DUALITY IN $\mathcal{N} = 2$ SUSY GAUGE THEORIES:
low-energy effective action and BPS spectra

Adel Bilal
Institute of Physics, University of Neuchâtel
rue Breguet 1, 2000 Neuchâtel, Switzerland
adel.bilal@unine.ch

Abstract

After an introduction to $\mathcal{N} = 2$ susy Yang-Mills theories, I review in some detail, for the SU(2) gauge group, how the low-energy effective action is obtained using duality and the constraints arising from the supersymmetry. Then I discuss how knowledge of this action, duality and certain discrete symmetries allow us to determine the spectra of stable BPS states at any point in moduli space. This is done for gauge group SU(2), without and with fundamental matter hypermultiplets which may have non-vanishing bare masses. In the latter case non-trivial four-dimensional CFTs arise at Argyres-Douglas type points.

1Lectures given at the Institut Henri Poincaré program “Strings, Supergravity and M-theory”, January 2001
1 Introduction

The central tool in modern string and M-theory certainly is duality. Duality has a long history, but it was only since the ground-breaking work by Seiberg and Witten in 1994 [1, 2] that it has become a useful tool and maybe even more: an organising principle and underlying symmetry of string/M-theory. Dualities were discovered and suspected in string theory even before, but it is certainly fair to say that it was in the context of $\mathcal{N} = 2$ supersymmetric Yang-Mills quantum field theories that their power was impressively revealed.

The prehistory of duality probably goes back to Dirac who observed that the source-free Maxwell equations are symmetric under the exchange of the electric and magnetic fields. More precisely, the symmetry is $E \rightarrow B, \ B \rightarrow -E$, or $F_{\mu\nu} \rightarrow \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$. To maintain this symmetry in the presence of sources, Dirac introduced, somewhat ad hoc, magnetic monopoles with magnetic charges $q_m$ in addition to the electric charges $q_e$, and showed that consistency of the quantum theory requires a charge quantization condition $q_m q_e = 2\pi n$ with integer $n$. Hence the minimal charges obey $q_m = \frac{2\pi}{q_e}$. Duality exchanges $q_e$ and $q_m$, i.e. $q_e$ and $\frac{2\pi}{q_e}$. Now recall that the electric charge $q_e$ also is the coupling constant. So duality exchanges the coupling constant with its inverse (up to the factor of $2\pi$), hence exchanging strong and weak coupling. This is the reason why physicists are so much interested in duality: the hope is to learn about strong-coupling physics from the weak-coupling physics of a dual formulation of the theory. Of course, in classical Maxwell theory we know all we may want to know, but this is no longer true in quantum electrodynamics.

Actually, quantum electrodynamics is not a good candidate for exhibiting a duality symmetry since there are no magnetic monopoles, but the latter naturally appear as solitons in spontaneously broken non-abelian gauge theories [3]. Unfortunately, electric-magnetic duality in its simplest form cannot be a symmetry of the quantum theory due to the running of the coupling constant (among other reasons). Indeed, if duality exchanges $g^2(\Lambda) \leftrightarrow \frac{1}{g^2(\Lambda)}$ at some scale $\Lambda$, in general this won’t be true at another scale. There are two ways out: either the coupling does not run, i.e. if the $\beta$-function vanishes as is the case in certain ($N = 4$) supersymmetric extensions of the Yang-Mills theory, or, if $\beta \neq 0$, we can have instead $g^2(\Lambda) \leftrightarrow \frac{1}{g^2(\Lambda_D)}$ where $\Lambda_D = \Lambda_D(\Lambda)$ is some dual scale which depends on the initial scale $\Lambda$ just in the right way to make the duality possible. The first possibility led Montonen and Olive [4] to conjecture that duality might be an exact symmetry of $N = 4$ susy Yang-Mills theory. A nice review of these ideas can be found in [5]. It is the second possibility that is realised for the low-energy effective action of $N = 2$ susy Yang-Mills theory, and it is this second case which will occupy the present review.

Let me insist that $\mathcal{N} = 2$ susy Yang-Mills theory is not duality invariant. For simplicity we will restrict to an $SU(2)$ gauge group, although other gauge groups can be discussed along the same lines. This $SU(2)$ is spontaneously broken down to $U(1)$, so that the gauge bosons are a massless “photon” and two massive “W-bosons” along with their superpartners. Since the “W-bosons” are heavy they can be integrated out along with all other heavy states when determining the low-energy effective action for the massless “photon”-multiplet. We refer to the underlying $SU(2)$ theory as the microscopic theory since it governs the UV behaviour, while the theory of the massless $U(1)$ degrees of freedom governs the effective IR dynamics. It is this latter theory in which duality is realised as follows. This low-energy effective action will turn out to have many gauge-inequivalent vacua determined by the scalar (“Higgs”) field expectation value, i.e. there is a moduli space. Let $g(\mu)$ be

\[^{2}\epsilon_{\mu\nu\rho\sigma} \text{ is the flat-space antisymmetric } \epsilon \text{-tensor with } \epsilon^{0123} = +1 \text{ and we use } \eta_{\mu\nu} \text{ with signature } (1, -1, -1, -1).\]

2
the effective coupling where the scale $\mu$ is given by the scalar vev. The duality then is

$$g^2(\mu) \leftrightarrow \frac{1}{g^2(\mu_D)} \quad (1.1)$$

What is duality good for? If it exchanges a coupling $g$ with its inverse, it will map weak coupling, say $g^2 < 0.1$ to very strong coupling, say $g^2 > 10$, and vice versa. But a coupling close to unity then again is mapped to a coupling close to unity. While often this is the more interesting case, it seems that precisely here duality might be useless. In fact, this is not so. As an illustration, let me recall that a somewhat similar duality symmetry appears in the two-dimensional Ising model where it exchanges the temperature with a dual temperature, thereby exchanging high and low temperature analogous to strong and weak coupling. It is useful to consider this example in slightly more detail:

**Kramers-Wannier duality of the 2D Ising model**

Consider the Ising model on a square lattice with $N$ sites. The partition function is

$$Z = \sum_{\sigma_i = \pm 1} \exp \left( \beta \sum_{<i,j>} J \sigma_i \sigma_j \right), \quad (1.2)$$

where $\beta = 1/T$. This can be rewritten as

$$Z(\beta) \equiv Z_{HT}(\beta) = \sum_{\sigma_i = \pm 1} \prod_{<i,j>} \cosh \beta J [1 + \sigma_i \sigma_j \tanh \beta J] \quad (1.3)$$

A little thinking shows that this yields the sum of all closed polygons $P$ drawn on the lattice (a polygon may have several disconnected pieces), each polygon having a weight factor $(\tanh \beta J)^{L(P)}$, where $L(P)$ is the length of the polygon in lattice units:

$$Z(\beta) = 2^N (\cosh \beta J)^{2N} \sum_{P} (\tanh \beta J)^{L(P)} \quad (1.4)$$

This is an appropriate expansion if $\tanh \beta J$ is small, i.e. at high temperatures $T = 1/\beta$. At low temperature instead, there will be domains of spins pointing in the same direction, separated from each other by domain boundaries which are polygons $P$ drawn on the dual lattice. For a square lattice the dual lattice is again the same square lattice (only displaced). The energy of such a configuration is is obtained from the ground state energy $E_0 = -2NJ$ by adding the contributions due to the domain boundaries:

$$E(P) = E_0 + 2JL(P) \quad (1.5)$$

so that the appropriate expansion of the partition function at low temperature is

$$Z(\beta) \equiv Z_{LT}(\beta) = e^{-\beta E_0} \sum_{P} (e^{-2\beta J})^{L(P)} \quad (1.6)$$

So there are two different expansions of the *same* Ising partition function. Now define a dual temperature by

$$e^{-2\beta_D J} = \tanh \beta_J \Leftrightarrow e^{-2\beta J} = \tanh \beta_D J \quad (1.7)$$

so that we can write using first (1.6) and then (1.4)

$$Z(\beta_D) = e^{-\beta_D E_0} \sum_{P} (\tanh \beta_J)^{L(P)} = 2^{-N} e^{-\beta_D E_0} (\cosh \beta J)^{-2N} Z(\beta). \quad (1.8)$$
This is an amazing functional relation for the partition function $Z$ which allows the determination of the critical temperature as follows: if there is a (single) phase transition at some critical value $\beta^*$ the free energy $F(\beta) = -\frac{1}{\beta} \log Z(\beta)$ must be singular at $\beta = \beta^*$. The prefactors on the r.h.s. of (1.8) will not change the singular behaviour of $F$, so we conclude from (1.8) that if $\log Z(\beta)$ is singular, so must be $\log Z(\beta_D)$. But this implies that $\beta_D$ must also be at the critical value: $\beta = \beta^* = \beta_D$, i.e. the critical temperature is the self-dual point. It is given by the solution of

$$e^{-2\beta^* J} = \tanh \beta^* J \Rightarrow \beta^* J = \frac{1}{2} \log(\sqrt{2} + 1) .$$

We see that for the Ising model, the sole existence of the duality symmetry leads to the exact determination of the critical temperature as the self-dual point. Historically this preceeded Onsager’s exact solution by a few years. One may view the existence of this self-dual point as the requirement that the dual high and low temperature regimes can be consistently “glued” together.

Note that the functional relation (1.8) not only gives the critical temperature but also allows us to obtain quite some information on the form of the partition function at any $N$ and $T$ as follows. We let $x = e^{-2\beta J}$ so that $e^{-2\beta_D J} = \frac{1-x}{1+x}$. If we define $Z(\beta) \equiv z(x)^N$ then (1.8) takes the form $z \left( \frac{1-\xi}{1+\xi} \right) = \frac{2\beta}{2\beta - 1} z(x)$. Substituting $z(x) = (1-x)f(\xi)$ with $\xi = \log \frac{x}{1-x}$ this simply becomes $f(-\xi) = f(\xi)$ so that $f$ is an even function of $\xi$, and the self-dual point is $\xi^* = 0$. Although this is not enough to completely determine the partition function, it gives valuable information relating the high and low-temperature behaviour.

Similarly, in the Seiberg-Witten theory, duality relates the behaviour of the effective action at strong coupling to its behaviour at weak coupling. Here however, as will be explained below, the requirement that both regimes can be consistently glued together is much stronger. While in the Ising model the “gluing” resulted in the determination of the critical point, in the $N = 2$ susy non-linear $\sigma$-models describing effective theories we have a holomorphicity requirement allowing for both regimes to be smoothly glued together. Adding some information on the asymptotic behaviours at weak and strong coupling then completely determines the full effective action for the light fields at any coupling.

Outline of the paper

Let me give an overview of the material covered in these lectures. Since $\mathcal{N} = 2$ supersymmetry plays a central role, I will spend some time and space in the next section to review those notions of supersymmetry that we will need. Particular emphasis will be on the $\mathcal{N} = 2$ super Yang-Mills theory and on susy non-linear $\sigma$-models describing effective theories. The reader who is familiar with these matters may want to skip part or all of this section. In section 3, I review the analysis of Seiberg and Witten in its simplest setting: $\mathcal{N} = 2$ susy YM theory with gauge group SU(2) without additional “matter” hypermultiplets. I will discuss the Wilsonian low-energy effective action corresponding to this microscopic $\mathcal{N} = 2$ super Yang-Mills action. The effective action describes the physics of the remaining massless $U(1)$ susy multiplet in terms of an a priori unknown function $\mathcal{F}(a)$ where $a$ is the vacuum expectation value of the scalar field. $\mathcal{N} = 2$ supersymmetry constrains $\mathcal{F}$ to be a (possibly multivalued) holomorphic function. Different vacuum expectation values $a$ lead to physically different theories, and we have a moduli space with a complex coordinate $u$ that is related to $a$ again by a (possibly multivalued) holomorphic function $a(u)$. Then I discuss how the function $\mathcal{F}$ can be obtained in certain asymptotic regimes, using asymptotic freedom of the microscopic theory as well as duality of the effective low-energy theory. Technically, the asymptotic behaviours are translated into monodromy matrices describing how the couple $(a(u), \partial \mathcal{F}(a)/\partial a)$ is transformed into itself as the
coordinate \( u \) goes once around the singular points of moduli space. Knowledge of the monodromy matrices and the asymptotics then allows to reconstruct the couple \((a(u), \partial F(a)/\partial a)\) everywhere. This can be inverted, at least in principle, to yield the function \( F(a) \) and hence all knowledge about the low-energy effective action. However, this is not all we want to know. The low-energy effective action only describes the dynamics of the massless particles, namely the “photons” supermultiplet. In addition there are the massive states, e.g. the analogues of the \( W^\pm \) supermultiplets, the magnetic monopoles, dyons, etc. Even if we don’t know their detailed dynamics, we can however determine their masses exactly at any point in moduli space, since they are BPS states and their masses are related to their charge quantum numbers and the functions \( a(u) \) and \( a_D(u) \). A more delicate question is to determine which BPS states are stable and exist in a given region of the moduli space. This requires the development of some simple new technique which was obtained in [6] and will be explained in section 4 for the simplest example of the SU(2) gauge theory without hypermultiplets. Section 5 then generalises these results to the more complicated cases where various hypermultiplets are present [7]. In particular, if these hypermultiplets have bare masses one encounters a host of new phenomena [8], in particular the existence of Argyres-Douglas points where several mutually non-local fields simultaneously become massless and where the theory is superconformal.

2 \( \mathcal{N} = 2 \) susy gauge theory

We begin by giving a rather detailed review of those notions of supersymmetry that will be useful in the following. There are many reviews on supersymmetry, some of them are [9, 10, 11]

2.1 \( \mathcal{N} = 1 \) superspace

A convenient and compact way to write actions for supersymmetric field theories is to introduce superspace and superfields, i.e. fields defined on superspace. This is particularly simple for unextended susy, so we will begin by looking at \( \mathcal{N} = 1 \) superspace and superfields. Then we have two plus two susy generators \( Q_\alpha \) and \( \overline{Q}_{\dot{\alpha}} \), as well as four generators \( P_\mu \) of space-time translations. There is one coordinate associated to each of them so that coordinates on superspace are \((x^\mu, \theta^\alpha, \overline{\theta}^{\dot{\alpha}})\) with \( \theta^\alpha \) and \( \overline{\theta}^{\dot{\alpha}} \) = 1, 2 anticommuting as usual. We will use superspace as a very efficient tool to formulate supersymmetric theories. We will not review the properties of superspace here but refer the reader to ref. [10] instead. Let us only remind the reader that a general superfield is a function of all the coordinates on superspace, that the supercovariant derivatives are defined as

\[
D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i \sigma^\mu_{\alpha \dot{\beta}} \overline{\theta}^{\dot{\beta}} \partial_\mu
\]

\[
\overline{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \overline{\theta}^{\dot{\alpha}}} + i \theta^\beta \sigma^\mu_{\beta \dot{\alpha}} \partial_\mu
\]

while the susy generators act as

\[
Q_\alpha = -i \frac{\partial}{\partial \theta^\alpha} - \sigma^\mu_{\alpha \dot{\beta}} \overline{\theta}^{\dot{\beta}} \partial_\mu
\]

\[
\overline{Q}_{\dot{\alpha}} = i \frac{\partial}{\partial \overline{\theta}^{\dot{\alpha}}} + \theta^\beta \sigma^\mu_{\beta \dot{\alpha}} \partial_\mu
\]

They satisfy the susy algebra, in particular

\[
\{Q_\alpha, Q_\beta\} = 2 \sigma^\mu_{\alpha \dot{\beta}} P_\mu = -2i \sigma^\mu_{\alpha \dot{\beta}} \partial_\mu
\]

(2.3)
These generators then act on an arbitrary superfield \( F \) as
\[
(1 + i\epsilon Q + i\epsilon^{-1}Q^\dagger)F(x^\mu, \theta^\alpha, \bar{\theta}^\dot{\alpha}) = F(x^\mu - i\epsilon\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\tau, \theta^\alpha + \epsilon^\alpha, \bar{\theta}^\dot{\alpha} + \bar{\epsilon}^\dot{\alpha}) \tag{2.4}
\]
and the susy variation of a superfield is of course defined as
\[
\delta_{\epsilon, \bar{\epsilon}} F = (i\epsilon Q + i\epsilon^{-1}Q^\dagger)F \tag{2.5}
\]
Since a general superfield contains too many component fields to correspond to an irreducible representation of \( \mathcal{N} = 1 \) susy, it will be very useful to impose susy invariant condition to lower the number of components. This can be done using the covariant derivatives \( D_\alpha \) and \( \overline{D}_\dot{\alpha} \) since they anticommute with the susy generators \( Q \) and \( \overline{Q} \). Then \( \delta_{\epsilon, \bar{\epsilon}} (D_\alpha F) = D_\alpha (\delta_{\epsilon, \bar{\epsilon}} F) \) and idem for \( \overline{D}_\dot{\alpha} \). It follows that \( D_\alpha F = 0 \) or \( \overline{D}_\dot{\alpha} F = 0 \) are susy invariant constraints one may impose to reduce the number of components in a superfield.

**Chiral superfields**

A chiral superfield \( \phi \) is defined by the condition
\[
\overline{D}_\dot{\alpha} \phi = 0 \tag{2.6}
\]
and an anti-chiral one \( \overline{\phi} \) by \( D_\alpha \overline{\phi} = 0 \). Introducing \( y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta} \) this is easily solved by
\[
\phi(y, \theta) = z(y) + \sqrt{2}\theta\psi(y) - \theta f(y) \tag{2.7}
\]
or Taylor expanding in terms of \( x, \theta \) and \( \bar{\theta} \):
\[
\phi(y, \theta) = z(x) + \sqrt{2}\theta \psi(x) + i\theta\sigma^\mu\overline{\theta}\overline{\partial}_\mu z(x) - \theta f(x) - \frac{i}{\sqrt{2}}\theta\theta\overline{\theta}\overline{\theta}\overline{\partial}_\mu\psi(x)\sigma^\mu\overline{\theta} - \frac{1}{4}\theta\theta\overline{\theta}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\partial}^2 z(x) \tag{2.8}
\]
and similarly for \( \overline{\phi} \). Physically, such a chiral superfield describes one complex scalar \( z \) and one Weyl fermion \( \psi \). The field \( f \) will turn out to be an auxiliary field. The susy variations of the component fields are given by
\[
\begin{align*}
\delta z &= \sqrt{2}\epsilon\psi \\
\delta \psi &= \sqrt{2}i\partial_\mu z\sigma^\mu\overline{\epsilon} - \sqrt{2}f\epsilon \\
\delta f &= \sqrt{2}i\partial_\mu \psi\sigma^\mu\overline{\tau} \tag{2.9}
\end{align*}
\]

**Vector superfields**

The \( \mathcal{N} = 1 \) supermultiplet of next higher spin is the vector multiplet. The corresponding superfield \( V(x, \theta, \overline{\theta}) \) is real and has the expansion
\[
V(x, \theta, \overline{\theta}) = C + i\theta\chi - i\overline{\theta}\overline{\chi} + \theta\sigma^\mu\overline{\theta}\nu_\mu + \frac{i}{2}\theta\theta(M + iN) - \frac{i}{2}\overline{\theta}\overline{\theta}(M - iN) + i\theta\theta\overline{\theta}(\frac{1}{2}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\partial}_\mu\alpha + i\theta\theta\overline{\theta}(D - \frac{1}{2}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\theta}\overline{\partial}^2 C) \tag{2.10}
\]
where all component fields only depend on \( x^\mu \). There are 8 bosonic components \( (C, D, M, N, \nu_\mu) \) and 8 fermionic components \( (\chi, \lambda) \). These are too many components to describe a single supermultiplet.
To reduce their number we make use of the supersymmetric generalisation of a gauge transformation. Note that the transformation $V \rightarrow V + \phi + \phi^\dagger$ with $\phi$ a chiral superfield, implies the component transformation $v_\mu \rightarrow v_\mu + \partial_\mu (2 \text{Im} z)$ which is an abelian gauge transformation. This shows that the transformation of $V$ indeed is the desired supersymmetric generalisation of gauge transformations. If this transformation is a symmetry of the theory then, by an appropriate choice of $\phi$, one can transform away the components $\chi, C, M, N$ and one component of $v_\mu$. This choice is called the Wess-Zumino gauge, and it reduces the vector superfield to

$$V_{WZ} = \theta \sigma^\mu \overline{\theta} v_\mu(x) + i \theta \theta \overline{\theta} \lambda(x) - i \theta \theta \theta \lambda(x) + \frac{1}{2} \theta \theta \theta \theta D(x).$$  \hspace{1cm} (2.11)$$

Since each term contains at least one $\theta$, the only non-vanishing power of $V_{WZ}$ is $V_{WZ}^2 = \theta \sigma^\mu \overline{\theta} \theta \sigma^\nu \overline{\theta} v_\mu v_\nu = \frac{1}{2} \theta \theta \theta \theta v_\mu v_\mu$ and $V_{WZ}^n = 0$, $n \geq 3$.

