The well-posedness of the stochastic nonlinear Schrödinger equations in $H^2(\mathbb{R}^d)$

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Abstract

The Cauchy problem for the stochastic nonlinear Schrödinger equation with a multiplicable noise is considered where the nonlinear term is of a power type and its coefficients are complex numbers. In particular, it is extremely important to consider the complex coefficients in the noise which cover non-conservative case, because they include measurement effects in quantum physics. The main purpose of this paper is to construct classical solutions in $H^2(\mathbb{R}^d)$ for the problem in question. The time local well-posedness in $L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ was investigated in the papers [6, 7]. In this paper we study the well-posedness in $H^2(\mathbb{R}^d)$ by making use of the rescaling approach as a main tool for dealing with the multiple noise, where we need to take advantage of a slight modification of the deterministic Strichartz estimate to fit into requirements under the setting of $H^2(\mathbb{R}^d)$. The other difficulty lies on the discussion on smoothness of functions in the nonlinear term, where the proof of time local well-posedness for the case of $H^2$-solutions does not go similarly as in the cases of $L^2$-solutions or $H^1$-solutions, because of the complexity in the computation of the nonlinear term with lower exponent $\alpha$. The techniques of Kato [24, 25] work well on this difficulty even for the stochastic equations. We use the stochastic Strichartz estimate [16, 21, 22] as well to deal with white noise which did not appear in the proof for $L^2$-solutions or $H^1$-solutions. We also discuss time-global solutions in $H^2(\mathbb{R}^d)$.

1 Introduction and main results

1.1 Stochastic and Deterministic nonlinear Schrödinger equations

We consider the Cauchy problem for the stochastic nonlinear Schrödinger equation (SNLS) with a multiplicative noise in the general spatial dimension $d \in \mathbb{N}$:

\[
\begin{cases}
    \text{id}X(t, \xi) = \Delta X(t, \xi)dt + \lambda |X(t, \xi)|^{\alpha-1}X(t, \xi)dt \\
    -i\mu(\xi)X(t, \xi)dt + iX(t, \xi)dW(t, \xi), & t \in (0, T), \xi \in \mathbb{R}^d, \\
    X(0, \xi) = x(\xi), & \xi \in \mathbb{R}^d,
\end{cases}
\]

(1.1)

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where the constants $\lambda = \pm 1$, $\alpha > 1$. The Wiener process $W$ and $\mu$ are the functions as follows, where $\mu_j \in \mathbb{C}$ is constant,

$$W(t, \xi) = \sum_{j=1}^{N} \mu_j e_j(\xi) \beta_j(t), \quad t \geq 0, \ \xi \in \mathbb{R}^d,$$

(1.2)

$$\mu(\xi) = \frac{1}{2} \sum_{j=1}^{N} |\mu_j|^2 e_j^2(\xi), \quad \xi \in \mathbb{R}^d.$$

(1.3)

The constant $N \in \mathbb{N} \cup \{+\infty\}$, and $e_j(\xi)$ is a real valued function. $\beta_j(t)$ is a real valued function independent Brownian motions with respect to a probability space $(\Omega, \mathcal{F}, P)$ with natural filtration $(\mathcal{F}_t)_{t \geq 0}$. This equation was introduced by [6, 7]. In this paper we assume $N < \infty$ same with [6, 7]. But our techniques easily go over to the case where $N = +\infty$ (i.e. infinite dimensional noise). We refer to [28, Remark 2.3.13] for details.

There are much amount of papers for results on the deterministic nonlinear Schrödinger equation which is given as in the case $\mu_j = 0$, $1 \leq j \leq N$ in (1.1), (1.2) and (1.3). We introduce the typical condition for the power of nonlinearity,

$$1 < \alpha < 1 + \frac{4}{(d-2s)^+},$$

(1.4)

for the time local well-posedness in $H^s(\mathbb{R}^d), s \geq 0$. Here,

$$\frac{1}{h^+} = \begin{cases} \frac{1}{h}, & h > 0, \\ \infty, & h \leq 0. \end{cases}$$

This condition is derived from a scaling argument, and actually the time local well-posedness were shown for $s = 1$ by Ginibre-Velo [19], for $s = 0$ by Tsutsumi [27], for $s = 2$ by Kato [24, 25] and for the general $0 \leq s < d/2$ by Cazenave-Weissler [17]. We call the case $s = 1$ charge class, $s = 0$ energy class respectively since there are the conservation laws for $L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ respective. We sometimes call the case $s = 2$ classical solution class since the nonlinear Schrödinger equation is the second order partial differential equation with the Laplacian in $x$.

So a natural question arise. Can we solve the stochastic nonlinear Schrödinger equation under the condition of (1.4)? There are some results. The conservative case (i.e. the special case of purely imaginary noise $\mu_j = i a_j, a_j \in \mathbb{R}, j = 1, 2, \ldots, N$) was studied in [11, 12]. In [11] local existence and uniqueness of solutions in $L^2(\mathbb{R}^d)$ were proved for $\alpha$ satisfying

$$\begin{cases} 1 < \alpha < 1 + \frac{4}{d}, & d = 1, 2, \\ 1 < \alpha < 1 + \frac{2}{d-1}, & d \geq 3, \end{cases}$$

(1.5)

and in [12] local existence and uniqueness of solutions in $H^1(\mathbb{R}^d)$ were proved for $\alpha$ satisfying

$$\begin{cases} 1 < \alpha < \infty, & d = 1, 2, \\ 1 < \alpha < 5, & d = 3, \\ 1 < \alpha < 1 + \frac{2}{d-1} \text{ or } 2 \leq \alpha < 1 + \frac{4}{d-2}, & d = 4, 5, \\ 1 < \alpha < 1 + \frac{2}{d-1}, & d \geq 6. \end{cases}$$

(1.6)
G. Da Prato, V. Barbu, M. Röckner solved the time local well-posedness in $L^2(\mathbb{R}^d)$ and $H^1(\mathbb{R}^d)$ under the full condition (1.4) by using the rescaling approach with additional condition (H1)$_s$.

See also [8, 13, 14, 15, 20, 23] for a study of the stochastic nonlinear Schrödinger equation.

### 1.2 Results by the rescaling approach

In this subsection we introduce the four results on the time local well-posedness for (1.1) by using the rescaling approach. The two results in Theorem 1.2 for $s = 0, 1$, are known, and the other two results for $s = 2$ with different settings in Theorem 1.3 and 1.4 are the main results in this paper.

To state those results precisely, we introduce an assumption on $\{e_j\}_{j=1}^N$ as follows. We assume some decay conditions.

(H1)$_s$ $e_j \in C^\infty_b(\mathbb{R}^d)$ such that

$$\lim_{|\xi| \to \infty} \zeta(\xi)|\partial^\gamma e_j(\xi)| = 0,$$

where $\gamma$ is a multi-index such that $|\gamma| \leq s + 2$, $1 \leq j \leq N$ and

$$\zeta(\xi) = \begin{cases} 1 + |\xi|^2, & \text{if } d \neq 2; \\ (1 + |\xi|^2)(\log(3 + |\xi|^2))^2, & \text{if } d = 2; \end{cases}$$

**Remark.** We remark that the assumption (H1)$_s$ is the almost same with [6, 7]. We assume $|\gamma| \leq 4$ on the derivative of $e_j$ though it was assumed $|\gamma| \leq 2$ in [6] and $|\gamma| \leq 3$ in [7].

We define the solvability for the stochastic equation.

**Definition 1.1.** Let $x \in H^s (s = 0, 1, 2)$ and let $\alpha > 1$. Fix $0 < T < \infty$. A solution of (1.1) is a pair $(X, \tau)$, where $\tau(\leq T)$ is an $(\mathcal{F}_t)$-stopping time, and $X = (X(t))_{t \in [0, T]}$ is an $H^s$-valued continuous ($\mathcal{F}_t$)-adapted process, such that $|X|^{\alpha - 1}X \in L^1(0, \tau; H^{s-2})$, $\mathbb{P}$-a.s., and it satisfies $\mathbb{P}$-a.s.

$$X(t) = x - \int_0^{t \wedge \tau} (i\Delta X(s) + \mu X(s) + \lambda i|X(s)|^{\alpha - 1}X(s))ds + \int_0^{t \wedge \tau} X(s)dW(s), \quad t \in [0, T], \quad (1.7)$$

as an equation in $H^{s-2}$.

We say that uniqueness for (1.1) holds in the function space $S$, if for any two solutions of (1.1) $(X_i, \tau_i)$, $X_i \in S$, $i = 1, 2$, it holds $\mathbb{P}$-a.s. that $X_1 = X_2$ on $[0, \tau_1 \wedge \tau_2]$.

We state the two known results for $s = 0$ and $s = 1$ which were shown in [6] Theorem 2.2 and [7] Theorem 2.1 respectively in the following theorem.

**Theorem 1.2** ($H^s$ local well-posedness for $s = 0, 1$).

Let $s = 0$ or 1. Assume (H1)$_s$. Let $\alpha$ satisfy (1.4). Then, for each $x \in H^s$ and $0 < T < \infty$, there is a sequence of local solutions $(X_n, \tau_n)$ of (1.1), $n \in \mathbb{N}$, where $\tau_n$ is a sequence of increasing stopping times. For every $n \geq 1$, it holds $\mathbb{P}$-a.s. that

$$X_n|_{[0, \tau_n]} \in C([0, \tau_n]; H^s) \cap L^\gamma(0, \tau_n; W^{s, \rho}),$$
and uniqueness holds in the function space \( C([0, \tau_n]; H^s) \cap L^\gamma(0, \tau_n; W^{s, \rho}) \) where \((\rho, \gamma)\) is any Strichartz pair. Moreover, defining \( \tau^*(x) = \lim_{n \to \infty} \tau_n \) and \( X = \lim_{n \to \infty} X_n \cdot 1_{[0, \tau^*(x))]}, \) for \( n \geq 1 \) and \( P\text{-}a.s. \ \omega \in \Omega, \) the map \( x \to X(\cdot, x, \omega) \) is continuous from \( H^s \) to \( L^\infty(0, \tau_n; H^s) \cap L^\gamma(0, \tau_n; W^{s, \rho}) \) where \((\rho, \gamma)\) is any Strichartz pair. Furthermore, we have the blowup alternative, that is, for \( P\text{-}a.s. \ \omega, \) if \( \tau_n(\omega) < \tau^*(x)(\omega), \ \forall n \in \mathbb{N}, \) then
\[
\lim_{t \to \tau^*(x)(\omega)} \|X(t)(\omega)\|_{H^s} = \infty.
\]

Now we state our main theorems. The main theorem is shown in the range of the following exponents.

\[
\begin{cases}
1 < \alpha < \infty, & 1 \leq d \leq 4, \\
1 < \alpha < 1 + \frac{2}{d-2}, & 2 \leq \alpha < 1 + \frac{4}{d-4}, \quad 5 \leq d \leq 7, \\
1 < \alpha < 1 + \frac{2}{d-2}, & d \geq 8.
\end{cases}
\]

**Theorem 1.3** *(\( H^2 \) local well-posedness).*

Let \( s = 2 \). Assume \((H1)_2\) with \( s = 2 \). Let \( \alpha \) satisfy (1.8). Then, for each \( x \in H^2 \) and \( 0 < T < \infty \), there is a sequence of local solutions \((X_n, \tau_n)\) of (1.1), \( n \in \mathbb{N} \), where \( \tau_n \) is a sequence of increasing stopping times. For every \( n \geq 1 \), it holds \( P\text{-}a.s. \) that
\[
X_n\vert_{[0, \tau_n]} \in C([0, \tau_n]; H^2),
\]
and uniqueness holds in the function space \( C([0, \tau_n]; H^2) \cap L^\gamma(0, \tau_n; W^{2, \rho}) \). Moreover, defining \( \tau^*(x) = \lim_{n \to \infty} \tau_n \) and \( X = \lim_{n \to \infty} X_n \cdot 1_{[0, \tau^*(x))]}, \) for \( n \geq 1 \) and \( P\text{-}a.s. \ \omega \in \Omega, \) the map \( x \to X(\cdot, x, \omega) \) is continuous from \( H^2 \) to \( L^\infty(0, \tau_n; H^s) \cap L^\gamma(0, \tau_n; W^{2, \rho}) \) where \((\rho, \gamma)\) is any Strichartz pair. Furthermore, we have the blowup alternative, that is, for \( P\text{-}a.s. \ \omega, \) if \( \tau_n(\omega) < \tau^*(x)(\omega), \ \forall n \in \mathbb{N}, \) then
\[
\lim_{t \to \tau^*(x)(\omega)} \|X(t)(\omega)\|_{H^2} = \infty.
\]

The proof of Theorem 1.3 is based on Kato’s technique [24, 25], in which the time derivative \( \partial_t W \) of the Wiener process \( W \) appears. This is known as white noise and does not belong in \( L^\infty(0, t; L^\infty) \). To avoid this problem, we use the stochastic Strichartz estimate, which is converted
to Itô integral by the rule of $\dot{W} dt = dW$. However, we note that the stochastic Strichartz estimate was not used in the setting of $L^2$-solutions, $H^1$-solutions, and Theorem 1.4 since $\partial_t W$ does not appear.

It is natural to consider the existence of global solutions. We introduce the known result in $H^1$ which is [8, Theorem 1.2].

