Motivic Milnor fibers and Jordan normal forms of Milnor monodromies *

Yutaka MATSUI† Kiyoshi TAKEUCHI‡

Abstract

By calculating the equivariant mixed Hodge numbers of motivic Milnor fibers introduced by Denef-Loeser, we obtain explicit formulas for the Jordan normal forms of Milnor monodromies. The numbers of the Jordan blocks will be described by the Newton polyhedron of the polynomial.

1 Introduction

In this paper, by using motivic Milnor fibers introduced by Denef-Loeser [4] and [5], we obtain explicit formulas for the Jordan normal forms of Milnor monodromies. Let $f(x) = \sum_{v \in Z^n} a_v x^v \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial on $\mathbb{C}^n$ such that the hypersurface $f^{-1}(0) = \{ x \in \mathbb{C}^n \mid f(x) = 0 \}$ has an isolated singular point at $0 \in \mathbb{C}^n$. Then by a fundamental theorem of Milnor [15], the Milnor fiber $F_0$ of $f$ at $0 \in \mathbb{C}^n$ has the homotopy type of bouquet of $(n-1)$-spheres. In particular, we have $H_j(F_0; \mathbb{C}) \simeq 0$ ($j \neq 0, n-1$).

Denote by $\Phi_{n-1,0} : H^{n-1}(F_0; \mathbb{C}) \sim \longrightarrow H^{n-1}(F_0; \mathbb{C})$ (1.1) the $(n-1)$-th Milnor monodromy of $f$ at $0 \in \mathbb{C}^n$. By the theory of monodromy zeta functions due to A’Campo [1] and Varchenko [26] etc., the eigenvalues of $\Phi_{n-1,0}$ were fairly well-understood. See Oka’s book [17] for an excellent exposition of this very important result. However to the best of our knowledge, it seems that the Jordan normal form of $\Phi_{n-1,0}$ is not fully understood yet. In this paper, we give a combinatorial description of the Jordan normal form of $\Phi_{n-1,0}$ by using motivic Milnor fibers (For a computer algorithm by Brieskorn lattices, see Schulze [22] etc.).

From now on, let us assume also that $f$ is convenient and non-degenerate at $0 \in \mathbb{C}^n$ (see Definitions 4.4 and 4.5). Note that the second condition is satisfied by generic polynomials $f$. Then we can describe the Jordan normal form of $\Phi_{n-1,0}$ very explicitly as follows. We call the convex hull of $\bigcup_{v \in \text{supp}(f)} \{ v + \mathbb{R}^n_+ \}$ in $\mathbb{R}^n_+$ the Newton polyhedron of $f$ and denote it by $\Gamma_+(f)$. Let $q_1, \ldots, q_l$ (resp. $\gamma_1, \ldots, \gamma'_{l'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_+(f)$ such that $q_i \in \text{Int}(\mathbb{R}^n_+)$ (resp. the relative interior $\text{rel.int}(\gamma_i)$ of $\gamma_i$ is contained in $\text{Int}(\mathbb{R}^n_+)$). For each $q_i$ (resp. $\gamma_i$), denote by $d_i > 0$ (resp. $e_i > 0$) its lattice distance

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†Department of Mathematics, Kinki University, 3-4-1, Kowakae, Higashi-Osaka, Osaka, 577-8502, Japan. E-mail: matsui@math.kindai.ac.jp
‡Institute of Mathematics, University of Tsukuba, 1-1-1, Tennodai, Tsukuba, Ibaraki, 305-8571, Japan. E-mail: takemicro@nifty.com
dist($q_i, 0$) (resp. dist($\gamma_i, 0$)) from the origin $0 \in \mathbb{R}^n$. For $1 \leq i \leq l'$, let $\Delta_i$ be the convex hull of $\{0\} \cup \gamma_i$ in $\mathbb{R}^n$. Then for $\lambda \in \mathbb{C} \setminus \{1\}$ and $1 \leq i \leq l'$ such that $\lambda^{e_i} = 1$ we set

$$n(\lambda)_i = \sharp \{v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i) \mid \text{ht}(v, \gamma_i) = k\} + \sharp \{v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i) \mid \text{ht}(v, \gamma_i) = e_i - k\},$$

where $k$ is the minimal positive integer satisfying $\lambda = \zeta_{e_i}^k (\zeta_{e_i} := \exp(2\pi \sqrt{-1}/e_i))$ and for $v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i)$ we denote by $\text{ht}(v, \gamma_i)$ the lattice height of $v$ from the base $\gamma_i$ of $\Delta_i$. Then in Section 4 we prove the following result which describes the number of Jordan blocks for each fixed eigenvalue $\lambda \neq 1$ in $\Phi_{n-1,0}$. Recall that by the monodromy theorem the sizes of such Jordan blocks are bounded by $n$.

**Theorem 1.1** Assume that $f$ is convenient and non-degenerate at $0 \in \mathbb{C}^n$. Then for any $\lambda \in \mathbb{C}^* \setminus \{1\}$ we have

(i) The number of the Jordan blocks for the eigenvalue $\lambda$ with the maximal possible size $n$ in $\Phi_{n-1,0}$: $H^{n-1}(F_0; \mathbb{C}) \overset{\sim}{\longrightarrow} H^{n-1}(F_0; \mathbb{C})$ is equal to $\sharp \{q_i \mid \lambda^{e_i} = 1\}$.

(ii) The number of the Jordan blocks for the eigenvalue $\lambda$ with size $n - 1$ in $\Phi_{n-1,0}$ is equal to $\sum_{i: \lambda^{e_i} = 1} n(\lambda)_i$.

Namely the Jordan blocks for the eigenvalues $\lambda \neq 1$ in the monodromy $\Phi_{n-1,0}$ are determined by the lattice distances of the faces of $\Gamma_+(f)$ from the origin $0 \in \mathbb{R}^n$. The monodromy theorem asserts also that the sizes of the Jordan blocks for the eigenvalue 1 in $\Phi_{n-1,0}$ are bounded by $n - 1$. In this case, we have the following result. Denote by $\Pi_f$ the number of the lattice points on the 1-skeleton of $\partial \Gamma_+(f) \cap \text{Int}(\mathbb{R}^n_+)$.

**Theorem 1.2** In the situation of Theorem 1.1 we have

(i) (van Doorn-Steenbrink [6]) The number of the Jordan blocks for the eigenvalue 1 with the maximal possible size $n - 1$ in $\Phi_{n-1,0}$ is $\Pi_f$.

(ii) The number of the Jordan blocks for the eigenvalue 1 with size $n - 2$ in $\Phi_{n-1,0}$ is equal to $2 \sum_{\gamma} l^*(\gamma)$, where $\gamma$ ranges through the compact faces of $\Gamma_+(f)$ such that $\dim \gamma = 2$ and $\text{rel.int}(\gamma) \subset \text{Int}(\mathbb{R}^n_+)$. In particular, this number is even.

Note that Theorem 1.2 (i) was previously obtained in van Doorn-Steenbrink [6] by using different methods. Roughly speaking, the nilpotent part for the eigenvalue 1 in the monodromy $\Phi_{n-1,0}$ is determined by the convexity of the hypersurface $\partial \Gamma_+(f) \cap \text{Int}(\mathbb{R}^n_+)$. Thus Theorems 1.1 and 1.2 generalize the well-known fact that the monodromies of quasi-homogeneous polynomials are semisimple. In fact, by our results in Sections 2 and 4 a general algorithm for computing all the spectral pairs of the Milnor fiber $F_0$ is obtained. This in particular implies that we can compute the Jordan normal form of $\Phi_{n-1,0}$ completely. Note that the spectrum of $F_0$ obtained in Saito [20] and Varchenko-Khovanskii [27] is not enough to deduce the Jordan normal form. Moreover, if any compact face of $\Gamma_+(f)$ is prime (see Definition 2.9) we obtain also a closed formula for the Jordan normal form. See Section 4 for the details.
This paper is organized as follows. In Section 2 we introduce some generalizations of the results of Danilov-Khovanskii [3] obtained in [14]. By them we obtain a general algorithm for computing the equivariant mixed Hodge numbers of non-degenerate toric hypersurfaces. In Section 3 we recall some basic definitions and results on motivic Milnor fibers introduced by Denef-Loeser [4] and [5]. Then in Section 4 by rewriting them in terms of the Newton polyhedron \( \Gamma_+(f) \) with the help of the results in Section 2 and [14], we prove various combinatorial formulas for the Jordan normal form of the Milnor monodromy \( \Phi_{n-1,0} \). Although our proof for the eigenvalue 1 in this paper is very different from the one in [14], our results in Section 4 are completely parallel to those for monodromies at infinity obtained in [14]. We thus find a striking symmetry between local and global. Finally, let us mention that in [7] the results for the other eigenvalues \( \lambda \neq 1 \) in this paper were already generalized to the monodromies over complete intersection subvarieties in \( \mathbb{C}^n \).

