Acceleration and localization of matter in a ring trap

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A toroidal trap combined with external time-dependent electric field can be used for implementing different dynamical regimes of wave motions. In particular, we show that dynamical and stochastic acceleration, localization and implementation of the Kapitza pendulum can be originated by means of proper choice of the external force.

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I. INTRODUCTION

Exploring different geometries of potentials trapping cold condensed atoms is of both fundamental and practical importance. Toroidal traps play a special role allowing for "infinite" atomic trajectories and for realization of quasi-one-dimensional (quasi-1D) regimes. These advantages are relevant for designing highly precise sensors based on matter wave interferometry \[1,2\] as well as for accurate study of such phenomena as superfluid currents, stability of sound waves, solitons and vortices in Bose-Einstein condensates (BEC's) \[3,4\]. Traps with circular geometry are also believed to be conceptually important for implementation of the main ideas of the accelerator physics at ultra-low temperatures \[2,5\] and, in particular, for acceleration of ultracold atoms \[6,7\]. In this last context existence of well localized wave packets, and thus attenuation of the dispersion, the latter being the intrinsic property of a quantum systems, is of primary importance. In the first experimental studies \[2\] it was shown that the dispersive spreading out \[1\] can be compensated by using betatron resonances in a storage ring. An alternative way of contra-balancing dispersion is also well known - it is nonlinearity, leading in quasi-1D regime to existence of bright and dark matter solitons (see e.g. \[4,7\] and \[8,9,10\], respectively). This issue has already been explored \[11\] from the point of view of acceleration of matter waves in a toroidal trap with help of a modulated optical lattice, which is known to be an efficient tool for acceleration of matter waves \[12\].

In this paper we propose two alternative ways of accelerating matter wave solitons - either by time varying or by stochastic external electric field. These new ways of soliton acceleration are especially relevant in view of radiative losses \[13\] and distortions \[12\] of solitons moving in optical lattices (the effects acquiring significance for long trajectories). At the same time, it turns out that the toroidal geometry of a trap confining a BEC allows one to realize a number of other dynamical regimes, like dynamical localization of solitons and solitonic implementation of the celebrated Kapitza pendulum. Theoretical description of all mentioned phenomena can be put witching the unique framework, based on the perturbation theory for solitons, what is done in the present paper. More specifically, in Sec. \[II\] we formulate the model and the main physical constraints determining its validity. In sections \[III\] and \[IV\] we describe how by applying external time-dependent electric field matter solitons can be accelerated in the usual sense and in the sense of the time increase of the velocity variance (the stochastic acceleration), respectively. In Sec. \[V\] we describe localized states of the matter in circular trap subject to external field, and in Sec. \[VI\] we show that a matter soliton affected by rapidly varying force represents an example of the Kapitza pendulum \[14\]. Summary and discussion of the results are given in the Conclusion.

II. SCALING AND THE EVOLUTION EQUATION

We assume that a BEC is loaded in a circular trap, which in cylindrical coordinates \(r = (\rho, \varphi, z)\) is described by \(V = V_0(\rho) + m \omega_z^2 z^2/2\), where \(\omega_z\) is the frequency of the magnetic trap in the \(z\)-direction, \(V_0(\rho)\) is the potential in the radial direction, forming the trap circular in the \((x,y)\)-plane, and \(m\) is the mass of an atom. We also suppose that the BEC is subject to external electric field with amplitude \(E_0\), which produces an additional potential \(V_{ext} = -\alpha' E_0^2/4\), where \(\alpha'\) is the polarizability of the atoms (see e.g. \[13\]). If the amplitude \(E_0\) or direction of the field vary along some direction, say, along the \(x\)-axis, smoothly on the scale of the trap radius \(R\), the potential energy \(V_{ext}\) can be expanded in the Taylor series and, after neglecting nonessential constant, be rewritten in the form \(V_{ext} = -\alpha x\), where \(\alpha = (\alpha'/4) \partial (E_0^2)/\partial x\)\(=0\) and we restricted the consideration only to the first term of the expansion. In order to realize one-dimensional geometry we require torus radius to be much larger than the core radius \(r_c\), what allows us to define a small parameter \(\varepsilon = r_c/R \ll 1\). In order to introduce quantitative characteristics, we consider the normalized ground state
\[ \phi \text{ of the eigenvalue problem} \]
\[ -\frac{\hbar^2}{2m} \frac{d}{d\rho} \left( \frac{d\phi}{d\rho} \right) + V_\text{e}(\rho)\phi = \varepsilon_\rho \phi, \quad \int_0^\infty \phi^2 \rho d\rho = 1 \]

