The Strongly Antimagic labelings of Double Spiders

Fei-Huang Chang
Division of Preparatory Programs for Overseas Chinese Students
National Taiwan Normal University
New Taipei City, Taiwan
cfh@ntnu.edu.tw

Pinhui Chin
Department of Mathematics
Tamkang University
New Taipei City, Taiwan
encorex63447@gmail.com

Wei-Tian Li†
Department of Applied Mathematics
National Chung-Hsing University
Taichung City, Taiwan
weitianli@nchu.edu.tw

Zhishi Pan‡
Department of Mathematics
Tamkang University
New Taipei City, Taiwan
zhishi.pan@gmail.com

September 17, 2018

Abstract

A graph $G = (V, E)$ is strongly antimagic, if there is a bijective mapping $f : E \rightarrow \{1, 2, \ldots, |E|\}$ such that for any two vertices $u \neq v$, not only $\sum_{e \in E(u)} f(e) \neq \sum_{e \in E(v)} f(e)$ and also $\sum_{e \in E(u)} f(e) < \sum_{e \in E(v)} f(e)$ whenever $\deg(u) < \deg(v)$, where $E(u)$ is the set of edges incident to $u$. In this paper, we prove that double spiders, the trees contains exactly two vertices of degree at least 3, are strongly antimagic.

*Grant number: MOST 104-2115-M-003-008-MY2
†Grant number: MOST-105-2115-M-005-003-MY2
‡Corresponding Author
1 Introduction

Suppose $G = (V, E)$ is a connected, finite, simple graph and $f : E \to \{1, 2, \ldots, |E|\} := [|E|]$ is a bijection. For each vertex $u$ of $V$, let $E(u)$ be the set of edges incident to $u$, and the vertex-sum $\varphi_f$ at $u$ is defined as $\varphi_f(u) = \sum_{e \in E(u)} f(e)$. The degree of $u$, denoted by $\text{deg}(u)$, is the capacity of $E(u)$, i.e. $\text{deg}(u) = |E(u)|$ and the leaf set is defined by $V_1 = \{u | \text{deg}(u) = 1, u \in V\}$. If $\varphi_f(u) \neq \varphi_f(v)$ for any two distinct vertices $u$ and $v$ of $V$, then $f$ is called an antimagic labeling of $G$.

The problem of finding antimagic labelings of graphs was introduced by Hartsfield and Ringel [5] in 1990. They proved that some special families of graphs, such as paths, cycles, complete graphs, are antimagic and put two conjectures. The conjectures have received much attention, but both conjectures remain open.

Conjecture 1 [5] Every connected graph with order at least 3 is antimagic.

The most significant progress of Conjecture 1 is a result of Alon, Kaplan, Lev, Roditty, and Yuster [1]. They proved that a graph $G$ with minimum degree $\delta(G) \geq c \log |V|$ for a constant $c$ or with maximum degree $\Delta(G) \geq |V| - 2$ is antimagic. They also proved that complete partite graph other than $K_2$ is antimagic.

Cranston [4] proved that for $k \geq 2$, every $k$-regular bipartite graph is antimagic. For non-bipartite regular graphs, Bérczi, Bernáth, Vizer [2] and Chang, Liang, Pan, Zhu [3] proved independently that every regular graph is antimagic.

Conjecture 2 [5] Every tree other than $K_2$ is antimagic.

For Conjecture 2, Kaplan, Lev, and Roditty [7] and Liang, Wong, and Zhu [9] showed that a tree with at most one vertex of degree 2 is antimagic. Recently, Shang [11] proved that a special family of trees, spiders, is antimagic. A spider is a tree formed from taking a set of disjoint paths and identifying one endpoint of each path together. Huang, in his thesis [10], also proved that spiders are antimagic. Moreover, the antimagic labellings $f$ given in [10] have the property: $\text{deg}(u) < \text{deg}(v)$ implies $\varphi_f(u) < \varphi_f(v)$. Given a graph $G$, if there exists an antimagic labeling $f$ satisfying the above property, the $f$ is called a strongly antimagic labeling of $G$. A graph $G$ is called strongly antimagic if it has a strongly antimagic labeling.

Finding a strongly antimagic labeling on a graph $G$ enables us to find an antimagic labeling of the supergraph of $G$. Let us describe such inductive method in Lemma 1 which is extracted from the ideas in [10]. For a graph $G$, let $V_k$ be the set of vertices of degree $k$ in $V(G)$. Assume that $V_1 = \{v_1, v_2, \ldots, v_i\}$, then we define $G \oplus V'_1 = (V(G) \cup V'_1, E(G) \cup E')$, where $V'_1 = \{v'_1, v'_2, \ldots, v'_i\}$ and $E' = \{v_1v'_1, v_2v'_2, \ldots, v_iv'_i\}$.

Lemma 1 For any connected graph $G$ with $V_1 \neq \emptyset$, if $G$ is strongly antimagic, then $G \oplus V'_1$ is strongly antimagic.
Proof. The proof of this lemma is similar to the proof of a corollary in [10]. Let $f$ be a strongly antimagic labeling of $G$ and $V_1 = \{v_1, v_2, \ldots, v_i\}$ with $\varphi_f(v_1) < \varphi_f(v_2) < \ldots < \varphi_f(v_i)$. We construct a bijective mapping $f' : E(G \oplus V'_1) \rightarrow [|E(G)| + i]$ as following.

$$f'(e) = \begin{cases} j, & \text{if } e = v_jv'_j \in E', 1 \leq j \leq i; \\ f(e) + i, & \text{if } e \in E. \end{cases}$$

For any vertices $u \in V(G) - V_1$, $v_j \in V_1$, and $v'_j \in V'_1$, the vertex sums under $f'$ are $\varphi_{f'}(u) = \varphi_f(u) + i \deg(u)$, $\varphi_{f'}(v_j) = \varphi_f(v_j) + j$, and $\varphi_{f'}(v'_j) = j$. By some calculations and comparisons, it is clear that $f'$ is a strongly antimagic labeling of $G \oplus V'_1$. ■

A double spider is a tree which contains exactly two vertices of degree greater than 2. It also can formed by first taking two sets of disjoint paths and one extra path, and then identifying an endpoint of each path in the two sets to the two endpoints of the extra path, respectively. In this paper, we manage to solve Conjecture 2 for double spiders. We have a stronger result:

**Theorem 2** Double spiders are strongly antimagic.

The rest of the paper is organized as follows. In Section 2, we give some reduction methods and classify the double spiders into four types. For the four types of the double spiders, we will prove that they are strongly antimagic by giving the labeling rules in four different lemmas. Hence, to prove our main theorem, it suffices to prove the lemmas. The proofs of the lemmas are presented in Section 3. However, we will only give the labeling rules and show the strongly antimagic properties for degree one and degree two vertices. For the comparisons between other vertices, we put all the details in Appendix. Some concluding remarks and problems will be proposed in Section 4.

## 2 Main Results

Given a double spider, we decompose its edge set into three subsets: The core path $P_{\text{core}}$, $L$, and $R$, where $P_{\text{core}}$ is the unique path connecting the two vertices of degree at least three, $L$ consists of paths with one endpoint of each path identified to an endpoint of $P_{\text{core}}$, and $R$ consists of paths with one endpoint of each path identified to the other endpoint of $P_{\text{core}}$. We denote the endpoints of $P_{\text{core}}$ by $v_l$ and $v_r$, respectively. Conventionally, we assume $L$ contains at least as many paths as $R$, hence $\deg(v_l) \geq \deg(v_r)$. See Figure 1 as an illustration. Note that two double spiders are isomorphic if their $L$ sets, $R$ sets, and the core paths are isomorphic. From now on, we denote a double spider by $DS(L, P_{\text{core}}, R)$. The complexity of finding an antimagic labeling of a double spider depends on the number of the paths and their lengths composing the double spider. So let us begin with reducing the number of paths of length one in $L$ and $R$. 

3
Lemma 3 Suppose $G = DS(L, P_{\text{core}}, R)$ contains some path $P$ of length one in $L \cup R$. Assume at least one of the following conditions holds.

1. $P \in R$, $\deg_G(v_l) \geq \deg_G(v_r) > 3$, and $DS(L, P_{\text{core}}, R \setminus \{P\})$ is strongly antimagic,
2. $P \in L$, $\deg_G(v_l) > \deg_G(v_r) \geq 3$, and $DS(L \setminus \{P\}, P_{\text{core}}, R)$ has a strongly antimagic labeling $f$ with $\varphi_f(v_l) > \varphi_f(v_r)$.

Then $G$ is strongly antimagic.

Proof. We only prove (1), since the proof of (2) is analogous. Let $f$ be a strongly antimagic labeling on $G' = DS(L, P_{\text{core}}, R \setminus \{P\})$, and $P = v_r v$. We create a bijective mapping $f^*$ from $E(G)$ to $|E(G)|$ on $G$ by

$$f^*(e) = \begin{cases} 
1, & \text{if } e = v_r v; \\
 f(e) + 1, & \text{if } e \in E(G'). 
\end{cases}$$

Since $\deg_{G'}(v_r) < \deg_{G'}(v_l)$, we have $\varphi_f(v_r) < \varphi_f(v_l)$, and

$$\varphi_{f^*}(v_r) = \varphi_f(v_r) + (\deg_{G'}(v_r) - 1) + f^*(v_r v)$$
$$= \varphi_f(v_r) + \deg_G(v_r)$$
$$< \varphi_f(v_l) + \deg_G(v_l) = \varphi_{f^*}(v_l).$$

It is clear that $f^*$ is a strongly antimagic labeling of $DS(L, P_{\text{core}}, R)$. $\blacksquare$

An odd path (rest. even path) is a path of odd (even) length. Now suppose $R$ consists of $a$ odd paths and $b$ even paths, and $L$ consists of $c$ odd paths with length greater than one, $d$ even paths, and $t$ odd paths of length one. By means of the following lemmas, Theorem 2 will be proved.

Lemma 4 If $\deg(v_l) = \deg(v_r) = 3$ then $DS(L, P_{\text{core}}, R)$ is strongly antimagic.

Lemma 5 If $\deg(v_l) > \deg(v_r) \geq 3$, $b = 0$, and $R$ has no odd path of length at least 3, then $DS(L, P_{\text{core}}, R)$ is strongly antimagic.
Lemma 6 If deg($v_l$) > deg($v_r$) ≥ 3, $b = 0$, and $R$ has at least one odd path of length at least 3, then $DS(L, P^{core}, R)$ is strongly antimagic.

Lemma 7 If deg($v_l$) > deg($v_r$) ≥ 3 and $b ≥ 1$, then $DS(L, P^{core}, R)$ is strongly antimagic.

