Two-parameter sums signatures and corresponding quasisymmetric functions

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Quasisymmetric functions have recently been used in time series analysis as polynomial features that are invariant under, so-called, dynamic time warping. We extend this notion to data indexed by two parameters and thus provide warping invariants for images. We show that two-parameter quasisymmetric functions are complete in a certain sense, and provide a two-parameter quasi-shuffle identity. A compatible coproduct is based on diagonal concatenation of the input data, leading to a (weak) form of Chen’s identity.

Keywords: quasisymmetric functions, warping invariants, matrix compositions, Hopf algebra, image analysis, signatures and data streams

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1 Introduction

Certain forms of signatures have proven beneficial as features in time series analysis. The iterated-integrals signature was introduced by Chen in the 1950s [Che54] for homological considerations on loop space. After applications in control theory, starting in the 1970s, [Fli76, Fli81] and rough path theory, starting in the 1990s, [Lyo98], it has in the last decade been successfully applied in machine learning tasks on time series [KBPA+19, KO19, DR19, KMFL20, TBO20]. As the name suggests, it applies to continuous objects, namely (smooth enough) curves in Euclidean space. For discrete time series to fit in the machinery, they have to undergo a (simple) interpolation step.

The iterated-sums signature, introduced in [DEFT20], forgoes this intermediate step and immediately works on the discrete-time object. This discrete perspective brings additional benefits: a broader class of features (even for one-dimensional time series, whose integral signature is trivial), flexibility in the underlying ground field [DEFT22], and a tight, new-found, connection to the theory of quasisymmetric functions [MR95] and dynamic time warping [SC78, BC94, KR05].

In the present work we will take the latter perspective and apply it to data indexed by two parameters, the canonical example being image data.

Related work

In data science, two recent works have very successfully applied iterated integrals to images. In [IL22] the classical, one-parameter, iterated-integrals signature is used for images (by working “row-by-row”), whereas certain multi-parameter iterated integrals are used in [ZLT22]. A principled extension of Chen’s iterated integrals, based on their original use in topology, is presented in [GLNO22].

More generally, the use of “signature-like” feature-maps has recently been extended to graphs [TLHO22, CDEFV22] and trees [CFC+21].

Notation

Throughout, \( \mathbb{N} = \{1, 2, \ldots\} \) denotes the strictly positive integers and \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \) denotes the non-negative integers. Let \((\mathbb{N}^2, \leq)\) denote the product poset (partially ordered
set), i.e. (here, and throughout, we denote tuples with bold letters)

\[ i \leq j \iff i_1 \leq j_1 \text{ and } i_2 \leq j_2. \]

For every matrix \( A \in M^{m \times n} \) with entries from an arbitrary set \( M \) let \( \text{size}(A) := (\text{rows}(A), \text{cols}(A)) := (m, n) \in \mathbb{N}^2 \) denote the number of rows and columns in \( A \) respectively. Let \( g \circ f : M \to P, m \mapsto g(f(m)) \) be the set-theoretic composition of functions \( f : M \to N \) and \( g : N \to P \). Let \( \mathbb{C} \) denote the complex number field.

### 1.1 Warping invariants motivate the signature

We briefly recall the notion of (classical, one-parameter) time warping invariants, as covered in [DEFT20]. For simplicity, we consider eventually-constant, \( \mathbb{C} \)-valued time series in discrete time,

\[ \text{evC}(\mathbb{N}, \mathbb{C}) := \{ x : \mathbb{N} \to \mathbb{C} | \exists n \in \mathbb{N} : x_i \neq x_j \implies i \leq n \} \].

Intuitively one might think of complex numbers as colored pixels, becoming especially valuable for visualization in the two-parameter case. Later in this paper, the complex numbers are actually replaced by the module \( \mathbb{K}^d \) over some arbitrary integral domain, covering the classical encoding of colors via \( \mathbb{R}^3 \). A single time warping operation is formalized by the mapping

\[ \text{warp}_k : \text{evC}(\mathbb{N}, \mathbb{C}) \to \text{evC}(\mathbb{N}, \mathbb{C}), \ (\text{warp}_k x)_i := \begin{cases} x_i & i \leq k \\ x_{i-1} & i > k, \end{cases} \]

which leaves all entries until the \( k \)-th unchanged, copies this value once, attaches it at position \( k + 1 \), and shifts all remaining successors by one.

For example, consider

\[ \text{warp}_2 \left( \begin{array}{cccc} 2 & 1 & 3 & 1 \\ \end{array} \right) = \begin{array}{cccc} 2 & 1 & 1 & 3 & 1 \\ \end{array} \]

where the dots on the right hand side indicate that all relevant information is provided, i.e., that the series has reached a constant and will not change again. A time warping invariant is a function from the set of time series to the complex numbers which remains unchanged under warping. An example of such an invariant is

\[ \varphi : \text{evC}(\mathbb{N}, \mathbb{C}) \to \mathbb{C}, \ x \mapsto x_1 - \lim_{t \to \infty} x_t \]  

where the limit exists, since \( x \) was assumed to be eventually constant. This invariant does not “see”, whether certain entries of a time series are repeated over and over again. Indeed, both the first entry as well as the limit can not be changed by any warping. In
the numerical example

\[ \varphi \left( \begin{array}{cccccc} 2 & 3 & 1 & 5 & 1 & 1 \\
\end{array} \right) = \varphi \left( \begin{array}{cccccc} 2 & 3 & 1 & 1 & 1 & 5 & 1 \\
\end{array} \right) = \varphi \left( \begin{array}{cccccc} 2 & 2 & 3 & 1 & 5 & 1 \\
\end{array} \right) = \varphi \left( \begin{array}{cccccc} 2 & 3 & 3 & 1 & 5 & 5 & 1 \\
\end{array} \right) = 2 - 1 \]

the initial time series is warped to three different representatives and yet still yields the same value under \( \varphi \).

Let us now move on to the two-parameter case, the setting of this paper. We denote by

\[ \text{evC}(\mathbb{N}^2, C) := \left\{ X : \mathbb{N}^2 \to C \mid \exists n \in \mathbb{N}^2 : X_i \neq X_j \implies i \leq n \right\} \]

the set of two-parameter functions which are eventually constant. A function from this set is a two-parameter analogue to a (classical, one-parameter) time series that is eventually constant and can be thought of as an image of arbitrary size, with its pixels being encoded by \( C \).

We define a single warping operation \( \text{warp}_{a,k} \) similar to the one-parameter case, except that we add a second index \( a \in \{1, 2\} \) indicating on which axis the warping takes place. For the axis \( a = 1 \) we obtain an operation on rows, i.e., at position \( k \) we copy a row and shift all remaining rows by one. Illustratively, for \( k = 2 \) we have

\[ \text{warp}_{1,2} \left( \begin{array}{cccccc}
2 & 1 & 3 & 1 & 1 \\
3 & 2 & 5 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{array} \right) = \begin{array}{cccccc}
2 & 1 & 3 & 1 & 1 \\
3 & 2 & 5 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
\end{array} \]

whereas for the axis \( a = 2 \) we get the warping of columns, illustrated by

\[ \text{warp}_{2,2} \left( \begin{array}{cccc}
2 & 1 & 3 & 2 & 2 \\
3 & 0 & 5 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 \\
\end{array} \right) = \begin{array}{cccc}
2 & 1 & 1 & 3 & 2 & 2 \\
3 & 0 & 0 & 5 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 \\
\end{array} \]

also at position \( k = 2 \). A formal definition of \( \text{warp}_{a,k} \) is provided in Section 2.4.

We call a function \( \psi : \text{evC}(\mathbb{N}^2, C) \to C \) an invariant to warping (in both directions independently), if it remains unchanged under warping, i.e.,

\[ \psi \circ \text{warp}_{a,k} = \psi \quad \forall (a,k) \in \{1, 2\} \times \mathbb{N}. \]
In Figure 1 we give a graphical illustration of eight two-parameter functions (i.e. images), that can be obtained from three initial functions by repeatedly applying warping.\footnote{In the sense of Definition 2.30 each of the eight images is equivalent to one of the initial three.} Hence, the value of any warping invariant will coincide on pairwise equivalent inputs.

As an example, consider the two-parameter analogon of Equation (1),

\begin{equation}
\Psi[t] : \ev C(\mathbb{N}^2, \mathbb{C}) \rightarrow \mathbb{C}, \ X \mapsto X_{1,1} - \lim_{s,t \to \infty} X_{s,t}
\end{equation}

which is a warping invariant. In fact it belongs to an entire class $\Psi$ of invariants constructed as follows.

An $\mathbb{N}_0$-valued matrix is called a matrix composition, if it has no zero lines or zero columns. For every eventually-zero $Z \in \ev C(\mathbb{N}^2, \mathbb{C})$, that is $\lim_{s,t \to \infty} Z_{s,t} = 0$, we define the two-parameter sums signature coefficient of $Z$ at the matrix composition $a$ via

\[
\langle \text{SS}(Z), a \rangle := \sum_{\iota_1 < \cdots < \iota_{\text{rows}(a)}} \prod_{s=1}^{\text{rows}(a)} \prod_{t=1}^{\text{cols}(a)} Z_{\iota_s \iota_t} \in \mathbb{C}.
\]

Note that this sum over strictly increasing chains $\iota$ and $\kappa$ is always finite since $Z$ is zero almost everywhere. A numerical illustration is provided in Example 2.7.

We collect the second ingredient for warping invariants. For every $X \in \ev C(\mathbb{N}^2, \mathbb{C})$ let...
\( \delta X \in \text{evC}(\mathbb{N}^2, \mathbb{C}) \) be defined via the forward difference operator

\[
(\delta X)_{i,j} := X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1} + X_{i,j}.
\]

Note that \( \delta X \) is eventually zero, since each index \( i \) for which \( X_i \) has no changing successors yields \((\delta X)_i = 0\). Putting all together, we consider a matrix composition \( a \) as an index and obtain an \( a \)-indexed family of functions

\[
\Psi_a : \text{evC}(\mathbb{N}^2, \mathbb{C}) \to \mathbb{C}, X \mapsto \langle \text{SS}(\delta X), a \rangle
\]

which are invariant to warping in both directions independently (Corollary 2.27). For example, we have

\[
\Psi_a \left( \begin{array}{cccc}
2 & 2 & 2 & 2 \\
2 & 3 & 3 & 2 \\
2 & 3 & 3 & 2 \\
2 & 3 & 3 & 3 \\
2 & 2 & 4 & 2 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{array} \right) = \Psi_a \left( \begin{array}{cccc}
2 & 2 & 2 & 2 \\
2 & 3 & 3 & 2 \\
2 & 3 & 3 & 2 \\
2 & 2 & 4 & 2 \\
2 & 2 & 4 & 2 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{array} \right)
\]

for every matrix composition \( a \). Note that the invariant from Equation (2) is indeed “signature-induced”, verified by Example 2.28.

In Corollary 2.29 and Theorem 2.32 we argue that the family \( \Psi \) is sufficiently large. In one dimension, they are precisely given by the polynomial invariants (Theorem 3.14). In Section 4.1 we provide a certain subfamily of \( \Psi \) which can be computed iteratively in linear time.

### 1.2 Illustrations of the quasi-shuffle identity and Chen’s identity

The sums signature introduced in the previous section, and properly defined in Definition 2.6, satisfies two crucial properties.

Firstly, polynomial expressions in its entry can be rewritten as linear expressions in other entries. For instance, the invariant (2), when squared, can be rewritten as the following linear combination

\[
\Psi[1] \cdot \Psi[1] = \Psi[2] + 2 \Psi[1,1] + 2 \Psi[1, 0] + 2 \Psi[0, 1].
\]

The two-parameter quasi-shuffle for matrix compositions, which appears on the right-hand side here, is defined in Definition 2.8. This then allows to formulate a quasi-shuffle identity for two-parameter signatures, Theorem 2.13. We put our results into context to the (classical, one-parameter) quasi-shuffle of words, and we suggest an algorithm (Section 4.2) for an efficient computation.

Secondly, the sums signature satisfies a kind of Chen’s identity. For two-parameter data, there is no single choice of concatenation, and we focus on the case of diagonal
concatenation. The general statement is formulated in Lemma 2.35 and Corollary 2.38, after properly introducing diagonal concatenation for functions with domain \( \mathbb{N}^2 \). Concerning the invariants \( \Psi_a \) from above, we obtain a formula to compute \( \Psi_a(X) \) with \( X \in \text{evC}(\mathbb{N}^2, \mathbb{C}) \) via certain subparts of \( X \) and \( a \), specified by the deconcatenations of \( X \) and \( a \) in the sense of Definitions 2.10, 2.33 and 2.36, respectively. As a numerical example, we can compute

\[
\Psi_a \begin{pmatrix}
7 & 2 & 2 & 2 & 2 & 0 \\
5 & 2 & 2 & 2 & 2 & 0 \\
2 & 2 & 2 & 3 & 3 & 2 \\
2 & 2 & 3 & 3 & 3 & 0 \\
2 & 2 & 2 & 4 & 2 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} = \sum_{\text{diag}(b,c)=a} \Psi_b \begin{pmatrix}
5 & 2 & 2 \\
2 & 2 & 2 \\
1 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix} \Psi_c \begin{pmatrix}
2 & 2 & 2 & 2 & 0 \\
2 & 3 & 3 & 2 & 0 \\
2 & 3 & 3 & 2 & 0 \\
2 & 2 & 3 & 3 & 0 \\
2 & 2 & 4 & 2 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

via the upper left and lower right subpart of \( X \), and all diagonal submatrices of composition \( a \). Note that this formula includes an empty composition \( e \) for which \( \Psi_e \) is constant with image \{1\}.

Section 2 introduces matrix compositions and two-parameter signatures, with its corresponding quasisymmetric functions presented in Section 3. Note that several details and proofs are postponed until Section 5.

2 Two-parameter sums signature

Let \( \mathbb{K} \) denote an integral domain, that is a non-zero, commutative ring which includes a multiplicative identity and which contains no zero divisors. For every set \( M \) let \( \mathbb{K}^M \) denote the \( \mathbb{K} \)-algebra of functions from \( M \) to \( \mathbb{K} \).

We will now replace the monoid \( (\mathbb{N}_0, +, 0) \) used for compositions from the previous section (and not to be confused with the role played by \( \mathbb{N} \) in the indexing poset of a function \( A : \mathbb{N}^2 \to \mathbb{K} \)) by an arbitrary commutative monoid \( (\mathfrak{M}, \star, \varepsilon) \). Apart from the non-negative integers, the monoid most relevant to us is the free commutative monoid generated by \( d \) letters \( 1, \ldots, d \). The latter is used in this article for the same purpose as in [DEFT22, DEFT20], to index monomials on a \( d \)-dimensional vector (the data at each point \( i \in \mathbb{N}^2 \)).

A matrix composition (with entries in \( \mathfrak{M} \)) is an element in the set of all matrices without \( \varepsilon \)-lines or \( \varepsilon \)-columns,

\[
\text{Cmp} := \text{Cmp}(\mathfrak{M}) := \left\{ a \in \mathfrak{M}^{m \times n} \middle| a_{i,*} \neq \varepsilon_{1 \times n} \text{ for } 1 \leq i \leq n \right. \\
\left. a_{*,j} \neq \varepsilon_{m \times 1} \text{ for } 1 \leq j \leq m \right\} \cup \{e\},
\]

\(^2\)Let \( \varepsilon_{s \times t} \) denote the \( \mathfrak{M} \)-valued \( s \times t \) matrix with only \( \varepsilon \) as its entries.
including the empty composition denoted by \( e \). Matrix compositions have already appeared as the building block of a certain Hopf algebra in [DHT02], but the Hopf algebra we construct in Section 2.1 is different. The following binary operation turns \( \text{Cmp} \) into a monoid.

**Definition 2.1.** For each pair of matrices \((a, b) \in \mathbb{M}^{m \times n} \times \mathbb{M}^{s \times t}\), define the block matrix\(^3\)

\[
\text{diag}(a, b) := \begin{bmatrix} a & \varepsilon_{m \times t} \\ \varepsilon_{n \times s} & b \end{bmatrix} \in \mathbb{M}^{(m+s) \times (n+t)}.
\]

With \( \text{diag}(e, a) := \text{diag}(a, e) := a \) for all \( a \in \text{Cmp} \) this defines a binary operation \( \text{diag} : \text{Cmp} \times \text{Cmp} \to \text{Cmp} \), which extends to any sequence of matrices \((a_i)_{1 \leq i \leq k}\) via\(^4\)

\[
\text{diag}(a_1, \ldots, a_k) := \text{diag}(\text{diag}(a_1, \ldots, a_{k-1}), a_k).
\]

A non-empty composition \( a \neq e \) is called **connected**, if \( a = \text{diag}(b, c) \) implies \( b = e \) or \( c = e \), i.e., if it has no non-trivial block matrix decomposition. Let \( \text{Cmp}^\text{con} \) denote the set of connected compositions.

**Example 2.2.** Let \((\mathbb{M}_3, \ast, \varepsilon)\) denote the free commutative monoid generated by three elements \(1\), \(2\) and \(3\). Then,

\[
\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \ast & 2 \\ 3 & 1 \end{bmatrix} \in \text{Cmp}^\text{con}
\]

and as a non-example,

\[
\text{diag}\left(\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & \varepsilon \\ \varepsilon & \varepsilon & 3 \end{bmatrix} \notin \text{Cmp}^\text{con}. \quad (3)
\]

**Lemma 2.3.** Every non-empty composition \( a \neq e \) has a unique **factorization into connected compositions**, i.e., for uniquely determined \( k \in \mathbb{N} \),

\[
a = \text{diag}(v_1, \ldots, v_k),
\]

where \( v \in \text{Cmp}^\text{con}_k \) is unique.

For illustration, compare Equation (3) or Example 2.11. We now consider formal power series over connected compositions of the form

\[
F := \sum_{a \in \text{Cmp}} c_a \ a \quad (4)
\]

\(^3\)Block matrices are allowed to be non-square, similar to the direct sum of two \( \mathbb{C} \)-valued matrices when considered as homomorphisms of \( \mathbb{C} \)-vector spaces.

\(^4\)Note that \( \text{diag} \) is associative, so we could equivalently take \( \text{diag}(a_1, \text{diag}(a_2, \ldots, a_k)) \). For convenience we also allow single inputs \( \text{diag}(a) := a \).
where \(c : \text{Cmp} \to \mathbb{K}\) provides coefficients from the integral domain. The set of all such series is denoted by

\[
\mathbb{K}\langle\langle \text{Cmp}_{\text{con}} \rangle\rangle \cong \mathbb{K}^{\text{Cmp}}.
\]

The notation \(\mathbb{K}\langle\langle \text{Cmp}_{\text{con}} \rangle\rangle\), indicating (the closure of) a free associative algebra over the “letters” \(\text{Cmp}_{\text{con}}\), is justified since matrix compositions can be considered as words, with respect to \(\text{diag}\) from Definition 2.1, over connected compositions, i.e.

\[
\text{Cmp} = \left\{ \text{diag}(a_1, \ldots a_k) \mid k \in \mathbb{N}, a \in \text{Cmp}_{k_{\text{con}}} \right\} \cup \{e\}.
\]

The **support** of a series (4) consists of all \(a \in \text{Cmp}\) with \(c_a \neq 0\). We denote by

\[
\mathbb{K}(\text{Cmp}_{\text{con}}) := \bigoplus_{a \in \text{Cmp}} \mathbb{K}a
\]

the subset of series with finite support, i.e., the direct sum of all modules \(\mathbb{K}a\). Each series (4) can be viewed as \(F \in \mathbb{K}^{\text{Cmp}}\) with

\[
F(a) := \langle F ; a \rangle := c_a
\]

and extends uniquely to a linear map \(F : \mathbb{K}\langle\langle \text{Cmp}_{\text{con}} \rangle\rangle \to \mathbb{K}\), i.e. we obtain a pairing

\[
\langle \cdot , \cdot \rangle : \mathbb{K}\langle\langle \text{Cmp}_{\text{con}} \rangle\rangle \times \mathbb{K}\langle\langle \text{Cmp}_{\text{con}} \rangle\rangle \to \mathbb{K}.
\]

We turn \(\mathbb{K}\langle\langle \text{Cmp}_{\text{con}} \rangle\rangle\) into a \(\mathbb{K}\)-algebra by equipping it with a linear and multiplicative structure

\[
\langle sF , a \rangle := s \langle F , a \rangle , \\
\langle F + G , a \rangle := \langle F , a \rangle + \langle G , a \rangle , \text{ and} \\
\langle F \cdot G , a \rangle := \sum_{\text{diag}(b, c) = a} \langle F , b \rangle \langle G , c \rangle
\]

where \(a \in \text{Cmp}\) and \(s \in \mathbb{K}\). The **constant series** are precisely the scaled empty compositions \(se\), including in particular the empty composition \(e\) as the neutral element with respect to multiplication.

Fixing a dimension \(d \in \mathbb{N}\), let \(\mathcal{M}_d\) denote the free commutative monoid generated by \(d\) elements \(1, \ldots , d\).