and define \( R_1 = \int_0^\infty \phi^2 \rho^2 d\rho, \quad R_2 = \left( \int_0^\infty \phi^2 \rho d\rho \right)^{-1/2}, \) and \( \lambda = \lambda_0^{-1/2}. \) In the case at hand \( \lambda \sim \sqrt{R_1} \sim e^{1/2} R \) and thus \( \lambda \ll R_1 \sim R_2 \sim R. \)

In the present paper we are interested in the dynamical properties of the BEC excitations along the trap are of order of \( \lambda, \) which is the well defined parameter and thus convenient for formulation of the constraints of the theory. Indeed, now we can estimate the kinetic energy of the longitudinal excitations as \( \varepsilon_\rho = \hbar^2/(2m\lambda^2) \) and require it to be much less than the kinetic energy of the transverse excitations, \( \varepsilon_\tau \sim \hbar^2/(2mr^2) \) (for the sake of simplicity here we assume that the size of the trap in \( z \)-direction is of order of the core radius: \( a_s = \sqrt{\hbar/m\omega_z} \sim r_c. \) Adding the requirement for the energy of the two-body interactions, which is estimated as \( \varepsilon_{\text{e}} = \hbar^2/(2ma^2) \) (where \( g = 4\pi\hbar^2a_s/m, \) \( a_s \) is the scattering length, \( n \sim N/V \) is a mean density, \( N \) is the total number of atoms and \( V \) is the effective volume occupied by the atoms and estimated as \( V \sim \pi L a_s, \) ) to be of order of \( \varepsilon_{\text{e}} \) and to be much less than \( \varepsilon_\rho \) (or more precisely, requiring \( \varepsilon_{\text{e}}/\varepsilon_\rho \sim \epsilon \) can we neglect in the leading order the transitions between the transverse energy levels \( \epsilon_0, \epsilon_1, \) and employ the multiple scale expansion \([8,10]\) for description of the quasi-one-dimensional evolution of the BEC. We also notice that subject to the assumptions introduced, one has the estimate \( N \sim e^{1/2}\Pi/(8|a_s|). \)

In order to get an insight on practical numbers, let us consider \( ^7\text{Li} \) atoms \((a_s = -2 \text{nm})\) in a trap with \( R = 100 \mu m, \) \( r_c = 5 \mu m \) and \( a_s = 10 \mu m. \) Then \( \epsilon = 0.05, \) the characteristic size of solitonic excitations is \( \lambda \approx 22 \mu m \) and the number of particles is estimated as \( N \approx 140. \) We emphasize, that these estimates indicate only an order of the parameters. Thus, for example, a condensate of \( 10^2 \div 10^3 \) lithium atoms satisfy the conditions of the theory.

We will be interested in managing soliton dynamics by means of weak (i.e not destroying solitons) electromagnetic field varying in time. Respectively, we consider a time-dependent and characterized by the estimate \( \alpha \sim \hbar^2/(mR_1 \lambda^2). \) Then, starting with the Gross-Pitaevskii equation, in which the external potential in cylindrical coordinates has the form \( V_{\text{ext}} = -\alpha \rho \cos(\varphi) \approx -\alpha R_1 \cos(\varphi), \) and using the multiple-scale expansion one ensures that the BEC macroscopic wave function in the leading order allows factorization

\[ \Psi = \pi^{-1/4} a_s^{-1/2} e^{-i(\omega_\tau + \omega_\rho) t/2} e^{-z^2/2a^2} \phi(t, \varphi), \]

