New Exact Operational Shifted Pell Matrices and Their Application in Astrophysics

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Abstract. In this work, the exact operational matrices for shifted Pell polynomials are achievable; so one can integrate and product the vector of basic functions s. The general form of the matrix of integration \( P \) is established, the dual matrix of integration \( Q \) is derived with general formulation, and the general form of the matrix derived from the product of two shifted Pell polynomials has been given. This idea is implemented on shifted Pell basis vector. Using such exact matrices, then the resident function of the equation is reached which can be written as \( R, P(x) \), where \( R \) is an algebraic equation vector and \( P(x) \) is the shifted Pell basis vector. The presented matrices can be utilized to find the approximate solution of differential equations, integral equation and the calculus of variations problems. An investigation for the convergence and error analysis of the proposed shifted Pell expansion is performed. Numerical treatment for problems in physics are included in this work to demonstrate the accuracy, easy to implement as well as accurate and satisfactory results with a small number of shifted Pell basis. Using operational matrices and the spectral technique are used together for solving Lane-Emden equation.

1. Introduction

Keywords: Shifted Pell polynomials, operation matrix, Lane-Emden equation, dual operation matrix, convergence analysis.

Approximating using an orthogonal family of functions has played an important role in the improvement of engineering, physical sciences, and mathematical analysis. In particular, classical orthogonal functions are applied extensively for the approximate solution of some problems together with spectral methods. For example, in [1], the collocation method was applied using sixth-kind Chebyshev polynomials for treating a class of fractional nonlinear quadratic integro-differential equations. An efficient approximate approach for fractional differential equations was proposed in [2-3] by applying a Tau-Collocation method. The authors in [4] investigated a computational scheme based Müntz polynomials and the collocation method for the approximate solution of fractional differential equations while Legendre collocation spectral technique was presented in [5] for numerically solving the high-order Volterra–Fredholm integro-differential equations.

Special attention has been given to different important orthogonal polynomials using spectral procedures, the explicit formula for the expansion coefficients of the integrals in terms of the original expansion coefficients of the polynomials are required. The new idea in this article, all the presented operational matrices are written in terms of power basis. The advantage has the new formulation of the operational matrices ids the easier computation of the coefficients. When reducing the original problem
to a set algebraic equations which are easy to solve. Moreover, only a small number of shifted Pell basis is required to reach a satisfactory results. The authors in [6] used Taylor wavelet method to solve Lane-Emden singular problems. Other techniques for solving Lane-Emden type equations are fractional Adomian decomposition [7], Legendre wavelets spectral technique [8], a general analytical solution [9], a special algorithm that produces an approximate polynomial solution [10], a modified Legendrespectral method [11], Haar wavelet collocation [12], and analytical solutions [13]. For more works, one can see [14-17].

The Pell polynomials have become interested in numerical analysis from partial and theoretical points of view. There are Pell and modified Pell polynomials. The research papers dealing with Pell polynomials include some results and properties of either Pell or modified Pell polynomials [18-24]. The best of our knowledge up to now no analytical expressions for the operational matrices of integral, dual and the product of shifted Pell polynomials are given yet in the literature. This motivates our interest in shifted Pell polynomials. Moreover, utilize of shifted Pell polynomials in solving Lane-Emden equation is also studied in the present article.

The organization of this work is as follows: the definition of shifted Pell polynomials is given in section 2 with some important properties. In section 3-5 state with proof, the three formulas expressing explicitly the integration, dual integral and the product of two shifted Pell polynomials. The convergence analysis and error estimates are studied in section 6. Section 7 gives numerical examples to explain the numerical spectral solution of a Lane-Emden equation. Finally, section 8 lists some concluding remarks.

2. Properties of Shifted Pell Polynomials

The fundamental property of the shifted Pell polynomials is that the function is expressed in terms of the independent variable $\tau$ or vice versa.

