MULTIPLICITY OF CLOSED REEB ORBITS ON PREQUANTIZATION BUNDLES

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Abstract. We establish multiplicity results for geometrically distinct contractible closed Reeb orbits of non-degenerate contact forms on a broad class of prequantization bundles. The results hold under certain index requirements on the contact form and are sharp for unit cotangent bundles of CROSS’s. In particular, we generalize and put in the symplectic-topological context a theorem of Duan, Liu, Long, and Wang for the standard contact sphere. We also prove similar results for non-hyperbolic contractible closed orbits and briefly touch upon the multiplicity problem for degenerate forms. On the combinatorial side of the question, we revisit and reprove the enhanced common jump theorem of Duan, Long and Wang, and interpret it as an index recurrence result.

Contents

1. Introduction 2  
2. Main results 3  
2.1. Multiplicity results for closed Reeb orbits 3  
2.2. Existence of non-hyperbolic periodic orbits 7  
2.3. The case of a degenerate form 8  
3. Preliminaries 9  
3.1. The Conley–Zehnder index for paths of symplectic matrices 9  
3.2. The Conley–Zehnder index of periodic orbits 10  
3.3. Equivariant symplectic homology 11  
3.4. Equivariant symplectic homology of prequantizations 12  
3.5. Local equivariant symplectic homology, resonance relation and Morse inequalities 15  
4. Index recurrence 16  
4.1. The index recurrence theorem 16  
4.2. Proof of Theorem 4.1 16  
5. Proofs of Theorems 2.1 and 2.7 20  
5.1. Proof of Theorem 2.1 20  
5.2. Proof of Theorem 2.7 27  
6. Multiplicity results and the contact Conley conjecture via contact homology 27  
6.1. Contact homology 27  
6.2. Multiplicity results 28  
6.3. Contact Conley conjecture 29  
References 29

1. Introduction

We establish multiplicity results for geometrically distinct contractible closed Reeb orbits of non-degenerate contact forms on a broad class of prequantization bundles. The results hold under certain index requirements on the contact form and are sharp for unit cotangent bundles of CROSS’s. In particular, we generalize and put in the symplectic-topological context a theorem of Duan, Liu, Long, and Wang for the standard contact sphere. We also prove similar results for non-hyperbolic contractible closed orbits and briefly touch upon the multiplicity problem for degenerate forms. On the combinatorial side of the question, we revisit and reprove the enhanced common jump theorem of Duan, Long and Wang, and interpret it as an index recurrence result.

2. Main results

2.1. Multiplicity results for closed Reeb orbits

2.2. Existence of non-hyperbolic periodic orbits

2.3. The case of a degenerate form

3. Preliminaries

3.1. The Conley–Zehnder index for paths of symplectic matrices

3.2. The Conley–Zehnder index of periodic orbits

3.3. Equivariant symplectic homology

3.4. Equivariant symplectic homology of prequantizations

3.5. Local equivariant symplectic homology, resonance relation and Morse inequalities

4. Index recurrence

4.1. The index recurrence theorem

4.2. Proof of Theorem 4.1

5. Proofs of Theorems 2.1 and 2.7

5.1. Proof of Theorem 2.1

5.2. Proof of Theorem 2.7

6. Multiplicity results and the contact Conley conjecture via contact homology

6.1. Contact homology

6.2. Multiplicity results

6.3. Contact Conley conjecture

References
1. Introduction

The main theme of this paper is the multiplicity problem for geometrically distinct contractible closed Reeb orbits of non-degenerate contact forms satisfying certain index conditions on a broad class of prequantization bundles. The multiplicity results established here apply to the unit cotangent bundles of CROSS’s (compact rank one symmetric spaces) for which they are sharp and to some other prequantization bundles. In particular, we generalize and put in the symplectic-topological context the main theorem from [14] on the multiplicity of simple closed Reeb orbits on the standard contact $S^{2n+1}$. On the combinatorial side of the question, we revisit and reprove the enhanced common jump theorem from [13] and interpret it as an index recurrence result along the lines of the index analysis from [21].

The multiplicity problem for geometrically distinct closed Reeb orbits originated in Hamiltonian dynamics, going back at least a hundred years. In its modern form, the question is about establishing a lower bound, ideally sharp, for the number of such orbits of a contact form $\alpha$ on a given contact manifold $(M^{2n+1}, \xi)$. The form $\alpha$ is usually required to meet some additional conditions playing both conceptual and technical roles. Here, for instance, we mainly focus on non-degenerate contact forms. Then a suitable homology theory associated with an action functional is utilized to detect closed Reeb orbits. In our case, this is the equivariant symplectic homology, i.e., essentially Floer theory.

The fundamental difficulty in the multiplicity problem, at least in dimensions $2n + 1 \geq 3$, lies not in the choice of homology theory but in distinguishing simple orbits from iterated ones. This difficulty already manifests itself in the classical problem of the existence of infinitely many simple closed geodesics for a Riemannian metric on $S^n$, which is wide open for $n \geq 2$. To get around this problem, one invariably has to impose restrictions on the index or action of closed Reeb orbits.

To illustrate the state of the art of multiplicity results for $2n + 1 > 3$, let us consider the simplest example of the standard contact structure $\xi$ on $S^{2n+1}$ without trying to give a comprehensive account even in this case. Hypothetically, every contact form $\alpha$ supporting $\xi$ has at least $n + 1$ simple closed Reeb orbits. This conjecture, however, is very far from being proved when $n \geq 2$. (See [11, 23, 33] for the proofs when $n = 1$.) In general, without any non-degeneracy or index/action assumptions, it is not even known if there is more than one simple closed Reeb orbit if $n \geq 2$. When $\alpha$ is non-degenerate, it is easy to see that there must be at least two such orbits (see, e.g., [25, 32]), but the existence of three simple orbits on, say, $S^5$ is already a difficult open question.

The situation changes dramatically once we impose further restrictions on the indices or actions of closed Reeb orbits. Putting action requirements aside, although these are also of considerable interest, we will focus on the index constraints which are more relevant to our goals here. In a series of papers starting with a groundbreaking work of Long and Zhu, [34, 35], various multiplicity results have been proved under what is usually referred to as the dynamical convexity assumption; see [2, 21, 26, 41, 42] and references therein. For $S^{2n+1}$ this is the requirement that all closed Reeb orbits have Conley–Zehnder index at least $n + 2$ and follows from geometrical convexity; see, e.g., [1, 21, 27]. (When the form is degenerate, one has to replace the Conley–Zehnder index by its lower semicontinuous extension.) Then it has been shown that a non-degenerate dynamically convex contact form on $S^{2n+1}$ must have at least $n + 1$ simple closed Reeb orbits and, without the non-degeneracy assumption, the number of orbits is at least $\lceil (n + 1)/2 \rceil + 1$. Some of these results and methods carry over to
other contact manifolds, e.g., to certain prequantization bundles, although then the notion of dynamical convexity gets more involved; cf. [2].

More recently, in [14], the existence of $n + 1$ simple closed Reeb orbits for non-degenerate forms on $S^{2n+1}$ was established under a condition which is less restrictive than dynamical convexity. This condition is that all closed Reeb orbits have positive mean index and there are no orbits with Conley-Zehnder index 0 when $n$ is odd and index 0 or $\pm 1$ when $n$ is even.

Our main goal in this paper is to extend this result to some other prequantization bundles including the unit cotangent bundles of CROSS’s. This is done in Theorem 2.1 and its corollaries; see Section 2.1. In particular, we establish the existence of at least two geometrically distinct closed geodesics for a bumpy Finsler metric on a CROSS; cf. [12]. We also show that many of the orbits found in the setting of Theorem 2.1 are non-hyperbolic; see Section 2.2 and, in particular, Theorem 2.10 and Corollary 2.11 generalizing the results from [13, 14]. Finally, in Section 2.3, for the sake of comparison we briefly touch upon the case where the contact form is degenerate.

The proof of Theorem 2.1, similarly to the proof of the multiplicity theorem in [14], hinges on a combinatorial result – the so-called enhanced common index jump theorem – enabling one to distinguish simple closed orbits from iterated ones. In Section 4, we revisit and reprove this theorem from the perspective of index recurrence; cf. [21, Sect. 5].

On the technical side, as has been mentioned above, the proof of our main theorem relies on the machinery of equivariant symplectic homology treated in a somewhat unconventional way following [21, Sect. 3]; see Section 3.3. This machinery necessitates certain fillability requirements or index lower bounds, which limit the class of prequantization bundles and contact forms in the main theorem. If the equivariant symplectic homology is replaced by contact homology, also used in a slightly non-standard form (see Section 6.1), the main theorem can be further generalized. This generalization is discussed in Section 6.2; see Theorem 6.2.

Another application of the variant of the contact homology from Section 6.1 is a refinement of the contact Conley conjecture originally proved in [22] and asserting the unconditional existence of infinitely many simple closed Reeb orbits on some prequantization bundles, not forced by homological growth. This is Theorem 6.4 in Section 6.3. We refer the reader to [20] for a detailed survey of the results on the Conley conjecture.

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2. Main results

2.1. Multiplicity results for closed Reeb orbits. Let $(M^{2n+1}, \xi)$ be a closed contact manifold satisfying $c_1(\xi)|_{\pi_2(M)} = 0$ and let $\alpha$ be contact form supporting the contact structure $\xi$. We call $\alpha$ index-positive (resp. index-negative) if the mean index $\hat{\mu}(\gamma)$ is positive (resp. negative) for every contractible periodic orbit $\gamma$ of $\alpha$ and index-definite when $\alpha$ is index-positive or index-negative. Note that these requirements are notably weaker than the standard notions of index positivity/negativity where, in, say, the positive case, the mean index is required to grow at least linearly with the action; cf. Lemma 3.3. However, the requirements become equivalent when the Reeb flow has only finitely many contractible simple closed orbits. The form $\alpha$ is said to be index-admissible if it has no closed orbits with index $2 - n$ or $2 - n \pm 1$ contractible in $M$. Below, as is customary, a non-degenerate periodic orbit $\gamma$ is called good if its Conley-Zehnder index $\mu(\gamma)$ has the same parity as that of the underlying simple closed orbit; see Section 3.2.
Throughout the paper we will focus on contact manifolds \((M^{2n+1}, \xi)\) which are prequantization circle bundles over closed integral symplectic manifolds \((B^{2n}, \omega)\), i.e., the first Chern class of the principle bundle \(M \to B\) is \(-[\omega]\). We will consider such contact manifolds which admit a “nice” symplectic filling and also the non-fillable ones. Accordingly, we will impose one of the following two conditions, \((F)\) and \((NF)\), in most of our results.

\((F)\)  
(i) The manifold \((M^{2n+1}, \xi)\) admits a strong symplectic filling \((W, \Omega)\) which is symplectically aspherical, i.e., \(\Omega|_{\pi_2(W)} = 0\) and \(c_1(TW)|_{\pi_2(W)} = 0\), and the map \(\pi_1(M) \to \pi_1(W)\) induced by the inclusion is injective.

(ii) The contact form \(\alpha\) is non-degenerate, index-definite and has no contractible good periodic orbits \(\gamma\) such that \(\mu(\gamma) = 0\) if \(n\) is odd or \(\mu(\gamma) \in \{0, \pm 1\}\) if \(n\) is even.

\((NF)\) We have \(c_1(\xi) = 0\) in \(H^2(M; \mathbb{Z})\) and \(B\) is spherically positive monotone. Furthermore, the contact form \(\alpha\) is non-degenerate, index-positive, index-admissible and has no contractible good periodic orbits \(\gamma\) such that \(\mu(\gamma) = 0\) if \(n\) is odd or \(\mu(\gamma) \in \{0, \pm 1\}\) if \(n\) is even.

Note that in the setting of Part (i) of \((F)\), \(B\) is necessarily spherically monotone. (We show this in the proof of Proposition 3.1.) Likewise, the condition that \(c_1(\xi) = 0\) from \((NF)\) implies via the Gysin exact sequence that \(c_1(TB) = \lambda[\omega]\) in \(H^2(B; \mathbb{Q})\) for some \(\lambda \in \mathbb{R}\), i.e., the symplectic manifold \((B, \omega)\) is positive or negative monotone in a very strong sense. (Then \(\lambda > 0\) since \(B\) is also spherically positive monotone.)

A word is also due on the role of the condition that \(\alpha\) is simultaneously index-positive and index-admissible in \((NF)\). This condition is equivalent to that all contractible periodic orbits have index greater than \(3 - n\) whenever the contact form is index-definite (more precisely, whenever the contact form has no contractible closed orbits with zero mean index). As a consequence, the positive equivariant symplectic homology of \(M\) is defined and well-defined without a filling of \((M, \alpha)\) when \((NF)\) holds; see Section 3.3 and [8, Sect. 4.1.2].

Our main result is Theorem 2.1 which, under some index conditions, establishes a sharp lower bound for the number of contractible closed Reeb orbits on certain prequantization \(S^1\)-bundles. In what follows, given a symplectic manifold \(B\), denote by \(\chi(B)\) its Euler characteristic and by

\[ c_B := \inf\{k \in \mathbb{Z}^+ \mid \exists S \in \pi_2(B) \text{ with } \langle c_1(TB), S \rangle = k\} \]

its minimal Chern number.

