A RECONSTRUCTION THEOREM FOR VARIETIES

MAX LIEBLICH AND MARTIN OLSSON

ABSTRACT. We show that varieties of dimension at least 2 over infinite fields are determined as abstract schemes by their Zariski topological spaces together with the rational equivalence relation on the set of effective divisors. This gives a universal Torelli theorem in the sense of Bogomolov and Tschinkel. The proof relies heavily on a rational version of the classical Fundamental Theorem of Projective Geometry.

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1. INTRODUCTION

The underlying topological space $|X|$ of a smooth projective variety $X$ over a field $k$ is a rather weak invariant. For example, for a smooth projective curve $X/k$ the topological space $|X|$ is determined by the cardinality of the set of points. As we discuss further below in Lemma 7.1, there are also many examples of homeomorphic smooth projective surfaces (over algebraic closures of finite fields)\footnote{in fact, examples of homeomorphic surfaces over fields of different characteristics} that are not isomorphic.

The present paper is a reflection on what additional structures on $|X|$ enable one to recover the scheme $X$ or of large parts of $X$ (for example, the birational equivalence class of $X$, or the codimension 1 structure of $X$). There is a substantial literature on related questions. In particular, we mention the work of Bogomolov–Korotiaev–Tschinkel\footnote{in fact, examples of homeomorphic surfaces over fields of different characteristics} and subsequent work of Zilber\footnote{in fact, examples of homeomorphic surfaces over fields of different characteristics}.

The main result of this paper, stated here in the introduction somewhat informally and with slightly stronger assumptions than in the body of the text, is the following.

**Main Theorem** (Universal Torelli, proper case). Let $X$ be a scheme satisfying the following conditions:

(i) $\Gamma(X, \mathcal{O}_X)$ is an infinite field.
(ii) $X$ is geometrically integral over $\Gamma(X, \mathcal{O}_X)$ and of dimension $\ge 2$. 
(iii) The map \( X \to \text{Spec} \Gamma(X, \mathcal{O}_X) \) is proper.
(iv) \( X \) is normal.

Then \( X \) is uniquely determined by the pair
\[
(|X|, c : X^{(1)} \to \text{Cl}(X)),
\]
where \( |X| \) is the underlying Zariski topological space, \( \text{Cl}(X) \) is the group of Weil divisor classes, \( X^{(1)} \) is the set of codimension 1 points of \( |X| \), and \( c \) is the map sending a codimension 1 point of \( X \) to its divisor class.

Equivalently, \( X \) is determined by its underlying topological space \( |X| \) and the rational equivalence relation on the set of effective divisors.

The full statement of the Main Theorem is Theorem 2.13 below. Note that a smooth geometrically connected projective variety over an infinite field satisfies the assumptions of the theorem.

As an example, observe that the theorem implies that for \( n \geq 4 \), any Zariski homeomorphism of hypersurfaces in \( \mathbb{P}^n_K \), with \( K \) a number field, that preserves degrees of divisors induces an isomorphism of the underlying \( \mathbb{Q} \)-schemes. This is the best one could hope for: the group \( \text{Gal}(K/\mathbb{Q}) \) acts on \( \mathbb{P}^n_K \) by degree-preserving Zariski homeomorphisms.

Remark 1.1. The Main Theorem is a Torelli-type result in the following sense. One can think of Torelli’s theorem as a statement about adding a small amount of geometric content to the cohomology of a variety in order to distinguish distinct algebraic structures on a fixed differentiable manifold. The Main Theorem starts with the Zariski topology, which already encodes some of the algebraic structure – for example, the algebraic cycles – and adds the class group, which encodes the finest possible “cohomological relation” among divisors.

The key to proving the Main Theorem is a rational form of the Fundamental Theorem of Projective Geometry, which we develop in Section 4. In Section 5, we leverage the incidence-definition of a Zariski open set of pencils in certain linear systems to recover the linear structure on set-theoretic rational equivalence classes using the rational fundamental theorem. This is inspired by work of Bogomolov and Tschinkel who used similar ideas to reconstruct function fields [2].

1.1. Conventions. In this paper, we freely use the theory of projective structures, as described in [6,7] and summarized in [2, Section 3]. We will not recapitulate the theory here.

Given a commutative monoid \((M, +)\), we will call an equivalence relation \( \Lambda \) on \( M \) a congruence relation if for all \( a, b, c, d \in M \), we have that \((a, c) \in \Lambda \) and \((b, d) \in \Lambda \) imply that \((a + c, b + d) \in \Lambda \).

For a vector space \( V \) over a field \( k \) we write \( PV \) for the projective space of lines in \( V \). This convention makes the discussion of classical projective geometry easier, though it conflicts with the conventions of EGA.

Given a variety \( X \) and a divisor \( D \), we will write \(|D|\) for the classical linear system of \( D \), that is \(|D| = \text{P} H^0(X, \mathcal{O}(D))\). We will write \(|D|^\vee\) for the dual projective space \( \text{P} H^0(X, \mathcal{O}(D))^\vee \) (i.e., the space of hyperplanes in \(|D|\)), which is the natural target for the induced rational map \( \nu_D : X \dashrightarrow |D|^\vee \). When the base field is algebraically closed,
the closed points of the image of $\nu_D$ correspond to the hyperplanes $H_x \subset |D|$, where $H_x = \{ E \in |D| : x \in E \}$.  

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2. **Divisorial Structures**

In this section we introduce the key structure that will ultimately be the subject of our main reconstruction theorem. Recall that, given a Zariski topological space $Z$, an effective divisor is a formal finite sum $\sum a_i x_i$, where each $x_i \in X$ is a point of codimension 1 and each $a_i$ is positive. We denote the set of effective divisors on $Z$ by $\text{Eff}(Z)$. When $X$ is a scheme, we will write $\text{Eff}(X)$ for $\text{Eff}(|X|)$.

**Definition 2.1.** A scheme $X$ is strongly integral if it is integral and $\Gamma(X, \mathcal{O}_X)$ is a field.

**Notation 2.2.** If $X$ is strongly integral, we will write $\kappa_X$ for the field $\Gamma(X, \mathcal{O}_X)$.

**Definition 2.3.** An absolute variety is a separated strongly integral scheme $X$ such that the canonical morphism $X \to \text{Spec}\,\kappa_X$ is of finite type with integral geometric fibers. An absolute variety $X$ has infinite constant field if $\kappa_X$ is infinite.

**Definition 2.4.** A normal separated $k$-scheme $X$ is divisorially proper over $k$ if for any reflexive sheaf $L$ of rank 1 we have that $\Gamma(X, L)$ is finite-dimensional over $k$.

**Lemma 2.5.** If a $k$-scheme $X$ is normal, separated, and divisorially proper over $k$ and $U \subset X$ is an open subscheme such that $\text{codim}(X \setminus U \subset X) \geq 2$ at every point, then $U$ is also divisorially proper over $k$.

**Proof.** Any reflexive sheaf $L$ of rank 1 on $U$ is the restriction of a reflexive sheaf $L'$ of rank 1 on $X$, and Krull's theorem tells us that the restriction map

$$\Gamma(X, L') \to \Gamma(U, L)$$

is an isomorphism of $k$-vector spaces. \hfill \Box

**Definition 2.6.** A absolute variety $X$ is definable if it is normal and divisorially proper over $\Gamma(X, \mathcal{O}_X)$. We say that $X$ is polarizable if it admits an ample invertible sheaf.

Write $\mathcal{V}_{\text{var}}$ for the category whose objects are absolute varieties and whose morphisms are open immersions $f : X \to Y$ such that $Y \setminus f(X)$ has codimension at least 2 in $Y$ at every point. We will write $\mathcal{D}_{\text{ef}} \subset \mathcal{V}_{\text{var}}$ for the full subcategory of definable schemes.

**Definition 2.7.** A divisorial structure is a pair $(Z, \Lambda)$ with $Z$ a Zariski topological space and $\Lambda$ a congruence relation on the monoid $\text{Eff}(Z)$.
**Definition 2.8** (Restriction of a divisorial structure). Suppose \( t := (Z, \Lambda) \) is a divisorial structure. Given an open subset \( U \subset Z \), the *restriction of \( t \) to \( U \)*, denoted \( t|_U \), is the divisorial structure \((U, \Lambda_{\text{Eff}(U)})\), where \( \Lambda_{\text{Eff}(U)} \) is the induced relation on the quotient monoid \( \text{Eff}(X) \to \text{Eff}(U) \).

In other words, if we let \( \text{Eff}(X) \to Q \) denote the quotient by \( \Lambda \), we define the congruence relation on \( \text{Eff}(U) \) by forming the pushout

\[
\begin{array}{ccc}
\text{Eff}(X) & \longrightarrow & Q \\
\downarrow & & \downarrow \\
\text{Eff}(U) & \longrightarrow & Q_U \\
\end{array}
\]

in the category of commutative monoids.

