Equivalences Induced by Infinitely Generated Silting Modules

Simion Breaz¹ · George Ciprian Modoi¹

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Abstract
We study equivalences induced by a complex \( P \), consisting of projectives and concentrated in degrees \(-1\) and \(0\), which is silting in the derived category \( D(R) \) of a ring \( R \).

Keywords
Silting module · Silting complex · Endomorphism ring · Endomorphism dg-algebra · dg-module · Derived functors

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1 Introduction

A torsion theory in an abelian category \( \mathcal{A} \) (e.g. \( \mathcal{A} = \text{Mod}(R) \) is the category of right \( R \)-modules) is a pair \( \tau = (\mathcal{T}, \mathcal{F}) \), such that the classes \( \mathcal{T} \) and \( \mathcal{F} \) satisfy that \( \text{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0 \), and for every \( X \in \mathcal{A} \) there is a short exact sequence \( 0 \rightarrow T \rightarrow X \rightarrow F \rightarrow 0 \), such that \( T \in \mathcal{T} \) and \( F \in \mathcal{F} \). Then \( \mathcal{T} \) and \( \mathcal{F} \) are called the torsion class, respectively the torsion free class of \( \tau \).

In the context of a triangulated category \( \mathcal{D} \) endowed with the shift functor \([-1] : \mathcal{D} \rightarrow \mathcal{D} \) (e.g. \( \mathcal{D} = D(R) \) the derived category of the category of \( R \)-modules), a \( t \)-structure is a pair \((A, B)\) of full subcategories if \( \mathcal{D} \) such that

1. \( \text{Hom}_{\mathcal{D}}(A, B[-1]) = 0 \).
2. \( A \subseteq A[-1] \) (or equivalently \( B[-1] \subseteq B \)).
3. For every \( X \in \mathcal{D} \) there is a triangle \( X' \rightarrow X \rightarrow X'' \rightarrow \), where \( X' \in A \) and \( X'' \in B[-1] \).

¹ Faculty of Mathematics and Computer Science 1, Babeș-Bolyai University, Mihail Kogălniceanu, 400084 Cluj–Napoca, Romania
The heart of a t-structure \((A, B)\) is defined to be the subcategory \(H = A \cap B\). We recall that the heart \(H\) is an abelian category. Note that the definition of a t-structure implies immediately that the inclusion functors \(A \to D\) and \(B \to D\) have a right, respectively a left adjoint. For more informations about torsion pairs and t-structures one can consult [19, Chapter I, Section 2].

One of the central results in Tilting Theory is the Tilting Theorem, [13, Theorem 3.5.1], which states that if \((T, F)\) the torsion theory generated by a finitely presented (i.e. classical) tilting right \(R\)-module \(T\) then there exists a torsion theory \((X, Y)\) in the category of right \(E\)-modules \((E\) is the endomorphism ring of \(T\)) and a pair of equivalences

\[
\text{Hom}_R(T, -) : T \cong Y : - \otimes_E T \quad \text{and} \quad \text{Ext}^1_R(T, -) : T \cong X : \text{Tor}_1^E(\_, T).
\]

Such a pair of equivalences is called a counter-equivalence. It was proven in [12] and [15] that the existence of a counter equivalence is strongly related to the existence of a classical tilting module which generates \(T\). In the case of infinitely generated tilting modules, versions of Tilting Theorem was formulated at the level of for derived category in [6] and [7]. The main idea is that every tilting module \(S\) is equivalent to a good tilting module \(T \in \text{Mod}(R)\) which induces an equivalence between the derived category \(D(R)\) and a subcategory of the derived category \(D(\text{End}(T))\). This equivalence also induces a counter equivalence at the level of module categories, i.e. the functors \(\text{Hom}_R(T, -)\) and \(\text{Ext}^1_R(T, -)\) are fully faithful, and the quasi-inverses of the functors are induced by \(- \otimes E T\) and \(\text{Tor}_1^E(\_, T)\), [6, 16, 17].

In order to be more precise, let us start with some settings and well–known definitions. In this paper all rings are unital, all categories and functors are additive, and all classes of objects are closed under isomorphisms. If \(R\) is a ring then \(\text{Mod}(R)\) denotes the category of right \(R\)-modules, and \(\text{D}(R)\) is the associated derived category of \(\text{Mod}(R)\). If \(\mathbb{P}\) is a complex, then \(H^n(\mathbb{P})\) denotes the \(n\)-th cohomology group associated to \(\mathbb{P}\). If \(C\) is a category and \(X\) is an object in \(C\) then \(\text{Add}(X)\) (resp. \(\text{add}(X)\)) denotes the class of all objects isomorphic to direct summands of (finite) direct sums of copies of \(X\). If \(F : C \to D\) is a functor then \(\text{Ker} F\) denotes the class of all objects \(X\) from \(C\) such that \(F(X) = 0\). If \(T\) and \(M\) are \(R\)-modules then \(M\) is \(T\)-generated if there exists an epimorphism \(T^{(1)} \to M\), and \(\text{Gen}(T)\) denotes the class of all \(T\)-generated modules.

Now we return to the case of an \(R\)-module \(T\), with \(E = \text{End}_R(T)\). If \(T\) is tilting, then the torsion theory \((T, F)\) associated with \(T\) has \(T = \text{Gen}(T)\). By [19, Chapter 1, Proposition 2.1] it induces a t-structure in the derived category \(D(R)\) of \(R\), whose heart \(H\) is equivalent to the category of right \(E\)-modules. This equivalence is realized by the derived Hom functor \(\text{RHom}_R(T, -)\), and its quasi-inverse is computed by using the derived tensor product. Conversely, it was proved that the heart of the t-structure associated to a torsion theory is equivalent to a module category if the torsion class is generated by a module \(T\) which has a projective presentation \(P^{-1} \to P^0 \to T \to 0\) such that the associated complex

\[
\mathbb{P} = \cdots \to 0 \to P^{-1} \to P^0 \to 0 \to \cdots
\]

has some special properties (it is compact and silting) in the derived category [14, 18, 26, 29, 30]. In particular, the support \(\tau\)-tilting modules introduced in [1] admit such a projective presentation.

Silting modules are generalizations of tilting ones and they were introduced in [5] as infinitely generated versions of support \(\tau\)-tilting modules. Further they are characterized as the modules of the form \(H^0(\mathbb{P})\), where \(\mathbb{P}\) is a two term silting complex. We refer to [20, 27], and [31] for various correspondences realized by such complexes. The main aim of the present paper is to study some equivalences induced by silting modules, providing a Silting Theorem, that is a correspondent for the Tilting Theorem. This can be useful
since for perfect or hereditary rings many torsion theories are generated by silting modules, [10], but there are many of them which are not generated by tilting modules, [3].

It was proved in [21, Theorem 3.8] that the Hom-covariant functor and the tensor functor induced by a support τ-tilting module define an equivalence as in the above described counter-equivalences. If $T$ is a silting module then it still induces a torsion pair $(\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = \text{Gen}(T)$. If $U$ is the annihilator ideal for $\mathcal{T}$ then $T$ is an $R/U$-tilting module (possible infinitely generated), hence the Tilting Theorem proved in [6] can be applied to deduce that $\text{Hom}_R(T, -) : \text{Mod-}R \rightarrow \text{Mod-}E$ induces an equivalence between $\mathcal{T}$ and its image with $- \otimes_E T$ as a quasi-inverse. But a direct application of the Tilting Theorem does not give us information for the whole class $\mathcal{F}$. For a support τ-tilting module $T$, the case when the covariant functors $\text{Ext}^1_R(T, -)$ and $\text{Tor}^1_F(-, T)$ induce an equivalence is characterized in [32].

If $R$ is hereditary, by [4, Proposition 5.2] it follows that the annihilator of a silting module is idempotent, and it is easy to see using [32, Theorem 2.1] that in the case of support τ-tilting modules the covariant functor $\text{Ext}^1_R(T, -)|_{\mathcal{F}}$ induces an equivalence with the quasi-inverse the functor $\text{Tor}^1_F(-, T)$ iff the module $T$ is tilting (at the level of derived categories the same conclusion can be obtained by using [31, Theorem A]).