**Theorem 2.1.** Let \((M^{2n+1}, \xi)\) be a prequantization \(S^1\)-bundle of a closed symplectic manifold \((B, \omega)\) such that \([\omega]|_{\pi_2(B)} \neq 0\) and \(c_B > n/2\) and, furthermore, \(H_k(B; \mathbb{Q}) = 0\) for every odd \(k\) or \(c_B > n\). Let \(\alpha\) be a contact form supporting \(\xi\) and assume that \(M\) and \(\alpha\) satisfy condition \((F)\) or \((NF)\). Then \(\alpha\) carries at least \(r_B\) geometrically distinct contractible periodic orbits, where

\[ r_B := \begin{cases} 
\chi(B) + 2 \dim H_n(B; \mathbb{Q}) & \text{if } n \text{ is odd} \\
\chi(B) + 4 \dim H_{n-1}(B; \mathbb{Q}) & \text{if } n \text{ is even}.
\end{cases} \]

**Remark 2.2.** Strictly speaking, the prequantization \((M^{2n+1}, \xi)\) is uniquely determined by a lift of the de Rham cohomology class of \(\omega\) to \(H^2(M; \mathbb{Z})\) but not, in general, by the de Rham cohomology class itself. The ambiguity in the lift is the torsion \(T = \text{Tors}(H^2(B; \mathbb{Z}))\), which by the universal coefficient theorem is also equal to \(\text{Tors}(H_1(B; \mathbb{Z}))\); cf. [22, Rmk. 2.3]. In what follows, we will tacitly assume that a lift is fixed and use the notation \([\omega]\) for either the lift or, depending on the context, the de Rham cohomology class, i.e., an element of \(H^2(M; \mathbb{Z})/T\).
Corollary 2.5. Let \((M^{2n+1}, \xi)\) be a prequantization \(S^1\)-bundle of a closed symplectic manifold \((B, \omega)\) such that \(\omega|_{\pi_2(B)} \neq 0\), \(c_B > n/2\) and \(H_k(B; \mathbb{Q}) = 0\) for every odd \(k\). Let \(\alpha\) be a contact form supporting \(\xi\). Assume that \(M\) and \(\alpha\) satisfy either condition \((F)\) or condition \((NF)\). Then \(\alpha\) carries at least \(r_B\) geometrically distinct contractible periodic orbits, where \(r_B\) is the total rank of \(H_s(B; \mathbb{Q})\).

Examples satisfying the hypotheses of Corollary 2.5 include the standard contact sphere \(S^{2n+1}\) and the unit cosphere bundle of a compact rank one symmetric space \((\text{CROSS})\). More precisely, \(S^{2n+1}\) is the prequantization of \(\mathbb{C}P^n\), and its obvious filling in \(\mathbb{R}^{2n+2}\) satisfies \((F)\). A compact rank one locally symmetric space \(N\) is a closed Riemannian manifold such that...
its curvature tensor is invariant under parallel transport and the maximal dimension of a flat totally geodesic submanifold is one. By the classification of symmetric spaces, a CROSS must be one of the following manifolds: $S^m$, $\mathbb{R}P^m$, $\mathbb{C}P^m$, $\mathbb{H}P^m$ and $CaP^2$; see [4] for details. Thus the filling of the unit cosphere bundle $S^*N$ given by the unit disk bundle $D^*N$ in $T^*N$ clearly meets the condition (F) unless $N$ is $S^2$ or $\mathbb{R}P^2$ (which are the only cases where $\pi_1(S^*N) \to \pi_1(D^*N)$ is not injective). However, in these cases it is well known that every Reeb flow has at least two simple closed orbits.

Every CROSS $N$ admits a metric such that all of its geodesics are periodic of the same minimal period; in other words, the geodesic flow generates a free circle action on $S^*N$. Thus the unit cosphere bundle $S^*N$ is the prequantization of a closed symplectic manifold $(B, \omega)$. Moreover, a homological computation shows that $H_k(B; \mathbb{Q}) = 0$ for every odd $k$; see [44, page 141]. In this case, the total rank $r_B$ of $H_*(B; \mathbb{Q})$ and the minimal Chern number $c_B$ are given in the following table.

| Prequantization | $r_B = \dim H_*(B; \mathbb{Q})$ | $c_B$ |
|-----------------|---------------------------------|-------|
| $S^{2n+1}$      | $n+1$                          | $n+1$ |
| $S^*S^2$ or $S^*\mathbb{R}P^2$ | $2$                          | $2$   |
| $S^*S^m$ or $S^*\mathbb{R}P^m$ with $m > 2$ even | $m$                      | $m-1$ |
| $S^*S^m$ or $S^*\mathbb{R}P^m$ with $m$ odd | $m+1$                     | $m-1$ |
| $S^*\mathbb{C}P^m$ | $(m+1)$                     | $m$   |
| $S^*\mathbb{H}P^m$ | $2m(m+1)$                   | $2m+1$ |
| $S^*CaP^2$      | $24$                          | $11$  |

Notice that the hypothesis on $c_B$ in Corollary 2.5 barely holds for $M = S^*\mathbb{C}P^m$, where $\dim B/4 = m - 1/2$ and $c_B = m$. We have the following consequence of Corollary 2.5, which was previously proved for the standard contact sphere by Duan, Liu, Long and Wang in [14] and for Finsler metrics on a simply connected CROSS by Duan, Long and Wang in [13].

**Corollary 2.6.** Let $(M, \xi)$ be either the standard contact sphere $S^{2n+1}$ or the unit cosphere bundle $S^*N$ of a CROSS and let $\alpha$ be a contact form supporting $\xi$. Assume that $\alpha$ satisfies condition (F). Then $\alpha$ has at least $r_B$ geometrically distinct periodic orbits, where $r_B$ is given by the table above.

The standard contact sphere and the unit cosphere bundle of a CROSS (with dimension bigger than two) satisfy the assumption (F) and therefore the only condition on the contact form in Corollary 2.6 is that it is index-definite and has no good contractible periodic orbits $\gamma$ such that $\mu(\gamma) = 0$ if $n$ is odd or $\mu(\gamma) \in \{0, \pm 1\}$ if $n$ is even. Furthermore, the prequantization bundles in Corollary 2.6 admit contact forms with precisely $r_B$ geometrically distinct periodic orbits. These contact forms are given by irrational ellipsoids and the Katok-Ziller Finsler metrics; [44]. This shows that the lower bound in Theorem 2.1 is sharp. To the best of our knowledge, all the examples of prequantization $S^1$-bundles admitting contact forms with finitely many simple closed Reeb orbits known so far satisfy the hypothesis that $H_* (B; \mathbb{Q})$ vanishes in odd degrees.

As an easy application of Theorem 2.1, we establish, with no index assumptions, the existence of at least two geometrically distinct contractible closed orbits for any non-degenerate contact form on manifolds as in Corollary 2.5 satisfying (F); see Section 5.2.

**Theorem 2.7.** Let $(M^{2n+1}, \xi)$ be a prequantization $S^1$-bundle of a closed symplectic manifold $(B, \omega)$ such that $\omega|_{\pi_2(B)} \neq 0$, $c_B > n/2$, and $H_k(B; \mathbb{Q}) = 0$ for every odd $k$. Assume that $M$
satisfies Part (i) of condition (F). Then every non-degenerate contact form \( \alpha \) supporting \( \xi \) has at least two geometrically distinct contractible periodic orbits.

This theorem combined with the above discussion implies the following corollary.

**Corollary 2.8.** Let \((M, \xi)\) be the standard contact sphere \( S^{2n+1} \) or the unit cosphere bundle \( S^N \) of a CROSS. Then every non-degenerate contact form supporting \( \xi \) has at least two geometrically distinct contractible periodic orbits.

The next result is closely related to a theorem of Duan, Long and Wang asserting the existence of two geometrically distinct closed geodesics for a bumpy metric on a simply connected manifold; [12].

**Corollary 2.9.** Every bumpy Finsler metric on a CROSS has at least two geometrically distinct closed geodesics.

There are also a few examples where \( M \) does not obviously meet the requirements of Part (i) of (F) but for a suitable form \( \alpha \) can satisfy (NF). Among these are the prequantizations of the following manifolds \((B, \omega)\): the complex Grassmannians \( \text{Gr}_C(2; m) \), \( \text{Gr}_C(3; 6) \) and \( \text{Gr}_C(3; 7) \), the monotone products \( \mathbb{CP}^m \times \mathbb{CP}^m \) (cf., [19, Sect. 1.2]) and also the monotone products \( \mathbb{CP}^m \times \text{Gr}_C^+ \mathbb{R}(2; m+3) \) where the second factor is a real oriented Grassmannian and its minimal Chern number is \( m + 1 \). For these manifolds \( B \) the lower bound \( r_B \) from Theorem 2.1 is sharp. (The reason is that \( B \) admits a Hamiltonian circle or torus action with isolated fixed points. Such an action has exactly \( r_B \), the sum of Betti numbers, fixed points and the required Reeb flow is then obtained by lifting a flow generating the action to \( M \).)

2.2. **Existence of non-hyperbolic periodic orbits.** The proof of Theorem 2.1 also yields the following multiplicity result concerning non-hyperbolic closed orbits when the contact form has finitely many geometrically distinct contractible closed orbits. Recall that a closed orbit is hyperbolic if its linearized Poincaré map has no eigenvalues on the unit circle.

**Theorem 2.10.** Let \((M^{2n+1}, \xi)\) be a prequantization \( S^1 \)-bundle of a closed symplectic manifold \((B, \omega)\) such that \( \omega|_{\pi_2(B)} \neq 0 \) and \( c_B > n/2 \) and, furthermore, \( H_k(B; \mathbb{Q}) = 0 \) for every odd \( k \) or \( c_B > n \). Let \( \alpha \) be a contact form supporting \( \xi \) with finitely many geometrically distinct contractible closed orbits. Assume that \( M \) and \( \alpha \) satisfy either condition (F) or condition (NF). Then \( \alpha \) carries at least \( r_B^{\text{non-hyp}} \) geometrically distinct contractible non-hyperbolic periodic orbits, where

\[
r_B^{\text{non-hyp}} := r_B - \dim H_n(B; \mathbb{Q}) = \begin{cases} 
\chi(B) + \dim H_n(B; \mathbb{Q}) & \text{if } n \text{ is odd} \\
\chi(B) + 4 \dim H_{n-1}(B; \mathbb{Q}) - \dim H_n(B; \mathbb{Q}) & \text{if } n \text{ is even.}
\end{cases}
\]

This result immediately follows from the proof of Theorem 2.1; see Remarks 5.2 and 5.3. Clearly, under the additional assumption that the contact form has finitely many geometrically distinct contractible closed orbits, all the applications of Theorem 2.1 have analogous statements replacing \( r_B \) by \( r_B^{\text{non-hyp}} \). For instance, when \( M \) is the standard contact sphere or
the unit cosphere bundle of a CROSS, a computation yields the following table:

| Prequantization | $r_{B}^{\text{non-hyp}}$ |
|-----------------|--------------------------|
| $S^{2n+1}$ with n even | n |
| $S^{2n+1}$ with n odd | n + 1 |
| $S^*S^m$ or $S^*\mathbb{R}P^m$ with m even | m |
| $S^*S^m$ or $S^*\mathbb{R}P^m$ with m odd | m(m + 1) |
| $S^*\mathbb{C}P^m$ | 2m(m + 1) |
| $S^*\mathbb{H}P^m$ | 24 |
| $S^*\text{Ca}P^2$ | 24 |

Thus we obtain the following corollary which, again, was previously proved for the standard contact sphere in [14] and for Finsler metrics on a simply connected CROSS in [13].

**Corollary 2.11.** Let $(M, \xi)$ be the standard contact sphere $S^{2n+1}$ or the unit cosphere bundle $S^*N$ of a CROSS. Let $\alpha$ be a contact form supporting $\xi$ satisfying the hypothesis (F) and having finitely many geometrically distinct contractible closed orbits. Then $\alpha$ has at least $r_{B}^{\text{non-hyp}}$ non-hyperbolic geometrically distinct contractible periodic orbits, where $r_{B}^{\text{non-hyp}}$ is given by the previous table.

2.3. The case of a degenerate form. It is interesting to compare Theorem 2.1 with the lower bounds one has without the non-degeneracy condition on the form $\alpha$. In this case, the index restrictions become much more severe and the lower bound $r$ on the number of simple closed Reeb orbits much weaker. In particular, $r$ depends only on the dimension of $M$ and the index lower bound but not on the topology of $B$. To be more precise, denote by $\mu_-$ the lower semicontinuous extension of the Conley–Zehnder index; see, e.g., [1, Sect. 3] or [21, Sect. 4.1.2]. We have the following result:

**Theorem 2.12.** Let $(M^{2n+1}, \xi)$ be a prequantization $S^1$-bundle of a closed symplectic manifold $(B, \omega)$ such that $\omega|_{\pi_2(B)} \neq 0$ and $c_B > n/2$ and, furthermore, $H_k(B; \mathbb{Q}) = 0$ for every odd $k$ or $c_B > n$. Assume, in addition, that $M$ satisfies Part (i) of condition (F) and the filling $W$ is exact. Let $\alpha$ be a contact form supporting $\xi$ such that $\mu_-(\gamma) \geq q$ for all, not necessarily simple, contractible closed Reeb orbits $\gamma$. Then $M$ carries at least $r$ geometrically distinct contractible closed Reeb orbits, where

$$r = \begin{cases} 
q - \lfloor (n + 1)/2 \rfloor & \text{when } n \text{ is even and } q \text{ is odd,} \\
q + 1 - \lfloor (n + 1)/2 \rfloor & \text{otherwise.}
\end{cases}$$

Here the result is void if $r \leq 0$. The main class of manifolds this theorem applies to is again the unit cotangent bundles of CROSS’s. For $S^*S^m$ (already considered in [21]) and $S^*\mathbb{R}P^m$ the theorem yields, depending on $q$, the existence of a number of geometrically distinct periodic orbits and of two such orbits for $S^*\mathbb{H}P^1$ when $q = 3$. The main limitation comes from the fact that $q$ cannot be larger than the minimal degree $d$ where the relevant symplectic homology for contractible orbits is non-trivial. For $S^*S^m$ and $S^*\mathbb{R}P^m$ (with $m > 2$), we have $d = m - 1$; for $S^*\mathbb{C}P^m$, $S^*\mathbb{H}P^m$ and $S^*\text{Ca}P^2$, we have $d = 1$, 3 and, respectively, 7; see [1]. Most likely, Theorem 2.12, in contrast with Theorem 2.1, is very far from being sharp. In fact, one can expect that a degenerate form necessarily has infinitely many simple closed Reeb orbits and, in particular, Theorem 2.1 holds without any non-degeneracy assumptions.

The proof of Theorem 2.12 uses Lusternik–Schnirelmann theory for the shift operator in equivariant symplectic homology developed in [21] and a variant of the index recurrence.
Theorem for a degenerate paths from [21, Sect. 5] or the common jump theorem from [34, 35]. The argument is essentially identical to the proofs of [21, Thm. 6.9 and Thm. 6.15] and we omit it. The requirement that \( W \) is exact is needed to ensure that the Hamiltonian action filtration of the symplectic homology agrees with the contact action.

3. Preliminaries

In this section we will review some basic concepts from the Conley-Zehnder index theory and equivariant symplectic homology used throughout the paper.

