ON THE SOBOLEV-POINCARÉ INEQUALITY OF CR-MANIFOLDS

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Abstract. The purpose is to study the CR-manifold with a contact structure conformal to the Heisenberg group. In our previous work [WY17], we have proved that if the $Q'$-curvature is nonnegative, and the integral of $Q'$-curvature is below the dimensional bound $c'_1$, then we have the isoperimetric inequality. In this paper, we manage to drop the condition on the nonnegativity of the $Q'$-curvature. We prove that the volume form $e^{4u}$ is a strong $A_\infty$ weight. As a corollary, we prove the Sobolev-Poincaré inequality on a class of CR-manifolds with integrable $Q'$-curvature.

1. Introduction

On a four dimensional manifold, the Paneitz operator $P_g$ and the Branson’s $Q$-curvature [Bra95] have many analogous properties as the Laplacian operator $\Delta_g$ and the Gaussian curvature $K_g$ on surfaces. The Paneitz operator is defined as

$$P_g = \Delta^2 + \delta\left(\frac{2}{3}R_g - 2Ric\right)d,$$

where $\delta$ is the divergence, $d$ is the differential, $R$ is the scalar curvature of $g$, and $Ric$ is the Ricci curvature tensor. The $Q$-curvature is defined as

$$Q_g = \frac{1}{12}\left\{-\Delta R + \frac{1}{4}R^2 - 3|E|^2, \right\}$$

where $E$ is the traceless part of $Ric$, and $| \cdot |$ is taken with respect to the metric $g$. The most important two properties for the pair $(P_g, Q_g)$ are that under the conformal change $g_w = e^{2w}g_0$,

1. $P_g$ transforms by $P_{g_w}(\cdot) = e^{-4w}P_{g_0}(\cdot)$;
2. $Q_g$ satisfies the fourth order equation

$$P_{g_0}w + 2Q_{g_0} = 2Q_{g_w}e^{4w}.$$

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As proved by Beckner [Bec93] and Chang-Yang [CY13], the pair \((P, Q)\) also appears in the Moser-Trudinger inequality for higher order operators.

On CR-manifold, it is a fundamental problem to study the existence and analogous properties of CR invariant operator \(P\) and curvature scalar invariant \(Q\). Graham and Lee [GL88] has studied a fourth-order CR covariant operator with leading term \(\Delta^2 + T^2\) and Hirachi [Hir93] has identified the \(Q\)-curvature which is related to \(P\) through a change of contact form. However, although the integral of the \(Q\)-curvature on a compact three-dimensional CR-manifold is a CR invariant, it is always equal to zero. And in many interesting cases when the CR three manifold is the boundary of a strictly pseudoconvex domains, the \(Q\)-curvature vanishes everywhere. As a consequence, it is desirable to search for some other invariant operators and curvature invariants on a CR-manifold that are more sensitive in the CR geometry. The work of Branson, Fontana and Morpurgo [BFM13] aims to find such a pair \((P', Q')\) on a CR sphere. Later, the definition of \(Q'\)-curvature is generalized to all pseudo-Einstein CR-manifolds by the work of Case-Yang [CY13] and that of Hirachi [Hir14]. The construction uses the strategy of analytic continuation in dimension by Branson [Bra95], restricted to the subspace of the CR pluriharmonic functions.

\[
P'_4 := \lim_{n \to 1} \frac{2}{n - 1} P_{4,n}|_P.
\]

Here \(P_{4,n}\) is the fourth-order covariant operator that exists for every contact form \(\theta\) by the work of Gover and Graham [GG05]. By [GL88], the space of CR pluriharmonic functions \(P\) is always contained in the kernel of \(P_4\).

In this paper, we want to explore the geometric meaning of this newly introduced conformal invariant \(Q'\)-curvature.

In Riemannian geometry, a classical isoperimetric inequality on a complete simply connected surface \(M^2\), called Fiala-Huber’s [Fia41], [Hub57] isoperimetric inequality

\[
Vol(\Omega) \leq \frac{1}{2(2\pi - \int_{M^2} K_g^+ dv_g)} Area(\partial \Omega)^2,
\]

where \(K_g^+\) is the positive part of the Gaussian curvature \(K_g\). Also \(\int_{M^2} K_g^+ dv_g < 2\pi\) is the sharp bound for the isoperimetric inequality to hold.

In [Wan15], we generalize the Fiala-Huber’s isoperimetric inequality to all even dimensions, replacing the role of the Gaussian curvature in dimension two by that of the \(Q\)-curvature in higher dimensions:
Let $(M^n, g) = (\mathbb{R}^n, g = e^{2u}|dx|^2)$ be a complete noncompact even dimensional manifold. Let $Q^+$ and $Q^-$ denote the positive and negative part of $Q_g$ respectively; and $dv_g$ denote the volume form of $M$. Suppose $g = e^{2u}|dx|^2$ is a “normal” metric, i.e.

