The Brauer group of Kummer surfaces and torsion of elliptic curves

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Introduction

In this paper we are interested in computing the Brauer group of K3 surfaces. To an element of the Brauer–Grothendieck group \( \text{Br}(X) \) of a smooth projective variety \( X \) over a number field \( k \), class field theory associates the corresponding Brauer–Manin obstruction, which is a closed condition satisfied by \( k \)-points inside the topological space of adelic points of \( X \), see [20, Ch. 5.2]. If such a condition is non-trivial, \( X \) is a counterexample to weak approximation, and if no adelic point satisfies this condition, \( X \) is a counterexample to the Hasse principle. The computation of \( \text{Br}(X) \) is thus a first step in the computation of the Brauer–Manin obstruction on \( X \).

Let \( k \) be an arbitrary field with a separable closure \( \overline{k} \), \( \Gamma = \text{Gal}(\overline{k}/k) \). Recall that for a variety \( X \) over \( k \) the subgroup \( \text{Br}_0(X) \subset \text{Br}(X) \) denotes the image of \( \text{Br}(k) \) in \( \text{Br}(X) \), and \( \text{Br}_1(X) \subset \text{Br}(X) \) denotes the kernel of the natural map \( \text{Br}(X) \to \text{Br}(\overline{X}) \), where \( \overline{X} = X \times_k \overline{k} \). In [22] we showed that if \( X \) is a K3 surface over a field \( k \) finitely generated over \( \mathbb{Q} \), then \( \text{Br}(X)/\text{Br}_0(X) \) is finite. No general approach to the computation of \( \text{Br}(X)/\text{Br}_0(X) \) seems to be known; in fact until recently there was not a single K3 surface over a number field for which \( \text{Br}(X)/\text{Br}_0(X) \) was known. One of the aims of this paper is to give examples of K3 surfaces \( X \) over \( \mathbb{Q} \) such that \( \text{Br}(X) = \text{Br}(\mathbb{Q}) \).

We study a particular kind of K3 surfaces, namely Kummer surfaces \( X = \text{Kum}(A) \) constructed from abelian surfaces \( A \). Let \( \text{Br}(X)_n \) denote the \( n \)-torsion subgroup of \( \text{Br}(X) \). Section 1 is devoted to the geometry of Kummer surfaces. We show that there is a natural isomorphism of \( \Gamma \)-modules \( \text{Br}(\overline{X}) \cong \text{Br}(\overline{A}) \) (Proposition 1.3). When \( A \) is a product of two elliptic curves, the algebraic Brauer group \( \text{Br}_1(X) \) often coincides with \( \text{Br}(k) \), see Proposition 1.4.

Section 2 starts with a general remark on the étale cohomology of abelian varieties which may be of independent interest (Proposition 2.2). It implies that if \( n \) is an odd integer, then for any abelian variety \( A \) the group \( \text{Br}(A)_n/\text{Br}_1(A)_n \) is canonically isomorphic to the quotient of \( H^2_\text{ét}(\overline{A}, \mu_n)^\Gamma \) by \( (\text{NS}(A)/n)^\Gamma \), where \( \text{NS}(A) \) is the Néron–Severi group (Corollary 2.3). For any \( n \geq 1 \) we prove that \( \text{Br}(X)_n/\text{Br}_1(X)_n \) is a subgroup of \( \text{Br}(A)_n/\text{Br}_1(A)_n \), and this inclusion is an equality for odd \( n \), see
Theorem 2.4. We deduce that the subgroups of elements of odd order of the transcendental Brauer groups $\text{Br}(X)/\text{Br}_1(X)$ and $\text{Br}(A)/\text{Br}_1(A)$ are naturally isomorphic.

More precise results are obtained in Section 3 in the case when $A = E \times E'$ is a product of two elliptic curves. In this case for any $n \geq 1$ we have

$$\text{Br}(A)_n/\text{Br}_1(A)_n = \text{Hom}_\Gamma(E_n, E'_n)/(\text{Hom}(E, E')/n)^\Gamma$$

(Proposition 3.3). This gives a convenient formula for $\text{Br}(X)_n/\text{Br}_1(X)_n$ when $n$ is odd. See Proposition 3.7 for the case $n = 2$.

In Section 4 we find many pairs of elliptic curves $E, E'$ over $\mathbb{Q}$ such that for $A = E \times E'$ the group $\text{Br}(A)/\text{Br}_1(A)$ is zero or a finite abelian 2-group. For example, if $E$ is an elliptic curve over $\mathbb{Q}$ such that for all primes $\ell$ the representation $\Gamma \to \text{Aut}(E_\ell) \simeq \text{GL}(2, \mathbb{F}_\ell)$ is surjective, then for $A = E \times E$ we have $\text{Br}(A) = \text{Br}_1(A)$, whereas $\text{Br}(\overline{A})^\Gamma \simeq \mathbb{Z}/2^m$ for some $m \geq 1$ (Proposition 4.3). This shows, in particular, that the Hochschild–Serre spectral sequence $H^p(k, H^q_\text{ét}(\overline{A}, G_m)) \Rightarrow H^{p+q}_\text{ét}(A, G_m)$ does not degenerate. For this $A$ the corresponding Kummer surface $\overline{X} = \text{Kum}(A)$ has trivial Brauer group $\text{Br}(\overline{X}) = \text{Br}(\mathbb{Q})$ (whereas $\text{Br}(\overline{X})^\Gamma \simeq \mathbb{Z}/2^m$ for some $m \geq 1$). Note that by a theorem of W. Duke [2] most elliptic curves over $\mathbb{Q}$ have this property, see the remark after Proposition 4.3.

In Section 5 we discuss the resulting infinitely many Kummer surfaces $X$ over $\mathbb{Q}$ such that $\text{Br}(X) = \text{Br}(\mathbb{Q})$, see (25-29) and Examples 3 and 4. In fact most Kummer surfaces $\text{Kum}(E \times E')$ over $\mathbb{Q}$ have trivial Brauer group, see Example A2 in Section 4. We also exhibit Kummer surfaces $X$ with an element of prime order $\ell \leq 13$ in $\text{Br}(X)$ which is not in $\text{Br}_1(X)$. Finally, we discuss the Brauer group of Kummer surfaces that do not necessarily have rational points.

In the follow up paper [6] with Evis Ieronymou we give an upper bound on the size of $\text{Br}(X)/\text{Br}_0(X)$, where $X$ is a smooth diagonal quartic surface in $\mathbb{P}^3_{\mathbb{Q}}$, and give examples when $\text{Br}(X) = \text{Br}(\mathbb{Q})$. The importance of K3 surfaces over $\mathbb{Q}$ such that $\text{Br}(X) = \text{Br}(\mathbb{Q})$ is that there is no Brauer–Manin obstruction, and so the Hasse principle and weak approximation for $\mathbb{Q}$-points can be tested by numerical experiments. It would be even more interesting to get theoretical evidence for or against the Hasse principle and weak approximation on such surfaces.

1 Picard and Brauer groups of Kummer surfaces over an algebraically closed field

Let $k$ be a field of characteristic zero with an algebraic closure $\overline{k}$, and the absolute Galois group $\Gamma = \text{Gal}(\overline{k}/k)$. Let $X$ be a smooth and geometrically integral variety over $k$, and let $\overline{X} = X \times_k \overline{k}$. Let $\text{Br}(X) = H^2_\text{ét}(X, G_m)$ be the Brauer group of $X$, and let $\text{Br}(\overline{X}) = H^2_\text{ét}(\overline{X}, G_m)$ be the Brauer group of $\overline{X}$. For any prime number $\ell$
the Kummer sequence

\[ 1 \rightarrow \mu_{\ell^n} \rightarrow G_m \rightarrow G_m \rightarrow 1 \]
gives rise to the exact sequence of abelian groups

\[ 0 \rightarrow \text{Pic}(X) \otimes \mathbb{Z}/\ell^n \rightarrow H^2_{\acute{e}t}(X,\mu_{\ell^n}) \rightarrow \text{Br}(X)_{\ell^n} \rightarrow 0, \]

and an exact sequence of \( \Gamma \)-modules

\[ 0 \rightarrow \text{Pic}(X) \otimes \mathbb{Z}/\ell^n \rightarrow H^2_{\acute{e}t}(X,\mu_{\ell^n}) \rightarrow \text{Br}(X)_{\ell^n} \rightarrow 0. \]  

(1)

If \( X \) is projective, then \( \text{Pic}(X) \otimes \mathbb{Z}/\ell^n = \text{NS}(X) \otimes \mathbb{Z}/\ell^n \), where \( \text{NS}(X) \) is the Néron–Severi group of \( X \). So in this case we have an exact sequence of \( \Gamma \)-modules

\[ 0 \rightarrow \text{NS}(X) \otimes \mathbb{Z}/\ell^n \rightarrow H^2_{\acute{e}t}(X,\mathbb{Z}/\ell^n) \rightarrow \text{Br}(X)_{\ell^n} \rightarrow 0. \]  

(2)

Passing to the projective limit in (2) we obtain an embedding of \( \Gamma \)-modules

\[ \text{NS}(X) \otimes \mathbb{Z}/\ell \hookrightarrow H^2_{\acute{e}t}(X,\mathbb{Z}/\ell(1)). \]

The Néron–Severi group of an abelian variety or a K3 surface is torsion free, so in these cases \( \text{NS}(X) \) is a submodule of \( H^2_{\acute{e}t}(X,\mathbb{Z}/\ell(1)) \).

**Remark.** Let \( \rho \) be the rank of \( \text{NS}(X) \), and let \( b_2 \) be the second Betti number of \( X \). It is known ([3], Cor. 3.4, p. 82; [11], Ch. 5, Remark 3.29, pp. 216–217) that the \( \ell \)-primary component \( \text{Br}(X)(\ell) \subset \text{Br}(X) \) is an extension of \( H^2_{\acute{e}t}(X,\mathbb{Z}/\ell(1))_{\text{tors}} \) by \((\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho}\). By Poincaré duality, if \( X \) is a surface such that \( \text{NS}(X) \) has no \( \ell \)-torsion, then \( \text{Br}(X)(\ell) \simeq (\mathbb{Q}_\ell/\mathbb{Z}_\ell)^{b_2-\rho} \). It follows that if \( X \) is an abelian variety or a K3 surface we have \( \text{Br}(X) \simeq (\mathbb{Q}/\mathbb{Z})^{b_2-\rho} \).

We write \( k[X] \) for the \( k \)-algebra of regular functions on \( X \), and \( k[X]^* \) for the group of invertible regular functions. We state the following well known fact for future reference.

**Lemma 1.1** Let \( X \) be a smooth and geometrically integral variety over \( k \), and let \( U \subset X \) be an open subset whose complement in \( X \) has codimension at least 2. Then the natural restriction maps

\[ k[X] \rightarrow k[U], \quad \text{Pic}(X) \rightarrow \text{Pic}(U), \quad \text{Br}(X) \rightarrow \text{Br}(U) \]

are isomorphisms.

**Proof** The first two statements are clear, and the last one follows from Grothendieck’s purity theorem, see [3], Cor. 6.2, p. 136. QED

For an abelian variety \( A \) we denote by \( A_n \) the kernel of the multiplication by \( n \) map \([n]: A \rightarrow A\). Let \( \iota \) be the antipodal involution on \( A \), \( \iota(x) = -x \). The set of fixed points of \( \iota \) is \( A_2 \).
Assume now that $A$ is an abelian surface. Let $A_0 = A \setminus A_2$ be the complement to $A_2$, and let $X_0 = A_0/\iota$. The surface $X_0$ is smooth and the morphism $A_0 \to X_0$ is a torsor under $\mathbb{Z}/2$. Let $X$ be the surface obtained by blowing-up the singular points of $A/\iota$. Then $X$ can be viewed as a smooth compactification of $X_0$; the complement to $X_0$ in $X$ is a closed subvariety of dimension 1 which splits over $\bar{k}$ into a disjoint union of 16 smooth rational curves with self-intersection $-2$. We shall call $X$ the Kummer surface attached to $A$, and write $X = \text{Kum}(A)$.

Let $A'$ be the surface obtained by blowing-up the subscheme $A_2$ in $A$, and let $\sigma : A' \to A$ be the resulting birational morphism. Let $\pi : A' \to X$ be the natural finite morphism of degree 2 ramified at $X \setminus X_0$ (cf. [14]). The set $A_2(\bar{k})$ is the disjoint union of $\Gamma$-orbits $\Lambda_1, \ldots, \Lambda_r$. One may view each $\Lambda_i$ as a closed point of $A$ with residue field $K_i$. Then $M_i = \sigma^{-1}(\Lambda_i)$ in $A'$ is the projective line $\mathbb{P}^1_{K_i}$ (cf. [9], Ch. III, Thm. 2.4 and Remark 2.5). It follows that $L_i = \pi(M_i)$ is also isomorphic to $\mathbb{P}^1_{K_i}$.

