Temporal-multiplexing interferometry applied to co-phased profilometry

MANUEL SERVIN,* MOISES PADILLA, AND GUILLERMO GARNICA
Centro de Investigaciones en Optica A.C., Loma del Bosque 115, 37150 Leon, Guanajuato, Mexico.
*mservin@cio.mx
https://www.cio.mx

Abstract: Fringe-projection profilometry with 1 camera and 1 fringe-projector is a well-known and widely used technique in optical metrology. Spatial-frequency multiplexing interferometry with several spatial-carriers having non-overlapping spatial-spectra is well known and productive in optical metrology. In this paper we propose temporal-multiplexing phase-shifting interferometry applied to profilometry. That is, instead of having fringe-patterns with well separated spatial-spectra, we propose instead to separate the fringe information in the temporal-spectra. In other words, we may have overlapping spatial-spectra, but separated in the temporal-spectra by frequency multiplexing. Using 1-camera and several fringe-projectors one minimizes the object shadows and specular reflections from the digitizing solid. Temporal multiplexing profilometry allows us to illuminate the object from several projectors turned-on simultaneously. In previous phase-shifting co-phased profilometry, the projectors were turned-on and off sequentially. As seen in this work temporal-multiplexing allow us to demodulate the several fringe-patterns without crosstalk from other simultaneously projected fringes. This is entirely analogous to having several television stations broadcasting simultaneously, each TV-transmitter having its own broadcasting frequency. A given TV-receiver tunes into a single TV-station and filter-out all other broadcasters. Following this analogy, each fringe-projector must have its own temporal broadcasting frequency to remain well separated from all other projectors in the time-spectra domain. In addition to the general theory presented, we assess its feasibility with experimental results.

Leon Guanajuato, Mexico, February 20, 2017

OCIS Codes: (120.0120) Instrumentation, measurement, and metrology; (120.2650) Fringe analysis; (120.5050) Phase measurement; (120.6650) Surface measurements, figure.

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1. Introduction

Fringe projection profilometry applied to digitize three dimensional (3D) objects has been used for a long time [1]. A basic fringe projection profilometer consists of a multimedia digital projector, a CCD digital camera to capture the phase-modulated linear fringes and a computer to phase-demodulate the fringe patterns [1]. The phase sensitivity of a fringe projection profilometer is proportional to the camera-projector sensitivity angle multiplied by the spatial frequency of the fringes [1]. If just a single carrier-frequency fringe pattern is digitalized, it can be phase-demodulated using the Fourier technique [1,2]. On the other hand, when higher resolution digitization is required, we need to process at least three phase-shifted fringe patterns using phase-shifting algorithms (PSAs) [1,2]. The introduction of the frequency transfer function (FTF) formalism for PSAs design has been a significant advance in phase-shifting phase-demodulation [2,3,5,6,7]. However new in temporal interferometry, the FTF formalism has been used in telecommunications engineering for at least 60 years [4]. The inverse Fourier transform of the FTF give us the impulse response of the filter and from it one can easily find the PSA which filters the desired analytic signal [2,3]. The analytic signal preserves the amplitude and the phase of the temporal interferograms in a complex-valued signal; this is paramount in co-phased profilometry [2,3,5-7]. As briefly reviewed in this work, in co-phased profilometry, N fringe-patterns are projected towards the digitizing object from different directions having equal sensitivity angles and equal spatial-carrier [5,7]. In this way, the complex-valued signals estimated from each fringe projection are co-phased. Having several co-phased analytic signals one simply adds them to obtain the solid’s digitalization with minimized shadows and specular reflections [5-7]. In addition to this, co-phased profilometry coupled to 2-steps temporal-unwrapping has substantially improved fringe-projection profilometry [7] with respect to the previous state of the art [8-22].