\[
u(x) = \frac{1}{c_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y)dv_g(y) + C;
\]

for some constant $C$. If

\[\alpha := \int_{M^n} Q^+ dv_g < c_n\]

where $c_n = 2^{n-2}(\frac{n-2}{2})! \pi^{\frac{n}{2}}$, and

\[\beta := \int_{M^n} Q^- dv_g < \infty,
\]

then $(M^n, g)$ satisfies the isoperimetric inequality with isoperimetric constant depending only on $n, \alpha$ and $\beta$. Namely, for any bounded domain $\Omega \subset M^n$ with smooth boundary,

\[|\Omega|_g^{\frac{n}{n-1}} \leq C(n, \alpha, \beta)|\partial \Omega|_g.
\]

In our previous paper [WY17], we have studied the $Q'$-curvature and $P'$ operator, and proved that if $(\mathbb{H}^1, e^u \theta)$ for pluriharmonic function $u$ is a complete CR-manifold with nonnegative $Q'$ curvature and nonnegative Webster scalar curvature at infinity, if in addition $Q'$ curvature satisfies

\[\int_{\mathbb{H}^1} Q' e^{4u} \theta \wedge d\theta < c_1',
\]

then $e^{4u}$ is an $A_1$ weight. Here $c_1'$ is the constant in the fundamental solution of $P'$ operator (See [WY17]). As a corollary, we have derived the isoperimetric inequality on CR-manifold $(\mathbb{H}^1, e^u \theta)$:

\[Vol(\Omega) \leq C Area(\partial \Omega)^{4/3}.
\]

Here the constant $C$ is controlled by $c_1' - \int_{\mathbb{H}^1} Q' e^{4u} \theta \wedge d\theta$. To prove this result, we notice that the class of pluriharmonic functions $P$ is the relevant subspace of functions for the conformal factor $u$.

The purpose of the current paper is two-fold. We will first study the case when $Q'$ curvature is negative. Then we will discuss the general case when $Q'$ curvature does not have a sign. The main results of the paper are stated in the following.
Theorem 1.1. Let \((\mathbb{H}^1, e^u\theta)\) be a complete CR-manifold, where \(\theta\) denotes the contact form on the Heisenberg group \(\mathbb{H}^1\) and \(u\) is a pluriharmonic function on \(\mathbb{H}^1\). If the \(Q'\)-curvature is negative, and the Webster scalar curvature is nonnegative at infinity. If

\[ (1.8) \quad \int_{\mathbb{H}^1} Q' e^{4u} \theta \wedge d\theta < \infty, \]

then \(e^{4u}\) is a strong \(A_{\infty}\) weight.

Note that \(e^{4u}\) is the volume form of this conformal metric, where 4 is the homogeneous dimension of \(\mathbb{H}^1\). The descriptions of \(A_1\) weight and strong \(A_{\infty}\) weight will be in Section 2.

We will then discuss the case when the \(Q'\)-curvature does not have a sign.

Theorem 1.2. Let \((\mathbb{H}^1, e^u\theta)\) be a complete CR-manifold, where \(\theta\) denotes the contact form on the Heisenberg group \(\mathbb{H}^1\) and \(u\) is a pluriharmonic function on \(\mathbb{H}^1\). If the Webster scalar curvature is nonnegative at infinity, If

\[ (1.9) \quad \alpha := \int_{\mathbb{H}^1} Q'^+ e^{4u} \theta \wedge d\theta < c'_1, \]

and

\[ (1.10) \quad \beta := \int_{\mathbb{H}^1} Q'^- e^{4u} \theta \wedge d\theta < \infty, \]

then \(e^{4u}\) is a strong \(A_{\infty}\) weight.

As a corollary of Franchi-Lu-Wheeden \([FLW95]\), we will show that \((\mathbb{H}^1, e^u\theta)\) satisfies Sobolev-Poincaré inequality. We remark that on a CR-manifold \((\mathbb{H}^1, e^u\theta)\), the David-Semme’s \([DS90]\) type of isoperimetric inequality is still an open question for strong \(A_{\infty}\) weights.

Theorem 1.3. Let \((\mathbb{H}^1, e^u\theta)\) satisfy the same assumptions as in Theorem 1.2. Let \(K\) be a compact subset of \(\Omega\). Then there exists \(r_0\) depending on \(K,\Omega,\) and \(\{X_j\}\) such that if \(B = B(x, r)\) is a ball with \(x \in K\) and \(0 < r < r_0\), and if \(e^{4u}\) is \(A_p\) weight for some \(1 \leq p < 4\). Let \(\mu(x) := e^{4u} dx, \nu(x) := e^{(4-p)u} dx\). Then

\[ (1.11) \quad \left( \frac{1}{\mu(B)} \int_B |f(x) - f_B|^q d\mu \right)^{1/q} \leq c r \left( \frac{1}{\nu(B)} \int_B |\nabla_b f(x)|^p d\nu \right)^{1/p}, \]

for any \(f \in \text{Lip}(\bar{B})\), with \(f_B = \frac{1}{\mu(B)} \int_B f(x) d\mu\). The constant \(c\) depends only on \(K, \Omega, \alpha, \beta, p\).
2. Preliminaries

On a Heisenberg group $\mathbb{H}^n$, one can also define the $A_p$ weight, in the same way as on the Euclidean space $\mathbb{R}^n$. For a nonnegative local integrable function $\omega$, we call it an $A_p$ weight $p > 1$, if for all balls $B$ in $\mathbb{H}^n$

$$\frac{1}{|B|} \int_B \omega(x) dx \cdot \left( \frac{1}{|B|} \int_B \omega(x)^{\frac{p'}{p}} dx \right)^{\frac{p}{p'}} \leq C < \infty.$$  

Here $\frac{1}{p'} + \frac{1}{p} = 1$. The constant $C$ is uniform for all $B$. The definition of $A_1$ weight is given by taking the limit process $p \to 1$. Namely, $\omega$ is called an $A_1$ weight, if

$$M\omega(x) \leq C\omega(x),$$

for almost all $x \in B$.

