COVARIANT LIE DERIVATIVES AND FRÖLICHER-NIJENHUIS BRACKET ON LIE ALGEBROIDS

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ABSTRACT: We define covariant Lie derivatives acting on vector-valued forms on Lie algebroids and study their properties. This allows us to obtain a concise formula for the Frölicher-Nijenhuis bracket on Lie algebroids.

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1. Introduction

The Frölicher-Nijenhuis calculus was developed in the seminal article [2] and extended to Lie algebroids in [10]. It has proven to be an indispensable tool of Differential Geometry. Indeed, different kinds of curvatures and obstructions to integrability are computed by the Frölicher-Nijenhuis bracket. For example, if \( J : TM \to TM \) is an almost-complex structure, then \( J \) is complex structure if and only if the Nijenhuis tensor \( N_J = \frac{1}{2} [J, J]_{FN} \) vanishes (this is the celebrated Newlander-Nirenberg theorem [9]). If \( F : TM \to TM \) is a fibrewise diagonalizable endomorphism with real eigenvalues and of constant multiplicity, then the eigenspaces of \( F \) are integrable if and only if \( [F, F]_{FN} = 0 \) (see [4]). Further, if \( P : TE \to TE \) is a projection operator on the tangent spaces of a fibre bundle \( E \to B \), then \( [P, P]_{FN} \) is a version of the Riemann curvature (see [5], page 78). Finally, given a Lie algebroid \( \mathcal{A} \) and \( N \in \Gamma(\mathcal{A}^* \otimes \mathcal{A}) \) such that \( [N, N]_{FN} = 0 \), one can construct a new (deformed) Lie algebroid \( \mathcal{A}_N \) (cf. [3, 6]). Moreover, Frölicher-Nijenhuis calculus is useful in geometric mechanics where it allows to give an intrinsic formulation of Euler-Lagrange equations. In this field, Lie algebroids have also been shown to be a useful tool to deal with systems with some kinds of symmetries.

In [8], P. Michor obtained a short expression for the Frölicher-Nijenhuis bracket on manifolds in terms of the covariant Lie derivatives. A formula for the Frölicher-Nijenhuis bracket on Lie algebroids in supergeometric language was obtained by P. Antunes in [1]. In this paper we define some operators relevant for Frölicher-Nijenhuis calculus in the setting of Lie algebroids, including the covariant Lie derivative, and study their properties. In this way
we are able to extend Michor’s formula for Frölicher-Nijenhuis bracket to Lie algebroids.

2. Covariant Lie derivative on Lie algebroids

Let $(\mathcal{A}, [\cdot , \cdot], \rho)$ be a Lie algebroid over a manifold $M$, and $E$ a vector bundle over $M$. We write $\Omega^k(\mathcal{A}, E) = \Gamma(\wedge^k \mathcal{A}^* \otimes E)$ for the space of skew-symmetric $E$-valued $k$-forms on $\mathcal{A}$. If $E = M \times \mathbb{R}$ is the trivial line bundle over $M$, we denote $\Omega^k(\mathcal{A}, E)$ by $\Omega^k(\mathcal{A})$.

We write $\Sigma_m$ for the permutation group on $\{1, \ldots, m\}$. For $k$ and $s$ such that $k + s = m$, we denote by $\text{Sh}_{k,s}$ the subset of $(k, s)$-shuffles in $\Sigma_m$. Thus $\sigma \in \text{Sh}_{k,s}$ if and only if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k), \quad \sigma(k + 1) < \cdots < \sigma(k + s).$$

