A construction for coisotropic subalgebras of Lie bialgebras

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1. Introduction

A Lie bialgebra [6] structure on a Lie algebra \((g, [\cdot, \cdot])\) is a degree 1 derivation \(\delta \) of \(\wedge^\ast g\) which squares to zero and satisfies \(\delta([X, Y]) = [\delta(X), Y] + [X, \delta(Y)]\). Dualizing \(\delta : g \to \wedge^2 g\) one obtains a Lie bracket on \(g^\ast\), encoding \(\delta\), so that the Lie algebra structures on \(g\) and \(g^\ast\) are compatible. The aim of this paper is to construct Lie subalgebras \(h\) of \(g\) with the property that \(h^\ast\), the subspace of \(g^\ast\) consisting of elements that vanish on \(h\), is a Lie subalgebra of \(g^\ast\). Such an \(h\) is called coisotropic subalgebra.

Our main result (Theorem 3.3) is a explicit and computationally friendly construction that works for Lie bialgebras arising from \(r\)-matrices. Recall that any \(r\)-matrix on a Lie algebra \(g\), i.e. any \(\pi \in \wedge^2 g\) such that \([\pi, \pi]\) is \(ad\)-invariant, gives rise to a Lie bialgebra by setting \(\delta = [\pi, \cdot]\). Our result can be phrased as follows:

**Theorem.** Let \(g\) be a Lie bialgebra arising from an \(r\)-matrix \(\pi\). Suppose \(X \in g\) satisfies

\[
[X, [X, \pi]] = \lambda[X, \pi]
\]

for some \(\lambda \in \mathbb{R}\). Then the image of the map \(g^\ast \to g\) given by contraction with \([X, \pi] \in \wedge^2 g\) is a coisotropic subalgebra of \(g\).

We remark that the coisotropic subalgebras that arise as in the theorem are all even dimensional, therefore they are by no means all coisotropic subalgebras. Using this theorem we produce in a straightforward way families of coisotropic subalgebras when \(g\) is one of the four classical simple complex Lie algebras or one of their split real forms.

Coisotropic subalgebras give rise to Lagrangian subalgebras of the Drinfeld double \(g \oplus g^\ast\) (hence also to Poisson homogeneous spaces [8]) via \(t \mapsto t + t^\circ\). \(L(g \oplus g^\ast)\), the variety of Lagrangian subalgebras of \(g \oplus g^\ast\), can be endowed with a Poisson structure [9]. It would be interesting to characterize the points of \(L(g \oplus g^\ast)\) which correspond to the coisotropic subalgebras we constructed. Notice that \(g \oplus g^\ast\) is isomorphic to the direct sum Lie algebra \(g \oplus g\) studied in [10] (see Remark 4.1). A further reason why coisotropic subalgebras are interesting is that they have a counterpart in the Hopf algebra setting after quantization [5].

Even though the above theorem is phrased entirely in terms of the Lie bialgebra \(g\), its proof involves the Poisson Lie group \(G\) integrating \(g\). The paper is organized as follows. In Section 2 for each \(g \in G\) we consider \(h^\delta\), the left translation to the identity of \(T_g \mathcal{O}\), where \(\mathcal{O}\) denotes the symplectic leaf through \(g\). If \(h^\delta\) is a Lie subalgebra of \(g\) then it is automatically a coisotropic subalgebra. In Section 3 we restrict our attention to Lie bialgebras arising from \(r\)-matrices and elements \(g\) of the form \(\exp(X)\), proving the theorem stated above. Section 4 is devoted to explicit examples in which \(g\) is a semi-simple Lie algebra. In the Appendix we present the geometric motivation that lead to considering the subspaces \(h^\delta\), namely pre-Poisson maps.

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2. Coisotropic subalgebras

We recall some notions from the theory of Poisson Lie groups; we refer to the expositions [17,15,16] for more details. Recall that a Poisson manifold is a manifold $P$ endowed with a bivector field $\Lambda \in \Gamma(\wedge^2 TP)$ satisfying $[\Lambda, \Lambda] = 0$, where $[\cdot, \cdot]$ denotes the Schouten bracket on multivector fields. We denote by $\Lambda^\#: T^*P \to TP$ the map given by contraction with $\Lambda$.

**Definition 2.1.** A Poisson Lie group is a Lie group $G$ equipped with a Poisson bivector $\Lambda$ such that the multiplication map $m: G \times G \to G$ is a Poisson map, or equivalently such that

$$A(gh) = (L_g)_*A(h) + (R_h)_*A(g) \quad \text{for all } g, h \in G. \quad (1)$$

To every element $g$ of the Poisson Lie group $G$ we associate a subspace of its Lie algebra $\mathfrak{g}$ as follows:

$$\mathfrak{h}^\#: = (\eta^\#)^g \mathfrak{g}^*,$$

where we use the short-hand notation

$$\eta^\#: = (L_g)_*A(g^{-1}) \in \wedge^2 \mathfrak{g}. \quad (3)$$

The subspace $\mathfrak{h}^\#$ is the left-translation to the identity of $T_{g^{-1}}\Theta$, where $\Theta$ denotes the symplectic leaf of $(G, \Lambda)$ through $g^{-1}$; in particular it is always even-dimensional. Notice that $(\eta^\#)^g: \mathfrak{g}^* \to \mathfrak{g}$ satisfies the identity

$$(L_g)_* \circ (A(g^{-1}))^\# = (\eta^\#)^g \circ (L_{g^{-1}})^*. \quad (2)$$

**Definition 2.2** ([17, Sec. 3.1]). Let $\mathfrak{g}$ be a Lie bialgebra. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called coisotropic\(^1\) if its annihilator $\mathfrak{h}^\circ$ is a Lie subalgebra of $\mathfrak{g}^\#$.

**Proposition 2.3.** Let $G$ be a Poisson Lie group and $g \in G$. If $\mathfrak{h}^\# \subset \mathfrak{g}$ is a Lie subalgebra then it is automatically a coisotropic subalgebra.

