COHOMOLOGY OF ALGEBRAIC GROUPS AND SULLIVAN’S MINIMAL MODELS

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ABSTRACT. Constructing an explicit cochain complex homomorphism which induces the Hochschild isomorphism on the rational cohomology of an algebraic group, we obtain explicit Sullivan’s minimal models of certain differential graded algebras defined on simplicial classifying spaces of torsion-free virtually polycyclic groups and we give de Rham homotopy theorem for simplicial complexes with torsion-free virtually polycyclic fundamental groups.

1. Introduction

Let $N$ be a simply connected nilpotent Lie group and $\mathfrak{n}$ be the Lie algebra of $N$. We suppose $N$ has a lattice $\Gamma$. We consider the nilmanifold $\Gamma \backslash N$. Then, considering the cochain complex $\bigwedge \mathfrak{n}^*$ of the Lie algebra as the space of the invariant differential forms, in [21], Nomizu proves that the canonical inclusion $\bigwedge \mathfrak{n}^* \subset A^*(\Gamma \backslash N)$ induces a cohomology isomorphism

$$H^*(\mathfrak{n}, \mathbb{R}) \cong H^*(\Gamma \backslash N, \mathbb{R})$$

where $A^*(\Gamma \backslash N)$ is the de Rham complex of $\Gamma \backslash N$. Nomizu’s theorem gives an important fact on the theory of Sullivan’s minimal model. We can say that the Differential graded algebra (shortly DGA) $\bigwedge \mathfrak{n}^*$ is the minimal model of $A^*(\Gamma \backslash N)$.

In [14], the author gives an extension of Nomizu’s theorem in view of the theory of Sullivan’s minimal model. Let $S$ be a simply connected solvable Lie group with a lattice $\Gamma$ and $\mathfrak{s}$ be the Lie algebra of $S$. We consider the solvmanifold $\Gamma \backslash S$. We can define the diagonalized adjoint representation $\text{Ad}_s : S \to \text{Aut}(\mathfrak{s})$ ([14 Construction 1.1.]). Take the complex Zariski-closure $D$ of $\text{Ad}_s(\Gamma)$. We consider the Hain’s DGA $A^*(\Gamma \backslash S, \mathbb{C}[D])$ which is the DGA of differential forms with values in the coordinate ring $\mathbb{C}[D]$ of the algebraic group $D$ as a local system over $\Gamma \backslash S$ (see [8]). Then as an extension of Nomizu’s theorem, in [14], the author construct an explicit minimal model of $A^*(\Gamma \backslash S, \mathbb{C}[D])$.

In [17], Lambe and Priddy give a simplicial and rational analogue of Nomizu’s theorem. For simply connected nilpotent Lie group $N$ with a lattice $\Gamma$, $\Gamma$ is a torsion-free finitely generated nilpotent group and a nilmanifold $\Gamma \backslash N$ is an aspherical manifold with the fundamental group $\Gamma$. Conversely any torsion-free finitely generated nilpotent group $\Gamma$ can be embedded in a simply connected nilpotent Lie group $N$ whose Lie algebra $\mathfrak{n}$ admits a $\mathbb{Q}$-structure $\mathfrak{n}_\mathbb{Q}$ (see [22]). In [17], considering the simplicial classifying space $B\Gamma$ of a torsion-free finitely generated nilpotent group

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Γ, Lambe and Priddy construct an explicit cochain complex homomorphism which induces a cohomology isomorphism
\[ H^\ast(n_\mathbb{Q}, \mathbb{Q}) \cong H^\ast(B\Gamma, \mathbb{Q}) \]
by using simplicial de Rham theory.

Now we are interested in a simplicial and rational analogue of author’s extended Nomizu’s theorem. For a simply connected solvable Lie group \( S \) with a lattice \( \Gamma \), \( \Gamma \) is a torsion-free polycyclic group and a solvmanifold \( \Gamma \backslash S \) is an aspherical manifold with the fundamental group \( \Gamma \). In this paper, we consider the simplicial classifying space \( B\Gamma \) of a torsion-free virtually polycyclic group \( \Gamma \).

Let \( K \) be a simplicial complex with \( \pi_1 K = \Gamma \). Let \( \rho : \Gamma \to T \) be a representation of \( \Gamma \) in a reductive \( \mathbb{Q} \)-algebraic group \( T \) with the Zariski-dense image. For the coordinate ring \( \mathbb{Q}[T] \) as a \( \Gamma \)-module, we define the DGA \( A^\ast_p(K, \mathbb{Q}[T]) \) which is a simplicial analogue of the DGA on a manifold which is constructed by Hain in [8].

Let \( G \) be a torsion-free virtually polycyclic group, \( \rho : \Gamma \to G \) a representation with the Zariski-dense image. It is known that \( \dim U \leq \text{rank } \Gamma \) where \( U \) is the unipotent radical of \( G \). We say that \( \rho : \Gamma \to G \) is a full representation if \( \dim U = \text{rank } \Gamma \). It is known that there exists a unique algebraic group \( G \) such that the centralizer \( Z_G(U) \) of \( G \) is contained in \( U \) and we have an injective full representation \( \rho : \Gamma \to G \) ([12, Appendix A]). We call such algebraic group \( G \) the algebraic hull of \( \Gamma \) and we call the unipotent radical of the algebraic hull \( G \) the unipotent hull of \( \Gamma \) and denote it by \( U_\Gamma \). Denote by \( \mathfrak{u}_\Gamma \) the Lie algebra of the unipotent hull \( U_\Gamma \) of \( \Gamma \). In this paper, we prove the following result.

