Derivation of Modified Input Estimation Filter Using Bayesian Framework

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Abstract. A new approach to derive the modified input estimation is proposed using Bayesian framework. Tracking manoeuvring targets is a tough problem and many algorithms are proposed to deal with it. Among these algorithms, modified input estimation (MIE) technique has been proved to be an effective method. MIE technique treats acceleration as additive input term in the state vector and estimates the original state and acceleration vectors simultaneously with a standard Kalman filter. Analogous to standard Kalman filter, however, the MIE filter equations which have a modification on Kalman gain is given without a cohesive derivation. Using a Bayesian framework, we will present a conceptually cohesive roadmap that starts at first principles and leads directly to derivation of MIE filter equations.

1. Introduction
The main task of target tracking is extracting useful information about the target’s state from observations, which are often noisy. A good model of the target will certainly facilitate this information extraction to a great extent [1]. Meanwhile, tracking algorithm based on a good model will perform better than any model-free tracking algorithm if the underlying model turns out to be a good one. On the contrary, when the model adopted by tracking algorithm is inconsistent with the underlying target model, the tracking filter tends to become unstable or even diverge. The manoeuvring of target poses a great challenge to the model-based filter due to the uncertainty of the model of target. Generally, two difference approaches are studied to handle this problem. One uses a non-manoeuvring target model to track a target moving at a constant velocity and then switches to a filter for an appropriate manoeuvring model, when the target manoeuvre is detected. This method is referred to as input estimation (IE) [2~7]. The other treats the change of plant as a Markovian parameter having a transition probability, which is referred as the interacting multiple model (IMM) [8~12].

Among IE techniques, the MIE filter proposed by Khaloozadeh and Karsaz. [13] Stands out due to its superiority performance. This method has provided a special augmentation in the state space model which considers both the state vector and an unknown acceleration vector as a new augmented state. The simulation results show that the MIE not only provides fast initial convergence rate, but it can also track a maneuvering target with fairly good accuracy. Although the MIE has some advantages of tracking maneuvering target, its step become smaller after some iterations, which will lead to decay in tracking maneuvering target. To overcome this particular problem, several methods were studied.
Bahari and Pariz employed a self-tuning approaches with variable forgetting factor whose value is determined using fuzzy logic in each iteration [14]. Paper [15] used the strong tracking technique which was proposed by Zhou DH [16] to enhance the tracking performance of MIE for high maneuvering target. However, none of these papers mentioned above can give a conceptually cohesive derivation of MIE equations, which is key to explain why the Kalman gain should be modified to achieve better results.

To give a conceptually cohesive derivation of MIE equations, the optimal filter is defined using Bayesian framework, the state estimation and its covariance matrix are expressed as Bayesian estimation. In this way, the concept is all clear and we will surely find out why MIE filter needs to modify the Kalman gain and how to modify it to achieve a better filter.

2. The framework of bayesian theory

2.1. Bayesian optimal estimation

The Bayesian approach is used to estimate the parameter $\theta$ which is a random variable and has a prior probability. We could say $\hat{\theta}$ is the optimal estimation if it would minimize the Bayesian mean square error (MSE) defined as

$$Bmse(\hat{\theta}) = E[(\theta - \hat{\theta})^2]$$

$$= \int \int (\theta - \hat{\theta})^2 p(x, \theta) dx \, d\theta$$

In (1), the expectation operator is with respect to the joint PDF $p(x, \theta)$. Using Bayes’ theorem, we can write

$$p(x, \theta) = p(\theta | x)p(x)$$

So that

$$Bmse(\hat{\theta}) = E[(\theta - \hat{\theta})^2]$$

$$= \int \int (\theta - \hat{\theta})^2 p(\theta | x)d\theta p(x)d\theta$$

Now since $p(x) \geq 0$ for all $x$, if the integral in brackets can be minimized for each $x$, then the Bayesian MSE will be minimized. Hence, differentiate the integral in the brackets, we achieve

$$\frac{\partial}{\partial \theta} \int (\theta - \hat{\theta})^2 p(\theta | x)d\theta$$

$$= \int \frac{\partial}{\partial \theta} (\theta - \hat{\theta})^2 p(\theta | x)d\theta$$

$$= -2 \int \theta p(\theta | x)d\theta + 2\hat{\theta} \int p(\theta | x)d\theta$$

let (4) equals zero, resulting in

$$\dot{\theta} = \int \theta p(\theta | x)d\theta$$

Or finally
\hat{\theta} = E(\theta | x) \hspace{1cm} (6)

