The canonical transformations of the dynamical multiparameter systems as recurrence relations for the models on the grating

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Abstract

The theory of recurrence relations of linear multi-component and multi-parameter systems on the basis of the canonical transformations theory of the dynamical systems’ sets is constructed. The parameters of the grating’s knots are defined from the condition of the invariance of the model under shifts along the grating. The connection with a zero curvature representation for models on the grating is installed. The examples of two- and three-parameter systems described by the hypergeometric functions $M(\alpha, \beta, t)$ and $M(\alpha, \beta, \xi, t)$ are considered in details. The canonical recurrence relations increasing and decreasing parameters $\{\alpha, \beta, \xi\}$ for solutions of the corresponding equations are constructed.

I. INTRODUCTION

Many equations of the theoretical physics especially of quantum mechanics are linear. In this connection there is a possibility of constructing solutions of these equations searching the canonical transformations of the linear dynamical systems (but this canonical transformations are different from those of quantum theory). These transformations can be formulated in terms of the creation and annihilation operators for some “quantum numbers” of the corresponding quantities that conserve their values. The indicated possibility was considered in [1], where the method of constructing the recurrence relation for searching for the eigenvalues and eigenfunctions of linear operators is proposed. This method is restricted by the case of one-dimensional systems described by the Lagrangian containing the spectral parameters in a potential part only.

In the present paper the generalization of this method for the case of the multidimensional systems and systems with the Lagrangian containing spectral parameters in a kinetic part too (for example, for the radial equation in hydrogen-like atom theory) is constructed.

The physical meaning of the considered canonical transformations for the classical theory differs from that for the quantum theory. The general point is that these transformations

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can be considered as translations on the multidimensional grating in the phase space. In the classical theory the parameters of the grating’s knots define the different configurations of considered class of dynamical systems (for example, the set of mechanical oscillators with different masses and spring rigidities). Conditions of the invariance under the canonical transformations along the grating define the admissible grating parameters and, therefore, solutions and a spectrum of oscillations.

In quantum theory the parameters of the knots give the possible states of the same quantum system. In its turn the translation along the grating is defined by the creation and annihilation operators stipulating the transition from one state to another.

The canonical recurrence transformations obtained in this way are the analogies of the known Beclund’s transformations for finite-dimensional dynamical systems. Note that they do not coincide by their form with the standard recurrence relations cited in handbooks. The difference between them is explained by the way of their construction. The main here is that the canonical recurrence relations conserve the symplectic structure of the corresponding dynamical systems in contrast with the standard ones. That is why from the standpoint of the theory of dynamical systems the canonical recurrence relations have a certain advantage. This relations can be reduced to the standard ones by the corresponding normalization of eigenfunctions.

II. A CANONICAL MAP

Let us consider the sets \( D = \{q, p, H\} \) and \( \tilde{D} = \{\tilde{q}, \tilde{p}, \tilde{H}\} \) of the s-dimensional dynamical systems with the generalized coordinates \( q = q(t) \equiv \{q^1(t), \ldots, q^s(t)\} \), \( \tilde{q} = \tilde{q}(t) \equiv \{\tilde{q}^1(t), \ldots, \tilde{q}^s(t)\} \), momenta \( p = p(t) \equiv \{p_1(t), \ldots, p_s(t)\} \), \( \tilde{p} = \tilde{p}(t) \equiv \{\tilde{p}_1(t), \ldots, \tilde{p}_s(t)\} \) and Hamiltonians \( H = H(p, q, t), \tilde{H} = \tilde{H}(\tilde{p}, \tilde{q}, t) \).

We shall call the mapping \( F : D \to \tilde{D} \) the canonical one, if it is defined by the set \( F = \{F\} \) of generating functions \( F = F(q, \tilde{q}, t) \), so that for an arbitrary system \((q, p, H) \in D\) and for some function \( F \in F \) there is the system \((\tilde{q}, \tilde{p}, \tilde{H}) \in \tilde{D}\), for which the relations

\[
\begin{align*}
p_i &= \frac{\partial F}{\partial q_i}, & \tilde{p}_i &= -\frac{\partial F}{\partial \tilde{q}_i}, & \tilde{H} &= H + \frac{\partial F}{\partial t} \quad (i, j, k, \ldots = 1, 2, \ldots, s),
\end{align*}
\]

are realized.

If \( D = \tilde{D} \), we shall say that given set of dynamical systems \( D \) is invariant under the canonical transformation \( F \).

