Region of fidelities for a $1 \rightarrow N$ universal qubit quantum cloner

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I. INTRODUCTION

A. Historical overview: the no-cloning theorem

In 1982 Nick Herbert published a paper about faster than light communication, based on quantum correlations. He called his project FLASH, an acronym for 'First Laser-Amplified Superluminal Hookup' [1]. Of course, the idea was incorrect but it was the source for Wootters and Żurek [2] and independently Dieks [3] to establish the so-called no-cloning theorem. The proof (we follow the version of the proof as in [4], [5], [6]) is the following: consider a cloning unitary operation $U_{cl}$ that is able to clone every given qubit. Before the copying process, we have states:

\[ |A⟩ = |x⟩|0⟩|M⟩, \]
\[ |B⟩ = |x’⟩|0⟩|M⟩, \]

where $|x⟩$, $|x’⟩$ denote ‘text’ to copy for each state, $|0⟩$ represents a ‘blank card’ and $|M⟩$ is a cloning machine state ($M$ denotes the cloning machine). Acting $U_{cl}$ on (1), we should get:

\[ U_{cl}|A⟩ = |x⟩|x⟩|M’⟩, \]
\[ U_{cl}|B⟩ = |x’⟩|x’⟩|M’⟩, \]

where $|M’⟩, |M”⟩$ represent new states of the machine.

Now let us consider a scalar product in both cases:

\[ ⟨A|B⟩ = ⟨x|x’⟩, \]

since $⟨0|0⟩ = ⟨M|M⟩ = 1$. On the other hand, for the second case, we have:

\[ ⟨U_{cl}A|U_{cl}B⟩ = (⟨x|x’⟩)^2⟨M’|M’⟩. \]

Comparing (3) and (4), we get that in the case of $0 < |⟨x|x’⟩| < 1$ scalar products do not match each other (but from the property of the unitary action, they should be the same). So, we have contradiction and it proves that one is not able to clone initially unknown quantum states. In general, the no-cloning theorem comes from the linearity of quantum mechanics [2], [3]. Note that the no-cloning theorem could be generalized [7] or even strengthened [8].

B. Beyond the no-cloning theorem: origins of quantum cloning

Although, the no-cloning theorem is fundamental for quantum physics, it is not of great use in practice. The no-cloning theorem states that one is not able to copy an arbitrary quantum state. However, it is well known that performing ideal operations in physics and especially in quantum physics is impossible. On the other hand, it is obvious that one can copy, though perhaps with a very bad quality. Thus it is crucial to know the ultimate bounds for the quality of copying.

Several years after Wootters and Żurek paper has been published, namely in 1996, Hillery and Bužek published a paper called 'Quantum copying: beyond the no-cloning theorem' [9]. It was the first time, when the above question regarding imperfect cloning has been formulated. Subsequently, the subject was a matter of wide research (see [7] and references therein for a comprehensive review). Here, we would like to point only a few (regarding the cloning of finite quantum states only). Quite soon, the Bužek-Hillery $1 \rightarrow 2$ (qubits) Quantum Cloning Machine (QCM) (for all formal definitions of quantum cloning machines, we refer to [7]), was generalized to the case $N_1 \rightarrow N_2$ first for qubits by Bruß et al. in [10], and by Gisin and Massar in [11], and then for arbitrary-dimensional states.
by Werner in [12], and Keyl and Werner in [13]. Thus
the family of symmetric Universal Quantum Cloning
Machines (UQCM) is, at present, well known. What
is more, the asymmetric UQCM were also very heav-
ily studied, let us mention here works of Braunstein
e et al. [14], Cerf [15], Fiurášek et al. [16], and Iblis-
dir et al. [17], [18]. To emphasize the relevance of all
these works, let us mention that formalism of asym-
metric cloning machines is important in the context of
quantum cryptography - it can be used in studies of
relations between the eavesdropper’s information gain
and the noise in the channel. Later, efforts were made
to unify these two kinds of QCM [19]. Of course, a lot
of questions and problems still need answers, for in-
stance, optimal state-dependent QCM (as an example,
see [20] or [21] - unfortunately, not many results are
known for this kind of QCM) or optimal asymmetric
QCM. There is also another ‘gap’ in quantum cloning:
interestingly, up to our best knowledge, one is lacking
a general result on an admissible region of fidelities (in
general not the optimal one) for universal asymmetric
1 → N quantum cloning machines; in our work, we
want to make progress in this direction, providing an
answer to this particular problem in the case of qubits.
Let us mention briefly that it has been studied partially,
for example in [16] (and partially in [17]), where expres-
sions for an optimal fidelities region for an asymmetric
1 → 3 quantum cloner were obtained (and some partial
results for a 1 → N cloner).

In this Letter, we shall consider a 1 → N universal
quantum cloning machine (quantum cloner) for qubits.
We first point out that this problem could be related to
singlet states by recalling a relation between fidel-
ities and singlet states. Using this fact, it turns out that
by the application of Schur - Weyl duality, our prob-
lem could be quite easily solved and it leads to plots of
ranges of fidelities for different irreducible representa-
tions of the symmetric group S_n (where n = N + 1).
The method of irreps (irreducible representations)
is ‘nice’ here, because it makes calculations a lot easier.
It allows us to decompose our initial Hilbert space into
blocks of smaller dimensions, connected to a given par-
tition λ (λ’s label irreps of a symmetric group), and
moreover, in our calculations, we can restrict ourself to
a pure state only, linked to a given block. After taking
the convex hull of the figures corresponding to all par-
titions, we can obtain the possible range of fidelities. In
our case-study example of the 1 → 3 cloner, we check
that our results are consistent with the existing meth-
ods for 1) symmetric cloners (for example, Keyl and
Werner work [13]), namely we obtain that in the case of
optimal, symmetric UQCM, result of all three fidelities
equals \frac{2}{3} is obtained, 2) an optimal asymmetric 1 → 3
cloner obtained in [16]. Finally, as a direct application
of our method, we study one particular example, namely
the case when one has \max_i (F_1 + F_3 = 2F_2).

This work is organized as follows. In Section IV we
start with showing that the quantum cloning problem
is equivalent to the entanglement sharing picture, so
that the cloning machine is equivalently represented by
a multipartite quantum state. Then, we formulate prob-
lem that we want to address in this Letter, namely, to
determine the region of allowed fidelities for our QCM.
What is more (and crucial for us), we show a transfor-
mation between Bell’s states that allows to use Schur -
Weyl duality. Section III is devoted to the group rep-
resentation theory, in particular, Schur - Weyl decom-
position. We present the basic formalism that leads to
Schur - Weyl duality. At the beginning, we introduce
SWAP operators and then we show how they lead to
Schur - Weyl decomposition. At the end of this sec-
tion, we show how Schur - Weyl duality is connected
with Young diagrams formalism. This section is strictly
mathematically oriented, but as it will be shown in the
next section, this formalism allows us to simplify our
problem a lot and it makes all calculations quite ele-
mentary. In Section V which is the key section of our
Letter, we present our main lemmas and theorems for
N-tuples of fidelities, especially, we show which form
of the multiparticle state representing QCM can be used,
and also how one can connect fidelity calculations with
the formalism of Young diagrams. It follows, that to de-
termine the admissible region of fidelities, it is enough
to consider overlaps of real vectors with the matrices
of irreps of the symmetric group. What is important,
all results are valid for N-tuples of fidelities, so they
describe an asymmetric 1 → N cloner in general. In
Section VI which is the case study of the 1 → 3 uni-
versal quantum cloner, we present how to obtain the
allowed region for triples of fidelities. We also show
that our results are consistent with: a) calculations for
a symmetric cloning case, predicted by the Werner’s
formula [12] and b) partially with results obtained in
[16], where optimal fidelities for this kind of UQCM
were presented. At the end, we give an application of
our model, namely we show that for any given triple of
fidelities from the allowed region, we could reconstruct
a state that gives rise to that fidelities. This technique
is actually quite general and could be applied to the N
number of clones. In Section VII we briefly explore our
approach with a higher number of clones, by studying
the case of a 1 → 4 UQCM.

