The orbit method solution for the deformed three coupled scalar fields

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Abstract

In this work, we present a deformed solutions starting from systems of three coupled scalar fields with super-potential $W(\phi_1, \phi_2, \phi_3)$ by orbit method. First, we deform the corresponding super-potential and obtain defect solutions. It is shown that how to construct new models altogether with its defect solutions in terms of the non-deformed model. Therefore, we draw the graph of super-potential and different fields in terms of $x$. So we observe that the graphs for deformed and non-deformed cases are changed by the scale.

Keywords: Three scalar fields; Deformation Method; Orbit solution

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1 Introduction

As we know the defect structures exist in different branches of physics, such as domain wall, kinks, vertices, monopoles, condensed matter and string theory. In higher space-times dimensions the defect structures are generated by real scalar fields such that the single real scalar field produce just single defect as kink-like and the double Sin-Gordon model may

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create two different defects. On the other hand, models containing two or more real scalar fields can give rise to at least two other classes of systems that produce defect that engender internal structure and those that support junctions of defects. Also two and three scalar fields describe the regular hexagonal network, Higgs model [1-5] and bent brane in five dimensions [6]. In other hand for field theories involving two and three real scalar fields, the mathematical problem concerning the integrability of equations of motion is much harder, as one deals with a system of two coupled second order nonlinear ordinary differential equations. Also the configuration space shows a distribution of minima that allows for a number of topological sectors. One way of simplifying the problem is to consider potentials belonging to large class corresponding to bosonic sector of supersymmetric theories. Such systems can be studied by super-potential. This super-potential lead us to consider all second order equations in terms of first order. All Bogomol’nyi-Prasad-Sommerfield (BPS) configuration can be described by this first order equation. For models with three interacting components, the solutions on each topological sector determine orbits in the configuration space, which can be expressed as a constraint equation \( C(\chi_1, \chi_2, \chi_3) = 0 \). The equations arising for three fields in deformation procedure more complicated than two and single field, so three deformation functions are required. It is difficult to solve this deformation equation, then we restrict the orbit in field space. Therefore, by deforming the first order equations for a three coupled field system we need to impose the orbit constraint. As we know the deformation method for the single and two coupled scalar fields are discussed by Bazeia and et al. But we are going to consider three coupled scalar fields in (1,1) dimensions in flat space-time, where metric is \( \eta_{\mu\nu} = (1, -1) \) [1-4]. The first-order differential equation and the static solutions such as topological solutions for stable states has been discussed by Refs. [7,8].

In present paper, we use deformation procedure to obtain defect solutions. This deformation plays an important role to investigate the energy of systems. For example in cosmological model, the energy density, pressure and equation of states can be controlled by the deformation method on the fields.

The above pretext give us motivation to discuss the deformation procedure. So the outline of the paper follows. In the next section three coupled scalar fields with an example is discussed and the analysis of the solution without deformation is shown. The deformation procedure for the three coupled scalar field with diagram is discussed in Sec.3. Also we compare two diagrams and show that the variations of the fields with respect to coordinates are changed by scale.

## 2 Three coupled scalar fields system

Lagrangian systems described by coupled scalar fields are gaining renewed attention. In the case of real fields, in particular, the work presented in [4] has introduced a specific class of systems of two coupled real scalar fields.

Let us to start with the lagrangian density with three coupled system,

\[
L = \frac{1}{2} \partial_{\mu} \phi_1 \partial^{\mu} \phi_1 + \frac{1}{2} \partial_{\mu} \phi_2 \partial^{\mu} \phi_2 + \frac{1}{2} \partial_{\mu} \phi_3 \partial^{\mu} \phi_3 - U(\phi_1, \phi_2, \phi_3),
\] (1)
where \( U = U(\phi_1, \phi_2, \phi_3) \) is in general a non-linear function of the three fields \( \phi_1, \phi_2 \) and \( \phi_3 \). Here we are using natural units, and so \( \hbar = 1 \) and the metric is such that \( x^\mu = (t, x) \) and \( x^\mu = (t, -x) \).

By using Euler-Lagrange equations, one can obtain the equations of motion as follows:

\[
\frac{\partial^2 \phi_1}{\partial x^2} = \frac{dU}{d\phi_1} \tag{2}
\]

\[
\frac{\partial^2 \phi_2}{\partial x^2} = \frac{dU}{d\phi_2} \tag{3}
\]

\[
\frac{\partial^2 \phi_3}{\partial x^2} = \frac{dU}{d\phi_3}.
\]

In order to obtain solution for the equation (2), we define super-potential function \( W = W(\phi_1, \phi_2, \phi_3) \) such that one may write the potential in terms of super potential,

\[
U(\phi_1, \phi_2, \phi_3) = \frac{1}{2} W_{\phi_1}^2 + \frac{1}{2} W_{\phi_2}^2 + \frac{1}{2} W_{\phi_3}^2, \tag{4}
\]

where \( W = W(\phi_1, \phi_2, \phi_3) \) is a smooth function and we have,

\[
W_{\phi_i} = \frac{\partial W}{\partial \phi_i}, \quad i = 1, 2, 3. \tag{5}
\]