**Proof.** Recall that, for every Poisson manifold $(P, \Lambda)$, there is a Lie bracket\(^2\) on the space of 1-forms, inducing a Lie algebra structure on $(T_p\Theta)^\#$ for each $p \in P$ (here $\Theta$ denotes the symplectic leaf through $p$). It is known that the space of left-invariant 1-forms on the Poisson Lie group $G$ is closed with respect to this bracket, and that evaluation at $e \in G$ is a Lie algebra isomorphism onto the Lie algebra $\mathfrak{g}^\#$ [17, Sect. 2.5]. In particular $(L_{g^{-1}})^\#: (T_{g^{-1}}\Theta)^\# \to \mathfrak{g}^*$ is a Lie algebra homomorphism, with image $(\mathfrak{h}^\#)^\circ$. Hence $(\eta^\#)^g$ is a Lie subalgebra of $\mathfrak{g}^\#$. \(\square\)

It would be interesting to study the set $\{ g \in G : \mathfrak{h}^\# \text{ is a Lie subalgebra} \}$. It is closed under inversion but is not a subgroup of $G$ (see Remark 3.7).

**Remark 2.4.** We are indebted to Jiang Hua Lu for pointing out the above simple proof of Proposition 2.3. In Appendix we present another proof, based on properties of the left translation $L_g$.

3. Poisson Lie groups arising from $r$-matrices

Let $(G, \Lambda)$ be a Poisson Lie group. In this section we determine elements $g \in G$ for which the subspace $\mathfrak{h}^\# \subset \mathfrak{g}$ of Eq. (2) is a Lie subalgebra, for Proposition 2.3 tells us that then it is a coisotropic subalgebra.

**Lemma 3.1.** If $[\eta^\#, \eta^\#] = 0 \in \wedge^3 \mathfrak{g}$ then $\mathfrak{h}^\#$ is a Lie subalgebra of $\mathfrak{g}$.

**Proof.** $[\eta^\#, \eta^\#] = 0$ iff $\mathfrak{h}^\#$, the right-invariant bivector on $G$ whose value at the identity is $\eta^\#$, is a Poisson bivector. In that case the symplectic distribution $(\eta^\#)^g T^*G = \mathfrak{h}^\#$ is involutive, and this is equivalent to $\mathfrak{h}^\#$ being a Lie subalgebra of $\mathfrak{g}$.

**Definition 3.2.** Let $\mathfrak{g}$ be a Lie algebra. An $r$-matrix is an element $\pi \in \wedge^2 \mathfrak{g}$ such that $[\pi, \pi]$ is $ad$-invariant.

It is known [7] that if $\pi$ is an $r$-matrix for the Lie algebra $\mathfrak{g}$ then $\Lambda := \frac{\pi}{\pi_{\mathfrak{g}}} - \frac{\pi_{\mathfrak{g}}}{\pi} \Lambda$ makes $G$, any Lie group integrating $\mathfrak{g}$, into a Poisson Lie group. From now on we restrict ourselves to such Poisson Lie groups. Notice that from definition (3) we get

$$\eta^\# = \pi - \text{Ad}_{\mathfrak{g}}\pi. \quad (4)$$

Now we are able to state the main result of this paper.

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\(^1\) A Lie subalgebra $\mathfrak{h}$ is coisotropic iff the connected subgroup $H$ integrating it is a coisotropic subgroup of $(G, \Lambda)$ (see for instance [5]).

Another equivalent characterization of the fact that $\mathfrak{h}$ is a coisotropic Lie subalgebra is the following: $\mathfrak{h}$ is a coisotropic submanifold of $\mathfrak{g}$, endowed with the linear Poisson structure induced by the Lie algebra $\mathfrak{g}^*$, and $\mathfrak{h}^\#$ is a coisotropic submanifold of the linear Poisson manifold $\mathfrak{g}^*$.

\(^2\) Indeed, $T^*P$ with this bracket and the bundle map $\Lambda^\#: T^*P \to TP$ forms a Lie algebroid [2].
Theorem 3.3. Let $G$ be a Poisson Lie group corresponding to an $r$-matrix $\pi$, $X \in \mathfrak{g}$, $g := \exp(X)$. Assume that
\[
[X, [X, \pi]] = \lambda[X, \pi]
\] for some $\lambda \in \mathbb{R}$.

Then $\mathfrak{h}^g$ is a coisotropic subalgebra of $\mathfrak{g}$. Further
\[
\mathfrak{h}^g = [X, \pi]^g.
\]

Proof. Notice that
\[
Ad_{\exp(X)^g} = e^{adX} = \pi + X, [X, \pi] + \frac{1}{2}[X, [X, \pi]] + \frac{1}{3!}[X, [X, [X, \pi]]] + \cdots = \pi + \frac{e^X - 1}{\lambda}[X, \pi].
\]

Therefore
\[
\eta^g = \pi - Ad_{\exp(X)^g} = \pi - \left(\pi + \frac{e^X - 1}{\lambda}[X, \pi]\right) = -\frac{e^X - 1}{\lambda}[X, \pi].
\]

Now we use twice the fact that $[\pi, [X, \pi]] = \frac{1}{2}[X, [\pi, \pi]] = 0$ (by the graded Jacobi identity) to show that
\[
[[X, \pi], [X, \pi]] = [X, [\pi, [X, \pi]]] = [\pi, [X, [X, \pi]]] = 0 - \lambda \cdot 0 = 0.
\]

This means that $[\eta^g, \eta^g] = 0$, and by Lemma 3.1 and Proposition 2.3 $\mathfrak{h}^g$ is a coisotropic subalgebra. The last part of the theorem follows since the function $\frac{e^X - 1}{\lambda}$ never vanishes. $\Box$

Remark 3.4. If $X \in \mathfrak{g}$ satisfies condition (5) then $A = \pi - \pi$ and $\mathfrak{H}^g$ (or $\mathfrak{h}^g$) are commuting Poisson structures on $G$. This follows at once from the computations of the proof of Theorem 3.3, noticing that $\eta^g$ is a multiple of $[X, \pi]$. Here as usual $g := \exp(X)$.

