ON GRAPHS WITH 2 TRIVIAL DISTANCE IDEALS

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Abstract. Distance ideals generalize the Smith normal form of the distance matrix of a graph. The family of graphs with 2 trivial distance ideals contains the family of graphs whose distance matrix has at most 2 invariants factors equal to 1. Here we give an infinite family of forbidden induced subgraphs for the graphs with 2 trivial distance ideals. These are also related with other well known graph classes.

1. Introduction

Through the paper all graphs will be considered connected. Let $G = (V, E)$ be a connected graph and $X_G = \{x_u : u \in V(G)\}$ a set of indeterminates. The distance $d_G(u,v)$ between the vertices $u$ and $v$ is the number of edges of a shortest path between them. Let $\text{diag}(X_G)$ denote the diagonal matrix with the indeterminates in the diagonal. The distance matrix $D(G)$ of $G$ is the matrix with rows and columns indexed by the vertices of $G$ where the $uv$-entry is equal to $d_G(u,v)$. Thus the generalized distance matrix $D(G, X_G)$ of $G$ is the matrix with rows and columns indexed by the vertices of $G$ defined as $\text{diag}(X_G) + D(G)$. Note we can recover the distance matrix from the generalized distance matrix by evaluating $X_G$ at the zero vector, that is, $D(G) = D(G, 0)$.

Let $\mathcal{R}[X_G]$ be the polynomial ring over a commutative ring $\mathcal{R}$ in the variables $X_G$. For all $i \in [n] := \{1, ..., n\}$, the $i$-th distance ideal $I^\mathcal{R}_i(G, X_G)$ of $G$ is the ideal, over $\mathcal{R}[X_G]$, given by $\langle \text{minors}_i(D(G, X_G)) \rangle$, where $n$ is the number of vertices of $G$ and $\text{minors}_i(D(G, X_G))$ is the set of the determinants of the $i \times i$ submatrices of $D(G, X_G)$.

Distance ideals were defined in [3] as a generalization of the Smith normal form of distance matrix and the distance spectra of graphs. Through this paper will be interested in the case when $\mathcal{R}$ is the ring of integer numbers. For this reason we will omit $\mathcal{R}$ in the notation of distance ideal.

1.1. Distance ideals and Smith normal form of distance matrix. Smith normal forms have been useful in understanding algebraic properties of combinatorial objects see [23]. For instance, computing the Smith normal form of the adjacency or Laplacian matrix is a standard technique used to determine the Smith group and the critical group of a graph, see [5, 21, 23].

Smith normal forms can be computed using row and column operations. In fact, Kannan and Bachem found in [20] polynomial algorithms for computing the
Lemma 3. Results. are not monotone under taking induced subgraphs. However, we have the following.

Distance ideals and induced subgraphs.

1.2. $I_3$ Lemma 4. the distance between any pair of vertices in $v_i, v_j$ in $V(H)$, there is a shortest path from $v_i$ to $v_j$ in $G$ which lies entirely in $H$. Then, $I_i^R(H, X_H) \subseteq I_i^R(G, X_G)$ for all $i \leq |V(H)|$, and $\Phi_R(H) \leq \Phi_R(G)$.

In particular we have the following.

Lemma 4. $\mathbb{R}$ Let $H$ be an induced subgraph of $G$ with diameter is 2, that is the distance between any pair of vertices in $H$ is at most 2. Then $I_i^R(H, X_H) \subseteq I_i^R(G, X_G)$ for all $i \leq |V(H)|$.

In fact, in $\mathbb{R}$ the family $\Lambda_1$ of graphs having only 1 trivial distance ideal was characterized in terms of induced forbidden subgraphs: $\{P_4, paw, diamond\}$-free graphs;
that are the graphs isomorphic to an induced subgraph of $K_{m,n}$ or $K_n$. Also in [3], the family of graphs having only 1 trivial distance ideal over $\mathbb{R}$ was characterized as: \{P_4, paw, diamond, C_4\}-free graphs, that are the graphs isomorphic to an induced subgraph of $K_{1,n}$ or $K_n$.

These families appear in other contexts. A graph is \textit{trivially perfect} if for every induced subgraph the stability number equals the number of maximal cliques. In [16, Theorem 2], Golumbic characterized trivially prefect graphs as \{P_4, C_4\}-free graphs. There are other equivalent characterization of this family, see [10, 22]. Therefore, graphs with 1 trivial distance ideal over $\mathbb{R}$ are a subclass of trivial perfect graphs.

The aim of this paper is to explore the properties of the family $\Lambda_2$ of graphs with at most two trivial distance ideals over $\mathbb{Z}$. In particular, we are going to find an infinite number of graphs that are forbidden for $\Lambda_2$. Let $F$ be the set of 17 graphs shown in Figure 1. In Section 2 we will prove that graphs in $\Lambda_2$ are \{F, odd-holes\}-free graphs, where \textit{odd-holes} are cycles of odd length greater or equal than 7.

One of the main applications in finding a characterization of $\Lambda_2$ is that it is an approach to obtain a characterization of the graphs with $\phi_2(G) = 2$ since they are contained in $\Lambda_2$. In [18, Theorem 3], it was proved that the distance matrix of trees
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has exactly 2 invariant factors equal to 1. Therefore,

trees ⊆ Λ_2 ⊆ \{F, odd-holes\}-free graphs.

Among the forbidden graphs for Λ_2 there are several graphs studied in other contexts, like bull and odd-holes studied in [11, 12] and [13], respectively. Other related family is the 3-leaf powers that was characterized in [15] as \{bull, dart, gem\}-free chordal graphs.

