Finite-size scaling in globally coupled phase oscillators with a general coupling scheme

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Received October 04, 2013; Revised November 21, 2013; Accepted January 08, 2014; Published February 21, 2014

We investigate the critical exponent of correlation size, related to synchronization transition, in globally coupled nonidentical phase oscillators. The critical exponent has so far been identified for sinusoidal coupling, but has not been fully studied for other coupling schemes. Herein, for a general coupling function including a negative second harmonic term in addition to the sinusoidal term, we numerically estimate the critical exponent of the correlation size, denoted by $\nu_+$, in a synchronized regime of the system by employing a non-conventional statistical quantity. First, we confirm that the estimated value of $\nu_+$ is approximately 5/2 for the sinusoidal coupling case, which is consistent with the well known theoretical result. Second, we show that the value of $\nu_+$ increases with an increase in the strength of the second harmonic term. Our result means that the universality of a critical exponent can break down in the globally coupled phase oscillators.

Subject Index A40, A55, A56, A57

1. Introduction

Universal phenomena in the realm of nature have been explored by many researchers in mathematical and physical science [1–4]. A critical exponent, which is invariant for a class of systems exhibiting collective behavior, has been a topic of primary interest [1–13]. In general, the value of a critical exponent, or a scaling law of an order parameter around a phase transition point, is usually independent of details of a system, although other features such as critical transition points are crucially dependent on such details [2]. These properties have been widely confirmed not only in mathematical models but also in experiments [2]. Populations of coupled rhythmic elements can exhibit synchronization and collective behavior via mutual interactions [3,4]. Such phenomena can be observed in a variety of systems such as chemical reactions [14], electrical circuits [15], and biological populations [16]. To elucidate the general properties of such phenomena, the phase description of systems has been widely used [3,4]. In particular, there have been many studies on globally coupled phase oscillators [3–6], defined as follows:

$$\dot{\theta}_j = \omega_j + \frac{K}{N} \sum_{k=1}^{N} h(\theta_k - \theta_j), \ j = 1, \ldots, N, \quad (1)$$
Fig. 1. Bifurcation diagram of the order parameter $R$ in model (1), where $h(x) = \sin x - 0.5 \sin 2x$ and $N = 8000$. The vertical axis $\langle R \rangle_t$ means the long-time average of $R(t)$. A supercritical bifurcation occurs.

where $\theta_j$ is the phase of the $j$th oscillator, $\omega_j$ is the natural frequency of the $j$th oscillator, $K > 0$ is the coupling strength, $h$ is the coupling function, and $N$ is the number of oscillators. When $h(x) = \sin x$, this model is referred to as the Kuramoto model [3]. One of the main issues in this model is a scaling property of the order parameter defined as follows [3]:

$$ R(t) \equiv \frac{1}{N} \left| \sum_{j=1}^{N} \exp(2\pi i \theta_j) \right|, \quad (2) $$

where $| \cdot |$ represents the absolute value. In the thermodynamic limit $N \to \infty$, the phase oscillator model (1) exhibits synchronization transition when the coupling strength $K$ surpasses a critical value $K_c$. This transition can be characterized by a change of the order parameter from zero to a non-zero value. We assume that the stationary state ($R(t) = 0$) in the incoherent regime supercritically bifurcates at the critical coupling strength $K = K_c$, above which the oscillators are synchronized [3–8,17]. The behavior of the order parameter is exemplified for a finite-size system in Fig. 1. The scaling property of the order parameter has been fully understood in the case where $N$ is infinite, for the sinusoidal coupling function as well as for general coupling functions [8,10–13]. However, it is less clear in finite-size systems, as described in the following paragraph. The aim of this study is to elucidate a scaling property of fluctuations in finite-size systems.