To construct kinetic terms for the vector field $v_\mu$ one must act on $V$ with the covariant derivatives $D$ and $\overline{D}$. Define

$$W_\alpha = -\frac{1}{4} DDD_\alpha V, \quad \overline{W}_\dot{\alpha} = -\frac{1}{4} D\overline{D}\dot{\alpha} V.$$  \hspace{1cm} (2.12)$$

(This is appropriate for abelian gauge theories and will be slightly generalized in the non-abelian case.) Since $D^3 = \overline{D}^3 = 0$, $W_\alpha$ is chiral and $\overline{W}_\dot{\alpha}$ antichiral. Furthermore it is clear that they behave as anticommuting Lorentz spinors. Note that they are invariant under the susy gauge transformation $V \rightarrow V + \phi + \phi^\dagger$. It is then easiest to use the WZ-gauge to compute $W_\alpha$. To facilitate things further, change variables to $y^\mu, \theta^\alpha, \overline{\theta}^{\dot{\alpha}}$. Then one finds

$$W_\alpha = -i \lambda_\alpha(y) + \theta_\alpha D(y) + i (\sigma^{\mu \nu} \theta)_{\alpha} f_{\mu \nu}(y) + \theta \theta (\sigma^\mu \partial_\mu \overline{\lambda}(y))_{\alpha}$$  \hspace{1cm} (2.13)$$

with

$$f_{\mu \nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$$  \hspace{1cm} (2.14)$$

being the abelian field strength associated with $v_\mu$.

Susy invariant actions

To construct susy invariant actions we now only need to make a few observations. First, products of superfields are of course superfields. Also, products of (anti) chiral superfields are still (anti) chiral superfields. Typically, one will have a superpotential $W(\phi)$ which is again chiral. This $W$ may depend on several different $\phi_i$. Using the $y$ and $\theta$ variables one easily Taylor expands

$$W(\phi) = W(z(y)) + \sqrt{2} \frac{\partial W}{\partial z_i} \psi_i(y) - \theta \theta \left( \frac{\partial W}{\partial z_i} f_i(y) + \frac{1}{2} \frac{\partial^2 W}{\partial z_i \partial z_j} \psi_i(y) \psi_j(y) \right)$$  \hspace{1cm} (2.15)$$

where it is understood that $\partial W/\partial z$ and $\partial^2 W/\partial z \partial z$ are evaluated at $z(y)$. The second and important observation is that any Lagrangian of the form

$$\int d^2 \theta d^2 \overline{\theta} F(x, \theta, \overline{\theta}) + \int d^2 \theta W(\phi) + \int d^2 \overline{\theta} [W(\phi)]^\dagger$$  \hspace{1cm} (2.16)$$

is automatically susy invariant, i.e. it transforms at most by a total derivative in space-time. The proof is very simple and can be found e.g. in \cite{11}.

As a first example consider an action for chiral superfields only

$$S = \int d^4 x d^2 \theta d^2 \overline{\theta} \phi_i \phi_i + \int d^4 x d^2 \theta W(\phi_i) + h.c.$$  \hspace{1cm} (2.17)$$
which in components gives
\[ S = \int d^4x \left[ \partial_\mu z_i |^2 - i \psi_i \sigma^\mu \partial_\mu \overline{\psi}_i + f^\dagger_i f_i \right. \]
\[ - \left. \frac{\partial W}{\partial z_i} f_i + h.c. - \frac{1}{2} \frac{\partial^2 W}{\partial z_i \partial z_j} \overline{\psi}_i \psi_j + h.c. \right] . \] \quad (2.18)

More generally, one can replace \( \phi_i^\dagger \phi_i \) by a (real) Kähler potential \( K(\phi_i^\dagger, \phi_j) \). This leads to the non-linear \( \sigma \)-model discussed later. In any case, the \( f_i \) have no kinetic term and hence are auxiliary fields. They should be eliminated by substituting their algebraic equations of motion
\[ f^\dagger_i = \left( \frac{\partial W}{\partial z_i} \right) \] \quad (2.19)
into the action, leading to
\[ S = \int d^4x \left[ \partial_\mu z_i |^2 - i \psi_i \sigma^\mu \partial_\mu \overline{\psi}_i - \left| \frac{\partial W}{\partial z_i} \right|^2 - \frac{1}{2} \frac{\partial^2 W}{\partial z_i \partial z_j} \overline{\psi}_i \psi_j - \frac{1}{2} \left( \frac{\partial^2 W}{\partial z_i \partial z_j} \right) \overline{\psi}_i \psi_j \right] . \] \quad (2.20)

We see that the scalar potential \( V \) is determined in terms of the superpotential \( W \) as
\[ V = \sum_i \left| \frac{\partial W}{\partial z_i} \right|^2 . \] \quad (2.21)

To construct an action for the vector superfield one remarks that, since \( W_\alpha \) is a chiral superfield, \( \int d^2 \theta W_\alpha W_\alpha \) will be a susy invariant Lagrangian. Its component expansion is obtained from the \( \theta\theta \)-term (F-term) of \( W_\alpha W_\alpha \):
\[ W_\alpha W_\alpha \big|_{\theta\theta} = -2i \lambda \sigma^\mu \partial_\mu \overline{\lambda} + D^2 - \frac{1}{2} (\sigma^{\mu\nu})_{\alpha\beta} (\sigma^{\rho\sigma})_{\alpha\beta} f_{\mu\nu} f_{\rho\sigma} , \] \quad (2.22)
where we used \( (\sigma^{\mu\nu})_{\alpha\beta} = \text{tr} \sigma^{\mu\nu} = 0 \). Furthermore, \( (\sigma^{\mu\nu})_{\alpha\beta}(\sigma^{\rho\sigma})_{\alpha\beta} = \frac{1}{2} (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) - \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} \) (with \( \epsilon^{0123} = +1 \)) so that
\[ \int d^2 \theta W_\alpha W_\alpha = -\frac{1}{2} f_{\mu\nu} f^{\mu\nu} - 2i \lambda \sigma^\mu \partial_\mu \overline{\lambda} + D^2 + \frac{i}{4} \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} . \] \quad (2.23)

Note that the first three terms are real while the last one is purely imaginary.

### 2.2 \( \mathcal{N} = 1 \) susy YM action

We will now look at the non-abelian generalisation and construct the action for \( \mathcal{N} = 1 \) super YM theory coupled to matter multiplets which are chiral multiplets. We need a slight generalization of the definition of \( W_\alpha \) to the non-abelian case. All members of the vector multiplet (the gauge boson \( v_\mu \) and the gaugino \( \lambda \)) necessarily are in the same representation of the gauge group, i.e. in the adjoint representation. The chiral fields can be in any representation of the gauge group, e.g. in the fundamental one. The non-abelian generalisation of the susy gauge transformation is
\[ e^{2gV} \rightarrow e^{i\Lambda^f} e^{2gV} e^{-i\Lambda^f} \leftrightarrow e^{-2gV} \rightarrow e^{i\Lambda} e^{-2gV} e^{-i\Lambda^f} \] \quad (2.24)
with Λ a chiral superfield and $g$ being the gauge coupling constant. This transformation can again be used to set $\chi, C, M, N$ and one component of $v_\mu$ to zero, resulting in the same component expansion (2.11) of $V$ in the Wess-Zumino gauge. From now on we adopt this WZ gauge. Then $V^n = 0, n \geq 3$. The same remains true if some $D_\alpha$ or $\overline{D}_\dot{\alpha}$ are inserted in the product, e.g. $V(D_\alpha V)V = 0$. One then simply has $e^{2gV} = 1 + 2gV + 2g^2V^2$. The superfields $W_\alpha$ are now defined as

$$W_\alpha = -\frac{1}{4} DD\left(e^{-2gV} D_\alpha e^{2gV}\right), \quad \overline{W}_\dot{\alpha} = +\frac{1}{4} DD\left(e^{2gV} \overline{D}_{\dot{\alpha}} e^{-2gV}\right),$$

which to first order in $V$ reduces to the abelian definition. The $W_\alpha$ now transform covariantly under the susy gauge transformations. The component expansion of $W_\alpha$ in WZ gauge is given by

$$\frac{1}{2g} W_\alpha = -i\lambda_\alpha(y) + \theta_\alpha D(y) + i(\sigma^{\mu\nu}\theta)_{\alpha} F_{\mu\nu}(y) + \theta(\sigma^\mu D_\mu \overline{\lambda}(y))_{\alpha}$$

where now

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu - ig[v_\mu, v_\nu]$$

and

$$D_\mu \overline{\lambda} = \partial_\mu \overline{\lambda} - ig[v_\mu, \overline{\lambda}].$$

The reader should not confuse the gauge covariant derivative $D_\mu$ neither with the super covariant derivatives $D_\alpha$ and $\overline{D}_{\dot{\alpha}}$, nor with the auxiliary field $D$.

The generators $T^a$ of the gauge group $G$ satisfy

$$[T^a, T^b] = if^{abc} T^c$$

with real structure constants $f^{abc}$. The field strength then is $F_{\mu\nu}^a = \partial_\mu v^a_\nu - \partial_\nu v^a_\mu + gf^{abc} v^b_\mu v^c_\nu$ and the gauge covariant derivative is $(D_\mu \lambda)^a = \partial_\mu \lambda^a + gf^{abc} v^b_\mu \lambda^c$. One then introduces the complex coupling constant

$$\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}$$

where $\Theta$ stands for the $\Theta$-angle. (We use a capital $\Theta$ to avoid confusion with the superspace coordinates $\theta$.) Then

$$\mathcal{L}_{\text{gauge}} = \frac{1}{32\pi} \text{Im} \left( \tau \int d^2 \theta \text{ Tr } W^a W_a \right) = \text{ Tr } \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda\sigma^\mu D_\mu \overline{\lambda} + \frac{1}{2} D^2 \right) + \frac{\Theta}{32\pi^2} g^2 \text{ Tr } F_{\mu\nu} \tilde{F}^{\mu\nu}$$

where

$$\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$$

is the dual field strength. The single term $\text{ Tr } W^a W_a$ has produced both, the conventionally normalised gauge kinetic term $-\frac{1}{4} \text{ Tr } F_{\mu\nu} F^{\mu\nu}$ and the instanton density $\frac{\tau}{32\pi^2} \text{ Tr } F_{\mu\nu} \tilde{F}^{\mu\nu}$ which multiplies the $\Theta$-angle!

We now add chiral (matter) multiplets $\phi^i$ transforming in some representation $R$ of the gauge group where the generators are represented by matrices $(T^a_R)^i_j$. Then

$$\phi^i \rightarrow (e^{i\Lambda})^i_j \phi^j, \quad \phi^i_\dagger \rightarrow \phi^j_\dagger \left(e^{-i\Lambda}\right)^j_i$$

9
or simply $\phi \to e^{i\Lambda} \phi$, $\phi^\dagger \to \phi^\dagger e^{-i\Lambda^\dagger}$ where $\Lambda = \Lambda^a T^a_R$ is understood. Then
\[
\phi^\dagger e^{2gV} \phi \equiv \phi^\dagger e^{2gV^w T^w_R} \phi \equiv \phi^\dagger_1 \left(e^{2gV}\right)^i_j \phi^j
\] (2.34)
is the gauge invariant generalisation of the kinetic term and
\[
\mathcal{L}_{\text{matter}} = \int d^2\theta d^2\bar{\theta} \, \phi^\dagger e^{2gV} \phi + \int d^2\theta \, W(\phi) + \int d^2\bar{\theta} \, [W(\phi)]^\dagger.
\] (2.35)
Working out the relevant superspace components yields
\[
\phi^\dagger e^{2gV} \phi \bigg|_{\theta \theta} = \left( D_\mu z \right)^\dagger D^\mu z - i\psi^\dagger \sigma^\mu D_\mu \bar{\psi} + f^\dagger f
\]
\[
+ i\sqrt{2}g z^\dagger \lambda \psi - i\sqrt{2}g \bar{\psi} \lambda z + g z^\dagger Dz + \text{total derivative}.
\] (2.36)
now with $D_\mu z = \partial_\mu z - ig \bar{\nu}^a T^a_R z$ and $D_\mu \psi = \partial_\mu \psi - ig \bar{\nu}^a T^a_R \psi$. This part of the Lagrangian contains the kinetic terms for the scalar fields $z^i$ and the matter fermions $\psi^j$, as well as specific interactions between the $z^i$, the $\psi^j$ and the gauginos $\lambda^a$. One has e.g. $z^\dagger (T^a_R)^\dagger \lambda^a \psi^j$. What happens to the superpotential $W(\phi)$? This must be a chiral superfield and hence must be constructed from the $\phi^\dagger$ alone. It must also be gauge invariant which imposes severe constraints on the superpotential. For the special case of $\mathcal{N} = 2$ e.g. it will turn out that no non-trivial superpotential is allowed. There is a last type of term that may appear in case the gauge group simply is $U(1)$ or contains $U(1)$ factors. These are the Fayet-Iliopoulos terms. Since we will be mainly interested in groups without $U(1)$ factors we will not discuss them here.

We can finally write the full $\mathcal{N} = 1$ Lagrangian, being the sum of (2.34) and (2.36):
\[
\mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}}
\]
\[
= \frac{1}{32\pi} \text{Im} \left( \tau \int d^2\theta \, \text{Tr} \, W^a W_a \right) + \int d^2\theta d^2\bar{\theta} \, \phi^\dagger e^{2gV} \phi + \int d^2\theta \, W(\phi) + \int d^2\bar{\theta} \, [W(\phi)]^\dagger
\]
\[
= \text{Tr} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda \sigma^\mu D_\mu \bar{\lambda} + \frac{1}{2} D^2 \right) + \frac{\theta^{2a}}{32\pi g^2} \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu}
\]
\[
+ \left( D_\mu z \right)^\dagger D^\mu z - i\psi^\dagger \sigma^\mu D_\mu \bar{\psi} + f^\dagger f + i\sqrt{2}g z^\dagger \lambda \psi - i\sqrt{2}g \bar{\psi} \lambda z + g z^\dagger Dz
\]
\[
- \frac{\partial W}{\partial z^i} f^i + h.c. - \frac{\partial^a W}{2 \partial z^i \partial z^j} \psi^i \psi^j + h.c. + \text{total derivative}.
\] (2.37)
The auxiliary field equations of motion are
\[
f^i_1 = \frac{\partial W}{\partial z^i}, \quad D^a = -g z^\dagger T^a z.
\] (2.38)
Substituting this back into the Lagrangian one finds
\[
\mathcal{L} = \text{Tr} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - i\lambda \sigma^\mu D_\mu \bar{\lambda} \right) + \frac{\theta^{2a}}{32\pi g^2} \text{Tr} F_{\mu\nu} \tilde{F}^{\mu\nu} + \left( D_\mu z \right)^\dagger D^\mu z - i\psi^\dagger \sigma^\mu D_\mu \bar{\psi}
\]
\[
+ i\sqrt{2}g z^\dagger \lambda \psi - i\sqrt{2}g \bar{\psi} \lambda z - \frac{1}{2} \frac{\partial^a W}{\partial z^i \partial z^j} \psi^i \psi^j - \frac{1}{2} \left( \frac{\partial^a W}{\partial z^i \partial z^j} \right)^\dagger \psi^i \psi^j - V(z^\dagger, z) + \text{total derivative},
\] (2.39)
where the scalar potential $V(z^\dagger, z)$ is given by
\[
V(z^\dagger, z) = f^\dagger f + \frac{1}{2} D^2 = \sum_i \left| \frac{\partial W}{\partial z^i} \right|^2 + \frac{g^2}{2} \sum_a \left| z^\dagger T^a z \right|^2.
\] (2.40)
2.3 \( \mathcal{N} = 1 \) susy non-linear sigma model

As long as one wants to formulate a fundamental, i.e. microscopic theory, one is guided by the principle of renomalisability. For a gauge theory this is quite restrictive and the only possibility is the YM theory formulated above. The only freedom lies in the choice of gauge group and matter content, i.e. the number of chiral multiplets and the representations of the gauge group under which they transform. Special choices will lead to extended supersymmetry, in particular \( \mathcal{N} = 2 \) susy which will be discussed below. There is also some freedom to choose the gauge invariant superpotential. Different choices will lead to different masses and Yukawa interactions.

In many cases, however, the theory one considers is an effective theory, valid at low energies only. Then renormalisability no longer is a criterion. The only restriction for such a low-energy effective theory is to contain no more than two (space-time) derivatives. Higher derivative terms are irrelevant at low energies. Thus we are led to study the supersymmetric non-linear sigma model. Another motivation comes from supergravity which is not renormalisable anyway. We will first consider the model for chiral multiplets only, and then extend the resulting theory to a gauge invariant one.

Chiral multiplets only

We start with the action

\[
S = \int d^4x \left( \int d^2\theta d^2\bar{\theta} \ K(\phi^i, \phi^i) + \int d^2\theta \ w(\phi^i) + \int d^2\bar{\theta} \ w^\dagger(\phi^i) \right). \tag{2.41}
\]

We have denoted the superpotential by \( w \) rather than \( W \). The function \( K(\phi^i, \phi^i) \) must be a real superfield, which will be the case if \( \mathcal{K}(z^i, \bar{z}^\dagger_j) = K(z^i, \bar{z}^\dagger_j) \). Derivatives with respect to its arguments will be denoted as

\[
K_i = \frac{\partial}{\partial z^i} K(z, \bar{z}^\dagger) \quad , \quad K^j = \frac{\partial}{\partial \bar{z}^j} K(z, \bar{z}^\dagger) \quad , \quad K_{ij} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} K(z, \bar{z}^\dagger) \tag{2.42}
\]

e tc. and similarly \( w_i = \frac{\partial}{\partial z^i} w(z) \), \( w_{ij} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} w(z) \) and \( w^i = [w_i]^\dagger \), \( w^{ij} = [w_{ij}]^\dagger \).

One has to expand the various terms in (2.41) and pick out the \( \theta \theta \bar{\theta} \bar{\theta} \) terms or the \( \theta \theta \) or \( \bar{\theta} \bar{\theta} \) terms. This is quite tedious and we refer to \[10\] for details. The result is

\[
\int d^2\theta \ w(\phi^i) + h.c. = \left( -w_i f^i - \frac{1}{2} w_{ij} \psi^i \psi^j \right) + h.c. \tag{2.43}
\]

and

\[
\int d^2\theta d^2\bar{\theta} \ K(\phi^i, \phi^i) = K^j_i \left( f^i f_j + \partial_\mu z^i \partial^\mu \bar{z}^j - \frac{i}{2} \psi^i \sigma^\mu \partial_\mu \bar{\psi}^j + \frac{i}{4} \partial_\mu \psi^i \sigma^\mu \bar{\psi}^j \right) + \frac{i}{4} K^k_{ij} \left( \psi^i \sigma^\mu \bar{\psi}^j \partial_\mu z^k + \psi^j \sigma^\mu \bar{\psi}^i \partial_\mu z^k - 2i \psi^i \psi^j f_k^i \right) + h.c. \tag{2.44}
\]

where the last term is a total derivative and hence can be dropped from the Lagrangian. Note that after discarding this total derivative, (2.44) no longer contains the “purely holomorphic” terms \( \sim K_{ij} \) or the “purely antiholomorphic” terms \( \sim K_{ij}^\dagger \). Only the mixed terms with at least one upper and one lower index remain. This shows that the transformation

\[
K(z, \bar{z}^\dagger) \to K(z, \bar{z}^\dagger) + g(z) + \bar{g}(z^\dagger) \tag{2.45}
\]
does not affect the Lagrangian. Moreover, the metric of the kinetic terms for the complex scalars is

\[ K_i^j = \frac{\partial^2}{\partial z^i \partial \overline{z}_j} K(z, z^\dagger). \]  

(2.46)

A metric like this obtained from a complex scalar function is called a Kähler metric, and the scalar function \( K(z, z^\dagger) \) the Kähler potential. The metric is invariant under Kähler transformations (2.43) of this potential. Thus one is led to interpret the complex scalars \( z^i \) as (local) complex coordinates on a Kähler manifold, i.e., the target manifold of the sigma-model is Kähler. The Kähler invariance (2.45) actually generalizes to the superfield level since

\[ K(\phi, \phi^\dagger) \rightarrow K(\phi, \phi^\dagger) + g(\phi) + \overline{g}(\phi^\dagger) \]  

(2.47)

does not affect the resulting action because \( g(\phi) \) is again a chiral superfield and its \( \theta \theta \overline{\theta} \theta \) component is a total derivative, see (2.8), hence \( \int d^2 \theta d^2 \overline{\theta} g(\phi) = \int d^2 \theta d^2 \overline{\theta} \overline{g}(\phi^\dagger) = 0. \)

Once \( K_i^j \) is interpreted as a metric it is straightforward to compute the affine connection and curvature tensor. They are given by

\[ \Gamma_{ij}^l = (K^{-1})_k^l K_i^k, \quad \Gamma_{li}^j = (K^{-1})_l^k K_i^j, \quad R_{ij}^{kl} = K_i^{kl} - K_i^{kn} (K^{-1})_m^k K_m^n. \]  

(2.48)

This allows us to rewrite various terms in the Lagrangian in a simpler and more geometric form. Define “Kähler covariant” derivatives of the fermions as

\[ D_\mu \psi^i = \partial_\mu \psi^i + \Gamma_{jk}^i \partial_\mu z^j \psi^k = \partial_\mu \psi^i + (K^{-1})_i^j K_j^k \partial_\mu z^j \psi^k \]

\[ D_\mu \overline{\psi}_j = \partial_\mu \overline{\psi}_j + \Gamma_{ij}^k \partial_\mu z^k \overline{\psi}_i = \partial_\mu \overline{\psi}_j + (K^{-1})_j^i K_i^k \partial_\mu z^k \overline{\psi}_i. \]  

(2.49)

The fermion bilinears in (2.44) then precisely are \( \frac{i}{2} K_i^j D_\mu \psi^i \sigma^\mu \overline{\psi}_j + h.c. \). The four fermion term is \( K_{ij}^{kl} \psi^i \psi^j \overline{\psi}_k \overline{\psi}_l \). The full curvature tensor will appear after we eliminate the auxiliary fields \( f^i \). To do this, we add the two pieces (2.44) and (2.43) of the Lagrangian to see that the auxiliary field equations of motion are

\[ f^i = (K^{-1})_j^i w^j - \frac{1}{2} \Gamma_{jk}^i \psi^j \psi^k. \]  

(2.50)

Substituting back into the sum of (2.44) and (2.43) we finally get the Lagrangian

\[ \int d^4 x \left[ \int d^2 \theta d^2 \overline{\theta} K(\phi, \phi^\dagger) + \int d^2 \theta w(\phi) + \int d^2 \overline{\theta} [w(\phi)]^\dagger \right] \]

\[ = \int d^4 x \left[ K_i^j \left( \partial_\mu z^i \partial^\mu \overline{z}_j + \frac{1}{2} D_\mu \psi^i \sigma^\mu \overline{\psi}_j - \frac{i}{2} \psi^i \sigma^\mu D_\mu \overline{\psi}_j \right) - (K^{-1})_j^i w_i w^j \right. \]

\[ - \left. \frac{1}{2} \left( w_{ij} - \Gamma_{ij}^k w_k \right) \psi^i \psi^j - \frac{1}{2} \left( w^{ij} - \Gamma_{ij}^k w^k \right) \overline{\psi}_i \overline{\psi}_j + \frac{1}{4} R_{ij}^{kl} \psi^i \psi^j \overline{\psi}_k \overline{\psi}_l \right]. \]  

(2.51)

Including gauge fields

The inclusion of gauge fields changes two things. First, the kinetic term \( K(\phi, \phi^\dagger) \) has to be modified so that, among others, all derivatives \( \partial_\mu \) are turned into gauge covariant derivatives as we did in the
previous subsection when we replaced $\phi^\dagger \phi$ by $\phi^\dagger e^{2gV} \phi$. Second, one has to add kinetic terms for the
gauge multiplet $V$. In the spirit of the $\sigma$-model, one will allow a susy Lagrangian leading to terms
of the form $f_{ab}(z)F^{a\mu\nu}_\mu F^{b\mu\nu}$ etc.

