Working directly with probabilities in quantum field theory

R Dickinson¹, J Forshaw¹ and P Millington²*

¹Consortium for Fundamental Physics, School of Physics and Astronomy, University of Manchester, Manchester M13 9PL, United Kingdom
²School of Physics and Astronomy, University of Nottingham, Nottingham NG7 2RD, United Kingdom
E-mail: robert.dickinson-2@manchester.ac.uk, jeff.forshaw@manchester.ac.uk, p.millington@nottingham.ac.uk

Abstract. We present a novel approach to computing transition probabilities in quantum field theory, which allows them to be written directly in terms of expectation values of nested commutators and anti-commutators of field operators, rather than squared matrix elements. We show that this leads to a diagrammatic expansion in which the retarded propagator plays a dominant role. As a result, one is able to see clearly how faster-than-light signalling is prevented between sources and detectors. Finally, we comment on potential implications of this approach for dealing with infra-red divergences.

1. Introduction

The utility of scattering-matrix theory has biased the development of techniques in quantum field theory towards the calculation of transition amplitudes. However, isolated transition amplitudes are not physical observables and they often suffer from artefacts that are eliminated only after we combine individual amplitudes into transition probabilities. For instance, the soft and collinear infra-red (IR) divergences occurring in gauge amplitudes are cancelled by the interference of virtual and real emissions by means of the Bloch-Nordsieck [1] and Kinoshita-Lee-Nauenberg theorems [2, 3] (in the case of spin-1) or the Weinberg soft graviton theorem [4] (in the case of spin-2). In non-Abelian gauge theories, unphysical polarisation states are removed by the contributions from Faddeev-Popov ghosts [5]. It seems reasonable therefore to pursue means of directly calculating transition probabilities that bypass the amplitude level altogether, the hope being that such artefacts never appear explicitly.

Another (not unrelated) artefact of dealing with amplitudes, and one that we will focus on in this note, is apparent a-causal behaviour, i.e. behaviour seemingly at odds with Einstein causality and the forbiddance of faster-than-light signalling. The archetypal example of such a signalling process is the famous Fermi two-atom problem [6]. Fermi considered two point-like atoms, A and B, separated by a distance $R$. Atom A is initially prepared in an excited state, and atom B is initially prepared in its ground state. Fermi calculated the probability that, after a time $T$, atom A would be found in its ground state, and atom B would be found in an excited state, following the exchange of a photon. Fermi claimed to prove that this probability

*Talk presented by P Millington.
Figure 1. The four amplitudes relevant to the Fermi problem. Fermi considered diagram (b) in isolation, and it can be shown that $|\langle b \rangle|^2 \neq 0$ for $T < R/c$. It is only in the combination $|\langle b \rangle|^2 + |\langle c \rangle|^2 + 2 \text{Re}\{\langle a \rangle\langle d \rangle^*\}$ that the a-causal parts cancel. Notice that this cancellation relies on the interference of diagrams involving only unobserved emissions.

is strictly zero for all times $T < R/c$. Fermi was in fact wrong, and one should expect a non-zero probability for all $T$, since the simultaneous measurement of the states of atoms A and B does not constitute a local measurement. Instead, we should ask for the probability that atom B is found in an excited state after a time $T$ without making any restriction on the states of atom A or the electromagnetic field [7, 8, 9]. (A recent introduction to the Fermi problem and an overview of its history can be found in Ref. [10].) As we will see, this local measurement is independent of the state of atom A for all times $T < R/c$, therefore being manifestly causal in the weak sense (see, e.g., Ref [11]).

The local measurement described above is fully inclusive over the states of atom A and the electromagnetic field. As such, we must account for unobserved emission of photons in the final state, and the four amplitudes relevant to the Fermi problem are shown in Fig. 1. By working directly at the level of the probability, we will find that we do not need to sum explicitly over these unobserved emissions [10]. This suggests that one may be able to write down semi-inclusive transition probabilities in which the Bloch-Nordsieck cancellation is applied implicitly, potentially having a significant impact on the way in which we deal with IR divergences in gauge theories. As a step towards this, we will conclude the present note by describing an example of a Fock-space projection operator for (semi-)inclusive observables (see Ref. [12]).