II. STATEMENT OF THE PROBLEM

Let us recast a question of cloning in an equivalent
picture of entanglement sharing (for simplicity, we fo-
cus here on our case study-example, namely the 1 → 3
UQCM - all calculation could be easily extended to the
case of N clones). Suppose that we have an initial state
described by the Bell state: |ψ^+⟩ = \sqrt{\frac{1}{2}} (|00⟩ + |11⟩).
We also have a cloning machine M, which could be
described by a completely positive, trace preserving
(CPTP) map \( \tilde{\Lambda} \) (note that the cloning map \( \tilde{\Lambda} \) applied to the second subsystem of the maximally entangled state \( |\psi^+\rangle \), when the first is untouched, produces, in general, mixed state that contains all the information about the map). As an output, we want to obtain \( N \) shares of our initial state. Our scheme is presented in Figure 1 (for our case-study example). Let us write a state obtained after an application of the cloning map to the half of \( |\psi^+\rangle \)

\[
\rho_{1234} = \frac{1}{2} \left( \mathbb{I} \otimes \tilde{\Lambda} \right) (|\psi^+\rangle \langle \psi^+|),
\]

where indexes 1, 2, 3 and 4 are connected to an initial state and clones, according to Figure 1.

We want to calculate an allowed region for singlet fractions \( F_{ii} \) (between the initial state and one of the three clones (copies)), denoted by \( [22] \)

\[
F_{12} = \langle \psi^{+}_{12} | \text{Tr}_{34} (\rho_{1234}) | \psi^{+}_{12} \rangle, \\
F_{13} = \langle \psi^{+}_{13} | \text{Tr}_{24} (\rho_{1234}) | \psi^{+}_{13} \rangle, \\
F_{14} = \langle \psi^{+}_{14} | \text{Tr}_{23} (\rho_{1234}) | \psi^{+}_{14} \rangle.
\]

Note that:

\[
F_{ii} = (\psi^{+}_{ii} | \text{Tr}_{ii} (\tilde{\rho}_{1234}) | \psi^{+}_{ii}),
\]

where \( \text{Tr}_{ii} \) means partial trace over all systems except \( i \), and \( |\psi^{+}_{ii}\rangle \) and \( \tilde{\rho}_{1234} \) are defined below. The vector \( |\psi^{+}_{ii}\rangle = U \otimes I |\psi^{+}\rangle \), \( |\psi^{-}\rangle \) is obtained after the action of \( -i\sigma_y \) on \( |\psi^+\rangle \):

\[
|\psi^{-}\rangle = -i\sigma_y |\psi^+\rangle = -i\sigma_y \sqrt{\frac{1}{2}} (|00\rangle + |11\rangle) = \\
= \sqrt{\frac{1}{2}} (|01\rangle - |10\rangle),
\]

where \( \sigma_y \) is one of the Pauli matrices: \( \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \).

Using (8) we can write that:

\[
|\psi^{-}_{11}\rangle = U \otimes I |\psi^{+}_{11}\rangle,
\]

where \( U = -i\sigma_y \). The state \( \tilde{\rho}_{1234} \) from Eq. (7) is obtained after the following transformation:

\[
\tilde{\rho}_{1234} = (U \otimes I) \left( (U \otimes I) |\psi^{+}_{11}\rangle \langle \psi^{+}_{11}| (U^t \otimes I) \right).
\]

The four-partites states \( \tilde{\rho}_{1234} \), with the constraint \( \tilde{\rho}_1 = I/2 \), are in one-to-one correspondence with cloning machines, and the cloning fidelity of a given machine is determined by fidelities \( F_{ii} \) of the corresponding state.

Note that for any channel \( \Lambda \), fidelity of the state \( (\mathbb{I} \otimes \tilde{\Lambda}) (|\psi^+\rangle \langle \psi^+|) \) can be related to the average fidelity of transmission of an initial state as follows \( [23] \):

\[
f = \frac{Fd + 1}{d + 1},
\]

where \( d \) is the dimension of the Hilbert space \( \mathcal{H} \cong \mathbb{C}^d \). Thus instead of cloning fidelities \( f \), we can consider singlet fractions \( F \), which we will further call simply fidelities, while the fidelity of cloning will we term the ’cloning fidelity’.

Werner \([12]\) provided the following formula for an optimal cloning fidelity of universal symmetric \( N_1 \to N_2 \) cloning machine:

\[
f_{N_1 N_2}(d) = \frac{N_1}{N_2} + \frac{(N_2 - N_1)(N_1 + 1)}{N_2(N_1 + d)}. \tag{12}
\]

In our case, when \( N_1 = 1, N_2 = 3 \) and \( d = 2 \), we obtain that the fidelity \( f \) for the universal, symmetric cloning machine should be equal to

\[
f = \frac{1}{3} + \frac{4}{9} = \frac{7}{9}. \tag{13}
\]

We can now formulate question for our case study-example: which values of triples of cloning fidelities \( (F_{12}, F_{13}, F_{14}) \) are allowed for a universal cloning machine? (Of course in general our question is the following: which values of \( N \)-tuple of cloning fidelities \( (F_{12}, F_{13}, \ldots, F_{1n}) \) are allowed for a universal cloning machine?) As said above, we shall address equivalent question: what values of triples of fidelities \( (F_{12}, F_{13}, F_{14}) \) are allowed for an arbitrary state of a maximally mixed first subsystem? In the next sections, the answer is presented.

