Delorme’s intertwining conditions for sections of homogeneous vector bundles on two- and three-dimensional hyperbolic spaces

Martin Olbrich · Guendalina Palmirotta

Received: 22 March 2022 / Accepted: 21 October 2022 / Published online: 5 December 2022
© The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract
The description of the Paley–Wiener space for compactly supported smooth functions $C_c^\infty(G)$ on a semi-simple Lie group $G$ involves certain intertwining conditions that are difficult to handle. In the present paper, we make them completely explicit for $G = \text{SL}(2, \mathbb{R})^d$ ($d \in \mathbb{N}$) and $G = \text{SL}(2, \mathbb{C})$. Our results are based on a defining criterion for the Paley–Wiener space, valid for general groups of real rank one, that we derive from Delorme’s proof of the Paley–Wiener theorem. In a forthcoming paper, we will show how these results can be used to study solvability of invariant differential operators between sections of homogeneous vector bundles over the corresponding symmetric spaces.

Keywords Paley–Wiener spaces · Symmetric spaces · Homogeneous vector bundles · Representation theory of semi-simple Lie groups · Intertwining operators

1 Introduction
Consider a Riemannian symmetric space of non-compact type $X = G/K$, where $G$ is a real connected semi-simple Lie group with finite center of non-compact type and $K \subset G$ is its maximal compact subgroup.

Arthur [1] as well as Delorme [6, Thm. 2] established a Paley–Wiener theorem for ($K$-finite) compactly supported smooth functions on $G$. Their results involved the so-called Arthur–Campoli and Delorme conditions, respectively. Later van den Ban and Souaifi [19] proved, without using the proof nor validity of any associated Paley–Wiener theorem, that the two compatibility conditions are equivalent.

In [16], we proved a topological Paley–Wiener(–Schwartz) theorem for sections of homogeneous vector bundles by adapting Delorme’s intertwining conditions for our purposes. We
considered the intertwining conditions in three levels, namely (Level 1) referred to Delorme’s condition in the setting of van den Ban and Souaifi [19] for the Fourier transform for functions on the group. (Level 2) corresponded to the conditions for sections of homogeneous vector bundles and (Level 3) stood for spherical functions. From the representation theoretical point of view in (Level 2), we fix a $K$-type from the right, where in (Level 3) $K$-types from both sides are fixed.

However, these intertwining conditions are very difficult to check in practice, even for special $K$-types.

The most important sources of such conditions are the Knapp–Stein [12], [13] and Zelobenko [23] intertwining operators, as well as the embedding of discrete series into principal series.

Therefore, in this article, we rewrite them in a more accessible way involving such intertwining operators and the Harish–Chandra $c$-functions, which we introduce in the first Sect. 2. Moreover, in Sect. 3, we show that a part of them is already sufficient for $G$ of real rank one (Theorem 4). This is already implicitly contained in Delorme’s proof of the Paley–Wiener theorem [6, Thm. 2]. For our proof, we essentially use an intermediate result of Delorme [6, Prop. 1] and his induction procedure [6, Prop. 2] on the length of minimal $K$-types of a generalized principal series representation.

To apply Theorem 4, one has to know more or less the complete composition series of reducible principle series representations, which is the case for the two special examples, $\text{SL}(2, \mathbb{R})$ and $\text{SL}(2, \mathbb{C})$, which are in the focus of the present paper.

In fact, for $G = \text{SL}(2, \mathbb{R})$, in Sect. 4, by drawing its principal series representations $H^\pm_{\infty, \lambda}$ by ‘box-pictures’ (Fig. 1), we can see in which closed $G$-submodule of $H^\pm_{\infty, \lambda}$ there is an intertwining condition in (Level 2) (Theorem 7). Afterward, we can deduce the corresponding results also for the other levels (Theorems 8 & 6) and even go beyond by illustrating the intertwining conditions for finite products of $\text{SL}(2, \mathbb{R})$ (Theorem 9). As a last example, in Sect. 5, we consider $G = \text{SL}(2, \mathbb{C})$. The description of its intertwining conditions (Theorems 13, 14 & 18) is more difficult than for the previous examples. We remark that $\text{SL}(2, \mathbb{R})$ and $\text{SL}(2, \mathbb{C})$ are locally isomorphic to $\text{SO}(1, n)$, $n = 2, 3$, respectively. In fact, based on Theorem 4, we have already checked that it is possible to obtain analogous results for all $n$. This will be the subject of a future paper. We are grateful to P. Delorme for mentioning to us that the results for odd $n$ (including $\text{SL}(2, \mathbb{C})$) can also be derived from his earlier Paley–Wiener theorem for groups with only one conjugacy class of Cartan subgroups [5] (instead of our Theorem 4).

This work is part of the second author’s doctoral dissertation [18]. In fact, our results can be used to study solvability of invariant differential operators between sections of homogeneous vector bundles on the corresponding symmetric spaces. This application was the main motivation for the present work. The way we expressed the intertwining conditions in (Level 3) is particularly adapted to the solvability questions. For instance, the already quite explicit description (in (Level 1)) of the intertwining conditions in [5] for groups with only one conjugacy class of Cartan subgroups does not fit yet that propose. In an upcoming paper [17], these solvability questions will be discussed in detail.

The present paper—as well as its follows up [17]—with its focus on the lowest dimensional groups $\text{SL}(2, \mathbb{R})$ and $\text{SL}(2, \mathbb{C})$ should be considered as a first test how Paley–Wiener theorems can be made sufficiently explicit to allow for applications like solvability of invariant differential operators. It is not yet clear to us how far our or similar methods will lead beyond the case $\text{SO}(1, n)$. We mention the difficulties that would come from the occurrence of higher derivatives of principal series representations in the intertwining conditions (which
in contrast to the case $SO(1, n)$ already appear for $SU(1, n)$) or of Knapp–Stein intertwining operators corresponding to non-minimal parabolics (for higher rank situations).

2 Intertwining conditions and operators

Let $G$ be a real connected semi-simple Lie group with finite center of non-compact type with Lie algebra $\mathfrak{g}$. We fix an Iwasawa decomposition $G = KAN$, where $K \subset G$ is a maximal compact subgroup with Lie algebra $\mathfrak{k}$, $A = \exp(\mathfrak{a})$ is abelian and $N$ is nilpotent. The quotient $X = G/K$ is a Riemannian symmetric space of non-compact type.

Let $M = Z_K(\mathfrak{a})$ be the centralizer of $A$ in $K$. Then, $P = MAN$ is a minimal parabolic subgroup of $G$. Let $(\sigma, E_\sigma) \in \hat{M}$ be a finite-dimensional irreducible representation of $M$ and $\lambda \in \mathfrak{a}_C^*$ the complexified dual of the Lie algebra of $A$. For $(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}_C^*$, consider $H^\sigma_{\lambda}$ the space of smooth vectors of the principal series representation $\sigma$ of $G$. We call this representation the $m$-th derived representation $\sigma_{\lambda}$, which is equipped with the natural Fréchet topology. We call this representation the $m$-th derived principal series representation of $G$. For similar slightly different definitions see also [6, Def. 3 (4.4)] or [19, Sect. 4.5].

**Definition 1** ($m$-th derived representation, [16, Def. 2]) For $\lambda \in \mathfrak{a}_C^*$, let $\text{Hol}_\lambda$ be the set of germs at $\lambda$ of $C^\infty$-valued holomorphic functions $\mu \mapsto f_\mu$ and $m_\lambda \subset \text{Hol}_\lambda$ the maximal ideal of germs vanishing at $\lambda$.

Denote by $H^\sigma_{[\lambda]}$ the set of germs at $\lambda$ of $H^\infty_{[\lambda]}$-valued holomorphic functions $\mu \mapsto \phi_\mu \in H^\infty_{[\lambda]}$ with $G$-action

$$(g\phi)_\mu = \pi_{\sigma,\mu}(g)\phi_\mu, \quad g \in G.$$  

For $m \in \mathbb{N}_0$, it induces a representation $\pi^{(m)}_{\sigma,\lambda}$ on the space

$$H^\infty_{\lambda,(m)} := H^\sigma_{[\lambda]}/m_\lambda^{m+1}H^\sigma_{[\lambda]},$$

(1)

which is equipped with the natural Fréchet topology. We call this representation the $m$-th derived principal series representation of $G$.

Let us first recall the intertwining conditions and corresponding Paley–Wiener theorems in the three levels from [16].

2.1 Intertwining conditions and Paley–Wiener theorem in (Level 1)

We denote by $\text{Hol}(\mathfrak{a}_C^*)$ the space of holomorphic functions on $\mathfrak{a}_C^*$ and by $\text{Hol}(\mathfrak{a}_C^*, \text{End}(H^\infty_{\lambda}))$ the space of maps $\mathfrak{a}_C^* \ni \lambda \mapsto \phi(\lambda) \in \text{End}(H^\infty_{\lambda})$ such that for $\phi \in H^\infty_{\lambda}$, the function $\lambda \mapsto \phi(\lambda)\phi \in H^\infty_{\lambda}$ is holomorphic.

**Definition 2** (Delorme’s intertwining condition in (Level 1), [16, Def. 3]) Let $\Xi$ be the set of all 3-tuples $(\sigma, \lambda, m)$ with $\sigma \in \hat{M}$, $\lambda \in \mathfrak{a}_C^*$ and $m \in \mathbb{N}_0$. Consider the $m$-th derived $G$-representation $H^\infty_{\lambda,(m)}$ defined in (1). For every finite sequence $\xi = (\xi_1, \xi_2, \ldots, \xi_s) \in \Xi^s, s \in \mathbb{N}$, we define the $G$-representation

$$H_\xi := \bigoplus_{i=1}^s H^\sigma_{\xi_i,(m_i)},$$
We consider proper closed \( G \)-subrepresentations \( W \subseteq H_\xi \). Such a pair \( (\xi, W) \) with \( \xi \in \Xi^\tau \) and \( W \subset H_\xi \) as above, is called an intertwining datum. Every function \( \phi \in \prod_{\sigma \in \hat{\mathcal{M}}} \text{Hol}(a^*_C, \text{End}(H^\sigma_\infty)) \) induces an element

\[
\phi_\xi \in \bigoplus_{i=1}^s \text{End}(H^\sigma_i, \lambda_i) \subset \text{End}(H_\xi).
\]

(D.a) We say that \( \phi \) satisfies Delorme’s intertwining condition, if \( \phi_\xi (W) \subseteq W \) for every intertwining datum \( (\xi, W) \).

Moreover, for \( r > 0 \), one can introduce a corresponding Paley–Wiener space

\[
PW_r(G) \subset \prod_{\sigma \in \hat{\mathcal{M}}} \text{Hol}(a^*_C, \text{End}(H^\sigma_\infty)),
\]

which is characterized by the usual growth condition (e.g., [6, Def. 3 (4.3)] or [16, Def. 4 (1.ii)]) and the intertwining condition (D.a). Let \( \overline{B}_r(o) \) be the preimage of the closed ball of radius \( r > 0 \) centered at \( o \in X \) under the projection \( p : G \to X \). Write by \( C^\infty_r(G) \) the space of smooth complex functions on \( G \) with support in \( \overline{B}_r(o) \). Then, by taking the union over all \( r > 0 \), and considering the Fourier transform of \( f \in C^\infty_c(G) \):

\[
\hat{\mathcal{M}} \times a^*_C \in (\sigma, \lambda) \mapsto \mathcal{F}_{\sigma, \lambda}(f) := \pi_{\sigma, \lambda}(f) = \int_G f(g)\pi_{\sigma, \lambda}(g) \, dg \in \bigoplus_{\sigma \in \hat{\mathcal{M}}} \text{Hol}(a^*_C, \text{End}(H^\sigma_\infty))
\]

we have the following isomorphism.

**Theorem 1** (Paley–Wiener Theorem in (Level 1), [16, Thm. 1]) For any \( r > 0 \), the Fourier transform

\[
C^\infty_r(G) \ni f \mapsto \mathcal{F}_{\sigma, \lambda}(f) \in PW_r(G), \quad (\sigma, \lambda) \in \hat{\mathcal{M}} \times a^*_C
\]

is a topological isomorphism between the Fréchet spaces \( C^\infty_r(G) \) and \( PW_r(G) \). \( \square \)

This is our reformulation of Delorme’s Paley–Wiener theorem [6, Thm. 2]. Note that Delorme preferred to state it in terms of all cuspidal parabolic subgroups. By Casselman’s subrepresentation theorem (see, e.g., [21, Thm. 3.8.3.]), it is clear that it remains true if we restrict to the minimal parabolic subgroup \( P \) (compare [19, Lem. 4.4]).

### 2.2 Intertwining conditions and Paley–Wiener theorems in (Level 2) and (Level 3)

Next, consider two finite-dimensional, not necessary irreducible, \( K \)-representations \( (\tau, E_\tau) \) and \( (\gamma, E_\gamma) \). We consider the space of smooth compactly supported sections of homogeneous vector bundles \( E_\tau \) over \( X \) by

\[
C^\infty_c(X, E_\tau) = \bigcup_{r>0} C^\infty_r(X, E_\tau)
\]

\[
\cong \bigcup_{r>0} \left\{ f : G \xrightarrow{C^\infty} E_\tau \mid f(gk) = \tau^{-1}(k)(f(g)), \quad \forall g \in K \text{ and } \supp(f) \subset \overline{B}_r(0) \right\}.
\]
The group $G$ acts on $C_c^\infty(X, \mathbb{E}_\tau)$ by left translations
\[ (g \cdot f)(g') = f(g^{-1}g'), \quad \forall g, g' \in G. \]

It is not difficult to see that we have the $G$-isomorphisms $C_c^\infty(X, \mathbb{E}_\tau) \cong C_c^\infty(G, \mathbb{E}_\tau)^K \cong [C_c^\infty(G) \otimes E_\tau]^K$. Moreover, we also consider the space of $(\gamma, \tau)$-spherical functions on $G$
\[
C_c^\infty(G, \gamma, \tau) = \bigcup_{r>0} C_r^\infty(G, \gamma, \tau)
\]
\[ := \bigcup_{r>0} \left\{ f : G \to \text{Hom}(E_\gamma, E_\tau) \mid f(k_1gk_2) = \tau(k_2)^{-1}f(g)\gamma(k_1)^{-1}, \right. \]
\[ \left. \forall k_1, k_2 \in K \text{ and } \text{supp}(f) \subset \overline{B}_r(0) \right\}. \]

Note that by taking topological linear duals (of sections of dual bundles), we obtain the spaces of distributional sections with compact support, i.e., in the spaces $C_c^\infty(X, \mathbb{E}_\tau)$ and $C_c^\infty(G, \gamma, \tau)$, respectively. We are particularly interested in distributional sections with compact support, i.e., in the spaces $C_c^\infty(X, \mathbb{E}_\tau)$ and $C_c^\infty(G, \gamma, \tau)$, respectively.

The Fourier transform for (distributional) sections of homogeneous vector bundles in (Level 2) and (Level 3) is given in the following definition:

**Definition 3** *(Fourier transforms, [16, Def. 5 & 6])* Let $g = k(g)\alpha(\gamma)(\eta)g(\in) KAN = G$ be the Iwasawa decomposition. For fixed $\lambda \in a_\mathbb{C}^*$ and $k \in K$, we define the function $e^\tau_{\lambda,k}$ by
\[
e^\tau_{\lambda,k} : G \to \text{End}(E_\tau) \cong E_\tau \otimes E_\tau
\]
\[ g \mapsto e^\tau_{\lambda,k}(g) := \tau(k(g^{-1}k))^{-1}a(g^{-1}k)^{-(\lambda+\rho)}, \quad (2) \]

where $\rho$ is the half sum of the positive roots of $(g, \alpha)$.

