6D conformal gravity

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Abstract
In the framework of an ordinary-derivative approach, conformal gravity in spacetime of dimension 6 is studied. The field content, in addition to conformal graviton field, includes two auxiliary rank-2 symmetric tensor fields, two Stueckelberg vector fields and one Stueckelberg scalar field. The gauge invariant Lagrangian with conventional kinetic terms and corresponding gauge transformations are obtained. One of the rank-2 tensor fields and the scalar field have a canonical conformal dimension. With respect to these fields, the Lagrangian contains, in addition to other terms, a cubic potential. Gauging away the Stueckelberg fields and excluding the auxiliary fields via equations of motion, the higher derivative Lagrangian of 6D conformal gravity is obtained. The higher derivative Lagrangian involves quadratic and cubic curvature terms. This higher derivative Lagrangian coincides with the simplest Weyl invariant density discussed in the earlier literature. Generalization of de Donder gauge conditions to 6D conformal fields is also obtained.

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1. Introduction
In view of their aesthetic features, conformal fields have attracted considerable interest for a long period of time (see [1–3]). In spacetime of dimension $d \geq 4$, conformal fields can be separated into two groups: fundamental fields and shadow fields. A field having Lorentz algebra spin $s$ and conformal dimension $\Delta = s + d - 2$ is referred to as the fundamental field, while a field having Lorentz algebra spin $s$ and dual conformal dimension $\Delta = 2 - s$ is referred to as the shadow field. In this paper, we deal only with shadow fields which will be referred to simply as conformal fields in what follows.

1 Lorentz algebra label $s$ is used for the description of the totally symmetric fields. To discuss so-called mixed-symmetry fields one needs to involve more labels of the Lorentz algebra. Discussion of conformal mixed-symmetry fields may be found in [2, 4].
The conformal fields are used, among other things, to discuss conformally invariant Lagrangians (see e.g. [1–3]). With the exception of some particular cases, Lagrangian formulation of the conformal fields involve higher derivatives and non-conventional kinetic terms. We also note that the higher derivative terms hide propagating degrees of freedom (d.o.f.) of conformal fields. In [5, 6] we developed an ordinary (not higher) derivative, gauge invariant Lagrangian formulation for free conformal fields. That is to say, our Lagrangians for bosonic fields do not involve higher than second-order terms in derivatives and have conventional kinetic terms.

In this paper, we discuss 6D conformal gravity using the framework of ordinary-derivative approach developed in [6]. The purpose of this paper is to develop an ordinary-derivative, gauge invariant, and Lagrangian formulation for interacting fields of 6D conformal gravity. Our approach to the interacting conformal 6D gravity can be summarized as follows.

1. We introduce additional field degrees of freedom, i.e. we extend the space of fields entering the standard 6D conformal gravity. In addition to conformal graviton field, our field content involves two rank-2 symmetric tensor fields, two vector fields and one scalar field. All additional fields are supplemented by appropriate gauge symmetries. We note that the vector fields and the scalar field turn out to be Stueckelberg fields, i.e. they are somewhat similar to the ones used in the gauge invariant approach to massive fields.

2. Our Lagrangian for interacting fields of the 6D conformal gravity does not contain higher than second-order terms in derivatives. To second order in fields, two-derivative contributions to the Lagrangian take the form of the standard kinetic terms of the scalar, vector, and tensor fields. Two derivative contributions also appear in the interaction vertices.

3. Gauge transformations of fields 6D conformal gravity do not involve higher than first-order terms in derivatives. Interacting independent one-derivative contributions to the gauge transformations take the form of the standard gauge transformations of the vector and tensor fields.

4. The gauge symmetries of our Lagrangian make it possible to match our approach with the higher derivative one, i.e. by an appropriate gauge fixing of the Stueckelberg fields and by solving some constraints we obtain the higher derivative formulation of the 6D conformal gravity. This implies that our approach retains propagating d.o.f. of the higher derivative 6D conformal gravity theory, i.e. our approach is equivalent to the higher derivative one, at least at the classical level.

As is well known, the Stueckelberg approach turned out to be successful for the study of theories involving massive fields (see e.g. [9]). In fact, all covariant formulations of string theories are realized using Stueckelberg gauge symmetries. Therefore, we expect that use of the Stueckelberg fields for studying conformal fields might be useful for developing new interesting formulations of the 6D conformal theory.

The rest of the paper is organized as follows.

Section 2 is devoted to the discussion of free spin-2 conformal field in 6D flat space. In section 2.1, we start with a brief review of the higher derivative formulation of free 6D conformal gravity. After this, in section 2.2, we review ordinary-derivative formulation of free 6D conformal gravity. We discuss various representation for the gauge invariant Lagrangian. We review gauge symmetries of the Lagrangian and realization of global conformal algebra symmetries on the space of gauge fields.

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2 An ordinary-derivative approach to interacting 4D conformal gravity was discussed in [6].

3 To realize those additional gauge symmetries, we adopt the approach of [5–8] which turns out to be most useful for our purposes.
In section 3, we describe the ordinary-derivative formulation of interacting theory of 6D conformal gravity. We discuss ordinary-derivative gauge invariant Lagrangian and its gauge symmetries.

In section 4, we show how the higher derivative Lagrangian of interacting 6D conformal gravity is obtained from our ordinary-derivative Lagrangian.

Section 5 suggests directions for future research.

In appendix A, we summarize our conventions and the notation. In appendix B, we present some details of the derivation of gauge invariant Lagrangian and the corresponding gauge transformations.

2. Free spin-2 conformal field in 6D flat space

To make contact with studies in the earlier literature we start with the presentation of the standard, i.e. higher derivative, formulation for the spin-2 conformal field propagating in 6D flat space. In the literature, such a field is often referred to as conformal Weyl graviton.

2.1. Higher-derivative formulation of a spin-2 conformal field

To discuss higher derivative and gauge invariant formulation of spin-2 conformal field one uses the rank-2 Lorentz algebra $so(5, 1)$ tensor field $\phi^{ab}$ having conformal dimension $\Delta_{\phi^{ab}} = 0$. The field $\phi^{ab}$ is symmetric, $\phi^{ab} = \phi^{ba}$, and traceful $\phi^{aa} \neq 0$. The higher derivative Lagrangian for the field $\phi^{ab}$ is given by

$$ L = \frac{1}{3} C_{abc} \Box C_{abc}, \quad (2.1) $$

where $C_{abc}^{lin}$ is the linearized Weyl tensor. Using representation of the Weyl tensor in terms of curvatures

$$ C_{abc}^{lin} = R^{abce} - \frac{1}{4} \left( \eta^{ae} R^{bc} - \eta^{be} R^{ac} + \eta^{ce} R^{ab} - \eta^{ae} R^{bc} \right) $$

and the Gauss–Bonnet relation

$$ R_{abc}^{lin} \Box R_{abc}^{lin} - 4 R_{abc}^{lin} \Box R_{abc}^{lin} + R_{abc}^{lin} \Box R_{abc}^{lin} = 0 \quad (\text{up to total derivative}), \quad (2.3) $$

we obtain the representation for Lagrangian (2.1) in terms of linearized Ricci curvatures,

$$ L = R_{abc}^{lin} \Box R_{abc}^{lin} - \frac{3}{10} R^{2}_{abc}^{lin}, \quad (2.4) $$

which is also useful for certain purposes. Using explicit representation of the Ricci curvatures in terms of the field $\phi^{ab}$

$$ R_{abc}^{lin} = \frac{1}{2} (\Box \phi^{ab} + \partial^a \partial^c \phi^{bc} + \partial^b \partial^c \phi^{ac} - \partial^a \partial^b \phi^{cc}), \quad (2.5) $$

$$ R_{lin} = \partial^a \partial^b \phi^{ab} - \Box \phi^{aa}, \quad (2.6) $$

leads to other well known form of the Lagrangian

$$ L = \frac{1}{4} \phi^{ab} \Box^3 P^{abce} \phi^{ce}, \quad (2.7) $$

where we use the notation as in [1]:

$$ P^{abce} \equiv \frac{1}{2} \left( \pi^{ac} \pi^{be} + \pi^{ae} \pi^{bc} \right) - \frac{1}{2} \eta^{ab} \pi^{1ce}, \quad \pi^{ab} \equiv \eta^{ab} - \frac{\partial^a \partial^b}{\Box}. \quad (2.8) $$

Lagrangian (2.1) is invariant under linearized diffeomorphism and Weyl gauge transformations

$$ \delta \phi^{ab} = \partial^a \xi^b + \partial^b \xi^a + \eta^{ab} \xi, \quad (2.9) $$

where $\xi^a$ and $\xi$ are the respective diffeomorphism and Weyl gauge transformation parameters.
We now discuss on-shell d.o.f. of 6D conformal gravity. To this end we use fields transforming in irreps of the $so(4)$ algebra. One can prove that on-shell d.o.f. are described by three rank-2 *traceless* symmetric tensor fields $\phi_{ij}^{k'},$ two vector fields $\phi_i^{k'},$ and one scalar field $\phi_0$:

$$
\begin{bmatrix}
\phi_{i2}^{j'} \\
\phi_{0}^{j'} \\
\phi_{1}^{j'} \\
\phi_0
\end{bmatrix}
$$

$i, j = 1, \ldots, 4$ (for details see appendix B in [6]). Total number of on-shell d.o.f. shown in (2.10) is given by

$$n = 36.$$  

We note that $n$ is decomposed in a sum of d.o.f. for fields given in (2.10) as

$$n = \sum_{k'=0} n(\phi_{ij}^{k'}) + \sum_{k'=\pm 1} n(\phi_i^{k'}) + n(\phi_0),$$  

(2.12)

$$n(\phi_{ij}^{k'}) = 9, \quad k' = 0, \pm 2;$$  

(2.13)

$$n(\phi_i^{k'}) = 4, \quad k' = \pm 1;$$  

(2.14)

$$n(\phi_0) = 1.$$  

(2.15)

### 2.2. Ordinary-derivative formulation of the spin-2 conformal field

We now review the ordinary-derivative formulation of the spin-2 conformal field in 6D flat space developed in [6]. In addition to results in [6], we discuss also two new representations for gauge invariant Lagrangian. One of the new representations turns out to be convenient for the generalization to theory of interacting spin-2 conformal field. Also, we present our results for de Donder like gauge conditions which have not been discussed in the earlier literature.

#### 2.2.1. Field content

To discuss ordinary-derivative and gauge invariant formulation of the spin-2 conformal field in 6D flat space we use three rank-2 tensor fields $\phi_{ab}^{k'},$ two vector fields $\phi_a^{k'},$ and one scalar field $\phi_0$:

$$
\begin{bmatrix}
\phi_{ab}^{a'} \\
\phi_0^{a'} \\
\phi_1^{a'} \\
\phi_0
\end{bmatrix}
$$

The fields $\phi_{ab}^{k'}, \phi_a^{k'},$ and $\phi_0$ are the respective rank-2 tensor, vector, and scalar fields of the Lorentz algebra $so(5, 1)$. Note that the tensor fields $\phi_{ab}^{k'}$ are symmetric, $\phi_{ab}^{k'} = \phi_{ba}^{k'},$ and traceful, $\phi_0^{k'} \neq 0.$ Fields in (2.16) have the conformal dimensions

$$\Delta \phi_{ab}^{k'} = 2 + k', \quad k' = 0, \pm 2,$$

$$\Delta \phi_a^{k'} = 2 + k', \quad k' = \pm 1,$$

$$\Delta \phi_0 = 2.$$  

(2.17)

4 Fields (2.10) are related to the non-unitary representation of the conformal algebra $so(6, 2).$ Discussion of unitary representations of the conformal algebra that are relevant for elementary particles may be found, e.g., in [10, 11].

5 Total d.o.f. given in (2.11) was found in [1]. Decomposition of $n$ (2.12) into irreps of the $so(4)$ algebra was carried out in [6] (see appendix B in [6]). The light-cone gauge approach used in [6] provides easy possibility to decompose the total $n$ into irreps of the $so(4)$ algebra. Discussion of other methods for counting propagating d.o.f. of higher derivative theories may be found in [12, 13].
2.2.2. Gauge invariant Lagrangian. We discuss three representations for Lagrangian in turn.

