CONSUMPTION-PORTFOLIO OPTIMIZATION AND FILTERING IN A HIDDEN MARKOV-MODULATED ASSET PRICE MODEL

YANG SHEN
Department of Mathematics and Statistics, York University
Toronto, Ontario M3J 1P3, Canada

TAK KUEN SIU
Department of Applied Finance and Actuarial Studies
Faculty of Business and Economics, Macquarie University
Sydney, NSW 2109, Australia

(Communicated by Alejandro Jofre)

ABSTRACT. We study a consumption-portfolio optimization problem in a hidden Markov-modulated asset price model with multiple risky assets, where model uncertainty is present. Under this modeling framework, the appreciation rates of risky shares are modulated by a continuous-time, finite-state hidden Markov chain whose states represent different modes of the model. We consider the situation where an economic agent only has access to information about the price processes of risky shares and aims to maximize the expected, discounted utility from intermediate consumption and terminal wealth within a finite horizon. The standard innovations approach in filtering theory is then used to transform the partially observed consumption-portfolio optimization problem to the one with complete observations. Robust filters of the chain and estimates of some other parameters are presented. Using the stochastic maximum principle, we derive a closed-form solution of an optimal consumption-portfolio strategy in the case of a power utility.

1. Introduction. Portfolio optimization is an important research topic in finance. Markowitz [16] pioneered the use of quantitative methods for an optimal portfolio selection problem and developed the celebrated mean-variance approach for portfolio optimization. He considered a single-period model and provided an elegant mathematical formulation of the portfolio selection problem, where the problem was reduced to a mean-variance optimization problem under the normality assumption for the return rates of individual securities. Merton [17, 18] considered the portfolio optimization problem in a continuous-time framework. Under the assumption that the dynamics of a risky asset follow a Geometric Brownian Motion (GBM) and some specific forms of utility functions, Merton derived a closed-form solution of an optimal portfolio strategy in a continuous-time setting. Although Merton’s approach to portfolio optimization problem produces beautiful theoretical results, there are some shortcomings from the practical perspective. For example, the GBM assumption is not consistent with many important ‘stylized’ features of assets’ returns, such as the asymmetry and heavy-tailedness of the distribution of returns,
time-varying conditional volatility, regime switches and others. However, this seems to be far from the reality, especially if one wishes to consider investment problems under economic uncertainties over a long time period, where structural changes in macroeconomic conditions may occur several times and cause fundamental changes in investment opportunity sets. Consequently, it may pay us dividends to consider a portfolio optimization problem in a more flexible model which can incorporate empirical features of the returns of risky assets and describe the stochastic evolution of investment opportunity sets.

Markovian regime-switching models have received a considerable attention from both academic researchers and market practitioners in economics and finance. Under regime-switching models, some model parameters or coefficients can change over time according to different states of a market or an economy which are described by the values of a state process, say, an underlying Markov chain. In econometrics and statistics, some early contributions on regime-switching models may be attributed to the works of Quandt [21], Goldfeld and Quandt [9], Tong [31, 32], and Hamilton [11]. There is quite a large literature on portfolio optimization in regime-switching models. Some of them include Zhou and Yin [37], Yin and Zhou [33], Sotomayor and Cadenillas [30], Zhang et al. [35], Yiu et al. [34], Shen and Siu [25] and Zhang et al. [36], to name a few.

It is worth pointing out that the works aforementioned on portfolio optimization in regime-switching models mostly assumed that the underlying Markov chain is observable. However, in reality, the underlying state of the economy, or the Markov chain, is not directly observable. The model, where the underlying Markov chain is unobservable, is usually called the hidden Markov model (HMM). Some works on portfolio optimization problem under the HMM include Rieder and Bäuerle [22], Sass and Haussmann [23], Putschöggl and Sass [20], Elliott et al. [6], and Elliott and Siu [7], Korn et al. [15], Siu [26, 27, 28, 29], among others. Indeed, the unobservability of the states introduces model uncertainty, which is an important issue that needs to be addressed in a portfolio optimization problem. The importance of model uncertainty was highlighted in many fields including economics, finance, insurance, statistics and engineering. There are two major approaches to model uncertainty, namely, the Bayesian approach and the robust approach. The Bayesian approach supposes that the “true” state of the underlying model is represented by a set of random quantities. The robust approach assumes that the modeler surrounds an approximating model by introducing a family of “neighborhood” models via perturbations. Schied et al. [24] gave a good survey on the theory of robust preferences and robust portfolio selection. Particularly, Schied et al. [24] considered a dynamic version of model uncertainty, where the uncertain drift of the risky asset price process was assumed to be stochastic and time-dependent. For a systematic account of robustness and model uncertainty in economic modeling, one may also refer to Hansen and Sargent [12].

In this paper, we investigate a consumption-portfolio optimization problem of an economic agent in a hidden Markov-modulated asset price model in presence of model uncertainty, where the “true” state or mode of the model evolves over time according to a continuous-time, finite-state, hidden Markov chain. This is called the HMM approach in Elliott and Siu [7], which may be thought of as a Bayesian approach to model uncertainty. We consider a financial market consisting of one risk-free bond and \( n \) risky shares. The price dynamics of the risky shares are modeled as a Markovian regime-switching, multi-dimensional, GBM. More specifically,
the appreciation rates of the risky shares are modulated by a hidden Markov chain and the random shocks of the shares are modeled by an \( n \)-dimensional standard Brownian motion. Consequently, the modeling framework considered here allows the flexibility that the price processes of the risky shares may be correlated. Besides investing in the bond and the shares, the agent can consume part of his/her wealth over time. We consider the situation where the agent aims to maximize the expected, discounted utility from intermediate consumption and terminal wealth only based on observations of the prices of the risky shares. As in the literature using the standard innovations approach in filtering theory, the problem with partial observations is transformed to the one with complete observations first. Then robust filters of the chain and estimates of some other parameters of the hidden Markov model with multiple correlated observations and Gaussian noises are presented based on the standard reference probability approach in filtering. Finally the stochastic maximum principle is applied to derive a closed-form solution to the problem in the case of a power utility.

The rest of this paper is organized as follows. The next section presents the model dynamics. Section 3 formulates the consumption-portfolio optimization problem under the hidden Markov model and discusses the standard innovations approach. In Section 4, robust filters and estimates are presented. The closed-form solution of the optimal consumption-portfolio strategy is derived in Section 5. The final section summarizes the paper. Some standard proofs are relegated in an Appendix.

2. The model dynamics. First of all, we present the basic notation to be used throughout the paper.

- \( \mathbb{R} \): the set of real numbers;
- \( C^\top \): the transpose of any vector or matrix \( C \);
- \( \text{tr}(C) \): the trace of a square matrix \( C \);
- \( \langle C, D \rangle \): the inner product of \( C \) and \( D \), that is \( \langle C, D \rangle := \text{tr}(C^\top D) \);
- \( ||C|| \): the Euclidean norm of \( C \), that is \( ||C||^2 = \langle C, C \rangle \);
- \( \text{diag}(C) \): the diagonal matrix with the elements of a vector \( C \) on the diagonal;
- \( 1_n \): the \( n \)-dimensional vector, where all entries are equal to 1, i.e., \( 1_n := (1, 1, \cdots, 1)^\top \in \mathbb{R}^n \).

To avoid confusion, only the vectors or matrices associated with different states of the hidden Markov chain will be denoted in boldfaced letters unless otherwise stated.

We follow Elliott and Siu [7] to introduce the HMM approach for model uncertainty in a continuous-time setting. Consider a complete probability space \( (\Omega, \mathcal{F}, \mathcal{P}) \), where \( \mathcal{P} \) is a real-world probability measure. Let \( \mathcal{T} := [0, T] \) be a finite-time horizon, where \( T < \infty \). To describe the transitions of the “true” state, or mode, of the model over time, we consider a continuous-time, finite-state, hidden Markov chain \( X := \{X(t) | t \in \mathcal{T} \} \) on \( (\Omega, \mathcal{F}, \mathcal{P}) \) taking values in a finite-state space \( \mathcal{S} := \{s_1, s_2, \cdots, s_N \} \). The states of the chain \( X \) represent different hidden states of the model. In practice, the “true” state of the model in force at a particular time is not observable and will have to be estimated by some observable market and economic information. In our case, the observable information is the price processes of the risky shares.

Without loss of generality, as in Elliott et al. [5], we identify the states of the chain with the set of standard unit vectors \( \mathcal{E} := \{e_1, e_2, \cdots, e_N \} \subset \mathbb{R}^N \), where the \( j^{th} \) component of \( e_i \) is the Kronecker delta \( \delta_{ij} \) for each \( i, j = 1, 2, \cdots, N \). This is called
the canonical state space representation of the chain. Let $\mathbf{A} := [a_{ij}]_{i,j=1,2,\ldots,N}$ be the rate matrix of the chain $\mathbf{X}$ under $\mathcal{F}$, where $a_{ij}$ is a constant transition intensity of the chain $\mathbf{X}$ from state ‘$e_j$’ to state ‘$e_i$’. Note that for each $i,j$, $a_{ij} \geq 0$ and $\sum_{i=1}^{N} a_{ij} = 0$, so $a_{ii} \leq 0$. We further assume $a_{ij} > 0$, so $a_{ii} < 0$. Then under the canonical representation of the state space $\mathcal{F}$, Elliott et al. [5] obtained the following semimartingale dynamics for the chain:

$$X(t) = X(0) + \int_0^t \mathbf{A}X(s)ds + \mathbf{M}(t), \quad t \in \mathcal{F},$$

(1)

where $\{\mathbf{M}(t) | t \in \mathcal{F}\}$ is an $\mathbb{R}^N$-valued, $(\mathbb{F}^\mathbf{X}, \mathcal{F})$-martingale. Here $\mathbb{F}^\mathbf{X} := \{\mathbb{F}^\mathbf{X}(t) | t \in \mathcal{F}\}$ is the right-continuous, $\mathcal{F}$-complete, natural filtration generated by the chain $\mathbf{X}$. The representation (1) describes how the hidden “true” state of the model evolves over time.

The financial market consists of one risk-free bond $B := \{B(t) | t \in \mathcal{F}\}$ and $n$ risky shares $S_k := \{S_k(t) | t \in \mathcal{F}\}$, $k = 1,2,\ldots,n$. Let $r$ be a constant, continuously compounded risk-free interest rate such that $r > 0$. Then the evolution of the price process of the bond $B$ over time is given by

$$dB(t) = rB(t)dt, \quad t \in \mathcal{F}, \quad B(0) = 1.$$  

(2)

Let $\mu_k := \{\mu_k(t) | t \in \mathcal{F}\}$ be the process describing the evolution of the appreciation rate of the share $S_k$ over time, for each $k = 1,2,\ldots,n$. Suppose that the chain $\mathbf{X}$ determines $\mu_k(t)$ as

$$\mu_k(t) := \langle \mu_k, X(t) \rangle, \quad t \in \mathcal{F}.$$

(3)

Here $\mu_k := (\mu_k^1, \mu_k^2, \cdots, \mu_k^n)^\top \in \mathbb{R}^N$ with $\mu_k^i > r$, for each $i = 1,2,\cdots,N$ and $k = 1,2,\cdots,n$; $\mu_k^i$ is the appreciation rate of the share $S_k$ corresponding to the $i$-th state of the model. The inner product $\langle \cdot, \cdot \rangle$ selects which component of the vector of the appreciation rate $\mu_k$ is in force at a particular time $t$ based on the hidden state of the model at this time. That is, $\langle \mu_k(t) = \mu_k^i \rangle$ if and only if $X(t) = e_i$, for each $k = 1,2,\cdots,n$ and $i = 1,2,\cdots,N$.