Alternatively, recall that the condition that an equivalence relation \( \Lambda \subset \text{Eff}(Z) \times \text{Eff}(Z) \) is a congruence relation is equivalent to the condition that \( \Lambda \) is a submonoid. The congruence relation on \( \text{Eff}(U) \) induced by \( \Lambda \) is simply the image of \( \Lambda \) under the surjective map

\[
\text{Eff}(Z) \times \text{Eff}(Z) \to \text{Eff}(U) \times \text{Eff}(U).
\]

**Definition 2.9** (Morphisms of divisorial structures). A *morphism of divisorial structures* \((Z, \Lambda) \to (Z', \Lambda')\) is an open immersion of topological spaces \( f : Z \to Z' \) such that

\[
\text{Eff}(f) : \text{Eff}(Z) \to \text{Eff}(Z')
\]

is a bijection and

\[
(\text{Eff}(f) \times \text{Eff}(f))(\Lambda) = \Lambda'.
\]

**Notation 2.10.** We will write \( \mathcal{T} \) for the category of divisorial structures.

**Definition 2.11.** The *divisorial structure* of an integral scheme \( X \) is the pair

\[
\tau(X) := (|X|, \Lambda_X),
\]

where \(|X|\) is the underlying Zariski topological space of \( X \) and

\[
\Lambda_X \subset \text{Eff}(X) \times \text{Eff}(X)
\]

is the rational equivalence relation on effective divisors.

**Remark 2.12.** The divisorial structure of an integral scheme \( X \) can be obtained from the data of the triple

\[
(|X|, \text{Cl}(X), c : X^{(1)} \to \text{Cl}(X)).
\]

Indeed by the universal property of a free monoid on a set giving the map \( c \) is equivalent to giving a map of monoids

\[
\text{Eff}(X) \to \text{Cl}(X),
\]

and the congruence relation defined by this map is precisely the equivalence relation given by rational equivalence. Conversely, from the equivalence relation on \( \text{Eff}(X) \) we obtain the class group as the group associated to the quotient of \( \text{Eff}(X) \) by the congruence relation and the map \( c \) is induced by the natural map \( X^{(1)} \to \text{Eff}(X) \).
Formation of the divisorial structures defines a diagram of categories

\[ \mathcal{D} \subset \mathcal{V}_{\text{Var}} \xrightarrow{\tau} \mathcal{T} \]  

The main result of this paper is the following.

**Theorem 2.13.** The functor \( \tau|_{\mathcal{D}} \) is fully faithful.

The proof of Theorem 2.13 will be given in Section 6 after some preliminary foundational work.

3. **Some remarks on divisors**

In this section we gather a few facts about divisors on normal varieties. Our main purpose is to demonstrate that some basic features of such varieties – such as the maximal factorial open subscheme – can be characterized purely in terms of the divisorial structure.

Fix a field \( k \). For a normal irreducible separated \( k \)-scheme \( X \) let

\[ q : \text{Eff}(X) \to \overline{\text{Eff}}(X) \]

denote the quotient monoid given by rational equivalence of divisors, so that \( \overline{\text{Eff}}(X) \) is the image of \( \text{Eff}(X) \) in \( \text{Cl}(X) \). Given a divisor \( D \) on \( X \), upon identifying \( |D| \) with the subset of effective divisors on \( X \) that are linearly equivalent to \( D \), we have a set-theoretic equality

\[ |D| = q^{-1}(q(D)). \]

In particular, the linear system is defined as a set by the map \( q \).

There is a reflexive sheaf of rank 1 canonically associated to \( D \) that we will write \( \mathcal{O}(D) \). Members of \( |D| \) are in bijection with sections \( \mathcal{O} \to \mathcal{O}(D) \) in the usual way. Recall that \( D \) is Cartier if and only if \( \mathcal{O}(D) \) is an invertible sheaf on \( X \).

**Lemma 3.1.** Let \( U \subset X \) be an open subscheme. Then the commutative diagram

\[
\begin{array}{ccc}
\text{Eff}(X) & \longrightarrow & \overline{\text{Eff}}(X) \\
\downarrow & & \downarrow \\
\text{Eff}(U) & \longrightarrow & \overline{\text{Eff}}(U)
\end{array}
\]

is a pushout diagram in the category of integral monoids.

**Proof.** If \( E_1, E_2 \in \text{Eff}(U) \) are two classes mapping to the same class in \( \overline{\text{Eff}}(U) \) then there exists a rational function \( f \in \overline{\text{Eff}}(U) \) such that

\[ \text{div}_U(f) = E_1 - E_2 \]

in \( \text{Div}(U) \). Then \( \text{div}_X(f) \in \text{Div}(X) \) maps to \( \text{div}_U(f) \) in \( \text{Div}(U) \), so if we write \( \text{div}_X(f) \) as

\[ \tilde{E}_1 - \tilde{E}_2, \]

where the divisors \( \tilde{E}_i \) are effective, then \( \tilde{E}_i \in \text{Eff}(X) \) are rationally equivalent divisors mapping to the \( E_i \). This shows that the equivalence relation on \( \text{Eff}(U) \) given by rational equivalence is the image of equivalence relation on \( \text{Eff}(X) \) given by the projection \( \text{Eff}(X) \to \overline{\text{Eff}}(X) \), which implies the lemma. \( \square \)
Corollary 3.2. If $X$ is an integral scheme and $U \subset X$ is an open subscheme then the divisorial structure $\tau(U)$ is canonically isomorphic to the restriction $\tau(X)|_U$ (see Definition 2.8).

Proof. The equivalence relation on $\text{Eff}(X)$ is the relation defined by the quotient map $\text{Eff}(X) \to \overline{\text{Eff}}(X)$. By Lemma 3.1, we see that the induced relation on $\tau(X)|_U$ is precisely the relation for $\tau(U)$, giving the desired result. □

Definition 3.3. Given an excellent scheme $X$, the Cartier locus of $X$ is the largest open subscheme $U \subset X$ that is factorial (i.e., such that every Weil divisor on $U$ is Cartier).

Proposition 3.4. Let $X$ be a normal irreducible quasi-compact separated scheme and let $D \subset X$ be a divisor.

1. If $|D|$ is basepoint free then $D$ is Cartier.
2. If $D$ is ample then $D$ is $Q$-Cartier.

Proof. Since $X$ is quasi-compact, if $D$ is ample we know that $|nD|$ is basepoint free for some $n$. Thus it suffices to prove the first statement. Given a point $x \in X$, choose $E \in |D|$ such that $x \not\in E$. This gives some section $s : \mathcal{O} \to \mathcal{O}(D)$. Restricting to the local ring $R = \mathcal{O}_{X,x}$, we see that $s_x : R \to \mathcal{O}(D)_x$ is an isomorphism in codimension 1 (for otherwise $E$ would be supported at $x$). Since $\mathcal{O}(D)$ is reflexive, it follows that $s_x$ is an isomorphism, whence $\mathcal{O}(D)$ is invertible in a neighborhood of $x$. Since this holds at any $x \in X$, we conclude that $\mathcal{O}(D)$ is invertible, as desired. □

Corollary 3.5. A normal irreducible separated scheme $X$ is factorial if and only if it is covered by open subschemes $U \subset X$ with the property that every divisor class on $U$ is basepoint free.

Proof. If $X$ is factorial, then any affine open covering has the desired property, since any Cartier divisor on an affine scheme is basepoint free. On the other hand, if $X$ admits such a covering, then we know that every divisor class on $X$ is locally Cartier, whence it is Cartier. □

Proposition 3.6. If $X$ is a normal $k$-variety then we can characterize the Cartier locus of $X$ as the union of all open subsets $U \subset X$ such that every divisor class on $U$ is basepoint free.

Proof. This is an immediately consequence of Corollary 3.5. □

The preceding discussion implies that various properties of a scheme $X$ and its divisors can be read off from the divisorial structure. We summarize this in the following.

Proposition 3.7. Let $X$ be a normal separated quasi-compact and irreducible scheme and let

$$\tau(X) = (|X|, \Lambda_X)$$

be the associated divisorial structure. Then

1. the property that $D \in \text{Eff}(X)$ has basepoint free linear system $|D|$ depends only on $\tau(X)$;
2. the property that $X$ is factorial depends only on $\tau(X)$.
(iii) the Cartier locus of $X$ depends only on $\tau(X)$;
(iv) the condition that a divisor $D$ is ample depends only on $\tau(X)$.

Proof. Let

$$q : \text{Eff}(X) \to \text{Eff}(X)$$

denote the quotient map defined by $\Lambda_X$, so that for $D \in \text{Eff}(X)$ we have $|D| = q^{-1}q(D)$. The condition that $|D|$ is base point free is the statement that for every $x \in |X|$ there exists $E \in |D|$ such that $x \notin E$. Evidently this depends only on $\tau(X)$, proving (i).

Likewise the condition that a divisor $D$ is ample is the statement that the open sets defined by elements of $|nD|$ for $n \geq 0$ give a base for the topology on $|X|$. Again this clearly only depends on $\tau(X)$, proving (iv).

Statement (ii) follows from Corollary 3.5 and Lemma 3.1, which implies that the divisorial structure $\tau(U)$ for an open subset $U \subset X$ is determined by $|U| \subset |X|$ and $\tau(X)$.

Finally (iii) follows from Proposition 3.6.

3.8. Our proof of Theorem 2.13 will ultimately rely on reducing to the projective case. For the remainder of this section, we record some results about polarizations that we will need later.

Given a definable scheme $X$, write $\text{Abpf}(X) \subset \text{Eff}(X)$ for the (possibly empty) submonoid of ample basepoint free effective divisors $D$ and $\text{Abpfi}(X) \subset \text{Abpf}(X)$ for the submonoid of divisors whose associated linear system defines an injective map $\nu_D : X \hookrightarrow |D|^\vee$.