**Theorem 1.5** ($H^1$ global well-posedness).

Assume (H1)$_1$. Let $\alpha$ satisfy $1 < \alpha < 1 + \frac{4}{(d-2)^r}$ if $\lambda = 1$, or $1 < \alpha < 1 + \frac{4}{d}$ if $\lambda = -1$. Then, for each $x \in H^1$ and $0 < T < \infty$, there exists a unique $H^1$-global solution $(X, T)$ of (1.1), such that

$$X \in L^\gamma(0, T; W^{1, \rho}), \ P\text{-a.s.}$$

where $(\rho, \gamma)$ is any Strichartz pair. Moreover, for $P$-a.s. $\omega$, the map $x \rightarrow X(\cdot, x, \omega)$ is continuous from $H^1$ to $L^\infty(0, T; H^1) \cap L^\gamma(0, T; W^{1, \rho})$ where $(\rho, \gamma)$ is any Strichartz pair.

The following theorem which corresponds to $H^2$-solution is new.

**Theorem 1.6** (The existence of global solutions in $H^2$).

Assume $d \leq 7$ and (H1)$_2$. Let $\alpha$ satisfy $2 \leq \alpha < 1 + \frac{4}{(d-4)^r}$. In addition, we assume that for $x \in H^2$, $0 < T < \infty$, the following holds.

$$||X||_{L^\infty(0, T; H^1)} + ||X||_{L^q(0, T; W^{1, p})} < \infty, \ P\text{-a.s.}$$

where $(p, q) = (\alpha + 1, \frac{4(\alpha - 1)}{d(\alpha - 1)})$. Then, there exists a unique $H^2$-global solution $(X, T)$ of (1.1).

If we combine the above theorem and $H^1$ global well-posedness, we have the following corollary.

**Corollary 1.7.** Assume $d \leq 7$ and (H1)$_2$. Let $\alpha$ satisfy $2 \leq \alpha < 1 + \frac{4}{(d-2)^r}$ if $\lambda = 1$, or $2 \leq \alpha < 1 + \frac{4}{d}$ if $\lambda = -1$. Then, there exists a unique $H^2$-global solution $(X, T)$ of (1.1).

## 2 Rescaling approach

The main tool to prove Theorem 1.3 and Theorem 1.4 is based on the rescaling approach as used in [6, 7]. We apply the rescaling transformation

$$X(t, \xi) = e^{W(t, \xi)} y(t, \xi).$$

(2.1)

By an application of Itô’s product formula, we see that $P$-a.s.

$$dX = e^W dy + e^W y dW + \tilde{\mu} e^W y dt,$$

where

$$\tilde{\mu}(\xi) = \frac{1}{2} \sum_{j=1}^N \nu_j^2 e_j^2(\xi).$$

(2.2)

We apply (2.1) to (1.1) to have

$$\begin{cases}
\frac{\partial y(t, \xi)}{\partial t} = A(t)y(t, \xi) - \lambda i e^{(\alpha-1)ReW(t, \xi)}|y(t, \xi)|^{\alpha-1} y(t, \xi), \\
y(0, \xi) = x(\xi),
\end{cases}$$

(2.3)
where

\[ A(t)y(t, \xi) = -ie^{-W} \Delta(e^Wy) - (\mu + \bar{\mu})y \]
\[ = -i(\Delta + b(t, \xi) \cdot \nabla + c(t, \xi))y(t, \xi), \quad \text{(2.4)} \]
\[ b(t, \xi) = 2\nabla W(t, \xi), \quad \text{(2.5)} \]
\[ c(t, \xi) = \sum_{j=1}^{d} (\partial_j W(t, \xi))^2 + \Delta W(t, \xi) - i(\mu(\xi) + \bar{\mu}(\xi)). \quad \text{(2.6)} \]

The definition of solutions to (2.3) are given in the following sense which is similar to Definition 1.1.

**Definition 2.1.** Let \( x \in H^s(\tau \leq T) \) and let \( \alpha > 1 \). Fix \( 0 < T < \infty \). A solution of (2.3) is a pair \((y, \tau)\), where \( \tau \leq T \) is an \((F_t)\)-stopping time, and \( y = (y(t))_{t \in [0, T]} \) is an \(H^s\)-valued continuous \((F_t)\)-adapted process, such that

\[ |y|^\alpha - 1 y \in L^1(0, \tau; H^{s-2}), \quad \text{P-a.s.,} \]

and it satisfies

\[ y(t) = x + \int_0^{t \wedge \tau} A(s)y(s)ds - \int_0^{t \wedge \tau} \lambda ie^{(\alpha-1)\text{Re}W(s)}|y(s)|^{\alpha-1}y(s)ds, \quad t \in [0, T], \quad \text{(2.7)} \]

as an equation in \( H^{s-2} \).

We say that uniqueness holds for (2.3) in the function space \( S \), if for any two solutions of (2.3) \((y_i, \tau_i), y_i \in S, \ i = 1, 2\), it holds \( \text{P-a.s.} \) that \( y_1 = y_2 \) on \([0, \tau_1 \wedge \tau_2]\).

The following theorem establishes the equivalence between the two definitions of solution to (1.1) and (2.3) respectively.

**Theorem 2.2.** For \( s = 0, 1, 2 \), the following holds.

1. Let \((y, \tau)\) be a solution of (2.3) in the sense of Definition 2.1. Set \( X := e^Wy \). Then \((X, \tau)\) is a solution of (1.1) in the sense of Definition 1.1.

2. Let \((X, \tau)\) be a solution of (1.1) in the sense of Definition 1.1. Set \( y := e^{-W}X \). Then \((y, \tau)\) is a solution of (2.3) in the sense of Definition 2.1.

**Proof.** The cases of \( L^2 \) and \( H^1 \) were proved in [6, 7]. Therefore, we prove the case of \( H^2 \). In the first case (1), since \( x \in H^2 \subset H^1 \) and \( y \) satisfies (2.7) in \( L^2 \subset H^{-1} \), Lemma 2.4 in [7] implies that \( (X, \tau) \) is a solution of (1.1) in the sense of Definition 1.1 in particular, \( X \) solve (1.7) in \( H^{-1} \). But, as \( y \in C([0, T]; H^2) \) and \( e^W \in C([0, T]; W^{2, \infty}) \), we deduce that \( X \in C([0, T]; H^2) \). Hence, the right hand side of (1.7) is in \( L^2 \), which implies that \( (X, \tau) \) is a solution of (1.1) in the sense of Definition 1.1 thereby completing the proof of (1). The proof for (2) follows analogously.

By the equivalence of two expressions of solutions via the rescaling transformation (2.1), Theorem 1.3, Theorem 1.4 and Theorem 1.6 are rewritten by Theorem 2.3, Theorem 2.4 and Theorem 2.5 below respectively.
Theorem 2.3. Assume (H1)$_2$. Let $\alpha$ satisfy (1.8). Then, for each $x \in H^2$ and $0 < T < \infty$, there is a sequence of local solutions $(y_n, \tau_n)$ of (2.3), $n \in \mathbb{N}$, where $\tau_n$ is a sequence of increasing stopping times. For every $n \geq 1$, it holds $\mathbb{P}$-a.s. that

$$y_n|_{[0, \tau_n]} \in C([0, \tau_n]; H^2),$$

and uniqueness holds in the function space $C([0, \tau_n]; H^2)$. Moreover, defining $\tau^*(x) = \lim_{n \to \infty} \tau_n$ and $y = \lim_{n \to \infty} y_n 1_{[0, \tau^*(x)]}$, for $n \geq 1$, $\mathbb{P}$-a.s. $\omega \in \Omega$ and $0 \leq s < 2$, the map $x \to y(\cdot, x, \omega)$ is continuous from $H^2$ to $L^\infty(0, \tau_n; H^s)$.

Theorem 2.4. Assume $d \leq 7$ and (H1)$_2$. Let $\alpha$ satisfy $2 \leq \alpha < 1 + \frac{4}{(d-4)^+}$. Then, for each $x \in H^2$ and $0 < T < \infty$, there is a sequence of local solutions $(y_n, \tau_n)$ of (2.3), $n \in \mathbb{N}$, where $\tau_n$ is a sequence of increasing stopping times. For every $n \geq 1$, it holds $\mathbb{P}$-a.s. that

$$y_n|_{[0, \tau_n]} \in C([0, \tau_n]; H^2) \cap L^\gamma(0, \tau_n; W^{2, \rho}),$$

(2.8)

and uniqueness holds in the function space $C([0, \tau_n]; H^2)$. Moreover, defining $\tau^*(x) = \lim_{n \to \infty} \tau_n$ and $y = \lim_{n \to \infty} y_n 1_{[0, \tau^*(x)]}$, for $n \geq 1$ and $\mathbb{P}$-a.s. $\omega \in \Omega$, the map $x \to y(\cdot, x, \omega)$ is continuous from $H^2$ to $L^\infty(0, \tau_n; H^2) \cap L^\gamma(0, \tau_n; W^{2, \rho})$ where $(\rho, \gamma)$ is any Strichartz pair. Furthermore, we have the blowup alternative, that is, for $\mathbb{P}$-a.s. $\omega$, if $\tau_n(\omega) < \tau^*(x)(\omega)$, $\forall n \in \mathbb{N}$, then

$$\lim_{t \to \tau^*(x)(\omega)} ||y(t)\omega||_{H^2} = \infty.$$

Theorem 2.5. Assume $d \leq 7$ and (H1)$_2$. Let $\alpha$ satisfy $2 \leq \alpha < 1 + \frac{4}{(d-4)^+}$. In addition, we assume that for $x \in H^2$, $0 < T < \infty$, the following holds.

$$||y||_{L^\infty(0,T; H^1)} + ||y||_{L^\alpha(0,T; W^{1, \rho})} < \infty, \mathbb{P}\text{-a.s.}$$

where $(p, q) = (\alpha + 1, \frac{4(\alpha+1)}{d(\alpha-1)})$. Then, there exists a unique $H^2$-global solution $(y, T)$ of (2.3).

3 Deterministic and Stochastic Strichartz estimates

For the proof of Theorem 2.3 and Theorem 2.4, we discuss the evolution operators and its Strichartz estimate which was shown in [6] and [7]. We modify those Strichartz estimates in order to deal with $H^2$ solution.

Lemma 3.1. Assume (H1)$_2$. For $\mathbb{P}$-a.s. $\omega$, the operator $A(t)$ defined in (2.3) generates evolution operators $U(t, s) = U(t, s, \omega)$, $0 \leq s \leq t \leq T$, in the spaces $H^2(\mathbb{R}^d)$. Moreover, for each $x \in H^2(\mathbb{R}^d)$, the process $[s, T] \ni t \mapsto U(t, s)x \in H^2(\mathbb{R}^d)$ is continuous and $(\mathcal{F}_t)$-adapted, hence progressively measurable with respect to the filtration $(\mathcal{F}_t)_{t \geq s}$.

Proof. The existence of the evolution operator $U$ generated by $A(t)$ is a direct consequence of the fact that, for (P-a.s.) every $\omega \in \Omega$, the Cauchy problem

$$\frac{dy}{dt} = A(t)y, \quad y(s) = x, \quad s \leq t < \infty,$$
for each \( x \in H^2(\mathbb{R}^d) \) has a unique solution \( y \in C([s,T];H^2(\mathbb{R}^d)) \) for all \( T > s \).

Indeed, by Theorem 1.1 in [18], under our assumptions on \( c \) and \( b \), for each \( x \in H^2 \) and \( f \in L^1(s,T;H^2) \), the Cauchy problem

\[
\begin{align*}
\frac{\partial u}{\partial t} + \Delta u + cu + b \cdot \nabla u + f &= 0 & \text{in } (s,T) \times \mathbb{R}^d, \\
u(s) &= x,
\end{align*}
\]

has a unique solution \( u \in C([s,T];H^2) \). Therefore, an evolution operator \( U(t,s) \in L(H^2,H^2) \) is defined by \( U(t,s)x = y(t), \ 0 \leq s \leq t \leq T \). For details, see Lemma 3.3 in [6] (see also [28]).

We give a precise definition of local smoothing space introduced in [20] as used in this paper.