3.1. The Conley–Zehnder index for paths of symplectic matrices. To every continuous path \( \Phi : [0, 1] \to \text{Sp}(2n) \) beginning at \( \Phi(0) = I \), one can associate the mean index \( \hat{\mu}(\Phi) \in \mathbb{R} \), a homotopy invariant of the path with fixed end-points. The mean index \( \hat{\mu}(\Phi) \) measures the total rotation angle of certain unit eigenvalues of \( \Phi(t) \) and \( \hat{\mu}(\Phi_s) = \text{const} \) for a family of paths \( \Phi_s \) as long as the eigenvalues of \( \Phi_s(1) \) remain constant. The resulting map \( \hat{\mu} : \hat{\text{Sp}}(2n) \to \mathbb{R} \) is a unique quasimorphism on the universal covering \( \hat{\text{Sp}}(2n) \) of \( \text{Sp}(2n) \) which is continuous and homogeneous, i.e.,

\[
\hat{\mu}(\Phi^k) = k\hat{\mu}(\Phi),
\]

and satisfies the normalization condition

\[
\hat{\mu}(\Phi_0) = 2 \quad \text{for} \quad \Phi_0(t) = \exp(2\pi\sqrt{-1}t) \oplus I_{2n-2}
\]

with \( t \in [0, 1] \); see [3]. The quasimorphism condition asserts that \( \hat{\mu} \) fails to be a homomorphism only up to a constant, i.e.,

\[
|\hat{\mu}(\Phi \Psi) - \hat{\mu}(\Phi) - \hat{\mu}(\Psi)| \leq C_n,
\]

where the constant is independent of \( \Phi \) and \( \Psi \), but may depend on \( n \). (In fact, one may be able to take \( C_n = 4n \); [43].) We refer the reader to [34, 39] for a very detailed discussion of the mean index. In this paper we use conventions and notation from [21, Sec. 4].

Assume next that the path \( \Phi \) is non-degenerate, i.e., by definition, all eigenvalues of the end-point \( A = \Phi(1) \) are different from one. We denote the set of such matrices \( A \in \text{Sp}(2n) \) by \( \text{Sp}^*(2n) \) and also denote the part of \( \hat{\text{Sp}}(2n) \) lying over \( \text{Sp}^*(2n) \) by \( \hat{\text{Sp}}^*(2n) \). It is not hard to see that \( A \) can be connected to a symplectic transformation with elliptic part equal to \( -I \) (if non-trivial) by a path \( \Psi \) lying entirely in \( \text{Sp}^*(2n) \). Concatenating this path with \( \Phi \), we obtain a new path \( \Phi' \). By definition, the Conley–Zehnder index \( \mu(\Phi) \in \mathbb{Z} \) of \( \Phi \) is \( \hat{\mu}(\Phi') \). One can show that \( \mu(\Phi) \) is well-defined, i.e., independent of \( \Psi \). The function \( \mu : \hat{\text{Sp}}^*(2n) \to \mathbb{Z} \) is locally constant, i.e., constant on connected components of \( \hat{\text{Sp}}^*(2n) \). In other words, \( \mu(\Phi_s) = \text{const} \) for a family of paths \( \Phi_s \) as long as \( \Phi_s(1) \in \text{Sp}^*(2n) \) for every \( s \). Furthermore, we call \( \Phi \) strongly non-degenerate if all its “iterations” \( \Phi^k \) are non-degenerate, i.e., none of the eigenvalues of \( \Phi(1) \) is a root of unity.

In the rest of this section we briefly discuss the properties of the Conley–Zehnder type indices which are essential for our purposes, referring the reader to, e.g., [34, 39] for the proofs. Below all paths are required to begin at \( I \) and are taken up to homotopy, i.e., as elements of \( \hat{\text{Sp}}(2n) \).

We start with three specific examples. For the path \( \Phi(t) = \exp(2\pi\sqrt{-1}t\lambda), \ t \in [0, 1] \), we have

\[
\hat{\mu}(\Phi) = 2\lambda \quad \text{and} \quad \mu(\Phi) = \text{sign}(\lambda)(2|\lambda| + 1) \quad \text{when} \ \lambda \notin \mathbb{Z}.
\]
Next, let $H$ be a non-degenerate quadratic form on $\mathbb{R}^{2n}$ with eigenvalues in the range $(-\pi, \pi)$. (The eigenvalues of a quadratic form $H$ on a symplectic vector space are by definition the eigenvalues of its Hamiltonian vector field $X_H = J \nabla H$, where $J$ is the matrix of the symplectic form.) The path $\Phi(t) = \exp(JHt)$, $t \in [0, 1]$, is the linear autonomous Hamiltonian flow generated by $H$. Then, with our conventions,

$$\mu(\Phi) = \frac{1}{2} \text{sgn}(H),$$

where $\text{sgn}(H)$ is the signature of $H$, i.e., the number of positive squares minus the number of negative squares in the diagonal form of $H$ with $\pm 1$ and $0$ on the diagonal. In addition, when $\Phi(1)$ is hyperbolic, we have

$$\mu(\Phi) = \hat{\mu}(\Phi).$$

Furthermore,

$$\mu(\Phi^{-1}) = -\mu(\Phi)$$

for any non-degenerate path $\Phi$. When $\varphi$ is a loop, we also have

$$\mu(\varphi \Phi) = \hat{\mu}(\varphi) + \mu(\Phi).$$

Finally, $\hat{\mu}$ and $\mu$ are additive under direct sum. Namely, for $\Phi \in \widetilde{\text{Sp}}(2n)$ and $\Psi \in \widetilde{\text{Sp}}(2n')$, we have

$$\hat{\mu}(\Phi \oplus \Psi) = \hat{\mu}(\Phi) + \hat{\mu}(\Psi) \quad \text{and} \quad \mu(\Phi \oplus \Psi) = \mu(\Phi) + \mu(\Psi),$$

where in the second identity we assumed that both paths are non-degenerate. The mean index and the Conley–Zehnder index are related by the inequality

$$|\hat{\mu}(\Phi) - \mu(\Phi)| < n$$

where $\Phi \in \widetilde{\text{Sp}}^+(2n)$. As a consequence,

$$\lim_{k \to \infty} \frac{\mu(\Phi^k)}{k} = \hat{\mu}(\Phi),$$

and hence the name “mean index” for $\hat{\mu}$.

### 3.2. The Conley–Zehnder index of periodic orbits.

Let $\gamma$ be a strongly non-degenerate periodic orbit of the Reeb vector field $R_\alpha$ and $\Psi: \gamma^* \xi \to S^1 \times \mathbb{R}^{2n}$ a symplectic trivialization of $\xi$ over $\gamma$. Denote by $\Psi_t: \xi(\gamma(t)) \to \mathbb{R}^{2n}$ the composition of $\Psi|_{\gamma^* \xi(t)}$ with the projection onto the second factor. Via this trivialization, the linearized Reeb flow gives rise to the symplectic path

$$\Phi(t) = \Psi_t \circ d\phi^t_\alpha(\gamma(0))|_\xi \circ \Psi^{-1}_0,$$

where $\phi^t_\alpha$ is the Reeb flow of $\alpha$. In this way, we define the Conley–Zehnder index and the mean index of $\gamma$ with respect to the trivialization $\Psi$ as

$$\mu(\gamma; \Psi) = \mu(\Phi) \quad \text{and} \quad \hat{\mu}(\gamma; \Psi) = \hat{\mu}(\Phi)$$

respectively. The Conley–Zehnder index and the mean index depend only on the homotopy class of $\Psi$. Indeed, if we choose another trivialization $\Upsilon: \gamma^* \xi \to S^1 \times \mathbb{R}^{2n}$ then we have the relation

$$\mu(\gamma; \Upsilon) = \mu(\gamma; \Phi) + 2\mu_{\text{Maslov}}(\Upsilon_t \circ \Phi_t^{-1}),$$

where $\mu_{\text{Maslov}}$ denotes the Maslov index which is a suitably chosen one of the two isomorphisms between $\pi_1(\text{Sp}(2n))$ and $\mathbb{Z}$. In particular, the parity of the index does not depend on the
The even/odd iterates of a periodic orbit are the same, i.e., for all $j, k \in \mathbb{N}$,

$$\mu(\gamma^{2j}; \Psi^{2j}) \equiv \mu(\gamma^{2k}; \Psi^{2k}) \quad \text{and} \quad \mu(\gamma^{2j-1}; \Psi^{2j-1}) \equiv \mu(\gamma^{2k-1}; \Psi^{2k-1}) \pmod{2}.$$ 

A periodic orbit of $\alpha$ is called good if its Conley–Zehnder index has the same parity as that of the index of the underlying simple closed orbit. (As has just been pointed out, the parity of the index does not depend on the choice of the trivialization of $\xi$.) A periodic orbit that is not good is called bad.

If $\gamma$ is contractible, there is a standard way to choose the trivialization $\Psi$ unique up to homotopy. Namely, consider a capping disk of $\gamma$, i.e., a smooth map $\sigma: D^2 \to M$, where $D^2$ is the two-dimensional disk, such that $\sigma|_{\partial D^2} = \gamma$. Choose a trivialization of $\sigma^*\xi$ and let $\Psi: \gamma^*\xi \to S^1 \times \mathbb{R}^{2n}$ be its restriction to the boundary, which gives a trivialization of $\xi$ over $\gamma$. Since $D^2$ is contractible, the homotopy class of $\Psi$ does not depend on the choice of the trivialization of $\sigma^*\xi$. Moreover, the condition that $c_1(\xi)|_{\pi_2(M)} = 0$ ensures that the homotopy class of $\Psi$ does not depend on the choice of $\sigma$ as well. Throughout the paper, whenever $\gamma$ is contractible, we denote by $\mu(\gamma)$ and $\hat{\mu}(\gamma)$ the Conley–Zehnder index and, respectively, the mean index of $\gamma$ with respect to the standard trivialization.

### 3.3. Equivariant symplectic homology

In this section we briefly recall several facts about positive equivariant symplectic homology, treating the subject from a slightly unconventional perspective.

Let first $(M, \xi)$ be a closed contact manifold and $(W, \Omega)$ be a strong symplectic filling of $M$ with $\Omega|_{\pi_2(W)} = 0 = c_1(TW)|_{\pi_2(W)}$. Furthermore, let $\alpha$ be a non-degenerate contact form on $M$ supporting the contact structure $\xi$. Then the positive equivariant symplectic homology $\text{SH}^{S^1,+}(W)$ with coefficients in $\mathbb{Q}$ is the homology of a complex $C_{\alpha}(\alpha)$ generated by the good closed Reeb orbits of $\alpha$; see [21, Prop. 3.3]. This complex is graded by the Conley–Zehnder index and filtered by the action. Furthermore, once we fix a free homotopy class of loops in $W$, the part of $C_{\alpha}(\alpha)$ generated by closed Reeb orbits in that class is a subcomplex. As a consequence, the entire complex $C_{\alpha}(\alpha)$ breaks down into a direct sum of such subcomplexes indexed by free homotopy classes of loops in $W$.

The differential in the complex $C_{\alpha}(\alpha)$, but not its homology, depends on several auxiliary choices, and the nature of the differential is not essential for our purposes. The complex $C_{\alpha}(\alpha)$ is functorial in $\alpha$ in the sense that a symplectic cobordism equipped with a suitable extra structure gives rise to a map of complexes. For the sake of brevity and to emphasize the obvious analogy with contact homology, we denote the homology of $C_{\alpha}(\alpha)$ by $H_{\alpha}(M)$ rather than $\text{SH}^{S^1,+}(W)$. The homology of the subcomplex formed by the orbits contractible in $W$ will be denoted by $H_{\alpha}^0(M)$. However, it is worth keeping in mind that $C_{\alpha}(\alpha)$ and hypothetically even the homology may depend on the choice of the filling $W$.

This description of the positive equivariant symplectic homology as the homology of $C_{\alpha}(\alpha)$ is not quite standard, but it is most suitable for our purposes. (We refer the reader to [21] for more details and further references and to [9, 40] for the original construction of the equivariant symplectic homology.) To see why $H_{\alpha}(M) := \text{SH}^{S^1,+}(W)$ can be obtained as the homology of a single complex generated by good closed Reeb orbits, let us first consider an admissible Hamiltonian $H$ on the symplectic completion of $W$ and focus on the orbits of $H$ with positive action. Such orbits are in a one-to-one correspondence with closed Reeb orbits $\gamma$ with action below a certain threshold $T$ depending on the slope of $H$. The $S^1$-equivariant Floer homology of $H$ is the homology of a Floer-type complex obtained from a non-degenerate
parametrized perturbation of $H$; [9, 40]. This complex is filtered by the action. The $E^1$-term of the resulting spectral sequence (over $\mathbb{Q}$) is generated by the good Reeb orbits of $\alpha$ with action below $T$. Now we can (canonically, once the generators are fixed) reassemble the differentials $\partial_r$ into a single differential $B$ on $CC^\ast(pH^q)$: $E^\infty = HF^S_t^\ast + (H)$. Roughly speaking, $\partial = \partial_1 + \partial_2 + \ldots$, where $\partial_r$ is suitably “extended” from $E^r$ to $E^1$. Moreover, this procedure respects the action filtration and is functorial with respect to continuation maps. Passing to the limit in $H$, we obtain the complex $CC^\ast(p\alpha^q)$ as the limit of the complexes $CC^\ast(pH^q)$; see [21, Sect. 2.5 and 3] for further details.

A remarkable observation by Bourgeois and Oancea in [8, Sect. 4.1.2] is that under suitable additional assumptions on the indices of closed Reeb orbits the positive equivariant symplectic homology is defined and well-defined even when $M$ does not have a symplectic filling. To be more precise, assume that $c_1(\xi)|_{\pi_2(M)} = 0$ and let $\alpha$ be a non-degenerate contact form on $M$ such that all of its closed contractible Reeb orbits have Conley–Zehnder index strictly greater than $3 - n$. Furthermore, under this assumption the proof of [21, Prop. 3.3] carries over essentially word-for-word, and hence again the positive equivariant symplectic homology of $M$ can be described as the homology of a complex $CC^\ast(\alpha)$ generated by good closed Reeb orbits of $\alpha$, graded by the Conley–Zehnder index and filtered by the action. The complex breaks down into the direct sum of subcomplexes indexed by free homotopy classes of loops in $M$. As in the fillable case, we will use the notation $HC^\ast(M)$ and $HC^\ast_0(M)$.