The main aim of this paper is to study the equivalences induced by a silting module associated to a silting complex $\mathbb{P}$, and to extend the results proved in [11] and [18] for the support τ-tilting case. In contrast with the tilting case, when we consider a silting object $\mathbb{P} \in \mathbf{D}(R)$, the module $T = H^0(\mathbb{P})$ does not carry all information we need since a silting complex is not quasi-isomorphic (i.e. it is not isomorphic in the derived category) to the corresponding silting module. Therefore we have to deal not only with the module $T$ but with the whole complex $\mathbb{P}$. In Section 2 are gathered necessary results about dg-modules over dg-algebras. In Section 3 we recall the definitions for silting modules and silting complexes, and some basic properties connected to the torsion theory $(\mathcal{T}, \mathcal{F})$ associated with such a module (complex). Next we construct a good silting module $T$ which generates $\mathcal{T}$. Therefore, every silting complex will be equivalent (in the sense that they induce the same torsion theory) to a good one. If $\mathbb{P} = P^{-1} \xrightarrow{\sigma} P^0$ is a silting complex, for which $H^0(\mathbb{P}) = T$, then we consider the right derived Hom functor and its left adjoint (namely the left derived tensor product) between the category $\mathbf{D}(R)$ and the derived category of the dg-endomorphism algebra $\text{DgEnd}_R(\mathbb{P})$ of $\mathbb{P}$. In Section 4 we state and prove the targeted Silting Theorem, first at the level of the derived categories, that is for a good silting complex of $R$-modules $\mathbb{P}$ we construct an equivalence between $\mathbf{D}(R)$ and a subcategory of $\mathbf{D}(B)$ where $B$ is a smart truncation of $\text{DgEnd}_R(\mathbb{P})$. Then we specialize the above equivalence, in order to obtain the so called, a silting counter equivalence between the torsion theory induced by a silting module and some subcategories of the torsion-free class, repsectively the torsion class of the torsion pair $(\mathcal{U}, \mathcal{V})$ in $\text{Mod}(\mathbb{E})$ which is defined by $\mathcal{U} = \text{Ker}(- \otimes_E T)$.

## 2 Preliminaries

We recall here some generalities about dg-algebras and the total derived functors between their derived categories. We will follow [24, 25], and [23] in these considerations.

Let $k$ be a commutative ring. Recall that a $\text{dg-algebra}$ is a $\mathbb{Z}$-graded $k$-algebra $B = \bigoplus_{i \in \mathbb{Z}} B^i$ endowed with a differential $d : B \rightarrow B$ such that $d^2 = 0$ which is homogeneous of degree 1, that is $d(B^i) \subseteq B^{i+1}$ for all $i \in \mathbb{Z}$, and satisfies the graded Leibniz rule:

$$d(ab) = d(a)b + (-1)^i ad(b), \text{ for all } a \in B^i \text{ and } b \in B.$$
A (by default, right) dg-module over $B$ is a $\mathbb{Z}$-graded module

$$M = \bigoplus_{i \in \mathbb{Z}} M^i$$

endowed with a $k$-linear square-zero differential $d : M \to M$, which is homogeneous of degree 1 and satisfies the graded Leibnitz rule:

$$d(xb) = d(x)b + (-1)^i xd(b), \text{ for all } x \in M^i \text{ and } b \in B.$$

Left dg-$B$-modules are defined similarly. A morphism of dg-$B$-modules is a $B$-linear map $f : M \to N$ compatible with gradings and differentials. In this way we obtain the category $\text{Mod}(B)$ of all dg-$B$-modules.

If $B$ is a dg-algebra, then the dual dg-algebra $B^{\text{op}}$ is defined as follows: as graded $k$-modules $B^{\text{op}} = B$, the multiplication is given by $ab = (-1)^{ij} ba$ for all $a \in B^i$ and all $b \in B^j$ and the differential $d : B^{\text{op}} \to B^{\text{op}}$ is the same as in the case of $B$. It is clear that a left dg-$B$-module $M$ is a right dg-$B^{\text{op}}$-module with the “opposite” multiplication $xa = (-1)^{ij} ax$, for all $a \in B^i$ and all $x \in M^j$, henceforth we denote by $\text{Mod}(B^{\text{op}})$ the category of left dg-$B$-modules.

For a dg-module $M \in \text{Mod}(B)$ and for all $n \in \mathbb{Z}$ we define the $n$-th cocycles, coboundaries, respectively cohomology $B^0$-modules by

$$Z^n(M) = \ker(M^n \xrightarrow{d} M^{n+1}), \ B^n(M) = \text{im}(M^{n-1} \xrightarrow{d} M^n), \text{ and}$$

$$H^n(M) = Z^n(M)/B^n(M).$$

Note that these formulas induce functors into the category of $B^0$-modules.

A morphism of dg-modules is called quasi-isomorphism if it induces isomorphisms in all cohomologies. A dg-module $M \in \text{Mod}(B)$ is acyclic if $H^n(M) = 0$ for all $n \in \mathbb{Z}$. A morphism of dg-$B$-modules $f : M \to N$ is called null–homotopic provided that there is a graded homomorphism $s : M \to N$ of degree $-1$ such that $f = sd + ds$. The homotopy category $\mathbf{K}(B)$ has the same objects as $\text{Mod}(B)$ and the morphisms are equivalence classes of morphism of dg-modules, up to homotopy. It is well–known that the homotopy category is triangulated. Moreover a null–homotopic morphism has zero cohomology, therefore the functors $H^n$ factor through $\mathbf{K}(B)$ for all $n \in \mathbb{Z}$.

The derived category $\mathbf{D}(B)$ is obtained from $\mathbf{K}(B)$ by formally inverting all quasi-isomorphisms. An object $U \in \mathbf{D}(B)$ is called cofibrant if for every acyclic dg-$B$-module $N$ we have $\text{Hom}_{\mathbf{K}(B)}(U, N) = 0$. This is equivalent to

$$\text{Hom}_{\mathbf{D}(B)}(U, M) = \text{Hom}_{\mathbf{K}(B)}(U, M)$$

for all dg-$B$-modules $M$. Dually we define fibrant objects.

For two dg-modules $M, N \in \text{Mod}(B)$ we consider the so called dg-Hom complex

$$\text{Hom}^\bullet_B(M, N) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}^n_B(M, N)$$

with $\text{Hom}^n_B(M, N) = \prod_{i \in \mathbb{Z}} \text{Hom}_B(M^i, N^{n+i})$, whose differentials are given by

$$d(f)(u) = d_N f(u) - (-1)^n f d_M(u) \text{ for all } f \in \text{Hom}^n_B(M, N).$$

In this way we obtain a new category, $\text{DgMod}(B)$ whose objects are the same as the objects of $\text{Mod}(B)$, that is dg-modules, but whose morphisms are dg-Hom complexes. Note that the morphisms in $\text{Mod}(B)$ and $\mathbf{K}(B)$ between the dg-modules $M$ and $N$, are exactly $Z^0 \text{Hom}^\bullet_B(M, N)$, respectively $H^0 \text{Hom}^\bullet_B(M, N)$. 

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Let now $A$ and $B$ be two dg-algebras and let $U$ be a dg-$B$-$A$-bimodule (that is $U$ is a dg-$B^{op} \otimes_k A$-module). In this situation, for every $X \in \text{Mod}(A)$ the dg-Hom complex $\text{Hom}^\bullet_A(U, X)$ becomes a dg-$B$-module, so we get a functor (the definition on morphisms is obvious)

$$\text{Hom}^\bullet_A(U, -) : \text{Mod}(A) \to \text{Mod}(B).$$

It induces the right derived Hom functor

$$\text{RHom}_A(U, -) : \text{D}(A) \to \text{D}(B),$$

where $\text{RHom}_A(U, X) = \text{Hom}^\bullet_A(U', X) \cong \text{Hom}^\bullet_A(U, X')$ where $U'$ is a cofibrant replacement of $U$ (that is, a cofibrant dg-$A$-module $U'$ together with a quasi-isomorphism $U' \to U$) and $X'$ is a fibrant replacement of $X$ (which is defined by duality), see [34, Theorem 12.1.1]. It was proved in [24, Theorem 3.1] that (co)fibrant replacements always exist in $\text{K}(A)$.