(a) (Level 2) For $f \in C_c^\infty(X, \mathbb{E}_\tau)$, the Fourier transformation is given by
\[
\mathcal{F}_\tau f(\lambda, k) = \int_G e^\tau_{\lambda,k}(g)f(g) \, dg = \int_{G/K} e^\tau_{\lambda,k}(g)f(g) \, dg, \quad (3)
\]
where the last equality makes sense, since the integrand is right $K$-invariant. Similarly, for distributional sections $T \in C_c^\infty(X, \mathbb{E}_\tau)$:
\[
\mathcal{F}_\tau T(\lambda, k) := \langle T, e^\tau_{\lambda,k} \rangle = T(e^\tau_{\lambda,k}) \in E_\tau,
\]
with $(\lambda, k) \in a_\mathbb{C}^* \times K/M$.

(b) (Level 3) The Fourier transformation for $f \in C_c^\infty(G, \gamma, \tau)$ is given by
\[
y^\tau_\gamma \mathcal{F}_\tau f(\lambda) := \int_G e^\tau_{\lambda,1}(g)f(g) \, dg, \quad \lambda \in a_\mathbb{C}^*. \quad (4)
\]
Similar, for $T \in C_c^\infty(G, \gamma, \tau)$, we have $y^\tau_\gamma \mathcal{F}_\tau T(\lambda) := \langle T, e^\tau_{\lambda,1} \rangle$.

Note that $\mathcal{F}_\tau(f), \mathcal{F}_\tau(T) \in \text{Hol}(a_\mathbb{C}^*, H^\tau_{\infty|m})$, where
\[
H^\tau_{\infty|m} := \{ f : K \to E_\tau \mid f(km) = \tau(m)^{-1}f(k) \}
\]
and that $y^\tau_\gamma \mathcal{F}_\tau(f), y^\tau_\gamma \mathcal{F}_\tau(T) \in \text{Hol}(a_\mathbb{C}^*, \text{Hom}_M(E_\gamma, E_\tau))$. We also remark that $\text{Hom}_K(E_\gamma, H^\tau_{\infty|m}) \cong \text{Hom}_M(E_\gamma, E_\tau)$, by Frobenius reciprocity. We will also consider $H^\tau_{\infty|m,\lambda}$ defined as in Definition 1 with $\sigma$ replaced by $\tau |_m$. 

\[ Springer \]
Definition 4 We define
\[ \text{Hom}_M(E_{\tau}, E_{\sigma})^\lambda := \text{Hol}(a_C^*, \text{Hom}_M(E_{\tau}, E_{\sigma})) / m_{\lambda}^{m+1} \] as in (1). For \( \tau \in \tilde{K} \) and an intertwining datum \((\xi, W)\), we set
\[
D_W^\tau := \left\{ t \in \bigoplus_{i=1}^s \text{Hom}_M(E_{\tau}, E_{\sigma_i})^{\lambda_i}_D \mid T = \text{Frob}^{-1}(t) \in \text{Hom}_K(E_{\tau}, W) \subset \text{Hom}_K(E_{\tau}, H_\xi) \right\}
\]
\[
\subset \bigoplus_{i=1}^s \text{Hom}_M(E_{\tau}, E_{\sigma_i})^{\lambda_i}_D.
\]
Here, \( \text{Frob}^{-1} \) is the inverse of the dual of the Frobenius reciprocity given by
\[
\text{Frob}^{-1}(t)(v)(k) = t \tau(k^{-1})v
\]
for \( t \in \text{Hom}_M(E_{\tau}, E_{\sigma}) \) and \( v \in E_{\tau} \). We have shown [16, Def. 7.8; Prop. 7 & Thm. 2] that Delorme’s intertwining condition \((D.a)\) corresponds to the following intertwining conditions in (Level 2) and (Level 3).

Definition 5 (Intertwining conditions in (Level 2) and (Level 3), [16, Thm. 2]) Let \( \Xi \) be as in Definition 2 and \( \Xi \) be the set of all tuples \((\lambda, m)\) with \( \lambda \in a_C^* \) and \( m \in \mathbb{N}_0 \). We define a map
\[
\Xi \longrightarrow \Xi, \quad \xi = (\sigma, \lambda, m) \mapsto \bar{\xi} = (\lambda, m).
\]
For \( s \in \mathbb{N} \) and \( \xi \in \Xi^s \), we have the corresponding element \( \bar{\xi} \in \Xi^s \).

(D.2) (Level 2) We say that \( \psi \in \text{Hol}(a_C^*, H^{\tau|_M}) \) satisfies the intertwining condition, if for each intertwining datum \((\xi, W)\) and each non-zero \( t = (t_1, t_2, \ldots, t_s) \in D_W^\tau \), the induced element \( \psi_{\bar{\xi}} \in \bigoplus_{i=1}^s H^{\tau|_M}_{\xi_i} \) satisfies
\[
t \circ \psi_{\bar{\xi}} = (t_1 \circ \psi_1, \ldots, t_s \circ \psi_s) \in W.
\]

(D.3) (Level 3) We say that \( \varphi \in \text{Hol}(a_C^*, \text{Hom}_M(E_{\gamma}, E_{\tau})) \) satisfies the intertwining condition, if for each intertwining datum \((\xi, W)\) and each nonzero \( t = (t_1, t_2, \ldots, t_s) \in D_W^\tau \), the induced element \( \varphi_{\bar{\xi}} \in \bigoplus_{i=1}^s \text{Hom}_M(E_{\gamma}, E_{\tau})_{\xi_i}^{\lambda_i} \) satisfies
\[
t \circ \varphi_{\bar{\xi}} = (t_1 \circ \varphi_1, \ldots, t_s \circ \varphi_s) \in D_W^\gamma.
\]

Example 1 ([16, Example 1]) Consider now \( s = 2 \) and \( m_1 = m_2 = 0 \). Let
\[
L : H^{|\sigma_1, \lambda_1|}_\infty \longrightarrow H^{|\sigma_2, \lambda_2|}_\infty
\]
be an intertwining operator between principal series representations.

The corresponding intertwining datum is given by \( \xi := ((\sigma_1, \lambda_1, 0), (\sigma_2, \lambda_2, 0)) \in \Xi^2 \) and \( W = \text{graph}(L) \subset H^{\sigma_1, \lambda_1}_\infty \oplus H^{\sigma_2, \lambda_2}_\infty \). Moreover, define
\[
L^\tau : \text{Hom}_M(E_{\tau}, E_{\sigma_1}) \longrightarrow \text{Hom}_M(E_{\tau}, E_{\sigma_2})
\]
by
\[
L^\tau(t)(v) = L(t \tau(t^{-1})v)(e) = L(\phi_v(t))(e),
\]
where the element $\phi_v(t) \in H^\sigma_\infty$ is given by the function $\phi_v(t)(k) := t \tau(k^{-1})v$, for $k \in K$, $v \in E_\tau$ and $t \in \text{Hom}_M(E_\tau, E_{\sigma_1})$. Then,

$$D^1_W = \{(t_1, t_2) \mid t_2 = l^\tau(t_1)\} = \{(t, l^\tau(t)) \mid t \in \text{Hom}_M(E_\tau, E_{\sigma_1})\} \subset \text{Hom}_M(E_\tau, E_{\sigma_1}) \oplus \text{Hom}_M(E_\tau, E_{\sigma_2}).$$

In this situation, we get the following intertwining conditions.

(D2.L) (Level 2) For each $t \in \text{Hom}_M(E_\tau, E_{\sigma_1})$, we have for $\psi(\lambda_i, \cdot) \in H^\sigma|_M\cdot$, $i = 1, 2$

$$L(t \circ \psi(\lambda_1, \cdot)) = l^\tau(t) \circ \psi(\lambda_2, \cdot).$$

(D3.L) (Level 3) For each $t \in \text{Hom}_M(E_\tau, E_{\sigma_1})$, we have for $\varphi(\lambda_i) \in \text{Hom}_M(E_\gamma, E_\tau)$, $i = 1, 2$

$$l^\gamma(t \circ \varphi(\lambda_1)) = l^\tau(t) \circ \varphi(\lambda_2).$$

Similar as in (Level 1), for $r > 0$, we introduce a Paley–Wiener space in:

(L2) $\mathcal{P}W_r,\tau(a^*_C \times K/M) \subset \text{Hol}(a^*_C, H^\sigma|_M)$

and

(L3) $\mathcal{P}W_\tau(a^*_C) \subset \text{Hol}(a^*_C, \text{Hom}_M(E_\gamma, E_\tau))$.

which is characterized by the usual growth condition [16, Def. (2.ii)_r and (3.ii)_r, resp.] and the corresponding intertwining condition (D.2) and (D.3), respectively. The Paley–Wiener–Schwartz space for distributional sections is denoted by $\mathcal{P}WS_\tau(a^*_C \times K/M)$, respectively

$$\mathcal{P}W_\tau(a^*_C).$$

It is characterised in the same way as above except that we replace the growth condition by a weaker one [16, Def. (2.iii) and (3.iii)_, resp.]. As in [16, Sect. 6], we equip the Paley–Wiener–Schwartz space with the inductive limit topology and the space of distributional sections $C^\infty_c(X, E_\tau)$ and $C^\infty_c(G, \gamma, \tau)$ with the strong dual topology. In [16], we derived the following.

**Theorem 2** (Topological Paley–Wiener–Schwartz theorem for sections in (Level 2) and (Level 3), [16, Thms. 3 & 4]) *The Fourier transform*

$$C^\pm\infty_c(X, E_\tau) \ni \psi \mapsto \mathcal{F}_\tau(\psi)(\lambda, k) \in \mathcal{P}W(S)_\tau(a^*_C \times K/M), \quad (\lambda, k) \in a^*_C \times K$$

is a topological isomorphism between $C^\pm\infty_c(X, E_\tau)$ and $\mathcal{P}W(S)_\tau(a^*_C \times K/M)$.

Moreover, by considering an additional $K$-representation $(\gamma, E_\gamma)$ with associated homogeneous vector bundle $E_\gamma$, then the Fourier transform

$$C^\pm\infty_c(G, \gamma, \tau) \ni \varphi \mapsto \gamma \mathcal{F}_\tau(\varphi)(\lambda) \in \gamma \mathcal{P}W(S)_\tau(a^*_C), \quad \lambda \in a^*_C$$

is a topological isomorphism between $C^\pm\infty_c(G, \gamma, \tau)$ and $\gamma \mathcal{P}W(S)_\tau(a^*_C)$. \square

In Example 1, we worked out the intertwining conditions coming from intertwining operators between principal series representations of $G$. We want to make them more explicit for the most important case, the Knapp–Stein intertwining operators, which we now recall.
2.3 Knapp–Stein intertwining operator

Let \( W_A := N_K(a)/M \) be the Weyl group. Note that \( W_A \) acts on \( \mathfrak{a}_C^* \) as well as on \( \widehat{M} \). Let \( w \in W_A \) be represented by \( m_w \in M' := N_K(a) \) and \( \sigma \in \widehat{M} \). We realize \( \sigma \) on the vector space \( E_\sigma \). We define a new representation \( w_\sigma \in \widehat{M} \) of \( M \) acting on \( E_\sigma \) by

\[ w_\sigma(m) := \sigma(m_w^{-1}mm_w), \quad m \in M. \]

Its equivalence class only depends on \( w \in W_A \) and not on the choice of \( m_w \).

**Definition 6** (Knapp–Stein intertwining operator, [12, 13] and [10, Chap. VI]) Let \( \Delta^+_a \) be the positive root system of \((g, a)\) corresponding to \( N \). Let \( \Delta^-_a := -\Delta^+_a \) and \( \overline{N} \) be the unipotent subgroup coming from the associated Iwasawa decomposition corresponding to \( \Delta^-_a \). Write \( N_w := N \cap w\overline{N}w^{-1} \). For \((\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_C^* \) with \((\text{Re}(\lambda), \alpha) > 0\), for all \( \alpha \in \Delta^+_a \cap w^{-1}\Delta^-_a \) and for a fixed representative \( m_w \in M' \), we define the intertwining operator:

\[ J_{w,\sigma,\lambda} : H^\sigma_\infty \rightarrow H^{w,\sigma,\lambda}_\infty \]

by the convergent integral

\[ J_{w,\sigma,\lambda}(\varphi(g)) := \int_{N_w} \varphi(gnm_w) \, dn, \quad g \in G, \varphi \in H^\sigma_\infty, \]

which depends holomorphically on \( \lambda \in \mathfrak{a}_C^* \). This operator has a meromorphic continuation to \( \mathfrak{a}_C^* \).

For later reference, let us state the intertwining conditions coming from the Knapp–Stein operators.

**Example 2** (Knapp–Stein intertwining condition in (Level 1)) For \( w \in W_A \) and fixed \((\sigma, \lambda) \in \widehat{M} \times \mathfrak{a}_C^* \), we consider the Knapp–Stein intertwining operator \( J_{w,\sigma,\lambda} \) as in Definition 6. Let \( \phi \in \prod_{\sigma \in \widehat{M}} \text{Hol}(\mathfrak{a}_C^*, \text{End}(H^\sigma_\infty)) \). Then, the condition

\[ J_{w,\sigma,\lambda} \circ \varphi(\sigma, \lambda) = \varphi(w\sigma, w\lambda) \circ J_{w,\sigma,\lambda} \tag{8} \]

is a special case of \((D.a)\) in Definition 2. Note that the corresponding intertwining datum \((\xi, W)\) is given by \( \xi = ((\sigma, \lambda, 0), (w\sigma, w\lambda, 0)) \) and \( W = \text{graph}(J_{w,\sigma,\lambda}) \subset H^{\sigma,\lambda}_\infty \oplus H^{w,\sigma,\lambda}_\infty \), (compare Example 1).

**Definition 7** (Harish-Chandra \( c \)-function, e.g., [15, Def. 3.8]) Let \( \tau \in \widehat{K} \), \( w \in W_A \), \( \overline{N}_w := \overline{N} \cap w^{-1}Nw \) and \( \lambda \in \mathfrak{a}_C^* \) with \((\text{Re}(\lambda), \alpha) > 0\), for all \( \alpha \in \Delta^+_a \cap w^{-1}\Delta^-_a \). The \( c \)-function is defined by

\[ c_{w,\tau}(\lambda) := \int_{\overline{N}_w} a(n)^{-\lambda+\rho}(\kappa(n)) \, d\overline{n} \in \text{End}_M(E_\tau), \]

which can be extended to a meromorphic function on \( \mathfrak{a}_C^* \).

Furthermore, we have \( c_{w,\tau}(\sigma, \lambda) := pr_\sigma \circ c_{w,\tau}(\lambda) \circ pr_\sigma \in \text{End}_M(E_\tau(\sigma)) \), where \( pr_\sigma : E_\tau \rightarrow E_\tau(\sigma) \) is the projection on the \( \sigma \)-isotypic component.

Consider now \( w \in W_A \) as a Weyl element with maximal length, then we set

\[ c_\tau(\lambda) := c_{w,\tau}(\lambda) \quad \text{and} \quad c_\tau(\sigma, \lambda) := c_{w,\tau}(\sigma, \lambda). \]

One can express the Knapp–Stein intertwining operators, by the Harish-Chandra \( c \)-functions [15, Lem. 3.12 & Satz 3.13]:

\[ J_{w,\sigma,\lambda}(\phi_v(t)) = \phi_v(t \circ \tau(m_w^{-1})c_{w^{-1},\tau}(-w\lambda)), \tag{9} \]
where \( \phi_v(t) \) is defined as in Example 1 for \( v \in E_\tau \) and \( t \in \text{Hom}_M(E_\tau, E_\sigma) \). Set \( J_{w,\tau,\lambda} := \tau(m_w) J_{w,\tau}^{M,\lambda} \), which is independent of the choice of \( w \in W_A \). Hence, this leads to the following statement.

