Abstract. We find good dynamical compactifications for arbitrary polynomial mappings of $\mathbb{C}^2$ and use them to show that the degree growth sequence satisfies a linear integral recursion formula. For maps of low topological degree we prove that the Green function is well behaved. For maps of maximum topological degree, we give normal forms.

Introduction

The theory of iteration of rational maps on complex projective varieties has recently seen the introduction of new analytic techniques for constructing invariant currents and measures of dynamical interest, through the work of Bedford-Diller [1], de Thélin-Vigny [13], Diller-Dujardin-Guedj [14, 15, 16, 17], Dinh-Sibony [19, 21, 23], Dujardin [25, 26], Guedj [36, 38], and others. These constructions, however, often require a good birational model in which the dynamical indeterminacy set has a relatively small size, so that the action on cohomology of the rational map is compatible with iteration.

Diller-Favre [14] proved the existence of such models for birational surface maps using the decomposition into blow-ups and blow-downs. There are no other general results, for two reasons. First, it is a delicate task to control a rational map near its indeterminacy set. Second, the indeterminacy set of a non-invertible map tends to grow very rapidly under iteration.

In this paper we prove the existence of good birational models for an important class of rational surface maps, namely polynomial maps.

**Theorem A.** Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be any polynomial mapping. Then there exists a projective compactification $X \supset \mathbb{C}^2$ with at worst quotient singularities and an integer $n \geq 1$ such that the lift $\tilde{F} : X \to X$ satisfies $\tilde{F}^{(j+n)*} = (\tilde{F}^*)^j \tilde{F}^n*$ on the Picard group $\text{Pic}(X)$ for all $j \geq 1$.

Fix an algebraic embedding $\mathbb{C}^2 \subset \mathbb{P}^2$. Define $\text{deg}(F)$ by the relation $F^* \mathcal{L} = \deg(F) \mathcal{L}$ for $\mathcal{L}$ a generator of $\text{Pic}(\mathbb{P}^2)$. Using Theorem A we prove

**Theorem B.** For any polynomial mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$, the sequence $(\deg(F^j))_{j \geq 0}$ satisfies an integral linear recursion formula.

This gives a positive answer to the main conjecture of Bellon-Viallet [6, §5] in our setting. Example [4,5] shows that there may not be a recursion of order one or two, even though the asymptotic degree $\lambda_1$ (see below) is always a quadratic integer [30]. In general, the degree growth of rational maps of projective space remains mysterious,
Despite recent works [2, 3, 4, 7, 39, 45], Hasselblatt and Propp [39] give examples of rational maps of \( \mathbb{C}^2 \) for which the degree growth does not satisfy any linear recursion formula. From this perspective, Theorem B is quite remarkable.

In many cases—and always after replacing \( F \) by \( F^2 \)—\( X \) is smooth and can be obtained from \( \mathbb{P}^2 \) after finitely many blowups at infinity. In these cases, Theorem B follows immediately from Theorem A.

Note that the compactifications that we consider here are different in nature from the ones studied by Hubbard, Papadopol and Veselov [42].

The conclusion in Theorem A is slightly weaker than the condition \( (\tilde{F}^j)^* = (\tilde{F}^*)^j \) on \( \text{Pic}(X) \) for all \( j \); the latter is often referred to as algebraic stability [17]. Its importance was first recognized by Fornæss and Sibony [32]. Guedj has conjectured [36, Remark 3.1] than any polynomial mapping of \( \mathbb{C}^2 \) is algebraically stable on some (smooth) compactification \( X \) of \( \mathbb{C}^2 \). While we suspect this may be too much to ask for, see Remark 6.1 Theorem A shows that the conjecture holds after replacing \( F \) by an iterate. In any case, Theorem A is sufficient for all known applications.

The proof of Theorem A is based on the valuative techniques developed in [30].

These give a framework for studying the dynamics induced by \( F \) on the set of divisors at infinity in all compactifications of \( \mathbb{C}^2 \). This set of divisors can be identified with a dense subset of a metrized \( \mathbb{R} \)-tree \( \mathcal{V}_0 \) consisting of all valuations on \( \mathbb{C}[x,y] \), centered at infinity and suitably normalized. There are however two difficulties in working directly with \( \mathcal{V}_0 \). First, a valuation in \( \mathcal{V}_0 \) is a local object, whereas we are interested in global properties of \( F \). Second, \( F \) might be not proper, and in this case it does not preserve \( \mathcal{V}_0 \). To remedy these problems, we introduced in [30] a subtree \( \mathcal{V}_1 \) of \( \mathcal{V}_0 \) consisting of valuations close enough to \(-\text{deg}\), see [15] for a formal definition. This subtree is a fundamental technical tool in our analysis. In op. cit., we showed that valuations in \( \mathcal{V}_1 \) still capture global information, that \( F \) induces a continuous map on \( \mathcal{V}_1 \), and we proved the existence of a locally attracting valuation \( \nu_* \in \mathcal{V}_1 \) that we called an eigenvaluation. Theorem A is a consequence of a detailed study of the global contracting properties of \( F \) on \( \mathcal{V}_1 \).

Denote by \( \lambda_1 := \lim_{n \to \infty} \text{deg}(F^n)^{1/n} \) the asymptotic degree of \( F \) [46] and by \( \lambda_2 \) the topological degree of \( F \). These degrees are invariant under conjugacy by polynomial automorphisms and satisfy \( \lambda_2 \leq \lambda_1^2 \). In the case \( \lambda_2 < \lambda_1^2 \), the Hilbert space methods of Boucksom and the authors [8] apply (see also the work of Hubbard-Papadopol [41], Cantat [11] and Manin [43]). We showed in [8] that \( \text{deg}(F^n) \sim \lambda_1^n \). Here we use these techniques to prove that \( \nu_* \) attracts all valuations in \( \mathcal{V}_1 \) (with at most one exception).

When \( \lambda_2 = \lambda_1^2 \), the Hilbert space technique loses some strength. However, this loss is compensated by the built-in rigidity of these maps. A more detailed study of the valuative dynamics allows us to show

**Theorem C.** Let \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) be a dominant polynomial mapping with \( \lambda_2 = \lambda_1^2 \). Then we are in one of the following mutually exclusive cases:

1. \( \text{deg}(F^n) \sim n\lambda_1^n \); then \( \lambda_1 \in \mathbb{N} \) and in suitable affine coordinates, \( F \) is a skew product of the form \( F(x,y) = (P(x), Q(x,y)) \), where \( \text{deg}(P) = \lambda_1 \) and \( Q(x,y) = A(x)y^{\lambda_1} + O(y^{\lambda_1-1}) \) with \( \text{deg}(A) \geq 1 \);
2. \( \text{deg}(F^n) \sim \lambda_1^n \); then there exists a projective compactification \( X \supset \mathbb{C}^2 \) with at most quotient singularities such that \( F \) extends to a holomorphic selfmap of \( X \).
Here, and throughout the paper, the expression “in suitable affine coordinates” means that the statement holds after conjugation by a polynomial automorphism of $\mathbb{C}^2$.

In suitable affine coordinates, $X$ can be chosen as a toric surface and we can give normal forms for all maps occurring in (2), see Section 5.3. When $\lambda_1$ is an integer, $X$ can be chosen as a weighted projective plane.

Theorem C extends the Friedland-Milnor classification [33] of polynomial automorphisms with $\lambda_1 = 1$. For automorphisms, only case (2) appears, and $X$ can be chosen as $\mathbb{P}^2$ or a Hirzebruch surface. There is an analogous statement in the general birational surface case, see [14, 35]: $\deg(F^n)$ is then either bounded or grows linearly or quadratically.

Our last result is another application of the dynamical compactifications. One of the basic problems in the iteration of rational maps is the construction and study of an ergodic measure of maximal entropy. When $\lambda_2 > \lambda_1$, such a measure can be defined as a limit of preimages of a generic point [32, 46, 47], and its basic ergodic properties are completely understood, see [9, 19, 22, 38]. In the case of maps with small topological degree $\lambda_2 < \lambda_1$, this construction fails. A different strategy has been proposed for constructing a dynamically interesting invariant measure, see [36]. One first constructs two positive closed $(1, 1)$ currents, invariant by pull-back and by push-forward, respectively. The measure is then obtained by taking their intersection. In our setting, the existence of these two currents follows from Theorem A, see [15]. The existence of their intersection, however, is not guaranteed in general except if one has a good control of the singularities of their potentials. We prove such a control for the pull-back invariant current.

Fix affine coordinates $(x, y)$ on $\mathbb{C}^2$ and set $\|(x, y)\| = \max\{1, |x|, |y|\}$.

**Theorem D.** Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a dominant polynomial mapping with $\lambda_2 < \lambda_1$. Then the limit

$$G^+(p) = \lim_{n \to \infty} \lambda_1^{-n} \log^+ \|F^n p\|$$

exists locally uniformly on $\mathbb{C}^2$, and defines a continuous, nonzero, non-negative plurisubharmonic function of logarithmic growth satisfying $G^+ \circ F = \lambda_1 G^+$. The support of the positive closed current $dd^c G^+$ is equal to $\partial K^+$ where $K^+ = \{G^+ = 0\}$. Further, for each $\varepsilon > 0$, there exists a constant $C > 0$, such that

$$\log^+ \|F^n p\| \leq (\lambda_2 + \varepsilon)^n (\log^+ \|p\| + C)$$

for all $n \geq 0$, and all $p \in K^+$.

Theorem D generalizes classical properties of the Green function both of polynomials in one variable and of Hénon maps [5, 31, 40]. On the locus $\{G^+ = 0\}$ the dynamics can exhibit various speeds of convergence towards infinity, see [38]. Note that (*) is in sharp contrast with the phenomenon described in [48].

The paper is organized as follows. In Sections 1 and 2 we discuss the relationship between compactifications and valuations, and study the induced dynamics on the space of valuations at infinity. We then turn to the proof of refined versions of Theorems A and B in the case $\lambda_2 < \lambda_1^2$: in Section 3 when the eigenvaluation is nondivisorial and in Section 4 when it is divisorial. Polynomial maps with $\lambda_2 = \lambda_1^2$ are handled in Section 5 where we prove Theorem C. Theorems A and B are proved in Section 6. Theorem D in
Section 7.1 where we also provide a list of examples of maps with \( \lambda_2 = \lambda_1 \). The paper ends with a short appendix outlining an adaptation of the necessary material from §8 to our setting.

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1. Geometry at infinity

We start by discussing compactifications of \( \mathbb{C}^2 \) together with valuations centered at infinity.

1.1. Admissible compactifications. We consider \( \mathbb{C}^2 \) equipped with a fixed embedding into \( \mathbb{P}^2 \).

Definition 1.1. An admissible compactification of \( \mathbb{C}^2 \) is a smooth projective surface \( X \) admitting a birational morphism \( \pi : X \to \mathbb{P}^2 \) that is an isomorphism above \( \mathbb{C}^2 \).

It follows that \( \pi \) is a composition of point blowups and that \( X \setminus \mathbb{C}^2 \) is a connected curve with simple normal crossings. The primes of \( X \) are the irreducible components of \( X \setminus \mathbb{C}^2 \). Every \( X \) contains a special prime \( L_{\infty} \), the strict transform of the line at infinity in \( \mathbb{P}^2 \).

1.2. Valuations [10, Appendix A]. Let \( R \) be the coordinate ring of \( \mathbb{C}^2 \). We define \( \mathcal{V}_0 \) as the set of valuations \( \nu : R \to (−\infty, +\infty] \) centered at infinity, i.e. \( \nu(L) < 0 \) for a generic affine function \( L \) on \( \mathbb{C}^2 \). If \( \nu \in \mathcal{V}_0 \) and \( X \) is an admissible compactification of \( \mathbb{C}^2 \), then the center of \( \nu \) on \( X \) is the unique scheme-theoretic point on \( X \) such that \( \nu \) is strictly positive on the maximal ideal of its local ring. Thus the center is either a prime of \( X \) or a point on \( X \setminus \mathbb{C}^2 \). We let \( \mathcal{V}_0 \) be the subset of \( \nu \in \mathcal{V}_0 \) that are normalized by \( \nu(L) = −1 \).

There are four kinds of valuations in \( \mathcal{V}_0 \) that we now describe.

First, if \( X \) is an admissible compactification of \( \mathbb{C}^2 \), then each prime \( E \) of \( X \) defines a divisorial valuation \( \text{ord}_E \in \mathcal{V}_0 \), the order of vanishing along \( E \). In particular, \( \text{ord}_{L_{\infty}} = −\deg \). Any valuation proportional to some \( \text{ord}_E \) will also be called divisorial. Thus a valuation is nondivisorial iff its center on every admissible compactification is a point. If we set \( b_E = −\text{ord}_E(L) \) for a generic affine function \( L \), then \( \nu_E := b_E^{-1} \text{ord}_E \) is normalized.

We denote by \( \mathcal{V}_{\text{div}} \) and \( \mathcal{V}_{\text{div}} \) the set of divisorial valuations in \( \mathcal{V}_0 \) and \( \mathcal{V}_0 \), respectively.

Second, we have irrational valuations. To define them, consider any two primes \( E, E' \) in \( X \) intersecting at a point \( p \) and local coordinates \((z, w)\) at \( p \) such that \( E = \{z = 0\} \) and \( E' = \{w = 0\} \). To any pair \((s, t) \in \mathbb{R}^2_+\) we attach the valuation \( \nu \) defined on the ring \( \mathcal{O}_p \) of holomorphic germs at \( p \) by \( \nu(\sum a_{ij}z^i w^j) = \min\{si + tj \mid a_{ij} \neq 0\} \); it does not depend on the choice of coordinates \((z, w)\). By first extending \( \nu \) to the common fraction field \( \mathbb{C}(X) \) of \( \mathcal{O}_p \) and \( R \), then restricting to \( R \), we obtain a valuation in \( \mathcal{V}_0 \), called quasimonomial. (It is monomial in the local coordinates \((z, w)\) at \( p \).) The valuation \( \nu \) is normalized iff \( sb_E + tb_{E'} = 1 \). It is divisorial iff either \( t = 0 \) or the ratio \( s/t \) is a rational number. Any divisorial valuation is quasimonomial. An irrational valuation is by definition a nondivisorial quasimonomial valuation.
Third, pick a point \( p \) on the line at infinity \( L_\infty \subset \mathbb{P}^2 \), a formal irreducible curve \( C \) at \( p \), not contained in \( L_\infty \), and a constant \( \gamma > 0 \). Then \( P \mapsto \gamma^{-1} \text{ord}_p(P|_C) \), for polynomials \( P \in R \), defines a valuation in \( \mathcal{V}_0 \) called a \emph{curve valuation}. It is normalized iff \( \gamma \) is chosen as the intersection number \((C \cdot L_\infty)\). Note that a curve valuation can take the value \(+\infty\) when the curve is algebraic.

In suitable affine coordinates \((x, y)\), a curve valuation can be computed using a Puiseux parameterization \( y = h(x^{-1}) \) of the curve: the value on a polynomial \( P \) is proportional to \( \text{ord}_{x=\infty} P(x, h(x^{-1})) \). Now replace the Puiseux series \( h \) by a formal series of the form \( h(\zeta) = \sum a_k\zeta^{\beta_k} \) with \( a_k \in \mathbb{C}^* \), and \( \beta_k \) an increasing sequence of rational numbers with unbounded denominators. Then \( P \mapsto \text{ord}_{x=\infty} P(x, h(x^{-1})) \) defines a valuation of the fourth and last kind, namely an infinitely singular valuation.

1.3. Tree structure \[30\] Appendix A. The space \( \mathcal{V}_0 \) of normalized valuations is equipped with a partial ordering: \( \nu \leq \mu \iff \nu(P) \leq \mu(P) \) for all \( P \in R \), naturally turning it into a rooted tree. In particular, every two elements \( \mu, \nu \in \mathcal{V}_0 \) admit a minimum \( \mu \wedge \nu \in \mathcal{V}_0 \). The valuation \( \deg \) is the minimal element of \( \mathcal{V}_0 \). For any admissible compactification \( X \), one can map a prime \( E \) to the normalized valuation \( \nu_E := b_E^{-1} \text{ord}_E \).

In this way, we get an embedding of the set of primes of \( X \) into \( \mathcal{V}_0 \). The partial ordering coming from \( \mathcal{V}_0 \) on the set of primes coincides with the one coming from the tree structure of the dual graph of \( X \setminus C^2 \).

The \emph{ends} of \( \mathcal{V}_0 \), i.e. the maximal elements in the partial ordering, are the curve and infinitely singular valuations, i.e. the valuations that are not quasimonomial.