The assumption that all contractible orbits have index greater than $3 - n$ is equivalent to that $\alpha$ is simultaneously index-positive and index-admissible (assuming that there is no contractible closed orbit with zero mean index), which are parts of the requirement $(NF)$. Indeed, index positivity implies that all contractible orbits have index greater than $-n$ and the condition that $\alpha$ is index-admissible rules out the orbits of index $1 - n, 2 - n$ and $3 - n$. (The converse is obvious if there is no contractible periodic orbit with zero mean index.) Hence in case $(NF)$ of Theorem 2.1 the positive equivariant symplectic homology of $M$ is defined and well-defined without a filling of $M$.

### 3.4. Equivariant symplectic homology of prequantizations

The next proposition shows how to compute the equivariant symplectic homology of a suitable prequantization in terms of the homology of the basis. This computation will be crucial throughout this work.

**Proposition 3.1.** Let $(M^{2n+1}, \xi)$ be a prequantization of a closed symplectic manifold $(B, \omega)$ with $\omega|_{\pi_2(B)} \neq 0$ and such that $H_k(B; \mathbb{Q}) = 0$ for every odd $k$ or $c_B > n$.

(a) Assume that $M$ satisfies the requirements from Part (i) of $(F)$. Then, $B$ is spherically monotone. When $B$ is spherically positive monotone, the positive equivariant symplectic homology for contractible periodic orbits of $M$ is given by

\[
HC^0_\ast(M) \cong \bigoplus_{m \in \mathbb{N}} H_{\ast-2mc_B+n}(B; \mathbb{Q}).
\] (3.2)

When $B$ is spherically negative monotone, we have

\[
HC^0_\ast(M) \cong \bigoplus_{m \in \mathbb{N}} H_{\ast+2mc_B-n}(B; \mathbb{Q}).
\] (3.3)

In particular, in both cases the homology is independent of the choice of the filling $W$ satisfying Part (i) of $(F)$.
(b) Alternatively, assume that $B$ is spherically positive monotone with $c_B \geq 2$ and, as in (NF), $c_1(\xi) = 0$ and $\alpha$ is a non-degenerate contact form on $(M, \xi)$ such that all closed Reeb orbits have index greater than $3 - n$. Then (3.2) also holds.

In other words, (3.2) asserts that $H^q_c(M)$ is obtained by taking an infinite number of copies of $H_{* - q}(B; \mathbb{Q})$ with grading shifted up by positive integer multiples of $2c_B$ and adding up the resulting spaces.

**Remark 3.2.** Note that while the only known spherically positive monotone manifold meeting the requirements $c_B > n$ is $\mathbb{C}P^n$, there are numerous negative monotone manifolds satisfying this condition, e.g., complete intersections of high degree. Also recall that in (b), we necessarily have $c_1(TB) = \lambda[\omega]$ in $H^2(B; \mathbb{Q})$ for some $\lambda \in \mathbb{R}$, i.e., the symplectic manifold $(B, \omega)$ is positive or negative monotone in a very strong sense. (Then $\lambda \geq 0$ since $B$ is also spherically positive monotone.) This follows from the condition that $c_1(\xi) = 0$ and the Gysin exact sequence.

It is worth pointing out that in Case (a) of Proposition 3.1 the conditions, although quite restrictive, are purely of topological nature and ultimately imposed only on the symplectic manifold $(B, \omega)$. The homology in this case is defined for any contact form and given by (3.2) or (3.3). On the other hand, in Case (b) the conditions are imposed on both the manifold and the contact form $\alpha$ and the homology is defined and satisfies (3.2) only when $\alpha$ meets those requirements. Finally, note that the requirement that $c_B \geq 2$ from (b) is automatically satisfied in the setting of Theorem 2.1 as a consequence of the assumptions $\omega_{\pi_2(B)} \neq 0$ and $c_B > n/2$. Indeed, then $c_B \geq 2$ when $n > 1$ and for $n = 1$ we necessarily have $B = S^2$ and hence $c_B = 2$.

**Proof of Proposition 3.1.** Let us focus first on Case (a). To show that $B$ is spherically monotone note that $TW|_M$ decomposes as the direct sum of $\xi$ and a trivial complex line bundle. Hence $c_1(\xi)$ is the image of $c_1(TW)$ in $H^2(M; \mathbb{Z})$ and, as a consequence, $c_1(\xi)$ is an aspherical class. Next, arguing by contradiction, assume that $B$ is not spherically monotone. Then $c_1(TB)$ and $[\omega]$ are linearly independent as maps from $\pi_2(B) \otimes \mathbb{Q}$ to $\mathbb{Q}$. Therefore, as is easy to see, there exists $S \in \pi_2(M)$ such that $\langle c_1(TB), S \rangle > 0$ but $\langle [\omega], S \rangle = 0$. The restriction of the prequantization bundle to $S$ is trivial and $S$ admits a lift $S'$ to $M$. Then, since $c_1(\xi)$ is the pull-back of $c_1(TB)$, we have

$$\langle c_1(\xi), S' \rangle = \langle c_1(TB), S \rangle > 0.$$  

This is impossible because $c_1(\xi)$ is aspherical.

For the sake of simplicity we will assume throughout the rest of the proof of Case (a) that $B$ is positive monotone. (When $B$ is negative monotone, the argument is similar up to some sign changes.) Then, as has been pointed out above, the positive equivariant symplectic homology is defined and well-defined for any contact form supporting $\xi$. Let $\alpha_0$ be a connection contact form on $(M, \xi)$. This form is not non-degenerate, but rather Morse-Bott non-degenerate. Let $a > 0$ be the rationality constant of $(B, \omega)$, i.e., the positive generator of $\langle \omega, \pi_2(B) \rangle \subset \mathbb{R}$. Then the action spectrum of $\alpha_0$ is $a\mathbb{N}$. Pick small non-overlapping intervals $I_m = [ma - \epsilon, ma + \epsilon]$ with $\epsilon > 0$.

A standard Morse-Bott type argument shows that

$$HC^{(m, 0)}(\alpha_0) \cong H_{* - 2mc_B + n}(B; \mathbb{Q}),$$

where on the left we have the filtered homology of $\alpha_0$ or to be more precise of a small non-degenerate perturbation $\alpha$ of $\alpha_0$; cf., e.g., [38] and also [5, 7] for a different approach.
Furthermore, the contractible positive equivariant symplectic homology of $\alpha$ can be viewed as the homology of a certain complex generated by good closed Reeb orbits; see Section 3.3. This complex is filtered by action, and the $E^1$-page of the resulting Morse–Bott spectral sequence is given by the right-hand side of (3.2). Namely,

$$E^1_{m,q} = \text{HC}^{l_m,0}_{m+q}(\alpha_0) \cong H_{m+q-2mc_B+n}(B; \mathbb{Q}).$$

The condition that $H_{\text{odd}}(B; \mathbb{Q}) = 0$ or $c_B > n$ readily implies that this spectral sequence collapses in the $E^1$-term: $E^1 = E^\infty = \text{HC}^0_{\ast}(M)$, which proves (3.2).

This argument applies in Case (b) word-for-word with one nuance. Namely, to carry out the Morse–Bott calculation for $\alpha_0$ we need to make sure that it admits an arbitrarily small non-degenerate perturbation $\alpha$ such that all good closed Reeb orbits of $\alpha$ have Conley–Zehnder index greater than $3 - n$. This is a consequence of the following lemma.

**Lemma 3.3.** Let $(M, \xi)$ be the prequantization $S^1$-bundle over $(B, \omega)$ with connection contact form $\alpha_0$ such that $\xi = \ker \alpha_0$. Assume that $B$ is spherically positive monotone, $\omega|_{\pi_2(B)} \neq 0$, and $c_1(\xi) = 0$. Let $\beta$ be a sufficiently small non-degenerate perturbation of $\alpha_0$. Then $\mu(\gamma) \geq 2c_B - n$ for every contractible closed Reeb orbit of $\beta$. Furthermore, there exists a constant $\Delta > 0$, independent of $\beta$, such that for all $\gamma$ we have

$$\hat{\mu}(\gamma) \geq \Delta \cdot T(\gamma),$$

where $T(\gamma)$ is the period (i.e., the action) of $\gamma$.

Note that $(B, \omega)$ is spherically positive monotone and $c_1(\xi) = 0$ whenever, for instance, $(B, \omega)$ is positive monotone over $\mathbb{Z}$, i.e., $c_1(TB) = \lambda[\omega]$ in $H^2(B; \mathbb{Z})$ for some integer $\lambda \geq 0$, where abusing notation we treat $[\omega]$ as its lift to $H^2(B; \mathbb{Z})$; cf. Remarks 2.2 and 3.2. Lemma 3.3 is the main point in the proof of Theorem 2.1 where it is essential that in (NF) $c_1(\xi) = 0$ as an element of $H^2(M; \mathbb{Z})$ and not only modulo torsion. The lemma is not entirely new and has several predecessors (see, e.g., [5, Sect. 2.2] or [18, Sect. 3] and references therein). However, we include a short detailed proof for the sake of completeness and because we think the argument is a good illustration of usefulness of the quasimorphism property of the mean index.

**Proof.** Contractible closed Reeb orbits of $\alpha_0$ comprise connected sets $P_m$ each of which is a principal $S^1$-bundle over $B$. The set $P_1$ is formed by the orbits with period $a$, where as above $a$ is the positive generator of $\langle \omega, \pi_2(B) \rangle \subset \mathbb{R}$. These orbits are not necessarily simple but they are “simple contractible orbits”. The orbits from $P_m$ are the $m$th iterations of the orbits in $P_1$. These orbits have mean index $2c_Bm$ and period $ma$.

Fix $T_0 > 0$. Then, when $\beta$ is sufficiently $C^2$-close to $\alpha_0$, every contractible closed Reeb orbit $\gamma$ of $\beta$ with action $T(\gamma) \leq T_0$ is close to one of the orbits in $P_m$ with $ma \leq T_0$. Hence

$$\mu(\gamma) > 2c_Bm - n \geq 2c_B - n \text{ and } \hat{\mu}(\gamma) \geq \frac{2c_B}{a'} \cdot T(\gamma) \geq \frac{c_B}{a} \cdot T(\gamma),$$

where we can take $a' > a > 0$ arbitrarily close to $a$ when $\beta$ is close to $\alpha_0$.

Since $c_1(\xi) = 0$, the determinant line bundle $\bigwedge^n_{\mathbb{R}} \xi$ is trivial. Fix a section of this line bundle. Then, using this section, we can define the mean index for all finite segments $\eta$ of Reeb orbits, not necessarily contractible or even closed, for any contact form on $(M, \xi)$; see, e.g., [16]. This index depends continuously on the initial condition and the contact form (in the $C^2$-topology), and for closed contractible Reeb orbits it agrees with the standard mean index defined in Section 3.2.
Pick \( m \) such that \( 2c_B m > C_n \), where \( C_n \) is the quasimorphism constant from (3.1), and fix \( b_0 \) with \( 2c_B m > b_0 > C_n \). By continuity, when \( \beta \) is sufficiently \( C^2 \)-close to \( \alpha_0 \), every segment of a Reeb orbit of \( \beta \) with action \( ma \) has mean index greater than \( b_0 \). (This is a consequence of the fact that the Reeb orbits of \( \alpha_0 \) with action \( ma \) have mean index \( 2c_B m \).

Then, by the quasimorphism property (3.1),
\[
\hat{\mu}(\eta) \geq bT(\eta) - c
\]
for all finite segments \( \eta \) of Reeb orbits of \( \beta \). Here we can take \( b = (b_0 - C_n) / ma > 0 \) and \( c \) depends on \( m \) and the section, but can be taken independent of \( \beta \). In particular, this inequality holds for all contractible closed Reeb orbits of \( \beta \) and all such orbits with sufficiently large action have large Conley–Zehnder index.

Let us now take \( \beta \) so close to \( \alpha_0 \) that (3.4) holds for all contractible closed Reeb orbits of \( \beta \) with action smaller than a large initial time \( T_0 \). To be more specific, fix a positive constant \( \Delta < \min\{b, c_B / a\} \). Then, as is easy to see, when \( T_0 \) is large enough (e.g., \( T_0 = c_B / (b - \Delta) \)), for all contractible closed Reeb orbits \( \gamma \) of \( \beta \) we have \( \mu(\gamma) \geq 2c_B - n \) and \( \hat{\mu}(\gamma) \geq \Delta \cdot T(\gamma) \). This concludes the proof of the lemma and of the proposition.

\( \Box \)

### 3.5. Local equivariant symplectic homology, resonance relation and Morse inequalities.

Let \( \gamma \) be an isolated closed Reeb orbit and denote by \( HC_\ast(\gamma) \) its local equivariant symplectic homology. For a non-degenerate orbit \( \gamma \), if \( \gamma \) is good \( HC_\ast(\gamma) = \mathbb{Q} \), concentrated in degree \( * = \mu(\gamma) \); \( HC_\ast(\gamma) = 0 \) if \( \gamma \) is bad. The Euler characteristic of \( \gamma \) is defined as
\[
\chi(\gamma) = \sum_{m \in \mathbb{Z}} (-1)^m \dim HC_m(\gamma).
\]
This sum is finite. When \( \gamma \) is non-degenerate, \( \chi(\gamma) = (-1)^{\mu(\gamma)} \) or \( \chi(\gamma) = 0 \) depending on whether \( \gamma \) is good or bad. The local mean Euler characteristic of \( \gamma \) is
\[
\hat{\chi}(\gamma) = \lim_{j \to \infty} \frac{1}{j} \sum_{k=1}^{j} \chi(\gamma^k).
\]
The limit above exists and is rational; see [17]. When \( \gamma \) is strongly non-degenerate, we have
\[
\hat{\chi}(\gamma) = \begin{cases} 
(-1)^{\mu(\gamma)} & \text{if } \gamma^2 \text{ is good} \\
(-1)^{\mu(\gamma)}/2 & \text{if } \gamma^2 \text{ is bad}.
\end{cases}
\]
Assume now that \( \alpha \) is index-positive/index-negative and has finitely many distinct simple contractible closed orbits \( \gamma_1, \ldots, \gamma_r \). (Here “simple” means that each \( \gamma_i \) is not an iterate of a contractible orbit.) This assumption ensures that the positive/negative mean Euler characteristic
\[
\chi_\pm(M) := \lim_{j \to \infty} \frac{1}{j} \sum_{m=0}^{j} (-1)^m b_{\pm m}
\]
is well defined, where \( b_m := \dim HC_{m}^0(M) \) is the \( m \)-th Betti number; see [17]. The mean Euler characteristic is related to local equivariant symplectic homology via the resonance relation
\[
\sum_{i=1}^{r} \frac{\hat{\chi}(\gamma_i)}{\hat{\mu}(\gamma_i)} = \chi_\pm(M),
\]
(3.5)
proved in [24, 31]. Here the right-hand side is \( \chi_+ \) when \( \alpha \) is index-positive and \( \chi_- \) when \( \alpha \) is index-negative.

