Lagrangian studies in convective turbulence

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We present high-resolution direct numerical simulations of turbulent three-dimensional Rayleigh-Bénard convection with a focus on the Lagrangian properties of the flow. The volume is a Cartesian slab with an aspect ratio of four bounded by free-slip planes at the top and bottom and with periodic side walls. The turbulence is inhomogeneous with respect to the vertical direction. This manifests in different lateral and vertical two-particle dispersion and in a dependence of the dispersion on the initial tracer position for short and intermediate times. Similar to homogeneous isotropic turbulence, the dispersion properties depend in addition on the initial pair separation and yield a short-range Richardson-like scaling regime of two-particle dispersion for initial separations close to the Kolmogorov dissipation length. The Richardson constant is about half the value of homogeneous isotropic turbulence. The multiparticle statistics is very close to the homogeneous isotropic case. Clusters of four Lagrangian tracers show a clear trend to form flat, almost coplanar objects in the long-time limit and deviate from the Gaussian prediction. Significant efforts have been taken to resolve the statistics of the acceleration components up to order four correctly. We find that the vertical acceleration is less intermittent than the lateral one. The joint statistics of the vertical acceleration with the local convective and conductive heat flux suggests that rising and falling thermal plumes are not associated with the largest acceleration magnitudes. It turns out also that the Nusselt number which is calculated in the Lagrangian frame converges slowly in time to the standard Eulerian one.

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I. INTRODUCTION

Turbulent convection is one of the best studied fundamental flows in fluid dynamics research \[1, 2\]. One reason is the large range of examples and applications in nature and technology for which a turbulent motion is initiated and sustained by heating a fluid from below and cooling from above. Almost all of these studies have been conducted in the Eulerian frame of reference. They were primarily focussed to the mechanisms of local \[3, 4, 5, 6\] and global \[7, 8, 9, 10\] turbulent heat transfer.

The Lagrangian perspective of turbulence, in which the fields are monitored along the trajectories of infinitesimal fluid parcels, has recently produced new insights into the local topology of fluid parcel tracks, the local strength of accelerations, and the statistics of time increments of turbulent fields \[11, 12\]. The progress is caused on the one hand by significant innovations in the experimental techniques, such as three-dimensional particle tracking \[13, 14, 15\] or acoustic methods \[16\]. On the other hand, direct numerical simulations of turbulence become now feasible that resolve three-dimensional Lagrangian turbulence at moderate and higher Reynolds numbers \[17, 18, 19, 20\]. Both, experiments and simulations, made a deeper understanding of the small-scale intermittency and its connection with large accelerations of fluid parcels possible.

Lagrangian investigations in convective turbulence are however rare. Several reasons can be given for this circumstance. First, on the experimental side it is desirable to monitor the temperature along the particle tracks beside the velocity components and the accelerations. Only recently, Gasteuil \textit{et al.} \[21\] constructed therefore a smart particle, that monitors velocity, temperature and orientation while moving through the cell. Due to the integrated power supply the particle diameter remained however larger than the thermal boundary layer thickness, such that the large-scale bulk motion can be monitored only. Second, it is also clear that the complexity of direct numerical simulations increases since the temperature field has to be advected in addition to the velocity. Temperature tracking along the tracer positions requires additional interpolations. Furthermore, one cannot return to simulations in a fully periodic cube, the so-called homogeneous Rayleigh-Bénard convection setup, since the periodicity in the direction of the mean temperature gradient causes a self-amplifying fluid motion. This was discussed in detail by Calzavarini \textit{et al.} \[22, 23\]. Third, the turbulence is inhomogeneous –at least in the vertical direction as in the following setup– and it is thus not clear which of findings from the homogeneous isotropic purely hydrodynamic turbulence pertain. For example, the height dependence of the statistics has to be considered additionally.

First numerical attempts have been made recently to study some aspects of the heat transfer and tracer dispersion in the Lagrangian framework of convective turbulence \[24\]. The motivation of the study can be condensed in one question: Which new insight into the nature of turbulent convection provides the complementary Lagrangian view? One result of \[24\] was to determine a mixing zone which is dominated by rising and falling
thermal plumes. This is done by combining acceleration and local convective heat flux statistics. The mixing zone starts right above the thermal boundary layer and extends several tens of the boundary layer thickness into the bulk of the cell. Thermal plumes are fragments of the thermal boundary layer that detach in the vicinity of the top and bottom isothermal planes. The existence of a mixing zone has been suggested in several Eulerian studies on the basis of other criteria, e.g. and was thus confirmed in the complementary Lagrangian frame of reference.

The present work extends the previous study into several directions. Beside the local convective, the local conductive heat flux is studied along the tracer tracks. It requires to monitor temperature gradient components. Furthermore, the analysis of the Lagrangian tracer dispersion is extended. In addition to the hydrodynamic case, we study the dependence of pair dispersion on the initial separation and the initial seeding position. As discussed in Refs., for the pure hydrodynamic case, higher order particle statistics requires to track little clusters of tracers. We provide here an analysis of the four-particle-statistics, where the tracers start out of groups of tetrahedra of different sidelengths and initial vertical positions.

The outline of the manuscript is as follows. In the next section the equations of motion, the numerical scheme, the Lagrangian tracer tracking and the turbulent heat transfer. In section III, some results of the Eulerian statistics of the temperature field are presented. This section is followed by sections on the Lagrangian particle dispersion, the acceleration statistics and the conductive and convective heat flux. We conclude with a short discussion of our results and will give a brief outlook to possible extensions of the present work.