Distance-hereditary graphs is another related family, defined by H oworka in [19]. A graph G is distance-hereditary if for each connected induced subgraph H of G and every pair u and v of vertices in H, d_H(u, v) = d_G(u, v). Distance-hereditary graphs are \{house, gem, domino, holes\}-free graphs, where holes are cycles of length greater or equal than 5, which intersects with Λ_2. Also, if H is a connected induced subgraph of a distance-hereditary graph G, then I^R_i(H, X_H) ⊆ I^R_i(G, X_G) for all i ≤ |V(H)|.

Previously, an analogous notion to the distance ideals but for the adjacency and Laplacian matrices was explored. These were called critical ideals, see [14]. They have been explored in [2, 6, 7, 8], and in [11, 14] it was found new connections in contexts different from the Smith group or Sandpile group like the zero-forcing number and the minimum rank of a graph. In this setting, the set of forbidden graphs for the family with at most k trivial critical ideals is conjectured to be finite, see [6, Conjecture 5.5]. It is interesting that for distance ideals this is not true.

2. Graphs with 2 trivial distance ideals

A graph G is forbidden for Λ_k if the (k + 1)-th distance ideal of G is trivial. In addition, a forbidden graph H for Λ_k is minimal if H does not contain a connected forbidden graph for Λ_k as induced subgraph, and any graph G containing H as induced subgraph, have that G id forbidden for Λ_k. The set of minimal forbidden graphs for Λ_k will be denoted by Forb_k.

Lemma 5. Let G ∈ F. Then, Φ(G) = 3 and no induced subgraph H of G has Φ(H) = 3.

Proof. This could be checked using Macaulay2 code [17] given in [3, Appendix]. □

Proposition 6. The graphs dart, house, gem, full-house, G_{6,8}, G_{6,10}, co-twin-C_5, G_{6,14} are in Forb_2.

Proof. The result follows by Lemmas 4 and 5 □

Lemma 7. bull graph is in Forb_2.

Proof. If there is no vertex u in G adjacent to the leaf vertices of bull, then distance in G between the two leaf vertices is 3. Therefore, by Lemma 3 Φ(G) ≥ Φ(bull) = 3, and G ∈ Forb_2. On the other hand, if the distance, in G, between the two leaf vertices of bull is 2, then the submatrix of D(G, X_G) associated with the vertices of bull is

\[
M = D(G, X_G)[V(bull)] = \begin{bmatrix}
  u & 2 & 2 & 1 \\
  v & 2 & 1 & 2 \\
  2 & x_1 & 1 & 1 \\
  2 & 1 & x_2 & 1 \\
  1 & 2 & 1 & x_3
\end{bmatrix},
\]
which have \( \langle \text{minors}_3(M) \rangle = \langle 1 \rangle \). From which follows that \( \langle 1 \rangle = \langle \text{minors}_3(M) \rangle \subseteq I_3(G, X_G) \), and thus \( G \in \text{Forb}_2 \).

**Lemma 8.** \( G_{6,5} \) is in \( \text{Forb}_2 \).

**Proof.** Let \( G \) be a graph containing \( G_{6,5} \) as induced subgraph, and let \( M \) be the submatrix of \( D(G, X_G) \) associated with the vertices of \( G_{6,5} \) that is described in Figure 3 and where \( d_G(v_0, v_3) = y_0 \), \( d_G(v_0, v_2) = y_1 \) and \( d_G(v_0, v_1) = y_2 \).

\[
\begin{bmatrix}
x_0 & y_2 & y_1 & y_0 & 2 & 1 \\
y_2 & x_1 & 2 & 2 & 1 & 2 \\
y_1 & 2 & x_2 & 1 & 1 & 2 \\
y_0 & 2 & 1 & x_3 & 1 & 2 \\
2 & 1 & 1 & 1 & x_4 & 1 \\
1 & 2 & 2 & 2 & 1 & x_5
\end{bmatrix}
\]

**Figure 3.** Submatrix associated with \( G_{6,5} \).

We have that \( \det(M[[1,2,5],\{0,3,4\}]) = -y_2+1 \) and \( \det(M[[1,2,4],\{0,3,5\}]) = -y_2+4 \). Thus \( (-y_2+1, -y_2+4) \subseteq \langle \text{minors}_3(M) \rangle \subseteq I_3(G, X_G) \). There are two cases either \( y_2 \) is equal to 2 or 3. In both cases we have \( 1 \in I_3(G, X_G) \), from which follows that \( G \in \text{Forb}_2 \). \( \square \)

**Lemma 9.** \( 5\text{-pan} \) is in \( \text{Forb}_2 \).

**Proof.** Let \( G \) be a graph having \( 5\text{-pan} \) as induced subgraph, and let \( M \) be the submatrix of \( D(G, X_G) \) associated with the vertices of \( 5\text{-pan} \) that is shown in Figure 4 and where \( d_G(v_0, v_1) = y_1 \) and \( d_G(v_0, v_2) = y_0 \).

\[
\begin{bmatrix}
x_0 & y_1 & y_0 & 2 & 2 & 1 \\
y_1 & x_1 & 1 & 2 & 1 & 2 \\
y_0 & 1 & x_2 & 1 & 2 & 2 \\
2 & 2 & 1 & x_3 & 2 & 1 \\
2 & 1 & 2 & 2 & x_4 & 1 \\
1 & 2 & 2 & 1 & 1 & x_5
\end{bmatrix}
\]

**Figure 4.** Submatrix associated with \( 5\text{-pan} \).

We have that \( \det(M[[2,3,4],\{1,2,5\}]) = 5-x_2 \), \( \det(M[[2,4,5],\{1,2,3\}]) = 3x_2-4 \) and \( \det(M[[0,1,2],[3,4,5]]) = -5 \). Since \( \langle 1 \rangle = \langle 5-x_2, 3x_2-4, -5 \rangle \subseteq \langle \text{minors}_3(M) \rangle \subseteq I_3(G, X_G) \), then \( G \in \text{Forb}_2 \). \( \square \)

