ISOPARAMETRIC HYPERSURFACES IN PRODUCT SPACES

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Abstract. In this paper, we characterize and classify the isoparametric hypersurfaces with constant principal curvatures in the product spaces $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, where $\mathbb{Q}^2_{c_i}$ is a space form with constant sectional curvature $c_i$, for $c_1 \neq c_2$.

1. Introduction

A hypersurface $M^n$ of a Riemannian manifold $\tilde{M}^{n+1}$ is said to be isoparametric if it has constant mean curvature as well as its nearby equidistant hypersurfaces (i.e., the correspondent mean curvatures depend only on the distance to $M$). Equivalently, we say that $M$ is isoparametric if it is the level set of some isoparametric function defined on $\tilde{M}$, Following M. Domínguez-Vázquez [6], the first notion of isoparametric function appeared in 1919 in the work of the Italian mathematician C. Somigliana [13], which deals with the relations between the Huygens principle and geometric optics. This study represented the beginning of an important research line in Differential Geometry, namely the isoparametric hypersurfaces studied by renowned mathematicians such as Beniamino Segre, Élie Cartan, and Tullio Levi-Civita.

When the ambient space is a space form, i.e., a simply connected complete Riemannian manifold with constant sectional curvature, the previous definition of isoparametric hypersurface is equivalent to saying that the hypersurface has constant principal curvatures (see [2] and [6]). However, in arbitrary ambient spaces of nonconstant curvature, the equivalence between isoparametric hypersurfaces and hypersurfaces with constant principal curvatures may no longer be true. For instance, Q. M. Wang, in [15], found examples of isoparametric hypersurfaces in complex projective spaces that do not have constant principal curvatures. For more examples, we refer [3], [4] and [9]. Recently, A. Rodríguez-Vázquez, in [12], found an example of a non-isoparametric hypersurface with constant principal curvatures. Another example was given by the authors, in a joint work with F. Guimarães [10].

In this work, we consider the Riemannian products of 2-dimensional space forms $\mathbb{Q}^2_{c_1} \times \mathbb{Q}^2_{c_2}$, with constant sectional curvatures $c_1$ and $c_2$, respectively, with $c_1 \neq c_2$, where $c_i = 1$, 0 or $-1$, $i = 1, 2$. The particular case where $c_1 = 1$ and $c_2 = 0$, that is, when the ambient space is $S^2 \times \mathbb{R}^2$, was considered by J. Julio-Batalla in [11] where he obtained a complete classification of isoparametric hypersurfaces with constant principal curvatures. Using some ideas developed by F. Urbano in [14], where isoparametric hypersurfaces of $S^2 \times S^2$ were classified, J. Julio-Batalla showed that if $\Sigma$ is an isoparametric hypersurface in $S^2 \times \mathbb{R}^2$, with constant principal curvatures...
and unit normal $N = N_1 + N_2$, then $|N_1|$ and $|N_2|$ are constant. The classification continues by showing that $|N_1| = 1$ and $|N_2| = 0$ or $|N_1| = 0$ and $|N_2| = 1$. Thus, the hypersurface families obtained are $S^2 \times \mathbb{R}$, $S^2 \times S^1(r)$ (for $r \in \mathbb{R}^+$), or $S^1(t) \times \mathbb{R}^2$ (for $t \in (0, 1]$).

In this paper, we extend and improve the results of [11] in the following sense. Considering the ambient space $Q^2_{c_1} \times Q^2_{c_2}$ with $c_1 \neq c_2$, we prove

**Theorem 1.** Let $\Sigma$ be an isoparametric hypersurface in $Q^2_{c_1} \times Q^2_{c_2}$, $c_1 \neq c_2$, and unit normal $N = N_1 + N_2$. Then the principal curvatures of $\Sigma$ are constant if and only if $|N_1|$ and $|N_2|$ are constant.

In addition to the converse of a result obtained by J. Julio-Batalla, which states that if $|N_1|$ and $|N_2|$ are constant, then $\Sigma$ has constant principal curvatures, Theorem 1 also provides the equivalence for the entire class of ambient spaces $Q^2_{c_1} \times Q^2_{c_2}$, with $c_1 \neq c_2$. To get this Theorem, we use the theory of Jacobi fields, based on the ideas developed by M. Domínguez-Vázquez and J. M. Manzano in [7], to analyze the extrinsic geometry of hypersurfaces parallel to $\Sigma$. It is interesting to note that Jacobi field theory allows us to obtain an alternative proof of J. Julio-Batalla’s result. Moreover, we obtain the following general classification of isoparametric hypersurfaces with constant principal curvatures in $Q^2_{c_1} \times Q^2_{c_2}$, $c_1 \neq c_2$, which includes the classification for $S^2 \times \mathbb{R}^2$ given in [11]:

**Theorem 2.** Let $\Sigma$ be an isoparametric hypersurface in $Q^2_{c_1} \times Q^2_{c_2}$, $c_1 \neq c_2$, with constant principal curvatures. Then, up to rigid motions, $\Sigma$ is an open subset of one of the following hypersurfaces:

a) $C^1(\kappa_j) \times Q^2_{c_2}$ or $Q^2_{c_1} \times C^1(\kappa_j)$, where $C^1(\kappa_j)$ is a complete curve with constant geodesic curvature $\kappa_j$ in $Q^2_{c_1}$.

b) $\Psi(\mathbb{R}^3) \subset \mathbb{H}^2 \times \mathbb{R}^2$, where $\Psi : \mathbb{R}^3 \to \mathbb{H}^2 \times \mathbb{R}^2$ is an immersion given by

$$
\Psi(t,u,v) = e^{-bt}(\alpha(u),0) + \left(\cosh(-bt),0,\sinh(-bt),V_0t\right) + \left(\bar{0},p_0 + W_0v\right),
$$

where $\mathbb{H}^2 \subset \mathbb{L}^3$ is given as the standard model of the hyperbolic space in the Lorentz 3-space $\mathbb{L}^3$, the curve $\alpha$ is given by $\alpha(u) = \left(\frac{u^2}{2}, u, -\frac{u^2}{2}\right)$, $V_0$ and $W_0$ are constant orthogonal vectors in $\mathbb{R}^2$ such that $||W_0|| = 1$ and $b = \sqrt{1 - ||V_0||^2}$.

Recall that, besides of the geodesics, the complete curves $C^1(\kappa_j) \subset Q^2_{c_2}$ with constant geodesic curvature are given by: $S^1(t) \subset S^2$ for $t \in (0, 1)$; circles, horocycles or hypercycles in $\mathbb{H}^2$; and $S^1(r) \subset \mathbb{R}^2$ for $r \in \mathbb{R}^+$. Regarding the hypersurfaces given in Theorem 2b), geometrically, $\Psi(\mathbb{R}^3)$ provides a hypersurface given as a family of geodesically parallel surfaces given by the products $C^1(1) \times \mathbb{R}$, where $C^1(1) \subset \mathbb{H}^2$ is a horocycle (see Remark 5).

The paper is organized as follows. In Section 2, we provide some preliminary concepts and notations that will be used throughout the work. Section 3 is devoted to the proof of Theorems 1 and 2. Using Jacobi field theory, we start by proving Theorem 1, which characterizes isoparametric hypersurfaces with constant principal curvatures in $Q^2_{c_1} \times Q^2_{c_2}$. Then, using Theorem 1, we classify these hypersurfaces by proving Theorem 2.
2. Preliminary notions and results

Before proving our main results, let us present some background content on complex and product structures, the Jacobi field theory and isoparametric functions.