II. NUMERICAL MODEL

A. Equations of motion and boundary conditions

The Boussinesq equations, i.e. the Navier-Stokes equations for an incompressible flow with an additional buoyancy term \( \alpha g \theta e_z \) and the advection-diffusion equation for the temperature field, are solved by a standard pseudospectral method for the three-dimensional case. The equations are given by

\[
\begin{align*}
\nabla \cdot \mathbf{u} &= 0, \\
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} &= -\nabla p + \nu \nabla^2 \mathbf{u} + \alpha g \theta e_z, \\
\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla) \theta &= \kappa \nabla^2 \theta + u_z \frac{\Delta T}{H}.
\end{align*}
\]

Here, \( \mathbf{u} \) is the turbulent velocity field, \( p \) the (kinematic) pressure field and \( \theta \) the temperature fluctuation field. The system parameters are: gravity acceleration \( g \), kinematic viscosity \( \nu \), thermal diffusivity \( \kappa \), vertical temperature gradient \( \Delta T/H \), and thermal expansion coefficient \( \alpha \). The temperature field is decomposed into a linear profile and fluctuations \( \theta \) about the profile

\[ T(x, t) = -\frac{\Delta T}{H}(z - H/2) + \theta(x, t). \]  

Since \( T \) is prescribed and constant at bottom and top boundaries \( z = 0 \) and \( z = H \), the condition \( \theta = 0 \) follows there. Here, \( \Delta T > 0 \). The dimensionless control parameters are the Prandtl number \( Pr \), the Rayleigh number \( Ra \), and the aspect ratio \( \Gamma \),

\[ Pr = \frac{\nu}{\kappa}, \]

\[ Ra = \frac{\alpha g H^3 \Delta T}{\nu \kappa}, \]

\[ \Gamma = \frac{L}{H}. \]  

The simulation domain is \( V = L \times L \times H = [0, \Gamma x] \times [0, \Gamma y] \times [0, \pi] \). In lateral directions \( x \) and \( y \), periodic boundary conditions are taken. In the vertical direction, free-slip boundary conditions are used which are given by

\[ u_z = \theta = 0 \quad \text{and} \quad \partial_x u_x = \partial_y u_y = 0. \]

The computational grid has a size of \( N_x \times N_y \times N_z = 2048 \times 2048 \times 513 \) points. For an aspect ratio \( \Gamma = 4 \), it is thus equidistant in all three space directions with a grid spacing \( \Delta x \). Time-stepping is done by a second-order predictor-corrector scheme. The production runs are conducted on one rack of the Blue Gene/P system which corresponds with 4096 MPI tasks. We use volumetric Fast Fourier Transforms based on the p3dfft package by D. Pekurovsky. The spectral resolution is \( k_{\max} \eta_K = 4.5 \) where \( k_{\max} = 2 \sqrt{2} \pi N_x/(3L_x) \). Quantity \( \eta_K = \nu^{3/4}/\langle \epsilon \rangle^{1/4} \) is the Kolmogorov scale with the mean energy dissipation rate \( \langle \epsilon \rangle \).

B. Lagrangian particle tracking

Lagrangian tracer particles follow the streamlines of the turbulent velocity field in correspondence with

\[ \dot{x} = \mathbf{u}(x(t), t). \]

For the majority of the analysis, we seeded \( 3 \times 10^5 \) tetrahedra aligned along the outer coordinate axes in the box, i.e. \( x_1 = x_0, \ x_2 = x_0 + \ell e_x, \ x_3 = x_0 + \ell e_y, \) and \( x_4 = x_0 + \ell e_z \). The vector \( x_0 \) is randomly chosen in the box. The tracer ensemble was divided into 6 groups with initial sidelengths of \( \ell = 1, 2, 4, 8, 16 \) and 32 grid spacings \( \Delta x \) which correspond with 0.5, 1, 2, 4, 8 and 16 \( \eta_K \). The two-particle dispersion analysis is consequently conducted for the three tracer pairs \( \{x_1, x_2\}, \{x_1, x_3\}, \) and \( \{x_1, x_4\} \) of each tetrahedron.

For the two- and multiparticle statistics, we run in addition a simulation with the following initial conditions:
agin \(x_1 = x_0, x_2 = x_0 + \ell e_x, x_3 = x_0 + \ell e_y,\) and \(x_4 = x_0 + \ell e_z.\) The \(x\) and \(y\) coordinates of the vector \(x_0\) are again randomly chosen. The vertical coordinate corresponds with \(z_0 = \delta_T/2, \delta_T, 10\delta_T, 20\delta_T\) and \(H/2.\) Here, we pick \(\ell = \eta_K/2\) and \(2\eta_K.\)

The Lagrangian particles are advanced in time simultaneously with the Boussinesq equations. The velocity, temperature and temperature gradient components at intergrid positions are calculated by trilinear interpolation. The full particle set is written out each 0.45 \(\tau_n.\) Here, \(\tau_n = \sqrt{u/\epsilon}\) is the Kolmogorov time. Accelerations along the Lagrangian tracks are calculated from three successive integration steps \((\Delta t = 0.006\tau_n)\) and the output interval is the same as for the particle positions, velocities, temperature, and temperature gradient. We thus gather Lagrangian statistics up to \(4.8 \times 10^8\) tracer particle events. Figure 1 illustrates the initial phase of the tracer dispersion. All tracers start from a \(x - y\) plane close to the bottom wall.

C. Turbulent heat transfer

The convective turbulence is studied for one parameter setting. The Rayleigh number is \(Ra = 1.2 \times 10^8,\) the Prandtl number \(Pr = 0.7\) and the aspect ratio \(\Gamma = 4.\) The response of the system is a turbulent heat transport as quantified by the dimensionless (Eulerian) Nusselt number which is given for a plane at fixed height \(z\) by

\[
N_u(z) = \frac{\langle u_z T \rangle_{A,t} - \kappa \partial_z \langle T \rangle_{A,t}}{\kappa \Delta T / H},
\]

where \(\langle \cdot \rangle_{A,t}\) denote averages in planes at \(z\) and with respect to time. The value of \(N_u(z)\) is constant and independent of \(z.\) The global Nusselt number is then defined as

\[
N_u = \frac{1}{H} \int_0^H N_u(z) dz = 1 + \frac{H}{\kappa \Delta T} \langle u_z T \rangle_{V,t},
\]

where \(\langle \cdot \rangle_{V,t}\) is a combined volume and time average. The Nusselt number for the present free-slip boundary case follows to \(N_u = 56.36 \pm 0.59\). Similar to Julien et al. [31], we find an enhanced turbulent heat transport in comparison to no-slip top and bottom plates. For Rayleigh numbers between \(9.8 \times 10^5\) and \(1.2 \times 10^8\), we fit the power law \(N_u = 0.166 \times Ra^{0.316}\) to the data.

