Minimizers of $L^2$-critical inhomogeneous variational problems with a spatially decaying nonlinearity in bounded domains

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Abstract

We consider the minimizers of $L^2$-critical inhomogeneous variational problems with a spatially decaying nonlinear term in an open bounded domain $\Omega$ of $\mathbb{R}^N$ which contains 0. We prove that there is a threshold $a^* > 0$ such that minimizers exist for $0 < a < a^*$ and the minimizer does not exist for any $a > a^*$. In contrast to the homogeneous case, we show that both the existence and nonexistence of minimizers may occur at the threshold $a^*$ depending on the value of $V(0)$, where $V(x)$ denotes the trapping potential. Moreover, under some suitable assumptions on $V(x)$, based on a detailed analysis on the concentration behavior of minimizers as $a \rightarrow a^*$, we prove local uniqueness of minimizers when $a$ is close enough to $a^*$.

Keywords: $L^2$-critical; Spatially decaying nonlinear; Minimizers; Concentration behavior; Local uniqueness.

1 Introduction

In this paper, we consider the following $L^2$-critical constraint inhomogeneous variational problem

$$e(a) := \inf_{u \in \mathcal{M}} E_a(u),$$

where the energy functional $E_a(u)$ contains a spatially decaying nonlinearity and is defined by

$$E_a(u) := \int_{\Omega} (|\nabla u|^2 + V(x)|u|^2)dx - \frac{a}{1 + \beta^2} \int_{\Omega} \frac{|u|^{2+2\beta^2}}{|x|^\beta}dx,$$

and the space $\mathcal{M}$ is defined as

$$\mathcal{M} := \{u \in H^1_0(\Omega) : \int_\Omega |u(x)|^2dx = 1\}. $$

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Here we assume that $N \geq 1$, $a > 0$, $0 < b < \min \{2, N\}$ and $\beta = \sqrt{\frac{2-b}{N}}$ is the $L^2$-critical exponent. Moreover, we suppose that the bounded domain $\Omega$ of $\mathbb{R}^N$ and the trapping potential $V(x) \geq 0$ satisfy the following assumptions respectively.

$(V_0)$. $\Omega$ is an open connected domain containing 0 and $\partial \Omega \in C^1$.

$(V_1)$. $V(x) \in C(\overline{\Omega})$ and $\min_{x \in \partial \Omega} V(x) = 0$.

From now on, we assume that $\Omega$ satisfies $(V_0)$ throughout this paper.

The variational problem (1.1) arises in various physical contexts, including the propagation of a laser beam in the optical fiber, Bose-Einstein condensates (BECs), and nonlinear optics (cf. [1, 3, 21]). When $b = 0$, (1.1) is a homogeneous constraint variational problem. For $b = 0$ and $N = 2$, Luo and Zhu proved that there exists a critical constant $a_0 > 0$ such that minimizers of $e(a)$ exist if and only if $0 < a < a_0$, and the limit behavior of minimizers as $a \searrow a_0$ is also analyzed in [23]. When $b > 0$, the variational problem (1.1) contains the inhomogeneous nonlinear term $K(x)|u|^{2+2\beta}$, where $K(x) = \frac{1}{|x|^\beta}$ admits $x = 0$ as a singular point. Compared with the homogeneous case where $b = 0$, we shall point out that our analysis on inhomogeneous problem (1.1) is more complex due to the singularity of $|x|^{-b}$.

If the bounded domain $\Omega$ is replaced by the whole space $\mathbb{R}^N$, the existence and nonexistence of minimizers for (1.1) and various quantitative properties of (1.1) were investigated extensively, see [3, 6, 7, 10, 13, 14, 16, 17, 22] and the references therein. Note that most of the problems in these studies are homogeneous. Recently, the inhomogeneous $L^2$-constrained variation problems were also analyzed in [6, 7], where the authors addressed the existence, nonexistence and limit behavior of minimizers mainly in the case that the inhomogeneous nonlinear term $f(x)|u(x)|^{2+2\beta}$ with $f(x) \in L^\infty(\mathbb{R}^N)$. More recently, Luo and Zhang in [22] analyzed the variational problem (1.1) defined in $\mathbb{R}^N$, where the authors made full use of the following nonlinear Schrödinger Equation

$$- \Delta u + u - |x|^{-b} |u|^{2\beta} u = 0 \quad \text{in} \quad \mathbb{R}^N. \quad (1.4)$$

Compared with [22], we emphasize that the bounded domain $\Omega$ is variable under the scaling transformation and the optimal constant of the Gagliardo-Nirenberg inequality for the bounded domain $\Omega$ is not attained. Therefore, we need to explore some new analytic approaches to overcome these difficulties.

It is well-known from [3, 9, 25] that (1.1) admits a unique positive radial solution for any $N \geq 1$. Such a positive solution of (1.4) is always denoted by $Q = Q(|x|)$ throughout this paper. Moreover, we cite from [10] the following Gagliardo-Nirenberg inequality

$$\frac{a^*}{1 + \beta^2} \int_{\mathbb{R}^N} |x|^{-b} |u(x)|^{2+2\beta} \, dx \leq \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, dx \left( \int_{\mathbb{R}^N} |u(x)|^2 \, dx \right)^{\beta^2} \quad \text{in} \quad H^1(\mathbb{R}^N), \quad (1.5)$$

where the constant $a^* > 0$ is given by

$$a^* := \|Q\|_{L^{2\beta}(\mathbb{R}^N)}^{2\beta}, \quad (1.6)$$

and the identity of (1.5) holds if and only if $u(x) = mn^\frac{N}{2} Q(nx)$ $(m, n \neq 0$ are arbitrary). Recall also from [13, Theorem 2.2] that $Q$ admits the following exponential decay

$$|Q(x)|, \quad |\nabla Q(x)| \leq C e^{-|x|} \quad \text{as} \quad |x| \to \infty. \quad (1.7)$$
Moreover, we can derive from (1.4) and (1.5) that $Q$ satisfies
\[
\int_{\mathbb{R}^N} |\nabla Q|^2 \, dx = \frac{1}{\beta^2} \int_{\mathbb{R}^N} |Q|^2 \, dx = \frac{1}{1 + \beta^2} \int_{\mathbb{R}^N} |x|^{-b} |Q|^{2+2\beta^2} \, dx.
\] (1.8)

We next define the linearized operator $L$ by
\[
L := -\Delta + 1 - (1 + 2\beta^2) |x|^{-b} Q^{2\beta^2} \text{ in } \mathbb{R}^N,
\] (1.9)
where $Q = Q(|x|) > 0$ is the unique positive solution of (1.4). It then follows from [13] that
\[
\ker(L) = \text{span}\{0\} \text{ if } N \geq 3. \quad (1.10)
\]

Stimulated by the works mentioned above, the main purpose of this paper is to address the existence and nonexistence of minimizers for (1.1), and analyze the local uniqueness of minimizers for a special class of $V(x)$.

Before stating our results, we first introduce the Gagliardo-Nirenberg type inequality: for any open bounded domain $\Omega \subset \mathbb{R}^N$, it holds
\[
\frac{a^*}{1 + \beta^2} \int_{\Omega} |x|^{-b} |u(x)|^{2+2\beta^2} \, dx \leq \int_{\Omega} |\nabla u(x)|^2 \, dx \left( \int_{\Omega} |u(x)|^2 \, dx \right)^{\beta^2}, \quad u \in H^1_0(\Omega), \quad (1.11)
\]
where the optimal constant $\frac{a^*}{1 + \beta^2}$ is not attained. The proof of (1.11) is left to the Appendix. Applying the inequality (1.11), we shall establish the following existence and nonexistence of minimizers for (1.1) under the general assumption ($V_1$).

**Theorem 1.1.** Let $N \geq 1$, $0 < b < \min\{2, N\}$, $\beta^2 = \frac{2-b}{N}$, and $V(x)$ satisfies ($V_1$), then

1. If $0 < a < a^* := \|Q\|_{L^2(\mathbb{R}^N)}^{2\beta^2}$, there exists at least one minimizer for (1.1).

2. If $a > a^*$, there is no minimizer for (1.1).

It is worth noting that the threshold $a^*$ stated in Theorem 1.1 is independent of the external potential $V(x)$ and the domain $\Omega$ as well. Moreover, in view of Theorem 1.1 it is therefore natural to wonder whether the minimizers of $e(a)$ exist for the case $a = a^*$. The following result presents that $e(a)$ may admit minimizers at the threshold $a^*$ for some trapping potentials $V(x)$ satisfying ($V_1$). Indeed, using the Gagliardo-Nirenberg inequality (1.11), one can check that $e(a^*) \geq 0$. On the other hand, taking the same trial function as in (2.8) below and letting $\tau \to \infty$, we can obtain that $e(a^*) \leq V(0)$. Therefore, we conclude from above that
\[
0 \leq e(a^*) \leq V(0),
\]
and our second result can be summarized as follows.

**Theorem 1.2.** Let $N \geq 1$, $0 < b < \min\{2, N\}$, $\beta^2 = \frac{2-b}{N}$, and $V(x)$ satisfies ($V_1$), then we have

1. If $0 \leq e(a^*) < V(0)$, then there exists at least one minimizer for $e(a^*)$.

2. If $V(0) = 0$, then there is no minimizer for $e(a^*)$. Moreover, $\lim_{a \nearrow a^*} e(a) = e(a^*) = V(0)$. 

3
In order to analyze the uniqueness of minimizers for $e(a)$ when $a$ is close enough to $a^*$, we first discuss the limit behavior of minimizers as $a \not\to a^*$. Let $u_a$ be a minimizer of $e(a)$. It then follows from the variational theory that $u_a$ satisfies the following nonlinear Schrödinger equation

$$-\Delta u_a + V(x)u_a - a|x|^{-b}|u_a|^{2\beta^2}u_a = \mu_au_a \text{ in } \Omega,$$

(1.12)

where $\mu_a \in \mathbb{R}$ is a suitable Lagrange multiplier associated to $u_a$.

For any minimizer $u_a$ of $e(a)$, it yields from [19, Theorem 6.17] that $E_a(u_a) = E_a(|u_a|)$, which further implies that $|u_a|$ is a minimizer of $e(a)$. By the strong maximum principle, we can derive from (1.12) that $|u_a| > 0$ holds in $\Omega$. Therefore, without loss of generality, we can restrict the minimizers of $e(a)$ to positive functions. Motivated by [12, 23], in order to analyze the limit behavior of $u_a$ as $a \not\to a^*$, some additional assumptions on $V(x)$ are required.

We shall assume that the external potential $V(x) \in C(\bar{\Omega})$ has $n \geq 1$ distinct zero points $x_i \in \Omega$ satisfying $V(x_i) = 0$ and $V(x) > 0$ for any $x \neq x_i$, where $i = 0, 1, \cdots, n - 1$. Considering Theorem [12, 23] (2), we suppose that $x_0 = 0$ is a minimum point of $V(x)$. More precisely, we suppose that there exist $p_i > 0$ and $C > 0$ such that

$$(V_2). \ V(x) = h(x)|x|^{p_0} \prod_{i=1}^{n-1} |x - x_i|^{p_i} \text{ with } C < h(x) < \frac{1}{C} \text{ for all } x \in \Omega, \text{ and } \lim_{x \to x_i} h(x) \text{ exists for all } 0 \leq i \leq n - 1.$$

For convenience, we denote

$$p := p_0 > 0,$$

(1.13)

and

$$\lambda := \left(\frac{p}{2} \int_{\mathbb{R}^N} |x|^p \overline{Q(x)}^2dx \lim_{x \to 0} (h(x) \prod_{i=1}^{n-1} |x - x_i|^{p_i})^{\frac{1}{p + 2}} \right) > 0.$$

(1.14)

By virtue of the conditions on $V(x)$, the main result of concentration behavior is stated as follows.

**Theorem 1.3.** Let $N \geq 1$, $0 < b < \min\{2, N\}$, $\beta^2 = \frac{2 - b}{N}$, $0 < a < a^*$, and $V(x)$ satisfies (V1) and (V2). Then for any positive minimizer $u_a$ of (1.1), we have

$$\lim_{a \not\to a^*} \left(\frac{(a^* - a)\|Q\|^2_2}{a^*\beta^2\lambda^{p+2}}\right)^{\frac{N}{p+2}} u_a \left(\left(\frac{(a^* - a)\|Q\|^2_2}{a^*\beta^2}\right)^\frac{1}{p+2} \frac{x}{\lambda} \right) = \frac{Q(x)}{\|Q\|_2} \quad (1.15)$$

strongly in $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, where $p > 0$ and $\lambda > 0$ are defined by (1.13) and (1.14) respectively.