Let $W := \{W(t) | t \in \mathcal{F}\} = \{(W_1(t), W_2(t), \cdots, W_n(t))^\top | t \in \mathcal{F}\}$ be an $n$-dimensional standard Brownian motion on $(\Omega, \mathcal{F}, \mathcal{P})$ with respect to its right-continuous, $\mathcal{P}$-complete, natural filtration. To simplify our discussion, we assume that the Brownian motion $W$ and the Markov chain $\mathbf{X}$ are stochastically independent under $\mathcal{F}$ throughout this paper.

Suppose that $\beta_{kd}$ is the volatility rate of the share $S_k$ corresponding to the random shock described by the Brownian motion $W_d$, for each $k = 1,2,\cdots,n$ and $d = 1,2,\cdots,n$. We assume that $\beta_{kd}$ is a positive constant instead of assuming that it is modulated by the chain. There are some technical difficulties in filtering when $\beta_{kd}$ is modulated by the chain. Furthermore, it is rather uneasy to interpret the information structure when a hidden-Markov modulated volatility is considered. For details, one may refer to, for example, Guo [10]. Then we assume that the price processes of the risky shares are governed by a hidden Markov-modulated multi-dimensional geometric Brownian motion as follows:

$$dS_k(t) = S_k(t)\left(\mu_k(t)dt + \sum_{d=1}^{n} \beta_{kd}dW_d(t)\right), \quad t \in \mathcal{F}, \quad S_k(0) = s_k > 0,$$

(4)

or equivalently,

$$dS_k(t) = S_k(t)(\mu_k(t)dt + \beta_kdW(t)), \quad t \in \mathcal{F}, \quad S_k(0) = s_k > 0,$$

(5)
where $\beta_k := (\beta_{k1}, \beta_{k2}, \cdots, \beta_{kn}) \in \mathbb{R}^n$, for each $k = 1, 2, \cdots, n$.

Write $S(t) := (S_1(t), S_2(t), \cdots, S_n(t))^\top \in \mathbb{R}^n$, for each $t \in \mathcal{T}$. We can rewrite Eq. (5) in the vector form:

$$dS(t) = \text{diag}(S(t))(\mu(t)dt + \beta dW(t)) , \quad t \in \mathcal{T} , \quad S(0) = s ,$$

where $\mu(t) := (\mu_1(t), \mu_2(t), \cdots, \mu_n(t))^\top \in \mathbb{R}^n$, $\beta := [\beta_{k1}]_{k,d=1,2,\cdots,n} \in \mathbb{R}^{n \times n}$ and $s := (s_1, s_2, \cdots, s_n)^\top \in \mathbb{R}^n$. Since this paper focuses on the consumption-portfolio optimization in a hidden Markov model, we only consider the complete market case. So we assume that the number of risky shares and the number of randomness are the same; otherwise, the market is incomplete. This will highlight how to use filtering techniques to deal with model uncertainty rather than how to tackle the incomplete market.

Throughout this paper, we assume that the following non-degeneracy condition is satisfied, i.e.,

$$\Theta := \beta \beta^\top \geq \delta I_{n \times n} ,$$

where $\delta$ is some positive constant and $I_{n \times n}$ is the $(n \times n)$-identity matrix.

Let $Y_k := \{Y_k(t) | t \in \mathcal{T}\}$ denote the logarithmic return of the share $S_k$, that is, $Y_k(t) := \ln(S_k(t)/S_k(0))$ for each $t \in \mathcal{T}$ and $k = 1, 2, \cdots, n$. By Itô’s differentiation rule, it is easy to see that

$$dY_k(t) = \left(\mu_k(t) - \frac{1}{2}\beta_k^2\right)dt + \beta_k dW(t) , \quad t \in \mathcal{T} , \quad Y_k(0) = 0 ,$$

where $\tilde{\beta}_k := ||\beta_k||$ and $|| \cdot ||$ denotes the Euclidean norm in $\mathbb{R}^n$.

Write $g_k(t) := \mu_k(t) - \frac{1}{2}\beta_k^2$, for each $t \in \mathcal{T}$ and $k = 1, 2, \cdots, n$. Then the return process $Y_k$ of the $k$-th share can be rewritten as:

$$dY_k(t) = g_k(t)dt + \beta_k dW(t) , \quad t \in \mathcal{T} , \quad Y_k(0) = 0 .$$

The vector of the return processes $Y := \{Y(t) | t \in \mathcal{T}\} = \{Y_1(t), Y_2(t), \cdots, Y_n(t)\}^\top | t \in \mathcal{T}\}$ is governed by the following stochastic differential equation:

$$dY(t) = g(t)ds + \beta dW(t) , \quad t \in \mathcal{T} , \quad Y(0) = 0_n ,$$

where $g(t) := (g_1(t), g_2(t), \cdots, g_n(t))^\top \in \mathbb{R}^n$ and $0_n := (0, 0, \cdots, 0)^\top \in \mathbb{R}^n$.

Note that both the appreciation rate $\mu_k$ and the Brownian motion $W_t$ are unobservable to the agent, for each $k = 1, 2, \cdots, n$ and $d = 1, 2, \cdots, n$. The agent can only observe the price process $S_k$ or the return process $Y_k$ of the risky share. As pointed out by Merton [19], estimates of appreciation rates based on realized returns data are noisy. Therefore, incorporating the model uncertainty of the appreciation rate is an important issue.

We now specify the information structure of our model. Let $\mathcal{F}^{S_k} := \{\mathcal{F}^{S_k}(t) | t \in \mathcal{T}\}$ and $\mathcal{F}^{Y_k} := \{\mathcal{F}^{Y_k}(t) | t \in \mathcal{T}\}$ be the right-continuous, $\mathcal{P}$-complete, natural filtrations generated by the processes $S_k$ and $Y_k$, respectively, for each $k = 1, 2, \cdots, n$. Since $\mathcal{F}^{S_k}$ and $\mathcal{F}^{Y_k}$ are equivalent, we choose to use $\mathcal{F}^{Y_k}$ as the observed information structure. For each $t \in \mathcal{T}$, let $\mathcal{F}^Y(t) := \mathcal{F}^{Y_1}(t) \vee \mathcal{F}^{Y_2}(t) \vee \cdots \vee \mathcal{F}^{Y_n}(t)$, be the minimal $\sigma$-field containing $\mathcal{F}^{Y_1}(t), \mathcal{F}^{Y_2}(t), \cdots, \mathcal{F}^{Y_n}(t)$; let $\mathcal{F}(t) := \mathcal{F}^Y(t) \vee \mathcal{F}^X(t)$, be the minimal $\sigma$-field containing $\mathcal{F}^Y(t)$ and $\mathcal{F}^X(t)$; write $\mathcal{F}^\mathbb{U} := \{\mathcal{F}^\mathbb{U}(t) | t \in \mathcal{T}\}$ and $\mathcal{G} := \{\mathcal{G}(t) | t \in \mathcal{T}\}$. Consequently, $\mathcal{F}^Y$, $\mathcal{F}^X$ and $\mathcal{G}$ describe the flows of observable information, hidden information and full information, respectively.
3. The consumption-portfolio optimization problem. In this section, we formulate the portfolio optimization problem of the economic agent. Denote by $\pi_k(t)$, $k = 1, 2, \cdots, n$, the amount of the agent’s wealth invested in the share $S_k$ at time $t$, and $c(t)$ the instantaneous amount of the agent’s wealth consumed per unit of time at time $t$. We call $\pi := \{\pi(t)|t \in \mathcal{T}\} = \{(\pi_1(t), \pi_2(t), \cdots, \pi_n(t))\}^{\top}|t \in \mathcal{T}\}$ and $c := \{c(t)|t \in \mathcal{T}\}$ a portfolio process and a consumption process of the agent.

Specifically, the following definition is standard in the existing literature:

**Definition 3.1.** A portfolio process $\pi$ is an $\mathbb{F}^Y$-progressively measurable, $\mathbb{R}^n$-valued process such that

$$
\int_0^T ||\pi(t)||^2 dt < \infty, \quad \mathcal{P}\text{-a.s.}
$$

A consumption process $c$ is an $\mathbb{F}^Y$-progressively measurable, nonnegative process such that

$$
\int_0^T c(t) dt < \infty, \quad \mathcal{P}\text{-a.s.}
$$

Denote by $V(t) := V_{\pi,c}(t)$ the total wealth of the agent at time $t$ corresponding to the pair of the portfolio and consumption processes $(\pi, c)$. The portfolio process $\pi$ is said to be $c$-financed if the amount of the agent’s wealth invested in the bond is $V(t) - \sum_{k=1}^n \pi_k(t)$ at time $t$, for each $t \in \mathcal{T}$. Suppose that (1) the stocks can be traded continuously over time; (2) there are no transaction costs, taxes, and short-selling constraints in trading; (3) the portfolio process $\pi$ is $c$-financed. Then the wealth process $V := \{V(t)|t \in \mathcal{T}\}$ is governed by

$$
\begin{align*}
    dV(t) &= [r(t)V(t) + \pi(t)^\top (\mu(t) - r1_n) - c(t)]dt + \pi(t)^\top \beta dW(t), \\
    V(0) &= v > 0.
\end{align*}
$$

We consider the consumption-portfolio optimization problem where the economic agent aims to maximize the expected, discounted utility from intermediate consumption and terminal wealth over the set of admissible portfolio and consumption processes with partial observations. The definitions of the admissible set and the utility function are given below.

**Definition 3.2.** A pair of portfolio and consumption processes $(\pi, c)$ is said to be admissible for the initial wealth $v > 0$ if the stochastic differential equation (11) admits a unique strong solution $V$ such that

$$
V(t) \geq 0, \quad t \in \mathcal{T}, \quad \mathcal{P}\text{-a.s.}
$$

Write $\mathcal{A}(v)$ for the space of admissible pairs $(\pi, c)$ associated with the initial wealth $v > 0$. Note that $V(t) \geq 0$ may be interpreted as a solvency constraint.

**Definition 3.3.** The function $U(\cdot) : (0, +\infty) \to \mathbb{R}$ is called a utility function if it is strictly increasing, strictly concave and of class $C^1$ and satisfies the following Inada conditions:

1. $\lim_{x \to 0^+} U'(x) = +\infty$;
2. $\lim_{x \to +\infty} U'(x) = 0$.