Let \( Q_{c_1}^2 \) and \( Q_{c_2}^2 \) be two 2-dimensional space forms with distinct constant sectional curvatures \( c_1 \) and \( c_2 \), respectively. For \( i = 1, 2 \), we denote by \( L_i \) the standard complex structure in \( Q_{c_i}^2 \). If \( Q_{c_2}^2 \) is the 2-dimensional sphere \( S^2 \) of curvature \( c_i = 1 \), \( L_i \) is given by

\[
L_i : TS^2 \rightarrow TS^2 \\
v \mapsto L_i(v) = p \times v,
\]

for \( p \in S^2 \), \( v \in T_pS^2 \), see \([5]\). When \( Q_{c_1}^2 \) is the hyperbolic space \( \mathbb{H}^2 \) of curvature \( c_i = -1 \), we will consider its standard Lorentzian model, i.e.,

\[
\mathbb{H}^2 = \{(x_1, x_2, x_3) \in \mathbb{L}^3 \mid -x_1^2 + x_2^2 + x_3^2 = -1 \text{ and } x_1 > 0\},
\]

where \( \mathbb{L}^3 \) is the 3-dimensional Minkowski space endowed with the Lorentzian cross product \( \times \), defined by

\[
(a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_3b_2 - a_2b_3, a_3b_1 - a_3b_1, a_1b_2 - a_2b_1).
\]

In this model, \( L_i \) is given by

\[
L_i : T\mathbb{H}^2 \rightarrow T\mathbb{H}^2 \\
v \mapsto L_i(v) = p \times v,
\]

for \( p \in \mathbb{H}^2 \), \( v \in T_p\mathbb{H}^2 \), see \([5]\) and \([3]\). Finally, if \( Q_{c_1}^2 \) is the space form \( \mathbb{R}^2 \) of curvature \( c_i = 0 \), \( L_i \) is defined by

\[
L_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\
v \mapsto L_i(q_1, q_2) = (-q_2, q_1),
\]

see \([11]\).

We endow \( Q_{c_1}^2 \times Q_{c_2}^2 \) with the standard product metric, denoted by \( \langle \cdot, \cdot \rangle \). Moreover, given \( Y \in T(Q_{c_1}^2 \times Q_{c_2}^2) \), we write \( Y = Y^{Q_{c_1}^2} + Y^{Q_{c_2}^2} \), where the components \( Y^{Q_{c_1}^2} \) and \( Y^{Q_{c_2}^2} \) of \( Y \) are given as its tangent parts to \( Q_{c_1}^2 \) and \( Q_{c_2}^2 \), respectively. We define on \( Q_{c_1}^2 \times Q_{c_2}^2 \) the complex structures

\[
J_1 = L_1 + L_2, \quad J_2 = L_1 - L_2,
\]

and we denote by \( \nabla \) and \( R \) its Levi-Civita connection and curvature tensor, respectively.

Now, let us consider the product structure \( P \) in \( Q_{c_1}^2 \times Q_{c_2}^2 \) defined by

\[
P(y^{Q_{c_1}^2} + y^{Q_{c_2}^2}) = y^{Q_{c_1}^2} - y^{Q_{c_2}^2},
\]

for any vector \( y \in T(Q_{c_1}^2 \times Q_{c_2}^2) \). Note that \( P \) satisfies

\[
P = -J_1J_2 = -J_2J_1.
\]

Moreover, \( P \) has the following properties:

\[
P^2 = I \quad (P \neq I), \quad \langle PY, Z \rangle = \langle Y, PZ \rangle, \quad \text{and} \quad (\nabla_Y P)(Z) = 0,
\]
for any vector field $Y, Z \in T(Q^2_{c_1} \times Q^2_{c_2})$. Using the product structure $P$, $\tilde{R}$ is given by

$$
\tilde{R}(V, W, Z, Y) = \frac{c_1}{4} \left\{ (V, PY + Y)\langle PW + W, Z \rangle - (W, PY + Y)\langle PV + V, Z \rangle \right\}
+ \frac{c_2}{4} \left\{ (V, PY - Y)\langle PW - W, Z \rangle - (W, PY - Y)\langle PV - V, Z \rangle \right\},
$$

where $V, W, Z, Y \in T(Q^2_{c_1} \times Q^2_{c_2})$, see [5].

Let $\Sigma^3 \subset Q^2_{c_1} \times Q^2_{c_2}$ be an oriented hypersurface with unit normal vector $N = N_1 + N_2$ and Levi-Civita connection $\nabla$. We define in $\Sigma^3$ a smooth function $C$ and a tangent vector field $X$ by

$$
(2.1) \quad C = \langle PN, N \rangle \quad \text{and} \quad X = PN - CN.
$$

Observe that $X$ is the tangential component of $PN$ and $|X|^2 = 1 - C^2$, which implies $-1 \leq C \leq 1$.

Using the curvature tensor of $Q^2_{c_1} \times Q^2_{c_2}$ and the vector field $X$ defined above, the Codazzi equation of $\Sigma$ is given by

$$
\nabla S(V, W, Z) - \nabla S(W, V, Z) = \tilde{R}(V, W, Z, N),
$$

where

$$
\tilde{R}(V, W, Z, N) = \frac{c_1}{4} \left\{ (V, PN + N)\langle PW + W, Z \rangle - (W, PN + N)\langle PV + V, Z \rangle \right\}
+ \frac{c_2}{4} \left\{ (V, PN - N)\langle PW - W, Z \rangle - (W, PN - N)\langle PV - V, Z \rangle \right\}
= \frac{c_1}{4} \left\{ (V, X)\langle PW + W, Z \rangle - (W, X)\langle PV + V, Z \rangle \right\}
+ \frac{c_2}{4} \left\{ (V, X)\langle PW - W, Z \rangle - (W, X)\langle PV - V, Z \rangle \right\},
$$

with $V, W, Z \in T\Sigma$.

In this work, we will use the Jacobi field theory to analyze the extrinsic geometry of hypersurfaces equidistant to the hypersurface $\Sigma$. In what follows, we will give a brief description of this theory. For more details, we refer to [16].

Given a hypersurface $\Sigma^n$ of a Riemannian manifold $\tilde{M}^{n+1}$ with unit normal vector field $N$, let $\varepsilon$ be a positive real number and, for $r \in (-\varepsilon, \varepsilon)$, consider the application

$$
(2.2) \quad \Phi_r : \Sigma^n \rightarrow \tilde{M}^{n+1}, \quad p \mapsto \exp_p(rN_p),
$$

where $\exp_p : T_p\tilde{M} \rightarrow \tilde{M}$ denotes the exponential map of $\tilde{M}^{n+1}$ at $p \in \Sigma$. For $\varepsilon > 0$ small enough, the map $\Phi_r$ is smooth and it parametrizes the parallel displacement of $\Sigma$ at an oriented distance $r$ in the direction $N$. The parallel hypersurface $\Phi_r(\Sigma)$ will be denoted by $\Sigma_r$.