Proof of Theorem 2. By Lemma 4, 5, 6, and 7 the remaining case we need to show is deg($v_l$) = deg($v_r$) ≥ 4. For such a double spider $DS(L, P^{core}, R)$, let $h$ be the minimum length of a path in $L \cup R$. Without loss of generality, we assume there is a $P_h$ in $R$. Consider the double spider $DS(L', P^{core}, R')$ that is obtained by recursively deleting the leaf sets of $DS(L, P^{core}, R)$ and of the resulting graphs $h - 1$ times. According to Lemma 1, we only need to show that $DS(L', P^{core}, R')$ is strongly antimagic. It is clear that $R'$ contains a path $P$ of length one. By Lemma 3 it is sufficient to show $G^* = DS(L', P^{core}, R' \setminus \{P\})$ is strongly antimagic. Now deg$_{G^*}(v_l) >$ deg$_{G^*}(v_r)$, by Lemma 5, 6, and 7 $G^*$ is strongly antimagic.

3 Proofs of the Remaining Lemmas

In this section, we are going to prove the Lemmas in last section. We will give the rules to label the double spiders in each proof. However, part of the work of checking the strongly antimagic property is moved to Appendix because of the tedious and complicated calculations.

To achieve the goal, we need to give all edges and vertices the informative names. Here we use $P_h$ to denote a path with length $h$, i.e. $P_h = u_0u_1u_2, \ldots, u_h$, which is not the common way but is helpful for us to simply the notation in our proof. In addition, for paths of the same length in $L$ (or $R$), we can interchange the labelings on the edges of one paths with those of another. Thus, only the length of a path matters, and we use the same notation to represent paths of the same length in $L$ or $R$. Now, let $DS(L, P^{core}, R)$ be a double spider with path parameters $a, b, c, d,$ and $t$ defined in Section 2, and let $s$ be the length of $P^{core}$. Then we name the vertices and edges on the paths as follows:

$L = \{P_{2s_1+1}, \ldots, P_{2s_2+1}, P_{2s_1}, \ldots, P_{2s_2}, P_{1}, \ldots, P_{1}\}$ with $s_1 ≥ 1$.

- $P_{2s_1+1} = v_1^{l,odd}i_{1}^{l,odd} \ldots i_{2s_1+1}^{l,odd}$ with $e_{i,j}^{l,odd} = e_{i,j-1}^{l,odd}i_{i,j}^{l,odd}$ and $e_{i,1}^{l,odd} = v_1i_{1}^{l,odd}$.
- $P_{2s_2} = v_1^{l,even}i_{1}^{l,even} \ldots i_{2s_2}^{l,even}$ with $e_{i,j}^{l,even} = e_{i,j-1}^{l,even}i_{i,j}^{l,even}$ and $e_{i,1}^{l,even} = v_1i_{1}^{l,even}$.

$R = \{P_{2r_1+1}, P_{2r_2+1}, \ldots, P_{2r_1}, P_{2r_2}, \ldots, P_{2r_2} \}$ with $y_1 ≤ \cdots ≤ y_b$.

- $P_{2r_1+1} = v_r^{r,odd}i_{1}^{r,odd} \ldots i_{2r_1+1}^{r,odd}$ with $e_{i,j}^{r,odd} = e_{i,j-1}^{r,odd}i_{i,j}^{r,odd}$ and $e_{i,1}^{r,odd} = v_r^{r,odd}$.
- $P_{2r_2} = v_r^{r,even}i_{1}^{r,even} \ldots i_{2r_2}^{r,even}$ with $e_{i,j}^{r,even} = e_{i,j-1}^{r,even}i_{i,j}^{r,even}$ and $e_{i,1}^{r,even} = v_r^{r,even}$.
• \( P_i^i = v_i v_i^i \) with \( e^i = v_i v_i^i \), \( 1 \leq i \leq t \).

A vertex (resp. edge) denoted as \( v_{ij}^{r, \text{odd}} \) (resp. \( e_{ij}^{l, \text{even}} \)) means that it is the \( j \)th vertex (resp. edge) of the \( i \)th odd (resp. even) path in \( R \) (resp. \( L \)). Observe that the index \( j \) of an edge of a path in \( R \) is increasing from \( v \), to the leaf of the path, but the index of that in \( L \) is reverse. An edge of a path is called an odd (or even) edge if the index \( j \) of the edge is odd (or even). Define the following quantities for the total number of odd (even) edges in some odd (even) paths. (The summation is zero if \( i = 0 \).)

\[
A_{i}^{\text{odd}} = \sum_{k=1}^{i}(x_k + 1), \quad A_{i}^{\text{even}} = \sum_{k=1}^{i} x_k, \quad B_{i}^{\text{odd}} = B_{i}^{\text{even}} = \sum_{k=1}^{i} y_k, \quad C_{i}^{\text{odd}} = \sum_{k=1}^{i} w_k, \quad C_{i}^{\text{even}} = \sum_{k=1}^{i} w_k.
\]

Let \( A^{\text{all}} = A_{a}^{\text{odd}} + A_{a}^{\text{even}}, \ldots, D^{\text{all}} = D_{d}^{\text{odd}} + D_{d}^{\text{even}} \). Then the total number of edges \( m = A^{\text{all}} + B^{\text{all}} + s + C^{\text{all}} + D^{\text{all}} + t \). We use \( [n]_{\text{odd}} \) and \( [n, n_2]_{\text{odd}} \) to denote the set of all odd integers in \( [n] \) and the set of all odd integers in \( \{n_1, n_1 + 1, \ldots, n_2\} \), respectively. The definitions of \( [n]_{\text{even}} \) and \( [n, n_2]_{\text{even}} \) are similar.

Let us begin with Lemma 6, which is the simplest one.

**Proof of Lemma 6.** We construct a bijective mapping \( f \) by assigning 1, 2, \ldots, \( m \) to the edges accordingly in the following steps. Some steps can be skipped if no such edges exist. Without loss of generality, \( x_a \geq 1 \).

**Step 1.** Label the odd edges of the odd paths in \( R \) by

\[
f(e_{ij}^{r, \text{odd}}) = \begin{cases} A_{i-1}^{\text{odd}} + \frac{i+1}{2}, & \text{for } i \in [a-1] \text{ and } j \in [2x_i + 1]_{\text{odd}} \\
A_{a-1}^{\text{odd}} + \frac{c}{2}, & \text{for } i = a \text{ and } j \in [2, 2x_a + 1]_{\text{odd}} \end{cases}
\]

We label the edge \( e_{r,1}^{r, \text{odd}} \) later in order to ensure that the vertex sum at \( v_r \) is large enough. Recall \( A_{a}^{\text{odd}} = 0 \).

**Step 2.** If \( c \geq 1 \), for \( i \in [c] \) and \( j \in [2w_i]_{\text{odd}} \), label the odd edges of the odd paths in \( L \) by

\[
f(e_{i,j}^{l, \text{odd}}) = A_{i}^{\text{odd}} - 1 + C_{i-1}^{\text{odd}} - (i - 1) + \frac{j+1}{2}.
\]

We also leave the \( c \) edges \( e_{i,2w_i+1}^{l, \text{odd}} \) for \( 1 \leq i \leq c \) to enlarge the vertex sum at \( v_l \).

**Step 3.** If \( s \geq 4 \), label the edges of \( P_{\text{core}} \) by,

\[
f(e_j) = A_{a}^{\text{odd}} - 1 + C_{c}^{\text{odd}} - c + \begin{cases} \frac{s-2}{2}, & \text{for } j \in [2, s-2]_{\text{even}}, \text{ when } s \text{ is even}; \\
\frac{s}{2}, & \text{for } j \in [3, s-2]_{\text{odd}}, \text{ when } s \text{ is odd}. \end{cases}
\]

In this step, we have labeled \( s_1 = \lfloor \frac{s-2}{2} \rfloor \) edges on the core path \( P_s \).

**Step 4.** If \( d \geq 1 \), for \( i \in [d] \) and \( j \in [2z_i]_{\text{odd}} \), label the odd edges of the even paths in \( L \) by

\[
f(e_{i,j}^{l, \text{even}}) = A_{a}^{\text{odd}} - 1 + C_{c}^{\text{odd}} - c + s_1 + D_{i-1}^{\text{odd}} + \frac{i+1}{2}.
\]

**Step 5.** If \( t \geq 1 \), for \( i \in [t] \), label the paths of length one in \( L \) by
\[ f(e^i) = A_u^{odd} - 1 + C_c^{odd} - c + s_1 + D_d^{odd} + i. \]

**Step 6.** For \( i \in [a] \) and \( j \in [2x_i]_{even} \), label the even edges of the odd paths in \( R \) by
\[ f(e_{i,j}^{r,odd}) = A_u^{odd} - 1 + C_c^{odd} - c + s_1 + D_d^{odd} + t + A_{i-1}^{even} + \frac{j}{2}. \]

**Step 7.** If \( c \geq 1, \text{ for } i \in [c] \) and \( j \in [2w_i]_{even} \), label the even edges of the odd paths in \( L \) by
\[ f(e_{i,j}^{l,odd}) = A_u^{all} - 1 + C_c^{odd} - c + s_1 + D_d^{odd} + t + C_{i-1}^{even} + \frac{j}{2}. \]

**Step 8.** If \( s \geq 2 \), label the edges in \( P_{core} \) by
\[ f(e_j) = A_u^{all} - 1 + C_c^{all} - c + s_1 + D_d^{odd} + t + \begin{cases} \frac{s+1-j}{2}, & \text{for } j \in [s]_{odd}, \text{ when } s \text{ is even.} \\ \frac{j}{2}, & \text{for } j \in [s]_{even}, \text{ when } s \text{ is odd.} \end{cases} \]

We now have \( s_2 \) unlabeled edges on \( P_{core} \), where \( s_2 = 1, \text{ if } s = 1 \) or \( s \) is even, otherwise \( s_2 = 2. \)

**Step 9.** If \( d \geq 1, \text{ for } i \in [d] \) and \( j \in [2z_i]_{even} \), label the even edges of the even paths in \( L \) by
\[ f(e_{i,j}^{l,even}) = A_u^{all} - 1 + C_c^{all} - c + s_1 + D_d^{odd} + t + D_{i-1}^{even} + \frac{j}{2}. \]

**Step 10.** Label the edge \( e_{a,1}^{r,odd} \) by
\[ f(e_{a,1}^{r,odd}) = A_u^{all} - 1 + C_c^{all} - c + s_1 + D_d^{odd} + t + 1 = m - c - s_2. \]

**Step 11.** If \( c \geq 1, \text{ for } i \in [c] \), label the edges \( e_{i,2w_i+1}^{l,odd} \) by
\[ f(e_{i,2w_i+1}^{l,odd}) = m - c - s_2 + i. \]

**Step 12.** Label the remaining edges in \( P_{core} \) by the following rules: If \( s = 1 \) or \( s \) is even, then let \( f(e_s) = m \); otherwise, let \( f(e_1) = m - 1 \) and \( f(e_s) = m. \)

We prove that \( f \) is strongly antimagic:

**Claim:** \( \varphi_f(u) > \varphi_f(v) \) for any \( u \in V_2 \) and \( v \in V_1 \).

Observe that, at Step 5, every pendant edge has been labeled, and for a vertex \( u \) in \( V_2 \), there is an unlabeled edge in \( E(u) \). This guarantees that \( \varphi_f(u) > \varphi_f(v) \) for every vertex \( v \in V_1 \).