Let \( D \) be a parametrized class of dynamical systems described by the Lagrangian

\[
L = L(q, \dot{q}, t, \lambda, \mu_a) = \frac{1}{2} \left[ g_{ij} \dot{q}^i \dot{q}^j - (U_{ij} - \lambda g_{ij}) q^i q^j \right],
\]

where \( \dot{q} = dq/dt, g_{ij} = g_{ij}(\mu_a, t), U_{ij} = U_{ij}(\mu_a, t) \) are some functions, \( \det |g_{ij}| \neq 0, \{\lambda, \mu_a\} \in D, D \) is some set of parameters, \( a = 1, \ldots, d \). The ordinary summation convention works here and on.

If we write the Lagrange-Euler equations for the system (2.2) in the form

\[
\dot{P}_k^l q^k \equiv g^{li} \left( -\frac{d}{dt} g_{ik} \frac{d}{dt} + U_{ik} \right) q^k = \lambda q^l
\]

\[
(2.3)
\]
Further we shall write the indices $K$ where we shall make a conclusion that $\lambda$ is a spectral parameter of the eigenvalue problem of the $d$-parametric set of the linear operators $\hat{P}_k^l (\mu_a, t)$. For the Hamiltonians of the set (2.2) we have

$$H(p, q, t, \lambda, \mu_a) = \frac{1}{2} g^{ij} p_i p_j - \frac{1}{2}(U_{ij} - \lambda g_{ij}) q^i q^j, \quad (2.4)$$

where

$$p_i = g_{ij} \dot{q}^j, \quad (g^{ik} g_{kj} = \delta_j^i). \quad (2.5)$$

The condition of the invariance of the set $\mathcal{D}$ of dynamical systems under the canonical transformations $\mathcal{F}$ is

$$\hat{H}(\tilde{p}, \tilde{q}, t, \tilde{\lambda}, \tilde{\mu}_a) = H(\tilde{p}, \tilde{q}, t, \tilde{\lambda}, \tilde{\mu}_a), \quad \forall \{\tilde{\lambda}, \tilde{\mu}_a\} \in \mathcal{D}. \quad (2.6)$$

Under sequential actions of the series of the canonical transformations a sequence of the systems of the type (2.2) and the parameters $\{\lambda_m, \mu_{n_a}|m, n_a = 1, 2, \ldots \} \in \mathcal{D}$ appears. Introduce the collective indices $K = (m, n_a)$, $\bar{K} = (\bar{m}, \bar{n}_a)$ and designate $(\lambda_m, \mu_{n_a}) \equiv \sigma_K$, $(\lambda_{\bar{m}}, \mu_{\bar{n}_a}) \equiv \bar{\sigma}_K$. Let the corresponding coordinates and momenta be $q_K, q_{\bar{K}}$ and $p_K, p_{\bar{K}}$. Further, $F_{KK}(q_K, q_{\bar{K}}, t)$ be the generating function of the canonical transformation $F: D \rightarrow D$, so that $\{q_K, p_K, H(p_K, q_K, t, \sigma_K)\} \rightarrow \{q_{\bar{K}}, p_{\bar{K}}, H(p_{\bar{K}}, q_{\bar{K}}, t, \bar{\sigma}_K)\}$. Then the conditions (2.1), (2.6) give the following equation for the generating function

$$\frac{\partial F_{KK}}{\partial t} + H\left(\frac{\partial F_{KK}}{\partial q_D}, q_D, \sigma_K, t\right) = H\left(\frac{\partial F_{KK}}{\partial q_{\bar{K}}}, q_{\bar{K}}, \bar{\sigma}_K, t\right). \quad (2.7)$$

Further we shall write the indices $K$ and $\bar{K}$ for the generating functions only, and for an arbitrary physical variable $f$ we shall write $f$ or $\tilde{f}$ instead of $f_K$ or $f_{\bar{K}}$ respectively. Then it is possible to rewrite the condition of the invariance (2.7) for the system (2.2) with the Hamiltonian (2.4) in the following form:

$$\frac{2}{\partial t} \frac{\partial F_{KK}}{\partial t} + g^{ij} \frac{\partial F_{KK}}{\partial q^i} \frac{\partial F_{KK}}{\partial q^j} - \tilde{g}^{ij} \frac{\partial F_{KK}}{\partial \tilde{q}^i} \frac{\partial F_{KK}}{\partial \tilde{q}^j} = (U_{ij} - \lambda g_{ij}) q^i q^j - (\tilde{U}_{ij} - \tilde{\lambda} \tilde{g}_{ij}) \tilde{q}^i \tilde{q}^j. \quad (2.8)$$

