Eigenfunction Statistics for a Point Scatterer on a Three-Dimensional Torus

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Abstract. In this paper, we study eigenfunction statistics for a point scatterer (the Laplacian perturbed by a delta-potential) on a three-dimensional flat torus. The eigenfunctions of this operator are the eigenfunctions of the Laplacian which vanish at the scatterer, together with a set of new eigenfunctions (perturbed eigenfunctions). We first show that for a point scatterer on the standard torus all of the perturbed eigenfunctions are uniformly distributed in configuration space. Then we investigate the same problem for a point scatterer on a flat torus with some irrationality conditions, and show uniform distribution in configuration space for almost all of the perturbed eigenfunctions.

1. Introduction

One of the key results in the field of Quantum Chaos is Schnirelman’s quantum ergodicity theorem [2,11,13], which asserts that quantum systems whose classical counterpart have chaotic dynamics are quantum ergodic, in the sense that for almost all eigenstates, the expectation values of observables converge to the phase space average, i.e. almost all eigenstates are equidistributed in phase space. An important case is when all expectation values converge to the phase space average—such behavior is called quantum unique ergodicity.

The opposite of chaotic systems in classical mechanics are systems with integrable dynamics, whose behavior is predictable over a long period of time. In this paper, we study eigenfunction statistics for a point scatterer on a three-dimensional flat torus, which is an intermediate system—its classical dynamics is close to integrable, yet the quantum system is substantially influenced by the scatterer, and therefore, shares some of the behavior of classically chaotic quantum systems. We study quantum ergodicity and quantum unique ergodicity in configuration space (rather than in full phase space), a notion which
is of growing interest in recent research, for example, in the field of control theory (cf. [8]).

A point scatterer is formally described by a quantum Hamiltonian

\[- \Delta + \alpha \delta_{x_0}\]

(1.1)

where \(\delta_{x_0}\) is the Dirac mass at \(x_0\) and \(\alpha\) is a coupling parameter. Mathematically it is realized as a self-adjoint extension of the Laplacian \(-\Delta\) acting on functions vanishing near \(x_0\) (see Sect. 2). Such extensions are parametrized by a phase \(\phi \in (-\pi, \pi]\), where \(\phi = \pi\) corresponds to the standard Laplacian (\(\alpha = 0\) in (1.1)). For \(\phi \neq \pi\), the eigenfunctions of the corresponding operator consist of eigenfunctions of the Laplacian which vanish at the scatterer, and new eigenfunctions (perturbed eigenfunctions).

In two dimensions, Rudnick and Ueberschär [10] proved quantum ergodicity in configuration space regarding the perturbed eigenfunctions of a point scatterer on the flat torus \(T^2 = \mathbb{R}^2/\mathbb{Z}L_0\) where \(L_0 = \mathbb{Z}(1/a, 0) \oplus \mathbb{Z}(a, 0)\) is (any) unimodular lattice, i.e. they proved that almost all of the perturbed eigenfunctions are uniformly distributed in configuration space. Our goal is to prove a similar result for a point scatterer on the three-dimensional torus, showing uniform distribution in configuration space for almost all (and hopefully all) of the perturbed eigenfunctions.

We remark that in dimensions four and greater, the Laplacian \(-\Delta\) acting on functions vanishing near \(x_0\) is essentially self-adjoint, so there are no non-trivial self-adjoint extensions in those cases.

The three-dimensional problem provides some essential differences from the two-dimensional case. For example, Weyl’s law for the three-dimensional torus, establishing the asymptotics of the counting function \(N(x)\) of eigenvalues of the Laplacian below \(x\), reads as \(N(x) \sim Cx^{3/2}\) for some constant \(C\), while in two dimensions we have \(N(x) \sim Cx\), so we deduce completely different bounds for the density of the perturbed eigenvalues in each case. Moreover, there are major differences in the behavior of the eigenvalues of the Laplacian on different three-dimensional tori (and therefore in the behavior of the perturbed eigenfunctions), so instead of a general theorem, we will investigate two main cases: the case of the standard three-dimensional flat torus, and the case of an irrational torus where the multiplicities of the corresponding eigenvalues of the Laplacian are bounded.

In the case of the standard torus \(T^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3\), the eigenvalues of the Laplacian are the integers which are sums of three squares, and the multiplicity of each eigenvalue is its number of representations as such sums, so we can use some arithmetic properties of sums of three squares and their number of representations to show a stronger result—we show that for a point scatterer on the standard torus, all of the perturbed eigenfunctions are uniformly distributed in configuration space. More precisely, for every \(\phi \in (-\pi, \pi)\), we will have a set of perturbed eigenvalues \(\Lambda_\phi\), with the corresponding \(L^2\)-normalized eigenfunctions \(g_\lambda (\lambda \in \Lambda_\phi)\). We prove the following theorem:
Theorem 1.1. Let $\mathbb{T}^3 = \mathbb{R}^3/2\pi \mathbb{Z}^3$ be the standard torus. Fix $\phi \in (-\pi, \pi)$. Then for all observables $a \in C^\infty (\mathbb{T}^3)$, \footnote{Consequently, Theorem 1.1 still holds for all observables which are Riemann integrable on $\mathbb{T}^3$. The same is true for Theorem 1.3.}

$$\int_{\mathbb{T}^3} a(x) |g_\lambda(x)|^2 \, dx \to \frac{1}{\text{area}(\mathbb{T}^3)} \int_{\mathbb{T}^3} a(x) \, dx$$

as $\lambda \to \infty$ along $\Lambda_\phi$.

Then we show a similar result for a point scatterer on an irrational torus, but with convergence only along a density one set in the set of the perturbed eigenvalues: consider the family of flat tori $\mathbb{T}^3 = \mathbb{R}^3/2\pi \mathbb{L}_0$, where

$$\mathbb{L}_0 = \mathbb{Z}(a,0,0) \oplus \mathbb{Z}(b,0) \oplus \mathbb{Z}(0,c)$$

is a lattice such that $1/a^2, 1/b^2, 1/c^2 \in \mathbb{R}$ are independent over $\mathbb{Q}$. We also demand that at least one of the ratios $b^2/a^2, c^2/a^2, c^2/b^2$ will be an irrational of finite type, as in the following definition.

Definition 1.2. An irrational $\alpha$ is said to be of finite type $\tau \in \mathbb{R}$, if $\tau$ is the supremum of all $\gamma$ for which $\lim_{q \to \infty} q^\gamma \|q\alpha\| = 0$, where $q$ runs through the positive integers. Here

$$\|t\| = \min_{n \in \mathbb{Z}} |t - n| = \min (\{t\}, \{-t\})$$

denotes the distance from $t$ to the nearest integer.

In particular, if $\alpha$ is an irrational of finite type $\tau$, then for every $\varepsilon > 0$, there exists a positive constant $c = c(\alpha, \varepsilon)$ such that $\|q\alpha\| \geq \frac{c}{q^{\tau+\varepsilon}}$ holds for all positive integers $q$. Also note that by Dirichlet’s Theorem we must have $\tau \geq 1$, and every algebraic irrational is of type 1 due to the theorem of Roth [9].

As in the case of the standard torus, for every $\phi \in (-\pi, \pi)$, we will have a set of perturbed eigenvalues $\Lambda_\phi$, with the corresponding $L^2$-normalized eigenfunctions $g_\lambda$, and we prove:

Theorem 1.3. Let $\mathbb{T}^3 = \mathbb{R}^3/2\pi \mathbb{L}_0$ be an irrational torus as defined above. Fix $\phi \in (-\pi, \pi)$. There is a subset $\Lambda_{\phi,\infty} \subseteq \Lambda_\phi$ of density one so that for all observables $a \in C^\infty (\mathbb{T}^3)$,

$$\int_{\mathbb{T}^3} a(x) |g_\lambda(x)|^2 \, dx \to \frac{1}{\text{area}(\mathbb{T}^3)} \int_{\mathbb{T}^3} a(x) \, dx$$

as $\lambda \to \infty$ along $\Lambda_{\phi,\infty}$.

2. Background

2.1. Point Scatterers on the Flat Torus

Let $\mathbb{T}^3 = \mathbb{R}^3/2\pi \mathbb{L}_0$ be a flat three-dimensional torus, where

$$\mathbb{L}_0 = \mathbb{Z}(a,0,0) \oplus \mathbb{Z}(b,0) \oplus \mathbb{Z}(0,c)$$

is a lattice.
We want to study the Schrödinger operator with a delta-potential on the flat three-dimensional torus $T^3$, formally given by

$$-\Delta + \alpha \delta_{x_0} \quad (2.1)$$

where $\Delta$ is the associated Laplacian on $T^3$, $\delta_{x_0}$ is the Dirac delta-function at the point $x_0$, and $\alpha \in \mathbb{R}$ is a coupling parameter.

We now give a rigorous mathematical description for the operator $(2.1)$ following [1,10]:

Consider the domain of $C^\infty$-functions which vanish in a neighborhood of $x_0$:

$$D_0 = C^\infty_0 (T^3 \setminus \{x_0\})$$

and denote $-\Delta_{x_0} = -\Delta|_{D_0}$, which is an operator in the Hilbert space $L^2(T^3)$. One finds that the adjoint of $-\Delta_{x_0}$ has as its domain $\text{Dom} (-\Delta_{x_0}^*)$ the Sobolev space $H^2(T^3 \setminus \{x_0\})$, which equals the space of $f \in L^2(T^3)$ for which there is some $A \in \mathbb{C}$ such that

$$-\Delta f + A\delta_{x_0} \in L^2(T^3).$$

For such $f$, there is some $B \in \mathbb{C}$ so that

$$f(x) = A \cdot \frac{-1}{4\pi |x-x_0|} + B + o(1), \quad x \to x_0.$$

One finds that the self-adjoint extensions of $-\Delta_{x_0}$ are parametrized by a phase $\phi \in (-\pi, \pi]$; denoting the corresponding operators by $-\Delta_{\phi,x_0}$, their domain is given by $f \in \text{Dom} (-\Delta_{\phi,x_0}^*)$ for which there is some $a \in \mathbb{C}$ so that

$$f(x) = a \left( \cos \frac{\phi}{2} \cdot \frac{-1}{4\pi |x-x_0|} + \sin \frac{\phi}{2} \right) + o(1), \quad x \to x_0.$$

The action of $-\Delta_{\phi,x_0}$ on $f \in \text{Dom} (-\Delta_{\phi,x_0})$ is then given by

$$-\Delta_{\phi,x_0} f = -\Delta f + A\delta_{x_0} = -\Delta f + a \cos \frac{\phi}{2} \delta_{x_0}. \quad (2.2)$$

Note that for $\phi = \pi$ we have

$$\text{Dom} (-\Delta_{\pi,x_0}) = H^2(T^3) = \{ f \in L^2(T^3) : -\Delta f \in L^2(T^3) \}$$

and

$$-\Delta_{\pi,x_0} f = -\Delta f$$

so this extension retrieves the standard Laplacian $-\Delta_\infty$ on the domain $H^2(T^3)$ (which is the unique self-adjoint extension of $-\Delta_{|C^\infty(T^3)}$).

The operator $-\Delta_\infty$ has a discrete spectrum; an orthonormal basis of eigenfunctions for $-\Delta_\infty$ consists of the functions

$$\frac{1}{\sqrt{\text{area}(T^3)}} e^\xi$$

where

$$e^\xi = \exp(i\xi \cdot (x-x_0))$$
and $\xi$ ranges over the dual lattice

$$
\mathcal{L} = \{ \xi \in \mathbb{R}^3 : \xi \cdot l \in \mathbb{Z} \ \forall l \in \mathcal{L}_0 \} = \mathbb{Z} \left( \frac{1}{a}, 0, 0 \right) \oplus \mathbb{Z} \left( 0, \frac{1}{b}, 0 \right) \oplus \mathbb{Z} \left( 0, 0, \frac{1}{c} \right).
$$

The corresponding eigenvalues are the norms $|\xi|^2$ of the vectors of the dual lattice $\mathcal{L}$; denote by $\mathcal{N}$ the set of these norms. In the case of the standard torus $\mathcal{L}_0 = \mathbb{Z}^3$ (and then $\mathcal{L} = \mathbb{Z}^3$) we have $\mathcal{N} = \mathcal{N}_3$, where $\mathcal{N}_3$ is the set of integers which are sums of three squares, and each eigenvalue is of multiplicity $r_3(n)$ which is the number of representations of $n = a^2 + b^2 + c^2$ with $a, b, c \in \mathbb{Z}$ integers.