**Definition 3.2.** Set \( B_0 = \{ |\xi| \leq 2 \} \), \( B_j = \{ 2^j \leq |\xi| \leq 2^{j+1} \} \), \( j = 1, \ldots \), and \( B_{<j} = \{ |\xi| \leq 2^j \} \). Let \( A_j = [0,T] \times B_j, \ j \geq 0 \), \( A_{<j} = [0,T] \times B_{<j}, \ j \geq 1 \). We consider a dyadic partition of unity of frequency, i.e. \( 1 = \sum_{k=-\infty}^{\infty} S_k(D) \). We say a function \( f \) is localized at frequency \( 2^k \), if \( \hat{f} \) is supported in \( \{ 2^{k-1} < |\xi| < 2^{k+1} \} \). The functions at frequency \( 2^k \) are measured using the norm

\[
\| u \|_{X_k(T)} = \| u \|_{L^2(A_0)} + \sup_{j > 0} \| |\xi|^{-1} u \|_{L^2(A_j)}, \ k \geq 0,
\]

\[
\| u \|_{X_k(T)} = 2^k \| u \|_{L^2(A_{<k})} + \sup_{j \geq -k} \| (|\xi| + 2^{-k})^{-1} u \|_{L^2(A_j)}, \ k < 0,
\]

where \( |\xi| = \sqrt{1 + |\xi|^2} \). Then the local smoothing space \( \tilde{X}_{[0,T]} \) defined by the norm

\[
\| u \|_{\tilde{X}_{[0,T]}^d} = \left( \| \xi \|_{L^2([0,T] \times \mathbb{R}^d)}^2 + \sum_{k=-\infty}^{\infty} 2^k \| S_k u \|_{X_k(T)}^2 \right)^{\frac{d}{2}}, \ d \neq 2,
\]

\[
\| u \|_{\tilde{X}_{[0,T]}^2} = \left( \| \xi \|_{L^2([0,T] \times \mathbb{R}^d)}^2 + \sum_{k=-\infty}^{\infty} 2^k \| S_k u \|_{X_k(T)}^2 \right)^{\frac{1}{2}}, \ d = 2.
\]

**Lemma 3.3.** (Deterministic Strichartz estimate)

Assume (H1)$_2$. Then for any \( T > 0 \), \( u_0 \in H^2 \) and \( f \in L^{q_2}(0,T;W^{2,q_2}) \), the solution of

\[
u(t) = U(t,0)u_0 + \int_0^t U(t,s)f(s)\,ds, \ 0 \leq t \leq T, \quad (3.1)
\]

satisfies the estimates

\[
\| u \|_{L^{q_1}(0,T;L^{p_1})} \leq C_T(\| u_0 \|_{L^2} + \| f \|_{L^{q_2}(0,T;L^{p_2})}), \quad (3.2)
\]

\[
\| u \|_{L^{q_1}(0,T;W^{1,p_1})} + C_T(\| u_0 \|_{H^1} + \| f \|_{L^{q_2}(0,T;W^{1,p_2})}), \quad (3.3)
\]

and

\[
\| u \|_{L^{q_1}(0,T;W^{2,p_1})} \leq C_T(\| u_0 \|_{H^2} + \| f \|_{L^{q_2}(0,T;W^{2,p_2})}), \quad (3.4)
\]
where \((p_1, q_1)\) and \((p_2, q_2)\) are Strichartz pairs, namely
\[
(p_i, q_i) \in [2, \infty] \times [2, \infty] : \frac{2}{q_i} = \frac{d}{2} - \frac{d}{p_i}, \quad \text{if} \quad d \neq 2,
\]
\[
(p_i, q_i) \in [2, \infty) \times (2, \infty) : \frac{2}{q_i} = \frac{d}{2} - \frac{d}{p_i}, \quad \text{if} \quad d = 2.
\]
Furthermore, the process \(C_t, \ t \geq 0\), can be taken to be \((\mathcal{F}_t)\)-progressively measurable, increasing and continuous.

**Proof.** The key estimates of the proof are the following known results, for \(T > 0\),
\[
\|u\|_{L^q_t(0,T;L^p)} \leq C_T(\|u_0\|_{L^2} + \|f\|_{L^p_t(0,T;L^q)} + \|\tilde{X}[0,T]\}), \quad (3.5)
\]
\[
\|\nabla u\|_{L^q_t(0,T;L^p)} \leq C_T(\|u_0\|_{H^1} + \|f\|_{L^p_t(0,T;W^{1,q})} + \|\tilde{X}[0,T]\}), \quad (3.6)
\]
Indeed, \((3.5)\) is shown in \([6]\) and \((3.6)\) is shown in \([7]\) by using \((3.5)\). \((3.2)\) and \((3.3)\) are direct conclusion from \((3.5)\) and \((3.6)\) respectively. So we give a proof of \((3.4)\) only. The proof is based on Theorem 1.13 and Proposition 2.3(a) in \([26]\). Let us use the notation \(D_t := -i\partial_t, \ D_j := -i\partial_j, \ 1 \leq j \leq d, \) to rewrite \((3.1)\) in the form
\[
D_t u = (D_j a^{jk} D_k + D_j \tilde{b}^j + \tilde{b}^j D_j + \tilde{c}) u - if,
\]
with \(a^{jk} = \delta_{jk}, \ \tilde{b}^j = -i\partial_j W_t\) and \(\tilde{c} = -\sum_{j=1}^d (\partial_j W)^2 + (\mu + \bar{\mu})i, \ 1 \leq j, k \leq d.\)

Direct computations show
\[
D_t \Delta u = (\Delta + D_j \tilde{b}^j + \tilde{b}^j D_j + \tilde{c}) \Delta u + 2(D_j \nabla \tilde{b}^j + \nabla \tilde{b}^j D_j + \nabla \tilde{c}) \nabla u + (D_j \Delta \tilde{b}^j + \Delta \tilde{b}^j D_j + \Delta \tilde{c}) u - i\Delta f. \quad (3.7)
\]
We regard \((3.7)\) as the equation for the unknown \(\Delta u\) and treat the lower order term \((D_j \nabla \tilde{b}^j + \nabla \tilde{b}^j D_j + \nabla \tilde{c}) \nabla u\) and \((D_j \Delta \tilde{b}^j + \Delta \tilde{b}^j D_j + \Delta \tilde{c}) u\) as equal terms with \(\Delta f\). This leads to
\[
\Delta u(t) = U(t,0) \Delta u_0 + \int_0^t U(t,s) [2i(D_j \nabla \tilde{b}^j(s) + \nabla \tilde{b}^j(s) D_j + \nabla \tilde{c}(s)) \nabla u + i(D_j \Delta \tilde{b}^j(s) + \Delta \tilde{b}^j(s) D_j + \Delta \tilde{c}(s)) u + \Delta f(s)] ds. \quad (3.8)
\]
Hence applying \((3.5), (3.6)\) to \((3.8)\) and then using Theorem 1.13 and Proposition 2.3(a) in \([26]\) to
control the lower order term, we derive that

\[ \| \Delta u \|_{L^{q_1}(0,T;L^{p_1}) \cap X_{[0,T]}}, \]

\[ \leq C_T \| \Delta u_0 \|_{L^2} + \| 2 i (D_j \nabla \tilde{b}_j + \nabla \tilde{b}_j \cdot D_j + \nabla \tilde{c}) \nabla u + \Delta f \|_{L^{2}_{\tilde{b}} (0,T;L^{p_2}) + \bar{X}_{[0,T]}}, \]

\[ \leq C_T \| \Delta u_0 \|_{L^2} + \| 2 i (D_j \nabla \tilde{b}_j + \nabla \tilde{b}_j \cdot D_j + \nabla \tilde{c}) \nabla u \|_{\bar{X}_{[0,T]}}, \]

\[ \quad + \| (D_j \Delta \tilde{b}_j + \Delta \tilde{b}_j \cdot D_j + \Delta \tilde{c}) u \|_{\bar{X}_{[0,T]}} + \| \Delta f \|_{L^{2}_{\tilde{b}} (0,T;L^{p_2})}, \]

\[ \leq C_T \left[ \| \Delta u_0 \|_{L^2} + \| \nabla u \|_{\bar{X}_{[0,T]}} + \| \tilde{b}_j \|_{H^1} + \| \Delta f \|_{L^{2}_{\tilde{b}} (0,T;L^{p_2})} \right], \] (3.9)

\[ \leq C_T \| \Delta u_0 \|_{L^2} + C_T \| \nabla u \|_{H^1} + \| \Delta f \|_{L^{2}_{\tilde{b}} (0,T;L^{p_2})} \]

\[ + C_T \| \Delta f \|_{L^{2}_{\tilde{b}} (0,T;L^{p_2})} + \| \Delta u_0 \|_{H^2} + \| \Delta f \|_{L^{2}_{\tilde{b}} (0,T;W^{1,p_2})} \]

\[ = C_T (C_T \kappa_T^1 + C_T \kappa_T^2 + 1) \left[ \| \Delta u_0 \|_{H^2} + \| \Delta f \|_{L^{2}_{\tilde{b}} (0,T;W^{1,p_2})} \right]. \]

here we are faced at the term of the third derivative of \( \tilde{b}_j \) which corresponds to the 4th derivative of \( W \) which means the 4th derivative of \( \varepsilon_j \). This together with (3.2) (see also [28]) yields the estimate (3.4).

The \( H^2 \) continuity follows from Strichartz estimate (3.4) in the usual way. Now, we set

\[ C_t = \sup \{ \| U(\cdot,0)u_0 \|_{L^{q_1}(0,t;L^{p_1})} \mid \| u_0 \|_{H^2} \leq 1 \} \]

\[ + \sup \left\{ \left\| \int_0^t U(\cdot,s)f(s)ds \right\|_{L^{q_1}(0,t;L^{p_1})} ; \| f \|_{L^{2}_{\tilde{b}} (0,t;L^{p_2})} = 1 \right\}. \] (3.10)

Then the asserted properties of \( C_t \), \( t \geq 0 \), follow analogously as in the proof of Lemma 4.1 in [6]. (see also [28]).

\[ \square \]

Remark. The estimate (3.5) holds only on the bounded interval \([0,T]\). See [6] Appendix for details. Therefore, the main result statement is that well-posedness is obtained whenever \( T > 0 \) is fixed.

Moreover, we also use the following stochastic Strichartz estimates. This evaluation was first proved by Brzeźniak and Millet [16] Theorem 3.10, Proposition 3.12 and Corollary 3.13] in the case \( p = q \) and then by Hornung [21 Proposition 2.3] and [22] in the general case. We denote by \( HS(\mathbb{R}^N;L^2(\mathbb{R}^d)) \) the space of Hilbert-Schmidt operators \( \Phi \) from \( \mathbb{R}^N \) into \( L^2(\mathbb{R}^d) \).

Lemma 3.4. (Stochastic Strichartz estimate)

Assume (H1)$_2$. Then for any \( T > 0 \), \( p > 1 \) and predictable process \( \Phi \in L^p(\Omega, L^2(0,T;HS(\mathbb{R}^N;L^2(\mathbb{R}^d)))) \), the solution of

\[ J_{[0,T]}(t) := \int_0^t U(t,s)\Phi(s)dW(s), \quad t \in [0,T], \]

satisfies the estimates

\[ \| J_{[0,T]} \|_{L^p(\Omega,L^q(0,T;L^p(\mathbb{R}^d)))} + \| J_{[0,T]} \|_{L^p(\Omega,L^{\infty}(0,T;L^2(\mathbb{R}^d)))}, \]

\[ \leq C_T \| \Phi \|_{L^p(\Omega,L^2(0,T;HS(\mathbb{R}^N;L^2(\mathbb{R}^d))))}, \]

where \( (p,q) \) is Strichartz pair.
4 Proof of Theorem 2.3 and Theorem 2.4

Remark. The proof of Theorem 2.3 is shown only for the case $1 < \alpha \leq 1 + \frac{3}{(d-2)\tau}$. For $3 \leq d \leq 7$, combining with Theorem 2.4 yields Theorem 2.3.

Proof of Theorem 2.3. Set $g(y) = |y|^{\alpha - 1}y$. We solve the weak equation (2.7) in the mild sense, namely

$$y = U(t, 0)x - \lambda t \int_0^t U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds. \quad (4.1)$$

We consider the following map

$$F(y)(t) = U(t, 0)x - \lambda t \int_0^t U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds.$$

The local solutions $\{(y_n, \tau_n)\}_{n \geq 1}$ of (2.3) will be constructed explicitly below.

Step 1. First, we find $(y_1, \tau_1)$. Choose the Strichartz pair $(p, q) = \left(\frac{4\alpha}{\alpha + 1}, \frac{8\alpha}{d(\alpha-1)}\right)$. Fix $\omega \in \Omega$ and consider $F$ on the set

$$\mathcal{Y}_{M_1}^{\tau_1} = \{y, \partial_t y, \Delta y \in L^\infty(0, \tau_1; L^2); ||y||_{str(\tau_1)} + ||\partial_t y||_{str(\tau_1)} + ||\Delta y||_{L^\infty(0, \tau_1; L^2)} \leq M_1\},$$

where $||y||_{str(t)} := ||y||_{L^\infty(0, t; L^2)} + ||y||_{L^q(0, t; L^p)}$, $\tau_1 = \tau_1(\omega) \in (0, T]$ and $M_1 = M_1(\omega) > 0$ are random variables. The distance is defined by $d(y_1, y_2) = ||y_1 - y_2||_{str}$. We differentiate with respect to $t$,

$$\partial_t \left( \int_0^t U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds \right)$$

$$= \int_0^t \partial_t U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds + U(t, t)e^{(\alpha-1)ReW(t)}g(y(t))$$

$$= \int_0^t A(t)U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds + e^{(\alpha-1)ReW(t)}g(y(t)). \quad (4.2)$$

Also, we change the variables

$$\int_0^t U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds = \int_0^t U(t, t-s)e^{(\alpha-1)ReW(t-s)}g(y(t-s))ds,$$

and from

$$\partial_s U(t, s)x = A(t)U(t, s)x, \quad t \geq s,$$

$$\partial_s U(t, s)x = -U(t, s)A(s)x, \quad t \geq s,$$

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we have
\[
\partial_t \left( \int_0^t U(t, s)e^{(\alpha-1)\text{Re}W(s)} g(y(s))ds \right)
\]
\[
= \partial_t \left( \int_0^t U(t, t-s)e^{(\alpha-1)\text{Re}W(t-s)} g(y(t-s))ds \right)
\]
\[
= \int_0^t \partial_t \left( U(t, t-s)e^{(\alpha-1)\text{Re}W(t-s)} g(y(t-s)) \right) ds + U(t, 0)e^{(\alpha-1)\text{Re}W(0)} g(y(0))
\]
\[
= \int_0^t \{ A(t)U(t, s) - U(t, s)A(s) \} e^{(\alpha-1)\text{Re}W(s)} g(y(s))ds
\]
\[
+ \int_0^t U(t, t-s)\partial_t \left( e^{(\alpha-1)\text{Re}W(t-s)} g(y(t-s)) \right) ds + \int_0^t U(t, s)e^{(\alpha-1)\text{Re}W(s)} (\partial_t g)(y(s))ds
\]
\[
+ U(t, 0)g(x)
\]
\[
=: 3 \sum_{j=1}^3 I_j + U(t, 0)g(x).
\]