Lemma 3.9. Suppose given two definable schemes $X$ and $Y$ and an isomorphism $\varphi : \tau(X) \to \tau(Y)$. If $X$ is polarizable and factorial then so is $Y$ and $\varphi$ induces a commutative diagram of monoids

$$
\begin{array}{ccc}
\text{Abpf}(X) & \longrightarrow & \text{Eff}(X) \\
\downarrow & & \downarrow \\
\text{Abpf}(Y) & \longrightarrow & \text{Eff}(Y)
\end{array}
$$

in which the vertical arrows are isomorphisms. If the constant fields of $X$ and $Y$ are algebraically closed, then the isomorphism $\text{Abpf}(X) \to \text{Abpf}(Y)$ restricts to an isomorphism $\text{Abpfi}(X) \to \text{Abpfi}(Y)$ so we obtain a commutative diagram

$$
\begin{array}{ccc}
\text{Abpf}(X) & \longrightarrow & \text{Abpf}(X) \\
\downarrow & & \downarrow \\
\text{Abpfi}(Y) & \longrightarrow & \text{Abpfi}(Y)
\end{array}
$$

Proof. Since $X$ is factorial all divisors are Cartier divisors. By Proposition 3.7, $Y$ is also factorial and polarizable, and the submonoid $\text{Abpf}$ is preserved, as claimed. Finally, when $\kappa_X$ and $\kappa_Y$ are algebraically closed, one can tell if $\nu_D$ is injective by seeing if the sets $H_x = \{ E \in |D| : x \in E \}$ are distinct for distinct closed points $x$; thus, $\text{Abpfi}(X)$, resp. $\text{Abpfi}(Y)$, is determined by $\tau(X)$, resp. $\tau(Y)$. (Note that it is not yet clear if $\kappa_X$ is determined by $\tau(X)$. This will be discussed in Section 4 and Section 6 below.)
Definition 3.10. Suppose $X$ is a definable scheme. An open subscheme $U \subset X$ will be called essential if $\text{codim}(X \setminus U \subset X) \geq 2$, $U$ is factorial, and $U$ is polarizable.

Note that if $U \subset X$ is essential, then the natural restriction map $\text{Eff}(X) \to \text{Eff}(U)$ is an isomorphism of monoids.

Lemma 3.11. If $X$ is a normal, separated, quasi-compact $k$-scheme then there is an open subscheme $U \subset X$ such that $\text{codim}(X \setminus U \subset X) \geq 2$ and $U$ is quasi-projective. In particular, any definable scheme $X$ contains an essential open subset $U \subset X$.

Proof. Working one connected component at a time, we may assume that $X$ is irreducible. By Chow’s lemma, there is a proper birational morphism $\pi : \tilde{X} \to X$ with $\tilde{X}$ quasi-projective. Since $X$ is normal, $\pi$ is an isomorphism in codimension $1$. Thus, $\tilde{X}$ and $X$ have a common open subset $U$ whose complement in $X$ has codimension at least $2$, and which is quasi-projective. Passing to the factorial locus yields the second statement. □

Lemma 3.12. Suppose $X$ and $Y$ are definable schemes and $\varphi : \tau(X) \to \tau(Y)$ is an isomorphism of divisorial structures. If $U \subset X$ is an essential open subset then $\varphi(U) \subset Y$ is an essential open subset and there is an induced isomorphism $\tau(U) \cong \tau(\varphi(U))$.

Proof. First note that since $\varphi$ induces a homeomorphism $|X| \to |Y|$, we have that $\text{codim}(X \setminus Y \subset X) = \text{codim}(Y \setminus \varphi(U) \subset Y)$. In particular, if $U$ is definable then so is $\varphi(U)$ by Lemma 2.5. By Lemma 2.8 we have isomorphisms

$$\tau(X)|_U \cong \tau(U)$$

and

$$\tau(Y)|_{\varphi(U)} \cong \tau(\varphi(U)).$$

On the other hand, $\varphi$ induces an isomorphism $\tau(X)|_U \cong \tau(Y)|_{\varphi(U)}$. The result thus follows from Lemma 3.9. □

4. The Fundamental Theorem of Definable Projective Geometry

In this section we discuss one of the most classical parts of algebraic geometry, the Fundamental Theorem of Projective Geometry, which says that isomorphisms of abstract projective spaces are induced by semilinear isomorphisms of their underlying vector spaces (over an isomorphism of the scalar fields). We will need a variant of this theorem in which one only knows distinguished subsets of “strongly definable” lines in the projective structures and one still wishes to produce a semilinear isomorphism between the underlying vector spaces that induces the isomorphism on a dense open subset. We prove such a theorem here. In Section 5 and Section 6 we explain how to use this theory to reconstruct varieties.

Definition 4.1. A strongly definable projective space is a triple $(k, V, U)$ consisting of an infinite field $k$, a $k$-vector space $V$, and a subset $U \subset \text{Gr}(1, P(V))(k)$ which is the $k$-points of a non-empty Zariski open subset of the space $\text{Gr}(1, P(V))$ of lines in the projective space $P(V)$. 8
**Definition 4.2.** The *sweep* of a strongly definable projective space \((k, V, U)\) is the set of \(k\)-points \(p \in \mathbb{P}(V)\) that lie on some line parametrized by \(U\). We will write this as \(S_U(\mathbb{P}(V))\).

In other words, a strongly definable projective space is a projective space together with a collection of lines that are declared “strongly definable”.

**Example 4.3.** Fix a projective \(k\)-variety \(X\) of dimension \(d\) at least 2. Given a closed subset \(Z \subset X\), we can associate the subspace \(V(Z) \subset |\mathcal{O}(1)|\) of divisors that contain \(Z\). The lines of the form \(V(Z)\) whose base locus has codimension 2 in \(X\) give an open subset of \(\text{Gr}(1, |\mathcal{O}(1)|)\) (see Section 3). These are the strongly definable lines.

The main goal of this section is to prove the following result.

**Theorem 4.4.** Suppose \((k_1, V_1, U_1)\) and \((k_2, V_2, U_2)\) are finite-dimensional strongly definable projective spaces. Given a bijection \(\varphi : \mathbb{P}(V_1) \to \mathbb{P}(V_2)\) that induces an inclusion \(\lambda : U_1 \to U_2\), there is an isomorphism \(\sigma : k_1 \to k_2\) and a \(\sigma\)-linear isomorphism \(\psi : V_1 \to V_2\) such that \(\mathbb{P}(\psi)\) agrees with \(\varphi\) on a Zariski-dense open subset of \(\mathbb{P}(V_1)\) containing the sweep of \((k_1, V_1, U_1)\).

**Proof.** This proof is very similar to the proof due to Emil Artin in the classical case, as described by Jacobson in [4, Section 8.4].

Let us begin by showing the existence of the isomorphism of fields \(\sigma : k_1 \to k_2\). The construction will be in several steps.

First we set up some basic notation. Let \(V\) be a vector space over a field \(k\). For a nonzero element \(v \in V\) let \([v] \in \mathbb{P}(V)\) denote the point given by the line spanned by \(v\). For \(P \in \mathbb{P}(V)\) write \(\ell_P \subset V\) for the line corresponding to \(P\), and for two points \(P, Q \in \mathbb{P}(V)\) write \(L_{P,Q} \subset \mathbb{P}(V)\) for the projective line connecting \(P\) and \(Q\). If \(P = [v]\) and \(Q = [w]\) then \(L_{P,Q}\) corresponds to the 2-dimensional subspace of \(V\) given by

\[
\text{Span}(v, w) := \{av + bw | a, b \in k\}.
\]

If \(L \subset \mathbb{P}(V)\) is a line and \(P, Q, R \in L\) are three pairwise distinct points then there is a unique \(k\)-linear isomorphism \(L \xrightarrow{\epsilon} \mathbb{P}^1\) sending \(P\) to 0, \(Q\) to 1, and \(R\) to \(\infty\). For a collection of data \((L, \{P, Q, R\})\) we therefore have a canonical identification

\[
\ell^{P,Q,R} : k \xrightarrow{\sim} L - \{R\}.
\]

In the case when \(L = L_{[v],[w]}\) for two distinct vectors \(v, w \in V - \{0\}\) we take \(P = [v]\), \(Q = [v + w]\), and \(R = [w]\). Then the identification of \(k\) with \(L - \{R\}\) is given by

\[
a \mapsto v + aw.
\]

Suppose given \((L, \{P, Q, R\})\) as above, and fix a basis vector \(v_P \in \ell_P\). Then one sees that there exists a unique basis vector \(v_R \in \ell_R\) such that \([v_P + v_R] = Q\). This observation enables us to relate the maps \(\ell^{P,Q,R}\) for different lines as follows.

Consider a second line \(L'\) passing through \(P\) and equipped with two additional points \(\{S, T\}\). We can then consider the two lines

\[
L_{T,R}, \quad \ell^{P,Q,R(a)} \circ \ell^{P,S,T(b)},
\]

which will intersect in some point

\[
\{O\} = L_{T,R} \cap \ell^{P,Q,R(a)} \circ \ell^{P,S,T(b)}.
\]
The situation is summarized in the following picture, where to ease notation we write simply $a$ (resp. $b$) for $\epsilon^{P,Q,R}(a)$ (resp. $\epsilon^{P,S,T}(b)$):

If we fix a basis element $v_P \in \ell_P$ we get by the above observation a basis vector $v_Q$ (resp. $v_R$, $v_S$, $v_T$) for $\ell_Q$ (resp. $\ell_R$, $\ell_S$, $\ell_T$), which in turn gives an identification

$$\epsilon^{[v_T],[v_T+v_R],[v_R]} : k \to L_{T,R} - \{R\}.$$  

An elementary calculation then shows that

$$O = \epsilon^{[v_T],[v_T+v_R],[v_R]}(-a/b).$$

In particular, if $a = b$ then the point $O$ is independent of the choice of $a$, and furthermore it follows from the construction that $O$ is also independent of the choice of the basis element $v_P$.