An important property of $A_p$ weight is the reverse Hölder inequality: if $\omega$ is an $A_p$ weight for some $p \geq 1$, then there exist an $r > 1$ and a $C > 0$ such that for all balls $B$

$$\left( \frac{1}{|B|} \int_B \omega^r dx \right)^{1/r} \leq \frac{C}{|B|} \int_B \omega dx.$$  

This would imply that any $A_p$ weight $\omega$ satisfies the doubling property: there exists a $C > 0$ s.t.

$$\int_{B(x_0,2r)} \omega(x) dx \leq C \int_{B(x_0,r)} \omega(x) dx$$

for all balls $B(x_0,r)$.

The notion of strong $A_\infty$ weight was first proposed by David and Semmes in [DS90]. Given a positive continuous weight $\omega$, we define

$$\delta_\omega(x,y) := \left( \int_{B_{x,y}} \omega(z) dz \right)^{1/n},$$

where $B_{x,y}$ is the ball with diameter $|x - y|$ that contains $x$ and $y$. On the other hand, we can define the geodesic distance with respect to the weight $\omega$ to be

$$d_\omega(x,y) := \inf_\gamma \int_\gamma \omega^{\frac{2}{n}}(s) ds.$$

Here $\gamma \subset B_{x,y}$ is a curve connecting $x, y$ such that the tangent vector is always contact. If $\omega$ is an $A_\infty$ weight, then it is easy to prove (see for example Proposition 3.12 in [Sem93])

$$d_\omega(x,y) \leq C\delta_\omega(x,y).$$
for all \(x, y \in \mathbb{H}^n\). If in addition, \(\omega\) also satisfies the reverse inequality (2.8)
\[
\delta_{\omega}(x, y) \leq Cd_{\omega}(x, y)
\]
then we say \(\omega\) is a strong \(A_\infty\) weight.

The product of an \(A_1\) weight and an \(A_\infty\) weight is an \(A_\infty\) weight. This can be proved using the same proof as in the Euclidean space.

3. CR-manifold with negative \(Q'\)-curvature

In this section, we will prove Theorem 1.1. It shows that for CR-manifolds with negative \(Q'\)-curvature, the integral of \(Q'\)-curvature controls the geometry in a very rigid way.

We first remark that since \(Q'(y)e^{4u(y)}\) is integrable, \(\log \frac{|y|}{|x-y|}Q'(y)e^{4u(y)}\) is also integrable in \(y\) for each fixed \(x \in \mathbb{H}^1\).

In this section, we consider the analytic property of \(e^{4u(x)}\). For simplicity, we denote it by \(\omega_2(x)\). We define \(\beta := \int_{\mathbb{H}^1} |Q'(y)e^{4u(y)}|dy < \infty\).

Recall that for a nonnegative continuous function \(\omega(x)\),
\[
d_{\omega}(x, y) := \left( \int_{B_{xy}} \omega(z)dz \right)^{\frac{1}{n}},
\]
\[
\delta_{\omega}(x, y) := \inf_{\gamma} \int_{\gamma} \omega^{\frac{1}{n}}(\gamma(s))ds,
\]
where \(B_{xy}\) is the ball with diameter \(|x - y|\) that contains \(x\) and \(y\), the infimum is taken over all contact curves (meaning that the tangent vector on each point of this curve is contact) \(\gamma \subset B_{xy}\) connecting \(x\) and \(y\), and \(ds\) is the arc length.

We want to prove \(\omega_2(x) := e^{4u(x)}\) is a strong \(A_\infty\) weight, i.e. there exists a constant \(C = C(\beta)\) such that
\[
\frac{1}{C(\beta)}d_{\omega_2}(x, y) \leq \delta_{\omega_2}(x, y) \leq C(\beta)d_{\omega_2}(x, y).
\]

Since the Webster scalar curvature is nonnegative at infinity, by Proposition in [WY17], \(u\) is normal. Thus
\[
u_{\lambda}(x) := u(\lambda x) = \frac{-1}{c'_1} \int_{\mathbb{H}^1} \log \frac{|y|}{|x-y|}Q'(y)e^{4u(y)}dy.
\]

We first observe that without generality we can assume \(|x - y| = 2\). This is because we can dilate \(u\) by a factor \(\lambda > 0\),
\[
u_{\lambda}(x) := u(\lambda x) = \frac{-1}{c'_1} \int_{\mathbb{H}^1} \log \frac{|y|}{|\lambda x-y|}Q'(y)e^{4u(y)}dy.
\]
By the change of variable, this is equal to
\[ -\frac{1}{c_1'} \int_{\mathbb{H}^1} \log \frac{|y|}{|x-y|} |Q'(\lambda y)| e^{4u(y)} \lambda^4 dy. \]

Notice $|Q'(\lambda y)| e^{4u(y)} \lambda^4$ is still an integrable function on $\mathbb{H}^1$, with integral equal to $\beta$. Thus by choosing $\lambda = \frac{2}{|x-y|}$, the problem reduces to proving inequality (3.1) for $u^\lambda$ and $|x-y| = 2$.