First, consider $I_1$. By (3.8), we have
\[
\{ A(t)U(t, s) - U(t, s)A(s) \} e^{(\alpha-1)\text{Re}W(s)} g(y(s))
\]
\[
= \{-i(\Delta + b(t) \cdot \nabla + c(t))U(t, s) + iU(t, s)(\Delta + b(s) \cdot \nabla + c(s)) \} e^{(\alpha-1)\text{Re}W(s)} g(y(s))
\]
\[
= -i(\Delta U(t, s) - U(t, s)\Delta) e^{(\alpha-1)\text{Re}W(s)} g(y(s))
\]
\[
+ \{-i(b(t) \cdot \nabla + c(t))U(t, s) + iU(t, s)(b(s) \cdot \nabla + c(s)) \} e^{(\alpha-1)\text{Re}W(s)} g(y(s))
\]
\[
= -i \int_s^t U(t, \tau)[2i(D_j \nabla \tilde{b}^j(\tau) + \nabla \tilde{b}^j(\tau)D_j + \nabla \tilde{c}(\tau))\nabla(e^{(\alpha-1)\text{Re}W(\tau)} g(y(\tau)))
\]
\[
+ i(D_j \Delta \tilde{b}^j(\tau)D_j + \Delta \tilde{c}(\tau))e^{(\alpha-1)\text{Re}W(\tau)} g(y(\tau))]d\tau
\]
\[
+ \{-i(b(t) \cdot \nabla + c(t))U(t, s) + iU(t, s)(b(s) \cdot \nabla + c(s)) \} e^{(\alpha-1)\text{Re}W(s)} g(y(s)).
\]

Therefore, we get
\[
I_1 = -i \int_0^t \int_s^t U(t, \tau)[2i(D_j \nabla \tilde{b}^j(\tau) + \nabla \tilde{b}^j(\tau)D_j + \nabla \tilde{c}(\tau))\nabla(e^{(\alpha-1)\text{Re}W(\tau)} g(y(\tau)))
\]
\[
+ i(D_j \Delta \tilde{b}^j(\tau)D_j + \Delta \tilde{c}(\tau))e^{(\alpha-1)\text{Re}W(\tau)} g(y(\tau))]d\tau ds
\]
\[
- i \int_0^t b(t) \cdot \nabla U(t, s)e^{(\alpha-1)\text{Re}W(s)} g(y(s))ds
\]
\[
- i \int_0^t c(t)U(t, s)e^{(\alpha-1)\text{Re}W(s)} g(y(s))ds
\]
\[
+ i \int_0^t U(t, s)b(s) \cdot \nabla(e^{(\alpha-1)\text{Re}W(s)} g(y(s)))ds
\]
\[
+ i \int_0^t U(t, s)c(s)e^{(\alpha-1)\text{Re}W(s)} g(y(s))ds.
\]
Next, consider $I_2$. Since $e^{(\alpha-1)ReW(t-s)}$ cannot be $t$-differentiable in the usual sense, it is interpreted in the sense of stochastic differential using Ito’s formula. In other words,

$$d(e^{(\alpha-1)ReW(t-s)}) = (\alpha - 1)e^{(\alpha-1)ReW(t-s)}dW(t) + (\alpha - 1)^2 \mu e^{(\alpha-1)ReW(t-s)}dt.$$ 

Substituting this into $I_2$, we get

$$I_2 = (\alpha - 1) \int_0^t U(t, t-s)e^{(\alpha-1)ReW(t-s)}g(y(t-s))dW(s)$$

$$+ (\alpha - 1)^2 \int_0^t U(t, t-s)\mu e^{(\alpha-1)ReW(t-s)}g(y(t-s))ds.$$

Therefore, put this into (4.3) to have

$$\partial_t \left( \int_0^t U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds \right)$$

$$= -i \int_0^t \int_s^t \left[ 2i(D_j \nabla \tilde{b}^j(\tau) + \nabla \tilde{b}^j(\tau)D_j + \nabla \tilde{c}^j(\tau))\nabla(e^{(\alpha-1)ReW(\tau)}g(y(\tau))) \right.$$

$$+ i(D_j \Delta \tilde{b}^j(\tau)D_j + \Delta \tilde{c}^j(\tau))e^{(\alpha-1)ReW(\tau)}g(y(\tau))]d\tau ds$$

$$- i \int_0^t b(t) \cdot \nabla U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds - i \int_0^t c(t)U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds$$

$$+ i \int_0^t U(t, s)b(s) \cdot \nabla(e^{(\alpha-1)ReW(s)}g(y(s)))ds + i \int_0^t U(t, s)c(s)e^{(\alpha-1)ReW(s)}g(y(s))ds$$

$$+ (\alpha - 1) \int_0^t U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))dW(s) + (\alpha - 1)^2 \int_0^t U(t, s)\mu e^{(\alpha-1)ReW(s)}g(y(s))ds$$

$$+ \int_0^t U(t, s)e^{(\alpha-1)ReW(s)}(\partial_t g)(y(s))ds + U(t, 0)g(x).$$

So

$$\partial_t F(y)(t) = A(t)U(t, 0)x - \lambda iU(t, 0)g(x)$$

$$- \lambda \int_0^t \int_s^t \left[ 2i(D_j \nabla \tilde{b}^j(\tau) + \nabla \tilde{b}^j(\tau)D_j + \nabla \tilde{c}^j(\tau))\nabla(e^{(\alpha-1)ReW(\tau)}g(y(\tau))) \right.$$

$$+ i(D_j \Delta \tilde{b}^j(\tau)D_j + \Delta \tilde{c}^j(\tau))e^{(\alpha-1)ReW(\tau)}g(y(\tau))]d\tau ds$$

$$- \lambda \int_0^t b(t) \cdot \nabla U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds - \lambda \int_0^t c(t)U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds$$

$$+ \lambda \int_0^t U(t, s)b(s) \cdot \nabla(e^{(\alpha-1)ReW(s)}g(y(s)))ds + \lambda \int_0^t U(t, s)c(s)e^{(\alpha-1)ReW(s)}g(y(s))ds$$

$$- \lambda i(\alpha - 1) \int_0^t U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))dW(s)$$

$$- \lambda i(\alpha - 1)^2 \int_0^t U(t, s)\mu e^{(\alpha-1)ReW(s)}g(y(s))ds$$

$$- \lambda i \int_0^t U(t, s)e^{(\alpha-1)ReW(s)}(\partial_t g)(y(s))ds,$$
where
\[
A(t)U(t, 0)x = -i(\Delta + b(t, \xi) \cdot \nabla + c(t, \xi))U(t, 0)x \\
= -i\Delta U(t, 0)x - ib(t, \xi) \cdot \nabla U(t, 0)x - i\epsilon(t, \xi)U(t, 0)x \\
= -i\Delta U(t, 0)x - 2i(\nabla W)(t, \xi) \cdot \nabla U(t, 0)x - i \sum_{j=1}^{d} (\partial_j W(t, \xi))^2 U(t, 0)x \\
= -i(\Delta W)(t, \xi)U(t, 0)x - \mu(\xi)U(t, 0)x - \bar{\mu}(\xi)U(t, 0)x.
\]

Therefore,
\[
\|\partial_t F(y)\|_{str(\tau_1)} \lesssim \|\Delta U(\cdot, 0)x\|_{L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p)} + \|\nabla U(\cdot, 0)x\|_{L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p)} \\
+ \|\sum_{j=1}^{d} (\partial_j W)^2 U(\cdot, 0)x\|_{L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p)} \\
+ \|\Delta W U(\cdot, 0)x\|_{L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p)} \\
+ \|\mu x\|_{L^2} + \|\bar{\mu} x\|_{L^2} + \|g(x)\|_{L^2} \\
+ \|\int_0^t \int_s^t U(t, \tau)2i(D_j \nabla \tilde{b}^j(\tau) + \nabla \tilde{b}^j(\tau) D_j + \nabla \bar{c}(\tau)) \nabla (e^{(\alpha-1)\Re W(\tau)} g(y(\tau))) \\
+ i(D_j \nabla \tilde{b}^j(\tau) D_j + \Delta \bar{c}(\tau)) e^{(\alpha-1)\Re W(\tau)} g(y(\tau))) d\tau ds\|_{L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p)} \\
+ \|\int_0^t b(t) \cdot \nabla U(t, s) e^{(\alpha-1)\Re W} g(y) ds\|_{L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p)} \\
+ \|\int_0^t c(t) U(t, s) e^{(\alpha-1)\Re W} g(y) ds\|_{L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p)} \\
+ \|\int_0^t U(t, s) b(s) \cdot \nabla (e^{(\alpha-1)\Re W} g(y)) ds\|_{L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p)} \\
+ \|\int_0^t U(t, s) c(s) e^{(\alpha-1)\Re W} g(y) ds\|_{L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p)} \\
+ \|\int_0^t U(t, s) e^{(\alpha-1)\Re W} g(y) dW\|_{L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p)} \\
+ \|\mu e^{(\alpha-1)\Re W} g(y)\|_{L^q(0, \tau_1; L^p')} \\
+ \|e^{(\alpha-1)\Re W} (\partial_t g)(y)\|_{L^q(0, \tau_1; L^p')}.
\]

We estimate each terms of the right-hand side as follows
\[
\|\Delta U(\cdot, 0)x\|_{L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p)} \lesssim \|x\|_{H^2}, \\
\|\mu x\|_{L^2} \lesssim \|x\|_{L^2} \lesssim \|x\|_{H^2}, \quad \|g(x)\|_{L^2} = \|x\|_{L^{2a}}^a \lesssim \|x\|_{H^2}^a.
\]
\[\| (\nabla W) \cdot \nabla U(\cdot, 0)x \|_{L^\infty(0, \tau_1; L^2)} \leq \| \nabla U \|_{L^\infty(0, \tau_1; L^\infty)} \| \nabla U(\cdot, 0)x \|_{L^\infty(0, \tau_1; L^2)} \leq \| x \|_{H^2},\]

\[\| \sum_{j=1}^{d} (\partial_j W)^2 U(\cdot, 0)x \|_{L^\infty(0, \tau_1; L^2)} \leq \| \sum_{j=1}^{d} (\partial_j W)^2 \|_{L^\infty(0, \tau_1; L^2)} \| U(\cdot, 0)x \|_{L^\infty(0, \tau_1; L^2)} \leq \| x \|_{L^2} \lesssim \| x \|_{H^2},\]

\[\| (\Delta W) U(\cdot, 0)x \|_{L^\infty(0, \tau_1; L^2)} \leq \| \Delta W \|_{L^\infty(0, \tau_1; L^\infty)} \| U(\cdot, 0)x \|_{L^\infty(0, \tau_1; L^2)} \leq \| x \|_{L^2} \lesssim \| x \|_{H^2},\]

\[\| \mu e^{(\alpha-1)ReW} g(y) \|_{L^{q'}(0, \tau_1; L^{p'})} \leq \| e^{(\alpha-1)ReW} g(y) \|_{L^{q'}(0, \tau_1; L^{p'})} \leq \| y \|_{L^{q'}(0, \tau_1; L^{p'})} \lesssim \tau_1^\alpha M_1^\alpha,\]

\[\| e^{(\alpha-1)ReW} (\partial_t g)(y) \|_{L^{q'}(0, \tau_1; L^{p'})} \lesssim \| e^{(\alpha-1)ReW} (\partial_t g)(y) \|_{L^{q'}(0, \tau_1; L^{p'})} \lesssim \| \partial_t y \|_{L^{q'}(0, \tau_1; L^{p'})} \lesssim \tau_1^\alpha M_1^\alpha,\]

\[\| b(t) \cdot \nabla U(t, s)e^{(\alpha-1)ReW} g(y)ds \|_{L^\infty(0, \tau_1; L^2)} \leq \| b \|_{L^\infty(0, \tau_1; L^\infty)} \| \nabla U(t, s)e^{(\alpha-1)ReW} g(y)ds \|_{L^\infty(0, \tau_1; L^2)} \leq \| g(y) \|_{L^{q'}(0, \tau_1; L^{p'})} + \| \nabla g(y) \|_{L^{q'}(0, \tau_1; L^{p'})} \lesssim \tau_1^\alpha M_1^\alpha,\]

\[\| c(t)U(t, s)e^{(\alpha-1)ReW} g(y)ds \|_{L^\infty(0, \tau_1; L^2)} \leq \| c \|_{L^\infty(0, \tau_1; L^\infty)} \| e^{(\alpha-1)ReW} g(y) \|_{L^{q'}(0, \tau_1; L^{p'})} \lesssim \tau_1^\alpha M_1^\alpha,\]

\[\| U(t, s)b(s) \cdot \nabla (e^{(\alpha-1)ReW} g(y))ds \|_{L^\infty(0, \tau_1; L^2)} \leq \| b \|_{L^\infty(0, \tau_1; L^\infty)} \| \nabla (e^{(\alpha-1)ReW} g(y))ds \|_{L^{q'}(0, \tau_1; L^{p'})} \lesssim \| g(y) \|_{L^{q'}(0, \tau_1; L^{p'})} + \| \nabla g(y) \|_{L^{q'}(0, \tau_1; L^{p'})} \lesssim \tau_1^\alpha M_1^\alpha,\]
\[ \| \int_0^t U(t, s)c(s)e^{(\alpha-1)ReW} g(y)ds \|_{L^\infty(0,\tau_1;L^2)\cap L^q(0,\tau_1;L^p)} \lesssim \|c\|_{L^\infty(0,\tau_1;L^\infty)} \|e^{(\alpha-1)ReW}\|_{L^\infty(0,\tau_1;L^\infty)} \|g(y)\|_{L^{p'}(0,\tau_1;L^{p'})} \lesssim \tau_1^{\theta} M_1^\alpha. \]

