Comments on Stability of KPV Metastable State

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ABSTRACT: Using the blackfold approach, we study the stability of the KPV (Kachru-Pearson-Verlinde) metastable state of anti-D3 branes at the tip of the Klebanov-Strassler throat. With regard to long-wavelength deformations observable to blackfold, in the regime $p/M \in (0, p_{\text{new}})$ with $p_{\text{new}} \approx 0.0801446$, we found no classical instabilities. However, in the regime $p/M \in (p_{\text{new}}, p_{\text{crit}})$ with $p_{\text{crit}} \approx 0.080488$ the original metastable threshold, we observe that the KPV state is tachyonic in the radial direction. We comment on the relation of this result to existing results on the stability of the KPV state.
1 Introduction

Understanding the dynamics of antibranes in fluxed background, particularly anti-D3 branes in Klebanov-Strassler background, has been of revived interest in recent years. This is due, in part, to the debate over the validity of the KKLT (Kachru, Kallosh, Linde, Trivedi) construction of de Sitter vacua [1], in which metastable state of anti-D3 branes remains a controversial prerequisite [2].

A brief review The KPV (Kachru-Pearson-Verlinde) metastable state [3] is a proposed configuration of anti-D3 branes at the tip of the Klebanov-Strassler [4] background. Originally in [3], it was argued that anti-D3 branes can polarise into a spherical NS5 brane, and that, in probe approximation, in the regime of \( p/M \) between 0 and \( p_{\text{crit}} \) with \( p_{\text{crit}} \approx 0.080488 \), the polarised anti-D3-NS5 brane balances its own “weight” with “electromagnetic” forces from the fluxes to form a metastable configuration.

Because of the singularities found when considering backreaction of anti-D3 branes to the throat [5], concerns about the existence of the metastable state were raised. It is important to note that the KPV metastable state is actually an anti-D3-NS5 state formed by the polarisation of anti-D3 branes under non-trivial fluxes as opposed to a state of

\(^1p\) denotes the number of the anti-D3 branes and \( M \) the strength of the Klebanov-Strassler background flux.

\(^2\)NS5 branes with dissolved anti-D3 brane charge.
localised anti-D3 branes. Nevertheless, the singularities still mean bad news for the KPV metastable state especially when [6] pointed out that the state claimed by KPV is outside of the regime of validity of probe analysis.

The first evidence for the existence of the KPV state came in the form of [7], where it was shown that there are no singularities if we consider a single anti-D3 brane in full string theory. Subsequently, [8] observed further that even in supergravity, singularities are not expected to appear once we consider extremal anti-D3-NS5 branes. Treating backreaction perturbatively through the blackfold approach, [9] showed evidence of the existence of the KPV metastable configuration exactly where no-go theorems are evaded. More precisely, it was found that the polarised anti-D3-NS5 branes could form a metastable state at the tip of the throat, and such metastable solution would disappear as soon as we heat up the polarised state sufficiently that it geometrically resembles localised black anti-D3 branes.

Our focus  It is important to note that the claim regarding metastability of the relevant anti-D3-NS5 state in [3] and subsequently in [9] is only with respect to some mode of deformations and not a general statement of stability. For example, in [9], only spatially homogeneous blackfolds were considered. This means spatially non-homogeneous deformations of the KPV state are not considered. For the purpose of cosmological model building through uplifting, we need not only that the metastable configuration exists but also that it is long lived. However, there is evidence suggesting that this might not be the case, at least for certain regimes of parameters.

In [6], from the perspective of localised anti-D3 branes, it was argued that there exists a direction along which the branes feel repulsive forces among themselves and destabilise away from the KPV state. This suggests that, in appropriate regime of parameters, the KPV metastable configuration suffers from fragmentation instability.

From the complementary perspective of anti-D3-NS5 branes, we shall study the stability properties of the KPV metastable state using the blackfold approach. Before presenting the results, let us stress what our analysis does not do. As blackfold is based on the idea of matched asymptotic expansion, one need to specify a seed metric as the description of the solution in the near zone. By choosing the stacked anti-D3-NS5 branes solution as the near zone seed, we have effectively ignore all brane splitting and fragmentation deformations. Moreover, as noted in [9], the analysis is only reliable with $p/M$ not close to zero, at which point the metastable is too close to the north pole and the localised anti-D3 perspective becomes the better description. Since the analysis in [6] is done from the localised anti-D3 branes perspective and the discovered instabilities are brane splitting instabilities, the blackfold results presented here should be thought of as complimentary and not contradictory to that of [6]. Another important caveat is that blackfold theory is an effective theory describing long-wavelength interactions only, so even if we see no instabilities in the blackfold analysis, it’s not guaranteed that there are no instabilities. For more discussion of the blackfold approach as an effective theory, see [10].
Our results  With the blackfold analysis, we obtain the following results. For the regime of $p/M \in (0,p_{\text{new}})$ with $p_{\text{new}} \approx 0.0801446$ where the dissolved D3 brane charge density $Q_3$ carried by the polarised state is negative, we observed no classical instabilities. On the other hand, for the regime of $p/M \in (p_{\text{new}},p_{\text{crit}})$ with $p_{\text{crit}} \approx 0.080488$ where $Q_3$ is positive, we found the KPV metastable configuration to be unstable in the radial direction. As the transition of $Q_3$ to positive occurs at $p_{\text{new}} < p_{\text{crit}}$, the KPV metastability is lost before reaching $p_{\text{crit}}$ previously found in [3] and [9]. As our results are obtained from first order blackfold, they are reliable in the limit of large $S^3$ where the leading order blackfold effects dominate (for the regime of validity see [9]).

In the KPV metastable configuration, the shift in $p_{\text{crit}}$ is small but we believe that it deserves our full attention because it emphasises a physical phenomenon that is often ignored when considering metastable antibranes. When constructing antibranes metastable state in fluxed throats, it’s the polarised branes (e.g. anti-D3-NS5 branes in the KPV case) that are metastable. Contrary to the case of localised antibranes where it’s trivial that the branes feel an “electromagnetic" force attracting them to the tip, when taking into account polarisation, it is possible for the antibranes to polarise into branes with positive dissolved flux and, thus, feel an “electromagnetic" force pushing them away from the tip. There exists other metastable antibranes configurations in which the shifting of metastable regime is more pronounced. In particular, for the case of anti-M2 branes at the tip of the CGLP throat [11, 12], as the regime corresponding to the dissolved charge positive is parametrically comparable to the range for which it is negative, this realisation reduces substantially the number of metastable configurations.

Outlook  Although in this paper we do not consider non-extremal branes, we would like to note the blackfold approach effectiveness in studying thermal effects as done in [9]. This is particularly relevant for us because, if the fragmentation instability exist for the extremal state, it would be interesting to study thermal effects and see if it resolves the instabilities. This possibility is one we would like to pursue in later works.

Outline of paper  The plan of the paper is as follows. A short derivation of the KPV metastable state from blackfold analysis is reviewed in section 2. The blackfold stability analysis of the KPV metastable state is presented in section 3. A description of the Klebanov-Strassler background near the apex is provided in appendix A for convenience of the readers. Details on the construction of the equivalent currents used in the KPV metastable state derivation is collected in appendix B. Lastly, the derivation of blackfold perturbation equations used in the stability analysis of the KPV metastable state is summarised in appendix C.

2  KPV Metastable State from Blackfold

Overview  Blackfold theory [13–15] is a long wavelength effective theory of gravity, conceptually based on the technique of matched asymptotic expansions. As a thorough dis-
discussion of blackfold and its application to antibranes metastable state has already been
given in [9] and [12], we shall not repeat it here. Nevertheless, let us briefly present the
fundamental of the blackfold argument for the existence of metastable antibranes.

Our master equations, the first order blackfold equations, are the constraint equations
of the backreacted metric and gauge fields that asymptote anti-D3-NS5 branes in the near
zone and the Klebanov-Strassler background in the far zone to first order in a derivative
expansion. Analogous to the fluid equations of the Fluid/Gravity correspondence [16],
because of the interplay between derivative expansion and constraint equations, the first
order blackfold equations will determine the zeroth order terms of the derivative expansion.
By explicitly solving the first order blackfold equations, we have proven the necessary
conditions for the existence of the KPV metastable state.

In general, one might be worried that solving the constraint equations alone does not
automatically guarantee a full solution. However, in all examples of matched asymptotic ex-
pansions that have been worked out in details (most notably [17]), the constraint equations
not only provide the necessary conditions but also the sufficient conditions for a regular
solution to first order in derivative expansion. It is therefore natural to conjecture that
there is a one to one correspondence between a solution of the blackfold equations and a
regular solution of the gravitational equations. This conjecture is almost analogous to the
statement in Fluid/Gravity that there is a one to one map between a solution of the fluid
equations and a regular solution of the gravitational equations.

