ON DISTRIBUTIONAL POINT VALUES AND BOUNDARY VALUES OF ANALYTIC FUNCTIONS

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Abstract. We give the following version of Fatou’s theorem for distributions that are boundary values of analytic functions. We prove that if \( f \in D'(a, b) \) is the distributional limit of the analytic function \( F \) defined in a region of the form \( (a, b) \times (0, R) \), if the one sided distributional limit exists, \( f(x_0 + 0) = \gamma \), and if \( f \) is distributionally bounded at \( x = x_0 \), then the Lojasiewicz point value exists, \( f(x_0) = \gamma \) distributionally, and in particular \( F(z) \to \gamma \) as \( z \to x_0 \) in a non-tangential fashion.

1. Introduction

The study of boundary values of analytic functions is an important subject in mathematics. In particular, it plays a vital role in the understanding of generalized functions \([1, 2, 4]\). As well known, the behavior of an analytic function at the boundary points is intimately connected with the pointwise properties of the boundary generalized function \([7, 9, 18, 19, 20]\) and the study of this interplay has often an Abelian-Tauberian character. There is a vast literature on Abelian and Tauberian theorems for distributions (see the monographs \([8, 12, 14, 20]\) and references therein).

In this article we present sufficient conditions for the existence of Lojasiewicz point values \([11]\) for distributions that are boundary values of analytic functions. The pointwise notions for distributions used in this paper are explained in Section \( 2 \). The following result by one of the authors is well known \([7]\):

\[\text{Suppose that } f \in D'(\mathbb{R}) \text{ is the boundary value of a function } F, \text{ analytic in the upper half-plane, that is, } f(x) = F(x + i0); \text{ if the distributional lateral limits } f(x_0 \pm 0) = \gamma_\pm \text{ both exist, then } \gamma_+ = \gamma_- = \gamma, \text{ and the distributional limit } f(x_0) \text{ exists and equals } \gamma.\]

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On the other hand, the results of [5] imply that there are distributions \( f(x) = F(x + i0) \) for which one distributional lateral limit exits but not the other. In Theorem 2 we show that the existence of one the distributional lateral limits may be removed from the previous statement if an additional Tauberian-type condition is assumed, namely, if the distribution is distributionally bounded at the point. We also show that when the distribution \( f \) is a bounded function near the point, then the distributional point value is of order 1. Furthermore, we give a general result of this kind for analytic functions that have distributional limits on a contour.

As an immediate consequence of our results, we shall obtain the following version of Fatou’s theorem [13, 15] for distributions that are boundary values of analytic functions.

**Corollary 1.** Let \( F \) be analytic in a rectangular region of the form \((a, b) \times (0, R)\). Suppose that \( f(x) = \lim_{y \to 0^+} F(x + iy) \) in \( \mathcal{D}'(a, b) \), that \( f \) is a bounded function near \( x_0 \in (a, b) \), and that the following average lateral limit exists

\[
\lim_{x \to x_0^+} \frac{1}{x - x_0} \int_{x_0}^{x} f(t) \, dt = \gamma.
\]

Then,

\[
\lim_{z \to x_0} F(z) = \gamma \quad (\text{angularly}).
\]

Finally, we remark that Theorem 4 below generalizes some of our Tauberian results from [18].

### 2. Preliminaries

We explain in this section several pointwise notions for distributions. There are several equivalent ways to introduce them. We start with the useful approach from [3]. Define the operator \( \mu_a \) on locally integrable complex valued functions in \( \mathbb{R} \) as

\[
\mu_a \{ f(t); x \} = \frac{1}{x - a} \int_{x_0}^{x} f(t) \, dt, \quad x \neq a,
\]

while the operator \( \partial_a \) is the inverse of \( \mu_a \),

\[
\partial_a (g) = ((x - a) g(x))'.
\]

Suppose first that \( f_0 = f \) is real. Then if it is bounded near \( x = a \), we can define

\[
\overline{f}_0(a) = \limsup_{x \to a} f(x), \quad f_0(a) = \liminf_{x \to a} f(x).
\]
Then \( f_1 = \mu_a (f) \) will be likewise bounded near \( x = a \) and actually
\[
\underline{f_0} (a) \leq f_1 (a) \leq \overline{f_1} (a) \leq \overline{f_0} (a)
\]
and, in particular, if \( f (a) = f_0 (a) \) exists, then \( f_1 (a) \) also exists and \( f_1 (a) = f_0 (a) \).