Let us denote the midpoint of $x$ and $y$ by $p_0$. And from now on, we adopt the notation $\lambda B := B(p_0, \lambda)$. Since $|x-y| = 2$, we have $B_{xy} = B(p_0, 1) = B$. We also define
\[ u_1(x) := -\frac{1}{c_1'} \int_{10B} \log \frac{|y|}{|x-y|} |Q'(y)| e^{4u(y)} dy, \]
and
\[ u_2(x) := -\frac{1}{c_1'} \int_{H^1 \setminus 10B} \log \frac{|y|}{|x-y|} |Q'(y)| e^{4u(y)} dy. \]

In the following lemma, we prove that when $z$ is close to $p_0$, the difference between $u_2(z)$ and $u_2(p_0)$ is controlled by $\beta$.

**Lemma 3.1.**
\[ |u_2(z) - u_2(p_0)| \leq \frac{\beta}{4c_1'} \]
for $z \in 2B$.

**Proof.**
\[ |u_2(z) - u_2(p_0)| = \frac{1}{c_1'} \left| \int_{\mathbb{H}^1 \setminus 10B} \log \frac{|y|}{|z-y|} |Q'(y)| e^{4u(y)} dy + \int_{\mathbb{H}^1 \setminus 10B} \log \frac{|y|}{|p_0-y|} |Q'(y)| e^{4u(y)} dy \right| \]
\[ \leq \frac{1}{c_1'} \int_{\mathbb{H}^1 \setminus 10B} \frac{1}{|1-t^*(p_0-y) + t^*(z-y)|} |Q'(y)| e^{4u(y)} dy, \]
for some $t^* \in [0, 1]$. Since $y \in \mathbb{H}^1 \setminus 10B$ and $z, p_0 \in 2B$,
\[ \frac{1}{|1-t^*(p_0-y) + t^*(z-y)|} \leq \frac{1}{8}, \]
$|u_2(z) - u_2(p_0)|$ is bounded by
\[ |z - p_0| \cdot \frac{1}{8c_1'} \int_{\mathbb{H}^1 \setminus 10B} |Q'(y)| e^{4u(y)} dy. \]
Note that for $z \in 2B$, $|z - p_0| \leq 2$. From this, (3.6) follows. □

Now we adopt some techniques used in [BHS04] for potentials to deal with the $\epsilon$-singular set $E_\epsilon$.

**Lemma 3.2. (Cartan’s lemma)** For the Radon measure $|Q'|(y)e^{4u(y)}dy$, given $\epsilon > 0$, there exists a set $E_\epsilon \subseteq \mathbb{H}^1$, such that

$$\mathcal{H}^1(E_\epsilon) := \inf_{E_\epsilon \subseteq \bigcup B_i} \left\{ \sum_i \text{diam } B_i \right\} < 10\epsilon$$

and for all $x \notin E_\epsilon$ and $r > 0$,

$$\int_{B(x,r)} |Q'|(y)e^{4u(y)}dy \leq \frac{r\beta}{\epsilon}.$$  

The proof of Lemma 1 follows from standard measure theory argument. Thus we omit it here.

**Proposition 3.3.** Given $\epsilon > 0$,

$$\mathcal{H}^1 \left( \left\{ x \in 10B : \left| -\frac{1}{c_1} \int_{10B} \log \frac{1}{|x-y|} |Q'(y)e^{4u(y)}dy| \right| > \frac{C_0\beta}{\epsilon} \right\} \right) < 10\epsilon.$$  

*Proof.* Fix $\epsilon > 0$. By Lemma 3.2, there exists a set $E_\epsilon \subseteq \mathbb{H}^1$, s.t. $\mathcal{H}^1(E_\epsilon) < 10\epsilon$ and for $x \notin E_\epsilon$ and $r > 0$

$$\int_{B(x,r)} |Q'|(y)e^{4u(y)}dy \leq \frac{r\beta}{\epsilon}.$$  

If we can show for some $C_0$

$$10B \setminus E_\epsilon \subseteq \left\{ x \in 10B : \left| -\frac{1}{c_1} \int_{10B} \log \frac{1}{|x-y|} |Q'(y)e^{4u(y)}dy| \right| \leq \frac{C_0\beta}{\epsilon} \right\},$$  