**Claim:** \( \varphi(u) \) are all distinct for \( u \in V_2 \).

For any two vertices \( u' \) and \( u'' \) in \( V_2 \), let \( E(u') = \{e_{u'}^1, e_{u'}^2\} \) and \( E(u'') = \{e_{u''}^1, e_{u''}^2\} \). Assume \( f(e_{u'}^1) < f(e_{u'}^2) \) and \( f(e_{u''}^1) < f(e_{u''}^2) \). Our labeling rules give that if \( f(e_{u'}^1) \leq f(e_{u''}^1) \), then \( f(e_{u'}^2) \leq f(e_{u''}^2) \), and at least one of the inequalities is strict. This guarantees that \( \varphi_f(u) \) are distinct for all \( u \in V_2 \).

For \( \varphi_f(v_l) > \varphi_f(v_r) > \varphi_f(u) \) for any \( u \in V_2 \), see Appendix.

The next is the proof of Lemma [7].
Proof of Lemma 7. We use similar concepts of the proof of Lemma 6 to create a bijection from $E$ to $[m]$. However, the rules will be a little more complicated than those in Lemma 6. In our basic principles, for each path in $L$ or $R$, edges of the same parity as the pendant edge on the same path should be labeled first in general, except for some edges incident to $v_i$ or $v_r$; and edges of different parities to the pendant edge on the same path will always be labeled after all pendant edges have been labeled (so that the vertex sum at a vertex of degree two can be greater than the vertex sum of a pendant vertex). Thus, when $L$ contains $P_1$’s, the labels of these edges will be less than the labels of the odd edges incident to $v_r$ on the even paths in $R$. This could lead to $\phi_f(v_i) < \phi_f(v_r)$ if there are too many $P_1$’s in $L$.

Once this happens, our solution is to switch the labeling order of some edges in the even paths in $R$. More precisely, we need to change the labeling orders of the edges for $\alpha$ even paths in $R$’s in $L$; and edges of different parities to the pendant edge on the same path will always be labeled after all pendant edges have been labeled (so that the vertex sum at a vertex of degree two can be greater than the vertex sum of a pendant vertex). Thus, when $L$ contains $P_1$’s, the labels of these edges will be less than the labels of the odd edges incident to $v_r$ on the even paths in $R$. This could lead to $\phi_f(v_i) < \phi_f(v_r)$ if there are too many $P_1$’s in $L$.

The followings are our labeling rules. Again, some steps can be skipped if no such edges exist.

**Step 1.** If $\beta > 0$, we first label the odd edges of $\beta$ $P_2$’s in $R$ and the edges of $\beta - 1$ $P_1$’s in $L$ by

$$f(e_{i,1}^{r,even}) = 2i - 1 \text{ for } i \in \beta \text{ and } f(e_i) = 2i \text{ for } i \in \beta - 1.$$ 

Previously, we should label the even edges of $P_2$ in $R$, but now we first label the odd edges of them and leave the pendant edges to be labeled later. Observe that for the label of each $e_{i,1}^{r,even}$ is an odd integer, and $f(e) > f(e_{i,1}^{r,even})$ for $i \in \beta - 1$. We define

$$\beta_1 = \max\{0, \beta - 1\}.$$

**Step 2.** If $\alpha > \beta = b_2$, for $i \in \beta + 1, \alpha$ and $j \in \{4, 2y_i\}_{\text{even}}$, label the even edges of the even paths in $R$ by

$$f(e_{i,j}^{r,even}) = \beta_1 + B_{i-1}^{even} - (i - (\beta + 1)) + \frac{i - 2}{2}.$$ 

Furthermore, if $b - 1 > \alpha$, we also label $e_{i,j}^{r,even}$ for $i \in [\alpha + 1, b - 1]$ and $j \in [2y_i]_{\text{even}}$ by

$$f(e_{i,j}^{r,even}) = \beta_1 + B_{i-1}^{even} - (\alpha - \beta) + \frac{j}{2}.$$ 

**Step 3.** If $a \geq 1$, for $i \in [a]$ and $j \in [2x_i + 1]_{\text{odd}}$, label the odd edges of odd paths in $R$ by

$$f(e_{i,j}^{r,odd}) = \beta_1 + B_{i-1}^{even} - (\alpha - \beta) + A_{i-1}^{odd} + \frac{j + 1}{2}.$$ 

**Step 4.** If $c \geq 1$, for $i \in [c]$ and $j \in [2w_i]_{\text{odd}}$, label the odd edges of odd paths in $L$ by

$$f(e_{i,j}^{l,odd}) = \beta_1 + B_{i-1}^{even} - (\alpha - \beta) + A_{i-1}^{odd} + C_{i-1}^{odd} - (i - 1) + \frac{i + 1}{2}.$$ 

8
We leave the edges $e_{i,2w_i+1}^{l,odd}$ of odd paths in $L$ to be labeled later to ensure that $v_l$ has a large vertex sum.

**Step 5.** If $s \geq 4$, label the edges of $P_{core}$ by

\[
f(e_j) = \beta_1 + B_{b-1}^{even} - (\alpha - \beta) + A_a^{odd} + C_c^{odd} - c + \frac{s}{2}, \quad \text{for } j \in [2, s-2]_{even}, \quad \text{when } s \text{ is even.}
\]

\[
f(e_j) = \beta_1 + B_{b-1}^{even} - (\alpha - \beta) + A_a^{odd} + C_c^{odd} - c + s_1 + \frac{i}{2}, \quad \text{for } j \in [3, s-2]_{odd}, \quad \text{when } s \text{ is odd.}
\]

Again, we have labeled $s_1 = \left\lfloor \frac{s-2}{2} \right\rfloor$ edges on the core path $P_s$ at this step. Next, we label the even edges of the $b$-th even path in $R$.

**Step 6.** For $j \in [2y_b]_{even}$, label the edges $e_{b,j}^{r,even}$ by

\[
f(e_{b,j}^{r,even}) = \beta_1 + B_{b-1}^{even} - (\alpha - \beta) + A_a^{odd} + C_c^{odd} - c + s_1 + \frac{i}{2}.
\]

**Step 7.** If $d \geq 1$, for $i \in [d]$ and $j \in [2z_1]_{odd}$, label the odd edges of the even paths in $L$ by

\[
f(e_{i,j}^{l,even}) = \beta_1 + B_{b-1}^{even} - (\alpha - \beta) + A_a^{odd} + C_c^{odd} - c + s_1 + D_{l-1}^{odd} + \frac{i+1}{2}.
\]

**Step 8.** If $\alpha > \beta = b_2$, for $i \in [\beta + 1, \alpha]$, label the edges $e_{i,1}^{r,even}$ in $R$ by

\[
f(e_{i,1}^{r,even}) = \beta_1 + B_{b-1}^{even} - (\alpha - \beta) + A_a^{odd} + C_c^{odd} - c + s_1 + D_{d}^{odd} + (i - \beta).
\]

Note that, for $\beta + 1 \leq i \leq \alpha$, $e_{i,2}^{r,even}$ and $e_{i,3}^{r,even}$ on $P_{2y_1}$ are two incident edges unlabeled yet.

**Step 9.** If $\beta \geq 1$, then $t > \beta$. Recall that we have labeled $\beta - 1$ paths of length one in $L$ at Step 1. Now label the remaining edges $e^i$ in $L$ by

\[
f(e^i) = \beta_1 + B_{b-1}^{even} + A_a^{odd} + C_c^{odd} - c + s_1 + D_{d}^{odd} + (i - \beta_1) \quad \text{for } i \in [\beta, t].
\]

**Step 10.** If $\beta \geq 1$, for $i \in [\beta]$, label the pendant edges of the $P_2$'s in $R$ by

\[
f(e_{i,2}^{r,even}) = B_{b-1}^{even} + A_a^{odd} + C_c^{odd} - c + s_1 + D_{d}^{odd} + t + (\beta + 1 - i).
\]

**Step 11.** If $\alpha > \beta$, for $i \in [\beta + 1, \alpha]$ and $j \in [3, 2y_1]_{odd}$, label the odd edges of the even paths in $R$ by

\[
f(e_{i,j}^{r,even}) = B_{b-1}^{even} + A_a^{odd} + C_c^{odd} - c + s_1 + D_{d}^{odd} + t + B_{i-1}^{odd} - (i - (\beta + 1)) + \frac{i-1}{2}.
\]

Moreover, if $b - 1 > \alpha$, for $i \in [\alpha + 1, b - 1]$ and $j \in [2y_1]_{odd}$, label $e_{i,j}^{r,even}$ by

\[
f(e_{i,j}^{r,even}) = B_{b-1}^{even} + A_a^{odd} + C_c^{odd} - c + s_1 + D_{d}^{odd} + t + B_{i-1}^{odd} - (\alpha - \beta) + \frac{i+1}{2}.
\]

**Step 12.** If $a \geq 1$, for $i \in [a]$ and $j \in [2x_1]_{even}$, label the even edges of odd paths in $R$ by

\[
f(e_{i,j}^{r,odd}) = B_{all} - y_b - (\alpha - \beta) + A_a^{odd} + C_c^{odd} - c + s_1 + D_{d}^{odd} + t + A_{i-1}^{even} + \frac{i}{2}.
\]
Step 13. If \( c \geq 1 \), for \( i \in \{c\} \) and \( j \in \{2w_1\}_{\text{even}} \), label the even edges of odd paths in \( L \) by
\[
f(e_{i,j}^\text{even}) = B^\text{all} - y_b - (\alpha - \beta) + A^\text{all} + C^\text{odd}_c - c + s_1 + D^\text{odd}_d + t + C^\text{even}_{i-1} + \frac{j}{2}.
\]

Step 14. If \( s \geq 2 \), label the edges in \( P_{\text{core}} \) by
\[
f(e_j) = B^\text{all} - y_b - (\alpha - \beta) + A^\text{all} + C^\text{all} - c + s + s_2 + D^\text{odd}_d + t + \frac{j+1}{2},
\]
\[
+ \begin{cases} \frac{s+1-j}{2}, & \text{for } j \in [s]_{\text{odd}}, \text{ when } s \text{ is even}. \\ \frac{j}{2}, & \text{for } j \in [s]_{\text{even}}, \text{ when } s \text{ is odd}. \end{cases}
\]

