Holomorphic curves in Shimura varieties

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We prove an hyperbolic analogue of the Bloch-Ochiai theorem about the Zariski closure of holomorphic curves in abelian varieties. We consider the case of non compact Shimura varieties completing the proof of the result for all Shimura varieties. The statement analysed was first formulated and proven by Ullmo and Yafaev for compact Shimura varieties.

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1 Introduction

The Bloch-Ochiai theorem [cf. Kob98, Chapter 9, 3.9.19] states that the Zariski closure of an holomorphic curve in an abelian variety is a coset of an abelian subvariety.

Theorem 1.1 (Bloch-Ochiai). Let $A$ be an abelian variety and $f : \mathbb{C} \to A$ a non-constant holomorphic map. Then the Zariski closure of $f(\mathbb{C})$ is a translate of an abelian subvariety.

In [UY16], Ullmo and Yafaev formulate and prove an analogue of this result for compact Shimura varieties.

Let $\mathcal{D}$ be a hermitian symmetric space realised as a bounded symmetric domain in $\mathbb{C}^n$ via the Harish-Chandra embedding$^1$, $G$ its isometry group and $\Gamma \subset G(\mathbb{R})$ an arithmetic lattice. Let $S = \Gamma \backslash \mathcal{D}$. Assume that $S$ is a component of a Shimura variety; in particular $G$ is defined over $\mathbb{Q}$ and $\Gamma$ is a congruence subgroup of $G(\mathbb{Q})$. Finally consider a holomorphic function $f : \mathbb{C}^m \to \mathbb{C}^n$ such that $f(\mathbb{C}^m) \cap \mathcal{D} \neq \emptyset$.

$^1$see Mok89, Chapter 4.
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**Theorem 1.2** ([UY16, Theorem 1.2]). Let $\pi: D \to S$ be the quotient map, $f$ as above and $V = f(\mathbb{C}^m) \cap D$. Assume $S$ is compact then the components of the Zariski closure $\text{Zar}(\pi(V))$ of $\pi(V)$ in $S$ are weakly special subvarieties of $S$.

For general definitions about Shimura varieties and weakly special subvarieties see [UY14] and the references contained there.

Along with the Bloch-Ochiai theorem the above result draws inspiration from the hyperbolic Ax-Lindemann theorem, first proven by Ullmo and Yafaev in [UY14] for compact Shimura varieties and then in general by Klingler, Ullmo, and Yafaev in [KUY16].

**Theorem 1.3** (Ax-Lindemann). Let $Y \subset D$ be an algebraic subset of $D^2$. Then the components of the Zariski closure $\text{Zar}(\pi(Y))$ are weakly special.

Our aim is to prove the result analogous to theorem 1.2 for all Shimura varieties (not necessarily compact), thus completing the proof of:

**Theorem 1.4** (Main Result). Let $\pi: D \to S$ be the quotient map, $f$ as above and $V = f(\mathbb{C}^m) \cap D$. Then the components of the Zariski closure $\text{Zar}(\pi(V))$ of $\pi(V)$ in $S$ are weakly special subvarieties of $S$.

As in [UY16], the proof follows the general lines of the proof of the hyperbolic Ax-Lindemann theorem. In particular, it relies on the theory of o-minimal structures and specifically on the use of Pila-Wilkie’s theorem on counting rational points in definable sets (see theorem 4.2).

The main steps of the proof are as follows. First we reduce to proving the result separately on several ‘branches’ $V_i$ of the portion of the image of $f$ contained in $D$ in such a way that each $V_i$ is definable in $\mathbb{R}_{an,exp}$. Then, we use toroidal compactifications of Shimura varieties, Pila-Wilkie’s and the Ax-Lindemann-Weierstrass theorem to prove that the Zariski closure of the image of $U_i$ contains a Zariski dense set of weakly special subvarieties. Here, the crucial part is Lemma 3.3, which asserts that the volume of the intersection between one of these definable curves $U_i$ in $D$ and a translate $\gamma F$ of a fixed fundamental domain $F$ for the action of $\Gamma$ on $D$ is bounded independently of $\gamma \in \Gamma$.

Finally, we conclude the proof of the main result of the paper using a result of Ullmo [Ull14, Théorème 1.3] and induction on the dimension.