In other words, the optimal estimator in terms of minimizing the Bayesian MSE is the mean of the posterior PDF \( p(\theta | x) \). The posterior PDF refers to the PDF of \( \theta \) after the data have been observed. In contrast, \( p(\theta) \) or

\[ p(\theta) = \int p(x, \theta) dx \hspace{1cm} (7) \]

may be thought of as the prior PDF of \( \theta \) before the data are observed.

In determining the MMSE estimator we first require the posterior PDF. We can use Bayes’ rule to determine it as

\[ p(\theta | x) = \frac{p(\theta | x)p(x)}{\int p(\theta | x)p(x) d\theta} \hspace{1cm} (8) \]

2.2. Properties of Gaussian PDF

We generalize the results of Bayesian theory by examining the properties of the Gaussian PDF. Assume \( x \) and \( y \) are jointly Gaussian, where \( x \) is \( k \times 1 \) and \( y \) is \( l \times 1 \), with mean vector \( \begin{bmatrix} E(x) \end{bmatrix} \) and partitioned covariance matrix

\[ C = \begin{bmatrix} C_{xx} & C_{xy} \\ C_{yx} & C_{yy} \end{bmatrix} \hspace{1cm} (9) \]

**Theorem 1:**

In this way, we can derive that

\[ E(y | x) = E(y) + C_{yx}C_{xx}^{-1}(x - E(x)) \hspace{1cm} (10) \]

\[ C_{yx} = C_{yy} - C_{yx}C_{xx}^{-1}C_{yx} \hspace{1cm} (11) \]

**Theorem 2:**

\( x - E(x | y) \) is independent with any linear transformation of \( y \).

**Theorem 3:**

\[ E(x | y, y_2) = E(x | y_1, \hat{y}_2) \hspace{1cm} (12) \]

in which

\[ \hat{y}_2 = y_2 - E(y_2 | y_1) \perp y_1 \hspace{1cm} (13) \]

considering the length of this paper, the derivation of the theorems mentioned above is omitted, anyone who is interested can refer to the book written by Steven M. Kay[17].
3. The review of MIE algorithm

3.1. Formulation of problem

We assume that the target moves in a two-dimension space. The state equation and the measurement equation are given as follows, respectively

\[ X(n+1) = F(n)X(n) + C(n)u(n) + G(n)w(n) \]  
\[ z(n) = H(n)X(n) + v(n) \]

where

- \( n \): Discrete time.
- \( X(n) \): State vector.
- \( u(n) \): the unknown acceleration vector.
- \( w(n) \): the state noise.
- \( z(n) \): the observation vector.
- \( v(n) \): the measurement noise.

\[ E\{v(n_i)\} = \begin{cases} R(n_i) & n_i = n_2 \\ 0 & n_i \neq n_2 \end{cases} \]
\[ E\{w(n_i)w^T(n_j)\} = \begin{cases} Q(n_i) & n_i = n_2 \\ 0 & n_i \neq n_2 \end{cases} \]
\[ E\{w(n)\} = 0, E\{v(n)\} = 0 \]
\[ E\{w(n)v(n)\} = 0 \]

\[ X(n) = [x(n) \quad v_x(n) \quad y(n) \quad v_y(n)]^T \]
\[ u(n) = [a_x(n) \quad a_y(n)]^T \]

- \( R(n) \): the measurement covariance matrix.
- \( Q(n) \): the process covariance matrix.

in which \( G(n), F(n), C(n), H(n) \) as functions of the update time \( T \) are expressed as

\[ F = \begin{bmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad G = \begin{bmatrix} T^2 / 2 & 0 \\ T & 0 \\ 0 & T^2 / 2 \\ 0 & T \end{bmatrix} \]
\[ C = \begin{bmatrix} T^2 / 2 & 0 \\ T & 0 \\ 0 & T^2 / 2 \\ 0 & T \end{bmatrix} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \]

