A REFLECTION TYPE PROBLEM FOR THE STOCHASTIC 2-D NAVIER-STOKES EQUATIONS WITH PERIODIC CONDITIONS

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Abstract
We prove the existence of a solution for the Kolmogorov equation associated with a reflection problem for 2-D stochastic Navier-Stokes equations with periodic spatial conditions and the corresponding stream flow in a closed ball of a Sobolev space of the torus $\mathbb{T}^2$.

1 Introduction

We consider here the 2-D stochastic Navier-Stokes equation for an incompressible non-viscous fluid

\[
\begin{cases}
\frac{dX}{dt} = -\nu \Delta X dt + (X \cdot \nabla)X dt = \nabla p dt + dW_t \\
\nabla \cdot X = 0
\end{cases}
\]

This equation is considered on a 2-D torus, that we identify with the square $\mathbb{T}^2 = [0, 2\pi] \times [0, 2\pi]$ and with periodic boundary conditions.

Here $\nu$ is the viscosity of the fluid, $X$ is the velocity field, $p$ is the pressure and $W$ is a cylindrical Wiener process.

If we denote by $\phi : \mathbb{T}^2 \to \mathbb{R}$ the corresponding stream function, that is

\[
X = \nabla^\perp \phi, \quad -\Delta \phi = \text{curl } X, \quad \phi(\xi + 2\pi) \equiv \phi(\xi)
\]

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where $\nabla^\perp = (-D_2, D_1)$, curl $X = D_2X_1 - D_1X_2$, $X = (X_1, X_2)$ we may rewrite (1) in terms of the stream function $\phi$ (see [1], [2])

$$d(\nabla^\perp \phi) - \nu \Delta \nabla^\perp \phi \, dt + (\nabla^\perp \phi \cdot \nabla) \nabla^\perp \phi \, dt = \nabla p \, dt + dW_t$$

(3)

and formulate for (1) the corresponding reflection problem on the set

$$K = \{ \phi \in H^{1-a}(T; \mathbb{R}^2) : \|\phi\|_{1-a} \leq \ell \}$$

(4)

where $H^{1-a}$ is the Sobolev space of order $1 - \alpha$ with $\alpha > \frac{3}{2}$, with respect to the natural Gibbs measure $\mu$ given by enstrophy (see Section 2 below.)

More precisely, we shall prove that the Kolmogorov equation associated with (1), (2) and (4) has at least one solution $\varphi : T^2 \to \mathbb{R}$. In terms of coordinates $u_j = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{ij\cdot\xi} \phi(\xi) \, d\xi$ this equation has the form

$$\left\{ \begin{array}{l}
\lambda \varphi - L \varphi = f \quad \text{in } \mathcal{K} \\
\frac{\partial \varphi}{\partial n} = 0 \quad \text{on } \partial \mathcal{K}.
\end{array} \right.$$  

(5)

where $L$ is the Kolmogorov operator

$$L \varphi(u) = \sum_{k \in \mathbb{Z}^2} \left[ \frac{1}{2k^2} D^2_k \varphi(u) - \nu k^2 u_k D_k \varphi(u) - B_k(u) D_k \varphi(u) \right],$$

(6)

defined on a space $\mathcal{F}C^2_b$ of cylindrical smooth functions. (The function $B_k$ is defined in (10).)

The main result of this work, Theorem 1 below, amounts to saying that the Neumann problem (5) has at least one weak solution $\varphi$, but the uniqueness of this solution remains open. It should be said that the uniqueness is still an open problem in the case $K = H^{1-a}$ and it is equivalent in the later case with the unique extension of operator $L$ from $\mathcal{F}C^2_b$ to an $m$-dissipative operator in $L^2(\mu)$ see [3]. We mention, however, that $L$ is essentially $m$-dissipative in $L^1(\mu)$ when the viscosity $\nu$ is sufficiently large (Stannat [11]). It should mention also that in this way the study of stochastic process $X = X_1$, reduces to a linear infinite dimensional equation in the space $H^{1-a}$ associated to the operator $L$.

There is a large number of works devoted to infinite dimensional stochastic reflection problems but most of them are, except a few notable works, concerned with Wiener processes $W$ with finite covariance. So the existence theory for (13) is still open.