$$P_0(\tau) = 0$$

$$P_1(\tau) = 1$$

$$P_2(\tau) = 4\tau - 2$$

$$P_3(\tau) = 16\tau^2 - 16\tau + 5$$

$$P_4(\tau) = 46\tau^3 - 96\tau^2 + 56\tau - 12$$

$$P_5(\tau) = 256\tau^4 - 512\tau^3 + 432\tau^2 - 176\tau + 29$$

and

$$1 = P_1(\tau)$$

$$\tau = \frac{1}{4}[P_2(\tau) + 2P_1(\tau)]$$

$$\tau^2 = \frac{1}{16}[P_3(\tau) + 4P_2(\tau) + 3P_1(\tau)]$$

$$\tau^3 = \frac{1}{64}[P_4(\tau) + 6P_3(\tau) + 10P_2(\tau) + 2P_1(\tau)]$$

The recurrence relation of the shifted Pell polynomials can be written as

$$P_{n+1}(\tau) = (4\tau - 2)P_n(\tau) + P_{n-1}(\tau)$$

(1)

where: $P_0(\tau) = 0$, $P_1(\tau) = 1$

The other property of the shifted Pell polynomials functions is that

$$P_n(\tau) = \frac{1}{4^n}(P_{n+1}(\tau) + P_{n-1}(\tau))$$

(2)

A function $f(\tau)$ that is square integrable on $x \in [0,1]$ can be expressed, exactly in terms of shifted Pell polynomials as

$$f(\tau) = \sum_{k=1}^{\infty} a_k P_k(\tau)$$

(3)

The infinite series in Eq. 3, can be truncated to a sufficient order $n$ to get

$$f(\tau) = \sum_{k=1}^{n} a_k P_k(\tau) = A_n^f P(\tau)$$

(4)
where: \( A_n = [a_1 \ a_2 \ \ldots \ a_n]^T \) is called shifted Pell spectrum of \( f(\tau) \) which makes orthogonal polynomials in general adequate for approximate computations in different complex applied mathematical problems.

\[ \overline{P}(\tau) = [\overline{P}_1(\tau) \ \overline{P}_2(\tau) \ \ldots \ \overline{P}_n(\tau)]^T \]

is called the shifted Pell polynomials vector. The coefficients \( a_k \) in Eq. 4 are given by the following

\[ a_k = < f(\tau), P_k(\tau) >, \ k = 1,2,\ldots,n \]  

(5)

3. Matrix Form Shifted Pell Polynomials

Shifted Pell polynomials can be represented in the matrix form as follows

\[ \overline{P}(\tau) = Z T \]  

(6)

where: \( \overline{P}(\tau) = [\overline{P}_1(\tau) \ \overline{P}_2(\tau) \ \ldots \ \overline{P}_n(\tau)]^T, T = (1 \ \tau \ \ldots \ \tau^n)^T \),

and the matrix \( Z \) is a \( n \times n \) matrix defined by

\[
Z = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \cdots & 0 \\
-2 & 4 & 0 & 0 & 0 & \cdots & 0 \\
5 & z_{21} & 4 & 0 & 0 & \cdots & 0 \\
-12 & z_{31} & z_{32} & 4^2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
z_{n0} & z_{n1} & z_{n2} & z_{n3} & z_{n4} & \cdots & 4^{n-1}
\end{pmatrix}
\]

The first element of each row is

\[ z_{i0} = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k 2^{n-2k} \binom{n-k}{k}, \ i = 0,1,\ldots,n \]

In addition, other elements are defined by

\[ z_{ij} = 4z_{i-1,j-1} + 2z_{i-1,j} + z_{i-2,j} \]

4. Product of Shifted Pell Polynomials

Before solving the nonlinear TFF equation, we recall Weierstrass theorem [21]:

Theorem 4.1: The product of two shifted Pell polynomials is given by the expression

\[ \overline{P}_n(\tau) \overline{P}_m(\tau) = \sum_{k=0}^{\min(n,m)-1} (-1)^k \overline{P}_{n+m-(2k+1)}(\tau), \ n,m > 1 \]  

(7)

where: \( \overline{P}_0(\tau) \overline{P}_m(\tau) = \overline{P}_0(\tau) \)

Proof

Take \( m = 1 \), then one can get \( \overline{P}_n(\tau) \overline{P}_1(\tau) = \overline{P}_n(\tau) \), this means that Eq. 7 is true for \( m = 1 \).