Let $\gamma_p : I \rightarrow \tilde{M}$ be the geodesic parametrized by arc length with $0 \in I \subset \mathbb{R}$, $\gamma_p(0) = p \in \Sigma$ and $\gamma_p(0) = N_p$. Let $\zeta_Y$ be the Jacobi field along $\gamma_p$ with initial conditions given by

$$
\zeta_Y(0) = Y, \quad \zeta_Y'(0) = -AY,
$$
We can assume that the open set the following orthonormal frame where $a, b$ are called isoparametric hypersurfaces. In this case, let us take in $\Sigma, r$ is parallel, the parallel hypersurface at an oriented distance $r$ is denoted by $\Sigma_r$. We first observe that, since $\Sigma$ is isoparametric and the product structure $P$ is parallel, the function $C$, defined on the family of parallel hypersurfaces, does not depend on the displacement parameter $r$, once $N(C) = 0$. In fact, since $C = \langle PN, N \rangle$ and $\nabla_N N = 0$, we have

$$N(C) = \langle \nabla_N N, PN \rangle + \langle N, P\nabla_N N \rangle = 0.$$  

Now we prove that $C$ is constant along $\Sigma$. Let us recall that $|C| \leq 1$. Consider the open set

$$U = \{ p \in \Sigma \mid C^2(p) < 1 \}.$$

We can assume that $U \neq \emptyset$, otherwise $C^2 = 1$ on $\Sigma$. In this case, let us take in $U$ the following orthonormal frame

$$B = \left\{ B_1 = \frac{X}{\sqrt{1 - C^2}}, B_2 = \frac{J_1 N + J_2 N}{\sqrt{2(1 + C)}}, B_3 = \frac{J_1 N - J_2 N}{\sqrt{2(1 - C)}} \right\},$$

where $A$ is the shape operator of $\Sigma$ associated with $N$. Then, a unit normal vector to $\Sigma_r$ at $\gamma_r(p)$ is given by $\gamma_r(p)$ and its correspondent shape operator satisfies

$$A_r \zeta_Y(r) = -\zeta'_Y(r).$$

If we write $\zeta_Y(r) = D(r) \tilde{Y}(r)$, where $D(r)$ is an endomorphism acting on $T_{\gamma_p(r)} \Sigma_r$ and $\tilde{Y}(r)$ is the parallel transport of $Y$ along $\gamma_r$, then we have

$$A_r = -(D' \circ D^{-1})(r).$$

Consequently, by the Jacobi formula, the mean curvature of the hypersurface $\Sigma_r$ is given by

$$h(r) = \frac{(\det D)'(r)}{n \det D(r)}.$$

Finally, we introduce the notion of isoparametric function. A non-constant smooth function $f : M^{n+1} \to \mathbb{R}$ is called isoparametric if the gradient and the Laplacian of $f$ satisfy

$$|\nabla f|^2 = a(f) \quad \text{and} \quad \Delta f = b(f),$$

where $a, b : I \subset \mathbb{R} \to \mathbb{R}$ are smooth functions. The smooth hypersurfaces $\Sigma_r = f^{-1}(r)$ for $r$ regular value of $f$ are called isoparametric hypersurfaces. In this case, the unit normal vector field is given by $N = \nabla_f$. We observe that, by the conditions under the gradient and the Laplacian given in the definition of an isoparametric function, $\Sigma_r$ has constant mean curvature for each $r$ (i.e., depending only on $r$) and $N$ is a geodesic field, see [6].

### 3. Proof of the main results

To prove Theorems 1 and 2 we combine the techniques developed by F. Urbano [14], J. Julio-Batalla [11], and Domínguez-Vázquez and Manzano [7].

**Proof of Theorem 1.** Let $\Sigma$ be an isoparametric hypersurface in $Q^{c_1} \times Q^{c_2}$ with $c_1 \neq c_2$ and unit normal $N = N_1 + N_2$. In order to prove Theorem 1 it is enough to show that the principal curvatures of $\Sigma$ are constant if and only if the function $C$, given in (2.1), is constant. In fact, as $|N_1|^2 = \frac{1 + c_1}{c_1 c_2}$ and $|N_2|^2 = \frac{1 + c_2}{c_1 c_2}$, it follows that $|N_1|$ and $|N_2|$ are constant if and only if $C$ is constant.

Recall that the family of hypersurfaces parallel to $\Sigma$ in the direction of $N$ is given by (2.2) and the parallel hypersurface at an oriented distance $r$ is denoted by $\Sigma_r$. We first observe that, since $\Sigma$ is isoparametric and the product structure $P$ is parallel, the function $C$, defined on the family of parallel hypersurfaces, does not depend on the displacement parameter $r$, once $N(C) = 0$. In fact, since $C = \langle PN, N \rangle$ and $\nabla_N N = 0$, we have

$$N(C) = \langle \nabla_N N, PN \rangle + \langle N, P\nabla_N N \rangle = 0.$$  

Now we prove that $C$ is constant along $\Sigma$. Let us recall that $|C| \leq 1$. Consider the open set

$$U = \{ p \in \Sigma \mid C^2(p) < 1 \}.$$  

We can assume that $U \neq \emptyset$, otherwise $C^2 = 1$ on $\Sigma$. In this case, let us take in $U$ the following orthonormal frame

$$B = \left\{ B_1 = \frac{X}{\sqrt{1 - C^2}}, B_2 = \frac{J_1 N + J_2 N}{\sqrt{2(1 + C)}}, B_3 = \frac{J_1 N - J_2 N}{\sqrt{2(1 - C)}} \right\},$$
Consequently, we also can extend the fields \( B \) since the curvature tensor formula of a manifold of constant sectional curvature, we get

where

Thus, we can extend the unit normal \( \Sigma \) by the definition of

Therefore, \( \xi_j(0) = B_j \) and \( \xi_j(0) = -AB_j \),

where \( A \) is the shape operator of \( \Sigma \) associated with \( N \).

Since these initial conditions are orthogonal to \( \dot{\gamma}_p(0) \), each Jacobi field \( \xi_j \) is also orthogonal to \( N_{\gamma_p(r)} = \dot{\gamma}_p(r) \) and, hence, it can be written as

for certain smooth functions \( b_{i_j} \) on \( (-\epsilon, \epsilon) \).

Let us observe that \( \nabla_N B_i = 0 \), for all \( i = 1, 2, 3 \). In fact, since \( N(C) = 0 \) and \( P \) is parallel, we have \( \nabla_N X = 0 \), which implies \( \nabla_N B_1 = 0 \). Furthermore, since \( J_i \) is also parallel, for \( i = 1, 2 \), we conclude that \( \nabla_N B_j = 0 \), \( j = 2, 3 \). Thus, we have, on the one hand,

On the other hand, if we denote by \( \tilde{R}^c \) the curvature tensor of \( \mathbb{Q}_2^2 \), we get

since \( X + PX = (1 - C)(N + PN) \) and \( X - PX = -(1 + C)(N - PN) \). Now, using the curvature tensor formula of a manifold of constant sectional curvature, we get

Therefore,

\[ \tilde{R}(\xi_j, \dot{\gamma}_p)\dot{\gamma}_p = \tilde{R}(\xi_j, N)N \]

\[ = b_{1_j}R(B_1, N)N + b_{2_j}R(B_2, N)N + b_{3_j}R(B_3, N)N \]

\[ = b_{2_j}c_1|N + PN|^2 4 B_2 + b_{3_j}c_2|N - PN|^2 4 B_3 \]

\[ = b_{2_j}c_1(1 + C) 2 B_2 + b_{3_j}c_2(1 - C) 2 B_3. \]

Since \( \xi_j \) is a Jacobi field, we have from (3.2) and (3.3) the following homogeneous linear system of ordinary differential equations

\[ b''_{1_j} = 0, \quad b''_{2_j} + \delta_1 b_{2_j} = 0, \quad b''_{3_j} + \delta_2 b_{3_j} = 0, \]
where \( \delta_1 = \frac{c_1(1+C)}{2} \) and \( \delta_2 = \frac{c_2(1-C)}{2} \).