Step 15. For \( j \in [2y_b]_{\text{odd}} \), label the odd edges of the \( b \)-th even path in \( R \) by
\[
f(e_{b,j}^\text{even}) = B^\text{all} - y_b - (\alpha - \beta) + A^\text{all} + C^\text{all} - c + s - s_2 + D^\text{odd}_d + t + \frac{j+1}{2}.
\]

Step 16. If \( d \geq 1 \), for \( i \in \{d\} \) and \( j \in \{2z_i\}_{\text{even}} \), label the even edges of the even paths in \( L \) by
\[
f(e_{i,j}^\text{even}) = B^\text{all} - (\alpha - \beta) + A^\text{all} + C^\text{all} - c + s - s_2 + D^\text{odd}_d + t + D^\text{even}_c + \frac{j}{2}.
\]

Step 17. If \( \alpha > \beta \), for \( i \in [\beta + 1, \alpha] \), label the unlabeled edges \( e_{i,2}^\text{even} \) by
\[
f(e_{i,2}^\text{even}) = B^\text{all} - (\alpha - \beta) + A^\text{all} + C^\text{all} - c + s - s_2 + D^\text{all} + t + (i - \beta).
\]

Step 18. If \( c \geq 1 \), for \( i \in \{c\} \), label \( e_{i,2w_i+1}^\text{odd} \) by
\[
f(e_{i,2w_i+1}^\text{odd}) = B^\text{all} + A^\text{all} + C^\text{all} - c + s - s_2 + D^\text{all} + t + i.
\]

Step 19. Label the remaining edges in \( P_{\text{core}} \) by the following rules: If \( s = 1 \) or \( s \) is even, then let \( f(e_s) = m \); otherwise, let \( f(e_1) = m - 1 \) and \( f(e_s) = m \).

Next, we prove that \( f \) is strongly antimagic.

Claim: \( \varphi_f(u) \geq \varphi_f(v) \) for any \( u \in V_2 \) and \( v \in V_1 \).

Observe that all pendent edges have been labeled at Step 9 or Step 10. For the former case, \( \beta = 0 \) and there is an unlabeled edge in \( E(u) \) for every \( u \in V_2 \) at the end of Step 8. Hence the claim holds. For the latter case, observe that the edge \( e_{i,1}^\text{even} \) has the largest label among all pendent edge. Hence the largest vertex sum of all pendent vertices is \( f(e_{i,1}^\text{even}) \). Let us check the vertex sum of a vertex in \( V_2 \). For \( 1 \leq i \leq \beta \), the vertex sum at \( v_{i,1}^\text{even} = f(e_{i,1}^\text{even}) + f(e_{i,2}^\text{even}) = (2i-1) + (B^\text{even}_b + A^\text{odd}_a + C^\text{odd}_c - c + s_1 + D^\text{odd}_d + t + (\beta + 1 - i)) \) is increasing in \( i \). For any other vertex \( u \in V_2 \), by our labeling rules, we can find one edge \( e' \in E(u) \) with \( f(e') > f(e_{\beta,1}^\text{even}) \) and the other edge \( e'' \in E(u) \) with \( f(e'') > f(e_{\beta,2}^\text{even}) \). Thus, the smallest vertex sum of a vertex in \( V_2 \) happens at \( v_{i,1}^\text{even} \), and is greater than the vertex sum of any pendent vertex.

Claim: \( \varphi(u) \) are all distinct for \( u \in V_2 \).

We have already showed that the vertex sums satisfy \( \varphi_f(v_{i,1}^\text{even}) < \varphi_f(v_{2,1}^\text{even}) < \ldots < \varphi_f(v_{\beta,1}^\text{even}) < \varphi_f(u) \) for \( u \in V_2 - \{v_{1,1}^\text{even}, v_{1,2}^\text{even}, \ldots, v_{\beta,1}^\text{even}\} \). For other vertices \( u' \) and
follows that we only need to show that the double spider is strongly antimagic for every label $\phi$. Assume the number of $\phi$-values is equal to the number of edges in $V_2$, let $E(u') = \{e_{uv}, e_{u''v}\}$ and $E(u'') = \{e_{u''v}, e_{u''v}\}$. Assume $f(e_{uv}) < f(e_{u''v})$ and $f(e_{u''v}) < f(e_{u''v})$. Our labeling rules give that if $f(e_{uv}) \leq f(e_{u''v})$, then $f(e_{u''v}) \leq f(e_{u''v})$, and at least one of the inequalities is strict. This guarantees that $\phi_f(u)$ are distinct for all $u \in V_2$. For $\phi_f(v_t) > \phi_f(v_r) > \phi_f(u)$ for any $u \in V_2$, see Appendix.

Proof of Lemma 4 and Lemma 5. First, We use Lemma 1 and Lemma 3 to do some reductions.

Given a double spider $DS(L, P_{\text{core}}, R)$ described in Lemma 4, let us first consider $h = \min\{j | P_j \in R \cup L\}$. Since $\deg(v_t) = \deg(v_r) = 3$, without loss of generality, we assume the number of $P_h$ in $R$ is greater than or equal to that in $L$. By Lemma 1, it follows that we only need to show that the double spider is strongly antimagic for $h = 1$.

Given a double spider $DS(L, P_{\text{core}}, R)$ described in Lemma 5, we remove all but two $P_j$'s in $R$. Moreover, if there are some $P_j$ in $L$ as well, we remove them as many as possible unless one of the three situations happens: $L$ contains no $P_1$'s, or $L$ consists of exactly two $P_1$'s, or $L$ consists of exactly one $P_1$ and one path of length at least two. By Lemma 3, if the resulting double spider is strongly antimagic, then $DS(L, P_{\text{core}}, R)$ is also strongly antimagic.

Every reduced double spider belongs to at least one of the three types:

(a) $\deg(v_t) = \deg(v_r) = 3$, $t = 2$, $a = 2$, and $x_1 = x_2 = 0$; or

(b) $\deg(v_t) = \deg(v_r) = 3$, $t \leq 1$, $a \geq 1$, and $x_1 = 0$; or

(c) $\deg(v_t) \geq \deg(v_r) = 3$, $t = 0$, $a = 2$, and $x_1 = x_2 = 0$.

Now we show each type of double spiders above is strongly antimagic. If a double spider is of type (a), then the total number of edges $m = s + 4$. When $s$ is odd, we give the labeling $f$ as follows: $f(e_j) = \frac{(j+1)-i}{2}$ for $j \in [s]_{\text{even}}$, $f(e_{1,1}) = \frac{s-1}{2} + 1$, $f(e_{2,1}) = \frac{s-1}{2} + 2$, $f(e_1) = \frac{s+1}{2} + 3$, $f(e^2) = \frac{s+1}{2} + 4$, and $f(e_j) = \frac{s+1}{2} + 4 + \frac{(s+2)-i}{2}$ for $j \in [s]_{\text{odd}}$. For this labeling, we have vertex sums $\phi_f(v_t) = 2s + 10$, $\phi_f(v_r) = \frac{3s+13}{2}$, $\phi_f(v_j) = \frac{3s+11-2j}{2}$ for $2 \leq j \leq s$, and $\phi_f(v) \in \{\frac{s+1}{2}, \frac{s+3}{2}, \frac{s+5}{2}, \frac{s+7}{2}\}$ if $\deg(v) = 1$.

When $s$ is even, we give the labeling $f$ as follows: $f(e_j) = \frac{1}{2}$ for $j \in [s]_{\text{even}}$, $f(e_1) = \frac{s+1}{2} + 3$, $f(e^2) = \frac{s+1}{2} + 4$, and $f(e_{j,1}) = \frac{s+1}{2} + j$ for $j \in [s]_{\text{odd}}$. For this labeling, we have vertex sums $\phi_f(v_t) = \frac{3s+20}{2}$, $\phi_f(v_r) = \frac{3s+10}{2}$, $\phi_f(v_j) = \frac{3s+8+2j}{2}$ for $j \in [2, s]$, and $\phi_f(v) \in \{\frac{s+1}{2}, \frac{s+3}{2}, \frac{s+5}{2}, \frac{s+7}{2}\}$ if $\deg(v) = 1$.

It is easy to see the labelings are strongly antimagic. For a double spider of type (b) or (c), we will give the labeling rules to label the edges by $1, 2, \ldots, m$ accordingly. Our rules will produce a strongly antimagic labeling except for the double spider is isomorphic to the following ones:

We construct a strongly labeling separately in the right graph of Figure [2].
Figure 2: The double spider with $L = \{P_3, P_1\}$, $R = \{2P_1\}$, $P_{\text{core}} = P_2$

Note that for the two types of double spiders, $R$ contains only two paths and one of them has length one. For convenience, we will denote the two paths in $R$ by $P_1 = v_1, v_2, \ldots, v_k$ and $P_k = v_r, v_r, \ldots, v_r$. The following are our rules to label the double spiders of type (b) and (c):

**Step 1.** If $k \geq 2$, label all even edges of $P_k$ in $R$ by

$$f(e_{r,j}) = \left\lfloor \frac{k-j}{2} \right\rfloor,$$

for $j \in [k]_{\text{even}}$.

**Step 2.** If $c \geq 1$, label all odd edges of $P_{2w_1+1}$ in $L$, except for $e_{1,2w_1+1}$, by

$$f(e_{i,j}) = \left\lfloor \frac{k}{2} \right\rfloor + \frac{j+i}{2},$$

for $j \in [2w_1-1]_{\text{odd}}$.

Moreover, we define $w' = -1$ when $c \geq 1$, otherwise $w' = 0$. Then we have $\left\lfloor \frac{k}{2} \right\rfloor + c + w' + D_{\text{odd}} + t \geq 1$.

**Step 3.** If $s \geq 4$, label the edges of $P_{\text{core}}$ by,

$$f(e_{j}) = \left\lfloor \frac{k}{2} \right\rfloor + C_{\text{odd}} + w' + \begin{cases} \frac{s-2}{2}, & \text{for } j \in [s-2]_{\text{even}} \text{, when } s \text{ is even,} \\
\frac{2}{2}, & \text{for } j \in [3, s-2]_{\text{odd}} \text{, when } s \text{ is odd.}
\end{cases}$$

As before, we labeled $s_1 = \left\lfloor \frac{s-2}{2} \right\rfloor$ edges of the core path $P_s$.

