GENERALIZATION OF A THEOREM OF ZIPPIN

P. HAJEK, TH. SCHLUMPRECHT, AND A. ZSÁK

Abstract. We generalize and prove a result which was first shown by Zippin [11], and was explicitly formulated by Benyamini in [2].

1. Introduction

In 1977, Zippin [11] proved a result, which was reformulated by Benyamini [2, page 27] as follows. For a Banach space $X$ and $\varepsilon > 0$ we denote by $\text{Sz}(X, \varepsilon)$ the $\varepsilon$-Szlenk index of $X$ whose definition will be recalled at the end of Section 3.

Theorem 1.1. Let $X$ be a Banach space with separable dual, and $\varepsilon > 0$. Let $F$ be a $w^*$-closed totally disconnected, $(1 - \varepsilon)$-norming subset of $B_{X^*}$, the unit ball of $X^*$.

Then there is a countable ordinal $\alpha < \omega^{\text{Sz}(X, \varepsilon/8)+1}$, and a subspace $Y$ of $C(F)$, which is isometrically isomorphic to $C[0, \alpha]$, such that for every $x \in X$ there is a $y \in Y$ with $\|i_F(x) - y\| \leq \varepsilon(1 - \varepsilon)^{-1}\|i_F(x)\|$, where $i_F : X \rightarrow C(F)$ denotes the embedding defined by $i_F(x)(f) = f(x)$ for $x \in X$ and $f \in F$.

The goal of our paper is to prove a generalization of this theorem which includes non-separable Banach spaces, and at the same time we provide a more conceptual proof of Zippin’s result (see Theorem 5.1 and Corollary 5.2 in Section 5).

The paper is organized as follows. In Section 1 we recall the definition of trees, and introduce the tree topology defined on the set $[T]$ of all branches of a tree $T$. This topology is generated by a basis consisting of clopen sets and, if $T$ has finitely many roots, $[T]$ with this topology is compact. Fragmentation indices of topological spaces with respect to a pseudo-metric will be recalled in Section 3. As a particular example we define the Szlenk index of a Banach space at the end of Section 3. Section 4 represents the heart of the proof of our main result (see Theorem 4.2) which might be interesting in its own right. It is shown that if $T$ is a tree and $d$ is a pseudo-metric on $[T]$ so that $[T]$ is fragmentable with respect to $d$, then for every $\varepsilon > 0$ there is a subset $\mathcal{N}$ of the basis of the topology of $[T]$ which forms a well-founded tree with respect to containment so that $\mathcal{N}$ (in this case equal to $[\mathcal{N}]$), with its tree topology is a quotient of $[T]$, and so that for all $M \in \mathcal{N}$ the $d$-diameter of $M \setminus \bigcup_{N \subseteq M, N \in \mathcal{N}} N$ is smaller than $\varepsilon$. In Section 5 we present and prove the generalization of Zippin’s theorem.

2. Trees and the tree topology

Let $T$ be a tree, which means that $T$ is a set with a reflexive partial order denoted by $\preceq$ (where we introduce the notation $x \prec y \iff x \preceq y$ and $x \neq y$), with the property that for each $t \in T$, the set of predecessors of $t$,

$$b_t = \{s \in T : s \preceq t\}$$
is finite and linearly ordered. An initial node of $T$ is a minimal element of $T$, i.e., an element $t \in T$ for which $b_t = \{t\}$. If $t \in T$ is not an initial node, then $b_t \setminus \{t\}$ is not empty and we call the maximum of $b_t \setminus \{t\}$ the direct predecessor of $t$. A successor of $u \in T$ is an element $w \in T$ so that $u \prec w$ and it is called a direct successor of $u$ if, moreover, there is no $v \in T$ with $u \prec v \prec w$, in other words if $u$ is the direct predecessor of $w$. The set of all direct successors of $u$ is denoted by $S_u$.

A branch of $T$ is a non-empty, linearly ordered subset of $T$ which is closed under taking direct predecessors.

Remark 2.1. We note that for a branch $b$ it follows that if $t \in b$ and $s \prec t$ then also $s \in b$, and

- either $b$ is finite, in which case there is a unique $t$ so that $b = b_t = \{s \in T : s \preceq t\}$ (take $t$ to be the maximal element of $b$),
- or $b$ is infinite, in which case $b$ is a maximal linearly ordered set in $T$.

Indeed, if $b$ is infinite, then we can find $t_j \in b$, $j \in \mathbb{N}$, with $t_1 \prec t_2 \prec t_3 \prec \cdots$ (choose $t_1 \in b$ arbitrarily, then $t_2 \in b \setminus b_{t_1}$, then $t_3 \in b \setminus b_{t_2}$, etc.). We claim that $b = \bigcup_{j=1}^{\infty} b_{t_j}$. Indeed, since $b$ is closed under taking predecessors, we have $\bigcup_{j=1}^{\infty} b_{t_j} \subset b$, and if there were an element $t \in b \setminus \bigcup_{j=1}^{\infty} b_{t_j}$, then $t_j \prec t$ for all $j \in \mathbb{N}$, and thus $b_t$ would be infinite, which is a contradiction. Finally we note that $\bigcup_{j=1}^{\infty} b_{t_j}$ is a maximal linearly ordered set, because if for some $t \in T \setminus \bigcup_{j=1}^{\infty} b_{t_j}$ the set $\bigcup_{j=1}^{\infty} b_{t_j} \cup \{t\}$ were also linearly ordered, it would follow as before that $t \not\prec t_j$ for all $j \in \mathbb{N}$, and would contradict the assumption that $b_t$ is finite. Conversely, if $(t_j)$ is any strictly increasing sequence in $T$ then $b = \bigcup_{j=1}^{\infty} b_{t_j}$ is an infinite branch.

We will identify a finite branch $b$ of $T$ with the element $t \in T$ such that $b = b_t$, and hence identify the tree $T$ with the set of all its finite branches. We call the set of all infinite branches the boundary of $T$ and denote it by $\partial T$. If $s \in T$ and $b \in \partial T$, we also write $s \prec b$ if $s \in b$. The set of all branches of $T$ is denoted by $[T]$. Thus, we have $[T] = T \cup \partial T$. The tree $T$ is called well-founded if $\partial T = \emptyset$. We define the ordinal index $o(T)$ of a well-founded tree $T$ as follows. For every subset $S$ of $T$ we put $S' = \{s \in S : s$ is not maximal in $S\}$. Note that since $T$ is well-founded, if $S$ is not empty then $S' \subset S$. Then we put $T^{(0)} = T$, and by transfinite induction for any ordinal $\alpha$ we define

$$T^{(\alpha)} = (T^{(\gamma)})' \text{ if } \alpha = \gamma + 1 \text{ for some } \gamma, \text{ and } T^{(\alpha)} = \bigcap_{\gamma < \alpha} T^{(\gamma)} \text{ if } \alpha \text{ is a limit ordinal}.$$ 

Since $T$ is assumed to be well-founded, it follows that

$$o(T) = \min \{\alpha \in \text{Ord} : T^{(\alpha)} = \emptyset\}$$

exists. Note that since the sets $T^{(\alpha)} \setminus T^{(\alpha+1)}$, $\alpha < o(T)$, are non-empty and pairwise disjoint, it follows that if $T$ is countable, then so is $o(T)$.

