A characterization of trace zero symmetric nonnegative 5x5 matrices

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Abstract
The problem of determining necessary and sufficient conditions for a set of real numbers to be the eigenvalues of a symmetric nonnegative matrix is called the symmetric nonnegative inverse eigenvalue problem (SNIPE). In this paper we solve SNIPE in the case of trace zero symmetric nonnegative 5 × 5 matrices.

1. Introduction
The problem of determining necessary and sufficient conditions for a set of complex numbers to be the eigenvalues of a nonnegative matrix is called the nonnegative inverse eigenvalue problem (NIEP). The problem of determining necessary and sufficient conditions for a set of real numbers to be the eigenvalues of a nonnegative matrix is called the real nonnegative inverse eigenvalue problem (RNIEP). The problem of determining necessary and sufficient conditions for a set of real numbers to be the eigenvalues of a symmetric nonnegative matrix is called the symmetric nonnegative inverse eigenvalue problem (SNIPE). All three problems are currently unsolved in the general case.

Loewy and London [6] have solved NIEP in the case of 3 × 3 matrices and RNIEP in the case of 4 × 4 matrices. Moreover, RNIEP and SNIPE are the same in the case of n × n matrices for n ≤ 4. This can be seen from papers by Fielder [2] and Loewy and London [6]. However, it has been shown by Johnson et al. [3] that RNIEP and SNIPE are different in general. More results about the general NIEP, RNIEP and SNIPE can be found in [1]. Other results about SNIPE in the case n = 5 can be found in [7] and [8].

Reams [9] has solved NIEP in the case of trace zero nonnegative 4 × 4 matrices. Laffey and Meehan [4] have solved NIEP in the case of trace zero nonnegative 5 × 5 matrices. In this paper we solve SNIPE in the case of trace zero symmetric nonnegative 5 × 5 matrices.
The paper is organized as follows: In Section 2 we present some notations. We also give some basic necessary conditions for a spectrum to be realized by a trace zero symmetric nonnegative $5 \times 5$ matrix. In Section 3 we state our main result without proof. In Section 4 we present some preliminary results that are needed for the proof of the main results. Finally, in Section 5 we prove our main result.

2. Notations and some necessary conditions
Let $\mathcal{R}$ be the set of trace zero nonnegative $5 \times 5$ matrices with real eigenvalues. Let $A \in \mathcal{R}$. We call a spectrum $\sigma = \sigma(A)$ a normalized spectrum if $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ with $1 = \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \geq -1$. Since $A$ has a zero trace $\sum_{i=1}^{5} \lambda_i = 0$.

Let $\mathcal{R}_s$ be the set of trace zero symmetric nonnegative $5 \times 5$ matrices. Then $\mathcal{R}_s \subset \mathcal{R}$.

Notice that for a normalized spectrum $\sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$:

$$\lambda_5 = \frac{1}{4} (\lambda_5 + \lambda_5 + \lambda_5 + \lambda_5) \leq \frac{1}{4} (\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) = -\frac{1}{4} \lambda_1 = -\frac{1}{4},$$

$$\lambda_2 + \lambda_3 + \lambda_4 = -\lambda_1 - \lambda_5 = -1 - \lambda_5 \geq -1 + \frac{1}{4} = -\frac{3}{4},$$

$$\lambda_2 + \lambda_3 + \lambda_4 = -\lambda_1 - \lambda_5 \leq -1 + 1 = 0.$$

Let $d = \lambda_2 + \lambda_3 + \lambda_4$. Then $d \in \left[-\frac{3}{4}, 0\right]$, $\lambda_4 = d - \lambda_2 - \lambda_3$ and $\lambda_5 = -d - 1$.

For the rest of this paper we shall use $x$ instead of $\lambda_2$ and $y$ instead of $\lambda_3$. We shall deal mainly with spectra of the form $\sigma = (1, x, y, d - x - y, -d - 1)$.

We are interested in finding necessary and sufficient conditions for $\sigma = (1, x, y, d - x - y, -d - 1)$ to be a normalized spectrum of a matrix $A \in \mathcal{R}_s$.

We start by looking at the necessary conditions that a normalized spectrum $\sigma = (1, x, y, d - x - y, -d - 1)$ imposes.

From $1 \geq x \geq y \geq d - x - y \geq -d - 1$ we get:

$$x \leq 1,$$

$$y \leq x,$$

$$y \leq -x + 2d + 1,$$

$$y \geq \frac{1}{2} (d - x).$$
Let the following points on the $xy$ plain be defined:

- $A = \left(\frac{1}{3}d, \frac{1}{3}d\right)$,
- $B = \left(d + \frac{1}{2}, d + \frac{1}{2}\right)$,
- $C = (3d + 2, -d - 1)$,
- $D = (1,2d)$,
- $E = \left(1, \frac{1}{2}d - \frac{1}{2}\right)$,
- $F = (d + 1, d)$,
- $G = \left(d + 1, -\frac{1}{2}\right)$,
- $H = (f(d), f(d))$,
- $I = \left(\frac{1}{2} + g(d), d + \frac{1}{2} - g(d)\right)$,
- $J = (2d + 1, 0)$,
- $O = (0,0)$,

where,

$$f(d) = \frac{2}{3}d - \frac{\sqrt{4d^2}}{3\sqrt{r(d)}} - \frac{3 \sqrt{r(d)}}{3\sqrt{4}},$$

$$r(d) = 4d^3 + 27d^2 + 27d + 3\sqrt{3}d^2(d + 1)(8d^2 + 27d + 27),$$

$$g(d) = \sqrt{\frac{\left(d - \left(-\frac{3}{4} + \frac{\sqrt{5}}{4}\right)\right)\left(d - \left(-\frac{3}{4} - \frac{\sqrt{5}}{4}\right)\right)}{2d + 1}}.$$

A few notes are in order:

1. $r(d)$ is real for $d \geq -1$ so $f(d)$ is real for $d \geq -1$ (except perhaps when $d = 0$),
2. $g(d)$ is real for $d \geq -\frac{3}{4} + \frac{\sqrt{5}}{4}$. 


For $d \in \left[ -\frac{3}{4}, -\frac{1}{3} \right]$ the above normalization conditions form the triangle $ABC$ in the $xy$ plane. Note that for $d = -\frac{3}{4}$ the triangle becomes a single point with coordinates \((-\frac{1}{4}, -\frac{1}{4})\).

For $d \in \left[ -\frac{1}{3}, 0 \right]$ the conditions form the quadrangle $ABDE$ in the $xy$ plane.

Another necessary condition we shall use is due to McDonald and Neumann [8] (Lemma 4.1).

**Theorem 1 (MN):** Let $A$ be a $5 \times 5$ irreducible nonnegative symmetric matrix with a spectrum $\sigma(A) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5$. Then $\lambda_2 + \lambda_5 \leq \text{trace}(A)$.