For the perturbed operator (2.2) with $\phi \neq \pi$ we still have the nonzero eigenvalues from the unperturbed problem ($0 \neq \lambda \in \sigma(-\Delta_\infty)$), with multiplicity decreased by one, as well as a new set $\Lambda = \Lambda_\phi$ of eigenvalues, each appearing with multiplicity one, which are the solutions to the equation

$$
\sum_{\xi \in \mathcal{L}} \left\{ \frac{1}{|\xi|^2 - \lambda} - \frac{|\xi|^2}{|\xi|^4 + 1} \right\} = c_0 \tan \frac{\phi}{2} \tag{2.3}
$$

where

$$
c_0 = \sum_{\xi \in \mathcal{L}} \frac{1}{|\xi|^4 + 1}
$$

with the corresponding eigenfunctions being multiples of the Green’s function

$$
G_\lambda(x, x_0) = (\Delta + \lambda)^{-1} \delta_{x_0}
$$

which is an element of $\text{Dom}(-\Delta_{x_0}^*)$ for every $\lambda \notin \sigma(-\Delta_\infty)$, and has the $L^2$-expansion

$$
G_\lambda(x, x_0) = -\frac{1}{8\pi^3} \sum_{\xi \in \mathcal{L}} \frac{\exp(i\xi \cdot (x - x_0))}{|\xi|^2 - \lambda}.
$$

(2.3) can be written as

$$
\sum_{n \in \mathcal{N}} r_{\mathcal{L}}(n) \left\{ \frac{1}{n - \lambda} - \frac{n}{n^2 + 1} \right\} = c_0 \tan \frac{\phi}{2}
$$

where

$$
r_{\mathcal{L}}(n) = \# \{ \xi \in \mathcal{L} : |\xi|^2 = n \}
$$

is the multiplicity of the norm $n$. The function

$$
F(\lambda) = \sum_{n \in \mathcal{N}} r_{\mathcal{L}}(n) \left\{ \frac{1}{n - \lambda} - \frac{n}{n^2 + 1} \right\}
$$

is meromorphic with simple poles in $n \in \mathcal{N}$, and $F|\mathbb{R}$ is strictly increasing between the poles, so if we label

$$
\mathcal{N} = \{ 0 = n_0 < n_1 < n_2 < \cdots \}
$$
then the new eigenvalues interlace between the elements of $N$, and we may denote the perturbed eigenvalues by $\lambda_k = \lambda_k^\phi$ so that

$$\lambda_0 < n_0 < \lambda_1 < n_1 < \lambda_2 < \cdots < n_k < \lambda_{k+1} < n_{k+1} < \ldots .$$

We say that a subset $\Lambda' = \{\lambda_j \} \subseteq \Lambda$ is of density $a (0 \leq a \leq 1)$ in $\Lambda$ if

$$\lim_{J \to \infty} \frac{1}{J} \# \{ k \in \mathbb{N} : j_k \leq J \} = a$$

or equivalently

$$\lim_{X \to \infty} \frac{\# \{ \lambda \in \Lambda' : \lambda \leq X \}}{\# \{ \lambda \in \Lambda : \lambda \leq X \}} = a.$$ 

Denote by $g_\lambda (x) := \frac{G_\lambda (x, x_0)}{\|G_\lambda\|_2}$ the $L^2$-normalized Green’s function.

2.2. Arithmetic Background

In this section we recall some basic arithmetic facts, that we will need to use in the proof of Theorem 1.1 for the standard torus.

By the famous theorem due to Legendre and Gauss (see [4] for example), the Diophantine equation

$$x_1^2 + x_2^2 + x_3^2 = n \quad (2.4)$$

has solutions in integers $x_i (i = 1, 2, 3)$ if and only if $n$ is not of the form $4^a (8k + 7)$ with $a \in \mathbb{Z}$, $a \geq 0$ and $k \in \mathbb{Z}$. Denote by $r_3 (n)$ the number of solutions to (2.4), then for all $n$, $r_3 (4^a n) = r_3 (n)$.

Equivalently, if we write $n = 4^a n_1$, with $4 \nmid n_1$, then $n$ is a sum of three squares if and only if $n_1 \not\equiv 7 (8)$, that is to say

$$N_3 = \{ n \in \mathbb{N} : n = 4^a n_1, 4 \nmid n_1 \Rightarrow n_1 \not\equiv 7 (8) \},$$

and $r_3 (n) = r_3 (n_1)$.

The fact that for all $n$: $r_3 (4^a n) = r_3 (n)$, follows from a simple lemma, that will be in use for us later:

Lemma 2.1. For every $\xi \in \mathbb{Z}^3$ and $a \geq 0$,

$$4^a \mid |\xi|^2 \iff \xi = 2^a \xi_1 \ (\xi_1 \in \mathbb{Z}^3).$$

Proof. If $\xi = 2^a \xi_1$ and $\xi_1 \in \mathbb{Z}^3$, then $|\xi|^2 = 4^a |\xi_1|^2$, so $4^a \mid |\xi|^2$.

The other direction is proved by induction on $a$: the case $a = 0$ is clear. Assume that the argument is true for $a$, and that $4^{a+1} \mid |\xi|^2$. Denoting by $\xi = (x, y, z)$ we get in particular that $x^2 + y^2 + z^2 \equiv 0 \ (4)$. Since clearly $x^2, y^2, z^2 \equiv 0, 1 \ (4)$, it follows that necessarily $x^2, y^2, z^2 \equiv 0 \ (4)$, so $x, y$ and $z$ are all even. If we write $x = 2x_1, y = 2y_1, z = 2z_1$, and define $\xi_0 = (x_1, y_1, z_1) \in \mathbb{Z}^3$, we get that $4^{a+1} \mid |\xi|^2 = |2\xi_0|^2 = 4 |\xi_0|^2$, so $4^a \mid |\xi_0|^2$. From the induction hypothesis $\xi_0 = 2^a \xi_1 \ (\xi_1 \in \mathbb{Z}^3)$, and we get that $\xi = 2^a \xi_0 = 2^{a+1} \xi_1$. □
Denote by $R_3(n)$ the number of primitive solutions to $(2.4)$, i.e. the number of solutions such that $\gcd(x_1, x_2, x_3) = 1$, then we have
\[ r_3(n) = \sum_{d^2|n} R_3 \left( \frac{n}{d^2} \right). \tag{2.5} \]

We will need some asymptotic bounds for $r_3(n)$. For an upper bound, assume that $n$ is a sum of three squares, and as before, write $n = 4^a n_1$ with $4 \nmid n_1$, so $n_1 \not\equiv 0, 4, 7 \pmod{8}$. We will use the following theorem of Gauss (see [4]):
\[ R_3(n) = \pi^{-1} G_n \sqrt{n} L(1, \chi) \tag{2.6} \]

with
\[
G_n = \begin{cases} 
0 & n \equiv 0, 4, 7 \pmod{8} \\
16 & n \equiv 3 \pmod{8} \\
24 & n \equiv 1, 2, 5, 6 \pmod{8} 
\end{cases}
\]

where $L(1, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m}$, and $\chi(m) = \left( \frac{-4n}{m} \right)$ (the Kronecker symbol, so $\chi$ is a quadratic character modulo $4n$).

From (2.6) we have $R_3(n_1) \asymp \sqrt{n_1} L(1, \chi)$ (here $\chi(m) = \left( \frac{-4n_1}{m} \right)$). To bound $L(1, \chi)$ from above, write
\[
L(1, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m} = \sum_{m=1}^{4n_1} \frac{\chi(m)}{m} + \sum_{4n_1+1}^{\infty} \frac{\chi(m)}{m}.
\]
Clearly
\[
\left| \sum_{m=1}^{4n_1} \frac{\chi(m)}{m} \right| \leq \sum_{m=1}^{4n_1} \frac{1}{m} \ll \log n_1
\]
and for the second sum, summation by parts yields
\[
\sum_{4n_1+1}^{\infty} \frac{\chi(m)}{m} \ll \int_{4n_1}^{\infty} \frac{s(t)}{t^2} \, dt
\]
where $s(t) = \sum_{k \leq t} \chi(k)$. But $|s(t)| \leq 4n_1$, so
\[
\left| \sum_{4n_1+1}^{\infty} \frac{\chi(m)}{m} \right| \ll 4n_1 \int_{4n_1}^{\infty} \frac{dt}{t^2} = 1
\]
and we conclude that $|L(1, \chi)| \ll \log n_1$, so $R_3(n_1) \ll \sqrt{n_1} \log n_1$. Note that if $d^2|n_1$, then $\frac{n_1}{d^2} \not\equiv 0, 4, 7 \pmod{8}$, so
\[
R_3 \left( \frac{n_1}{d^2} \right) \ll \sqrt{\frac{n_1}{d}} \log \left( \frac{n_1}{d} \right) \leq \sqrt{n_1} \log n_1
\]
and using (2.5) we get that
\[ r_3(n) = r_3(n_1) = \sum_{d^2|n_1} R_3\left(\frac{n_1}{d^2}\right) \]
\[ \ll n_1^{1/2} \log n_1 \sum_{d^2|n_1} 1 \]
\[ \leq n_1^{1/2} \log n_1 \sum_{d|n_1} 1 \]
\[ \ll_\varepsilon n_1^{1/2+\varepsilon} \]
\[ \leq n^{1/2+\varepsilon}. \]

We cannot have a lower bound for \( r_3(n) \) for every \( n \), so assume now that
\[ n \not\equiv 0, 4, 7 \pmod{8}. \]

Again, from (2.6) we have
\[ r_3(n) \geq R_3(n) \asymp \sqrt{nL(1, \chi)} \]
and by Siegel’s theorem [12]: \( L(1, \chi) \gg_\varepsilon n^{-\varepsilon} \), so
\[ r_3(n) \gg_\varepsilon n^{1/2-\varepsilon}. \]

3. The Standard Torus

3.1. Bounds for the Green’s Function and Truncation

We begin with the proof of Theorem 1.1, so here \( \mathbb{T}^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3, \mathcal{L} = \mathbb{Z}^3 \), and \( \mathcal{N} = \mathcal{N}_3 \).

We first want to give a lower bound for the \( L^2 \)-norm of the Green’s function \( G_\lambda \):

**Lemma 3.1.** For every \( \lambda \in \Lambda \), we have
\[ \|G_\lambda\|_2^2 \gg \lambda^{1/2-\varepsilon}. \]

**Proof.** Note that
\[ \|G_\lambda\|_2^2 \asymp \sum_{\xi \in \mathbb{Z}^3} \frac{1}{(|\xi|^2 - \lambda)^2} = \sum_{n=0}^{\infty} \frac{r_3(n)}{(n - \lambda)^2}. \]

Take \( n_0 > \lambda, n_0 \equiv 1 \pmod{8}, n_0 - \lambda \leq 10 \), then
\[ \sum_{n=0}^{\infty} \frac{r_3(n)}{(n - \lambda)^2} \geq \frac{r_3(n_0)}{(n_0 - \lambda)^2} \gg r_3(n_0) \gg n_0^{1/2-\varepsilon} > \lambda^{1/2-\varepsilon}. \]

We will now use a truncation procedure.
For $L > 0$, denote by
\[ G_{\lambda, L} = -\frac{1}{8\pi^3} \sum_{|\xi|^2 - \lambda < L} \frac{\exp(i\xi \cdot (x - x_0))}{|\xi|^2 - \lambda} \]
the truncated Green’s function, and let $g_{\lambda, L}$ be the $L^2$-normalized truncated Green’s function:
\[ g_{\lambda, L} = \frac{G_{\lambda, L}}{\|G_{\lambda, L}\|_2}. \]

We have the following approximation:

**Lemma 3.2.** Let $L = \lambda^{\delta}$, $0 < \delta < 1/4$. Then $\|g_{\lambda} - g_{\lambda, L}\|_2 \to 0$ as $\lambda \to \infty$.

**Proof.** Clearly
\[
\|g_{\lambda} - g_{\lambda, L}\|_2 = \left\| \frac{G_{\lambda}}{\|G_{\lambda}\|_2} - \frac{G_{\lambda, L}}{\|G_{\lambda, L}\|_2} \right\|_2
\leq \frac{\|G_{\lambda} - G_{\lambda, L}\|_2}{\|G_{\lambda}\|_2} + \|G_{\lambda, L}\|_2 \left[ \frac{1}{\|G_{\lambda}\|_2} - \frac{1}{\|G_{\lambda, L}\|_2} \right]
\leq 2 \frac{\|G_{\lambda} - G_{\lambda, L}\|_2}{\|G_{\lambda}\|_2}.
\]

Using Lemma 3.1 we conclude that
\[
\|g_{\lambda} - g_{\lambda, L}\|_2 = \frac{\|G_{\lambda, L}\|_2}{\|G_{\lambda}\|_2} (1 + o(1))
\]
which tends to zero (for $\varepsilon > 0$ small enough) since $\delta > 0$. \qed

We conclude that the $L^2$-norm of the truncated Green’s function $G_{\lambda, L}$ is asymptotically equivalent to the $L^2$-norm of the non-truncated function $G_{\lambda}$:

**Lemma 3.3.** Let $L = \lambda^{\delta}$, $0 < \delta < 1/4$. Then
\[
\|G_{\lambda, L}\|_2 = \|G_{\lambda}\|_2 (1 + o(1)).
\]
Proof. This follows from (3.2), since
\[ \frac{\|G_{\lambda,L}\|_2 - \|G_{\lambda}\|_2}{\|G_{\lambda}\|_2} \leq \frac{\|G_{\lambda} - G_{\lambda,L}\|_2}{\|G_{\lambda}\|_2} \to 0 \]
as \( \lambda \to \infty \). \qed

We turn to prove the next approximation:

Lemma 3.4. Let \( L = \lambda^\delta \), \( 0 < \delta < 1/4 \). For every \( f \in C^\infty (\mathbb{T}^3) \), we have
\[ |\langle fg_\lambda, g_\lambda \rangle - \langle fg_{\lambda,L}, g_{\lambda,L} \rangle| \to 0 \]
as \( \lambda \to \infty \), so
\[ \langle fg_{\lambda,L}, g_{\lambda,L} \rangle \to 0 \Rightarrow \langle fg_\lambda, g_\lambda \rangle \to 0 \]
as \( \lambda \to \infty \).

