ISOCAPACITY ESTIMATES FOR HESSIAN OPERATORS

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Abstract. Through a new powerful potential-theoretic analysis, this paper is devoted to discovering the geometrically equivalent isocapacity forms of Chou-Wang’s Sobolev type inequality and Tian-Wang’s Moser-Trudinger type inequality for the fully nonlinear $1 \leq k \leq \frac{n}{2}$ Hessian operators.

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1. HESSIAN SOBOLEV THROUGH ISOCAPACITARY INEQUALITIES

1.1. Sobolev type inequalities for Hessian operators. Unless a special remark is made, from now on, $\Omega$ is a bounded smooth domain in the $n$-dimensional Euclidean space $\mathbb{R}^n$ with $n \geq 2$. Let $u$ be a $C^2$ real-valued function on $\Omega$. For each integer $k \in [1, n]$, the $k$-Hessian operator $F_k$ is defined as

$$F_k[u] = S_k(\lambda(D^2u)) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},$$

2010 Mathematics Subject Classification. Primary 35J60, 35J70, 35J96, 31C15, 31C45, 53A40.

Key words and phrases. Hessian operators; Isocapacitary inequalities; Hessian capacities.

Project supported by NSERC of Canada as well as by URP of Memorial University, Canada.
where \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is the vector of the eigenvalues of the real symmetric Hessian matrix \([D^2u]\). In particular, one has:

\[
F_k[u] = \begin{cases} 
\Delta u = \text{the Laplace operator} & \text{as } k = 1; \\
a \text{ fully nonlinear operator} & \text{as } 1 < k < n; \\
\det(D^2u) = \text{the Monge-Ampère operator} & \text{as } k = n.
\end{cases}
\]

Here and henceforth, the following facts should be kept in mind: for \( 1 < k < n \) each \( F_k[u] \) is degenerate elliptic for any \( k \)-convex or \( k \)-admissible function \( u \), denoted by \( u \in \Phi^k(\Omega) \), namely, any \( C^2(\Omega) \) function \( u \) enjoying

\[
F_j[u] = 0 \quad \text{on } \Omega \quad \forall \quad j = 1, 2, \ldots, k.
\]

Moreover, if \( \Phi^0_0(\Omega) \) stands for the class of all functions \( u \in \Phi^k(\Omega) \) with zero value on the boundary \( \partial \Omega \) of \( \Omega \), then \( \Phi^0_0(\Omega) \neq \emptyset \) amounts to that \( \partial \Omega \) is \((k-1)\)-convex, i.e., the \( j \)-th mean curvature

\[
H_j(\partial \Omega, x) = \sum_{1 \leq i_1 < \ldots < i_j \leq n-1} \kappa_{i_1}(x) \ldots \kappa_{i_j}(x) \quad \forall \quad j = 1, \ldots, k - 1
\]

of the boundary \( \partial \Omega \) at \( x \) is nonnegative, where \( \kappa_1(x), \ldots, \kappa_{n-1}(x) \) are the principal curvatures of \( \partial \Omega \) at the point \( x \); see for example [3, 14, 11, 20, 18, 23, 12].

As a natural generalization of the well-known case \( k = 1 \), the following Sobolev type inequalities indicate that \( \Phi^k_0(\Omega) \) can embed into some integrable function spaces; see Wang [22], Chou [7, 8], and Tian-Wang [18] for details.

**Theorem 1.1.** Let

\[
\begin{cases} 
1 \leq k \leq n; \\
u \in \Phi^k_0(\Omega); \\
\|u\|_{\Phi^k_0(\Omega)} = \left( \int_{\Omega} (-u) F_k[u] \right)^{1/(k+1)}.
\end{cases}
\]

(i) If \( 1 \leq k < \frac{n}{2} \) and \( 1 \leq q \leq k^* = \frac{n(k+1)}{n-2k} \), then there is a positive constant \( c(n, k, q, |\Omega|) \) depending only on \( n, k, q, \) and the volume \(|\Omega|\) of \( \Omega \) such that the Sobolev type inequality

\[
\|u\|_{L^q(\Omega)} \leq c(n, k, q, |\Omega|) \|u\|_{\Phi^k_0(\Omega)}
\]

holds, where for \( q = k^* \) the best constant in the last estimate is obtained via letting \( \Omega \to \mathbb{R}^n \) by the function

\[
u(x) = (1 + |x|^2)^{\frac{2k-n}{2k}}.
\]

Moreover, for \( k = \frac{n}{2} \) and \( 0 < q < \infty \), there is a positive constant \( c(n, q, \text{diam}(\Omega)) \) depending only on \( n, q, \) and the diameter \( \text{diam}(\Omega) \) of \( \Omega \) such that the Sobolev type inequality

\[
\|u\|_{L^q(\Omega)} \leq c(n, q, \text{diam}(\Omega)) \|u\|_{\Phi^k_0(\Omega)}
\]

holds.
(ii) If $k = \frac{n}{2}$, then there is a positive constant $c(n, k, \text{diam}(\Omega))$ depending only on $n, k$ and $\text{diam}(\Omega)$ such that the Moser-Trudinger type inequality
\[
\sup_{0 < \|u\|_{\Phi^k_0(\Omega)} < \infty} \int_{\Omega} \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi^k_0(\Omega)}} \right)^{\beta} \right) \leq c(n, k, \text{diam}(\Omega))
\]
holds, where
\[
\begin{cases}
0 < \alpha \leq \alpha_0 = n \left( \frac{\omega_n}{k} \left( \frac{n-1}{k-1} \right) \right)^{\frac{n}{k}}; \\
1 \leq \beta \leq \beta_0 = 1 + \frac{2}{n}; \\
\omega_n = \text{the surface area of the unit sphere in } \mathbb{R}^{n+1}.
\end{cases}
\]

(iii) If $\frac{n}{2} < k \leq n$, then there is a positive constant $c(n, k, \text{diam}(\Omega))$ depending only on $n, k$ and $\text{diam}(\Omega)$ such that the Morrey-Sobolev type inequality
\[
\|u\|_{L^{\infty}(\Omega)} \leq c(n, k, \text{diam}(\Omega)) \|u\|_{\Phi^k_0(\Omega)}
\]
holds.

1.2. Statement of Theorems 1.2-1.3. Since the Morrey-Sobolev type inequality in Theorem 1.1 (iii) is relatively independent (cf. [17]), a natural question comes up: what is the geometrically equivalent form of Theorem 1.1 (i)-(ii)? To answer this question, we need the so-called $k$-Hessian capacity that was introduced by Trudinger-Wang [21] in a way similar to the capacity defined by Bedford-Taylor in [2] for the plurisubharmonic functions. To be more precise, if $K$ is a compact subset of $\Omega$, then the $[1, n] \ni k$ Hessian capacity of $K$ with respect to $\Omega$ is determined by
\[
cap_k(K, \Omega) = \sup \left\{ \int_K F_k[u] : u \in \Phi^k(\Omega), -1 < u < 0 \right\};
\]
and hence for an open set $O \subset \Omega$ we define
\[
cap_k(O, \Omega) = \sup \left\{ \cap_k(K, \Omega) : \text{compact } K \subset O \right\};
\]
whence giving the definition of $\cap_k(E, \Omega)$ for an arbitrary set $E \subset \Omega$:
\[
cap_k(E, \Omega) = \inf \left\{ \cap_k(O, \Omega) : \text{open } O \text{ with } E \subset O \subset \Omega \right\}.
\]