Loewy and McDonald [7] extended this result to any $5 \times 5$ nonnegative symmetric matrix (not just irreducible).

In our case $\sigma = (1, x, y, d - x - y, -d - 1)$ so we get the necessary condition $x \leq d + 1$.

The MN condition is already met when $d \in \left[ -\frac{3}{4}, -\frac{1}{2} \right]$, while it limits the realizable set to the quadrangle $ABFG$ when $d \in \left[ -\frac{1}{2}, 0 \right]$.

For $d \in \left[ -\frac{3}{4} + \frac{\sqrt{5}}{4}, 0 \right]$ let $P$ be the 5-vertex shape $AHIFG$ such that all its edges but $HI$ are straight lines and the edge $HI$ is described by the curve $\gamma: [0,1] \to \mathbb{R}^2$, where

$$\gamma(t) = (x(t), h(x(t))),$$

$$x(t) = (1 - t)f(d) + t \left( d + \frac{1}{2} + g(d) \right),$$

$$h(t) = -\frac{1}{2} (t - d) + \frac{1}{2} \sqrt{\frac{t^3 + dt^2 - d^2 t - 4d - 4d^2 - d^3}{t - d}}.$$ 

We shall show later that $h(t)$ is real for $t \in \left[ f(d), d + \frac{1}{2} + g(d) \right]$, $h(f(d)) = f(d)$ and $h \left( d + \frac{1}{2} + g(d) \right) = d + \frac{1}{2} - g(d)$, so that $\gamma(0) = H, \gamma(1) = I$ which makes $P$ well defined.
Let
\[ r(d) = 4d^3 + 27d^2 + 27d - 3\sqrt{3}d^2(d + 1)(8d^2 + 27d + 27). \]

For \( d < 0 \)
\[ r(d) = d\left(4d^2 + 27d + 27 + 3\sqrt{3}(d + 1)(8d^2 + 27d + 27)\right), \]
\[ r(d) = \frac{16d^5}{4d^2 + 27d + 27 + 3\sqrt{3}(d + 1)(8d^2 + 27d + 27)} \]

Note that \( \lim_{d \to 0^-} \frac{r(d)}{d^5} = \frac{16}{54} \) which means \( r(d) = O(d^5) \), so \( \lim_{d \to 0^-} f(d) = 0 \).

Moreover, \( g(0) = \frac{1}{2} \) and when \( d = 0 \) we have \( h(t) = 0 \) for \( t \geq 0 \). Therefore, when \( d = 0 \) we get \( H = 0 = A \) and \( I = J = F = D \) so \( P \) becomes the triangle \( AFG = OJG \).

See Appendix A for the orientation of the points \( A - J, O \) in the plain for different values of \( d \).

### 3. Statement of main result

The following theorem completely solves SNIEP in the case of trace zero symmetric nonnegative \( 5 \times 5 \) matrices.

**Theorem 2 (Main result):** A necessary and sufficient condition for \( \sigma = (1, x, y, d - x - y, -d - 1) \) to be a normalized spectrum of a matrix \( A \in \mathbb{R} \) is:

a) \((x, y)\) lies within the triangle \( ABC \) for \( d \in \left[-\frac{3}{4}, -\frac{1}{2}\right] \),

b) \((x, y)\) lies within the quadrangle \( ABFG \) for \( d \in \left[-\frac{1}{2}, -\frac{3}{4} + \frac{\sqrt{5}}{4}\right] \),

c) \((x, y)\) lies within \( P \) for \( d \in \left[-\frac{3}{4} + \frac{\sqrt{5}}{4}, 0\right] \).

An immediate corollary of Theorem 2 is:

**Theorem 3:** Let \( \sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \) and let \( s_k = \sum_{i=1}^{5} \lambda_i^k \). Suppose \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \). Then \( \sigma \) is a spectrum of a matrix \( A \in \mathbb{R} \) if and only if the following conditions hold:

1. \( s_1 = 0 \),
2. \( s_3 \geq 0 \),
3. \( \lambda_2 + \lambda_5 \leq 0 \).

### 4. Preliminary results

In order to prove Theorem 2 we need several results.
The first result is due to Fiedler [2], which extended a result due to Suleimanova [10]:

**Theorem 4 (Fiedler):** Let \( \lambda_1 \geq 0 \geq \lambda_2 \geq \cdots \geq \lambda_n \) and \( \sum_{i=1}^{n} \lambda_i \geq 0 \). Then there exists a symmetric nonnegative \( n \times n \) matrix \( A \) with a spectrum \( \sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), where \( \lambda_1 \) is its Perron eigenvalue.

The second result is due to Loewy [5]. Since this result is unpublished we shall give Loewy's proof.

**Theorem 5 (Loewy):** Let \( n \geq 4, \lambda_1 \geq \lambda_2 \geq 0 \geq \lambda_3 \geq \cdots \geq \lambda_n, \sum_{i=1}^{n} \lambda_i \geq 0 \). Suppose \( K_1, K_2 \) is a partition of \( \{3,4,\ldots,n\} \) such that \( \lambda_1 \geq -\sum_{i \in K_1} \lambda_i \geq -\sum_{i \in K_2} \lambda_i \). Then there exists a nonnegative symmetric \( n \times n \) matrix \( A \) with a spectrum \( \sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n) \), where \( \lambda_1 \) is its Perron eigenvalue.

For the proof of Theorem 5 we shall need another result of Fiedler [2]:

**Theorem 6 (Fiedler):** Let the following conditions hold:

1. \( (\alpha_1, \alpha_2, \ldots, \alpha_m) \) is realizable by a nonnegative symmetric \( m \times m \) matrix with \( \alpha_1 \) as its Perron eigenvalue,
2. \( (\beta_1, \beta_2, \ldots, \beta_n) \) is realizable by a nonnegative symmetric \( n \times n \) matrix with \( \beta_1 \) as its Perron eigenvalue,
3. \( \alpha_1 \geq \beta_1 \),
4. \( \varepsilon \geq 0 \).

Then \( (\alpha_1 + \varepsilon, \beta_1 - \varepsilon, \alpha_2, \ldots, \alpha_m, \beta_2, \ldots, \beta_n) \) is realizable by a nonnegative symmetric \( (m+n) \times (m+n) \) matrix with \( \alpha_1 + \varepsilon \) as its Perron eigenvalue.

**Proof of Theorem 5 (Loewy):** Let \( \varepsilon = \lambda_1 + \sum_{i \in K_1} \lambda_i \). Then by the assumptions \( \varepsilon \geq 0 \). We deal with two cases.

If \( \lambda_1 - \varepsilon \geq \lambda_2 + \varepsilon \) then by the assumptions the set \( \{\lambda_1 - \varepsilon\} \cup \{\lambda_i | i \in K_1\} \) meets the conditions of Theorem 4. Thus, there exists a symmetric nonnegative matrix whose eigenvalues are \( \{\lambda_1 - \varepsilon\} \cup \{\lambda_i | i \in K_1\} \) with \( \lambda_1 - \varepsilon \) as its Perron eigenvalue.

