Odd Pfaffian Forms

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Abstract
On any odd-dimensional oriented Riemannian manifold we define a volume form called the *odd Pfaffian* through a certain invariant polynomial with integral coefficients in the curvature tensor. We prove an intrinsic Chern–Gauss–Bonnet formula for incomplete edge singularities in terms of the odd Pfaffian on the fibers of the boundary fibration. The same method produces a Chern–Gauss–Bonnet formula for complete, non-compact manifolds with fibered boundaries in the sense of Mazzeo–Melrose, this time involving the odd Pfaffian of the base of the fibration. We deduce obstructions for certain Riemannian metrics on a total space of a fibration to be realized as the model at infinity of a flat metric with conical, edge or fibered boundary singularities.

Keywords Gauss-Bonnet · Transgression · Pfaffian · Degenerate metric · Edge degeneration · Fibered boundary

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1 Introduction

Gauss–Bonnet formulas in singular geometric contexts abound in mathematical literature, we mention here for instance (Albin 2007; Allendoerfer and Weil 1943; Bröcker and Kuppe 2000; Cheeger et al. 1984, 1986; Dai and Wei 2007; Dutertre 2008; Fu 1994; McMullen 2017; Rataj and Zähle 2003; Rosenberg 1985; Satake 1957; Walter 1975). With a few notable exceptions, most of those theorems treat the case of singular sets embedded in a smooth Riemannian manifold $M$, typically $M = \mathbb{R}^n$, since by the Nash embedding theorem all Riemannian manifolds are isometrically embeddable in some euclidean space. In this article we look at a different type of degeneration, for which the techniques of the “embedded” situation do not apply. Namely, we consider a compact differentiable manifold $M$ with boundary, endowed with a Riemannian metric which is smooth in the interior and degenerates at the boundary following certain precise patterns. Examples of such degenerate metrics include the so called incomplete edge metrics, for instance any Riemannian metric in the complement of a submanifold, and also the fibered boundary metrics, a class of complete metrics including the generalized Taub-NUT metrics on $\mathbb{R}^4$.

Double Forms and the Odd Pfaffian

We set the stage with our own algebraic treatment of the Gauss–Bonnet formula on compact oriented manifolds $(M^{2k}, g)$ using the formalism of double forms (see Labbi 2007 for an ample discussion and more references about double forms):

$$ (2\pi)^k \chi(M) = \int_M \text{Pf}(g), \quad \text{Pf}(g) = \frac{1}{k!} B_g \left( (R^g)^k \right). $$
Here $R^g \in \Lambda^2(M) \otimes \Lambda^2(M)$ is the curvature form of $g$, a double form of bi-degree $(2, 2)$, and $B_g$ is the Berezin integral, or contraction with the volume form of $g$ in the second component. When $M$ has a nonempty boundary $(N, h)$, essentially as a consequence of the second Bianchi identity, Allendoerfer and Weil (1943) and Chern (1944, 1945) isolated the boundary correction term:

$$(2\pi)^k \chi(M) = \int_M \text{Pf}(g) - \sum_{j=0}^{k-1} \frac{(-1)^{k+j(2k-2j-3)!}}{j!(2k-2j-1)!} \int_{\partial M} B_h \left( (R^h)^j \wedge \Pi^{k-2j-1} \right).$$

(1.1)

In this formula $\Pi \in \Lambda^1(N) \otimes \Lambda^1(N)$ is the second fundamental form of the boundary, a double form of bi-degree $(1, 1)$, $B_h$ is the Berezin integral with respect to $h$, and

$$(-1)!! := 1, \quad (2n-1)!! := 1 \cdot 3 \cdot \ldots \cdot (2n-1) \text{ for } n \geq 1.$$

This algebraic way of writing the Gauss–Bonnet integrand on the boundary is well-suited for generalizations. Motivated by (1.1), we define the odd Pfaffian form of a $2k-1$-dimensional Riemannian manifold $(N, h)$ in terms of the curvature form $R^h \in \Lambda^2 \otimes \Lambda^2$ and the metric tensor $h \in \Lambda^1 \otimes \Lambda^1$.

**Definition 1.1** For every oriented $2k-1$-dimensional Riemannian manifold $(N, h)$ define

$$\text{Pf}^{\text{odd}}(h) := \sum_{j=0}^{k-1} (-1)^{k+j(2k-2j-3)!} B_h \left( \frac{(R^h)^j \wedge h^{2k-2j-1}}{j!(2k-2j-1)!} \right) \in \Lambda^{2k-1}(N).$$

In any orthonormal frame, $\text{Pf}^{\text{odd}}$ is a polynomial with integral coefficients in the entries of the curvature form $R$. Up to a constant, this form appears already, in a different presentation, in the work of Albin (Albin 2007, Eq. (7.12)) as the boundary correction term in the Gauss–Bonnet formula for scattering metrics. It consists of a linear combination with integral coefficients of the Lipschitz–Killing curvatures (Definition 4.3). As explained in Sect. 4, the odd Pfaffian is in fact the transgression of the Pfaffian for any slice $\{r\} \times N$ on the cone $(-\epsilon, 0) \times N$ with the metric $dr^2 \oplus r^2 h$.

**Edge Singularities**

The first type of metrics analyzed here are the incomplete edge metrics. This means we have an (oriented) compact manifold with boundary $M$ together with a fibration structure of the boundary $\pi : \partial M \to B$ over a compact manifold $B$. Fix a boundary-defining function $r$ for the boundary. The (singular) metric in a collar neighborhood of $\partial M = \{r = 0\}$ has the form

$$g = dr^2 \oplus g(r), \quad g(r) = r^2 g^V \oplus \pi^* g^B \quad (1.2)$$
where \( g^B \) is a metric on \( B \), \( g^V \) is a Riemannian metric on the fibers and the splitting is induced by an Ehresmann connection. Even in this first analysis we allow \( g^V \) to vary with \( r \) but still converging to some true metric at \( r = 0 \).

We prove that a Gauss–Bonnet formula holds on such manifolds and we compute the contribution of the singular locus \( \partial M \) in terms of the geometric data, essentially the Pfaffian of the base and the odd Pfaffian of the fibers. Due to its importance in geometric applications, we review the (perturbed) conical case separately (see Theorem 4.6).

**Theorem 1.2** Let \((M^{2k}, g)\) be a manifold with edge singularities with \( g \) as in (1.2).

(a) If \( \dim(B) \) is odd,

\[
\chi(M) = \frac{1}{(2\pi)^k} \int_M \text{Pf}^g.
\]

(b) If \( \dim(B) \) is even,

\[
(2\pi)^k \chi(M) = \int_M \text{Pf}^g - \int_B \left( \text{Pf}(g^B) \int_{\partial M/B} \text{Pf}^{\text{odd}}(g^V) \right).
\]

When we allow horizontal variations of the metric, i.e., \( g^B \) varies with \( r \), we obtain certain additional terms (see Theorem 5.9).

The computation is based on two observations. First, the second fundamental form of a slice is the Lie derivative of the metric in the direction of the normal geodesic flow \( \partial_r \). Secondly, we describe explicitly the decomposition of the curvature form of a Riemannian submersion into its horizontal, mixed and vertical components with respect to the second variable when seen as a double form.

**Manifolds with Fibered Boundaries**

The same method used for edge metrics leads to a Gauss–Bonnet formula for a different type of degeneracy. Following Mazzeo and Melrose (1998), a non-compact Riemannian manifold \((M, g)\) is called with fibered boundary if it has a finite number of ends which are modeled on \((1, \infty) \times N\) with the metric \( g := dr^2 \oplus g^V \oplus r^2 \pi^* g^B \) for \( r \gg 1 \). We assume here that \( N \rightarrow B \) is a fiber bundle with a fixed Ehresmann connection with respect to which the extension of \( g^V \) to \( N \) is defined. It is not hard to see that such a metric is complete. (These metrics were studied in depth by Vaillant (2001) under the name \( \phi \)-metrics.) Let \( F \) be a generic fiber of \( \pi \), \( b := \dim B \) and \( f := 2k - 1 - b \) the dimension of \( F \).

**Theorem 1.3** Let \((M^{2k}, g)\) be a manifold with fibered boundary.

(a) If \( b \) is even,

\[
\chi(M) = \frac{1}{(2\pi)^k} \int_M \text{Pf}^g.
\]
(b) If \( b \) is odd,

\[
(2\pi)^k \chi(M) = \int_M Pf^g + (2\pi)^{f/2} \chi(F) \int_B Pf^{odd}(g^B).
\]  

(1.3)

Compared with Theorem 1.2 there are two significant differences: the odd Pfaffian appears now in the base, not in the fibers; and the sign in front of the transgression has changed.

**Prior Results on Gauss–Bonnet for Edge and for Fibered-Boundary Metrics**

The Chern–Gauss–Bonnet formulæ for incomplete edge metrics and for fibered boundary metrics stated above in terms of the odd Pfaffian are new in full generality, even in the model case. The Gauss–Bonnet problem for fibered boundary metrics was previously studied by Albin (2007) and also by Dai and Wei (2007). Theorem 1.3 can be seen as an extension of their partial results. Albin gives a formula in the case where either the fiber or the base of the boundary fibration reduce to a point, while for \( \dim(M) = 4 \), Dai and Wei give the formula when the fiber is a point, i.e., for “large conical” metrics, better known as scattering metrics by the Melrose school. Note that Dai–Wei also state a formula in the general case, claiming the vanishing of the transgression term from (1.3). This claim holds true for even-dimensional \( B \), but is incorrect when the base is odd-dimensional, as noted also in Zerouali (2017). (They apply this result in dimension four when the fiber is a circle, hence their results concerning Hitchin–Thorpe inequalities on blow-ups of the Taub-NUT space are not adversely affected by this issue.)

We should mention, besides the thesis paper of Albin cited above, previous related results obtained by Rosenberg (1985) and Grieser (2003). The main statement from Rosenberg (1985) can be seen as a particular case of Theorem 1.3. Our “conical” Gauss–Bonnet Theorem 4.6 recovers Theorem 1.4 from Grieser (2003), albeit with a slightly stronger differentiability condition on the metric.

We should also point out the paper of Atiyah and Lebrun (2013), who obtained an elegant formula for the Euler characteristic when \( M \) is 4-dimensional and the boundary fibration is a circle bundle of constant fiber length. Their Theorem 2.1 is a particular case of Theorem 1.2 where the fiberwise integral of the odd Pfaffian boils down to the (constant) length of the circle fibers.

**Perturbations of the Model Degenerate Metrics and Transgressions**

In our view, one of the rewarding outcomes of the present paper is being able to extend the statements from model metrics to the large classes of perturbations described in (1.2). We show that if the perturbations of \( g \) are of second order (in the sense made precise in Definition 7.6), then the formulæ from Theorems 1.2 and 1.3 remain valid.

When dealing with (product-type) model metrics, one can take advantage of certain symmetries in order to perform the computations, like being able to isolate various components of the curvature and second fundamental form. This does not seem to be
the case in the presence of perturbations, obstructing a direct computational approach. In compensation, algebraic properties of transgression forms are fundamental for the proofs given here and allow us to use arguments of topological flavor in places where geometric computations seem overly complicated.

We devote a first section to proving such properties. An alternative exposition of the same topic can be found in Brüning and Ma (2006), where preparatory results similar to the ones in Sect. 2 are proved in a more general context for the transgression of the Euler class of vector bundles with metric connections. Since neither of the presentations is a subset of the other, we chose to include ours which is based, arguably, on more elementary considerations.

Recall that, given an Euclidean vector bundle \( E \to B \) of rank \( 2k \) endowed with two metric connections \( \nabla_1, \nabla_2 \), there exists a canonical form \( TPf(\nabla_1, \nabla_2) \) satisfying

\[
Pf(\nabla_1) - Pf(\nabla_2) = dTPf(\nabla_1, \nabla_2) \in \Omega^{2k}(B).
\]

It is known since Chern (1945) that the boundary integrand in the standard Gauss–Bonnet Theorem can be described as such a transgression form. So at first it might seem unremarkable that the correction term in Gauss–Bonnet Theorem for first-order perturbations (see below) of the model metric is a transgression form integrated over the boundary. However, one should keep in mind that due to the degeneracy of the metric, there is a priori no well-defined connection along the singular locus, let alone two of them.

We analyze perturbations of the model degenerate metrics, both for incomplete edge metrics and for complete fibered boundary metrics. The methods to treat the two cases are similar and we only outline here the treatment of the non-complete (edge) case. One natural approach would be to follow the ideas first introduced by Melrose in the general context of the \( b \)-calculus (Melrose et al. 1993; Melrose and Wunsch 2004), and employ as background the edge tangent bundle, transferring all geometric structures onto it. Nevertheless, since the edge tangent bundle is isomorphic (albeit non-canonically) to the tangent bundle, rather than relying explicitly on this natural notion we prefer to work here with an endomorphism \( \varphi \in \text{End}(TM) \) which has, given the choice of a boundary defining function \( r \), the following expression in a collar neighborhood of \( \partial M \):

\[
\varphi(v, w) = (rv, w),
\]

i.e., \( \varphi \) acts as multiplication by \( r \) on the vertical component of the fiber bundle \( \partial M \to B \) and leaves the horizontal and the normal components unchanged. (Of course, the edge tangent bundle remains hidden behind the curtain.)

The endomorphism \( \varphi \) is an isomorphism in the interior but not at \( r = 0 \). It is easy to see that the pull-back

\[
g^\varphi(\cdot, \cdot) := g(\varphi^{-1}(\cdot), \varphi^{-1}(\cdot))
\]

of the model degenerate metric \( g \) extends to a smooth metric on \( TM \). Consequently, we consider perturbations \( \tilde{g} \) of \( g \) that preserve this property. In fact, a perturbation \( \tilde{g} \) of \( g \) is a degenerate metric that satisfies

\[ \diamond \text{Springer} \]
\[ \tilde{g}^\varphi = g^\varphi + \alpha(\cdot, \cdot) \]

for certain smooth symmetric bilinear form \( \alpha \) which vanishes at least to order 1 at \( r = 0 \). We call the perturbation to be of order \( j \geq 1 \) if \( \alpha \in O(r^j) \).

The main result that allows the investigation of Gauss–Bonnet formulas for perturbations of model metrics is the next theorem which should be compared with extension results for the Levi-Civita connection in the context of \( \phi \)-geometry (see Vaillant 2001, Prop. 1.5).

**Theorem 1.4** Let \( \nabla^g, \nabla^{\tilde{g}} \) be the Levi-Civita connections of the edge degenerate metric \( g \) and a first-order perturbation \( \tilde{g} \). Then \( \varphi \nabla^g \varphi^{-1} \) and \( \varphi \nabla^{\tilde{g}} \varphi^{-1} \) extend to smooth connections on \( TM \). If \( \tilde{g} \) is a second-order perturbation, then the restriction of these connections to \( r = 0 \) coincide:

\[ \varphi \nabla^{\tilde{g}} \varphi^{-1}\big|_{r=0} = \varphi \nabla^g \varphi^{-1}\big|_{r=0}. \] (1.4)

We use an “abstract” version of the Christoffel coefficients formula which reduces this theorem to proving the smooth extension at \( r = 0 \) of the Levi-Civita connection for the model metric \( g \). It is exactly property (1.4) that allows one to conclude that Theorem 1.2 holds for second-order perturbations.

A consequence that is easy to miss of Theorem 1.4 is that even for first-order perturbations \( \tilde{g} \) of the model metrics \( g \) one still has a Gauss–Bonnet formula of type

\[ (2\pi)^k \chi(M) = \int_M \text{Pf}^{\tilde{g}} + \int_B \gamma \] (1.5)

for some geometric term \( \gamma \) which is itself the result of integration over the fibers \( \partial M \rightarrow B \) of a geometric quantity which takes the guise of a transgression form as follows. Let

\[ \nabla^1 := \varphi \nabla^{\tilde{g}} \varphi^{-1}\big|_{r=0}, \quad \nabla^0 := \varphi \nabla^g \varphi^{-1}\big|_{r=0}, \]

be the two connections on \( TM|_{\partial M} \) whose existence is guaranteed by Theorem 1.4. The restriction \( \nabla^0 \) has a particularly simple geometric description (see Corollary 7.2). Then the following Gauss–Bonnet formula holds:

**Theorem 1.5** Let \( \tilde{g} \) be a first-order perturbation of a model edge metric \( g = dr^2 \oplus r^2 g^V \oplus \pi^* g^B \). Then

\[ (2\pi)^k \chi(M) = \int_M \text{Pf}^{\tilde{g}} - \int_B \left( \text{Pf}(g^B) \int_{\partial M/B} \text{Pf}^{\text{odd}}(g^V) \right) - \int_{\partial M} \text{TPf}(\nabla^0, \nabla^1). \]

The form \( \text{Pf}(g^B) \) is zero, by definition, when \( \dim B \) is odd.
Note that the sum of the two boundary terms is itself a transgression form.
In the particular case when the degeneration is of first order with respect to a conical metric, we are able to give a geometric expression for the boundary contribution in the spirit of the classical Gauss–Bonnet formula. Let
\[
G_{j,2k-1}^{\partial M} := \frac{1}{j!(2k-1-2j)!} B_g \left((R^N)^j \wedge (\Pi^g)^{2k-1-2j}\right).
\]
where the second fundamental form \(\Pi^g\) is defined via \(\nabla^1\) above.