Consider now a strongly definable projective space $(k,V,U)$, and let $L_0 \subset \mathbb{P}(V)$ be a strongly definable line with three points $P, Q, R \in L$. Fix $a \in k$ so we have a point

$$\epsilon^{P,Q,R}(a) \in L_0.$$

Let $M_P$ denote the scheme classifying data $(L, \{S,T\})$, where $L$ is a line through $P$ and $\{S,T\}$ is a set of two additional points on $L$. The scheme $M_P$ has the following description. The point $P$ corresponds to a line $\ell_P \subset V$ and the set of lines passing through $P$ is given by $\mathbb{P}(V/\ell_P)$. If $\mathcal{L} \to \mathbb{P}(V/\ell_P)$ denotes the universal line in $\mathbb{P}(V)$ passing through $P$ then there is an open immersion

$$M_P \subset \mathcal{L} \times_{\mathbb{P}(V/\ell_P)} \mathcal{L},$$

whence $M_P$ is smooth and geometrically connected. We see that the $k$-points of $M_P$ are dense (since $k$ is infinite).

**Lemma 4.5.** Fix $a \in k$. There exist a nonempty open subset $U_{P,a} \subset M_P$ such that if $(L, \{S,T\})$ is a line through $P$ with two points corresponding to a $k$-point of $U_{P,a}$ then the lines

$$L_{P,T}, \ L_{T,R}, \ L_{\epsilon^{P,Q,R}(a),\epsilon^{P,S,T}(a)}$$

are all strongly definable.

**Proof.** Let $Q_0 \in M_P$ denote the point corresponding to $(L_0, \{Q,R\})$. The procedure of assigning one of the lines in (4.5.1) to a pointed line $(L, \{S,T\})$ is a map

$$q : M_P \to \text{Gr}(1, \mathbb{P}(V)).$$

Note that the image of this map contains the point corresponding to the line $L_0$, and therefore the inverse image $q^{-1}(U)$ is nonempty. Since $M_P$ is integral it follows that
the intersection of the preimages of $U$ under the three maps defined by (4.5.1) is nonempty. 

With these preparations we can now proceed with the proof of Theorem 4.4. Proceeding with the notation of the theorem, let us first define the map $\sigma : k_1 \rightarrow k_2$. Choose a strongly definable line $L_0 \subset P(V_1)$ together with three points $P, Q, R \in L_0$ such that $\varphi(L_0) \subset P(V_2)$ is also a strongly definable line. We then get a map

$$k_1 \xrightarrow{\epsilon^{P,Q,R}} L_0 - \{R\} \xrightarrow{\varphi} \varphi(L_0) - \{\varphi(R)\} \xrightarrow{\left(\epsilon^{\varphi(P),\varphi(Q),\varphi(R)}\right)^{-1}} k_2,$$

which we temporarily denote by $\sigma^{(L_0,\{P,Q,R\})}$.

**Claim 4.6.** The map $\sigma^{(L_0,\{P,Q,R\})}$ is independent of $(L_0, \{P, Q, R\})$.

**Proof.** Let $(L'_0, \{P', Q', R'\})$ be a second strongly definable line with three points. Given $a \in k_1$, we will show that

$$\sigma^{(L_0,\{P,Q,R\})}(a) = \sigma^{(L'_0,\{P',Q',R'\})}(a).$$

From the definition, we see that this holds for $a = 0$ and $a = 1$, so we assume that $a \neq 0$ in what follows. First consider the case when $P = P'$. By Lemma 4.5 we can find a line $L$ with two points $\{S, T\}$ such that the lines (4.5.1) are all strongly definable, as well as the lines (4.5.1) obtained by replacing $(L_0, \{P, Q, R\})$ with $(L'_0, \{P, Q, R'\})$.

The picture in Figure 1 is taken by $\varphi$ to the corresponding picture in $P(V_2)$. Looking at the intersection point it follows that

$$\sigma^{(L_0,\{P,Q,R\})}(a) = \sigma^{(L_0,\{P,S,T\})}(a) = \sigma^{(L'_0,\{P',Q',R'\})}(a).$$

It follows, in particular, that the map $\sigma^{(L_0,\{P,Q,R\})}$ is independent of the points $Q$ and $R$. Since $\sigma^{(L_0,\{R,Q,P\})}$ is the composition of $\sigma^{(L_0,\{P,Q,R\})}$ with the inversion map it follows that the map $\sigma^{(L_0,\{P,Q,R\})}$ is independent of the triple $\{P, Q, R\}$, so we get a well-defined map $\sigma^{L_0} : k_1 \rightarrow k_2$. Now for a second strongly definable line $L'_0$ the intersection

$$P := L_0 \cap L'_0$$

is a point on both lines so we can apply the preceding discussion with the two lines $L_0$ and $L'_0$ and $Q, R$ and $Q', R'$ chosen arbitrarily to deduce the independence of the choice of $(L_0, \{P, Q, R\})$. 

Let us write the map of Claim 4.6 as $\sigma : k_1 \rightarrow k_2$.

**Claim 4.7.** The map $\sigma$ is an isomorphism of fields.

**Proof.** First note that by construction the map $\sigma$ sends 1 to 1 and is compatible with the inversion map $a \mapsto a^{-1}$. Indeed the statement that $\sigma(1) = 1$ is immediate from the construction and the compatibility with the inversion map can be seen as follows. Let $\iota_j : k_j^x \rightarrow k_j^x$ ($k = 1, 2$) denote the map $a \mapsto a^{-1}$, and let $(L, \{P, Q, R\})$ be a strongly definable line with three marked points. Write $L^x$ (resp. $\varphi(L)^x$) for $L - \{P, R\}$ (resp. $\varphi(L) - \{\varphi(P), \varphi(R)\}$). Then by the independence of the choice of marked line in the
definition of $\sigma$, we have that the diagram

\[
\begin{array}{c}
k_1 \xrightarrow{\sigma} k_1 \\
\downarrow \quad \downarrow \\
k_2 \xrightarrow{\sigma} k_2 \\
\end{array}
\begin{array}{c}
\xrightarrow{\varepsilon^{P,Q,R}(P)} L^x \\
\downarrow \quad \downarrow \\
\xrightarrow{\sigma(P)(Q),\sigma(R)} \varphi(L)^x \\
\end{array}
\begin{array}{c}
\xrightarrow{\varepsilon^{R,Q,P}(R)} \\
\downarrow \\
\varphi(R,Q,P) \\
\end{array}
\begin{array}{c}
\varepsilon^{P,Q,R}(Q) \\
\downarrow \\
\varepsilon^{P,Q,R}(P) \\
\end{array}
\end{array}
\]

commutes. The compatibility with the multiplicative structure again follows from contemplating Figure 1, and the observation that by construction the map $\sigma$ takes 1 to 1. Indeed given $a, b \in k_1^\times$ such that all the lines in Figure 1 are strongly definable, we must have

\[
\sigma(-a/b) = -\sigma(a)/\sigma(b)
\]

since this fraction is given by the point $O$. Since the condition of being strongly definable is open, the fact that for any strongly definable $(L, \{P, Q, R\})$ the line through $\varepsilon^{P,Q,R}(a)$ and $\varepsilon^{P,Q,R}(b)$ is strongly definable implies that the same is true after deforming $(L, \{P, Q, R\})$. This we get the formula (4.7.1) for all $a$ and $b$. In particular, taking $b = 1$ we get that $\sigma(-a) = -\sigma(a)$ for all $a$, and since $\sigma$ is compatible with the inversion maps we get that

\[
\sigma(ab) = \sigma(a)\sigma(b)
\]

for all $a, b \in k^\times$. Since 0 is also taken to 0 by $\sigma$ we in fact get this formula for all $a, b \in k$.

For the verification of the compatibility with additive structure, consider a marked line $(L, \{P, Q, R\})$. Let $S$ be a point not on the line and let $T$ be a third point on $L_{S,R}$. The lines $L_{P,T}$ and $L_{Q,S}$ intersect in a point we call $V$, and then the line $L_{V,R}$ intersects $L_{P,S}$ in a point we call $W$. This is summarized in the following picture, where we write simply $a$ (resp. $b$) for $\varepsilon^{P,Q,R}(a)$ (resp. $\varepsilon^{S,T,R}(b)$).

![Figure 2](image-url)
A straightforward calculation done by choosing a basis \( v_R \in \ell_R \) then shows that the point of intersection marked with the larger bullet is the point

\[
e^{W;V;R}(a + b).
\]

To prove that \( \sigma \) is compatible with the additive structure it suffices to show that for any \( a, b \in k \) there exist a pointed line \((L, \{ P, Q, R \})\) and points \( S \) and \( T \) such that all the lines in Figure 2 are strongly definable. This is done by an argument similar to the one proving Lemma 4.5.

Now that we have constructed the isomorphism \( \sigma \), it remains to construct the map \( \psi : V_1 \to V_2 \).

