A Simple Test
for Non-Gaussianity in CMBR Measurements

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Abstract

We propose a set of statistics $S_q$ for detecting non-gaussianity in CMBR anisotropy data sets. These statistics are both simple and, according to calculations over a space of linear combinations of three-point functions, nearly optimal at detecting certain types of non-gaussian features. We apply $S_3$ to the UCSB SP91 experiment and find that the mean of the four frequency channels is by this criterion strongly non-gaussian. Such an observation would be highly unlikely in a gaussian theory with a small coherence angle, such as standard ($n = 1$, $\Omega_b = .05$, $h = .5$, $\Lambda = 0$) inflation. We cannot conclude that the non-gaussianity is cosmological in origin, but if we assume it due instead to foreground contamination or instrumental effects, and remove the points which are clearly responsible for the non-gaussian behavior, the rms of the remaining fluctuations is too small for consistency with standard inflation at high confidence. Further data are clearly needed however, before definitive conclusions may be drawn. We also generalize the ideas behind this statistic to non-gaussian features that might be detected in other experimental schemes.

subject headings: cosmic microwave background radiation, cosmology
1. Introduction

Many experiments, current and proposed, are dedicated to measuring fluctuations in the Cosmic Microwave Background Radiation (CMBR). These measurements promise a strong experimental test of theories of structure formation in the early universe, as each theory predicts a distinct magnitude and form for CMBR fluctuations. In inflationary models, the structure generation mechanism is linear, resulting in a gaussian pattern of fluctuations, completely characterised by its power spectrum (see e.g. Efstathiou 1990). By contrast, in theories based on symmetry breaking and field ordering (e.g. cosmic strings and textures) nonlinear dynamics lead to a non-gaussian anisotropy pattern, due in part to horizon-sized topological defects at the epoch of last scattering (Kaiser & Stebbins 1984; Turok & Spergel 1990; Bennett & Rhie 1992; Coulson, Pen, & Turok 1993; Pen, Spergel, & Turok 1993).

CMBR measurements have not yet discriminated among different structure formation theories to the extent that one might have hoped. This is in part because the measurements are still far from perfect. COBE has a low signal to noise ratio and large angular smoothing scale (Smoot et. al. 1992; Ganga et. al. 1993), while other experiments are more accurate but cover only a small region of the sky (e.g. Gaier et. al. 1992; Schuster, et. al. 1993; Meinhold et. al. 1993; Gundersen et. al. 1993; Cheng et. al. 1993). But it is also because most theories include parameters \( (n, h, \Omega, \Omega_B, \Lambda, \text{ (Tensor/Scalar)}) \) which can be adjusted to modify the power spectrum. Such adjustments do not however alter the more fundamental gaussian or non-gaussian character of the theories, which may be a more powerful discriminator. This paper is aimed at finding statistics which focus on this basic question. Other recent papers which propose statistical tests for non-gaussianity are Luo & Schramm (1993) and Moessner, Perivaropoulos & Brandenberger (1993).

We are attempting to extract information from very small data sets, obtained in very difficult experiments. This is of course quite hazardous: it is unlikely that the idealized assumptions we shall make about the experimental errors are correct. Any effect we see may well be due to foreground sources or systematic instrumental effects, rather than non-gaussian cosmology. Nevertheless it is an interesting exercise to see how much may be learned, in principle, from experiments of the type currently being undertaken. And this may also serve as a guide to what kind of experimental effort would be most informative in the future. At the very least, we can make rigorous a process which is often performed by eye: the identification of data points which are inconsistent with gaussian theories and...
must be thrown out as contaminated if these theories are to be believed.

The various non-gaussian field ordering theories predict different characteristic forms on the microwave sky - for example, linelike discontinuities for strings, hot and cold spots for textures. However they all predict regions of sharp gradient, separated by a characteristic scale of order the horizon at last scattering, from one to a few degrees depending on the reionisation history of the universe (Pen et al. 1993; Coulson et al. 1993). In this paper we shall discuss statistics which are sensitive to degree scale large-gradient regions, and use them to discriminate between gaussian and non-gaussian theories.

We shall follow tradition and use as our canonical ‘straw man’ gaussian theory the ‘standard’ inflationary model, with parameters $n = 1$, $h = .5$, $\Omega = 1$, $\Omega_B = .05$, $\Lambda = 0$, and negligible tensor mode contribution. This theory has a coherence angle of order 15’, substantially smaller than the scales degree scale experiments probe. The results we get are similar to those obtained assuming uncorrelated gaussian noise at each point differenced on the sky. So while we shall find rather strong evidence against such theories from the UCSB SP91 data, we expect that the constraints would be weaker for gaussian theories with a large coherence angle, such as inflationary theories where there is a large tensor mode contribution (Crittenden et al. 1993).