We remark that the limit (1.15) is defined in $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ in the sense that $u_a(x) \equiv 0$ for all $x \in \mathbb{R}^N \setminus \Omega$. Because the limit equation (1.14) has no translation invariance, it is worth pointing out that $u_a$ must concentrate at the origin rather than any global minimum point of $V(x)$ as $a \not\to a^*$. Moreover, Theorem 1.3 gives explicitly concentration rates of $u_a$ as $a \not\to a^*$, that is

$$u_a(x) \approx \frac{1}{\|Q\|_2} \left(\frac{a^*\beta^2\lambda^{p+2}}{(a^* - a)\|Q\|^2_2}\right)^{\frac{N}{p+2}} Q\left(\left(\frac{a^*\beta^2}{(a^* - a)\|Q\|^2_2}\right)^\frac{1}{p+2} \lambda x \right) \text{ as } a \not\to a^*,$$
where \( f(a) \approx g(a) \) means that \( f/g \to 1 \) as \( a \nearrow a^* \).

Theorem 1.3 is proved by the blow up analysis, for which we need to derive the exact blow up rate of minimizers as \( a \nearrow a^* \). We shall reach this purpose by analyzing the refined energy estimates of \( e(a) \) as \( a \nearrow a^* \). However, due to the singularity of \(|x|^{-b}\) and the variability of the scaling transformation of the bounded domain \( \Omega \), we remark that the standard elliptic regularity theory is used with caution when investigating the \( L^\infty \)-uniform convergence of (1.15).

Motivated by \([5, 11, 20]\) and the references therein, based on Theorem 1.3, we finally investigate the uniqueness of positive minimizers for \( e(a) \) as \( a \nearrow a^* \) under some suitable assumptions on \( V(x) \).

Theorem 1.4. Let \( N \geq 3 \), \( 0 < b < \min\{2, \frac{N}{2}\} \), \( \beta^2 = \frac{2-b}{N} \) and assume that \( V(x) \) satisfies \((V_1)\) and \((V_2)\). Then there exists a unique positive minimizer of \( e(a) \) as \( a \nearrow a^* \).

We note that if the non-degeneracy property (1.10) still holds for all \( N \geq 1 \), then the restriction \( N \geq 3 \) in Theorem 1.4 can be removed. In addition, the condition \( 0 < b < \min\{2, \frac{N}{2}\} \) guarantees that the integral (4.25) makes sense. Inspired by \([5, 11, 20]\), we shall prove Theorem 1.4 by constructing Pohozaev identity, which is widely used in the existing literature of studying the real-valued elliptic PDEs. However, the calculations involved in the proof are more complicated due to the existence of inhomogeneous nonlinear term.

This paper is organized as follows. In Section 2 we shall prove Theorem 1.1 and Theorem 1.2 on the existence and nonexistence of minimizers for \( e(a) \). In Section 3, we first establish some energy estimates of positive minimizers for \( e(a) \) as \( a \nearrow a^* \), after which we complete the proof of Theorem 1.3. By establishing local Pohozaev identity, we then prove Theorem 1.4 on the local uniqueness of positive minimizers in Section 4. In the Appendix, we shall sketch the proof of Proposition 2.1.

2 Existence of minimizers

The main purpose of this section is to prove Theorem 1.1 and Theorem 1.2 on the existence and nonexistence of minimizers for \( e(a) \). The proofs of Theorem 1.1 and Theorem 1.2 are based on the following Gagliardo-Nirenberg type inequality in any given open bounded domain.

**Proposition 2.1.** Suppose \( \Omega \subset \mathbb{R}^N \) is an open bounded domain, then we have

\[
\frac{a^*}{1+\beta^2} \int_{\Omega} |x|^{-b}|u(x)|^{2+2\beta^2} \, dx \leq \int_{\Omega} |\nabla u(x)|^2 \, dx \left( \int_{\Omega} |u(x)|^2 \, dx \right)^{\beta^2}, \quad u \in H^1_0(\Omega),
\]

where the optimal constant \( \frac{1+\beta^2}{a^*} \) is not attained.

Since the proof of Proposition 2.1 is long and similar to the proof of \([12, Proposition 2.1]\), we leave it to the Appendix for simplicity. Making full use of Proposition 2.1 we then complete the proofs of Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** (1). We first prove that \( e(a) \) admits at least one minimizer for all \( 0 < a < a^* \). Indeed, for any fixed \( 0 < a < a^* \) and \( u \in \mathcal{M} \), we derive from the
nonnegativity of \( V(x) \) and the Gagliardo-Nirenberg inequality \( (2.1) \) that

\[
E_a(u) = \int_\Omega (|\nabla u(x)|^2 + V(x)|u(x)|^2) dx - \frac{a}{1 + \beta^2} \int_\Omega |x|^{-b}|u(x)|^{2+2\beta^2} dx
\]

\[
\geq (1 - \frac{a}{a^*}) \int_\Omega |\nabla u(x)|^2 dx + \int_\Omega V(x)|u(x)|^2 dx
\]

\[
\geq (1 - \frac{a}{a^*}) \int_\Omega |\nabla u(x)|^2 dx \geq 0,
\]

which implies that \( E_a(u) \) is bounded uniformly from below. Choose a minimizing sequence \( \{u_n\} \subset H^1_0(\Omega) \) satisfying \( \|u_n\|_2^2 = 1 \) and \( \lim_{n \to \infty} E_a(u_n) = e(a) \). It then follows from \( (2.2) \) that \( \int_\Omega |\nabla u_n|^2 dx \) is bounded uniformly for \( n \). Since the embedding \( H^1_0(\Omega) \hookrightarrow L^q(\Omega) \) \( (2 \leq q < 2^*) \) is compact, there exists a subsequence, still denoted by \( \{u_n\} \), such that for some \( u \in H^1_0(\Omega) \),

\[
u_n \rightharpoonup u \text{ weakly in } H^1_0(\Omega) \quad \text{and} \quad u_n \to u \text{ strongly in } L^q(\Omega), \quad 2 \leq q < 2^*. \tag{2.3}\]

This further implies that \( u \in \mathcal{M} \).

As for the convergence of nonlinear term, motivated by \( (2.2) \), we claim that

\[
\lim_{n \to \infty} \int_\Omega |x|^{-b}|u_n(x)|^{2+2\beta^2} dx = \int_\Omega |x|^{-b}|u(x)|^{2+2\beta^2} dx. \tag{2.4}\]

Actually, for any \( 1 < \bar{s} < \frac{N}{s-1} \), applying the H"older inequality then yields that

\[
\left| \int_\Omega |x|^{-b}|u_n(x)|^{2+2\beta^2} dx - \int_\Omega |x|^{-b}|u(x)|^{2+2\beta^2} dx \right|
\]

\[
\leq \int_\Omega |x|^{-b}|u_n(x)|^{2+2\beta^2} dx - |u(x)|^{2+2\beta^2} dx
\]

\[
\leq \left( \int_\Omega |x|^{-b\bar{s}} dx \right)^{\frac{1}{\bar{s}}} \left( \int_\Omega |u_n(x)|^{2+2\beta^2} dx - |u(x)|^{2+2\beta^2} dx \right)^{\frac{1}{s}}
\]

\[
\leq C \left( \int_\Omega |u_n(x)|^{2+2\beta^2} dx - |u(x)|^{2+2\beta^2} dx \right)^{\frac{1}{s}}, \tag{2.5}\]

where \( s, \bar{s} > 1 \) and \( \bar{s} = \frac{s}{s-1} \). In addition,

\[
\left( \int_\Omega |u_n|^{2+2\beta^2} dx \right)^{\frac{1}{s}} \leq C \left( \int_\Omega |u_n - u|^{\bar{s}} \left( |u_n|^{1+2\beta^2} + |u|^{1+2\beta^2} \right) dx \right)^{\frac{1}{\bar{s}}}
\]

\[
\leq C \|u_n - u\|_{L^s(\Omega)} \left( \|u_n|^{1+2\beta^2} + |u|^{1+2\beta^2} \right) \|L^{1+2\beta^2}(\Omega)
\]

\[
\leq C \|u_n - u\|_{L^s(\Omega)} \left( \|u_n|^{1+2\beta^2} + |u|^{1+2\beta^2} \right) \|L^{1+2\beta^2}(\Omega), \tag{2.6}\]

where

\[
\frac{1}{\sigma} + \frac{1 + 2\beta^2}{\tau} = \frac{1}{s} < \frac{N - b}{N}. \tag{2.7}\]
On the other hand, for \( N = 1, 2 \), it is obvious that there exist \( \sigma, \tau \in [2, \infty) \) such that
\[
\frac{1}{\sigma} + \frac{1 + 2\beta^2}{\tau} = \frac{1}{\sigma} < \frac{N - b}{N};
\]
For \( N \geq 3 \), one can check that
\[
\frac{N - 2}{2N} + \frac{N - 2}{2N}(1 + 2\beta^2) = \frac{(N - 2)(N + 2 - b)}{N^2} < \frac{N - b}{N},
\]
which implies that there exist suitable constants \( \sigma, \tau \in [2, \frac{2N}{N - 2}] \) satisfying (2.7). Combining this with the convergence of \( u_n \) in (2.3), we deduce that
\[
\|u_n\|_{L^2(\Omega)}^{1 + 2\beta^2} + \|u\|_{L^2(\Omega)}^{1 + 2\beta^2} \leq C \quad \text{and} \quad \|u_n - u\|_{L^\tau(\Omega)} \to 0 \quad \text{as} \quad n \to \infty.
\]
From the above facts, the claim (2.4) is therefore proved.

Consequently, following the weak lower semicontinuity and (2.4), we deduce that
\[
E_a(u) \geq e(a) = \lim_{n \to \infty} E_a(u_n) \geq E_a(u),
\]
which indicates that \( u \) is a minimizer of \( e(a) \) and Theorem 1.1 (1) is thus proved.

(2). We next show that there is no minimizer for \( e(a) \) once \( a > a^* \). Since \( \Omega \) is an open bounded domain of \( \mathbb{R}^N \) and contains 0, there exists an open ball \( B_{2R}(0) \subset \Omega \) centered at an inner point 0, where \( R > 0 \) is sufficiently small. Choose a nonnegative cutoff function \( \varphi(x) \in C_0^\infty(\mathbb{R}^N) \) such that \( \varphi(x) = 1 \) for \( |x| \leq R \) and \( \varphi(x) = 0 \) for \( |x| \geq 2R \). Set for all \( \tau > 0 \),
\[
\Phi_\tau(x) = A_\tau \frac{\tau^N}{\|Q\|_2} \varphi(\frac{x}{\tau})Q(\tau x), \quad x \in \Omega, \tag{2.8}
\]
where \( A_\tau > 0 \) is chosen such that \( \int_\Omega \Phi_\tau(x)^2 \, dx = 1 \). According to the exponential decay of \( Q \), we then obtain from (2.8) that
\[
\frac{1}{|A_\tau|^2} = \frac{1}{\|Q\|_2^2} \int_{\mathbb{R}^N} \varphi^2(\frac{x}{\tau})Q^2(x) \, dx = 1 + O(\tau^{-\alpha}) \quad \text{as} \quad \tau \to \infty, \tag{2.9}
\]
where \( O(\tau^{-\alpha}) := m(t) \) denotes the function \( m(t) \) satisfying \( \lim_{t \to \infty} |m(t)|t^s = 0 \) for all \( s > 0 \). Moreover, following (1.7) and (1.8), some calculations yield that
\[
\int_\Omega |\nabla \Phi_\tau(x)|^2 \, dx - \frac{a}{1 + \beta^2} \int_\Omega |x|^{-b} |\Phi_\tau(x)|^{2+2\beta^2} \, dx
\]
\[
= \frac{\tau^2 A_\tau^2}{\|Q\|_2^2} \int_{\mathbb{R}^N} |\nabla Q(x)|^2 \, dx - \frac{a}{1 + \beta^2} \frac{\tau^2 A_\tau^{2+2\beta^2}}{\|Q\|_2^{2+2\beta^2}} \int_{\mathbb{R}^N} |x|^{-b} |Q(x)|^{2+2\beta^2} \, dx + O(\tau^{-\alpha})
\]
\[
= \frac{\tau^2 A_\tau^2}{\|Q\|_2^2} \int_{\mathbb{R}^N} |\nabla Q(x)|^2 \, dx - \frac{a \tau^2 A_\tau^{2+2\beta^2}}{\|Q\|_2^{2+2\beta^2}} \int_{\mathbb{R}^N} |\nabla Q(x)|^2 \, dx + O(\tau^{-\alpha})
= \frac{\tau^2}{\|Q\|_2^2} \left( 1 - \frac{a}{a^*} \right) \int_{\mathbb{R}^N} |\nabla Q(x)|^2 \, dx + O(\tau^{-\alpha}) \quad \text{as} \quad \tau \to \infty. \tag{2.10}
\]
On the other hand, since \( x \mapsto V(x)\varphi^2(x) \) is bounded in \( \Omega \), it follows from [19, Theorem 1.8] that
\[
\lim_{\tau \to \infty} \int_\Omega V(x) |\Phi_\tau(x)|^2 \, dx = V(0). \tag{2.11}
\]
We then conclude from (2.10) and (2.11) that
\[ e(a) \leq E_a(\Phi_{\tau}(x)) \]
\[ = \int_{\Omega} |\nabla \Phi_{\tau}(x)|^2 dx - \frac{a}{1 + \beta^2} \int_{\Omega} |x|^{-b} |\Phi_{\tau}(x)|^{2+2\beta^2} dx + \int_{\Omega} V(x) |\Phi_{\tau}(x)|^2 dx \]
\[ = \frac{\tau^2}{\|Q\|_2^2} (1 - \frac{a}{a^*}) \int_{\mathbb{R}^N} |\nabla Q(x)|^2 dx + \int_{\Omega} V(x) |\Phi_{\tau}(x)|^2 dx + O(\tau^{-\infty}) \]
\[ \rightarrow -\infty \quad \text{as} \quad \tau \rightarrow \infty, \]
which implies that \( e(a) = -\infty \) as soon as \( a > a^* \) and the nonexistence of minimizers is therefore proved. This completes the proof of Theorem 1.1.

