0, 1$. The former is IR stable, while the latter is IR unstable. Hence, for weak disorder order, $\gamma$ is an irrelevant coupling, while it becomes relevant when its value is beyond some critical one $\gamma_c$, and we get $\gamma_c = 1$ by extrapolating our one-loop RG equations to strong disorder strength. The RG flow of $\alpha$ and $\gamma$ with $N = 1$ and $\eta = 1$ is shown in Fig. 4(a). Near the non-Gaussian fixed point, $(\alpha, \gamma) = (0, 1)$, the critical line is $\gamma = 1$. For
weak disorder, we conclude that the critical properties of the clean system remain intact. By increasing the disorder strength, a QPT will occur. Since $\alpha = 0$ at the non-Gaussian fixed point, we expect that this disorder-driven transition lies in the same universality class as found in the disordered non-interacting WSM, and the WSM phase becomes the DM phase at strong disorder strength.

For $\Gamma = \gamma_3$, the RG equations have two fixed points: the Gaussian fixed point $(\alpha, \gamma) = (0, 0)$ and the non-Gaussian fixed point $(\alpha, \gamma) = \left(\frac{3N}{2(N-2)}, \frac{N+1}{N-2}\right)$ when $N > 2$, and have only one fixed point – the Gaussian one when $N = 1, 2$. The Gaussian fixed point is IR stable, while the non-Gaussian fixed point, if it exists, is IR unstable. Hence, at weak disorder, the critical properties of the clean system remain intact. The non-Gaussian fixed point describes a disorder-driven transition and the system turns into a disorder-dominated phase at strong disorder strength when $N > 2$. The RG flow of $\alpha$ and $\gamma$ with $N = 1$ and $\eta = 1$ is shown in Fig. [4](b).

For $\Gamma = i\eta_0\gamma$, the RG flow of $\alpha$ and $\gamma$ with $N = 1$ and $\eta = 1$ is shown in Fig. [3](c). The RG equations have two fixed points: the Gaussian fixed point $(\alpha, \gamma) = (0, 0)$ and the non-Gaussian fixed point $(\alpha, \gamma) = \left(\frac{3N}{2(N-1)}, 9\right)$. Similar to the random chemical potential, the Gaussian fixed point is IR stable, while the non-Gaussian fixed point is IR unstable. Hence, at weak disorder, the critical properties of the clean system remain intact. The non-Gaussian fixed point describes a disorder-driven transition. At strong disorder strength, the system turns into a disorder-dominated phase. By extrapolating our one-loop RG equations to the strong disorder strength, the boson-fermion coupling $\alpha$ will flow to the strong-coupling regime. We expect that the quasiparticles will acquire a gap through dynamical chiral symmetry breaking and the system would become an insulator. However, we cannot rule out the possibility that this disorder-driven transition belongs to the same universality class as found in Ref. [24] and the system is in the DM phase at strong disorder strength.

Finally, for $\Gamma = \sigma_{12}$, the only fixed point is the Gaussian one, which is IR stable. Thus, the critical properties of the clean system remain intact at weak disorder. The RG flow of $\alpha$ and $\gamma$ with $N = 1$ and $\eta = 1$ is shown in Fig. [4](d). It is still possible to have a disorder-driven transition at strong disorder strength. However, this situation cannot be captured by the one-loop RG equations.

V. CONCLUSIONS AND DISCUSSIONS

In this work, we investigate the nature of the magnetic phase transition in the interacting WSM by proposing an effective theory describing the critical region. In terms of the perturbative RG method, we find that the critical properties are of the mean-field type and the Lorentz symmetry emerges at low energies so that $\nu = 1/2$ and $z = 1$. We notice that in a recent work, the QPT to symmetry-breaking phases for an interacting WSM has been studied from a RG analysis on the purely fermionic model. Interestingly, the correlation length exponent $\nu = 1/2$ is also obtained, which is based on the $1/n$ expansion where $n = 1$ corresponds to the WSM. We also study the effects of weak disorder on this QPT, and show the stability of this QCP against weak disorder. The situation where the strong-coupling phase exhibits the CDW or excitonic ordering can also be studied in a similar way by introducing the corresponding order parameter and treating the order parameter and the fermionic quasiparticles on equal footing.