We topologize \( \mathcal{V}_0 \) by declaring \( \nu_n \to \nu \iff \nu_n(P) \to \nu(P) \) for all \( P \). This topology is compact and admits two important characterizations. First, it is the weakest topology such that the natural retraction map \( r_I : \mathcal{V}_0 \to I \) is continuous for any given closed segment \( I \subset \mathcal{V}_0 \). Second, given any admissible compactification \( X \) and any point \( p \in X \setminus C^2 \), let \( U(p) \subset \mathcal{V}_0 \) be the set of valuations whose center on \( X \) is \( p \). Then the topology on \( \mathcal{V}_0 \) is the weakest one in which \( U(p) \subset \mathcal{V}_0 \) is always open. Each of the four subsets of divisorial, irrational, curve and infinitely singular valuations is dense in \( \mathcal{V}_0 \).

There is a unique, decreasing, upper semicontinuous \emph{skewness} function \( \alpha : \mathcal{V}_0 \to [\infty, 1] \) satisfying \( \alpha(\deg) = 1 \) and \( [\alpha(\nu_E) - \alpha(\nu_{E'})] = (b_E b_{E'})^{-1} \) whenever \( E \) and \( E' \) are intersecting primes in some admissible compactification \( X \). See \[30\] §A.1 and \[29\] §6.6.3, §6.8. One has \( b_E^2 \alpha(\nu_E) \in \mathbb{Z} \), see Lemma A.2.

Similarly, there is a unique, increasing, lower semicontinuous \emph{thinness} function \( A : \mathcal{V}_0 \to [-2, \infty] \) such that \( A(\nu_E) = a_E/b_E \), where \( a_E = 1 + \text{ord}_E(dx \wedge dy) \) and \( b_E = -\text{ord}_E(L) \) as above, see \[30\] §A.1. An irrational valuation has irrational skewness and thinness.

1.4. The subtree \( \mathcal{V}_1 \). Define \( \mathcal{V}_1 \) as the set of valuations \( \nu \in \mathcal{V}_0 \) with skewness \( \alpha(\nu) \geq 0 \) and thinness \( A(\nu) \leq 0 \). Then \( \mathcal{V}_1 \) is a subtree of \( \mathcal{V}_0 \) of crucial importance to our study. Its properties are spelled out in detail in \[30\] Theorem A.7. Here we note the fundamental fact that any quasimonomial valuation in \( \mathcal{V}_1 \) is dominated by a \emph{pencil valuation}.

To define the latter, consider an affine curve \( C = \{ P = 0 \} \subset C^2 \) with \emph{one place at infinity}, that is, the closure of \( C \) in \( \mathbb{P}^2 \) is an irreducible curve intersecting the line at infinity in a single point and is locally irreducible there. Consider the pencil \( \{ C \} \) consisting of the affine curves \( C_\lambda = \{ P = \lambda \} \subset C^2 \) for \( \lambda \in \mathbb{C} \). It is a theorem by Moh (see \[44\], \[10\]) that \( C_\lambda \) has one place at infinity for every \( \lambda \in \mathbb{C} \). The (normalized) pencil valuation
Lemma 1.4 we only have to show that $\forall k > m$ of $C$ with the curve $\{Q = \rho\}$ for generic $\rho \in C$. See [31, §A.2] for more details.

This crucial fact that any quasimonomial valuation in $V_1$ is dominated by a pencil valuation is the affine analog of the (easier) local result that any quasimonomial valuation in the valuative tree is dominated by a curve valuation, see [28, Proposition 3.20].

A pencil valuation $\nu_{C|}$ is divisorial but does not itself belong to $V_1$ in general. Indeed, we have $\alpha(\nu_{C|}) = 0$ and $A(\nu_{C|}) = (2g - 1)/\deg(C)$, where $g$ is the geometric genus of the closure of a generic curve $C_\lambda$ in the pencil. Thus $\nu_{C|}$ belongs to $V_1$ iff $C_\lambda \simeq C$ for generic $\lambda$. We then call $\nu_{C|}$ rational pencil valuation.

It follows that if $\nu$ is a quasimonomial valuation in $V_1$, then either $\alpha(\nu) > 0$ or $\nu$ is a rational pencil valuation.

1.5. Tight compactifications. Associated to the subtree $V_1$ is an important class of compactifications of $C^2$.

Definition 1.2. An admissible compactification $X$ of $C^2$ is tight if the normalized divisorial valuations associated to the primes of $X$ belong to the subtree $V_1$ of $V_0$.

Remark 1.3. An admissible compactification $X$ is tight iff $\text{ord}_E(dx \wedge dy) < 0$ and $\text{ord}_E(P) \leq 0$ for any prime $E$ of $X$ and any polynomial $P \in C[x,y]$. The second condition is equivalent to $Z_{\text{ord}_E}$ being nef, see [A.2] this implies that the nef and psef cones of an admissible compactification $X$ of $C^2$ are simplicial whenever $Z_{\text{ord}_E}$ is nef for every prime $E$ of $X$. A compactification of $C^2$ associated to a curve with one place at infinity as defined in [10] always has the latter property and is tight iff the curve is rational.

Lemma 1.4. Let $X$ be a tight compactification of $C^2$. Pick a point $p \in X \setminus C^2$ and let $X'$ be the admissible compactification of $C^2$ obtained by blowing up $p$. Then $X'$ is tight iff $p$ does not lie on a unique prime of $X$, whose associated normalized divisorial valuation has skewness zero or thinness zero.

Proof. Let $\nu \in V_0$ be the divisorial valuation associated to the blowup of $p$. Then $X'$ is tight iff $\nu \in V_1$. First assume $p$ is the intersection point between two primes $E_1, E_2$ of $X$ with associated divisorial valuations $\nu_1, \nu_2 \in V_0$. Then $\nu$ lies in the segment between $\nu_1$ and $\nu_2$. Since $\nu_2 \in V_1$ and $V_1$ is a subtree, we get $\nu \in V_1$, so that $X'$ is tight.

Now assume $p$ lies on a single prime $E$ of $X$, with associated divisorial valuation $\nu_E \in V_1$. In this case, $\nu > \nu_E$. Let $b_E = -\text{ord}_E(L)$ as above. Then $A(\nu) - A(\nu_E) = 1/b_E$ and $\alpha(\nu) - \alpha(\nu_E) = -1/b_E^2$. On the other hand, $b_E^2 \alpha(\nu_E), b_E A(\nu_E) \in \mathbb{Z}$. Hence $\nu \in V_1$ iff $A(\nu_E) < 0$ and $\alpha(\nu_E) > 0$. □

Corollary 1.5. Let $\nu \in V_1$ and define a sequence of admissible compactifications $(X_m)_{m \geq 0}$ of $C^2$ as follows: $X_0 = P^2$ and $X_{m+1}$ is obtained from $X_m$ by blowing up the center $p_m$ of $\nu$ on $X_m$. Then $X_m$ is tight for all $m$.

Proof. In view of Lemma 1.4 we only have to show that $p_m$ never lies on a unique prime $E$ of $X_m$ whose associated normalized valuation $\nu_E \in V_0$ has skewness zero or thinness zero. But if it did, the valuation $\nu_k \in V_0$ associated to $p_k$ would satisfy $\nu_k > \nu_E$ for all $k > m$. This would imply $\nu > \nu_E$ so that $\alpha(\nu) < 0$ or $A(\nu) > 0$, contradicting $\nu \in V_1$. □
2. Valuative dynamics

Consider a dominant polynomial mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$.

2.1. Induced map on valuations. Let $\nu \in \tilde{\mathcal{V}}_0$ be a valuation centered at infinity and set $d(F, \nu) := -\nu(F^*L) \geq 0$ for a generic affine function $L$. Define a valuation $F_*\nu$ by $F_*\nu = 0$ if $d(F, \nu) = 0$ and $F_*\nu(P) = \nu(F^*P)$ if $d(F, \nu) > 0$. In the former case, note that $\nu(F^*P) = 0$ for a generic polynomial. In the latter case, $F_*\nu := F_*\nu/d(F, \nu)$ is a well-defined normalized valuation in $\tilde{\mathcal{V}}_0$.

We have the following valuative criterion for properness of maps. It is a consequence of [30, Proposition 7.2].

**Proposition 2.1.** When $F$ is not proper, one can find a divisorial valuation $\nu$ such that $d(F, \nu) = 0$. When $F$ is proper, there exists a constant $c > 0$ such that $d(F, \nu) \geq c$ for all $\nu \in \mathcal{V}_0$.

When $\nu \in \tilde{\mathcal{V}}_{\text{div}}$ is divisorial and $d(F, \nu) > 0$, the valuation $\nu' := F_*\nu \in \tilde{\mathcal{V}}_{\text{div}}$ is also divisorial. More precisely, given any two admissible compactifications $X, X'$ of $\mathbb{C}^2$ such that the centers of $\nu$ and $\nu'$ are primes $E$ and $E'$ of $X$ and $X'$, respectively, we can write $\nu = t \text{ord}_E$ and $\nu' = t' \text{ord}_{E'}$. Then $t'/t$ is the coefficient of $E$ in $F^*E'$, where $\tilde{F} : X \to X'$ is the lift of $F$.

The following result, proved in [30, §7.2] allows us to do dynamics on the subtree $\mathcal{V}_1$ of $\mathcal{V}_0$, even when $F$ is not proper.

**Proposition 2.2.** We have $d(F_\cdot, \cdot) > 0$ on $\mathcal{V}_1$ and $F_\bullet$ leaves $\mathcal{V}_1$ invariant.

The map $F_\bullet$ preserves the tree structure on $\mathcal{V}_1$ in the sense that small segments are mapped homeomorphically onto small segments. See [30, Theorem 7.4] for a precise statement.

2.2. Eigenvaluations. Define the asymptotic degree of $F$ by $\lambda_1 := \lim_{n \to \infty} (\deg F^n)^{1/n}$. The following result was proved in [30, §7.3].

**Proposition 2.3.** There exists a valuation $\nu_*$ belonging to the subtree $\mathcal{V}_1$ of $\mathcal{V}_0$ such that $F_*\nu_* = \lambda_1 \nu_*$. In particular, $\alpha(\nu_*) \geq 0$ and $d(F, \nu_*) = \lambda_1$.

Such a valuation is called an eigenvaluation. The proof is based on the fact that $F_\bullet$ preserves the tree structure on $\mathcal{V}_1$. We also proved that the eigenvaluation admits a small basin of attraction. Using the techniques of [8] as described in the appendix, we can strengthen these conclusions considerably under the assumption $\lambda_2 < \lambda_1^2$, where $\lambda_2$ is the topological degree of $F$.

**Theorem 2.4.** Assume $\lambda_2 < \lambda_1^2$.

(a) the valuation $\nu_*$ in Proposition 2.3 is the unique valuation $\nu \in \mathcal{V}_0$ with $\alpha(\nu) \geq 0$ and $F_*\nu = \lambda_1 \nu$;

(b) if $\nu \in \mathcal{V}_0$ and $\alpha(\nu) > 0$, then $F_*^n\nu \to \nu_*$ in $\mathcal{V}_0$ as $n \to \infty$;

(c) there exists at most one $\nu \in \mathcal{V}_0$ with $\alpha(\nu) = 0$ such that $F_*^n\nu \not\to \nu_*$ as $n \to \infty$; this $\nu$ must satisfy $F_*\nu = \nu$.

Hence it makes sense to refer to $\nu_*$ as the eigenvaluation when $\lambda_2 < \lambda_1^2$. 
Proof. The proof invokes the spectral properties of the operators $F^*$ and $F_*$ on the space $L^2(\mathcal{X})$ as discussed in Appendix A.

For any $\nu \in \mathcal{V}_0$ let $Z_\nu \in W(\mathcal{X})$ be the associated Weil class, see Section A.2. Then $F_*Z_\nu = Z_{F_*\nu}$. When $\nu \not\in \mathcal{V}_0$ and $\alpha(\nu) \geq 0$, Lemma A.3 shows $Z_\nu$ is nef, hence in $L^2(\mathcal{X})$. Theorem A.3 therefore implies that $F_*\nu = \lambda_1\nu$ if $Z_\nu = c\theta$ for some $c > 0$. In fact, $c = 1$ since $(\theta \cdot L) = (Z_\nu \cdot L) = 1$, and so $\nu = \nu_*$. This proves (a), and that $\theta = Z_{\nu_*}$.

To prove (b) and (c), pick any $\nu \in \mathcal{V}_0$ with $\alpha(\nu) \geq 0$. It follows from Theorem A.8 that $\lambda_1^{-n} F_*^n Z_\nu \to (Z_\nu \cdot \theta^*) \theta_\nu = (Z_\nu \cdot \theta^*) Z_{\nu_*}$ as $n \to \infty$. Thus $F_*^n \nu \to \nu_*$ as long as $(Z_\nu \cdot \theta^*) > 0$. Since $Z_\nu$ is nef and $(\theta^* \cdot \theta^*) = 0$, the Hodge inequality implies $(Z_\nu \cdot \theta^*) \geq 0$ with equality if $Z_\nu = c\theta^*$ for some $c > 0$. In the latter case, $\nu$ is uniquely determined, $\alpha(\nu) = (Z_\nu \cdot Z_\nu) = 0$ and $F_*\nu = \nu$ as $F_*\theta = (\lambda_2/\lambda_1)\theta^*$. \hfill $\Box$ 

**Proposition 2.5.** The asymptotic degree $\lambda_1$ is a quadratic integer. Moreover, when $\lambda_2 < \lambda_1^2$, we are in one of the following two cases:

(i) $\lambda_1 \in \mathbb{N}$ and $\nu_*$ is divisorial or infinitely singular;
(ii) $\lambda_1 \notin \mathbb{Q}$ and $\nu_*$ is irrational.

Proof. The fact that $\lambda_1$ is a quadratic integer is contained in [30, Theorem A']. An outline of the proof goes as follows.

When $\nu_*$ is divisorial, it follows from the discussion above Proposition 2.1 that $\lambda_1 = d(F, \nu_*) \in \mathbb{N}$.

When $\nu_*$ is not divisorial, Theorem 7.7 in [30] provides local normal forms of $F$ at some point at infinity of some suitable admissible compactification of $\mathcal{C}^2$. In the infinitely singular case, one sees by inspection that $\lambda_1$ is an integer.

If instead $\nu_*$ is irrational, then the local normal form at $p$ is monomial, and $\lambda_1$ is the spectral radius of a $2 \times 2$ matrix $M$ having nonnegative integer coefficients. Suppose $\lambda_1 \in \mathbb{Q}$. Then the other eigenvalue $\lambda_1'$ of $M$ is also rational. Now $\nu_*$ being irrational means $M$ has an eigenvector $(u, v) \in \mathbb{R}^2_+$ with $u/v$ irrational. This is only possible if $\lambda_1' = 1$. But then the local topological degree of $F$ at $p$ equals $\det M = \lambda_1^2$, contradicting $\lambda_2 < \lambda_1^2$. \hfill $\Box$

**Proposition 2.6.** When $\lambda_2 < \lambda_1$, the eigenvaluation $\nu_*$ cannot be divisorial.

Proof. Assume $\nu_*$ is divisorial. Pick an admissible compactification $X$ of $\mathcal{C}^2$ such that $\nu_*$ is proportional to $\text{ord}_E$ for some prime $E$ of $X$. Then the rational lift $\tilde{F} : X \dashrightarrow X$ maps $E$ onto itself, and the eigenvalue $\lambda_1$ is the coefficient of $E$ in $\tilde{F}^* E$. This coefficient is dominated by the topological degree of $F$ in a neighborhood of $E$. Hence $\lambda_1 \leq \lambda_2$. \hfill $\Box$

In the proof of Theorem A, the case when $\nu_*$ is divisorial and also an end in $\mathcal{V}_1$ needs special treatment. It occurs exactly when $\alpha(\nu_*) = 0 > A(\nu_*)$, that is, $\nu_*$ is a rational pencil valuation; or when $\alpha(\nu_*) > 0 = A(\nu_*)$. As the next results show, the dynamics is then quite particular.

**Proposition 2.7.** Assume that $\nu \in \mathcal{V}_1$ is a rational pencil valuation such that $F_*\nu = \lambda\nu$ for some $\lambda > 0$. Then $\lambda$ is an integer and $F$ is conjugate by a polynomial automorphism of $\mathcal{C}^2$ to a skew product of the form $F(x, y) = (P(x), Q(x, y))$ with $\deg_y Q = \lambda$.