We now display two very simple examples.

Example 3.5. Let $\mathfrak{g} = su(2)$, so that for a suitable basis we have $[e_1, e_2] = e_3$, $[e_2, e_1] = 1$, $[e_3, e_1] = e_2$, and take the $r$-matrix $\pi = 2e_2 \wedge e_3$ as in [17, Ex. 2.10]. Then the only elements of $su(2)$ that satisfy Eq. (5) are the multiples $X$ of $e_1$, and applying (6) we see that they all give $\mathfrak{h}^{\exp(X)} = \{0\}$.

Example 3.6. Let $\mathfrak{g} = sl(2, \mathbb{R})$, with basis
\[
e_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Then $[e_1, e_2] = e_3$, $[e_2, e_1] = e_1$, $[e_3, e_1] = -e_2$, and $\pi = 2e_2 \wedge e_3$ is an $r$-matrix [17, Ex. 2.9]. The vectors $X$ of $sl(2, \mathbb{R})$ that satisfy Eq. (5) are exactly those of the form $\alpha e_1 + \beta (e_2 + e_3)$ (the upper triangular matrices) and $\alpha e_1 + \beta (e_2 - e_3)$ (the lower triangular matrices). Applying Theorem 3.3 we obtain coisotropic subalgebras $span\{e_1, e_2 - e_3\}$, $span\{e_1, e_2 + e_3\}$, and $[0]$. Using (3) one can compute directly all the elements $g \in G = SL(2, \mathbb{R})$ for which $[\eta^g, \eta^g] = 0$: those of the form $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ 1 & a^{-1} \end{pmatrix}$. By Lemma 3.1 and Proposition 2.3 these group elements $g$ give rise to a coisotropic subalgebra of $g$. The first class of elements $g$ with $b \neq 0$ all give rise to $span\{e_1, e_2 - e_3\}$, the second class of elements $g$ with $c \neq 0$ all give rise to $span\{e_1, e_2 + e_3\}$, and the diagonal matrices give rise to the trivial subalgebra $[0]$, i.e. we obtain exactly the same coisotropic subalgebras as above.

Remark 3.7. We show that $\{g \in G : \mathfrak{h}^g$ is a Lie subalgebra$\}$ is closed under the inversion map but not under multiplication. Indeed notice that $\eta^{g^{-1}} = -Ad_{g^{-1}} \eta^g$ by (1), so $\mathfrak{h}^{g^{-1}} = Ad_{g^{-1}} \mathfrak{h}^g$, and since $Ad_{g^{-1}}$ is a Lie algebra isomorphism the first statement follows.

To show the second statement consider $\mathfrak{g} = sl(2, \mathbb{R})$ as in Example 3.6. The elements $g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ of $G = SL(2, \mathbb{R})$ have the property that $\mathfrak{h}^g$ and $\mathfrak{h}^h$ are Lie subalgebras, by Example 3.6. However $\mathfrak{h}^{gh} = \pi - Ad_{gh} \pi = 2(e_1 \wedge e_2 + 2e_2 \wedge e_3 - e_1 \wedge e_3)$, implying that $\mathfrak{h}^{gh}$ is not a Lie subalgebra of $\mathfrak{g}$.

4. Examples: semi-simple complex Lie algebras

In this section we consider the standard Lie bialgebra structure on a semi-simple complex Lie algebra, and out of its roots, using Theorem 3.3 we construct families of coisotropic subalgebras. We write down explicitly3 the resulting families for the classical simple Lie algebras $sl(n + 1, \mathbb{C})$, $so(2n + 1, \mathbb{C})$, $sp(2n, \mathbb{C})$, $so(2n, \mathbb{C})$ and for their split real forms $sl(n + 1, \mathbb{R})$, $so(n + 1, \mathbb{R})$, $sp(2n, \mathbb{R})$, $so(n, n)$. We refer to [1, Ch. 2.6], to [12] and to [13] for background material about semi-simple complex Lie algebras and their real forms.

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3 One reason for doing this is that we were not able to find any explicit families of examples of coisotropic subalgebras in the literature.
Let \( g \) be a semi-simple Lie algebra over \( \mathbb{C} \), and fix a Cartan subalgebra \( h \). There is a decomposition \( g = h \oplus \oplus_{\alpha \in R} g^\alpha \) where \( g^\alpha \) denotes the one dimensional eigenspace for the adjoint action of \( h \) associated to the "eigenvalue" \( \alpha \in h^* \). The set \( R \subset h^* \) is called root system; make a choice \( R_+ \) of positive roots. For each \( \alpha \in R_+ \) choose non-zero \( e_\alpha \in g^\alpha \) and \( f_\alpha \in g^{-\alpha} \).

Then an \( r \)-matrix is given by

\[
\pi := \sum_{\alpha \in R_+} \lambda_\alpha e_\alpha \wedge f_\alpha
\]

where \( \lambda_\alpha := \frac{1}{B(e_\alpha, f_\alpha)} \) \cite[Ex. 2.10]{10}. Notice that, since the subspaces \( g^\alpha \) are one dimensional and the Killing form \( B \) is \( \mathbb{C} \)-bilinear, the above \( r \)-matrix depends only on the choice of Cartan subalgebra.

**Remark 4.1.** As above let \( g \) be a semi-simple complex Lie algebra, Evens and Lu \cite{10} \cite[Sec. 2.1]{11} consider the direct sum Lie algebra \( g \oplus g \) endowed with the pairing\(^4\) \( \langle x_1 + y_1, x_2 + y_2 \rangle = \frac{1}{2} B(x_1, y_1) - \frac{1}{2} B(x_2, y_2) \) where \( B \) is the Killing form of \( g \). They study the variety \( \mathcal{L}(g \oplus g) \) of Lagrangian subalgebras, and endow it with interesting Poisson structures.