The purpose of this section is to provide a brief derivation of the KPV metastable
state from the blackfold approach. Various aspects of anti-D3-NS5 blackfold, including the
recovery of the KPV metastable state, have already been discussed in [9]. Nevertheless, we
find it useful to revisit the starting point of our stability analysis. We will also take this
opportunity to state our conventions, provide some relevant details and explanations, and
fix some typos in the literature.

Conventions

1. The signature is mostly plus \((-++...\)).

2. The type IIB supergravity gauge invariant field strength are denoted as

\[
\tilde{F}_{q+2} = F_{q+2} - H_3 \wedge C_{q-1} .
\]  (2.1)

3. Electric currents appear with a \(-\) sign in the forced Maxwell equations:

\[
d \ast F_{p+2} = -16\pi G \, J_{p+1}
\]  (2.2)

4. Magnetic currents appear with a \(+\) sign in the forced Maxwell equations:

\[
d F_{p+2} = 16\pi G \, j_{n-q-3}
\]  (2.3)
Background  We refer readers to Appendix A for a complete description of the Klebanov-Strassler background near the apex. For the purpose of deriving the KPV metastable state, we shall only present here the metric and the flux components that contribute to the derivation. As the dilaton of the Klebanov-Strassler solution is a constant, we shall set $g_s = 1$ for our convenience. In the rescaled coordinates system presented in [9], the Klebanov-Strassler metric near the apex is given by

$$
g_{\mu\nu} dx^\mu dx^\nu = M b_0^2 \left( -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + dr^2 + d\psi^2 + \sin^2 \psi \left( d\omega^2 + \sin^2 \omega d\phi \right) + r^2 \left( d\omega^2 + \sin^2 \omega d\phi \right) \right) \quad (2.4)$$

and relevant fluxes are given

$$F_3 = 2M \sin^2 \psi \sin \omega \ d\psi \wedge d\omega \wedge d\phi + ... \quad (2.5)$$

$$H_7 = -2M^3 b_0^4 \sin^2 \psi \sin \omega \ dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\psi \wedge d\omega \wedge d\phi + ... \quad (2.6)$$

where $b_0^2 \approx 0.93266$ and the dots refer to components of the flux that do not contribute in our derivation.

Branes & currents  S-dualising the widely-known description of the D3-D5 bound state in flat space (given in [18] for example), we obtain the description of the flat space D3-NS5 branes. In the string frame, the metric is given by

$$ds^2 = D^{-1/2} \left( -f dt^2 + D \left( (dx^1)^2 + (dx^2)^2 \right) + \sum_{i=3}^5 (dx^i)^2 \right) + H D^{-1/2} \left( f^{-1} dr^2 + r^2 d\Omega_3^2 \right) \quad (2.7)$$

with

$$f = 1 - \frac{r_0}{r^2}, \quad D = \left( \sin^2 \theta H^{-1} + \cos^2 \theta \right)^{-1} \quad (2.8)$$

$$H = 1 + \frac{r_0^2 \sinh^2 \alpha}{r^2} \quad (2.9)$$

The dilaton:

$$e^{2\phi} = H D^{-1} \quad (2.10)$$

The RR fields strength:

$$\tilde{F}_3 = F_3 = -2r_h^2 r^{-3} D H^{-2} \sin \theta \cos \theta \ dx^1 \wedge dx^2 \wedge dr \quad (2.11)$$

$$\tilde{F}_5 = 2r_h^2 r^{-3} H^{-2} \sin \theta \ dt \wedge dx^3 \wedge dx^4 \wedge dx^5 \wedge dr$$

The NSNS field strength:

$$H_3 = -2r_h^2 \cos \theta \ d\Omega_3 \quad (2.12)$$

where $d\Omega_3$ is the volume form of a unit 3 sphere. For the purpose of describing the KPV metastable state, we shall only be interested in the extremal limit of the D3-NS5 bound state, which is obtained by taking the limit $r_0 \to 0, \alpha \to \infty$ in such a way that we can define the extremal horizon radius $r_h \equiv r_0 \sinh \alpha$. Let us note that if one is interested in studying thermal effects, one should keep both parameters $r_0$ and $\alpha$. 

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As demonstrated in the blackfold literature, the first order blackfold equations can be obtained as the conservation equations of equivalent sources induced by the branes onto the background in the far zone. Therefore, we shall go to the far zone and ask what equivalent sources can mimic the effects of these extremal D3-NS5 branes. In the interest of time and space, let us relegate the details of this process to Appendix B and simply present the results here. We have the equivalent energy-stress tensor

\[ T^{ab} = C \left( -r_h^2 \sin^2 \theta (\gamma^{ab} - v^a v^b - w^a w^b) - r_h^2 \cos^2 \theta \gamma^{ab} \right) \]  

(2.13)

and the equivalent currents\(^4\)

\[ J_2 = C r_h^2 \sin \theta \cos \theta v \wedge w \]  

(2.14)

\[ J_4 = C r_h^2 \sin \theta \ast (v \wedge w) \]  

(2.15)

\[ j_6 = -C r_h^2 \cos \theta \ast 1 \]  

(2.16)

where \( C = \frac{\Omega_3}{8 \pi G}, \theta \in \left(0, \frac{\pi}{2}\right)\), and \( \ast \) is the worldvolume Hodge dual operator. For later convenience, let us define the charge density carried by the \( J_4 \) current as

\[ Q_3 = -C r_h^2 \sin \theta \]  

(2.17)

**Blackfold equations**  In the blackfold set-up of extremal anti-D3-NS5 branes in Klebanov-Strassler background, the variables of the system are

\[ r, \tilde{\omega}, \tilde{\phi}, \psi, r_h, \tan \theta, v^a, w^a \]  

(2.18)

The variables \( r, \tilde{\omega}, \tilde{\phi}, \psi \) are the embedding degrees of freedom of the anti-D3-NS5 branes to the background. The variables \( r_h, \tan \theta, v^a, w^a \) are the characteristic degrees of freedom describing the horizon length, the charge distribution, and the flow of the dissolved charge.

As noted in the introduction, the first order blackfold equations will describe the zeroth order terms in the derivative expansion of the metric and gauge fields that asymptote the stacked anti-D3-NS5 branes at the tip of Klebanov-Strassler throat, we can already fix variables \( r, \tilde{\omega}, \tilde{\phi}, v^a, w^a \) (see equations (2.30)-(2.31) for detailed expressions). Therefore, we only have to worry about variables \( r_h, \psi, \tan \theta \) which, because of the static and spatially homogeneous conditions, are constant with respect to the worldvolume coordinates.

In our conventions, the first order anti-D3-NS5 blackfold equations are given by:

1. The sourced energy-momentum tensor conservation equations

\[ \nabla_a T^{a\mu} = \mathcal{F}^\mu \]  

(2.19)

\[^4\text{Compared to the D3-NS5 description above, as detailed in appendix B, we have made the reparametrisation } \theta \rightarrow \theta - \pi \text{ for convenience.}\]
with the force term given by

\[ F^\mu = -\frac{1}{6!} H_7^{\mu a_1...a_6} j_{a_1...a_6} + \frac{1}{2!} F_3^{\mu a_1 a_2} J_{a_1 a_2} + \frac{3}{4!} H_3^{\mu a_1 a_2} C_2 a_3 a_4 J_{a_1...a_4} + \frac{1}{4!} \tilde{F}_5^{\mu a_1...a_4} J_{a_1...a_4} \] (2.20)

We note that for the purpose of describing the KPV metastable state, the terms with \( H_3 \) and \( \tilde{F}_5 \) are not relevant because they vanish at the tip of the throat. Nevertheless, as they will play a role when we consider perturbations away from the tip, we present them explicitly here.

2. The current conservation equations

\[ d * j_6 = 0 \] (2.21)
\[ d * J_4 + * j_6 \wedge F_3 = 0 \] (2.22)
\[ d * J_2 + H_3 \wedge * J_4 = 0 \] (2.23)

where \( * \) is the 6-dimensional Hodge dual of the worldvolume directions.