**Theorem 1.1.** For some representation \( \Gamma \to T \) in certain reductive \( \mathbb{Q} \)-algebraic group \( G \) with the Zariski-dense image, we have an explicit DGA homomorphism \( \mathfrak{u}_\Gamma^* \to A^\ast_p(B\Gamma, \mathbb{Q}[T]) \) which induces a cohomology isomorphism. Hence \( \mathfrak{u}_\Gamma^* \) is the minimal model of \( A^\ast_p(B\Gamma, \mathbb{Q}[T]) \).

Moreover, extending the constructions of minimal models of simplicial classifying spaces to Borel constructions, we show the de Rham homotopy theorem for simplicial complexes with torsion-free virtually polycyclic fundamental groups.

**Theorem 1.2.** Let \( K \) be a simplicial complex whose fundamental group is a torsion-free virtually polycyclic group \( \Gamma \). We suppose that \( \bigoplus_{p \geq 2} \pi_p K \otimes \mathbb{Q} \) is finite dimensional. Then for some representation \( \Gamma \to T \) in some reductive \( \mathbb{Q} \)-algebraic group \( T \) with the Zariski-dense image, the minimal model \( \mathfrak{v}^* \) of the DGA \( A^\ast_p(K, \mathbb{Q}[T]) \) satisfies the following conditions:

- The sub-DGA \( \mathfrak{v}^1 \) which is generated by the elements of degree 1 is the dual of the Lie algebra of the unipotent hull \( U_\Gamma \) of \( \Gamma \).
- For each \( p \geq 2 \), we have an isomorphism
  \[ \mathfrak{v}^p \cong \text{Hom}(\pi_p K \otimes \mathbb{Q}, \mathbb{Q}) \]

**Key techniques.** To construct minimal models, we use ”invariant differential forms” on \( \mathbb{Q} \)-algebraic groups. Let \( G \) be a \( \mathbb{Q} \)-algebraic group and \( V \) a rational \( G \)-module. We consider the rational cohomology \( H^\ast(G, V) \). For a splitting \( G = T \ltimes U \) such that \( U \) is the unipotent radical of \( G \) and \( T \) is a maximal reductive subgroup of \( G \) (see [20]), in [12, Theorem 5.2], Hochschild showed that there exists a natural isomorphism

\[ H^\ast(u, V)^T \cong H^\ast(G, V) \]
where $H^\ast(u, V)^T$ is the space of $T$-invariant elements of the cohomology of the Lie algebra $u$ of $U$ with values in the $u$-module $V$. In this paper, we construct an explicit cochain complex map $\theta : (\bigwedge u^* \otimes V)^T \to C^\ast(G, V)^G$ which induces the Hochschild isomorphism where $(\bigwedge u^* \otimes V)^T$ is the space of $T$-invariant elements of the cochain complex of the Lie algebra $u$ with values in $V$ and $C^\ast(G, V)^G$ is the standard cochain complex for the rational cohomology $H^\ast(G, V)$. By this construction, for a group $\Gamma$ with a representation $\Gamma \to G$, we can construct an explicit cochain complex map 

$$\left(\bigwedge u^* \otimes V\right)^T \to A^\ast_p(B\Gamma, V)$$

where $B\Gamma$ is the simplicial construction of the classifying space of $\Gamma$. For our main results, this map is very important.

2. COHOMOLOGY OF ALGEBRAIC GROUPS

The purpose of this section is to construct an explicit cochain complex homomorphism which induces the Hochschild isomorphism on the rational cohomology of an algebraic group. For this construction, we are inspired by the simplicial construction of Van Est isomorphism on the continuous cohomology of a Lie group (see [23]). Let $G$ be a $\mathbb{Q}$-algebraic group. For a rational $G$-module $V$, we define the rational cohomology $H^\ast(G, V) = \text{Ext}^\ast_G(\mathbb{Q}, V)$ as [12] and [10]. We have the standard resolution. We denote by $C^p(G, V)$ the set of the $V$-valued rational functions on $G \times \cdots \times G$ with the left-$G$-action. Consider the sequence

$$V \longrightarrow C^0(G, V) \overset{d}{\longrightarrow} C^1(G, V) \overset{d}{\longrightarrow} \cdots$$

such that the first map $V \to C^0(G, V)$ is the embedding as constant functions and $d : C^p(G, V) \to C^{p+1}(G, V)$ is given by

$$d\phi(g_0, \ldots, g_{p+1}) = \sum (-1)^i \phi(g_0, \ldots, \hat{g}_i, \ldots, g_{p+1})$$

for $\phi \in C^p(G, V)$, $g_0, \ldots, g_{p+1} \in G$. Then the rational cohomology $H^\ast(G, V)$ is the cohomology of the cochain complex $C^\ast(G, V)^G$. For the unipotent radical $U$ of $G$ and a maximal reductive subgroup $T$, we have a splitting $G = T \ltimes U$ ([20]). Let $u$ be the Lie algebra of $U$. Hochschild showed that we have an isomorphism

$$H^\ast(u, V)^T \cong H^\ast(G, V).$$

The purpose of this section is to represent this isomorphism as a cochain complex homomorphism.