Since $\sigma : A' \to A$ is a monoidal transformation with smooth centre, the induced maps $\sigma^* : \text{Br}(A) \to \text{Br}(A')$ and $\sigma^* : \text{Br}(A) \to \text{Br}(\bar{A})$ are isomorphisms by [3], Cor. 7.2, p. 138. Let $Y \subset A'$ be an open subset containing $A_0$. The composition of injective maps

$$
\text{Br}(A) \xrightarrow{\sigma^*} \text{Br}(A') \to \text{Br}(Y) \to \text{Br}(A_0)
$$

is an isomorphism by Lemma 1.1, and the same is true after the base change from $k$ to $\bar{k}$. It follows that the following restriction maps are isomorphisms:

$$
\text{Br}(A') \xrightarrow{\sigma^*} \text{Br}(Y), \quad \text{Br}(\bar{A}) \xrightarrow{\sigma^*} \text{Br}(\bar{Y}).
$$

This easily implies that the natural homomorphisms $\text{Br}_1(A) \to \text{Br}_1(A') \to \text{Br}_1(Y)$ are isomorphisms. We also obtain isomorphisms

$$
\text{Br}(A_n)/\text{Br}_1(A)_n \xrightarrow{\sigma^*} \text{Br}(A'_n)/\text{Br}_1(A')_n \xrightarrow{\sigma^*} \text{Br}(Y)_n/\text{Br}_1(Y)_n.
$$

Throughout the paper, we will freely use these isomorphisms, identifying the corresponding groups.

**Proposition 1.2** Let $X_1 \subset X$ be the complement to the union of some of the irreducible components of $X \setminus X_0$ (that is, some of the lines $L_i$). Then there is an exact sequence

$$
0 \to \text{Br}(X) \to \text{Br}(X_1) \to \bigoplus_i K_i^* / K_i^{*2},
$$

where the sum is over $i$ such that $L_i \subset X \setminus X_1$. In particular, the restriction map $\text{Br}(X) \to \text{Br}(X_1)$ induces an isomorphism of the subgroups of elements of odd order. The restriction map $\text{Br}(X) \xrightarrow{\sigma^*} \text{Br}(X_1)$ is an isomorphism of $\Gamma$-modules.

**Proof** Let $Y = \pi^{-1}(X_1)$. From Grothendieck’s exact sequence ([3], Cor. 6.2, p. 137) we obtain the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \to & \text{Br}(X) & \to & \text{Br}(X_1) & \to & \bigoplus_i H^1(L_i, \mathbb{Q}/\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Br}(A') & \xrightarrow{\sigma^*} & \text{Br}(Y) & \to & \bigoplus_i H^1(M_i, \mathbb{Q}/\mathbb{Z})
\end{array}
$$

4
where both sums are over \( i \) such that \( L_i \subset X \setminus X_1 \). Recall that the restriction map \( \text{Br}(A') \to \text{Br}(Y) \) is an isomorphism by (3), hence the right bottom arrow is zero. Let \( \text{res}_{L_i} : \text{Br}(X_1) \to H^1(L_i, \mathbb{Q}/\mathbb{Z}) \) and \( \text{res}_M : \text{Br}(Y) \to H^1(M, \mathbb{Q}/\mathbb{Z}) \) be the residue maps from (6). The double covering \( \pi : A' \to X \) is ramified at \( L_i \), thus \( \text{res}_{L_i}(\pi^* \alpha) = 2 \text{res}_{L_i}(\alpha) \). But this is zero, so that \( \text{res}_{L_i}(\alpha) \) belongs to the injective image of \( H^1_{\text{et}}(\mathbb{P}^1_{K_i}, \mathbb{Z}/2) \) in \( H^1_{\text{et}}(\mathbb{P}^1_{K_i}, \mathbb{Q}/\mathbb{Z}) \). Since \( H^1_{\text{et}}(\mathbb{P}^1_{K_i}, \mathbb{Z}/2) = 0 \) we deduce from the Hochschild–Serre spectral sequence

\[
H^p(K_i, H^q_{\text{et}}(\mathbb{P}^1_{K_i}, \mathbb{Z}/2)) \Rightarrow H^p_{\text{et}}(\mathbb{P}^1_{K_i}, \mathbb{Z}/2)
\]

that \( H^1(\mathbb{P}^1_{K_i}, \mathbb{Z}/2) = K_i^*/K_i^{*2} \). This establishes the exact sequence (5).

The same theorem of Grothendieck [3, Cor. 6.2] gives an exact sequence

\[
0 \to \text{Br}(X) \to \text{Br}(X_1) \to \oplus_i H^1(L_i \times_k \overline{k}, \mathbb{Q}/\mathbb{Z}) = 0.
\]

Since \( L_i \times_k \overline{k} \) is a disjoint union of finitely many copies of \( \mathbb{P}^1_{\overline{k}} \), and \( H^1_{\text{et}}(\mathbb{P}^1_{\overline{k}}, \mathbb{Q}/\mathbb{Z}) = 0 \), this implies the last statement of the proposition. QED

**Proposition 1.3** The natural map \( \pi^* : \text{Br}(\overline{X}_0) \to \text{Br}(\overline{A}_0) \) is an isomorphism, so that the composed map \( (\sigma^*)^{-1} \pi^* : \text{Br}(X) \to \text{Br}(\overline{A}) \) is an isomorphism of \( \Gamma \)-modules.

**Proof** Since \( \pi : A_0 \to X_0 \) is a torsor under \( \mathbb{Z}/2 \) we have the Hochschild–Serre spectral sequence [11, Thm. III.2.20]

\[
H^p(\mathbb{Z}/2, H^q_{\text{et}}(\overline{A}_0, \mathbb{G}_m)) \Rightarrow H^p_{\text{et}}(\overline{X}_0, \mathbb{G}_m).
\]

Let us compute a first few terms of this sequence. By Lemma 1.1 we have

\[
\overline{k}[A_0]^* = \overline{k}^*, \quad \text{Pic}(\overline{A}_0) = \text{Pic}(\overline{A}), \quad \text{Br}(\overline{A}_0) = \text{Br}(\overline{A}).
\]

Since \( k \) has characteristic 0, and \( \mathbb{Z}/2 \) acts trivially on \( \overline{k}^* \), the Tate cohomology group \( H^0(\mathbb{Z}/2, \overline{k}^*) \) is trivial. We have \( H^1(\mathbb{Z}/2, \overline{k}^*) = \mathbb{Z}/2 \). By the periodicity of group cohomology of cyclic groups we obtain \( H^2(\mathbb{Z}/2, \overline{k}^*) = 0 \).

We have an exact sequence of \( \Gamma \)-modules

\[
0 \to A'(\overline{k}) \to \text{Pic}(\overline{A}) \to \text{NS}(\overline{A}) \to 0,
\]

where \( A' \) is the dual abelian surface. The torsion-free abelian group \( \text{NS}(\overline{A}) \) embeds into \( H^2_{\text{et}}(\overline{A}, \mathbb{Z}_l(1)) \), and since the antipodal involution \( \iota \) acts trivially on \( H^2_{\text{et}}(\overline{A}, \mathbb{Z}_l(1)) \), it acts trivially on \( \text{NS}(\overline{A}) \), too. Thus \( H^1(\mathbb{Z}/2, \text{NS}(\overline{A})) = 0 \), so that \( H^1(\mathbb{Z}/2, \text{Pic}(\overline{A})) \) is the image of \( H^1(\mathbb{Z}/2, A'(\overline{k})) \). Since \( \iota \) acts on \( A' \) as multiplication by \(-1\), we have

\[
H^1(\mathbb{Z}/2, A'(\overline{k})) = A'(\overline{k})/(1 - \iota)A'(\overline{k}) = 0.
\]

Putting all this into the spectral sequence and using Proposition 1.2 with \( X_1 = X_0 \) we obtain an embedding \( \text{Br}(\overline{X}) \hookrightarrow \text{Br}(\overline{A}) \). In order to prove that this is an
isomorphism, it suffices to check that the corresponding embeddings \( \text{Br}(X)_{\ell^n} \hookrightarrow \text{Br}(A)_{\ell^n} \) are isomorphisms for all primes \( \ell \) and all positive integers \( n \). It is well known that \( b_2(X) = 22, b_2(A) = 6 \) and \( \rho(X) = \rho(A) + 16 \), see, e.g., [15] or [14]. From this and the remark before Lemma 1.1 follows that the orders of \( \text{Br}(X)_{\ell^n} \) and \( \text{Br}(A)_{\ell^n} \) are the same. This finishes the proof. QED

**Remark 1** The same spectral sequence gives an exact sequence of \( \Gamma \)-modules

\[
0 \to \mathbb{Z}/2 \to \text{Pic}(X_0) \to \text{Pic}(A) \to 0,
\]

where \( \text{Pic}(A) \) is the \( \iota \)-invariant subgroup of \( \text{Pic}(A) \). From (7) we deduce the exact sequence

\[
0 \to \mathbb{Z}/2 \to \text{Pic}(X_0)_{\text{tors}} \to A_2^t \to 0.
\]  

Let \( Z^{16} \subset \text{Pic}(X) = \text{NS}(X) \) be the lattice generated by the classes of the 16 lines. Its saturation \( \Pi \) in \( \text{NS}(X) \) is called the Kummer lattice. In other words, \( \Pi \) is the subgroup of \( \text{NS}(X) \) consisting of linear combinations of the classes of the 16 lines with coefficients in \( \mathbb{Q} \). Since \( \text{NS}(X)/Z^{16} = \text{Pic}(X_0) \), we have \( \Pi/Z^{16} = \text{Pic}(X_0)_{\text{tors}} \).

**Remark 2** Considering (7) modulo torsion and taking into account that \( \text{NS}(A) = \text{NS}(A)_{\text{tors}} = 0 \), we obtain an isomorphism

\[
\text{NS}(X)/\Pi = \text{Pic}(X_0)/\text{Pic}(X_0)_{\text{tors}} \xrightarrow{\sim} \text{NS}(A).
\]

In other words, we have an exact sequence of \( \Gamma \)-modules

\[
0 \to \Pi \to \text{NS}(X) \xrightarrow{\sigma^* \pi^*} \text{NS}(A) \to 0.
\]

**Remark 3** Recall that \( A_2 \) acts on \( X \) and \( X_0 \) compatibly with its action on \( A \) by translations, moreover, the morphisms \( \pi \) and \( \sigma \) are \( A_2 \)-equivariant. Thus the isomorphism \( (\sigma^*)^{-1} \pi^* : \text{Br}(X) \to \text{Br}(A) \) is also \( A_2 \)-equivariant. Since translations of an abelian variety act trivially on its cohomology, the exact sequence (2) shows that the induced action of \( A_2 \) on \( \text{Br}(X) \) is trivial. We conclude that the induced action of \( A_2 \) on \( \text{Br}(X) \) is also trivial.

Let us now assume that \( X \) is the Kummer surface constructed from the abelian surface \( A = E \times E' \), where \( E \) and \( E' \) are elliptic curves. For a divisor \( D \) we write \([D]\) for the class of \( D \) in the Picard group.

Let \( C \subset A \) be a curve, and let \( p : C \to E, p' : C \to E' \) be the natural projections. Then \( p_\circ p' : \text{Pic}^0(E) \to \text{Pic}^0(E') \) defines a homomorphism \( E \to E' \). This gives a well known isomorphism of Galois modules

\[
\text{NS}(A) = \mathbb{Z}[e] \oplus \mathbb{Z}[e'] \oplus \text{Hom}(E, E'),
\]
where \( e = E \times \{0\} \) and \( e' = \{0\} \times E' \), and the \( \Gamma \)-module \( \text{Hom}(E, E') \) is realised inside \( \text{NS}(\overline{A}) \) as the orthogonal complement to \( \mathbb{Z}[e] \oplus \mathbb{Z}[e'] \) with respect to the intersection pairing.

For a curve \( C \subset A \) we denote by \( \sigma^{-1}C \subset A' \) its strict transform in \( A' \) (i.e. the Zariski closure of \( C \cap A_0 \) in \( A' \)). In particular, if \( C \) does not contain a point of order 2 in \( A \), then \( \sigma^{-1}C \) does not meet the corresponding line in \( A' \), and hence \( \pi(\sigma^{-1}C) \) does not meet the corresponding line in \( X \).