Proof. Let \( f \in C^\infty (\mathbb{T}^3) \). We have
\[ |\langle fg_\lambda, g_\lambda \rangle - \langle fg_{\lambda,L}, g_{\lambda,L} \rangle| \leq |\langle fg_\lambda, g_\lambda - g_{\lambda,L} \rangle| + |\langle f (g_\lambda - g_{\lambda,L}), g_{\lambda,L} \rangle|. \]
The Cauchy–Schwarz inequality implies that
\[ |\langle fg_\lambda, g_\lambda - g_{\lambda,L} \rangle| \leq \|fg_\lambda\|_2 \|g_\lambda - g_{\lambda,L}\|_2 \leq \|f\|_\infty \|g_\lambda - g_{\lambda,L}\|_2. \]
From the same reason
\[ |\langle f (g_\lambda - g_{\lambda,L}), g_{\lambda,L} \rangle| \leq \|f (g_\lambda - g_{\lambda,L})\|_2 \leq \|f\|_\infty \|g_\lambda - g_{\lambda,L}\|_2, \]
but by Lemma 3.2 we know that
\[ \|g_\lambda - g_{\lambda,L}\|_2 \to 0 \]
as \( \lambda \to \infty \). It follows that if we have \( \langle fg_{\lambda,L}, g_{\lambda,L} \rangle \to 0 \), then
\[ |\langle fg_\lambda, g_\lambda \rangle| \leq |\langle fg_{\lambda,L}, g_{\lambda,L} \rangle| + |\langle fg_\lambda, g_\lambda \rangle - \langle fg_{\lambda,L}, g_{\lambda,L} \rangle| \to 0. \]
so
\[ \langle fg_\lambda, g_\lambda \rangle \to 0 \]
as \( \lambda \to \infty \). \qed

3.2. Powers of 4

We want to divide the elements of \( \mathcal{N}_3 \) into two kinds: those which are divisible by a high power of 4, and those which are not.

Fix \( 0 \neq \zeta \in \mathbb{Z}^3 \), and write \( |\zeta|^2 = n_\zeta = 4^{a_\zeta} n_1^\zeta \), with \( 4 \nmid n_1^\zeta \).

We make the following definition:

Definition 3.5. Define
\[ \mathcal{N}_0^\zeta = \{ n \in \mathcal{N}_3 : n = 4^{a} n_1, 4 \nmid n_1 \Rightarrow a > a_\zeta \}, \]
the set of elements which are divisible by a high power of 4, and define
\[ \mathcal{N}_1^\zeta = \{ n \in \mathcal{N}_3 : n = 4^{a} n_1, 4 \nmid n_1 \Rightarrow a \leq a_\zeta \}, \]
the complement set in \( \mathcal{N}_3 \).
The following observation will be useful:

**Lemma 3.6.** For every \( \xi \in \mathbb{Z}^3 \), if \( 2 \langle \xi, \zeta \rangle = |\zeta|^2 \), then \( |\xi|^2 \in \mathcal{N}_1^\zeta \).

**Proof.** Let \( \xi \in \mathbb{Z}^3 \) such that \( 2 \langle \xi, \zeta \rangle = |\zeta|^2 \), and write \( |\xi|^2 = n = 4^a n_1 \), with \( 4 \nmid n_1 \). By Lemma 2.1, \( \xi = 2^a \xi_1 \), with \( |\xi_1|^2 = n_1 \), and \( \zeta = 2^a \zeta_1 \), with \( |\zeta_1|^2 = 4 n_1^\zeta \). Therefore we get that

\[ 2^{a+a_\zeta+1} \langle \xi_1, \zeta_1 \rangle = |\zeta|^2 = 4^a n_1^\zeta \]

so

\[ 2^{a-a_\zeta+1} \langle \xi_1, \zeta_1 \rangle = n_1^\zeta \]

and since \( 4 \nmid n_1^\zeta \), we get that \( a - a_\zeta + 1 \leq 1 \), so \( a \leq a_\zeta \), and \( |\xi|^2 \in \mathcal{N}_1^\zeta \). \( \square \)

**Corollary 3.7.** For every \( \xi \in \mathbb{Z}^3 \), if \( |\xi|^2 \in \mathcal{N}_0^\zeta \), then \( 2 \langle \xi, \zeta \rangle - |\zeta|^2 \geq 1 \).

**Proof.** Let \( \xi \in \mathbb{Z}^3 \) such that \( |\xi|^2 \in \mathcal{N}_0^\zeta \). From Lemma 3.6 we have \( 2 \langle \xi, \zeta \rangle - |\zeta|^2 \neq 0 \), and since \( 2 \langle \xi, \zeta \rangle - |\zeta|^2 \) is an integer, we get that \( 2 \langle \xi, \zeta \rangle - |\zeta|^2 \geq 1 \). \( \square \)

For every \( \lambda \in \Lambda \), define \( n_\lambda \) to be the element of \( \mathcal{N}_3 \) which is closest to \( \lambda \) (if there are two elements with the same distance from \( \lambda \), take the smallest of them). Note that since the elements of \( \Lambda \) interlace between the elements of \( \mathcal{N}_3 \), and since for every \( n \neq 0, 4, 7 \) (8) we have \( n \in \mathcal{N}_3 \), we conclude that for every \( \lambda \in \Lambda \) we have \( |n_\lambda - \lambda| \leq 1.5 \), and in particular \( n_\lambda \sim \lambda \).

We conclude this section with the following lemma:

**Lemma 3.8.** Assume that \( n_\lambda \in \mathcal{N}_0^\zeta \). Then for every \( \xi \in \mathbb{Z}^3 \):

\[ \left| |\xi|^2 - \lambda \right| < \frac{1}{2} \implies \left| |\xi|^2 - \lambda \right| > \frac{1}{2}. \]

**Proof.** Let \( \xi \in \mathbb{Z}^3 \), and assume that \( \left| |\xi|^2 - \lambda \right| < \frac{1}{2} \).

It clearly follows that \( |\xi|^2 = n_\lambda \in \mathcal{N}_0^\zeta \). By Corollary 3.7 we get that:

\[ \left| |\xi|^2 - \lambda \right| = \left| |\xi|^2 - \lambda - 2 \langle \xi, \zeta \rangle + |\zeta|^2 \right| \geq 2 |\xi| - |\zeta|^2 - \left| |\xi|^2 - \lambda \right| > \frac{1}{2}. \]

\( \square \)

### 3.3. Proof of Theorem 1.1

We are now in condition to prove Theorem 1.1. We will need an estimate for the number of integral points inside some strips on three-dimensional spheres:

**Lemma.** Let \( L = \lambda^\delta \), \( 0 < \delta < 1/4 \). For every \( 0 \neq \zeta \in \mathbb{Z}^3 \), \( C_1, C_2 \) and \( n \) such that \( |n - \lambda| < C_1 L \), we have

\[ \# \left\{ \eta \in \mathbb{Z}^3 : |\eta|^2 = n, |\langle \eta, \zeta \rangle| < C_2 L \right\} \ll_{C_1,C_2,\zeta,\varepsilon} Ln^{\varepsilon}. \]

This is Lemma A.1 in the Appendix, see there for a proof.
The following main proposition will easily imply Theorem 1.1:

**Proposition 3.9.** For every $0 \neq \zeta \in \mathbb{Z}^3$, we have

$$\langle e_\zeta g_\lambda, g_\lambda \rangle \to 0$$

as $\lambda \to \infty$.

**Proof.** Let $L = \lambda^\delta$, $0 < \delta < 1/4$. By Lemma 3.4, it suffices to show that

$$\langle e_\zeta g_\lambda, g_\lambda \rangle \to 0$$

as $\lambda \to \infty$. Note that

$$\langle e_\zeta G_\lambda, G_\lambda \rangle \asymp \sum \frac{1}{|\xi - \zeta|^2 - \lambda \left|\xi^2 - \lambda\right| \left(\left|\xi^2 - \lambda\right|\right)}$$

and therefore

$$|\langle e_\zeta G_\lambda, G_\lambda \rangle| \ll \sum \frac{1}{|\xi - \zeta|^2 - \lambda \left|\xi^2 - \lambda\right|} = \sum^1 \frac{1}{|\xi - \zeta|^2 - \lambda \left|\xi^2 - \lambda\right|}$$

$$+ \sum^2 \frac{1}{|\xi - \zeta|^2 - \lambda \left|\xi^2 - \lambda\right|}$$

where in $\sum^1$ the summation is over $\xi \in \mathbb{Z}^3$ such that:

$$|\xi^2 - \lambda| < L, \quad |\langle \xi, \zeta \rangle| \geq L,$$

and in $\sum^2$ the summation is over $\xi \in \mathbb{Z}^3$ such that:

$$|\xi^2 - \lambda| < L, \quad |\langle \xi, \zeta \rangle| < L.$$

Note that

$$|\xi - \zeta|^2 - \lambda \geq 2 |\langle \xi, \zeta \rangle| - |\xi^2 - \lambda| - |\zeta|^2$$

so if $|\xi^2 - \lambda| < L, \quad |\langle \xi, \zeta \rangle| \geq L$, then

$$|\xi - \zeta|^2 - \lambda \geq 2L - L - |\zeta|^2 \gg L,$$

and hence

$$\sum^1 \frac{1}{\left|\xi - \zeta\right|^2 - \lambda \left|\xi^2 - \lambda\right|} \ll \frac{1}{L} \sum^1 \frac{1}{\left|\xi^2 - \lambda\right|} \leq \frac{1}{L} \sum_{|\xi^2 - \lambda| < L} \frac{1}{\left|\xi^2 - \lambda\right|}. $$
Cauchy–Schwarz gives
\[
\sum_{|\xi|^2-\lambda<L} \frac{1}{|\xi|^2 - \lambda} \ll \|G_{\lambda, L}\|_2 \left( \sum_{|\xi|^2-\lambda<L} 1 \right)^{1/2} = \|G_{\lambda, L}\|_2 \left( \sum_{n-\lambda<L} r_3(n) \right)^{1/2} \ll \|G_{\lambda, L}\|_2 \left( \sum_{n-\lambda<L} n^{1/2+\epsilon} \right)^{1/2} \ll \|G_{\lambda, L}\|_2 L^{1/2} \lambda^{1/4+\epsilon/2}
\]
and therefore
\[
\sum^1_{|\xi|\neq \lambda} \frac{1}{|\xi| - \lambda} \ll \frac{\|G_{\lambda, L}\|_2 \lambda^{1/4+\epsilon/2}}{L^{1/2}} = \|G_{\lambda, L}\|_2 \lambda^{-\delta/2+1/4+\epsilon/2} \ll \|G_{\lambda}\|_2 \lambda^{-\delta/2+1/4+\epsilon/2}.
\]
For the estimation of \(\sum^2\), remember that we defined \(n_\lambda\) to be the element of \(N_3\) which is closest to \(\lambda\) (and if there are two elements with the same distance from \(\lambda\), we take \(n_\lambda\) to be the smallest of them). We distinguish between two cases: whether \(n_\lambda \in N_0^\xi\) or \(n_\lambda \in N_1^\xi\).

