Factorization law for two lower bounds of concurrence

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We study the dynamics of two lower bounds of concurrence in bipartite quantum systems when one party goes through an arbitrary channel. We show that these lower bounds obey the factorization law similar to that of [Konrad et al., Nat. Phys. 4, 99 (2008)]. We also, discuss the application of this property, in an example.

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I. INTRODUCTION

Entanglement, one of the important features of quantum systems which does not exist classically, has been known as a key resource for some quantum computation and information processes. But the entanglement of a system changes due to its unavoidable interactions with environment. To study the entanglement changes, one needs to make use of an entanglement measure in order to specify the entanglement amount of a system. Unfortunately, most of the measures having been proposed for quantification of entanglement can not be computed easily, have been introduced for these entanglement measures. Using these bounds, one can estimate the amount of entanglement.

In Ref. [1] Konrad et al. have provided a factorization law for concurrence which is one of the remarkable entanglement measures. They have shown that the concurrence of a two qubit state, when one of its qubits goes through an arbitrary quantum channel, is equal to the product of its initial concurrence and concurrence of the maximally entangled state undergoing the effect of the same quantum channel. Then Li et al. [2] have shown that the generalization of the above factorization law to arbitrary dimensional bipartite states only leads to an upper bound for the concurrence of the system. If, beside this upper bound we have a lower bound obeying a similar factorization law, then we can make better use of this useful dynamical property. So it will be valuable to seek for such entanglement lower bounds.

In section II, we introduce the concurrence and some of its lower bounds. Next, in section III, we briefly review the results of Ref. [1, 2]. Then, in sections IV and V, we investigate the factorization property of the lower bounds introduced in section II. In section VI, as an application, we discuss an example. Finally, we give some conclusions in section VII.

II. CONCURRENCE AND SOME OF ITS LOWER BOUNDS

For a pure bipartite state \( |\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \), concurrence is defined as [3]:

\[
C(\Psi) = \sqrt{2(\langle \Psi | \Psi \rangle^2 - tr\rho_r^2)},
\]

where \( \rho_r \) is the reduced density operator obtained by tracing over either subsystems A or B. Concurrence of \( |\Psi\rangle \) can also be written in terms of the expectation value of an observable with respect to two identical copies of \( |\Psi\rangle \) [4-6]:

\[
C(\Psi) = \sqrt{AB \langle \Psi | A' B' \langle A | A \rangle B \langle | A | B \rangle B'}} = 4P_{A'A'} \otimes P_{B'B'},
\]

where \( P_{A'A'} \) (\( P_{B'B'} \)) is the projector onto the antisymmetric subspace of \( \mathcal{H}_A \otimes \mathcal{H}_{A'} \) (\( \mathcal{H}_B \otimes \mathcal{H}_{B'} \)). A possible decomposition of \( \mathcal{A} \) is

\[
\mathcal{A} = \sum_{i<j, m<n} |\chi_{ij, mn}\rangle \langle \chi_{ij, mn}|,
\]

where |\( |i\rangle \) and |\( |j\rangle \) are two different members of an orthonormal basis of the A (B) subsystem (instead of the index \( \alpha \) in reference [3], we use the indexes \( ij, mn \) because it seems most convenient for the future usage).

For mixed states, the concurrence is defined as follows [4]:

\[
C(\rho) = \min_{\{\psi_k\}} \sum_k p_k C(\psi_k),
\]

\[
\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|, \quad p_k \geq 0, \quad \sum_k p_k = 1,
\]
where the minimum is taken over all decompositions of ρ into pure states |Ψ_k⟩. Like most of the other entanglement measures, C(ρ) can not be computed in general, i.e., in general, one can not find the optimal decomposition of ρ minimizing Eq. (11). Any numerical effort for finding the optimal decomposition, is equivalent to find an upper bound for C(ρ). So, some lower bounds have been introduced for C(ρ) (e.g. [12, 13]).

