The Common Prefix Problem On Trees

Sreyash Kenkre  Sundar Vishwanathan

Department Of Computer Science & Engineering,
IIT Bombay, Powai-400076, India.
{srek,sundar}@cse.iitb.ac.in

Abstract

We present a theoretical study of a problem arising in database query optimization [1], which we call as The Common Prefix Problem. We present a \((1 - o(1))\) factor approximation algorithm for this problem, when the underlying graph is a binary tree. We then use a result of Feige and Kogan [2] to show that even on stars, the problem is hard to approximate.

1 Problem

Let \(T\) be a tree with \(V\) as its vertex set and \(E\) as its edge set. Let each vertex \(v\) be associated with a set of labels \(S_v\), taken from an alphabet \(\Sigma\). Suppose that the vertices \(v\) and \(u\) are adjacent and their corresponding labels are given permutations \(P_v\) and \(P_u\). We define the benefit of the edge \(uv\) as the length of the largest common prefix, denoted by \(P_v \wedge P_u\). The goal is to maximize the total benefit by permuting the labels associated with each vertex appropriately. More precisely, find permutations \(P_1, P_2, \ldots, P_{|V|}\), so as to maximize \(\sum_{uv \in E} |P_u \wedge P_v|\). The corresponding decision problem is known to be \(NP - Complete\) [1]. It can be solved in polynomial time if the tree is a path, and a \(1/2\)-factor approximation is known for the case of a binary tree [1]. In this paper give a \((1 - o(1))\) factor algorithm for this problem on binary trees. We then study the problem when the underlying graph is a star \((K_{1,r})\) and prove a hardness of approximation result by relating this problem to the Maximum Edge Biclique
problem. Throughout the paper we assume that the size of the alphabet $\Sigma$ is a constant.

2 Optimal Recursion For Trees

In this section we give a recursion to optimally solve Common Prefix on trees. This recursion may run in exponential time. In the next section we will run this on sufficiently small trees to get the $(1 - o(1))$ factor algorithm.

We observe that the labels that are common to all vertices can always be put as prefixes to the permutations associated with the vertices. If the first label in the permutation associated with each vertex is the same, then we have a label common to all vertices. Hence, once the common labels are removed, there will be an edge with zero benefit in the optimal. This we can delete from the tree $T$, and recurse as follows.

$$OPT_{CP}(T) = |\cap_{v \in V} S_v| + \max_{e \in E} [OPT_{CP}(T_1) + OPT_{CP}(T_2)]$$  \hspace{1cm} (1)

where $T_1$ and $T_2$ are the two connected components of $T \setminus e$. However solving this recursion may involve steps exponential in the number of nodes for example, on a complete binary tree of size $n$. The recursion-(1) can be implemented as a dynamic program for trees which have a polynomially bounded number of subtrees, for example, paths. We show that binary trees of height $\log \log n$ also have this property.

Claim: The total number of subtrees in a binary tree of height $\log \log n$ is at most $n^2$.

Proof: The total number of nodes in a binary tree of height $h$ is at most $2^h$. Connecting each subset of the vertices to the root yields a subtree containing the root, so there are at most $2^{2h}$ such subtrees. Thus the total number of subtrees in a binary tree of height $h$ is at most

$\begin{align*}
2^h + 22^{2h-1} + 2^22^{2h-2} + \ldots + 2^h \\
= 2^{2h+1} - 2^h
\end{align*}$
If \( h \) equals \( \log \log n \), we get the desired result.

It follows that the recursion (1) can be solved optimally in time \( O(n^2) \) on binary trees of height \( \log \log n \). We use this to give a \( (1 - \frac{1}{\log \log n}) \) factor approximation for Common Prefix on binary trees.