Similarly, for a triple $(k, l, s)$, such that $k + l + s = m$, we denote by $\text{Sh}_{k,l,s}$ the subset of $(k, l, s)$-shuffles in $\Sigma_m$, that is the set of permutations $\sigma$, such that

$$\sigma(1) < \sigma(2) < \cdots < \sigma(k), \quad \sigma(k + 1) < \cdots < \sigma(k + l),$$

$$\sigma(k + l + 1) < \cdots < \sigma(k + l + s).$$

For a $k$-form $\omega \in \Omega^k(\mathcal{A})$ and $\phi \in \Omega^p(\mathcal{A}, E)$, we define the form $\omega \wedge \phi \in \Omega^{k+p}(\mathcal{A}, E)$ by

$$(\omega \wedge \phi)(Z_1, \ldots, Z_{p+k}) = \sum_{\sigma \in \text{Sh}_{k,p}} (-1)^{\sigma} \omega(Z_{\sigma(1)}, \ldots, Z_{\sigma(k)}) \phi(Z_{\sigma(k+1)}, \ldots, Z_{\sigma(k+p)}).$$

Here and everywhere in this paper $Z_1, \ldots, Z_{p+k}$ denote arbitrary sections of the Lie algebroid $\mathcal{A}$. If $E = M \times \mathbb{R}$ is the trivial line bundle over $M$, we denote $\wedge$ by $\wedge$, and $\Omega^*(\mathcal{A})$ becomes a commutative graded algebra with the multiplication given by $\wedge$. Further, note that $\Omega^*(\mathcal{A}, E)$ is an $\Omega^*(\mathcal{A})$-module with the action given by $\wedge$. For any $\omega \in \Omega^k(\mathcal{A})$ we define the operator $\epsilon_\omega$ on $\Omega^*(\mathcal{A}, E)$ by

$$\epsilon_\omega : \Omega^*(\mathcal{A}, E) \rightarrow \Omega^{*+k}(\mathcal{A}, E)$$

$$\phi \mapsto \omega \wedge \phi$$

Sometimes, given a operator $A$ we will use $\omega \wedge A$ as an alternative notation for $\epsilon_\omega A$. 
Let $\phi \in \Omega^p(\mathcal{A}, \mathcal{A})$. For any vector bundle $E$ over $M$, we define the operator $i_\phi$ on $\Omega^*(\mathcal{A}, E)$ by

$$(i_\phi \psi) (Z_1, \ldots, Z_{p+k}) = \sum_{\sigma \in \text{Sh}_{p,k}} (-1)^\sigma \psi \left( \phi(Z_{\sigma(1)}, \ldots, Z_{\sigma(p)}), Z_{\sigma(p+1)}, \ldots, Z_{\sigma(p+k)} \right)$$

where $\psi \in \Omega^{k+1}(\mathcal{A}, E)$.

We say that $\nabla : \Gamma(\mathcal{A}) \times \Gamma(E) \to \Gamma(E)$ is an $\mathcal{A}$-connection on $E$ (see [7]) if

1) $\nabla_X$ is an $\mathbb{R}$-linear endomorphism of $\Gamma(E)$;
2) $\nabla s$ is a $\mathcal{C}^\infty(M)$-linear map from $\Gamma(\mathcal{A})$ to $\Gamma(E)$;
3) $\nabla_X(fs) = (\rho(X)f)s + f\nabla_X s$ for any $f \in \mathcal{C}^\infty(M)$, $X \in \Gamma(\mathcal{A})$, and $s \in \Gamma(E)$.

The curvature of an $\mathcal{A}$-connection $\nabla$ is defined by

$$R(X, Y)s := \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X, Y]}s.$$  

It is easy to check that $R$ is tensorial and skew-symmetric in the first two arguments, thus we can consider $R$ as an element of $\Omega^2(\mathcal{A}, \text{End}(E))$, where $\text{End}(E)$ is the endomorphism bundle of $E$.