We write the \( \overline{k} \)-points in \( E_2 \) as \( \{o, 1, 2, 3\} \) with the convention that \( o \) is the origin of the group law, and similarly for \( E'_2 \). The divisors \( \{i\} \times E', E \times \{j\} \), where \( i \in E_2 \), \( j \in E'_2 \), are \( \iota \)-invariant, thus there are rational curves \( s_j \) and \( l_i \) in \( \overline{X} \) such that \( \pi \) induces double coverings

\[
\sigma^{-1}(E \times \{j\}) \rightarrow s_j, \quad \sigma^{-1}(\{i\} \times E') \rightarrow l_i.
\]

Let \( l_{ij} \) be the line in \( \overline{X} \) corresponding to the 2-division point \( (i, j) \in A_2 \). Note that \( \sigma_* \pi^* \) sends \([s_j], [l_i], [l_{ij}]\) to \([e], [e'], 0\), respectively. Finally, let

\[
f_1 = 2l_o + l_{oo} + l_{o1} + l_{o2} + l_{o3}, \quad f_2 = 2s_o + l_{oo} + l_{1o} + l_{2o} + l_{3o}.
\]

Consider the following 9-element Galois-invariant subsets of \( \text{NS}(\overline{X}) \):

\[
\Lambda = \{[l_{ij}]\}, \quad \Sigma = \{[f_1], [f_2], [l_o], [l_i], [s_j]\},
\]

where \( i \) and \( j \) take all values in \( \{1, 2, 3\} \). Let \( N_\Lambda \) (resp. \( N_\Sigma \)) be the subgroup of \( \text{NS}(\overline{X}) \) generated by \( \Lambda \) (resp. by \( \Sigma \)).

**Proposition 1.4** Let \( A = E \times E' \), where \( E \) and \( E' \) are elliptic curves, and let \( X = \text{Kum}(A) \).

(i) The set \( \Lambda \) (resp. \( \Sigma \)) is a \( \Gamma \)-invariant basis of \( N_\Lambda \) (resp. of \( N_\Sigma \)). There is an exact sequence of \( \Gamma \)-modules

\[
0 \rightarrow N_\Lambda \oplus N_\Sigma \rightarrow \text{NS}(\overline{X}) \rightarrow \text{Hom}(E, E') \rightarrow 0. \tag{11}
\]

(ii) We have \( \text{Br}_1(X) = \text{Br}(k) \) in each of the following cases:

- \( E \) and \( E' \) are not isogenous over \( \overline{k} \),
- \( E = E' \) is an elliptic curve without complex multiplication over \( \overline{k} \),
- \( E = E' \) is an elliptic curve which, over \( \overline{k} \), has complex multiplication by an order \( \mathcal{O} \) of an imaginary quadratic field \( K \), that is, \( \text{End}(E) = \mathcal{O} \), and, moreover, \( \text{H}^1(k, \mathcal{O}) = 0 \) (for example, \( K \subset k \)).

**Proof** (i) Relations (3.8) on p. 3217 of [5] easily imply that all 16 classes \([l_{ij}]\) are in \( N_\Lambda + N_\Sigma \). Then relation (3.9) of loc. cit. shows that \([s_o]\) also belongs to \( N_\Lambda + N_\Sigma \). Recall that \([\Pi : \mathbb{Z}^{16}] = 2^5 \), see Remark 1 above. It is easy to deduce that the
Kummer lattice $\Pi$ is generated by the 16 classes $[l_{ij}]$, together with the differences $[s_i] - [s_j]$ and $[l_i] - [l_j]$ for all possible pairs $(i,j)$. Thus $\Pi \subseteq N_\Lambda + N_\Sigma$.

A straightforward computation of the intersection pairing shows that the lattice generated by $[f_1]$, $[f_2]$ and the classes of 16 lines is freely generated by these elements, and so is of rank 18. This lattice is contained in $N_\Lambda + N_\Sigma$, hence $N_\Lambda + N_\Sigma$ is also of rank 18 and is freely generated by $\Lambda \cup \Sigma$, so that $N_\Lambda + N_\Sigma = N_\Lambda \oplus N_\Sigma$. We have $\sigma_\ast \pi_\ast ([s_j]) = [e]$ and $\sigma_\ast \pi_\ast ([l_i]) = [e']$ for any $i$ and $j$. It follows from the exact sequence (9) that $(N_\Lambda \oplus N_\Sigma)/\Pi$ is a $\Gamma$-submodule of $\text{NS}(\mathcal{A})$ generated by $[e]$ and $[e']$. We now obtain (11) from (10).

(ii) If $E$ and $E'$ are not isogenous, then $\text{Hom}(E, E') = 0$. If $E = E'$ is an elliptic curve without complex multiplication, then $\text{Hom}(E, E') = \mathbb{Z}$ is a trivial $\Gamma$-module. In the last case $\text{Hom}(E, E') = 0$, so in all cases we have $H^1(k, \text{Hom}(E, E')) = 0$. By Shapiro’s lemma $H^1(k, N_\Lambda) = H^1(k, N_\Sigma) = 0$, thus $H^1(k, \text{NS}(\mathcal{X})) = 0$ follows from the long exact sequence of Galois cohomology attached to (11).

The surface $\mathcal{X}$ has $k$-points, for example, on $\overline{l}_\infty \simeq \mathbb{P}^1_k$. This implies that the natural map $\text{Br}(k) \to \text{Br}(\mathcal{X})$ has a retraction, and hence is injective. The same holds for the natural map $H^1_{\text{et}}(k, \mathbb{G}_m) \to H^1_{\text{et}}(\mathcal{X}, \mathbb{G}_m)$. Now from the Hochschild–Serre spectral sequence $H^p(k, H^q_{\text{et}}(\mathcal{X}, \mathbb{G}_m)) \Rightarrow H^{p+q}_{\text{et}}(X, \mathbb{G}_m)$ we obtain a split exact sequence

$$0 \to \text{Br}(k) \to \text{Br}_1(\mathcal{X}) \to H^1(k, \text{Pic}(\mathcal{X})) \to 0.$$  

Since $\text{Pic}(\mathcal{X}) = \text{NS}(\mathcal{X})$, this finishes the proof. QED

2 On étale cohomology of abelian varieties and Kummer surfaces

We refer to [20, Ch. 2] for a general introduction to torsors.

Let $A$ be an abelian variety over $k$, and let $n \geq 1$. Let $\mathcal{T}$ be the $A$-torsor with structure group $A_n$ defined by the multiplication by $n$ map $[n] : A \to A$. Let $[\mathcal{T}]$ be the class of $\mathcal{T}$ in $H^1_{\text{et}}(A, A_n)$, and let $[\overline{\mathcal{T}}]$ be the image of $[\mathcal{T}]$ under the natural map $H^1_{\text{et}}(A, A_n) \to H^1_{\text{et}}(\overline{A}, A_n)^\Gamma$. The cup-product defines a Galois-equivariant bilinear pairing

$$H^1_{\text{et}}(\overline{A}, A_n) \times \text{Hom}(A_n, \mathbb{Z}/n) \to H^1_{\text{et}}(\overline{A}, \mathbb{Z}/n).$$

Pairing with $[\overline{\mathcal{T}}]$ gives a homomorphism of $\Gamma$-modules

$$\tau_A : \text{Hom}(A_n, \mathbb{Z}/n) \to H^1_{\text{et}}(\overline{A}, \mathbb{Z}/n).$$

The following lemma is certainly well known, and is proved here for the convenience of the reader.

**Lemma 2.1** $\tau_A$ is an isomorphism of $\Gamma$-modules.
Proof The two groups have the same number of elements, hence it is enough to prove the injectivity. A non-zero homomorphism \( \alpha : A_n \to \mathbb{Z}/n \) can be written as the composition of a surjection \( \beta : A_n \to \mathbb{Z}/m \) where \( m | n, m \neq 1 \), followed by the injection \( \mathbb{Z}/m \hookrightarrow \mathbb{Z}/n \). The induced map \( H^1_{et}(\mathbb{A}, \mathbb{Z}/m) \to H^1_{et}(\mathbb{A}, \mathbb{Z}/n) \) is injective, hence if \( [T] \cup \alpha = [\alpha, T] = 0 \), then the \( \mathbb{A} \)-torsor \( \beta \ast T \) under \( \mathbb{Z}/m \) is trivial.

A trivial \( \mathbb{A} \)-torsor under a finite group is not connected, whereas the push-forward \( \beta \ast T \) is canonically isomorphic to the quotient \( \mathbb{A}/\text{Ker}(\beta) \), which is connected. This contradiction shows that sending \( \alpha \) to \( [\alpha, T] \) defines an injective homomorphism of abelian groups \( \text{Hom}(A_n, \mathbb{Z}/n) \to H^1_{et}(\mathbb{A}, \mathbb{Z}/n) \). The lemma is proved. QED

Proposition 2.2 Let \( A \) be an abelian variety over \( k \), and let \( m, n \geq 1 \) and \( q \geq 0 \) be integers such that \((n, q!)=1\). Then the natural group homomorphism

\[
H^q_{et}(A, \mu_n \otimes m) \to H^q_{et}(\mathbb{A}, \mu_n \otimes m)
\]

has a section, and hence is surjective.

Proof We break the proof into three steps.

Step 1. Let \( M \) be a free \( \mathbb{Z}/n \)-module of rank \( d \) with a basis \( \{e_i\}_{i=1}^d \), and let \( M^* \) be the dual \( \mathbb{Z}/n \)-module with the dual basis \( \{f_i\}_{i=1}^d \). For each \( q \geq 1 \) we have the identity map

\[
\text{Id}_{\wedge^q M} \in \text{End}_{\mathbb{Z}/n}(\wedge^q M) = \wedge^q M \otimes_{\mathbb{Z}/n} \wedge^q M^*, \quad \text{Id}_{\wedge^q M} = \sum (e_{i_1} \wedge \ldots \wedge e_{i_q}) \otimes (f_{i_1} \wedge \ldots \wedge f_{i_q}),
\]

where \( i_1 < \ldots < i_q \). The multiplication law \( (a \otimes b) \cdot (a' \otimes b') = (a \wedge a') \otimes (b \wedge b') \) turns the ring

\[
\bigoplus_{q \geq 0} \wedge^q M \otimes_{\mathbb{Z}/n} \wedge^q M^*
\]

into a commutative \( \mathbb{Z}/n \)-algebra. A straightforward calculation shows that

\[
(\text{Id}_M)^q = q! \text{Id}_{\wedge^q M}.
\]

Step 2. Recall that the cup-product defines a canonical isomorphism

\[
\wedge^q H^1_{et}(\mathbb{A}, \mathbb{Z}/n) \cong H^q_{et}(\mathbb{A}, \mathbb{Z}/n).
\]

We have a natural homomorphism of \( \mathbb{Z}/n \)-modules

\[
H^q_{et}(\mathbb{A}, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} \wedge^q (A_n) \to H^q_{et}(\mathbb{A}, \wedge^q (A_n)),
\]

and since the abelian group \( \wedge^q (A_n) \) is a product of copies of \( \mathbb{Z}/n \), this is clearly an isomorphism.
Write $M = H^1_{\text{ét}}(\mathfrak{A}, \mathbb{Z}/n)$ and use Lemma 2.1 to identify $A_n$ with $M^*$. We obtain an isomorphism of $\mathbb{Z}/n$-modules

$$H^q_{\text{ét}}(\mathfrak{A}, \wedge^q(A_n)) = H^q_{\text{ét}}(\mathfrak{A}, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} \wedge^q(A_n) = \wedge^q M \otimes_{\mathbb{Z}/n} \wedge^q M^*. \quad (13)$$

The cup-product in étale cohomology gives rise to the map

$$H^1_{\text{ét}}(\mathfrak{A}, A_n)^{\otimes q} \to H^q_{\text{ét}}(\mathfrak{A}, A_{n^q}) \to H^q_{\text{ét}}(\mathfrak{A}, \wedge^q(A_n)),$$

and we denote by $\wedge^q[\mathcal{T}] \in H^q_{\text{ét}}(\mathfrak{A}, \wedge^q(A_n))^\Gamma$ the image of the product of $q$ copies of $[\mathcal{T}]$. Now (12) says that the isomorphism (13) identifies $\wedge^q[\mathcal{T}]$ with $q! \text{Id}_{\wedge^q M}$.