where \( \omega_\tau = 2\varepsilon_\rho /\hbar \) and \( \phi(t, \varphi) \) solves the nonlinear Schrödinger equation, which we write in terms of \( A = \sqrt{|g|/\sqrt{2\pi\hbar^2}a_\omega}, \quad \zeta = R_2 \varphi/\lambda, \) and \( \tau = \hbar t/m\lambda^2 \)

\[ i \frac{\partial A}{\partial \tau} = -\frac{1}{2} \frac{\partial^2 A}{\partial \zeta^2} - \cos(\kappa \zeta) f(\tau) A + \sigma |A|^2 A. \]  

Here \( \sigma = 4\alpha a_s, \) \( f(\tau) \equiv mR_1 \lambda^2 \alpha(t)/\hbar^2 \) and \( \kappa = \lambda/R_2 \sim \sqrt{\epsilon}. \) We choose the scaling such in a way that all terms in \( \epsilon \) are of the unity order, and in particular \( A = O(1). \) This can be done, taken into account the normalization

\[ \int_0^L |A|^2 d\zeta = 2\sqrt{2\pi a_s N} /\kappa a_s, \]  

\( \lambda = 2\pi/\kappa, \) which follows from the normalization condition for the order parameter \( \int |\Psi|^2 dz = N, \) and considering \( N \sim a_s /|a_s|, \) what is of order of \( 10^3, \) in a typical experimental setting.

Eq. \( 3 \) is subject to periodic boundary conditions \( A(\zeta, \tau) = A(\zeta + L, \tau). \)

### III. ACCELERATION OF BRIGHT MATTER SOLITONS BY TIME-DEPENDENT EXTERNAL FORCE

First we consider the acceleration, \( \gamma, \) which can be achieved due to the potential \( V_\text{e} \) properly dependent on time. An order of magnitude of \( \gamma \) can be estimated by taking into account that Eq. \( 3 \) makes sense provided that all terms are of the unity order. In the physical units this gives \( \gamma \sim \hbar^2/(m^2 \lambda^3). \) Then, recalling the above example of the lithium condensate we estimates \( \gamma \sim 7 \text{mm/s}^2. \) What is of order of the acceleration announced in \([11]\). It however does not provide the best estimate in our case, because it is based on the 1D model, while lowering dimensionality imposes constrains on the atomic density and consequently on the amplitude of the applied force.

To describe the physics of the phenomenon we consider a BEC with a negative scattering length \( (\sigma = -1). \) Then a "bright soliton" solution of Eq. \( 3 \) at \( f(\tau) \equiv 0 \) (or more precisely a periodic solution mimicking a bright soliton in an infinite 1D system) which moves with a constant velocity \( v_0 \) can be written down as follows\([8]\)

\[ A_s = e^{-i(\omega_\tau t) + v_0^2/2) t} e^{i\omega_\rho \tau} \eta(k) \sin(\eta(k)(\zeta - v_0 \tau)) \cos(\eta(k)(\zeta - v_0 \tau)), \]

Here \( \sin(x, k) \) is the Jacobi elliptic function \([10], \) \( k \) is the elliptic modulus parameterizing the solution. The frequency and the amplitude are given by: \( \omega(k) = (k^2/2 - 1) \eta(k) \) and \( \eta(k) = 2K(k)/L \) \([K(k) \text{ is the complete elliptic integral of the first kind}]. \) The velocity of the soliton is quantized \( v_0 = 2\pi n /L \) with \( n \) being integer.

To ensure that the solution \( A_s \) satisfies the scaling relations imposed above, we notice that the size of the soliton can be estimated as \( \pi/K(k) \) and its smallness implies that \( k \) is close to unity. In that case we obtain the estimates \( 1 - k^2 \sim 16\exp(-2\pi/\sqrt{\epsilon}) \) and \( \sin(\eta(k)(\zeta - v_0 \tau), k) \approx 1/\cosh(\eta(k)(\zeta - v_0 \tau)). \)
In the limit $k \to 1$ quantization of the velocity does not play any significant role. We verified this numerically. For example, for $L = 10$ deviation of the initial velocity from the quantized one produces appreciable effect on dynamics during intervals $\tau \lesssim 100$ only if $k \lesssim 0.99$.