III. MATHEMATICAL INTRODUCTION: SCHUR - WEYL DECOMPOSITION

In this section we introduce necessary mathematical tools from group theory. We are especially focused on Schur-Weyl duality \([24]\).
Consider a unitary representation of a permutation group \( S_n \) acting on the \( n \)-fold tensor product of complex spaces \( \mathbb{C}^d \), so our full Hilbert space is \( \mathcal{H} \cong (\mathbb{C}^d)^{\otimes n} \). For a fixed permutation \( \pi \in S_n \) a unitary transformation \( V_\pi \) is given by
\[
V_\pi (|i_1\rangle \otimes \ldots \otimes |i_n\rangle) = |i_{\pi(1)}\rangle \otimes \ldots \otimes |i_{\pi(n)}\rangle.
\] (14)
where \(|i_1\rangle, \ldots, |i_n\rangle\) is a standard basis in \((\mathbb{C}^d)^{\otimes n}\). The space of rank-\( n \) tensors can be also considered as a representation space for a general linear group \( \text{GL}(d, \mathbb{C}) \). Let \( U \in \text{GL}(d, \mathbb{C}) \), thus, this induces in the tensor product \((\mathbb{C}^d)^{\otimes n}\) the following transformation
\[
U^{\otimes n} (|i_1\rangle \otimes \ldots \otimes |i_n\rangle) = U|i_1\rangle \otimes \ldots \otimes U|i_n\rangle.
\] (15)
A key property is that these two representations turn out to be each other commutants. Any operator on \((\mathbb{C}^d)^{\otimes n}\) that commutes with all \( U^{\otimes n}, \forall U \in \text{GL}(d, \mathbb{C}) \), is a linear combination of permutation matrices \( V_\pi \). Conversely, any operator commuting with all permutation matrices \( V_\pi, \forall \pi \in S_n \), is a linear combination of \( U^{\otimes n} \). This duality is called Schur-Weyl duality. It was shown (see, for example, \([25]\) or \([26]\)) that there always exists some basis called the Schur basis which gives decomposition of \( V_\pi \) and \( U^{\otimes n} \) into irreducible representations (irreps) simultaneously. Thanks to this, the space \((\mathbb{C}^d)^{\otimes n}\) can be decomposed into irreducible representations of \( S_n \)
\[
(\mathbb{C}^d)^{\otimes n} \cong \bigoplus_{\lambda \vdash n} \mathcal{H}^H_\lambda \otimes \mathcal{H}^S_\lambda,
\] (16)
where \( \lambda \) labels inequivalent irreps of \( S_n \) and \( \mathcal{H}^H_\lambda \) is the multiplicity space. It is called Schur-Weyl decomposition. The labels \( \lambda \) are allowed partitions of some natural number \( n \). Every partition is a sequence \( \lambda = (\lambda_1, \ldots, \lambda_r) \) satisfying
\[
\forall i \lambda_i \geq 0, \quad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r, \quad \sum_{i=1}^r \lambda_i = n,
\] (17)
where \( r \in \{1, \ldots, n\} \). Every such partition corresponds to some diagram, which is called the Young diagram \([27]\). Here are few examples of Young diagrams for \( n = 4, 6 \) and 3 respectively:

\[
\begin{array}{cccc}
\lambda = (2, 2), & \lambda = (3, 2, 1), & \lambda = (1, 1, 1)
\end{array}
\]
In this Letter we are interested in representations on the symmetric part \( \mathcal{H}^S_\lambda \). For example the SWAP operator \( \mathcal{V}_\pi \) can be decomposed, due to Schur - Weyl decomposition, in the following way:
\[
\mathcal{V}_\pi = \bigoplus_{\lambda} I_{r(\lambda)} \otimes \mathcal{V}^\lambda_{\pi},
\] (18)
where \( \pi \in S_n \) and \( r(\lambda) \) is the dimension of a unitary part. The operators \( \mathcal{V}^\lambda_{\pi} \) are irreducible representations of \( S_n \). Thanks to the above-mentioned method, we can decompose \( U^{\otimes n} \)-invariant states in the following way:
\[
\rho_{1\ldots n} = \bigoplus_{\lambda} \mathcal{I}_{r(\lambda)} \otimes \bar{\rho}^\lambda.
\] (19)
Note that fidelities with singlet states are invariant under averaging over \( U \otimes U \) transformations. Therefore the \( N \)-tuple of fidelities is invariant under an application of transformation \( U^{\otimes n} \) to the state, so we can always use density operators that are commutant of \( U^{\otimes n} \) i.e. they are of the form \([19]\).

IV. EXPRESSION FOR AN ALLOWED REGION OF \( N \)-TUPLES OF FIDELITIES.

In this section we provide a general formula for an allowed region of \( N \)-tuples of fidelities in terms of overlaps of pure states with irreducible representations of \( S_n (n = N + 1) \). This is contained in Theorem \([2]\). We further show in Lemma \([3]\) that one can restrict attention to pure states with real coefficients, since they determine the admissible region of fidelities.

**Lemma 1.** Fidelity \( F_{1k} \) as defined in \([7]\) is of the form
\[
F_{1k} = \sum_{\lambda} F_{1k}^\lambda,
\] (20)
where
\[
F_{1k}^\lambda = \frac{1}{2} - \frac{1}{2} \text{Tr} \left( \rho^\lambda \mathcal{V}^\lambda_{(1k)} \right),
\] (21)
The lower index \( (1k) \) means a permutation that swaps 1 and \( k \), and \( \rho^\lambda \)’s are arbitrary normalized states on partition \( \lambda \).

**Proof.** From the definition of a fidelity we can write
\[
F_{1k} = \langle \psi_{1k} | \rho_{1k} | \psi_{1k} \rangle = \text{Tr} (\rho_{1k} | \psi_{1k} \rangle \langle \psi_{1k} |),
\] (22)
where \(| \psi_{1k} \rangle \langle \psi_{1k} | = \frac{1}{2} (\text{id}_{1k} - \mathcal{V}_{1k})\), \( \rho_{1k} = \text{Tr}_{1\ldots n} \rho_{1\ldots n} \) and \( \text{Tr} \) denote partial trace over all systems except 1 and \( k \). Expanding \(22\), we obtain:
\[
F_{1k} = \text{Tr} \left( \frac{1}{2} \rho_{1k} (\text{id}_{1k} - \mathcal{V}_{1k}) \right) = \text{Tr} \left( \frac{1}{2} \rho_{1k} - \frac{1}{2} \rho_{1k} \mathcal{V}_{1k} \right)
\]
\[
= \frac{1}{2} - \frac{1}{2} \text{Tr} \left( \mathcal{V}_{1k} \rho_{1\ldots n} \right).
\] (23)
Now we can use Schur - Weyl decomposition to represent \( \mathcal{V}_{1k} \) and \( \rho_{1\ldots n} \):\[
\mathcal{V}_{1k} = \bigoplus_{\lambda} \mathcal{I}_{r(\lambda)} \otimes \mathcal{V}^\lambda_{(1k)}, \quad \rho_{1\ldots n} = \bigoplus_{\lambda} \mathcal{I}_{r(\lambda)} \otimes \bar{\rho}^\lambda,
\] (24)
Inserting (24) into (23), we have:

\[
F_{lk} = \frac{1}{2} - \frac{1}{2} \text{Tr} \left( \left( \bigoplus_{\lambda} \tau_\lambda (\tilde{\rho}) \otimes \tilde{\rho}^\lambda \right) \left( \bigoplus_{\mu} \tau_\mu (\mu) \otimes \tilde{\sigma}^\lambda_{1k} \right) \right) = \sum_{\lambda} \left( \frac{1}{2} - \frac{1}{2} \text{Tr}(\rho^\lambda \tilde{\sigma}^\lambda_{1k}) \right). \tag{25}
\]

Equation (25) could be rewritten as:

\[
F_{lk} = \sum_{\lambda} F_{lk}^\lambda, \tag{26}
\]

where \(F_{lk}^\lambda = \frac{1}{2} - \frac{1}{2} \text{Tr}(\rho^\lambda \tilde{\sigma}^\lambda_{1k})\), and \(\rho^\lambda = d_\lambda \tilde{\rho}^\lambda\) and \(d_\lambda\) stands for the dimension of a given partition.