According to Labutin’s computation in [14] (4.16)-(4.17), we see that if $B_\rho \subset \mathbb{R}^n$ is used to represent an open ball centered at the origin with radius $\rho > 0$ and if $0 < r < R < \infty$, then there is a constant $c(n, k) > 0$ depending only on $n, k$ such that
\[
\cap_k(B_r, B_R) = \begin{cases}
c(n, k) \left( r^{\frac{n}{2}} - R^{\frac{n}{2}} \right)^{-k} \text{ for } 1 \leq k < \frac{n}{2}; \\
c(n, k) \left( \log \frac{R}{r} \right)^{\frac{n}{2}} \text{ for } 1 \leq k = \frac{n}{2}.
\end{cases}
\]
Moreover, $\cap_k(\cdot, \Omega)$ has the following metric properties (cf. [14] Lemma 4.1):
Theorem 1.2. Let (8.8)-(8.9)
for the case
\[ k \]
capacitary inequalities for the \( k \) and a nonnegative Randon measure
\[ k \]
and a new characterization of
\[ k \]
Theorem 1.3. Given an origin-centered Euclidean ball \( \Omega \subset \mathbb{R}^n \), \( 1 \leq k \leq \frac{n}{2} \), and a nonnegative Randon measure \( \mu \) on \( \Omega \), let
\[ \tau(\mu, \Omega, t) = \inf \{ \text{cap}_k(K, \Omega) : \text{compact } K \subset \Omega \text{ with } \mu(K) \geq t \} \quad \forall \quad t > 0. \]
be the $k$-Hessian capacitary minimizing function with respect to $\mu$.

(i) If $1 \leq k \leq \frac{n}{2}$, then

$$\sup \left\{ \frac{\|u\|_{L^q(\Omega,\mu)}}{\|u\|_{\Phi_k^0(\Omega)}} : u \in \Phi_k^0(\Omega) \cap C^2(\Omega), \ 0 < \|u\|_{\Phi_k^0(\Omega)} < \infty \right\} < \infty$$

holds when and only when

$$\sup_{t>0} \left( \frac{\|u\|_{L^q(\Omega,\mu)}}{t} \right)^{\frac{k+1}{q}} < \infty \quad \text{for} \quad k + 1 \leq q < \infty;$$

$$\int_0^\infty \left( \frac{\|u\|_{L^q(\Omega,\mu)}}{t} \right)^{\frac{k+1}{q}} dt < \infty \quad \text{for} \quad 1 < q < k + 1.$$

(ii) If $k = \frac{n}{2}$, then

$$\sup \left\{ \frac{\|u\|_{L^q(\Omega,\mu)}}{\|u\|_{\Phi_k^0(\Omega)}} : u \in \Phi_k^0(\Omega) \cap C^2(\Omega), \ 0 < \|u\|_{\Phi_k^0(\Omega)} < \infty \right\} < \infty$$

holds when and only when

$$\sup_{t>0} t \exp \left( \frac{\alpha}{\tau(\mu,\Omega,t)} \right) < \infty,$$

where

$$\|u\|_{L^p(\Omega,\mu)} = \int_\Omega \varphi(u) \, d\mu;$$

$$\varphi(u) = \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi_k^0(\Omega)}} \right)^\beta \right);$$

$$0 < \alpha < \alpha_0 = n \left( \frac{n}{2} \left( \frac{n-1}{k-1} \right) \right)^\frac{1}{n};$$

$$1 \leq \beta \leq \beta_0 = 1 + \frac{2}{n};$$

$$\omega_n = \text{the surface area of the unit sphere in } \mathbb{R}^{n+1}.$$

Often referred to as trace estimates (due to the fact that $\mu$ lives on $\Omega$ and may be the surface measure on a smooth submanifold of $\Omega$), the results in Theorem 1.3 will be proved in §5 through the $k$-Hessian capacitary weak and strong type estimates for $\|\cdot\|_{\Phi_k^0(\Omega)}$ presented in §4. Here, it is worth pointing out that the case $k = 1$ of Theorem 1.3 can be read off from the case $p = 2$ of Maz’ya’s [16, Theorem 8.5 & Remark 8.7] (related to the Nirenberg-Sobolev inequality [5, Lemma VI.3.1]), and the case $q = k + 1$ of Theorem 1.3 leads to a kind of Cheeger’s inequality - for $k = 1$ see also [6, 5, Theorem VI.1.2], and [24].

Remark 1.4. Two more comments are in order:

(i) Upon adapting the relatively natural capacity of a compact $K \subset \Omega$ for $k$-Hessian operator below (cf. [2])

$$\operatorname{cap}_{k,3}(K,\Omega) = \inf \left\{ \|u\|_{\Phi_k^0(\Omega)}^{k+1} : u \in \Phi_k^0(\Omega) \cap C^2(\Omega), \ u|_K \leq -1, \ u \leq 0 \right\},$$

$$\text{cap}_{k,3}(K,\Omega) = \frac{1}{\alpha \left( \frac{|u|}{\|u\|_{\Phi_k^0(\Omega)}} \right)^\beta},$$

where

$$\|u\|_{L^p(\Omega,\mu)} = \int_\Omega \varphi(u) \, d\mu;$$

$$\varphi(u) = \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi_k^0(\Omega)}} \right)^\beta \right);$$

$$0 < \alpha < \alpha_0 = n \left( \frac{n}{2} \left( \frac{n-1}{k-1} \right) \right)^\frac{1}{n};$$

$$1 \leq \beta \leq \beta_0 = 1 + \frac{2}{n};$$

$$\omega_n = \text{the surface area of the unit sphere in } \mathbb{R}^{n+1}.$$
we can see that Theorem 1.3 without assuming that $\Omega$ is an origin-centered Euclidean ball, still hold with $\text{cap}_{k}^{m}(\cdot,\Omega)$ being replaced by $\text{cap}_{k,3}^{m}(\cdot,\Omega)$.

(ii) While going along with demonstrating Theorems 1.2-1.3, we will introduce the required notation. But here, we only write $c(a,b,c,d)$ for different constants (in different lines) depending only on $a, b, c, d$ - for instance - $X \leq c(a,b,c,d) Y \leq \tilde{c}(a,b,c,d) Z$ means that there exist two positive constants $c(a,b,c,d)$ and $\tilde{c}(a,b,c,d)$ depending only on $a, b, c, d$ such that $X \leq c(a,b,c,d) Y \leq \tilde{c}(a,b,c,d) Z$.

2. Four alternatives to $\text{cap}_{k}(\cdot,\Omega)$

The purpose of this section is to define four new types of the $k$-Hessian capacity with $1 \leq k \leq \frac{n}{2}$ and then to establish their relations with $\text{cap}_{k}(\cdot,\Omega)$.

