A type of multiple integral
with log-gamma function

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In this paper, we give a general formula for the multiple integral
\[ I = \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \cdots dx_n. \]
As an application, the integral \( I \) with \( f(x) = \log \Gamma(x) \) is evaluated for all \( n \in \mathbb{N} \). The subsidiary computational challenges are interesting in their own right.

1. Introduction

A general idea, when faced with a multiple integral, is to lower its dimension. A well-known example, (see [Chang and Shi 2003], for instance) is the \( n \)-dimensional integral
\[ \int f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \cdots dx_n, \quad \text{(1-1)} \]
which can be simplified to a one-dimensional integral
\[ \frac{1}{(n-1)!} \int_0^1 t^{n-1} f(t) \, dt. \]
However, to the best of our knowledge, a similar integral,
\[ I = \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \cdots dx_n, \quad \text{(1-2)} \]
has no such formula.

The aim of this paper is to find a formula for the above integral \( I \) and apply it to the special case when \( f(x) = \log \Gamma(x) \). The main results are as follows. A general formula of \( I \) is obtained in Theorem 4.1.

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**Theorem 4.1.** The integral $I$ satisfies

\[
I = \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \ldots \, dx_n
\]

\[
= \frac{1}{(n-1)!} \sum_{m=1}^{n} \int_0^1 G_m(t) f(t + m - 1) \, dt,
\]

where

\[
G_m(t) = \sum_{i=1}^{m} (-1)^{i-1} (t + m - i)^{n-1} \binom{n}{i-1}.
\]

When $f(x) = \log \Gamma(x)$, the value of $I$ is given in Theorem 5.1. The main challenge of the proof is to find appropriate combinatorial identities to simplify $I$.

**Theorem 5.1.**

\[
I = I(n) = \int_0^1 \int_0^1 \cdots \int_0^1 \log \Gamma(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \ldots \, dx_n
\]

\[
= \frac{1}{2} \log(2\pi) - \frac{n-1}{2} H_n + \sum_{k=2}^{n-1} \frac{(-1)^{n+k+1} k^n}{n!} \binom{n-1}{k} \log k,
\]

where the last sum is missing when $n = 2$ and $H_n = \sum_{k=1}^{n} 1/k$.

The paper is organized as follows. In Sections 2 and 3, we explain the main ideas by using the cases $n = 2$ and 3. One can see from Figures 1 and 2 how we cut the square and the cube so that the integral $I$ over each subset becomes a simple one-dimensional integral. In Section 4, a formula of $I$ is derived in Theorem 4.1, and in Section 5, we evaluate $I$ when $f(x) = \log \Gamma(x)$.

### 2. The case $n = 2$

When $n = 2$, the integral $I$ becomes $\int_0^1 \int_0^1 f(x + y) \, dx \, dy$, where the integral domain is a unit square. Let $t = x + y$. The unit square can then be divided into two domains, $D_1$ and $D_2$ as in Figure 1, where

\[
D_1 = \{(x, y) : 0 \leq x + y \leq 1, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1\},
\]

\[
D_2 = \{(x, y) : 1 \leq x + y \leq 2, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1\}.
\]

The following lemma shows that $t^1_0 \int_0^1 f(x + y) \, dx \, dy$ is the sum of two one-dimensional integrals.
Lemma 2.1.
\[
\int_0^1 \int_0^1 f(x+y) \, dx \, dy = \iint_{D_1} f(x+y) \, dx \, dy + \iint_{D_2} f(x+y) \, dx \, dy
\]
\[
= \int_0^1 tf(t) \, dt + \int_0^1 (1-t)f(t+1) \, dt. \quad (2-1)
\]

Proof. It is clear that
\[
\int_0^1 \int_0^1 f(x+y) \, dx \, dy = \iint_{D_1} f(x+y) \, dx \, dy + \iint_{D_2} f(x+y) \, dx \, dy.
\]
We first consider \( \iint_{D_1} f(x+y) \, dx \, dy \). Note that \( t = x+y \), and consider the transformation \( (x, y) \mapsto (x, t) \). It is clear that the Jacobian is 1. Then
\[
\iint_{D_1} f(x+y) \, dx \, dy = \int_0^1 \int_0^t f(t) \, dx \, dt = \int_0^1 tf(t) \, dt. \quad (2-2)
\]
For the integral over domain \( D_2 \), we set \( x_1 = 1-x \) and \( y_1 = 1-y \). Then \( (x_1, y_1) \in D_1 \) and
\[
\iint_{D_2} f(x+y) \, dx \, dy = \iint_{D_1} f(2-x_1-y_1) \, dx_1 \, dy_1
\]
\[
= \int_0^1 tf(2-t) \, dt. \quad (2-3)
\]
If one sets \( u = 1-t \), it follows that \( \int_0^1 tf(2-t) \, dt = \int_0^1 (1-u)f(u+1) \, du \). Then
\[
\iint_{D_2} f(x+y) \, dx \, dy = \int_0^1 (1-u)f(u+1) \, du. \quad (2-4)
\]
Then, identity (2-1) follows by identities (2-2) and (2-4). \( \square \)

Figure 1. Domains \( D_1 \) and \( D_2 \).
3. The case $n = 3$

When $n = 3$, the integral domain of $I$ is a unit cube. The main idea is to cut the unit cube into several simplexes so that we can apply the integral formula (1-1) over each one.

Let $E = \{(x, y, z) : 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 0 \leq z \leq 1\}$ be the unit cube. Set

\[E_1 = \{(x, y, z) : 0 \leq x + y + z \leq 1, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 0 \leq z \leq 1\},\]
\[E_2 = \{(x, y, z) : 1 \leq x + y + z \leq 2, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 0 \leq z \leq 1\},\]
\[E_3 = \{(x, y, z) : 2 \leq x + y + z \leq 3, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 0 \leq z \leq 1\}.