It has been shown that

\[ \text{ALB}_{ij,mn}(ρ) \equiv \min_{\{p_k, |Ψ_k⟩\}} \sum_k p_k \langle χ_{ij,mn}|Ψ_k⟩|Ψ_k⟩, \quad (5) \]

is a lower bound of concurrence (ALB is the abbreviation of the Algebraic Lower Bound) [4, 7]. ALB_{ij,mn}(ρ) can be computed analytically;

\[ \text{ALB}_{ij,mn}(ρ) = \max \{0, \mathcal{A}_{ij,mn}^1 - \sum_{l>1} \mathcal{A}_{ij,mn}^l \} \quad (4) \]

\[ \mathcal{A}_{ij,mn}^l \quad \text{are the singular values of matrix} \quad T_{ij,mn}^l \quad \text{in decreasing order.} \]

\[ T_{ij,mn}^l \quad \text{entries are defined as} \quad T_{ij,mn}^l \equiv \sqrt{\rho_{ij,mn} |Ψ_{ij,mn}⟩⟨Ψ_{ij,mn}|}, \quad \text{where} \quad |Ψ_{ij,mn}⟩ \quad \text{and} \quad \lambda_r \quad \text{are eigenvectors and eigenvalues of ρ, respectively.} \]

Another lower bounds of concurrence are those introduced in reference [9]. In this reference, it has been shown that in terms of two identical copies of an arbitrary mixed state ρ_{AB} we have

\[ C^2(ρ_{AB}) \geq \text{MLB}^2_{(k)ij,mn}(ρ) \equiv \text{tr} (ρ_{AB} \otimes ρ_{A′B′}V(k)_{ij,mn}), \quad k = 1, 2, \]

\[ V(1)_{ij,mn} = 4P_{A′A′}^{ij} \otimes (P_{B′-mn} - P_{B′B′}^j), \]

\[ V(2)_{ij,mn} = 4 \left( P_{A′A′}^{ij} - P_{A′A′}^{ij+} \right) \otimes P_{B′-mn}. \]

(6)

(MLB is the abbreviation of the Measurable lower bound) where

\[ 2P_{A′A′}^{ij} = (|ij⟩ - |ji⟩)(⟨ij⟩ - ⟨ji⟩) \quad \text{and} \quad 2P_{A′A′}^{ij+} = (|ij⟩ + |ji⟩)(⟨ij⟩ + ⟨ji⟩) + 2|ii⟩(⟨ii⟩ + 2|jj⟩), \]

operate on \( H_A \otimes H_A \), whereas

\[ 2P_{B′-mn} = (|mm⟩ - |nm⟩)(⟨mm⟩ - ⟨nm⟩) \quad \text{and} \quad 2P_{B′-mn} = (|mm⟩ + |nm⟩)(⟨mm⟩ + ⟨nm⟩) + 2|nn⟩(⟨nn⟩ + 2|nm⟩) \]

operate on \( H_B \otimes H_B \). \(|m|, |n|, |m| + |n|, |m| - |n| \) were introduced in Eq. (3). The above expression means that measuring \( V(k)_{ij,mn} \) on two identical copies of ρ, i.e. \( ρ \otimes ρ \), gives us a measurable lower bound on \( C^2(ρ) \).

In reference [10], another lower bound of concurrence was introduced. There, it was shown that:

\[ τ(ρ) = \sum_{i<j, m<n} C_{ij,mn}^2(ρ) \leq C^2(ρ), \]

\[ C_{ij,mn}(ρ) = \min_{\{p_k, |Ψ_k⟩\}} \sum_k p_k \langle χ_{ij,mn}|L_{A,ij} ⊗ L_{B,mn}|Ψ_k⟩|Ψ_k⟩, \quad (7) \]

where \( L_{A,ij} \) and \( L_{B,mn} \) are the generators of SO(d_A) and SO(d_B) respectively (d_A d_B is the dimension of \( H_A(H_B) \)), and \(|Ψ_k⟩ \) is the complex conjugate of \(|Ψ_k⟩ \) in the computational basis. In this basis \( L_{A,ij} \) and \( L_{B,mn} \) are

\[ \begin{align*}
L_{A,ij} &= |i⟩A⟨j| - |j⟩A|i⟩, \\
L_{B,mn} &= |m⟩B⟨n| - |n⟩B|m⟩.
\end{align*} \quad (8) \]

III. FACTORIZATION OF THE CONCURRENCE

According to the Schmidt decomposition, any pure bipartite state \(|Ψ⟩, |Ψ⟩ ∈ H_A ⊗ H_B, \) can be expressed as

\[ |Ψ⟩ = \sum_{i=1}^d \sqrt{ω_i}|α_iβ_i⟩, \quad 0 ≤ ω_i ≤ 1, \quad \sum_{i=1}^d ω_i = 1, \quad (9) \]

where \( d = \min(d_A, d_B). \)