**Step 4.** If $d \geq 1$, for $i \in [d]$ and $j \in [2z_i]_{\text{odd}}$, label the odd edges of $P_{2z_i}$ by

$$f(e_{i,j}) = \left\lfloor \frac{k}{2} \right\rfloor + C_{\text{odd}} + w' + s_1 + D_{i-1} + \frac{j+i}{2}.$$ 

Next, we label edges of the paths of length one in $L$ and $R$. We have to slightly adjust the labeling orders for different cases. Let

$$t' = \begin{cases} 1, & \text{if } t = 1, d = 1 \text{ or } t = 1, s = 2, c = 1, w_1 = 1, k \geq 2; \\
0, & \text{otherwise;}
\end{cases}$$

Observe that if $t' = 1$, then $\left\lfloor \frac{k}{2} \right\rfloor + D_{d} \geq 1$. 

```plaintext
Figure 2: The double spider with L = {P3, P1}, R = {2P1}, Pcore = P2

Note that for the two types of double spiders, R contains only two paths and one of them has length one. For convenience, we will denote the two paths in R by P1 = v1, v2, ..., vk and Pk = v_r, v_r, ..., v_r. The following are our rules to label the double spiders of type (b) and (c):

**Step 1.** If k ≥ 2, label all even edges of Pk in R by

f(e_r,j) = \left\lfloor \frac{k-j}{2} \right\rfloor, for j ∈ [k]_{even}.

**Step 2.** If c ≥ 1, label all odd edges of P_{2w_1+1} in L, except for e_{1,2w_1+1}, by

f(e_{i,j}) = \left\lfloor \frac{k}{2} \right\rfloor + \frac{j+i}{2}, for j ∈ [2w_1-1]_{odd},

and for i ∈ [2, c] and j ∈ [2w_i+1]_{odd}, let

f(e_{i,j}) = \left\lfloor \frac{k}{2} \right\rfloor + C_{\text{odd}} c + \frac{j}{2} + 1 + \frac{i}{2}.

Moreover, we define w' = -1 when c ≥ 1, otherwise w' = 0. Then we have \left\lfloor \frac{k}{2} \right\rfloor + c + w' + D_{\text{odd}} + t ≥ 1.

**Step 3.** If s ≥ 4, label the edges of P_{\text{core}} by,

f(e_{j}) = \left\lfloor \frac{k}{2} \right\rfloor + C_{\text{odd}} + w' + \begin{cases} \frac{s-2}{2}, & \text{for } j ∈ [s-2]_{\text{even}}, \text{ when } s \text{ is even.} \\
\frac{2}{2}, & \text{for } j ∈ [3, s-2]_{\text{odd}}, \text{ when } s \text{ is odd.}
\end{cases}

As before, we labeled s_1 = \left\lfloor \frac{s-2}{2} \right\rfloor edges of the core path P_s.

**Step 4.** If d ≥ 1, for i ∈ [d] and j ∈ [2z_i]_{\text{odd}}, label the odd edges of P_{2z_i} by

f(e_{i,j}) = \left\lfloor \frac{k}{2} \right\rfloor + C_{\text{odd}} + w' + s_1 + D_{i-1} + \frac{j+i}{2}.

Next, we label edges of the paths of length one in L and R. We have to slightly adjust the labeling orders for different cases. Let

\begin{align*}
t' = \begin{cases} 1, & \text{if } t = 1, d = 1 \text{ or } t = 1, s = 2, c = 1, w_1 = 1, k \geq 2; \\
0, & \text{otherwise;}
\end{cases}
\end{align*}

Observe that if t' = 1, then \left\lfloor \frac{k}{2} \right\rfloor + D_{d} \geq 1.
```
Step 5. We label $e^1$ (it does not exist if $t = 0$) and $e^r_{1,1}$ in different order according to the number $t'$. If $t' = 1$, we label $e^r_{1,1}$ and $e^1$ by

$$f(e^r_{1,1}) = \left\lfloor \frac{k}{2} \right\rfloor + C_c^{\text{odd}} + w' + s_1 + D_d^{\text{odd}} + 1.$$  \hfill (1)

$$f(e^1) = \left\lfloor \frac{k}{2} \right\rfloor + C_c^{\text{odd}} + w' + s_1 + D_d^{\text{odd}} + 2.$$  

Else, $t' = 0$, then we label $e^1$ and $e^r_{1,1}$ by

$$f(e^1) = \left\lfloor \frac{k}{2} \right\rfloor + C_c^{\text{odd}} + w' + s_1 + D_d^{\text{odd}} + t.$$  \hfill (2)

$$f(e^r_{1,1}) = \left\lfloor \frac{k}{2} \right\rfloor + C_c^{\text{odd}} + w' + s_1 + D_d^{\text{odd}} + t + 1.$$  

In this step, $f(e^1)$ is undefined when $t = 0$.

Step 6. Label all the odd edges of $P_k$, $j \in [k]_{\text{odd}}$, in $R$ by

$$f(e^r_{2,j}) = \left\lfloor \frac{k}{2} \right\rfloor + C_c^{\text{odd}} + w' + s_1 + D_d^{\text{odd}} + 1 + t + \left\lceil \frac{k+1-j}{2} \right\rceil.$$  \hfill (3)

Step 7. If $c \geq 1$, for $i \in [c]$ and $j \in [2w]_{\text{even}}$, label the even edges of $P_{2w^1+1}$ in $L$ by

$$f(e^l_{i,j}) = k + 1 + C_c^{\text{odd}} + w' + s_1 + D_d^{\text{odd}} + t + C_{i-1}^{\text{even}} + \frac{j}{2}.$$  

Step 8. If $s \geq 2$, label the edges in $P_{\text{core}}$ by

$$f(e_j) = k + 1 + C_c^{\text{all}} + w' + D_d^{\text{odd}} + s_1 + t + \begin{cases}  
\frac{s+1-j}{2}, & \text{for } j \in [s]_{\text{odd}}, \text{ when } s \text{ is even;} \\
\frac{1}{2}, & \text{for } j \in [s]_{\text{even}}, \text{ when } s \text{ is odd}; 
\end{cases}$$  \hfill (4)

Let $s_2$ be the number of unlabeled edges on $P_{\text{core}}$. So $s_2 = 1$, if $s = 1$ or $s$ is even, otherwise $s_2 = 2$.

Step 9. If $d \geq 1$, for $j \in [2z]_{\text{even}}$ and $i \in [d]$, label the even edges of $P_{2z_i}$ in $L$ by

$$f(e^l_{i,j}) = k + 1 + C_c^{\text{all}} + w' + s - s_2 + D_d^{\text{odd}} + t + D_{i-1}^{\text{even}} + \frac{j}{2}.$$  

Step 10. If $c \geq 1$, label the edge $e^l_{1,2w^1+1}$ left at Step 2 by

$$f(e^l_{1,2w^1+1}) = m - s_2.$$  

Step 11. Label the remaining edges in $P_{\text{core}}$ by the following rules:

If $s = 1$ or $s$ is even, then let $f(e_s) = m$; otherwise, let $f(e_1) = m - 1$ and $f(e_s) = m$.

We prove $f$ is a strongly antimagic labeling.

Claim: $\varphi_f(v) > \varphi_f(u)$ for any $v \in V_2$ and $u \in V_1$.

Observe that either all pendent edges were labeled before Step 6, or there exists exactly one pendent edge labeled at Step 6, when $k$ is odd and $k \geq 3$. In the former case, for every $v \in V_2$, there is an edge in $E(v)$ not labeled yet at the beginning at Step 6. This promises that $\varphi_f(v) > \varphi_f(u)$ for any $u \in V_1$. In the latter case, we label the pendent edge of $P_k$ by $\left\lfloor \frac{k}{2} \right\rfloor + C_c^{\text{odd}} + w' + s_1 + D_d^{\text{odd}} + t + 2$ at Step 6, and it is equal to $\varphi_f(v^r_{2,k})$. 

13
Moreover, every vertex \( v \in V_2 \), except for \( v_{2,k-1}^r \), is incident to an edge of label greater than \( \left\lfloor \frac{k}{2} \right\rfloor + C_{c,d}^{\text{odd}} + \sum s_1 + D_{d}^{\text{odd}} + t + 2 \). This also leads \( \varphi_f(v) > \varphi_f(u) \) for any vertex \( v \in V_2 \) and \( u \in V_1 \).

**Claim:** \( \varphi(u) \) are all distinct for \( u \in V_2 \).

For any two vertices \( u' \) and \( u'' \) in \( V_2 \), let \( E(u') = \{ e_{u'1}, e_{u'2} \} \) and \( E(u'') = \{ e_{u''1}, e_{u''2} \} \). Assume \( f(e_{u'1}) < f(e_{u'2}) \) and \( f(e_{u''1}) < f(e_{u''2}) \). Our labeling rules give that if \( f(e_{u'1}) \leq f(e_{u''1}) \), then \( f(e_{u'2}u') \leq f(e_{u''2}u''), \) and at least one of the inequalities is strict. This guarantees that \( \varphi_f(u) \) are distinct for all \( u \in V_2 \).

For \( \varphi_f(v_l) > \varphi_f(v_r) > \varphi_f(u) \) for any \( u \in V_2 \), see Appendix. \( \blacksquare \)

## 4 Conclusion and Future Work

In general, given an antimagic graph \( G \), there exist many antimagic labelings on \( G \). Some of the labelings are not strongly antimagic. Thus, finding a strongly antimagic labeling of a graph could be more difficult than finding a general antimagic labeling. In fact, we do not know if there exists a strongly antimagic labeling for every antimagic graph. However, if a graph is strongly antimagic, then we can use Lemma 1 to construct a larger graph which is not only antimagic but also strongly antimagic. It would be helpful to tackle the antimagic labeling problem if we have more constructive methods like that. For example, Lemma 1 can be generalized to the following theorem.

**Theorem 8** Let \( G \) be a strongly antimagic graph and \( V_k = \{ v \in V \mid \deg(v) = k \} \). If for each vertex in \( V_k \), we attach an edge to it, then the resulting graph is also strongly antimagic.

The proof of the above theorem is exactly the same as Lemma 1. First add \( |V_k| \) to the label of each edge in \( E \) when the strongly antimagic labeling is given, then label the new edges by \( 1, \ldots, |V_k| \) according to the order of the vertex sums of the vertices in \( V_k \). For antimagic graphs, we ask the following questions.

**Question 1** Does there exist a strongly antimagic labellings for every antimagic graph?

In 2008, Wang and Hsiao \[13\] introduced the \( k \)-antimagic labeling on a graph \( G \), which is a bijection \( f \) from \( E(G) \) to \( \{ k + 1, \ldots, k + |E(G)| \} \) for an integer \( k \geq 0 \) such that the vertex sums \( \varphi_f(v) \) are distinct over all vertices. We call a graph \( k \)-antimagic if it has a \( k \)-antimagic labeling. The purpose of studying such kind of labelings is to apply them for finding the antimagic labelings of the Cartesian product of graphs. Wang and Hsiao also pointed out that if the antimagic labeling \( f \) of a graph \( G \) has the property that the order of vertex sums is consistent with the order of degrees, then \( G \) is \( k \)-antimagic for any \( k \geq 0 \). This property on the vertex sums is exactly the same definition of the strongly antimagic labeling in our article. In fact, all the \( k \)-antimagic labelings studied in \[13\] are derived from the strongly antimagic labeling of the graph with a translation on
labels. Hence all those $k$-antimagic labelings have the “strong property”: $\varphi_f(v) < \varphi_f(u)$ whenever $\deg(u) < \deg(v)$.