From the current conservation equations, we can define the conserved Page charges \( Q_3 \) and \( Q_5 \) that keep track of the number of anti-D3 branes and NS5 branes:

\[ Q_5 = * j_6 = C r_h^2 \cos \theta \] (2.24)
\[ Q_3 = \int_{S^2} * (J_4 + *(j_6 \wedge C_2)) \] (2.25)
\[ = -4\pi \left( C r_h^2 \sin \theta M b_0 \sin^2 \psi + C r_h^2 \cos \theta M (\psi - \frac{1}{2} \sin 2\psi) \right) \] (2.26)

It follows immediately that we can write \( \tan \theta \) as

\[ \tan \theta = \frac{1}{b_0^2 \sin^2 \psi} \left( \frac{\pi p}{M} - \left( \psi - \frac{1}{2} \sin 2\psi \right) \right) \] (2.27)

where we have made the identification that

\[ \frac{-Q_3}{4\pi Q_5} = \frac{\pi p}{M} \] (2.28)

Note also that the conserved charge \( Q_5 \) fixes \( r_h \) in term of \( \tan \theta \).

From the energy-momentum tensor conservation equations, after some algebraic acrobatics, we can write all variables in term of \( \psi \) and obtain the equation

\[ \cot \psi - \frac{1}{b_0^2} \sqrt{1 + \tan^2 \theta} - \frac{1}{b_0^2} \tan \theta = 0 \] (2.29)

Integrating equation (2.29) gives us the KPV potential originally obtained from the DBI approach in [3].
KPV metastable state  We can numerically determine that equation (2.29) has metastable solutions for $0 < p/M < p_{\text{crit}}$ where $p_{\text{crit}} \approx 0.080488$. These metastable solutions are the KPV metastable states. For our convenience later, let us note down some explicit information of the configuration.

With respect to our variables, the KPV metastable states are specified by

$$r = 0, \quad \psi = \psi_0, \quad \tan \theta = \frac{1}{b_0^2 \sin^2 \psi_0} \left( \frac{\pi p}{M} - \psi_0 + \frac{1}{2} \sin(2\psi_0) \right)$$ (2.30)

$$r_h = \sqrt{\frac{Q_5}{C \cos \theta(\psi_0)}}, \quad v^a \partial_a = \frac{1}{\sqrt{g_s M b_0 \sin \psi_0 \sin \omega}} \partial_\psi, \quad w^a \partial_a = \frac{1}{\sqrt{g_s M b_0 \sin \psi_0 \sin \omega}} \partial_\phi$$ (2.31)

where $\psi_0$ is the metastable solution of

$$\cot \psi - \frac{1}{b_0^2} \sqrt{1 + \tan^2 \theta} - \frac{1}{b_0^2} \tan \theta = 0$$ (2.32)

We note also the induced metric on the worldvolume of the anti-D3-NS5 branes

$$\gamma_{ab} d\sigma^a d\sigma^b = M b_0^2 \left( -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + \sin^2 \psi_0 \left( d\omega^2 + \sin^2 \omega d\phi \right) \right) ,$$ (2.33)

the non-zero components of the worldvolume Christoffel symbol $\Theta^g_{bc}$

$$\Theta^\phi_{\omega \psi} = \Theta^\phi_{\omega \phi} = \cot \omega, \quad \Theta^\omega_{\phi \phi} = -\cos \omega \sin \omega ,$$ (2.34)

the relevant anti-D3-NS5 branes’ normal covectors

$$n^{(1)}_\mu dx^\mu = \sqrt{M b_0} d\psi \quad n^{(2)}_\mu dx^\mu = \frac{\sqrt{M b_0}}{\sqrt{2}} dr ,$$ (2.35)

and the non-zero component of the extrinsic curvature $K^\rho_{ab}$

$$K^\omega_{\psi \psi} = -\cos \psi_0 \sin \psi_0 \quad K^\omega_{\phi \phi} = -\cos \psi_0 \sin \psi_0 \sin^2 \omega .$$ (2.36)

This concludes the review of the KPV metastable state from the blackfold approach. We refer readers to [9] for more information on the derivation and other aspects of the KPV metastable state from the blackfold approach.

3 Stability of KPV Metastable State

The goal of this section is to look for unstable deformations of the KPV metastable states within all the deformations that are amendable to the blackfold approach. Starting with the blackfold description of the KPV metastable configuration, we perturb it by varying slightly all its variables. As the blackfold equations are the constraint equations of the gravitational theory, a necessary condition for the perturbed configuration to be a legitimate solution of type IIB supergravity, the perturbations would need to obey the blackfold perturbation equations. We shall see that in the regime $p/M \in (0, p_{\text{new}})$ where $Q_3$ is negative, unstable deformations are not allowed by the blackfold equations. In the regime $p/M \in (p_{\text{new}}, p_{\text{crit}})$ where $Q_3$ is positive, the KPV metastable state is tachyonic in the radial direction.
3.1 Perturbation parameters

To introduce perturbations to our system, we shall vary slightly the variables of the configuration around their KPV metastable values. Explicitly, we have

\[ r = 0 + \delta r, \quad \psi = \psi_0 + \delta \psi, \quad r_h = \frac{Q_5}{C \cos \theta(\psi_0)} + \delta r_h, \]  

(3.1)

\[ \tan \theta = \frac{1}{b_0^2 \sin^2 \psi_0} \left( \frac{\pi p}{M} - \psi_0 + \frac{1}{2} \sin(2\psi_0) \right) + \delta \tan \theta \]  

(3.2)

\[ v^a \partial_a = \frac{1}{\sqrt{g_s M b_0 \sin \psi_0}} \partial_\omega + \delta v^a \partial_a, \]  

(3.3)

\[ w^a \partial_a = \frac{1}{\sqrt{g_s M b_0 \sin \psi_0 \sin \omega}} \partial_\phi + \delta w^a \partial_a \]  

(3.4)

where all variations are functions of the worldvolume coordinates, e.g. \( \delta r_h(\sigma) \). Note that for the transverse \( S^2 \) direction, we only consider deformations in the radial direction\(^5\). To simplify our syntax, from here on we shall denote the variable values at the KPV metastable by the variables, e.g. \( \psi_0 \) will be denoted as \( \psi \), the value of \( \tan \theta \) at KPV is denoted as \( \tan \theta \), etc.

Our goal now is to make use of symmetries and constraints to minimise the number of parameters we work with while still preserve all the relevant information for the stability analysis. Firstly, because of the Lorentz symmetry of the blackfold equations and the original KPV configuration, without loss of generality, we can consider the variations involving the \( t \) direction only instead of the full Minkowskian coordinates \( t, x^1, x^2, x^3 \). Secondly, using the unitary constraints on \( v \) and \( w \), i.e. \( v^a v_a = w^a w_a = 1 \), we can show that

\[ \delta v^\omega = - \frac{\cos \psi}{\sqrt{M b_0 \sin^2 \psi}} \delta \psi \]  

(3.5)

\[ \delta w^\varphi = - \frac{\cos \psi}{\sqrt{M b_0 \sin^2 \psi \sin \omega}} \delta \psi \]  

(3.6)

Thirdly, we note that in our set-up, we use \( v \) and \( w \) together as orthogonal vectors to specify the anti-D3 charge flow inside the NS5 branes. As a result, it’s obvious that we have a rotational gauge symmetry here. Making use of this gauge symmetry along with the orthogonality constraint, i.e. \( v^a w_a = 0 \), we can set

\[ \delta v^\varphi = \delta w^\omega = 0 \]  

(3.7)

With the simplifications noted above, our relevant variation parameters are

\[ \delta r(t, \omega, \varphi), \delta \psi(t, \omega, \varphi), \delta r_h(t, \omega, \varphi), \delta \tan \theta(t, \omega, \varphi), \]  

(3.8)

\[ \delta v^t(t, \omega, \varphi), \delta v^\omega(t, \omega, \varphi), \delta w^t(t, \omega, \varphi), \delta w^\varphi(t, \omega, \varphi) \]  

(3.9)

where \( \delta v^\omega, \delta w^\varphi \) can be written in term of \( \delta \psi \) as expressed in (3.5)-(3.6).

\(^5\)One can turn on the variations in the transverse angles \( \tilde{\omega}, \tilde{\varphi} \) and consider orbital motion of the anti-D3-NS5 branes around the throat but because of radiation effects, we expect them to be stable. Nevertheless, it’s important to note that the inclusion of variations in transverse angles will not change our results regarding radial instabilities.
3.2 Blackfold perturbation equations

In this subsection, we shall describe the blackfold perturbation equations. We relegate much details of the perturbation equations to appendix C because they are not extremely illuminating. Here, we shall only discuss the final form of these equations.

3.2.1 Conservative Currents & Charges

From the blackfold current conservation equations (2.21)-(2.23), we obtain the following perturbation equations.

1. From the $j_6$ conservation equation, we obtain:

$$\partial_a \delta Q_5 = 0 \quad (3.10)$$

where $Q_5 = -Cr_h^2 \cos \theta$. This means $\delta Q_5$ is a constant of motion. Recall that $Q_5$ keeps track of the number of NS5 branes. As we are interested in the dynamical stability of the KPV metastable configuration, we shall impose that $\delta Q_5$ vanishes.

