A Note concerning Subsingular Vectors and Embedding Diagrams of the N=2 Superconformal Algebras

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ABSTRACT

Subsingular vectors of the N=2 superconformal algebras were discovered, and examples given, in 1996. Shortly afterwards Semikhatov and Tipunin claimed to have obtained a complete classification of the N=2 subsingular vectors in the paper ‘The Structure of Verma Modules over the N=2 Superconformal algebra’, hep-th/9704111, published in CMP 195 (1998) 129. Surprisingly, the only explicit examples of N=2 subsingular vectors known at that time did not fit into their classification. All the results presented in that paper, including the classification of subsingular vectors, were based on the following assumptions: i) The authors claimed that there are only two different types of submodules in N=2 Verma modules, overlooking from the very beginning indecomposable ‘no-label’ singular vectors, that had been discovered a few months before, and clearly do not fit into their two types of submodules, and ii) The authors claimed to have constructed ‘non-conventional’ singular vectors with the property of generating the two types of submodules maximally, i.e. with no subsingular vectors left outside. In this note we prove that both assumptions are incorrect. These facts also affect profoundly the results presented in several other publications, especially the papers: ‘On the Equivalence of Affine sl(2) and N=2 ....’, by Semikhatov, hep-th/9702074, ‘Embedding Diagrams of N=2 Verma Modules ....’, by Semikhatov and Sirota, hep-th/9712102, and ‘All Singular Vectors of the N=2 ....’, by Semikhatov and Tipunin, hep-th/9604176 (last revised version in September 98).

October 1999

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1 Introduction and Notation

The Topological N=2 superconformal algebra was deduced in 1990 as the symmetry algebra of two-dimensional topological conformal field theory (TCFT) [21]. It was the last N=2 superconformal algebra to be discovered and in fact can be obtained from the Neveu-Schwarz N=2 algebra by modifying the stress-energy tensor by adding the derivative of the U(1) current, a procedure known as topological twist [21] [22]. It reads

\[
\begin{align*}
\{\mathcal{L}_m, \mathcal{L}_n\} &= (m-n)\mathcal{L}_{m+n}, &\{\mathcal{H}_m, \mathcal{H}_n\} &= \frac{c}{4}m\delta_{m+n,0}, \\
\{\mathcal{L}_m, \mathcal{G}_n\} &= (m-n)\mathcal{G}_{m+n}, &\{\mathcal{H}_m, \mathcal{G}_n\} &= \mathcal{G}_{m+n}, \\
\{\mathcal{L}_m, \mathcal{Q}_n\} &= -n\mathcal{Q}_{m+n}, &\{\mathcal{H}_m, \mathcal{Q}_n\} &= -\mathcal{Q}_{m+n}, \\
\{\mathcal{G}_m, \mathcal{Q}_n\} &= 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{c}{6}(m^2 + m)\delta_{m+n,0},
\end{align*}
\]

where \(\mathcal{L}_m\) and \(\mathcal{H}_m\) are the bosonic generators corresponding to the stress-energy tensor (Virasoro generators) and the U(1) current, respectively, and \(\mathcal{G}_m\) and \(\mathcal{Q}_m\) are the spin-2 and spin-1 fermionic generators, the latter being the modes of the BRST-current. The eigenvalues of the bosonic zero modes \((\mathcal{L}_0, \mathcal{H}_0)\) correspond to the conformal weight and the U(1) charge of the states. In a Verma module these eigenvalues split conveniently as \((\Delta + l, h + q)\) for secondary states, where \(l\) and \(q\) are the level and the relative charge of the state and \((\Delta, h)\) are the conformal weight and the charge of the primary state on which the secondary is built. The ‘topological’ central charge \(c\) is the central charge corresponding to the Neveu-Schwarz N=2 algebra. The determinant formula for this algebra has remained unpublished until very recently [7], although the ‘chiral determinant formula’, which applies to chiral Verma modules was published in 1997 [3].

Due to the existence of the fermionic zero modes \(\mathcal{G}_0\) and \(\mathcal{Q}_0\) this algebra has two sectors: the \(\mathcal{G}\)-sector (states annihilated by \(\mathcal{G}_0\)) and the \(\mathcal{Q}\)-sector (BRST-invariant states annihilated by \(\mathcal{Q}_0\)), in analogy with the (+)-sector and the (−)-sector of the Ramond N=2 algebra, due to the fermionic zero modes \(\mathcal{G}^+_0\) and \(\mathcal{G}^-_0\). However, the two sectors do not provide the complete description – and this is true for the Ramond N=2 algebra as well – since there are also states which do not belong to any of the sectors [1] [6] [7]. That is, not all Verma modules and submodules decompose into the two sectors, but there are also indecomposable states, in particular indecomposable singular vectors. To see this one only needs to inspect the anticommutator of the fermionic zero modes \(\{\mathcal{G}_0, \mathcal{Q}_0\} = 2\mathcal{L}_0\) acting on a given state \(|\chi\rangle\). If the conformal weight of \(|\chi\rangle\) is different from zero; i.e. \(\mathcal{L}_0|\chi\rangle = (\Delta + l)|\chi\rangle \neq 0\), then \(|\chi\rangle\) can be decomposed into a state \(|\chi\rangle^G\) annihilated by \(\mathcal{G}_0\), but not by \(\mathcal{Q}_0\), that we refer as \(\mathcal{G}_0\)-closed and a state \(|\chi\rangle^Q\) annihilated by \(\mathcal{Q}_0\), but not by \(\mathcal{G}_0\), that we refer as \(\mathcal{Q}_0\)-closed:

\[
|\chi\rangle = \frac{1}{2\Delta}\mathcal{Q}_0\mathcal{G}_0|\chi\rangle + \frac{1}{2\Delta}\mathcal{G}_0\mathcal{Q}_0|\chi\rangle = |\chi\rangle^Q + |\chi\rangle^G.
\]

If the conformal weight of \(|\chi\rangle\) is zero, however, one only obtains \((\mathcal{G}_0\mathcal{Q}_0 + \mathcal{Q}_0\mathcal{G}_0)|\chi\rangle = 0\), which is satisfied in four different ways: i) The state is \(\mathcal{G}_0\)-closed, \(|\chi\rangle = |\chi\rangle^G\), and \(\mathcal{G}_0\mathcal{Q}_0|\chi\rangle^G = 0\), ii) The state is \(\mathcal{Q}_0\)-closed, \(|\chi\rangle = |\chi\rangle^Q\), and \(\mathcal{Q}_0\mathcal{G}_0|\chi\rangle^Q = 0\), iii) The state is chiral, \(|\chi\rangle = |\chi\rangle^{G,Q}\), annihilated
by both $G_0$ and $Q_0$, and iv) The state is indecomposable ‘no-label’, $|\chi\rangle = |\chi\rangle$, not annihilated by any of the fermionic zero modes.

In what follows we will use the standard definition of highest weight vectors and singular vectors for conformal algebras, i.e. they are the states with lowest conformal weight (lowest energy) in the Verma modules and in the null submodules, respectively, and therefore are annihilated by all the positive modes of the generators of the algebra (the lowering operators); i.e. $L_{n \geq 1}|\chi\rangle = H_{n \geq 1}|\chi\rangle = G_{n \geq 1}|\chi\rangle = Q_{n \geq 1}|\chi\rangle = 0$. Hence these annihilation conditions will be referred to as the conventional, standard highest weight (h.w.) conditions. Singular vectors that are not generated by acting with the algebra on other singular vectors are called primitive, otherwise they are called secondary singular vectors.