The energy spectrum associated with these configurations could be written as [9-10],

\[
E = \frac{1}{2} \int_{-\infty}^{+\infty} dx \sum_{i=1}^{3} \left[ \left( \frac{d\phi_i}{dx} \right)^2 + W_{\phi_i}^2 \right], \tag{6}
\]

we can rewrite it also as following,

\[
E = E_{BPS} + \frac{1}{2} \int_{-\infty}^{+\infty} dx \sum_{i=1}^{3} \left[ \frac{d\phi_i}{dx} - W_{\phi_i} \right]^2. \tag{7}
\]

Here also we set the BPS energy,

\[
E_{BPS} = |\Delta W| = |W(\phi_i(\infty)) - W(\phi_i(-\infty))| \tag{8}
\]

This procedure shows that the energy is minimized to,

\[
E = E_{BPS}. \tag{9}
\]

As we have already learned from [4], we impose conditions,

\[
\frac{d\phi_i}{dx} = W_{\phi_i}, \quad i = 1, 2, 3. \tag{10}
\]
In this case we see that the energy gets to its lower bound $E_{BPS}$, and the above first-order equation (9) solve the corresponding equations of motion (2).

Now we consider special example of three coupled scalar fields which describes the regular hexagonal network. Let us consider the system defined by the following super-potential,

$$W(\phi_1, \phi_2, \phi_3) = \phi_1 - \frac{\phi_1^3}{3} - r\phi_1(\phi_2^2 + \phi_3^2), \quad (10)$$

In this case the first-order equations become,

$$\begin{align*}
\frac{d\phi_1}{dx} &= 1 - \phi_1^2 - r(\phi_2^2 + \phi_3^2) \\
\frac{d\phi_2}{dx} &= -2r\phi_1\phi_2 \\
\frac{d\phi_3}{dx} &= -2r\phi_1\phi_3
\end{align*}, \quad (11)$$

As we know the exact solution for the above equations is not clear, so we shall use the following elliptical orbit procedure [6,9,11],

$$\phi_1^2 + \frac{\phi_2^2}{r-2} + \frac{\phi_3^2}{r-2} = 1, \quad (12)$$

finally three-field static solution for the system (11) is,

$$\begin{align*}
\phi_1(x) &= \pm \tanh(2rx) \\
\phi_2(x) &= \pm \sqrt{\frac{1}{r} - 2\cos(\theta)sech(2rx)} \\
\phi_3(x) &= \pm \sqrt{\frac{1}{r} - 2\sin(\theta)sech(2rx)}
\end{align*}, \quad (13)$$

So we have a three-field model and its general orbit equation depending on two parameters $r$ and $\theta$ (the arbitrary phase).

We recall that there are several orbits for solving the equation (11). But here we shall consider the condition $0 < r < \frac{1}{2}$ and the orbit equation (12).

By putting equation (13) in (10) and (3) the corresponding super-potential and potential in terms of $x$ are given by,

$$W(x) = \frac{2}{3} \tanh(2rx)[1 + (3r - 1)sech^2(2rx)], \quad (14)$$

$$U(x) = 2rsech^2(2rx)[(3r - 1)sech^2(2rx) + 1 - 2r]. \quad (15)$$
3 Deformation procedure and orbit solution

We are going to apply deformation procedure for three couple scalar fields. As we know the deformation method for two and single field were discussed in Ref. s. [9, 12, 13]. First we transform three initial scalar fields \( \phi_1(x), \phi_2(x) \) and \( \phi_3(x) \) into the form of scalar fields \( \chi_1(x), \chi_2(x) \) and \( \chi_3(x) \) respectively.

In order to deform three scalar fields we introduce the following deformed function:

\[
\phi_i(x) = f_i(\chi_i), \quad i = 1, 2, 3, \quad (16)
\]

where non-deformed function \( \phi_i(x) \) and deformed function \( \chi_i(x) \) are differentiable and invertible. We can write the inverse function as \( \chi_i(x) = f_i^{-1}(\phi_i) \) for \( i = 1, 2, 3 \). For simplicity the deformed functions are just function of single field. The essential condition for the deformation functions are given by [12],

\[
\left[ \frac{d f_1(\chi_1)}{d \chi_1}, \frac{d f_2(\chi_2)}{d \chi_2}, \frac{d f_3(\chi_3)}{d \chi_3} \right]_{\text{orbit}}.
\]

which is just the orbit-based deformation procedure. Also, we note that in Ref. [12] they applied this procedure in two fields system and we will apply for three fields system where \( \phi_1(x) = f_1(\chi_1), \phi_2(x) = f_2(\chi_2) \) and \( \phi_3(x) = f_3(\chi_3) \).