We use, from now on, \(\mathcal{M}_d\) as the underlying monoid of \(\text{Cmp} = \text{Cmp}(\mathcal{M}_d)\).

**Definition 2.4.** Let \(\text{weight} : \mathcal{M}_d \to \mathbb{N}_0\) denote the homomorphism of monoids, uniquely determined by \(\text{weight}(j) = 1\) for all \(j \in \{1, \ldots , d\}\). For every \(a \in \text{Cmp}\) let

\[
\text{weight}(a) := \sum_{n \leq \text{size}(a)} \text{weight}(a_n).
\]
This yields an $\mathbb{N}_0$-grading of the $K$-module

$$K\langle \text{Cmp}_{\text{con}} \rangle = K\mathbb{e} \oplus \bigoplus_{i \in \mathbb{N}} \left( \bigoplus_{a \in \text{Cmp} \atop \text{weight}(a) = i} K a \right)$$

(5)

with respect to weight. Note that (5) is also connected, i.e. its zero-weight component is one-dimensional. This grading is compatible with the diagonal operation from Definition 2.1.

**Lemma 2.5.** For all $a, b \in \text{Cmp}$,

$$\text{weight}(\text{diag}(a, b)) = \text{weight}(a) + \text{weight}(b).$$

For every $d$-dimensional data point $z \in \mathbb{K}^d$ let $z^{(i)} : \mathcal{M}_d \to \mathbb{K}$ denote its evaluation homomorphism, where $z^{(i)} := z_j$ extends uniquely via the universal property of $\mathcal{M}_d$.

**Definition 2.6.** For $Z : \mathbb{N}_2 \to \mathbb{K}^d$ and $\ell, r \in \mathbb{N}_0^2$ define the two-parameter sums signature $SS_{\ell, r}(Z) \in K\langle \text{Cmp}_{\text{con}} \rangle$ via its coefficients of $a \in \text{Cmp}$,

$$\langle SS_{\ell, r}(Z), a \rangle := \sum_{\ell_1 < \kappa_1 < \cdots < \kappa_{\text{rows}(a)} \leq r_1} \prod_{s=1}^{\text{rows}(a)} \prod_{t=1}^{\text{cols}(a)} Z_{\ell_s, \kappa_t}^{(a_{s,t})} \in \mathbb{K}.$$

If $Z$ has finite support, that is $Z_i \neq 0_d$ for only finitely many $i \in \mathbb{N}^2$, we also write $SS(Z) := \lim_{r \to \infty} SS_{0, r}^{\ell, r}(Z)$.

If $d = 1$, one has $\mathcal{M}_1 \cong \mathbb{N}_0$ as monoids, and thus obtains back the definition of the two-parameter sums signature from the introduction.

**Example 2.7.** Let $d = 1$ and $\mathbb{K} = \mathbb{C}$ as in the introduction. If, for instance, $a$ is of format $1 \times 1$, then the product reduces to a single factor $Z_{\ell_1, \kappa_1}$ raised to the power of $a_{1,1}$. For instance

$$\langle SS \left( \begin{array}{ccc} 2 & 1 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \end{array} \right), \left[ \begin{array}{c} 1 \end{array} \right] \rangle = 2^1 + 1^1 + 3^1 + 1^1$$

is simply the sum of all non-zero entries from the input function.

If $a$ is of format $2 \times 2$, then one considers all matrices

$$\begin{bmatrix} Z_{\ell_1, \kappa_1} & Z_{\ell_1, \kappa_2} \\ Z_{\ell_2, \kappa_1} & Z_{\ell_2, \kappa_2} \end{bmatrix} \in \mathbb{C}^{2 \times 2}$$
with increasing $\iota_1 < \iota_2$ and $\kappa_1 < \kappa_2$, raises it entrywise to the power of $a$, multiplies its entries, and sums the resulting products over all constellations of independent $\iota$ and $\kappa$. For instance, in

$$\langle SS \rangle \begin{pmatrix} 1 & 2 & 0 \\ 0 & 7 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \\ \vdots \end{pmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 2 \\ 1 & 1 \end{bmatrix}$$

only the two constellations $\iota_1 = \kappa_1 = 1$, $\kappa_2 = 2$ and either $\iota_2 = 2$ or $\iota_2 = 3$ yield non-zero summands.

2.1 The Hopf algebra of matrix compositions

We shall now equip the $\mathbb{K}$-module $\mathbb{K}\langle \text{Cmp}_{\text{con}} \rangle$ with another product by introducing a two-parameter version of the well-known quasi-shuffle from [Hof99]. This novel product (Definition 2.8) is clearly different from the classical quasi-shuffle over $\mathbb{K}\langle \text{Cmp}_{\text{con}} \rangle$. We call the latter one-parameter quasi-shuffle, recalled in Section 2.2 to compute our two-parameter generalization using recursion.

As in the classical setting, let $q\text{Sh}(m,s;j)$ with $m, s, j \in \mathbb{N}$ denote the set of surjections $q : \{1, \ldots, m + s\} \to \{1, \ldots, j\}$ such that $q(1) < \ldots < q(m)$ and $q(m + 1) < \ldots < q(m + s)$.

**Definition 2.8.** The two-parameter quasi-shuffle product is the bilinear mapping $\mathbb{K}\langle \text{Cmp}_{\text{con}} \rangle \times \mathbb{K}\langle \text{Cmp}_{\text{con}} \rangle \to \mathbb{K}\langle \text{Cmp}_{\text{con}} \rangle$ determined on compositions $(a, b) \in \mathfrak{M}^{m \times n} \times \mathfrak{M}^{s \times t}$ via $e \preceq e := e$, $a \preceq e := e \preceq a := a$ and

$$a \preceq b := \sum_{j,k \in \mathbb{N}} \sum_{p \in q\text{Sh}(m,s;j)} \sum_{q \in q\text{Sh}(n,t;k)} \begin{bmatrix} c_{p,q}^{1,1} & \cdots & c_{p,q}^{1,k} \\ \vdots & \ddots & \vdots \\ c_{p,q}^{j,1} & \cdots & c_{p,q}^{j,k} \end{bmatrix}$$

where

$$c_{x,y}^{p,q} := \star \quad \text{diag}(a, b)_{u,v} \in \mathfrak{M}.$$  \hspace{1cm} (6)

**Example 2.9.** The quasi-shuffle of $\begin{bmatrix} 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \end{bmatrix}$ from $\mathfrak{M}^{1 \times 1}$ involves surjections only in
qSh(1, 1; 1) and qSh(1, 1; 2), i.e.,

\[
\begin{bmatrix}
1 \\
2
\end{bmatrix} \uplus \begin{bmatrix}
1 \\
2
\end{bmatrix} = \begin{bmatrix}
1 \\
2
\end{bmatrix} + \begin{bmatrix}
2 \\
1
\end{bmatrix} + \begin{bmatrix}
1 \\
2
\end{bmatrix} + \begin{bmatrix}
2 \\
1
\end{bmatrix} + \begin{bmatrix}
2 \\
1
\end{bmatrix} + \begin{bmatrix}
2 \\
1
\end{bmatrix}.
\]

Larger examples are provided in Section 2.2, after relating two-parameter quasi-shuffles to the one-parameter setting.

Note that if \( j < \max(m, s) \) or \( m + s < j \), then qSh\((m, s; j)\) is empty, i.e., the sum in Definition 2.8 is guaranteed to be finite.

We now endow \( \mathbb{K}\langle\text{Cmp}_\text{con}\rangle \) with a coproduct, turning it into a graded, connected bialgebra (Theorem 5.24) and hence into a Hopf algebra. The coproduct will be seen to be compatible with a certain “concatenation” of the input data, leading to a (weak) form of Chen’s identity (Section 2.5).

**Definition 2.10.** The deconcatenation coproduct \( \Delta : \mathbb{K}\langle\text{Cmp}_\text{con}\rangle \to \mathbb{K}\langle\text{Cmp}_\text{con}\rangle \otimes 2 \) is defined on non-empty basis elements \( a \in \text{Cmp} \) as

\[
\Delta a := a \otimes e + e \otimes a + \sum_{\alpha=1}^{a-1} \text{diag}(v_1, \ldots, v_{\alpha}) \otimes \text{diag}(v_{\alpha+1}, \ldots, v_a) = \sum_{\text{diag}(b,c)=a} b \otimes c.
\]

Here, \( v \) is a factorization of composition \( a \) into connected components \( v_\alpha \) according to Lemma 2.3. Furthermore, let \( \Delta e := e \otimes e \).

**Example 2.11.** For the factorization of the \( 4 \times 5 \) composition

\[
a = \begin{bmatrix}
1 & 2 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 2 \ \varepsilon \ 1 \ \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 1 \ \varepsilon
\end{bmatrix} = \text{diag}\left(\begin{bmatrix}1 & 2 \end{bmatrix}, 1, \begin{bmatrix}2 & 1 \ 2 \ 1 \ \varepsilon \ \varepsilon
\end{bmatrix}\right)
\]

into connected compositions,

\[
\Delta a = a \otimes e + \begin{bmatrix}
1 & 2 & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 \ \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 3 \ \varepsilon
\end{bmatrix} \otimes \begin{bmatrix}
2 & 1 \ \varepsilon \\
3 & \varepsilon
\end{bmatrix} + \begin{bmatrix}
1 & 2 \\
\varepsilon & \varepsilon
\end{bmatrix} \otimes \begin{bmatrix}
3 & \varepsilon \ 2 \ \varepsilon \\
3 & \varepsilon \ 2 \ \varepsilon
\end{bmatrix} + e \otimes a.
\]

This coproduct is coassociative with canonical counit map \( \epsilon : \mathbb{K}\langle\text{Cmp}_\text{con}\rangle \to \mathbb{K} \), where \( \epsilon(e) := 1 \) and \( \epsilon(a) := 0 \) for all \( a \neq e \). Furthermore, it is graded with respect to (5) due to Lemma 2.5. Note that this coproduct is dual to the free associative product \( \text{diag} \) from Definition 2.1.
Theorem 2.12. If \( K \) is a commutative \( \mathbb{Q} \)-algebra, then
\[
(K, \langle \text{Cmp}_{\text{con}} \rangle, \eta, \Delta, \epsilon)
\]
is a graded, connected Hopf algebra with canonical unit map \( \eta : K \to K \langle \text{Cmp}_{\text{con}} \rangle \) determined through \( \eta(1) := e \). The antipode map \( A : K \langle \text{Cmp}_{\text{con}} \rangle \to K \langle \text{Cmp}_{\text{con}} \rangle \) is defined on basis elements via
\[
A(a) := \sum_{(i_1, \ldots, i_\ell) \in C(k)} (-1)^\ell \text{diag}(v_1, \ldots, v_{i_1}) \overset{\alpha_1}{\overleftarrow{\alpha_1}} \cdots \overset{\alpha_k}{\overleftarrow{\alpha_k}} \text{diag}(v_{(i_1+\cdots+i(\ell-1)+1)}, \ldots, v_k),
\]
where \( a = \text{diag}(v_1, \ldots, v_k) \) is according to Lemma 2.3 and \( C \) contains all (classical) compositions of length \( k \in \mathbb{N} \). In particular, \((K, \langle \text{Cmp}_{\text{con}} \rangle, +, \overset{\alpha_i}{\overleftarrow{\alpha_i}}, 0, e)\) is free.

Proof. In Corollaries 5.13, 5.15 and 5.16 (Section 5.2) we prove that the two-parameter quasi-shuffle is associative, commutative, graded and that its codomain is indeed \( K \langle \text{Cmp}_{\text{con}} \rangle \). The proof of the bialgebra relation requires further machinery and is provided in Theorem 5.24. The remaining follows since every graded, connected bialgebra is a Hopf algebra where the antipode is explicit, e.g. [HGKK10, Proposition 3.8.8] or [CP21, Remark 4.1.2, Example 4.1.3]. Freeness is a classical consequence, compare for instance [CP21, Theorem 4.4.2] or [GR14, Theorem 1.7.29].

2.2 Quasi-shuffle identity

Our first results concern the verification that \( SS \) is a proper sums signature. By this we mean that it carries the two main properties mentioned in the introduction, i.e., it satisfies Chen’s identity (Section 2.5) and the quasi-shuffle identity, with the quasi-shuffle product taken from Definition 2.8.

Theorem 2.13. For all \( Z : \mathbb{N}^2 \to K^d \), bounds \( \ell, r \in \mathbb{N}^2 \) and \( a, b \in \text{Cmp} \),
\[
\langle SS_{\ell r}(Z), a \rangle \langle SS_{\ell r}(Z), b \rangle = \langle SS_{\ell r}(Z), a \overset{\alpha}{\overleftarrow{\alpha}} b \rangle.
\]
The proof is postponed until Section 3.2. In Section 5.2 we relate the two-parameter quasi-shuffle to classical one-parameter operations on columns and rows, resulting in an algorithm for its efficient computation. Here we give a brief explanation and a first illustration of the method (Example 2.15), in order to make the product more transparent to the reader.

Equip the free associative algebra \( K(\mathcal{M}^i) \) generated by column vectors of size \( i \) with a (classical, one-parameter) quasi-shuffle of columns defined recursively via
\[
e \overset{\alpha}{\overleftarrow{\alpha}} w := w \overset{\alpha}{\overleftarrow{\alpha}} e := w, \text{ and } \ \hspace{1cm} av \overset{\alpha}{\overleftarrow{\alpha}} bw := a(v \overset{\alpha}{\overleftarrow{\alpha}} bw) + b(av \overset{\alpha}{\overleftarrow{\alpha}} w) + (a \star b)(v \overset{\alpha}{\overleftarrow{\alpha}} w)
\]
for all monomials \( v, w \in K(\mathcal{M}^i) \) and letters \( a, b \in \mathcal{M}^i \). Here \( \ast \) operates entrywise on the vectors. Analogously let \( \tilde{\text{w}}_1 \) be the (classical, one-parameter) \textbf{quasi-shuffle of rows} in \( K(\mathcal{M}_1^{j \times j}) \).

\textbf{Example 2.14.} A classical, one-parameter quasi-shuffle of columns from \( \mathcal{M}_3^2 \) is

\[
\begin{bmatrix}
1 \\
3
\end{bmatrix}_{\tilde{\text{w}}_2}
\begin{bmatrix}
\varepsilon \\
2
\end{bmatrix} =
\begin{bmatrix}
1 & \varepsilon \\
3 & 2
\end{bmatrix} +
\begin{bmatrix}
\varepsilon & 1 \\
2 & 3
\end{bmatrix} \in K(\mathcal{M}_3^2),
\]

similar for columns of size three,

\[
\begin{bmatrix}
1 \\
\varepsilon \\
2
\end{bmatrix}_{\tilde{\text{w}}_2}
\begin{bmatrix}
\varepsilon \\
3
\end{bmatrix} =
\begin{bmatrix}
1 & \varepsilon \\
\varepsilon & 2 \\
\varepsilon & 3
\end{bmatrix} +
\begin{bmatrix}
\varepsilon & 1 \\
2 & \varepsilon \\
3 & \varepsilon
\end{bmatrix} \in K(\mathcal{M}_3^3).
\]

\[ (7) \]

A quasi-shuffle of monomials in rows from \( \mathcal{M}_3^{1 \times 2} \) is

\[
\begin{bmatrix}
1 & \varepsilon \\
\varepsilon & 2 \\
\varepsilon & 3
\end{bmatrix}_{\tilde{\text{w}}_1}
\begin{bmatrix}
\varepsilon \\
2
\end{bmatrix} =
\begin{bmatrix}
1 & \varepsilon \\
\varepsilon & 2 \\
\varepsilon & 3
\end{bmatrix} +
\begin{bmatrix}
\varepsilon & 2 \\
1 & \varepsilon \\
1 & 3
\end{bmatrix} +
\begin{bmatrix}
\varepsilon & 2 \\
2 & \varepsilon \\
3 & \varepsilon
\end{bmatrix} \in K(\mathcal{M}_3^3).
\]

\[ (8) \]

and thus an element in the free algebra \( K(\mathcal{M}_3^{1 \times 2}) \).

Note that \( K(\mathcal{M}^i) \) can be considered as a set of linear combinations in \( M^i_{j \times j} \), where \( j \in \mathbb{N} \) is varying. Section 4.2 provides a full description of the algorithm for computing the two-parameter quasi-shuffle. We close this section by applying it step by step.

\textbf{Example 2.15.} First, we compute the column quasi-shuffle of the input \( a \in \mathcal{M}_m^{n \times n} \) and \( b \in \mathcal{M}_s^{t \times t} \), when lifted to monomials in columns of length \( m + s \). This step is illustrated for instance in Equation (7) with \( m = n = t = 1 \), \( s = 2 \) and \( d = 3 \). From the resulting polynomial we take each of its monomials, and decompose it in two \( \mathcal{M}_3 \)-valued matrices \( c \) and \( d \) of shape \( m \times j \) and \( s \times j \), respectively. In the running example this would be

\[
\begin{bmatrix}
1 & \varepsilon \\
\varepsilon & 2 \\
\varepsilon & 3
\end{bmatrix} +
\begin{bmatrix}
\varepsilon & 2 \\
1 & \varepsilon \\
1 & 3
\end{bmatrix} +
\begin{bmatrix}
\varepsilon & 2 \\
2 & \varepsilon \\
3 & \varepsilon
\end{bmatrix} \in K(\mathcal{M}_3^3),
\]

\[ (9) \]

from which we would take the first degree \( j = 2 \) monomial and decompose it into

\[
\begin{bmatrix}
1 & \varepsilon \\
\varepsilon & 2 \\
\varepsilon & 3
\end{bmatrix} \in \mathcal{M}_3^{1 \times 2} \times \mathcal{M}_3^{2 \times 2}.
\]

We now interpret \( c \) and \( d \) as monomials of rows, and compute its row quasi-shuffle as illustrated in Equation (8). The resulting monomials are then the first five terms of the
quasi-shuffle

\[
\begin{pmatrix}
1 \\
\varepsilon
\end{pmatrix} \uplus \begin{pmatrix}
2 \\
\varepsilon
\end{pmatrix} = \begin{pmatrix}
1 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
2 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
\varepsilon \\
1
\end{pmatrix} + \begin{pmatrix}
2 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
\varepsilon \\
3
\end{pmatrix} + \begin{pmatrix}
2 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
1 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
2 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
\varepsilon \\
3
\end{pmatrix} + \begin{pmatrix}
\varepsilon \\
1
\end{pmatrix} + \begin{pmatrix}
1 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
3 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
\varepsilon \\
1
\end{pmatrix} + \begin{pmatrix}
2 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
3 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
\varepsilon \\
1
\end{pmatrix} + \begin{pmatrix}
1 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
2 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
3 \\
\varepsilon
\end{pmatrix} + \begin{pmatrix}
\varepsilon \\
1
\end{pmatrix}
\in \mathbb{K}(\text{Cmp}_{\text{con}}).
\]

The remaining ten summands result with the remaining two monomials from Equation (9), when plugged into the quasi-shuffle of rows as described in detail above for the first monomial.

2.3 Invariance to zero insertion

Before addressing warping invariants from the introduction in more detail, we introduce a second invariant that is closely related. Instead of a warping operation, we consider the insertion of zero rows and columns. We show in Theorem 2.20 that the two-parameter signature is invariant under the latter, and furthermore, that it characterizes whether two elements are equal up to insertion of zeros. This result is then used in Section 2.4 concerning warping invariants. Let

\[
\text{evZ} := \text{evZ}(\mathbb{N}^2, \mathbb{K}^d) := \left\{ X : \mathbb{N}^2 \to \mathbb{K}^d \mid \exists n \in \mathbb{N}^2 : X_i \neq 0, \implies i \leq n \right\}
\]

denote the set of all functions from the index set \(\mathbb{N}^2\) to the \(\mathbb{K}\)-module \(\mathbb{K}^d\) which are eventually zero, i.e., of finite support.

Let \(\text{Zero}_{a,k} : \text{evZ} \to \text{evZ}\) be the zero insertion operation which puts a zero row or column (specified by axis \(a \in \{1, 2\}\)) at position \(k \in \mathbb{N}\) via

\[
(\text{Zero}_{a,k} X)_i :=
\begin{cases}
X_i & i_a < k \\
0_d & i_a = k \\
X_{i-e_a} & i_a > k.
\end{cases}
\]

Example 2.16. As in the introduction with \(d = 1\) and \(\mathbb{K} = \mathbb{C}\),

\[
\text{Zero}_{2,2} \circ \text{Zero}_{1,2}
\begin{pmatrix}
5 & 1 & 0 \\
3 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon
\end{pmatrix}
\in \text{evZ}.
\]
Lemma 2.17.

1. $\text{Zero}_{a,k}$ is an injective endomorphism of modules for all $(a, k) \in \{1, 2\} \times \mathbb{N}$.

2. $\text{Zero}_{1,k} \circ \text{Zero}_{2,j} = \text{Zero}_{2,j} \circ \text{Zero}_{1,k}$ for all $k, j$.

3. $\text{Zero}_{a,k} \circ \text{Zero}_{a,j} = \text{Zero}_{a,j} \circ \text{Zero}_{a,k-1}$ for $k > j$ and all $a$.