2. Causality

Causality is built into quantum field theory \textit{ab initio} through the vanishing of the equal-time commutator of field operators, in the case of bosons, or anti-commutator of field operators, in the case of fermions. This is the so-called \textit{microcausality condition}. For example, given a real scalar field $\phi_x$, we have

$$\{\phi_x, \phi_y\} = 0 \quad \forall (x-y)^2 < 0 \quad \text{(space-like)}.$$ (1)

However, it is the Feynman propagator

$$\Delta_{xy}^{(F)}(x,y) = \Delta^{(F)}(x,y) = \frac{1}{2} \text{sgn}(x_0-y_0) \langle \{\phi_x, \phi_y\} \rangle + \frac{1}{2} \langle \{\phi_x, \phi_y\} \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik(x-y)}}{k^2 - E_k^2 + i\epsilon}$$ (2)

that is ubiquitous in $S$-matrix theory and it has support over space-like separations due to the anti-commutator term $\langle \{\phi_x, \phi_y\} \rangle$. In Eq. (2), $E_k = \sqrt{k^2 + m^2}$ is the on-shell energy (for a field of mass $m$), and $\text{sgn}(x_0-y_0)$ is the signum function, which may be expressed in terms of unit-step functions as $\text{sgn}(x_0-y_0) \equiv \Theta(x_0-y_0) - \Theta(y_0-x_0)$. 


The fact that causality is not manifest at the amplitude level can be understood directly in terms of the Feynman prescription ($k_0^2 \to k_0^2 + i\epsilon$, $\epsilon = 0^+$), since this choice of analytic continuation allows us to Wick rotate between Minkowski space-time, which has causal structure, and four-dimensional Euclidean space, which does not. It is perhaps even more disconcerting that the $a$-causal part of the Feynman propagator actually corresponds to on-shell propagation, that is the part that we would normally associate with real particles. If causality is to be manifest then it is surely the retarded and advanced propagators

$$\Delta^{(R)}_{xy} = \Delta^{(A)}_{yx} = \frac{1}{i} \Theta(x_0 - y_0) \langle [\phi_x, \phi_y] \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot(x-y)}}{k_0^2 - E_k^2 + i\epsilon \text{sgn}(k_0)}$$

that should play the dominant role, since these have support only over light- and time-like separations. We therefore require that the quantity $\lambda(y) \equiv x_0 \approx y$, where $\approx$ means $|x_0 - y_0| < \epsilon$. Taking a local interaction Hamiltonian density $H_z^{\text{int}}(\lambda) = \lambda \phi_z^2/4!$, we imagine promoting the coupling $\lambda$ to one of two space-time dependent functions $\lambda_x \equiv \lambda(x)$ or $\lambda_x' \equiv \lambda'(x)$. If the coupling functions $\lambda_x$ and $\lambda_x'$ coincide for all $x_0$ earlier than some time $t$, the product $S[\lambda'] S^\dagger[\lambda]$ must be independent of the behaviour of these functions for all times $x_0 < t$ by unitarity. Letting $\lambda_x' = \lambda_x + \delta\lambda_x$, where the infinitesimal variation $\delta\lambda_x$ is only non-zero for $x_0 > t$, we expand

$$S[\lambda'] S^\dagger[\lambda] = S[\lambda] S^\dagger[\lambda] + \frac{\delta S[\lambda]}{\delta \lambda_y} \frac{\delta S^\dagger[\lambda]}{\delta \lambda_y} S^\dagger[\lambda] \quad \text{and} \quad \frac{\delta S[\lambda]}{\delta \lambda_y} \frac{\delta S^\dagger[\lambda]}{\delta \lambda_y} S^\dagger[\lambda]$$

in which the integral over $y$ (for $y_0 > t$) in the first functional variations is left implicit. If this product is to be consistent with causality, Eq. (5) must still be independent of the behaviour of the coupling function for all times $x_0 < t < y_0$ and, by Lorentz covariance, for all $x$ and $y$ that are space-like separated. We therefore require that the quantity

$$\frac{\delta}{\delta \lambda_x} \left[ \frac{\delta S[\lambda]}{\delta \lambda_y} \frac{\delta S^\dagger[\lambda]}{\delta \lambda_y} S^\dagger[\lambda] \right]$$