In contrast, spatial frequency multiplexing has been used in interferometry and profilometry for years [8]. In spatial frequency-multiplexing, several linear fringe-patterns, each one having non-overlapping spatial-spectra, are projected into the object. The spatial spectral-lobes must be well separated (non overlapped), to phase demodulate them using spatial quadrature-filters [11-22]. In temporal-multiplexing however each projected fringe-pattern must oscillates at a unique temporal frequency, in this way all fringe-projectors can be turned-on simultaneously. As shown here, in temporal-multiplexing one must use as many temporal-frequencies as fringe-projectors simultaneously illuminate the object under analysis.
This is entirely analogous to public radio broadcasting television (TV). Several TV-stations may transmit simultaneously, but each one must have its own broadcasting frequency [4]. Therefore, each TV-receiver tunes into a single TV-station flatly rejecting all other TV-stations. Even though frequency multiplexing in telecommunications engineering is more than 100 years old [4], as far as we know temporal multiplexing has not been explored yet in phase-shifting profilometry. In this paper, the digitizing 3D object is simultaneously illuminated from several fringe-projectors simultaneously. Each fringe-projector must have its own temporal phase-shifting step (broadcasting frequency). In this way, one may design temporal multiplexed PSAs capable of tuning-in at different temporal frequencies. Therefore the main objective of this paper is the synthesis of temporal-multiplexed PSAs for fringe projection profilometry when several projectors illuminate the object simultaneously from different directions. Previous non-multiplexed phase-shifting algorithms (PSAs) were designed to operate at a single temporal-frequency [2,3]. But thanks to the FTF formalism [2], we can now easily design PSAs which operate at several temporal-frequencies (several phase-steps) simultaneously. This generates new interferometric phase-demodulation possibilities for designing temporal-multiplexed phase-shifting algorithms (PSAs).

2. Mathematical model for 2-temporal-frequencies multiplexed fringes

The digitizing spherical metallic solid used in the experiments reported herein is shown in Fig. 1(a). The experimental set-up for co-phased profilometry using 2-projectors and 1-camera is shown in Fig. 1(b).

![Fig. 1 Panel (a) shows the spheric metallic solid being digitized; this is a regular photography of the object, not a digital 3D rendering. Panel (b) shows the co-phased profilometer composed by 2-projectors and 1-camera. In (b) the shadows of projector-1 are covered by projector-2 and vice-versa. The phase-sensitivity angle of this profilometer is $\theta$.](image)

The digitized phase-modulated fringes for standard 1-camera, 1-projector is given by

$$I_i(x, y, t) = a(x, y) + b(x, y) \cos \left[ g h(x, y) + u_0 x + \omega_i t \right]; \quad g = u_0 \tan(\theta); \quad t \in \mathbb{R}.$$  \hspace{1cm} (1)

This fringe image is phase-modulated by $gh(x, y)$; the temporal carrier is $\omega_i$; the spatial carrier is $u_0$, and $\theta$ is the sensitivity angle. The background illumination is $a(x, y)$ and the fringes’ contrast is $b(x, y)$. Finally the digitizing solid is $z = h(x, y)$.

With 2-temporal-frequencies and using the set-up in Fig. 1(b) composed by 1-camera and 2 simultaneous fringe-projectors our previous mathematical model now modifies to,

$$I(x, y, t) = a + b_1 \cos \left[ g h(x, y) + u_0 x + \omega_1 t \right] + b_2 \cos \left[ -g h(x, y) + u_0 x + \omega_2 t \right].$$  \hspace{1cm} (2)

The 2 fringe-projectors are multiplexed by two temporal-frequencies $(\omega_1, \omega_2)$. This is shown in Fig. 2. The red trace represents the temporal fringes $b_1 \cos \left[ g h(x, y) + u_0 x + \omega_1 t \right]$, while the
blue trace represents the temporal fringes \( b_t \cos \left[ -g h(x, y) + u_0 x + \omega_2 t \right] \). Finally the lower trace represents their sum \( I(x, y, t) \) in Eq. (2).

The fringes modulated by \( b_1(x, y) \) displace in the \( x \)-direction at a phase-velocity of \( (\omega_1) \) radians/image, while the fringes modulated by \( b_2(x, y) \) move at a phase-velocity of \( (\omega_2) \) radians/image.

Taking the temporal Fourier transform, the multiplexed fringes with amplitudes \( b_1/2 \) and \( b_2/2 \) are now well-separated whenever \( \omega_1 \neq \omega_2 \) (see Fig. 4):

\[
I(x, y, \omega) = a \delta(\omega) + \frac{b_1}{2} \left[ e^{i \left(g h + u_0 x\right)} \delta(\omega - \omega_1) + e^{-i \left(g h + u_0 x\right)} \delta(\omega + \omega_1) \right] +
\]

\[
\frac{b_2}{2} \left[ e^{i \left(-g h + u_0 x\right)} \delta(\omega - \omega_2) + e^{-i \left(-g h + u_0 x\right)} \delta(\omega + \omega_2) \right].
\]

This Fourier time-frequency spectrum is graphically represented in Fig. 4.
Fig. 4. Fourier spectrum for 2-temporal-multiplexed simultaneous fringe-projections using 2-temporal-frequencies \( \omega_1 \) and \( \omega_2 \). The vertical arrows represent Dirac deltas.