### 3 \( (1 - O(1)) \) Factor Algorithm

Consider a binary tree \( T \), of height \( h \) on \( n \) vertices rooted at vertex \( r \). We split \( T \) into sets \( A_1, A_2, \ldots, A_{\log \log n} \), each consisting of subtrees of height at most \( \log \log n \). \( A_1 \) consists of the subtrees obtained by deleting the edges joining vertices from heights \( i \log \log n - 1 \) and \( i \log \log n \) for \( 1 \leq i \leq \left\lfloor \frac{h}{\log \log n} \right\rfloor \). \( A_2 \) consists of subtrees obtained by deleting the edges joining vertices from heights \( i \log \log n \) and \( i \log \log n + 1 \) for \( 0 \leq i \leq \left\lfloor \frac{h}{\log \log n} \right\rfloor \) and so on. Each \( A_i \) consists of vertex disjoint subtrees of height at most \( \log \log n \). Since each \( A_i \) contains no more than \( n \) subtrees, we can solve Common Prefix on each \( A_i \) optimally. We denote the optimal value for \( A_i \) by \( \text{OPT}_{CP}(A_i) \). Note that each edge occurs in all but one of the \( A_i \)'s. Let \( b_e \) denote the benefit of the edge \( e \) in the optimal, and let \( A \) denote the maximum of all \( \text{OPT}_{CP}(A_i) \)'s. Then from the preceding discussion we have,

\[
(\log \log n - 1) \text{OPT}_{CP} = (\log \log n - 1) \sum_{e \in E} b_e \\
= \sum_{e \in A_1} b_e + \sum_{e \in A_2} b_e + \ldots + \sum_{e \in A_{\log \log n}} b_e \\
\leq \text{OPT}_{CP}(A_1) + \text{OPT}_{CP}(A_2) + \ldots + \text{OPT}_{CP}(A_{\log \log n}) \\
\leq (\log \log n) A.
\]

We thus have a factor \( (1 - \frac{1}{\log \log n}) \) algorithm for binary trees by taking the maximum of the \( A_i \)'s. Since a binary tree of height \( \log \log n \) has at most \( n^2 \) subtrees, and each \( A_i \) can have at most \( \frac{h}{\log \log n} \) trees, and since there are \( \log \log n \) \( A_i \)'s, the total time taken for this algorithm is \( O(\frac{h}{\log \log n} n^2 \log \log n) = O(n^3) \).
Note that we can trade the approximation factor for running time as follows. For fixed \( \epsilon < 1 \), take \( N = \lceil \frac{1}{\epsilon} \rceil \). Now, instead of taking subtrees of height at most \( \log \log n \) in the \( A_i \)'s take them to be of height at most \( N \). We can use the recursion (1) to solve for the subtrees of height at most \( N \) in time \( O(n2^N) \). Using the same analysis as above, we get a \((1 - \epsilon)\) factor algorithm that runs in \( O(n\epsilon^22^{\lceil \frac{1}{\epsilon} \rceil + 1}) \) time.

4 Common Prefix on Stars

In this section we prove that the \textit{Common Prefix} problem on stars is equivalent to a problem of finding large nested neighborhoods in bipartite graphs. We shall use this in the next section to prove a hardness of approximation result for \textit{Common Prefix}. Consider the following problem.

\textbf{Definition 1 Nested Neighborhoods :} Given a bipartite graph \( G = (U, V, E) \) with \( U \) and \( V \) as its bipartition and \( E \) as its edge set, find subsets \( U' \subseteq U \) and \( V' \subseteq V \), such that the elements of \( U' \) can be ordered as \( u_1, u_2, \ldots, u_{|U'|} \), with \( V' \cap \Gamma(u_1) \supseteq \Gamma(u_2) \cap V' \supseteq \cdots \supseteq \Gamma(u_{|U'|}) \cap V' \), and such that \( |\Gamma(u_1) \cap V'| + |\Gamma(u_2) \cap V'| + \cdots + |\Gamma(u_{|U'|}) \cap V'| \) is maximized.

Note that the above problem is independent of whether we choose the subset from \( U \) or from \( V \), since \( V' \) can be labeled to get a feasible solution of the same cost. We show that this problem is equivalent to the \textit{Common Prefix} problem on stars.

