Theory of Generalized Bernoulli-Hurwitz Numbers for the Algebraic Functions of Cyclotomic Type

YOSHIHiro ÔNISHI

1. Introduction.
Let \( C \) be the projective curve of genus \( g \) defined by

\[
y^2 = x^{2g+1} - 1 \quad \text{or} \quad y^2 = x^{2g+1} - x.
\]

The unique point of \( C \) at infinity is denoted by \( \infty \). Let consider the integral

\[
u = \int_{\infty}^{x} \frac{x^{g-1}dx}{2y}.
\]

This is an integral of a differential of first kind on \( C \) which does not vanish at \( \infty \). The integral converges everywhere. If \( g = 0 \) the inverse function of (1.2) is \(-1/\sin^2(u)\). As is well-known, if \( g = 1 \) the inverse function of (1.2) is just the Weierstrass function \( \wp(u) \) with \( \wp'(u)^2 = 4\wp(u)^3 - 4 \) (or \( \wp'(u)^2 = 4\wp(u)^3 - 4 \)).

The Bernoulli numbers \( \{B_{2n}\} \) are the coefficients of the Laurent expansion of \(-1/\sin^2(u)\) at \( u = 0 \):

\[
\frac{-1}{\sin^2(u)} = \frac{1}{u^2} - \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n} B_{2n}}{2n} \frac{u^{2n-2}}{(2n-2)!}.
\]

The Hurwitz numbers \( \{H_{4n}\} \) are the coefficients of the expansion

\[
\wp(u) = \frac{1}{u^2} + \sum_{n=1}^{\infty} \frac{2^{4n} H_{4n}}{4n} \frac{u^{4n-2}}{(4n-2)!}.
\]

These kinds of numbers are quite important, especially, in number theory. Because the inverse function on the neighborhood of \( u = 0 \) (1.2) can not extend globally with respect to \( u \) if \( g > 1 \), most of mathematician never considered (1.2) for \( g > 1 \), usually based on the classical idea of Jacobi, and worked in several variable functions.

Surprisingly, for the case \( g > 1 \), the Laurent coefficients of the inverse function of (1.2) have properties which properly resemble von Staudt-Clausen’s theorem and Kummer’s congruence for Bernoulli numbers ([C], [vS], [K]) and such theorems for Hurwitz numbers ([H2], [L]). To explain these facts is the aim of this paper.

Detailed and extended exposition should be referred to [Ô].
2. Main results.
Here we describe only for the curve $C$ defined by

$$y^2 = x^5 - 1.$$ 

We consider the integrals

\begin{equation}
(2.1)\quad u_1 = \int_\infty^{(x,y)} \frac{dx}{2y}, \quad u_2 = \int_\infty^{(x,y)} \frac{x dx}{2y}
\end{equation}

of the elements of a natural base of the differentials of first kind. Since these integrals are converges, there exists a inverse function $(x(u), y(u))$ of $(x, y) \mapsto u = (u_1, u_2)$ on the range of all values $u$ of (2.1). On a neighborhood of $u = (0,0)$, the functions $x(u)$ and $y(u)$ are functions of the second variable $u_2$ only. Hence we have the following differential equation:

\begin{equation}
(2.2)\quad \frac{du_2}{dx} = \frac{x}{2y}.
\end{equation}

If we denote $\frac{dx}{du_2} = x'(u)$, then

\begin{equation}
(2.3)\quad x'(u) = \frac{2y}{x}.
\end{equation}

After squaring the two sides and substituting $y(u)^2 = x(u)^5 - 1$, by removing the denominator, we have

\begin{equation}
(2.4)\quad x(u)^2 x'(u)^2 = 4x(u)^5 - 4
\end{equation}

This (2.4) is just a good analogy of

$$\psi''(u) = 6\psi(u)^2 \quad \text{(or } \psi''(u) = 6\psi(u)^2 - 2),$$

obtained by $y'(u)^2 = 4\psi(u)^3 - 1$ (or $y'(u)^2 = 4\psi(u)^3 - 4\psi(u)$). Indeed, if we define the numbers $C_{10n}$ and $D_{10n}$ by

\begin{equation}
(2.5)\quad x(u) = \frac{1}{u_2^2} + \sum_{n=1}^{\infty} \frac{C_{10n}u_2^{10n-2}}{10n(10n)!},
\end{equation}

\begin{equation}
\quad y(u) = \frac{-1}{u_2^5} + \sum_{n=1}^{\infty} \frac{D_{10n}u_2^{10n-5}}{10n(10n)!},
\end{equation}

then we have two Theorems 2.7 and 2.8 below. Here, using the property that

\begin{equation}
(2.6)\quad x(-\zeta u_1, -\zeta^2 u_2) = \zeta x(u_1, u_2), \quad y(-\zeta u_1, -\zeta^2 u_2) = -y(u_1, u_2),
\end{equation}