Let $A^\ast_u(U)$ be the algebraic de Rham complex of the algebraic variety $U$. Consider the coordinate ring $\mathbb{Q}[U]$ of $U$. By the spectral sequence as [9] Proposition 3.4, we have $H^1(G, \mathbb{Q}[U] \otimes W) = H^1(U, \mathbb{Q}[U] \otimes W)^T = 0$ for any finite dimensional rational $G$-module $W$. This implies that $\mathbb{Q}[U]$ is an injective $G$-module (see [13]). We have $A^\ast_u(U) \cong \text{Hom}_\mathbb{Q}(\bigwedge u, \mathbb{Q}[U])$ and hence $A^\ast_u(U) \otimes V$ is an injective $G$-module (see [10]). Since the exponential map $\exp : u \to U$ is an isomorphism of $\mathbb{Q}$-algebraic variety, we have $H^0(A^\ast_u(U) \otimes V) = V$ and $H^\ast(A^\ast_u(U) \otimes V) = 0$ for $\ast > 0$. Hence the sequence

$$V \longrightarrow A^0_u(U) \otimes V \overset{d}{\longrightarrow} A^1_u(U) \otimes V \overset{d}{\longrightarrow} \cdots$$

is an injective resolution.
By a splitting $G = T \ltimes U$, we have the homomorphism $\alpha : G \rightarrow \text{Aut}(U) \ltimes U$. We define the map

$$\sigma^p(\cdot)(\cdot) : \underbrace{G \times \cdots \times G}_p \times \mathbb{Q}^p \rightarrow U$$

such that $\sigma^0(g_0)(0) = \alpha(g_0)e$, $\sigma^1(g_0, g_1)(t_1) = \alpha(g_0)\exp((1-t_1)\log(g_0^{-1}g_1)e)$ and inductively

$$\sigma^p(g_0, \ldots, g_p)(t_1, \ldots, t_p) = \alpha(g_0)\exp((1-t_1)\log\sigma^p(g_0^{-1}g_1, \ldots, g_0^{-1}g_p)(t_2, \ldots, t_p)).$$

It is known that the exponential map $\exp : u \rightarrow U$ is an isomorphism of $\mathbb{Q}$-algebraic variety and $\log : U \rightarrow u$ is the inverse. Hence the map

$$\sigma^p : \underbrace{G \times \cdots \times G}_p \times \mathbb{Q}^p \rightarrow U$$

is a homomorphism of $\mathbb{Q}$-algebraic variety such that for any $(t_1, \ldots, t_p)$, the map

$$\sigma^p(\cdot)(t_1, \ldots, t_p) : \underbrace{G \times \cdots \times G}_p \rightarrow U$$

is a $G$-equivariant map. We note

$$\sigma^p(g_0, \ldots, g_p)(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{p-1}) = \sigma^{p-1}(g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_p)(t_1, \ldots, t_{p-1}).$$

For $(g_0, \ldots, g_p)$ and $\omega \in A_n^*(U) \otimes V$, considering the map $\sigma^p(g_0, \ldots, g_p)(\cdot) : \mathbb{Q}^p \rightarrow U \cong u$ which is a homomorphism of $\mathbb{Q}$-algebraic variety, we have the $\mathbb{Q}$-algebraic differential form $\sigma^p(g_0, \ldots, g_p)^*\omega$ for parameters $(t_1, \ldots, t_p) \in \mathbb{Q}^p$. We regard $\sigma^p(g_0, \ldots, g_p)^*\omega$ as a $\mathbb{Q}$-polynomial differential form on $\mathbb{R}^p$. We define the map $\theta : A_n^*(U) \otimes V \rightarrow C^p(G, V)$ such that

$$\theta(\omega)(g_0, \ldots, g_p) = \int_\Delta \otimes \delta V \sigma^p(g_0, \ldots, g_p)^*\omega$$

where

$$\Delta = \{(1-t_1 - \cdots - t_p, t_1, \ldots, t_p)|0 \leq t_i \leq 1\}$$

is the standard $p$-simplex for the parameters $(t_1, \ldots, t_p)$. Then we can easily show that the map $\theta : A_n^*(U) \otimes V \rightarrow C^*(G, V)$ is $G$-equivariant cochain complex homomorphism by Stokes' theorem. (cf. [23, Section 3])

Now we have $(A_n^*(U) \otimes V)^G = (\bigwedge u^* \otimes V)^T$ where $\bigwedge u^* \otimes V$ is the cochain complex of the Lie algebra $u$ with values in the $u$-module $V$. Consider the restriction $\theta : (\bigwedge u^* \otimes V)^T \rightarrow C^*(G, V)^G$. Then the induced map $\theta : H^*(u, V)^T \rightarrow H^*(G, V)$ is identified with the map $\text{Ext}^*_\mathbb{Q}(\mathbb{Q}, V) \rightarrow \text{Ext}^*_\mathbb{Q}(\mathbb{Q}, V)$ induced by the identity map $V \rightarrow V$. Hence we have the following result.

**Theorem 2.1.** The map $\theta : (\bigwedge u^* \otimes V)^T \rightarrow C^*(G, V)^G$ induces a cohomology isomorphism

$$H^*(u, V)^T \cong H^*(G, V).$$

**Remark 2.1.** Let $f : G_1 \rightarrow G_2$ be a surjective homomorphism between $\mathbb{Q}$-algebraic groups $G_1$ and $G_2$ with decompositions $G_1 = T_1 \ltimes U_1$ and $G_2 = T_2 \ltimes U_2$. We can restrict $f : U_1 \rightarrow U_2$. By the definition, we have

$$f \circ \sigma^p(g_0, \ldots, g_p)(\cdot) = \sigma^p(f(g_0), \ldots, f(g_p))(\cdot).$$
Hence the pullbacks $f^* : (\wedge u_2^* \otimes V)^T \rightarrow (\wedge u_1^* \otimes f^* V)^T$ and $f^* : C^*(G_2, V)^G \rightarrow C^*(G_1, f^* V)^G$ commutes with the maps $\theta_1 : (\wedge u_1^* \otimes f^* V)^T \rightarrow C^*(G_1, f^* V)^G_2$, and $\theta_2 : (\wedge u_2^* \otimes V)^T \rightarrow C^*(G_2, V)^G_2$. Theorem 2.1 and so the pullback $f^* : (\wedge u_2^* \otimes V)^T \rightarrow (\wedge u_1^* \otimes f^* V)^T$ induces the map $f^* : H^*(G_2, V) \rightarrow H^*(G_1, f^* V)$ which is defined functorially.