First, assume that \(n_\lambda \in N_0^\xi\). By Lemma 3.8, for every \(\xi \in \mathbb{Z}^3\),
\[
|\xi|^2 - \lambda < \frac{1}{2} \implies |\xi - \xi_0|^2 - \lambda > \frac{1}{2}.
\]
Hence we can write
\[
\sum^2_{|\xi|\neq \lambda} \frac{1}{|\xi| - \lambda} = \sum^3_{|\xi|\neq \lambda} \frac{1}{|\xi| - \lambda} + \sum^4_{|\xi|\neq \lambda} \frac{1}{|\xi| - \lambda} + \sum^5_{|\xi|\neq \lambda} \frac{1}{|\xi| - \lambda}
\]
where in \(\sum^3\) the summation is over \(\xi \in \mathbb{Z}^3\) such that:
\[
\frac{1}{2} \leq |\xi|^2 - \lambda < L, \quad |\langle \xi, \xi \rangle| < L, \quad |\xi - \xi|^2 - \lambda \geq \frac{1}{2},
\]
in \(\sum^4\) the summation is over \(\xi \in \mathbb{Z}^3\) such that:
\[
|\xi|^2 - \lambda < \frac{1}{2}, \quad |\langle \xi, \xi \rangle| < L, \quad |\xi - \xi|^2 - \lambda > \frac{1}{2},
\]
and in $\sum^5$ the summation is over $\xi \in \mathbb{Z}^3$ such that:

$$\frac{1}{2} \leq |\xi|^2 - \lambda < L, \quad |\langle \xi, \zeta \rangle| < L, \quad |\xi - \zeta|^2 - \lambda < \frac{1}{2}.$$

Using Lemma A.1 we have

$$\sum^3 \frac{1}{|\xi - \zeta|^2 - \lambda| |\xi|^2 - \lambda|} \ll \sum^3 \frac{1}{|\xi|^2 - \lambda| L \epsilon} \ll L^2 \lambda \epsilon = \lambda^{2\delta + \epsilon}.$$

For the second sum, we get by Cauchy–Schwarz and Lemma A.1

$$\sum^4 \frac{1}{|\xi - \zeta|^2 - \lambda| |\xi|^2 - \lambda|} \ll \left( \sum^4 \frac{1}{|\xi|^2 - \lambda|^2} \right)^{1/2} \left( \sum^4 1 \right)^{1/2} \ll \|G_{\lambda, L}\|_2 \left( \sum^4 1 \right)^{1/2} \ll \|G_{\lambda, L}\|_2 (Ln^\epsilon)^{1/2} \ll \|G_{\lambda, L}\|_2 L^{1/2} \lambda^{\epsilon/2} = \|G_{\lambda, L}\|_2 \lambda^{\delta/2 + \epsilon/2}.$$

For the third sum we note that for $|\xi|^2 - \lambda < L, |\langle \xi, \zeta \rangle| < L$ (for every large enough $L$) we have:

$$|\xi - \zeta|^2 - \lambda = |\xi|^2 - \lambda - 2 \langle \xi, \zeta \rangle + |\zeta|^2 \leq |\xi|^2 - \lambda + 2 |\langle \xi, \zeta \rangle| + |\zeta|^2 < 3L + |\zeta|^2 \leq 4L$$

and

$$|\langle \xi - \zeta, \zeta \rangle| = |\langle \xi, \zeta \rangle - |\zeta|^2| \leq |\langle \xi, \zeta \rangle| + |\zeta|^2 < 2L,$$

so

$$\sum^2 \frac{1}{|\xi - \zeta|^2 - \lambda|^2} \leq \sum^6 \frac{1}{|\eta|^2 - \lambda|^2}$$

where in $\sum^6$ the summation is over $\eta \in \mathbb{Z}^3$ such that

$$|\eta|^2 - \lambda < 4L, \quad |\langle \eta, \zeta \rangle| < 2L.$$
We get that
\[ \sum_{5}^{1} \frac{1}{|\xi - \zeta|^{2} - \lambda} \ll \sup_{5}^{1} \frac{1}{|\xi - \zeta|^{2} - \lambda} \]
\[ \ll \left( \sum_{5}^{1} \frac{1}{|\xi - \zeta|^{2} - \lambda} \right)^{1/2} \left( \sum_{5}^{1} \right)^{1/2} \]
\[ \ll \left( \sum_{2}^{1} \frac{1}{|\xi - \zeta|^{2} - \lambda} \right)^{1/2} \left( \sum_{5}^{1} \right)^{1/2} \]
\[ \ll \left( \sum_{6}^{1} \frac{1}{|\eta|^{2} - \lambda} \right)^{1/2} \left( \sum_{5}^{1} \right)^{1/2} \]
\[ \ll \|G_{\lambda}\|_{2} \left( \sum_{5}^{1} \right)^{1/2} \]

but since \(|\langle \xi, \zeta \rangle| < L\) implies that \(|\langle \xi - \zeta, \zeta \rangle| < 2L\), and since \(|\xi - \zeta|^{2} - \lambda| < 1/2\) implies that \(|\xi - \zeta|^{2} = n_{\lambda}\), Lemma A.1 yields
\[ \sum_{5}^{1} \ll \sum_{|\eta|^{2} = n_{\lambda}}^{1} \ll Ln_{\lambda}^{\epsilon} \ll L\lambda^{\epsilon} = \lambda^{\delta + \epsilon}, \]
so
\[ \sum_{5}^{1} \frac{1}{|\xi - \zeta|^{2} - \lambda} \ll \|G_{\lambda}\|_{2} \lambda^{\delta/2 + \epsilon/2}. \]

We conclude that
\[ \sum_{2}^{1} \frac{1}{|\xi - \zeta|^{2} - \lambda} \ll \lambda^{2\delta + \epsilon} + \|G_{\lambda L}\|_{2} \lambda^{\delta/2 + \epsilon/2} \]
\[ + \|G_{\lambda}\|_{2} \lambda^{\delta/2 + \epsilon/2} \]
\[ \ll \lambda^{2\delta + \epsilon} + 2 \|G_{\lambda}\|_{2} \lambda^{\delta/2 + \epsilon/2}. \]

Now, assume that \(n_{\lambda} \in \mathcal{N}^{\epsilon}_{\lambda}\). Cauchy–Schwarz yields
\[ \sum_{2}^{1} \frac{1}{|\xi - \zeta|^{2} - \lambda} \]
\[ \ll \left( \sum_{2}^{1} \frac{1}{|\xi|^{2} - \lambda} \right)^{1/2} \left( \sum_{2}^{1} \frac{1}{|\xi - \zeta|^{2} - \lambda} \right)^{1/2} \]
\[ \ll \| G_{\lambda,L} \|_2 \left( \sum_2^2 \frac{1}{|\xi - \zeta|^2 - \lambda} \right)^{1/2} \]
\[ \ll \| G_{\lambda,L} \|_2 \left( \sum_6^6 \frac{1}{|\eta|^2 - \lambda} \right)^{1/2} \]
\[ = \| G_{\lambda,L} \|_2 \left( \sum_7^7 \frac{1}{|\eta|^2 - \lambda}^2 + \sum_8^8 \frac{1}{|\eta|^2 - \lambda} \right)^{1/2} \]

where in \( \sum_7 \) the summation is over \( \eta \in \mathbb{Z}^3 \) such that:

\[ |\eta|^2 - \lambda < 4L, \quad |\langle \eta, \zeta \rangle| < 2L, \quad |\eta|^2 \neq n_{\lambda}, \]

and in \( \sum_8 \) the summation is over \( \eta \in \mathbb{Z}^3 \) such that:

\[ |\eta|^2 - \lambda < 4L, \quad |\langle \eta, \zeta \rangle| < 2L, \quad |\eta|^2 = n_{\lambda}. \]

For \( |\eta|^2 \neq n_{\lambda} \), we clearly have \( |\eta|^2 - \lambda \geq \frac{1}{2} \), so using Lemma A.1, we get

\[ \sum_7^7 \frac{1}{|\eta|^2 - \lambda} \ll \sum_7^7 1 \ll \sum_{|n - \lambda| < 4L} L \lambda \ll \lambda^{2\delta + \varepsilon}. \]

For the last sum, we use Lemma A.1 again to get

\[ \frac{\sum_8^8 \frac{1}{|\eta|^2 - \lambda}}{\| G_{\lambda,L} \|_2^2} \ll \frac{L \lambda \varepsilon}{(n_{\lambda} - \lambda)^2} \ll \frac{L \lambda \varepsilon}{G_{\lambda,L}^2 \| G_{\lambda,L} \|_2^2} \ll \frac{L \lambda \varepsilon}{\sum_{n=0}^\infty \frac{r_3(n)}{(n-\lambda)^2}}. \]

\forall n_0 \text{ we have } \sum_{n=0}^\infty \frac{r_3(n)}{(n-\lambda)^2} \geq \frac{r_3(n_0)}{(n_0-\lambda)^2}, \text{ so }

\[ \frac{L \lambda \varepsilon}{\sum_{n=0}^\infty \frac{r_3(n)}{(n-\lambda)^2}} \leq \frac{L \lambda \varepsilon}{r_3(n_{\lambda})}. \]

If we write \( n_{\lambda} = 4^a n_1 \), with \( 4 \nmid n_1 \), then since \( n_{\lambda} \in \mathcal{N}_3^\zeta \) we know that \( n_{\lambda} \leq 4^{a+1} n_1 \), so

\[ \frac{L \lambda \varepsilon}{r_3(n_{\lambda})} = \frac{L \lambda \varepsilon}{r_3(n_1)} \ll \frac{L \lambda \varepsilon}{n_1^{1/2 - \varepsilon}} \]
\[ \ll \frac{L \lambda \varepsilon}{n_1^{1/2 - \varepsilon}} \ll \frac{L \lambda \varepsilon}{\lambda^{1/2 - \varepsilon}} \]
\[ = L \lambda^{-1/2 + 2\varepsilon} = \lambda^{\delta - 1/2 + 2\varepsilon} \]
and therefore
\[ \sum_{n=1}^{8} \frac{1}{|\eta|^2 - \lambda} \ll \|G_\lambda\|_2^2 \lambda^{-1/2 + 2\varepsilon}, \]
so now we conclude that
\[ \sum_{n=1}^{2} \frac{1}{|\xi - \zeta|^2 - \lambda} \ll \|G_{\lambda,L}\|_2 \left( \lambda^{2\delta + \varepsilon} + \|G_{\lambda}\|_2 \lambda^{-1/2 + 2\varepsilon} \right)^{1/2} \]
\[ \leq \|G_{\lambda,L}\|_2 \left( \lambda^{\delta/2 + 1/4 + \varepsilon} + \|G_{\lambda}\|_2 \lambda^{-1/2 + 1/4 + \varepsilon} \right) \]
\[ \ll \|G_{\lambda}\|_2 \left( \lambda^{\delta/2 + 1/4 + \varepsilon} + \|G_{\lambda}\|_2 \lambda^{-1/2 + 1/4 + \varepsilon} \right). \]

Either way we got that
\[ \sum_{n=1}^{2} \frac{1}{|\xi - \zeta|^2 - \lambda} \ll \lambda^{2\delta + \varepsilon} + 2 \|G_{\lambda}\| \lambda^{\delta/2 + \varepsilon/2} \]
\[ + \|G_{\lambda}\|_2 \lambda^{\delta/2 + 1/4 + \varepsilon} + \|G_{\lambda}\|_2 \lambda^{-1/2 + 1/4 + \varepsilon}, \]
so we have
\[ |\langle e_\xi g_{\lambda,L}, g_{\lambda,L} \rangle| = \frac{|\langle e_\xi G_{\lambda,L}, G_{\lambda,L} \rangle|}{\|G_{\lambda,L}\|_2^2} \ll \frac{\lambda^{2\delta + \varepsilon}}{\|G_{\lambda}\|_2^2} + \frac{\lambda^{-\delta/2 + 1/4 + \varepsilon/2} + 2 \lambda^{\delta/2 + \varepsilon/2} + \lambda^{\delta + \varepsilon/2}}{\|G_{\lambda}\|_2} \]
\[ + \lambda^{\delta/2 - 1/4 + \varepsilon} \]
\[ \ll \frac{\lambda^{2\delta + \varepsilon}}{\lambda^{1/2 - \varepsilon}} + \frac{\lambda^{-\delta/2 + 1/4 + \varepsilon/2} + 2 \lambda^{\delta/2 + \varepsilon/2} + \lambda^{\delta + \varepsilon/2}}{\lambda^{1/4 - \varepsilon/2}} \]
\[ + \lambda^{\delta/2 - 1/4 + \varepsilon} \]
\[ = \lambda^{2\delta - 1/2 + 2\varepsilon} + \lambda^{-\delta/2 + \varepsilon} + 3 \lambda^{\delta/2 - 1/4 + \varepsilon} + \lambda^{\delta - 1/4 + \varepsilon}, \]
and since \(0 < \delta < 1/4\), the proposition follows. \( \square \)

Theorem 1.1 now easily follows by the density of trigonometric polynomials in \(C^\infty (T^3)\) in the uniform norm:

**Theorem.** For every \(a \in C^\infty (T^3)\), we have
\[ \int_{T^3} a(x) |g_{\lambda}(x)|^2 \, dx \to \frac{1}{\text{area}(T^3)} \int_{T^3} a(x) \, dx \]
as \(\lambda \to \infty\) along \(\Lambda\).
Proof. Let \( P(x) = \sum_{|\zeta| \leq J} p_\zeta \epsilon_\zeta (x) \) be a trigonometric polynomial. From Proposition 3.9 we have

\[
\langle Pg_{\lambda}, g_{\lambda} \rangle = \sum_{|\zeta| \leq J} p_\zeta \langle \epsilon_\zeta g_{\lambda}, g_{\lambda} \rangle = p(0, 0, 0) = \frac{1}{\text{area } (T^3)} \int_{T^3} P(x) \, dx
\]
as \( \lambda \to \infty \).