Let $M \in \text{Mod}(B)$. There exists a natural grading on the usual tensor product $M \otimes_B U$, which can be described as:

$$M \otimes^\bullet_B U = \bigoplus_{n \in \mathbb{Z}} M \otimes_B^n U,$$

where $M \otimes_B^n U$ is the quotient of $\bigoplus_{i \in \mathbb{Z}} M_i \otimes_B^n U_{n-i}$ by the submodule generated by $m \otimes bu - mb \otimes u$ where $m \in M_i$, $u \in U^j$ and $b \in B^{n-i-j}$, for all $i, j \in \mathbb{Z}$. Together with the differential

$$d(m \otimes u) = d(m) \otimes u + (-1)^j m \otimes d(u),$$

for all $m \in M^i$, $u \in U^j$, we obtain a functor $- \otimes^\bullet_B U : \text{Mod}(B) \to \text{Mod}(A)$, and further a triangle functor $- \otimes^\bullet_B U : \text{K}(B) \to \text{K}(A)$. The left derived tensor product

$$- \otimes^L_B U : \text{D}(B) \to \text{D}(A)$$

is defined by $Y \otimes^L_B U = Y' \otimes^\bullet_B U \cong Y \otimes^\bullet_B U'$, where $Y'$ and $U'$ are cofibrant replacements for $Y$ and $U$ in $\text{K}(B)$ and $\text{K}(B^{op})$ respectively.

A dg-algebra $B = \bigoplus_{i \in \mathbb{Z}} B^i$ is called (homologically) non-positive if $B^i = 0$ (respectively $H^i(B) = 0$) for $i > 0$.

### 3 Two Term Silting Complexes

#### 3.1 Silting Modules and Silting Complexes

Let $R$ be a unital ring. If $P^{-1} \xrightarrow{\sigma} P^0$ is a morphism between projective right $R$-modules then the defect of $\sigma$ is defined as the functor

$$\text{Def}_\sigma(-) = \text{Coker}(\text{Hom}_R(\sigma, -)) : \text{Mod}(R) \to \text{Ab}.$$ 

We will denote by $\mathcal{D}_\sigma$ the kernel (on objects) of $\text{Def}_\sigma$, i.e. the class of all modules $L \in \text{Mod}(R)$ such that every morphism $\alpha : P^{-1} \to L$ can be extended to a morphism $P^0 \to L$.

We recall from [5] that a right $R$-module $T$ is silting with respect to a projective resolution $P^{-1} \xrightarrow{\sigma} P^0 \to T \to 0$ if

(s) $\text{Gen}(T) = \mathcal{D}_\sigma$.

It is easy to see that $\text{Gen}(T)$ is closed under direct sums and epimorphic images. Using [9, Proposition 4], it follows that $\text{Ker}(\text{Def}_{\sigma})$ is closed under extensions. Therefore, if $T$ is
silting with respect to $\sigma$ then the class $T = \text{Gen}(T) = \mathcal{D}_\sigma$ is a torsion class. We will denote by $\tau = (T, \mathcal{F})$ the induced torsion theory in $\text{Mod}(R)$.

In this case we associate to $\sigma$ the complex

$$\mathbb{P} = \cdots \to 0 \to P^{-1} \xrightarrow{\sigma} P^0 \to 0 \to \cdots$$

of projective modules, and we note that $T$ is silting with respect to $\sigma$ if and only if $\mathbb{P}$ is a silting complex of projective modules (cf. [5, Theorem 4.9]), i.e.

(S1) $\mathbb{P}^{(I)} \in \mathbb{P}^{\perp_{\geq 0}}$ for all sets $I$, and

(S2) the homotopy category $\mathcal{K}^b(\text{Proj}(R))$ is the smallest triangulated subcategory of $\mathcal{D}(R)$ containing $\text{Add}(\mathbb{P})$.

where

$$\mathbb{P}^{\perp_{\geq 0}} = \{ Y \in \mathcal{D}(R) \mid \text{Hom}_{\mathcal{D}(R)}(\mathbb{P}, Y[n]) = 0 \text{ for all positive integers } n \}.$$

If $\mathbb{P}$ satisfies only the condition (S1) then it is called presilting.

**Remark 3.1.1** In the literature a complex as before is also called 2-term silting complex, in order to emphasize that it contains only two non-zero entries. There are also defined $n$-term silting complexes, which are complexes with $n$ non-zero entries satisfying (S1) and (S2).

We refer to [2] for a recent survey on this subject. However in what follows we entirely stick to the case of a 2-term silting complex, hence we drop the expression “2-term” from our considerations.

The following lemma is straightforward. It records connections between the functors induced by $T$ and $\mathbb{P}$.

**Lemma 3.1.2** Let $\mathbb{P} \in \mathcal{D}(R)$ be a complex induced by a morphism $\sigma : P^{-1} \to P^0$ between projective modules, and denote $T = H^0(\mathbb{P})$. Then for every $M \in \text{Mod}_R$ there are canonical isomorphisms:

1. $\text{Hom}_{\mathcal{D}(R)}(\mathbb{P}, M) \cong \text{Hom}_R(T, M)$;
2. $\text{Hom}_{\mathcal{D}(R)}(\mathbb{P}, M[1]) \cong \text{Def}_\sigma(M)$;
3. $T \cong \text{Hom}_{\mathcal{D}(R)}(R, \mathbb{P})$.

From [5, Theorem 4.6] we extract the following useful result:

**Lemma 3.1.3** If $\mathbb{P}$ is a silting complex then

$$\text{Add}(\mathbb{P}) = \{ X \in \mathbb{P}^{\perp_{\geq 0}} \mid \text{Hom}_{\mathcal{D}(R)}(X, Y[1]) = 0 \text{ for all } Y \in \mathbb{P}^{\perp_{\geq 0}} \}.$$
Proof (1)⇒(2) Let $I = \text{Hom}_{D(R)}(P, R[1])$, and we consider a triangle

$$R \to Q \to P(I) \to R[1]$$

induced by the canonical $\text{Add}(P)$-precovering $P(I) \to R[1]$ (this means that $\text{Hom}_{D(R)}(P, \beta)$ is an epimorphism). Applying the functor $\text{Hom}_{D(R)}(P, -)$ to the above triangle, we obtain the exact sequence of $k$-modules

$$\text{Hom}_{D(R)}(P, R[i]) \to \text{Hom}_{D(R)}(P, Q[i]) \to \text{Hom}_{D(R)}(P, P(I)[i]) \to$$

for all $i \geq 0$. Since $\text{Hom}_{D(R)}(P, \beta)$ is an epimorphism, $\text{Hom}_{D(R)}(P, P(I)[i]) = 0$ for all $i > 0$, and $\text{Hom}_{D(R)}(P, R[i]) = 0$ for all $i \geq 2$, it follows that $Q \in \oplus_{i > 0} \text{Add}(P)$.

Let $Y \in \oplus_{i > 0} \text{Add}(P)$. By [5, Theorem 4.9] we know that $H^i(Y) = 0$ for all $i > 0$. Then $\text{Hom}_{D(R)}(R, Y[1]) = 0$. Since $\text{Hom}_{D(R)}(P(I), Y[1]) = 0$, it follows that $\text{Hom}_{D(R)}(Q, Y[1]) = 0$.

Therefore, we can apply Lemma 3.1.3 to obtain that $Q \in \text{Add}(P)$.

(2)⇒(1) From the existence of the triangle $R \to P' \to P'' \to R[1]$, it follows that $P$ is a generator for $D(R)$. Now the conclusion follows from [5, Theorem 4.9].

Remark 3.1.5 The exact sequence $R \to H^0(P') \to H^0(P'') \to 0$ induced in cohomology by the triangle $R \to P' \to P'' \to R[1]$ is a $\text{Gen}(T)$-preenvelope for $R$. Therefore, Proposition 3.1.4 is the triangulated version of [5, Proposition 3.11].

3.2 Good Silting Complexes

Using the same technique as in [6, Proposition 3.1] we obtain the following

Corollary 3.2.1 Let $T$ be a silting module with respect to a morphism $\sigma$, and let $P$ be the silting complex associated to $\sigma$. Then there exists a silting complex $Q$ such that

1. there exists a triangle $R \to Q \to Q' \to R[1]$ such that $Q'$ is a direct summand of $Q$;
2. the silting module $H^0(Q)$ generates the same torsion theory as $T$.