We have an obvious commutative diagram of $\Gamma$-equivariant pairings, where the vertical arrows are isomorphisms:

$$\begin{array}{ccc}
\wedge^q M \otimes_{\mathbb{Z}/n} \wedge^q M^* & \times & \text{Hom}(\wedge^q M^*, \mathbb{Z}/n) \\
\downarrow & & \downarrow \\
H^q_{\text{ét}}(\mathfrak{A}, \mathbb{Z}/n) \otimes_{\mathbb{Z}/n} \wedge^q M^* & \times & \text{Hom}(\wedge^q M^*, \mathbb{Z}/n) \\
\downarrow & & \downarrow \\
H^q_{\text{ét}}(\mathfrak{A}, \wedge^q(A_n)) & \times & \text{Hom}(\wedge^q(A_n), \mathbb{Z}/n) \\
\end{array} \quad (14)$$

The pairing with the $\Gamma$-invariant element $\wedge^q[\mathcal{T}]$ gives a homomorphism of $\Gamma$-modules

$$\text{Hom}(\wedge^q(A_n), \mathbb{Z}/n) \longrightarrow H^q_{\text{ét}}(\mathfrak{A}, \mathbb{Z}/n),$$

which is $q!$ times the identity of $\wedge^q M = \text{Hom}(\wedge^q M^*, \mathbb{Z}/n)$. By assumption $q!$ is invertible in $\mathbb{Z}/n$, so this is an isomorphism of $\Gamma$-modules. Tensoring with the $\Gamma$-module $\mu_n^{\otimes m}$ we obtain an isomorphism of $\Gamma$-modules

$$\text{Hom}(\wedge^q(A_n), \mu_n^{\otimes m}) \longrightarrow H^q_{\text{ét}}(\mathfrak{A}, \mu_n^{\otimes m}).$$

Therefore, pairing with $\wedge^q[\mathcal{T}]$ gives rise to an isomorphism of abelian groups

$$\text{Hom}_\Gamma(\wedge^q(A_n), \mu_n^{\otimes m}) \longrightarrow H^q_{\text{ét}}(\mathfrak{A}, \mu_n^{\otimes m})^\Gamma. \quad (14)$$

**Step 3.** The cup-product in étale cohomology gives rise to the map

$$H^1_{\text{ét}}(A, A_n)^{\otimes q} \to H^q_{\text{ét}}(A, A_{n^q}) \to H^q_{\text{ét}}(A, \wedge^q(A_n)),$$

and we denote by $\wedge^q[\mathcal{T}] \in H^q_{\text{ét}}(A, \wedge^q(A_n))$ the image of the product of $q$ copies of $[\mathcal{T}]$. There is a natural pairing of abelian groups

$$H^q_{\text{ét}}(A, \wedge^q(A_n)) \times \text{Hom}(\wedge^q(A_n), \mu_n^{\otimes m}) \to H^q_{\text{ét}}(A, \mu_n^{\otimes m}).$$

Pairing with $\wedge^q[\mathcal{T}]$ induces a map $\text{Hom}_\Gamma(\wedge^q(A_n), \mu_n^{\otimes m}) \to H^q_{\text{ét}}(A, \mu_n^{\otimes m})$ such that the composition

$$\text{Hom}_\Gamma(\wedge^q(A_n), \mu_n^{\otimes m}) \to H^q_{\text{ét}}(A, \mu_n^{\otimes m}) \to H^q_{\text{ét}}(\mathfrak{A}, \mu_n^{\otimes m})^\Gamma$$

is the isomorphism (14). This proves the proposition. QED

In some cases the condition $(n, q!)$ = 1 can be dropped, see Corollary 3.2 below.
Corollary 2.3 Let $n$ be an odd integer. Then the images of the groups $\text{Br}(A)_n$ and $H^2_{\text{ét}}(\overline{A}, \mu_n)^\Gamma$ in $\text{Br}(\overline{A})_n$ coincide, so that we have an isomorphism

$$\text{Br}(A)_n/\text{Br}_1(A)_n \simeq H^2_{\text{ét}}(\overline{A}, \mu_n)^\Gamma/(\text{NS}(\overline{A})/n)^\Gamma.$$ 

Proof. The Kummer sequences for $A$ and $\overline{A}$ give rise to the following obvious commutative diagram with exact rows, cf. (2):

$$
0 \to (\text{NS}(\overline{A})/n)^\Gamma \to H^2_{\text{ét}}(\overline{A}, \mu_n)^\Gamma \to \text{Br}(\overline{A})_n^\Gamma \\
\uparrow \downarrow \uparrow \downarrow \\
H^2_{\text{ét}}(A, \mu_n) \to \text{Br}(A)_n \to 0
$$

The downward arrow is the section of Proposition 2.2. Both statements follow from this diagram. QED

Theorem 2.4 Let $A$ be an abelian surface, and let $X = \text{Kum}(A)$. Then $\pi^*$ defines an embedding

$$\text{Br}(X)_n/\text{Br}_1(X)_n \hookrightarrow \text{Br}(A)_n/\text{Br}_1(A)_n,$$

which is an isomorphism if $n$ is odd. The subgroups of elements of odd order of the transcendental Brauer groups $\text{Br}(X)/\text{Br}_1(X)$ and $\text{Br}(A)/\text{Br}_1(A)$ are isomorphic.

Proof By Proposition 1.3 we have the commutative diagram

$$
\begin{array}{ccc}
\text{Br}(X)_n & \to & \text{Br}(A)_n \\
\downarrow & & \downarrow \\
\text{Br}(\overline{X})_n & \hookrightarrow & \text{Br}(\overline{A})_n
\end{array}
$$

which implies the desired embedding. Now assume that $n$ is odd. We can write

$$\text{Br}(A)_n = \text{Br}(A)_n^+ \oplus \text{Br}(A)_n^-,$$

where $\text{Br}(A)_n^+$ (resp. $\text{Br}(A)_n^-$) is the $\iota$-invariant (resp. $\iota$-antiinvariant) subgroup of $\text{Br}(A)_n$. The involution $\iota$ acts trivially on $H^2_{\text{ét}}(\overline{A}, \mu_{\ell^m})$ for any $\ell$ and $m$, hence by (2) it also acts trivially on $\text{Br}(\overline{A})$. It follows that for odd $n$ the image of $\text{Br}(A)_n^-$ in $\text{Br}(\overline{A})$ is zero. This gives an isomorphism

$$\text{Br}(A)_n/\text{Br}_1(A)_n = \text{Br}(A)_n^+/\text{Br}_1(A)_n^+.$$ 

Thm. 1.4 of [6] states that if $Y \to X$ is a finite flat Galois covering of smooth geometrically irreducible varieties with Galois group $G$, and $n$ is coprime to $|G|$, then the natural map $\text{Br}(X)_n \to \text{Br}(Y)_n^G$ is an isomorphism. We apply this to the double covering $\pi : A' \to X$. Taking into account the isomorphism $\text{Br}(A) = \text{Br}(A')$ we obtain the following commutative diagram

$$
\begin{array}{ccc}
\text{Br}(X)_n & \to & \text{Br}(A')_n \\
\downarrow & & \downarrow \\
\text{Br}(\overline{X})_n & \hookrightarrow & \text{Br}(\overline{A})_n
\end{array}
$$

Our first statement follows. The second statement follows from the first one once we note that an element of odd order in $\text{Br}(X)/\text{Br}_1(X)$ comes from $\text{Br}(X)_n$ for some odd $n$. QED
3 The case of product of two elliptic curves

We now assume that $A = E \times E'$ is the product of two elliptic curves. In this case we can prove the same statement as in Corollary 2.3 but without the assumption on $n$.

The Künneth formula (see [11], Cor. VI.8.13) gives a direct sum decomposition of $\Gamma$-modules

\[ H^2_{\text{ét}}(A, Z/n) = H^2_{\text{ét}}(E, Z/n) \oplus H^2_{\text{ét}}(E', Z/n) \oplus H^2_{\text{ét}}(A, Z/n)_{\text{prim}}, \]

where

\[ H^2_{\text{ét}}(A, Z/n)_{\text{prim}} = H^1_{\text{ét}}(E, Z/n) \otimes H^1_{\text{ét}}(E', Z/n) \]

is the primitive subgroup of $H^2_{\text{ét}}(A, Z/n)$. On twisting with $\mu_n$ we obtain the decomposition of $\Gamma$-modules

\[ H^2_{\text{ét}}(A, \mu_n) = Z/n \oplus Z/n \oplus H^2_{\text{ét}}(A, \mu_n)_{\text{prim}}, \]

where

\[ H^2_{\text{ét}}(A, \mu_n)_{\text{prim}} = H^1_{\text{ét}}(E, Z/n) \otimes H^1_{\text{ét}}(E', \mu_n). \]

The canonical isomorphism $\tau_E : \text{Hom}(E_n, Z/n) \rightarrow H^1_{\text{ét}}(E, Z/n)$ from Lemma 2.1 gives an isomorphism of $\Gamma$-modules

\[ H^2_{\text{ét}}(A, \mu_n)_{\text{prim}} = \text{Hom}(E_n \otimes E_n', \mu_n). \]

Using the Weil pairing we obtain a canonical isomorphism

\[ H^1_{\text{ét}}(E', \mu_n) = \text{Hom}(E_n', \mu_n) = E_n'. \]

Combining all this gives canonical isomorphisms of $\Gamma$-modules

\[ H^2_{\text{ét}}(A, \mu_n) = Z/n \oplus Z/n \oplus H^2_{\text{ét}}(A, \mu_n)_{\text{prim}} = Z/n \oplus Z/n \oplus \text{Hom}(E_n, E_n'). \]  \hspace{1cm} (17)

Let $p : A \rightarrow E$ and $p' : A \rightarrow E'$ be the natural projections. The multiplication by $n$ map $E \rightarrow E$ defines an $E$-torsor $\mathcal{T}$ with structure group $E_n$. We define the $E'$-torsor $\mathcal{T}'$ similarly. The pullbacks $p^* \mathcal{T}$ and $p'^* \mathcal{T}'$ are $A$-torsors with structure groups $E_n$ and $E_n'$, respectively. Let $[\mathcal{T}] \boxtimes [\mathcal{T}']$ be the product of $p^* \mathcal{T}$ and $p'^* \mathcal{T}'$ under the pairing

\[ H^1_{\text{ét}}(A, E_n) \times H^1_{\text{ét}}(A, E_n') \rightarrow H^2_{\text{ét}}(A, E_n \otimes E_n'). \]

Consider the natural pairing

\[ H^2_{\text{ét}}(A, E_n \otimes E_n') \times \text{Hom}_\Gamma(E_n \otimes E_n', \mu_n) \rightarrow H^2_{\text{ét}}(A, \mu_n). \]
Let
\[ \xi : \text{Hom}_\Gamma(E_n \otimes E'_n, \mu_n) \to H^2_{\text{ét}}(A, \mu_n) \]
be the map defined by pairing with \([T] \boxtimes [T']\).

The following map is defined by the base change from \(k\) to \(\overline{k}\) followed by the Künneth projector to the primitive subgroup:
\[ \eta : H^2_{\text{ét}}(A, \mu_n) \to H^2_{\text{ét}}(\overline{A}, \mu_n)^\Gamma \to H^2_{\text{ét}}(\overline{A}, \mu_n)_{\text{prim}} = \text{Hom}_\Gamma(E_n \otimes E'_n, \mu_n). \]

**Lemma 3.1** We have \(\eta \circ \xi = \text{Id}\). In particular, \(\eta\) has a section, and hence is surjective.

**Proof** We must check that the composed map
\[ \text{Hom}(E_n \otimes E'_n, \mathbb{Z}/n) \to H^2_{\text{ét}}(\overline{A}, \mathbb{Z}/n) \to H^1_{\text{ét}}(\overline{E}, \mathbb{Z}/n) \otimes H^1_{\text{ét}}(\overline{E}', \mathbb{Z}/n) \quad (18) \]
defined by pairing with the image of \([T] \boxtimes [T']\) in \(H^2_{\text{ét}}(\overline{A}, E_n \otimes E'_n)\) followed by the Künneth projector to the primitive subgroup, is the isomorphism
\[ \tau_E \otimes \tau_{E'} : \text{Hom}(E_n \otimes E'_n, \mathbb{Z}/n) \longrightarrow H^1_{\text{ét}}(\overline{E}, \mathbb{Z}/n) \otimes H^1_{\text{ét}}(\overline{E}', \mathbb{Z}/n) \]
(cf. Lemma 2.1). Note that the first arrow in (18) is \(\tau_E \otimes \tau_{E'}\) followed by the composed map
\[ H^1_{\text{ét}}(\overline{E}, \mathbb{Z}/n) \otimes H^1_{\text{ét}}(\overline{E}', \mathbb{Z}/n) \to H^1_{\text{ét}}(\overline{A}, \mathbb{Z}/n) \otimes H^1_{\text{ét}}(\overline{A}, \mathbb{Z}/n) \to H^2_{\text{ét}}(\overline{A}, \mathbb{Z}/n), \quad (19) \]
where the first arrow is \(p^* \otimes p'^*\), and the second one is the cup-product. By [11, Cor. VI.8.13] the composition of (19) with the Künneth projector is the identity, hence the composed map in (18) is \(\tau_E \otimes \tau_{E'}\). QED

**Corollary 3.2** For \(A = E \times E'\) and any \(n \geq 1\) the natural map
\[ H^2_{\text{ét}}(A, \mu_n) \to H^2_{\text{ét}}(\overline{A}, \mu_n)^\Gamma \]
has a section, and hence is surjective.