We can rewrite this \(|Ψ⟩ \) as \(|Ψ⟩ = (M ⊗ I)|φ^+⟩ \) where \(|φ^+⟩ = \sum_{i=1}^d 1/\sqrt{d} |α_iβ_i⟩ \) is a maximally entangled state and \( M = \sqrt{d} \sum_{i=1}^d \sqrt{ω_i}|α_i⟩⟨α_i|. \)

Assume that the second part of this state goes through an arbitrary channel \( S \), then this state transforms to \(|ρ'⟩ = (M ⊗ I)|φ^+⟩ \)

\[ = tr[(1 ⊗ S)|Ψ⟩⟨Ψ|] \quad \text{where} \quad ρ_S = (1 ⊗ S)|φ^+⟩⟨φ^+|, \quad p' = tr[(M ⊗ I)ρ_S(M1 ⊗ I)], \quad p'' = tr[(1 ⊗ S)|φ^+⟩⟨φ^+|] \quad \text{and} \quad p = p''p'. \]

By using these relations, for any two-qubit state \(|Ψ⟩ \), Konrad et al. [11] have proved the following factorization law [12]:

\[ C[(1 ⊗ S)|Ψ⟩⟨Ψ|] = C[(1 ⊗ S)|φ^+⟩⟨φ^+|C(Ψ). \quad (10) \]

The right hand side of the above equation is factorized into two independent parts. The first part is the concurrence of \(|φ^+⟩ \) after going through the channel \((1 ⊗ S) \), which is independent of the initial state \(|Ψ⟩ \), and the second part is the concurrence of the initial state \(|Ψ⟩ \) (before going into the channel). So, if we know the concurrence of \(|φ^+⟩ \), after one of its qubits goes through a channel \( S \), we know, up to the factor \( C(Ψ) \), the concurrence of any arbitrary state \(|Ψ⟩ \) undergoing true the same quantum channel.

For higher dimensional bipartite systems, Lie et al. [2] have shown that the above equality changes to the following inequality

\[ C[(1 ⊗ S)|Ψ⟩⟨Ψ|] ≤ \frac{d_B}{2} C[(1 ⊗ S)|φ^+⟩⟨φ^+|C(Ψ). \quad (11) \]

For the \( d_A × 2 \) dimensional states, we have the equality instead of the inequality in the above relation. But, in general, the concurrence of \((1 ⊗ S)|φ^+⟩⟨φ^+| \) provides only an upper bound for \( C[(1 ⊗ S)|Ψ⟩⟨Ψ|] \). We point out that in relations (10) and (11), instead of \(|φ^+⟩ \), we can use any other maximally entangled state. It is also interesting to investigate similar relations for the lower bounds of concurrence. In the next section, we study the factorization property of the lower bounds introduced in the previous section.
In order to obtain the last equality, we have used \((M_A^\dagger \otimes M_A^\dagger) P^{AA'}(M_A \otimes M_A) = d^2 \omega_j \omega_j P^{AA'}\) where \(P^{AA'}\) is written in the Schmidt basis, i.e. we choose \(|i\rangle = |\alpha_i\rangle\) and \(|j\rangle = |\alpha_j\rangle\) in construction of \(P^{AA'}\). Also, writing \(I_{BB'}^m\) and \(P_{BB'}^m\) in the Schmidt basis, we have \(MLB_{(ij),mn}(\rho) = 4 \omega_j \delta_{im} \delta_{jn}\). Using this relation, Eq. (12) can be written in the form:

\[
MLB_{(ij),mn}(1 \otimes |\Psi\rangle \langle \Psi|) = \frac{d^2}{4} MLB_{(ij),mn}(1 \otimes |\phi^+\rangle \langle \phi^+|) MLB_{(ij),ij}\langle |\Psi\rangle |(13)
\]

The above equation (which is our main result) is similar to Eq. (10), so \(MLB_{(ij),mn}(\rho)\) have the same factorization property as concurrence, i.e. knowing the effect of \((1 \otimes |\Psi\rangle\langle \Psi|)\) on the \(MLB_{(ij),mn}(\rho)\) when the initial state is \(|\Phi^+\rangle\), we know this effect for any other initial state \(|\Psi\rangle\), up to a factor \(MLB_{(ij),ij}\langle |\Psi\rangle |\).