In the sequence, we describe the initial conditions of the system (3.4). Firstly, as \( \xi_j(0) = B_j \), we get

\[
\begin{align*}
  b_{11}(0) &= 1, \quad b_{12}(0) = 0, \quad b_{13}(0) = 0, \\
  b_{21}(0) &= 0, \quad b_{22}(0) = 1, \quad b_{23}(0) = 0, \\
  b_{31}(0) &= 0, \quad b_{32}(0) = 0, \quad b_{33}(0) = 1.
\end{align*}
\]  

(3.5)

Secondly, let the shape operator of \( \Sigma \) be determined by the relations \( AB_i = \sigma_1 B_1 + \sigma_2 B_2 + \sigma_3 B_3 \), for certain smooth functions \( \sigma_{ij} \). Since \( A \) is symmetric, we have \( \sigma_{12} = \sigma_{21}, \quad \sigma_{13} = \sigma_{31} \) and \( \sigma_{32} = \sigma_{23} \). Furthermore, taking into account that \( \xi_j' = \nabla_N \xi_j = -A\xi_j \), we obtain

\[
\begin{align*}
  b_{11}'(0) &= -\sigma_{11}, \quad b_{12}'(0) = -\sigma_{21}, \quad b_{13}'(0) = -\sigma_{31}, \\
  b_{21}'(0) &= -\sigma_{12}, \quad b_{22}'(0) = -\sigma_{22}, \quad b_{23}'(0) = -\sigma_{23}, \\
  b_{31}'(0) &= -\sigma_{13}, \quad b_{32}'(0) = -\sigma_{32}, \quad b_{33}'(0) = -\sigma_{33}.
\end{align*}
\]  

(3.6)

With the initial conditions (3.5) and (3.6), the solution of system (3.4) is given by

\[
\begin{align*}
  b_{11}(r) &= -\sigma_{11} r + 1, \\
  b_{12}(r) &= -\sigma_{12} r, \\
  b_{13}(r) &= -\sigma_{13} r, \\
  b_{21}(r) &= -\sigma_{22} S_{\delta_1}(r), \\
  b_{22}(r) &= -\sigma_{22} S_{\delta_1}(r) + C_{\delta_1}(r), \\
  b_{23}(r) &= -\sigma_{22} S_{\delta_1}(r), \\
  b_{31}(r) &= -\sigma_{13} S_{\delta_2}(r), \\
  b_{32}(r) &= -\sigma_{32} S_{\delta_2}(r), \\
  b_{33}(r) &= -\sigma_{33} S_{\delta_2}(r) + C_{\delta_2}(r),
\end{align*}
\]  

(3.7)

where we consider the auxiliary functions

\[
S_{\delta_1}(r) = \begin{cases} 
\frac{1}{\sqrt{\delta_1}} \sinh(r\sqrt{-\delta_1}) & \text{if } \delta_1 < 0, \\
\frac{1}{\sqrt{\delta_1}} \sin(r\sqrt{\delta_1}) & \text{if } \delta_1 > 0,
\end{cases}
\]

\[
C_{\delta_1}(r) = \begin{cases} 
\cosh(r\sqrt{-\delta_1}) & \text{if } \delta_1 < 0, \\
\cos(r\sqrt{\delta_1}) & \text{if } \delta_1 > 0.
\end{cases}
\]

for \( i \in \{1, 2\} \).

For every \( r \), the shape operator \( A_r \) of \( \Sigma_r \) with respect to the normal \( \gamma'_p(r) \) is given by (2.3), where \( D(r) \) is linear endomorphism of \( T_{\gamma_p(r)} \Sigma_r \), determined by the relations

\[
D(r) B_j(\gamma_p(r)) = \xi_j(r), \quad D'(r) B_j(\gamma_p(r)) = \xi'_j(r).
\]

Considering the orthonormal basis \( \{ B_1(\gamma_p(r)), B_2(\gamma_p(r)), B_3(\gamma_p(r)) \} \) of \( T_{\gamma_p(r)} \Sigma_r \), the matrix form of the operator \( D(r) \) is given by

\[
D(r) = \begin{pmatrix}
  b_{11}(r) & b_{12}(r) & b_{13}(r) \\
  b_{21}(r) & b_{22}(r) & b_{23}(r) \\
  b_{31}(r) & b_{32}(r) & b_{33}(r)
\end{pmatrix},
\]

(3.8)

From now on, our strategy is given as follows. Firstly, we are going to get explicitly the formulas of \( \det D(r) \) and \( \frac{d}{dr}(\det D(r)) \) in terms of the functions \( b_{ij} \) and its derivatives. Secondly, will apply such formulas to construct

\[
f(r) = \frac{d}{dr}(\det D(r)) + 3h(r) \det D(r),
\]
which vanishes identically on \((-\epsilon, \epsilon)\), by equation (2.24). Finally, we will use the fact that \(f \equiv 0\) as well as its derivatives to obtain some algebraic relations between the components of \(A\) on the basis \(\{B_i\}_{i=1}^{3}\) and the function \(C\).

From (3.7) and (3.8), we have that

\[
\det D(r) = A_1 r S_{\delta_1}(r) S_{\delta_2}(r) + A_2 r S_{\delta_1}(r) C_{\delta_2}(r) + A_3 r S_{\delta_2}(r) C_{\delta_1}(r) + A_4 r S_{\delta_1}(r) C_{\delta_2}(r)
\]

where

\[
A_1 = -\det A, \quad A_2 = \sigma_{11} \sigma_{22} - \sigma_{13}^2, \quad A_3 = \sigma_{11} \sigma_{33} - \sigma_{13}^2, \quad A_4 = \sigma_{22} \sigma_{33} - \sigma_{23}^2.
\]

Now, taking into account that \(S'_{\delta_1}(r) = C_{\delta_1}(r)\) and \(C'_{\delta_1}(r) = -\delta_1 S_{\delta_1}(r)\), we obtain

\[
\frac{d}{dr} (\det D(r)) = A_1 \left( S_{\delta_1}(r) S_{\delta_2}(r) + r C_{\delta_1}(r) S_{\delta_2}(r) + r S_{\delta_1}(r) C_{\delta_2}(r) \right)
\]

Thus, the function \(f\) is given explicitly as

\[
f(r) = A_1 \left( S_{\delta_1}(r) S_{\delta_2}(r) + r C_{\delta_1}(r) S_{\delta_2}(r) + r S_{\delta_1}(r) C_{\delta_2}(r) \right)
\]

As \(f \equiv 0\), so is its derivative. Then, taking the derivative in (3.10) and applying at \(r = 0\), we obtain the following relation:

\[
0 = f'(0) = 2(A_2 + A_3 + A_4) - 9h^2(0) + 3h'(0) - (\delta_1 + \delta_2),
\]

where \(h(0)\) is the mean curvature of \(\Sigma\).
Note that $A_i$, $\delta_i$, $h(0)$ and $h'(0)$, depend only, in principle, of the base point $p \in \Sigma$. However, by assumption, $\Sigma$ is isoparametric and hence, $h(0)$ and $h'(0)$ are constants throughout $\Sigma$, that is, it is independent of the chosen base point $p \in \Sigma$ of normal geodesic $\gamma_p$.