Let $U_1$ and $U_2$ be the utility functions of intermediate consumption and terminal wealth of the agent, respectively. Suppose that $\rho$ is a positive constant, which represents the discount rate of the agent. The objective of the agent is to maximize the
expected, discounted utility from intermediate consumption and terminal wealth:

\[
J(v, \pi, c) = E^v \left[ \int_0^T e^{-\rho t} U_1(c(t)) dt + e^{-\rho T} U_2(V(T)) \right],
\]

subject to the wealth process (11) over the class

\[
A_1(v) := \left\{ (\pi, c) \in A(v) \mid E^v \left[ \int_0^T e^{-\rho t} U_1^-(c(t)) dt + e^{-\rho T} U_2^-(V(T)) \right] < \infty \right\},
\]

where \( U_i^- \) denotes the negative part of \( U_i \), for each \( i = 1, 2 \), i.e., \( U_i^- := \max\{-U_i, 0\} \). Here \( E^v[\cdot] \) is the conditional expectation with respect to \( \mathcal{P} \) given \( V(0) = v \).

Since the investor only has access to the information about the share prices, the consumption-portfolio optimization problem under the hidden Markov model is a stochastic control problem with partial observations. In this paper, we employ the standard innovations approach in filtering theory to transform the problem with partial information to the one with complete information. Particularly, using the separation principle, the two problems, namely, the filtering problem and the consumption-portfolio optimization problem are separated, so that these two problems can be solved independently.

For any integrable, \( \mathcal{G} \)-adapted, process \( \xi := \{ \xi(t) \mid t \in \mathcal{T} \} \), let \( \hat{\xi} := \{ \hat{\xi}(t) \mid t \in \mathcal{T} \} \) be the \( \mathcal{F}^Y \)-optional projection of \( \xi \) under \( \mathcal{P} \), so that for each \( t \in \mathcal{T} \)

\[
\hat{\xi}(t) := E[\xi(t) \mid \mathcal{F}^Y(t)], \quad \mathcal{P} - \text{a.s.}
\]

That is, \( \hat{\xi}(t) \) is a version of the conditional expectation of \( \xi(t) \) with respect to \( \mathcal{P} \) given \( \mathcal{F}^Y(t) \). It is known that the \( \mathcal{F}^Y \)-optional projection takes care of the measurability in \( (t, \omega) \in \mathcal{T} \times \Omega \).

Define an \( \mathcal{F}^Y \)-adapted process \( \hat{W} := \{ \hat{W}(t) \mid t \in \mathcal{T} \} \) by putting

\[
\hat{W}(t) = W(t) + \int_0^t \beta^{-1}(g(s) - \hat{g}(s)) ds, \quad t \in \mathcal{T},
\]

where \( \hat{g}(t) := (\hat{g}_1(t), \hat{g}_2(t), \cdots, \hat{g}_n(t))^\top \in \mathbb{R}^n \) and \( \hat{g}_k(t) := \hat{\mu}_k(t) - \frac{1}{2} \hat{\beta}_k^2 \). This is called an innovations process, which is often used in filtering theory. Indeed, using a version of stochastic Fubini’s theorem, it can be shown that \( \hat{W} \) is an \( (\mathcal{F}^Y, \mathcal{P}) \)-standard Brownian motion, (see, for example, Fleming and Rishel [8], Kallianpur [14], and Elliott [14]).

It is then not difficult to see that under \( \mathcal{P} \), the return process and the wealth process can be written in terms of the filtered estimate \( \hat{g}(t) \) of \( g(t) \) and the innovations process \( \hat{W}(t) \) as follows:

\[
dY(t) = \hat{g}(t) dt + \beta d\hat{W}(t),
\]

and

\[
dV(t) = \left[ rV(t) + \pi(t)^\top (\hat{\mu}(t) - r1_n) - c(t) \right] dt + \pi(t)^\top \beta d\hat{W}(t).
\]

Therefore, the consumption-portfolio optimization problem with partial observations is transformed to the one with complete observations:

\[
\begin{align*}
\text{maximize} & \quad J(v, \pi, c) = E^v \left[ \int_0^T e^{-\rho t} U_1(c(t)) dt + e^{-\rho T} U_2(V(T)) \right], \\
\text{subject to} & \quad (\pi, c) \in A_1(v) \quad \text{and} \quad (V(t), \pi, c) \text{ satisfy } [15].
\end{align*}
\]
4. Filtering. To discuss the consumption-portfolio optimization problem (10), robust filters of the hidden Markov chain and estimates of some other model parameters should first be derived. Then some techniques in stochastic optimal control theory can be employed to find an optimal pair of portfolio and consumption processes. We present robust filters of the hidden Markov chain and estimates of other parameters in Subsections 4.1 and 4.2, respectively. The filtering results to be presented in this section are standard and are derived using standard reference probability approach to filtering, see, for example, Elliott et al. [5]. Since the proofs of results in this section are standard, they are all placed in an Appendix. Notice that only robust filters of the hidden Markov chain will be used in expressions for optimal portfolio and consumption processes (see Theorem 5.1), specifically in the optional projection of \( \hat{\mu} \), i.e., \( \tilde{\mu} \), while estimates of other parameters will not be needed there. The readers may skip Subsection 4.2 and jump to Section 5 if they wish.

4.1. Robust filters of hidden Markov chain. For each \( k = 1, 2, \ldots, n \) and \( i = 1, 2, \ldots, N \), let \( g_k^i = \mu_k - \frac{1}{2} \tilde{\Sigma}_k^i \). Write \( g_k := (g_k^1, g_k^2, \ldots, g_k^N)\top \in \mathbb{R}^N \). Then under \( \mathcal{P} \), the return process of the \( k \)-th share can be represented as

\[
Y_k(t) = \int_0^t \langle g_k, X(s) \rangle ds + \beta_k W(t) .
\]

For notational simplicity, two processes \( \widetilde{Y}_k := \{ \widetilde{Y}_k(t) | t \in \mathcal{F} \} \) and \( \widetilde{W}_k := \{ \widetilde{W}_k(t) | t \in \mathcal{F} \} \) are defined by putting

\[
\widetilde{Y}_k(t) := \tilde{\beta}^{-1}_k Y_k(t) ,
\]

and

\[
\widetilde{W}_k(t) := \tilde{\beta}^{-1}_k \beta_k W(t) ,
\]

for each \( k = 1, 2, \ldots, n \).

Write \( \bar{Y}(t) := (\bar{Y}_1(t), \bar{Y}_2(t), \ldots, \bar{Y}_n(t))\top \in \mathbb{R}^n \), \( \bar{W}(t) := (\bar{W}_1(t), \bar{W}_2(t), \ldots, \bar{W}_n(t))\top \in \mathbb{R}^n \), for each \( t \in \mathcal{F} \). It is easy to see that \( \bar{W}(t) \) is an \( n \)-dimensional correlated Brownian motion with the constant covariance matrix

\[
\Sigma := [\tilde{\beta}^{-1}_k \tilde{\beta}^{-1}_l \langle \beta_k, \beta_l \rangle]_{k,l=1,2,\ldots,n} \in \mathbb{R}^{n \times n} .
\]

Since \( \Sigma \) is a covariance matrix, it must be a positive-semidefinite, symmetric matrix. Thus there exists a unique square root of \( \Sigma \), which is also a positive semidefinite matrix (see Ben-Israel and Greville [1]). We denote this unique square root of \( \Sigma \) by \( \Sigma^{\frac{1}{2}} \). The relationship between \( \Theta = [\langle \beta_k, \beta_l \rangle]_{k,l=1,2,\ldots,n} \) (see Eq. (7)) and \( \Sigma \) implies that the non-degeneracy condition is also satisfied for \( \Sigma \) and \( \Sigma^{\frac{1}{2}} \). Therefore, the covariance matrix \( \Sigma \) and its square root \( \Sigma^{\frac{1}{2}} \) are invertible. We denote by

\[
\Sigma^{-1} = [\psi_{kl}]_{k,l=1,2,\ldots,n} \in \mathbb{R}^{n \times n} ,
\]

and

\[
\Sigma^{-\frac{1}{2}} = [\phi_{kl}]_{k,l=1,2,\ldots,n} \in \mathbb{R}^{n \times n} .
\]

Note that both \( \Sigma^{-1} \) and \( \Sigma^{-\frac{1}{2}} \) are symmetric, i.e.,

\[
\psi_{kl} = \psi_{lk} ,
\]

and

\[
\phi_{kl} = \phi_{lk} ,
\]

for each \( k, l = 1, 2, \ldots, n \).
for each $k,l = 1,2,\cdots, n$.

From (17), we have

$$
\tilde{Y}_k(t) = \int_0^t (\tilde{g}_k, \tilde{X}(s))\, ds + \tilde{W}_k(t) ,
$$

(20)

where $\tilde{g}_k := (\tilde{g}_{k1}, \tilde{g}_{k2}, \cdots, \tilde{g}_{kN})^\top \in \mathbb{R}^N$ and $\tilde{g}_{ki}^l := \tilde{\beta}_k^i g_{ki}^l$, for each $k = 1,2,\cdots, n$ and $i = 1,2,\cdots, N$. Rewriting (20) in the vector form gives

$$
\tilde{Y}(t) = \int_0^t \tilde{g}X(s)\, ds + \tilde{W}(t) ,
$$

(21)

where $\tilde{g} := [\tilde{g}_k]_{k=1,2,\cdots,n; i=1,2,\cdots,N} \in \mathbb{R}^{n\times N}$.

Consider the following $\mathcal{G}$-adapted process $K := \{ K(t) | t \in \mathcal{T} \}$ defined by

$$
K(t) := \exp \left[ - \int_0^t (\tilde{g}X(s))^\top \Sigma^{-1} d\tilde{W}(s) - \frac{1}{2} \int_0^t (\tilde{g}X(s))^\top \Sigma^{-1} (\tilde{g}X(s)) \, ds \right] .
$$

(22)

Since $\beta$ is a constant matrix and the chain $X$ has a finite number of states, the Novikov condition for the stochastic exponential in (22) is satisfied. Then, $K$ is a $(\mathcal{G}, \mathcal{F})$-martingale, and hence $\mathbb{E}[K(T)] = 1$.

Consequently a reference probability measure $\overline{\mathcal{F}}$ equivalent to $\mathcal{F}$ is defined by setting

$$
\frac{d\overline{\mathcal{F}}}{d\mathcal{F}}|_{\mathcal{F}(T)} := K(T) .
$$

By Girsanov’s theorem for the correlated Brownian motion (see Jeanblanc et al. [13]), the process $\{ \tilde{Y}(t) | t \in \mathcal{T} \}$ is an $n$-dimensional, $(\mathcal{G}, \overline{\mathcal{F}})$-Brownian motion with the covariance matrix $\Sigma$. Since $\{ \tilde{W}(t) | t \in \mathcal{T} \}$ and $\{ X(t) | t \in \mathcal{T} \}$ are independent under $\mathcal{F}$, the probability law of the chain remains unchanged under changing the measures from $\mathcal{F}$ to $\overline{\mathcal{F}}$.