When external force is applied, $f(\tau) \neq 0$, the velocity is not preserved any more, what manifests itself in evolution of the momentum $P = (1/2i) \int_0^L (A_{\bar{\zeta}}A - A\bar{A}_{\zeta})$ (here $\bar{A}$ stands for complex conjugation of $A$) according to the law:

$$\frac{dP}{d\tau} = -f(\tau) \int_0^L \cos(k\zeta) \frac{\partial |A|^2}{\partial \zeta} d\zeta. \quad (6)$$

The external field, however, does not affect the norm: $N = \int_0^L |A|^2 d\zeta = \text{const.}$ It follows from (3) that in the adiabatic approximation the solution of the perturbed equation (3) can be searched in the form

$$A = e^{-i\omega (k + v(\tau)^2/2) + iV(\tau)\zeta} \eta \frac{\partial n(\eta(\zeta - X(\tau)), k)}{\partial X}. \quad (7)$$

where $V(\tau) = dX(\tau)/d\tau$ is the time-dependent velocity of the soliton and $X(\tau)$ is the coordinate of the soliton center. Substituting (7) in (3) and taking into account the parity of the functions in the integrand as well as the fact that all of them are periodic with the same period $L$, we obtain the equation for the soliton coordinate

$$\frac{d^2 X}{d\tau^2} = -\kappa C(k) f(\tau) \sin(\kappa X). \quad (8)$$

Here $C(k) = \frac{K(k)}{2\pi E(k)} \int_0^{2\pi} \cos(\theta) d\theta \left\{ \frac{K(k)}{\pi} \delta(k, k) \right\} d\theta$ and it is taken into account that $N = 2\eta E(k)$, where $E(k)$ is the complete elliptic integral of the second kind.

Depending on the choice of the function $f(\tau)$, Eq. (8) describes different dynamical regimes. Now we are interested in acceleration which occurs during the rotational movement of the soliton in the trap (i.e., $X$ is a growing function). We illustrate this acceleration using an example of the simplest step-like dependence $f(\tau)$. To this end we assume that initially the soliton is centered at $X(0) = 0$ and require $f(\tau)$ to be a constant $f_0$ for time intervals such that the soliton coordinates $X(\tau) \in I_p$ and to be zero for $X(\tau) \notin I_p$ where the intervals $I_p$ are given by $I_p = [(p + \frac{1}{2})L, (p + 1)L]$ with $p = 0, 1, \ldots$. Then, as it is clear, the acceleration of the soliton, which is given by the right hand side of (8), is positive for all times. The above requirement introduces natural splitting of the temporal axis in the set of intervals $T_l = [\tau_l, \tau_{l+1}]$ ($l = 0, 1, \ldots$), with $\tau_0 = 0$, such that $f(\tau) = 0$ for $\tau \in T_0$ and $f(\tau) = f_0$ for $\tau \in T_{2p+1}$ (here $X(\tau_l) = Ll/2$. Thus, our task is to find $\tau_l$. This can be done by taking into account that during each of the "odd" intervals $T_{2p+1}$ Eq. (8) describes conservative nonlinear oscillator, the solution for which is well known. During "even" intervals $T_{2p}$ the motion is free (with a constant velocity) what means that the time $T_{2p}$ necessary for the soliton to cross an interval $[pL, (p + \frac{1}{2})L]$ is

$$T_{2p} = \tau_{2p+1} - \tau_p = L/(2v_{2p}), \quad (10)$$

where $v_{2p}$ is the velocity in the point $pL$. During the time interval $T_{2p+1}$ the soliton has to cross the interval $I_p$. From this condition we obtain:

$$T_{2p+1} = \tau_{2p+2} - \tau_{2p+1} = \frac{\sqrt{2}K(\sqrt{2}E_0/(H_{2p+1} + E_0))}{\kappa \sqrt{H_{2p+1} + E_0}}, \quad (11)$$