Proof. As in [30, §7.4] this follows from the Line Embedding Theorem. \hfill $\Box$
Proposition 2.8. Assume $\lambda_1 > 1$. If there exists a valuation $\nu_* \in \mathcal{V}_0$ (not necessarily divisorial) with $F_* \nu_0 = \lambda_1 \nu_*$ and $\alpha(\nu_*) > 0 = A(\nu_*)$, then $F$ is a counterexample to the Jacobian conjecture.

Proof. The change of variables formula implies $A(F_* \nu) = A(\nu) + \nu(JF)$ for all $\nu \in \mathcal{V}_1$, where $JF$ is the Jacobian determinant of $F$, see [30, Lemma 7.6]. Applying this to $\nu = \nu_*$ yields $\nu_* (JF) = 0$. As $\alpha(\nu_*) > 0$, this is only possible if $JF$ is constant. But if $F$ were a polynomial automorphism, we would have $\alpha(\nu_*) = (\theta_*(F) \cdot \theta_*(F)) = (\theta^*(F^{-1}) \cdot \theta^*(F^{-1})) = 0$. $\square$

2.3. Examples of eigenvaluations. Curve valuations do not belong to $\mathcal{V}_1$, hence can never be eigenvaluations. As the following examples show, any other type of valuation can occur.

Example 2.9. Assume that the extension to $\mathbb{P}^2$ of $F : \mathbb{C}^2 \to \mathbb{C}^2$ does not contract the line at infinity. Then the divisorial valuation $- \deg$ is an eigenvaluation.

Example 2.10. Any rational pencil valuation appears as an eigenvaluation. Indeed, in suitable affine coordinates, the valuation is associated to the pencil $x = \text{const}$ and is the eigenvaluation of a suitable skew product $F(x, y) = (P(x), Q(x, y))$.

Example 2.11. Pick positive integers $a, b, c, d$ such that $\Delta := (a + d)^2 - 4(ad - bc) > 0$. Then the $2 \times 2$ matrix $M$ with entries $a, b, c, d$ has two real eigenvalues $t > 1$ and $t' < t$. The eigenvalue $t$ admits an eigenvector $(u, v) \in \mathbb{R}^2$ that we can normalize by the condition $\min \{u, v\} = -1$. The map $F(x, y) := (x^a y^b, x^c y^d)$ has topological degree $\lambda_2 = |ad - bc|$ and asymptotic degree $\lambda_1 = t$. It admits a unique eigenvaluation which is the monomial valuation with weights $u$ on $x$ and $v$ on $y$. When $\Delta$ is not a square, the eigenvaluation is irrational. Otherwise it is divisorial.

One may also perturb $F$ by adding terms of sufficiently low order. After doing so, $\lambda_1, \lambda_2$ and the eigenvaluation remain unchanged.

Example 2.12. The eigenvaluation $\nu_*$ of a polynomial automorphism $F$ with $\lambda_1 > 1$ is always infinitely singular. Indeed, the eigenclass $\theta_* \in L^2(\mathcal{X})$ can be written $\theta_*(F) = \theta^*(F^{-1})$, hence $\alpha(\nu_*) = (\theta_*(F) \cdot \theta_*(F)) = 0$. Thus $\nu_*$ cannot be irrational, as that would imply $\alpha(\nu_*) \notin \mathbb{Q}$. It cannot be divisorial by Proposition 2.3. Hence it is infinitely singular. Proposition 2.5 now implies that the asymptotic degree $\lambda_1$ is an integer, a fact which also follows from the Friedland-Milnor classification [33].

It would be interesting to investigate the relation between the eigenvaluation $\nu_*$, the solenoids constructed by Hubbard et al. [11, 42], and the singularity of the Green function of $F$ at infinity; see also [29, Proposition 6.9].

Example 2.13. The argument above shows more generally that the eigenvaluation of any polynomial mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$ with $\lambda_2 < \lambda_1 \in \mathbb{N}$ must be infinitely singular. The condition of $\lambda_1$ being an integer is satisfied, for instance, if $F$ defines an algebraically stable map on $\mathbb{P}^2$, in which case $\lambda_1 = \deg F$ [32]

3. Stability when $\lambda_2 < \lambda_1^2$: the nondivisorial case

Our aim is now to prove precise versions of Theorem A and B in the case when the eigenvaluation is nondivisorial. Recall that we are always in this situation when $\lambda_2 < \lambda_1$, see Proposition 2.6.
Theorem 3.1. Let \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) be a dominant polynomial mapping with \( \lambda_2 < \lambda_1^2 \) such that the eigenvaluation \( \nu_\ast \) is nondivisorial. Then there exists a tight compactification \( X \) of \( \mathbb{C}^2 \), a point \( p \in X \setminus \mathbb{C}^2 \) and local coordinates \((z, w)\) at \( p \) such that the lift \( \tilde{F} : X \to X \) defines a superattracting fixed point germ at \( p \) taking one of the following forms:

(a) \( \tilde{F}(z, w) = (z^n w^b, z^c w^d) \), where \( a, b, c, d \in \mathbb{N} \) and the \( 2 \times 2 \) matrix \( M \) with entries \( a, b, c, d \) has spectral radius \( \lambda_1 \); locally at \( p \) we have \( X \setminus \mathbb{C}^2 = \{zw = 0\} \);

(b) \( \tilde{F}(z, w) = (z^\lambda_1, \mu z^\mu w + P(z)) \), where \( c \geq 1, \mu \in \mathbb{C}^* \) and \( P \neq P(0) = 0 \) is a polynomial; locally at \( p \) we have \( X \setminus \mathbb{C}^2 = \{z = 0\} \).

Moreover, if \( F \) is not conjugate to a skew product by a polynomial automorphism of \( \mathbb{C}^2 \), there exists \( n \geq 1 \) such that each prime of \( X \) is contracted to \( p \) by \( F^n \). When \( F \) is conjugate to a skew product, the same conclusion holds for all primes of \( X \) with the exception of a single prime invariant by \( \tilde{F} \).

Remark 3.2. The skew product case in Theorem 3.1 does not occur when \( \lambda_2 < \lambda_1 \). Indeed, if \( F(x, y) = (P(x), Q(x, y)) \), then \( \lambda_2 = \deg P \cdot \deg_y Q \geq \max\{\deg P, \deg_y Q\} = \lambda_1 \).

Corollary 3.3. Under the assumptions of Theorem 3.1 for any \( \nu \in \mathcal{V}_1 \), there exists \( n = n(\nu) \) such that the sequence \((d(F^j, \nu))_{j \geq n}\) satisfies an integral linear recursion formula of order 1 or 2. In particular, \((\deg F^j)_{j \geq n}\) satisfies such a recursion formula for \( n \) large enough.

Proof of Theorem 3.1. Since \( \nu_\ast \) is nondivisorial, its center on any admissible compactification of \( \mathbb{C}^2 \) is a point. We may therefore define an infinite sequence \((X_m)_{m \geq 0}\) of admissible compactifications of \( \mathbb{C}^2 \) by setting \( X_0 = \mathbb{P}^2 \) and letting \( X_{m+1} \) be the blowup of \( X_m \) at the center \( p_m \in X_m \) of \( \nu_\ast \). Let \( \nu_0, \ldots, \nu_m \) be the normalized divisorial valuations associated to the primes of \( X_m \). As the eigenvaluation \( \nu_\ast \) lies in \( \mathcal{V}_1 \), the compactification \( X_m \) is tight by Corollary 1.5. Hence, for any \( j \), either \( \alpha(\nu_j) > 0 \) or \( \nu_j \) is a rational pencil valuation.

We claim that there exists \( m \geq 0 \) such that the lift \( \tilde{F}_m : X_m \to X_m \) defines a superattracting fixed point germ at \( p_m \), taking one of the forms (a) or (b) above. This follows from the proof of Theorem 7.7 in [30]; let us briefly indicate how to proceed.

When \( \nu_\ast \) is infinitely singular, there exists an infinite subsequence \((m_j)_{j \geq 1}\) such that \( X_{m_j} \setminus \mathbb{C}^2 \) is locally irreducible at \( p_{m_j} \). The corresponding valuations \( \nu_{m_j+1} \) increase to \( \nu_\ast \) as \( j \to \infty \). Consider the segment \( I_j := [\nu_{m_j+1}, \nu_\ast] \) in \( \mathcal{V}_1 \) and the corresponding open neighborhood \( U_j \) of \( \nu_\ast \) in \( \mathcal{V}_0 \) consisting of valuations whose tree retraction to the closed segment \( \overline{T}_j \) belongs to \( I_j \). The attracting properties of \( \nu_\ast \) imply \( F_\ast I_j \subset I_j \) and \( F_\ast U_j \subset U_j \) for large \( j \). Now \( U_j \) is the set of valuations whose center on \( X_{m_j} \) equals \( p_{m_j} \), so this means that \( \tilde{F}_{m_j} \) defines a holomorphic fixed point germ at \( p_{m_j} \) which in fact is rigid in the sense of [27]. The normal form in (b) follows from the classification in [27]. A direct inspection shows that \( \tilde{F}_{m_j} \) is superattracting at \( p_{m_j} \).

If instead \( \nu_\ast \) is irrational, then for all large \( m \), \( p_m \) is the intersection of two primes of \( X_m \). Now \( F_\ast \) preserves the tree structure on \( \mathcal{V}_0 \); if \( I \) is any small open segment in \( \mathcal{V}_0 \) containing \( \nu_\ast \) and \( I \) a small subsegment, then \( F_\ast \) is a homeomorphism of \( I \) onto its image \( F_\ast I \subset I \). Moreover, \( F|_{I_0} \) is contracting at \( \nu_\ast \) in the skewness metric. This means that if \( I \) is sufficiently symmetric around \( \nu_\ast \), then \( F_\ast I \subset I \). (The symmetry condition is only necessary when \( F_\ast \) is order-reversing on \( I_0 \).) Let \( \mu_m, \mu'_m \in \mathcal{V}_0 \) be the
normalized divisorial valuations associated to the primes of \(X_m\) containing \(p_m\) and set \(I_m = \frac{[\mu_m, \nu_m]}{[\mu_m, \nu'_m]}\). By repeating the arithmetic argument as in [30] Lemma 5.6, one shows that there exists an infinite subsequence \((m_j)_{j \geq 1}\) such that \(I_{m_j}\) is sufficiently symmetric so that \(F_{m_j} \subseteq I_{m_j}\). For \(j\) large enough we will also have \(F_{m_j} \subseteq U_{m_j}\), where \(U_m\) is the set of valuations whose center on \(X_m\) equals \(p_m\). Thus \(\tilde{F}_{m_j}\) defines a holomorphic fixed point germ at \(p_{m_j}\), which by invoking [27] can be put in the monomial form (a) above. Finally \(\tilde{F}_{m_j}\) must be superattracting at \(p_{m_j}\) or else one of the primes of \(X_{m_j}\) containing \(p_{m_j}\) would be an eigenvaluation for \(F^2\) with eigenvalue \(\lambda_1^2\), contradicting Theorem 2.4 (a).

In both the irrational and infinitely singular case we have found a tight compactification \(X\) of \(C^2\) and a point \(p \in X \setminus C^2\) such that the lift \(\tilde{F} : X \to X\) defines a superattracting fixed point germ at \(p\). The point \(p\) defines an open subset \(U(p)\) of \(\mathcal{V}_0\); a valuation \(\nu \in \mathcal{V}_0\) belongs to \(U(p)\) iff the center of \(\nu\) on \(X\) equals \(p\).

Now pick any prime \(E\) of \(X\) and let \(\nu_E \in \mathcal{V}_0\) be the associated divisorial valuation. We have \(\alpha(\nu_E) \geq 0\) since \(X\) is tight. If \(\alpha(\nu_E) > 0\) then Theorem 2.4 (b) shows that \(F^n \nu_E \subset U(p)\) for \(n \geq 1\). This means that \(F^n\) contracts \(E\) to \(p\). If instead \(\alpha(\nu_E) = 0\), then \(\nu_E\) must be a rational pencil valuation. Theorem 2.4 (c) shows that \(\tilde{F}^n\) contracts \(E\) to \(p\) for \(n \geq 1\) unless \(F\nu_E = \nu_E\). In the latter case, \(F\) is conjugate to a skew product by Proposition 2.7.

**Proof of Corollary 2.3** Write \(\nu_j = F_j^*(\nu)\) for \(j \geq 0\). Then \(d(F^j, \nu) = \prod_{i=0}^{j-1} d(F, \nu_i)\). We may assume \(F^* \nu \to \nu_\ast\) as \(n \to \infty\), since otherwise \(F^n \nu = \nu\) and \(d(F^j, \nu) = d(F, \nu)^j\).

If \(\nu_\ast\) is infinitely singular, then \(d(F, \cdot) \equiv \lambda_1\) in a neighborhood of \(\nu_\ast\). Since \(\nu_j \to \nu_\ast\) as \(j \to \infty\), we get \(d(F^j+1, \nu) = \lambda_1 d(F^j, \nu)\) for \(j \geq 0\).

When instead \(\nu_\ast\) is irrational, we use the local monomial form in Theorem 3.1 (b). Let \(E_z = \{z = 0\}\) and \(E_w = \{w = 0\}\) be the primes of \(X\) containing \(p\) and write \(b_z = -\text{ord}_{E_z}(L), b_w = -\text{ord}_{E_w}(L)\) for a generic affine function \(L\) on \(C^2\). For \(j \geq n\), set \(s_j := F_j^*(\nu)(z) > 0, t_j := F_j^*(\nu)(w) > 0\). Then \((s_{j+1}, t_{j+1}) = M(s_j, t_j)\). Now \(d(F^j, \nu) = b_z s_j + b_w t_j\), so this easily implies that \((d(F^j, \nu))_{j \geq n}\) satisfies an integral linear recursion formula of order at most two.

**4. Stability when \(\lambda_2 < \lambda_1^2\): the divisorial case**

Next we prove Theorem A and B in the case when the eigenvaluation \(\nu_\ast\) is divisorial. Recall that this implies \(\lambda_2 \geq \lambda_1\). We distinguish between two subcases: \(\nu_\ast\) may or may not be an end in the tree \(\mathcal{V}_1\).

When \(\nu_\ast\) is an end, either \(F\) is conjugate to a skew product; or \(F\) is a counterexample to the Jacobian conjecture, see Propositions 2.7 and 2.8.

**Theorem 4.1.** Let \(F : C^2 \to C^2\) be a dominant polynomial mapping with \(\lambda_2 < \lambda_1^2\). Assume that the eigenvaluation \(\nu_\ast\) is divisorial and an end in \(\mathcal{V}_1\). Then there exists a tight compactification \(X\) of \(C^2\), a prime \(E_\ast\) of \(X\), a point \(p\) on \(E_\ast\), and an integer \(n \geq 1\) such that the lift \(\tilde{F} : X \to X\) maps \(E_\ast\) onto \(E_\ast\) and defines a holomorphic fixed point germ at \(p\); and we are in one of the following situations:

(i) for each prime \(E \neq E_\ast\) of \(X\), either \(\tilde{F}^n(E) = E_\ast\), or \(\tilde{F}^n\) contracts \(E\) to \(p\);
(ii) $F$ is conjugate to a skew product by a polynomial automorphism of $\mathbb{C}^2$ and the properties in (i) hold for all primes $E$ of $X$ with the exception of a single prime invariant by $\tilde{F}$.

**Corollary 4.2.** Under the assumptions of Theorem 4.1, for any $\nu \in \mathcal{V}$, there exists $n = n(\nu)$ such that the sequence $(d(F^j, \nu))_{j \geq n}$ satisfies an integral linear recursion formula of order 1 or 2. In particular, $(\deg F^j)_{j \geq n}$ satisfies such a recursion formula for $n$ large enough.

When $\nu_*$ is not an end, so that $\alpha(\nu_*) > 0 > A(\nu_*)$, the result is slightly less precise and the proof more subtle.

**Theorem 4.3.** Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a dominant polynomial mapping with $\lambda_2 < \lambda_1^2$. Assume that the eigenvaluation $\nu_* \in \mathcal{V}_1$ is divisorial and not an end in $\mathcal{V}_1$. Then there exists a tight compactification $X$ of $\mathbb{C}^2$, a prime $E_*$ of $X$ and an integer $n \geq 1$ such that the lift $\tilde{F} : X \to X$ maps $E_*$ onto $E_*$, and we are in one of the following situations:

(i) for each prime $E \neq E_*$ of $X$, either $\tilde{F}^n(E) = E_*$, or $\tilde{F}^n$ contracts $E$ to a point on $E_*$ at which all iterates of $\tilde{F}$ are holomorphic;

(ii) $F$ is conjugate to a skew product by a polynomial automorphism of $\mathbb{C}^2$ and the properties in (i) hold for all primes $E$ of $X$ with the exception of a single prime invariant by $\tilde{F}$.