Now we are in position to formulate the main theorem of this section:

**Theorem 2.** The set \(\mathcal{F}\) of admissible vectors of fidelities \(\{F_{12}, \ldots, F_{1n}\}\) is of the form

\[
\mathcal{F} = \text{conv} \left( \bigcup_{\lambda} \mathcal{F}^\lambda \right), \tag{27}
\]

where \(\text{conv}\) stands for a convex hull, the union runs over all irreps of \(S_n\) and

\[
\mathcal{F}^\lambda = \left\{ \left( F_{12}^\lambda, \ldots, F_{1n}^\lambda \right) : |\psi\rangle \in \mathbb{C}^{d_\lambda} \right\}, \tag{28}
\]

where \(F_{lk}^\lambda\) are of the form: \(F_{lk}^\lambda = \frac{1}{2} - \frac{1}{2} \langle \psi | \tilde{\sigma}^\lambda_{lk} | \psi \rangle\), and where \(|\psi\rangle\) is a pure state.

**Proof.** At the beginning, let us consider the following mapping:

\[
\bar{F} : P \rightarrow \mathcal{R}^N \tag{29}
\]

which maps states \(\rho_{1\ldots n} \in P\) (\(P\) stands for a convex set of all states \(\rho_{1\ldots n}\)) into the \(N\)-tuples \((F_{12}, \ldots, F_{1n}) \in \mathcal{R}^N\). Explicitly, we have

\[
\bar{F}(\rho_{1\ldots n}) = [F_{12}(\rho_{1\ldots n}), \ldots, F_{1n}(\rho_{1\ldots n})]. \tag{30}
\]

This mapping is affine (Lemma 9), i.e. is of the form:

\[
\bar{F}(\rho_{1\ldots n}) = \bar{F}(\rho_{1\ldots n}) + \tilde{C}, \tag{31}
\]

where \(\bar{F} : P \rightarrow \mathcal{R}^N\) is linear. Indeed, one can see that RHS of (30) could be written as:

\[
F_{12}(\rho_{1\ldots n}) = \frac{1}{2} \text{Tr} \left( (V_{(1)\ldots(n)} - V_{(12)}) \rho_{1\ldots n} \right) = \frac{1}{2} - \frac{1}{2} \text{Tr}(V_{(12)} \rho_{1\ldots n}),
\]

\[
\vdots
\]

\[
F_{1n}(\rho_{1\ldots n}) = \frac{1}{2} \text{Tr} \left( (V_{(1)\ldots(n)} - V_{(1n)}) \rho_{1\ldots n} \right) = \frac{1}{2} - \frac{1}{2} \text{Tr}(V_{(1n)} \rho_{1\ldots n}). \tag{32}
\]

Comparing (31) and (32), we obtain that in our case \(\tilde{C} = \left[ \frac{1}{2}, \ldots, \frac{1}{2} \right] \) and \(\bar{F}(\rho_{1\ldots n}) = -\left[ \text{Tr}(V_{(12)} \rho_{1\ldots n}), \ldots, \text{Tr}(V_{(1n)} \rho_{1\ldots n}) \right] \), which is obviously linear with respect to \(\rho_{1\ldots n}\).

One can note that in general \(\tilde{\rho}^\lambda\) from Eq. (24) is a mixed state, but according to Lemma 9 we can take the mapping of extreme points of \(\rho_{1\ldots n}\) of the form:

\[
\rho_{1\ldots n} = \bigoplus_{\lambda} \tau_\lambda (\mu) \otimes \rho^\lambda = \bigoplus_{\lambda} \frac{1}{d_\lambda} \tau_\lambda (\mu) \otimes d_\lambda \tilde{\rho}^\lambda
\]

\[
= \bigoplus_{\lambda} \frac{1}{d_\lambda} \tau_\lambda (\mu) \otimes \rho^\lambda = \bigoplus_{\lambda} p_\lambda \left( \frac{\tau_\lambda (\mu)}{d_\lambda} \otimes \tilde{\rho}^\lambda \right), \tag{33}
\]

where \(\tilde{\rho}^\lambda = \frac{\rho^\lambda}{Tr \rho^\lambda}\), so namely, it is normalized, and \(p_\lambda = Tr \rho^\lambda\). We can now eigen-decompose \(\tilde{\rho}^\lambda\):

\[
\tilde{\rho}^\lambda = \sum_{i=1}^{n_\lambda} |\psi_i^\lambda\rangle \langle \psi_i^\lambda|, \tag{34}
\]

where \(\sum_{\lambda} p_\lambda = 1\). Inserting Eq. (34) into Eq. (33), we get that:

\[
\bigoplus_{\lambda} \sum_{i=1}^{n_\lambda} p_\lambda |\psi_i^\lambda\rangle \langle \psi_i^\lambda| \left( \frac{\tau_\lambda (\mu)}{d_\lambda} \otimes |\psi_i^\lambda\rangle \langle \psi_i^\lambda| \right),
\]

we see from (35) that extreme points are of the form:

\[
\rho_{\text{extreme}} = \frac{\tau_\lambda (\mu)}{d_\lambda} \otimes |\psi_i^\lambda\rangle \langle \psi_i^\lambda|, \tag{36}
\]

where \(\lambda\) runs over all irreps from \(S_n\) and \(|\psi_i^\lambda\rangle\) is an arbitrary state form the space of irrep linked to a given partition \(\lambda\). Inserting Eq. (36) into Eq. (25), we obtain the desired result.

We note that to determine the allowed region of fidelities, it is enough to consider only vectors of real coefficients.

**Lemma 3.** To generate a convex hull of the allowed region of fidelities, it is sufficient to consider pure states of real coefficients only.

**Proof.** We need to show that, in our case, a kind of majorization of complex pure states by real ones occurs. To prove that, note that our operators \(\tilde{\nu}^\lambda_{(1)}\) (43) are symmetric and real1, so they could be written in a general
where $k = \dim \mathcal{H}_\lambda^S$. Now let us write a density matrix of a pure state $|\psi\rangle = (a_1, a_2, \ldots, a_k)^T$ with complex values (letter $C$) corresponds to the word ‘complex’:

$$\rho^\lambda_C = \begin{pmatrix} |a_1|^2 & a_1 a_2 & \cdots & a_1 a_k \\ a_1 a_2 & |a_2|^2 & \cdots & a_2 a_k \\ \vdots & \vdots & \ddots & \vdots \\ a_1 a_k & a_2 a_k & \cdots & |a_k|^2 \end{pmatrix}. \tag{38}$$

Next, let us rewrite $\text{Tr} \left( \rho^\lambda \overline{\nu}^\lambda_{(1i)} \right)$ from Lemma 1 using (38) and (37), as follows:

$$\text{Tr} \left( \rho^\lambda \overline{\nu}^\lambda_{(1i)} \right) = \sum_{j=1}^k |a_j|^2 v_{jj} + \sum_{j,s=1}^k (a_j a_s + \bar{a}_j \bar{a}_s) v_{js} = \sum_{j=1}^k |a_j|^2 v_{jj} + 2 \sum_{j,s=1}^k \text{Re} (a_j a_s) v_{js}. \tag{39}$$

We see that calculating the trace $\text{Tr} \left( \rho^\lambda \overline{\nu}^\lambda_{(1i)} \right)$ with the operator $\rho^\lambda_C$ is equivalent to calculating the trace $\text{Tr} \left( \rho^\lambda \overline{\nu}^\lambda_{(1i)} \right)$ with the following matrix $\rho^\lambda_{R-C}$:

$$\rho^\lambda_{R-C} = \begin{pmatrix} |a_1|^2 & \text{Re}(a_1 a_2) & \cdots & \text{Re}(a_1 a_k) \\ \text{Re}(a_1 a_2) & |a_2|^2 & \cdots & \text{Re}(a_2 a_k) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Re}(a_1 a_k) & \text{Re}(a_2 a_k) & \cdots & |a_k|^2 \end{pmatrix}. \tag{40}$$

where the new index $R - C$ means that we are left only with real numbers. We shall now show, that $\rho^\lambda_{R-C}$ is positive-semidefinite.