then

$$\mathcal{H}^1 \left( \left\{ x \in 10B : \left| -\frac{1}{c_1} \int_{10B} \log \frac{1}{|x-y|} |Q'(y)e^{4u(y)}dy| \right| > \frac{C_0\beta}{\epsilon} \right\} \right) \leq \mathcal{H}^1(E_\epsilon) < 10\epsilon.$$
To prove (3.10), we notice for \( x \in 10B \setminus E, r = 2^{-j} \cdot 10 \), (3.9) implies
\[
\left| \frac{-1}{c_1} \int_{10B} \log \frac{1}{|x-y|} |Q'(y)e^{4u(y)}dy| \right| \\
\leq \frac{1}{c_1} \sum_{j=-1}^{\infty} \left| \int_{B(x,2^{-j} \cdot 10) \setminus B(x,2^{-(j+1)} \cdot 10)} \log \frac{1}{|x-y|} |Q'(y)e^{4u(y)}dy| \right| \\
\leq \frac{1}{c_1} \sum_{j=-1}^{\infty} \left( \max\{ |\log 2^{-j}|, |\log 2^{-(j+1)}| \} + \log 10 \right) \\
\int_{B(x,2^{-j} \cdot 10) \setminus B(x,2^{-(j+1)} \cdot 10)} |Q'(y)e^{4u(y)}dy| \\
\leq \frac{1}{c_1} \sum_{j=-1}^{\infty} \left( \max\{ |\log 2^{-j}|, |\log 2^{-(j+1)}| \} + \log 10 \right) \cdot \frac{2^{-j} \cdot 10^\beta}{\epsilon} \\
\leq C_0 \frac{\beta}{\epsilon}
\]
By Lemma 3.1
\[
\int_{2B} e^{4u(z)} dz = \int_{2B} e^{4u_1(z)} e^{4u_2(z)} dz \\
\leq e^{\frac{\beta}{c_1}} e^{4u_2(p_0)} \int_{2B} e^{4u_1(z)} dz.
\] (3.15)

To estimate \( u_1 \), by definition \( \beta_{10} := \int_{10B} |Q'(y)e^{4u(y)}|dy \leq \beta < \infty \). If \( \beta_{10} = 0 \), then \( u_1(z) = 0 \) and \( \bar{c} := \frac{1}{c_1} \int_{10B} \log |y||Q'(y)e^{4u(y)}|dy = 0 \). So (3.12) follows immediately. If \( \beta_{10} \neq 0 \), \( |Q'(y)e^{4u(y)}|_{\beta_{10}} \) is a nonnegative probability measure on \( 10B \). Hence by Jensen’s inequality
\[
\int_{2B} e^{4u_1(z)} dz \leq e^{\frac{\beta_{10}}{c_1}} \int_{10B} |Q'(y)e^{4u(y)}|_{\beta_{10}} dy d\omega_2(z).
\] (3.16)

Since \( z \in 2B \) and \( y \in 10B \),
\[
\int_{2B} |z - y|^{\frac{4\beta_{10}}{c_1}} dz \leq C.
\] (3.17)

From this, we get
\[
\int_{2B} e^{4u_1(z)} dz \leq Ce^{4\bar{c}} \int_{10B} |Q'(y)e^{4u(y)}|_{\beta_{10}} dy = Ce^{4\bar{c}}.
\] (3.18)

Plugging it to (3.15), we finish the proof of the proposition. \( \square \)

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let us assume \( \omega_2 := e^{4u} \) is an \( A_p \) weight for some large \( p \), with bounds depending only on \( \beta \). The proof of this fact follows that of Proposition 5.1 in [Wan15]. So we omit it here. By the reverse Hölder’s inequality for \( A_p \) weights, it is easy to prove (see for example Proposition 3.12 in [Sem93]),
\[
\delta_{\omega_2}(x, y) \leq C_2(\beta) d_{\omega_2}(x, y).
\]

Hence we only need to prove the other side of the inequality:
\[
\delta_{\omega_2}(x, y) \geq C_3(\beta) d_{\omega_2}(x, y),
\] (3.19)

for some constant \( C_3(\beta) \). By Proposition 3.3, for a given \( \epsilon > 0 \), there exists a Borel set \( E_\epsilon \subseteq \mathbb{R}^1 \), such that
\[
\mathcal{H}^1(E_\epsilon) \leq 10\epsilon,
\] (3.20)

and for \( z \in 10B \setminus E_\epsilon \), according to (3.10)
\[
|\hat{u}_1(z)| \leq \frac{C_0}{\epsilon} \beta.
\] (3.21)
Here
\[ \hat{u}_1(z) := \frac{-1}{c'_1} \int_{10B} \log \frac{1}{|x-y|} |Q'(y)| e^{4u(y)} dy. \]

With this, we claim the following estimate.

**Claim:** Suppose \( H^1(E_\epsilon) < 10 \epsilon \) with \( \epsilon \leq \frac{1}{20} \). Then the length of \( \gamma \setminus E_\epsilon \) with respect to the metric of Heisenberg group \( \mathbb{H}^1 \) satisfies

\[ (3.22) \quad \text{length } (\gamma \setminus E_\epsilon) > \frac{3}{2}, \]

where \( \gamma \subset B_{xy} \) is a curve connecting \( x \) and \( y \).