Subsingular vectors are also null but they do not satisfy the h.w. conditions, becoming singular, that is annihilated by all the positive generators, in the quotient of the Verma module by a submodule, however. As a consequence they are located outside that particular submodule (otherwise they would disappear after taking the quotient), although descending to it necessarily by the action of the lowering operators (so that they descend to ‘nothing’ once the submodule is set to zero). This implies that the singular vectors cannot reach the subsingular vectors going upwards by the action of the negative, rising operators, whereas the subsingular vectors can reach the singular vectors going downwards by the action of the positive, lowering operators.

Subsingular vectors for the $N=2$ algebras were discovered in 1996 in ref. [2] and the first examples for the case of the Topological $N=2$ algebra were published in January 1997 in ref. [1], together with the classification of all possible types of singular vectors taking into account the relative $U(1)$ charge and the annihilation conditions with respect to the fermionic zero modes $G_0$ and $Q_0$ (the BRST-invariance properties). This classification resulted in: 4 different types of singular vectors for chiral Verma modules built on chiral highest weight vectors $|0, h\rangle^{G,Q}$, 20 different types of singular vectors for generic (standard) Verma modules (10 types built on $G_0$-closed h.w. vectors $|\Delta, h\rangle^G$ and 10 types built on $Q_0$-closed h.w. vectors $|\Delta, h\rangle^Q$) and 9 different types of singular vectors for no-label Verma modules built on no-label indecomposable h.w. vectors $|0, h\rangle$. In generic Verma modules one can find $G_0$-closed, $Q_0$-closed, chiral and no-label singular vectors. In chiral and no-label Verma modules, however, only $G_0$-closed and $Q_0$-closed singular vectors can exist, with the exception of the chiral singular vectors at level zero in no-label Verma modules (curiously, in chiral and no-label Verma modules neither chiral nor no-label singular vectors exist). For the case of the generic Verma modules built on $G_0$-closed h.w. vectors $|\Delta, h\rangle^G$, which are of special importance for the discussion that follows, the possible types of singular vectors one can find are given by the following table:

|                  | $q = -2$ | $q = -1$ | $q = 0$ | $q = 1$ |
|------------------|----------|----------|---------|---------|
| $G_0$-closed     | $|\chi\rangle^{(-1)}_l^G$ | $|\chi\rangle^{(0)}_l^G$ | $|\chi\rangle^{(1)}_l^G$ |         |
| $Q_0$-closed     | $|\chi\rangle^{(-1)}_l^Q$ | $|\chi\rangle^{(0)}_l^Q$ |         | $-$     |
| chiral           | $|\chi\rangle^{(-1)}_l^{G,Q}$ | $|\chi\rangle^{(0)}_l^{G,Q}$ | $-$     |         |
| no-label         | $|\chi\rangle^{(-1)}_l$ | $|\chi\rangle^{(0)}_l$ | $-$     |         |
In ref. [1] all singular vectors (i.e. $4 + 20 + 9$) were written down explicitly at level 1. This classification was proved to be rigorous later in ref. [3], where also the maximal dimensions of the corresponding singular vector spaces were given (1, 2 or 3 depending on the type of singular vector). Regarding subsingular vectors, in ref. [1] all the subsingular vectors in generic Verma modules that become singular in the chiral Verma modules were written down at levels 2 and 3. To understand this one has to take into account that chiral Verma modules are nothing but the quotient of generic Verma modules with zero conformal weight, $\Delta = 0$, by the submodules generated by the level-zero singular vectors (which are present in all generic Verma modules with $\Delta = 0$).

Three months after ref. [1] was published in hep-th/9701041 (January 97), the paper ‘The Structure of Verma Modules over the $\mathcal{N}=2$ Superconformal Algebra’, by Semikhatov and Tipunin, appeared in hep-th/9704111[10]. In this paper the authors considered also the Topological $\mathcal{N}=2$ algebra, that they called ‘the only’ $\mathcal{N}=2$ algebra\footnote{Although the Topological and the Neveu-Schwarz $\mathcal{N}=2$ algebras are related by the topological twists, and the Neveu-Schwarz and the Ramond $\mathcal{N}=2$ algebras are related by the spectral flows, their representation theories are different. This has not been appreciated by the authors who claim that the corresponding Verma modules are isomorphic, creating some confusion. In addition, the authors also claim that the Topological and Neveu-Schwarz $\mathcal{N}=2$ algebras are related by the spectral flows, creating more confusion. Furthermore there is also the ‘twisted’ $\mathcal{N}=2$ algebra (not to be confused with the ‘twisted topological’, i.e. the Topological algebra) which is not connected to the other three algebras.}. All the analysis and the results presented in this paper were based on the following assumptions (without proofs):

i) In the $\mathcal{N}=2$ Verma modules there are only two types of submodules. In particular, in the generic Verma modules built on $\mathcal{G}_0$-closed h.w. vectors (called ‘massive’ Verma modules) one can find the two types, denoted as ‘massive’ (large) and ‘topological’ (small) submodules.

ii) These two types of submodules are maximally generated (i.e. without letting any null states outside, like subsingular vectors) by some ‘non-conventional singular vectors’, constructed by the authors in refs. [1], [2] which satisfy ‘twisted’ h.w. conditions and coincide with the conventional singular vectors only in the case of ‘zero twist’. In more intuitive terms one can think of the ‘non-conventional’ singular vectors simply as certain null states which, unlike the conventional singular vectors, are not located at the bottom of the submodules, that is, they are not the null states with lowest conformal weight, except for the case of ‘zero twist’.

Let us notice already that, although ref. [1] appeared in the bibliography given by the authors, the classification of Verma modules (generic, no-label and chiral), with their possible existing types of singular vectors, was overlooked. Most surprising, from the very beginning they ignored no-label singular vectors in the generic Verma modules that they consider (called ‘massive’ as we pointed out before), as shown in table (1.3), which clearly do not fit into the two types of submodules described by the authors.

Based on these assumptions the authors presented a ‘complete’ classification of subsingular vectors (without giving explicit examples) where, surprisingly, the subsingular vectors given in ref. [1], which were the only explicit examples written down so far for the Topological algebra, did not fit. In what follows, in subsections 2.1 and 2.2, we will show that:

i) In generic (‘massive’) Verma modules one can find at least four different types of submodules. Two of them fit, in principle, into the description of ‘massive’ and ‘topological’ submodules given
by Semikhatov and Tipunin. The other two types do not fit, clearly, into that description.

ii) The subsingular vectors written down in ref. [1] do not fit into the classification presented by the authors in ref. [10], providing in fact a proof that the ‘non-conventional singular vectors’ do not generate maximal submodules since one can find subsingular vectors outside which are pulled inside the submodule by the action of the positive lowering operators.

Afterwards, in subsection 2.3, we will argue that the ‘non-conventional singular vectors’, constructed by Semikhatov and Tipunin in the papers [11] and [12], are poorly defined objects. They are supposed to be related to some special types of singular vectors of the affine \( \hat{sl}(2) \) algebra via an isomorphism [16][17]. We will also make some remarks about this isomorphism and conclude that it is far from satisfactory. The results discussed in this note affect profoundly the results presented by Semikhatov and collaborators in several other publications. This applies very specially to the classification of \( \text{N}=2 \) embedding diagrams proposed (although never properly done) by Semikhatov and Sirota in the paper [15], as we point out in subsection 2.4. In section 3 we will make some final remarks.