vanish for all $x \lesssim y$, i.e. for all $x$ causally preceding or space-like separated from $y$. This gives the Bogoliubov-Shirkov causality condition [13]:

$$\frac{\delta^2 S[\lambda]}{\delta \lambda_x \delta \lambda_y} S^\dagger[\lambda] + \frac{\delta S[\lambda]}{\delta \lambda_y} \frac{\delta S^\dagger[\lambda]}{\delta \lambda_x} = 0 \quad \forall \ x \lesssim y$$
which can be satisfied only through the cancellation of terms originating from $S$ (time-ordered) and $S^\dagger$ (anti-time-ordered).

Making use of the explicit form of the $S$-matrix operator in Eq. (4), Eq. (7) can be written

$$T \left[ \frac{\partial H_{x}^{\text{int}}}{\partial \lambda_x} \frac{\partial H_{y}^{\text{int}}}{\partial \lambda_y} S \right] S^\dagger - T \left[ \frac{\partial H_{y}^{\text{int}}}{\partial \lambda_y} \frac{\partial H_{x}^{\text{int}}}{\partial \lambda_x} S \right] T \left[ \frac{\partial H_{x}^{\text{int}}}{\partial \lambda_x} S \right] = 0 \quad \forall x \prec y ,$$

where $T$ is the anti-time-ordering operator. At lowest order in the coupling function, this requires

$$\Theta(x_0 - y_0) \left[ \frac{\partial H_{x}^{\text{int}}}{\partial \lambda_x}, \frac{\partial H_{y}^{\text{int}}}{\partial \lambda_y} \right] = 0 \quad \forall x \prec y ,$$

and, on substituting for the interaction Hamiltonian density, we obtain the constraint

$$\frac{i}{(3!)^2} \Delta_{xy}^{(R)} \phi_x \phi_y^3 + \frac{1}{(2!)^3} \Delta_{xy}^{(R)} \phi_x^2 \phi_y^2 - \frac{i}{3!} \Delta_{xy}^{(R)} \phi_x \phi_y - \frac{1}{4!} \Delta_{xy}^{(R)}^4 = 0 \quad \forall x \prec y .$$

This is automatically satisfied due to the presence of the retarded propagators,† which have originated from the commutator in Eq. (9). Proceeding to higher orders in the coupling, the combination of time- and anti-time-ordered products will give rise to a series of nested commutators of the Hamiltonian density that is again consistent with causality. In the next section, we will show that this series can be obtained straightforwardly at the level of expectation values by application of the Baker-Campbell-Hausdorff formula.

We remark that the Bogoliubov-Shirkov causality condition [Eq. (7)] can be expressed in terms of the largest time equation [14], of which Eq. (10) is an example. This hints at the fundamental relationship between unitarity and causality. This connection has recently been emphasised [15] by means of the tree-loop duality relations [16, 17], and it can be made explicit (see Ref. [18]) through the Kobes-Semenoff unitarity cutting rules [19, 20, 21] of the Schwinger-Keldysh closed-time-path (or in-in) formalism [22, 23].

### 3. Sources and Detectors

We now turn out attention to the signalling between sources and detectors. For simplicity, we introduce a scalar analogue of the Fermi two-atom problem [10], which consists of a real scalar field $\phi$ that interacts with two static “atoms” $S$ and $D$, fixed at positions $x^S$ and $x^D$, respectively. Working in the interaction picture, the free part of the Hamiltonian is

$$H_0 = \sum_n \omega_n^S |n^S\rangle \langle n^S| + \sum_n \omega_n^D |n^D\rangle \langle n^D| + \int d^3x \left( \frac{i}{2} (\partial_t \phi)^2 + \frac{i}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right) ,$$

where the states $\{|n^X\rangle\}$ form a complete set of bound states for atom $X \in \{S, D\}$. The interaction Hamiltonian is

$$H_{\text{int}}(t) = M^S(t) \phi(x^S, t) + M^D(t) \phi(x^D, t) ,$$

$$M^X(t) \equiv \sum_{m \neq n} \mu_{mn}^X e^{i\omega_{mn}^X t} |m^X\rangle \langle n^X| .$$

†We have made use of the result (valid for all integer $n, m \geq 1$)

$$[\phi_n^x, \phi_m^y] = \sum_{k=1}^{\min(n, m)} (-1)^{k+1} \frac{n! m!}{k!(n-k)!(m-k)!} [\phi_k^x, \phi_n^y]^k [\phi_k^y, \phi_m^x]^k [\phi_k^x, \phi_m^y]^k .$$
where $\omega_{m}^{X} \equiv \omega_{m}^{X} - \omega_{m}^{X}$ and $\mu_{m}^{X}$ are monopole moments.