Let’s discuss the matter Lagrangian first. Since

$$\phi \rightarrow e^{i\Lambda} \phi, \quad \phi^\dagger \rightarrow \phi^\dagger e^{-i\Lambda^\dagger}, \quad e^{2gV} \rightarrow e^{i\Lambda^\dagger} e^{2gV} e^{-i\Lambda}$$

one sees that

$$\phi^\dagger e^{2gV} \rightarrow \phi^\dagger e^{2gV} e^{-i\Lambda}.$$  

(2.52)

Then the combination $(\phi^\dagger e^{2gV})_i \phi^i$ is gauge invariant and the same is true for any real (globally)
$G$-invariant function $K(\phi^i, \phi^i)$ if the argument $\phi^i$ is replaced by $(\phi^\dagger e^{2gV})_i$. We conclude that if $w(\phi^i)$
is a $G$-invariant function of the $\phi^i$, i.e. if

$$w_i(T^a)_i^j \phi^j = 0 \quad a = 1, \ldots \text{dim}G$$

(2.54)

then

$$\mathcal{L}_{\text{matter}} = \int d^2 \theta d^2 \overline{\theta} \ K \left( \phi^i, (\phi^\dagger e^{2gV})_i \right) + \int d^2 \theta \ w(\phi^i) + \int d^2 \overline{\theta} \ [w(\phi^i)]^\dagger$$

is supersymmetric and gauge invariant. To obtain the component expansion again is a bit lengthy.
The result is

$$\mathcal{L}_{\text{matter}} = K^j_i \left[ f_j^i f_j^i + (D_\mu z)^i_j (D^\mu z)_j^i - \frac{i}{2} \overline{\psi}^i \sigma^\mu \overline{D}_\mu \psi_j^i + \frac{i}{2} \overline{\psi}^i \sigma^\mu \psi_j^i \right]$$

$$+ \ \frac{1}{2} K^k_j \overline{\psi}^i \psi^j f_k^i + \text{h.c.} + \frac{1}{4} K^k_l \overline{\psi}^i \psi^j \overline{\psi}^k \psi^l$$

$$- \ \left( w_i f^i + \frac{1}{2} w_i \overline{\psi}^i \psi^j \right) + \text{h.c.}$$

$$+ \ i \sqrt{2} g K^j_i z_i^j \lambda \psi^j - \ i \sqrt{2} g K^j_i \overline{\psi}^i \lambda z^j + g z_i^j DK_i^j.$$  

(2.56)

Here all gauge indices have been suppressed, e.g. $z_i^j \equiv z_i^j T_R^j z^j X^a \equiv (z_i^j)_M (T_R^j)_M^N (z^j)^N X^a$ where

$(T_R^1)_M^N$ are the matrices of the representation carried by the matter fields $(z^j)^N$ and $(\lambda^j)^N$. The
derivatives $\overline{D}_\mu$ acting on the fermions are gauge and Kähler covariant, i.e.

$$\overline{D}_\mu \psi^i = \partial_\mu \psi^i - i g v^a_\mu T^a_R \psi^i + \Gamma^i_{jk} \partial_\mu z^j \psi^k$$

$$\overline{D}_\mu \overline{\psi}_j = \partial_\mu \overline{\psi}_j - i g v^a_\mu T^a_R \overline{\psi}_j + \Gamma^k_{ij} \partial_\mu z^k \overline{\psi}_i.$$  

(2.57)

To discuss the generalisation of the gauge kinetic Lagrangian (2.31), recall that $W_\alpha$ is defined by
(2.25) and in WZ gauge it reduces to (2.20). Note that any power of $W$ never contains more than two
derivatives, so we could consider a susy Lagrangian of the form $\int d^2 \theta H(\phi^i, W_\alpha)$ with an arbitrary
$G$-invariant function $H$. We will be slightly less general and take

$$\mathcal{L}_{\text{gauge}} = \frac{1}{16 g^2} \int d^2 \theta \ f_{ab}(\phi^i) W^{\alpha a} W^b_\alpha + \text{h.c.}$$  

(2.58)

with $f_{ab} = f_{ba}$ transforming under $G$ as the symmetric product of the adjoint representation with itself. To get back the standard Lagrangian (2.31) one only needs to take $\frac{1}{g^2} f_{ab} = \frac{1}{4 \pi g^2} \text{Tr} T^a T^b$, so
that \( f_{ab} \) is identified with a matrix of generalised effective coupling constants. Expanding (2.58) in components is straightforward and yields

\[
\mathcal{L}_{\text{gauge}} = \text{Re} f_{ab}(z) \left( -\frac{1}{4} F_{\mu
u}^a F^{b\mu\nu} - i \lambda^a \sigma^\mu D_\mu \bar{\chi}^b + \frac{1}{2} D^a D^b \right) - \frac{1}{4} \text{Im} f_{ab}(z) F_{\mu
u}^a \bar{F}^{b\mu\nu} \\
+ \frac{1}{2} f_{ab,i}(z) \left( \sqrt{2} i \bar{\psi}_i \chi^b D^b - \sqrt{2} \lambda^a \sigma_{\mu\nu} \psi^i F_{\mu\nu}^b + \lambda^a \lambda^b f^i \right) + h.c. \quad (2.59)
\]

where \( f_{ab,i} = \frac{\partial}{\partial z} f_{ab} (z) \) etc.

The full Lagrangian is given by \( \mathcal{L} = \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}} \). The auxiliary field equations of motion are

\[
f^i = (K^{-1})^i_j \left( w^j - \frac{1}{2} K^j_{kl} \psi^k \psi^l - \frac{1}{4} (f_{ab,j})^i_{\bar{k}} \bar{\chi}^k \right) \\
D^a = -(\text{Re} f}_{ab}^{-1} \left( g z^i T^b K^i + \frac{1}{2\sqrt{2}} f_{bc,i} \psi^i \lambda^c - \frac{1}{2\sqrt{2}} (f_{bc,i})^\dagger \psi_i \bar{\chi} \right) . \quad (2.60)
\]

It is straightforward to substitute this into the Lagrangian \( \mathcal{L} \) and we will not write the result explicitly. Let us only note that the scalar potential is given by

\[
V(z, z^\dagger) = (K^{-1})^i_j w_i w^j + \frac{g^2}{2} (\text{Re} f}_{ab}^{-1} (z^i \bar{T}^b K^i)(z^i \bar{T}^b K^i) . \quad (2.61)
\]

### 2.4 \( \mathcal{N} = 2 \) susy gauge theories

The \( \mathcal{N} = 2 \) multiplets with helicities not exceeding one are the massless \( \mathcal{N} = 2 \) vector multiplet and the hypermultiplet. The former contains an \( \mathcal{N} = 1 \) vector multiplet and an \( \mathcal{N} = 1 \) chiral multiplet, altogether a gauge boson, two Weyl fermions and a complex scalar, while the hypermultiplet contains two \( \mathcal{N} = 1 \) chiral multiplets. The \( \mathcal{N} = 2 \) vector multiplet is necessarily massless while the hypermultiplet can be massless or be a short (BPS) massive multiplet. In this section we will concentrate on the \( \mathcal{N} = 2 \) vector multiplet. The \( \mathcal{N} = 2 \) susy algebra is

\[
\{Q^I_\alpha, \overline{Q}^I_{\dot{\alpha}}\} = 2 \sigma^\mu_{\alpha\dot{\alpha}} P_\mu g^{IJ} , \quad (2.62)
\]

\[
\{Q^I_\alpha, Q^J_\beta\} = 2 \sqrt{2} \epsilon_{\alpha\beta} e^{IJ} Z , \quad (2.63)
\]

\[
\{\overline{Q}^I_{\dot{\alpha}}, \overline{Q}^J_{\dot{\beta}}\} = 2 \sqrt{2} \epsilon_{\dot{\alpha}\dot{\beta}} e^{IJ} Z^* . \quad (2.64)
\]

In order to construct susy multiplets one combines these susy generators into fermionic harmonic oscillator operators. Positivity of the corresponding Hilbert space then requires

\[
m \geq \sqrt{2} |Z| . \quad (2.65)
\]

If the bound is satisfied, there are 2 combinations out of the 4 susy generators that yield zero norm states. Hence these combinations should be set to zero, and effectively we are left with only half the susy generators. Thus if the bound is satisfied we have short multiplets with only four helicity states, while otherwise we have long multiplets with 16 helicity states. A short multiplet is called
a BPS multiplet. For such a BPS state, the relation \( m = \sqrt{2} |Z| \) between the mass and the central charge is very powerful \([12]\), since it allows for an exact determination of the mass, once its central charge is known.

\[ \mathcal{N} = 2 \] super Yang-Mills

Given the decomposition of the \( \mathcal{N} = 2 \) vector multiplet into \( \mathcal{N} = 1 \) multiplets, we start with a Lagrangian being the sum of the \( \mathcal{N} = 1 \) gauge and matter Lagrangians \((2.31)\) and \((2.36)\). At present, however, all fields are in the same \( \mathcal{N} = 2 \) multiplet and hence must be in the same representation of the gauge group, namely the adjoint representation. The \( \mathcal{N} = 1 \) matter Lagrangian \((2.36)\) then becomes

\[
\mathcal{L}_{\text{matter}}^{\mathcal{N}=1} = \int d^2 \theta d^2 \bar{\theta} \ Tr \phi^i e^{2gV} \phi = Tr \left[ (D_\mu z)^\dagger D^\mu z - i \psi^a \sigma^\mu D_\mu \bar{\psi} + f^a f \right]
+ i \sqrt{2} g z^\dagger \{ \lambda, \psi \} - i \sqrt{2} g \{ \bar{\psi}, \lambda \} z + g D[z, z^\dagger] \tag{2.66}
\]

where now

\[
z = z^a T^a, \quad \psi = \psi^a T^a, \quad f = f^a T^a, \quad a = 1, \ldots, \dim G
\]

in addition to \( \lambda = \lambda^a T^a \), \( D = D^a T^a \), \( v_\mu = v_\mu^a T^a \). The commutators or anticommutators arise since the generators in the adjoint representation are given by

\[
(T_{ad})_{bc} = -if_{abc}
\]

and we normalise the generators by

\[
Tr T^a T^b = \delta^{ab}
\]

so that

\[
z^\dagger \lambda \psi \rightarrow z^\dagger_b \lambda^a (T_{ad})_{bc} \psi^c = -i z^\dagger_b \lambda^a f_{abc} \psi^c = i z^\dagger_b f_{bac} \lambda^a \psi^c
\]

\[
= z^\dagger_b \lambda^a \psi^c Tr T^b [T^a, T^c] = Tr z\dagger \{ \lambda, \psi \}
\]

and

\[
z^\dagger D z \rightarrow z^\dagger_b D^a (T_{ad})_{bc} z^c = -i f_{abc} z^\dagger_b D^a z^c = -Tr D[z^\dagger, z] = Tr D[z, z^\dagger]. \tag{2.71}
\]

We now add \((2.66)\) to the \( \mathcal{N} = 1 \) gauge lagrangian \( \mathcal{L}_{\text{gauge}}^{\mathcal{N}=1} \) and obtain

\[
\mathcal{L}_{\text{YM}}^{\mathcal{N}=2} = \frac{1}{32 \pi} \text{Im} \left( \tau \int d^2 \theta \ Tr W^a W_\alpha \right) + \int d^2 \theta d^2 \bar{\theta} \ Tr \phi^i e^{2gV} \phi
\]

\[
= Tr \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - i \lambda \sigma^\mu D_\mu \bar{\lambda} - i \psi^a \sigma^\mu D_\mu \bar{\psi} + (D_\mu z)^\dagger D^\mu z \right)
+ \frac{g}{32 \pi^2} \int d^2 \theta d^2 \bar{\theta} \ Tr F_{\mu \nu} \bar{F}^{\mu \nu} + \frac{1}{2} D^2 + f^a f^a \right.
\]

\[
+ i \sqrt{2} g z^\dagger \{ \lambda, \psi \} - i \sqrt{2} g \{ \bar{\psi}, \lambda \} z + g D[z, z^\dagger]. \tag{2.72}
\]

A necessary and sufficient condition for \( \mathcal{N} = 2 \) susy is the existence of an SU(2)_R symmetry that rotates the two supersymmetry generators \( Q^a_1 \) and \( Q^2_\alpha \) into each other. As follows from the construction of the supermultiplet in section 2, the same symmetry must act between the two fermionic fields \( \lambda \) and \( \psi \). Now the relative coefficients of \( \mathcal{L}_{\text{gauge}}^{\mathcal{N}=1} \) and \( \mathcal{L}_{\text{matter}}^{\mathcal{N}=1} \) in \((2.72)\) have been chosen precisely in such a way to have this SU(2)_R symmetry: the \( \lambda \) and \( \psi \) kinetic terms have the same coefficient, and
the Yukawa couplings \( z \{ \lambda, \psi \} \) and \( \{ \overline{\psi}, \overline{\lambda} \} z \) also exhibit this symmetry. The Lagrangian (2.72) is indeed \( \mathcal{N} = 2 \) supersymmetric.

Note that we have not added a superpotential. Such a term (unless linear in \( \phi \)) would break the \( \text{SU}(2)_R \) invariance and not lead to an \( \mathcal{N} = 2 \) theory.

The auxiliary field equations of motion are simply

\[
 f^a = 0 \quad , \quad D^a = -g [z, z^\dagger]^a
\]

leading to a scalar potential

\[
 V(z, z^\dagger) = \frac{1}{2} g^2 \text{Tr} \left( [z, z^\dagger] \right)^2 .
\]

This scalar potential is fixed and a consequence solely of the auxiliary \( D \)-field of the \( \mathcal{N} = 1 \) gauge multiplet.

**Effective \( \mathcal{N} = 2 \) gauge theories**

The above \( \mathcal{N} = 2 \) super YM theory is renormalisable and constitutes the asymptotically free microscopic theory we want to study below. However, we will be even more interested in studying the effective low-energy action for the light degrees of freedom. As discussed above for the non-linear \( \sigma \)-model, if one considers effective theories, disregarding renormalisability, one may allow more general gauge and matter kinetic terms and start with an appropriate sum of (2.55) (with \( w(\phi^i) = 0 \)) and (2.58). It is clear however that the functions \( f_{ab} \) cannot be independent from the Kähler potential \( K \). Indeed, the \( \text{SU}(2)_R \) symmetry equates \( \text{Re} f_{ab} \) with the Kähler metric \( K^a_b \). It turns out that this requires the following identification

\[
 \frac{16\pi}{(2g)^2} f_{ab}(z) = -i \frac{\partial^2}{\partial z_a \partial z_b} \mathcal{F}(z) \equiv -i \mathcal{F}_{ab}(z)
\]

\[
 \frac{16\pi}{(2g)^2} K(z, z^\dagger) = -i z_a^\dagger \frac{\partial}{\partial z_a} \mathcal{F}(z) + \text{h.c.} \equiv -\frac{i}{2} z_a^\dagger \mathcal{F}_a(z) + i \frac{1}{2} [\mathcal{F}_a(z)]^\dagger z_a
\]

where the holomorphic function \( \mathcal{F}(z) \) is called the \( \mathcal{N} = 2 \) prepotential. We have pulled out a factor \( \frac{16\pi}{(2g)^2} \) for later convenience. Also, we again absorb the factor \( 2g \) into the normalisation of the field. This makes sense since \( \text{Im} \mathcal{F}_{ab} \) will play the role of an effective generalised coupling. Hence we set

\[
 2g = 1 .
\]

Then the full general \( \mathcal{N} = 2 \) Lagrangian is

\[
 \mathcal{L}_{\text{eff}}^{\mathcal{N}=2} = \frac{1}{16\pi} \int d^2 \theta \mathcal{F}_{ab}(\phi) W^{\alpha \alpha} W^b_{\alpha} + \frac{1}{32\pi i} \int d^2 \theta d^2 \theta \left( \phi^i e^V \right)^a \mathcal{F}_a(\phi) + \text{h.c.}
\]

\[
 = \frac{1}{16\pi} \text{Im} \left[ \frac{1}{2 i} \int d^2 \theta \mathcal{F}_{ab}(\phi) W^{\alpha \alpha} W^b_{\alpha} + \int d^2 \theta d^2 \theta \left( \phi^i e^V \right)^a \mathcal{F}_a(\phi) \right] .
\]

Note that with the Kähler potential \( K \) given by (2.73), the Kähler metric is proportional to \( \text{Im} \mathcal{F}_{ab} \) as required by \( \text{SU}(2)_R \):

\[
 K^b_a = \frac{1}{16\pi} \text{Im} \mathcal{F}_{ab} = \frac{1}{32\pi i} \left( \mathcal{F}_{ab} - \mathcal{F}_{ab}^\dagger \right) .
\]

The component expansion follows from the results of the previous section on the non-linear \( \sigma \)-model, using the identifications (2.73) and (2.78), and taking vanishing superpotential \( w(\phi) \). In particular, the scalar potential is given by (cf (2.61))

\[
 V(z, z^\dagger) = -\frac{1}{2\pi} (\text{Im} \mathcal{F})_{ab}^{-1} [z^\dagger, \mathcal{F}_c(z) T^c]^a [z^\dagger, \mathcal{F}_d(z) T^d]_b .
\]
Let us insist that the full effective $\mathcal{N} = 2$ action written in (2.77) is determined by a single holomorphic function $\mathcal{F}(z)$. Holomorphicity will turn out to be a very strong requirement. Finally note that $\mathcal{F}(z) = \frac{1}{2} \tau \text{Tr} z^2$ gives back the standard Yang-Mills Lagrangian (2.72).

3 Seiberg-Witten duality in $\mathcal{N} = 2$ susy SU(2) gauge theory

3.1 Low-energy effective action of $\mathcal{N} = 2$ $SU(2)$ YM theory

Following Seiberg and Witten [1] we want to study and determine the low-energy effective action of the $\mathcal{N} = 2$ susy Yang-Mills theory with gauge group SU(2). The latter theory is the microscopic theory which controls the high-energy behaviour. It was discussed in the previous subsection and its Lagrangian is given by (2.72). This theory is renormalisable and well-known to be asymptotically free. The low-energy effective action will turn out to be quite different. Generalisations to bigger gauge groups have been extensively discussed in the literature, see e.g. [13, 14], but here we will restrict ourselves to the simplest case of SU(2). For other reviews, see e.g. [12].

Low-energy effective actions

There are two types of effective actions. One is the standard generating functional $\Gamma[\varphi]$ of one-particle irreducible Feynman diagrams (vertex functions). It is obtained from the standard renormalised generating functional $W[\varphi]$ of connected diagrams by a Legendre transformation. Momentum integrations in loop-diagrams are from zero up to a UV-cutoff which is taken to infinity after renormalisation. $\Gamma[\varphi] \equiv \Gamma[\mu, \varphi]$ also depends on the scale $\mu$ used to define the renormalized vertex functions.