2 The Facts

2.1 Different types of submodules

In the recent paper [7] the determinant formulae for the Topological \( \text{N}=2 \) algebra were presented as well as a detailed analysis of the singular vectors corresponding to the roots of the determinants. In addition it was proved – both theoretically and with explicit examples – that in generic Verma modules one can find four different types of submodules just by taking into account the size and the shape at the bottom of the submodules. Now we will see that two of these types do not fit into the ‘massive’ and ‘topological’ submodules claimed to be the only existing types of submodules by Semikhatov and Tipunin. The argument goes as follows. The determinant formula for all the generic Verma modules – either with two h.w. vectors \(|\Delta, h\rangle^G\) and \(|\Delta, h - 1\rangle^Q\) (\(\Delta \neq 0\)) or with only one h.w. vector \(|0, h\rangle^G\) or \(|0, h - 1\rangle^Q\) – reads

\[
\det(M^T_l) = \prod_{2 \leq r,s \leq 2l} (f_{r,s})^{2P(l-r,s)} \prod_{0 \leq k \leq l} (g^+_{k})^{2P_k(l-k)} \prod_{0 \leq k \leq l} (g^{-}_{k})^{2P_{k}(l-k)}, \tag{2.1}
\]

where

\[
f_{r,s}(\Delta, h, t) = -2t\Delta + th - h^2 - \frac{1}{4}t^2 + \frac{1}{4}(s-tr)^2, \quad r \in \mathbb{Z}^+, \ s \in 2\mathbb{Z}^+ \tag{2.2}
\]

and

\[
g^\pm_{k}(\Delta, h, t) = 2\Delta \mp 2kh - tk(k \mp 1), \quad 0 \leq k \in \mathbb{Z}, \tag{2.3}
\]

---

2 The Verma modules built on \( G_0 \)-closed h.w. vectors and the ones built on \( Q_0 \)-closed h.w. vectors are not the same for zero conformal weight \( \Delta = 0 \) because in this case there is only one h.w. vector at the bottom of the Verma module together with one singular vector. In some sense there is only one sector in the Verma modules with \( \Delta = 0 \), as happens for the Verma modules of the Ramond \( \text{N}=2 \) algebra for \( \Delta = \mathbb{C}/24 \).
defining the parameter $t = (3 - c)/3$. For $c \neq 3$ ($t \neq 0$) one can factorize $f_{r,s}$ as

$$f_{r,s}(\Delta, h, t \neq 0) = -2t(\Delta - \Delta_{r,s}), \quad \Delta_{r,s} = -\frac{1}{2t}(h - h^{(0)}_{r,s})(h - \hat{h}_{r,s}), \quad (2.4)$$

with

$$h^{(0)}_{r,s} = \frac{t}{2}(1 + r) - \frac{s}{2}, \quad r \in \mathbb{Z}^+, \quad s \in 2\mathbb{Z}^+, \quad (2.5)$$

$$\hat{h}_{r,s} = \frac{t}{2}(1 - r) + \frac{s}{2}, \quad r \in \mathbb{Z}^+, \quad s \in 2\mathbb{Z}^+. \quad (2.6)$$

For all values of $c$ one can factorize $g^+_k$ and $g^-_k$ as

$$g^\pm_k(\Delta, h, t) = 2(\Delta - \Delta^\pm_k), \quad \Delta^\pm_k = \pm k(h - h^\pm_k), \quad (2.7)$$

with

$$h^\pm_k = \frac{t}{2}(1 \mp k), \quad k \in \mathbb{Z}^+ \quad (2.8)$$

The partition functions are defined by

$$\sum_N P_k(N)x^N = \frac{1}{1 + x^k} \sum_n P(n)x^n = \frac{1}{1 + x^k} \prod_{0<r \in \mathbb{Z}, \ 0<m \in \mathbb{Z}} \frac{(1 + x^r)^2}{(1 - x^m)^2}. \quad (2.9)$$

The fact that $2P(0) = 2P_k(0) = 2$ indicates that the singular vectors come two by two at the same level, in the same Verma module. Generically one is in the $G$-sector, annihilated by (at least) $G_0$, while the other is in the $Q$-sector, annihilated by (at least) $Q_0$. The roots of the quadratic vanishing surface $f_{r,s}(\Delta, h, t) = 0$ and of the vanishing planes $g^\pm_k(\Delta, h, t) = 0$ are related to the corresponding roots of the determinant formula for the Neveu-Schwarz N=2 algebra via the topological twists. These transform the standard h.w. vectors of the Neveu-Schwarz N=2 algebra into $G_0$-closed h.w. vectors of the Topological N=2 algebra. As a consequence, under the topological twists, the Neveu-Schwarz singular vectors are transformed into the singular vectors of the $G$-sector of the Topological algebra (see refs. [1][3] for a detailed account of the twisting and untwisting of primary states and singular vectors).

It is easy to check, by counting of states, that the partitions $2P(l - \frac{r^2}{2})$, exponents of $f_{r,s}$ in the determinant formula, correspond to complete Verma submodules of generic type, whereas the partitions $2P_k(l - k)$, exponents of $g^\pm_k$ in the determinant formula, correspond to incomplete Verma submodules. Furthermore, as pointed out before, taking into account also the shape at the bottom one can distinguish four types of submodules. Two of these types correspond to the

3The exponents of the determinant formulae for the Neveu-Schwarz and for the Ramond N=2 algebras also show the same behaviour: the exponents of the quadratic surfaces $f^A_{r,s}$ and $f^P_{r,s}$ correspond to complete Verma submodules and the exponents of the planes $g^A_k$ and $g^P_k$ correspond to incomplete Verma submodules, where we use the notation of ref. [3]. Although this fact is very well known and is even explained explicitly in ref. [23], abundantly cited in ref. [10], Semikhatov and Tipunin believed that they themselves had discovered that for N=2 algebras some submodules are ‘large’, of the same size of the Verma module itself, and some submodules are ‘small’. See the note at the end of ref. [10] where they ask for credit for this fact referring to a paper by M. Dörzapf.
quadratic vanishing surfaces $f_{r,s}(\Delta, h, t) = 0$, a third type corresponds to the vanishing planes $g_{r,s}^\pm(\Delta, h, t) = 0$, and the fourth type corresponds to the 'no-label' submodules that one finds in certain intersections of $f_{r,s}(\Delta, h, t) = 0$ and $g_{r,s}^\pm(\Delta, h, t) = 0$, as we will explain.

The two types that correspond to the quadratic vanishing surfaces $f_{r,s}(\Delta, h, t) = 0$, have therefore the same size (complete generic Verma submodules). The difference between them consists of the shape at the bottom, where they both have (in the most general case $\Delta \neq 0$) two uncharged singular vectors at level $l = \frac{r^2 + s^2}{2}$: $|\chi\rangle_1^{(0)G}$ in the $G$-sector and $|\chi\rangle_1^{(0)Q}$ in the $Q$-sector\footnote{For the case $\Delta = 0$ there is only one h.w. vector in the Verma module and therefore only one of the singular vectors can be described as 'uncharged' while the other must necessarily be described as charged with respect to the unique h.w. vector. These details are however irrelevant for the present discussion.}. An important remark now is that if one chooses as h.w. vector of the Verma module only the quadratic vanishing surfaces $f_{r,s}(\Delta, h, t) = 0$, then the singular vector $|\chi\rangle_1^{(0)Q}$ in the $Q$-sector is necessarily described as a negatively charged state $|\chi\rangle_1^{(-1)Q}$ built on the h.w. vector $|\Delta, h\rangle^G$. As shown in Figure I, in the most general case the bottom of the submodule consists of two singular vectors connected by one or two horizontal arrows corresponding to $Q_0$ and/or $G_0$. There is only one arrow if one of the singular vectors is chiral, i.e. of type $|\chi\rangle_1^{(0)G,Q}$ instead, what happens generically for $\Delta = -l$. These submodules fit, in principle, into the description of 'massive' submodules given by Semikhatov and Tipunin. Namely, 'massive' submodules are supposed to correspond to the uncharged roots $f_{r,s}(\Delta, h, t) = 0$, they have the same size as the generic ('massive') Verma module and they have two states at the bottom connected through $Q_0$ and/or $G_0$, one of these states being the $G_0$-closed uncharged singular vector $|\chi\rangle_1^{(0)G}$ (they do not mention the possibility that this singular vector may be chiral for $\Delta = -l$, though).