**Definition 1.** A distribution \( f \in \mathcal{D}'(\mathbb{R}) \) is called distributionally bounded at \( x = a \) if there exist \( n \in \mathbb{N} \) and \( f_n \in \mathcal{D}'(\mathbb{R}) \), continuous and bounded in a pointed neighborhood \((a - \varepsilon, a) \cup (a, a + \varepsilon)\) of \( a \), such that \( f = \partial_a^n f_n \).

If \( f_0 \) is distributionally bounded at \( x = a \), then there exists a unique distributionally bounded distribution near \( x = a \), \( f_1 \), with \( f_0 = \partial_a f_1 \). Therefore, \( \partial_a \) and \( \mu_a \) are isomorphisms of the space of distributionally bounded distributions near \( x = a \). Given \( f_0 \) we can form a sequence of distributionally bounded distributions \( \{f_n\}_{n=-\infty}^{\infty} \) with \( f_n = \partial_a f_{n+1} \) for each \( n \in \mathbb{Z} \).

We say that \( f \) has the distributional point value \( \gamma \) in the sense of Lojasiewicz [11, 10] and write
\[
f (a) = \gamma \quad (L),
\]
if there exists \( n \in \mathbb{N} \), the order of the point value, such that \( f_n \) is continuous near \( x = a \) and \( f_n (a) = \gamma \).

It can be shown [3, 8, 11, 14] that \( f (a) = \gamma \) (L) if and only if
\[
\lim_{\varepsilon \to 0} f (a + \varepsilon x) = \gamma,
\]
distributionally, that is, if and only if
\[
(2.1) \quad \lim_{\varepsilon \to 0^+} \langle f (a + \varepsilon x), \phi (x) \rangle = \gamma \int_{-\infty}^{\infty} \phi (x) \, dx,
\]
for each \( \phi \in \mathcal{D}(\mathbb{R}) \). On the other hand, if \( f \) is distributionally bounded at \( x = a \) then \( \langle f (a + \varepsilon x), \phi (x) \rangle \) is bounded as \( \varepsilon \to 0 \).

We can also consider distributional lateral limits [11, 17]. We say that the distributional lateral limit \( f (a + 0) \) (L) as \( x \to a \) from the right exists and equals \( \gamma \), and write
\[
f (a + 0) = \gamma \quad (L),
\]
if (2.1) holds for all \( \phi \in \mathcal{D}(\mathbb{R}) \) with support contained in \((0, \infty)\). The distributional lateral limit from the left \( f (a - 0) \) (L) is defined in a similar fashion.

Observe also that if \( f = \partial_a^n f_n \), and \( f_n \) is bounded near \( x = a \), then \( f (a + 0) \) (L) exists, and equals \( \gamma \), if and only if \( f_n (a + 0) = \gamma \) (L).
These notions have straightforward extensions to distributions defined in a smooth contour of the complex plane. A natural extension of this pointwise notions for distributions is the so called quasiasymptotic behavior of distributions, explained, e.g., in [14, 16, 20].

3. Boundary values and distributional point values

We shall need the following well known fact [1]. We shall use the notation \( \mathbb{H} \) for the half plane \( \{ z \in \mathbb{C} : \Im z > 0 \} \).

**Lemma 1.** Let \( F \) be analytic in the half plane \( \mathbb{H} \), and suppose that the distributional limit \( f(x) = F(x + i0) \) exists in \( \mathcal{D}'(\mathbb{R}) \). Suppose that there exists an open, non-empty interval \( I \) such that \( f \) is equal to the constant \( \gamma \) in \( I \). Then \( f = \gamma \) and \( F = \gamma \).