**Proof of Claim.** Let \( P \) be the projection map from points in \( B_{xy} \) to the contact line segment \( I_{xy} \) between \( x \) and \( y \). Since the Jacobian of the projection map is less or equal to 1,

\[ (3.23) \quad \text{length } (\gamma \setminus E_\epsilon) \geq \text{length } (P(\gamma \setminus E_\epsilon)) = m(P(\gamma \setminus E_\epsilon)), \]

where \( m \) is the arc length measure on the line segment \( I_{xy} \). Notice \( P(\gamma) = I_{xy} \), and \( P(\gamma) \setminus P(E_\epsilon) \) is a subset of \( P(\gamma \setminus E_\epsilon) \). Therefore

\[ (3.24) \quad m(P(\gamma \setminus E_\epsilon)) \geq m(P(\gamma)) - m(P(E_\epsilon)) = 2 - m(P(E_\epsilon)). \]

Now by assumption, \( H^1(E_\epsilon) < 10 \epsilon \), so \( H^1(\gamma \cap E_\epsilon) < 10 \epsilon \). Hence there is a covering \( \cup_i B_i \) of \( \gamma \cap E_\epsilon \), so that

\[ \sum_i \text{diam } B_i < 10 \epsilon. \]

This implies that \( \cup_i P(B_i) \) is a covering of the set \( P(\gamma \cap E_\epsilon) \) and

\[ \sum_i \text{diam } P(B_i) = \sum_i \text{diam } B_i \leq 10 \epsilon. \]

Thus \( m(P(E_\epsilon)) = H^1(P(E_\epsilon)) < 10 \epsilon < \frac{1}{2} \), by choosing \( \epsilon \leq \frac{1}{20} \). Plug it to (3.24), and then to (3.23). This completes the proof of the claim.

We now continue the proof of Theorem 1.1. Since \( \gamma \subset B \), then by Lemma 3.1

\[ (3.25) \quad \int_\gamma e^{u_-(\gamma(s))} ds = \int_\gamma e^{(u_1+u_2)(\gamma(s))} ds \geq \tilde{c} \int_\gamma e^{\hat{u}_1(\gamma(s))} ds. \]

Here \( \tilde{c} \) is the constant defined in Proposition 3.4. Let \( \epsilon = \frac{1}{20} \). By (3.21),

\[ |\hat{u}_1(z)| \leq 20C_0 \beta \]

for \( z \in 10B \setminus E_\epsilon \). Thus

\[ (3.26) \quad \int_\gamma e^{\hat{u}_1(\gamma(s))} ds \geq e^{-20C_0 \beta} \text{length } (\gamma \setminus E_\epsilon). \]
By (3.22), it is bigger than
\[ \frac{3}{2} e^{-20C_0 \beta}. \]
Therefore
\[ (3.27) \quad \int_{\gamma} e^{u-(\gamma(s))} ds \geq \frac{3}{2} e^{-20C_0 \beta} e^{u_2(p_0)} e^{\tilde{c}} = C_4(\beta) e^{u_2(p_0)} e^{\tilde{c}} \]
for \( C_4(\beta) = \frac{3}{2} e^{-20C_0 \beta}. \) By inequality (3.27) and Proposition 3.4, we conclude for any curve \( \gamma \subset B_{xy} \) connecting \( x \) and \( y \), there is a \( C_3 = C_3(\beta) \) such that
\[ (3.28) \quad \int_{\gamma} e^{u-(\gamma(s))} ds \geq C_3(\beta) \left( \int_{B_{xy}} e^{4u-(z)} dz \right)^{\frac{1}{4}}. \]
This implies inequality (3.19) and thus completes the proof of Theorem 1.1.

4. \( Q' \)-CURVATURE WITHOUT A SIGN

In this section, we consider CR-manifold on which the \( Q' \)-curvature does not have a sign any more. Suppose \( (\mathbb{H}^1, e^{2u} \theta) \) satisfies that
\[ (4.1) \quad \alpha := \int_{\mathbb{H}^1} Q'^+ e^{4u} \theta \wedge d\theta < c_1', \]
\[ (4.2) \quad \beta := \int_{\mathbb{H}^1} Q'^- e^{4u} \theta \wedge d\theta < \infty. \]
Suppose also that the Webster scalar curvature is nonnegative at infinity.

By Theorem 1.1, \( e^{4u} \) is a strong \( A_\infty \) weight. By Theorem 1.4 in [WY17], \( e^{4u} \) is an \( A_1 \) weight.

**Proposition 4.1.** Assume \( \omega_1 \) is an \( A_1 \) weight, \( \omega_2 \) is a strong \( A_\infty \) weight. If \( \omega_1^r \omega_2 \) for some \( r \in \mathbb{R} \) is an \( A_\infty \) weight, then \( \omega_1^r \omega_2 \) a strong \( A_\infty \) weight.

**Remark 4.2.** The proposition for the Euclidean space has been proved in [Sem93]. We prove here the proposition for Heisenberg groups.