Similarly, the set \( \{\lambda_2 + \varepsilon\} \cup \{\lambda_i | i \in K_2\} \) meets the conditions of Theorem 4, so there exists a symmetric nonnegative matrix whose eigenvalues are \( \{\lambda_2 + \varepsilon\} \cup \{\lambda_i | i \in K_2\} \), where \( \lambda_2 + \varepsilon \) is its Perron eigenvalue. As \( \lambda_1 - \varepsilon \geq \lambda_2 + \varepsilon \) and \( \varepsilon \geq 0 \) all the conditions of Theorem 6 are met and therefore the set \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) is realizable by a nonnegative symmetric \( n \times n \) matrix with \( \lambda_1 \) as its Perron eigenvalue.
If \( \lambda_1 - \varepsilon < \lambda_2 + \varepsilon \) then let \( \delta = \frac{1}{2} (\lambda_1 - \lambda_2) \geq 0 \). We have \( \varepsilon > \frac{1}{2} (\lambda_1 - \lambda_2) = \delta \) and \( \lambda_1 - \delta = \lambda_2 + \delta = \frac{1}{2} (\lambda_1 + \lambda_2) \geq 0 \). Also \( \lambda_1 - \delta + \sum_{i \in K_1} \lambda_i = \varepsilon - \delta > 0 \), so the set \( \{ \lambda_1 - \delta \} \cup \{ \lambda_i | i \in K_1 \} \) meets the conditions of Theorem 4. We infer that there exists a symmetric nonnegative matrix whose eigenvalues are \( \{ \lambda_1 - \delta \} \cup \{ \lambda_i | i \in K_1 \} \) with \( \lambda_1 - \delta \) as its Perron eigenvalue. Also,

\[
\lambda_2 + \delta + \sum_{i \in K_2} \lambda_i \geq \lambda_2 + \delta + \sum_{i \in K_1} \lambda_i = \lambda_2 + \delta + \varepsilon - \lambda_1 = \delta + \varepsilon - 2\delta = \varepsilon - \delta > 0.
\]

Therefore, the set \( \{ \lambda_2 + \delta \} \cup \{ \lambda_i | i \in K_2 \} \) meets the conditions of Theorem 4. Thus, there exists a symmetric nonnegative matrix whose eigenvalues are \( \{ \lambda_2 + \delta \} \cup \{ \lambda_i | i \in K_2 \} \) with \( \lambda_2 + \delta \) as its Perron eigenvalue. Again, all the conditions of Theorem 6 are met and therefore the set \( \{ \lambda_1, \lambda_2, ..., \lambda_n \} \) is realizable by a nonnegative symmetric \( n \times n \) matrix with \( \lambda_1 \) as its Perron eigenvalue.

This completes the proof of the theorem.

Let \( d \in \left[ -\frac{1}{2}, 0 \right] \) and let \( \sigma = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \). Define \( s_k = \sum_{i=1}^{5} \lambda_i^k \). A necessary condition for \( \sigma \) to be a spectrum of a nonnegative 5 \( \times \) 5 matrix \( A \) is \( s_k \geq 0 \) for \( k = 1, 2, 3, ..., \). This easily follows from the fact that \( s_k = \text{trace}(A^k) \).

We shall investigate the properties of \( s_k \) for \( \sigma = (1, x, y, d - x - y, -d - 1) \) when \((x, y)\) lies within the triangle \( OBJ \). Note that for \( d \in \left[ -\frac{1}{2}, 0 \right] \) the triangle \( OBJ \) is contained in the triangle \( ABC \).

**Lemma 1:** Let \( d \in \left[ -\frac{1}{2}, 0 \right] \) and a positive integer \( k \) be set. Let \( s_k = 1 + x^k + y^k + (d - x - y)^k + (-d - 1)^k \) where \((x, y)\) lies within the triangle \( OBJ \). The following table summarizes the minimum and maximum values of \( s_k \) over the triangle \( OBJ \):

| \( k \) | Even | Odd |
|---|---|---|
| Minimum achieved at | \((0,0)\) | \((d + \frac{1}{2}, d + \frac{1}{2})\) |
| Minimum value | \(1 + d^k + (-d - 1)^k\) | \(1 + 2(d + \frac{1}{2})^k + 2(-d - 1)^k\) |
| Maximum achieved at | \((2d + 1,0)\) | \((0,0)\) |
| Maximum value | \(1 + (2d + 1)^k + 2(-d - 1)^k\) | \(1 + d^k + (-d - 1)^k\) |
Proof: First we note that for \( d = -\frac{1}{2} \) the triangle \( OBJ \) becomes a single point with coordinates \((0,0)\). In this case \( s_k = 1 + 2\left(-\frac{1}{2}\right)^k \), so for even \( k \) we have

\[
    s_k = 1 + \left(\frac{1}{2}\right)^{k-1} = 1 + \left(-\frac{1}{2}\right)^k + \left(-\left(-\frac{1}{2}\right) - 1\right)^k
    = 1 + \left(2\left(-\frac{1}{2}\right) + 1\right)^k + 2\left(-\left(-\frac{1}{2}\right) - 1\right)^k,
\]

and for odd \( k \) we have

\[
    s_k = 1 - \left(\frac{1}{2}\right)^{k-1} = 1 + 2\left(-\frac{1}{2} + \frac{1}{2}\right)^k + 2\left(-\left(-\frac{1}{2}\right) - 1\right)^k
    = 1 + \left(-\frac{1}{2}\right) + \left(-\left(-\frac{1}{2}\right) - 1\right)^k.
\]

The triangle \( OBJ \) is characterized by the following inequalities:

\[
    y \leq x,
\]

\[
    y \leq -x + 2d + 1,
\]

\[
    y \geq 0.
\]

We investigate \( s_k \) within the triangle by looking at lines of the form \( y = wx \), where \( w \in [0,1] \). These lines cover the entire triangle. For a given \( w \) we have \( x \in \left[0, \frac{2d + 1}{w + 1}\right] \).

\[
    s_k = 1 + x^k + y^k + (d - x - y)^k + (-d - 1)^k
    = 1 + (w^k + 1)x^k + (d - (w + 1)x)^k + (-d - 1)^k,
\]

\[
    \frac{\partial s_k}{\partial x} = k(w^k + 1)x^{k-1} - k(w + 1)(d - (w + 1)x)^{k-1}.
\]