**Proof** We have a canonical map \(p^* : H^2_{\text{ét}}(E, \mu_n) \to H^2_{\text{ét}}(A, \mu_n)\). By Künneth decomposition and Lemma 3.1 it is enough to check that
\[ H^2_{\text{ét}}(E, \mu_n) \to H^2_{\text{ét}}(\overline{E}, \mu_n)^\Gamma \]
has a section (and similarly for \(E'\)). The Kummer sequences for \(E\) and \(\overline{E}\) give a commutative diagram
\[
\begin{array}{ccc}
0 & \to & \mathbb{Z}/n \\
& & \uparrow \\
& \overset{\sim}{\to} & H^2_{\text{ét}}(\overline{E}, \mu_n) \\
& & \uparrow \\
0 & \to & \text{Pic}(E)/n \\
& & \overset{\sim}{\to} \\
& & H^2_{\text{ét}}(E, \mu_n)
\end{array}
\]
The left vertical arrow is given by the degree map $\text{Pic}(E) \to \mathbb{Z}$. It has a section that sends $1 \in \mathbb{Z}/n$ to the class of the neutral element of $E$ in $\text{Pic}(E)/n$. QED

**Remark.** This shows that if an abelian variety $A$ is a product of elliptic curves, then the condition on $n$ in Proposition 2.2 is superfluous.

Recall that the natural map $\text{Hom}(E, E')/n \to \text{Hom}(E_n, E'_n)$ is injective [12, p. 124]. Write $\text{Hom}(E, E') = \text{Hom}_\Gamma(E, E')$ for the group of homomorphisms $E \to E'$.

**Proposition 3.3** For $A = E \times E'$ we have a canonical isomorphism of $\Gamma$-modules

$$\text{Br}(A)_n = \text{Hom}(E_n, E'_n)/(\text{Hom}(E, E')/n),$$

and a canonical isomorphism of abelian groups

$$\text{Br}(A)_n/\text{Br}_1(A)_n = \text{Hom}_\Gamma(E_n, E'_n)/(\text{Hom}(E, E')/n).$$

**Proof** The Kummer sequences for $A$ and $\overline{A}$ give rise to the commutative diagram

$$
\begin{array}{cccccc}
0 & \to & \text{NS}(\overline{A})/n & \to & H^2_{\text{et}}(\overline{A}, \mu_n) & \to & \text{Br}(\overline{A})_n & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & H^2_{\text{et}}(A, \mu_n) & \to & \text{Br}(A)_n & \to & 0
\end{array}
$$

Using (10) and (17) we rewrite this diagram as follows:

$$
\begin{array}{cccccc}
0 & \to & \text{Hom}(E, E')/n & \to & \text{Hom}(E_n, E'_n) & \to & \text{Br}(\overline{A})_n & \to & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & H^2_{\text{et}}(A, \mu_n) & \to & \text{Br}(A)_n & \to & 0
\end{array}
$$

The upper row here is the first isomorphism of the proposition. From Lemma 3.1 we deduce the commutative diagram

$$
\begin{array}{cccccc}
0 & \to & (\text{Hom}(E, E')/n)^\Gamma & \to & \text{Hom}_\Gamma(E_n, E'_n) & \to & \text{Br}(\overline{A})_n^\Gamma \\
& & \uparrow & & \uparrow & & \uparrow & & \\
& & H^2_{\text{et}}(A, \mu_n) & \to & \text{Br}(A)_n & \to & 0
\end{array}
$$

where the left upward arrow is $\eta$, and the downward arrow is $\xi$. The second isomorphism of the proposition is a consequence of the commutativity of this diagram. QED

Until the end of this section we assume that $n = 2$ and the points of order 2 of $E$ and $E'$ are defined over $k$, i.e. $E_2 \subset E(k)$ and $E'_2 \subset E'(k)$. The above considerations can then be made more explicit. (This construction was previously used in [21], Appendix A.2, see also [5], Sect. 3.2). In this case

$$\text{Br}(A)_2 = \text{Br}(\overline{A})_2^\Gamma = \text{Br}(A)/\text{Br}_1(A).$$
Using the Weil pairing the map $\xi$ gives rise to the map $E_2 \otimes E'_2 \to \text{Br}(A)_2$ whose image maps surjectively onto $\text{Br}(\overline{A})_2$. The elements of $\text{Br}(A)_2$ obtained in this way can be given by symbols as follows. The curves $E$ and $E'$ can be given by their respective equations

$$y^2 = x(x - a)(x - b), \quad v^2 = u(u - a')(u - b'),$$

where $a$ and $b$ are distinct non-zero elements of $k$, and similarly for $a'$ and $b'$. The multiplication by 2 torsor of $E$ is contained in $W$.

Using the Weil pairing the map $\iota_\ast$ sends $(x, y)$ to $(x, -y)$, and $(u, v)$ to $(u, -v)$, hence the Kummer surface $X = \text{Kum}(A)$ is given by the affine equation

$$z^2 = x(x - a)(x - b)y(y - a')(y - b').$$

The four resulting Azumaya algebras on $A$ are written as follows:

$$((x - \mu)(x - b), (u - \nu)(u - b')), \quad \mu \in \{0, a\}, \quad \nu \in \{0, a'\}. \quad (20)$$

We note that the specialisation of any of these algebras at the neutral element of $A$ is $0 \in \text{Br}(k)$. By the above, the classes of the algebras (20) in $\text{Br}(\overline{A})$ generate $\text{Br}(\overline{A})_2$.

The antipodal involution $\iota$ sends $(x, y)$ to $(x, -y)$, and $(u, v)$ to $(u, -v)$, hence the Kummer surface $X = \text{Kum}(A)$ is given by the affine equation

$$z^2 = x(x - a)(x - b)y(y - a')(y - b').$$

We denote by $A_{\mu,\nu}$ the class in $\text{Br}(k(X))$ generated by the corresponding symbol (20).

For $A = E \times E'$ it is convenient to replace $X_0 \subset X$ by a larger open subset. Let us denote by $E^0_2$ the set of $k$-points of $E$ of exact order 2; in other words, $E_2$ is the disjoint union of $\{0\}$ and $E^0_2$. Define $W \subset X$ as the complement to the 9 lines that correspond to the points of $E^0_2 \times E^0_2$. The line $l_{oo} = \pi(\sigma^{-1}(0))$, where $0 \in A(k)$ is the neutral element, is contained in $W$. Choose a $k$-point $Q$ on $l_{oo}$, and denote by $\text{Br}(W)^0$ the subgroup of $\text{Br}(W)$ consisting of the elements that specialise to 0 at $Q$. Since $\text{Br}(\overline{F}_k) = \text{Br}(k)$, we see that $\text{Br}(W)^0$ is the kernel of the restriction map $\text{Br}(W) \to \text{Br}(l_{oo})$, hence $\text{Br}(W)^0$ does not depend on the choice of $Q$.

**Lemma 3.4** We have $A_{\mu,\nu} \in \text{Br}(W)^0_2$ for any $\mu \in \{0, a\}$ and $\nu \in \{0, a'\}$.

**Proof** We have $A_{\mu,\nu} \in \text{Br}(W)$ by [21], Lemma A.2. Every $A_{\mu,\nu}$ lifts to an element of $\text{Br}(A)$ with value 0 at the neutral element of $A$, hence $A_{\mu,\nu} \in \text{Br}(W)^0$. Since $A_{\mu,\nu}$ is the class of a quaternion algebra, we have $A_{\mu,\nu} \in \text{Br}(W)^0_2$. QED

The map $(\mu, \nu) \mapsto A_{\mu,\nu}$ defines a group homomorphism $\omega : E_2 \otimes E'_2 \to \text{Br}(W)^0_2$.

**Proposition 3.5** Assume one of the conditions of Proposition 1.4 (ii). Then we have

(i) $\text{Br}_1(W) = \text{Br}(k)$;
(ii) $\text{Im}(\omega) = \text{Br}(W)^0_2$;
(iii) $\text{Ker}(\omega) = \text{Hom}(\overline{E}, \overline{E}')/2$. 

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Proof (Cf. [21], App. A2.) (i) We have \( \overline{k}[W]^* = \overline{k}^* \), as it follows from \( \overline{k}[A_0]^* = \overline{k}^* \). We also have \( \text{Br}_0(W) = \text{Br}(k) \) since \( W \) has a \( k \)-point. Then the Hochschild–Serre spectral sequence \( H^n(k, H^0(W, G_m)) \Rightarrow H^{n+1}(W, G_m) \) shows that it is enough to prove \( H^1(k, \text{Pic}(W)) = 0 \). In the notation of Proposition 1.4 we have \( \text{Pic}(W) = \text{Pic}(\overline{X})/\text{Nm} \), hence there is an exact sequence of \( \Gamma \)-modules analogous to (11):

\[
0 \to N_\Sigma \to \text{Pic}(W) \to \text{Hom}(\overline{E}, \overline{E'}) \to 0.
\]

By Shapiro’s lemma \( H^1(k, N_\Sigma) = 0 \), thus, under the assumptions of Proposition 1.4 (ii), \( H^1(k, \text{Pic}(W)) = 0 \) follows from the long exact sequence of Galois cohomology.

(ii) Let \( \mathfrak{A} \subset \text{Br}(W) \) be the four-element set \( \{A_{\mu, \nu}\} \), and let \( \overline{\mathfrak{A}} \) be the image of \( \mathfrak{A} \) in \( \text{Br}(\overline{X}) \). By Proposition 1.2 we have \( \text{Br}(W) = \text{Br}(\overline{X}) \), thus we can think of \( \overline{\mathfrak{A}} \) as a subset of \( \text{Br}(\overline{X}) \). The image of \( \overline{\mathfrak{A}} \) under the isomorphism \( (\sigma^*)^{-1} \pi^*: \text{Br}(\overline{X}) \to \text{Br}(A) \) from Proposition 1.3 generates \( \text{Br}(A)_2 \), hence \( \overline{\mathfrak{A}} \) generates \( \text{Br}(\overline{W})_2 \). Therefore, any \( \alpha \in \text{Br}(W)_2 \) can be written as

\[
\alpha = \beta + \sum_{\mu, \nu} \delta_{\mu, \nu} A_{\mu, \nu},
\]

where \( \delta_{\mu, \nu} \in \{0, 1\} \), and \( \beta \in \text{Br}_1(W) \) has value zero at \( Q \). It remains to apply (i).

(iii) By part (i) the natural map \( \text{Br}(W)^0 \to \text{Br}(\overline{W}) \) is injective, and we have just seen that the latter group is naturally isomorphic to \( \text{Br}(A) \). Now our statement follows from the first formula of Proposition 3.3. QED

We now calculate the residues of the \( A_{\mu, \nu} \) at the 9 lines of \( X \setminus W \) (cf. Proposition 1.2 and its proof).

Lemma 3.6 The residues of \( A_{a, a'} \), \( A_{a, 0} \), \( A_{0, a'} \), \( A_{00} \), at the lines \( l_{00}, l_{0, a'}, l_{a, 0}, l_{a, a'} \), written in this order, are the classes in \( k^*/k^{*2} \) represented by the entries of the following matrix:

\[
\begin{pmatrix}
1 & ab & a'b' & -aa' \\
ab & 1 & aa' & a'(a' - b') \\
a'b' & aa' & 1 & a(a - b) \\
-aa' & a'(a' - b') & a(a - b) & 1
\end{pmatrix}
\]

(22)

For any \( \mu \in \{0, a\} \) and \( \nu \in \{0, a'\} \) the product of residues of \( A_{\mu \nu} \) at the three lines \( l_{ij} \), \( i \neq 0 \), \( j \neq 0 \), with fixed first or second index, is \( 1 \in k^*/k^{*2} \).