Although fluctuations of the order parameter in the Kuramoto model have been intensively studied for the past two decades [18–25], those in the model (1) with a more general coupling function $h(x)$ have not been fully understood. In particular, for a general coupling scheme other than the sinusoidal one, the critical exponents of statistical quantities have been little reported, except for that of a non-conventional statistical quantity [24,25]. For the system in the thermodynamic limit $N \to \infty$, the critical exponent of the order parameter is different between the Kuramoto model with sinusoidal coupling and the model (1) including both sinusoidal and second harmonic coupling terms. In the latter model, the critical exponent is the same independently of the coupling strength of the second harmonic term [8,10–13]. This fact indicates a universal property of a class of coupled oscillator systems with first and second harmonic terms. In finite-size systems, however, it is unclear whether the coupling scheme affects the critical exponents of statistical quantities. We show that a critical exponent characterizing fluctuations in finite-size systems crucially depends on the strength of the
second harmonic term. Although more than a few models are known to exhibit non-universal critical behavior, characterized by a continuous variation of critical exponents with a change in the model parameters \([2,26,27]\), such a phenomenon is commonly considered to be exceptional, at least in equilibrium systems \([2,26–29]\). Even for such a system, it is known that the ratio between the critical exponent considered in the present paper and that of an order parameter \([8,10–13]\) is usually constant \([2,26–29]\). In contrast, our result indicates that the traditional universality described above can break down in the globally coupled phase oscillators \((1)\).

The correlation size is one of the statistical quantities used in studying critical phenomena, corresponding to correlation length, which is often used for locally coupled systems \([2]\). It roughly represents the number of oscillators that are almost synchronized but not completely. Critical phenomena can occur only in the infinite system-size limit \(N \to \infty\). The critical exponent of the correlation size is also defined in such a limit. On the other hand, it is well known that finite-size scaling enables one to estimate the value of the critical exponent by using statistical quantities in a finite-size system \([2]\). In the present paper, we employ a non-conventional statistical quantity to evaluate the critical exponent of the correlation size, \(\nu^+\), in the synchronized regime of the phase oscillator model \((1)\) with finite large \(N\). This is because it is not easy to compute the value of \(\nu^+\) by using the critical exponent of the order parameter \([23]\). The statistical quantity that we use is denoted by \(D\), which is the diffusion coefficient of the temporal integration of the order parameter, multiplied by system size \(N\) \([24,25]\). Using the statistical quantity \(D\), we perform a finite-size scaling analysis \([2]\). First, we confirm that the estimated value of \(\nu^+\) is approximately 5/2 for the sinusoidal coupling \(h(x) = \sin x\), which is consistent with the well known theoretical result \([20,30]\). Second, we consider a general coupling function including a negative second harmonic term in addition to the sinusoidal term, i.e. \(h(x) = \sin x - q \sin 2x\) with \(q > 0\). Numerical results show that the value of \(\nu^+\) increases with an increase in the strength \(q\) of the second harmonic term.

### 2. Statistical quantity for the finite-size scaling analysis

We consider the diffusion coefficient of the time integral of \(R(t)\) in Eq. \((2)\), which characterizes long-term fluctuations in \(R(t)\) \([24,25]\). The variance of \(\int_0^T R(s) \, ds\) is given by

\[
\sigma^2(t) \equiv N \lim_{T \to \infty} \frac{1}{T} \int_{t_0=0}^{t_0=T} \left( \int_{s=t_0}^{s=t+t_0} R(s) \, ds - \langle R \rangle_t \right)^2 \, dt_0,
\]

where \(\langle R \rangle_T\) represents the long-term average, defined as follows:

\[
\langle R \rangle_T \equiv \lim_{T \to \infty} \frac{1}{T} \int_0^T R(t) \, dt.
\]

The following diffusion law then holds:

\[
D = \lim_{T \to \infty} \sigma^2(t)/2t.
\]

The value of \(D\) characterizes how \(\int_0^T R(s) \, ds\) differs from its mean value \(\langle R \rangle_T\). The value of \(D\) is estimated by fitting \(\sigma^2(t)\) with a line of slope \(2D\) after excluding a transient period, as shown in Fig. 2. Different initial conditions may yield different values of \(D\) because of multistability \([31]\), as seen in Fig. 2. However, the dependence of \(D\) on the initial conditions is sufficiently small, as shown in Fig. 3. This result implies that the property on fluctuations is similar for multiple coexisting solutions. All of them transit almost simultaneously from a non-synchronous state to a synchronous one when the coupling strength \(K\) exceeds the critical value \(K = K_c\) for a large system size \(N\) \([3–8,17,32]\).
Fig. 2. The time evolutions of the variance $\sigma^2(t)$ of the integrated order parameter in the coherent regime of the Kuramoto model where $h(x) = \sin x$, $N = 1000$, and $K = 1.635 > K_c = 1.59 \ldots$. These two evolutions are generated for different initial conditions. The value of $\sigma^2(t)$ increases linearly with slope $2D$ after a transient period in both cases. Because of multistability, the value of $D$ can differ depending on the initial conditions. In this study, to measure the value of $D$, we compute $\sigma^2(t)$ for the period of $t \in [0, 1000]$, then fit the data by the diffusion law $\sigma^2(t) \sim 2Dt$ in the period of $t \in [180, 1000]$.