4. For every $A \in \text{evZ}$ there exists a uniquely determined zero insertion normal form $\text{NF}_{\text{Zero}}(A) \in \text{evZ}$ such that

$$A = \text{Zero}_{a_q,k_q} \circ \cdots \circ \text{Zero}_{a_1,k_1} \circ \text{NF}_{\text{Zero}}(A)$$

with suitable $a_i \in \{1, 2\}$, $k_i \in \mathbb{N}$, and where $\text{NF}_{\text{Zero}}(A)$ has no zero column before a non-zero column, and no zero row before a non-zero row.

Most proofs in this and in the remaining subsections are postponed until Section 5.1. For an illustration of normal forms, compare the argument in Example 2.16.

Definition 2.18. A mapping $\phi : \text{evZ} \to \mathbb{K}$ is invariant to inserting zero (in both directions independently\(^5\)), if

$$\phi \circ \text{Zero}_{a,k} = \phi \quad \forall (a, k) \in \{1, 2\} \times \mathbb{N}. \quad (10)$$

With Lemma 2.17 one obtains a partition of $\text{evZ}$ into classes of functions which are equal up to insertion of zeros.

Lemma 2.19. The relation $\sim_{\text{Zero}}$ defined via

$$A \sim_{\text{Zero}} B :\Leftrightarrow \text{NF}_{\text{Zero}}(A) = \text{NF}_{\text{Zero}}(B)$$

is an equivalence relation on $\text{evZ}$.

It turns out that the two-parameter sums signature characterizes the resulting equivalence classes.

Theorem 2.20. For $A, B \in \text{evZ}$,

$$A \sim_{\text{Zero}} B \iff \text{SS}(A) = \text{SS}(B).$$

Its proof (provided in Section 5.1) is similar to [DEFT22, Theorem 3.15] after generalization for two parameters.

---

\(^5\)Invariance to simultaneous insertion of zeros would demand $\phi \circ \text{Zero}_{2,k} \circ \text{Zero}_{1,k} = \phi$ for all $k \in \mathbb{N}$. 

16
2.4 Invariance to warping

In this subsection we show that zero insertion invariants and warping invariants translate back and forth, specified in Corollary 2.26. Modulo constants, this results in Theorem 2.32, a full characterization of when two time series are equal up to warping.

From the introduction, recall the set of functions

\[ \text{evC} := \text{evC}(\mathbb{N}^2, \mathbb{K}^d) := \left\{ X : \mathbb{N}^2 \to \mathbb{K}^d \mid \exists n \in \mathbb{N}^2 : X_i \neq X_j \implies i, j \leq n \right\} \]

which are eventually constant. Note that \( \text{evZ} \subseteq \text{evC} \). Let \( \text{warp}_{a,k} : \text{evC} \to \text{evC} \) be a single warping operation which inserts a copy of the \( k \)-th row or column (specified by axis \( a \in \{1, 2\} \)) via

\[(\text{warp}_{a,k} X)_i := \begin{cases} X_i & i_a \leq k \\ X_{i-e_a} & i_a > k. \end{cases}\]

For illustrations, recall the introduction in Section 1.1 with \( d = 1 \). We collect analogous properties as in Lemma 2.17, in particular a normal form with respect to warping.

**Lemma 2.21.**

1. \( \text{warp}_{a,k} \) is an injective endomorphism of modules for all \( a \in \{1, 2\}, k \in \mathbb{N} \).
2. \( \text{warp}_{1,k} \circ \text{warp}_{2,j} = \text{warp}_{2,j} \circ \text{warp}_{1,k} \) for all \( k, j \in \mathbb{N} \).
3. \( \text{warp}_{a,k} \circ \text{warp}_{a,j} = \text{warp}_{a,j} \circ \text{warp}_{a,k-1} \) for \( k > j \) and \( a \in \{1, 2\} \).
4. For every \( A \in \text{evC} \) there is a unique warping normal form \( \text{NF}_{\text{warp}}(A) \in \text{evC} \) such that

\[ A = \text{warp}_{a_q,k_q} \circ \cdots \circ \text{warp}_{a_1,k_1} \circ \text{NF}_{\text{warp}}(A) \]

with suitable \( a_i \in \{1, 2\}, k_i \in \mathbb{N} \) and where \( \text{NF}_{\text{warp}}(A) \) has no warped columns and rows before the constant part.

In Lemma 2.24 we show how warping and zero insertion relate to each other. For this, recall the difference operator \( \delta : \text{evC} \to \text{evZ} \) with

\[(\delta X)_{i,j} := X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1} + X_{i,j} \quad (11)\]

explained also in the introduction.

Factorizing its kernel, we obtain an isomorphism of \( \mathbb{K} \)-modules as follows.

**Lemma 2.22.**

1. \( \delta \) is a surjective homomorphism of \( \mathbb{K} \)-modules with
2. \( \ker(\delta) = \{ X \mid X_i = X_j \ \forall i, j \in \mathbb{N}^2 \} \),
3. and linear right inverse

\[ \varsigma : \text{evZ} \to \text{evC}, \ Z \mapsto \left( \sum_{i \leq s} \sum_{j \leq t} Z_{s,t} \right)_{i,j} \]

4. which yields a normal form modulo constants 

\[ \text{NF}_{\ker(\delta)}(X) := \varsigma \circ \delta(X) \text{ as a representative of } X + \ker(\delta) \in \text{evC}/\ker(\delta) \cong \text{evZ}. \]

Example 2.23. For \( d = 1 \) and \( K = \mathbb{C} \),

\[ \varsigma \circ \delta \begin{pmatrix} 7 & 3 & 2 & 2 \\ 5 & 3 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ \vdots \end{pmatrix} = \varsigma \begin{pmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} 5 & 1 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 0 \\ \vdots \end{pmatrix} \]

is in normal form according to Lemma 2.22.

The three normal forms which were introduced in Lemmas 2.17, 2.21 and 2.22 are related as follows.

**Lemma 2.24.** For all \((a, k) \in \{1, 2\} \times \mathbb{N}\),

1. \( \delta \circ \text{warp}_{a,k} = \text{Zero}_{a,k} \circ \delta \),
2. \( \varsigma \circ \text{Zero}_{a,k} = \text{warp}_{a,k} \circ \varsigma \),
3. \( \delta \circ \text{NF}_{\text{warp}} = \text{NF}_{\text{Zero}} \circ \delta \),
4. \( \varsigma \circ \text{NF}_{\text{Zero}} = \text{NF}_{\text{warp}} \circ \varsigma \),
5. \( \text{NF}_{\ker(\delta)} \circ \text{warp}_{a,k} = \text{warp}_{a,k} \circ \text{NF}_{\ker(\delta)} \), and
6. \( \text{NF}_{\text{warp}} \circ \text{NF}_{\ker(\delta)} = \text{NF}_{\ker(\delta)} \circ \text{NF}_{\text{warp}} \).

**Definition 2.25.** A mapping \( \psi : \text{evC} \to K \) is

1. invariant modulo constants, if

\[ \psi(X) = \psi(X + C) \quad \forall X, C \in \text{evC} \text{ with constant } C, \quad (12) \]

2. and invariant to warping (in both directions independently \(^6\)), if

\[ \psi \circ \text{warp}_{a,k} = \psi \quad \forall(a, k) \in \{1, 2\} \times \mathbb{N}. \quad (13) \]

\(^6\)Invariance to simultaneous warping would demand \( \psi \circ \text{warp}_{2,k} \circ \text{warp}_{1,k} = \psi \) for all \( k \in \mathbb{N} \). Thinking of image data, this operation corresponds to scaling uniformly in both directions, whereas the one we are interested in allows independent scaling for both directions.
With transitivity, both statements can be reformulated using normal forms, i.e., Equation (12) is equivalent to $\psi \circ \text{NF}_{\ker(\delta)} = \psi$, and Equation (13) holds if and only if $\psi \circ \text{NF}_{\text{warp}} = \psi$.

Warping and zero insertion invariants translate back and forth via the difference operator $\delta$ and its right inverse $\varsigma$.

**Corollary 2.26.**

1. If $\phi : \text{evZ} \rightarrow \mathbb{K}$ is invariant to inserting zero, then $\phi \circ \delta$ is invariant to warping.
2. If $\psi : \text{evZ} \rightarrow \mathbb{K}$ is invariant to warping, then $\psi \circ \varsigma$ is invariant to inserting zero.

**Proof.** Using Lemma 2.24, part 1. follows with

$$\phi \circ \delta = \phi \circ \text{Zero}_{a,k} \circ \delta = \phi \circ \delta \circ \text{warp}_{a,k}$$

and analogously, part 2. with $\psi \circ \varsigma = \psi \circ \varsigma \circ \text{Zero}_{a,k}$ for all $(a,k) \in \{1,2\} \times \mathbb{N}$. \hfill \(\Box\)

We now define the family of invariants $\Psi$ motivated in the introduction.

**Corollary 2.27.** For every $a \in \text{Cmp}$,

$$\Psi_a : \text{evC} \rightarrow \mathbb{K}, \ X \mapsto \langle \text{SS}(\delta X), a \rangle$$

is invariant modulo constants and warping in both directions independently.

**Proof.** With $\delta \circ \text{NF}_{\text{Zero}} = \delta \circ \varsigma \circ \delta = \delta$ holds Equation (12), whereas Equation (13) is an immediate consequence of Theorem 2.20 and the first part of Corollary 2.26. \hfill \(\Box\)

**Example 2.28.** With the summation rule (compare Lemma 5.3),

$$\sum_{i=1}^{s} \sum_{j=1}^{t} (\delta X)_{i,j} = X_{s+1,t+1} - X_{1,t+1} - X_{s+1,1} + X_{1,1}$$

for all $X \in \text{evC}$ and $(s,t) \in \mathbb{N}^2$,

$$\Psi^{[1]}(X) = \langle \text{SS} \circ \delta(X), [1] \rangle = X_{1,1} - \lim_{s,t \rightarrow \infty} X_{s,t}$$

as it was claimed in the introduction.

Invariants according to Definition 2.25 form a $\mathbb{K}$-subalgebra of $\mathbb{K}^{\text{evC}}$. The family $\Psi$ is chosen sufficiently large in the sense, its linear span is closed under products.

**Corollary 2.29.** The $\mathbb{K}$-module $\text{span}_{\mathbb{K}}(\Psi)$ is a $\mathbb{K}$-subalgebra of $\mathbb{K}^{\text{evC}}$.

**Proof.** Follows immediately with the quasi-shuffle identity (Theorem 2.13). \hfill \(\Box\)
**Definition 2.30.** Consider the union of the equivalence relation modulo \( \ker(\delta) \) from Lemma 2.22 and \( \sim_{\text{Zero}} \) from Lemma 2.19. Denote its transitive closure by \( \sim \), which is an equivalence relation modulo constants or warping.

**Example 2.31.** For \( d = 1 \) and \( K = \mathbb{C} \),

\[
\begin{array}{cccc}
7 & 3 & 2 & 2 \\
5 & 3 & 2 & 2 \\
2 & 2 & 2 & 2 \\
\end{array}
\sim
\begin{array}{cccc}
5 & 1 & 0 \\
3 & 1 & 0 \\
0 & 0 & 0 \\
\end{array}
\sim
\begin{array}{cccc}
5 & 5 & 1 & 0 \\
3 & 3 & 1 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

where the first equivalence is due to Example 2.23, and the second is resulting from two warping operations on the second row and column. Note that the first function is equivalent to the third due to the transitive closure used in Definition 2.30.

Since warping and addition of constants commute (Lemma 2.24), we obtain a normal form modulo constants or warping \( \text{NF}_{\sim} := \text{NF}_{\ker(\delta)} \circ \text{NF}_{\text{warp}} \).

**Theorem 2.32.** For \( X, Y \in \text{evC} \),

\[
X \sim Y \iff \text{NF}_{\sim}(X) = \text{NF}_{\sim}(Y) \\
\iff \text{SS} \circ \delta(X) = \text{SS} \circ \delta(Y)
\]

**Proof.** We sketch only the second equivalence, which is valid with Theorem 2.20,

\[
\delta \circ \text{NF}_{\sim} = \delta \circ \varsigma \circ \delta \circ \text{NF}_{\text{warp}} = \text{NF}_{\text{Zero}} \circ \delta,
\]

and since \( \delta \) is injective when restricted on \( \text{NF}_{\ker(\delta)}(\text{evC}) \).

With Theorem 2.32 we obtain for every \( X, Y \in \text{evC} \),

\[
X \sim Y \iff \Psi_{a}(X) = \Psi_{a}(Y) \quad \forall a \in \text{Cmp},
\]

i.e., \( \Psi \) is expressive enough to decide whether \( X \) and \( Y \) are equivalent. In Section 3 we show under further assumptions on \( K \), that \( \text{span}(\Psi) \) contains precisely the so-called polynomial invariants from \( K^{\text{evC}} \).

**2.5 Chen’s identity**

We now return to Chen’s identity, the second main property of signature-like objects. In this section we present **Chen’s identity with respect to diagonal deconcatenation**,\n
\[
\forall A, B \in \text{evZ} : \text{SS}(A \odot B) = \text{SS}(A) \cdot \text{SS}(B),
\]

where the first equivalence is due to Example 2.23, and the second is resulting from two warping operations on the second row and column. Note that the first function is equivalent to the third due to the transitive closure used in Definition 2.30.
where the product of signatures is in $\mathbb{K}\langle\langle \text{Cmp}_{\text{con}} \rangle\rangle$ and the product $\odot$ is defined below. It provides an algebraic relation between input functions and their signatures, allowing to compute the signature of the concatenation $X := A \odot B$ (Definitions 2.33 and 2.36) via the signatures of $A$ and $B$ alone. We relate this result (Lemma 2.35) to warping invariants in Corollary 2.38.

For convenience, we extend the notion of size, from matrices to $\text{evC}$ via

$$\text{size}(X) := (\text{rows}(X), \text{cols}(X)) := \min \left\{ n \in \mathbb{N}^2 \mid X_i \neq X_j \implies i, j \leq n \right\}.$$ 

With $\text{evZ} \subseteq \text{evC}$ this implicitly defines a size for eventually-zero functions, specified further in Lemma 5.1. We now define diagonal concatenation in the range of the difference operator $\delta$.

**Definition 2.33.** The binary operation $\odot : \text{evZ} \times \text{evZ} \to \text{evZ}$ sends $(A, B)$ to its diagonal concatenation,

$$A \odot B := A + \text{Zero}_{1,1}^{\text{rows}(A)} \circ \text{Zero}_{2,1}^{\text{cols}(A)}(B).$$

In Lemma 5.6 we verify that $(\text{evZ}, \odot)$ is a non-commutative semigroup, i.e., that $\odot$ is associative.

**Example 2.34.** For $d = 1$ and $\mathbb{K} = \mathbb{C}$,

\[
\begin{bmatrix}
2 & 7 & 0 \\
2 & 5 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} \odot 
\begin{bmatrix}
2 & 2 & 0 \\
1 & 4 & 0 \\
0 & 0 & 0 \\
\end{bmatrix} = 
\begin{bmatrix}
2 & 7 & 0 & 0 & 0 & 0 \\
2 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 2 & 0 & 0 \\
0 & 0 & 1 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \in \text{evZ}.
\]

With this concatenation, we can formulate Chen’s identity for eventually-zero functions.

**Lemma 2.35.** For all $A, B \in \text{evZ}$ and $\ell, r \in \mathbb{N}_0^2$ with $\ell \leq \text{size}(A) \leq r$,

$$\langle SS_{\ell,r}(A \odot B), a \rangle = \sum_{\text{diag}(b,c) = a} \langle SS_{\ell,\text{size}(A)}(A \odot B), b \rangle \langle SS_{\text{size}(A),r}(A \odot B), c \rangle$$

or equivalently, Equation (14).

The proof is provided in Section 5.1. When functions in $\text{evC}$ are considered as pictures, one might think of a different notion of concatenation. In Definition 2.36 we “continue” the initial lower right picture in all those data points that are not captured by the initial upper left function.

**Definition 2.36.** The binary operation $\boxdot : \text{evC} \times \text{evC} \to \text{evC}$ sends $(X, Y)$ to its concatenation along the diagonal,

$$X \boxdot Y := \text{NF}_{\ker(\delta)}(X) + \text{warp}_{1,1}^{\text{rows}(X)} \circ \text{warp}_{2,1}^{\text{cols}(X)}(Y).$$
In Lemma 5.6 we verify that \((evC, \otimes)\) is a non-commutative semigroup, i.e., that \(\otimes\) is associative. We furthermore show that \(\delta\) and \(\varsigma\) are semigroup homomorphisms to and from \((evZ, \otimes)\) respectively.

**Example 2.37.** For \(d = 1\) and \(K = \mathbb{C}\),

\[
\begin{array}{ccc}
2 & 7 & 2 \\
2 & 5 & 2 \\
2 & 2 & 2 \\
\end{array}
\begin{array}{ccc}
2 & 2 & 0 \\
1 & 4 & 0 \\
0 & 0 & 0 \\
\end{array}
\begin{array}{ccc}
2 & 7 & 2 & 2 & 0 \\
2 & 5 & 2 & 2 & 0 \\
2 & 2 & 2 & 2 & 0 \\
1 & 1 & 1 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{array}
\in evC.
\]

**Corollary 2.38.** For all \(X, Y \in evC\),

\[
SS(\delta(X \otimes Y)) = SS(\delta(X)) \cdot SS(\delta(Y)).
\]

**Proof.** Follows immediately with Lemma 5.6 and Equation (14). \(\square\)

### 3 Two-parameter quasisymmetric functions

For the entire section, let \(\text{Cmp}\) be the set of compositions with entries in \(\mathbb{N} = \mathbb{N}_0\). Let \((x_i)_{i \in \mathbb{N}^2}\) be a family of symbols and

\[
\mathbb{K}[x]_{<\infty} := \{f \in \mathbb{K}[x] \mid \deg(f) < \infty\}
\]

be the \(\mathbb{K}\)-algebra of all power series in \(x\) which have finite degree.

**Definition 3.1.** Call \(f \in \mathbb{K}[x]_{<\infty}\) a **two-parameter quasisymmetric function**, if for all matrix compositions \(a \in \text{Cmp}\) and increasing chains \(\iota_1 < \ldots < \iota_{\text{rows}(a)}\) and \(\kappa_1 < \ldots < \kappa_{\text{cols}(a)}\) the coefficients of

\[
\prod_{s=1}^{\text{rows}(a)} x_{a_s}^{\iota_s} \text{ and } \prod_{s=1}^{\text{rows}(a)} x_{a_s}^{\iota_s, \kappa_t}
\]

are equal in \(f\). Let \(\text{QSym}^{(2)}\) denote the set of all two-parameter quasisymmetric functions.

It is clear that \(\text{QSym}^{(2)} = \bigoplus_{d \in \mathbb{N}_0} \text{QSym}^{(2)}_{d}\) is a graded \(\mathbb{K}\)-module with homogeneous components \(\text{QSym}^{(2)}_{d} = \{f \in \text{QSym}^{(2)} \mid \deg(f) = d\}\). Corollary 3.5 shows that \(\text{QSym}^{(2)}\) is in fact a graded \(\mathbb{K}\)-subalgebra of \(\mathbb{K}[x]_{<\infty}\).
Lemma 3.2. $\text{QSym}^{(2)}$ is a free $\mathbb{K}$-module with monomial basis $B$ consisting of all

$$M_a := \sum_{i_1 < \cdots < i_{\text{rows}(a)} \atop \kappa_1 < \cdots < \kappa_{\text{cols}(a)}} \prod_{s=1}^{\text{rows}(a)} \prod_{t=1}^{\text{cols}(a)} x_{i_s,\kappa_t}$$

for non-empty $a \in \text{Cmp}$ and $M_e := 1 \in \mathbb{K}$.

Proof. For $B_d := \{M_a \mid a \in \text{Cmp} \text{ with } \sum_i \text{size}(a) a_i = d\}$ and monomorphism

$$\pi : \text{QSym}^{(2)}_d \to \text{span}_\mathbb{K} \left( \prod_{s=1}^{\text{rows}(a)} \prod_{t=1}^{\text{cols}(a)} x_{a_s,t} \mid a \in \text{Cmp} \right) =: V,$$

the image $\pi(B_d)$ is linear independent by construction, and therefore so is $B_d$. It suffices to show that $\text{QSym}^{(2)}_d$ is generated by $B_d$. For $f \in \text{QSym}^{(2)}_d$ let $\pi(f) := \sum_v \lambda(v) v \in V$ with $\lambda : \{v \in V \mid v \text{ monomial} \} \to \mathbb{K}$ of finite support. With $f = \sum_v \lambda(v) M_{a(v)}$ follows $\text{QSym}^{(2)}_d = \text{span}_\mathbb{K} B_d$ for unique $a(v)$ such that $M_{a(v)}$ contains $v$. \qed

Example 3.3.

$$M_1[1] = \sum_{i_1, \kappa_1} x_{i_1, \kappa_1}, \quad M_{\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}} = \sum_{i_1 < i_2 \atop \kappa_1 < \kappa_2} x_{i_1, \kappa_1} x_{i_2, \kappa_2} x_{i_2, \kappa_2}^2,$$

$$M_{\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}} = \sum_{i_1 < i_2 \atop \kappa_1 < \kappa_2} x_{i_1, \kappa_2} x_{i_2, \kappa_1} = \sum_{i_1 < i_2 \atop \kappa_1 > \kappa_2} x_{i_1, \kappa_1} x_{i_2, \kappa_2}, \quad M_{\begin{bmatrix} 2 & 1 \end{bmatrix}} = \sum_{i_1, \kappa_1 < \kappa_2} x_{i_1, \kappa_1}^2 x_{i_1, \kappa_2}.$$

3.1 Closure property under multiplication

We verify the quasi-shuffle identity for two-parameter quasisymmetric functions.