First note that we can choose a basis \( e_1, \ldots, e_n \) for \( V_1 \) with the property that the span of \( e_i \) and \( e_j \) is a strongly definable line for any \( i \neq j \). Define \( e'_1, \ldots, e'_n \in V_2 \) as follows. For \( e'_1 \) we take any basis element in \( \ell_{\varphi([e_1])} \). Now for each \( e_i, i \geq 2 \), the line in \( P(V_1) \) associated to the plane \( \text{Span}(e_1, e_i) \) is strongly definable, and therefore the image under \( \varphi \) is a strongly definable line and contains the points \( \varphi([e_1]), \varphi([e_i]), \) and \( \varphi([e_1 + e_i]) \). The choice of the representative \( e'_i \) for \( \varphi([e_1]) \) defines a representative \( e'_i \) for \( \varphi([e_i]) \) such that \( \varphi([e_1 + e_i]) = e'_1 + e'_i \). Consider the map

\[
\gamma : V_1 \to V_2
\]
defined by

\[
\gamma(a_1 e_1 + \cdots + a_n e_n) := \sigma(a_1) e'_1 + \cdots + \sigma(a_n) e'_n.
\]

**Claim 4.8.** For general \( (a_1, \ldots, a_n) \) we have

\[
\varphi([a_1 e_1 + \cdots + a_n e_n]) = [\gamma(a_1 e_1 + \cdots + a_n e_n)].
\]

**Proof.** By the construction of \( \sigma \), if for each \( 2 \leq i \leq n \) the vectors

\[
(4.8.1) \quad a_1 e_1 + \cdots + a_{i-1} e_{i-1}, \ a_i e_i
\]

span a strongly definable line, then we get by induction on \( i \) that

\[
\varphi([a_1 e_1 + \cdots + a_i e_i]) = [\gamma(a_1 e_1 + \cdots + a_i e_i)].
\]

Now for each \( i \) the map sending a vector \( (a_1, \ldots, a_n) \) to the span of the elements \( (4.8.1) \) defines a map

\[
A \to G(1, P(V_1))
\]

whose image meets \( U_1 \). Taking the common intersection of the preimages of \( U_1 \) under these maps, we get a nonempty open subset \( A^c \subset A \) of tuples \( (a_1, \ldots, a_n) \in A(k_1) \) for which the vectors \( (4.8.1) \) span a strongly definable line.

As a consequence, the map \( \gamma \) defined above is uniquely associated to \( \varphi \) and is thus independent of the general choice of basis \( e_1, \ldots, e_n \).

To complete the proof of Theorem 4.4 it suffices to show that \( P(\gamma) \) agrees with \( \varphi \) on the entire sweep of \( (k_1, V_1, U_1) \). By the above remark, to show this for a particular point \( p \), it suffices to work with any general basis. To prove this we show that given a point \( p \in S_{U_1}(P_{k_1}(V_1)) \) there exists a basis \( e_1, \ldots, e_n \) for \( V_1 \) as above for which \( p \) lies in the resulting subset \( A^c \). Reviewing the above construction, one sees that it suffices to show that we can find a basis \( e_1, \ldots, e_n \) for \( V_1 \) such that the following hold:

(i) \( p \) is the point corresponding to the line spanned by \( e_1 \).
(ii) Any two elements \( e_i \) and \( e_j \), with \( i \neq j \), span a strongly definable line.
(iii) For any \( 2 \leq i \leq n \) the vectors
\[
e_1 + \cdots + e_{i-1}, \ e_i
\]
span a strongly definable line.

For this start by choosing \( e_1 \) so that (i) holds. Since \( p \) lies in the sweep we can then find \( e_2 \) such that \( e_1 \) and \( e_2 \) span a strongly definable line. Now observe that given \( 2 \leq r \leq n \) and a basis \( e_1, \ldots, e_r \) satisfying (ii) and (iii) with \( i, j \leq r \) we can find \( e_{r+1} \) such that (ii) and (iii) hold with \( i, j \leq r + 1 \). Indeed a general choice of vector in \( V_1 \) will do for \( e_{r+1} \) since for given fixed vector \( v_0 \) lying in the sweep there is a nonempty Zariski open subset of vectors \( w \) such that \( w \) and \( v_0 \) span a strongly definable line.

This completes the proof of the Theorem. \( \square \)

5. Definable subspaces in linear systems

Fix a definable absolute variety \( X \) with infinite constant field. Let \( P := |D| \) be the linear system associated to an effective divisor \( D \).

**Definition 5.1.** A subspace \( V \subset P \) is definable if there is a subset \( Z \subset X \) such that
\[
V = V(Z) := \{ E \in P \mid Z \subset E \}.
\]

**Remark 5.2.** If \( Z \subset X \) is a subset and \( Z' \subset X \) is the closure of \( Z \) then \( V(Z) = V(Z') \).

When considering definable subspaces it therefore suffices to consider subspaces defined by closed subsets.

**Remark 5.3.** Note that \( V(Z) \) is the projective space associated to the kernel of the restriction map
\[
H^0(X, \mathcal{O}_X(D)) \to H^0(Z_{\text{red}}, \mathcal{O}_X(D)|_{Z_{\text{red}}}),
\]
where we write \( Z_{\text{red}} \subset X \) for the reduced subscheme associated to the subspace \( Z \subset |X| \).

**Lemma 5.4.** Suppose \( V = V(Z) \) is a non-empty definable subset of a basepoint free linear system \( P \) on \( X \). Then there is an ascending chain of closed subsets
\[
Z = Z_1 \subsetneq \cdots \subsetneq Z_n
\]
such that the induced chain
\[
V(Z) = V(Z_1) \supsetneq \cdots \supsetneq V(Z_n)
\]
is a full flag of linear subspaces ending in a point.

**Proof.** By induction, it suffices to produce \( Z_2 \supsetneq Z_1 = Z \) such that \( V(Z_2) \subsetneq V(Z_1) \) has codimension 1.

Since \( P \) is base point free we have a morphism \( \pi : X \to \mathbb{P}^V \), where \( \mathbb{P}^V \) denotes the dual projective space of \( P \). Given a subset \( Z \subset X \) the space \( V(Z) \) is the same as the space \( V(\langle \pi(Z) \rangle) \), the space of hyperplanes containing the linear span of \( \pi(Z) \) in \( \mathbb{P}^V \). If \( \langle \pi(Z) \rangle = \mathbb{P}^V \) then \( V(Z) = \emptyset \), contrary to our assumption. Thus, there must be a point \( x \in X \setminus \pi^{-1}(\langle \pi(Z) \rangle) \) (or if \( X = \pi^{-1}(\langle \pi(Z) \rangle) \) then \( \pi \) factors through a morphism \( \pi' : X \to \langle \pi(Z) \rangle \subset \mathbb{P}^V \) which implies that \( \langle \pi(Z) \rangle = \mathbb{P}^V \).

\[\text{14}\]
We claim that we can find a linear space $L \subset \mathbb{P}^r$ containing $\langle \pi(Z) \rangle$ such that the space of hyperplanes containing $L$ has codimension 1 in the space of hyperplanes containing $Z$ and $\pi^{-1}(L)$ has nonempty intersection with $X \setminus \pi^{-1}(\langle \pi(Z) \rangle)$. This will conclude the proof. Indeed setting $Z_2 = \pi^{-1}(L)$ gives $V(Z_2) \subset V(Z_1)$ of codimension 1, as desired.

To verify the claim, note that since the space parametrizing such $L \supset \langle \pi(Z) \rangle$ is rational and $k$ is infinite, it suffices to verify the claim after base changing to an algebraic closure of $k$. We may therefore assume that we have a $k$-point $x \in X \setminus \pi^{-1}(\pi(Z))$. In this case we have

$$\dim \langle Z \cup \{x\} \rangle = \dim(Z) + 1.$$  

Since the linear system $P$ is base point free it follows that $V(Z \cup \{x\}) \subset V(Z)$ has codimension 1.

**Observation 5.5.** There is a subtle point in the proof of Lemma 5.4: we are not assuming that $\pi^{-1}(L)$ is reduced. However, we are assuming that

$$\pi^{-1}(L) \cap (X \setminus \pi^{-1}(\pi(Z))) \neq \emptyset,$$

so that the linear span must grow, and thus the topological condition suffices to reduce the size of the space. Since the space of sections vanishing set-theoretically on a subscheme $Y$ (i.e., $V(Y_{\text{red}})$) is possibly larger than the space vanishing on the scheme $Y$ (i.e., $V(Y)$), we see that set-theoretic vanishing on $\pi^{-1}(L) \cap X$ must be a codimension 1 condition (since the same is true for scheme-theoretic vanishing along $\pi^{-1}(L) \cap X$, and the spaces are already set-theoretically different because $\pi^{-1}(L) \cap (X \setminus \pi^{-1}(\pi(Z))) \neq \emptyset$).

**Corollary 5.6.** Given a basepoint free linear system $P$ on $X$, the definable lines in $P$ are precisely those definable subsets with more than one element that are minimal with respect to inclusions of definable subsets.

**Proof.** By Lemma 5.4, any definable set of higher dimension contains a definable line. □

**Lemma 5.7.** Let $\ell \subset \mathbb{P}(V)$ be a line corresponding to a two-dimensional subspace $T \subset V$. Let $Z' \subset X$ be the maximal reduced closed subscheme of the intersection of the zero-loci of elements of $T$. Then $\ell$ is definable if and only if the dimension of the kernel

$$K := \ker(\mathcal{H}^0(X, \mathcal{L}) \to \mathcal{H}^0(Z', \mathcal{L}|_{Z'}))$$

is equal to 2.