3.2. The MIE algorithm

MIE algorithm was proposed by Khaloozadeh and Karsaz. The authors treat the acceleration as an additive input term in the corresponding state equation. The formulation of this method is as follows:

\[ \begin{bmatrix} X(n+1) \\ u(n+1) \end{bmatrix} = \begin{bmatrix} F & C \\ 0 & I \end{bmatrix}\begin{bmatrix} X(n) \\ u(n) \end{bmatrix} + \begin{bmatrix} G \\ 0 \end{bmatrix}w(n) \]
Define $X_{\text{aug}}(n+1), F_{\text{aug}}, G_{\text{aug}}, W_{\text{aug}}$ as

$$
X_{\text{aug}}(n+1) = \begin{bmatrix} X(n+1) \\ u(n+1) \end{bmatrix}, \quad F_{\text{aug}} = \begin{bmatrix} F & C \\ 0 & I \end{bmatrix}
$$

$$
G_{\text{aug}} = \begin{bmatrix} G \\ 0 \end{bmatrix}, \quad W_{\text{aug}} = w
$$

(17)

then the maneuvering target process model turns into non-maneuvering process model as

$$
X_{\text{aug}}(n+1) = F_{\text{aug}} X_{\text{aug}}(n) + G_{\text{aug}}(n) W_{\text{aug}}(n)
$$

(18)

Similarly, the innovative posteriori measurement $z(n+1)$ could be written as

$$
z(n+1) = HX(n+1) + v(n+1)
$$

$$
= H \left[ FX(n) + Cu(n) + Gw(n) \right] + v(n+1)
$$

$$
= \begin{bmatrix} HF & HC \end{bmatrix} \begin{bmatrix} X(n) \\ u(n) \end{bmatrix} + HGw(n) + v(n+1)
$$

(19)

Define the new augmented measurement equation components as

$$
Z_{\text{aug}}(n) = z(n+1), \quad H_{\text{aug}} = \begin{bmatrix} HF & HC \end{bmatrix}
$$

$$
V_{\text{aug}}(n) = HGw(n) + v(n+1)
$$

(20)

Achieving the augmented measurement equation

$$
Z_{\text{aug}}(n) = H_{\text{aug}} X_{\text{aug}}(n) + V_{\text{aug}}(n)
$$

(21)

We write (18) and (21) together, achieving the augmented process and measurement equations

$$
\begin{cases}
X_{\text{aug}}(n+1) = F_{\text{aug}} X_{\text{aug}}(n) + G_{\text{aug}}(n) W_{\text{aug}}(n) \\
Z_{\text{aug}}(n) = H_{\text{aug}} X_{\text{aug}}(n) + V_{\text{aug}}(n)
\end{cases}
$$

(22)

Although (22) has the form of standard Kalman filter, we should always be careful that the augmented process noise and the measurement noise are time correlation. The covariance matrix of the augmented process noise $W_{\text{aug}}(n)$ and $V_{\text{aug}}(n)$ as

$$
E \left[ 
\begin{bmatrix} W_{\text{aug}}(n_1) \\ V_{\text{aug}}(n_1) \end{bmatrix} \begin{bmatrix} W_{\text{aug}}(n_1)^T & V_{\text{aug}}(n_1)^T \end{bmatrix} \right] = \begin{bmatrix}
Q_{\text{aug}}(n_1) & T_{\text{aug}}(n_1) \\
T_{\text{aug}}^T(n_1) & R_{\text{aug}}(n_1)
\end{bmatrix}, \quad n_1 = n_2 \\
0, \quad n_1 \neq n_2
$$

(23)

Where
\[ X_{\text{Aug}}(n | n) = E[X_{\text{Aug}}(n) | Z(n)] 
= E(X_{\text{Aug}}(n)) + \text{cov}(X_{\text{Aug}}(n), Z_{\text{Aug}}(n)) \times 
[\text{cov}(Z_{\text{Aug}}(n), Z_{\text{Aug}}(n))]^{-1} \times 
[Z_{\text{Aug}}(n) - E(Z_{\text{Aug}}(n))] \]  

where the \text{cov}() means the operation of acquiring covariance. Defined the innovation as

\[ \hat{\mathbf{Z}}_{\text{Aug}}(n) = Z_{\text{Aug}}(n) - \hat{X}_{\text{Aug}}(n | n-1) \]