Our objective is to estimate the two analytic signals at frequencies \( (\omega_1, \omega_2) \). To this end, we need to filter-out the signals at \( \omega = \{-\omega_2, -\omega_1, 0, \omega_2\} \) in order to keep \( (b_1 / 2) e^{i(\omega_1 + \omega_2)} \delta(\omega - \omega_1) \) at \( \omega = \omega_1 \). Conversely, we have to filter-out the signals at \( \omega = \{-\omega_2, -\omega_1, 0, \omega_2\} \) to keep \( (b_2 / 2) e^{i(\omega_1 + \omega_2)} \delta(\omega - \omega_2) \) at \( \omega = \omega_2 \). Therefore we need to design 2-PSAs, one tuned at \( \omega = \omega_1 \) and another one tuned at \( \omega = \omega_2 \).

3. FTF design for 2-PSAs using 5-step time-multiplexed fringe-patterns

Here we use two temporal frequencies namely: \( \omega_3 = 2\pi / 5 \) and \( 2\omega_3 = 4\pi / 5 \). As Fig. 5 shows, using the FTF formalism [2,3,7] we need at least four first-order zeroes. One FTF must be tuned at \( \omega_3 = 2\pi / 5 \) and the other one at \( 2\omega_3 \), these are,

\[
H_1(\omega) = \left(1 - e^{i\omega}\right)\left[1 - e^{i(\omega - \omega_3)}\right]\left[1 - e^{i(\omega - 2\omega_3)}\right];
\]

\[
H_2(\omega - \omega_3) = \left[1 - e^{i(\omega - \omega_3)}\right]\left[1 - e^{i(\omega - 2\omega_3)}\right]\left[1 - e^{i(\omega - 3\omega_3)}\right];
\]

(4)

(5)

Figure 5 shows that \( H_1(\omega) \) is tuned to \( (b_1 / 2)H_3(\omega_3) e^{i(\omega + \omega_3)} \delta(\omega - \omega_3) \), while \( H_2(\omega - \omega_3) \) is tuned to \( (b_2 / 2)H_3(\omega_3) e^{i(\omega + \omega_3)} \delta(\omega - 2\omega_3) \).

Fig. 5. Fringes’ spectrum and FTFs of 5-temporal-multiplexed PSAs tuned at \( \omega_3 \) and \( 2\omega_3 \).

To find the impulse responses of \( H_1(\omega) \) and \( H_2(\omega - \omega_3) \), we take their inverse Fourier transforms to obtain,

\[
F^{-1}[H_1(\omega_3)] = \delta(t) + \delta(t-1)e^{i\omega_3} + \delta(t-2)e^{2i\omega_3} + \delta(t-3)e^{3i\omega_3} + \delta(t-4)e^{4i\omega_3},
\]

\[
F^{-1}[H_2(\omega - \omega_3)] = \delta(t) + \delta(t-1)e^{i\omega_3} + \delta(t-2)e^{2i\omega_3} + \delta(t-3)e^{3i\omega_3} + \delta(t-4)e^{4i\omega_3};
\]

(6)

(7)

With \( \omega_3 = 2\pi / 5 \). Therefore the two temporal-multiplexed PSAs are given by [2]:

\[
\frac{b_1}{2} H_1(\omega_3) e^{i[\theta + \omega_3]} = I(0) + I(1)e^{i\omega_3} + I(2)e^{2i\omega_3} + I(3)e^{3i\omega_3} + I(4)e^{4i\omega_3};
\]

(8)
\[
\frac{b_z}{2} H_s(\omega_z) e^{i [g(h(x,y)) + \omega_z t]} = I(0) + I(1)e^{i \omega_1 t} + I(2)e^{i \omega_2 t} + I(3)e^{i \omega_3 t} + I(4)e^{i \omega_4 t}.
\] (9)

The spatial coordinates \((x, y)\) were omitted for clarity. Please note that we are using the same fringe data \(\{I(0), I(1), \ldots, I(4)\}\) for both 5-step PSAs tuned at \(\omega_5 = 2\pi/5\) and \(2\omega_5\). We then remove their spatial carriers \(\exp(iu_{0,x})\) and finally add them (co-phased) as,

\[
\frac{H_s(\omega_5)}{2} \left\{ b_1 e^{i [g(h(x,y)) + \omega_5 t]} e^{i u_{0,x}} + b_2 e^{i [g(h(x,y)) - \omega_5 t]} e^{-i u_{0,x}} \right\} = \frac{H_s(\omega_5)}{2} \left\{ b_1 + b_2 \right\} e^{i g(h(x,y))}.
\] (10)