III. EULERIAN TEMPERATURE STATISTICS

Figure 2 displays the mean temperature profile as a function of height. The total temperature can take values between \(-\Delta T/2\) and \(\Delta T/2\) only. As typical for higher Rayleigh numbers, the jump of mean profile to zero is observed across a thin layer, the thermal boundary layer. The inset magnifies the vicinity of the bottom plate. The
thickness of the thermal boundary layer is defined as

$$\delta_T = \frac{H}{2N_u}. \quad (12)$$

For $z = 0$, the conductive part of $\chi$ contributes to $N_u$ only and we can set $N_u = -\kappa \partial_z \langle T \rangle_{A,t}|_{z=0}/(\kappa \Delta T/H)$. This leads to $\delta_T = -\Delta T/(2\partial_z \langle T \rangle_{A,t}|_{z=0})$ and to the geometric derivation of the thermal boundary layer thickness (as indicated in the inset of Fig. 3). In contrast to the no-slip case, we have $\langle u_x \rangle_{A,t} = \langle u_y \rangle_{A,t} = \langle u_z \rangle_{A,t} = 0$. Consequently, no velocity boundary layer is present. The Taylor microscale Reynolds number $R_\lambda = \sqrt{5/(3\nu)} (\langle u_i^2 \rangle)^{1/2} \approx 143$.

Figure 3 shows an instantaneous snapshot of the total temperature field $T(x, t)$. Contour plots in two side-planes and close to the top and bottom planes are shown. We observe a typical feature of thermal convection – the ridge-like maxima which correspond with thermal plumes that detach randomly. They form a skeleton which is advected by the flow close to the boundaries. The plumes coincide with local maxima of the thermal dissipation rate field (see Fig. 4 and compare it with Fig. 3) which is defined as

$$\chi(x, t) = \kappa (\nabla T'(x, t))^2. \quad (13)$$

The definition contains the temperature fluctuations which are given by

$$T'(x, t) = T(x, t) - \langle T(z) \rangle_{A,t}. \quad (14)$$

The probability density function (PDF) of $T'$ is shown in Fig. 5. We compare the PDF of data taken from the whole slab volume with the Gaussian statistics in the top figure. Similar to findings for turbulent convection in
closed cylindrical vessels with solid walls the temperature field statistics deviates from Gaussian \[6\]. The analysis can be refined. The inset of the top panel shows therefore vertical profiles of the plane- and time-averaged flatness, \(F = \langle T'^{4}\rangle_{A,t}/\langle T'^{2}\rangle_{A,t}^{2}\). The flatness differs clearly from the Gaussian value of 3 in all parts of the convection cell. In addition we plot the profile of the plane- and time-averaged skewness \(S = \langle T'^{3}\rangle_{A,t}/\langle T'^{2}\rangle_{A,t}^{3/2}\) in the same inset. The magnitude of the skewness peaks at about 5\(\tau\) which is well inside the plume mixing zone \[24\]. Both profiles agree also qualitatively with those by Kerr \[7\] and by Ref. \[6\] which have been conducted with no-slip top and bottom boundaries. In the bottom panel of Fig. \[5\] we show the PDF of the temperature fluctuations in four different planes (see the legend). The PDF in the midplane comes closest to a Gaussian profile. Our data suggest that the free-slip boundary conditions lead to smaller deviations from Gaussianity compared to the no-slip case. It should also be noted that for strong rotation of the cell about the vertical coordinate the temperature fluctuations are Gaussian for both, no-slip and free-slip boundary conditions, as reported by Julien et al. \[31\].

IV. LAGRANGIAN PARTICLE DISPERSION

A. Two-particle dispersion

The Eulerian framework analysis of turbulent convection demonstrated already that the flow is indeed inhomogeneous with respect to the vertical direction. Furthermore, we recall that the Lagrangian tracer motion is constrained between \(z = 0\) and \(H\) since both walls cannot be penetrated. One motivation to study the dispersion in three-dimensional turbulent convection is therefore to verify if the classical Richardson dispersion law \[33\] can be also observed for the present case. Recall that the Richardson dispersion law follows from a solution of a diffusion problem which assumes a homogeneous and isotropic turbulent state. It states that, given two particle tracks, \(x_i(t)\) and \(x_j(t)\) with \(x_i = (x_i, y_i, z_i)\), the distance vector \(R(t) = x_2(t) - x_1(t)\) will follow

\[
\langle R^2(t) \rangle_L = g_{3d} \langle \epsilon \rangle t^3,
\]

where \(g_{3d}\) is a universal constant of \(O(1)\). The symbol \(\langle \cdot \rangle_L\) denotes an average over Lagrangian particle tracks. We decompose the relative tracer motion into a lateral and vertical contribution in order to separate homogeneous and inhomogeneous directions. The distance vector can be written as

\[
R(t) = R_{xy}(t) + R_z(t) e_z.
\]

The lateral two-particle dispersion is given by \(\langle R^2_{xy}(t) \rangle - \langle R^2_{xy}(0) \rangle_L\) where the average is taken over \(6 \times 10^5\) particle pairs. Here,

\[
R_{xy} = |x_2(t) - x_1(t)|e_x + |y_2(t) - y_1(t)|e_y.
\]

FIG. 6: (Color online) Vertical and lateral particle pair dispersion as a function of the initial pair separation. (a) Compensated lateral dispersion for different initial separations as indicated in the legend. The dashed lines correspond with Eq. \[18\] for \(R_0 = \eta_K/2\) and \(R_0 = \eta_K\) and with Eq. \[16\] for \(R_0 = 8\eta_K\) and \(R_0 = 16\eta_K\). (b) Compensated vertical dispersion. Again, Eqs. \[18\] and \[19\] are fitted to the data for the same initial separations. (c) Same data as in (b) without compensation by \(\langle \epsilon \rangle t^3\). The vertical dispersion is constrained between the planes and levels thus off at larger times. The square of the cell height, \(H^2\), and the Kolmogorov time scale, \(\tau_\eta\), are indicated. All axes are given in decadic logarithm. Tracer pairs are seeded initially across the whole volume.