The interplay between electron-electron interactions and disorder is not clear for the WSM. Our work sheds some light on the global phase diagram, as shown in Fig. [4]. At weak disorder strength and weak short-range repulsive interactions, the WSM phase is stable. By increasing the disorder strength, a QPT from the WSM phase to a disorder-dominated phase may occur for the random chemical potential, the random vector potential, or the random chiral chemical potential (for $N > 2$) by extrapolating our one-loop RG equations. For other types of disorder respecting the chiral symmetry, the WSM phase may be still stable for strong disorder strength.

On the other hand, by increasing the interaction strength, a QPT from the WSM phase to a symmetry-breaking phase (the SDW phase in our case) occurs. This QPT belongs to a non-Gaussian universality class within the framework consisting only of the Weyl fermions. Nevertheless, according to our analysis, this QPT turns into a Gaussian universality class by introducing the order-parameter fluctuations. Furthermore, the critical properties of this QCP are immune to weak disorder.

By increasing the disorder strength along the phase boundary between the WSM and the SDW phases, we expect the existence of a multicritical point at which the WSM, the DM, and the SDW phases meet each other. The nature of this multicritical point is not clear at this moment. Based on the global topology of the phase diagram, there are two possibilities when both interaction and disorder strengths are strong: (a) The simplest scenario is that there is a direct continuous phase transition from the DM phase to the SDW phase. (b) It is possible that there is an unknown intermediate phase existing between the DM and the SDW phase. The nature of this intermediate phase is unclear. However, it may be insulating due to the dynamical chiral symmetry breaking by extrapolating our one-loop RG equations. If this is indeed the case, then the transition between this phase and the DM phase should be a continuous one. To firmly answer whether the above speculations are true or not, one way is to find an effective field theory which addresses the critical properties of the multicritical point directly. This is beyond the scope of the present paper, and much work remains to be done to clarify these issues.
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1 A.M. Turner and A. Vishwanath, Beyond Band Insulators: Topology of Semi-metals and Interacting Phases, arXiv:1301.0330.
2 P. Hosur and X.L. Qi, C. R. Physiqu, 14, 857 (2013).
3 O. Vafek and A. Vishwanath, Annu. Rev. Condens. Matter Phys. 5, 83 (2014).
4 A.H. Castro Neto, F. Guinea, N.M.R. Peres, K.S. Novoselov, and A.K. Geim, Rev. Mod. Phys. 81, 109, (2009).
5 M.I. Katsnelson, Graphene: Carbon in Two Dimensions, Cambridge University Press, Cambridge, 2012.
6 M.Z. Hasan and C.L. Kane, Rev. Mod. Phys. 82, 3045 (2010).
7 X.L. Qi and S.C. Zhang, Rev. Mod. Phys. 83, 1057 (2011).
8 W. Witeczak-Krempa, Y.-B. Kim, Phys. Rev. B 85, 045124 (2012).
9 G. Chen and M. Hermele, Phys. Rev. B 86, 235129 (2012).
10 T.T. Heikkila, N.B. Kopnin, and G.E. Volovik, JETP Lett. 94, 233 (2011).