### 2 Preliminary notions and results

In this section, we recall our results in [14, Section 2] which will be used in this paper. They are slight generalizations of the results in Danilov-Khovanskii [3].

#### Definition 2.1

Let \( g(x) = \sum_{v \in \mathbb{Z}^n} a_v x^v \ (a_v \in \mathbb{C}) \) be a Laurent polynomial on \( (\mathbb{C}^*)^n \).

(i) We call the convex hull of \( \text{supp}(g) := \{ v \in \mathbb{Z}^n \mid a_v \neq 0 \} \subset \mathbb{R}^n \) the Newton polytope of \( g \) and denote it by \( NP(g) \).

(ii) For \( u \in (\mathbb{R}^n)^* \), we set \( \Gamma(g; u) := \{ v \in NP(g) \mid \langle u, v \rangle = \min_{w \in NP(g)} \langle u, w \rangle \} \).

(iii) For \( u \in (\mathbb{R}^n)^* \), we define the \( u \)-part of \( g \) by \( g^u(x) := \sum_{v \in \Gamma(g; u)} a_v x^v \).

#### Definition 2.2 ([9])

Let \( g \) be a Laurent polynomial on \( (\mathbb{C}^*)^n \). Then we say that the hypersurface \( Z^* = \{ x \in (\mathbb{C}^*)^n \mid g(x) = 0 \} \) of \( (\mathbb{C}^*)^n \) is non-degenerate if for any \( u \in (\mathbb{R}^n)^* \) the hypersurface \( \{ x \in (\mathbb{C}^*)^n \mid g^u(x) = 0 \} \) is smooth and reduced.

In the sequel, let us fix an element \( \tau = (\tau_1, \ldots, \tau_n) \in T := (\mathbb{C}^*)^n \) and let \( g \) be a Laurent polynomial on \( (\mathbb{C}^*)^n \) such that \( Z^* = \{ x \in (\mathbb{C}^*)^n \mid g(x) = 0 \} \) is non-degenerate and invariant by the automorphism \( l_\tau := (\mathbb{C}^*)^n \xrightarrow{\tau} (\mathbb{C}^*)^n \) induced by the multiplication by \( \tau \). Set \( \Delta = NP(g) \) and for simplicity assume that \( \dim \Delta = n \). Then there exists \( \beta \in \mathbb{C} \) such that \( l_\tau^* g = g \circ l_\tau = \beta g \). This implies that for any vertex \( v \) of \( \Delta = NP(g) \) we have \( \tau v = \tau_1^{n_1} \cdots \tau_n^{n_n} = \beta \). Moreover by the condition \( \dim \Delta = n \) we see that \( \tau_1, \tau_2, \ldots, \tau_n \) are roots of unity. For \( p, q \geq 0 \) and \( k \geq 0 \), let \( h^{p,q} (H_k^c(Z^*; \mathbb{C})) \) be the mixed Hodge number of \( H_k^c(Z^*; \mathbb{C}) \) and set

\[
e^{p,q}(Z^*) = \sum_k (-1)^k h^{p,q}(H_k^c(Z^*; \mathbb{C})) \tag{2.1}
\]

as in [3]. The above automorphism of \( (\mathbb{C}^*)^n \) induces a morphism of mixed Hodge structures \( l_\tau^* : H_k^c(Z^*; \mathbb{C}) \xrightarrow{\tau} H_k^c(Z^*; \mathbb{C}) \) and hence \( \mathbb{C} \)-linear automorphisms of the \( (p,q) \)-parts \( H_k^c(Z^*; \mathbb{C})^{p,q} \) of \( H_k^c(Z^*; \mathbb{C}) \). For \( \alpha \in \mathbb{C} \), let \( h^{p,q}(H_k^c(Z^*; \mathbb{C}))_\alpha \) be the dimension of the \( \alpha \)-eigenspace \( H_k^c(Z^*; \mathbb{C})^{p,q}_\alpha \) of this automorphism of \( H_k^c(Z^*; \mathbb{C})^{p,q} \) and set

\[
e^{p,q}(Z^*)_\alpha = \sum_k (-1)^k h^{p,q}(H_k^c(Z^*; \mathbb{C}))_\alpha \tag{2.2}
\]
We call \( e^{p,q}(Z^*)_\alpha \) the equivariant mixed Hodge numbers of \( Z^* \). Since we have \( \ell^r_r = \text{id}_{Z^*} \) for some \( r \gg 0 \), these numbers are zero unless \( \alpha \) is a root of unity. Obviously we have

\[
e^{p,q}(Z^*) = \sum_{\alpha \in \mathbb{C}} e^{p,q}(Z^*)_\alpha, \quad e^{p,q}(Z^*)_\alpha = e^{q,p}(Z^*)_{\alpha^*}. \tag{2.3}
\]

In this setting, along the lines of Danilov-Khovanskii \footnote{3} we can give an algorithm for computing these numbers \( e^{p,q}(Z^*)_\alpha \) as follows. First of all, as in \footnote{3 Section 3} we have the following result.

**Proposition 2.3 (\footnote{14, Proposition 2.6})** For \( p, q \geq 0 \) such that \( p + q > n - 1 \), we have

\[
e^{p,q}(Z^*)_\alpha = \begin{cases} (-1)^{n+p+1} \binom{n}{p+1} & (\alpha = 1 \text{ and } p = q), \\ 0 & (\text{otherwise}), \end{cases} \tag{2.4}
\]

(we used the convention \( \binom{a}{b} = 0 \) \((0 \leq a < b)\) for binomial coefficients).

For a vertex \( w \) of \( \Delta \), consider the translated polytope \( \Delta^w := \Delta - w \) such that \( 0 < \Delta^w \) and \( \tau^w = 1 \) for any vertex \( v \) of \( \Delta^w \). Then for \( \alpha \in \mathbb{C} \) and \( k \geq 0 \) set

\[
\ell^*(k\Delta)_\alpha = \{ v \in \text{Int}(k\Delta^w) \cap \mathbb{Z}^n \mid \tau^v = \alpha \} \in \mathbb{Z}_+: = \mathbb{Z}_{\geq 0}. \tag{2.5}
\]

We can easily see that these numbers \( \ell^*(k\Delta)_\alpha \) do not depend on the choice of the vertex \( w \) of \( \Delta \). We define a formal power series \( P_\alpha(\Delta; t) = \sum_{i \geq 0} \varphi_{\alpha,i}(\Delta) t^i \) by

\[
P_\alpha(\Delta; t) = (1 - t)^{n+1} \left\{ \sum_{k \geq 0} \ell^*(k\Delta)_\alpha t^k \right\}. \tag{2.6}
\]

Then we can easily show that \( P_\alpha(\Delta; t) \) is actually a polynomial as in \footnote{3 Section 4.4].

**Theorem 2.4 (\footnote{14, Theorem 2.7})** In the situation as above, we have

\[
\sum_q e^{p,q}(Z^*)_\alpha = \begin{cases} (-1)^{p+n+1} \binom{n}{p+1} + (-1)^{n+1} \varphi_{\alpha,n-p}(\Delta) & (\alpha = 1), \\ (-1)^{n+1} \varphi_{\alpha,n-p}(\Delta) & (\alpha \neq 1). \end{cases} \tag{2.7}
\]

By Proposition 2.3 and Theorem 2.4 we can now calculate the numbers \( e^{p,q}(Z^*)_\alpha \) on the non-degenerate hypersurface \( Z^* \subset (\mathbb{C}^*)^n \) for any \( \alpha \in \mathbb{C} \) as in \footnote{3 Section 5.2}. Indeed for a projective toric compactification \( X \) of \( (\mathbb{C}^*)^n \) such that the closure \( \overline{Z^*} \) of \( Z^* \) in \( X \) is smooth, the variety \( \overline{Z^*} \) is smooth projective and hence there exists a perfect pairing

\[
H^{p,q}(\overline{Z^*}; \mathbb{C})_\alpha \times H^{n-1-p,n-1-q}(\overline{Z^*}; \mathbb{C})_{\alpha^*} \longrightarrow \mathbb{C} \tag{2.8}
\]

for any \( p, q \geq 0 \) and \( \alpha \in \mathbb{C}^* \) (see for example \footnote{28 Section 5.3.2}2). Therefore, we obtain equalities \( e^{p,q}(Z^*)_\alpha = e^{n-1-p,n-1-q}(\overline{Z^*})_{\alpha^*} \) which are necessary to proceed the algorithm in \footnote{3 Section 5.2}. We have also the following analogue of \footnote{3 Proposition 5.8}.