Next, we shall make full use of the various Gagliardo-Nirenberg type inequalities to prove Theorem 1.2 by contradiction.

**Proof of Theorem 1.2.** (1). Applying the Gagliardo-Nirenberg inequality (2.1), one can check that \( e(a^*) \geq 0 \) is bounded from below, and thus there exists a minimizing sequence \( \{u_n\} \) of \( e(a^*) \) such that \( e(a^*) = \lim_{n \to \infty} E_{a^*}(u_n) \). To prove Theorem 1.2 (1), the argument similar to that of proving Theorem 1.1 (1), it is sufficient to prove that \( \{u_n\} \) is bounded uniformly in \( H^1_0(\Omega) \). On the contrary, suppose that \( \{u_n\} \) is unbounded in \( H^1_0(\Omega) \), then there exists a subsequence of \( \{u_n\} \), still denoted by \( \{u_n\} \), such that
\[ \|u_n\|_{H^1_0(\Omega)} \to \infty \quad \text{as} \quad n \to \infty, \]

Define now
\[ \varepsilon_n^{-2} := \int_{\Omega} |\nabla u_n(x)|^2 dx, \]
then by (2.13), we have that \( \varepsilon_n \to 0 \) as \( n \to \infty \). In view of the above facts, we next define
\[ w_n(x) := \begin{cases} \frac{\varepsilon_n^N}{\varepsilon_n^2} u_n(\varepsilon_n x) & \text{if} \quad x \in \Omega_n := \{x \in \mathbb{R}^N : \varepsilon_n x \in \Omega\}, \\ 0 & \text{if} \quad x \in \mathbb{R}^N \setminus \Omega_n, \end{cases} \]
then
\[ \int_{\mathbb{R}^N} |\nabla w_n(x)|^2 dx = \int_{\mathbb{R}^N} |w_n(x)|^2 dx = 1. \]
This implies that \( w_n \) is bounded uniformly in \( H^1(\mathbb{R}^N) \). According to the embedding theorem, we can extract a subsequence if necessary such that
\[ w_n \rightharpoonup w_0 \quad \text{weakly in} \quad H^1(\mathbb{R}^N) \quad \text{and} \quad w_n \to w_0 \quad \text{strongly in} \quad L^q_{loc}(\mathbb{R}^N), \quad 2 \leq q < 2^* \]
for some \( w_0 \in H^1(\mathbb{R}^N) \).

Next, we claim that
\[ \lim_{n \to \infty} w_n = \frac{\beta^N Q(\beta x)}{\|Q\|_2} \quad \text{strongly in} \quad H^1(\mathbb{R}^N). \]
Indeed, by the definition of $w_n$, we deduce from the Gagliardo-Nirenberg inequality (1.5) that

$$C \geq e(a^*) = \lim_{n \to \infty} E_{a^*}(u_n)$$

$$= \lim_{n \to \infty} \left\{ \int_{\Omega} \left( |\nabla u_n|^2 - \frac{a^*}{1 + \beta^2} |x|^{-b} |u_n|^{2+2\beta^2} + V(x)|u_n(x)|^2 \right) dx \right\}$$

$$= \lim_{n \to \infty} \left\{ \frac{1}{\varepsilon^2_n} \int_{\mathbb{R}^N} \left[ |\nabla w_n|^2 - \frac{a^*}{1 + \beta^2} |x|^{-b} |w_n|^{2+2\beta^2} \right] dx + \int_{\Omega} V(\varepsilon_n x) |w_n(x)|^2 dx \right\}$$

$$\geq 0.$$  \hspace{1cm} (2.18)

Therefore, we derive from (2.16), the fact $\varepsilon_n \to 0$ as $n \to \infty$ and above that

$$\lim_{n \to \infty} \frac{a^*}{1 + \beta^2} \int_{\mathbb{R}^N} |x|^{-b} |w_n(x)|^{2+2\beta^2} dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\nabla w_n(x)|^2 dx = 1. \hspace{1cm} (2.19)$$

We then prove

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |x|^{-b} |w_n(x)|^{2+2\beta^2} dx = \int_{\mathbb{R}^N} |x|^{-b} |w_0(x)|^{2+2\beta^2} dx. \hspace{1cm} (2.20)$$

Indeed,

$$\left| \int_{\mathbb{R}^N} |x|^{-b} |w_n(x)|^{2+2\beta^2} dx - \int_{\mathbb{R}^N} |x|^{-b} |w_0(x)|^{2+2\beta^2} dx \right|$$

$$\leq \int_{B_R(0)} |x|^{-b} \left| w_n |^{2+2\beta^2} - |w_0|^{2+2\beta^2} \right| dx + \int_{\mathbb{R}^N \setminus B_R(0)} |x|^{-b} \left| w_n |^{2+2\beta^2} - |w_0|^{2+2\beta^2} \right| dx$$

$$:= A_n + B_n. \hspace{1cm} (2.21)$$

For any $\varepsilon > 0$, we can choose $R$ large enough such that $R > \varepsilon^{-\frac{1}{b}}$, then we derive that

$$B_n = \int_{\mathbb{R}^N \setminus B_R(0)} |x|^{-b} \left| w_n |^{2+2\beta^2} - |w_0|^{2+2\beta^2} \right| dx \leq C \varepsilon. \hspace{1cm} (2.22)$$

On the other hand, the convergence in (2.4) indicates that

$$\lim_{n \to \infty} A_n = \lim_{n \to \infty} \int_{B_R(0)} |x|^{-b} \left| w_n |^{2+2\beta^2} - |w_0|^{2+2\beta^2} \right| dx = 0. \hspace{1cm} (2.23)$$

Combining this with (2.22), (2.20) is therefore proved. It then follows from (2.19) and (2.20) that

$$\frac{a^*}{1 + \beta^2} \int_{\mathbb{R}^N} |x|^{-b} |w_0|^{2+2\beta^2} dx = \lim_{n \to \infty} \frac{a^*}{1 + \beta^2} \int_{\mathbb{R}^N} |x|^{-b} |w_n|^{2+2\beta^2} dx = 1, \hspace{1cm} (2.24)$$

which implies $w_0 \neq 0$. 

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Moreover, combining the Gagliardo-Nirenberg inequality (1.5) and the fact that \( w_n \rightharpoonup w_0 \) weakly in \( H^1(\mathbb{R}^N) \), we derive that
\[
0 = \lim_{n \to \infty} \varepsilon^2_n e(a^*) \\
\geq \lim_{n \to \infty} \int_{\mathbb{R}^N} \left[ |\nabla w_n(x)|^2 - \frac{a^*}{1 + \beta^2} |x|^{-b}|w_n(x)|^{2+2\beta^2} \right] dx \\
\geq \int_{\mathbb{R}^N} |\nabla w_0(x)|^2 dx - \frac{a^*}{1 + \beta^2} \int_{\mathbb{R}^N} |x|^{-b}|w_0(x)|^{2+2\beta^2} dx \\
\geq \left[ 1 - \left( \int_{\mathbb{R}^N} |w_0(x)|^2 dx \right)^{\frac{\beta^2}{2}} \right] \int_{\mathbb{R}^N} |\nabla w_0(x)|^2 dx. \tag{2.25}
\]
Since \( \|w_0\|_{L^2(\mathbb{R}^N)} \leq 1 \), the above inequality implies that
\[
\int_{\mathbb{R}^N} |w_0(x)|^2 dx = 1, \quad \int_{\mathbb{R}^N} |\nabla w_0(x)|^2 dx = \frac{a^*}{1 + \beta^2} \int_{\mathbb{R}^N} |x|^{-b}|w_0(x)|^{2+2\beta^2} dx = 1. \tag{2.26}
\]
As a consequence, we conclude that the Gagliardo-Nirenberg inequality (1.5) is achieved by \( w_0 \) and hence \( w_0 = \frac{\beta^N}{\|Q\|_2^2} Q(\beta x) \) in view of (2.26) and (1.8). By the norm preservation, we finally conclude that \( w_n \) converges to \( w_0 = \frac{\beta^N}{\|Q\|_2^2} Q(\beta x) \) strongly in \( H^1(\mathbb{R}^N) \) and the claim (2.17) is thus proved.

Based on (2.17), applying Fatou’s lemma, one can deduce that
\[
e(a^*) \geq \lim_{n \to \infty} \int_{\Omega_n} V(\varepsilon_n x)|w_n(x)|^2 dx \geq \int_{\mathbb{R}^N} V(0)|w_0(x)|^2 dx = V(0), \tag{2.27}
\]
which however contradicts the assumption \( e(a^*) < V(0) \). Hence the minimizing sequence \( \{u_n\} \) of \( e(a^*) \) is bounded uniformly in \( H^1_0(\Omega) \) and the proof of Theorem 1.2 (1) is accomplished.

(2). We next consider the case \( a = a^* \) and \( V(0) = 0 \). In this case, the argument of proving (2.10) and (2.11) yields that \( e(a^*) \leq V(0) = 0 \). Additionally, it follows from (2.22) that \( e(a^*) \geq 0 \), we then conclude that \( e(a^*) = 0 \). Suppose now that there exists a minimizer \( u \) for \( e(a^*) \), we would thus have
\[
\int_{\Omega} |\nabla u(x)|^2 dx = \frac{a^*}{1 + \beta^2} \int_{\Omega} |x|^{-b}|u(x)|^{2+2\beta^2} dx, \tag{2.28}
\]
which contradicts the fact that the optimal constant in Proposition 2.1 is not attained. Therefore there is no minimizer for \( e(a^*) \) in this case.

Moreover, taking the same test function as in (2.3), we have \( V(0) = \inf_{x \in \Omega} V(x) \leq \lim_{\alpha \nearrow a^*} e(a) \leq V(0) \) and hence \( \lim_{\alpha \nearrow a^*} e(a) = V(0) = e(a^*) \). The proof of Theorem 1.2 is therefore finished. \( \square \)

3 Mass concentration as \( a \nearrow a^* \)

In this section, we shall analyze the mass concentration behavior of minimizers for \( e(a) \) as \( a \nearrow a^* \) in the case that \( V(x) \) satisfies the additional condition \((V_2)\). Let \( u_\alpha \) be a
positive minimizer of \( e(a) \), by variational theory, \( u_a \) solves the following Euler-Lagrange equation

\[
- \Delta u_a(x) + V(x)u_a(x) = \mu_a u_a(x) + a u_a^{1+2\beta^2}(x)|x|^{-b} \quad \text{in } \Omega, \tag{3.1}
\]

where \( \mu_a \in \mathbb{R} \) is a suitable Lagrange multiplier satisfying

\[
\mu_a = e(a) - \frac{a\beta^2}{1 + \beta^2} \int_\Omega |x|^{-b}|u_a|^{2+2\beta^2} \, dx. \tag{3.2}
\]

Motivated by [20, 23], we begin with the following energy estimate.