**Question 2** Is there a $k$-antimagic graph but not $(k + 1)$-antimagic?

Note that if the answer of Question 2 is yes for some graph $G$, then every $k$-antimagic labeling on $G$ does not have the above strong property on the vertex sums and the degrees. Moreover, $G$ is a negative answer for Question 1 if $k = 0$.

**Remark.** There is a different version of $k$-antimagic labeling studied in [6]. They consider injections from $E(G)$ to $\{1, 2, ..., |E(G)| + k\}$ such that all vertex sums are pairwise distinct.

Recall that the set $V_k$ of a graph consists of vertices of degree $k$. For any graph, let $V_{\geq 3}$ be the set of vertices of degree at least three. Kaplan, Lev and Roditty [7] proved that for a tree, if the set $|V_2| \leq 1$, then it is antimagic. Our strongly antimagic double spiders together with the known results on spiders and paths can be rephrased as following: For a tree, if the set $|V_{\geq 3}| \leq 2$, then it is antimagic. If we have both large $V_2$ and $V_{\geq 3}$ in the tree, then the problem turns out to be more difficult. We explain the reasons. Note that $|V_1|$ must be larger than $|V_{\geq 3}|$ by the simple fact that the average degree of a tree is less than two. Hence, large $V_2$ and $V_{\geq 3}$ leads to large $V_2$ and $V_1$. If we label the edges at random, then the vertex sum of a vertex in $V_2$ has fifty percent likelihood to be smaller than $|E|$, which is very likely to coincide with the vertex sums of vertices in $V_1$. A very recently result [8] is that for a caterpillar, if $|V_1| \geq \frac{1}{2}(3(|V_2| + |V_{\geq 3}| + 1))$, then it is antimagic. Until the paper is completed, we do not have an affirmative answer of Conjecture 2 for all caterpillars yet.

**Acknowledgment**

The first and third authors would like to thank Alfréd Rényi Institution of Mathematics for host on August, 2017, in Hungary.

**References**

[1] N. Alon, G. Kaplan, A. Lev, Y. Roditty, and R. Yuster, *Dense graphs are antimagic*, J. Graph Theory, 47 (2004), 297-309.

[2] K. Bérczi, A. Bernáth, and M. Vizer, *Regular graphs are antimagic*, The Electronic Journal of Combinatorics 22 (2015), paper P3.34

[3] F. Chang, Y.-C. Liang, Z. Pan, X. Zhu, *Antimagic labeling of regular graphs*, J. Graph Theory 82 (2016), 339-349.

[4] D. W. Cranston, *Regular bipartite graphs are antimagic*, J. Graph Theory 60 (2009), 173-182.
[5] N. Hartsfield and G. Ringel, *Pearls in Graph Theory*, Academic Press, INC., Boston, 1990, pp. 108-109, Revised version 1994.

[6] D. Hefetz, *Anti-magic graphs via the Combinatorial Nullstellensatz*, J Graph Theory 50 (2005), 263-272

[7] G. Kaplan, A. Lev and Y. Roditty, *On zero-sum partitions and antimagic trees*, Discrete Math. 309 (2009), 2010-2014.

[8] A. Lozano, M. Mora and C. Seara, *Antimagic Labeling of Caterpillars*, ArXiv1708.00624

[9] Y. Liang, T. Wong and X. Zhu, *Anti-magic labeling of trees*, Discrete Math. 331 (2014), 9-14.

[10] T.-Y. Huang, *Antimagic Labeling on Spiders*, Master Thesis, Department of Mathematics, National Taiwan University.(2015)

[11] J.-L. Shang, *Spiders are antimagic*, Ars Combinatoria, 118 (2015), 367-372.

[12] T. Wong and X. Zhu. *Antimagic labelling of vertex weighted graphs*, Journal of Graph Theory. 70(3), (2012), 348V350.

[13] T.-M. Wang and C. C. Hsiao, *On anti-magic labeling for graph products*, Discrete Math. 308(16), (2008), 3624V3633.
5 Appendix

5.1 Rest of the Proof of Lemma 4 and Lemma 5

Claim: \( \varphi_f(v_r) > \varphi_f(u) \) for any \( u \in V_2 \).

Let \( u_2 \) be the vertex in \( V_2 \) with the largest vertex sum. If \( s = 1 \), we have

\[
f(e^r_{1,1}) + f(e^r_{2,1}) \geq m - 1,
\]

by Equalities (1) or (2), and (3). Moreover, \( s = 1 \) implies that we label \( m \) to the core edge which incident to \( v_r \) and \( v_l \). So \( \varphi_f(u_2) \leq (m - 1) + (m - 2) < 2m - 1 \leq f(e_1) + f(e^r_{1,1}) + f(e^r_{2,1}) = \varphi_f(v_r) \).

If \( s \geq 2 \), then \( u_2 = v_s \) since it is incident to \( e_s \), the last labeled edge. By Equalities (3) and (4), we have

\[
f(e_{s-1}) = f(e^r_{2,1}) + C_c^{\text{even}} + \begin{cases} 1, & \text{if } s \text{ is even;} \\ \frac{s-1}{2}, & \text{if } s \text{ is odd;} \end{cases}
\]

Recall that \( \left\lfloor \frac{k}{2} \right\rfloor + c + w' + D_d^{\text{odd}} + t > 0 \). In Equation (2), we have

\[
f(e^r_{1,1}) \geq \left\lfloor \frac{k}{2} \right\rfloor + C_c^{\text{odd}} + w' + s_1 + D_d^{\text{odd}} + t + 1 \\
= s_1 + C_c^{\text{even}} + 1 + \left( \left\lfloor \frac{k}{2} \right\rfloor + c + w' + D_d^{\text{odd}} + t \right) \\
> s_1 + C_c^{\text{even}} + 1 \geq f(e_{s-1}) - f(e^r_{2,1}),
\]

and hence

\[
\varphi_f(v_r) = f(e^r_{1,1}) + f(e^r_{2,1}) + f(e_s) > f(e_{s-1}) + f(e_s) = \varphi_f(u_2).
\]

Claim: \( \varphi_f(v_l) > \varphi_f(v_r) \).

When \( t' = 1 \), we have \( f(e^1) = f(e^r_{1,1}) + 1 \) by the rules in Step 5. Thus, \( \varphi_f(v_r) = f(e_s) + f(e^r_{1,1}) + f(e^r_{2,1}) = (m - 1) + f(e^1) + f(e^r_{2,1}) \). Note that \( m - 1 \) is assigned to an edge \( e \) at Step 9, or Step 10, or Step 11. So, \( e \in E(v_l) \setminus \{e^1\} \).

If \( f(e_1) = m - 1 \) (\( s \) is odd and greater than 3), then there exists an edge in \( E(v_l) \) labeled at Step 9 or 10, whose label is \( m - 2 \) and greater than \( f(e^r_{2,1}) \). So

\[
\varphi_f(v_l) > (m - 1) + f(e^1) + f(e^r_{2,1}) = \varphi_f(v_r).
\]

If \( f(e) = m - 1 \) for some \( e \in E(v_l) \setminus \{e_1, e^1\} \), then \( e_1 \) is labeled at Step 11 when \( s = 1 \) or it is labeled at Step 8 when \( s \) is even. In the former case, we have \( f(e^r_{2,1}) < m - 1 \) and hence

\[
\varphi_f(v_l) \geq f(e^1) + 2m - 1 > f(e^r_{2,1}) + f(e^r_{1,1}) + m = \varphi_f(v_r).
\]
In the latter case, we have $f(e_1) > f(e'_{2,1})$, and hence

$$\varphi_f(v_t) \geq f(e') + f(e_1) + m - 1 > f(e'_{2,1}) + f(e'_{1,1}) + m = \varphi_f(v_r).$$

When $t' = 0$, recall that $\deg(v_t) = c + d + t + 1 \geq 3$. We classify the possible values of $c$, $d$, and $t$.

Case 1. $c + d \geq 2$.

Subcase 1.1. $d \geq 1$.

Then we can pick two edges $e', e'' \in E(v_t)$ labeled at Step 9 and Step 10, whose labels are both greater than $f(e'_{2,1})$ and $f(e'_{1,1})$. If $f(e_1) \in \{m, m - 1\}$, then

$$\varphi_f(v_t) \geq f(e') + f(e'') + f(e_1) > f(e'_{2,1}) + f(e'_{1,1}) + f(e_s) = \varphi_f(v_r).$$

If $f(e_1) \not\in \{m, m - 1\}$, then we use $f(e_1) > f(e'_{1,1})$, and hence

$$\varphi_f(v_t) \geq (m - 1) + (m - 2) + f(e_1) > m + f(e'_{2,1}) + f(e'_{1,1}).$$

Subcase 1.2. $d = 0$.

If $s = 1$, we have $f(e_{c,2w+1}^{l,odd}) + 1 = f(e'_{1,1})$. Moreover, $f(e_{1,2w+1}^{l,odd} - f(e'_{2,1}) \geq 2$ since $C_{c,even}^{even} \geq 2$. Therefore,

$$\varphi_f(v_t) \geq f(e_s) + f(e_{1,2w+1}^{l,odd}) + f(e_{c,2w+1}^{l,odd})$$

$$> f(e_s) + f(e'_{2,1}) + f(e'_{1,1}) = \varphi_f(v_r).$$

If $s$ is odd and greater than 1, we have $f(e_{c,2w+1}^{l,odd}) + \frac{s - 3}{2} + 1 = f(e'_{1,1})$ and $f(e_{1,2w+1}^{l,odd} - f(e'_{2,1}) \geq C_{c,even}^{even} + \frac{s - 1}{2} + 1$. Thus,

$$\varphi_f(v_t) \geq f(e_1) + f(e_{1,2w+1}^{l,odd}) + f(e_{c,2w+1}^{l,odd})$$

$$\geq (m - 1) + [f(e_{1,1}) - \frac{s - 3}{2} + 1] + [f(e_{2,1}^{r,odd} + C_{c,even}^{even} + \frac{s - 1}{2} + 1]$$

$$> f(e_s) + f(e'_{2,1}) + f(e'_{1,1}) = \varphi_f(v_r).$$

If $s$ is even, we have $f(e_{c,2w+1}^{l,odd}) + \frac{s - 2}{2} + 1 = f(e_{1,1}^{r,odd})$, and by Equalities (3) and (4), we have $f(e_1) - f(e_{2,1}) = \frac{s}{2} + C_{c,even}^{even} \geq \frac{s}{2} + 2$. Then

$$\varphi_f(v_t) \geq f(e_1) + f(e_{1,2w+1}^{l,odd}) + f(e_{c,2w+1}^{l,odd})$$

$$\geq (f(e_{2,1}^{r,odd}) + \frac{s}{2} + 2) + [f(e_{1,1}^{r,odd}) - \frac{s - 2}{2} + 1] + (m - 1)$$

$$> f(e_s) + f(e'_{2,1}) + f(e'_{1,1}) = \varphi_f(v_r).$$
Case 2. $c + d = 1$.