**Proposition 3** (Knapp–Stein intertwining condition in (Level 2) and (Level 3))

(a) (Level 2) Let \( \psi \in \text{Hol}(a^*_C, H^\tau_{\text{flow}}) \) satisfying (D.2). Then, we have

\[
J_{w,\tau,\lambda} \psi(\lambda, \cdot) = c_{w^{-1},\tau}(-w\lambda) \psi(w\lambda, \cdot), \quad \lambda \in a^*_C, \ w \in W_A. \tag{10}
\]

(b) (Level 3) Let \( \varphi \in \text{Hol}(a^*_C, \text{Hom}_M(E_\gamma, E_\tau)) \) satisfying (D.3). Then, we have

\[
\varphi(\lambda) \gamma(m_w^{-1}) c_{w^{-1},\tau}(-w\lambda) = \tau(m_w^{-1}) c_{w^{-1},\tau}(-w\lambda) \circ \varphi(w\lambda), \quad \lambda \in a^*_C, \ w \in W_A. \tag{11}
\]

**Proof** We consider the operator \( I^\tau : \text{Hom}_M(E_\tau, E_\sigma) \to \text{Hom}_M(E_\tau, E_\sigma) \) as in Example 1 associated by \( L = J_{w,\sigma,\lambda} \). Then, (9) says that

\[
I^\tau(t) = t \circ \tau(m_w^{-1}) \circ c_{w^{-1},\tau}(-w\lambda).
\]

Now the proposition follows from (6) and (7), as in Example 1. \( \square \)

**Example 3** (a) Let \( \tau \) be a trivial one-dimensional representation. Then, \( C^\infty_c(X, E_\tau) = C^\infty_c(X) \) and \( H^\tau_{\text{flow}} = C^\infty(K/M) \). Helgason showed in [8, Thm. 5.1.] that \( \beta \in \text{Hol}(a^*_C, C^\infty(K/M)) \) belongs to the Paley–Wiener space if, and only, if it satisfies the usual growth condition and the intertwining condition:

\[
\int_{K/M} e^\tau_{w,\lambda,k}(g) \beta(w\lambda, k) dk = \int_{K/M} e^\tau_{-\lambda,k}(g) \beta(\lambda, k) dk, \quad w \in W_A. \tag{12}
\]

It is not difficult to show that Helgason’s intertwining condition is equivalent to (10). In fact, for \( \lambda \in a^*_C \), consider the Poisson transform (e.g., [8] or [15, Def. 3.2]) \( P_\lambda : C^\infty(K/M) \to C^\infty_c(X) \) given by \( P_\lambda(f)(g) := \int_K e^\tau_{-\lambda,k}(g) f(k) dk, \) for \( g \in G \). Then, Helgason’s condition (12) can be expressed in terms of Poisson transform

\[
P_\lambda \circ \beta_\lambda = P_w \lambda \circ \beta_{w\lambda}, \quad w \in W_A, \ \lambda \in a^*_C,
\]

where \( \beta_\lambda := \beta(\lambda, \cdot) \). The result now follows from the functional equation of the Poisson transform [15, Satz 3.15].

(b) Let \( \tau \) and \( \gamma \) be two trivial one-dimensional representations. Consider a function \( \beta \in \text{Hol}(a^*_C) \) which satisfies the usual growth condition. Helgason [9, Thm. 7.1] and Gangolli [7] proved that \( \beta \in \gamma PW_{\tau}(a^*_C) \), if and only, if

\[
\beta(\lambda) = \beta(w\lambda), \quad \text{for} \ \lambda \in a^*_C, \ w \in W_A.
\]

This condition is equivalent to (11) in the case \( \gamma, \tau \) are trivial.

### 3 Sufficient intertwining conditions for rank one

We want to reduce the amount of intertwining data in (D,a) of Definition 2 to a minimum. In this section, we assume that \( G \) has real rank one. In this situation, the set of positive restricted roots \( \Delta_\alpha^+ \) consists of at most two elements, namely \( \alpha \) and possibly \( 2\alpha \). The Weyl group is reduced to \( \{-1, 1\} \) acting on \( a^*_C \) by multiplication.

We have the following special intertwining conditions. An irreducible unitary representation \( (\pi, E_\pi) \) of \( G \) is called a representation of the discrete series if there is a \( G \)-invariant
embedding \( E_{\pi} \hookrightarrow L^2(G) \). Here, \( L^2(G) \) denote the space of all square integrable functions with respect to invariant measure \( dg \) on \( G \). Write \( \hat{G}_d \) for the set of equivalence classes of discrete series representations of \( G \). Let \( H_{\pi} \) be any Hilbert space, where the representation \( \pi \in \hat{G}_d \) is realized. Let \( H_{\pi}^\infty \subset H_{\pi} \) be the corresponding space of smooth vectors. For every representation of the discrete series \( \pi \in \hat{G}_d \), we choose an embedding

\[
i_{\pi}: H_{\pi}^\infty \hookrightarrow H_{\pi}^\sigma,\lambda_{\pi}
\]

into some principal series representation (Casselman’s subrepresentation theorem [21, Thm. 3.8.3.] and Casselman’s and Wallach’s globalization theorem [22, Chap. 11]) and set

\[
W_{\pi} := i_{\pi}(H_{\pi}^\infty) \subset H_{\pi}^\sigma,\lambda_{\pi}.
\]

It is a closed \( G \)-invariant subspace. Hence, the condition

\[
\phi(\sigma_{\pi}, \lambda_{\pi})(W_{\pi}) \subset W_{\pi}, \quad \pi \in \hat{G}_d
\]

is also of the form \((D, a)\), with \( s = 1 \) and \( m = 0 \), and it permits us to define

\[
\phi(\pi) := \phi(\sigma_{\pi}, \lambda_{\pi})|_{W_{\pi}} \in \text{End}(W_{\pi}).
\]

For \( r > 0 \), we define the ‘special’ Paley–Wiener space \( PW_r^+(G) \) by replacing Delorme’s intertwining condition \((D, a)\) by the conditions \((8)\) and \((13)\), only.

Let \( w \in W_A \) be the non-trivial element. For \( \lambda \in \mathfrak{a}_C^s \) with \((\text{Re}(\lambda), \alpha) > 0\), let \( m \in \mathbb{N}_0 \) be the maximal order of the zeros of \( J_{w,\sigma,\mu}(f_{\mu}) \) at \( \mu = \lambda \), where \( \mu \mapsto f_{\mu} \in H_{\pi}^\sigma,\mu \) runs over all germs of holomorphic functions at \( \lambda \) with \( f_{\lambda} \neq 0 \).

We consider the induced operator

\[
J_{w,\sigma,\lambda}^{(m-1)}: \ H_{\infty}^{\sigma,\lambda}_{\pi},(m-1) \longrightarrow H_{\infty}^{\sigma,\lambda}_{\pi},(m-1)
\]

and its kernel \( \text{Ker}(J_{w,\sigma,\lambda}^{(m-1)}) \subset H_{\infty}^{\sigma,\lambda}_{\pi},(m-1) \). By convention, we set \( H_{\infty}^{\sigma,\lambda}_{\pi},(-1) := \{0\} \), for \( m = 0 \). Notice that due to condition \((8)\), we have for \( \phi^{(m-1)}(\sigma, \lambda) \in \text{End}(H_{\pi}^\sigma,\lambda) \):

\[
\phi^{(m-1)}(\sigma, \lambda)(\text{Ker}(J_{w,\sigma,\lambda}^{(m-1)})) \subset \text{Ker}(J_{w,\sigma,\lambda}^{(m-1)}) \subset H_{\infty}^{\sigma,\lambda}_{\pi},(m-1), \quad (\sigma, \lambda) \in \hat{M} \times \mathfrak{a}_C^s.
\]

Let \( \tau \in \hat{K} \) with highest weight \( \mu_{\tau} \in i\mathfrak{t}^* \), where \( t \subset \mathfrak{k} \) is the Lie algebra of a maximal torus \( T \subset K \). We define

\[
2\rho_c := \sum_{\alpha \in \Delta^+(\mathfrak{k},\mathfrak{t})} \alpha \in i\mathfrak{t}^*,
\]

being the sum of all positive roots of \( t_c \) in \( \mathfrak{k} \). For a \( K \)-representation \( V \), we denote by \( V(\tau) \) its corresponding isotypic component. For \( \sigma \in \hat{M} \) and \( \pi \in \hat{G}_d \), we define \( |\sigma|, |\pi| \in [0, \infty] \) by

\[
|\sigma| := \min_{\{\tau \mid H_{\infty}^{\sigma}(\tau) \neq \{0\}\}} ||\mu_{\tau} + 2\rho_c|| \quad \text{and} \quad |\pi| := \min_{\{\tau \mid H_{\infty}^{\pi}(\tau) \neq \{0\}\}} ||\mu_{\tau} + 2\rho_c||,
\]

i.e., \(|\sigma|, |\lambda|\) are the lengths of ‘the’ minimal \( K \)-type \( \tau \) of \( H_{\infty}^{\sigma} \) and \( H_{\infty}^{\pi} \), respectively [6, Sect. 1.3]. Denote by \( B(\sigma), B(\pi) \subset \hat{K} \) the finite set of all minimal \( K \)-types of \( H_{\infty}^{\sigma}, H_{\infty}^{\pi} \).

**Example 4** Let \( G = \text{SL}(2,\mathbb{R}) \) and \( K = \text{SO}(2) \) its maximal compact subgroup. With the notations introduced in Sect. 4, we have that \( \hat{K} \cong \mathbb{Z} \) and \( \rho_c = 0 \).

(i) \( M = \{\pm 1\} \), thus

- if \( \sigma \) is trivial, then \( B(\sigma) = \{0\} \subset \mathbb{Z} \) (trivial \( K \)-type), and \( |\sigma| = 0 \),

\( \Box \) Springer
if $\sigma$ is non-trivial, then $B(\sigma) = \{+1, -1\} \subset \mathbb{Z}$ and $|\sigma| = 1$.

(ii) Let $\pi = D_k, k \in \mathbb{Z}\setminus\{0\}$ be the discrete series representation of $G$ parametrized as in Theorem 5, then

$$B(D_k) = \begin{cases} 
\{k + 1\}, & k > 0, \\
\{k - 1\}, & k < 0 
\end{cases}$$

and $|\pi| = |k| + 1$.

The following result tells us that in case of real rank one, the intertwining conditions (8) and (13) with an additional ‘vanishing’ condition are sufficient for the characterization of the image of the Fourier transform.

**Theorem 4** Let $r_{kR}(G) = 1$. For $r > 0$, let $\mathcal{A}$ be a linear closed and $K \times K$-invariant subspace of $PW_r(G)$ satisfying $\mathcal{F}_{\sigma, \lambda}(C_r^\infty(G)) \subset \mathcal{A}$ and the following condition:

(D.b) Let $\sigma \in \hat{M}$ and $\phi \in \mathcal{A}$ such that

(i) $\phi(\sigma', \lambda) = 0$, for all $\sigma' \in \hat{M}$ with $|\sigma'| > |\sigma|$ and $\lambda \in a_C^*$,

(ii) $\phi(\pi) = 0$, for all $\pi \in \hat{G}_d$ with $|\pi| > |\sigma|$.

Then, for all $\lambda \in a_C^*$ with $(\text{Re}(\lambda), \alpha) > 0$, $\phi$ induces the zero-operator on $\text{Ker}(J_{w, \sigma, \lambda}^{(m-1)})$:

$$\phi^{(m-1)}(\sigma, \lambda) \mid_{\text{Ker}(J_{w, \sigma, \lambda}^{(m-1)})} = 0.$$

Here, $(m-1)$ depends on $(\sigma, \lambda)$ as defined above (15) and $\phi(\pi)$ is defined in (14).

Then,

$$\mathcal{A} = PW_r(G) \cong \mathcal{F}_{\sigma, \lambda}(C_r^\infty(G)).$$

**Proof of Theorem 4** By Delorme’s Paley–Wiener Theorem 1, we already know that $PW_r(G) \cong \mathcal{F}_{\sigma, \lambda}(C_r^\infty(G))$ is a closed and $K \times K$-invariant subspace of $PW_r^+(G)$. Therefore, it suffices to show that $\mathcal{F}_{\sigma, \lambda}(C_r^\infty(G)) \subset \mathcal{A}$ is dense. Thus, for every $K \times K$-finite element $\phi \in \mathcal{A}$, we need to find a function $f \in C_r^\infty(G)_{K \times K}$ such that

$$\pi_{\sigma, \lambda}(f) = \phi(\sigma, \lambda), \quad \forall (\sigma, \lambda) \in \hat{M} \times a_C^*.$$

Let $\phi \in A_{K \times K}$. It is given by a collection $(\phi_\sigma, \sigma \in \hat{M})$. By $K \times K$-finiteness, only finitely many $\phi_\sigma$ are nonzero. Similar, by $K \times K$-finiteness, $\phi(\pi) = 0$, for all but finitely many $\pi \in \hat{G}_d$. Indeed, for any given $K$-type $\tau$, there are only finitely many $\pi \in \hat{G}_d$, with $H_\pi^\infty(\tau) \neq 0$ (e.g., [21, Cor. 7.7.3]).

We define $l(\phi) \in [0, \infty)$ by

$$l(\phi) := \max\{|\sigma|, |\pi| \mid \sigma \in \hat{M}, \phi_\sigma \neq 0; \pi \in \hat{G}_d, \phi(\pi) \neq 0\}.$$

We can now imitate the inductive proof of Prop. 2 in Delorme’s paper [6]. Assume, as induction hypothesis, that for all $\psi \in \mathcal{A}$ with $l(\psi) < l(\phi)$, there are $f \in C_r^\infty(G)$ with $\mathcal{F}(f) = \psi$. We enumerate

$$\{\sigma \in \hat{M} \mid |\sigma| = (\phi)\} = \{\sigma_1, \ldots, \sigma_n\} \cup \{w\sigma_1, \ldots, w\sigma_n\}$$

and

$$\{\pi \in \hat{G}_d \mid |\pi| = (\phi)\} = \{\pi_1, \ldots, \pi_s\}.$$
Condition (ii) together with (8) says, in particular, that $\phi_{n_i}$ belongs to a space that Delorme denotes by $K_{\alpha_1}$, [6, Def. 1]. Strictly speaking Delorme has a condition for $(\Re(\lambda), \alpha) \geq 0$. But if $rk_s(G) = 1$, only $(\Re(\lambda), \alpha) > 0$ matters. Note, that $\phi(\pi_f)$ belongs automatically to $K_{\pi_j}$. We can apply Prop. 1 together with Eq. (1.38) of Delorme’s paper [6], to deduce the existence of $f_1, f_2, \ldots, f_n \in C^\infty_r(G)$ and $g_1, g_2, \ldots, g_s \in C^\infty_r(G)$ with

$$\pi_{\alpha,\lambda}(f_i) = \phi(\sigma_i, \lambda), \quad i \in \{1, 2, \ldots, n\},$$

$$\pi_f(g_j) = \phi(\pi_j), \quad j \in \{1, 2, \ldots, s\},$$

for $\lambda \in a^*_n$. Moreover, the discussion after Eq. (3.9) in [6, p.1018], makes clear that we can choose the $f_i$ and $g_j$ such that

(i) $l(\mathcal{F}(f_i)) = l(\mathcal{F}(g_j)) = l(\phi)$, $\forall i \in \{1, 2, \ldots, n\}, j \in \{1, 2, \ldots, s\}$ and

(ii) $\pi_{\sigma_i,\lambda}(f_i) = 0$, $\forall k \neq i$,

(iii) $\pi_{\sigma_i,\lambda}(g_j) = 0$, $\forall i, j$,

(iv) $\pi_f(f_i) = 0$, $\forall i, j$,

(v) $\pi_f(g_j) = 0$, $\forall k \neq j$.

Now, we set

$$\psi := \phi - \sum_{i=1}^n \mathcal{F}(f_i) - \sum_{j=1}^s \mathcal{F}(g_j).$$

Then, by (i)-(v) we have $l(\psi) < l(\phi)$. Thus, by induction hypothesis $\psi = \mathcal{F}(f_0)$. We conclude that $\phi = \mathcal{F}(f)$ with $f = f_0 + f_1 + \cdots + f_n + g_1 + g_2 + \cdots + g_s$. \hfill \Box

\textbf{Remark 1} The result above can be extended to higher real rank. The extension involves representations induced from all cuspidal parabolic subgroups $P$ as well as the Knapp–Stein intertwining operator for them.