\[ = H_{\text{Aug}}(n)X_{\text{Aug}}(n) + V_{\text{Aug}}(n) - 
H_{\text{Aug}}(n)E[X_{\text{Aug}}(n) | Z_{\text{Aug}}(n-1)] \]

thus, (27) can be written as

\[ \hat{X}_{\text{Aug}}(n | n) = E[X_{\text{Aug}}(n) | Z(n)] 
= E(X_{\text{Aug}}(n)) + \text{cov}(X_{\text{Aug}}(n), Z_{\text{Aug}}(n)) \times 
[\text{cov}(Z_{\text{Aug}}(n), Z_{\text{Aug}}(n))]^{-1} \times \hat{\mathbf{Z}}_{\text{Aug}}(n) \]

**Step 1: state predicting**

\[ \hat{X}_{\text{Aug}}(n+1 | n) = E[X_{\text{Aug}}(n+1) | Z_{\text{Aug}}(n)] \]

Using (10), (30) can be expanded as

\[ \hat{X}_{\text{Aug}}(n+1 | n) = E(X_{\text{Aug}}(n+1)) + 
\text{cov}(X_{\text{Aug}}(n+1), Z_{\text{Aug}}(n)) \times 
[\text{cov}(Z_{\text{Aug}}(n), Z_{\text{Aug}}(n))]^{-1} \hat{\mathbf{Z}}_{\text{Aug}}(n) \]

in which

\[ Q_{\text{Aug}}(n) = E\{W_{\text{Aug}}(n)W_{\text{Aug}}^T(n)\} 
= E\{w(n)w^T(n)\} = Q \]

\[ R_{\text{Aug}}(n) = E\{V_{\text{Aug}}(n)V_{\text{Aug}}^T(n)\} 
= H(n)G(n)Q(n)G(n)^T H(n)^T + R(n) \]

\[ T_{\text{Aug}}(n) = E\{W_{\text{Aug}}(n)V_{\text{Aug}}^T\} 
= QG^T(n)H^T(n) \]
\[ E(X_{\text{deg}}(n+1)) = E(F_{\text{deg}}X_{\text{deg}}(n) + G_{\text{deg}}(n)W_{\text{deg}}(n)) = F_{\text{deg}}E(X_{\text{deg}}(n)) \] (32)

\[ \text{cov}(X_{\text{deg}}(n+1), Z_{\text{deg}}(n)) = \text{cov}(F_{\text{deg}}X_{\text{deg}}(n) + G_{\text{deg}}(n)W_{\text{deg}}(n), Z_{\text{deg}}(n)) = F_{\text{deg}} \text{cov}(X_{\text{deg}}(n), Z_{\text{deg}}(n)) + G_{\text{deg}}(n)\text{cov}(W_{\text{deg}}(n), Z_{\text{deg}}(n)) \] (33)

Substitute (32), (33) in (31), achieving

\[ \hat{X}_{\text{deg}}(n+1|n) = F_{\text{deg}}E(X_{\text{deg}}(n)) + F_{\text{deg}} \text{cov}(X_{\text{deg}}(n), Z_{\text{deg}}(n)) \times \left[ \text{cov}(Z_{\text{deg}}(n), Z_{\text{deg}}(n)) \right]^{-1} \hat{Z}_{\text{deg}}(n) + G_{\text{deg}}(n)\text{cov}(W_{\text{deg}}(n), Z_{\text{deg}}(n)) \times \left[ \text{cov}(Z_{\text{deg}}(n), Z_{\text{deg}}(n)) \right]^{-1} \hat{Z}_{\text{deg}}(n) \] (34)

Incorporating (29), (34) can be simplified to

\[ \hat{X}_{\text{deg}}(n+1|n) = F_{\text{deg}} \hat{X}_{\text{deg}}(n|n) + G_{\text{deg}}(n)\text{cov}(W_{\text{deg}}(n), Z_{\text{deg}}(n)) \times \left[ \text{cov}(Z_{\text{deg}}(n), Z_{\text{deg}}(n)) \right]^{-1} \hat{Z}_{\text{deg}}(n) \] (35)