Note that the imposition $\delta Q_5 = 0$ automatically fixes $\delta r_h$ in term of $\delta \tan \theta$

$$\delta r_h = \frac{1}{2} r_h \cos \theta \sin \theta \delta \tan \theta. \quad (3.11)$$

2. From the $J_4$ conservation equation, we obtain:

$$\sin \omega \left( Mb_0 \sin^2 \psi \partial_\omega \tan \theta + 2 M b_0^3 \tan \theta \cos \psi \sin \psi \partial_\psi \delta \psi + 2 M \sin^2 \psi \partial_\psi \delta \psi \right)$$

$$= \left( \tan \theta \sqrt{M} b_0 \sin \psi \partial_\omega \delta w_t \right) + \left( \tan \theta \sqrt{M} b_0 \sin \psi \partial_\omega \left( \sin \omega \delta w_t \right) \right) \quad (3.12)$$

Integrating over $\omega$ and $\varphi$ and enforcing the periodicity conditions

$$\delta w_t|_{\varphi=0} = \delta w_t|_{\varphi=2\pi} \quad (3.13)$$

we obtain

$$\partial_a \delta Q_3 = 0 \quad (3.14)$$

where

$$\delta Q_3 = -Q_5 M b_0^2 \sin^2 \psi \int d\omega d\varphi \sin \omega \left( \delta \tan \theta + 2 \left( \tan \theta \cot \psi + \frac{1}{b_0^2} \right) \delta \psi \right) \quad (3.15)$$

This means $\delta Q_3$ is a constant of motion. In a similar fashion to how the $Q_5$ charge keep track of the number of NS5 branes, the $Q_3$ charge keeps track of the number of anti-D3 branes. As we are interested in the dynamical stability of the KPV metastable configuration, we shall impose that $\delta Q_3 = 0$. However, note that unlike the $Q_5$, the imposition $\delta Q_3 = 0$ doesn’t automatically guarantee the satisfaction of the current perturbation equation.

3. From the $J_2$ conservation equation, we obtain:

$$\cot \theta \cos^2 \theta \partial_\omega \delta \tan \theta + \sqrt{M} b_0 \sin \psi \partial_\omega \delta v^t = 0 \quad (3.16)$$

$$\cot \theta \cos^2 \theta \partial_\varphi \delta \tan \theta + \sqrt{M} b_0 \sin \psi \sin \omega \partial_\varphi \delta w^t = 0 \quad (3.17)$$

$$\partial_\varphi \delta v^t - \partial_\omega \left( \sin \omega \delta w^t \right) = 0 \quad (3.18)$$
3.2.2 Energy-momentum conservation equations

As evidenced by many blackfold papers, it’s convenient to project the blackfold energy-momentum conservation equations along the worldvolume directions to obtain the intrinsic equations and along the orthogonal directions to obtain the extrinsic equations [14]. These equations are given by

\[ \nabla_a T^{ab} = F^b \quad (3.19) \]
\[ T^{ab} K_{ab}^{(i)} = \mathcal{F}^{(i)} \quad (3.20) \]

where \( K_{ab}^{(i)} = K_{ab} n_{\mu}^{(i)} \) and \( F^b, \mathcal{F}^{(i)} \) are the appropriately projected values of the force term \( \mathcal{F}^\mu \) (2.20). From these equations, we can derive the following blackfold perturbation equations:

1. The \( t \) intrinsic perturbation equation
   \[
   \partial_t \delta \tan \theta + \frac{\sqrt{M}b_0}{\sin \psi} \tan \theta \left( \partial_\omega \delta v^t + \frac{1}{\sin \omega} \partial_\varphi \delta w^t + \cot \omega \delta v^t \right) + 2 \left( \cot \psi \tan \theta + \frac{1}{b_0^2} \right) \partial_t \delta \psi = 0 \quad (3.21)
   \]

2. The \( \omega \) intrinsic perturbation equation
   \[
   \sqrt{M}b_0 \sin \psi \tan^2 \theta \partial_t \delta v^t + \sin \theta \cos \theta \partial_\omega \delta \tan \theta = 0 \quad (3.22)
   \]

3. The \( \varphi \) intrinsic perturbation equation
   \[
   \sqrt{M}b_0 \sin \psi \sin \omega \tan^2 \theta \partial_t \delta w^t + \sin \theta \cos \theta \partial_\varphi \delta \tan \theta = 0 \quad (3.23)
   \]

4. The \( \psi \) extrinsic perturbation equation
   \[
   -\frac{1}{\sin^2 \psi} \delta \psi - \frac{1}{b_0^2} (1 + \sin \theta) \delta \tan \theta + \frac{1}{2} (1 + \tan^2 \theta) (\partial_t)^2 \delta \psi - \frac{1}{2 \sin^2 \psi} \nabla^2 \delta \psi = 0 \quad (3.24)
   \]
   where \( \nabla^2 \) is the Laplacian, i.e. \( \nabla^2 = (\partial_\omega)^2 + 1/\sin^2 \omega (\partial_\varphi)^2 + \cot \omega \partial_\omega \)

5. The \( r \) extrinsic perturbation equation
   \[
   (\partial_t)^2 \delta r - \frac{\cos^2 \theta}{\sin^2 \psi} \nabla^2 \delta r = -\mathcal{A}M^2 \sin \theta \delta r \quad (3.25)
   \]
   where \( \mathcal{A} \approx 1.79304 \) is made up of the constants \( b_0 \) in (2.4) and \( a_0 \) in (A.9).

3.3 Stability analysis

Immediately from the expressions of the blackfold perturbation equations above, we see that the \( \delta r \) variation decouples from other variations and is controlled only by equation (3.25). From (3.25), we can easily see that in the regime where \( \sin \theta < 0 \) or equivalently \( Q_3 \) positive, the \( \delta r \) deformation is tachyonic. On the other hand, in the regime where \( \sin \theta > 0 \)
or equivalently $Q_3$ negative, the $\delta r$ deformation is damped. See Figure 1, for a plot of the value of the charge density $Q_3$ of KPV metastable states with different values of $p/M$. In Figure 1, we shall also present an analogous plot for the KP metastable state of anti-M2 branes at the tip of the CGLP throat.

As we have found that $\delta r$ variation is damped for $p/M \in (0, p_{\text{new}})$ where $Q_3$ is negative, let us consider other variations in this regime. Firstly, note that as $\delta r, \delta \tan \theta, \delta v^t, \delta w^t$ can all be expressed in term of $\delta \psi, \delta \tan \theta, \delta v^t, \delta w^t$, we only have to worry about these four variations. Secondly, as the blackfold perturbation equations do not mix momentum modes, it’s more convenient to consider each momentum mode individually. We have

\[
\delta \psi(t, \omega, \varphi) = \varepsilon e^{-i\lambda t} S_\psi(\lambda, \omega, \varphi) \\
\delta \tan \theta(t, \omega, \varphi) = \varepsilon e^{-i\lambda t} S_{\tan \theta}(\lambda, \omega, \varphi) \\
\delta v^t(t, \omega, \varphi) = \varepsilon e^{-i\lambda t} S_{v^t}(\lambda, \omega, \varphi) \\
\delta w^t(t, \omega, \varphi) = \varepsilon e^{-i\lambda t} S_{w^t}(\lambda, \omega, \varphi)
\]

Substituting in the momentum modes, the intrinsic equations of $\omega$ and $\varphi$ (3.22)-(3.23) yields

\[
-i\lambda S_{v^t}(\lambda, \omega, \varphi) = -\kappa \partial^\omega S_{\tan \theta}(\lambda, \omega, \varphi) \\
-i\lambda \sin \omega S_{w^t}(\lambda, \omega, \varphi) = -\kappa \partial^\varphi S_{\tan \theta}(\lambda, \omega, \varphi)
\]

where

\[
\Lambda = \frac{\cot \theta \cos^2 \theta}{\sqrt{M} b_0 \sin \psi} > 0
\]

Now there two things that can happen.
1. If $\lambda = 0$, we have $S_{\tan \theta}(\lambda, \omega, \varphi)$ is a constant in $\omega, \varphi$.

2. If $\lambda \neq 0$, we have

$$S_{\nu'}(\lambda, \omega, \varphi) = \frac{A}{i\lambda} \partial_{\omega} S_{\tan \theta}(\lambda, \omega, \varphi)$$

$$\sin \omega S_{\nu''}(\lambda, \omega, \varphi) = \frac{A}{i\lambda} \partial_{\varphi} S_{\tan \theta}(\lambda, \omega, \varphi)$$

In the case where $\lambda = 0$, considering equations (3.15) and (3.24), we can show that there are no allowed perturbations in the regime of $p/M \in (0, p_{\text{new}})$.