### III. A RECURRENCE RELATIONS AS A CANONICAL MAP

Let us consider the problem of searching for the solutions of Eq. (2.8). The canonical map, that converts the linear dynamical systems into linear ones, must be linear. That is why the generating function $F_{KK}$ must be a quadratic function of the generalized coordinates $q, \tilde{q}$. Therefore we search for $F_{KK}$ in the form

$$F_{KK} = \frac{1}{2} \left(2 \gamma_{ij} q^i \tilde{q}^j - b_{ij} q^i q^j - c_{ij} \tilde{q}^i \tilde{q}^j\right), \quad (3.1)$$

where $\gamma_{ij}, b_{ij}, c_{ij}$ are some functions. Substituting Eq. (3.1) for Eq. (2.8) and equating the coefficients at $q^i q^j, q^i \tilde{q}^j, \tilde{q}^i \tilde{q}^j$, we obtain
\[ 2\dot{\gamma}_{rs} - g^{ij}(\gamma_{ir}b_{js} + \gamma_{jr}b_{is}) = -\ddot{g}^{ij}(\gamma_{is}c_{jr} + \gamma_{js}b_{ir}), \]  
\[ -\dot{b}_{rs} + g^{ij}b_{ir}b_{js} + \lambda g_{rs} - U_{rs} = \ddot{g}^{ij}\gamma_{ir}\gamma_{js}, \]  
\[ -\dot{c}_{rs} + g^{ij}\gamma_{ir}\gamma_{js} = \ddot{g}^{ij}c_{ir}c_{js} - \ddot{U}_{rs} + \ddot{\lambda}g_{rs}. \]

If we find one of the particular solutions \( \gamma_{ij}, b_{ij}, c_{ij} \), then, using Eqs. (2.1), (2.5), (3.1), it will be easy to construct the binomial recurrence relations for the eigenfunctions \( q(t) \) and \( \tilde{q}(t) \) of the set of operators \( \hat{P}_k^l \)

\[ \left( b_{ij} + g_{ij} \frac{d}{dt} \right) q^{ij} = \gamma_{ij}\tilde{q}^{ij}, \]  
\[ \left( c_{ij} - \ddot{g}_{ij} \frac{d}{dt} \right) \tilde{q}^{ij} = \gamma_{ij}q^{ij}. \]

The obtained relations, increasing and decreasing \( K \), are the analogies of the well-known Beclund’s transformations [3] for finite-dimensional systems. They are also the generalization of the known relations of the factorization method [3], [4]. It means, that Eqs. (3.5), (3.6) install the correlation between the canonical transformation method and factorization one.

The algebraic trinomial recurrence relations can be obtained considering the sum of the generating functions of the two sequential transformations

\[ F_{KK\hat{K}} = F_{K\hat{K}} + F_{\hat{K}K}. \]

According to Eq. (2.1) we have \( \partial F_{KK\hat{K}}/\partial \tilde{q} = 0 \), from which the linear algebraic recurrence relation

\[ \gamma_{ij}\tilde{q}^j = (c_{ij} + \ddot{b}_{ij})\tilde{q}^j + \ddot{\gamma}_{ij}\tilde{q}^j = 0, \]  
follows. Here \( \ddot{b}_{ij}, \ddot{c}_{ij}, \ddot{\gamma}_{ij} \) are the coefficients of a quadratic form of the type (3.1) for the generating function \( F_{KK\hat{K}} \).

The generating function \( F_{\hat{K}K} \) of the canonical transformation \( (q, p) \rightarrow (\tilde{q}, \tilde{p}) \), which is a composition of the canonical transformations \( (q, p) \rightarrow (\tilde{q}, \tilde{p}) \) and \( (\tilde{q}, \tilde{p}) \rightarrow (\tilde{q}, \tilde{p}) \), can be constructed excluding the intermediate state \( \tilde{q} \) from Eq. (3.7) by means of Eq. (3.8). It makes possible to simplify the procedure of searching for the generating function of the transformation \( K \rightarrow \hat{K} \) decomposing it into more simple steps of the calculating the elementary generating functions of the transformations: \( K = \{m, n_1, \ldots\} \rightarrow \{m+1, n_1, \ldots\} \rightarrow \{m+1, n_1+1, \ldots\} \rightarrow \cdots \rightarrow \hat{K} \), and after this to construct their composition.

From the standpoint of classical mechanics we consider here the canonical transformation theory of an infinite-dimensional system with the action

\[ S = \frac{1}{2} \int dt \sum_K \left\{ g^{ij}q^i_K q^j_K + (U_{ij} - \lambda_m g_{ij})q^i_K q^j_K \right\}, \]

describing a multicomponent model on a \( (d+1) \)-dimensional grating. The values of the parameters \( \sigma_K = \{\lambda_m, \mu_{n_1}, \ldots, \mu_{n_d}\} \) that define grating knots are determined from the
condition of an invariance of the dynamical system under grating translations induced by
the canonical transformations of the system.