**Corollary 4.4.** Under the assumptions of Theorem 4.3, there exist $l \geq 1$ and, for any $\nu \in \mathcal{V}$, $n = n(\nu)$ such that the sequence $(d(F^{lj+k}, \nu))_{j \geq 0}$ satisfies an integral linear recursion formula of order 1 or 2 for any $k \geq n$. In particular, $(\deg F^{lj+k})_{j \geq 0}$ satisfies such a recursion formula for any sufficiently large $k$.

**Example 4.5.** Define $F(x, y) = (x(x - y^2), x + y)$ and set $a_j = \deg(F^j)$. Then $a_0 = 1$, $a_1 = 3$, $a_2 = 6$ and $a_j = a_{j-1} + a_{j-2} + 2a_{j-3}$ for $j \geq 3$. One checks that $\lambda_1 = 2$, $\lambda_2 = 3$, and that $(a_j)_\infty$ does not satisfy any integral recursion formula of order smaller than three.

**Proof of Theorem 4.4.** The proof is similar to the infinitely singular case of Theorem 3.1. Let $X_0$ be the smallest admissible compactification such that the center of $\nu_*$ on $X_0$ is a prime $E_*$ of $X_0$: it is obtained from $\mathbb{P}^2$ by successively blowing up the center of $\nu_*$, so $X_0$ is tight by Corollary 1.3. Inductively define a sequence $(X_m)_{m \geq 0}$ of tight compactifications by letting $X_{m+1}$ be the blowup of $X_m$ at the unique intersection point of (the strict transform of) $E_*$ and the unique prime $E_m \neq E_*$ of $X_m$ intersecting $E_*$. Let $\nu_m \in \mathcal{V}_0$ be the divisorial valuation associated to $E_m$. Then $(\nu_m)_{m \geq 0}$ form a sequence of valuations increasing to $\nu_*$. Set $I_m = [\nu_m, \nu_*]$. As in [30] §7.3 we see that $F_*$ maps $I_m$ into itself for large $m$. Moreover, we have $F_*U_m \subset U_m$, where $U_m \subset \mathcal{V}_0$ is the set of valuations whose center on $X_m$ equals $p_m$. Set $X = X_m$ and $p = p_m$. Then the lift $\tilde{F} : X \to X$ maps $E_*$ onto itself and defines a holomorphic fixed point germ at $p$.

Let $U$ be the open neighborhood of $\nu_*$ in $\mathcal{V}$ consisting of valuations whose center on $X$ is contained in $E_*$. Pick a prime $E \neq E_*$ of $X$. If $F_*\nu_E = \nu_E$, then $E$ is unique with this property, $\nu_E$ is a rational pencil valuation and $F$ is conjugate to a skew product. For all other primes $E$ we have $F_*^{n}\nu_E \notin U$ for $n \gg 1$. If $F_*^{n}\nu_E = \nu_*$, then $F^{n}(E) = E_*$. Otherwise $E$ is contracted by $F^n$ to $p \in E_*$. This completes the proof. □
**Proof of Corollary 4.2** We can use the same proof as of Corollary 3.3 in the irrational case. Indeed, let $E$ and $E_*$ be the primes of $X$ containing $p$ and write $E = \{ z = 0 \}$, $E_* = \{ w = 0 \}$ for coordinates $(z, w)$ at $p$. We may perhaps not arrange that $F$ is monomial in $(z, w)$, but if we set $s_j := F_j^* \nu(z) > 0$, $t_j := F_j^* \nu(w) > 0$, we will nevertheless have $(s_{j+1}, t_{j+1}) = M(s_j, t_j)$ for some $2 \times 2$ matrix $M$ with nonnegative integer entries, see [30, Theorem 7.4]

**Proof of Theorem 4.3** Since $\nu_*$ is not an end in $V_1$, we have $\alpha(\nu_*) > 0 > A(\nu_*)$. Let $X_0$ be the smallest admissible compactification such that the center of $\nu_*$ on $X_0$ is a prime $E_*$ of $X_0$: it is obtained from $\mathbb{P}^2$ by successively blowing up the center of $\nu_*$, so $X_0$ is tight by Corollary 1.5.

**Lemma 4.6.** There exists a tight compactification $X$ of $\mathbb{C}^2$ dominating $X_0$ such that the lift $\tilde{F} : X \longrightarrow X$ is holomorphic at all periodic points of $\tilde{F}|_{E_*}$, where $E_*$ denotes the center of $\nu_*$ on $X$.

Using this lemma we now conclude the proof. Let $U$ be the open neighborhood of $\nu_*$ in $V$ consisting of valuations whose center on $X$ is contained in $E_*$. Pick a prime $E \neq E_*$ of $X$. If $F_* \nu_E = \nu_F$, then $E$ is unique with this property, $\nu_F$ is a rational pencil valuation and $F$ is conjugate to a skew product. For all other primes $E$ we have $F_*^n \nu_E \in U$ for $n \gg 1$. If $F_*^n \nu_E = \nu_*$, then $\tilde{F}^n(E) = E_*$. Otherwise $E$ is contracted by $\tilde{F}^n$ to a point $p \in E_*$. By increasing $n$ we can assume that the orbit of $p$ under $\tilde{F}|_{E_*}$ is either periodic or infinite. In the first case, $\tilde{F}$ is holomorphic at $p$ by Lemma 4.6. In the second case, we can by increasing $n$ assume that the orbit does not intersect the indeterminacy set of $\tilde{F}$. This completes the proof.

**Proof of Lemma 4.6** Write $\tilde{F}_0$ for the lift of $F_0$ to $X_0$. Let $Z \subset E_*$ be the (finite) set of periodic points of $\tilde{F}_0|_{E_*}$ whose orbit contains an indeterminacy point of $\tilde{F}_0$. First assume for simplicity that $Z$ consists of a single periodic orbit $p_0, p_1, \ldots, p_l = p_0$ of length $l \geq 1$ for $\tilde{F}_0|_{E_*}$.

Let $\mu_k \in V_1$ be the divisorial valuation associated to the blowup of $X_0$ at $p_k$, $0 \leq k \leq l$. The segment $J_k := [\mu_k, \nu_*] \in V_1$ has length $(b_k m_k)^{-1}$ in the skewness metric, where $b_* = -\text{ord}_{E_*} L$, $b_k = -\text{ord}_{p_k} L$ for a generic affine function $L$ on $\mathbb{C}^2$.

For large positive integers $m_0, m_1, \ldots, m_l = m_0$ to be determined shortly, define valuations $\nu_k$, $0 \leq k < l$ as follows: blow up $p_k$, then successively $m_k$ times blow up the intersection point between the (strict transform of) $E_*$ and the previously obtained exceptional divisor. The segment $I_k := [\nu_k, \nu_*] \subset J_k$ then has length $(b_*(b_k + m_k b_*))^{-1}$ in the skewness metric.

For $m_k$ large, the segment $I_k$ is small enough so that $F_*$ maps $I_k$ homeomorphically onto a subsegment of $J_{k+1}$. Moreover, when the segments are parameterized by skewness, $F_*$ is given by a Möbius map with nonnegative integer coefficients, see [30, Theorem 7.4]. Thus the one-sided derivative of $F_*$ on $I_k$ at $\nu_*$ is a well defined rational number $s_k > 0$.

The key fact is now that the Möbius property above implies that the iterate $F_*^l$ maps the segment $I_0$ into itself and that either $F_*^l = \text{id}$ or $F_*^l$ is a contraction on $I_0$, see [30, Lemma 5.5]. The former case is impossible by Theorem 2.4. Hence we conclude that $\prod_{k=0}^{l-1} s_k < 1$. 

**Dynamical Compactifications of $\mathbb{C}^2$**
Pick $\varepsilon > 0$ such that $\prod_{k=0}^{l-1} s_k \leq (1 - 2\varepsilon)^l$. We may then pick the integers $m_k$ (with $m_0 = m_l$) above arbitrarily large so that

$$\frac{s_k}{m_kb_s + b_k} \leq (1 - \varepsilon) \frac{1}{m_{k+1}b_s + b_{k+1}} \quad \text{for } 0 \leq k < l; \quad (4.1)$$

we just need to make $m_{k+1}/m_k$ slightly smaller than $1/s_k$. By the definition of $s_k$, (4.1) implies that $F_k$ maps $I_k$ into $I_{k+1}$. Let $U_k$ be the open subset of $V_0$ consisting of valuations whose tree retraction to the closed segment $I_k$ is contained in $I_k$. Then $F_k$ maps $U_k$ into $U_{k+1}$ for $0 \leq k < l$, assuming the $m_k$’s are large enough (again, by convention, $U_l = U_0$).

Let $X$ be the smallest admissible compactification of $\mathbb{C}^2$ dominating $X_0$ such that the center of $\nu_k$ is one-dimensional for $0 \leq k < l$. Then $X$ is obtained from $X_0$ by performing all the blowups mentioned above, so $X$ is tight and the morphism $X \to X_0$ induced by the identity on $\mathbb{C}^2$ is an isomorphism above $X_0 \setminus Z$.

The center of $\nu_k$ on $X$ intersects (the strict transform of) $E_s$ at some point $q_k$ and the open set $U_k$ above exactly consists of the valuations in $V_0$ whose center on $X$ equals $q_k$. Thus $F_k U_k \subset U_{k+1}$ translates into the lift $\tilde{F} : X \to X$ of $F$ being holomorphic at $q_k$.

This completes the proof when $Z$ consists of a single periodic orbit. In general, there are several orbits, but we can handle them one at a time.

Proof of Corollary 4.4. There is a finite subset $Z \subset E_\ast$ such that if $p \in E_\ast \setminus Z$, we have $d(F, \cdot) \equiv \lambda_1$ on the open set $U(p) \subset V_0$ of valuations whose center on $X$ is $p$. We may pick $l \geq 1$ such that any periodic orbit of $\tilde{F}|E_s$ intersecting $Z$ has order dividing $l$.

As in the proof of Corollary 3.3 we may assume $F^n \nrightarrow \nu_\ast$ as $n \to \infty$. Similarly, we may assume $F^n \nrightarrow \nu_\ast$ for all $n$.

Pick $n = n(\nu)$ so that the center of $F^n \nu$ on $X$ is a point $p_k \in E_s$ for $k \geq n$. After increasing $n$ we may assume that the orbit $p_n, p_{n+1}, \ldots$ is either disjoint from $Z$, or periodic of order (dividing) $l$. In the first case, $d(F^{k+j+1}, \nu) = \lambda_1 d(F^{k+j}, \nu)$ for any $j \geq 0$. In the second case we conclude the proof as in Corollary 3.3. □

5. Maximum topological degree: $\lambda_2 = \lambda_1^2$

Next we turn to maps with maximum topological degree $\lambda_2 = \lambda_1^2$. As we do not have an analog of Theorem 4.8 at our disposal, we base our analysis on a detailed description of the dynamics of $F_k$ on $V_1$ using tree arguments.

5.1. Dynamics on $V_1$. The results of this section form the basis for the proof of Theorems A, B and C in the case $\lambda_2 = \lambda_1^2$. Define

$$T_F := \{ \nu \in V_1 \mid F_n \nu = \nu \}. \quad (5.1)$$

This set is nonempty by Proposition 2.3. The following three results summarize the structure of $T_F$ and its dynamical significance.

Proposition 5.1. Suppose $\deg F^n/\lambda_1^n$ is unbounded. Then $T_F = \{ \nu_\ast \}$ is a singleton, where $\nu_\ast$ is a rational pencil valuation, and $F^n \nrightarrow \nu_\ast$ on $V_1$ as $n \to \infty$. Moreover, $F$ is not proper, $\deg F^n \sim n\lambda_1^n$ and there exist affine coordinates in which $F(x, y) = (P(x), A(x)y^{\lambda_1} + O_x(y^{\lambda_1-1}))$, where $\deg P = \lambda_1$ and $\deg A \geq 1$. □
Proposition 5.2. Suppose deg $F^n$ is bounded. Then $F$ is a polynomial automorphism of $\mathbb{C}^2$. In suitable affine coordinates, either

(a) $F$ is an affine map and $-\deg \in T_F$; or
(b) $F$ is a skew product of the form $F(x) = (ax + b, cy + P(x))$, where $a, c \in \mathbb{C}^*$, $b \in \mathbb{C}$; we may then assume $T_F = [\nu_0, \nu_1]$, where $\nu_1$ is associated to the pencil $x = \text{const}$ and $\nu_0$ is a monomial valuation satisfying $\nu_0(y) = -1$, $\nu_0(x) = -1/q$, where $q = \deg P > 1$.

Proposition 5.3. Suppose deg $F^n/\lambda_1^n$ is bounded and $\lambda_1 > 1$. Then $F$ is proper. Moreover:

(a) either $T_F$ consists of a single quasimonomial valuation $\nu_s \in V_1$ with $\alpha(\nu_s) > 0$; or $T_F$ is a closed segment in $V_1$ whose endpoints are divisorial valuations;
(b) $T_F = T_{F^2}$ for $n \geq 2$, and either $T_F = T_{F^2}$ or $T_F$ is a singleton, lying in the interior of $T_{F^2}$;
(c) for $\nu \in V_1$, $F^{2n}\nu \rightarrow r(\nu)$ as $n \rightarrow \infty$, where $r : V_1 \rightarrow T_{F^2}$ is the natural retraction;
(d) in suitable affine coordinates, all the valuations in $T_{F^2}$ are monomial.

The convergence in (c) holds in a strong sense: $F^{2n}\nu \rightarrow r(\nu)$ weakly and $A(F^{2n}\nu) \rightarrow A(r(\nu))$.

The proofs of these results are given in Section 5.4.

5.2. Proof of Theorem C. If deg$(F^n)/\lambda_1^n$ is unbounded, then we are in case (1) by Proposition 5.1.

If deg$(F^n)$ is bounded, then we pick suitable affine coordinates as in Proposition 5.2. When $F$ is affine, it extends holomorphically to $X = \mathbb{P}^2$. When $F$ is a skew product as in (b), it extends holomorphically to the Hirzebruch surface $X = F_q$. Indeed, we can view $F_q$ as a toric surface associated to the complete fan generated by the vectors $(1,0)$, $(0,1)$, $(0,-1)$ and $(-q,-1)$ in $\mathbb{R}^2$, see [34, pp.6–8]. Then $X$ is a compactification of $\mathbb{C}^2$ and $X \setminus \mathbb{C}^2$ is a union of two rational curves, corresponding to the centers of $\nu_0$ and $\nu_1$ on $X$. As $\nu_0$ and $\nu_1$ are totally invariant under $F$, the lift of $F$ to $F_q$ is holomorphic.

Finally, when deg$(F^n)/\lambda_1^n$ is bounded but $\lambda_1 > 1$, we apply Proposition 5.3. Hence we may assume that $T_{F^2}$ consists of monomial valuations. If $T_F$ contains a divisorial valuation $\nu_s$, then $\nu_s(x) = -p/q$, $\nu_s(y) = -1$, where $q \geq p \geq 1$ and gcd$(p,q) = 1$. Let $X := X_{p,q}$ be the toric surface associated to the complete fan generated by the vectors $(1,0)$, $(0,1)$ and $(-p,-q)$ in $\mathbb{R}^2$. Then $X$ has at worst quotient singularities and is in fact a weighted projective plane, see below. Note that $X \setminus \mathbb{C}^2$ is a single, totally invariant, rational curve. As $\nu_s$ is totally invariant under $F$, $F$ lifts to a holomorphic selfmap of $X$.

If $T_F$ contains no divisorial valuation, then $T_F = \{\nu_s\}$, where $\nu_s$ is an irrational valuation belonging to the interior of $T_{F^2}$. Note that $d(F, \cdot)$ cannot be locally constant at $\nu_s$, or else the tangent map of $F$ at $\nu_s$ (see below) would be the identity and $T_F = T_{F^2}$. Thus $\lambda_1 = d(F, \nu_s)$ is irrational.

Pick any divisorial valuation $\nu \in T_{F^2}$ with $\nu < \nu_s$, and set $\nu' = F_\nu \nu$. Then $\nu' > \nu_s$ and $\nu, \nu'$ are both totally invariant under $F_\nu$. We have $\nu(y) = \nu'(y) = -1$, $\nu(x) = -p/q$, $\nu'(x) = -p'/q'$ for some integers with $q \geq p \geq 1$, $q' > p' \geq 1$ and gcd$(p,q) = \gcd(p',q') = 1$. Define $X$ to be the toric surface associated to the complete fan generated by $(1,0)$, $(0,1)$, $(-p,-q)$, $(-p',-q')$ in $\mathbb{R}^2$. Then $X$ has at worst quotient singularities, and
5.3. Weighted projective spaces and normal forms. Suppose \( \lambda_2 = \lambda_1^2, \deg F^n/\lambda_1^n \) is bounded and \( \lambda_1 > 1 \).