**Definition 2.1.** Suppose
\[
\begin{cases}
1 \leq k \leq \frac{n}{2}; \\
1_{E} \text{ stands for the characteristic function of } E \subset \Omega.
\end{cases}
\]
First, for a compact $K \subset \Omega$ let
\[
\begin{align*}
\text{cap}_{k,1}(K,\Omega) & = \sup \left\{ \int_{K} F_{k}[u] : u \in \Phi_{0}^{k}(\Omega) \cap C^{2}(\Omega), \ -1 < u < 0 \right\}; \\
\text{cap}_{k,2}(K,\Omega) & = \inf \left\{ \int_{\Omega} F_{k}[u] : u \in \Phi_{0}^{k}(\Omega) \cap C^{2}(\Omega), \ u \leq -1_{K} \right\}; \\
\text{cap}_{k,3}(K,\Omega) & = \inf \left\{ -\int_{\Omega} u F_{k}[u] : u \in \Phi_{0}^{k}(\Omega) \cap C^{2}(\Omega), \ u \leq -1_{K} \right\}; \\
\text{cap}_{k,4}(K,\Omega) & = \sup \left\{ -\int_{K} u F_{k}[u] : u \in \Phi_{0}^{k}(\Omega) \cap C^{2}(\Omega), \ -1 < u < 0 \right\}.
\end{align*}
\]
Second, for an open set $O \subset \Omega$ and $j = 1, 2, 3, 4$ set
\[\text{cap}_{k,j}(O,\Omega) = \sup \left\{ \text{cap}_{k,j}(K,\Omega) : \text{ compact } K \subset O \right\}.\]
Third, for a general set $E \subset \Omega$ and $j = 1, 2, 3, 4$ put
\[\text{cap}_{k,j}(E,\Omega) = \inf \left\{ \text{cap}_{k,j}(K,\Omega) : \text{ open } O \text{ with } E \subset O \subset \Omega \right\}.\]

**Lemma 2.2.** Suppose $1 \leq k \leq \frac{n}{2}$. Let $\Omega$ be the Euclidean ball $B_{r}$ of radius $r$ centered at the origin. If $K$ is a compact subset of $\Omega$, then
\[
\text{cap}_{k,j}(K,\Omega) = \begin{cases}
\int_{K} F_{k}[R_{k}(K,\Omega)] & \text{ as } j = 1; \\
\int_{K} (-R_{k}(K,\Omega)) F_{k}[R_{k}(K,\Omega)] & \text{ as } j = 4,
\end{cases}
\]
where
\[R_{k}(K,\Omega)(x) = \lim_{y \to x} \sup_{u \in \Phi_{0}^{k}(\Omega)} \left\{ u(y) : u \in \Phi_{0}^{k}(\Omega), \ u \leq -1_{K} \right\}\]
is the regularised relative extremal function associated with $K \subset \Omega$. 

Proof. As showed in [14], the function \( x \mapsto R_k(K, \Omega)(x) \) is upper semicontinuous, is of \( C^2(\Omega) \), and is the viscosity solution of the following Dirichlet problem:

\[
\begin{aligned}
F_k[u] &= 0 \quad \text{in } \Omega \setminus K; \\
u &= -1 \quad \text{on } \partial K; \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Moreover,

\[
cap_k(K, \Omega) = \int_K F_k[R_k(K, \Omega)].
\]

Note that \( R_k(K, \Omega) \) is in \( \Phi^k_0(\Omega) \cap C^2(\Omega) \). So, from Definition 2.1 it follows that

\[
cap_{k,1}(K, \Omega) = \int_K F_k[R_k(K, \Omega)].
\]

To see the desired formula for \( j = 4 \), let \( u \in \Phi^k_0(\Omega) \cap C^2(\Omega) \). Then, for any \( \epsilon \) there exists a function \( v \in \Phi^k_0(\Omega) \cap C^2(\Omega) \) satisfying \( v = (1 + \epsilon)u \), such that

\[
(1 + \epsilon)^{k+1}cap_{k,4}(K, \Omega) = (1 + \epsilon)^{k+1}\sup \left\{ \int_K (-u)F_k[u] : u \in \Phi^k_0(\Omega) \cap C^2(\Omega), -1 < u < 0 \right\}
\]

\[
= \sup \left\{ \int_K (-v)F_k[v] : v \in \Phi^k_0(\Omega) \cap C^2(\Omega), -1 - \epsilon < v < 0 \right\}.
\]

Since \( R_k(K, \Omega) > -1 - \epsilon \) in \( K \), we have

\[
(1 + \epsilon)^{-(k+1)} \int_K (-R_k(K, \Omega))F_k[R_k(K, \Omega)] \leq cap_{k,4}(K, \Omega).
\]

Letting \( \epsilon \to 0 \), we obtain

\[
\int_K (-R_k(K, \Omega))F_k[R_k(K, \Omega)] \leq cap_{k,4}(K, \Omega).
\]

To reach the reversed one of the last inequality, let \( \{K_i\} \) be a decreasing open set with smooth boundary in \( \Omega \) and provide

\[K_{i+1} \subset K_i \subset \Omega \quad \& \quad \bigcup_{i=1}^{\infty} K_i = K.\]

Then, using the regularity of \( \partial K_i \) we define

\[u_i = R_k(K_i, \Omega) \in C(\Omega).\]

According to [19] Lemma 2.1, we have the following monotonicity: if

\[
\begin{aligned}
u, v &\in \Phi^k(\Omega) \cap C^2(\Omega); \\
u &\geq v \quad \text{in } \Omega; \\
u &= v \quad \text{on } \partial \Omega,
\end{aligned}
\]

then

\[
\int_\Omega F_k[u] \leq \int_\Omega F_k[v],
\]
whence getting
\[ \int_K F_k[u] \leq \int_{\{u_i < u\}} F_k[u] \leq \int_{\Omega} F_k[u] \leq \int_{\Omega} F_k[u_i]. \]

Letting \( i \to \infty \) in the last inequality yields that
\[ \int_K (-u)F_k[u] \leq \int_K (-R_k(K,\Omega))F_k[R_k(K,\Omega)] \]
holds for any \( u \in \Phi^k_0(\Omega) \cap C^2(\Omega) \) with \(-1 < u < 0\). As a consequence, we get
\[ \int_K (-R_k(K,\Omega))F_k[R_k(K,\Omega)] \geq \text{cap}_{k,4}(K,\Omega), \]
thereby completing the argument. \( \square \)

**Theorem 2.3.** Suppose \( 1 \leq k \leq \frac{n}{2} \). Let \( \Omega \) be the Euclidean ball \( B_r \) of radius \( r \) centered at the origin. If \( E \subset \Omega \), then
\[ \text{cap}_{k}(E,\Omega) = \text{cap}_{k,j}(E,\Omega) \quad \forall \quad j = 1, 2, 3, 4. \]

**Proof.** By Definition 2.1 it is enough to prove that if \( E = K \) is a compact subset of \( \Omega \) then
\[ \text{cap}_{k,1}(K,\Omega) \leq \text{cap}_{k,2}(K,\Omega) \leq \text{cap}_{k,3}(K,\Omega) \leq \text{cap}_{k,4}(K,\Omega) \leq \text{cap}_{k,1}(K,\Omega). \]

To do so, note first that the inequalities
\[
\begin{cases}
\text{cap}_{k,4}(K,\Omega) \leq \text{cap}_{k,1}(K,\Omega); \\
\text{cap}_{k,2}(K,\Omega) \leq \text{cap}_{k,3}(K,\Omega),
\end{cases}
\]
just follow from Definition 2.1. Next, an application of Lemma 2.2 yields
\[ \text{cap}_{k,1}(K,\Omega) = \text{cap}_{k}(K,\Omega) = \int_K F_k[R_k(K,\Omega)] = \int_{\Omega} F_k[R_k(K,\Omega)]. \]

Thus, from the definition of \( R_k(K,\Omega) \) and the monotonicity described in the proof of Lemma 2.2 it follows that for any \( u \in \Phi^k_0(\Omega) \cap C^2(\Omega) \) satisfying \( u|_K \leq -1 \) and \( u < 0 \) one has
\[ \int_{\Omega} F_k[R_k(K,\Omega)] \leq \int_{\Omega} F_k[u]. \]

Upon minimizing the right-hand side of the last inequality we obtain
\[ \text{cap}_{k,1}(K,\Omega) = \int_{\Omega} F_k[R_k(K,\Omega)] \leq \text{cap}_{k,2}(K,\Omega). \]