At time $t = 0$, the system is prepared with atom $D$ in its ground state ($|g^{D}\rangle$), the field $\phi$ in its vacuum state ($|0^{\phi}\rangle$) and atom $S$ in an incoherent superposition of its ground and excited states ($|g^{S}\rangle$ and $|p^{S}\rangle$). This corresponds to an initial density operator

$$\rho_{0} \equiv \rho(0) = \gamma |i_{p}\rangle\langle i_{p}| + (1 - \gamma) |i_{g}\rangle\langle i_{g}|, \quad |i_{p(g)}\rangle = |p(g)\rangle \otimes |g^{D}\rangle \otimes |0^{\phi}\rangle,$$

where $\gamma \in \mathbb{R}$ parametrises the admixture of the initial states of the source atom $S$. The measurement (at some later time $t = T$) is effected by an operator $E$, which projects into the subspace of states in which the detector atom $D$ is in its excited state ($|q^{D}\rangle$) but with no restriction on the states of atom $S$ or the field $\phi$:

$$E \equiv E^{S} \otimes E^{D} \otimes \mathcal{E} = |q^{D}\rangle\langle q^{D}| \otimes \mathbb{I}^{\phi}.$$  

The probability of the local measurement outcome “atom $D$ excited” is then

$$\mathbb{P}(D^{*}) = \text{Tr}(E \rho_{T}) = \text{Tr}(U_{T,0}^{\dagger} E U_{T,0} \rho_{0}), \quad (15)$$

where

$$U_{T,0} = T \exp \left[ -i \int_{0}^{T} dt' H_{\text{int}}(t') \right] \quad (16)$$

is the time-evolution operator. It is important to draw a clear distinction between the probability of the measurement outcome in Eq. (15) and the sensitivity of the detector to the initial preparation of the system. The latter tells us about our ability to signal between the source and detector atoms, and it is given by

$$\sigma_{pg} \equiv \frac{d \mathbb{P}(D^{*})}{d \gamma} = \mathbb{P}_{p}(D^{*}) - \mathbb{P}_{g}(D^{*}), \quad \mathbb{P}_{p(g)}(D^{*}) \equiv \langle i_{p(g)}| U_{T,0}^{\dagger} E U_{T,0} |i_{p(g)}\rangle.$$

By applying the Baker-Campbell-Hausdorff formula to Eq. (17), the combination of time-ordered (from $U$) and anti-time-ordered exponentials (from $U^{\dagger}$) gives rise to an infinite series of nested commutators $[10, 18, 24, 25]$. This can be written in the form $[10, 18, 24, 25]$

$$\mathbb{P}_{p,g} = \sum_{j=0}^{\infty} \int_{0}^{T} dt_{1} dt_{2} \ldots dt_{j} \Theta_{12 \ldots j} \langle i_{p,g} | F_{j} | i_{p,g} \rangle; \quad (18)$$

where $F_{0} \equiv E$,

$$F_{j} = \frac{1}{T} \left[ F_{j-1}, H_{\text{int}}(t_{j}) \right] = \frac{1}{T} \left[ F_{j-1}, M_{j}^{S} \phi_{j}^{S} + M_{j}^{D} \phi_{j}^{D} \right], \quad (19)$$

and $\Theta_{i,j,k, \ldots}$ is equal to 1 for $t_{i} > t_{j} > t_{k} > \ldots$ and 0 otherwise. We use the short-hand notation $M_{j}^{X} \equiv M^{X}(t_{j})$ and $\phi_{j}^{X} \equiv \phi(x^{X}, t_{j})$. By defining