**Lemma 10.** \( G_{6,7} \) is in \( \text{Forb}_2 \).

**Proof.** Let \( G \) be a graph having \( G_{6,7} \) as induced subgraph, and let \( M \) be the submatrix of \( D(G, X_G) \) associated with the vertices of \( G_{6,7} \) that is shown in Figure 5 and where \( d_G(v_0, v_1) = y_4 \), \( d_G(v_0, v_2) = y_3 \), \( d_G(v_0, v_5) = y_2 \), \( d_G(v_1, v_4) = y_1 \) and \( d_G(v_2, v_4) = y_0 \).

\[
\begin{bmatrix}
x_0 & y_1 & y_3 & y_2 & y_0 & 2 & 1 \\
y_1 & x_2 & 1 & x_3 & x_4 & 1 & 2 \\
y_3 & 1 & xx & 1 & 2 & 1 \\
y_2 & 1 & 2 & 2 & x_4 & 1 \\
y_0 & 2 & 2 & 2 & 1 & x_5
\end{bmatrix}
\]

**Figure 5.** Submatrix associated with \( G_{6,7} \).
Let
\[ I = \{ y_0y_2 + 2y_0 - y_1y_2 - 2y_1, y_0y_3 - 2y_0 - 4y_3 + 5, \\
y_0y_4 - 2y_0 - 4y_4 + 5, 3y_0 - 3y_1, \\
2y_1y_2 - 2y_1 - y_2 - 3y_4 + 4, y_1y_3 - 2y_1 - 4y_3 + 5, \\
y_1y_4 - 2y_1 - 4y_4 + 5, y_2y_3 + 6y_2y_4 - 8y_2 + 2y_3 - 3y_2^2 + 2, \\
7y_2y_4 - 8y_2 - 3y_4^2 + 2y_4 + 2, 3y_3 - 3y_4 \} , \]
and
\[ J = \{ x_0 - 2y_2y_4 + 2y_2 + y_3y_4 - 2y_3, x_1x_2 - 5x_1 - 5x_2 + 9, \\
x_1x_3 - 2x_1 - x_3 + 2, x_1y_0 - x_1 - 2y_0 + y_1 + 1, \\
x_1y_2 - x_1 - y_2 + 1, x_1y_3 - 2x_1 - 8y_3 + 7y_4 + 2, \\
3x_1 - 3, x_2x_3 - 2x_2 - x_3 + 2, x_2y_1 - x_2 + y_0 - 2y_1 + 1, \\
x_2y_2 - x_2 - y_2 + 1, x_2y_4 - 2x_2 + 7y_3 - 8y_4 + 2, 3x_2 - 3, \\
x_3y_0 - 2x_3 - 2y_0 + 7, x_3y_1 - 2x_3 - 2y_1 + 7, \\
x_3y_2 - x_3y_4 - 4y_2 + 2y_4 + 2, x_3y_3 - x_3y_4 - 2y_3 + 2y_4, \\
2x_3y_4 - x_3 - y_4 - 4, x_4 + y_0y_1 - y_0 - 4y_1 + 2, \\
x_5 - 3y_1y_2 + 3y_1 - 2y_2 + 6y_4 - 6 \} . \]

It can be checked that the lexicographic Gröbner basis of the ideal \( \langle \text{minors}_3(M) \rangle \) is equal to \( (I \cup J) \). Furthermore, let \( d \) be a vector such that the \( i \)-th entry is taken over a possible values of \( y_i \), that is, \( d_0 \in \{ 2, 3 \} \), \( d_1 \in \{ 2, 3 \} \), \( d_2 \in \{ 2, 3 \} \), \( d_3 \in \{ 2, 3, 4 \} \) and \( d_4 \in \{ 2, 3, 4 \} \). Note that if \( \gcd(I_{y\in d}) \) is equal to 1, then \( \langle \text{minors}_3(M) \rangle = (1) \), which implies that \( I_3(G, X_G) \) is trivial. It can be verified that \( \gcd(I_{y\in d}) \) is equal to 1, except for the following vectors: \( (2, 2, 2, 2, 2), (2, 2, 3, 2, 2), (2, 2, 3, 3, 3) \) and \( (3, 3, 3, 3, 3) \).

Consider \( p = x_3y_0 - 2x_3 - 2y_0 + 7 \in J \) and \( q = x_3y_2 - x_3y_4 - 4y_2 + 2y_4 + 2 \in J \). Since \( p|_{y_0=2} = 3 \) and \( q|_{y_2=2, y_4=2} = -2 \), then in the case when \( d = (2, 2, 2, 2, 2) \), \( \gcd(I_{y\in d}) = 1 \), and \( I_3(G, X_G) \) is trivial. Since \( q|_{y_2=3, y_4=3} = -4 \), then in the case when \( d = (2, 2, 3, 3, 3) \), \( \gcd(I_{y\in d}) = 1 \), and \( I_3(G, X_G) \) is trivial.

Let us consider the case associated with vector \( (2, 2, 3, 2, 2) \). In particular, \( d_1 = 2 = d_G(v_1, v_4) \) implies there exists a vertex \( u \in G \) adjacent with \( v_1 \) and \( v_4 \). Let \( M' \)
be the submatrix of $D(G, X_G)$ associated with the vertices $V(G_{6,7}) \cup \{u\}$, that is

$$M' = \begin{bmatrix} x_0 & 2 & 2 & 1 & 3 & a \\ 2 & 2 & 1 & 2 & 2 & 1 \\ 1 & x_2 & 2 & 2 & 1 & c \\ 2 & 2 & 2 & x_3 & 1 & 1 & d \\ 1 & 2 & 2 & 1 & x_4 & 2 & 1 \\ 3 & 1 & 1 & 1 & 2 & x_5 & f \\ a & 1 & c & d & 1 & f & x_u \end{bmatrix} ,$$

Since $\langle \text{minors}_3(M') \rangle = \langle 1 \rangle$, then $I_3(G, X_G)$ is trivial.