We may also notice that both \(H_s(\omega)\) and \(H_s(\omega - \omega_5)\) have the same signal-to-noise (S/N) power-ratio gain \([2,3]\) with respect to the raw fringe data, that is,

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ |H_s(\omega_5)|^2 \right\} d\omega = 5; \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ |H_s(\omega - \omega_5)|^2 \right\} d\omega = 5.
\] (11)

Without temporal-multiplexing, one would have needed 10-phase-shifted fringes, 5 for each projector sequentially turned-on and off, obtaining the same signal-to-noise (S/N) power-ratio gain with respect to the raw-data fringes (Eq. (2)) \([2]\).

4. Experiment with 2 projectors and 2 temporal-frequencies multiplexing

Here we show 2 simultaneous fringe-projections having fully overlapped spatial-frequency spectra (Fig. 3). Figure 6 shows the 5 phase-shifted fringe-patterns obtained with simultaneous projection from the left and right-projectors. The 2 frequencies are \(\omega_5 = 2\pi/5\) and \(2\omega_5 = 4\pi/5\) radians/image. In Fig. 6 the fringes modulated by \(b_1(x,y)\) travel at a phase-velocity of \(\omega_5 = 2\pi/5\) radians/image in the \(x\)-direction, while the fringes modulated by \(b_2(x,y)\) travel at a phase-velocity of \(2\omega_5\) radians/image.

\[I(0)\]
\[I(1)\]
\[I(2)\]
\[I(3)\]
\[I(4)\]

Fig. 6. The five temporal-multiplexed phase-shifted fringes obtained with simultaneous left and right projectors turned-on. The temporal-multiplexed frequencies are \(\omega_5 = 2\pi/5\) and \(2\omega_5 = 4\pi/5\) radians/image. Using our TV-stations analogy, each projector has its own broadcasting temporal-frequency; 2-projectors need 2-temporal-frequencies \(\omega_5\) and \(2\omega_5\). The spatial-carriers \(u_{0,x}\) from the right and left projectors are equal. As seen in the previous section, the 2 PSAs tuned at \(\omega_0 = 2\pi/5\) and \(2\omega_0\) are:

\[
\frac{b_1}{2} H_s(\omega_0) e^{i g(h(x,y))} = e^{-i u_{0,x}} \left[ I(0) + I(1)e^{i \omega_0 t} + I(2)e^{i 2\omega_0 t} + I(3)e^{i 3\omega_0 t} + I(4)e^{i 4\omega_0 t} \right].
\] (12)

\[
\frac{b_2}{2} H_s(\omega_0) e^{-i g(h(x,y))} = e^{-i u_{0,x}} \left[ I(0) + I(1)e^{i \omega_0 t} + I(2)e^{i 2\omega_0 t} + I(3)e^{i 3\omega_0 t} + I(4)e^{i 4\omega_0 t} \right].
\] (13)
These analytic signals have different amplitudes $h_1(x, y)/2$ and $h_2(x, y)/2$ because the 2 projectors cast different shadows and glare over the digitizing solid. Finally we add these 2 signals to have a full co-phased description of the object without shadows and glare:

$$H_a(\omega) \left[ \frac{h_1(x, y)}{2} e^{i \phi h(x, y)} + \frac{h_2(x, y)}{2} e^{-i \phi h(x, y)} \right] = \frac{H_a(\omega)}{2} [h_1(x, y) + h_2(x, y)] e^{i \phi h(x, y)}. \quad (14)$$

The operator $[*]$ stands for the complex conjugate. In co-phased profilometry the region where $h_1(x, y) = 0$ is different from the region where $h_2(x, y) = 0$. In other words, we obtain a well-defined analytic signal $\exp[i \phi h(x, y)]$ whenever $[h_1(x, y) + h_2(x, y)] \gg 0$. Therefore the co-phased sum effectively minimizes the shadow and glare regions where we would be otherwise be unable to recover a well-defined object phase.

![Fig. 7. Wrapped-phases estimated at: (a) $\omega = 2\pi/5$, and at (b) $2\omega = 4\pi/5$. Panel (c) shows their co-phased sum. Panels (a) and (b) show the shadow-regions where the modulated phase is undefined (just phase-noise is obtained). The co-phased object-phase is shown in (c); note that the self-occluding noisy shadows have been eliminated in Panel (c).](image)

We must remark that the wrapped phases in Fig. 7(a)-7(b) are shown just for illustrative purposes. Only the co-phased wrapped sum in Fig. 7(c) is required to obtain the 3D digital rendering shown in Fig. 8.