Assume that Eq. 7 is true for \( m \), that is

\[ \overline{P}_n(\tau) \overline{P}_m(\tau) = \sum_{k=0}^{\min(n,m)-1} (-1)^k \overline{P}_{n+m-(2k+1)}(\tau) \]

Multiply both sides of this equation by \((4\tau - 2)\), yields

\[ \overline{P}_n(\tau)(4\tau - 2) \overline{P}_m(\tau) = (4\tau - 2) \sum_{k=0}^{\min(n,m)-1} (-1)^k \overline{P}_{n+m-(2k+1)}(\tau) \]  

(8)

Using the important recurrence relation for shifted Pell polynomials

\[ (4\tau - 2) \overline{P}_n(\tau) = \overline{P}_{n+1}(\tau) - \overline{P}_{n-1}(\tau), \]  

this will lead to

\[
\overline{P}_n(\tau)[\overline{P}_{m+1}(\tau) - \overline{P}_{m-1}(\tau)] = \sum_{k=0}^{\min(n,m)-1} (-1)^k [\overline{P}_{n+m-2k}(\tau) - \overline{P}_{n+m-(2k-2)}(\tau)]
\]

\[
\overline{P}_n(\tau)\overline{P}_{m+1}(\tau) = \overline{P}_n(\tau)\overline{P}_{m-1}(\tau) + \sum_{k=0}^{\min(n,m)-1} (-1)^k [\overline{P}_{n+m-2k}(\tau) - \overline{P}_{n+m-(2k-2)}(\tau)]
\]
\[ P_n(\tau)P_{m+1}(\tau) = \sum_{k=0}^{\min(n,m-1)-1} (-1)^k P_{n+m-(2k+2)}(\tau) \]

\[ + \sum_{k=0}^{\min(n,m)-1} (-1)^k P_{n+m-2k}(\tau) \sum_{k=0}^{\min(n,m)-1} (-1)^k P_{n+m-(2k+2)}(\tau) \]

From Eq. 9, it is easy to obtain the desired result.

That is

\[ P_n(\tau)P_{m+1}(\tau) = \sum_{k=0}^{\min(n,m-1)-1} (-1)^k P_{n+m-(2k+2)}(\tau) \]

5. Operation Matrix of Integration

The structure of shifted Pell operational matrix of integration is given. Take \( n = 5 \), in this case

\[ P(\tau) = \begin{bmatrix} P_1(\tau) & P_2(\tau) & P_3(\tau) & P_4(\tau) & P_5(\tau) \end{bmatrix}^T \]

By integrating the vector \( P(\tau) \) from 0 to \( \tau \) and representing them in matrix form, yields

\[ \int_0^\tau P_1(t) dt = \begin{bmatrix} \frac{1}{2} \ 1/4 \ 0 \ 0 \end{bmatrix} P(\tau) \]

\[ \int_0^\tau P_2(t) dt = \begin{bmatrix} -5/8 \ 0 \ 1/8 \ 0 \end{bmatrix} P(\tau) \]

\[ \int_0^\tau P_3(t) dt = \begin{bmatrix} 14/12 \ 1/12 \ 0 \ 1/12 \end{bmatrix} P(\tau) \]

\[ \int_0^\tau P_4(t) dt = \begin{bmatrix} 34/16 \ 0 \ 1/16 \ 0 \end{bmatrix} P(\tau) \]

\[ \int_0^\tau P_5(t) dt = \begin{bmatrix} -82/20 \ 0 \ 0 \ 1/20 \ 0 \end{bmatrix} P(\tau) + \frac{1}{20} P_6(\tau) \]

Thus

\[ \int_0^\tau P(t) dt = R_{5\times5} P(\tau) + P_6(\tau) \]

where: \( R_{5\times5} = \begin{bmatrix} \frac{1}{2} & 1/4 & 0 & 0 & 0 \\ 5/8 & 0 & 1/8 & 0 & 0 \\ -14/12 & 1/12 & 0 & 1/12 & 0 \\ 34/16 & 0 & 1/16 & 0 & 1/16 \\ -82/20 & 0 & 0 & 1/20 & 0 \end{bmatrix} \)

\[ P_6(\tau) = \begin{bmatrix} 0 & 0 & 0 & 1/20 \end{bmatrix}^T \]