\section{The case $G = \text{SL}(2, \mathbb{R})$ and beyond}

We consider $G = \text{SL}(2, \mathbb{R}) = \left\{g := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{R}) \mid \det(g) = 1 \right\}$, the special linear group of $\mathbb{R}^2$. It has dimension three. We fix the Iwasawa decomposition $G = KAN$, where

$$K = \text{SO}(2) = \left\{k_\theta := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}, \quad A = \left\{a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \mid t \in \mathbb{R} \right\},$$

$$N = \left\{n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \right\}.$$

$G$ is a connected and simple Lie group with maximal compact subgroup $K$. Clearly, $K$ is isomorphic to the unit circle $\mathbb{S}^1$. Hence,

$$\widehat{K} = \{\delta_n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}, \quad \delta_n(k_\theta) := e^{inx} \in \text{GL}(1, \mathbb{C}) \cong \mathbb{C} \setminus \{0\}.$$

The representation space $E_{\delta_n}$ is one-dimensional and equal to $\mathbb{C}$. We sometimes denote the $K$-representation $\delta_n$ simply by $n$. If $H = \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \in \mathfrak{a}$, then the positive root $\alpha$ is given by $\alpha(H) = 2t$ and $\rho(H) = t$. We identify $a^*_n$ with $\mathbb{C}$ via $z \alpha \mapsto z$. In particular, $\rho \mapsto \frac{1}{2}$.
Since $M = \{\pm \text{Id}\}$, we have $\hat{M} \cong \mathbb{Z}/2\mathbb{Z}$. Let $\{\pm\}$ be the trivial and $\{-\}$ the non-trivial element of $\hat{M}$. For $\sigma = \{\pm\} \in \hat{M}$ and $\lambda \in \mathbb{C} \cong a_{\mathbb{C}}^*$, we write $(\pi_{\pm, \lambda}, H_{\infty}^{\pm, \lambda})$ for the principal series representations of $G$. Its restriction to $K$ is the set of Fourier series on $S^1$ with only nonzero even or odd Fourier coefficients

$$H_{\infty}^{\pm} = \{ f \in C^\infty(K, \mathbb{C}) \mid f \text{ even or odd} \} \cong \bigoplus_{n \text{ even or odd}} \delta_n.$$ 

In order to write down the composition series of $H_{\infty}^{\pm, \lambda}$, we will denote an exact, non-splitting module sequence

$$0 \to A \to B \to C \to 0$$

shortly by a ‘boxes-picture’

$$B = \begin{array}{c}
C \\
A
\end{array}.$$ 

A proof of the following classical result can be found for example in [21, 5.6] or in [14, Ch. VI]. Note that the referenced proofs are also valid for $G$-representations of smooth vectors instead of $(g, K)$-modules, if we apply Casselman’s and Wallach’s globalization theorem [22, Prop. 11].

**Theorem 5** (Structure of principal series representations of $SL(2, \mathbb{R})$) The principal series representation $H_{\infty}^{\pm, \lambda}$ of $SL(2, \mathbb{R})$ is reducible if and only if

$$\lambda \in I_\pm := \begin{cases}
\frac{1}{2} + \mathbb{Z}, & \sigma = +, \\
\mathbb{Z}, & \sigma = -.
\end{cases}$$

For $\lambda = \frac{k}{2} \in I_\pm, k \in \mathbb{N}$, we have

$$H_{\infty}^{\pm, -\frac{k}{2}} = \begin{array}{c}
D_{-k} \oplus D_k \\
F_k
\end{array}, \quad H_{\infty}^{\pm, \frac{k}{2}} = \begin{array}{c}
F_k \\
D_{-k} \oplus D_k
\end{array}$$

where $F_k := \bigoplus_{j=-1}^{k-1} \delta_{2j}$ is the finite-dimensional irreducible $SL(2, \mathbb{R})$-representation of dimension $k$ and $D_{\pm k}$ is the space of vectors of a representation of discrete series, which is characterized by the $K$-type decomposition $D_{\pm k} = \bigoplus_{j \geq 0} \delta_{\pm(k+1+2j)}$. Furthermore, for $\lambda = 0$, we have

$$H_{\infty}^0 = \begin{array}{c}
D_- \oplus D_+
\end{array}$$

where $D_\pm = \bigoplus_{j \geq 0} \delta_{\pm(1+2j)}$ are the limits of the discrete series. \(\square\)

**Remark 2** Let $W_\lambda$ be a proper closed invariant $G$-submodule of $H_{\infty}^{\pm, \lambda}$ for $\lambda \in I_\pm$. Then, Theorem 5 tells us that then, one can observe that

- for $\lambda > 0$, $W_\lambda \in \{ D_{-k}, D_k, D_{-k} \oplus D_k \}, k = 2\lambda$,

- for $\lambda < 0$, $W_\lambda \in \{ F_k, F_k \}, k = 2\lambda$,

- while for $\lambda = 0$ and $\sigma = -$, $W_\lambda \in \{ D_+, D_- \}$. 

\(\copyright\) Springer
To describe the intertwining conditions for \( G = \text{SL}(2, \mathbb{R}) \) in the three levels, we need some preparation. The Harish–Chandra \( c \)-function for \( G \) is denoted by \( c_n(\lambda) \) for \( n \in \mathbb{Z} \). Due to Cohn [4, App. 1], it is given explicitly in terms of gamma function \( \Gamma(\cdot) \) by the formula (for a suitable normalization of the Haar measure \( d\tilde{\nu} \)):

\[
c_n(\lambda) = c_{-n}(\lambda) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda + \frac{1+n}{2})\Gamma(\lambda + \frac{1-n}{2})}, \quad \lambda \in a_+^*.
\]

(16)

Let \( n \equiv m \pmod{2} \). Using the gamma function recurrence formula

\[
\Gamma(\lambda + a) = (\lambda + (a - 1))\Gamma(\lambda + (a - 1)), \quad a \in \mathbb{Z}, \lambda \in a_+^*
\]

(17)

repeatedly, we find the following expression of the quotient of the \( c \)-functions:

\[
c_n(\lambda) = \frac{\Gamma(\lambda + \frac{1+m}{2})\Gamma(\lambda + \frac{1-m}{2})}{\Gamma(\lambda + \frac{1+n}{2})\Gamma(\lambda + \frac{1-n}{2})} = \begin{cases} 
1, & \text{for } |n| = |m|, \\
\frac{(\lambda - |n|-1)(\lambda - |n|+1)...(\lambda - |n|+1)}{(\lambda + |m|-1)(\lambda + |m|+1)...(\lambda + |m|+1)}, & \text{for } |n| > |m|, \\
\frac{(\lambda + |m|-1)(\lambda + |m|+1)...(\lambda + |m|+1)}{(\lambda - |m|-1)(\lambda - |m|+1)...(\lambda - |m|+1)}, & \text{for } |n| < |m|. 
\end{cases}
\]

(18)

Note that the quotient has zeros in \( \{\frac{|n|-1}{2}, \frac{|n|-3}{2}, \ldots, \frac{|m|+1}{2}\} \) and poles in \( \{-\frac{|n|-1}{2}, -\frac{|n|-3}{2}, \ldots, -\frac{|m|+1}{2}\} \) for \( |n| > |m| \), and inversely for \( |n| < |m| \). We know by (11) that the matrix coefficient of the Knapp–Stein intertwining operator \( J_{w,\pm,\lambda} : H_{\pm,\lambda} \to H_{\pm,\lambda} \) with respect to the Fourier decomposition of \( H_{\pm,\lambda} \) is denoted by \( c_n(\lambda) \) (up to sign).

**Theorem 6** (Intertwining conditions in (Level 1)) For \( r > 0 \), let \( A \) be the space of all \( \phi \in \prod_{\sigma \in \tilde{M}} \text{Hol}(a_+^*, \text{End}(H_{\sigma,\pm,\lambda})) \) such that \( \phi \) satisfies the corresponding growth condition as well as the two intertwining conditions (8) and

\[(D.b^*) \quad \phi \text{ leaves every proper closed } G\text{-submodule } W_\lambda \text{ of } H_{\pm,\lambda} \text{, listed in Remark 2, invariant.} \]

Then, \( A \) satisfies the conditions of Theorem 4, this means that \( A = PW_r(G) \).

**Proof** Note first, that the space \( A \) is \( K \times K \)-invariant and closed. We have that (\( D.b \)) of Theorem 4 gives a condition for each \( \sigma \in \tilde{M} = \{\pm\} \). Let us first consider \( \sigma = + \) \( \in \tilde{M} \).

By Example 4, we have \( |+| = 0 \) and \( |\pi| = |k| + 1 > 0 \). Now let \( \phi \in \mathcal{A} \) satisfying the assumption (\( D.b \)) (ii), i.e., in particular

\[
\phi^{(0)}\left( +, \frac{k}{2} \right)|_{D_-k \oplus D_k} = 0, \quad k \in 2\mathbb{Z} + 1.
\]

(19)

Let us check that:

(a) for \( \text{Re}(\lambda) > 0 \), the intertwining operator \( J_{-,+}\lambda \) has zeros of order at most one.
(b) the kernel of \( J_{+,+}\lambda \) is equal to 0 or \( D_-k \oplus D_k \) for \( \text{Re}(\lambda) > 0 \).

Consider the \( K \)-type \( n \in 2\mathbb{Z} \) and the Harish–Chandra \( c \)-function \( c_n \) as in (16). If \( n = 0 \), then \( c_0(\lambda) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(\lambda)}{\Gamma(\lambda + \frac{1}{2})} \) and we see that \( c_0(\lambda) \) has no zeros and no poles for \( \text{Re}(\lambda) > 0 \). Thus, instead of \( c_n \), we can consider the quotient

\[
\frac{c_n(\lambda)}{c_0(\lambda)} = \frac{\Gamma(\lambda + \frac{1}{2})^2}{\Gamma(\lambda + \frac{1+n}{2})\Gamma(\lambda + \frac{1-n}{2})} = \frac{(\lambda - \frac{|n|-1}{2})\cdots(\lambda - \frac{1}{2})}{(\lambda + \frac{|n|-1}{2})\cdots(\lambda + \frac{1}{2})}.
\]
It has zeros \( \lambda \in \left\{ \frac{1}{2}, \cdots, \frac{|n|-1}{2} \right\} \) of first order. Due to (9), we know that the intertwining operator \( J_{w,+,\lambda} \) is in relation with the e-function. If on all \( K \)-types, we have zeros of first order, then \( J_{w,+,\lambda} \) should also have zeros of first order. Hence, \( J_{w,+,\lambda} \) has zeros of at most order one, this proves the first assertion (a) of the claim.

Concerning (b), we need to check for which \( K \)-type \( n \), the quotient \( \frac{c_n(\lambda)}{c_0(\lambda)} \) has a zero, for fixed \( \text{Re}(\lambda) > 0 \). It is clear that if \( \lambda \notin I_+ \), then \( \frac{c_n(\lambda)}{c_0(\lambda)} \) has no zeros, i.e., that Ker\( (J_{w,+,\lambda}) = 0 \).

For fixed \( \lambda = \frac{k}{2} \), \( k \in \mathbb{Z} + 1 \), the quotient \( \frac{c_n(\lambda)}{c_0(\lambda)} \) has zeros if, and only if, \( n \) is a \( K \)-type of \( D_{-k} \) and \( D_k \), i.e., Ker\( (J_{w,+,\lambda}) = D_{-k} \oplus D_k \). Thus, this implies (b).

By (b) and (19), we have that the operator \( \phi^{(0)}(+, \lambda) \) annihilates Ker\( (J_{w,+,\lambda}) \) for \( \sigma = + \) \( \in \hat{M} \) and \( \text{Re}(\lambda) > 0 \). By (a), we have that the order \( n \) is equal to one; thus, this condition is sufficient.

By arguing in a similar way as above for \( \sigma = - \) \( \in \hat{M} \) with \( | - | = 1 \), and \( \pi = D_k \in \hat{G}_d \), \( k \in \mathbb{Z} \setminus \{0\} \), with \( |\pi| = |k| + 1 \), we can conclude that \( \mathcal{A} \) satisfies the condition \( (D.b) \) of Theorem 4.

Now let us move to (Level 2).

**Theorem 7** (Intertwining conditions in (Level 2)) Let \( m \in \mathbb{Z} \) be a \( K \)-type, then, \( \psi \in \text{Hol}(\mathfrak{a}_C^+, H_{\infty}^{|m|}) \) satisfies the intertwining condition \( (D.2) \) of Definition 5 if and only if

1. \( J_{w,m,\lambda}(\lambda, \cdot) = \mathfrak{c}_m(\lambda)\psi(-\lambda, \cdot) \) for all \( \lambda \in \mathfrak{a}_C^+ \),
2. \( \psi(\lambda, \cdot) \in W_\lambda, \lambda \in I_{\pm} \), where \( W_\lambda \) is the invariant \( G \)-submodule of \( H_{\infty}^{\pm,\lambda} \) represented by the blue boxes in Fig. 1. Here the choice of \( \{\pm\} \) depends on the parity of \( m \).

Notice that, if \( W_\lambda \) is the whole colored blue box, then there are no intertwining conditions.

**Proof** We need to show that the conditions (2.a) and (2.b) correspond to the condition \( (D.2) \) in Definition 5. In fact, by Proposition 3(a), we have that (2.a) is a special case of (D.2).

Concerning (2.b), condition \( (D.2) \) says that for each \( W \) we have an intertwining condition corresponding to \( (D.b') \) in Theorem 6. Now we need to extract in which of these \( W_\lambda \), there is an intertwining condition. If the \( K \)-type \( m \) is in a closed \( G \)-submodule \( W_\lambda \) of \( H_{\infty}^{\pm,\lambda} \), then \( D_m^W \) is one-dimensional. Hence, by \( (D.2) \), \( \psi \) has values in this \( G \)-submodule \( W_\lambda \). By \( (D.b') \) in Theorem 6, we thus take the smallest closed proper invariant \( G \)-submodule of them. Otherwise, if the \( K \)-type is not in a closed \( G \)-submodule \( W_\lambda \) of \( H_{\infty}^{\pm,\lambda} \), then \( D_m^W \) is \{0\} and thus there are no intertwining conditions. Consequently, we obtain the boxes-pictures in Fig. 1.

The final step will be to move to (Level 3). Note that \( \text{Hom}_M(E_n, E_m) = \{0\} \), if \( n \equiv m \) (mod 2).

**Definition 8** Let \( n \equiv m \) (mod 2). We define the polynomial \( q_{n,m} \) in \( \lambda \in \mathfrak{a}_C^+ \) with values in \( \text{Hom}_M(E_n, E_m) \cong \mathbb{C} \) by

\[
q_{n,m}(\lambda) := \begin{cases} 
1, & \text{if } n = m \\
(\lambda + \frac{|m|+1}{2})(\lambda + \frac{m}{2}) \cdots (\lambda + \frac{-1}{2}), & \text{if } |n| > |m| \text{ and same signs} \\
(\lambda - \frac{|m|+1}{2})(\lambda - \frac{m}{2}) \cdots (\lambda - \frac{-1}{2}), & \text{if } |n| < |m| \text{ and same signs} \\
(\lambda + \frac{|m|-1}{2})(\lambda + \frac{|m|-3}{2}) \cdots (\lambda + \frac{1}{2}), & \text{else, with different signs} 
\end{cases}
\]

(20)
for \( m = 0 \):

\[
\begin{array}{cccccccc}
\ldots & 
\begin{array}{c}
\text{(boxes)}
\end{array} & 
\begin{array}{c}
\text{(pictures)}
\end{array} & 
\ldots
\end{array}
\]

\[ \lambda \]

for \( m \in 2 \mathbb{Z} \):

\( m > 0 \):

\[
\begin{array}{cccccccc}
\ldots & 
\begin{array}{c}
\text{(boxes)}
\end{array} & 
\begin{array}{c}
\text{(pictures)}
\end{array} & 
\ldots
\end{array}
\]

\[ \lambda \]

\( m < 0 \):

\[
\begin{array}{cccccccc}
\ldots & 
\begin{array}{c}
\text{(boxes)}
\end{array} & 
\begin{array}{c}
\text{(pictures)}
\end{array} & 
\ldots
\end{array}
\]

\[ \lambda \]

for \( m \in 2 \mathbb{Z} + 1 \):

\( m > 0 \):

\[
\begin{array}{cccccccc}
\ldots & 
\begin{array}{c}
\text{(boxes)}
\end{array} & 
\begin{array}{c}
\text{(pictures)}
\end{array} & 
\ldots
\end{array}
\]

\[ \lambda \]

\( m < 0 \):

\[
\begin{array}{cccccccc}
\ldots & 
\begin{array}{c}
\text{(boxes)}
\end{array} & 
\begin{array}{c}
\text{(pictures)}
\end{array} & 
\ldots
\end{array}
\]

\[ \lambda \]

Fig. 1 Boxes-pictures for \( G = \text{SL}(2, \mathbb{R}) \)

**Theorem 8** (Intertwining conditions in (Level 3)) Let \( n \equiv m \pmod{2} \) be two \( K \)-types. Then, \( \varphi \in \text{Hol}(a^\mathbb{C}_\mathbb{C}, \text{Hom}_M(E_n, E_m)) \) satisfies the intertwining condition \((D.3)\) of Definition 5 if, and only if, there exists an even holomorphic function \( h \in \text{Hol}(\lambda^2) \) such that

\[
\varphi(\lambda) = h(\lambda) \cdot q_{n,m}(\lambda), \quad \lambda \in a^\mathbb{C}_\mathbb{C},
\]

where \( q_{n,m} \) is the polynomial \((20)\).