Similarly, the vertical dispersion is given by \(\langle R^2_z(t) \rangle - \langle R^2_z(0) \rangle_L\). The dispersion in each space direction would result with a weight of \(1/3\) in homogeneous isotropic turbulence. In order to compare our pair dispersion results with the predictions for isotropic turbulence, we will introduce two weight factors, \(C_{xy} = 2/3\) for the lateral motion and \(C_z = 1/3\) for the vertical one.

Figure 6 displays both dispersion processes with respect to time for six different initial pair separations as explained in section II C. The two-particle dispersion is given by a compensated plot in panels (a) and (b) of the figure. The graphs are normalized by \(\langle \epsilon \rangle t^3\) to capture a
Richardson-like scaling as a plateau. The initial ballistic behavior at small separations causes then an algebraic decay with $t^{-1}$. Following Sawford et al. [20], we fit the following two relations to our data at small times

$$\frac{\langle R_m^2(t) - R_m^2(0) \rangle_L}{\langle \epsilon \rangle t^3} = \frac{C_m}{3} \left( \frac{R_m(0)}{\eta_K} \right)^2 \frac{\tau_m}{t} \quad (18)$$

if $R_m(0) \ll \eta_K$ and

$$\frac{\langle R_m^2(t) - R_m^2(0) \rangle_L}{\langle \epsilon \rangle t^3} = \frac{11C_m}{3} C \left( \frac{R_m(0)}{\eta_K} \right)^{2/3} \frac{\tau_m}{t} \quad (19)$$

if $\eta_K \ll R_m(0) \ll L$. Here $L$ is the outer scale of turbulence and $C \approx 2$ [20]. Index $m$ stands for the lateral terms, $xy$, or the vertical term, $z$. The agreement with [18] for the initial Kolmogorov and sub-Kolmogorov separations is reasonable. For larger initial separations we use [19]. The larger the initial separation the better agree prediction and data. We fitted the two smallest and largest initial separations only. None of the initial separations is neither much smaller nor much larger than the Kolmogorov scale which explains the slight deviations of the numerical results from the laws [18] and [19].

As discussed for example in Refs. [20][22], the establishment of a Richardson-like regime depends sensitively on the initial separation between the tracers. Indeed, for one of the six different separations the lateral dispersion curve passes through a small plateau with a Richardson constant $g_{xy} \approx 0.25$. This is observed for an initial separation of $R_{xy}(0) = 2\eta_K$, (see solid line in Fig. (a)). The re-translation of the proportionality constant $g_{xy}$ into a three-dimensional homogeneous isotropic turbulence case is obtained by

$$\tilde{g}_{xy} = \frac{g_{xy}}{C_{xy}} = \frac{3}{2} g_{xy} \approx 0.375. \quad (20)$$

The proportionality constant is smaller than the value $\tilde{g}_{xy} \approx 0.5 - 0.6$ for homogeneous isotropic turbulence [18],[19],[20]. In Ref. [24], it was already shown that the PDF of the lateral particle pair distance can be fitted to the stretched exponential form of Richardson [32], however not to the Gaussian shape as suggested by Batchelor [34].

Figures (b) and (c) display the vertical dispersion. In panel (b), we repeat the compensated plot of panel (a) and show the fits to (18). A plateau is observed now for initial separations between $1\eta_K$ and $2\eta_K$. The resulting constant is $g_z \approx 0.05$ (see solid line in Fig. (b)). If one combines the lateral and vertical dispersion, the Richardson constant for the turbulent convection follows to

$$g_{xy} + g_z \approx 0.3, \quad (21)$$

which is less than our earlier estimate of $\tilde{g}_{xy} \approx 0.375$ and $g_{zd} \approx 0.5 - 0.6$. A smaller Richardson constant corresponds with a stronger correlated pair motion. Such behavior can be attributed to the presence of rising and falling thermal plumes - a feature that is absent in isotropic turbulence. Additionally, it is known that the plumes can cluster and form a large scale circulation [2]. Figure (c) demonstrates that the vertical dispersion is constrained by the top and bottom planes. The vertical contribution $\langle R_z^2(t) - R_z^2(0) \rangle_L$ to the pair dispersion levels off. Eventually the lateral dispersion contributes solely to the long-time behavior.

The specifics of the present inhomogeneous flow is that not only the initial pair separation, but also the initial seeding position is important. This brings us to the second series of particle dispersion studies where tracer pairs with fixed distance in different horizontal planes of the slab are seeded (see section II C for details) and shorter
simulations for about half the duration are rerun. Figure 7 summarizes our findings. We picked five initial seeding heights: two in the boundary layer, two in the plume mixing zone and the center plane. While panels (a) and (b) are for $R_0 = \eta_K/2$, panels (c) and (d) are for $R_0 = 2\eta_K$. The latter is the separation that yielded a short Richardson-like range in Fig. 6(a).

Figures 6(a) and (c) show that the lateral dispersion curves of the tracer subgroups differ in magnitude. The local slope is however nearly the same for all subsets. The situation is slightly different for the vertical dispersion: while the seeding in the center plane causes a gradual variation of the local slope of the dispersion curve (see Figs. 6(b) and (d)), the seeding in the thermal boundary layer leads to significant differences after the initial ballistic period. The same result is observed in Fig. 6(d). The reason is that the tracer pairs probe then the detachment of the boundary layer fragments to full extent. This is not the case when starting in the bulk of the cell. It can also be observed that a plateau (which would imply Richardson-like scaling) depends sensitively on the initial separation and seeding height.