**Theorem 1.6** Let \(g\) be a first-order perturbation of a conical metric \(dr^2 \oplus r^2 g^N\). Then
\[
(2\pi)^k \chi(M) = \int_M Pf^g - \sum_{j=0}^{k-1} (-1)^j (2j-1)!! \int_{\partial M} G_{k-1-j,2k-1}^{\partial M}
\]
Similar results, proved with the same techniques, hold for first and second order perturbations of manifolds with fibered boundary (see Sect. 8).

The notions of model degenerate metrics studied here, together with their perturbations, depend on the choice of a boundary-defining function \(x\). A model edge degenerate metric with respect to such a function \(x\) will look more complicated with respect to a different choice \(x'\). We refer to the work of Grieser (2011), which solves completely the conic case, and of Joshi (2000) dealing with the \(b\)-case. In this work we assume the boundary-defining function \(x\) to be fixed once and for all, leaving open the quest for an optimal choice of \(x\).

**Orbifolds**

A natural example of first-order perturbation of a model edge metric is the complement of a submanifold \(B\) in a Riemannian manifold \(M\) when one lifts the original metric to the oriented blow-up of \(B\). The integral of the transgression form from Theorem 1.5 vanishes in this case, reflecting a basic topological fact:
\[
\chi(M \setminus B) = \chi(M) - \chi(B).
\]
The situation becomes more interesting when we blend in isometric actions of finite groups. If \(M\) is a Riemannian orbifold with singularities locally modeled on quotients of type \(N/G\) where \(G\) acts freely on \(N \setminus \text{Fix}_G(N)\) and \(\text{Fix}_G(N)\) is a smooth submanifold locus, we obtain the following Gauss–Bonnet formula for orbifolds:

**Theorem 1.7** Let \(\hat{M}\) be a compact Riemannian orbifold of dimension \(2k\) with simple singularities along \(Z \subset \hat{M}\) and let \(g\) be the Riemannian metric on \(\hat{M} \setminus Z\). Then
\[
\chi(\hat{M}) = \frac{1}{(2\pi)^k} \int_{\text{Int} \hat{M}} Pf^g + \sum_{Z_i \in \text{Fix}(\hat{M})} \chi(Z_i) \frac{|G_i| - 1}{|G_i|}
\]
where \(\text{Fix}(\hat{M})\) is the set of connected components of the singular locus of \(\hat{M}\).
It turns out that Theorem 1.7 is a particular case of Satake (1957, Theorem 2) which expresses the Euler–Satake characteristic as the integral of the Pfaffian in the complement of the fixed point locus.

After a first version of this article was posted online we noticed that using the same methods we can recover also Melrose Gauss–Bonnet Theorem, by using the result of Joshi (2000) from which one obtains directly that exact $b$-metrics are second order perturbations of certain model $b$-metrics.

**Historical Notes**

Our intention is to give below a broad picture of the huge development of results directly related to Gauss–Bonnet. Voluntary or involuntary omissions are obviously inevitable.

The Gauss–Bonnet formula for polygonal surfaces embedded in Euclidean 3-space was discovered almost 200 years ago by Gauss, Binet and Bonnet. The standard textbook formula for closed surfaces in $\mathbb{R}^3$, expressing the Euler characteristic in terms of the integral of the Gaussian curvature, was stated and proved by von Dyck (1888) at the end of the nineteenth century. The modern history of its generalizations can be found in the nice survey (Wu 2008). The integrand in higher dimensions was discovered in the 1920s by Heinz Hopf for hypersurfaces in Euclidean space, by interpreting the Pfaffian of the curvature as the pullback through the Gauss map of the volume form from the sphere. The validity of Hopf’s formula for manifolds of arbitrary codimension embedded in $\mathbb{R}^N$ was proved by Allendoerfer (1940), building on work of Weyl. Allendoerfer and Weil (1943) not only proved the validity of Hopf’s formula in the abstract (non-embedded) case, but also gave the correction term for a manifold with boundary. They went even further and produced a formula valid for a topological manifold with boundary which is a Riemannian polyhedron, i.e., boundary points have neighborhoods which are differentially modeled on convex cones in $\mathbb{R}^n$ and there exists a globally defined smooth Riemannian metric on the resulting differentiable polyhedron. Their theorem is in some sense at the crossroad of what we call embedded/non-embedded situation. Soon afterwards, Chern (1944, 1945) gave intrinsic proofs of the Allendoerfer–Weil result for compact smooth Riemannian manifolds, both with and without boundary. Chern’s articles have been immensely influential in differential geometry and topology, leading to his name being frequently appended to the Gauss–Bonnet theorem.

It is perhaps worth mentioning here that the Gauss–Bonnet–Chern formula, together with Hirzbruch’s signature theorem and the Hirzbruch–Riemann–Roch formula, constituted the main motivating examples behind the celebrated Atiyah–Singer index theorem.

With regard to more modern developments, the generalization of the Allendoerfer–Weil theorem of Walter (1975) on compact locally convex subsets of Riemannian manifolds anticipates the techniques coming from Geometric Measure Theory with applications to the integral geometry of subanalytic cycles promoted by Fu (1994). Incidentally, Walter already uses double forms, a notion which he credits to Bourbaki. Ideas from stratified Morse theory have also been used successfully in the context...
of integral geometry of tamed sets (Bröcker and Kuppe 2000). Melrose et al. (1993) proved a Gauss–Bonnet Theorem for \( b \)-exact metrics as a corollary to his celebrated \( b \)-Index Theorem. More recently, an enhanced version of the Allendoerfer–Weil theorem was used by McMullen (2017) to compute the volume of the moduli space of \( n \)-pointed Riemann surfaces of genus 0. Probabilistic interpretations and proofs of Gauss–Bonnet have been given by Nicolaescu and Savale (2017). Other important works related to the topic of this paper are cited in the bibliography.

2 The Transgressions of the Pfaffian: General Facts

We include here a series of general facts, more or less well-known, about the transgression of the Pfaffian. There exist various incarnations of the transgression form (compare for example, Chern 1945; Gilkey 1995; Nicolaescu 2007; Walter 1975) and one of the purposes of this section is to bring them under the same umbrella in order to simplify the presentation in the sequel. Another purpose is to put together a collection of formulas relating transgressions for different metrics and different metric connections which will turn out to be essential for our computations in the degenerate metric setting.

2.1 Transgressions and Connections

Let \( E \rightarrow M \) be an oriented Euclidean vector bundle of rank \( 2k \) over a manifold \( M \). Every connection \( \nabla \) compatible with the metric gives rise to a closed form of degree \( 2k \) on \( M \), the Pfaffian, associated to the curvature tensor \( F(\nabla) := d\nabla \circ \nabla \), locally a skew-symmetric matrix of 2-forms. If \( F(\nabla)_{ij} := \langle F(\nabla)_{sj}, s_i \rangle \) in a local orthonormal basis \( \{s_1, \ldots, s_{2k}\} \) of \( E \) then

\[
\operatorname{Pf}(\nabla) := \frac{1}{2^{k}k!} \sum_{\sigma \in S_{2k}} \varepsilon(\sigma) F(\nabla)_{\sigma(1)\sigma(2)} \wedge \cdots \wedge F(\nabla)_{\sigma(2k-1)\sigma(2k)}. \tag{2.1}
\]

In the next section we will define the Pfaffian intrinsically via double forms, proving its gauge independence. What is special about the Pfaffian compared to other invariant polynomials is that it vanishes in the presence of a non-zero parallel section in \( E \).

Given a smooth path of metric connections \( \alpha^{\nabla} := (\nabla^t)_{t \in [0,1]} \), one can construct a transgression form \( \operatorname{TPf}(\alpha^{\nabla}) \) which satisfies

\[
d\operatorname{TPf}(\alpha^{\nabla}) = \operatorname{Pf}(\nabla^1) - \operatorname{Pf}(\nabla^0). \tag{2.2}
\]

The construction goes as follows. On the oriented Euclidean vector bundle \( \pi_2^* E \rightarrow [0, 1] \times M \) (where \( \pi_2 : [0, 1] \times M \rightarrow M \) is the projection) consider the connection \( \tilde{\nabla} := \frac{d}{dt} + \nabla^t \) which acts on a section \( (s_t)_{t \in [0,1]} \) of \( \pi_2^* E \) as follows:

\[
\tilde{\nabla}s_t = dt \otimes \frac{\partial s_t}{\partial t} + (\nabla^t s_t).
\]
Consider the Pfaffian $\text{Pf}(\tilde{\nabla})$ which is a closed form and use the homotopy formula for $H := \text{id}_{[0,1] \times M}$ and $\text{Pf}(\tilde{\nabla})$ to conclude that (2.2) is valid with

$$\text{TPf}(\alpha) := \int_{[0,1]} \text{Pf}(\tilde{\nabla}),$$


the integral being over the fibers of the projection $\pi_2$.

**Example 2.1** Suppose $(M, g)$ is a Riemannian manifold with boundary of even dimension. Then the Euclidean vector bundle $TM|_{\partial M} \to \partial M$ is endowed with two metric connections. One is the Levi-Civita connection $\nabla^1 := \nabla^M$ on $M$ and the other one is “the cylindrical connection” $\nabla^0 := d \oplus \nabla_{\partial M}$ where we use the splitting

$$TM|_{\partial M} = \mathbb{R}v \oplus T\partial M$$

induced by the unit normal $v$. Notice that $\text{Pf}(\nabla^0) = 0$ since $v$ is a parallel section and the curvature splits into a direct sum of factors, one of which is zero. We use the affine path of connections $\nabla^s := (1 - s)\nabla^1 + s\nabla^2$ to construct the Chern transgression $\text{TPf}(\nabla^M) = \text{TPf}^g$ associated to the metric $g$. This is the form which appears in the Gauss–Bonnet formula.

If the splitting (2.3) is extended to a neighborhood $U$ of $\partial M$ (e.g. via minus the gradient of the distance function to $\partial M$), the identity $dT\text{Pf}^g = \text{Pf}^g$ on $U$ is valid on $U$.

**Remark 2.2** For the reverse path $-\alpha$ defined via $-\alpha(t) := \alpha(1 - t)$ one has:

$$\text{TPf}(-\alpha) = -\text{TPf}(\alpha).$$

Indeed, one uses the orientation-reversing diffeomorphism

$$[0, 1] \times M \to [0, 1] \times M, \quad (t, m) \to (1 - t, m)$$

while fiberwise integration is sensitive to the orientation.

**Proposition 2.3** For two smooth paths $\alpha$ and $\beta$ of metric connections with $\alpha(i) = \beta(i)$, $i = 0, 1$ there exists a form $\text{TPf}(\alpha, \beta)$ of degree $2k - 2$ such that:

$$\text{TPf}(\alpha) - \text{TPf}(\beta) = d\text{TPf}(\alpha, \beta).$$

**Proof** Let $\mathcal{A}$ be the space of affine connections compatible with the metric. It is an affine space modeled on $\Gamma(M; T^*M \otimes \text{End}^-(E))$. Let $\Box := [0, 1] \times [0, 1]$. Consider the smooth family of connections

$$\tilde{\alpha} \beta : \Box \to \mathcal{A}, \quad \tilde{\alpha} \beta(s, t) = (1 - s)\alpha(t) + s\beta(t).$$
On the vector bundle $\pi_2^*E \to \square \times M$ (where $\pi_2: \square \times M \to M$ is the projection) consider the connection $\hat{\nabla} := \frac{d}{ds} + \frac{d}{dt} + \tilde{\alpha}\beta(s, t)$ which acts on a smooth section $u: \square \to \Gamma(M; E)$ of $\pi_2^*E$ via

$$\hat{\nabla} s = ds \otimes \frac{\partial u}{\partial s} + dt \otimes \frac{\partial u}{\partial t} + \left[(1 - s)\alpha^\nabla(t) + s\beta^\nabla(t)\right](u).$$

Applying Stokes formula on $\square$ to the smooth closed form $\text{Pf}(\hat{\nabla}) \in \Lambda^*(\square \times M)$ we obtain

$$-d \int_\square \text{Pf}(\hat{\nabla}) = \int_{\partial \square} \text{Pf}(\hat{\nabla})$$

where integration is really integration over the fibers of the projections $\square \times M \to M$ and $(\partial \square) \times M \to M$. Now $\partial \square$ consists of two constant paths of connections for $t = 0$ and $t = 1$, while for $s = 0$ and $s = 1$ by definition the integral on the right hand side gives the transgressions induces by $\alpha^\nabla$ and $\beta^\nabla$. Taking into account the orientations, we get (2.4) with

$$\text{TPf}(\alpha^\nabla, \beta^\nabla) := - \int_\square \text{Pf}(\hat{\nabla}).$$

\[\Box\]

**Notation** For two metric connections $\nabla^0$ and $\nabla^1$ on $E$ we denote by $\text{TPf}(\nabla^0, \nabla^1)$ the transgression form induced by the affine path $(1 - s)\nabla^0 + s\nabla^1$.

If $\nabla^0$ is obtained from $\nabla^1$ through a section $s: M \to E$ of norm 1 by using the splitting

$$E = \mathbb{R}s \oplus \langle s \rangle^\perp$$

with $\nabla^0 := d \oplus P\nabla^1 P$, $P$ being the orthogonal projection on $\langle s \rangle^\perp$ then we set $\text{TPf}(\nabla^1, s) := \text{TPf}(\nabla^0, \nabla^1)$. We will use the same notation even if $s$ is only defined along a submanifold $B$ (or boundary) of $M$ with the understanding that the splitting (2.5) holds only along $B$, $\nabla^0$ is a connection on $E|_B \to B$ and consequently $\text{TPf}$ is a form on $B$.

If $s$ is clear from the context, we use $\text{TPf}(\nabla^1)$ for $\text{TPf}(\nabla^1, s)$. If the connection $\nabla^1$ is the Levi-Civita connection of a metric $g$, then we use $\text{TPf} g$ for $\text{TPf}(\nabla^1)$, like in Example 2.1.

**Proposition 2.4** For any 4 metric connections $\nabla^i$, $0 \leq i \leq 3$, there exists a form $\gamma$ such that

$$\text{TPf}(\nabla^0, \nabla^1) + \text{TPf}(\nabla^1, \nabla^2) + \text{TPf}(\nabla^2, \nabla^3) + \text{TPf}(\nabla^3, \nabla^0) = d\gamma.$$
Proof Put $\nabla^i$ in cyclic order at the vertices of a smooth map $\theta : \Box \to A$ which on the edges of $\Box$ gives the affine path connecting $\nabla^i$ and $\nabla^{i+1}$. The proof goes on as in Proposition 2.3.

Proposition 2.5 Let $M$ be a Riemannian manifold (with or without boundary). Let $\nabla^0$ and $\nabla^1$ be two metric connections and $s : M \to E$ a smooth section of norm 1. Then there exists a $(2k-2)$-form $\gamma$ such that the following equality of pairs holds:

$$ (\text{Pf}(\nabla^1), -\text{TPf}(\nabla^1, s)) - (\text{Pf}(\nabla^0), -\text{TPf}(\nabla^0, s)) = (- d\text{TPf}(\nabla^1, \nabla^0), \text{TPf}(\nabla^1, \nabla^0) + d\gamma). $$

If $s$ is only defined along a submanifold (or boundary) $B$ of $M$ then the same relation holds with the second components restricted to $B$.

Proof The equality in the first component is clear by (2.2) and Remark 2.2.

For the second component, let $\nabla^0_c := d \oplus P\nabla^0 P$ and $\nabla^1_c := d \oplus P\nabla^1 P$, where $P$ is the projection onto $(s)^\perp$. Apply Proposition 2.4 to the connections $\nabla^0_c$, $\nabla^0$, $\nabla^1$, $\nabla^1_c$ to get:

$$ \text{TPf}(\nabla^0, s) - \text{TPf}(\nabla^1, s) + \text{TPf}(\nabla^1_c, \nabla^0_c) = -\text{TPf}(\nabla^0, \nabla^1) + d\gamma = \text{TPf}(\nabla^1, \nabla^0) + d\gamma. $$

But $\text{TPf}(\nabla^1_c, \nabla^0_c) = 0$ because $s$ is simultaneously parallel for $\nabla^0_c$ and $\nabla^1_c$ hence $\text{Pf}(\tilde{\nabla})$ vanishes on the affine segment of connections from $\nabla^0_c$ to $\nabla^1_c$.

Proposition 2.5 has a topological interpretation. Suppose that $s$ is a unit section of $E_{|\partial M}$. Each pair $(\text{Pf}(\nabla^i), -\text{TPf}(\nabla^i, s))$ is closed in $\Omega^{2k}(M, \partial M) := \Omega^{2k}(M) \oplus \Omega^{2k-1}(\partial M)$ for the differential

$$ d(\omega, \gamma) := (-d\omega, \iota^*\omega + d\gamma). $$

Proposition 2.5 says that two such pairs determine the same relative cohomology class. In the compact case, this was proved in Cibotaru (2017) by showing that such a pair is Lefschetz dual to the zero locus of a generic extension of $s$ to $M$. In the classical case, when $s$ is the unit normal of $\partial M$ this is also a consequence of Chern–Gauss–Bonnet (Chern 1944) since the map

$$ (\omega, \gamma) \to \int_M \omega + \int_{\partial M} \gamma $$

gives an isomorphism $H^{\dim M}(M, \partial M) \cong \mathbb{R}$.