Define a transformed process $Z := \{ Z(t) | t \in \mathcal{T} \} = \{(Z_1(t), Z_2(t), \cdots, Z_n(t))^\top | t \in \mathcal{T} \}$ by putting

$$
Z(t) := \Sigma^{-\frac{1}{2}} \tilde{Y}(t) .
$$

(23)

It can be shown that $Z$ is an $n$-dimensional, $(\mathcal{G}, \overline{\mathcal{F}})$-standard Brownian motion since $\Sigma^{-\frac{1}{2}}$ is the rotation matrix. It is obvious that the transformed process $\{ Z(t) | t \in \mathcal{T} \}$ is an $\mathcal{F}^Y$-adapted process. Thus adopting $Z$ as the new observation process will simplify the analysis.

Consider a $\mathcal{G}$-adapted, real-valued process $\Lambda := \{ \Lambda(t) | t \in \mathcal{T} \}$ defined by

$$
\Lambda(t) := 1 + \int_0^t \Lambda(s)(\tilde{g}X(s))^\top \Sigma^{-1} d\tilde{Y}(s)
$$

$$
= 1 + \int_0^t \Lambda(s)(\tilde{g}X(s))^\top \Sigma^{-\frac{1}{2}} dZ(s) .
$$

(24)

Here $\Lambda$ is a $(\mathcal{G}, \overline{\mathcal{F}})$-(local)-martingale. Indeed, $\Lambda$ is a $(\mathcal{G}, \overline{\mathcal{F}})$-martingale. In addition, it is not difficult to show that

$$
\Lambda(t) = \exp \left[ \int_0^t (\tilde{g}X(s))^\top \Sigma^{-\frac{1}{2}} dZ(s) - \frac{1}{2} \int_0^t (\tilde{g}X(s))^\top \Sigma^{-1} (\tilde{g}X(s)) \, ds \right] .
$$

(25)
\[
= \exp \left[ \sum_{k,l=1}^{n} \left( \int_{0}^{t} \langle \tilde{g}_k, X(s) \rangle \phi_{kl} dZ_l(s) - \frac{1}{2} \int_{0}^{t} \langle \tilde{g}_k, X(s) \rangle \psi_{kl} \langle \tilde{g}_l, X(s) \rangle ds \right) \right],
\]

(26)

and

\[
\Lambda(t) K(t) = 1, \quad t \in \mathcal{T}.
\]

For convenience of presentation, we shall use Eqs. (25) and (26) interchangeably in the sequel.

The real-world probability measure \( \mathcal{P} \) can then be re-constructed from the reference probability measure \( \mathcal{F} \) on \( \mathcal{G}(T) \) as follows:

\[
\frac{d\mathcal{P}}{d\mathcal{F}}|_{\mathcal{G}(T)} := \Lambda(T).
\]

(27)

It can be seen from Eqs. (21) and (23) that under \( \mathcal{P} \) the observation process \( Z \) depends on the chain \( X \).

Using a version of the Bayes’ rule, for any \( \mathcal{G} \)-adapted, integrable process \( \xi = \{ \xi(t) | t \in \mathcal{T} \} \),

\[
\tilde{\xi}(t) := \mathbb{E}[\xi(t)|\mathcal{F}^Y(t)] = \frac{\mathbb{E}[\Lambda(t)\xi(t)|\mathcal{F}^Y(t)]}{\mathbb{E}[\Lambda(t)|\mathcal{F}^Y(t)]} = \frac{\sigma(\xi(t))}{\sigma(1)},
\]

where \( \mathbb{E}[\cdot] \) is the expectation taken under the reference measure \( \mathcal{F} \); \( \sigma(\xi(t)) \) and \( \sigma(1) \) are the optional projections of \( \Lambda \xi := \{ \Lambda(t)\xi(t) | t \in \mathcal{T} \} \) and \( \Lambda \) on the observed filtration \( \mathcal{F}^Y \) under \( \mathcal{F} \), respectively. Here \( \sigma(\xi(t)) \) is called an unnormalized, or Zakai form, of the filter of \( \xi(t) \).

The following proposition gives the Zakai stochastic differential equation governing the evolution of the unnormalized filter \( \sigma(X(t)) \) of the chain \( X \). To simplify our notation, we write

\[
q(t) := \sigma(X(t)), \quad t \in \mathcal{T}.
\]

**Proposition 1.** Let \( \tilde{B}_k := \text{diag}(\tilde{g}_k) \), the diagonal matrix with entities given by the components of \( \tilde{g}_k \), for each \( k = 1, 2, \cdots, n \). Then, the unnormalized filter \( q(t) \) of \( X(t) \) is governed by the Zakai stochastic differential equation:

\[
q(t) = q(0) + \int_{0}^{t} Aq(s)ds + \sum_{k,l=1}^{n} \int_{0}^{t} \tilde{B}_k q(s)\phi_{kl} dZ_l(s).
\]

(28)

**Remark 1.** With a slight abuse of notation, let \( \tilde{B} := \text{diag}(\tilde{g}) \), the three-dimensional diagonal array with non-zero entries given by the components in the matrix \( \tilde{g} \). Note that each entry on the diagonal of \( \tilde{B} \) is a vector \( \tilde{g}^i := (\tilde{g}_{1i}, \tilde{g}_{2i}, \cdots, \tilde{g}_{ni})^\top \in \mathbb{R}^n \), for each \( i = 1, 2, \cdots, N \). Then, Eq. (28) can be also written as

\[
q(t) = q(0) + \int_{0}^{t} Aq(s)ds + \int_{0}^{t} (\tilde{B} \otimes \Sigma^{-\frac{1}{2}})dZ(s)q(s),
\]

(29)

where \( \otimes \) is the tensor product of appropriate dimension.

Although Eq. (29) looks apparently simpler than Eq. (28), it may not be convenient due to the multiplication with the three-dimensional array \( \tilde{B} \). Here we adopt
The gauge transformation matrix \( \Gamma \) for the transformed process of the Zakai filter. For each gauge transformation technique of Clark [3] to simplify the stochastic differential integrals, which may not be easy to implement in practice. So we now employ the array may lead to confusion in the sequel.

The Zakai stochastic differential equation in Proposition 1 involves stochastic integrals. Since \( \Gamma \) is always a matrix valued, stochastic differential equation:

\[
\text{d}(\bar{Y}(t)) = (\bar{g}^i)^\top \Sigma^{-1} \text{d}Y(t) - \frac{1}{2} (\bar{g}^i)^\top \Sigma^{-1} \bar{g}^i \text{d}t,
\]

Hence the gauge transformation matrix \( \Gamma(t) \) is defined as the following matrix:

\[
\Gamma(t) := \text{diag}(\gamma^1(t), \gamma^2(t), \ldots, \gamma^N(t)) , \quad t \in \mathcal{T}.
\]

Since \( \gamma^i(t) > 0 \), for each \( i = 1, 2, \ldots, N \) and \( t \in \mathcal{T} \), the inverse of \( \Gamma(t) \) exists. Write, for each \( t \in \mathcal{T} \), \( (\Gamma(t))^{-1} \) for the inverse of \( \Gamma(t) \).

Applying Itô’s differentiation rule gives

\[
d(\gamma^i(t))^{-1} = -(\gamma^i(t))^{-1} (\bar{g}^i)^\top \Sigma^{-\frac{1}{2}} \text{d}Z(t) - (\bar{g}^i)^\top \Sigma^{-1} \bar{g}^i \text{d}t
\]

Hence the gauge transformation matrix \( \Gamma := \{ \Gamma(t)|t \in \mathcal{T} \} \) follows the matrix-valued, stochastic differential equation:

\[
d(\Gamma(t))^{-1} = - (\Gamma(t))^{-1} (\bar{B} \otimes \Sigma^{-\frac{1}{2}}) dZ(t) + (\Gamma(t))^{-1} (\bar{B} \otimes \Sigma^{-1} \otimes \bar{B}) dt
\]

A transformed process \( \mathbf{q} := \{ \mathbf{q}(t)|t \in \mathcal{T} \} \) is then defined by

\[
\mathbf{q}(t) := (\Gamma(t))^{-1} \mathbf{q}(t) , \quad t \in \mathcal{T}.
\]

The following proposition gives the path-wise linear ordinary differential equation for the transformed process \( \mathbf{q} \).

**Proposition 2.** \( \mathbf{q} \) satisfies the following path-wise linear ordinary differential equation:

\[
\frac{d \mathbf{q}(t)}{dt} = (\Gamma(t))^{-1} \mathbf{A} \mathbf{q}(t) \quad \mathbf{q}(0) = \mathbf{q}(0).
\]

**Remark 2.** Note that the gauge transformation matrix \( \Gamma(t) \) is stochastic. Consequently the path-wise linear ordinary differential equation (31) is not deterministic.
4.2. Filtered estimates of other parameters. In this subsection, we discuss a robust, filter-based, EM algorithm to estimate the unknown drift parameters \( \mu^t := (\mu^1, \mu^2, \cdots, \mu^N)^T \in \mathbb{R}^N \), \( i = 1, 2, \cdots, N \), and the intensity parameters \( a_{ij} \) of the chain \( X, i, j = 1, 2, \cdots, N \). To obtain these recursive estimates, the dynamics of some measure-valued quantities that are useful for the derivations of these estimates are first presented. For each \( t \in \mathcal{T} \) and each \( i, j = 1, 2, \cdots, N \), let \( J_{ij}(t) \) be the number of jumps of the chain \( X \) from \( e_i \) to \( e_j \) up to time \( t \). That is,

\[
J_{ij}(t) := \int_0^t \langle X(s-), e_i \rangle \langle dX(s), e_j \rangle .
\]

For each \( t \in \mathcal{T} \) and each \( i = 1, 2, \cdots, N \), suppose that \( O^i(t) \) is the occupation time of the chain \( X \) in state \( e_i \) up to time \( t \). That is,

\[
O^i(t) := \int_0^t \langle X(s), e_i \rangle ds .
\]

Furthermore, for each \( t \in \mathcal{T} \), \( k, l = 1, 2, \cdots, n \) and \( i = 1, 2, \cdots, N \), write

\[
H^i_{kl}(t) := \int_0^t \langle X(s), e_i \rangle \tilde{\beta}_k \psi_{ki} \tilde{\beta}_l^{-1} ds ,
\]

\[
H^i_{lk}(t) := \int_0^t \langle X(s), e_i \rangle \tilde{\beta}_k^{-1} \psi_{kl} \tilde{\beta}_l ds ,
\]

and

\[
H^i_{kl}(t) := \int_0^t \langle X(s), e_i \rangle \tilde{\beta}_k^{-1} \psi_{kl} \tilde{\beta}_l^{-1} ds .
\]

Clearly, from [18], we have

\[
H^i_{kl}(t) = H^i_{lk}(t) , \quad H^i_{kl}(t) = H^i_{lk}(t) . \tag{32}
\]

For each \( t \in \mathcal{T} \), \( k, l = 1, 2, \cdots, n \) and \( i = 1, 2, \cdots, N \), let \( G^i_{kl}(t) \) be the level integral with respect to \( \tilde{\beta}_k^{-1} \phi_{kl} Z(t) \) corresponding to the state \( e_i \) up to time \( t \). That is,

\[
G^i_{kl}(t) := \int_0^t \langle X(s), e_i \rangle \tilde{\beta}_k^{-1} \phi_{kl} dZ(s) .
\]

The following theorem gives the stochastic differential equations governing the measure-valued quantities \( \sigma(X(t)J_{ij}(t)) \), \( \sigma(X(t)O^i(t)) \), \( \sigma(X(t)H^i_{kl}(t)) \), \( \sigma(X(t)H^i_{lk}(t)) \), \( \sigma(X(t))H^i_{kl}(t) \) and \( \sigma(X(t))G^i_{kl}(t) \), which are Zakai-type filtered estimates of \( J_{ij}(t), O^i(t), H^i_{kl}(t), H^i_{lk}(t), H^i_{kl}(t) \) and \( G^i_{kl}(t) \), respectively.