Given an $\mathcal{A}$-connection on a vector bundle $E$, we define the covariant exterior derivative on $\Omega^*(\mathcal{A}, E)$ by

$$(d^\nabla \phi)(Z_1, \ldots, Z_{p+1}) = \sum_{\sigma \in \text{Sh}_{1,p}} (-1)^\sigma \nabla^E_{Z_{\sigma(1)}} \left( \phi(Z_{\sigma(2)}, \ldots, Z_{\sigma(p+1)}) \right)$$

$$- \sum_{\sigma \in \text{Sh}_{2,p-1}} (-1)^\sigma \phi \left( [Z_{\sigma(1)}, Z_{\sigma(2)}], Z_{\sigma(3)}, \ldots, Z_{\sigma(p+1)} \right).$$

Note that $d^\nabla$ is related to the curvature $R$ of $\nabla^E$ by the formula

$$((d^\nabla)^2 \phi)(Z_1, \ldots, Z_{p+2}) = \sum_{\sigma \in \text{Sh}_{2,p}} (-1)^\sigma R(Z_{\sigma(1)}, Z_{\sigma(2)}) \left( \phi(Z_{\sigma(3)}, \ldots, Z_{\sigma(p+2)}) \right).$$

**Definition 1.** A derivation of degree $k$ on $\Omega^*(\mathcal{A}, E)$ is a linear map $D : \Omega^*(\mathcal{A}, E) \to \Omega^{*-1+k}(\mathcal{A}, E)$ such that

$$D(\omega \wedge \phi) = \overline{D}(\omega) \wedge \phi + (-1)^kp \omega \wedge D(\phi)$$

for all $\omega \in \Omega^p(\mathcal{A})$ and $\phi \in \Omega^*(\mathcal{A}, E)$, where $\overline{D} : \Omega^*(\mathcal{A}) \to \Omega^*(\mathcal{A})$ is some map.
For any derivation $D$ on $\Omega^*(\mathcal{A}, E)$ and $\alpha \in \Omega^*(\mathcal{A})$, we have

$$[D, \epsilon_\alpha] = \epsilon_{D\alpha}.$$ 

In particular, the map $D$ is unique for a given derivation $D$ on $\Omega^*(\mathcal{A}, E)$. Let $\omega_1 \in \Omega^{p_1}(\mathcal{A}), \omega_2 \in \Omega^{p_2}(\mathcal{A})$. From the following computation

$$D((\omega_1 \wedge \omega_2) \triangledown \phi) = D(\omega_1 \wedge \omega_2) \triangledown \phi + (-1)^{k(p_1+p_2)} \omega_1 \wedge \omega_2 \triangledown D(\phi)$$

$$D(\omega_1 \triangledown (\omega_2 \triangledown \phi)) = D(\omega_1) \wedge \omega_2 \triangledown \phi + (-1)^{k_1} \omega_1 \triangledown D(\omega_2 \triangledown \phi)$$

one can see that $\triangledown D$ is a derivation on $\Omega^*(\mathcal{A})$.

It is easy to check that for any given $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$, $i_\phi$ is a derivation of degree $k - 1$, and $d^\triangledown$ is a derivation of degree 1 on $\Omega^*(\mathcal{A}, E)$. The **covariant Lie derivative** $\mathcal{L}_\phi^\triangledown$ is defined as the **graded commutator** $[i_\phi, d^\triangledown] = i_\phi d^\triangledown + (-1)^k d^\triangledown i_\phi$. The graded commutator of two derivations of degree $k$ and $l$ is a derivation of degree $k + l$. In particular, $\mathcal{L}_\phi^\triangledown$ is a derivation of degree $k$ for any $\phi \in \Omega^k(\mathcal{A}, \mathcal{A})$.

Suppose we have an $\mathcal{A}$-connection $\triangledown$ on $\mathcal{A}$. We will say that $\triangledown$ is torsion-free if $\triangledown_X Y - \triangledown_Y X = [X, Y]$ for all $X, Y \in \Gamma(\mathcal{A})$. On every algebroid $(\mathcal{A}, \{\, , \, \}, \rho)$, there exists a torsion-free $\mathcal{A}$-connection. Namely, one can take an arbitrary bundle metric on $\mathcal{A}$ and the associated Levi-Civita connection on $\mathcal{A}$. Given $\mathcal{A}$-connections $\triangledown^\mathcal{A}$ on $\mathcal{A}$ and $\triangledown^E$ on $E$, we define $\triangledown_X s \in \Omega^p(\mathcal{A}, E)$ for every $s \in \Omega^p(\mathcal{A}, E)$ by

$$(\triangledown_X s)(Z_1, \ldots, Z_p) := \triangledown^E_X(s(Z_1, \ldots, Z_p)) - \sum_{i=1}^p s(Z_1, \ldots, \triangledown^A_X Z_i, \ldots, Z_p).$$