To summarize this part, Richardson-like dispersion appears for a very small range of scales in the present flow. Similar to previous studies, we confirm that the initial pair dispersion depends sensitively on both, the initial separation and seeding height.

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B. Multiparticle statistics

The Lagrangian statistics of higher-order moments requires to follow more than two Lagrangian tracers simultaneously. In the following, we will focus to the four-particle case. The tracers are initially seeded at the edges of tetrahedra as discussed in section II C. The distortion of such a small particle cluster by the turbulence has been studied for the pure hydrodynamic case in Refs. 19, 27, 28, 35. The original motivation for such analysis was to get a deeper geometrical insight into the formation of front-like structures in scalar turbulence: in the vicinity of steep scalar gradients small particle clusters become co-planar. Furthermore, since the cluster evolution probes the whole range of scales of turbulence, one hopes to disentangle systematically correlated large-scale advection from decorrelated small-scale motion. The presence of thermal plumes in convection will alter the deformation of the cluster at small times. It is however open, what will be observed in the long-time limit.

The particle tracks $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$ can be transformed into the center-of-mass coordinate

$$r(t) = \frac{1}{4} \sum_{i=1}^{4} x_i(t),$$

and the three relative coordinates (which are of interest here)

$$\rho_1(t) = \frac{1}{\sqrt{2}} [x_2(t) - x_1(t)]$$
$$\rho_2(t) = \frac{1}{\sqrt{6}} [2x_3(t) - x_1(t) - x_2(t)]$$
$$\rho_3(t) = \frac{1}{\sqrt{12}} [3x_4(t) - x_1(t) - x_2(t) - x_3(t)].$$

The radius of gyration follows in this frame to $R_g = \sqrt{\rho_1^2 + \rho_2^2 + \rho_3^2}$. The shape evolution of the particle cluster is monitored by the following moment-of-inertia tensor

$$g^{ab} = \sum_{i=1}^{3} \rho_i^{a} \rho_i^{b},$$

where $a, b = x, y, z$ is the component index of the vector $\rho_i$ and $i = 1, 2, 3$. The real eigenvalues $g_1 \geq g_2 \geq g_3 \geq 0$ quantify the shape of the particle cluster. Isotropic objects correspond with $g_1 = g_2 = g_3$, cigar-shaped clusters with $g_1 \gg g_2 \approx g_3$ and pancake-shaped clusters with $g_1 \approx g_2 \gg g_3$. Figure 8 shows the time evolution of the three eigenvalues for one specific 4-particle cloud. The eigenvalues are normalized and given by

$$I_k = \frac{g_k}{\sum_{m=1}^{3} g_m} \quad \text{for} \quad k = 1, 2, 3.$$

Thus $0 \leq I_k \leq 1$. One can observe, that the eigenvalue variations become smoother with increasing time.

![Figure 8](image-url)
A convergence of the cluster to an almost coplanar object is observable for larger times, as quantified by the small value of $I_3$. It will turn out now that this example displays a typical long-time behavior.

In Fig. 9 we show the time evolution of the Lagrangian ensemble average of the normalized eigenvalues. Data for different initial sidelengths of the tetrahedra are compared. The tetrahedra are seeded across the whole volume. Similar to the two-point measure, the initial deformation of the clusters depends sensitively on the side-length of the tetrahedron. The smaller the initial side-length the stronger the initial stretching of the cluster to a cigar-shaped object (for $t \lesssim 15\tau_n$). After about $87\tau_n$ all curves collapse and the mean values remain almost unchanged. Our data yield $\langle I_1 \rangle_L = 0.84$, $\langle I_2 \rangle_L = 0.15$, and $\langle I_3 \rangle_L = 0.01$. Surprisingly, the obtained mean values are very close to the findings of Biferale et al. [19] and Hackl et al. [36] for homogeneous isotropic turbulence. In Ref. [19] two-point measure was excluded that had two points too close or too far of each other such that their values are not directly comparable with the present ones. Recent three-dimensional particle tracking experiments by Lüthi et al. [37] report an $\langle I_2 \rangle_L$ which is also close to 0.16.

Since the large scales are probed in the long-term limit by the particle clusters, this agreement suggests that convective and isotropic turbulence on this scale do not differ significantly. The relative motion within a cluster is insensitive to whether the tracer particles are swept by large vortex structures or by large-scale circulation. Consequently, our findings suggest that the constrained vertical motion (and thus the inhomogeneity) is not important in the diffusive long-time limit of the cluster dynamics.

One can expect that for times $t > 10^2 \tau_n$, the tracers advance independently of each other. The present long-time means are compared with the result of a joint Gaussian distribution of the relative coordinates, $p(\rho_1, \rho_2, \rho_3) \sim \exp[-(\rho_1^2 + \rho_2^2 + \rho_3^2)]$. This ansatz results to $\langle I_1 \rangle_G = 0.75$, $\langle I_2 \rangle_G = 0.22$, and $\langle I_3 \rangle_G = 0.03$ for the three-dimensional case which is obtained by Monte-Carlo simulations [28]. The reported mean of $\langle I_2 \rangle_L = 0.15$ is smaller than the Gaussian value.

Figure 10 reports the dependence of the shape evolution from the initial position $z_0$. The effects remain small, but systematic. The closer the starting position of the tetrahedron to the boundary plane, the faster it converges into the final quasistatic state. Again the stretching and deformation is most efficient when the particle cluster passes through the mixing zone right above the thermal boundary layer.

Figure 11 displays the PDFs of $I_k$ for five instants $t > 87\tau_n$. The plots highlight two aspects. First, there is still a big variety in the amplitudes of individual $I_k$ although their means remain almost unchanged. Second, a very slow drift in the tails is present which is indicated by the arrows in Fig. 8(a) and (c).