Furthermore, observe that

$$9h^2(0) = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2(\sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33}),$$

and

$$tr(A^2) = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 + 2(\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{23}^2).$$

Thus, by the definitions of the $A_i's$ in (3.9), we have $2(A_2 + A_3 + A_4) - 9h^2(0) = -tr(A^2)$. Substituting in (3.11), we get

$$tr(A^2) = 3h'(0) - (\delta_1 + \delta_2),$$

where $\delta_1 + \delta_2 = \frac{1}{2}(C(c_1 - c_2) + c_1 + c_2)$.

Therefore, if $\Sigma$ has constant principal curvatures $\mu_1$, $\mu_2$, $\mu_3$, then $tr(A^2) = \mu_1^2 + \mu_2^2 + \mu_3^2$ is constant and hence, $C$ is constant, since $c_1 \neq c_2$.

Conversely, suppose $C$ is constant. Since the gradient of the function $C$ is given by $\nabla C = -2A(X)$ (see [Lemma 1, [14]]), then $A(X) = 0$. Therefore, $\sigma_{1j} = \sigma_{j1} = 0$, for all $j = 1, 2, 3$. Thus, we have $A_1 = A_2 = A_3 = 0$ and we can rewrite (3.11) as

$$0 = 2A_4 - 9h^2(0) + 3h'(0) - (\delta_1 + \delta_2),$$

and, as a consequence, we have that $A_4$ is constant.

Moreover, as $\sigma_{1j} = \sigma_{j1} = 0$, the characteristic polynomial $Q_A$ of $A$ is given by

$$Q_A(\lambda) = -\lambda^3 + 3h(0)\lambda^2 - A_4\lambda.$$  

Therefore, since $A_4$ is constant, it follows that the principal curvatures of $\Sigma$ are constant. \hfill \Box

**Proof of Theorem 3.** Let $\Sigma$ be an isoparametric hypersurface in $\mathbb{Q}_{e_1}^2 \times \mathbb{Q}_{e_2}^2$ with constant principal curvatures. By Theorem 1, we have that $C$ is constant. If $C = 1$ we have $PN = N$, and thus, $N = (N_1, 0)$. If $C = -1$ we have $PN = -N$, and then, $N = (0, N_2)$. In such cases, $\Sigma$ is an open subset of $C^1(\kappa_j) \times \mathbb{Q}_{e_2}^2$ or $\mathbb{Q}_{e_1}^2 \times C^1(\kappa_j)$, respectively, where $C^1(\kappa_j)$ is a complete curve in $\mathbb{Q}_{e_2}^2$, of constant geodesic curvature $\kappa_j$. In fact, let us suppose that $N = (N_1, 0)$, then $\Sigma$ is an open subset of $C^1 \times \mathbb{Q}_{e_2}^2$, where $C^1$ is a regular curve in $\mathbb{Q}_{e_1}^2$. Let $\psi$ be a parametrization by arc length of $C^1$, with unit normal vector $n_\psi = \pm N_1$. Let $\{e_1, e_2, e_3\}$ an orthonormal frame in $C^1 \times \mathbb{Q}_{e_2}^2$, with $e_1 = \psi'$ and $\{e_2, e_3\}$ an orthonormal basis in $\mathbb{Q}_{e_2}^2$. If we denote the shape operator of $\Sigma$ by $A$, considering without loss of generality that $N_1 = n_\psi$, we have

$$Ae_1 = -\nabla_{e_1} N_1 = -\nabla^\mathbb{Q}_{e_1}^2 n_\psi = \kappa_j \psi' = \kappa_j e_1,$$

$$Ae_2 = -\nabla_{e_2} N_1 = 0,$$

$$Ae_3 = -\nabla_{e_3} N_1 = 0.$$

Therefore, the curvature $\kappa_j$ of $C^1$ is a principal curvature of $\Sigma$, which implies that $\kappa_j$ is constant. The case where $N = (0, N_2)$ is analogous.

In the sequence, we are going to prove that, if $|C| < 1$, the only remaining possibility is the case when one $e_i$ is negative. Therefore, in what follows, let us
assume that $C \in (-1, 1)$. In this case, as in the proof of Theorem \[1\] let us consider the frame

$$B = \begin{cases} B_1 = \frac{X}{\sqrt{1 - C^2}}, & B_2 = \frac{J_1 N + J_2 N}{\sqrt{2(1 + C)}}, & B_3 = \frac{J_1 N - J_2 N}{\sqrt{2(1 - C)}} \end{cases},$$

and the function $f$ given in (3.10). Again, taking derivatives in (3.10) and applying them at $r = 0$, we obtain the following relations:

\begin{align*}
(3.12) & \quad 0 = f'(0) = 2(A_2 + A_3 + A_4) - 9\sigma^2(0) + 3h'(0) - (\delta_1 + \delta_2), \\
(3.13) & \quad 0 = f''(0) = 6A_1 + 6h(0)(A_2 + A_3 + A_4) - 18h'(0)h(0) + 2\sigma_{11}(\delta_1 + \delta_2) \\
& \quad + 2\sigma_{22}\delta_2 + 2\sigma_{33}\delta_1 + 3h''(0),
\end{align*}

where the functions $A_i$, $i = 1, \ldots, 4$, are given in (3.9).

Let us recall that as $C$ is constant we have $\sigma_{11} = \sigma_{11} = 0$, which imply that $A_1 = A_2 = A_3 = 0$. Moreover, since $h(0)$ is the mean curvature of $\Sigma$, we also conclude that

$$3h(0) = \sigma_{22} + \sigma_{33}.$$  

Thus, we can rewrite (3.12) and (3.13) as follows:

\begin{align*}
(3.14) & \quad 0 = 2(\sigma_{22}\sigma_{33} - \sigma_{23}^2) - 9\sigma^2(0) + 3h'(0) - (\delta_1 + \delta_2), \\
(3.15) & \quad 0 = 6h(0)(\sigma_{22}\sigma_{33} - \sigma_{23}^2) - 18h'(0)h(0) + 2\sigma_{22}\delta_2 + 2\sigma_{33}\delta_1 + 3h''(0).
\end{align*}

Combining (3.14), (3.15) and (3.16), we have that

$$2\sigma_{33}(\delta_1 - \delta_2) + 3h(0)(\delta_1 + \delta_2) + 6h(0)\delta_2 + 27h''(0) - 27h'(0)h(0) + 3h''(0) = 0.$$

Note that $(\delta_1 - \delta_2) = \frac{1}{2}(c_1 - c_2 + C(c_1 + c_2)) \neq 0$, since $C \in (-1, 1)$ and $c_1 \neq c_2$. Therefore $\sigma_{33}$ is constant and hence, from (3.14) and (3.15), we have that $\sigma_{22}$ and $\sigma_{33}$ are also constant.