Then $E = E_1 \cup E_2 \cup E_3$ and the integral $I$ satisfies

\[
I = \int_0^1 \int_0^1 \int_0^1 f(x + y + z) \, dx \, dy \, dz = \int_{E_1} f(x + y + z) \, dx \, dy \, dz + \int_{E_2} f(x + y + z) \, dx \, dy \, dz + \int_{E_3} f(x + y + z) \, dx \, dy \, dz.
\]

Using formula (1-1), it follows that $\int_{E_1} f(x + y + z) \, dx \, dy \, dz = \frac{1}{2} \int_0^1 t^2 f(t) \, dt$. The difficult parts are the integrals over $E_2$ and $E_3$. The following lemma explains how to simplify these two integrals to one-dimensional integrals.

**Lemma 3.1.**

\[
\int_0^1 \int_0^1 \int_0^1 f(x + y + z) \, dx \, dy \, dz = \frac{1}{2} \int_0^1 t^2 f(t) \, dt + \frac{1}{2} \int_0^1 \left(-2t^2 + 2t + 1\right) f(t+1) \, dt + \frac{1}{2} \int_0^1 (1-t)^2 f(t+2) \, dt. \tag{3-1}
\]

**Proof.** We introduce the transformation $(x, y, z) \mapsto (x, y, t)$. By formula (1-1),

\[
\int_{E_1} f(x + y + z) \, dx \, dy \, dz = \frac{1}{2} \int_0^1 t^2 f(t) \, dt. \tag{3-2}
\]

Note that integral (3-2) can be applied to calculate the integral over $E_3$. Let $x_1 = 1 - x$, $y_1 = 1 - y$ and $z_1 = 1 - z$. The integral over $E_3$ becomes

\[
\int_{E_3} f(x + y + z) \, dx \, dy \, dz = \int_{E_1} f(3 - x_1 - y_1 - z_1) \, dx_1 \, dy_1 \, dz_1 = \frac{1}{2} \int_0^1 t^2 f(3-t) \, dt. \tag{3-3}
\]
Figure 2. Region $E_{20}$ and its partition: $E_2, E_{21}, E_{22}, E_{23}$.

If one sets $u = 1 - t$, it implies that $\frac{1}{2} \int_0^1 t^2 f(3-t) \, dt = \frac{1}{2} \int_0^1 (1-u)^2 f(2+u) \, du$. Hence,

$$\int_{E_3} f(x + y + z) \, dx \, dy \, dz = \frac{1}{2} \int_0^1 (1-t)^2 f(t+2) \, dt. \quad (3-4)$$

By equalities (3-2) and (3-4), it is sufficient to show that

$$\int_{E_2} f(x + y + z) \, dx \, dy \, dz = \frac{1}{2} \int_0^1 (-2t^2 + 2t + 1) f(t+1) \, dt. \quad (3-5)$$

Consider the domain

$$E_{20} = \{(x, y, z) : 1 \leq x + y + z \leq 2, \ 0 \leq x \leq 2, \ 0 \leq y \leq 2, \ 0 \leq z \leq 2\}.$$  

Similar to Figure 1, we can cut $E_{20}$ into 4 different domains, $E_2, E_{21}, E_{22}$ and $E_{23}$, so that the integral over each domain can be handled easily. A picture of this partition is shown in Figure 2.

$$E_2 = \{(x, y, z) : 1 \leq x + y + z \leq 2, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 0 \leq z \leq 1\},$$

$$E_{21} = \{(x, y, z) : 1 \leq x + y + z \leq 2, \ 1 \leq x \leq 2, \ 0 \leq y \leq 1, \ 0 \leq z \leq 1\},$$

$$E_{22} = \{(x, y, z) : 1 \leq x + y + z \leq 2, \ 0 \leq x \leq 1, \ 1 \leq y \leq 2, \ 0 \leq z \leq 1\},$$

$$E_{23} = \{(x, y, z) : 1 \leq x + y + z \leq 2, \ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ 1 \leq z \leq 2\},$$

where $E_{20} = E_2 \cup E_{21} \cup E_{22} \cup E_{23}$.

Again by using formula (1-1), the integral over $E_{20}$ is

$$\int_{E_{20}} f(x + y + z) \, dx \, dy \, dz = \int_1^2 \frac{1}{2} t^2 f(t) \, dt = \frac{1}{2} \int_0^1 (t+1)^2 f(t+1) \, dt. \quad (3-6)$$
On the other hand, the integral over $E_{20}$ satisfies
\[
\int_{E_{20}} f(x + y + z) \, dx \, dy \, dz
= \int_{E_{21}} f(x + y + z) \, dx \, dy \, dz + \int_{E_{22}} f(x + y + z) \, dx \, dy \, dz
+ \int_{E_{23}} f(x + y + z) \, dx \, dy \, dz + \int_{E_2} f(x + y + z) \, dx \, dy \, dz. \quad (3-7)
\]
By the definitions of $E_{21}$, $E_{22}$ and $E_{23}$, it is clear that
\[
\int_{E_{21}} f(x + y + z) \, dx \, dy \, dz = \int_{E_{22}} f(x + y + z) \, dx \, dy \, dz = \int_{E_{23}} f(x + y + z) \, dx \, dy \, dz.
\]
So we only need to consider $\int_{E_{21}} f(x + y + z) \, dx \, dy \, dz$. Let $\tilde{x} = x - 1$; then by equality (3-2),
\[
\int_{E_{21}} f(x + y + z) \, dx \, dy \, dz = \int_{E_1} f(\tilde{x} + y + z + 1) \, d\tilde{x} \, dy \, dz
= \frac{1}{2} \int_0^1 t^2 f(t + 1) \, dt. \quad (3-8)
\]
Therefore, (3-6), (3-7) and (3-8) imply that
\[
\int_{E_2} f(x + y + z) \, dx \, dy \, dz
= \int_{E_{20}} f(x + y + z) \, dx \, dy \, dz - 3 \int_{E_{21}} f(x + y + z) \, dx \, dy \, dz
= \frac{1}{2} \int_0^1 (t + 1)^2 f(t + 1) \, dt - \frac{3}{2} \int_0^1 t^2 f(t + 1) \, dt
= \frac{1}{2} \int_0^1 (-2t^2 + 2t + 1) f(t + 1) \, dt,
\]
which shows equality (3-5). \hfill \square