Here we used Hölder’s inequality with \[ 1 - \frac{1}{p} = \frac{\alpha - 1}{2\alpha} + \frac{1}{p}, \quad 1 - \frac{1}{q} = \theta + \frac{1}{q}, \] and \( q = \frac{8\alpha}{d(\alpha-1)} > 2 \) from the assumption of \( \alpha \), and so \( \theta > 0 \). We consider 8th terms in (4.4).

\[ \begin{align*}
\| \int_0^t \int_s^t U(t, \tau)\left[ 2i(D_j \nabla \tilde{b}^j(\tau) + \nabla \tilde{b}^j(\tau)D_j + \nabla \tilde{c}(\tau))\nabla (e^{(\alpha-1)ReW(\tau)} g(y(\tau))) \\
+ i(D_j \Delta \tilde{b}^j(\tau)D_j + \Delta \tilde{c}(\tau)e^{(\alpha-1)ReW(\tau)} g(y(\tau))) \right] \|_{L^\infty(0,\tau_1;L^2)\cap L^q(0,\tau_1;L^p)} \\
\lesssim \| \int_0^t \| e^{(\alpha-1)ReW(s)} g(y(s))\|_{H^1}ds \|_{L^1(0,\tau_1;H^1)} \lesssim \| g(x) - g(y)\|_{L^\infty(0,\tau_1;H^1)} + \| g(x)\|_{H^1} \\
\leq \| g(x) - g(y)\|_{L^\infty(0,\tau_1;H^1)} + \tau_1^{\theta} M_1^\alpha.
\end{align*} \]

Since we see \[ \| y(t) - y(s)\|_{L^2} = \left\| \int_s^t (\partial_t y)(\tau)d\tau \right\|_{L^2} \leq \int_s^t \| (\partial_t y)(\tau)\|_{L^2}d\tau \leq |t - s| \| \partial y\|_{L^\infty(0,\tau_1;L^2)} \leq M_1 |t - s|, \] \( y \) is Lipschitz continuous. Also, by the interpolation inequality, we have \[ \| y(t) - y(s)\|_{L^{2\alpha}} \leq \| y(t) - y(s)\|_{L^2}^{\alpha} |y(t) - y(s)|_{H^{1-\alpha}}^{1-\alpha} \lesssim M_1^{\alpha} |t - s|^\alpha M_1^{1-\alpha} = M_1 |t - s|^{\alpha}, \] \( 4.7 \)

\[ \| \nabla (y(t) - y(s))\|_{L^{2\alpha}} \lesssim \| \nabla (y(t) - y(s))\|_{L^2}^{\alpha} \| \nabla (y(t) - y(s))\|_{H^1}^{1-\alpha} \lesssim \| y(t) - y(s)\|_{B^\alpha} |y(t) - y(s)|_{H^{1-\alpha}}^{1-\alpha} \lesssim M_1^{\alpha} |t - s|^{\alpha'} M_1^{1-\alpha'} = M_1 |t - s|^{\alpha'}, \] \( \alpha' \leq \alpha, \)

where we use Sobolev’s embedding in \( 1 < \alpha \leq 1 + \frac{2}{(d-2)^2} \). So, \( y \) is Hölder continuous. Therefore, \[ \| g(x) - g(y)\|_{L^\infty(0,\tau_1;H^{1})} \lesssim (\| y\|_{L^{2\alpha}}^{\alpha-1} + \| y\|_{L^\infty(0,\tau_1;L^{2\alpha})}^{\alpha-1} |x - y|\|_{L^\infty(0,\tau_1;W^{1,2\alpha})} \lesssim (\| x\|_{H^{2\alpha}}^{\alpha-1} + M_1^{\alpha-1}) M_1^{\alpha}. \]

We consider the term of the stochastic integral. By the definition of \( W \), we have
\[
\int_0^t U(t, s)e^{(\alpha-1)ReW} g(y)dW = \int_0^t U(t, s)\Phi(s)d\tilde{W}(s), \quad \text{P-a.s.,}
\]
where
\[ \tilde{W} = (\beta_1, \ldots, \beta_N), \]
\[ \Phi(t)(v) = \sum_{j=1}^{N} g(y_j) e_j \langle v, f_j \rangle, \quad v \in \mathbb{R}^N, \]
\[ (f_j)_{1 \leq j \leq N} : \text{an orthonormal basis in } \mathbb{R}^N. \]

Then, by the stochastic Strichartz estimates, we have
\[
\mathbb{E} \left\{ \left\| \int_{t}^{\tau} U(t, s)e^{(\alpha-1)\text{Re}W} g(y) dW \right\|_{L^\infty(0, \tau; L^2) \cap L^q(0, \tau; L^p)} \right\}^\rho
\leq \mathbb{E} \left\{ \left\| \Phi(s) \right\|_{L^2(0, \tau; HS(\mathbb{R}^N; L^2(\mathbb{R}^d)))} \right\}^\rho.
\]
The right hand side is finite since we can estimate
\[
\mathbb{E} \left\{ \left\| \Phi(s) \right\|_{L^2(0, \tau; HS(\mathbb{R}^N; L^2(\mathbb{R}^d)))} \right\}^\rho
\leq \mathbb{E} \left\{ \left\| g(y) \right\|_{L^2(0, \tau; L^2(\mathbb{R}^d))} \right\}^\rho
\leq \mathbb{E} \left\{ \left\| y^{\alpha-1} \right\|_{L^\infty(0, \tau; L^2(\mathbb{R}^d))} \right\}^\rho
\leq \mathbb{E} \left\{ \left\| y \right\|_{L^2(0, \tau; L^2(\mathbb{R}^d))} \right\}^\rho
\leq \mathbb{E} \left\{ \left\| y \right\|_{L^2(0, \tau; L^2(\mathbb{R}^d))} \right\}^\rho
\leq \mathbb{E} \left\{ \left\| 1_{R_1} M_1^{\alpha} \right\| \right\}^\rho
\leq \left( \mathbb{E} \left\| 1_{R_1} M_1^{2\alpha} \right\| \right)^\frac{1}{2} \leq T^{\rho} (\mathbb{E} \left\| M_1^{2\alpha} \right\|)^\frac{1}{2} < \infty.
\]
Here, \( M_1 \) is determined later to be \( (\mathbb{E} \left\| M_1^{2\alpha} \right\|)^\frac{1}{2} < \infty. \) Therefore,
\[
\left\| \int_{t}^{\tau} U(t, s)e^{(\alpha-1)\text{Re}W} g(y) dW \right\|_{L^\infty(0, \tau; L^2) \cap L^q(0, \tau; L^p)} \leq C_T < \infty, \quad \text{P-a.s.}
\]
In the long run, the following holds,
\[
\left\| \partial_t F(y) \right\|_{str(\tau_1)} \lesssim C_T + \left\| x \right\|_{H^2} + \left\| x \right\|_{H^2}^\alpha + \tau_1^\alpha M_1^\alpha + (\left\| x \right\|_{H^2}^{\alpha-1} + M_1^{\alpha-1}) M_1 \tau_1^\alpha.
\]
Similarly,
\[
\left\| F(y) \right\|_{str(\tau_1)} \lesssim \left\| x \right\|_{H^2} + \left\| (\alpha-1)\text{Re}W \right\|_{L^{q'}(0, \tau; L^{q'})} \lesssim \left\| x \right\|_{H^2} + \tau_1^\alpha M_1^\alpha.
\]
Therefore,
\[
\left\| \Delta F(y) \right\|_{str(\tau_1)} + \left\| \partial_t F(y) \right\|_{str(\tau_1)} \lesssim C_T + \left\| x \right\|_{H^2} + \left\| x \right\|_{H^2}^\alpha + \tau_1^\alpha M_1^\alpha + (\left\| x \right\|_{H^2}^{\alpha-1} + M_1^{\alpha-1}) M_1 \tau_1^\alpha.
\]
Next, we estimate \( \left\| \Delta F(y) \right\|_{L^\infty(0, \tau; L^2)}. \) We write \( G(y(t)) = e^{(\alpha-1)\text{Re}W(t)} g(y(t)) \) in the followings. From (1.2) and (1.3), we have
\[
\int_{0}^{t} U(t, s)A(s)G(y(s)) ds = U(t,0)G(y(0)) - G(y(t)) + \int_{0}^{t} U(t, s)(\partial_t G)(y(s)) ds.
\]
By

\[ U(t, s)A(s)G(y(s)) = -iU(t, s)(\Delta + b(s, \xi) \cdot \nabla + c(s, \xi))G(y(s)) \]

\[ = -iU(t, s)(b(s, \xi) \cdot \nabla + c(s, \xi))G(y(s)) - i\Delta U(t, s)G(y(s)) \]

\[ + i \int_s^t U(t, \tau)[2i(D_j \nabla \tilde{b}^j(\tau) + \nabla \tilde{b}^j(\tau)D_j + \nabla \tilde{c}(\tau))\nabla G(y(\tau))] \]

\[ + i(D_j \Delta \tilde{b}^j(\tau)D_j + \Delta \tilde{c}(\tau))G(y(\tau))]d\tau, \]

we have

\[ \Delta \int_0^t U(t, s)G(y(s))ds = iU(t, 0)G(y(0)) - iG(y(t)) + i \int_0^t U(t, s)(\partial_t G)(y(s))ds \]

\[ - \int_0^t U(t, s)b(s, \xi) \cdot \nabla G(y(s))ds - \int_0^t U(t, s)c(s, \xi)G(y(s))ds \]

\[ - \int_0^t \int_s^t U(t, \tau)[2i(D_j \nabla \tilde{b}^j(\tau) + \nabla \tilde{b}^j(\tau)D_j + \nabla \tilde{c}(\tau))\nabla G(y(\tau))] \]

\[ + i(D_j \Delta \tilde{b}^j(\tau)D_j + \Delta \tilde{c}(\tau))G(y(\tau))]d\tau ds. \]
Therefore,

\[ ||\Delta F(y)||_{L^\infty(0,\tau_1;L^2)} \lesssim ||x||_{H^2} + ||U(t,0)G(y(0)) - G(y(t))||_{L^\infty(0,\tau_1;L^2)} \]

\[ + \left| \left| \int_0^t U(t,s)(\partial_t G)(y)ds \right| \right|_{L^\infty(0,\tau_1;L^2)} \]

\[ + \left| \left| \int_0^t U(t,s)b(s,\xi) \cdot \nabla G(y)ds \right| \right|_{L^\infty(0,\tau_1;L^2)} \]

\[ + \left| \left| \int_0^t U(t,s)c(s,\xi)G(y)ds \right| \right|_{L^\infty(0,\tau_1;L^2)} \]

\[ + \left| \left| \int_0^t \int_s^t U(t,\tau)[2i(D_j \nabla \tilde{b}^j(\tau) + \nabla \tilde{b}^j(\tau)D_j + \nabla \tilde{c}(\tau))\nabla G(y(\tau))] \right. \]

\[ + i(D_j \Delta \tilde{b}^j(\tau)D_j + \Delta \tilde{c}(\tau))G(y(\tau))]d\tau ds \right|_{L^\infty(0,\tau_1;L^2)} \]

\[ \lesssim ||x||_{H^2} + ||U(t,0)G(y(0)) - G(y(t))||_{L^\infty(0,\tau_1;L^2)} \]

\[ + \left| \left| \int_0^t U(t,s)(\partial_t G)(y)ds \right| \right|_{L^\infty(0,\tau_1;L^2)} \]

\[ + ||b||_{L^\infty(0,\tau_1;L^\infty)}||\nabla G(y)||_{L^{r'}(0,\tau_1;L^{r'})} \]

\[ + ||c||_{L^\infty(0,\tau_1;L^\infty)}||G(y)||_{L^{r'}(0,\tau_1;L^{r'})} \]

\[ + \left| \left| \int_0^t \left| \int_s^t U(t,\tau)[2i(D_j \nabla \tilde{b}^j(\tau) + \nabla \tilde{b}^j(\tau)D_j + \nabla \tilde{c}(\tau))\nabla G(y(\tau))] \right. \right| \right|_{L^\infty(0,\tau_1;L^2)} \]

\[ \lesssim ||x||_{H^2} + ||U(t,0)G(y(0)) - G(y(t))||_{L^\infty(0,\tau_1;L^2)} \]

\[ + \left| \left| \int_0^t U(t,s)(\partial_t G)(y)ds \right| \right|_{L^\infty(0,\tau_1;L^2)} + ||\nabla G(y)||_{L^{r'}(0,\tau_1;L^{r'})} \]

\[ + ||G(y)||_{L^{r'}(0,\tau_1;L^{r'})} + ||g(y)||_{L^\infty(0,\tau_1;H^1)}. \]  

(4.8)

We estimate 3rd, 4th and 5th terms in the last line

\[ \int_0^t U(t,s)(\partial_t G)(y(s))ds = (\alpha - 1) \int_0^t U(t,s)e^{(\alpha - 1)ReW(s)}g(y(s))dW(s) \]

\[ + (\alpha - 1)^2 \int_0^t U(t,s)e^{(\alpha - 1)ReW(s)}g(y(s))ds, \]

\[ ||\nabla G(y)|| \leq |e^{(\alpha - 1)ReW}||[(\alpha - 1)|\nabla W||g(y)| + |\nabla g(y)||]. \]