3. Simplicial de Rham theory

We denote by $A^p_\sigma(n)$ the $\mathbb{Q}$-DGA which is generated by $t_0, \ldots, t_n$ of degree $0$ and $dt_0, \ldots, dt_n$ of degree $1$ with the relations $t_0 + \cdots + t_n = 1$ and $dt_0 + \cdots + dt_n = 0$. We can regard $A^p_\sigma(n)$ as the $\mathbb{Q}$-polynomial de Rham complex on the standard $n$-simplex $\Delta^n$. We define the map $\int_{\Delta^n} : A^p_\sigma(n) \rightarrow \mathbb{Q}$ by the ordinary Riemannian integral. Let $K$ be a simplicial complex with a universal covering complex $\tilde{K}$. We denote by $n(\sigma)$ the dimension of a simplex $\sigma \in K$. Let $V$ be a $\mathbb{Q}$-vector space which is a $\pi_1(K)$-module. We denote by $A^p(K, V)$ the space of collections $\{\omega_\sigma \in A^p_\sigma(n(\sigma)) \otimes V\}_{\sigma \in \tilde{K}}$ such that:

- $\{\omega_\sigma\}_{\sigma \in K}$ are compatible under restrictions to faces i.e. $i^* \omega_\sigma = \omega_\tau$ for the inclusion $i : \tau \rightarrow \sigma$ of a face.
- $\{\omega_\sigma\}_{\sigma \in K}$ are invariant under the $\pi_1(K)$-action i.e. $\gamma \cdot \omega_\gamma \sigma = \omega_\sigma$ for any $\gamma \in \pi_1(K)$.

We call $A^p_\sigma(K, V)$ the $\mathbb{Q}$-polynomial de Rham complex of $K$ with values in local system $V$. The space $A^p_\sigma(K, V)$ with the exterior derivation is a cochain complex. Let $C^*(K, V) = (C^*(\tilde{K}) \otimes V)^T$ be the cochain complex of simplicial cochains with values in the local system $V$. Define the map $\iota : A^p_\sigma(K, V) \rightarrow C^n(K, V)$ such that for $\sigma \in K$ with $n(\sigma) = n$

$$\iota(\{\omega\})(\sigma) = \int_{\Delta^n} \otimes \text{id}_V(\omega_\sigma).$$

Then, this map is a cochain complex homomorphism and this map induces a cohomology isomorphism (see [7] Chapter 9, [17] Chapter 12-14).

Let $V = \lim_{\rightarrow} V_i$ for an inductive system of finite dimensional $\pi_1(K)$-modules. We define

$$A^p_\sigma(K, V) = \lim_{\rightarrow} A^p_\sigma(K, V_i)$$

Let $T$ be a reductive $\mathbb{Q}$-algebraic group and $\rho : \pi_1(K) \rightarrow T$ a representation. Consider the coordinate ring $\mathbb{Q}[T]$ of $T$. Then as a $(T, T)$-bimodule, we have

$$\mathbb{Q}[T] = \bigoplus V_\alpha \otimes V_\alpha$$

such that $\{V_\alpha\}$ is a set of isomorphism classes of irreducible right $T$-module ([6] Proposition 3.1]). We regard $\mathbb{Q}[T]$ as a $\pi_1(K)$-module by $\rho$. Then we have

$$A^p_\sigma(K, \mathbb{Q}[T]) = \bigoplus A^p_\sigma(K, V_\alpha^*) \otimes V_\alpha$$

and it is a DGA.

Let $\Gamma$ be a discrete group. For a $\Gamma$-module $V$, we define the group cohomology $H^*(\Gamma, V) = \text{Ext}_\Gamma^*(\mathbb{Q}, V)$. We have the standard resolution. We denote by $C^p(\Gamma, V)$ the set of the $V$-valued functions on $\Gamma \times \cdots \times \Gamma$ with the left-$\Gamma$-action. Consider the sequence

$$V \rightarrow C^0(\Gamma, V) \xrightarrow{d} C^1(\Gamma, V) \xrightarrow{d} \cdots$$
We define the map $\iota$ such that the first map $V \to C^0(\Gamma, V)$ is the embedding as constant functions and $d : C^p(\Gamma, V) \to C^{p+1}(\Gamma, V)$ is given by
\[
d\phi(\gamma_0, \ldots, \gamma_{p+1}) = \sum (-1)^{i} \phi(\gamma_0, \ldots, \hat{\gamma}_i, \ldots, \gamma_{p+1})
\]
for $\phi \in C^p(\Gamma, V)$, $\gamma_0, \ldots, \gamma_{p+1} \in \Gamma$. Then the group cohomology $H^*(\Gamma, V)$ is the cohomology of the cochain complex $C^*(\Gamma, V)^{\Gamma}$. A representation $\rho : \Gamma \to G$ induces a homomorphism $\rho^* : H^*(G, V) \to H^*(\Gamma, V)$. By the map $\theta : (\bigwedge u^* \otimes V)^{T} \to C^*(G, V)^{G}$ as Section 2, we give a geometric representation of $\rho^* : H^*(G, V) \to H^*(\Gamma, V)$.