In this case, $t = 1$ by the fact $d + c + t = \deg(v_l) \geq 3$ and reduction.

Subcase 2.1. $c = 1$ and $d = 0$.

Since the special case of $c = 1, d = 0, t = 1, w_1 = 1, s = 2$ and $k = 1$ has been handled separately as illustrated in Figure 2, we may assume at least one of the conditions $w_1 \geq 2, s \neq 2,$ and $k \geq 2$ holds. Because $t = 1$, we have $f(e^1) + 1 = f(e_{1,1}^{r,odd})$.

If $s = 1$, by Equality (3), $f(e_{1,2w_1+1}^{l,odd}) - f(e_{2,1}^{r}) = w_1 + 1 \geq 2$. Then

$$\varphi_f(v_l) = f(e_{1,2w_1+1}^{l,odd}) + f(e^1) + f(e_1) > f(e_{2,1}^{r}) + f(e_{1,1}^{l,odd}) + f(e_1) = \varphi_f(v_r).$$

If $s$ is odd and greater than one, then $f(e_1) = m - 1$. By Equalities (3) and (4), we have $f(e_{1,2w_1+1}^{l,odd}) - f(e_{2,1}^{r}) = w_1 + \frac{s + 1}{2} \geq 3$. Then

$$\varphi_f(v_l) = f(e_{1,2w_1+1}^{l,odd}) + f(e^1) + f(e_1) > f(e_{2,1}^{r}) + f(e_{1,1}^{l,odd}) + f(e_s) = \varphi_f(v_r).$$

If $s$ is even, then $f(e_{1,2w_1+1}^{l,odd}) = m - 1$. By Equalities (3) and (4), we have $f(e_1) - f(e_{2,1}^{r}) = w_1 + \frac{s}{2} \geq 3$. Then

$$\varphi_f(v_l) = f(e_{1,2w_1+1}^{l,odd}) + f(e^1) + f(e_1) > f(e_{2,1}^{r}) + f(e_{1,1}^{l,odd}) + f(e_s) = \varphi_f(v_r).$$

Subcase 2.2. $c = 0$ and $d = 1$.

This cannot happen since $t = 1$ and $d = 1$ will imply $t' = 1$.

5.2 Rest of the Proof of Lemma 6

The conditions $\deg(v_r) \geq 3$ and $b = 0$ imply $a \geq 2$. Without loss of generality, assume the length of the $a$-th odd path in $L$ is at least 3. Since $\deg(v_l) \geq 4, s \geq 1, 2x_a + 1 \geq 3$, and $2x_1 + 1 \geq 1$, we have the total number of edges $m \geq D_{d}^{even} + z_d + 7$.

We make some observations.

- At Step 5, if $t \geq 2$, we have
  \[
  f(e^t) + f(e^{t-1}) = (A_a^{odd} - 1 + C_c^{odd} - c + s_1 + D_d^{odd} + t) \\
  + (A_a^{odd} - 1 + C_c^{odd} - c + s_1 + D_d^{odd} + t - 1) \\
  = (A^{all} + a - 2) + (C^{all} - c) + 2s_1 + D^{all} + 2(t - 1) \\
  = m + a - c + t - (2s_1 - s - 3) \\
  \]

- At Step 9, if $d \geq 2$, we have
  \[
  f(e_{d,2z_d}^{l,even}) \geq m - c - 3 \text{ and } f(e_{d-1,2z_{d-1}}^{l,even}) \geq m - c - z_d - 3. 
  \]
• At Step 10, if \( s \) is even, then
\[
f(e_1) = f(e_{a,1}^{r,\text{odd}}) - D_d^{\text{even}} - 1,
\]
and, by the order we labeled the edges \( e_{a,1}^{r,\text{odd}}, e_{s-1}, \) and \( e_{c,2w_c}^{l,\text{odd}} \), we have
\[
f(e_{a,1}^{r,\text{odd}}) > f(e_{s-1}) > f(e_{c,2w_c}^{l,\text{odd}}).
\]
Moreover, we have
\[
f(e_{a,1}^{r,\text{odd}}) = \begin{cases} m - c - 1, & \text{if } s = 1 \text{ or } s \text{ is even.} \\ m - c - 2, & \text{otherwise}. \end{cases}
\]

• At Step 11, if \( c \geq 2 \), we have
\[
f(e_{c,2w_c+1}^{l,\text{odd}}) \geq m - 2 \text{ and } f(e_{c-1,2w_c-1+1}^{l,\text{odd}}) \geq m - 3.
\]

• At Step 12, if \( s \) is odd, \( f(e_1) = f(e_{a,1}^{r,\text{odd}}) + c + 1 \). With Equality (7), we have
\[
f(e_1) = \begin{cases} f(e_{a,1}^{r,\text{odd}}) + c + 1, & \text{if } s \text{ is odd.} \\ f(e_{a,1}^{r,\text{odd}}) - D_d^{\text{even}} - 1, & \text{if } s \text{ is even.} \end{cases}
\]

Claim: \( \phi_f(v_r) > \phi_f(u) \) for any \( u \in V_2 \).
Let \( u_2 \) be the vertex of the largest vertex sum in \( V_2 \). Then, we have
\[
u_2 = \begin{cases} v_s, & \text{if } s > 1. \\ v_{c,2w_c+1}^{l,\text{odd}}, & \text{if } s = 1, c > 0. \\ v_{a,1}^{r,\text{odd}}, & \text{if } s = 1, c = 0. \end{cases}
\]
By Inequality (3) and \( f(e_s) = m > f(e_{c,2w_c+1}^{l,\text{odd}}) > f(e_{a,2}^{r,\text{odd}}) \), the vertex sum at \( v_r \) is
\[
\phi_f(v_r) = \sum_{i=1}^{a-1} f(e_{i,1}^{r,\text{odd}}) + f(e_{a,1}^{r,\text{odd}}) + f(e_s) > f(e_{a,1}^{r,\text{odd}}) + f(e_s)
= \begin{cases} f(e_{s-1}) + f(e_s) = \phi_f(v_s), & \text{if } s > 1; \\ f(e_{c,2w_c}^{l,\text{odd}}) + f(e_{c,2w_c+1}^{l,\text{odd}}) = \phi_f(v_{c,2w_c+1}^{l,\text{odd}}), & \text{if } s = 1, c > 0; \\ f(e_{a,1}^{r,\text{odd}}) + f(e_{a,2}^{r,\text{odd}}) = \phi_f(v_{a,1}^{r,\text{odd}}), & \text{if } s = 1, c = 0; \end{cases}
\]

Claim: \( \phi_f(v_l) > \phi_f(v_r) \).
Recall that \( \deg(v_l) > \deg(v_r) \geq 3 \), and for any \( e \in E(v_l) \), we have \( f(e) > f(e_{i,1}^{r,\text{odd}}) \) for \( 1 \leq i \leq a - 1 \). Thus, if we can find three edges in \( E(v_l) \) such that the sum of the labels is not less than the sum of of the maximal two labels of the edges in \( E(v_r) \), namely \( f(e_s) + f(e_{a,1}^{r,\text{odd}}) \), then we are done. Recall that \( \deg(v_l) = c + d + t + 1 \). The choice of the three edges in \( E(v_l) \) depends on the values of \( c, d, \) and \( t \):
Case 1. \( c \geq 2 \)

By Inequality (10) and Equality (11), and \( m \geq D_d^{\text{even}} + z_d + 7 \),

\[
f(e_{c,2w+1}^{l,\text{odd}}) + f(e_{c-1,2w+1}^{l,\text{odd}}) + f(e_1) \geq (m - 2) + (m - 3) + \left(f(e_{a,1}^{r,\text{odd}}) - D_d^{\text{even}} - 1\right) = (m + f(e_{a,1}^{r,\text{odd}})) + (m - D_d^{\text{even}} - 6) > f(e_s) + f(e_{a,1}^{r,\text{odd}}).
\]

Case. 2 \( c = 1 \)

Subcase 2.1. \( d \geq 1 \).

By Inequalities (6) and (10), and Equality (11),

\[
f(e_{c,2w+1}^{l,\text{odd}}) + f(e_{d,2z_d}^{l,\text{even}}) + f(e_1) \geq (m - 2) + (m - c - 3) + \left(f(e_{a,1}^{r,\text{odd}}) - D_d^{\text{even}} - 1\right) = (m + f(e_{a,1}^{r,\text{odd}})) + (m - D_d^{\text{even}} - 7) > f(e_s) + f(e_{a,1}^{r,\text{odd}}).
\]

Subcase 2.2. \( d = 0 \).

By Inequality (10) and Equality (11),

\[
f(e_{c,2w+1}^{l,\text{odd}}) + f(e) + f(e_1) \geq (m - 2) + \left(A_a^{\text{odd}} - 1 + C_c^{\text{odd}} - c + s_1 + D_d^{\text{odd}} + t\right) + \left(f(e_{a,1}^{r,\text{odd}}) - D_d^{\text{even}} - 1\right) = (m + f(e_{a,1}^{r,\text{odd}})) + (A_a^{\text{odd}} + C_c^{\text{odd}} + s_1 + t - 5) > f(e_s) + f(e_{a,1}^{r,\text{odd}}).
\]

Case 3. \( c = 0 \).

Subcase 2.1. \( d \geq 2 \).