Let us consider the case where $\lambda \neq 0$. Plugging in the momentum modes, using equations (3.33)-(3.34), the blackfold perturbation equations reduce to a set of 3 non-trivial equations of $S_{\tan \theta}(\lambda, \omega, \varphi)$ and $S_{\psi}(\lambda, \omega, \varphi)$.

$$\frac{\cos^2 \theta}{\lambda^2} \frac{1}{\sin^2 \psi} \nabla^2 S_{\tan \theta}(\lambda, \omega, \varphi) + S_{\tan \theta}(\lambda, \omega, \varphi) + 2 \left( \cot \psi \tan \theta + \frac{1}{b_0^2} \right) S_{\psi}(\lambda, \omega, \varphi) = 0 \quad (3.35)$$

$$\int d\omega d\varphi \sin \omega \left( S_{\tan \theta}(\lambda, \omega, \varphi) \right) + 2 \left( \tan \theta \cot \psi + \frac{1}{b_0^2} \right) S_{\psi}(\lambda, \omega, \varphi) = 0 \quad (3.36)$$

$$- \frac{1}{\sin^2 \psi} S_{\psi}(\lambda, \omega, \varphi) - \frac{1}{b_0^2} (1 + \sin \theta) S_{\tan \theta}(\lambda, \omega, \varphi) - \frac{\lambda^2}{2 \cos^2 \theta} S_{\psi}(\lambda, \omega, \varphi)$$

$$- \frac{1}{2 \sin^2 \psi} \nabla^2 S_{\psi}(\lambda, \omega, \varphi) = 0 \quad (3.37)$$

As spherical harmonic modes are not mixed by our equations, let us decompose $S_{\tan \theta}(\lambda, \omega, \varphi)$ and $S_{\psi}(\lambda, \omega, \varphi)$ into spherical harmonics:

$$S_{\psi}(\lambda, \omega, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (S_{\psi})_l^m Y_l^m(\omega, \varphi) \quad (3.38)$$

$$S_{\tan \theta}(\lambda, \omega, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (S_{\tan \theta})_l^m Y_l^m(\omega, \varphi) \quad (3.39)$$

where the dependence on $\lambda$ is implicit in $(S_{\psi})_l^m$ and $(S_{\tan \theta})_l^m$. Substituting these spherical harmonics into (3.35)-(3.37), making use of the identity $\nabla^2 Y_l^m = -l(l + 1) Y_l^m$, it’s easy to see that equation (3.36) is trivially satisfied by (3.35) and we are left with a linear system of 2 equations and 2 unknowns:

$$\left( 1 - l(l + 1) \frac{\cos^2 \theta}{\lambda^2} \frac{1}{\sin^2 \psi} \right) (S_{\tan \theta})_l^m + 2 \left( \cot \psi \tan \theta + \frac{1}{b_0^2} \right) (S_{\psi})_l^m = 0 \quad (3.40)$$

$$\frac{1}{b_0^2} (1 + \sin \theta) (S_{\tan \theta})_l^m + \left( \frac{1}{\sin^2 \psi} + \frac{\lambda^2}{2 \cos^2 \theta} - l(l + 1) \frac{1}{2 \sin^2 \psi} \right) (S_{\psi})_l^m = 0 \quad (3.41)$$

This linear system has non-trivial solutions when the determinant of the associated matrix vanishes. As $\lambda \neq 0$, we can write such equation into a quadratic equation of $\lambda^2$:

$$a\lambda^4 + b\lambda^2 + c = 0 \quad (3.42)$$
where

\[ a = \frac{1}{2 \cos^2 \theta} \]  
\[ b = \frac{1}{\sin^2 \psi} - l(l + 1) \frac{1}{\sin^2 \psi} - \frac{2}{b_0^2} \left( \cot \psi \tan \theta + \frac{1}{b_0^2} \right) (1 + \sin \theta) \]  
\[ c = -l(l + 1) \cos^2 \theta \frac{1}{\sin^4 \psi} \left( 1 - \frac{l(l + 1)}{2} \right) \]  

The roots of this quadratic equation is given by

\[ \lambda^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]  

Given the lengthy computations in the appendices, it’s remarkable that our stability analysis boils down a simple linear system of \((S_\psi)_l^m\) and \((S_{\tan \theta})_l^m\). A sanity check one can do is to make sure that this equation includes the description of the homogeneous metastable perturbation mode previously determined in [3]. Considering the KPV potential \(V_{\text{eff}}(\psi)\) given in equation (45) of [3], expanding around the metastable position, we obtain the equation of motion

\[ \int d\lambda e^{-i\lambda \psi} \left( -\frac{1}{\sin^2 \psi} + \frac{2}{b_0^2} (1 + \sin \theta) \left( \cot \psi \tan \theta + \frac{1}{b_0^2} \right) - \frac{1}{2} \lambda^2 (1 + \tan^2 \theta) \right) \delta \psi(\lambda) = 0 \]  

(3.47)

It’s easy to check that for this perturbation, \(\lambda^2\) given by

\[ \lambda^2 = \frac{2}{1 + \tan^2 \theta} \left( -\frac{1}{\sin^2 \psi} + \frac{2}{b_0^2} (1 + \sin \theta) \left( \cot \psi \tan \theta + \frac{1}{b_0^2} \right) \right) \]  

(3.48)

is positive for \(p/M \in (0, p_{\text{crit}})\) and, thus, the perturbation mode is (meta)stable. Returning to the linear system (3.40)-(3.41), it’s obvious that if we restrict our attention to homogeneous perturbations, i.e. \(l = 0\), we reproduce the previously known equation of motion and metastable momentum mode.

Let us consider the momentum modes of the perturbations around the KPV state given by equation (3.46) for different values of \(l\). For \(l = 0\), as we expect, we have one allowed perturbation mode and it has \(\lambda^2\) given by (3.48). Using the fact that the expression in (3.48) is positive, it’s easy to see that for \(l = 1\), we have one allowed perturbation mode which has \(\lambda^2\) positive. For \(l \geq 2\), we have two allowed perturbation modes which both have \(\lambda^2\) positive. Therefore, we conclude that in the regime \(p/M \in (0, p_{\text{new}})\) where \(Q_3\) is negative, with regards to variations observable to blackfold, unstable perturbations are not allowed by the first order blackfold constraints.

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A Klebanov-Strassler near apex

For the purpose of examining the stability of the KPV metastable state, we shall be interested in the description of the Klebanov-Strassler background near the apex. To avoid confusion, it’s important to remember that the discussion in this subsection is in the notation of [19], unlike the rest of the paper which is using the notation of [9].

Let us note down the basis used for the conic part of the Klebanov-Strassler background as defined in [19].

\begin{align}
g_1 &= -\sin \theta_1 d\phi_1 - \cos \psi \sin \theta_2 d\phi_2 + \sin \psi d\theta_2 \\
g_2 &= \frac{d\theta_1 - \sin \psi \sin \theta_2 d\phi_2 - \cos \psi d\theta_2}{\sqrt{2}} \\
g_3 &= -\sin \theta_1 d\phi_1 + \cos \psi \sin \theta_2 d\phi_2 - \sin \psi d\theta_2 \\
g_4 &= \frac{d\theta_1 + \sin \psi \sin \theta_2 d\phi_2 + \cos \psi d\theta_2}{\sqrt{2}} \\
g_5 &= d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2
\end{align}

where \( \psi \) is a special angular coordinate going from 0 to \( 4\pi \), \( \theta \) and \( \phi \) are the standard \( S^2 \) spherical coordinate. As we are only interested in the near apex description, let us write the metric and gauge fields to leading order in \( \tau \) for all legs. For convenience, let us also set \( \alpha' = 1 \).

