AN OPTIMAL LOWER CURVATURE BOUND FOR CONVEX HYPERSURFACES IN RIEMANNIAN MANIFOLDS

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Abstract. It is proved that a convex hypersurface in a Riemannian manifold of sectional curvature $\geq \kappa$ is an Alexandrov’s space of curvature $\geq \kappa$. This theorem provides an optimal lower curvature bound for an older theorem of Buyalo.

The purpose of this paper is to provide a reference for the following theorem:

**Theorem 1.** Let $M$ be a Riemannian manifold with sectional curvature $\geq \kappa$. Then any convex hypersurface $F \subset M$ equipped with the induced intrinsic metric is an Alexandrov’s space with curvature $\geq \kappa$.

Here is a slightly weaker statement:

**Theorem 2.** [Buyalo] If $M$ is a Riemannian manifold, then any convex hypersurface $F \subset M$ equipped with the induced intrinsic metric is locally an Alexandrov’s space.

In the proof of Theorem 2 in [Buyalo], the (local) lower curvature bound depends on (local) upper as well as lower curvature bounds of $M$. We show that the approach in [Buyalo] can be modified to give Theorem 1.

**Definition 3.** A locally Lipschitz function $f$ on an open subset of a Riemannian manifold is called $\lambda$-concave ($\lambda \in \mathbb{R}$) if for any unit-speed geodesic $\gamma$, the function $f \circ \gamma(t) - \lambda t^2/2$ is concave.

**Lemma 4.** Let $f : \Omega \to \mathbb{R}$ be a $\lambda$-concave function on an open subset $\Omega$ of a Riemannian manifold. Then there is a sequence of nested open domains $\Omega_i$, with $\Omega_i \subset \Omega_j$ for $i < j$ and $\bigcup_i \Omega_i = \Omega$, and a sequence of smooth $\lambda_i$-concave functions $f_i : \Omega_i \to \mathbb{R}$ such that

(i) on any compact subset $K \subset \Omega$, $f_i$ converges uniformly to $f$;
(ii) $\lambda_i \to \lambda$ as $i \to \infty$.

This lemma is a slight generalization of [Greene–Wu, Theorem 2] and can be proved exactly the same way.

**Proof of Theorem 1.** Without loss of generality one can assume that

(a) $\kappa \geq -1$,
(b) \( F \) bounds a compact convex set \( C \) in \( M \),
(c) there is a \((-2)\)-concave function \( \mu \) defined in a neighborhood of \( C \) and \(|\mu(x)| < 1/10\) for any \( x \in C \),
(d) there is unique minimal geodesic between any two points in \( C \).

(If not, rescale and pass to the boundary of the convex piece cut by \( F \) from a small convex ball centered at \( x \in F \), taking \( \mu = -10 \text{dist}^2 \)).

Consider the function \( f = \text{dist}_F \). By Rauch comparison (as in [Petersen, 11.4.8]), for any unit-speed geodesic \( \gamma \) in the interior of \( C \), \((f \circ \gamma)''\) is bounded in the support sense by the corresponding value in the model case (where \( M = \mathbb{H}^2 \) and \( F \) is a geodesic). In particular,

\[(f \circ \gamma)'' \leq f \circ \gamma.\]

Therefore \( f + \varepsilon \mu \) is \((-\varepsilon)\)-concave in \( \Omega_{\varepsilon} = C \cap f^{-1}(0, \varepsilon) \). Take \( K_{\varepsilon} = f^{-1}([\frac{1}{3} \varepsilon, \frac{2}{3} \varepsilon]) \cap C \). Applying lemma 4, we can find a smooth \((-\frac{\varepsilon}{2})\)-concave function \( f_{\varepsilon} \) which is arbitrarily close to \( f + \varepsilon \mu \) on \( K_{\varepsilon} \) and which is defined on a neighborhood of \( K_{\varepsilon} \). Take a regular value \( \vartheta_{\varepsilon} \approx \frac{3}{5} \varepsilon \) of \( f_{\varepsilon} \). (In fact one can take \( \vartheta_{\varepsilon} = \frac{2}{3} \varepsilon \), but it requires a little work.) Since \(|\mu(C)| < 1/10\), the level set \( F_{\varepsilon} = f_{\varepsilon}^{-1}(\vartheta_{\varepsilon}) \) will lie entirely in \( K_{\varepsilon} \). Therefore \( F_{\varepsilon} \) forms a smooth closed convex hypersurface. By the Gauss formula, the sectional curvature of the induced intrinsic metric of \( F_{\varepsilon} \) is \( \geq \kappa \). \( F_{\varepsilon} \) bounds a compact convex set \( C_{\varepsilon} \), where \( F_{\varepsilon} \to F \), \( C_{\varepsilon} \to C \) in Hausdorff sense as \( \varepsilon \to 0 \). By property (4), the restricted metrics from \( M \) to \( C, C_{\varepsilon} \) are intrinsic, and so \( C_{\varepsilon} \) is an Alexandrov space with \( F_{\varepsilon} \) as boundary, that converges in Gromov–Hausdorff sense to \( C \). It follows from [Petrunin, Theorem 1.2] (compare [Buyalo, Theorem 1]) that \( F_{\varepsilon} \) equipped with its intrinsic metric converges in Gromov–Hausdorff sense to \( F \) equipped with its intrinsic metric. Therefore \( F \) is an Alexandrov space with curvature \( \geq \kappa \).

\[\square\]

Remark 5. We are not aware of any proof of theorem 1 which is not based on the Gauss formula. (Although if \( M \) is Euclidean space, there is a beautiful purely synthetic proof in [Milka].) Finding such a proof would be interesting on its own, and also could lead to the generalization of theorem 1 to the case when \( M \) is an Alexandrov space.

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