Let \( \varepsilon > 0 \). For every \( a \in C^\infty (T^3) \), there exists a trigonometric polynomial \( P \) such that \( \|a - P\|_\infty < \varepsilon \). Thus for every large enough \( \lambda \in \Lambda \)

\[
\left| \langle ag_{\lambda}, g_{\lambda} \rangle - \frac{1}{\text{area } (T^3)} \int_{T^3} a(x) \, dx \right| \\
\leq |\langle ag_{\lambda}, g_{\lambda} \rangle - \langle Pg_{\lambda}, g_{\lambda} \rangle| + \left| \langle Pg_{\lambda}, g_{\lambda} \rangle - \frac{1}{\text{area } (T^3)} \int_{T^3} P(x) \, dx \right| \\
+ \left| \frac{1}{\text{area } (T^3)} \int_{T^3} P(x) \, dx - \frac{1}{\text{area } (T^3)} \int_{T^3} a(x) \, dx \right| \\
< 2 \|a - P\|_\infty + \varepsilon < 3\varepsilon.
\]

\[\square\]

4. The Irrational Torus

4.1. Basic Setup

Let \( T^3 = \mathbb{R}^3 / 2\pi L_0 \) be a flat three-dimensional torus, where

\[ L_0 = \mathbb{Z}(a, 0, 0) \oplus \mathbb{Z}(0, b, 0) \oplus \mathbb{Z}(0, 0, c) \]
is a lattice, such that \( 1/a^2, 1/b^2, 1/c^2 \in \mathbb{R} \) are independent over \( \mathbb{Q} \). We also demand that at least one of the ratios \( b^2/a^2, c^2/a^2, c^2/b^2 \) will be an irrational of finite type \( \tau \), as in Definition 1.2 (without loss of generality assume it to be \( c^2/a^2 \)).

The norm of a lattice vector \( \xi = (\xi_1/a, \xi_2/b, \xi_3/c) \in L \) is

\[ \xi_1^2/a^2 + \xi_2^2/b^2 + \xi_3^2/c^2 \]

so if \( \eta = (\eta_1/a, \eta_2/b, \eta_3/c) \) is another vector of \( L \) of the same norm, we have

\[ (\xi_1^2 - \eta_1^2) / a^2 + (\xi_2^2 - \eta_2^2) / b^2 + (\xi_3^2 - \eta_3^2) / c^2 = 0 \]

and since \( 1/a^2, 1/b^2, 1/c^2 \) are independent over the rationals we get that \( \eta_i = \pm \xi_i \) for \( 1 \leq i \leq 3 \).

We conclude that for \( n \in \mathcal{N} \) we have \( r_L(n) = 1, 2, 4 \) or \( 8 \).

Weyl’s law for the torus, establishing the asymptotics of the counting function \( N(x) \) of eigenvalues below \( x \), is equivalent to counting the number of points of the standard lattice \( \mathbb{Z}^3 \) in an ellipsoid:
The trivial bound on the remainder term is $\theta = 1$. We will need a bound $\theta < 1$, such as the bound due to Hlawka [5] using Poisson summation which translates to $\theta = 3/4$.

Note that since the multiplicities $r_L(n)$ are bounded, we have

$$\# \{ \lambda \in \Lambda : \lambda \leq X \} \approx N(X) \approx X^{3/2}.$$ 

We will need to analyze the spacing between the elements of $\Lambda$. We first notice that for most of the elements, the nearest neighbor cannot be too far:

For $\varepsilon > 0$, define

$$\Lambda_1^\varepsilon = \left\{ \lambda \in \Lambda : \lambda \geq 1, \left( \lambda, \lambda^{1/2+\varepsilon} \right) \cap \Lambda \neq \emptyset \right\}.$$ 

We claim that this is a density one set in $\Lambda$. To show this, define

$$B_1 = \Lambda \setminus \Lambda_1 = \{ \lambda \in \Lambda : \lambda < 1 \} \cup \left\{ \lambda \in \Lambda : \lambda \geq 1, \left( \lambda, \lambda^{1/2+\varepsilon} \right) \cap \Lambda = \emptyset \right\}.$$ 

We want to show that $B_1$ is a density zero set in $\Lambda$, that is,

$$\# \{ \lambda \in B_1 : \lambda \leq X \} = o(\# \{ \lambda \in \Lambda : \lambda \leq X \}) = o\left(X^{3/2}\right).$$ 

We have $\# \{ \lambda \in \Lambda : \lambda < 1 \} = O(1)$, so we only need to check that

$$\# \{ \lambda \in \tilde{B}_1 : \lambda \leq X \} \leq X^{3/2-\varepsilon}$$

where

$$\tilde{B}_1 = \left\{ \lambda \in \Lambda : \lambda \geq 1, \left( \lambda, \lambda^{1/2+\varepsilon} \right) \cap \Lambda = \emptyset \right\}.$$ 

Indeed, the intervals $(\lambda, \lambda^{1/2+\varepsilon}), \lambda \in \tilde{B}_1$ are disjoint, and therefore

$$\# \{ \lambda \in \tilde{B}_1 : \lambda \leq X \} \cdot X^{-1/2+\varepsilon} \leq \text{meas} \left( \bigcup_{\lambda \in \tilde{B}_1} \left( \lambda, \lambda^{1/2+\varepsilon} \right) \right) \leq \text{meas} ((1, X+1)) = X$$

so $\# \{ \lambda \in \tilde{B}_1 : \lambda \leq X \} \leq X^{3/2-\varepsilon}$.

4.2. Lattice Points in Thin Spherical Shells

For $0 < L \leq 1$, define $A(\lambda, L)$ to be the set of lattice points in the spherical shell $\lambda - L < |\xi|^2 < \lambda + L$:

$$A(\lambda, L) = \left\{ \xi \in \mathcal{L} : \lambda - L < |\xi|^2 < \lambda + L \right\}.$$ 

Define

$$\tilde{A}(\lambda, L) = (\lambda - L, \lambda + L) \cap \Lambda.$$ 

We want to show that for $L = \lambda^{-\delta}$, $0 < \delta < \min \{(1 - \theta)/2 - \varepsilon, 1/\tau - \varepsilon\}$ (for $\varepsilon > 0$ small enough there is such $\delta$, since $\theta < 1$), we have a density one set in
A such that \( \# \tilde{A} (\lambda, 3L) \leq L \lambda^{1/2 + 2 \varepsilon} \) for every element \( \lambda \) of this set. In order to show this, we need the following lemma:

**Lemma 4.1.** Let \( L = \lambda^{-\delta}, 0 < \delta < \min \{ (1 - \theta) / 2 - \varepsilon, 1 / \tau - \varepsilon \} \). We have

\[
\sum_{\lambda \in \Lambda_1} \# A (\lambda, 3L) \leq X^{2 - \delta + \varepsilon}.
\]

**Proof.** For every \( \lambda \in \Lambda_1 \), \( X < \lambda \leq 2X \), choose \( \xi \in \mathcal{L} \) such that \( |\xi|^2 \) is the smallest element in \( \mathcal{N} \) greater than \( \lambda \). Since \( \lambda \in \Lambda_1 \) and \( \delta < (1 - \theta) / 2 - \varepsilon < 1 / \tau - \varepsilon \) we know that \( \xi \in A (\lambda, X^{-\delta}) \), and we conclude that

\[
\# \tilde{A} (\lambda, 3X^{-\delta}) \leq \# A (\lambda, 3X^{-\delta}) \leq \# A \left( |\xi|^2, 4X^{-\delta} \right) = \sum_{\eta \in \mathcal{L}} \frac{1}{|\eta|^2 - |\xi|^2 < 4X^{-\delta}}.
\]

Hence

\[
\sum_{\lambda \in \Lambda_1} \# A (\lambda, 3L) \leq \sum_{\lambda \in \Lambda_1} \# A (\lambda, 3X^{-\delta}) \leq \sum_{\xi, \eta \in \mathcal{L}} \sum_{\lambda \in \Lambda_1} \frac{1}{|\eta|^2 - |\xi|^2 < 4X^{-\delta}}.
\]

Thus we want to bound the number of solutions of the RHS of (4.1), which is a quadratic Diophantine inequality with a shrinking target, by \( O \left( X^{2 - \delta + \varepsilon} \right) \).

We transform the problem into linear Diophantine inequality as follows: Write \( \xi = (\xi_1 / \alpha, \xi_2 / b, \xi_3 / c) \), \( \eta = (\eta_1 / \alpha, \eta_2 / b, \eta_3 / c) \), then

\[
|\xi|^2 - |\eta|^2 = \frac{1}{a^2} (\xi_1^2 - \eta_1^2) + \frac{1}{b^2} (\xi_2^2 - \eta_2^2) + \frac{1}{c^2} (\xi_3^2 - \eta_3^2) = \frac{z_1}{a^2} + \frac{z_2}{b^2} + \frac{z_3}{c^2}
\]

where \( z_j = \xi_j^2 - \eta_j^2 \) is of size \( |z_j| \leq CX \) for some constant \( C \).

Assume first that for every \( 1 \leq j \leq 3, \xi_j^2 \neq \eta_j^2 \); so \( z_j \neq 0 \). The number of solutions to

\[
z_j = \xi_j^2 - \eta_j^2 = (\xi_j - \eta_j) (\xi_j + \eta_j)
\]

is bounded by the number of divisors of \( z_j \) which is \( O \left( z_j^\varepsilon \right) = O \left( X^\varepsilon \right) \). Thus we have

\[
\# \left\{ \xi, \eta \in \mathcal{L} : |\xi|^2, |\eta|^2 \leq 4X, \xi_j^2 \neq \eta_j^2, |\xi_j^2 - |\eta_j|^2| < 4X^{-\delta} \right\}
\]

\[
\leq X^\varepsilon \cdot \# \left\{ (z_1, z_2, z_3) \in \mathbb{Z}^3 : 1 \leq |z_j| \leq CX, \frac{z_1}{a^2} + \frac{z_2}{b^2} + \frac{z_3}{c^2} < 4X^{-\delta} \right\}.
\]

Denote

\[
A_X = \left\{ (z_1, z_2, z_3) \in \mathbb{Z}^3 : 1 \leq |z_j| \leq CX, \frac{z_1}{a^2} + \frac{z_2}{b^2} + \frac{z_3}{c^2} < 4X^{-\delta} \right\}.
\]
We will show that $\#A_X \ll X^{2-\delta}$: first note that for every $(z_1, z_2, z_3) \in A_X$, assuming that $X$ is large enough we have

$$\left| \frac{c^2}{a^2} z_1 + \frac{c^2}{b^2} z_2 + z_3 \right| < \frac{1}{4}$$

so $z_3$ is uniquely determined by the values of $z_1, z_2$, and we have

$$\left\| \frac{c^2}{a^2} z_1 + \frac{c^2}{b^2} z_2 \right\| < 4X^{-\delta}$$

so

$$\#A_X \leq \# \left\{ (z_1, z_2) \in \mathbb{Z}^2 : 1 \leq |z_j| \leq CX, \left\| \frac{c^2}{a^2} z_1 + \frac{c^2}{b^2} z_2 \right\| < 4X^{-\delta} \right\}$$

$$= \sum_{1 \leq |z_j| \leq CX} \sum_{0 \leq j \leq 1} \# \left\{ z_1 \in \mathbb{N} : z_1 \leq |CX|, \left\| (-1)^j \frac{c^2}{a^2} z_1 + \frac{c^2}{b^2} z_2 \right\| < 4X^{-\delta} \right\}$$

$$\ll \sum_{1 \leq |z_j| \leq CX} \sum_{0 \leq j_1, j_2 \leq 1} \# \left\{ z_1 \in \mathbb{N} : z_1 \leq |CX|, \left\{ (-1)^{j_1} \frac{c^2}{a^2} z_1 + (-1)^{j_2} \frac{c^2}{b^2} z_2 \right\} < 4X^{-\delta} \right\}.$$ 

In the Appendix (A.2), we show that for every sequence $x_n = \alpha n + \beta$, where $\alpha \in \mathbb{R}$ is an irrational of finite type $\tau$ and $\beta \in \mathbb{R}$, we have an upper bound for the discrepancy $D_N$ of the sequence $(x_n)$:

$$D_N \leq cN^{-1/\tau+\varepsilon}$$

where $c = c(\alpha, \varepsilon)$ is a constant which does not depend on $\beta$ (formula (A.7)).

For every $z_2$, using this bound for the sequence $x_n = \alpha n + \beta$, with $\alpha = \pm c^2/a^2$ (which is an irrational of finite type $\tau$) and $\beta = \pm (c^2/b^2) z_2$, we get that (since $\delta < 1/\tau - \varepsilon$)

$$\# \left\{ z_1 \in \mathbb{N} : z_1 \leq |CX|, \left\{ \pm \frac{c^2}{a^2} z_1 \pm \frac{c^2}{b^2} z_2 \right\} < 4X^{-\delta} \right\} \ll X^{1-\delta}$$

so we conclude that

$$\#A_X \ll X^{2-\delta}.$$ 

Assume now that exactly one of the $z_j$ equals zero: without loss of generality assume that $z_1 = 0$ and $z_2, z_3 \neq 0$. Since for $j = 2, 3$ the number of solutions to $z_j = \xi_j^2 - \eta_j^2$ is $O(X^\varepsilon)$, and the number of solutions to $\xi_1^2 = \eta_1^2$ is $O(1)$ we conclude that the number of solutions of the RHS of (4.1) (under our assumption) is bounded by

$$X^{1/2+\varepsilon} \cdot \# \left\{ (z_2, z_3) \in \mathbb{Z}^2 : 1 \leq |z_j| \leq CX, \left| \frac{z_2}{b^2} + \frac{z_3}{c^2} \right| < 4X^{-\delta} \right\}.$$ 

Denote

$$B_X = \left\{ (z_2, z_3) \in \mathbb{Z}^2 : 1 \leq |z_j| \leq CX, \left| \frac{z_2}{b^2} + \frac{z_3}{c^2} \right| < 4X^{-\delta} \right\}.$$
Note that for every \((z_2, z_3) \in B_X\), assuming that \(X\) is large enough we have

\[
\left| \frac{c^2}{b^2 z_2^2 + z_3} \right| < \frac{1}{4}
\]

so \(z_3\) is uniquely determined by the value of \(z_2\), and therefore \(#B_X \ll X\).