Proof We write \( \text{res}_{ij} \) for the residue at \( l_{ij} \). The local ring \( O \subset k(X) \) of \( l_{ij} \) is a discrete valuation ring with valuation \( \text{val}: k(X)^* \to \mathbb{Z} \). For \( f, g \in O \setminus \{0\} \) the residue of \( (f, g) \) at \( l_{ij} \) is computed by the following rule: if \( \text{val}(f) = \text{val}(g) = 0 \), then \( \text{res}_{ij}(f, g) \) is trivial, and if \( \text{val}(f) = 0, \text{val}(g) = 1 \), then \( \text{res}_{ij}(f, g) \) is the class in \( k(l_{ij})^*/k(l_{ij})^{*2} \) of the reduction of \( f \) modulo the maximal ideal of \( O \). In our case this class will automatically be in \( k^*/k^{*2} \).
Let us calculate the residues of $A_{00} = (x(x - b), x(y - b'))$. Using the above rule we obtain

$$\text{res}_{0,a'}(A_{00}) = a'(a' - b'), \quad \text{res}_{a,0}(A_{00}) = a(a - b), \quad \text{res}_{a,a'}(A_{00}) = 1.$$ 

Using equation (21) and the relation $(r, -r) = 0$ for any $r \in k(X)^*$ we can write

$$A_{00} = (x(x - b), -(x - a)(y - a')).$$

The residue of $A_{00}$ at $l_{00}$ is then the value of $-(x - a)(y - a')$ at $x = y = 0$, that is, $-aa'$. We thus checked the last row of (22). The residue of $A_{00}$ at $l_{0,a'}$ is the class of $a(b' - a')$, which shows that the product of residues of $A_{00}$ at $l_{00}, l_{0,a'}$ and $l_{0,b'}$ is 1. The calculations in all other cases are quite similar. QED

**Question.** Is there a conceptual explanation of the symmetry of (22)?

Let $r$ be the rank of $\text{Hom}(E, E')$, and let $d$ be the dimension of the kernel of the homomorphism

$$\text{Hom}(E_2, E_2') = E_2 \otimes E_2' \simeq (\mathbb{Z}/2)^4 \to (k^*/k^2)^4$$

given by the matrix (22).

**Proposition 3.7** Let $X = \text{Kum}(E \times E')$, where $E$ and $E'$ are elliptic curves with rational 2-torsion points. Assume one of the conditions of Proposition 1.4 (ii). Then

$$\dim F_2 \text{Br}(X) / \text{Br}(k) = d - r.$$ 

In particular, if $E = E'$ and $d = 1$, then $\text{Br}(X) = \text{Br}(k)$.

**Proof** This follows from Proposition 3.5 (ii) and (iii), and Lemma 3.6. QED

## 4 Brauer groups of abelian surfaces

In the rest of this paper we discuss abelian surfaces of the following types:

(A) $A = E \times E'$, where the elliptic curves $E$ and $E'$ are not isogenous over $\overline{k}$.

(B) $A = E \times E$, where $E$ has no complex multiplication over $\overline{k}$.

(C) $A = E \times E$, where $E$ has complex multiplication over $\overline{k}$.

**Case A.** In case A the Néron–Severi group $\text{NS}(\overline{A})$ is freely generated by the classes $E \times \{0\}$ and $\{0\} \times E'$, hence $H^2_{\text{et}}(\overline{A}, \mu_n)$ is the direct sum of $\Gamma$-modules $\text{NS}(\overline{A}) / n \oplus \text{Br}(\overline{A})_n$, and we have

$$\text{Br}(\overline{A})_n = \text{Hom}(E_n, E'_n).$$  \hspace{1cm} (23)
Proposition 4.1 Let $E$ be an elliptic curve such that the representation of $\Gamma$ in $E_\ell$ is a surjection $\Gamma \to \text{GL}(E_\ell)$ for every prime $\ell$. Let $E'$ be an elliptic curve with complex multiplication over $\overline{k}$, which has a $k$-point of order 6. Then for $A = E \times E'$ we have $\text{Br}(\overline{A})^\Gamma = 0$.

Proof Since $\text{Br}(\overline{A})$ is a torsion group it is enough to prove that for every prime $\ell$ we have $\text{Br}(\overline{A})_\ell^\Gamma = \text{Hom}_\Gamma(E_\ell, E'_\ell) = 0$.

By assumption $E'$ has complex multiplication by some imaginary quadratic field $K$. Thus there exists an extension $k'/k$ of degree at most 2 such that the image of $\text{Gal}(\overline{k}/k')$ in $\text{Aut}(E'_\ell)$ is abelian. Thus the image of $\Gamma$ in $\text{Aut}(E'_\ell)$ is a solvable group.

We note that for $\ell \geq 5$ the group $\text{GL}(2, \overline{F}_\ell)$ is not solvable. This implies that $E$ has no complex multiplication over $\overline{k}$. It follows that $E$ and $E'$ are not isogenous over $\overline{k}$.

The $\Gamma$-module $E_\ell$ is simple, hence any non-zero homomorphism of $\Gamma$-modules $E_\ell \to E'_\ell$ must be an isomorphism. This gives a contradiction for $\ell \geq 5$. If $\ell = 2$ or $\ell = 3$, the curve $E'$ has a $k$-point of order $\ell$, so that $E'_\ell$ is not a simple $\Gamma$-module, which is again a contradiction. QED

Example A1 Let $k = \mathbb{Q}$, let $E$ be the curve $y^2 = x^3 + 6x - 2$ of conductor $2^63^3$, and let $E'$ be the curve $y^2 = x^3 + 1$ with the point $(2,3)$ of order 6. It follows from [18], 5.9.2, p. 318, that the conditions of Proposition 4.1 are satisfied.

Example A2 Here is a somewhat different construction for case A, again over $k = \mathbb{Q}$. Let us call a pair of elliptic curves $(E, E')$ non-exceptional if for all primes $\ell$ the image of the Galois group $\Gamma = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\text{Aut}(E_\ell) \times \text{Aut}(E'_\ell)$ is as large as it can possibly be, that is, it is the subgroup of $\text{GL}(2, \mathbb{F}_\ell) \times \text{GL}(2, \mathbb{F}_\ell)$ given by the condition $\det(x) = \det(x')$. This implies $\text{Hom}_\Gamma(E_\ell, E'_\ell) = 0$, so that $\text{Br}(\overline{A})^\Gamma = 0$, where $A = E \times E'$. For example, let $E$ be the curve $y^2 + y = x^3 - x$ of conductor 37, and let $E'$ be the curve $y^2 + y = x^3 + x^2$ of conductor 43. The curve $E$ has multiplicative reduction at 37, whereas $E'$ has good reduction, therefore $E$ and $E'$ are not isogenous over $\overline{\mathbb{Q}}$. By the remark on page 329 of [18] the pair $(E, E')$ is non-exceptional. In fact, most pairs $(E, E')$ are non-exceptional in a similar sense to the remark after Proposition 4.3 (Nathan Jones, see [7]).

We now explore some other constructions providing an infinite series of examples when $\text{Br}(\overline{A})^\Gamma$ has no elements of odd order. Later we shall show that for such abelian surfaces $A$ we often have $\text{Br}(\text{Kum}(A)) = \text{Br}(\mathbb{Q})$, see Example 3 in Section 5.

Proposition 4.2 Let $E$ be an elliptic curve over $\mathbb{Q}$ such that $\text{val}_5(j(E)) = -2^m$ and $\text{val}_7(j(E)) = -2^n$, where $m$ and $n$ are non-negative integers. Let $E'$ be an elliptic curve over $\mathbb{Q}$ with good reduction at 5 and 7, and with rational 2-torsion, i.e. $E'_2 \subset E'(\mathbb{Q})$. Then $E$ and $E'$ are not isogenous over $\overline{\mathbb{Q}}$, and $\text{Hom}_\Gamma(E_\ell, E'_\ell) = 0$ for any prime $\ell \neq 2$. If $A = E \times E'$, then $\text{Br}(\overline{A})^\Gamma$ is a finite abelian 2-group. 18
Proof Since $j(E)$ is not a 5-adic integer, $E$ has potential multiplicative reduction at 5. But $E'$ has good reduction at 5, so $E$ and $E'$ are not isogenous over $\overline{\mathbb{Q}}$.

Let $p = 5$. Our assumption implies that there exists a Tate curve $\tilde{E}$ over $\mathbb{Q}_p$ such that $E \times \mathbb{Q} \mathbb{Q}_p$ is the twist of $\tilde{E}$ by a quadratic or trivial character

$$\chi : \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \{\pm 1\}.$$ 

Consider the case when $\ell \neq 5$. Let $K$ be the extension of $\mathbb{Q}_p$ defined as follows: if $\chi$ is trivial or unramified, then $K = \mathbb{Q}_p$, and if $\chi$ is ramified, then $K \subset \overline{\mathbb{Q}}_p$ is the invariant subfield of $\text{Ker}(\chi)$. Let $p$ be the maximal ideal of the ring of integers of $K$. We note that in both cases the residue field of $K$ is $\mathbb{F}_p$.

Since $\tilde{E}$ is a Tate curve, the $\ell$-torsion $\tilde{E}_\ell$ contains a Galois submodule isomorphic to $\mu_\ell$. Then the quotient $\tilde{E}_\ell/\mu_\ell$ is isomorphic to the trivial Galois module $\mathbb{Z}/\ell$. Hence there is a basis of $E_\ell$ such that the image of $\text{Gal}(\overline{\mathbb{Q}}_p/K)$ in $\text{Aut}(E_\ell) \simeq GL(2, \mathbb{F}_\ell)$ is contained in the subgroup of upper-triangular matrices. Let $q_\ell$ be the multiplicative period of $\tilde{E}$. Since $\text{val}_p(q_\ell) = -\text{val}_p(j(E))$ is not divisible by the odd prime $\ell$, the image of the inertia group $I(p)$ in $\text{Aut}(E_\ell)$ contains $\text{Id} + N$ for some nilpotent $N \neq 0$, see [17], Ch. IV, Section 3.2, Lemma 1. Thus $E_\ell$ has exactly one non-zero $\text{Gal}(\overline{\mathbb{Q}}_p/K)$-invariant subgroup $C \neq E_\ell$. As a $\text{Gal}(\overline{\mathbb{Q}}_p/K)$-module, $C$ is isomorphic to $\mu_\ell$ if $K \neq \mathbb{Q}_p$, and to $\mu_\ell$ twisted by the unramified character $\chi$ if $K = \mathbb{Q}_p$. The $\text{Gal}(\overline{\mathbb{Q}}_p/K)$-module $E_\ell/C$ is isomorphic to $\mathbb{Z}/\ell$ if $K \neq \mathbb{Q}_p$, and to $\mathbb{Z}/\ell$ twisted by $\chi$ if $K = \mathbb{Q}_p$. In particular, the $\text{Gal}(\overline{\mathbb{Q}}_p/K)$-modules $C$ and $E_\ell/C$ are isomorphic if and only if $K$ contains a primitive $\ell$-th root of unity.

Suppose that there exists a non-zero homomorphism of $\text{Gal}(\overline{\mathbb{Q}}_p/K)$-modules $\phi : E_\ell \to E'_\ell$. Since $E'$ has good reduction at $p$, the inertia $I(p)$ acts trivially on $E'_\ell$; in particular $\phi$ is not an isomorphism. Then $\text{Ker}(\phi) = C$, and $E'_\ell$ contains a $\text{Gal}(\overline{\mathbb{Q}}_p/K)$-submodule isomorphic to $\mathbb{Z}/\ell$ (when $\chi$ is ramified) or $\mathbb{Z}/\ell$ twisted by $\chi$ (when $\chi$ is unramified or trivial). In the first case let $E'' = E'$, and in the second case let $E''$ be the quadratic twist of $E'$ by $\chi$. Then $E''(K)$ contains a point of order $\ell$, so that $E''(K)$ contains a finite subgroup of order $4\ell$. Since $E'$ has good reduction at $p$, the curve $E''$ also has good reduction at $p$. Then the group of $\mathbb{F}_p$-points on the reduction has at least 12 elements, which contradicts the Hasse bound, according to which an elliptic curve over $\mathbb{F}_p$ cannot have more than $p + 2\sqrt{p} + 1$ points, see [8].

It remains to consider the case $\ell = 5$. The above arguments work equally well with $p = 7$. We obtain a contradiction with the Hasse bound since no elliptic curve over $\mathbb{F}_7$ can contain as many as $4\ell = 20$ rational points.

The last statement of (i) follows from formula (23). QED

Example A3 As a curve $E$ in this proposition one can take any curve with equation $y^2 = x(x - a)(x - b)$, where $a$ and $b$ are distinct non-zero integers such that exactly one of the numbers $a$, $b$, $a - b$ is divisible by 5, exactly one is divisible by 7, and none are divisible by 25 or 49. (Then a standard computation shows that $\text{val}_5(j(E)) =$
3.3 it is enough to prove the following lemma:

Proof (i) We note that the argument in the proof of Proposition 4.1 shows that the prime where the second arrow comes from the injection $\mathbb{Z}/\ell \hookrightarrow \mathbb{Z}/\ell$ Proposition 4.3 $\text{Br}(A) = \text{End}(E_n)/\mathbb{Z}/n$, \hspace{1cm} (24)

where $\mathbb{Z}/n$ is the subring of scalars in $\text{End}(E_n)$. 