We point out that, although our result is independent of the realisation of the symmetric domain $D$ uniformising $S^3$, we use in a crucial way that there is a bounded realisation. Indeed this allows us to reduce the proof to the definable sets $V_i$ and is again used in a fundamental way in the proof of the above cited Lemma 3.3.

To stress further the importance of the boundedness of $D$, we point out that questions related to the Bloch-Ochiai theorem in the abelian setting were investigated using o-minimal techniques by Ullmo and Yafaev in [UY17]. In this setting the authors were not able to prove the full Bloch-Ochiai theorem with the present techniques; this is ultimately due to the fact that the symmetric space uniformizing an abelian variety of dimension $d$ is $\mathbb{C}^d$ which has no bounded realisation.

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2 An algebraic subset of $D$ is a component of the intersection of an algebraic subset of $\mathbb{C}^m$ with $D$

3 see [Ull14] for the definition of realisation of a symmetric domain.
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2 Preliminaries

First we fix some notation.

- Let $(G, X)$ be a Shimura datum and let $D$ be a connected component of $X$. $D$ is an hermitian symmetric domain which we realise as a bounded hermitian symmetric domain in the holomorphic tangent space $p \cong \mathbb{C}^n$ at a point $x \in D$ via the Harish-Chandra embedding.

- Let $G(\mathbb{Q})^+$ be the stabiliser of $D$ in $G(\mathbb{Q})$ and $\Gamma \subset G(\mathbb{Q})^+$ a neat arithmetic subgroup; we may assume there is a faithful finite dimensional representation $\rho : G \to GL(E)$ defined over $\mathbb{Q}$ and a lattice $E_{\mathbb{Z}} \subset E$ such that $\Gamma = G(\mathbb{Z}) = G(\mathbb{Q}) \cap GL(E_{\mathbb{Z}})$.

- Let $\Sigma \subset D$ be a Siegel set for the action of $\Gamma$ such that there exists a finite set $J \subset G(\mathbb{Q})$ such that $J.\Sigma = \mathcal{F}$ is a fundamental set for the action of $\Gamma$.

Let $f : \mathbb{C}^m \to \mathbb{C}^n$ be an holomorphic map such that $f(\mathbb{C}^m) \cap D \neq \emptyset$. We decompose

$$f^{-1}(f(\mathbb{C}^m) \cap D) = \coprod_{i \in I} U_i$$

(2.1)

as a disjoint union of connected components. By definition of $U_i$,

$$\overline{f(U_i)} \cap \partial X \neq \emptyset;$$

(2.2)

hence there exists $R_i$ such that $\overline{f(U_i \cap B(0, R_i))} \cap \partial X \neq \emptyset$.

By analytic continuation, it follows that the Zariski closures $Zar(\pi \circ f(U_i))$ and $Zar(\pi \circ f(U_i \cap B(0, R_i)))$ are equal. Let $V_i = f(U_i \cap B(0, R_i))$ and $W_i = Zar(\pi(V)) \subset S$. Following [UY16], we will deduce our main result 1.4 from the following theorem.

**Theorem 2.1.** There exists a positive dimensional semialgebraic set $X$ in $G(\mathbb{R})$ such that

$$X \cdot f(B(0, R_i) \cap U_i) \subset \pi^{-1}(W_i).$$

(2.3)
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Following [UY16], we now briefly describe how the main result follows from the above theorem. Let \( P \in V_i \), let \( X \subset G(\mathbb{R}) \) be a maximal semialgebraic subset such that \( X \cdot P \subset \pi^{-1}(W_i) \). By the above theorem this has positive dimension and by [PT13, Lemma 4.1], it is a complex algebraic subset. By the Ax-Lindemann theorem 1.3, the Zariski closure \( \text{Zar}(\pi(X \cdot P)) \subset W_i \) is a union of weakly special subvariety. Hence for each point of \( P \in \pi(V_i) \) there is a weakly special subvariety \( Y \) such that \( P \in Y \subset W_i \). This proves the following

**Theorem 2.2.** \( W_i = \text{Zar}(\pi(V_i)) \) contains a Zariski dense subset of weakly special subvarieties.

Now we proceed by induction on the dimension of \( W_i \) to show that theorem 2.2 implies the main result. The case of dimension zero is trivial since all points are weakly special. If \( W_i \) is special we are done, otherwise by [Ull14, Théorème 1.3], it follows that the smallest special subvariety \( S' \subset S \) containing \( W_i \) can be decomposed as a product \( S' = S_1 \times S_2 \) with both factors non trivial, such that

\[
W_i = S_1 \times V' \tag{2.4}
\]

for some subvariety \( V' \subset S_2 \).