Fig. 3. Probability density function of $D$ estimated by using 300 different initial conditions in the model (1), where $h(x) = \sin x$ and $K = 1.635 > K_c = 1.59 \ldots$. The horizontal axis is normalized by $\langle D \rangle_e$, where $\langle \cdot \rangle_e$ means the ensemble average over 300 different initial conditions. We numerically obtained 300 values of $D$ for each $N$ and estimated the probability density function of $D$ by means of kernel density estimation. The standard deviation of $D$ is sufficiently small for each $N$.

3. Finite-size scaling analysis

In this section, we perform a finite-size scaling analysis to estimate the critical exponent in the system (1) with $h(x) = \sin x - q \sin 2x$. The scaling hypothesis [2] states that any quantity $A$, which shows critical divergence at $K = K_c$, is scaled for $K > K_c$ as follows:

$$N^{-\nu_+ / \nu_1} A = \Psi((K - K_c)N^{1/\nu_1}),$$

(6)
where \( \nu_+ \) and \( r_+ \) represent the critical exponent of the correlation size and that of \( A \) in the synchronized regime, respectively. The function \( \Psi \) is approximated as \( \Psi(x) \sim ux^{-r_+} + x^{-s} \) for large \( x \), where \( s > r_+ \) and \( u \) is a constant [2], so that the orders with respect to system size \( N \) in both sides of Eq. (6) are consistent. It has previously been shown that the value of \( D \) goes to zero as \( N \to \infty \) in a synchronized regime of the phase oscillator model (1) with a general coupling function \( h(x) \) [24]. This means that the critical exponent of \( D \) in the synchronized regime is equal to 0. Therefore, we replace \( A \) by \( D \) and then substitute \( r_+ = 0 \) into Eq. (6). Furthermore, we assume \( u = 0 \) in the above approximation form of \( \Psi(x) \) because fluctuations vanish in the strong coupling limit \( K \to \infty \). As a result, we obtain

\[
D \sim (K - K_c)N^{1/\nu_+} - s, \tag{7}
\]

which means that, if the finite-size effect is not so strong, (i) \( D \) decreases in a power law fashion with system size \( N \) for a fixed value of \( K \), and (ii) the numerically computed values of \( D \) can fall on a straight line against \((K - K_c)N^{1/\nu_+}\) in a log–log plot.

In numerical simulations, the natural frequencies \( \omega_j \) of the individual oscillators are chosen to satisfy

\[
j/(N + 1) = \int_{-\infty}^{\omega_j} g(\tilde{\omega})d\tilde{\omega}, \tag{8}
\]

where \( g(\tilde{\omega}) \) is the Gaussian distribution with mean zero and variance one. To avoid a situation in which the finite-size effect yields an erroneous value of \( \nu_+ \), we limit the range of \( K \) to \( K_c < K_- < K < K_+ \). We set the lower value \( K_- \) such that \( D \) decreases in a power law fashion with system size \( N \) for \( K > K_- \). The upper value \( K_+ \) is set so that the power law decay of \( D \) is well kept within the range \( K_- < K < K_+ \) in order to avoid a large variation in the estimated values of \( \nu_+ \). For estimation of \( \nu_+ \), we use a bootstrap method, a well known efficient method for estimation of fluctuations [33] as follows. For an initial condition \( \theta_j(0) \) for \( j = 1, \ldots, N \), a time evolution of \( \sigma^2(t) \) is obtained as shown in Fig. 2 and a value of \( D \) is calculated by Eq. (5). We obtain 100 simulation results of \( D \) values from “100 different initial conditions”. Then we repeatedly choose one of the 100 simulation results \( M \) times. Note that the same simulation result can be chosen more than once. The chosen values are the \( M \) overlapping samples of \( D \) values from the 100 simulation results. We average the \( M \) samples. Next, we estimate the value of \( \nu_+^{(M)} \) that minimizes the mean squared error between the averaged values of \( D \) and \((K - K_c)N^{1/\nu_+}\) in a log–log plot, where \( \nu_+^{(M)} \) represents the estimated value of \( \nu_+ \) from the averaged values of \( D \) over the \( M \) samples. By repeating the above procedure 10,000 times, we obtain the estimated values of the mean and the standard deviation of \( \nu_+^{(M)} \).