$$E_{X}^{\ldots k} \equiv \frac{1}{T} \left[ E_{X}^{\ldots}, M_{k}^{X} \right], \quad E_{X}^{\ldots k} \equiv \left\{ E_{X}^{\ldots}, M_{k}^{X} \right\}, \quad (20a)$$

$$\varepsilon_{X}^{\ldots k} \equiv \frac{1}{T} \left[ \varepsilon_{X}^{\ldots}, \phi_{k}^{X} \right], \quad \varepsilon_{X}^{\ldots k} \equiv \left\{ \varepsilon_{X}^{\ldots}, \phi_{k}^{X} \right\}, \quad (20b)$$

we can introduce a convenient “undercircle” notation $[10]$

$$E_{kl}^{\varepsilon_{kl}} \equiv E_{kl}\varepsilon_{kl} + E_{kl}^{\varepsilon_{kl}} + E_{kl}^{\varepsilon_{kl}} + E_{kl}^{\varepsilon_{kl}}, \quad (21)$$
which allows us to write the n-th order operator appearing in Eq. (18) as

\[ F_n = 2^{-n} \sum_{a=0}^{n} E^S_{(1...a} E^D_{a+1...n)} E^{(S...D)}_{(1...a+a+1...n)}. \]  

(22)

The parentheses indicate a summation over all unique ordered permutations of the indices, e.g.,

\[ E^S_{(12} E^D_{3)} = E^S_{12} E^D_{3} + E^S_{3} E^D_{12} + E^S_{13} E^D_{2}. \]

In the effect operator [Eq. (14)], we have made no restriction on the states of atom S or the field \( \phi \), fixing \( E^S = 1 \) and \( E = 1 \). We may then show that:

(i) \( E^S_{k...} = 0 \), such that the first index on \( E^S \) must always be underlined, with \( E^S_{k} = 2M^S_{k} \).

(ii) \( E^{X...}_{1...} = 0 \) and \( E^{XYZ...}_{123...} = 0 \), such that the ‘1’ index (the latest time) is never underlined on \( E^{X...}_{1...} \) and, more generally, any \( E^{...} \) operator vanishes when its first k indices consist of more non-underlined than underlined indices for any k.

These restrictions allow us to eliminate a large proportion of the terms in Eq. (22).

The first non-vanishing contributions to Eq. (17) arise from the fourth-order operator

\[ F_4 = \frac{1}{16} \left( E^D_{1234} E^{DDDD}_{1234} + E^D_{12} E^{DDDD}_{3412} + E^D_{13} E^{DDDD}_{2413} + E^D_{14} E^{DDDD}_{2314} \right. \]

\[ + E^D_{12} E^D_{34} E^{DDDD}_{1234} + E^D_{13} E^D_{24} E^{DDDD}_{1234} + E^D_{14} E^D_{23} E^{DDDD}_{1234} \]

\[ + E^D_{12} E^D_{34} E^{DDDD}_{12} + E^D_{13} E^D_{24} E^{DDDD}_{13} + E^D_{14} E^D_{23} E^{DDDD}_{14} \right) \cdot \cdot \cdot . \]

(23)

Substituting this expression into Eq. (17), the leading-order result for the sensitivity of atom D to the preparation of the system is [10]

\[ \frac{d \Phi(D^\gamma)}{d \gamma} = 2 \sum_{n} \int_{t_1 > t_2 > t_3 > t_4} |\mu_{pn}^S|^2 |\mu_{qg}^D|^2 \]

\[ \times \left[ \cos(\omega_{pn}^S t_4) \left( \sin(\omega_{pn}^S t_3) \Delta^{DS}_{24} + \cos(\omega_{pn}^S t_3) \Delta^{DS}_{24} \right) \Delta^{DS}_{13} \right. \]

\[ + \cos(\omega_{pq}^D t_1) \left( \sin(\omega_{pq}^D t_2) \Delta^{DS}_{14} + \cos(\omega_{pq}^D t_2) \Delta^{DS}_{14} \right) \Delta^{DS}_{23} \]

\[ + \cos(\omega_{pq}^D t_1) \left( \sin(\omega_{pq}^D t_2) \Delta^{DS}_{14} + \cos(\omega_{pq}^D t_2) \Delta^{DS}_{14} \right) \Delta^{DS}_{12} \]

\[ + \sin(\omega_{pq}^D t_1) \left( \cos(\omega_{pq}^D t_1) \Delta^{SD}_{34} + \sin(\omega_{pq}^D t_1) \Delta^{SD}_{34} \right) \Delta^{DS}_{12} \right] + \cdot \cdot \cdot , \]