It is easy to check that for any $s \in \Omega^k(\mathcal{A}, E)$, $X \in \Gamma(\mathcal{A})$, and a torsion-free $\mathcal{A}$-connection on $\mathcal{A}$, we have $\mathcal{L}_X^\mathcal{A} s = \triangledown_X s + i_{\triangledown_X s}$ and $\triangledown X = d^\triangledown X$. In other words $\triangledown_X = \mathcal{L}_X^\mathcal{A} - i_{d^\triangledown X}$. Motivated by this relation, we define for $\phi \in \Omega^p(\mathcal{A}, \mathcal{A})$ an operator $\triangledown_\phi$ on $\Omega^*(\mathcal{A}, E)$ by

$$\triangledown_\phi := \mathcal{L}_\phi^\mathcal{A} - (-1)^p i_{d^\triangledown \phi}. \quad (2)$$

Note that $\triangledown_\phi$ depends on two connections: an $\mathcal{A}$-connection on $E$ and a torsion-free $\mathcal{A}$-connection on $\mathcal{A}$. Since $\triangledown_\phi$ is a linear combination of two derivations of degree $p$, we see that $\triangledown_\phi$ is a derivation of degree $p$. The following proposition shows that for $s \in \Omega^*(\mathcal{A}, E)$ the map $\triangledown s : \Omega^*(\mathcal{A}, \mathcal{A}) \to \Omega^* (\mathcal{A}, E)$ is a homomorphism of $\Omega^*(\mathcal{A})$-modules.
Proposition 2. For any $\omega \in \Omega^p(A), \phi \in \Omega^k(A,A)$, and $s \in \Omega^*(A,E)$, we have

$$\nabla_{\omega \wedge \phi} s = (\omega \wedge \nabla \phi) s = \epsilon_\omega \nabla \phi s = \omega \nabla(\nabla \phi s).$$

Proof: The equation

$$L_{\omega \wedge \phi} = \nabla_{\omega \wedge \phi} s = (\omega \wedge \nabla \phi) s = \epsilon_\omega \nabla \phi s = \omega \wedge (\nabla \phi s)$$

implies that $\omega \wedge L_{\nabla \phi} = L_{\omega \wedge \phi} - (\nabla - 1)^{k+p} \omega \wedge \nabla \phi$. Now we have

$$\omega \wedge (\nabla \phi) = \omega \wedge L_{\nabla \phi} - (\nabla - 1)^{p+k} \omega \wedge \nabla \phi$$

Thus $[\nabla \phi, i_\psi] = i_{\nabla \phi \psi} - (\nabla - 1)^{k(l-1)} \nabla i_{\phi \psi}$. (3)

Theorem 3. Let $\nabla$ be a torsion-free $A$-connection on $A$ and $\nabla^E$ be an $A$-connection on a vector bundle $E$. For $\phi \in \Omega^k(A,A)$ and $\psi \in \Omega^l(A,A)$ we have on $\Omega^*(A,E)$

$$[\nabla \phi, i_\psi] = i_{\nabla \phi \psi} - (\nabla - 1)^{k(l-1)} \nabla i_{\phi \psi}.$$  (4)