First, let us consider the Kuramoto model, i.e. \( q = 0 \). The value of \( \nu_+^{(100)} \) is given as \( \nu_+^{(100)} \sim 2.49 \) by using the method explained above. Figure 4(a) shows the value of \( D \) averaged over 100 overlapping samples for each pair of \( (N, K) \). The same data points shown in Fig. 4(b) are plotted against \((K - K_c)N^{1/2.49}\) in a log–log plot. We emphasize that all the data points for the different system size \( N \) fall on one curve by the transformation. The estimated exponent \( \nu_+^{(100)} \sim 2.49 \) is consistent with the analytical result: \( \nu_+ = 5/2 \) [20,30]. Therefore, our method can estimate the exact value of \( \nu_+ \) well.

Next, we investigate the model (1) with the general coupling function with \( q = 0.1 \). Figure 5(a) shows the averaged values of \( D \) over 100 overlapping samples. The estimated value of \( \nu_+^{(100)} \) is given by \( \nu_+^{(100)} \sim 2.95 \), as shown in Fig. 5(b). This means that the value of \( \nu_+ \) in this model is larger than that in the Kuramoto model.
Fig. 4. Estimation of $v_+$ in the Kuramoto model. (a) $\langle D \rangle_e$ vs. $(K - K_c)$, where $\langle \cdot \rangle_e$ means the ensemble average over 100 overlapping samples. (b) $\langle D \rangle_e$ vs. $(K - K_c)N^{1/2.49}$ in a log–log plot, which implies $\hat{v}_+^{(100)} \sim 2.49$. The two left endpoints for $N = 1000$ were not used for the line fitting. All the data points for the different system size $N$ fall on one curve by the transformation, although the parameter values were obtained by line fitting.

Fig. 5. Estimation of $v_+$ in the phase oscillator model (1) with $h(x) = \sin x - 0.1 \sin 2x$. (a) $\langle D \rangle_e$ vs. $(K - K_c)$, where $\langle \cdot \rangle_e$ means the ensemble average over 100 overlapping samples. (b) $\langle D \rangle_e$ vs. $(K - K_c)N^{1/2.95}$ in a log–log plot, which implies $\hat{v}_+^{(100)} \sim 2.95$. The two left endpoints for $N = 1000$ were not used for the line fitting. All the data points for the different system size $N$ fall on one curve by the transformation, although the parameter values were obtained by line fitting.

Note that $M = 100$ samples are enough to estimate the accurate value of $v_+$ because the standard deviation of the estimated values of $\hat{v}_+^{(100)}$ is sufficiently small, as shown in Fig. 6.

Finally, we examine the relationship between the exponent $v_+$ and the strength $q$ of the negative second harmonic term. The mean of $\hat{v}_+^{(100)}$ is likely to increase with an increase in the value of $q$ as shown in Fig. 7(a). In addition, we obtain the distributions of $\hat{v}_+^{(1)}$ as shown in Fig. 7(b). The standard deviation of $\hat{v}_+^{(1)}$ is not so large and the mean value of $\hat{v}_+^{(1)}$ is also likely to increase with an increase in the value of $q$. These results imply that the critical exponent $v_+$ depends on the strength of the second harmonic term of the coupling function.
Fig. 6. The standard deviation of the values of $\hat{\nu}^{(M)}$ estimated from $M$ samples, where $5 \leq M \leq 100$. (a) The case of $q = 0$. (b) The case of $q = 0.1$. In both cases, the standard deviation is very small for a sufficiently large $M$, such as $M = 100$.