One more interesting possibility connected with the method of the inverse scattering
problem is shown below by the examples of more simple dynamical systems.

For the particular case of the set of \(d\)-component three-parametrical dynamical systems
\((s = 1)\) let us denote: \(g_{11} \equiv m(\nu, t), U_{11} \equiv m(\nu, t)U(\mu, t)\), where \(\mu, \nu\) are parameters. Write
the Lagrangian \((2.2)\) in the form

\[
L = \frac{m(\nu, t)}{2} \left[ \dot{q}^2 + (U(\mu, t) - \lambda)q^2 \right].
\] (3.9)

It is convenient to find the generating function \(F_{K\tilde{K}}\) in such a form

\[
F_{K\tilde{K}} = mS, \quad S = \frac{1}{2a}(2\gamma q\tilde{q} - bq^2 - cq^2),
\] (3.10)

where \(\gamma\) is some unknown constant, \(a, b, c\) are desired functions of \(t\). The conditions \((2.1)\)
may be rewritten in the form

\[
\dot{q} = \frac{\partial S}{\partial q}, \quad \dot{\tilde{q}} = -\frac{m}{\tilde{m}} \frac{\partial S}{\partial \tilde{q}}, \quad \tilde{H} = H + \frac{\partial}{\partial t}(mS).
\] (3.11)

The recurrence relations

\[
\left( b + a \frac{d}{dt} \right) q = \gamma \tilde{q}, \quad \left( c - a m \frac{d}{m dt} \right) \tilde{q} = \gamma q
\] (3.12)

and equation for the function \(S\)

\[
\frac{2}{m} \frac{\partial}{\partial t}(mS) + \left( \frac{\partial S}{\partial q} \right)^2 - \frac{m}{\tilde{m}} \left( \frac{\partial S}{\partial t} \right)^2 - (\lambda - U)q^2 - \frac{\tilde{m}}{m}(\tilde{\lambda} - \tilde{U})q^2 = 0.
\] (3.13)

follow from Eq.\((2.6)\), \((3.11)\). Eqs. \((3.2)-(3.4)\) for determining \(a, b, c, \gamma\) acquire the form

\[
\dot{a} = \frac{\dot{m}}{m} a + \frac{m}{\tilde{m}} c - b,
\] (3.14)

\[
\frac{m}{\tilde{m}} \dot{c} + \dot{b} = (\lambda - \tilde{\lambda} + \tilde{U} - U)a,
\] (3.15)

\[
a \dot{b} = (\lambda - U)a^2 + \frac{m}{\tilde{m}}(bc - \gamma^2).
\] (3.16)

The system of equations \((3.14)-(3.15)\) is linear with respect to the unknown functions \(a, b, c\)
and does not contain the constant \(\gamma\), while the equation \((3.16)\) is quadratic and contains \(\gamma\).
The procedure of the searching for the solutions of the system \((3.14)-(3.16)\) consists in the
finding a more simple particular solution of the homogeneous linear undetermined system
\((3.14)-(3.13)\) for which the quadratic equation \((3.16)\) is satisfied at the some unknown
parameters \(\{\lambda, \tilde{\mu}, \tilde{\nu}\}\). Then the constant \(\gamma\) will be determined.

5
The algebraic trinomial recurrence relations (without derivatives) and their compositions can be found by the way analogous to obtaining Eq. (3.8) (see [1] also). As a result we shall have
\[
\frac{\gamma_{K+1}}{a_K} q_K - \left( \frac{c_{K+1}+1}{a_K} + \frac{b_{K+1}+1}{a_{K+1}} \right) q_{K+1} + \frac{\gamma_{K+1}+2}{a_{K+1}} q_{K+2} = 0, \tag{3.17}
\]
where \(K\) is the collective index of the totality of the parameters \(\{\lambda_m, \mu_n, \nu_l\}\), i.e. if \(K = \{m, n, l\}\) then \(K+1\) means that one of the indices \(m, n, l\) is increased by one. The coefficients in Eq. (3.17) correspond to that of the generating functions \(S_{K+1}\) and \(S_{K+2}\) of the type (3.10) of the transformations \(q_K \rightarrow q_{K+1}\) and \(q_{K+1} \rightarrow q_{K+2}\).

Writing the similar equations for the sequence of the transformations \(q_{K+1} \rightarrow q_{K+2} \rightarrow q_{K+3}\) and excluding the coordinate \(q_{K+2}\) by means of Eq. (3.17), we obtain the two relations, which can be written in the matrix form
\[
Q_{K+2} = B_K Q_K, \tag{3.18}
\]
where
\[
Q_K = \begin{pmatrix} q_K \\ q_{K+1} \end{pmatrix}, \quad Q_{K+2} = \begin{pmatrix} q_{K+2} \\ q_{K+3} \end{pmatrix}, \tag{3.19}
\]
and \(B_K\) is a \(2 \times 2\) matrix, which is expressed through the coefficients of the quadratic forms \(\{S_{K+1}, \ldots, S_{K+2, K+3}\}\) of the above sequence of the transformations; its explicit form is not indicated here because of its inconvenience.