On the other hand, we are going to use Codazzi equation to compute $X(\sigma_{22})$, $X(\sigma_{23})$ and $X(\sigma_{33})$. As each $J_i$ is parallel and $A(X) = 0$ (since $\nabla C = -2A(X)$), we have $\nabla_X B_j = 0$ for all $j = 1, 2, 3$. In this way, since

$$X(\sigma_{ij}) = X(A(B_i, B_j)) = \nabla A(X, B_i, B_j)$$

it follows from the Codazzi equation that

\begin{align*}
X(\sigma_{22}) & = \nabla A(B_2, X, B_2) + \frac{c_1}{4} \{ \langle X, X \rangle \langle PB_2 + B_2, B_2 \rangle - \langle B_2, X \rangle \langle PX + X, B_2 \rangle \} \\
& \quad + \frac{c_2}{4} \{ \langle X, X \rangle \langle PB_2 - B_2, B_2 \rangle - \langle B_2, X \rangle \langle PX - X, B_2 \rangle \} \\
& = -C \langle AB_2, AB_2 \rangle + \langle PAB_2, AB_2 \rangle + \frac{c_1|X|^2}{2} \\
& = \frac{c_1(1 - C^2)}{2} + (1 - C)\sigma_{22}^2 - (1 + C)\sigma_{23}^2, \\
X(\sigma_{23}) & = \nabla A(B_2, B_3, B_3) + \frac{c_1}{4} \{ \langle X, X \rangle \langle PB_2 + B_2, B_3 \rangle - \langle B_2, X \rangle \langle PX + X, B_3 \rangle \} \\
& \quad + \frac{c_2}{4} \{ \langle X, X \rangle \langle PB_2 - B_2, B_3 \rangle - \langle B_2, X \rangle \langle PX - X, B_3 \rangle \} \\
& = -C \langle AB_2, AB_3 \rangle + \langle PAB_2, AB_3 \rangle \\
& = (1 - C)\sigma_{22}\sigma_{33} - (1 + C)\sigma_{23}\sigma_{33},
\end{align*}
\[ X(\sigma_{33}) = \nabla A(B_3, X, B_3) + \frac{c_1}{4} \{ \langle X, X \rangle \langle PB_3 + B_3, B_3 \rangle - \langle B_3, X \rangle \langle PX + X, B_3 \rangle \} \\
+ \frac{c_2}{4} \{ \langle X, X \rangle \langle PB_3 - B_3, B_3 \rangle - \langle B_3, X \rangle \langle PX - X, B_3 \rangle \} \\
= -C(AB_3, AB_3) + \langle PAB_3, AB_3 \rangle - \frac{c_2|X|^2}{2} \\
= \frac{c_2(C^2 - 1)}{2} + (1 - C)\sigma_{23}^2 -(1 + C)\sigma_{33}^2. \]

Therefore,

\begin{align*}
(3.17) & \quad \frac{c_1(1 - C^2)}{2} + (1 - C)\sigma_{22}^2 - (1 + C)\sigma_{23}^2 = 0, \\
(3.18) & \quad \frac{c_2(C^2 - 1)}{2} + (1 - C)\sigma_{23}^2 -(1 + C)\sigma_{33}^2 = 0, \\
(3.19) & \quad (1 - C)\sigma_{22}\sigma_{23} - (1 + C)\sigma_{23}\sigma_{33} = 0.
\end{align*}

Let us show that \( \sigma_{23} = 0 \). Suppose by contradiction that \( \sigma_{23} \neq 0 \). From (3.19), we have

\[ (3.20) \quad (1 - C)^2\sigma_{22}^2 - (1 + C)^2\sigma_{33}^2 = 0. \]

Now, multiplying (3.17) by \( 1 - C \) and (3.18) by \( 1 + C \), we have

\begin{align*}
(3.21) & \quad \frac{c_1(1 - C)(1 - C^2)}{2} + (1 - C)^2\sigma_{22}^2 -(1 - C^2)\sigma_{23}^2 = 0, \\
(3.22) & \quad \frac{c_2(1 + C)(C^2 - 1)}{2} + (1 - C^2)\sigma_{23}^2 -(1 + C)^2\sigma_{33}^2 = 0.
\end{align*}

Adding (3.21) to (3.22) and using (3.20), we get

\[ (3.23) \quad c_1(1 - C) = c_2(1 + C), \]

Since \( C \in (-1, 1) \) and \( c_1 \neq c_2 \), we have a contradiction. Therefore \( \sigma_{23} = 0 \).

If \( \sigma_{23} = 0 \), the system given by equations (3.17), (3.18) and (3.19) is reduced to

\[ (3.24) \quad \sigma_{22}^2 = -\frac{c_1(1 + C)}{2}, \quad \sigma_{33}^2 = -\frac{c_2(1 - C)}{2}. \]

Observe that the only possibility of solving (3.24) is to consider that one \( c_i \) is negative and the other is zero. Then, without loss of generality, let us assume from now on that \( c_1 = -1 \) and \( c_2 = 0 \). Thus, the previous computation shows us that \( \sigma_{ij} = 0 \), for \( i \neq j \) and \( \sigma_{11} = \sigma_{33} = 0 \). Therefore, we conclude that \( \{B_1, B_2, B_3\} \) must be a frame of principal directions of \( \Sigma \), with principal curvatures

\[ \mu_1 = 0, \quad \mu_2 = \pm \sqrt{\frac{1 + C}{2}}, \quad \mu_3 = 0. \]

In what follows, we consider the case when \( \mu_2 = \sqrt{\frac{1 + C}{2}} \). The shape operator \( A \) and the tangential component of the product structure \( P^T \) are given, with respect to the frame \( B \), respectively by