Fig. 7. Relationship between the estimated exponent $\hat{\nu}^{(M)}$ and the strength $q$ of the negative second harmonic term in the phase oscillator model (1) with $h(x) = \sin x - q \sin 2x$. (a) The crosses indicate the mean values of $\hat{\nu}^{(100)}$. The circles and the error bars indicate the averages and the standard deviations of the distributions of $\hat{\nu}^{(1)}$, respectively. (b) Distributions of $\hat{\nu}^{(1)}$ for each $q$.

4. Discussion

First, we have confirmed that the critical exponent $\nu_+$ is estimated as $\nu_+ \sim 5/2$ in the Kuramoto model. Next, our numerical simulations have shown that, when the coupling function possesses a negative second harmonic term with strength $q$ in addition to the sinusoidal one, $\nu_+ > 5/2$ and $\nu_+$ increases with $q$. This result is consistent with the following property. Near the synchronization transition point $K = K_c$, the order parameter $R$ is scaled as $R \sim (1 + 1/q)(K - K_c) + O((K - K_c)^2)$ [8]. In addition, fluctuations of a system are generally larger for a smaller value of $R$ [2]. These facts imply that the larger the value of $q$ is, the more slowly the fluctuations of this system decay with the coupling strength $K$. In other well known systems such as the Ising model, the differential of the order parameter $R$ with respect to $K - K_c$ at the critical transition point $K = K_c$ is infinite [2]. In such a case, the functional form of the order parameter $R$ influences the value of $\nu_+$ [2]. In contrast,
the order parameter $R$ linearly increases with respect to $K - K_c$ near the critical transition point $K = K_c$ in the present case. As a result, the slope can influence the value of $\nu_+$. We explain why our method yields a good estimation for the critical exponent $\nu_+$. To estimate the value of $\nu_+$, our numerical simulations have selected the parameter region $K > K_c$ such that the solution shows coherent behavior and the scaling property $D \sim O(1/N^\alpha)$ is satisfied. As shown in Fig. 8, $D$ shows its peak in the coherent regime. We have excluded the parameter region around the peak of $D$ that virtually reflects the feature of the incoherent solution due to the finite-size effect [2]. Such a parameter region can be easily specified by using $D$ because the scaling property of $D$ with respect to $N$ in the coherent regime has been revealed [24,25]. The choice of the parameter region is crucial, because the value of the critical exponent can differ depending on whether the system behavior is coherent or incoherent [24,25]. Note that the accuracy of the estimation is usually better when using the parameter region closer to the critical point in other well known systems [2]. The standard numerical method to obtain the value of a critical exponent [28,29] seems not to be valid in the present case where the critical exponent is not the same between the coherent and incoherent regimes.

We have deterministically generated the natural frequencies $\omega_j$ of the individual oscillators [18,31,34–37]. Another popular method is to randomly generate them [20,21,36]. We have not confirmed that the result is unchanged for the random generation case because of the computational cost. However, we can expect that the generating methods do not influence the value of the critical exponent $\nu_+$. Our estimation has yielded $\nu_+ \sim 2.49$ for the system with sinusoidal coupling. This estimation is consistent with the theoretical result, $\nu_+ = 5/2$, for the system with randomly generated natural frequencies [20]. The same critical exponent value was previously obtained in other types of networks consisting of phase oscillators with random natural frequencies [20,30,38,39]. The consistency of the critical exponent value suggests that the above systems belong to the same universality class. Further, in previous studies of the Kuramoto model [18,21] and phase oscillator models in random networks [39], it was shown that the generating method of natural frequencies does not influence the values of critical exponents.
Fig. 9. Relationship between the sum of the estimated exponents $\hat{\gamma}^{(M)}_+ + \hat{\zeta}^{(M)}_+$ and the strength $q$ of the negative second harmonic term in the phase oscillator model (1) with $h(x) = \sin x - q \sin 2x$, where $\hat{\gamma}^{(M)}_+$ and $\hat{\zeta}^{(M)}_+$ were defined and obtained in the same way as $\hat{\nu}^{(100)}_+$ was in Sect. 3. (a) The crosses indicate the mean values of $\hat{\gamma}^{(100)}_+ + \hat{\zeta}^{(100)}_+$. The circles and the error bars indicate the averages and the standard deviations of the distributions of $\hat{\gamma}^{(1)}_+ + \hat{\zeta}^{(1)}_+$, respectively. (b) Distributions of $\hat{\gamma}^{(1)}_+ + \hat{\zeta}^{(1)}_+$ for different values of $q$.