Finally, let us consider the case associated with vector $(3, 3, 3, 3)$. Since $d_G(v_0, v_1) = 3$, then there exists a path $v_0, v, u, v_1$ such that $u \neq v_4$, since otherwise $d_G(v_4, v_1)$ would be equal to 2. Let $M'$ be the submatrix of $D(G, X_G)$ associated with the vertices $V(G_{6,7}) \cup \{u\}$, that is

$$M' = \begin{bmatrix} x_0 & 3 & 3 & 2 & 1 & 3 & 2 \\ 3 & x_1 & 1 & 2 & 3 & 1 & 1 \\ 3 & 1 & x_2 & 2 & 3 & 1 & c \\ 2 & 2 & 2 & x_3 & 1 & 1 & d \\ 1 & 3 & 3 & 1 & x_4 & 2 & e \\ 3 & 1 & 1 & 1 & 2 & x_5 & f \\ 2 & 1 & c & d & 1 & e & f & x_u \end{bmatrix} ,$$

Since $\langle \text{minors}_3(M') \rangle = \langle 1 \rangle$, then $I_3(G, X_G)$ is trivial.

Lemma 11. $G_{6,9}$ is in $\text{Forb}_2$.

Proof. Let $G$ be a graph containing $G_{6,9}$ as induced subgraph, and let $M$ be the submatrix of $D(G, X_G)$ associated with the vertices of $G_{6,9}$ that is shown in Figure 6 and where $d_G(v_1, v_2) = y_0$ and $d_G(v_0, v_2) = y_1$.

$$\begin{bmatrix} x_0 & 2 & y_1 & 2 & 2 & 1 \\ 2 & x_1 & y_0 & 2 & 2 & 1 \\ y_1 & y_0 & x_2 & 1 & 1 & 2 \\ 2 & 2 & 1 & x_3 & 1 & 1 \\ 2 & 2 & 1 & 1 & x_4 & 1 \\ 1 & 1 & 2 & 1 & 1 & x_5 \end{bmatrix} ,$$

![Figure 6. Submatrix associated with $G_{6,9}$](image)

We have $\det(M[\{1, 4, 5\}, \{1, 3, 4\}]) = 4 - y_0$ and $\det(M[\{0, 4, 5\}, \{1, 2, 3\}]) = 4 - y_1$. Since $y_0, y_1 \in \{2, 3\}$, then when one of $y_0$ or $y_1$ is equal to 3, $\langle \text{minors}_3(M) \rangle = \langle 1 \rangle$ and $I_3(G, X_G)$ is trivial. On the other hand, $\det(M[\{1, 4, 5\}, \{0, 3, 5\}]) = 1 - 2x_3$. Thus, when one of $y_0$ or $y_1$ is equal to 2, then $(2, 1 - 2x_3) = \langle 1 \rangle$. In all cases we have $1 \in I_3(G, X_G)$, from which follows that $G \in \text{Forb}_2$.

Lemma 12. co-twin-house is in $\text{Forb}_2$.

Proof. Let $G$ be a graph having co-twin-house as induced subgraph, and let $M$ be the submatrix of $D(G, X_G)$ associated with the vertices of co-twin-house that is shown in Figure 7 and where $d_G(v_0, v_1) = y_2$, $d_G(v_0, v_4) = y_1$ and $d_G(v_1, v_5) = y_0$. 

![Figure 7. Submatrix associated with co-twin-house](image)
Let

\[ I = \{ y_0y_1 - 2y_0 - 2y_1 + 3, y_2 - 5 \} , \]

and

\[ J = \{ x_0 + y_1 - 6, x_1 + y_0 - 6, x_2x_3 - 2x_2 - 2x_3 + 3, \]
\[ x_2y_0 - x_2 - y_0 + 1, x_2y_1 - x_2 - y_1 + 1, 3x_2 - 3, \]
\[ x_3y_0 - x_3 - y_0 + 1, x_3y_1 - x_3 - y_1 + 1, 3x_3 - 3, \]
\[ x_4 + y_1 - 3, x_5 + y_0 - 3 \} . \]

It can be verified that the lexicographic Gröbner basis of the ideal \( \langle \text{minors}_3(M) \rangle \) is equal to \( \langle I \cup J \rangle \). Let \( d \) be a vector such that the \( i \)-th entry is taken over a possible values of \( y_i \). Thus \( d_0, d_1 \in \{2, 3\} \) and \( d_2 \in \{2, 3, 4\} \). Note that if \( \gcd(I_{|y=d}) \) is equal to 1, then \( \langle \text{minors}_3(M) \rangle = \{1\} \), which implies that \( I_3(G, X_G) \) is trivial. It can be verified that \( \gcd(I_{|y=d}) \) is equal to 1 for all valid vectors, except for the following vectors: \( (3, 3, 2) \) and \( (3, 3, 3) \).