**Proposition 3.** \( \sigma(X(t)J_{ij}(t)) \), \( \sigma(X(t)O^i(t)) \), \( \sigma(X(t)H^i_{kl}(t)) \), \( \sigma(X(t)H^i_{lk}(t)) \), \( \sigma(X(t))H^i_{kl}(t) \) and \( \sigma(X(t))G^i_{kl}(t) \) are governed by the following stochastic differential equations:

\[
\sigma(X(t)J_{ij}(t)) = \int_0^t \langle q(s), e_i \rangle a_{ij} e_j ds + \int_0^t A \sigma(X(s)J_{ij}(s)) ds
\]

\[+ \sum_{k_0, l_0=1}^n \int_0^t \tilde{B}_{k_0} \sigma(X(s)J_{ij}(s)) \phi_{k_0 l_0} dZ_{l_0}(s) , \tag{33}\]

\[
\sigma(X(t)O^i(t)) = \int_0^t \langle q(s), e_i \rangle e_0 ds + \int_0^t A \sigma(X(s)O^i(s)) ds
\]
σ(\mathbf{X}(t)H_{kl}^t(t)) = \int_0^t \langle \mathbf{q}(s), \mathbf{e}_i \rangle \mathbf{e}_i \tilde{\beta}_k \psi_{kl}^t \tilde{\beta}_l^{-1} ds + \int_0^t A \sigma(\mathbf{X}(s)H_{kl}^t(s)) ds \\
+ \sum_{k_0, l_0=1}^n \int_0^t \mathbf{B}_{k_0} \sigma(\mathbf{X}(s)H_{kl}^t(s)) \phi_{k_0 l_0} dZ_{l_0}(s) \ , \quad (35)

σ(\mathbf{X}(t)H_{kl}^t(t)) = \int_0^t \langle \mathbf{q}(s), \mathbf{e}_i \rangle \mathbf{e}_i \tilde{\beta}_k \psi_{kl}^t \tilde{\beta}_l^{-1} ds + \int_0^t A \sigma(\mathbf{X}(s)H_{kl}^t(s)) ds \\
+ \sum_{k_0, l_0=1}^n \int_0^t \mathbf{B}_{k_0} \sigma(\mathbf{X}(s)H_{kl}^t(s)) \phi_{k_0 l_0} dZ_{l_0}(s) \ , \quad (36)

σ(\mathbf{X}(t)G_{kl}^t(t)) = \int_0^t \langle \mathbf{q}(s), \mathbf{e}_i \rangle \mathbf{e}_i \tilde{\beta}_k \phi_{kl} \psi_{kl}^t \tilde{\beta}_l^{-1} ds + \int_0^t A \sigma(\mathbf{X}(s)G_{kl}^t(s)) ds \\
+ \sum_{k_0, l_0=1}^n \int_0^t \mathbf{B}_{k_0} \sigma(\mathbf{X}(s)G_{kl}^t(s)) \phi_{k_0 l_0} dZ_{l_0}(s) \\
+ \sum_{k=1}^n \int_0^t \langle \mathbf{q}(s), \mathbf{e}_i \rangle \tilde{\beta}_k \psi_{kl} \phi_{kl} \psi_{kl}^t \tilde{\beta}_l^{-1} ds \ . \quad (38)

Similarly, the gauge transformation technique can be applied to simplify the stochastic differential equations (33)-(38) and to derive the corresponding path-wise linear ordinary differential equations for the transformed processes of the measure-valued quantities. These transformed, measure-valued, processes, denoted as \( \bar{\sigma}(\mathbf{X}(t)J^{ij}(t)), \bar{\sigma}(\mathbf{X}(t)O^i(t)), \bar{\sigma}(\mathbf{X}(t)H_{kl}^t(t)), \bar{\sigma}(\mathbf{X}(t)H_{kl}^t(t)), \bar{\sigma}(\mathbf{X}(t)H_{kl}^t(t)) \) and \( \bar{\sigma}(\mathbf{X}(t)G_{kl}^t(t)) \), are defined by

\[
\bar{\sigma}(\mathbf{X}(t)J^{ij}(t)) := (\mathbf{\Gamma}(t))^{-1} \sigma(\mathbf{X}(t)J^{ij}(t)) , \quad \bar{\sigma}(\mathbf{X}(t)O^i(t)) := (\mathbf{\Gamma}(t))^{-1} \sigma(\mathbf{X}(t)O^i(t)) , \\
\bar{\sigma}(\mathbf{X}(t)H_{kl}^t(t)) := (\mathbf{\Gamma}(t))^{-1} \sigma(\mathbf{X}(t)H_{kl}^t(t)) , \quad \bar{\sigma}(\mathbf{X}(t)H_{kl}^t(t)) := (\mathbf{\Gamma}(t))^{-1} \sigma(\mathbf{X}(t)H_{kl}^t(t)), \\
\bar{\sigma}(\mathbf{X}(t)H_{kl}^t(t)) := (\mathbf{\Gamma}(t))^{-1} \sigma(\mathbf{X}(t)H_{kl}^t(t)) , \quad \bar{\sigma}(\mathbf{X}(t)G_{kl}^t(t)) := (\mathbf{\Gamma}(t))^{-1} \sigma(\mathbf{X}(t)G_{kl}^t(t)).
\]

The following proposition gives the dynamics of these transformed, measure-valued, processes.

**Proposition 4.** \( \bar{\sigma}(\mathbf{X}(t)J^{ij}(t)), \bar{\sigma}(\mathbf{X}(t)O^i(t)), \bar{\sigma}(\mathbf{X}(t)H_{kl}^t(t)), \bar{\sigma}(\mathbf{X}(t)H_{kl}^t(t)), \bar{\sigma}(\mathbf{X}(t)H_{kl}^t(t)) \) and \( \bar{\sigma}(\mathbf{X}(t)G_{kl}^t(t)) \) are governed by the following equations:

\[
\bar{\sigma}(\mathbf{X}(t)J^{ij}(t)) = \int_0^t \langle \mathbf{q}(s), \mathbf{e}_i \rangle a_{ji} \mathbf{e}_j ds + \int_0^t A \bar{\sigma}(\mathbf{X}(s)J^{ij}(s)) ds , \quad (39)
\]
\[
\bar{\sigma}(\mathbf{X}(t)O^i(t)) = \int_0^t \langle \mathbf{q}(s), \mathbf{e}_i \rangle \mathbf{e}_i ds + \int_0^t A \bar{\sigma}(\mathbf{X}(s)O^i(s)) ds , \quad (40)
\]
\[
\bar{\sigma}(\mathbf{X}(t)H_{kl}^t(t)) = \int_0^t \langle \mathbf{q}(s), \mathbf{e}_i \rangle \mathbf{e}_i \tilde{\beta}_k \psi_{kl} \tilde{\beta}_l^{-1} ds + \int_0^t A \bar{\sigma}(\mathbf{X}(s)H_{kl}^t(s)) ds , \quad (41)
\]
Using the EM algorithm, the revised, or updated, estimates of \( J^{ij} \), \( O^i \), \( H_{kl}^i \), \( H_{kl}^j \), \( G_{kl}^i \) and \( G_{kl}^j \) can then be determined by taking the inner products with \( \mathbf{1}_N \). That is,

\[
\sigma(J^{ij}(t)) = \langle \mathbf{1}_N \rangle \sigma(X(t)J^{ij}(t)), \quad \sigma(O^i(t)) = \langle \mathbf{1}_N \rangle \sigma(X(t)O^i(t)), \quad \sigma(H_{kl}^i(t)) = \langle \mathbf{1}_N \rangle \sigma(X(t)H_{kl}^i(t)), \quad \sigma(H_{kl}^j(t)) = \langle \mathbf{1}_N \rangle \sigma(X(t)H_{kl}^j(t)), \quad \sigma(G_{kl}^i(t)) = \langle \mathbf{1}_N \rangle \sigma(X(t)G_{kl}^i(t)), \quad \sigma(G_{kl}^j(t)) = \langle \mathbf{1}_N \rangle \sigma(X(t)G_{kl}^j(t)),
\]

for each \( t \in T \), \( k, l = 1, 2, \cdots, n \) and \( i, j = 1, 2, \cdots, N \). From (42), it is clear that

\[
\sigma(H_{kl}^i(t)) = \sigma(H_{kl}^j(t)), \quad \sigma(H_{kl}^i(t)) = \sigma(H_{kl}^j(t)).
\]

For each \( t \in T \) and \( i, j = 1, 2, \cdots, N \), write

\[
\Pi^i(t) := \left[ \sigma(H_{kl}^i(t)) \right]_{k,l=1,2,\cdots,n} \in \mathbb{R}^{n \times n},
\]

and

\[
\Xi^i(t) := \left[ \sum_{l=1}^{n} \left( \frac{1}{2} \sigma(G_{kl}^i(t)) + \frac{1}{2} \sigma(H_{kl}^i(t)) \right) \right]_{k,l=1,2,\cdots,n} \in \mathbb{R}^n.
\]

Using the expectation-maximization (EM) algorithm, the revised, or updated, estimates of \( a_{ij} \) and \( \mu^i = (\mu_{1}^i, \mu_{2}^i, \cdots, \mu_{n}^i)^{\top} \in \mathbb{R}^n \), for each \( i, j = 1, 2, \cdots, N \), are presented in the following theorem. The basic idea of the EM algorithm is to split the estimation of a model containing hidden quantities into two steps. The first step is the expectation step which evaluates the expectation of pseudo-likelihood function. The second step is the maximization step which aims to determine an optimal estimate of the unknown parameter by maximizing the expectation of the pseudo-likelihood function. The use of pseudo-likelihood function in the EM algorithm seems to be quite standard and has been discussed in, for example, Elliott et al. [5].