If exactly two of the \(z_j\) equal zero, for instance \(z_1 = z_2 = 0\), then for large enough \(X\) we must have \(z_3 = 0\), so the number of solutions of the RHS of (4.1) (under our assumption) is \(O(X^{3/2})\). Since \(\delta < (1 - \theta)/2 - \varepsilon < 1/2\) the lemma is proved.  

Define

\[
\Lambda_2 = \Lambda_2^{\varepsilon, \delta} = \left\{ \lambda \in \Lambda_1 : \# \tilde{A} (\lambda, 3L) \leq L\lambda^{1/2 + 2\varepsilon} \right\}.
\]

We will show that this is a density one set in \(\Lambda_1\) (and hence in \(\Lambda\)):

Define

\[
B_2 = \Lambda_1 \setminus \Lambda_2 = \left\{ \lambda \in \Lambda_1 : \# \tilde{A} (\lambda, 3L) > L\lambda^{1/2 + 2\varepsilon} \right\}.
\]

We will check that

\[
\# \{ \lambda \in B_2 : \lambda \leq X \} \ll X^{3/2 - \varepsilon}.
\]

Indeed, from Lemma 4.1 we have

\[
\# \{ \lambda \in B_2 : X < \lambda \leq 2X \} \cdot X^{1/2 + 2\varepsilon - \delta} < \sum_{\lambda \in B_2} \# \tilde{A} (\lambda, 3L)
\]

\[
\leq \sum_{\lambda \in \Lambda_1} \# \tilde{A} (\lambda, 3L)
\]

\[
\ll X^{2 - \delta + \varepsilon}
\]

so there exist \(C > 0\) and \(M > 0\) such that for all \(X \geq M/2\)

\[
\# \{ \lambda \in B_2 : X < \lambda \leq 2X \} \leq CX^{3/2 - \varepsilon}. \quad (4.2)
\]

Note that

\[
\# \{ \lambda \in B_2 : \lambda \leq X \} = \sum_{k=0}^{\infty} \# \{ \lambda \in B_2 : X/2^{k+1} < \lambda \leq X/2^k \}
\]

(and actually the summation over \(k\) is finite). From (4.2), for every \(k \geq 0\) such that \(X/2^{k+1} \geq M/2\) (so \(k \leq \lfloor \log_2 (X/M) \rfloor\)), we have

\[
\# \{ \lambda \in B_2 : X/2^{k+1} < \lambda \leq X/2^k \} \leq C \left( \frac{X}{2^{k+1}} \right)^{3/2 - \varepsilon}
\]
so for $X \geq M$ we have

$$
\sum_{k=0}^{\infty} \sum_{k=0}^{[\log_2(X/M)]} \# \{ \lambda \in B_2 : X/2^{k+1} < \lambda \leq X/2^k \} 
\leq CX^{3/2-\varepsilon} \sum_{k=0}^{[\log_2(X/M)]} \frac{1}{2^{(k+1)(3/2-\varepsilon)}} + \sum_{k=0}^{[\log_2(X/M)]} \# \{ \lambda \in B_2 : \lambda \leq M \}
\ll X^{3/2-\varepsilon}
$$

as we claimed.

4.3. Bounds for the Green’s Function and Truncation

We first give a lower bound for the $L^2$-norm of the Green’s function $G_\lambda$:

**Lemma 4.2.** For every $\lambda \in \Lambda_2$, we have

$$
\|G_\lambda\|_2^2 \gg \lambda^{1-2\varepsilon}.
$$

**Proof.** Take $\lambda_0 \in (\lambda, \lambda + \lambda^{-1/2+\varepsilon}) \cap \Lambda$. Let $n_0$ be some norm such that $\lambda < n_0 < \lambda_0$. We have

$$
\|G_\lambda\|_2^2 \asymp \sum_{\xi \in \mathcal{L}} \frac{1}{(|\xi|^2 - \lambda)^2} = \sum_{n \in \mathcal{N}} \frac{r_\mathcal{L}(n)}{(n - \lambda)^2} 
\asymp \sum_{n \in \mathcal{N}} \frac{1}{(n - \lambda)^2} 
\geq \frac{1}{(n_0 - \lambda)^2} 
> \frac{1}{(\lambda_0 - \lambda)^2} > \lambda^{1-2\varepsilon}.
$$

We will now use a truncation procedure.

Recall that for $0 < L \leq 1$ we defined

$$
A(\lambda, L) = \{ \xi \in \mathcal{L} : \lambda - L < |\xi|^2 < \lambda + L \}
$$

as the set of lattice points in the spherical shell $\lambda - L < |x|^2 < \lambda + L$. We denote by

$$
G_{\lambda, L} = -\frac{1}{8\pi^3} \sum_{\xi \in A(\lambda, L)} \frac{\exp(i\xi \cdot (x - x_0))}{|\xi|^2 - \lambda}
$$
the truncated Green’s function, and let \( g_{\lambda, L} \) be the \( L^2 \)-normalized truncated Green’s function:

\[
g_{\lambda, L} = \frac{G_{\lambda, L}}{\|G_{\lambda, L}\|_2}.
\]

Lemma 4.3. For \( 0 < \delta < \min \{ (1 - \theta) / 2 - \epsilon, 1 / \tau - \epsilon \} \), \( L = \lambda^{-\delta} \), we have \( \|g_{\lambda} - g_{\lambda, L}\|_2 \to 0 \) as \( \lambda \to \infty \) along \( \Lambda_2 \).

Proof. As in (3.1), we get that

\[
\|g_{\lambda} - g_{\lambda, L}\|_2 \leq 2 \|G_{\lambda} - G_{\lambda, L}\|_2.
\]

We have

\[
\|G_{\lambda} - G_{\lambda, L}\|_2 \approx \sum_{||\xi| - \lambda| \geq L} \frac{1}{(|\xi|^2 - \lambda)^2}.
\]

We recall how to evaluate lattice sums using summation by parts:

Let \( 0 = n_0 < n_1 < n_2 < \ldots \) be the set of norms, and

\[
N(t) = \sum_{n_k \leq t} r_L(n_k).
\]

Then for a smooth function \( f(t) \) on \([n_A+1, n_B]\) we have

\[
\sum_{n_A < |\xi| \leq n_B} f(|\xi|^2) = N(n_B) f(n_B) - N(n_A) f(n_{A+1}) - \int_{n_{A+1}}^{n_B} f'(t) N(t) \, dt.
\]

But since

\[
N(x) = \frac{4}{3} \pi abc x^{3/2} + O(x^\theta)
\]

and since

\[
\frac{4}{3} \pi abc n_{A+1}^{3/2} = N(n_{A+1}) + O(n_{A+1}^\theta) = N(n_A) + r_L(n_{A+1}) + O(n_{A+1}^\theta)
\]

\[
= N(n_A) + O(n_{A+1}^\theta)
\]

\[
= \frac{4}{3} \pi abc n_A^{3/2} + O(n_{A+1}^\theta)
\]
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(and therefore \( n_A \approx n_{A+1} \), so \( \frac{4}{3} \pi abc n_{A+1}^{3/2} = \frac{4}{3} \pi abc n_A^{3/2} + O\left(n_A^2\right) \), we have

\[
\sum_{n_A < |\xi|^2 \leq n_B} f(|\xi|^2) = \left( \frac{4}{3} \pi abc n_B^{3/2} + O\left(n_B^2\right) \right) f(n_B)
- \left( \frac{4}{3} \pi abc n_A^{3/2} + O\left(n_A^2\right) \right) f(n_{A+1})
- \int_{n_{A+1}}^{n_B} f'(t) \left( \frac{4}{3} \pi abct^{3/2} + O\left(t^\theta\right) \right) dt
= \left( \frac{4}{3} \pi abc n_B^{3/2} + O\left(n_B^2\right) \right) f(n_B)
- \left( \frac{4}{3} \pi abc n_A^{3/2} + O\left(n_A^2\right) \right) f(n_{A+1})
- \frac{4}{3} \pi abc n_B^{3/2} f(n_B) + \frac{4}{3} \pi abc n_A^{3/2} f(n_{A+1})
+ \frac{4}{3} \pi abc \cdot \frac{3}{2} \int_{n_{A+1}}^{n_B} f(t) t^{1/2} dt - \int_{n_{A+1}}^{n_B} f'(t) O\left(t^\theta\right) dt
= 2\pi abc \int_{n_{A+1}}^{n_B} f(t) t^{1/2} dt
+ O\left(n_B^2 f(n_B) + n_A^2 f(n_{A+1})\right)
+ O\left( \int_{n_{A+1}}^{n_B} |f'(t)| t^\theta dt \right).
\]

Now

\[
\sum_{|\xi|^2 - \lambda \geq L} \frac{1}{\left(|\xi|^2 - \lambda\right)^2} = \frac{1}{\lambda^2} + \sum_{0 < |\xi|^2 \leq \lambda - L} \frac{1}{\left(|\xi|^2 - \lambda\right)^2} + \sum_{\lambda + L \leq |\xi|^2} \frac{1}{\left(|\xi|^2 - \lambda\right)^2}.
\]

Applying (4.3) with \( f(t) = 1/(t - \lambda)^2 \), once with \( n_A = n_0 = 0 \) and \( n_B \leq \lambda - L < n_{B+1} \) and then with \( n_A < \lambda + L \leq n_{A+1} \) and \( n_B = \infty \) gives

\[
\sum_{n_0 < |\xi|^2 \leq \lambda - L} \frac{1}{\left(|\xi|^2 - \lambda\right)^2} = 2\pi abc \int_{n_1}^{n_B} \frac{t^{1/2}}{(t - \lambda)^2} dt
+ O\left( \frac{n_B^2}{(n_B - \lambda)^2} \right) + O\left( \int_{n_1}^{n_B} \frac{t^\theta}{(\lambda - t)^3} dt \right).
\]
Note that
\[\int_{n_1}^{n_B} \frac{t^{1/2}}{(t-\lambda)^2} dt \leq n_B^{1/2} \int_{n_1}^{n_B} \frac{1}{(t-\lambda)^2} dt \leq \lambda^{1/2} \int_{n_1}^{n_B} \frac{1}{(t-\lambda)^2} dt \]
\[= \lambda^{1/2} \left( \frac{1}{\lambda-n_B} - \frac{1}{\lambda-n_1} \right) \leq \lambda^{1/2} \frac{1}{L} \leq \frac{\lambda^\theta}{L^2} \]

also
\[\frac{n_B^\theta}{(n_B-\lambda)^2} \leq \frac{\lambda^\theta}{L^2} \]

and
\[\int_{n_1}^{n_B} \frac{t^\theta}{(\lambda-t)^3} dt \leq n_B^\theta \int_{n_1}^{n_B} \frac{1}{(\lambda-t)^3} dt \leq \lambda^\theta \int_{n_1}^{n_B} \frac{1}{(\lambda-t)^3} dt \]
\[= \lambda^\theta \left( \frac{1}{(\lambda-n_B)^2} - \frac{1}{(\lambda-n_1)^2} \right) \leq \frac{\lambda^\theta}{2L^2} \]

so
\[\sum_{n_0<|\xi|^2\leq\lambda-L} \frac{1}{(|\xi|^2-\lambda)^2} \ll \frac{\lambda^\theta}{L^2}. \]

For the second sum, we have
\[\sum_{\lambda+L\leq|\xi|^2} \frac{1}{(|\xi|^2-\lambda)^2} = 2\pi abc \int_{n_A+1}^{\infty} \frac{t^{1/2}}{(t-\lambda)^2} dt \]
\[+ O\left( \frac{n_A^\theta}{(n_A+1-\lambda)^2} \right) + O\left( \int_{n_A+1}^{\infty} \frac{t^\theta}{(t-\lambda)^3} dt \right). \]

Note that
\[\int_{n_A+1}^{\infty} \frac{t^{1/2}}{(t-\lambda)^2} dt = \int_{n_A+1-\lambda}^{\infty} \frac{(s+\lambda)^{1/2}}{s^2} ds \]
\[\leq \int_L^{\infty} \frac{(s+\lambda)^{1/2}}{s^2} ds \]
\[= \int_{\lambda}^{\infty} \frac{(s+\lambda)^{1/2}}{s^2} ds + \int_{\lambda}^{\infty} \frac{(s+\lambda)^{1/2}}{s^2} ds \]
\[
\ll \lambda^{1/2} \int_{L}^{\lambda} \frac{1}{s^2} ds + \int_{\lambda}^{\infty} \frac{1}{s^{3/2}} ds \\
\ll \frac{\lambda^{1/2}}{L} \leq \frac{\lambda^\theta}{L^2}
\]
also
\[
\frac{n_A^{\theta}}{(n_{A+1} - \lambda)^2} \ll \frac{\lambda^\theta}{L^2}
\]
and
\[
\int_{n_{A+1}}^{\infty} \frac{t^\theta}{(t - \lambda)^3} dt = \int_{n_{A+1}}^{\infty} \frac{(s + \lambda)^\theta}{s^3} ds \\
\leq \int_{L}^{\infty} \frac{(s + \lambda)^\theta}{s^3} ds \\
= \int_{L}^{\lambda} \frac{(s + \lambda)^\theta}{s^3} ds + \int_{\lambda}^{\infty} \frac{(s + \lambda)^\theta}{s^3} ds \\
\ll \lambda^\theta \int_{L}^{\lambda} \frac{1}{s^3} ds + \int_{\lambda}^{\infty} \frac{1}{s^{3-\theta}} ds \ll \frac{\lambda^\theta}{L^2}
\]
so
\[
\sum_{\lambda + L \leq |\xi|^2} \frac{1}{(|\xi|^2 - \lambda)^2} \ll \frac{\lambda^\theta}{L^2}.
\]
We conclude that
\[
\|G_\lambda - G_{\lambda,L}\|^2 \ll \frac{\lambda^\theta}{L^2}
\]
and therefore by Lemma 4.2
\[
\|g_\lambda - g_{\lambda,L}\|_2^2 \ll \frac{\lambda^{-1+\theta+2\varepsilon}}{L^2} = \lambda^{2\delta-1+\theta+2\varepsilon}.
\]
Since \(\delta < (1 - \theta)/2 - \varepsilon\) this tends to zero.