**Lemma 3.1.** Under the assumptions of Theorem 1.3, we have the following energy estimate

\[
0 \leq e(a) \leq \frac{p + 2}{p} \lambda^2 \left( \frac{1}{\|Q\|_2^2} \right)^{\frac{2}{p+2}} \left( \frac{a^* - a}{a^* \beta^2} \right)^{\frac{p}{p+2}} \quad \text{as } a \searrow a^*, \tag{3.3}
\]

where \( p > 0 \) and \( \lambda > 0 \) are defined by (1.13) and (1.14) respectively.

**Proof.** Since \( \Omega \) containing 0 is an open bounded domain of \( \mathbb{R}^N \), there exists an open ball \( B_{2R}(0) \subset \Omega \), where \( R > 0 \) is small enough. Choose the same trial function as in (2.8), then it follows from (2.9) and the exponential decay of \( Q \) that

\[
\int_\Omega V(x)\Phi_\tau^2(x) \, dx = \frac{A^2 \tau^N}{\|Q\|_2^2} \int_{B_{2R}(0)} V(x)\varphi^2(x)Q^2(\tau x) \, dx
\]

\[
= \frac{A^2}{\|Q\|_2^2} \int_{B_{2R}(0)} V(x)\varphi^2(\frac{x}{\tau})Q^2(\tau x) \, dx
\]

\[
\leq \frac{A^2}{\|Q\|_2^2} \int_{B_{\sqrt{\tau R}}(0)} V(x)Q^2(\tau x) \, dx + Ce^{-\sqrt{\tau R}}
\]

\[
= \frac{2}{p\|Q\|_2^2} \lambda^{p+2} \tau^{-p} + o(\tau^{-p}) \quad \text{as } \tau \to \infty.
\]

Therefore, we derive from (1.8), (2.10) and above that

\[
e(a) \leq E_a(\Phi_\tau) = \int_\Omega |\nabla \Phi_\tau(x)|^2 \, dx - \frac{a}{1 + \beta^2} \int_\Omega |x|^{-b}|\Phi_\tau(x)|^{2+2\beta^2} \, dx + \int_\Omega V(x)|\Phi_\tau(x)|^2 \, dx
\]

\[
= \frac{\tau^2}{\|Q\|_2^2} (1 - \frac{a}{a^*}) \int_{\mathbb{R}^N} |\nabla Q(x)|^2 \, dx + \int_\Omega V(x)|\Phi_\tau(x)|^2 \, dx + O(\tau^{-\infty})
\]

\[
\leq \frac{(a^* - a)\tau^2}{a^*\|Q\|_2^2} \int_{\mathbb{R}^N} |\nabla Q(x)|^2 \, dx + \frac{2}{p\|Q\|_2^2} \lambda^{p+2} \tau^{-p} + o(\tau^{-p})
\]

\[
= \frac{(a^* - a)\tau^2}{a^*\beta^2} + \frac{2}{p\|Q\|_2^2} \lambda^{p+2} \tau^{-p} + o(\tau^{-p}) \quad \text{as } \tau \to \infty.
\]

By choosing \( \tau = \lambda \left[ \frac{(a^* - a)\|Q\|_2^2}{(a^* \beta^2)} \right]^{\frac{1}{p+2}} \), one can check that

\[
e(a) \leq \frac{p + 2}{p} \lambda^2 \left( \frac{1}{\|Q\|_2^2} \right)^{\frac{2}{p+2}} \left( \frac{a^* - a}{a^* \beta^2} \right)^{\frac{p}{p+2}} \quad \text{as } a \searrow a^*.
\]

Combining this with (2.2), we complete the proof of Lemma 3.1.

We then establish the following crucial lemma, which is a weak version of Theorem 1.3.
Lemma 3.2. Under the assumptions of Theorem 1.3 and suppose $u_a$ be a positive minimizer of $e(a)$. Define
\[
\varepsilon_a := \left( \frac{1}{\int_\Omega |\nabla u_a|^2 \, dx} \right)^{-\frac{1}{2}},
\]
and
\[
w_a(x) := \varepsilon_a^N u_a(\varepsilon_a x), \quad x \in \Omega_a,
\]
where $\Omega_a := \{ x \in \mathbb{R}^N : \varepsilon_a x \in \Omega \}$. We then have the following
1. $\varepsilon_a \to 0$ as $a \nearrow a^*$ and $\lim_{a \nearrow a^*} \Omega_a = \mathbb{R}^N$.
2. $w_a(x) \to \frac{\beta \varepsilon_a^N Q(\beta x)}{\|Q\|_2^2}$ strongly in $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ as $a \nearrow a^*$ in the sense that $w_a(x) \equiv 0$ for all $x \in \mathbb{R}^N \setminus \Omega_a$.
3. There exist positive constants $R_0 > 0$ and $C > 0$ independent of $a$ such that
\[
|w_a(x)|, \quad |\nabla w_a(x)| \leq C e^{-\frac{\beta |x|}{2}} \quad \text{in} \quad \Omega_a \setminus B_{R_0}(0) \quad \text{as} \quad a \nearrow a^*.
\]

Proof. 1. By contradiction, assume that there exists a sequence $\{a_k\}$ satisfying $a_k \nearrow a^*$ as $k \to \infty$ such that the sequence $\{u_{a_k}\}$ is bounded uniformly in $H^1_0(\Omega)$. By applying the compact embedding $H^1_0(\Omega) \to L^q(\Omega)$ for $2 \leq q < 2^*$, we can choose a subsequence of $\{u_{a_k}\}$, still denoted by $\{u_{a_k}\}$, and $u_0 \in H^1_0(\Omega)$ such that
\[
u_{a_k} \to u_0 \quad \text{weakly} \quad \text{in} \quad H^1_0(\Omega) \quad \text{and} \quad \nu_{a_k} \to u_0 \quad \text{strongly} \quad \text{in} \quad L^q(\Omega), \quad 2 \leq q < 2^*.
\]
Combining the above convergence with (3.4) gives that
\[
0 = e(a^*) \leq E_{a^*}(u_0) \leq \lim_{k \to \infty} E_{a_k}(u_{a_k}) = \lim_{k \to \infty} e(a_k) = 0,
\]
which indicates that $u_0$ is a minimizer of $e(a^*)$. This however contradicts Theorem 1.2 (2), which further implies that $\varepsilon_a \to 0$ as $a \nearrow a^*$. By the definition of $\Omega_a$, we deduce that $\lim_{a \nearrow a^*} \Omega_a = \mathbb{R}^N$ and the proof of Lemma 3.2 (1) is complete.

2. Note from Theorem 1.2 (2) that $0 = e(a^*) = \lim_{a \nearrow a^*} e(a)$, thus $\{u_a\}$ is also a minimizing sequence of $e(a^*)$. In a similar way to (2.15), we define the zero continuation of $w_a(x)$ as follows
\[
\tilde{w}_a(x) := \begin{cases} \nu_a(x) & \text{if} \quad x \in \Omega_a, \\ 0 & \text{if} \quad x \in \mathbb{R}^N \setminus \Omega_a. \end{cases}
\]
Since $\lim_{a \nearrow a^*} \Omega_a = \mathbb{R}^N$, combining this with the exponential decay of $Q(x)$, in order to prove Lemma 3.2 (2), we just need to prove that
\[
\tilde{w}_a(x) \to \frac{\beta \varepsilon_a^N Q(\beta x)}{\|Q\|_2^2} \quad \text{strongly} \quad \text{in} \quad H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \quad \text{as} \quad a \nearrow a^*.
\]
Firstly, we can derive from (3.4), (3.5) and (3.7) that
\[
\int_{\mathbb{R}^N} |\nabla \tilde{w}_a|^2 \, dx = \int_{\mathbb{R}^N} |\tilde{w}_a|^2 \, dx = 1 \quad \text{for any} \quad a \nearrow a^*.
\]
Hence $\tilde{w}_a$ is bounded uniformly for $a$ in $H^1(\mathbb{R}^N)$. Passing to a subsequence if necessary, we obtain that

$$\tilde{w}_a \to \tilde{w}_0 \text{ weakly in } H^1(\mathbb{R}^N) \text{ and } \tilde{w}_a \to \tilde{w}_0 \text{ strongly in } L^q_{\text{loc}}(\mathbb{R}^N), \ 2 \leq q < 2^*$$

holds for some $\tilde{w}_0 \in H^1(\mathbb{R}^N)$. Note that the proof of the claim (2.17) does not rely on the assumption $0 \leq e(a^*) < V(0)$. As a result, the similar argument of proving (2.17) yields that

$$\tilde{w}_a(x) \to \frac{\beta^\frac{N}{2} Q(\beta x)}{\|Q\|_2} \text{ strongly in } H^1(\mathbb{R}^N) \text{ as } a \nearrow a^*. \quad (3.9)$$

Next, we claim that

$$\tilde{w}_a(x) \to \frac{\beta^\frac{N}{2} Q(\beta x)}{\|Q\|_2} \text{ strongly in } L^\infty(\mathbb{R}^N) \text{ as } a \nearrow a^*. \quad (3.10)$$

For $N = 1$, the above claim is trivial due to the fact that the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ is continuous. We then consider the case $N \geq 2$. We first prove that

$$\tilde{w}_a(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly for } a \nearrow a^*. \quad (3.11)$$

In fact, we derive from (3.8) that for sufficiently large $R > 0$,

$$\int_{\mathbb{R}^N \setminus B_R(0)} |\tilde{w}_a(x)|^2 dx \to 0 \text{ as } a \nearrow a^*.$$ 

Combining this with the definition of $\tilde{w}_a(x)$, it then yields from Lemma 3.2 (1) that

$$\int_{\Omega_a \setminus B_R(0)} |w_a(x)|^2 dx \to 0 \text{ as } a \nearrow a^*. \quad (3.12)$$

On the other hand, recall from (3.1) and (3.5) that $w_a$ satisfies the following equation

$$-\Delta w_a + \varepsilon_a^2 V(\varepsilon_a x)w_a = \varepsilon_a^2 \mu_a w_a + aw_a^{1+2\beta^2} |x|^{-b} \quad \text{in } \Omega_a, \quad (3.13)$$

where $\mu_a \in \mathbb{R}$ is a suitable lagrange multiplier satisfying (3.2). In view of (3.2) and the proof of (2.11), we can obtain that

$$\varepsilon_a^2 \mu_a = \varepsilon_a^2 e(a) - \varepsilon_a^2 \frac{a \beta^2}{1 + \beta^2} \int_{\Omega_a} |x|^{-b} |u_a(x)|^{2+2\beta^2} dx$$

$$= \varepsilon_a^2 e(a) - \frac{a \beta^2}{1 + \beta^2} \int_{\Omega_a} |x|^{-b} |w_a(x)|^{2+2\beta^2} dx$$

$$= \varepsilon_a^2 e(a) - \frac{a \beta^2}{1 + \beta^2} \int_{\mathbb{R}^N} |x|^{-b} |\tilde{w}_a(x)|^{2+2\beta^2} dx \to -\beta^2 \text{ as } a \nearrow a^*. \quad (3.14)$$

It then follows from (3.13) and (3.14) that

$$-\Delta w_a(x) - c(x)w_a(x) \leq 0 \text{ in } \Omega_a \text{ as } a \nearrow a^*, \quad (3.15)$$

where $c(x) = aw_a^{2\beta^2} |x|^{-b}$. Similar to the proof of Lemma B.2 in [22], one can check that $c(x) \in L^m(\Omega_a)$ where $m \in \left( \frac{N}{2}, \frac{N^2}{2N+2\beta^2-4} \right)$ if $N \geq 3$ and $m \in (1, \frac{2}{3})$ if $N = 2$. Applying
De Giorgi-Nash-Moser theory [18, Theorem 4.1] to [3.15], we conclude that there exists $C > 0$ independent of $a$ such that

$$
\max_{B_1(\rho)} w_a(x) \leq C \left( \int_{B_2(\rho)} |w_a(x)|^2 \, dx \right)^{\frac{1}{2}} \text{ as } a \to a^*, \tag{3.16}
$$

where $\rho \in \Omega_a$ is arbitrary and $l > 0$ is small enough satisfying $B_2(\rho) \subset \Omega_a$. In view of (3.12), (3.16) and the definition of $\tilde{w}_a(x)$, (3.11) is therefore proved.