Therefore, we have the following

\[ ||\nabla G(y)||_{L^{r'}(0,\tau_1;L^{r'})} \lesssim ||g(y)||_{L^{r'}(0,\tau_1;L^{r'})} + ||\nabla g(y)||_{L^{r'}(0,\tau_1;L^{r'})} \lesssim \tau_1^q M_1^a, \]

\[ ||G(y)||_{L^{r'}(0,\tau_1;L^{r'})} \lesssim ||g(y)||_{L^{r'}(0,\tau_1;L^{r'})} \lesssim \tau_1^q M_1^a, \]

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Therefore, \( \therefore \)

So, \( \therefore \)

First, we estimate the first terms in the last line of (4.9). From (4.5), we have

\[
\|g(y)\|_{L^\infty(0, \tau_1; L^2)} \lesssim \|\mu e^{(\alpha-1)\Re W}g(y)\|_{L^\infty(0, \tau_1; L^{2'})} \lesssim \|g(y)\|_{L^{2'}(0, \tau_1; L^2')} \lesssim \tau_1^\theta \alpha_1^\alpha.
\]

Also, the 6th term and the term in the stochastic integral are estimated in the same as before, i.e.

\[
\|g(y)\|_{L^\infty(0, \tau_1; H^1)} \lesssim \tau_1^\theta \alpha_1^\alpha + (\|x\|_{H^2}^{-1} + \alpha_1^\alpha - 1) \alpha_1^\alpha,
\]

\[
\left\| \int_0^t U(t, s) \mu e^{(\alpha-1)\Re W} g(y) ds \right\|_{L^\infty(0, \tau_1; L^2)} \leq C_T < \infty, \ \text{P.-a.s.}
\]

Next, we estimate the second terms of the last line of (4.8).

\[
\|U(t, 0)G(y(0)) - G(y(t))\|_{L^\infty(0, \tau_1; L^2)} \leq \|U(t, 0)G(x) - G(x)\|_{L^\infty(0, \tau_1; L^2)} + \|G(x) - G(y(t))\|_{L^\infty(0, \tau_1; L^2)}. \tag{4.9}
\]

First, we estimate the first terms in the last line of (4.9). From (4.9), we have \( G(x) = e^{(\alpha-1)\Re W} g(x) \in L^\infty(0, \tau_1; L^2) \). We estimate

\[
\|U(t, 0)G(x) - G(x)\|_{L^\infty(0, \tau_1; L^2)} = \|(U(t, 0) - I)e^{(\alpha-1)\Re W} g(x)\|_{L^\infty(0, \tau_1; L^2)} \to 0.
\]

Also,

\[
\|U(t, 0)G(x) - G(x)\|_{L^\infty(0, \tau_1; L^2)} = \|(U(t, 0) - I)e^{(\alpha-1)\Re W} g(x)\|_{L^\infty(0, \tau_1; L^2)} \leq \|U(t, 0) - I\|_0 \|e^{(\alpha-1)\Re W} g(x)\|_{L^\infty(0, \tau_1; L^\infty)} \|g(x)\|_{L^2} \lesssim \|g(x)\|_{L^2} \lesssim \|x\|_{H^2}.
\]

Next, we estimate the second terms in the last line of (4.9). By (4.6) and (4.7), we have

\[
\|G(x) - G(y(t))\|_{L^\infty(0, \tau_1; L^2)} \lesssim \|g(x) - g(y)\|_{L^\infty(0, \tau_1; L^2)} \lesssim (\|x\|_{L^2}^{-1} + \|y\|_{L^\infty(0, \tau_1; L^{2\alpha})}^{-1}) \|x - y\|_{L^\infty(0, \tau_1; L^{2\alpha})} \lesssim (\|x\|_{H^2}^{-1} + \alpha_1^\alpha - 1) \alpha_1^\alpha.
\]

So,

\[
\|\Delta F(y)\|_{L^\infty(0, \tau_1; L^2)} \lesssim C_T + \|x\|_{H^2} + \|x\|_{H^2}^\alpha + \tau_1^\alpha \alpha_1^\alpha.
\]

Therefore,

\[
\|F(y)\|_{\text{str}(\tau_1)} + \|\partial_t F(y)\|_{\text{str}(\tau_1)} + \|\Delta F(y)\|_{L^\infty(0, \tau_1; L^2)} \lesssim C_{\tau_1}(C_T + \|x\|_{H^2} + \|x\|_{H^2}^\alpha + \tau_1^\alpha \alpha_1^\alpha + (\|x\|_{H^2}^{-1} + \alpha_1^\alpha - 1) \alpha_1^\alpha),
\]

where \( C_{\tau_1} \) is a constant that depends on \( \tau_1 \).

We shall choose \( M_1 \) and \( \tau_1 \) to obtain \( F(Y_{M_1}^{\tau_1}) \subset Y_{M_1}^{\tau_1} \) such that

\[
C_{\tau_1}(C_T + \|x\|_{H^2} + \|x\|_{H^2}^\alpha + \tau_1^\alpha \alpha_1^\alpha + (\|x\|_{H^2}^{-1} + \alpha_1^\alpha - 1) \alpha_1^\alpha) \leq M_1.
\]

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To this end, we define the real-valued continuous, \((\mathcal{F}_t)\)-adapted process
\[
Z_t^{(1)} = 2^{\alpha - 1} C_t^\alpha (C_T + ||x||_{H^2} + ||x||_{H^2}^{\alpha - 1} (t^\theta + t^\alpha) + C_t t^\alpha ||x||_{H^2}^{\alpha - 1}, t \in [0, T],
\]
choose the \((\mathcal{F}_t)\)-stopping time as
\[
\tau_1 = \inf \left\{ t \in [0, T]; Z_{t}^{(1)} > \frac{1}{2} \right\} \land T,
\]
and set \(M_1 = 2C_{\tau_1} (C_T + ||x||_{H^2} + ||x||_{H^2}^{\alpha - 1}).\) Then it follows that \(Z_{\tau_1}^{(1)} \leq \frac{1}{2}\) and \(F(\mathcal{Y}_{M_1}^{\tau_1}) \subset \mathcal{Y}_{M_1}^{\tau_1} \).
Moreover, since \(|g(y_1) - g(y_2)| \leq \alpha (||y_1||_{\alpha - 1} + ||y_2||_{\alpha - 1})|y_1 - y_2|\), for \(y_1, y_2 \in \mathcal{Y}_{M_1}^{\tau_1}\)
\[
||F(y_1) - F(y_2)||_{\text{str} (\tau_1)} \leq C_{\tau_1} (||y_1||_{L^\infty (0, \tau_1; L^{2\alpha})} + ||y_2||_{L^\infty (0, \tau_1; L^{2\alpha})}) ||y_1 - y_2||_{L^p (0, \tau_1; L^p)}
\]
\[
\leq C_{\tau_1} \alpha ||M_1^{\alpha - 1}||_{L^p (0, \tau_1; L^p)}
\]
which implies that \(F\) is a contraction in \(L^p (0, \tau_1; L^p)\). Since \(\mathcal{Y}_{M_1}^{\tau_1}\) is a complete metric subspace in \(L^\infty (0, \tau_1; L^2)\), Banach’s fixed point theorem yields a unique \(y \in \mathcal{Y}_{M_1}^{\tau_1}\) with \(y = F(y)\) on \([0, \tau_1]\).
Consequently, setting \(y_1(t) := y(t \land \tau_1), t \in [0, T]\), we deduce that \((y_1, \tau_1)\) is a local solution of \(\text{(2.3)}\), such that \(y_1(t) = y_1(t \land \tau_1), t \in [0, \tau_1],\) and \(y_1|_{[0, \tau_1]} \in C([0, \tau_1]; H^2)\).

**Step 2.** We use an induction argument. Suppose that at the \(n\)th step we have a local solution \((y_n, \tau_n)\) of \(\text{(2.3)}\), such that \(\tau_n \geq \tau_{n-1}, y_n(t) = y_n(t \land \tau_n), t \in [0, T]\), and \(y_n|_{[0, \tau_n]} \in C([0, \tau_n]; H^2)\).
We construct \((y_{n+1}, \tau_{n+1})\). Set
\[
\mathcal{Y}_{M_{n+1}}^{\sigma_n} = \{ z, \partial_t z, \Delta z \in L^\infty (0, \sigma_n; L^2); ||z||_{\text{str} (\sigma_n)} + ||\partial_t z||_{\text{str} (\sigma_n)} + ||\Delta z||_{L^\infty (0, \sigma_n; L^2)} \leq M_{n+1}\},
\]
and define the map \(F_n\) by
\[
F_n(z)(t) = U(\tau_n + t, \tau_n) y_n(\tau_n) - \lambda t \int_0^t U(\tau_n + t, \tau_n + s)(e^{(\alpha - 1)ReW(\tau_n + s)}g(z(s))) ds. \tag{4.10}
\]

Analogous calculations as in Step 1. show that for \(z \in \mathcal{Y}_{M_{n+1}}^{\sigma_n}\)
\[
||F_n(z)||_{\text{str} (\sigma_n)} + ||\partial_t F_n(z)||_{\text{str} (\sigma_n)} + ||\Delta F_n(z)||_{L^\infty (0, \sigma_n; L^2)}
\]
\[
\leq C_{\tau_n + \sigma_n} (C_T + ||y_n(\tau_n)||_{H^2} + ||y_n(\tau_n)||_{H^2}^{\alpha - 1} + \sigma_n M_{n+1}^{\alpha - 1} + (||y_n(\tau_n)||_{H^2}^{\alpha - 1} + M_{n+1}^{\alpha - 1})M_{n+1} \sigma_n^{\alpha}).
\]

We shall choose \(M_{n+1}\) and \(\sigma_n\) to obtain \(F_n(\mathcal{Y}_{M_{n+1}}^{\sigma_n}) \subset \mathcal{Y}_{M_{n+1}}^{\sigma_n}\) such that
\[
C_{\tau_n + \sigma_n} (C_T + ||y_n(\tau_n)||_{H^2} + ||y_n(\tau_n)||_{H^2}^{\alpha - 1} + \sigma_n M_{n+1}^{\alpha - 1} + (||y_n(\tau_n)||_{H^2}^{\alpha - 1} + M_{n+1}^{\alpha - 1})M_{n+1} \sigma_n^{\alpha}) \leq M_{n+1}.
\]

To this end, we define the real-valued continuous, \((\mathcal{F}_{\tau_n + t})\)-adapted process
\[
Z_t^{(n+1)} = 2^{\alpha - 1} C_{\tau_n + t}^\alpha (C_T + ||y_n(\tau_n)||_{H^2} + ||y_n(\tau_n)||_{H^2}^{\alpha - 1} (t^\theta + t^\alpha) + C_{\tau_n + t} t^\alpha ||y_n(\tau_n)||_{H^2}^{\alpha - 1}, t \in [0, T],
\]
choose the \((F_{\tau_{n+1}})\)-stopping time as

\[
\sigma_n = \inf \left\{ t \in [0, T - \tau_n]; Z_i^{(n+1)} > \frac{1}{2} \right\} \land (T - \tau_n),
\]

and set \(M_{n+1} = 2C_{\tau_n + \sigma_n}(C_T + \|y_n(\tau_n)\|_{H^2} + \|y_n(\tau_n)\|_{H^2}^\alpha)\). Then it follows that \(Z_{\sigma_n}^{(n+1)} \leq \frac{1}{2}\) and \(F_n(\mathcal{Y}_{\alpha\omega}^\sigma M_{n+1}) \subset \mathcal{Y}_{\alpha\omega}^\sigma M_{n+1}\).

Moreover, since \(|g(z_1) - g(z_2)| \leq \alpha(|z_1|^{\alpha-1} + |z_2|^{\alpha-1})|z_1 - z_2|\), for \(z_1, z_2 \in \mathcal{Y}_{\alpha\omega}^\sigma M_{n+1}\)

\[
||F_n(z_1) - F_n(z_2)||_{str(\sigma_n)} \leq C_{\tau_n + \sigma_n} ||g(z_1) - g(z_2)||_{L^\gamma(0, \sigma_n; L^\rho)}
\]

\[
\leq C_{\tau_n + \sigma_n} ||z_1|^{\alpha-1}||_{L^\infty(0, \sigma_n; L^{2\alpha})} + ||z_2|^{\alpha-1}||_{L^\infty(0, \sigma_n; L^{2\alpha})}||z_1 - z_2||_{L^\gamma(0, \sigma_n; L^\rho)}
\]

\[
\leq C_{\tau_n + \sigma_n} \sigma_n^{\alpha-1} ||z_1 - z_2||_{L^\gamma(0, \sigma_n; L^\rho)}
\]

\[
\leq \frac{1}{2} ||z_1 - z_2||_{L^\gamma(0, \sigma_n; L^\rho)},
\]

which implies that \(F_n\) is a contraction from \(L^\infty(0, \sigma_n; L^2)\) to the same space.

Set \(\tau_{n+1} = \tau_n + \sigma_n\). Then, similarly to the proof of Lemma 4.2 in [6], we can show \(\tau_{n+1}\) is an \((F_t)_{t \geq 0}\)-stopping time. By Banach’s fixed point theorem, there exists a unique \(z_{n+1} \in \mathcal{Y}_{\alpha\omega}^\sigma M_{n+1}\)

satisfying \(z_{n+1} = F_n(z_{n+1})\) on \([0, \sigma_n]\). We define

\[
y_{n+1}(t) = \begin{cases} 
  y_n(t), & t \in [0, \tau_n]; \\
  z_{n+1}((t - \tau_n) \land \sigma_n), & t \in (\tau_n, T].
\end{cases}
\]

It follows from the definition of \(F\) and \(F_n\) that \(y_{n+1} = F(y_{n+1})\) on \([0, \tau_{n+1}]\).