V. ACCELERATION STATISTICS

The top panel of Fig. 12 shows the PDFs of the three acceleration components. Each component is given in units of the corresponding root-mean-square value. As expected, the distributions of the two lateral components coincide. Table 1 provides the quantitative details of the acceleration statistics and lists for example the skewness $S(a_k) = \langle a_k^3 \rangle / \langle a_k^2 \rangle^{3/2}$ and the flatness $F(a_k) = \langle a_k^4 \rangle / \langle a_k^2 \rangle^2$ with $k = x, y$ or $z$. The numbers for the lateral components are almost identical. The vertical acceleration component has a smaller flatness which is in line with a sparser tail of the corresponding PDF. The bottom panel of the same figure provides the statistical convergence test of the fourth-order moments where the product $w^4 p(w)$ is plotted vs. $w$ with $w = a_k / a_{k, rms}$. It reflects the fundamental difficulty to gather reliable statistics for higher-order moments in Lagrangian turbu-
can rationalize the large lateral accelerations. The sup-
anti-cyclones close to the thermal boundary layer which
section that the vorticity is concentrated in cyclones and
lateral accelerations and the fattest tails are found close
to the top and bottom planes. It will turn out in the next
order moments display almost no scatter (not shown).
Maxima and mininma are given in units of the gravity
acceleration $g$.

Recall that this analysis is conducted over a set of $4.4 \times 10^8$ events. The area which is occupied by the scatter of the graphs in the tails of the PDFs determines the error bar of the fourth-order moment (and consequently of the flatness). We checked that the second and third-order moments display almost no scatter (not shown). The issue of statistical convergence has been discussed for turbulence measurements in a swirling flow \[38, 39\] and in numerical simulations of homogeneous isotropic turbulence \[43\]. Our values for the flatness $F(a_k)$ of the lateral flatness are of the same magnitude as those reported in \[38\].

Figure 13 refines the statistical analysis of the acceleration components. Due to the vertical inhomogeneity, we report the height dependence of the acceleration statistics for one lateral and the vertical component, respectively, and plot contours of the joint PDF $p(a_z, z)$. The largest lateral accelerations and the fattest tails are found close to the top and bottom planes. It will turn out in the next section that the vorticity is concentrated in cyclones and anti-cyclones close to the thermal boundary layer which can rationalize the large lateral accelerations. The support of the PDF decreases monotonically to the center plane. We will get back to this point later in the text when discussing the role of the vertical vorticity component in connection with plume detachments. In contrast to the result for the lateral accelerations, the support of the joint PDF $p(a_z, z)$ shows no significant variation with height. It shrinks to zero in the boundary planes (since is $u_z \equiv 0$) and grows rapidly up to about the thermal boundary layer thickness. The slight asymmetry of the inner contour lines (for the largest probability density levels) corresponds with the rising plumes which detach from the bottom plane at $z = 0$ and have $a_z > 0$ and with falling plumes at $z = H$ for which $a_z < 0$. Note also that the support of all pdfs is the same in the center of the

![FIG. 11: (Color online) Probability density functions of the normalized eigenvalues $I_k$ of the moment-of-inertia tensor (a) $I_1$, (b) $I_2$, and (c) $I_3$. The time instants at which the data have been analysed are for $t > 87 \tau_0$ as given in the legend in (a). The vertical dashed lines mark the long-time averages $\langle I_k \rangle_L$. The arrows in the upper and lower panel indicate that there is still a slight drift in the tails although the means in Fig. 10 remain nearly unchanged. The tetrahedra are seeded again in the whole volume initially.](image1)

![FIG. 12: (Color online) Probability density function (PDF) of acceleration components $a_k$ with $k = x, y$ and $z$. Upper panel: PDF plots. Each component is normalized by its corresponding root-mean-square value. Lower panel: Statistical convergence test for the 4th moment of the acceleration. The total number of events which as been included for the analysis are $4.4 \times 10^8$.](image2)

| $\sqrt{\langle a_k^2 \rangle}$ | $\max(a_k)$ | $\min(a_k)$ | $S(a_k)$ | $F(a_k)$ |
|-----------------------------|------------|------------|--------|--------|
| $a_x$                       | 1.17       | 1001       | -574   | 0.093  | 63.4 ($\pm$ 16) |
| $a_y$                       | 1.16       | 559        | -540   | -0.156 | 64.4 ($\pm$ 16) |
| $a_z$                       | 1.10       | 518        | -680   | -0.119 | 30.3 ($\pm$ 11) |

TABLE I: Root-mean-square values, total maximum/minimum amplitudes, skewness $S(a_k) = \langle a_k^3 \rangle / \langle a_k^2 \rangle^{3/2}$ and flatness $F(a_k) = \langle a_k^3 \rangle / \langle a_k^2 \rangle^2$ of the acceleration components. The error bars of $F(a_k)$ have been obtained by measuring the area of the scatter in the lower panel of Fig. 12. Maxima and minima are given in units of the gravity acceleration $g$. |
The important result of this section is that there is different strong intermittency for the vertical and lateral accelerations in thermal convection. It is caused by the higher level of intermittency in and close to the thermal boundary layer. As a consequence, we will have to take a closer look at the mechanisms of local heat transfer in the vicinity of the top and bottom planes. This is done in the next section.