Since \( (g, \{ \cdot, \cdot \}) \) is a Lie bialgebra, \( g \oplus g^* \) admits a Lie algebra structure known as **Drinfeld double**, for which the natural pairing is \( ad \)-invariant \cite[Sec. 2.3]{17}. If \( \mathfrak{t} \subset g \) is a coisotropic subalgebra, then \( \mathfrak{t} \oplus \mathfrak{t}^* \subset g \oplus g^* \) is a Lagrangian subalgebra.

There is an isomorphism of Lie algebras

\[
g \oplus g^* \cong g \oplus g
\]

preserving the pairings. As a consequence, coisotropic subalgebras of \( g \) give rise to points of \( \mathcal{L}(g \oplus g) \), which as seen above is an interesting and well-studied variety.

Eq. (8) follows from \cite[Prop. 1.5]{18} (see also \cite[Prop. 2.1]{21}). We reproduce the proof for completeness. Recall that a Manin triple consists of a Lie algebra with an \( ad \)-invariant non-degenerate symmetric pairing and a decomposition into two Lagrangian subalgebras. There is a bijection between Manin triples and Lie bialgebras \cite[Thm. 2.3.2]{14}, \( g \oplus g, \) together with the diagonal \( g_0 \) and

\[
\{(h + v, -h + w) : h \in h, v \in \oplus_{\alpha \in R_+} g^\alpha, w \in \oplus_{\alpha \in R_+} g^{-\alpha}\},
\]

forms a Manin triple. The corresponding Lie bialgebra consists of the Lie algebra \( g \) with the derivation of \( \wedge^* g \) obtained dualizing the Lie bracket on (9). A computation shows that this derivation is exactly \( \{\cdot, \cdot\} \). Hence the Drinfeld double \( g \oplus g^* \) of the Lie bialgebra \( (g, \{\cdot, \cdot\}) \) is isomorphic to \( g \oplus g \) by a pairing-preserving map, showing (8).

**Lemma 4.2.** Let \( X \in g \) and assume that for all \( \alpha \in R_+ \)

\[
(1) \ [X, [X, e_\alpha]] \wedge f_\alpha = 0 \\
(2) \ [X, [X, f_\alpha]] \wedge e_\alpha = 0 \\
(3) \ [X, e_\alpha] \wedge [X, f_\alpha] = 0.
\]

Then \( X \) satisfies condition (5) (with \( \lambda = 0 \)).

**Proof.** We compute

\[
[X, \pi] = \sum_{\alpha \in R_+} \lambda_\alpha ([X, e_\alpha] \wedge f_\alpha + e_\alpha \wedge [X, f_\alpha]),
\]

so

\[
[X, [X, \pi]] = \sum_{\alpha \in R_+} \lambda_\alpha ([X, [X, e_\alpha]] \wedge f_\alpha + [X, e_\alpha] \wedge [X, f_\alpha]) + e_\alpha \wedge [X, f_\alpha])
\]

each term of which vanishes by our assumptions. \( \square \)

**Proposition 4.3.** Let \( \beta \in R_+ \) satisfy this condition:

\[
(\alpha + Z \beta) \cap R \text{ does not contain a string of 3 consecutive elements.}
\]

Then \( e_\beta \) and \( f_\beta \) satisfy condition (5).

**Proof.** We check that \( X = e_\beta \) satisfies the assumptions of Lemma 4.2; the proof for \( f_\beta \) is similar. Let \( \alpha \in R \).

Suppose that \( [e_\beta, [e_\beta, e_\alpha]] \neq 0 \). Then \( \alpha, \alpha + \beta \) and \( \alpha + 2\beta \) form a string of 3 consecutive elements in \( (\alpha + Z \beta) \cap (R \cup \{0\}) \).

Since the intersection of \( R \) with any line through the origin is either empty or of the form \( (\alpha, -\alpha) \) \cite[Prop. 2.20]{1} it follows that \( \beta = -\alpha \). So \( [e_\beta, [e_\beta, e_\alpha]] \) is a multiple of \( f_\alpha \), and assumption (1) of Lemma 4.2 is satisfied.

Similarly, if \( [e_\beta, [e_\beta, f_\alpha]] \neq 0 \), then \(-\alpha, -\alpha + \beta \) and \(-\alpha + 2\beta \) form a string of 3 consecutive elements in \( (\alpha + Z \beta) \cap (R \cup \{0\}) \), so we must have \( \beta = \alpha \). So \( [e_\beta, [e_\beta, f_\alpha]] \) is a multiple of \( e_\alpha \), and assumption (2) of Lemma 4.2 is satisfied.

At most one of \( \alpha + \beta \) or \( \alpha - \beta \) lie in \( R \); if they both did then \( (\alpha - \beta, \alpha, \alpha + \beta) \) would be a string of 3 consecutive elements in \( (\alpha + Z \beta) \cap R \), contradicting our assumption. If \( \alpha - \beta \notin R \) then either \( \alpha - \beta = 0 \), in which case \( [e_\alpha, e_\beta] = 0 \), or \( [e_\alpha, f_\beta] \in g^{\alpha - \beta} = \{0\} \). A similar reasoning holds for \( \alpha + \beta \), so we conclude that assumption (3) of Lemma 4.2 holds. \( \square \)

\(^4\) They actually consider any non-zero multiple of the Killing form, not just \( \frac{1}{2} \).
Corollary 4.4. Assume the notation above and assume that \( \beta \in R_+ \) satisfy condition (10). Let \( g_\mathbb{R} \) denote \( g \) viewed as a real Lie algebra. Then \( [e_\beta, \pi]^*g_\mathbb{R}^* \) and \( [f_\beta, \pi]^*g_\mathbb{R}^* \)

- are coisotropic subalgebras of \( g_\mathbb{R} \)
- their complexifications are coisotropic subalgebras of the complex Lie bialgebra \( g \).

Proof. The first statement follows from Proposition 4.3 and applying Theorem 3.3 to \( g_\mathbb{R} \).