Consider \( d = 0 \). As \( x \geq 0 \) and \( w \geq 0 \), for even \( k \) we have

\[
    \frac{\partial s_k}{\partial x} = k((w^k + 1) + (w + 1)^k)x^{k-1} \geq 0,
\]

and for odd \( k \) we have

\[
    \frac{\partial s_k}{\partial x} = k((w^k + 1) - (w + 1)^k)x^{k-1} \leq 0.
\]
Next assume \(-\frac{1}{2} \leq d < 0\). The derivative is zero when
\[ k(w^k + 1)x^{k-1} = k(w + 1)(d - (w + 1)x)^{k-1}. \]
We know \(d - (w + 1)x \neq 0\). Otherwise, we must have \(x = 0\) and therefore \(d = 0\), which is a contradiction to our assumption. Then, as \(0 \leq w \leq 1\), we have
\[
\left( \frac{x}{d - (w + 1)x} \right)^{k-1} = \frac{w + 1}{w^k + 1} \geq 1.
\]
Define
\[ h_1(w, k) = \frac{k}{k-1} \sqrt[1-k]{\frac{w + 1}{w^k + 1}} \geq 1. \]
Then, for every \(k\) the derivative is zero at
\[
x = \frac{d}{(w + 1)h_1(w, k)} + 1 \leq 0.
\]
For odd \(k\) the derivative is zero also at
\[
x = \frac{d}{(w + 1)h_1(w, k) - 1} \leq 0.
\]
In the triangle \(OBJ\) we have \(x \geq y \geq 0\), so the derivative of \(s_k\) has the same sign for \(x \in \left[0, \frac{2d+1}{w+1}\right]\). The derivative at \(x = 0\) equals \(-k(w + 1)d^{k-1}\). Therefore, for odd (even) \(k\) the derivative is negative (positive).

Thus, for \(d \in \left[-\frac{1}{2}, 0\right]\) and \(w \in [0,1]\) we proved the following: For odd (even) \(k\) the maximum (minimum) value of \(s_k\) over the line \(y = wx\) is achieved at \(x = 0\) and the minimum (maximum) value of \(s_k\) over the line \(y = wx\) is achieved at \(x = \frac{2d+1}{w+1}\).

At \(x = 0\) we have \(s_k = 1 + d^k + (-d - 1)^k\) and at \(x = \frac{2d+1}{w+1}\) we have \(s_k = h_2(w, d, k)\), where
\[
h_2(w, d, k) = 1 + (w^k + 1) \left(\frac{2d + 1}{w + 1}\right)^k + 2(-d - 1)^k.
\]
The derivative of \(h_2\) with respect to \(w\) is
\[
\frac{\partial h_2}{\partial w} = \left(-k \frac{w^k + 1}{(w + 1)^{k+1}} + k \frac{w^{k-1}}{(w + 1)^k}\right)(2d + 1)^k.
\]
Since we already dealt with the case \(d = -\frac{1}{2}\) we can assume \(d \neq -\frac{1}{2}\). Therefore, the derivative is zero when
\[ \frac{kw^{k-1}}{(w+1)^k} = \frac{w^k + 1}{(w+1)^{k+1}} \]

\[ (w+1)w^{k-1} = w^k + 1, \]

\[ w^{k-1} = 1. \]

So, for every \( k \) the derivative is zero at \( w = 1 \). For odd \( k \) the derivative is zero also at \( w = -1 \). In any case, the derivative has the same sign in the range \((-1, 1)\) of \( w \).

Since we assume \(-\frac{1}{2} < d \leq 0\) at \( w = 0 \) the derivative is negative and its value is \(-k(2d + 1)^k\). Thus, the minimum of \( h_2 \) is achieved at \( w = 1 \) and its value there is \( 1 + 2(d + \frac{1}{2})^k + 2(-d - 1)^k \). The maximum of \( h_2 \) is achieved at \( w = 0 \) and its value there is \( 1 + (2d + 1)^k + 2(-d - 1)^k \).

This completes the proof of the lemma.

**Lemma 2:** Let \( s_3 = 1 + x^3 + y^3 + (d - x - y)^3 + (-d - 1)^3 \). Then,

1. If \( d \in \left[-\frac{1}{2}, -\frac{3}{4} + \frac{\sqrt{5}}{4}\right] \) then \( s_3 \geq 0 \) within the triangle \( OBJ \),
2. If \( d \in \left[-\frac{3}{4} + \frac{\sqrt{5}}{4}, 0\right] \) then \( s_3 \geq 0 \) within the 4-vertex shape \( OHIJ \), which is formed by the intersection of the shape \( P \) with the triangle \( OBJ \).

**Proof:** Note that

\[ s_3 = 1 + x^3 + y^3 + (d - x - y)^3 + (-d - 1)^3 = 3(d - x)y^2 + 3(2dx - d^2 - x^2)y + 3(dx^2 - d^2x - d - d^2). \]

By Lemma 1 the minimum value over the triangle \( OBJ \) is

\[ 1 + 2 \left(d + \frac{1}{2}\right)^3 + 2(-d - 1)^3 = -3d^2 - \frac{9}{2}d - \frac{3}{4} \]

\[ = -3 \left(d - \left(-\frac{3}{4} + \frac{\sqrt{5}}{4}\right)\right) \left(d - \left(-\frac{3}{4} - \frac{\sqrt{5}}{4}\right)\right). \]

Therefore, for \( d \in \left[-\frac{1}{2}, -\frac{3}{4} + \frac{\sqrt{5}}{4}\right] \) we have \( s_3 \geq 0 \) over the entire triangle \( OBJ \).

For \( d \in \left[-\frac{3}{4} + \frac{\sqrt{5}}{4}, 0\right] \) we shall find when \( s_3 \geq 0 \).

When \( d = 0 \) we get \( s_3 = -3xy(x + y) \). As \( x \geq y \geq 0 \) in the triangle we conclude that \( s_3 \geq 0 \) if and only if \( y = 0 \). We already know that the when \( d = 0 \) the shape \( P \)
becomes the triangle \( OJG \), so the intersection with the triangle \( OBJ \) is the line \( OJ \). Therefore, the lemma is proved in this case.

Assume that \( d < 0 \). Then \( x \geq 0 > d \) so \( s_3 \) is a quadratic function in the variable \( y \).

The roots of \( s_3 \) as a function of \( y \) are:

\[
y_1 = -\frac{1}{2}(x - d) + \frac{1}{2} \sqrt{\frac{x^3 + dx^2 - d^2x - 4d - 4d^2 - d^3}{x - d}}.
\]

\[
y_2 = -\frac{1}{2}(x - d) - \frac{1}{2} \sqrt{\frac{x^3 + dx^2 - d^2x - 4d - 4d^2 - d^3}{x - d}}.
\]

Note that \( y_1 \) is exactly the function \( h(t) \) defined at the beginning of this paper with \( t \) replaced by \( x \).