It also happens, however, for $\Delta = -l$, $t = -\frac{\Delta}{n}$, $n = 1,..,r$, that the two singular vectors at the bottom of the submodule are chiral both, and therefore disconnected from each other, as shown in Fig. I. Consequently these 'chiral-chiral' submodules, of the same size as the 'massive' Verma modules and corresponding also to the uncharged roots $f_{r,s}(\Delta, h, t) = 0$, contain two disconnected pieces at the bottom and as a result do not fit into the description of 'massive' submodules. Nor do they fit into the description of two 'topological' (smaller) submodules since these correspond to the charged roots $g_{r,s}^\pm(\Delta, h, t) = 0$ with charged singular vectors $|\chi\rangle_1^{(1)G}$ or $|\chi\rangle_1^{(-1)G}$ at the bottom of the submodules, as we will see.

Let us stress that the existence of 'chiral-chiral' submodules was obvious since January 1997 when the whole set of singular vectors of the Topological algebra at level 1 was written down in ref. [1]. For example, the chiral singular vectors $|\chi\rangle_1^{(q)G,Q}$ at level 1 built on $G_0$-closed h.w. vectors $|\Delta, h\rangle^G$ (which are the only h.w. vectors considered by Semikhatov and Tipunin) were shown to be:

$$|\chi\rangle_1^{(0)G,Q} = (-2 \mathcal{L}_{-1} + \mathcal{G}_{-1} Q_0)|-1, -1\rangle^G,$$

$$|\chi\rangle_1^{(-1)G,Q} = (\mathcal{L}_{-1} Q_0 + \mathcal{H}_{-1} Q_0 + Q_{-1})|-1, \frac{6 - c}{3}\rangle^G. \quad \text{(2.11)}$$

For $c = 9$ ($t = -2$) these two chiral singular vectors are together in the same generic ('massive')
Verma module built on the h.w. vector $|−1, −1⟩^G$. Hence these results already prove the existence of ‘chiral-chiral’ submodules at level 1.

\[
f_{r,s}(\Delta, \mathbf{h}, t) = 0, \quad \Delta \neq -l
\]
\[
f_{r,s}(\Delta, \mathbf{h}, t) = 0, \quad \Delta = -l, \quad t \neq -\frac{n}{2}
\]
\[
f_{r,s}(\Delta, \mathbf{h}, t) = 0, \quad \Delta = -l, \quad t = -\frac{n}{2}
\]

The singular vectors corresponding to the series $f_{r,s}(\Delta, \mathbf{h}, t) = 0$ belong to two different types of submodules of the same size (complete Verma submodules). In the first type, as shown in the figures on the left and in the center, the two singular vectors at the bottom of the submodules are connected by $\mathcal{G}_0$ and/or $\mathcal{Q}_0$, depending on whether $\Delta \neq -l$ or $\Delta = -l$, $t \neq -\frac{n}{2}$, $n = 1, .., r$ (for which one of the singular vectors is chiral). In the second type, corresponding to $\Delta = -l$, $t = -\frac{n}{2}$, $n = 1, .., r$, the two singular vectors are chiral and therefore disconnected from each other, as shown in the figure on the right.

Finally, the fourth type of submodules, shown also in Fig. II, corresponds to the ‘no-label’ submodules. These are the widest submodules, with four singular vectors at the bottom, generated by ‘no-label’ singular vectors. These are primitive singular vectors that only exist for discrete values of $\Delta, \mathbf{h}, t$, in Verma modules where there are intersections, at the same level $l$, of singular vectors corresponding to the series $f_{r,s}(\Delta, \mathbf{h}, t) = 0$ with singular vectors corresponding to one of the series
The values of $c$ for which no-label singular vectors exist are $c = \frac{2k-6}{2}$, corresponding to $t = \frac{2}{k}$. These results have been proved in ref. [8] although the existence of no-label singular vectors was proved in January 1997 in ref. [1], since they were explicitly written down at level 1 (shortly afterwards no-label singular vectors were written down at level 2 in ref. [3] – this was even advertised in the abstract of the paper – where it was proved that no-label singular vectors of the Topological algebra correspond to subsingular vectors of the Neveu-Schwarz $N=2$ algebra under the topological twists).

The action of $G_0$ and $Q_0$ on a no-label singular vector $|\chi\rangle_l^{(q)}$ produce three secondary singular vectors (one $G_0$-closed, one $Q_0$-closed and one chiral) which cannot ‘come back’ to the no-label singular vector by acting with $G_0$ and $Q_0$:

$$Q_0 |\chi\rangle_l^{(q)} \rightarrow |\chi\rangle_l^{(q-1)Q}, \quad G_0 |\chi\rangle_l^{(q)} \rightarrow |\chi\rangle_l^{(q+1)G}, \quad G_0 Q_0 |\chi\rangle_l^{(q)} \rightarrow |\chi\rangle_l^{(q)G,Q}. \quad (2.12)$$

It happens that one of these singular vectors corresponds to the series $f_{r,s}(\Delta, h, t) = 0$, another one corresponds to the series $g_k^+(\Delta, h, t) = 0$, and the remaining one corresponds to both series. Hence the bottom of the no-label submodules is connected, generated by the no-label singular vector and consists of four singular vectors: the primitive no-label singular vector and the three secondary singular vectors. Obviously, these submodules are wider than the ‘massive’ submodules (twice wider at the bottom, in fact) and do not fit into the description of ‘massive’ and ‘topological’ submodules.

In Fig. III one can see the case of an uncharged no-label singular vector $|\chi\rangle_l^{(0)}$ with the three corresponding secondary singular vectors. The uncharged no-label singular vector $|\chi\rangle_l^{(0)}$ at level 1, built on a $G_0$-closed h.w. vector $|\Delta, h\rangle^G$, together with the three secondary singular vectors that it generates at level 1 by the action of $G_0$ and $Q_0$ read:

$$|\chi\rangle_{1,-1,-1,t=2}^G = (\mathcal{L}_1 - \mathcal{H}_1)|-1,-1, t=2\rangle^G, \quad (2.13)$$
The no-label singular vector only exists for \( t = 2 \) (\( c = -3 \)) whereas the three secondary singular vectors are just the particular cases, for \( t = 2 \), of the one-parameter families of singular vectors of the corresponding types, which exist for all values of \( t \).

\[
|\chi\rangle^{(1)G}_{1,|-1,-1, t=2} = \mathcal{G}_0 |\chi\rangle^{(0)}_{1,|-1,-1, t=2} = 2\mathcal{G}_{-1}|-1,-1, t = 2\rangle^G, \tag{2.14}
\]

\[
|\chi\rangle^{(-1)Q}_{1,|-1,-1, t=2} = Q_0 |\chi\rangle^{(0)}_{1,|-1,-1, t=2} = (\mathcal{L}_{-1}Q_0 - \mathcal{H}_{-1}Q_0 - Q_{-1})|--1,-1, t = 2\rangle^G, \tag{2.15}
\]

\[
|\chi\rangle^{(0)G,Q}_{1,|-1,-1, t=2} = \mathcal{G}_0Q_0 |\chi\rangle^{(0)}_{1,|-1,-1, t=2} = 2(-2\mathcal{L}_{-1} + \mathcal{G}_{-1}Q_0)|-1,-1, t = 2\rangle^G. \tag{2.16}
\]

The no-label singular vector only exists for \( t = 2 \) (\( c = -3 \)) whereas the three secondary singular vectors are just the particular cases, for \( t = 2 \), of the one-parameter families of singular vectors of the corresponding types, which exist for all values of \( t \).