As a concrete example of a degree scale CMBR measurement, we shall analyse the UCSB SP91 experiment (Gaier et al. 1992; Schuster et al. 1993). This is a ‘single difference’ experiment; to generate a single data point, the beam moves in a sinusoidal pattern, with the antenna temperature integrated antisymmetrically. The result approximates the first spatial derivative of the fluctuations. A set of results consist of nine to fifteen data points (temperature differences), on an arc of constant declination on the sky with 2.1 degrees separation between points. To correct for atmospheric and other drifts, a best-fit line is removed. The Schuster et al. (1993) data set currently has the lowest error per pixel ($5 \times 10^{-6}$ for the four channel average) reported for any CMBR anisotropy measurement. Other ground-based experiments share many of these features, although some integrate their intensities in such a way as to approximate the second or third derivative of the fluctuations rather than the first and are called double- or triple- difference experiments. Depending on how many spatial derivatives an experiment takes, a gradient region will leave certain ‘signature’ forms on the data, as shown in Figure 1.

It might seem that a sharp gradient would span too few points to make these characteristic patterns visible, especially for higher derivatives with complicated signatures.
However, the derivatives are taken by sweeping the beam on the sky or by combining data from adjacent instrument positions. These methods give rise to an instrument response function broad enough that the characteristic signatures will be visible even for an infinitely sharp discontinuity in the CMBR.

A final caveat should be added regarding our use of ‘classical’ confidence intervals, as opposed to Bayesian measures of the relative probability of different theories. These are notoriously difficult to interpret. We have done so mainly out of expediency - it is far easier to construct realisations of gaussian theories than for nongaussian theories. When enough non-gaussian maps are available, the following procedure may be preferable. If one is comparing theory $A$ to theory $B$, the change in the relative odds following a measurement of a continuous observable $X$ is given by the ‘Bayes factor’

$$\frac{P(X|A)}{P(X|B)}$$

where probability of observing $X$ in the interval $dX$ is $P(X|A)dX$ according to theory $A$ and similarly for theory $B$. As we mention in the conclusions, preliminary results indicate that according to this, more simply interpretable test, the nongaussian theories may be significantly favored by the UCSB SP91 data.

2. Choosing Statistics

All current theories of the origin of structure produce fluctuations in the form of a stationary random process. Any such process may be completely characterized by the set of all $n$-point correlation functions. On a one-dimensional data set these may be estimated as

$$C_{0r_1r_2\ldots r_{n-1}} \equiv \frac{1}{(N - r_{n-1})} \sum_{i=1}^{N-r_{n-1}} (x_i)(x_{i+r_1})\cdots(x_{i+r_{n-1}})$$

where, by convention, $0 \leq r_1 \leq r_2 \leq \cdots \leq r_{n-1}$

We shall assume that the data set of interest has zero mean, as in UCSB SP91, where a best-fit line is substracted as explained above. We shall adopt the convention of normalizing the data set to unit variance, in order to concentrate on the shape and not the amplitude of the signal.

The one-point function $C_0$ is identically zero, and the two-point function $C_{00}$ is identically one (because of our unit-variance convention), and so the first nontrivial correlation is $C_{0i}$, the two point correlation function at scale $i$. This does contain information about
the spectrum of fluctuations — it is the Fourier transform of the power spectrum — but it is no help in distinguishing Gaussian from non-Gaussian data.

The three-point function $C_{0ij}$ is a much more promising test for non-Gaussianity. For Gaussian noise, $\langle C_{0ij} \rangle = 0$ for all $i$ and $j$ (although for finite data sets there will be random fluctuations about the expected value of zero.) For non-Gaussian skies we expect non-zero three-point functions. For example, consider the three-point function $C_{000}$. A data set containing a positive ‘bump’ has several outlying high points, and thus positive skewness. A downward-pointing bump will lead to negative skewness. Either way, the high absolute value of the skewness could be used to distinguish a data set drawn from a Gaussian model from one drawn from a region containing a bump (which, from fig. 1, is a likely signature of non-Gaussianity in a single-difference experiment.)

As we shall show, skewness is not a very powerful statistic for reliably detecting non-Gaussianity in a noisy experiment. We can improve on its performance in two ways.

First, if the CMBR is non-Gaussian, a ‘bump’ marking a gradient region may span two or more adjacent points. Even if the region of steep gradient on the sky were infinitely sharp, it would register in at least two data points because of the instrument’s response function. Skewness fails to take advantage of these correlations among closely neighboring points (obviously, since skewness is invariant under spatial scrambling of the data points.) We can remedy this shortcoming by combining several adjacent points; for example, to look for bumps of width on the order of $q$, define

$$S_q \equiv \frac{1}{(N-q+1)} \sum_{i=1}^{N-q+1} \left( \frac{x_i + x_{i+1} + \cdots + x_{i+q-1}}{q} \right)^3$$

This statistic responds much more sharply to several adjacent high points than to the same number of high points scattered randomly over the data set, so it better distinguishes actual non-Gaussian bumps from noise. The absolute value of $S_q$ will be near zero if no bump exists and strongly non-zero if there is a single bump.