Assume that \( T_F \) contains a divisorial valuation \( \nu_* \), with \( \nu_*(x) = -p/q, \nu_*(y) = -1 \), where \( q \geq p \geq 1 \) are relatively prime integers. We saw that \( F \) is holomorphic on the toric surface \( X_{p,q} \). Conversely any polynomial map of \( \mathbb{C}^2 \) which extends as a holomorphic map to \( X_{p,q} \) satisfies \( \lambda_2 = \lambda_1^2 \), and \( \deg F^n/\lambda_1^n \) is bounded. The surface \( X_{p,q} \) is the weighted projective space with homogeneous coordinates \([x : y : z] \sim [\lambda^p x : \lambda^q y : \lambda^z] \) for all \( \lambda \in \mathbb{C}^* \), see [33, p.35], [24] or [12]. For any polynomial \( P \), let \( P_+ \) be its \( \nu_* \)-leading homogenous part, i.e. the sum of all monomials \( a_{ij} x^i y^j \) in \( P \) such that \(- (p/q + j) = \nu_*(P)\). Then a polynomial map \( F = (P, Q) \) is holomorphic on \( X_{p,q} \) iff \( P_+ \) and \( Q_+ \) have no common zeroes on the weighted projective line \([x : y] \sim [\lambda^p x : \lambda^q y] \), i.e. iff \( P_+(x^p, y^p) \) and \( Q_+(x^q, y^p) \) have no common zeroes in \( \mathbb{C}^2 \setminus \{0\} \).

Note that there exist polynomial maps of \( \mathbb{C}^2 \) which extend to a unique \( X_{p,q} \). For such an example, pick any \( \lambda_1 \) divisible by \( p \) and \( q \), write \( \lambda_1 = pa = qb \) and take \( F(x, y) = (P, Q) \) with \( P_+ = \alpha x^{\lambda_1} + \beta y^{p-b}, Q_+ = \gamma x^{p-a} + \delta y^{\lambda_1}, \) where \( \alpha \beta \gamma \delta \neq 0 \). When \( \lambda_1 \) is not divisible by \( \text{lcm}(p, q) \), \( T_F \) is not reduced to a singleton.

Pick \( p, q, p', q' \) any two pairs of relatively prime integers with associated monomial valuations \( \nu \) and \( \nu' \) and \( p'/q' > p/q \). Then there exists a polynomial map of \( \mathbb{C}^2 \) for which \( T_F \) is precisely the segment of monomial valuations \([\nu, \nu']\). Take \( \lambda_1 \) divisible by \( \text{lcm}(p, q, p', q') \), write \( \lambda_1 = pa = qb = p'a' = q'b' \) and define \( F(x, y) = (\alpha x^{\lambda_1} + \beta y^{p-b} + C_0, \gamma x^{p-a} + \delta y^{\lambda_1} + \gamma x^{p'a'} + C_1) \) with \( \alpha \beta \gamma \delta \neq 0 \), and \( C_0, C_1 \in \mathbb{C} \).

Finally if \( T_F \) contains no divisorial valuation, then \( \lambda_1 \not\in \mathbb{N} \) and \( T_F \) consists of a single irrational monomial valuation \( \nu_* \). We may assume \( \nu_*(x) = -t, \nu_*(y) = -1 \), where \( t \in (0, 1) \) is irrational. This leads to

\[
P_+ = \alpha y^{p-b} \quad \text{and} \quad Q_+ = \beta x^{p-a}
\]

where \( b, c \in \mathbb{N}, 0 < b < c, bc = \lambda_2 = \lambda_1^2, t = \sqrt{b/c} \) and \( \alpha, \beta \in \mathbb{C}^* \).

5.4. Proofs of Propositions 5.1, 5.3. The arguments utilize the tree structure of \( \mathcal{V}_1 \) in much more detail than other parts of this paper. In particular, we need to exploit the relationship between the parameterizations \( \alpha \) and \( A \) on the tree \( \mathcal{V}_0 \) as explained in [30, Appendix A]. There is an increasing, lower semicontinuous multiplicity function \( m : \mathcal{V}_0 \to \mathbb{N}^* \cup \{+\infty\} \) such that \( A(\nu) = -2 - \int_0^{\nu} \deg m(\mu) \, d\nu(\mu) \) for all \( \nu \in \mathcal{V}_0 \), see [30, Theorem A.4]. The multiplicity of any quasimonomial valuation is finite, whereas infinitely singular valuations have infinite multiplicity.

Write \( JF \) for the Jacobian determinant of \( F \). The multiplicity function will be primarily exploited through the following Jacobian formula, see [30, Lemma 7.6]

\[
A(\nu) + \nu(JF) = d(F, \nu) A(F_* \nu).
\]

We start by proving some general facts about the set \( T_F \) defined in (5.1).

**Lemma 5.4.** The set \( T_F \) is nonempty. For every \( \nu \in T_F \), \( F_* \nu = \lambda_1 \nu \) and \( F^* Z_\nu = F_* Z_\nu = \lambda_1 Z_\nu \). If \( F \) is proper, every \( \nu \in T_F \) is totally invariant under \( F_* \).
Proof. By Proposition 2.3, there exists \( \nu \in V_1 \) with \( F_*\nu = \lambda_1\nu \), hence \( T_F \) is nonempty. For any such \( \nu \), we have \( F_*Z_\nu = \lambda_1 Z_\nu \) by Lemma A.6. The condition \( \lambda_2 = \lambda_1^2 \) and the Hodge Index Theorem then imply \( F_*Z_\nu = \lambda_1 Z_\nu \). When \( F \) is proper, the latter equation implies that \( \nu \) is totally invariant by Proposition A.7.

Now pick any \( \mu \in T_F \). We must prove that \( F_*\mu = \lambda_1\mu \). In any case, \( F_*\mu = \lambda\mu \) for some \( \lambda > 0 \). Pick \( \nu \in V_1 \) such that \( F_*\nu = \lambda_1\nu \). We may assume \( \nu \neq \mu \). By what precedes and by (A.1), \( \lambda_1(\mu \wedge \nu) = (Z_\mu \cdot F_*Z_\nu) = (F_*Z_\mu \cdot Z_\nu) = \lambda\alpha(\mu \wedge \nu) \). Since \( \mu \neq \nu \), \( \alpha(\mu \wedge \nu) > 0 \) and so \( \lambda = \lambda_1 \). \( \square \)

Proof of Proposition 5.3. Every valuation \( \nu \in T_F \) must have \( \alpha(\nu) = 0 \), or else \( \deg(F^n)/\lambda_1^n \) would be bounded by \( 1/\alpha(\nu) \). By Theorem B’ and §7.4 in [30], there exists a rational pencil valuation \( \nu_\ast \in T_F \). Moreover, in suitable affine coordinates, \( \nu_\ast \) corresponds to the pencil \( x = \text{const} \) and \( F \) takes the required form. Hence \( \deg F^n \sim n\lambda_1^n \).

Now note that since \( F_*Z_{\nu_\ast} = \lambda_1 Z_{\nu_\ast} \), we have

\[
\alpha(F^n \nu \wedge \nu_\ast) = \frac{\lambda_1^n}{d(F^n, \nu)} \alpha(\nu \wedge \nu_\ast)
\]

(5.3)

for any \( \nu \in V_1 \) and \( n \geq 1 \).

Apply this to \( n = 1 \) and \( \nu < \nu_\ast \) close to \( \nu_\ast \). Then \( F_*\nu < \nu_\ast \), so \( d(F, \nu) < \lambda_1 \) or else \( \nu \in T_F \). Hence \( d(F, \nu) \) is nonconstant near \( \nu_\ast \). By [30] Proposition 7.2 there exists \( \nu_0 \in V_0, \nu_0 > \nu_\ast \) such that \( F_*Z_{\nu_0} = c\mathcal{L} \) where \( c \geq 0 \) and \( \mathcal{L} \) is the class of a line. Then \( c = (F_*Z_{\nu_0} \cdot Z_{\nu_\ast}) = \lambda_1\alpha(\nu_0 \wedge \nu_\ast) = 0 \). In particular, \( F \) is not proper, see Proposition 2.1.

It remains to prove that \( F^n \nu \to \nu_\ast \) as \( n \to \infty \) for every \( \nu \in V_1 \). This will in particular imply \( T_{F^n} = \{ \nu_\ast \} \) for all \( n \geq 1 \). When \( \alpha(\nu) > 0 \), this follows from (5.3), since \( d(F^n, \nu) \geq \deg(F^n)\alpha(\nu) \), so suppose \( \alpha(\nu) = 0 \) and set \( \nu' = F_*\nu \). If \( \alpha(\nu') > 0 \), then again \( F^n \nu \to \nu_\ast \), so suppose \( \alpha(\nu') = 0 \). Consider the nef Weil class \( Z_{\nu'} \) and note that \( (F_*Z_{\nu'} \cdot Z_{\nu}) = 0 \). By the Hodge Index Theorem, \( F_*Z_{\nu'} = cZ_{\nu'} \), where \( c = \lambda_2/d(F, \nu) \). When \( \lambda_2 > 0 \), \( \nu_0 > \nu_\ast \) be the valuation with \( F_*Z_{\nu_0} = 0 \) considered above. Then \( 0 = (F_*Z_{\nu_0} \cdot Z_{\nu'}) = (Z_{\nu_0} \cdot F_*Z_{\nu'}) = c\alpha(\nu_0 \wedge \nu') \), which implies \( \nu' = \nu_\ast \), completing the proof. \( \square \)

Proof of Proposition 5.2. Note that \( \lambda_1 = \lambda_2 = 1 \). It suffices to prove that \( F \) is proper. Indeed, then \( F \) is a polynomial automorphism, and all the assertions follow from the Friedland-Milnor classification [33].

Now, if \( F \) were not proper, by Proposition 2.1 we could find a divisorial valuation \( \nu_0 \in V_0 \) such that \( F_*Z_{\nu_0} = 0 \). Hence, for any \( \nu \in T_F, 0 = (F_*Z_{\nu_0} \cdot Z_{\nu}) = \lambda_1\alpha(\nu_0 \wedge \nu) \), so that \( \alpha(\nu) = 0 \) and \( \nu_0 \geq \nu \). Thus \( F \) would be of the form \( (ax + b, C(x)y + D(x)) \) in suitable coordinates. As \( F \) is nonproper, \( \deg C \geq 1 \), contradicting that \( \deg F^n \) is bounded. \( \square \)

Next we turn to Proposition 5.3, which is significantly harder to prove than the previous two propositions. Assume therefore, for the rest of Section 5.4 that \( \lambda_2 = \lambda_1^2 > 1 \) and that \( \deg F^n/\lambda_1^n \) is bounded. Then \( F \) is proper as follows from the proof of Proposition 5.2. Hence, by Lemma 5.4 every \( \nu \in T_F \) is totally invariant under \( F_* \).

To continue the proof, we recall the definition of the (tree) tangent map of \( F \) at any valuation \( \nu \in V_0 \), see [30] §3. Declare two segments of the form \( [\nu_1, \nu] \) and \( [\nu_2, \nu] \) to be equivalent if they have nonempty intersection. An equivalence class is called a tangent
vector at \( \nu \) and the set \( T\nu \) of tangent vectors the tangent space at \( \nu \). If \( \vec{v} \) is a tangent vector, we denote by \( U(\vec{v}) \) the open set of all valuations determining \( \vec{v} \). These open sets form a basis for the weak topology on \( \nu_0 \) \[28\] Theorem 5.1. As \( F_* \) preserves the tree structure it naturally induces a surjective selfmap \( F : T\nu \to T\nu \), the tangent map, for any eigenvaluation \( \nu \in T_F \).

When \( \nu \in T_F \) is infinitely singular, \( T\nu \) is a singleton, so \( F \equiv \text{id} \). When \( \nu \in T_F \) is irrational, \( T\nu \) consists of two tangent vectors and \( F^2 \equiv \text{id} \). If instead \( \nu \in T_F \) is divisorial and \( X \) is an admissible compactification for which the center of \( \nu \) is a prime \( E \) of \( X \), then there exists a canonical identification of \( E \) with the tangent space \( T\nu \) at \( \nu \) as follows. For any point \( p \in E \), all valuations centered at \( p \) determine the same tangent vector \( \vec{v}_p \) at \( \nu \). Conversely all valuations in \( U(\vec{v}) \) are centered along a connected subspace intersecting \( E \) in a single point \( p(\vec{v}) \), see \[28\] Theorem B.1. With this identification, \( F \) can be viewed as a rational selfmap of \( E \cong \mathbb{P}^1 \).

**Lemma 5.5.** Assume \( \nu_* \in T_F \) and consider a tangent vector \( \vec{v} \) at \( \nu_* \) represented by a valuation with \( \alpha > 0 \). If \( \vec{v} \) is totally invariant by \( F \), then \( F_* \equiv \text{id} \) on a small segment representing \( \vec{v} \).

**Lemma 5.6.** Assume \( \nu_* \in T_F \) is divisorial but not a rational pencil valuation. Then \( F_* U(\vec{v}) = U(F\vec{v}) \) for any tangent vector \( \vec{v} \) at \( \nu_* \).

With these two lemmas at hand, we continue the proof of Proposition \[5.3\]

First suppose \( T_F = \{\nu_*\} \) is a singleton. Then \( \nu_* \) cannot be infinitely singular by Lemma \[5.5\]. Neither can it be a rational pencil valuation, since then \( \deg F^n \sim n\lambda_1^n \). Hence \( \nu_* \) is quasimonomial with \( \alpha(\nu_*) > 0 \).

If \( T_F \) is not a singleton, pick two distinct valuations \( \nu_1, \nu_2 \in T_F \). The set \( [\nu_1, \nu_2] \cap T_F \) is clearly closed, and Lemmas \[5.5\] and \[5.6\] show that it is open. Hence \( [\nu_1, \nu_2] \subset T_F \), so \( T_F \) is a subtree of \( V_1 \). Suppose \( \nu \) is a branch point of \( T_F \). Then \( \nu \) is divisorial and \( \alpha(\nu) > 0 \). Each branch of \( T_F \) emanating from \( \nu \) corresponds to a tangent vector \( \vec{v} \) which is totally invariant by the tangent map \( F \) at \( \nu \), since a valuation in \( T_F \) is totally invariant. Now \( F \) can be identified with a rational map on \( \mathbb{P}^1 \) whose degree equals \( \lambda_2/d(F, \nu) = \lambda_1 > 1 \). Hence \( F \) admits at most two totally invariant tangent vectors. This gives a contradiction. We have shown that \( T_F \) is a non-empty closed segment of \( V_1 \).

Suppose \( T_F \subseteq T_{F_2} \) and that \( T_F \) is not a singleton. We can then find a valuation \( \nu \) which is an interior point of \( T_{F_2} \) but an endpoint of \( T_F \). Consider the tangent map \( F \) at \( \nu \). We see that \( F^2 \) admits two totally invariant tangent vectors, exactly one of which is (totally) invariant by \( F \). This is a contradiction. Similarly, \( T_F \) cannot be a singleton consisting of an endpoint of \( T_{F_2} \). An analogous argument shows that \( T_{F^n} = T_F \) for all \( n \geq 3 \), establishing (b).

Now suppose that \( T_F \) is a nontrivial segment, and consider a subsegment \( I = ]\nu_1, \nu_2[ \subset T_F \), that is totally ordered, i.e. \( \nu_1 < \nu_2 \). We claim that the multiplicity function \( m \) is constant on \( I \). This will imply that the endpoints of \( T_F \) are divisorial (rather than infinitely singular), and hence complete the proof of (a). The Jacobian formula \[5.2\] gives \( \nu(JF) = (\lambda_1 - 1)A(\nu) \), so the function \( \nu \to \nu(JF) \) is piecewise affine on \( I \) (with respect to skewness) with slope \( m(\nu)(\lambda_1 - 1) \). Now \( \nu \to \nu(JF) \) is concave on \( I \) \[30\] §A.4, whereas \( \nu \mapsto m(\nu) \) is nondecreasing on \( I \). Thus \( m \) is constant on \( I \), as required.
Next we turn to (c). We may replace $F$ by $F^2$ so that $\mathcal{T}_F = \mathcal{T}_{F^2}$. Pick $\nu \in \mathcal{V}_1 \setminus \mathcal{T}_F$ and write $\nu_s := r(\nu) \in \mathcal{T}_F$. Then $\nu_s$ is divisorial and $\alpha(\nu_s) > 0$. We need to show that $F^n_\nu \nu \to \nu_s$ as $n \to \infty$. Denote by $\vec{v}$ the tangent vector at $\nu_s$ represented by $\nu$. Note that $U(\vec{v}) \cap \mathcal{T}_F = \emptyset$. If $\vec{v}$ is not preperiodic, then for $n$ large enough, the functions $\mu \mapsto \mu(JF)$ and $\mu \mapsto d(F, \mu)$ are both constant on $U(\vec{v})$, where $\vec{v}_n := F^n \vec{v}$, see [30]. Proposition 3.4. By Lemma 5.6, $F^n_\nu \nu \in U(\vec{v}_n)$ for all $n$, and the Jacobian formula (5.2) implies $|A(F^n_\nu \nu) - A(\nu_s)| \sim \lambda_1^{-n} - 0$. This implies $F^n_\nu \nu \to \nu_s$ in the weak topology.