Actually using the theorem of Privalov [15, Cor 6.14] it is easy to see that if \( F \) is analytic in the half plane \( \mathbb{H} \), \( f(x) = F(x + i0) \) exists in \( \mathcal{D}'(\mathbb{R}) \), and there exists a subset \( X \subset \mathbb{R} \) of non-zero measure such that the distributional point value \( f(x_0) \) exists and equals \( \gamma \) if \( x_0 \in X \), then \( f = \gamma \) and \( F = \gamma \).

Our first result is for bounded analytic functions.

**Theorem 1.** Let \( F \) be analytic and bounded in a rectangular region of the form \( (a,b) \times (0,R) \). Set \( f(x) = \lim_{y \to 0^+} F(x + iy) \) in \( \mathcal{D}'(a,b) \), so that \( f \in L^\infty(a,b) \). Let \( x_0 \in (a,b) \) be such that

\[
(3.1) \quad f(x_0 + 0) = \gamma \quad (L)
\]

exists. Then the distributional point value

\[
(3.2) \quad f(x_0) = \gamma \quad (L)
\]

also exists. In fact, the point value is of the first order, and thus

\[
(3.3) \quad \lim_{x \to x_0} \frac{1}{x-x_0} \int_{x_0}^{x} f(t) \, dt = \gamma.
\]

**Proof.** We shall first show that it is enough to prove the result if the rectangular region is the upper half-plane \( \mathbb{H} \). Indeed, let \( C \) be a smooth simple closed curve contained in \( (a,b) \times [0,R] \) such that \( C \cap (a,b) = [x_0 - \eta, x_0 + \eta] \), and which is symmetric with respect to the line \( \Re z = x_0 \). Let \( \varphi \) be a conformal bijection from \( \mathbb{H} \) to the region enclosed by \( C \) such that the image of the line \( \Re z = x_0 \) is contained in \( \Re z = x_0 \), so that, in particular, \( \varphi(x_0) = x_0 \). Then (3.1) - (3.3) hold if and only if the corresponding equations hold for \( f \circ \varphi \).

Therefore we may assume that \( a = -\infty \), and \( b = R = \infty \). In this case, \( f \) belongs to the Hardy space \( H^\infty \), the closed subspace of \( L^\infty(\mathbb{R}) \) consisting of the boundary values of bounded analytic functions on \( \mathbb{H} \).
Let $f_\varepsilon(x) = f(x_0 + \varepsilon x)$. Clearly, the set $\{f_\varepsilon : \varepsilon > 0\}$ is weak* bounded (as a subset of the dual space $(L^1(\mathbb{R}))' = L^\infty(\mathbb{R})$) and, consequently, a relatively weak* compact set. If $\{\varepsilon_n\}_{n=0}^\infty$ is a sequence of positive numbers with $\varepsilon_n \to 0$ such that the sequence $\{f_{\varepsilon_n}\}_{n=0}^\infty$ is weak* convergent to $g \in L^\infty(\mathbb{R})$, then $g = \gamma$, since $g \in H^\infty$, and $g(x) = \gamma$ for $x > 0$. In fact, the condition (3.1) means that
\[
\int_0^\infty g(x)\psi(x)dx = \lim_{n \to \infty} \int_0^\infty f_{\varepsilon_n}(x)\psi(x)dx = \gamma \int_0^\infty \psi(x)dx,
\]
for all $\psi \in \mathcal{D}(0, \infty)$, which yields the claim. Since any sequence $\{f_{\varepsilon_n}\}_{n=0}^\infty$ with $\varepsilon_n \to 0$ has a weak* convergent subsequence, and since that subsequence converges to the constant function $\gamma$, we conclude that $f_\varepsilon \to \gamma$ in the weak* topology of $L^\infty(\mathbb{R})$. Furthermore, (3.3) follows by taking $x = x_0 + \varepsilon$ and $\phi(t) = \chi_{[0,1]}(t)$, the characteristic function of the unit interval, in the limit $\lim_{\varepsilon \to 0} \langle f_\varepsilon(t), \phi(t) \rangle = \gamma \int_{-\infty}^\infty \phi(t)\,dt$.