We define the acyclic simplicial complex $E\Gamma$ with the free discontinuous $\Gamma$-action so that:

- For integers $n \geq 0$, simplices of $E\Gamma$ are standard $n$-simplices $\Delta_{(\gamma_0, \ldots, \gamma_n)}$ indexed by $\Gamma^{n+1}$.

\[
(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)_{(\gamma_0, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_n)} \in \Delta_{(\gamma_0, \ldots, \gamma_{i-1}, \gamma_{i+1}, \ldots, \gamma_n)}
\]
is identified with
\[
(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n)_{(\gamma_0, \ldots, \gamma_n)} \in \Delta_{(\gamma_0, \ldots, \gamma_n)}.
\]

- For $\gamma \in \Gamma$, the action is given by
\[
\gamma \cdot (t_1, \ldots, t_n)_{(\gamma_0, \ldots, \gamma_n)} = (t_1, \ldots, t_n)_{(\gamma_0 \gamma, \ldots, \gamma_n \gamma)}.
\]

We define $B\Gamma$ the quotient of $E\Gamma$ by the $\Gamma$-action. Then the simplicial complex $B\Gamma$ is an Eilenberg-Maclane space $K(\Gamma, 1)$. For a finite dimensional $\Gamma$-module $V$, we consider the $\mathbb{Q}$-polynomial de Rham complex $A_p^*(B\Gamma, V)$ of $B\Gamma$ with values in local system $V$. Define the map $\iota : A^p_n(B\Gamma, V) \to C^*(\Gamma, V)^{\Gamma}$ as
\[
\iota(\{\omega_{\sigma}\}_{\sigma \in B\Gamma})_{(\gamma_0, \ldots, \gamma_n)} = \int_{\Delta_{(\gamma_0, \ldots, \gamma_n)}} \omega_{\Delta_{(\gamma_0, \ldots, \gamma_n)}}.
\]
Since we can identify $C^*(\Gamma, V)^{\Gamma}$ with the cochain complex $C^*(B\Gamma, V)$, the map $\iota : A^p_n(B\Gamma, V) \to C^p(\Gamma, V)^{\Gamma}$ induces a cohomology isomorphism
\[
H^*(A^p_n(B\Gamma, V)) \cong H^*(\Gamma, V).
\]
We define the map $\psi : (\bigwedge u^* \otimes V)^{T} \to A^p_n(B\Gamma, V)$ such that
\[
\psi(\omega) = \{\sigma^p(\rho(\gamma_0), \ldots, \rho(\gamma_p))^* \omega\}_{\Delta_{(\gamma_0, \ldots, \gamma_n)}}
\]
where $\sigma^p(\rho(\gamma_0), \ldots, \rho(\gamma_p))^* \omega$ is defined in Section 2. By the $G$-invariance of $\omega \in (\bigwedge u^* \otimes V)^{T}$ and the relation
\[
\sigma^p(g_0, \ldots, g_p)(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_{p-1})
\]
\[
= \sigma^{p-1}(g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_p)(t_1, \ldots, t_{p-1}),
\]
we actually have $\psi(\omega) \in A^p_n(B\Gamma, V)$. Then the map $\psi : (\bigwedge u^* \otimes V)^{T} \to A^p_n(B\Gamma, V)$ is a cochain complex homomorphism.
We have the commutative diagram

\[
\begin{array}{c}
(\Lambda u^* \otimes V)^T \quad \psi \quad C^*(G,V)^G \\
A^*_p(B\Gamma, V) \quad \rho^* \quad C^p(\Gamma, V)^T
\end{array}
\]

Hence we have:

**Corollary 3.1.** The induced map \( \psi^* : H^*(u,V)^T \to H^*(A^*_p(B\Gamma, V)) \) is identified with the map \( \rho^* : H^*(G,V) \to H^*(\Gamma, V) \).

Consider \( \mathbb{Q}[T] \) as a \( \Gamma \)-module. Then we have \((\Lambda u^* \otimes \mathbb{Q}[T])^T = \Lambda u^*\) and we have the DGA map \( \psi : \Lambda u^* \to A^*_p(B\Gamma, \mathbb{Q}[T]) \) such that the induced map \( \psi^* : H^*(u,\mathbb{Q}) \to H^*(A^*_p(B\Gamma, \mathbb{Q}[T])) \) is identified with the map

\[
\rho^* : \bigoplus H^*(G, V^*_\alpha) \otimes V_\alpha \to \bigoplus H^*(\Gamma, V^*_\alpha) \otimes V_\alpha.
\]

### 4. Preliminary on minimal models

From this section, we consider the theory of Sullivan’s minimal models. There are many references ([4, 5, 6, 10, 19, 24]). See these references for details.

**Definition 4.1.** Let \( k \) be a field of characteristic 0. A DGA \( A^* \) with a differential \( d \) defined over \( k \) is minimal if the following conditions hold.

- \( A^* \) is a free graded algebra \( \Lambda V^* \) generated by a graded \( k \)-vector space \( V^* \) with \( * \geq 1 \).
- \( V^* \) admits a basis \( \{x_i\} \) which is indexed by a well-ordered set such that \( dx_i \in \Lambda \langle\{x_j\}\rangle_{j<i} \).
- \( d(V^*) \subset \Lambda^2 V^* \).

Let \( n^* \) be a nilpotent Lie algebra. Then the DGA \( \Lambda n^* \) which is the dual of \( n^* \) is a minimal DGA.

**Definition 4.2.** Let \( A^* \) be a DGA with \( H^0(A^*) = k \). A minimal DGA \( \mathcal{M} \) is the minimal model of \( A^* \) if there is a DGA homomorphism \( \mathcal{M} \to A^* \) which induces a cohomology isomorphism.