When $\vec{v}$ is preperiodic, we may assume it is fixed but not totally invariant under $F$. Let $I := [\nu_s, \nu]$ and $\Omega := \{\mu \in I \mid F^n_\mu \mu \to \nu_s\}$. As in the proof of Lemma 4.6, $F_\nu$ is given on $I$ by a (piecewise) Möbius transformation with non-negative integer coefficient fixing $\nu_s$. Hence $\Omega$ contains a small neighborhood of $\nu_s$ in $I$, and is therefore open.

**Lemma 5.7.** Let $\nu_1, \nu_2 \in \mathcal{V}_1$ be comparable (i.e. $\nu_1 \leq \nu_2$ or $\nu_2 \leq \nu_1$) valuations with $\alpha(\nu_i) > 0$, $i = 1, 2$. Then

$$|A(F_\nu \nu_2) - A(F_\nu \nu_1)| \leq \frac{4}{\alpha(\nu_1)\alpha(\nu_2)}|A(\nu_2) - A(\nu_1)|$$

(5.4)

for any dominant polynomial mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$.

This result implies the (local) equicontinuity of the family $(F^n_\nu)$ on $\Omega \cap \{\alpha > 0\}$, hence $\Omega \cap \{\alpha > 0\}$ is closed. When $\alpha(\nu) > 0$, we conclude that $\nu \in \Omega$. Otherwise, $\alpha(\nu) = 0$. To complete the proof, we only need to show that $\alpha(\nu_n) > 0$ for some $n \geq 1$, where $\nu_n = F^n_\nu \nu$.

Suppose to the contrary that $\alpha(\nu_n) = 0$ for all $n \geq 1$. As in the proof of Proposition 5.1 we get $F^n_\nu \alpha \nu_n = c_n \alpha \nu$, where $c_n > 0$. By Proposition A.7 this shows that $\nu$ is the only preimage of $\nu_n$ under $F^n_\nu$. Let $\vec{v}_n$ be the tangent vector at $\nu_s$ represented by $\nu_n$. By Lemma 5.6 $\vec{v}$ is the only preimage of $\vec{v}_n$ under $F^n$. If $n \geq 2$, this implies that $\vec{v}$ is totally invariant under $F^2$, contradicting Lemma 5.5 since $U(\vec{v}) \cap \mathcal{T}_{F^2} = \emptyset$. This proves (c).

Finally we consider the monomialization statement in (d). The starting point is

**Lemma 5.8.** A quasimonomial valuation $\nu \in \mathcal{V}_1$ is monomial in some affine coordinates iff $A(\nu) + m(\nu)\alpha(\nu) < 0$.

Define $\nu_s$ as the minimal element in $\mathcal{T}_{F^s}$. We claim that, in suitable affine coordinates, $\nu_s$ becomes a monomial valuation. In view of Lemma 5.8 it suffices to prove that $A(\nu_s) + m(\nu_s)\alpha(\nu_s) < 0$. We assume $\nu_s \not= -\deg$. By Lemma 5.5 and the minimality of $\nu_s$, the tangent vector at $\nu_s$ represented by $-\deg$ is not totally invariant under the tangent map. Hence we can find a small segment $I \subset [-\deg, \nu_s]$ and another small segment $I'$ such that $I \cap I' = \{\nu_s\}$ and $F_\nu$ maps $I$ homeomorphically onto $I'$.

We use the Jacobian formula (5.2). If the segments are chosen small enough, we have $A = -m_s\alpha + \delta$ on $I'$, where $m_s = m(\nu_s)$ and $\delta \in \mathbb{R}$. We must prove that $\delta < 0$. The right hand side of (5.2) can be written

$$-m_s d(F, \nu)\alpha(F_\nu \nu) + \delta d(F, \nu) = -m_s(Z_{F_\nu \nu} \cdot Z_{\nu_s}) + \delta d(F, \nu) =$$

$$= -m_s(Z_{\nu_s} \cdot F^* Z_{\nu_s}) + \delta d(F, \nu) = -m_s\lambda_1\alpha(\nu_s) + \delta d(F, \nu).$$

As the left hand side in (5.2) is strictly increasing in $\nu$ we get $\delta < 0$. The
Hence $T_{F^2}$ contains a monomial valuation. If $T_{F^2} = \{\nu_s\}$ is a singleton, then $T_F = \{\nu_s\}$, $\nu_s$ is divisorial and $\alpha(\nu_s) > 0$, and there is nothing left to do. Assume therefore that $T_{F^2}$ is a nontrivial segment that contains at least one monomial valuation.

First suppose that $T_{F^2}$ contains a unique monomial valuation $\nu_s$; this is then necessarily divisorial, given by $\nu(x) = -p/q$, $\nu(y) = -1$, where $q \geq p \geq 1$ and $\gcd(p, q) = 1$. We wish to change coordinates so that $T_{F^2}$ contains a nontrivial segment of monomial valuations. For this, it suffices to prove that $p = 1$. Indeed, there then exists a valuation $\nu_1 \in T_{F^2}$ with $\nu_1 > \nu_s$, $\nu_1(x) = -1/q$, $\nu_1(y) = -1$ and $\nu_1(y + ax^q) > -1$ for some $a \in \mathbb{C}^*$. After a change of coordinates by a shear of the form $(x, y) \mapsto (x, y + ax^q)$, $T_{F^2}$ will then contain a monomial subsegment.

To prove $p = 1$, pick $\nu_1 > \nu_s$ such that $\nu_1 \in T_{F^2}$ and such that the multiplicity is constant, equal to $q$, on $I = [\nu_s, \nu_1]$. The Jacobian formula (5.2) applied to $F^{2n}$ on $I$ yields $\nu(\partial F^{2n}/\partial \nu) = (\lambda_{2n}^2 - 1)\nu(\nu)$, so $\nu \mapsto \nu(\partial F^{2n}/\partial \nu)$ is affine on $I$ with slope $q(\lambda_{2n}^2 - 1)$. This implies $\deg(\partial F^{2n}/\partial \nu) \geq q(\lambda_{2n}^2 - 1)$. Applying the Jacobian formula to $-\deg$ gives $\deg(\partial F^{2n}/\partial \nu) = -\deg(\partial F^{2n}/\partial \nu(\partial F^{2n}/\partial \nu)) - 2$. The retraction of $-\deg$ on $T_{F^2}$ equals $\nu_s$, so $A(\partial F^{2n}/\partial \nu(\partial F^{2n}/\partial \nu)) \mapsto A(\nu_s) = -1 \neq p/q$ as $n \to \infty$. On the other hand, $\deg(\partial F^{2n}/\partial \nu) \leq d(\partial F^{2n}/\partial \nu)/\alpha(\nu) = \lambda_{2n}^2 q/p$. Altogether this gives $q/p + 1 \geq q$, so $p = 1$.

We may therefore assume that $T_{F^2}$ contains a nontrivial segment consisting of monomial valuations. We saw that the multiplicity on any totally ordered open subsegment of $T_{F^2}$ is constant, hence $T_{F^2}$ is a segment which is the union of a segment of monomial valuations $[\nu_0, \nu_1]$, $\nu_0 \leq \nu_1$ and a totally ordered segment whose minimum is $\nu_0$. We may assume $\nu_0(x) = -p/q$, $\nu_0(y) = -1$, where $q \geq p \geq 1$ and $\gcd(p, q) = 1$. If $p > 1$, then the above argument shows that $\nu_0$ is an endpoint in $T_{F^2}$, yielding $T_{F^2} = [\nu_0, \nu_1]$. Hence assume $p = 1$ and that $T_{F^2} \neq [\nu_0, \nu_1]$. We proceed by induction on $q$. If $q = 1$, then $\nu_0 = -\deg$ and we may change coordinate by an affine map of the form $(x, y) \mapsto (x, y + ax)$. The valuations in $[\nu_0, \nu_1]$ remain monomial, but $T_{F^2}$ contains a new monomial valuation $\nu_1'$, which we may assume maximal. Thus $T_{F^2} = [\nu_1', \nu_1]$ and we are done. If $q > 1$, then we change coordinates by a shear of the form $(x, y) \mapsto (x, y + ax^q)$. The valuations in $[\nu_0, \nu_1]$ remain monomial, but $\nu_0$ is no longer the minimal monomial valuation in $T_{F^2}$. By the inductive hypothesis we may now make $T_{F^2}$ monomial. This completes the proof of Proposition 5.3.

5.5. Proofs of Lemmas. Finally we prove the lemmas used above.

Proof of Lemma 5.3. Pick a segment $I$ in $\nu_0$ representing the totally invariant tangent vector at $\nu_s$. If $\nu \in I$ is close enough to $\nu_s$, then $\nu' := F \circ \nu \in I$. As the tangent vector is totally invariant, $\nu$ is then the only preimage of $\nu'$ under $F \circ \nu$. Write $d = d(F, \nu)$, $\alpha(\nu) = \alpha(\nu')$. Then $F \circ \nu = d \circ \nu$ and since $F$ is proper $F^* \nu = \frac{\lambda_2}{\lambda_1} \circ \nu$. On the one hand, this gives

$$\lambda_2 \nu = \lambda_2(\nu) \circ \nu = (F^* \nu) \circ (F^* \nu) = \frac{\lambda_2}{\lambda_1} \circ \nu = \frac{\lambda_2}{\lambda_1} \circ \nu.$$

On the other hand, we get

$$\frac{\lambda_2}{d} \circ \nu = (F^* \nu) \circ (F^* \nu) = \lambda_1 \circ \nu.$$

Now either $\nu, \nu' > \nu_s$ or $\nu, \nu' < \nu_s$. In both cases, we easily deduce, using $\lambda_2 = \lambda_1^2$ and $\alpha(\nu_s) > 0$, that $\lambda_1 = d$ and $\alpha = \alpha'$. Hence $\nu' = F \circ \nu = \nu$, which completes the proof.
Proof of Lemma 5.6. Pick a tight compactification $X_0$ of $\mathbb{C}^2$ on which the center of $\nu_*$ is a prime $E_0$ of $X$. Let $V \subset \text{Pic}(X_0)$ be the subspace spanned by all primes $E \not= E_0$. Then $V$ lies in the orthogonal complement of the class $Z_0$ determined by the condition $Z_0 \cdot Z = \text{ord}_{E_0}(Z)$ for all $Z$. But $\alpha(\nu_*) > 0$, hence $Z_0^2 > 0$, see Lemma A.2. Thus the intersection form is negative definite on $V$. We may therefore contract all primes $E \not= E_0$: we get a normal (singular) surface $X_1$ on which the center of $\nu_*$ is a prime $E_1$, and the lift $\tilde{F} : X_1 \to X_1$ is holomorphic. Pick a tangent vector $\tilde{v}$ at $\nu_*$. It corresponds to a point $p \in E_1$: the set $U(\tilde{v})$ is exactly the set of valuations $\nu \in V_0$ centered at $p$. The map $\tilde{F}$ induces a finite germ $(X_1, p) \to (X_1, \tilde{F}(p))$ hence $F_*$ maps $U(\tilde{v})$ onto $U(F(\tilde{v}))$. □

Proof of Lemma 5.7. The proof is based on the Jacobian formula (5.2). Write $\alpha_i = \alpha(\nu_i) > 0$, $A_i = A(\nu_i) < 0$, $A_i' = A(F_*\nu_i) < 0$, $d_i = d(F, \nu_i) > 0$ and $J_i = \nu_i(JF) < 0$ for $i = 1, 2$ and write $d = \deg F$. The functions $\nu \mapsto -d(F, \nu)$ and $\nu \mapsto \nu(JF)$ define tree potentials on $V_0$ in the sense of [30] §A.4. Hence $|d_2 - d_1| \leq d|\alpha_2 - \alpha_1| \leq d|A_2 - A_1|$, $|J_2 - J_1| \leq (\deg JF)|\alpha_2 - \alpha_1| \leq (2d - 2)|A_2 - A_1|$ and $d_i \geq d\alpha_i$. Moreover, $|A_1 + J_1| \leq 2d$. Thus (5.2) gives

$$|A_1' - A_1'| = \left| \frac{A_2 + J_2}{d_2} - \frac{A_1 + J_1}{d_1} \right| = \left| \frac{(A_2 + J_2)(d_1 - d_2)}{d_1d_2} + \frac{(A_2 - A_1)(J_2 - J_1)}{d_1} \right| \leq \frac{2}{\alpha_1\alpha_2} \left| A_2 - A_1 \right|,$$

which completes the proof since $0 < \alpha_2 \leq 1$. □

Proof of Lemma 5.8. The proof is based on [30] Appendix A. By the Line Embedding Theorem it is sufficient to prove that $A(\nu) + m(\nu)\alpha(\nu) < 0$ iff $\nu \in V_1$ is dominated by a rational pencil valuation.

Suppose $C$ is a curve with one place at infinity whose associated pencil valuation $\nu_{\mid C}$ dominates $\nu$. Then $\alpha(\nu_{\mid C}) = 0$ and $\alpha \leq 0$ on $[\nu, \nu_{\mid C}]$, so

$$A(\nu_{\mid C}) - A(\nu) = -\int_{\nu_{\mid C}} m(\mu)\, d\alpha(\mu) \geq -\int_{\nu} m(\nu)\, d\alpha(\mu) = m(\nu)\alpha(\nu).$$

By [30] Proposition A.4, the pencil $|C|$ is rational iff $A(\nu_{\mid C}) < 0$. Hence, $A(\nu) + m(\nu)\alpha(\nu) < 0$ if $|C|$ is rational. On the other hand, we may always pick the curve $C$ with $\deg(C) = m(\nu)$. Then the multiplicity $m$ is constant equal to $m(\nu)$ on $[\nu, \nu_{\mid C}]$. Thus equality holds above, and we get $A(\nu_{\mid C}) = A(\nu) + m(\nu)\alpha(\nu)$. So when $A(\nu) + m(\nu)\alpha(\nu) < 0$, $|C|$ is rational. □

6. Proofs of Theorems A and B

Fix an embedding $\mathbb{C}^2 \subset \mathbb{P}^2$ and consider an arbitrary polynomial mapping $F : \mathbb{C}^2 \to \mathbb{C}^2$. We start by making a few general remarks.

First, if $X$ is an admissible compactification of $\mathbb{C}^2$ and the lift $\tilde{F} : X \to X$ satisfies $\tilde{F}^{(j+n)*} = F^* \tilde{F}^{n*}$ on Pic($X$) for $j \geq 1$, then $\text{deg}(F^{j+n}) = (\tilde{F}^{(j+n)*}) \cdot \mathcal{L} = (F^* \tilde{F}^{n*}) \cdot \mathcal{L} \mathcal{L}$ with $\mathcal{L} \in \text{Pic}(X)$ the (pull-back of the) class of a line in $\mathbb{P}^2$. Hence $(\text{deg}(F^j))_{j \geq 1}$ satisfies the linear recurrence relation determined by the characteristic polynomial of the linear
map \( \tilde{F}^* : \text{Pic}(X) \to \text{Pic}(X) \). In the basis given by the primes of \( X \), \( \tilde{F}^* \) can be expressed with integer coefficients. Hence Theorem B follows from Theorem A in this case.

Second, if \( F : \mathbb{C}^2 \to \mathbb{C}^2 \) is not dominant, its image is a point or a curve. In either case, we pick an admissible compactification \( X \) of \( \mathbb{C}^2 \) such that the map \( X \to \mathbb{P}^2 \) induced by \( F \) is holomorphic. One can then check that the lift \( \tilde{F} : X \to X \) is also holomorphic. This establishes Theorems A and B in the nondominant case.

Third, if \( F : X \to Y, G : Y \to Z \) are dominant rational maps between surfaces, and \( F^*, G^* \) denotes the action on the respective Picard groups of these surfaces, then \( F^* \circ G^* = (G \circ F)^* \) iff no curve in \( X \) is contracted by \( F \) to an indeterminacy point of \( G \).