Theorem 3.4. For all $a, b \in \text{QSym}^{(2)}$,

$$M_a M_b = \sum_{j, k \in \mathbb{N}} \sum_{p \in q\text{Sh(rows}(a), \text{rows}(b); j)} \sum_{q \in \text{Sh(cols}(a), \text{cols}(b); k)} M_{c^{p,q}}$$

with $c^{p,q} \in \text{Cmp}$ according to Equation (6).
Proof. The proof is inspired by [GR14, Proposition 5.1.3.] and generalizes it for two parameters. It uses the matrix notation for surjections introduced in Remark 5.8. Consider

\[
M_a M_b = \sum_{\ell_1 < \ldots < \ell_{\text{rows}(a)}} \sum_{\kappa_1 < \ldots < \kappa_{\text{cols}(a)}} (\prod_{i=1}^{\text{rows}(a)} \prod_{k=1}^{\text{cols}(a)} a_{i,k} x_{i,\kappa_k}) (\prod_{s=1}^{\text{rows}(b)} \prod_{t=1}^{\text{cols}(b)} b_{s,t} x_{\sigma_s,\tau_t})
\]

\[
= \sum_{c \in \text{Cmp} \mu_1 < \ldots < \mu_{\text{rows}(c)}} N_{\mu,\nu}^{\text{rows}(c)} \prod_{m=1}^{\text{rows}(c)} \prod_{n=1}^{\text{cols}(c)} x_{m,n}^{\text{cols}(c)}
\]

where \( N_{\mu,\nu}^{\text{rows}(c)} \in \mathbb{N} \) is the number of all 4-tuples

\[
(\iota, \kappa, \sigma, \tau) \in \mathbb{N}^{\text{rows}(a)} \times \mathbb{N}^{\text{cols}(a)} \times \mathbb{N}^{\text{rows}(b)} \times \mathbb{N}^{\text{cols}(b)}
\]

such that \( \iota, \kappa, \sigma \) and \( \tau \) are strictly increasing with

\[
(\prod_{i=1}^{\text{rows}(a)} \prod_{k=1}^{\text{cols}(a)} a_{i,k} x_{i,\kappa_k}) (\prod_{s=1}^{\text{rows}(b)} \prod_{t=1}^{\text{cols}(b)} b_{s,t} x_{\sigma_s,\tau_t}) = \prod_{m=1}^{\text{rows}(c)} \prod_{n=1}^{\text{cols}(c)} x_{m,n}^{\text{cols}(c)}
\]

The central argument is that \( N_{\mu,\nu}^{\text{rows}(c)} \) is also the number of matrix pairs

\[
(P, Q) \in \text{QSH(rows}(a), \text{rows}(b); \text{rows}(c)) \times \text{qSh(cols}(a), \text{cols}(b); \text{cols}(c))
\]

such that

\[
P \text{ diag}(a, b) Q^\top = c.
\]

To show this, we construct a bijection \( \varphi \) from all 4-tuples in (16) with (17) to the set of all matrix pairs (18) with (19).

Concerning the construction of \( \varphi \), assume \((\iota, \kappa, \sigma, \tau)\) satisfies (17). Then, for every \((i, k) \leq \text{size}(a)\) there is a unique \((p(i), q(i))\) such that \((\iota_i, \kappa_k) = (\mu_{p(i)}, \nu_{q(i)})\).

Furthermore, for every \((s, t) \leq \text{size}(b)\) there is a unique \((p'(s), q'(t))\) such that \((\sigma_s, \tau_t) = (\mu'_{p'(s)}, \nu'_{q'(t)})\). We define

\[
P := \begin{bmatrix} P_1 & P_2 \end{bmatrix} := \begin{bmatrix} e_{p(1)} & \cdots & e_{p(\text{rows}(a))} \\ e_{p'(1)} & \cdots & e_{p'(\text{rows}(b))} \end{bmatrix}
\]

for rows,

\[
Q := \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} := \begin{bmatrix} e_{q(1)} & \cdots & e_{q(\text{cols}(a))} \\ e_{q'(1)} & \cdots & e_{q'(\text{cols}(b))} \end{bmatrix}
\]

for columns and set \( \varphi(\iota, \kappa, \sigma, \tau) := (P, Q)\). Note that both \( P \) and \( Q \) are right-invertible since (17) requires for all \( m \) and \( n \), either \( x_{\mu_m,\nu_n} = x_{\iota_i,\kappa_k} \) or \( x_{\mu_m,\nu_n} = x_{\sigma_s,\tau_t} \) with suitable
where and analogously, \( Q_{\text{Sym}} \)

**Corollary 3.5.** \( Q_{\text{Sym}}^{(2)} \) is a graded \( \mathbb{K} \)-subalgebra of \( \mathbb{K}[x]_{<\infty} \).
As in the one-parameter setting, e.g., [MR95, p.969], we can endow $\text{QSym}^{(2)}$ with a deconcatenation-type coproduct. Let $I := \mathbb{N} \cup \mathbb{N}$ be the disjoint union of two copies of $\mathbb{N}$, totally ordered by setting $\mathbb{N} < \mathbb{N}$. Let $(x_i)_{i \in \mathbb{N}^2}$ and $(y_i)_{i \in \mathbb{N}^2}$ be two sets of variables and define $z$ indexed by $I \times I$ as

$$ z_i := \begin{cases} x_i & i \in \mathbb{N}^2 \\ y_i & i \in \mathbb{N}^2 \\ 0 & \text{elsewhere.} \end{cases} $$

Any element $f \in \text{QSym}^{(2)}$ can be evaluated at the variables $z$ and can be written as

$$ f(z) = \sum_{k=1}^{\ell} g_k(x) h_k(y), $$

for some uniquely determined two-parameter quasisymmetric functions $g_k$ and $h_k$, where $1 \leq k \leq \ell$. Then

$$ \Delta_{\text{QSym}^{(2)}} f := \sum_{k=1}^{\ell} g_k \otimes h_k. $$

**Example 3.6.** For composition $a = \text{diag}(\begin{bmatrix} 1 & 1 \\ 3 \end{bmatrix}) \in \mathbb{N}_0^{2 \times 3}$,

$$ M_a(z) = \sum_{\ell_1 < \ell_2} z_{\ell_1, \ell_2}^2 z_{\ell_1, \kappa_2} z_{\ell_2, \kappa_3} + \sum_{\kappa_1 < \kappa_2 < \kappa_3} z_{\ell_1, \kappa_1} z_{\ell_1, \kappa_2} z_{\ell_2, \kappa_3} + \sum_{\ell_1 < \ell_2} z_{\ell_1, \kappa_1} z_{\ell_1, \kappa_2} z_{\ell_2, \kappa_3} $$

$$ = M_a(x) + M[2 \ 1](x) M[3 \ 1](y) + M_a(y). $$

We omit the proof of the following result.

**Theorem 3.7.** $\text{QSym}^{(2)}$, with induced the product of power series, the coproduct $\Delta_{\text{QSym}^{(2)}}$ and obvious unit and counit is a graded bialgebra, and hence a Hopf algebra. It is isomorphic to the Hopf algebra of Theorem 2.12, with isomorphism given by the linear extension of

$$ K\langle \text{Cmp}_\text{con} \rangle \rightarrow \text{QSym}^{(2)} \quad a \mapsto M_a. $$

### 3.2 Polynomial invariants

**Definition 3.8.** Let $\text{zero}_{a,k} \in \text{End}(K[[x]]_{<\infty})$ be the “free analogue” of zero insertion defined via

$$ \text{zero}_{a,k}(x_i) := \begin{cases} x_i & i_a < k \\ 0 & i_a = k \\ x_{(i-a)} & i_a > k. \end{cases} $$
Example 3.9. For all \( t \in \mathbb{N} \), \( \text{zero}_{1,2}(x_{1,t} + x_{2,t} + x_{3,1} x_{4,1}) = x_{1,t} + x_{2,1} x_{3,1} \).

We evaluate power series to set-theoretic functions with domain \( \text{evZ} = \text{evZ}(\mathbb{N}^2, \mathbb{K}) \), i.e., we assume \( d = 1 \) for the entire section. Let \( \text{eval} \) be the evaluation homomorphism of a formal power series \( f \in \mathbb{K}[x]_{<\infty} \) to its related

\[
\text{eval}(f) : \text{evZ} \to \mathbb{K},
\]

uniquely determined by

\[
\text{eval}(x_i)(X) := X_i
\]

for \( i \in \mathbb{N}^2 \). For convenience, we also write \( f(X) := \text{eval}(f)(X) \) for all \( f \) and \( X \).

With this evaluation, we can prove the quasi-shuffle identity for the two-parameter sums signature.

Proof. This follows from \( f(X) = \text{eval} \circ \text{zero}_{a,k}(f)(X) = f(\text{Zero}_{a,k}X) \) for all \( a, k \) and \( X \), since \( \text{eval} \circ \text{zero}_{a,k} \) is a homomorphism and

\[
\text{eval} \circ \text{zero}_{a,k}(x_i)(X) = \text{eval}(x_i)(\text{Zero}_{a,k}X)
\]

for every \( i \in \mathbb{N}^2 \).

The converse of Lemma 3.10 is not true in general.

Lemma 3.10. If \( f \in \mathbb{K}[x]_{<\infty} \) with \( \text{zero}_{a,k}(f) = f \) for all \( (a, k) \in \{1, 2\} \times \mathbb{N} \), then \( \text{eval}(f) \) is invariant to inserting zero (in both directions independently).

Proof. This follows from

\[
f(X) = \text{eval} \circ \text{zero}_{a,k}(f)(X) = f(\text{Zero}_{a,k}X)
\]

for all \( a, k \) and \( X \), since \( \text{eval} \circ \text{zero}_{a,k} \) is a homomorphism and

\[
\text{eval} \circ \text{zero}_{a,k}(x_i)(X) = \text{eval}(x_i)(\text{Zero}_{a,k}X)
\]

for every \( i \in \mathbb{N}^2 \).

The converse of Lemma 3.10 is not true in general.

Example 3.11. Let \( \mathbb{K} \) be the field of two elements. The polynomial \( f = x_{4,1}^2 - x_{4,1} \) induces the constant zero function \( \text{eval}(f) = 0 \), but \( \text{zero}_{1,2}(f) = x_{3,1}^3 - x_{3,1} \neq f \).

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Theorem 3.12. Let $f \in \mathbb{K}[x]_{<\infty}$. Then:

$$f \in \text{QSym}^{(2)} \text{ if and only if } f = \text{zero}_{a,k}(f) \text{ for all } a, k.$$ 

Proof. The proof is analogous to [DEFT22, Theorem 3.11.] when generalized for two parameters. For every $a \in \text{Cmp}$,

$$\text{zero}_{a,k}(M_a) = \sum_{1 \leq m \leq \text{rows}(a)} \sum_{1 \leq n \leq \text{cols}(a)} \text{zero}_{1,m}^{\text{rows}(a)} \text{zero}_{2,n}^{\text{cols}(a)} \prod_{s=1}^{\text{rows}(a)} \prod_{t=1}^{\text{cols}(a)} x_{s,t}^{a_{s,t}} = M_a$$

thus $\text{QSym}^{(2)} \subseteq \{f \mid \text{zero}_{a,k}(f) = f \ \forall a, k\}$.

Conversely, consider every $f \in \text{QSym}^{(2)}$ as a function from the free monoid generated by $(x_i)_{i \in \mathbb{N}^2}$ to the coefficient ring $\mathbb{K}$. Then,

$$f \left( \prod_{s=1}^{\text{rows}(a)} \prod_{t=1}^{\text{cols}(a)} x_{s,t}^{a_{s,t}} \right) = f \left( \prod_{s=1}^{\text{rows}(a)} \prod_{t=1}^{\text{cols}(a)} x_{s,t}^{a_{s,t}} \right)$$

for all compositions $a \in \text{Cmp}$ and strictly increasing chains $0 = \iota_0 < \iota_1 < \cdots < \iota_{\text{rows}(a)}$ and $0 = \kappa_0 < \kappa_1 < \cdots < \kappa_{\text{cols}(a)}$. \hfill \Box

Definition 3.13. We call $\psi : \text{evC} \rightarrow \mathbb{K}$ from Definition 2.25 a polynomial invariant\footnote{Note that this is a mere naming convention, i.e., $\psi$ is not a polynomial function itself, e.g., Example 2.28. However, Equations (12) and (13) forbid polynomial functions per se, so there is no conflict in naming.}, if it is induced by a formal power series $f \in \mathbb{K}[x]_{<\infty}$ of finite degree, i.e.,

$$\psi(X) = f(X) \ \forall X \in \text{evZ} \subseteq \text{evC}. \quad (22)$$

In this case, Equation (22) guarantees that $\psi$ is determined by a formal power series on the entire domain evC.

Theorem 3.14. Let $\mathbb{K}$ be an infinite field. A mapping $\psi : \text{evC} \rightarrow \mathbb{K}$ is a
polynomial invariant according to Definition 3.13,

if and only if

it is induced by the two-parameter sums signature precomposed with the difference
operator, i.e., there is \( w \in K \langle Cmp_{\text{con}} \rangle \) such that

\[
\psi(X) = \langle SS(\delta X), w \rangle \quad \forall X \in evC.
\]

In order to prove this, we recall properties of multivariate polynomials. For every truncation level \( t \in \mathbb{N}^2 \) let the polynomial \( t \)-truncation

\[
\text{trunc}_t : K[[x]]_{<\infty} \to K[x_i \mid i \leq t]
\]

be a homomorphism of \( K \)-algebras which projects all \( x_i \) with \( i \not\leq t \) to zero. In particular, \( \text{eval} \) defines an evaluation of truncated \( \text{trunc}_t(f) \) to its polynomial function such that

\[
(\text{trunc}_t f)(X) = f(X)
\]  

(23)

for all \( X \in ev\mathbb{Z} \) with \( \text{size}(X) \leq t \). We now identify multivariate polynomials with its corresponding functions.

Lemma 3.15. Let \( K \) be an infinite field.

1. If \( f \in K[[x]]_{<\infty} \) with \( f(X) = 0 \) for all \( X \in ev\mathbb{Z} \), then \( f = 0 \).
2. If \( f, g \in K[[x]]_{<\infty} \) with \( f(X) = g(X) \) for all \( X \in ev\mathbb{Z} \), then \( f = g \).

Proof. The second part follows from the first. For every \( f \neq 0 \) there is a truncation level \( t \in \mathbb{N}^2 \) such that \( \text{trunc}_t(f) \neq 0 \), and thus there is \( X \in K^{t_1 \times t_2} \) such that \( (\text{trunc}_t f)(X) \neq 0 \). Considering \( X \) as an element \( X \in ev\mathbb{Z} \) via

\[
X_i := \begin{cases} X_i & \text{if } i \leq t \\ 0 & \text{elsewhere,} \end{cases}
\]

we obtain \( f(X) = (\text{trunc}_t f)(X) \neq 0 \).

Corollary 3.16.

1. If \( f \in \text{QSym}^{(2)} \), then \( \text{eval}(f) \) is invariant to insertion of zeros in both directions independently.
2. If \( K \) is an infinite field, then the converse of part 1. is also true.

Proof. Part 1. is an immediate consequence of Lemma 3.10 and Theorem 3.12. Part 2. additionally uses Lemma 3.15.
Proof of Theorem 3.14. The backward direction is covered by Corollary 2.27 since concatenation of polynomials with formal power series remains a formal power series. We show that for any \( \psi \) which is

1. invariant to warping in both directions independently,
2. invariant modulo constants, and
3. polynomial, i.e., satisfies Equation (22),

there is a \( w \in \mathbb{K}\langle \text{Cmp}_{\text{con}} \rangle \) such that \( \psi(X) = \langle \text{SS}(\delta X), w \rangle \) for all \( X \in \text{evC} \). With Corollary 2.26, \( \psi \circ \varsigma \) is zero insertion invariant and clearly induced by a formal power series of finite degree, hence there is

\[
w = \sum_{1 \leq i \leq m} \lambda_i a_i \in \mathbb{K}\langle \text{Cmp}_{\text{con}} \rangle
\]

with \( \lambda_i \in \mathbb{K} \) and \( a_i \in \text{Cmp} \) such that

\[
\text{eval} \left( \sum_{1 \leq i \leq m} \lambda_i M_{a_i} \right) = \psi \circ \varsigma
\]

with Lemma 3.2 and Corollary 3.16. With \( \text{eval}(M_{a_i})(\delta X) = \langle \text{SS}(\delta X), a_i \rangle \) and invariance modulo constants,

\[
\psi(X) = \psi \circ \varsigma \circ \delta(X) = \langle \text{SS}(\delta X), w \rangle
\]

for all \( X \in \text{evC} \) as claimed. \( \square \)

4 Algorithmic considerations

4.1 Iterated two-parameter sums

For every eventually-constant \( X \in \text{evC} \), the computation of \( \Psi_{\alpha}(X) \) from Corollary 2.27 involves differences \( Z = \delta(X) \), and the coefficient of the two-parameter sums signature \( \text{SS}(Z) \) when tested at composition \( \alpha \).

Taking differences can be performed in linear time, i.e., the evaluation of \( \delta(X) \) according to Equation (11) requires \( O(\text{rows}(X) \cdot \text{cols}(X)) \) arithmetic operations for every \( X \in \text{evC} \).

The naive evaluation of \( \langle \text{SS}(Z), \alpha \rangle \) with \( (Z, \alpha) \in \text{evZ} \times \text{Cmp} \) however, sums over all pairs of increasing chains with lengths \( \text{cols}(\alpha) \) and \( \text{rows}(\alpha) \), respectively. In general, each of those resulting summands is a product with \( \text{rows}(\alpha) \cdot \text{cols}(\alpha) \) factors, leading to an upper complexity bound of

\[
O(\text{rows}(\alpha) \cdot \text{cols}(\alpha) \cdot \left( \frac{\text{rows}(X)}{\text{rows}(\alpha)} \right) \cdot \left( \frac{\text{cols}(X)}{\text{cols}(\alpha)} \right)) \tag{24}
\]

arithmetic operations for evaluating \( \Psi_{\alpha}(X) \).
In the current section we investigate a certain subclass of matrix compositions for which the coefficients of the two-parameter sums signature \(SS(Z), a\) can be evaluated in linear time. This subclass can be considered as a chain of connected \(1 \times 1\) compositions, for which iterative methods similar to those from the one-parameter setting [DEFT22] remain possible. For this we define three binary chaining operations, in particular covering block diagonal matrices from Definition 2.1.

**Definition 4.1.** For \(a \in \{0, 1, 2\}\) let

\[
\text{chain}_a : \text{Cmp} \times \text{Cmp} \to \text{Cmp}
\]

be a **chaining operation** where \((a, b) \in \text{Cmp}^2\) maps to \(\text{chain}_0(a, b) := \text{diag}(a, b),\)

\[
\begin{bmatrix}
a_{1,1} & \cdots & a_{1,\text{cols}(a)} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & & \vdots \\
a_{\text{rows}(a)-1,1} & \cdots & a_{\text{rows}(a)-1,\text{cols}(a)} & 0 & \cdots & 0 \\
0 & \cdots & 0 & b_{1,1} & \cdots & b_{1,\text{cols}(b)} \\
\vdots & \ddots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & b_{\text{rows}(b),1} & \cdots & b_{\text{rows}(b),\text{cols}(b)}
\end{bmatrix}
\]

with an overlapping at axis \(a = 1\), and

\[
\text{chain}_1(a, b) :=
\begin{bmatrix}
a_{1,1} & \cdots & a_{1,\text{cols}(a)-1} & a_{1,\text{cols}(a)} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & & \vdots \\
a_{\text{rows}(a),1} & \cdots & a_{\text{rows}(a),\text{cols}(a)-1} & a_{\text{rows}(a),\text{cols}(a)} & 0 & \cdots & 0 \\
0 & \cdots & 0 & b_{1,1} & b_{1,2} & \cdots & b_{1,\text{cols}(b)} \\
\vdots & \ddots & \vdots & \vdots & & \vdots \\
0 & \cdots & 0 & b_{\text{rows}(b),1} & b_{\text{rows}(b),2} & \cdots & b_{\text{rows}(b),\text{cols}(b)}
\end{bmatrix}
\]

with an analogous overlapping at axes \(a = 2\). Regarding empty compositions, we set \(\text{chain}_a(e, b) := \text{chain}_a(b, e) := b\) for all \((a, b) \in \{0, 1, 2\} \times \text{Cmp}\).