Proof (1) We start with a triangle $R \xrightarrow{\alpha} P' \xrightarrow{\beta} P'' \to R[1]$. If $Q = P' \oplus P''$ and $Q' = P' \oplus P'' \oplus \oplus P''(\omega)$ then we have a triangle

$$R \xrightarrow{\alpha \oplus 0} Q \xrightarrow{\beta \oplus 1 \oplus 0} Q' \to R[1].$$

It is easy to see that $Q$ is partial silting, hence Lemma 3.1.4 proves that $Q$ is a silting complex.

(2) Since $\text{Hom}_{D(R)}(Q'[-1], M) = 0$ for all $M \in \text{Gen}(T)$, it follows that $\text{Gen}(T) \subseteq \text{Gen}(H^0(Q))$. The converse inclusion is obvious, so we conclude that $\text{Gen}(T) = \text{Gen}(H^0(Q))$.

A torsion theory $(\mathcal{T}, \mathcal{F})$ in $\text{Mod-}R$ is called silting torsion theory if there exists a silting module $S$ such that $\mathcal{T} = \text{Gen}(S)$. By Corollary 3.2.1 we know that there exists a silting
complex $\mathbb{P}$ such that the silting module $T = H^0(\mathbb{P})$ generates the class $\mathcal{T}$ and there exists a triangle

$$R \to \mathbb{P}^n \to \mathbb{P}' \to R[1]$$

in $\mathbf{D}(R)$ such that $\mathbb{P}' \in \text{add}(\mathbb{P})$. Such a complex will be called a good silting complex.

Example 3.2.2 Every compact silting object is good. Indeed, if $\mathbb{P}$ is silting compact, then we can suppose that $\mathbb{P}^{-1}$ and $\mathbb{P}^0$ are finitely generated. Therefore, it is not hard to see that in the proof of Proposition 3.1.4 we can find a finite set $I$ and a triangle

$$R \to Q \xrightarrow{\beta} P(I) \to R[1]$$

such that $Q \in \text{Add}(\mathbb{P})$. Since the class of compact objects is closed under extensions, it follows that $Q$ is compact, so $Q \in \text{add}(\mathbb{P})$, hence $\mathbb{P}$ is a good, cf. also [11, Corollary 3.3].

3.3 Derived Functors Induced by Silting Complexes

By [23, Example 2.1 a)] we observe that the ordinary ring $R$ can be viewed as a dg-algebra concentrated in degree 0. Therefore, a dg-module over $R$ is a complex of ordinary (right) $R$-modules, hence $\text{Mod}(R)$ is the category of all complexes of $R$-modules. We can identify $\mathbf{D}(R) = \mathbf{D}(R)$, and we view $\mathbb{P}$ as an $R$-dg-module. The complex $\mathbb{P}$ is cofibrant because it is a bounded complex with projective entries. Therefore $\text{RHom}_R(\mathbb{P}, -) = \text{Hom}_R^\bullet(\mathbb{P}, -)$.

By [23, Example 2.1.b)] $\mathbb{P}$ induces a dg-algebra $\text{DgEnd}_R(\mathbb{P}) = \text{Hom}_R^\bullet(\mathbb{P}, \mathbb{P})$, called the endomorphism dg-algebra of $\mathbb{P}$.

Let us observe that $\text{RHom}_R(R, \mathbb{P})$ is the complex

$$\cdots \to 0 \to \text{Hom}_R(R, P^{-1}) \xrightarrow{\sigma \circ -} \text{Hom}_R(R, P^0) \to 0 \cdots$$

which is concentrated in the degrees $-1$ and 0. This complex has a canonical structure as a dg-module over the dg-algebra $\text{DgEnd}_R(\mathbb{P})$. Therefore $\mathbb{P}$ becomes a $\text{dg-End}_R(\mathbb{P})$-$R$-bimodule and consequently it induces the right derived covariant functors

$$\text{RHom}_R(\mathbb{P}, -) : \mathbf{D}(R) \rightleftarrows \mathbf{D}(\text{DgEnd}_R(\mathbb{P})) : - \otimes_{\text{DgEnd}_R(\mathbb{P})}^\mathbb{L}.\mathbb{P}.$$
a dg-$B$-module by restriction of scalars. As in [34, Section 12.4] we do not distinguish nota-
tionally between such a dg module seen as $\text{DgEnd}_R(\mathcal{P})$-module or a $B$-module. Moreover restriction of scalars functor is an equivalence with the quasi-inverse the induction functor, that is the derived tensor product $\mathcal{P} \otimes^L_B$. Composing this equivalence with the previous adjoint pair and using the associativity, up to a natural equivalence, of the derived tensor product, we get an adjoint pair:

$$\text{RHom}_R(\mathcal{P}, -) : \mathsf{D}(R) \leftrightarrows \mathsf{D}(B) : - \otimes^L_B \mathcal{P}.$$  

**Lemma 3.3.1** The following statements are true:

1. the functors $\text{RHom}_R(\mathcal{P}, -) : \mathsf{D}(R) \leftrightarrows \mathsf{D}(B) : - \otimes^L_B \mathcal{P}$ are triangle functors;
2. $- \otimes^L_B \mathcal{P}$ is a left adjoint for $\text{RHom}_R(\mathcal{P}, -)$;
3. $B \otimes^L_B \mathcal{P} \cong \mathcal{P}$;
4. $\text{RHom}_{B^{op}}(B, \mathcal{P}) \cong \mathcal{P}$.

**Proof** For (1) see [35, Proposition 24.4]. For (2) and (3) see [35, Proposition 21.4] completed by [35, Example 24.5].

## 4 The Silting Theorem

### 4.1 The Setting and Some Basic Properties

We are ready to fix some objects and homomorphisms which will be used in the following.

Let $k$ be a commutative ring, and $R$ a $k$-algebra. We will use the following fixed objects, morphisms, and torsion pairs:

- $\mathcal{P} = \cdots \to 0 \to P^{-1} \xrightarrow{\sigma} P^0 \to 0 \to \cdots$ is a good silting complex (hence $P^{-1}$ and $P^0$ are projective right $R$-modules);
- $T = \text{Coker}(\sigma) = H^0(\mathcal{P})$ is the corresponding silting module;
- the torsion pair generated by $T$ in $\text{Mod}(R)$ is denoted by $\tau = (T, F)$;
- we will denote by $\mathcal{H}(\tau)$ be the heart of the $t$-structure associated to $\tau = (T, F)$, i.e. the category of all objects $X \in \mathsf{D}(R)$ which lie in triangles $F[1] \to X \to M \to F$, where $F \in \mathcal{F}$ and $M \in \mathcal{I}$;
- $\mathcal{E} = \text{End}_{\mathsf{D}(R)}(\mathcal{P})$ is the endomorphism ring of $\mathcal{P}$ in the derived category of $\text{Mod}(R)$;
- we consider the torsion pair $(\mathcal{U}, \mathcal{V})$ in $\text{Mod}(\mathcal{E})$, where

$$\mathcal{U} = \{ X \in \text{Mod}(\mathcal{E}) \mid X \otimes_{\mathcal{E}} T = 0 \};$$

- we fix a triangle

$$\begin{array}{c}
\xymatrix{ R & R[1] \\
\ar^{\alpha \ast} & \ar_{\beta \ast} \mathcal{P} \ar_{\gamma} \ar \ar_{\delta} & \mathcal{P} & \mathcal{P} \ar_{\varepsilon} & \mathcal{P} \ar_0 \mathcal{P} \ar_1} \\
\end{array}$$

such that $\mathcal{P}' \in \text{add}(\mathcal{P})$.
- $\text{DgEnd}_R(\mathcal{P})$ will denote the endomorphism dg-algebra associated to $\mathcal{P}$.

**Remark 4.1.1** Applying the functor $\text{Hom}_{\mathsf{D}(R)}(-, \mathcal{P})$ on the triangle $(\dagger)$ we obtain the exact sequence of left $\mathcal{E}$-modules

$$\text{Hom}_{\mathsf{D}(R)}(\mathcal{P}', \mathcal{P}) \xrightarrow{\beta \ast} \text{Hom}_{\mathsf{D}(R)}(\mathcal{P} \ast, \mathcal{P}) \xrightarrow{\alpha \ast} \text{Hom}_{\mathsf{D}(R)}(R, \mathcal{P}) \to 0.$$
Therefore, the above exact sequence is a projective presentation for the left \( E \)-module \( T \cong \text{Hom}_{D(R)}(R, \mathbb{P}) \). In this setting it will be useful to consider the defect functor associated to the tensor product

\[
Z_{\beta^*} = \ker(- \otimes_E \beta^*)
\]

induced by \( \beta^* \).