Let $T$ be an arbitrary tree. We define a locally compact topology on $[T]$, the tree topology on $[T]$, as follows. For $t \in T$ we put

$$U_t = \{b \in [T] : t \in b\}$$

and let the tree topology be generated by the set

$$\{U_t, [T] \setminus U_t : t \in T\}.$$ 

We call $[T]$ with its tree topology the tree space of $T$. 
Note that for $s,t \in T$, either $U_s \subset U_t$, or $U_t \subset U_s$, or $U_t \cap U_s = \emptyset$. Indeed, using the properties of trees and the definition of branches, it follows that if $s \prec t$, then $U_t \subset U_s$, if $t \sim s$, then $U_s \subset U_t$, and if $s$ and $t$ are incomparable then $U_t \cap U_s = \emptyset$. It follows that

$$B = \left\{ U_t \setminus \bigcup_{j=1}^{n} U_{s_j} : n \in \mathbb{N}_0, \ s_1, s_2, \ldots, s_n \in S_t \right\} \cup \{\emptyset\}$$

is stable under taking finite intersections, and thus is a basis of the tree topology consisting of clopen sets. We note that for $b \in \partial T$,

$$U_b = \{ U_t : t \in b \}$$

is a neighbourhood basis of $b$, and for a finite branch $b = b_t$,

$$U_b = \left\{ U_t \setminus \bigcup_{s \in F} U_s : F \subset S_t \text{ finite} \right\}$$

is a neighbourhood basis of $b$. In particular, if $t$ has only finitely many direct successors, then the singleton $\{b_t\}$ is clopen.

**Proposition 2.2.** $[T]$ is a locally compact Hausdorff space, and $[T]$ is compact if and only if $T$ has finitely many initial nodes.

**Proof.** If $b$ and $b'$ are two different branches, and, say, without loss of generality $t \in b \setminus b'$ then $U_t$ is a neighbourhood of $b$ and $[T] \setminus U_t$ a neighbourhood of $b'$. Thus $[T]$ is Hausdorff.

In order to show that $[T]$ is locally compact, and also the claimed equivalence, it is enough to show that $U_t$ is compact for each $t \in T$. Thus, let $\mathcal{U}$ be an open cover of $U_t$, and assume $\mathcal{U}$ has no finite subcover of $U_t$. Then there is a $t_1 \succ t$ so that $\mathcal{U}$ has no finite subcover of $U_{t_1}$. Indeed, since $t \in U_t$, one of the elements of $\mathcal{U}$ must contain a subset of the form $U_t \setminus \bigcup_{j=1}^{n} U_{s_j}$ with $n \in \mathbb{N}$ and each $s_j \in S_t$, which means that for one of the $s_i$ it follows that $\mathcal{U}$ has no finite subcover of $U_{s_i}$. Inductively, we can find an increasing sequence $t \prec t_1 \prec t_2 \prec \ldots$ so that $\mathcal{U}$ has no finite subcover of $U_{t_i}$, $i = 1, 2, \ldots$. Let $b = \bigcup_{i=1}^{\infty} b_{t_i}$ be the branch generated by these $t_i$. Since $b \in U_t$, there must be a $U \in \mathcal{U}$ for which $b \in U$. This implies that there must be an $s \in b$ so that $b \in U_s \subset U$. For large enough $n \in \mathbb{N}$ we have $t_n \succ s$, and thus $U_{t_j} \subset U$ for all $j \geq n$, which is a contradiction. \hfill \Box

We shall call a tree $T$ **compact** if the corresponding tree space $[T]$ is compact, i.e., when $T$ has finitely many initial nodes.

**Remark 2.3.** Branches of $T$ are certain subsets of $T$. Thus we can think of the tree space $[T]$ as a subset of $\{0,1\}^T$. The tree topology of $[T]$ is simply the restriction to $[T]$ of the product topology of $\{0,1\}^T$.

We next recall the definition of the Cantor–Bendixson index of a compact topological space $K$. For a closed set $F \subset K$ we put

$$d(F) = \{ \xi \in F : \xi \text{ is not an isolated point in } F \}.$$  

We put $d_0(K) = K$. By transfinite induction we define $d_\alpha(K)$ for ordinals $\alpha$ as follows: $d_\alpha(K) = d(d_{\gamma+1}(K))$ if $\alpha = \gamma + 1$ for some $\gamma$, and $d_\alpha(K) = \bigcap_{\gamma < \alpha} d_\gamma(K)$ if $\alpha$ is a limit ordinal.

It follows that there must be an ordinal $\alpha_0$ for which $d_{\alpha_0}(K) = d_{\alpha_0+1}(K) = d_{\alpha_0+2}(K) = \ldots$, and if in that case $d_{\alpha_0}(K) = \emptyset$, we define the **Cantor–Bendixson index of $K$** to be

$$\text{CB}(K) = \min \{ \alpha \in \text{Ord} : d_\alpha(K) = \emptyset \},$$

otherwise we put $\text{CB}(K) = \infty$. 
Now we assume that $T$ is a well-founded tree with finitely many initial nodes, and we want to compare $\omega(T)$ with $\text{CB}(T)$. Since every maximal element in any subset $S$ of $T$ is isolated in $S$, it follows that

$$\text{CB}(T) \leq \omega(T).$$

It follows from results in general topology that if $K$ is a non-empty, countable, compact space, then $\text{CB}(K) = \beta + 1$ for a countable ordinal $\beta$, and $|d_\beta(K)| = n$ for some $0 < n < \omega$, and moreover $K$ is homeomorphic to the ordinal interval $[0, \omega^\beta \cdot n]$. We next give a direct proof of this fact in the special case when $K$ is a well-founded, countable, compact tree with its tree topology. We shall also prove the converse that every countable successor ordinal (and thus every countable, compact topological space) is homeomorphic to a well-founded, countable, compact tree.

**Theorem 2.4.** Let $(T, \preceq)$ be a countable, well-founded tree. Then there is an ordinal $\beta \leq \omega^{\omega(T)}$ such that $T$ with its tree topology is homeomorphic to the ordinal interval $[0, \beta]$.

Conversely, given a countable ordinal $\beta$, the interval $[0, \beta]$ is homeomorphic to a countable, well-founded tree.

**Proof.** For $t \in T$ let $\alpha(t)$ be the ordinal $\alpha < \omega(T)$ such that $t \in [T]^{(\alpha)} \setminus [T]^{(\alpha+1)}$. We first show that for each $t \in T$ there is an ordinal $\beta \leq \omega^{\alpha(t)}$ and a homeomorphism $\varphi: U_t \to [0, \beta]$ with $\varphi(t) = \beta$. We shall proceed by induction on $\alpha(t)$.

If $\alpha(t) = 0$, then $U_t = \{t\}$, so the claim follows. Now assume that $\alpha = \alpha(t) > 0$. Then $S_t \neq \emptyset$ and by definition of $\alpha(t)$, if $s \in S_t$, then $s \notin [T]^{(\alpha)}$, and thus $\alpha(s) < \alpha$. Enumerate $S_t$ as a (finite or infinite) sequence $s_1, s_2, \ldots$. By induction hypothesis, each clopen set $U_{s_n}$ is homeomorphic to $[0, \beta_n]$ for some ordinal $\beta_n \leq \omega^{\alpha(s_n)}$. Set $\gamma_0 = 0$ and $\gamma_n = \gamma_{n-1} + \beta_n + 1$ for $n \geq 1$. Let $\beta = \sup_{n \geq 1}(\gamma_n + 1)$. Since $\beta_n < \omega^\alpha$ for all $n$, it follows that $\beta \leq \omega^\alpha$. For each $n \geq 1$, the interval $[\gamma_{n-1}, \gamma_n)$ is order-isomorphic to $[0, \beta_n)$. Since $[0, \beta)$ is the disjoint union of the clopen intervals $[\gamma_{n-1}, \gamma_n)$, it follows that $\bigcup_n U_{s_n}$ is homeomorphic to $[0, \beta]$. Let $\varphi$ be such a homeomorphism. We extend $\varphi$ to $U_t$ by setting $\varphi(t) = \beta$. Under $\varphi$ the basic neighbourhood $U_t \setminus \bigcup_{1 \leq k \leq n} U_{s_k}$ of $t$ corresponds to the basic neighbourhood $[\gamma_n, \beta]$ of $\beta$. Hence $\varphi: U_t \to [0, \beta]$ is a homeomorphism with $\varphi(t) = \beta$. This completes the proof of the induction step.