![Diagram](image)

**Fig. III.** The uncharged no-label singular vector \( |\chi\rangle^{(0)}_l \) at level \( l \), built on the h.w. vector \( |-l, h\rangle^G \), is the primitive singular vector generating the three secondary singular vectors at level \( l \): \( |\chi\rangle^{(1)G}_l = \mathcal{G}_0|\chi\rangle^{(0)}_l \), \( |\chi\rangle^{(-1)Q}_l = Q_0|\chi\rangle^{(0)}_l \) and \( |\chi\rangle^{(0)G,Q}_l = \mathcal{G}_0Q_0|\chi\rangle^{(0)}_l \). These cannot generate the no-label singular vector by acting with the algebra. However, they are the singular vectors detected by the determinant formula, corresponding to the series \( f_{r,s}(\Delta, h, t) = 0 \) (\( |\chi\rangle^{(-1)Q}_l \) and \( |\chi\rangle^{(0)G,Q}_l \)) and the series \( g^+_k(\Delta, h, t) = 0 \) (\( |\chi\rangle^{(1)G}_l \) and \( |\chi\rangle^{(0)G,Q}_l \)).

Two important observations come now in order. First, a given submodule may not be completely generated by the singular vectors at the bottom. These could generate only a submodule of the whole (maximal) submodule, in which case one or more subsingular vectors generate the missing parts. Second, we have seen that there are four different types of submodules that may appear in generic Verma modules, distinguished by their size and/or the shape at the bottom of the submodule. A more accurate classification of the submodules, however, should take into account also the shape of the whole submodule, including the possible existence of subsingular vectors. For this reason we claim that there are at least four different types of submodules in generic Verma modules.

We have shown so far that the two types of submodules proposed by Semikhatov and Tipunin – ‘massive’ and ‘topological’ submodules – correspond, in principle, to the submodules of the first and third types, respectively, whereas the submodules of the second and fourth types (‘chiral-chiral’
and ‘no-label’ submodules) have been overlooked by the authors. (As a matter of fact, it is not clear if chiral uncharged singular vectors $|\chi\rangle_{l}^{(0)G,Q}$ fit in the framework of the authors at all since they are completely ignored). Furthermore the authors claim that the ‘massive’ and ‘topological’ submodules are generated maximally (i.e. completely, without letting any null states outside) by the non-conventional singular vectors that they constructed in refs. [11] [12].

Let us notice that the very existence of no-label singular vectors already disproves this claim because no-label singular vectors (and many of their descendants) can be viewed as pieces left outside from the ‘topological’ and ‘massive’ submodules. To be precise, no-label submodules contain one ‘topological’ and one ‘massive’ submodule, starting at the bottom, but the no-label singular vector, and many of its descendants, do not belong to these submodules. As a result the ‘massive’ and ‘topological’ submodules together do not build the whole no-label submodule. Consequently, the non-conventional singular vectors together do not generate the maximal no-label submodule (not even the bottom!). In the next subsection we will give another argument showing in a different way that the non-conventional singular vectors do not generate maximal submodules: one can find subsingular vectors outside of them!

### 2.2 The classification of subsingular vectors

In order to understand the results presented by Semikhatov and Tipunin in ref. [10] (and in several other publications) we have to make two important remarks concerning the presentation of the conventional singular vectors by the authors. First, the authors claim that in the conventional approach the h.w. conditions imposed on the h.w. vectors and on any singular vector must include the annihilation by $G_{0}$ (eq.(2.11) in ref. [10]). This statement is incorrect since in the conventional approach, for the (super)conformal algebras, one defines the h.w. vectors and singular vectors (often called simply null vectors) as the states with lowest conformal weight (lowest energy) in the Verma modules and submodules, respectively. As a result, in most Verma modules and submodules of the Ramond and the Topological N=2 algebras (they are isomorphic in fact [7]) there are two sectors degenerated in energy, the + and $-$ sectors for the Ramond algebra and the $G$ and $Q$ sectors for the Topological algebra, the corresponding states annihilated by the fermionic zero modes $G_{0}^{+}$ or $G_{0}^{-}$ and $G_{0}$ or $Q_{0}$, respectively [2] [21] [22] [23] [24] [25] [26] [27] [28] [29] [7]. That is, at the bottom of most Verma modules and submodules of the Ramond and of the Topological N=2 algebras there are two h.w. vectors and two singular vectors, respectively, the fermionic zero modes interpolating between them. In addition, one can find indecomposable singular vectors not annihilated by any of the fermionic zero modes, that also must be called singular vectors following the conventional definition [1] [8] [6] [7].

Second, let us also notice that to break the symmetry between the $G$ and the $Q$ sectors, regarding the singular vectors of the $Q$-sector simply as descendant states (non-singular) of ‘the singular vectors’ of the $G$-sector, leads to confusion in the case of zero conformal weight $\Delta + l = 0$. The reason is that for $\Delta + l = 0$ the $Q_{0}$-closed (non-chiral) singular vectors $|\chi\rangle_{l=-\Delta}^{(q)Q}$ are in fact the primitive ones generating the secondary singular vectors of the $G$-sector, which are necessarily chiral of type $|\chi\rangle_{l=-\Delta}^{(q+1)G,Q}$ (see the details in ref. [7], Appendix A). In the conventions used by Semikhatov and Tipunin, however, the vectors $|\chi\rangle_{l=-\Delta}^{(q)Q}$ are not singular by definition. As a result,
since they are not descendant states of ‘the singular vector’ \( |\chi\rangle^{(q+1)G,Q}_{l=-\Delta} \), but the other way around, the singular vectors of the \( Q \)-sector \( |\chi\rangle^{(q)Q}_{l=-\Delta} \) must be called \textit{subsingular vectors} instead. For similar reasons, the indecomposable ‘no-label’ singular vectors must also be called subsingular vectors (they are not descendants of the singular vectors of the \( G \)-sector, but the other way around, and they are not singular by definition).

Now we will see that the explicit examples of subsingular vectors given in ref. \cite{1}, which are singular in the chiral Verma modules, do not fit into the complete classification of subsingular vectors presented by Semikhatov and Tipunin in ref. \cite{10}. As a consequence we will deduce that the non-conventional singular vectors constructed in refs. \cite{11} \cite{12} do not generate maximal submodules. The authors classified the generic Verma modules built on \( G_0 \)-closed h.w. vectors (‘massive’ Verma modules) according whether they have zero, one, two or more singular vectors from the uncharged and/or charged series associated to the roots of the determinant formula (in our notation \( f_{r,s} (\Delta, h, t) = 0 \) and/or \( g^{\pm}_{k} (\Delta, h, t) = 0 \)). In every case they applied the assumption that there are only two types of submodules – ‘massive’ and ‘topological’ – and these are generated maximally by the non-conventional singular vectors constructed in refs. \cite{11} \cite{12}. Namely, one ‘twisted topological’ non-conventional singular vector (where they mean twisted by the spectral flows) is assumed to generate maximally one ‘topological’ submodule whereas one ‘twisted massive’ non-conventional singular vector is assumed to generate maximally one ‘massive’ submodule.

As pointed out before, these objects are null states that in general are not located at the bottom of the submodules unlike the conventional singular vectors. In fact, in the cases when they lie at the bottom then they coincide with the conventional singular vectors. An important remark is that the ‘twisted topological’ h.w. conditions satisfied by the ‘twisted topological’ non-conventional singular vectors reduce to the chirality h.w. conditions (i.e. annihilation by \( G_0, Q_0 \) and by all the positive generators) in the case of the twist parameter equal to zero. As a result, the ‘zero twist topological’ non-conventional singular vectors coincide with the chiral \textit{charged} conventional singular vectors at the bottom of the ‘topological’ submodules. The claim that the non-conventional singular vectors generate maximal submodules implies that acting with the lowering and raising operators of the algebra one generates whole submodules without any null states left outside, such as subsingular vectors and descendants of them, that can be pulled inside the submodules by the action of the algebra.