Proposition 2.6 Let $(M, g)$ be a manifold, $\pi : E \to M$ a Euclidean vector bundle with metric connection $\nabla$, and $s_0, s_1 : M \to S(E)$ sections in the sphere bundle of $E$. Suppose there exists a homotopy $(s_t)_{t \in [0, 1]} : M \to S(E)$ between the two sections. Then there exists a smooth form $\eta$ such that:

$$ \text{TPf}(\nabla, s_1) - \text{TPf}(\nabla, s_0) = d\eta. $$
Proof Let \( \tau \) be the tautological section of \( \pi^* E \to S(E) \). The corresponding “tautological” transgression \( \text{TPf}(\pi^* \nabla, \tau) \in \Omega^*(S(E)) \) satisfies:

\[
(s_t)^* \text{TPf}(\pi^* \nabla, \tau) = \text{TPf}(\nabla, s_t), \quad (\forall) \ t \in [0, 1];
\]

\[
d\text{TPf}(\pi^* \nabla, \tau) = \pi^* \text{Pf}(\nabla).
\]

The homotopy formula for the homotopy \( H := (s_t)_{t \in [0,1]} : [0, 1] \times M \to S(E) \) and \( \omega = \text{TPf}(\pi^* \nabla, \tau) \) implies that

\[
\text{TPf}(\nabla, s_1) - \text{TPf}(\nabla, s_0) = d \int_{[0,1]} H^* \omega + \int_{[0,1]} dH^* \omega.
\]

But \( dH^* \omega = \pi_2^* \text{Pf}(\nabla) \) where \( \pi_2 : [0, 1] \times M \to M \) is the projection. The fiber integral over the fibers of \( \pi_2 \) of any form of type \( \pi_2^* \eta \) is zero.

Proposition 2.6 implies the following refinement of Proposition 2.5.

**Proposition 2.7** Let \( M \) be a manifold with or without boundary, let \( \nabla^0 \) and \( \nabla^1 \) be two metric connections on the Euclidean vector bundle \( E \) and \( (s_t)_{t \in [0,1]} : M \to S(E) \) a smooth homotopy. Then there exists a \((2k - 2)\)-form \( \gamma \) such that:

\[
(\text{Pf}(\nabla^1), -\text{TPf}(\nabla^1, s_1)) - (\text{Pf}(\nabla^0), -\text{TPf}(\nabla^0, s_0)) = (-d\text{TPf}(\nabla^1, \nabla^0), \text{TPf}(\nabla^1, \nabla^0) + d\gamma).
\]

If the homotopy is defined only along a submanifold (or boundary) \( B \) then the second components are defined only over \( B \).

**2.2 Transgressions and Metrics**

On an Euclidean vector bundle \( V \) of rank \( 2k \), it is convenient to identify the space of skew-symmetric endomorphisms \( \text{End}^- (V) \) with \( \Lambda^2 V^* \) by the rule:

\[
\text{End}^- (V) \ni A \mapsto a_A (v, w) := \langle v, Aw \rangle = -\langle Av, w \rangle.
\]

Notice that on \( \mathbb{R}^2, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \) goes to \( e_1^* \wedge e_2^* \). The Pfaffian of \( A \) is defined by

\[
\text{Pf}(A) = \frac{1}{k!} \langle a_A^\wedge k, \text{vol}_{V^*} \rangle \in \mathbb{R}.
\]

In any orthonormal basis of \( V \), \( \text{Pf} \) is a polynomial with integral coefficients in the entries of \( A \).

Clearly this definition can be extended to endomorphisms \( A \in \mathcal{A} \otimes \text{End}^- (V) \) with values in any algebra \( \mathcal{A} \), with the inner product acting only on the \( \Lambda^* V \) component. Then \( \text{Pf}(A) \in \mathcal{A} \). In this note, \( \mathcal{A} \) will be the algebra of differential forms on a manifold.

If \( \nabla \) is a metric connection on a Euclidean vector bundle \( E \) of rank \( 2k \), from the curvature tensor \( F(\nabla) \in \Gamma(\Lambda^2 T^* M \otimes \text{End}^- (E)) \) we get a form of degree 2 with
values in $\Lambda^2 E^*$ called the curvature form and denoted here by the same symbol. Explicitly:

$$F(\nabla) : \Lambda^2 T M \otimes \Lambda^2 E \to \mathbb{R}, \quad F(\nabla)(X, Y; Z, W) = \left( Z, \left[ [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \right] W \right).$$

Then $F(\nabla)^k \in \Lambda^{2k} T^* M \otimes \Lambda^{2k} E^*$, and $\text{Pf}(F(\nabla)) \in \Omega^{2k}(M)$. This definition agrees with (2.1). The operation of contraction with the volume element in the second component is sometimes called Berezin integral. Double forms, i.e., sections of $\Lambda^* T^* M \otimes \Lambda^* E^*$, form an algebra.

From now on we take $E = TM$. Let $g_0, g_1$ be two Riemannian metrics on $M$, and $\nabla^{g_0}, \nabla^{g_1}$ the corresponding Levi-Civita connections. We want to find an explicit primitive of the difference $\text{Pf}(R^{g_1}) - \text{Pf}(R^{g_0})$. Set $g_s = (1-s)g + sg_1$, a 1-parameter family of Riemannian metrics on $M$, and define a Riemannian metric on $X := [0, 1] \times M$ as a generalized cylinder (Bär et al. 2005):

$$G = ds^2 + g_s.$$

It is easy to see that for every $x \in M$, the intervals $[0, 1] \times \{x\}$ are geodesics in $X$. Therefore, parallel transport on $X$ along these intervals preserves the orthogonal complement to $\partial_s$, i.e., $TM$. We get for each $s$ a vector bundle isometry

$$\tau_s : (TM, g_0) \to (TM, g_s).$$

We identify in this way for all $s$ the Euclidean vector bundles with metric connections $(TM, g_s, \nabla^{g_s})$ with $(TM, g_0, \nabla^s)$, where $\nabla^s = \tau_s^{-1} \nabla^{g_s} \tau_s$. Clearly such an identification preserves the Pfaffian of the curvature:

$$\text{Pf}(R^{g_s}) = \text{Pf}(R^s),$$

where $R^s = F(\nabla^s)$ is the curvature of $\nabla^s$. Write

$$\text{Pf}(R^{g_1}) = \text{Pf}(R^{g_0}) + \int_0^1 \frac{d}{ds} \text{Pf}(R^{g_s}) ds = \text{Pf}(R^{g_0}) + \int_0^1 \frac{d}{ds} \text{Pf}(R^s) ds.$$

The advantage of the second expression over the first is that now we work in a fixed Euclidean vector bundle $(TM, g_0)$ endowed with a family in $s$ of metric connections $\nabla^s$, and the coefficients of the Pfaffian polynomial depend on the metric but not on the connection. We compute

$$\partial_s \text{Pf}(R^s) \otimes \text{vol}_{g_0} = \frac{1}{k!} \partial_s \left( (R^s)^k \right) = \frac{1}{(k-1)!} \mathring{R}^s \wedge \left( (R^s)^{k-1} \right).$$

It is well-known that $\mathring{R}^s$ is $d^{\nabla^s}$-exact: indeed, let $u, v$ be vector fields on $X$ tangent to $M$ and parallel in the $\partial_s$ direction. For every vector field $Y$ on $M$ constant in $s$ (i.e.,
\[ [\partial_s, Y] = 0, \text{ write} \]
\[ \langle \nabla^s_Y u, v \rangle = \langle \nabla^0_Y u, v \rangle + \langle \theta^s(Y)u, v \rangle. \]

Then \( \hat{\nabla}^s = \hat{\theta}^s \) and so \( \hat{R}^s = d\nabla^s \hat{\theta}^s \). From the second Bianchi identity, \( d\nabla^s R^s = 0 \), so
\[ \dot{R}^s \wedge (R^s)^{k-1} = d\nabla^s (\hat{\theta}^s \wedge (R^s)^{k-1}). \]

For every double form \( \mu \in \Lambda^* M \otimes \Lambda^2 M \), write \( \mu = B_{g_0} \mu \otimes \text{vol}_{g_0} \), where \( B_{g_0} \) is the Berezin integral with respect to \( g_0 \). Since \( \text{vol}_{g_0} \) is parallel, we have \( d\nabla^s \mu = d(B_{g_0} \mu) \otimes \text{vol}_{g_0} \). Hence
\[ \frac{\partial}{\partial s} \text{Pf}(R^s) = \frac{1}{(k-1)!} d \left( B_{g_0} \left( \hat{\theta}^s \wedge (R^s)^{k-1} \right) \right). \]

It follows that
\[ \text{Pf}(R^{g_1}) = \text{Pf}(R^{g_0}) + \frac{1}{(k-1)!} d \left( \int_0^1 B_{g_0} \left( \hat{\theta}^s \wedge (R^s)^{k-1} \right) \right). \]

**Proposition 2.8** Let \( \alpha^\nabla(s) := \nabla^s \) be the above family of \( g_0 \)-compatible connections. Then
\[ \frac{1}{(k-1)!} \int_0^1 B_{g_0} \left( \hat{\theta}^s \wedge (R^s)^{k-1} \right) = \text{TPf}(\alpha^\nabla). \]

**Proof** Let \( \tilde{\nabla} := \frac{d}{ds} + \nabla^s \) be the connection on \( \pi_2^* TM \) used in the previous subsection. By definition \( \text{TPf}(\alpha^\nabla) = \int_{[0,1]} \text{Pf}(\tilde{\nabla}) \), where the integration is over the fibers of \( \pi_2 : X \to M \).

Every form \( \gamma \) on \( X \) is a sum of type \( ds \wedge \omega_s + \eta_s \) where \( (\omega_s)_{s \in [0,1]} \) and \( (\eta_s)_{s \in [0,1]} \) are smooth families of smooth forms on \( M \). Fiber integration kills the component \( \eta_s \) which does not contain the volume form of the fiber. In other words:
\[ \int_{[0,1]} \gamma = \int_{[0,1]} ds \wedge \omega_s = \int_0^1 \omega_s ds = \int_0^1 t_{\partial_s} \gamma ds. \]

The integrals \( \int_0^1 (\cdot) ds \) is to be understood as integrals of functions (of \( s \)) with values in \( \Lambda^* T_p M \) for \( p \in M \). We need to compute \( t_{\partial_s} (\text{Pf}(\tilde{\nabla})) \). First notice that \( F(\tilde{\nabla}) = ds \wedge \tilde{\nabla}^s + F(\nabla^s) \). Then
\[ F(\tilde{\nabla})^k = k ds \wedge \tilde{\nabla}^s \wedge F(\nabla^s)^{k-1} + F(\nabla^s)^k. \]

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The contraction operation $\iota_{(\cdot)}$ can be defined equally well on forms with values in an algebra. Then
\[
\iota_{\partial_s} (\text{Pf} (\tilde{\nabla})) = \frac{1}{k!} \iota_{\partial_s} (B_{g_0} (F (\tilde{\nabla})^k)) = \frac{1}{k!} B_{g_0} (\iota_{\partial_s} [F (\tilde{\nabla})^k])
\]
where $\iota_{\partial_s}$ acts by definition only on the first component of a double form$^1$. The second equality holds because $B_{g_0}$ acts on the second component only of the double form. By (2.7),
\[
\iota_{\partial_s} (\text{Pf} (\tilde{\nabla})) = \frac{1}{(k-1)!} B_{g_0} (\tilde{\theta}^s \wedge F (\nabla^s)^{k-1}).
\]

$\square$

3 Gauss–Bonnet on Manifolds with Boundary

This section contains a proof of the well-known version of Gauss–Bonnet on manifolds with boundary proved by Allendoerfer–Weil and Chern 80 years ago. While both versions of Gauss–Bonnet have received many proofs, the main idea we use in this section seems natural and delivers a direct proof. It will appear again in the degenerate metric case.

Briefly, the generalization of Gauss–Bonnet to manifolds with boundary where the metric is of product type near the boundary is a triviality and of course there is no contribution from the boundary. In order to find the “defect” in the non-product metric case we use parallel transport to produce tangent bundle isometries between a product metric and the one we are interested in (origins of this idea can be traced to Bourguignon and Gauduchon 1992; Gilkey 1995). Then we use properties of transgressions. The boundary integrand we obtain is not obviously equal to the standard one obtained by Chern (1945) and we clarify at the end of the section why this is the case. We use the formalism of double forms which simplifies the presentation to a certain extent.

Let $g$ be a smooth metric on a compact manifold $M^{2k}$ with boundary $\partial M$. Let $R^h \in \Lambda^2 \partial M \otimes \Lambda^2 \partial M$ be the curvature form of the boundary with respect to the induced metric $h$ and $\Pi$ the second fundamental form of $\partial M \hookrightarrow M$. Our convention here is the following:
\[
\Pi(X, Y) = -\langle \nabla_X \nu, Y \rangle
\]
where $\nu$ is the exterior unit normal. We will use the symbol $\Pi$ also for the $(1, 1)$ double form on $\partial M$ determined by $\Pi$. We denote by $\text{Pf}^g$ the Pfaffian of $g$ and by $\text{TPf}^g$ the transgression form on $\partial M$ constructed from $\nabla^g$ and $d \oplus \nabla^h$ (see Example 2.1) where $\nabla^g$ and $\nabla^h$ are the Levi-Civita connections on $M$ and $\partial M$ respectively. We give a direct proof of the Allendoerfer–Weil–Gauss–Bonnet–Chern (Chern 1945) formula for manifolds with boundary using the formalism of double forms.

$^1$ The curvature form is in general a section of $\Lambda^2 T^* M \otimes \Lambda^2 E^*$. 

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Proof of the Gauss–Bonnet–Chern formula (1.1) Let \( g_1 := g \). Using the unit geodesic flow normal to the boundary, we can write \((M, g)\) as a generalized cylinder (Bär et al. 2005) near the boundary:

\[
g = dt^2 + h(t),
\]

where \( h(t) \) is a smooth family of symmetric 2-tensors on \( \partial M \), and \( h(0) \) is a metric. Take \( g_0 \) to be any metric which in the same product decomposition near the boundary looks like

\[
g_0 = dt^2 + h_0(t),
\]
i.e., \( g_0 \) is of product type near the boundary and induces the same metric \( h_0 \) on \( \partial M \) as \( g \). By the Gauss–Bonnet formula for product-type metrics (obtained by doubling the manifold for example) and the transgression formula (2.6), we get

\[
(2\pi)^k \chi(M) = \int_M Pf(R_{g_0}) = \int_M Pf(R_g) - \frac{1}{(k - 1)!} \int_0^1 \int_{\partial M} B_{g_0} \left( \dot{\theta}^s \wedge (R^s)^{k-1} \right) \tag{3.1}
\]

Notice that all metrics \( g_s \) coincide on \( T M \big|_{\partial M} \). One consequence is that all bundle isometries \( \tau_s \) when restricted to \( T M \big|_{\partial M} \) are equal to the identity. Hence every Levi-Civita connection \( \nabla_{g_s} \) when restricted to \( T M \big|_{\partial M} \) is equal to \( \nabla^s \) and all are metric compatible whether we refer to \( g_0 \) or \( g \). By Proposition 2.8 the integral on the boundary in (3.1) is in fact equal to \( \iota^* TPf(\alpha \nabla) \) where \( \iota^* : \partial M \to M \) is the inclusion and \( \alpha \nabla(s) = \nabla^s \). By Proposition 2.3 when integrating over the boundary, it does not matter what path of connections one takes between the first and the last connection so we might as well take the segment. To complete the proof of (1.1) we still have to identify explicitly the transgression term from (3.1).

First, the Berezin integrals with respect to \( g \) and to \( h \) at the boundary are related

\[
B_g(dt \wedge \mu) = B_h(\mu)
\]

for every form \( \mu \in \Lambda^{2k-1} \partial M \).

The difference \( \nabla^{g_s} - \nabla^{g_0} \) is a \( \text{End}^{-}(TM) \)-valued 1-form. Define \( \theta^s \in \Lambda^1(\partial M) \otimes \text{End}^{-}(TM \big|_{\partial M}) \) as the pull-back of this 1-form to the boundary. We claim that \( \theta^s \), viewed as a \((1, 2)\) double form, equals

\[
\theta^s = (1 \otimes dt) \wedge s \Pi^s. \tag{3.2}
\]

Indeed, notice that \( \langle \nabla^s_X Y, Z \rangle = \langle \nabla^X_{\dot{t}} Y, Z \rangle \) for all \( X, Y, Z \in T \partial M \) as \( g_s \equiv h \) on \( T \partial M \). Moreover \( \langle \nabla^s_X \partial_t, \partial_t \rangle = 0 \) for all \( s \) and \( X \in T \partial M \). Hence with respect to the decomposition \( T M \big|_{\partial M} = \mathbb{R} \partial_t \oplus T \partial M \) and the corresponding decomposition of \( \text{End}^{-}(TM \big|_{\partial M}) \), the only non-zero components of \( \theta^s \) are off-diagonal. Then for
\[ X, Y \in T \partial M \]

\[
(\theta^s_X(Y), \partial_t) = \Pi^s_X(X, Y) = -\langle \nabla^s_X \partial_t, Y \rangle = -\frac{1}{2} \langle \mathcal{L}_{\partial_t} g_X \rangle(X, Y)
\]

\[
= -\frac{s}{2} h'(0)(X, Y) = -\frac{s}{2} \mathcal{L}_{\partial_t} g(X, Y) = s \Pi_X^s(X, Y).
\]

(3.3)

where we used Lemma 3.1 in the first line. Notice that (3.3) is a rewriting of (3.2).