Now choose \( \tilde{e}_\alpha \in g^\alpha \) and \( \tilde{f}_\alpha \in g^{\alpha^*} \) to be part of a Chevalley basis [1, Ch. 2.6] of \( g_\mathbb{R} \), so that

\[
g_0 := \{ h \in h : \alpha(h) \in \mathbb{R} \text{ for all } \alpha \in R_+ \} \oplus_{\alpha \in R_+} \text{span}_\mathbb{R}[\tilde{e}_\alpha, \tilde{f}_\alpha]
\]

is a Lie subalgebra of \( g_\mathbb{R} \), namely a split real form of \( g \) [13, p. 296]. Since \( \pi \in \wedge^2 g_0 \) and \( \tilde{e}_\beta \in g_0 \), applying Theorem 3.3 to \( g_0 \) we deduce that \( [e_\beta, \pi]^*g_0^* \) is a coisotropic subalgebra of \( g_0 \). The complexification of \( [e_\beta, \pi]^*g_0^* = [\tilde{e}_\beta, \pi]^*g_\mathbb{R}^* \) coincides with the complexification of \( [e_\beta, \pi]^*g_\mathbb{R}^* \), hence the second statement follows. \( \square \)

Our main references for the computation of the examples below are [12, part III] and [20]. Two remarks about the derivation of the examples are in order.

Remark 4.5. (1) We use the fact that the Killing form \( B(A_1, A_2) \) is a non-zero real multiple of \( Tr(A_1A_2) \) [12, Ex. 14.36]. Since the elements \( e_\alpha \) and \( f_\alpha \) we choose are always real matrices, the bivector \( \pi \) is also real, and the coisotropic subalgebras of \( g_\mathbb{R} \) we obtain are also coisotropic subalgebras of \( g \cap \text{Mat}(n, \mathbb{R}) \), which agrees with the split real form of \( g \).

(2) The coisotropic subspace associated to \( f_\beta \) will be obtained just applying the transposition map to the one associated to \( e_\beta \). Indeed in all the examples below the transposition map \( \bullet^T \) is an anti-homomorphism of \( g \) which switches the \( e_\alpha \)’s and the \( f_\alpha \)’s, so it maps \( \pi \) to \( -\pi \) and \( [e_\beta, \pi] \) to \( [f_\beta, \pi] \).

Example 4.6 (A0). Let \( g = \mathfrak{so}(n+1, \mathbb{C}) \) with Cartan subalgebra \( h \) given by the diagonal matrices, so that as roots we obtain \( R = \{ L_i - L_j \}_{(i \neq j)} \subset \mathbb{R}^{n+1} \), where \( L_1, \ldots, L_{n+1} \) denotes the standard basis of \( \mathbb{R}^{n+1} \). It is easy to check that all roots satisfy assumption (10).

For a root \( \alpha = L_i - L_j \) with \( i < j \) we choose \( e_\alpha := E_{ij} \in g^{L_i - L_j} \) and \( f_\alpha := E_{ji} \in g^{L_j - L_i} \), where \( E_{ij} \) denotes the matrix with 1 in the \((i, j)\)-entry and zeros elsewhere. We have \( \pi \sim \sum_{i<j} E_{ji} \wedge E_{ij} \), where “\( \sim \)” means “is a non-zero real multiple of”. Fix a root \( \beta = L_i - L_j \) with \( i < j \). A computation shows that

\[
[E_{ij}, \pi] \sim \left( \sum_{i<k<j} E_{ik} \wedge E_{kj} \right) \wedge E_{kj} = 2 \sum_{i<k<j} E_{ik} \wedge E_{kj} - E_{ij} \wedge (H_i - H_j),
\]

where \( H_i := E_{ii}, \) so for all \( i < j \) we obtain a coisotropic subalgebra of \( g \) spanned by

\[
\{ E_{ij}, H_i - H_j, \{ E_{ik}\}_{i<k<j} \text{ and } \{ E_{ik}\}_{i<k<j} \}.
\]

For instance, letting \( n = 2 \) and taking \( e_\beta = E_{13} \) leads to the coisotropic subalgebra

\[
\left\{ \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & -a & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.
\]

The coisotropic subalgebra we obtain from \( f_\beta = E_{ji} \) \((i < j)\) is spanned by

\[
\{ E_{ji}, H_i - H_j, \{ E_{jk}\}_{i<k<j} \text{ and } \{ E_{kj}\}_{i<k<j} \}.
\]

All of the above are also coisotropic subalgebras of the split real form \( \mathfrak{so}(n+1, \mathbb{R}) \).

Example 4.7 (B0). Let \( g = \mathfrak{so}(2n+1, \mathbb{C}) \), with Cartan subalgebra given by the diagonal matrices. Then \( R = \{ \pm L_i \pm L_j \}_{(i < j)} \cup \{ \pm L_i \} \subset \mathbb{R}^n \). The roots that satisfy assumption (10) are exactly those of the form \( \pm L_i \pm L_j \) \((i < j)\).

The root space of a root \( L_i - L_j \) \((i \neq j)\) is spanned by \( Y_{ij} = E_{i,j+n} - E_{j,n+i} \), the one of \( -L_i + L_j \) is spanned by \( Z_{ij} = E_{n+j,i} - E_{n+i,j} \). Finally, the root space of \( L_i \) is spanned by \( U_i = E_{i,2n+1} - E_{2n+1,i} \) and the one of \( -L_i \) is spanned by \( V_i = E_{n+i,2n+1} - E_{2n+1,i} \). As earlier, \( E_{ij} \) denotes the matrix with 1 in the \((i, j)\)-entry and zeros elsewhere. The r-matrix of Eq. (7) satisfies

\[
\pi \sim \frac{1}{2} \left( \sum_{i<j} X_{ij} \wedge X_{ji} - \sum_{i<j} Y_{ij} \wedge Z_{ij} - \sum_i U_i \wedge V_i \right).
\]