**Example 4.2.**

\[
\text{chain}_1 \left( \text{chain}_0 \left( \begin{bmatrix} 1 \\ 2 \times 3 \end{bmatrix}, \begin{bmatrix} 4 \\ \epsilon \end{bmatrix} \right) \right) = \begin{bmatrix} 1 \\ \epsilon \end{bmatrix} \begin{bmatrix} 2 & \epsilon \\ \epsilon & 3 \\ 4 \end{bmatrix} \in \text{Cmp}.
\]

**Lemma 4.3.** The chaining operation is interassociative, i.e., for all \((a, b, c) \in \text{Cmp}^3\) and \((a, b) \in \{0, 1, 2\}^2\),

\[
\text{chain}_a(a, \text{chain}_b(b, c)) = \text{chain}_b(\text{chain}_a(a, b), c).
\]

In particular \((\text{Cmp}, \text{chain}_0)\) is a non-commutative monoid for all \(a \in \{0, 1, 2\}\). For axis \(a \in \{1, 2\}\) we define the \(K\)-linear **cumulative sum** \(\text{cumsum}_a : \text{evC} \to \text{evC}\) via

\[
(\text{cumsum}_a X)_i := \sum_{j=1}^{i_{1_a}} X_{i_1 + (j - i_{1_a})e_a}
\]

\[ \text{(25)} \]

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and we recall the zero insertion operation \( \text{Zero}_{a,1} : \text{evC} \rightarrow \text{evC} \) with
\[
(\text{Zero}_{a,1} X)_i := \begin{cases} 
0_d & i_a = 1 \\
X_{i-e_a} & i_a > 1 
\end{cases}
\] (26)
from Section 2.3. For convenience we set \( \text{Zero}_{0,1} := \text{Zero}_{1,1} \circ \text{Zero}_{2,1} \), \( \text{Zero}_{a} := \text{Zero}_{a,1} \) for all \( a \in \{0, 1, 2\} \) and \( \text{cumsum}_0 := \text{cumsum}_1 \circ \text{cumsum}_2 \).

**Lemma 4.4.** For \((Z, b) \in \text{evZ} \times \text{Cmp}\) let \( A \in \text{evC} \) with
\[
A_r := \langle SS_{0, r}(Z), b \rangle
\] denote the two-parameter sums signature of \( Z \) tested at \( b \). Then,
\[
\langle SS_{0, r}(Z), \text{chain}_a(b, \lfloor \lambda \rfloor) \rangle = \text{cumsum}_a(Z^{(\lambda)} \cdot \text{Zero}_a A)_r
\] (27)
for all \( 1 \times 1 \) compositions \( \lfloor \lambda \rfloor \in \text{Cmp} \) and \( r \leq \text{size}(Z) \).

Both \( \text{cumsum}_a \) and \( \text{Zero}_a \) can be computed in linear time, that is in \( O(\text{rows}(Z) \cdot \text{cols}(Z)) \), leading to an linear evaluation of Equation (27). Iteratively, this yields an efficient method to compute two-parameter signature coefficients for chained matrix compositions, denoted by
\[
M_d^{\text{chain}} \subseteq \text{Cmp}.
\]
This set is defined to be the smallest set containing all \( 1 \times 1 \) compositions which is closed under \( \text{chain}_a \) for all \( a \in \{0, 1, 2\} \). With Lemma 4.3,
\[
M_d^{\text{chain}} = \left\{ \bigcap_{1 \leq t \leq \ell} \text{chain}_a \left( \bullet, \left[ \lambda_t \right] \right), \left[ \lambda_0 \right] \right\} \quad \ell \in \mathbb{N}, \lambda_0 \in M_d \quad a \in \{0, \ldots, 2\}^\ell \quad \lambda \in \left( M_d \setminus \{ \varepsilon \} \right)^\ell
\] (28)
can be thought of as sequential objects.

**Theorem 4.5.** For every \((a, Z) \in M_d^{\text{chain}} \times \text{evZ}\), the entire matrix
\[
\left( \langle SS_{0, r}(Z), a \rangle \right)_{r \leq \text{size}(Z)}
\] can be evaluated in linear time, i.e., requires
\[
O \left( \text{rows}(a) \cdot \text{cols}(a) \cdot \text{rows}(Z) \cdot \text{cols}(Z) \right)
\] arithmetic operations. In particular, so is \( \langle SS(Z), a \rangle \).
Proof. With Equation (28) one can write
\[ a = \bigcup_{1 \leq t \leq \ell} \text{chain}_{\mathcal{a}_t}(\lambda_t)(\lambda_0). \]

With \( h_t(U) := (\text{Zero}_{\mathcal{a}_t} \circ \text{cumsum}_{\mathcal{a}_t})(Z^{(\lambda_t)} \cdot U) \) follows inductively
\[ \langle SS_{0:r}(Z), a \rangle = \left( \text{cumsum}_0(Z^{(\lambda_r)} \cdot (h_{t-1} \circ \cdots \circ h_1)(1_{evC})) \right) r \]
for all \( r \leq \text{size}(Z) \).

We note that for arbitrary \( \lambda, \mu, \nu \in \mathcal{M}_d \) with \( \lambda \neq \varepsilon \neq \nu \), the following 2 \( \times \) 2 matrices
\[
\begin{bmatrix}
\lambda & \varepsilon \\
\mu & \nu
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
\lambda & \mu \\
\varepsilon & \nu
\end{bmatrix}
\]
are all in \( \mathcal{M}_{d^{chain}} \). Chaining along the anti-diagonal leads to jet another class of matrix compositions similar to Equation (28). The only 2 \( \times \) 2 matrices which are not covered by methods similar to Theorem 4.5 are
\[
\begin{bmatrix}
\lambda & \mu \\
\xi & \nu
\end{bmatrix}
\]
with \( \lambda, \mu, \nu, \xi \in \mathcal{M} \setminus \{\varepsilon\} \) all being non-trivial. We will momentarily see that for this type of matrix we can still do better than the naive (in this case quadratic) cost explained in Equation (24).

Lemma 4.6. For every \( Z \in evZ \) and 2 \( \times \) 2 composition \( a \), the entire matrix
\[
\langle SS_{0:r}(Z), a \rangle_{r \leq \text{size}(Z)}
\]
can be evaluated in \( O(\text{rows}(Z)^2 \cdot \text{cols}(Z)) \) arithmetic operations.

Proof. For all \( (2, 2) \leq r \leq \text{size}(Z) \),
\[
\langle SS_{0:r}(Z), a \rangle = \sum_{t_2=2}^{r_1} \sum_{\kappa_2=2}^{r_2} Z_{t_2=2, \kappa_2=2}^{(a_{2,2})} \sum_{t_1=1}^{r_2-1} \sum_{\kappa_1=1}^{r_2-1} Z_{t_1=1, \kappa_1=1}^{(a_{1,2})} Z_{t_2=2, \kappa_1=1}^{(a_{2,1})}. 
\]

Lemma 4.7. If \( \mathbb{K} \) is the Boolean semiring and \( Z \in evC \) with \( \text{size}(Z) = (T, T) \), then
\[
\langle SS(Z), \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \rangle
\]
can be computed in \( O(T^\omega) \), where \( \omega \) is the matrix-multiplication exponent.
Proof. Consider $Z$ as a $\{0, 1\}$-valued matrix in $Z \in \mathbb{Z}^{T \times T}$. Calculate

$$Z \cdot Z^\top,$$

at cost $O(T^\omega)$. The result contains an entry 2 that is not on the diagonal, if and only if there exist $\iota_1 < \iota_2$ and $\kappa_1 < \kappa_2$ with

$$Z_{\iota_1, \kappa_1} \land Z_{\iota_1, \kappa_2} \land Z_{\iota_2, \kappa_1} \land Z_{\iota_2, \kappa_2}$$

being true. This is exactly what has to be checked in order to calculate (29). \qed

4.2 Two-parameter quasi-shuffle

We provide an algorithm to compute two-parameter quasi-shuffles of matrix compositions efficiently. A step-by-step illustration is given in Example 2.15. Detailed explanations and a mathematical verification of soundness follows in Lemma 5.14.

**Input:** Nonempty matrix compositions $a, b \in \text{Cmp}$.

**Output:** List of matrix compositions $L$ which sums up to the two-parameter quasi-shuffle $\sum_{s \in L} s = a \text{ qs } b \in K^{\langle \text{Cmp}_{\text{con}} \rangle}$.

```python
def main(a ∈ M^{m×n}, b ∈ M^{s×t}):
    L, P ← [[], col_qsh(a ∈ M^{s×n}, [e_{m×t}])]
    for p ∈ P do
        c · d ← p where c ∈ M^{m×j} and d ∈ M^{s×j}
        L.extend(row_qsh(c, d))
    return L

def col_qsh(c, d ∈ K^{(M^{m+s})}):
    if c = e or d = e then
        return [c · d]
    a · v, b · w ← c, d where a, b ∈ M^{m+s}
    return a · col_qsh(v, b · w) + b · col_qsh(a · v, w) + (a * b) · col_qsh(v, w)

def row_qsh(c, d ∈ K^{(M^{1×j})}):
    return [x^T where x ∈ col_qsh(c^T, d^T)]
```

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5 Technical details and proofs

5.1 Signature properties

We provide omitted details and proofs from Sections 2.3 to 2.5. In Section 2.5 we extend the notion of size to $X \in \text{evC}$, via

$$\text{size}(X) := \left(\text{rows}(X), \text{cols}(X)\right) := \min \left\{ n \in \mathbb{N}^2 \mid X_i \neq X_j \implies i, j \leq n \right\}, \quad (30)$$

where the minimum is taken over the poset $(\mathbb{N}^2, \leq)$. Lemma 5.1 guarantees its existence. Note that each eventually-zero function can be described as a matrix after “filling it up with zeros”. We show that Equation (30) relates to the size of matrices as expected.

Lemma 5.1.

1. The minimum in Equation (30) exists.
2. For all $Z \in \text{evZ}$,

$$\text{size}(Z) = \min \{ n \in \mathbb{N}^2 \mid Z_i \neq 0 \implies i \leq n \}$$

Proof. For all $X \in \text{evC}$, the maxima in

$$k := \left(\max \left\{ m \in \mathbb{N} \mid X_{m, \bullet} \neq X_{m+1, \bullet} \right\}, \max \left\{ n \in \mathbb{N} \mid X_{\bullet, n} \neq X_{\bullet, n+1} \right\}\right)$$

exist. Furthermore, $k$ satisfies the implication $X_i \neq X_j \implies i, j \leq k$ and is clearly minimal with this property. Part 2. follows with $\lim_{n \to \infty} Z_n = 0_2$. \qed

Proof of Lemma 2.22. The difference $\delta$ is clearly well-defined, i.e., $\delta(\text{evC}) \subseteq \text{evZ}$ and linear. From Lemma 5.3 we get $\varsigma(\text{evZ}) \subseteq \text{evZ} \subset \text{evC}$.

Linearity,

$$\lambda (\varsigma X)_{i,j} + (\varsigma Y)_{i,j} = \lambda \left( \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} X_{s,t} \right) + \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} Y_{s,t} = \sum_{s=1}^{\infty} \sum_{t=1}^{\infty} \lambda X_{s,t} + Y_{s,t}$$

$$= \varsigma(\lambda X + Y)_{i,j} \quad \forall (i, j, \lambda, X, Y) \in \mathbb{N}^2 \times \mathbb{K} \times \text{evZ}^2,$$

where the sums are all finite since $\text{evZ}$ is a $\mathbb{K}$-module. Moreover, $\varsigma$ is a linear right inverse of $\delta$. Indeed,

$$\delta(\varsigma Z)_{i,j} = \sum_{i+1 \leq s} \left( \sum_{j+1 \leq t} Z_{s,t} - \sum_{j \leq t} Z_{s,t} \right) + \sum_{i \leq s} \left( \sum_{j \leq t} Z_{s,t} - \sum_{j+1 \leq t} Z_{s,t} \right) = Z_{i,j}$$
for all \((i, j, Z) \in \mathbb{N}^2 \times \text{ev}Z\). In particular, \(\delta\) is surjective. Regarding 2., if \(X \in \ker(\delta)\), then
\[
0 = X_{\text{rows}(X) + 1, \text{cols}(X) + 1} - X_{\text{rows}(X), \text{cols}(X) + 1} + X_{\text{rows}(X) + 1, \text{cols}(X)}
\]
and hence \(X_{\text{rows}(X), \text{cols}(X)} = \lim_{i \to \infty} X_i\). Recursively one obtains \(X_j = \lim_{i \to \infty} X_i\) for all \(j \in \mathbb{N}^2\). Part 4. follows from the homomorphism theorem for \(K\)-modules.

Furthermore, we show that \(\delta\) and \(\varsigma\) do not change the size of its input.

**Lemma 5.2.**

1. For all \(X \in \text{ev}C\), \(\text{size}(\delta X) = \text{size}(X)\).
2. For all \(Z \in \text{ev}Z\), \(\text{size}(\varsigma Z) = \text{size}(Z)\).

**Proof.** For part 1., note that \((\delta X)_i = 0\) for all \(i \not\in \text{size}(X)\), hence \(\text{size}(\delta X) \leq \text{size}(X)\). Assuming \(\text{size}(\delta X) \neq \text{size}(X)\), then \((\delta X)_{\text{cols}(X), t} = 0\) or \((\delta X)_{t, \text{rows}(X)} = 0\) for all \(t \in \mathbb{N}\). We pursue the first case, the second yields a similar contradiction. With \((\delta X)_{\text{cols}(X), t} = 0\) for all \(t\) and
\[
X_{\text{cols}(X), n} \neq X_{\text{cols}(X) + 1, n} = X_{\text{cols}(X) + 1, s}
\]
for suitable \(n \leq \text{rows}(X)\) and arbitrary \(s \in \mathbb{N}\) follows \(X_{\text{cols}(X), n} = X_{\text{cols}(X), s}\) for all \(s\). In particular this implies \((\delta X)_{\text{size}(X)} = X_{\text{size}(X)} = X_{\text{cols}(X), \text{rows}(X) + 1} \neq 0\). This also implies 2., since \(\text{size}(Z) = \text{size}(\delta(\varsigma Z)) = \text{size}(\varsigma Z)\) for every \(Z \in \text{ev}C\).

Restricted to eventually-zero functions, \(\delta\) and \(\varsigma\) are isomorphisms, as the following statement implies.

**Lemma 5.3.**

1. \(\varsigma(\text{ev}Z) \subseteq \text{ev}Z\).
2. \(\forall X \in \text{ev}Z \subseteq \text{ev}C : \varsigma(\delta Z) = Z\).

**Proof.** With Lemma 5.1, \(\text{size}(Z) = \text{size}(\varsigma Z)\) and
\[
\varsigma(Z)_{\text{rows}(Z), \text{cols}(Z) + 1} = \sum_{s = \text{rows}(Z)}^{\infty} \sum_{t = \text{cols}(Z) + 1}^{\infty} Z_{s, t} = 0,
\]
hence \((\varsigma Z)_n = 0\) for all \(n \not\in \text{size}(Z)\), i.e., \(\varsigma Z \in \text{ev}Z\). Part 2. follows with
\[
(\varsigma(\delta Z))_{i, j} = \sum_{s = 1}^{\text{rows}(Z)} \sum_{t = j}^{\text{cols}(Z)} Z_{s + 1, t + 1} - Z_{s + 1, t} - Z_{s, t + 1} + Z_{s, t}
\]
and \(\text{size}(Z) = \text{size}(\delta Z)\).
Proof of Lemma 2.21, parts 1., 2. and 3. The mapping $\text{warp}_{a,k} : \text{evC} \to \text{evC}$ from Section 2.4 is well-defined (compare Lemma 5.4 for $\text{warp}_{a,k}(\text{evC}) \subseteq \text{evC}$), linear with

$$
(warp_{a,k} \lambda X + Y)_i = \begin{cases} 
(\lambda X + Y)_i & i_a \leq k \\
(\lambda X + Y)_{i-e_a} & i_a > k.
\end{cases}
$$

$$
= \lambda (warp_{a,k} X)_i + (warp_{a,k} Y)_i
$$

for all $(\lambda, X, Y) \in K \times \text{evC}^2$ and injective since $\ker(warp_{a,k}) = 0$. Part 2. follows with

$$
(warp_{1,k}(warp_{2,j} X))_i \begin{cases} 
(warp_{2,j} X)_i & i_1 \leq k \\
(warp_{2,j} X)_{(i_1-1,i_2)} & i_1 > k
\end{cases}
$$

$$
= \begin{cases} 
X_i & i_1 \leq k \land i_2 \leq j \\
X_{(i_1,i_2-1)} & i_1 \leq k \land i_2 > j
\end{cases}
$$

$$
= \begin{cases} 
(warp_{1,k} X)_i & i_2 \leq j \\
(warp_{1,k} X)_{(i_1,i_2-1)} & i_2 > j
\end{cases}
$$

$$
= (warp_{1,k}(warp_{2,j} X))_i \forall (i,j,k,X) \in \mathbb{N}^4 \times \text{evC},
$$

and part 3. with

$$
(warp_{a,k}(warp_{a,j} X))_i \begin{cases} 
(warp_{a,j} X)_i & i_a \leq k \\
(warp_{a,j} X)_{i-e_a} & i_a > k
\end{cases}
$$

$$
= \begin{cases} 
X_i & i_a \leq k \land i_a \leq j < k \\
X_{i-e_a} & i_a \leq k \land i_a > j
\end{cases}
$$

$$
= \begin{cases} 
X_{i-e_a} & k \leq i_a \land i_a - 1 = (i-e_a)_a \leq j < k \\
X_{i-2e_a} & i_a > k \land i_a - 1 = (i-e_a)_a > j
\end{cases}
$$

$$
= \begin{cases} 
X_i & i_a \leq j < k \land i_a \leq k - 1 \\
X_{i-e_a} & i_a > j \land i_a - 1 = (i-e_a)_a \leq k - 1
\end{cases}
$$

$$
= \begin{cases} 
X_{i-e_a} & i_a \leq j < k \land i_a - 1 = (i-e_a)_a \leq j - 1 \\
X_{i-2e_a} & i_a > j \land i_a - 1 = (i-e_a)_a > k - 1 > j - 1
\end{cases}
$$

$$
= \begin{cases} 
(warp_{a,k-1} X)_i & i_a \leq j \\
warp_{a,k-1} X)_{i-e_a} & i_a > j
\end{cases}
$$

$$
= (warp_{a,j}(warp_{a,k-1} X))_i
$$

where $(a,i,j,k,X) \in \{1, 2\} \times \mathbb{N}^4 \times \text{evC}$ and $k > j$. \qed
Lemma 5.4. For all \((a, k, X) \in \{1, 2\} \times \mathbb{N} \times \text{evC},\)

\[
\text{size}(\text{warp}_{a,k} X) = \begin{cases} 
\text{rows}(X) + 1, & a = 1 \land k \leq \text{rows}(X), \\
\text{cols}(X) + 1, & a = 2 \land k \leq \text{cols}(X), \\
\text{size}(X), & \text{elsewhere}.
\end{cases}
\]

Proof. Assume \(a = 1\) and \(\text{rows}(X) \geq k\). For all \(m > \text{rows}(X)\),

\[
(\text{warp}_{1,k} X)_{m+1, \bullet} = X_{m, \bullet} = X_{m+1, \bullet} = (\text{warp}_{1,k} X)_{m+2, \bullet},
\]

which implies \(\text{rows}(\text{warp}_{1,k} X) \leq \text{rows}(X) + 1\). With

\[
(\text{warp}_{1,k} X)_{\text{rows}(X)+1, \bullet} = X_{\text{rows}(X), \bullet} \neq X_{\text{rows}(X)+1, \bullet} = (\text{warp}_{1,k} X)_{\text{rows}(X)+2, \bullet}
\]

follows equality. Furthermore

\[
(\text{warp}_{1,k} X)_{\bullet, n} = (\text{warp}_{1,k} X)_{\bullet, n+1} \iff X_{\bullet, n} \neq X_{\bullet, n+1}
\]

for all \(n \in \mathbb{N}\), hence \(\text{cols}(\text{warp}_{1,k} X) = \text{cols}(X)\). The case \(a = 2\) and \(\text{cols}(X) \geq k\) is treated similar. The following cases cover all remaining possibilities.

1. If \(a = 2\) and \(k > \text{cols}(X)\), then

\[
(\text{warp}_{2,k} X)_{i,j} = \begin{cases} 
X_{i,j} & j \leq k \\
X_{i,j-1} = X_{i,j} & j > k > \text{cols}(X),
\end{cases}
\]

thus \(\text{size}(\text{warp}_{2,k} X) = \text{size}(X)\).