where $H_{2p+1} = v_{2p+1}^2/2 + E_0$ is the energy of the soliton in the point $(p + 1/2)L$, $E_0 = C(k)f_0$, and $v_{2p+1}$ is the soliton velocity in the same point. At the end of the interval $T_{2p+1}$ the soliton velocity is given by $v_{2(p+1)} = \sqrt{2(H_{2p+1} + E_0)}$. Thus one computes that after $p$ rotations the soliton acquires the velocity $v_{2(p+1)}$, which can be obtained from the recurrent relation: $v_{2(p+1)} = \sqrt{v_{2p}^2 + 4C(k)f_0}$.

In Fig. 1 a,b we compare the solution, obtained from the perturbation theory, Eq. (8), with numerical simulation of Eq. (9) for $f_0 = 0.3$. Nevertheless during the numerical simulation we used the values for $T_{2p}$ and $T_{2p+1}$ (Eqs. (10)–(11) obtained for the case of adiabatic approximation. It follows form the results presented that the dashed and solid lines perfectly match until $\tau \approx 50.0$. At larger times appreciable discrepancy appears. It occurs due to failure of the adiabatic approximation and can be removed by introducing temporal corrections to $T_{2p}$ and $T_{2p+1}$. This naturally leads to an optimization problem, which requires numerical approach and goes beyond the scope of the present work. Finally we notice, that for the above example of $^7$Li condensate the obtained acceleration is $0.36 \text{mm/s}^2$.

Comparison of the panels a and b in Fig. 1 shows that for $k \approx 1$ quantization of the velocity is not important,
what is also confirmed by the evolution of the solitonic forms depicted in the panels Fig.1 c-e.

IV. STOCHASTIC ACCELERATION OF MATTER SOLITONS

Now we concentrate on another dynamical regime – on the stochastic acceleration – where increase of the velocity of a matter soliton in a toroidal trap is achieved by applying a fluctuating external field. To this end holding all conditions of the applicability of the model [3], we consider the case of a stochastic force $f(\tau)$, which is delta-correlated Gaussian process with characteristics $\langle f(\tau) \rangle = 0$ and $\langle f(\tau)f(\tau') \rangle = D\delta(\tau - \tau')$ (the angular brackets stand for the stochastic averaging and $D$ is the dispersion). Now the dynamics can be described in terms of the distribution function

$$\mathcal{P}(V, \Phi, \tau) = \langle \delta(\Phi - \Phi(\tau))\delta(V - V(\tau)) \rangle,$$

(12)

where $\Phi(\tau) \equiv \kappa X$ is the angular coordinate of the soliton, $\Phi(\tau)$ and $V(\tau)$ with explicit time dependence stand for the soliton coordinates obtained from the dynamical equations while $\Phi$ and $V$ are considered as independent variables. The distribution function solves the Fokker-Planck equation, which is obtained by the standard procedure (see e.g. [17]):

$$\frac{\partial \mathcal{P}}{\partial \tau} = -V \frac{\partial \mathcal{P}}{\partial \Phi} + \tilde{D} \sin^2(\Phi) \frac{\partial^2 \mathcal{P}}{\partial V^2}.$$  

(13)

Here $\tilde{D} = \kappa^4 C^2 (k) D$ is the diffusion coefficient. Due to the circular geometry of the trap Eq. (13) is considered on the interval $-\pi < \Phi < \pi$ with the periodic boundary conditions $\mathcal{P}(V, \Phi - \pi, \tau) = \mathcal{P}(V, \Phi + \pi, \tau)$ with respect to $\Phi$ and zero boundary conditions with respect to $V$: $\mathcal{P} \to 0$ as $V \to \pm \infty$. The normalization condition for the probability density reads: $\int_{-\infty}^{\infty} dV \int_{-\pi}^{\pi} d\Phi \mathcal{P} = 1$.