Substitute (25), (26) in (35), getting

\[ \hat{X}_{\text{deg}}(n+1|n) = F_{\text{deg}} \hat{X}_{\text{deg}}(n|n) + G_{\text{deg}}(n)T_{\text{deg}}(n) \times \left[ \text{cov}(Z_{\text{deg}}(n), Z_{\text{deg}}(n)) \right]^{-1} \hat{Z}_{\text{deg}}(n) \] (36)

Where

\[ \text{cov}(Z_{\text{deg}}(n), Z_{\text{deg}}(n)) = E[\hat{Z}_{\text{deg}}(n)\hat{Z}_{\text{deg}}(n)^\top] \]

\[ = E\left[\left(H_{\text{deg}}(n)[X_{\text{deg}}(n) - \hat{X}_{\text{deg}}(n|n-1)] + V_{\text{deg}}(n)\right)^\top\left[H_{\text{deg}}(n)[X_{\text{deg}}(n) - \hat{X}_{\text{deg}}(n|n-1)] + V_{\text{deg}}(n)\right]\right] \]

\[ = H_{\text{deg}}(n)P_{\text{deg}}(n|n-1)H_{\text{deg}}(n)^\top + R_{\text{deg}}(n) \] (37)
Define the covariance of innovation as

\[ S_{aug}(n) = H_{aug}(n)P_{aug}(n | n-1)H_{aug}^T + R_{aug}(n) \]  

(38)

thus, (36) turns into

\[ \hat{X}_{aug}(n+1 | n) = F_{aug}\hat{X}_{aug}(n | n) + \\
G_{aug}(n)T_{aug}(n)S_{aug}^{-1}(n)\tilde{Z}_{aug}(n) \]  

(39)

**Step 2: state predicting covariance**

Assume the initial state covariance is \( P_{aug}(n | n) \), it can be expressed in Bayesian form as

\[ P_{aug}(n | n) = E\{[X_{aug}(n)-\hat{X}_{aug}(n | n)]\} \times \\
[X_{aug}(n)-\hat{X}_{aug}(n | n)]^T \]  

(40)

Define the state estimation error and state prediction error as

\[ e(n | n) = X_{aug}(n) - \hat{X}_{aug}(n | n) \]  

(41)

\[ e(n+1 | n) = X_{aug}(n+1) - \hat{X}_{aug}(n+1 | n) \]  

(42)

Thus, (40) turns into

\[ P_{aug}(n | n) = E\{e(n | n)e(n | n)^T\} \]  

(43)

The state prediction covariance \( P_{aug}(n+1 | n) \) can be expressed as

\[ P_{aug}(n+1 | n) = E\{e(n+1 | n)e(n+1 | n)^T\} \]  

(44)

using (22) and (36), (42) can be written as

\[ e(n+1 | n) = F_{aug}X_{aug}(n) + G_{aug}(n)W_{aug}(n) \\
- F_{aug}\hat{X}_{aug}(n | n) - G_{aug}(n)T_{aug}(n)S_{aug}^{-1}(n)\tilde{Z}_{aug}(n) \\
= F_{aug}e(n | n) + G_{aug}(n)W_{aug}(n) - \\
G_{aug}(n)T_{aug}(n)S_{aug}^{-1}(n)\tilde{Z}_{aug}(n) \]  

(45)

So,
\[ P_{aug}(n+1 \mid n) = E\{e(n+1 \mid n)e(n+1 \mid n)\} \]

\[ = E\{[F_{aug}e(n \mid n) + G_{aug}(n)W_{aug}(n) - \\
G_{aug}(n)T_{aug}(n)S_{aug}^{-1}(n)\tilde{Z}_{aug}(n)]\times \\
[F_{aug}e(n \mid n) + G_{aug}(n)W_{aug}(n) - \\
G_{aug}(n)T_{aug}(n)S_{aug}^{-1}(n)\tilde{Z}_{aug}(n)]^T \} \]

\[ = F_{aug}P_{aug}(n \mid n)F_{aug}^T + G_{aug}(n)Q_{aug}(n)G_{aug}^T(n) + \\
G_{aug}(n)T_{aug}(n)S_{aug}^{-1}(n)T_{aug}^T(n)G_{aug}^T(n) \tag{46} \]