Since \( \nabla^s = \nabla^0 + \theta^s \) we get that \( R^s = R^0 + d^{\nabla^0} \theta^s + \theta^s \circ \theta^s \) where we use the symbol \( \circ \) instead of the more popular \( \wedge \) in order to distinguish it from the product for double forms.

On one hand, \( R^0 = 0 \oplus R^h \) with respect to \( TM \rvert_{\partial M} = \mathbb{R} \partial_t \oplus T \partial M \). Hence as \((2, 2)\) forms on \( \partial M \) one has \( R^0 = R^h \). Second, \( d^{\nabla^0} \theta^s \) also respects this decomposition so \( d^{\nabla^0} \theta^s \) will be a 2-form with non-zero values only on the anti-diagonal blocks of \( \text{End}^{-1} (TM \rvert_{\partial M}) \). It follows that, when writing \( d^{\nabla^0} \theta^s \) as a double form, the second component will always contain a \( dt \). But \( \theta^s \circ \theta^s \) also contains a \( dt \) in its second component. So in \( \theta^s \wedge (R^s)^{k-1} \) this product vanishes.

We are left with turning \( \theta^s \circ \theta^s \) into a double form. If \( \{ \partial_t, e_2, \ldots, e_n \} \) is an oriented orthonormal basis for \( TM \) at a point \( p \in \partial M \) then at \( p, \theta^s \) is a skew-symmetric matrix with non-zero terms only along the first line and the first column. In fact \( \theta^s_{i} = s \Pi^s(\cdot, e_i), i \geq 2 \) and

\[
(\theta^s \circ \theta^s)_{i j} = -s^2 \Pi^s(\cdot, e_i) \wedge \Pi^s(\cdot, e_j), \quad i < j.
\]

This represents the \((2, 2)\) double form

\[
-s^2 \sum_{2 \leq i < j} \Pi^s(\cdot, e_i) \wedge \Pi^s(\cdot, e_j) \otimes e_i^* \wedge e_j^*.
\]

On the other hand

\[
\Pi^s \wedge \Pi^s = \left( \sum_{i \geq 2} \Pi^s(\cdot, e_i) \otimes e_i^* \right) \wedge \left( \sum_{i \geq 2} \Pi^s(\cdot, e_i) \otimes e_i^* \right)
\]

\[
= 2 \sum_{2 \leq i < j} \Pi^s(\cdot, e_i) \wedge \Pi^s(\cdot, e_j) \otimes e_i^* \wedge e_j^*.
\]

Hence \( \theta^s \circ \theta^s = -\frac{s^2}{2} \Pi^s \wedge \Pi^s \), and so the integrand over \( \partial M \) in (3.1) is

\[
\frac{1}{(k - 1)!} \int_0^1 B_{g^s} \left( (1 \otimes dt) \wedge \Pi^s \wedge \left( R^h - \frac{s^2}{2} (\Pi^s)^2 \right)^{k-1} \right) ds
\]

\[
= \frac{1}{(k - 1)!} B_h \left( \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{(-1)^j}{2^j} \frac{1}{2j+1} (\Pi^s)^{2j+1} \wedge (R^h)^{k-1-j} \right).
\]

\[ \square \]
The next simple Lemma is quite well-known, and will be widely used in this article.

**Lemma 3.1** Let $TM$ be endowed with a metric $G$ and corresponding Levi-Civita connection $\nabla$. Let $X \in \Gamma(TM)$ be a vector field such that $X^\sharp$ is a closed 1-form (e.g. if $X$ is gradient). Then

$$G(\nabla_Y X, Z) = \frac{1}{2}(L_X G)(Y, Z).$$

**Proof** Directly from the Koszul formula one has

$$2G(\nabla_Y X, Z) = (L_X G)(Y, Z) + dX^\sharp(Y, Z).$$

By hypothesis the second term vanishes.

**Remark 3.2** Not only that the integral over $\partial M$ of $TPfg$ equals the integral on $\partial M$ of the right hand side of (3.1) but the integrands themselves coincide. This is because the Levi-Civita connection for $g_s = (1 - s)g_0 + sg$, when restricted to $TM|_{\partial M}$ coincides with $(1 - s)\nabla^{g_0} + s \nabla^g$. This follows from $g_s \equiv g_0$ on $TM|_{\partial M}$ for all $s$ and from the Koszul formula which always gives:

$$\langle \nabla_X^{g_s} Y, Z \rangle_{g_s} = (1 - s)\langle \nabla_X^{g_0} Y, Z \rangle_{g_0} + s \langle \nabla_X^{g_1} Y, Z \rangle_g.$$

**Remark 3.3** Let

$$B_h \left( (R^h)^j \wedge II^{2k-1-2j} \right) \frac{j!(2k - 1 - 2j)!}{j!(2k - 1 - 2j)!} =: G^h_{j,2k-1}.$$

Then the integral of the transgression form has the following aesthetically pleasing form

$$\sum_{j=0}^{k-1} (-1)^j (2j - 1)!! \int_{\partial M} G^h_{j-1-j,2k-1}.$$

**Example 3.4** The Gauss–Bonnet formula 1.1 applied to the unit disk $D^{2n} \subset \mathbb{R}^{2n}$ anticipates that

$$\frac{1}{(2\pi)^n} \int_{S^{2n-1}} TPfg = -1.$$
Using that \( \text{vol}(S^{2k-1}) = \frac{2\pi^k}{(k-1)!} \) we get
\[
\sum_{j=0}^{k-1} c(j, k) \int_{S^{2n-1}} B_h( ((R^h)^j \wedge \Pi^{2k-2j-1}) )
\]
\[
= - \sum_{j=0}^{k-1} \frac{(-1)^{k-1-j}}{2^{k-1-j}} \frac{1}{j!} \frac{1}{(k-1-j)!} \frac{1}{2k-2j-1} \frac{(2k-1)!}{2j} \text{vol}(S^{2k-1})
\]
\[
= - \frac{(2k-1)!}{2^{k-1}} \frac{2\pi^k}{[(k-1)!]^2} \sum_{j=0}^{k-1} (-1)^{j} \binom{k-1}{j} \frac{1}{2j+1}.
\]

Notice that
\[
\sum_{j=0}^{k-1} (-1)^{j} \binom{k-1}{j} \frac{1}{2j+1} = \int_0^1 (1-x^2)^{k-1} \, dx = \int_0^{\pi/2} (\cos \theta)^{2k-1} \, d\theta = \frac{2^{2k-2} [(k-1)!]^2}{(2k-1)!}.
\]

Hence
\[
\frac{1}{(2\pi)^k} \sum_{j=0}^{k-1} c(j, k) \int_{S^{2n-1}} B_h( ((R^h)^j \wedge \Pi^{2k-2j-1}) ) = -1.
\]

Remark 3.5 The integrand in (1.1) on \( \partial M \) coincides with Chern’s integrand (Chern 1944). Chern’s transgression, which lives on the spherical bundle \( SM \), can be written (see for example Walter 1975) as
\[
\Phi := - \sum_{j=0}^{k-1} a_i A_j, \quad a_i = [(2\pi)^k i! (2k-2i-1)!]^{-1}, \quad A_i = (\pi^* R)^i \wedge I \wedge (DI)^{2k-2i-1}.
\]

In (3.4), \( R \) is the curvature form on \( M \), \( I : SM \rightarrow \pi^* TM \) is the tautological section seen as a 0-form on \( SM \) with values in \( \pi^* TM \) and \( DI = (\pi^* \nabla) I \) is the covariant derivative seen as a 1-form with values in \( \pi^* TM \). Hence one works in the algebra of forms on \( SM \) with values in \( \Lambda^* \pi^* TM \). Wedging with \( I \) kills the normal component in any product \( DI^{2k-2h-1} \) and also in \( (\pi^* R)^i \).

Given a hypersurface \( N \) oriented by the normal \( \nu \) one has that \( \nu^*(I \wedge DI) = \nu \wedge \nu^*(DI) \) actually equals \( -\nu \wedge \Pi_N \) where \( \Pi_N : TN \rightarrow TN \) is the second fundamental form seen as the endomorphism \( -\nabla \nu \). Moreover \( \nu^* R \) is the tangential component of the curvature tensor of \( M \) restricted to \( N \). Let \( \Pi := \Pi_N \) and \( R^N \) the curvature form on \( N \). Gauss Equation gives
\[
\nu^* R = R^N - \frac{1}{2} \Pi \wedge \Pi.
\]

\footnote{The negative sign in front of the sum is there so that \( d\Pi = \pi^* Pf^g \).}
Therefore
\[-\nu^*(A_i) = (R^N - 1/2 \Pi \wedge \Pi)^i \wedge \nu \wedge \Pi^{2k-2i-1}\]
and we must check that
\[
\nu^*\Pi = \sum_{i=0}^{k-1} \sum_{j=0}^{i} \frac{1}{j! \cdot 1 \cdot 3 \cdots (2k - 2i - 1)} \frac{(-1)^j(i)}{2j} \Pi^{2k-2(i-j)-1}(R^N)^{i-j}
\]
\[
= \sum_{j=0}^{k-1} \frac{(k-1)}{(k-1)!} \frac{1}{2j + 1} \frac{(-1)^j}{2j} \Pi^{2j+1}(R^N)^{k-1-j} =: TPf^N.
\]
This equality follows from the elementary identity of double factorials
\[
\sum_{j=0}^{p} (-1)^j \frac{(2p)!!}{(2j)!!(2p - 2j + 1)!!} = \frac{(-1)^p}{2p + 1}.
\]

4 Conical Manifolds

Let $N$ be a compact oriented manifold, possibly disconnected. A conical singularity modeled on $N$ is a Riemannian metric on $(-\epsilon, 0) \times N$ of the form
\[g_c = dr^2 \oplus f^2(r) \cdot h(r)\]
where $h(r)$ is a smooth family of Riemannian metrics on $N$ down to $r = 0$, and $f : (-\epsilon, 0] \to [0, \infty)$ is a function with the following properties
(i) $f$ is smooth on $(-\epsilon, 0)$;
(ii) $f$ vanishes only at 0;
(iii) $f$ is $C^1$ at 0.

Notice that, as a consequence of the hypotheses, $f'(0) \leq 0$.

Definition 4.1 When $h(r) \equiv h$ is constant and $f(r) = -\theta r$ with $\theta > 0$ we call the conical singularity a geometric cone of inclination $\theta$.

The smoothness at $r = 0$ of $h(r)$ needs to be emphasized. There are two equivalent formulations for this property:
(1) The metric $dr^2 \oplus h(r)$ is the restriction to $(-\epsilon, 0) \times N$ of a smooth metric on $(-\epsilon, \epsilon) \times N$;
(2) The family $(-\epsilon, 0) \ni r \mapsto h(r) \in C^\infty(N, T^*N \otimes T^*N)$ has a limit at $r = 0$ together with all its derivatives in $r$. 

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Definition 4.2 An oriented manifold with conical-type singularities is a Riemannian manifold \((M, g)\) such that there exists a compact set \(K\) and an orientation preserving diffeomorphism \(\varphi : M \setminus K \cong (-\epsilon, 0) \times N\) such that on \(M \setminus K\):

\[
g = \varphi^* g_c.
\]

We now define some polynomials in the curvature of a Riemannian manifold \((N, h)\) of dimension \(n\) using the Berezin integral \(B_h\) where \(h = h(0)\).

Definition 4.3 The Lipschitz–Killing curvature (see Labbi 2007 or Morvan 2008) of level \(j\) is, up to a normalization constant, the following form of degree \(n\) on \(N\):

\[
P_{j,n}(h) = \frac{1}{j!(n-2j)!} B_h \left( (R^h)^j \land h^{n-2j} \right).
\]

Like the Pfaffian, in any orthonormal base the form \(P_{j,n}\) is a polynomial with integral coefficients in the components of \(R^h\). The Lipschitz–Killing curvatures are familiar objects and they appear in Weyl’s tube formula.

Example 4.4 Here are a few examples of Lipschitz–Killing curvatures:

\[
P_{0,n}(h) = \text{vol}_h, \quad P_{1,n}(h) = \frac{1}{2} \text{scal}_h \cdot \text{vol}_h, \quad P_{k,2k} = \text{Pf}(R^h).
\]

Remark 4.5 Let \(\tilde{N} := (-\epsilon, 0) \times N\) be a geometric cone of inclination \(c > 0\). Then the transgression form for each slice \(\{r\} \times N\) does not depend on \(r\). Indeed the Levi-Civita connection and the cylindrical connection obtained from it are the same for \(T \tilde{N} \mid_{\{r\} \times N}\) irrespective of \(r\). Denote this transgression form by \(\text{TPf}(N, h, c)\). For the inclination \(c = 0\), set \(\text{TPf}(N, h, c) = 0\).

We prove now the main result of this section.

Theorem 4.6 Let \((M^{2k}, g)\) be an oriented manifold with conical-type singularities modeled on a possibly disconnected manifold \(N\) with induced metric \(h\). Then

\[
(2\pi)^k \chi(M) = \int_M \text{Pf}^g - \int_N \text{TPf}(N, h, -f'(0)) \tag{4.2}
\]

\[
= \int_M \text{Pf}^g + \sum_{j=0}^{k-1} [f'(0)]^{2k-2j-1} \tilde{c}(k - 1 - j) \int_N P_{j,2k-1}(h) \tag{4.3}
\]

with

\[
\tilde{c}(l) = (-1)^l \cdot (2l - 1)!!.
\]
Proof For each \( r \in (-\epsilon, 0) \), let \( M_r \) be the complement of \( \varphi^{-1}((r, 0) \times N) \). It is a compact manifold with boundary and therefore (1.1) applies to it:

\[
(2\pi)^k \chi(M_r) = \int_{M_r} \text{Pf}^g - \int_{\partial M_r} T\text{Pf}^g.
\]

Clearly all \( M_r \) are homotopic to each other so the left hand side does not change with \( r \). We will show that

\[
\lim_{r \to 0} \int_{\partial M_r} T\text{Pf}^g = - \left( \sum_{j=0}^{k-1} [f'(0)]^{2k-1-2j} c(k-1-j) \int_N P_{j,2k-1}(h) \right).
\]

This will also prove the convergence of \( \int_{M_r} \text{Pf}^g \) when \( r \to 0 \). (In Sect. 7 we prove the stronger statement that \( \text{Pf}^g \) is a smooth form on \( M \), including at the boundary.)

The first observation is that the Levi-Civita connection \( \partial M_r \) with the metric \( h_1(r) := f(r)^2 h(r) \) is the same as the Levi-Civita connection for the metric \( h(r) \), hence as operators

\[
R^{h_1(r)} = f^2(r) R^{h(r)}
\]

due to the metric dependence of the identification \( \text{End}^{-}(V) \simeq \Lambda^2 V^* \).

One is left computing the evolution of \( \text{II}' \) for \( \partial M_r \). Since \( \partial_r \) is a gradient vector field we apply Lemma 3.1 again:

\[
\text{II}'(X, Y) = -\langle \nabla^g_X \partial_r, Y \rangle = - \frac{1}{2} L_{\partial_r} (dr^2 + f^2(r) h(r))
= - \left[ f'(r) f(r) h(r) + \frac{f^2(r)}{2} h'(r) \right].
\]

We also have \( B_{h_1(r)} = (f(r))^{1-2k} B_{h(r)} \) and so

\[
B_{h_1(r)} \left( (R^{h_1(r)})^j \wedge (\text{II}')^{2k-1-2j} \right) = - f'(r)^{2k-1-2j} B_{h(r)} \left( (R^{h(r)})^j \wedge h(r)^{2k-1-2j} \right) + o(f(r)).
\]

Multiply this with \( c(j, k) = \frac{c(k-1-j)}{j!(2k-1-2j)!} \), take the sum in \( j \) and the limit \( r \to 0 \) to get (4.3).

To see that (4.2) is true, recall (for example Remark 3.2) that \( \int_N T\text{Pf}(N, h, c) \) can be computed also as a sum of integrals over \( N \) of products \( \text{II}^{2k-1-2j} \wedge R^j \) where \( \text{II} \) and \( R \) are the second fundamental form respectively the curvature form of a slice of a geometric cone. But for such a geometric cone, \( \text{II} \) is a multiple of the metric, and the computations go as before. \( \square \)
We notice thus that for an odd-dimensional manifold the total Lipschitz–Killing curvatures can be recovered as coefficients of the integral of a certain transgression. We state this separately.