Using these assumptions and simple geometrical arguments, the authors deduced in which cases the \textit{conventional} singular vectors at the bottom of the submodules do not generate maximal submodules, the remaining pieces outside being generated by subsingular vectors. In some of these cases the authors gave general expressions for the subsingular vectors. The subsingular vectors given by us in ref. \cite{1} must correspond necessarily to the ones described by the authors in the case ‘codimension-2 charge-massive’, given \cite{1} in Proposition 3.9, for \( n = 0 \), since they are located in Verma modules with one charged chiral singular vector (at level zero, what gives \( n = 0 \)) and one uncharged \( G_0 \)-closed singular vector (and its companion in the \( Q \)-sector that is ignored by the authors). In the notation of the authors, who draw the Verma modules upside-down, the charged singular vector is

\footnote{The authors themselves claimed that the subsingular vectors given in ref. \cite{1} were described by Proposition 3.9, case \( n = 0 \) \cite{23}, although they did not explicitly mention this in the last revised, published version of ref. \cite{10}. See a comment in the Final Remarks regarding this issue.}
both a conventional ‘top-level’ singular vector and a non-conventional ‘twisted topological’ charged singular vector \(|E(n)\rangle_{ch}\) with twist parameter \(n = 0\) (i.e. the non-conventional singular vector is at the bottom of the submodule so that it coincides with the conventional singular vector). The uncharged \(G_0\)-closed singular vector is described as the conventional ‘top-level’ uncharged singular vector in the ‘massive’ submodule generated by the ‘massive’ singular vector \(|S(r, s)\rangle\), and is denoted as \(|s\rangle\).

For this case, and in fact for all cases ‘described’ by Proposition 3.9, the authors deduced that a subsingular vector \(|Sub\rangle\) must exist inside the maximal massive submodule generated by \(|S(r, s)\rangle\) in the sense that \(|Sub\rangle\) is located outside the non-maximal submodule generated by the conventional uncharged singular vector \(|s\rangle\), becoming singular once \(|s\rangle\) is set to zero. This implies that the subsingular vector \(|Sub\rangle\) is ‘pushed down’ (‘up’ in the authors figures) by the action of the lowering operators inside the non-maximal submodule generated by \(|s\rangle\), so that setting this submodule to zero is equivalent to push down the vector to nothing, i.e. the subsingular vector becomes singular. Observe that in this case the subsingular vector \(|s\rangle\) is set to zero is equivalent to push down the vector to nothing, i.e. the subsingular vector becomes singular. Observe that in this case the subsingular vector \(|Sub\rangle\), once it reaches \(|s\rangle\) by the action of the lowering operators, cannot go down (‘up’) anymore since \(|s\rangle\) is the conventional singular vector at the bottom of the submodule annihilated by all the lowering operators. In other words, if the subsingular vector \(|Sub\rangle\) becomes singular when \(|s\rangle\) is set to zero, then acting with the lowering operators on \(|Sub\rangle\) it cannot be pulled down beyond the level of \(|s\rangle\), getting in fact ‘stuck’ in \(|s\rangle\) (up to constants).

The subsingular vectors at level 3 given by us in ref. [1] do not follow the behaviour described by Proposition 3.9, however. Rather, they are pulled down beyond the uncharged conventional singular vector \(|s\rangle\) that one finds at level 2 and, in fact, they can be pulled down until the very end, i.e. level zero, becoming singular only when the charged chiral singular vector \(|E(0)\rangle_{ch}\) at level zero is set to zero. As a consequence, these subsingular vectors do not become singular when \(|s\rangle\) is set to zero, what implies that they are not pulled inside the submodule generated by \(|s\rangle\) by the action of the lowering operators (see Fig. IV), and therefore they are not located inside the maximal massive submodule supposed to be generated by the ‘massive’ singular vector \(|S(r, s)\rangle\). But these subsingular vectors are neither located inside the submodule generated by \(|E(0)\rangle_{ch}\) since they do not disappear when \(|E(0)\rangle_{ch}\) is set to zero, becoming singular rather. In other words, as shown in Fig. IV, these subsingular vectors are pulled inside the submodule generated by \(|E(0)\rangle_{ch}\) by acting with the lowering operators. This implies that the submodule generated by the non-conventional singular vector \(|E(0)\rangle_{ch}\) is not maximal, in contradiction with the claims of Semikhatov and Tipunin.

One example given in ref. [1] is the subsingular vector \(|Sub\rangle_{3}^{(1)}\) at level 3 with charge \(q = 1\) built on the \(G_0\)-closed h.w. vector \(|\Delta, h\rangle^G\) with conformal weight \(\Delta = 0\) and U(1) charge \(h = 2\):

\[
|Sub\rangle_{3}^{(1)} = \left\{ \frac{3 - c}{24} \mathcal{L}_{-1} \mathcal{G}_{-1} - \frac{3}{4} \mathcal{L}_{-1} \mathcal{G}_{-2} - \frac{1}{4} \mathcal{L}_{-2} \mathcal{G}_{-1} + \frac{c + 9}{4(c - 3)} \mathcal{H}_{-2} \mathcal{G}_{-1} + \frac{27 - c}{4(3 - c)} \mathcal{G}_{-3} + \frac{6}{c - 3} \mathcal{H}_{-1} \mathcal{G}_{-2} + \frac{3}{4} \mathcal{H}_{-1} \mathcal{L}_{-1} \mathcal{G}_{-1} + \frac{3}{3 - c} \mathcal{H}^2 \mathcal{G}_{-1} \right\} |0, 2\rangle^G.
\]

Acting with \(Q_1\) on this vector one does not hit the conventional uncharged singular vector \(|s\rangle\) at level 2 but one reaches the state

\[
\left\{ \frac{c - 12}{12} \mathcal{L}_{-1} \mathcal{G}_{-1} + \frac{3(11 - c)}{4(3 - c)} \mathcal{G}_{-2} + \frac{3(11 - c)}{4(c - 3)} \mathcal{H}_{-1} \mathcal{G}_{-1} \right\} Q_0 |0, 2\rangle^G,
\]  

(2.17)
which is a non-singular descendant of the level zero charged singular vector $|E(0)\rangle_{ch} = Q_0 |0, 2\rangle^G$. That is, $|Sub\rangle_3^{(1)}$ is pulled inside the submodule generated by $|E(0)\rangle_{ch}$ by the action of $Q_1$. Acting further with $L_1$ one reaches the state $G^{-1}Q_0 |0, 2\rangle^G$ at level 1 which, again, is not singular. Acting with $Q_1$ on this state one reaches finally the level zero chiral charged singular vector: $Q_1L_1Q_1 |Sub\rangle_3^{(1)} = Q_0 |0, 2\rangle^G = |E(0)\rangle_{ch}$.