First representation for the Lagrangian. This representation found in [6] is given by

\[
\mathcal{L} = \frac{1}{2} \phi^{ab} (E_{\text{EH}} \phi_{-2})^{ab} + \frac{1}{2} \phi^{00} (E_{\text{EH}} \phi_0)^{ab} + \phi_0^a (E_{\text{Max}} \phi_{-1})^a + \frac{1}{2} \phi_0 \Box \phi_0 + \phi_0^a \partial^b \chi_0^{ab} + \phi_0^a \partial^b \chi_2^{ab} - \frac{1}{2} \chi_0^{ab} \phi_2^{ab},
\]

(2.18)

where \( E_{\text{EH}} \) and \( E_{\text{Max}} \) are the respective second-order Einstein–Hilbert and Maxwell operators,

\[
(E_{\text{EH}} \phi)^{ab} = \Box \phi^{ab} - \partial_a \partial^b \phi^{ab} + \partial_a \phi^{b0} + \partial_b \phi^{a0} + \eta^{ab} \Box \phi^{cc},
\]

(2.22)

\[
(E_{\text{Max}} \phi)^a = \Box \phi^a - \partial_a \phi^b.
\]

(2.23)

Thus, we see that two-derivative contributions to Lagrangian (2.18) takes the form of standard second-order kinetic terms for the respective rank-2 tensor fields, vector fields and scalar field. Note also that besides the two-derivative contributions, the Lagrangian involves one-derivative contributions and derivative-independent mass-like contributions.

Second representation for the Lagrangian. The second representation has not been discussed in the earlier literature. For the reader’s convenience, we discuss this representation because it allows us to introduce de Donder like gauge conditions for 6D conformal gravity. This is to say that Lagrangian (2.18) can be represented as (up to total derivative)

\[
\mathcal{L} = \frac{1}{2} \phi^{ab} \Box \phi_{-2} - \frac{1}{2} \phi^{0a} \Box \phi_0 - \frac{1}{2} \phi^{ab} \Box \phi_{00} - \frac{1}{2} \phi^{00} \Box \phi_0 + \phi_0^a \phi_{-1}^a + \frac{1}{2} \phi_0 \Box \phi_0 + C_a^{a_1} C_a^{a_2} C_a^{a_3} + C_0 C_2 - \frac{1}{2} \phi_2^{0a} \phi_0^{ab} + \frac{1}{2} \phi_2^{0a} \phi_0^{ab} + \frac{1}{2} \phi_0^{ab} \phi_2^{0a} - \frac{1}{2} \phi_0^a \phi_2^{0a},
\]

(2.24)

where quantities \( C_a^{a_1}, C_a^{a_2}, C_a^{a_3} \), which we refer to as conformal de Donder divergences, are given by

\[
C_a^{a_1} = \partial^b \phi^{ab}_{-2} - \frac{1}{2} \partial^a \phi^{bb}_{-2} + \phi_0^a,
\]

(2.25)

\[
C_a^{a_1} = \partial^b \phi^{0b} - \frac{1}{2} \partial^a \phi^{0b} + \phi_0^a,
\]

(2.26)

\[
C_a^{a_3} = \partial^b \phi^{ab} - \frac{1}{2} \partial^a \phi^{bb},
\]

(2.27)

\[
C_0 = \partial^a \phi_0^a + \frac{1}{2} \phi_0^a + u \phi_0,
\]

(2.28)

\[
C_2 = \partial^a \phi_2^a + \frac{1}{2} \phi_2^a.
\]

(2.29)

We note that it is the conformal de Donder divergencies that define de Donder like gauge conditions for our conformal 6D fields,

\[
C_{k'} = 0, \quad k' = -1, 1, 3, \quad \text{conformal de Donder gauge conditions.}
\]

(2.30)

\( \text{de Donder gauges turn out to be useful for study of various dynamical systems. Recent discussion of the standard de Donder-Feynman gauge for massless fields may be found in [14–16]. Applications of modified de Donder gauges for massless and massive AdS fields [17] to studying the AdS/CFT correspondence may found in [18].} \)

5
The fields described by Lagrangian (2.24) are related to non-unitary representation of the conformal algebra. Fields with \( k' = 0 \) have kinetic terms with correct signs. The remaining fields with \( k' \neq 0 \) can be collected into the vector and tensor doublets, \( \phi_{-1}^a, \phi_0^a \) and \( \phi_{-2}^{ab}, \phi_2^{ab} \). We note then that vector (tensor) doublet describes one vector (tensor) field with wrong sign of kinetic term and one vector (tensor) field with correct sign of the kinetic term.

**Third representation for the Lagrangian.** Finally, we discuss representation for free Lagrangian (2.18) which turns to be most adapted for generalization to interacting 6D conformal gravity. This is to say that Lagrangian (2.18) can be represented as (up to total derivative)

\[
\mathcal{L} = \sum_{a=1}^{6} \mathcal{L}_a, \quad (2.31)
\]

\[
\mathcal{L}_1 = -\phi^{ab}_{-1} \hat{G}^{(ab)}_{\text{lin}}, \quad (2.32)
\]

\[
\mathcal{L}_2 = -\frac{1}{4} \partial^c \phi_{0}^{ab} \partial^c \phi_{0}^{ab} + \frac{1}{8} \partial^c \phi_0^{ac} \partial^c \phi_0^{bc} + \frac{1}{2} C_1^a C_1^a, \quad (2.33)
\]

\[
\mathcal{L}_3 = -\frac{1}{2} F_{ab}(\phi_{-1}) F_{ab}(\phi_{1}), \quad (2.34)
\]

\[
\mathcal{L}_4 = -\frac{1}{2} \partial^a \phi_0 \partial^a \phi_0, \quad (2.35)
\]

\[
\mathcal{L}_5 = \phi_1^a \partial^b \chi_{ab}, \quad (2.36)
\]

\[
\mathcal{L}_6 = -\frac{1}{2} \phi_2^{ab} \chi_0^{ab}, \quad (2.37)
\]

\[
\hat{G}^{(ab)}_{\text{lin}} = G^{ab}_{\text{lin}} + \frac{1}{2} (\partial^a \phi_{-1}^b + \partial^b \phi_{-1}^a) - \eta^{ab} \partial^c \phi_{-1}^c, \quad (2.38)
\]

\[
G^{ab}_{\text{lin}} = R^{ab}_{\text{lin}}(\phi_{-2}) - \frac{1}{2} \eta^{ab} R_{\text{lin}}(\phi_{-2}), \quad (2.39)
\]

\[
C_1^a \equiv \partial^b \phi_0^a - \frac{1}{2} \partial^a \phi_{0}^{bb}, \quad (2.40)
\]

\[
F_{ab}(\phi_{k}) = \partial^a \phi_{k}^b - \partial^b \phi_{k}^a, \quad k' = \pm 1, \quad (2.41)
\]

where \( \chi_{0}^{ab} \) is defined in (2.19). The linearized Ricci curvatures for the field \( \phi_{-2}^{ab} \) in (2.39) are obtained by substituting the field \( \phi_{-2}^{ab} \) in the respective expressions (2.5) and (2.6). Note that linearized Einstein tensor \( G^{ab}_{\text{lin}} \) (2.39) can be represented by using operator \( E_{\text{lin}} \) (2.22) as

\[
G^{ab}_{\text{lin}} = -\frac{1}{4} (E_{\text{lin}} \phi_{-2})^{ab}. \quad (2.42)
\]

Also, we note that shifted linearized Einstein tensor \( \hat{G}^{ab} \) (2.38) can be expressed in terms of the corresponding shifted linearized Ricci curvatures

\[
\hat{G}^{ab}_{\text{lin}} = \hat{R}^{ab}_{\text{lin}} - \frac{1}{2} \eta^{ab} \hat{R}_{\text{lin}}, \quad (2.43)
\]

where the shifted linearized curvatures are defined by relations

\[
\hat{R}^{abce}_{\text{lin}} = R^{abce}_{\text{lin}} + \eta^{ac} \psi_{le}^{be} - \eta^{bc} \psi_{le}^{ae} + \eta^{be} \psi_{le}^{ac} - \eta^{ae} \psi_{le}^{bc}, \quad (2.44)
\]

\[
R^{abce}_{\text{lin}} = \frac{1}{2} (\partial^a \partial^c \phi_{-2}^{be} + \partial^b \partial^c \phi_{-2}^{ae} - \partial^b \partial^e \phi_{-2}^{ac} + \partial^a \partial^e \phi_{-2}^{bc}), \quad (2.45)
\]
\[
\hat{R}_{\text{lin}}^{ab} = R_{\text{lin}}^{ab} + 4 \phi_{\text{lin}}^{ab} + \eta^{ab} \phi_{\text{lin}}^{cc},
\]
(2.46)
\[
\hat{R}_{\text{lin}} = R_{\text{lin}} + 10 \phi_{\text{lin}}^{cc},
\]
(2.47)
\[
\phi_{\text{lin}}^{ab} = g \partial^a \phi^b, \quad q = \frac{1}{2},
\]
(2.48)
\[
\hat{R}^{ab} = \hat{R}^{ac} b, \quad \hat{R} = \hat{R}^{aa}.
\]
(2.49)

2.2.3. Gauge transformations. We now discuss gauge symmetries of Lagrangian (2.18). To this end we introduce the gauge transformation parameters,
\[
\xi^a_{-3} \quad \xi^0_{-2} \quad \xi^a_1 \quad \xi_0
\]
(2.50)
Conformal dimensions of the gauge transformation parameters are given by
\[
\Delta_{\xi^a_{-3}} = 2 + k', \quad k' = -3, -1, 1,
\]
\[
\Delta_{\xi^0_{-2}} = 2 + k', \quad k' = -2, 0.
\]
(2.51)
The gauge transformation parameters \(\xi^a_{\pm k'}\) and \(\xi^0_{\pm k'}\) are the respective vector and scalar fields of the Lorentz algebra \(so(5, 1)\). The Lagrangian is invariant under the gauge transformations
\[
\delta \phi_{\text{lin}}^{ab} = \partial^a \xi^b_{-3} + \partial^b \xi^a_{-3} + \frac{1}{2} \eta^{ab} \xi^c_{-2},
\]
(2.52)
\[
\delta \phi_{0}^{ab} = \partial^a \xi^b_{1} + \partial^b \xi^a_{1} + \frac{1}{4} \eta^{ab} \xi^c_{0},
\]
(2.53)
\[
\delta \phi^a_1 = \partial^a \xi^0_{1} + \partial^0 \xi^a_{1},
\]
(2.54)
\[
\delta \phi^a_0 = -u \xi^a_0,
\]
(2.55)
\[
\delta \phi_0 = -u \xi^0_0,
\]
(2.56)
where \(u\) is given in (2.21).

2.2.4. Realization of conformal algebra symmetries. In 6D space-time, the conformal algebra \(so(6, 2)\) referred to the basis of Lorentz algebra \(so(5, 1)\) consists of translation generators \(P^a\), conformal boost generators \(K^a\), dilatation generator \(D\) and generators of the Lorentz algebra \(so(5, 1)\) denoted by \(J^{ab}\). We assume the following normalization for commutators of the conformal algebra\(^7\):
\[
[D, P^a] = -P^a, \quad [P^a, J^{bc}] = \eta^{ab} P^c - \eta^{ac} P^b,
\]
\[
[D, K^a] = K^a, \quad [K^a, J^{bc}] = \eta^{ab} K^c - \eta^{ac} K^b,
\]
\[
[P^a, K^b] = \eta^{ab} D - J^{ab},
\]
\[
[J^{ab}, J^{bc}] = \eta^{bc} J^{ab} + 3 \text{ terms}.
\]
(2.58)
Let \(\phi\) denote a field propagating in the flat spacetime. Let a Lagrangian for the free field \(\phi\) be conformal invariant. This implies, that the Lagrangian is invariant with respect to transformation (invariance of the Lagrangian is assumed to be up to total derivative)
\[
\delta_{\bar{G}} \phi = \bar{G} \phi,
\]
(2.59)
\(^7\) Note that in our approach only \(so(5, 1)\) symmetries are realized manifestly. The \(so(6, 2)\) symmetries could be realized manifestly by using ambient space approaches (see e.g. [19–21]).
where a realization of the conformal algebra generators $\hat{G}$ in terms of differential operators acting on $\phi$ takes the form

$$P^a = \partial^a,$$  \hspace{1cm} (2.60)

$$J^{ab} = x^a \partial^b - x^b \partial^a + M^{ab},$$  \hspace{1cm} (2.61)

$$D = x^a \partial^a + \Delta,$$  \hspace{1cm} (2.62)

$$K^a = K^a_{\Delta,M} + R^a,$$  \hspace{1cm} (2.63)

$$K^a_{\Delta,M} \equiv -\frac{1}{2} x^b x^c \partial^a + x^a D + M^{ab} x^b.$$  \hspace{1cm} (2.64)

In (2.62)–(2.64), $\Delta$ is operator of conformal dimension, $M^{ab}$ is spin operator of the Lorentz algebra. Action of $M^{ab}$ on fields of the Lorentz algebra is well known and for rank-2 tensor, vector, and scalar fields considered in this paper is given by

$$M^{ab} \phi^{ce} = \eta^{ae} \phi^{cb} + \eta^{ac} \phi^{be} - (a \leftrightarrow b),$$

$$M^{ab} \phi^{c} = \eta^{ac} \phi^{b} - (a \leftrightarrow b),$$

$$M^{ab} \phi = 0.$$  \hspace{1cm} (2.65)