And, similar to the proof of Lemma 6.2 in [6], \(y_{n+1}\) is an adapted to \((F_t)_{t \geq 0}\) in \(H^2\). Hence, \((y_{n+1}, \tau_{n+1})\) is a local solution of \((2.3)\), such that \(y_{n+1}(t) = y_{n+1}(t \land \tau_{n+1}), t \in [0, T],\) and \(y_{n+1}[0, \tau_{n+1}] \subset C([0, \tau_{n+1}]; H^2)\). Starting from Step 1 and repeating the procedure in Step 2, we finally construct a sequence of local solutions \((y_n, \tau_n)\), \(n \in \mathbb{N}\), where \(\tau_n\) are increasing stopping times and \(y_{n+1} = y_n\), on \([0, \tau_n]\).

To prove the uniqueness, for any two local solutions \((\tilde{y}_1, \sigma_1), i = 1, 2\), define \(\iota = \sup\{t \in [0, \sigma_1 \land \sigma_2]; \tilde{y}_1 = \tilde{y}_2\) on \([0, t]\}\). Suppose that \(P(t < \sigma_1 \land \sigma_2) > 0\). For \(\omega \in \{t < \sigma_1 \land \sigma_2\}\), we have \(\tilde{y}_1(\omega) = \tilde{y}_2(\omega)\) on \([0, \iota(\omega)]\) by the continuity in \(H^2\), and for \(t \in [0, \sigma_1 \land \sigma_2(\omega) - \iota(\omega)]\)

\[
||\tilde{y}_1(\omega) - \tilde{y}_2(\omega)||_{L^\infty(\iota(\omega), \iota(\omega) + \iota(\omega) + t; L^2) + ||\tilde{y}_1(\omega) - \tilde{y}_2(\omega)||_{L^\gamma(\iota(\omega), \iota(\omega) + \iota(\omega) + t; L^\rho)}
\]

\[
= ||F(\tilde{y}_1(\omega)) - F(\tilde{y}_2(\omega))||_{L^\infty(\iota(\omega), \iota(\omega) + \iota(\omega) + t; L^2) + ||F(\tilde{y}_1(\omega)) - F(\tilde{y}_2(\omega))||_{L^\gamma(\iota(\omega), \iota(\omega) + \iota(\omega) + t; L^\rho)}
\]

\[
\leq C_{\iota(\omega) + \iota(\omega)} ||\tilde{y}_1(\omega) - \tilde{y}_2(\omega)||_{L^\infty(\iota(\omega), \iota(\omega) + \iota(\omega) + t; L^\rho)}
\]

\[
\leq C_{\iota(\omega)} ||\tilde{y}_1(\omega)||_{L^\infty(\iota(\omega), \iota(\omega) + \iota(\omega) + t; L^{2\alpha})} + ||\tilde{y}_2(\omega)||_{L^\infty(\iota(\omega), \iota(\omega) + \iota(\omega) + t; L^{2\alpha})}||\tilde{y}_1(\omega) - \tilde{y}_2(\omega)||_{L^\gamma(\iota(\omega), \iota(\omega) + \iota(\omega) + t; L^\rho)}
\]

\[
\leq C_{\iota(\omega) + \iota(\omega) + \iota(\omega)} ||\tilde{y}_1(\omega) - \tilde{y}_2(\omega)||_{L^\gamma(\iota(\omega), \iota(\omega) + \iota(\omega) + t; L^\rho)},
\]

where \(\tilde{M}(t) := ||\tilde{y}_1(\omega)||_{L^\infty(\iota(\omega), \iota(\omega) + \iota(\omega) + t; H^2) + ||\tilde{y}_2(\omega)||_{L^\infty(\iota(\omega), \iota(\omega) + \iota(\omega) + t; H^2)} \rightarrow 0\) as \(t \rightarrow 0\). Therefore, with \(t\) small enough we deduce that \(\tilde{y}_1(\omega) = \tilde{y}_2(\omega)\) on \([\iota(\omega), \iota(\omega) + t],\) hence \(\tilde{y}_1(\omega) = \tilde{y}_2(\omega)\) on \([0, \iota(\omega) + t],\)
Finally, prove the continuous dependence. Suppose that \( x_m \to x \) in \( H^2 \) and let \((y_m, (\tau_m^i))_{n \in \mathbb{N}}, \tau^* (x_m)\) be the unique local solutions of (2.3) corresponding to the initial data \( x_m \), \( m \geq 1 \). Since \( |x_m|_{H^2} \leq \|x\|_{H^2} + 1 \) for \( m \geq m_1 \) with \( m_1 \) large enough, we modify \( \tau_1(\leq T) \) in the proof of Proposition [2.3] by

\[
\tau_1 = \inf \{ t \in [0, T] \colon 2^{\alpha-1}C_T^\alpha (C_T + (\|x\|_{H^2} + 1) + (\|x\|_{H^2} + 1)^{\alpha-1}(t^\theta + t^{\alpha}) + Ct^\alpha(\|x\|_{H^2} + 1)^{\alpha-1} > \frac{1}{2} \} \land T,
\]

such that \( \tau_1 \) is independent for \( m \geq m_1 \). Hence,

\[
\tilde{R} := \sup_{m \geq m_1} \|y_m\|_{L^\infty(0, \tau_1; H^2)} < \infty, \text{ } \text{P-a.s.}
\]

We first prove the continuous dependence on initial data on the interval \([0, \tau_1]\).

(i) Claim 1: \( \|y_m - y\|_{L^\infty(0, \tau_1; L^2)} \to 0 \), as \( m \to \infty \).

From

\[
y_m - y = U(t, 0)(x_m - x) - \lambda \int_0^t U(t, s)e^{(\alpha-1)ReW(s)}(g(y_m(s)) - g(y(s))) ds,
\]

taking \( t \) small and independently of \( m \geq m_1 \), we have

\[
\|y_m - y\|_{L^\infty(0, t; L^2)} \leq \|x_m - x\|_{L^2} + \|g(y_m) - g(y)\|_{L^\infty(0, t; L^{2\alpha} \cap L^\infty(0, t; L^{2\alpha}))} + \|y\|_{L^\infty(0, t; L^{2\alpha})} \|y_m - y\|_{L^\infty(0, t; L^{2\alpha})}.
\]

Therefore,

\[
\|y_m - y\|_{L^\infty(0, t; L^2)} \leq C_{r_1} (\|x_m - x\|_{L^2} + t^\theta \tilde{R}^{\alpha-1}) \|y_m - y\|_{L^\infty(0, t; L^{2\alpha})}.
\]

So, if we choose \( t \) satisfy \( C_{r_1} t^\theta \tilde{R}^{\alpha-1} \leq \frac{1}{2} \), we obtain

\[
\|y_m - y\|_{L^\infty(0, t; L^2)} \leq 2C_{r_1} \|x_m - x\|_{L^2}.
\]

Since \( t \) is independent of \( m \geq m_1 \),

\[
\|y_m - y\|_{L^\infty(0, t; L^2)} \to 0, \text{ as } m \to \infty.
\]

By repeating this a finite number of times, we can show the claim 1.

(ii) Claim 2: \( \|y_m - y\|_{L^\infty(0, \tau_1; H^s)} \to 0 \), as \( m \to \infty \), \( 0 < s < 2 \).

From the interpolation inequality,

\[
\|y_m - y\|_{L^\infty(0, \tau_1; H^s)} \leq \|y_m - y\|_{L^\infty(0, \tau_1; L^2)}^{1-\theta} \|y_m - y\|_{L^\infty(0, \tau_1; H^2)}^{\theta} \leq \|y_m - y\|_{L^\infty(0, \tau_1; L^2)}^{1-\theta} \|y_m - y\|_{L^\infty(0, \tau_1; H^2)}^{\theta} \leq \tilde{R}^{1-\theta} \|y_m - y\|_{L^\infty(0, \tau_1; L^2)}^{\theta}.
\]
where, Now, since we have that \( \tau \) depending on \( \|y(\tau_2)\|_{H^2} \). Therefore, we can show the continuous dependence on \([0, \tau_2]\). Reiterating this procedure, we then obtain increasing stopping times \( \tau_n \), depending on \( \|y(\tau_{n-1})\|_{H^2} \), such that continuous dependence holds on every \([0, \tau_n]\). Therefore, for \( n \geq 1, \) \( \mathbb{P} \)-a.s. \( \omega \in \Omega \) and \( 0 \leq s < 2 \), the map \( x \to y(\cdot, x, \omega) \) is continuous from \( H^2 \) to \( L^\infty(0, \tau_n; H^s) \).

Proof of Theorem 2.4 First, for any Strichartz pair \( (p, \gamma) \), we show \( y_n\|_{[0, \tau_n]} \in L^\gamma(0, \tau_n; W^{2,p}) \).

Where, \( y = U(t, 0)x - \lambda i \int_0^t U(t, s)e^{(\alpha-1)ReW(s)}g(y(s))ds, \)

and consider the map

\[
F(y)(t) = U(t, 0)x - \lambda i \int_0^t U(t, s)(e^{(\alpha-1)ReW(s)}g(y(s)))ds, \quad t \in [0, T].
\]

(4.11)

Let us first consider the case \( d = 4, 6, 7 \). Choose the Strichartz pair \( (p, q) = \left( \frac{d(\alpha+1)}{d+2\alpha-2}, \frac{4(\alpha+1)}{(d-4)(\alpha-1)} \right) \). By Strichartz estimates in Lemma 3.3,

\[
\|F(y)\|_{L^p(0, T; W^{2,p})} \lesssim \|x\|_{H^2} + \|e^{(\alpha-1)ReW}g(y)\|_{L^q'(0, T; W^{2,p'})}.
\]

(4.12)

And we have that

\[
\|e^{(\alpha-1)ReW}g(y)\|_{L^q'(0, T; W^{2,p'})} \lesssim \|y\|_{L^q'(0, T; L^p')} + \|y\|_{L^q'(0, T; L^p')}^{\alpha-1}\|\nabla y\|_{L^q'(0, T; L^p')} + \|y\|_{L^q'(0, T; L^p')}^{\alpha-2}\|\nabla y\|_{L^q'(0, T; L^p')}.
\]

(4.13)

To estimate the right-hand side, since

\[
|\nabla g(y)| \leq \alpha |y|^{\alpha-1}|\nabla y|,
\]

\[
|\nabla (e^{(\alpha-1)ReW}g(y))| \leq |e^{(\alpha-1)W}||y|^{\alpha-1}|\nabla W||g(y)| + |\nabla g(y)|,
\]

\[
|\Delta g(y)| \lesssim |y|^{\alpha-2}\|\nabla y\|^2 + |y|^{\alpha-1}|\Delta y|,
\]

\[
|\Delta (e^{(\alpha-1)ReW}g(y))| \lesssim |e^{(\alpha-1)W}||\nabla W|^2|g(y)| + |e^{(\alpha-1)W}||\Delta W||g(y)| + |e^{(\alpha-1)W}||\nabla W||\Delta g(y)| + |e^{(\alpha-1)W}||\Delta g(y)|.
\]

From Hölder’s inequality and the Sobolev imbedding it follows that

\[
\|y\|_{L^q'(0, T; L^p')} \lesssim T^\theta \|y\|_{L^q(0, T; L^{\alpha-1})} \lesssim T^\theta \|y\|_{L^q(0, T; W^{2,p})} \leq T^\theta \|y\|_{L^q(0, T; W^{2,p})} \|y\|_{L^q(0, T; W^{2,p})},
\]

(4.14)
Thus, inserting (4.14), (4.15), (4.16), (4.17) into (4.13) and (4.12) yields that

\[ ||y||^\alpha L'(0,T;L^p) \leq T^\theta ||y||^\alpha L'(0,T;L^p) \]

\[ \lesssim T^\theta ||y||^\alpha L'(0,T;W^{2,p}) ||y|| L^q(0,T;W^{2,p}), \]  

(4.15)

\[ ||y||^\alpha \nabla y || L'(0,T;L^p) \leq T^\theta ||y||^\alpha L'(0,T;L^p) \]

\[ \lesssim T^\theta ||y||^\alpha L'(0,T;W^{2,p}) ||y|| L^q(0,T;W^{2,p}), \]  

(4.16)

\[ ||y||^\alpha \nabla y || L'(0,T;L^p) \leq T^\theta ||y||^\alpha L'(0,T;L^p) \]

\[ \lesssim T^\theta ||y||^\alpha L'(0,T;W^{2,p}) ||y|| L^q(0,T;W^{2,p}), \]  

(4.17)

Thus, inserting (4.14), (4.15), (4.16), (4.17) into (4.13) and (4.12) yields that

\[ ||F(y)|| L^q(0,T;W^{2,p}) \lesssim ||x|| H^2 + T^\theta ||y||^\alpha L^q(0,T;W^{2,p}). \]  

(4.18)

Fix \( \omega \in \Omega \) and consider \( F \) on the set

\[ \mathcal{Y}_{M_1} = \{ y \in L^\infty(0, \tau_1; H^2) \cap L^q(0, \tau_1; W^{2,p}) : ||y||L^\infty(0,\tau_1;H^2) + ||y||L^q(0,\tau_1;W^{2,p}) \leq M_1 \}. \]

Then, suggested to the proof of Theorem 2.3. We can conclude \( F \) is a contraction from \( L^\infty(0, \tau_1; L^2) \cap L^q(0, \tau_1; L^p) \) to the same space, which yields a unique \( y \in \mathcal{Y}_{M_1} \) with \( y = F(y) \) on \([0, \tau_1]\). We also have \( y_1|_{[0,\tau_1]} \in L^\gamma(0, \tau_1; W^{2,p}) \) by using Strichartz estimate [3.4]. The same is true for \( n \geq 2 \).