Let \((G',X')\) be the sub-Shimura datum of \((G,X)\) associated to \( S' \). The above decomposition induces a decomposition of the adjoint datum \((G'^{\text{adj}},X'^{\text{adj}})\) as a product \( (G_1,X_1) \times (G_2,X_2) \) such that, for \( i = 1, 2 \), \( S_i = \Gamma_i \backslash \mathcal{D}_i \) for some suitable arithmetic subgroup \( \Gamma_i \) of \( G_i(\mathbb{Q})^+ \). We can realise both \( \mathcal{D}_i \) as bounded symmetric domains inside their holomorphic tangent spaces \( p_i \). Then we can write \( f : \mathbb{C}^m \to p_1 \times p_2 \subset p \) as \( f = (f_1,f_2) \). It now follows that \( V' \) is exactly the Zariski closure of \( \pi \circ f_2(U_i) \). By theorem 2.2, \( V' \) contains a Zariski dense set of weakly special subvarieties and by the inductive hypothesis it is weakly special.

We now recall two results about the structure of \( \mathcal{D} \) at the boundary we will need later.

**Proposition 2.3 ([AMRT10, Chapter III.4]).** Given a boundary component \( F \subset \overline{\mathcal{D}} \), its normaliser \( N(F) \) in \( G \) is a parabolic subgroup and can be decomposed as follows

\[
N(F) = (G_h(F)G_l(F)M(F))(V(F)U(F)),
\]

where

- \( R(F) = (G_h(F)G_l(F)M(F)) \) is a Levi factor of \( N(F) \) and the product is direct modulo a finite central group
- \( W(F) = (V(F)U(F)) \) is the unipotent radical of \( N(F) \)
- \( U(F) \) is the center of \( W(F) \) and is a real vector space
- \( V(F) = W(F)/U(F) \) is also a real vector space of even dimension \( 2l \)
- \( G_h(F) \) modulo a finite center is \( \text{Aut}^0(F) \), all the other factor act trivially
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- $G_l(F)$ modulo a finite center acts on $U(F)$ by inner automorphisms, the other factors commute with $U(F)$
- $M(F)$ is compact

**Proposition 2.4** ([KUY16, Proposition 3.2]). Fix a boundary component $F \subset \overline{D}$. Define

$$D_F = \bigcup_{g \in U(F)_{\mathbb{C}}} g \cdot D \subset D^c.$$  

There is an holomorphic isomorphism $j : D_F \to U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F$. This isomorphism realises $D$ as a Siegel domain of the third kind $D \simeq \{ (x, y, t) \in U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \mid \text{Im}(x) + l_t(y, y) \in C(F) \}$ where $C(F)$ is a self-adjoint convex cone in $U(F)$ homogeneous under $G_l(F)$ and $l_t : \mathbb{C}^l \times \mathbb{C}^l \to U(F)$ is a symmetric bilinear form varying real-analytically with $t \in F$.

Let $\Sigma \subset D$ be a Siegel set for the action of $\Gamma$, as above. Then $\Sigma$ is covered by a finite number of open subsets $\Theta$ having the following properties. For each $\Theta$ there is a cone $\sigma$ with $\sigma \subset C(F)$, a point $a \in C(F)$, relatively compact subsets $U'$, $Y'$ and $F'$ of $U(F)$, $\mathbb{C}^l$ and $F$ respectively such that the set $\Theta$ is of the form

$$\Theta \simeq \{ (x, y, t) \in U(F)_{\mathbb{C}} \times \mathbb{C}^l \times F \mid \text{Re}(x) \in U', \ y \in Y', \ t \in F' \text{ and } \text{Im}(x) + l_t(y, y) \in \sigma + a \}.$$  

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In this section we restrict attention to holomorphic curves; hence, we set $m = 1$ and consider holomorphic maps $f : \mathbb{C} \to \mathbb{C}^n = \mathfrak{p}$ be a holomorphic map. Maintaining the same notation as in the last section, we fix some $i \in I$ and let $C = f(U_i \cap B(0, R_i))$.