Using this, it is not difficult to see that in the case \( \lambda_2 < \lambda_1^2 \), Theorem A follows directly from Theorems 3.1, 4.1 and 4.3. We obtain Theorem B as a consequence, in view of the argument above, but we have in any case established stronger versions in Corollaries 3.3, 4.2 and 4.4.

From now on, assume \( F \) is dominant and \( \lambda_2 = \lambda_1^2 \). We shall freely use the results in Section 5.

If \( \deg F^n/\lambda_1^n \) is unbounded, then by Proposition 5.1 there is a unique rational pencil valuation \( \nu_1 \) such that \( F^n_\nu : \nu \to \nu_1 \) for every \( \nu \in \mathcal{V}_1 \). We may then proceed exactly as in Section 4 and prove analogs of Theorem 4.1 and Corollary 4.2.

If instead \( \deg F^n/\lambda_1^n \) is bounded, then we have already seen in the proof of Theorem C that \( F \) lifts to a holomorphic selfmap of a suitable compactification \( X \) of \( \mathbb{C}^2 \), proving Theorem A. However, this compactification need not be smooth or dominate the given compactification \( \mathbb{P}^2 \supset \mathbb{C}^2 \), so Theorem B does not immediately follow.

First assume \( \lambda_1 = 1 \) so that \( \deg F^n \) is bounded. Then \( F \) is in particular birational. The argument by Diller-Favre [14, Theorem 0.1] gives us an admissible compactification \( X \) of \( \mathbb{C}^2 \) such that the lift \( \tilde{F} : X \to X \) is algebraically stable. Thus Theorems A and B hold in this case.

Finally assume \( \lambda_1 > 1 \) and \( \deg F^n/\lambda_1^n \) is bounded and consider the set \( \mathcal{T}_{F^2} \) of eigenvaluations for \( F^2 \). We consider three cases. In the first case, \( \mathcal{T}_{F^2} = \mathcal{T}_F = \{ \nu_1 \} \) is a singleton. Then Proposition 5.3 shows that \( F^n_\nu : \nu \to \nu_1 \) for any \( \nu \in \mathcal{V}_1 \). We may then proceed exactly as in Section 4 and prove analogs of Theorem 4.3 and Corollary 4.4. This gives a precise version of Theorem A (with \( X \) a tight—hence smooth—compactification of \( \mathbb{P}^2 \)) and Theorem B.

In the second case, \( \mathcal{T}_{F^2} = \mathcal{T}_F = \{ \nu_1, \nu_2 \} \) is a segment, where \( \nu_1 \) and \( \nu_2 \) are divisorial. We can then proceed essentially as in Section 4. Namely, let \( X_0 \) be the minimal admissible compactification of \( \mathbb{C}^2 \) such that the centers of \( \nu_1 \) and \( \nu_2 \) are one-dimensional. We can then make further blowups to arrive at a tight compactification \( X \) such that for any prime \( E \) of \( X \) that is the center of a divisorial valuation in \( \mathcal{T}_F \), the lift \( \tilde{F} : X \to X \) is holomorphic at any periodic point of \( \tilde{F}^E \). In view of Proposition 5.3 (c) this shows that there exists \( n \geq 1 \) such that the following holds for any prime \( E \) of \( X \): either \( E \) is the center of a valuation in \( \mathcal{T}_F \) and then \( \tilde{F}E = E \); or \( F^n \) contracts \( E \) onto a point at which all iterates of \( X \) are holomorphic. Thus a precise form of Theorem A holds.

We can also prove a precise version of Theorem B as in Corollary 4.4 in this setting; in particular, the sequence \( (d(F^j, \nu)_{j \geq n(\nu)}) \) satisfies an integral linear recursion formula for all \( \nu \in \mathcal{V}_1 \).
In the third and final case, \( T_{F^2} = [\nu_1, \nu_2] \) is a nontrivial segment and \( T_F = \{ \nu_3 \} \) is a singleton. Again we can prove a precise version of Theorem B as in Corollary 4.4; it suffices to apply the result just proved to \( F^2 \).

Remark 6.1. One can check that for \( F(x, y) = (y^3, x^2) \) there is no toric admissible compactification of \( X \supset C^2 \) such that the lift of \( F \) to \( X \) satisfies the properties in Theorem A. However, \( F \) is holomorphic on \( \mathbb{P}^1 \times \mathbb{P}^1 \). We conjecture that by adding suitable lower degree terms to \( F \), no smooth compactification of \( C^2 \) (admissible or not) will do.

7. SMALL TOPOLOGICAL DEGREES: \( \lambda_2 \leq \lambda_1 \)

Here we prove Theorem D and provide examples of maps with \( \lambda_2 = \lambda_1 \).

7.1. Proof of Theorem D. Let \( F : C^2 \to C^2 \) be a dominant polynomial mapping with \( \lambda_2 < \lambda_1 \). Define

\[
G^+(p) := \limsup_{n \to \infty} \lambda_1^{-n} \log^+ \| F^np \|.
\]

We shall see momentarily that the \( \limsup \) is in fact a limit (and ultimately even a locally uniform one), but let us first establish

Lemma 7.1. We have \( G^+ \leq C_1 \log^+ \| \cdot \| + C_2 \) on \( C^2 \) for some constants \( C_1, C_2 > 0 \).

Proof. By Theorem B' in [30] there exists a psh function \( U \) on \( C^2 \) and a constant \( C > 0 \) such that \( U \circ F \leq \lambda_1 U \) on \( C^2 \) and such that \( C^{-1} \log \| \cdot \| \leq U \leq C \log \| \cdot \| \) outside a compact subset of \( C^2 \). This easily implies the lemma. \( \square \)

The key step in the proof of Theorem D is to find a sort of filtration for the dynamics, similar to the one in the case of polynomial automorphisms, see e.g. [5, 33, 40].

Lemma 7.2. For every \( \varepsilon > 0 \) there exists an integer \( n_0 \geq 1 \), a constant \( C > 0 \) and a partition \( C^2 = V \cup V' \) with \( V^+ \) open, \( FV^+ \subset V^+ \) and such that:

(i) the \( \limsup \) defining \( G^+ \) is a locally uniform limit on \( V^+ \); \( G^+ \) is pluriharmonic and strictly positive there;

(ii) For \( p \in V \) we have \( \log^+ \| F^{n_0} p \| \leq (\lambda_2 + \varepsilon)^{n_0} \log^+ \| p \| + C \).

Proof of Theorem D. Apply Lemma 7.2 with \( 0 < \varepsilon < \lambda_1 - \lambda_2 \). Set \( U^+ = \bigcup_{n \geq 0} F^{-n} V^+ \) and \( K^+ := C^2 \setminus U^+ \). Then \( U^+ \) is open and \( G^+ \) is pluriharmonic and strictly positive there. On the other hand, the estimate in (ii) implies \( G^+ \equiv 0 \) on \( K^+ \). Thus \( G^+ \) is everywhere defined and satisfies \( G^+ \circ F = \lambda_1 G^+ \).

Let \( G^{++} \) be the upper semicontinuous regularization of \( G^+ \). We shall prove that \( G^{++} = 0 \) on \( K^+ \). Lemma 7.1 implies \( G^{++} \leq C_1 \log^+ \| \cdot \| + C_2 \) on \( C^2 \) for some constants \( C_1, C_2 > 0 \). Now pick \( p \in K^+ \). Then \( F^k p \in V \) for all \( k \geq 0 \) so using the estimate in Lemma 7.2 (ii) we get

\[
G^{++} \circ F^k (p) \leq C_3 (\lambda_2 + \varepsilon)^{kn_0} (\log^+ \| p \| + C_4)
\]

for suitable constants \( C_3, C_4 > 0 \) and all \( k \geq 1 \). Now the equality \( G^+ \circ F = \lambda_1 G^+ \) yields the inequality \( G^{++} \circ F \geq \lambda_1 G^{++} \) (we have equality outside the curves contracted by \( F \)), so (7.1) gives

\[
G^{++}(p) \leq C_3 \left( \frac{\lambda_2 + \varepsilon}{\lambda_1} \right)^{kn_0} (\log^+ \| p \| + C_4) \to 0 \text{ as } k \to \infty.
\]
Hence $G^{++} \equiv G^+ \equiv 0$ on $K^+$. We may now argue as in the proof of [5, Proposition 3.4] to prove that $G^+$ is continuous and psh on $\mathbb{C}^2$, that the limit defining $G^+$ is locally uniform on $\mathbb{C}^2$, and that the support of the current $dd^cG^+$ equals $\partial K^+$.

The estimate in Theorem D follows from the corresponding estimate in Lemma 7.2(ii). This completes the proof. □

The filtration in Lemma 7.2 is constructed using a well chosen compactification of $\mathbb{C}^2$.

**Lemma 7.3.** For every $\varepsilon > 0$ there exists an integer $n_0 \geq 1$, an admissible compactification $X$ of $\mathbb{C}^2$ and a decomposition $X \setminus \mathbb{C}^2 = Z^+ \cup Z^-$ into (reducible) curves $Z^+$, $Z^-$ without common components such that:

(i) if $E$ is any irreducible component of $Z^-$ and $L$ is a generic affine function on $\mathbb{C}^2$ then

\[
\text{ord}_E(F^{n_0}L) \leq (\lambda_2 + \varepsilon)^{n_0} \text{ord}_E(L); \tag{7.2}
\]

(ii) there exists a point $p \in Z^+ \setminus Z^-$ such that $\tilde{F}^{n_0}$ is holomorphic in a neighborhood of $Z^+$, $\tilde{F}^{n_0}(Z^+) = \{p\}$, $\tilde{F}$ is holomorphic at $p$, $\tilde{F}(p) = p$; and there exist local coordinates $(z, w)$ at $p$ in which $\tilde{F}$ takes a simple normal form as in Theorem 7.2:

(a) if $Z^+$ is locally reducible at $p$, then $Z^+ = \{zw = 0\}$ and $\tilde{F}(z, w) = (z^aw^b, z^cw^d)$, where $a, b, c, d \in \mathbb{N}$ and the $2 \times 2$ matrix $M$ with entries $a, b, c, d$ has spectral radius $\lambda_1$;

(b) if $Z^+$ is locally irreducible at $p$, then $Z^+ = \{z = 0\}$ and $\tilde{F}(z, w) = (z^{\lambda_1}, \mu z^c w + P(z))$, where $c \geq 1$, $\mu \in \mathbb{C}^*$, and $P$ is a nonconstant polynomial with $P(0) = 0$.

The admissible compactification $X$ will not be tight in general. We first show how to deduce Lemma 7.2 from Lemma 7.3 then prove the latter lemma.

**Proof of Lemma 7.2.** Apply Lemma 7.3 after having decreased $\varepsilon$ slightly. We can find a small neighborhood $\Omega_p$ of $p$ of the form $\{|z| < \delta_1, |w| < \delta_2\}$ such that $\tilde{F}\Omega_p \subset \Omega_p$. Set $V_p = \Omega_p \cap \mathbb{C}^2$, $V^+ := F^{-n_0}(\Omega_p) \cap \mathbb{C}^2 = F^{-n_0}V_p$ and $V = \mathbb{C}^2 \setminus V^+$. Then $V^+ \subset \mathbb{C}^2$ is open and $FV^+ \subset V^+$.

In local coordinates $(z, w)$ near a prime $E = \{z = 0\}$ of $X$, the function $\log^+ \| \cdot \|$ in $\mathbb{C}^2$ equals $-\text{ord}_E(L) \log |z| + O(1)$. Similarly, in local coordinates $(z, w)$ at the intersection point between two primes $E = \{z = 0\}$ and $E' = \{w = 0\}$ we have $\log^+ \|(z, w)\| = -\text{ord}_{E'}(L) \log |w| + O(1)$.

Note that $F^{-n_0}(\Omega_p)$ contains a neighborhood of $Z^+$ in $X$. Estimate (ii) in Lemma 7.2 is therefore a consequence of (7.2).

It remains to prove (i). For this we use the normal forms in Lemma 7.3. Suppose we are in case (a). Then $\log^+ \|(z, w)\| = -s \log |z| - t \log |w| + \varphi(z, w)$ in $V_p$ for some constants $s, t > 0$ and a bounded function $\varphi$. It then follows easily that $\lambda_1^{-n} \log^+ \| F^n \|$ converges locally uniformly on $V_p$ to $G^+ = -s' \log |z| - t' \log |w|$, where $s', t' > 0$, (the vector $(s', t')$ is proportional to the eigenvector with eigenvalue $\lambda_1$ of $M^t$.) Hence $G^+$ is pluriharmonic and strictly positive in $V_p$. Since $F^{n_0}V^+ \subset V_p$, the same properties must hold in $V^+$. This completes the proof in case (a). Case (b) is similar and left to the reader. □
Proof of Lemma 7.3. We apply Theorem 3.1. We shall only treat case (a) of that theorem, case (b) being similar.

Thus we have an admissible (tight) compactification $X_0$ of $\mathbb{C}^2$, two primes $E_1$, $E_2$ of $X_0$, intersecting in a point $p$, such that the lift $\tilde{F}_0 : X_0 \rightarrow X_0$ defines an attracting holomorphic fixed point germ at $p$ given by $\tilde{F}_0(z, w) = (z^a w^b, z^c w^d)$ in suitable local coordinates $(z, w)$. Write $Z_0 = E_1 \cup E_2$. Set $\gamma := \max_{i=1,2} \text{ord}_{\tilde{F}_0} (L)$, where $L$ is a generic affine function on $\mathbb{C}^2$. Pick $n_0 \geq 1$ large enough so that $\gamma \lambda_2^{n_0} < (\lambda_2 + \varepsilon)^{n_0}$.

By increasing $n_0$ if necessary, we can also assume that all valuations in the segment $I = [-\text{deg}, \nu_s]$ are mapped by $F^{n_0}$ into the open set $U(p) \subset V_0$. Here, as before, $U(p)$ consists of valuations whose center in $X_0$ is the point $p$. Indeed, $U(p)$ is $F_s$-invariant as $F$ is holomorphic at $p$, and by Theorem 2.3 (b), $F^n \nu$ eventually falls into $U(p)$ for $n$ large enough for any $\nu \in I$.

We can now find an admissible compactification $X$ of $\mathbb{C}^2$ dominating $X_0$ such that $F^{n_0}$ lifts to a holomorphic map $G : X \rightarrow X_0$. Since $\tilde{F}_0$ was holomorphic at $p$, we may assume that the strict transforms of $E_1$ and $E_2$ in $X$ still intersect in a point $p$. We use the notation $E_1$, $E_2$ and $p$ also for these objects on $X$.

Let $Z^+ \subset X \setminus \mathbb{C}^2$ be the connected component of $G^{-1}(p)$ containing $p$. Our choice of $n_0$ implies that all primes whose associated normalized divisorial valuation in $V_0$ is dominated by the eigenvaluation lie in $Z^+$. In particular the center $L_{\infty}$ of $-\text{deg}$ in $X$ belongs to $Z^+$. Let $Z^-$ be the union of the primes of $X$ not in $Z^+$. Let $\tilde{F} : X \rightarrow X$ be the rational lift of $F$. It is easy to see that (ii) holds. The key point is to establish (i).

The dual graph of $X \setminus \mathbb{C}^2$ being a tree and $Z^+$ being connected imply that each connected component $W$ of $Z^-$ contains a unique irreducible component $E = E_W$ intersecting $Z^+$. Moreover, since $L_{\infty} \subset Z^+$, the normalized divisorial valuations in $V_0$ associated to the irreducible components of $W$ all dominate (in the partial ordering on $V_0$) the normalized divisorial valuation associated to $E_W$. Thus it suffices to verify (i) for $E = E_W$.

Now $G$ must map $E$ onto one of $E_1$ or $E_2$, say $G(E) = E_1$. The restriction of $G$ to a neighborhood of $E$ has topological degree at most $\lambda_2^{n_0}$. This implies in particular that the coefficient of $E$ in the divisor $G^* E_1$ is at most $\lambda_2^{n_0}$. But this coefficient is easily seen to be

$$\frac{\text{ord}_E (F^{n_0} \ast L)}{\text{ord}_{E_1} (L)} \geq \gamma^{-1} \text{ord}_E (F^{n_0} \ast L)$$

for a generic affine function $L$ on $\mathbb{C}^2$. This proves (7.2) since $\text{ord}_E (L) \geq 1$ and $\gamma \lambda_2^{n_0} < (\lambda_2 + \varepsilon)^{n_0}$. \hfill \Box

7.2. Examples with $\lambda_2 = \lambda_1$. Very few surface maps with $\lambda_2 = \lambda_1$ have been described in the literature. We provide here a (presumably incomplete) list of examples. The mappings in (1) below appear in [20, §4]. A classification of quadratic polynomial maps with $\lambda_2 = \lambda_1$ is given in [27]. Note that Proposition 2.3 implies that the eigenvaluation of a map with $\lambda_1 = \lambda_2 > 1$ is always divisorial or infinitely singular.