**Theorem 4.1.** Using the EM algorithm, the revised, or updated, estimates of \( a_{ij} \), for each \( t \in T \), \( i, j = 1, 2, \cdots, N \) and \( i \neq j \), are given by

\[
\hat{a}_{ij} = \frac{\sigma(J^{ij}(t))}{\sigma(O^i(t))}.
\]
Furthermore, if $\Pi^i(t)$ is nonsingular, for each $t \in T$ and $i = 1, 2, \cdots, N$, the estimate of $\mu^i$ is given by

$$\hat{\mu}^i = (\Pi^i(t))^{-1}\Xi^i(t).$$  \hspace{1cm} (46)

**Remark 3.** Theorem 4.1 gives unique estimates of the appreciation rates $\mu^i$ under the assumption that $\Pi^i(t)$ is nonsingular. In general, the system

$$\Pi^i(t) \mu^i = \Xi^i(t)$$  \hspace{1cm} (47)

may not have a unique solution or there may be no solution at all. However, it is known that the Moore-Penrose inverse (pseudoinverse) for any matrix always exists and is unique. Let $(\Pi^i(t))^\dagger$ be the Moore-Penrose inverse of $\Pi^i(t)$, for each $t \in T$ and $i = 1, 2, \cdots, N$. Using $(\Pi^i(t))^\dagger$, one may consider

$$(\Pi^i(t))^\dagger\Xi^i(t)$$  \hspace{1cm} (48)

as a “best-fit” solution to the system of linear equations (47), for each $t \in T$ and $i = 1, 2, \cdots, N$.

Recall that in regression analysis, a vector $\xi^i \in \mathbb{R}^n$ is called a least squares solution to (47) if

$$||\Pi^i(t)\xi^i - \Xi^i(t)|| \leq ||\Pi^i(t)\xi^i - \Xi^i(t)||, \quad \forall \xi^i \in \mathbb{R}^n,$$

for each $t \in T$ and $i = 1, 2, \cdots, N$. Among all least squares solutions, a vector $\hat{\mu}^i$ is called a minimum-norm least squares solution to (47) if

$$||\hat{\mu}^i|| < ||\xi^i||,$$

for all other least squares solutions $\xi^i$. One may refer to Definition 2.1.1 in Campbell and Meyer [2] for more details about the minimum-norm least squares solution, which is called the minimal least squares solution therein. It follows from Theorem 2.1.1 in Campbell and Meyer [2] that (48) is the minimum-norm least squares solution to (47), i.e.,

$$\hat{\mu}^i = (\Pi^i(t))^\dagger\Xi^i(t).$$

It is well known that $(\Pi^i(t))^\dagger = (\Pi^i(t))^{-1}$ if $\Pi^i(t)$ is nonsingular. Thus when $\Pi^i(t)$ is nonsingular, the minimum-norm least squares solution coincides with (46).

To compute the estimates $\hat{a}_{ij}$ and $\hat{\mu}^i$, for each $i, j = 1, 2, \cdots, N$, one can implement a robust filter bank including the stochastic differential equations described by Propositions 2 and 4. Then these estimates can be computed by the standard procedures of the robust-filter-based EM algorithm in Elliott et al. [5].

5. The maximum principle approach. In this section, we adopt the maximum principle approach to discuss the “filtered” consumption-portfolio optimization problem of the agent. With the robust filters and estimates presented in the previous section, we only need to solve a stochastic optimal control problem with complete observations. We derive a closed-form solution of the optimal consumption-portfolio strategy when the agent has a power utility.

We now define two auxiliary admissible sets $A^i(v)$ and $A^i_1(v)$, where $A^i(v)$ and $A^i_1(v)$ are the same as $A(v)$ and $A_1(v)$, respectively, except that the wealth process $V$ is allowed to be negative, i.e., $V(t) \in \mathbb{R}, t \in T$, $\mathcal{F}$-a.s., in $A^i(v)$ and $A^i_1(v)$.
Rather than solving Problem (16) directly, we consider an auxiliary consumption-portfolio problem first:

\[
\begin{cases}
\text{maximize} & J(v, \pi, c) = E^v \left[ \int_0^T e^{-\rho t} U_1(c(t)) dt + e^{-\rho T} U_2(V(T)) \right], \\
\text{subject to} & (\pi, c) \in A'_1(v) \quad \text{and} \quad (V(t), \pi, c) \text{ satisfy } (15).
\end{cases}
\] (49)

Since no state constraint is imposed in Problem (49), we can apply the sufficient maximum principle directly. Indeed, we shall show that the original problem (16) and the auxiliary problem (49) have the same solution after Problem (49) is solved.

To pave the way for the maximum principle approach, we restate the problem in the language of the stochastic control. Recall that the controlled state process and the performance functional of the consumption-portfolio optimization problem (49) are given by (15) and (12), respectively. In this case, the Hamiltonian of the problem is

\[ H(t, v, \pi, c, p, q) = e^{-\rho t} U_1(c) + [rv + \pi^\top (\hat{\mu}(t) - r1_n) - c]p + (\pi^\top \beta)q^\top. \] (50)

The adjoint processes \( p := \{p(t) | t \in \mathcal{T}\} \) and \( q := \{q(t) | t \in \mathcal{T}\} \) form the solution pair of the following adjoint equation:

\[ dp(t) = -rp(t)dt + q(t)d\hat{W}(t), \quad p(T) = e^{-\rho T} U'_2(V(T)). \] (51)

Note that our consumption-portfolio problem is a special case of the stochastic control problem. Furthermore, from the condition that \( U_1 \) and \( U_2 \) are concave functions, we see that both the terminal cost and the Hamiltonian are concave. Therefore, the sufficient maximum principle can be applied in our problem. To make the problem solvable and explicit, we consider the power utility functions for the running and terminal costs. That is

\[ U_1(c(t)) = \frac{(c(t))^{1-\gamma}}{1-\gamma}, \quad U_2(V(T)) = \frac{(V(T))^{1-\gamma}}{1-\gamma}, \] (52)

where \( 0 < \gamma < 1 \).

To find a solution \((p, q)\) to (51), we now consider the following trial solution

\[ p(t) = e^{-\rho t} (V(t))^{-\gamma} f(t), \] (53)

where \( f \) is a continuously differentiable function with terminal condition \( f(T) = 1 \). It remains to determine the function \( f(t) \).

In what follows, let \( \pi^* := \{\pi^*(t) | t \in \mathcal{T}\} \) and \( c^* := \{c^*(t) | t \in \mathcal{T}\} \) be candidates of the optimal portfolio and consumption processes, respectively. Suppose that \( V^* := \{V^*(t) | t \in \mathcal{T}\}, p^* := \{p^*(t) | t \in \mathcal{T}\} \) and \( q^* := \{q^*(t) | t \in \mathcal{T}\} \) are the state process and the adjoint processes corresponding to \( \pi^* \) and \( c^* \).

From the sufficient maximum principle, maximizing \( H(t, V^*(t), \pi, c, p^*(t), q^*(t)) \) with respect to \( \pi \) and \( c \) leads to the following first-order conditions

\[ e^{-\rho t} U'_1(c^*(t)) - p^*(t) = 0, \] (54)

and

\[ (\hat{\mu}(t) - r)p^*(t) + \beta q^*(t)^\top = 0. \] (55)

Solving (54) gives

\[ c^*(t) = [e^{\rho t} p^*(t)]^{-\frac{1}{\gamma}}. \] (56)
To obtain the expression of the function \( f \), we set \( \pi(t) := \pi^*(t) \), \( c(t) := c^*(t) \), \( V(t) := V^*(t) \), \( p^*(t) := p^*(t) \) and \( q(t) := q^*(t) \) in (15), (51) and (53). Then applying Itô’s differentiation rule to (53) yields

\[
dp^*(t) = e^{-\rho t} \left\{ \left[ -\rho (V^*(t))^{-\gamma} f(t) + (V^*(t))^{-\gamma} f'(t) \right] dt - \gamma (V^*(t))^{-\gamma-1} f(t) dV^*(t) + \frac{1}{2} \gamma (\gamma + 1) (V^*(t))^{-\gamma-2} f(t) \pi^*(t) \theta^*(t) dt \right\}
\]

where

\[
\alpha(t) := -\rho - \gamma r - \gamma \pi^*(t)^\top \big( \hat{\mu}(t) - r 1_n \big) (V^*(t))^{-1}
\]

\[
+ \frac{1}{2} \gamma (\gamma + 1) \pi^*(t)^\top \theta^*(t) (V^*(t))^{-2} .
\]

Comparing the coefficients of (57) with those of (51), we have

\[
f'(t) + (\alpha(t) + r) f(t) + \gamma (f(t))^{\gamma-1} = 0 ,
\]

\[
q^*(t) = -e^{-\rho t} (V^*(t))^{-\gamma-1} f(t) \pi^*(t)^\top \beta .
\]

Moreover, substituting (53) and (60) into (55) gives the optimal portfolio process

\[
\pi^*(t) = \Theta^{-1} (\hat{\mu}(t) - r 1_n) \frac{V^*(t)}{\gamma} .
\]

Combining with (58), we have

\[
\alpha(t) + r = -\rho + (1 - \gamma) r + \frac{1 - \gamma}{2\gamma} \theta(t) ,
\]

where

\[
\theta(t) = (\hat{\mu}(t) - r 1_n)^\top \Theta^{-1} (\hat{\mu}(t) - r 1_n) .
\]

Using the change of variable \( f(t) = (h(t))^\gamma \) leads to

\[
h'(t) + \eta(t) h(t) + 1 = 0 ,
\]

\[
h(T) = 1 ,
\]

where

\[
\eta(t) := \frac{\alpha(t) + r}{\gamma} .
\]

Solving (62) gives

\[
h(t) = e^{\int_t^T \eta(s) ds} + \int_t^T e^{-\int_s^T \eta(\zeta) d\zeta} ds ,
\]

and hence

\[
f(t) = \left[ e^{\int_t^T \eta(s) ds} + \int_t^T e^{-\int_s^T \eta(\zeta) d\zeta} ds \right]^\gamma .
\]

Substituting (53) and (64) into (66) gives the optimal consumption process

\[
e^*(t) = \frac{V^*(t)}{h(t)} .
\]
Furthermore, substituting \( \pi^*(t) \) and \( c^*(t) \) into \([15]\) gives

\[
dV^*(t) = V^*(t) \left\{ r + \frac{1}{\gamma} \theta(t) - \frac{1}{h(t)} \right\} dt + \frac{1}{\gamma} (\bar{\mu}(t) - r1_n)^\top (\beta^\top)^{-1} d\bar{W}(t) \right\}.
\] (66)

Solving gives

\[
V^*(t) = v \exp \left\{ \int_0^t \left[ r + \frac{2\gamma - 1}{2\gamma^2} \theta(s) - \frac{1}{h(s)} \right] ds 
+ \int_0^t \frac{1}{\gamma} (\bar{\mu}(s) - r1_n)^\top (\beta^\top)^{-1} d\bar{W}(s) \right\}.
\] (67)

Clearly, the wealth process \( V^* \) is non-negative. Then, the pair of optimal portfolio and consumption processes \((\pi^*, c^*)\) is not only in \( A_1(v) \), but also in \( A'_1(v) \). More precisely,

\[
(\pi^*, c^*) \in A_1(v) \subset A'_1(v).
\] (68)

Indeed, the pair \((\pi^*, c^*)\) is also the optimal control of the original problem \([16]\). Otherwise, we can construct a contradiction. Suppose that there exists an admissible pair \((\pi^{**}, c^{**})\) in \( A_1(v) \), different from \((\pi^*, c^*)\), such that

\[
J(v, \pi^*, c^*) < J(v, \pi^{**}, c^{**}).
\] (69)

Note that the pair \((\pi^{**}, c^{**})\) is also in the admissible set \( A'_1(v) \). Since \((\pi^*, c^*)\) is the optimal control of the auxiliary problem \([16]\), we have

\[
J(v, \pi^*, c^*) = \max_{(\pi, c) \in A'_1(v)} J(v, \pi, c) \geq J(v, \pi^{**}, c^{**}).
\] (70)

Evidently, \([69]\) contradicts with \([70]\). Then we must have

\[
J(v, \pi^*, c^*) \geq J(v, \pi, c), \quad \forall (\pi, c) \in A_1(v).
\]

Consequently,

\[
J(v, \pi^*, c^*) = \max_{(\pi, c) \in A_1(v)} J(v, \pi, c).
\]

In other words, \((\pi^*, c^*)\) forms the optimal control pair of portfolio and consumption for both the auxiliary and original problems.