**Fig. IV.** When the charged level zero singular vector $|E(0)\rangle_{ch} = Q_0 |0, 2\rangle^G$ is set to zero, the generic (‘massive’) Verma module $V(|0, 2\rangle^G)$ is divided by the submodule generated by this singular vector. As a result one obtains the incomplete, chiral Verma module $V(|0, 2\rangle^G, Q)$ built on the chiral h.w. vector $|0, 2\rangle^G, Q$. The subsingular vector $|Sub\rangle_3^{(1)}$ at level 3 is outside the submodule generated by $|E(0)\rangle_{ch}$, being pulled inside by the action of the lowering operators. Consequently, the submodule generated by the non-conventional ‘topological’ charged singular vector $|E(0)\rangle_{ch}$ which being at the bottom of the submodule coincides with the conventional chiral singular vector $Q_0 |0, 2\rangle^G$ is not maximal since there is (at least) one subsingular vector left outside. This subsingular vector becomes singular, therefore, in the chiral Verma module $V(|0, 2\rangle^G, Q)$ obtained after the quotient. Inside the submodule generated by $|E(0)\rangle_{ch}$ one finds the uncharged $G_0$-closed singular vector $|s\rangle$ (and its companion in the $Q$-sector that is not indicated). The subsingular vector $|Sub\rangle_3^{(1)}$ is not pulled inside the submodule generated by $|s\rangle$ by the lowering operators and therefore it does not become singular once $|s\rangle$ is set to zero. As a result $|Sub\rangle_3^{(1)}$ does not belong to the ‘massive’ submodule, supposed to be generated by the non-conventional ‘massive’ singular vector $|S(r, s)\rangle$, having $|s\rangle$ at the bottom.

This example not only proves that Proposition 3.9 is incorrect, as $|Sub\rangle_3^{(1)}$ does not become singular when $|s\rangle$ is set to zero, and that the subsingular vectors presented in ref. [10] (the only examples known at that time!) do not fit into the ‘complete’ classification of subsingular vectors given by Semikhatov and Tipunin in ref. [10]. As we have just discussed, this example also proves that the non-conventional topological singular vector $|E(0)\rangle_{ch} = Q_0 |0, 2\rangle^G$ (which is located at the bottom of the submodule and therefore coincides with the conventional chiral singular vector) does NOT generate a maximal submodule since the subsingular vector $|Sub\rangle_3^{(1)}$ is outside this submodule, being pulled inside by the action of the lowering operators. This example disproves the claim of Semikhatov and Tipunin that their non-conventional ‘massive’ and ‘topological’ singular vectors generate maximal submodules with no space left outside for subsingular vectors. Indeed, we have shown that the subsingular vector $|Sub\rangle_3^{(1)}$ is neither generated by the ‘massive’ singular vector $|S(r, s)\rangle$ nor by the ‘topological’ singular vector $|E(0)\rangle_{ch}$, nor by both of them together. As a result, this example proves that the non-conventional ‘massive’ and ‘topological’ singular vectors constructed by Semikhatov and Tipunin do not generate maximal submodules.
2.3 The non-conventional singular vectors and the isomorphism $N=2 \leftrightarrow \hat{sl}(2)$

We will argue now that the non-conventional singular vectors of Semikhatov and Tipunin, apart from the fact that they do not generate maximal submodules, are poorly defined objects of unclear meaning (except for some simple cases).

On the one hand, the key idea underlying the construction of these ‘non-conventional’ singular vectors in refs. [11] and [12] was the misconception that the spectral flows map h.w. vectors to h.w. vectors (for any value of the parameter $\theta$!) transforming the Verma modules into isomorphic Verma modules, consequently. (The fact that the spectral flows do not map h.w. vectors to h.w. vectors, except for some specific values of $\theta$, was already pointed out by Schwimmer and Seiberg just a few lines after they wrote down the spectral flows in ref. [30], abundantly cited by Semikhatov and Tipunin. An exhaustive analysis of this issue for the even and the odd spectral flows can be found in ref. [3]). To be precise, the authors claimed that the spectral flows do not act only on the states but also on the h.w. conditions in such a way that the h.w. vectors always remain h.w. vectors under any spectral flow transformation. (Observe that transforming the states and the observables, like the h.w. conditions, at the same time is equivalent to not doing any transformation at all, just a redefinition of the states in the same Verma module). One can see this misconception abundantly used in ref. [11]. In the last version of ref. [12] (Sept. 1998), however, the corresponding misleading statements have been removed everywhere in the paper. Intriguingly enough, all the final expressions and results have remained the same.

On the other hand, the ‘non-conventional’ singular vectors are constructed out of ‘continued’ operators or ‘intertwiners’, $g(a, b)$ and $q(a, b)$, that generalize the products of fermionic modes $G_0 G_{a+1} G_{a+2} \ldots G_{a+N}$ and $Q_0 Q_{a+1} Q_{a+2} \ldots Q_{a+N}$, respectively, to a complex number of factors. The authors claim that this procedure is analogous to the analytical continuation of Malikov-Feigin-Fuchs for affine Lie algebras [31], for which complex exponents of the generators are used. We disagree with this view because the complex exponents for the affine Lie algebras, used also by Kent for the Virasoro algebra [32], involve only well defined generators of the algebra under study. The intertwiners $g(a, b)$ and $q(a, b)$, however, are made out of a continuum of ‘generators’ of types $G_\alpha$ and $Q_\alpha$, with $\alpha$ a complex number, that do not belong to the Topological $N=2$ algebra (1.1) under study, except for $\alpha$ integer, mixing in fact a continuum of different $N=2$ algebras. Furthermore, the authors postulate a number of properties and results for $g(a, b)$ and $q(a, b)$ that they call a ‘consistent’ set of algebraic rules. In this set, however, important commutators are absent: $[g(a, b), Q_\alpha]$, $[q(a, b), G_\alpha]$ and $[g(a, b), q(a', b')]$.

Finally, underlying the construction of the ‘non-conventional’ singular vectors and the idea that they generate ‘maximal’ submodules, is the authors claim [16] [17] [15] that the ‘topological’ and ‘massive’ non-conventional singular vectors of ‘the N=2 algebra’ (see footnote 1) are in one-to-

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6This is also the reason why they believed that the Verma modules of the Neveu-Schwarz and the Ramond $N=2$ algebras were isomorphic. We tried to warn the authors, unsuccessfully, about their wrong use of the spectral flows. We also suggested to them to use the involutive automorphism $A$ that we had deduced in ref. [4], eq. (2.8), acting as $L_m \rightarrow L_m - m H_m$, $H_m \rightarrow -H_m - \frac{c}{2}$, $G_n \rightarrow Q_n$ and $Q_n \rightarrow G_n$, with $A^{-1} = A$. This suggestion was followed by the authors, but without citing our work, as the reader can see in ref. [12], eq. (2.7), and in ref. [3], eq. (2.17).
one correspondence with the conventional and ‘relaxed’ singular vectors of the affine \( \hat{\mathfrak{sl}}(2) \) algebra, respectively. The ‘relaxed’ h.w. vectors and singular vectors have neither clear meaning nor intrinsic interest from the point of view of the affine \( \hat{\mathfrak{sl}}(2) \) algebra. They are not annihilated by any of the zero modes \( J_0 \), unlike the conventional singular vectors, and they satisfy a very strong constraint (apparently in order to mimic some fermionic properties of ‘the N=2 algebra’). They have been introduced by the authors with the unique purpose to create an isomorphism between the Verma modules and submodules of ‘the N=2 algebra’ and the Verma modules and submodules of the affine \( \hat{\mathfrak{sl}}(2) \) algebra. The authors claim that the ‘topological’ and ‘massive’ submodules generated by the ‘topological’ and ‘massive’ singular vectors of ‘the N=2 algebra’ are isomorphic to the conventional and ‘relaxed’ submodules generated by the conventional and ‘relaxed’ singular vectors of the affine \( \hat{\mathfrak{sl}}(2) \) algebra, respectively. The crucial point now is that the latter must be maximal submodules since in the affine \( \hat{\mathfrak{sl}}(2) \) algebra subsingular vectors are not supposed to exist. As a consequence, the submodules generated by the ‘topological’ and ‘massive’ non-conventional singular vectors of ‘the N=2 algebra’ must be also maximal. But we have proved in the previous subsection, using a counterexample, that these submodules are not maximal. Therefore there are only two alternatives: either the isomorphism N=2 ↔ \( \hat{\mathfrak{sl}}(2) \) proposed by the authors fails or there exist subsingular vectors in the Verma-like modules (conventional and ‘relaxed’ Verma modules) of the affine \( \hat{\mathfrak{sl}}(2) \) algebra. We believe that the isomorphism fails for two reasons. First, as we pointed out in the paragraphs above, the N=2 non-conventional singular vectors, which are the N=2 counterpart of the isomorphism, are objects of very unclear nature. In our opinion, it is not even clear that they belong to any Verma modules and/or submodules of the Topological N=2 algebra, as claimed by the authors. Second, the isomorphism already fails, in its present form, because in the ‘relaxed’ Verma modules, which are the \( \hat{\mathfrak{sl}}(2) \) partners of the generic ‘massive’ N=2 Verma modules, no submodules have been found that could be the partners of the ‘chiral-chiral’ or of the ‘no-label’ N=2 submodules. As a matter of fact, it seems a highly non-trivial task to find the \( \hat{\mathfrak{sl}}(2) \) partners of the no-label singular vectors (that in the notation used by the authors should be called no-label subsingular vectors instead, as we have explained before). We think that these partners simply do not exist.