Here following the way developed in [5], [6], we will treat instead of (1) its associated Kolmogorov equation which as noted in Introduction will lead to an infinite dimensional Neumann problem on the convex $K$. (The Kolmogorov equation [6] in the special case $K = H^{1-a}$ was previously studied by Flandoli and Gozzi [9].)

Previous results on infinite dimensional reflection problems, starting from [10] are essentially concerned with reversible systems. We believe that the present paper is the first attempt to study non symmetric infinite dimensional Kolmogorov operators with Neumann boundary conditions.

### 2 The functional setting

Consider the Sobolev space of order $p \in \mathbb{R}$ defined by

$$H^p = \left\{ y(\xi) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}^2} u_j e^{ij\cdot\xi} : \sum_{j \in \mathbb{Z}^2} j^{2p} |u_j|^2 < +\infty \right\}.$$
where \( j = (j_1, j_2) \) and \( Z_+^2 = \{ j \in Z^2 : j_1 > 0 \text{ or } j_1 = 0, j_2 > 0 \} \). We set also \( Z_0^2 = Z^2 \setminus \{(0,0)\} \), \( j^2 = j_1^2 + j_2^2 \) and set \( u = \{u_j\}_{j \in Z_0^2} \), \( u_j = \bar{u} - j \) for \( j \in Z_0^2 \setminus Z_+^2 \). The space \( H^p \) is a complex Hilbert space with the scalar product

\[
\langle y_1, y_2 \rangle_p = \sum_{j \in Z_+^2} j^{2p} \langle y_1(j), \bar{y}_2 \rangle, \quad y_j = \frac{1}{2\pi} \int_{T^2} y(\xi) e^{ij \xi} \, d\xi.
\]

Consider the Gibbs measure \( \mu = \mu_\nu \) given by the enstrophy, that is

\[
d\mu(u) = \prod_{j \in Z_+^2} d\mu^j(u_j), d\mu^j(z) = \nu_j^4 \exp\left(-\frac{1}{2} \nu_j^4 |z|^2 \right) \, dx \, dy, z = x + i \, y.
\]

We recall (see [1], [3]) that for \( \alpha > 0 \) we have

\[
\int_H |u|^{2-\alpha} \, d\mu(u) < \infty,
\]

and so the probability measure \( \mu \) is supported by \( H^p \), \( p < 1 \). For each \( q \geq 1 \) we denote the space \( L^q(\Lambda, \mu) \) by \( L^q_\mu \).

We denote by \( H^{1,2}(H^\delta, \mu) \) the completion of the space \( \mathcal{F}C^2_\delta \) in the norm

\[
\| \varphi \|_{L^{1,2}(H^\delta, \mu)}^2 = \sum_{j \in Z_0^2} |j|^{2\delta} \int_K |D_j \varphi|^2 \, d\mu + \int_K |\varphi|^2 \, d\mu.
\]

Given a closed convex subset \( K \subset H^\delta \) with smooth boundary we denote by \( H^{1,2}(K, \mu) \) the space \( \{ \varphi \mid \varphi \in H^{1,2}(H^\delta, \mu) \} \) with the norm

\[
\| \varphi \|_{H^{1,2}(K, \mu)}^2 = \sum_{j \in Z_0^2} |j|^{2\delta} \int_K |D_j \varphi|^2 \, d\mu + \int_K |\varphi|^2 \, d\mu.
\]

There is a standard way (see [1], [2]) to reduce equation (1) to a differential equation in \( H^{1-\alpha} \) we briefly present below. Namely applying the curl operator into (3) we get for \( \psi = \text{curl} \, X \) the equation

\[
d\psi - \nu \Delta \psi \, dt + \text{curl} \, \left[ (\nabla^+ \phi \cdot \nabla^\bot \phi) \right] \, dt = d \, \text{curl} \, W_t.
\]