Of course, $S_q$ is not equal to a simple three-point function, but except for the treatment of points near the edges of a data set, it is equivalent to a linear combination of three-point functions. We will expand on this point later, when we show that $S_3$ is nearly optimal, among all linear combinations of a certain set of three-point functions, at detecting bumps of width near three.

A second way to improve the performance of almost any statistic which detects bumps in a data set is to apply it not to the entire data set but to shorter subsets or ‘windows’ of
length $L \leq N$. The final statistic $S_{q;L}$ is defined as the absolute value of the most extreme (positive or negative) $S_q$ found in any of the $N - L + 1$ possible window positions. Use of these ‘sliding windows’ improves the statistic’s performance for several reasons. Most importantly, it prevents a positive gradient region in one part of the data from cancelling a negative gradient in another region (by isotropy, both signs are equally likely to occur.) The procedure also reduces the effects of noise on the statistic’s probability distribution by concentrating on only a few points around each gradient signature.

If the data set contains a bump with some number $p$ of adjacent ‘strong’ (highly positive or highly negative) points, we expect the best results when $L \approx p + 2(q - 1)$. This allows the window to contain every group of $q$ adjacent points which includes at least one ‘strong’ point, and no groups of $q$ points with no ‘strong’ point. $S_q$ is most sensitive to bumps with about $q$ ‘strong’ points. We generally choose $p = q - 1$, so

$$L = 3(q - 1)$$

We will show that both in monte carlo runs and on actual experimental data, statistics perform much better on sliding windows of about this scale than on entire data sets. For long data sets, one might better consider the probability distribution of $S_q$ over window positions, rather than the maximal value, to reduce sensitivity to a few extreme points.

Our favored choice of statistic will be $S_{3;6}$, as $q = 3$ will be sensitive to bumps only slightly wider than the experimental response function, and can thus detects gradient regions whose width (relative to the two-degree scale set by the experiment) is fairly low but non-zero. Equation (3) then sets the window length of $L = 6$.

3. Monte Carlo Results

To test the relative power of different statistics, we devised a monte carlo technique based on the UCSB SP91 experiment. A large number of trial data sets $\{x_i : 1 \leq i \leq N\}$ were generated. Typically, $N = 13$ to match the UCSB SP91 experiment. Half these data sets were generated from a ‘null’ gaussian model and half from a ‘bumpy’ non-gaussian model.

The null data sets could be generated in either of two ways. G. Efstathiou (private communication 1993) generously provided 1000 sets of 13 points, based on his computer-generated ‘standard inflation-plus-CDM’ skies and his simulation of the properties of the UCSB SP91 experiment. Alternatively, null sets could be generated by simply drawing $N$ independent, random points from a gaussian distribution. These methods returned similar
results (in that most interesting statistics $S(\{x_i\})$ calculated from the $N$ points had similar distributions for the two null models.) The statistics are apparently not greatly affected either by correlations arising from the CMBR’s power spectrum (no great surprise, since the two-degree scale of this experiment is on the low-frequency side of the power spectrum for this theory, and taking a spatial derivative further shifts the spectrum toward high spatial frequencies) or by correlations arising from the instrument’s response function (again no surprise; the signature that we’re looking for is symmetric and thus orthogonal to the antisymmetric response function. Were we searching for point sources rather that discontinuities, response–function–induced correlations would be a more powerful confounding factor.)

The other half of the data sets were generated from a non-gaussian ‘bumpy’ model. A single bump, centered at some random location $n_0$ within the data set, was laid down:

$$x_n = e^{-\alpha(n-n_0)^2}$$

where we typically used $\alpha = 0.5$, corresponding to a bump with full-width-half-max of 2.8 pixels. Incidentally, $n_0$ is not necessarily an integer; the center of the bump can lie between pixels. Independent gaussian noise was then added to each point to simulate instrument noise. In both the null and ‘bumpy’ models, the each data set $\{x_i\}$ was normalized to zero mean and unit variance.

For each statistic $S$ which we want to investigate, we can calculate probability distributions of $S$ in the ‘null’ and ‘bumpy’ models. If $S$ is a powerful detector of non-gaussianity, there should be little or no overlap between the two distributions. Figure 2 shows these distributions for our favorite statistic, $S_{3,6}$, using a signal-to-noise ratio of 1.25 (about the same level as seen in the UCSB SP91 data set). There is indeed very little overlap: the value of $S_{3,6}$ calculated on a ‘bumpy’ data set typically exceeds the values of all but a small fraction of the null sets. This “small fraction” varies from one ‘bumpy’ set to another, but its average value is 1.2% (from now on, we’ll refer to this as “a mean significance of 1.2%”.

The performance degrades if instead of using sliding windows, we simply calculate $S_3$ on the entire data set at once; the mean significance rises from 1.2% to 3.4%. For comparison, figure 3 shows the distributions of absolute values of skewness (calculated in sliding windows of width 6) of data sets from the null and ‘bumpy’ models. The overlap is tremendous; at this noise level, skewness could never reliably distinguish the two models. Performance is even worse if we calculate skewness on the whole data set instead of on
a sliding subset. Evidently, $S_3$ is a much more powerful detector of non-gaussian bumps than is skewness.