**Remark 4.1.2** As in Section 3.3, we consider \( B \) the smart truncation of \( \text{DgEnd}_R(\mathbb{P}) \). Since \( B \) is non-positive, we apply [22, Proposition 2.1] to observe that the standard \( t \)-structure \( \mathcal{H}(B) \) exists in \( D(B) \) (that is the subcategory of \( D(B) \) consisting of objects concentrated in degree 0). The heart of this \( t \)-structure in \( D(B) \) is denoted by \( \mathcal{H}(B) \). It is easy to see that \( H^0(B) = \mathbb{E} \), and it follows that \( H^0 : \mathcal{H}(B) \to \text{Mod}(\mathbb{E}) \) is an equivalence.

**Remark 4.1.3** Applying the (contravariant) triangle functor \( \text{RHom}_R(-, \mathbb{P}) \) to the triangle \((\dagger)\) above, we obtain a triangle in \( D(B^{\text{op}}) \):

\[
(\ddagger) B' \xrightarrow{\beta^\circ} B^n \to \mathbb{P} \to B'[1]
\]

where the entries of this triangle are identified as \( B' = \text{RHom}_R(\mathbb{P}', \mathbb{P}) \in \text{add}(B), B^n = \text{RHom}_R(\mathbb{P}, \mathbb{P})^n \cong \text{RHom}_R(\mathbb{P}^n, \mathbb{P}), \beta^\circ = \text{RHom}_R(\beta, \mathbb{P}) \), and \( \text{RHom}_R(R, \mathbb{P}) \cong \mathbb{P} \).

**Remark 4.1.4** If we view \( \mathbb{E} \) as a dg-agebra concentrated in degree 0, then there is an obvious homomorphism of dg-algebras \( p : B \to \mathbb{E} \). Using [34, Theorem 12.4.23(1)], \( p \) induces the extension and the restriction of scalar functors

\[
p^*_s = - \otimes^L_B \mathbb{E} : D(B) \rightleftarrows D(\mathbb{E}) : p_*
\]

and \( p_* \) is the right adjoint of \( p^* \). Note that the restriction of \( p^* \) to \( \mathcal{H}(B) \) coincides with the restriction of \( H^0 \) to \( \mathcal{H}(B) \). Therefore, the restriction of \( p_* \) at \( \text{Mod}(\mathbb{E}) \) is a quasi-inverse of the equivalence \( H^0 : \mathcal{H}(B) \to \text{Mod}(\mathbb{E}) \).

**Lemma 4.1.5** If \( X \in D(R) \) is a complex concentrated in \(-1 \) and 0 then

\[
H^0(\text{RHom}_R(\mathbb{P}, X)) = \text{Hom}_{D(R)}(\mathbb{P}, X).
\]

Moreover, the following are equivalent

(a) \( X \in \mathcal{H}(\tau) \);
(b) \( \text{RHom}_R(\mathbb{P}, X) \in \mathcal{H}(B) \).

**Proof** The complex \( X \in \mathcal{H}(\tau) \) is isomorphic to a complex

\[
\cdots \to 0 \to X^{-1} \xrightarrow{\alpha} X^0 \to 0 \to \cdots
\]

which is concentrated in \(-1 \) and 0 such that \( \ker(\alpha) \in \mathcal{F} \) and \( \text{coker}(\alpha) \in \mathcal{T} \). Then \( \text{RHom}_R(\mathbb{P}, X) \) is the complex

\[
0 \to \text{Hom}_R(P^0, X^{-1}) \xrightarrow{(-) \circ \sigma} \text{Hom}_R(P^{-1}, X^{-1}) \times \text{Hom}_R(P^0, X^0) \to \text{Hom}_R(P^{-1}, X^0) \to 0,
\]

and the first conclusion can be obtained by a direct computation.
Lemma 4.1.6

Let $f: P^0 \to X^{-1}$ be an $R$-morphism such that $f \sigma = 0$ and $\alpha f = 0$. From $f \sigma = 0$ it follows that there exists $g: T \to X^{-1}$ such that $f = g\pi$, where $\pi: P^0 \to T$ is the cokernel of $\sigma$. Since $\pi$ is an epimorphism, it follows by $\alpha f = 0$ that $\alpha g = 0$. Then $g$ factorizes through a morphism $T \to \text{Ker}(\alpha)$. But $\text{Ker}(\alpha) \in \mathcal{F}$, and we obtain $g = 0$, hence $f = 0$.

Using similar techniques it follows that $(\alpha \circ - \circ (\sigma-\sigma))$ is surjective. Then $R\text{Hom}_R(\mathbb{P}, X)$ is in fact isomorphic to the complex concentrated in 0 which is represented by

$$H^0(R\text{Hom}_R(\mathbb{P}, X)) = \text{Hom}_{\mathcal{D}(R)}(\mathbb{P}, X).$$

(b)$\Rightarrow$(a) Let $u: \text{Ker}(\alpha) \to X^{-1}$ be the inclusion map. Suppose that $\text{Ker}(\alpha) \notin \mathcal{F}$. Then there is a nonzero morphism $g: T \to \text{Ker}(\alpha)$. It is easy to see that $u\pi : P^0 \to X^{-1}$ is a nonzero morphism which belongs to the kernel of $\left(\begin{array}{c}
\alpha \\
\alpha \circ -
\end{array}\right)$, a contradiction. Therefore, $\text{Ker}(\alpha) \in \mathcal{F}$.

Let $K = \text{Coker}(\alpha)$, and denote by $p: X^0 \to K$ the canonical surjection. We will prove that $K \in \mathcal{D}_\sigma$. If $f: P^{-1} \to K$ is a morphism, it can be lifted to a morphism $g: P^{-1} \to X^0$ such that $f = pg$. Since $(\alpha \circ - \circ (\sigma-\sigma))$ is surjective, there exist morphisms $h^i: P^i \to X^i$, $i \in \{-1, 0\}$, such that $g = ah^{-1} - h^0\sigma$. It follows that $f = -ph^0\sigma$, hence $K \in \mathcal{D}_\sigma$. Since $\mathcal{D}_\sigma = \mathcal{T}$, the proof is complete.

\[ \square \]

We recall that applying the functor $\text{Hom}_{\mathcal{D}(R)}(-, \mathbb{P})$ to the triangle $(\dagger)$ we obtain the exact sequence of left $\mathbb{E}$-modules

$$\text{Hom}_{\mathcal{D}(R)}(\mathbb{P}^0, \mathbb{P}) \xrightarrow{\beta^*} \text{Hom}_{\mathcal{D}(R)}(\mathbb{P}^1, \mathbb{P}) \xrightarrow{\alpha^*} \text{Hom}_{\mathcal{D}(R)}(R, \mathbb{P}) \to 0.$$

**Lemma 4.1.6** Let $Y$ be an object in $\mathcal{H}(B)$.

1. The restrictions of the functors

$$H^0(Y \otimes_B^L R\text{Hom}_R(-, \mathbb{P})) \text{ and } H^0(Y \otimes_{\mathbb{E}} \text{Hom}_{\mathcal{D}(R)}(-, \mathbb{P})$$

are naturally isomorphic.