A quite different object is the Wilsonian effective action $S_W[\mu, \varphi]$. It is defined as $\Gamma[\mu, \varphi]$, except that all loop-momenta are only integrated down to $\mu$ which serves as an infra-red cutoff. In theories with massive particles only, there is no big difference between $S_W[\mu, \varphi]$ and $\Gamma[\mu, \varphi]$ (as long as $\mu$ is less than the smallest mass). When massless particles are present, as is the case for gauge theories, the situation is different. In particular, in supersymmetric gauge theories there is the so-called Konishi anomaly which can be viewed as an IR-effect. Although $S_W[\mu, \varphi]$ depends holomorphically on $\mu$, this is not the case for $\Gamma[\mu, \varphi]$ due to this anomaly.

The SU(2) case, moduli space

We want to determine the Wilsonian effective action in the case where the microscopic theory is the SU(2), $\mathcal{N} = 2$ super Yang-Mills theory. As explained above, classically this theory has a scalar potential $V(z) = \frac{1}{2} g^2 \text{tr} ([z^\dagger, z])^2$ as given in (2.74). Unbroken susy requires that $V(z) = 0$ in the vacuum, but this still leaves the possibilities of a vacuum with non-vanishing $z$ provided $[z^\dagger, z] = 0$. We are interested in determining the gauge inequivalent vacua. A general $z$ is of the form $z(x) = \frac{1}{2} \sum_{j=1}^3 (a_j(x) + ib_j(x)) \sigma_j$ with real fields $a_j(x)$ and $b_j(x)$ (where I assume that not all three $a_j$ vanish, otherwise exchange the roles of the $a_j$’s and $b_j$’s in the sequel). By a SU(2) gauge transformation one can always arrange $a_1(x) = a_2(x) = 0$. Then $[z, z^\dagger] = 0$ implies $b_1(x) = b_2(x) = 0$ and hence, with $a = a_3 + ib_3$, one has $z = \frac{1}{2} a \sigma_3$. Obviously, in the vacuum $a$ must be a constant. Gauge transformation from the Weyl group (i.e. rotations by $\pi$ around the 1- or 2-axis of SU(2)) can still change $a \rightarrow -a$, so $a$ and $-a$ are gauge equivalent, too. The gauge invariant quantity describing inequivalent vacua is $\frac{1}{2} a^2$, or $\text{tr} z^2$, which is the same, semiclassically. When quantum fluctuations are important this is no longer so. In the sequel, we will use the following definitions for $a$ and $u$:

$$u = \langle \text{tr} z^2 \rangle, \quad \langle z \rangle = \frac{1}{2} a \sigma_3.$$

(3.1)
The complex parameter $u$ labels gauge inequivalent vacua. The manifold of gauge inequivalent vacua is called the moduli space $\mathcal{M}$ of the theory. Hence $u$ is a coordinate on $\mathcal{M}$, and $\mathcal{M}$ is essentially the complex $u$-plane. We will see in the sequel that $\mathcal{M}$ has certain singularities, and the knowledge of the behaviour of the theory near the singularities will eventually allow the determination of the effective action $S_W$.

Clearly, for non-vanishing $\langle z \rangle$, the $SU(2)$ gauge symmetry is broken by the Higgs mechanism, since the $z$-kinetic term $|D_\mu z|^2$ generates masses for the gauge fields. With the above conventions, $\psi^b$, $b = 1, 2$ become massive with masses given by $\frac{1}{2} m^2 = g^2 |a|^2$, i.e. $m = \sqrt{2} g |a|$. Similarly due to the $z, \lambda, \psi$ interaction terms, $\psi^b, \lambda^b$, $b = 1, 2$ become massive with the same mass as the $\psi^b$, as required by supersymmetry. Obviously, $\psi^3, \psi^3$ and $\lambda^3$, as well as the mode of $z$ describing the fluctuation of $z$ in the $\sigma_3$-direction, remain massless. These massless modes are described by a Wilsonian low-energy effective action which has to be $N = 2$ supersymmetry invariant, since, although the gauge symmetry is broken, $SU(2) \rightarrow U(1)$, the $N = 2$ susy remains unbroken. Thus it must be of the general form (2.77) where the indices $a, b$ now take only a single value $(a, b = 3)$ and will be suppressed since the gauge group is $U(1)$. Also, in an abelian theory there is no self-coupling of the gauge boson and the same arguments extend to all members of the $N = 2$ susy multiplet: they do not carry electric charge. Thus for a $U(1)$-gauge theory, from (2.77) we simply get

$$\frac{1}{16\pi} \int d^4x \left[ \frac{1}{2} \left( \int d^2\theta \bar{F}''(\phi) W^\alpha W_\alpha + \int d^2\theta d^2\theta' \phi^a \bar{F}'(\phi) \right) \right]. \quad (3.2)$$

Metric on moduli space

As shown in (2.78), the Kähler metric of the present $\sigma$-model is given by $K_{\sigma\bar{\sigma}} = \frac{1}{16\pi} \text{Im} F''(z)$. By the same token this defines the metric in the space of (inequivalent) vacuum configurations, i.e. the metric on moduli space as ($\bar{a}$ denotes the complex conjugate of $a$)

$$ds^2 = \text{Im} F''(a)d\bar{a}d\bar{a} = \text{Im} \tau(a)d\bar{a} \quad (3.3)$$

where $\tau(a) = F''(a)$ is the effective (complexified) coupling constant according to the remark after eq. (2.58). The $\sigma$-model metric $K_{\sigma\bar{\sigma}}$ has been replaced on the moduli space $\mathcal{M}$ by $(16\pi$ times) its expectation value in the vacuum corresponding to the given point on $\mathcal{M}$, i.e. by $\text{Im} F''(a)$.

The question now is whether the description of the effective action in terms of the fields $\phi, W$ and the function $F$ is appropriate for all vacua, i.e. for all value of $u$, i.e. on all of moduli space. In particular the kinetic terms or what is the same, the metric on moduli space should be positive definite, translating into $\text{Im} \tau(a) > 0$. However, a simple argument shows that this cannot be the case: since $F(a)$ is holomorphic, $\text{Im} \tau(a) = \text{Im} \frac{\partial^2 F(a)}{\partial a^2}$ is a harmonic function and as such it cannot have a minimum, and hence (on the compactified complex plane) it cannot obey $\text{Im} \tau(a) > 0$ everywhere (unless it is a constant as in the classical case). The way out is to allow for different local descriptions: the coordinates $a, \bar{a}$ and the function $F(a)$ are appropriate only in a certain region of $\mathcal{M}$. When a singular point with $\text{Im} \tau(a) \rightarrow 0$ is approached one has to use a different set of coordinates $\bar{a}$ in which $\text{Im} \tau(\bar{a})$ is non-singular (and non-vanishing). This is possible provided the singularity of the metric is only a coordinate singularity, i.e. the kinetic terms of the effective action are not intrinsically singular, which will be the case.

Asymptotic freedom and the one-loop formula

18
Classically the function $\mathcal{F}(z)$ is given by $\frac{1}{2}\tau_{\text{class}} z^2$. The one-loop contribution has been determined by Seiberg [16]. The combined tree-level and one-loop result is

$$\mathcal{F}_{\text{pert}}(z) = \frac{i}{2\pi} z^2 \ln \frac{z^2}{\Lambda^2}.$$  

Here $\Lambda^2$ is some combination of $\mu^2$ and numerical factors chosen so as to fix the normalisation of $\mathcal{F}_{\text{pert}}$. Note that due to non-renormalisation theorems for $\mathcal{N} = 2$ susy there are no corrections from two or more loops to the Wilsonian effective action $S_W$ and (3.4) is the full perturbative result. There are however non-perturbative corrections that will be determined below.

For very large $a$ the dominant contribution when computing $S_W$ from the microscopic SU(2) gauge theory comes from regions of large momenta ($p \sim a$) where the microscopic theory is asymptotically free. Thus, as $a \to \infty$ the effective coupling constant goes to zero, and the perturbative expression (3.4) for $\mathcal{F}$ becomes an excellent approximation. Also $u \sim \frac{1}{2} a^2$ in this limit. Thus

$$\mathcal{F}(a) \sim \frac{i}{2\pi} a^2 \ln \frac{a^2}{\Lambda^2}$$

and

$$\tau(a) \sim \frac{i}{\pi} \left( \ln \frac{a^2}{\Lambda^2} + 3 \right) \quad \text{as} \quad u \to \infty.$$  

(3.5)

Note that due to the logarithm appearing at one-loop, $\tau(a)$ is a multi-valued function of $a^2 \sim 2u$. Its imaginary part, however, $\text{Im} \tau(a) \sim \frac{1}{\pi} \ln \frac{|a|^2}{\Lambda^2}$ is single-valued and positive (for $a^2 \to \infty$).

### 3.2 Duality

As already noted, $a$ and $\bar{a}$ do provide local coordinates on the moduli space $M$ for the region of large $u$. This means that in this region $\phi$ and $W^\alpha$ are appropriate fields to describe the low-energy effective action. As also noted, this description cannot be valid globally, since $\text{Im} \mathcal{F}''(\phi)$, being a harmonic function, must vanish somewhere, unless it is a constant - which it is not. Duality will provide a different set of (dual) fields $\phi_D$ and $W^\alpha_D$ that provide an appropriate description for a different region of the moduli space.

**Duality transformation**

Define a dual field $\phi_D$ and a dual function $\mathcal{F}_D(\phi_D)$ by

$$\phi_D = \mathcal{F}'(\phi), \quad \mathcal{F}'_D(\phi_D) = -\phi.$$  

(3.6)

These duality transformations simply constitute a Legendre transformation $\mathcal{F}_D(\phi_D) = \mathcal{F}(\phi) - \phi \phi_D$. Using these relations, the second term in the $\phi$ kinetic term of the action can be written as

$$\text{Im} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \phi^+ \mathcal{F}'(\phi) = \text{Im} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, (-\mathcal{F}'_D(\phi_D))^+ \phi_D$$

$$= \text{Im} \int d^4x \, d^2\theta \, d^2\bar{\theta} \, \phi_D \mathcal{F}'_D(\phi_D).$$  

(3.7)

We see that this second term in the effective action is invariant under the duality transformation.

Next, consider the $\mathcal{F}''(\phi)W^\alpha W_\alpha$-term in the effective action (3.2). While the duality transformation on $\phi$ is local, this will not be the case for the transformation of $W^\alpha$. Recall that $W$ contains

---

3 One can check from the explicit solution below that one indeed has $\frac{1}{2}u^2 - u = \mathcal{O}(1/u)$ as $u \to \infty$. 

---
the U(1) field strength $F_{\mu\nu}$. This $F_{\mu\nu}$ is not arbitrary but of the form $\partial_\mu v_\nu - \partial_\nu v_\mu$ for some $v_\mu$. This can be translated into the Bianchi identity $\frac{1}{2} \epsilon^{\mu
u\rho\sigma} \partial_\rho F_{\sigma\alpha} \equiv \partial_\mu \tilde{F}^{\mu\nu} = 0$. The corresponding constraint in superspace is $\text{Im} (D_\alpha W^\alpha) = 0$. In the functional integral one has the choice of integrating over $V$ only, or over $W^\alpha$ and imposing the constraint $\text{Im} (D_\alpha W^\alpha) = 0$ by a real Lagrange multiplier superfield which we call $V_D$:

$$\int D\mathcal{V} \exp \left[ \frac{i}{32\pi} \text{Im} \int d^4x \, d^2\theta \, F''(\phi) W^\alpha W_\alpha \right]$$

$$\simeq \int DWDV_D \exp \left[ \frac{i}{32\pi} \text{Im} \int d^4x \left( \int d^2\theta \, F''(\phi) W^\alpha W_\alpha + \frac{1}{2} \int d^2\theta \, d^2\bar{\theta} \, V_D D_\alpha W^\alpha \right) \right]$$

(3.8)

Observe that

$$\int d^2\theta \, d^2\bar{\theta} \, V_D D_\alpha W^\alpha = - \int d^2\theta \, d^2\bar{\theta} \, D_\alpha V_D W^\alpha = + \int d^2\theta \, \tilde{D}^2(D_\alpha V_D W^\alpha)$$

$$= \int d^2\theta \,(\tilde{D}^2D_\alpha V_D)W^\alpha = -4 \int d^2\theta \,(W_D)_\alpha W^\alpha$$

(3.9)

where we used $\tilde{D}_\beta W^\alpha = 0$ and where the dual $W_D$ is defined from $V_D$ by $(W_D)_\alpha = -\frac{1}{4} \tilde{D}^2 D_\alpha V_D$, as appropriate in the abelian case. Then one can do the functional integral over $W$ and one obtains

$$\int D\mathcal{V} \exp \left[ \frac{i}{16\pi} \text{Im} \int d^4x \, d^2\theta \left( -\frac{1}{F''(\phi)} W^\alpha W_\alpha \right) \right].$$

(3.10)

This reexpresses the ($N = 1$) supersymmetrized Yang-Mills action in terms of a dual Yang-Mills action with the effective coupling $\tau(a) = F''(a)$ replaced by $-\frac{1}{\tau(a)}$. Recall that $\tau(a) = \frac{\theta(a)}{2\pi} + \frac{4\pi i}{g^2(a)}$, so that $\tau \to -\frac{1}{\tau}$ generalizes the inversion of the coupling constant discussed in the introduction. Also, it can be shown that the replacement $W \to W_D$ corresponds to replacing $F_{\mu\nu} \to \tilde{F}_{\mu\nu}$, the electromagnetic dual, so that the manipulations leading to (3.10) constitute a duality transformation that generalizes the old electromagnetic duality of Montonen and Olive. Expressing the $-\frac{1}{F''(\phi)}$ in terms of $\phi_D$ one sees from (3.6) that $F''_D(\phi_D) = -\frac{d\phi_D}{d\phi_D} = -\frac{1}{F''(\phi)}$ so that

$$-\frac{1}{\tau(a)} = \tau_D(a_D).$$

(3.11)

The whole action can then equivalently be written as

$$\frac{1}{16\pi} \text{Im} \int d^4x \left[ \frac{1}{2} \int d^2\theta \, F''_D(\phi_D) W^\alpha W_\alpha + \int d^2\theta \, d^2\bar{\theta} \, \phi_D^+ \phi_D \right].$$

(3.12)

The duality group

To discuss the full group of duality transformations of the action it is most convenient to write it as

$$\frac{1}{16\pi} \text{Im} \int d^4x \, d^2\theta \, \frac{d\phi_D}{d\phi} W^\alpha W_\alpha + \frac{1}{32i\pi} \int d^4x \, d^2\theta \, d^2\bar{\theta} \left( \phi_D^+ \phi_D - \phi_D^+ \phi_D \right).$$

(3.13)

While we have shown in the previous subsection that there is a duality symmetry

$$\begin{pmatrix} \phi_D \\ \phi \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \phi_D \\ \phi \end{pmatrix},$$

(3.14)
the form (3.13) shows that there also is a symmetry
\[
\left( \begin{array}{c} \phi_D \\ \phi \end{array} \right) \rightarrow \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} \phi_D \\ \phi \end{array} \right), \quad b \in \mathbb{Z}. \tag{3.15}
\]
Indeed, in (3.13) the second term remains invariant since \(b\) is real, while the first term gets shifted by
\[
\frac{b}{16\pi} \text{Im} \int d^4x \: d^2\theta \: W^\alpha W_\alpha = -\frac{b}{16\pi} \int d^4x \: F_{\mu\nu} \tilde{F}^{\mu\nu} = -2\pi b\nu \tag{3.16}
\]
where \(\nu \in \mathbb{Z}\) is the instanton number. Since the action appears as \(e^{iS}\) in the functional integral, two actions differing only by \(2\pi\mathbb{Z}\) are equivalent, and we conclude that (3.15) with integer \(b\) is a symmetry of the effective action. The transformations (3.14) and (3.15) together generate the group \(Sl(2, \mathbb{Z})\). This is the group of duality symmetries.

Note that the metric (3.3) on moduli space can be written as
\[
ds^2 = \text{Im} \left( da_D d\bar{a} \right) = \frac{i}{2} \left( da_D D - da_D d\bar{a} \right) \tag{3.17}
\]
where \(a_D = \partial \mathcal{F}(a)/\partial a\), and that this metric obviously also is invariant under the duality group \(Sl(2, \mathbb{Z})\).

Monopoles, dyons and the BPS mass spectrum
At this point, I will have to add a couple of ingredients without much further justification and refer the reader to the literature for more details.

In a spontaneously broken gauge theory as the one we are considering, typically there are solitons (static, finite-energy solutions of the equations of motion) that carry magnetic charge and behave like non-singular magnetic monopoles (for a pedagogical treatment, see Coleman’s lectures [17]). The duality transformation (3.14) constructed above exchanges electric and magnetic degrees of freedom, hence electrically charged states, as would be described by hypermultiplets of our \(\mathcal{N} = 2\) supersymmetric version, with magnetic monopoles.

As for any theory with extended supersymmetry, there are long and short (BPS) multiplets in the present \(\mathcal{N} = 2\) theory. Small (or short) multiplets have 4 helicity states and large (or long) ones have 16 helicity states. As discussed earlier, massless states must be in short multiplets, while massive states are in short ones if they satisfy the BPS condition \(m^2 = 2|Z|^2\), or in long ones if \(m^2 > 2|Z|^2\). Here \(Z\) is the central charge of the \(\mathcal{N} = 2\) susy algebra. The states that become massive by the Higgs mechanism must be in short multiplets since they were before the symmetry breaking and the Higgs mechanism cannot generate the missing \(16 - 4 = 12\) helicity states. The heavy gauge bosons\(^4\) have masses \(m = \sqrt{2}|a| = \sqrt{2}|Z|\) and hence \(Z = a\). This generalises to all purely electrically charged states as \(Z = an_e\) where \(n_e\) is the (integer) electric charge. Duality then implies that a purely magnetically charged state has \(Z = a_D(-nm)\) where \(nm\) is the (integer) magnetic charge. A state with both types of charge, called a dyon, has \(Z = an_e - a_D nm\) since the central charge is additive. All this applies to states in short multiplets, so-called BPS-states. The mass formula for these states then is
\[
m^2 = 2|Z|^2, \quad Z = an_e - a_D nm = (n_e, nm) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} a_D \\ a \end{array} \right) = \eta \left( \begin{array}{c} n_e \\ nm \end{array} \right), \left( \begin{array}{c} a_D \\ a \end{array} \right) \tag{3.18}
\]
\(^4\) Again, to conform with the Seiberg-Witten normalisation, we have absorbed a factor of \(g\) into \(a\) and \(a_D\), so that the masses of the heavy gauge bosons now are \(m = \sqrt{2}|a|\) rather than \(\sqrt{2g}|a|\).
where $\eta$ is the standard symplectic product such that for any $Sl(2, \mathbb{Z}) \equiv Sp(2, \mathbb{Z})$ transformation $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ acting on $\begin{pmatrix} a_D \\ a \end{pmatrix}$ one has

$$
\eta \left( \begin{pmatrix} n_e \\ n_m \end{pmatrix}, M \begin{pmatrix} a_D \\ a \end{pmatrix} \right) = \eta \left( \begin{pmatrix} M^{-1} n_e \\ M^{-1} n_m \end{pmatrix}, \begin{pmatrix} a_D \\ a \end{pmatrix} \right) .
$$

(3.19)

It is then clear that under such an $Sl(2, \mathbb{Z})$ transformation $M$ the charge vector gets transformed to $M^{-1} (n_m, n_e) = (n'_m, n'_e)$ with integer $n'_e$ and $n'_m$. In particular, one sees again at the level of the charges that the transformation (3.14) exchanges purely electrically charged states with purely magnetically charged ones. It can be shown (section 4.1) that precisely those BPS states are stable for which $n_m$ and $n_e$ are relatively prime, i.e. for stable states $(n_m, n_e) \neq (qm, qn)$ for integer $m, n$ and $q \neq \pm 1$.

### 3.3 Singularities and Monodromy

In this section we will study the behaviour of $a(u)$ and $a_D(u)$ as $u$ varies on the moduli space $M$. Particularly useful information will be obtained from their behaviour as $u$ is taken around a closed contour. If the contour does not encircle certain singular points to be determined below, $a(u)$ and $a_D(u)$ will return to their initial values once $u$ has completed its contour. However, if the $u$-contour goes around these singular points, $a(u)$ and $a_D(u)$ do not return to their initial values but rather to certain linear combinations thereof: one has a non-trivial monodromy for the multi-valued functions $a(u)$ and $a_D(u)$.

**The monodromy at infinity**

This is immediately clear from the behaviour near $u = \infty$. As already explained in section 3.4, as $u \to \infty$, due to asymptotic freedom, the perturbative expression for $F(a)$ is valid and one has from (3.4) for $a_D = \partial F(a) / \partial a$

$$
a_D(u) = \frac{i}{\pi} a \left( \ln \frac{a^2}{\Lambda^2} + 1 \right) , \quad u \to \infty .
$$

(3.20)

Now take $u$ around a counterclockwise contour of very large radius in the complex $u$-plane, often simply written as $u \to e^{2\pi i} u$. This is equivalent to having $u$ encircle the point at $\infty$ on the Riemann sphere in a *clockwise* sense. In any case, since $u = \frac{i}{2} a^2$ (for $u \to \infty$) one has $a \to -a$ and

$$
a_D \to \frac{i}{\pi} (-a) \left( \ln \frac{e^{2\pi i} a^2}{\Lambda^2} + 1 \right) = -a_D + 2a
$$

(3.21)

or

$$
\begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} \to M_\infty \begin{pmatrix} a_D(u) \\ a(u) \end{pmatrix} , \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} .
$$

(3.22)

Clearly, $u = \infty$ is a branch point of $a_D(u) \sim \frac{i}{\pi} \sqrt{2u} \left( \ln \frac{u}{\Lambda^2} + 1 \right)$. This is why this point is referred to as a singularity of the moduli space.