VI. LAGRANGIAN CONVECTIVE AND CONDUCTIVE HEAT FLUX

A. Lagrangian heat flux

The local heat flux contributions can be probed in the Lagrangian frame of reference. We adopt therefore definition \[10\] and calculate the local Lagrangian conductive and convective heat contributions. They are given by

\[
Nu(x) = Nu_{\text{conv}}(x) + Nu_{\text{cond}}(x)
\]

\[
= \frac{H}{\kappa \Delta T} (u_z(x)T(x) - \kappa \partial_z T(x)) ,
\]

along the Lagrangian tracks \(x(t)\). Figure 14 shows typical time traces of \(Nu_{\text{conv}}\) and \(Nu_{\text{cond}}\) along two tracers. They display a big variability with respect to time. Even negative values are possible for both contributions. As expected, the convective term has a significantly larger magnitude than the conductive one. The latter is dominant in the thermal boundary layers where the thermal dissipation rate \(\chi\) has the largest magnitude. Figure 15 displays the PDFs of the convective and conductive contributions gathered along the tracks of the whole tracer ensemble. In addition, we display the results for the products \(u_z T\) and \(u_y T\). All quantities are shown in units of their root-mean-square values. While the PDFs for \(u_z T\) and \(u_y T\) are symmetric, those of \(u_x T\) and \(\kappa \partial_z T\) are strongly skewed. This reflects the vertical net transfer of heat through the volume. The Lagrangian average \(Nu_L = \langle Nu(x) \rangle_{L,t}\) results to

\[
Nu_L = 1 + \frac{H}{\kappa \Delta T} (u_z T)_{L,t} .
\]

We have directly verified from the corresponding PDF in Fig. 15(c) that the mean of the Lagrangian conductive heat transfer is with 1.04 very close to one. Furthermore, it is found that \(Nu_L < Nu\). This is in contrast to the experimental findings for the smart particle probe of Gasteuil et al. [21] where \(Nu < Nu_L\). The reason for this difference might be due to the finite extension of the thermal boundary layer. We have compared the result for the full record (F) with that of a smaller subset (S) which is one third of set (F) and the results differed by 6.5 % only (see also Fig. 15). A time-resolved analysis shows that \(Nu_L(t) = \langle Nu(x,t) \rangle_L\) relaxes slowly to the Eulerian value of \(Nu\). The tracers which have been seeded randomly or at particular heights at the beginning have to pass a kind of "thermalization" process.

Our result sheds interesting light on the joint velocity-temperature sampling properties of the Lagrangian tracers. First, it is known that for the present geometry a large-scale circulation is present [40, 41]. Tracers will preferentially follow the circulation motion in the convection layer. The large-scale circulation can carry a fraction of the total heat transport only, as has been analyzed recently with a Proper Orthogonal Decomposition [42].
FIG. 15: (Color online) Probability density functions (PDF) of the conductive and conductive heat flux contributions along
the Lagrangian tracer tracks. (a) Lateral convective contributions \(u_x T\) and \(u_y T\). (b) Vertical convective contribution \(u_z T\). (c) Conductive contribution \(-\partial T/\partial z\). All quantities are normalized by their root-mean-square values. The analysis is conducted over two different data sets: the full record (F) and a third of it (S).

Second, we observe high-amplitude events of the vertical vorticity \(\omega_z = \partial_x u_y - \partial_y u_x\) close to the maxima of the thermal dissipation rate \(\chi\). This is shown in Figs. 16 and 17 where contour plots of slice snapshots of \(\chi\) and \(\omega_z\) at a height \(z = \delta T\) are compared. The local maxima in the dissipation rate plot (16) reproduce the skeleton of plume sheets. In their vicinity, we observe cyclones and anti-cyclones that are generated in connection with

FIG. 16: (Color online) Contour plot of the decadic logarithm of the thermal dissipation rate \(\log_{10}(\chi(x, y, z = \delta T, t_0))\). The logarithmic contour spacing is chosen in order to highlight the small amplitude events.

FIG. 17: (Color online) Contour plot of the vertical vorticity component \(\omega_z(x, y, z = \delta T, t_0)\). Data correspond to those in Fig. 16.

the detachment of plume fragments. This observation is in line with results in \[\text{Ref.}\] for the non-rotating and with \[\text{Ref.}\] for the rotating case. Exactly these cyclones and anti-cyclones cause the large lateral positive and negative accelerations as seen in the PDFs in Fig. 12. Fig. 18 plots vertical profiles of means of both quantities. While the root-mean-square of the vertical vorticity component varies weakly, the thermal dissipation rate is strongly peaked in and close to the boundary layer. It is also known from studies in homogeneous isotropic turbulence \[\text{Ref.}\] that the Lagrangian tracers are not very frequently trapped in the core of such vortex structures.
this analysis and study the correlations between the vertical velocity component and the (total) temperature in relation to the acceleration. Figure 19 displays the joint statistics of the vertical and lateral accelerations and the products of velocity and temperature fluctuations, \( u_z T \).

In order to highlight the statistical correlation between the two variables of the joint PDF we divide the joint PDF by the two single quantity PDFs,

\[
\Pi(a_z, u_z T) = \frac{p(a_z, u_z T)}{p(a_z)p(u_z T)}.
\]

The top panel of Fig. 19 shows the joint statistics for \( a_z \) and \( u_z T \). A pronounced maximum at larger accelerations and values of \( u_z T \) is found. They can be related to coherent structures, such as vorticity tubes in the bulk of the slab or the cyclones/anti-cyclones in the boundary layer. The correlation between vertical acceleration and the product \( u_z T \) is weaker. The local amplitudes of the joint PDF found at the outer boundaries of the support, at moderate acceleration amplitudes and larger values of \( u_z T \). In Ref. [24], we reported the same behavior for \( p(a_z, u_z T') \) and identified rising plumes \( (a_z > 0, u_z T' > 0) \) and falling plumes \( (a_z < 0, u_z T' > 0) \). Recirculations around rising and falling plumes have to form due to the incompressibility of the fluid. They were related to maxima in the halfplane \( u_z T' < 0 \). These are only some of the possible scenarios which can be assigned to strong correlations in the joint statistics. The firm conclusion which we can draw from the present analysis and the one in [24] is that plumes (and therefore the vertical convective flux events) are not connected with the largest vertical accelerations. The detachment of plumes is a more gradual process.