**Step 3: state updating**

\[ \hat{X}_{aug}(n+1 \mid n+1) = E(X_{aug}(n+1) \mid Z_{aug}^{n+1}) \]

\[ = E(X_{aug}(n+1) \mid Z_{aug}) + \]

\[ \text{cov}(X_{aug}(n+1), Z_{aug}(n+1))\times S_{aug}^{-1}(n+1)\tilde{Z}_{aug}(n+1) \tag{47} \]

in which

\[ \text{cov}(X_{aug}(n+1), Z_{aug}(n+1) \mid Z_{aug}) \]

\[ = E\{(X_{aug}(n+1) - \hat{X}_{aug}(n+1 \mid n))\times \\
(Z_{aug}(n+1) - \hat{Z}_{aug}(n+1 \mid n))\} \]

\[ = E\{e(n+1 \mid n)\tilde{Z}_{aug}(n+1) \} \tag{48} \]

where

\[ \tilde{Z}_{aug}(n+1) = Z_{aug}(n+1) - \hat{Z}_{aug}(n+1 \mid n) \]

\[ = H_{aug}(n+1)X_{aug}(n+1) + \\
V_{aug}(n+1) - H_{aug}(n+1)\hat{X}_{aug}(n+1) \]

\[ = H_{aug}(n+1)e(n+1 \mid n) + V_{aug}(n+1) \tag{49} \]

Then, (48) turns into

\[ \text{cov}(X_{aug}(n+1), Z_{aug}(n+1) \mid Z_{aug}^{n}) \]

\[ = E\{e(n+1 \mid n)\tilde{Z}_{aug}^n(n+1) \} \]

\[ = P_{aug}(n+1 \mid n)H_{aug}^T(n+1) \tag{50} \]

Substitute (50) in (47)

\[ \hat{X}_{aug}(n+1 \mid n+1) = \hat{X}_{aug}(n+1 \mid n) + \\
P_{aug}(n+1 \mid n)H_{aug}^T(n+1)S_{aug}^{-1}(n+1)\tilde{Z}_{aug}(n+1) \tag{51} \]

We have derived the expression for \( \hat{X}_{aug}(n+1 \mid n) \), thus
\[ \hat{X}_{\text{aug}}(n+1|n+1) = \hat{X}_{\text{aug}}(n+1|n) + \]
\[ P_{\text{aug}}(n+1|n)H_{\text{aug}}^T \times S_{\text{aug}}^{-1}(n+1) \hat{Z}_{\text{aug}}(n+1) \]
\[ = F_{\text{aug}} \hat{X}_{\text{aug}}(n|n) + G_{\text{aug}}(n)T_{\text{aug}}(n) \times \]
\[ S_{\text{aug}}^{-1}(n) \hat{Z}_{\text{aug}}(n) + P_{\text{aug}}(n+1|n)H_{\text{aug}}^T(n+1) \times \]
\[ S_{\text{aug}}^{-1}(n+1) \hat{Z}_{\text{aug}}(n+1) \]

**Step 4 state updating covariance**

\[ P_{\text{aug}}(n+1|n+1) = E[(X_{\text{aug}}(n+1) - \hat{X}_{\text{aug}}(n+1|n+1))^T] \times \]
\[ [X_{\text{aug}}(n+1) - \hat{X}_{\text{aug}}(n+1|n+1)]^T \]  \hspace{1cm} (52)

Substitute (52) in (53), achieving

\[ P_{\text{aug}}(n+1|n+1) = E[(X_{\text{aug}}(n+1) - \hat{X}_{\text{aug}}(n+1|n+1))^T] \times \]
\[ [X_{\text{aug}}(n+1) - \hat{X}_{\text{aug}}(n+1|n)+ n1]^T \]
\[ = E[(X_{\text{aug}}(n+1) - \hat{X}_{\text{aug}}(n+1|n) - \sum_{i=0}^{n1} H_{\text{aug}}i(n+1)S_{\text{aug}}^{-1}(n+1) \hat{Z}_{\text{aug}}(n+1)]^T \times \]
\[ [X_{\text{aug}}(n+1) - \hat{X}_{\text{aug}}(n+1|n) - \sum_{i=0}^{n1} H_{\text{aug}}i(n+1)S_{\text{aug}}^{-1}(n+1) \hat{Z}_{\text{aug}}(n+1)]^T \]
\[ = P_{\text{aug}}(n+1|n) - P_{\text{aug}}(n+1|n) \times \]
\[ H_{\text{aug}}^T(n+1)S_{\text{aug}}^{-1}(n+1)H_{\text{aug}}(n+1)P_{\text{aug}}(n+1|n) \]