**How many singularities?**

Can $u = \infty$ be the only singular point? Since a branch cut has to start and end somewhere, there must be at least one other singular point. Following Seiberg and Witten, I will argue that one actually
needs three singular points at least. To see why two cannot work, let’s suppose for a moment that there are only two singularities and show that this leads to a contradiction.

Before doing so, let me note that there is an important so-called U(1)$_R$-symmetry in the classical theory that takes $z \to e^{2i\alpha}z$, $\phi \to e^{2i\alpha}\phi$, $W \to e^{i\alpha}W$, $\theta \to e^{i\alpha}\theta$, $\bar{\theta} \to e^{i\alpha}\bar{\theta}$, thus $d^2\theta \to e^{-2i\alpha}d^2\theta$, $d^2\bar{\theta} \to e^{-2i\alpha}d^2\bar{\theta}$. Then the classical action is invariant under this global symmetry. More generally, the action will be invariant if $\mathcal{F}(z) \to e^{4i\alpha}\mathcal{F}(z)$. This symmetry is broken by the one-loop correction and also by instanton contributions. The latter give corrections to $\mathcal{F}$ of the form $z^2 \sum_{k=1}^{\infty} c_k (\Lambda^2 / z^2)^{2k}$, and hence are invariant only for $(e^{4i\alpha})^{2k} = 1$, i.e. $\alpha = \frac{2\pi n}{8}$, $n \in \mathbb{Z}$. Hence instantons break the U(1)$_R$-symmetry to a discrete $\mathbb{Z}_8$. The one-loop corrections behave as $\frac{i}{2\pi} z^2 \ln \frac{z^2}{\Lambda^2} \to e^{4i\alpha} \left( \frac{i}{2\pi} z^2 \ln \frac{z^2}{\Lambda^2} - \frac{2\alpha}{\pi} z^2 \right)$.

As before one shows that this classical change is irrelevant as long as $4\alpha = n$ or $\alpha = \frac{2\pi n}{8}$. Under this $\mathbb{Z}_8$-symmetry, $z \to e^{i\pi n/2}z$, i.e. for odd $n$ one has $z^2 \to -z^2$. The non-vanishing expectation value $u = \langle \text{tr } z^2 \rangle$ breaks this $\mathbb{Z}_8$ further to $\mathbb{Z}_4$. Hence for a given vacuum, i.e. a given point on moduli space there is only a $\mathbb{Z}_4$-symmetry left from the U(1)$_R$-symmetry. However, on the manifold of all possible vacua, i.e. on $\mathcal{M}$, one has still the full $\mathbb{Z}_8$-symmetry, taking $u$ to $-u$. Said differently, the quotient $\mathbb{Z}_8 / \mathbb{Z}_4 = \mathbb{Z}_2$ acts as a symmetry on moduli space mapping the theory at $u$ to the theory at $-u$.

Due to this global symmetry $u \to -u$, singularities of $\mathcal{M}$ should come in pairs: for each singularity at $u = u_0$ there is another one at $u = -u_0$. The only fixed points of $u \to -u$ are $u = \infty$ and $u = 0$. We have already seen that $u = \infty$ is a singular point of $\mathcal{M}$. So if there are only two singularities the other must be the fixed point $u = 0$.

If there are only two singularities, at $u = \infty$ and $u = 0$, then by contour deformation (“pulling the contour over the back of the sphere”)[5] one sees that the monodromy around 0 (in a counterclockwise sense) is the same as the above monodromy around $\infty$: $M_0 = M_\infty$. But then $a^2$ is not affected by any monodromy and hence is a good global coordinate, so one can take $u = \frac{1}{2} a^2$ on all of $\mathcal{M}$, and furthermore one must have

$$a_D = \frac{i}{\pi} a \left( \ln \frac{a^2}{\Lambda^2} + 1 \right) + g(a)$$
$$a = \sqrt{2u}$$

(3.23)

where $g(a)$ is some entire function of $a^2$. This implies that

$$\tau = \frac{da_D}{da} = \frac{i}{\pi} \left( \ln \frac{a^2}{\Lambda^2} + 3 \right) + \frac{dg}{da}.$$  

(3.24)

The function $g$ being entire, Im $\frac{dg}{da}$ cannot have a minimum (unless constant) and it is clear that Im $\tau$ cannot be positive everywhere. As already emphasized, this means that $a$ (or rather $a^2$) cannot be a good global coordinate and (3.23) cannot hold globally. Hence, two singularities only cannot work.

The next simplest choice is to try 3 singularities. Due to the $u \to -u$ symmetry, these 3 singularities are at $\infty, u_0$ and $-u_0$ for some $u_0 \neq 0$. In particular, $u = 0$ is no longer a singularity of the quantum moduli space. To get a singularity also at $u = 0$ one would need at least four singularities at $\infty, u_0, -u_0$ and 0. As discussed later, this is not possible, and more generally, exactly 3 singularities seems to be the only consistent possibility.

---

[5] It is well-known from complex analysis that monodromies are associated with contours around branch points. The precise form of the contour does not matter, and it can be deformed as long as it does not meet another branch point. Our singularities precisely are the branch points of $a(u)$ or $a_D(u)$. 23
So there is no singularity at \( u = 0 \) in the quantum moduli space \( \mathcal{M} \). Classically, however, one precisely expects that \( u = 0 \) should be a singular point, since classically \( u = \frac{1}{2} a^2 \), hence \( a = 0 \) at this point, and then there is no Higgs mechanism any more. Thus all (elementary) massive states, i.e. the gauge bosons \( v_\mu^1, v_\mu^2 \) and their susy partners \( \psi^1, \psi^2, \lambda^1, \lambda^2 \) become massless. Thus the description of the light fields in terms of the previous Wilsonian effective action should break down, inducing a singularity on the moduli space. As already stressed, this is the classical picture. While \( a \rightarrow \infty \) leads to asymptotic freedom and the microscopic SU(2) theory is weakly coupled, as \( a \rightarrow 0 \) one goes to a strong coupling regime where the classical reasoning has no validity any more, and \( u \neq \frac{1}{2} a^2 \). By the BPS mass formula (3.18) massless gauge bosons still are possible at \( a = 0 \), but this does no longer correspond to \( u = 0 \).

So where has the singularity due to massless gauge bosons at \( a = 0 \) moved to? One might be tempted to think that \( a = 0 \) now corresponds to the singularities at \( u = \pm u_0 \), but this is not the case as I will show in a moment. The answer is that the point \( a = 0 \) no longer belongs to the quantum moduli space (at least not to the component connected to \( u = \infty \) which is the only thing one considers). This can be seen explicitly from the form of the solution for \( a(u) \) given in the next section.

The strong coupling singularities

Let’s now concentrate on the case of three singularities at \( u = \infty, u_0 \) and \( - u_0 \). What is the interpretation of the (strong-coupling) singularities at finite \( u = \pm u_0 \)? One might first try to consider that they are still due to the gauge bosons becoming massless. However, as Seiberg and Witten point out, massless gauge bosons would imply an asymptotically conformally invariant theory in the infrared limit and conformal invariance implies \( u = \langle \text{tr } z^2 \rangle = 0 \) unless \( \text{tr } z^2 \) has dimension zero and hence would be the unity operator - which it is not. So the singularities at \( u = \pm u_0 \) (\( \neq 0 \)) do not correspond to massless gauge bosons.

There are no other elementary \( \mathcal{N} = 2 \) multiplets in our theory. The next thing to try is to consider collective excitations - solitons, like the magnetic monopoles or dyons. Let’s first study what happens if a magnetic monopole of unit magnetic charge becomes massless. From the BPS mass formula (3.18), the mass of the magnetic monopole is

\[
m^2 = 2 |a_D|^2 \tag{3.25}
\]

and hence vanishes at \( a_D = 0 \). We will see that this produces one of the two strong-coupling singularities. So call \( u_0 \) the value of \( u \) at which \( a_D \) vanishes. Magnetic monopoles are described by hypermultiplets \( H \) of \( \mathcal{N} = 2 \) susy that couple locally to the dual fields \( \phi_D \) and \( W_D \), just as electrically charged “electrons” would be described by hypermultiplets that couple locally to \( \phi \) and \( W \). So in the dual description we have \( \phi_D, W_D \) and \( H \), and, near \( u_0 \), \( a_D \sim \langle \phi_D \rangle \) is small. This theory is exactly \( \mathcal{N} = 2 \) susy QED with very light electrons (and a subscript \( D \) on every quantity). The latter theory is not asymptotically free, but has a \( \beta \)-function given by

\[
\mu \frac{d}{d \mu} g_D = \frac{g_D^3}{8 \pi^2} \tag{3.26}
\]

where \( g_D \) is the coupling constant. But the scale \( \mu \) is proportional to \( a_D \) and \( \frac{4 \pi i}{g_D^2(a_D)} \) is \( \tau_D \) for \( \theta_D = 0 \) (of course, super QED, unless embedded into a larger gauge group, does not allow for a non-vanishing theta angle). One concludes that for \( u \approx u_0 \) or \( a_D \approx 0 \)

\[
a_D \frac{d}{d a_D} \tau_D = - \frac{i}{\pi} \Rightarrow \tau_D = - \frac{i}{\pi} \ln a_D \ . \tag{3.27}
\]
Since \( \tau_D = \frac{d(-a)}{da_D} \) this can be integrated to give

\[
a \approx a_0 + \frac{i}{\pi} a_D \ln a_D \quad (u \approx u_0)
\]  

(3.28)

where we dropped a subleading term \(-\frac{i}{\pi} a_D\). Now, \(a_D\) should be a good coordinate in the vicinity of \(u_0\), hence depend linearly on \(u\). One concludes

\[
a_D \approx c_0(u - u_0) \quad , \quad a \approx a_0 + \frac{i}{\pi} c_0(u - u_0) \ln(u - u_0).
\]  

(3.29)

From these expressions one immediately reads the monodromy as \(u\) turns around \(u_0\) counterclockwise, \(u - u_0 \to e^{2\pi i}(u - u_0)\):

\[
\begin{pmatrix} a_D \\ a \end{pmatrix} \to \begin{pmatrix} a_D \\ a - 2a_D \end{pmatrix} = M_{u_0} \begin{pmatrix} a_D \\ a \end{pmatrix} \quad , \quad M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.
\]  

(3.30)

Note that the magnetic monopole \(\begin{pmatrix} n_e \\ n_m \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) is invariant under this monodromy, i.e. it is an eigenvector of \(M_{u_0}\) with unit eigenvalue.

To obtain the monodromy matrix at \(u = -u_0\) it is enough to observe that the contour around \(u = \infty\) is equivalent to a counterclockwise contour of very large radius in the complex plane. This contour can be deformed into a contour encircling \(u_0\) and a contour encircling \(-u_0\), both counterclockwise.

It follows the factorisation condition on the monodromy matrices:

\[
M_{\infty} = M_{u_0} M_{-u_0}
\]  

(3.31)

and hence

\[
M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}.
\]  

(3.32)

What is the interpretation of this singularity at \(u = -u_0\)? As discussed above, using the \(Sl(2, \mathbb{Z})\) invariance of \(Z\), the monodromy transformation \(\begin{pmatrix} a_D \\ a \end{pmatrix} \to M \begin{pmatrix} a_D \\ a \end{pmatrix}\) can be interpreted as changing the magnetic and electric quantum numbers as \(\begin{pmatrix} n_e \\ n_m \end{pmatrix} \to M^{-1} \begin{pmatrix} n_e \\ n_m \end{pmatrix}\). The state of vanishing mass responsible for a singularity should be invariant under the monodromy, and hence be an eigenvector of \(M\) with unit eigenvalue. We already noted this for the magnetic monopole. Similarly, the eigenvector of (3.32) with unit eigenvalue is \(\begin{pmatrix} n_e \\ n_m \end{pmatrix} = (1, 1)\). This is a dyon. Thus the singularity at \(-u_0\) is interpreted as being due to a \((1, 1)\) dyon becoming massless.

More generally, \(\begin{pmatrix} n_e \\ n_m \end{pmatrix}\) is the eigenvector with unit eigenvalue\(^7\) of

\[
M(n_e, n_m) = \begin{pmatrix} 1 - 2n_m n_e & 2n_e^2 \\ -2n_m^2 & 1 + 2n_m n_e \end{pmatrix}
\]  

(3.33)

\(^6\) One might want to try a more general dependence like \(a_D \approx c_0(u - u_0)^k\) with \(k > 0\). This leads to a monodromy in \(Sl(2, \mathbb{Z})\) only for integer \(k\). The factorisation condition below, together with the form of \(M(n_m, n_e)\) also given below, then imply that \(k = 1\) is the only possibility.

\(^7\) There is an ambiguity concerning the ordering of \(M_{u_0}\) and \(M_{-u_0}\) which will be resolved below.

\(^8\) Of course, the same is true for any \((qn_m, qn_e)\) with \(q \in \mathbb{Z}\), but according to the discussion in section 4.3 on the stability of BPS states, states with \(q \neq \pm 1\) are not stable.
which is the monodromy matrix that should appear for any singularity due to a massless dyon with charges \((n_m, n_e)\). Note that \(M_\infty\) as given in (3.22) is not of this form, since it does not correspond to a hypermultiplet becoming massless.

One notices that the relation (3.31) does not look invariant under \(u \rightarrow -u\), i.e. \(u_0 \rightarrow -u_0\) since \(M_{u_0}\) and \(M_{-u_0}\) do not commute. The apparent contradiction with the \(\mathbb{Z}_2\)-symmetry is resolved by the following remark. The precise definition of the composition of two monodromies as in (3.31) requires a choice of base-point \(u = P\) (just as in the definition of homotopy groups). Using a different base-point, namely \(u = -P\), leads to

\[
M_\infty = M_{-u_0}M_{u_0}
\]

(3.34) instead. Then one would obtain \(M_{-u_0} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}\), and comparing with (3.33), this would be interpreted as due to a \((-1, 1)\) dyon. Thus the \(\mathbb{Z}_2\)-symmetry \(u \rightarrow -u\) on the quantum moduli space also acts on the base-point \(P\), hence exchanging (3.31) and (3.34). At the same time it exchanges the \((1, 1)\) dyon with the \((-1, 1)\) dyon.

Does this mean that the \((1, 1)\) or \((-1, 1)\) dyons play a privileged role? Actually not. If one first turns \(k\) times around \(\infty\), then around \(u_0\), and then \(k\) times around \(\infty\) in the opposite sense, the corresponding monodromy is

\[
M_\infty^{-k}M_{u_0}M_\infty^k = \begin{pmatrix} 1 - 4k & 8k^2 \\ -2 & 1 + 4k \end{pmatrix} = M(2k, 1)
\]

(3.35)

and similarly

\[
M_\infty^{-k}M_{-u_0}M_\infty^k = \begin{pmatrix} -1 - 4k & 2 + 8k + 8k^2 \\ -2 & 3 + 4k \end{pmatrix} = M(2k + 1, 1).
\]

(3.36)

So one sees that these monodromies correspond to dyons with \(n_m = 1\) and any \(n_e \in \mathbb{Z}\) becoming massless. Similarly one has e.g. \(M_{u_0}^{-k}M_{-u_0}M_{u_0}^{-k} = M(2k - 1, -1)\), etc.

Let’s come back to the question of how many singularities there are. Suppose there are \(p\) strong coupling singularities at \(u_1, u_2, \ldots, u_p\) in addition to the one-loop perturbative singularity at \(u = \infty\). Then one has a factorisation analogous to (3.31):

\[
M_\infty = M_{u_1}M_{u_2} \ldots M_{u_p}
\]

(3.37)

with \(M_{u_i} = M(n_m^{(i)}, n_e^{(i)})\) of the form (3.33). It thus becomes a problem of number theory to find out whether, for given \(p\), there exist solutions to (3.37) with integer \(n_m^{(i)}\) and \(n_e^{(i)}\). For several low values of \(p > 2\) it has been checked that there are no such solutions, and it seems likely that the same is true for all \(p > 2\).

### 3.4 The solution

Recall that our goal is to determine the exact non-perturbative low-energy effective action, i.e. determine the function \(F(z)\) locally. This will be achieved, at least in principle, once we know the functions \(a(u)\) and \(aD(u)\), since one then can invert the first to obtain \(a(u)\), at least within a certain domain of the moduli space. Substituting this into \(aD(u)\) yields \(aD(a)\) which upon integration gives the desired \(F(a)\).

So far we have seen that \(aD(u)\) and \(a(u)\) are single-valued except for the monodromies around \(\infty, u_0\) and \(-u_0\). As is well-known from complex analysis, this means that \(aD(u)\) and \(a(u)\) are really
multi-valued functions with branch cuts, the branch points being $\infty$, $u_0$ and $-u_0$. A typical example is $f(u) = \sqrt{u} F(a, b, c; u)$, where $F$ is the hypergeometric function. The latter has a branch cut from 1 to $\infty$. Similarly, $\sqrt{u}$ has a branch cut from 0 to $\infty$ (usually taken along the negative real axis), so that $f(u)$ has two branch cuts joining the three singular points 0, 1 and $\infty$. When $u$ goes around any of these singular points there is a non-trivial monodromy between $f(u)$ and one other function $g(u) = u^d F(a', b', c'; u)$. The three monodromy matrices are in (almost) one-to-one correspondence with the pair of functions $f(u)$ and $g(u)$.

In the physical problem at hand one knows the monodromies, namely

\[ M_{\infty} = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}, \quad M_{u_0} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \quad M_{-u_0} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \] (3.38)

and one wants to determine the corresponding functions $a_D(u)$ and $a(u)$. As will be explained, the monodromies fix $a_D(u)$ and $a(u)$ up to normalisation, which will be determined from the known asymptotics (3.20) at infinity.

The precise location of $u_0$ depends on the renormalisation conditions which can be chosen such that $u_0 = 1$. Assuming this choice in the sequel will simplify somewhat the equations. If one wants to keep $u_0$, essentially all one has to do is to replace $u \pm 1$ by $u \pm u_0$. The differential equation approach

This approach to determining $a_D$ and $a$ was first exposed in [18]. Monodromies typically arise from differential equations with periodic coefficients. This is well-known in solid-state physics where one considers a Schrödinger equation with a periodic potential:

\[ \left[ -\frac{d^2}{dx^2} + V(x) \right] \psi(x) = 0 \ , \ V(x + 2\pi) = V(x) \ . \] (3.39)

There are two independent solutions $\psi_1(x)$ and $\psi_2(x)$. One wants to compare solutions at $x$ and at $x + 2\pi$. Since, due to the periodicity of the potential $V$, the differential equation at $x + 2\pi$ is exactly the same as at $x$, the set of solutions must be the same. In other words, $\psi_1(x + 2\pi)$ and $\psi_2(x + 2\pi)$ must be linear combinations of $\psi_1(x)$ and $\psi_2(x)$:

\[ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (x + 2\pi) = M \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (x) \] (3.40)

where $M$ is a (constant) monodromy matrix.

The same situation arises for differential equations in the complex plane with meromorphic coefficients. Consider again the Schrödinger-type equation

\[ \left[ -\frac{d^2}{d\xi^2} + V(\xi) \right] \psi(\xi) = 0 \] (3.41)

with meromorphic $V(\xi)$, having poles at $\xi_1, \ldots, \xi_p$ and (in general) also at $\infty$. The periodicity of the previous example is now replaced by the single-valuedness of $V(\xi)$ as $\xi$ goes around any of the poles of $V$ (with $\xi - \xi_i$ corresponding roughly to $e^{i\pi}$). So, as $\xi$ goes once around any one of the $\xi_i$, the differential equation (3.41) does not change. So by the same argument as above, the two solutions

---

9 The constant energy has been included into the potential, and the mass has been normalised to $\frac{1}{2}$. 

27
\(\psi_1(\xi)\) and \(\psi_2(\xi)\), when continued along the path surrounding \(\xi_i\) must again be linear combinations of \(\psi_1(\xi)\) and \(\psi_2(\xi)\):
\[
\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (\xi + e^{2\pi i} (\xi - \xi_i)) = M_i \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} (\xi)
\]
(3.42)
with a constant \(2 \times 2\)-monodromy matrix \(M_i\) for each of the poles of \(V\). Of course, one again has the factorisation condition (3.37) for \(M_0\). It is well-known, that non-trivial constant monodromies correspond to poles of \(V\) that are at most of second order. In the language of differential equations, (3.41) then only has regular singular points.