4. The general case

In this section, we give a general formula for
\[
I = \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \ldots \, dx_n
\]
in Theorem 4.1. In order to prove it, we first find a recursive formula for $I$ in Theorem 4.3. The proof of Theorem 4.1 then follows by Theorem 4.4 and Theorem 4.3.
Theorem 4.1. The integral $I$ satisfies
\[
I = \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \cdots dx_n
\]
\[
= \frac{1}{(n-1)!} \sum_{m=1}^{n} \int_0^1 G_m(t) f(t + m - 1) \, dt,
\]
where
\[
G_m(t) = \sum_{i=1}^{m} (-1)^{i-1}(t + m - i)^{n-1}\left(\begin{array}{c}n \\ i-1\end{array}\right).
\]

The idea is to divide the $n$-dimensional unit box into $n$ different polyhedrons and the integral $I$ over each polyhedron can be simplified to a one-dimensional integral by applying the ideas in the 2D or 3D cases. The $n$ different polyhedrons are defined as follows:

$K_1 = \{(x_1, x_2, \ldots, x_n): 0 \leq x_1 + x_2 + \cdots + x_n \leq 1,\]
\[0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 1, \quad \ldots \quad 0 \leq x_n \leq 1\},$

$K_2 = \{(x_1, x_2, \ldots, x_n): 1 \leq x_1 + x_2 + \cdots + x_n \leq 2,\]
\[0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 1, \quad \ldots \quad 0 \leq x_n \leq 1\},$

$\vdots$

$K_n = \{(x_1, x_2, \ldots, x_n): n-1 \leq x_1 + x_2 + \cdots + x_n \leq n,\]
\[0 \leq x_1 \leq 1, \quad 0 \leq x_2 \leq 1, \quad \ldots \quad 0 \leq x_n \leq 1\}.$

By formula (1-1), the integral over $K_1$ satisfies the following proposition.

Proposition 4.2. \[
\int_{K_1} f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \cdots dx_n = \frac{1}{(n-1)!} \int_0^1 t^{n-1} f(t) \, dt.
\]

Let \[
I_m = \int_{K_m} f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \cdots dx_n, \quad m = 1, 2, \ldots, n.
\]

It is obvious that $I = \sum_{m=1}^{n} I_m.$ Then the integral $I$ reduces to the calculation of each $I_m$ ($1 \leq m \leq n$). Define \[
J_{s,m} = \int_{K_s} f(x_1 + \cdots + x_n + m-s) \, dx_1 \, dx_2 \cdots dx_n, \quad (4-2)
\]
where $s$ is an integer and $1 \leq s \leq m.$ Note that $J_{m,m} = I_m.$ For any $1 \leq s \leq m-1,$ $J_{s,m}$ can be calculated by $I_s.$ The following theorem shows that $I_m$ satisfies a recursive formula.
Theorem 4.3.

\[ I_m = \frac{1}{(n-1)!} \int_0^1 (t + m - 1)^{n-1} f(t + m - 1) \, dt \]

\[-a_1 J_{1,m} - a_2 J_{2,m} - \cdots - a_{m-1} J_{m-1,m}. \quad (4-3)\]

where

\[ a_i = \left( \frac{m+n-i-1}{n-1} \right), \quad i = 1, 2, \ldots, m-1. \]

Proof. We consider the region

\[ K_{m0} = \{(x_1, x_2, \ldots, x_n) : m - 1 \leq x_1 + x_2 + \cdots + x_n \leq m, \quad 0 \leq x_1 \leq m, \quad 0 \leq x_2 \leq m, \quad \ldots, \quad 0 \leq x_n \leq m\}. \]

By Proposition 4.2,

\[ \int_{K_{m0}} f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \cdots \, dx_n \]

\[ = \frac{1}{(n-1)!} \int_{m-1}^m t^{n-1} f(t) \, dt \]

\[ = \frac{1}{(n-1)!} \int_0^1 (t + m - 1)^{n-1} f(t + m - 1) \, dt. \quad (4-4)\]

We define the subset \( K_{i_1 i_2 \ldots i_n} \subset K_{m0} \) as follows:

\[ K_{i_1 i_2 \ldots i_n} = \{(x_1, x_2, \ldots, x_n) : m - 1 \leq x_1 + x_2 + \cdots + x_n \leq m, \quad i_1 - 1 \leq x_1 \leq i_1, \quad i_2 - 1 \leq x_2 \leq i_2, \quad \ldots, \quad i_n - 1 \leq x_n \leq i_n\}, \]

where \( i_1, i_2, \ldots, i_n \in [1, m] \) are positive integers. It is easily seen that the intersection of any two subsets \( K_{i_1 i_2 \ldots i_n} \) only happens on their boundaries. We then classify all possible \( K_{i_1 i_2 \ldots i_n} \) so that the integral over each one can be evaluated easily. Note that by definition, \( K_{1,1,\ldots,1} = K_m \). To find the integral over \( K_m \), we need to subtract the integrals over all the other nonempty subsets \( K_{i_1 i_2 \ldots i_n} (i_1, i_2, \ldots, i_n \in [1, m]) \) from \( \int_{K_{m0}} f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \cdots \, dx_n. \)

The first step is to determine when \( K_{i_1 i_2 \ldots i_n} (i_1, i_2, \ldots, i_n \in [1, m]) \) is nonempty. For any set \( K_{i_1 i_2 \ldots i_n} \), let

\[ \tilde{x}_1 = x_1 - (i_1 - 1), \quad \tilde{x}_2 = x_2 - (i_2 - 1), \quad \ldots, \quad \tilde{x}_n = x_n - (i_n - 1). \quad (4-5)\]