Following the above argument and the exponential decay of $Q(x)$, to complete the proof of (3.10), it is enough to show that

$$
\tilde{w}_a(x) \to \frac{\beta^\frac{N}{2} Q(\beta x)}{||Q||_2} \text{ strongly in } L^\infty_{loc}(\mathbb{R}^N) \text{ as } a \to a^*. \tag{3.17}
$$

Note that $\lim_{a \to a^*} \Omega_a = \mathbb{R}^N$, for any $R > 0$, it holds $B_{R+1}(0) \subset \Omega_a$ as $a \to a^*$. Recall that $w_a$ satisfies (3.13), we denote

$$
G_a(x) := \varepsilon_a^2 \mu_a w_a(x) - \varepsilon_a^2 V(\varepsilon_a x) w_a(x) + aw_a^{1+2\beta} |x|^{-b},
$$

then

$$
-\Delta w_a(x) = G_a(x) \text{ in } \Omega_a. \tag{3.18}
$$

Since $\int_{\Omega_a} |w_a(x)|^2 \, dx = 1$, it follows from (3.16) that

$$
w_a(x) \text{ is bounded uniformly in } L^\infty(\Omega_a). \tag{3.19}
$$

Therefore, for any $r \in \left( \frac{N}{2}, \frac{N}{b} \right)$, $w_a^{1+2\beta} |x|^{-b}$ is bounded uniformly in $L^r(B_{R+1}(0))$, which implies that $G_a(x)$ is bounded uniformly in $L^r(B_{R+1}(0))$. It then follows from [15, Theorem 9.11] that

$$
\|w_a(x)\|_{W^{2,r}(B_R(0))} \leq C \left( \|w_a(x)\|_{L^r(B_{R+1}(0))} + \|G_a(x)\|_{L^r(B_{R+1}(0))} \right), \tag{3.20}
$$

where $C > 0$ is independent of $a$ and $R$. Therefore, we can obtain that $w_a(x)$ is also bounded uniformly in $W^{2,r}(B_R(0))$. Since $r > \frac{N}{2}$ and the embedding $W^{2,r}(B_R(0)) \hookrightarrow L^\infty(B_R(0))$ is compact, see [15, Theorem 7.26], we deduce that there exists a subsequence $\{w_{a_k}\}$ of $\{w_a\}$ such that

$$
\lim_{a_k \to a^*} \tilde{w}_{a_k}(x) = \lim_{a_k \to a^*} w_{a_k}(x) = \tilde{w}_0(x) \text{ strongly in } L^\infty(B_R(0)). \tag{3.21}
$$

In view of (3.9) and the fact that $R > 0$ is arbitrary, we deduce that

$$
\lim_{a_k \to a^*} \tilde{w}_{a_k}(x) = \frac{\beta^\frac{N}{2} Q(\beta x)}{||Q||_2} \text{ strongly in } L^\infty_{loc}(\mathbb{R}^N). \tag{3.22}
$$

Moreover, because the above convergence is independent of what subsequence we choose, we conclude that $\tilde{w}_a(x) \to \frac{\beta^\frac{N}{2} Q(\beta x)}{||Q||_2}$ in $L^\infty_{loc}(\mathbb{R}^N)$ as $a \to a^*$ and (3.17) is proved. Combining this with (3.11) that (3.10) holds true. As a result, by the definition of $\tilde{w}_a(x)$, we derive from (3.9) and (3.10) that Lemma 3.2 (2) is proved.
3. Following (3.11), (3.13) and (3.14), we obtain that there exists $R_0 > 0$ large enough such that

$$- \Delta w_a(x) + \frac{\beta^2}{4} w_a(x) \leq 0 \quad \text{in} \quad \Omega \setminus B_{R_0}(0) \quad \text{as} \quad a \nearrow a^*.$$ \hfill (3.23)

By the comparison principle, comparing $w_a(x)$ with $Ce^{-\frac{\beta |x|}{2}}$, we get that

$$w_a(x) \leq Ce^{-\frac{\beta |x|}{2}} \quad \text{in} \quad \Omega \setminus B_{R_0}(0) \quad \text{as} \quad a \nearrow a^*,$$ \hfill (3.24)

where $C > 0$ is independent of $a$. Besides, applying the local elliptic estimate \[15\] (3.15)] to (3.18) yields that

$$|\nabla w_a(x)| \leq Ce^{-\frac{\beta |x|}{2}} \quad \text{in} \quad \Omega \setminus B_{R_0}(0) \quad \text{as} \quad a \nearrow a^*.$$ \hfill (3.25)

Therefore, the exponential decay (3.6) is then proved and we complete the proof of Lemma 3.2. \hfill \Box

**Proof of Theorem 1.3.** By the definition of $w_a(x)$, some direct calculations yield that

$$e(a) = E_a(u_a)$$

$$= \int_{\Omega} |\nabla u_a(x)|^2 dx - \frac{a}{1 + \beta^2} \int_{\Omega} |x|^{-b} \left| u_a^{2+2\beta^2}(x) \right| dx + \int_{\Omega} V(x)|u_a(x)|^2 dx$$

$$= \frac{1}{\varepsilon_a^2} \left( \int_{\Omega_a} |\nabla w_a(x)|^2 dx - \frac{a^*}{1 + \beta^2} \int_{\Omega_a} |x|^{-b} w_a^{2+2\beta^2}(x) dx \right)$$

$$+ \frac{a^* - a}{(1 + \beta^2)^2 \varepsilon_a^2} \int_{\Omega_a} |x|^{-b} w_a^{2+2\beta^2}(x) dx + \int_{\Omega_a} V(\varepsilon_a x)|w_a(x)|^2 dx$$

$$\geq \frac{a^* - a}{(1 + \beta^2)^2 \varepsilon_a^2} \int_{\Omega_a} |x|^{-b} w_a^{2+2\beta^2}(x) dx + \int_{\Omega_a} V(\varepsilon_a x)|w_a(x)|^2 dx.$$ \hfill (3.26)

Based on Lemma 3.2 (2) and (3.24), one can check that

$$\int_{\Omega_a} V(\varepsilon_a x)|w_a(x)|^2 dx$$

$$= \int_{B_{\varepsilon_a x}(0) \setminus \Omega_a} V(\varepsilon_a x)|w_a(x)|^2 dx + \int_{\Omega_a \setminus B_{\varepsilon_a x}(0)} V(\varepsilon_a x)|w_a(x)|^2 dx$$

$$= \int_{B_{\varepsilon_a x}(0)} h(\varepsilon_a x)|\varepsilon_a x|^p \prod_{i=1}^{n-1} |x_i - x_i|^p |w_a(x)|^2 dx + o(\varepsilon_a^p)$$ \hfill (3.27)

$$= \left( \frac{\varepsilon_a}{\beta} \right)^p \frac{1}{\|Q\|^2} \int_{\mathbb{R}^N} |x|^p Q^2(x) dx \lim_{x \to 0} \left( h(x) \prod_{i=1}^{n-1} |x_i - x_i|^p \right) + o(\varepsilon_a^p)$$

$$= \left( \frac{\varepsilon_a}{\beta} \right)^p \frac{2\lambda^{p+2}}{p\|Q\|^2} + o(\varepsilon_a^p) \quad \text{as} \quad a \nearrow a^*.$$ \hfill (3.28)

Using the similar argument of the proof for (2.19), we can deduce that

$$\frac{a^* - a}{(1 + \beta^2)^2 \varepsilon_a^2} \int_{\Omega_a} |x|^{-b} w_a^{2+2\beta^2}(x) dx = \left[ 1 + o(1) \right] \frac{(a^* - a)}{a^* \varepsilon_a^2} \quad \text{as} \quad a \nearrow a^*.$$ \hfill (3.29)
In view of the above facts, we conclude that
\[
e(a) \geq \frac{a^* - a}{a^* \varepsilon_a^2} + \left(\frac{\varepsilon_a}{\beta}\right)^p \frac{2\lambda p + 2}{p} \|Q\|_2^p + o\left(\frac{a^* - a}{\varepsilon_a^2}\right)
\]
\[
\geq \frac{p + 2}{p} \lambda^2 \left(\frac{1}{\|Q\|_2^2}\right)^{\frac{2}{p + 2}} \left(\frac{a^* - a}{a^* \beta^2}\right)^{\frac{2}{p + 2}} + o\left(\frac{a^* - a}{\beta^2}\right)^{\frac{2}{p + 2}} \text{ as } a \nearrow a^*.
\] (3.29)

Combining this with the upper energy estimate (3.3), we obtain that
\[
\lim_{a \nearrow a^*} e(a) \left(\frac{a^* - a}{\beta^2}\right)^{\frac{2}{p + 2}} = \frac{p + 2}{p} \lambda^2 \left(\frac{1}{\|Q\|_2^2}\right)^{\frac{2}{p + 2}},
\] (3.30)
and the second equality of (3.29) holds if and only if
\[
\lim_{a \nearrow a^*} \frac{\varepsilon_a}{(a^* - a)^{\frac{2}{p + 2}}} = \frac{1}{\lambda} \left(\frac{\|Q\|_2^2}{a^* \beta^2}\right)^{\frac{2}{p + 2}}.
\] (3.31)

Applying Lemma 3.2, we then conclude that
\[
\lim_{a \nearrow a^*} \left(\frac{a^* - a}{a^* \beta^2}\right)^{\frac{2}{p + 2}} u_a \left(\left(\frac{(a^* - a)}{a^* \beta^2}\right)^{\frac{2}{p + 2}} x\right) = \frac{Q(x)}{\|Q\|_2^2}
\] (3.32)
strongly in $H^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and the proof of Theorem 1.3 is thus complete.

4 Local uniqueness of positive minimizers

Following the $L^\infty$-uniform convergence and exponential decay of previous section, this section is concerned with the analysis of the local uniqueness of positive minimizers for $e(a)$ as $a \nearrow a^*$. Here and in the sequel, we always denote $u_k$ to be a positive minimizer of $e(a_k)$ for convenience, where $a_k \nearrow a^*$ as $k \to \infty$. By the variational theory, there exists a Lagrange multiplier $\mu_k \in \mathbb{R}$ satisfying
\[
\mu_k = e(a_k) - \frac{a_k \beta^2}{1 + \beta^2} \int_\Omega |x|^{-b} |u_k|^{2 + 2\beta^2} dx,
\] (4.1)
such that $u_k$ solves the following Euler-Lagrange equation
\[
- \Delta u_k(x) + V(x) u_k(x) = \mu_k u_k(x) + a_k u_k^{1+2\beta^2}(x)|x|^{-b} \text{ in } \Omega.
\] (4.2)

Under the assumptions of Theorem 1.4, we also define
\[
\alpha_k := \frac{1}{\lambda} \left(\frac{(a^* - a_k)}{a^* \beta^2}\right)^{\frac{1}{p + 2}} > 0.
\] (4.3)

It follows from (3.31) that
\[
\lim_{k \to \infty} \frac{\varepsilon_{a_k}}{\alpha_k} = 1,
\] (4.4)
which combined with (3.31) yields that
\[
\mu_k \alpha_{2k}^2 \to -\beta^2 \text{ as } k \to \infty.
\] (4.5)
Based on the above argument, we now prove the local uniqueness of positive minimizers for \( e(a) \) as \( a \nearrow a^* \).