Relation (2.63) implies that conformal boost transformations can be presented as

$$\delta K^a \phi = \delta K^a_{\Delta,M} \phi + \delta R^a \phi.$$  \hspace{1cm} (2.66)

Explicit representation for the action of operator $K^a_{\Delta,M}$ (2.64) is easily obtained from the relations given above. This is to say that the rank-2 tensor, vector, and scalar fields considered in this paper transform as

$$\delta K^a_{\Delta,M} \phi^{bc} = K^a_{\Delta,\phi} \phi^{bc} + M^{ab} \phi^{f}_{\Delta} + M^{ac} \phi^{bc}_{\Delta}, \quad k' = 0, \pm 2,$$  \hspace{1cm} (2.67)

$$\delta K^a_{\Delta,M} \phi^{b} = K^a_{\Delta,\phi} \phi^{b} + M^{ab} \phi^{f}, \quad k' = \pm 1,$$  \hspace{1cm} (2.68)

$$K^a_{\Delta,M} \equiv -\frac{1}{2} x^b x^c \partial^a + x^a (x \partial + \Delta),$$

$$M^{abc} \equiv \eta^{ab} \phi^{c} - \eta^{ac} \phi^{b}.$$  \hspace{1cm} (2.69)

Thus, all that remains is to find explicit representation for operator $R^a$ in (2.63). The operator $R^a$ depends on the derivative $\partial^a$ and does not depend on the spacetime coordinates $x^a$. $[P^a, R^b] = 0$. In the standard formulation of the conformal fields, the operator $R^a$ is equal to zero, while in the ordinary-derivative approach we discuss in this paper, the operator $R^a$ is non-trivial. This implies that, in the framework of ordinary-derivative approach, the complete description of the conformal fields requires finding not only gauge invariant Lagrangian but also the operator $R^a$. Realization of the operator $R^a$ on a space of gauge fields (2.16) is given by

$$\delta R^a \phi^{bc} = 0,$$  \hspace{1cm} (2.70)

$$\delta R^a \phi^{b}_{0} = -2(\eta^{ab} \phi^{c}_{-1} + \eta^{ac} \phi^{b}_{-1}) + \eta^{bc} \phi^{a}_{-1} - 4\partial^a \phi^{bc}_{-2},$$  \hspace{1cm} (2.71)

$$\delta R^a \phi^{b}_{2} = -4(\eta^{ab} \phi^{c}_{1} + \eta^{ac} \phi^{b}_{1}) + 2\eta^{bc} \phi^{a}_{1} - 4\partial^a \phi^{bc}_{0},$$  \hspace{1cm} (2.72)

$$\delta R^a \phi^{b}_{-1} = 4\phi^{ab}_{-2},$$  \hspace{1cm} (2.73)

$$\delta R^a \phi^{b}_{1} = 2\phi^{ab}_{0} - 2\partial^a \phi^{b}_{0} - 2\partial^a \phi^{b}_{-1}.$$  \hspace{1cm} (2.74)
\[ \delta R_a \phi_0 = 2 \omega \phi_{-1}. \]  

From (2.70)–(2.75), we see the operator \( R^a \) maps the gauge field with conformal dimension \( \Delta_1 \) into the ones having conformal dimension less than \( \Delta_1 \). This is to say that the realization of the operator \( R^a \) given in (2.70)–(2.75) can schematically be represented as:

\[ \phi_{ab} \rightarrow \phi_{ab}^0 \oplus \phi_{0} \oplus \partial \phi_{ab}^{0}, \quad \phi_{-2} \rightarrow 0. \]  

\[ \phi_{-1} \rightarrow \phi_{-1} \oplus \partial \phi_{a}^{0}. \]  

2.2.5. Interrelation of the ordinary-derivative and the higher derivative approaches. From (2.55)–(2.57), we see that both vector fields \( \phi_a^{\pm 1} \) and the scalar field \( \phi_0 \) transforms as Stueckelberg (Goldstone) fields under the gauge transformations, i.e. these fields can be gauged away by using the gauge symmetries. Gauging away the vector fields and the scalar field,

\[ \phi_a^{\pm 1} = 0, \quad \phi_0 = 0, \]  

we see that our Lagrangian (2.31) takes the simplified form

\[ L = -\phi_{ab} G_{ab}^{lin} - \frac{1}{3} \partial^c \phi_0^a \partial^c \phi_0^b + \frac{1}{8} \partial^c \phi_0^a \partial^c \phi_0^b + \frac{1}{2} C_a^c C_b^c - \frac{1}{2} \phi_{ab} \chi_{ab}. \]  

Now using equations of motion for the rank-2 tensor field \( \phi_{ab}^{2} \) obtained from Lagrangian (2.79) we find the equation

\[ \phi_{0}^{ab} - \eta^{ab} \phi_{0}^{cc} = -2 G_{ab}^{lin}, \]  

which has the obvious solution

\[ \phi_{0}^{ab} = -2 R_{ab}^{lin} + \frac{1}{2} \eta^{ab} R_{lin}. \]  

3. Interacting 6D conformal gravity

We begin our discussion of interacting theory of 6D conformal gravity with the description of a field content. Field content of the interacting theory is simply obtained by promoting the Minkowski space free fields (2.16) to the fields in curved spacetime described by the metric tensor field \( g_{\mu \nu} \). As usual, this metric tensor field is considered to be a conformal graviton field. As we have already said, the field \( \phi_{ab}^{2} \) describes excitation of the conformal graviton, i.e. in the interacting theory, the field \( \phi_{ab}^{2} \) is related to the metric tensor field \( g_{\mu \nu} \). Also note that, instead of the metric-like approach to conformal gravity, we prefer to use the frame-like approach, i.e. we use the vielbein field \( e^\mu_a, g_{\mu \nu} = e^\mu_a e^\nu_a \) and fields carrying tangent-flat indices, \( \phi^a, \phi^{ab} \), which are related to fields carrying base manifold indices \( \phi^\mu, \phi^{\mu \nu} \), by the standard relations \( \phi^a = e^a_\mu \phi^\mu, \phi^{ab} = e^a_\mu e^b_\nu \phi^{\mu \nu} \) (for details of our notation see appendix A). Also, following the commonly used nomenclature, we use notation \( b^a \) in place of the field \( \phi_a^{0} \). To summarize,

\[ \text{Realization of the operator } R^a \text{ on space of on-shell fields can be obtained using group theoretical methods, while the realization of } R^a \text{ on space of gauge fields requires the use of the gauge invariant approach.} \]
the field content we use to develop the ordinary-derivative approach to the interacting 6D conformal gravity is given by
\begin{equation}
\begin{aligned}
e^a_{\mu} & \phi^a_{0b} \phi^b_{01} \\
a^a & \phi^a_{00}
\end{aligned}
\end{equation}
(3.1)

For field $\phi$ having the Weyl dimension $\Delta^w$, we define local Weyl transformations in the usual way,
\begin{equation}
\delta \phi = \Delta^w \phi \sigma \phi,
\end{equation}
(3.2)

where $\sigma$ is the Weyl gauge transformation parameter. Using this convention, the Weyl dimensions of the fields are given by 
\begin{equation}
\begin{aligned}
\Delta^w_{\epsilon_e} &= -1, \\
\Delta^w_{\phi^a_{0b}} &= 2 + k', \\
\Delta^w_{\phi^1_{a}} &= 3, \\
\Delta^w_{\phi^0_0} &= 2.
\end{aligned}
\end{equation}
(3.3)

Gauge transformation of the compensator field $b^e$ involves the gradient term (see below), but for constant $\sigma$ the field $b^e$ transforms as in (3.2) with $\Delta^w_{b^e} = 1$. With this convention for the Weyl dimension of the field $b^e$, we note that conformal dimensions of fields not carrying base manifold indices, $\phi^a_{0b}, \phi^b_{01}, b^e, \phi^1_{a}, \phi^0_0$, (2.17) are equal to their Weyl dimensions (3.3).

We now discuss gauge invariant Lagrangian for interacting fields (3.1). The Lagrangian we find is given by
\begin{equation}
\mathcal{L} = \sum_{a=1}^{8} \mathcal{L}_a,
\end{equation}
(3.4)

\begin{equation}
e^{-1} \mathcal{L}_1 = -\phi^{ab}_{0} \tilde{G}^{(ab)},
\end{equation}
(3.5)

\begin{equation}
e^{-1} \mathcal{L}_2 = -\frac{1}{4} \mathcal{D}^e \phi^b_{0c} \mathcal{D}^e \phi^c_{0b} + \frac{1}{8} \mathcal{D}^e \phi^{ab}_{0b} \mathcal{D}^e \phi^{ab}_{0a} + \frac{1}{2} \mathcal{C}^e_{\mathcal{C}^e}
- \frac{1}{2} \tilde{R}^{ab}_{0b} \phi^b_{00} + \frac{1}{2} \tilde{R}^{ab}_{0a} \phi^c_{0c} - \frac{1}{2} \tilde{R}^{ab}_{0b} \phi^b_{00} + \left( \frac{1}{8} \phi^{ab}_{0a} \phi^{ab}_{0b} - \frac{1}{4} \phi^{ab}_{0a} \phi^{ab}_{0b} \right) \tilde{R},
\end{equation}
(3.6)

\begin{equation}
e^{-1} \mathcal{L}_3 = -\frac{1}{2} \mathcal{F}^{ab}_{(b)} \mathcal{F}^{ab}_{(1)},
\end{equation}
(3.7)

\begin{equation}
e^{-1} \mathcal{L}_4 = -\frac{1}{2} \mathcal{D}^e \phi^b_{00} \mathcal{D}^e \phi^a_{0a},
\end{equation}
(3.8)

\begin{equation}
e^{-1} \mathcal{L}_5 = \phi^{a}_{0} \mathcal{D}^b \chi^a_{ab},
\end{equation}
(3.9)

\begin{equation}
e^{-1} \mathcal{L}_6 = -\frac{1}{2} \phi^{a}_{0} \chi^a_{ab},
\end{equation}
(3.10)

\begin{equation}
e^{-1} \mathcal{L}_7 = \frac{1}{4} \phi^{ab}_{0} T^{ab} - \frac{u}{8} \phi^{ab}_{0} \mathcal{F}^{ab},
\end{equation}
(3.11)

9 The symmetric and antisymmetric parts of the gauge field associated with the conformal boosts are related to the field $\phi^{ab}_{0}$ and the field strength $\mathcal{F}^{ab}_{(b)}$, respectively (see (3.24)). Also, we note that the parameter $\xi^a_{ab}$ (see (3.29)) is related to conformal boosts gauge transformation parameter.

10 In [1], conformal dimension is referred to as canonical dimension.
\( e^{-1} L_8 = \frac{1}{4} \phi_0^{ab} \phi_0^{bc} \phi_0^{ca} - \frac{5}{16} \phi_0^{ab} \phi_0^{ac} \phi_0^{bc} + \frac{1}{16} (\phi_0^a)^3 \)

\( - \frac{u}{8} \phi_0^{ab} \phi_0^{ac} \phi_0^d - \frac{3}{16} \phi_0^{ab} \phi_0^{bc} \phi_0^a - \frac{3}{16 \mu} \phi_0^3, \)  

(3.12)

e \equiv \det e_\mu^\nu,  

(3.13)

\( \hat{G}^{(ab)} \equiv G^{ab} + \frac{1}{2} (D^a b^b + D^b b^a) + \frac{1}{4} b^a b^b - \eta^{ab} \left( D^a b^c - \frac{3}{8} b^c b^a \right), \)  

(3.14)

\( G^{ab} \equiv R^{ab} - \frac{1}{2} \eta^{ab} R, \)  

(3.15)

\( C_0^a \equiv D_b^{ab} \phi_0^b - \frac{1}{2} D^a \phi_0^{bb} \)  

(3.16)

\( \chi_0^{ab} \equiv \phi_0^{ab} - \eta^{ab} \phi_0^{cc} - u \eta^{ab} \phi_0, \)  

(3.17)

\( T^{ab} \equiv F^{au} F^{bc} - \frac{1}{4} \eta^{ab} F^{ce} F^{ce}, \)  

(3.18)

\( F^{ab} \equiv D^a b^b - D^b b^a. \)  

(3.19)

where \( u \) is defined in (2.21). A complete description of our notation may be found in appendix A. Here we mention the most important notation.