Finally, by the definitions of \( R_k(K,\Omega) \) and \( \text{cap}_{k,3}(K,\Omega) \), we achieve
\[ \text{cap}_{k,3}(K,\Omega) \]
\[ \leq \int_{\Omega} (-R_k(K,\Omega))F_k[R_k(K,\Omega)] \]
\[ = \int_K (-R_k(K,\Omega))F_k[R_k(K,\Omega)]. \]
thereby finding
\[ \text{cap}_{k,3}(K, \Omega) \leq \text{cap}_{k,4}(K, \Omega). \]

\[ \square \]

**Corollary 2.4.** Let \( \Omega \) be the Euclidean ball \( B_r \) of radius \( r \) centered at the origin. If \( E \subset \Omega \), then
\[ \text{cap}_1(E, \Omega) = \inf \left\{ \int_\Omega |Du|^2 : u \in W^{1,2}(\Omega), \ u \geq 1_E \right\} \equiv 2\text{-cap}(E, \Omega), \]
where \( W^{1,2}(\Omega) \) stands for the Sobolev space of all functions whose distributional derivatives are in \( L^2(\Omega) \).

**Proof.** Thanks to the well-known metric properties of the Wiener capacity \( 2\text{-cap}(\cdot, \Omega) \) (cf. [15, Chapter 2]), we only need to check that
\[ \text{cap}_1(E, \Omega) = 2\text{-cap}(E, \Omega), \]
for all compact \( E \subset \Omega \).

Since \( F_1[u] = \Delta u \), for any \( u \in \Phi_0^k(\Omega) \cap C^2(\Omega) \) with \( u \leq -1_E \) we apply an integration-by-part to obtain
\[ \int_\Omega (-u) F_1[u] = \int_\Omega (-u) \Delta u = \int_\Omega |Du|^2 = \int_\Omega |D(-u)|^2. \]

Upon considering the unique solution of the Dirichlet problem:
\[ \begin{cases} F_1[u] = \Delta u = 0 & \text{in } \Omega \setminus E; \\ -u = 1 & \text{on } \partial E; \\ u = 0 & \text{on } \partial \Omega, \end{cases} \]
we get
\[ \text{cap}_{1,3}(E, \Omega) = \int_\Omega (-R) F_k[R] = \int_\Omega |D(-R)|^2 = 2\text{-cap}(E, \Omega), \]
whence reaching the conclusion via Theorem 2.3. \[ \square \]

3. ISOCAPACITARY INEQUALITIES

3.1. **Proof of Theorem 1.2 (i).** Step (i). We start with proving that if \( E \subset B_r \) and \( 1 \leq k < \frac{d}{2} \), then there is a constant \( c(n, k, q, \Omega) > 0 \) depending only on \( n, k, q, \) and \( |\Omega| \) such that
\[ |E|^{\frac{1}{k+1}} \leq c(n, k, q, \Omega)(\text{cap}_k(E, B_r)). \]

Without losing generality, we may assume that \( E \) is a compact set in \( B_r \).

Now, by Theorem 1.1 (i), we have that if \( 1 \leq q \leq k^* \) then
\[ \|u\|_{L^q(B_r)} \leq c(n, k, q, r)\|u\|_{\Phi_0^k(B_r)} \quad \forall \ u \in \Phi_0^k(B_r), \]
where \( c(n, k, q, r) > 0 \) is a constant depending only on \( n, k, q, r \).

Since \( R_k(E, B_r) \in \Phi_0^k(B_r) \), from the definition of \( \|\cdot\|_{\Phi_0^k(B_r)} \) it follows that
\[ \|R_k(E, B_r)\|_{L^q(B_r)} \leq c(n, k, q, r) \left( \int_{B_r} (-R_k(E, B_r)) F_k[R_k(E, B_r)] \right)^{\frac{1}{k+1}}. \]
In other words, Theorem 2.3 is employed to derive
\[ \|R_k(E, B_r)\|_{L^q(B_r)} \leq c(n, k, q, r) \left( \text{cap}_k(E, B_r) \right)^{\frac{1}{k+1}}. \]

Thus, by the definition of \( R_k(E, B_r) \), we achieve
\[
\frac{|E|}{n} \leq \left( \int_E |R_k(E, B_r)|^q \right)^{\frac{1}{q}} \leq \left( \int_{B_r} |R_k(E, B_r)|^q \right)^{\frac{1}{q}} \leq \|R_k(E, B_r)\|_{L^q(B_r)}^{k+1} \leq \left( c(n, k, q, r) \right)^{k+1} \text{cap}_k(E, B_r).
\]

\textbf{Step (i)2.} Next, we verify that if \( E \subset \Omega \) and \( 1 \leq k < \frac{n}{2} \) then there is a constant \( c(n, k, q, |\Omega|) > 0 \) depending only on \( n, k, q \), and \( |\Omega| \) such that
\[
\frac{|E|}{n} \leq c(n, k, q, |\Omega|) \left( \text{cap}_k(E, \Omega) \right)^n.
\]

Without losing generality, we may assume that \( E \) is a compact subset of \( \Omega \) and \( \Omega \) contains the origin. Then there exists a ball \( B_r \) centered at the origin with radius \( \text{diam}(\Omega) \) such that \( \Omega \subset B_r \).

Since \( 1 \leq k < \frac{n}{2} \), by \textbf{Step (i)1} and [14, Lemma 4.1(ii)], we obtain
\[
\frac{|E|}{n} \leq c(n, k, q, |\Omega|) \left( \text{cap}_k(E, \Omega) \right)^n,
\]
as desired.

\textbf{Step (i)3.} Particularly, for \( q = \frac{n(k+1)}{n-2k} \) we make the following analysis. Suppose \( E \) is a compact set contained in \( B_r - \) a ball centered at the origin with radius \( r > 0 \). We claim that if \( 1 \leq k < \frac{n}{2} \) then there is a constant \( c(n, k) > 0 \) depending only on \( n, k \) such that
\[
\frac{|E|}{n} \leq c(n, k) \text{cap}_k(E, \mathbb{R}^n).
\]

In fact, according to Dai-Bao’s paper [10], there exists a unique viscosity solution to the Dirichlet problem stated in the proof of Lemma 2.2. Such a solution guarantees that there exists a unique \( R_k(E, \mathbb{R}^n) \) satisfying
\[
R_k(E, \mathbb{R}^n) = \lim_{r \to \infty} R_k(E, B_r).
\]

Now, by the previous \textbf{Step (i)1}, we have that if \( q = k^* \) then
\[
\frac{|E|}{n} \leq c(n, k, r) \text{cap}_k(E, B_r),
\]
whence reaching the above claim through letting \( r \to \infty \) in the last estimate.

Now, using the same argument for \textbf{Step (i)2}, we get
\[
\frac{|E|}{n} \leq c(n, k) \text{cap}_k(E, \mathbb{R}^n) \leq c(n, k) \text{cap}_k(E, \Omega).
\]
Step (i). Following the above argument plus applying [14, Lemma 4.1(ii)], Theorem [1.1 (ii)] and Theorem 2.3 we can readily find that

\[ |E|^{\frac{k+1}{q}} \leq c(n, k, q, \text{diam}(\Omega)) \text{cap}_k(E, \Omega) \]

holds for \( k = \frac{n}{2} \) and \( 1 \leq q < \infty \).