Proof: First we check the claim for $\phi = X \in \Gamma(A)$ and $\psi = Y \in \Gamma(A)$. Let $s \in \Omega^{p+1}(A,E)$. We get

$$(\nabla_X i_Y s)(Z_1, \ldots, Z_p) = \nabla^E_X (s(Y, Z_1, \ldots, Z_p)) - \sum_{t=1}^p s(Y, Z_1, \ldots, \nabla_X Z_t, \ldots, Z_p)$$

$$= (\nabla_X s)(Y, Z_1, \ldots, Z_p) + \epsilon s(\nabla_X Y, Z_1, \ldots, Z_p)$$

$$= (i_Y \nabla_X s)(Z_1, \ldots, Z_p) + (i_{\nabla_X Y} s)(Z_1, \ldots, Z_p).$$

Thus $[\nabla_X, i_Y] = i_{\nabla_X Y}$. Since (4) is additive in $\phi$ and $\psi$, it is enough to prove it for $\phi = \alpha A, \psi = \beta A$, where $\alpha \in \Omega^k(A)$, $\beta \in \Omega^l(A)$, and $X, Y \in \Gamma(A)$. 
Repeatedly using Proposition 2 and \([\nabla_X, i_Y] = i_Y \nabla_X\), we get
\[
[\nabla_{\alpha \pi X}, i_{\beta \pi Y}] = [\alpha \wedge \nabla_X, \beta \wedge i_Y] = [\epsilon_\alpha, \beta \wedge i_Y] \nabla_X + \epsilon_\alpha [\nabla_X, \beta \wedge i_Y] \\
= (-1)^{kl} \epsilon_\beta [\epsilon_\alpha, i_Y] \nabla_X + \epsilon_\alpha [\nabla_X, \epsilon_\beta] i_Y + \epsilon_\alpha \epsilon_\beta [\nabla_X, i_Y] \\
= (-1)^{kl-1} \epsilon_\beta \epsilon_\alpha i_Y \nabla_X + \epsilon_\alpha \epsilon_\beta \nabla_X i_Y + \epsilon_\alpha \epsilon_\beta i_Y \nabla_X \\
= i_\alpha \nabla_X \beta \pi Y + \alpha \wedge \beta \pi_X \nabla_X Y + (-1)^{(k-1)l} \nabla_{\beta \wedge i_Y \alpha \pi X} \\
= i_\alpha \nabla_X (\beta \pi Y) + (-1)^{(k-1)l} \nabla_{\beta \wedge i_Y \alpha \pi X} \\
= i_\alpha \nabla_X (\beta \pi Y) + (-1)^{(k-1)l} \nabla_{i_\beta \pi Y \alpha \pi X}.
\]

To formulate the next result, we extend the definition of \(R\) by defining for any \(\phi \in \Omega^k(A, A)\) and \(\psi \in \Omega^l(A, A)\) the form \(R(\phi, \psi) \in \Omega^{k+l+1}(A, A)\) as follows
\[
R(\phi, \psi)(Y_1, \ldots, Y_{k+l+1}) = \\
= \sum_{\sigma \in \text{Sh}_{k,l,1}} R(\phi(Y_{\sigma(1)}, \ldots, Y_{\sigma(p)}), \psi(Y_{\sigma(p+1)}, \ldots, Y_{\sigma(p+q)})) Y_{\sigma(p+q+1)}.
\]

**Theorem 4.** Let \(\nabla\) be a torsion-free \(A\)-connection on \(A\) and \(\nabla^E\) a flat \(A\)-connection on a vector bundle \(E\) over \(M\) (i.e. \(\nabla^E\) is a representation of \(A\)). Then for any \(\phi \in \Omega^k(A, A), \psi \in \Omega^l(A, A)\), we have the following equality on \(\Omega^*(A, E)\)
\[
[\nabla_\phi, \nabla_\psi] = \nabla_{\nabla_\phi \psi} - (-1)^{kl} \nabla_{\psi \phi} - i_{R(\phi, \psi)}.
\]