**Proof of Theorem 1.4.** Suppose that there are two different positive minimizers \( u_{1,k} \) and \( u_{2,k} \) of \( e(a_k) \), where \( a_k \nearrow a^* \) as \( k \to \infty \). It then follows from (4.2) that \( u_{i,k} \) solves the following equation

\[
- \Delta u_{i,k}(x) + V(x)u_{i,k}(x) = \mu_{i,k} u_{i,k}(x) + a_k u_{i,k}^{1+2\beta^2}(x)|x|^{-b} \quad \text{in } \Omega, \quad i = 1, 2, \quad (4.6)
\]

where \( \mu_{i,k} \in \mathbb{R} \) is the suitable Lagrange multiplier. Define

\[
\bar{u}_{i,k}(x) := \frac{\alpha_k}{\beta} \|Q\|_2 u_{i,k}(x) \quad \text{and} \quad \bar{u}_{i,k}(x) := \bar{u}_{i,k}(\frac{\alpha_k}{\beta} x), \quad i = 1, 2. \quad (4.7)
\]

Note from (4.2) that \( \bar{u}_{i,k} \) satisfies

\[
- \alpha_k^2 \Delta \bar{u}_{i,k}(x) + \alpha_k^2 V(x) \bar{u}_{i,k}(x) = \mu_{i,k} \alpha_k^2 \bar{u}_{i,k}(x) + \frac{a_k \alpha_k \beta^2 - b}{a^*} |x|^{-b} u_{1,k}^{1+2\beta^2}(x) \quad \text{in } \Omega, \quad (4.8)
\]

and \( \bar{u}_{i,k}(x) \) satisfies

\[
- \Delta \bar{u}_{i,k}(x) + \frac{\alpha_k^2}{\beta^2} V(x) \bar{u}_{i,k}(x) = \frac{\mu_{i,k} \alpha_k^2}{\beta^2} \bar{u}_{i,k}(x) + \frac{a_k}{a^*} |x|^{-b} \bar{u}_{i,k}^{1+2\beta^2}(x) \quad \text{in } \Omega_k, \quad (4.9)
\]

where \( \Omega_k := \{ x \in \mathbb{R}^N : \frac{\alpha_k}{\beta} x \in \Omega \} \to \mathbb{R}^N \) as \( k \to \infty \) in view of Lemma 3.2 (1) and (4.4).

From now on, when necessary we shall extend \( \bar{u}_{i,k}(x) \) to \( \mathbb{R}^N \) by setting \( \bar{u}_{i,k}(x) \equiv 0 \) on \( \mathbb{R}^N \setminus \Omega_k \). It then follows from Section 3 that \( \bar{u}_{i,k}(x) \) is bounded uniformly in \( L^\infty(\mathbb{R}^N) \) and \( \bar{u}_{i,k}(x) \to Q(x) \) uniformly in \( \mathbb{R}^N \) as \( k \to \infty \). Moreover, by the exponential decay (3.6), one can check that there exist \( C > 0 \) and \( R_1 > 0 \) such that

\[
|\bar{u}_{i,k}(x)|, |\nabla \bar{u}_{i,k}(x)| \leq Ce^{-\frac{C}{k}} \quad \text{in } \Omega_k \setminus B_{R_1}(0) \quad \text{as } k \to \infty, \quad (4.10)
\]

where \( C > 0 \) is independent of \( k \).

Since \( u_{1,k} \equiv u_{2,k} \), we consider

\[
\bar{\xi}_k(x) = \frac{u_{2,k}(x) - u_{1,k}(x)}{\|u_{2,k} - u_{1,k}\|_{L^\infty(\Omega)}} = \frac{\bar{u}_{2,k}(x) - \bar{u}_{1,k}(x)}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty(\Omega)}}. \quad (4.11)
\]

Stimulated by [4], we first claim that for any \( x_0 \in \Omega \), there exists a small constant \( \delta > 0 \) such that

\[
\int_{\partial B_{\delta}(x_0)} \left[ \alpha_k^2 |\nabla \bar{\xi}_k|^2 + \frac{\beta^2}{2} |\bar{\xi}_k|^2 + \alpha_k^2 V(x)|\bar{\xi}_k|^2 \right] dS = O(\alpha_k^N) \quad \text{as } k \to \infty. \quad (4.12)
\]

In fact, denote

\[
\bar{D}_{k}^{s-1}(x) := \frac{\bar{u}_{2,k}(x) - \bar{u}_{1,k}(x)}{\bar{u}_{2,k} - \bar{u}_{1,k}},
\]

\[
\frac{\int_0^1 \frac{1}{s\beta^n} \left[t \bar{u}_{2,k} + (1 - t) \bar{u}_{1,k} \right]^s dt}{\bar{u}_{2,k} - \bar{u}_{1,k}} = \int_0^1 \left[t \bar{u}_{2,k} + (1 - t) \bar{u}_{1,k} \right]^{s-1} dt, \quad (4.13)
\]
then we obtain from (4.8) and (4.11) that \( \tilde{\xi}_k \) satisfies

\[-a_k^2 \Delta \tilde{\xi}_k + C_k(x) \tilde{\xi}_k = \bar{g}_k(x) \text{ in } \Omega, \quad (4.14)\]

where the coefficients

\[\bar{C}_k(x) := -\mu_{1,k} a_k^2 - \frac{a_k \alpha_k (1 + 2\beta^2)}{a^*} |x|^{-b} \tilde{D}_k^{2\beta^2} + \alpha_k^2 V(x), \quad (4.15)\]

and

\[\bar{g}_k(x) := \frac{\bar{u}_{2,k} (\mu_{2,k} - \mu_{1,k}) a_k^2}{\|u_{2,k} - \bar{u}_{1,k}\|_{L^\infty(\Omega)}} \int \Omega |x|^{-b} (u_{2,k}^{2+2\beta^2} - u_{1,k}^{2+2\beta^2}) dx \]

\[= \frac{a_k \bar{u}_{2,k} \alpha_k^2}{(1 + \beta^2) \|u_{2,k} - \bar{u}_{1,k}\|_{L^\infty(\Omega)}} \int \Omega |x|^{-b} (u_{2,k}^{2+2\beta^2} - u_{1,k}^{2+2\beta^2}) dx \]

\[= \frac{2a_k \bar{u}_{2,k} \bar{\beta}^{N+4-b}}{\alpha_k^N \|Q\|^{2(1+\beta^2)}} \int \Omega |x|^{-b} \tilde{D}_k^{1+2\beta^2} dx. \quad (4.16)\]

Multiplying (4.14) by \( \tilde{\xi}_k \) and integrating over \( \Omega \), we obtain that

\[
\alpha_k^2 \int \Omega |\nabla \tilde{\xi}_k|^2 dx - \mu_{1,k} \alpha_k^2 \int \Omega |\tilde{\xi}_k|^2 dx + \alpha_k^2 \int \Omega V(x)|\tilde{\xi}_k|^2 dx \\
= \frac{a_k \alpha_k (1 + 2\beta^2)}{a^*} \int \Omega |x|^{-b} \tilde{D}_k^{2\beta^2} |\tilde{\xi}_k|^2 dx \\
- \frac{2a_k \alpha_k \bar{\beta}^{N+4-b}}{\alpha_k^N \|Q\|^{2(1+\beta^2)}} \int \Omega \bar{u}_{2,k} \tilde{\xi}_k dx \int \Omega |\tilde{\xi}_k|^{-b} \tilde{D}_k^{1+2\beta^2} dx \\
\leq \frac{a_k \alpha_k (1 + 2\beta^2)}{a^*} \int \Omega |x|^{-b} \tilde{D}_k^{2\beta^2} dx \\
+ \frac{2a_k \alpha_k \bar{\beta}^{N+4-b}}{\alpha_k^N \|Q\|^{2(1+\beta^2)}} \int \Omega \bar{u}_{2,k} dx \int \Omega |x|^{-b} \tilde{D}_k^{1+2\beta^2} dx \\
\leq C\alpha_k^N \text{ as } k \to \infty,
\]

where we used the fact that \( \tilde{\xi}_k \) and \( \bar{u}_{i,k}(x) \) are bounded uniformly in \( k \) and \( \bar{u}_{i,k}(x) \) decays exponentially as \( |x| \to \infty \). Recall from (4.5) that \( \alpha_k^2 \mu_{i,k} \to -\beta^2 \) as \( k \to \infty \), the above estimate further implies that there exists a constant \( C_1 > 0 \) such that

\[I := \alpha_k^2 \int \Omega |\nabla \tilde{\xi}_k|^2 dx + \frac{\beta^2}{2} \int \Omega |\tilde{\xi}_k|^2 dx + \alpha_k^2 \int \Omega V(x)|\tilde{\xi}_k|^2 dx < C_1 \alpha_k^N \text{ as } k \to \infty. \quad (4.17)\]

We then conclude from [4, Lemma 4.5] that for any \( x_0 \in \Omega \), there exist constants \( \delta > 0 \) and \( C_2 > 0 \) such that

\[\int_{\partial B_\delta(x_0)} \left[ \alpha_k^2 |\nabla \tilde{\xi}_k|^2 + \frac{\beta^2}{2} |\tilde{\xi}_k|^2 + \alpha_k^2 V(x)|\tilde{\xi}_k|^2 \right] dS \leq C_2 I \leq C_1 C_2 \alpha_k^N \text{ as } k \to \infty,
\]

and the claim (4.12) is hence proved.
Under the assumptions of Theorem 1.4 we complete the proof by deriving a contradiction through the following three steps.

**Step 1.** There exists a constant $b_0$ such that up to a subsequence if necessary, $\tilde{\xi}_k(\frac{a_k}{\beta} x) \to \xi_0(x)$ in $C_{loc}(\mathbb{R}^N)$ as $k \to \infty$, where

$$\xi_0 = b_0 \left( \frac{N}{2} Q + x \cdot \nabla Q \right).$$

(4.18)

To prove (4.18), we denote

$$\tilde{D}_k^{s-1}(x) := \tilde{D}_k^{s-1}(\frac{a_k}{\beta} x) = \int_0^1 \left[ t\tilde{u}_{2,k} + (1-t)\tilde{u}_{1,k} \right]^{s-1} dt,$$

(4.19)

and

$$\xi_k(x) := \xi_k(\frac{a_k}{\beta} x),$$

(4.20)

then $\xi_k$ satisfies

$$-\Delta \xi_k(x) + C_k(x)\xi_k(x) = g_k(x) \quad \text{in} \quad \Omega_k,$$

(4.21)

where $\lim_{k \to \infty} \Omega_k = \mathbb{R}^N$ and the coefficients

$$C_k(x) = -\frac{a_k(1+2\beta^2)}{a^*|x|^b} \tilde{D}_k^{2\beta^2}(x) - \frac{a_k^2}{\beta^2} \mu_{1,k} + \frac{a_k^2}{\beta^2} V \left( \frac{\alpha_k x}{\beta} \right),$$

(4.22)

and

$$g_k(x) = \frac{\tilde{u}_{2,k}}{\beta^2} \frac{\alpha_k^2 (\mu_{2,k} - \mu_{1,k})}{\|\tilde{u}_{2,k} - \tilde{u}_{1,k}\|_{L^\infty(\Omega_k)}} = -\frac{2a_k \tilde{u}_{2,k}}{\|Q\|_2^{2(1+2\beta^2)}} \int_{\Omega_k} |x|^{-b}\xi_k(x) \tilde{D}_k^{1+2\beta^2}(x) dx.$$  

(4.23)

Recall that $\tilde{u}_{i,k}(x)$ is bounded uniformly in $L^\infty(\mathbb{R}^N)$, we then obtain that $|x|^{-b}\tilde{D}_k^{2\beta^2}(x)$ is bounded uniformly in $L^r_{loc}(\mathbb{R}^N)$, where $r \in (\frac{N}{2}, \frac{N}{2})$. Since $\|\xi_k\|_{L^\infty(\Omega_k)} \leq 1$, the standard elliptic regularity (cf. [15]) then yields that $\|\xi_k\|_{W^{1,r}_{loc}(\Omega_k)} \leq C$ and thus $\|\xi_k\|_{C^{0,\alpha}_{loc}(\Omega_k)} \leq C$ for some $\alpha \in (0,2-b)$, where the constant $C > 0$ is independent of $k$. Therefore, there exist a subsequence $a_k$, still denoted by $a_k$, and a function $\xi_0(x)$ such that $\xi_k(x) \to \xi_0(x)$ in $C_{loc}(\mathbb{R}^N)$ as $k \to \infty$. Applying Lemma 3.2 and (4.5), direct calculations yield from (4.22) and (4.23) that

$$C_k(x) \to 1 - (1+2\beta^2)|x|^{-b}Q^{2\beta^2}(x) \quad \text{in} \quad C_{loc}(\mathbb{R}^N) \quad \text{as} \quad k \to \infty,$$

and

$$g_k(x) \to \frac{2Q(x)\beta^2}{\|Q\|_2^2} \int_{\mathbb{R}^N} |x|^{-b}\xi_0 Q^{1+2\beta^2}(x) dx \quad \text{in} \quad C_{loc}(\mathbb{R}^N) \quad \text{as} \quad k \to \infty.$$  