3.2. Proof of Theorem 1.2 (ii). Step (ii). Partially motivated by [1, 9, 25], we begin with a slight improvement of the Moser-Trudinger inequality stated in Theorem 1.1 (ii): if \( k = \frac{n}{2} \) then there is a constant \( c(n) > 0 \) depending only on \( n \) such that

\[ \sup_{0 < \|u\|_{\Phi_0^k(\Omega)}} \int_\Omega \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}} \right)^{\beta} \right) \leq c(n) (\text{diam}(\Omega))^n, \]

where \( \alpha, \beta \) are the constants determined in Theorem 1.1 (ii).

Without loss of generality, we may assume that \( \Omega \) contains the origin. Then there exists a ball \( B_r \) centered at the origin with radius \( \text{diam}(\Omega) \) such that \( \Omega \subset B_r \). Following the argument for [18, Theorem 1.2], we have that for any radial function \( u = u(s) \) in \( \Phi_0^k(B_r) \) there exists a ball \( B_{\hat{r}} \subset \mathbb{R}^{n+1} \) with radius \( \hat{r} = r^{\frac{2n}{n+2}} \) and a radial function \( v(s) = u(s^{\frac{n+2}{2n}}) \) in \( \Phi_0^k(B_{\hat{r}}) \) such that

\[ \int_\Omega \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}} \right)^{\beta} \right) \leq (\frac{n+2}{2n})^{\alpha \beta} \int_{B_{\hat{r}}} \exp \left( \frac{\alpha}{c_0^\beta} \left( \frac{|v|}{\|v\|_{L^{n+1}(B_{\hat{r}})}} \right)^{\beta} \right) \leq c(n) |B_{\hat{r}}| \leq c(n) \hat{r}^{\frac{n}{2}+1} \leq c(n) r^n, \]

where

\[ c_0^\beta = \left( \frac{\omega_{n-1}}{k \omega_{n/2}} \right)^{\frac{n-1}{k-1} \left( \frac{2n}{n+2} \right)^{\frac{k}{k+1}}}. \]

Thus, by [18, Lemma 3.2], we achieve

\[ \sup \left\{ \int_\Omega \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}} \right)^{\beta} \right) : u \in \Phi_0^k(\Omega) \& 0 < \|u\|_{\Phi_0^k(\Omega)} < \infty \right\} \leq \sup \left\{ \int_\Omega \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi_0^k(\Omega)}} \right)^{\beta} \right) : u \in \Phi_0^k(\Omega) \text{ is radial} \right\} \leq c(n) (\text{diam}(\Omega))^n, \]

as desired.
Step (ii). We utilize the last step to check the remaining part of Theorem 1.2 (ii). Since $k = \frac{n}{2}$, by Lemmas 2.2, 3.2 and Theorem 2.3 we have

$$|E| \exp \left( \frac{\alpha}{(\text{cap}_k(E, B_r))^\frac{\beta}{k+1}} \right)$$

$$|E| \exp \left( \frac{\alpha}{(\text{cap}_{k,3}(E, B_r))^\frac{\beta}{k+1}} \right)$$

$$\leq \sup \left\{ \int_E \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi^k_0(B_r)}} \right) \beta : u \in \Phi^k_0(B_r) \right) \right\}$$

$$\leq c(n)(\text{diam}(B_r))^n,$$

i.e.,

$$\frac{\alpha}{(\text{cap}_k(E, \Omega))^\frac{\beta}{k+1}} \leq \frac{\alpha}{(\text{cap}_{k,3}(E, B_r))^\frac{\beta}{k+1}} \leq \ln \left( c(n)|E|^{-1}(\text{diam}(\Omega))^n \right).$$

Now, a simple calculation gives the desired inequality.

4. Capacitary weak and strong type estimates for $\Phi^k_0(\Omega)$

In a way different from proving the capacitary weak and strong type estimates for the Wiener capacity $2\text{cap}(\cdot, \Omega)$, we establish the following $k$-Hessian capacitary weak and strong type inequalities.

**Theorem 4.1.** Suppose that $\Omega$ is an origin-centered Euclidean ball. If $u \in \Phi^k_0(\Omega) \cap C^2(\Omega)$ and $1 \leq k \leq \frac{n}{2}$, then one has:

(i) the capacitary weak type inequality

$$\text{cap}_k \left( \{ x \in \Omega : |u(x)| \geq t \} \right) \leq t^{-(k+1)}\|u\|_{\Phi^k_0(\Omega)}^{k+1} \quad \forall \ t > 0;$$

(ii) the capacitary strong type inequality

$$\int_0^\infty t^k \text{cap}_k \left( \{ x \in \Omega : |u(x)| \geq t \}, \Omega \right) dt \leq c(n, k)\|u\|_{\Phi^k_0(\Omega)}^{k+1},$$

where $c(n, k) > 0$ is a constant depending only on $n, k$.
Proof. (i) For $t > 0$ let $v = t^{-1}u$. By Theorem 2.3 we obtain

$$
\text{cap}_k(\{x \in \Omega : |v(x)| \geq 1\})
= \sup \left\{ \int_{\{|v| \geq 1\}} (-f)F_k[f] : f \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}), -1 < f < 0 \right\}
= \int_{\{|v| \geq 1\}} (-R(|v| \geq 1, \Omega))F_k[R(|v| \geq 1, \Omega)]
\leq \int_{\Omega} (-R(|v| \geq 1, \Omega))F_k[R(|v| \geq 1, \Omega)]
\leq \int_{\Omega} (-v)F_k[R(|v| \geq 1, \Omega)]
\leq \int_{\Omega} (-v)F_k[v],
$$

thereby getting

$$
\text{cap}_k(\{x \in \Omega : |u(x)| \geq t\}) \leq t^{-(k+1)} \int_{\Omega} (-u)F_k[u].
$$

(ii) For $t > 0$ let $M_t = \{x \in \Omega : |u(x)| \geq t\}$. Without loss of generality, we may assume $\|u\|_{\Phi_0^k(\Omega)} < \infty$, and then define a normed set function (cf. [4])

$$
\phi(E) \equiv \phi(E, \Omega) = \frac{\int_E (-u)F_k[u]}{\|u\|_{\Phi_0^k(\Omega)}^{k+1}} \forall \ E \subset \Omega.
$$

Note that

$$
E_1 \cap E_2 = \emptyset \Rightarrow \phi(E_1 \cup E_2) = \phi(E_1) + \phi(E_2).
$$

Applying [13] Theorem 2.2-Corollary 2.3, we can find a non-negative measure $\psi$ defined on $\Omega$ and a positive constant $c_n$ depending only on $n$ such that

$$
\begin{cases}
\phi(E) \leq \psi(E) & \forall \ E \subset \Omega; \\
\psi(\Omega) \leq c_n.
\end{cases}
$$
Consequently, for a given constant \( a > 1 \) we estimate

\[
\int_0^\infty \phi(M_t \setminus M_{at}) \frac{dt}{t} \leq \int_0^\infty \psi(M_t \setminus M_{at}) \frac{dt}{t} = \int_0^\infty \int_0^{at} d\psi(M_s) \frac{dt}{t} = -(\ln a) \int_0^\infty d\psi(M_s) = \psi(M_0) \ln a \leq \psi(\Omega) \ln a \leq c_n \ln a,
\]

whence deriving

\[
\int_0^\infty \| u_{1 M_t \setminus M_{at}} \|^{k+1} \frac{dt}{t} \leq c_n (\ln a) \| u \|^{k+1}_{\Phi_0^k(\Omega)}.
\]