By Inequality (6) and Equality (11),

\[
f(e_{d,2z_d}^{l,\text{even}}) + f(e_{d-1,2z_d-1}^{l,\text{even}}) + f(e_1) \geq (m - 3) + (m - z_d - 3) + \left(f(e_{a,1}^{r,\text{odd}}) - D_d^{\text{even}} - 1\right) = (m + f(e_{a,1}^{r,\text{odd}})) + (m - D_d^{\text{even}} - z_d - 7) \geq f(e_s) + f(e_{a,1}^{r,\text{odd}}).
\]

Subcase 2.2. \( d = 1 \).
Then we have \( t \geq 2 \). By Inequality (6), Equality (11), and \( A_a^{\text{odd}} \geq 3 \),
\[
    f(e_{d,2d}^{\text{even}}) + f(e') + f(e_1) \geq (m - 3) + (A_a^{\text{odd}} - 1 + C_c^{\text{odd}} - c + s_1 + D_d^{\text{odd}} + t) + (f(e_{a,1}^{r,\text{odd}}) - D_d^{\text{even}} - 1)
    \geq (m + f(e_{a,1}^{r,\text{odd}})) + (A_a^{\text{odd}} + s_1 + t - 5)
    \geq f(e_s) + f(e_{a,1}^{r,\text{odd}}).
\]

Subcase 2.3. \( d = 0 \)

Then \( t \geq 3 \). If \( s \) is odd, by Equalities (5) and (11), and \( a \geq 2 \),
\[
    f(e') + f(e^{t-1}) + f(e_1) \geq m + t + a - c - 6 + (f(e_{a,1}^{r,\text{odd}}) + c + 1)
    = (m + f(e_{a,1}^{r,\text{odd}})) + (t + a - 5)
    \geq f(e_s) + f(e_{a,1}^{r,\text{odd}}).
\]

If \( s \) is even, by Equality (5),
\[
    f(e') + f(e^{t-1}) + f(e_1) \geq m + t + a - c - 5 + (f(e_{a,1}^{r,\text{odd}}) - D_d^{\text{even}} - 1)
    = (m + f(e_{a,1}^{r,\text{odd}})) + (t + a - 6)
    \geq f(e_s) + f(e_{a,1}^{r,\text{odd}}) + (t + a - 6).
\]

The quantity \( t + a - 6 \) in the above inequality is negative only if \( t = 3 \) and \( a = 2 \). However, we have \( f(e^1) \geq (A_a^{\text{odd}} - 1) + 1 \geq 3 \) and \( f(e_{1,1}^{r,\text{odd}}) = 1 \). So
\[
    \varphi_f(v_t) = f(e^1) + f(e') + f(e^{t-1}) + f(e_1)
    \geq 3 + (m + f(e_{a,1}^{r,odd})) + (t + a - 6)
    > 1 + f(e_s) + f(e_{a,1}^{r,odd})
    = f(e_{1,1}^{r,odd}) + f(e_s) + f(e_{a,1}^{r,odd})
    = \varphi_f(v_r).
\]

5.3 Rest of the Proof of of Lemma 7

We make some observations.

- From Step 1, Step 8, and Step 9, we have
  \[
  f(e^i) > f(e_{i,1}^{r,\text{even}}) \quad \text{for } i \in [\alpha], \quad \text{and } f(e^i) > f(e_{i'}^{r,\text{odd}}) \quad \text{for } i \in [\beta, t] \quad \text{and } i' \in [a].
  \] (12)

- From Step 16 and Step 18, we have
  \[
  f(e) > f(e').
  \] (13)

for \( e \in \{e_{i,2d_{i+1}}, i \in [c]\} \cup \{e_{i,2d_{i}}, i \in [d]\} \) and \( e' \in E(v_r) \setminus \{e_s\} \).
• At Step 9, if \( t \geq 2 \), we have
\[
f(e^t) = B_e^{even} + A_u^{odd} + C_v^{odd} - c + s_1 + D_d^{odd} + t, \quad \text{if } t \geq 1,
\]
and
\[
f(e^{t-1}) = B_e^{even} + A_u^{odd} + C_v^{odd} - c + s_1 + D_d^{odd} + t - 1, \quad \text{if } t \geq 2.
\]

• At Step 14, if \( s \) is even, then we have
\[
f(e_1) = m - y_b - (\alpha - \beta) - c - D_d^{even} - 1.
\]

• At Step 15, after labeling \( e_{b,1}^{r,even} \), we have
\[
f(e_{b,1}^{r,even}) = m - D_d^{even} - (y_b - 1) - (\alpha - \beta) - c - \begin{cases} 1, & \text{if } s = 1 \text{ or } s \text{ is even,} \\ 2, & \text{if } s \geq 3 \text{ and is odd.} \end{cases}
\]
Moreover, when \( s \geq 2 \)
\[
f(e_{b,1}^{r,even}) > f(e_{s-1}).
\]

• By the order we labeled edges on \( E \), we have
\[
f(e_{b,1}^{r,even}) > f(e_{c,2w_1}^{l,odd}) > f(e_{a,3}^{r,even}) > f(e_{d,2z_1-1}^{r,even}) > f(e_{b,2y_b}^{r,even}).
\]

**Claim:** \( \varphi_f(v_r) > \varphi_f(u) \) for any \( u \in V_2 \).

Let \( u_2 \) be the vertex in \( V_2 \) with the largest vertex sum. If \( s = 1 \), then
\[
\varphi_f(u_2) = \begin{cases} \varphi_f(v_{c,2w_1+1}^{l,odd}) = f(e_{c,2w_1}^{l,odd}) + f(e_{c,2w_1}^{l,odd}), & \text{if } c > 0; \\ \varphi_f(v_{a,2}^{l,even}) = f(e_{a,3}^{r,even}) + f(e_{a,2}^{r,even}), & \text{if } c = 0, (\alpha - \beta) > 0; \\ \varphi_f(v_{d,2z_1}^{l,even}) = f(e_{d,2z_1}^{l,even}) + f(e_{d,2z_1}^{l,even}), & \text{if } c = 0, (\alpha - \beta) = 0, d > 0; \\ \varphi_f(v_{b,2y_b-1}^{l,even}) = f(e_{b,2y_b-1}^{l,even}) + f(e_{b,2y_b-1}^{l,even}), & \text{otherwise.} 
\end{cases}
\]

By Inequality (19) and \( f(e_1) = m \), we have
\[
\varphi_f(v_r) > f(e_{b,1}^{r,even}) + f(e_1) > \varphi_f(u_2).
\]

If \( s \geq 2 \), then \( u_2 = v_s \). By Inequality (18), we have \( \varphi_f(v_r) > \varphi_f(u_2) \).

**Claim:** \( \varphi_f(v_1) > \varphi_f(v_r) \).

The idea is similar to that in the proof of Lemma 6. We will choose \( k + 1 \) edges in \( E(v_1) \) and \( k \) edges in \( E(v_r) \) such that the sum of the labels of the \( k + 1 \) edges in \( E(v_1) \) is not less than the sum of the labels of the \( k \) edges in \( E(v_r) \). Moreover, for other edges \( e' \in E(v_1) \) and \( e'' \in E(v_1) \) which are not chosen, \( f(e') > f(e'') \) holds.

Case 1. \( s = 1 \).

If \( t \leq 1 \), by Inequalities (12) and (13), we have \( \varphi_f(v_1) > \varphi_f(v_r) \).

If \( t \geq 2 \), by Equalities (14), (15), and (17), we have
\[
f(e^t) + f(e^{t-1}) = m + a - c + t - 2 > f(e_{b,1}^{r,even}).
\]

With Inequalities (12) and (13), \( \varphi_f(v_1) > \varphi_f(v_r) \).
Case 2. \( s \geq 2 \).

If \( t = 0 \), we have either \( f(e_{c,2w_1+1}^{\text{odd}}) + f(e_1) > f(e_s) \) or \( f(e_{d,2z_1}^{\text{even}}) + f(e_1) > f(e_s) \); and if \( t = 1 \), by Equalities (14) and (16), \( f(e^t) + f(e_1) > f(e_s) \) holds. With Inequalities (12) and (13), we have \( \varphi_f(v_t) > \varphi_f(v_r) \).

For \( t \geq 2 \), note \( f(e^t) + f(e^{t-1}) > f(e_{b,r}^{\text{even}}) \) and if \( s \) is odd, then \( f(e_1) = m - 1 \). So \( f(e^t) + f(e^{t-1}) + f(e_1) \geq f(e_s) + f(e_{b,r}^{\text{even}}) \). With Inequalities (12) and (13), we have \( \varphi_f(v_t) > \varphi_f(v_r) \). If \( s \) is even, we need compare more edges. First we have \( f(e_1) = f(e_{b,r}^{\text{even}}) + 1 \) and, by Equalities (14) and (15), \( f(e^t) + f(e^{t-1}) = m + a - c + t - 3 \).

Subcase 2.1. \( c \geq 1 \).

We compare the sum of the labels of the edges \( e_{c,2w_1+1}^{\text{odd}}, e^t, e^{t-1} \), an \( e_1 \) in \( E(v_t) \) and the sum of maximal three labels of edges in \( E(v_r) \). Let

\[
r = \max\{f(e) \mid e \in E(v_r) \setminus \{e_s, e_{b,r}^{\text{even}}\}\}.
\]

Then \( f(e_{c,2w_1+1}^{\text{odd}}) - r > c + 3 \), and hence

\[
f(e_{c,2w_1+1}^{\text{odd}}) + f(e^t) + f(e^{t-1}) + f(e_1) > m + f(e_{b,r}^{\text{even}}) + r.
\]

With Inequalities (12) and (13), \( \varphi_f(v_t) > \varphi_f(v_r) \) holds.

Subcase 2.2. \( c = 0 \).

If \( t > 3 \) or \( a \geq 1 \), then \( m + a - c + t - 3 \geq m + 1 \) and \( f(e^t) + f(e^{t-1}) + f(e_1) \geq f(e_s) + f(e_{b,r}^{\text{even}}) \). The remaining cases are \( t = 2 \) and \( a = 0 \), or \( t = 3 \) and \( a = 0 \). Note that \( a = 0 \) implies \( b \geq 2 \), because \( \deg(v_r) \geq 3 \). If \( d > 0 \), no matter \( t = 2 \) or \( t = 3 \), \( f(e_{d,2z_1}^{\text{even}}) - f(e_{b-1,1}^{\text{even}}) > 2 \). So

\[
f(e_{d,2z_1}^{\text{even}}) + f(e^t) + f(e^{t-1}) + f(e_1) > m + f(e_{b,r}^{\text{even}}) + f(e_{b-1,1}^{\text{even}}).
\]

With Inequalities (12) and (13), \( \varphi_f(v_t) > \varphi_f(v_r) \).

If \( d = 0 \), by \( \deg(v_t) > \deg(v_r) \), we have \( t = 3 \) and \( b = 2 \). Hence \( \beta = 1 \). Then \( f(e^{t-2}) = f(1) = 3 \) and \( f(e_{1,1}^{r,\text{even}}) = 1 \). Therefore,

\[
\varphi_f(v_t) = f(e^t) + f(e^{t-1}) + f(e^{t-2}) + f(e_1) > m + f(e_{b,r}^{\text{even}}) + f(e_{b-1,1}^{r,\text{even}}) = \varphi_f(v_r).
\]