First we find when \( y_1, y_2 \) are real. As \( x > d \) we require

\[
h_3(x) = x^3 + dx^2 - d^2x - 4d - 4d^2 - d^3 \geq 0.
\]

The derivative of \( h_3 \) with respect to \( x \) is

\[
\frac{\partial h_3}{\partial x} = 3x^2 + 2dx - d^2 = 3(x + d) \left( x - \frac{1}{3} d \right).
\]

So \( h_3 \) has a local minimum at \( x = -d \) and a local maximum at \( x = \frac{1}{3} d \). Since \( d < 0 \) we get

\[
h_3 \left( \frac{1}{3} d \right) = -\frac{32}{27} d^3 - 4d^2 - 4d = -4d \left( \frac{8}{27} d^2 + d + 1 \right) \geq 0,
\]

\[
h_3(-d) = -4d^2 - 4d = -4d(d + 1) \geq 0.
\]

Therefore, \( h_3 \) has a single real root \( x_3(d) \leq \frac{1}{3} d < 0 \). For \( x \geq x_3(d) \) we have \( h_3(x) \geq 0 \).

By using the formula for the roots of a cubic equation we get

\[
x_3(d) = -\frac{1}{3} d + \frac{2\sqrt[3]{4d^2}}{3\sqrt[3]{\Delta(d)}} + \frac{\sqrt[3]{2\sqrt[3]{\Delta(d)}}}{3},
\]

where,

\[
\Delta(d) = 4d^3 + 27d^2 + 27d + 3\sqrt{3}d^2(d + 1)(8d^2 + 27d + 27).
\]
Note that \( \Delta(d) < 0 \) for \(-1 \leq d < 0\). Otherwise, \( x_3(d) \geq -\frac{1}{3}d > 0 > \frac{1}{3}d \geq x_3(d) \), which is a contradiction. Also note that \( \Delta(d) \) is exactly the function \( r(d) \) defined at the beginning of this paper.

We conclude that \( y_1, y_2 \) are real for \( x \geq x_3(d) \). In that case \( y_1 \geq y_2 \). As \( x > d \) we have \( y_2 \leq 0 \) and when \( y \) is in the range \([y_2, y_1]\) we have \( s_3 \geq 0 \).

Next we check for what values of \( x \) the function \( y_1 \) intersects the triangle \( OBJ \).

Notice that:

\[
h_3(x) = (x - d)^3 + 4d(x + 1)(x - d - 1).
\]

Inside the triangle \( OBJ \) we have \( 0 \leq x \leq 2d + 1 \). As \( d < 0 \) we get \( x - d - 1 \leq d < 0 \). Therefore, \( h_3(x) \geq (x - d)^3 \geq 0 \). We conclude that \( y_1 \geq 0 \).

We first check when \( y_1 \leq x \):

\[
x \geq -\frac{1}{2}(x - d) + \frac{1}{2} \sqrt{\frac{x^3 + dx^2 - d^2x - 4d - 4d^2 - d^3}{x - d}},
\]

\[
(\frac{3}{2}x - \frac{1}{2}d)^2 \geq \frac{x^3 + dx^2 - d^2x - 4d - 4d^2 - d^3}{4(x - d)},
\]

\[
4(\frac{3}{2}x - \frac{1}{2}d)^2 (x - d) - (x^3 + dx^2 - d^2x - 4d - 4d^2 - d^3) \geq 0,
\]

\[
8x^3 - 16dx^2 + 8d^2x + 4d + 4d^2 \geq 0.
\]

Let \( h_4(x) = 8x^3 - 16dx^2 + 8d^2x + 4d + 4d^2 \).

The derivative of \( h_4 \) with respect to \( x \) is

\[
\frac{\partial h_4}{\partial x} = 24x^2 - 32dx + 8d^2 = 24(x - d) \left( x - \frac{1}{3}d \right).
\]

At \( x = d \) there is a local maximum of \( h_4 \) and its value there is

\[
h_4(d) = 4d + 4d^2 = 4d(d + 1) < 0.
\]

Therefore, \( h_4 \) has a single real root.

The derivative is positive for \( x > \frac{1}{3}d \). Also, as \( -\frac{3}{4} + \frac{\sqrt{15}}{4} \leq d < 0 \),

\[
h_4(0) = 4d + 4d^2 = 4d(d + 1) < 0,
\]

\[12\]
\[ h_4 \left( d + \frac{1}{2} \right) = 4d^2 + 6d + 1 = 4 \left( d - \left( -\frac{3}{4} + \frac{\sqrt{5}}{4} \right) \right) \left( d - \left( -\frac{3}{4} - \frac{\sqrt{5}}{4} \right) \right) \geq 0. \]

We conclude that there is a single root of \( h_4 \) in the range \( \left[ 0, d + \frac{1}{2} \right] \), which we denote by \( x_4(d) \). By using the formula for the roots of a cubic equation we get

\[ x_4(d) = \frac{2}{3} d - \frac{\sqrt{4d^2 - 3\Delta(d)}}{3\sqrt{3}}, \]

where \( \Delta(d) \) is as before. Also note that \( x_4(d) \) is exactly the function \( f(d) \) defined at the beginning of this paper.

We found that \( y_1 \leq x \) for \( -\frac{3}{4} + \frac{\sqrt{5}}{4} \leq d < 0 \) and \( x \in \left[ x_4(d), d + \frac{1}{2} \right] \). In other words, for \( x \in \left[ f(d), d + \frac{1}{2} \right] \) the only pairs \((x, y)\) that lie in the triangle and meet the condition \( s_3 \geq 0 \) are those that have \( 0 \leq y \leq h(x) \). In particular, as we stated at the beginning of this paper:

\[ h(f(d)) = y_1(x_4(d)) = x_4(d) = f(d). \]

Next we check when \( y_1 \leq -x + 2d + 1 \):

\[ -x + 2d + 1 \geq -\frac{1}{2} (x - d) + \frac{1}{2} \sqrt{\frac{x^3 + dx^2 - d^2x - 4d - 4d^2 - d^3}{x - d}}, \]

\[ \left( 1 + \frac{3}{2} \frac{x}{2} \right)^2 \geq \frac{x^3 + dx^2 - d^2x - 4d - 4d^2 - d^3}{4(x - d)}, \]

\[ 4 \left( 1 + \frac{3}{2} \frac{x}{2} \right)^2 (x - d) - (x^3 + dx^2 - d^2x - 4d - 4d^2 - d^3) \geq 0, \]

\[ -4(2d + 1)x^2 + 4(2d + 1)^2x - 8d^2(d + 1) \geq 0. \]

Let

\[ h_5(x) = -4(2d + 1)x^2 + 4(2d + 1)^2x - 8d^2(d + 1). \]

Note that \( h_5 \) is a quadratic since we assume \( d \geq -\frac{3}{4} + \frac{\sqrt{5}}{4} \). The roots of \( h_5 \) are:

\[ p_1(d) = d + \frac{1}{2} + \sqrt{\frac{\left( d - \left( -\frac{3}{4} + \frac{\sqrt{5}}{4} \right) \right) \left( d - \left( -\frac{3}{4} - \frac{\sqrt{5}}{4} \right) \right)}{2d + 1}}, \]
\[ p_2(d) = d + \frac{1}{2} - \frac{\left( d - \left( -\frac{3}{4} + \frac{\sqrt{5}}{4} \right) \right) \left( d - \left( -\frac{3}{4} - \frac{\sqrt{5}}{4} \right) \right)}{2d + 1}. \]

Note that \( p_1(d) = d + \frac{1}{2} + g(d) \), where \( g(d) \) is the function defined at the beginning of this paper.