Now, if

\[
\tilde{u} = \max \left\{ \frac{t-u}{(a-1)t}, -1 \right\},
\]

then

\[
\begin{aligned}
\tilde{u} &\in \Phi_0^k(M_t); \\
\tilde{u}1_{M_{at}} &\leq -1,
\end{aligned}
\]

and hence

\[
\begin{aligned}
\| \tilde{u} \|^{k+1}_{\Phi_0^k(M_t)} &= \int_{M_t} (-\tilde{u}) F_k[\tilde{u}] \\
&= k^{-1} \int_{M_t} \tilde{u}_i \tilde{u}_j F_k^{ij} [D^2 \tilde{u}] \\
&= k^{-1} \int_{M_t \setminus M_{at}} \left( \frac{u}{(a-1)t} \right)_i \left( \frac{u}{(a-1)t} \right)_j F_k^{ij} \left[ D^2 \frac{u}{(a-1)t} \right] \\
&\leq \int_{M_t \setminus M_{at}} \left( -\frac{u}{(a-1)t} \right) F_k \left[ \frac{u}{(a-1)t} \right] \\
&= (a-1)^{-k-1} t^{-k-1} \int_{M_t \setminus M_{at}} (-u) F_k[u],
\end{aligned}
\]

where

\[
\begin{aligned}
F_k^{ij}[A] &= \frac{\partial}{\partial a_{ij}} F_k[A]; \\
D^2 f &= A = \{ a_{ij} \}.
\end{aligned}
\]
Using the definition of $\text{cap}_k(\cdot, \Omega)$, we obtain

$$
\int_0^\infty \frac{t^{k+1}\text{cap}_k(M_{at}, M_t)}{t} dt \\
\leq \int_0^\infty \frac{t^{k+1}\|\tilde{u}\|_{\Phi^k_0(M_t)}}{t} dt \\
\leq \int_0^\infty (a-1)^{-(k+1)} \left( \int_{M_t \setminus M_{at}} (\omega)F_k[u] \right) dt \\
\leq c_n (\ln a) (a-1)^{-(k+1)}\|u\|_{\Phi^k_0(\Omega)}^{k+1}.
$$

In particular, if $\lambda = at$, then a combination of $M_t \subset \Omega$, Theorem 2.3 and Theorem 1.1 (ii) derives

$$
\int_0^\infty \lambda^k \text{cap}_k(\{x \in \Omega : |u| > \lambda\}, \Omega) d\lambda \\
\leq \int_0^\infty (at)^k \text{cap}_k(M_{at}, M_t) d(at) \\
\leq c_n a^{k+1} (\ln a) (a-1)^{-(k+1)}\|u\|_{\Phi^k_0(\Omega)}^{k+1}.
$$

\[ \Box \]

5. Analytic vs geometric trace inequalities

5.1. **Proof of Theorem 1.3 (i).** In what follows, we always let

\[
\begin{align*}
\{ 1 \leq k &\leq \frac{n}{2}; \\
u &\in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega}); \\
M_t = \{ x \in \Omega : |u(x)| \geq t \} \quad &\forall \quad t > 0.
\end{align*}
\]

**Step (i)1.** For $k+1 \leq q < \infty$ let

\[
C_1 \equiv \sup_{t > 0} \frac{t^{k+1}}{\tau(\mu, \Omega, t)} < \infty.
\]

Then

$$
\mu(K) \frac{1}{q} \leq C_1^{\frac{1}{q}} (\text{cap}_k(K, \Omega))^{\frac{k+1}{q+1}} \quad \forall \quad \text{compact} \quad K \subset \Omega.
$$
An application of Theorem 4.1 (ii) yields that for any $u \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega})$,
\begin{align*}
\int_{\Omega} |u|^q \, d\mu &= \int_0^\infty \mu(M_\lambda) \, d\lambda^q \\
&\leq C_1^{\frac{q}{k+1}} \int_0^\infty \left(\text{cap}_k(M_\lambda, \Omega)\right)^{\frac{q}{k+1}} \, d\lambda^q \\
&\leq q(k+1)^{-1} C_1^{\frac{q}{k+1}} \|u\|_{\Phi^k_0(\Omega)}^{q-k} \int_0^\infty \text{cap}_k(M_\lambda, \Omega) \, d\lambda^{k+1} \\
&\leq q(k+1)^{-1} C_1^{\frac{q}{k+1}} c(n, k) \|u\|_{\Phi^k_0(\Omega)}^q.
\end{align*}
This gives
\[ C_2 \equiv \sup \left\{ \frac{\|u\|_{L^q(\Omega, \mu)}}{\|u\|_{\Phi^k_0(\Omega)}} : u \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega}) \text{ with } 0 < \|u\|_{\Phi^k_0(\Omega)} < \infty \right\} < \infty. \]

Conversely, assume $C_2 < \infty$. An application of the H"older inequality with $q' = \frac{q}{q-k}$ implies
\begin{align*}
t \mu(M_t) &\leq \int_{M_t} |u| \, d\mu(M_t) \\
&\leq \|u\|_{L_q(\Omega, \mu)} (\mu(M_t))^{\frac{1}{q'}} \\
&\leq C_2 \|u\|_{\Phi^k_0(\Omega)} (\mu(M_t))^{\frac{1}{q'}},
\end{align*}
and thus
\[ \sup_{t > 0} t \left(\mu(M_t)\right)^{\frac{1}{q'}} \leq C_2 \|u\|_{\Phi^k_0(\Omega)}. \]

Now, taking
\[ \begin{cases} t = 1; \\
u \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega}); \\
|u| \geq 1_K \text{ for any compact } K \subset \Omega,
\end{cases} \]
we obtain
\[ \left(\mu(K)\right)^{\frac{1}{q'}} \leq C_2 \|u\|_{\Phi^k_0(\Omega)} \leq C_2 (\text{cap}_k(K, \Omega))^{\frac{k}{k+1}}, \]
whence reaching $C_1 \leq C_2^{k+1}$.

**Step (i) \_2.** For $1 < q < k+1$ let
\begin{align*}
I_{k,q}(\mu) &\equiv \int_0^\infty \left( t^{\frac{k+1}{q}} (\tau(\mu, \Omega, t))^{-1} \right)^{\frac{q}{k+1-q}} t^{-1} \, dt; \\
S_{k,q}(\mu, u) &\equiv \sum_{j=-\infty}^\infty \frac{\left(\mu(M_{2j+1}(u)) - \mu(M_{2j+1}(u))\right)^{\frac{k+1}{k+1-q}}}{(\text{cap}_k(M_{2j}(u)))^{\frac{k+1}{k+1-q}}}. 
\end{align*}
If $I_{k,q}(\mu) < \infty$, then the elementary inequality

$$a^c + b^c \leq (a + b)^c \quad \forall \quad a, b \geq 0 \quad \& \quad c \geq 1$$

implies

$$S_{k,q}(\mu, u)$$

$$= \sum_{j=0}^{\infty} \left( \mu(M_{2j}(u)) - \mu(M_{2j+1}(u)) \right) \frac{k+1}{k+1-q} \left( \text{cap}_k(M_{2j}(u), \Omega) \right) \frac{q}{k+1-q}$$

$$\leq \sum_{j=0}^{\infty} \left( (\mu(M_{2j}(u)) - \mu(M_{2j+1}(u))) \frac{k+1}{k+1-q} \left( \tau(\mu, \Omega, \mu(M_{2j})) \right) \right) \frac{q}{k+1-q}$$

$$\leq \sum_{j=0}^{\infty} \mu(M_{2j}(u)) \frac{k+1}{k+1-q} - \mu(M_{2j+1}(u)) \frac{k+1}{k+1-q} \left( \tau(\mu, \Omega, \mu(M_{2j})) \right) \frac{q}{k+1-q}$$