Let \( c_m := \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \dim HC_m(\gamma_k^i) \) be the \( m \)-th Morse type number and define \( m_{\text{min}} := \inf\{m \in \mathbb{Z} \mid c_m \neq 0\} \) and \( m_{\text{max}} := \sup\{m \in \mathbb{Z} \mid c_m \neq 0\} \). When \( \alpha \) is non-degenerate, \( c_m \) is simply the number of good orbits of index \( m \). Furthermore, \( m_{\text{min}} > -\infty \) if \( \alpha \) is index-positive and \( m_{\text{max}} < \infty \) if \( \alpha \) is index-negative. We have the Morse inequalities

\[
c_m - c_{m-1} + \cdots + c_{m_{\text{min}}} \geq b_m - b_{m-1} + \cdots + b_{m_{\text{min}}}, \tag{3.6}
\]

for every \( m \geq m_{\text{min}} \) if \( \alpha \) is index-positive, and

\[
c_m - c_{m+1} + \cdots + c_{m_{\text{max}}} \geq b_m - b_{m+1} + \cdots + b_{m_{\text{max}}},
\]

for every \( m \leq m_{\text{max}} \) if \( \alpha \) is index-negative. We note that these inequalities are notably stronger than the inequalities \( c_m \geq b_m \).

\section{Index Recurrence}

\subsection{The index recurrence theorem}

A crucial ingredient for distinguishing simple and iterated orbits in the proof of Theorem 2.1 is the following combinatorial result addressing the index behavior under iterations. This result can be deduced from the so-called enhanced common index jump theorem due to Duan, Long and Wang [13] (see also [34, 35]), but we will give a different, self-contained proof along the lines of the argument from [21, Thm. 5.1].

\begin{thm}
Let \( \Phi_1, \ldots, \Phi_r \) be a finite collection of strongly non-degenerate elements of \( \hat{\text{Sp}}(2n) \) with \( \hat{\mu}(\Phi_i) > 0 \) for all \( i \). Then for any \( \eta > 0 \) and any \( \ell_0 \in \mathbb{N} \), there exist two integer sequences \( d^\pm_j \to \infty \) and two sequences of integer vectors \( \vec{k}^\pm_j = (k_{ij}^\pm, \ldots, k_{ij}^\pm) \) with all components going to infinity as \( j \to \infty \), such that for all \( i \) and \( j \), and all \( \ell \in \mathbb{Z} \) in the range \( 1 \leq |\ell| \leq \ell_0 \), we have

(i) \( |\hat{\mu}(\Phi_i^{k_{ij}^\pm}) - d^\pm_j| < \eta \) with the equality \( \hat{\mu}(\Phi_i^{k_{ij}^\pm}) = \mu(\Phi_i^{k_{ij}^\pm}) = d^\pm_j \) whenever \( \Phi_i(1) \) is hyperbolic,

(ii) \( \mu(\Phi_i^{k_{ij}^\pm+\ell}) = d^\pm_j + \mu(\Phi_i^\ell) \), and

(iii) \( \mu(\Phi_i^{k_{ij}^\pm}) - d^-_j = -\mu(\Phi_i^{k_{ij}^\pm}) - d^+_j \).

Furthermore, for any \( N \in \mathbb{N} \) we can make all \( d^\pm_j \) and \( k_{ij}^\pm \) divisible by \( N \).
\end{thm}

The condition that \( \hat{\mu}(\Phi_i) > 0 \) for all \( i \) can be relaxed, but the theorem, as is, is sufficient for our purposes.

In the assertion and the proof of the theorem we follow closely [21, Sect. 5]. (The new point is (iii): the rest is contained in, e.g., [21, Thm. 5.1].) Note that it suffices to find just one pair \( \vec{k}^\pm = (k_{ij}^1, \ldots, k_{ij}^\pm) \) of iteration vectors and one pair \( d^\pm \), both divisible by any given \( N \) — and this is the form of the theorem we actually use here. Once \( \vec{k}^\pm = \vec{k}^\pm \) and \( d^\pm \) are found we can replace \( N \) by \( pN \), where \( p \) is a sufficiently large integer, and repeat the process to find \( \vec{k}^\pm \) and \( d^\pm \), and so on.

\subsection{Proof of Theorem 4.1}

We first establish the case of a single path \( \Phi \), i.e., \( r = 1 \), and then show how to modify the argument for a finite collection of paths.
4.2.1. The case of $r = 1$. Let $\Phi = \Phi_1 \in \Sp(2n)$. Throughout the argument we suppress $i$ in the notation, i.e., we write $k$ for $k_{11}$ or $\tilde{k}_1$, etc. To prove the theorem in this setting, we will consider two subcases depending on the end-map $\Phi(1)$ and then derive the general case from additivity. Fix $\eta > 0$ and $\ell_0 \in \mathbb{N}$. Without loss of generality, we can assume that $\eta < 1/2$.

**Subcase A**: $\Phi(1)$ is hyperbolic. Set $d_k^+ = \hat{\mu}(\phi^k)$ for any $k \in \mathbb{N}$. Clearly, (i) is automatically satisfied. Furthermore, $\phi^k$ is non-degenerate for all $k \in \mathbb{N}$ and $\hat{\mu}(\phi^k) = \mu(\phi^k)$. Hence we have

$$
\mu(\phi^{k+\ell}) = \hat{\mu}(\phi^k) + \hat{\mu}(\phi^\ell) = \mu(\phi^k) + \mu(\phi^\ell).
$$

Thus (i)–(iii) hold for all $k^\pm = k \in \mathbb{N}$ and all $\ell$, with $d_k^2 = d_k$. To make $d$ and $k$ divisible by $N$, it suffices to just take $k$ divisible by $N$.

**Subcase B**: $\Phi(1)$ is elliptic. Let $\exp(\pm 2\pi \sqrt{-1}\lambda_q)$, for $q = 1, \ldots, n$, be the eigenvalues of $\Phi(1) \in \Sp(2n)$, where $|\lambda_q| < 1$. (The choice of sign for $\lambda_q$ is not essential, but when the eigenvalues are distinct it is convenient to assume that $\exp(2\pi \sqrt{-1}\lambda_q)$ are the eigenvalues of the first kind; see, e.g., [39].) Since $\Phi(1)$ is strongly non-degenerate, all $\lambda_q$ are irrational. Set

$$
\epsilon_0 = \min_{0 < \epsilon \leq \epsilon_0} \min_q \|\lambda_q\ell\| > 0,
$$

where $\| \cdot \|$ stands for the distance to the nearest integer. Let $\epsilon > 0$ be so small that

$$
\epsilon \leq \epsilon_0 \quad \text{and} \quad n\epsilon < \eta.
$$

It is easy to see that there exists $k > 0$ such that for all $q$ we have

$$
\|\lambda_qk\| < \epsilon \leq \epsilon_0.
$$

Indeed, consider the positive semi-orbit $\Gamma_+ = \{k\tilde{\lambda} \mid k \in \mathbb{N}\} \subset \mathbb{T}^n$ where $\tilde{\lambda} \in \mathbb{T}^n$ is the collection of eigenvalues of $\Phi(1)$. As is well known, the closure $\Gamma$ of $\Gamma_+$ is a subgroup of $\mathbb{T}^n$. Hence $\Gamma_+$ contains points arbitrarily close to the unit in $\mathbb{T}^n$ and, in particular, there exist infinitely many points $k\tilde{\lambda} \in \Gamma_+$ in the $2\pi\epsilon$-neighborhood of the unit. Clearly, for any $N \in \mathbb{N}$ we can also make $k$ divisible by $N$. (To see this, it suffices to replace the semi-orbit $\Gamma_+$ by $\{kN\tilde{\lambda} \mid k \in \mathbb{N}\}$.)

Let $d$ be the nearest integer to $\hat{\mu}(\phi^k)$. Then

$$
[d - \hat{\mu}(\phi^k)] \leq n\epsilon < \eta,
$$

and hence (i) is satisfied. (This also shows that $d$ is unambiguously defined.) Furthermore, replacing as above the semi-orbit $\Gamma_+$ by $\{kN\tilde{\lambda} \mid k \in \mathbb{N}\}$ we can also make $d$ divisible by $N$.

It is shown in [21, Sect. 5.2.1, Subcase C] that the inequalities (4.2) and (4.3) imply (ii). For the sake of completeness we recall here the argument. Observe first that a small perturbation of $\Phi$ does not effect individual terms in these inequalities for fixed $k$ and $\ell$. Thus, by altering $\Phi$ slightly, we can ensure that all eigenvalues $\lambda_q$ are distinct. Then we can write $\Phi$, up to homotopy, as the product of a loop $\varphi$ and the direct sum of paths $\Psi_q = \exp(2\pi \sqrt{-1}\lambda_q\ell) \in \Sp(2)$ for a suitable choice of signs of $\lambda_q$; see, e.g., [39, Sect. 3]. The loop $\varphi$ contributes $k\hat{\mu}(\varphi)$ to $\mu(\phi^k)$ and hence we only need to prove (ii) when $\varphi = I$.

Then, for any $k$,

$$
\mu(\phi^k) = \sum_q \mu(\Psi_q).
$$

Next, observe that by (4.1) and (4.2) we have

$$
d = \sum_q \lfloor \hat{\mu}(\Psi_q) \rfloor,
$$

where $\lfloor \cdot \rfloor$ denotes the greatest integer less than or equal to $\cdot$. This formula is crucial in the proof of the theorem.
where \([\cdot]\) denotes the nearest integer. Thus it suffices to prove (ii) and (iii) for each path \(\Psi_q\) individually when we set \(d_q = [\hat{\mu}(\Psi_q)]\). However, with (4.1) and (4.2) in mind, (ii) for \(\Psi_q\) easily follows from the definition.

Now we need to find \(k^-\) satisfying (ii) for a suitable choice of \(d^-\). To this end, observe that for any \(\delta > 0\) the system of inequalities

\[
\|\lambda_q(k^- + k^+)\| < \delta
\]

has infinitely many solutions \(k^- \in \mathbb{N}\), where \(k^+ := k\). This is again a consequence of the fact that \(\Gamma_+\) is dense in the group \(\Gamma\), and hence contains points arbitrarily close to \(-k^+\tilde{\lambda}\) in \(\mathbb{T}^n\). It is also clear that \(k^-\) can be made divisible by any given integer. Using (4.2), let us take \(\delta\) so small that \(\|k^+\lambda_q\| + \delta < \epsilon\).

Then (4.2) is still satisfied for \(k = k^-\). Let \(d^-\) be the nearest integer to \(\hat{\mu}(\Phi^{k^-})\). Then (4.2) and (4.3) hold for \(k^-\) and \(d^-\) and hence so does (ii). Finally, since \(k^-\tilde{\lambda}\) is close to \(-k^+\tilde{\lambda}\) in the torus \(\mathbb{T}^n\), we have

\[
\mu(\Phi^{k^-}) - d^- = \mu(\Phi^{-k^+}) + d^+ = -(\mu(\Phi^{k^+}) - d^+),
\]

which proves (iii).

**Putting Subcases A–B together.** Let us decompose \(\Phi\) into the direct sum of two paths \(\Phi_A\) and \(\Phi_B\) such that \(\Phi_A(1)\) is hyperbolic and \(\Phi_B(1)\) is elliptic. (It is easy to see that we can always do this up to homotopy.) We take \(k^\pm\) as in Subcase B and adjust \(d^\pm\) by adding \(\hat{\mu}(\Phi_A^{k^\pm})\). It is clear that (i)–(iii) hold for this choice of \(k^\pm\) and \(d^\pm\) and that, in addition, we can make \(k^\pm\) and \(d^\pm\) divisible by any integer.

4.2.2. **The general case:** \(r \geq 1\). Let \(\Phi_1, \ldots, \Phi_r\) be a finite collection of elements in \(\widetilde{\text{Sp}}(2n)\). As above, each of these paths can be decomposed into a sum of paths with hyperbolic end-points and elliptic end-points. Then it is easy to see that it suffices to prove the theorem when all \(\Phi_i(1)\) are elliptic. For the general case follows again by additivity.

Denote by \(\exp\left(\pm 2\pi\sqrt{-1}\lambda_{iq}\right)\) the eigenvalues of \(\Phi_i\) with \(|\lambda_{iq}| < 1\) and set \(\Delta_i = \hat{\mu}(\Phi_i) > 0\). (The choice of the sign of \(\lambda_{iq}\) is immaterial at the moment, but again when all eigenvalues are distinct it is convenient to assume that \(\exp(2\pi\sqrt{-1}\lambda_{iq})\) are the eigenvalues of the first kind.) Given \(\epsilon > 0\), consider the system of inequalities

\[
\begin{align*}
&|k_i\lambda_{iq}| < \epsilon & \text{for all } i \text{ and } q, \\
&|k_1\Delta_1 - k_i\Delta_i| < \frac{1}{16} & \text{for } i = 2, \ldots, r,
\end{align*}
\]

where we treat the integer vector \(\vec{k} = (k_1, \ldots, k_r) \in \mathbb{Z}^r\) as a variable. Introducing additional integer variables \(c_{iq}\), we can rewrite (4.5) in the form

\[
|k_i\lambda_{iq} - c_{iq}| < \epsilon.
\]

With this in mind, the system of equations (4.5) and (4.6), or equivalently (4.6) and (4.7), has one fewer equation than the number of variables. By Minkowski’s theorem (see, e.g., [10]), there exists a non-zero solution \(\vec{k} = (k_1, \ldots, k_r)\) of (4.6) and (4.7). Now it follows from (4.6) and the assumption that \(\Delta_i > 0\) that all \(k_i\) are non-zero and have the same sign. Hence, replacing if necessary \(\vec{k}\) by \(-\vec{k}\), we can ensure that \(k_i > 0\). Moreover, we can make all \(k_i\) divisible by any fixed integer \(N\).
Note also that by (4.6) we have
\[ \left| \sum_q c_{1q} - \sum_q c_{iq} \right| < \frac{1}{16} + 2r \epsilon. \]
If \( \epsilon < 1/4r \), this inequality is satisfied only when the left-hand side is zero.