**Proposition 2.5 (\footnote{14, Proposition 2.8})** For any \( \alpha \in \mathbb{C} \) and \( p > 0 \) we have

\[
e^{p,0}(Z^*)_\alpha = e^{0,p}(Z^*)_{\alpha^*} = (-1)^{n-1} \sum_{\Gamma \subset \Delta, \dim \Gamma = p+1} \ell^*(\Gamma)_\alpha. \tag{2.9}
\]
The following result is an analogue of [3, Corollary 5.10]. For $\alpha \in \mathbb{C}$, denote by $\Pi(\Delta)_\alpha$ the number of the lattice points $v = (v_1, \ldots, v_n)$ on the 1-skeleton of $\Delta^w = \Delta - w$ such that $\tau^v = \alpha$, where $w$ is a vertex of $\Delta$.

**Proposition 2.6 ([14, Proposition 2.9])** In the situation as above, for any $\alpha \in \mathbb{C}^*$ we have

$$e^{0,0}(Z^*)_\alpha = \begin{cases} (-1)^{n-1}(\Pi(\Delta)_{1} - 1) & (\alpha = 1), \\ (-1)^{n-1}\Pi(\Delta)_{\alpha - 1} & (\alpha \neq 1). \end{cases} \quad (2.10)$$

For a vertex $w$ of $\Delta$, we define a closed convex cone $\text{Con}(\Delta, w)$ by $\text{Con}(\Delta, w) = \{r \cdot (v-w) \mid r \in \mathbb{R}_+, v \in \Delta\} \subset \mathbb{R}^n$.

**Definition 2.7 ([3])** Let $\Delta$ and $\Delta'$ be two $n$-dimensional integral polytopes in $(\mathbb{R}^n, \mathbb{Z}^n)$. We denote by $\text{som}(\Delta)$ (resp. $\text{som}(\Delta')$) the set of vertices of $\Delta$ (resp. $\Delta'$). Then we say that $\Delta'$ majorizes $\Delta$ if there exists a map $\Psi: \text{som}(\Delta') \longrightarrow \text{som}(\Delta)$ such that $\text{Con}(\Delta, \Psi(w)) \subset \text{Con}(\Delta', w)$ for any vertex $w$ of $\Delta'$.

For an integral polytope $\Delta$ in $(\mathbb{R}^n, \mathbb{Z}^n)$, we denote by $X_{\Delta}$ the toric variety associated with the dual fan of $\Delta$ (see Fulton [8] and Oda [16] etc.). Recall that if $\Delta'$ majorizes $\Delta$ there exists a natural morphism $X_{\Delta'} \longrightarrow X_{\Delta}$.

**Proposition 2.8 ([14, Proposition 2.12])** Let $\Delta$ and $\Delta'$ be as above. Assume that an $n$-dimensional integral polytope $\Delta'$ in $(\mathbb{R}^n, \mathbb{Z}^n)$ majorizes $\Delta$ by the map $\Psi: \text{som}(\Delta') \longrightarrow \text{som}(\Delta)$. Then for the closure $Z^*_1$ of $Z^*$ in $X_{\Delta'}$, we have

$$\sum_q e^{p,q}(Z^*_1) = \sum_{\Gamma < \Delta'} (-1)^{\dim \Gamma + p + 1} \left\{ \binom{\dim \Gamma}{p+1} - \binom{b_\Gamma}{p+1} \right\} + \sum_{\Gamma < \Delta'} (-1)^{\dim \Gamma + 1} \sum_{i=0}^{\min\{b_\Gamma, p\}} \binom{b_\Gamma}{i} \left(-1\right)^i \varphi_{1, \dim \Gamma - 1}^{\Psi(\Gamma)}(\Psi(\Gamma)), \quad (2.11)$$

where for $\Gamma < \Delta'$ we set $b_\Gamma = \dim \Gamma - \dim \Psi(\Gamma)$.

**Definition 2.9** Let $\Delta$ be an $n$-dimensional integral polytope in $(\mathbb{R}^n, \mathbb{Z}^n)$.

(i) (see [3, Section 2.3]) We say that $\Delta$ is prime if for any vertex $w$ of $\Delta$ the cone $\text{Con}(\Delta, w)$ is generated by a basis of $\mathbb{R}^n$.

(ii) (see [14, Definition 2.10]) We say that $\Delta$ is pseudo-prime if for any 1-dimensional face $\gamma < \Delta$ the number of the 2-dimensional faces $\gamma' < \Delta$ such that $\gamma < \gamma'$ is $n-1$.

By definition, prime polytopes are pseudo-prime. Moreover any face of a pseudo-prime polytope is again pseudo-prime. For $\alpha \in \mathbb{C} \setminus \{1\}$ and a face $\Gamma < \Delta$, set $\varphi_\alpha(\Gamma) = \sum_{i=0}^{\dim \Gamma} \varphi_{\alpha, i}(\Gamma)$. Then as in [3, Section 5.5 and Theorem 5.6] we obtain the following result.
Proposition 2.10 ([14, Corollary 2.15]) Assume that $\Delta = NP(g)$ is pseudo-prime. Then for any $\alpha \in \mathbb{C} \setminus \{1\}$ and $r \geq 0$, we have

$$\sum_{p+q=r} e^{p,q}(Z^*)_{\alpha} = (-1)^{n+r} \sum_{\Gamma \prec \Delta} \left\{ \sum_{\Gamma' \prec \Gamma} (-1)^{\dim \Gamma'} \tilde{\varphi}_{\alpha}(\Gamma') \right\}. \quad (2.12)$$

The following lemma will be used later.

Lemma 2.11 Let $\gamma$ be a d-dimensional prime polytope. Then for any $0 \leq p \leq d$ we have

$$\sum_{\Gamma \prec \gamma} (-1)^{\dim \Gamma} \left( \dim \Gamma \right)_p = \sum_{\Gamma \prec \gamma} (-1)^{d+\dim \Gamma} \left( \dim \Gamma \right) \quad (2.13)$$

Proof. For a polytope $\Delta$, denote the number of the $j$-dimensional faces of $\Delta$ by $f_{\Delta,j}$ and set $f_{\Delta,-1} = 1$. Let $\gamma^\vee$ be the dual polytope of $\gamma$. Then $\gamma^\vee$ is simplicial and we have $f_{\gamma^\vee,j} = f_{\gamma,d-1-j}$ for any $0 \leq j \leq d$. Hence (2.13) follows from the Dehn-Sommerville equations (see [23] etc.) for simplicial polytopes. \qed

3 Motivic Milnor fibers

In [4] and [5] Denef and Loeser introduced motivic Milnor fibers. In this section, we recall their definition and basic properties. Let $f \in \mathbb{C}[x_1, x_2, \ldots, x_n]$ be a polynomial such that the hypersurface $f^{-1}(0) = \{x \in \mathbb{C}^n \mid f(x) = 0\}$ has an isolated singular point at $0 \in \mathbb{C}^n$. Then by a fundamental theorem of Milnor [15], for the Milnor fiber $F_0$ of $f$ at 0 we have $H^j(F_0; \mathbb{C}) \cong 0 \ (j \neq 0, n - 1)$. Denote by $\Phi_{n-1,0} : H^{n-1}(F_0; \mathbb{C}) \to H^{n-1}(F_0; \mathbb{C})$ the $(n-1)$-th Milnor monodromy of $f$ at $0 \in \mathbb{C}^n$. Let $\pi : X \to \mathbb{C}^n$ be an embedded resolution of $f^{-1}(0)$ such that $\pi^{-1}(0)$ and $\pi^{-1}(f^{-1}(0))$ are normal crossing divisors in $X$. Let $D_1, D_2, \ldots, D_m$ be the irreducible components of $\pi^{-1}(0)$ and denote by $Z$ the proper transform of $f^{-1}(0)$ in $X$. For $1 \leq i \leq m$ denote by $a_i > 0$ the order of the zero of $g := f \circ \pi$ along $D_i$. For a non-empty subset $I \subset \{1, 2, \ldots, m\}$ we set $D_I = \gcd(a_i)_{i \in I} > 0$, $D_I = \bigcap_{i \in I} D_i$ and

$$D_I^0 = D_I \setminus \left\{ \left( \bigcup_{i \notin I} D_i \right) \cup Z \right\} \subset X. \quad (3.1)$$