2. If \(k > \text{rows}(X)\) and \(a = 1\), then analogously to part 1., \(\text{warp}_{1,k} X = X\).

3. The case \(k > \text{rows}(X)\) and \(k > \text{cols}(X)\) is covered by either part 1. or 2.

\[
\square
\]

Proof. (of the remaining part 4. from Lemma 2.21.) We use the parts 2. and 3. to rewrite every

\[
A = \text{warp}_{a_q,k_q} \circ \cdots \circ \text{warp}_{a_1,k_1}(B)
\]

with \((a,k,A,B) \in \{1,2\}^q \times \mathbb{N}^q \times \text{evC}^2\) into the form

\[
A = \text{warp}_{2,j_m} \circ \cdots \circ \text{warp}_{2,j_1} \circ \text{warp}_{1,i_t} \circ \cdots \circ \text{warp}_{1,i_1}(B)
\]

with strictly increasing chains \(i_1 < \cdots < i_t \leq \text{rows}(A)\) and \(j_1 < \cdots < j_m \leq \text{cols}(A)\).

With this and Lemma 5.4, there exists the minimum

\[
\min \left\{ \text{size}(B) \mid A = \text{warp}_{a_q,k_q} \circ \cdots \circ \text{warp}_{a_1,k_1}(B), (a,k,B) \in \{1,2\}^q \times \mathbb{N}^q \times \text{evC} \right\}
\]

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over the poset \((\mathbb{N}^2, \preceq)\) for every \(A \in \text{evC}\). Let \(B, B' \in \text{evC}\) such that the minimum is reached with

\[
A = \text{warp}_{2,j_m} \circ \cdots \circ \text{warp}_{2,j_1} \circ \text{warp}_{1,i} \circ \cdots \circ \text{warp}_{1,i_1}(B)
\]

\[
= \text{warp}_{2,j_m} \circ \cdots \circ \text{warp}_{2,j'_1} \circ \text{warp}_{1,i'_1} \circ \cdots \circ \text{warp}_{1,i_1}(B'),
\]

where the strictly increasing chains \(i, i'\) and \(j, j'\) are of same length due to Lemma 5.4 and \(\text{size}(B) = \text{size}(B')\). If \(i_1 < i'_1 < \cdots < i'_{\ell}\), then \(A_{i_1,\bullet} = A_{i_1+1,\bullet}\) by definition, and hence, there exists \(B'' := \text{warp}_{1,i_1}(B') \in \text{evC}\) such that

\[
A = \text{warp}_{2,j_m} \circ \cdots \circ \text{warp}_{2,j'_1} \circ \text{warp}_{1,i'_1} \circ \cdots \circ \text{warp}_{1,i_1}(B''),
\]

contradicting that \(\text{size}(B) = \text{size}(B')\) is minimal. Therefore \(i_1 = i'_1\), thus with injectivity of \(\text{warp}_{1,i_1}\) and parts 2. and 3.,

\[
A' = \text{warp}_{2,j_m} \circ \cdots \circ \text{warp}_{2,j_1} \circ \text{warp}_{1,\iota - 1} \circ \cdots \circ \text{warp}_{1,\iota - 1}(B)
\]

\[
= \text{warp}_{2,j'_m} \circ \cdots \circ \text{warp}_{2,j'_1} \circ \text{warp}_{1,i'_1 - 1} \circ \cdots \circ \text{warp}_{1,i'_1 - 1}(B')
\]

for \(A' \in \text{evC}\) with \(\text{warp}_{1,i_1}(A') = A\). Recursively we obtain \(B = B'\). \(\square\)

Clearly \(\text{Zero}_{a,k} : \text{evZ} \to \text{evZ}\) from Section 2.3 is well-defined. We use the following relations to show properties of \(\text{Zero}_{a,k}\) via the properties of \(\text{warp}_{a,k}\).

**Proof of Lemma 2.24, parts 1. and 2.** Let \(a = 1\) and \(X \in \text{evC}\). Then,

\[
(\text{Zero}_{1,k}(\delta X))_{i,j} = \begin{cases} 
X_{i+1,j+1} - X_{i+1,j} - X_{i,j+1} + X_{i,j} & i < k \\
X_{i,j+1} - X_{i,j} - X_{i,j+1} + X_{i,j} & i = k \\
X_{i,j+1} - X_{i,j} - X_{i-1,j+1} + X_{i-1,j} & i > k
\end{cases}
\]

\[
= \delta(\text{warp}_{1,k} X)_{i,j} \quad \forall (i, j, k, X) \in \mathbb{N}^3 \times \text{evC}.
\]

The case \(a = 2\) is treated similar. With this and Lemmas 2.22 and 5.3 follows part 2. via

\[
\zeta \circ \text{Zero}_{a,k} = \zeta \circ \text{Zero}_{a,k} \circ \delta \circ \zeta = \zeta \circ \delta \circ \text{warp}_{a,k} \circ \zeta = \text{warp}_{a,k} \circ \zeta
\]

for every \((a, k) \in \{1,2\} \times \mathbb{N}\). \(\square\)

**Lemma 5.5.** For all \((a, k, Z) \in \{1,2\} \times \mathbb{N} \times \text{evZ}\),

\[
\text{size}(\text{Zero}_{a,k} Z) = \begin{cases} 
(\text{rows}(Z) + 1, \text{cols}(Z)) & a = 1 \land k \leq \text{rows}(Z), \\
(\text{rows}(Z), \text{cols}(Z) + 1) & a = 2 \land k \leq \text{cols}(Z), \\
\text{size}(Z) & \text{elsewhere}.
\end{cases}
\]
Proof. Follows with Lemma 5.4, size \( \circ \varsigma = \text{size} \)
\[
\text{size} \circ \text{Zero}_{a,k} = \text{size} \circ \text{Zero}_{a,k} \circ \delta \circ \varsigma = \text{size} \circ \delta \circ \text{warp}_{a,k} \circ \varsigma = \text{size} \circ \text{warp}_{a,k} \circ \varsigma
\]
for all \((a,k) \in \{1,2\} \times \mathbb{N}\). \(\square\)

Proof of Lemma 2.17. With Lemmas 2.22 and 2.24,
\[
\text{Zero}_{a,k} = \text{Zero}_{2,j} \circ \delta \circ \varsigma = \delta \circ \text{warp}_{2,j} \circ \varsigma
\]
is linear for all \((a,k) \in \{1,2\} \times \mathbb{N}\),
\[
\text{Zero}_{1,k} \circ \text{Zero}_{2,j} = \delta \circ \text{warp}_{1,k} \circ \text{warp}_{2,j} \circ \varsigma
\]
for all \((j,k) \in \mathbb{N}^2\) with Lemma 2.21, and analogously, part 3. With Lemma 5.5, we get a minimal \(\delta(\text{NF}_{\text{warp}}(\varsigma(Z))) \in \text{evZ}\) for every \(Z \in \text{evZ}\) with
\[
Z = \delta(\varsigma(Z)) = \delta \circ \text{warp}_{2,j_m} \circ \cdots \circ \text{warp}_{2,j_1} \circ \text{warp}_{1,i_\ell} \circ \cdots \circ \text{warp}_{1,i_1}(\text{NF}_{\text{warp}}(\varsigma(Z)))
\]
for increasing chains \(i_1 < \cdots < i_\ell\) and \(j_1 < \cdots < j_m\). Uniqueness follows analogously to Lemma 2.21 via injective \(\text{Zero}_{a,k}\) and previous parts 2. and 3. \(\square\)

Proof of the remaining parts from Lemma 2.24. For \(A \in \text{evC}\), let \(a,k\) such that
\[
\delta(A) = \delta \circ \text{warp}_{a,q,k_q} \circ \cdots \circ \text{warp}_{a_1,k_1} \circ \text{NF}_{\text{warp}}(A)
\]
\[
= \text{Zero}_{a,q,k_q} \circ \cdots \circ \text{Zero}_{a_1,k_1} \circ \delta \circ \text{NF}_{\text{warp}}(A),
\]
hence part 3. via uniqueness. With this follows also
\[
\varsigma \circ \text{Nf}_{\text{Zero}} = \varsigma \circ \text{Nf}_{\text{Zero}} \circ \delta \circ \varsigma = \varsigma \circ \delta \circ \text{Nf}_{\text{warp}} \circ \varsigma,
\]
and thus part 4. since \(\varsigma \circ \delta(X) = X\) for all \(X \in \text{evZ} \subseteq \text{evC}\). Part 5. holds via
\[
\text{NF}_{\ker(\delta)} \circ \text{warp}_{a,k} = \varsigma \circ \text{Zero}_{a,k} \circ \delta = \text{warp}_{a,k} \circ \text{NF}_{\ker(\delta)},
\]
resulting in part 6. analogously as in 3. \(\square\)

We show that \((\text{evC}, \varnothing)\) is a semigroup. Together with Lemma 5.7, \(\delta\) is a surjective homomorphism of semigroups.

Lemma 5.6. For all \(X, Y, Z \in \text{evC}\),
\begin{enumerate}
\item rows\((X \varnothing Y) = \text{rows}(X) + \text{rows}(Y),
\item cols\((X \varnothing Y) = \text{cols}(X) + \text{cols}(Y),
\end{enumerate}
3. \((X \boxplus Y) \boxplus Z = X \boxplus (Y \boxplus Z)\), and
4. \(\delta(X \boxplus Y) = \delta(X) \circ \delta(Y)\).

**Proof.** Clearly
\[
\left(\text{rows}(X), \text{cols}(X)\right) \leq \left(\text{rows}(X \boxplus Y), \text{cols}(X \boxplus Y)\right).
\]
We assume \(n \leq \left(\text{rows}(X) + \text{rows}(Y), \text{cols}(X) + \text{cols}(Y)\right)\) with \(n_1 > \text{rows}(X) + \text{rows}(Y)\). Then
\[
(X \boxplus Y)_n = \left(\text{NF}_{\ker(\delta)}(X) + \text{warp}_{1,1}^{\text{rows}(X)} \left(\text{warp}_{2,1}^{\text{cols}(X)}(Y)\right)\right)_n
= \text{NF}_{\ker(\delta)}(X)_n + Y_{(n_1 - \text{rows}(X), m)} = \lim_{i \to \infty} Y_i
\]
for suitable \(m \in \mathbb{N}\). The remaining case \(n_2 > \text{cols}(X) + \text{cols}(Y)\) is treated similar, hence
\[
\left(\text{rows}(X) + \text{rows}(Y), \text{cols}(X) + \text{cols}(Y)\right) \leq \left(\text{rows}(X \boxplus Y), \text{cols}(X \boxplus Y)\right).
\]
Equality follows with
\[
\left(\text{rows}(X), \text{cols}(X)\right) < n \implies (X \boxplus Y)_n = Y_n.
\]
This shows part 1. and 2. Lemmas 2.21, 2.22 and 2.24 yield
\[
(X \boxplus Y) \boxplus Z = \text{NF}_{\ker(\delta)}(X) + \text{warp}_{1,1}^{\text{rows}(X)} \left(\text{warp}_{2,1}^{\text{cols}(X)}(Y)\right) + \text{warp}_{1,1}^{\text{rows}(X \boxplus Y)} \left(\text{warp}_{2,1}^{\text{cols}(X \boxplus Y)}(Z)\right)
= X \boxplus (Y \boxplus Z),
\]
i.e. associativity of \(\boxplus\). Part 4. follows with
\[
\delta(X \boxplus Y) = \delta \circ \varsigma \circ \delta(X) + \text{Zero}_{1,1}^{\text{rows}(\delta X)} \left(\text{Zero}_{2,1}^{\text{cols}(\delta Y)}(\delta Y)\right) = \delta(X) \circ \delta(Y),
\]
and Lemma 5.1.

Also \(\odot\) is associative and \(\varsigma\) an injective homomorphism of semigroups.

**Lemma 5.7.** For all \(X, Y, Z \in \text{evC}\),

1. \(\text{rows}(X \odot Y) = \text{rows}(X) + \text{rows}(Y)\),

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2. \( \text{cols}(X \otimes Y) = \text{cols}(X) + \text{cols}(Y) \),
3. \( (X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \), and
4. \( \varsigma(X \otimes Y) = \varsigma(X) \sqcup \varsigma(Y) \).

**Proof.** Part 1 follows with Lemmas 5.1 and 5.6 via

\[
\text{rows}(X \otimes Y) = \text{rows} \left( \delta (\varsigma(X) \otimes \delta (\varsigma(Y))) \right)
\]

\[
= \text{rows} \left( \delta (\varsigma(X) \sqcup \varsigma(Y)) \right) = \text{rows}(\varsigma(X)) + \text{rows}(\varsigma(Y)) = \text{rows}(X) + \text{rows}(Y),
\]

similarly part 2. Associativity is similar as in Lemma 5.6, and with

\[
\varsigma(X \otimes Y) = \varsigma \circ \delta \circ \varsigma(X) + \text{warp}^{\text{rows}(\varsigma(X))} \left( \text{warp}^{\text{cols}(\varsigma(X))}(\varsigma(Y)) \right) = \varsigma(X) \sqcup \varsigma(Y)
\]

follows part 4. \( \square \)

With now prove Chen’s identity with respect to diagonal concatenation.

**Proof of Lemma 2.35.** Let \( j = \text{size}(A) \) and \( Z = A \otimes B \). For every \( a \in \text{Cmp} \) with \( \text{size}(a) = (m, n) \) follows

\[
\langle \text{ISS}_{E_{\eta_t}}(Z), a \rangle = \sum_{\ell_1 < \ell_2 < \ldots < \ell_m \leq r_1} \prod_{s=1}^{m} \prod_{l=1}^{n} Z_{\ell_s, \kappa_l}^{(a_{s,l})}
\]

\[
= \sum_{0 \leq u \leq m} \sum_{0 \leq v \leq n} \prod_{s=1}^{m} \prod_{l=1}^{n} Z_{\ell_s, \kappa_l}^{(a_{s,l})}
\]

\[
= \sum_{\text{diag}(b,c)=a} \left( \sum_{\ell_1 < \ell_2 < \ldots < \ell_m \leq j_1} \prod_{s=1}^{u} \prod_{l=1}^{v} Z_{j_s, \kappa_l}^{(b_{s,l})} \left( \sum_{\ell_1 < \ell_2 < \ldots < \ell_m \leq r_1} \prod_{s=1}^{m} \prod_{l=1}^{v} Z_{j_s, \kappa_l}^{(c_{s,l})} \right) \right)
\]

\[
+ \sum_{\ell_1 < \ell_2 < \ldots < \ell_m \leq j_1} \prod_{s=1}^{m} \prod_{l=1}^{n} Z_{\ell_s, \kappa_l}^{(a_{s,l})}
\]

where the summands in Equation (31) are always zero whenever

\[
a = \begin{bmatrix}
    b & u \\
    v & c
\end{bmatrix}
\]

with \( \varepsilon_{ux(n-v)} \neq u \) or \( \varepsilon_{(m-u)xv} \neq v \) for \( 1 \leq u \leq m-1 \) and \( 1 \leq v \leq n-1 \). \( \square \)
Proof of Theorem 2.20. The forward direction is discussed in Corollary 3.16 for the case $d = 1$. As usual, the case $d > 1$ is analogous. We show the backwards direction. If $\text{size}(X) \neq \text{size}(Y)$, then $SS(X) \neq SS(Y)$ and $X \neq Y$. It therefore suffices to assume $SS(X) = SS(Y)$, $X = \text{NF}_s(X)$, $\text{size}(X) = \text{size}(Y) = (S, T)$, and to verify $X = Y$. Define the matrix composition $a(j, X) \in M_d^{S \times T}$ with entries
\[
a(j, X)_{s,t} := \begin{cases} j & \text{if } X_{s,t}^{(j)} \neq 0, \\ i & \text{if there exists } i \leq d, \text{ chosen minimally, such that } X_{s,t}^{(i)} \neq 0, \\ \varepsilon & \text{elsewhere}. \end{cases}
\]
This matrix is a composition since $(X_{s,t})_{s \leq S, t \leq T}$ has no zero lines (columns treated analogously), and therefore, for every column $X_{\bullet, t}$ with $t \leq T$ there exist $i \leq d$ and $s \leq S$ such that $X_{s,t}^{(i)} \neq 0$ and $a(j, X)_{s,t} = i$. Furthermore, since $\mathbb{K}$ has no zero divisors,
\[
\langle SS(X), a(j, X) \rangle \neq 0
\]
for all $j \leq d$. Assume there is $j \leq d$ with $a(j, X) \neq a(j, Y)$, then there exist $(s, t) \leq (S, T)$ with either
\[
\begin{cases} a(j, X)_{s,t} = j \neq i = a(j, Y)_{s,t} \text{ or} \\ a(j, X)_{s,t} = i > k = a(j, Y)_{s,t} \text{ or} \\ a(j, X)_{s,t} = i \neq \varepsilon = a(j, Y)_{s,t} \text{ for suitable } i \leq d, \end{cases}
\]
after possibly exchanging the role of $X$ and $Y$. With the minimality condition in Equation (32),
\[
0 \neq \langle SS(X), a(j, X) \rangle = \langle SS(Y), a(j, X) \rangle = 0,
\]
thus $a(j) := a(j, X) = a(j, Y)$. In particular holds $X_{s,t}^{(j)} = 0 \iff Y_{s,t}^{(j)} = 0$. If $X_{s,t}^{(j)} \neq 0$
then
\[
X_{s,t}^{(j)} \prod_{s=1}^{S} \prod_{t=1}^{T} X_{s,t}^{(a(j)_{s,t})} = Y_{s,t}^{(j)} \prod_{s=1}^{S} \prod_{t=1}^{T} Y_{s,t}^{(a(j)_{s,t})} = Y_{s,t}^{(j)} \prod_{s=1}^{S} \prod_{t=1}^{T} X_{s,t}^{(a(j)_{s,t})},
\]
thus $X_{s,t}^{(j)} = Y_{s,t}^{(j)}$, by the cancelation property\(^8\) of $\mathbb{K}$. \qed

5.2 Two-parameter quasi-shuffle

We provide omitted details and proofs from Sections 2.1, 2.2 and 4.2. We define a more convenient description of the two-parameter quasi-shuffle via surjections, acting on columns or rows of compositions. For $1 \leq i \leq j$, let $e_i \in \mathbb{N}_0^j$ encode the $i$-th standard column defined via the Kronecker delta $(e_i)_k := \delta_{i,k}$.

Consider the following set of non-negative integer matrices
\[
\text{QSH}(m, s; j) := \left\{ e_{\kappa_1} \cdots e_{\kappa_m} \right\} \in \mathbb{N}_0^{j \times (m+s)} \text{ right invertible}.
\]
\(^8\)The cancelation property is satisfied via the quotient field of commutative rings without zero-divisors.
Remark 5.8. $\text{QSH}(m, s; j)$ is in one-to-one correspondence\(^9\) with $\text{qSh}(m, s; j)$.

For every $P \in \mathbb{N}_0^{j \times m}$ and matrix composition $a \in \mathcal{M}^{m \times n}$ let $Pa \in \mathcal{M}^{j \times n}$ denote the action on the row space of $a$,

$$(Pa)_{i, \nu} := \bigstar_{1 \leq \mu \leq m} a_{\mu, \nu}^P, \quad (i, \nu) \leq (j, n).$$

Analogously, let $aQ^\top \in \mathcal{M}^{m \times k}$ denote the action on the column space via $Q \in \mathbb{N}_0^{k \times n}$.

Example 5.9. The surjection $q \in \text{qSh}(2, 2; 3)$ with $q(1) = 1$, $q(2) = 3$, $q(3) = 2$ and $q(4) = 3$ defines the following action on rows,

$$
\begin{bmatrix}
1 & 1 & 2 & 4 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix} = 
\begin{bmatrix}
1 & 1 & 4 \\
1 & 3 \\
2
\end{bmatrix} \in \mathcal{M}_4^3.
$$

With this language we can bring the matrix (6) of products over preimages from Definition 2.8 in a convenient form. Note that this viewpoint is also valid for the classical one-parameter setting.

Lemma 5.10.

1. For all $a \in \mathcal{M}^{m+s}$ and $p \in \text{qSh}(m, s; j)$ with $P = \varphi(p)$,

$$
\begin{bmatrix}
\bigstar_{u \in p^{-1}(1)} a_u \\
\vdots \\
\bigstar_{u \in p^{-1}(j)} a_u
\end{bmatrix} = Pa \in \mathcal{M}^j.
$$

2. For all $b \in \mathcal{M}^{(m+s) \times (n+t)}$ and $q \in \text{qSh}(n, t; k)$ with $Q = \varphi(q)$,

$$
\left( \bigstar_{u \in q^{-1}(x)} b_{u, v} \right)_{x, y} = Pb Q^\top \in \mathcal{M}_j^k.
$$

Lemma 5.11.