For the \(MLB_{(ij),mn}(\rho)\), we obtain exactly the same result as above if instead of the second part, the first part of the state \(|\Psi\rangle\) goes through the channel \(S\).

Now we discuss the factorization property of \(ALB_{ij,mn}(\rho)\). We use a similar method as Ref. 4, namely, at first we restrict ourselves to those cases where \(\rho_S\) is a pure state i.e. \(\rho_S = |\psi\rangle \langle \psi|\). In this cases \(\rho'\) is also a pure state i.e. \(\rho' = |\psi'\rangle \langle \psi'|\) = \((M \otimes 1)(|\psi\rangle \langle \psi|)(M \otimes 1))\). From Eq. (9) we have

\[
pALB_{ij,mn}(\rho') = p(|\chi_{ij,mn}\langle \psi'\rangle|^2) = d^2 \omega_j \omega_j ALB_{ij,mn}(\rho_S), \tag{14}
\]

where we used \((M \otimes 1)(|\chi_{ij,mn}\langle \chi_{ij,mn}|(M \otimes 1) = d^2 \omega_j \omega_j |\chi_{ij,mn}\rangle \langle \chi_{ij,mn}|\) and \(|\chi_{ij,mn}\rangle\) is written in the Schmidt basis. Using \(ALB_{ij,mn}(\rho) = 4 \omega_j \delta_{im} \delta_{jn}\), we obtain

\[
pALB_{ij,mn}(\rho') = \frac{d}{2} ALB_{ij,ij}(|\Psi\rangle) ALB_{ij,mn}(\rho_S). \tag{15}
\]

Next, we consider the general case where \(\rho_S\) is a mixed state. Corresponding to any pure state decomposition of \(\rho_S\) as \(\rho_S = \sum_k p_k |\psi_k\rangle \langle \psi_k|\), there exist a pure state decomposition for \(\rho'\) in terms of pure states \(|\psi_k'\rangle = \langle \psi_k' |\langle \psi_k'| \rangle\). From this relation we have

\[
|\langle \psi_k'| \rangle| = \langle \psi_k' |\langle \psi_k'| \rangle\rangle = \langle \psi_k' |\langle \psi_k'| \rangle\rangle = d \sqrt{\omega_j \omega_j} ALB_{ij,mn}(\rho_S). \tag{16}
\]

In the cases where \(M^{-1}\) exists, i.e. when in Eq. (9) all \(\omega_i\) we have \(\omega_i \neq 0\), as for the \(d_A \times 2\) dimensional systems (the case of the separable initial states is not of interest), corresponding to any pure state decomposition for \(\rho'\), there is a pure state decomposition for \(\rho_S\) and vice versa, namely, for any \(|\psi_k'\rangle\) in the expression \(\rho' = \sum_k p_k |\psi_k'\rangle \langle \psi_k'|\) we have \(|\psi_k\rangle = \sqrt{p}(M^{-1} \otimes 1)|\psi_k'|\) such that \(\rho_S = \sum_k p_k |\psi_k\rangle \langle \psi_k|\). So, if the \(\rho_S = \sum_k p_k |\psi_k\rangle \langle \psi_k|\) is the optimal decomposition for \(ALB_{ij,mn}(\rho_S)\) then \(\sum_k p_k |\psi_k\rangle \langle \psi_k|\) is the optimal pure state decomposition of \(\rho'\) for \(ALB_{ij,mn}(\rho')\). Therefore, in Eq. (16) we have an equality instead of the inequality.
V. FACTORIZATION OF THE LOWER BOUND OF SQUARED CONCURRENCE ($\tau$)

In Ref. [3] Liu et al. have shown that $\tau$ (Eq. (17)), for a $d \times d$ bipartite quantum state, obeys the relation

$$
\tau((1 \otimes S)|\Psi\rangle\langle\Psi|) \leq \frac{d^2}{4} \tau((1 \otimes S)|\phi^+\rangle\langle\phi^+|)C^2(\Psi).
$$

(17)

The above relation is the factorization law for $\tau$ similar to the Eq. (11) which is for the concurrence itself.