Let us denote the total phase in this case by $e^{i\phi_j}$ for $j = 1, \ldots, k$, so we are left with $\text{Re} (e^{i\phi_j}) = \text{Re} (\cos \phi_j) + i \sin \phi_j = \cos \phi_j$. We then obtain

$$\rho^\lambda_{R-C} = \begin{pmatrix} |a_1|^2 & |a_1 a_2|^2 & \cdots & |a_1 a_k|^2 \\ |a_1 a_2|^2 & |a_2|^2 & \cdots & |a_2 a_k|^2 \\ |a_1 a_k|^2 & |a_2 a_k|^2 & \cdots & |a_k|^2 \\ 1 & \cos(\phi_1 - \phi_2) & \cdots & \cos(\phi_1 - \phi_k) \\ \cos(\phi_1 - \phi_2) & 1 & \cdots & \cos(\phi_2 - \phi_k) \\ \vdots & \vdots & \ddots & \vdots \\ \cos(\phi_1 - \phi_k) & \cos(\phi_2 - \phi_k) & \cdots & 1 \end{pmatrix} = A \bullet C, \tag{41}$$

where $\phi_j$ are phases connected with a given $|a_j|$ and by $\bullet$ we denote a Hadamard product of two matrices.

It is easy to show that matrix $C$ can be rewritten in terms of vectors $|\omega_j\rangle$ i.e. $C_{ij} = \langle \omega_i | \omega_j \rangle$ for $i, j = 1, 2, \ldots, k$. Thanks to this operation we can conclude that $C$ is a square Gramian matrix. From (39) we know that every square Gramian matrix is positive-semidefinite.

The matrix $A$ is of the form $|\phi_j\rangle \langle \phi_j|$, with $|\phi_j\rangle = \sum_i |a_i| |i\rangle$, so it is positive-semidefinite. Now using Fact 10 we obtain that our operator $\rho^\lambda_{R-C}$ is also positive-semidefinite.

Because $\rho^\lambda_{R-C}$ is real and symmetric we get from Fact 11 that the matrix $\rho^\lambda_{R-C}$ possesses real eigenvectors, so indeed it is a mixture of real pure states.

\[ \Box \]

V. CASE STUDY: A REGION FOR TRIPLES OF FIDELITIES

A. Partitions and transpositions

In our case-study example we have four particles ($n = 4$) which means that allowed partitions are $\lambda_1 = (4), \lambda_2 = (3,1), \lambda_3 = (2,2), \lambda_4 = (2,1,1)$ and $\lambda_5 = (1,1,1,1)$. Because we want to consider only qubits here, so in our case $d = 2$, and hence $\lambda$ runs over binary partitions only or, equivalently, over Young diagrams with two rows (for other diagrams the multiplicity space $\mathcal{H}_{\lambda}^{\cup}$ becomes zero-dimensional). Thanks to this, we are left only with $\lambda_1 = (4), \lambda_2 = (3,1), \lambda_3 = (2,2), \lambda_4 = (2,1,1)$ and $\lambda_5 = (1,1,1,1)$. For the partition $\lambda_2 = (3,1)$ unitary representations of transpositions $T(1,2), T(1,3), T(1,4)$, which we shall need, are $[30]$:

$$\overline{\nu}_{(12)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \overline{\nu}_{(13)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \tag{43}$$

$$\overline{\nu}_{(14)} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} \\ -\frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{2} & \frac{1}{2} \end{pmatrix}. \tag{44}$$

For the partition $\lambda_3 = (2,2)$ unitary transposition representations $T(1,2), T(1,3), T(1,4)$, which we shall need, are $[30]$:...
Note that for the partition $\lambda_1$ we have a trivial representation, so it is not reported here, explanation is provided later.

B. Partial result: fidelity regions for each partition

According to Lemma 9 all extreme points are just the convex hull of the figure from Figure 2. It allows us to use pure states only. In our situation, we use states which are of the form $|\psi^{(2,2)}\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ and $|\psi^{(3,1)}\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $a_i \in \mathcal{R}$, so they are real pure states (see Lemma 3). Each of this state generates a pure state $\rho^{(2,2)}$ of the form $\begin{bmatrix} a_2^2 & a_1a_2 & a_1a_2 \\ a_1a_2 & a_1^2 & a_1a_3 \\ a_1a_3 & a_1a_3 & a_2^2 \end{bmatrix}$, where $a_1^2 + a_2^2 = 1$, for the partition $\lambda = (2,2)$, and $\rho^{(3,1)} = \begin{bmatrix} a_1^2 & a_1a_2 & a_1a_3 \\ a_1a_2 & a_2^2 & a_2a_3 \\ a_1a_3 & a_2a_3 & a_3^2 \end{bmatrix}$, where $a_1^2 + a_2^2 + a_3^2 = 1$, for $\lambda = (3,1)$ respectively. Inserting $\rho^{(2,2)}$ and $\rho^{(3,1)}$ into Equation (20) from Lemma 1, we obtain the plot in Figure 2 with values $F_{12}$, $F_{13}$ and $F_{14}$ on axes. Now we want to give an explanation, why the partition $\lambda = (4)$ is not included in figures. It can be shown that in this case, all fidelities are equal to 0, and since we do not want to ‘blur’ the main message of this Letter, this partition is excluded from Figure 2. But, strictly speaking, in the next step, the convex hull also with this point should be created. This should lead to a ‘larger’ possible region for fidelities. Of course, it is not a hard task from the computational point of view, since taking a convex hull of some point and some figure is quite an elementary operation.

Remark 4 Note that this does not mean that cloning fidelities are equal to 0 and then that the partition $\lambda = (4)$ should be excluded because perfect universal anti-cloning is forbidden by the laws of quantum mechanics. In our case singlet fractions are equal to 0 and fidelities are equal $\frac{1}{2}$ (see Eq. (11) and the comment in the paragraph below it).

At the end of this paragraph we present also explicit formulas for fidelities for every irrep labeled by partitions $\lambda_2$ and $\lambda_3$.