We now finish the proof of the first half of the theorem as follows. We join a root to $T$ by adding a new element $r$ to $T$ and by declaring $r < t$ for all $t \in T$. Put $T^* = T \cup \{r\}$. An easy induction shows that $(T^{(\alpha)})^* = (T^*)^{(\alpha)}$ for all $\alpha \leq \omega(T)$, and hence in $T^*$ we have $\alpha(r) = \omega(T)$. By our initial claim, there is an ordinal $\beta \leq \omega^{\omega(T)}$ and a homeomorphism $\varphi: T^* \to [0, \beta]$ such that $\varphi(r) = \beta$. Thus, $T$ is homeomorphic to the interval $[0, \beta]$, as required.

For the converse statement, it is enough to prove that for all $\beta < \omega$ there is a countable, well-founded tree $(T, \preceq)$ with one initial node $r$ and a homeomorphism $\varphi: T \to [0, \beta]$ with $\varphi(r) = \beta$. Indeed, it then follows that the interval $[0, \beta]$ is homeomorphic to $T \setminus \{r\}$ with the subspace topology which is easily seen to be the same as its tree topology. We proceed by induction on $\beta$.

For $\beta = 0$ we take $T$ to be the tree with one element. If $T$ is a suitable tree for some $\beta$, then $T^*$, i.e., $T$ with a new root adjoined, works for $\beta + 1$. Let us now assume that $\beta$ is a countable limit ordinal. Then there is an increasing sequence $(\gamma_n)_{1 \leq n < \omega}$ of successor ordinals with $\beta = \sup \gamma_n$. Set $\gamma_0 = 0$ and for each $1 \leq n < \omega$ choose $\beta_n$ with $\gamma_n = \gamma_{n-1} + \beta_n + 1$. Since $\beta_n < \beta$, by induction hypothesis, there is a well-founded, countable tree $(T_n, \preceq_n)$ with one initial node $s_n$ homeomorphic to the interval $[\gamma_{n-1}, \gamma_n)$. Let $(T, \preceq)$ be the disjoint
union of the $T_n$ together with a new element $r$ such that for all $s, t \in T$ we have $s \preceq t$ if and only if either $s, t \in T_n$ and $s \preceq_n t$ for some $1 \leq n < \omega$, or $s = r$. Then $T$ is a well-founded, countable tree with one initial node $r$, and moreover $S_r = \{s_n : 1 \leq n < \omega\}$ and $U_{s_n} = T_n$ for each $n$. As we have seen in the proof of the first half of the theorem, in this situation there is a homeomorphism $\varphi : T \to [0, \beta]$ with $\varphi(r) = \beta$. □

Remarks 2.5. Since for $\beta > 0$ the interval $[0, \beta]$ is compact if and only if $\beta$ is a successor ordinal, it follows from Theorem 2.4 that a non-empty, countable, well-founded, compact tree $T$ is homeomorphic to $[0, \beta]$ for some ordinal $\beta < \omega^\om(T)$.

Another consequence of the above theorem is that a countable, well-founded tree $(T, \preceq)$ has a well-ordering, and the corresponding order topology is the same as the tree topology. This ordering can be described explicitly as follows. Adjoin a root $r$ to $T$, and set $T^* = T \cup \{r\}$. This allows us to refer to the set of $\preceq$-minimal elements of $T$ as $S_r$. For each $t \in T^*$ fix a well-ordering $\prec$ of $S_t$ with order type at most $\omega$. We then extend $\prec$ to a linear ordering of $T$ as follows. For $s, t \in T$, we let $s \prec t$ if and only if either $t \prec s$, or there exist $u \in T^*$ and $v, w \in S_u$ such that $v \preceq s$, $w \preceq t$ and $v < w$.

We next give an example of an uncountable compact space that can also be realized as a tree space. This example will be important later.

Example 2.6. Let $D$ be the Cantor set, i.e., the set $D = \{0, 1\}^N$ endowed with the product topology of the discrete topology on $\{0, 1\}$. Denote by $[N]$, $[N]^<\omega$ and $[N]^\omega$ the subsets of $\mathbb{N}$, the finite subsets of $\mathbb{N}$, and the infinite subsets of $\mathbb{N}$, respectively. Each element of $D$ can be identified in a canonical way with an element of $[N]$, and vice versa, by letting $A_{\sigma} = \{n : \sigma_n = 1\}$ for $\sigma = (\sigma_n) \in D$. Via this identification $[N]$ becomes a compact, metrizable space.

Now we will show that this topology on $[N]$ is identical with the tree topology on the branches of a tree $T$. Indeed, let $T = [N]^<\omega$, on which we consider the tree structure given by extension. For $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_l\}$, both sets written in increasing order, we say that $B$ is an extension of $A$, or that $A$ is an initial segment of $B$, and write $A \prec B$, if $l > m$ and $a_i = b_i$ for $i = 1, 2, \ldots, m$. This turns $T$ into a tree whose only initial node is $\emptyset$ and it is easy to see that $[T] = [N]$ and $\partial T = [N]^\omega$. Here we identify any $A = \{a_1, a_2, \ldots\} \in [N]^\omega$ (written in increasing order) with the branch $b = \{a_1, a_2, \ldots, a_n\} : n = 0, 1, 2, \ldots\}$.

If $A \in [N]^\omega$, then $S_A = \{A \cup \{n\} : n > \max(A)\}$, and $U_A = \{B \in [N] : B \nsubseteq A\}$. For $n \geq \max(A)$ we put

$$U_{A,n} = \{C \supseteq A : C \setminus A \subseteq [n + 1, \infty)\}$$

(thus $U_{A,\max(A)} = U_A$). Then $B = \{U_{A,n} : A \in [N]^\omega, n \geq \max(A)\}$ is a basis of the product topology consisting of clopen sets.

By definition, $U_A$ is also clopen in the tree topology. If $\mathcal{F} \subseteq S_A$ is finite, then

$$U_A \setminus \bigcup_{B \in \mathcal{F}} U_B = \{D \nsubseteq A : \forall B \in \mathcal{F} \max(B) \neq \min(D \setminus A)\}.$$

Thus, if we put $N = \max\{\max(B) : B \in \mathcal{F}\}$, it follows that

$$U_{A,N} = U_A \setminus \bigcup_{n = \max(A) + 1}^N U_{A \cup \{n\}} \subseteq U_A \setminus \bigcup_{B \in \mathcal{F}} U_B \subseteq U_A.$$

This implies that the product topology on $[N]$ and the tree topology coincide.
We conclude this section with a well-known folklore result in topology. For the convenience of the reader we include the proof.

**Lemma 2.7.** Let $K$ be a compact Hausdorff space, $\varepsilon > 0$, and $f_1: K \to \mathbb{R}$ a function such that every point of $K$ has a neighbourhood on which the oscillation of $f_1$ is at most $\varepsilon$. Then there is a continuous function $f: K \to \mathbb{R}$ such that $|f(x) - f_1(x)| \leq \varepsilon$ for all $x \in K$.