Fortunately, for general \( n \), operational matrix of integration for shifted Pell polynomials has a regular expression, so no need to pre-calculate the corresponding operational matrix \( R \) and \( R \) for different \( n \) when dealing with the problems

\[ \int_0^\tau P(t) dt = R_{n\times n} P(\tau) + P_{n+1}(\tau) \]

where: \( R \) is \( n \times n \) matrix given by
\[
\begin{pmatrix}
\delta & \frac{1}{4} & 0 & 0 & \cdots & 0 \\
-\delta & 0 & \frac{1}{8} & 0 & \cdots & 0 \\
\delta & \frac{1}{12} & 0 & \frac{1}{12} & \cdots & 0 \\
-\delta & 0 & \frac{1}{16} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\delta & 0 & 0 & \frac{1}{4n} & \cdots & \frac{1}{4n}
\end{pmatrix}
\]

\[\mathbf{P}(\tau) = [\mathbf{P}_1(\tau) \quad \mathbf{P}_2(\tau) \quad \cdots \quad \mathbf{P}_n(\tau)]^T\]

\[\mathbf{P}_{n+1}(\tau) = [0 \quad 0 \quad \cdots \quad 0 \quad \frac{1}{4n}]^T\]

\[\delta = \mathbf{P}_{n+1}(0) + \mathbf{P}_{n-1}(0)\]

\[\delta = \left\{ \begin{array}{ll}
\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} 2^{n-2k} \binom{n-k}{k} & \text{n odd} \\
-\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} 2^{n-2k} \binom{n-k}{k} & \text{n even}
\end{array} \right. \]

6. Dual Operational Matrix

The dual operational matrix of \(\mathbf{P}(\tau)\) is presented in this section. The integration of the cross product is taken of two shifted Pell basis vectors; the results is \((n + 1) \times (n + 1)\) dimensional matrix which is indicated by

\[H = \int_0^1 \mathbf{P}(\tau)\mathbf{P}(\tau)d\tau\] (10)

The matrix \(H\) is called the dual operational matrix of \(\mathbf{P}(\tau)\) and will be calculated as follows

The integrals of the products of shifted Pell basis functions is

\[\int_0^1 \mathbf{P}_n(\tau)\mathbf{P}_m(\tau)d\tau = \sum_{k=0}^{\min(n,m)} \sum_{k=1}^{(-1)^{k+1} \mathbf{P}_{n+m-(2k+1)}(\tau)d\tau}
= \sum_{k=1}^{\min(n,m)} (-1)^{k+1} \int_0^1 \mathbf{P}_{n+m-(2k+1)}(\tau)d\tau
\]

Since

\[\int_0^1 \mathbf{P}(\tau)d\tau = \left\{ \begin{array}{ll}
\sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} 2^{n-2k} \binom{n-k}{k} & \text{n odd} \\
0 & \text{n even}
\end{array} \right. \]

Therefore, we have \(H_1 = \left( \begin{array}{ccc}
\mathbf{P}_1\mathbf{P}_1 & \mathbf{P}_1\mathbf{P}_2 & \mathbf{P}_1\mathbf{P}_3 \\
\mathbf{P}_2\mathbf{P}_1 & \mathbf{P}_2\mathbf{P}_2 & \mathbf{P}_2\mathbf{P}_3 \\
\mathbf{P}_3\mathbf{P}_1 & \mathbf{P}_3\mathbf{P}_2 & \mathbf{P}_3\mathbf{P}_3
\end{array} \right)\)
7. Convergence Analysis and Error Estimate

The approximate function of \( f(\tau) \) using shifted Pell polynomials is as follows

\[
f_n(\tau) = \sum_{i=1}^{n+1} a_i \overline{P}_i(\tau)
\]

where:

\[
a_i = \langle f(\tau), \overline{P}_n(\tau) \rangle = \int_0^1 f(\tau) \overline{P}_n(\tau) d\tau
\]

Since every continuous function on the closed interval is a bounded function, thus there is a constant \( \epsilon \) such that \( |f(\tau)| \leq \epsilon \forall \tau \in [0,1] \) (13)

Therefore, Eq. 12 leads to

\[
|a_n| \leq \epsilon \int_0^1 |\overline{P}_n(\tau)| d\tau 
\]

\[
\leq \epsilon \left| \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} (4\tau - 2)^{n-2k} \right| d\tau 
\]

\[
\leq \epsilon \left| \sum_{k=0}^{[\frac{n}{2}]} \binom{n-k}{k} \sum_{i=0}^{n-2k} (-1)^{n-i} \binom{n-2k}{i} 2^{n-2k} |(2^{4+i})| d\tau 
\]

\[
\int_0^1 |\overline{P}_n(\tau)| d\tau \leq \sum_{k=0}^{[\frac{n}{2}]} 2^{n-2k} \prod_{l=0}^{k-1} \frac{n-k-L}{k-L} \text{ for odd } n \text{ and } \int_0^1 |\overline{P}_n(\tau)| d\tau = 0 \text{ for even } n.
\]

This will lead to the following result

\[
|a_n| \leq \epsilon \sum_{k=0}^{[\frac{n}{2}]} 2^{n-2k} \prod_{l=0}^{k-1} \frac{n-k-L}{k-L}
\]

8. Application of the Shifted Pell Operational Matrices

This section describes the application of the presented operational matrices of shifted Pell polynomials for solving some problems arising in physics throughout the following two suggested examples.