2. There are natural isomorphisms of $R$-modules

$$H^0(Y \otimes_B^L \mathbb{P}) \cong H^0(Y \otimes_{\mathbb{E}} T \text{ and } H^{-1}(Y \otimes_B^L \mathbb{P}) \cong \text{Ker}(H^0(Y \otimes_{\mathbb{E}} \beta^*)].$$

3. For all $i \notin \{-1, 0\}$ we have $H^i(Y \otimes_B^L \mathbb{P}) = 0$.

**Proof** (1) Let $X = \cdots \to 0 \to X^{-1} \xrightarrow{f} X^0 \to 0 \to \cdots$ be a complex from $\text{add}(\mathbb{P})$, where $X^{-1}$ and $X^0$ are projective $R$-modules. We replace the complex $R\text{Hom}_R(X, \mathbb{P})$ by its smart truncation

$$\cdots \to 0 \to \text{Hom}_R(X^0, P^{-1}) \xrightarrow{\Phi} Z^0(R\text{Hom}_R(X, \mathbb{P})) \to 0 \to \cdots,$$

where

$$Z^0(R\text{Hom}_R(X, \mathbb{P})) = \left\{(\alpha^{-1}, \alpha^0) \in \text{Hom}(X^{-1}, P^{-1}) \times \text{Hom}(X^0, P^0) \mid \sigma\alpha^{-1} = \alpha^0\rho\right\}.$$
Since \( Y \in \mathcal{H}(B) \), we can replace it by \( p_* H^0(Y) \). The homomorphism of dg-algebras \( p : B \to \mathbb{E} \) from Remark 4.1.4 induces a ring homomorphism \( p : B^0 \to \mathbb{E} \). It follows that we suppose that \( Y \) is a complex concentrated in 0 and \( Y^0 \) is the restriction along the homomorphism \( p : B^0 \to \mathbb{E} \) of the \( \mathbb{E} \)-module \( H^0(Y) \). Moreover \( \text{Coker}(\Phi) = \text{Hom}_{\text{Der}}(X, \mathbb{P}) \).

Since \( X \in \text{add}(\mathbb{P}) \), it follows that \( \text{RHom}_R(X, \mathbb{P}) \in \text{add}(B) \) is cofibrant. It follows, by using the definition of \( Y \otimes^L_B \text{RHom}_R(X, \mathbb{P}) \) that the functors

\[
H^0(Y \otimes^L_B \text{RHom}_R(X, \mathbb{P})) \text{ and } H^0(Y) \otimes^L_B \text{Hom}_R(X, \mathbb{P})
\]

are naturally isomorphic.

Using [8, Proposition II.2] we observe that, in order to complete the proof, it is enough to prove that \( \text{Ker}(p) \) is contained in the annihilators of the \( B^0 \) modules \( H^0(Y) \) and \( \text{Hom}_R(X, \mathbb{P}) \). For \( H^0(Y) \) this is obvious since \( H^0(Y) \) is a module obtained via the restriction of scalars functor.

Let \( (\alpha^{-1}, \alpha^0) \in \text{Ker}(p) \). It follows that there exists \( s : P^0 \to P^{-1} \) such that \( \alpha^0 = \sigma s \) and \( \alpha^{-1} = s \sigma \). Note \( B^0 \) acts on \( \text{Hom}_R(X, \mathbb{P}) \) via the composition of maps (of complexes): \( (\alpha^{-1}, \alpha^0)(\beta^{-1}, \beta^0) = (\alpha^{-1} \beta^{-1}, \alpha^0 \beta^0) \) (here \( (\beta^{-1}, \beta^0) \) represents the homotopy class of \( (\beta^{-1}, \beta^0) \)). It follows that \( \text{Ker}(p) \text{Hom}_R(X, \mathbb{P}) = 0 \), and the proof is complete.

(2) We apply the triangle functor \( Y \otimes^L_B \to \) to the triangle \((\dagger)\) from Remark 4.1.3. We get a triangle

\[
Y \otimes^L_B B' \to Y \otimes^L_B B^n \to Y \otimes^L_B \mathbb{P} \to Y \otimes^L_B B'[1].
\]

Since \( Y \otimes^L_B B \cong Y \), it follows that \( Y \otimes^L_B B' = Y \otimes^L_B \text{RHom}_R(\mathbb{P}', \mathbb{P}) \) and \( Y \otimes^L_B B^n = Y \otimes^L_B \text{RHom}_R(\mathbb{P}^n, \mathbb{P}) \) are elements from \( \text{add}(Y) \). It follows that we have the following exact sequence of \( k \)-modules

\[
0 = H^{-1}(Y \otimes^L_B B^n) \to H^{-1}(Y \otimes^L_B \mathbb{P}) \to H^0(Y \otimes^L_B B') \to H^0(Y \otimes^L_B B^n) \to H^0(Y \otimes^L_B \mathbb{P}) \to H^1(Y \otimes^L_B B') = 0.
\]

By (1) this induces the exact sequence

\[
0 \to H^{-1}(Y \otimes^L_B \mathbb{P}) \to H^0(Y) \otimes^L_\mathbb{E} \text{Hom}_R(\mathbb{P}', \mathbb{P}) \xrightarrow{H^0(Y) \otimes^L_\mathbb{E} \text{Hom}_R(\mathbb{P}', \mathbb{P})} H^0(Y) \otimes^L_\mathbb{E} \text{Hom}_R(\mathbb{P}', \mathbb{P}) \to H^0(Y) \otimes^L_B \mathbb{P} \to 0,
\]

and the conclusion is now clear.

(3) This is a consequence of the proof of (2).

\[\square\]

**Remark 4.1.7** By the statement (2) in the above lemma it follows that the functor \( Z_{p^*} = \text{Ker}(\cdot \otimes^L \mathbb{P}^*) \) defined in Section 4.1 acts actually between \( \text{Mod}(\mathbb{E}) \) and \( \text{Mod}(R) \). Using a similar proof as in [9, Proposition 4], it follows that it plays a similar role with the role the functor \( \text{Tor}_1^R(\cdot, T) \) for the case when \( T \) is of flat dimension at most 1. We recall that if \( T \) is a tilting module then its flat dimension as a left \( \text{End}(T) \)-module is at most 1.

### 4.2 The Silting Theorem for Derived Categories

In the following Theorem we will describe the equivalence induced by a silting complex at the level of derived categories.
Theorem 4.2.1 Let $\mathbb{P}$ be a good silting complex. As in Setting Section 4.1, we denote by $\tau = (\mathcal{T}, \mathcal{F})$ the torsion pair generated by $T = H^0(\mathbb{P})$ in $\text{Mod}(R)$, and by $\mathcal{H}(\tau)$ the heart of the $t$-structure associated to $\tau$. Moreover, let $B$ be the smart truncation of $DgEnd_R(\mathbb{P})$ defined in Section 3.3 (see also Remark 4.1.2). We consider the full subcategories
\[ \mathcal{K} = \text{Ker}(- \otimes^L_B \mathbb{P}) \subseteq D(R), \]
and
\[ \mathcal{K}^\perp = \{ Y \in D(B) \mid \text{Hom}_{D(B)}(X, Y[n]) = 0 \text{ for all } X \in \mathcal{K} \text{ and } n \in \mathbb{Z} \}. \]

Then
1. the functor $R\text{Hom}_R(\mathbb{P}, -)$ induces an equivalence
\[ R\text{Hom}_R(\mathbb{P}, -) : D(R) \cong \mathcal{K}^\perp, \]
and $- \otimes^L_B \mathbb{P} : \mathcal{K}^\perp \to D(R)$ is a quasi-inverse for $R\text{Hom}_R(\mathbb{P}, -)$;
2. the restrictions of these functors to $\mathcal{H}(\tau)$ and $\mathcal{H}(B) \cap \mathcal{K}^\perp$ induce an equivalence
\[ R\text{Hom}_R(\mathbb{P}, -) : \mathcal{H}(\tau) \cong \mathcal{H}(B) \cap \mathcal{K}^\perp : - \otimes^L_B \mathbb{P}. \]

Proof 
1. Let us denote by $\gamma$ and $\delta$ the unit, respectively the counit, associated to the adjunction $(- \otimes^L_B \mathbb{P}) \dashv R\text{Hom}_R(\mathbb{P}, -)$. Then the map $\gamma_B : B \to R\text{Hom}_R(\mathbb{P}, B \otimes^L_B \mathbb{P})$ is an isomorphism, and the triangle $(\dagger)$ implies that $R$ lies in the smallest thick subcategory containing $\mathbb{P}$. Therefore the condition 4) from [28, Theorem 6.4] holds true. By the (equivalent) condition 3) of the above cited Theorem it follows that $R\text{Hom}_R(\mathbb{P}, -)$ is fully faithful, hence $\delta : R\text{Hom}_R(\mathbb{P}, -) \otimes^L_B \mathbb{P} \to 1_{D(R)}$ is an isomorphism. From the adjunction isomorphism $\text{Hom}_{D(R)}(Y, R\text{Hom}_R(\mathbb{P}, X)) \cong \text{Hom}_{D(R)}(Y \otimes^L_B \mathbb{P}, X)$ we obtain $R\text{Hom}_R(\mathbb{P}, D(R)) \subseteq \mathcal{K}^\perp$.