**Proof:** First we prove (5) for \(\phi = X, \psi = Y \in \Gamma(A)\). For \(s \in \Omega^p(A)\), we get
\[
(\nabla_X \nabla_Y s)(Z_1, \ldots, Z_p) = \nabla_X^E(\nabla_Y^E s(Z_1, \ldots, Z_p)) - \sum_{s=1}^p \nabla_Y^E s(Z_1, \ldots, \nabla_X Z_s, \ldots, Z_p) \\
= \nabla_X^E \nabla_Y^E (s(Z_1, \ldots, Z_p)) - \sum_{s=1}^p \nabla_X^E (s(Z_1, \ldots, \nabla_Y Z_s, \ldots, Z_p)) \\
- \sum_{s=1}^p \nabla_Y^E (s(Z_1, \ldots, \nabla_X Z_s, \ldots, Z_p)) + \sum_{s=1}^p s(Z_1, \ldots, \nabla_Y \nabla_X Z_s, \ldots, Z_p) \\
+ \sum_{s \neq t} s(Z_1, \ldots, \nabla_Y Z_t, \ldots, \nabla_X Z_s, \ldots, Z_p).
\]
By anti-symmetrization of the above formula in \( X \) and \( Y \) and using that \( \nabla^E \) is flat, we get

\[
[\nabla_X, \nabla_Y] s(Z_1, \ldots, Z_p) = \nabla^E_{[X,Y]}(s(Z_1, \ldots, Z_p)) - \sum_{s=1}^{p} s(Z_1, \ldots, [\nabla_X, \nabla_Y] Z_s, \ldots, Z_p).
\]

Further

\[
(\nabla_{\nabla_X Y} - \nabla_{\nabla_Y X}) s(Z_1, \ldots, Z_p) = \nabla^E_{\nabla_X Y - \nabla_{\nabla_Y X}}(s(Z_1, \ldots, Z_p))
- \sum_{s=1}^{p} s(Z_1, \ldots, (\nabla_{\nabla_X Y} - \nabla_{\nabla_Y X}) Z_s, \ldots, Z_p).
\]

Taking the difference of the last two formulas and using the definition of \( R \) and that \( \nabla \) torsion-free, we have

\[
(([\nabla_X, \nabla_Y] - \nabla_{\nabla_X Y} + \nabla_{\nabla_Y X}) s)(Z_1, \ldots, Z_p) = (-iR(X,Y)s)(Z_1, \ldots, Z_p).
\]

Since (5) is additive in \( \phi \) and \( \psi \), it is enough to prove it for \( \phi = \alpha \wedge X \) and \( \psi = \beta \wedge Y \), where \( \alpha \in \Omega^k(A) \), \( \beta \in \Omega^l(A) \), and \( X, Y \in \Gamma(A) \). Using the already proved case and Proposition 2, we get

\[
[\nabla_{\alpha \wedge X}, \nabla_{\beta \wedge Y}] = [\alpha \wedge \nabla_X, \beta \wedge \nabla_Y] = [\epsilon_{\alpha}, \beta \wedge \nabla_Y] \nabla_X + \epsilon_{\alpha}[\nabla_X, \beta \wedge \nabla_Y]
= (-1)^{kl} \epsilon_{\beta}[\epsilon_{\alpha}, \nabla_Y] \nabla_X + \epsilon_{\alpha}[\nabla_X, \epsilon_{\beta}] \nabla_Y + \epsilon_{\alpha} \epsilon_{\beta}[\nabla_X, \nabla_Y]
= -(-1)^{kl} \epsilon_{\beta} \epsilon_{\nabla_Y} \alpha \wedge \nabla_X + \epsilon_{\alpha} \epsilon_{\nabla_X} \beta \wedge \nabla_Y + \epsilon_{\alpha} \epsilon_{\beta}(\nabla_{\nabla_X Y} - \nabla_{\nabla_Y X} - iR(X,Y)).
\]

Repeatedly using Proposition 2, we see that \([\nabla_{\alpha \wedge X}, \nabla_{\beta \wedge Y}]\) can be written as \( \nabla_{\theta} + i\tau \), where

\[
\theta = -(-1)^{kl} \beta \wedge \nabla_Y \alpha \wedge X + \alpha \wedge \nabla_X \beta \wedge Y + \alpha \wedge \beta \wedge \nabla_X Y - \alpha \wedge \beta \wedge \nabla_Y X
= \alpha \wedge \nabla_X (\beta \wedge Y) - (-1)^{kl} (\beta \wedge \nabla_Y (\alpha \wedge X)) = \nabla_{\phi} \psi - (-1)^{kl} \nabla_{\psi} \phi
\]
and

\[
\tau = -\alpha \wedge \beta \wedge R(X, Y) = -R(\alpha \wedge X, \beta \wedge Y) = -R(\phi, \psi).
\]