Suppose \( G = (U, V, E) \) is an instance of \textit{Nested Neighborhoods}. Consider a star \( T \) with leaf nodes corresponding to the vertices in \( U \) and a vertex \( r \notin U \) as the non-leaf vertex. We treat the vertex set \( V \) as a set of labels to be assigned to vertices of \( T \). The vertex \( r \) is given the entire set \( V \) as its set of labels, while each of the remaining vertices \( u \in U \) is assigned the label set \( \Gamma(u) \subseteq V \). We thus have a \textit{Common Prefix} instance on \( T \). If \( u_1, u_2, \ldots, u_{|U'|} \) and \( V' \) is feasible for \textit{Nested Neighborhoods} on \( G \), then we can construct a feasible solution for \textit{Common Prefix} on \( T \), with the same cost, by choosing a permutation of \( V \) that has the labels of \( \Gamma(u_{|U'|}) \cap V' \) first, followed by those of \( \Gamma(u_{|U'|-1}) \cap V' \setminus \Gamma(u_{|U'|}) \) and so on. Thus the \textit{Nested Neighborhoods} problem reduces to the \textit{Common Prefix} problem on stars.
Conversely, if $T$ is star in an instance of Common Prefix, with $\Sigma$ as the label set of the non-leaf vertex $r$ and $\Sigma_i$ as the label set of each leaf $u_i$, then we construct a Nested Neighborhoods instance as follows. The bipartition has the vertex sets $U$, which consists of all the leaf nodes of $T$, and $V$ which consists of the set of labels $\Sigma$ on $r$. A vertex $u_i \in U$ is connected by an edge to a vertex $v_s \in V$, if the corresponding label $s \in \Sigma$ belongs to the label set $\Sigma_i$ of $u_i$. Using an argument similar to that in the previous paragraph, it can be shown that each feasible solution to Common Prefix on $T$ has a corresponding feasible solution to Nested Neighborhoods on $G$, with the same cost. We thus have the following result.

**Theorem 1** The Nested neighborhoods problem is equivalent to the Common Prefix problem on an appropriate star.

We note that these are approximation preserving reduction. From now on, we deal with the Nested Neighborhoods problem.

### 5 Edge Bicliques Problem

Let $G = (U,V,E)$ be a bipartite graph with $U$ and $V$ as its bipartition and $E$ as its set of edges. If $B$ is a subset of the vertex set $(U \cup V)$, the subgraph induced by $B$ is said to be a biclique if $uv \in E$ for all $u \in B \cap U$ and $v \in B \cap V$. The Maximum Edge Biclique (EBCS) problem asks for a subgraph of a given bipartite graph, which is a biclique and has the largest number of edges.

**Lemma 1** Let $G = (U,V,E)$ be a bipartite graph, and let $\text{OPT}_{EBCS}$ and $\text{OPT}_{CP}$ be the optimal values of the EBCS and the Nested Neighborhoods problem on $G$. Then $\text{OPT}_{EBCS} \leq \text{OPT}_{NN}$.

**Proof:** Suppose that $U' = \{u_1, u_2, \ldots, u_k\} \subseteq U$ and $V' = \{v_1, v_2, \ldots, v_l\} \subseteq V$ is a biclique. Since $\Gamma(u_1) \cap V' = \Gamma(u_2) \cap V' = \ldots = \Gamma(u_k) \cap V'$, this corresponds to a feasible solution of the Nested Neighborhoods problem, with the same cost. \hfill $\Box$
Note that the above proof shows the stronger result that every feasible solution to EBCS has a corresponding feasible solution to Nested Neighborhoods with at least as much cost.

**Lemma 2** Let $G = (U, V, E)$ be a bipartite graph. If it has a feasible solution to Nested Neighborhoods of cost $c$, then $G$ contains a biclique with at least $\frac{c}{H_n}$ edges, where $H_n$ denotes the $n^{th}$ harmonic number and $|U| = n$.