Conversely, if $Y \in \mathcal{K}^\perp$, we have $\delta_{Y \otimes^L_B \mathbb{P}}(\gamma_Y \otimes^L_B \mathbb{P}) = 1_{Y \otimes^L_B \mathbb{P}}$. Since $R\text{Hom}_R(\mathbb{P}, -)$ is fully faithful, it follows that $\delta_{Y \otimes^L_B \mathbb{P}}$ is an isomorphism. Then $\gamma_Y \otimes^L_B \mathbb{P}$ is an isomorphism. Therefore, completing $\gamma_Y$ to a triangle
\[ Z \xrightarrow{\alpha} Y \xrightarrow{\gamma_Y} R\text{Hom}_R(\mathbb{P}, Y \otimes^L_B \mathbb{P}) \xrightarrow{\beta} Z[1], \]
it follows $Z \otimes^L_B \mathbb{P} = 0$. This implies that $Z \in \mathcal{K}$, hence $\alpha = 0$ and $\alpha[1] = 0$. It follows that $\beta$ is a split epimorphism. Since $R\text{Hom}_R(\mathbb{P}, Y \otimes^L_B \mathbb{P}) \in \mathcal{K}^\perp$ and $Z[1] \in \mathcal{K}^\perp$, this is possible only if $Z = 0$. Then $\gamma_Y$ is an isomorphism.

Therefore $R\text{Hom}_R(\mathbb{P}, D(R)) \subseteq \mathcal{K}^\perp$. This shows that the functors
\[ R\text{Hom}_R(\mathbb{P}, -) : D(R) \cong D(B) \cap \mathcal{K}^\perp : - \otimes^L_B \mathbb{P} \]
induce mutually inverse equivalences.

2. Using Lemma 4.1.5 it follows that for every $X \in \mathcal{H}(\tau)$ we have $R\text{Hom}_R(\mathbb{P}, X) \in \mathcal{H}(B)$.

Conversely, let $Y \in \mathcal{H}(B) \cap \mathcal{K}^\perp$. By using Lemma 4.1.6, we observe that $Y \otimes^L_B \mathbb{P}$ is a complex concentrated in $-1$ and $0$. Since $R\text{Hom}_R(\mathbb{P}, Y \otimes^L_B \mathbb{P}) \cong Y$, we can apply Lemma 4.1.5 one more time to conclude that $Y \otimes^L_B \mathbb{P} \in \mathcal{H}(\tau)$, and the proof is complete.

\[ \square \]

4.3 The Silting Counter Equivalence

We have seen in Theorem 4.2.1 that $H^0 R\text{Hom}_R(\mathbb{P}, -)$ is an equivalence between $\mathcal{H}(\tau)$ and an abelian subcategory of $\text{Mod}(\mathbb{E})$. From [19, Chapter I, Corollary 2.2], the pair $\langle \mathcal{F}[1], \mathcal{T} \rangle$
is a torsion pair in the abelian category $\mathcal{H}(\tau)$. It induces a torsion pair in the abelian subcategory $H^0 \text{RHom}_R(\mathcal{P}, \mathcal{H}(\tau))$ of Mod$(\mathcal{E})$, where $\mathcal{E} = \text{End}_{D(R)}(\mathcal{P})$. In the following we will describe, as in the tilting case, this torsion pair by using a natural torsion pair induced by $\mathcal{P}$ on Mod$(\mathcal{E})$.

**Theorem 4.3.1** Let $\mathcal{P}$ be a silting complex. We denote by $\tau = (T, F)$ the torsion pair generated by $T = H^0(\mathcal{P})$ in Mod$(R)$, and by $\mathcal{H}(\tau)$ the heart of the $t$-structure associated to $\tau$. If $B$ is the smart truncation of $\text{DgEnd}_R(\mathcal{P})$ defined in Section 3.3, $\mathcal{E} = \text{End}_{D(R)}(\mathcal{P})$, and $p : B \to \mathcal{E}$ is the canonical homomorphism of dg-algebras, denote, as in Remark 4.1.4, by $p^* = - \otimes^L_B \mathcal{E} : D(B) \leftrightarrow D(\mathcal{E}) : p_*$ the extension and the restriction of scalar functors induced by $p$. We consider the full subcategory $p^*(\mathcal{K})^\perp = \{ Y \in D(\mathcal{E}) \mid \text{Hom}_{D(\mathcal{E})}(X, Y[n]) = 0 \text{ for all } X \in p^*(\mathcal{K}) \text{ and } n \in \mathbb{Z} \}$ of $D(\mathcal{E})$, where $\mathcal{K} = \text{Ker}(- \otimes^L_B \mathcal{P}) \subseteq D(B)$.

The following statements are true.

1. The functor $\text{Hom}_{D(R)}(\mathcal{P}, -) : \mathcal{H}(\tau) \to \text{Mod}(\mathcal{E}) \cap p^*(\mathcal{K})^\perp$ induces an equivalence of categories, whose quasi-inverse is $(- \otimes^L_B \mathcal{P}) \circ p_*$.

2. If $(\mathcal{U}, \mathcal{V})$ is the torsion pair in $\text{Mod}(\mathcal{E})$ induced by the torsion class $\mathcal{U} = \{ X \in \text{Mod}(\mathcal{E}) \mid X \otimes_T T = 0 \}$, the restrictions of the above functors induce the equivalences

(a) $\text{Def}_\sigma(-) = \text{Hom}_{D(R)}(\mathcal{P}, -[1]) : \mathcal{F} \leftrightarrow \mathcal{U} \cap p^*(\mathcal{K})^\perp : Z_{\sigma^*}(-)$, and

(b) $\text{Hom}_R(T, -) = \text{Hom}_{D(R)}(\mathcal{P}, -) : \mathcal{T} \leftrightarrow \mathcal{V} \cap p^*(\mathcal{K})^\perp : - \otimes_T T$.

In order to prove this result we need the following

**Lemma 4.3.2** Using the above notations we have

$$H^0(\mathcal{H}(B) \cap \mathcal{K}^\perp) = \text{Mod}(\mathcal{E}) \cap p^*(\mathcal{K})^\perp.$$ 

**Proof** Let $Y \in \mathcal{H}(B) \cap \mathcal{K}^\perp$. We use the adjunction $p^* \dashv p_*$ and the fact that $H^0$ is an equivalence with the inverse $p_*$, in order to obtain:

$$\text{Hom}_{D(\mathcal{E})}(p^*(\mathcal{K}), H^0(Y)) \cong \text{Hom}_{D(B)}(\mathcal{K}, p_*H^0(Y)) \cong \text{Hom}_{D(B)}(\mathcal{K}, Y) = 0,$$

so $H^0(Y) \in p^*(\mathcal{K})^\perp$.

Conversely, for $Z \in \text{Mod}(\mathcal{E}) \cap p^*(\mathcal{K})^\perp$, we denote $Y = p_*(Z) \in \mathcal{H}(B)$ and we have $H^0(Y) \cong Z$. Then

$$\text{Hom}_{D(B)}(\mathcal{K}, Y) \cong \text{Hom}_{D(B)}(\mathcal{K}, p_*(Z)) \cong \text{Hom}_{D(\mathcal{E})}(p^*(\mathcal{K}), Z) = 0,$$

and it follows that $Y \in \mathcal{K}^\perp$.

**Proof** of Theorem 4.3.1
(1) By Lemma 4.1.5 it follows that the restrictions of the functors $\text{H}^0(\text{RHom}_{\mathcal{R}}(\mathbb{P}, -))$ and $\text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, -)$ to $\mathcal{H}(\tau)$ coincide. Now the conclusion follows from Theorem 4.2.1 and Lemma 4.3.2 since we have

$$\text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, \mathcal{H}(\tau)) = \text{H}^0(\text{RHom}_{\mathcal{R}}(\mathbb{P}, \mathcal{H}(\tau))) = \text{H}^0(\mathcal{H}(B) \cap \mathcal{K}^\perp) = \text{Mod}(\mathbb{E}) \cap p^*(\mathcal{K})^\perp.$$

(2) Note that if $X \in \mathcal{H}(\tau)$ then the exact sequence associated to $X$ which is induced by the torsion pair $(\mathcal{F}[1], \mathcal{T})$ is

$$0 \to \text{H}^{-1}(X)[1] \to X \to \text{H}^0(X) \to 0.$$

Hence $X \in \mathcal{F}[1]$ (respectively $X \in \mathcal{T}$) if and only if $\text{H}^0(X) = 0 (\text{H}^{-1}(X) = 0)$.