Now, we expand \( \phi \) in Fourier series

\[
\phi(t, \xi) = \frac{1}{2\pi} \sum_{j \in Z_0^2} u_j(t) e^{ij \cdot \xi}
\]

and take \( W \) to be the cylindrical Wiener process

\[
W_t = \frac{1}{2\pi} \sum_{j \in Z_0^2} |j|^{-1} \nabla^\bot (e^{ij \cdot \xi}) W_j(t)
\]

where \( \{W_j\}_{j \in Z_0^2} \) are independent Brownian motions in a probability space \( \Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0} \). We note that

\[
\text{curl} \, W_t = -\frac{1}{2\pi} \sum_{j \in Z_0^2} |j| e^{ij \cdot \xi} W_j(t)
\]
By (7) we have
\[ \psi(t, \xi) = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}_2} j^2 u_j(t) e^{ij \cdot \xi}, \quad \Delta \psi(t, \xi) = -\frac{1}{2\pi} \sum_{j \in \mathbb{Z}_2} (j^2)^2 u_j(t) e^{ij \cdot \xi} \]
and (see [2])
\[ \text{curl } (\nabla \phi \cdot \nabla \phi) = \sum_{j \in \mathbb{Z}_2} j^2 B_j(u). \]

Then (1) reduces to
\[ du_j(t) + \nu j^2 u_j(t) dt - B_j(u(t)) dt = |j|^{-1} dW_j(t). \]  

Equation (9) can be written in
\[ H = \{k^{1-a} u_k\}_{k \in \mathbb{Z}_2^d}, \quad W_l = \{|j|^{-1} W_j(t)\}_{j \in \mathbb{Z}_2^d}, \quad Qv = \{k^{3+a} v_k\}_{k \in \mathbb{Z}_2^d}. \]

Moreover, the measure \( \mu \) is infinitesimally invariant for \( B \) (see [1], [7].)

Equation (9) can be written in \( H^{1-a} \) as
\[ du + \nu QAu dt - Bu dt = dW, \]  

where
\[ Au = \{k^{-(1+a)} u_k\}_{k \in \mathbb{Z}_2^d}, \quad W_l = \{|j|^{-1} W_j(t)\}_{j \in \mathbb{Z}_2^d}, \quad Qv = \{k^{3+a} v_k\}_{k \in \mathbb{Z}_2^d}. \]

We recall (see [1]) that \( A \) is a Hilbert-Schmidt operator on \( H^2 \) and \( \|Au\|_2 = \|u\|_{1-a}. \)

Now, we associate with (12) the stochastic variational inequality
\[ du + \nu QAu dt - B(u) dt + R \partial I_K(u) dt \ni dW_t \]  

where \( Rv = \{k^{-2a} v_k\}_{k \in \mathbb{Z}_2^d}, K \) is a smooth closed and convex subset of \( H = H^{1-a} \) and \( \partial I_K : K \to 2^H \) is the normal cone to \( K \). Formally (13) can be written as
\[
\begin{cases}
    du(t) + \nu QAu(t) dt - Bu(t) dt = dW_t & \text{in } \{t \mid u(t) \in \hat{K}\} \\
    du(t) + \nu QAu(t) dt - Bu(t) dt + \lambda(t) n_K(u(t)) = dW_t & \text{in } \{t \mid u(t) \in \partial K\} \\
    u(t) \in K & \forall t \geq 0
\end{cases}
\]

where \( \lambda(t) \geq 0 \) and \( n_K(u) \) is the unit exterior normal to \( \partial K \).
Coming back to equation (1) and taking into account (2) the variational inequality (13) can be rewritten in terms of the velocity field \( X \) under the form

\[
\begin{cases}
    dX - \nu \Delta X \, dt + (X \cdot \nabla)X \, dt + N_{\mathcal{K}}(X) \, dt \ni \nabla p \, dt + dW_t \\
    \nabla \cdot X = 0, X = 0 \text{ on } \partial \Omega
\end{cases}
\]  

(14)

where \( N_{\mathcal{K}}(X) \) is the normal cone to the closed convex set \( \mathcal{K} \) of \( \{X \in (L^2(0, 2\pi))^2; \nabla \cdot X = 0, X(0) = X(2\pi)\} \) defined by,

\[
\mathcal{K} = \{X : \{(\phi, e^{-ij \cdot \xi})_{(j \in \mathbb{Z}^2)}\}_{j \in \mathbb{Z}^2} \in \mathcal{K}, \ \phi = (-\Delta)^{-1} \text{curl } X \}.
\]