(a) For the rank-\( s \) field \( \phi^{b_1...b_s} \) having the Weyl dimension \( \Delta_\phi^w \), the covariant derivative \( D^a \) is defined to be

\[ D^a \phi^{b_1...b_s} = \hat{D}^a \phi^{b_1...b_s} + \Delta_\phi^w q b^a \phi^{b_1...b_s}, \]  

(3.20)

where \( \hat{D}^a \) is a covariant derivative with the shifted Lorentz connection \( \hat{\omega}^{ab}_\mu \),

\[ \hat{D}^a \phi^{b} = e^{\mu a} \partial_\mu \phi^{b} + \hat{\omega}^{abc} \phi^{c}, \]  

(3.21)

\[ D^a \phi^{b} = e^{\nu a} \partial_\nu \phi^{b} + \omega^{abc} \phi^{c}, \]  

(3.22)

\[ \hat{\omega}^{abc} = \omega^{abc} + q (\eta^{as} b^b - \eta^{ab} b^c), \]  

(3.23)

while \( D^a \) is a covariant derivative with the standard Lorentz connection \( \omega^{ab}_\mu \).

(b) Field strength \( F^{ab}(\phi) \) for vector field \( \phi^a \) is defined to be

\[ F^{ab}(\phi) \equiv D^a \phi^b - D^b \phi^a. \]  

(3.24)

Note that for the compensator field \( b^a \) the field strength \( F^{ab} \) becomes the standard one (3.19):

\[ F^{ab}(b) = F^{ab}. \]  

(3.25)

(c) Curvature \( \hat{R}^{abcde} \) is defined for the shifted connection \( \hat{\omega}^{ab}_\mu \) as

\[ \hat{R}^{abcde} = \hat{\partial}_e \hat{\omega}^{abc} - \hat{\partial}_c \hat{\omega}^{ab} + \hat{\omega}^{ace} \hat{\omega}^{b} - \hat{\omega}^{ace} \hat{\omega}^{b} \]  

(3.26)

\[ \hat{R}^{abcde} = c^{aa} c^{bb} \hat{R}^{a}_{\mu \nu \rho \sigma}, \]  

(3.27)

Ricci curvatures \( R^{ab}, R \) are defined as \( R^{ab} = R^{cabh}, R = R^{aa} \), where \( R^{abcde} = c^{aa} c^{bb} \hat{R}^{a}_{\mu \nu \rho \sigma} \) and \( R^{a}_{\mu \nu \rho \sigma} \) is the curvature for the standard Lorentz connection \( \omega^{ab}_\mu(e) \). We note that the shifted Einstein tensor is defined in a usual way

\[ \hat{G}^{ab} = \hat{R}^{ab} - \frac{1}{2} \eta^{ab} \hat{R}, \quad \hat{R}^{cb} = \hat{R}^{cab}, \quad \hat{R} = \hat{R}^{aa}. \]  

(3.28)
A few remarks are in order.

(i) Contributions to interacting Lagrangian (3.4) denoted by $L_1, L_3, L_4, L_5$ are obtained by covariantization of the flat derivative, $\partial^a \rightarrow D^a$, and the linearized Einstein tensor, $G^a_{ab} \rightarrow \hat{G}^a_{ab}$, in the respective contributions $L_1, L_3, L_4, L_5$ to flat Lagrangian (2.31). Note also that contribution $L_6$ (2.37) in the flat Lagrangian is promoted to the interacting Lagrangian without any changes (see (3.10)).

(ii) Comparing $L_2$ (3.6) that enters interacting Lagrangian (3.4) and the respective $L_2$ (2.33) that enters flat Lagrangian (2.31), we see that $L_2$ in (3.6) involves the additional contributions of first order in the shifted curvatures and second order in the field $\phi_{ab}^0$. We note that the $L_2|_{b^a=0}$ part of $L_2$ (3.6) is obtained simply by expanding the Einstein–Hilbert Lagrangian $\sqrt{g} R(g)$ with $g_{\mu\nu} = g_{\mu\nu} + \phi_{0,\mu\nu}$ as a power series in the field $\phi_{0,\mu\nu}$ and taking terms of the second order in the field $\phi_{0,\mu\nu}$, where $\phi_{0,\mu\nu} = \epsilon_{\mu a}\epsilon_{\nu b}\phi_0^{ab}$. So, we see that the tensor field $\phi_0^{ab}$ looks like an excitation of the graviton field.

(iii) An interesting contribution which is absent in flat Lagrangian (2.31) and involved in interacting Lagrangian (3.4) is governed by the contribution denoted by $L_7$ (3.11). From (3.11), we see that the tensor field $\phi_{ab}^0$ is coupled to the energy–momentum tensor of the compensator field $b^a$, i.e. we see that the tensor field $\phi_{ab}^0$ again exhibits some properties of a excitation of the graviton field.

(iv) The remaining contribution which is absent in the flat Lagrangian and enters the interacting Lagrangian is governed by the contribution denoted by $L_8$ (3.12). From (3.12), we see that the tensor field $\phi_{ab}^0$ and the scalar field $\phi^0$ lead to the appearance of a cubic potential. As is well known, massless fields in flat space, with exception of a scalar field, do not allow cubic vertices without derivatives (see e.g. [22]). Also, we know that cubic vertices having derivative-independent contributions are allowed for arbitrary spin massive fields in flat space (see e.g. [23]). Appearance of the derivative-independent cubic potential for the tensor field $\phi_{ab}^0$ implies that the tensor field $\phi_{ab}^0$ exhibits some features of the massive field. Taking the discussion above into account, we see that, on the one hand the tensor field $\phi_{ab}^0$ exhibits some features of excitation of the graviton field, i.e. massless spin-2 field, and on the other hand, the field $\phi_{ab}^0$ exhibits some features of massive field.

Gauge transformations. We now discuss gauge symmetries of Lagrangian (3.4). Because we promote all gauge symmetries of free fields of 6D conformal gravity to the interacting fields we introduce the same amount of gauge transformation parameters as in the free theory (see (2.50)): 

\begin{align}
\xi^a, \xi^{a \prime}, \xi_1, \xi_1^{a \prime}, \xi_2, \xi_2^{a \prime}, \xi_3, \xi_3^{a \prime} 
\end{align}

(3.29)

We note that Weyl transformations of gauge transformation parameters (3.29) are given by

\begin{align}
\Delta^w_{\xi^a} &= 2 + k', \quad k' = -3, -1, 1, \\
\Delta^w_{\xi_2} &= 2 + k', \quad k' = -2, 0.
\end{align}

(3.30)

Note also that because we are using a frame-like description we assume, as usual, the standard Lorentz gauge transformations of our fields in (3.1):

\begin{align}
\delta_{\lambda} e^a_{\mu} &= \lambda^{ab} e^b_{\mu}, \quad \delta_{\lambda} \phi^{ab}_{k} = \lambda^{ac} \phi_{k}^{cb} + \lambda^{bc} \phi_{k}^{ca}, \quad k' = 0, 2, \\
\delta_{\lambda} b^a &= \lambda^{ac} b^c, \quad \delta_{\lambda} \phi_{1}^{a} = \lambda^{ac} \phi_{1}^{c}, \quad \delta_{\lambda} \phi_{0} = 0,
\end{align}

(3.31)

$\lambda^{ab} = -\lambda^{ba}$. We now discuss gauge transformations associated with gauge transformations parameters (3.29) in turn.
The $\xi^a_\mu$ transformation (diffeomorphism transformations). In our approach, the gauge transformation parameter which is responsible for diffeomorphism transformations is denoted by $\xi^a_\mu$. The diffeomorphism transformations take the standard form
\begin{equation}
\delta \xi^a_\mu = \xi^a_\nu \partial^\nu \xi^\mu + e^a_\mu \partial_\mu \xi^\nu, \tag{3.32}
\end{equation}
\begin{equation}
\delta \xi^a_\nu \phi^b_{k'} = \xi^a_\nu \partial \phi^b_{k'}, \quad k' = 0, 2, \tag{3.33}
\end{equation}
\begin{equation}
\delta \xi^a_\nu b^a = \xi^a_\nu \partial b^a, \tag{3.34}
\end{equation}
\begin{equation}
\delta \xi^a_\nu \phi^a_1 = \xi^a_\nu \partial \phi^a_1, \tag{3.35}
\end{equation}
\begin{equation}
\delta \xi^a_\nu \phi_0 = \xi^a_\nu \partial \phi_0, \tag{3.36}
\end{equation}
\begin{equation}
\xi^a_\nu \partial \equiv \xi^a_\nu \partial_\mu, \quad \xi^a_\mu \equiv e^{a\mu} \xi^a_\mu. \tag{3.37}
\end{equation}

$\xi^a_\mu$ gauge transformations. In our approach, gauge transformations associated with the parameter $\xi^a_\mu$ are simultaneously realized as the gradient gauge transformation of the spin-2 field $\phi^{ab}_0$ and the Stueckelberg gauge transformation of the compensator $b^a$. The $\xi^a_\mu$ gauge transformations receive interaction-dependent corrections in the interacting theory. This is to say that the $\xi^a_\mu$ gauge transformations take the form
\begin{equation}
\delta \xi^a_\mu e^a_\mu = 0, \tag{3.38}
\end{equation}
\begin{equation}
\delta \xi^a_\nu \phi^{ab}_0 = \mathcal{D}^a \xi^{ab}_\nu + \mathcal{D}^b \xi^{ab}_\nu, \tag{3.39}
\end{equation}
\begin{equation}
\delta \xi^a_\nu \phi^{ab}_2 = \mathcal{L}_{\xi^a_\nu} \phi^{ab}_0 + \phi^{ab}_0 \xi^{ab}_\nu + \phi^{ab}_1 \xi^{ab}_\nu - \frac{1}{2} \eta^{ab} \phi^c_1 \xi^c_\nu, \tag{3.40}
\end{equation}
\begin{equation}
\delta \xi^a_\nu b^a = - \xi^a_\mu, \tag{3.41}
\end{equation}
\begin{equation}
\delta \xi^a_\nu \phi^a_1 = - \frac{1}{2} \phi^{ab}_0 \xi^{ab}_\nu - \frac{1}{2} F^{ab} \xi^{ab}_\nu + \frac{1}{2} \phi^b_0 \xi^{ab}_\nu, \tag{3.42}
\end{equation}
\begin{equation}
\delta \xi^a_\nu \phi_0 = 0, \tag{3.43}
\end{equation}
where the action of Lie derivative $\mathcal{L}_{\xi^a_\nu}$ on the field $\phi^{ab}_0$ is defined to be
\begin{equation}
\mathcal{L}_{\xi^a_\nu} \phi^{ab}_0 \equiv \xi^a_\nu \mathcal{D}^a \phi^{ab}_0 + \mathcal{D}^a \xi^c_\nu \phi^{ab}_0 + \mathcal{D}^b \xi^c_\nu \phi^{a\nu}_0. \tag{3.44}
\end{equation}
Comparing these gauge transformations with free theory gauge transformations given in (2.53), (2.54) and (2.56), we see that there are two types of interaction-dependent contributions. The first ones given in (3.39) are obtained simply by the covariantization of the flat derivatives in free theory transformations, $\partial \rightarrow \mathcal{D}^a$, (see (2.53)). The remaining contributions given in (3.40) and (3.42) are obtained in due course of building both the interacting gauge invariant Lagrangian and the corresponding gauge transformations.
ξ1 gauge transformations. In our approach, gauge transformations associated with the parameter ξ1 are simultaneously realized as the gradient gauge transformation of the spin-2 field φ2 and the Stueckelberg gauge transformation of the vector field φ1. In the interacting theory, the ξ1 gauge transformations take the form

\[
\begin{align*}
\delta_{\xi_1} e^\mu_\nu &= 0, \\
\delta_{\xi_1} \phi^a_0 &= 0, \\
\delta_{\xi_1} \phi^a_2 &= D^a \xi_1^b + D^b \xi_1^a, \\
\delta_{\xi_1} b^a &= 0, \\
\delta_{\xi_1} \phi^a_1 &= -\xi_1^a, \\
\delta_{\xi_1} \phi_0 &= 0.
\end{align*}
\] (3.45)

Comparing these gauge transformations with free theory gauge transformations given in (2.54) and (2.56), we see that all that is required for the generalization of free theory ξ1 gauge transformations is to make covariantization of the gradient gauge transformation of the spin-2 field φ2 (see (2.54) and (3.47)).