From the other hand the differential equation (3.12) can be rewritten in these terms as
\[
\frac{dQ_K}{dt} = A_K Q_K, \tag{3.20}
\]
where
\[
A_K = \frac{1}{a_K m_{K+1}} \begin{pmatrix} -m_{K+1} b_{K,K} & m_{K+1} \gamma_{K,K+1} \\ -m_K \gamma_{K,K+1} & m_K c_{K+1,K+1} \end{pmatrix}. \tag{3.21}
\]

The conditions of compatibility of Eqs. (3.18)–(3.19)
\[
\frac{dB_K}{dt} + B_K A_K - A_{K+2} B_K = 0 \tag{3.22}
\]
are equations for the coefficients \(a_K, b_K, c_K, \gamma_K\) of the sequence of the generating functions \(S_{K,K+1}\) and represent zero curvature conditions for the models on the grating [2]. They are the conditions of an applicability of the inverse scattering problem method for integrable models. In our case they are realized as a consequence of the equations of the type (3.14)–(3.16).

Thus the regular way of the constructing zero curvature conditions, which can be generalized for nonlinear dynamical integrable systems, follows from our analysis.
IV. THE CONFLUENT HYPERGEOMETRIC EQUATION

Many equations of quantum mechanics under the separation of variables and after isolation of an angular part lead to the second order ordinary differential equations, which can be reduced to the hypergeometric (or confluent hypergeometric) equation. In this connection it is important to consider the canonical transformation theory of the equations of such a type in general case.

Let us consider construction of the canonical recurrence relations for the confluent hypergeometric equation

\[ t\ddot{q} + (\beta - t)\dot{q} - \alpha q = 0, \quad (4.1) \]

where \( \alpha, \beta \) are some parameters. It is the Lagrange-Euler equation for a dynamical system with the Lagrangian

\[ L = \frac{1}{2} t^\beta e^{-t}(\dot{q}^2 + \frac{\alpha}{t} q^2). \quad (4.2) \]

Comparing Eq.(4.2) with Eq.(3.9), we can conclude that

\[ \lambda = 0, \quad \nu = \beta, \quad \mu = \alpha, \quad U(\alpha, t) = \frac{\alpha}{t}, \quad m(\beta, t) = t^\beta e^{-t}. \]

Therefore the system (3.14)–(3.16) has the form

\[ \dot{a} = \frac{m}{\tilde{m}} a + \frac{m}{\tilde{m}} c - b, \quad \frac{m}{\tilde{m}} \dot{c} + \dot{b} = \frac{\tilde{\alpha} - \alpha}{t} a \quad (4.3) \]

\[ \dot{ab} = -\frac{\alpha}{t} a^2 + \frac{m}{\tilde{m}} (bc - \gamma^2) \quad (4.4) \]

According to the above-mentioned, we shall find the generating functions of the elementary canonical transformations a) \( \{ \alpha, \beta \} \to \{ \tilde{\alpha}, \beta \} \), b) \( \{ \alpha, \beta \} \to \{ \alpha, \tilde{\beta} \} \) only. Besides note that Eqs.(4.3)–(4.4) can be written as equations with rational on \( t \) coefficients. Therefore it is naturally to find the solution in a class of rational functions.

A. A canonical transformation \( \{ \alpha, \beta \} \to \{ \tilde{\alpha}, \beta \} \)

In this case \( \beta = \tilde{\beta}, \ m = \tilde{m} \) and the system (4.3)–(4.4) acquires the form

\[ \dot{a} = (\frac{\beta}{t} - 1)a + c - b, \quad \dot{c} + \dot{b} = \frac{\tilde{\alpha} - \alpha}{t} a, \quad (4.5) \]

\[ \dot{ab} = bc - \gamma^2 - a^2 \frac{\alpha}{t}. \quad (4.6) \]

We shall obtain the simplest rational solution of the undetermined system (4.3) by assumption \( a = t \). From (4.3) we shall find
where \( b_0 \) is the integration constant. Substituting Eq.(4.7) for Eq.(4.6) and equating the coefficients at the same powers of \( t \), one obtains

\[
b_0 = \alpha, \quad \tilde{\alpha} = \alpha + 1, \quad \gamma = \sqrt{\alpha(\alpha - \beta + 1)}
\]  

(4.8)