Let us consider the case associated with vector \( (3, 3, 3) \). Since \( d_G(v_0, v_1) = 3 \), then there exists a path \( v_0, v, u, v_1 \) such that \( v \neq v_5 \), since otherwise \( d_G(v_5, v_1) \) would be equal to 2. Let \( M' \) be the submatrix of \( D(G, X_G) \) associated with the vertices \( V(\text{co-twin-house}) \cup \{v\} \), that is

\[
M' = \begin{bmatrix}
  x_0 & 3 & 2 & 2 & 3 & 1 & 1 \\
  3 & x_1 & 2 & 2 & 1 & 3 & 2 \\
  2 & 2 & x_2 & 1 & 1 & 1 & c \\
  2 & 2 & 1 & x_3 & 1 & 1 & d \\
  3 & 1 & 1 & 1 & x_4 & 2 & e \\
  1 & 3 & 1 & 1 & 2 & x_5 & f \\
  1 & 2 & c & d & e & f & x_v
\end{bmatrix},
\]

Since \( \langle \text{minors}_3(M') \rangle = \{1\} \), then \( I_3(G, X_G) \) is trivial.
Finally, let us consider the case associated with vector $(3, 3, 2)$. Since $d_G(v_0, v_1) = 2$, then there exists a vertex $u \in G$ adjacent with $v_0$ and $v_1$. Let $M'$ be the submatrix of $D(G, X_G)$ associated with the vertices $V(\text{co-twin-house}) \cup \{u\}$, that is

$$M' = \begin{bmatrix} x_0 & 2 & 2 & 2 & 3 & 1 & 1 \\ 2 & x_1 & 2 & 2 & 1 & 3 & 1 \\ 2 & 2 & x_2 & 1 & 1 & 1 & c \\ 2 & 2 & 1 & x_3 & 1 & 1 & d \\ 3 & 1 & 1 & 1 & x_4 & 2 & e \\ 1 & 3 & 1 & 1 & 2 & x_5 & f \\ 1 & 1 & c & d & e & f & x_u \end{bmatrix},$$

It can be seen that $\langle \text{minors}_3(M') \rangle$ is equal to

$$\langle x_0, x_1, x_2 + 2, x_3 + 2, x_4, x_5, c + 1, d + 1, e + 1, f + 1, x_u + 2, 3 \rangle.$$  

From which follows that if $u$ is adjacent with $x_2$, $x_3$, $x_4$ or $x_5$, then one of $c + 1$, $d + 1$, $e + 1$ or $f + 1$ is equal to 2, and 1 would be in $I_3(G, X_G)$. So, suppose $u$ is not adjacent with neither $x_2$, $x_3$, $x_4$ nor $x_5$. Thus $e = f = 2$. However, $c$ and $d$ could be 2 or 3. Note that if one of $c$ or $d$ is equal to 3, then 1 would be in $I_3(G, X_G)$. Thus, assume $c$ and $d$ are equal to 2. Therefore, there exists a vertex $v$ adjacent with $u$ and $x_2$. Let $M''$ be the submatrix of $D(G, X_G)$ associated with the vertices $V(\text{co-twin-house}) \cup \{u, v\}$, that is

$$M'' = \begin{bmatrix} x_0 & 2 & 2 & 2 & 3 & 1 & 1 & a \\ 2 & x_1 & 2 & 2 & 1 & 3 & 1 & b \\ 2 & 2 & x_2 & 1 & 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & x_3 & 1 & 1 & 2 & d \\ 3 & 1 & 1 & 1 & x_4 & 2 & 2 & e \\ 1 & 3 & 1 & 1 & 2 & x_5 & 2 & f \\ 1 & 1 & 2 & 2 & 2 & x_u & 1 \\ a & b & 1 & d & e & f & 1 & x_v \end{bmatrix},$$

The result follows since $\langle \text{minors}_3(M'') \rangle = \langle 1 \rangle$. □

**Lemma 13.** $G_{6,12}$ is in $\text{Forb}_2$. 

**Proof.** Let $G$ be a graph having $G_{6,12}$ as induced subgraph, and let $M$ be the submatrix of $D(G, X_G)$ associated with the vertices of $G_{6,12}$ that is shown in Figure 8 and where $d_G(v_0, v_1) = y_0$.

$$\begin{bmatrix} x_0 & 2 & 2 & 2 & y_0 & 1 \\ 2 & x_1 & 2 & 2 & 1 & 1 \\ 2 & 2 & x_2 & 1 & 1 & 1 \\ 2 & 2 & 1 & x_3 & 1 & 1 \\ y_0 & 1 & 1 & 1 & x_4 & 2 \\ 1 & 1 & 1 & 1 & 2 & x_5 \end{bmatrix},$$

**Figure 8.** Submatrix associated with $G_{6,12}$.

We have that $\det(M[\{0, 3, 4\}, \{1, 2, 5\}]) = 3$ and $\det(M[\{0, 1, 2\}, \{3, 4, 5\}]) = 1 - y_0$. Since $y_0 \in \{2, 3\}$, then $\langle 1 \rangle = (3, 1 - y_0) \subseteq I_3(G, X_G)$, from which follows that $G \in \text{Forb}_2$. □
Lemma 14. $G_{6,15}$ is in $\text{Forb}_2$.

Proof. Let $G$ be a graph having $G_{6,15}$ as induced subgraph, and let $M$ be the submatrix of $D(G, X_G)$ associated with the vertices of $G_{6,15}$ that is shown in Figure 9 and where $d_G(v_0, v_2) = y_3$, $d_G(v_0, v_3) = y_2$, $d_G(v_1, v_2) = y_1$ and $d_G(v_1, v_3) = y_0$.

Let

$$I = \{x_0x_1 + x_1 + x_0y_0 + 2x_0 + y_0 + y_2 + 1, \]
$$

$$x_0y_1 + 2x_0 + y_1 + y_3 + 1, x_1y_2 + 2x_1 + y_0 + y_2 + 1, \]
$$

$$x_1y_3 + 2x_1 + y_1 + y_3 + 1, x_2 + y_1y_3 + 2y_1 + 2y_3 + 2, \]
$$

$$x_3 + y_0y_2 + 2y_0 + 2y_2 + 2, x_4 + 1, x_5 + 1, x_6 + 1\},$$

and

$$J = \{y_0y_1 + 2y_0 + 2y_1 + 1, y_0y_3 + 2y_0 + 2y_3 + 1, \]
$$

$$y_1y_2 + 2y_1 + 2y_2 + 1, y_2y_3 + 2y_2 + 2y_3 + 1, 3\}.\]

It can be verified that the lexicographic Gr"{o}bner basis of the ideal $\langle \text{minors}_3(M) \rangle$ is equal to $(I \cup J)$. Let $d$ be a vector such that the $i$-th entry is taken over a possible value of $y_i$, that is, $d_i \in \{2, 3\}$ for $i \in \{0, \ldots, 3\}$. Since $\gcd(J|_{y=d}) = 1$, for any possible vector $d$, then $\langle \text{minors}_3(M) \rangle = \langle 1 \rangle$. Which implies $I_3(G, X_G)$ is trivial. Therefore by previous lemmas we have graphs in $\Lambda_2$ are $F$-free. Now, we are going to prove that graphs in $\Lambda_2$ are odd-holes-free.