8.1 Application to Lane-Emden Type Equation

Consider the following Lane-Emden singular initial value problem equation

\[
f''(\tau) + \frac{2}{\tau} f'(\tau) + f(\tau) = \tau^2 + 7, \ f(0) = 1, \ f'(0) = 0, \ \tau \in (0,1]
\]

with the exact \( f(\tau) = \tau^2 + 1 \) (15)

Assume that the unknown function \( f''(\tau) \) is approximated by \( n \) basis of the shifted Pell polynomials as below

\[
f''(\tau) = \sum_{i=1}^{n} a_i \overline{P}_i(\tau) = a^T \overline{P}(\tau)
\]

where:

\[
a = [a_1 \ldots a_n]^T
\]
For approximate the solution $f'(\tau)$ and $f(\tau)$ two-times integration are applied on both sides Eq. 16, yields

$$\int_0^\tau f''(t)dt = f'(\tau) - f'(0) = a^T \mathbf{I} \mathbf{P}(\tau)$$  \hspace{1cm} (17)

and

$$\int_0^\tau f'(t)dt = f(\tau) - f(0) = \int_0^\tau a^T \mathbf{I} f(t)dt$$

or

$$f(\tau) = a^T \mathbf{I}^2 \mathbf{P}(\tau) + V^T \mathbf{P}(\tau)$$  \hspace{1cm} (18)

(Note that $1 = V^T \mathbf{P}(\tau)$

Eq. 18 can be rewritten as follows

$$f(\tau) = (a^T \mathbf{I}^2 + V^T) \mathbf{P}(\tau) = L^T \mathbf{P}(\tau)$$  \hspace{1cm} (19)

Using the relationship between shifted Pell polynomials and the powers $1 = \tau, \ldots, \tau^n$ one can obtain

$$\tau = V^T \mathbf{P}(\tau)$$  \hspace{1cm} (20)

$$\tau^3 + 7\tau = Z^T \mathbf{P}(\tau)$$  \hspace{1cm} (21)

$$\tau f(\tau) = a^T V_1 \mathbf{P}(\tau)$$  \hspace{1cm} (22)

$$\tau f(\tau) = a^T V_2 \mathbf{P}(\tau)$$  \hspace{1cm} (23)

Eq. 14 can be rewritten as

$$a^T V_1 \mathbf{P}(\tau) + 2a^T \mathbf{I} \mathbf{P}(\tau) + a^T V_2 \mathbf{P}(\tau) = Z^T \mathbf{P}(\tau)$$

That is $a^T V_1 + a^T V_2 = Z^T \mathbf{P}(\tau)$  \hspace{1cm} (24)

Eq. 24 yields a set of algebraic equations which can be solved for $a$.

For the case $n = 2$, one can get

$$f_2(\tau) = \mathbf{P}_1(\tau) + \frac{1}{4\left(\frac{1}{2} a_1 - \frac{5}{8} a_2\right)}(\mathbf{P}_2(\tau) - 2\mathbf{P}_1(\tau)) + \frac{1}{32} a_1(\mathbf{P}_3(\tau) - 5\mathbf{P}_1(\tau))$$

$$+ \frac{1}{8} a_2(\mathbf{P}_4(\tau) + \mathbf{P}_2(\tau) + 14\mathbf{P}_1(\tau))$$

or

$$f_2(\tau) = a^T \mathbf{P}(\tau)$$

where: $a^T = \left[\left(1 + \frac{3}{2} a_1 - \frac{16}{96} a_2\right), \frac{1}{8} a_1 - \frac{14}{96} a_2, \frac{1}{32} a_1, \frac{1}{96} a_2\right]$.