To test our assertion that each statistic $S_q$ is most sensitive to bumps of width of about $q$, we performed monte carlo runs like those described above for a variety of ‘bumpy’ models with bumps of different widths. For each model, we measured the average significance obtained using $S_{q:3(q−1)}$ for $q = 1,..5$. The results are shown in figure 4, which plots mean significance vs. full-width-half-max of the bump model for each of the five statistics. As expected, each statistic $S_q$ reaches its maximum power (lowest mean significance) for bumps of full-width-half-max near $q$ (or slightly higher.)

4. Optimal Statistics

We have justified the statistic $S_{q:3(q−1)}$ by an incremental process, starting with skewness, the simplest detector of non-gaussianity, and modifying it to counter its obvious shortcomings. Our monte carlo results showed that the resulting statistic is a much better detector of non-gaussian ‘bumps’ than is skewness, but we would like to go further and show that it is optimal or near-optimal for this job, at least over certain classes of related statistics.

The procedure of calculating $S_{q:3(q−1)}$ can be separated into three steps. First we convolve the data set with a square tooth of width $q$ (a function equal to one at $q$ adjacent points, and zero elsewhere). Next we take the third power of the convolved data points. Finally we add the results for each connected subset or ‘window’ of length $3(q−1)$ within the data set, and take the most extreme value of $S_q$ as our final statistic $S_{q:3(q−1)}$. For each of these three steps, we can investigate whether a modification of the procedure would produce stronger results.

4.1 Optimal Choice of Convolution Function

Our choice to convolve the data with a square tooth function amounts to a filtered deconvolution about the gradient signature we are searching for (in this case a bump), with the high-frequency components suppressed. It is not better to use the rigorous deconvolution function of the signature we are seeking; this method is notoriously vulnerable to high frequency noise. But it is worthwhile to see whether convolving the data with some other function, rather than the arbitrarily chosen square tooth, would produce a better statistic.

As mentioned before, $S_q$ is nearly equivalent to a linear combination of several three-
point functions. Except for its treatment of points near the window edges, $S_3$ is proportional to

$$C_{000} + 2(C_{001} + C_{011}) + 2C_{012} + (C_{002} + C_{022}) \quad (4)$$

To the extent that this approximation holds, the search for the optimal convolution function is equivalent to a search for the optimal linear combination of three-point functions. We define the generalized three-point function $S_{G3}$ by:

$$S_{G3} \equiv C_{000} + t(C_{001} + C_{011}) + u(C_{012}) + v(C_{002} + C_{022}) \quad (5)$$

$C_{001}$ and $C_{011}$ share the same coefficient for reasons of symmetry, as do $C_{002}$ and $C_{022}$. There is no coefficient before $C_{000}$ because an overall multiplicative constant does not affect a statistic’s ability to distinguish between distributions of different shapes.

$S_{G3}$ includes all six of the three-point functions which involve no more than three adjacent points at a time. Wider-ranging three-point functions, such as $C_{013}$, are not included because we are attempting to generalize $S_3$, which searches most powerfully for bumps spanning 2 or 3 points.

To estimate the optimal coefficients $t$, $u$, and $v$, the procedure is as follows. We first adopt two simple analytic models of null (gaussian) and ‘bumpy’ (non-gaussian) distributions. We then calculate the mean of $S_{G3}$ for the bumpy model, and its mean and variance for the null model. We define the ‘bump-resolving power’ $R$ as the distance, measured in standard deviations of the null model, between the means of the null and bumpy models:

$$R \equiv \frac{< S_{G3} >_{bump} - < S_{G3} >_{null}}{\sqrt{< S_{G3}^2 >_{null} - < S_{G3}^2 >_{null}^2}} \quad (6)$$

Finally we maximize $R$ as a function of $t$, $u$, and $v$. The quantity $R$ is not the most accurate measure one could think of, but is at least straightforwardly calculable.

To simplify the calculation, we work in an infinitely long window of length $N \to \infty$, instead of the window of length $L = 6$ which we shall use in practice. This underscores the fact that choosing a statistic (like $S_q$) and choosing a window size are two separate ideas; the idea of sliding windows is not specific to $S_q$ but improves the performance of almost any statistic.
4.1.1 Null Model:

The null model consists of \( N \) points drawn from a gaussian distribution of zero mean and unit variance, with one important modification: each data set is zero mean. If \( \{y_i\} \) are a set of independent points drawn from a gaussian distribution,

\[
x_i = y_i - \frac{1}{N} \sum_{j=1}^{N} y_j
\]

This sounds like a trivial change, especially for large data sets which tend to have means very close to zero anyway, but the explicit normalization makes a surprisingly large difference in the calculation even as \( N \to \infty \). We do not explicitly normalize each set to unit variance, because it can be shown to make no difference in the \( N \to \infty \) limit.