![Fig. 8. The unwrapped phase shown wrapped in Fig. 7(c), and its 3D digital-rendering of the digitizing solid in Fig.1. The gray-coded phase-values have been used as texture.](image)

5. Seven-step PSAs for 2 projectors and 2 frequencies multiplexed-fringes

Here we still keep 2 projectors and 2 temporal-frequencies. But now we are increasing from 5 to 7 the phase-shifted fringe-pattern number. Higher number of phase-shifted fringes improves the (S/N) and harmonic rejection.

The 7 step phase-shifted, 2 frequencies ($\omega_1, 2\omega_1$), temporal-multiplexed fringes are,

$$I(x, y, t) = \{a + b_1 \cos[g h + u_x t + \omega_1 n] + b_2 \cos[-g h + u_x t + 2\omega_1 n] \} \delta(t - n); \; \omega_1 = \frac{2\pi}{7}, \quad (15)$$
being \( n = \{0,1,\ldots,6\} \). Three out-of-seven temporal-multiplexed phase-shifted fringe-patterns are shown in Fig. 9. The temporal Fourier-spectrum is equal to that shown in Fig. 4 except that in this case \( \omega_1 = \omega_4, \omega_2 = 2\omega_4 \).

As before we need two FTFs, namely \( H_1(\omega) \) and \( H_2(\omega - \omega_4) \). The \( H_1(\omega) \) have spectral zeros at \( \omega = \{-2\omega_4,-\omega_4,0,2\omega_4\} \) to isolate \( (b_1/2)e^{-i(\omega_4+n\omega_4)}\delta(\omega-\omega_4) \). And \( H_2(\omega - \omega_4) \) filter-out the deltas at \( \omega = \{-2\omega_4,-\omega_4,0,\omega_4\} \) to keep the signal \( (b_2/2)e^{-i(-\omega_4+n\omega_4)}\delta(\omega-2\omega_4) \). These two FTFs are given by:

\[
H_1(\omega) = \prod_{n=0}^{5} \left[ 1 - e^{-i(\omega+n\omega_4)} \right],
\]

(16)

\[
H_2(\omega - \omega_4) = \prod_{n=0}^{5} \left[ 1 - e^{-i(\omega_4+n\omega_4)} \right],
\]

(17)

The graphs of \( H_1(\omega) \) and \( H_2(\omega - \omega_4) \) are shown in Fig. 10. The signal-to-noise (S/N) power-ratio gains (with respect to the raw-data fringes) for \( H_1(\omega) \) and \( H_2(\omega - \omega_4) \) are [2,3],

\[
\frac{\int_{-\pi}^{\pi} |H_1(\omega)|^2 d\omega}{\frac{1}{2\pi} \int_{-\pi}^{\pi} |H_1(\omega)|^2 d\omega} = 7;
\]

\[
\frac{\int_{-\pi}^{\pi} |H_2(\omega - \omega_4)|^2 d\omega}{\frac{1}{2\pi} \int_{-\pi}^{\pi} |H_2(\omega - \omega_4)|^2 d\omega} = 7.
\]

(18)

Therefore \( H_1(\omega) \) and \( H_2(\omega - \omega_4) \) have both the same signal-to-noise power-ratio gain [2,3].
Taking the inverse Fourier transforms \( F^{-1}[H_z(\omega)] \) and \( F^{-1}[H_z(\omega - \omega_r)] \) the 7-step impulse responses of the PSAs are obtained as,

\[
F^{-1}[H_z(\omega)] = \sum_{n=0}^{6} \delta(t-n)e^{j\omega_r n}, \\
F^{-1}[H_z(\omega - \omega_r)] = \sum_{n=0}^{6} \delta(t-n)e^{2j\omega_r n}.
\]

(19) \hspace{2cm} (20)

From these impulse responses one obtains the two phase-shifting algorithms (PSAs) as [2],

\[
A_1(x, y)e^{jgk(x,y)} = e^{-j\omega_r x} \left[ \sum_{n=0}^{6} e^{jn\omega_r} I(x, y, n) \right], \\
A_2(x, y)e^{-jgk(x,y)} = e^{-j\omega_r x} \left[ \sum_{n=0}^{6} e^{2jn\omega_r} I(x, y, n) \right].
\]

(21) \hspace{2cm} (22)

Fig. 11. (a) Shows the recovered phase at temporal-frequency \( \omega_r=2\pi/7 \). (b) Shows the wrapped-phase at temporal-frequency \( 2\omega_r \). (c) Shows the wrapped phase of the co-phased sum. In panel (c) the co-phase sum is well-defined over the entire object.