We have restricted ourselves here by the positive value of the radical. As a result we have

\[
F_{\alpha,\beta|\alpha+1,\beta} = \frac{1}{2} t^{\beta-1} e^{-t} \left( 2\sqrt{\alpha(\alpha - \beta + 1)} \tilde{q} \alpha q^2 - (t + \alpha - \beta + 1) \tilde{q}^2 \right)
\]  

(4.9)

for the generating function (3.10). The canonical recurrence transformations (3.12) have the form

\[
\left( \alpha + t \frac{d}{dt} \right) q = \sqrt{\alpha(\alpha - \beta + 1)} \tilde{q}, \quad \left( t + \alpha - \beta + 1 - t \frac{d}{dt} \right) \tilde{q} = \sqrt{\alpha(\alpha - \beta + 1)} q.
\]  

(4.10)

To determine the connection with the standard recurrence relations and thus with the confluent hypergeometric function \( M(\alpha, \beta, t) \), we shall make the following substitutions

\[
q = y(\alpha) \sqrt{N(\alpha)}, \quad \tilde{q} = y(\alpha) \sqrt{N(\alpha + 1)}, \quad y(\alpha) \equiv M(\alpha, \beta, t)
\]  

(4.11)

in Eq.(4.10). From the condition of coincidence of the obtained recurrence relations with the standard ones [4] (rather with their consequence)

\[
\left( \alpha + t \frac{d}{dt} \right) y(\alpha) = \alpha y(\alpha + 1), \quad \left( \alpha - \beta + t - 1 - t \frac{d}{dt} \right) y(\alpha + 1) = (\alpha - \beta + 1) y(\alpha)
\]  

(4.12)

the functional equation for the normalization multiplier \( N(\alpha) \)

\[
\alpha N(\alpha) = (\alpha - \beta + 1) N(\alpha + 1)
\]  

(4.13)

follows. It can be solved by using the gamma-function’s property \( \Gamma(z + 1) = z\Gamma(z) \) from where the substitutions

\[
\alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)}, \quad \alpha - \beta + 1 = \frac{\Gamma(\alpha - \beta + 2)}{\Gamma(\alpha - \beta + 1)}.
\]  

(4.14)

follow. Using these substitutions, equation (4.13) can be reduced to the form

\[
\frac{\Gamma(\alpha - \beta + 1)}{\Gamma(\alpha)} N(\alpha) = \frac{\Gamma(\alpha - \beta + 2)}{\Gamma(\alpha + 1)} N(\alpha + 1) = \frac{\Gamma(\alpha - \beta + 3)}{\Gamma(\alpha + 2)} N(\alpha + 2) = \cdots.
\]  

(4.15)

The additional equations arise owing to considering the next elementary steps by the parameter \( \alpha \). From here it is easy to see that \( \Gamma(\alpha - \beta + 1) N(\alpha)/\Gamma(\alpha) = C_1 = \text{const} \). Therefore

\[
N(\alpha) = C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta + 1)}.
\]

Thus, we have obtained the “canonical” solution of the confluent hypergeometrical equation in the form

\[
q(t) = C_1 M(\alpha, \beta, t) \sqrt{\frac{\Gamma(\alpha)}{\Gamma(\alpha - \beta + 1)}}.
\]  

(4.16)
B. A canonical transformation \{α, β\} → \{α, \tilde{β}\}

In this case α = \tilde{α} and the system (4.3)–(4.4) can be rewritten in the form

\[ t^{\tilde{β}−\tilde{β}}c - b = \dot{α} + (1 - \frac{β}{t})a, \quad t^{\tilde{β}−\tilde{β}}\dot{c} + \dot{b} = 0, \]  

(4.17)

\[ a\dot{b} + \frac{α}{t}a^2 + t^{β−\tilde{β}}(γ^2 - bc) = 0. \]  

(4.18)

We shall obtain the simplest rational solution of the inhomogeneous undetermined system (4.17) by taking \(a = 1\). Then a particular solution of the obtained nonhomogeneous system has the form

\[ c = \frac{β}{β − β + 2} t^{β−β−1}, \quad b = \frac{\tilde{β} − β − 1}{β − β + 2} \frac{β}{t}. \]  

(4.19)

From the quadratic equation (4.18) we find

\[ \tilde{β} = β + 1, \quad γ = \sqrt{β − α}. \]  

(4.20)

For the generating function and canonical recurrence transformations we have

\[ F_{α, β | α+1, β+1} = \frac{1}{2} t^{β} e^{-t} \left( 2\sqrt{β − α} q\tilde{q} + q^2 + β\tilde{q}^2 \right), \]  