**Proof** By Theorem 7, it is sufficient to prove that the conditions \((2.a)\) and \((2.b)\) correspond to \((21)\). In particular, we want to show that \((\lambda, k_\theta) \mapsto \varphi(\lambda)e^{in\theta}\) satisfies \((2.b)\) if, and only if, \( \varphi \) has zeros at the zeros of the polynomial \( q_{n,m} \). From Theorem 7 \((2.b)\), we know that the invariant \( G \)-submodules \( W_\lambda \) are represented by the boxes-pictures in Fig. 1. Thus, we need to check, where the \( K \)-type \( n \) is not in the colored blue invariant \( G \)-submodule \( W_\lambda \). We leave
it to the reader to check that this happens exactly at the zeros of \( q_{n,m} \). Thus, we can deduce that \( \varphi \) is of the form (21) with \( h \) an arbitrary holomorphic function.

Concerning the correspondence between the conditions (2.a) and (21), by Proposition 3(b), we observe that (2.a) corresponds to (D.3):

\[
(-1)^{(m-n)/2} \frac{c_n(\lambda)}{c_m(\lambda)} \varphi(\lambda) = \varphi(-\lambda), \quad \lambda \in \mathfrak{a}_C^*, n, m \in \mathbb{Z}. \tag{22}
\]

By using Definition 8, we observe that \( \frac{q_{n,m}(-\lambda)}{q_{n,m}(\lambda)} = (-1)^{(m-n)/2} \frac{c_n(\lambda)}{c_m(\lambda)} \), for \( \lambda \in \mathfrak{a}_C^*, n, m \in \mathbb{Z} \). Hence, we obtain

\[
\frac{\varphi(\lambda)}{q_{n,m}(\lambda)} = \frac{\varphi(-\lambda)}{q_{n,m}(-\lambda)}.
\]

This means that (22) is satisfied if and only if \( h(\lambda) = h(-\lambda) \) for \( \lambda \in \mathfrak{a}_C^* \). \( \square \)

We now have completely determined the Paley–Wiener–(Schwartz) spaces for \( G = \text{SL}(2, \mathbb{R}) \) in (Level 2) and (Level 3).

### 4.1 The case \( G = \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \)

Now let

\[
G := G' \times G' = \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}).
\]

Since \( K' = \text{SO}(2) \) is a maximal compact subgroup of \( G', K := K' \times K' \) is maximal compact in \( G \).

The irreducible representations of \( K \) are given by pairs \( (n_1, n_2), n_i \in \hat{K} \cong \mathbb{Z} \) [20, Sect. 2.36]. More precisely, we denote by a tuple of integers \( n := (n_1, n_2) \in \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}^2 \cong \hat{K} \) the \( K \)-representations on the vector space \( E_n := E_{n_1} \otimes E_{n_2} \cong \mathbb{C} \) with action \( [n_1, n_2] = n_1(k_1) \otimes n_2(k_2) \). The associated homogeneous line bundle over \( X := \mathbb{X} \times \mathbb{X} \) is denoted by \( \mathbb{E}_n \). For \( l, n \in \hat{K} \), we observe that \( \text{Hom}_M(E_l, E_n) = \{0\} \), if \( l_1 \not\equiv n_1 \) (mod 2) or \( l_2 \not\equiv n_2 \) (mod 2). Note that \( \mathfrak{a}_C^* \cong \mathbb{C} \times \mathbb{C} \). By using Definition 8, we define for \( l, n \in \mathbb{Z}^2 \)

\[
l_1 \equiv n_1 \) (mod 2), \quad l_2 \equiv n_2 \) (mod 2),
\]

the polynomial \( q_{l,n} \) given by

\[
q_{l,n}(\lambda_1, \lambda_2) := q_{l_1,n_1}(\lambda_1) \cdot q_{l_2,n_2}(\lambda_2), \quad (\lambda_1, \lambda_2) \in \mathbb{C} \times \mathbb{C} \cong \mathfrak{a}_C^*, \tag{24}
\]

where \( q_{l_1,n_1}, i = 1, 2 \), is the ‘intertwining’ polynomial (20).

**Theorem 9** (Intertwining condition in (Level 3)) Let \( l, n \in \mathbb{Z}^2 \) be two tuples of integers satisfying (23). Then, \( \varphi \in \text{Hol}(\mathfrak{a}_C^*), \text{Hom}_M(E_l, E_n) \) satisfies the intertwining condition (D.3) of Definition 5 if and only if there exists an holomorphic function \( h \in \text{Hol}(\lambda_1^2, \lambda_2^2) \), i.e., \( h(\lambda_1, \lambda_2) = h(-\lambda_1, -\lambda_2) = h(\lambda_1, -\lambda_2), \) such that

\[
\varphi(\lambda_1, \lambda_2) := h(\lambda_1, \lambda_2) \cdot q_{l,n}(\lambda_1, \lambda_2). \tag{25}
\]

To prove Theorem 9, we first need a density argument, which permits us to approximate the even holomorphic function \( h \) in (25) by even polynomials.

**Lemma 10** Consider the subset \( P \) of polynomial functions in \( \text{Hol}(\lambda_1^2, \lambda_2^2) \). Then, \( P \subset \text{Hol}(\lambda_1^2, \lambda_2^2) \) is dense with respect to uniform convergence on compact subsets of \( \mathbb{C}^2 \).
Proof Let \( h(\lambda_1, \lambda_2) \in \text{Hol}(\lambda_1^2, \lambda_2^2) \). Consider the Taylor series at the point \( 0 = (0, 0) \) in two variables \((\lambda_1, \lambda_2) \in \mathbb{a}_C^n\):
\[
\sum_{\alpha} a_{\alpha} \lambda^{\alpha} = \sum_{\alpha_1, \alpha_2} a_{\alpha_1, \alpha_2} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2},
\]
where \( a_{\alpha} \) are constants and the sum runs over multi-indices \( \alpha = (\alpha_1, \alpha_2), \alpha_j \in \mathbb{N}_0 \). Let \( |\alpha| = \alpha_1 + \alpha_2 \). Note that \( a_{\alpha_1, \alpha_2} = 0 \), if \( \alpha_1 \) or \( \alpha_2 \) is odd. Thus, the Taylor polynomials \( \sum_{|\alpha| \leq k} a_{\alpha} \lambda^{\alpha} \) belong to \( P \). The Taylor polynomials converge locally uniformly to \( h \) for \( k \) going to infinity. \( \square \)

Next, by using the Iwasawa decomposition of \( g = (g_1, g_2) \in G \), the ‘exponential’ function \( e_{\lambda,k}^{n} \) can be rewritten as follows:

**Proposition 11** For fixed \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{a}_C^n \) and \( k = (k_1, k_2) \in K \), the function \( e_{\lambda,k}^{n} \) in \( C^\infty(G) \) defined as in Definition 3 is a product of the corresponding functions on \( G' \):
\[
e_{\lambda,k}^{n}(g_1, g_2) = e_{\lambda_1,k_1}^{n_1}(g_1) \cdot e_{\lambda_2,k_2}^{n_2}(g_2), \quad (g_1, g_2) \in G.
\]

**Proof** Consider the Iwasawa decomposition of \( g = (g_1, g_2) = (n'_1 a_1 k'_1, n'_2 a_2 k'_2) = n'ak' \in G \) so that \( n' = (n'_1, n'_2) \in N, a = (a_1, a_2) \in A \) and \( k' = (k'_1, k'_2) \in K \). One can easily deduce that for \( \lambda = (\lambda_1, \lambda_2) \in \mathbb{a}_C^n \), we get
\[
e^{(\lambda+\rho) \log(a)} = e^{(\lambda_1+\rho) \log(a_1)+(\lambda_2+\rho) \log(a_2)} = a_1^{\lambda_1+\rho} \cdot a_2^{\lambda_2+\rho}.
\]

Hence, for \( g \in G \), we then have
\[
e_{\lambda,k}^{n}(g) = e_{\lambda,k}^{n}(g_1, g_2) = a_1^{\lambda_1+\rho} a_2^{\lambda_2+\rho} e_{\lambda_1,k_1}^{n_1}(k_1, k_2) = a_1^{\lambda_1+\rho} a_2^{\lambda_2+\rho} n_1(k_1)n_2(k_2)
\[
= a_1^{\lambda_1+\rho} a_2^{\lambda_2+\rho} e_{\lambda_1,k_1}^{n_1}(k_1) e_{\lambda_2,k_2}^{n_2}(k_2)
\[
= e_{\lambda_1,k_1}^{n_1}(g_1) \cdot e_{\lambda_2,k_2}^{n_2}(g_2).
\]

Let \( I_{PW_{n,H}}(\mathbb{a}_C^n) = \{ \varphi \in \text{Hol}(\mathbb{a}_C^n, \text{Hom}_M(E_l, E_n)) \mid \varphi \text{ satisfies } (D.3) \} \) be the ‘pre’-Paley–Wiener–Schwartz space. Note that
\[
I_{PW_{n}}(\mathbb{a}_C^n) \subset I_{PW_{n,H}}(\mathbb{a}_C^n) \subset I_{PW_{n,H}}(\mathbb{a}_C^n).
\]

**Proof of Theorem 9** It suffices to show that
(a) every function \( \varphi \in I_{PW_{n}}(\mathbb{a}_C^n) \) is of the form (25) and
(b) (inversely) if \( \varphi \) is of the form (25), then it is in \( I_{PW_{n,H}}(\mathbb{a}_C^n) \).

Let \( \varphi \in I_{PW_{n}}(\mathbb{a}_C^n) \). By the Paley–Wiener theorem 2, there exists \( f \in C_c^\infty(G, l, n) \) with \( I_{\mathcal{F}_n}(f) = \varphi \). By Fubini’s theorem and Proposition 11, we have
\[
I_{\mathcal{F}_n}(f)(\lambda) = \int_{G'} e_{\lambda_1,1}^{n_1}(g_1) \left( \int_{G'} e_{\lambda_2,1}^{n_2}(g_2) f(g_1, g_2) \, dg_2 \right) \, dg_1 = \int_{G'} e_{\lambda_1,1}^{n_1}(g_1) f_{\lambda_2}(g_1) \, dg_1 = I_{1_{\mathcal{F}_n}} f_{\lambda_2}(\lambda_1),
\]
(26)
where we set \( \tilde{f}_{\lambda_2}(g_1) := \int_G e^{\alpha n_2} f(g_1, g_2) \, dg_1 \) for some distribution \( f \). Note that \( \tilde{f}_{\lambda_2} \in C_c^\infty(X', \mathbb{H}_n) \). Similarly, if we fix \( \lambda_1 \in a_C^* \), we have \( \tilde{f}_{\lambda_1}(g_2) := \int_G e^{\alpha n_1} f(g_1, g_2) \, dg_1 \in C_c^\infty(X', \mathbb{H}_n) \) and \( i\mathcal{F}_n(f)(\lambda) = i_2 \mathcal{F}_n \tilde{f}_{\lambda_1}(\lambda_2) \). Thus, by Theorem 8, the Fourier transform has the form:

\[
i\mathcal{F}_n(f)(\lambda) = h_{\lambda_2}(\lambda_1) \cdot q_{l_1, n_1} = h_{\lambda_1}(\lambda_2) \cdot q_{l_2, n_2}(\lambda_2),
\]

where \( h_{\lambda_2} \) is an even holomorphic function in \( \lambda_2 \in \mathbb{C} \). In view of (24), we deduce that there exists \( h \in \text{Hol}(\mathbb{C}) \) such that

\[
\varphi(\lambda) = i\mathcal{F}_n f(\lambda) = h(\lambda_1, \lambda_2) \cdot q_{l_1, n}(\lambda_1, \lambda_2), \quad \lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2,
\]

as desired.

Concerning (b), let \( \varphi \) of the form (25). Then, for \( (\lambda_1, \lambda_2) \in \mathbb{C}^2 \)

\[
\varphi(\lambda_1, \lambda_2) = h(\lambda_1, \lambda_2) q_{l_1, n_1}(\lambda_1) q_{l_2, n_2}(\lambda_2).
\]

By Lemma 10, we can approximate \( h \) by linear combinations of products of monomials of the form \( \lambda_1^{\alpha_1} \) and \( \lambda_2^{\alpha_2} \), where \( \alpha_1 \) and \( \alpha_2 \) are even. By Theorem 8 and the Paley–Wiener–Schwartz theorem 2,

\[
\lambda^{\alpha} q_{l_1, n_1}(\lambda_1) = i_1 \mathcal{F}_n(f_i)(\lambda_1) \in i_1PWS_{n_1}(a_C^*)
\]

for some distribution \( f_i \in C_c^\infty(G', l_1, n_1) \). Consider now the tensor product of these two distributions:

\[
f_1 \otimes f_2 \in C_c^\infty(G, l, n).
\]

By the computations involving Fubini’s theorem from the beginning of the proof, we obtain that

\[
i\mathcal{F}_n(f_1 \otimes f_2)(\lambda) = i_1 \mathcal{F}_n(f_1)(\lambda_1) \cdot i_2 \mathcal{F}_n(f_2)(\lambda_2) = \lambda^{\alpha} q_{l_1, n}(\lambda).
\]

Hence \( \lambda^{\alpha} q_{l_1, n} \in iPWS_{n, H}(a_C^*) \subset iPWS_{n, H}(a_C^*) \). Since \( iPWS_{n, H}(a_C^*) \subset \text{Hol}(a_C^*) \) is closed with respect to uniform convergence on compact subsets, we conclude that \( \varphi = h \cdot q_{l_1, n} \in iPWS_{n, H}(a_C^*) \).

By Theorem 9, we have explicitly determined the Paley–Wiener(–Schwartz) spaces for \( G = \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \) in (Level 3).

Moreover, all the previous results can be generalized to \( G = \text{SL}(2, \mathbb{R})^d, d \geq 2 \).

5 The case \( G = \text{SL}(2, \mathbb{C}) \)

Let \( G = \text{SL}(2, \mathbb{C}) = \{ g \in \text{GL}(2, \mathbb{C}) \mid \det(g) = 1 \} \) be the special linear group of \( \mathbb{C}^2 \) with maximal compact subgroup

\[
K = \text{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \in \text{GL}(2, \mathbb{C}) \mid |\alpha|^2 + |\beta|^2 = 1 \right\}.
\]

Note that \( K \) is homeomorphic to the 3-sphere and therefore simply connected. Furthermore, we can take \( A \subset \text{SL}(2, \mathbb{R}) \subset G \) as in Sect. 4 and

\[
N := \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{C} \right\}.
\]

We identify \( a_C^* \cong \mathbb{C} \) by sending \( \lambda \) to \( \lambda(H) \), where

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in a.
\]

Note that \( \rho(H) = 2 \).