Hence, we derive from (4.21) that $\xi_0$ solves

$$\mathcal{L}\xi_0 = -\Delta \xi_0 + \xi_0 - (1+2\beta^2)|x|^{-b}Q^{2\beta^2}\xi_0 = -\frac{2Q(x)\beta^2}{\|Q\|_2^2} \int_{\mathbb{R}^N} |x|^{-b}\xi_0 Q^{1+2\beta^2} \quad \text{in} \quad \mathbb{R}^N. $$

(4.24)
we conclude from the non-degeneracy of $\mathcal{L}$ in (1.10) that \( L(\frac{N}{2}, Q + x \cdot \nabla Q) = -2Q \), then the last equality has used the following fact integration by parts to get that
\[
\delta > 0
\]

On the other hand, recall from [22, Lemma C.1] that \( \|u_{i,k}(x)\|_{W^{2,r}_0(\Omega_k)} \leq C \), where \( r \in (\frac{N}{2}, \frac{N}{b}) \) and \( C > 0 \) is independent of \( k \). Since \( 0 < b < \min\{2, \frac{N}{2}\} \), we further obtain that \( \|\tilde{u}_{i,k}(x)\|_{H^2_{0}(\Omega_k)} \leq C \). Applying the Cauchy-Schwarz inequality and the Young’s inequality, it follows that
\[
\begin{align*}
\alpha^2_k &\int_{B_\delta(0)} (x \cdot \nabla \tilde{u}_{i,k}(x)) \Delta \tilde{u}_{i,k}(x) dx \\
= &\alpha^2_k \int_{B_\delta(0)} (x \cdot \nabla \tilde{u}_{i,k}(\frac{\beta}{\alpha_k}x)) \Delta \tilde{u}_{i,k}(\frac{\beta}{\alpha_k}x) dx \\
= &\beta^{N-2} \int_{B_{\frac{\beta\delta}{\alpha_k}}(0)} (x \cdot \nabla \tilde{u}_{i,k}(x)) \Delta \tilde{u}_{i,k}(x) dx \\
\leq &\frac{\delta\alpha^{N-1}_k}{\beta^{N-2}} \int_{B_{\frac{\beta\delta}{\alpha_k}}(0)} |\nabla \tilde{u}_{i,k}(x)| |\Delta \tilde{u}_{i,k}(x)| dx \\
\leq &\frac{\delta\alpha^{N-1}_k}{2\beta^{N-2}} \left( \int_{B_{\frac{\beta\delta}{\alpha_k}}(0)} |\nabla \tilde{u}_{i,k}(x)|^2 dx + \int_{B_{\frac{\beta\delta}{\alpha_k}}(0)} |\Delta \tilde{u}_{i,k}(x)|^2 dx \right) \\
\leq &\frac{\delta\alpha^{N-1}_k}{2\beta^{N-2}} \|\tilde{u}_{i,k}(x)\|_{H^2_{0}(\Omega_k)}^2 \leq C,
\end{align*}
\]
where \( \delta > 0 \) is sufficiently small such that \( B_{\delta}(0) \subset \Omega \). Therefore, we can use the integration by parts to get that
\[
\begin{align*}
-\alpha^2_k &\int_{B_\delta(0)} (x \cdot \nabla \tilde{u}_{i,k}) \Delta \tilde{u}_{i,k} dx \\
= &-\alpha^2_k \int_{\partial B_\delta(0)} \frac{\partial \tilde{u}_{i,k}}{\partial \nu} (x \cdot \nabla \tilde{u}_{i,k}) dS + \alpha^2_k \int_{B_\delta(0)} \nabla \tilde{u}_{i,k} \cdot \nabla (x \cdot \nabla \tilde{u}_{i,k}) dx \\
= &\alpha^2_k \int_{\partial B_\delta(0)} \left[ -\frac{\partial \tilde{u}_{i,k}}{\partial \nu} (x \cdot \nabla \tilde{u}_{i,k}) + \frac{1}{2} (x \cdot \nu |\nabla \tilde{u}_{i,k}|^2 \right] dS + \frac{(2-N)\alpha^2_k}{2} \int_{B_\delta(0)} |\nabla \tilde{u}_{i,k}|^2 dx \\
= &\alpha^2_k \int_{\partial B_\delta(0)} \left[ -\frac{\partial \tilde{u}_{i,k}}{\partial \nu} (x \cdot \nabla \tilde{u}_{i,k}) + \frac{1}{2} (x \cdot \nu |\nabla \tilde{u}_{i,k}|^2 + \frac{(2-N)}{4} \left( \nabla \tilde{u}_{i,k} \cdot \nu \right) dS \\
- &\frac{(2-N)}{2} \int_{B_\delta(0)} \left[ \alpha^2_k V(x) \tilde{u}_{i,k}^2 - \alpha^2_k \mu_{i,k} \tilde{u}_{i,k}^2 + \frac{a_k b^2 - b^2 \tilde{u}_{i,k}^{2(1+\beta^2)}}{a^* |x|^b} \right] dx,
\end{align*}
\]
where the last equality has used the following fact
\[
\alpha^2_k \int_{B_\delta(0)} |\nabla \tilde{u}_{i,k}|^2 = \frac{\alpha^2_k}{2} \int_{\partial B_\delta(0)} (\nabla \tilde{u}_{i,k} \cdot \nu) dS \\
- \int_{B_\delta(0)} \left[ \alpha^2_k V(x) \tilde{u}_{i,k}^2 - \alpha^2_k \mu_{i,k} \tilde{u}_{i,k}^2 + \frac{a_k b^2 - b^2 \tilde{u}_{i,k}^{2(1+\beta^2)}}{a^* |x|^b} \right] dx.
\]
On the other hand, multiplying (4.8) by \((x \cdot \nabla \bar{u}_{i,k})\) and integrating over \(B_3(0)\), where \(\delta > 0\) is small as before, we deduce that

\[
-\alpha_k^2 \int_{B_3(0)} (x \cdot \nabla \bar{u}_{i,k}) \Delta \bar{u}_{i,k} dx
= \alpha_k^2 \int_{B_3(0)} \left[ \mu_{i,k} - V(x) \right] \bar{u}_{i,k} (x \cdot \nabla \bar{u}_{i,k}) dx
+ \frac{a_k \alpha_k^b \beta^{2-b}}{a^*} \int_{B_3(0)} \bar{u}_{i,k}^{1+2\beta^2} |x|^{-b} (x \cdot \nabla \bar{u}_{i,k}) dx
= -\alpha_k^2 \int_{B_3(0)} \bar{u}_{i,k}^2 \left\{ N \left[ \mu_{i,k} - V(x) \right] - [x \cdot \nabla V(x)] \right\} dx
+ \frac{\alpha_k^2}{2} \int_{\partial B_3(0)} \bar{u}_{i,k}^2 \left[ \mu_{i,k} - V(x) \right] (x \cdot \nu) dS
+ \frac{a_k \alpha_k^b \beta^{2-b}}{2a^*(1 + \beta^2)} \int_{\partial B_3(0)} \bar{u}_{i,k}^{2(1+\beta^2)} \frac{(x \cdot \nu)}{|x|^b} dS
- \int_{B_3(0)} \bar{u}_{i,k}^{2(1+\beta^2)} \frac{N - b}{|x|^b} dx.
\]

(4.27)

Substituting (4.25) into (4.27), one can derive that

\[
\alpha_k^2 \int_{\partial B_3(0)} \left[ -\frac{\partial \bar{u}_{i,k}}{\partial \nu} (x \cdot \nabla \bar{u}_{i,k}) + \frac{1}{2} (x \cdot \nu) |\nabla \bar{u}_{i,k}|^2 + \frac{(2 - N)}{4} (\nabla \bar{u}_{i,k}^2 \cdot \nu) \right] dS
= \int_{B_3(0)} \left\{ \alpha_k^2 \left[ -\mu_{i,k} + V(x) + \frac{1}{2} (x \cdot \nabla V(x)) \right] \bar{u}_{i,k}^2 \right. \left. - \frac{a_k \alpha_k^b \beta^{4-b}}{a^*(1 + \beta^2)} \bar{u}_{i,k}^{2(1+\beta^2)} |x|^{-b} \right\} dx + I_i,
\]

where \(I_i\) satisfies

\[
I_i = \frac{\alpha_k^2}{2} \int_{\partial B_3(0)} \bar{u}_{i,k}^2 \left[ \mu_{i,k} - V(x) \right] (x \cdot \nu) dS
+ \frac{a_k \alpha_k^b \beta^{2-b}}{2a^*(1 + \beta^2)} \int_{\partial B_3(0)} \bar{u}_{i,k}^{2(1+\beta^2)} \frac{(x \cdot \nu)}{|x|^b} dS.
\]

(4.29)

Note from (4.1) that

\[
\mu_{i,k} \alpha_k^2 \int_{\Omega} \bar{u}_{i,k}^2 dx + \frac{a_k \alpha_k^b \beta^{4-b}}{a^*(1 + \beta^2)} \int_{\Omega} \bar{u}_{i,k}^{2(1+\beta^2)} |x|^{-b} dx = \frac{\|Q\|_{2}^2 \alpha_k^{N+2}}{\beta^N} e(a_k).
\]

We then deduce from above that

\[
-\alpha_k^2 \int_{B_3(0)} \left[ V(x) + \frac{1}{2} [x \cdot \nabla V(x)] \right] \bar{u}_{i,k}^2 dx
= \frac{\alpha_k^2}{2} \int_{\partial B_3(0)} (x \cdot \nabla \bar{u}_{i,k}) dS - \frac{\alpha_k^2}{2} \int_{\partial B_3(0)} (x \cdot \nu) \nabla \bar{u}_{i,k}^2 dS
- \frac{2 - N}{4} \alpha_k^2 \int_{\partial B_3(0)} \nabla \bar{u}_{i,k} \cdot \nu dS + \mu_{i,k} \alpha_k^2 \int_{\Omega \setminus B_3(0)} \bar{u}_{i,k}^2 dx
+ \frac{a_k \alpha_k^b \beta^{1-b}}{a^*(1 + \beta^2)} \int_{\Omega \setminus B_3(0)} \bar{u}_{i,k}^{2(1+\beta^2)} |x|^{-b} dx,
\]

(4.28)
which further implies that

$$- \alpha_k^2 \int_{B_k(0)} \left[ V(x) + \frac{1}{2} |x \cdot \nabla V(x)| \right] (\bar{u}_{1,k} + \bar{u}_{2,k}) \xi_k = T_k,$$

(4.30)

here $T_k$ satisfies

$$T_k = \frac{I_2 - I_1}{\| \bar{u}_{2,k} - \bar{u}_{1,k} \|_{L^\infty(\Omega)}} - \frac{\alpha_k^2}{2} \int_{\partial B_k(0)} (x \cdot \nu) \left[ (\nabla \bar{u}_{2,k} + \nabla \bar{u}_{1,k}) \cdot \nabla \xi_k \right] dS + \frac{2 - N}{4} \alpha_k^2 \left[ \int_{\partial B_k(0)} (\nabla \bar{u}_{2,k} + \nabla \bar{u}_{1,k}) \cdot \nabla \xi_k dS \right]$$

$$+ \int_{\partial B_k(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) (\nabla \xi_k \cdot \nu) dS$$

$$+ a_k \int_{\Omega \setminus B_k(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) \xi_k dx + \frac{(\mu_{2,k} - \mu_{1,k}) \alpha_k^2}{\| \bar{u}_{2,k} - \bar{u}_{1,k} \|_{L^\infty(\Omega)}} \int_{\Omega \setminus B_k(0)} \bar{u}_{1,k}^2 dx$$

$$+ \frac{2a_k \beta^4 - b}{a^*} \int_{\partial B_k(0)} D_k^{1+2\beta} |x|^{-b} \xi_k dS,$$

where

$$\frac{I_2 - I_1}{\| \bar{u}_{2,k} - \bar{u}_{1,k} \|_{L^\infty(\Omega)}} = \frac{a_k \beta^2 - b}{a^*} \int_{\partial B_k(0)} (x \cdot \nu) |x|^{-b} D_k^{1+2\beta} \xi_k dS$$

$$- \frac{\alpha_k^2}{2} \int_{\partial B_k(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) \xi_k V(x) (x \cdot \nu) dS + \frac{\mu_{2,k} \alpha_k^2}{2} \int_{\partial B_k(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) \xi_k (x \cdot \nu) dS$$

$$+ \frac{(\mu_{2,k} - \mu_{1,k}) \alpha_k^2}{2 \| \bar{u}_{2,k} - \bar{u}_{1,k} \|_{L^\infty(\Omega)}} \int_{\partial B_k(0)} \bar{u}_{1,k}^2 (x \cdot \nu) dS.$$