Note that in this case $a_1 = 2$ and $a_2 = 0$, as a result

$$f_2(\tau) = \frac{19}{16} \mathbf{P}_1(\tau) + \frac{1}{4} \mathbf{P}_2(\tau) + \frac{1}{16} \mathbf{P}_3(\tau)$$

8.2 Application to Emden-Fowler Type Equation

Consider the Emden-Fowler equation studied by [25],

$$f''(\tau) + \frac{k}{\tau} f'(\tau) + \tau^r f^{(n)}(\tau) = 0, \quad n \in \mathbb{N} \cup \{0\}, \quad \tau \in (0,1]$$  \hspace{1cm} (25)

together with the initial conditions

$$f(0) = 1, \quad f'(0) = 0$$  \hspace{1cm} (26)

Eq. 25 governed by the conditions in Eq. 26 has $f(\tau) = \frac{\sin \tau}{\tau}$ as exact solution when $\tau = 0$, $n = 1$ and $k = 1$. The application of the operational shifted Pell matrices which is proposed in the previous sections yields the following approximate solutions corresponding to $n = 1$ and $n = 3$, respectively

$$f_1(\tau) = 0.96875 \mathbf{P}_1(\tau) - 0.041667 \mathbf{P}_2(\tau) - 0.010417 \mathbf{P}_3(\tau)$$

and

$$f_3(\tau) = 0.96855 \mathbf{P}_1(\tau) - 0.04115 \mathbf{P}_2(\tau) - 0.009733 \mathbf{P}_3(\tau) + 0.00026 \mathbf{P}_4(\tau) + 0.000033 \mathbf{P}_5(\tau)$$

Table 1 lists the results in the case of $n = 1, 3$ using shifted Pell operational matrices and the results obtained by applying orthonormal Bernstein polynomials developed in [25]. With some comparisons aiming to illustrate the applicability and accuracy of the proposed shifted Pell operational matrices procedure. Assume $\varepsilon_1$ and $\varepsilon_2$ denote the error in maximum norm corresponding to the approximate solution $f_1(\tau)$ and $f_3(\tau)$ respectively, that is

$$\varepsilon_1 = \max_{\tau \in [0,1]} |f(\tau) - f_1(\tau)| \quad \text{and} \quad \varepsilon_2 = \max_{\tau \in [0,1]} |f(\tau) - f_3(\tau)|$$
Figure 1 ensures that the obtained approximate solution using shifted Pell operational matrices is closer to the exact one with few basis shifted Pell. Figure 2 plots the absolute error of Example 2. The results in Table 1 illustrate that the obtained numerical solution with few terms of the presented shifted Pell expansion is more accurate than the results in [25]. The advantage of the proposed method is demonstrated when compared with other approximate method.

**Table 1 Approximate and exact values for example 2**

| $\tau$ | Shifted Pell $n = 1$ | Shifted Pell $n = 3$ | Exact $\sin \tau$ | OBP [25] $n = 3$ | OBP [25] $n = 5$ |
|--------|---------------------|---------------------|-------------------|----------------|----------------|
| 0.1    | 0.998333            | 0.99833             | 0.99834166        | 0.99837        | 0.99831        |
| 0.2    | 0.993333            | 0.99334             | 0.99334654        | 0.99340        | 0.99323        |
| 0.3    | 0.985000            | 0.98506             | 0.985067356       | 0.98509        | 0.98505        |
| 0.4    | 0.973333            | 0.97354             | 0.97354856        | 0.97352        | 0.97354        |
| 0.5    | 0.958333            | 0.95885             | 0.958851077       | 0.95881        | 0.95885        |
| 0.6    | 0.940000            | 0.94108             | 0.941070789       | 0.94104        | 0.94108        |
| 0.7    | 0.918333            | 0.92033             | 0.920310982       | 0.92031        | 0.92033        |
| 0.8    | 0.893333            | 0.89674             | 0.896695114       | 0.89672        | 0.89672        |
| 0.9    | 0.865000            | 0.87046             | 0.870363233       | 0.87037        | 0.87037        |

**Figure 1:** Different solution of example 2.

**Figure 2:** Absolute Error of example 2 in case $n = 1, 3$. 
Conclusion

In the present work, the definition of shifted Pell polynomials is presented first with some important properties. Then a novel general procedure for calculating operational matrices corresponding to the shifted Pell polynomials is presented. They are operational matrix of integration, operation matrix of product and dual operational matrix. These matrixes can be applied together with spectral methods to solve problems arising in physics called Lane-Emden equation. The efficiency of the proposed technique is shown from the obtained results. The aim of such technique is to get an effective algorithm that is suitable the digital computers by reducing the underlying Lane-Emden equation to algebraic equations.

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