From this, as in Lemmas 3.3, 3.4, we get the next two lemmas:

**Lemma 4.4.** Let \(0 < \delta < \min \{(1 - \theta)/2 - \varepsilon, 1/\tau - \varepsilon\}\), \(L = \lambda^{-\delta}\), we have
\[
\|G_{\lambda,L}\|_2 = \|G_\lambda\|_2 (1 + o(1))
\]
as \(\lambda \to \infty\) along \(\Lambda_2\). and
Lemma 4.5. Let $f \in C^\infty (\mathbb{T}^3)$ and $0 < \delta < \min \{(1 - \theta) / 2 - \varepsilon, 1/\tau - \varepsilon\}$, $L = \lambda^{-\delta}$, we have

$$|\langle fg, g \rangle - \langle fg_L, g_L \rangle| \to 0$$

as $\lambda \to \infty$ along $\Lambda_2$, so

$$\langle fg_L, g_L \rangle \to 0 \Rightarrow \langle fg, g \rangle \to 0$$

as $\lambda \to \infty$ along $\Lambda_2$.

4.4. A Density One Set

Let $0 \neq \zeta \in \mathcal{L}$, and denote $\zeta = (\zeta_1/a, \zeta_2/b, \zeta_3/c)$. Assume that $\zeta_3 \neq 0$ (the other cases are symmetric).

Define

$$S_\zeta = \left\{ \xi \in \mathcal{L} : |\xi|^2 \geq 1, \left| 2 \langle \xi, \zeta \rangle - |\zeta|^2 \right| < 1/4c^2 \right\}.$$

We prove now a simple upper bound for the number of elements in $S_\zeta$ up to $X$:

Lemma 4.6. We have

$$\# \left\{ \xi \in S_\zeta : |\xi|^2 \leq X \right\} \ll X.$$

Proof. For every $\xi \in S_\zeta$ such that $|\xi|^2 \leq X$, denote $\xi = (\xi_1/a, \xi_2/b, \xi_3/c)$.

For $1 \leq i \leq 3$ we have $|\xi_i| < X^{1/2}$, and

$$\left| 2 \langle \xi, \zeta \rangle - |\zeta|^2 \right| = \left| c_1 a^2 (2\xi_1 - \zeta_1) + c_2 b^2 (2\xi_2 - \zeta_2) + c_3 c^2 (2\xi_3 - \zeta_3) \right| < 1/4c^2$$

so

$$\left| \frac{c_1^2}{a^2} c_1 (2\xi_1 - \zeta_1) + \frac{c_2^2}{b^2} c_2 (2\xi_2 - \zeta_2) + c_3 c^2 (2\xi_3 - \zeta_3) \right| < 1/4$$

and we see that $\zeta_3 (2\xi_3 - \zeta_3)$ is uniquely determined by the values of $\xi_1, \xi_2$, and since $\zeta_3 \neq 0$, we get that $\xi_3$ is uniquely determined by the values of $\xi_1, \xi_2$. Since $|\xi_1|, |\xi_2| < X^{1/2}$, we conclude that

$$\# \left\{ \xi \in S_\zeta : |\xi|^2 \leq X \right\} \ll X$$

as claimed. \qed

For $0 \neq \zeta \in \mathcal{L}$, define

$$A_\zeta = \Lambda_2 \setminus \Lambda_2 = \{ \lambda \in \Lambda_2 : A(\lambda, L) \cap S_\zeta = \emptyset \}$$

(recall that $L = \lambda^{-\delta}$).

We claim that this is a density one set in $\Lambda_2$ (and hence in $\Lambda$), i.e. if we denote

$$B_\zeta = B_\zeta^\varepsilon = \Lambda_2 \setminus \Lambda_\varepsilon = \{ \lambda \in \Lambda_2 : A(\lambda, L) \cap S_\zeta \neq \emptyset \}$$

then

Lemma 4.7. We have $\{ \lambda \in B_\zeta : \lambda \leq X \} = o \left( X^{3/2} \right)$. 

Proof. Define \( \mathcal{N}_\zeta \subseteq \mathcal{N} \) to be the set of norms \(|\xi|^2\) of \( \xi \in S_\zeta \). We have
\[
\# \{ n \in \mathcal{N}_\zeta : n \leq 2X \} \asymp \# \left\{ \xi \in S_\zeta : |\xi|^2 \leq 2X \right\} \ll X.
\]
We have a map \( \iota : B_\zeta \to \mathcal{N}_\zeta \) defined by \( \iota(\lambda) \) being the closest element \( n \in \mathcal{N}_\zeta \) to \( \lambda \); if there are two such elements, i.e. \( n_- < \lambda < n_+ \) with \( n_\pm \in \mathcal{N}_\zeta \) and \( n_+ - \lambda = \lambda - n_- \), then set \( \iota(\lambda) = n_+ \).

We have \( \#\iota^{-1}(n) \ll n^{1/2-\delta+2\varepsilon} \). Indeed, first note that
\[
\#\iota^{-1}(n) \leq \# \left\{ \lambda \in \Lambda_2 : \exists \xi \in S_\zeta \cap A(\lambda, L), |\xi|^2 = n \right\} \ll \#\tilde{A}(\lambda_0, 3\lambda_0^{-\delta}) \leq \lambda_0^{1/2-\delta+2\varepsilon} \ll n^{1/2-\delta+2\varepsilon}.
\]
We conclude that
\[
\# \left\{ \lambda \in B_\zeta : \lambda \leq X \right\} \leq \sum_{\substack{n \in \mathcal{N}_\zeta \\text{such that} \\, n \leq 2X}} \iota^{-1}(n)
\ll X^{1/2-\delta+2\varepsilon} \cdot \# \left\{ n \in \mathcal{N}_\zeta : n \leq 2X \right\}
\ll X^{3/2-\delta+2\varepsilon}
\]
proving our claim. \( \square \)

From the last lemma we conclude that:

**Lemma 4.8.** We have \( \langle e_\zeta g_{\lambda, L}, g_{\lambda, L} \rangle \to 0 \) as \( \lambda \to \infty \) along \( \lambda \in \Lambda_\zeta \).

**Proof.** We have
\[
\langle e_\zeta G_{\lambda, L}, G_{\lambda, L} \rangle \asymp \sum_{\xi \in A(\lambda, L)} \frac{1}{(|\xi|^2 - \lambda) \left( |\xi - \zeta|^2 - \lambda \right)}.
\]
Note that
\[
|\xi - \zeta|^2 - \lambda = |\xi|^2 - \lambda - 2 \langle \xi, \zeta \rangle + |\zeta|^2 \geq |2 \langle \xi, \zeta \rangle - |\zeta|^2| - |\xi|^2 - \lambda|
\]
and since \( \lambda \in \Lambda_\zeta \) we have \( S_\zeta \cap A(\lambda, L) = \emptyset \), so
\[
|2 \langle \xi, \zeta \rangle - |\zeta|^2| \geq \frac{1}{4c^2}.
\]
We get that for large enough \( \lambda \)
\[
|\xi - \zeta|^2 - \lambda \geq \frac{1}{4c^2} - L \gg 1
\]
and therefore
\[ |\langle e_\zeta G_\lambda, L, g_\lambda, L \rangle| \ll \sum_{\xi \in A(\lambda, L)} \frac{1}{(|\xi|^2 - \lambda)} \ll \|G_\lambda\|_2 \left( \sum_{\xi \in A(\lambda, L)} 1 \right)^{1/2}. \]

Since \( \lambda \in \Lambda_\zeta \subseteq \Lambda_2 \), and since the multiplicities \( r_L(n) \) are bounded, we have
\[ \sum_{\xi \in A(\lambda, L)} 1 = \# A(\lambda, L) \ll \# \tilde{A}(\lambda, 3L) \leq L^{1/2 + 2\varepsilon} \]
so
\[ |\langle e_\zeta g_\lambda, L, g_\lambda, L \rangle| \ll \left\| G_\lambda \right\|_2 \lambda^{1/4 - \delta/2 + \varepsilon} \ll \left\| G_\lambda \right\|_2 \frac{\lambda^{1/4 - \delta/2 + \varepsilon}}{\left\| G_\lambda \right\|_2} \ll \lambda^{-1/4 - \delta/2 + 2\varepsilon} \]
which tends to zero as \( \lambda \to \infty \) (for \( \varepsilon > 0 \) small enough).

We conclude from Lemma 4.5 that
\[ \langle e_\zeta g_\lambda, g_\lambda \rangle \to 0 \]
as \( \lambda \to \infty \) along \( \lambda \in \Lambda_\zeta \).

4.5. Proof of Theorem 1.3
We now use a diagonalization argument to prove Theorem 1.3:

**Theorem.** There is a density one subset \( \Lambda_\infty \subseteq \Lambda \) so that for every observable \( a \in C^\infty (T^3) \), we have
\[ \langle ag_\lambda, g_\lambda \rangle \to \frac{1}{\text{area} (T^3)} \int_{T^3} a(x) \, dx \]
as \( \lambda \to \infty \) along \( \Lambda_\infty \).

**Proof.** For \( J \geq 1 \), let \( \Lambda_J \subseteq \Lambda \) be of density one so that for all \( |\zeta| \leq J \),
\[ \langle e_\zeta g_\lambda, g_\lambda \rangle \to 0 \]as \( \lambda \to \infty \) along \( \Lambda_J \), and in particular for every trigonometric polynomial \( P_J(x) = \sum_{|\zeta| \leq J} p_\zeta e_\zeta(x) \) we have
\[ \langle P_J g_\lambda, g_\lambda \rangle \to \frac{1}{\text{area} (T^3)} \int_{T^3} P_J(x) \, dx \quad (4.4) \]
along \( \Lambda_J \).

We can assume that \( \Lambda_{J+1} \subseteq \Lambda_J \) for each \( J \). Choose \( M_J \) so that \( M_J \uparrow \infty \) as \( J \to \infty \), and so that for all \( X > M_J \)
\[ \frac{\# \{ \lambda \in \Lambda_J : \lambda \leq X \}}{\# \{ \lambda \in \Lambda : \lambda \leq X \}} \geq 1 - \frac{1}{2^J} \]
and let \( \Lambda_\infty \) be such that \( \Lambda_\infty \cap [M_J, M_{J+1}] = \Lambda_J \cap [M_J, M_{J+1}] \) for all \( J \). Then \( \Lambda_\infty \cap [0, M_{J+1}] \) contains \( \Lambda_J \cap [0, M_{J+1}] \) and therefore \( \Lambda_\infty \) has density one in \( \Lambda \), and (4.4) holds for \( \lambda \in \Lambda_\infty \).
The theorem now follows from the density of trigonometric polynomials in $C^\infty(T^3)$ in the uniform norm (as in the proof of Theorem 1.1).

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Appendix A.

A.1. Integral Points in Spherical Strips

We estimate the number of integral points inside some strips on 3D spheres. The strategy is to estimate the integral points on every circle in the strip: a simple substitution reduces the problem to counting integral points on two-dimensional ellipses, which could be treated by some basic algebraic number theory.