Fix \( \ell_0 \) and \( \eta > 0 \) which we assume to be sufficiently small (e.g., \( \eta < 1/4 \)). Similarly to Subcase B, set
\[ \epsilon_0 = \min_{0 < \epsilon \leq \ell_0} \min_{i,q} \| \lambda_{iq} \ell \| > 0, \]
and let \( \epsilon > 0 \) be so small that again
\[ \epsilon \leq \epsilon_0 \quad \text{and} \quad n \epsilon < \eta. \]

By (4.6) we have
\[ |k_i \Delta_i - k_{i'} \Delta_{i'}| < \frac{1}{8} \]
for all \( i \) and \( i' \), and \( \| k_i \Delta_i \| < \eta \) by the first group of inequalities (4.5). Thus \( k_i \Delta_i \) is \( \eta \)-close, for all \( i \), to the same integer
\[ d = [k_i \Delta_i] = \sum_q c_{iq} = \sum_q c_{1q}. \]
In other words, (i) is satisfied for this choice of \( d \). Furthermore, for every \( i \), condition (4.2) is met for \( \lambda_{iq} \), and hence (ii) holds for all \( \Phi \). As in Subcase B, we set \( d^+ = d \) and \( \tilde{k}^+ = \tilde{k} \).

Note that so far it would be sufficient to take \( 1/8 \) as the right-hand side in (4.6).

Our next goal is to find \( d^- \) and \( \tilde{k}^- \). To this end consider the inequalities
\[ |k_i' \Delta_i - c_{pq}'| < \delta \quad \text{for all } i \quad \text{and} \quad q, \]
(4.8)
\[ |k_{i'} \Delta_{i'} - k_i \Delta_i| < \frac{1}{16} \quad \text{for } i = 2, \ldots, r, \]
(4.9)
where (4.8) can again be written in the form
\[ |k_i' \lambda_{iq} - c_{pq}'| < \delta \]
(4.10)
for some integer variables \( c_{pq}' \). For any \( \delta > 0 \), the system of inequalities (4.9) and (4.10) has a non-zero solution \( \tilde{k}' \) by Minkowski’s theorem. The same argument as above shows that we can take \( k_i' > 0 \) for all \( i \) and, in fact, we can make \( k_i' \) arbitrarily large. In particular, we can ensure that
\[ k_i^- := k_i' - k_i^+ > 0. \]
Then we have
\[ \| (k_i^- + k_i^+) \lambda_{iq} \| < \delta \quad \text{for all } i \quad \text{and} \quad q \]
and
\[ |k_i^- \Delta_i - k_i^- \Delta_i| < \frac{1}{8} \quad \text{for } i = 2, \ldots, r, \]
where to obtain the last inequality we used (4.6) and (4.9). Let us now assume that \( \delta > 0 \) is so small that \( \| k_{iq}^+ \lambda_{iq} \| + \delta < \epsilon \) for all \( i \) and \( q \). Then we also have
\[ \| k_{iq}^- \lambda_{iq} \| < \epsilon \quad \text{for all } i \quad \text{and} \quad q, \]
i.e., (4.9) holds for \( \tilde{k}^- \).

To summarize, \( \tilde{k}^- \) satisfies (4.5) and (4.6) with \( 1/16 \) replaced by \( 1/8 \), which is sufficient for our purposes. Setting
\[ d^- = [k_i^- \Delta_i] = \sum_q (c_{iq}' - c_{iq}) \]
we conclude that (i) and (ii) hold for \( \vec{k} \). It is also clear that we can make, if necessary, all \( k_i \) divisible by an arbitrary constant \( N \).

Finally, (iii) also holds for each \( \Phi_i \) individually just as in Subcase B since the vector \( \vec{k} \) is close to \( \vec{k}^\pm \) modulo the integer lattice. This concludes the proof of Theorem 4.1. \( \square \)

5. Proofs of Theorems 2.1 and 2.7

5.1. Proof of Theorem 2.1. Let us focus on the case where \( \alpha \) is index-positive, for the argument in the index-negative case is similar. The main tool used in the proof is the positive equivariant symplectic homology. Recall from Section 3.3 that both of the conditions (F) and (NF) ensure that this homology (for \( M \) or the filling) with integer grading is defined and, as Proposition 3.1 shows, given by (3.2). Then the proof is the same in both cases of the theorem, (F) and (NF), and relies only on the condition shared by these cases that \( \alpha \) has no good contractible periodic orbits \( \gamma \) such that \( \mu(\gamma) = 0 \) if \( n \) is odd or \( \mu(\gamma) \in \{0, \pm 1\} \) if \( n \) is even. We should note that the argument also uses several ideas from [13, 14].

Starting the proof, assume that \( \alpha \) has finitely many distinct contractible simple closed orbits \( \gamma_1, \ldots, \gamma_r \). (Here, as in Section 3.5, “simple” means that each \( \gamma_i \) is not an iterate of a contractible orbit.) Our goal is to establish the lower bound on \( r \) asserted by the theorem. Define

\[
\ell_0 = \max \min \{k_0 \in \mathbb{N} | \mu(\gamma_i^{k+\ell}) \geq \mu(\gamma_i^k) + 2n + 1 \text{ for all } k \geq 1 \text{ and } \ell \geq k_0\}. 
\]

By Theorem 4.1, given \( N \in \mathbb{N}, \eta > 0 \) and \( \ell_0 \) as above we have two sequences of integer vectors \((d_j^+ \pm, k_1^j \pm, \ldots, k_r^j \pm)\) satisfying conditions (i), (ii) and (iii) and such that all \( d_j^\pm, k_1^j \pm, \ldots, k_r^j \pm \) are divisible by \( N \). As has been mentioned before, we will only need one such vector from each sequence. Hence set

\[
(d, k_1, \ldots, k_r) := (d_1^+, k_1^+ 1, \ldots, k_r^+ 1) \text{ and } (d', k_1', \ldots, k_r') := (d_1^-, k_1^- 1, \ldots, k_r^- 1).
\]

The following lemma, giving an expression for the truncated mean Euler characteristic (see Section 3.5), is one of the key steps in the proof; cf. [2, Sublemma 5.2].

Lemma 5.1. The numbers \( N \) and \( \eta \) can be chosen such that \( d = 2sc_B \) for some integer \( s \) and

\[
\sum_{i=1}^{r} k_i \hat{\chi}(\gamma_i) = \sum_{i=1}^{r} k_i \chi(\gamma_i) = d\chi(M) = (-1)^n s\chi(B).
\]

The same holds for \( d', k_1', \ldots, k_r' \).

Proof. Let \( N \) be any (positive) integer multiple of \( 2c_B \). The first equality follows from the (strong) non-degeneracy of \( \gamma_1, \ldots, \gamma_r \) since the numbers \( k_i \) are even. It is easy to see from (3.2) that

\[
\chi(M) = (-1)^n \frac{\chi(B)}{2c_B},
\]
which implies the third equality. To prove the second equality, take $\eta$ sufficiently small such that $\eta|\chi(M)| < 1$. Using the resonance relation (3.5), we conclude that

$$
d\chi(M) = \sum_{i=1}^{r} \frac{d\hat{\chi}(\gamma_i)}{\hat{\mu}(\gamma_i)} = \sum_{i=1}^{r} k_i \hat{\chi}(\gamma_i) + \sum_{i=1}^{r} \left( \frac{(d - k_i \hat{\mu}(\gamma_i))}{\hat{\mu}(\gamma_i)} \hat{\chi}(\gamma_i) \right).
$$

By property (i) of Theorem 4.1 and, again, (3.5),

$$
\left| \sum_{i=1}^{r} \frac{(d - \hat{\mu}(\gamma_i^{k_i}))}{\hat{\mu}(\gamma_i)} \hat{\chi}(\gamma_i) \right| < \eta \left| \sum_{i=1}^{r} \frac{\hat{\chi}(\gamma_i)}{\hat{\mu}(\gamma_i)} \right| = \eta|\chi(M)| < 1.
$$

Note that by our choice of $N$ the numbers $d\chi(M)$ and $k_i \hat{\chi}(\gamma_i)$ for all $i$ are integers. Therefore,

$$
d\chi(M) = \sum_{i=1}^{r} k_i \hat{\chi}(\gamma_i).
$$

Obviously, the same argument works for $d', k_1', \ldots, k_r'$.

Let us now break down the proof of Theorem 2.1 into two cases, according to the parity of $n$.

**Case 1: $n$ is odd.**

Fix $N$ and $\eta$ as in Lemma 5.1. (In particular, $N$ is even.) Clearly, $\eta$ can be chosen so small that the vector $(d, k_1, \ldots, k_r) = (d^1, k_1^1, \ldots, k_r^1)$ given by Theorem 4.1 satisfies

$$
\mu(\gamma_i^{k_i-\ell}) = d - \mu(\gamma_i^\ell),
$$

and

$$
|\mu(\gamma_i^{k_i}) - d| \leq n
$$

for every $1 \leq i \leq r$ and $1 \leq \ell \leq \ell_0$. (Here, since $N$ is even, the integers $d$ and $k_i$ are even. One can also assume that $k_i > \ell_0$ for all $i$.) Observe that for each periodic orbit $\gamma_i$, there are four types of iterates outside $\gamma_i$:

- (A) $\gamma_i^{k_i-\ell}$ with $\ell > \ell_0$;
- (B) $\gamma_i^{k_i-\ell}$ with $1 \leq \ell \leq \ell_0$;
- (C) $\gamma_i^{k_i+\ell}$ with $1 \leq \ell \leq \ell_0$;
- (D) $\gamma_i^{k_i+\ell}$ with $\ell > \ell_0$.

Let us analyze the contributions of these iterates to the Morse type numbers defined by the alternating sum

$$
\sum_{m=m_{\text{min}}}^{d} (-1)^m c_m,
$$

where, as in Section 3.5, $c_m$ is the number of good closed orbits of index $m$ and $m_{\text{min}} > -\infty$ is the smallest integer with $c_m \neq 0$. First, class (A) iterates have index $< d$ and hence all good
orbits here contribute to (5.4). Indeed, by definition of $\ell_0$, we have $\mu(\gamma_i^{k_i}) \geq \mu(\gamma_i^{k_i-\ell}) + 2n + 1$ for every $\ell > \ell_0$ which, combined with (5.3), implies that $\mu(\gamma_i^{k_i-\ell}) \leq d - n - 1$ for all $\ell > \ell_0$. Class (D) orbits do not contribute to (5.4) since $\mu(\gamma_i^{k_i+\ell}) \geq \mu(\gamma_i^{k_i}) + 2n + 1 \geq d + n + 1$ for every $\ell > \ell_0$, where the last inequality again follows from (5.3).

In order to understand the contributions from classes (B) and (C), let us further divide each of them into two subclasses:

(B1) $\gamma_i^{k_i-\ell}$ with $1 \leq \ell \leq \ell_0$ if $\mu(\gamma_i^{\ell}) \geq 0$,

(B2) $\gamma_i^{k_i-\ell}$ with $1 \leq \ell \leq \ell_0$ if $\mu(\gamma_i^{\ell}) < 0$,

and

(C1) $\gamma_i^{k_i+\ell}$ with $1 \leq \ell \leq \ell_0$ if $\mu(\gamma_i^{\ell}) \geq 0$,

(C2) $\gamma_i^{k_i+\ell}$ with $1 \leq \ell \leq \ell_0$ if $\mu(\gamma_i^{\ell}) < 0$.

Now all of the good orbits in class (B1) contribute to (5.4), while class (B2) makes no contribution to (5.4). Indeed, by (5.1), $\mu(\gamma_i^{k_i-\ell}) = d - \mu(\gamma_i^{\ell})$ which is $\leq d$ whenever $\mu(\gamma_i^{\ell}) \geq 0$ and $> d$ whenever $\mu(\gamma_i^{\ell}) < 0$.

The key to dealing with class (C1) is the condition that $\alpha$ has no good contractible periodic orbits of index zero. (This is the main point where this condition is used.) Then for all good iterates $\gamma_i^{k_i+\ell}$ in class (C1) $\mu(\gamma_i^{\ell}) > 0$ and, by (5.2), $\mu(\gamma_i^{k_i+\ell}) = d + \mu(\gamma_i^{\ell}) > d$ whenever $\gamma_i^{k_i+\ell}$ is good. Finally, all of the good orbits from class (C2) contribute to (5.4) since $\mu(\gamma_i^{\ell}) < 0$ and $\mu(\gamma_i^{k_i+\ell}) = d + \mu(\gamma_i^{\ell}) < d$ by (5.2). (Above $\mu(\gamma_i^{k_i+\ell})$ and $\mu(\gamma_i^{\ell})$ have the same parity since the integers $k_i$ are even.)

To summarize, all good orbits from classes (A), (B1) and (C2) have index $\leq d$ and contribute to (5.4), and good orbits from classes (D), (B2) and (C1) have index $> d$ and make no contribution to (5.4). Define

$$c_2^+ = \sum_{i=1}^{r} \#\{1 \leq \ell \leq \ell_0 \mid \mu(\gamma_i^{\ell}) < 0, \gamma_i^{k_i+\ell} \text{ is even and } \mu(\gamma_i^{\ell}) \text{ is even}\}$$

and

$$c_2^- = \sum_{i=1}^{r} \#\{1 \leq \ell \leq \ell_0 \mid \mu(\gamma_i^{\ell}) < 0, \gamma_i^{k_i+\ell} \text{ is odd and } \mu(\gamma_i^{\ell}) \text{ is even}\}.$$  (5.5)

Consider now $\sum_{m=m_{\min}}^{\infty} (-1)^m c_m$ to which all good orbits from classes (A)–(D) and the collection $\{\gamma_i^{k_i}\}$ contribute. In particular, the contributions of class (B2) and class (C2) iterates are respectively $c_2^- - c_2^-$ and $c_2^+ - c_2^-$. Viewing (5.4) as

$$\sum_{m=m_{\min}}^{d} (-1)^m c_m = \sum_{m=m_{\min}}^{\infty} (-1)^m c_m - \sum_{m>d} (-1)^m c_m,$$

with the above discussion in mind, we have

$$\sum_{m=m_{\min}}^{d} (-1)^m c_m = \sum_{i=1}^{r} \left( \sum_{\ell=1}^{k_i} \chi(\gamma_i^{\ell}) + \frac{c_2^+ - c_2^+}{(C2)} - \frac{c_2^- - c_2^-}{(B2)} - \sum_{\mu(\gamma_i^{k_i}) > d} \chi(\gamma_i^{k_i}) \right).$$  (5.7)

Here, as indicated by the underbraces, the first term on the right-hand side comes from the iterates in classes (A) and (B) and the iterate $\gamma_i^{k_i}$, the second term comes from class (C2) iterates, and the third term cancels out the contribution of class (B2) orbits to the first term.
Finally, the last term eliminates the contribution to the first term of the orbits $\gamma_{i_i}^{k_i}$ with index greater than $d$.