ξ−2 gauge transformations (Weyl gauge transformations). In our approach, the gauge transformation parameter responsible for Weyl gauge transformations is denoted by ξ−2. To make contact with the commonly used notation we introduce the parameter σ by the relation

\[
\sigma = -\frac{1}{4} \xi_{-2}.
\] (3.51)

Using Weyl dimensions of our fields (3.3), we write down the standard Weyl gauge transformations for our fields

\[
\begin{align*}
\delta_{\xi_{-2}} e^\mu_\nu &= -\sigma e^\mu_\nu, \\
\delta_{\xi_{-2}} \phi^a_0 &= \left(2 + k'\right) \sigma \phi^a_{k'}, \quad k' = 0, 2, \\
\delta_{\xi_{-2}} b^a &= D^a \xi_{-2} + \sigma b^a, \\
\delta_{\xi_{-2}} \phi^a_1 &= 3\sigma \phi^a_1, \\
\delta_{\xi_{-2}} \phi_0 &= 2\sigma \phi_0.
\end{align*}
\] (3.52)

ξ0 gauge transformations. In our approach, gauge transformations associated with parameter ξ0 are simultaneously realized as the gradient gauge transformation for the spin-1 field φ1 and the Stueckelberg transformation of the scalar field φ0. In the interacting theory, the ξ0 gauge transformations take the form

\[
\begin{align*}
\delta_{\xi_0} e^\mu_\nu &= 0, \\
\delta_{\xi_0} \phi^a_0 &= \frac{1}{2} \eta_{ab} \xi_0, \\
\delta_{\xi_0} \phi^a_2 &= -\frac{1}{2} \phi^a_0 \xi_0, \\
\delta_{\xi_0} b^a &= 0, \\
\delta_{\xi_0} \phi^a_1 &= D^a \xi_0, \\
\delta_{\xi_0} \phi_0 &= -u \xi_0.
\end{align*}
\] (3.57)

Comparing these gauge transformations with free theory gauge transformations given in (2.54) and (2.56), we see that the generalization of free theory ξ0 gauge transformations is arrived in two steps: (i) by covariantization of the gradient gauge transformations of spin-1 field φ1 (see (2.56) and (3.61)); (ii) by modification of the gauge transformation of field φ2 (see (2.54) and (3.59)).
Matching of linearized background gauge symmetries of interacting theory and global symmetries of free theory. Let us recall the definition of linearized background gauge symmetries. Consider a gauge transformation for the interacting field $\Phi$:

$$\delta \Phi = G_a(\Phi) \xi^a. \quad (3.63)$$

If $\tilde{\Phi}$ is a solution to equations of motion, then gauge transformation that respects this solution is realized by using the gauge transformation parameters $\tilde{\xi}^a$ satisfying the equations

$$G_a(\tilde{\Phi}) \tilde{\xi}^a = 0. \quad (3.64)$$

Using the field expansion $\Phi = \Phi + \phi$, the linearized background gauge transformations are then defined as

$$\delta \phi = \partial_\Phi G_a(\tilde{\Phi}) \tilde{\xi}^a \phi, \quad (3.65)$$

where $\partial_\Phi$ stands for a functional derivative. As is well known, the linearized background gauge transformations are interrelated with global symmetries of the corresponding flat theory. We now demonstrate this interrelation for the case of 6D conformal gravity.

We note that solution to 6D conformal gravity equations of motions corresponding to the flat space background is given by

$$\tilde{e}_a^{\mu} = \delta_a^{\mu}, \quad \tilde{\phi}^{ab} = 0, \quad \tilde{\phi}^{a}_{\mu} = 0, \quad \tilde{\phi}^{\mu} = 0. \quad (3.66)$$

Collecting all gauge transformations

$$\delta = \delta_{\xi_3} + \delta_{\xi_1} + \delta_{\xi_2} + \delta_{\xi_0} + \delta_{\lambda}, \quad (3.67)$$

and using notation $\tilde{\Phi}$ for background fields in (3.66), we now look for gauge transformations that respect solution given in (3.66)

$$\delta \tilde{\Phi} = 0. \quad (3.68)$$

To discuss the solution to equations in (3.68) we use the following notation. Solution to gauge transformation parameter $\xi$ that corresponds to the symmetry generator $G$ will be denoted as $\tilde{\xi}^G$. In our case there are the following set of symmetry generators $G = P^a, J^{ab}, D, K^a$. We now write solutions to gauge transformation parameters corresponding to these symmetry generators.

**Poincaré translations,**

$$\tilde{\xi}^{P^a}_{\mu} = \eta^{ab}, \quad \tilde{\xi}^{P^a}_{\nu} = 0, \quad \tilde{\xi}^{P^a}_{\mu} = 0, \quad \tilde{\xi}^{P^a}_{\nu} = 0. \quad (3.69)$$

**Lorentz rotations,**

$$\tilde{\xi}^{J_{ab}} = 2 \eta^{[b} x^{a]}, \quad \tilde{\xi}^{J_{ab}} = 0, \quad \tilde{\xi}^{J_{ab}} = 0, \quad \tilde{\xi}^{J_{ab}} = 2 \eta^{[b} x^{a]}, \quad (3.70)$$

**Dilatation,**

$$\tilde{\xi}^{aD} = x^a, \quad \tilde{\xi}^{aD} = 0, \quad \tilde{\xi}^{aD} = 0, \quad \tilde{\xi}^{aD} = 0. \quad (3.71)$$
Conformal boosts,
\[
\begin{align*}
\xi^{bK^c} & = -\frac{1}{2} x^b \eta_{cb} + \chi^a x^b, & \bar{\xi}^{bK^c} & = -4 \eta^{ab}, & \bar{\xi}^{bK^c} & = 0, \\
\xi_{\bar{z}^{-1}} & = -4 x^a, & \bar{\xi}^{K^c} & = 0, & \bar{\lambda}^{bc,K^c} & = 2 \eta^{[bc} x^{c]}.
\end{align*}
\] (3.72)

We now note that the linearized background gauge symmetries with gauge transformation parameters given in (3.69)–(3.72) correspond to the respective Poincaré translation, Lorentz, dilatation, and conformal boost symmetries of 6D conformal gravity in the flat space. To demonstrate this, we introduce fields of 6D conformal gravity in the flat space background (3.68):
\[
\begin{align*}
\phi_{ab}^{2}, & \quad \phi_{0}^{ab}, & \quad \phi_{2}^{ab}, & \quad \phi_{a}^{1}, & \quad \phi_{0}^{a}, & \quad \phi_{0}^{0}, & \quad \phi_{0}^{0}, & \quad \phi_{a}^{1}.
\end{align*}
\] (3.73)
where \(\phi_{ab}^{2}\) appears in the small field expansion of the vielbein field \(e_{\mu}^{a}\), while \(\phi_{a}^{0}\) is identified with the compensator field in the flat space background:
\[
e_{\mu}^{a} = \delta_{a}^{\mu} + \frac{1}{2} \phi_{ab}^{2} \eta_{\mu b}, & \quad b = \phi_{a}^{0}.
\] (3.74)

Note that expansion for \(e_{\mu}^{a}\) (3.74) implies that we use Lorentz gauge transformation (3.31) to obtain the symmetric tensor field \(\phi_{ab}^{2}\). Now using gauge transformation parameters given in (3.69)–(3.71) and general relation given in (3.65), we make sure that linearized background gauge symmetries with gauge transformation parameters in (3.69)–(3.71) coincide precisely with the respective Poincaré and dilatation symmetries of free 6D conformal theory discussed in section 2.2. Matching of the conformal boost symmetries turns out to be more interesting. This is to say, the linearized background gauge symmetries with gauge transformation parameters given in (3.72) take the form given in (2.66) with the same \(K_{a}^{\alpha,\beta}\) transformations as in (2.67) and the following \(R^{a}\) transformations:
\[
\begin{align*}
\delta_{R^{a}} \phi_{bc}^{0} & = 0, \\
\delta_{R^{a}} \phi_{bc}^{0} & = -2 \eta^{ab} \phi_{c}^{1} - 2 \eta^{ac} \phi_{b}^{1} + 2 \eta^{bc} \phi_{a}^{1} - 4 \phi_{a}^{b} \phi_{c}^{0} - 2 \phi_{bc}^{0} + 2 \phi_{ab}^{0}, \\
\delta_{R^{a}} \phi_{bc}^{0} & = -4 \eta^{ab} \phi_{c}^{1} - 4 \eta^{ac} \phi_{b}^{1} + 2 \eta^{bc} \phi_{a}^{1} - 4 \phi_{a}^{b} \phi_{c}^{0} - 2 \phi_{bc}^{0} + 2 \phi_{ab}^{0}, \\
\delta_{R^{a}} \phi_{b}^{1} & = 2 \phi_{b}^{0}, \\
\delta_{R^{a}} \phi_{b}^{0} & = 2 \phi_{0}^{b} - 2 \eta^{ab} u \phi_{0} - 2 F^{ab}(\phi_{..}), \\
\delta_{R^{a}} \phi_{0}^{0} & = 0.
\end{align*}
\] (3.75)–(3.79)

Comparing \(R^{a}\) transformations in (2.70)–(2.75) and the ones in (3.75)–(3.80), we notice some differences in \(R^{a}\) transformations for the fields \(\phi_{bc}^{0}\), \(\phi_{b}^{1}\), \(\phi_{0}^{a}\), and \(\phi_{0}^{0}\). Explanation of these differences is obvious: global transformations of gauge fields are defined up to gauge transformations. Introducing notation for the gauge transformation parameters
\[
\begin{align*}
\xi_{\bar{z}^{-1}}^{K^{c}} & = 2 \phi_{0}^{0}, & \xi_{\bar{z}^{-1}}^{K^{c}} & = 2 \phi_{a}^{1},
\end{align*}
\] (3.81)
and using notation \(\delta_{\bar{R}^{a}}\) and \(\delta_{R^{a}}\) for the respective \(R^{a}\) transformation in (2.70)–(2.75) and (3.75)–(3.80), we note that \(R^{a}\) transformations given in (2.70)–(2.75) and the ones in (3.75)–(3.80) are related by the gauge transformations:
\[
\begin{align*}
\delta_{R^{a}} \phi_{bc}^{0} & = \delta_{\bar{R}^{a}} \phi_{bc}^{0} + \phi_{ab}^{0} \xi_{\bar{z}^{-1}}^{K^{c}} + \phi_{bc}^{0} \xi_{\bar{z}^{-1}}^{K^{c}} + \frac{1}{2} \eta_{\bar{z}^{-1}}^{bc} \xi_{\bar{z}^{-1}}^{K^{c}}, \\
\delta_{R^{a}} \phi_{b}^{1} & = \delta_{\bar{R}^{a}} \phi_{b}^{1} - \xi_{\bar{z}^{-1}}^{K^{c}}, \\
\delta_{R^{a}} \phi_{0}^{a} & = \delta_{\bar{R}^{a}} \phi_{0}^{a} + \phi_{a}^{b} \xi_{\bar{z}^{-1}}^{K^{c}}, \\
\delta_{R^{a}} \phi_{0}^{0} & = \delta_{\bar{R}^{a}} \phi_{0}^{0} - u \xi_{\bar{z}^{-1}}^{K^{c}}.
\end{align*}
\] (3.82)–(3.85)

Thus, we see that the conformal boost transformations also match.
4. Higher-derivative Lagrangian of interacting 6D conformal gravity

Our ordinary-derivative Lagrangian can be used for the derivation of the higher derivative Lagrangian of interacting 6D conformal gravity. The higher derivative Lagrangian of interacting theory can be obtained by following the procedure we used for the derivation of the higher derivative Lagrangian of free 6D conformal gravity in section 2.2. We now proceed to the details of the derivation.

From gauge transformations (3.41), (3.49) and (3.62), we see that the vector fields $b^a$, $\phi^a_1$ and the scalar field $\phi_0$ transform as Stueckelberg fields and can therefore be gauged away by fixing the Stueckelberg gauge symmetries. Gauging away the vector fields and the scalar field, $b^a = 0$, $\phi^a_1 = 0$, $\phi_0 = 0$, (4.1)

we see that our Lagrangian (3.4) takes the simplified form

$$L = L_1 + L_2 + L_6 + L_8,$$

(4.2)

and

$$e^{-1} L_1 = -\phi^{ab}_{\mu} G^{ab}_{\mu},$$

(4.3)

$$e^{-1} L_2 = -\frac{1}{4} D^a \phi^b_{\mu} D^a \phi^c_{\mu} + \frac{1}{8} D^a \phi^{bb}_{\mu} D^a \phi^c_{\mu} + \frac{1}{2} C^a_1 C^a_1$$

$$- \frac{1}{2} R^{abc} \phi^b_{\mu} \phi^c_{\mu} + \frac{1}{2} R^{ab} \phi^a_{\mu} \phi^b_{\mu} - \frac{1}{2} R^{ab} \phi^a_{\mu} \phi^b_{\mu} + \frac{1}{2} (\phi_0^{ab}_{\mu} \phi_0^{ab} - \frac{1}{2} \phi_0^{ab}_{\mu} \phi_0^{ab}) R,$$

(4.4)

$$e^{-1} L_6 = -\frac{1}{16} G^{ab}_{\mu} \chi^{ab}_{\mu}.$$ (4.5)

$$G^{ab}_{\mu} \equiv R^{ab} - \frac{1}{2} \eta^{ab} R.$$

(4.6)

$$C^a_1 \equiv D^b \phi^c_{\mu} \phi^a_{\mu} - \frac{5}{16} \phi^a_{\mu} \phi^b_{\mu} \phi^c_{\mu} + \frac{1}{16} (\phi_0^{ab}_{\mu})^3,$$

(4.7)

$$\chi^{ab}_{\mu} \equiv \phi^{ab}_{\mu} - \eta^{ab} \phi_0^{cc}.$$

(4.8)

We note that the Lagrangian (4.2) depends on the rank-2 tensor field $\phi^{ab}_{\mu}$ linearly (see expressions for $L_1$ and $L_6$). Using equations of motion for the field $\phi^{ab}_{\mu}$ obtained from Lagrangian (4.2) we find the equation

$$\phi_0^{ab}_{\mu} - \eta^{ab} \phi_0^{cc} = -2 G^{ab}_{\mu},$$

(4.9)

which has the obvious solution

$$\bar{\phi}_0^{ab}_{\mu} = -2 R^{ab}_{\mu} + \frac{1}{2} \eta^{ab} R.$$

(4.10)

Plugging solution $\bar{\phi}_0^{ab}_{\mu}$ (4.11) into Lagrangian (4.2) we obtain the higher derivative Lagrangian

$$e^{-1} L = R^{ab} D^2 R^{ab} - \frac{3}{10} R D^2 R - 2 R^{cabe} R^{ab} R^{ce} - \frac{1}{2} R^{ab} R^{ab} R + \frac{3}{25} R^3.$$ (4.12)

This higher derivative Lagrangian should be invariant under Weyl gauge transformations

$$\delta e^a_{\mu} = -\sigma e^a_{\mu}.$$ (4.13)

We have checked directly that the Lagrangian is indeed invariant under Weyl gauge transformations (4.13). Note that using various identities for $R^3$ terms (see e.g. [24]), the Lagrangian can be expressed in terms of the Weyl tensor and Ricci curvatures.