(24)

where \( t_{ij} \equiv t_i - t_j \), and we have defined the retarded and Hadamard propagators

\[ \Delta^{X Y(R)}_{jk} \equiv \frac{1}{\gamma} \Theta_{jk} \langle \phi^X_j, \phi^Y_k \rangle , \]

(25a)

\[ \Delta^{X Y(H)}_{jk} \equiv \langle \{ \phi^X_j, \phi^Y_k \} \rangle . \]

(25b)

Most importantly, we see that the latest time on the source atom is always connected to a later time on the detector atom by a retarded propagator. As a result, Eq. (24) is strictly zero if the source and detector atoms are space-like separated, and faster-than-light signalling is manifestly prohibited. The expectation value of \( F_4 \) is illustrated diagrammatically in Fig. 2. A full exposition of the diagrammatic rules for this system is presented in Ref. [10]; this includes results for the expectation values of general nestings of commutators and anti-commutators of field operators, as well as the rules for computing the expectation values of the atom operators.

We remark that the above approach can be generalised to include additional fields, as well as their mutual and self-interactions; local interactions that involve spatial and temporal derivatives of the fields; and spatially extended sources and detectors (see Ref. [10]). In all of these cases, the manifest causality of local measurements persists.
Figure 2. Diagrammatic representation of $\mathcal{F}_4$, describing the evolution of the density operator forwards in time due to the exchange of a $\phi$ quantum. The initial time $t = 0$ is at the bottom of the diagrams and the time of the measurement $t = T$ is at the top of the diagrams. The solid vertical lines track the source (blue) and detector (green) atoms. Orange dashed lines with arrows are associated with retarded propagators $\Delta_{ij}^{XY}(R)$, and those without arrows are associated with Hadamard propagators $\Delta_{ij}^{XY}(H)$. Each vertex is associated with a monopole moment $\mu_{mn}$.

4. Implications for (semi-)inclusive observables

In order to obtain a causal result in Sec. 3, it was necessary to sum inclusively over the unobserved final states of the would-be photon field $\phi$ (and source atom $S$). Doing otherwise would have required us to sample the state of the field everywhere simultaneously, which would not constitute a local measurement. However, if we compare Figs. 1 and 2, we notice that the unobserved emissions necessarily accounted for in the amplitude-level calculation never appeared explicitly in the probability-level calculation. This can be traced back to the simple fact that, at the probability level, the effect operator corresponding to a fully inclusive measurement is just the unit operator, which trivially commutes with everything else, dropping out of the calculation entirely. The import of this observation is the following: by working directly with probabilities, we do not have to keep track of and calculate the individual amplitudes for all possible unobserved emissions in the final state.

Notwithstanding the potential implications of this for treating IR divergences, probability-level calculations allow us to introduce projection operators for (semi-)inclusive measurements, which do not have simple analogues at the amplitude level [12]. As an example, the effect operator that sums inclusively over final-state radiation with momenta below some scale $\mu$ is

$$E_\mu = :e^{-N_\mu}:, \quad N_\mu = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2E_k} \theta(|k| - \mu) a_k^\dagger a_k.$$  

(26)

The colons indicate normal ordering, and $a_k^\dagger$ and $a_k$ are the usual scalar creation and annihilation operators. In the limit $\mu \to \infty$, the operator is fully inclusive over final-state radiation:

$$\lim_{\mu \to \infty} E_\mu = I,$$  

(27a)

$$\lim_{\mu \to \infty} \langle 0 | \phi(x_1)\phi(x_2)E_\mu\phi(x_3)\phi(x_4)|0 \rangle = \langle 0 | \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0 \rangle,$$  

(27b)

where the appearance of the four-point amplitude is a consequence of the optical theorem. In
the limit $\mu \to 0$, the operator projects out the vacuum exclusively (see, e.g., Ref. [26]):

$$
\lim_{\mu \to 0} E_\mu = \left| 0 \right> \left< 0 \right|,
$$

(28a)

$$
\lim_{\mu \to 0} \left< 0 \right| \phi(x_1) \phi(x_2) E_\mu \phi(x_3) \phi(x_4) |0\right> = \left< 0 \right| \phi(x_1) \phi(x_2) |0\right> \left< 0 \right| \phi(x_3) \phi(x_4) |0\right> .
$$

(28b)

For finite $\mu$, Eq. (26) defines an operator form of the Sudakov factor [27]. Such an object could not be written down at the amplitude level.