(2(a) Fix an object $X \in \mathcal{H}(\tau)$. The natural map $\delta_X : \text{RHom}_{\mathcal{R}}(\mathbb{P}, X) \otimes_B \mathbb{P} \to X$ is an isomorphism. By Lemma 4.1.6, we have the isomorphisms

$$\text{H}^0(X) \cong \text{H}^0(\text{RHom}_{\mathcal{R}}(\mathbb{P}, X) \otimes_B \mathbb{P}) \cong \text{H}^0(\text{RHom}_{\mathcal{R}}(\mathbb{P}, X)) \otimes_{\mathbb{E}} \mathcal{T}$$

$$= \text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, X) \otimes_{\mathbb{E}} \mathcal{T}.$$

As we have seen, $X \in \mathcal{F}[1]$ if and only if $\text{H}^0(X) = 0$, which is further equivalent to say that $\text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, X) \in \text{Ker}(- \otimes_{\mathbb{E}} \mathcal{T}) = \mathcal{U}$. Therefore the equivalence from (1) induces the equivalence

$$\text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, [-1]) : \mathcal{F} \to \mathcal{U} \cap p^*(\mathcal{K})^\perp,$$

whose quasi-inverse is $p_*([-1]) \otimes_B \mathbb{P}$.

Moreover, for every $Z \in \mathcal{U} \cap p^*(\mathcal{K})^\perp$, since $p_*(Z) \otimes_B \mathbb{P}$ is concentrated in degree $-1$, we obtain from Lemma 4.1.6(2) the natural isomorphisms

$$\text{H}^{-1}(p_*(Z) \otimes_B \mathbb{P}) \cong \text{Ker}(\text{H}^0 p_* \otimes_{\mathbb{E}} \beta^*) \cong \text{Ker}(Z \otimes_{\mathbb{E}} \beta^*).$$

Therefore, the restrictions of functors $p_*(-) \otimes_B \mathbb{P}$ and $Z \beta^*$ to $\mathcal{U} \cap p^*(\mathcal{K})^\perp$ are natural isomorphic, and the proof is complete.

(2(b) Let $E$ be the endomorphism ring of $T$, and $\phi : \mathbb{E} \to E$ be the canonical surjective ring homomorphism.

If $M \in \mathcal{T}$ then the right $\mathbb{E}$-module $\text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, M)$ is the module induced by the restriction of scalars along $\phi$ of the $E$-module $\text{Hom}_{\mathcal{R}}(T, M)$. Moreover, if $X \in \text{Mod}(\mathbb{E})$ is a module such that $X \otimes_{\mathbb{E}} T = 0$, then the induced $E$-module $X' = X \otimes_{\mathbb{E}} E$ has the property $X' \otimes_E T = 0$. By [5, Proposition 3.2] we conclude that $T$ is tilting as an $R/\text{Ann}(T)$-module. It follows from the tilting theorem proved in [6, Theorem 4.5] that $\text{Hom}_{\mathcal{E}}(X', \text{Hom}_{\mathcal{R}}(T, M)) = 0$. Using the canonical adjunction isomorphisms, we obtain the equality $\text{Hom}_{\mathbb{E}}(X, \text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, M)) = 0$, hence $\text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, M) \in \mathcal{V} \cap p^*(\mathcal{K})^\perp$.

Let $X \in \mathcal{V} \cap p^*(\mathcal{K})^\perp$. Then there exists an object $L \in \mathcal{H}(\tau)$ such that $\text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, L) = X$. Since $(\mathcal{F}[1], \mathcal{T})$ is a torsion pair in $\mathcal{H}(\tau)$, there exists a short exact sequence in $\mathcal{H}(\tau)$ of the form $0 \to F[1] \to L \to M \to 0$, where $F \in \mathcal{F}$ and $M \in \mathcal{T}$. We apply the functor $\text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, -) : \mathcal{H}(\tau) \to \text{Mod}(\mathbb{E})$ to this exact sequence. Since $\text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, -)$ is an equivalence of categories from $\mathcal{H}(\tau)$ to a full subcategory of $\text{Mod}(\mathbb{E})$, we obtain the short exact sequence

$$0 \to \text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, F[1]) \to \text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, L) \to \text{Hom}_{\mathcal{D}(\mathcal{R})}(\mathbb{P}, M) \to 0.$$
in \( \text{Mod}(\mathbb{E}) \). But \( \text{Hom}_{\text{D}(R)}(\mathbb{P}, F[1]) \in \mathcal{U} \) and \( \text{Hom}_{\text{D}(R)}(\mathbb{P}, L) \cong X \in \mathbb{V} \). This implies that \( \text{Hom}_{\text{D}(R)}(\mathbb{P}, L) \to \text{Hom}_{\text{D}(R)}(\mathbb{P}, M) \) is an isomorphism, hence \( L \cong M \) belongs to \( \mathcal{T} \).

It follows that \( \text{Hom}_{\text{D}(R)}(\mathbb{P}, T) = \mathcal{V} \cap p^*(\mathcal{K})^\perp \). Applying Lemma 4.1.6 it is easy to see that for every \( X \in \mathcal{V} \cap p^*(\mathcal{K})^\perp \) the complex \( p_*(X) \otimes_B^L \mathbb{P} \) is concentrated in 0. Moreover, we have \( H^0(p_*(X) \otimes_B^L \mathbb{P}) = X \otimes_E T \). Therefore, the functor \( - \otimes_E T \) is a quasi-inverse of the functor \( \text{Hom}_R(T, -) = \text{Hom}_{\text{D}(R)}(\mathbb{P}, -) : \mathcal{T} \to \mathcal{V} \cap p^*(\mathcal{K})^\perp \).

\begin{corollary}
If \( \mathbb{P} \) is a compact silting complex then we have the equivalences:

(a) \( \text{Hom}_{\text{D}(R)}(\mathbb{P}, -) : \mathcal{H}(\tau) \rightleftharpoons \text{Mod}(\mathbb{E}) : - \otimes^L \mathbb{P} \),

(b) \( \text{Hom}_R(T, -) = \text{Hom}_{\text{D}(R)}(\mathbb{P}, -) : \mathcal{T} \rightleftharpoons \mathcal{V} : - \otimes_E T = - \otimes_E \mathcal{T} \), and

(c) \( \text{Def}_\mathcal{H}(-) = \text{Hom}_{\text{D}(R)}(\mathbb{P}, [-1]) : \mathcal{F} \rightleftharpoons \mathcal{U} : [-1] \otimes_B^L \mathbb{P} = \mathbb{Z}_{\mathbb{P}} \), where \( \mathbb{Z}_{\mathbb{P}} \) is computed with respect a (fixed) triangle of the form (\( \dagger \)).

\end{corollary}

\textbf{Proof} As we have seen in Example 3.2.2, the compact silting complex \( \mathbb{P} \) is good, hence we can use Theorems 4.2.1 and 4.3.1.

We apply \( \text{RHom}_R(-, \mathbb{P}) \) to the triangle \( P^{-1} \to P^0 \to \mathbb{P} \to P^{-1}[1] \), and we obtain a triangle of left \( B \)-modules \( X \to Y \to B \to X[1] \), with \( X, Y \in \text{add}(\mathbb{P}) \). If \( Z \in \mathcal{K} \) then \( X \otimes_B^L Z = Y \otimes_B^L Z = 0 \). It follows that \( Z \cong B \otimes_B^L Z = 0 \), and the proof is complete.

\begin{remark}
The compact case was discovered in [18, Theorem 2.15], where the authors proved directly that \( \text{Hom}_{\text{D}(R)}(\mathbb{P}, -) : \mathcal{H}(\tau) \to \text{Mod}(\mathbb{E}) \) is fully faithful. Our approach has the advantage that we are able to compute the quasi-inverse of \( \text{Hom}_{\text{D}(R)}(\mathbb{P}, -) \).

\end{remark}

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