Remark Let $A = E \times E$, where $E$ is an elliptic curve without complex multiplication over $\overline{\mathbb{F}}$, such that the image of $\Gamma$ in $\text{Aut}(E_2)$ is $\text{GL}(2, \mathbb{F}_2)$. It is easy to check that $\text{Br}(A)^\Gamma = (\text{End}(E_2)/\mathbb{Z}/2)^\Gamma$ has order 2; in fact, the non-zero element of this group can be represented by a symmetric $2 \times 2$-matrix $S$ over $\mathbb{F}_2$ such that $S^3 = \text{Id}$. Thus the 2-primary component of $\text{Br}(A)^\Gamma$ is finite cyclic. In this example the map $H^2_0(\overline{A}, \mathbb{Z}/2)^\Gamma \to \text{Br}(\overline{A})^\Gamma_2$ is zero. By the second formula of Proposition 3.3 the map $\text{Br}(A)_2 \to \text{Br}(\overline{A})_2^\Gamma$ is not surjective. The following proposition shows that the non-zero element of $\text{Br}(\overline{A})^\Gamma_2$ does not belong to the image of the map $\text{Br}(A) \to \text{Br}(\overline{A})^\Gamma$.

Proposition 4.3 Let $A = E \times E$, where $E$ is an elliptic curve such that for every prime $\ell$ the image of $\Gamma$ in $\text{Aut}(E_\ell)$ is $\text{GL}(2, \mathbb{F}_\ell)$. Then we have

(i) $\text{Br}(A) = \text{Br}_1(A)$;

(ii) $\text{Br}(A)^\Gamma \simeq \mathbb{Z}/2^m$ for some $m \geq 1$.

Proof (i) We note that the argument in the proof of Proposition 4.1 shows that the curve $E$ has no complex multiplication. In view of the second formula of Proposition 3.3 it is enough to prove the following lemma:

Lemma 4.4 Let $G \subset \text{GL}(2, \mathbb{Z}_\ell)$ be a subgroup that maps surjectively onto $\text{GL}(2, \mathbb{F}_\ell)$. Let $\text{Mat}_2(\mathbb{Z}/\ell^n)$ be the abelian group of $2 \times 2$-matrices with entries in $\mathbb{Z}/\ell^n$, and let $\text{Mat}_2(\mathbb{Z}/\ell^n)^G$ be the subgroup of matrices commuting with the image of $G$ in $\text{GL}(2, \mathbb{Z}/\ell^n)$. Then for any positive integer $n$ we have $\text{Mat}_2(\mathbb{Z}/\ell^n)^G = \mathbb{Z}/\ell^n \cdot \text{Id}$.

Proof We proceed by induction starting with the obvious case $n = 1$. Suppose we know the statement for $n$, and need to prove it for $n+1$. Consider the exact sequence of $G$-modules

$$0 \to \text{Mat}_2(\mathbb{Z}/\ell) \to \text{Mat}_2(\mathbb{Z}/\ell^{n+1}) \to \text{Mat}_2(\mathbb{Z}/\ell^n) \to 0,$$

where the second arrow comes from the injection $\mathbb{Z}/\ell = \ell^n \mathbb{Z}/\ell^{n+1} \hookrightarrow \mathbb{Z}/\ell^{n+1}$, and the third one is the reduction modulo $\ell^n$. By induction assumption $\text{Mat}_2(\mathbb{Z}/\ell^n)^G = $
\[ \mathbb{Z}/\ell^n \cdot \text{Id}. \] Thus, the map \( \text{Mat}_2(\mathbb{Z}/\ell^{n+1})^G \to \text{Mat}_2(\mathbb{Z}/\ell^n)^G \) is surjective, and every element in \( \text{Mat}_2(\mathbb{Z}/\ell^{n+1})^G \) is the sum of a scalar multiple of \( \text{Id} \) and an element of \( \text{Mat}_2(\mathbb{Z}/\ell)^G \). But \( \text{Mat}_2(\mathbb{Z}/\ell)^G = \mathbb{Z}/\ell \cdot \text{Id} \), and so the lemma, and hence also part (i) of the proposition, are proved.

(ii) For an odd prime \( \ell \) we have a direct sum decomposition of \( \Gamma \)-modules \( \text{End}(E_\ell) = \mathbb{Z}/\ell \oplus \text{Br}(\mathcal{A})_\ell \), where \( \text{Br}(\mathcal{A})_\ell \) is identified with the group of endomorphisms of trace zero. Our assumption implies that \( \text{Br}(\mathcal{A})_\ell^\Gamma = 0 \). The remark before the proposition shows that \( \text{Br}(\mathcal{A})^\Gamma \) is a finite cyclic 2-group. QED

**Remark.** By a theorem of W. Duke [2] ‘almost all’ elliptic curves over \( \mathbb{Q} \) satisfy the assumption of Proposition 4.3. More precisely, if \( y^2 = x^3 + ax + b \) is the unique equation for \( E \) such that \( a, b \in \mathbb{Z} \) and \( \gcd(a^3, b^2) \) does not contain twelfth powers, the height \( H(E) \) of \( E \) is defined to be \( \max(|a|^3, |b|^2) \). For \( x > 0 \) write \( C(x) \) for the set of elliptic curves \( E \) over \( \mathbb{Q} \) (up to isomorphism) such that \( H(E) \leq x^6 \), and \( \mathcal{E}(x) \) for the set of curves in \( C(x) \) for which there exists a prime \( \ell \) such that the image of \( \Gamma \) in \( \text{Aut}(E_\ell) \) is not equal to \( \text{GL}(2, \mathbb{F}_\ell) \). Then \( \lim_{x \to +\infty} |\mathcal{E}(x)|/|C(x)| = 0 \). By Proposition 1.4 and Theorem 2.4 this implies that for most Kummer surfaces

\[
  z^2 = (x^3 + ax + b)(y^3 + ay + b)
\]

we have \( \text{Br}(X) = \text{Br}(\mathbb{Q}) \). In particular, there are infinitely many such surfaces.

**Proposition 4.5** Let \( E \) be an elliptic curve over \( \mathbb{Q} \) satisfying the assumptions of Proposition 4.2. Then \( E \) has no complex multiplication, and \( \text{End}_\Gamma(E_\ell) \) is the subring of scalars \( \mathbb{F}_\ell \cdot \text{Id} \subset \text{End}(E_\ell) \) for any prime \( \ell \neq 2 \). If \( A = E \times E \), then \( \text{Br}(\mathcal{A})^\Gamma \) is a finite abelian 2-group.

**Proof** The curve \( E \) has no complex multiplication because \( j(E) \) is not an algebraic integer. Let \( \ell \) be an odd prime, \( \ell \neq p = 5 \), and let \( \phi \) be a non-zero endomorphism of the \( \Gamma \)-module \( E_\ell \) such that \( \text{Tr}(\phi) = 0 \). From the proof of Proposition 4.2 we know that there exists a nilpotent \( N \neq 0 \) in \( \text{End}(E_\ell) \) such that the image of \( I(p) \) in \( \text{Aut}(E_\ell) \) contains \( \text{Id} + N \). Since \( \phi \) is an endomorphism of the \( I(p) \)-module \( E_\ell \), it commutes with \( N \), and it follows from \( \text{Tr}(\phi) = 0 \) that \( \phi \) is also nilpotent. As was explained in the proof of Proposition 4.2, the existence of such an endomorphism \( \phi \) implies that \( \text{Gal}(\overline{\mathbb{Q}}_p/K) \)-modules \( \mathbb{Z}/\ell \) and \( \mu_\ell \) are isomorphic. However, \( K \) does not contain non-trivial roots of 1 of order \( \ell \) when \( p = 5 \) and \( \ell \neq 5 \) is odd, because the residue field of \( K \) is \( \mathbb{F}_5 \). This contradiction shows that \( \text{End}_\Gamma(E_\ell) \) is the subring of scalars \( \mathbb{Z}/\ell \subset \text{End}(E_\ell) \). If \( \ell = 5 \) we repeat these arguments with \( p = 7 \) taking into account that \( \mathbb{Q}_7 \), and hence \( K \), does not contain non-trivial 5-th roots of 1. QED

**Example B1** Let \( A = E \times E \), where \( E/\mathbb{Q} \) is an elliptic curve such that the representation of \( \Gamma \) in \( E_\ell \) is a surjection \( \Gamma \to \text{GL}(E_\ell) \) for every odd prime \( \ell \). Then \( E \) has no complex multiplication over \( \overline{\mathbb{Q}} \) (see the proof of Proposition 4.1). Then
let $\ell$. Theorem 4.6 shows that in each of these two cases we have $\text{Br}(\mathcal{A}) = \text{Br}(\mathbb{Q})$ (see Example 4 in the next section).

Case C. Lastly, we would like to consider the case when $A = E \times E$, where $E$ is an elliptic curve with complex multiplication.

Theorem 4.6 Let $E$ be an elliptic curve over $\mathbb{Q}$ with complex multiplication, and let $\ell$ be an odd prime such that $E$ has no rational isogeny of degree $\ell$, i.e., $E_{\ell}$ does not contain a Galois-invariant subgroup of order $\ell$. Let $G_{\ell}$ be the image of $\Gamma = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in $\text{Aut}(E_{\ell})$. Then $G_{\ell}$ is nonabelian, the order $|G_{\ell}|$ is not divisible by $\ell$, and the centralizer of $G_{\ell}$ in $\text{End}(E_{\ell})$ is $\mathbb{F}_{\ell} = \mathbb{Z}/\ell\mathbb{Z}$.

The theorem remains true if one replaces $\mathbb{Q}$ by any real number field (with the same proof).

Proof of Theorem Suppose that $E$ has complex multiplication by an order $\mathcal{O}$ of an imaginary quadratic field $K$, that is, $\text{End}(\overline{E}) = \mathcal{O}$. We start with the observation that $\ell$ is unramified in $\mathcal{O}$, or, equivalently, the 2-dimensional $\mathbb{F}_{\ell}$-algebra $\mathcal{O}/\ell \subset \text{End}(E_{\ell})$ has no nilpotents. Indeed, if the radical of $\mathcal{O}/\ell$ is non-zero, it is an $\mathbb{F}_{\ell}$-vector space of dimension 1, and so is spanned by one element. Its kernel in $E_{\ell}$ is a Galois-invariant cyclic subgroup of order $\ell$. We assumed that such subgroups do not exist, so this is a contradiction.

Therefore, $\mathcal{O}/\ell$ is either $\mathbb{F}_{\ell} \oplus \mathbb{F}_{\ell}$ or the field $\mathbb{F}_{\ell^2}$. In the first case $(\mathcal{O}/\ell)^*$ is a split Cartan subgroup of order $(\ell - 1)^2$, whereas in the second case it is a non-split Cartan subgroup of order $\ell^2 - 1$, so that $\ell$ does not divide $|(\mathcal{O}/\ell)^*|$. On the other hand, the image of $\text{Gal}(\overline{\mathcal{O}}/K)$ in $\text{Aut}(E_{\ell})$ commutes with the Cartan subgroup $(\mathcal{O}/\ell)^*$, and so belongs to $(\mathcal{O}/\ell)^*$. Since $\text{Gal}(\overline{\mathcal{O}}/K)$ is a subgroup of $\Gamma$ of index 2, we conclude that the order of $G_{\ell}$ divides $2|(\mathcal{O}/\ell)^*|$ and so is not divisible by $\ell$.

The group $G_{\ell}$ contains an element $c$ corresponding to the complex conjugation. Any $z \in \mathcal{O} \setminus \ell \mathcal{O}$ such that $\text{Tr}(z) = 0$ anticommutes with complex conjugation. Since $\ell$ is odd, the non-zero image of $z$ in $\mathcal{O}/\ell$ anticommutes with $c$. Thus $c$ is not a scalar; in particular, $c$ has exact order 2. If $G_{\ell}$ is abelian, then both eigenspaces of $c$ in $E_{\ell}$ are Galois-invariant cyclic subgroups of order $\ell$, but these do not exist. This implies that $G_{\ell}$ is nonabelian.

Finally, the absence of Galois-invariant order $\ell$ subgroups in $E_{\ell}$ implies that the $G_{\ell}$-module $E_{\ell}$ is simple, so the centralizer of $G_{\ell}$ in $\text{End}(E_{\ell})$ is $\mathbb{F}_{\ell}$. QED

Corollary 4.7 Let $A = E \times E$, where $E$ is an elliptic curve over $\mathbb{Q}$ with complex multiplication, and let $\ell$ be an odd prime such that $E$ has no rational isogeny of degree $\ell$. Then $\text{Br}(\mathcal{A})^\ell_{\ell} = 0$. 