### 5. Concluding remarks

In this note, we have outlined a method for directly calculating transition probabilities in quantum field theory in terms of the expectation values of nested commutators and anti-commutators of field operators. These transition probabilities are manifestly causal and have the interesting feature that unobserved emissions are accounted for implicitly in (semi-)inclusive measurements. The ability to sum implicitly over soft emissions may have important implications for how we deal with IR divergences in gauge theories, and this motivates the further development of technologies for evaluating these probabilities. Such technology may, for instance, exploit the connection between causality and unitarity to which we have alluded, making use of the unitarity cutting rules of the closed-time-path formalism and building upon ideas presented in Ref. [18].

### Acknowledgments

The work of PM is supported by STFC grant ST/L000393/1. PM would like to thank the organisers and participants of DICE2016 for their hospitality, questions and comments.

### References

[1] Bloch F and Nordsieck A 1937 Phys. Rev. 52 54–9
[2] Kinoshita T 1962 J. Math. Phys. 3 650–77
[3] Lee T D and Nauenberg M 1964 Phys. Rev. 133 B1549–62
[4] Weinberg S 1965 Phys. Rev. 140 B516–24
[5] Faddeev L D and Popov V N 1967 Phys. Lett. B 25 29–30
[6] Fermi E 1932 Rev. Mod. Phys. 4 87–132
[7] Shirokov M I 1966 Yad. Fiz. 4 1077 [1967 Sov. J. Nucl. Phys. 4 774]
[8] Ferretti B 1968 in Old and New Problems in Elementary Particles ed Puppi G (New York: Academic Press)
[9] Power E A and Thirunamachandran T 1997 Phys. Rev. A 56 3395–408
[10] Dickinson R, Forshaw J and Millington P 2016 Phys. Rev. D 93 065054
[11] Dickinson R, Forshaw J and Millington P 2017 Preprint arXiv:1702.04131 [hep-th]
[12] Bogolubov N N and Shirkov D V 1959 Introduction to the Theory of Quantized Fields (New York: Interscience) pp 200–4
[13] ’t Hooft G and Veltman M J G 1974 NATO Sci. Ser. B 4 177–322
[14] Tomboulis E T 2017 Preprint arXiv:1703.07052 [hep-th]
[15] Catani S, Gieleberg T, Krauss F, Rodrigo G and Winter J-C 2008 J. High Energy Phys. 09 (2008) 065
[16] Bierenbaum I, Catani S, Draggiotis P and Rodrigo G 2010 J. High Energy Phys. 10 (2010) 073
[17] Dickinson R, Forshaw J, Millington P and Cox B 2014 J. High Energy Phys. 06 (2014) 049
[18] Kobes R L and Semenoff G W 1985 Nucl. Phys. B 260 714–46
[19] Kobes R L and Semenoff G W 1986 Nucl. Phys. B 272 329–64
[20] Kobes R 1991 Phys. Rev. D 43 1269–82
[21] Schwinger J S 1961 J. Math. Phys. 2 407–32
[22] Keldysh L V 1964 Zh. Eksp. Teor. Fiz. 47 1515–27 [1965 Sov. Phys. JETP 20 1018–26]
[23] Cliche M and Kempf A 2010 Phys. Rev. A 81 012330
[24] Franson J D and Donegan M M 2002 Phys. Rev. A 65 052107
[25] Louisell W H 1973 Quantum Statistical Properties of Radiation (New York: John Wiley & Sons) p 158
[26] Sudakov V V 1956 Sov. Phys. JETP 3 65–71 [Zh. Eksp. Teor. Fiz. 30 87–95]