Although there is a lot of literature on conformal gravity, we did not find a discussion of the Lagrangian (4.12). We note however that all Weyl invariant densities for 6D conformal...
theory were presented in [25] (see also [26–28] 11). Up to total derivative, in 6D conformal gravity theory, there are three Weyl invariant densities constructed out the Weyl tensor, Ricci curvatures, and covariant derivative. In [25], the simplest combination of those three invariants, which involves $R_{ab}D^2 R^{ab}$ term, has been found (see Eq.(4.12) in [25]). It turns out that it is this simplest invariant that coincides with our Lagrangian in (4.12) 12.

As a side of remark, we note that the remaining two Weyl invariant densities also can be lifted to our gauge-invariant approach, i.e. it is possible to build the invariants which respect all gauge symmetries of our approach. To this end we introduce the new curvature

$$R^{abce} = R^{abce} + h_1 (\eta^{ac} R^{be} - \eta^{bc} R^{ae} + \eta^{be} R^{ac} - \eta^{ae} R^{bc}),$$

(4.14)

$$q = 1/4,$$ and note that under $\xi_{-2}$, $\xi_{-1}$, and $\xi_0$ the gauge transformations new curvature (4.14) transforms as

$$\delta_{\xi_{-2}} R^{abce} = 2\sigma R^{abce}, \quad \delta_{\xi_{-1}} R^{abce} = 0, \quad \delta_{\xi_0} R^{abce} = 0,$$

(4.16)

From these relations, we see that the curvature $R^{abce}$ has the Weyl dimension equal to 2 and this curvature is invariant under the $\xi_{-1}$ and $\xi_1$ gauge transformations. Also, we note that gauging away Stueckelberg fields (4.1) and using solution (4.11), the curvature $R^{abce}$ becomes the standard Weyl tensor

$$R^{abce} \big|_{b=0} = \phi^{ab}, \quad R^{ab} = \phi^{ab} + \frac{1}{2}\phi^{ab} \bar{\phi}^{ab} = 0,$$

(4.18)

We see however that the curvature $R^{abce}$ does not respect $\xi_0$ gauge symmetry (4.17). The general curvature that respects the $\xi_0$ gauge symmetry can be built as follows:

$$R^{abce} = R^{abce} + h_1 (\eta^{ac} R^{be} - \eta^{bc} R^{ae} + \eta^{be} R^{ac} - \eta^{ae} R^{bc}) + h_2 (\eta^{ac} \eta^{be} - \eta^{ae} \eta^{bc}) R + h_3 (\eta^{ac} \eta^{be} - \eta^{ae} \eta^{bc}) \phi_0,$$

(4.19)

where $R^{ab} = R^{cahb}$, $R = R^{aa}$ and coefficients $h_1$, $h_2$, $h_3$ satisfy the equation

$$1 + 10 h_1 + 30 h_2 + 2 \frac{n}{q} h_3 = 0,$$

(4.20)

and $n$ is given in (2.21). Equation (4.20) is simply obtained by requiring the curvature $R^{abce}$ to be invariant under $\xi_0$ gauge transformations (3.57)–(3.62). Thus, curvature (4.19) has the desired properties:

$$\delta_{\xi_{-2}} R^{abce} = 2\sigma R^{abce}, \quad \delta_{\xi_{-1}} R^{abce} = 0, \quad \delta_{\xi_0} R^{abce} = 0, \quad \delta_{\xi_1} R^{abce} = 0,$$

(4.21)

Using the curvature $R^{abce}$, we can construct the remaining two gauge invariant densities in a straightforward way. For instance, we can consider the invariants

$$e^{1/2} R^{abce} R^{be} R_{ab}, \quad e^{1/2} R^{abce} R^{be} R_{ab}.$$

(4.22)

11 A discussion of interesting methods for constructing Weyl invariant densities may be found in [29, 30]. Classification of all the six-derivative Lagrangians in arbitrary dimensions such that the trace of the resulting field equations are at most of order 3 may be found in [31]. Study of the effective 6D conformal gravity may be found in [32].

12 In (4.12), our signs in front of terms involving odd number of the Ricci tensors and Ricci scalars are opposite to the ones in [25]. Perhaps these sign differences can be explained by the different conventions used for the definition of the Ricci tensor and Ricci scalar in our paper and in [25]. Our curvature conventions are $R^a_{\mu
u p} = \partial_\nu \Gamma^a_{\mu p} - \cdots$, $R_{\mu
u} = R^a_{\mu
u a}$, $R = R^a_{\mu a}$. 

18
In our gauge invariant approach, these invariants are counterparts of the well-known Weyl invariants appearing in the standard approach to 6D conformal theory,

\[ eC^{ab}_{ce} f_{ef} C^{fg}_{ab} + eC^{ab}_{ce} f_{ef} C^{ge}_{ab}, \]

(4.23)
i.e. by using Stueckelberg gauge conditions (4.1) and plugging solution (4.11) into invariants (4.22), we obtain the respective Weyl invariant densities given in (4.23).

Thus, the remaining two invariants (4.22) respect all gauge symmetries of our gauge invariant approach. The important difference of these two invariants as compared to our Lagrangian (3.4) is that these two invariants involve the higher derivatives, while our Lagrangian involves only the ordinary derivatives\(^{13}\). At the present time, we do not know the representation of invariants (4.22) in terms of the ordinary derivatives. It is not obvious that such representation can be constructed without adding new fields to our field content in (3.1).

We finish with remark that some special representatives of the general curvature (4.19) might be interesting in various contexts. For instance, solution to equation (4.20) given by

\[ h_1 = -\frac{1}{4}, \quad h_2 = \frac{1}{20}, \quad h_3 = 0, \]

(4.24)
leads to the traceless tensor

\[ R^{abce}_{\text{weyl}} = R^{abce} - \frac{1}{4} (\eta^{ac} R^{be} - \eta^{bc} R^{ae} + \eta^{be} R^{ac} - \eta^{ae} R^{bc}) + \frac{1}{20} \left( \eta^{ac} \eta^{be} - \eta^{ae} \eta^{bc} \right) \mathcal{R}, \]

(4.25)
which can be considered as counterpart of the Weyl tensor in our approach. Another solution to equation (4.20) given by

\[ h_1 = 0, \quad h_2 = 0, \quad h_3 = \frac{q}{2u}, \]

(4.26)
leads to the curvature

\[ R^{abce}_{\text{scal}} = R^{abce} + \frac{q}{2u} (\eta^{ac} \eta^{be} - \eta^{ae} \eta^{bc}) \phi_0, \]

(4.27)
The curvature \( R^{abce}_{\text{scal}} \) has the following interesting property. If we introduce the corresponding Einstein tensor

\[ G^{ab}_{\text{scal}} = R^{ab}_{\text{scal}} - \frac{1}{2} \eta^{ab} R_{\text{scal}}, \]

(4.28)
where \( R^{ab}_{\text{scal}} = R^{ac}_{\text{scal}} f_{ca}, \ R_{\text{scal}} = R_{\text{scal}}, \) then it turns out that the \( L_1 \) and \( L_6 \) parts of the Lagrangian (3.4) can be collected as

\[ L_1 + L_6 = -\phi_{ab}^{(ab)} G^{(ab)}_{\text{scal}}. \]

(4.29)
Because the field \( \phi_{ab}^{(ab)} \) does not appear in the remaining contributions to Lagrangian (3.4), equations of motion for this field can be represented as

\[ G^{(ab)}_{\text{scal}} = 0. \]

(4.30)

\(^{13}\) In [33] (see section 8.2), authors conjectured that two invariants given in (4.23) do not deform gauge algebra in the standard higher derivative approach to conformal 6D gravity. The fact that our remaining two invariants (4.22) respect our Stueckelberg gauge symmetries seems to be in agreement with this conjecture.
5. Conclusions

In this paper, we applied the ordinary-derivative approach, developed in [6], to the study of 6D interacting conformal gravity. The results presented here should have a number of interesting applications and generalizations. Let us comment on some of them.

(i) As we have already mentioned, the gauge symmetries of our Lagrangian make it possible to match our approach with the standard one, i.e. by an appropriate gauge fixing of the Stueckelberg fields and solving the constraints, we obtain the higher derivative formulation of the 6D conformal gravity. This implies that, at least at the classical level, our 6D conformal gravity theory is equivalent to the standard one. In this respect it would be interesting to investigate the quantum equivalence of our theory and the standard one. We note that our formulation provides new interesting possibilities for investigation of quantum behaviour of conformal gravity. The first step in studying quantum behaviour of conformal gravity is a computation of one-loop effective action. One powerful method for the computation of the one-loop effective action of Einstein gravity is based on the use of so-called \( \Delta_2 \) algorithm [34–36] (see also [37] 14). In order to investigate quantum properties of fourth-derivative 4D Weyl gravity, this algorithm was generalized to the so-called \( \Delta_4 \)-algorithm (see [39] and references therein). We note, however, that since our approach does not involve higher derivatives and formulated in terms of conventional kinetic terms we can use the standard \( \Delta_2 \) algorithm for the investigation of the one-loop effective action.

(ii) In addition to the local Weyl and diffeomorphism symmetries that enter the standard approach to conformal gravity, our approach involves gauge symmetries for two rank-2 tensor fields and some amount of Stueckelberg gauge symmetries. In other words, we deal with extended gauge algebra. In this respect, it would be interesting to analyse the general solution of the Wess–Zumino consistency condition for our gauge algebra along the lines in [30].

(iii) The results in this paper provide the complete ordinary-derivative description of interacting 6D conformal gravity. It would be interesting to apply these results to the study of supersymmetric conformal field theories [40–44] in the framework of an ordinary-derivative approach. The first step in this direction would be understanding how the supersymmetries are realized in the framework of our approach.

(iv) The BRST approach is one of the powerful approaches to the analysis of various aspects of relativistic dynamics (see e.g. [45–50]). This approach turned out to be successful for application to string theory. We believe therefore that use of this approach for the study of conformal fields might also be helpful for the better understanding of conformal gravity theory.

(v) In recent years, there were interesting developments in the studying mixed-symmetry fields [51–59] that are invariant with respect to anti-de Sitter or Minkowski spacetime symmetries. It would be interesting to apply methods developed in [51–59] to the study of interacting 6D conformal mixed-symmetry fields15. There are various other interesting approaches in the literature which could be used to discuss the ordinary-derivative formulation of 6D conformal fields. This is to say, various recently developed interesting formulations in terms of unconstrained fields in flat space may be found e.g. in [60–63].

14 Generalization of methods in [34–36] to quantum effective actions for brane-induced gravity models may be found in [38].