Moreover we set

$$Z_I^0 = \left( D_I \setminus \left( \bigcup_{i \notin I} D_i \right) \right) \cap Z \subset X. \quad (3.2)$$

Then, as in [5, Section 3.3], we can construct an unramified Galois covering $\tilde{D}_I^0 \to D_I^0$ of $D_I^0$ as follows. First, for a point $p \in D_I^0$ we take an affine open neighborhood $W \subset X \setminus \left\{ \bigcup_{i \notin I} D_i \cup Z \right\}$ of $p$ on which there exist regular functions $\xi_i \ (i \in I)$ such that $D_i \cap W = \{\xi_i = 0\}$ for any $i \in I$. Then on $W$ we have $g = f \circ \pi = g_{1,W}(g_{2,W})^{d_I}$, where we set $g_{1,W} = g \prod_{i \in I} \xi_i^{-a_i}$ and $g_{2,W} = \prod_{i \in I} \xi_i^{2d_I}$. Note that $g_{1,W}$ is a unit on $W$ and $g_{2,W} : W \to \mathbb{C}$ is a regular function. It is easy to see that $D_I^0$ is covered by such affine open subsets $W$. Then as in [5, Section 3.3] by gluing the varieties

$$\tilde{D}_{I,W}^0 = \{(t, z) \in \mathbb{C}^* \times (D_I^0 \cap W) \mid t^{d_I} = (g_{1,W})^{-1}(z)\} \quad (3.3)$$
together in the following way, we obtain the variety $\tilde{D}_f^0$ over $D_f^0$. If $W'$ is another such open subset and $g = g_{1,W'}(g_{2,W'})^{d_2}$ is the decomposition of $g$ on it, we patch $\tilde{D}_{I,W}^0$ and $\tilde{D}_{I,W'}^0$ by the morphism $(t, z) \mapsto (g_{2,W'}(z)(g_{2,W})^{-1}(z) \cdot t, z)$ defined over $W \cap W'$. Now for $d \in \mathbb{Z}_{>0}$, let $\mu_d \simeq \mathbb{Z}/d\mathbb{Z}$ be the multiplicative group consisting of the $d$-roots in $\mathbb{C}$. We denote by $\hat{\mu}$ the projective limit $\lim_{\rightarrow d}$ of the projective system $\{\mu_i\}_{i \geq 1}$ with morphisms $\mu_{id} \rightarrow \mu_i$ given by $t \mapsto t^d$. Then the unramified Galois covering $\tilde{D}_f^0$ of $D_f^0$ admits a natural $\mu_d$-action defined by assigning the automorphism $(t, z) \mapsto (\zeta_d t, z)$ of $\tilde{D}_f^0$ to the generator $\zeta_d := \exp(2\pi\sqrt{-1}/d_1) \in \mu_d$. Namely the variety $\tilde{D}_f^0$ is equipped with a good $\hat{\mu}$-action in the sense of Denef-Loeser [5, Section 2.4]. Note that also the variety $Z_f^0$ is equipped with the trivial good $\hat{\mu}$-action. Following the notations in [5], denote by $M_C^{\hat{\mu}}$ the ring obtained from the Grothendieck ring $K_0^{\hat{\mu}}(\text{Var}_C)$ of varieties over $\mathbb{C}$ with good $\hat{\mu}$-actions by inverting the Lefschetz motive $\mathbb{L} \simeq \mathbb{C} \in K_0^{\hat{\mu}}(\text{Var}_C)$. Recall that $\mathbb{L} \in K_0^{\hat{\mu}}(\text{Var}_C)$ is endowed with the trivial action of $\hat{\mu}$.

**Definition 3.1 (Denef and Loeser [4] and [5])** We define the motivic Milnor fiber $S_{f,0} \in M_c^{\hat{\mu}}$ of $f$ at $0 \in \mathbb{C}^n$ by

$$S_{f,0} = \sum_{l \neq 0} \left\{ (1 - L)^{d-1}[D_f^0] + (1 - L)^{d}[Z_f^0] \right\} \in M_c^{\hat{\mu}}. \quad (3.4)$$

As in [5] Section 3.1.2 and 3.1.3], we denote by $HS^{\text{mon}}$ the abelian category of Hodge structures with a quasi-unipotent endomorphism. Let $K_0(\text{HS}^{\text{mon}})$ be its Grothendieck ring. Then as in [5], to the cohomology groups $H^j(F_0; \mathbb{C})$ and the semisimple parts of their monodromy automorphisms, we can naturally associate an element

$$[H_f] \in K_0(\text{HS}^{\text{mon}}). \quad (3.5)$$

To describe the element $[H_f] \in K_0(\text{HS}^{\text{mon}})$ in terms of $S_{f,0} \in M_c^{\hat{\mu}}$, let

$$\chi_h : M_c^{\hat{\mu}} \rightarrow K_0(\text{HS}^{\text{mon}}) \quad (3.6)$$

be the Hodge characteristic morphism defined in [5] which associates to a variety $Z$ with a good $\mu_d$-action the Hodge structure

$$\chi_h([Z]) = \sum_{j \in \mathbb{Z}} (-1)^j [H^j_c(Z; \mathbb{Q})] \in K_0(\text{HS}^{\text{mon}}) \quad (3.7)$$

with the actions induced by the one $z \mapsto \exp(2\pi\sqrt{-1}/d)z$ ($z \in Z$) on $Z$. Then we have the following fundamental result.

**Theorem 3.2 (Denef-Loeser [4, Theorem 4.2.1])** In the Grothendieck group $K_0(\text{HS}^{\text{mon}})$, we have

$$[H_f] = \chi_h(S_{f,0}). \quad (3.8)$$

For $[H_f] \in K_0(\text{HS}^{\text{mon}})$ also the following result due to Steenbrink [24] and Saito [19], [21] is fundamental.
Theorem 3.3 (Steenbrink [24] and Saito [19, 21]) In the situation as above, we have

(i) Let $\lambda \in \mathbb{C}^* \setminus \{1\}$. Then we have $e^{p,q}([H_f])_\lambda = 0$ for $(p, q) \notin [0, n-1] \times [0, n-1]$.
Moreover for $(p, q) \in [0, n-1] \times [0, n-1]$ we have

$$e^{p,q}([H_f])_\lambda = e^{n-1-q,n-1-p}([H_f])_\lambda.$$  
(3.9)

(ii) We have $e^{p,q}([H_f])_1 = 0$ for $(p, q) \notin \{(0,0)\} \cup ([1, n-1] \times [1, n-1])$ and $e^{0,0}([H_f])_1 = 1$. Moreover for $(p, q) \in [1, n-1] \times [1, n-1]$ we have

$$e^{p,q}([H_f])_1 = e^{n-q,n-p}([H_f])_1.$$  
(3.10)

We can check these symmetries of $e^{p,q}([H_f])_\lambda$ by calculating $\chi_h(S_{f,0}) \in K_0(HS^{\text{mon}})$ explicitly by our methods (see Section 4) in many cases. Since the weights of $[H_f] \in K_0(HS^{\text{mon}})$ are defined by the monodromy filtration, we have the following result.

Theorem 3.4 In the situation as above, we have

(i) Let $\lambda \in \mathbb{C}^* \setminus \{1\}$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue $\lambda$ with sizes $\geq k$ in $\Phi_{n-1,0} : H^{n-1}(F_0; \mathbb{C}) \to H^{n-1}(F_0; \mathbb{C})$ is equal to

$$(-1)^{n-1} \sum_{p+q=n-2+k,n-1+k} e^{p,q} (\chi_h(S_{f,0}))_\lambda.$$  
(3.11)

(ii) For $k \geq 1$, the number of the Jordan blocks for the eigenvalue $1$ with sizes $\geq k$ in $\Phi_{n-1,0}$ is equal to

$$(-1)^{n-1} \sum_{p+q=n-1+k,n+k} e^{p,q} (\chi_h(S_{f,0}))_1.$$  
(3.12)

4 Jordan normal forms of Milnor monodromies

Our methods in [13] can be applied also to the Jordan normal forms of local Milnor monodromies. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial such that the hypersurface $\{x \in \mathbb{C}^n \mid f(x) = 0\}$ has an isolated singular point at $0 \in \mathbb{C}^n$.

Definition 4.1 Let $f(x) = \sum_{v \in \mathbb{Z}_+^n} a_v x^v \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial on $\mathbb{C}^n$.

(i) We call the convex hull of $\bigcup_{v \in \text{supp}(f)} \{v + \mathbb{R}_+^n\}$ in $\mathbb{R}_+^n$ the Newton polyhedron of $f$
and denote it by $\Gamma_+(f)$.