**Corollary 4.7** For a geometric cone modeled on \((N, h)\) of inclination \(\theta\) with \(\dim N = n, n \text{ odd}\), the following holds:

\[
\int_N \text{TPf}(N, h, \theta) = \sum_{j=0}^{(n-1)/2} \theta^{n-2j} \tilde{c} \left( \frac{n-1}{2} - j \right) \int_N P_{j,n}(h).
\]

**Proof** The function is in this case \(f(r) = -\theta r\). \(\square\)

**Remark 4.8** The odd Pfaffian from the Introduction can in fact be seen as a transgression:

\[
Pf^{\text{odd}}(h) = \text{TPf}(N, h, 1).
\]

Notice that in the case when \(N = S^{2n-1}\) with the round metric we get:

\[
\int_{S^{2n-1}} \text{TPf}(S^{2n-1}, \text{round}, 1) = (2\pi)^n.
\]

One can compare this with Example 3.4. The difference in sign has to do with the fact that \(S^{2n-1}\) seen as a geometric cone is oriented with the inner normal first since that is the direction of \(\partial r\) that points towards the “singularity”.

**Remark 4.9** We can construct a manifold with boundary \(\tilde{M} := M \cup (-\epsilon, 0] \times N / \sim\) where the identification is made via the diffeomorphism \(\varphi\) of Definition 4.2 in an obvious way. The degenerated conical metric \(g\) induces a pseudo-distance on \(\tilde{M}\) in which the (pseudo) distance between any two points on \(\partial \tilde{M}\) is zero. Collapsing the boundary of \(\tilde{M}\) to a point gives a metric space \(\hat{M}\) which is homeomorphic to the one point compactification of \(M\). Then

\[
\chi(\hat{M}) = 1 + \chi(M).
\]

If the singular space \(\hat{M}\) is the focus of the analysis, then we can say that the singularity, or the point at \(\infty\) contributes to the Euler characteristic with the quantity

\[
1 + \frac{1}{(2\pi)^k} \sum_{j=0}^{k-1} f'(0)^{2k-1-2j} \tilde{c}(k-1-j) \int_N P_{j,2k-1}(h).
\]

**Example 4.10** In the case \(k = 1\) the contribution of the singularity is (recall that \(f'(0) \leq 0, \tilde{c}(0, 1) = 1\))

\[
1 + \frac{f'(0)}{2\pi} \text{length}_h(N).
\]
This fits with two opposite examples. The first is a closed surface $S$ embedded in $\mathbb{R}^3$ with a cuspidal singularity. Then $f'(0) = 0$. The geometric contribution to the Euler characteristic of the cusp is 1 which is the area of the half unit sphere divided by $2\pi$. The half unit sphere is the normal cycle of the cusp, or the solid angle described by the variation of a unit normal to each surface of a family of smooth surfaces contained in the bounded region of $S$ and converging to $S$.

The other example is when $N = S^1$ with the round metric and $f'(0) = -1$. Then $\hat{M}$ is a closed surface with smooth metric (see Petersen 2006, p. 13, Prop.1) and the contribution of the removable singularity vanishes, recovering Gauss–Bonnet for $\hat{M}$ in this case.

## 5 Edge Manifolds: The Model Metrics

Let $N$ be an $n$-dimensional closed, oriented manifold. Assume $\pi: N \to B$ is a locally trivial fiber bundle with vertical bundle $VN$ and suppose $\pi$ is endowed with an Ehresmann connection $E \in \text{Hom}(TN, VN)$ that induces a decomposition

$$TN = VN \oplus \pi^*TB.$$ 

An (incomplete) edge singularity modeled on $(N, \pi, E)$ is a metric on $(-\epsilon, 0) \times N$ of the type $dr^2 \oplus r^2 g^V \oplus \pi^* g^B$ where $g^V$ and $g^B$ are metrics on $VN$ and $TB$ respectively. More generally, a model edge metric will be any metric of type:

$$g_e = dr^2 \oplus r^2 g^V(r) \oplus \pi^* g^B$$

where $g^V(r)$ is a smooth family of metrics down to $r = 0$. We set:

$$g^V := g^V(0), \quad g^N := g^V \oplus \pi^* g^B. \quad (5.1)$$

The Levi-Civita connection $\nabla^N$ of the metric $g^N$ on $N$ induces a connection $\nabla^VN$ on $VN$ via $E \nabla^N E$. We will call it the orthogonal projection. Clearly $\nabla^VN$, restricted to each fiber $N_b$ is the Levi-Civita connection of that fiber for the metric $g^V$.

**Definition 5.1** A manifold with edge singularities is a smooth manifold $M$ with a Riemannian metric $g$ such that there exists a compact set $K$ and a diffeomorphism $\varphi: M \setminus K \to (-\epsilon, 0) \times N$, such that on $M \setminus K$:

$$g = \varphi^* g_e.$$ 

**Proof of Theorem 1.2** As in the conical case, the Euler characteristic of $M_r$ is constant and equal to $\chi(M)$. So it is enough to prove the convergence of the integrals of transgression forms in (1.1) for the slices $\partial M_r \simeq \{r\} \times N$.

We will use the following terminology for double forms of type $(2, 2)$ on $N$. A form is called (purely) horizontal if its second component belongs to $\Gamma(\pi^* \Lambda^2 T^*B)$. It is called (purely) vertical its second component belongs to $\Gamma(\Lambda^2 V^* N)$. It is a mixed
form if its second component belongs to $\Gamma(\pi^* T^* B \otimes V^* N \oplus V^* N \otimes \pi^* T^* B)$. Clearly every $(2, 2)$ form can be written as a sum of a purely horizontal, a purely vertical and a mixed form.

The technical part of the proof is to decompose the curvature form of the slice $\{r\} \times N$ for the metric $g_r := r^2 g^V(r) \oplus \pi^* g^B$ into its horizontal, vertical, and mixed components. This is the object of Proposition 5.8 below, according to which the curvature $F(\nabla^g_r)$ for the slice $\{r\} \times N$ with metric $r^2 g^V(r) \oplus \pi^* g^B$ decomposes as follows:

$$F(\nabla^g_r) = (A_0 + A_2 r^2 + A_4 r^4) + r^2 (C_2 + r^2 C_4) + r^2 (D_2 + r^2 D_4) = X(r) + r^2 Y(r)$$

where $A_0, A_2, A_4, C_2, C_4, D_2, D_4$ are geometric quantities which depend smoothly on $r$ down to $r = 0$, and are constant when $g^V$ is constant in $r$. Moreover, for all $i, A_i$ is purely horizontal, $D_i$ is purely vertical and $C_i$ is mixed. We have $A_0 = \pi^* F(\nabla^B)$ and $D_2 = F(\nabla^B_r)$, and this is all we need for subsequent computations. Then

$$X(r) := A_0 + A_2 r^2 + A_4 r^4, \quad Y(r) := C_2 + D_2 + r^2 (C_4 + D_4)$$

is a convenient separation of the terms.

Applying Lemma 3.1 yet again, we conclude that

$$II' = -\left( r g^V(r) + \frac{r^2}{2} \dot{g}^V(r) \right) =: -r Z$$

where $Z$ is a vertical $(1, 1)$ double form. Let $b := \text{dim} B$ and $f := 2k - 1 - b$ be the dimension of the fiber of $\pi$. For the Berezin integrals, one has (taking into account that $r$ is negative):

$$B_{g_r}(\cdot) = \frac{1}{(-r)^f} B_{g^N}(\cdot)$$

Then

$$F(\nabla^g) \wedge (II')^{2k-1-2j} = \sum_{i=0}^j \binom{j}{i} (-r)^{2k-1-2i} X^i \wedge (Y^{j-i} Z^{2k-1-2j}).$$

Hence

$$B_{g_r} \left( F(\nabla^g) \wedge (II')^{2k-1-2j} \right) = \sum_{i} \binom{j}{i} (-r)^{b-2i} B_{g^N} \left( X^i \wedge (Y^{j-i} Z^{2k-1-2j}) \right).$$

(5.2)

Notice that $X^i$ is a purely horizontal double form of bi-degree $(2i, 2i)$, hence it vanishes if $2i > b$. On the other hand, for $2i < b$ all forms $\omega = B_{g^N}(X^i \wedge (Y^{j-i} Z^{2k-1-2j}))$ have a finite limit when $r \to 0$. Therefore only the term $2i = b$ survives in the sum (5.2) when $r \to 0$. 

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In conclusion, if \( b \) is odd, the limit is 0. If \( b \) is even, we get in the limit
\[
\binom{j}{b/2} B_{g^N} \left( X(0)^{b/2} Y(0)^{j-b/2} Z(0)^{2k-1-2j} \right).
\]
Now \( Y(0) = C_2 + D_2 \) and \( C_2 \) is a mixed term. Since \( X(0)^{b/2} \) is a purely horizontal form of maximal bi-degree, wedging with it will kill all terms from \( Y(0)^{j-b/2} \) containing a horizontal component. Hence only \( D_2^{j-b/2} \) will survive. We are left with
\[
\binom{j}{b/2} B_{g^N} \left( (\pi^* F(\nabla B))^{b/2} F(\nabla VN)^{j-b/2} (gV)^{2k-1-2j} \right).
\]
Multiplying with \( c(j, k) \), integrating and summing over \( 0 \leq i := j-b/2 \leq (f-1)/2 \) gives the result, since \( k - j - 1 = (f-1)/2 - i \).

**Example 5.2** Let \( \pi : E \to B \) be a Euclidean vector bundle of rank \( 2k \) endowed with a metric connection \( \nabla \). Then \( \pi^* \nabla \) and the tautological section \( \tau \) determine on \( SE := \{ v \in E \mid |v| = 1 \} \) a transgression form \( TPf(\pi^* \nabla, \tau) \) of degree \( 2k-1 \) with the property:
\[
\frac{1}{(2\pi)^k} \int_{SE/B} TPf(\pi^* \nabla, \tau) \equiv 1 \tag{5.3}
\]
when the fibers of \( SE \to B \) are oriented via the interior normals. This reduces immediately to Example 3.4 (see also Remark 4.8).

Suppose now that \( B \) is a compact submanifold of a closed Riemannian manifold \( \hat{M} \), both of even dimension. The normal bundle \( vB \) inherits a metric which is obviously a model edge metric with \( N = S(vB) \). Assume for the moment that the normal exponential map induces an isometry \( D_\epsilon (vB) \to U \) onto a neighborhood \( U \) of \( B \) where \( D_\epsilon (\cdot) \) is the disk bundle of radius \( \epsilon \). Let \( M^\circ := \hat{M} \setminus B \). Then using (5.3), Theorem 1.2 together with the classical Gauss–Bonnet turns into
\[
\chi(M^\circ) = \chi(\hat{M}) - \chi(B). \tag{5.4}
\]
Clearly this relation is also a topological consequence of Mayer–Vietoris for the cover \( \{ M^\circ, U \} \) of \( \hat{M} \). The same identity holds when \( \dim B \) is odd, albeit in that case \( \chi(B) = 0 \).

It turns out that 1.2 continues to hold *ad litteram* if \( B \) is totally geodesic recovering once again (5.4). In the general case, we will turn the tables around. We will see in Theorem (7.9) that the metric on \( M^\circ \), which, by the way, can be seen as the interior of the oriented blow-up of \( B \) is, in a neighborhood of the boundary, a first order perturbation of the model metric on \( (-\epsilon, 0] \times S(vB) \). Once we will know that a Gauss–Bonnet formula (1.5) holds for such perturbations, the topological statement (5.4) will serve to conclude that the integral over \( B \) equals \( \chi(B) \).

**Example 5.3** Here is a more general situation when the integral of the transgression form is independent of the fiber. Let \( P \to B \) be a principal bundle with structure group
Suppose \( G \) acts by isometries on a Riemannian manifold \( F \). Let \( N := P \times_G F \) be the associated fiber bundle over \( B \) (This is another way of saying that the fiber bundle with fiber \( F \) has transition maps taking values in \( G \subset \text{Isom}(F) \)). Then the vertical bundle \( VN \) inherits a Riemannian metric, since \( VN \simeq P \times_G TF \) with \( G \) acting on \( TF \) via the differentials of the isometries. Since \( TF \) has a metric to start with and \( G \) preserves it, we obtain a metric on \( VN \).

Any \( G \)-principal connection \( \omega \in \Omega^1(P; g) \) gives rise to a parallel transport via isometries between the fibers of \( N \to B \). Clearly the transgression form \( \text{TPf}(N_b, g^{N_b}, 1) \) of a fiber \( N_b \) obtained from the conical metric \( dr^2 \oplus r^2 g^{N_b} \) on \((-1, 1) \times N \) depends only on the isometry class of the metric \( g^{N_b} \). Therefore in the situation when all the fibers are isometric, the integral will be constant.

**Remark 5.4** One might ask what happens when \( \dim M = 2k + 1 \) is odd with an edge singularity. If we look at \( M_r \) which is a compact manifold of odd dimension with boundary then by Lefschetz Duality one gets that \( \chi(M_r) = \frac{1}{2} \chi(\partial M_r) \).

Now, \( \chi(\partial M_r) = \int_{\partial M_r} \text{Pf}(\nabla^{gr}) \) is constant with respect to \( r \). If one uses as above the decomposition of \( F(\nabla^{gr}) \) into its horizontal, mixed and vertical components then for \( B \) even dimensional one gets

\[
\chi(N) = \lim_{r \to 0} \frac{1}{(2\pi)^k (2k)!} \int_{\partial M_r} F(\nabla^{gr})^{2k} = \int_B \text{Pf}(g^B) \int_{N/B} \text{Pf}(g^V) = \chi(B) \chi(F)
\]

while for odd dim \( B \) one gets zero. We recover thus a Riemannian-geometric proof of the multiplicativity of Euler characteristic in fibrations.

### 5.1 The Curvature Form of a Riemannian Submersion

In order to completely describe the decomposition of the curvature form \( F(\nabla^{gr}) \) into its vertical, horizontal and mixed components, we set \( u := r^2 \), and consider the adiabatic deformation of the metric on \( N \):

\[
h_u := g_u^V \oplus u^{-1} \pi^* g^B.
\]

In this section we are interested in \( uh_u \) but then in terms of curvature forms one has:

\[
F(\nabla^{uh_u}) = u F(\nabla^{h_u})
\]

since the Levi-Civita connection of \( uh_u \) and \( h_u \) are the same. The reason for working with \( h_u \) is that we can make use of the results of Berline et al. (1992, Ch. 10).

To begin with, let us notice that the family of vertical connections \( \nabla^{VN}(u) \) resulting from the projections of the Levi-Civita connections \( \nabla^{h_u} \) has a limit \( \nabla^{VN}(0) := \lim_{u \to 0} \nabla^{VN}(u) \) and this limit is the projection of the Levi-Civita connection of \( g^N \) (see (5.1)) onto \( VN \). This follows from the Koszul formula (see also Prop. 10.2 in Berline et al. 1992).
Define, using the Ehresmann connection, the following family of connections on $TN \to N$:

$$\nabla^\oplus_u := \nabla^{VN}(u) \oplus \pi^*\nabla^B \quad \rightarrow \quad \nabla^{\ominus} := \nabla^{VN} \oplus \pi^*\nabla^B.$$  

**Remark 5.5** One should not confuse $\nabla^{HN}$, the result of projecting $\nabla^{h_u}$ onto $HN$, with $\pi^*\nabla^B$.

For $u \neq 0$, let $\tau_u : \Lambda^2 T^*N \to \text{End}^{-}(TN)$ be the bundle morphism:

$$\tau_u(\omega_1 \wedge \omega_2)(\xi) = \omega_2(\xi)\omega^u_1 - \omega_1(\xi)\omega^u_2.$$  

The notation $\omega^u$ represents the $h_u$-metric dual. Notice that $\tau_u$ is the inverse of $(\tau_u)^{-1} : \text{End}^{-}(TN) \to \Lambda^2 T^*N$:

$$(\tau_u)^{-1}(A)(\xi_1, \xi_2) = h_u(\xi_1, A\xi_2).$$  

We can write (see Prop. 10.6 in Berline et al. 1992):

$$(\nabla^{h_u} - \nabla^{\ominus}_u) = \tau_u(\omega_u)$$  

for $u \neq 0$, where $\omega_u : TN \to \Lambda^2 T^*N$ is defined by

$$\omega_u(X)(Y, Z) = \hat{S}_u(X, Y, Z) - \hat{S}_u(X, Z, Y) - \hat{Q}_u(X, Z, Y) + \hat{Q}_u(X, Y, Z) - \hat{Q}_u(Y, Z, X).$$  

We recall the definitions of $\hat{S}_u$ and $\hat{Q}_u$ (both differ by a sign compared with Section 10.1 in Berline et al. 1992):

$$\hat{Q}_u \in \Gamma(HN^* \otimes HN^* \otimes VN^*), \quad \hat{Q}_u(X, Y, Z) = \frac{1}{2} g^V_u([X, Y]^v, Z),$$  

$$\hat{S}_u \in \Gamma(VN^* \otimes VN^* \otimes HN^*), \quad \hat{S}_u(X, Y, Z) = g^V_u(Y, [Z, X]^v) - (\nabla^{VN}(u))^v Z X.$$  

where superscript $^v$ indicates projection onto the vertical component. Notice that both $\hat{Q}_u$ and $\hat{S}_u$ have well-defined limits when $u \to 0$. We conclude that $\omega_u$ has a well-defined limit $\omega_0$ when $u \to 0$.