This is the reflection problem to the boundary of \( \mathcal{K} \) on the oblique normal direction \( N_{\mathcal{K}}(x) \). In the special case of \( K \) given by (4) its meaning is that the stream value \( \phi \) of the fluid is constrained to the set \( \|\phi\|_{1-\alpha} \leq \ell \) and when \( \phi \) reaches the boundary \( \partial K \) in the dynamic of fluid arises a convective acceleration oriented toward interior of \( K \) along an oblique direction. Indeed we have by definition of the normal cone \( N_{\mathcal{K}}(X) \),

\[
N_{\mathcal{K}}(X) = \left\{ \eta \in (L^2(0, 2\pi))^2; \int_0^{2\pi} \int_0^{2\pi} \eta(\xi) (X(\xi) - Y(\xi)) d\xi \geq 0 \ \forall Y \in \mathcal{K} \right\}
\]

Recalling that by (2), (7),

\[
X = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}^2} j^i u_j e^{ij \cdot \xi}
\]

and setting

\[
\eta = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}^2} j^i \eta_j e^{ij \cdot \xi}, \quad Y = \frac{1}{2\pi} \sum_{j \in \mathbb{Z}^2} j^i v_j e^{ij \cdot \xi}
\]

where \( \{\eta_j\}, \{v_j\} \in H^{1-\alpha} \), we see that

\[
N_{\mathcal{K}}(X) = \left\{ \eta; \sum_{j \in \mathbb{Z}^2} |j|^2 \eta_j (\tilde{u}_j - \tilde{v}_j) \geq 0, \forall \{v_j\} \in K \right\}
\]

On the other hand, the normal cone \( N_{\mathcal{K}}(u) \) to \( K \) in \( H^{1-\alpha} \) is given by

\[
N_{\mathcal{K}}(u) = \left\{ \tilde{\eta} = \{\tilde{\eta}_j\}; \sum_{j \in \mathbb{Z}^2} j^{2(1-\alpha)} \tilde{\eta}_j (\tilde{u}_j - \tilde{v}_j) \geq 0, \forall \tilde{u} = \{u_j\} \in K \right\}
\]

Hence

\[
N_{\mathcal{K}}(X) = \left\{ \eta; (\eta, e^{ij \cdot \xi})_{(j \in \mathbb{Z}^2)} = \eta_j = j^{-2\alpha} \tilde{\eta}_j; \{\tilde{\eta}_j\} \in N_{\mathcal{K}}(u) \right\}
\]

and taking into account (13) and definition of \( \mathcal{K} \) this yields (14) as claimed.

3 The Kolmogorov equation

Consider the Kolmogorov operator \( L \) corresponding to (9) which is defined by (6) on the space \( \mathcal{F}_{2} C^2_b \) of cylindrical \( C^2 \)-functions

\[
\mathcal{F}_{2} C^2_b = \{\varphi = \varphi(u_{j_1}, u_{j_2}, \ldots, u_{j_n}) : n \geq 1, j_1, j_2, \ldots, j_n \in \mathbb{Z}^2_0, \varphi \in C^2_b(C^n)\}.
\]
We recall (see e.g., [1], [2], [3]) that the measure $\mu$ is invariant for operator $L$. As noticed earlier the essential $m$-dissipativity of $L$ in the space $L^2(\mu)$ is still an open problem.

Our aim here is to study the Neumann problem

$$\begin{aligned}
\begin{cases}
\lambda \varphi - L \varphi = f & \text{in } K \\
\frac{\partial \varphi}{\partial n} = 0 & \text{on } \partial K =: \Sigma
\end{cases}
\end{aligned}
$$

considered in some generalized sense to be precised below.

**Definition 1.** The function $\varphi : K \to \mathbb{R}$ is said to be weak solution to (15) if

$$\int_K |\varphi|^2 \, d\mu < \infty, \quad \sum_{j \in \mathbb{Z}^2} j^{-2} \int_K |D_j \varphi|^2 \, d\mu < \infty,$$

and

$$\lambda \int_K \varphi \psi \, d\mu + \frac{1}{2} \sum_{j \in \mathbb{Z}^2} j^{-2} \int_K D_j \varphi D_j \psi \, d\mu - \sum_{j \in \mathbb{Z}^2} \int_K B_j(u) D_j \psi(u) \varphi(u) \, d\mu(u) = \int_K f \psi \, d\mu$$

for all real valued $\psi \in \mathcal{F}C^2_b$.