Given a root \( \beta = L_i - L_j \) \((i < j)\), a lengthy but straightforward computation shows

\[
[X_{ij}, \pi] \sim -2 \sum_{i<k<j} (X_{ik} \wedge X_{kj}) + X_{ij} \wedge (H_i - H_j).
\]
So for all \( i < j \) we obtain a coisotropic subalgebra spanned by

\[
\{X_{ik}, X_{kj}\}_{(i < k < j)}, \quad X_{ij}, \quad H_i - H_j
\]

where \( H_i := E_{i,i} - E_{n+i,n+i} \in \mathfrak{h} \). The negative root vector \( f_\beta = X_\beta \) delivers the coisotropic subalgebra spanned by

\[
\{X_{ik}, X_{jk}\}_{(i < k < j)}, \quad X_{ij}, \quad H_i - H_j
\]

If instead we pick a root \( \beta = L_i + L_j \) (with \( i < j \)) we obtain

\[
[Y_{ij}, \pi] = -2 \sum_{i < k < j} (X_{ik} \wedge Y_{kj}) + 2 \sum_{j < k} (X_{jk} \wedge Y_{ki}) + Y_{ij} \wedge (H_i - H_j) + 2U_i \wedge U_j,
\]

giving rise to a coisotropic subalgebra spanned by

\[
\{X_{ik}, X_{jk}\}_{(i < k < j)}, \quad Y_{ij}, \quad H_i - H_j, \quad U_i, \quad U_j.
\]

The root \(-(L_i + L_j)\) (with \( i < j \)) delivers the coisotropic subalgebra spanned by

\[
\{X_{ik}, Z_{kj}\}_{(i < k < j)}, \quad (X_{jk}, Z_{ki})_{(j < k < i)}, \quad Z_{ij}, \quad H_i - H_j, \quad V_i, \quad V_j.
\]

All of the above are also coisotropic subalgebras of the split real form \( \mathfrak{so}(n + 1, n) \).

**Example 4.8 (C\(_n\)).** Let \( g = \mathfrak{sp}(2n, \mathbb{C}) \). Then, choosing the diagonal matrices as Cartan subalgebra, \( R = \{ \pm L_i \pm L_j \} \subset \mathbb{R}^n \). The only roots that satisfy assumption (10) are those of the form \( \pm 2L_i \).

For \( i \neq j \) the root space of a root \( L_i - L_j \) is spanned by \( X_{ij} = E_{i,j} - E_{n+i,n+j} \), as in **Example 4.7**: the root space of a root \( L_i + L_j \) is spanned by \( Y_{ij} = E_{i,n+j} + E_{j,n+i} \), the one of \(-L_i - L_j \) is spanned by \( Z_{ij} = E_{n+i,j} + E_{n+j,i} \). Finally, the root space of \( 2L_i \) is spanned by \( U_i = E_{i,n+i} \) and the one of \(-2L_i \) is spanned by \( V_i = E_{n+i,i} \). We obtain the \( r \)-matrix

\[
\pi \sim \frac{1}{2} \sum_{i < j} X_{ij} \wedge X_{ji} + \frac{1}{2} \sum_{i < j} Y_{ij} \wedge Z_{ij} + \sum_{i} U_i \wedge V_i.
\]

Let us consider the root \( 2L_i \). A computation shows

\[
[U_i, \pi] \sim \sum_{i < k} (Y_{ik} \wedge X_{ik}) + U_i \wedge H_i,
\]

where \( H_i := E_{ii} - E_{n+i,n+i} \), so as coisotropic subspace we obtain the span of

\[
\{Y_{ik}, X_{ik}\}_{i < k}, \quad U_i, \quad H_i.
\]

For instance, when \( n = 2 \), taking \( e_\beta = U_2 = E_{24} \) and \( e_\beta = U_1 = E_{13} \) we obtain the coisotropic subalgebras of \( \mathfrak{sp}(4, \mathbb{C}) \)

\[
\left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -a & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} a & c & b & d \\ 0 & 0 & d & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & -c & 0 \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}.
\]

For the root \(-2L_i \), whose root space is spanned by \( V_i \), as coisotropic subspace we obtain the span of

\[
\{Z_{ik}, X_{ik}\}_{i < k}, \quad V_i, \quad H_i.
\]

All of the above are also coisotropic subalgebras of the split real form \( \mathfrak{sp}(2n, \mathbb{R}) \).

**Example 4.9 (D\(_n\)).** Let \( g = \mathfrak{so}(2n, \mathbb{C}) \). Then \( R = \{ \pm L_i \pm L_j \}_{i < j} \subset \mathbb{R}^n \), and the same computation as in **Example 4.7** shows that all roots satisfy assumption (10). The root spaces of \( L_i - L_j, L_i + L_j \) and \(-L_i - L_j \) are spanned by elements \( X_{ij}, Y_{ij} \) and \( Z_{ij} \) defined by the same formulae as in **Example 4.7**, and the \( r \)-matrix of Eq. (7) satisfies

\[
\pi \sim \frac{1}{2} \left( \sum_{i < j} X_{ij} \wedge X_{ji} - \sum_{i < j} Y_{ij} \wedge Z_{ji} \right)
\]

(it consists of the first two summands of the \( r \)-matrix for the \( B_n \) case).

The same computations as in **Example 4.7** show that (with \( i < j \)) from the root \( L_i - L_j \) we obtain the coisotropic subalgebras spanned by

\[
\{X_{ik}, X_{kj}\}_{(i < k < j)}, \quad X_{ij}, \quad H_i - H_j
\]
and
\[ \{X_{ki}, X_{kj}\}_{i<k<j}, \quad X_{ji}, \quad H_i - H_j \]
whereas from the root \( L_i + L_j \) we obtain the coisotropic subalgebras spanned by
\[ \{X_{ik}, Y_{kj}\}_{i<k<j}, \quad \{X_{kk}, Y_{kk}\}_{j<k}, \quad Y_{ij}, \quad H_i - H_j \]
and
\[ \{X_{ki}, Z_{kj}\}_{i<k<j}, \quad \{X_{ji}, Z_{ji}\}_{i<j}, \quad Z_{ji}, \quad H_i - H_j \]
From now on we consider only the case when the map $\phi$ of Lemma A.2 is a diffeomorphism. Then $D_\phi := E(\phi^{-1}(y))$ defines a singular distribution on $P_2$ which measures how $\phi$ fails to be a Poisson map.