**Proof.** Let \( \delta_2(\cdot, \cdot) \) and \( \delta_{12}(\cdot, \cdot) \) be the quasidistance associated to \( \omega_2 \) and \( \omega_1^r \omega_2 \) respectively. Let \( x_1, ..., x_k \in \mathbb{H}^1 \) such that \( x_j \in B(x_1, 2|x_k - x_1|) \) for all \( j \). Notice that it suffices to prove
\[ (4.3) \quad \delta_{12}(x_1, x_k) \leq C \sum_{j=1}^{k-1} \delta_{12}(x_j, x_{j+1}). \]
Let $B = B_{x_1, x_k}$ and $B_j = B_{x_j, x_{j+1}}$. Since $x_j \in B(x_1, 2|x_k - x_1|)$ for all $j$, $B_j \subset 100B$ for all $j$. By definition $\delta_{12}$,

\begin{equation}
\delta_{12}(x_j, x_{j+1})
\end{equation}

By Proposition 4.1, in order to prove Theorem 1.2, we only need to show that $e^{4u}$ is an $A_\infty$ weight. In other words, we need to show $e^{4u}$ is an $A_p$ weight for some $p$.

**Proposition 4.3.** Suppose $(H^1, e^{2u} \theta)$ satisfies the same assumptions as in Theorem 1.2. Then $e^{4u}$ is an $A_p$ weight for some $p$. The $A_p$ bound depends only on the integral of $Q'$ curvature.

**Proof.**

\begin{equation}
u(x) = \frac{1}{c_1'} \int_{H^1} \log \frac{|y|}{|x-y|} Q'(y)e^{4u(y)} dy\end{equation}

with assumptions (1.3) and (1.4). By Theorem 1.4 in [WY17], $e^{4u}$ is an $A_1$ weight, so there is a uniform constant $C = C(\alpha)$, so that for all $x_0 \in H^1$ and $r > 0$

\begin{equation}
\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} e^{4u}(y) dy \leq C(\alpha) e^{4u}(x_0).
\end{equation}

So for all $x \in B(x_0, r)$

\begin{equation}
\frac{1}{|B(x_0, r)|} \int_{B(x_0, r)} e^{4u}(y) dy \leq \frac{1}{|B(x_0, r)|} \int_{B(x, 2r)} e^{4u}(y) dy
\end{equation}

\begin{equation}
= \frac{2^4}{|B(x, 2r)|} \int_{B(x, 2r)} e^{4u}(y) dy \leq C(\alpha) e^{4u}(x).
\end{equation}

Namely, for all ball $B$ in $H^1$ and $x \in B$,

\begin{equation}
\frac{1}{|B|} \int_{B} e^{4u}(y) dy \leq C(\alpha) e^{4u}(x).
\end{equation}

We observe that $e^{-4\varepsilon u_-(x)}$ is also an $A_1$ weight for $\varepsilon = \varepsilon(\beta) << 1$. In fact,

\begin{equation}
e^{-4\varepsilon u_-(x)} = e^{-4\varepsilon} \int_{H^1} \log \frac{|y|}{|x-y|} Q^-(y) e^{4u(y)} dy
\end{equation}

$Q^-(y) e^{4u(y)} \geq 0$ and $\int_{H^1} \varepsilon Q^-(y) e^{4u(y)} dy < c_1$ if $\varepsilon$ is small enough. Thus by Theorem 1.4 in [WY17], $e^{-4\varepsilon u_-(x)}$ is an $A_1$ weight. As (4.8), we have

\begin{equation}
\frac{1}{|B|} \int_{B} e^{-4\varepsilon u_-(y)} dy \leq C(\beta) e^{-4\varepsilon u_-(x)}
\end{equation}
for all ball $B$ in $\mathbb{H}^1$ and all $x \in B$. Choose $1 < p < \infty$ such that $\epsilon = p'/p$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Using $e^{4u} = e^{4u_+} \cdot e^{4u_-}$, we get

$$
\left( \int_B e^{4u(x)} \, dx \right)^{\frac{1}{p}} \left( \int_B (e^{4u(x)})^{-\frac{1}{p'}} \, dx \right)^{\frac{1}{p'}}
$$

(4.11)

$$
\left( \int_B e^{4u_+} \cdot (e^{-4u_-})^{-\frac{1}{p'}} \, dx \right)^{\frac{1}{p}} \left( \int_B (e^{4u_+})^{-\frac{1}{p'}} \cdot e^{-4u_-} \, dx \right)^{\frac{1}{p'}}.
$$

By (4.10), if $p$ is large enough and thus $\epsilon$ is small enough, then

$$
(e^{-4\epsilon u_-})^{-\frac{1}{p'}} \leq \left( \frac{1}{C(\beta)|B|} \int_B e^{-4\epsilon u_-} \, dx \right)^{-\frac{1}{p'}}.
$$

So

(4.12)

$$
\left( \int_B e^{4u_+} \cdot (e^{-4\epsilon u_-})^{-\frac{1}{p'}} \, dx \right)^{\frac{1}{p}} \leq \left( \int_B e^{4u_+} \, dx \right)^{\frac{1}{p}} \left( \frac{1}{C(\beta)|B|} \int_B e^{-4\epsilon u_-} \, dx \right)^{-\frac{1}{p'}}
$$

$$
\left( \int_B e^{4u_+} \, dx \right)^{\frac{1}{p}} \left( \frac{1}{C(\beta)|B|} \int_B e^{-4\epsilon u_-} \, dx \right)^{-\frac{1}{p'}}.
$$