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Proof It follows from Theorem 4.6 that the Γ-module $\text{End}(E_\ell)$ is semisimple, hence $H^2_{\text{ét}}(\mathbb{A}, \mu_\ell) = (\mathbb{Z}/\ell)^2 \oplus \text{End}(E_\ell)$ is also semisimple. Thus $H^2_{\text{ét}}(\mathbb{A}, \mu_\ell) = \text{NS}(\mathbb{A})/\ell \oplus \text{Br}(\mathbb{A})_\ell$ is a direct sum of Γ-modules. Since the identity in $\text{End}(E_\ell)$ corresponds to the diagonal in $E \times E$, it is contained in $\text{NS}(\mathbb{A})/\ell$. By Theorem 4.6 we have $H^2_{\text{ét}}(\mathbb{A}, \mu_\ell) \subset \text{NS}(\mathbb{A})/\ell$, so that $\text{Br}(\mathbb{A})_\ell = 0$. QED

Example C1 Let $A = E \times E$, where $E$ is the curve $y^2 = x^3 - x$ with complex multiplication by $\mathbb{Z}[(\sqrt{-1})]$. An application of sage [23] gives that every isogeny of prime degree $E \to E'$ defined over $\mathbb{Q}$ is the factorization by a subgroup of $E(\mathbb{Q})_{\text{tors}} = E_2$. Hence $\text{Br}(\mathbb{A})_\ell = 0$ for every odd prime $\ell$.

Example C2 Let $A = E \times E$, where $E$ is the curve $y^2 = x^3 - 1$ with complex multiplication by $\mathbb{Z}[(\frac{1 + \sqrt{-3}}{2})]$. An application of sage gives that every isogeny of prime degree $E \to E'$ over $\mathbb{Q}$ is the factorization by a subgroup of $E(\mathbb{Q})_{\text{tors}} \simeq \mathbb{Z}/6$. Hence $\text{Br}(\mathbb{A})_\ell = 0$ for every prime $\ell \geq 5$.

5 Brauer groups of Kummer surfaces

Example 1 Let $k = \mathbb{Q}$. Examples A1 and A2 show that the Kummer surfaces $X$ given by the following affine equations have trivial Brauer group $\text{Br}(X) = \text{Br}(\mathbb{Q})$:

$$z^2 = (x^3 + 6x - 2)(y^3 + 1), \quad (25)$$
$$z^2 = (4x^3 - 4x + 1)(4y^3 + 4y^2 + 1). \quad (26)$$

In both examples we have $\text{Br}(\mathbb{X})_\ell = 0$.

Example 2 Other examples can be obtained using Proposition 4.3 in conjunction with Theorem 2.4. For example, for the following Kummer surface $X$ we also have $\text{Br}(X) = \text{Br}(\mathbb{Q})$, whereas $\text{Br}(\mathbb{X})_\ell \simeq \mathbb{Z}/2^m$ for some $m \geq 1$:

$$z^2 = (x^3 + 6x - 2)(y^3 + 6y - 2). \quad (27)$$

The interest of the following series of examples is that for them the image of $\text{Br}(A)$ in $\text{Br}(\mathbb{X})$ contains $\text{Br}(\mathbb{A})_2$, so in order to prove the triviality of $\text{Br}(X)$ we need to compute the residues at the nine lines in $X \setminus W$.

Example 3 Let $X$ be the Kummer surface over $\mathbb{Q}$ with affine equation

$$z^2 = x(x - a)(x - b)y(y - a')(y - b'),$$

such that $a = 5 + 35m$, $b = 7 + 35n$, where $m, n \in \mathbb{Z}$, $m$ is not congruent to 2 modulo 5, $n$ is not congruent to 4 modulo 7, and $a' = 35m' + 1$, $b' = 35n' + 2$ for any $m', n' \in \mathbb{Z}$. We have $X = \text{Kum}(E \times E')$, where the elliptic curves $E$ and $E'$ are as in Example A3. Since $X(\mathbb{Q}) \neq \emptyset$ we see that $\text{Br}(\mathbb{Q})$ is a direct factor of $\text{Br}(X)$.
By Propositions 4.2 and 1.4 (ii) to show that \( \text{Br}(X) = \text{Br}(\mathbb{Q}) \) it is enough to prove that every element of \( \text{Br}(X) \) of order 2 is algebraic. By Proposition 3.7 we need to compute the dimension \( d \) of the kernel of the matrix (22). Considering the first two entries in each row, and taking their valuations at 5 and 7 immediately shows that no product of some of the rows of (22) is trivial. Thus \( d = 0 \), hence \( \text{Br}(X) = \text{Br}(\mathbb{Q}) \).

**Example 4** Let \( X = \text{Kum}(E \times E) \), where \( E \) is as in Example 3, or the elliptic curve with conductor 24 or 40 mentioned in Example B1. In the latter case \( X \) is given by one of the following equations:

\[
\begin{align*}
z^2 &= (x - 1)(x - 2)(x + 2)(y - 1)(y - 2)(y + 2), \\
z^2 &= (x + 1)(x + 2)(x - 3)(y + 1)(y + 2)(y - 3).
\end{align*}
\]  
(28)  
(29)

One checks that the dimension of the kernel of (22) is 1, so that \( \text{Br}(X) = \text{Br}(\mathbb{Q}) \) by Proposition 3.7.

**Kummer surfaces without rational points.** There is a more general construction of Kummer surfaces than the one previously considered. Let \( c \) be a 1-cocycle of \( \Gamma \) with coefficients in \( A_2 \) so that \([c] \in H^1(k, A_2)\). All quasi-projective varieties and Galois modules acted on by the \( k \)-group scheme \( A_2 \) can be twisted by \( c \). The twist of \( A \) is a principal homogeneous space \( A^c \), also called a 2-covering of \( A \). The action of \( A_2 \) on \( A \) by translations descends to an action of \( A_2 \) on \( X = \text{Kum}(A) \), so we obtain a twisted Kummer surface \( X^c \) together with morphisms \( A^c \leftarrow A^c \rightarrow X^c \).

For example, if \( A = E \times E' \), a 2-covering \( C \) of \( E \) is given by \( y^2 = f(x) \), where \( f(x) \) is a separable polynomial of degree 4, and a 2-covering \( C' \) of \( E' \) is given by a similar equation \( y^2 = g(x) \), then the twisted Kummer surface \( X^c \) is given by the affine equation

\[
z^2 = f(x)g(y).
\]

The Hasse principle on such surfaces over number fields was studied in [21].

**Proposition 5.1** Suppose that for every integer \( n > 1 \) we have \( H^2_{\text{ét}}(A, \mu_n)^\Gamma = (\text{NS}(A)/n)^\Gamma \) (this condition is satisfied when \( \text{Br}(A)^\Gamma = 0 \)). Then \( \text{Br}(X^c) = \text{Br}_1(X^c) \) for any \([c] \in H^1(k, A_2)\).

**Proof** By Remark 3 after Proposition 1.3, \( A_2(\bar{k}) \) acts trivially on \( \text{Br}(A) \) and on \( \text{Br}(\bar{X}) \), thus we have the following isomorphisms of \( \Gamma \)-modules:

\[
\text{Br}(X^c) \simeq \text{Br}(\bar{X}) \simeq \text{Br}(\bar{A}) \simeq \text{Br}(A).
\]

Translations act trivially on étale cohomology groups of \( \bar{A} \), hence we have a canonical isomorphism of \( \Gamma \)-modules \( H^2_{\text{ét}}(\bar{A}, \mu_n) \simeq H^2_{\text{ét}}(\bar{A}, \mu_n) \). In the commutative diagram

\[
\begin{array}{ccc}
H^2_{\text{ét}}(\bar{A}, \mu_n)^\Gamma & \to & \text{Br}(\bar{A})^\Gamma_n \\
\uparrow & & \uparrow \\
H^2_{\text{ét}}(A^c, \mu_n) & \to & \text{Br}(A^c)_n
\end{array}
\]
the bottom arrow is surjective, and the top arrow is zero by assumption. It follows that $\text{Br}(A^c) = \text{Br}_1(A^c)$. We conclude by Theorem 2.4. QED

This proposition in conjunction with Proposition 4.1 gives many examples of twisted Kummer surfaces $X^c$ such that $\text{Br}(X^c) = \text{Br}_1(X^c)$.

**Kummer surfaces with non-trivial transcendental Brauer group.** Let $E$ be an elliptic curve over $\mathbb{Q}$. As pointed out by Mazur [10, p. 133] the elliptic curves $E'$ such that the Galois modules $E_\ell$ and $E'_\ell$ are symplectically isomorphic correspond to $\mathbb{Q}$-points on the modular curve $X(\ell)$ twisted by $E_\ell$. Thus for $\ell \leq 5$ there are infinitely many possibilities for $E'$ due to the fact that the genus of $X(\ell)$ is zero, see [19], [16]. Examples of pairs of non-isogenous elliptic curves with isomorphic Galois modules $E_\ell \simeq E'_\ell$ for $\ell = 7, 11$ and 13 can be found in [4] and [1]. Let $X = \text{Kum}(E \times E')$. For $\ell > 2$ our results imply that $\text{Br}(X)$ contains an element of order $\ell$ whose image in $\text{Br}(\overline{X})$ is non-zero.

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**References**

[1] H. Darmon. Serre’s conjectures. In: *Seminar on Fermat’s Last Theorem* (Toronto, 1993–1994) CMS Conf. Proc. 17, Amer. Math. Soc., 1995, 135–153.

[2] W. Duke. Elliptic curves with no exceptional primes. *C. R. Acad. Sci. Paris* Sér. I Math. 325 (1997) 813–818.

[3] A. Grothendieck. Le groupe de Brauer. In: *Dix exposés sur la cohomologie des schémas*. North-Holland, 46–188.

[4] E. Halberstadt. Sur la courbe modulaire $X_{\text{ndép}}(11)$. *Experiment. Math.* 7 (1998) 163–174.

[5] D. Harari and A.N. Skorobogatov. Non-abelian descent and the arithmetic of Enriques surfaces. *Int. Math. Res. Not.* 52 (2005) 3203–3228.
[6] E. Ieronymou, A.N. Skorobogatov and Yu.G. Zarhin. On the Brauer group of diagonal quartic surfaces. (With an appendix by P. Swinnerton-Dyer) J. London Math. Soc., to appear. arXiv:0912.2865

[7] N. Jones. Pairs of elliptic curves with maximal Galois representations at all primes. Preprint, 2010.

[8] A. Knapp. *Elliptic curves*. Princeton University Press, 1992.

[9] Yu.I. Manin. *Cubic Forms*. North-Holland, 1986.

[10] B. Mazur. Rational isogenies of prime degree (with an appendix by D. Goldfeld). *Invent. Math.* 44 (1978) 129–162.

[11] J.S. Milne. *Étale cohomology*. Princeton University Press, 1980.

[12] J.S. Milne. Abelian varieties. In: *Arithmetic geometry*, G. Cornell, J.H. Silverman, eds. Springer-Verlag, 1986.

[13] D. Mumford. *Abelian varieties*. Oxford University Press, 1970.

[14] V.V. Nikulin. Kummer surfaces. *Izv. Akad. Nauk SSSR Ser. Mat.* 39 (1975) 278–293. English translation: *Math. USSR Izvestia* 9 (1975) 261-275.

[15] I.I. Piatetskii-Shapiro and I.R. Shafarevich. Torelli’s theorem for algebraic surfaces of type K3. *Izv. Akad. Nauk SSSR Ser. Mat.* 35 (1971) 530–572. English translation: *Math. USSR Izvestia* 5 (1971) 547–588.

[16] K. Rubin and A. Silverberg. Mod 2 representations of elliptic curves. *Proc. Amer. Math. Soc.* 129 (2001) 53–57.

[17] J-P. Serre. *Abelian ℓ-adic representations and elliptic curves*. With the collaboration of W. Kuyk and J. Labute. Revised reprint of the 1968 original. *Research Notes in Mathematics* 7 A.K. Peters, Ltd., 1998.

[18] J-P. Serre. Propriétés galoisiennes des points d’ordre fini des courbes elliptiques. *Inv. Math.* 15 (1972) 259–331.

[19] A. Silverberg. Explicit families of elliptic curves with prescribed mod $N$ representations. In: *Modular forms and Fermat’s last theorem*, G. Cornell, J.H. Silverman, G. Stevens, eds. Springer-Verlag, 1997, 447–461.

[20] A. Skorobogatov. *Torsors and rational points*. Cambridge University Press, 2001.

[21] A.N. Skorobogatov and Sir Peter Swinnerton-Dyer. 2-descent on elliptic curves and rational points on certain Kummer surfaces. *Adv. Math.* 198 (2005) 448–483.
[22] A.N. Skorobogatov and Yu.G. Zarhin. A finiteness theorem for the Brauer group of abelian varieties and K3 surfaces. *J. Alg. Geom.* **17** (2008) 481–502.

[23] W.A. Stein et al., *Sage Mathematics Software*. [http://www.sagemath.org](http://www.sagemath.org).

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