**Definition A.4.** Given a diffeomorphism $\phi: (P_1, A_1) \rightarrow (P_2, A_2)$ between Poisson manifolds, the deficit distribution associated to $\phi$ is the singular distribution on $P_2$ given by

$$D = \{(A_2 - \phi_yA_1) \mathbb{I} : \xi \in T_yP_2\}.$$ 

The deficit distribution $D$ singles out an interesting subalgebra of $C^\infty(P_2)$:

**Lemma A.5.** Let $\phi: (P_1, A_1) \rightarrow (P_2, A_2)$ be a diffeomorphism. Then the set of $D$-invariant functions $\{f : df|_{D_\phi} = 0 \text{ for all } y \in P_2\}$ coincides with

$$\{f : \phi^*(f, \phi^*g) = \{\phi^*f, \phi^*g\} \text{ for all } g \in C^\infty(P_2)\},$$

and is a Poisson subalgebra of $C^\infty(P_2)$.

**Proof.** Expressing $D$ in terms of Hamiltonian vector fields we have $D = \{X_{\phi^*g} - \phi_*X_{\phi^*g} : g \in C^\infty(P_2)\}$. The claimed equality follows from

$$df(X_{\phi^*g} - \phi_*X_{\phi^*g}) = \{f, g\}_y - d\phi^{-1}(0)(\phi^*f)X_{\phi^*g} = (\phi^*(f, g) - \{\phi^*f, \phi^*g\})\phi^{-1}(y)$$

for all $y \in P_2$.

To show that (11) is a Poisson subalgebra we compute for $D$-invariant functions $f$ and $\tilde{f}$ on $P_2$ and for $g \in C^\infty(P_2)$ that

$$\phi^*\{f, \tilde{f}\} = \phi^*\{f, \phi^*g, \phi^*\tilde{f}\} = \{\phi^*f, \phi^*\tilde{f}, \phi^*g\}.$$ 

Hence using twice the Jacobi identity we obtain

$$\phi^*\{f, \tilde{f}\}, g\} = \phi^*\{f, g\}, \phi^*\tilde{f}\} + \phi^*\{f, \tilde{f}, g\} = \{\phi^*f, \phi^*\tilde{f}, \phi^*g\} = \{\phi^*f, \phi^*\tilde{f}, \phi^*g\} = \{\phi^*f, \phi^*\tilde{f}, \phi^*g\}.$$ 

Summarizing the above results we have

**Corollary A.6.** A diffeomorphism $\phi: (P_1, A_1) \rightarrow (P_2, A_2)$ is a pre-Poisson map iff $A_2 - \phi_*A_1$ is a constant rank bivector on $P_2$, i.e. if $D$ is a smooth constant rank distribution on $P_2$. If $D$ is integrable and the leaf space $P_2/D$ is smooth, then $P_2/D$ has a Poisson structure induced by the projection map $\pi: P_2 \rightarrow P_2/D$. In this case the composition $\pi \circ \phi: P_1 \rightarrow P_2/D$ is a Poisson map.

**Proof.** $\phi$ is a pre-Poisson map by Lemma A.2. By the second part of Lemma A.5 the $D$-invariant functions on $P_2$ form a Poisson subalgebra of $C^\infty(P_2)$, so $P_2/D$ has an induced Poisson structure. By the first part of Lemma A.5 in particular $\phi^*\{f, \tilde{f}\} = \{\phi^*f, \phi^*\tilde{f}\}$ for all $D$-invariant functions $f, \tilde{f}$ on $P_2$, so $\pi \circ \phi$ is a Poisson map.

Now let $G$ be a Poisson Lie group and $g \in G$. The subspace $\mathfrak{g}^\circ$ defined in Section 2 generates the deficit distribution of the left translation $L_g: G \rightarrow G$.

**Lemma A.7.** (a) $L_g: G \rightarrow G$ is a pre-Poisson map.

(b) Its deficit distribution is $\mathfrak{g}^\circ$, the right-invariant distribution obtained translating $\mathfrak{g}^\circ \subset T_eG$.

**Proof.** (a) By Corollary A.6 we have to show that $A - (L_g)_*A$ is a constant rank bivector on $G$. This bivector field at the point $k \in G$ is

$$A(k) - (L_g)_*A(g^{-1}k) = -(L_g)_*(R_e)_*A(g^{-1}) = -(R_e)_*\eta^\circ,$$

where we have used (1) applied to $A(g^{-1}k)$ in the first equality. In other words $A - (L_g)_*A = -\eta^\circ$, which obviously has constant rank.

(b) Using (a) we see that the deficit distribution is $[A - (L_g)_*A]^\circ T^*G = [\eta^\circ]^\circ T^*G = \mathfrak{g}^\circ$. 

The observations above allow for an alternative, perhaps more geometric, proof of Proposition 2.3.

**Alternative proof of Proposition 2.3.** For any $f_1, f_2 \in C^\infty(G)$ and $X \in \mathfrak{g}$ we have [17, Ch. 2.3]

$$\{df_1, df_2\}, X\} = X\{f_1, f_2\},$$

(13)

Any element of $(\mathfrak{g}^\circ)^\circ$ can be realized as $df$ where $f$ is a function on $G$ which is invariant along the integrable distribution obtained right-translating $\mathfrak{g}^\circ$. This distribution coincides with the deficit distribution of $L_g: G \rightarrow G$ by Lemma A.7(b). Hence, if $f_1$ and $f_2$ are invariant functions, by Lemma A.5 $\{f_1, f_2\}$ is also invariant. Therefore the right hand side of (13) vanishes for all $X \in \mathfrak{g}^\circ$, from which we deduce that $[df_1, df_2] \in (\mathfrak{g}^\circ)^\circ$. \(\square\)
We conclude with two remarks on Poisson actions.