Then \( K_{i_1 i_2 \ldots i_n} \) becomes

\[ \tilde{K}_{i_1 i_2 \ldots i_n} = \{ (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) : m + n - \alpha - 1 \leq \tilde{x}_1 + \tilde{x}_2 + \cdots + \tilde{x}_n \leq m + n - \alpha, \]

\[ 0 \leq \tilde{x}_1 \leq 1, \quad 0 \leq \tilde{x}_2 \leq 1, \quad \ldots, \quad 0 \leq \tilde{x}_n \leq 1 \}.
where \( \alpha = i_1 + i_2 + \cdots + i_n \). Let \( s = m + n - \alpha \). It is clear that \( K_{i_1 i_2 \ldots i_n} \cong \bar{K}_{i_1 i_2 \ldots i_n} = K_s \). Since \( m + n - s = \sum_{j=1}^{n} i_j \geq n \), it follows that \( s \leq m \). Note that if \( s = m \), by equality (4-2), \( J_{m,m} = I_m \). If \( s = 0 \), \( K_{i_1 i_2 \ldots i_n} \cong \bar{K}_{i_1 i_2 \ldots i_n} = \{0\} \), and if \( s < 0 \), \( K_{i_1 i_2 \ldots i_n} \cong \bar{K}_{i_1 i_2 \ldots i_n} = \emptyset \). So we only need to consider the case \( 1 \leq s \leq m - 1 \). For any given \( s \in [1, m - 1] \), it follows that

\[
\int_{K_{i_1 i_2 \ldots i_n}} f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \ldots \, dx_n
\]

\[
= \int_{\bar{K}_{i_1 i_2 \ldots i_n}} f(\tilde{x}_1 + \cdots + \tilde{x}_n + i_1 + \cdots + i_n - n) \, d\tilde{x}_1 \ldots \, d\tilde{x}_n
\]

\[
= \int_{K_s} f(x_1 + \cdots + x_n + m - s) \, dx_1 \ldots \, dx_n
\]

\[
= J_{s,m}. \quad (4-6)
\]

It implies that the subsets \( K_{i_1 i_2 \ldots i_n} \) \((i_1, i_2, \ldots, i_n \in [1, m], i_1 + i_2 + \cdots + i_n \neq n)\) with nonzero measure can be classified into \( m - 1 \) classes. In each class, every element is identical to some subset \( K_s \) after a shifting transformation in (4-5): \((x_1, x_2, \ldots, x_n) \mapsto (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)\).

Next step is to fix \( m \) and \( s \) \((1 \leq s \leq m - 1)\), and find out how many subsets are identical to \( K_s \). Since \( s = m + n - (i_1 + i_2 + \cdots + i_n) \), we have

\[
m + n - s = i_1 + i_2 + \cdots + i_n, \quad \text{where } i_1, i_2, \ldots, i_n \text{ are positive integers} \quad (4-7)
\]

The number of positive integer solutions \((i_1, i_2, \ldots, i_n)\) for (4-7) is \({m+n-s-1 \choose n-1}\). It follows that the total number of subsets identical to \( K_s \) \((s \in [1, m - 1])\) is

\[
as_s = {m+n-s-1 \choose n-1}. \quad (4-8)
\]

Therefore, by equalities (4-4), (4-6) and (4-8), \( I_m \) satisfies

\[
I_m = \int_{K_m} f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \ldots \, dx_n
\]

\[
= \frac{1}{(n-1)!} \int_0^1 (t + m - 1)^{n-1} f(t + m - 1) \, dt
\]

\[
- a_1 J_{1,m} - a_2 J_{2,m} - \cdots - a_{m-1} J_{m-1,m}. \quad (4-9)
\]

where \( a_s \) \((s = 1, \ldots, m - 1)\) is defined by (4-8).

By using the cases \( n = 2 \) and \( 3 \), we can show by induction that

\[
I_m = \frac{1}{(n-1)!} \int_0^1 G_m(t) f(t + m - 1) \, dt, \quad (4-10)
\]
where $G_m(t)$ is a polynomial. It follows that

$$J_{s,m} = \int_{K_s} f(x_1 + \cdots + x_n + m - s) \, dx_1 \cdots dx_n$$

$$= \frac{1}{(n-1)!} \int_0^1 G_s(t) f(t + m - 1) \, dt. \quad (4-11)$$

where $s$ is an integer and $1 \leq s \leq m$. The integral $I$ satisfies

$$I = \int_0^1 \int_0^1 \cdots \int_0^1 f(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \cdots dx_n = \sum_{m=1}^n I_m. \quad (4-12)$$

In order to find a formula for $I$, we only need to compute the polynomial $G_m(t)$ in equality $(4-10)$ for all $1 \leq m \leq n$. For $m = 1, 2$ and $3$, a direct calculation shows that

$$G_1(t) = t^{n-1},$$

$$G_2(t) = (t + 1)^{n-1} - \binom{n}{1} t^{n-1}. \quad (4-13)$$

By Theorem 4.3 and equality $(4-11),$

$$G_3(t) = (t + 2)^{n-1} - \binom{n+1}{n-1} G_1(t) - \binom{n}{n-1} G_2(t)$$

$$= (t + 2)^{n-1} - \binom{n}{1} (t + 1)^{n-1} + \binom{n}{2} t^{n-1}. $$

Similarly,

$$G_4(t) = (t + 3)^{n-1} - \binom{n}{1} (t + 2)^{n-1} + \binom{n}{2} (t + 1)^{n-1} - \binom{n}{3} t^{n-1}. $$

It is reasonable to believe that $G_m(t)$ follows a pattern. The following theorem actually proves this fact.