**Theorem 4.3** ([24]). For a DGA \( A^* \) with \( H^0(A^*) = k \), the minimal model of \( A^* \) exists and it is unique up to DGA isomorphism.

**Lemma 4.4.** We suppose the following settings:

- \( \mathcal{M}^* = \Lambda V^* \) is a minimal DGA with a differential \( d \) such that \( V^1 = 0 \) and for each \( p \geq 2 \) \( V^p \) is finite dimensional.
- \( \mathcal{U} \) is a unipotent \( k \)-algebraic group with the Lie algebra \( u \).
- \( \mathcal{U} \) acts on the DGA \( \mathcal{M}^* \) rationally.

We consider the cochain complex \( \Lambda u^* \otimes \mathcal{M}^* \) of the Lie algebra \( u \) with values in the module \( \mathcal{M}^* \) with the differential \( d_1 \). For the differential \( d_2 \) on \( \Lambda u^* \otimes \mathcal{M}^* \) which is the extension of the differential on \( \mathcal{M}^* \), we consider \( \Lambda u^* \otimes \mathcal{M}^* \) as the total complex of the double complex

\[
\left( \Lambda u^* \otimes \mathcal{M}^*, d_1, d_2 \right)
\]

such that the component of degree \((p, q)\) is \( \Lambda^p u^* \otimes \mathcal{M}^q \).

Then \( \Lambda u^* \otimes \mathcal{M}^* \) is a minimal DGA.
Proof. By the assumption, for each \( p \geq 2 \), \( \mathcal{M}^p \) is finite dimensional rational \( U \)-module. Since \( U \) is unipotent, the Lie algebra \( u \) is nilpotent and \( V^p \) is a nilpotent \( u \)-module. Hence by Engel’s theorem, we can take a basis \( \{ m^p_1, \ldots, m^p_{r_p} \} \) of \( \mathcal{M}^p \) such that

\[
dm^p_i = \sum_{s=1}^{i-1} A_{is}m^p_{i-s} + C_i
\]

with \( A_{is} \in u^* \) and \( C_i \in \sum_{q+r=p+1} \mathcal{M}^q \cdot \mathcal{M}^r. \) By the nilpotency, we have a basis \( \{ m^1_1, \ldots, m^1_{l_1} \} \) of \( u^* \) such that \( dm^1_i \in \bigwedge^q(m^1_1, \ldots, m^1_{i-1}). \) Hence, choosing generators of \( \mathcal{M}^* \) from \( \{ m^1_1 \} \), we can prove the lemma.

\[
\square
\]

5. Minimal models of classifying spaces of torsion-free virtually polycyclic groups

Let \( \Gamma \) be a torsion-free virtually polycyclic group, \( G \) a \( \mathbb{Q} \)-algebraic group and \( \rho : \Gamma \to G \) a representation with the Zariski-dense image. It is known that we have \( \dim U \leq \operatorname{rank} \Gamma \) where \( U \) is the unipotent radical of \( G \). We say that \( \rho : \Gamma \to G \) is a full representation if \( \dim U = \operatorname{rank} \Gamma \).

**Theorem 5.1** ([15]). If \( \rho : \Gamma \to G \) is an injective full representation, then for any rational \( G \)-module \( V \) the induced map \( \rho^*: H^*(G,V) \to H^*(\Gamma,V) \) is an isomorphism.

It is known that there exists a unique algebraic group \( G \) such that the centralizer \( Z_G(U) \) is contained in \( U \) and we have an inective full representation \( \rho : \Gamma \to G \) ([11 Appendix A.]). We call such algebraic group \( G \) the algebraic hull of \( \Gamma \) and we call the unipotent radical of the algebraic hull \( G \) the unipotent hull of \( \Gamma \) and denote it by \( U_{\Gamma} \). Denote by \( u_{\Gamma} \) the Lie algebra of the unipotent hull \( U_{\Gamma} \) of \( \Gamma \). Take \( G \) the algebraic hull of \( \Gamma \) and a splitting \( G = T \ltimes U_{\Gamma} \) for a maximal reductive subgroup \( T \). Then by Corollary [13], we have the following fact.

**Theorem 5.2.** The map \( \psi : (\bigwedge u_{\Gamma}^* \otimes V)^T \to A^*_p(B\Gamma,V) \) induces a cohomology isomorphism.

Consider \( \mathbb{Q}[T] \) as a \( \Gamma \)-module.

**Theorem 5.3.** We have an explicit DGA map \( \bigwedge u_{\Gamma}^* \to A^*_p(B\Gamma,\mathbb{Q}[T]) \) which induces a cohomology isomorphism. Hence \( \bigwedge u_{\Gamma}^* \) is the minimal model of \( A^*_p(B\Gamma,\mathbb{Q}[T]) \).

**Remark 5.1.** Let \( S \) be a simply connected solvable Lie group with a lattice \( \Gamma \) and \( s \) be the Lie algebra of \( S \). Take \( G \) the algebraic hull of \( \Gamma \) and a splitting \( G = T \ltimes U_{\Gamma} \) for a maximal reductive subgroup \( T \). Then the map \( \Gamma \to T \) is identified with the diagonalized adjoint representation \( \operatorname{Ad}_s : \Gamma \to \operatorname{Aut}(s) \) as ([14] Construction 1.1.] (see [14] Section 2)). Hence Theorem 5.3 is a simplicial and rational analogue of extended Nomizu’s theorem for solvmanifolds.