In order to calculate \( R \) from equation (6), we need to know \( < S_{G3} > \) and \( < S_{G3}^2 > \) for this model. \( < S_{G3} > \) is clearly zero, and using other symmetry properties,

\[
< S_{G3}^2 > = < C_{000}^2 > + 2t^2 < C_{001}^2 > + u^2 < C_{012}^2 > + 2v^2 < C_{002}^2 >
\]

So we need to calculate expectations such as

\[
< C_{000}^2 > = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \langle x_i^3 x_j^3 \rangle
\]

for a gaussian distribution with unit variance.

Expectations such as \( \langle x_i^3 x_j^3 \rangle \) can be calculated using Wick’s theorem, starting with the fundamental 2-point expectation \( \langle x_i x_j \rangle = \delta_{ij} - N^{-1} \). The \( N^{-1} \) term is due to the normalization of the data \( \{x_i\} \) to zero mean. The results are:

\[
< C_{000}^2 > = 6N^{-1} \quad < C_{001}^2 > = 2N^{-1} \quad < C_{002}^2 > = 2N^{-1} \quad < C_{012}^2 > = N^{-1}
\]

All other terms in the expression for \( < S_{G3}^2 > \) are zero, so

\[
< S_{G3}^2 >_{null} \to (6 + 4t^2 + u^2 + 4v^2)N^{-1} \quad \text{as} \; N \to \infty
\] (7)
4.1.2 Bumpy Model:

To represent non-gaussian, ‘bumpy’ data sets we take

\[ x_n = ay_n + bm_n \]

where the ‘noise’ \( y_n \) is drawn from a zero mean, unit variance normal distribution, and \( m_n \) is the underlying ‘bumpy’ model:

\[ m_n = g \left( e^{-\alpha(n-n_0)^2} - c \right) \quad \text{where} \quad c = \frac{1}{N} \sqrt{\frac{\alpha}{\pi}} \quad g = N^{\frac{1}{2}} \left( \frac{2\alpha}{\pi} \right)^{\frac{1}{4}} \]

The bump center \( n_0 \) is chosen randomly and is not necessarily an integer. The constants \( g \) and \( c \) are chosen so that, as \( N \to \infty \), \( m_n \) will also have zero mean and unit variance (when averaged over all \( n_0 \)). The purpose of \( a \) and \( b \) is to set the signal-to-noise ratio: \( \text{SNR} = \frac{b}{a} \), and \( a^2 + b^2 = 1 \). It is straightforward to check that all terms involving the noise term give zero in \( S_{G3} \) (this would not be true for higher moments). Averaging over \( n_0 \) converts all sums to integrals, which in the limit \( N \to \infty \) yield the results:

\[ < C_{000} > = b^3 N^{\frac{1}{2}} \left( \frac{8\alpha}{9\pi} \right)^{\frac{1}{4}} < C_{001} > = < C_{000} > e^{-\frac{2}{3}\alpha} \]

\[ < C_{002} > = < C_{000} > e^{-\frac{8}{3}\alpha} \quad < C_{001} > = < C_{000} > e^{-2\alpha} \]

For \( \alpha = 0.5 \) (corresponding to a bump with full-width-half-max of about 2.8 data points, about the scale we hope to detect with \( S_{G3} \)),

\[ < S_{G3} >_{\text{bump}} = b^3 N^{\frac{1}{2}} [0.6133 + 0.8789t + 0.2256u + 0.3233v] \quad (8) \]

Now we use (6) to estimate the statistic’s power to distinguish gaussian from non-gaussian models:

\[ R \equiv \frac{< S_{G3} >_{\text{bump}}}{\sqrt{< S_{G3}^2 >_{\text{null}}}} = b^3 N^{\frac{1}{2}} \frac{0.6133 + 0.8789t + 0.2256u + 0.3233v}{\sqrt{6 + 4t^2 + u^2 + 4v^2}} \]

The optimal values of \( t, u, \) and \( v \) are those which maximize \( R \); a numerical search for these yields the optimized three-point statistic:

\[ S_{G3} = C_{000} + 2.15(C_{001} + C_{011}) + 2.21C_{012} + 0.79(C_{002} + C_{022}) \quad (9) \]

As we hoped, this result is similar to (4), confirming that our original combination of three-point functions (or equivalently, our choice to convolve the data set with a square
tooth) is probably among the most powerful methods. Since this calculation was approxi-
mate, we checked it with more precise monte carlo runs, which confirmed that no other no other choice of convolution function gives dramatically better results. We have settled on the choice of a square tooth as the best combination of simplicity and power.

4.2 Optimal choice of power

After convolving the data set with a square tooth of width $q$, $S_q$ requires us to sum the third powers of the convolved data points. We should investigate whether taking some power other than the third would give better results. The higher the power, the more emphasis is given to the most extreme points in the data set (after convolution.) Emphasizing extreme points has the advantage of reducing the effects of noise, since it prevents several small bumps, caused by noise, from matching the effect of a large bump in the signal. The drawback is that for a real signal, the highest points will have neighbors which are also higher than average, since neither a physical gradient region on the sky nor the instrument’s response function have perfectly sharp edges. High points due to extreme values of noise will not in general have unusual neighbors. So focusing too heavily on extreme points throws away information which would help distinguish a physical signal from noise. The optimal choice of power is that which balances these two competing effects.