(4.21)

\[ \left( -1 + \frac{d}{dt} \right) q = \sqrt{β − α} \tilde{q}, \quad - \left( β + t \frac{d}{dt} \right) \tilde{q} = \sqrt{β − α} q. \]  

(4.22)

The recurrence relations increasing and decreasing \(β\) that follow from the recurrence relations of the handbook [4]

\[ β \left( -1 + \frac{d}{dt} \right) y(β) = (β − α)y(β + 1), \quad - \left( β + t \frac{d}{dt} \right) y(β + 1) = βy(β), \]  

(4.23)

where \(y(β) \equiv M(α, β, t)\), can be obtained from Eq. (4.22) by the above procedure of the constructing of Eq. (4.12)–(4.16). As a result one has

\[ q(t) = C_2 \frac{Γ(β)}{\sqrt{Γ(β − α) M(α, β, t)}}. \]  

(4.24)
V. THE HYPERGEOMETRIC EQUATION

As it was mentioned, some “radial equations” of the quantum theory are reduced to the hypergeometric-like equations. Therefore consider a dynamical system described by the hypergeometrical equation

\[ t(1-t)\ddot{q} - ((\alpha + \beta + 1)t - \zeta)\dot{q} - \alpha\beta q = 0, \quad (5.1) \]

where \(\alpha, \beta, \zeta\) are some parameters, as another application of the developed theory. It is the Lagrange-Euler equation of a dynamical system with the Lagrangian

\[ L = \frac{1}{2} t^\zeta(1-t)^{\alpha+\beta-\zeta+1} \left( \dot{q}^2 + \frac{\alpha\beta}{t(1-t)} q^2 \right). \quad (5.2) \]

Comparing Eq.(5.2) with Eqs.(2.2), (3.9) we have:

\[ \lambda = 0, \quad \mu_1 = \alpha, \quad \mu_2 = \beta, \quad \mu_3 = \zeta, \]

\[ U = \alpha\beta t(1-t)^{\alpha+\beta-\zeta+1}. \quad (5.3) \]

The system (5.2) contains the three parameters \(\alpha, \beta, \zeta\). Therefore it is necessary to find the generating functions of three canonical transformations. Due to the symmetry between the parameters \(\alpha\) and \(\beta\), it is sufficient to search for the generating functions of the transformations a) \(\{\alpha, \beta, \zeta\} \rightarrow \{\tilde{\alpha}, \beta, \zeta\}\) and b) \(\{\alpha, \beta, \zeta\} \rightarrow \{\alpha, \beta, \tilde{\zeta}\}\) only.

A. A canonical transformation \(\{\alpha, \beta, \zeta\} \rightarrow \{\tilde{\alpha}, \beta, \zeta\}\)

For the elementary transformation \(\{\alpha, \beta, \zeta\} \rightarrow \{\tilde{\alpha}, \beta, \zeta\}\) the system (3.14)–(3.16) can be rewritten the form

\[ (1-t)^{\alpha-\tilde{\alpha}} c - b = \dot{a} - \left( \frac{\zeta}{t} - \frac{\alpha + \beta - \zeta + 1}{1-t} \right) a, \quad (5.4) \]

\[ (1-t)^{\alpha-\tilde{\alpha}} \dot{c} + \dot{b} = \frac{\tilde{\alpha} - \alpha}{t(1-t)} \beta a, \quad (5.5) \]

\[ \alpha \dot{b} = -\frac{\alpha\beta}{t(1-t)} a^2 + (1-t)^{\alpha-\tilde{\alpha}} (bc - \gamma^2). \quad (5.6) \]

Similarly to the first case of the previous example we obtain the simplest rational solution of the homogeneous undetermined system (5.4), (5.5) at \(a = t\). Then the obtained inhomogeneous system has the following particular solution

\[ b = \alpha - \frac{\tilde{\alpha} - \alpha - 1}{\tilde{\alpha} - \alpha - 2} \frac{\alpha + \beta - \zeta + 1}{1-t}, \quad c = -\left( \beta + \frac{\alpha + \beta - \zeta + 1}{\tilde{\alpha} - \alpha - 2} \frac{1}{1-t} \right)(1-t)^{\tilde{\alpha}-\alpha}. \quad (5.7) \]

In this case the quadratic equation (5.6) is satisfied at
\[ \alpha = \tilde{\alpha} + 1, \quad \gamma = \sqrt{\alpha(\alpha - \zeta + 1)}. \]  

(5.8)

So for the generating function we have

\[
F_{\alpha, \beta, \zeta|\alpha + 1, \beta, \zeta} = \frac{1}{2} t^{\zeta-1}(1-t)^{\alpha+\beta-\zeta+1} \left[ 2\sqrt{\alpha(\alpha - \zeta + 1)q\tilde{q} - \alpha q^2 - (\alpha - \zeta + 1 + \beta t)\tilde{q}^2} \right].
\]  