**Proof.** By assumption, the family of open subsets of $K$ has a neighbourhood on which the oscillation of $f_1$ is at most $\varepsilon$. Let $\varphi_1, \ldots, \varphi_n$ be a partition of unity subordinate to the cover $U_1, \ldots, U_n$. Thus, $\varphi_j: K \to [0, 1]$ is a continuous function whose support is contained in $U_j$ for each $j = 1, \ldots, n$ such that $\sum_{j=1}^n \varphi_j(x) = 1$ for all $x \in K$. For each $j = 1, \ldots, n$ fix $x_j \in U_j$, and define $f: K \to \mathbb{R}$ by setting $f(x) = \sum_{j=1}^n f_1(x_j) \varphi_j(x)$, $x \in K$. Then $f$ is continuous and

$$|f(x) - f_1(x)| \leq \sum_{j=1}^n \varphi_j(x) |f_1(x_j) - f_1(x)| \leq \varepsilon$$

for all $x \in K$, as required. \(\square\)

### 3. Fragmentation indices

In this section we recall some well known notation and results on fragmentation indices. All of the results below, and much more, may be found in books on topology and descriptive set theory (for example [H]). Nevertheless, for better reading, we would like to recall the results we will need here. We also do this because we consider fragmentations of topological spaces with respect to pseudo-metrics, and not only metrics.

**Definition 3.1.** Let $(X, \mathcal{T})$ be a topological space and $d(\cdot, \cdot)$ a pseudo-metric on $X$. We say that $(X, \mathcal{T})$ is $d$-fragmentable if for all non-empty closed subsets $F$ of $X$ and all $\varepsilon > 0$ there is an open set $U \subset X$ so that $U \cap F \neq \emptyset$ and $d$-diam($U \cap F$) < $\varepsilon$.

The following statement is a well known corollary of the Baire Category Theorem.

**Theorem 3.2.** Let $(X, \mathcal{T})$ be a Polish space (i.e., separable and completely metrizable), and let $d$ be a pseudo-metric on $T$ so that all closed $d$-balls, $B_r(x) = \{y \in X: d(x, y) \leq r\}$ with $r > 0$ and $x \in X$, are closed in $X$ with respect to $\mathcal{T}$.

Then $(X, \mathcal{T})$ is $d$-fragmentable if and only if $(X, d)$ is separable.

**Proof.** "$\Rightarrow$" Let $F \subset X$ be $\mathcal{T}$-closed and $\varepsilon > 0$. Choose $D \subset F$ dense in $(F, d)$ and countable, and then note that $F$ can be written as the countable union of $\mathcal{T}$-closed sets in the following way:

$$F = \bigcup_{a \in D} F \cap B_{\varepsilon}(a) .$$

By the Baire Category Theorem there must therefore be an $a \in D$ so that $F \cap B_{\varepsilon}(a)$ has a non-empty interior with respect to the subspace topology defined by $\mathcal{T}$ on $F$.

"$\Leftarrow$" Assume that $(X, d)$ is not separable. We need to find $\varepsilon > 0$ and a $\mathcal{T}$-closed set $F$ in $X$ which has the property that $d$-diam($U \cap F$) $\geq \varepsilon$ for all $\mathcal{T}$-open $U \subset X$ with $U \cap F \neq \emptyset$.

Since $(X, d)$ is not separable, we find an uncountable $A \subset X$ and an $\varepsilon > 0$ so that $d(x, z) > \varepsilon$ for all $x \neq z$ in $A$. Set

$$B = \{x \in A : \exists U \in \mathcal{T} \text{ such that } x \in U \text{ and } U \cap A \text{ is countable}\} .$$

Then $\mathcal{U} = \{U \in \mathcal{T} : U \cap A \text{ countable}\}$ is an open cover for $B$, and hence it has a countable subcover $\mathcal{V}$. (Here we are using the fact that a Polish space is second countable, and hence
so are all its subsets.) It follows that $B = \bigcup_{V \in V}(B \cap V)$ is countable. Consider the $\mathcal{T}$-closed subset $F = A \setminus B^\mathcal{T}$ of $X$. Let $U$ be a $\mathcal{T}$-open subset of $X$ with $U \cap F \neq \emptyset$. Then $U \cap (A \setminus B) \neq \emptyset$, and hence $U \cap (A \setminus B)$ is uncountable by the definition and countability of $B$. It follows that $U \cap (A \setminus B)$ has at least two elements which, by definition of $A$, implies that $d$-diam$(U \cap F) \geq \varepsilon$.

\begin{proof}
We put $F$, $\varepsilon$ and from \cite[Theorem 6.9]{7} that Frag($F$) is a closed set, implies that Frag($F$) is uncountable. Consider “$\infty$” to be outside of the class of ordinals. Secondly, we define the $\varepsilon$-derivative of $F$ of order $\alpha$, denoted $F^{(\alpha)}_\varepsilon$, by transfinite induction:

$$F^{(0)}_\varepsilon = F, \quad F^{(\alpha)}_\varepsilon = (F^{(\alpha)}_\varepsilon)^{\gamma}_\varepsilon \text{ if } \alpha = \gamma + 1, \text{ and } F^{(\alpha)}_\varepsilon = \bigcap_{\gamma < \alpha} F^{(\gamma)}_\varepsilon \text{ if } \alpha \text{ is a limit ordinal.}$$

Let $(X, \mathcal{T})$ be a topological space, $d(\cdot, \cdot)$ a pseudo-metric on $X$, $F$ be a $\mathcal{T}$-closed subset of $X$, and $\varepsilon > 0$. First we note that if $F^{(\alpha)}_\varepsilon = F^{(\alpha+1)}_\varepsilon$ then $F^{(\alpha)}_\varepsilon = F^{(\beta)}_\varepsilon$ for all $\beta > \alpha$. It follows that if $F^{(\alpha)}_\varepsilon \neq F^{(\alpha+1)}_\varepsilon$, then $\beta \mapsto F^{(\beta)}_\varepsilon \setminus F^{(\beta+1)}_\varepsilon$ defines an injection on $\alpha$ into the power set of $F$, which is not possible for $\alpha$ sufficiently large. Therefore there must be a minimal ordinal $\alpha_0$ for which

$$F^{(\alpha_0)}_\varepsilon = F^{(\alpha_0+1)}_\varepsilon = F^{(\alpha_0+2)}_\varepsilon = \ldots .$$

We put $F^{(\infty)}_\varepsilon = F^{(\alpha_0)}_\varepsilon$. If $(X, \mathcal{T})$ is $d$-fragmentable, then $F^{(\infty)}_\varepsilon = \emptyset$.

We define the $\varepsilon$-fragmentation index of $F$ with respect to $d$ by

$$\text{Frag}(d, F, \varepsilon) = \text{Frag}(F, \varepsilon) = \begin{cases} \min\{\beta \in \text{Ord} : F^{(\beta)}_\varepsilon = \emptyset\} & \text{if } F^{(\infty)}_\varepsilon = \emptyset, \\ \infty & \text{if not.} \end{cases}$$

Here we consider “$\infty$” to be outside of the class of ordinals. Secondly, we define the fragmentation index of $F$ with respect to $d$ by

$$\text{Frag}(d, F) = \text{Frag}(F) = \sup_{\varepsilon > 0} \text{Frag}(F, \varepsilon)$$

with $\text{Frag}(F) = \infty$ if for some $\varepsilon > 0$ we have $\text{Frag}(F, \varepsilon) = \infty$.

\begin{remark}
Assume that $(X, \mathcal{T})$ is $d$-fragmentable and that $F \subset X$ is compact. Let $\varepsilon > 0$. Then it follows for a limit ordinal $\alpha$ for which $F^{(\gamma)}_\varepsilon \neq \emptyset$ whenever $\gamma < \alpha$ that also $F^{(\alpha)}_\varepsilon \neq \emptyset$. Therefore Frag$(d, F, \varepsilon)$ will always be a successor ordinal.