We look at the curvature tensors now. We get:

$$F(\nabla^{h_u}) = F(\nabla^{\ominus}_u) + [\nabla^{\ominus}_u, \tau_u(\omega_u)] + \tau_u(\omega_u) \wedge \tau_u(\omega_u). \quad (5.5)$$  

Notice that for a fixed $u$, $\nabla^{\ominus}_u$ is $h_u$-metric compatible since $\nabla^{VN}(u)$ preserves $g^V(u)$ and $\pi^*\nabla^B$ preserves $\pi^*g^B$. As a consequence, the morphism $\tau_u : \Lambda^2 T^*N \to \text{End}^{-}(TN)$ is parallel with respect to the connection $\nabla^{\ominus}_u$ for every $u$. Therefore

$$[\nabla^{\ominus}_u, \tau_u(\omega_u)] = \tau_u(\nabla^{\ominus}_u \omega_u). \quad (5.6)$$
where on the right $\nabla_u^\oplus$ is the extension on tensors of $\nabla_u^\oplus$. It preserves the type of a double form, i.e., it takes purely horizontal to purely horizontal, etc.

Due to the fact that $\nabla^{HN} \neq \pi^*\nabla^B$, $\omega_u$ is not a mixed form, which means that $\tau_u(\omega_u)$ has a certain diagonal component. In fact we can write:

$$\omega_u = \tilde{\omega}_u + \omega_u^h$$  \hspace{1cm} (5.7)

where $\tilde{\omega}_u$ is made exclusively of mixed terms while $\omega_u^h$ is a purely horizontal term with:

$$\tilde{\omega}_u := (\tau_u)^{-1}(\nabla^h_u - \nabla^{VN}(u) \oplus \nabla^{HN}(u))$$

and

$$\omega_u^h := (\tau_u)^{-1}(\nabla^{VN}(u) \oplus \nabla^{HN}(u) - \nabla_u^\oplus).$$

We used $\nabla^{HN}(u)$ for the horizontal orthogonal projection of $\nabla^h_u$ which does not coincide with $\pi^*\nabla^B$. Instead, we have the following.

**Lemma 5.6** Let $\pi : P \rightarrow B$ be a Riemannian submersion, and $\nabla^{HP}$ the orthogonal projection of the Levi-Civita connection onto $HP \simeq \pi^*TB$. Let

$$\Omega : HP \times HP \rightarrow VP,$$

$$\Omega(X, Y) = P^{VP}[X, Y]$$

be the curvature of the Ehresmann connection (a bundle morphism), and $\tilde{\Omega} : VP \times HP \rightarrow HP$ the unique bundle morphism that satisfies

$$(\tilde{\Omega}(X, Y), Z) = (X, \Omega(Y, Z)), \hspace{1cm} (\forall)Z \in \Gamma(HP).$$

Then, for all $X \in \Gamma(TP)$, $Y \in \Gamma(HP)$,

$$\nabla_X^{HP}Y - (\pi^*\nabla^B)_X Y = \frac{1}{2}\Omega(P^{HP}(X), Y) - \frac{1}{2}\tilde{\Omega}(P^{VP}(X), Y).$$

In particular

$$\langle \nabla_X^{HP}Y, Z \rangle - \langle \pi^*\nabla^B_X Y, Z \rangle = -\frac{1}{2}\langle P^{VP}(X), [Y, Z] \rangle, \hspace{1cm} (\forall)Y, Z \in \Gamma(HP).$$

**Proof** It is well-known (see Petersen 2006, pp. 82) that if $X$ and $Y$ are horizontal lifts of vector fields $\bar{X}, \bar{Y}$ on $B$ then

$$\nabla^P_X Y = \pi \circ \nabla^B_{\bar{X}} \bar{Y} + \frac{1}{2}\Omega(X, Y).$$

In other words, for this kind of vector fields we have:

$$\nabla^P_X Y = (\pi^*\nabla^B)_X Y + \frac{1}{2}\Omega(X, Y).$$  \hspace{1cm} (5.8)
It is easy to extend Eq. (5.8) to vector fields \( X = fX_1 \) and \( Y = gY_1 \) where \( X_1 \) and \( Y_1 \) are horizontal lifts and \( f, g \in C^\infty(P) \). This means that (5.8) holds for all \( X, Y \in \Gamma(HP) \).

On the other hand, for \( X \in \Gamma(VP) \) and \( Y, Z \) horizontal lifts, one has

\[
2\langle \nabla^P_X Y, Z \rangle = \langle [X, Y], Z \rangle - \langle [Y, Z], X \rangle + \langle [Z, X], Y \rangle + X(Y, Z) = -\langle [Y, Z], X \rangle,
\]

the reason being that \( [X, Y] = 0 = [Z, X] \) (see Lemma 10.7 in Berline et al. 1992).

Since in this case \( \pi^* \nabla^B X Y = 0 \) we get

\[
\langle \nabla^P_X Y, Z \rangle - \langle \pi^* \nabla_X Y, Z \rangle = -\frac{1}{2} \langle [Y, Z], X \rangle
\]

and the relation holds also for \( Y = gY_1 \) and \( Z = hZ_1 \) with \( Y_1 \) and \( Z_1 \) horizontal lifts and \( g, h \in C^\infty(P) \). This means that (5.9) holds for all \( X \in \Gamma(VP), Y, Z \in \Gamma(HP) \).

According to Lemma 5.6, for \( X \in \Gamma(TN) \) and \( Y, Z \in \Gamma(HN) = \Gamma(\pi^*TB) \) we have:

\[
\left( \left( \nabla^{VN}(u) \oplus \nabla^{HN}(u) - \nabla^u \right)_X Y, Z \right) = \omega^h_u(X)(Y, Z) = -1/2\langle \pi^* \nabla^{VN}_X (P^{VN}(X), [Y, Z]) \rangle
\]

and thus \( \omega^h_u \) has a finite limit when \( u \to 0 \). Since \( \omega_u \) has a limit we deduce from (5.7) that \( \tilde{\omega}_u \) has a limit when \( u \to 0 \). We conclude that

\[
\nabla^u \omega_u = \nabla^u \omega^h_u + \nabla^u \omega^h_u
\]

is a decomposition into a purely mixed term and a purely horizontal term since \( \nabla^u \) preserves the type of the form. Both sides have a well-defined limit when \( u \to 0 \).

In order to control \( (\tau_u) \circ (\tau_u(\omega_u) \wedge \tau_u(\omega_u)) \) we need to take a closer look at \( \tau_u \). Since for every \( \eta \in \Omega^1(TN) \) we have

\[
\eta^u = (\eta^v)^u + u(\eta^h)^u
\]

where the decomposition \( \eta = \eta^v + \eta^h \) is independent of \( u \) and \( \eta^u \) is the \( g^V_u \)-metric dual while \( \eta^h \) is the \( \pi^*g^B \)-metric dual, we get:

\[
\tau_u = \tau^u_0 + u\tau^r_0,
\]

where

\[
\begin{align*}
\tau^u_0 : \Lambda^2 T^* N &\to \text{Hom}(TN, VN), \\
\tau^r_0 : \Lambda^2 T^* N &\to \text{Hom}(TN, HN), \\
\tau^u_0(\omega_1 \wedge \omega_2)(\xi) &\equiv \omega_2(\xi)(\omega_1^v)^u_1 \omega_1(\xi)(\omega_1^h)^u_1 \\
\tau^r_0(\omega_1 \wedge \omega_2)(\xi) &\equiv \omega_2(\xi)(\omega_1^v)^h_1 \omega_1(\xi)(\omega_1^h)^h_1.
\end{align*}
\]

Remark 5.7 Notice that

\[\square\]
If $\omega_1$ or $\omega_2$ is horizontal, then $\tau_0^u(\omega_1 \land \omega_2)\xi = 0$ for $\xi$ vertical.

If $\omega_1$ or $\omega_2$ is vertical then $\tau_0^u(\omega_1 \land \omega_2)\xi = 0$ for $\xi$ horizontal.

If $\xi$ is vertical but $\omega_1$ and $\omega_2$ are both horizontal then $\tau_0^u(\omega_1 \land \omega_2)\xi = 0$.

Clearly $\tau_0(u)$ has a finite limit as $u \to 0$. Define $\gamma_u : \Lambda^2 TN \to \Lambda^2 T^* N$:

$$\gamma_u := (\tau_u)^{-1}(\tau_u(\omega_u) \land \tau_u(\omega_u)).$$

More explicitly,

$$\gamma_u(a_1, a_2)(\xi_1, \xi_2) = h_u(\xi_1, \eta_u(\omega_u(a_1))\eta_u(\omega_u(a_2)) - \tau_u(\omega_u(a_2))\tau_u(\omega_u(a_1))\eta_u(\omega_u(1))\eta_u(\omega_u(2)))$$

$$= h_u(\tau_u(\omega_u(a_2))\eta_1, \tau_u(\omega_u(a_1))\eta_2) - h_u(\tau_u(\omega_u(a_1))\eta_1, \tau_u(\omega_u(a_2))\eta_2)$$

$$= g_u^V(\tau_0^u(\omega_u(a_2))\eta_1, \tau_0^u(\omega_u(a_1))\eta_2) - g_u^V(\tau_0^u(\omega_u(a_1))\eta_1, \tau_0^u(\omega_u(a_2))\eta_2)$$

$$+ u \left[ \pi^* g_B(\tau_0^u(\omega_u(a_2))\eta_1, \tau_0^u(\omega_u(a_1))\eta_2) \right]$$

$$- \pi^* g_B(\tau_0^u(\omega_u(a_1))\eta_1, \tau_0^u(\omega_u(a_2))\eta_2) \right].$$

The last equality follows from the fact that $\tau_0(u)$ takes values in $VN$ and $\tau_0^u$ takes values in $HN$.

We define $(\omega \land \omega)_0^u : \Lambda^2 TN \to \Lambda^2 T^* N$,

$$(\omega \land \omega)_0^u := g_u^V(\tau_0^u(\omega_u(a_2))\eta_1, \tau_0^u(\omega_u(a_1))\eta_2) - g_u^V(\tau_0^u(\omega_u(a_1))\eta_1, \tau_0^u(\omega_u(a_2))\eta_2).$$

By Remark 5.7, $\tau_0^u$ will take mixed forms and purely horizontal forms into endomorphisms which vanish on vertical vectors. Recall (5.7) by which $\omega_u$ is a sum of mixed terms and purely horizontal terms. It follows that $\tau_0^u(\omega_u(a_2))\xi$ is zero for $\xi$ vertical. We conclude that $(\omega \land \omega)_0^u$ is a purely horizontal form.

Define also $(\omega \land \omega)_0^1(u) : \Lambda^2 TN \to \Lambda^2 T^* N$ by

$$(\omega \land \omega)_0^1(u) := \pi^* g_B(\tau_0^1(\omega_u(a_2))\eta_1, \tau_0^1(\omega_u(a_1))\eta_2) - \pi^* g_B(\tau_0^1(\omega_u(a_1))\eta_1, \tau_0^1(\omega_u(a_2))\eta_2).$$

Then

$$\gamma_u = (\omega \land \omega)_0^u + u(\omega \land \omega)_0^1(u). \quad (5.10)$$

We will use the same notation $F(\nabla^{h_u})$ for the curvature forms $(\tau_u)^{-1}(F(\nabla^{h_u}))$ and $F(\nabla^{\oplus u})$ for $(\tau_u)^{-1}(F(\nabla^{\oplus u}))$. From (5.5), (5.6) and (5.10) we get the following equality of (2, 2) double forms:

$$F(\nabla^{s_u}) = F(\nabla^{\oplus u}) + \nabla^{\oplus u} \omega_u + (\omega \land \omega)_0^1(u) + u(\omega \land \omega)_0^1(u).$$

The matrix decomposition $F(\nabla^{\oplus u}) = F(\nabla^{VN}(u)) \oplus F(\pi^* \nabla^B)$ translates into the equality of (2, 2) double forms for the metric $h_u$:

$$F(\nabla^{\oplus u}) = F(\nabla^{VN}(u)) + u^{-1} \pi^* F(\nabla^B).$$
We finally look at the decomposition for \((\omega \wedge \omega)'_0(u)\). Use (5.7) to get

\[(\omega \wedge \omega)'_0(u) = A^1_u + A^2_u + A^3_u + A^4_u,
\]

where

\[
\begin{align*}
A^1_u(a_1, a_2)(\xi_1, \xi_2) &= \pi^* g^B (\tau'_0(\omega_u(a_2))\xi_1, \tau'_0(\omega_u(a_1))\xi_2) - \pi^* g^B (\tau'_0(\omega_u(a_1))\xi_1, \tau'_0(\omega_u(a_2))\xi_2) \\
A^2_u(a_1, a_2)(\xi_1, \xi_2) &= \pi^* g^B (\tau'_0(\omega_u^b(a_2))\xi_1, \tau'_0(\omega_u^b(a_1))\xi_2) - \pi^* g^B (\tau'_0(\omega_u^b(a_1))\xi_1, \tau'_0(\omega_u^b(a_2))\xi_2) \\
A^3_u(a_1, a_2)(\xi_1, \xi_2) &= \pi^* g^B (\tau'_0(\omega_u^\alpha(a_2))\xi_1, \tau'_0(\omega_u^\alpha(a_1))\xi_2) - \pi^* g^B (\tau'_0(\omega_u^\alpha(a_1))\xi_1, \tau'_0(\omega_u^\alpha(a_2))\xi_2) \\
A^4_u(a_1, a_2)(\xi_1, \xi_2) &= \pi^* g^B (\tau'_0(\omega_u^\beta(a_2))\xi_1, \tau'_0(\omega_u^\beta(a_1))\xi_2) - \pi^* g^B (\tau'_0(\omega_u^\beta(a_1))\xi_1, \tau'_0(\omega_u^\beta(a_2))\xi_2).
\end{align*}
\]

Now \(A^1_u\) is purely vertical, \(A^4_u\) is purely horizontal, and moreover one can check that \(A^2_u\) and \(A^3_u\) are mixed. We have thus proved the following

**Proposition 5.8** The following equality of (2, 2) double forms holds

\[F(\nabla^h u) = \left[u^{-1} \pi^* F(\nabla^B) + \nabla^\omega u^b + (\omega \wedge \omega)'_0 + u A^4_u \right] + \left[\nabla^\omega u^b + u A^2_u + u A^3_u \right] + \left[F(\nabla^V B(u)) + u A^1_u \right]
\]

where the sums in square brackets represent the purely horizontal, mixed, or purely vertical components. All terms dependent on \(u\) have a finite limit when \(u \to 0\).

From \(F(\nabla^u^h u) = u F(\nabla^h u)\) one gets the corresponding decomposition for \(F(\nabla^u^h u)\).

### 5.2 Horizontal Variations of the Model Metric

We close this section by discussing what happens when the model metric has the following structure:

\[g_e = dr^2 \oplus r^2 g^V(r) \oplus \pi^* g^B(r) \tag{5.11}\]

with \(g^B(r)\) a smooth family of metrics on \((-\epsilon, 0]\). Various types of perturbations will be considered in Sect. 7. By reasoning exactly as in the proof of Theorem 1.2 one can compute the limits of transgression forms. In order to state the result, we need to introduce more notation.

Let \((g_r)_{r \in (-\epsilon, \epsilon)}\) be a smooth family of metrics on a smooth manifold \(B\) of dimension \(b\). Let \(g := g_0\) and \(\dot{g} := \frac{\partial g}{\partial r}(0)\) and denote:

\[Q_{i,b}(g_r) := \frac{1}{i!(b-2i)!} B_{g_b} \left(R^i \wedge g^{b-2i}\right).
\]

**Theorem 5.9** On a manifold with incomplete edge singularities of type \((5.11)\),

\[
(2\pi)^k \chi(M) = \int_M Pf^g - \sum_{(i,j) \in A_{k,b}} (-1)^{2k-b} \tilde{c}(k - j - 1) \int_B \left(Q_{i,b}(g^B(r)) \int_{N/B} P_{j,f} \left(g^V\right)\right)
\]
where

\[ A_{k,b} := \{(i, j) \mid 0 \leq i \leq j \leq k - 1, \ i \leq b/2\}. \]

**Proof** One writes \( II = -(rZ + T) \) where \( T = \dot{g}^B \) and notices first that \( ZT = TZ \). Then one ends up with a sum for fixed \( 0 \leq j \leq k - 1 \)

\[
\sum_i \sum_l (-1)^{f+l} r^{b-(2i+l)} \binom{j}{i} \binom{2k-2j-1}{l} B_g^N \left( X^i(0)T^l(0)Y^{j-i}(0)Z^{2k-2j-1-l}(0) \right)
\]

where \( X(0) \) and \( T(0) \) are purely horizontal. Only when \( 2i+l = b \) one gets something non-trivial. Multiply by \( c(j, k) \) and sum to get the desired formula. \( \square \)

**Corollary 5.10** If \( \dot{g}^B(0) \equiv 0 \) one recovers the formula of Theorem 1.2.

Anticipating Sect. 7, Theorem 5.9 is an example of a Gauss–Bonnet formula for first order perturbations of the model metric

\[
dr^2 \oplus r^2 g^V (r) \oplus \pi^* g^B(0)
\]

in the sense of Definition 7.6.