(1) $F = (A(x), Q(x, y))$ with $A$ affine and $\text{deg}_y Q \geq 1$. Then $\lambda_1 = \lambda_2 = \text{deg}_y Q$, and $F$ is proper iff for any fixed $x$, $\text{deg}_y Q(x, y) = \text{deg}_y Q$. One can show that any proper polynomial map with $\lambda_1 = \lambda_2 > 1$ whose eigenvaluation is divisorial takes this form in suitable affine coordinates.
(2) \( F = (P(x), A(x, y)) \) with \( \deg_y(A) = 1 \). Then \( \lambda_1 = \lambda_2 = \deg(P) \), and \( F \) is proper iff \( A(x, y) = ay + B(x) \), \( a \neq 0 \).

(3) \( F = (\lambda x P + a, \mu y P + b) \), or \( F = (x P + a, (x + y)P + b) \) with \( P = P(x, y) \) of degree \( d - 1 \), and \( a, b \in \mathbb{C}, \lambda, \mu \in \mathbb{C}^* \). Then \( \lambda_1 = \lambda_2 = d \), the eigenvaluation is \( -\deg \) which is divisorial, and \( F \) is not proper.

(4) \( F = (ay^p + P(x), x^q) \) with \( \deg P = pq, a \in \mathbb{C} \). Then \( \lambda_1 = \lambda_2 = pq \), \( F \) is proper and has an infinitely singular eigenvaluation.

(5) \( F = (x + aP, y + bP) \) with \( P = P_d + 1.0.t, d = \deg(P) \geq 2 \) and \( P_d(a, b) \neq 0 \).

Then \( F \) is proper and \( \lambda_1 = \lambda_2 = \deg P \).

(6) \( F = (ax + by + c, P(x, y)) \) with \( \deg P = \deg_y P \geq 2 \). Then \( \lambda_1 = \lambda_2 = \deg P \).

(7) \( F = (P(y), ax + b + Q(y)) \) with \( d = \deg P = \deg Q \geq 2 \). Then \( \lambda_1 = \lambda_2 = d \).

There seems to be no general conjecture or approach for studying the ergodic properties of these maps.

**Appendix A. The Riemann-Zariski space at infinity**

In this appendix we briefly develop the necessary material needed for the proof of Theorem 2.4. Most of the discussion is completely analogous to the one in the paper [8], to which we refer for details. See also [11, 43].

The main new consideration is the construction and study of the Weil class \( Z_\nu \) associated to a valuation \( \nu \) centered at infinity.

### A.1. Weil and Cartier classes.

The set of admissible compactifications of \( \mathbb{C}^2 \) defines an inverse system: \( X' \) dominates \( X \) if the birational map \( X' \to X \) induced by the identity on \( \mathbb{C}^2 \) is a morphism. Typically we then identify the primes of \( X \) with their strict transforms in \( X' \). Formally, the Riemann-Zariski space (of \( \mathbb{P}^2 \) at infinity) is defined as \( \mathcal{X} := \lim X \). (The only difference to [8] is that here we never blow up points in \( \mathbb{C}^2 \).)

Our concern is with classes on \( \mathcal{X} \) rather than \( \mathcal{X} \) itself. Given \( X \) we let \( \text{NS}(X) \) be the vector space of \( \mathbb{R} \)-divisors on \( X \) modulo numerical equivalence. Then \( \text{NS}(X) \simeq \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \). When \( X' \) dominates \( X \), the associated morphism \( \mu : X' \to X \) induces linear maps \( \mu_* : \text{NS}(X') \to \text{NS}(X) \) and \( \mu^* : \text{NS}(X) \to \text{NS}(X') \) satisfying \( \mu_* \mu^* = \text{id} \).

The space of Weil classes on \( \mathcal{X} \) is \( W(\mathcal{X}) := \lim \text{NS}(X) \). We equip it with the projective limit topology. Concretely, a Weil class \( \beta \in W(\mathcal{X}) \) is given by a collection of classes \( \beta_X \in \text{NS}(X) \), the *incarnation* of \( \beta \) on \( X \), compatible under pushforward.

A class \( \beta \in \text{NS}(X) \) for a fixed \( X \) defines a Weil class whose incarnation in any compactification \( X' \) dominating \( X \) is the pullback of \( \beta \) to \( X' \). Such a Weil class is called a Cartier class. It is determined in \( X \). Formally, the set of Cartier classes is \( \text{C}(\mathcal{X}) := \lim \text{NS}(X) \).

The intersection pairing on each \( X \) extends to a nondegenerate pairing \( W(\mathcal{X}) \times C(\mathcal{X}) \to \mathbb{R} \). In particular, we have an inner product on \( C(\mathcal{X}) \). By the Hodge index theorem, this is of Minkowski type, allowing us to define the completion \( L^2(\mathcal{X}) \).

A Weil class is nef if all its incarnations are nef. The class of a line in \( \mathbb{P}^2 \) defines a nef Cartier class \( L \) on \( \mathcal{X} \). The set of nef classes forms a closed convex cone in \( W(\mathcal{X}) \). Any nef Weil class belongs to \( L^2(\mathcal{X}) \).
A.2. Classes and valuations. Every class in $\text{NS}(X)$ admits a unique representation as a divisor with support at infinity, i.e. a real-valued function on the set of primes of $X$. Hence a Weil class $Z$ on $X$ can be identified with a homogeneous function on $\hat{V}_{\text{div}}$: its value at $\nu$ will be denoted $\nu(Z)$. For example, $\text{ord}_E(\mathcal{L}) = b_E$, where $\mathcal{L} \in W(X)$ is the class of a line on $\mathbb{P}^2$.

Given a valuation $\nu \in \hat{V}_0$ we define a Weil divisor $Z_\nu \in W(X)$ as follows: $\text{ord}_E(Z_\nu) = t \nu E_\text{ord}(\nu \wedge \nu_E)$ where $\alpha$ denotes skewness and $\nu = t\nu, \nu \in \hat{V}_0$. When $\nu$ is divisorial, $Z_\nu$ is Cartier, see Lemma A.2 below.

**Lemma A.1.** The assignment $\nu \mapsto Z_\nu$ defines a continuous embedding of $\hat{V}_0$ onto a closed subset of $W(X)$.

**Proof.** After unwinding definitions, the statement boils down to the topology on $\hat{V}_0$ being the weakest topology such that $\nu \mapsto \alpha(\nu \wedge \nu_E)$ is continuous for all $\nu \in \hat{V}_{\text{div}}$. This in turn follows from the characterization of the topology on $\hat{V}_0$ in terms of the tree structure.

**Lemma A.2.** We have $(Z_\nu \cdot W) = \nu(W)$ for $\nu \in \hat{V}_{\text{div}}$ and $W \in C(X)$. In particular, for any two valuations $\mu, \nu \in \hat{V}_0$ one has

$$(Z_{\nu} \cdot Z_{\mu}) = (\nu \wedge \mu) \in [-\infty, 1].$$

**Proof.** To prove $(Z_\nu \cdot W) = \nu(W)$, we pick $X$ such that $W \in C(X)$ is determined in $X$. By linearity we may assume $W = E$ is a prime of $X$. What we seek to prove is then a special case of a more general formula $(Z \cdot E) = b_E \Delta g(\nu_E)$ for any $Z \in \text{NS}(X)$, where $g = g_Z$ is the function on $\hat{V}_0$ defined by $Z$ and $\Delta g$ is its tree Laplacian as defined (up to a sign) in [30] Section A.4. Indeed, $Z = Z_\nu$ as above is chosen so that $\Delta g_Z = \delta_\nu$.

Let $E_1, \ldots, E_n$ be the primes of $X$ intersecting $E$ properly and write $b_i = b_{E_i}, \nu_i = \nu_{E_i}$. Assume $E \neq L_\infty$ for simplicity. Then $0 = (\mathcal{L} \cdot E) = b_E(E \cdot E) + \sum_i b_i$ and $(Z \cdot E) = b_E g(\nu_E)(E \cdot E) + \sum_i b_i(g(\nu_i))$. Subtracting, and rearranging using $|\alpha(\nu_i) - \alpha(\nu_E)| = (b_e b_E)^{-1}$ yields $(Z \cdot E) = b_E \sum_i (g(\nu_i) - g(\nu_E))/|\alpha(\nu_i) - \alpha(\nu_E)|$, which equals $b_E \Delta g(\nu_E)$. A similar computation works in the case $E = L_\infty$, and completes the proof of the relation $(Z_{\nu} \cdot W) = \nu(W)$. This relation and the definition of $Z_\nu$ and $Z_\mu$ imply (A.1).

For the last statement it suffices to observe that by the non-degeneracy and unimodularity of the intersection form on $\text{Pic}(X)$, there exists a unique integral class $Z$ satisfying $(Z \cdot W) = \text{ord}_E(W)$ for any $W \in \text{Pic}(X)$.

**Lemma A.3.** If $\nu \in \hat{V}_0$ then $Z_\nu$ is nef iff $\alpha(\nu) \geq 0$.

**Proof.** If $Z_\nu$ is nef, then (A.1) shows that $\alpha(\nu) = (Z_{\nu} \cdot Z_{\nu}) \geq 0$. Conversely, if $\alpha(\nu) \geq 0$, then the definition of $Z_\nu$ shows that $Z_\nu \geq 0$ as a function on $\hat{V}_{\text{div}}$ and that $\nu \mapsto Z_\nu$ is decreasing. The nef cone in $W(X)$ being closed, it suffices by Lemma A.1 to consider the case when $\nu$ is divisorial.

Hence assume $\nu = b_E^{-1} \text{ord}_E$ is divisorial, where $E$ is a prime in some admissible compactification $X$. We must show that $(Z_\nu \cdot C) \geq 0$ for every irreducible curve $C$ in $X$. If $C$ is a prime of $X$, then this is clear by the above. If instead $C$ is the closure of a curve $\{P = 0\}$ in $\mathbb{C}^2$, then $(Z_\nu \cdot C) = -\nu(P)$ which is nonnegative since $\alpha(\nu) \geq 0$. 


Remark A.4. One can also show that $Z_\nu \in L^2(\mathfrak{X})$ iff $\alpha(\nu) > -\infty$.

A.3. Functoriality. Let $F : C^2 \to C^2$ be a dominant polynomial mapping. Following [8] we define actions of $F$ by pushforward and pullback on classes on the Riemann-Zariski space $\mathfrak{X}$. The two key facts are 1) given any admissible compactification $X'$, there exists another admissible compactification $X$ such that the lift $X \to X'$ of $F$ is holomorphic, and 2) given any admissible compactification $X$ and any prime $E$ of $X$, either $F$ maps $E$ onto a point or a curve in $C^2$, or there exists another compactification $X'$ and a prime $E'$ of $X'$ such that the lift $X \to X'$ of $F$ maps $E$ onto $E'$.

To begin with, we have natural actions $F^* : C(\mathfrak{X}) \to C(\mathfrak{X})$ and $F_* : W(\mathfrak{X}) \to W(\mathfrak{X})$. For example, if $\beta \in W(\mathfrak{X})$ is a Weil class, the incarnation of $F_\ast \beta \in W(\mathfrak{X})$ on a given admissible compactification $X'$ is the push-forward of $\beta_X \in NS(X)$ by the map $X \to X'$ induced by $F$ for any $X$ such that this is holomorphic.

As in [8], the pushforward (resp. pullback) preserves (resp. extends to) $L^2$-classes. We obtain bounded operators $F_* : L^2(\mathfrak{X}) \to L^2(\mathfrak{X})$ and $(F_* \beta \cdot \gamma) = (\beta \cdot F^* \gamma)$ for $\beta, \gamma \in L^2(\mathfrak{X})$. These operators preserve nef classes. We have $F_* F^* = \lambda_2 \cdot \text{id}$ on $L^2(\mathfrak{X})$ where $\lambda_2$ is the topological degree of $F$.

Remark A.5. Note that $F$ defines a dominant rational selfmap of $\mathbb{P}^2$, so the constructions in [8] apply, but they may not yield the same result as above. For example, consider the map $F(x, y) = (x^2, xy)$, which contracts the line $x = 0$ to the origin. If $\mathcal{L}$ is the class of a line in $\mathbb{P}^2$, then $F_* \mathcal{L} = 2 \mathcal{L}$ with $F_*$ as above, whereas the image of $\mathcal{L}$ under the pushforward considered in [8] is a Cartier class determined in the blowup of $\mathbb{P}^2$ at the origin.

Lemma A.6. We have $F_* Z_\nu = Z_{F_* \nu}$ for any $\nu \in \hat{V}_0$.

Proof. By continuity and homogeneity it suffices to prove this when $\nu = \text{ord}_E$ is divisorial, associated to a prime $E$ of some admissible compactification $X$ of $C^2$. Then we can find another admissible compactification $X'$ such that the lift $\tilde{F} : X \to X'$ is holomorphic and such that $\tilde{F}(E)$ is either 1) a prime $E'$ of $X'$ or 2) a point or a curve in $C^2$.

In the first case, $F_* \nu = k \text{ord}_{E'}$, where $k$ is the coefficient of $\tilde{F}^* E'$ in $E$. For any Cartier class $W \in C(\mathfrak{X})$ we then have $(F_* Z_\nu \cdot W) = (Z_{\nu} \cdot F^* W) = \text{ord}_E(F^* W) = k \text{ord}_{E'}(W) = (Z_{F_* \nu} \cdot W)$.

In the second case, we have $F_* \nu = 0$ by definition and the same computation above shows that $(F_* Z_\nu \cdot W) = 0$ for all $W \in C(\mathfrak{X})$. Indeed, $W$ can be represented as a divisor supported at infinity. Hence $F_* Z_\nu = 0$.

The situation for the pull-back is more delicate. For simplicity we state a result only in the proper case.

Proposition A.7. Suppose $F$ is a proper polynomial map. Then any $\nu \in V_0$ admits at most $\lambda_2$ preimages in $V_0$ under $F_*$, and one can write:

$$F^* Z_\nu = \sum_{F_* \mu = \nu} a(\mu) Z_{\mu},$$

for some positive constants $a(\mu)$.
Proof. Let $K$ be the function field of $\mathbb{C}^2$. The field extension $[K : F^*K]$ has degree $\lambda_2$. By standard valuation theory, see [19], for any valuation $\nu$ on $K$ there exists at most $\lambda_2$ valuations $\mu$ such that $F_*\mu = \nu$. This implies the first assertion.

As $F$ is proper, $F_*$ is an open surjective map for the weak topology on $\mathcal{V}_0$. Note also that if (A.2) is true, then one has the uniform bound $\sum_{F^{-1}_*\nu} a(\mu) = (Z_\nu \cdot F_*\mathcal{L}) \leq \deg(F)\alpha(F_*(-\deg)) \leq \deg F$. Hence we can reduce the proof to the divisorial case by continuity. The proof is then of the same flavor as the proof of Lemma A.6 and left to the reader.

A.4. Dynamics. Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a dominant polynomial mapping. Denote its topological degree by $\lambda_2 \geq 1$ and set $\lambda_1 := \lim_{n \to \infty} (\deg F_n)^{1/n}$. The techniques of [8] can be easily adapted to prove the following result, which corresponds to parts of Theorem 3.2 and Theorem 3.5 in that paper.

Theorem A.8. Assume $\lambda_2 < \lambda_1^2$. Then there exist nonzero nef Weil classes $\theta_*, \theta^* \in \mathcal{L}^2(\mathcal{X})$ such that $F^*\theta^* = \lambda_1 \theta^*$ and $F_*\theta_* = \lambda_1 \theta_*$. They are unique up to scaling and we may normalize them by $(\theta_* \cdot \mathcal{L}) = (\theta_* \cdot \theta^*) = 1$. Then $(\theta^* \cdot \theta^*) = 0$, $F_*\theta^* = (\lambda_2/\lambda_1)\theta^*$ and for any Weil class $\theta \in \mathcal{L}^2(\mathcal{X})$, we have

$$\frac{1}{\lambda_1^n} F^{*n}\theta \to (\theta \cdot \theta_*)\theta^* \quad \text{and} \quad \frac{1}{\lambda_1^n} F^{*n}\theta \to (\theta \cdot \theta^*)\theta_* \quad \text{as } n \to \infty.$$ 

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