If $(X, \mathcal{T})$ is a Polish space (and thus second countable) that is $d$-fragmentable, then for a closed $F \subset X$ and $\varepsilon > 0$ it follows from the fact that $F^{(\beta)}_\varepsilon \subsetneq F^{(\alpha)}_\varepsilon$ for $\alpha < \beta \leq \text{Frag}(F, \varepsilon)$ and from \cite[Theorem 6.9]{7} that Frag$(F, \varepsilon) < \omega_1$, where $\omega_1$ denotes the first uncountable ordinal, and thus also that Frag$(F) < \omega_1$.

As an important example we consider the Szlenk index of a Banach space, which we will introduce for general Banach spaces, not only for separable ones. We call a Banach space $X$ an Asplund space if every separable subspace of $X$ has separable dual. This is not the original definition of Asplund \cite{1}, but proven to be equivalent to it. The following equivalence is stated in \cite{4} and gathers the results from \cite{1, 8, 2}.
Theorem 3.5. [6, Theorem 11.8, p. 486] Let \((X, \|\cdot\|)\) be a Banach space. Then the following assertions are equivalent:

(i) \(X\) is an Asplund space.

(ii) \(B_{X^*}\) with the \(w^*\)-topology is \(\|\cdot\|\)-fragmentable. Here \(\|\cdot\| = \|\cdot\|_{X^*}\) denotes the dual norm on \(X^*\), and \(\|\cdot\|\)-fragmentable means \(d\)-fragmentable, where \(d\) is the induced metric defined by \(d(x^*, y^*) = \|x^* - y^*\|\) for \(x^*, y^* \in X^*\).

Assume that \(X\) is an Asplund Banach space. For a \(w^*\)-closed subset \(F\) of \(B_{X^*}\) and \(\varepsilon > 0\) we denote the \(\varepsilon\)-fragmentation index of \(F\) with respect to \(\|\cdot\|_{X^*}\) by \(Sz(F, \varepsilon)\) and call it the \(\varepsilon\)-Szlenk index of \(F\). The Szlenk index of \(F\) is then \(Sz(F) = \sup_{\varepsilon > 0} Sz(F, \varepsilon)\). The \(\varepsilon\)-Szlenk index of \(X\) is then defined to be \(Sz(B_{X^*}, \varepsilon)\) and denoted by \(Sz(X, \varepsilon)\), and the Szlenk index of \(X\) is \(Sz(X) = \sup_{\varepsilon > 0} Sz(X, \varepsilon) = Sz(B_{X^*})\). Note that by Theorem 3.5 above, \(X\) is an Asplund space if and only if all these indices are ordinal numbers.

Remark 3.6. Let \(K\) be a compact topological space. By identifying the elements of \(K\) with their Dirac measure, we can think of \(K\) as a compact subset of \(B_{C(K)^*}\) which 1-norms the elements of \(C(K)\). It is then easy to see that \(CB(K) = Sz(K, \varepsilon) = Sz(K)\) for all \(0 < \varepsilon < 2\). It follows therefore that \(CB(K) = Sz(K) \leq Sz(C(K))\). In general it is not true that \(Sz(K) = Sz(C(K))\). Nevertheless, in [10] Theorem C for the case of separable dual, and in [3] Theorem 1.1 for the general case, it was shown that if \(X\) is a Banach space and \(B \subset B_{X^*}\) is compact and 1-norming for \(X\), then

\[
Sz(X) = \min \{ \omega^\alpha : \omega^\alpha \geq Sz(B) \}
\]

if \(X\) is an Asplund space, and \(Sz(X) = Sz(B) = \infty\) otherwise.

4. Fragmentation of \([T]\)

Throughout this section we fix a tree \(T\) and a pseudo-metric \(d(\cdot, \cdot)\) on its tree space \([T]\) which, we recall, is the set of all branches of \(T\) equipped with the tree topology. We assume that \(T\) is compact, i.e., that it has finitely many initial nodes or, equivalently, that its tree space \([T]\) is compact. We also assume that \([T]\) is \(d\)-fragmentable. This situation arises in the following important example which we will later use.

Example 4.1. Consider the space \(C([T])\) of continuous functions on \([T]\) for our compact tree \(T\). Let \(X\) be a closed subspace of \(C([T])\) and assume that \(X\) is an Asplund space. For \(b_1, b_2 \in [T]\) set

\[
d(b_1, b_2) = \sup_{x \in B_X} |x(b_1) - x(b_2)|.
\]

Then \(d(\cdot, \cdot)\) is a pseudo-metric on \([T]\) and the map sending \(b \in [T]\) to the Dirac measure at \(b\) restricted to \(X\) is an isometry of \(([T], d)\) into \((B_{X^*}, \|\cdot\|)\). It follows from Theorem 3.5 above that \([T]\) is \(d\)-fragmentable.

We now fix an \(\varepsilon > 0\), and let \(\eta\) be the ordinal so that \(\text{Frag}(d, [T], \varepsilon) = \eta + 1\). We abbreviate \([T]^{(\alpha)} = [T]^{(\alpha)}_{\varepsilon}\) for \(\alpha \in \text{Ord}\). Let \(B\) be the family of basic open subsets of \([T]\), i.e., the sets of the form \(N = U_t \setminus \bigcup_{s \in F} U_s\), where \(t \in T\) and \(F\) is a finite (possibly empty) subset of \(S_t\). Note that \(t\) and \(F\) are uniquely determined by \(N\). Indeed, \(t\) is the least element of \(N\), and then \(F\) is the complement in \(S_t\) of the set of minimal elements of \(N \setminus \{t\}\). We say \(N\) is of type I if \(F = \emptyset\), otherwise we say \(N\) is of type II. Note that \(B\) is partially ordered by containment: \(M \preceq N\) if and only if \(M \supseteq N\). However, in general, \(B\) is not a tree. The following theorem is the main result of this section.
Theorem 4.2. Let $T, d, \varepsilon, \eta$ and $B$ be as above. Then there exists a subset $N$ of $B$ which is a well-founded tree under containment such that

$$d \text{-diam} \left( M \setminus \bigcup_{N \in S_M} N \right) < \varepsilon$$

for each $M \in \mathcal{N}$ (where, as before, $S_M$ denotes the set of direct successors of $M$ in the tree $(\mathcal{N}, \supset)$), and the ordinal index $\alpha(\mathcal{N})$ of $\mathcal{N}$ satisfies $\alpha(\mathcal{N}) \leq \lambda + 2n + 2$, where $\eta = \lambda + n$, $\lambda$ is a limit ordinal and $n < \omega$. Moreover, $\bigcup\mathcal{N} = [T]$, and $\mathcal{N}$ has finitely many initial nodes.

Proof. For $b \in [T]$ let $\alpha(b)$ be the ordinal $\alpha \leq \eta$ such that $b \in [T]^{(\alpha)} \setminus [T]^{(\alpha + 1)}$. Then $b$ has a neighbourhood whose intersection with $[T]^{(\alpha)}$ has $d$-diameter less than $\varepsilon$. If there exists a type I neighbourhood of $b$ with that property (which is the case if $b \in \partial T$), then there exists a least $s \in b$ such that $d \text{-diam}(U_s \cap [T]^{(\alpha)}) < \varepsilon$, and in this case we set $N_b = U_s$. Otherwise $b$ is necessarily a finite branch $b_t$ for some $t \in T$ and $d \text{-diam}(U_t \cap [T]^{(\alpha)}) \geq \varepsilon$. In this case there is a minimal (with respect to inclusion), finite, non-empty subset $F$ of $S_t$ such that the $d$-diam $\left( \bigcup_{s \in F} U_s \cap [T]^{(\alpha)} \right) < \varepsilon$. Note that $F$ is not necessarily unique: we simply choose one such minimal $F$ and set $N_b = U_t \setminus \bigcup_{s \in F} U_s$. We do this for every $b \in [T]$ and set $\mathcal{N} = \{ N_b : b \in [T] \}$. For $N \in \mathcal{N}$ we let $\alpha(N) = \alpha(b)$ where $b \in [T]$ is such that $N = N_b$.