\[
A = \begin{pmatrix}
0 & 0 & 0 \\
0 & \sigma_{22} & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad P^T = \begin{pmatrix}
-C & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}.
\]
Since $P$ and $J_i$ are parallel, we have that the Levi-Civita connection $\nabla$ of $\mathbb{H}^2 \times \mathbb{R}^2$ is given by
\[
\begin{align*}
\nabla_{B_1} B_1 &= 0, & \nabla_{B_2} B_3 &= 0, & \nabla_{B_3} B_2 &= 0, \\
\nabla_{B_2} B_1 &= -\sqrt{\frac{1-C}{2}} B_2, & \nabla_{B_1} B_2 &= \frac{PN+N}{\sqrt{2(1+C)}}, & \nabla_{B_3} B_1 &= 0, \\
\nabla_{B_1} B_3 &= 0.
\end{align*}
\]
Note that $[B_1, B_3] = [B_2, B_3] = 0$. Now, let $\lambda$ a function such that
\[
B_1(\lambda) = -\lambda \sqrt{\frac{1-C}{2}}, \quad B_2(\lambda) = 0 \quad \text{and} \quad B_3(\lambda) = 0.
\]
In this way, we have
\[
[B_1, \lambda B_2] = (B_1(\lambda) + \lambda \sqrt{\frac{1-C}{2}}) B_2 = 0
\]
and $[\lambda B_2, B_3] = 0$. Therefore, there is a parametrization $\Psi: \Omega \subset \mathbb{R}^3 \rightarrow \Sigma$, where $\Omega$ is an open subset of $\mathbb{R}^3$ with coordinates $(t, u, v)$, such that
\[
\Psi_t = B_1, \quad \Psi_u = \lambda B_2 \quad \text{and} \quad \Psi_v = B_3.
\]
Now, we are going to construct the parametrization $\Psi$. Since $\Psi_v = B_3$, $B_3$ has no component in $\mathbb{H}^2$, and $\nabla_{B_3} B_3 = 0$, i.e., $B_3$ is a geodesic field of $\mathbb{H}^2 \times \mathbb{R}^2$, when we integrate it with respect to $v$, we have
\[
\Psi = \left( \Psi^\mathbb{H}_2(t, u), \beta(t, u) + B_3 v \right),
\]
where $\Psi^\mathbb{H}_2$ is the component of $\Psi$ in $\mathbb{H}^2$.
Before integrating with respect to the variable $u$, we first observe that $B_2$ has no component in $\mathbb{R}^2$ and
\[
\nabla_{B_2} B_2 = \nabla_{B_2}^\mathbb{H}_2 B_2^\mathbb{H}_2 = \frac{PN+N}{\sqrt{2(1+C)}}
\]
Therefore,
\[
\langle \nabla_{B_2}^\mathbb{H}_2 B_2^\mathbb{H}_2, \nabla_{B_2}^\mathbb{H}_2 B_2^\mathbb{H}_2 \rangle = \frac{1}{2(1+C)} \left( 2(PN, N) + (PN, PN) + (N, N) \right) = \frac{1}{2(1+C)}(2C+2) = 1,
\]
that is, if $\varphi$ is a curve parametrized by arc length, with $\varphi' = B_2^\mathbb{H}_2$, then the geodesic curvature $k_g$ of $\varphi$ is $k_g = 1$, and hence $\varphi$ is a horocycle. Up to rigid motions, $\varphi$ is given by
\[
\varphi(u) = \left( 1 + \frac{u^2}{2}, u, -\frac{u^2}{2} \right).
\]
As $\Psi_u = \left( \Psi^\mathbb{H}_2, \beta_u \right) = \lambda B_2$, it follows that $\beta$ does not depend on $u$. Thus,
\[
\Psi^\mathbb{H}_2 = \lambda B_2 = \lambda(t) (u, 1, -u, 0, 0),
\]
and $B_2(\lambda) = B_3(\lambda) = 0$. When we integrate $\Psi^\mathbb{H}_2$ with respect to $u$, we have
\[
\Psi^\mathbb{H}_2(t, u) = \lambda(t) \left( \frac{u^2}{2}, u, -\frac{u^2}{2} \right) + \Lambda(t),
\]
where \( \Lambda(t) \) is a smooth curve in \( \mathbb{H}^2 \). Hence,

\[
\Psi(t, u, v) = \left( \lambda(t) \alpha(u) + \Lambda(t), \beta(t) + B_3 v \right),
\]

with \( \alpha(u) = \left( \frac{u}{2}, u, -\frac{u}{2} \right) \).

Finally, we integrate \( B_1 = \Psi_t = \left( \lambda'(t) \alpha(u) + \Lambda'(t), \beta'(t) \right) \). Since \( \nabla_B B_1 = 0 \), \( B_1 \) is also a geodesic field of \( \mathbb{H}^2 \times \mathbb{R}^2 \). Therefore, \( \beta(t) = p_0 + V_0 t \). Considering \( \gamma(t) = \lambda(t) \alpha(u) + \Lambda(t) \), we have \( \Psi_t = \left( \gamma'(t), V_0 \right) = B_1 \), with \( V_0 = B_1^{\mathbb{R}^2} \). It follows by the definition of \( B_1 \) that \( \|B_1^{\mathbb{R}^2}\| = \sqrt{1 + C} \). As \( \|\gamma'\|^2 + \|B_1^{\mathbb{R}^2}\|^2 = 1 \), we get

\[
\|\gamma'\| = \|B_1^{\mathbb{R}^2}\| = \sqrt{\frac{1 - C}{2}}.
\]

Note that

\[
\frac{D\gamma'}{dt} = \frac{d\gamma'}{dt} - \|B_1^{\mathbb{R}^2}\|^2 \gamma = \alpha(u) \left( \lambda''(t) - \|B_1^{\mathbb{R}^2}\|^2 \lambda(t) \right) + \Lambda''(t) - \|B_1^{\mathbb{R}^2}\|^2 \Lambda(t).
\]

Since \( \gamma \) is a geodesic in \( \mathbb{H}^2 \), we have that

\[
\lambda''(t) - \|B_1^{\mathbb{R}^2}\|^2 \lambda(t) = 0 \quad \text{and} \quad \Lambda''(t) - \|B_1^{\mathbb{R}^2}\|^2 \Lambda(t) = 0,
\]

and hence \( \lambda(t) \) and \( \Lambda(t) \) are given by

\[
\begin{align*}
\lambda(t) &= b_1 \cosh(rt) + b_2 \sinh(rt), \\
\Lambda(t) &= V_1 \cosh(rt) + V_2 \sinh(rt),
\end{align*}
\]

where \( r = \pm\|B_1^{\mathbb{R}^2}\| \), \( b_i \) are real constants and \( V_i \) orthonormal vectors. If \( \Lambda = (\Lambda_1, \Lambda_2, \Lambda_3) \), using \( \langle \gamma, \gamma \rangle = -1 \), we obtain the following polynomial equation in \( u \):

\[
(\lambda - (\Lambda_1 + \Lambda_3))u^2 + 2\Lambda_2 u = 0,
\]

that is,

\[
\lambda - (\Lambda_1 + \Lambda_3) = 0 \quad \text{and} \quad \Lambda_2 = 0.
\]

Therefore, if \( V_1 = (v_{11}, v_{12}, v_{13}) \) and \( V_2 = (v_{21}, v_{22}, v_{23}) \), we have \( v_{12} = v_{22} = 0 \), \( b_1 = v_{11} + v_{13} \) and \( b_1 = v_{21} + v_{23} \). Now, writing \( V_1 = (\cosh(a_1), 0, \sinh(a_1)) \) and \( V_2 = (\sinh(a_1), 0, \cosh(a_1)) \), we get \( b_1 = b_2 = e^{a_1} \). Thus, we conclude that

\[
\lambda(t) = e^{rt},
\]

\[
\Lambda(t) = \left( \cosh(rt), 0, \sinh(rt) \right).
\]

From \( 3.26 \), it follows that

\[
r + \sqrt{\frac{1 - C}{2}} = 0.
\]

Thus, we obtain that \( r = -\|B_1^{\mathbb{R}^2}\| \), and therefore

\[
\lambda(t) = e^{-\|B_1^{\mathbb{R}^2}\| t},
\]

\[
\Lambda(t) = \left( \cosh(-\|B_1^{\mathbb{R}^2}\| t), 0, \sinh(-\|B_1^{\mathbb{R}^2}\| t) \right).
\]
Writing \( b = \| B_1^2 \| = \sqrt{1 - \| B_1^2 \|^2} = \sqrt{1 - \| V_0 \|^2} \) and \( W_0 = B_3 \), when we replace (3.27) in (3.26), we obtain the parametrization (1.1).