For the irreducible complex representations of \( K \) we have (e.g., [21, Sect. 5.7])

\[
\hat{K} = \{ \delta_n \mid n \in \mathbb{N}_0 \} \cong \mathbb{N}_0 \text{ with } d_{\delta_n} := \dim(\delta_n) = n + 1.
\]
The tensor product of two irreducible $K$-representations decomposes into irreducibles according to the classical Clebsch–Gordan rule (e.g., [21, 5.7.1 (1)]):

$$\delta_n \otimes \delta_m = \bigoplus_{0 \leq j \leq \min(n,m)} \delta_{n+m-2j}, \quad n, m \in \mathbb{N}_0.$$  \hspace{1cm} (28)

In addition, $M = Z_K(A) = \{ m_{\theta} : \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \mid \theta \in \mathbb{R} \}$ is abelian and a maximal torus in $K$. We parametrize $\hat{M} := \{ \sigma_l : l \in \mathbb{Z} \} \cong \mathbb{Z}$ by the integers with $\sigma_l(m_{\theta}) = e^{il\theta} \in \mathbb{U}(1)$. Moreover, let $\chi_n : M \to \mathbb{C}$ denote the character of the finite-dimensional irreducible representation $(\delta_n, E_n)$ of $K$. Then, the Weyl character formula for $m_{\theta} \in M$ (e.g., [10, Chap. V.6])

$$\chi_n(m_{\theta}) = \frac{e^{i(n+1)\theta} - e^{-i(n+1)\theta}}{e^{i\theta} - e^{-i\theta}} = \frac{\sin((n+1)\theta)}{\sin(\theta)} = e^{-in\theta} + e^{-i(n-2)\theta} + \ldots + e^{in\theta}$$

tells us that the weights of $(\delta_n, E_n)$ have the form $-n, -(n-2), \ldots, n-2, n$, each with multiplicity one. The following important result describes the reducibility of the principal series representations of $SL(2, \mathbb{C})$. We refer, for example, to Wallach’s book [21, Sect. 5.7] for a proof. Note that Wallach’s proof is also valid for $G$-representation on smooth vectors (see remark before Theorem 5).

**Theorem 12** (Structure of principal series representation of $SL(2, \mathbb{C})$) The principal series representations $H_{\infty}^{\sigma, \lambda}$ of $SL(2, \mathbb{C})$ is reducible if and only if $\lambda$ is real and

$$|\lambda| > |\sigma|, \quad |\lambda| - |\sigma| \text{ even integer.}$$

In this case, for $\lambda > 0$, there is a unique irreducible subrepresentation $R_{\sigma, \lambda}$ of each $H_{\infty}^{\sigma, \lambda}$. Then, we have

$$H_{\infty}^{-\sigma, -\lambda} = \begin{array}{c|c} R_{\sigma, \lambda} & F_{m,n} \\ \hline F_{m,n} & R_{\sigma, \lambda} \end{array} \quad H_{\infty}^{\sigma, \lambda} = \begin{array}{c} F_{m,n} \\ R_{\sigma, \lambda} \end{array}$$

where $m = \frac{\sigma + \lambda}{2} - 1$, $n = \frac{\lambda - \sigma}{2} - 1$ and $F_{m,n}$ is an irreducible finite-dimensional $G$-representation that is isomorphic to $\delta_m \otimes \delta_n$ as a $K$-representation. Moreover, there is an intertwining operator

$$L_{\sigma, \lambda} : H_{\infty}^{-\lambda, -\sigma} \longrightarrow H_{\infty}^{\sigma, \lambda}$$

so that $\ker(J_{w, \sigma, \lambda}) = \text{im}(L_{\sigma, \lambda}) = R_{\sigma, \lambda}$. In particular $R_{\sigma, \lambda}$ is isomorphic to $H_{\infty}^{\lambda, -\sigma}$. Here, $J_{w, \sigma, \lambda} : H_{\infty}^{\sigma, \lambda} \longrightarrow H_{\infty}^{-\sigma, -\lambda}$ denotes the Knapp–Stein intertwining operator defined in Definition 6 with $w = -1$ and $m_w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Furthermore, from the intertwining operator $L_{\sigma, \lambda}$, we can deduce the existence of further intertwining operators:
Fig. 2 Principal series representations, where the colored one indicate the intertwining relations that occurs between each others with the same colors.

Note that the operators $L_{\sigma,\lambda}$, $\tilde{L}_{\sigma,\lambda}$, $L'_{\sigma,\lambda}$ and $\tilde{L}'_{\sigma,\lambda}$ are precisely the Zelobenko operators (also called BGG-operators) for $G = \text{SL}(2, \mathbb{C})$ [2, 3, 23]. In Fig. 2, we illustrate the principal series representations (with regular integral infinitesimal character) in a grid, where the horizontal axis represents the values of $\lambda \in a^*_\mathbb{C}$ and the vertical one the values of $\sigma \in \hat{M}$. Note that inside the region $\{\pm |\sigma| > \lambda\}$, we have the irreducible principal series representations $H_{\infty}^{-\lambda, -\sigma}$, respectively, $H_{\infty}^{\lambda, \sigma}$ colored in gray and outside the reducible ones, colored in black.

**Example 5** Fix $(\sigma, \lambda) \in \hat{M} \times a^*_\mathbb{C}$ such that $\lambda - |\sigma| \in 2\mathbb{N}$, we have the special condition

$$L_{\sigma,\lambda} \circ \phi(-\lambda, -\sigma) = \phi(\sigma, \lambda) \circ L_{\sigma,\lambda}. \quad (29)$$
Theorem 13 (Intertwining condition in (Level 1)) For $r > 0$, let $A$ be the space of all

$$\phi \in \prod_{\sigma \in \hat{M}} \text{Hol}(a^\sigma_C, \text{End}(H^\sigma_\infty))$$

such that $\phi$ satisfies the corresponding growth condition as well as the two intertwining conditions (8) and (29). Then, $A$ satisfies the conditions of Theorem 4, this means that $A = PW_r(G)$.

Before we proceed with the proof of Theorem 13, let us first state the explicit expression of the Harish-Chandra c-function [4, App. 2] for $G = SL(2, \mathbb{C})$, which is given by the following formula, for $|\sigma| \leq n$, $\sigma \equiv n \pmod{2}$:

$$c_n(\sigma, \lambda) := \frac{\Gamma\left(\frac{1}{2}(\lambda + \sigma)\right)\Gamma\left(\frac{1}{2}(\lambda - \sigma)\right)}{\Gamma\left(\frac{1}{2}(\lambda + n + 2)\right)\Gamma\left(\frac{1}{2}(\lambda - n)\right)}, \quad \lambda \in a^\sigma_C.$$  

Consider an additional, not necessarily distinct, $K$-type $m$ and fix $\lambda \in a^\sigma_C$. Then, by using repeatedly the relation (17), we obtain for $n \equiv m \pmod{2}$, the following quotient:

$$\frac{c_n(\sigma, \lambda)}{c_m(\sigma, \lambda)} = \frac{\Gamma\left(\frac{1}{2}(\lambda + m + 1)\right)\Gamma\left(\frac{1}{2}(\lambda - m)\right)}{\Gamma\left(\frac{1}{2}(\lambda + n + 1)\right)\Gamma\left(\frac{1}{2}(\lambda - n)\right)} = \begin{cases} 
1, & \text{if } n = m \\
\frac{(\lambda + m)(\lambda + m - 2)(\lambda + m - 4)\cdots(\lambda + n + 2)}{(\lambda - n)(\lambda - (n + 2))\cdots(\lambda - (n + 2))^2}, & \text{if } n < m \\
\frac{(\lambda - n)(\lambda - (n + 2))\cdots(\lambda - (n + 2))\lambda(n + 2)\cdots(\lambda + m + 2)}{(\lambda + n)(\lambda + n - 2)\cdots(\lambda + m + 2)}, & \text{if } n > m. 
\end{cases} \quad (30)$$

Hence, we can directly see that the quotient has zeros in $\{-m, -m + 2, \ldots, n - 2\}$ and poles in $\{m, m + 2, \ldots, n + 2\}$ for $n < m$ and inversely for $n > m$.

Proof of Theorem 13. $A$ is a $K \times K$-invariant closed linear subspace. We proceed similar as in the proof of Theorem 6. Note that $\hat{G}_d = \emptyset$. Consider $\phi \in A$ such that for $\sigma \in \hat{M}$, the assumption $(D.b)(i)$ of Theorem 4 is satisfied:

$$\phi_{\sigma'} = 0 \text{ for all } \sigma' \in \hat{M} \text{ with } |\sigma'| > |\sigma|. \quad (31)$$

Analogous to the proof of Theorem 6, we need to check that:

(a) for $\text{Re}(\lambda) > 0$, the intertwining operator $J_{w, \sigma, \lambda}$ has a zero of order at most one.

(b) the kernel of $J_{w, \sigma, \lambda}$ is equal to $0$ or $R^{\sigma, \lambda}$ for $\text{Re}(\lambda) > 0$.

The minimal $K$-type of the principal series representation $H^{\sigma, \lambda}_{C, \infty}$ is $n = |\sigma|$; hence, its Harish-Chandra c-function (5) $c_{\sigma, \sigma}$ is regular for $\text{Re}(\lambda) > 0$. Let $n \in \hat{K}$, then we see that for $n \geq |\sigma|$ and for $\text{Re}(\lambda) > 0$:

$$\frac{c_n(\sigma, \lambda)}{c_\sigma(\sigma, \lambda)} = \frac{(\lambda - n)(\lambda - (n + 2))(\lambda - (n + 4))\cdots(\lambda - (\sigma + 2))}{(\lambda + n)(\lambda + n - 2)(\lambda + n - 4)\cdots(\lambda + \sigma + 2)}$$

is also regular and has no poles, but zeros $\lambda \in [n, n + 2, \ldots, \sigma + 2]$ of first order. Hence, due (9), $J_{w, \sigma, \lambda}$ has zeros of order one, this proves the first assertion (a) of the claim.

By Theorem 12, we have $\text{Ker}(J_{w, \sigma, \lambda}) = \text{Im}(L_{\sigma, \lambda}) = R^{\sigma, \lambda}$, thus, this implies (b).

Now, by putting everything together and using the intertwining condition (29) as well as (31), we have for each $\text{Re}(\lambda) > 0$ with $|\lambda| - |\sigma| \in 2\mathbb{N}$ that

$$\phi_{\sigma}(\lambda) \circ L_{\sigma, \lambda} \circ L_{\sigma, \lambda}^{-\lambda}(\sigma) = 0.$$
Thus, by (b) and the assumption, we deduce that the operator $\phi_\sigma^{(0)}(\lambda)$ annihilates $\text{Ker}(J_{w,\sigma,\lambda}) = \text{Im}(L_{\sigma,\lambda}) = R^{n,\lambda}$ for $|\lambda| > |\sigma|$ and $|\lambda| - |\sigma|$ even. Moreover, by (a), this condition is sufficient since the order $m$ is one. This completes the proof. \hfill $\square$

Now let us move to (Level 2) and state the corresponding intertwining conditions there. In fact, this will determine explicitly the Paley–Wiener(–Schwartz) spaces for $G = SL(2, \mathbb{C})$ in (Level 2). In order to distinguish between representation spaces of $K$ and $M$, we denote $E_{\delta_n}$ by $E_n$, while the one-dimensional space $E_{\sigma_1}$ is denoted by $C_1$.

**Theorem 14** (Intertwining condition in (Level 2)) Let $n \in \mathbb{N}_0$ be a $K$-type and $k, l \in \mathbb{M} \cong \mathbb{Z}$ so that $n \geq |k|, |l|, l > |k|$ and $k \equiv l \equiv n (\text{mod } 2)$. Consider the operator

$$l^n_{k,l} : \text{Hom}_M(E_n, C_{-l}) \rightarrow \text{Hom}_M(E_n, C_k)$$

defined as in Example 1 corresponding to $L = L_{k,l}$ and $\tau = \delta_n$. Then, $\psi \in \text{Hol}(a_C^*, H_{\infty}^{1|l})$ satisfies the intertwining condition (D.2) of Definition 5 if, and only if,

$$J_{w,n,\lambda} \psi(\lambda) = \begin{pmatrix} c_n(n, \lambda) & 0 & \cdots & 0 \\ 0 & c_n(n - 2, \lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n(-n, \lambda) \end{pmatrix} \psi(-\lambda) \text{ for all } \lambda \in a_C^*,$$

(2.a) $J_{w,n,\lambda} \psi(\lambda) = \begin{pmatrix} c_n(n, \lambda) & 0 & \cdots & 0 \\ 0 & c_n(n - 2, \lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c_n(-n, \lambda) \end{pmatrix} \psi(-\lambda) \text{ for all } \lambda \in a_C^*,$$

(2.b) $L_{k,l}(t \circ \psi(-k)) = l^n_{k,l}(t) \circ \psi(l) \text{ for all } \lambda \in a_C^* \text{ and } t \in \text{Hom}_M(E_n, E_{-l}).$

**Proof** By Proposition 3, we have that (2.a) corresponds to (6); hence, it corresponds to the intertwining condition (D.2) in Definition 5. Similar for (2.b), which is, by Example 1 (6), a special intertwining condition of (D.2). Hence, we have equivalence between the conditions (2.a), (2.b) and (D.2). \hfill $\square$

In order to move to (Level 3), let us first consider the case where the two $K$-types $(n, E_n)$ and $(m, E_m)$ are equal and then progress to the case of distinct $K$-types.

**5.1 Initial case: the $K$-types $m$ and $n$ are equal**

**Definition 9** Let $(m, E_m)$ be a fixed $K$-type and $k = -m, -(m-2), \ldots, m-2, m$. We define $\mathcal{A}_m$ as the space of all elements in $\text{Hol}(a_C^*, \text{End}_M(E_m))$, which are given by holomorphic functions $\varphi_k : a_C^* \rightarrow \mathbb{C}$, ordered to a $(m + 1) \times (m + 1)$ diagonal matrix with respect to $M$-weight vectors as a basis of $E_m$:

$$\varphi := \begin{pmatrix} \varphi_m(\lambda) & 0 & \cdots & 0 \\ 0 & \varphi_{m-2}(\lambda) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_{-m}(\lambda) \end{pmatrix} \in \text{Hol}(a_C^*, \text{End}_M(E_m))$$

such that

$$\varphi_k(\lambda) = \varphi_{-k}(-\lambda), \text{ for } \lambda \in a_C^*$$

$$\varphi_k(l) = \varphi_l(k), \text{ for } k \equiv l \equiv m (\text{mod } 2) \text{ and } |k|, |l| \leq m. \quad (32)$$

Note that $\mathcal{A}_m \subset \text{Hol}(a_C^*, \text{End}_M(E_m))$ is an algebra. Let $PWS_m, H(a_C^*)$ be the pre-Paley–Wiener–Schwartz space consisting of all $\varphi \in \text{Hol}(a_C^*, \text{End}_M(E_m))$ satisfying the intertwining conditions (D.3) of Definition 5.
Theorem 15  With the previous notations, we have that
\[ mA_m \cong mPW_{\mathcal{A}m,H}(a^C_\mathbb{C}). \]

**Proof** We need to check that the intertwining conditions (32) of \( mA_m \) correspond to the intertwining condition (D.3) of \( mPW_{\mathcal{A}m,H}(a^C_\mathbb{C}) \). More precisely, by Example 1, it suffices to show that the intertwining condition (7) for \( L = J_{w,\alpha,\lambda} \) and \( L = L'_{\alpha,\lambda} \) corresponds to (32).