Lemma 15. Let $n \geq 3$. Then, $\Phi(C_{2n+1}) \geq 3$ and no connected induced subgraph $H$ of $G$ has $\Phi(H) = 3$.

Proof. When $n = 3$, we have

$$\det(D(C_7, X_{C_7})[[0, 1, 2], [4, 5, 6]]) = 2$$

and

$$\det(D(C_7, X_{C_7})[[1, 2, 4], [3, 5, 6]]) = 5.$$ 

Since $\gcd(2, 5) = 1$, then $I(C_7, X_{C_7})$ is trivial. When $n \geq 4$, we have the following. Let $C_{2n+1}$ be the cycle with vertex set $V(C_{2n}) = \{v_i : i \in \{0, \ldots, 2n + 1\}$ and edge set $E(C_{2n+1}) = \{v_{i-1}v_i : i \in \{2n\} \cup \{v_{2n}, v_0\}$. Consider the submatrix $D(C_{2n+1}, X_{C_{2n+1}})\{[0, 1, 2], [n-1, n, n+1]\}$:

$$M = \begin{bmatrix}
 n-1 & n & n \\
 n-2 & n-1 & n \\
 n-3 & n-2 & n-1
\end{bmatrix}$$

Figure 9. Submatrix associated with $G_{6,15}$. 

Since $\gcd(2, 5) = 1$, then $I(C_7, X_{C_7})$ is trivial. When $n \geq 4$, we have the following. Let $C_{2n+1}$ be the cycle with vertex set $V(C_{2n}) = \{v_i : i \in \{0, \ldots, 2n + 1\}$ and edge set $E(C_{2n+1}) = \{v_{i-1}v_i : i \in \{2n\} \cup \{v_{2n}, v_0\}$. Consider the submatrix $D(C_{2n+1}, X_{C_{2n+1}})\{[0, 1, 2], [n-1, n, n+1]\}$:
Since $\det(M) = -1$, it follows that $I(C_{2n+1}, X_{C_{2n+1}})$ is trivial. Finally, any connected induced subgraph of $C_{2n+1}$ is a path. In [18] Theorem 3, it was proved that the third invariant factor of the Smith normal form of the distance matrix of a tree is 2. Therefore by Corollary \ref{cor:2}, $\Phi(P_k) \leq 2$. \hfill $\Box$

**Lemma 16.** For $n \geq 3$, $C_{2n+1}$ is in $\text{Forb}_2$.

*Proof.* It only remains to prove that for $n \geq 3$ if $G$ is a graph having $C_{2n+1}$ as induced subgraph, then $I_3(G, X_G)$ is trivial. First note that case $n = 3$ is true.

**Claim 17.** $C_7$ is minimal in $\text{Forb}_2$.

*Proof.* Let $G$ be a graph having $C_7$ as induced subgraph, and let $M$ be the submatrix of $D(G, X_G)$ associated with the vertices of $C_7$, that is

$$
\begin{bmatrix}
 x_0 & y_6 & y_5 & 2 & 2 & 1 & 1 \\
 y_6 & x_1 & 1 & 2 & 1 & y_4 & 2 \\
 y_5 & 1 & x_2 & 1 & 2 & 2 & y_3 \\
 2 & 2 & 1 & x_3 & y_2 & 1 & y_1 \\
 2 & 1 & 2 & y_2 & x_4 & y_0 & 1 \\
 1 & y_4 & 2 & 1 & y_0 & x_5 & 2 \\
 1 & 2 & y_3 & y_1 & 1 & 2 & x_6
\end{bmatrix},
$$

where $d_G(v_0, v_1) = y_6$, $d_G(v_0, v_2) = y_5$, $d_G(v_1, v_5) = y_4$, $d_G(v_2, v_6) = y_3$, $d_G(v_3, v_5) = y_2$, $d_G(v_3, v_6) = y_1$ and $d_G(v_4, v_5) = y_0$. We have that $\det(M[[\{3, 4, 5\}, \{0, 1, 2\}]] = 3 - 2y_4$ and $\det(M[[\{4, 5, 6\}, \{0, 1, 2\}]] = 2y_3y_4 - y_3 - 2y_4 - 2$. Since $y_3, y_4 \in \{2, 3\}$, then $\langle 1 \rangle = \{3 - 2y_4, 2y_3y_4 - y_3 - 2y_4 - 2\} \subseteq I_3(G, X_G)$, from which follows that $G \in \text{Forb}_2$. \hfill $\Box$

We are going to proceed by contraposition. Assume $n \geq 4$ is the first odd positive integer where there exists a graph $G$ with $C_{2n+1}$ as induced subgraph such that $I_3(G, X_G)$ is not trivial. Thus, for $k \in \{7, \ldots, 2n - 1\}$, if a graph $H$ has $C_k$ as induced subgraph, then $I_3(H, X_G)$ is trivial.

For simplicity we are going to denote $C_{2n+1}$ as $C$. There are two cases.