The answer is not obvious and is clearly model-dependent, so we turn again to monte carlo results. Consider a class of statistics

$$S^{(p)}_{3;6} = \frac{1}{(N-2)} \sum_{i=1}^{N-2} \left( \frac{x_i + x_{i+1} + x_{i+2}}{3} \right)^p$$

These statistics are calculated in sliding windows of width 6; they differ from $S_{3;6}$ only in the use of the $p$-th power rather than the third. We investigated their ability to distinguish two different ‘bumpy’ models from white noise. One model, described in the earlier section on monte carlo results, used bumps of gaussian profile with full-width-half-max of 2.8 data points. The other model used bumps of a square tooth profile with three adjacent, equally high points randomly placed in the data set. Gaussian white noise was added to both models at a signal-to-noise ratio of 1.25. For the gaussian-profile model, we expect the resolving strength to peak at some finite power $p$, while the square bumps should be best resolved at very high $p$ since the argument for lower powers applies only to bumps in which points near the bump have non-zero expectations.

The results are shown in Figure 5, which plots mean significance of detection vs. $p$. For detection of the gaussian-profile bumps, the optimal power was $p = 3$. For the square
bumps, higher powers are always better, as expected. Again, neither gradient regions on
the CMBR nor instrument response functions are expected to have sharp cutoffs, so we
view the gaussian profiles as more physically realistic than the square ones, and continue
to use \( p = 3 \). However, the results show that \( p = 3 \) is only slightly better than other
nearby choices, so we will not hesitate to use \( p = 4 \) later in the paper when we generalize
the statistic to search for other types of gradient signatures (because even powers will be
more convenient than odd ones).

Also, we should note that \( S^{(p)}_{3,6} \) becomes very simple as \( p \rightarrow \infty \). Our statistic is
then equivalent (in its relative ranking of different data sets, which is the only thing that
matters) to simply convolving the data with a square tooth of width three, and then
choosing the most extreme point. The ‘sliding window’ becomes irrelevant in this limit.
Readers who feel that this extra simplicity is worth sacrificing some power may prefer to
use this straightforward procedure.

### 4.3 Optimizing Window Size

The final choice that we have made is to use windows of length \( L = 3(q - 1) \) (equation
(3)). There is not much of interest to say about this choice. We gave a rough justification
earlier, and monte carlo runs confirm that it is the best or nearly the best length for a
wide range of \( q \) (when searching for bumps of full-width-half-max near \( q \).)

### 5. Experimental Results

We focus on a run of the UCSB SP91 experiment which observed 13 points in four
frequency channels (Gaier et. al. 1992). Cosmological fluctuations should be frequency-
independent, so we can average the four channels to better distinguish cosmological fluctu-
tuations from instrument noise and, possibly, from astrophysical and atmospheric effects.
The mean of the four channels is shown in figure 5.

We applied the statistic \( S_{3,6} \) to data from each of the four channels and to their mean.
We compared the results to G. Efstathiou’s 1000 simulations of UCSB SP91’s view of
standard inflation skies (after adding gaussian noise at the estimated experimental level,
and removing a best-fit line, as was done to the actual UCSB SP91 data.) Channels 1 and
2 (the two lowest-frequency channels, covering 25-27.5 GHz and 27.5-30 GHz respectively)
appear highly non-gaussian. Both achieve 0.1% significance (only 1 of the 1000 inflationary
skies gave as large a value of \( S_3 \).) The average of all four channels does nearly as well, at
0.2% significance. Channels 3 and 4 were not conclusively non-gaussian.
By contrast, if we had used pure skewness, we would find the mean of the four channels non-gaussian at only 5% significance. If we had used skewness on the whole data set, instead of with sliding windows, we would conclude the mean of the four channels to be non-gaussian at only 11% significance.

We chose $S_3$ (rather than some other $S_q$) as the preferred statistic for analyzing data sets because we expect the perceived size of the gradient regions to be not much larger than the lower limit set by the instrument response function (two or three pixels.) For comparison, Figure 6 shows the significance levels at which all four channels, as well as their mean, can be shown non-gaussian by the various $\{S_q\}$, with $q$ ranging from 1 to 13. $S_{3,6}$ provides the strongest overall results, although $S_{2,3}$ and $S_{4,9}$ both outperform it on individual channels.