\begin{align}
ds_{10}^2 &= \frac{\epsilon^{4/3}}{2^{1/3} \alpha_0^{1/2} g_s M \alpha'} (-a_0)(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2) \\
&\quad + a_0^{1/2} \alpha^{-1/3}(g_s M \alpha') \left( \frac{1}{2} d\tau^2 + \frac{1}{2} (g_5)^2 + (g_3)^2 + (g_4)^2 + \frac{1}{4} \tau^2 (g_1)^2 + (g_2)^2) \right) \quad (A.6)
\end{align}

\begin{align}
F_3 &\approx \frac{M}{2} \left( g_5 \wedge g_3 \wedge g_4 + \frac{\tau^2}{12} g_5 \wedge g_1 \wedge g_2 + \frac{\tau}{6} d\tau \wedge (g_1 \wedge g_3 + g_2 \wedge g_4) \right) \quad (A.7)
\end{align}

\begin{align}
H_3 &\approx \frac{g_s M}{2} \left( \left( \frac{\tau^2}{4} + O(\tau^4) \right) d\tau \wedge g_1 \wedge g_2 + \frac{1}{3} d\tau \wedge g_3 \wedge g_4 + \frac{\tau}{6} g_5 \wedge (g_1 \wedge g_3 + g_2 \wedge g_4) \right) \quad (A.8)
\end{align}

\begin{align}
F_5 &\approx \frac{g_s M^2}{36} \tau g_1 \wedge g_2 \wedge g_3 \wedge g_4 \wedge g_5 + \frac{\epsilon^{8/3}}{3^{4/3} g_0} \tau dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\tau \quad (A.9)
\end{align}

where \( a_0 \approx 0.71805 \). To make contact with the notations of the rest of the paper, the coordinates \( x^0, x^1, x^2, x^3, \tau \) are respectively, up to some scaling, equivalent to the coordinates \( t, x^1, x^2, x^3, r \) used in the rest of the paper. The description of the conic part are expressed here with the basis \( g_1, g_2, g_3, g_4, g_5 \), in the rest of the paper it is described with coordinates \( \psi, \omega, \varphi, \tilde{\omega}, \tilde{\varphi} \).
B Maxwell Charges & Equivalent Currents

Let us start from the type IIB supergravity action in Einstein frame

\[
I = \frac{1}{16\pi G} \int_{\mathcal{M}_{10}} \left[ *R - \frac{1}{2} d\phi \wedge *d\phi - \frac{1}{2} e^{-\phi} H_3 \wedge *H_3 - \frac{1}{2} \sum_{q=-1,1} e^{\frac{3-\phi}{2}} \tilde{F}_{q+2} \wedge *\tilde{F}_{q+2} \right]
\]

\[
- \frac{1}{16\pi G} \int_{\mathcal{M}_{10}} \left[ \frac{1}{4} \tilde{F}_5 \wedge *\tilde{F}_5 + \frac{1}{2} C_4 \wedge H_3 \wedge F_3 \right]
\]  

(B.1)

where \( \tilde{F}_{q+2} = F_{q+2} - H_3 \wedge C_{q-1} \).

For the purpose of constructing equivalent currents, we are interested in the notion of Maxwell charges [20]. The key idea for the Maxwell charges is that the Chern-Simon terms in the equation of motion can be thought of as a source for the gauge field. For example, let us look at the equation of motion for the \( C_4 \) field in type IIB:

\[
d * \tilde{F}_5 - H_3 \wedge F_3 = -16\pi G * J_4
\]  

(B.2)

In this case, the Maxwell current is given by

\[
d * \tilde{F}_5 = -16\pi G * J_4^{Maxwell} = -16\pi G * J_4 + H_3 \wedge F_3
\]  

(B.3)

where the sign and factors in front of \( J_4^{Maxwell} \) is to make sure it is compatible with our conventions of \( J_4 \). The Maxwell charge can be computed easily from Gauss’s law of the \( \tilde{F}_5 \) flux and, thus, can be interpreted as the monopole source that will reproduce the \( \tilde{F}_5 \) flux far away.

**Equivalent currents of anti-D3-NS5 branes** Turning our attention to the case of D3-NS5 branes, we have the relevant forced Maxwell equations are

\[
d * \tilde{F}_3 = -16\pi G * J_2^{Maxwell}
\]  

(B.4)

\[
d * \tilde{F}_5 = -16\pi G * J_4^{Maxwell}
\]  

(B.5)

\[
d * H_7 = 16\pi G * j_6^{Maxwell}
\]  

(B.6)

The idea is that we do not know the exact expressions of these Maxwell currents, however, we can mimic their effects far away by using Maxwell charges to construct a set of equivalent currents. Adopting the convention that \( Q = \int *J \), using the description of D3-NS5 branes in (2.7)-(2.12), we obtain the Maxwell charges

\[
Q_1^{Maxwell} = Vol_4 \ Cr_5^2 \ sin \ theta \ cos \ theta
\]  

(B.7)

\[
Q_3^{Maxwell} = Vol_2 \ Cr_5^2 \ sin \ theta
\]  

(B.8)

\[
Q_5^{Maxwell} = -Cr_5^2 \ cos \ theta
\]  

(B.9)
Requiring that they reproduce the same Maxwell charges at \( r \to \infty \), our equivalent currents can now be easily constructed. These are\(^6\)

\[
J_2^\text{equiv} = Cr_h^2 \sin \theta \cos \theta \ v \wedge w \tag{B.10}
\]
\[
J_4^\text{equiv} = Cr_h^2 \sin \theta \ast (-v \wedge w) \tag{B.11}
\]
\[
J_6^\text{equiv} = -Cr_h^2 \cos \theta \ast (-1) \tag{B.12}
\]

where \( v, w \) are orthogonal vectors used to describe the distribution of the dissolved D3 charge.

In the description of D3-NS5 branes above, we have not restricted the range of \( \theta \in (0, 2\pi) \). For the construction of antibranes metastable, we are interested in anti-D3-NS5 branes, which corresponds to the range \( \theta \in (\pi, 3\pi/4) \) of our description\(^7\). For convenience, we can do a reparametrisation \( \theta \to \theta - \pi \) to bring it to the regime \( \theta \in (0, \pi/2) \). In the new \( \theta \), our currents are given by

\[
J_2 = Cr_h^2 \sin \theta \cos \theta \ v \wedge w \tag{B.13}
\]
\[
J_4 = Cr_h^2 \sin \theta \ast (v \wedge w) \tag{B.14}
\]
\[
j_6 = -Cr_h^2 \cos \theta \ast (1) \tag{B.15}
\]

where we have drop the superscript \textit{equiv} for syntactical simplicity.

\[\text{C Blackfold Perturbation Equations}\]

In this appendix, we shall derive the blackfold perturbation equations for deformations around the KPV metastable state. We start with a discussion of embedding geometry and computations of some useful variational expressions. Subsequently, we introduce variations to the blackfold equations and simplify the results to arrive at the perturbation equations used in the main text. For further discussion on embedding geometry and blackfold perturbation equation, see [21–23].

\[\text{C.1 Useful definitions & formulae}\]

\textbf{Definitions} \hspace{1em} Given a manifold \( \mathcal{M} \) and a submanifold \( \mathcal{W} \) defined by the embedding \( X^\mu(\sigma^a) \), we can define the induced metric

\[
\gamma_{ab} \equiv \partial_a X^\mu \partial_b X^\nu g_{\mu \nu} \tag{C.1}
\]

the tangential projector

\[
h^{\mu \nu} \equiv \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \tag{C.2}
\]

and the orthogonal projector

\[
\perp_{\mu \nu} \equiv g_{\mu \nu} - h_{\mu \nu} \tag{C.3}
\]

\(^6\)The currents here are localised (\( \delta \) function) currents in the full 10 dimensional picture.

\(^7\)The statement that anti-D3-NS5 branes are described by \( \theta \) in the regime of \( (\pi, 3\pi/4) \) is only true for neutral background where Maxwell charges and Page charges are the same.
For convenience, let us define the object $\partial_{a}X^{\mu}$ as

$$\partial_{a}X^{\mu} \equiv g_{\mu\nu}\gamma^{ab}\partial_{b}X^{\nu}$$  \hspace{1cm} (C.4)$$

then the pullback of a general tensor from $M$ to $W$ is given by

$$T^{a_{1}a_{2}\ldots a_{n}}_{b_{1}b_{2}\ldots b_{m}} \equiv \partial^{a_{1}}X_{\mu_{1}}\ldots \partial_{b_{1}}X^{\nu_{1}}\ldots T^{\mu_{1}\ldots \mu_{n}}_{\nu_{1}\ldots \nu_{m}}$$  \hspace{1cm} (C.5)$$

Let us define also the extrinsic curvature

$$K_{\rho\mu\nu} \equiv h^{\sigma}_{\mu}\nabla_{\rho}h_{\sigma} = -h^{\sigma}_{\mu}\nabla_{\rho}h_{\sigma}$$  \hspace{1cm} (C.6)$$

where $\nabla_{a}$ acts only on the $b$ index of $\partial_{b}X^{\rho}$: $\nabla_{a}(\partial_{b}X^{\rho}) = \partial_{a}(\partial_{b}X^{\rho}) - \Theta^{\rho}_{ab}\partial_{a}X^{\rho}$ with $\Theta^{\rho}_{ab}$ the Christoffel symbols of the induced metric $\gamma$.