Now, we show that $ALB_{ij,mn}(\rho)$ is closely related to $\tau$; For an arbitrary $|\Psi\rangle$, according to the definition of $|\chi_{ij,mn}\rangle$ in Eq. (3), it can be seen that $|\langle\Psi|L_{A,ij} \otimes L_{B,mn}|\Psi\rangle| = |\langle\chi_{ij,mn}|\Psi\rangle|$. So from Eq. (5), we have

$$
ALB_{ij,mn}(\rho) = \min_{\{p_k,|\Psi_k\rangle\}} \sum_k p_k |\chi_{ij,mn}|\langle\Psi_k|\rangle\langle\Psi_k|\rangle
$$

and so:

$$
\tau(\rho) = \sum_{i,j,m<n} ALB_{ij,mn}^2(\rho).
$$

(20)

Therefore, from the Eq. (12) of Ref. [3] and Eq. (19) we deduce that the Eq. (12) of Ref. [3], i.e.

$$
C_{ij,mn}^2((1 \otimes S)|\Psi\rangle\langle\Psi|)
$$

$$
= \frac{d^2}{4} \left( \sum_{i,j,m<n} C_{ij,kl}|\langle\Psi|\rangle C_{kl,mn}|(1 \otimes S)|\phi^+\rangle\langle\phi^+|\right)^2.
$$

(21)

and so the Eq. (15) of the same reference, i.e.

$$
\tau((1 \otimes S)|\Psi\rangle\langle\Psi|) \geq \frac{2dn}{d-1} \frac{d^2}{4} \tau((1 \otimes S)|\phi^+\rangle\langle\phi^+|)C^2(|\Psi|).
$$

(22)

where $\eta = \min_{\{p,r\}} \omega_p \omega_r$ for any pair $p < r$ satisfying $\omega_p \omega_r \neq 0$, dose not hold in general.

VI. EXAMPLE

Consider a two-qutrit system which one of its qutrit interacts with an environment. The time evolution of this system is given by the following Master equation:

$$
\dot{\rho} = L\rho, \quad L = L_A \otimes L_B,
$$

(23)

FIG. 1. Time evolution of the $MLB_{1,12}^2$ when the initial state of the system is $|\phi^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ for the cases(a) spontaneous decay(dashed line)(b) decoherence(solid line).

where $L_B$, for a one-qutrit $\rho_B$, is

$$
L_B = \frac{\Gamma}{2} (2\gamma \rho_B^\dagger \gamma - \rho_B \gamma^\dagger \gamma - \gamma^\dagger \gamma \rho_B).
$$

$\Gamma$ is the decay constant and $\gamma$ is a coupling operator characterizing the dynamics of system. For $\gamma = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ the Eq. (23) represents the spontaneous decay of the system and for $\gamma = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ the Eq. (23) represents the system’s decoherence (13).

In order to evaluate the entanglement dynamics of this system, we use the $MLB_{1,12}^2(\rho)$ (which is a lower bound of squared concurrence). Fig. 1 shows the time evolution of $MLB_{1,12}^2(\rho)$ for the case $i = 1, j = 2, m = 1$ and $n = 2$, when the initial state of the system is $|\phi^+\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$ (for other value of $i,j,m$ and $n$, $MLB_{1,12}^2(\rho)$ dose not give better estimate for entanglement). From this figure and using Eq. (13), we can deduce the behavior of $MLB_{1,12}^2(\rho)$ for any initial states of the form $|\psi\rangle = a|00\rangle + b|11\rangle + c|22\rangle$. For any such initial state, the ability of the $MLB_{1,12}^2(\rho)$ in detecting the entanglement of $\rho' = (1 \otimes S)|\Psi\rangle\langle\Psi|$ is determined by the ability of $MLB_{1,12}^2(\rho)$ in detecting the entanglement of $\rho_S = (1 \otimes S)|\phi^+\rangle\langle\phi^+|$, which is shown in Fig. 1. Also, the amount of the lower bound $MLB_{1,12}^2(\rho)$ is, up to a factor, equal to $MLB_{1,12}^2(\rho_S)$.

VII. CONCLUSIONS

We have studied the dynamics of two lower bounds of bipartite concurrence introduced in Eq. (5) and Eq.
In Eq. 13, we have shown that for arbitrary bipartite quantum states, $MLB_{ij,mn}(\rho)$ obeys the factorization law similar to that of Eq. 10 for the concurrence. In an example, we have discussed the application of this factorization law in determining the behavior of the $MLB_{ij,mn}(\rho)$ in estimating the entanglement of the system. Also, we have shown that the $ALB_{ij,mn}(\rho)$ obeys a similar factorization law for concurrence as Eq. 11.

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