For the converse, suppose that \( \Sigma \) is parametrized by (1.1). Since
\[
\Psi_t = -b \left( e^{-bt}(\alpha(u), \bar{0}) + \left( \sinh(-bt), 0, \cosh(-bt), -\frac{V_0}{b} \right) \right),
\]
\[
\Psi_u = e^{-bt}(\alpha'(u), 0),
\]
\[
\Psi_v = (\bar{0}, W_0),
\]
we conclude that a unit normal vector field \( N \) to \( \Sigma \) is given by
\[
N = -\| V_0 \| \left( e^{-bt}(\alpha(u), \bar{0}) + \left( \sinh(-bt), 0, \cosh(-bt) \right), \frac{b}{\| V_0 \|^2} \right).
\]

Denoting by \( \tilde{D} \) the covariant derivative in \( L^3 \), we obtain
\[
\tilde{D}_{\Psi_t} N = b \| V_0 \| \left( e^{-bt} \alpha(u) + \left( \cosh(-bt), 0, \sinh(-bt) \right), \bar{0} \right)
= b \| V_0 \| \Psi^{12},
\]
\[
\tilde{D}_{\Psi_u} N = -\| V_0 \| e^{-bt}(\alpha'(u), 0)
= -\| V_0 \| \Psi_u,
\]
\[
\tilde{D}_{\Psi_v} N = 0.
\]
It follows immediately from the derivatives above and the parametrization \( \Psi \) that
\[
\langle \tilde{D}_{\Psi_u} N, \Psi^{12} \rangle = \langle \tilde{D}_{\Psi_v} N, \Psi^{12} \rangle = 0 \quad \text{and} \quad \langle \tilde{D}_{\Psi_t} N, \Psi^{12} \rangle = -b \| V_0 \|.
\]

Therefore, since \( \tilde{\nabla}_V W = \tilde{D}_V W + \langle \tilde{D}_V W, \Psi^{12} \rangle \Psi^{12} \), we get
\[
\tilde{\nabla}_{\Psi_t} N = 0,
\]
\[
\tilde{\nabla}_{\Psi_u} N = -\| V_0 \| \Psi_u,
\]
\[
\tilde{\nabla}_{\Psi_v} N = 0,
\]
that is, \( \Sigma \) has principal curvatures \( \mu_1 = 0 \), \( \mu_2 = \| V_0 \| \) and \( \mu_3 = 0 \). Finally, since
\[
PN = -\| V_0 \| \left( e^{-bt}(\alpha(u), \bar{0}) + \left( \sinh(-bt), 0, \cosh(-bt) \right), -\frac{b}{\| V_0 \|^2} \right)
\]
and \( b = \sqrt{1 - \| V_0 \|^2} \), it follows that
\[
C = \langle PN, N \rangle
= \| V_0 \|^2 \left( e^{-2bt}u^2 + e^{-bt} \left( -\frac{u^2}{2} \sinh(-bt) - \frac{u^2}{2} \cosh(-bt) \right) - \sinh^2(-bt) + \cosh^2(-bt) - \frac{b^2}{\| V_0 \|^2} \right)
= \| V_0 \|^2 \left( 1 - \frac{b^2}{\| V_0 \|^2} \right)
= 2\| V_0 \|^2 - 1,
\]
that is, \( \| V_0 \| = \sqrt{\frac{1 + C}{2}} \). Thus, we conclude the proof of the theorem. \( \square \)
Remark 3.1. Following the notation established in the proof of Theorem 2, let us provide a geometric description of the hypersurface given by the parametrization $\Psi$. Note that a unit normal vector to the horocycle

$$\varphi(u) = \left(1 + \frac{u^2}{2}, u, -\frac{u^2}{2}\right),$$

is given by

$$n(u) = \left(\frac{u^2}{2}, u, 1 - \frac{u^2}{2}\right).$$

Fixing $u, v \in \mathbb{R}$, let us consider in $\mathbb{H}^2 \times \mathbb{R}^2$ the following geodesic parametrized by arc length

$$\gamma(t) = \left(\cosh(\omega t)\varphi(u) + \sinh(\omega t)n(u), g(v) + V_0t\right),$$

where $g(v) = p_0 + W_0v$ is a geodesic in $\mathbb{R}^2$ with normal vector $V_0$. Since

$$\gamma'(t) = \left(\omega \sinh(\omega t)\varphi(u) + \omega \cosh(\omega t)n(u), V_0\right),$$

it follows that

$$1 = ||\gamma'(t)||^2 = \omega^2 + ||V_0||^2,$$

which implies $\omega = \pm \sqrt{1 - ||V_0||^2} = \pm b$. Considering $\omega = -b$, we get

$$\gamma(t) = e^{-bt}(\alpha(u), 0) + \left(\cosh(-bt), 0, \sinh(-bt), V_0t\right)$$

$$+ (0, p_0 + W_0v).$$

Varying the parameters $(t, u, v) \in \mathbb{R}^3$, the construction above provides exactly the parametrization $\Psi$. Therefore, the hypersurface $\Psi(\mathbb{R}^3)$ is a family of geodesically parallel surfaces of $\mathbb{H}^2 \times \mathbb{R}^2$, given by products of horocycles in $\mathbb{H}^2$ and straight lines in $\mathbb{R}^2$.

References

[1] J. Berndt, S. Console, and C. E. Olmos. Submanifolds and holonomy. Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, second edition, 2016.
[2] E. Cartan. Familles de surfaces isoparamétriques dans les espaces à courbure constante. Ann. Mat. Pura Appl., 17(1):177–191, 1938.
[3] J. C. Díaz-Ramos and M. Domínguez-Vázquez. Inhomogeneous isoparametric hypersurfaces in complex hyperbolic spaces. Mathematische Zeitschrift, 271(3):1037–1042, 2012.
[4] J. C. Díaz-Ramos and M. Domínguez-Vázquez. Isoparametric hypersurfaces in Damek–Ricci spaces. Advances in Mathematics, 239:1–17, 2013.
[5] F. Dillen and D. Kowalczyk. Constant angle surfaces in product spaces. Journal of Geometry and Physics, 62(6):1414–1432, 2012.
[6] M. Domínguez-Vázquez. An introduction to isoparametric foliations. Preprint. Available at http://xstuxnet.usc.es/miguel/teaching/jae2018.html, 2018.
[7] M. Domínguez-Vázquez and J. M. Manzano. Isoparametric surfaces in $E(\kappa, \tau)$-spaces. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 22(1):269–285, 2021.
[8] D. Gao, J. Van der Veken, A. Wijffels, and B. Xu. Lagrangian surfaces in $\mathbb{H}^2 \times \mathbb{H}^2$. arXiv:2106.13975, 2021.
[9] J. Ge, Z. Tang, and W. Yan. A filtration for isoparametric hypersurfaces in riemannian manifolds. Journal of the Mathematical Society of Japan, 67(3):1179–1212, 2015.
[10] F. Guimarães, J. B. M. dos Santos, and J. P. dos Santos. Isoparametric hypersurfaces of riemannian manifolds as initial data for the mean curvature flow. arXiv:2206.02635, 2022.
[11] J. Julio-Batalla. Isoparametric functions on $\mathbb{R}^n \times \mathbb{M}^m$. Diff. Geom. and its Appl., 60:1–8, 2018.
[12] A. Rodríguez-Vázquez. A nonisoparametric hypersurface with constant principal curvatures. *Proc. Amer. Math. Soc.*, 147(12):5417–5420, 2019.

[13] C. Somigliana. Sulle relazione fra il principio di huygens e l’ottica geometrica, also. *Atti Acc. Sc. Torino*, pages 974–979, 1918.

[14] F. Urbano. On hypersurfaces of $S^2 \times S^2$. *Comm. Anal. Geom.*, 27(6):1381–1416, 2019.

[15] Q. M. Wang. Isoparametric hypersurfaces in complex projective spaces. In *Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980)*, pages 1509–1523. Sci. Press Beijing, Beijing, 1982.