This finishes the proof. \( \blacksquare \)

Note that the connection \( \nabla^\rho_X f := \rho(X)f \) defined on the trivial line bundle \( M \times \mathbb{R} \to M \) is obviously flat. Thus (5) holds on \( \Omega^*(A) \), if \( \nabla \) is defined via \( \nabla^\rho \) and any torsion-free connection on \( A \).
3. The Frölicher-Nijenhuis bracket on Lie algebroids

In [10], Nijenhuis defined the Frölicher-Nijenhuis bracket on Lie algebroids of \( \phi \in \Omega^k(A, A) \) and \( \psi \in \Omega^l(A, A) \) by an equality of operators on \( \Omega^*(A) \) equivalent to

\[
[L^\nabla_{\phi}, i_\psi] = i_{[\phi, \psi]_{FN}} - (-1)^{k(l-1)}L^\nabla_{i_{\psi}\phi}.
\]  

(6)

He also obtained a formula for computing \( [\phi, \psi]_{FN} \). In the next theorem we give an alternative formula using the covariant Lie derivatives, which extends the one obtained in [8] to the Lie algebroids setting.

**Theorem 5.** Let \( \phi \in \Omega^k(A, A) \) and \( \psi \in \Omega^l(A, A) \). Suppose \( \nabla \) be a torsion-free \( A \)-connection on \( A \). Then

\[
[\phi, \psi]_{FN} = L^\nabla_{\phi} \psi - (-1)^{kl}L^\nabla_{\psi} \phi.
\]

**Proof:** By (2) we have

\[
[L^\nabla_{\phi}, i_\psi] = [\nabla_\phi + (-1)^k i_{d\nabla_\phi}, i_\psi] = [\nabla_\phi, i_\psi] + (-1)^k [i_{d\nabla_\phi}, i_\psi].
\]

Hence, using (3) and (4) we get

\[
[L^\nabla_{\phi}, i_\psi] = i_{\nabla_\phi \psi} - (-1)^{k(l-1)}\nabla_{i_{\phi}\psi} + (-1)^k i_{d\nabla_\phi \psi} - (-1)^{kl} i_{i_{\psi}d\nabla_\phi}.
\]

Next, using (2) in the second summand we have

\[
[L^\nabla_{\phi}, i_\psi] = -(-1)^{k(l-1)} \left(L^\nabla_{i_{\phi}\psi} - (-1)^{k+l-1} i_{d\nabla_{i_{\phi}\psi}}\right) + i_{\nabla_{i_{\phi}} \psi} + (-1)^k i_{i_{d\nabla_\phi} \psi} - (-1)^{kl} i_{i_{\psi}d\nabla_\phi}.
\]

Notice that the subscripts of \( L^\nabla \) in (6) and in the above formula are the same. Hence, due to the injectivity of \( \phi \mapsto i_\phi \), we get by comparing the subscripts of \( i \) that

\[
[\phi, \psi]_{FN} = (-1)^{k(l-1)}(-1)^{k+l-1}d\nabla_{i_{\phi}\psi} + \nabla_{i_{\phi}} \psi + (-1)^k i_{d\nabla_\phi} \psi - (-1)^{kl} i_{\psi}d\nabla_\phi = \nabla_{i_{\phi}} \psi + (-1)^k i_{d\nabla_\phi} \psi - (-1)^{kl} (i_{\psi}d\nabla_\phi - (-1)^{l-1}d\nabla_{i_{\phi}} \psi)
\]

Finally, using the definitions of \( \nabla_{i_{\phi}} \) and of \( L^\nabla_{\psi} \) we get the claimed result. ■
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