In our physical problem, the two multivalued functions \(a_D(\xi)\) and \(a(\xi)\) have 3 singularities with non-trivial monodromies at \(-1, +1\) and \(\infty\). Hence they must be solutions of a second-order differential equation (3.41) with the potential \(V\) having (at most) second-order poles precisely at these points. The general form of this potential is
\[
V(\xi) = -\frac{1}{4} \left[ \frac{1 - \lambda_1^2}{(\xi + 1)^2} + \frac{1 - \lambda_2^2}{(\xi - 1)^2} - \frac{1 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2}{(\xi + 1)(\xi - 1)} \right]
\]
(3.43)
with double poles at \(-1, +1\) and \(\infty\). The corresponding residues are \(-\frac{1}{4}(1 - \lambda_1^2), -\frac{1}{4}(1 - \lambda_2^2)\) and \(-\frac{1}{4}(1 - \lambda_3^2)\). Without loss of generality, I assume \(\lambda_i \geq 0\). The corresponding differential equation (3.41) is well-known in the mathematical literature since it can be transformed into the hypergeometric differential equation. The transformation to the standard hypergeometric equation is readily performed by setting
\[
\psi(\xi) = (\xi + 1)^{\lambda_1}(\xi - 1)^{\lambda_2} \left[ \frac{1}{2} x \right] f \left( \frac{\xi + 1}{2} \right).
\]
(3.44)
One then finds that \(f\) satisfies the hypergeometric differential equation
\[
x(1 - x) f''(x) + [c - (a + b + 1)x] f'(x) - a b f(x) = 0
\]
(3.45)
with
\[
a = \frac{1}{2}(1 - \lambda_1 - \lambda_2 + \lambda_3), \quad b = \frac{1}{2}(1 - \lambda_1 - \lambda_2 - \lambda_3), \quad c = 1 - \lambda_1.
\]
(3.46)
The solutions of the hypergeometric equation (3.45) can be written in many different ways due to the various identities between the hypergeometric function \(F(a, b, c; x)\) and products with powers, e.g. \((1 - x)^{c-a-b} F(c - a, c - b, c; x)\), etc. A convenient choice for the two independent solutions is the following
\[
f_1(x) = (-x)^{-a} F(a, a + 1 - c, a + 1 - b; \frac{1}{2})
\]
\[
f_2(x) = (1 - x)^{c-a-b} F(c - a, c - b, c + 1 - a - b; 1 - x).
\]
(3.47)
f_1 and \(f_2\) correspond to Kummer’s solutions denoted \(u_3\) and \(u_6\) [19]. The choice of \(f_1\) and \(f_2\) is motivated by the fact that \(f_1\) has simple monodromy properties around \(x = \infty\) (i.e. \(\xi = \infty\)) and \(f_2\) has simple monodromy properties around \(x = 1\) (i.e. \(\xi = 1\)), so they are good candidates to be identified with \(a(\xi)\) and \(a_D(\xi)\).

One can extract a great deal of information from the asymptotic forms of \(a_D(\xi)\) and \(a(\xi)\). As \(\xi \to \infty\) one has \(V(\xi) \sim -\frac{1}{4} \frac{1 - \lambda_3^2}{\xi^2}\), so that the two independent solutions behave asymptotically as
\[\text{Additional terms in } V \text{ that naively look like first-order poles (\(\sim \frac{1}{\xi - 1} \) or \(\frac{1}{\xi + 1}\)) cannot appear since they correspond to third-order poles at } \xi = \infty.\]
\( \xi^{\frac{1}{2}(1+\lambda_3)} \) if \( \lambda_3 \neq 0 \), and as \( \sqrt{\xi} \) and \( \sqrt{\xi} \ln \xi \) if \( \lambda_3 = 0 \). Comparing with (3.23) (with \( u \to \xi \)) we see that the latter case is realised. Similarly, with \( \lambda_3 = 0 \), as \( \xi \to 1 \), one has \( V(\xi) \sim -\frac{1}{4} \left( \frac{1-\lambda_2^2}{(\xi-1)^2} - \frac{1-\lambda_2^2-\lambda_3^2}{2(\xi-1)} \right) \), where I have kept the subleading term. From the logarithmic asymptotics (3.29) one then concludes \( \lambda_2 = 1 \) (and from the subleading term also \( -\frac{\lambda_2^2}{8} = \frac{i}{\pi a_0} \)). The \( \mathbb{Z}_2 \)-symmetry \( (\xi \to -\xi) \) on the moduli space then implies that, as \( \xi \to -1 \), the potential \( V \) does not have a double pole either, so that also \( \lambda_1 = 1 \). Hence we conclude

\[
\lambda_1 = \lambda_2 = 1 \quad \text{and} \quad \lambda_3 = 0 \quad \Rightarrow \quad V(\xi) = -\frac{1}{4} \frac{1}{(\xi+1)(\xi-1)} \tag{3.48}
\]

and \( a = b = -\frac{1}{2} \), \( c = 0 \). Thus from (3.44) one has \( \psi_{1,2}(\xi) = f_{1,2} \left( \frac{u+1}{2} \right) \). One can then verify that the two solutions

\[
a_D(u) = i\psi_2(u) = i\frac{u-1}{2} F \left( \frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{1-u}{2} \right) \\
a(u) = -2i\psi_1(u) = \sqrt{2}(u+1)^{\frac{1}{2}} F \left( -\frac{1}{2}, \frac{1}{2}, 1; \frac{2}{u+1} \right) \tag{3.49}
\]

indeed have the required monodromies (3.38), as well as the correct asymptotics.

It might look as if we have not used the monodromy properties to determine \( a_D \) and \( a \) and that they have been determined only from the asymptotics. This is not entirely true, of course. The very fact that there are non-trivial monodromies only at \( \infty, +1 \) and \( -1 \) implied that \( a_D \) and \( a \) must satisfy the second-order differential equation (3.41) with the potential (3.43). To determine the \( \lambda_i \) we then used the asymptotics of \( a_D \) and \( a \). But this is (almost) the same as using the monodromies since the latter were obtained from the asymptotics.

Using the integral representation of the hypergeometric function, the solution (3.49) can be nicely rewritten as

\[
a_D(u) = \frac{\sqrt{2}}{\pi} \int_1^{u} \frac{dx}{x} \frac{\sqrt{x-u}}{\sqrt{x^2-1}}, \quad a(u) = \frac{\sqrt{2}}{\pi} \int_{-1}^{1} \frac{dx}{x} \frac{\sqrt{x-u}}{\sqrt{x^2-1}} \tag{3.50}
\]

One can invert the second equation (3.49) to obtain \( u(a) \), within a certain domain, and insert the result into \( a_D(u) \) to obtain \( a_D(a) \). Integrating with respect to \( a \) yields \( F(a) \) and hence the low-energy effective action. I should stress that this expression for \( F(a) \) is not globally valid but only on a certain portion of the moduli space. Different analytic continuations must be used on other portions.

The approach using elliptic curves
In their paper [1], Seiberg and Witten do not use the differential equation approach just described, but rather introduce an auxiliary construction: a certain elliptic curve by means of which two functions with the correct monodromy properties are constructed. I will not go into details here, but simply sketch this approach.

To motivate their construction \textit{a posteriori}, we notice the following: from the integral representation (3.50) it is natural to consider the complex \( x \)-plane. More precisely, the integrand has square-root branch cuts with branch points at \( +1, -1, u \) and \( \infty \). The two branch cuts can be taken to run from \( -1 \) to \( +1 \) and from \( u \) to \( \infty \). The Riemann surface of the integrand is two-sheeted with the two sheets connected through the cuts. If one adds the point at infinity to each of the two sheets, the topology of the Riemann surface is that of two spheres connected by two tubes (the cuts), i.e. a torus. So one sees that the Riemann surface of the integrand in (3.50) has genus one. This is the elliptic curve considered by Seiberg and Witten.
As is well-known, on a torus there are two independent non-trivial closed paths (cycles). One cycle (\(\gamma_2\)) can be taken to go once around the cut \((-1, 1)\), and the other cycle (\(\gamma_1\)) to go from 1 to \(u\) on the first sheet and back from \(u\) to 1 on the second sheet. The solutions \(a_D(u)\) and \(a(u)\) in (3.50) are precisely the integrals of some suitable differential \(\lambda\) along the two cycles \(\gamma_1\) and \(\gamma_2\):

\[
\begin{align*}
a_D &= \oint_{\gamma_1} \lambda, & a &= \oint_{\gamma_2} \lambda, & \lambda &= \frac{\sqrt{2}}{2\pi} \frac{\sqrt{x-u}}{\sqrt{x^2-1}} dx.
\end{align*}
\]

These integrals are called period integrals. They are known to satisfy a second-order differential equation, the so-called Picard-Fuchs equation, that is nothing else than our Schrödinger-type equation (3.41) with \(V\) given by (3.48).

How do the monodromies appear in this formalism? As \(u\) goes once around \(+1, -1\) or \(\infty\), the cycles \(\gamma_1, \gamma_2\) are changed into linear combinations of themselves with integer coefficients:

\[
\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \rightarrow M \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}, \quad M \in SL(2, \mathbb{Z}).
\]

(3.52)

This immediately implies

\[
\begin{pmatrix} a_D \\ a \end{pmatrix} \rightarrow M \begin{pmatrix} a_D \\ a \end{pmatrix}
\]

(3.53)

with the same \(M\) as in (3.52). The advantage here is that one automatically gets monodromies with integer coefficients. The other advantage is that

\[
\tau(u) = \frac{da_D/du}{da/du}
\]

(3.54)

can be easily seen to be the \(\tau\)-parameter describing the complex structure of the torus, and as such is guaranteed to satisfy \(\text{Im} \, \tau(u) > 0\) which was the requirement for positivity of the metric on moduli space.

To motivate the appearance of the genus-one elliptic curve (i.e. the torus) \(a\) \(a\) priori - without knowing the solution (3.50) from the differential equation approach - Seiberg and Witten remark that the three monodromies are all very special: they do not generate all of \(SL(2, \mathbb{Z})\) but only a certain subgroup \(\Gamma(2)\) of matrices in \(SL(2, \mathbb{Z})\) congruent to 1 modulo 2. Furthermore, they remark that the \(u\)-plane with punctures at 1, \(-1, \infty\) can be thought of as the quotient of the upper half plane \(H\) by \(\Gamma(2)\), and that \(H/\Gamma(2)\) naturally parametrizes (i.e. is the moduli space of) elliptic curves described by

\[
y^2 = (x^2 - 1)(x-u).
\]

(3.55)

Equation (3.55) corresponds to the genus-one Riemann surface discussed above, and it is then natural to introduce the cycles \(\gamma_1, \gamma_2\) and the differential \(\lambda\) from (3.50). The rest of the argument then goes as I just exposed.

Summary

Let’s summarise what we have learnt so far. We have seen realised a version of electric-magnetic duality accompanied by a duality transformation on the expectation value of the scalar (Higgs) field, \(a \leftrightarrow a_D\). There is a manifold of inequivalent vacua, the moduli space \(\mathcal{M}\), corresponding to different Higgs expectation values. The duality relates strong coupling regions in \(\mathcal{M}\) to the perturbative region of large \(a\) where the effective low-energy action is known asymptotically in terms of \(\mathcal{F}\). Thus duality
allows us to determine the latter also at strong coupling. The holomorphicity condition from \( \mathcal{N} = 2 \) supersymmetry then puts such strong constraints on \( \mathcal{F}(a) \), or equivalently on \( a_D(u) \) and \( a(u) \) that the full functions can be determined solely from their asymptotic behaviour at the strong and weak coupling singularities of \( \mathcal{M} \).

4 The spectrum of stable BPS states: pure SU(2) without hypermultiplets

Knowing the low-energy effective action of the \( \mathcal{N} = 2 \) gauge theory allows us to study the dynamics of the light degrees of freedom. This certainly is quite an achievement. One may want to go further, however. The heavy, massive fields all must be BPS states since otherwise the multiplets contain spins exceeding one. Hence they must satisfy the BPS bound relating their masses to their charges. Studying their detailed dynamics is a difficult problem, in most cases well beyond what can be done. It is already a non-trivial question to study their existence and stability as the effective coupling changes, i.e. as one moves around in the moduli space. This problem though has been solved with somewhat surprising results. In this section I will review this solution in the simplest case corresponding to the theory studied in the previous section: gauge group SU(2) and no elementary hypermultiplets. In the next section, I will review the results for the more involved cases where massless or massive elementary hypermultiplets are present.

4.1 BPS states, charge lattice and curve of marginal stability

Recall that the \( \mathcal{N} = 2 \) susy algebra has long and short representations and for short representations (BPS states) the BPS bound must be satisfied:

\[
m = \sqrt{2}|Z| = \sqrt{2}|n_e a(u) - n_m a_D(u)|.
\]

We have seen that the central charge \( Z \) can be written in terms of the standard symplectic invariant \( \eta(p, \Omega) \) of \( p = (n_e, n_m) \) and \( \Omega = (a_D, a) \) which is such that \( \eta(Gp, G\Omega) = \eta(p, \Omega) \) for any \( G \in Sp(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z}). \)

For fixed quantum numbers \( (n_e, n_m) \) a BPS state has the minimal mass and must be stable. The question then is whether it can decay into two (or more) other states such that the charge quantum numbers are conserved. Take the example of a dyon with \( n_e = 1 \) and \( n_m = 1 \). By charge conservation this could decay into a monopole \( (n_e, n_m) = (0, 1) \) and a W-boson \( (n_e, n_m) = (1, 0) \). Kinematically however this is impossible, since the sum of the masses of the latter is larger than the mass of the initial dyon. To discuss the general case one draws the charge lattice in the complex plane as follows. Generically, for a given point \( u \) in moduli space, \( a \) and \( a_D \) are two complex numbers such that \( a_D/a \notin \mathbb{R} \) and all possible central charges form a lattice in the complex plane generated by \( a \) and \((-a_D)\), see Fig. 1. Each lattice point corresponds to an a priori possible BPS state \( (n_e, n_m) \) whose mass is simply its euclidean distance from the origin. For the above example of the \((1, 1)\) dyon it is clear by the triangle inequality of elementary geometry, that the sum of the masses of the decay products \((1, 0)\) and \((0, 1)\) would be larger than the mass of the \((1, 1)\) dyon. Obviously, decay into non-BPS states is even more impossible since they would have even larger masses for the same charge quantum numbers. Hence the \((1, 1)\) dyon is stable. The same argument applies to all BPS states.
Figure 1: The lattice of central charges for generic $a_D$ and $a$.

$(n_e, n_m)$ such that $(n_e, n_m) \neq q(n, m)$ with $n, m, q \in \mathbb{Z}, q \neq \pm 1$: states with $n_e$ and $n_m$ relatively prime are stable.

The preceding argument fails if

$$w(u) \equiv \frac{a_D(u)}{a(u)} \in \mathbb{R},$$

since then the lattice collapses onto a single line and decays of otherwise stable BPS states become possible. It is thus of interest to determine the set of all such $u$, i.e.

$$\mathcal{C} = \{ u \in \mathbb{C} \mid w(u) = \frac{a_D(u)}{a(u)} \in \mathbb{R} \},$$

which is called the curve of marginal stability [1, 21]. Given the explicit form of $a_D(u)$ and $a(u)$ it is straightforward to determine $\mathcal{C}$ numerically [6], see Fig. 2, although it can also be done analytically [21]. The precise form of the curve however is irrelevant for our purposes. What is important is that

as $u$ varies along the curve, $a_D/a$ takes all values in $[-1, 1]$. More precisely, if we call $\mathcal{C}^\pm$ the parts of $\mathcal{C}$ in the upper and lower half $u$ plane, then

$$\frac{a_D(u)}{a} \in [-1, 0] \text{ for } u \in \mathcal{C}^+, \quad \frac{a_D(u)}{a} \in [0, 1] \text{ for } u \in \mathcal{C}^-.$$
\[ \frac{a_D}{a}(u) \in [0,1] \quad \text{for } u \in \mathcal{C}^- , \]

with the value being discontinuous at \( u = -1 \) due to the cuts of \( a_D \) and \( a \) running along the real axis from \(-\infty\) to +1. More precisely, \( \frac{a_D}{a}(u) \) increases monotonically from \(-1\) at \( u = -1 + i\epsilon \) to +1 at \( u = -1 - i\epsilon \) as one follows the curve clockwise. Obviously \( \frac{a_D}{a} = 0 \) at \( u = 1 \).

The curve \( \mathcal{C} \) separates the moduli space into two distinct regions: inside the curve and outside the curve, see Fig. 2. If two points \( u \) and \( u' \) are in the same region, i.e. if they can be joined by a path not crossing \( \mathcal{C} \) then the spectrum of BPS states (by which we mean the set of quantum numbers \((n_e, n_m)\) that do exist) is necessarily the same at \( u \) and \( u' \). Indeed, start with a given stable BPS state at \( u \). Then imagine deforming the theory adiabatically so that the scalar field \( \phi \) slowly changes its vacuum expectation value and \( \langle \text{tr} \phi^2 \rangle \) moves from \( u \) to \( u' \). In doing so, the BPS state will remain stable and it cannot decay at any point on the path. Hence it will also exist at \( u' \). If, however, \( u \) and \( \tilde{u} \) are in different regions so that the path joining them must cross the curve \( \mathcal{C} \) somewhere, then the initial BPS state will no longer be stable as one crosses the curve and it can decay. Hence the spectrum at \( u \) and \( \tilde{u} \) need not be the same.

As an example, consider the possible decay of the W boson \((1, 0)\) when crossing the curve on \( \mathcal{C}^+ \) at a point where \( a_D/a = r \) with \( r \) any real number between \(-1\) and 0. Charge conservation alone allows for the reaction

\[ (1, 0) \rightarrow (1, -1) + (0, 1) . \]  

(4.5)

On \( \mathcal{C}^+ \), and only on \( \mathcal{C}^+ \), we also have the equality of masses, thanks to

\[ |a + a_D| + |a_D| = |a| (|1 + r| + |r|) = |a| (1 + r - r) = |a| . \]  

(4.6)

Had one crossed the curve in the lower half plane instead, \( r \) would have been between 0 and +1 and the dyon \((1, -1)\) would have been described as \((1, 1)\) (see below), and eq. (4.6) would have worked out correspondingly.

Since the region of moduli space outside the curve contains the semi-classical domain \( u \rightarrow \infty \), we refer to this region as the semi-classical or weak-coupling region \( \mathcal{R}_W \) and to the region inside the curve as the strong-coupling region \( \mathcal{R}_S \). We call the corresponding spectra also weak and strong-coupling spectra \( S_W \) and \( S_S \). This terminology is used due to the above-explained continuity of the spectra throughout each of the two regions. Nevertheless, the physics close to the curve is always strongly coupled even in the so-called weak-coupling region.

### 4.2 The main argument and the weak-coupling spectrum

The important property of the curve \( \mathcal{C} \) of marginal stability is

- **P1**: Massless states can only occur on the curve \( \mathcal{C} \).

The proof is trivial: If we have a massless state at some point \( u \), it necessarily is a BPS state, hence \( m(u) = 0 \) implies \( n_e a(u) - n_m a_D(u) = 0 \) which can be rewritten as \( (a_D/a)(u) = n_e/n_m \). But \( n_e/n_m \) is a real number, hence \( (a_D/a)(u) \) is real, and thus \( u \in \mathcal{C} \). Indeed the points \( u = \pm 1 \) where the magnetic monopole and the dyon \((\pm 1, 1)\) become massless are on the curve. The converse statement obviously also is true:

- **P2**: A BPS state \((n_e, n_m)\) with \( n_e/n_m \in [-1, 1] \) becomes massless somewhere on the curve \( \mathcal{C} \).
Of course, it will become massless precisely at the point \( u \in \mathcal{C} \) where \((a_D/a)(u) = n_e/n_m\). Strictly speaking, in its simple form, this only applies to BPS states in the weak-coupling region, since the description of BPS states in the strong-coupling region is slightly more involved as shown below. Let me now state the main hypothesis.

- **H**: A state becoming massless always leads to a singularity of the low-energy effective action, and hence of \((a_D/a)(u)\). The Seiberg-Witten solution for \((a_D(u), a(u))\) is correct and there are only two singularities at finite \(u\), namely \(u = \pm 1\).

Then the argument we will repeatedly use goes like this: If a certain state would become massless at some point \(u\) on moduli space, it would lead to an extra singularity which we know cannot exist. Hence this state either is the magnetic monopole \(\pm(0,1)\) or the \(\pm(1,1)\) dyon and \(u = \pm 1\), or this state cannot exist.

As an immediate consequence we can show that the weak-coupling spectrum cannot contain BPS states with \(|n_m| > |n_e| > 0\). Indeed, for such a state, \(-1 < n_e/n_m < 1\) and it would be massless at the point \(u\) on \(\mathcal{C}\) where \((a_D/a)(u) = n_e/n_m\). Since \(|n_m| > |n_e| > 0\) it is neither the monopole \((n_e = 0)\) nor the \((\pm 1,1)\) dyon, hence it cannot exist.

To determine which states are in \(\mathcal{S}_W\) one uses a global symmetry. Taking \(u \rightarrow e^{2\pi i} u\) along a path outside \(\mathcal{C}\) does not change the theory since one comes back to the same point of moduli space, and hence must leave \(\mathcal{S}_W\) invariant. But it induces a monodromy transformation

\[
\begin{pmatrix} n_e \\ -n_m \end{pmatrix} \rightarrow M_\infty \begin{pmatrix} n_e \\ -n_m \end{pmatrix}, \quad M_\infty = \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix}.
\]

In other words, \(M_\infty \mathcal{S}_W = \mathcal{S}_W\). Now, we know that \(\mathcal{S}_W\) contains at least the two states that are responsible for the singularities, namely \((0,1)\) and \((1,1)\) together with their antiparticles \((0,-1)\) and \((-1,-1)\). Applying \(M_\infty^{-1}\) on these two states generates all dyons \((n, \pm 1), n \in \mathbb{Z}\). This was already clear from [1]. But now we can just as easily show that there are no other dyons in the weak-coupling spectrum. If there were such a state \(\pm(k,m)\) with \(|m| \geq 2\), then applying \(M_\infty^n, n \in \mathbb{Z}\), there would also be all states \(\pm(k-2nm,m)\). The latter would become massless somewhere on \(\mathcal{C}\) if \((k-2nm/m) = (k/m) - 2n \in [-1,1]\). Since there is always such an \(n \in \mathbb{Z}\), this state, and hence \(\pm(k,m)\) cannot exist in \(\mathcal{S}_W\). Finally, the W boson which is part of the perturbative spectrum is left invariant by \(M_\infty\): \(M_\infty(1,0) = -(1,0)\), where the minus sign simply corresponds to the antiparticle. Hence we conclude

\[
\mathcal{S}_W = \{\pm(1,0), \pm(n,1), n \in \mathbb{Z}\}.
\]

This result was already known from semi-classical considerations on the moduli space of multi-monopole configurations [22, 23], but it is nice to rederive it in this particularly simple way. Now let us turn to the new results of [6] concerning the strong-coupling spectrum.