**Remark A.8.** The considerations of Lemma A.7 can be extended to locally free left Poisson actions (i.e., actions for which \( \sigma : G \times P \rightarrow P \) is a Poisson map, where \( G \times P \) is equipped with the product Poisson structure). In this case we obtain:

(a) For all \( g \in G \), \( \sigma_g : P \rightarrow P \) is a pre-Poisson map.

(b) The deficit distribution of \( \sigma_g \) is generated by the infinitesimal action of \( \mathfrak{h}_g \subset \mathfrak{g} \).

If \( \mathfrak{h}_g \) is a Lie subalgebra of \( \mathfrak{g} \) and \( P/H^g \) is a smooth manifold, where \( H^g \) the connected subgroup of \( G \) integrating \( \mathfrak{h}_g \), then \( P/H^g \) has a Poisson structure for which the projection map \( \pi : P \rightarrow P/H^g \) is Poisson. This is a well-known fact, see [19, Thm. 6] or [17, Prop. 3.4]. **Corollary A.6** in addition tells us that \( \pi \circ \sigma_g : P \rightarrow P/H^g \) is also a Poisson map.

**Remark A.9.** Consider the action by left multiplication \( G \) on itself, and let \( g \in G \) so that \( \mathfrak{h}_g \) is a Lie subalgebra of \( \mathfrak{g} \). Then \( H^g \setminus G \) (if smooth), together with the action of \( G \) by right multiplication, is a right Poisson homogeneous space (i.e., \((H^g \setminus G) \times G \rightarrow H^g \setminus G \) is a transitive right action and a Poisson map). Further both the projection \( \pi \) and \( \pi \circ L_g : G \rightarrow H^g \setminus G \) are Poisson maps which are equivariant for the \( G \)-actions by right multiplication.

**References**

[1] A. Arvanitoyeorgos, An Introduction to Lie Groups and the Geometry of Homogeneous Spaces, in: Student Mathematical Library, vol. 22, American Mathematical Society, Providence, RI, 2003, Translated from the 1999 Greek original and revised by the author.

[2] A. Cannas da Silva, A. Weinstein, Geometric Models for Noncommutative Algebras, in: Berkeley Mathematics Lecture Notes, vol. 10, American Mathematical Society, Providence, RI, 1999.

[3] A.S. Cattaneo, M. Zambon, Pre-Poisson submanifolds, in: Travaux mathématiques., Trav. Math., XVII, Univ. Luxemb., Luxembourg, 2007, pp. 61–74.

[4] A.S. Cattaneo, M. Zambon, Coisotropic embeddings in Poisson manifolds, Trans. Amer. Math. Soc. 361 (7) (2009) 3721–3746.

[5] N. Ciccoli, Quantization of co-isotropicsubgroups, Lett. Math. Phys. 42 (2) (1997) 123–138.

[6] V.G. Drinfel’d, Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang–Baxter equations, Dokl. Akad. Nauk SSSR 268 (2) (1983) 285–287.

[7] V.G. Drinfel’d, Quantum groups, in: Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 798–820.

[8] V.G. Drinfel’d, On Poisson homogeneous spaces of Poisson–Lie groups, Teoret. Mat. Fiz. 95 (2) (1993) 226–227.

[9] S. Evens, J.-H. Lu, On the variety of Lagrangian subalgebras, I, Ann. Sci. École Norm. Sup. (4) 34 (5) (2001) 631–668.

[10] S. Evens, J.-H. Lu, On the variety of Lagrangian subalgebras, II, Ann. Sci. École Norm. Sup. (4) 39 (2) (2006) 347–379.

[11] S. Evens, J.-H. Lu, Poisson geometry of the Grothendieck resolution of a complex semisimple group, Mosc. Math. J. 7 (4) (2007) 613–642, 766.

[12] W. Fulton, J. Harris, Representation Theory, in: Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course. Readings in Mathematics.

[13] A.W. Knapp, Lie Groups Beyond an Introduction, second ed., in: Progress in Mathematics, vol. 140, Birkhäuser Boston Inc., Boston, MA, 2002.

[14] L.I. Korogodski, Y.S. Soibelman, Algebras of Functions on Quantum Groups. Part I, in: Mathematical Surveys and Monographs, vol. 56, American Mathematical Society, Providence, RI, 1998.

[15] Y. Kosmann-Schwarzbach, Lie bialgebras, Poisson Lie groups and dressing transformations, in: Integrability of Nonlinear Systems, (Pondicherry, 1996), in: Lecture Notes in Phys., vol. 495, Springer, Berlin, 1997, pp. 104–170.

[16] C. Laurent-Gengoux, M. Stienon, P. Xu, Lectures on Poisson groupoids, 2007. arXiv:0707.2405.

[17] J.-H. Lu, Multiplicative and affine Poisson structures on Lie groups, 1990, Ph.D. Thesis, U.C. Berkeley. Available at http://hkumath.hku.hk/~jhlu/publications.html.

[18] N.Y. Reshetikhin, M.A. Semenov-Tian-Shansky, Quantum R-matrices and factorization problems, J. Geom. Phys. 5 (4) (1989) 533–550, 1988.

[19] M.A. Semenov-Tian-Shansky, Dressing transformations and Poisson group actions, Publ. Res. Inst. Math. Sci. 21 (6) (1985) 1237–1260.

[20] V.S. Varadarajan, Lie Groups, Lie Algebras, and their Representations, in: Graduate Texts in Mathematics, vol. 102, Springer-Verlag, New York, 1984, Reprint of the 1974 edition.

[21] M. Yakimov, Symplectic leaves of complex reductive Poisson–Lie groups, Duke Math. J. 112 (3) (2002) 453–509.