(ii) The union of the compact faces of $\Gamma_+(f)$ is called the Newton boundary of $f$ and
 denoted by $\Gamma_f$.

(iii) We say that $f$ is convenient if $\Gamma_+(f)$ intersects the positive part of any coordinate
axis in $\mathbb{R}_+^n$.

Definition 4.2 ([9]) We say that a polynomial $f(x) = \sum_{v \in \mathbb{Z}_+^n} a_v x^v$ ($a_v \in \mathbb{C}$) is non-
degenerate at $0 \in \mathbb{C}^n$ if for any face $\gamma \prec \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$ the complex
hypersurface $\{x \in (\mathbb{C}^*)^n \mid f_\gamma(x) = 0\}$ in $(\mathbb{C}^*)^n$ is smooth and reduced, where we set
$f_\gamma(x) = \sum_{v \in \gamma \cap \mathbb{Z}_+^n} a_v x^v$. 

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Recall that generic polynomials having a fixed Newton polyhedron are non-degenerate at $0 \in \mathbb{C}^n$. From now on, we always assume also that $f = \sum_{v \in \mathbb{Z}_+^n} a_v x^v \in \mathbb{C}[x_1, \ldots, x_n]$ is convenient and non-degenerate at $0 \in \mathbb{C}^n$. For each face $\gamma < \Gamma_+(f)$ such that $\gamma \subset \Gamma_f$, let $d_\gamma > 0$ be the lattice distance of $\gamma$ from the origin $0 \in \mathbb{R}^n$ and $\Delta_\gamma$ the convex hull of $\{0\} \cup \gamma$ in $\mathbb{R}^n$. Let $\mathbb{L}(\Delta_\gamma)$ be the $(\dim \gamma + 1)$-dimensional linear subspace of $\mathbb{R}^n$ spanned by $\Delta_\gamma$ and consider the lattice $M_\gamma = \mathbb{Z}^n \cap \mathbb{L}(\Delta_\gamma) \simeq \mathbb{Z}^{\dim \gamma - 1}$ in it. Then we set $T_{\Delta_\gamma} := \text{Spec}(\mathbb{C}[M_\gamma]) \simeq (\mathbb{C}^*)^{\dim \gamma + 1}$. Moreover let $\mathbb{L}(\gamma)$ be the smallest affine linear subspace of $\mathbb{R}^n$ containing $\gamma$ and for $v \in M_\gamma$ define their lattice heights $\text{ht}(v, \gamma) \in \mathbb{Z}$ from $\mathbb{L}(\gamma)$ in $\mathbb{L}(\Delta_\gamma)$ so that we have $\text{ht}(0, \gamma) = d_\gamma > 0$. Then to the group homomorphism $M_\gamma \longrightarrow \mathbb{C}^*$ defined by $v \mapsto \zeta_{d_\gamma}^{\text{ht}(v, \gamma)}$ we can naturally associate an element $\tau_\gamma \in T_{\Delta_\gamma}$. We define a Laurent polynomial $g_\gamma = \sum_{v \in M_\gamma} b_v x^v$ on $T_{\Delta_\gamma}$ by

$$b_v = \begin{cases} a_v & (v \in \gamma), \\ -1 & (v = 0), \\ 0 & \text{(otherwise)}. \end{cases} \quad \text{(4.1)}$$

Then we have $NP(g_\gamma) = \Delta_\gamma$, $\text{supp}(g_\gamma) \subset \{0\} \cup \gamma$ and the hypersurface $Z^*_\Delta_\gamma = \{ x \in T_{\Delta_\gamma} \mid g_\gamma(x) = 0 \}$ is non-degenerate by [14, Proposition 5.3]. Moreover $Z^*_{\Delta_\gamma} \subset T_{\Delta_\gamma}$ is invariant by the multiplication $t_{\tau_\gamma} : T_{\Delta_\gamma} \xrightarrow{\sim} T_{\Delta_\gamma}$ by $\tau_\gamma$, and hence we obtain an element $[Z^*_{\Delta_\gamma}]$ of $\mathcal{M}_C^\mu$. Let $\mathbb{L}(\gamma)' \simeq \mathbb{R}^{\dim \gamma}$ be a linear subspace of $\mathbb{R}^n$ such that $\mathbb{L}(\gamma) = \mathbb{L}(\gamma)' + w$ for some $w \in \mathbb{Z}^n$ and set $\gamma' = \gamma - w \subset \mathbb{L}(\gamma)'$. We define a Laurent polynomial $g'_\gamma = \sum_{v \in \mathbb{L}(\gamma)' \cap \mathbb{Z}^n} b'_v x^v$ on $T(\gamma) := \text{Spec}(\mathbb{C}[\mathbb{L}(\gamma)' \cap \mathbb{Z}^n]) \simeq (\mathbb{C}^*)^{\dim \gamma}$ by

$$b'_v = \begin{cases} a_{v+w} & (v \in \gamma'), \\ 0 & \text{(otherwise)}. \end{cases} \quad \text{(4.2)}$$

Then we have $NP(g'_\gamma) = \gamma'$ and the hypersurface $Z^*_\gamma = \{ x \in T(\gamma) \mid g'_\gamma(x) = 0 \}$ is non-degenerate. We define $[Z^*_\gamma] \in \mathcal{M}_C^\mu$ to be the class of the variety $Z^*_\gamma$ with the trivial action of $\mu$. Finally let $S_\gamma \subset \{1, 2, \ldots, n\}$ be the minimal subset of $\{1, 2, \ldots, n\}$ such that $\gamma \subset \{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \mid y_i = 0 \text{ for any } i \notin S_\gamma\} \simeq \mathbb{R}^{n-S}$ and set $m_\gamma := \dim \gamma - 1 \geq 0$. Then as in the same way as [14, Theorem 5.7] we obtain the following theorem.

**Theorem 4.3** In the situation as above, we have

(i) In the Grothendieck group $K_0(\text{HS}^{\text{mon}})$, we have

$$\chi_h(S_{f,0}) = \sum_{\gamma \in \Gamma_f} \chi_h((1 - \mathbb{L})^{m_\gamma} \cdot [Z^*_\Delta_\gamma]) + \sum_{\gamma \in \Gamma_f} \chi_h((1 - \mathbb{L})^{m_\gamma+1} \cdot [Z^*_\gamma]). \quad \text{(4.3)}$$

(ii) Let $\lambda \in \mathbb{C}^* \setminus \{1\}$ and $k \geq 1$. Then the number of the Jordan blocks for the eigenvalue $\lambda$ with sizes $\geq k$ in $\Phi_{n-1,0} : H^{n-1}(F_0 ; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0 ; \mathbb{C})$ is equal to

$$(-1)^{n-1} \sum_{p+q=n-k,n-1+k} \left\{ \sum_{\gamma \subset \Gamma_f} \chi_h((1 - \mathbb{L})^{m_\gamma} \cdot [Z^*_\Delta_\gamma])) \lambda \right\}. \quad \text{(4.4)}$$
(iii) For $k \geq 1$, the number of the Jordan blocks for the eigenvalue 1 with sizes $\geq k$ in $\Phi_{n-1,0}$ is equal to

\[ (-1)^{n-1} \sum_{p+q=n-1+k,n+k} \left\{ \sum_{\gamma_i \in \Gamma_f} e^{p,q}(\chi_h((1 - L)^{m_{\gamma_i}} \cdot [Z_{\gamma_i}^0])) \right\}_1 + \sum_{\gamma_i \in \Gamma_f, \dim \gamma_i \geq 1} e^{p,q}(\chi_h((1 - L)^{m_{\gamma_i} + 1} \cdot [Z_{\gamma_i}^0])) \left\}_1 . \]

(4.5)