**Proof.** First suppose $\ell$ is definable, so we can write $\ell = V(Z)$ for some closed subset $Z \subset |X|$, which we view as a subscheme with the reduced structure. Then by definition $\ell$ is the projective subspace of $\mathbb{P}(V)$ associated to the kernel of the map

$$\mathcal{H}^0(X, \mathcal{L}) \to \mathcal{H}^0(Z, \mathcal{L}|_Z),$$

which must therefore equal $T$. In particular, we have $Z \subset Z'$, which implies that

$$T \subset K \subset \ker(\mathcal{H}^0(X, \mathcal{L}) \to \mathcal{H}^0(Z, \mathcal{L}|_Z)).$$

It follows that $K = T$, and, in particular, $K$ has dimension 2. Conversely, if $K$ has dimension 2 then we have $T = K$ and $\ell = V(Z')$. □
Lemma 5.8. Suppose $P(V)$ is a basepoint free linear system on $X$. Let $F_1, F_2 \in V$ be two linearly independent vectors with zero loci $Z_1$ and $Z_2$. Assume that

1. $Z_1$ is reduced;
2. the natural map $H^0(X, O_X) \to H^0(Z_1, O_{Z_1})$ is an isomorphism;
3. the intersection $Z := Z_1 \cap Z_2$ is reduced and does not contain any components of the $Z_i$.

Then the line in $P(V)$ spanned by $F_1$ and $F_2$ is definable.

Proof. We have short exact sequences

$$0 \to O_X \xrightarrow{F_1} L \xrightarrow{\rho} L|_{Z_1} \to 0,$$

and

$$0 \to O_{Z_1} \xrightarrow{F_2} L|_{Z_1} \xrightarrow{\rho|_{Z_1}} L|_Z \to 0,$$

where the second sequence is exact because $Z_1$ is reduced and $Z$ does not contain any components of $Z_1$. From these sequences we see that if $K$ denotes the kernel of the map

$$H^0(X, L) \to H^0(Z, L|_Z)$$

then there is a short exact sequence

$$0 \to k \cdot F_2 \to K \to H^0(Z_1, O_{Z_1}).$$

Since $Z_1$ is reduced and connected the right term of this sequence is 1-dimensional, and since $K$ contains the span of $F_1$ and $F_2$ it follows that $K$ is 2-dimensional. The result therefore follows from Lemma 5.7. □

Lemma 5.9. Let $P$ be an ample basepoint free linear system on $X$ and assume the dimension of $X$ is at least 2. Then the set of definable lines in $P$ is dense and constructible in the Grassmanian $Gr(1, P)$.

Proof. The constructibility of the set of definable lines follows from the semicontinuity of kernel dimension of a map of sheaves and Lemma 5.7. To see that the set of definable lines in $Gr(1, P)$ is dense, note that by Bertini’s theorem, as discussed in [3, 3.4.10, 3.4.14], for a general choice of linearly independent elements in $V$ the conditions of Lemma 5.8 are met. (Note that here we use the fact that $X \to \text{Spec} \kappa_X$ has integral geometric fibers, since we need to know that a general section is geometrically reduced and connected in order to conclude that its ring of constants is equal to $\kappa_X$). □

Example 5.10. In general the set of definable lines in $Gr(1, P)$ is not open. An explicit example is the following.

Consider three $k$-points $A, B, C \in P_k^2$, say $A = [0 : 0 : 1]$, $B = [0 : 1 : 0]$, and $C = [1 : 0 : 0]$. For a line $L \subset P_k^2$ passing through $A$ set

$$T_L := \{ F \in H^0(P_k^2, O_{P_k^2}(2)) | V(F) \text{ passes through } A, B, C \text{ and is tangent to } L \text{ at } A \}.$$ 

Concretely if $X, Y,$ and $Z$ are the coordinates on $P_k^2$ and $L$ is given by

$$aX + bY = 0,$$
then $T_L$ is given by

$$T_L = \{ aXY + b(\alpha X + \beta Y)Z | a, b \in j \}.$$ 

In particular, $T_L$ gives a line $\ell_L$ in $\mathbb{P}^2$.

If $\alpha$ and $\beta$ are nonzero then $L$ does not pass through $B$ and $C$ and the set-theoretic base locus of $T_L$ is equal to $\{ A, B, C \}$ and the space of degree two polynomials passing through these three points has dimension 3. Therefore for such $L$ the line $\ell_L$ is not definable.

However, for $L$ the lines $X = 0$ or $Y = 0$ the line $\ell_L$ is definable. Indeed in this case the set-theoretic base locus of $\ell_L$ is given by the union of the line $L$ together with a third point not on the line, from which one sees that $T_L$ is definable.

Letting $\alpha$ and $\beta$ vary we obtain a 1-parameter family of lines $\mathbb{P}^1 \simeq \Sigma \subset \text{Gr}(1, |O_{\mathbb{P}^2}(2)|)$ whose general member is not definable but with two points giving definable lines. It follows that the definable locus is not open in this case.

5.11. The base locus of a subset $E \subset \mathbb{P}(V)$, denoted $\mathcal{B}(E) \subset |X|$, is defined to be the intersection of the divisors in $E$. If a line $\ell \subset \mathbb{P}(V)$ is spanned by two elements $F, G \in V$ then $\mathcal{B}(\ell)$ is simply the common intersection of the zero-loci of $F$ and $G$.

**Definition 5.12.** A line $\ell \subset \mathbb{P}(V)$ is strongly definable if $\ell$ is definable and if the base locus of $\ell$ has codimension 2 in $X$.

**Lemma 5.13.** Let $P$ be an ample basepoint free linear system on $X$ and assume $\dim(X) \geq 2$. Then the set of strongly definable lines is open in the Grassmanian $\text{Gr}(1, P)$.

**Proof.** It is clear that the set of strongly definable lines is constructible, since by Lemma 5.9 the set of definable lines is constructible and the condition on the codimension of the base locus is clearly a constructible condition. It therefore suffices to show that the set of strongly definable lines is closed under generization. Note that, since $X \to \text{Spec} \, \kappa_X$ has integral geometric fibers, a line is (strongly) definable if and only if it is (strongly) definable after extending the base field, whence definability is a geometric condition. This permits us to study the scheme of such lines over arbitrary base rings.

For this let $R$ be a complete discrete valuation ring with residue field $k$ and field of fractions $K$, and let $\ell_R \subset \mathbb{P}(V)_R$ be a line over $R$ such that the closed fiber $\ell_k \subset \mathbb{P}(V)$ is strongly definable. We claim that then the generic line $\ell_K \subset \mathbb{P}(V_K)$ is also strongly definable.

Choose an $R$-basis $F_R, G_R \in V_R$ for the submodule defining $\ell_R$, and let $Z_R \subset X_R$ denote the scheme-theoretic intersection of the zero-loci of $F_R$ and $G_R$. We claim that $Z_R$ is set-theoretically equal to the closure of the generic fiber $Z_K \subset X_K$. Indeed by Krull’s Hauptidealsatz each irreducible component of $Z_R$ has dimension greater than or equal to $\dim(X) - 1$, and since the closed fiber of $Z_R$ has no component of dimension $d - 1$ (because $\ell_k$ is strongly definable) it follows that each irreducible component of $Z_R$ is set-theoretically equal to the closure of its generic fiber. We therefore have

$$Z_{K, \text{red}} \leftrightarrow Z_{R, \text{red}} \leftrightarrow Z_{k, \text{red}}$$

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with $j$ a dense open immersion. It follows that we get an induced inclusion
\[
\ker(H^0(X_R, \mathcal{L}_R) \to H^0(Z_{k,\text{red}}, \mathcal{L})) \hookrightarrow \ker(H^0(X_R, \mathcal{L}_R) \to H^0(Z_{k,\text{red}}, \mathcal{L})).
\]
If $\Lambda$ denotes the kernel of the left side then $\Lambda$ is a saturated submodule of $H^0(X_R, \mathcal{L}_R)$ and therefore $\Lambda/p\Lambda$ injects into
\[
\ker(H^0(X_R, \mathcal{L}_R) \to H^0(Z_{k,\text{red}}, \mathcal{L})).
\]
It follows that $\Lambda$ has rank 2, and in particular $\ell_K$ has dimension 2. By Lemma 5.7 it follows that $\ell_K$ is definable. Furthermore, the base locus must have codimension 2 since this holds on the closed fiber. □

**Lemma 5.14.** If $\operatorname{dim}(X) \geq 2$, then every ample basepoint free linear system $P$ contains a strongly definable line.

**Proof.** By Lemma 5.8 it suffices to show that a general element $D \in P$ is geometrically reduced and connected and that for two general elements $D, E \in P$ the intersection $D \cap E$ is reduced and $D$ and $E$ have no common component of intersection. This follows from Bertini’s theorem as given in [3, 3.4.10, 3.4.14]. □

**Remark 5.15.** Let us summarize the main consequences of the results in this section. Starting with a projective normal geometrically integral scheme $X$ over an infinite field $k$ we can consider the associated divisorial structure $\tau(X) = (|X|, \Lambda_X)$. From the divisorial structure we can extract several key pieces of information.

(i) The basepoint free and ample effective divisors and their linear systems are determined by $\tau(X)$. This was discussed in Proposition 3.7.

(ii) For an ample basepoint free linear system $P$ the set of definable lines in $P$ is by Corollary 5.6 characterized as those definable subsets with more than one element minimal with respect to inclusion. This set depends only on the divisorial structure.

(iii) For an ample basepoint free linear system $P$ the set of strongly definable lines is characterized as the set of definable lines for which the base locus has codimension 2. This again depends only on the divisorial structure. Moreover, if $\operatorname{dim}(X) \geq 2$ then the set of strongly definable lines is the $k$-points of a dense open subset of $\operatorname{Gr}(1, P)$.