These roots are real since we deal with the case \( d \geq -\frac{3}{4} + \frac{\sqrt{5}}{4} \). Also, \( p_1(d) \geq d + \frac{1}{2} \geq p_2(d) \), and \( h_5(x) \geq 0 \) if and only if \( x \in [p_2(d), p_1(d)] \).

Finally, since \( h_5(2d + 1) = -8d^2(d + 1) < 0 \) and \( 2d + 1 \geq d + \frac{1}{2} \) we get that \( p_1(d) \leq 2d + 1 \).

We found that \( y_1 \leq -x + 2d + 1 \) for \( -\frac{3}{4} + \frac{\sqrt{5}}{4} \leq d < 0 \) and \( x \in \left[ d + \frac{1}{2}, p_1(d) \right] \). In other words, for \( x \in \left[ d + \frac{1}{2}, d + \frac{1}{2} + g(d) \right] \) the only pairs \( (x, y) \) that lie in the triangle and meet the condition \( s_3 \geq 0 \) are those that have \( 0 \leq y \leq h(x) \). In particular, as we stated at the beginning of this paper:

\[ h \left( d + \frac{1}{2} + g(d) \right) = y_1(p_1(d)) = -p_1(d) + 2d + 1 = d + \frac{1}{2} - g(d). \]

This completes the proof of the lemma.

The final result we shall need is:

**Lemma 3**: Define the following symmetric \( 5 \times 5 \) matrix families for \( d \in \left[ -\frac{1}{2}, 0 \right] \):

\[
A(x) = \begin{pmatrix}
0 & 0 & f_1 & 0 & f_1 \\
0 & 0 & 0 & 0 & g_1 \\
f_1 & 0 & 0 & g_1 & -d \\
0 & 0 & g_1 & 0 & 0 \\
f_1 & g_1 & -d & 0 & 0
\end{pmatrix},
\]

where,

\[
f_1 = f_1(x, d) = \frac{1}{\sqrt{2}}(x + 1)(d + 1 - x),
\]

\[
g_1 = g_1(x, d) = \sqrt{x(x - d)}.
\]
\[
B(x, y) = \frac{1}{2u^2} \begin{pmatrix}
0 & 0 & 2u^3 & 0 & 2u^3 \\ 0 & 0 & 0 & (d + 1)y & v \\
2u^3 & 0 & 0 & v & \frac{1}{3}s_3 \\
0 & (d + 1)y & v & 0 & 0 \\
2u^3 & v & \frac{1}{3}s_3 & 0 & 0
\end{pmatrix},
\]

where,
\[
s_3 = s_3(x, y, d) = 1 + x^3 + y^3 + (d - x - y)^3 + (-d - 1)^3,
\]
\[
u = u(x, y, d) = \sqrt{(w_1 - x)(x - w_2)(x + y)}
\]
\[
v = v(x, y, d)
\]
\[
= \sqrt{(x + y)(x + 1)(x - d)(x - d - 1)(x + y + 1)(x + y - d - 1)}
\]
\[
w_1 = w_1(x, y, d) = \frac{1}{2}d + \frac{1}{2}\sqrt{d^2 + \frac{4(d + 1)x}{x + y}}
\]
\[
w_2 = w_2(x, y, d) = \frac{1}{2}d - \frac{1}{2}\sqrt{d^2 + \frac{4(d + 1)x}{x + y}}.
\]

Then
1. \(A(x)\) is nonnegative for any \(x\) within the line segment \(OJ\),
2. If \(d \in \left[-\frac{1}{2}, -\frac{3}{4} + \frac{\sqrt{5}}{4}\right]\) then \(B(x, y)\) is nonnegative for any pair \((x, y)\) with \(y > 0\) within the triangle \(OBJ\),
3. If \(d \in \left[-\frac{3}{4} + \frac{\sqrt{5}}{4}, 0\right]\) then \(B(x, y)\) is nonnegative for any pair \((x, y)\) with \(y > 0\) within the 4-vertex shape \(OHIJ\), which is formed by the intersection of the shape \(P\) with the triangle \(OBIJ\),
4. The spectrum of \(A(x)\) is \(\sigma = (1, x, 0, d - x, -d - 1)\) for any \(x\) within the line segment \(OJ\),
5. The spectrum of \(B(x, y)\) is \(\sigma = (1, x, y, d - x - y, -d - 1)\) for any pair \((x, y)\) with \(y > 0\) within the triangle \(OBIJ\).

\textbf{Proof}: A point \((x, y)\) within the triangle \(OBIJ\) satisfies the following conditions:
\[
y \leq x,
\]
\[
y \leq -x + 2d + 1,
\]
\[
y \geq 0.
\]
Using these conditions and the fact that $d \leq 0$ it is trivial to show that all the matrix elements of $A(x)$ are nonnegative.

For $y > 0$ we get from the above conditions $0 < x < 2d + 1$. Also, as $y > 0$ we have $d \neq -\frac{1}{2}$. Since $-\frac{1}{2} < d \leq 0$, we get

$$d + 1 + d(2d + 1) = (2d + 1)^2 - 2d(d + 1) \geq (2d + 1)^2 > (2d + 1)x.$$}

Therefore,

$$\frac{d + 1}{x - d} > 2d + 1.$$} Using this result and by the second triangle condition

$$y \leq -x + 2d + 1 < -x + \frac{d + 1}{x - d},$$

$$x + y < \frac{d + 1}{x - d},$$

$$4x(x - d) < 4x \left(\frac{d + 1}{x + y}\right),$$

$$(2x - d)^2 < d^2 + \frac{4(d + 1)x}{x + y},$$

$$0 < 2x - d < \sqrt{d^2 + \frac{4(d + 1)x}{x + y}},$$

$$x < \frac{1}{2}d + \frac{1}{2}\sqrt{d^2 + \frac{4(d + 1)x}{x + y}} = w_1.$$} Therefore, we showed that $x < w_1$ for a point $(x, y)$ with $y > 0$ within the triangle $OBJ$. By the triangle conditions, $y > 0$, and $-\frac{1}{2} < d \leq 0$ it can be easily shown that $u > 0$ and that all the matrix elements of $B(x, y)$, apart from $\frac{1}{3}s_3$, are nonnegative. By Lemma 2 we know that $s_3 \geq 0$ within the given regions for which $B(x, y)$ is claimed to be nonnegative.