**Theorem 4.4.**

$$G_m(t) = \sum_{i=1}^m (-1)^{i-1} (t + m - i)^{n-1} \binom{n}{i-1}. \quad (4-14)$$

**Proof.** The proof is based on the recursive formula $(4-3)$ in Theorem 4.3 and the identity $(4-11)$. By formula $(4-3),$

$$I_m = \frac{1}{(n-1)!} \int_0^1 (t + m - 1)^{n-1} f(t + m - 1) \, dt - \sum_{i=1}^{m-1} a_i J_{i,m}$$

$$= \frac{1}{(n-1)!} \int_0^1 G_m(t) f(t + m - 1) \, dt,$$
where

\[ G_m(t) = (t + m - 1)^{n-1} - \sum_{i=1}^{m-1} a_i G_i(t), \quad \text{and} \quad a_i = \binom{m+n-i-1}{n-1}. \quad (4-15) \]

We show this theorem by induction. It is clear that formula (4-14) of \( G_m(t) \) holds for \( m = 1 \). Assume that it holds for any \( 1 \leq m \leq k \). We need to show that formula (4-14) also holds for \( m = k + 1 \).

By (4-15) and the induction assumption, the polynomial \( G_{k+1}(t) \) satisfies

\[ G_{k+1}(t) = (t + k)^{n-1} + \sum_{i=1}^{k} \binom{k+1+n-i-1}{n-1} \sum_{j=1}^{i} (-1)^j (t + i - j)^{n-1} \binom{n}{j-1}. \quad (4-16) \]

By formula (4-14), we can consider each \( G_m(t) (1 \leq m \leq k) \) as a polynomial of \( (t + m - j)^{n-1} \) \((j = 1, 2, \ldots, m)\) with coefficient \((-1)^{j-1} \binom{n}{j-1}\). Then identity (4-16) implies that the coefficient of \( (t + p)^{n-1} \) in \( G_{k+1}(t) \) is

\[ L_p(G_{k+1}(t)) = \sum_{i=p+1}^{k} \binom{k+1+n-i-1}{n-1} (-1)^{i-p} \binom{n}{i-p-1}, \quad (4-17) \]

where \( p \in [0, k - 1] \) is an integer. Similarly, \( G_k(t) \) satisfies

\[ G_k(t) = (t + k - 1)^{n-1} + \sum_{i=1}^{k-1} \binom{k+n-i-1}{n-1} \sum_{j=1}^{i} (-1)^j (t + i - j)^{n-1} \binom{n}{j-1}. \]

and the coefficient of \( (t + p)^{n-1} \) \((p \in [0, k - 2])\) in \( G_k(t) \) is

\[ \sum_{i=p+1}^{k-1} \binom{k+n-i-1}{n-1} (-1)^{i-p} \binom{n}{i-p-1}. \quad (4-18) \]

Note that \( G_k(t) = \sum_{i=1}^{k} (-1)^{i-1} (t + k - i)^{n-1} \binom{n}{i-1} \). It follows that

\[ \sum_{i=p+1}^{k-1} \binom{k+n-i-1}{n-1} (-1)^{i-p} \binom{n}{i-p-1} = (-1)^{k-p-1} \binom{n}{k-p-1}. \quad (4-19) \]
If $p \neq 0$, let $q = p - 1$. By identity (4-19), the coefficient of $(t + p)^{n-1}$ in (4-17) satisfies
\[
L_p(G_{k+1}(t)) = \sum_{i=p+1}^{k} \binom{k+1+n-i-1}{n-1}(-1)^{i-p}\binom{n}{i-p-1}
= \sum_{i=q+2}^{k} \binom{k+1+n-i-1}{n-1}(-1)^{i-q-1}\binom{n}{i-q-2}
= \sum_{i=q+1}^{k-1} \binom{k+n-i-1}{n-1}(-1)^{i-q}\binom{n}{i-q-1}
= (-1)^{k-q-1}\binom{n}{k-q-1} = (-1)^{k-p}\binom{n}{k-p}.
\] (4-20)

Identity (4-20) holds for all integers $p \in [1, k - 1]$. It remains to consider the case when $p = 0$.

If $p = 0$, by (4-17), the coefficient of $t^{n-1}$ in $G_{k+1}(t)$ is
\[
L_0(G_{k+1}(t)) = \sum_{i=1}^{k} \binom{k+1+n-i-1}{n-1}(-1)^{i}\binom{n}{i-1}.
\] (4-21)

Next, we show that $L_0(G_{k+1}(t)) = (-1)^k\binom{n}{k}$. Note that by the binomial theorem, the coefficient of the term $x^{k+1}$ in $(1 + x)^{-n}(1 + x)^n$ is
\[
\sum_{i=0}^{k} (-1)^i\binom{n+i-1}{i}\binom{n}{k-i}
= \sum_{i=0}^{k} (-1)^i\binom{n+i-1}{n-1}\binom{n}{k-i}
= \sum_{j=1}^{k+1} \binom{k+1+n-j-1}{n-1}(-1)^{k+1-j}\binom{n}{j-1} (j = k + 1 - i)
= (-1)^{k+1}\left(L_0(G_{k+1}(t)) + (-1)^{k+1}\binom{n}{k}\right).
\] (4-22)

On the other hand, for a nonnegative integer $k$, the coefficient of the term $x^{k+1}$ in $(1 + x)^{-n}(1 + x)^n = 1$ is always 0. Hence, (4-22) implies that
\[
L_0(G_{k+1}(t)) = (-1)^k\binom{n}{k}.
\] (4-23)
Therefore, by identities (4-20) and (4-23), it follows that
\[
G_{k+1}(t) = (t + k)^{n-1} + \sum_{p=0}^{k-1} (-1)^{k-p} \binom{n}{k-p} (t + p)^{n-1}
\]
\[
= \sum_{i=1}^{k+1} (-1)^{i-1} (t + k + 1 - i)^{n-1} \binom{n}{i-1}.
\] (4-24)

This concludes the proof. \( \square \)

5. Application to log-gamma function

In this section, we consider the integral of log-gamma function

\[
I = \int_0^1 \int_0^1 \ldots \int_0^1 \log \Gamma(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \ldots \, dx_n.
\] (5-1)

The integral of log-gamma function has its own importance in many parts of mathematics [Amdeberhan et al. 2011; Choi and Srivastava 2005]. Actually, the case when \( n = 2 \) is a problem proposed by Ovidiu Furdui [2010] in the Problems and Solutions section of *The College Mathematics Journal*, and one of its solutions is proposed by Geng-zhe Chang [2011]. When it comes to general dimension \( n \), it is quite a challenge to evaluate it.