Finally, we conclude this section with a theorem which presents the optimal portfolio and consumption processes and the associated wealth process of the original problem \([16]\).

**Theorem 5.1.** The optimal portfolio and consumption processes and the associated wealth process of Problem \([16]\) are given by

\[
\pi^*(t) = \Theta^{-1}(\bar{\mu}(t) - r1_n) \frac{V^*(t)}{\gamma}, \quad c^*(t) = \frac{V^*(t)}{h(t)},
\]

and

\[
V^*(t) = v \exp \left\{ \int_0^t \left[ r + \frac{2\gamma - 1}{2\gamma^2} \theta(s) - \frac{1}{h(s)} \right] ds 
+ \int_0^t \frac{1}{\gamma} (\bar{\mu}(s) - r1_n)^\top (\beta^\top)^{-1} d\bar{W}(s) \right\},
\]

where

\[
\bar{\mu}(t) = \frac{\mu(t)q(t)}{\Gamma(t)q(t), 1N},
\]

\[
\theta(t) = (\bar{\mu}(t) - r)^\top \Theta^{-1}(\bar{\mu}(t) - r),
\]
and
\[
    h(t) = \exp \left\{ \int_t^T \frac{1}{\gamma} \left[ -\rho + (1 - \gamma) r + \frac{1 - \gamma}{2\gamma} \theta(s) \right] ds \right\} 
    + \int_t^T \exp \left\{ - \int_s^t \frac{1}{\gamma} \left[ -\rho + (1 - \gamma) r + \frac{1 - \gamma}{2\gamma} \theta(s) \right] ds \right\} \left( \frac{1}{\gamma} \right) ds .
\]

6. Conclusion. We have investigated a consumption-portfolio optimization problem in a hidden Markov model with multiple risky assets in the paradigm of model uncertainty. Robust filters of the chain and estimates of some other parameters of the model have been provided. We have derived a closed-form solution of the optimal portfolio-consumption strategy via the stochastic maximum principle when the agent has a power utility.

Acknowledgments. We would like to thank the referee for helpful comments. Tak Kuen Siu would like to thank the Australian Research Council.

Appendix.

Proof of Proposition 1. Applying Itô’s differentiation rule to \( \Lambda(t)X(t) \) gives
\[
    \Lambda(t)X(t) = \Lambda(0)X(0) + \int_0^t X(s)d\Lambda(s) + \int_0^t \Lambda(s)dX(s) + [X, \Lambda](t) 
    = \Lambda(0)X(0) + \sum_{k,l} \int_0^t X(s)\Lambda(s) (\bar{g}_k, X(s)) \phi_{kl} dZ_l(s) 
    + \int_0^t \Lambda(s)AX(s)ds + \int_0^t \Lambda(s)dM(s) .
\]
Since \( Y \) and \( M \) are independent under \( \mathbb{F} \), conditioning both sides of the above equation on \( \mathcal{F}_Y(t) \) under \( \mathbb{F} \) makes the stochastic integral with respect to \( M \) vanish. This completes the proof. \( \square \)

Proof of Proposition 2. Note that \( \bar{q}(0) = q(0) = E[X(0)|\mathcal{F}_Y(0)] = E[X(0)] \).
Since diagonal matrices commute, applying Itô’s differentiation rule to \( \bar{q}(t) = (\Gamma(t))^{-1}q(t) \) gives
\[
    \bar{q}(t) = q(0) + \int_0^t (\Gamma(s))^{-1}d\bar{q}(s) + \int_0^t d(\Gamma(s))^{-1}q(s) + [\Gamma^{-1}, q](t) 
    = q(0) + \int_0^t (\Gamma(s))^{-1}A\bar{q}(s)ds + \int_0^t (\Gamma(s))^{-1}(\bar{B} \otimes \Sigma^{-\frac{1}{2}})dZ(s)q(s) 
    - \int_0^t (\Gamma(s))^{-1}(\bar{B} \otimes \Sigma^{-\frac{1}{2}})dZ(s)\bar{q}(s) + \int_0^t (\Gamma(s))^{-1}(\bar{B} \otimes \Sigma^{-1} \otimes \bar{B})d\bar{q}(s) 
    - \int_0^t (\Gamma(s))^{-1}(\bar{B} \otimes \Sigma^{-1} \otimes \bar{B})d\bar{q}(s) 
    = q(0) + \int_0^t (\Gamma(s))^{-1}A\Gamma(s)\bar{q}(s)ds .
\]
This completes the proof. \( \square \)
Proof of Proposition 3. We shall only derive the dynamics for \( \sigma(X(t)G^i(t)) \). The derivations of \( \sigma(X(t)J^{-1}(t)) \), \( \sigma(X(t)H^i_{k_l}) \), \( \sigma(X(t)H^i_{k_1}) \) and \( \sigma(X(t)H^i) \) are similar. By Itô’s differentiation rule,

\[
X(t)G^i(t) = \int_0^t X(s)\langle X(s), e_i \rangle \beta_{k_l}^{-1} \phi_{k_l} dZ_l(s) + \int_0^t G^i_k(s)AX(s)ds + \int_0^t G^i_k(s)dM(s)
\]

Again, applying Itô’s differentiation rule to \( \Lambda(t)X(t)G^i_k(t) \) gives

\[
\Lambda(t)X(t)G^i_k(t) = \int_0^t \Lambda(s)\langle X(s), e_i \rangle e_i \beta_{k_l}^{-1} \phi_{k_l} dZ_l(s) + \int_0^t \Lambda(s)G^i_k(s)AX(s)ds + \int_0^t \Lambda(s)G^i_k(s)dM(s)
\]

where

\[
\sum_{k_0, l_0=1}^n \int_0^t X(s)G^i_k(s)\Lambda(s) \langle \tilde{g}_{k_0}, X(s) \rangle \phi_{k_l} l_0 dZ_{l_0}(s)
\]

\[
= \sum_{k_0, l_0=1}^n \int_0^t \bar{B}_{k_0} X(s)G^i_k(s)\Lambda(s) \phi_{k_l} l_0 dZ_{l_0}(s)
\]

\[
= \int_0^t (\bar{B} \otimes \Sigma^{-\frac{1}{2}})dZ(s)X(s)G^i_k(s)\Lambda(s)
\]

and

\[
\sum_{k_0=1}^n \int_0^t \Lambda(s)\langle X(s), e_i \rangle X(s)\beta_{k_l}^{-1} \phi_{k_l} \langle \tilde{g}_{k_0}, X(s) \rangle \phi_{k_l} l_0 e_i ds
\]

\[
= \sum_{k_0=1}^n \int_0^t \Lambda(s)\langle X(s), e_i \rangle \beta_{k_l}^{-1} \phi_{k_l} \tilde{g}_{k_0} \phi_{k_l} l_0 e_i ds
\]

Then conditioning both sides on \( \mathcal{F}^Y(t) \) with respect to \( \mathcal{F} \) gives the desired result. \( \square \)

Proof of Proposition 2. We shall only prove Eq. (44). The proofs of Eqs. (39) - (43) are similar.

Applying Itô’s differentiation rule to \( \sigma(X(t)G^i_k(t)) = (\Gamma(t))^{-1}\sigma(X(t)G^i_k(t)) \) gives

\[
\sigma(X(t)G^i_k(t)) = \int_0^t (\Gamma(s))^{-1}\langle q(s), e_i \rangle e_i \beta_{k_l}^{-1} \phi_{k_l} dZ_l(s) + \int_0^t (\Gamma(s))^{-1}A\sigma(X(s)G^i_k(s))ds
\]
Proof of Theorem 4.1. Suppose we are given a set of parameters \( \hat{\theta} \). Using the stochastic integration by parts and Proposition 2, we have:

\[
\sum_{k_0, l_0=1}^{n} \int_{0}^{t} (\Gamma(s))^{-1} \mathbf{B}_{k_0} \sigma(X(s)G_{kl}(s)) \phi_{k_0l_0} dZ_{l_0}(s)
+ \sum_{k_0=1}^{n} \int_{0}^{t} (\Gamma(s))^{-1} \langle q(s), e_i \rangle \bar{\beta}_{k}^{-1} \phi_{kl} \mathbf{g}_{k_0}(s) \phi_{k_0l} ds
\]

\[- \int_{0}^{t} (\Gamma(s))^{-1} \sigma(X(s)G_{kl}(s))(\mathbf{B} \otimes \Sigma^{-1}) dZ(s)
+ \sum_{k_0=1}^{n} \int_{0}^{t} \mathbf{B}_{k_0} (\Gamma(s))^{-1} \mathbf{B}_{l_0} \sigma(X(s)G_{kl}(s)) \psi_{k_0l_0} ds
\]

\[- \int_{0}^{t} (\Gamma(s))^{-1} \sigma(X(s)G_{kl}(s))(\mathbf{B} \otimes \Sigma^{-1} \otimes \mathbf{B}) ds
- \sum_{k_0=1}^{n} \int_{0}^{t} \mathbf{B}_{k_0} (\Gamma(s))^{-1} \langle q(s), e_i \rangle \phi_{k_0l} \bar{\beta}_{k}^{-1} \phi_{kl} ds e_i .
\]

Using the stochastic integration by parts and Proposition 2,

\[
\int_{0}^{t} \langle \mathbf{q}(s), e_i \rangle \mathbf{e}_i \bar{\beta}_{k}^{-1} \phi_{kl} dZ_{l}(s)
= \langle \mathbf{q}(t), e_i \rangle \bar{\beta}_{k}^{-1} \phi_{kl} Z_{l}(t) e_i - \int_{0}^{t} \langle (\Gamma(s))^{-1} A \Gamma(s) \mathbf{q}(s), e_i \rangle \bar{\beta}_{k}^{-1} \phi_{kl} Z_{l}(s) ds e_i .
\]

This completes the proof. \( \square \)

\[\text{Proof of Theorem 4.1} \] Suppose we are given a set of parameters \( \hat{\theta} := \{ \pi_{ij}, \pi^t | i, j = 1, 2, \cdots, N \} \) as the starting values for unknown parameters of the hidden Markov model, \( \theta := \{ a_{ij}, \mu^t | i, j = 1, 2, \cdots, N \} \). Note that the estimates \( \hat{a}_{ij} \) of the diagonal element \( a_{ii} \) of the rate matrix \( \mathbf{A} \) can be computed by noticing that \( \sum_{j=1}^{N} \hat{a}_{ji} = 0 \), for each \( i = 1, 2, \cdots, N \). So we do not need to estimate \( a_{ii} \).