**Proof:** Let $U'$ and $V'$ be a feasible solution to the Nested Neighborhoods problem of cost $c$, with $\{u_1, u_2, \ldots, u_k\} = U'$ and such that $\Gamma(u_1) \cap V' \supseteq \Gamma(u_2) \cap V' \supseteq \ldots \supseteq \Gamma(u_k) \cap V'$. Each vertex subset of the form $u_1, u_2, \ldots, u_i$ along with $V' \cap \bigcap_{j=1}^{i} \Gamma(u_i)$ forms a biclique. It is easy to see that if the largest biclique in the subgraph $P$, induced by $U' \cup V'$, contains $u_i$, then it also contains all vertices $u_j$ for $j \leq i$. Let $\epsilon$ be the size of the largest biclique in $P$ and let $y_i$ denote $|\Gamma(u_i) \cap V'|$. The biclique induced by $u_1, u_2, \ldots, u_i$ and $V' \cap \bigcap_{j=1}^{i} \Gamma(u_i)$ has $i \times y_i$ edges. Hence, for each $i = 1, \ldots, k$, $y_i \leq \epsilon/i$. We now have

$$c = y_1 + y_2 + \ldots + y_k$$

$$\leq \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}\right) \epsilon$$

$$\leq H_n \epsilon.$$

This proves the lemma. □

Combining lemma (1) and lemma (2) we get the following.

$$OPT_{EBCS} \leq OPT_{NN} \leq H_n OPT_{EBCS}$$

There are graphs for which the inequality on the right is tight. Consider the bipartite graph $G = (U, V, E)$, with $U = \{u_1, u_2, \ldots, u_n\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ and the edges defined by the relation $\Gamma(u_i) = \{v_1, v_2, \ldots, v_n\}$. It is easily seen that every edge occurs in the optimal solution to Nested Neighborhoods. Thus $OPT_{NN} = n + (n/2) + (n/3) + \ldots + (n/n) = nH_n$. Further, if $k$ is the largest index of a vertex in $U$ in an optimal solution to $EBCS$, then every vertex $u_i$ is in the optimal for $i \leq k$, so that $OPT_{EBCS} = k(n/k) = n$. Thus $\frac{OPT_{NN}}{OPT_{EBCS}} = H_n$ for this graph.
6 Hardness of Common Prefix on Stars

We will need the following result of Feige and Kogan.

**Theorem 2 Feige-Kogan** [2]

*If the maximum edge biclique problem can be approximated within a factor of $2^{(\log n)\delta}$ for every constant $\delta > 0$, then 3-SAT can be solved in time $2^{n^{3/4+\epsilon}}$ for every constant $\epsilon > 0$."

Suppose that there is an algorithm that approximates Nested Neighborhoods on stars within a factor of $\alpha$, i.e. if it returns the value $A$, then $OPT_{CP} \leq A \leq \alpha OPT_{NN}$. Then using lemma-(2), we know that the bipartite graph contains a feasible solution to $EBCS$ of size $A'$, such that $A \leq H_n A'$. We then get an $\alpha/H_n$ factor algorithm for $EBCS$, since

$$A' \geq \frac{A}{H_n} \geq \frac{\alpha}{H_n} OPT_{NN} \geq \frac{\alpha}{H_n} OPT_{EBCS}.$$ 

Thus, using theorems (1) and (2), we get the following hardness result.

**Theorem 3** If the Common Prefix problem for stars can be approximated within a factor of $2^{(\log n)\delta-\log\log n}$ for every constant $\delta > 0$, then 3-SAT can be solved in time $2^{n^{3/4+\epsilon}}$ for every constant $\epsilon > 0$.

7 Acknowledgments

We thank Ravindra Guravannavar for posing this problem.
References

[1] Ravindra Guravannavar, S. Sudarshan, Ajit A. Diwan, Ch. Sobhan Babu, *Reducing Order Enforcement Cost in Complex Query Plans*. Manuscript, November 2006. Available at [http://arxiv.org/abs/cs.DB/0611094](http://arxiv.org/abs/cs.DB/0611094)

[2] Uriel Feige, Shimon Kogan, *Hardness of Approximation of The Balanced Complete Bipartite Subgraph Problem*. Manuscript, May 2004. Available at [http://research.microsoft.com/theory/feige/](http://research.microsoft.com/theory/feige/)