Similarly, by (4.8)

$$
(e^{4u_+})^{-\frac{1}{p'}} \leq \left( \frac{1}{C(\alpha)|B|} \int_B e^{4u_+} \, dx \right)^{-\frac{1}{p'}}.
$$

So

(4.13)

$$
\left( \int_B (e^{4u_+})^{-\frac{1}{p'}} \cdot e^{-4\epsilon u_-} \, dx \right)^{\frac{1}{p'}} \leq \left( \frac{1}{C(\alpha)|B|} \int_B e^{4u_+} \, dx \right)^{-\frac{1}{p'}} \left( \int_B e^{-4\epsilon u_-} \, dx \right)^{\frac{1}{p'}}.
$$

Applying (4.12) to (4.13) in (4.11), we have

(4.14)

$$
\left( \int_B e^{4u(x)} \, dx \right)^{\frac{1}{p}} \left( \int_B (e^{4u(x)})^{-\frac{1}{p'}} \, dx \right)^{\frac{1}{p'}} \leq \left( \frac{1}{C|B|} \right)^{-\frac{1}{p}} \left( \frac{1}{C|B|} \right)^{\frac{1}{p'}} = C|B|
$$

for $p >> 1$. This shows that $e^{4u(x)}$ is an $A_p$ weight for $p >> 1$. The bound $C$ depends only on $\alpha$ and $\beta$. \(\square\)

5. Proof of Theorem 1.3

Theorem 5.1. [FLW95, Theorem 2] Let $\{X_j\}$ be a family of vector fields that satisfies Hörmander’s condition. Let $K$ be a compact subset of $\Omega$. Then there exists $r_0$ depending on $K$, $\Omega$ and $\{X_j\}$ such that if $B = B(x, r)$ is a ball with $x \in K$ and $0 < r < r_0$, and if $1 \leq p < q < \infty$
and $\omega_1, \omega_2$ are weights satisfying the balance condition (5.2) for $B$, with $\omega_1 \in A_p(\Omega, \rho, dx)$ and $\omega_2$ doubling, then
\begin{equation}
\frac{1}{\omega_2(B)} \int_B |f(x) - f_B|^q \omega_2(x) dx \leq c r \frac{1}{\omega_1(B)} \int_B |X f(x)|^p \omega_1(x) dx
\end{equation}
for any $f \in \text{Lip}(\bar{B})$, with $f_B = \omega_2(B)^{-1} \int_B f(x) \omega_2(x) dx$. The constant $c$ depends only on $K, \Omega, \{X_j\}$ and the constants in the conditions imposed on $\omega_1, \omega_2$.

The balance condition is stated as follows: for two weight functions $\omega_1, \omega_2$ on $\Omega$ and $1 \leq p < q < \infty$, a ball $B$ with center in $K$ and $r(B) < r_0$:
\begin{equation}
\frac{r(I)}{r(J)} \left( \frac{\omega_2(I)}{\omega_2(J)} \right)^{1/q} \leq c \left( \frac{\omega_1(I)}{\omega_1(J)} \right)^{1/p}
\end{equation}
for all metric balls $I, J$ with $I \subset J \subset B$.

**Proof.** of Theorem 1.3. It is obvious that $X_1 := \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}$, $X_2 := \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial t}$ on the Heisenberg group $\mathbb{H}^1$ satisfy the Hömander’s condition. Let us take $\omega_1(x) = e^{(n-p)s(x)}$, $\omega_2(x) = e^{nu(x)}$, $q = \frac{np}{n-p}$.

We only need to check condition (5.2). Namely, we need to show
\begin{equation}
\left( \frac{r(I)}{r(J)} \right)^{n-p} \frac{\int_I \omega_2 dx}{\int_J \omega_2 dx} \leq c \left( \frac{\int_I \omega_2^{\frac{n-p}{n}} dx}{\int_J \omega_2^{\frac{n-p}{n}} dx} \right)
\end{equation}
This is true because $0 \leq \frac{n-p}{n} < 1$ and $\omega_2 = e^{nu}$ is a strong $A_\infty$ weight, thus it is an $A_\infty$ weight. In fact, for any $A_\infty$ weight $w$, $0 \leq s < 1$, by the result of Strömberg-Wheeden [SW85]
\begin{equation}
\left( \frac{1}{|B|} \int_B w(x)^s dx \right)^{\frac{1}{s}} \leq C \frac{1}{|B|} \int_B w(x) dx.
\end{equation}
On the other hand, by Hölder’s inequality,
\begin{equation}
\frac{1}{|B|} \int_B w(x) dx \leq \left( \frac{1}{|B|} \int_B w(x)^s dx \right)^{\frac{1}{s}}
\end{equation}
Therefore by taking $s = \frac{n-p}{n}$, (5.3) holds.

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