These results show that the UCSB SP91 data is strongly non-gaussian, but non-gaussian data does not necessarily imply non-gaussian cosmology. The data sets contain a visible spike spanning about 2 pixels (clearly visible in figure 5.) This could be the signature of a sharp gradient generated by a non-gaussian cosmological model, but there are several other possibilities. These include galactic foreground sources, extragalactic but noncosmological sources (unlikely; such sources could not easily match the spatial structure of the data), or a systematic instrumental effect such as sidelobe pickup. With UCSB SP91’s limited range of frequency (25-35 GHz) there is not enough spectral information to reliably distinguish astrophysics from cosmology (i.e., by fitting to the spectra of synchrotron or bremsstrahlung radiation). However, if a gaussian theory with small correlations on two-degree scales, like standard inflation, is to be believed, we must assume that the signal in points number 7 and 8 is non-cosmological, and remove those points from the data set. The fluctuations of the remaining eleven points may then be used to impose constraints on the cosmological fluctuations. We removed a best-fit line from the remaining points, because the line already removed from the scan must be assumed invalid if two points were contaminated. The remaining points have a quite reasonable chi-squared of 9.9, quite reasonable for 9 degrees of freedom. We then compared the r.m.s. to those of the standard inflation simulated data sets (with points 7 and 8 likewise removed, and a new best-fit line subtracted from the remaining points). After this procedure, only 1 of the 1000 simulated data sets had an r.m.s. as low as that of the mean of UCSB SP91’s four channels. We performed the same procedure, removing points 6, 7, 8, and 9 (since points near the bump may also be suspect) and found the UCSB SP91 data was quieter than all but 6 of the
1000 simulations.

If we are to believe the UCSB SP91 data, the standard inflation theory is caught on the horns of a dilemma. If points 7 and 8 of the UCSB SP91 data are of cosmological origin, the shape of the data is highly non-gaussian and thus inconsistent with the theory. If the two points are contaminated, the measured r.m.s. is too low at high confidence. Other gaussian models can be tested in a similar way. Before claiming that we have rejected any cosmological model, we must wait to see if these methods demonstrate non-gaussianity in other experiments.

6. Other Experiments

\( S_{q;3(q-1)} \) can be applied without modification to any single-difference experiment. We recommend \( q = 3 \) unless the experiment has a very short distance (well under a degree) between data points, in which case one should try several larger values of \( q \) to search for high-gradient regions typical of field ordering theories.

For double- and triple-difference experiments, the characteristic ‘bump’ marking gradient regions will be replaced by more complicated signatures representing higher derivatives of this sudden gradient (as seen in Figure 1). The ideas developed here, with some modification, should apply to these experiments as well. Recall that the procedure of \( S_q \) involves convolving the data set with a ‘square tooth’ of width \( q \), then adding the third powers of the convolved data points within each sliding window (see the section on optimal statistics for a discussion of these steps.) For more complicated signatures, we need to modify both the convolution function and the choice of the third power.

The square tooth was a natural choice for the convolution function because it approximates the ‘bump’ signature form which we are looking for. We will continue to convolve with a function or width \( q \) that approximates the signature form being sought. Unfortunately, extra complications arise when the gradient–signature being sought crosses zero (as it does for all but single-difference experiments.) The statistics \( \{ S_q \} \) designed to search for bumps with width of about \( q \) data points, were fairly powerful for a wide range of other widths as well. But when searching for a signature which changes sign, a mismatch of widths can leave the statistic searching for a form which is orthogonal to that actually present, thus cancelling the result. The more zero-crossings the signature contains, the more critical it becomes to use an accurate width. This requires knowledge not only of the instrument response function but also of the expected width of the gradient region on the sky. The search for non-gaussianity becomes uncomfortably theory-specific.
Even if the width is chosen perfectly, the convolution of a signature function with an approximation of itself will yield several adjacent points which are strongly non-zero but alternate in sign as rapidly as the signature itself does. If we added the third (or any odd) power of these convolved points, they would cancel one another. A simple solution is to raise the convolved data to the fourth power rather than the third, suffering a slight decrease in resolving power but gaining robustness. For example, in a third-derivative experiment, such as Dragovan et. al.’s Python (Dragovan et. al. 1993), a sharp gradient might be best resolved by a statistic of the form

\[ S \sim \sum_j (x_j - 2x_{j+1} + x_{j+2})^4 \]

The coefficients of \( x_j \), \( x_{j+1} \), and \( x_{j+2} \) are fairly obvious guesses, matched to the expected form of the data (Figure 1). To detect wider gradient regions or other signature forms, simply use an appropriate approximation to the shapes shown in Figure 1; for example, a bump of width four would respond to

\[ S \sim \sum_j (x_j - x_{j+1} - x_{j+2} + x_j)^4 \]

while in a double-difference experiment, gradient–signature regions with width of about three data points would respond well to

\[ S \sim \sum_j (x_j - x_{j+2})^4 \]

There is no need to modify the ‘sliding window’ scheme as we look for more intricate signature forms; windows of length \( 3(q - 1) \) still work quite well.