(1) when $d_C(u, v) = d_G(u, v)$ for each pair $u, v \in V(C)$, and

(2) there is at least a pair $u, v \in V(C)$ such that $d_G(u, v) < d_C(u, v)$.

In case (1), it follows by Lemma \ref{lem:3} that $I_3(C, X_C)$ is trivial, and then by Lemma \ref{lem:4} $I_3(G, X_G)$ is also trivial; which is not possible.

For case (2), take $u, v \in V(C)$ such that $d_G(u, v) < d_C(u, v)$, and $d_G(u, v)$ is minimum over all pairs in $C$. Let $v_0, \ldots, v_{2n+1}$ denote the vertices in $V(C)$ such that $v_k$ is adjacent only with $v_{k-1}$ and $v_{k+1}$. We abuse of notation to refer $v_k$ to $v_k \mod 2n+1$.

**Claim 18.** If $w \in V(G) \setminus V(C)$ is adjacent with $v_k \in V(C)$, then $w$ is not adjacent with neither $v_{k-1}$ nor $v_{k+1}$.

*Proof.* Suppose $w$ is adjacent with $v_{k+1}$. There are two cases: either $w$ is adjacent with $v_{k+2}$ or not. Consider the first case. Vertex $w$ cannot be adjacent with $v_{k-1}$ nor $v_{k+3}$, because otherwise the vertices $w, v_k, v_{k+1}, v_{k+2}, v_{k+3}$ or the vertices $w, v_{k-1}, v_k, v_{k+1}, v_{k+2}$ would induce a gem, and by Proposition \ref{prop:6} $I_3(G, X_G)$ would be trivial. Therefore, the vertices $w, v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}$ will induce a co-twin-house. Thus by Lemma \ref{lem:12} $I_3(G, X_G)$ would be trivial, which is impossible. Now consider the case when $w$ is not adjacent with $v_{k+2}$. There are two possible scenarios: either $w$ is adjacent with $v_{k-1}$ or not. In the first one, we also have
to consider whether \(w\) is adjacent with \(v_{k-2}\) or not. If \(w\) is adjacent with \(v_{k-2}\), then the vertices \(w, v_{k-2}, v_{k-1}, v_k v_{k+1}\) induce a gem, which is not possible. If \(w\) is not adjacent with \(v_{k-2}\), then the vertices \(w, v_{k-2}, v_{k-1}, v_k, v_{k+1}\) will induce a co-twin-house. Therefore, \(w\) is not adjacent with \(v_{k-1}\). And thus, the vertices \(w, v_{k-1}, v_k, v_{k+1}, v_{k+2}\) induce a bull; which is not possible since Lemma 7 implies \(I_3(G, X_G)\) must be trivial. Then, \(w\) is not adjacent with \(v_{k+1}\). Analogously, we can obtain that \(w\) is not adjacent with \(v_{k-1}\). □

Claim 19. If \(w \in V(G) \setminus V(C)\) is adjacent with \(v_k \in V(C)\), then \(w\) is not adjacent with neither \(v_{k-3}\) nor \(v_{k+3}\).

Proof. Suppose \(w\) is adjacent with \(v_{k+3}\). By Claim 18, \(w\) is not adjacent with \(v_{k+1}\), \(v_{k+2}\) nor \(v_{k+4}\). Then, the vertices \(w, v_k, v_{k+1}, v_{k+2}, v_{k+3}, v_{k+4}\) induce a 5-pan. Thus by Lemma 9, \(I_3(G, X_G)\) is trivial, which is impossible.

Next claim follows since \(n\) is the first integer greater or equal than 4 for which there exists a graph \(G\) with \(C_{2n+1}\) as induced subgraph such that \(I_3(G, X_G)\) is not trivial.

Claim 20. If \(w \in V(G) \setminus V(C)\) is adjacent with \(v_k \in V(C)\), then \(w\) is not adjacent with \(v_{k-1}\) nor \(v_{k+l}\), where \(l \in \{5, 7, \ldots, 2n-1\}\).

Proof. Assume \(w\) is adjacent with \(v_{k+5}\). By applying Claims 18 and 19 to \(v_k\) and \(v_{k+5}\), we have \(w\) is not adjacent with \(v_{k+1}\), \(v_{k+2}\), \(v_{k+3}\) and \(v_{k+4}\). Then, \(w, v_k, v_{k+1}, v_{k+2}, v_{k+3}, v_{k+4}\) induce a \(C_7\) which has at least 3 trivial distance ideals. That is not possible. The results follows by applying recursively the argument. □

Claims 18, 19 and 20 imply that if a vertex \(w \in V(G) \setminus V(C)\) is adjacent with \(v_k \in V(C)\), then \(w\) cannot be adjacent with \(v_{k-1}, v_{k-3}, \ldots, v_{k+1}\) neither with \(v_{k+1}, v_{k+3}, \ldots, v_{k-1}\). That is not other thing that next result.

Claim 21. If \(w \in V(G) \setminus V(C)\) is adjacent with \(v_k \in V(C)\), then \(w\) is not adjacent with any other vertex in \(V(C) \setminus \{v_k\}\).

On the other hand, since \(u\) and \(v\) are two vertices in \(V(C)\) such that \(d_G(u, v) < d_C(u, v)\), then there exists a path \(P = u, u_1, \ldots, u_l, v\) of minimum length \(l = d_G(u, v)\). We might assume \(u_k \notin V(C)\) for each \(k \in \{1, \ldots, l-1\}\). For simplicity, we also relabel the vertices of \(C\) such that \(u = v_0\) and \(v = v_m\). By Claim 21 for \(k \in \{1, \ldots, l-1\}\), vertex \(u_k\) is adjacent with at most one vertex in \(V(C)\). In particular, the length \(l\) cannot be 2. See Figure 10.