Note that, since $\mu(\gamma_{i_i}^{k_i-\ell})$ and $\mu(\gamma_{i_i}^{k_i+\ell})$ have the same parity, $c_-^e = c_+^e$ and $c_-^o = c_+^o$. Thus the second and third terms on the right-hand side of equation (5.7) cancel each other out and we arrive at

$$
\sum_{m=m_{\text{min}}}^{d} (-1)^m c_m = \sum_{i=1}^{r} \sum_{\ell=1}^{k_i} \chi(\gamma_{i_i}^{\ell}) - \sum_{i=1}^{r} \sum_{\mu(\gamma_{i_i}^{k_i})>d} \chi(\gamma_{i_i}^{k_i}).
$$

(5.8)

Define

$$
r_+^e = \#\{1 \leq i \leq r | \pm(\mu(\gamma_{i_i}^{k_i}) - d) > 0, \gamma_{i_i}^{k_i} \text{ is good and } \mu(\gamma_{i_i}^{k_i}) \text{ is even}\}
$$

(5.9)

and

$$
r_+^o = \#\{1 \leq i \leq r | \pm(\mu(\gamma_{i_i}^{k_i}) - d) > 0, \gamma_{i_i}^{k_i} \text{ is good and } \mu(\gamma_{i_i}^{k_i}) \text{ is odd}\}.
$$

(5.10)

Notice that the last term in (5.8) is given by $r_+^e - r_+^o$. Then, if we write $d = 2sc_B$, equation (5.8), together with Lemma 5.1, yields the relation

$$
-s\chi(B) - r_+^e + r_+^o = \sum_{m=m_{\text{min}}}^{d} (-1)^m c_m
$$

$$
\geq \sum_{m=m_{\text{min}}}^{d} (-1)^m b_m
$$

$$
= -s\chi(B) + \sum_{i=0}^{n-1} (-1)^i \dim H_i(B; \mathbb{Q}),
$$

where we have used the assumption that $n$ is odd. The inequality follows from the Morse inequalities (3.6) and the last equality follows from (3.2) using the hypothesis that $c_B > n/2$. Hence

$$
r_+^o \geq \sum_{i=0}^{n-1} (-1)^i \dim H_i(B; \mathbb{Q}).
$$

(5.11)

Now, we claim that

$$
r_+^e \geq \sum_{i=0}^{n-1} (-1)^i \dim H_i(B; \mathbb{Q}).
$$

(5.12)

In order to prove this, observe that applying Theorem 4.1 to $N$, $\eta$ and $\ell_0$ as above, we obtain positive integers $(d', k'_1, \ldots, k'_{d'}) = (d_1^{-}, k_1^{-}, \ldots, k_{r_1}^-)$ such that

$$
\mu(\gamma_{i_i}^{k_i'-\ell}) = d' - \mu(\gamma_{i_i}^{\ell}),
$$

$$
\mu(\gamma_{i_i}^{k_i'+\ell}) = d' + \mu(\gamma_{i_i}^{\ell}),
$$

and

$$
\mu(\gamma_{i_i}^{k_i'}) = d' - (\mu(\gamma_{i_i}^{k_i}) - d),
$$

(5.13)

for every $1 \leq i \leq r$ and $1 \leq \ell \leq \ell_0$. Arguing as before, we arrive at the equation

$$
\sum_{m=m_{\text{min}}}^{d'} (-1)^m c_m = \sum_{i=1}^{r} \sum_{\ell=1}^{k'_i} \chi(\gamma_{i_i}^{\ell}) - r_+^e + r_+^o,
$$

(5.14)
where, similarly to (5.9) and (5.10),
\[ r_{p}^{q} = \#\{1 \leq i \leq r \mid \pm (\mu(\gamma_{i}^{K_{r}}) - d') > 0, \gamma_{i}^{K_{r}} \text{ is good and } \mu(\gamma_{i}^{K_{r}}) \text{ is even}\} \]
and
\[ r_{o}^{p} = \#\{1 \leq i \leq r \mid \pm (\mu(\gamma_{i}^{K_{r}}) - d') > 0, \gamma_{i}^{K_{r}} \text{ is good and } \mu(\gamma_{i}^{K_{r}}) \text{ is odd}\}. \]
Notice that, due to (5.13), \( r_{p}^{0} = r_{o}^{0} \) and \( r_{p}^{0} = r_{o}^{0} \). Therefore, if we write \( d' = 2s'c_{B} \) for some integer \( s' \), equation (5.14), together with Lemma 5.1, gives rise to the relation
\[ -s'\chi(B) - r_{p}^{q} + r_{o}^{p} = \sum_{m=m_{\text{min}}}^{d'} (-1)^{m}c_{m} \]
\[ \quad \geq \sum_{m=m_{\text{min}}}^{d'} (-1)^{m}b_{m} \]
\[ = -s'\chi(B) + \sum_{i=0}^{n-1} (-1)^{i} \dim H_{i}(B; \mathbb{Q}), \]
where the assumptions that \( n \) is odd and \( c_{B} > n/2 \) have once more entered the picture. In particular, (5.12) holds.

Since \( r_{p}^{0} \) and \( r_{o}^{0} \) count two disjoint sets of orbits, in view of (5.11) and (5.12), we must have at least \( j \) distinct contractible simple closed orbits, say, \( \gamma_{1}, \ldots, \gamma_{j} \), where
\[ j := 2 \sum_{i=0}^{n-1} (-1)^{i} \dim H_{i}(B; \mathbb{Q}) = \chi(B) + \dim H_{n}(B; \mathbb{Q}) \]
as an immediate consequence of Poincaré duality and the assumption that \( n \) is odd. We claim that (good) iterates of these orbits have index different from \( d \) and hence do not contribute to \( \text{HC}_{0}(M) \). Indeed, since \( \mu(\gamma_{i}^{\ell}) \neq 0 \) for every \( 1 \leq i \leq r \) and \( \ell \in \mathbb{N} \) such that \( \gamma_{i}^{\ell} \) is good, we infer from (5.1), (5.2) and the definition of \( \epsilon_{0} \) that
\[ \mu(\gamma_{i}^{\ell}) \neq d \]
for every \( \ell \neq k_{i} \) and \( 1 \leq i \leq r \) such that \( \gamma_{i}^{\ell} \) is good. Therefore, only the orbits \( \gamma_{1}^{k_{1}}, \ldots, \gamma_{r}^{k_{r}} \) can contribute to \( \text{HC}_{0}(M) \). However, the definition of \( r_{p}^{0} \) given by (5.10) implies that \( \mu(\gamma_{i}^{k_{r}}) \neq d \) for all \( 1 \leq i \leq j \). Hence
\[ r \geq j + \dim \text{HC}_{0}(M) \geq j + \dim H_{n}(B; \mathbb{Q}), \]
where the second inequality follows from (3.2). Finally, we conclude that
\[ r \geq 2 \sum_{i=0}^{n-1} (-1)^{i} \dim H_{i}(B; \mathbb{Q}) + \dim H_{n}(B; \mathbb{Q}) = \chi(B) + 2 \dim H_{n}(B; \mathbb{Q}), \]
which proves Theorem 2.1 when \( n \) is odd.

**Remark 5.2.** It is clear from the definition of \( r_{p}^{q} \) and item (i) of Theorem 4.1 that the orbits \( \gamma_{1}, \ldots, \gamma_{j} \) are non-hyperbolic. In other words, if \( \alpha \) has finitely many distinct contractible simple closed orbits then at least \( \chi(B) + \dim H_{n}(B) \) of them are non-hyperbolic. This establishes Theorem 2.10 when \( n \) is odd.
**Case 2: n is even.**

As in the previous case, fix $N$ and $\eta$ as in Lemma 5.1, and an integer vector $(d, k_1, \ldots, k_v) = (d_1^v, k_1^v, \ldots, k_v^v)$ as in Theorem 4.1. The argument is very similar to the one for odd $n$. Namely, we consider, for each periodic orbit $\gamma_i$, the same classes of iterates $(A), (B), (C)$ and $(D)$, and study their contributions to the Morse type numbers defined by

\[ \sum_{m=m_{\min}}^{d+1} (-1)^m c_m. \] (5.15)

Due to the same index reasons as in Case 1, all good orbits in class $(A)$ contribute to (5.15) and class $(D)$ orbits do not contribute to (5.15). To deal with classes $(B)$ and $(C)$, we again consider four subclasses, although this time the index breakpoint is $-1$, rather than 0:

- $(B1)$ $\gamma_i^{k_i - \ell}$ with $1 \leq \ell \leq \ell_0$ if $\mu(\gamma_i^\ell) \geq 1$,
- $(B2)$ $\gamma_i^{k_i - \ell}$ with $1 \leq \ell \leq \ell_0$ if $\mu(\gamma_i^\ell) < -1$,

and

- $(C1)$ $\gamma_i^{k_i + \ell}$ with $1 \leq \ell \leq \ell_0$ if $\mu(\gamma_i^\ell) \geq -1$,
- $(C2)$ $\gamma_i^{k_i + \ell}$ with $1 \leq \ell \leq \ell_0$ if $\mu(\gamma_i^\ell) < -1$.

As before, all of the good orbits from class $(B1)$ contribute to (5.15) and class $(B2)$ makes no contribution. This is because, by (5.1), $\mu(\gamma_i^{k_i - \ell}) = d - \mu(\gamma_i^\ell)$, which is $\leq d + 1$ whenever $\mu(\gamma_i^\ell) \geq -1$ and $> d + 1$ whenever $\mu(\gamma_i^\ell) < -1$.

At this point recall that when $n$ is even $\alpha$ is assumed to have no contractible good periodic orbits of index 0 or $\pm 1$. (As in the case of an odd $n$, this is the key point where this assumption is utilized.) Hence, for all good iterates $\gamma_i^{k_i + \ell}$ in class $(C1)$, $\mu(\gamma_i^\ell) > 1$ and, by (5.2), $\mu(\gamma_i^{k_i + \ell}) = d + \mu(\gamma_i^\ell) > d + 1$ whenever $\gamma_i^{k_i + \ell}$ is good. As a result, class $(C1)$ does not contribute to (5.15). Finally, all of the good orbits from class $(C2)$ contribute to (5.15) since $\mu(\gamma_i^\ell) < -1$ and $\mu(\gamma_i^{k_i + \ell}) = d + \mu(\gamma_i^\ell) < d - 1$ by (5.2).

Thus, as in Case 1, all of the good orbits from classes $(A)$, $(B1)$ and $(C2)$ have index $\leq d + 1$ and contribute to (5.15), and good orbits from classes $(D)$, $(B2)$ and $(C1)$ have index $> d + 1$ and make no contribution to (5.15). Similarly to (5.5) and (5.6), define

\[ c_+^\ell = \sum_{i=1}^{r} \{ 1 \leq \ell \leq \ell_0 | \mu(\gamma_i^\ell) < -1, \gamma_i^{k_i + \ell} \text{ is good and } \mu(\gamma_i^\ell) \text{ is even} \} \]

and

\[ c_-^\ell = \sum_{i=1}^{r} \{ 1 \leq \ell \leq \ell_0 | \mu(\gamma_i^\ell) < -1, \gamma_i^{k_i + \ell} \text{ is good and } \mu(\gamma_i^\ell) \text{ is odd} \}. \]

Consider again $\sum_{m=m_{\min}}^{\infty} (-1)^m c_m$ to which all good orbits from classes $(A)$–$(D)$ and the collection $\{\gamma_i^k\}$ contribute. In particular, contributions of classes $(B2)$ and $(C2)$ are, respectively, $c_-^\ell - c_-^\ell$ and $c_+^\ell - c_+^\ell$. With the above discussion in mind, viewing (5.15) as $\sum_{m=m_{\min}}^{\infty} (-1)^m c_m - \sum_{m>d+1} (-1)^m c_m$, we obtain

\[ \sum_{m=m_{\min}}^{d+1} (-1)^m c_m = \sum_{i=1}^{r} \left( \sum_{\ell=1}^{k_i} \chi(\gamma_i^\ell) + c_+^\ell - c_-^\ell - (c_+^\ell - c_-^\ell) - \sum_{\mu(\gamma_i^\ell) > d+1} \chi(\gamma_i^\ell) \right). \]
We again have $c_i^c = c_i^e$ and $c_i^o = c_i^o$. Therefore,

$$\sum_{m=m_{\text{min}}}^{d+1} (-1)^m c_m = \sum_{i=1}^r \sum_{\ell=1}^{d_i} \chi(\gamma_i^\ell) - \sum_{i=1}^r \sum_{\mu(\gamma_i^k)>d+1} \chi(\gamma_i^k).$$

(5.16)

Similarly to (5.9) and (5.10), define

$$r_\pm^e = \#\{1 \leq i \leq r \mid \pm(\mu(\gamma_i^k) - d) > 1, \gamma_i^k \text{ is good and } \mu(\gamma_i^k) \text{ is even}\}$$

(5.17)

and

$$r_\pm^o = \#\{1 \leq i \leq r \mid \pm(\mu(\gamma_i^k) - d) > 1, \gamma_i^k \text{ is good and } \mu(\gamma_i^k) \text{ is odd}\}.$$  