### 4.3 The \(\mathbb{Z}_2\) symmetry

As discussed above, the classical susy \(SU(2)\) Yang-Mills theory has a \(U(1)_R\) R-symmetry acting on the scalar \(\phi\) as \(\phi \rightarrow e^{2i\alpha}\phi\) so that \(\phi\) has charge two. In the quantum theory this global symmetry is anomalous, and it is easy to see from the explicit form of the one-loop and instanton contributions to the low-energy effective action (i.e. to \(\phi\)) that only a discrete subgroup \(\mathbb{Z}_8\) survives, corresponding to phases \(\alpha = \frac{2\pi}{8}k, k \in \mathbb{Z}\). Hence under this \(\mathbb{Z}_8\) one has \(\phi^2 \rightarrow (-)^k \phi^2\). This \(\mathbb{Z}_8\) is a symmetry of the quantum action and of the Hamiltonian, but a given vacuum with \(u = (\text{tr} \phi^2) \neq 0\) is invariant only under the \(\mathbb{Z}_4\) subgroup corresponding to even \(k\). The quotient (odd \(k\)) is a \(\mathbb{Z}_2\) acting as
Although a given vacuum breaks the full $\mathbb{Z}_8$ symmetry, the broken symmetry (the $\mathbb{Z}_2$) relates physically equivalent but distinct vacua. In particular, the mass spectra at $u$ and at $-u$ must be the same. This means that for every BPS state $(n_e, n_m)$ that exists at $u$ there must be some BPS state $(\tilde{n}_e, \tilde{n}_m)$ at $-u$ having the same mass:

$$|\tilde{n}_e a(-u) - \tilde{n}_m a_D(-u)| = |n_e a(u) - n_m a_D(u)|.$$  \hspace{1cm} (4.9)

This equality shows that there must exist a matrix $G \in Sp(2, \mathbb{Z})$ such that

$$\begin{pmatrix} \tilde{n}_e \\ \tilde{n}_m \end{pmatrix} = \pm G \begin{pmatrix} n_e \\ n_m \end{pmatrix},$$

$$\begin{pmatrix} a_D \\ a \end{pmatrix}(-u) = e^{i\omega} G \begin{pmatrix} a_D \\ a \end{pmatrix}(u)$$  \hspace{1cm} (4.10)

where $e^{i\omega}$ is some phase. Indeed, from the explicit expressions of $a_D$ and $a$ one finds, using standard relations between hypergeometric functions, that

$$G = G_{W,\epsilon} \equiv \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}, \hspace{1cm} e^{i\omega} = e^{-i\pi\epsilon/2}$$  \hspace{1cm} (4.11)

where $\epsilon = \pm 1$ according to whether $u$ is in the upper or lower half plane. The subscript $W$ indicates that this is the matrix to be used in the weak-coupling region, while for the strong-coupling region there is a slight subtlety to be discussed soon. We have just shown that for any BPS state $(n_e, n_m)$ existing at $u$ (in the weak-coupling region) with mass $m$ there exists another BPS state $(\tilde{n}_e, \tilde{n}_m) = \pm G_{W,\epsilon}(n_e, n_m)$ at $-u$ with the same mass $m$. Now, since both $u$ and $-u$ are outside the curve $\mathcal{C}$, they can be joined by a path never crossing $\mathcal{C}$, see Fig. 3, and hence the BPS state $(\tilde{n}_e, \tilde{n}_m)$ must also exist at $u$, although with a different mass $\tilde{m}$. So we have been able to use the broken symmetry to infer the existence of the state $(\tilde{n}_e, \tilde{n}_m)$ at $u$ from the existence of $(n_e, n_m)$ at the same point $u$ of moduli space. Starting from the magnetic monopole $(0,1)$ at $u$ in the upper half plane (outside $\mathcal{C}$) one deduces the existence of all dyons $(n,1)$ with $n \geq 0$. Taking similarly $u$ in the lower half plane (again outside $\mathcal{C}$) one gets all dyons $(n,1)$ with $n \leq 0$. The W boson $(1,0)$ is invariant under $G_{W,\epsilon}$. Once again, one generates exactly the weak-coupling spectrum $S_W$ of (4.8), and clearly $G_{W,\epsilon} S_W = S_W$.

![Figure 3: Taking $u$ to $u' = -u$ in the weak-coupling region $\mathcal{R}_W$ without crossing the cuts on $(-\infty, 1]$](image-url)
4.4 The strong-coupling spectrum

It is in the strong-coupling region that this $Z_2$ symmetry will show its full power. Here $M_\infty$ no longer is a symmetry, since a monodromy circuit around infinity can be deformed all through the weak-coupling region but it cannot cross $\mathcal{C}$ into the strong-coupling region since the state that is taken along this circuit may well decay upon crossing the curve $\mathcal{C}$. The relations (4.10) and (4.11) expressing $a_D(-u), a(-u)$ in terms of $a_D(u), a(u)$ nevertheless remain true. What needs to be reexamined is the relation between $\tilde{n}_e, \tilde{n}_m$ and $n_e, n_m$. This is due to the fact that there is a cut of the function $a(u)$ running between $-1$ and $1$, separating the strong-coupling region $\mathcal{R}_S$ into two parts, $\mathcal{R}_{S+}$ and $\mathcal{R}_{S-}$, as shown in Fig. 2. As a consequence, the same BPS state is described by two different sets of integers in $\mathcal{R}_{S+}$ and $\mathcal{R}_{S-}$. If we call the corresponding spectra $\mathcal{S}_{S+}$ and $\mathcal{S}_{S-}$ then we have

$$\mathcal{S}_{S-} = M_1^{-1} \mathcal{S}_{S+}, \quad \begin{pmatrix} n'_e \\ n'_m \end{pmatrix} = M_1^{-1} \begin{pmatrix} n_e \\ n_m \end{pmatrix},$$

$$M_1^{-1} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}. \quad (4.12)$$

This change of description is easily explained: take a BPS state $(n_e, n_m) \in \mathcal{S}_{S+}$ at a point $u \in \mathcal{R}_{S+}$ and transport it to a point $u' \in \mathcal{R}_{S-}$, see Fig. 4. In doing so, its mass varies continuously and nothing dramatic can happen since one does not cross the curve $\mathcal{C}$. Hence, as one crosses from $\mathcal{R}_{S+}$ into $\mathcal{R}_{S-}$, the functions $a_D$ and $a$ must also vary smoothly, which means that at $u' \in \mathcal{R}_{S-}$ one has the analytic continuation of $a_D(u)$ and $a(u)$. But this is not what one calls $a_D$ and $a$ in $\mathcal{R}_{S-}$. Rather, these analytic continuations $\tilde{a}_D(u')$ and $\tilde{a}(u')$ are related to $a_D(u')$ and $a(u')$ by the monodromy matrix around $u = 1$ which is $M_1$ as

$$\begin{pmatrix} \tilde{a}_D(u') \\ \tilde{a}(u') \end{pmatrix} = M_1 \begin{pmatrix} a_D(u') \\ a(u') \end{pmatrix}. \quad (4.13)$$

Hence the mass of the BPS state at $u'$ is

$$\sqrt{2}|n_e \tilde{a}(u') - n_m \tilde{a}_D(u')| = \sqrt{2}|n'_e a(u') - n'_m a_D(u')|$$

where $n'_e, n'_m$ are given by eq. (4.12).

Figure 4: Taking $u$ to $u' = -u$ inside the strong-coupling region $\mathcal{R}_S$ one has to cross the cut on $[-1, 1]$.

As a consequence of the two different descriptions of the same BPS state, the $G$-matrix implementing the $Z_2$ transformation on the spectrum has to be modified. As before, from the existence of $(n_e, n_m)$ at $u \in \mathcal{R}_{S+}$ one concludes the existence of a state $G_{W+}(n_e, n_m)$ at $-u \in \mathcal{R}_{S-}$. This
same state must then also exist at \( u \) but is described as \( M_1 G_{W,+}(n_e, n_m) \). Had one started with a \( u \in \mathcal{R}_{S,-} \) the relevant matrix would have been \( M_1^{-1} G_{W,-} \). Hence, in the strong-coupling region \( G_{W,\pm} \) is replaced by
\[
G_{S,\epsilon} = (M_1)^\epsilon G_{W,\epsilon} = \begin{pmatrix} 1 & \epsilon \\ -2\epsilon & -1 \end{pmatrix},
\]
and again one concludes that the existence of a BPS state \((n_e, n_m)\) at \( u \in \mathcal{R}_{S,\epsilon} \) implies the existence of another BPS state \( G_{S,\epsilon}(n_e, n_m) \) at the same point \( u \). The important difference now is that \( G_{S,\epsilon}^2 = -1 \),
\[
(4.15)
\]
so that applying this argument twice just gives back \((-n_e, -n_m)\). But this is the antiparticle of \((n_e, n_m)\) and always exists together with \((n_e, n_m)\). As far as the determination of the spectrum is concerned we do not really need to distinguish particles and antiparticles. In this sense, applying \( G_{S,\epsilon} \) twice gives back the same BPS state. Hence in the strong-coupling region, all BPS states come in pairs, or \( \mathbb{Z}_2 \) doublets (or quartets if one counts particles and antiparticles separately):
\[
\pm \begin{pmatrix} n_e \\ n_m \end{pmatrix} \in \mathcal{S}_{S^+} \Leftrightarrow \pm G_{S,\epsilon} \begin{pmatrix} n_e \\ n_m \end{pmatrix} = \pm \begin{pmatrix} n_e + n_m \\ -2n_e - n_m \end{pmatrix} \in \mathcal{S}_{S^+}
\]
and similarly for \( \mathcal{S}_{S^-} \). An example of such a doublet is the magnetic monopole \((0, 1)\) and the dyon \((1, -1) = -(1, 1)\) which are the two states becoming massless at the \( \mathbb{Z}_2 \)-related points \( u = 1 \) and \( u = -1 \). Note that in \( \mathcal{S}_{S^-} \) the monopole is still described as \((0, 1)\) while the same dyon is described as \((1, 1)\). It is now easy to show that this is the only doublet one can have in the strong-coupling spectrum. Indeed, one readily sees that either \( n_e/n_m \equiv r \) is in \([-1, 0]\) or \((n_e + n_m)/(-2n_e - n_m) = -(r + 1)/(2r + 1)\) is in \([-1, 0]\). This means that one or the other member of the \( \mathbb{Z}_2 \) doublet \((1, 1)\) becomes massless somewhere on \( \mathcal{C}^+ \), the part of the curve \( \mathcal{C} \) that can be reached from \( \mathcal{R}_{S^+} \). But as already repeatedly argued, the only states ever becoming massless are the magnetic monopole \((0, 1)\) and the dyon \((1, -1)\). Hence no other \( \mathbb{Z}_2 \) doublet can exist in the strong-coupling spectrum and we conclude that
\[
\mathcal{S}_{S^+} = \{ \pm (0, 1), \pm (-1, 1) \} \quad \Leftrightarrow \quad \mathcal{S}_{S^-} = \{ \pm (0, 1), \pm (1, 1) \}.
\]

\( \bullet \) P3 : The strong-coupling spectrum consists of only those BPS states that are responsible for the singularities. All other weak-coupling, i.e. semi-classical BPS states must and do decay consistently into them when crossing the curve \( \mathcal{C} \).

We have shown above the example of the decay of the W boson, cf. eq. (4.5), but it is just as simple to show consistency of the other decays [6].

When adding massless quark hypermultiplets next, we will see that the details of the spectrum change, however, the conclusion P3 will remain the same.

5 Generalisation to \( \mathcal{N} = 2 \) susy QCD : including hypermultiplets

5.1 Massless hypermultiplets

We will continue to consider only the gauge group SU(2) as studied in [2]. First, in this subsection, we will also restrict ourselves to the case of vanishing bare masses of the quark hypermultiplets. The
number of hypermultiplets is usually referred to as the number of flavours $N_f$. Although these quark hypermultiplets have no bare masses in the original Lagrangian, they get physical masses through the Higgs mechanism much like the W-bosons. These masses are given by the same BPS mass formula as in the previous section.

We will be very qualitative and describe only the results, referring the reader to [7] for details. The main difference with respect to the previous case of pure Yang-Mills theory is that now the BPS states carry representations of the flavour group which is the covering group of $SO(2N_f)$, namely $SO(2)$ for one flavour, $Spin(4) = SU(2) \times SU(2)$ for two flavours, and $Spin(6) = SU(4)$ for three flavours. We will present each of the three cases separately. In all cases:

- There is a curve of marginal stability diffeomorphic to a circle and going through all (finite) singular points of moduli space.
- The BPS spectra are discontinuous across these curves.
- The strong-coupling spectra (inside the curves) contain only those BPS states that can become massless and are responsible for the singularities. They form a multiplet (with different masses) of the broken global discrete symmetry, except for $N_f = 3$ where there is no such symmetry.
- All other semi-classical BPS states must and do decay consistently when crossing the curves.
- The weak-coupling, i.e. semi-classical BPS spectra, contain no magnetic charges larger than one for $N_f = 0, 1, 2$ and no magnetic charges larger than two for $N_f = 3$.

It is useful to slightly change conventions for $a(u)$ and $n_e$: we henceforth replace

$$a(u) \to a(u)/2, \quad n_e \to 2n_e,$$

so that the W-bosons now have $(n_e, n_m) = (2, 0)$. This is useful since the hypermultiplets correspond to “quarks” in the fundamental representation of SU(2) and hence have half the charge of the gauge bosons which are in the adjoint. With the new conventions the “quarks” have integer rather than half-integer charges. Also, the spectra of the pure gauge theory obtained above now read:

$$S_W = \{ \pm (2,0), \pm (2n,1), \, n \in \mathbb{Z} \} \quad (5.2)$$

$$S_{S^+} = \{ \pm (0,1), \pm (-2,1) \} \leftrightarrow S_{S^-} = \{ \pm (0,1), \pm (2,1) \} \quad (5.3)$$

According to Seiberg and Witten [2] there are 3 singularities at finite points of the Coulomb branch of the moduli space. They are related by a global discrete $Z_3$ symmetry. This $Z_3$ is the analogue of the $Z_2$ symmetry discussed previously. Its origin is slightly more complicated, however, since the original $Z_{12}$ is due to a combination of a $Z_6$ coming from the anomalous $U(1)_R$ symmetry and of the anomalous flavour-parity of the $O(2N_f)$ flavour group. In any case, the global discrete symmetry of the quantum theory is $Z_{12}$. The vacuum with non-vanishing value of $u = \langle \text{tr} \, \phi^2 \rangle$ breaks this to $Z_4$. The quotient $Z_3$ acting as $u \to e^{\pm 2\pi i/3} u$ then is a symmetry relating different but physically equivalent vacua. The three singular points are due to a massless monopole $(0,1)$, a massless dyon $(-1,1)$ and another massless dyon $(-2,1)$. Again there is a curve of marginal stability that was obtained from the explicit expressions for $a_D(u)$ and $a(u)$ [7]. It is almost a circle, and of course, it goes through the three singular points, see Fig. 5, where we also indicated the various cuts and correspondingly different portions $R_{S^+}, R_{S^-}, R_{S^0}$ of the strong-coupling region $R_S$. So here one needs to introduce three different descriptions of the same strong-coupling BPS state. The corresponding spectra are denoted $S_{S^+}, S_{S^0}$ and $S_{S^-}$. We will not give $a_D(u)$ and $a(u)$ explicitly here, but refer the reader
Figure 5: The curve of marginal stability and the three different portions of the strong-coupling region separated by the cuts, for $N_f = 1$

to ref [4]. The ratio $a_D/a$ increases monotonically from $-2$ to $+1$ as one goes along the curve in a clockwise sense, starting at the point where $(-2, 1)$ is massless. Then using exactly the same type of arguments as we did before, one obtains the weak and strong-coupling spectra. All states in the latter now belong to a single $\mathbb{Z}_3$ triplet, containing precisely the three states responsible for the singularities. Denoting a BPS state by $(n_e, n_m)_S$ where $S$ is the $SO(2)$ flavour charge, and denoting its antiparticle $(-n_e, -n_m)_{-S}$ simply by $-(n_e, n_m)_S$, one finds [7]

\[
S_{W} = \left\{ \pm(2, 0)_0, \pm(1, 0)_1, \pm(2n, 1)_{1/2}, \pm(2n + 1, 1)_{-1/2}, n \in \mathbb{Z} \right\}
\]

\[
S_{S0} = \left\{ \pm(0, 1)_{1/2}, \pm(-1, 1)_{-1/2}, \pm(1, 0)_{1/2} \right\}
\]

(5.4)

with states in $S_{S+}$ or $S_{S-}$ related to the description in $S_{S0}$ by the appropriate monodromy matrices:

$S_{S+} = \left( \begin{array}{cc} 2 & 1 \\ -1 & 0 \end{array} \right) S_{S0}$ and $S_{S-} = \left( \begin{array}{cc} 1 & 0 \\ 1 & 1 \end{array} \right) S_{S0}$. One sees that the state $(1, 0)_{1/2}$ in $S_{S0}$ corresponds to $(2, -1)_{1/2}$ in $S_{S+}$ or to $(1, 1)_{1/2}$ in $S_{S-}$ and is the one responsible for the third singularity. Also note that $S_W$ contains the W-boson $(2, 0)_0$ and the quark $(1, 0)_1$ as well as dyons with all integer $n_e$. All decays across the curve $C$ are consistent with conservation of the mass and of all quantum numbers, i.e. electric and magnetic charges, as well as the $SO(2)$ flavour charge. For example, when crossing $C$ into $R_{S0}$, the quark decays as $(1, 0)_1 \rightarrow (0, 1)_{1/2} + (1, -1)_{1/2}$.

$N_f = 2$

This case is very similar to the pure Yang-Mills case. The global discrete symmetry acting on the Coulomb branch of moduli space is again $\mathbb{Z}_2$ and the curve of marginal stability is exactly the same, cf. Fig. 2, with the singularities again due to a massless magnetic monopole $(0, 1)$ and a massless dyon $(1, 1)$. Note however, that this is in the new normalisation where the W boson is $(2, 0)$. So this dyon has half the electric charge of the W, contrary to what happened for $N_f = 0$. With the present normalisation one finds the weak and strong-coupling spectra as

\[
S_{W} = \left\{ \pm(2, 0), \pm(1, 0), \pm(n, 1), n \in \mathbb{Z} \right\}
\]

\[
S_{S+} = \left\{ \pm(0, 1), \pm(-1, 1) \right\}
\]

(5.5)
and all decays across $\mathcal{C}$ are again consistent with all quantum numbers. For the quark one has e.g. $(1, 0) \rightarrow (0, 1) + (1, -1)$ with the flavour representations of $SU(2) \times SU(2)$ working out as $(2, 2) = (2, 1) \otimes (1, 2)$.

$N_f = 3$

In this case the global symmetry of the action is $\mathbb{Z}_4$ and a given vacuum is invariant under the full $\mathbb{Z}_4$. Consequently, there is no global discrete symmetry acting on the Coulomb branch of the moduli space. There are two singularities [2], one due to a massless monopole, the other due to a massless dyon $(-1, 2)$ of magnetic charge 2. The existence of magnetic charges larger than 1 is a novelty of $N_f = 3$. The curve of marginal stability again goes through the two singular points. It is a shifted and rescaled version of the corresponding curve for $N_f = 0$. Due to the cuts, again we need to introduce two different descriptions of the same strong-coupling BPS state. The variation of $a_D/a$ along the curve $\mathcal{C}$ is from $-1$ to $-1/2$ on $\mathcal{C}^+$ and from $-1/2$ to 0 on $\mathcal{C}^-$. Luckily, this is such that we do not need any global symmetry to determine the strong-coupling spectrum. For the weak-coupling spectrum, one uses the $M_{\infty}$ symmetry. One finds

$$S_W = \{ \pm (2, 0), \pm (1, 0), \pm (n, 1), \pm (2n + 1, 2), n \in \mathbb{Z} \}$$

$$S_{S^+} = \{ \pm (1, -1), \pm (-1, 2) \}$$

with $(1, -1) \in S_{S^+}$ corresponding to $(0, 1) \in S_{S^-}$, so this is really the magnetic monopole.