6. Borel constructions and polyccyclic de Rham homotopy theorem

Let \( K \) be a simplicial complex with \( \pi_1 K = \Gamma \). Take a universal covering \( \tilde{K} \). Then we have a minimal model \( \mathcal{M}^* = \bigwedge V^* \) of \( A^*_p(\tilde{K}) \) with a map \( j : \mathcal{M}^* \to A^*_p(\tilde{K}) \) which induces a cohomology isomorphism where \( V^* \) is a graded vector space such that \( V^p \cong \operatorname{Hom}(\pi_p \tilde{K} \otimes \mathbb{Q}, \mathbb{Q}) \). We consider the Borel construction \( ET \ltimes \Gamma \tilde{K} \) which is the quotient of \( ET \times \tilde{K} \) by the diagonal \( \Gamma \)-action. By the functorial property of the
minimal model, we have the $\Gamma$-action on $\mathcal{M}^*$ such that the map $j : \mathcal{M}^* \to A_p^*(\tilde{K})$ is $\Gamma$-equivariant. Let $\rho : \Gamma \to G$ be a representation in a $\mathbb{Q}$-algebraic group $G$. By the $\Gamma$-action on $\mathcal{M}^*$, we have the representation $\Gamma \to \text{Aut}(\mathcal{M}^*)$ where $\text{Aut}(\mathcal{M}^*)$ is the group of the DGA automorphisms of $\mathcal{M}^*$. We assume that $V^*$ is finite dimensional (equivalently $\oplus_{p \geq 2} \pi_p K \otimes \mathbb{Q}$ is finite dimensional). Then, by the minimality of $\mathcal{M}^*$, we can regard $\text{Aut}(\mathcal{M}^*)$ as a $\mathbb{Q}$-algebraic group (see [24, Theorem 6.1]. We consider the representation $\rho_{\mathcal{M}} : \Gamma \to G \times \text{Aut}(\mathcal{M}^*)$ which is the direct sum of $\rho : \Gamma \to G$ and $\Gamma \to \text{Aut}(\mathcal{M}^*)$. Take $G_{\mathcal{M}}$ the Zariski-closure of $\rho_{\mathcal{M}}(\Gamma)$. Then $\mathcal{M}^*$ is a rational $G_{\mathcal{M}}$-module. Take a splitting $G_{\mathcal{M}} = T_{\mathcal{M}} \times U_{\mathcal{M}}$ for the unipotent radical $U_{\mathcal{M}}$ and a maximal reductive subgroup $T_{\mathcal{M}}$. Let $\mathfrak{u}_{\mathcal{M}}$ be the Lie algebra of the unipotent radical $U_{\mathcal{M}}$. For a finite-dimensional rational $G_{\mathcal{M}}$-module $V$, we consider the cochain complex $(\bigwedge \mathfrak{u}_{\mathcal{M}} \otimes \mathcal{M}^* \otimes V)^{T_{\mathcal{M}}}$ and the cochain complex homomorphism

$$
\psi : (\bigwedge \mathfrak{u}_{\mathcal{M}} \otimes \mathcal{M}^* \otimes V)^{T_{\mathcal{M}}} \to (A_p^*(ET) \otimes \mathcal{M}^* \otimes V)^{\Gamma}
$$

as Section 3. By the map $j : \mathcal{M}^* \to A_p^*(\tilde{K})$, we have the map

$$(A_p^*(ET) \otimes \mathcal{M}^* \otimes V)^{\Gamma} \to (A_p^*(ET) \otimes A_p^*(\tilde{K}) \otimes V)^{\Gamma}.
$$

By the projections $ET \times \tilde{K} \to ET$ and $ET \times \tilde{K} \to K$, we have the map

$$(A_p^*(ET) \otimes A_p^*(\tilde{K}) \otimes V)^{\Gamma} \to (A_p^*(ET \times \tilde{K}) \otimes V)^{\Gamma}.
$$

Taking the composition, we have the map

$$
\Psi : (\bigwedge \mathfrak{u}_{\mathcal{M}} \otimes \mathcal{M}^* \otimes V)^{T_{\mathcal{M}}} \to (A_p^*(ET \times \tilde{K}) \otimes V)^{\Gamma} = A_p^*(ET \times K, V).
$$

Consider the two differentials $d_1$ and $d_2$ on $(\bigwedge \mathfrak{u}_{\mathcal{M}} \otimes \mathcal{M}^* \otimes V)^{T_{\mathcal{M}}}$ such that $d_1$ is the restriction of the differential on $\bigwedge \mathfrak{u}_{\mathcal{M}} \otimes \mathcal{M}^* \otimes V$ as a cochain complex of the Lie algebra $\mathfrak{u}_{\mathcal{M}}$ and $d_2$ is the extension of the differential on the minimal model $\mathcal{M}^*$. Then

$$
\left( (\bigwedge \mathfrak{u}_{\mathcal{M}} \otimes \mathcal{M}^* \otimes V)^{T_{\mathcal{M}}}, d_1, d_2 \right)
$$

is a double complex such that the component of degree $(p, q)$ is $(\bigwedge^p \mathfrak{u}_{\mathcal{M}} \otimes \mathcal{M}^q \otimes V)^{T_{\mathcal{M}}}$.