---

2 All plots are obtained using Mathematica software.
• Fidelities for the partition $\lambda = (2, 2)$:

$$F^{\lambda_1}_{12} = \frac{1}{2} \left( 1 - a_1^2 + a_2^2 + a_3^2 \right),$$

$$F^{\lambda_1}_{13} = \frac{1}{2} \left( 1 + a_1^2 - a_2^2 + \sqrt{3} a_1 a_2 \right),$$

$$F^{\lambda_1}_{14} = \frac{1}{2} \left( 1 + a_1^2 - a_2^2 - \sqrt{3} a_1 a_2 \right).$$

(45)

• Fidelities for the partition $\lambda = (3, 1)$:

$$F^{\lambda_2}_{12} = \frac{1}{2} \left( 1 - a_1^2 + a_2^2 + a_3^2 \right),$$

$$F^{\lambda_2}_{13} = \frac{1}{2} \left( 1 + a_1^2 + a_2^2 + a_3 + \sqrt{3} a_1 a_2 \right),$$

$$F^{\lambda_2}_{14} = \frac{1}{2} \left( 1 + a_1^2 + a_2^2 + 5 a_3^2 + a_1 a_2 \left( \frac{2 \sqrt{2} a_2 a_3}{3} - 2 \sqrt{\frac{2}{3}} a_1 a_3 \right) \right).$$

(46)

C. The main result: an allowed region for fidelities

Since, we have two partitions $(2, 2)$ and $(3, 1)$ and corresponding pure states $\rho^{(2, 2)}$ and $\rho^{(3, 1)}$, we obtain two figures in Figure 2. But, we are interested in situation, where we can obtain a general answer to our question from Section II, namely in situation where we have a mixture of both partitions: $\sum_{\lambda} p_{\lambda} F^{\lambda}_{13}$. To solve this task, we can construct a convex hull of figures from Figure 2. It is presented in Figure 3. One can also see that our Figures 2 and 3 are invariant under rotation of angle $2\pi/3$ along straight line $F_{12} = F_{13} = F_{14}$. This corresponds to three conjugacy classes.$^3$

Remark 5 Because of the properties of the cloning map $\Lambda$ (see Sec. II) all possible convex mixtures of the partitions shown in Fig. 3 are possible and produce a correct quantum cloner, i.e., a trace preserving completely positive map.

D. Comparison with other models

As we mentioned before, this particular $UQCM$ - an asymmetric $1 \to 3$ cloner - was studied before in [16] and in [17]. It was shown there how to obtain an optimal region for this particular cloner. Plotting Eq. (38) for $d = 2$ together with a constraint from Eq. (32) from [16] and then using Eq. (11) (from this work) to obtain fidelities, one can check that then the optimal cloner corresponds to a region for fidelities obtained for the

$^3$Two permutations $\pi$ and $\pi'$ are conjugate iff $\pi = \sigma \cdot \pi' \cdot \sigma^{-1}$, where $\sigma$ is also a permutation and $\sigma, \pi, \pi' \in S_n$. 

FIG. 3: The convex hull of figures that corresponds to the partition $(2, 2)$ and $(3, 1)$, based on Figure 2. On axes values of $F_{12}, F_{13}$ and $F_{14}$ are reported. On each plot, three coordinates are presented from the set: $A = (0, 1, 0), B = (1, 1, 0)$ and $C = (0, 0, 0)$, and $C' = (1, 1, 1)$. Two different views are presented.
partition \( \lambda = (3, 1) \) (but first some unknowns at this moment constraint should be applied, leading to a cut-off of the plot of the partition \( \lambda = (3, 1) \), so it will correspond to the result from [16]). It is an important remark that one should keep in mind - in our work optimal cloners are not considered, instead, the whole possible region for a given \( N \)-tuple of fidelities is obtained that later somehow could be optimized by ‘throwing away’ some partitions and establishing some constrains. What is more, authors of the paper [16] also proved the optimality of the asymmetric \( 1 \to 2 \) quantum cloner using their technique. This result is also in accordance with our results, one can check that using our method, by taking valid partitions from the symmetric group \( S(3) \), the same result could be obtained. In general, expressions for possible fidelities in both cases (so our and from [16]) look quite similar (compare for example ‘our’ Eq. (46) with Eq. (38) from [16]), but because of the constraint from Eq. (32), the optimal fidelities could be obtained, not the full region like in our work, so it would be worthy to explore the direct connection between these two approaches in the future.

E. Special case: symmetric cloning

It can also be shown that in the case of symmetric cloning, the maximal possible value of the triple \((F_{12}, F_{13}, F_{14})\) is \((\frac{2}{3}, \frac{2}{3}, \frac{1}{3})\), so it is in accordance with the formula obtained by Keyl et al. [13] and Werner [12] (first using Equation (11)), namely we are able to find such a point \((F_{12}, F_{13}, F_{14})\) that corresponds to the case of the optimal symmetric fidelity \(F\). To find the maximal possible fidelity (for a triple \((F_{12}, F_{13}, F_{14})\)) which corresponds to the case of the optimal symmetric cloning, one method is to find a plane equation which includes the ellipse (which corresponds to the partition \((2, 2)\)). It can be shown that it is given by the formula \(F_{12} + F_{13} + F_{14} = \frac{5}{2}\). It can be obtained, for example, by taking coordinates of points from Figure 2 and then calculating the plane equation according to the following well-known formula:

\[
\begin{bmatrix}
 x - x_1 \\ y - y_1 \\ z - z_1 \\
 x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \\
 x_3 - x_2 \\ y_3 - y_1 \\ z_3 - z_1
\end{bmatrix} = 0. \tag{47}
\]

So, of course we have to identify \(x, y, z\) with \(F_{12}, F_{13}\) and \(F_{14}\) first, and then insert \((47)\) values of three points \(P_1(F_{12}, F_{13}, F_{14}), P_2(F_{12}, F_{13}, F_{14})\) and \(P_3(F_{12}, F_{13}, F_{14})\) from the ellipse and solve equation which is then obtained.

F. Applications

The above-mentioned approach allows us to reconstruct states from every subspace \(\mathcal{H}_\lambda^S\) which satisfies not only the symmetric case \((F_1 = F_2 = F_3)\) (like in a typical QCMI) but also more general relations between fidelities. In this section we reconstruct states belonging to the subspaces corresponding to partitions \(\lambda_1 = (2, 2), \lambda_2 = (3, 1)\) and satisfying the following formula

\[
\max_{F_1} (F_1 + F_3 = 2F_2). \tag{48}
\]

a) Partition \(\lambda_1 = (2, 2)\)

In this case we have two constrains \(F_1 + F_3 = 2F_2\) and \(a_1^2 + a_2^2 = 1\), so after solving this system of equations we obtain four allowed pairs of solutions

\[
\begin{align*}
(a_1, a_2)_{(1)} &= \left(\frac{\sqrt{2} + \sqrt{3}}{4}, -\frac{\sqrt{2} - \sqrt{3}}{4}\right), \\
(a_1, a_2)_{(2)} &= \left(-\frac{\sqrt{2} - \sqrt{3}}{4}, \frac{\sqrt{2} + \sqrt{3}}{4}\right), \\
(a_1, a_2)_{(3)} &= \left(-\frac{\sqrt{2} + \sqrt{3}}{4}, \frac{\sqrt{2} - \sqrt{3}}{4}\right), \\
(a_1, a_2)_{(4)} &= \left(\frac{\sqrt{2} - \sqrt{3}}{4}, -\frac{\sqrt{2} + \sqrt{3}}{4}\right).
\end{align*}
\tag{49}
\]