Note that this definition does not depend on the choice of $b$. Indeed, we have

$$\alpha(N) = \max \left\{ \beta \leq \eta : N \cap [T]^{(\beta)} \neq \emptyset \right\} = \min \left\{ \beta : d \text{-diam} \left( N \cap [T]^{(\beta)} \right) < \varepsilon \right\}.$$ 

We now prove two simple facts. Recall that we identify $t \in T$ with the finite branch $b_t$. So we will sometimes write $N_t$ instead of $N_{b_t}$.

Lemma 4.3. Let $M_1, M_2 \in \mathcal{N}$. Then either $M_1 \subset M_2$ or $M_1 \supset M_2$ or $M_1 \cap M_2 = \emptyset$.

Proof. Let us first note that if $N \in \mathcal{N}$ is of type II, and thus of the form $N = U_t \setminus \bigcup_{s \in F} U_s$ for a unique $t \in T$ and finite, non-empty $F \subset S_t$, then for $b \in [T]$ we have $N = N_b$ if and only if $b = b_t$. It follows that if $N_1$ and $N_2$ in $\mathcal{N}$ are both of type II and of the form $N_1 = U_t \setminus \bigcup_{s \in F_1} U_s$ and $N_2 = U_t \setminus \bigcup_{s \in F_2} U_s$, then $N_1 = N_{b_t} = N_2$.

For each $i = 1, 2$, choose $t_i \in T$ and finite $F_i \subset S_{t_i}$, such that $M_i = U_{t_i} \setminus \bigcup_{s \in F_i} U_s$ (where the $F_i$ could be empty, and thus $M_i$ be of type I). If $t_1$ and $t_2$ are incomparable, then $M_1 \cap M_2 \subset U_{t_1} \cap U_{t_2} = \emptyset$. If $t_1 = t_2$ and one of $F_1$ and $F_2$ is empty, then $M_1 \subset M_2$ or $M_1 \supset M_2$. If $t_1 = t_2$ and both $F_1$ and $F_2$ are non-empty, then $M_1$ and $M_2$ are type II neighbourhoods, and hence, by the remark at the beginning of the proof, $b = b_{t_1} = b_{t_2}$ is the unique branch such that $M_1 = M_2 = N_{b_t}$.

Finally, assume that $t_1$ and $t_2$ are comparable and distinct. We may without loss of generality assume that $t_1 < t_2$. Let $s$ be the unique direct successor of $t_1$ with $s \not\leq t_2$. Then either $s \in F_1$, and thus $M_1 \cap M_2 = \emptyset$, or $s \not\in F_1$, and then $M_1 \supset M_2$. \hfill \Box

Before the next lemma, we observe the following consequence of $\mathbf{3}$. If $M, N \in \mathcal{N}$ and $M \supset N$, then $\alpha(M) \geq \alpha(N)$.

Lemma 4.4. Let $M, N \in \mathcal{N}$. Assume that $M \supset N$ and $\alpha(M) = \alpha(N)$. Then $M$ is of type II and $N$ is of type I.

Proof. Set $\alpha = \alpha(M) = \alpha(N)$, and choose branches $b, c \in [T]$ such that $M = N_b$ and $N = N_c$. Assume for a contradiction that $M$ is of type I. Then $M = U_t$ for some $t \in b$, and so the $d$-diameter of $U_t \cap [T]^{(\alpha)}$ is less than $\varepsilon$. Since $M \supset N$, we have $t \in c$, and thus by the definition of $N_c$, we have $N_c = U_s$ with $s = \min \{ r \in c : d \text{-diam}(U_r \cap [T]^{(\alpha)}) < \varepsilon \}$. 

But this implies that $s = t$ and we must have $M = U_t = N$, which is a contradiction. Thus $b$ is a finite branch $b_0$, say, and $M = U_t \setminus \bigcup_{s \in F} U_s$ for some non-empty, finite set $F \subset S_t$. Since $M \supseteq N$, there must be an $s \in S_t \setminus F$ such that $c \in U_s$. Since $U_s \subset M$, it follows that $d$-diam$(U_s \cap [T]^{(c)}) < \varepsilon$. Hence $N = U_s$, and so $N$ is of type 1.

We shall make use the following immediate consequence of Lemma 4.4. Given $M, N, P \in \mathcal{N}$, if $M \supseteq N \supseteq P$, then $\alpha(M) > \alpha(P)$.

Continuation of the proof of Theorem 4.2. It follows from Lemma 4.3 that for $M \in \mathcal{N}$ the set $b_M = \{ N \in \mathcal{N} : N \supseteq M \}$ is linearly ordered. Write $M$ as $M = U_t \setminus \bigcup_{s \in F} U_s$ with $t \in T$ and $F \subset S_t$ finite. To see that $b_M$ is finite, observe that if $N_1 \supseteq N_2 \supseteq N_3 \supseteq \ldots$ in $\mathcal{N}$, then $\alpha(N_1) \supseteq \alpha(N_2) \supseteq \ldots$, and hence this sequence of ordinals is eventually constant. Lemma 4.3 shows that this is not possible.

We will now prove the stated upper bound on $\alpha(\mathcal{N})$. Fix a limit ordinal $\alpha$ and assume that

$$\mathcal{N}^{(\alpha)} \subset \{ N \in \mathcal{N} : \alpha(N) \geq \alpha \} .$$

We show by induction that $\mathcal{N}^{(\alpha+2m)} \subset \{ N \in \mathcal{N} : \alpha(N) \geq \alpha + m \}$ for all $m < \omega$. The case $m = 0$ is our base assumption. Now let $M \in \mathcal{N}^{(\alpha+2m+1)}$. Then $M \supseteq N \supseteq P$ for some $N \in \mathcal{N}^{(\alpha+2m+1)}$ and $P \in \mathcal{N}^{(\alpha+2m)}$. By induction hypothesis we have $\alpha(P) \geq \alpha + m$, and hence, by Lemma 4.3 we have $\alpha(M) \geq \alpha + m + 1$. It remains to show that (4) does in fact hold for all limit ordinals $\alpha$. This can be done by an easy induction argument. As we go from $\alpha$ to $\alpha + \omega$ in the induction step, we use the previous fact about finite ordinals. If $\alpha = \sup I$, where $I$ is the set of limit ordinals strictly smaller than $\alpha$, then

$$\mathcal{N}^{(\alpha)} = \bigcap_{\gamma \in I} \mathcal{N}^{(\gamma)} \subset \bigcap_{\gamma \in I} \{ N \in \mathcal{N} : \alpha(N) \geq \gamma \} = \{ N \in \mathcal{N} : \alpha(N) \geq \alpha \} .$$

We next establish the statement concerning $d$-diameters. Fix $M \in \mathcal{N}$ and let $\alpha = \alpha(M)$. If $b \in M \setminus [T]^{(c)}$, then $\alpha(b) < \alpha$. Thus $\alpha(N_b) < \alpha(M)$, and since $N_b \cap M$ contains $b$, we must have $N_b \not\subseteq M$ using Lemma 4.3 and 3. It follows that $b \in N$ for some $N \in S_M$. We have proved that $M \setminus \bigcup_{N \in S_M} N \subset M \cap [T]^{(c)}$, which shows that $d$-diam$(M \setminus \bigcup_{N \in S_M} N) < \varepsilon$.