Consider the total complex $((\bigwedge \mathfrak{u}_{\mathcal{M}} \otimes \mathcal{M}^* \otimes V)^{T_{\mathcal{M}}}, d = d_1 + d_2)$. Then the map

$$
\Psi : (\bigwedge \mathfrak{u}_{\mathcal{M}} \otimes \mathcal{M}^* \otimes V)^{T_{\mathcal{M}}} \to (A_p^*(ET \times \tilde{K}) \otimes V)^{\Gamma} = A_p^*(ET \times \tilde{K}, V)
$$

is a cochain complex homomorphism. Consider the Serre spectral sequence $E_{r,s}^s$ for the simplicial flat fiber bundle $ET \times \tilde{K} \to B\Gamma$ such that the filtration is given by

$$
F_p^\Gamma \left( C^t(ET \times \tilde{K}) \otimes V \right)^\Gamma
$$

$$
= \left\{ c \in \left( C^t(ET \times \tilde{K}) \otimes V \right)^\Gamma : c(\sigma) = 0, \forall \sigma \in ET \times \tilde{K} \text{ s.t. } c(\sigma) \in \tilde{K}(r), r \geq t - p + 1 \right\}
$$
where \( \epsilon_2 : ET \times \tilde{K} \to \tilde{K} \) is the projection. We also consider the spectral sequence \( E_{p}^{*} \Gamma \) of the double complex
\[
\left( \bigwedge u_{M}^{*} \otimes M^{*} \otimes V \right)^{T_{M}}, d_1, d_2
\]
which is given by the filtration
\[
F^{p}\text{Tot}^{t} = \bigoplus_{r \geq p} \left( \bigwedge u_{M}^{*} \otimes M^{*} \otimes V \right)^{T_{M}}
\]
Then the map
\[
\Psi : \left( \bigwedge u_{M}^{*} \otimes M^{*} \otimes V \right)^{T_{M}} \to \left( A_{p}^{*}(E\Gamma \times \tilde{K}) \otimes V \right)^{\Gamma} = A^{*}(E\Gamma \times \tilde{K}, V)
\]
induces a spectral sequence homomorphism \( E_{2}^{*} \Gamma \to E_{2}^{*} \Gamma \). Since \( j : M^{*} \to A_{p}^{*}(\tilde{K}) \) induces a cohomology isomorphism, by Corollary \([3, 1]\) the map \( E_{2}^{*} \Gamma \to E_{2}^{*} \Gamma \) at \( E_{2} \)-term is identified with the map
\[
H^{*}(G_{M}, H^{*}(M^{*}) \otimes V) \to H^{*}(\Gamma, H^{*}(M^{*}) \otimes V).
\]
Suppose that \( \Gamma \) is torsion-free virtually polycyclic. Take \( \rho : \Gamma \to G \) the algebraic hull of \( \Gamma \). Then the representation \( \rho_{M} : \Gamma \to G_{M} \) is injective and full, the unipotent radical \( U_{M} \) is isomorphic to the unipotent hull \( U_{\Gamma} \), \( u_{M} = u_{\Gamma} \) and by the result in \([15]\), for any rational \( G_{M} \)-module \( W \), the map \( H^{*}(G_{M}, W) \to H^{*}(\Gamma, W) \) is an isomorphism. Hence the map
\[
H^{*}(G_{M}, H^{*}(M^{*}) \otimes V) \to H^{*}(\Gamma, H^{*}(M^{*}) \otimes V)
\]
as above is an isomorphism and hence the map \( E_{2}^{*} \Gamma \to E_{2}^{*} \Gamma \) is an isomorphism. By \([15]\) Theorem 3.5], we have the following result.

**Theorem 6.1.** If \( \Gamma \) is a torsion-free virtually polycyclic group, then the map
\[
\Psi : \left( \bigwedge u_{M}^{*} \otimes M^{*} \otimes V \right)^{T_{M}} \to \left( A_{p}^{*}(E\Gamma \times \tilde{K}) \otimes V \right)^{\Gamma} = A_{p}^{*}(E\Gamma \times \tilde{K}, V)
\]
induces cohomology isomorphism.

Consider \( \mathbb{Q}[T] \) as a \( \Gamma \)-module. Then \( \left( \bigwedge u_{\Gamma} \otimes M^{*} \otimes \mathbb{Q}[T_{M}] \right)^{T_{M}} = \bigwedge u_{\Gamma} \otimes M^{*} \). Hence we have the following result.

**Theorem 6.2.** If \( \Gamma \) is a torsion-free virtually polycyclic group, then we have an explicit DGA map \( \bigwedge u_{\Gamma} \otimes M^{*} \to A_{p}^{*}(E\Gamma \times \tilde{K}, \mathbb{Q}[T_{M}]) \) which induces a cohomology isomorphism. Hence \( \bigwedge u_{\Gamma} \otimes M^{*} \) is the minimal model of \( A_{p}^{*}(E\Gamma \times \tilde{K}, \mathbb{Q}[T_{M}]) \).

The map \( E\Gamma \times \tilde{K} \to \tilde{K}/\Gamma = K \) induces a cohomology isomorphism \( H^{*}(K, V) \to H^{*}(E\Gamma \times \Gamma \tilde{K}, V) \) for any local system \( V \). Hence we have the following de Rham homotopy theorem.

**Theorem 6.3.** Let \( K \) be a simplicial complex whose fundamental group is a torsion-free virtually polycyclic group \( \Gamma \). We suppose that \( \bigoplus_{p \geq 2} \pi_{p}K \otimes \mathbb{Q} \) is finite dimensional. Then for some representation \( \Gamma \to T \) in some reductive \( \mathbb{Q} \)-algebraic group \( T \) with the Zariski-dense image, the minimal model \( \bigwedge V^{*} \) of the DGA \( A_{p}^{*}(K, \mathbb{Q}[T]) \) satisfies the following conditions:

- The sub-DGA \( \bigwedge V^{1} \) which is generated by the elements of degree 1 is the dual of the Lie algebra of the unipotent hull \( U_{\Gamma} \) of \( \Gamma \).
For each $p \geq 2$, we have an isomorphism

$$V^p \cong \text{Hom}(\pi_p K \otimes \mathbb{Q}, \mathbb{Q}).$$

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