Organize (30), (46), (52), (53), getting the optimal input filter

\[ \hat{X}_{\text{aug}}(n+1|n) = F_{\text{aug}} \hat{X}_{\text{aug}}(n|n) + \]
\[ G_{\text{aug}}(n)T_{\text{aug}}(n)S_{\text{aug}}^{-1}(n) \hat{Z}_{\text{aug}}(n) \]
\[ P_{\text{aug}}(n+1|n) = F_{\text{aug}} \hat{X}_{\text{aug}}(n|n)F_{\text{aug}}^T + \]
\[ G_{\text{aug}}(n)Q_{\text{aug}}(n)G_{\text{aug}}^T(n) + \]
\[ G_{\text{aug}}(n)T_{\text{aug}}(n)S_{\text{aug}}^{-1}(n)T_{\text{aug}}^T(n)G_{\text{aug}}^T(n) \]

\[ \hat{X}_{\text{aug}}(n+1|n+1) = F_{\text{aug}} \hat{X}_{\text{aug}}(n|n) + \]
\[ G_{\text{aug}}(n)T_{\text{aug}}(n)S_{\text{aug}}^{-1}(n) \hat{Z}_{\text{aug}}(n) + \]
\[ P_{\text{aug}}(n+1|n)H_{\text{aug}}^T(n+1)S_{\text{aug}}^{-1}(n+1)H_{\text{aug}}(n+1)P_{\text{aug}}(n+1|n) \]
\[ P_{\text{aug}}(n+1|n+1) = P_{\text{aug}}(n+1|n) - P_{\text{aug}}(n+1|n) \times \]
\[ H_{\text{aug}}^T(n+1)S_{\text{aug}}^{-1}(n+1)H_{\text{aug}}(n+1)P_{\text{aug}}(n+1|n) \]
It should be noted that the state in time \((n+1)\) are related to the innovations both in time \(n\) and time \((n+1)\). Because of the innovation in time \(n\) is not defined in the process of recursion, we have to make some approximations. If we ignore the effect of innovation in state prediction, and let the innovation in time \(n\) equal to the innovation in time \((n+1)\), then (55) can be simplified to

\[
\hat{X}_{\text{aug}}(n+1 | n) = F_{\text{aug}} \hat{X}_{\text{aug}}(n | n)
\]

\[
P_{\text{aug}}(n+1 | n) = F_{\text{aug}} P_{\text{aug}}(n | n) F_{\text{aug}}^T + G_{\text{aug}}(n) Q_{\text{aug}}(n) G_{\text{aug}}^T(n)
\]

\[
\hat{X}_{\text{aug}}(n+1 | n+1) = F_{\text{aug}} \hat{X}_{\text{aug}}(n | n) + 
\[G_{\text{aug}}(n) T_{\text{aug}}(n) + P_{\text{aug}}(n+1 | n) H_{\text{aug}}^T \] \times
\]

\[S_{\text{aug}}^{-1}(n+1) \hat{Z}_{\text{aug}}(n+1) \]

\[
P_{\text{aug}}(n+1 | n+1) = P_{\text{aug}}(n+1 | n) - 
\[P_{\text{aug}}(n+1 | n) H_{\text{aug}}^T S_{\text{aug}}^{-1}(n+1) \times \]

\[H_{\text{aug}} P_{\text{aug}}(n+1 | n) \]

which is consistent with the MIE filter proposed by Khaloozadeh and Karsaz. We have to clarify that the MIE filter is a suboptimal filter because of the approximations during the derivation.

5. Conclusion
This paper presents a new approach to derive the MIE filter. First, the optimal filter is defined using the framework of Bayesian theory. Then, a cohesive derivation of the modified input estimation is given using the Bayesian theory, which is not mentioned in the original paper or the other studies concerning the MIE estimation. Finally, some approximations have to be made to achieve the suboptimal filter which is consistent with the MIE filter proposed by Khaloozadeh and Karsaz.

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