6. **THE UNIVERSAL TORELLI THEOREM**

In this section we prove Theorem 2.13. Suppose $X$ and $Y$ are definable schemes of dimension at least 2 with infinite constant fields. We need to show that given an isomorphism $\varphi : \tau(X) \to \tau(Y)$, there is a unique isomorphism of schemes $f : X \to Y$ such that $\tau(f) = \varphi$.

6.1. **Reduction to the quasi-projective case.**

**Lemma 6.1.1.** If $X$ is a separated Noetherian scheme then for any point $x \in X$ we have that
\[
\overline{\{x\}} = \bigcap \overline{\{y\}},
\]
the intersection taken over all points $y \in X$ of codimension at most 1 such that $x \in \overline{\{y\}}$.
Proof. By treating each component of $X$ separately and intersecting the final result, we may assume that $X$ is integral with function field $\kappa(X)$. Since $X$ is separated, the inclusion $x \in X$ is uniquely determined by the inclusion $\mathfrak{O}_{X,x} \subset \kappa(X)$. The points $y$ such that $\{x\} \subset \{y\}$ correspond to those points such that the local ring $\mathfrak{O}_{X,y}$ contains $\mathfrak{O}_{X,x}$, as subrings of $\kappa(X)$. Letting $I_y \subset \mathfrak{O}_{X,x}$ denote the ideal of $\{y\}$ in $\mathfrak{O}_{X,x}$ (i.e., the intersection $m_y \cap \mathfrak{O}_{X,x}$ in $\kappa(X)$), we see that we wish to prove that $m_x = \sqrt{\sum_i I_y}$ as $\mathfrak{O}_{X,x}$-modules. Writing $m_x = (\alpha_1, \ldots, \alpha_n)$, we see that $\{x\} = \cap Z(\alpha_i)$ in $|\text{Spec} \mathfrak{O}_{X,x}|$. By the Krull Hauptidealsatz, each $Z(\alpha_i)$ is a union of codimension 1 closed subschemes that contain $x$. Taking for $\{y_i\}$ the set of all codimension 1 points that occur among the $Z(\alpha_i)$, we have that $m_x = \sqrt{\sum_i I_y}$, as desired. \qed

Lemma 6.1.2. Suppose $f, g : |X| \to |Y|$ are homeomorphisms of the underlying spaces of two separated normal Noetherian schemes. Given an open subset $U \subset |X|$ containing all points of codimension 1, if $f|_U = g|_U$ then $f = g$.

Proof. By Lemma 6.1.1, we can characterize any point $x \in X$ as the unique generic point of an irreducible intersection of closures of codimension $\leq 1$ points. But $f$ and $g$ establish the same bijection on the sets of points of codimension $\leq 1$, and, since they are homeomorphisms, therefore the same bijections on the closures of those points. The result follows. \qed

Lemma 6.1.3. Suppose $X$ and $Y$ are normal separated Noetherian schemes, $U \subset X$ and $V \subset Y$ are dense open subschemes with complements of codimension at least 2. Suppose $f : |X| \to |Y|$ is a homeomorphism of Zariski topological spaces such that $f(U) = V$ and $f|_U$ is the underlying map of an isomorphism $\tilde{f}_U : U \to V$ of schemes. Then $\tilde{f}_U$ extends to a unique isomorphism of schemes $\tilde{f} : X \to Y$ whose underlying morphism of topological spaces is $f$.

Proof. Let us first show that $\tilde{f}_U$ extends to a morphism of schemes $\tilde{f} : X \to Y$. If $W_1, W_2 \subset X$ are two open subsets and $\tilde{f}_{W_i} : W_i \to Y$ ($i = 1, 2$) are morphisms of schemes such that $\tilde{f}_{W_i}$ and $\tilde{f}_U$ agree on $W_i \cap U$, then since $Y$ is separated the morphisms $\tilde{f}_{W_1}$ and $\tilde{f}_{W_2}$ agree on $W_1 \cap W_2$. To extend $\tilde{f}_U$ it therefore suffices to show that $\tilde{f}_U$ extends locally on $X$. In particular, by covering $X$ by open subsets of the form $\tilde{f}^{-1}(\text{Spec}(A))$ for affines $\text{Spec}(A) \subset Y$, we are reduced to proving the existence of an extension in the case when $Y = \text{Spec}(A)$ is affine. In this case, to give a morphism of schemes $X \to \text{Spec} A$, it suffices to give a morphism of rings $A \to \Gamma(X, \mathcal{O}_X)$. By Krull’s theorem, $\Gamma(U, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X)$. Thus, the morphism $\tilde{f}_U : U \to \text{Spec} A$ extends uniquely to a morphism $\tilde{f} : X \to \text{Spec} A$, and we get the desired extension $\tilde{f}$.

Applying the same argument to the inverse of $f$, and using that $X$ is separated, we see that in fact $\tilde{f}$ is an isomorphism. In particular, its underlying map of topological spaces is a homeomorphism and agrees with $f$ on $|U|$. We conclude by Lemma 6.1.2 that $|\tilde{f}| = f$. \qed

6.1.4. From this we get that in order to prove Theorem 2.13 it suffices to prove it assuming that $X$ is quasi-projective. Indeed by Lemma 3.12 there are essential open subsets $U \subset X$ and $V \subset Y$ such that $V = \varphi(U)$ and $\tau$ induces an isomorphism $\tau(U) \to \tau(V)$. If we know the result in the quasi-projective case then the homeomorphism
\(|U| \to |V|\) induced by \(\varphi\) extends to an algebraic isomorphism \(f_U : U \to V\) such that \(\tau(f) = \varphi\)\(_U\). By Lemma 6.1.3, \(f\) extends uniquely to an isomorphism of schemes \(f : X \to Y\) such that \(\tau(f) = \varphi\).

### 6.2. The quasi-projective case.

**6.2.1.** For remainder of the proof we assume furthermore that \(X\) is quasi-projective. Let \(\mathcal{O}_X(1)\) denote a very ample invertible sheaf on \(X\) and for \(m \geq 1\) let
\[
+ : [\mathcal{O}_X(1)]^m \to [\mathcal{O}_X(m)]
\]
denote the addition map on divisors.

**Lemma 6.2.2.** For a general point \(p\) of \([\mathcal{O}_X(1)]^m\) the point \(+p\) lies on a strongly definable line in \([\mathcal{O}_X(m)]\).

**Proof.** Let \(\overline{X}\) be the projective closure of \(X\) in the embedding given by \(\mathcal{O}_X(1)\). Note that \(\overline{X}\) is also geometrically integral. Indeed if \(j : X \hookrightarrow \overline{X}\) is the inclusion then the map \(\mathcal{O}_{\overline{X}} \to j_*\mathcal{O}_X\) is injective, and remains injective after base field extension. Since \(X\) is geometrically integral it follows that \(\overline{X}\) is as well.

By Bertini’s theorem [3, 3.4.14], for a general choice of \(p \in \mathcal{O}_{\overline{X}}(1)^m = \mathcal{O}_X(1)^m\) and a general choice of \(D \in \mathcal{O}_X(m) = \mathcal{O}_X(m)\), we have that the pair \(+p\) and \(D\) satisfy the conditions of Lemma 5.8. Moreover, the base locus \(B\) of such a pair has the property that \(B \cap X\) is dense in \(B\), so that \(V(B) = V(B \cap X)\). It follows that \(+p\) and \(D\) span a strongly definable line in \([\mathcal{O}_X(1)]\), as desired. \(\square\)

**Lemma 6.2.3.** Let \(X\) be an integral scheme of dimension \(d\) of finite type over \(k\) and let \(\mathcal{O}_X(1)\) be an ample invertible sheaf on \(X\). Let \(z \in X\) be a regular closed point. Then there exists an integer \(m_0\) such that for every \(m \geq m_0\) and any \(d + 1\) general elements \(D_1, \ldots, D_{d+1} \in [\mathcal{O}_X(m)]\) containing \(z\) we have
\[
\{z\} = |D_1| \cap |D_2| \cap \cdots \cap |D_{d+1}|.
\]

**Proof.** Let
\[
b : B \to X
\]
denote the blowup of \(z\), and let \(\mathcal{L}\) be the ample invertible sheaf given by \(b^*\mathcal{O}_X(1)(-E)\), where \(E\) is the exceptional divisor. Then elements of \([\mathcal{L}^m]\) map to elements of \([\mathcal{O}_X(m)]\) which pass through \(z\). Now choose \(m_0\) such that \(\mathcal{L}^m\) is very ample for \(m \geq m_0\). Then by [5, 6.11(1)] we have that the intersection of \(d + 1\) general elements of \([\mathcal{L}^m]\) is empty for \(m \geq m_0\), and that general \(D \in [\mathcal{L}^m]\) is irreducible. \(\square\)

**Corollary 6.2.4.** Let \(X\) be an integral \(k\)-scheme and \(\mathcal{O}_X(1)\) a very ample invertible sheaf on \(X\). Given a regular closed point \(z \in X\), we have that
\[
(6.2.4.1) \quad \{z\} = \bigcap |D| \subset |X|,
\]
the intersection taken over all irreducible divisors \(D\) in \([\mathcal{O}_X(m)]\) for all \(m\).