After the preparation of Theorem 4.1 in Section 4, we can evaluate the integral (5-1). A nice formula is given in Theorem 5.1.

**Theorem 5.1.**

\[
I = I(n) = \int_0^1 \int_0^1 \ldots \int_0^1 \log \Gamma(x_1 + x_2 + \cdots + x_n) \, dx_1 \, dx_2 \ldots \, dx_n
\]
\[
= \frac{1}{2} \log(2\pi) - \frac{n-1}{2} H_n + \sum_{k=2}^{n-1} \frac{(-1)^{n+k+1} k n}{n!} \binom{n-1}{k} \log k,
\] (5-2)

where the last sum is missing when \( n = 2 \) and \( H_n = \sum_{k=1}^{n} 1/k \).

The proof of this theorem is based on Theorem 4.1 and several combinatorial identities in Jihuai Shi’s book [2009].
Note that \( \Gamma(t+1) = t \Gamma(t) \) and \( G_m(t) = \sum_{i=1}^{m} (-1)^{i-1} (t + m - i)^{n-1} \binom{n}{i-1} \). By Theorem 4.1, the integral \( I \) becomes

\[
I = \frac{1}{(n-1)!} \sum_{m=1}^{n} \int_{0}^{1} G_m(t) \log \Gamma(t + m - 1) \, dt \\
= \frac{1}{(n-1)!} \int_{0}^{1} \sum_{m=1}^{n} G_m(t) \log \Gamma(t) \, dt \\
+ \frac{1}{(n-1)!} \int_{0}^{1} \sum_{k=2}^{n} \sum_{m=k}^{n} G_m(t) \log(t + k - 2) \, dt. \tag{5-3}
\]

Several combinatorial identities are introduced to simplify (5-3).

**Lemma 5.2.**

\[
\sum_{m=k}^{n} G_m(t) = (n-1)! - \sum_{m=1}^{k-1} \binom{n-1}{k-m-1} (-1)^{k-m-1} (t + m - 1)^{n-1},
\]

and when \( k = 1, \sum_{m=1}^{n} G_m(t) = (n-1)! \).

**Proof.** Note that \( G_m(t) = \sum_{i=1}^{m} (-1)^{i-1} (t + m - i)^{n-1} \binom{n}{i-1} \). It follows that

\[
\sum_{m=1}^{k} G_m(t) = \sum_{m=1}^{k} \sum_{i=1}^{m} (-1)^{i-1} (t + m - i)^{n-1} \binom{n}{i-1} \\
= \sum_{m=1}^{k} \sum_{i=0}^{k-m} (-1)^{i} \binom{n}{i} (t + m - 1)^{n-1}. 
\]

By the combinatorial identity \( \sum_{i=0}^{m} (-1)^i \binom{n}{i} = (-1)^m \binom{n-1}{m} \) \( (m < n) \), we have

\[
\sum_{m=1}^{k} \sum_{i=0}^{k-m} (-1)^{i} \binom{n}{i} (t + m - 1)^{n-1} = \sum_{m=1}^{k} (-1)^{m-1} \binom{n-1}{k-m} (t + m - 1)^{n-1}. 
\]

Hence,

\[
\sum_{m=1}^{k} G_m(t) = \sum_{m=1}^{k} (-1)^{m-1} \binom{n-1}{k-m} (t + m - 1)^{n-1}. 
\]
In the case when $k = n$, the combinatorial identity $\sum_{k=0}^{n} (-1)^k \binom{n}{k} (x+n-k)^n = n!$ implies

$$\sum_{m=1}^{n} G_m(t) = \sum_{m=1}^{n} \binom{n-1}{n-m} (-1)^{n-m} (t+m-1)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k (t+n-1-k)^{n-1} = (n-1)!.$$  

Therefore,

$$\sum_{m=k}^{n} G_m(t) = \sum_{m=1}^{n} G_m(t) - \sum_{m=1}^{k-1} G_m(t) = (n-1)! - \sum_{m=1}^{k-1} \binom{n-1}{k-m-1} (-1)^{k-m-1} (t+m-1)^{n-1}. \quad \square$$

Let

$$T_k = \sum_{m=1}^{k} \binom{n-1}{k-m} (-1)^{k-m} (t+m-1)^{n-1} = \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m (t+k-m-1)^{n-1}.$$  

Then

$$\sum_{m=k}^{n} G_m(t) = (n-1)! - T_{k-1}.$$  

By applying Lemma 5.2, (5-3) becomes

$$I = \int_{0}^{1} \log \Gamma(t) \, dt + \int_{0}^{1} \sum_{k=0}^{n-2} \log(t+k) \, dt - \frac{1}{(n-1)!} \int_{0}^{1} \sum_{k=1}^{n-1} T_k \log(t+k-1) \, dt = \frac{1}{2} \log(2\pi) + (n-1) \log(n-1) - n + 1 - \frac{1}{(n-1)!} \int_{0}^{1} \sum_{k=1}^{n-1} T_k \log(t+k-1) \, dt.$$  

(5-4)

Then, the calculation of $I$ reduces to the calculation of

$$\int_{0}^{1} \sum_{k=1}^{n-1} T_k \log(t + k - 1) \, dt.$$
Note that $T_1 = t^{n-1}$ and

$$
\int_0^1 T_k \log(t + k - 1) \, dt \\
= \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \int_0^1 (t + k - m - 1)^{n-1} \log(t + k - 1) \, dt.
$$