The characteristic polynomial of $A(x)$ is:
\[ p_A(z) = z^5 - (2g_1^2 + 2f_1^2 + d^2)z^3 + 2df_1^2z^2 + (g_1^4 + 2f_1^2g_1^2)z \]
\[ = z^5 - (2x(x - d) + (x + 1)(d + 1 - x) + d^2)z^3 \]
\[ + d(x + 1)(d + 1 - x)z^2 \]
\[ + (x^2(x - d)^2 + x(x - d)(x + 1)(d + 1 - x))z \]
\[ = z^5 - (x^2 - dx + d^2 + d + 1)z^3 + (-dx^2 + d^2x + d^2 + d)z^2 \]
\[ + (x^2 + dx^2 - d^2x + dx)z \]
\[ = (z - 1)(z - x)(z - (d - x))(z - (d - 1)). \]

This shows that the spectrum of \( A(x) \) is \( \sigma = (1, x, 0, d - x, -d - 1) \).

We shall now compare the coefficients of the characteristic polynomial of \( B(x, y) \) and the coefficients of the polynomial
\[ q(z) = (z - 1)(z - x)(z - y)(z - (d - x - y))(z - (d - 1)) \]
\[ = z^5 + q_4z^4 + q_3z^3 + q_2z^2 + q_1z + q_0. \]

As before let
\[ s_k = 1 + x^k + y^k + (d - x - y)^k + (-d - 1)^k. \]

Obviously, \( s_1 = 0 \) and \( q_4 = 0 \).

By the Newton identities:
\[ 2q_3 = -q_4s_1 - s_2 = -s_2, \]
\[ -3q_2 = q_3s_1 - q_4s_2 + s_3 = s_3, \]
\[ 4q_1 = -q_2s_1 - q_3s_2 - q_4s_3 - s_4 = -q_3s_2 - s_4, \]
\[ -5q_0 = q_1s_1 + q_2s_2 + q_3s_3 + q_4s_4 + s_5 = q_2s_2 + q_3s_3 + s_5. \]

We therefore get:
\[ q_3 = -\frac{1}{2}s_2, \]
\[ q_2 = -\frac{1}{3}s_3, \]
\[ q_1 = -\frac{1}{4}s_4 + \frac{1}{8}s_2^2, \]
\[ q_0 = \frac{1}{6}s_2s_3 - \frac{1}{5}s_5. \]

The characteristic polynomial of \( B(x, y) \) is:
\[ p_B(z) = z^5 + p_3z^3 + p_2z^2 + p_1z + p_0, \]
where,

\[ p_3 = -\left(\frac{(d+1)y}{2u^2}\right)^2 - 2\left(\frac{v}{2u^2}\right)^2 - 2u^2 - \left(\frac{s_3}{6u^2}\right)^2 = -\frac{1}{4u^4}(d+1)^2y^2 + 2v^2 + 8u^6 + \frac{1}{9}s_3^2, \]

\[ p_2 = -2u^2\frac{s_3}{6u^2} = -\frac{1}{3}s_3 = q_2, \]

\[ p_1 = -2\frac{(d+1)y}{2u^2}\left(\frac{v}{2u^2}\right)^2 s_3 + \frac{1}{6u^2}(d+1)^2y^2 u^2 \]

\[ + \left(\frac{(d+1)y}{2u^2}\right)^2 \left(\frac{s_3}{6u^2}\right)^2 + 2\left(\frac{v}{2u^2}\right)^2 u^2 \]

\[ = \frac{1}{16u^8}\left(\frac{-2}{3}(d+1)vy^2s_3 + v^4 + 8(d+1)^2y^2u^6 \right) + \frac{1}{9}(d+1)^2y^2s_3^2 + 8v^2u^6, \]

\[ p_0 = -2\left(\frac{(d+1)y}{2u^2}\right)^2 u^2 + 2\left(\frac{(d+1)y}{2u^2}\right)^2 s_3 \]

\[ = \frac{1}{4u^4}\left(-(d+1)yv^2 + \frac{1}{3}(d+1)^2y^2s_3\right). \]

Substituting \( w_1 \) and \( w_2 \) in the function \( u \) gives:

\[ u = \sqrt{\frac{1}{2}\left((d-x)(x+y) + d + 1\right)} = \frac{1}{\sqrt{2}}\left(-x^2 - xy + dx + dy + d + 1\right). \]

The following expressions are polynomials in \( x \) and \( y \) since \( u \) and \( v \) appear only with even powers. Therefore, it is straightforward to check that

\[ -4u^4p_3 = (d+1)^2y^2 + 2v^2 + 8u^6 + \frac{1}{9}s_3^2 = 2u^4s_2 = -4u^4q_3, \]

\[ 16u^8p_1 = -\frac{2}{3}(d+1)vy^2s_3 + v^4 + 8(d+1)^2y^2u^6 + \frac{1}{9}(d+1)^2y^2s^2 = u^8(-4s_4 + 2s_2^2) = 16u^8q_1, \]

\[ 4u^4p_0 = -(d+1)yv^2 + \frac{1}{3}(d+1)^2y^2s_3 = 4u^4\left(\frac{1}{6}s_2s_3 - \frac{1}{5}s_5\right) = 4u^4q_0. \]

Since \( u > 0 \) we get that all the coefficient of \( q(z) \) and \( p_B(z) \) are equal, so the spectrum of \( B(x,y) \) is \( \sigma = (1, x, y, d - x - y, -d - 1). \)

This completes the proof of the lemma.
5. Proof of main result

Proof of Theorem 2 (Main result): We shall prove the theorem by considering three different cases:

1. \( x \leq 0 \),
2. \( x > 0 \) and \( y \leq 0 \),
3. \( x > 0 \) and \( y > 0 \).

Note that the triangle \( ABC \) for \( d \in \left[-\frac{3}{4}, -\frac{1}{2}\right] \) is fully covered by case 1 and case 2.

For \( d \in \left[-\frac{1}{2}, -\frac{3}{4} + \frac{\sqrt{5}}{4}\right] \) case 3 is the triangle \( OBJ \) without the edge \( OJ \). For \( d \in \left[-\frac{3}{4} + \frac{\sqrt{5}}{4}, 0\right] \) case 3 is the 4-vertex shape \( OHIJ \), which is formed by the intersection of the shape \( P \) with the triangle \( OBJ \), without the edge \( OJ \). Note that when \( d = 0 \) this intersection is empty so case 3 is never happens.

Let \( d \in \left[-\frac{3}{4}, -\frac{1}{2}\right] \). We already know that if \( \sigma = (1, x, y, d - x - y, -d - 1) \) is a normalized spectrum of a matrix \( A \in \mathbb{R} \) then \((x, y)\) must lie within the triangle \( ABC \). This proves the necessity of the condition.

To prove sufficiency, assume that \((x, y)\) lies within the triangle \( ABC \). As mentioned before, case 3 is impossible, so we are left with the other two cases.