### 6 Manifolds with Fibered Boundary

The computations of the previous section allow us to address the Gauss–Bonnet problem for another class of metrics. Assume again that \( N \) fibers over \( B \), that we fix an Ehresmann connection, a family of vertical metrics on the fibers and a metric \( g^B \) on \( B \). The model fibered boundary metric on \((1, \infty) \times N\) defined by this data is

\[
g_e^\infty := dr^2 \oplus g^V \oplus r^2 \pi^* g^B.
\]

We consider Riemannian manifolds \((M, g)\) (called *manifolds with fibered boundary* for which there exists a diffeomorphism \( \varphi : M \setminus K \to (1, \infty) \times N \) outside a compact set \( K \) such that

\[
g = \varphi^* g_e^\infty.
\]

**Proposition 6.1** Manifolds with fibered boundary are complete.

**Proof** Outside a relatively compact set, \( M \) is isometric to \([r, \infty) \times N\) endowed with the metric \( g_e^\infty \) for some \( r \in \mathbb{R} \). The projection onto \([r, \infty)\) is proper because \( N \) is compact. Moreover, this projection clearly decreases lengths of vectors, hence of curves, hence it decreases distances (it is Lipschitz of constant 1). This is enough to imply that \([r, \infty) \times N\) is a complete metric space, hence \( M \) is also complete. \( \square \)
Proof of Theorem 1.3} The computations are similar to Theorem 1.2 and based also on Proposition 5.8 where we set $u^{-1} = r^2$. Let $g_r := g^V \oplus r^2 \pi^* g^B = h_u$ be the metric of the slice. Write the decomposition in purely horizontal, mixed and purely vertical terms as:

\[ F(\nabla g_r) = (r^2 A_2 + A_0 + r^{-2} A_{-2}) + (C_0 + r^{-2} C_{-2}) + (D_0 + r^{-2} D_{-2}) \]

where $A_2 = \pi^* F(\nabla B)$, $D_0 = F(\nabla VN)$. Then

\[ (\Pi')^{2k-1-2j} = -r^{2k-1-2j} (\pi^* g^B)^{2k-1-2j} \]

and $B_{g_r}(\cdot) = r^{-b} B_{g_s}(\cdot)$ where $g^N = g^V \oplus \pi^* g^B$. Hence

\[ B_{g_r} \left( F(\nabla g_r)^j \wedge (\Pi')^{2k-1-2j} \right) = -r^{f-2j} B_{g_s} \left( F(\nabla g_r)^j \wedge (\pi^* g^B)^{2k-1-2j} \right). \]

We look at the term $(r^2 A_2)^l$ for some $l \leq j$ in the expansion of $F(\nabla g_r)$. Now the horizontal component of the product $F(\nabla g_r)^j \wedge (\pi^* g^B)^{2k-1-2j}$ cannot have degree bigger than $b$ in order to be non-zero. Hence

\[ 2l + 2k - 1 - 2j \leq b \iff 2l + f - 2j \leq 0. \]

All the other terms in the expansion of $F(\nabla g_r)$ contribute with non-positive powers of $r$. Hence in the expansion of $r^{f-2j} B_{g_s} \left( F(\nabla g_r)^j \wedge (\pi^* g^B)^{2k-1-2j} \right)$ one ends up only with non-positive powers of $r$.

If $b$ is even, the inequalities are strict so all terms will vanish when $r \to \infty$. If $b$ is odd, collecting the terms that correspond to $2l = 2j - f$ (which incidentally forces $j \geq f/2$) we get (1.3).

\[ \square \]

**Corollary 6.2** Let $(M, g)$ be a manifold with fibered boundary. If the base $B$ of the boundary fibration $N \to B$ is an odd-dimensional sphere with the round metric, then

\[ \chi(M) - \chi(F) = \frac{1}{(2\pi)^k} \int_M \text{Pf}^g. \]

**Proof** The normal $\partial_r$ points outside the sphere. The computations of Example 3.4 apply (see also Remark 4.8). This fits with the example when $M = \mathbb{R}^n$ and $F$ reduced to a point.

\[ \square \]

**7 Edge Manifolds: Perturbations of the Model Metrics**

There is one familiar situation not entirely covered by the models of Sect. 5, namely that of a submanifold $B$ in a Riemannian manifold $(M, g)$. The spherical normal bundle $N := S \nu B$ inherits a fiber bundle structure over $B$ and an Ehresmann connection, induced by the Levi-Civita connection as follows. Let $\pi : TN \to N$ be the natural projection. The Levi-Civita connection induces a connection on $\nu B$ and therefore one
obtains a splitting $TvB = \pi^*vB \oplus \pi^*TB$ into vertical and horizontal components where $\pi : vB \to B$ is the natural projection. Now $S(vB) \subset vB$ is a hypersurface whose unit normal vector is vertical (i.e., it belongs to $\pi^*vB$) relative to the previous decomposition. It follows that $TS(vB)$ splits into the direct sum of $\tau^\perp \subset \pi^*vB$ (the orthogonal complement of the tautological section of $\pi^*vB \to S(vB)$) and $\pi^*TB$.

On both $TB$ and the normal vector bundle $vB \to B$ there are metrics induced by $g$, hence $(-\epsilon, 0) \times \mathcal{N}$ inherits an edge singularity metric. However, the original metric $g$ in a neighborhood of $B$ is not necessarily isometric to a model metric in the sense defined in Sect. 5 since the normal exponential map that gives rise to a tubular neighborhood for $B$ is only an “infinitesimal” isometry at the 0 section.

It is therefore natural to consider perturbations of the model edge metrics of Sect. 5. We will consider a differentiable edge manifold, meaning a compact manifold $M$ with boundary $\mathcal{N}$, such that $\pi : \mathcal{N} \to B$ is a locally trivial fibration. Moreover we assume the following data given:

(a) a boundary defining function $r : M \to (-\epsilon, 0]$;
(b) an Ehresmann connection on $\mathcal{N}$, i.e., a splitting $\mathcal{T}\mathcal{N} = V\mathcal{N} \oplus \pi^*TB$

We can use $r$ in order to produce a collar neighborhood $U$ of $\mathcal{N}$ diffeomorphic with $(-\epsilon, 0] \times \mathcal{N}$ such that the obvious diagram commutes:

\[
\begin{array}{ccc}
U & \xrightarrow{R} & (-\epsilon, 0] \times \mathcal{N} \\
\sim & & \\
\downarrow{r} & & \downarrow{p_1} \\
(-\epsilon, 0] & & \\
\end{array}
\]

The differential of $R$ gives a diffeomorphism between $TM|_U$ and $\mathbb{R} \oplus \pi_2^*\mathcal{T}\mathcal{N}$, where $\pi_2 : (-\epsilon, 0] \times \mathcal{N} \to \mathcal{N}$ is the projection on the second factor.

For our purposes, the edge manifold $M$ in the neighborhood $U$ will be identified with $(-\epsilon, 0] \times \mathcal{N}$ while the tangent bundle to $M$ in a neighborhood $U$ will be identified with $\mathbb{R} \oplus \pi_2^*\mathcal{T}\mathcal{N}$. The unit generator of $\mathbb{R}$ in this identification will be denoted $\partial_r$.

For the sake of brevity, we denote $U := (-\epsilon, 0] \times \mathcal{N}$.

Consider the vector bundles $F := V\mathcal{N}$ and $F' := \pi^*TB \oplus \mathbb{R}$ over $\mathcal{N}$. The Ehresmann connection induces a splitting

\[\mathbb{R} \oplus \mathcal{T}\mathcal{N} \simeq F \oplus F'.\]

We use the projection $\pi_2 : (-\epsilon, 0] \times \mathcal{N} \to \mathcal{N}$ to pull-back this bundle to $U$ but rather than writing $\pi_2^*F, \pi_2^*F'$ we keep the notation $F, F'$. We have thus in the neighborhood $U$ a splitting

\[TM|_U \simeq F \oplus F' \quad (7.1)\]
The fundamental object of this section is the following bundle endomorphism defined in terms of the splitting (7.1):

\[ \varphi : T\mathcal{M} \big|_U \to T\mathcal{M} \big|_U, \quad F \oplus F' \ni (v, w) \mapsto (\varphi(v, w)) \]

Clearly, \( \varphi \) is a bundle isomorphism only along \( U^c := U \setminus N \), i.e., for \( r \neq 0 \).

The model edge degenerate metric is throughout this section:

\[ h := dr^2 \oplus r^2 g^V \oplus \pi^* g^B. \]

The bilinear map

\[ h^\varphi : T\mathcal{M} \big|_{U^c} \times T\mathcal{M} \big|_{U^c} \to \mathbb{R}, \quad h^\varphi(Y', Z') := h(\varphi^{-1}(Y'), \varphi^{-1}(Z')) \]

extends as a non-degenerate metric on \( U \), and \( \varphi \) becomes a bundle isometry for \( r \neq 0 \). Indeed,

\[ h^\varphi = dr^2 \oplus g^V \oplus \pi^* g^B. \]

**Theorem 7.1** The Levi-Civita connection \( \nabla^h \) of the model metric has the property that \( \varphi \nabla^h \varphi^{-1} \) extends to a \( h^\varphi \)-metric connection down to \( r = 0 \).

**Proof** We will compare the Levi-Civita connection of \( \nabla^h \) with the following connection:

\[ \nabla' := d \oplus \left( \frac{\partial}{\partial r} + \frac{1}{r} \right) dr + \nabla^{VN} \oplus \pi^* \pi^* \nabla^B \] (7.2)

on the vector bundle \( T\mathcal{M} \big|_{U^c} = \mathbb{R} \oplus \pi^*_{\mathcal{V}} VN \oplus \pi^*\pi^* TB \) where \( \pi_2 : (-\epsilon, 0] \times N \to N \) is the projection.

In (7.2), the connection \( \nabla^{VN} \) is the projection of the Levi-Civita connection of any slice \( \{r\} \times N \) onto \( \pi^*_{\mathcal{V}} VN \). It does not depend on \( r \) since the projection of the Levi-Civita connection of a Riemannian submersion onto the vertical bundle does not depend on the choice of the horizontal metric (Prop. 10.2 in Berline et al. 1992), while the Levi-Civita connection of the slice \( \{r\} \times N \) is the same for the metrics \( r^2 g^V \oplus \pi^* g^B \) and \( g^V \ominus r^{-2} \pi^* g^B \).

We emphasize that the differential operator \( \frac{\partial}{\partial r} + \frac{1}{r} \) acts on families of sections

\[ (Y_r)_{r \in (-\epsilon, 0]} \in \Gamma(VN) \]

which can alternatively be seen as sections of \( \pi^*_{\mathcal{V}} VN \) (where \( \pi_2 : (-\epsilon, 0] \times N \to N \) is the projection), while \( \nabla^{VN} \) is used to differentiate only in the \( TN \) directions.

It follows from the Koszul relation (see (7.4) and (7.5)) that the \( \pi^*_{\mathcal{V}} VN \) component of \( \nabla' \) is actually the orthogonal projection of \( \nabla^h \) onto \( \pi^*_{\mathcal{V}} VN \). This implies that \( \nabla' \) is \( h \)-compatible (as \( \pi^* \nabla^B \) is clearly \( \pi^* g^B \)-compatible). As a consequence, \( \varphi \nabla' \varphi^{-1} \) is \( h^\varphi \)-compatible.
It is easy to check that $\varphi \nabla' \varphi^{-1}$ extends smoothly to $r = 0$ since $\nabla^\text{VN}$ commutes with multiplication by $r^{-1}$ and
\[
\frac{\partial Y_r}{\partial r} + \frac{Y_r}{r} = \frac{1}{r} \frac{\partial (r Y_r)}{\partial r}.
\]
Moreover, $\varphi (d \oplus \pi^* \nabla^B) \varphi^{-1} = d \oplus \pi^* \nabla^B$, since $\varphi$ acts as the identity on $F'$.

In order for the 1-form $\eta := \nabla^h - \nabla'$ to have the property that $\varphi \eta(X) \varphi^{-1}$ extends smoothly for every choice of $X \in \Gamma(T\mathcal{M}|_U)$, it is enough that in the decomposition
\[
\eta(X) := \begin{pmatrix} A_1(X) & A_2(X) \\ A_3(X) & A_4(X) \end{pmatrix} : F \oplus F' \rightarrow F \oplus F' \tag{7.3}
\]
the blocks $A_1(X)$, $A_4(X)$, $r A_2(X)$ and $r^{-1} A_3(X)$ extend smoothly all the way down to $r = 0$.

Clearly $A_1 \equiv 0$ since the orthogonal projections of $\nabla^h$ and $\nabla'$ on $F$ coincide.

Then metric compatibility implies for $Y \in \Gamma(F')$ and $Z \in \Gamma(F)$
\[
r^2 \langle A_2(X)(Y), Z \rangle_{\text{VN}} = \langle A_2(X)(Y), Z \rangle_h = -\langle Y, A_3(X)(Z) \rangle_h
\]
\[
= - \langle Y, A_3(X)(Z) \rangle_{F'} = - \langle A_3^T(X)(Y), Z \rangle_{\text{VN}}
\]
where the transpose $A_3^T$ is computed with respect to the metric $h^\nu$, independent of $r$.

Hence
\[
r A_2(X) = - r^{-1} A_3^T(X).
\]

Thus, it is enough to prove that $r^{-1} A_3(X)$ extends smoothly to $r = 0$.

To see that the remaining relations hold we look again at the Koszul relation:
\[
2 \langle \nabla^h_X Y, Z \rangle_h = \langle [X, Y], Z \rangle_h - \langle [Y, Z], X \rangle_h + \langle [Z, X], Y \rangle_h
\]
\[
+ X \langle Y, Z \rangle_h + Y \langle Z, X \rangle_h - Z \langle X, Y \rangle_h.
\]

For $X = \partial_r$, $Y = Y_r \in \Gamma(\pi^*_2 \text{VN})$, $Z = Z_r \in \Gamma(\pi^*_2 \text{VN})$,
\[
2 r^2 \langle \nabla_{\partial_r} Y, Z \rangle_{g^\text{VN}} = r^2 \left( \frac{\partial Y}{\partial r}, Z \right)_{g^\text{VN}} - r^2 \left( \frac{\partial Z}{\partial r}, Y \right)_{g^\text{VN}} + \frac{\partial}{\partial r} \left[ r^2 \langle Y, Z \rangle_{g^\text{VN}} \right].
\]

We end up with
\[
2 r^2 \langle \nabla_{\partial_r} Y, Z \rangle_{g^\text{VN}} = 2 r^2 \left( \frac{\partial Y}{\partial r}, Z \right)_{g^\text{VN}} + 2 r \langle Y, Z \rangle_{g^\text{VN}}.
\]
Hence
\[
\nabla_{\partial_r} Y = \frac{\partial Y}{\partial r} + \frac{Y}{r}
\]  

(7.4)

Taking \( X = X_r \in \Gamma(\pi^*_2TN) \) with \( Y = Y_r \in \Gamma(\pi^*_2VN) \), \( Z = Z_r \in \Gamma(\pi^*_2VN) \) then clearly
\[
\langle \nabla^h X Y, Z \rangle = \langle \nabla^V X Y, Z \rangle.
\]  

(7.5)

One verifies easily that the orthogonal projection of \( \nabla^h \) onto \( \mathbb{R} \), the tangent bundle of the foliation via integral curves of \( \partial_r \), is the trivial connection \( d \).

Recall that \( \pi^*\Pi^B \) is not the orthogonal projection of \( \nabla^h \) onto \( \pi^*TB \cong HN \). Let \( \nabla^HN \) be this projection. It follows from Lemma 5.6 for the Riemannian submersion \( M|_U \to B \) that for \( X \in \Gamma(\mathbb{R} \oplus \pi^*_2TN) \) and \( Y, Z \in \Gamma(\pi^*_2\pi^*TB) \):
\[
\langle \nabla^h X Y, Z \rangle_h = \langle \nabla^HN X Y, Z \rangle_h - \frac{1}{2} \langle P^{\mathbb{R} \oplus VN}(X), [Y, Z] \rangle_{gVN}
\]

and the right hand side is smooth at \( r = 0 \). This describes the bottom block diagonal component of \( A_4(X) \) in (7.3) relative to the decomposition \( F' = \mathbb{R} \oplus \pi^*TB \). The other diagonal block of \( A_4 \) is obviously 0. The off-diagonal terms of the skew-symmetric \( A_4(X) \) are of type
\[
\langle \nabla_X \partial_r, Y \rangle_h \text{ and its negative } \langle \nabla_X Y, \partial_r \rangle_h
\]

where \( X \in \Gamma(\mathbb{R} \oplus \pi^*_2TN) \), \( Y \in \Gamma(\pi^*_2HN) \). For \( X = \partial_r \) one gets obviously 0 and Lemma 3.1 gives for \( X \in \Gamma(\pi^*_2TN) \):
\[
\langle \nabla_X \partial_r, Y \rangle_h = \frac{1}{2} (L_{\partial_r} h)(X, Y) = r \langle X, Y \rangle_{VH} = 0.
\]

In other words, if \( \tilde{\Omega} : VN \times HN \to HN \) is the morphism induced by the curvature \( \Omega \) of the Ehresmann connection of the Riemannian submersion \( \pi : N \to B \) with the metric \( g^V \oplus \pi^*g^B \) as in Lemma 5.6, then for \( X \in \Gamma(\mathbb{R} \oplus \pi^*_2TN) \), \( Y \in \Gamma(F') \) one has:
\[
A_4(X)(Y) = -\frac{r^2}{2} \tilde{\Omega}(P^{VN}(X), P^{HN}(Y)).
\]
Finally, for \( Y \in \Gamma(\pi^*_2 VN) \), \( Z \in \Gamma(F') \) and \( X \in \Gamma(\mathbb{R} \oplus \pi^*_2 TN) \) we compute
\[
\langle A_3(X)(Y), Z \rangle_h = \langle \nabla^h_X Y, Z \rangle_h.
\]
For \( X = \partial_r, Z \in \pi^*_2 HN \) one gets from the Koszul formula
\[
2 \langle \nabla_{\partial_r} Y, Z \rangle_h = \langle \partial_r Y, Z \rangle_h - \langle \partial_r Z, Y \rangle_h = 0. \quad (7.6)
\]
The vanishing holds also for \( X = \partial_r \), \( Z = \partial_r \).