This equation shows that the variation of non-deformed functions are equivalent to the deformed functions. Finally the deformed super-potential \( W(\chi_1, \chi_2, \chi_3) \) from eqs. (4) and (9) will be as;

\[
W = \int W_{\phi_i} d \chi_i.
\]

With the help of essential condition and eq. (3) we have,

\[
U = \frac{U}{(\frac{d f_i(\chi_i)}{d \chi_i})^2}, \quad (19)
\]

where \( U \) deformed potential in terms of deformed super-potential is,

\[
U = \sum_{i=1}^{3} \frac{1}{2} W_{\chi_i}^2.
\]

The energy of deformed defects can be written by following expression,

\[
\mathcal{E}_{BPS} = \Delta W|_{-\infty}^{+\infty}, \quad (21)
\]

where \( \mathcal{E} \) is deformed BPS energy.
Now we apply deformation method for super-potential (10). To start, we shall introduce the deformed function $\chi_1$,

$$\phi_1 = f_1(\chi_1) = \tan(\chi_1), \quad (22)$$

and from eq. (9) $\chi_1$ is given by,

$$\chi_1 = \arctan(\phi_1). \quad (23)$$

By using the following expression

$$\frac{\phi_3}{\phi_2} = c = \tan \theta$$

the orbit equation (12) will be as,

$$\phi_1^2 + \frac{(1 + c^2)}{r - 2} \phi_2^2 = 1. \quad (24)$$

In order to obtain the deformation fields $\chi_2$ and $\chi_3$, we use Eqs. (16) and (17), so we shall have;

$$\chi_2 = \int \frac{d\chi_1}{d\phi_1} d\phi_2, \quad \chi_3 = \int \frac{d\chi_1}{d\phi_1} d\phi_3. \quad (25)$$

Here we use equations (23), (24) and (13) the corresponding deformed fields $\chi_2$ and $\chi_3$ are respectively;

$$\chi_2 = \sqrt{\frac{1}{2r} - \cos(\theta) \text{arctanh}(\frac{1}{\sqrt{2}} \text{sech}(2rx))}, \quad (26)$$

$$\chi_3 = \sqrt{\frac{1}{2r} - \sin(\theta) \text{arctanh}(\frac{1}{\sqrt{2}} \text{sech}(2rx))}. \quad (27)$$

The deformation functions also will be the following;

$$\phi_2 = f_2(\chi_2) = \sqrt{2\left(\frac{1}{r} - 2\right) \cos(\theta) \text{tanh}\left(\frac{\text{sec}(\theta)}{\sqrt{\frac{1}{2r} - 1}} \chi_2\right)} \quad (28)$$

$$\phi_3 = f_3(\chi_3) = \sqrt{2\left(\frac{1}{r} - 2\right) \sin(\theta) \text{tanh}\left(\frac{\text{csc}(\theta)}{\sqrt{\frac{1}{2r} - 1}} \chi_3\right)}. \quad (29)$$
The deformed super-potential and deformed potential from equations (18) and (19) are given by:

\[ W = r \tanh(4rx), \]  
(30)
and

\[ U = \frac{2r \text{sech}^2(2rx) [(3r - 1) \text{sech}^2(2rx) + 1 - 2r]}{(1 + \tanh^2(2rx))^2}. \]  
(31)

Finally we can say the topological solutions of non-deformed and deformed equations are compared together by drawing their graphs see Figs. (1) and (2).

![Graphs of fields](image)

Figure 1: Left hand: Graphs of the fields \( \phi_1(x) \) (line) and \( \chi_1(x) \) (point), Middle: Graphs of the fields \( \phi_2(x) \) (line) and \( \chi_2(x) \) (point), Right hand: Graphs of the fields \( \phi_3(x) \) (line) and \( \chi_3(x) \) (point), all of them are for \( r = 0.25 \) and \( \theta = 45 \)

Fig. (1) shows that the plots of non-deformed fields \( \phi_1 \) and deformed \( \chi_1 \) which are drawn as kink. Plots of non-deformed fields \( \phi_2 \) and \( \phi_3 \) and deformed fields \( \chi_2 \) and \( \chi_3 \) are lumps. We see that the graphs of three fields for non-deformed and deformed are same. In that case they are just changed by scale. Fig. (2) shows plots of non-deformed and deformed super-potentials. Similarly, we see that the variation of two cases are the same, though by scale are different.

4 Conclusion

In this paper, we first have introduced three coupled scalar fields \( \phi_1, \phi_2 \) and \( \phi_3 \) [11] and obtained topological solutions by orbit method. Next we have deformed the initial fields as \( \chi_1, \chi_2 \) and \( \chi_3 \), also the topological solutions for deformed field are obtained by the orbit method. The solution of deformed and non-deformed three coupled scalar fields lead us to compare these two cases. So, we have shown that the variation of fields for two cases with respect to coordinates in Figs.(1) and (2) are same and different just by scale. Consequently,
different orbit may be lead to different full deformed models and solutions. As we know the deformed fields similar to non-deformed fields are the form of kink and lump solutions, so we have to choose a deformation function such as Eq. (22). Therefore we could consider Eq. (22) in different forms, it may be interesting for the future work.

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