It is readily seen by (11) that (14) makes sense for all $\psi \in \mathcal{F}C^2_b$.

**Theorem 1.** Assume that $\alpha > \frac{3}{2}$ and

$$K = \{u \in H^{1-\alpha} : |u|_{1-\alpha} \leq \ell\}$$

then for each real valued $f \in L^2(K, \mu)$ problem (5) has at least one weak solution $\varphi \in H^{1,2}_{-\alpha}(K, \mu)$ and the following estimates hold

$$\lambda \int_K |\varphi|^2 \, d\mu + \frac{1}{2} \sum_{j \in \mathbb{Z}^2} j^{-2} \int_K |D_j \varphi|^2 \, d\mu \leq C \int_K |f|^2 \, d\mu$$

and

$$\int_K |\varphi|^2 \, d\mu \leq \frac{1}{\lambda^2} \int_K |f|^2 \, d\mu.$$
4 Proof of Theorem 1

To prove Theorem 1 we consider the approximating equation

$$\lambda \varphi_\epsilon - L \varphi_\epsilon + \sum_{j \in \mathbb{Z}^2_0} j^{-4} \beta_j^\epsilon D_j \varphi_\epsilon = f,$$

where $L$ is given by (6) and

$$\beta^\epsilon(u) = \frac{1}{\epsilon} (u - \Pi_K u) = \frac{u}{\epsilon} \left(1 - \frac{\ell}{|u|_{1-\alpha}}\right), \quad u \in H.$$  

(Here $\Pi_K$ is the projection on $K$.) We introduce also the measure

$$d\mu_\epsilon(u) = \prod_k e^{-\frac{\epsilon |u_k|^2}{2}} d\mu_k(u)$$

and note that

$$D_j \left( e^{-\frac{\epsilon |u_k|^2}{2}} \right) = - j^4 \beta_j^\epsilon(u) e^{-\frac{\epsilon |u_k|^2}{2}}.$$  

It should be mentioned that equation (21) in spite of its apparent simplicity is still unsolvable for all $f \in L^2(\mu)$ and the reason is that as mentioned earlier we don’t know whether the operator $L$ is essentially $m$-dissipative. In order to circumvent this we shall define just a weak solution concept for (21) and prove the existence of such a solution.

**Definition 2.** The function $\varphi_\epsilon : H = H^{1-\alpha} \rightarrow \mathbb{R}$ is said to be weak solution to equation (21) if the following conditions hold,

$$\int \varphi_\epsilon^2 d\mu_\epsilon < \infty, \quad \sum_{k \in \mathbb{Z}^2_0} k^{-2} \int |D\varphi_\epsilon|^2 d\mu_\epsilon < \infty$$

and

$$\lambda \int \varphi_\epsilon \psi d\mu_\epsilon + \sum_{k \in \mathbb{Z}^2_0} k^{-2} \int_H D_k \varphi_\epsilon D_k \psi d\mu_\epsilon +$$

$$+ \sum_{k \in \mathbb{Z}^2_0} \int B_k(u) D_k \varphi_\epsilon \psi d\mu_\epsilon = \int f \psi d\mu_\epsilon$$

for all real valued cylindrical functions $\psi \in \mathcal{F}C^2_h$.

We note that Definition 2 is in the spirit of Definition 1 and that if $\varphi_\epsilon$ is a smooth solution to (21) then we see by (21) via integration by parts that $\varphi_\epsilon$ satisfies also (23). We note that

$$\sum_{k \in \mathbb{Z}^2_0} \int B_k(u) D_k \varphi_\epsilon \psi d\mu_\epsilon =$$

$$- \sum_{k \in \mathbb{Z}^2_0} \int B_k(u) D_k \psi \varphi_\epsilon d\mu_\epsilon - \sum_{k \in \mathbb{Z}^2_0} \int \psi \varphi_\epsilon [D_k B_k(u) + k^4 B_k(u) \beta_k^\epsilon] d\mu_\epsilon =$$

$$- \sum_{k \in \mathbb{Z}^2_0} \int B_k(u) \varphi_\epsilon D_k \psi d\mu_\epsilon$$
because by enstrophy invariance we have (see e.g., [1], [2])
\[ \sum_{k \in \mathbb{Z}^2} k^4 \tilde{u}_k B_k(u) \equiv 0, \quad D_k B_k(u) \equiv 0, \quad \forall k \in \mathbb{Z}^2_0, \]  
(25)
and
\[ \beta^\epsilon_k(u) = \frac{u_k}{\epsilon} \left( 1 - \frac{\ell}{|u|_{1-a}} \right), \quad \forall k \in \mathbb{Z}^2_0. \]  
(26)