1. For all compositions $(a, b) \in \mathcal{M}^{m \times n} \times \mathcal{M}^{s \times t}$,

$$
a \uplus b = \sum_{j, k \in \mathbb{N}} \sum_{P \in \text{QSH}(m, s; j)} \sum_{Q \in \text{QSH}(n, t; k)} P \text{diag}(a, b)Q^\top.
$$

\(^9\)Throughout, the bijection $\varphi$ from $\text{qSh}(m, s; j)$ to $\text{QSH}(m, s; j)$ is omitted. For $q \in \text{qSh}(m, s; j)$ the matrix $\varphi(q) := [e_{q(1)} \cdots e_{q(m)} \cdots e_{q(m+s)}]$ is right invertible by construction and thus contained in $\text{QSH}(m, s; j)$. The converse direction is similar. Compare also Lemma 5.10.
2. For monomials \( a, b \in K\langle M^{1 \times n} \rangle \) with \( \deg(a) = m \) and \( \deg(b) = s \),
\[
a \boxtimes_1 b = \sum_{j \in \mathbb{N}} \sum_{P \in \text{QSH}(m,s;j)} P \begin{bmatrix} a \\ b \end{bmatrix}
\]

3. For monomials \( c, d \in K\langle M^{m} \rangle \) with \( \deg(c) = n \) and \( \deg(d) = t \),
\[
c \boxtimes_2 d = \sum_{k \in \mathbb{N}} \sum_{Q \in \text{QSH}(n,t;k)} \begin{bmatrix} c & d \end{bmatrix} Q^T
\]

In Corollaries 5.13 and 5.15 we show that \((K(\text{Cmp}_{\text{con}}), +, \boxtimes, 0, e)\) is a commutative algebra. Lemma 5.14 brings the two-parameter quasi-shuffle into relation to the well-known one-parameter setting from \([\text{EFFM}17, \text{Hof}99]\). In particular one can use the recursive characterization in the one-parameter setting for an efficient evaluation of Lemma 5.11 relying not on surjections. As a preliminary consideration, we recall concatenation of surjections for one-parameter quasi-shuffles, encoded as matrices.

**Lemma 5.12.**
1. For all \( Q \in \text{QSH}(m,s;j) \) exist unique block matrices \( Q_1 \in U(j,m) \) and \( Q_2 \in U(j,s) \) such that \( Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \) where
\[
U(j,\ell) := \left\{ \begin{bmatrix} e_{\iota_1} & \ldots & e_{\iota_\ell} \end{bmatrix} \in \mathbb{N}_0^{j \times \ell} \mid \iota_1 < \ldots < \iota_\ell \right\}.
\]
2. \( U(j,\ell) \cdot U(\ell,p) \subseteq U(j,p) \) where the product is taken entry-wise.
3.
\[
\text{QSH}(m, s, u, k) := \left\{ \begin{bmatrix} e_{\iota_1} & \ldots & e_{\iota_m} & e_{\kappa_1} & \ldots & e_{\kappa_s} & e_{\mu_1} & \ldots & e_{\mu_u} \end{bmatrix} \middle| \begin{array}{c}
\iota_1 < \ldots < \iota_m \\
\kappa_1 < \ldots < \kappa_s \\
\mu_1 < \ldots < \mu_u
\end{array} \right\} \quad \text{right invertible}
\]
\[
= \left\{ \begin{bmatrix} F_1 & F_2 A_1 & F_2 A_2 \end{bmatrix} \mid A \in \text{QSH}(s,u;j), F \in \text{QSH}(m,j;k) \right\}
\]
\[
= \left\{ \begin{bmatrix} C_1 P_1 & C_1 P_2 & C_2 \end{bmatrix} \mid P \in \text{QSH}(m,s;j), C \in \text{QSH}(j,u;k) \right\}
\]
\[
\subseteq \mathbb{N}_0^{k \times (m+s+u)}.
\]

The proof follows from the classical, one-parameter setting.

**Corollary 5.13.** The two-parameter quasi-shuffle \( \boxtimes \) is commutative and associative.
Proof. Let \( a \in \mathcal{M}^{m \times n} \) and \( b \in \mathcal{M}^{s \times t} \) be compositions. Commutativity follows from

\[
a \sqcup b = \sum_{[P_2, P_1]} \sum_{[Q_2, Q_1]} P_2 b Q_2^T \star P_1 a Q_1^T = b \sqcup a.
\]

For composition \( c \in \mathcal{M}^{u \times v} \) follows

\[
a \squplus (b \sqcup c) = a \squplus \left( \sum_{A \in \mathcal{QSH}(s,n;j)} \sum_{B \in \mathcal{QSH}(t,v;x)} \begin{bmatrix} A_1 & A_2 \end{bmatrix} \text{diag}(b, c) \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} \right)
\]

\[
= \sum_{A} \sum_{B} \sum_{F \in \mathcal{QSH}(m,j; k)} \sum_{G \in \mathcal{QSH}(n,x; y)} \begin{bmatrix} F_1 & F_2 \end{bmatrix} \text{diag}(a, \begin{bmatrix} A_1 & A_2 \end{bmatrix}) \begin{bmatrix} 0 & 0 \\ 0 & A_1 \\ A_2 \end{bmatrix} \begin{bmatrix} \text{diag}(b, c) \\ I_m \end{bmatrix} \begin{bmatrix} B_1^T \\ B_2^T \end{bmatrix} \begin{bmatrix} G_1^T \\ G_2^T \end{bmatrix}
\]

\[
= \sum_{F} \sum_{G} \begin{bmatrix} F_1 & F_2 A_1 & F_2 A_2 \end{bmatrix} \text{diag}(a, b, c) \begin{bmatrix} \text{diag}(b, c) \\ \text{diag}(b, c) \\ \text{diag}(b, c) \end{bmatrix} \begin{bmatrix} G_1^T \\ (G_2 B_1)^T \\ (G_2 B_2)^T \end{bmatrix}
\]

\[
= \sum_{D} \sum_{Q} \sum_{C \in \mathcal{QSH}(j,u;k)} \sum_{P \in \mathcal{QSH}(m,s;j)} \sum_{Q \in \mathcal{QSH}(n,t;x)} \begin{bmatrix} C_1 P_1 & C_1 P_2 & C_2 \end{bmatrix} \text{diag}(a, b, c) \begin{bmatrix} \text{diag}(b, c) \\ \text{diag}(b, c) \\ \text{diag}(b, c) \end{bmatrix} \begin{bmatrix} (Q_1 D_1)^T \\ (Q_2 D_2)^T \\ Q_2^T \end{bmatrix}
\]

\[
= \left( \sum_{P} \sum_{Q} \begin{bmatrix} P_1 & P_2 \end{bmatrix} \text{diag}(a, b) \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \right) \sqcup (a \sqcup b) \sqcup c = (a \sqcup b) \sqcup c.
\]

\[\square\]

With the following statement we verify that the algorithm in Section 4.2 is sound.

Lemma 5.14. Let \( a \in \mathcal{M}^{m \times n} \) and \( b \in \mathcal{M}^{s \times t} \) be compositions.

1. With the blocks arising in the following quasi-shuffle of columns

\[
\begin{bmatrix} a \\ b \end{bmatrix} \sqcup \begin{bmatrix} \varepsilon_{m \times t} \\ \varepsilon_{s \times n} \end{bmatrix} = \sum_{[Q_1, Q_2] \in \mathcal{QSH}(n,t;k)} \begin{bmatrix} a Q_1^T \\ b Q_2^T \end{bmatrix} \in K(\mathcal{M}^{m+s}),
\]

we have

\[
a \sqcup b = \sum_{[Q_1, Q_2]} a Q_1^T \sqcup b Q_2^T.
\]

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2. With the blocks arising in the following quasi-shuffle of rows

\[
\begin{bmatrix}
a & \varepsilon_{m \times t}
\end{bmatrix}_{\sqcup_1} \varepsilon_{s \times n} \begin{bmatrix} b \end{bmatrix} = \sum_{[P_1 P_2] \in \text{QSH}(m, s,j)} \begin{bmatrix} P_1 a & P_2 b \end{bmatrix} \in \mathbb{K}(M^{1 \times (n+t)}),
\]

we have

\[
a \sqcup_1 b = \sum_{[P_1 P_2]} P_1 a \sqcup_2 P_2 b.
\]

Proof.

\[
a \sqcup_1 b = \sum_{[P_1 P_2]} \left( \sum_{Q} \begin{bmatrix} P_1 a & P_2 b \end{bmatrix} Q^\top \right) = \sum_{[Q_1 Q_2]} \sum_{P} \begin{bmatrix} a Q_{1}^\top \ b Q_{2}^\top \end{bmatrix}
\]

Corollary 5.15. The two-parameter quasi-shuffle is closed, i.e., \( a \sqcup_1 b \in \mathbb{K}(\text{Cmp}_{\text{con}}) \) for all matrix compositions \( a, b \in \text{Cmp} \).

Proof. In every summand

\[
\begin{bmatrix} P_1 a & P_2 b \end{bmatrix} \text{ from } \begin{bmatrix} a & \varepsilon_{m \times t} \end{bmatrix}_{\sqcup_1} \varepsilon_{s \times n} \begin{bmatrix} b \end{bmatrix}
\]

there does not appear an \( \varepsilon \)-column, and therefore not in \( P_1 a \) or \( P_2 b \), and thus not in \( P_1 a \sqcup_1 P_2 b \) from \( a \sqcup_1 b \). An analogous argument yields with part 1. of Lemma 5.14 that \( a \sqcup_1 b \) contains no \( \varepsilon \)-rows.

Corollary 5.16. The two-parameter quasi-shuffle is graded, i.e., for all \( a, b \in \text{Cmp} \) with \( i = \text{weight}(\text{diag}(a,b)) \),

\[
a \sqcup_1 b \in \bigoplus_{c \in \text{Cmp}} \mathbb{K}c, \text{ weight}(c) = i
\]

Proof. This follows with Lemmas 2.5 and 5.10.

5.3 Bialgebra structure of matrix compositions

We show Theorem 5.24, which leads to Theorem 2.12. The following case study illustrates the central argument in the special case, where all compositions are connected.

Lemma 5.17. For all \( a, b \in \text{Cmp}_{\text{con}} \), \( P \in \text{QSH(rows}(a), \text{rows}(b); j) \) and \( Q \in \text{QSH(cols}(a), \text{cols}(b); k) \), exactly one of the following holds:

1. \( P \text{diag}(a, b)Q^\top \) is connected, and hence

\[
\Delta(P \text{diag}(a, b)Q^\top) = P \text{diag}(a, b)Q^\top \otimes e + e \otimes P \text{diag}(a, b)Q^\top.
\]
2. \( P \text{diag}(a, b)Q^\top \) is not connected, \( j = \text{rows}(a) + \text{rows}(b), k = \text{cols}(a) + \text{cols}(b) \), 
\[
P = I_j, \quad Q = I_k, \quad \text{and}
\[
\Delta(P \text{diag}(a, b)Q^\top) = \text{diag}(a, b) \otimes e + a \otimes b + e \otimes \text{diag}(a, b).
\]

3. \( P \text{diag}(a, b)Q^\top \) is not connected, 
\[
P = \begin{bmatrix} 0 & I_{\text{rows}(b)} \\ I_{\text{rows}(a)} & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & I_{\text{cols}(a)} \\ I_{\text{cols}(b)} & 0 \end{bmatrix},
\]
and 
\[
\Delta(P \text{diag}(a, b)Q^\top) = \text{diag}(b, a) \otimes e + b \otimes a + e \otimes \text{diag}(b, a).
\]

Example 5.18. In the setting of Example 2.15 with 
\[
(a, b) = \left( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right) \in \mathcal{M}_3^{1 \times 1} \times \mathcal{M}_3^{1 \times 2},
\]
the only summands in \( a \otimes b \) which are not connected are \( \text{diag}(a, b) \) and 
\[
\text{diag}(b, a) = \begin{bmatrix} e_3 & e_1 & e_2 \end{bmatrix} \text{diag}(a, b) \begin{bmatrix} e_2 & e_1 \end{bmatrix}^\top.
\]

Proof of Lemma 5.17. Let \( (m, n) = \text{size}(a) \) and \( (s, t) = \text{size}(b) \). Let \( 1 \leq \mu_1 < \cdots < \mu_m \leq j \) and \( 1 \leq \nu_1 < \cdots < \nu_s \leq j \) be such that 
\[
P = \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} \epsilon_{\mu_1} & \cdots & \epsilon_{\mu_m} & \epsilon_{\nu_1} & \cdots & \epsilon_{\nu_s} \end{bmatrix} \in N_0^{j \times (m+s)}
\]
according to Lemma 5.12, and similarly 
\[
Q = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \in N_0^{k \times (n+t)}
\]
with increasing chains \( \iota \) and \( \kappa \). For \( i \geq 2 \) and \( \omega \in \{ \mu, \iota \} \) let \( \overline{\omega} := \omega_i - \omega_{i-1} - 1 \). In the composition 
\[
P \text{diag}(a, b)Q^\top = P_1aQ_i^\top \ast P_2bQ_2^\top \in \mathcal{M}_j^{1 \times k}, \tag{33}
\]
consider the factor derived from the connected matrix composition \( a \), 
\[
P_1aQ_i^\top = \begin{bmatrix}
\epsilon_{(\mu_1-1) \times (\iota_1-1)} & \epsilon_{(\mu_1-1) \times 1} & \epsilon_{(\mu_1-1) \times 2} & \cdots & \epsilon_{(\mu_1-1) \times (k-\iota_s)} \\
\epsilon_{1 \times (\iota_1-1)} & a_{1,1} & \epsilon_{1 \times 2} & \cdots & a_{1,s} & \epsilon_{1 \times (k-\iota_s)} \\
\epsilon_{\iota_2 \times (\iota_1-1)} & \epsilon_{\iota_2 \times 1} & \epsilon_{\iota_2 \times 2} & \cdots & \epsilon_{\iota_2 \times (k-\iota_s)} \\
\epsilon_{1 \times (\iota_1-1)} & a_{2,1} & \epsilon_{1 \times 2} & \cdots & a_{2,s} & \epsilon_{1 \times (k-\iota_s)} \\
\epsilon_{\iota_3 \times (\iota_1-1)} & \epsilon_{\iota_3 \times 1} & \epsilon_{\iota_3 \times 2} & \cdots & \epsilon_{\iota_3 \times (k-\iota_s)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\epsilon_{1 \times (\iota_1-1)} & a_{m,1} & \epsilon_{1 \times 2} & \cdots & a_{m,s} & \epsilon_{1 \times (k-\iota_s)} \\
\epsilon_{(\mu_m-1) \times (\iota_1-1)} & \epsilon_{(\mu_m-1) \times 1} & \epsilon_{(\mu_m-1) \times 2} & \cdots & \epsilon_{(\mu_m-1) \times (k-\iota_s)}
\end{bmatrix}.
\]

It satisfies the following property: 
\[
P_1aQ_i^\top = \text{diag}(x_1, y_1) \tag{34}
\]
with non-empty $x_1$ and $y_1$, then $x_1$ has all entries equal to $\varepsilon$ or $y_1$ does.

The same argument holds for the connected matrix composition $b$ and

$$P_2 b Q_2^\top = \text{diag}(x_2, y_2). \quad (35)$$

If the composition $P \text{diag}(a, b) Q^\top = \text{diag}(x, y)$ is not connected with non-empty compositions $x$ and $y$, then the factors from Equation (33) must have a block factorization, and thus

$$P \text{diag}(a, b) Q^\top = \text{diag}(x_1 \ast x_2, y_1 \ast y_2)$$

with (34), (35) and where $\text{size}(x) = \text{size}(x_1) = \text{size}(x_2)$. If $y_1$ is $\varepsilon$-valued, then $x_1 = a$, $x_2$ is $\varepsilon$-valued, thus $y_2 = b$, and therefore we are in case 2. If instead $x_1$ is $\varepsilon$-valued, then $y_1 = a$, $y_2$ is $\varepsilon$-valued, $x_2 = b$, and hence we obtain case 3. \hfill $\Box$

Let $\tau : \mathbb{K}(\text{Cmp}_{\text{con}}) \otimes \mathbb{K}(\text{Cmp}_{\text{con}}) \rightarrow \mathbb{K}(\text{Cmp}_{\text{con}}) \otimes \mathbb{K}(\text{Cmp}_{\text{con}})$ denote the flip homomorphism, uniquely determined by $\tau(a \otimes b) := b \otimes a$ for all $a, b \in \text{Cmp}$.

**Corollary 5.19.** For $a, b \in \text{Cmp}_{\text{con}}$,

$$(\psi \otimes \psi) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta \otimes \Delta)(a \otimes b) = \Delta \circ \psi \circ (a \otimes b).$$

**Proof.** With Lemma 5.17,

$$\Delta \circ \psi \circ (a \otimes b) = \sum_{P, Q} \Delta(P \text{diag}(a, b) Q^\top)$$

$$= a \otimes b + b \otimes a + \sum_{P, Q} \left(P \text{diag}(a, b) Q^\top \otimes e + e \otimes P \text{diag}(a, b) Q^\top\right)$$

$$= (\psi \otimes \psi)(a \otimes e \otimes e \otimes b + e \otimes b \otimes a \otimes e + a \otimes b \otimes e \otimes e + e \otimes e \otimes a \otimes b)$$

verifies the bialgebra relation in the case where $a$ and $b$ are connected. \hfill $\Box$

We now generalize Lemma 5.17 for arbitrary matrix compositions, visualized in Example 5.22 and in Figure 2 with decomposition length $a = 2$ and $b = 1$. For this, let

$$u_\alpha := \sum_{1 \leq r \leq \alpha} u_r$$

denote the cumulative sum\(^{10}\) of $u \in \mathbb{N}^a$ at index $0 \leq \alpha \leq a$. For fixed $a, b \in \text{Cmp}$ with decompositions into connected compositions $a = \text{diag}(v_1, \ldots, v_a)$ and $b = \text{diag}(w_1, \ldots, w_b)$ let

$$s_{a, b} := \{(\alpha, \beta) \in \mathbb{N}_0^2 \mid 0_2 \leq (\alpha, \beta) \leq (a, b) \land 0 \neq \alpha + \beta \neq a + b\}.$$

\(^{10}\)The empty sum yields $u_0 := 0$. 

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For every \((\alpha, \beta) \in s_{a,b}\) we call

\[
P = \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} e_{\mu_1} & \cdots & e_{\mu_\text{rows}(a)} & 0_j_1 & \cdots & 0_j_1 \\ 0_{j_2} & \cdots & 0_{j_2} & e_{(\mu_{(2a+1)}-j_1)} & \cdots & e_{(\mu_{\text{rows}(a)}-j_1)} \end{bmatrix} \in \text{QSH(rows}(a), \text{rows}(b), j) \]

\((\alpha, \beta)\)-decomposable, if\(^{11,12}\)

\[
P_1 = \begin{bmatrix} e_{\mu_1} & \cdots & e_{\mu_{2a}} & 0_j_1 & \cdots & 0_j_1 \\ 0_{j_2} & \cdots & 0_{j_2} & e_{(\mu_{(2a+1)}-j_1)} & \cdots & e_{(\mu_{\text{rows}(a)}-j_1)} \end{bmatrix} =: \begin{bmatrix} P^{(\alpha)}_1 \\ P^{(\beta)}_1 \end{bmatrix} \tag{36}
\]

and

\[
P_2 = \begin{bmatrix} e_{\nu_1} & \cdots & e_{\nu_{2\beta}} & 0_j_1 & \cdots & 0_j_1 \\ 0_{j_2} & \cdots & 0_{j_2} & e_{(\nu_{(2\beta+1)}-j_1)} & \cdots & e_{(\nu_{\text{rows}(b)}-j_1)} \end{bmatrix} =: \begin{bmatrix} P^{(\alpha)}_2 \\ P^{(\beta)}_2 \end{bmatrix} \tag{37}
\]

with suitable \(j_1, j_2 \in \mathbb{N}\), where \((u_{\alpha}, v_{\alpha}) = \text{size}(v_{\alpha})\) and \((\omega_{\beta}, w_{\beta}) = \text{size}(w_{\beta})\) denote the block sizes in \(a\) and \(b\), respectively.

**Lemma 5.20.** If \(P\) is \((\alpha, \beta)\)-decomposable, then its block decomposition

\[
P = \begin{bmatrix} P^{(\alpha)}_1 & P^{(\beta)}_1 \\ P^{(\alpha)}_2 & P^{(\beta)}_2 \end{bmatrix} \tag{38}
\]

is uniquely determined. Furthermore, if \(P\) is also \((\alpha', \beta')\)-decomposable with \(\text{size}(P^{(\alpha)}_{11}) = \text{size}(P^{(\alpha')}_{11})\), then \((\alpha, \beta) = (\alpha', \beta') \in s\).

**Proof.** Due to Lemma 5.12, \(P\) decomposes uniquely into \(P_1\) and \(P_2\). Since \(P\) is right invertible, it contains no zero rows. Therefore, with \((36)\) and \((37)\), \(j_1\) and \(j_2 = j - j_1\) are unique. The second part follows since \(\mu\) and \(\nu\) are strictly increasing. \(\square\)

**Lemma 5.21.** Let \(a = \text{diag}(v_1, \ldots, v_a)\) and \(b = \text{diag}(w_1, \ldots, w_b)\) be decompositions into connected compositions. For fixed

\[
(P, Q) \in \text{QSH(rows}(a), \text{rows}(b); j) \times Q \in \text{QSH(cols}(a), \text{cols}(b); k),
\]

the set

\[
\{(x, y) \in \text{Cmp}^2 \mid x \neq e \neq y \land \text{diag}(x, y) = P \text{ diag}(a, b)Q^\top \}
\]

is in one-to-one correspondence to the set of **splittings**

\[
\mathcal{S}_{P, Q}^{a,b} := \{(\alpha, \beta) \in s_{a,b} \mid P = \begin{bmatrix} P^{(\alpha)}_1 & P^{(\beta)}_1 \\ P^{(\alpha)}_2 & P^{(\beta)}_2 \end{bmatrix} \land Q = \begin{bmatrix} Q^{(\alpha)}_1 & Q^{(\beta)}_1 \\ Q^{(\alpha)}_2 & Q^{(\beta)}_2 \end{bmatrix} \},
\]

\(^{11}\)Note that the notation does not distinguish between standard columns \(e_i\) from \(\mathbb{N}_0^{11}\) and \(\mathbb{N}_0^{12}\).