Proof. Since (ii) and (iii) follow from (i) and Theorem 3.4, it suffices to prove (i). The proof is very similar to the one in Varchenko [20]. Let $\Sigma_1$ be the dual fan of $\Gamma_+(f)$ in $\mathbb{R}^n_+$ and $\Sigma$ its smooth subdivision. Denote by $X_\Sigma$ the smooth toric variety associated to $\Sigma$ (see Fulton [8] and Oda [16] etc.). Since the union of the cones in $\Sigma$ is $\mathbb{R}^n_+$, there exists a proper morphism $\pi: X_\Sigma \to \mathbb{C}^n$. By the convenience of $f$, we can construct the smooth fan $\Sigma$ without subdividing the cones contained in $\partial \mathbb{R}^n_+$ (see [17, Lemma (2.6), Chapter II]). Then $\pi$ induces an isomorphism $X_\Sigma \setminus \pi^{-1}(0) \cong \mathbb{C}^n \setminus \{0\}$. Moreover, by the non-degeneracy at $0 \in \mathbb{C}^n$ of $f$, the proper transform $Z$ of the hypersurface $\{x \in \mathbb{C}^n \mid f(x) = 0\}$ in $X_\Sigma$ is smooth and intersects $T$-orbits in $\pi^{-1}(0)$ transversally. Let $D_1, \ldots, D_m$ be the toric divisors in $\pi^{-1}(0) \subset X_\Sigma$. For a non-empty subset $I \subset \{1, 2, \ldots, m\}$ we set $D_I = \bigcap_{i \in I} D_i$ and

\[ D_I^\circ = D_I \setminus \left( \bigcup_{\iota \in I} D_{\iota} \right) \subset X_\Sigma \]

(4.6)

and define its unramified Galois covering $\tilde{D}_I^\circ$ as in Section 3. Moreover we set

\[ Z_I^0 = \left( D_I \setminus \bigcup_{\iota \in I} D_{\iota} \right) \cap Z \subset X_\Sigma \]

(4.7)

and denote by $[Z_I^0] \in \mathcal{M}_{\mathbb{C}}^\ell$ the class of the variety $Z_I^0$ with the trivial action. Then, unlike the global object $S_{f,0}^\circ$ in [14], Denef-Loeser’s “local” motivic Milnor fiber $S_{f,0}$ contains not only $(1 - L)^{t-1}[\tilde{D}_I^\circ]$ but also $(1 - L)^{t-1}[Z_I^0]$ (see Definition 3.1). These new elements yield the second term in the right hand side of (4.5). Finally, in the Grothendieck group $K_0(\text{HS}^{\text{mon}})$ we can rewrite $\chi_h(S_{f,0})$ in terms of the dual fan $\Sigma_1$ (i.e. in terms of $\Gamma_+(f)$) as in the same way as the proof of [14, Theorem 5.7 (i)]. This completes the proof. \qed

Let $q_1, \ldots, q_l$ (resp. $\gamma_1, \ldots, \gamma_{\ell'}$) be the 0-dimensional (resp. 1-dimensional) faces of $\Gamma_+(f)$ such that $q_i \in \text{Int}(\mathbb{R}^n_+)$ (resp. rel.int($\gamma_i$) $\subset \text{Int}(\mathbb{R}^n_+)$). Here rel.int($\cdot$) stands for the relative interior. For each $q_i$ (resp. $\gamma_i$), denote by $d_i > 0$ (resp. $e_i > 0$) the lattice distance dist$(q_i, 0)$ (resp. dist$(\gamma_i, 0)$) of it from the origin $0 \in \mathbb{R}^n$. For $1 \leq i \leq \ell'$, let $\Delta_i$ be the convex hull of $\{0\} \cup \gamma_i$ in $\mathbb{R}^n$. Then for $\lambda \in \mathbb{C} \setminus \{1\}$ and $1 \leq i \leq \ell'$ such that $\lambda^{e_i} = 1$ we set

\[ n(\lambda)_i = \sharp\{v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i) \mid \text{ht}(v, \gamma_i) = k\} + \sharp\{v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i) \mid \text{ht}(v, \gamma_i) = e_i - k\} , \]

(4.8)

where $k$ is the minimal positive integer satisfying $\lambda = c_{e_i}^k$ and for $v \in \mathbb{Z}^n \cap \text{rel.int}(\Delta_i)$ we denote by $\text{ht}(v, \gamma_i)$ the lattice height of $v$ from the base $\gamma_i$ of $\Delta_i$. As in the same way as [14, Theorem 5.9], by using Propositions 2.5 and 2.6 and Theorem 4.3 (ii), we obtain the following theorem.
Theorem 4.4 In the situation as above, for \( \lambda \in \mathbb{C}^* \setminus \{1\} \), we have

(i) The number of the Jordan blocks for the eigenvalue \( \lambda \) with the maximal possible size \( n \) in \( \Phi_{n-1,0} \) is equal to \( \sharp \{ q_i \mid \lambda^{q_i} = 1 \} \).

(ii) The number of the Jordan blocks for the eigenvalue \( \lambda \) with size \( n-1 \) in \( \Phi_{n-1,0} \) is equal to \( \sum_{i} \chi_{e_i=1} n(\lambda)_i \).

Note that by Theorem 4.3 and our results in Section 2 we can always calculate the whole Jordan normal form of \( \Phi_{n-1,0} \). From now on, we shall rewrite Theorem 4.3(ii) more explicitly in the case where any face \( \gamma \) \( \prec \Gamma_j \) such that \( \gamma \subset \Gamma_j \) is prime (see Definition 2.9(i)). Recall that by Proposition 2.3 for \( \lambda \in \mathbb{C}^* \setminus \{1\} \) and a face \( \gamma \prec \Gamma_j \) such that \( \gamma \subset \Gamma_j \) we have \( e^{p,q}(Z_{\Delta_{\gamma}})_{\lambda} = 0 \) for any \( p, q \geq 0 \) such that \( p + q > \dim \Delta_{\gamma} - 1 = \dim \gamma \).

So the non-negative integers \( r \geq 0 \) such that \( \sum_{p+q=r} e^{p,q}(Z_{\Delta_{\gamma}})_{\lambda} \neq 0 \) are contained in the closed interval \([0, \dim \gamma]\) \( \subset \mathbb{R} \).

Definition 4.5 For a face \( \gamma \prec \Gamma_j \) such that \( \gamma \subset \Gamma_j \) and \( k \geq 1 \), we define a finite subset \( J_{\gamma,k} \subset [0, \dim \gamma] \cap \mathbb{Z} \) by

\[
J_{\gamma,k} = \{ 0 \leq r \leq \dim \gamma \mid n - 2 + k \equiv r \mod 2 \}. \tag{4.9}
\]

For each \( r \in J_{\gamma,k} \), set

\[
d_{k,r} = \frac{n - 2 + k - r}{2} \in \mathbb{Z}_+. \tag{4.10}
\]

If a face \( \gamma \prec \Gamma_j \) such that \( \gamma \subset \Gamma_j \) is prime, then the polytope \( \Delta_{\gamma} \) is pseudo-prime (see Definition 2.9(ii)). Then by Proposition 2.10 for \( \lambda \in \mathbb{C}^* \setminus \{1\} \) and an integer \( r \geq 0 \) such that \( r \in [0, \dim \gamma] \) we have

\[
\sum_{p+q=r} e^{p,q}(\chi_{h([Z_{\Delta_{\gamma}}])})_{\lambda} = (-1)^{\dim \gamma + r + 1} \sum_{\substack{\Gamma \subset \Delta_{\gamma} \\dim \Gamma = r + 1}} \left\{ \sum_{\substack{\Gamma' \subset \Gamma \\dim \Gamma' = r + 1}} (-1)^{\dim \Gamma'} \varphi_{\lambda}(\Gamma') \right\}. \tag{4.11}
\]

For simplicity, we denote this last integer by \( e(\gamma, \lambda)_r \). Then by Theorem 4.3(ii) we obtain the following result.

Theorem 4.6 Assume that any face \( \gamma \prec \Gamma_j \) such that \( \gamma \subset \Gamma_j \) is prime. Let \( \lambda \in \mathbb{C}^* \setminus \{1\} \) and \( k \geq 1 \). Then the number of the Jordan blocks for the eigenvalue \( \lambda \) with sizes \( \geq k \) in \( \Phi_{n-1,0} \): \( H^{n-1}(F_0; \mathbb{C}) \xrightarrow{\sim} H^{n-1}(F_0; \mathbb{C}) \) is equal to

\[
(-1)^{n-1} \sum_{\gamma \subset \Gamma_j} \left\{ \sum_{r \in J_{\gamma,k}} (-1)^{d_{k,r}} \left( m_{\gamma} \right) d_{k,r} \cdot e(\gamma, \lambda)_r + \sum_{r \in J_{\gamma,k+1}} (-1)^{d_{k+1,r}} \left( m_{\gamma} \right) d_{k+1,r} \cdot e(\gamma, \lambda)_r \right\}, \tag{4.12}
\]

where we used the convention \( \binom{a}{b} = 0 \) (0 \( \leq a < b \)) for binomial coefficients.