15 Unfolded form of equations of motion for conformal mixed-symmetry fields is studied in [4]. Higher-derivative Lagrangian formulation of the mixed-symmetry conformal fields was recently developed in [2].
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Appendix A. Notation

Flat spacetime notation. Our conventions are as follows. Coordinates in the flat spacetime are denoted by $x^a$, while $\partial_a$ stands for derivative with respect to $x^a$, $\partial_a \equiv \partial/\partial x^a$. Vector indices of the Lorentz algebra $so(5, 1)$ take the values $a, b, c, e = 0, 1, \ldots, 5$. To simplify our expressions we drop the flat metric $\eta_{ab}$ in scalar products, i.e. we use $X^a Y^b \equiv \eta_{ab} X^a Y^b$. We use operators constructed out of the coordinates and derivatives,

$$\Box = \partial^a \partial_a, \quad x \partial = x^a \partial_a. \quad (A.1)$$

Curved spacetime notation. We use spacetime base manifold indices $\mu, \nu, \rho, \sigma = 0, 1, \ldots, 5$ and tangent-flat vectors indices of the $so(5, 1)$ algebra $a, b, c, e, f = 0, 1, \ldots, 5$. Base manifold coordinates are denoted by $x^\mu$, while $\partial_\mu$ denote the respective derivatives, $\partial_\mu \equiv \partial/\partial x^\mu$. We use notation $e_a^\mu$ and $\omega_{ab}^{\mu(e)}$ for the respective vielbein and the Lorentz connection. The contravariant tensor field carrying base manifold indices, $\phi_{\mu_1 \ldots \mu_s}$, is related with tensor field carrying the tangent-flat indices, $\phi_{a_1 \ldots a_s}$, in a standard way $\phi_{a_1 \ldots a_s} \equiv e_{a_1}^\mu \ldots e_{a_s}^\mu \phi_{\mu_1 \ldots \mu_s}$. The covariant derivative $D_\mu$, acting on the vector field

$$D_\mu \phi^a = \partial_\mu \phi^a + \omega_{\mu a}^{\mu(e)} \phi^b,$$ \hspace{1cm} (A.2)

satisfies the standard commutator

$$[D_\mu, D_\nu] \phi^a = R_{\mu
u}^{ab} \phi^b. \quad (A.3)$$

Instead of $D_\mu$, we prefer to use a covariant derivative with the flat indices $D^a$,

$$D_a \equiv e_a^\mu D_\mu, \quad D^a = \eta^{ab} D_b, \quad (A.4)$$

where $e_a^\mu$ is inverse of the vielbein, $e_a^\mu e_b^\mu = \delta_a^b$.

For field $\phi^a$ with Weyl dimension $\Delta_\phi^a$, covariant derivative defined in (3.20) satisfies the commutator

$$[D^a, D^b] \phi^c = \tilde{R}^{abc} \phi^e + q \Delta_\phi^a F^{ab}(b) \phi^c, \quad q = 1/4, \quad (A.5)$$

where the shifted curvature and field strength $F^{ab}$ are defined in (3.26) and (3.19) respectively. Weyl gauge transformations defined in (3.51)–(3.56) lead to the following transformation rules:

$$\delta \omega^{abc} = \sigma \omega^{abc} + \eta^{ac} D^b \sigma - \eta^{ab} D^c \sigma, \quad (A.6)$$

$$\delta \tilde{R}^{abc} = 2 \sigma \tilde{R}^{abc}, \quad (A.7)$$

$$\delta \tilde{\omega}^{abc} = \sigma \tilde{\omega}^{abc}, \quad (A.8)$$

$$\delta F^{ab}(\phi) = (\Delta_\phi^w + 1) \sigma F^{ab}(\phi), \quad (A.9)$$

$$\delta F^{ab} = 2 \sigma F^{ab}, \quad (A.10)$$

$$\delta (D^a \phi^{a_1 \ldots a_s}) = (\Delta_\phi^w + 1) D^a \phi^{a_1 \ldots a_s}. \quad (A.11)$$
These relations imply the following Weyl dimensions:

\[
\Delta^w_{Rabc} = 2, \quad \Delta^w_{\phi abc} = 1, \quad \Delta^w_{F ab} = 2, \quad \Delta^w_{\phi(\phi)} = \Delta^w_{\phi} + 1.
\]  

(B.12)

Bianchi identities for the shifted curvature and field strength (3.19):

\[
\mathcal{D}^f \hat{R}^{abce} + \text{cycl.perms.}(f\, ab) = 0, \tag{B.13}
\]

\[
\mathcal{D}^a F^{bc} + \text{cycl.perms.}(abc) = 0, \tag{B.14}
\]

can be obtained in a usual way. Using explicit relations for the curvatures and the Einstein tensor

\[
\hat{R}^{abce} = R^{abce} + \eta^{ac} \phi^{be} - \eta^{bc} \phi^{ae} + \eta^{be} \phi^{ac} - \eta^{ae} \phi^{bc}, \tag{B.15}
\]

\[
\hat{R}^{ab} = \hat{R}^{cabc}, \quad \hat{R} = \hat{R}^{aa}, \tag{B.16}
\]

\[
\hat{R}^{ab} = R^{ab} + 4 \phi^{ab} + \eta^{ab} \phi^{cc}, \tag{B.17}
\]

\[
\hat{R} = R + 10 \phi^{aa}, \tag{B.18}
\]

\[
\hat{G}^{ab} = \hat{R}^{ab} - \frac{1}{2} \eta^{ab} \hat{R}, \tag{B.19}
\]

\[
\phi^{ab} = q D^ab + q^2 b^a b^b - \frac{1}{2} q^2 \eta^{ab} b^2, \tag{B.20}
\]

\[
\phi^{cc} = q Dc - 2 q^2 b^2, \quad q = 1/4, \tag{B.21}
\]

we obtain various useful identities

\[
\hat{R}^{abce} - \hat{R}^{cabe} = q(\eta^{ac} F^{be} - \eta^{bc} F^{ae} + \eta^{be} F^{ac} - \eta^{ae} F^{bc}), \tag{B.22}
\]

\[
\hat{R}^{ab} - \hat{R}^{ba} = F^{ab}, \tag{B.23}
\]

\[
\mathcal{D}^f \hat{R}^{abce} = \mathcal{D}^f R^{abe} - \mathcal{D}^b R^{ace}, \tag{B.24}
\]

\[
\mathcal{D}^f \hat{R}^{cabe} = \mathcal{D}^f R^{abe} - \mathcal{D}^b R^{ace} + q(\mathcal{D}^f F^{ab} - \eta^{ae} \mathcal{D}^f F^{cb} + \eta^{be} \mathcal{D}^f F^{ca}), \tag{B.25}
\]

\[
\mathcal{D}^b \hat{R}^{ab} = \frac{1}{2} \mathcal{D}^b \hat{R}, \tag{B.26}
\]

\[
\mathcal{D}^b \hat{R}^{ba} = \frac{1}{2} \mathcal{D}^b \hat{R} + \mathcal{D}^b F^{ba}, \tag{B.27}
\]

\[
\mathcal{D}^b \hat{G}^{ab} = 0, \quad \mathcal{D}^b \hat{G}^{ba} = \mathcal{D}^b F^{ba}, \quad \mathcal{D}^b \hat{G}^{(ab)} = \frac{1}{2} \mathcal{D}^b F^{ba}. \tag{B.28}
\]

Throughout this paper, symmetrization and antisymmetrization of the indices are normalized as \((ab) = \frac{1}{2}(ab + ba), [ab] = \frac{1}{2}(ab - ba)\).
Appendix B. Derivation of interacting gauge invariant Lagrangian

In this appendix, we outline some details of the derivation of gauge invariant Lagrangian given in (3.4) and the corresponding gauge transformations. We divide our derivation in seven steps which we now discuss in turn.

**Step 1.** We begin with the discussion of contribution to Lagrangian (3.4) denoted by \( \mathcal{L}_1 \) (3.5). This contribution is simply obtained from the contribution to free theory Lagrangian \((2.31)\) also denoted by \( \mathcal{L}_1 \) (2.32) in a straightforward way. Namely \( \mathcal{L}_1 \) (3.5) is obtained from \( \mathcal{L}_1 \) (2.32) by requiring \( \mathcal{L}_1 \) (3.5) to be invariant under Weyl gauge transformations given in (3.52)–(3.56). All that is required to respect those gauge transformations is to replace the linearized shifted Einstein tensor and curvatures given in \((2.43)–(2.49)\) by the corresponding complete shifted Einstein tensor and curvatures given in \((3.28), (A.17)\) and \((A.18)\).

**Step 2.** We covariantize the flat \( \xi^a \), gauge transformation of the field \( \phi^{ab} \) (2.53) by making replacement \( \partial^a \to \partial^a + \xi^a \), while the flat \( \xi^a \), gauge transformation of the field \( b^a \) (2.55) is not changed,

\[
\delta \xi_a \phi_0^{ab} = \partial^a \xi_b + \partial^b \xi_a, \quad \delta \xi_a b^a = -\xi_a. \tag{B.1}
\]

Also, making the covariantization \( \partial^a \to \partial^a + \xi^a \) in contribution to flat Lagrangian denoted by \( \mathcal{L}_2 \) (2.33), we introduce

\[
e^{-1} \mathcal{L}_{2x} = -\frac{1}{2} \partial^a \phi^{bc}_0 \partial^b \phi^{ac}_0 + \frac{1}{8} \partial^a \phi^{bb}_0 \partial^b \phi^{aa}_0 + \frac{1}{2} C_i^a C_i^a, \tag{B.3}
\]

where the covariantized \( C_i^a \) is defined as in (3.16). After this, we consider gauge variation of \( \mathcal{L}_{2x} \) under gauge transformations \((B.1)\) and \((B.2)\). In the gauge variation, we find unwanted terms proportional to the shifted curvatures \( \bar{R}^{abce}, \bar{R}^{ab}, \bar{R} \) which cannot be cancelled by modification of gauge transformations of the field entering our field content \((3.1)\). Our observation is that those unwanted terms can be cancelled by adding to \( \mathcal{L}_{2x} \) the following contribution:

\[
e^{-1} \mathcal{L}_{2x} = -\frac{1}{2} R^{abce} \phi^{ab}_0 \phi^{ce}_0 + \frac{1}{2} R^{ab} \phi^{bc}_0 \phi^{ac}_0 - \frac{1}{2} \bar{R}^{ab} \phi^{ab}_0 \phi^{ab}_0 + \left( \frac{1}{8} \phi^{aa}_0 \phi^{bb}_0 - \frac{1}{2} \phi^{ab}_0 \phi^{ab}_0 \right) \bar{R}. \tag{B.4}
\]

This is to say that variation of \( \mathcal{L}_2 = \mathcal{L}_x + \mathcal{L}_{2x} \) under gauge transformations \((B.1)\) and \((B.2)\) takes the form

\[
\delta \xi_a \mathcal{L}_2 = \delta \xi_a \mathcal{L}_x + \delta \xi a \mathcal{L}_2, \tag{B.5}
\]

\[
\delta \phi^{ab}_0 \mathcal{L}_2 = \delta \phi^{ab}_0 \mathcal{L}_x + \delta \phi^{ab}_0 \mathcal{L}_2, \tag{B.6}
\]

\[
e^{-1} \delta \phi^{ab}_0 \mathcal{L}_2 \bigg|_{\mathcal{G}} = \mathcal{G}^{ab} \mathcal{L}_x \phi^{00}_0. \tag{B.7}
\]

\[
e^{-1} \delta \xi^a \mathcal{L}_2 \bigg|_{\mathcal{L}_2} = -2q F^{ac} F^{bh} (\xi^a) \phi^{ab}_0 + 2q F^{ab} \xi^b (\partial \phi_0)^a - q F^{ab} \xi^b (\partial \phi_0)^c + 2q D^a F^{ab} \phi^{bc}_0 + \frac{q}{2} D^a F^{ab} \phi^{bc}_0 \phi_0^0, \tag{B.8}
\]

\[
e^{-1} \delta \phi^{ab}_0 \mathcal{L}_2 = 2q \xi^b \phi^{ab}_0 (\partial^c \phi_0^{ac} - \partial^c \phi_0^{cc})
\]

where the Lie derivative \( \mathcal{L}_{\xi_a} \phi^{00}_0 \) entering variation \((B.7)\) is defined in \((3.44)\). From these relations, we see that the remaining terms involving the shifted curvatures are proportional to
From these relations, we see that variation proportional to $\phi_{ab}$:

$$
\delta_{\xi^a} \phi_{ab}^2 = L_{\xi^a} \phi_{ab}^0.
$$

(B.10)

Namely, it is easy to see that variation of $L_1$ (3.5) under gauge transformation of $\phi_{ab}^0$ given in (B.10) cancels variation in (B.7).