we know that only the terms in (2.5) appear. The first Theorem is
Theorem 2.7 For each of $C_{10n}$ and $D_{10n}$, there exist integers $G_{10n}$ and $H_{10n}$ such that

$$C_{10n} = \sum_{p \equiv 1 \mod 5 \atop p \neq 1 \atop p \mid 10n} \frac{A_p^{10n/(p-1)}}{p} + G_{10n},$$

$$D_{10n} = \sum_{p \equiv 1 \mod 5 \atop p \neq 1 \atop p \mid 10n} \frac{(4!-1 \mod p) \ A_p^{10n/(p-1)}}{p} + H_{10n};$$

where

$$A_p = (-1)^{(p-1)/10} \left( \frac{(p-1)/2}{(p-1)/10} \right).$$

This is a generalization of von Staudt-Clausen’s theorem. The second theorem is

Theorem 2.8 For any prime $p \equiv 1 \mod 5$, and positive integers $a$ and $n$ such that $10n - 2 \geq a$, if $(p-1) / 10n$, then

$$\sum_{r=0}^{a} (-1)^r \binom{a}{r} A_p^{a-r} \frac{C_{10n+r(p-1)}}{10n + r(p-1)} \equiv 0 \mod p^a,$$

$$\sum_{r=0}^{a} (-1)^r \binom{a}{r} A_p^{a-r} \frac{D_{10n+r(p-1)}}{10n + r(p-1)} \equiv 0 \mod p^a;$$

where

$$A_p = (-1)^{(p-1)/10} \left( \frac{(p-1)/2}{(p-1)/10} \right).$$

This is a natural generalization of Kummer’s original congruence in [K] and of such a congruence for Hurwitz numbers ([L], p.190, (20)). To prove Theorem 2.8, we need

Theorem 2.9 For any prime number $p \equiv 1 \mod 5$ and any positive integer $n$, if $p - 1 \not\mid 10n$ then $C_{10n}/10n$ and $D_{10n}/10n$ belong to $\mathbb{Z}_p$.

3. About Proof of Theorems.

We could not use any addition or multiplication formulae in the classical theory of Abelian or Jacobin varieties. To prove Theorem 2.7 we use a technique based on a method of L. Carlitz [Car1], and to prove Theorem 2.9 we use a new technique that is an improved method from that of Carlitz. Theorem 2.8 is rather easily shown by Theorem 2.8. By using his method, Carlitz succeeded to prove Hurwitz’s theorem outside the prime 2 in his paper [Car1]. Furthermore, although he was trying to find a generalization of von Staudt-Clausen’s type theorem, Kummer’s
type congruence for hyperelliptic functions, he could not succeed. Our results is not only for hyperelliptic curve but also for any algebraic curves of type

\[ y^a = x^b - 1, \quad \text{or} \quad y^2 = x^b - x \]

with \( \gcd(a, b) = 1 \).

The detailed proof and a lot of numerical examples are given in [Ō].

REFERENCES

[Car1] L. Carlitz, *The coefficients of the reciprocal of a series*, Duke Math. J., 8 (1941), 689-700.
[Car2] L. Carlitz, *Some properties of Hurwitz series*, Duke Math. J., 16 (1949), 285-295.
[Car3] L. Carlitz, *Congruences for the coefficients of the Jacobi elliptic functions*, Duke Math. J., 16 (1949), 297-302.
[Car4] L. Carlitz, *Congruences for the coefficients of hyperelliptic and related functions*, Duke Math. J., 19 (1952), 329-337.
[Cl] T. Clausen, *Lehrsatz aus einer Abhandlung über die Bernoullischen Zahlen*, Astron. Nachr. 17 (1840), 325-330.
[H1] A. Hurwitz, *Über die Entwicklungskoeffizienten der leminiskatischen Funktionen*, Nachr. Acad. Wiss. Göttingen, (1897), 273-276 (Werke, Bd.II, pp.338-341).
[H2] A. Hurwitz, *Über die Entwicklungskoeffizienten der leminiskatischen Funktionen*, Math. Ann., 51 (1899), 196-226 (Werke, Bd.II, pp.342-373).
[K] E.E. Kummer, *Über eine allgemeine Eigenschaft der rationalen Entwicklungskoeffizienten einer bestimmten Gattung analytischer Functionen*, J. für die reine und angew. Math. 41 (1851), 368-372.
[L] H. Lang, *Kummersche Kongruenzen für die normierten Entwicklungskoeffizienten der Weierstrasschen \( \wp \)-Funktionen*, Abh. Math. Sem. Hamburg 33 (1969), 183-196.
[Ō] Y. Ōnishi, *Theory of Generalized Bernoulli-Hurwitz Numbers in the Algebraic Functions of Cyclotomic Type (in Japanese, 87 pages)*, downloadable from http://jinsha2.hss.iwate-u.ac.jp/~onishi/ (2003).
[vS] K.G.C. von Staudt, *Beweis einer Lehrsatzes, die Bernoullischen Zahlen betreffend*, J. Reine Angew. Math. 21 (1840), 372-374.