**Variation of induced metric**  Hitting $\delta$ to the definition of $\gamma$ in (C.1), we obtain the expression

$$\delta\gamma_{ab} = \partial_{b}X^{\mu}\partial_{c}X^{\nu}\left(\nabla_{\mu}(\delta X^{a}g_{\alpha\nu}) + \nabla_{\nu}(\delta X^{a}g_{\alpha\mu})\right)$$  \hspace{1cm} (C.8)$$

Note that when we embed a surface without edges in a higher dimensional background, we have the reparametrisation symmetry (diffeomorphism) of the surface. As a result, we only have to worry about the variations of the transverse scalars $\delta X_{\mu}^{a}$ defined by $\partial^{a}X_{\mu}\delta X_{\mu}^{a} = 0$ because variations along the brane directions are only gauge deformations and can be cancelled by a reparametrisation of the worldvolume coordinates. Therefore, we can make a gauge choice and write the variation of $\delta\gamma_{ab}$ as

$$\delta\gamma_{ab} = \partial_{b}X^{\mu}\partial_{c}X^{\nu}\left(\nabla_{\mu}(\delta X_{\mu}^{a}g_{\alpha\nu}) + \nabla_{\nu}(\delta X_{\mu}^{a}g_{\alpha\mu})\right)$$  \hspace{1cm} (C.9)$$

Making use of equation (C.7), the gauge choice where $\partial^{a}X_{\mu}\delta X_{\mu}^{a} = 0$, and the fact our background metric is diagonal, we can write

$$\delta\gamma_{ab} = -2K^{\rho}_{ab}(\delta X_{\mu}^{a}g_{\alpha\rho})$$  \hspace{1cm} (C.10)$$

Using the identity $\gamma_{ab}\gamma^{bc} = \delta_{a}^{e}$, we can easily deduce that

$$\delta\gamma^{ab}_{\rho} = 2K^{ab}_{\rho}(\delta X_{\mu}^{a}g_{\alpha\rho})$$  \hspace{1cm} (C.11)$$

**Variation of normal vectors**  We note that the normal vectors are implicitly defined by

$$\partial_{a}X^{\mu}n_{\mu}^{(i)} = 0$$  \hspace{1cm} (C.12)$$

$$n_{\rho}^{(i)}n_{\rho}^{(j)} = \delta^{(i)}_{(j)}$$  \hspace{1cm} (C.13)$$
Hitting $\delta$ to both equations yields respectively the variation of $n^{(i)}_{\rho}$ along the worldvolume directions and normal to the worldvolume directions.

\[
\begin{align*}
  h_{\rho}^\sigma \delta \left(n^{(i)}_{\rho} \right) &= -\partial^\sigma X^\alpha \left( \partial_\alpha \delta X^\rho_{\perp} \right) n^{(i)}_{\rho} - h^{\alpha \sigma} n^{(i)}_{\rho} \partial_\gamma g_{\rho \sigma} \delta X^\gamma_{\perp} \\
  \perp^\rho \delta \left(n^{(i)}_{\rho} \right) &= -\frac{1}{2} \partial_\gamma g_{\rho \sigma} \delta X^\gamma_{\perp} \perp^\alpha n^{(i)}_{\rho}
\end{align*}
\] (C.14) (C.15)

All together, we have

\[
\delta \left(n^{(i)}_{\rho} \right) = -h^{\alpha \mu} \partial_\mu \delta X^\rho_{\perp} n^{(i)}_{\rho} - h^{\alpha \sigma} n^{(i)}_{\rho} \partial_\gamma g_{\rho \sigma} \delta X^\gamma_{\perp} - \frac{1}{2} \partial_\gamma g_{\rho \sigma} \delta X^\gamma_{\perp} \perp^\alpha n^{(i)}_{\rho} + n^{(i)}_{\rho} \delta X^\gamma_{\perp} \partial_\gamma g_{\rho \sigma}
\] (C.16)

Using the relationship $n^{(i)}_{\rho} = n^{(i)\alpha} g_{\alpha \rho}$, we can derive the expression

\[
\delta \left(n^{(i)}_{\rho} \right) = -h^{\alpha \mu} \partial_\mu \delta X^\rho_{\perp} n^{(i)}_{\rho} - h^{\alpha \sigma} n^{(i)}_{\rho} \partial_\gamma g_{\rho \sigma} \delta X^\gamma_{\perp} - \frac{1}{2} \partial_\gamma g_{\rho \sigma} \delta X^\gamma_{\perp} \perp^\sigma n^{(i)}_{\rho} + n^{(i)}_{\rho} \delta X^\gamma_{\perp} \partial_\gamma g_{\rho \sigma}
\] (C.17)

**Variation of extrinsic curvature** Hitting $\delta$ to the expression of $K_{ab \rho}$ in (C.7), we obtain the expression

\[
\delta K_{ab \rho} = \perp_{\mu} \partial_\alpha \partial_\beta \delta X^\mu_{\perp} - \Gamma_{ab \rho}^\sigma \delta \left( \partial_\sigma X^\rho \right) + \perp_{\mu} \Gamma_{\mu \nu}^\sigma \delta \left( \partial_\alpha X^\nu \partial_\beta X^\nu \right) + \perp_{\mu} \delta \left( \Gamma_{\mu \nu}^\sigma \right) \partial_\alpha X^\mu \partial_\beta X^\nu - \delta \left( \partial_\sigma X^\nu \right) \partial_\alpha X^\mu \partial_\beta X^\nu \Gamma_{\mu \sigma}^\nu \partial_\gamma X^\rho - \delta \left( \partial_\sigma X^\nu \right) \partial_\alpha \partial_\beta X^\rho \Gamma_{\mu \sigma}^\nu \partial_\gamma X^\nu - \delta \left( \partial_\nu X^\alpha \right) \partial_\rho \partial_\nu \partial_\sigma X^\rho
\] (C.18)

Considering the variation of the projected extrinsic curvature $K_{ab \perp}^{(i)}$, we have

\[
\delta \left(K_{ab \perp}^{(i)} \right) = \delta \left(K_{ab \rho} n^{(i)}_{\rho} \right) = \delta \left(K_{ab \rho} \right) n^{(i)}_{\rho} + K_{ab \rho} \delta \left(n^{(i)}_{\rho} \right)
\] (C.19)

Making use of results in (C.17) and (C.18) along with some algebraic manipulations, we can write

\[
\delta \left(K_{ab \perp}^{(i)} \right) = n^{(i)}_{\rho} \nabla_\alpha \left( \partial_\beta \delta X^\rho \right) + \Gamma_{ab \rho}^{\mu} \delta \left( \partial_\alpha X^\mu \partial_\beta X^\nu \right) + n^{(i)}_{\rho} \partial_\alpha \Gamma_{\nu \rho}^{\mu} \partial_\beta X^\nu + \frac{1}{2} K_{ab \rho} \left( n^{(i)}_{\rho} \partial_\beta \delta X^\rho \right)
\] (C.20)

**Variation of anti-D3-NS5 blackfold energy-momentum tensor** Hitting $\delta$ to the expression of $T^{ab}$ in (2.13), we obtain the expression

\[
\delta T^{ab} = -Q_5 \left( \sin \theta \delta \tan \theta \right) \gamma^{ab} - Q_5 \frac{1}{\cos \theta} \left( 2 K_{ab \rho} \delta X^\rho \right) + Q_5 \left( \delta (v^a) v^b + v^a \delta (v^b) + \delta (w^a) w^b + w^a \delta (w^b) \right) \tan \theta \sin \theta + Q_5 \left( v^a v^b + w^a w^b \right) \sin \theta \delta (\tan \theta) + Q_5 \left( v^a v^b + w^a w^b \right) \sin \theta \cos^2 \theta \delta (\tan \theta)
\] (C.21)

We can also provide the expressions of the variations of other blackfold currents. However, as the blackfold currents either appear with a Hodge dual or coupled to the background flux, their full explicit variational expressions are not needed.
C.2 Current conservation equations

Recall from (2.21)-(2.23) the blackfold current conservation equations

\[
\begin{align*}
    d \ast j_6 & = 0 \quad \text{(C.22)} \\
    d \ast J_4 - \ast j_6 \wedge F_3 & = 0 \quad \text{(C.23)} \\
    d \ast J_2 + H_3 \wedge \ast J_4 & = 0 \quad \text{(C.24)}
\end{align*}
\]

1. Considering the \( j_6 \) conservation equation, we can easily show that it gives rise to the perturbation equation

\[
\partial_a \delta Q_5 = 0 \quad \text{(C.25)}
\]

where we have used \( \delta(\ast j_6) = \delta Q_5 \).