With \( A_1 = (b_1 / 2)H_z(\omega_r) \) and \( A_2 = (b_2 / 2)H_z(\omega_r) \). Finally, the searched co-phased analytic signal is given by the sum:

\[
A(x, y)e^{jgk(x,y)} = A_1(x, y)e^{jgk(x,y)} + \left[ A_2(x, y)e^{-jgk(x,y)} \right] = \left[ A_1(x, y) + A_2(x, y) \right] e^{jgk(x,y)}.
\]

(23)

Note that turning on-and-off each projector sequentially (as in [7]) would require 7 fringe-images per projector to obtain the same \((S/N)\) gain, totalizing 14 fringe-images.

Fig. 12. Panel (a) shows the 7 step co-phased unwrapped-phase. (b) Shows a 3D digital-rendering of the co-phased, temporal-multiplexed reconstruction. In (b) we added the object image as texture and a black circle around the object for easier visualization.
The wrapped-phases for 7 temporal-multiplexed fringes are shown in Fig. 11. Figure 11(c) shows the co-phased sum of the phases at Fig. 11(a) and Fig. 11(b). The unwrapped-phase for 2 projectors and 2 temporal-frequencies multiplexing with 7 phase-steps are shown in Fig. 12.

6. Multiplexing of 4 fringe-projectors with 4 temporal-frequencies

As mentioned, temporal-multiplexing needs as many temporal-frequencies as simultaneous fringe-projectors are used. Therefore each projector must have its own temporal broadcasting frequency. This should not be confused with the extra phase-steps for just 2 projectors and 2 temporal-frequencies seen previously.

Figure 13 shows a schematic of the experimental set-up needed for 4 co-phased fringe-projectors aimed towards the digitizing solid. Our mathematical model for temporal-multiplexed phase-shifted fringe-images using the frequencies \{\omega_0, 2\omega_0, 3\omega_0, 4\omega_0\} is,

\[
I(x, y, t) = a + b_1 \cos [g h(x, y) + u_0 x + \omega_0 t] + b_2 \cos [-g h(x, y) + u_0 x + 2\omega_0 t] + 
\]

\[
b_3 \cos [g h(x, y) + v_0 y + 3\omega_0 t] + b_4 \cos [-g h(x, y) + v_0 y + 4\omega_0 t]; \omega_0 = \frac{2\pi}{9}. \tag{24}
\]

Being \(t = \{0,1,\ldots,8\}\). Each fringe-pattern in Eq. 24 has its own broadcasting frequency \{\omega_0, 2\omega_0, 3\omega_0, 4\omega_0\}. So we need 4 PSAs each one tuned to \{\omega_0, 2\omega_0, 3\omega_0, 4\omega_0\} respectively. Again the spatial-frequencies are equal \(u_0 = v_0\). The fringes modulated by \(b_1(x, y)\) and \(b_2(x, y)\) move in the x-direction at a phase-velocity of \(\omega_0\) and \(2\omega_0\) radians/image respectively. The fringes modulated by \(b_3(x, y)\) and \(b_4(x, y)\) displace in the y-direction at a phase-velocity of \(3\omega_0\) and \(4\omega_0\) radians/image respectively.

![Fig. 13. Schematic of the geometry for 4 fringe-projectors aimed towards the solid located at the origin. The profilometer’ sensitivity angle is \(\theta\) and the spatial-frequencies for the 4 projectors (P_1,P_3,P_2,P_4) are the same. The fringes are projected perpendicular to the projector’s directions.](image)