We know from Proposition 3 with \( m = \gamma = \tau \in \mathbb{N}_0 \) that
\[ \delta_m(m_w)^{-1} \phi(\lambda) \delta_m(m_w) = \phi(-\lambda), \quad \lambda \in a^C_\mathbb{C}, \phi \in \text{Hol}(a^C_\mathbb{C}, \text{End}_M(E_m)). \]

Note that the complex hull of \( a \) is the sum of \( a \) and \( m \). This means that the Weyl group \( W_A \) acts on \( a \) as well as on \( m \) by \(-1\) (thus also on \( \text{im} \) by \(-1\)). Therefore, the matrix diag(\( \phi_m(\lambda), \ldots, \phi_m(-\lambda) \)) is reversed by conjugation by \( \delta_m(m_w) \), i.e., we get diag(\( \phi_m(-\lambda), \ldots, \phi_m(-\lambda) \)). Hence, \( \phi_{-k}(-\lambda) = \phi_k(\lambda), \) for all \( |k| \leq m \) and \( \lambda \in a^C_\mathbb{C}. \)

Let \( |l|, |k| \leq m \) and \( k \equiv l \equiv m \) (mod 2). Let \( I_{k,l}^m \) as in Theorem 14. We know that for \( t \in \text{Hom}_M(E_m, \mathbb{C}_-l) \) we have by (7) for \( m = \gamma = \tau \in \mathbb{N}_0 \) and \( l > |k| \)
\[ I_{k,l}^m(t \circ \phi(-k)) = I_{k,l}^m(t \circ \phi(l)). \]

Let \( t \in \text{Hom}_M(E_m, \mathbb{C}_{-l}) \) and \( t' \in \text{Hom}_M(E_m, \mathbb{C}_l) \) be such that \( \phi_{-l} = t \circ \phi \) and \( \phi_k = t' \circ \phi \). We have that \( I_{k,l}^m(t(l)) = c \cdot t'(l) \), where \( c \in \mathbb{C} \). Note that \( c \) is non-zero. In fact, by Theorem 12, the intertwining operator \( L_{k,l} \) on the \( K \)-type \( m \) is not zero, hence \( I_{k,l}^m \) too. Consequently, we have
\[ I_{k,l}^m(t \circ \phi(-k)) = I_{k,l}^m(t \circ \phi(l)) \iff c \cdot \phi_{-l}(-k) = c \cdot \phi_k(l) \iff \phi_{-l}(-k) = \phi_k(l). \]

(33)

For \( 0 \leq l < |k| \) we consider \( L_{l,|k|}^1 \) instead. Combined with \( \phi_{-l}(-k) = \phi(l) \), we obtain (33) for every pair \( k, l \) as above. This completes the proof. \( \square \)

We also consider the corresponding situation in polynomial functions:
\[ mPol_m := \{ \phi \in \text{Pol}(a^C_\mathbb{C}, \text{End}_M(E_m)) \mid \phi \text{ satisfies (32)} \}. \]

It follows from Theorem 15 that \( mPol_m \) is equal to the vector space \( mPW_{\mathcal{A}m,0}(a^C_\mathbb{C}) \). We will sometimes write elements of \( mPol_m \) and \( mA_m \) as functions of two variables:
\[ \phi(\lambda, k) := \phi_k(\lambda). \]

We consider the subalgebra of \( mPol_m \) generated by \( \lambda^2 + k^2 \). It is the subalgebra generated by the Fourier image of the Casimir operator and isomorphic to \( \text{Pol}(\mathbb{C}) \). Thus, we can view \( mPol_m \) as a \( \text{Pol}(\mathbb{C}) \)-module. Similarly, \( mA_m \) has the structure of \( \text{Hol}(\mathbb{C}) \)-module. Here, \( h \in \text{Hol}(\mathbb{C}) \) acts on \( mA_m \) by
\[ (h \cdot \phi)_k(\lambda) = h(\lambda^2 + k^2)\phi_k(\lambda), \quad h \in \text{Hol}(\mathbb{C}), \phi \in mA_m. \]

**Theorem 16** The algebra \( mA_m \) is a free \( \text{Hol}(\mathbb{C}) \)-module with the \( m+1 \) generators \( (k\lambda)^l \in mPol_m \subseteq mA_m, l = 0, \ldots, m. \) Furthermore, we have
\[ mA_m \cong \text{Hol}(\mathbb{C}) \otimes_{\text{Pol}(\mathbb{C})} mPol_m. \]

(34)

Analogously, \( mPol_m \) is a free \( \text{Pol}(\mathbb{C}) \)-module with the same generators as \( mA_m \).

Note that Theorem 16 also tells that the two elements \( \lambda^2 + k^2 \) and \( kl \) generate \( mPol_m \) as an algebra. Observe also that \( mPol_m \) is isomorphic to \( \mathcal{D}_G(E_m, E_m) \), the set of all invariant differential operators \( D : C^\infty(X, E_m) \rightarrow C^\infty(X, E_m) \) \([16, \text{Sect. 7}].\)
Proof Consider \( \varphi \in \mathcal{A}_m \). It is sufficient to show the existence and the uniqueness of holomorphic functions \( h_0, \ldots, h_m \in \text{Hol}(\mathbb{C}) \) so that

\[
\varphi_k(\lambda) := \sum_{l=0}^{m} h_l(\lambda^2 + k^2) \cdot (k\lambda)^l, \quad \text{for } k = -m, \ldots, m \tag{35}
\]

and

\[
\varphi \in \text{Pol}_m \text{ implies } h_l \in \text{Pol}(\mathbb{C}), \quad \text{for } l = 0, \ldots, m. \tag{36}
\]

Then, \( \mathcal{A}_m \) is a free \( \text{Hol}(\mathbb{C}) \)-module with generators \( (k\lambda)^l, l = 0, \ldots, m \). Similarly, for \( \text{Pol}_m \). Note that \( \text{Hol} \otimes_{\text{Pol}} \text{Pol} \cong \text{Hol} \). Since there are \( m + 1 \) free generators, we have

\[
\mathcal{A}_m \cong \text{Hol}(\mathbb{C})^{m+1} \text{ and } \text{Pol}_m \cong \text{Pol}(\mathbb{C})^{m+1}.
\]

Thus, also (34) follows.

For the existence of \( h_l \in \text{Hol}(\mathbb{C}), l = 0, \ldots, m \), we proceed by a step two induction on \( m \in \mathbb{N}_0 \).

For \( m = 0 \), we have

\[
\mathcal{A}_0 = \{ \varphi_0 \in \text{Hol}(\mathbb{C}) \mid \varphi_0(\lambda) = \varphi_0(-\lambda), \forall \lambda \in \mathbb{C} \}.
\]

We see immediately that there is exactly one holomorphic function \( h_0 \) such that \( \varphi_0(\lambda) = h_0(\lambda^2) \). \( \varphi_0 \) is a polynomial function if and only if \( h_0 \) is one.

For \( m = 1 \), we have that \( \mathcal{A}_1 = \{ \varphi_1, \varphi_{-1} \in \text{Hol}(\mathbb{C}) \mid \varphi_{-1}(\lambda) = \varphi_{1}(-\lambda) \forall \lambda \in \mathbb{C} \} \). \( \varphi_1 \) can be decomposed into an even and an odd part as follows:

\[
\varphi_1(\lambda) = \varphi_{1}^{\text{even}}(\lambda) + \varphi_{1}^{\text{odd}}(\lambda) = h_0(\lambda^2 + 1) + \lambda h_1(\lambda^2 + 1).
\]

Then, \( \varphi_{-1}(\lambda) = \varphi_{1}(-\lambda) = h_0(\lambda^2 + 1) - \lambda h_1(\lambda^2 + 1) \). Hence, this leads us to desired relation \( \varphi_k(\lambda) = h_0(\lambda^2 + k^2) + (k\lambda)h_1(\lambda^2 + k^2) \) for \( k = \pm 1 \). \( h_0 \) and \( h_1 \) are polynomials if and only if \( \varphi_{\pm1} \) are polynomials as well.

Assume now the existence of \( h_l \) satisfying (35) and (36) for \( m \) replaced by \( m - 2 \). Let \( \varphi \in \mathcal{A}_m \). We have

\[
\overline{\varphi} := \text{diag}(\varphi_{m-2}, \varphi_{m-4}, \ldots, \varphi_{-(m-4)}, \varphi_{-(m-2)}) \in \mathcal{A}_{m-2}
\]

so that, by induction hypothesis, there exists \( h_l \in \text{Hol}(\mathbb{C}) \) with

\[
\varphi_k(\lambda) = \sum_{l=0}^{m-2} h_l(\lambda^2 + k^2)(k\lambda)^l, \quad \text{for } |k| \leq m - 2,
\]

and \( h_l \in \text{Pol}(\mathbb{C}) \), if \( \varphi \in \text{Pol}_m \). Consider

\[
\tilde{\varphi} := \text{diag}(\tilde{\varphi}_m, \varphi_{m-2}, \ldots, \varphi_{-(m-2)}, \tilde{\varphi}_{-m}) \in \mathcal{A}_m
\]

with

\[
\tilde{\varphi}_{\pm m}(\lambda) := \sum_{l=0}^{m-2} h_l(\lambda^2 + m^2)(\pm m\lambda)^l.
\]

By taking the difference of \( \varphi \) and \( \tilde{\varphi} \), we get that

\[
\varphi - \tilde{\varphi} = \text{diag}(\varphi^+_m, 0, \ldots, 0, \varphi^+_{-m}) \in \mathcal{A}_m,
\]
where we have set $\varphi_{\pm m}(\lambda) := \varphi_{\pm m}(\lambda) - \varphi_{\pm m}(\lambda)$. Notice that $\varphi_{\pm m}(l) = 0$ for $|l| \leq m - 2$, $l \equiv m \pmod{2}$. We introduce the polynomial function

$$p_m(\lambda, k) := \prod_{|l| \leq m - 2, l \equiv m \pmod{2}} (k - l)(\lambda - l) \in \mathbb{P}_m \mathbb{P}_m.$$  

Note that $p_m(\lambda, k) \equiv 0$ for $|k| \leq m - 2$, $k \equiv m \pmod{2}$. Moreover, if $k = m$, then $p_m(\lambda, m) = c_m \prod_{|l| \leq m - 2, l \equiv m \pmod{2}} (m - l)$, where $c_m$ is a non-zero constant depending on the integer $m$. We conclude that there exist $h_0^+ \cdot h_1^+ \in \mathbb{P}(\mathbb{C})$, if $\varphi \in \mathbb{P}_m \mathbb{P}_m$ such that

$$\varphi^+_m(\lambda) = [h_0^+ (\lambda^2 + m^2) + h_1^+ (\lambda^2 + m^2)(m \lambda)]p_m(\lambda, m).$$

This implies that $(\varphi - \bar{\varphi})(\lambda, k) = [h_0^+ (\lambda^2 + k^2) + h_1^+ (\lambda^2 + k^2)(k \lambda)]p_m(\lambda, k)$. In addition, $(k - \lambda)(\lambda - l)(k + l)(\lambda + l) = (k^2 - l^2)(\lambda^2 - l^2) = (k \lambda)^2 - l^2(\lambda^2 + k^2) + l^4$ and thus $p_m(\lambda, k)$ is of the form:

$$p_m(\lambda, k) = (k \lambda)^{m-1} + \sum_{l \equiv m-1 \pmod{2}}^{m-3} p_l^m(\lambda^2 + k^2)(k \lambda)^l, \quad (37)$$

where $p_l^m$ are certain polynomials. We obtain

$$(\varphi - \bar{\varphi})(\lambda, k) = h_0^+ (\lambda^2 + k^2)(k \lambda)^{m-1} + h_0^+ (\lambda^2 + k^2) \sum_{l \equiv m-1 \pmod{2}}^{m-3} p_l^m(\lambda^2 + k^2)(k \lambda)^l$$

$$+ h_1^+ (\lambda^2 + k^2)(k \lambda)^{m-1} + h_1^+ (\lambda^2 + k^2) \sum_{l \equiv m-1 \pmod{2}}^{m-3} p_l^m(\lambda^2 + k^2)(k \lambda)^l+1.$$

This implies that $\varphi - \bar{\varphi}$ is of the desired form; hence, $\varphi$.

Concerning the uniqueness, we need to show that

$$\sum_{l=0}^{m} h_l(\lambda^2 + k^2)(k \lambda)^l = 0, \forall |k| \leq m, \lambda \in \mathbb{C}^*, \text{ implies } h_l = 0, l = 0, \ldots, m. \quad (38)$$

We proceed again by a two-step induction on $m \in \mathbb{N}_0$. For the initial cases $m = 0, 1$, the assertion is clear, see above. Assume that (38) holds true for $m - 2$ and let us prove that it holds for $m$. Let $\varphi \in \mathbb{P}_m \mathbb{P}_m$. By using the polynomial function (37), we have for $|k| \leq m - 2$:

$$0 = \sum_{l=0}^{m} h_l(\lambda^2 + k^2)(k \lambda)^l - [h_{m-1}(\lambda^2 + k^2) + (k \lambda) h_m(\lambda^2 + k^2) p_m(\lambda, k)]$$

$$= \sum_{l=0}^{m-2} h_l(\lambda^2 + k^2)(k \lambda)^l - h_{m-1}(\lambda^2 + k^2) \sum_{l \equiv m-1 \pmod{2}}^{m-3} p_l^m(\lambda^2 + k^2)(k \lambda)^l$$

$$- h_m(\lambda^2 + k^2) \sum_{l \equiv m-1 \pmod{2}}^{m-3} p_l^m(\lambda^2 + k^2)(k \lambda)^l+1.$$
in $m-2A_{m-2}$. By induction hypothesis, this implies that for $l \leq m-2$:

$$h_l(\mu) = \begin{cases} h_{m-1}(\mu) p^m_l(\mu), & l \equiv m - 1 \pmod{2} \\ h_m(\mu) p^m_{l-1}(\mu), & l \equiv m \pmod{2} \end{cases}$$

Then, for $k = m$, this implies

$$0 = \sum_{l=0}^m h_l(\lambda^2 + m^2)(k\lambda)^l = [h_{m-1}(\lambda^2 + m^2) + h_m(\lambda^2 + m^2)(m\lambda)]p_m(\lambda, m).$$

Since $p_m(\lambda, m)$ is not identical zero on $\lambda \in \mathfrak{a}^*_C$, we obtain that $h_{m-1}$ and $h_m$ are zero. Hence, $h_l = 0$, for $l \leq m$. This completes the proof. \hfill \Box

### 5.2 General case: the $K$-types $n$ and $m$ are distinct

Consider now two distinct $K$-types $(n, E_n)$ and $(m, E_m)$. Since $\text{Hom}_M(E_n, E_m) = 0$, for $n \neq m \pmod{2}$, we assume throughout the section that $n \equiv m \pmod{2}$. We define

$$nA_m := n \text{PWS}_{m,H}(\mathfrak{a}^*_C) \subset \text{Hol}(\mathfrak{a}^*_C, \text{Hom}_M(E_n, E_m))$$

and $n\text{Pol}_m := n \text{PWS}_{m,0}(\mathfrak{a}^*_C)$. Recall that $n \text{PWS}_{m,H}(\mathfrak{a}^*_C)$ denotes the pre-Paley–Wiener–Schwartz space and consists of all $\varphi \in \text{Hol}(\mathfrak{a}^*_C, \text{Hom}_M(E_n, E_m))$ satisfying the intertwining condition (D.3) of Definition 5. Moreover,

$$n \text{PWS}_{m,0}(\mathfrak{a}^*_C) = n \text{PWS}_{m,H}(\mathfrak{a}^*_C) \cap \text{Pol}(\mathfrak{a}^*_C, \text{Hom}_M(E_n, E_m)).$$

Note that after a choice of bases in $E_n, E_m$ consisting of $M$-weight vectors, we can write elements of $\text{Hol}(\mathfrak{a}^*_C, \text{Hom}_M(E_n, E_m))$ in the following way:

$$\varphi := \begin{cases} \left( \begin{array}{cccc} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \varphi_n(\lambda) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varphi_{-n}(\lambda) \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{array} \right)_{(m+1)\times(n+1)} & \text{if } n < m, \\
\left( \begin{array}{ccccccc} 0 & \cdots & \varphi_m(\lambda) & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \varphi_{-m}(\lambda) & \cdots & 0 \end{array} \right)_{(m+1)\times(n+1)} & \text{if } n > m. \end{cases}$$

Note that $n\text{Pol}_m$ is isomorphic to the set of all $G$-invariant differential operators $D_G(\mathfrak{E}_n, \mathfrak{E}_m)$ [16, Sect. 7]. We start with the case $|n - m| = 2$. 