11 S.-Y. Xu, I. Belopolski, N. Alidoust, M. Neupane, G. Bian, C.L. Zhang, R. Shankar, G.Q. Chang, Z.J. Yuan, C.-C. Lee, S.-M. Huang, H. Zheng, J. Ma, D.S. Sanchez, B.K. Wang, A. Bansil, F.C. Chou, P.P. Shibayev, H. Lin, S. Jia, and M.Z. Hasan, Science 349, 613 (2015).
12 B.Q. Lv, H.M. Weng, B.B. Fu, X.P. Wang, H. Miao, J. Ma, P. Richard, X.C. Huang, L.X. Zhao, G.F. Chen, Z. Fang, X. Dai, T. Qian, and H. Ding, Phys. Rev. X 5, 031013 (2015).
13 H.B. Nielsen, and M. Ninomiya, Nucl. Phys. B185, 20 (1981); ibid, 193, 173 (1981).
14 J. Maciejko and R. Nandkishore, Phys. Rev. B 90, 055126 (2014).
15 P. Goswami and S. Chakravarty, Phys. Rev. Lett. 107, 196803 (2011).
16 P. Hosur, S.A. Parameswaran, and A. Vishwanath, Phys. Rev. Lett. 108, 046602 (2012).
17 H. Iseobe and N. Nagaosa, Phys. Rev. B 86, 165127 (2012); ibid, 87, 205138 (2013).
18 H. Wei, S.-P. Chao, and V. Aji, Phys. Rev. Lett. 109, 196403 (2012) and Phys. Rev. B 89, 235109 (2014).
19 Z. Wang and S.-C. Zhang, Phys. Rev. B 87, 161107(R) (2013).
20 L.J. Zhai, P.H. Chou and C.-Y. Mou, Phys. Rev. B 94, 125135 (2016).
21 E. Fradkin, Phys. Rev. B 33, 3257 (1986); ibid, 33, 3263 (1986).
22 H. Shapourian and T.L. Hughes, Phys. Rev. B 93, 075108 (2016).
23 B. Roy, R.J. Slager, V. Jurić, arXiv:1610.08973
24 B. Sbierski, G. Pohl, E.J. Bergholtz, and P.W. Brouwer, Phys. Rev. Lett. 113, 026602 (2014).
25 S.V. Syzranov, L. Radzihovsky, and V. Gurarie, Phys. Rev. Lett. 114, 166601 (2015).
26 K. Kobayashi, T. Ohtsuki, K.-I. Imura, and I. F. Herbut, Phys. Rev. Lett. 112, 016402 (2014).
27 B. Sbierski, E.J. Bergholtz, and P.W. Brouwer, Phys. Rev. B 92, 115145 (2015).
28 J.H. Pixley, P. Goswami, and S. Das Sarma, Phys. Rev. Lett. 115, 076601 (2015).
29 J. H. Pixley, P. Goswami, and S. Das Sarma, Phys. Rev. B 93, 085103 (2016).
30 S.V. Syzranov, P.M. Ostrovsky, V. Gurarie, and L. Radzihovsky, Phys. Rev. B 93, 155113 (2016).
31 S. Bera, J.D. Sau, and B. Roy, Phys. Rev. B 93, 201302(R) (2016).
32 T. Louvet, D. Carpentier, and A.A. Fedorenko, Phys. Rev. B 94, 220201 (2016).
33 J.H. Pixley, Y.Z. Chou, P. Goswami, D.A. Huse, R. Nandkishore, L. Radzihovsky, and S. Das Sarma, arXiv:1701.00783.
34 B. Sbierski, K.S.C. Decker, and P.W. Brouwer, Phys. Rev. B 94, 220202(R) (2016).
35 K.Y. Yang, Y.M. Lu, and Y. Ran, Phys. Rev. B 84, 075129 (2011).
36 P. Delplace, J. Li, and D. Carpentier, Euro. Phys. Lett. 97, 67004 (2012).
37 J.A. Hertz, Phys. Rev. B 14, 1165 (1976); T. Moriya, Spin Fluctuations in Itinerant Electron Magnetism, Springer, Berlin (1985); A.J. Millis, Phys. Rev. B 48, 7183 (1993).
38 H. Lohneysen, A. Rosch, M. Vojta, and P. Wolfe, Rev. Mod. Phys. 79, 1015 (2007).
39 A. Abanov, A.V. Chubukov, and J. Schmalian, Adv. Phys. 52, 119 (2003); M.A. Metlitski and S. Sachdev, Phys. Rev. B 82, 075128 (2010).
40 Y. Schattner, S. Lederer, A. Kivelson, and E. Berg, Phys. Rev. X 6, 031028 (2016).
41 T. Stauber, F. Guinea, and M.A.H. Vozmediano, Phys. Rev. B 71, 041406(R) (2005).
42 Y. Huh and S. Sachdev, Phys. Rev. B 78, 064512 (2008).
43 J. Wang, Phys. Rev. B 87, 054511 (2013).
44 B. Roy, P. Goswami, and V. Jurić, arXiv: 1610.05762.