Therefore, we have proved the case \( d = 5, 6, 7 \).

If \( d = 4 \), choose the Strichartz pair \((p, q) = (\frac{2(\alpha+2)}{\alpha+1}, \alpha + 2)\). Then Hölder’s inequality and Sobolev’s imbedding give

\[ ||y||^\alpha L'(0,T;L^p) \leq T^\theta ||y||^\alpha L'(0,T;L^p) \]

\[ \lesssim T^\theta ||y||^\alpha L'(0,T;W^{2,p}) ||y|| L^q(0,T;W^{2,p}), \]  

(4.19)

\[ ||y||^\alpha \nabla y || L'(0,T;L^p) \leq T^\theta ||y||^\alpha L'(0,T;L^p) \]

\[ \lesssim T^\theta ||y||^\alpha L'(0,T;W^{2,p}) ||y|| L^q(0,T;W^{2,p}), \]  

(4.20)

\[ ||y||^\alpha \nabla y || L'(0,T;L^p) \leq T^\theta ||y||^\alpha L'(0,T;L^p) \]

\[ \lesssim T^\theta ||y||^\alpha L'(0,T;W^{2,p}) ||y|| L^q(0,T;W^{2,p}), \]  

(4.21)

\[ ||y||^\alpha \nabla y || L'(0,T;L^p) \leq T^\theta ||y||^\alpha L'(0,T;L^p) \]

\[ \lesssim T^\theta ||y||^\alpha L'(0,T;W^{2,p}) ||y|| L^q(0,T;W^{2,p}), \]  

(4.22)
Then, to obtain this, we use (3.8) to derive that for $t$

$$
\Delta(y_m-y) = U(t,0)\Delta(x_m-x) + \int_0^t U(t,s)\{2i(D_j\nabla\bar{b}_j(s) + \nabla\bar{b}_j(s)D_j + \nabla\bar{c}(s))
\times \nabla(y_m-y) + i(D_j\Delta\bar{b}_j(s) + \Delta\bar{b}_j(s)D_j + \Delta\bar{c}(s))(y_m-y)
- \lambda i\Delta[e^{(\alpha-1)ReW}(g(y_m(s)) - g(y(s)))\}]ds,
$$

(4.26)
where $g(y) = |y|^\alpha - 1 y$. We note that, by Proposition 2.3(a) in [26], (3.5) and (3.6), we obtain

$$
||i(D_j \Delta \tilde{b}^j + \Delta \tilde{b}^j D_j + \Delta \tilde{c})(y_m - y)||_{\tilde{X}[0,t]}
\lesssim ||y_m - y||_{\tilde{X}[0,t]}
\lesssim ||x_m - x||_{L^2} + |||e^{(a-1)\Re(W)}(g(y_m) - g(y))||_{L^q(0,t;L^{p'})}
\lesssim ||x_m - x||_{L^2} + \theta||y_m - y||_{L^q(0,t;L^{p'})},
$$

(4.27)

$$
||2i(D_j \nabla \tilde{b}^j + \nabla \tilde{b}^j D_j + \nabla \tilde{c})\nabla (y_m - y)||_{\tilde{X}[0,t]}
\lesssim ||\nabla (y_m - y)||_{\tilde{X}[0,t]}
\lesssim ||x_m - x||_{H^1} + |||e^{(a-1)\Re(W)}(g(y_m) - g(y))||_{L^q(0,t;W^{1,p'})}
\lesssim ||x_m - x||_{H^1} + \theta||y_m - y||_{L^q(0,t;W^{1,p'})},
$$

(4.28)

where $\tilde{X}[0,t]$ is the local smoothing space (see Definition 3.2). Applying (3.5), (3.6) to (4.26), we derive by (4.27) and (4.28),

$$
||\Delta y_m - \Delta y||_{L^q(0,t;L^2)} + |||\Delta y_m - \Delta y||_{L^q(0,t;L^p)}
\lesssim ||\Delta x_m - \Delta x||_{L^2} + ||2i(D_j \nabla \tilde{b}^j + \nabla \tilde{b}^j D_j + \nabla \tilde{c})(y_m - y)||_{\tilde{X}[0,t]}
+ ||i(D_j \Delta \tilde{b}^j + \Delta \tilde{b}^j D_j + \Delta \tilde{c})(y_m - y)||_{\tilde{X}[0,t]}
+ ||\lambda \Delta e^{(a-1)\Re(W)}(g(y_m) - g(y))||_{L^q(0,t;L^{p'})}
\lesssim ||x_m - x||_{H^1} + \theta||y_m - y||_{L^q(0,t;W^{1,p'})} + ||\Delta g(y_m) - \Delta g(y)||_{L^q(0,t;L^{p'})}.
$$

(4.29)

As regards the last term on the right hand side of (4.29), we note that $\nabla g(y) = F_1(y)\nabla y + F_2(y)\nabla \overline{y}$, where $F_1(y) = \frac{a+1}{2}|y|^{a-1}$ and $F_2(y) = \frac{a+1}{2}|y|^{a-3}y^2$. Then

$$
\Delta g(y_m) - \Delta g(y) = \nabla F_1(y_m)\nabla y + F_1(y_m)\Delta y_m + \nabla F_2(y_m)\nabla \overline{y} + F_2(y_m)\Delta \overline{y}_m
- \nabla F_1(y)\nabla y - F_1(y)\Delta y - \nabla F_2(y)\nabla \overline{y} - F_2(y)\Delta \overline{y}
= F_1(y_m)[\Delta y_m - \Delta y] + [F_1(y_m) - F_1(y)]\Delta y
+ F_2(y_m)[\Delta \overline{y}_m - \Delta \overline{y}] + [F_2(y_m) - F_2(y)]\Delta \overline{y}
+ \nabla F_1(y_m)[\nabla y_m - \nabla y] + [\nabla F_1(y_m) - \nabla F_1(y)]\nabla y
+ \nabla F_2(y_m)[\nabla \overline{y}_m - \nabla \overline{y}] + [\nabla F_2(y_m) - \nabla F_2(y)]\nabla \overline{y}
= \sum_{k=1}^8 I_k.
$$

Since $|I_1| + |I_3| \lesssim |y_m|^{a-1}||\Delta y_m - \Delta y||$, we have

$$
||I_1 + I_3||_{L^q(0,t;L^{p'})} \lesssim \bar{R}^{a-1}\theta||y_m - y||_{L^q(0,t;W^{1,p'})},
$$

(4.30)

And, since $|I_5| + |I_7| \lesssim |y_m|^{a-2}||\nabla y_m - \nabla y||$, we have

$$
||I_5 + I_7||_{L^q(0,t;L^{p'})} \lesssim \bar{R}^{a-2}\theta||y_m - y||_{L^q(0,t;W^{1,p'})}.
$$

(4.31)
Therefore, by (4.29), (4.30) and (4.31),
\[
\|y_m - y\|_{L^\infty(0,t;H^2)} + \|y_m - y\|_{L^q(0,t;W^{2,p})} \\
\lesssim \|x_m - x\|_{H^2(t)} + \theta \|y_m - y\|_{L^q(0,t;W^{1,p})} + \tilde{R}^{\alpha-1}t^\theta \|y_m - y\|_{L^q(0,t;W^{2,p})} \\
+ \tilde{R}^{\alpha-2}t^\theta \|y_m - y\|_{L^q(0,t;W^{2,p})} + \|I_2 + I_4 + I_6 + I_8\|_{L^{p'}(0,t;L^{p'})}. 
\] (4.32)

Also, similar to the proof of Theorem 1.2 of [7], it follows that
\[
\|I_2\|_{L^{p'}(0,t;L^{p'})} + \|I_4\|_{L^p(0,t;L^p)} + \|I_6\|_{L^{p'}(0,t;L^{p'})} + \|I_8\|_{L^{p'}(0,t;L^{p'})} \to 0, \text{ as } m \to \infty. 
\] (4.33)

Thus, (4.24), (4.32), (4.33) through (4.25) follow. Reiterating this procedure in finite steps we obtain (4.25) on [0, \tau_1].

Now, since \(y_n(\tau_1) \to y(\tau_1)\) in \(H^2\), similarly we can get the above results to \([\tau_1, \tau_2\) with \(\tau_2\) depending on \(\|y(\tau_1)\|_{H^2}\). Therefore, we can show the continuous dependence on \([0, \tau_2\]. Reiterating this procedure, we then obtain increasing stopping times \(\tau_n\), depending on \(\|y(\tau_{n-1})\|_{H^2}\), such that continuous dependence holds on every \([0, \tau_n\]. Therefore, for \(n \geq 1\) and \(P\)-a.s. \(\omega \in \Omega\), the map \(x \to y(\cdot, x, \omega)\) is continuous from \(H^2\) to \(L^\infty(0, \tau_n; H^2) \cap L^q(0, \tau_n; W^{2,p})\). The same is true for any Strichartz pair \((\rho, \gamma)\).

Finally, prove the blowup alternative. Suppose that
\[
P(M^* < \infty; \tau_n < \tau^*(x), \ \forall n \in \mathbb{N}) > 0,
\]
where
\[
M^* := \sup_{t \in [0, \tau^*(x))] \|y(t)\|_{H^2}.
\]

Define
\[
Z_t = 2 \cdot 3^{\alpha-1}C_t^\alpha(M^*)^{\alpha-1}t^\theta, \ \ t \in [0, T], \\
\sigma := \inf \left\{ t \in [0, T] : Z_t > \frac{1}{3} \right\} \land T.
\]

For \(\omega \in \{M^* < \infty; \tau_n < \tau^*(x), \ \forall n \in \mathbb{N}\}, \) since \(\tau_n(\omega) < T, \ \forall n \in \mathbb{N}\), by the definition of \(\sigma_n\) in Step 2, we have
\[
\sigma_n(\omega) = \inf \left\{ t \in [0, T - \tau_n(\omega)) : 2 \cdot 3^{\alpha-1}C_{\tau_n+t}^\alpha\|y(\tau_n)\|_{H^2}^{\alpha-1}t^\theta > \frac{1}{3} \right\} \land (T - \tau_n).
\]

Notice that, for every \(n \geq 1\), \(\|y(\tau_n(\omega))\|_{H^2} \leq M^*, \ C_{\tau_n(\omega)+t} \leq C_{T+t}. \) It follows that \(Z_t(\omega) \geq Z_t^{(n+1)}(\omega), \) therefore \(\sigma_n(\omega) > \sigma(\omega) > 0. \) Hence \(\tau_{n+1}(\omega) = \tau_n(\omega) + \sigma_n(\omega) > \tau_n(\omega) + \sigma(\omega), \) which implies \(\tau_{n+1}(\omega) > \tau_1(\omega) + n\sigma(\omega)\) for every \(n \geq 1\), contradicting the fact that \(\tau_n(\omega) \leq T. \) This completes the proof.

5 Proof of Theorem 2.5

We prove the time global well-posedness in \(H^2\) for the equation with power condition \(2 \leq \alpha \leq 1 + \frac{4}{(s-2)^+}\) which correspond to the condition for \(H^1\) well-posedness. This is so called persistence argument in NLS.
Proof of Theorem \ref{thm:main}. From Theorem \ref{thm:main}, for any $x \in H^2$ and $0 < T < \infty$, there exists a local solution $(y, \tau^*(x))$ where $\tau^*(x)$ is the maximum existence time of $y$. Assume that $\tau^*(x) < T$. Then, there exists $\delta$ such that $\tau^*(x) - \delta < \tau^*(x) + \delta < T$. Fix $\omega \in \Omega$. We estimate the map in (\ref{nonlinear}). From Strichartz estimate, we have

$$
\|F(y)\|_{L^\infty(\tau^*(x) - \delta, \tau^*(x) + \delta; H^2)} \lesssim \|x\|_{H^2} + \|y\|_{L^\infty(\tau^*(x) - \delta, \tau^*(x) + \delta; W^{2,p})}^{\alpha - 1} \|y\|_{L^\infty(\tau^*(x) - \delta, \tau^*(x) + \delta; W^{2,p'})}.
$$

Thus, by Sobolev’s embedding theorem, we have

$$
\|y\|_{L^\infty(\tau^*(x) - \delta, \tau^*(x) + \delta; W^{2,p'})} \lesssim \|y\|_{L^\infty(\tau^*(x) - \delta, \tau^*(x) + \delta; L^p)} \|\nabla y\|_{L^\infty(\tau^*(x) - \delta, \tau^*(x) + \delta; L^{p'})}
$$

In the similar way, we estimate the difference to obtain

$$
\|F(y_1) - F(y_2)\|_{L^\infty(\tau^*(x) - \delta, \tau^*(x) + \delta; H^2)} \lesssim (2\varepsilon)^{\theta} C(T) \|y_1 - y_2\|_{L^\infty(\tau^*(x) - \delta, \tau^*(x) + \delta; W^{2,p})}.
$$

Thus, if $\varepsilon$ is sufficiently small, $F$ is a contraction map. Therefore, it contradicts the definition of the maximal existence time $\tau^*(x)$.

\[ \square \]

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Declarations

Conflicts of interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Data Availability Statements

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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