Notice that the last term in (5.16) is given by $r_\pm^e - r_\pm^o$. Then, with the assumption $n$ is even in mind, setting $d = 2s c_B$ and using Lemma 5.1, we turn (5.16) into

$$s\chi(B) - r_\pm^e + r_\pm^o = \sum_{m=m_{\text{min}}}^{d+1} (-1)^m c_m
\leq \sum_{m=m_{\text{min}}}^{d+1} (-1)^m b_m
= s\chi(B) - \sum_{i=0}^{n-2} (-1)^i \dim H_i(B; \mathbb{Q}).$$

The inequality is due to the Morse inequalities (3.6) with the direction reversed since $d + 1$ is odd, and the last equality follows from (3.2) using the hypothesis that $c_B > n/2$. Hence

$$r_\pm^e \geq \sum_{i=0}^{n-2} (-1)^i \dim H_i(B; \mathbb{Q}).$$

Arguing similarly to the case where $n$ is odd, it is not hard to see that we also have

$$r_\pm^o \geq \sum_{i=0}^{n-2} (-1)^i \dim H_i(B; \mathbb{Q}).$$

Since $r_\pm^e$ and $r_\pm^o$ correspond to two disjoint collections of simple orbits, these two inequalities imply that we must have at least $j$ distinct contractible simple closed orbits, say, $\gamma_1, \ldots, \gamma_j$, where

$$j := 2 \sum_{i=0}^{n-2} (-1)^i \dim H_i(B; \mathbb{Q})
= \chi(B) + 2 \dim H_{n-1}(B; \mathbb{Q}) - \dim H_n(B; \mathbb{Q}).$$

(5.18)

Here the equality is due to Poincaré duality and the assumption that $n$ is even. Observe that iterates of these orbits do not contribute to $H^0_*(M)$ in degrees $* = d, d \pm 1$. Indeed, since $\mu(\gamma_i^\ell) \notin \{-1, 0, 1\}$ for every $1 \leq i \leq r$ and $\ell \in \mathbb{N}$ such that $\gamma_i^\ell$ is good, we infer from (5.1), (5.2) and the definition of $\ell_0$ that $\mu(\gamma_i^\ell) \notin \{d-1, d, d+1\}$
for all \( \ell \neq k_i \) and \( 1 \leq i \leq r \) such that \( \gamma_i^\ell \) is good. Thus only the orbits \( \gamma_1^{k_1}, \ldots, \gamma_r^{k_r} \) can contribute to \( \oplus_{m=d-1}^{d+1} \text{HC}_m^0(M) \). However, it follows from the definition of \( r' \) given by (5.17) that \( \mu(\gamma_i^{k_i}) \notin \{d-1, d, d+1\} \) for every \( 1 \leq i \leq j \). Hence

\[
r \geq j + \sum_{m=d-1}^{d+1} \dim \text{HC}_m^0(M).
\] (5.19)

By (3.2) and Poincaré duality, we also have

\[
\sum_{m=d-1}^{d+1} \dim \text{HC}_m^0(M) \geq \sum_{m=n-1}^{n+1} \dim H_m(B; \mathbb{Q}) = 2 \dim H_{n-1}(B; \mathbb{Q}) + \dim H_n(B; \mathbb{Q}).
\] (5.20)

Finally, combining (5.18), (5.19) and (5.20), we obtain

\[
r \geq \chi(B) + 4 \dim H_{n-1}(B; \mathbb{Q}),
\]

which establishes Theorem 2.1 when \( n \) is even.

Remark 5.3. By item (i) of Theorem 4.1, the above argument shows that if \( n \) is even and \( \alpha \) has finitely many distinct contractible simple closed orbits, then at least \( \chi(B) + 4 \dim H_{n-1}(B) - \dim H_n(B) \) of them are non-hyperbolic. This proves Theorem 2.10 when \( n \) is even.

5.2. Proof of Theorem 2.7. By Proposition 3.1, \( B \) is necessarily positive or negative spherically monotone. We will prove the theorem in the positive monotone case; the argument in the negative monotone case is similar. Arguing by contradiction, assume that \( \alpha \) has only one contractible simple closed orbit \( \gamma \). Note that the assumption \( c_B > n/2 \) implies that \( 2c_B \geq n + 1 \) if \( n \) is odd and \( 2c_B \geq n + 2 \) if \( n \) is even. Therefore, by the isomorphism (3.2), \( \text{HC}_m^0(M) = 0 \) for every \( m < 1 \) if \( n \) is odd or \( m < 2 \) if \( n \) is even. Moreover, there exists a sequence \( m_i \to \infty \) such that \( \text{HC}_{m_i}(M) \neq 0 \) for every \( i \). As consequence, \( \bar{\mu}(\gamma) > 0 \) and every good iterate of \( \gamma \) must have index \( \geq 1 \) if \( n \) is odd or \( \geq 2 \) if \( n \) is even. This implies that \( \alpha \) is index-positive and has no good contractible closed orbits \( \gamma^k \) such that \( \mu(\gamma^k) = 0 \) if \( n \) is odd or \( \mu(\gamma^k) \in \{0, \pm 1\} \) if \( n \) is even. Thus \( M \) and \( \alpha \) satisfy condition (F) and so Theorem 2.1 applies. This contradicts the assumption that \( \alpha \) has only one contractible simple closed orbit.

6. Multiplicity results and the contact Conley conjecture via contact homology

In this section we discuss a generalization of Theorem 2.1 relying on a variant of hybrid cylindrical-linearized contact homology. As another application of this homological construction we state a refinement of the contact Conley conjecture proved in [22].

6.1. Contact homology. Let \((M^{2n+1}, \xi)\) be a contact manifold and let \( \alpha \) be a non-degenerate contact form supporting \( \xi \). For the sake of simplicity, we assume that \( c_1(\xi) = 0 \). The differential graded algebra \((\mathfrak{A}(M, \alpha), d_\alpha)\) underlying the full rational contact homology is a graded commutative algebra generated by good closed Reeb orbits of \( \alpha \); see [6, 15]. With our dimension conventions, the grading is given by \( |\gamma| = \mu(\gamma) + n - 2 \). Assume furthermore that \((M, \xi)\) admits a non-degenerate index-admissible contact form \( \beta \). Without loss of generality, we may assume that \( \beta < \alpha \), i.e., \( \beta = f\alpha \) where \( 0 < f < 1 \). Hence we have a cylindrical cobordism from \((M, \beta)\) to \((M, \alpha)\) in the symplectization of \( M \), resulting in a homomorphism \( \Phi_\beta: (\mathfrak{A}(M, \alpha), d_\alpha) \to (\mathfrak{A}(M, \beta), d_\beta) \) of differential graded algebras.
Since $\beta$ is index-admissible, $(\mathfrak{A}(M, \beta), d_\beta)$ has a unique “trivial” augmentation $\epsilon_0$ determined by the requirement that the only monomial of degree zero for which $\epsilon_0 \neq 0$ is 1. Composing $\epsilon_0$ with $\Phi_\beta$, we obtain the augmentation

$$\epsilon_\beta = \epsilon_0 \circ \Phi_\beta: (\mathfrak{A}(M, \alpha), d_\alpha) \to \mathbb{Q}.$$ 

Note that, as is easy to see, $\epsilon_\beta(\gamma) = 0$ whenever $\gamma$ is not contractible.

A routine argument shows that the linearized homology of $(\mathfrak{A}(M, \beta), d_\beta)$ with respect to $\epsilon_\beta$ is independent of $\beta$ and $\alpha$; cf. [6, 15]. This is the “hybrid” homology we will use in this section but, for the sake of simplicity, we will still refer to this homology as the cylindrical contact homology of $\alpha$. The main advantage of this construction over the standard cylindrical contact homology is that the homology is defined for all non-degenerate contact forms: the form $\alpha$ need not be index-admissible. The only requirement is that $(M, \xi)$ admits one index-admissible form. It is essential that this homology can still be viewed as the homology of a complex freely generated by good closed Reeb orbits of $\alpha$. The complex is graded by $|\gamma|$ or $\mu(\gamma)$ and filtered by the contact action. Furthermore – and this is essential for what follows – the complex is also graded by the free homotopy class of $\gamma$ just as the standard cylindrical contact homology complex. (This is a consequence of the fact that $\epsilon_\beta$ vanishes on non-contractible orbits.)

Remark 6.1. The foundational aspects of the contact homology theory are still to be fully laid down. We refer the reader to [28, 29, 30] for the polyfold approach to this theory and to [36, 37] for the virtual cycle approach and further references.

6.2. Multiplicity results. Using contact homology we have the following refinement of Theorem 2.1.

**Theorem 6.2.** Let $(M^{2n+1}, \xi)$ be a prequantization $S^1$-bundle of a closed symplectic manifold $(B, \omega)$ such that $\omega|_{\pi_2(B)} \neq 0$ and $c_1(\xi) = 0$. Then $(B, \omega)$ is necessarily monotone and we require that $c_B > n/2$ when it is positive monotone and $c_B \geq n$ when it is negative monotone. Assume, furthermore, that $H_k(B; \mathbb{Q}) = 0$ for every odd $k$ or $c_B > n$. Let $\alpha$ be an index-definite non-degenerate contact form on $(M, \xi)$ having no contractible good periodic orbits $\gamma$ with $\mu(\gamma) = 0$ if $n$ is odd or with $\mu(\gamma) \in \{0, \pm 1\}$ if $n$ is even. Then $\alpha$ carries at least $r_B$ geometrically distinct contractible periodic orbits, where

$$r_B := \begin{cases} 
\chi(B) + 2 \dim H_n(B; \mathbb{Q}) & \text{if } n \text{ is odd} \\
\chi(B) + 4 \dim H_{n-1}(B; \mathbb{Q}) & \text{if } n \text{ is even}.
\end{cases}$$

A few words about the proof. The requirement that $c_1(\xi) = 0$ guarantees via the Gysin exact sequence that $(B, \omega)$ is monotone, i.e., $c_1(TB) = \lambda[\omega]$ in $H^2(B; \mathbb{Q})$ for some $\lambda \in \mathbb{R}$; cf. Remark 3.2. As in the proof of Proposition 3.1, we first need to show that the contact homology of $(M, \xi)$ is defined and given by (3.2) or (3.3) depending on whether $B$ is positive ($\lambda \geq 0$) or negative ($\lambda < 0$) monotone. To this end, it is sufficient to show that a non-degenerate perturbation $\beta$ of a connection contact form $\alpha_0$ is index-admissible.

When $B$ is positive monotone, this is an immediate consequence of Lemma 3.3 together with the requirement that $c_B > n/2$. (In fact, it is enough to assume that $c_B \geq 2$ which follows from $c_B > n/2$ when $n \geq 2$ and holds automatically under the conditions of the theorem when $n = 1$ since $c_{S^2} = 2$.) Then (3.2) follows exactly as in the proof of Proposition 3.1 using now the Morse–Bott calculations of contact homology from [5].

When $B$ is negative monotone, the situation is similar. In the notation from the proof of Lemma 3.3, for every $T_0 > 0$, all closed Reeb orbits $\gamma$ of a sufficiently small perturbation $\beta$
of \(\alpha_0\) with \(T(\gamma) \leq T_0\) have \(\mu(\gamma) \leq n - 2c_B\). Then, an argument completely similar to the proof of the lemma and using the quasimorphism property of the mean index shows that the same is true for all orbits when \(\beta\) is sufficiently closed to \(\alpha_0\). Thus \(\beta\) is index-admissible when \(n - 2c_B < 1 - n\), or equivalently \(c_B \geq n\), and (3.3) follows again from the results in [5] and the action filtration spectral sequence.

The proof is then finished exactly in the same way as the proof of Theorem 2.1; see Section 5.1. (In the SFT framework developed in [37], the gap in this argument is a Morse–Bott calculation of contact homology similar to [5] or [38].)

There is a broad class of symplectic manifolds to which Theorem 6.2 applies, while Theorem 2.1 does not. Among these are, for instance, negative monotone symplectic manifolds with large \(c_B\), e.g., complete intersections of high degree. These manifolds can have \(H_{odd}(B; \mathbb{Q}) \neq 0\). A simple example is the product of a complete intersection of a sufficiently high degree and a symplectically aspherical manifold.

Remark 6.3. It is clear from the discussion above that there is also a refinement of Theorem 2.10 relying on contact homology. Namely, under the assumptions of Theorem 6.2, if the contact form \(\alpha\) has finitely many geometrically distinct contractible closed orbits then it carries at least \(r_{B_{non-hyp}}\) geometrically distinct contractible non-hyperbolic periodic orbits.

6.3. **Contact Conley conjecture.** Another application of our definition of the cylindrical contact homology is a refinement of the contact Conley conjecture originally proved in [22]. Namely, we have

**Theorem 6.4 (Contact Conley Conjecture).** Let \(M \to B\) be a prequantization bundle and let \(\alpha\) be a contact form on \(M\) supporting the standard (co-oriented) contact structure \(\xi\) on \(M\). Assume that

(i) \(B\) is aspherical, i.e., \(\pi_r(B) = 0\) for all \(r \geq 2\), and
(ii) \(c_1(\xi) \in H^2(M; \mathbb{Q})\) is atoroidal.

Then the Reeb flow of \(\alpha\) has infinitely many simple closed orbits with contractible projections to \(B\). Assume furthermore that the Reeb flow has finitely many closed Reeb orbits in the free homotopy class \(f\) of the fiber and that these orbits are weakly non-degenerate. Then for every sufficiently large prime \(k\) the Reeb flow of \(\alpha\) has a simple closed orbit in the class \(f^k\).

The new point here, as compared to [22, Thm. 2.1], is that the form \(\alpha\) is not required to be index-admissible. The proof of Theorem 6.4 is essentially identical to the proof of [22, Thm. 2.1] and the only difference is that with our definition of cylindrical contact homology we need to ensure the existence of just one non-degenerate index-admissible contact form. In fact, every sufficiently small non-degenerate perturbation \(\beta\) of a connection contact form \(\alpha_0\) on \(M \to B\) is index admissible. This is an immediate consequence of, e.g., [18, Prop. 3.1] asserting that for every contractible closed Reeb orbit \(\gamma\) of \(\beta\), we have

\[|\hat{\mu}(\gamma)| \geq O(T(\gamma)),\]

where \(T(\gamma)\) is the period of \(\gamma\). (The proof of this fact is somewhat similar to the proof of Lemma 3.3. As in the proof of Theorem 6.2, in the contact homology framework from [37] the missing part in this argument is a Morse–Bott calculation of contact homology.)

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30 VIKTOR GINZBURG, BAŞAK GÜREL, AND LEONARDO MACARINI

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