Lemma A.1. Let $L = \lambda^\delta$, $0 < \delta < 1/4$. For every $0 \neq \zeta \in \mathbb{Z}^3, C_1, C_2$ and $n$ such that $|n - \lambda| < C_1 L$, we have

$$\# \left\{ \eta \in \mathbb{Z}^3 : |\eta|^2 = n, |\langle \eta, \zeta \rangle| < C_2 L \right\} \ll_{C_1, C_2, \zeta, \varepsilon} L n^\varepsilon.$$  

Proof. Denote $\zeta = (\zeta_1, \zeta_2, \zeta_3)$, and without loss of generality we can assume that $\zeta_3 \neq 0$. For $\eta = (x, y, z)$ with $|\eta|^2 = n$, $|\langle \eta, \zeta \rangle| = m$, $|m| < C_2 L$, we have $x^2 + y^2 + z^2 = n$, $\zeta_1 x + \zeta_2 y + \zeta_3 z = m$. Since $\zeta_3 \neq 0$ we get $z = \frac{m - \zeta_1 x - \zeta_2 y}{\zeta_3}$, and substitution gives:

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \quad (A.1)$$

where

$$a = \zeta_1^2 + \zeta_3^2,$$
$$b = \zeta_1 \zeta_2,$$
$$c = \zeta_2^2 + \zeta_3^2,$$
$$d = -\zeta_1 m,$$
$$e = -\zeta_2 m,$$
$$f = -\zeta_3^2 n + m^2.$$  

Note that $c > 0$, $ac - b^2 = \zeta_3^2 (\zeta_1^2 + \zeta_2^2 + \zeta_3^2) > 0$.

Completing the square using the fact that $c \neq 0$, we get:

$$(ac - b^2) x^2 + (bx + cy)^2 + 2cdx + 2cey + cf = 0.$$  

Setting $y' = bx + cy$ (and then $y = \frac{y' - bx}{c}$), we get that the number of integer solutions to equation (A.1) is bounded by the number of integer solutions to the equation

$$(ac - b^2) x^2 + y^2 + 2(cd - be) x + 2ey + cf = 0 \quad (A.2)$$
Completing the square again, we get
\[(ac - b^2) x^2 + (y + e)^2 + 2(cd - be) x + cf - e^2 = 0.\]
Setting \(y' = y + e\), we get that the number of integer solutions to equation \((A.2)\) is equal to the number of integer solutions to
\[(ac - b^2) x^2 + y^2 + 2(cd - be) x + cf - e^2 = 0. \quad (A.3)\]
Completing the square for the last time using the fact that \(ac - b^2 \neq 0\), we get
\[((ac - b^2) x + (cd - be))^2 + (ac - b^2) y^2 + (ac - b^2) (cf - e^2) - (cd - be)^2 = 0.\]
Setting \(x' = (ac - b^2) x + (cd - be)\), we get that the number of integer solutions to Eq. \((A.3)\) is bounded by the number of integer solutions to
\[x^2 + (ac - b^2) y^2 = (ac - b^2) (-cf + e^2) + (cd - be)^2. \quad (A.4)\]
Denote \(ac - b^2 = t^2 D\), where \(D > 0\) is squarefree, and
\[k = (ac - b^2) (-cf + e^2) + (cd - be)^2.\]
We get
\[x^2 + D (ty)^2 = k\]
and setting \(y' = ty\), we get that the number of integer solutions to Eq. \((A.4)\) is bounded by the number of integer solutions to
\[x^2 + Dy^2 = k\]
i.e. by the number \(r_D (k)\) of representations of an integer \(k\) by the quadratic form \(x^2 + Dy^2\). Now we claim that
\[r_D (k) \leq 6 \tau (k) \quad (A.5)\]
where \(\tau (k)\) is the number of divisors of \(k\). Since
\[\tau (k) \ll \varepsilon k^\varepsilon \ll_{C_1, C_2, \zeta, \varepsilon} n^\varepsilon\]
we conclude that
\[
\# \left\{ \eta \in \mathbb{Z}^3 : |\eta|^2 = n, \ \langle \eta, \zeta \rangle < C_2 L \right\} \ll_{C_1, C_2, \zeta, \varepsilon} Ln^\varepsilon
\]
as claimed.

The estimate \((A.5)\) follows from factorization into prime ideals in the ring of integers \(A\) of the imaginary quadratic extension \(\mathbb{Q} (\sqrt{-D})\). Indeed, given any prime \(p\), consider the principal ideal \(\langle p \rangle\) in \(A\). Then (cf. [7]) either \(\langle p \rangle\) is a prime ideal, or \(\langle p \rangle = \mathcal{P}_1 \mathcal{P}_2\), where \(\mathcal{P}_1, \mathcal{P}_2\) are prime ideals (and not necessarily different). The fundamental theorem of arithmetic yields
\[k = \prod_{\langle q_j \rangle \text{is prime}} q_j^{\beta_j} \prod_{\langle p_1 \rangle = \mathcal{P}_{1,1} \mathcal{P}_{1,2}} p_1^{\alpha_1}
\]
so we get the unique factorization
\[\langle k \rangle = \prod \langle q_j \rangle^{\beta_j} \prod \mathcal{P}_{1,1}^{\alpha_1} \mathcal{P}_{1,2}^{\alpha_1}\]
Each representation of $k = x^2 + Dy^2$ corresponds to a decomposition of the principal ideal

$$\langle k \rangle = \langle x + y\sqrt{-D} \rangle \langle x - y\sqrt{-D} \rangle$$

where $N (\langle x + y\sqrt{-D} \rangle) = N (\langle x - y\sqrt{-D} \rangle) = k$, so from the uniqueness of the factorization we get that

$$\langle x + y\sqrt{-D} \rangle = \prod \langle q_j \rangle^{\beta_j/2} \prod P_{i,1}^{\gamma_i} P_{i,2}^{\alpha_i - \gamma_i}$$

$$\langle x - y\sqrt{-D} \rangle = \prod \langle q_j \rangle^{\beta_j/2} \prod P_{i,1}^{\alpha_i - \gamma_i} P_{i,2}^{\gamma_i}$$

where $0 \leq \gamma_i \leq \alpha_i$. From this follows that the number of possibilities for $\langle x + y\sqrt{-D} \rangle$ is bounded by $\prod (1 + \alpha_i) \leq \tau (k)$. But $\langle x + y\sqrt{-D} \rangle = \langle x' + y'\sqrt{-D} \rangle$ if and only if $x + y\sqrt{-D}$ and $x' + y'\sqrt{-D}$ are associates in $A$, and since the number of units in $A$ is at most 6, we get that $r_D (k) \leq 6 \tau (k)$. □

**A.2. Discrepancy**

We prove some results from the theory of uniform distribution modulo 1. Most of this section is adapted from [6].

Let us recall the definition of discrepancy:

**Definition A.2.** Let $(x_n)$ be a sequence of real numbers. The number

$$D_N = \sup_{0 \leq a < b \leq 1} \left| \sum_{n=1}^{N} e^{2\pi i x_n (a+b/N)} - (b-a) \right|$$

is called the discrepancy of the given sequence.

A useful upper bound for the discrepancy is given by the theorem of Erdős–Turán [3]:

**Theorem A.3.** There exists an absolute constant $C$, such that for any real numbers $x_1, \ldots, x_N$ and for any positive integer $m$, we have

$$D_N \leq C \left( \frac{1}{m} + \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h x_n} \right| \right).$$

Let $\alpha \in \mathbb{R}$ be an irrational of finite type $\tau$, and let $\beta \in \mathbb{R}$. Define $x_n = \alpha n + \beta$. By the theorem of Erdős-Turán, the discrepancy of $(x_n)$ is bounded for any positive integer $m$ by
\[ D_N \ll \frac{1}{m} + \frac{1}{N} \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h(n\alpha + \beta)} \right| \]

\[ = \frac{1}{m} + \frac{1}{N} \sum_{h=1}^{m} \frac{1}{h} \left| \sum_{n=1}^{N} e^{2\pi i h\alpha} \right| \]

\[ = \frac{1}{m} + \frac{1}{N} \sum_{h=1}^{m} \frac{1}{h} \left| 1 - e^{2\pi i hN\alpha} \right| \]

\[ \leq \frac{1}{m} + \frac{1}{N} \sum_{h=1}^{m} \frac{1}{h} \frac{2}{\left| 1 - e^{2\pi i h\alpha} \right|} \]

\[ = \frac{1}{m} + \frac{1}{N} \sum_{h=1}^{m} \frac{1}{h} \frac{1}{\sin \pi h\alpha} \]

\[ \leq \frac{1}{m} + \frac{1}{2N} \sum_{h=1}^{m} \frac{1}{h} \frac{1}{\|h\alpha\|}. \quad (A.6) \]

From here we could continue by

\[ \frac{1}{m} + \frac{1}{2N} \sum_{h=1}^{m} \frac{1}{h} \frac{1}{\|h\alpha\|} \ll \frac{1}{m} + \frac{1}{2N} \sum_{h=1}^{m} h^{\tau + \varepsilon - 1} \ll \frac{1}{m} + \frac{m^{\tau + \varepsilon}}{N} \]

and choosing \( m = \lfloor N^{1/(1+\tau)} \rfloor \) we could deduce that \( D_N = O\left( N^{-1/(1+\tau)+\varepsilon} \right) \).

But with a very little effort we can get a better estimate for the RHS of (A.6):

**Lemma A.4.** Let \( \alpha \) be an irrational of finite type \( \tau \), and let \( m \) be a positive integer. Then for every \( \varepsilon > 0 \), there exists a positive constant \( c = c(\alpha, \varepsilon) \) such that

\[ \sum_{h=1}^{m} \frac{1}{\|h\alpha\|} \leq cm^{\tau + \varepsilon}. \]

**Proof.** For \( 0 \leq q_1 < q_2 \leq m \), we have

\[ \|q_2\alpha \pm q_1\alpha\| = \|(q_2 \pm q_1) \alpha\| \geq \frac{c}{(q_2 \pm q_1)^{\tau + \varepsilon/2}} \geq \frac{c}{(2m)^{\tau + \varepsilon/2}}. \]

But \( \|q_1\alpha\| = |q_1\alpha - n_1| \) for some integer \( n_1 \), and \( \|q_2\alpha\| = |q_2\alpha - n_2| \) for some integer \( n_2 \), and hence

\[ \|q_2\alpha\| - \|q_1\alpha\| = \|q_2\alpha - n_2\| - |q_1\alpha - n_1| \]

\[ \geq \frac{c}{(2m)^{\tau + \varepsilon/2}}. \]

This implies that in each of the intervals

\[ \left[ 0, \frac{c}{(2m)^{\tau + \varepsilon/2}} \right], \left[ \frac{c}{(2m)^{\tau + \varepsilon/2}}, \frac{2c}{(2m)^{\tau + \varepsilon/2}} \right], \ldots, \left[ \frac{mc}{(2m)^{\tau + \varepsilon/2}}, \frac{(m+1)c}{(2m)^{\tau + \varepsilon/2}} \right] \]
there is at most one number of the form \( \| h\alpha \|, 1 \leq h \leq m \), with no such number in the first interval. Therefore

\[
\sum_{h=1}^{m} \frac{1}{\| h\alpha \|} \leq \sum_{h=1}^{m} \frac{(2m)^{\tau+\varepsilon/2}}{hc} \leq \sum_{h=1}^{m} \frac{1}{h} \leq \tilde{c}m^{\tau+\varepsilon}.
\]

\[
\square
\]

**Corollary A.5.** Let \( \alpha \) be an irrational of finite type \( \tau \), and let \( m \) be a positive integer. Then for every \( \varepsilon > 0 \), there exists a positive constant \( c = c(\alpha, \varepsilon) \) such that

\[
\sum_{h=1}^{m} \frac{1}{h \| h\alpha \|} \leq cm^{\tau-1+\varepsilon}.
\]

**Proof.** Define \( S(t) = \sum_{h \leq t} \frac{1}{\| h\alpha \|} \).

From Lemma A.4 we have \( S(t) = S([t]) \leq c[t]^{\tau+\varepsilon} \leq ct^{\tau+\varepsilon} \). Using partial summation we conclude that

\[
\sum_{h=1}^{m} \frac{1}{h \| h\alpha \|} = \frac{S(m)}{m} + \int_{1}^{m} \frac{S(t)}{t^2} dt
\]

\[
\leq cm^{\tau-1+\varepsilon} + c \int_{1}^{m} \frac{dt}{t^2 - \tau - \varepsilon}
\]

\[
= c \left( m^{\tau-1+\varepsilon} + \frac{m^{\tau-1+\varepsilon} - 1}{\tau - 1 + \varepsilon} \right)
\]

\[
\leq \tilde{c}m^{\tau-1+\varepsilon}.
\]

\[
\square
\]

Substituting Corollary A.5 in (A.6), we conclude that \( D_N \ll \alpha, \varepsilon \frac{1}{m} + \frac{m^{\tau-1+\varepsilon}}{N} \), and choosing \( m = \lfloor N^{1/\tau} \rfloor \), we get that there exists a positive constant \( c = c(\alpha, \varepsilon) \) such that

\[
D_N \leq cN^{-1/\tau+\varepsilon}. \tag{A.7}
\]

Note that the constant \( c \) does not depend on \( \beta \).

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