Fidelities corresponding to the above pairs are

\[
\begin{align*}
(F_{11}^{\lambda_1}, F_{21}^{\lambda_1}, F_{31}^{\lambda_1})_{(1)} &= \left(\frac{1}{4}(2 - \sqrt{3}), \frac{1}{4}(2 + \sqrt{3})\right), \\
(F_{11}^{\lambda_1}, F_{21}^{\lambda_1}, F_{31}^{\lambda_1})_{(2)} &= \left(\frac{1}{4}(2 + \sqrt{3}), \frac{1}{4}(2 - \sqrt{3})\right), \\
(F_{11}^{\lambda_1}, F_{21}^{\lambda_1}, F_{31}^{\lambda_1})_{(3)} &= \left(\frac{1}{4}(2 - \sqrt{3}), \frac{1}{4}(2 + \sqrt{3})\right), \\
(F_{11}^{\lambda_1}, F_{21}^{\lambda_1}, F_{31}^{\lambda_1})_{(4)} &= \left(\frac{1}{4}(2 + \sqrt{3}), \frac{1}{4}(2 - \sqrt{3})\right).
\end{align*}
\tag{50}
\]

We can see that the maximal fidelity \(F_{11}^{\lambda_1}\) can be obtained for a pair (2) and (4), so our reconstructed states are the following

\[
\begin{align*}
\rho_{(2)}^{\lambda_1} &= \frac{1}{4} \begin{pmatrix} 2 - \sqrt{3} & 1 \2 + \sqrt{3} \1 - \sqrt{3} \1 + \sqrt{3} \end{pmatrix}, \\
\rho_{(4)}^{\lambda_1} &= \frac{1}{4} \begin{pmatrix} 2 - \sqrt{3} & 1 \1 + \sqrt{3} \2 + \sqrt{3} \1 + \sqrt{3} \end{pmatrix}.
\end{align*}
\tag{51}
\]

Now note something more general. Thanks to Lemma [1] we have \(F_{11}^{\lambda_1} = a_1^2\) and then \(a_1^2 = 1 - a_2^2 = 1 - F_{11}^{\lambda_1}\), so we can express every states \(\rho_{\lambda_1} \in \mathcal{H}_\lambda^S\) in terms of the fidelity \(F_{11}^{\lambda_1}\)

\[
\rho_{\lambda_1} = \begin{pmatrix}
\frac{1 - F_{11}^{\lambda_1}}{2F_{11}^{\lambda_1}} & \pm \frac{\sqrt{F_{11}^{\lambda_1}(1 - F_{11}^{\lambda_1})}}{F_{11}^{\lambda_1}} \\
\pm \frac{\sqrt{F_{11}^{\lambda_1}(1 - F_{11}^{\lambda_1})}}{F_{11}^{\lambda_1}} & \frac{1 - F_{11}^{\lambda_1}}{2F_{11}^{\lambda_1}}
\end{pmatrix}. \tag{52}
\]
For example, the sign ‘+’ in (52) corresponds to states obtained from pairs \((a_4,b_4)\) and \((a_3,b_3)\) while the sign ‘−’ corresponds to states obtained from pairs \((a_1,b_1)\), \((a_2,b_2)\).

b) Partition \(\lambda_2 = (3,1)\)

For the partition \(\lambda_2 = (3,1)\) we have a more complicated situation. Here we have three parameters \(a_1,a_2,a_3\) but only two constrains \(F_1 + F_3 = 2F_2\) and \(a_1^2 + a_2^2 + a_3^2 = 1\), so we first need to eliminate the parameter \(a_1\) and then express the parameter \(a_3\) as a function of \(a_2\). Numerically, we find that the fidelity \(F_1\) is maximal for the following values of parameters \(a_1,a_2,a_3\)

\[
(a_1,a_2,a_3)_{(1)} = (0.114, 0.318, 0.941),
\]

\[
(a_1,a_2,a_3)_{(2)} = (-0.114, -0.318, -0.941),
\]

\[
(a_1,a_2,a_3)_{(3)} = (0.114, -0.318, -0.941),
\]

\[
(a_1,a_2,a_3)_{(4)} = (-0.114,0.318,0.941).
\]

Corresponding states are

\[
\rho_{\lambda_2}^{(1)} = \rho_{\lambda_2}^{(2)} = \begin{pmatrix}
0.013 & 0.036 & 0.107 \\
0.036 & 0.101 & 0.299 \\
0.107 & 0.299 & 0.886
\end{pmatrix},
\]

\[
\rho_{\lambda_2}^{(3)} = \rho_{\lambda_2}^{(4)} = \begin{pmatrix}
0.013 & -0.036 & -0.107 \\
-0.036 & 0.101 & 0.299 \\
-0.107 & 0.299 & 0.886
\end{pmatrix}.
\]

Note that fidelities in this case are \((F_{12}^{\lambda_2}, F_{32}^{\lambda_2}, F_{33}^{\lambda_2}) = (0.886,0.556,0.220)\). The numbers for our numerical data are obtained by Wolfram Mathematica Alpha [31]. Most likely they are indeed optimal.

VI. Fidelities Formulas for a \(1 \to 4\) Universal Quantum Cloner

In this section we briefly report formulas for fidelities of a \(1 \to 4\) UQCM for every allowed irrep \(\lambda\) for an \(S(5)\) symmetric group in the case of qubits. In this particular example we are left with three partitions only i.e. \(\lambda_1 = (5)\), \(\lambda_2 = (4,1)\) and \(\lambda_3 = (3,2)\). For the partition \(\lambda_2\) unitary representations of transpositions \(T(1,2), T(1,3), T(1,4), T(1,5)\), which we shall with a normalization condition \(\sum_{i=1}^{4} a_i^2 = 1\).

Now, assuming that our pure state is of the form \(|\psi(1,4)\rangle = [a_1,a_2,a_3,a_4]^T\), we obtain from Lemma 1 the following formulas for fidelities:

\[
F_{12}^{\lambda_2} = \frac{1}{2} \left( 1 - a_1^2 - a_2^2 + a_2^2 + a_4^2 \right),
\]

\[
F_{13}^{\lambda_2} = \frac{1}{2} \left( 1 - a_1^2 - a_2^2 + a_2^2 + \sqrt{3}a_3a_4 - a_2^2 \right),
\]

\[
F_{14}^{\lambda_2} = \frac{1}{2} \left( 1 - a_1^2 + a_2^2 + 2\sqrt{2}a_2a_3 - \frac{5a_3^2}{6} + 2\sqrt{\frac{3}{5}}a_2a_4 + a_2a_4 - a_4^2 \right),
\]

\[
F_{15}^{\lambda_2} = \frac{1}{2} \left( 1 + a_1^2 + \frac{1}{2} \sqrt{5}a_2a_3 - 11a_2^2 - \sqrt{2}a_1a_3 + \frac{a_2a_3}{3\sqrt{2}} - \frac{5a_3^2}{6} + \sqrt{\frac{1}{5}}a_4a_4 + a_2a_4 + a_2a_4 + a_2a_4 - a_4^2 \right),
\]