The 9 temporal-multiplexed phase-shifted fringe-images (Eq. (24)) have the following spectrum (see Fig. 14),
\[ I(x, y, \omega) = a(x, y)\delta(\omega) + \frac{b_1(x, y)}{2} \left[ e^{i(gk_1\omega_0)} \delta(\omega - \omega_0) + e^{-i(gk_1\omega_0)} \delta(\omega + \omega_0) \right] + \]
\[ \frac{b_2(x, y)}{2} \left[ e^{i(-gk_1\omega_0)} \delta(\omega - 2\omega_0) + e^{-i(-gk_1\omega_0)} \delta(\omega + 2\omega_0) \right] + \]
\[ \frac{b_3(x, y)}{2} \left[ e^{i(gk_1\omega_0)} \delta(\omega - 3\omega_0) + e^{-i(gk_1\omega_0)} \delta(\omega + 3\omega_0) \right] + \]
\[ \frac{b_4(x, y)}{2} \left[ e^{i(-gk_1\omega_0)} \delta(\omega - 4\omega_0) + e^{-i(-gk_1\omega_0)} \delta(\omega + 4\omega_0) \right] \quad \omega_0 = \frac{2\pi}{9}. \]

Figure 14 shows graphically the temporal Fourier-spectra composed by 9 Dirac-deltas of the 4 temporal-multiplexed fringe-patterns in Eq. (24).

Figure 15 shows in a sequential way the 4 fringe projections over the spheric metallic object to clearly see the shadows and glare cast by each projector separately.

The simultaneous projected-fringes are shown in Fig. 16 along with its overlapped Fourier spatial-spectra. As Fig. 16(a) shows, 4 fringe-projections better cover the shadows and specular reflections of the object [5,7]. The spatial-spectrum in Fig. 16(b) can only be separated by temporal-multiplexing.
Panel (a) shows one out-of-nine temporal-multiplexed phase-shifted fringes simultaneously illuminated by the 4 projectors in Eq. (24). The shadows and glare are better covered by 4 projectors. Panel (b) shows the overlapped Fourier spatial-spectrum which precludes their spatial-filtering. This spectrum is however well separated by the temporal-frequencies \((\omega_1, 2\omega_1, 3\omega_1, 4\omega_1)\).

We need to isolate the analytic signals at \(\{\delta(\omega - \omega_b), \delta(\omega - 2\omega_b), \delta(\omega - 3\omega_b), \delta(\omega - 4\omega_b)\}\) while filtering-out (deleting) all other spectral components using the following 4 FTFs:

\[
H_s(\omega) = \prod_{n=0}^{2} \left[ 1 - e^{-i(\omega + n\omega_b)} \right], \quad H_s(\omega - \omega_b) = \prod_{n=0}^{3} \left[ 1 - e^{-i(\omega + (n+1)\omega_b)} \right],
\]
\[
H_s(\omega - 2\omega_b) = \prod_{n=0}^{3} \left[ 1 - e^{-i(\omega + (n+2)\omega_b)} \right], \quad H_s(\omega - 3\omega_b) = \prod_{n=0}^{3} \left[ 1 - e^{-i(\omega + (n+3)\omega_b)} \right], \quad \omega_b = \frac{2\pi}{9}.
\]

Figure 17 shows that \(H_s(\omega)\) keeps only the signal at \(\delta(\omega - \omega_b)\); \(H_s(\omega - \omega_b)\) isolates the signal at \(\delta(\omega - 2\omega_b)\); \(H_s(\omega - 2\omega_b)\) keeps only the signal at \(\delta(\omega - 3\omega_b)\), finally \(H_s(\omega - 3\omega_b)\) isolates the analytic signal located at \(\delta(\omega - 4\omega_b)\).

Fig. 17. Minimum 9 steps 4 FTFs for temporal-multiplexing 4 fringe-projectors. Panel (a) shows the standard 9 step least-squares FTF [2]. Panel (b), (c) and (d) show the frequency-shifted FTFs by temporal-frequencies \((2\omega_1, 3\omega_1, 4\omega_1)\) respectively. The vertical arrows represent Dirac deltas.

The \((S/N)\) power-ratio gain of the 4 FTFs: \(H_s(\omega)\), \(H_s(\omega - \omega_b)\), \(H_s(\omega - 2\omega_b)\) and \(H_s(\omega - 3\omega_b)\) equals nine [2,3,7],
\[
\frac{|H_s(\omega_0)|^2}{\frac{1}{2\pi} \int_{-\pi}^{\pi} |H_s(\omega)|^2 \, d\omega} = \frac{|H_s(\omega_1)|^2}{\frac{1}{2\pi} \int_{-\pi}^{\pi} |H_s(\omega-\omega_1)|^2 \, d\omega} = \ldots = \frac{|H_s(\omega_9)|^2}{\frac{1}{2\pi} \int_{-\pi}^{\pi} |H_s(\omega-3\omega_9)|^2 \, d\omega} = 9. \tag{27}
\]