**Definition 3.1.** Recall we fixed a faithful finite dimensional representation $\rho : G \to GL(E)$; for any $\gamma \in \Gamma$ write $\rho(\gamma) = (\gamma_{i,j})_{i,j}$. For any $\phi \in \text{End}(E_{\mathbb{R}})$ define

$$|\phi|_{\infty} = \max_{i,j} |\phi_{i,j}|.$$  

Moreover, define the height of $\gamma \in \Gamma$ as

$$H(\gamma) = \max(1, |\gamma|_{\infty}).$$  

Finally, let $T > 0$, define

$$N_C(T) = \# \{ \gamma \in \Gamma \mid \gamma \cdot \mathcal{F} \cap C \neq \emptyset \text{ and } H(\gamma) \leq T \}$$

The aim of this section is to prove the following result.
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**Theorem 3.2.** There exist constants $c_1, c_2 > 0$ such that for all $T > 0$ sufficiently large

$$N_C(T) \geq c_1 T^{c_2}.$$  

In the proof of theorem 3.2 for algebraic curves given in [KUY16], the only part that the relies on the curve being algebraic is the analogue of the following lemma.

**Lemma 3.3.** There exists a positive constant $c_3$ such that for any $\gamma \in \Gamma$

$$\text{Vol}_{\gamma C}(\gamma C \cap \mathcal{F}) \leq c_3,$$  

(3.2)

where $\text{Vol}_{\gamma C}$ is the volume with respect to the riemannian metric on $\gamma C$ induced by the metric on $\mathcal{D}$.

**Proof.** By definition, $\mathcal{F} = J.\Sigma$ for a finite subset $J \subset G(\mathbb{Q})$ and some Siegel subset $\Sigma \subset \mathcal{D}$. Hence it is sufficient to prove the theorem for the Siegel set $\Sigma$. In turn, every Siegel set is covered by a finite number of open subsets $\Theta$ as in proposition 2.4, so it is sufficient to prove that

$$\text{Vol}_{\gamma C}(\gamma C \cap \Theta) \leq c_4$$  

(3.3)

for some constant $c_4 > 0$.

Let $\omega$ be the natural Kähler form on $\mathcal{D}$, then

$$\text{Vol}_{\gamma C}(\gamma C \cap \Theta) = \int_{\gamma C \cap \Theta} \omega$$  

(3.4)

On $\mathcal{D}_F$ we have the Poincaré metric defined by

$$\omega_F = \sum \frac{dx_i \wedge d\overline{x}_i}{\text{Im}(x_i)^2} + \sum dy_j \wedge d\overline{y}_j + \sum df_k \wedge d\overline{f}_k.$$  

(3.5)

By a result of Mumford [Mum77, Theorem 3.1], there is a constant $c_5$ such that on $\mathcal{D}$

$$\omega \leq c_5 \omega_F.$$  

(3.6)

Hence

$$\text{Vol}_{\gamma C}(\gamma C \cap \Theta) \leq \int_{\gamma C \cap \Theta} \omega_F.$$  

(3.7)

Now let $w$ be a coordinate between $x_i, y_j$ or $f_k$ and $g \in G(\mathbb{R})$, denote by $p_{g,w} : g.C \to \mathbb{C}$ the projection to the $w$ axis. Let $w_0 \in \mathbb{C}$ and define

$$n_{g,C,w}(w_0) = \text{number of points in } g.C \cap p_w^{-1}(w_0) \text{ counted with multiplicity.}$$  

(3.8)

Consider the set

$$W = \{(z_0, g, w_0) \in U_i \cap B(0, R_i) \times G(\mathbb{R}) \times \mathbb{C} | p_{g,w} \circ f(z_0) = w_0\}.$$  

(3.9)