We repeated the joint statistical analysis for the conductive part, \( -\kappa \partial_z \bar{T}(x) \). Results are summarized in Fig. 20. The contour plots display now

\[
\Pi(a_z, -\partial_z \bar{T}) = \frac{p(a_z, -\partial_z \bar{T})}{p(a_z)p(-\partial_z \bar{T})}.
\]

We keep in mind from [10] and [26] that upward conductive heat transport events have a negative sign, i.e. \(-|\partial_z \bar{T}|\). While the lower panel of Fig. 20 includes the Lagrangian data of the whole volume, the upper panel of the same figure excludes events in the thermal boundary layer up to the beginning of the plume mixing zones. This zone was identified and studied in [24] and starts for the present parameter setting at a height \( H/16 \). The extended tail in the lower panel can thus be related to largest gradients (and thus largest thermal dissipation rate amplitudes) in and above the thermal boundary layer. The asymmetry to negative \( a_z \) for the largest gradients can be interpreted as tracer decelerations which are present when temperature gradients are formed. Negative accelerations (or decelerations) seem to be frequently related to stagnation-point flow topologies, those flows which can steepen the temperature field to large gradients.

B. Joint statistics of Lagrangian heat flux and acceleration

The thermal plumes detach permanently from the thermal boundary layer and can be identified as regions in which the product \( u_z T' > 0 \) [21, 24]. Here we extend this analysis and study the correlations between the vertical velocity component and the (total) temperature in relation to the acceleration. Figure 19 displays the joint statistics of the vertical and lateral accelerations and the products of velocity and temperature fluctuations, \( u_z T \).

In order to highlight the statistical correlation between the two variables of the joint PDF we divide the joint PDF by the two single quantity PDFs,

\[
\Pi(a_z, u_z T) = \frac{p(a_z, u_z T)}{p(a_z)p(u_z T)}.
\]

The top panel of Fig. 19 shows the joint statistics for \( a_z \) and \( u_z T \). A pronounced maximum at larger accelerations and values of \( u_z T \) is found. They can be related to coherent structures, such as vorticity tubes in the bulk of the slab or the cyclones/anti-cyclones in the boundary layer. The correlation between vertical acceleration and the product \( u_z T \) is weaker. The local amplitudes of the joint PDF found at the outer boundaries of the support, at moderate acceleration amplitudes and larger values of \( u_z T \). In Ref. [24], we reported the same behavior for \( p(a_z, u_z T') \) and identified rising plumes \( (a_z > 0, u_z T' > 0) \) and falling plumes \( (a_z < 0, u_z T' > 0) \). Recirculations around rising and falling plumes have to form due to the incompressibility of the fluid. They were related to maxima in the halfplane \( u_z T' < 0 \). These are only some of the possible scenarios which can be assigned to strong correlations in the joint statistics. The firm conclusion which we can draw from the present analysis and the one in [24] is that plumes (and therefore the vertical convective flux events) are not connected with the largest vertical accelerations. The detachment of plumes is a more gradual process.

We repeated the joint statistical analysis for the conductive part, \( -\kappa \partial_z \bar{T}(x) \). Results are summarized in Fig. 20. The contour plots display now

\[
\Pi(a_z, -\partial_z \bar{T}) = \frac{p(a_z, -\partial_z \bar{T})}{p(a_z)p(-\partial_z \bar{T})}.
\]

We keep in mind from [10] and [26] that upward conductive heat transport events have a negative sign, i.e. \(-|\partial_z \bar{T}|\). While the lower panel of Fig. 20 includes the Lagrangian data of the whole volume, the upper panel of the same figure excludes events in the thermal boundary layer up to the beginning of the plume mixing zones. This zone was identified and studied in [24] and starts for the present parameter setting at a height \( H/16 \). The extended tail in the lower panel can thus be related to largest gradients (and thus largest thermal dissipation rate amplitudes) in and above the thermal boundary layer. The asymmetry to negative \( a_z \) for the largest gradients can be interpreted as tracer decelerations which are present when temperature gradients are formed. Negative accelerations (or decelerations) seem to be frequently related to stagnation-point flow topologies, those flows which can steepen the temperature field to large gradients.
FIG. 20: (Color online) Joint probability density function of the vertical acceleration component $a_z/a_{z,rms}$ and $w = -\partial_z T/(\partial_z T)_{rms}$. Top panel: Tracer positions in the bulk have been considered only. Bottom panel: Tracer positions in the whole volume are included. The contour spacing is in decadic logarithm.

VII. SUMMARY AND DISCUSSION

The focus of the present work was on Lagrangian aspects of turbulent convection. The results can be summarized as follows. The study of pair and multiparticle dispersion yields qualitatively similar results compared to the homogeneous isotropic case. Although the scaling behavior of second order moments is sensitive to initial separations and seeding heights, initial separations close to the Kolmogorov length result in a short Richardson-like scaling range. Interestingly, we reproduce the same long-time limits for the particle cluster shapes as in isotropic turbulence despite turbulent convection is inhomogeneous. This limit deviates from the Gaussian value. Our results suggest that the dispersion laws can be obtained for more complex flows than isotropic homogeneous turbulence. The proportionality constants are however different which can be attributed to qualitatively different turbulence structures such as thermal plumes in convection. They affect the vertical dispersion more significantly than the lateral dispersion.

The inhomogeneity of the convective turbulence manifests in less intermittent statistics of the vertical acceleration component compared to the lateral ones. Thermal plumes are not coupled with the strongest accelerations.

We find that the Lagrangian Nusselt number converges slowly to the Eulerian value. A closer inspection of this point could be done in several steps: first, to disentangle the large-scale circulation from the turbulent background, as done in [12], and to combine such a study with a Lagrangian analysis. Second, the present study indicates also that the Nusselt number relaxes faster with a growing number of tracers, in other words the sampling of joint velocity-temperature statistics improves. Our observation is also related to recent experimental and numerical studies at very large Rayleigh numbers in which the existence and growth of a so-called superconducting core is discussed, which is in line with a decreasing importance of the large flow circulation for growing Rayleigh numbers. A decreasing importance of a coherent large-scale circulation might lead again to a faster convergence $N_{u_L} \rightarrow N_u$. More studies of the Lagrangian frame of high-Rayleigh number turbulence are thus necessary.

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