(4.32)

We next estimate the right hand side of (4.31). We first consider $\frac{I_2 - I_1}{\| \bar{u}_{2,k} - \bar{u}_{1,k} \|_{L^\infty(\Omega)}}$. Applying the Hölder inequality, it follows from the exponential decay of $\bar{u}_{i,k}(x)$ and (4.12) that

$$\frac{a_k \beta^2 - b}{a^*} \int_{\partial B_k(0)} (x \cdot \nu) |x|^{-b} D_k^{1+2\beta} \xi_k dS$$

$$- \frac{\alpha_k^2}{2} \int_{\partial B_k(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) \xi_k V(x) (x \cdot \nu) dS$$

$$+ \frac{\mu_{2,k} \alpha_k^2}{2} \int_{\partial B_k(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) \xi_k (x \cdot \nu) dS$$

(4.33)

Moreover, we deduce from (4.16) that

$$\frac{|\mu_{2,k} - \mu_{1,k}| \alpha_k^2}{\| \bar{u}_{2,k} - \bar{u}_{1,k} \|_{L^\infty(\Omega)}} \leq \frac{2a_k \beta^N + 4 - b}{a_k \alpha_k^N - b} \int_{\Omega} |x|^{-b} D_k^{1+2\beta} \xi_k \leq C \text{ as } k \to \infty,$$

(4.34)
which combined with (4.33) yield that
\[
\frac{I_2 - I_1}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty(\Omega)}} = o(e^{-\frac{c_{\delta}}{\alpha_k}}) \text{ as } k \to \infty. \tag{4.35}
\]

In addition, using the Hölder inequality, we derive from (4.10) and (4.12) that
\[
a_k^2 \int_{\partial B_\delta(0)} |(x \cdot \nabla \bar{u}_{2,k})(\nu \cdot \nabla \xi_k)|dS
\leq a_k \left( \int_{B_\delta(0)} |\nabla \bar{u}_{2,k}|^2 dS \right)^{\frac{1}{2}} \left( \alpha_k^2 \int_{\partial B_\delta(0)} |\nabla \xi_k|^2 dS \right)^{\frac{1}{2}}
\leq Ca_k^{\frac{N}{2} + 1} e^{-\frac{c_{\delta}}{\alpha_k}} \text{ as } k \to \infty,
\]
where \( C > 0 \) is independent of \( k \). Similarly, one can get that
\[
a_k^2 \int_{\partial B_\delta(0)} |(x \cdot \nabla \bar{u}_{1,k})(x \cdot \nabla \xi_k)|dS + a_k^2 \int_{\partial B_\delta(0)} |(x \cdot \nabla \bar{u}_{2,k} + \nabla \bar{u}_{1,k}) \cdot \nabla \xi_k|dS
\]
\[
+ a_k^2 \int_{\partial B_\delta(0)} \left| (\bar{u}_{1,k} + \bar{u}_{2,k})(\nu \cdot \nabla \xi_k) \right|dS
\]
\[
+ a_k^2 \int_{\partial B_\delta(0)} \nu \cdot (\nabla \bar{u}_{2,k} + \nabla \bar{u}_{1,k}) \xi_k |dS \leq Ca_k^{\frac{N}{2} + 1} e^{-\frac{c_{\delta}}{\alpha_k}} \text{ as } k \to \infty.
\]

On the other hand, since \(|\xi_k|\) and \(|\mu_{i,k}a_k^2|\) are bounded uniformly in \( k \), we deduce that
\[
|x| \int_{\Omega \setminus B_\delta(0)} (\bar{u}_{2,k} + \bar{u}_{1,k}) \xi_k dx + \frac{(\mu_{2,k} - \mu_{1,k})a_k^2}{\|\bar{u}_{2,k} - \bar{u}_{1,k}\|_{L^\infty(\Omega)}} \int_{\Omega \setminus B_\delta(0)} \bar{u}_{1,k}^2 dx
\]
\[
+ \frac{2a_k \alpha_k^{\beta/2}(\beta - 1)}{a^*} \int_{\Omega \setminus B_\delta(0)} D_k^{1+2\beta/2} \left| \int_{B_\delta(0)} \bar{u}_{1,k} \xi_k dx \right| = o(e^{-\frac{c_{\delta}}{\alpha_k}}) \text{ as } k \to \infty.
\]

We finally conclude from above that
\[
T_k = o(e^{-\frac{c_{\delta}}{\alpha_k}}) \text{ as } k \to \infty,
\]
and we obtain from (4.30) that
\[
-\alpha_k^2 \int_{B_\delta(0)} \left[ V(x) + \frac{1}{2} |x \cdot \nabla V(x)| \right] (\bar{u}_{1,k} + \bar{u}_{2,k}) \xi_k = o(e^{-\frac{c_{\delta}}{\alpha_k}}) \text{ as } k \to \infty. \tag{4.36}
\]

Furthermore, under the assumptions of Theorem 1.4, one can check that
\[
V(x) = [C_0 + o(1)]|x|^p, \quad \partial V(x)/\partial x_j = [C_0 + o(1)] \frac{\partial |x|^p}{\partial x_j} \text{ as } |x| \to 0,
\]
where \( j = 1, \ldots, N \), and \( C_0 \) is a positive constant satisfying \( \lim_{|x| \to 0} V(x)/|x|^p = C_0 \). Therefore, for \( x \in B_\delta(0) \), where \( \delta > 0 \) is small enough, we have
\[
V(x) + \frac{1}{2} |x \cdot \nabla V(x)| = [C_0 + o(1)] \left[ |x|^p + \frac{1}{2} (x \cdot \nabla |x|^p) \right] = [C_0 + o(1)] \frac{2 + p}{2} |x|^p,
\]
where the last equality follows from the fact that \( x \cdot \nabla |x|^p = p|x|^p \). We then derive that

\[
o(e^{-\frac{C\delta}{\alpha_k}}) = -\alpha_k^2 \int_{B_{\delta}(0)} \left[ V(x) + \frac{1}{2}(x \cdot \nabla V(x)) \right] (\tilde{u}_{1,k} + \tilde{u}_{2,k}) \xi_k dx
\]

\[
= -\alpha_k^2 \int_{B_{\delta}(0)} |C_0 + o(1)| \frac{2 + p}{2} |x|^p (\tilde{u}_{1,k}(x) + \tilde{u}_{2,k}(x)) \xi_k dx
\]

\[
= -\frac{\alpha_k^{2+2N+p}}{\beta^N+p} \int_{B_{\beta\delta}(0)} \left[ C_0 + o(1) \right] \frac{2 + p}{2} |y|^p (\tilde{u}_{1,k}(y) + \tilde{u}_{2,k}(y)) \xi_k(y) dy
\]

as \( k \to \infty \).

Note that \( \tilde{u}_{i,k}(x) = \tilde{u}_{i,k}(\frac{\alpha_k}{\beta} x) \rightarrow Q(x) \) uniformly in \( \mathbb{R}^N \) as \( k \to \infty \), we deduce from (4.18) and (4.37) that

\[
0 = \int_{\mathbb{R}^N} |x|^p Q(x) \xi_0(x) dx
\]

\[
= b_0 \int_{\mathbb{R}^N} |x|^p Q(x) \left( \frac{N}{2} Q + x \cdot \nabla Q \right) dx
\]

\[
= \frac{N b_0}{2} \int_{\mathbb{R}^N} |x|^p Q^2(x) dx + \frac{b_0}{2} \int_{\mathbb{R}^N} |x|^p (x \cdot \nabla Q^2(x)) dx
\]

\[
= -\frac{p b_0}{2} \int_{\mathbb{R}^N} |x|^p Q^2(x) dx,
\]

which implies that \( b_0 = 0 \), i.e., \( \xi_0 \equiv 0 \).

**Step 3.** \( \xi_0 \equiv 0 \) cannot occur.

Let \( y_k \) be a point satisfying \( |\xi_k(y_k)| = 1 \). Since \( \tilde{u}_{i,k}(x) \) and \( Q(x) \) decay exponentially as \( k \to \infty \), applying the maximum principle to (4.21) yields that \( |y_k| \leq C \) uniformly in \( k \). Therefore, we have \( \xi_k \rightarrow \xi_0 \neq 0 \) in \( C_{\text{loc}}(\mathbb{R}^N) \), which however contradicts the fact that \( \xi_0 \equiv 0 \). This completes the proof of Theorem 1.4.

\[
\square
\]

**A Appendix.**

In this section, we shall prove Proposition 2.1 on the Gagliardo-Nirenberg type inequality defined in any open bounded domain.

**Proof of Proposition 2.1.** For any given open bounded domain \( \Omega \subset \mathbb{R}^N \), stimulated by [24], we define

\[
a^\ast(\Omega) := \inf_{\{u \neq 0, u \in H_0^1(\Omega)\}} \Upsilon(u),
\]

where

\[
\Upsilon(u) := \frac{\int_{\Omega} |\nabla u(x)|^2 dx \left( \int_{\Omega} |u(x)|^2 dx \right)^{\beta^2}}{\int_{\Omega} |x|^{-\beta} |u(x)|^{2+2\beta^2} dx}.
\]

In this section, we shall prove Proposition 2.1 on the Gagliardo-Nirenberg type inequality defined in any open bounded domain.

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\]
To complete the proof, we only need to prove that \( a^*(\Omega) = a^* := \|Q\|_{L^2(\mathbb{R}^N)}^{2\beta^2} \) and the infimum \( \text{(A.1)} \) is not attained.

We first recall from [10] the following Gagliardo-Nirenberg inequality:

\[
\frac{a^*}{1 + \beta^2} \int_{\mathbb{R}^N} |x|^{-b}|u(x)|^{2+2\beta^2} dx \leq \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \left( \int_{\mathbb{R}^N} |u(x)|^2 dx \right)^{\beta^2} \text{ in } H^1(\mathbb{R}^N).
\]  

(A.3)

It is easy to infer that \( a^* \leq a^*(\Omega) \), since \( H^0_0(\Omega) \subset H^1(\mathbb{R}^N) \). We next prove that \( a^* \geq a^*(\Omega) \). Since \( 0 \in \Omega \) is an open bounded domain in \( \mathbb{R}^N \), there exists an open ball \( B_{2R}(0) \subset \Omega \) centered at an inner point \( 0 \), where \( R > 0 \) is small enough. Choose a nonnegative cutoff function \( \varphi \in C_0^\infty(\mathbb{R}^N) \) such that \( \varphi(x) = 1 \) for \( |x| \leq R \) and \( \varphi(x) = 0 \) for \( |x| \geq 2R \). For \( \tau > 0 \) and \( A_{\tau} > 0 \), set

\[
\Phi_{\tau}(x) = A_{\tau} \frac{\tau^N}{\|Q\|_2^2} \varphi(x)Q(\tau x), \quad x \in \Omega,
\]

(A.4)

where \( Q = Q(|x|) \) is the unique positive radial solution of (1.4) and \( A_{\tau} > 0 \) is chosen such that \( \int_{\Omega} |\Phi_{\tau}(x)|^2 dx = 1 \). One can check that

\[
\frac{a^*(\Omega)}{1 + \beta^2} \leq \Upsilon(\Phi_{\tau}) \rightarrow \frac{a^*}{1 + \beta^2} \text{ as } \tau \rightarrow \infty, \quad \text{(A.5)}
\]

which implies that \( a^* \geq a^*(\Omega) \). Therefore, we conclude that \( a^* = a^*(\Omega) \).

Finally, we prove that the infimum \( \text{(A.1)} \) is not attained. On the contrary, suppose \( \Upsilon(u) \) attains its infimum at some nonzero \( u_0 \in H^0_0(\Omega) \), it then implies that the equality \( \text{(A.3)} \) is attained at \( \bar{u}(x) \in H^1(\mathbb{R}^N) \), where \( \bar{u}(x) = u_0(x) \) for \( x \in \Omega \) and \( \bar{u}(x) \equiv 0 \) for \( x \in \mathbb{R}^N \setminus \Omega \). However, this contradicts the fact that the equality of \( \text{(A.3)} \) holds if and only if \( u(x) = mn^2Q(nx) \) (\( m, n \neq 0 \) are arbitrary). Therefore, the proof of Proposition 2.1 is complete.

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Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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