Multiplying Eq. (13) by $V$ and $\Phi$ and integrating over $V$ and $\Phi$ one readily obtains that the average velocity and angular position of the soliton are constants, which for the sake of simplicity will be considered zeros, i.e. $\langle V \rangle = 0$ and $\langle \Phi \rangle = 0$. Next, multiplying (13) by $V^2$, $\Phi^2$ and $V\Phi$ and performing the integration one obtains the equations of the second momenta. They are not closed for all conditions of the applicability of the model (3), but increase of the velocity invariance deviates form the linear, as it would happen for the Brownian diffusion in the momentum space, what happens because the diffusion coefficient in the Fokker-Planck equation (13) is not a constant, but depends on the angular variable. However, due to the diffusion one can expect that the phase distribution will tend to homogeneous, i.e. that $\mathcal{P} \to 1/(2\pi)$ as $\tau \to \infty$. In this formal limit one obtains that $\langle V(\Phi) \rangle \to 0$, $\langle \sin^2(\Phi) \rangle \to 1/2$ and hence $\langle V^2 \rangle \to \tilde{D}\tau$. In other words, the system (14)-(16), describes random walk which in the limit of large time, approaches the Brownian diffusion in the velocity space. In that limit the stochastic acceleration, which can be defined as $\tilde{\gamma} = d\sqrt{\langle V^2 \rangle}/d\tau$, would tend to zero according to the law $\tilde{\gamma} \propto \tau^{-1/2}$.

In order to check the above predictions and reveal other features of the stochastic dynamics of a soliton in a ring trap we solved numerically Eq. (13) subject to the initial condition $\mathcal{P}(V, \Phi, 0) = \langle \delta(\Phi)\delta(V) \rangle$. The results is summarized in Fig. 2. In the panel a one observes the predicted monotonic growth of the mean velocity with time, which slightly different from the linear law. In the panel b one can see that the stochastic acceleration $\tilde{\gamma}$ is a monotonically decreasing function, which at sufficiently large times tends to zero. In particular, at $\tau \gtrsim 15$ the decreasing of the acceleration with time can be well approximated by the predicted law $\tilde{\gamma} \propto \tau^{-1/2}$, as it is shown by dashed curves in the panel b of Fig. 2 (it was verified that in at the same times $\langle \sin^2(\Phi) \rangle \approx 1/2$, what is in agreement with the analytical predictions). Also Fig. 2 b shows that the stochastic acceleration is larger for larger $D$. The physical explanation of this last fact is simple: the acceleration is generated by the stochastic force, whose intensity is determined by the dispersion $D$.
V. LOCALIZATION OF MATTER INDUCED BY THE EXTERNAL FIELD

Let us now turn to localized states of a matter in a toroidal trap and concentrate on the states generated by the constant external electric field, i.e. by \( f(\tau) = f_0 \). Respectively, we look for stationary solutions of Eq. (3) in the form \( A = e^{-i\omega\tau}A(\zeta) \) and obtain for \( A(\zeta) \) the equation:

\[
- \frac{1}{2} \frac{d^2 A}{d\zeta^2} - f_0 \cos(\kappa\zeta)A + \sigma |A|^2 A = \omega A
\]

which is subject to periodic boundary conditions \( A(\zeta, \tau) = A(\zeta + L, \tau) \).

Several lowest branches of the numerically obtained solutions of Eq. (17) are shown in Fig. 3. The lowest branch approaches zero at the frequency \( \omega_0 \approx -0.143 \) (it is interesting to mention that this frequency coincides with the lowest gap edge of the spectrum of the Mathieu equation (17) considered on the whole axis), where the amplitude of the nonlinear periodic mode is small and it transforms into the linear periodic Bloch mode at the lowest gap edge. Such a behavior of the branch is similar to that of the strongly localized modes in a BECs (it is interesting to mention that this frequency coincides with the lowest gap edge of the spectrum of the Mathieu equation (17) considered on the whole axis), where the amplitude of the nonlinear periodic mode is small and it transforms into the linear periodic Bloch mode at the lowest gap edge. Such a behavior of the branch is similar to that of the strongly localized modes in a BECs (it is interesting to mention that this frequency coincides with the lowest gap edge of the spectrum of the Mathieu equation (17) considered on the whole axis), where the amplitude of the nonlinear periodic mode is small and it transforms into the linear periodic Bloch mode at the lowest gap edge.