The flavour symmetry group is $SU(4)$, and the quark $(1, 0)$ is in the representation $6$, the W boson $(2, 0)$ and the dyons of magnetic charge two are singlets, while the dyons $(n, 1)$ of magnetic charge one are in the representation $4$ if $n$ is even and in $\bar{4}$ if $n$ is odd. Antiparticles are in the complex conjugate representations of $SU(4)$. Again, all decays across the curve $\mathcal{C}$ are consistent with all quantum numbers, and in particular with the $SU(4)$ Clebsch-Gordan series. As an example, consider again the decay of the quark, this time as $(1, 0) \rightarrow 2 \times (0, 1) + (1, -2)$. The representations on the l.h.s. and r.h.s. are $6$ and $4 \otimes 4 \otimes 1$. Since $4 \otimes 4 = 6 \oplus 10$ this decay is indeed consistent. All other examples can be found in [4].

5.2 Massive hypermultiplets, RG flows and superconformal points

Introducing bare masses for the quark hypermultiplets adds a non-negligible technical complication to the previous stability analysis: the BPS mass formula now is modified and becomes

$$m_{\text{BPS}}(u) = \sqrt{2} \left| n_m a_D(u) - n_e a(u) + \sum_i s_i \frac{m_i}{\sqrt{2}} \right|,$$

where the $s_i$ are integers or half-integers which correspond to constant parts of the physical baryonic charges [3, 24]. Indeed the fractional fermion numbers $S_i(u)$ are non-trivial sections over the moduli space. While their $u$-dependent part already is included in the relevant $a(u)$ and $a_D(u)$, for each type of BPS state there is a constant part $s_i$ that cannot be consistently removed by shifting the $a(u)$ or $a_D(u)$. The bottom line is that there are $N_f$ non-vanishing quantum number $s_i$ for each BPS multiplet and they appear in the BPS mass formula multiplying the bare masses $m_i$ of the quarks. This implies that there is not a single curve of marginal stability, but an infinity, one for each decay mode.
While this complicates a lot the analysis of the spectra of stable BPS states, it also opens the way to studying much richer systems like e.g. larger gauge groups. The $m_i$ appear in the BPS mass formula as additional parameters, much as the coordinate on moduli space $u$, so understanding this case opens the door to studying higher rank gauge groups where the moduli space has complex dimension larger than one.

Another phenomenon which can and does occur is the appearance of superconformal points. As one varies the masses, the singularities move around in moduli space and for certain special values of the masses several singularities coincide. At these points, several, mutually non-local states have vanishing BPS mass and the theory is superconformal. This will be discussed below.

Again, in this subsection, we only give a rather brief summary of this complex situation, and refer the reader to [8] for more details.

**Decay curves**

It follows from the above BPS mass formula (5.7) and the same type of reasoning as in the previous section that a BPS state is stable against any decay of the type

$$(n_e, n_m)_{s_i} \to k \times (n'_e, n'_m)_{s'_i} + l \times (n''_e, n''_m)_{s''_i}$$

(5.8)

$(k, l \in \mathbb{Z})$ unless this satisfies at the same time the conservation of charges and of the total BPS mass:

$$n_e = kn'_e + ln''_e , \quad n_m = kn'_m + ln''_m , \quad s_i = ks'_i + ls''_i \quad \Rightarrow \quad Z = k Z' + l Z''$$

(5.9)

and

$$|Z| = |k Z'| + |l Z''|$$

(5.10)

with obvious notations for $Z'$ and $Z''$. If all bare masses $m_i$ are equal, due to the SU($N_f$) flavour symmetry, only the sum $s = \sum_i s_i$ is relevant and needs to be conserved. We see that a decay that satisfies the charge conservations (5.9) is possible only if

$$\frac{Z'}{Z} \equiv \zeta \in \mathbb{R},$$

(5.11)

and moreover if it is kinematically possible, i.e. if

$$0 \leq k \zeta \leq 1.$$  

(5.12)

For the case of vanishing bare masses, $m_i = 0$ condition (5.11) reduces to $\Re m \frac{a_{\phi(u)}}{a(u)} = 0$ which yields a single curve $C^0$ on the Coulomb branch independent of the initial state $(n_e, n_m)_{s_i}$ considered. For non-vanishing bare masses however, we have a whole family of possible decay curves. Moreover, a priori, there is a different family of such curves for each BPS state. As an example consider a dyon with $n_m = 1$. Then condition (5.11) reads

$$\Re m \frac{n'_m a_D - n'_e a + \sum_i s'_i \frac{m}{\sqrt{2}}}{a_D - n_e a + \sum_i s'_i \frac{m}{\sqrt{2}}} = 0 \quad \Leftrightarrow \quad \Re m \frac{- (n'_e - n'_m n_e) a + \sum_i (s'_i - n'_m s_i) \frac{m}{\sqrt{2}}}{a_D - n_e a + \sum_i s'_i \frac{m}{\sqrt{2}}} = 0.$$  

(5.13)

For fixed $n_e$ and $s_i$, this is an $N_f$-parameter family of curves with rational parameters $r_i = (n'_e - n'_m n_e)/(s'_i - n'_m s_i)$. Even though there are some relations between the possible quantum numbers $n'_e$.
and \( s'_i, n'_m \) there are still many possible values of \( r_i \) and we expect a multitude of curves of marginal stability on the Coulomb branch of moduli space resulting in a rather chaotic situation. Fortunately not all of these curves satisfy the additional criterion (5.12). In particular, for the case of \( N_f = 2 \) with equal bare masses, where one expects a different one-parameter family of curves labelled by \( r = (n'_e - n'_m n_e)/(s' - n'_m s), s = s_1 + s_2 \), for each BPS state, it turned out \([3]\) that only one or two such curves in each family are relevant, i.e. satisfy the additional criterion (5.12). Hence the set of all relevant curves for all BPS states are nicely described by a single set of curves \( \mathcal{C}_{2n}^f, n \in \mathbb{Z} \), and rather than having a chaotic situation one gets a very clear picture of which states exist in which region of the Coulomb branch.

One particularly simple case is the decay of states with \( \sum s_i m_i = 0 \) into states with \( \sum s'_i m_i = 0 \). The corresponding decay curves all are given by \( \Im m \omega_0(a(u)) = 0 \), i.e. they all coincide with the curve \( \mathcal{C}^0 \). This is quite an important case, and this curve \( \mathcal{C}^0 \) still plays a privileged rôle, even for non-zero bare masses.

Note that if we had considered decays into three independent BPS states, \( (n'_e, n'_m, s_i) \rightarrow k \times (n'_e, n'_m, s'_i) + l \times (n''_e, n''_m, s''_i) + q \times (n''_e, n''_m, s'''_i) \), we would have two conditions: eq. (5.11) would be supplemented by \( \frac{Z''}{Z} \in \mathbb{R} \), so that such “triple” decays can only occur at the intersection points of two curves. When transporting a BPS state along a path from one region to another, the path can always be chosen so as to avoid such intersection points. Hence, triple decays are irrelevant for establishing the existence domains of the BPS states. Obviously, “quadruple” and higher decays, if possible at all, are just as irrelevant.

In order to determine the BPS spectra at any point on the Coulomb branch, it is most helpful to use the following reasonable claim:

- **P4**: At any point of the Coulomb branch of a theory having \( N_f \) flavours with bare masses \( m_j, 1 \leq j \leq N_f \), the set of stable BPS states is included into the set of stable BPS states of the \( m_j = 0 \) theory at weak coupling.

Note that the Coulomb branch of the \( m_j = 0 \) theory is separated into two regions, one containing all the BPS states stable at weak coupling, and the other at strong coupling containing a finite subset of the BPS states stable at weak coupling \([3, 4]\). One simple consequence of the claim (P4) is that the set of stable BPS states cannot enlarge when one goes from the \( N_f \) to the \( N_f - 1 \) theory following the RG flow which is what one naturally expects. This is perfectly consistent with the spectra determined for zero bare masses in \([5, 7]\). Another consequence, which played a prominent rôle in the work of ref \([8]\), is that the possible decay reactions between BPS states are then extremely constrained and thus the number of relevant curves of marginal stability enormously decreased.

The detailed analysis is still quite complicated and lengthy and will not presented here. We refer instead to the original paper \([8]\).

\( N_f = 2 \) with \( m_1 = m_2 \equiv m \)

As an example we present the situation for the theory with two hypermultiplets, \( N_f = 2 \), having equal bare masses \( m_1 = m_2 = m \). There are now 3 singularities, all on the real axis. We call them \( \sigma_i \) with \( \sigma_1 \leq \sigma_2 \leq \sigma_3 \). One has to distinguish two regimes according to whether \( m \) is smaller or larger than a certain critical value which is \( \Lambda_2, \Lambda_2 \) being the relevant dynamically generated mass scale. Here, we will focus on \( m < \Lambda_2 \).

We have assembled all the relevant decay curves into Fig. \([3]\) that sketches their relative positions and indicates the BPS states that decay across these curves. All curves go through \( \sigma_3 \), while the other intersection point with the real axis depends on the curve: they are \( \sigma_2, \sigma_1 \) and certain points...
$x_{2n}$, $n = 1, 2, \ldots$

There are several types of states: first, we have the states that become massless at the singularities. These are $(0, 1)_0$ and, due to the cuts described differently in the two half planes, $(0, 1)_0$ and $(-1, 1)_{\pm 1}$ in the upper, and $(0, 1)_0$ and $(1, 1)_{\pm 1}$ in the lower half plane. These states exist everywhere (throughout the corresponding half plane).

Second, we have the other dyons of $n_e = \pm 1$, the quarks and the W-boson. These states decay on curves in the inner, strong coupling region of the Coulomb branch of moduli space: The W-boson decays on $C^\infty$, the quark $(1, 0)_{-1}$ on $C^+_0$ and the quark $(1, 0)_1$ on the innermost curve $C^-_0$, while the dyons $(\epsilon, 1)_{-e}$ decay on $C^+_0$ and the dyons $(\epsilon, 1)_e$ on $C^-_0$.

Third, we have the dyons with $|n_e| \geq 2$. As discussed above, among these one must distinguish two sorts: those that will survive the RG flow $m \to \infty$ to the pure gauge theory and those that do not. The dyons that will survive this RG flow are $(2n + 1, 1)_1$ in the upper half plane and $(2n + 1, 1)_{-1}$ in the lower half plane. These dyons $(n \neq -1, 0)$ all decay on the curve $C^\infty$ which thus plays a privileged role. The other dyons, namely $(2n, 1)_0$ $(n \neq 0)$, and $(2n + 1, 1)_{-1}$ in the upper and $(2n + 1, 1)_1$ in the lower half plane $(n \neq -1, 0)$ decay on curves $C_{2k}^\pm$, $k \neq 0$ (where $2k$ equals $|n_e|$, $|n_e| + 1$ or $|n_e| - 1$). There are only two states that decay on each of these curves $C_{2k}^\pm$, $k \neq 0$. These curves move more and more outwards as $m$ is increased. Also, as $|k|$ gets bigger (i.e. the $|n_e|$ of the corresponding dyons increase) these curves more and more reach out towards the semiclassical region. Conversely, as $m \to 0$, all curves flow towards a single curve, say $C^\infty$.

There are a couple of other points worth mentioning. First remark, that the whole picture is compatible with the $CP$ transformation $(n_e, n_m)_s \to (-n_e, n_m)_{-s}$ under reflection by the real $u$-axis. Second, since all curves go through the singularity $\sigma_3$, i.e. all existence domains touch $\sigma_3$, it follows that at this point all BPS states exist. The same is true for the points $u$ that lie on the part of the real $u$ line to the right of $\sigma_3$. Indeed, as $|n_e|$ is increased, the corresponding dyon curves leaving $\sigma_3$ to the right with an ever smaller slope get closer and closer to any given point on the real interval $(\sigma_3, \infty)$ but never touch it.

Finally we note that the whole picture is perfectly consistent: if a BPS state decays across a given curve, the decay products are also BPS states that must exist in the region considered, i.e. on both sides of the curve. Indeed, this is always the case. As an example, consider the dyons $(2n, 1)_0$ $(n \geq 1)$. In the upper half plane they decay on the curves $C_{2n}^+$ into the dyons $(2n - 1, 1)_1$ and the quark $(1, 0)_{-1}$. These dyons $(2n - 1, 1)_1$ exist everywhere in the upper half plane outside $C^\infty$, while the quark $(1, 0)_{-1}$ exists everywhere outside $C^-_0$, and in particular in the vicinity of the decay curves of $(2n, 1)_0$ considered.

### Superconformal points

As the mass $m$ is increased and gets closer to $\Lambda_2/2$, the singularities $\sigma_2$ and $\sigma_3$ approach each other and eventually coincide at $m = \Lambda_2/2$. There we have a superconformal point. As $m$ is increased beyond, the singularities separate again but correspond to different states that become massless there. The analysis of the decay curves and stable states then is different from, but analogous to the case $m < \Lambda_2/2$, and we refer the reader to \footnote{Note that according to the way $a_D$ and $a$ are defined here, the dyons $(2n, 1)$ of the $N_f = 0$ theory correspond to $(2n + 1, 1)_{\pm 1}$ in the massive $N_f = 2$ theory.}. Let us now look at $m = \Lambda_2/2$.

Call $M_1$ the monodromy matrices around $\sigma_1$ for $m < \Lambda_2/2$, and $M'_1$ the monodromy matrices around $\sigma_1$ for $m > \Lambda_2/2$. Clearly, $M_1 = M'_1$ since the singularity $\sigma_1$ is not affected by the collision of $\sigma_2$ and $\sigma_3$. Also the product of the monodromies around $\sigma_2$ and $\sigma_3$ should not be affected. Such a
statement however needs to be made with care since the precise definition of the monodromy matrices
depends on the analytic structure, i.e. how one arranges the different cuts along the real axis. With
an appropriate convention however, one has
\[
M_{sc} = M_3 M_2 = M'_3 M'_2.
\] (5.14)

\(M_{sc}\) then is the monodromy around the collapsed singularity \(\sigma_2 = \sigma_3\) at the superconformal point.
This monodromy matrix is
\[
M_{sc} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \equiv -S \in SL(2, \mathbb{Z}),
\] (5.15)
so that \(S\)-duality must be a symmetry at the superconformal point.

Which are the massless states at the superconformal point? For \(m < \Lambda_2/2\) the massless states
are \((-1,1)_{-1}\) and \((1,1)_1\) at \(\sigma_2\) and \(\sigma_3\) respectively. For \(m > \Lambda_2/2\) the massless states are \((0,1)_0\) and \((1,0)_1\) at \(\sigma_2\) and \(\sigma_3\) respectively. In each case the massless states at \(\sigma_2\) and \(\sigma_3\) are exchanged by
\(M_{sc} = -S\). At the superconformal point one might expect to have two massless states, but then
the question would be which ones. It turns out that actually all of these states, \((-1,1)_{-1}, (1,1)_1, (0,1)_0\)
and \((1,0)_1\) exist at the superconformal point and are massless. Furthermore, all other BPS states
also exist at this point, but are heavy.

One can extract quite a lot of information about the superconformal theory simply from studying
the monodromy matrices. We refer the interested reader to [8].

References

[1] N. Seiberg and E. Witten, Electric-magnetic duality, monopole condensation, and confinement
in \(N = 2\) supersymmetric Yang-Mills theory, Nucl. Phys. B426 (1994) 19, hep-th/9407087.

[2] N. Seiberg and E. Witten, Monopoles, duality and chiral symmetry breaking in \(N = 2\) super-
symmetric QCD, Nucl. Phys. B431 (1994) 484, hep-th/9408099.

[3] G. ’t Hooft, Nucl. Phys. B79 (1974) 276;
A.M. Polyakov, JETP Letters 20 (1974) 194.

[4] C. Montonen and D. Olive, Magnetic monopoles as gauge particles?, Phys. Lett. 72B (1977)
117.

[5] D. Olive, Exact electromagnetic duality, Nucl. Phys. Proc. Suppl. 45A (1996) 88, hep-th/9508089.

[6] F. Ferrari and A. Bilal, The strong-coupling spectrum of Seiberg-Witten theory, Nucl. Phys.
B469 (1996) 387, hep-th/9602082.

[7] A. Bilal and F. Ferrari, Curves of marginal stability, and weak and strong-coupling BPS spectra
in \(N = 2\) supersymmetric QCD, Nucl. Phys. B480 (1996) 589, hep-th/9605101.

[8] A. Bilal and F. Ferrari, The BPS spectra and superconformal points in massive \(N = 2\) super-
symmetric QCD, Nucl. Phys. B516 (1998) 175, hep-th/9706145.
[9] M.F. Sohnius, *Introducing supersymmetry*, Phys. Rep. **128** (1985) 39; P. Fayet and S. Ferrara, *Supersymmetry*, Phys. Rep. **32C** (1977) 1.

[10] A. Bilal, *Introduction to supersymmetry*, Lectures given at the Gif summer school 2000, hep-th/0101055.

[11] J.-P. Derendinger, *Globally Supersymmetric Theories in Four and Two Dimensions*, in: Proceedings of the Hellenic School of Particle Physics, Corfu, Greece, September 1989, edited by G. Zoupanos and N. Tracas, (World Scientific, Singapore, 1990) and www.unine.ch/phys/hepth/welcome.html.

[12] E. Witten and D. Olive, *Supersymmetry algebras that include topological charges*, Phys. Lett. **78B** (1978) 97.

[13] A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, *Simple singularities and N = 2 supersymmetric Yang-Mills theory*, Phys. Lett. **B344** (1995) 169, hep-th/9411048; A. Klemm, W. Lerche and S. Theisen, *Nonperturbative effective actions of N = 2 supersymmetric gauge theories*, Int. J. Mod. Phys. **A11** (1996) 1929, hep-th/9505150.

[14] P.C. Argyres and A.E. Faraggi, *The vacuum structure and spectrum of N = 2 supersymmetric SU(N) gauge theory*, Phys. Rev. Lett. **74** (1995) 3931, hep-th/9411057; U.H. Danielsson and B. Sundborg, *The moduli space and monodromies of N = 2 supersymmetric SO(2r + 1) Yang-Mills theory*, Phys. Lett. **B358** (1995) 273, hep-th/9504102; A. Brandhuber and K. Landsteiner, *On the monodromies of N = 2 supersymmetric Yang-Mills theory with gauge group SO(2N)*, Phys. Lett. **B358** (1995) 73, hep-th/9507008.

[15] C. Gomez and R. Hernandez, Electric-magnetic duality and effective field theories, Lectures given at Advanced School on Effective Theories, Almunecar, Spain, hep-th/9510023; L. Alvarez-Gaumé and S.F. Hassan, *Introduction to S duality in N = 2 supersymmetric gauge theories: a pedagogical review of the work of Seiberg and Witten*, Fortsch. Phys. **45** (1997) 159, hep-th/9701069.

[16] N. Seiberg, *Supersymmetry and non-perturbative beta functions*, Phys. Lett. **B206** (1988) 75.

[17] S. Coleman, *Classical lumps and their quantum descendants*, in: Aspects of Symmetry, Cambridge University Press, 1985.

[18] A. Bilal, *Duality in N=2 susy SU(2) Yang-Mills theory: A pedagogical introduction to the work of Seiberg and Witten*, hep-th/9601007; NATO ASI Series B, Physics Vol 364, eds G ’t Hooft et al, Plenum Press (1997) 21.

[19] A. Erdelyi et al, *Higher Transcendental Functions*, Vol 1, McGraw-Hill, New York, 1953.

[20] P.C. Argyres, A.E. Faraggi and A.D. Shapere, *Curves of marginal stability in N = 2 super-QCD*, hep-th/9505190; A. Fayyazuddin, *Some comments on N = 2 supersymmetric Yang-Mills*, Mod. Phys. Lett. **A10** (1995) 2703, hep-th/9504120.
[21] M. Matone, *Koebe 1/4-theorem and inequalities in N = 2 super-QCD*, Phys. Rev. D53 (1996) 7354, [hep-th/9506181](http://arxiv.org/abs/hep-th/9506181).

[22] A. Sen, *Dyon-monopole bound states, self-dual harmonic forms on the multi-monopole moduli space, and Sl(2, Z) invariance in string theory*, Phys. Lett. 329B (1994) 217, [hep-th/9402032](http://arxiv.org/abs/hep-th/9402032).

[23] S. Sethi, M. Stern and E. Zaslow, *Monopole and dyon bound states in N = 2 supersymmetric Yang-Mills theories*, Nucl. Phys. B457 (1995) 484, [hep-th/9508117](http://arxiv.org/abs/hep-th/9508117).

[24] F. Ferrari, *Charge fractionisation in N = 2 supersymmetric QCD*, Phys. Rev. Lett. 78 (1997) 795, [hep-th/9609101](http://arxiv.org/abs/hep-th/9609101).
Figure 6: Shown are a sketch of the relative positions of the relevant decay curves for $m < \frac{N}{2}$ (for not too large $|n_e|$) as well as the BPS states that decay across these curves. Three states do not decay anywhere and still are present in the innermost region inside $C_0^-$. They are described as $(0,1)_0$ and $(-1,1)_{\pm1}$ in the upper, and as $(0,1)_0$ and $(1,1)_{\pm1}$ in the lower half plane. Note that, in reality, the angles at which the curves meet the real axis at the points $x_k$ are slightly different from what they appear to be in the Figure: indeed, the curves $C_{-k-2}^-$, resp. $C_k^+$, in the upper half plane are the smooth continuations of the curves $C_{-k}^-$, resp. $C_{-k+2}^-$, in the lower half plane, in agreement with the monodromy around infinity.