(56)
Lemma 1. The following formulas for fidelities:

\[
\psi^{\lambda_3}_{(12)} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix},
\]

\[
\tilde{\psi}^{\lambda_3}_{(13)} = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & -\frac{\sqrt{3}}{2} & 0 & 0
\end{bmatrix},
\]

\[
\psi^{\lambda_3}_{(14)} = \begin{bmatrix}
1 & -\frac{\sqrt{3}}{2} & -\sqrt{\frac{3}{2}} & 0 \\
-\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & -\frac{\sqrt{3}}{2}
\end{bmatrix},
\]

\[
\tilde{\psi}^{\lambda_3}_{(15)} = \begin{bmatrix}
-\frac{1}{3} & -\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 \\
-\frac{\sqrt{3}}{2} & \frac{5}{6} & -\frac{1}{2\sqrt{3}} & 0 \\
0 & 0 & -\sqrt{\frac{3}{2}} & 1 \\
0 & 0 & \sqrt{\frac{3}{2}} & -\frac{1}{2}
\end{bmatrix}.
\]

Assuming that our pure state in this partition has a form \(|\psi^{(3,2)}\rangle\) = \([a_1, a_2, a_3, a_4, a_5]^T\) we obtain from Lemma [30] the following formulas for fidelities:

\[
F^{\lambda_3}_{12} = \frac{1}{2} \left( 1 - a_1^2 - a_2^2 - a_3^2 - a_4^2 + a_5^2 \right),
\]

\[
F^{\lambda_3}_{13} = \frac{1}{2} \left( 1 - a_1^2 + \frac{a_2^2}{2} + \sqrt{3}a_2a_3 - \frac{a_3^2}{2} + \frac{a_5^2}{2} + \sqrt{3}a_4a_5 - \frac{a_2^2}{2} \right),
\]

\[
F^{\lambda_3}_{14} = \frac{1}{2} \left( 1 + \frac{a_1^2}{3} + 2\sqrt{\frac{2}{3}}a_1a_2 - \frac{5a_2^2}{6} + 2\sqrt{\frac{2}{3}}a_1a_3 + \frac{a_2a_3}{\sqrt{3}} - \frac{a_3^2}{2} + \frac{a_4^2}{2} - \sqrt{3}a_4a_5 - \frac{a_5^2}{2} \right),
\]

\[
F^{\lambda_3}_{15} = \frac{1}{2} \left( 1 + \frac{a_1^2}{3} - \frac{a_2^2}{3} + \frac{a_3^2}{2} - \sqrt{\frac{2}{3}}a_1a_3 - 2a_2a_3 - \frac{a_3^2}{2} + \frac{a_4^2}{\sqrt{3}} + \frac{2a_2a_4 - a_3a_4 - a_5^2}{2} + \sqrt{2a_1a_5 - a_2a_5 - \frac{a_5^2}{2} \right),
\]

with a normalization condition \(\sum_{i=1}^{5} a_i^2 = 1\). After taking the convex hull, the allowed region for fidelities could be obtained also for this quantum cloner. Of course, the state reconstruction technique from Section [VF] could also be used in this case. At the end of this section, let us note that here we also do not consider partition \(\lambda_1\) (see Section [VF]).

VII. CONCLUSIONS

We have shown that by using representation theory, especially Young diagrams, action of the universal \(1 \rightarrow N\) quantum cloning machine could be described. The method of irreps is quite powerful, because it allows us to decompose (usually big) Hilbert space into blocks (linked with a given partition \(\lambda\)) of smaller dimensions which, of course, are easier to deal with. For example, in our case-study example \(1 \rightarrow 3\) UQCM, the Hilbert space \(\mathbb{C}^{16}\) is decomposed into blocks of dimension 1, 2 and 3 respectively. What is more, using our model, fidelity expressions are quite easy to obtain, one only needs to know representations of all possible irreps for a given symmetric group.

We have also shown that the convex hull could be made to obtain full knowledge about our model, in the sense that it gives raise to the full possible range of fidelities. After careful studies, also optimal UQCM could be found. What is more, we have shown that by restricting ones attention only to real pure states in each of the block, the full answer to our initial question is obtained. We point out that our UQCM gives correct results for the case of the symmetric cloning and we have also proved that the optimal value, in the symmetric case, could be obtained. What is more important, our approach allows to reconstruct any given state connected to a given \(N\)-tuple of fidelities.

Of course, in future, it would be interesting to extend our method in such a way that also qudits could be described. What is more, it would be interesting to search for method that allows to obtain the optimal \(N\)-tuple of fidelities - comparison of our results with these obtained in [16] may suggest that the optimal fidelities always correspond to only one partition (with some cutoffs) \(\lambda\), since for cases \(1 \rightarrow 2\) and \(1 \rightarrow 3\) we have that in the first case, the optimal fidelity region corresponds to \(\lambda = (2,1)\), in the second one to \(\lambda = (3,1)\). Also it is worthy to explore the direct relation between these techniques, so namely ‘our’ technique and this reported in [16].

Note added: After the completion of this paper we became aware that similar results for a \(1 \rightarrow 3\) universal quantum cloner have been reported in [32].

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If by \( \Omega \) (dimensional space and by \( E \) Lemma 9.

Suppose that we have an affine map \( L \) Definition 6.

A set \( Z \) is convex iff \[33\]:

\[ \forall x, y \in \Omega \quad \exists \alpha \in [0, 1] \quad \alpha x + (1 - \alpha) y \in Z. \]

To prove our results we need the following definitions and lemmas:

**Definition 6.** A set \( Z \) is convex iff \[33\]:

\[ \forall x, y \in \Omega \quad \exists \mu \in [0, 1] \quad \mu x + (1 - \mu) y \in Z. \]

**Definition 7.** \( L(\cdot) : x \to y \) is affine map iff \( \exists y \forall x \quad L(x) = \tilde{L}(x) + a \), where \( \tilde{L} \) is linear.

Now let us present two useful lemmas

**Lemma 8.** Suppose that \( y, y' \in L(\Omega) \), then:

\[ ay + (1 - a)y' \in L(\Omega) \] (59)

(it means that \( L(\Omega) \) is a convex set).

**Proof.** Let us write:

\[ L(x) = y = L(\sum p_i x_i) = \sum p_i \tilde{L}(x_i) + a, \]

\[ L(x') = y' = L(\sum q_i x'_i) = \sum q_i \tilde{L}(x'_i) + a, \]

where we use the fact that \( L \) is affine and \( x = \sum p_i x_i, \)

\( x' = \sum q_i x'_i \) for some \( x_i, x'_i \in E(\Omega) \). Inserting (60) into (59) we get:

\[ a(\sum p_i \tilde{L}(x_i) + a) + (1 - a)(\sum q_i \tilde{L}(x'_i) + a) \]

\[ = aa + (1 - a)a + \sum \tilde{L}(p_i x_i) + (1 - a) \sum \tilde{L}(q_i x'_i) \]

\[ = a + \tilde{L}(ax + (1 - a)x') = L(ax + (1 - a)x') \in L(\Omega). \] (61)

This ends the proof of lemma.

**Lemma 9.** Suppose that we have an affine map \( L : X \to Y \).

If by \( \Omega \) we denote a convex subset of \( X \), where \( X \) is a finite dimensional space and by \( E(\Omega) \) a set of extreme points of \( \Omega \),

then \( L(\Omega) \) is a convex set and \( L(E(\Omega)) \) can reproduce set \( L(\Omega) \) after taking the convex hull, i.e.:

\[ L(\Omega) = \text{conv } L(E(\Omega)). \] (62)

**Proof.** We can write \( L(E(\Omega)) \subseteq L(\Omega) \), because we know that \( E(\Omega) \subseteq \Omega \). This together with Lemma (8) ends our proof.
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