$$\leq c(n, k, q) \int_0^\infty (\tau(\mu, \Omega, s))^{-\frac{q}{k+1-q}} ds \frac{k+1}{k+1-q}$$

$$\leq c(n, k, q) I_{k,q}(\mu).$$

Therefore, by the Hölder inequality and Theorem 4.1, we have

$$\|u\|_{L^q(\Omega, \mu)}^q = \int_{\Omega} |u|^q d\mu$$

$$= \int_0^\infty t^q d\mu(M_t(u))$$

$$\leq \sum_{j=-\infty}^{\infty} \left( \mu(M_{2j}(u)) - \mu(M_{2j+1}(u)) \right) \frac{2^j}{2^j}$$

$$\leq (S_{k,q}(\mu, u)) \frac{k+1}{k+1-q} \left( \sum_{j=-\infty}^{\infty} 2^{j(k+1)} \text{cap}_k(M_{2j(k+1)}(u)) \right) \frac{q}{k+1-q}$$

$$\leq (S_{k,q}(\mu, u)) \frac{k+1}{k+1-q} \left( \int_0^\infty \text{cap}_k(M_{\lambda}(u), \Omega) d\lambda^{k+1} \right) \frac{q}{k+1-q}$$

$$\leq c(n, k, q)(S_{k,q}(\mu, u)) \frac{k+1-q}{k+1} \|u\|^q_{\Phi^q_{0}(\Omega)}$$

$$\leq c(n, k, q)(I_{k,q}(\mu)) \frac{k+1-q}{k+1} \|u\|^q_{\Phi^q_{0}(\Omega)},$$

whence getting

$$C^q_2 \leq c(n, k, q)(I_{k,q}(\mu)) \frac{k+1-q}{k+1}.$$ 

Conversely, suppose $C_2 < \infty$. Then

$$\sup_{t>0} t^{\frac{1}{q}} \leq \|u\|_{L^q(\Omega, \mu)} \leq C_2 \|u\|_{\Phi^q_{0}(\Omega)}$$
holds for any \( u \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega}) \). According to the definition of \( \tau(\mu, \Omega, t) \), for each integer \( j \) there exist a compact set \( K_j \subset \Omega \) and a function \( u_j \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega}) \) such that

\[
\begin{align*}
\text{cap}_k(K_j, \Omega) & \leq 2\tau(\mu, \Omega, 2^j); \\
\mu(K_j) & > 2^j; \\
u_j & \leq -1_{K_j}; \\
2^{-1} \|u_j\|^{k+1}_{\Phi^k_0(\Omega)} & \leq \text{cap}_k(K_j, \Omega).
\end{align*}
\]

Now, for integers \( i, m \) with \( i < m \) let

\[
\begin{align*}
u_{i,m} &= \sup_{i \leq j \leq m} \gamma_j \hat{f}_j; \\
\gamma_j &= \left( \frac{2^j}{\kappa(n, 2^j)} \right)^{\frac{1}{k+1-q}}.
\end{align*}
\]

Then \( u_{i,m} \) is a function in \( \Phi^k_0(\Omega) \cap C^2(\bar{\Omega}) \) – this follows from an induction and the easily-checked fact below

\[
\max\{u_1, u_2\} = \frac{u_1 + u_2 + |u_1 - u_2|}{2} \in \Phi^k_0(\Omega) \cap C^2(\bar{\Omega}).
\]

Consequently,

\[
\|u_{i,m}\|^{k+1}_{\Phi^k_0(\Omega)} \leq c(n, k) \sum_{j=i}^{m} \gamma_j^{k+1} \|u_j\|^{k+1}_{\Phi^k_0(\Omega)} \leq c(n, k) \sum_{j=i}^{m} \gamma_j^{k+1} \tau(\mu, \Omega, 2^j).
\]

Observe that for \( i \leq j \leq m \), one has

\[
u_{i,m}(x) \leq \gamma_j \quad \forall \quad x \in K_j.
\]

Therefore,

\[
2^j < \mu(K_j) \leq \mu \left(M_{\gamma_j}(u_{i,m})\right).
\]
This in turn implies
\[
\| u_{i,m} \|_{\Phi_0^k(\Omega)}^q \geq C_2^{-q} c(n, k, q) \int_\Omega |u_{j,m}|^q d\mu
\]
\[
\geq C_2^{-q} \int_0^\infty \left( \inf \{ t : \mu(M_t(u_{i,m})) \leq s \} \right)^q ds
\]
\[
\geq C_2^{-q} \sum_{j=1}^m \left( \inf \{ t : \mu(M_t(u_{i,m})) \leq 2^j \} \right)^q 2^j
\]
\[
\geq C_2^{-q} \sum_{j=1}^m \gamma_j 2^j
\]
\[
\geq C_2^{-q} c(n, k, q) \left( \frac{\sum_{j=1}^m \gamma_j 2^j}{\sum_{j=1}^m (\gamma_j)^k+1} \right) \| u_{i,m} \|_{\Phi_0^k(\Omega)}^q
\]
\[
\geq C_2^{-q} c(n, k, q) \left( \frac{\sum_{j=1}^m 2^{j(k+1)} (\tau(\mu, \Omega, 2^j))^{-\frac{q}{k+1-q}}}{\sum_{j=1}^m 2^\frac{j(k+1)}{k+1} (\tau(\mu, \Omega, 2^j))^{-\frac{q}{k+1-q}}} \right) \| u_{i,m} \|_{\Phi_0^k(\Omega)}^q
\]
\[
\geq C_2^{-q} c(n, k, q) \left( \sum_{j=1}^m 2^{j(k+1)} (\tau(\mu, \Omega, 2^j))^{-\frac{q}{k+1-q}} \right)^{\frac{k+1-q}{k+1}} \| u_{i,m} \|_{\Phi_0^k(\Omega)}^q.
\]

Consequently,
\[
I_{k,q}(\mu) \leq \lim_{i \to -\infty} \lim_{m \to \infty} \sum_{j=1}^m 2^{j(k+1)(k+1)-q} (\tau(\mu, \Omega, 2^j))^{-\frac{q}{k+1-q}} < \infty.
\]

5.2. Proof of Theorem 1.3 (ii). In the sequel, for
\[
\begin{cases}
  k = \frac{n}{2} ; \\
  u \in \Phi_0^k(\Omega) \cap C^2(\bar{\Omega}) ; \\
  M_t(u) = \{ x \in \Omega : |u(x)| \geq t \} \quad \forall \quad t > 0.
\end{cases}
\]

For convenience, rewrite the previous quantity \( C_1 \) as
\[
C_1(n, k, q, \mu, \Omega) \equiv \sup_{t > 0} \frac{t^{\frac{k+1}{q}}}{\tau(\mu, \Omega, t)}.
\]

If
\[
C_3(n, k, \alpha, \beta, \mu, \Omega) \equiv \sup_{t > 0} t \exp \left( \frac{\alpha}{(\tau(\mu, \Omega, t))^{\frac{1}{k+1}}} \right) < \infty,
\]
then for $\tilde{q} \geq k + 1$,

$$C_1(n, k, \tilde{q}, \mu, \Omega)$$

$$= \sup_{t > 0} \left( \frac{\tilde{q}^\beta}{\alpha \beta} \left( \frac{\alpha}{\tilde{q}^{\beta}} \right) \right)^{k+1}$$

$$= \sup_{t > 0} \left( \frac{\tilde{q}^\beta}{\alpha \beta} \left( \frac{\alpha}{\tilde{q}^{\beta}} \right) \right)^{k+1}$$

$$\leq \left( \frac{\tilde{q}}{\alpha \beta} \right)^{k+1} \sup_{t > 0} \left( t^\beta \exp \left( \frac{\alpha}{\tilde{q}^{\beta}} \right) \right)^{k+1}$$

$$= \left( \frac{\tilde{q}}{\alpha \beta} \right)^{k+1} \left( C_3(n, k, \mu, \Omega) \right)^{k+1}.$$  