**Proof.** This follows from Lemma 6.2.3. \(\square\)

**Proposition 6.2.5.** Suppose \(X\) and \(Y\) are definable schemes of dimension at least 2 with infinite constant fields, and assume that \(X\) is polarizable. Given an isomorphism \(\varphi : \tau(X) \to \tau(Y)\), the associated homeomorphism \(|X| \to |Y|\) extends to an isomorphism \(X \to Y\).
**Proof.** Let \( D \in \text{Abpf}(X) \) be a divisor with \( \mathcal{O}_X(D) = \mathcal{O}_X(1) \) and let \( \mathcal{O}_Y(1) \) denote \( \mathcal{O}_Y(\varphi(D)) \). After possibly taking a power of our choice of polarization, we may assume that \( \mathcal{O}_X(1) \) and \( \mathcal{O}_Y(1) \) are very ample and that for all \( m \geq 1 \) the multiplication maps

\[
\Gamma(X, \mathcal{O}_X(1))^\otimes m \twoheadrightarrow \Gamma(X, \mathcal{O}_X(m))
\]

and

\[
\Gamma(Y, \mathcal{O}_Y(1))^\otimes m \twoheadrightarrow \Gamma(Y, \mathcal{O}_Y(m))
\]

are surjective. Note that we are not asserting that we can detect very ampleness from \( \tau(X) \) and \( \tau(Y) \), just that we know that such a multiple must exist, so we are free to choose one.

By Remark [5.15](iii), for each \( m > 0 \) the sets of strongly definable lines are dense and open in the Grassmannians of \( |\mathcal{O}_X(m)| \) and \( |\mathcal{O}_Y(m)| \), and thus by Theorem [4.4](ii) there is an isomorphism \( \sigma_m : \kappa_X \to \kappa_Y \) and a \( \sigma \)-linear isomorphism \( \gamma_m : |\mathcal{O}_X(m)| \to |\mathcal{O}_Y(m)| \) that agrees with \( \varphi \) on a Zariski dense open subscheme \( U \subset |\mathcal{O}_X(m)| \).

Consider the diagram of addition maps

\[
\begin{array}{ccc}
|\mathcal{O}_X(m)| & \xrightarrow{\gamma_m} & |\mathcal{O}_Y(m)| \\
|\mathcal{O}_X(1)|^\times m & \xrightarrow{\gamma^m} & |\mathcal{O}_Y(1)|^\times m
\end{array}
\]

Since a general sum of divisors in \( \mathcal{O}(1) \) lies on a strongly definable pencil in \( \mathcal{O}(m) \) by Lemma [6.2.2](ii), we see that the associated diagram

\[
(6.2.5.1)
\]

of multi-semi-linear maps commutes over the \( \kappa \)-points of a Zariski-dense open in each (multi-)projective space. We conclude that diagram \((6.2.5.1)\) commutes. In addition, since diagram \((6.2.5.1)\) comes from an associated commutative diagram of schemes, we see upon computing the action on global functions that \( \sigma_m = \sigma_1 \) for all \( m \).

We will call this map \( \sigma \).

Choosing a lift

\( \tilde{\gamma}_1 : \Gamma(X, \mathcal{O}_X(1)) \to \Gamma(Y, \mathcal{O}_Y(1)) \)

yields an induced \( \sigma \)-linear isomorphism of graded rings

\[
\gamma^\times : A_X := \text{Sym}^\bullet \Gamma(X, \mathcal{O}_X(1)) \to A_Y : \text{Sym}^\bullet \Gamma(Y, \mathcal{O}_Y(1))
\]

that is uniquely defined up to scalars. Consider the ideal sheaves of \( X \) and \( Y \) in the canonical embeddings

\[
\nu_X : X \to |\mathcal{O}_X(1)|^\vee
\]

and

\[
\nu_Y : Y \to |\mathcal{O}_Y(1)|^\vee.
\]

These are homogeneous ideal sheaves \( I_X \subset A_X \) and \( I_Y \subset A_Y \) cutting out the closures of \( X \) and \( Y \). We claim that \( \gamma^\times(I_X) = I_Y \). To show this, it suffices to show that the degree \( m \) components are the same for all \( m \).
Consider the diagram

\[
\begin{array}{ccc}
\Gamma(X, \mathcal{O}_X(m)) & \xrightarrow{\bar{\gamma}_m} & \Gamma(Y, \mathcal{O}(m)) \\
\downarrow p_X & & \downarrow p_Y \\
(A_X)_m = \Gamma(X, \mathcal{O}_X(1)) \otimes_m \Gamma(Y, \mathcal{O}_Y(1)) \otimes_m = (A_Y)_m & & \\
\uparrow & & \uparrow \\
\Gamma(X, \mathcal{O}_X(1)) \times_m & \xrightarrow{\bar{\gamma} \times_m} & \Gamma(Y, \mathcal{O}_Y(1)) \times_m
\end{array}
\]

arising as follows. The vertical arrows are the natural multiplication maps, and the induced linear maps from the universal property of \( \otimes \). The arrow \( \bar{\gamma}_m \) is a lift of \( \gamma_m \).

By the commutativity of diagram (6.2.5.1), we see that this diagram commutes (up to suitably scaling \( \bar{\gamma}_m \)), which implies that \( \gamma_m((I_X)_m) = (I_Y)_m \), as desired.

In summary, we have shown that there exist an isomorphism of graded rings

\[ \bar{\gamma} : \oplus_{m \geq 0} \Gamma(X, \mathcal{O}_X(m)) \to \oplus_{m \geq 0} \Gamma(Y, \mathcal{O}_Y(m)) \]

such that the isomorphism induced by \( \bar{\gamma} \) in degree \( m \)

\[ |\mathcal{O}_X(m)|^y \to |\mathcal{O}_Y(m)|^y \]

is equal to the map given by \( \varphi \). It is useful to elucidate this last compatibility further.

Let

\[ \nu_X, m : X \to |\mathcal{O}_X(m)|^y, \quad \nu_Y, m : Y \to |\mathcal{O}_Y(m)|^y \]

be the canonical embeddings. For \( D \in |\mathcal{O}_X(m)| \) let \( \delta_D \subset |\mathcal{O}_X(m)|^y \) denote the image of those elements \( \lambda \in \Gamma(X, \mathcal{O}_X(m)) \) which vanish on \( D \). Then

\[ D = \{ x \in X | \nu_{X,m}(x) \in \delta_D \} \]

Letting \( \overline{X} \subset |\mathcal{O}_X(1)|^y \) and \( \overline{Y} \subset |\mathcal{O}_Y(1)|^y \) denote the Zariski closures, we then have a commutative diagram

\[
\begin{array}{ccc}
\overline{X} & \xrightarrow{f} & \overline{Y} \\
\downarrow i & & \downarrow j \\
\text{Proj}(\oplus_{m \geq 0} \Gamma(X, \mathcal{O}_X(m))) & \xrightarrow{\bar{\gamma}} & \text{Proj}(\oplus_{m \geq 0} \Gamma(Y, \mathcal{O}_Y(m))) \\
\downarrow & & \downarrow \\
|\mathcal{O}_X(1)|^y & \xrightarrow{\gamma} & |\mathcal{O}_Y(1)|^y
\end{array}
\]

where the horizontal morphisms are isomorphisms.

Moreover, by the commutativity of this diagram, \( f \) acts on the generic points of irreducible members of \( |\mathcal{O}_X(m)| \) in the same way as \( \varphi \). By Lemma 6.2.4, we conclude that \( f \) acts the same as \( \varphi \) on every regular closed point of \( X \). Since \( |X| \) is a Zariski topological space, it follows that \( \varphi \) and \( f \) have the same action on \( |X|^{\text{reg}} \), the regular locus of \( X \). This implies that there are open subschemes \( U \subset X \) and \( V \subset Y \) such that

1. \( \text{codim}(U^c \subset X) \geq 2 \),
2. \( \text{codim}(V^c \subset Y) \geq 2 \),
3. \( f \) induces an isomorphism \( f|_U : U \to V \), and
(4) $f|_U$ induces $\varphi|_U$ on topological spaces.

By Lemma 6.1.3, it follows that $\varphi$ is algebraizable to a unique isomorphism $f : X \to Y$, showing that $\tau$ is fully faithful. □

This completes the proof of Theorem 2.13.

7. COUNTEREXAMPLES TO WEAKER STATEMENTS

One might wonder if the congruence relation on $\text{Eff}(X)$ is really necessary, or if the Zariski topological space itself might suffice to capture $X$. In one direction, we have the following.

Lemma 7.1. Given two primes $p$ and $q$ and two smooth projective surfaces $X$ over $\mathbb{F}_p$ and $Y$ over $\mathbb{F}_q$ each of Picard number 1, any homeomorphism between a curve in $X$ and a curve in $Y$ extends to a homeomorphism $|X| \to |Y|$.

Proof. The proof of the Lemma is essentially contained in the proof of the final Proposition of [8], once one notes that the topological space of such a surface satisfies the axioms laid out in [8, Corollary 1] (even though they are stated there only for $\mathbb{P}^2$). □

Remark 7.2. In particular, there are examples of surfaces of general type that are homeomorphic to the projective plane. Even more bizarrely, one has that $\mathbb{P}^2_{\mathbb{F}_p}$ is homeomorphic to $\mathbb{P}^2_{\mathbb{F}_q}$ for any primes $p$ and $q$. In other words, the linear equivalence relation in the divisorial structure is necessary. The proofs of [8] are heavily reliant on working over the algebraic closure of a finite field.

On the other hand, we don’t know what happens over other fields. It is particularly interesting to consider the complex numbers.

Question 7.3. Suppose $X$ and $Y$ are smooth projective varieties of dimension $n$ over the complex numbers such that the Zariski topological spaces $|X|$ and $|Y|$ are homeomorphic. Are $X$ and $Y$ isomorphic as abstract schemes?

We don’t know the answer for any class of varieties with the exception of curves, where the answer is always “no”. One might also consider similar questions replacing the Zariski topological space with the étale homotopy types.

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