(5.9)

The corresponding recurrence transformations have the form

\[
\left( \alpha + t \frac{d}{dt} \right) q = \sqrt{\alpha(\alpha - \zeta + 1)}\tilde{q},
\]

\[
\left( \alpha - \zeta + 1 - \beta t - t(1-t) \frac{d}{dt} \right) \tilde{q} = \sqrt{\alpha(\alpha - \zeta + 1)} q.
\]  

(5.10)

The standard recurrence transformation [4]

\[
\left( \alpha + t \frac{d}{dt} \right) y(\alpha) = \alpha y(\alpha + 1),
\]

\[
\left( \alpha - \zeta + 1 + \beta t - t(1-t) \frac{d}{dt} \right) \tilde{q} = \sqrt{\alpha(\alpha - \zeta + 1)} q,
\]  

(5.11)

where \( y(\alpha) \equiv M(\alpha, \beta, \zeta, t) \) — hypergeometric function, can be obtained by making such the substitution

\[
q(t) = A_1 \sqrt{\frac{\Gamma(\alpha)}{\Gamma(\alpha - \zeta + 1)}} M(\alpha, \beta, \zeta, t)
\]  

(5.12)

in Eq.(5.10).

**B. A canonical transformation** \( \{\alpha, \beta, \zeta\} \rightarrow \{\alpha, \beta, \tilde{\zeta}\} \)

In this case the system (3.14)–(3.16) acquires the form

\[
t^{\zeta-\tilde{\zeta}}(1-t)^{\tilde{\zeta}-\zeta} c - b = \dot{a} - \left( \frac{\zeta}{t} - \frac{\alpha + \beta - \zeta + 1}{1-t} \right) a,
\]

\[
t^{\zeta-\tilde{\zeta}}(1-t)^{\tilde{\zeta}-\zeta} \dot{c} + \dot{b} = 0,
\]  

(5.13)

\[
\dot{a} \dot{b} = -\frac{\alpha \beta}{t(1-t)} a^2 + t^{\zeta-\tilde{\zeta}}(1-t)^{\tilde{\zeta}-\zeta}(b - \gamma^2).
\]  

(5.14)

We obtain the simplest solution of the system (5.13) at \( a = 1-t \). Then a particular solution (5.13) acquires the form

\[
b = \frac{\zeta}{\zeta - \zeta - 2} \frac{1-t}{t} + \frac{\zeta}{t} - \alpha - \beta, \quad c = \frac{\zeta}{\zeta - \zeta - 2} t^{\zeta-\zeta-1}(1-t)^{\tilde{\zeta}-\zeta+1}.
\]  

(5.15)
The quadratic equation (5.14) leads to the following constraints
\[ \tilde{\zeta} = \zeta + 1, \quad \gamma = \sqrt{(\zeta - \alpha)(\beta - \zeta)}. \] (5.16)

Therefore for the generating function and the recurrence relations we find
\[ F_{\alpha, \beta, \zeta \mid \alpha, \beta, \zeta + 1} = \frac{1}{2} t^\zeta (1 - t)^{\alpha + \beta - \zeta} \left( 2 \sqrt{(\zeta - \alpha)(\beta - \zeta)} q \tilde{q} + (\alpha + \beta - \zeta) q^2 + \zeta \tilde{q}^2 \right), \] (5.17)

\[ \left( \zeta - \alpha - \beta + (1 - t) \frac{d}{dt} \right) q = \sqrt{(\zeta - \alpha)(\beta - \zeta)} \tilde{q}, \]
\[ - \left( \zeta + t \frac{d}{dt} \right) \tilde{q} = \sqrt{(\zeta - \alpha)(\beta - \zeta)} q. \] (5.18)

Making the substitution
\[ q(t) = A_2 \sqrt{\frac{\Gamma(\zeta - \alpha) \Gamma(\zeta - \beta)}{\Gamma(\zeta)}} M(\alpha, \beta, \zeta, t) \] (5.19)
we reduce Eq.(5.18) to the relations which follow from the standard recurrence formulae [4] for the hypergeometric function \( M(\alpha, \beta, \zeta, t) \)
\[ \left( \zeta - \alpha - \beta + (1 - t) \frac{d}{dt} \right) y(\zeta) = \frac{(\zeta - \alpha)(\zeta - \beta)}{\zeta} y(\zeta + 1), \]
\[ \left( \zeta + t \frac{d}{dt} \right) y(\zeta + 1) = \zeta y(\zeta), \] (5.20)
where \( y(\zeta) \equiv M(\alpha, \beta, \zeta, t). \)
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