Fig. 10. The ratio between the sum of the estimated exponents $\hat{\gamma}^{(M)}_+ + \hat{\zeta}^{(M)}_+$ and $\hat{\nu}^{(M)}_+$, for several values of strength $q$ of the negative second harmonic term in the phase oscillator model (1) with $h(x) = \sin x - q \sin 2x$. It seems that the value is invariant and approximately 1.2 for each $q$. (a) The crosses indicate the mean values of $(\hat{\gamma}^{(100)}_+ + \hat{\zeta}^{(100)}_+)/(\hat{\nu}^{(100)}_+)$. The circles and the error bars indicate the averages and the standard deviations of the distributions of $(\hat{\gamma}^{(1)}_+ + \hat{\zeta}^{(1)}_+)/(\hat{\nu}^{(1)}_+)$, respectively. (b) Distributions of $(\hat{\gamma}^{(1)}_+ + \hat{\zeta}^{(1)}_+)/(\hat{\nu}^{(1)}_+)$ for different values of $q$.

Acknowledgements

This research is supported by a Grant-in-Aid for Scientific Research (A) (20246026) from MEXT of Japan, and by the Aihara Innovative Mathematical Modelling Project, the Japan Society for the Promotion of Science (JSPS) through the “Funding Program for World-Leading Innovative R&D on Science and Technology (FIRST Program)”, initiated by the Council for Science and Technology Policy (CSTP).

Note added in proof. After the submission of this paper for publication, we noticed that the results of the numerical simulations, presented in this study, enable us to estimate the values of other critical exponents. Let us define the variance of the order parameter as $V := N \langle (R(t) - \langle R \rangle)^2 \rangle$, and the autocorrelation function of
the order parameter as \( C(t) := N((R(t + t_0) - ⟨R⟩)(R(t_0) - ⟨R⟩))_z \), where \( ⟨·⟩_t \) represents the time average over the period from \( t_0 = 0 \) to \( t_0 \to \infty \). Let us also denote by \( τ \) the half-decay time of \( C(t) \), i.e. \( C(τ) = C(0)/2 \). It is well known that, in the vicinity of the synchronization transition point \( K = K_c \) of the coherent regime, \( V \sim |K - K_c|^{−γ_+} \) and \( τ \sim |K - K_c|^{−z_+} \) for a sufficiently large system size \( N \), where \( γ_+ \) and \( z_+ \) are the critical exponents [1,2]. According to the previous study [24], also around \( K = K_c \), it follows that \( D \sim |K - K_c|^{−(γ_+ + z_+)} N^{-ν} \) with a certain positive constant \( ν \) for a sufficiently large system size \( N \). This scaling law suggests that the slope of the straight lines in Figs. 4(b) and 5(b) approximately corresponds to the value of \( −(γ_+ + z_+) \). By computing the value of the slope, we can see that the value of \( γ_+ + z_+ \) increases with \( q \), as shown in Fig. 9. This simulation result also implies \( γ_+ + z_+ \sim 3 \) for the Kuramoto model of \( q = 0 \), which is consistent with the following physical argument. According to the hyperscaling relation [1,2], \( γ_+ = 3/2 \) [20].

The relationship of \( γ_+ = z_+ \) is assumed to hold, which is known in the incoherent regime of the Kuramoto model [18,21] and in mean-field type models of thermal phase transitions [1,2]. On the other hand, as we have described in the introduction, it is known that the ratio between the critical exponent \( ν_+ \) and other critical exponents can often be constant [2,26–29]. We plot the ratio between \( ν_+ \) and \( γ_+ + z_+ \) in Fig. 10. It appears that \( (γ_+ + z_+)/ν_+ \sim 1.2 \) for each \( q \), as predicted by the value in the Kuramoto model of \( q = 0 \). We are grateful to Prof. N. Hatano for his valuable comment, which led to this note.

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