The impulse responses corresponding to these 4-FTFs are,

\[
F^{-1}[H_s(\omega)] = \sum_{n=0}^{8} \delta(t-n)e^{i\omega_0 n}, \quad F^{-1}[H_s(\omega-\omega_0)] = \sum_{n=0}^{8} \delta(t-n)e^{i2\omega_0 n},
\]

\[
F^{-1}[H_s(\omega-2\omega_0)] = \sum_{n=0}^{8} \delta(t-n)e^{i3\omega_0 n}, \quad F^{-1}[H_s(\omega-3\omega_0)] = \sum_{n=0}^{8} \delta(t-n)e^{i4\omega_0 n}. \tag{28}
\]

The spatial-carriers are equal \(v_0 = u_0\). So the 4 searched 9-steps PSAs are given by:

\[
A_i(x, y)e^{igb(x,y)} = e^{-i\omega_0} \left[ \sum_{n=0}^{8} I(x, y, n)e^{i\omega_0 n} \right],
\]

\[
A_2(x, y)e^{-igb(x,y)} = e^{i\omega_0} \left[ \sum_{n=0}^{8} I(x, y, n)e^{2\omega_0 n} \right], \tag{29}
\]

\[
A_3(x, y)e^{igb(x,y)} = e^{-i\omega_0} \left[ \sum_{n=0}^{8} I(x, y, n)e^{3\omega_0 n} \right],
\]

\[
A_4(x, y)e^{-igb(x,y)} = e^{i\omega_0} \left[ \sum_{n=0}^{8} I(x, y, n)e^{4\omega_0 n} \right]; \quad \omega_0 = \frac{2\pi}{9}.
\]

With \( A_1(x, y) = (b_1 / 2)H_1(\omega_0) \), \( A_2(x, y) = (b_2 / 2)H_2(\omega_0) \) and \( A_3(x, y) = (b_3 / 2)H_3(\omega_0) \), and \( A_4(x, y) = (b_4 / 2)H_4(\omega_0) \). Finally the co-phased sum of these 4 analytic signals is:

\[
A(x, y)e^{gi(h(x,y))} = [A_1(x, y) + A_2^*(x, y) + A_3(x, y) + A_4^*(x, y)]e^{gi(h(x,y))}. \tag{30}
\]

In co-phased profilometry, one obtains a well-defined object-phase \(g(h(x,y))\) whenever the magnitude of \(|A_1 + A_2^* + A_3 + A_4^*| > 0\). This means that all the object self-occluding shadows and glare are covered-up by the 4 fringe-projections. Please note that we need at least 9 temporal-multiplexed fringe-patterns to perform this measurement. For comparison, co-phased fringe-projection profilometry with 4 projectors sequentially turned on-and-off would require 9x4=36 fringe-patterns to obtain the same (S/N) power-ratio gain with respect to the raw-data fringes \[2,3\]. Further generalization to higher number of fringe-projectors and/or higher-order PSAs is just a matter of mathematical induction.

7. **Summary**

Here we have shown how to design temporal multiplexed PSAs for co-phased fringe-projection profilometry. We have shown that turning-on all the co-phased projectors and using temporal-multiplexing one may reduce by half or more the number of fringe-images while maintaining the same (S/N) power-ratio gain. Our first experiment used 5 phase-step images with 2 simultaneous fringe-projectors and temporal-frequency \((\omega_0 = 2\pi / 5, 2\omega_0)\). As it is well known, the phase estimation improves by increasing the number of phase-shifted images \[2\]. In particular our next experiment also used 2-projectors and 2-temporal-frequencies \((\omega_0 = 2\pi / 7, 2\omega_0)\); we have increased to 7 fringe-patterns images. With temporal-multiplexing we are not only saving half or more fringe-patterns images, but
simultaneous fringe projection may be achieved much faster than turning on-and-off the fringe projectors sequentially as did before [5-7].

Finally we have generalized temporal-frequency multiplexing to 4 fringe-projectors turned-on simultaneously; each projector having a different illumination direction; each projector having also their own temporal frequency \((\omega_1 = 2\pi / 9, \omega_2, 3\omega_3, 4\omega_4)\). We have seen that we need at least 9 fringe-patterns to separate the 4 co-phased analytic-signals. Again without temporal-multiplexing one would require \(4 \times 9 = 36\) fringe-patterns (9-fringe-images per projector) to obtain the same signal-to-noise (S/N) power-ratio gain.

Acknowledgments

The authors acknowledge Cornell University for hosting arXiv e-print repository, and the Optical Society of America for allowing the contributors to post their manuscript at arXiv.