For the moreover part observe that $b \in N_b$ for all $b \in [T]$, and thus $[T] = \bigcup \mathcal{N}$. Since $[T]$ is assumed to be compact, there is a finite cover of $[T]$ by some elements $N_1, N_2, \ldots, N_k$ of $\mathcal{N}$. By Lemma 4.3 it follows that there can be at most $k$ initial nodes of $\mathcal{N}$.

Lemma 4.5. Let $\mathcal{N}$ be defined as in the proof of Theorem 4.2. For $M \in \mathcal{N}$ set $\tilde{M} = M \setminus \bigcup_{N \in S_M} N$. Then the sets $\tilde{M}, M \in \mathcal{N}$, are pairwise disjoint, and for each $b \in [T]$ there is an $M \in \mathcal{N}$ so that $b \in \tilde{M}$. Thus $\{ \tilde{M} : M \in \mathcal{N} \}$ is a partition of $[T]$.

Proof. Let $M, N \in \mathcal{N}$ with $\tilde{M} \neq \tilde{N}$. If $M \cap N = \emptyset$, then it is clear that $\tilde{M} \cap \tilde{N} = \emptyset$. Thus, by Lemma 4.3 we may assume that $N \subseteq M$. This means, since $\mathcal{N}$ is a tree, that there is an $M' \in S_M$ with $N \subseteq M'$, which yields our first claim.

Since $[T] = \bigcup \mathcal{N}$, and since $\mathcal{N}$ is a well-founded tree, there is a smallest (with respect to inclusion, or maximal in the order of $\mathcal{N}$) $M \in \mathcal{N}$ so that $b \in M$. This means that $b \not\in N$ for any $N \in S_M$, which implies our second claim.
By Lemma 4.4, we can define a map \( q: [T] \to \mathcal{N} \) by letting \( q(b) \) be the unique \( M \in \mathcal{N} \) such that \( b \in M \).

**Proposition 4.6.** The map \( q: [T] \to \mathcal{N} \) defined above is onto. The quotient topology on \( \mathcal{N} \) induced by \( q \) coincides with the tree topology of \( \mathcal{N} \).

**Proof.** Let \( M \in \mathcal{N} \). We need to show that \( \tilde{M} \neq \emptyset \). Choose \( b \in [T] \) with \( M = N_b \), and set \( \alpha = \alpha(b) = \alpha(M) \). Let \( N \in S_M \). Then \( \alpha(N) \leq \alpha \) by (3). If \( \alpha(N) < \alpha \), then \( N \) is disjoint from \( [T]^{(\alpha)} \), and hence \( b \notin N \). If \( \alpha(N) = \alpha \), then \( M \) is of type II and \( N \) is of type I by Lemma 4.4. It follows that \( b = b_t \) for some \( t \in T \), and \( N \supseteq U_s \) for some \( s \in T_{lt} \). But this means that \( t \notin U_s \), and thus again, we have \( b = b_t \notin N \). This shows that \( b \in \tilde{M} \), and so \( M = q(b) \) is in the image of \( q \).

We next observe that \( q \) is continuous when \( \mathcal{N} \) is given the tree topology. Indeed, let us fix \( M \in \mathcal{N} \) and \( b \in [T] \), and set \( N = q(b) \). Then \( b \in M \) if and only if \( M \supseteq N \). Thus the inverse image under \( q \) of the basic clopen set \( U_M \) (in the tree topology of \( (\mathcal{N}, \supseteq) \), i.e., \( U_M = \{ N \in \mathcal{N} : N \supseteq M \} \) in \( \mathcal{N} \) is the clopen subset \( M \) of \( [T] \). It follows that the quotient topology of \( \mathcal{N} \) is finer than the tree topology. Since the quotient topology is compact and the tree topology is Hausdorff, it follows that these two topologies coincide, as claimed. \( \square \)

5. **Zippin’s theorem**

We now present our main result.

**Theorem 5.1.** Let \( X \) be an Asplund space, \( (T, \preceq) \) be a tree with finitely many initial nodes, and \( i: X \to C([T]) \) be an isometric embedding. Then for all \( \varepsilon > 0 \) there exist a well-founded, compact tree \( S \) with ordinal index \( o(S) < Sz(X, \varepsilon/2) + \omega \) and an isometric copy \( Y \) of \( C(S) \) in \( C([T]) \) such that for all \( x \in X \) there exists \( y \in Y \) with \( \| i(x) - y \| \leq \varepsilon \| x \| \).

**Proof.** Consider the pseudo-metric \( d \) on \( [T] \) defined as follows.

\[
d(b, c) = \sup_{x \in B_X} |i(x)(b) - i(x)(c)|, \quad b, c \in [T].
\]

We identify \( b \in [T] \) with its Dirac measure \( \delta_b \). Note that the dual map \( i^* \) sends \( [T] \) onto a \( w^* \)-closed, 1-norming subset of \( B_{X^*} \), and

\[
\| i^*(\delta_b) - i^*(\delta_c) \|_{X^*} = \sup_{x \in B_X} |i(x)(b) - i(x)(c)| = d(b, c) \quad \text{for all } b, c \in [T].
\]

Fix \( \varepsilon > 0 \). It follows from above that the \( \frac{\varepsilon}{2} \)-fragmentation index of \( [T] \) with respect to \( d \) is equal to \( Sz(i^*([T]), \frac{\varepsilon}{2}) \leq Sz(X, \frac{\varepsilon}{2}) \). (See Example 4.1)

Theorem 1.2 applied to \( \frac{\varepsilon}{2} \), provides us with a well-founded, compact tree \( \mathcal{N} \) of basic clopen subsets of \( [T] \) with \( o(\mathcal{N}) < \text{Frag}(d, [T], \frac{\varepsilon}{2}) + \omega = Sz(i^*([T]), \frac{\varepsilon}{2}) + \omega \leq Sz(X, \frac{\varepsilon}{2}) + \omega \)

such that

\[
d\text{-diam}(M \setminus \bigcup_{N \in S_M} N) < \frac{\varepsilon}{2}
\]

for all \( M \in \mathcal{N} \). By Proposition 4.6, we also have a quotient map \( q: [T] \to \mathcal{N} \), which is continuous with respect to the tree topologies of \( [T] \) and \( \mathcal{N} \). Thus, we have an isometric embedding \( q^*: C(\mathcal{N}) \to C([T]) \) given by \( f \mapsto f \circ q \). Let \( Y \) be the image of \( q^* \). We will now show that \( Y \) is \( \varepsilon \)-close to \( i(X) \) which will prove the theorem with \( S = \mathcal{N} \).

Let \( x \in B_X \) and \( g = i(x) \). Then \( g \) is a continuous function on \( [T] \) whose oscillation on \( \tilde{M} = M \setminus \bigcup_{N \in S_M} N \) is less than \( \frac{\varepsilon}{2} \) for all \( M \in \mathcal{N} \). Indeed, we have

\[
|g(b) - g(c)| = |i(x)(b) - i(x)(c)| \leq d(b, c) < \frac{\varepsilon}{2} \quad \text{for all } b, c \in \tilde{M} \text{ and } M \in \mathcal{N}.
\]
For each $M \in \mathcal{N}$ fix a point $x_M \in \tilde{M}$ and set $f_1(M) = g(x_M)$. This defines a function $f_1: \mathcal{N} \to \mathbb{R}$. We will now show that $f_1$ is not far from being continuous, and hence it is not far from a continuous function.