Also, applying the Hölder inequality for $\tilde{q} \geq k + 1$, we get

$$\int_{\Omega} \exp \left( \alpha \left( \frac{|u|}{\|u\|_{\Phi^0_k(\Omega)}} \right)^\beta \right) d\mu$$

$$= \sum_{i=1}^{\infty} \int_{\Omega} \frac{\alpha^i}{i!} \left( \frac{|u|}{\|u\|_{\Phi^0_k(\Omega)}} \right)^{\beta i} d\mu$$

$$= \sum_{i < \frac{k+1}{\beta}} \int_{\Omega} \frac{\alpha^i}{i!} \left( \frac{|u|}{\|u\|_{\Phi^0_k(\Omega)}} \right)^{\beta i} d\mu + \sum_{i \geq \frac{k+1}{\beta}} \int_{\Omega} \frac{\alpha^i}{i!} \left( \frac{|u|}{\|u\|_{\Phi^0_k(\Omega)}} \right)^{\beta i} d\mu$$

$$\leq S_1 + S_2,$$

where

$$S_1 = \sum_{i < \frac{k+1}{\beta}} \int_{\Omega} \frac{\alpha^i}{i!} \left( \frac{|u|}{\|u\|_{\Phi^0_k(\Omega)}} \right)^{\beta i} d\mu$$

$$S_2 = \sum_{i \geq \frac{k+1}{\beta}} \int_{\Omega} \frac{\alpha^i}{i!} \left( \frac{|u|}{\|u\|_{\Phi^0_k(\Omega)}} \right)^{\beta i} d\mu.$$  

Next, we control $S_1$ and $S_2$ from above. As in the last subsection, we have that for any $u \in \Phi^0_k(\Omega) \cap C^2(\Omega)$ and integer $m \geq k + 1$,

$$\int_{\Omega} |u|^m d\mu \leq (C_1(n, k, m, \mu, \Omega))^m \frac{m}{\alpha \beta} c(n, k) \|u\|_{\Phi^0_k(\Omega)}^m.$$  

This, along with the previously-verified inequality

$$C_1(n, k, \tilde{q}, \mu, \Omega) \leq \left( \frac{\tilde{q}}{\alpha \beta} \right)^{k+1} \left( C_3(n, k, \mu, \Omega) \right)^{k+1} \forall \tilde{q} \geq k + 1,$$
gives

\[
S_1 \leq \sum_{i < \frac{k+1}{\beta}} \frac{\alpha^i}{i!} \left( \mu(\Omega) \right)^{1 - \frac{\beta_i}{\beta}} \left( (C_1(n, k, \bar{q}, \mu, \Omega))^{\frac{1}{\beta}} c(n, k) \right)^{\beta_i} < \infty.
\]

Meanwhile, Theorem 4.1 is utilized to derive

\[
S_2 = \sum_{i \geq \frac{k+1}{\beta}} \frac{\alpha^i}{i!} ||u||_{\Phi^k_0(\Omega)}^\beta i \int_\Omega |u|^{\beta_i} d\mu
\]

\[
= \sum_{i \geq \frac{k+1}{\beta}} \frac{\alpha^i}{i!} ||u||_{\Phi^k_0(\Omega)}^\beta i \int_0^\infty \mu(M_t) dt^{\beta_i}
\]

\[
= \sum_{i \geq \frac{k+1}{\beta}} \frac{\alpha^i}{i!} \int_0^\infty \left( \frac{\mu(M_t)}{||u||_{\Phi^k_0(\Omega)}^{\beta_i}} \right) \left( \frac{\mu(M_t)}{(cap_k(M_t, \Omega))^{\beta_i}} \right) dt^{\beta_i}
\]

\[
\leq \sum_{i \geq \frac{k+1}{\beta}} \frac{\alpha^i}{i!} \int_0^\infty \frac{cap_k(M_t, \Omega)}{t^{\beta_i - k - 1}} \left( \frac{\mu(M_t)}{||u||_{\Phi^k_0(\Omega)}^{\beta_i}} \right) \left( \frac{\mu(M_t)}{(cap_k(M_t, \Omega))^{\beta_i}} \right) dt^{\beta_i}
\]

\[
\leq \frac{\alpha \beta}{k+1} \int_0^\infty \sum_{i=0}^\infty \frac{\alpha^i}{i!} \left( \frac{\mu(M_t)}{cap_k(M_t, \Omega)^{\beta_i}} \right) cap_k(M_t, \Omega) ||u||_{\Phi^k_0(\Omega)}^{(k+1)} dt^{k+1}
\]

\[
\leq \frac{\alpha \beta}{k+1} \int_0^\infty \left( \mu(M_t) \exp \left( \frac{\alpha}{cap_k(M_t, \Omega)^{k+1}} \right) \right) \left( \frac{cap_k(M_t, \Omega)}{||u||_{\Phi^k_0(\Omega)}^{k+1}} \right) dt^{k+1}
\]

\[
\leq \alpha \beta (k+1)^{-1} C_3(n, k, \alpha, \beta, \mu, \Omega) ||u||_{\Phi^k_0(\Omega)}^{(k+1)} \int_0^\infty \left( cap_k(M_t, \Omega) \right) dt^{k+1}
\]

\[
\leq \alpha \beta (k+1)^{-1} c(n, k) C_3(n, k, \alpha, \beta, \mu, \Omega).
\]

Now, putting the estimates for $S_1$ and $S_2$ together, we obtain

\[
C_4 \equiv \sup \left\{ ||u||_{L^2(\Omega, \mu)} : u \in \Phi^k_0(\Omega) \cap C^2(\Omega) \text{ with } ||u||_{\Phi^k_0(\Omega)} > 0 \right\} < \infty.
\]

Conversely, if $C_4 < \infty$, then for any $u \in \Phi^k_0(\Omega) \cap C^2(\Omega)$ with $||u||_{\Phi^k_0(\Omega)} > 0$ one always has

\[
\int_\Omega \exp \left( \alpha \left( \frac{|u|}{||u||_{\Phi^k_0(\Omega)}} \right)^\beta \right) d\mu \leq C_4.
\]

Note that for any compact set $K \subset \Omega$ there exists a function $R(K, \Omega)$ such that

\[
\begin{cases}
R(K, \Omega) \in \Phi^k_0(\Omega) \cap C^2(\overline{\Omega}); \\
|R(K, \Omega)| \geq 1_K.
\end{cases}
\]
So, we get

\[
\mu(K) \exp \left( \frac{\alpha}{\left( \text{cap}_k(K, \Omega) \right)^{\frac{1}{k+1}}} \right) \\
\leq \int_K \exp \left( \frac{\alpha}{\left( \text{cap}_k(K, \Omega) \right)^{\frac{1}{k+1}}} \right) d\mu \\
\leq \int_\Omega \exp \left( \alpha \left( \frac{|R(K, \Omega)|}{\|R(K, \Omega)\|_{\Phi_k^0(\Omega)}} \right)^\beta \right) d\mu \\
\leq C_4,
\]

whence achieving \( C_3(n, k, \alpha, \beta, \mu, \Omega) \leq C_4 \).

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