Let \(C^1\) be the induced subgraph obtained by the vertices \(\{v_0, v_1, \ldots, v_{m-1}, v_m\} \cup V(P)\), and let \(C^2\) be the induced subgraph obtained by \(\{v_0, v_{-1}, \ldots, v_{m+1}, v_m\} \cup V(P)\). Note that either \(C^1\) or \(C^2\) has an odd number of vertices. Let us assume without loss of generality \(C^1\) is the side with an odd number of vertices. In \(C^1\) we want to find an induced cycle with an odd number of vertices. In fact, if \(C^1\) is already an induced cycle of odd length, then this cannot be of length 3 or 5. Otherwise \(P = v_0 u_1 v_m\) consists of three vertices, this is because \(P\) have less vertices than the path \(v_0, v_1, \ldots, v_m\). Which is impossible since by Claim 21 every vertex in \(V(P) \setminus \{v_0, v_m\}\) is adjacent with at most one vertex in \(C\). Therefore \(C^1\) is an odd induced cycle of length less than \(2n+1\) and greater or equal than 7. Which is also impossible, because otherwise \(I_3(G, X_G)\) is trivial.

So, suppose there is at least one vertex in \(V(P) \setminus \{v_0, v_m\}\) that is adjacent with a vertex in \(P' = v_1 v_2 \cdots v_{m-1}\). Let \(u_k \in P\) be the first vertex from \(v_0\) that is
adjacent with a vertex \( v_j \in P' \). Note that \( u_k \) cannot be adjacent with a vertex in \( V(P) \setminus \{u_{k-1}, u_k, u_{k+1}\} \), otherwise \( P \) would not be of minimum length. Also we have the next result which can be proved analogously to Claim 18.

**Claim 22.** Let \( v_j \in V(C^1) \setminus V(P) \). If \( v_j \) is adjacent with \( u_k \in V(P) \), then \( v_j \) is not adjacent with \( u_{k-1} \) nor \( u_{k+1} \).

Thus, Claims 19 and 22 imply that \( C^1 \) does not contain a \( C_3 \) as induced subgraph.

The edge \( u_k v_j \) divides \( C^1 \) in two induced subgraphs: an induced cycle \( C^3 \) with vertex set \( \{v_0, v_1, \ldots, v_j, u_k, u_{k-1}, \ldots, u_1\} \), and the induced subgraph obtained by the vertices \( u_k, v_j, v_{j+1}, \ldots, v_m, u_{l-1}, \ldots, u_{k+1} \). One of these induced subgraphs have an odd number of vertices.

If the cycle is the induced subgraph of odd length, then we are almost done. This cycle \( C^3 \) cannot have length 3 or greater than 7, since otherwise \( G \) would have three trivial distance ideals. We only need to consider when \( C^3 \) has length 5. But there are only two possibilities: either \( V(C^3) \) is equal to \( v_0, v_1, u_3, u_2, u_1 \) or \( v_0, v_1, v_2, u_2, u_1 \). However, in both cases the vertices \( v_0, v_1, u_3, u_2, u_1, u_4 \) and \( v_1, v_0, v_1, u_2, u_1 \) would induce a 5-pan. Which is not possible since otherwise \( G \) would have three trivial distance ideals. Then the vertices \( u_k, v_j, v_{j+1}, \ldots, v_m, u_{l-1}, \ldots, u_{k+1} \) induce a subgraph with an odd number of vertices.

Assume that, and take the first vertex, say \( u_r \), in \( P \) from \( u_k \) that is adjacent with a vertex, say \( v_s \), in \( P' \setminus \{v_0, v_1, \ldots, v_{j-1}\} \). If such vertex does not exist we are done with a similar argument than in previous paragraph. Therefore assume, \( v_s \) is adjacent with \( u_r \) such that \( r > k \) and \( s \geq j \). Again from the edge \( u_r v_s \) we obtain two induced subgraphs where one has odd length: let \( C^4 \) be the subgraph induced cycle by the vertices \( u_k, u_{k+1}, \ldots, u_r, v_s, v_{s-1}, \ldots, v_j \), and the other subgraph is the induced by \( u_r, v_s, v_{s+1}, \ldots, v_m, u_{l-1}, \ldots, u_{r+1} \). If \( C^4 \) has odd length, again we only need to consider when it has length 5. There are three possible cases: either \( r = k + 1 \), \( r = k + 2 \) or \( r = k + 3 \). See Figure 11.

In case \( r = k + 1 \), \( u_k \) is different from \( v_j \) since \( u_k \) is already adjacent with \( v_j \). Consider \( u_{k-1} \). By Claim 22 \( v_j \) is not adjacent with \( u_{k-1} \). However, it is possible that at most one of the vertices \( v_{j+1} \) or \( v_{j+1} + 1 \) is adjacent with \( u_{k-1} \). In any of the three possibilities we would obtain a \( 5 - \text{pan} \) or \( G_{6,10} \) as induced subgraph og \( G \), which is impossible.

![Figure 10. A drawing of the induced subgraph by \( C \) and \( P \).](image-url)
Similarly, in cases $r = k + 2$ and $r = k + 3$, by considering vertex $v_{j-1}$, we will obtain that $C^4$ and $v_{j-1}$ induce a $5 - \text{pan}$ or $G_{6,10}$. Which is impossible. Therefore, $C^4$ is of even length.

Continuing inductively with the next vertex in $P'$ after $v_r$ that is adjacent with a vertex $v_t$ with $t \geq s$, we will obtain that there is an odd cycle in $C^4$ of length greater or equal than 5. And we will get a contradiction. \hfill \Box

Thus by previous lemmas we have our main result.

**Theorem 23.** Graphs in $\Lambda_2$ are $\{F, \text{odd-holes}\}$-free.

The remaining difficult question is: are all $\{F, \text{odd-holes}\}$-free graphs in $\Lambda_2$?

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