2.4 The N=2 embedding diagrams

The same misleading ideas used by Semikhatov and Tipunin to obtain the degeneration patterns and classification of subsingular vectors in ref. [10], were used afterwards by Semikhatov and Sirota in ref. [15], where they presented a prescription to obtain a classification of N=2 embedding diagrams. That is, they applied again the assumption that in N=2 Verma modules there are submodules of exactly two different types – the (twisted) ‘massive’ and the (twisted) ‘topological’ ones and arbitrary sums thereof – and the assumption that these submodules are generated maximally by the (twisted) ‘massive’ and the (twisted) ‘topological’ non-conventional singular vectors. Hence

\footnote{The precise claim is that the ‘twisted topological’ and ‘twisted massive’ singular vectors and submodules of ‘the N=2 algebra’ are isomorphic to the ‘twisted conventional’ and ‘twisted relaxed’ singular vectors and submodules of the affine \( \hat{\mathfrak{sl}}(2) \) algebra, respectively, where twisted refers to the corresponding spectral flows. For convenience we have dropped the word ‘twisted’ in all this paragraph.}
it seems that there is no need to add any more comments to this issue. Nevertheless, there are two important remarks to be added.

First, the authors did not present a single N=2 embedding diagram in [15] (in spite of their claims). Instead they presented a classification of embedding diagrams of the $\hat{sl}(2)$ Verma-like structures, for which there are only two types of submodules and these are generated maximally by their corresponding singular vectors. Then the authors proposed the isomorphism that we have discussed in the previous subsection, between the N=2 Verma modules and submodules and the $\hat{sl}(2)$ Verma-like modules and submodules, and finally the reader was supposed to apply this isomorphism to obtain, after working for several weeks, the sought N=2 embedding diagrams. (In the abstract of the paper, the first sentence reads however: ‘We classify and explicitly construct the embedding diagrams of Verma modules over the N=2 supersymmetric extension of the Virasoro algebra’).

Second, the authors did not compare their ‘classification’ of N=2 embedding diagrams with the most complete classification done so far on this issue, due to Dörrzapf in 1995 [9] (Ph. D. thesis, Cambridge). The authors did not even mention this important work of Dörrzapf, abundantly referred, however, in some references cited by the authors [8].

3 Final Remarks

In January 97 our paper [1] appeared in hep-th. As was indicated in the abstract, all the singular vectors at level 1 of the Topological N=2 algebra were presented, as well as the subsingular vectors which become singular in chiral Verma modules at levels 2 and 3. Semikhatov and Tipunin only needed to look (literally!) at the singular vectors at level 1 to realize straightforwardly that their classification of submodules was incomplete as there is no place left for ‘no-label’ singular vectors neither for ‘chiral-chiral’ pairs of uncharged singular vectors. Moreover, the authors only needed to check the few examples of subsingular vectors, which were the only examples known in the literature, to find out that their classification of subsingular vectors was incorrect and their claim that the ‘non-conventional’ singular vectors generate maximal submodules was also incorrect. These incorrect claims they published however three months after, in April 97 in ref. [10]. Furthermore in August 97 they sent to hep-th a revised version of that paper, accepted for publication in Comm. Math. Phys., in which they added several misleading claims about ‘our’ subsingular vectors in ref. [1]. The most disturbing claim was that they were subsingular (instead of singular) in the chiral Verma modules.

Because of these incorrect claims we contacted the editors of Comm. Math. Phys. who very kindly stopped the publication of the paper [10] in order for the authors to revise the paper. We

8This is most intriguing because the classification of N=2 embedding diagrams in the thesis of Dörrzapf was well known to Semikhatov as it was shown to him in Durham during the academic year 1995-96 and he even borrowed the thesis for a few days. In addition, Semikhatov sent an e-mail to Dörrzapf in December 1997 where he admitted to know the work on N=2 embedding diagrams in his thesis [34]. In spite of these facts Semikhatov has kept ignoring the classification of N=2 embedding diagrams by Dörrzapf also in the papers [8], [12] and [10] that followed ref. [15].
also pointed out, among several other remarks, that: i) All the results presented in [10] were based on the assumption that there are only two different types of submodules in N=2 Verma modules, without giving any proof for this strong claim, ii) Our classification of h.w. vectors was completely overlooked, in particular no-label h.w. vectors were ignored, and iii) Our subsingular vectors, which are singular in chiral Verma modules, did not fit into the classification of subsingular vectors given by the authors.

The authors replied to the editors that they had in fact misclassified the subsingular vectors in ref. [1], but however they were described by Proposition 3.9 case \( n = 0 \). (In this note we have proved that this is not the case). They also assured that they did not take into account no-label singular vectors because they do not exist in generic (‘massive’) Verma modules (in ref. [1] we had proved, however, that no-label singular vectors only exist in generic Verma modules and one can see this also in the explicit examples at level 1). They also assured that the spectral flows transform not only the states but also the h.w. conditions in such a way that the h.w. vectors are always mapped into h.w. vectors, for any spectral flow transformation (they also assured that the Verma modules of the Neveu-Schwarz N=2 algebra are isomorphic to the Verma modules of the Ramond N=2 algebra for this reason). In spite of this misleading reply the paper was finally published, although with several improvements.

In November 98 we explained the content of this note to the authors, in particular we gave them the proof, step by step, that their ‘topological’ and ‘massive’ non-conventional singular vectors do not generate maximal submodules. In spite of this the authors (who never replied to us) still published, in Nucl. Phys. B, the paper [18] with the usual assumptions that there are only two types of submodules,... For example one can read in the second paragraph below eq. (3.6) the statement: ‘...these singular vectors (the topological ones) generate maximal submodules, which is crucial for the resolution to have precisely the form (3.2) ...’.

Finally we would like to point out that the facts discussed in this note affect drastically the results presented in refs. [10] – [18] and less importantly also the results presented in ref. [19].

Acknowledgements

We thank B.L. Feigin and A. Taormina for encouraging us to write this letter and make public our disagreements with the work of Semikhatov and collaborators about the N=2 superconformal algebras. We would like to thank B.L. Feigin also for elucidating to us his involvement in the papers [10], [18] and [19] and his commitment to the results. This clarifies the puzzling lack of rigour observed by us, and by several other colleagues, in these papers. Finally, we are indebted to A. Jaffe and T. Miwa, as editors of Comm. Math. Phys. for their help in removing the misleading claims about ‘our subsingular vectors’ from the final version of the paper [10], by A. Semikhatov and Y. Tipunin, published in CMP.

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