We observe that, since $\mathcal{D}$ is bounded and $C$ is an analytic subset of $\mathcal{D}$, the projection $p_{g,w}$ is definable in $\mathbb{R}_{an}$. This along with the definability of all other maps involved, implies
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that $S$ is a definable family over $G(\mathbb{R}) \times \mathbb{C}$. It is a consequence of the cell decomposition theorem (cf. [Dri98, Chapter 3, Corollary 3.6]) that the number of definably connected components of the fibres of a definable set, hence, in this case, their cardinality, is uniformly bounded by a constant $c_w$. We now observe that the fibre of $W$ over a point $(g, w_0) \in G(\mathbb{R}) \times \mathbb{C}$ is the set $f^{-1}(p_{g,w_0}^{-1}(w_0))$ whose cardinality is exactly $n_{g,C,w}(w_0)$. Hence

$$n_{\gamma C,w}(w_0) \leq c_w \quad (3.10)$$

for all $w_0 \in \mathbb{C}$ and all $\gamma \in \Gamma$. Let $c_6$ be the maximum of $c_w$ with $w$ equal to $x_i$, $y_j$ or $f_k$, then

$$\text{Vol}_{\gamma C}(\gamma C \cap \Theta) \leq c_5 \left( \sum \int_{p_{x_i}(\Theta)} n_{\gamma C} \left( p_{x_i}^{-1}(x_i) \right) \frac{dx_i \wedge d\overline{x_i}}{\text{Im}(x_i)^2} + \sum \int_{p_{y_j}(\Theta)} n_{\gamma C} \left( p_{y_j}^{-1}(y_j) \right) dy_j \wedge d\overline{y_j} + \sum \int_{p_{f_k}(\Theta)} n_{\gamma C} \left( p_{f_k}^{-1}(f_k) \right) df_k \wedge d\overline{f_k} \right) \quad (3.11)$$

Now we observe that from the description of $\Theta$, the projection $p_{x_i}(\Theta)$ is contained in a finite union of usual fundamental domains in the upper half plane, which have finite hyperbolic area. Moreover, if $w$ is one of $y_j$ or $f_k$, then, again from the description of $\Theta$, it follows that $p_{w}(\Theta)$ is relatively compact in the plane and hence has finite Euclidean area.

This result allows us to follow the proof used in [KUY16] for algebraic curves and apply it to our o-minimal setting. For the convenience of the reader, we briefly recall the proof of theorem 3.2. First we report some results from [KUY16].

**Lemma 3.4 ([KUY16, Lemma 5.4]).** Let $x_0 \in D$ be a base point. There exists a constant $c_7$ such that for any $g \in G(\mathbb{R})$ the following inequality holds

$$\log (c_7 |g|_{\infty}) \leq d(g,x_0,x_0). \quad (3.12)$$

**Lemma 3.5 ([KUY16, Lemma 5.5]).** Let $\mathcal{F}$ be the fundamental domain for the action of $\Gamma$ fixed in the previous section. There exists a positive constant $c_8$ such that for all $\gamma \in \Gamma$ and for all $u \in \gamma \mathcal{F}$

$$H(\gamma) \leq c_8 |u|_{\infty}^n. \quad (3.13)$$

**Theorem 3.6 ([HT02]).** Let $C$ be a complex analytic curve in $D$. For any point $x_0 \in C$ there exist positive constants $c_9$ and $c_{10}$ such that for any positive real number $R$ one has

$$\text{Vol}_C(C \cap B(x_0, R)) \geq c_9 \exp (c_{10} R) \quad (3.14)$$


where \( \text{Vol}_C \) is the volume with respect to the Riemannian metric on \( C \) induced by the metric on \( D \) and \( B(x_0, R) \) is the geodesic ball in \( D \) of center \( x_0 \) and radius \( R \).

We can now finish the proof of theorem 3.2.