By combining the proof of [3, Theorem 5.6] and [14, Proposition 2.14] with Theorem 4.3(iii), if any face \( \gamma \prec \Gamma_j \) such that \( \gamma \subset \Gamma_j \) is prime we can also describe the Jordan blocks for the eigenvalue 1 in \( \Phi_{n-1,0} \) by a closed formula. Since this result is rather involved, we omit it here.
Remark 4.7 Our results above are different from the previous ones due to Danilov [2] and Tanabé [25]. For example, in [2] and [25] they assume a stronger condition that the Newton polyhedron $\Gamma_+(f)$ itself is prime. We could weaken their condition, because our Propositions 2.13 and 2.14 and Proposition 2.10 are generalizations of the corresponding results in [3] to pseudo-prime polytopes.

We can also obtain the corresponding results for the eigenvalue 1 by rewriting Theorem 4.3 (iii) more simply as follows.

**Theorem 4.8** In the situation of Theorem 4.3, for $k \geq 1$ the number of the Jordan blocks for the eigenvalue 1 with sizes $\geq k$ in $\Phi_{n-1,0}$ is equal to

$$(-1)^{n-1} \sum_{p+q=n-2-k, n-1-k} \left\{ \sum_{\gamma \subset \Gamma_f} e^{p,q} \left( \chi_h \left( (1-L)^{m_{\gamma}} [Z_{\Delta,\gamma}^*] \right) \right) \right\}. \quad (4.13)$$

As in the same way as [14, Theorems 5.11 and 5.12], by using Propositions 2.5 and 2.6 and Theorem 4.8, we obtain the following corollary. Denote by $\Pi_f$ the number of the lattice points on the 1-skeleton of $\Gamma_f \cap \text{Int}(\mathbb{R}^n_+)$. Also, for a compact face $\gamma \subset \Gamma_+(f)$ we denote by $l^*(\gamma)$ the number of the lattice points on rel.int($\gamma$).

**Corollary 4.9** In the situation as above, we have

(i) (van Doorn-Steenbrink [6]) The number of the Jordan blocks for the eigenvalue 1 with the maximal possible size $n-1$ in $\Phi_{n-1,0}$ is $\Pi_f$.

(ii) The number of the Jordan blocks for the eigenvalue 1 with size $n-2$ in $\Phi_{n-1,0}$ is equal to $2 \sum \gamma l^*(\gamma)$, where $\gamma$ ranges through the compact faces of $\Gamma_+(f)$ such that $\text{dim}\gamma = 2$ and $\text{rel.int}(\gamma) \subset \text{Int}(\mathbb{R}^n_+)$.\n
Note that Corollary 4.9 (i) was previously obtained in van Doorn-Steenbrink [6] by different methods. Theorem 4.8 asserts that by replacing $\Gamma_+(f)$ with the Newton polyhedron at infinity $\Gamma_\infty(f)$ in [11], [13] and [14] etc. the combinatorial description of the local monodromy $\Phi_{n-1,0}$ is the same as that of the global one $\Phi_{n-1}^\infty$ obtained in [14, Theorem 5.7 (iii)]. Namely we find a beautiful symmetry between local and global. Theorem 4.8 can be deduced from the following more precise result.

**Theorem 4.10** In the situation as above, for any $0 \leq p, q \leq n-2$ we have

$$\sum_{\gamma \subset \Gamma_f} e^{p,q} \left( \chi_h \left( (1-L)^{m_{\gamma}} [Z_{\Delta,\gamma}^*] \right) \right)_1 = \sum_{\gamma \subset \Gamma_f} e^{p+1,q+1} \left( \chi_h \left( (1-L)^{m_{\gamma}} [Z_{\Delta,\gamma}^*] + (1-L)^{m_{\gamma}+1} [Z_{\gamma}^*] \right) \right)_1. \quad (4.14)$$

We can easily see that Theorem 4.10 follows from Proposition 4.11 below. For $[V] \in K_0(HS^\text{mon})$, let $e([V])_1 = \sum_{p,q=0}^{\infty} e^{p,q}([V])_1 t_1^p t_2^q$ be the generating function of $e^{p,q}([V])_1$ as in [3].
Proposition 4.11 We have
\[ \sum_{\gamma \subset \Gamma_f} e \left( \chi_h \left( \left( 1 - L \right)^{m-1} \left[ Z_{\gamma}^* + [Z_{\gamma}] \right] \right) \right) = 1 - (t_1 t_2)^n. \] (4.15)

From now on, we shall prove Proposition 4.11. First, we apply Proposition 2.8 to the case where \( \Delta = \Delta_\gamma \) for a face \( \gamma \) of \( \Gamma_+(f) \) such that \( \gamma \subset \Gamma_f \). Let \( \gamma' \) be a prime polytope in \( \mathbb{R}^{\dim \gamma} \) which majorizes \( \gamma \) and consider the Minkowski sum \( \gamma'' = \gamma + \gamma' \) (resp. \( \square_{\gamma''} := \Delta_\gamma + \gamma' \)) in \( \mathbb{R}^{\dim \gamma} \) (resp. \( \mathbb{R}^{\dim \gamma + 1} \)). Then \( \square_{\gamma''} \) is a \((\dim \gamma + 1)\)-dimensional truncated pyramid whose top (resp. bottom) is \( \gamma' \) (resp. \( \gamma'' \)) (see Figure 1 below). In particular, \( \square_{\gamma''} \) is prime. Since the dual fan of \( \gamma'' \) coincides with that of \( \gamma' \), the prime polytope \( \gamma'' \) majorizes \( \gamma \).

By extending \( \Psi \) to a morphism \( \widetilde{\Psi} : \text{som}(\square_{\gamma''}) \rightarrow \text{som}(\Delta_\gamma) \) as
\[ \widetilde{\Psi}(w) = \begin{cases} \Psi(w) & (w \in \text{som}(\gamma'')), \\ \{0\} & (w \in \text{som}(\gamma')) \end{cases}, \] (4.16)
we see that the prime polytope \( \square_{\gamma''} \) majorizes \( \Delta_\gamma \).

\[ \frac{\gamma'}{\square_{\gamma''}} \]

Figure 1

Proposition 4.12 For the closure \( \overline{Z_{\gamma}}^r \) of \( Z_{\gamma}^r \) in \( X_{\square_{\gamma''}} \), we have
\[ \sum_{q} e^{p,q}(\overline{Z_{\gamma}}^r)_1 = \sum_{\tau \prec \gamma''} (-1)^{\dim \tau + p} \binom{\dim \tau}{p}. \] (4.17)

Proof. It suffices to rewrite Proposition 2.8 in this case. For a face \( \Gamma \) of \( \square_{\gamma''} \), we set \( b_\Gamma = \dim \Gamma - \dim \widetilde{\Psi}(\Gamma) \). Note that the set of faces of \( \square_{\gamma''} \) consists of those of \( \gamma' \) and \( \gamma'' \) and side faces. Each side face of \( \square_{\gamma''} \) is a truncated pyramid \( \square_{\tau} \) whose bottom is \( \tau \prec \gamma'' \). Since \( \dim \square_{\tau} = \dim \Gamma + 1 \) and \( b_{\square_{\tau}} = b_{\tau} \) for \( \tau \prec \gamma'' \), we have
\[ \sum_{\Gamma \prec \square_{\gamma''}} (-1)^{\dim \Gamma + p + 1} \left\{ \binom{\dim \Gamma}{p + 1} - \binom{b_{\Gamma}}{p + 1} \right\} = \sum_{\tau \prec \gamma''} (-1)^{\dim \tau + p} \binom{\dim \tau}{p} \] (4.18)
and
\[ \sum_{\Gamma \prec \square_{\gamma''}} (-1)^{\dim \Gamma + 1} \sum_{i=0}^{\min\{b_{\Gamma},p\}} \binom{b_{\Gamma}}{i} (-1)^i \varphi_{1,\dim \widetilde{\Psi}(\Gamma)-p+i}(\widetilde{\Psi}(\Gamma)) = \sum_{\tau \prec \gamma''} (-1)^{\dim \tau + 1} \sum_{i=0}^{\min\{b_{\tau},p\}} \binom{b_{\tau}}{i} (-1)^i \varphi_{1,\dim \widetilde{\Psi}(\tau)-p+i}(\widetilde{\Psi}(\tau)) - \varphi_{1,\dim \widetilde{\Psi}(\square_{\tau})-p+i}(\widetilde{\Psi}(\square_{\tau})), \] (4.19)
where the faces $\tau$ of the top $\gamma'$ of $\square_{\gamma''}$ are neglected by the condition $\dim \Psi(\tau) = 0$. By $\Psi(\square_{\gamma}) = \Delta_{\Psi(\tau)}$ and Lemma 4.13 below, the last term is equal to 0. □

**Lemma 4.13** For any face $\gamma$ of $\Gamma_+ (f)$ such that $\gamma \subset \Gamma_f$, we have

$$\varphi_{1,j+1}(\Delta_{\gamma}) = \varphi_{1,j}(\gamma).$$

**(Proof.** By the relation $t^*((k + 1)\Delta_{\gamma})_1 - t^*(k\Delta_{\gamma})_1 = t^*(k\gamma)_1$ ($k \geq 0$) we have

$$P_1(\Delta_{\gamma}; t) = tP_1(\gamma; t).$$

By comparing the coefficients of $t^{j+1}$ in both sides, we obtain (4.20). □

The following proposition is a key in the proof of Proposition 4.11.