**Proposition 1.** For each \( f \in L^2(\mu) \), \( \lambda > 0 \) equation (19) has at least one weak solution \( \varphi_\epsilon \) which satisfies the estimates
\[ \int |\varphi_\epsilon|^2 d\mu_\epsilon \leq \frac{1}{\lambda^2} \int |f|^2 d\mu, \quad \forall \epsilon > 0, \]  
(27)
\[ \sum_{k \in \mathbb{Z}^2_0} k^{-2} \int |D_k \varphi_\epsilon|^2 d\mu_\epsilon \leq C \int |f|^2 d\mu, \quad \forall \epsilon > 0. \]  
(28)

**Proof.** We shall use the Galerkin scheme for equation (21). Namely, we introduce the finite dimensional approximation \( B^n_k \) of \( B_k \) (see [1])
\[ B^n_k(u) = \sum_{j,k \in I_n} \left[ \frac{1}{k^4} (k^4 \cdot j)(k \cdot j) - \frac{1}{2} k^4 \cdot j \right] u_k u_{j-k} \]
and \( I_n = \{ m \in \mathbb{Z}^2_0 : 0 < |m| \leq n \} \).
Then \( B^n = \{ B^n_k(u) \}_{k \in I_n} \), like \( B \), has the properties (25) and the operator
\[ L_n \varphi = \sum_{j \in I_n} \left[ \frac{1}{2j^2} D_j^2 \varphi - \nu j^2 u_j D_j \varphi \right], \]
defined on the space of smooth functions \( \varphi = \varphi(u_1, u_2, \ldots, u_n) \) has the invariant measure \( \mu^n = \prod_{j \leq n} \mu_j \).
Then we consider the equation
\[ \lambda \varphi^n_\epsilon - L_n \varphi^n_\epsilon + \sum_{k \in I_n} B^n_k D_k \varphi^n_\epsilon + \sum_{k \in I_n} k^{-4}(\beta^n_k)^\epsilon D_k \varphi^n_\epsilon = f, \quad \text{in } H_n \]  
(29)
where \( (\beta^n_k)^\epsilon = \frac{1}{\epsilon} \left( 1 - \frac{\ell}{|u|_{1-a}} \right) u_k \) and \( H_n = \{ u_j : j \in I_n \} \).
By standard existence theory for Kolmogorov equations associated with stochastic differential equations, the equation (29) has a unique solution \( \varphi^n_\epsilon \) which is precisely the function
\[ \varphi^n_\epsilon(u^0) = \mathbb{E} \int_0^\infty e^{-\lambda t} f(X^n_\epsilon(t, u^0)) dt, \]
and \( X^n_\epsilon = \{ u^n_j : j \in I_n \} \) is the solution to stochastic equation (see [3])
\[ du^n_j + \nu j^2 u^n_j dt - B^n_j(u^n) dt = \frac{1}{|I|} dW_j, \quad j \in I_n, \]
\[ u^n_j(0) = u^0_j, \quad j \in I_n. \]
We may assume therefore that \( \varphi_\epsilon \) is smooth and so multiplying (29) by \( \varphi^n_\epsilon \) and integrating with respect to the measure
\[ \mu^n_\epsilon = \prod_{k \in I_n} e^{-\frac{k^4}{\epsilon}} \mu_k, \]
we obtain that
\[
\lambda \int |\varphi^n_\varepsilon|^2 d\mu_\varepsilon + \frac{1}{2} \sum_{k \in I_n} k^{-2} \int |D_k \varphi^n_\varepsilon|^2 d\mu_\varepsilon + \frac{1}{2} \sum_{k \in I_n} B^n_k(\bar{u}_k) |\varphi^n_\varepsilon|^2 d\mu_\varepsilon = \int f \varphi^n_\varepsilon d\mu_\varepsilon. \tag{30}
\]