**Proof of Theorem 3.2.** Choose a base point \( x_0 \in C \), let \( c_7 \) and \( c_8 \) the constants given by lemma 3.4 and lemma 3.5 and consider the intersection \( C \cap B(x_0, R) \) of \( C \) with the geodesic ball of centre \( x_0 \) and radius \( R = \log \left( \frac{c_7 c_9}{c_8^{1/n}} T^{1/n} \right) \). On the one hand, we have by theorem 3.6

\[
\text{Vol}_C(C \cap B(x_0, R)) \geq \frac{c_7 c_9}{c_8^{1/n}} T^{1/n}. \tag{3.15}
\]

On the other hand, by lemma 3.4 and lemma 3.5

\[
B(x_0, \log R) \subseteq \left\{ g \cdot x_0 \mid g \in G(\mathbb{R}), |g|_{\infty} \leq r^{1/n}s^{1/n} \right\}
\subseteq \bigcup_{\gamma \in \Gamma, H(\gamma) \leq T} \gamma F.
\tag{3.16}
\]

Hence, by lemma 3.3

\[
\text{Vol}_C(C \cap B(x_0, \log R)) \leq \sum_{\gamma \in \Gamma, H(\gamma) \leq T} \text{Vol}_{\gamma^{-1}C(\gamma^{-1}C \cap F)} \leq N_C(T) c_3 \tag{3.17}
\]

We conclude comparing the lower bound and the upper bound

\[
\frac{c_7 c_9}{c_8^{1/n}} T^{1/n} \leq N_C(T) c_3. \tag{3.18}
\]

\[\square\]

4 Proof of Theorem 2.1

We use the same notation as in section 2; that is, we let \( f : \mathbb{C}^m \to \mathbb{C}^n \) be an holomorphic map such that \( f(\mathbb{C}^m) \cap D \neq \emptyset \). \( U_i \) are the connected components of \( f^{-1}(D) \). Fix an \( i \) and set \( U = U_i \); let \( R > 0 \) be such that \( f(U \cap B(0, R)) \cap \partial X \neq \emptyset \). Let \( V = f(U \cap B(0, R)) \). Finally, let \( W = \text{Zar}(\pi(V)) \subset S \).

First of all we note that, by definition, \( V \) is definable in the o-minimal structure \( \mathbb{R}_{an} \).

**Lemma 4.1.** Consider the set

\[\Sigma(W) = \left\{ g \in G(\mathbb{R}) \mid \dim(g.V \cap F) = \dim(V) \right\}. \tag{4.1}\]

Then the set \( \Sigma(W) \) is definable in \( \mathbb{R}_{an,exp} \). For all \( g \in \Sigma(W) \), \( g.V \subseteq \pi^{-1}(W) \). Moreover, define

\[\Sigma'(W) = \left\{ g \in G(\mathbb{R}) \mid V \cap g^{-1}.F \neq \emptyset \right\}. \tag{4.2}\]

Then

\[\Sigma(W) \cap \Gamma = \Sigma'(W) \cap \Gamma. \tag{4.3}\]
Proof. The set $\Sigma(W)$ is definable in $\mathbb{R}_{an,exp}$ because all sets and maps involved in its definition are\(^4\). The second assertion follows by analytic continuation. Finally the equality
\[
\Sigma(W) \cap \Gamma = \Sigma'(W) \cap \Gamma 
\] (4.4)
follows form the fact that $\pi^{-1}(W)$ is $\Gamma$-invariant. \(\Box\)

We now recall the Pila-Wilkie counting theorem.

**Theorem 4.2** ([PW06]). Let $S \subset \mathbb{R}^n$ be a set definable in the o-minimal structure $\mathbb{R}_{an,exp}$. Let $H : \mathbb{Q}^n \to \mathbb{R}$ be the usual multiplicative height and denote by $N_S(T)$ the number of rational points in $S$ of height at most $T$. If there exist constants $c, \varepsilon > 0$ such that $N_S(T) > cT^\varepsilon$ for all $T$ sufficiently large, then $S$ contains a positive dimensional semialgebraic subset.

Let
\[
N_{\Sigma(W)}(T) = \{ \gamma \in \Gamma \cap \Sigma(W) \mid H(\gamma) \leq T \}. 
\] (4.5)

From theorem 3.2, we see that $N_{\Sigma(W)}(T) \geq c_1 T^{c_2}$, for some constants $c_1, c_2 > 0$. Combining this with theorem 4.2, we get a semialgebraic set $X \subset \Sigma(W)$. Finally from lemma 4.1 we get that $X.V \subset \pi^{-1}(W)$, thus proving theorem 2.1.

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\(^4\)For the definability of the uniformisation map see [KUY16, Section 4].
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