Let \( \mathcal{P}_\theta \) and \( \mathcal{P}_\hat{\theta} \) be the probability measures under which the model has the sets of parameters \( \theta \) and \( \hat{\theta} \), respectively. To change the set of parameters from \( \hat{\theta} \) to \( \theta \), we introduce the following pseudo-likelihood function:

\[
\frac{d \mathcal{P}_\theta}{d \mathcal{P}_\hat{\theta}} (\xi(t)) := \prod_{i,j=1,i\neq j}^{N} K_{ij}(t) \exp \left\{ \sum_{k,l=1}^{n} \int_{0}^{t} \left( \langle \mathbf{g}_k, \mathbf{X}(s) \rangle - \langle \mathbf{f}_k, \mathbf{X}(s) \rangle \right) \phi_{kl} dZ_{l}(s) \right\}
- \frac{1}{2} \sum_{k,l=1}^{n} \int_{0}^{t} \left( \langle \mathbf{g}_k, \mathbf{X}(s) \rangle \psi_{kl} (\mathbf{g}_l, \mathbf{X}(s)) \right) ds .
\]
where $\bar{g}_k := (\bar{g}_1, \bar{g}_2, \ldots, \bar{g}_N)^\top \in \mathbb{R}^N$,

and

$$
K_{ij}(t) := \left( \frac{\mu_{ji}}{\sigma_{ji}} \right)^{J^{ji}(t)} \exp \left( \int_0^t (\bar{a}_{ji} - a_{ji}) \langle X(s), e_i \rangle \, ds \right).
$$

Then the pseudo-log-likelihood function is given by

$$
\ln \left( \frac{d\mathcal{P}_0}{d\mathcal{P}_\theta} \bigg|_{\theta(t)} \right) = \sum_{i,j=1,i\neq j}^N (J^{ij}(t) (\ln a_{ij} - \ln \pi_{ij}) + \int_0^t (\bar{a}_{ij} - a_{ij}) \langle X(s), e_i \rangle \, ds) + \sum_{k,l=1}^n \int_0^t (\bar{g}_k, X(s)) - \langle \bar{g}_k, X(s) \rangle \psi_{kl} dZ_t(s) - \frac{1}{2} \sum_{k,l=1}^n \int_0^t (\bar{g}_k, X(s)) \psi_{kl} \langle \bar{g}_l, X(s) \rangle \, ds + \frac{1}{2} \sum_{k,l=1}^n \int_0^t \langle \bar{g}_k, X(s) \rangle \psi_{kl} \langle \bar{g}_l, X(s) \rangle \, ds.
$$

Conditioning both sides on $\mathcal{F}^Y(t)$ gives

$$
Q(\theta, \bar{\theta}) := \mathbb{E} \left[ \ln \left( \frac{d\mathcal{P}_0}{d\mathcal{P}_\theta} \bigg|_{\theta(t)} \right) \bigg| \mathcal{F}^Y(t) \right] = \sum_{i,j=1,i\neq j}^N \left( \sigma(J^{ij}(t)) \ln a_{ij} - a_{ij} \sigma(O^{ij}(t)) \right)
$$

$$
+ \sum_{i=1}^N \sum_{k,l=1}^n \left( \mu_{ki} \sigma(G^{ki}_i(t)) - \frac{\mu_{ki}^2}{2} \sigma(H^{ki}_i(t)) + \frac{\mu_{ki}^2}{4} \sigma(H^{ki}_i(t)) \right)
$$

$$
+ R(\bar{\theta}),
$$

where $R(\bar{\theta})$ does not involve the parameter $\theta$.

Taking the partial derivatives of $Q(\theta, \bar{\theta})$ with respect to $a_{ij}$ and $\mu_{ki}$ gives

$$
\frac{\sigma(J^{ij}(t))}{a_{ij}} - \sigma(O^{ij}(t)) = 0,
$$

(A1)

and

$$
\sum_{l=1}^n \sigma(G^{kl}_i(t)) - \frac{1}{2} \sum_{l=1}^n \mu_{ki} \left( \sigma(H^{kl}_i(t)) + \sigma(H^{kl}_i(t)) \right)
$$

$$
+ \frac{1}{4} \sum_{l=1}^n \left( \sigma(H^{kl}_i(t)) + \sigma(H^{kl}_i(t)) \right)
$$
\[
\sum_{i=1}^{n} \sigma(G_{ik}(t)) - \sum_{i=1}^{n} \mu_i \sigma(H_{ik}(t)) + \frac{1}{2} \sum_{i=1}^{n} \sigma(H_{ik}^2(t)) = 0 ,
\]
(A2)
for each \(i, j = 1, 2, \cdots, N\) and \(k = 1, 2, \cdots, n\). We can rewrite Eq. (A2) in the following matrix form:
\[
\Pi^i(t) \mu^i = \Xi^i(t) ,
\]
(A3)
which leads to the desired result immediately.

REFERENCES
[1] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, 2nd edition, Springer, New York, 2003.
[2] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*, Pitman, London, 1979.
[3] J. M. C. Clark, *The design of robust approximations to the stochastic differential equations for nonlinear filtering*, in *Communications Systems and Random Process Theory* (ed. J. K. Skwirzynski), NATO Advanced Study Inst. Ser., Ser. E: Appl. Sci., No. 25, Sijthoff & Noordhoff, Alphen aan den Rijn, 1978, 721–734.
[4] R. J. Elliott, *Stochastic Calculus and Applications*, Springer, Berlin-Heidelberg-New York, 1982.
[5] R. J. Elliott, L. Aggoun and J. B. Moore, *Hidden Markov Models: Estimation and Control*, Springer, Berlin-Heidelberg-New York, 1994.
[6] R. J. Elliott, T. K. Siu and A. Badescu, *On mean-variance portfolio selection under a Markovian regime-switching model*, *Economic Modelling*, 27 (2010), 678–686.
[7] R. J. Elliott and T. K. Siu, *An HMM approach for optimal investment of an insurer*, *International Journal of Robust Nonlinear Control*, 22 (2012), 778–807.
[8] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, Berlin-Heidelberg-New York, 1975.
[9] S. M. Goldfeld and R. E. Quandt, *A Markov model for switching regressions*, *Journal of Econometrics*, 1 (1973), 3–16.
[10] X. Guo, *Information and option pricings*, *Quantitative Finance*, 1 (2001), 38–44.
[11] J. D. Hamilton, *A new approach to the economic analysis of nonstationary time series and the business cycle*, *Econometrica*, 57 (1989), 357–384.
[12] L. P. Hansen and T. J. Sargent, *Robustness*, Princeton University Press, Princeton, 2008.
[13] M. Jeanblanc, M. Yor and M. Chesney, *Mathematical Methods for Financial Markets*, Springer, New York, 2009.
[14] G. Kallianpur, *Stochastic Filtering Theory*, Springer, Berlin-Heidelberg-New York, 1980.
[15] R. Korn, T. K. Siu and A. Badescu, *Asset allocation for a DC pension fund under regime-switching environment*, *European Actuarial Journal*, 1 (2011), 361–377.
[16] H. M. Markowitz, *Portfolio selection*, *The Journal of Finance*, 7 (1952), 77–91.
[17] R. C. Merton, *Lifetime portfolio selection under uncertainty: the continuous-time model*, *The Review of Economics and Statistics*, 51 (1969), 247–257.
[18] R. C. Merton, *Optimal consumption and portfolio rules in a continuous time model*, *Journal of Economic Theory*, 3 (1971), 373–413.
[19] R. C. Merton, *On estimating the expected return on the market*, *Journal of Financial Economics*, 8 (1980), 323–361.
[20] W. Putschlogl and J. Sass, *Optimal consumption and investment under partial information*, *Decisions in Economics and Finance*, 31 (2008), 137–170.
[21] R. E. Quandt, *The estimation of parameters of linear regression system obeying two separate regimes*, *Journal of the American Statistical Association*, 55 (1958), 873–880.
[22] U. Rieder and N. Bäuerle, *Portfolio optimization with unobservable Markov-modulated drift process*, *Journal of Applied Probability*, 42 (2005), 362–378.
[23] J. Sass and U.G. Haussmann, *Optimizing the terminal wealth under partial information: the drift process as a continuous time Markov chain*, *Finance and Stochastics*, 8 (2004), 553–577.
[24] A. Schied, H. Föllmer and S. Weber, *Robust preferences and robust portfolio choice*, *Handbook of Numerical Analysis*, 15 (2009), 29–87.
[25] Y. Shen and T. K. Siu, *Asset allocation under stochastic interest rate with regime switching*, *Economic Modelling*, 29 (2012), 1126–1136.
[20] T. K. Siu, Long-term strategic asset allocation with inflation risk and regime switching, *Quantitative Finance*, 11 (2011), 1565–1580.

[27] T. K. Siu, A BSDE approach to risk-based asset allocation of pension funds with regime switching, *Annals of Operations Research*, 201 (2012), 449–473.

[28] T. K. Siu, A BSDE approach to optimal investment of an insurer with hidden regime switching, *Stochastic Analysis and Applications*, 31 (2013), 1–18.

[29] T. K. Siu, A stochastic flows approach for asset allocation with hidden economic environment, *International Journal of Stochastic Analysis*, 2015 (2015), Article ID 462524, 11pp.

[30] L. R. Sotomayor and A. Cadenillas, Explicit solution of consumption-investment problems in financial markets with regime switching, *Mathematical Finance*, 9 (2009), 251–279.

[31] H. Tong, On a threshold model, in *Pattern Recognition and Signal Processing* (ed. C. H. Chen), NATO ASI Series E: Applied Sc., No. 29, Sijthoff & Noordhoff, The Netherlands, 1978, 575–586.

[32] H. Tong, *Threshold Models in Non-linear Time Series Analysis*, Springer-Verlag, Berlin, 1978.

[33] G. Yin and X. Zhou, Markowitz’s mean-variance portfolio selection with regime switching: From discrete-time models to their continuous-time limits, *IEEE Transactions Automatic Control*, 49 (2004), 349–360.

[34] K. F. C. Yiu, J. Liu, T. K. Siu and W. K. Ching, Optimal portfolios with regime switching and value-at-risk constraint, *Automatica*, 46 (2010), 979–989.

[35] X. Zhang, T. K. Siu and Q. Meng, Portfolio selection in the enlarged Markovian regime-switching market, *SIAM Journal on Control and Optimization*, 48 (2010), 3368–3388.

[36] X. Zhang, R. J. Elliott and T. K. Siu, A stochastic maximum principle for a Markov regime-switching jump-diffusion model and its application to finance, *SIAM Journal on Control and Optimization*, 50 (2012), 964–990.

[37] X. Zhou and G. Yin, Markowitz’s mean-variance portfolio selection with regime switching: A continuous-time model, *SIAM Journal on Control and Optimization*, 42 (2003), 1466–1482.

Received July 2013; revised June 2015.

E-mail address: skyshen87@gmail.com
E-mail address: ktksiu2005@gmail.com