On the other hand, taking into account that by (25) we have
\[
\sum_{k \in I_n} k^4 B^n_k \bar{u}_k \equiv 0, \quad D_k B^n_k \equiv 0, \quad \forall k \in Z_0^2,
\]
and it follows as in (24) that
\[
\sum_{k \in I_n} \int B^n_k(\bar{u}) D^n_\varepsilon |\varphi^n_\varepsilon|^2 d\mu_\varepsilon = 0
\]
and so by (30) we have that
\[
\lambda \int |\varphi^n_\varepsilon|^2 d\mu_\varepsilon + \frac{1}{2} \sum_{k \in I_n} k^{-2} \int |D_k \varphi^n_\varepsilon|^2 d\mu_\varepsilon = 0
\]
and letting \( n \) tend to infinity into the weak form of (29), that is
\[
\lambda \int \varphi^n_\varepsilon \psi d\mu_\varepsilon + \frac{1}{2} \sum_{k \in I_n} k^{-2} \int D_k \varphi^n_\varepsilon D_k \psi d\mu_\varepsilon = \int f \psi d\mu_\varepsilon \tag{31}
\]
Hence, on a subsequence, again denoted by \( \{n\} \) we have for \( n \to \infty \)
\[
\varphi^n_\varepsilon \to \varphi_\varepsilon \quad \text{weakly in } L^2(\mu_\varepsilon) \quad \tag{32}
\]
\[
\{D_k \varphi^n_\varepsilon\} \to \{D_k \varphi_\varepsilon\} \quad \text{weakly in } L^2(\mu_\varepsilon) \tag{33}
\]
and letting \( n \) tend to infinity into the weak form of (29), that is
\[
\lambda \int \varphi^n_\varepsilon \psi d\mu_\varepsilon + \frac{1}{2} \sum_{k \in I_n} k^{-2} \int D_k \varphi^n_\varepsilon D_k \psi d\mu_\varepsilon = \int f \psi d\mu_\varepsilon \tag{34}
\]
and recalling that \( \{B^n_k\} \) is strongly convergent to \( \{B_k\} \) in \( L^2(\mu) \) (see Lemma 1.3.2 in [7]) we infer that \( \varphi_\varepsilon \) is solution to (21) as claimed. Estimates (27), (28) follow by (31), (32), (33). This complete the proof of Proposition 1.

**Proof** of Theorem 1 (continued). Let \( \varphi_\varepsilon \) be a solution to (19). By estimates (27), (28) we have for \( \varepsilon \to 0 \)
\[
\varphi^n_\varepsilon \to \varphi \quad \text{weakly in } L^2(K, \mu),
\]
\[
\{D_k \varphi^n_\varepsilon\} \to \{D_k \varphi\} \quad \text{weakly in } L^2(K, \mu; H^2).
\]
Then, letting \( \varepsilon \) tend to zero into (23) we see that \( \varphi \) satisfies (17) for all \( \psi \in \mathcal{P}C^2_b \). Estimates (19), (20) follow by (27), (28). This completes the proof. □
Remark 2. Letting $\epsilon$ tend to zero into (29) it follows via integration by parts formula by a similar argument as in [5] that $\varphi^n_\epsilon \to \varphi^n$, $D_j \varphi^n_\epsilon \to D_j \varphi^n$ in $L^2(H_n, \mu)$ where $\varphi^n$ is the solution to Neumann boundary value problem

$$\begin{cases}
\lambda \varphi^n - \nu \Delta \varphi^n + B^n(u_n) \cdot D \varphi^n = f & \text{in } \hat{K}_n \\
\frac{\partial \varphi^n}{\partial n_x} = 0 & \text{on } \partial K_n.
\end{cases}$$

where $K_n = K \cup H_n$. Moreover, by elliptic regularity, $\varphi^n \in H^2(\hat{K}_n)$. On the other hand, it is clear by the above energetic estimates in $H^{1-\alpha}$ that for $n \to \infty \{\varphi^n\}$ is convergent to a weak solution $\varphi$ to (15). However, this solution is not necessarily that given by approximating process $\varphi_\epsilon$.

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