The triangle \( ABC \) conditions are:

\[
\begin{align*}
    y & \leq x, \\
    y & \leq -x + 2d + 1, \\
    y & \geq \frac{1}{2} (d - x).
\end{align*}
\]

If \( x \leq 0 \) then the conditions of Theorem 4 are satisfied and therefore \( \sigma = (1, x, y, d - x - y, -d - 1) \) is realized by a symmetric nonnegative \( 5 \times 5 \) matrix \( A \). Since the sum of the elements of \( \sigma \) are zero, then \( A \in \mathbb{R} \).

Note that this proof is valid for \( d \in \left[-\frac{3}{4}, 0\right] \).

If \( x > 0 \) and \( y \leq 0 \) then let \( \sigma = (1, x, y, d - x - y, -d - 1) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \) and \( K_1 = \{3, 5\}, K_2 = \{4\} \). We have \( \sum_{i=1}^{5} \lambda_i = 0, \sum_{i \in K_1} \lambda_i = y - d - 1 \) and \( \sum_{i \in K_2} \lambda_i = d - x - y \).
Assume that \(- \sum_{i \in K_1} \lambda_i < - \sum_{i \in K_2} \lambda_i\). In that case \(d + 1 - y < x + y - d\). Therefore, \(2y > -x + 2d + 1\). As \(y \leq 0\) we have \(2y \leq y\). Using the second triangle condition we get \(-x + 2d + 1 \geq y \geq 2y > -x + 2d + 1\), which is a contradiction and therefore, \(- \sum_{i \in K_1} \lambda_i \geq - \sum_{i \in K_2} \lambda_i\).

As we assumed that \(d \leq - \frac{1}{2}\) we have \(3d + 2 \leq -d\). From the second and third triangle conditions we have \(x \leq 3d + 2\) and \(y \geq \frac{1}{2}(d - x)\). Therefore, \(y \geq \frac{1}{2}(d - x) \geq \frac{1}{2}d - \frac{1}{2}(3d + 2) \geq \frac{1}{2}d + \frac{1}{2}d = d\). This means that \(1 \geq d + 1 - y = - \sum_{i \in K_1} \lambda_i\).

All the conditions of Theorem 5 are satisfied and therefore we conclude that \(\sigma = (1, x, y, d - x - y, -d - 1)\) is realized by symmetric nonnegative \(5 \times 5\) matrix \(A\). Since the sum of the elements of \(\sigma\) are zero, then \(A \in \mathbb{R}\).

Let \(d \in \left[-\frac{1}{2}, -\frac{3}{4} + \frac{\sqrt{5}}{4}\right]\). We already know that if \(\sigma = (1, x, y, d - x - y, -d - 1)\) is a normalized spectrum of a matrix \(A \in \mathbb{R}\) then \((x, y)\) must lie within the quadrangle \(ABFG\). This proves the necessity of the condition.

To prove sufficiency, assume that \((x, y)\) lies within the quadrangle \(ABFG\). We deal with case 1 and case 2 and leave case 3 for later.

The quadrangle \(ABFG\) conditions are:

\[
\begin{align*}
    y & \leq x, \\
    y & \leq -x + 2d + 1, \\
    y & \geq \frac{1}{2}(d - x), \\
    x & \leq d + 1.
\end{align*}
\]

Obviously these are triangle \(ABC\) conditions plus the MN condition.

Case 1 is already proved as noted above.

Case 2 is proved using Theorem 5 as before. We assume \(x > 0\) and \(y \leq 0\). If \(x > 0\) and \(y \leq 0\) then let \(\sigma = (1, x, y, d - x - y, -d - 1) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\) and \(K_1 = \{3,4\}, K_2 = \{5\}\). We have \(\sum_{i=1}^{5} \lambda_i = 0, \sum_{i \in K_1} \lambda_i = d - x, \sum_{i \in K_2} \lambda_i = -d - 1\).

First assume that \(x > 2d + 1\). As \(d \geq -\frac{1}{2}\) we get \(x > 0\).

By the fourth quadrangle inequality \(x \leq d + 1\), so \(1 \geq x - d = - \sum_{i \in K_1} \lambda_i\).
By $x > 2d + 1$ we get $-\sum_{i \in K_1} \lambda_i = x - d > d + 1 = -\sum_{i \in K_2} \lambda_i$.

All the conditions of Theorem 5 are satisfied and therefore we conclude that $\sigma = (1, x, y, d - x - y, -d - 1)$ is realized by symmetric nonnegative $5 \times 5$ matrix $A$. Since the sum of the elements of $\sigma$ are zero, then $A \in \tilde{\mathcal{R}}$.

Next assume that $0 < x \leq 2d + 1$.

As $d \leq 0$ we have $1 \geq d + 1 = -\sum_{i \in K_2} \lambda_i$.

By $x \leq 2d + 1$ we get $-\sum_{i \in K_1} \lambda_i = x - d \leq d + 1 = -\sum_{i \in K_2} \lambda_i$.

Again, all the conditions of Theorem 5 are satisfied (with the roles of $K_1, K_2$ switched) and therefore $\sigma = (1, x, y, d - x - y, -d - 1)$ is realized by symmetric nonnegative $5 \times 5$ matrix $A$. Since the sum of the elements of $\sigma$ are zero, then $A \in \tilde{\mathcal{R}}$.

Note that this proof is valid for $d \in \left[-\frac{1}{2}, 0\right]$.

Let $d \in \left[-\frac{3}{4} + \frac{\sqrt{5}}{4}, 0\right]$. By Lemma 2 we know that if $\sigma = (1, x, y, d - x - y, -d - 1)$ is a normalized spectrum of a matrix $A \in \tilde{\mathcal{R}}$ then $(x, y)$ must lie within the shape $P$. This proves the necessity of the condition.

To prove sufficiency, assume that $(x, y)$ lies within the shape $P$. Case 1 and case 2 are already proved for this range of $d$ as noted above.

By Lemma 3 we know that for any pair $(x, y)$ which meets case 3 for $d \in \left[-\frac{1}{2}, 0\right]$ there is a matrix $B(x, y) \in \tilde{\mathcal{R}}$ with a spectrum $\sigma = (1, x, y, d - x - y, -d - 1)$. Therefore, we proved sufficiency.

This completes the proof of the theorem.

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**Appendix A**
The following figures show the points defined in Section 2 for different values of $d$. 

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Figure 1 – $d \in \left(-\frac{3}{4}, -\frac{1}{2}\right)$

Figure 2 – $d = -\frac{1}{2}$

Figure 3 – $d \in \left(-\frac{1}{2}, -\frac{1}{3}\right)$
Figure 4 – $\mathbf{d} = -\frac{1}{3}$

Figure 5 – $\mathbf{d} \in \left( -\frac{1}{3}, -\frac{3}{4} + \frac{\sqrt{5}}{4} \right)$

Figure 6 – $\mathbf{d} = -\frac{2}{4} + \frac{\sqrt{5}}{4}$
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