We have performed Monte Carlo simulations which confirm that statistics such as these are much more powerful than simple skewness or kurtosis at detecting the signatures of sharp gradient regions in double- and triple-difference experiments. However, there is another option. Double- and triple-difference results are often constructed in stages, starting with single-difference data and combining adjacent points. It may be best to look for regions of sharp gradient in the original, single-difference data, where they will appear as simple bumps and can be detected by the comparatively robust statistics \( \{S_q\} \). The disadvantage to this approach is that single-difference results may be more vulnerable to systematic errors; we cannot predict in general which approach will work best for all experiments.
7. Conclusions

We have proposed a class of statistics which should be quite powerful in detecting a wide range of non-gaussian features in one-dimensional data sets. Their greatest potential vulnerability is that gaussian data with significant correlations on the scale of the spacing between data sets may be hard to distinguish from non-gaussian forms. This could occur in cosmological models with unusually strong power spectra at large angular scales (Crittenden et. al. 1993), in experiments with a short distance between data points, or in cases where correlations introduced by the instrument’s response function are similar in form to the non-gaussian ‘signature’ being sought. Any of these factors may make conclusions harder to draw, but they should not lead to false rejections of a gaussian theory, as long as the null data sets accurately model the gaussian theory and the instrument’s properties.

Our analysis of the UCSB SP91 experiment indicates that the CMBR anisotropy is inconsistent with ‘standard’ inflation. If, as the theory predicts, the CMBR fluctuations are gaussian, the nongaussianity of the data must be due to foreground contamination. If we discard the contaminated points (those responsible for the non-gaussian shape) we find a level of fluctuations significantly smaller than the theory predicts. One might worry about possible ‘conspiracies’ here - if a high fluctuation in the CMBR actually contributed to the nongaussian ‘bump’, we would throw it away when we removed the contaminated points. Could this not bias us towards low amplitudes in the remaining points? No, not in a theory like standard inflation, because the remaining points are so weakly correlated with the removed points, that they still provide a fair sample. Of course, like any other conclusions drawn from the still very limited data on CMBR anisotropy, our conclusion requires confirmation from further experimental results.

It is also important to check whether current non-gaussian theories are ‘nongaussian enough’ to account for a data set like USCB SP91. We have calculated our statistic $S_{3;6}$ on a small number of sky maps produced by simulations of cosmic texture (Coulson et. al. 1993), with the preliminary result that the distribution of values of $S_{3;6}$ is indeed substantially broader than that of standard inflation. Further simulation results will soon allow a more definite conclusion, along the lines indicated in the introduction (equation (1)).

We conclude that either

i) The CMBR fluctuations are nongaussian, or gaussian with a larger coherence angle than standard inflation. In either case the anisotropy pattern will hold valuable new information
about the mechanisms of structure formation.

ii) The fluctuations are gaussian with a small coherence angle, and small, too small for the
standard inflationary theory, but are overlaid with significant foreground contamination.

iii) The UCSB SP91 data set is not a representative sample of the microwave sky.

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Figure Captions
**Figure 1:** The characteristic ‘signatures’ which would result from a step function or region of very sharp gradient in the CMBR, for single- through triple-difference experiments (that is, experiments whose results approximate the first through the third derivatives of the CMBR intensity.) The purpose of this paper is to develop methods of detecting these signatures against a background of instrument and other noise.

**Figure 2:** Probability distributions of the statistic $S_{3;6}$ (that is, $S_3$ as defined in (2), calculated in ‘sliding windows’ of length six) for the ‘bumpy’ and null models described in the section on Monte Carlo results. The two distributions show very little overlap, so $S_{3;6}$ appears to be powerful at detecting certain types of non-gaussianity.

**Figure 3:** Probability distributions of skewness, calculated in ‘sliding windows’ of length six, for the same two models used in Fig. 2. There is considerable overlap; skewness is much less powerful than $S_{3;6}$ at distinguishing between data sets drawn from these two models.

**Figure 4:** The average significance levels achieved by the statistics $S_{q;3(q-1)}$ for $q = 1\ldots5$, when searching for bumps of various widths (full-width-half-max ranges from 1 to 10.) We see that to detect bumps of width $\sim t$ data points, it is best to choose $q$ roughly equal to (or slightly lower than) $t$.

**Figure 5:** The average significance levels achieved by the statistics $S_{3;6}^{(p)}$ defined in (10) as a function of the power $p$, when used to discriminate gaussian noise from two different nongaussian models. One model employs bumps of a gaussian profile, the other bumps of a square tooth profile. We consider the former more physically reasonable, and consequently adopt the power $p = 3$ in our subsequent analysis.

**Figure 6:** A set of results from the UCSB SP91 experiment. This shows the temperature offset $\delta T$, averaged over all frequency channels, for each of 13 points separated by 2.1 degrees. UCSB SP91 is a single-difference experiment, so these values actually represent (roughly) the difference between the CMBR intensities at two different points. The ‘spike’ visible at points 7 and 8 thus suggests a region of sharp gradient in the CMBR, unless the data are contaminated at these points.

**Figure 7:** The significance levels at which each of UCSB SP91’s four channels, as well as the mean of these channels, can be shown non-gaussian by each of the statistics $S_{q;3(q-1)}$ for $1 \leq q \leq 13$. $S_{3;6}$ shows the best overall performance.