When $k > 1$,

$$
\int_0^1 (t + k - m - 1)^{n-1} \log(t + k - 1) \, dt \\
= \frac{(k-m)^n \log k - (k-m-1)^n \log(k-1)}{n} - \int_0^1 \frac{(t + k - m - 1)^n}{n(t + k - 1)} \, dt \\
= \frac{(k-m)^n - (-m)^n}{n} \log k - \frac{(k-m-1)^n - (-m)^n}{n} \\
- \frac{1}{n} \sum_{r=1}^{n} \frac{k^r - (k-1)^r}{r} \left(\binom{n}{r}\right)(-m)^{n-r}.
$$

Let $S_1(1) = 0$,

$$
S_1(k) = \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \left(\frac{(k-m)^n - (-m)^n}{n} \log k - \frac{(k-m-1)^n - (-m)^n}{n} \log(k-1)\right),
$$

and

$$
S_2(k) = \frac{1}{n} \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \sum_{r=1}^{n} \frac{k^r - (k-1)^r}{r} \left(\binom{n}{r}\right)(-m)^{n-r}.
$$

It follows that

$$
\int_0^1 \sum_{k=1}^{n-1} T_k \log(t + k - 1) \, dt = \sum_{k=1}^{n-1} S_1(k) - \sum_{k=1}^{n-1} S_2(k). \quad (5-5)
$$

The next lemma calculates $\sum_{k=1}^{n-1} S_1(k)$.

**Lemma 5.3.**

$$
\sum_{k=1}^{n-1} S_1(k) = \frac{1}{n} \sum_{k=2}^{n-2} \binom{n-1}{k} (-1)^k (k)^n \log k + \frac{\log(n-1)}{n} (n! (n-1) - (n-1)^n).
$$
Proof. Note that $S_1(1) = 0$.

\[
\sum_{k=1}^{n-1} S_1(k)
= \sum_{k=1}^{n-1} \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \left( \frac{(k-m)^n - (-m)^n}{n} \log k - \frac{(k-m-1)^n - (-m)^n}{n} \log(k-1) \right)
= \frac{1}{n} \sum_{k=2}^{n-1} \binom{n-1}{k} (-1)^k (-k)^n \log k
+ \frac{1}{n} \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m \left( (n-m-1)^n - (-m)^n \right) \log(n-1).
\]

Using the combinatorial identity $\sum_{k=0}^{n} \binom{n}{k} (-1)^k (x-k)^{n+1} = (x-n/2)(n+1)!$, we have

\[
\sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m (n-1-m)^n = \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m (n-1-m)^n = \frac{n-1}{2} n!,
\]
and

\[
\sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m (-m)^n
= \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m (-m)^n - (-1)^{n-1} (1-n)^n = (n-1)^n - \frac{n-1}{2} n!.
\]

Hence,

\[
\sum_{k=1}^{n-1} S_1(k) = \frac{1}{n} \sum_{k=2}^{n-1} \binom{n-1}{k} (-1)^k (-k)^n \log k + \frac{\log(n-1)}{n} \left( n! (n-1) - (n-1)^n \right).
\]

The following lemma calculates $\sum_{k=1}^{n-1} S_2(k)$. Here we only give the result. For reader’s convenience, the proof of it is given in the Appendix.

Lemma 5.4.

\[
\sum_{k=1}^{n-1} S_2(k) = (n-1)! (n-1) - \frac{n-1}{2} H_n (n-1)!,
\]

where $H_n = \sum_{k=1}^{n} 1/k$.

Using Lemma 5.3 and Lemma 5.4, we can prove Theorem 5.1 below.
Proof of Theorem 5.1. Let \( H_n = \sum_{k=1}^{n} 1/k \). By identity (5-5), Lemma 5.3 and Lemma 5.4, we have that

\[
\int_0^1 \sum_{k=1}^{n-1} T_k \log(t + k - 1) \, dt = \sum_{k=1}^{n-1} S_1(k) - \sum_{k=1}^{n-1} S_2(k) = \frac{1}{n} \sum_{k=2}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \log k + \frac{\log(n-1)}{n} \left( n! (n-1) - (n-1)^n \right) - (n-1)! (n-1) + \frac{n-1}{2} H_n(n-1)!. \tag{5-6}
\]

By identities (5-4) and (5-6), it follows that

\[
I = \frac{1}{2} \log(2\pi) + (n-1) \log(n-1) - n + 1 - \frac{1}{(n-1)!} \int_0^1 \sum_{k=1}^{n-1} T_k \log(t+k-1) \, dt = \frac{1}{2} \log(2\pi) - \frac{n-1}{2} H_n + \frac{1}{n!} \sum_{k=2}^{n-1} \binom{n-1}{k} (-1)^{k+n+1} k^n \log k. \tag*{□}
\]

When \( n = 2, 3 \) and 4, the values of the integral \( I \) are

\[
I(2) = -\frac{3}{4} + \frac{1}{2} \log(2\pi), \quad I(3) = \frac{1}{2} \log(2\pi) + \frac{4}{3} \log 2 - \frac{11}{6}, \quad I(4) = \frac{1}{2} \log(2\pi) - 2 \log 2 + \frac{27}{8} \log 3 - \frac{25}{8}.
\]

Appendix.

For reader’s convenience, the proof of Lemma 5.4 is given here.