2. Considering the \( J_4 \) conservation equation, we note that it can be rewritten as

\[
\begin{align*}
    d \ast \bar{J}_4 & = 0 \quad \text{(C.26)}
\end{align*}
\]

where

\[
\bar{J}_4 = J_4 + \ast (\ast j_6 \wedge C_2) \quad \text{(C.27)}
\]

Using the expression

\[
\delta(\ast \bar{J}_4) = -\sin \omega \left( Q_5 M b_0^2 \sin^2 \psi \delta \tan \theta + 2 Q_5 M b_0^2 \tan \theta \cos \psi \sin \psi \delta \psi \\
+ 2 Q_5 M \sin^2 \psi \delta \psi \right) d\omega \wedge d\varphi \\
- \left( Q_5 \tan \theta \sqrt{M b_0} \sin \psi \delta w \right) d\omega \wedge dt + \left( Q_5 \tan \theta \sqrt{M b_0} \sin \psi \sin \omega \delta v \right) d\varphi \wedge dt
\]

we can easily derive the perturbation equation

\[
\begin{align*}
\sin \omega \left( M b_0^2 \sin^2 \psi \partial_\theta \delta \tan \theta + 2 M b_0^2 \tan \theta \cos \psi \sin \psi \partial_\psi \delta \psi + 2 M \sin^2 \psi \delta \psi \right) \\
= \left( \tan \theta \sqrt{M b_0} \sin \psi \partial_\varphi \delta w \right) + \left( \tan \theta \sqrt{M b_0} \sin \psi \partial_\omega \left( \sin \omega \delta v \right) \right)
\end{align*}
\]

(C.28)

3. Considering the \( J_2 \) conservation equation, using the expression

\[
\begin{align*}
\delta(\ast J_2) = -Q_5 (\cos^3 \theta \delta \tan \theta) \sqrt{-\gamma} \left( v^\omega w^\varphi dt \wedge \ldots \wedge dx_3 \right) \\
+ 2 Q_5 \sin \theta (\sqrt{-\gamma} \gamma \omega \gamma \omega \psi \delta \psi) \left( v^\omega w^\varphi dt \wedge \ldots \wedge dx_3 \right) \\
- Q_5 \sin \theta \gamma \left( \delta v^i \wedge dx_1 \wedge \ldots \wedge dx_3 \wedge d\omega + v^\omega \delta w^i dx_1 \wedge \ldots \wedge dx_3 \wedge d\varphi \\
+ \delta v^\omega w^\varphi dt \wedge \ldots \wedge dx_3 + v^\omega \delta w^\varphi dt \wedge \ldots \wedge dx_3 \right)
\end{align*}
\]

(C.30)
we can easily obtain the set of perturbation equations
\[
cot \theta \cos^2 \theta \partial_\omega \delta \tan \theta + \sqrt{M} b_0 \sin \psi \partial_t \delta v^t = 0 \tag{C.31}
\]
\[
cot \theta \cos^2 \theta \partial_\omega \delta \tan \theta + \sqrt{M} b_0 \sin \psi \sin \omega \partial_t \delta w^t = 0 \tag{C.32}
\]
\[
\partial_\psi \delta v^t - \partial_\omega (\sin \omega \partial_t \delta w^t) = 0 \tag{C.33}
\]

### C.3 Energy-momentum conservation equations

Recall from (3.19)-(3.20), the intrinsic and extrinsic blackfold equations
\[
\nabla_a T^{ab} = \mathcal{F}^b \tag{C.34}
\]
\[
T^{ab} K_{(i)}^{ab} = F^{(i)} \tag{C.35}
\]
where \( K_{ab}^{(i)} = K_{ab} \rho_{\rho}^{(i)} \) and \( F^{(i)} \), \( F^{(i)} \) are the appropriately projected value of \( F^\mu \). It’s important to remember that \( F^\mu \) is a shorthand for the force terms coming from the coupling of the currents to the fluxes and not simply a 1-vector.

#### C.3.1 Intrinsic equation

The intrinsic blackfold perturbation equations are given by
\[
\delta \left( \nabla_a T^{ab} \right) = \delta F^b \tag{C.36}
\]
Making use of the identity
\[
\delta \Theta_{ac}^b = \frac{1}{2} \gamma^{bd} (\nabla_a \delta \gamma_{cd} + \nabla_c \delta \gamma_{ad} - \nabla_d \delta \gamma_{ac}) \tag{C.37}
\]
we can write the intrinsic blackfold perturbation equation into the more useful form
\[
\nabla_a \delta T^{ab} - T^{cd} \nabla_c (K^\rho (\delta X^a g_{\omega \rho}) - 2 T^{ac} \nabla_a \left( K^b_{c} \delta (\delta X^a g_{\omega \rho}) \right) + T^{ac} \nabla^b (K_{ac}^\rho (\delta X^a g_{\omega \rho})) = \delta F^b \tag{C.38}
\]
Expressing the variation of the induced force term as \( \delta (F^b) = \delta (\gamma^{\alpha \beta} \partial_a X^\nu g_{\mu \nu} F^\alpha) \), substituting in the appropriate expressions, with some algebraic manipulations, we obtain

1. The \( t \) intrinsic perturbation equation
\[
\partial_t \delta \tan \theta + \frac{\sqrt{M} b_0}{\sin \psi} \tan \theta \left( \partial_\omega \delta v^t + \frac{1}{\sin \omega} \partial_\varphi \delta w^t + \cot \omega \delta v^t \right) + 2 \left( \cot \psi \tan \theta + \frac{1}{b_0^2} \right) \partial_t \delta \psi = 0 \tag{C.39}
\]
2. The \( \omega \) intrinsic perturbation equation
\[
\sqrt{M} b_0 \sin \psi \tan^2 \theta \partial_t \delta v^t + \sin \theta \cos \partial_\omega \delta \tan \theta = 0 \tag{C.40}
\]
3. The \( \varphi \) intrinsic perturbation equation
\[
\sqrt{M} b_0 \sin \psi \sin \omega \tan^2 \theta \partial_t \delta w^t + \sin \theta \cos \partial_\varphi \delta \tan \theta = 0 \tag{C.41}
\]
C.3.2 Extrinsic equation

The extrinsic blackfold perturbation equations are given by

\[
\delta \left( T^{ab} K_{ab}^{(i)} \right) = \delta F^{(i)} \tag{C.42}
\]

Making use of the results in (C.20) and (C.21), we can write the extrinsic blackfold perturbation equations into the more useful form

\[
\begin{align*}
\delta T^{ab} K_{ab}^{(i)} + n^{(i)}_\mu T^{ab} \nabla_a \left( \partial_b X^\mu \right) + n^{(i)}_\sigma T^{ab} \Gamma^\sigma_{\mu
u} \delta \left( \partial_a X^\mu \partial_b X^\nu \right) \\
+ n^{(i)}_\sigma T^{ab} \partial_a X^\mu \partial_b X^\nu \delta \left( \partial_a X^\mu_\perp \partial_\alpha X^\nu_\perp \right) \\
+ \frac{1}{2} T^{ab} K_{ab}^{(i)} \partial_\alpha X^\mu_\perp \partial_\gamma g_{\alpha \rho} = \delta F^{(i)} \tag{C.43}
\end{align*}
\]

Similar to the intrinsic case, we would like to express the variation of the orthogonal force term as

\[
\delta F^{(i)} = \delta (F^{\mu \nu} n^{(i)}_{\mu}) = \delta F^{\mu} n^{(i)}_{\mu} + F^{\mu} \delta n^{(i)}_{\mu},
\]

substitute in the appropriate expressions, and simplify the resulting equations. The difference is only that we need a lot more patience.

In the end, we obtain

1. For \( n^{(1)} = d\psi \), the \( \psi \) extrinsic perturbation equation

\[
- \frac{1}{\sin^2 \psi} \delta \psi - \frac{1}{b_0} \left( 1 + \sin \theta \right) \delta \tan \theta + \frac{1}{2} \left( 1 + \tan^2 \theta \right) (\partial_t)^2 \delta \psi - \frac{1}{2} \frac{\sin^2 \psi}{\sin^2 \psi} \nabla^2 \delta \psi = 0 \tag{C.44}
\]

where \( \nabla^2 \) is the Laplacian, i.e. \( \nabla^2 = (\partial_\omega)^2 + 1/\sin^2 \omega (\partial_\varphi)^2 + \cot \omega \partial_\omega \).

2. For \( n^{(2)} = dr \), the \( r \) extrinsic perturbation equation

\[
(\partial_t)^2 \delta r - \frac{\cos^2 \theta}{\sin^2 \psi} \nabla^2 \delta r = -\mathcal{A} M^2 \sin \theta \delta r \tag{C.45}
\]

where \( \mathcal{A} \approx 1.79304 \).

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