**Proposition 4.14** For any face $\gamma$ of $\Gamma_+ (f)$ such that $\gamma \subset \Gamma_f$, we have

$$e(\chi_h([Z_{\Delta_{\gamma}}^*] + [Z_{\gamma}^*]))_1 = (t_1t_2 - 1)^{\dim \gamma}.$$  

**(Proof.** It is enough to prove

$$e^{p,q}(Z_{\gamma}^*)_1 + e^{p,q}(Z_{\Delta_{\gamma}}^*)_1 = (-1)^{\dim \gamma + p} \binom{\dim \gamma}{p} \cdot \delta_{p,q},$$

where $\delta_{p,q}$ is Kronecker’s delta. We consider the closure $\overline{Z_{\Delta_{\gamma}}}$ of $Z_{\Delta_{\gamma}}$ in $X_{\square_{\gamma''}}$. Then by the proofs of Propositions 2.8 and 4.12 we have

$$e^{p,q}(\overline{Z_{\Delta_{\gamma}}})_1 = \sum_{\tau \prec \gamma''} \left\{ e^{p,q}((C^*)^{b_{\tau}} \times Z_{\Psi(\tau)}^*)_1 + e^{p,q}((C^*)^{b_{\tau}} \times Z_{\Psi(\square_{\gamma})}^*)_1 \right\}$$

$$= \sum_{\tau \prec \gamma''} \sum_{i=0}^{b_{\tau}} \binom{b_{\tau}}{i} (-1)^{i + b_{\tau}} \left\{ e^{p-q-i}(Z_{\Psi(\tau)}^*)_1 + e^{-i,q-i}(Z_{\Psi(\square_{\gamma})}^*)_1 \right\}. \hspace{1cm} (4.24)$$

Let us prove (4.23) by induction on $\dim \gamma$. In the case $\dim \gamma = 0$, we can prove (4.23) easily by Propositions 2.3 and 2.6. Assume that for any $\sigma \subset \Gamma_f$ such that $\dim \sigma < \dim \gamma$ (4.23) holds. Then by $b_{\gamma''} = 0$ and (4.23) we have

$$e^{p,q}(\overline{Z_{\Delta_{\gamma}}})_1 = e^{p,q}(Z_{\gamma}^*)_1 + e^{p,q}(Z_{\Delta_{\gamma}}^*)_1 + \delta_{p,q} \sum_{\tau \preceq \gamma''} (-1)^{\dim \tau + p} \binom{\dim \tau}{p}. \hspace{1cm} (4.26)$$

In the case $p + q > \dim \gamma$, by Proposition 2.3 we have

$$e^{p,q}(\overline{Z_{\Delta_{\gamma}}})_1 = \delta_{p,q} \sum_{\tau \preceq \gamma''} (-1)^{\dim \tau + p} \binom{\dim \tau}{p}. \hspace{1cm} (4.27)$$

Therefore, also in the case $p + q < \dim \gamma$, by the Poincaré duality for $\overline{Z_{\Delta_{\gamma}}}$ ($\square_{\gamma''}$ is prime) and Lemma 2.11, we have

$$e^{p,q}(\overline{Z_{\Delta_{\gamma}}})_1 = e^{\dim \gamma - p, \dim \gamma - q}(\overline{Z_{\Delta_{\gamma}}})_1$$

$$= \delta_{p,q} \sum_{\tau \preceq \gamma''} (-1)^{\dim \tau + \dim \gamma - p} \binom{\dim \tau}{\dim \gamma - p} \binom{\dim \gamma - p}{p}. \hspace{1cm} (4.29)$$

$$= \delta_{p,q} \sum_{\tau \preceq \gamma''} (-1)^{\dim \tau + p} \binom{\dim \tau}{p}. \hspace{1cm} (4.30)$$

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In the case \( p + q = \text{dim} \gamma \), by Proposition \ref{proposition:4.12} and the previous results we have
\[
e^{p,q}(Z^*_{\Delta,\gamma})_1 = \sum_{q'} e^{p,q'}(Z^*_{\Delta,\gamma})_1 - (1 - \delta_{p,q})e^{p,p}(Z^*_{\Delta,\gamma})_1 \quad (4.31)
\]
\[
= \delta_{p,q} \sum_{\tau < \gamma} (-1)^{\dim \tau + p} \binom{\dim \tau}{p} . \quad (4.32)
\]
By (4.26), we obtain (4.23) for any \( p, q \). \( \Box \)

Now we can finish the proof of Proposition \ref{proposition:4.11} as follows. By Proposition \ref{proposition:4.14}, we have
\[
\sum_{\gamma \subset \Gamma_f} \chi_h \left( (1 - L)^{m_{\gamma} + 1}([Z^*_{\Delta,\gamma}] + [Z^*_f]) \right)_1 = \sum_{\gamma \subset \Gamma_f} (1 - t_1 t_2)^{m_{\gamma} + 1}(t_1 t_2 - 1)^{\text{dim} \gamma} \quad (4.33)
\]
\[
= \sum_{l=1}^{n} (1 - t_1 t_2)^l \sum_{i S_i = l} (-1)^{\text{dim} \gamma} \quad (4.34)
\]
\[
= \sum_{l=1}^{n} (1 - t_1 t_2)^l \binom{n}{l} (-1)^{l-1} \quad (4.35)
\]
\[
= 1 - (t_1 t_2)^n . \quad (4.36)
\]

\[\begin{align*}
\text{Remark 4.15} & \quad \text{Following the proof of [14, Theorem 5.16], we can easily give another proof to the Steenbrink conjecture which was proved by Varchenko-Khovanskii [27] and Saito [20] independently. For an introduction to this conjecture, see an excellent survey in Kulikov [10] etc.}
\end{align*}\]

\[\begin{align*}
\text{Remark 4.16} & \quad \text{For a polynomial map } f: \mathbb{C}^n \rightarrow \mathbb{C}, \text{ it is well-known that there exists a finite subset } B \subset \mathbb{C} \text{ such that the restriction}
\end{align*}\]
\[
\mathbb{C}^n \setminus f^{-1}(B) \rightarrow \mathbb{C} \setminus B \quad (4.37)
\]
of \( f \) is a locally trivial fibration. We denote by \( B_f \) the smallest such subset \( B \subset \mathbb{C} \). For a point \( b \in B_f \), take a small circle \( C_b(b) = \{ x \in \mathbb{C} \mid |x - b| = \varepsilon \} \) \( (0 < \varepsilon < 1) \) around \( b \) such that \( B_f \cap \{ x \in \mathbb{C} \mid |x - b| \leq \varepsilon \} = \{ b \} \). Then by the restriction of \( \mathbb{C}^n \setminus f^{-1}(B_f) \rightarrow \mathbb{C} \setminus B_f \) to \( C_b(b) \subset \mathbb{C} \setminus B_f \) we obtain a geometric monodromy automorphism \( \Phi^b_j: f^{-1}(b + \varepsilon) \sim \rightarrow f^{-1}(b + \varepsilon) \) and the linear maps
\[
\Phi^b_j: H^j(f^{-1}(b + \varepsilon); \mathbb{C}) \sim \rightarrow H^j(f^{-1}(b + \varepsilon); \mathbb{C}) \quad (j = 0, 1, \ldots) \quad (4.38)
\]
associated to it. The eigenvalues of \( \Phi^b_j \) were studied in [13, Sections 3 and 4] etc. If \( f \) is tame at infinity, as in [14 Section 4] we can introduce a motivic Milnor fiber \( S^b_j \in \mathcal{M}^b \) along the central fiber \( f^{-1}(b) \) to calculate the numbers of the Jordan blocks for the eigenvalues \( \lambda \neq 1 \) in \( \Phi^b_{n-1} \). This result can be easily obtained by using the proof of Sabbah [18 Theorem 13.1]. It would be an interesting problem to construct a motivic object to calculate the eigenvalue 1 part of \( \Phi^b_{n-1} \).
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