VI. MATTER SOLITON AS A KAPITZA PENDULUM

As the final example of nontrivial dynamics of a matter soliton in a toroidal trap we consider dynamical localization induced by a rapidly oscillating force \( f(\tau) = f_0 [\nu + \cos(\Omega \tau)] \). In this case the solitonic motion mimics the famous Kapitza pendulum, which acquires an additional stable point due rapid oscillation of the pivot [14].

Assuming that the physical conditions of the validity of the quasi-1D approximation [9] holds and that the frequency \( \Omega \) is large enough, i.e. \( \Omega^2 \gg \kappa^2 C(k) f_0 \), one can perform the standard analysis (see e.g. [14]), i.e. look for a solution of (3) in a form \( X(\tau) + \xi(\tau) \) where \( \xi \) is small, \( |\xi| \ll |X| \), and rapidly varying, and provide averaging over rapid oscillations. Then one arrives at the equation

\[
d^2 X/d\tau^2 = -\partial U/\partial X \quad \text{with the effective potential}
\]

\[
U = -C(k) f_0 \left[ \nu \cos(\kappa X) + \kappa^2 C(k) f_0/8\Omega^2 \cos(2\kappa X) \right].
\]

If the condition \( \kappa^2 C(k) f_0/(2\Omega^2 \nu) > 1 \) is met, the effective potential \( U \) possesses two stable points: \( X = 0 \) (\( \Phi = 0 \)) and \( X = L/2 \) (\( \Phi = \pi \)). So, it opens the possibility for the new type of soliton moving around the new stable point. Two typical trajectories of the soliton, obtained by numerical integration of Eq. (3), are presented in Fig. 4. One of the trajectories displays oscillations around the new equilibrium point, while the other one shows the large oscillations around the equilibrium point \( \Phi = 2\pi \) started with the same initial data but in the case where \( \Phi = \pi \) is not an equilibrium any more. The amplitude of large oscillations decay with time because of energy losses of the soliton in the nonconservative system.

VII. CONCLUSIONS

In the present paper we have shown that dynamics of a matter soliton in a toroidal trap, well reproducing one-dimensional geometry, can be very efficiently governed by time varying external electric field. In particular, such regimes like dynamical acceleration, stochastic acceleration, localization and implementation of the Kapitza pendulum can be realized by proper choices of the time dependence of the external force.

Experimental detection of the acceleration can be implemented either by direct imaging of the atomic cloud,
FIG. 4: (Color online) The angular coordinate of the soliton center vs time for the soliton motion affected by the rapidly oscillating external force, obtained numerically from Eq. (3) with parameters $L = 10.0$, $n = 0.01$, $\sigma = -1$, $f_0 = 0.15$, $\Omega = 2.0$, and $k = 0.99999$.

which is well localized in space and has well specified trajectory, or by measurement of the atomic distribution in the momentum space displaying shift of the maximum towards higher kinetic energies. Alternatively, one can study the evolution of the atomic cloud releasing from the trap (by switching the trap off) after some period of accelerating motion. The respective dynamics will be a spreading out cloud whose center of mass is moving with the acquired velocity.

The obtained results were based on the one-dimensional model, although deduced using the multiple-scale method and thus mathematically controllable. This means that a number of problem, are still left open. One of them is the limitation on the soliton velocity, and thus acceleration, introduced by lowering the space dimension, which appears when the solitonic kinetic energy becomes comparable with the transverse kinetic energy. Another limitation on the soliton acceleration emerges from the fact of the velocity quantization, when the radius of the ring trap is not large enough. We also left for further studies the diversity of localized stationary atomic distributions supported by the external filed, indicating only the lowest modes. We thus believe that the richness of the phenomena which can be observed by simple combination of the trap geometry and varying external field will stimulate new experimental and theoretical studies.

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