We can now prove our main result, a distributional extension of Theorem 1.

**Theorem 2.** Let $F$ be analytic in a rectangular region of the form $(a, b) \times (0, R)$. Suppose $f(z) = \lim_{y \to 0^+} F(x + iy)$ in the space $\mathcal{D}'(a, b)$. Let $x_0 \in (a, b)$ such that $f(x_0 + 0) = \gamma (L)$. If $f$ is distributionally bounded at $z = x_0$ then $f(x_0) = \gamma (L)$. Furthermore, $F(z) \to \gamma$ as $z \to x_0$ in an angular fashion.

**Proof.** There exists $n \in \mathbb{N}$ and a function $f_n$ bounded in a neighborhood of $x_0$ such that $f = \partial_{x_0}^n f_n$; notice that $f(x_0) = \gamma (L)$ if and only if $f_n(x_0) = \gamma (L)$. But $f_n(x) = F_n(x + i0)$ distributionally, where $F_n$ is analytic in $(a, b) \times (0, R)$; here $F_n$ is the only angularly bounded solution of $F(z) = \partial_{x_0}^n F_n(z)$ (derivatives with respect to $z$). Clearly, $f_n(x) = F_n(x + i0)$. Since $f_n$ is bounded near $x = x_0$, $F_n$ is also bounded in a rectangular region of the form $(a_1, b_1) \times (0, R_1)$, where $x_0 \in (a_1, b_1)$. Clearly $f_n(x_0 + 0) = \gamma (L)$, so the Theorem 1 yields $f_n(x_0) = \gamma (L)$, as required. Finally, the fact that $F(z) \to \gamma$ as $z \to x_0$, angularly, is a consequence of the existence of the distributional point value, as shown in [6, 16].

Observe that in general the result (3.3) does not follow if $f$ is not bounded but just distributionally bounded near $x_0$.

We may use a conformal map to obtain the following general form of the Theorem 2.

**Theorem 3.** Let $C$ be a smooth part of the boundary $\partial \Omega$ of a region $\Omega$ of the complex plane. Let $F$ be analytic in $\Omega$, and suppose that
\( f \in \mathcal{D}'(\mathbb{C}) \) is the distributional boundary limit of \( F \). Let \( \xi_0 \in \mathbb{C} \) and suppose that the distributional lateral limit \( f(\xi_0 + 0) = \gamma \) (L) exists and \( f \) is distributionally bounded at \( \xi = \xi_0 \), then \( f(\xi_0) = \gamma \) (L) and \( F(z) \) has non-tangential limit \( \gamma \) at the boundary point \( \xi_0 \).

We also immediately obtain the following Tauberian theorem. As mentioned at the Introduction, it generalizes some Tauberian results by the authors from [18].

**Theorem 4.** Let \( F \) be analytic in a rectangular region of the form \( (a, b) \times (0, R) \). Suppose \( f(x) = \lim_{y \to 0^+} F(x + iy) \) in the space \( \mathcal{D}'(a, b) \). Let \( x_0 \in (a, b) \) such that the distributional limit \( \lim_{y \to 0^+} F(x_0 + iy) = \gamma \) (L) exists. If \( f \) is distributionally bounded at \( x = x_0 \) then \( f(x_0) = \gamma \) (L) and the angular (ordinary) limit exists: \( \lim_{z \to x_0} F(z) = \gamma \).

**Proof.** If we consider the curve \( C \) to be the union of the segments \( (a, x_0] \) and \( [x_0, iR) \), then the distributional lateral limit of the boundary value of \( F \) on \( C \) exists and equals \( \gamma \) as we approach \( x_0 \) from the right along \( C \) and so the Theorem 3 yields that the distributional limit from the left, which is nothing but \( f(x_0 - 0) \) (L), also exists and equals \( \gamma \). Then the Theorem 2 gives us that \( f(x_0) = \gamma \) (L). The existence of the angular limit of \( F(z) \) as \( z \to x_0 \) then follows. \( \square \)

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