Lemma 5.4.

\[
\sum_{k=1}^{n-1} S_2(k) = (n-1)! (n-1) - \frac{n-1}{2} H_n(n-1)!,
\]

where \( H_n = \sum_{k=1}^{n} 1/k \).
Proof. Note that
\[
\sum_{k=1}^{n-1} S_2(k) = \frac{1}{n} \sum_{k=1}^{n-1} \left( \sum_{m=0}^{k-1} \binom{n-1}{m} (-1)^m \sum_{r=1}^{n} \frac{k^r - (k-1)^r}{r} \binom{n}{r} (-m)^{n-r} \right)
\]

\[
= -\frac{1}{n} \sum_{k=1}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r} + \frac{1}{n} \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m \sum_{r=1}^{n} \frac{(-m)^{n-r} (n-1)^r}{r} \binom{n}{r}.
\]

Let
\[
R_1 = -\frac{1}{n} \sum_{k=1}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \sum_{r=1}^{n} \frac{(-1)^r}{r} \binom{n}{r} \tag{A-1}
\]

and
\[
R_2 = \frac{1}{n} \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m \sum_{r=1}^{n} \frac{(-m)^{n-r} (n-1)^r}{r} \binom{n}{r}. \tag{A-2}
\]

Then
\[
\sum_{k=1}^{n-1} S_2(k) = R_1 + R_2. \tag{A-3}
\]

By applying the combinatorial identities
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k (x-k)^{n+1} = \left( x - \frac{n}{2} \right) (n+1)! \quad \text{and} \quad \sum_{k=1}^{n} \frac{(-1)^k}{k} \binom{n}{k} = H_n,
\]

the sum $R_1$ can be simplified to
\[
R_1 = \frac{1}{n} \sum_{k=1}^{n-2} \binom{n-1}{k} (-1)^k (-k)^n \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \binom{n}{r}
\]

\[
= \frac{H_n}{n} \left( (n-1)^n - \frac{n-1}{2} n! \right). \tag{A-4}
\]

To simplify $R_2$, we apply the combinatorial identity
\[
\sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} \binom{n}{k} (1 - (1-x)^k) = \sum_{k=1}^{n} \frac{x^k}{k},
\]
and it follows that

\[
\sum_{r=1}^{n} \frac{(-m)^{n-r}(n-1)^{r}}{r} \binom{n}{r} (n-r)^{r} = (-m)^{n} \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \binom{n}{r} \left( 1 - \frac{m-n+1}{m} \right)^{r} = (-m)^{n} \left( \sum_{r=1}^{n} \frac{1}{r} \left( \frac{m-n+1}{m} \right)^{r} - \sum_{r=1}^{n} \frac{(-1)^{r+1}}{r} \binom{n}{r} \right) = \sum_{r=1}^{n} \frac{1}{r} \left( m-n+1 \right)^{r} m^{n-r} \left( -1 \right)^{n} \left( -m \right)^{n} H_{n}.
\]

Recalling the formula of \( R_{2} \) in (A-2), we have

\[
nR_{2} = \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^{m+n} \sum_{k=1}^{n} \frac{1}{k} \left( m-n+1 \right)^{k} m^{n-k} - H_{n} \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^{m} (-m)^{n} - \frac{n-1}{2} n!
\]

(A-5)

By the combinatorial identity \( \sum_{k=0}^{n} \binom{n}{k} (-1)^{k} (x-k)^{n+1} = (x-n/2)(n+1)! \), we see that

\[
\sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^{m} (-m)^{n} = (n-1)^{n} - \frac{n-1}{2} n!
\]

(A-6)

We then simplify \( \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^{m+n} \sum_{k=1}^{n} \frac{1}{k} \left( m-n+1 \right)^{k} m^{n-k} \). Note that

\[
\sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^{m+n} \sum_{k=1}^{n} \frac{1}{k} \left( m-n+1 \right)^{k} m^{n-k} = \sum_{k=1}^{n} \frac{(-1)^{n}}{k} \left( \sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^{m} \sum_{i=0}^{k} \binom{k}{i} m^{n-k+i} (n-1)^{k-i} (-1)^{k-i} \right).
\]

(A-7)

Let

\[
P(m) = \sum_{i=0}^{k} \binom{k}{i} m^{n-k+i} (n-1)^{k-i} (-1)^{k-i}.
\]
We apply the combinatorial identity \( \sum_{k=0}^{n} (-1)^k (\binom{n}{k}) P(k) = 0 \) for any polynomial \( P(k) \) with \( \deg P(k) < n \), and it follows that

\[
\sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^m \sum_{i=0}^{k} \binom{k}{i} m^{n-k+i} (n-1)^{k-i} (-1)^{k-i} = \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m P(m) = \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m P(m) - (-1)^{n-1} P(n-1) = \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m (-k(n-1)m^{n-1} + m^n). \quad (A-8)
\]

By the combinatorial identity \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x + n - k)^n = n! \), we have

\[
-k(n-1) \sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m m^{n-1} = k(n-1)(-1)^n(n-1)!. \]

By the combinatorial identity \( \sum_{k=0}^{n} \binom{n}{k} (-1)^k (x-k)^{n+1} = (x-n/2)(n+1)! \), we see that

\[
\sum_{m=0}^{n-1} \binom{n-1}{m} (-1)^m m^n = (-1)^{n-1} \frac{n-1}{2} n!. \]

Then equality (A-7) becomes

\[
\sum_{m=0}^{n-2} \binom{n-1}{m} (-1)^{m+n} \sum_{k=1}^{n} \frac{1}{k} (m-n+1)^k m^{n-k} = \sum_{k=1}^{n} \frac{(-1)^n}{k} \left( k(n-1)(-1)^n(n-1)! + (-1)^{n-1} \frac{n-1}{2} n! \right) = n!(n-1) - \frac{n-1}{2} n! H_n, \quad (A-9)
\]

where \( H_n = \sum_{k=1}^{n} 1/k \).

Hence, by equalities (A-9) and (A-6), \( nR_2 \) in (A-5) can be simplified to

\[
nR_2 = n!(n-1) - (n-1)^n H_n. \quad (A-10)
\]

That is,

\[
R_2 = (n-1)! (n-1) - \frac{H_n}{n} (n-1)^n. \quad (A-11)
\]
Therefore, by equalities (A-3), (A-4) and (A-11), it follows that
\[
\sum_{k=1}^{n-1} S_2(k) = R_1 + R_2
\]
\[
= \frac{H_n}{n} \left( (n-1)^n - \frac{n-1}{2} n! \right) + (n-1)!(n-1) - \frac{H_n}{n} (n-1)^n
\]
\[
= (n-1)!(n-1) - \frac{n-1}{2} H_n (n-1)!,
\]
where \( H_n = \sum_{i=1}^{n} 1/i \).  

\[\square\]

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