\(^{12}\)In the cases \(\alpha \in \{0, a\}\) we have \(P^{(0)}_{11} = \begin{bmatrix} 0_j_1 & \cdots & 0_j_1 \end{bmatrix}, P^{(a)}_{11} = \begin{bmatrix} e_{\mu_1} & \cdots & e_{\mu_{\text{rows}(a)}} \end{bmatrix} \in \mathbb{N}_0^{j_1 \times \text{rows}(a)}\) and \(P^{(0)}_{21} = \begin{bmatrix} e_{(\mu_1-1)} & \cdots & e_{(\mu_{\text{rows}(a)}-1)} \end{bmatrix}, P^{(a)}_{21} = \begin{bmatrix} 0_{j_2} & \cdots & 0_{j_2} \end{bmatrix} \in \mathbb{N}_0^{j_2 \times \text{rows}(a)}\).
where $P$ and $Q$ are simultaneously $(\alpha, \beta)$-decomposable with (36), (37),
\[
\begin{bmatrix}
Q_{11}^{(\alpha)} \\
Q_{21}^{(\alpha)}
\end{bmatrix} = \begin{bmatrix}
e_{k_1} & \cdots & e_{k_{w(a)}} & 0 & \cdots & 0 \\
0 & \cdots & 0 & e_{(i_{(w(a)+1)}-k_1)} & \cdots & e_{(\text{cols}(a)-k_1)}
\end{bmatrix}
\]
(39)
and
\[
\begin{bmatrix}
Q_{12}^{(\beta)} \\
Q_{22}^{(\beta)}
\end{bmatrix} = \begin{bmatrix}
e_{k_1} & \cdots & e_{k_{w(b)}} & 0 & \cdots & 0 \\
0 & \cdots & 0 & e_{(i_{(w(b)+1)}-k_1)} & \cdots & e_{(\text{cols}(b)-k_1)}
\end{bmatrix}
\]
(40)
for suitable $k_1, k_2 \in \mathbb{N}$ and increasing $(i, k) \in \mathbb{N}^{\text{cols}(a)} \times \mathbb{N}^{\text{cols}(b)}$.

**Example 5.22.**

1. In Example 5.18 we have $a, b \in \text{Cmp}_{\text{con}}$, i.e., both $a$ and $b$ have a trivial decomposition of length $a = b = 1$. If $P = \begin{bmatrix} e_3 & e_1 & e_2 \end{bmatrix}$ and $Q = \begin{bmatrix} e_2 & e_1 \end{bmatrix}$, then
\[
\mathcal{E}_{P,Q} = \{(0, 1)\}
\]
where
\[
\begin{bmatrix}
P_{11}^{(\alpha)} & P_{12}^{(\beta)} \\
P_{21}^{(\alpha)} & P_{22}^{(\beta)}
\end{bmatrix} = \begin{bmatrix}
0_{j_1 \times 1} & I_2 \\
I_1 & 0_{j_2 \times 2}
\end{bmatrix}, \quad
\begin{bmatrix}
Q_{11}^{(\alpha)} & Q_{12}^{(\beta)} \\
Q_{21}^{(\alpha)} & Q_{22}^{(\beta)}
\end{bmatrix} = \begin{bmatrix}
0_{k_1 \times 1} & I_1 \\
I_1 & 0_{k_2 \times 1}
\end{bmatrix}
\]
with $(j_1, j_2) = (2, 1)$ and $(k_1, k_2) = (1, 1)$, that is case 3. of Lemma 5.17. The only other candidate $(\alpha, \beta) = (1, 0) \in \mathfrak{s}_{1,1}$ is not a splitting, since $\mu_1 = 2$ and $\mu_1 = 3$, i.e. (36) is not satisfied.

2. For a case with $a \not\in \text{Cmp}_{\text{con}}$, consider
\[
(a, b) = \left(\begin{array}{cccc}
1 & 2 & \varepsilon & \varepsilon \\
\varepsilon & 3 & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & 4 & 5
\end{array}\right), \quad \left(\begin{array}{cc}
\varepsilon & 6 \\
7 & 8
\end{array}\right) \in \mathfrak{M}_3^{3 \times 4} \times \mathfrak{M}_8^{2 \times 2},
\]
(41)
Then, with $P = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \end{bmatrix}$ and $Q = \begin{bmatrix} e_1 & e_2 & e_5 & e_6 & e_3 & e_4 \end{bmatrix}$,
\[
P \text{ diag}(a, b) Q^\top = \begin{bmatrix}
1 & 2 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 6 & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 7 & 8 & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 4 & 5
\end{bmatrix}
\]
which has two splittings $\mathcal{E}_{P,Q}^{a,b} = \{(1, 0), (1, 1)\}$ via
\[
(P, Q) = \left(\begin{bmatrix} e_1 & e_2 & 0_2 \\
0 & 3 & e_3 & e_4 \end{bmatrix}, \begin{bmatrix} 0_2 & 0_2 \\
0_4 & 0_4 & e_3 & e_4 \end{bmatrix}, \begin{bmatrix} e_1 & e_2 & 0_2 \\
0 & 2 & e_1 & e_2 \end{bmatrix}\right)
\]
\[
= \left(\begin{bmatrix} e_1 & e_2 & 0_4 \\
0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} e_3 & e_4 \\
0_2 & 0_2 & e_1 & e_2 \end{bmatrix}, \begin{bmatrix} e_1 & e_2 & 0_4 \\
0 & 2 & 0_2 & e_1 & e_2 \end{bmatrix}\right).
\]
Figure 2: For some terms appearing in the quasi-shuffle of \( a \) and \( b \) from Example 5.22.2 (which lead to right invertible matrices \( P \) and \( Q \)), all splittings due to the Lemma are given.

3. With \((a, b)\) as in (41), \( P = I_5 \) and \( Q = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{bmatrix}, \)

\[
P \text{ diag}(a, b) Q^\top = \begin{bmatrix} 1 & 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 4 & 5 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 6 \\ \varepsilon & \varepsilon & \varepsilon & 7 & 8 \\ \varepsilon & \varepsilon & \varepsilon & 8 & \end{bmatrix}
\]

is connected. The constellation \((0, 1) \in s_{2,1}\) is not a splitting with \( P_{1,1} = 1 \). Also for the remaining cases \(1 \leq \alpha \leq 2,\)

\[
Q = \begin{bmatrix} e_1 & e_2 & 0_2 & 0_2 \\ 0_3 & 0_3 & e_3 & e_4 \\ 0_3 & 0_3 & e_3 & e_4 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 \\ 0 & 0 & 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

is not \((\alpha, \beta)\)-decomposable. This follows since with \( \nu_{2,1} = 2 \) and \( \nu_{2,2} = 4 \), \( Q_2 \) is not of the form (40). Therefore, \( \mathcal{S}_{P, Q}^{a,b} \) is empty.

Proof of Lemma 5.21. If \((\alpha, \beta) \in \mathcal{S}_{P, Q}^{a,b}\) then \( P \) and \( Q \) are \((\alpha, \beta)\)-decomposable with
\( P_{21}^{(\alpha)} aQ_{11} - P_{12}^{(\beta)} bQ_{12}^{(\beta)} = \varepsilon_{j2 \times k1} \) and \( P_{11}^{(\alpha)} aQ_{21}^{(\alpha)} - P_{12}^{(\beta)} bQ_{22}^{(\beta)} = \varepsilon_{j1 \times k2} \), hence

\[
P \text{diag}(a, b)Q^T = \begin{bmatrix}
P_{11}^{(\alpha)} aQ_{11}^{(\alpha)} + P_{12}^{(\beta)} bQ_{12}^{(\beta)} & P_{11}^{(\alpha)} aQ_{21}^{(\alpha)} - P_{12}^{(\beta)} bQ_{22}^{(\beta)} \\
P_{21}^{(\alpha)} aQ_{11}^{(\alpha)} - P_{22}^{(\beta)} bQ_{12}^{(\beta)} & P_{21}^{(\alpha)} aQ_{21}^{(\alpha)} + P_{22}^{(\beta)} bQ_{22}^{(\beta)}
\end{bmatrix}
\]

is not connected. Different splittings in \( \mathcal{C}_{P, Q}^{a, b} \) yield different block decompositions \( (42) \) due to Lemma 5.20.

Conversely, if

\[
P \text{diag}(a, b)Q^T = \begin{bmatrix}
P_{11}^{(\alpha)} aQ_{11}^{(\alpha)} + P_{12}^{(\beta)} bQ_{12}^{(\beta)} & P_{11}^{(\alpha)} aQ_{21}^{(\alpha)} - P_{12}^{(\beta)} bQ_{22}^{(\beta)} \\
P_{21}^{(\alpha)} aQ_{11}^{(\alpha)} - P_{22}^{(\beta)} bQ_{12}^{(\beta)} & P_{21}^{(\alpha)} aQ_{21}^{(\alpha)} + P_{22}^{(\beta)} bQ_{22}^{(\beta)}
\end{bmatrix}
\]

with size(\( x \)) = \( (j_1, k_1) \), then analogous to Lemma 5.17, all \( s \)-indexed factors

\[
\begin{bmatrix} e_{\mu(s-1)+1} & \cdots & e_{\mu s} \\ e_{\nu(s-1)+1} & \cdots & e_{\nu s} \end{bmatrix} =: \text{diag}(x_1^{(s)}, y_1^{(s)})
\]
decompose with size(\( x_1^{(s)} \)) = \( (j_1, k_1) \). Furthermore, \( x_1^{(s)} \) has all entries equal to \( \varepsilon \) or \( y_1^{(s)} \) does. With strictly increasing \( \mu \) and \( \nu \), there is a unique \( 0 \leq \alpha \leq a \) such that \( x_1^{(1)}, \ldots, x_1^{(\alpha)} \) have entries different from \( \varepsilon \), and \( x_1^{(\alpha+1)}, \ldots, x_1^{(a)} \) have all entries equal to \( \varepsilon \). Therefore, \( P_1 \) and \( Q_1 \) are of shape \( (36) \) and \( (39) \), respectively. Analogously one obtains \( P_2 \) and \( Q_2 \) of shape \( (37) \) and \( (40) \) via factors derived from \( w_t \) and uniquely determined \( 0 \leq \beta \leq b \). Note \( (\alpha, \beta) \in \mathfrak{s}_{a, b} \) since \( x \neq e \neq y \).

If \( P \text{diag}(a, b)Q^T = \text{diag}(x', y') \) has a second decomposition with resulting splitting \( (\alpha', \beta') \in \mathfrak{s}_{a, b} \), then without loss of generality \( \text{rows}(x) < \text{rows}(x') \) and \( \text{cols}(x) < \text{cols}(x') \). With \( v_s, w_t \in \text{Cmp}_{\text{con}} \) this implies \( \alpha < \alpha' \) or \( \beta < \beta' \). \( \square \)

**Remark 5.23.** Let \( a, b, v \) and \( w \) be as in Lemma 5.21. For all \( (\alpha, \beta) \in \mathfrak{s}_{a, b} \) there is a one-to-one correspondence between

1. the set of all

\[
( P, Q ) \in \bigcup_{j, k \in \mathbb{N}} \text{QSH}(\text{rows}(a), \text{rows}(b); j) \times \text{QSH}(\text{cols}(a), \text{cols}(b); k)
\]

which are simultaneously \( (\alpha, \beta) \)-decomposable, and

2. the set of 4-tuples

\[
\left[ P_{11}^{(\alpha)} P_{12}^{(\beta)}, P_{21}^{(\alpha)} P_{22}^{(\beta)}, Q_{11}^{(\alpha)} Q_{12}^{(\beta)}, Q_{21}^{(\alpha)} Q_{22}^{(\beta)} \right]
\]
from
\[ \bigcup_{j_1,j_2,k_1,k_2 \in \mathbb{N}} \mathbb{N}_{j_1 \times \text{rows}(\text{diag}(a,b))} \times \mathbb{N}_{j_2 \times \text{rows}(\text{diag}(a,b))} \times \mathbb{N}_{k_1 \times \text{cols}(\text{diag}(a,b))} \times \mathbb{N}_{k_2 \times \text{cols}(\text{diag}(a,b))} \]

with right invertible components of the form (36), (37), (39) and (40).

We now verify the bialgebra property in order to conclude Theorem 2.12.

**Theorem 5.24.** \((\mathbb{K} / \text{Cmp}_{\text{con}}, \varrho, \imath, \eta, \Delta, \varepsilon)\) is a graded, connected bialgebra.

**Proof.** For fixed \(a, b \in \text{Cmp}\) let \(v \in \text{Cmp}_{\text{con}}^a\) and \(w \in \text{Cmp}_{\text{con}}^b\) denote the decompositions into connected compositions with block sizes \((u_\alpha, v_\alpha) = \text{size}(v_\alpha)\) and \((\omega_\beta, w_\beta) = \text{size}(w_\beta)\), respectively. Then, with Lemma 5.21,

\[
\Delta \circ \imath (a \otimes b) = \sum_{P, Q} \Delta(P \text{diag}(a, b)Q^\top) \\
= \sum_{P, Q} \left( P \text{diag}(a, b)Q^\top \otimes e + e \otimes P \text{diag}(a, b)Q^\top + \sum_{x, y \in \text{Cmp}(e)} x \otimes y \right) \\
(42) = \sum_{P, Q} \left( \sum_{P, Q} P \text{diag}(a, b)Q^\top \right) \otimes e + e \otimes \left( \sum_{P, Q} P \text{diag}(a, b)Q^\top \right) \\
+ \sum_{P, Q} \sum_{(\alpha, \beta) \in \mathbb{C}_{P, Q}} P^{(\alpha)}_{11} aQ_{11}^{(\alpha)} \otimes P^{(\beta)}_{12} bQ_{12}^{(\beta)} + P^{(\alpha)}_{21} aQ_{21}^{(\alpha)} \otimes P^{(\beta)}_{22} bQ_{22}^{(\beta)} \\
= a \otimes b \otimes e + e \otimes a \otimes b \\
+ \sum_{P, Q} \sum_{(\alpha, \beta) \in \mathbb{C}_{P, Q}} P^{(\alpha)}_{11} aQ_{11}^{(\alpha)} \otimes P^{(\beta)}_{12} bQ_{12}^{(\beta)} + P^{(\alpha)}_{21} aQ_{21}^{(\alpha)} \otimes P^{(\beta)}_{22} bQ_{22}^{(\beta)} \\
= a \otimes b \otimes e + e \otimes a \otimes b \\
+ \sum_{(\alpha, \beta) \in \mathbb{C}_{P, Q}} P^{(\alpha)}_{11} aQ_{11}^{(\alpha)} \otimes P^{(\beta)}_{12} bQ_{12}^{(\beta)} + P^{(\alpha)}_{21} aQ_{21}^{(\alpha)} \otimes P^{(\beta)}_{22} bQ_{22}^{(\beta)} \\
(43) = (\imath \otimes \imath) \circ (\text{id} \otimes \tau \otimes \text{id}) \circ (\Delta \otimes \Delta)(a \otimes b),
\]

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where the summation over all tuples

\[
\left( \begin{bmatrix} \mathbf{P}_1^{\alpha} \\
\mathbf{P}_1^{\beta} \end{bmatrix}, \begin{bmatrix} \mathbf{Q}_1^{\alpha} \\
\mathbf{Q}_1^{\beta} \end{bmatrix} \right)
\]

yields

\[
\sum_{[\mathbf{P}_1^{\alpha} \mathbf{P}_1^{\beta}]} \mathbf{P}_1^{\alpha} \mathbf{Q}_1^{\alpha} \mathbf{P}_1^{\beta} \mathbf{Q}_1^{\beta} = \sum_{\mathbf{P}_1 \in \text{QSH}(\alpha, \omega; j_1)} \tilde{\mathbf{P}}_1 \mathbf{Q}_1 \mathbf{P}_1^{\alpha} \mathbf{Q}_1^{\beta} \mathbf{P}_1^{\beta} \mathbf{Q}_1^{\beta} = \text{diag}(\mathbf{v}_1, \ldots, \mathbf{v}_\alpha, \mathbf{w}_1, \ldots, \mathbf{w}_\beta), \quad (43)
\]

and in the same manner,

\[
\sum_{[\mathbf{P}_2^{\alpha} \mathbf{P}_2^{\beta}]} \mathbf{P}_2^{\alpha} \mathbf{Q}_2^{\alpha} \mathbf{P}_2^{\beta} \mathbf{Q}_2^{\beta} = \text{diag}(\mathbf{v}_{\alpha+1}, \ldots, \mathbf{v}_\alpha, \mathbf{w}_{\beta+1}, \ldots, \mathbf{w}_\beta), \quad (44)
\]

for the second component in the tensor product.

\section{Conclusions and outlook}

In Definition 2.6 we introduce the two-parameter sums signature of a multi-dimensional function \( Z : \mathbb{N}^2 \rightarrow \mathbb{K}^d \). It stores all polynomial warping invariants (Corollary 2.27, Section 3.2) as a linear functional on the linear span of matrix compositions.

These invariants are compatible with a Hopf algebra on matrix compositions: they satisfy a “quasi-shuffle identity”, which means that the sums signature is a character with respect to the quasi-shuffle (Theorem 2.13). Compatibility with a concatenation-type coproduct is encoded in a (weak) form of Chen’s identity, Lemma 2.35.

The underlying Hopf algebra of matrix compositions is, in the case of a one-dimensional ambient space \((d = 1)\), isomorphic to a sub-Hopf algebra of formal power series. These are akin to quasisymmetric functions, where the underlying poset of the natural numbers has been replaced by the poset \( \mathbb{N}^2 \). We therefore call them two-parameter quasisymmetric functions.

There remain several directions for future work.

- The recent extension of Chen’s iterated integrals in [GLNO22] has properties similar to our two-parameter sums signature. In which sense does the discrete setting converge to the continuous? How are shuffles and quasi-shuffles related to each other? Is there a continuous version of diagonal concatenation that leads to Chen’s identity with respect to diagonal deconcatenation?

- In this article we restrict to \( p = 2 \) parameters. Many results remain true for \( p > 2 \), when adjusting the row and column operations to tensor operations over
arbitrary axes. This quickly leads to notational clutter. Can the language of restriction species (and decomposition spaces) or $B_\infty$-structures simplify the proofs of the bialgebra properties?

- For two parameters, there are several operations that can take on the role of concatenation (and thus deconcatenation). We have presented a “diagonal” concatenation, with a resulting (weak) notion of Chen’s identity.

  Mutatis mutandis, one could instead concatenate along the anti-diagonal. In which sense are those concepts equivalent (if at all)? Alternatively, concatenation along just one axis (as in [GLNO22]) yields a modified Chen’s relation when restricted to certain matrix compositions. Is there a combination of several Chen-like formulas, which results in an efficient dynamic programming method?

- Regarding applications, it will be interesting to investigate how the sums that we propose as feature for, say, images, compare to the integral features used in [ZLT22]. We believe that, as in the one-parameter case, sums (instead of integrals) might provide a richer set of features.

- We introduced two-parameter quasisymmetric functions as a natural generalization of the (classical, one-parameter) quasisymmetric functions. We presented a basis which corresponds to the monomial basis in the one-parameter case. Are there relevant analogons of other bases, for example the fundamental basis? Closely related, what is a suitable notion of refinement for matrix compositions?

- We provided an iterative evaluation procedure of signature coefficients for certain matrix compositions $\mathfrak{M}_d^{\text{chain}}$. Similar methods (with linear complexity) would be possible for matrix compositions based on chaining on the anti-diagonal. What is the general complexity for arbitrary matrix compositions?

- The quasi-shuffle that we introduce can be considered as two interacting (one-parameter) quasi-shuffles of rows and columns, Lemma 5.14. It is therefore plausible, and in fact true, that our product possesses a tridendriform structure (in fact, there is one for the row-point-of-view and one for the column-point-of-view). Moreover, these two tridendriform structures combine into an ennea structure [Ler04], with $3^2 = 9$ operations.

  There is already an ennea structure on one-parameter quasi-shuffles, by pinning the last and the first letter (compare [AL04, Example 1.8]). “Tensoring” this gives a (maybe interesting) structure with $9^2 = 81$ operations.

- The Hopf algebra from Theorem 2.12 guaranteed that the algebra of matrix compositions if free. Is there a free generating set akin to Lyndon words?
Statements and Declarations

Competing Interests

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