\hfill Springer
Proposition 17 Let $m$ be a $K$-type. There exists a unique operator of first order $q_m^+$ in $m \text{Pol}_{m+2}$ (resp. $q_m^-$ in $m+2 \text{Pol}_m$), up to normalization. It corresponds to

$$
(\lambda + m + 2) \cdot \begin{pmatrix}
0 \cdots 0 \\
1 \cdots 0 \\
\vdots & \ddots & \vdots \\
0 \cdots 1 \\
0 \cdots 0
\end{pmatrix}_{(m+2) \times m}
$$

(resp. $(\lambda - (m + 2)) \cdot \begin{pmatrix}
0 & d(m, m) & \cdots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & d(m, -m) & 0
\end{pmatrix}_{m \times (m+2)}$

under some appropriate basis choice. Here $d(m, k) = (m + 2)^2 - k^2$.

Proof Consider the symmetric algebra $S(p)$ of $p \cong \mathbb{R}^3$ over $\mathbb{C}$. It follows from the Poincaré–Birkhoff–Witt theorem and the description of invariant differential operators in terms of $U(q)$ that the associated graded module of $m \text{Pol}_{m+2}$ is isomorphic to $[S(p) \otimes \text{Hom}_M(E_m, E_{m+2})]^K$ (e.g., [15, Folgerung 2.5] or [18]). Using the Clebsch-Gordan rule (28) we obtain

$$[S^{=1}(p) \otimes \text{Hom}(E_m, E_{m+2})]^K \cong [(S^0(p) \oplus S^1(p)) \otimes (E_2 \oplus E_4 \oplus \cdots \oplus E_{m+2})]^K$$

$$\cong [S^1(p) \otimes E_2]^K,$$

where $[S^1(p) \otimes E_2]^K$ is one-dimensional. Note that $S^1(p) \cong p^*_{m,1} \cong E_2$. This means that $m \text{Pol}_{m+2}$ contains exactly one element $q_m^+$ of filter degree one (up to normalization). After an appropriate choice of basis, we can find basis element such that

$$q_m^+(\lambda) = \begin{cases} 
\hat{c}(m, k), & \text{or} \\
\lambda + c(m, k), & \text{at least for one k}.
\end{cases}$$

Here, $c(m, k), \hat{c}(m, k)$ are constants depending on $k$ and $m$. Consider $\varphi_k \in m \text{A}_{m+2}$. By Proposition 3 (b) and formula (30), we have that

$$\varphi_{-k}(-\lambda) = (-1)^{(m+2-m)/2} \frac{c_{m+2}(k, \lambda)}{c_{m}(k, \lambda)} \varphi_k(\lambda) = (-1)^{(m+2-m)/2} \frac{\Gamma(\frac{k}{2} + m + 2)}{\Gamma(\frac{k}{2} + m + 2)} \Gamma(\frac{k}{2} - m - 1)} \varphi_k(\lambda)$$

$$= (-1)^{(m+2-m)/2} \frac{(\lambda - (m + 2))}{(\lambda + (m + 2))} \varphi_k(\lambda)$$

$$= \frac{(-\lambda + (m + 2))}{(\lambda + (m + 2))} \varphi_k(\lambda).$$

Since $\varphi_{-k}$ has no singularities, we conclude that $\varphi_k$ has a zero at $-(m + 2)$. Applying this to $\varphi = q_m^+$, we see that

$$q_m^+_{m,k}(\lambda) = \begin{cases} 
0, & \\
\lambda + (m + 2).
\end{cases}$$

Analogously, there exists exactly one (up to normalization) $q_m^- \in m+2 \text{A}_m$ of degree 1. We consider $q_m^+ q_m^- \in m+2 \text{A}_{m+2}$. Then, $q_m^+ q_m^-(\lambda, \pm(m+2)) = 0$. This implies that $q_m^+ q_m^-(\pm(m+2), k) = 0$ for $|k| \leq m, k \equiv m \pmod{2}$ of degree 2. Hence, $q_m^+ q_m^-(\lambda, k) = d(m, k)(\lambda^2 -$
\( (m + 2)^2 \) for some constants \( d(m, k) \). Since \( q_m^- = c \cdot (q_m^\dagger)^* \), where \(^*\) stands for the adjoint \( p^*(\lambda) := p(-\overline{\lambda})^* \), we have \( d(m, k) \neq 0 \) for at least one \( k \). In addition, by (32)

\[
d(m, k) = \begin{cases} 
    d(m, 0) \cdot \frac{(m+2)^2-k^2}{(m+2)^2-k^2}, & \text{if } m \text{ even} \\
    d(m, 1) \cdot \frac{(m+2)^2-k^2}{(m+2)^2-k^2}, & \text{if } m \text{ odd}.
\end{cases}
\]

We conclude (up to normalization) that \( q^+_m(\lambda, k) = \lambda + c(m, k) = \lambda + (m + 2)^2 \) and

\[
q^-_m(\lambda, k) = ((m + 2)^2 - k^2)(\lambda - (m + 2)).
\]

\[\square\]

Now we consider general \( n, m \in \mathbb{N}_0, n \equiv m \pmod{2} \).

**Definition 10** Consider \( q^\pm_m \) and \( q^\pm_n \) as in Proposition 17. We define the polynomial \( q_{n,m} \in \mathcal{P}_{n,m} \) by

\[
q_{n,m} = \begin{cases} 
    q^+_{m-2} \cdot q^+_{m-4} \cdot q^+_{m-2} \cdot q^+_n, & \text{if } n < m \\
    q^-_{m+2} \cdot q^-_{m+4} \cdot q^-_{m+2} \cdot q^-_n, & \text{if } n > m \\
    \text{Id.} & \text{if } n = m.
\end{cases}
\]

Finally, the following theorem gives explicitly the Paley–Wiener(–Schwartz) spaces in (Level 3) for \( G = \text{SL}(2, \mathbb{C}) \).

**Theorem 18** (Intertwining condition in (Level 3)) Let \( n, m \in \mathbb{N}_0 \) be two \( K \)-types, which are not necessarily distinct, and let \( l := \min(n, m) \). Then, \( \mathcal{P}_{n,m} \) (resp. \( \mathcal{A}_m \)) is a free \( \mathcal{P}_l \) (resp. \( \mathcal{A}_l \))-module with generator \( q_{n,m} \). This means that there exists a unique function \( h \in \mathcal{A}_l \) such that

\[
n \mathcal{A}_m \ni \varphi(\lambda) = \begin{cases} 
    h(\lambda)q_{n,m}(\lambda), & \text{if } m < n \\
    q_{n,m}(\lambda)h(\lambda), & \text{if } m > n
\end{cases}
\]

for \( \lambda \in \mathfrak{a}_C^n \).

Moreover, if \( L = \max(n, m) \), then \( \mathcal{P}_{n,m} \) (resp. \( \mathcal{A}_m \)) is a \( L \mathcal{P}_l \) (resp. \( L \mathcal{A}_l \))-module generated by \( q_{n,m} \).

**Proof** Consider the case \( m < n \), it suffices to prove that

(a) there exists a unique \( h \in \mathcal{A}_m \) such that \( \varphi = h \cdot q_{n,m} \in \mathcal{A}_m \),

(b) there exists a \( \tilde{h} \in \mathcal{A}_n \) such that \( \varphi = q_{n,m} \cdot \tilde{h} \in \mathcal{A}_n \).

The polynomial case as well as \( m > n \) can be proved in a similar way. Consider

\[
q_{n,m} \mathcal{A}_m \subset \{ g \in \mathcal{A}_n \mid g_k = 0, \forall |k| > m \} \subset \mathcal{A}_n.
\]

We also have zeros between the lines \(-m\) and \(m\), this means \( g_k(0) = 0, k = \pm(m + 2), \pm(m + 4), \ldots, \pm n \). Let \( \varphi \in \mathcal{A}_m \). Every component of \( g = q_{m,n} \cdot \varphi \) has zeros as described above and \( q_{m,n} \) has zeros only at negative \( \lambda \). It follows that every component \( \varphi \) has zeros at \( m + 2, \ldots, n \). Hence, there exists a unique \( h \in \text{Hol}(\mathfrak{a}_C^n, \text{End}(E_m)) \) so that

\[
\varphi = h \cdot q_{n,m}.
\]

Since \( g \) satisfies the intertwining conditions \( g_k(l) = g_l(k), g_k(\lambda) = g_{-k}(-\lambda) \), and

\[
q_{n,m}(k, l)q_{n,m}(k, \lambda) = q_{m,n}(\lambda, k)q_{n,m}(k, -\lambda),
q_{n,m}(k, l)q_{n,m}(k, l) = q_{m,n}(l, k)q_{n,m}(l, k)
\]
we see that 

\[ h_l(k) = h_k(l) \text{ and } h_k(\lambda) = h_{-k}(-\lambda) \]  

for \( k \equiv l \equiv m \pmod{2} \), \(|k| \leq m\) and \( \lambda \in \mathfrak{a}_C^\ast \). This proves (a).

Concerning (b), we know from (a) that \( \varphi = h \cdot q_{m,n} \). Thus, we need to find

\[ \tilde{h} = \text{diag}(\tilde{h}_n, \ldots, \tilde{h}_{m+2}, \tilde{h}_m, \ldots, \tilde{h}_{-(m-2)}, \ldots, \tilde{h}_{-n}) \in {}_nA_n \]

with \( \tilde{h}_k = h_k \) for \(|k| \leq m\). Note that the crucial condition to be satisfied is \( \tilde{h}_k(l) = \tilde{h}_l(k) \).

Then, \( \varphi = {}_nq_{m,n} \cdot \tilde{h} \). By using interpolation polynomials, we define recursively

\[
\begin{align*}
\tilde{h}_{m+2}(\lambda) &:= \sum_{i=-m}^{m+2} \tilde{h}_i(m+2) \prod_{l=-m}^{m+2} \left( \frac{\lambda - l}{i - l} \right) \\
\tilde{h}_{m+4}(\lambda) &:= \sum_{i=-(m+2)}^{m+4} \tilde{h}_i(m+4) \prod_{l=-(m+2)}^{m+2} \left( \frac{\lambda - l}{i - l} \right) \\
&\vdots \\
\tilde{h}_{n-2}(\lambda) &:= \sum_{i=-(n-4)}^{n-4} \tilde{h}_i(n-2) \prod_{l=-(n-4)}^{n-2} \left( \frac{\lambda - l}{i - l} \right) \\
\tilde{h}_n(\lambda) &:= \sum_{i=-(n-2)}^{n-2} \tilde{h}_i(n) \prod_{l=-(n-2)}^{n-2} \left( \frac{\lambda - l}{i - l} \right),
\end{align*}
\]

and \( \tilde{h}_{-k}(\lambda) := \tilde{h}_k(-\lambda) \). Then, \( h \in {}_nA_n \).

We observe that an analogue of Theorem 16, in particular relation (34), is also true in general for distinct \( K \)-type \( n \) and \( m \).

**Corollary 19** With the notations above, \( {}_nA_m \cong \text{Hol}(\mathbb{C}) \otimes \text{Pol}(\mathbb{C}) \otimes {}_nP_{m,n} \).

**Proof** As a consequence of Theorem 18, the \( \text{Hol}(\mathbb{C}) \)-module \( {}_nA_m \) is isomorphic to \( {}_lA_l \), and \( {}_nP_m \) is isomorphic to \( {}_lP_l \) as \( \text{Pol}(\mathbb{C}) \)-module. Then, we can apply Theorem 16 and obtain the desired result.

**Acknowledgements** The second author is supported by the Luxembourg National Research Fund under the Project PRIDE15/10949314/GSM.

**Data availability** Availability of data and materials not applicable to this article as no datasets were generated or analyzed during the current study.

**References**

1. Arthur, J.: A Paley–Wiener theorem for real reductive groups. Acta Math. **150**, 1–89 (1983). https://doi.org/10.1007/BF02392967
2. Bernstein, I.N., Gel’fand, I.M., Gel’fand, S.I.: Structure of representations generated by vectors of highest weight. Funct. Anal. Appl. **5**, 1–8 (1971). https://doi.org/10.1007/BF01075841
3. Bernstein, I.N., Gel’fand, I.M., Gel’fand, S.I.: A certain category of \( g \)-modules. Funct. Anal. Appl. **10**, 87–92 (1976). https://doi.org/10.1007/BF01077933
4. Cohn, L.: Analytic Theory of the Harish–Chandra C-Function, Lecture Notes in Mathematics, vol. 429. Springer-Verlag (1974). https://doi.org/10.1007/bfb0064335
5. Delorme, P.: Théorème de type Paley–Wiener pour les groupes de Lie semi-simple réels avec une seule classe de conjugaison de sous groupes de Cartan, (in French). J. Funct. Anal. 47, 26–63 (1982). https://doi.org/10.1016/0022-1236(82)90099-4
6. Delorme, P.: Sur le théorème de Paley–Wiener d’Arthur. Ann. Math. 162, 987–1029 (2005). https://doi.org/10.4007/annals.2005.162.987. (in French)
7. Gangolli, R.: On the Plancherel formula and the Paley–Wiener theorem for spherical functions on semisimple Lie groups. Ann. Math. 93, 159–165 (1971). https://doi.org/10.2307/1970758
8. Helgason, S.: Geometric Analysis on Symmetric Spaces, vol. 39, 2nd edn. AMS Mathematical Surveys Monographs (1989). https://doi.org/10.1090/surv/039
9. Helgason, S.: Groups and Geometric Analysis, Integral Geometry, Invariant Differential Operators and Spherical Functions. AMS Mathematical Surveys Monographs, vol. 83 (2000)
10. Knapp, A.W.: Lie Groups Beyond an Introduction, Progress in Mathematics, 2nd edn. Birkhäuser. (2002) https://doi.org/10.1007/978-1-4757-2453-0
11. Knapp, A.W.: Representation Theory of Semisimple Groups: An Overview Based on Examples (PMS-36). Princeton University Press (2016). https://doi.org/10.1515/9781400883974
12. Knapp, A.W., Stein, E.M.: Intertwining operators for semisimple groups. Ann. Math. 93, 489–578 (1971). https://doi.org/10.2307/1970887
13. Knapp, A.W., Stein, E.M.: Intertwining operators for semisimple groups II. Invent. Math. 60, 9–84 (1980)
14. Lang, S.: SL_2(R), Graduate Texts in Mathematics, vol. 105. Springer-Verlag (1975). https://doi.org/10.1007/978-1-4612-5142-2
15. Olbrich, M.: Die Poisson–Tranformation für Homogene Vektorbündel, Doctoral Dissertation, HU Berlin (1995) (in German)
16. Olbrich, M., Palmirotta, G.: A topological Paley–Wiener–Schwartz theorem for sections over homogeneous vector bundles on G/K, preprint. (2022). arXiv:2202.06905
17. Olbrich, M., Palmirotta, G.: Solvability of invariant systems of differential equations on \( \mathbb{H}^2 \) and beyond, preprint. (2022). arXiv:2206.01835
18. Palmirotta, G.: Solvability of Systems of Invariant Differential Equations on Symmetric Spaces G/K, Doctoral Dissertation, University of Luxembourg. (2021). http://hdl.handle.net/10993/50041
19. van den Ban, E.P., Souaifi, S.: A comparison of Paley–Wiener theorems. J. Reine Angew. Math. 695, 99–149 (2014). https://doi.org/10.1515/crelle-2012-0105
20. Wallach, N.R.: Harmonic Analysis on Homogeneous Spaces, Pure and Applied Mathematics. Marcel Dekker Inc. (1973)
21. Wallach, N.R.: Real Reductive Groups I, Pure and Applied Mathematics. Academic Press (1988)
22. Wallach, N.R.: Real Reductive Groups II, Pure and Applied Mathematics. Academic Press (1992)
23. Zelobenko, D.P.: Operators of discrete symmetry for reductive Lie groups. Math. USSR-Izvestiya 10, 1003–1029 (1976). https://doi.org/10.1070/IM1976v010n05ABEH001823

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.