Fix $M \in \mathcal{N}$. Since the oscillation of $g$ on $\tilde{M}$ is smaller than $\frac{\varepsilon}{2}$, it follows that
\[
\left\{(b,c) \in M \times M : |g(b) - g(c)| \geq \frac{\varepsilon}{2}\right\} \subset \bigcup_{N \in S_M} (M \times N \cup N \times M),
\]
and thus by compactness of the left-hand set, there is a finite set $F_M \subset S_M$ so that
\[
\left\{(b,c) \in M \times M : |g(b) - g(c)| \geq \frac{\varepsilon}{2}\right\} \subset \bigcup_{N \in F_M} (M \times N \cup N \times M),
\]
and hence
\[
|g(b) - g(c)| < \frac{\varepsilon}{2} \quad \text{for all } b,c \in M \setminus \bigcup_{N \in F_M} N.
\]

Since for $P \in \mathcal{N}$ if $P \in U_M \setminus \bigcup_{N \in F_M} U_N$, then $x_P \in \tilde{P} \subset M \setminus \bigcup_{N \in F_M} N$, it follows from (5) above that for all $P,Q \in U_M \setminus \bigcup_{N \in F_M} U_N$, we have
\[
|f_1(P) - f_1(Q)| = |g(x_P) - g(x_Q)| < \frac{\varepsilon}{2}.
\]

We have shown that in the compact space $\mathcal{N}$ every point has a neighborhood on which the oscillation of the function $f_1$ is at most $\frac{\varepsilon}{2}$. An application of Lemma 2.7 now yields a continuous function $f: \mathcal{N} \to \mathbb{R}$ which is $\frac{\varepsilon}{2}$-close to $f_1$. We complete the proof by showing that $y = ^*a(f)$ is $\varepsilon$-close to $g = i(x)$. Indeed, given $b \in [T]$, for $M = g(b)$ we have $b, x_M \in \tilde{M}$, and hence
\[
|y(b) - g(b)| = |f(M) - g(b)| \leq |f(M) - f_1(M)| + |g(x_M) - g(b)| \leq \varepsilon,
\]
as required.

From Theorem 5.1 the following (slight) sharpening of Zippin’s Theorem follows. Recall that $D$ denotes the Cantor set.

**Corollary 5.2.** Let $X$ be a Banach space with separable dual and $i: X \to C(D)$ an isometric embedding (which always exists). Then for all $\varepsilon > 0$ there is a countable ordinal $\alpha < \omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega}}}}}}}}$ and a subspace $Y$ of $C(D)$ isometric to $C[0,\alpha]$ such that for all $x \in X$ there exists $y \in Y$ with $\|i(x) - y\| \leq \varepsilon\|x\|$.

**Proof.** Since $X$ is separable, we can think of it as a subspace of $C(D)$. As explained in Example 2.4, $D$ can be seen as the set of all branches of a tree $T$ endowed with the tree topology. Applying now Theorem 5.1 we obtain a well-founded, compact tree $S$ with ordinal index $\omega(S) < \omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega}}}}}}}$ and an isometric copy $Y$ of $C(S)$ in $C(D)$ such that for all $x \in X$ there exists $y \in Y$ with $\|i(x) - y\| \leq \varepsilon\|x\|$. It follows from the proof of Theorem 5.1 that $S$ is a subset of the basis $B$ of $D$ consisting of clopen sets, so in particular $S$ is countable. By Theorem 2.4 and by the subsequent remark, there is a countable ordinal $\alpha$ such that $S$ is homeomorphic to $[0,\alpha]$, and moreover
\[
\alpha < \omega^{\omega(S)} < \omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega^{\omega}}}}}}} + \omega,
\]
as claimed.

We note the following corollary of Theorem 5.1.
Corollary 5.3. Let $T$ be a tree with finitely many initial nodes. If $T$ is well-founded, then $C(T)$ is an Asplund space. Conversely, if $C([T])$ is an Asplund space, then there is a well-founded, compact tree $S$ homeomorphic to $[T]$.

Proof. Assume that $T$ is well-founded. It follows that the ordinal index $o(T)$ of $T$ exists. As explained in Remark 3.6, we can identify elements $t \in T$ with their Dirac measure $\delta_t$, and hence view $T$ as a $w^*$-closed, 1-norming subset of $B_{C(T)^*}$. Since $\|\delta_s - \delta_t\| = 2$ for $s \neq t$ in $T$, it follows from $\Pi$ for $0 < \varepsilon < 2$ that $\text{Sz}(T) = \text{Sz}(T, \varepsilon) = \text{CB}(T) \leq o(T)$. Thus, in particular, $\text{Sz}(T) \neq \infty$, and hence it follows from $\Pi$ that $C(T)$ is Asplund.

Now assume that $C([T])$ is Asplund. The we apply Theorem 5.1 to $X = C([T]) \subset C([T])$ and $\varepsilon = \frac{1}{2}$ and find a well-founded, compact tree $S$ and a closed subspace $Y$ of $C([T])$ as in the statement. Since now every element of $C([T])$ has to be close to an element of $Y$, it follows that $Y = C([T])$. Since $Y$ is isometric to $C(S)$, it follows from the Banach–Stone Theorem that $[T]$ is homeomorphic to $S$. □

References

[1] E. Asplund, Fréchet differentiability of convex functions, Acta Math. 121 (1968), 31–47.
[2] Y. Benyamini, An extension theorem for separable Banach spaces, Israel J. of Math., 29 (1978), 24–30.
[3] R. Causey, The Szlenk index of injective tensor products and convex hulls, J. Funct. Anal. 272 (2017), no. 8, 3375 – 3409.
[4] P. Dodos, Banach spaces and descriptive set theory: selected topics. Lecture Notes in Mathematics. Springer-Verlag, Berlin (2010).
[5] R. Engelking, General topology. Second edition. Sigma Series in Pure Mathematics, 6. Heldermann Verlag, Berlin, 1989.
[6] M. Fabian, P. Habala, P. Hájek, V. Montesinos, and V. Zizler, Banach Space Theory: The Basis for Linear and Nonlinear Analysis, Canadian Math. Soc. Books in Mathematics, Springer-Verlag, New York (2011).
[7] A. Kechris, Classical descriptive set theory. Graduate Texts in Mathematics, 156. Springer-Verlag, New York (1995).
[8] I. Namioka and R. R. Phelps, Banach spaces which are Asplund spaces, Duke Math. J. 42 (1968), 735 – 750.
[9] R. R. Phelps, Convex functions, monotone operators and differentiability, Lecture Notes in Mathematics 1364, Springer (1989).
[10] Th. Schlumprecht, On Zippin’s embedding theorem of Banach spaces into Banach spaces with bases, Adv. Math. 274 (2015), 833 – 880.
[11] M. Zippin, The separable extension problem, Israel J. Math. 26 (1977), 372–387.

Institute of Mathematics, Academy of Science of the Czech Republic, Žitná 25 115 67 Prague 1, Czech Republic, and Faculty of Electrical Engineering, Czech Technical University in Prague, Zikova 4, 166 27, Prague
E-mail address: hajek@mail.cas.cz

Department of Mathematics, Texas A&M University, College Station, TX 77843, USA and Faculty of Electrical Engineering, Czech Technical University in Prague, Zikova 4, 166 27, Prague
E-mail address: schlump@math.tamu.edu

Peterhouse, Cambridge, CB2 1RD, UK
E-mail address: a.zsak@dpmms.cam.ac.uk