**Step 3.** We now consider variation of contributions to the Lagrangian denoted by $L_3$ (3.7) and $L_5$ (3.9) under gauge transformations given in (B.1) and (B.2). Using notation $\delta_{\phi^a,\xi^a} L_3$ and $\delta_{\phi^a,\xi^a} L_5$ for the respective variations of $L_3$ and $L_5$ under $\xi^a$ gauge transformation of the field $b^a$, and notation $\delta_{\phi^a,\xi^a} L_5$ for variation of $L_5$ under $\xi^a$ gauge transformation of the field $\phi_{ab}^0$ we find

$$
e^{-1}(\delta_{\phi^a,\xi^a} L_3 + \delta_{\phi^a,\xi^a} L_5) = \left(\phi_{ab}^1 \psi_{-} + \phi_{c}^b \psi_{-} \phi_{a} - \frac{1}{2} \eta^{ab} \phi_{a} \right) \tilde{G}_{ab},$$

(B.11)

$$
e^{-1} \delta_{\phi^a,\xi^a} L_5 = \phi_{ab}^1 \left(\phi_{ab}^0 \psi_{-} + \frac{1}{4} \phi_{c}^b \psi_{-} \phi_{a} + \frac{1}{2} \phi_{b} \right).$$

(B.12)

From these relations, we see that variation proportional to $\tilde{G}_{ab}$ (B.11) can be cancelled by modifying $\xi^a$ gauge transformation of the field $\phi_{ab}^0$:

$$
\delta_{\xi^a} \phi_{ab}^2 = \phi_{ab}^2 + \phi_{ab}^1 \psi_{-} - \frac{1}{2} \eta^{ab} \phi_{a} \phi_{a}.
$$

(B.13)

Namely, it is easy to see that variation of $L_1$ (3.5) under gauge transformation of $\phi_{ab}^0$ (B.13) cancels variation in (B.11). Collecting results in (B.10) and (B.13), we find complete $\xi^a$ gauge transformation of the field $\phi_{ab}^0$:

$$
\delta_{\xi^a} \phi_{ab}^2 = L_{\xi^a} \phi_{ab}^0 + \phi_{ab}^1 \psi_{-} + \phi_{ab}^0 \psi_{-} - \frac{1}{2} \eta^{ab} \phi_{a} \phi_{a}.
$$

(B.14)

**Step 4.** Using notation $\delta_{\phi^a,\xi^a} L_6$ for variation of $L_6$ under $\xi^a$ gauge transformation of field $\phi_{ab}^0$ (B.14), we find

$$
\delta_{\phi^a,\xi^a} L_6 = \delta_{\phi^a,\xi^a} L_6 [\phi_{ab}^0, \phi_{ab}^0] \phi_{a} + \delta_{\phi^a,\xi^a} L_6 [\phi_{ab}^0, \phi_{a}^0],
$$

(B.15)

$$
e^{-1} \delta_{\phi^a,\xi^a} L_6 [\phi_{ab}^0, \phi_{ab}^0] \phi_{a} = -\left(\phi_{ab}^1 \phi_{ab}^2 D^{\psi} \xi^b + \phi_{ab}^0 \phi_{ab}^2 D^{\psi} \xi^b + \phi_{ab}^0 \phi_{ab}^0 D^{\psi} \xi^b, \frac{1}{2} \phi_{ab}^0 \phi_{ab}^0 D^{\psi} \xi^b \right)$$

$$
-\phi_{ab}^0 \phi_{c}^b \phi_{c}^e \phi_{a} + \phi_{ab}^0 \phi_{ab}^0 \phi_{a} \phi_{a} + \phi_{ab}^0 \phi_{ab}^0 \phi_{a} \phi_{a} + \phi_{ab}^0 \phi_{ab}^0 \phi_{a} \phi_{a}$$

(B.16)

$$
e^{-1} \delta_{\phi^a,\xi^a} L_6 [\phi_{ab}^0, \phi_{a}^0] \phi_{a} = -\phi_{ab}^0 \left(\phi_{ab}^0 \psi_{-} + \frac{1}{4} \phi_{ab}^0 \psi_{-} \phi_{a} + \frac{1}{2} \phi_{a} \right).$$

(B.17)

Comparing (B.12) and (B.17), we find the cancellation

$$
\delta_{\phi^a,\xi^a} L_5 + \delta_{\phi^a,\xi^a} L_6 [\phi_{ab}^0, \phi_{a}^0] = 0.
$$

(B.18)

We proceed to the next step of our procedure noting that variations that remain to be cancelled are given in (B.8), (B.9), and (B.16).

**Step 5.** We now consider $F_{ab}$ depending variation given in (B.8). This variation can be cancelled by adding new contributions to Lagrangian and modifying $\xi^a$ gauge transformations of the field $\phi_{ab}^0$. Note that, in flat conformal gravity, the field $\phi_{a}^0$ is not transformed under $\xi^a$.
gauge transformations (see (2.56)). This is to say, in interacting conformal gravity, we consider the following new contributions to Lagrangian and $\xi_1$ gauge transformation of the field $\phi_0^a$:

$$e^{-1} \mathcal{L}_7 = c_1 F^{ac} F^{ab} \phi_0^b + c_2 F^{ab} F^{ab} \phi_0^c + c_3 F^{ab} F^{ab} \phi_0^0,$$

where coefficients $c_{1,2,3}$ and $f_1, f_2, f_3, f_4$ remain to be determined. To this end we compute variations of $\mathcal{L}_3$ (3.7) and $\mathcal{L}_5$ (3.9) under $\xi_1$ gauge transformation of the field $\phi_1^a$ (B.20),

$$e^{-1} \delta \mathcal{L}_3 F |_{\phi_0^a} = f_3 F^{ab} F^{bc} \xi_1^c + f_1 D^a F^{ab} \phi_0^b + f_3 D^a F^{ab} \xi_1^b - f_5 D^a F^{ab} \phi_0^b, \quad (B.21)$$

$$\mathcal{L}_5 = \phi_1^a, \mathcal{L}_5 \rightarrow \phi_1^a, \mathcal{L}_5 |_{\phi_0^a}, \phi_0^a,$$

$$e^{-1} \delta \mathcal{L}_5=F |_{\phi_0^a} = f_2 F^{ab} \xi_1^b (D \phi_0^a) + f_2 F^{ab} \xi_1^b D^a \phi_0^c - f_2 u F^{ab} \xi_1^b D^a \phi_0^0, \quad (B.23)$$

Requiring the $F^{ab}$ depending terms to cancel gives the equations

$$e^{-1} \delta \mathcal{L}_7 = -2c_1 D^a F^{ab} F^{bc} \xi_1^c + 2c_2 \left(2c_2 - \frac{c_1}{2}\right) F^2 D \xi_1^c - 2c_1 F^{ac} F^{ab} (\xi_1^c) \phi_0^{ab} + 4c_3 D^a F^{ab} \xi_1^b D^a \phi_0^c + 4c_2 D^a F^{ab} \xi_1^b \phi_0^c + 4c_3 F^{ab} \phi_0^b \phi_0^c + 4c_3 F^{ab} \phi_0^b \phi_0^c.$$

Step 6. Variations that remain to be cancelled are given in (B.9), (B.16) and (B.28). All these variations involve terms of the second order in the fields $\phi_0^{ab}$ and $\phi_0$. Note that the variation of $\mathcal{L}_4$ (3.8) under $\xi_1^a$ gauge transformation also gives terms of the second order in the field $\phi_0$.

$$e^{-1} \delta \mathcal{L}_4 = -\frac{1}{2} \phi_0^b D^b \xi_1^a.$$

We note that variations (B.9), (B.16) and (B.28) can be cancelled by adding new contributions to Lagrangian without any additional modification of $\xi_1^a$ gauge transformations of the fields. This is to say that we consider the following new contributions to Lagrangian:

$$e^{-1} \mathcal{L}_8 = p_1 \phi_0^{ab} \phi_0^{bc} \phi_0^{ca} + p_2 \phi_0^{ab} \phi_0^{bc} \phi_0^{ca} + p_3 \phi_0^{ab} \phi_0^c \phi_0^c + p_4 \phi_0^a \phi_0^b \phi_0^b + p_5 \phi_0^a \phi_0^b + p_7 \phi_0^2 \phi_0^2 \phi_0^2.$$

(B.30)
Computing
\[ e^{-1} \delta_{\xi_a} \mathcal{L}_8 = 6 p_1 (\phi_0^{ab})_{\xi_a} + p_2 \phi_0^{ab} \phi_0^{ab} D^2 \xi_a + 2 p_2 (\phi_0^{ab})_{\xi_a} + 6 p_3 \phi_0^{aa} \phi_0^{bb} D \xi_a \\
+ 4 p_4 \phi_0^{ab} \phi_0^{ab} D^2 \xi_a + 2 p_5 \phi_0^{ab} D \xi_a + 4 p_7 \phi_0^{aa} \phi_0^{bb} D \xi_a, \]
we see that Lagrangian (3.4) is invariant under the gauge transformations.

We now consider the remaining gauge variations to be cancelled by covariantization of the corresponding gauge transformations of free fields. Using gauge transformations (B.31) and requiring
\[ \delta_{\xi_a} \mathcal{L}_2 + \delta_{\xi_a} \mathcal{L}_4 + \delta_{\xi_a} \mathcal{L}_5 |_{\phi_0^{ab} \phi_0^{ab}} + \delta_{\xi_a} \mathcal{L}_6 |_{\phi_0^{aa} \phi_0^{bb}} + \delta_{\xi_a} \mathcal{L}_8 = 0, \]
we obtain
\[ p_1 = \frac{1}{4}, \quad p_2 = - \frac{5}{16}, \quad p_3 = \frac{1}{16}, \quad p_4 = - \frac{u}{8}, \quad p_5 = - \frac{3}{16}, \quad p_7 = 0. \]

Thus, we see that, with exception of the coefficient \( p_6 \), all the remaining coefficients entering cubic potential (B.30) are fixed by \( \xi_a^0 \) gauge symmetries. Note also that all variations of the Lagrangian \( \mathcal{L} \), which are proportional to the gauge transformation parameter \( \xi_a^0 \), have been cancelled.

**Step 7.** The coefficient \( p_6 \) is fixed by considering \( \xi_0 \) gauge transformations. With exception of the field \( \phi_0^{ab} \), \( \xi_0 \) gauge transformations given in (3.57)–(3.62) are obtained by covariantization of the corresponding gauge transformations of free fields. Using gauge transformations in (3.57)–(3.62) we check the relations
\[ \delta_{\xi_0} \mathcal{L}_{2x} + \mathcal{L}_4 + \mathcal{L}_5 = 0, \quad \delta_{\xi_0} \mathcal{L}_2 + \mathcal{L}_7 = 0, \]
where we use the decomposition \( \mathcal{L}_2 = \mathcal{L}_{2x} + \mathcal{L}_{2z} \) (see (4.4), (B.3), (B.4)). We note that \( \mathcal{L}_{2x} \) (B.4) is not invariant under \( \xi_0 \) gauge transformation
\[ e^{-1} \delta_{\xi_0} \mathcal{L}_{2x} = - \frac{1}{2} \phi_0^{ab} \mathcal{G}^{ab} \xi_0. \]
It is easy to see that this gauge variation can be cancelled by modifying \( \xi_0 \) gauge transformation of the field \( \phi_0^{ab} \),
\[ \delta_{\xi_0} \phi_0^{ab} = \frac{1}{2} \phi_0^{ab} \xi_0. \]

We now consider the remaining gauge variations to be cancelled
\[ e^{-1} \delta_{\xi_0} \mathcal{L}_6 = - \frac{1}{4} \phi_0^{ab} \phi_0^{ab} \xi_0 + \frac{1}{4} \phi_0^{ab} \phi_0^{ab} \xi_0 + \frac{u}{4} \phi_0^{aa} \phi_0^{bb} \xi_0 - 3 \left( \frac{3}{16} + u p_6 \right) \phi_0^{ab} \xi_0, \]
\[ e^{-1} \delta_{\xi_0} \mathcal{L}_6 = - \frac{1}{4} \phi_0^{ab} \phi_0^{ab} \xi_0 - \frac{1}{4} \phi_0^{ab} \phi_0^{ab} \xi_0 - \frac{u}{4} \phi_0^{aa} \phi_0^{bb} \xi_0. \]

Requiring \( \delta_{\xi_0} \mathcal{L}_6 = 0 \), we obtain \( p_6 = - \frac{3}{16u} \).

Thus, with exception of \( \xi_0 \) gauge transformations, we have checked the gauge invariance of our Lagrangian with respect to all gauge transformations. The \( \xi_0 \) gauge transformations of interacting theory (3.45)–(3.50) are simply obtained by covariantization, \( \delta \phi^a \to \mathcal{D}^a \), of the ones of flat theory (2.52)–(2.57). In doing so, we note that only the contributions to Lagrangian denoted by \( \mathcal{L}_1, \mathcal{L}_3, \mathcal{L}_5, \mathcal{L}_6 \) are changed under \( \xi_0 \) gauge transformations. Using the easily derived relations
\[ \delta_{\xi_0} \mathcal{L}_1 + \mathcal{L}_3 = 0, \quad \delta_{\xi_0} \mathcal{L}_5 + \mathcal{L}_6 = 0, \]
we see that Lagrangian (3.4) is invariant under the \( \xi_0 \) gauge symmetries. This finishes our procedure of building the gauge invariant Lagrangian and the corresponding gauge transformations.
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