For \( X \in \Gamma(\pi^*_2 VN), Z \in \Gamma(\pi^*_2 HN) \) we get the relation:
\[
\langle A_3(X)(Y), Z \rangle_{HN} = -\langle Y, \rho VN (\nabla^h_X Z) \rangle_h = -r^2 \langle Y, P^VN ([Z, X]) - \nabla^V Z X \rangle_{VM}. \quad (7.7)
\]
For \( X \in \Gamma(\pi^*_2 HN), Z \in \Gamma(\pi^*_2 HN) \) we get the curvature of the Ehresmann connection:
\[
\langle A_3(X)(Y), Z \rangle_{HN} = -\langle Y, P^VN (\nabla^h_X Z) \rangle_h = -r^2 \langle Y, \Omega(X, Z) \rangle_{VN}. \quad (7.8)
\]
It is now clear from (7.6), (7.7), (7.8) and (7.9) that \( \frac{A_3(X)}{r} \) extends for any smooth vector fields \( X, Y : (-\epsilon, 0) \times N \to TM|_U \).

**Corollary 7.2** The restriction to \( TM|_{\partial M} = \mathbb{R} \oplus \pi^*_2 VN \oplus \pi^*_2 \pi^*T B \) (i.e., to \( r = 0 \)) of the extended connection \( \varphi \nabla^h \varphi^{-1} \) coincides with the connection
\[
\left( \begin{array}{c}
\frac{\partial}{\partial r} dr + \nabla VN \\
\pi^* \nabla B
\end{array} \right) + \left( \begin{array}{c|cc}
0 & \langle \bullet, \cdot \rangle_{VN} & 0 \\
\langle \bullet, \cdot \rangle_{VN} & 0 & 0 \\
0 & 0 & 0
\end{array} \right) \quad (7.10)
\]
where the matrix represents a 1-form (the \( \bullet \) entry) with values in \( \text{End}(\mathbb{R} \oplus \pi^*_2 VN \oplus \pi^*_2 \pi^*T B) \).

**Proof** The only non-trivial term in the difference \( \varphi (\nabla^h - \nabla') \varphi^{-1} \) comes from relation (7.7).

**Corollary 7.3** The Pfaffian \( \text{Pf}(\nabla^h) \) is a smooth form on \( M \) down to the boundary \( \{ r = 0 \} \).

**Proof** The map \( \varphi : (TM|_{U^c}, h) \to (TM|_{U^c}, h^\varphi) \) is a bundle isometry. Hence on \( U^c \), \( \text{Pf}(\nabla^h) \) is, up to a sign, equal to \( \text{Pf}(\varphi \nabla^h \varphi^{-1}) \).
We consider now a perturbation $g$ of $h$, i.e., a bilinear and symmetric form on $TM$ that is degenerate only along $N$ in a sense made precise in Definition 7.6.

Clearly there exists an $h$-symmetric endomorphism $C \in \Gamma(\text{End}(TM|_{U_c}))$ such that

$$g(X, Y) = h(CX, Y) = h(X, CY), \quad (\forall) X, Y \in TM|_{U_c}.\]$$

The next Lemma linking the two Levi-Civita connections is fundamental for our computations.

**Lemma 7.4** (Christoffel formula) Let $\nabla^h$ and $\nabla^g$ be the corresponding Levi-Civita connections on $TM|_{U_c}$. Then the 1-form $\omega : TM|_{U_c} \to \text{End}(TM|_{U_c})$ defined by

$$\omega(X)(Y) = \nabla^g_X Y - \nabla^h_X Y,$$

satisfies:

$$h(C\omega(X)(Y), Z) = \frac{1}{2} \left( h((\nabla^h_X Y, Z) + h((\nabla^h_Y C)X, Z) - h((\nabla^h_Z C)X, Y) \right).$$

**Proof** Notice first that due to the symmetry of the Levi-Civita connections one has:

$$\omega(X)(Y) = \omega(Y)(X) \quad (7.11)$$

and therefore $C\omega(X)(Y) = C\omega(Y)(X)$. Then from

$$Xh(Y, CZ) = h(\nabla^h_X Y, CZ) + h(Y, \nabla^g_X (CZ)) \text{ and}$$

$$Xg(Y, Z) = g(\nabla^g_X Y, Z) + g(Y, \nabla^g_X Z)$$

which translates into

$$Xh(Y, CZ) = h(\nabla^g_X Y, CZ) + h(Y, C\nabla^g_X Z)$$

one gets by subtraction:

$$h(\nabla^h_X Y - \nabla^g_X Y, CZ) = h(Y, C\nabla^g_X Z - \nabla^h_X (CZ)).$$

Taking $\nabla^h_X (CZ) = C(\nabla^h_X Z) + (\nabla^h_X C)(Z)$ we get:

$$h(\omega(X)(Y), CZ) + h(Y, C\omega(X)(Z)) = h(Y, (\nabla^h_X C)(Z)).$$

or

$$\omega(X)^T C + C\omega(X) = \nabla^h_X C. \quad (7.12)$$
Notice that the system (7.11) and (7.12) has a unique solution for $C\omega(X)$ due to the well-known fact that a trilinear map which is symmetric in the first two variables and anti-symmetric in the last two variables is zero. Finding this solution is simple linear algebra.

We know already that $\varphi\nabla^h\varphi^{-1}$ extends to $r = 0$. Let

$$C = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix} : \begin{bmatrix} F \\ F' \end{bmatrix} \rightarrow \begin{bmatrix} F \\ F' \end{bmatrix}$$

be the block decomposition of $C$. Then

$$C^\varphi := \varphi C \varphi^{-1} = \begin{bmatrix} C_1 & rC_2 \\ r^{-1}C_3 & C_4 \end{bmatrix}$$

is symmetric with respect to the $h^\varphi$-metric. In other words $C_3 = r^2 C_2^T$ where the transpose is computed with respect to $h^\varphi$. We have the following obvious remark.

**Lemma 7.5** If $\varphi C \varphi^{-1}$ extends smoothly to $TM|_U$, then $g^\varphi(\cdot, \cdot) := g(\varphi^{-1}(\cdot), \varphi^{-1}(\cdot))$ extends and

$$g^\varphi(\cdot, \cdot) = h^\varphi(C^\varphi \cdot, \cdot).$$

The morphism $\varphi C \varphi^{-1}$ controls the degenerations we are interested in. By Lemma 7.5, saying that

$$C = I + f(r)\varphi^{-1}D\varphi$$

where $D$ is smooth at $r = 0$ and $h^\varphi$-symmetric and $f$ smooth and vanishing at 0 is equivalent to

$$g^\varphi(\cdot, \cdot) = h^\varphi(\cdot, \cdot) + f(r)\alpha(\cdot, \cdot)$$

for some $\alpha(\cdot, \cdot)$ smooth, bilinear and symmetric on $TM|_U$.

**Definition 7.6** A perturbation of first (respectively second) order of $h$ is a bilinear, positive, symmetric $g : TM|_U \times TM|_U$ such that the endomorphism $C$ above satisfies:

$$C = I + r\varphi^{-1}D\varphi, \text{ resp. } C = I + r^2\varphi^{-1}D\varphi,$$

where $D$ is a smooth endomorphism of $TM|_U$, symmetric in the $h^\varphi$ metric.

Equivalently for $p = 1$ (resp. $p = 2$)

$$g^\varphi(\cdot, \cdot) = h^\varphi(\cdot, \cdot) + r^p\alpha(\cdot, \cdot)$$

where $\alpha$ is bilinear, symmetric and smooth on $TM|_U$.
Lemma 7.7

\[ \varphi(\nabla h C)\varphi^{-1} = (\varphi^{h} \varphi^{-1})(\varphi^{C} \varphi^{-1}). \]

**Proof** It follows from the next equalities that hold for any \( X \) and \( Y \):

\[ \varphi(\nabla h C)\varphi^{-1}(Y) = \varphi(\nabla h C)\varphi^{-1}(Y) - \varphi C(\nabla h (\varphi^{-1}(Y))) \]

\[ (\varphi C)^{-1}(\varphi C\varphi^{-1}) = \varphi(\nabla h (\varphi^{-1}(Y))) - \varphi C\varphi^{-1}(\varphi C\varphi^{-1}(Y)) \].

\[ \square \]

Theorem 7.8 Let \( g \) be a perturbation of a model edge metric \( h \).

(i) For perturbations of first order, the connection \( \varphi^{g} \varphi^{-1} \) extends at \( r = 0 \).

(ii) For perturbations of second order the connection the extension of \( \varphi^{g} \varphi^{-1} \) coincides on \( TM |_{\delta M} \) with \( \varphi^{h} \varphi^{-1} \).

**Proof** Let \( C^{\varphi} := \varphi C\varphi^{-1}, Y' := \varphi(Y), Z' := \varphi(Z), \nabla^{\varphi} := \varphi^{h} \varphi^{-1}, h^{\varphi}(-, -) := h(\varphi^{-1}(-), \varphi^{-1}(-)) \), and \( \omega(X)^{\varphi} := \varphi \omega(X)\varphi^{-1} \). The Christoffel formula (Lemma 7.4) can be written using Lemma 7.7:

\[ 2h^{\varphi}(C^{\varphi} \omega(X)^{\varphi}(Y'), Z') = h^{\varphi}(\nabla^{\varphi} X C^{\varphi}(Y'), Z') \]

\[ + h^{\varphi} \left( \left( \nabla^{\varphi} \nabla_{\varphi^{-1} Y'}^{\varphi} C^{\varphi} \right)(\varphi(X)), Z' \right) - h^{\varphi} \left( \left( \nabla^{\varphi} \nabla_{\varphi^{-1} Z'}^{\varphi} C^{\varphi} \right)(\varphi(X)), Y' \right). \] (7.13)

We deduce from this formula that, in order to show that \( \varphi \omega(X)\varphi^{-1} \) extends for perturbations of first order, it is enough to show that

\[ \nabla^{\varphi}_{\varphi^{-1} Y'} C^{\varphi} = \nabla^{\varphi}_{\varphi^{-1} Y'} (rD) \]

extends for all choices of \( Y' \), since the first term in the sum (r.h.s. of (7.13)) extends anyway.

The only situation when the extension is not a priori clear is when \( Y' \in \Gamma(\pi_{2}^{*}N) \). Then \( \varphi^{-1}(Y') = \frac{Y'}{r} \). But we can use now that \( Y'(r) = 0 \) and therefore

\[ \nabla^{\varphi}_{\varphi^{-1} Y'} (rD) = \nabla^{\varphi}_{Y'} (D), \]

and the later term extends.

Since \( C^{\varphi} \to 0 \) when \( r \to 0 \), in order to show that \( \varphi \omega(X)\varphi^{-1} \) extends by 0 for perturbations of second order we need to check that

\[ \lim_{r \to 0} \nabla^{\varphi}_{X} C^{\varphi} = \lim_{r \to 0} \nabla^{\varphi}_{X} (r^{2} D) = 0 \]

\[ \lim_{r \to 0} \nabla^{\varphi}_{\varphi^{-1} Y'} C^{\varphi} = \lim_{r \to 0} \nabla^{\varphi}_{\varphi^{-1} Y'} (r^{2} D) = 0 \]
for all choices of $X$ and $Y$. If either $X = Y = \partial_r$ then since $\varphi^{-1}(\partial_r) = \partial_r$ the two limits are identical and clearly equal to 0. When $Y' \in \Gamma(\pi_2^* VN)$ then the same idea as in the first order perturbations apply.

\[ \square \]

7.1 The Riemannian Metric in a Neighborhood of a Submanifold

The purpose of this section is to prove that the degenerate metric on the oriented blow-up space of a submanifold inside a Riemannian manifold is a first-order perturbation of a canonical model edge degenerate metric.

Let $B \subset M$ be a compact submanifold in a Riemannian manifold $(M, g)$. Let $\nu B \subset TX|_B$ be the normal bundle, $\pi : S(\nu B) \to B$ the unit sphere bundle inside $\nu B$, and

$$\exp : S(\nu B) \times [0, \infty) \to M, \quad (v_x, r) \mapsto \exp_x(rv_x)$$

the geodesic exponential map in normal directions to $B$. This map defines a diffeomorphism from $S(\nu B) \times (0, \epsilon)$ to the complement of $B$ inside its $\epsilon$-neighborhood. The function $r$ becomes the distance function to $B$. In fact, replacing the $\epsilon$-neighborhood of $B$ with $S(\nu B) \times [0, \epsilon)$ amounts precisely to constructing the (real) blow-up of $M$ along $B$.

The normal bundle $\nu B$ inherits itself a metric which makes the canonical projection $\pi : \nu B \to B$ a Riemannian submersion. The Ehresmann connection here is just the normal connection on $B$ induced from the Levi-Civita connection of $M$. One can use the blow-down map:

$$\exp : [0, \epsilon) \times S(\nu B) \to M$$

which is a diffeomorphism for $r \neq 0$ in order to endow $[0, \epsilon) \times S(\nu B)$ with a degenerate metric $g_1$. Clearly there exist a model edge degenerate metric $h_1$ on $[0, \epsilon) \times S(\nu B)$ of type:

$$dr^2 \oplus r^2 g^V \oplus \pi^* g^B$$

where $g^V$, the metric on $VS(\nu B) \subset \pi^* \nu B$ is induced by pulling back the metric $g|_{\nu B}$. The decomposition is relative to the Ehresmann connection mentioned earlier.

We will consider the first-order perturbation

$$\hat{h}_1(r) = h_1 - 2r \Pi \quad \text{(7.14)}$$

where $\Pi : S(\nu B) \to \text{Bil}(\pi^* T^* B)$ is the second fundamental form

$$\Pi_{W_0}(X_0, Y_0) := g(\nabla_{X_0} Y, W_0), \quad W_0 \in S(\nu B), \quad X_0, Y_0 \in T_b B$$

with $Y$ vector field along $B$ such that $Y(0) = Y_0$. 

\[ \square \] Springer
Theorem 7.9 Let $B \subset M$ be a compact submanifold in a Riemannian manifold $(M, g)$. Then the degenerate metric $g_1$ on $[0, \epsilon) \times S(vB)$ is a second order perturbation of the metric $h_1$ defined in (7.14).

Proof By the Gauss Lemma, $R := \partial_r$ is a geodesic field orthogonal to the slices $\{r\} \times S(vB)$, and therefore $g_1 = dr^2 \oplus g_1(r)$. We need only look at $g_1(r)$ on $T(SvB)$. The metric $g_1(r)$ is obtained via the map:

$$\exp^r : SvB \to M, \quad (p, v) \mapsto \exp_p(rv), \quad g_1(r)(\cdot, \cdot) := g(d \exp^r(\cdot), d \exp^r(\cdot)).$$

We use curves $W : (-\epsilon, \epsilon) \to S(vB)$ with $\gamma(s) := \pi(W(s))$ where $\pi : S(vB) \to B$ is the projection in order to represent tangent vectors of $S(vB)$. Let then

$$f(r) := g_1(r)(W_1'(0), W_2(0)) = g(\partial_s \exp^r(W_1(s))|_{s=0}, \partial_s \exp^r(W_2(s))|_{s=0}).$$

Notice that

$$J_i(r) := \partial_s \exp^r(W_i(s))|_{s=0}$$

are Jacobi vector fields, along the geodesics $r \to \exp_{\gamma_i(0)}(rW_i(0))$. We will assume that $W_1(0) = W_2(0) = (b, W_0) \in S(v_B B)$.

It is good to keep in mind that

(1) there exists a splitting

$$TS(vB) = \pi^*TB \oplus VS(vB)$$

induced by the normal connection $\nabla^v$ of $vB$; consequently, the derivative $W'(s)$ decomposes as:

$$W'(s) = (\gamma'(s), (\gamma^*\nabla^v)\partial_s W|_{s=0}) \in T_{\gamma(s)}B \oplus v_{\gamma(s)}B$$

where $\gamma^*\nabla^v$ is the pull-back connection on $\gamma^*vB \to (-\epsilon, \epsilon)$.

(2) via the same splitting we have an injective morphism of vector bundles over $S(vB)$:

$$TS(vB) \hookrightarrow \pi^*TB \oplus VvB = \pi^*TB \oplus \pi^*vB = \pi^*(TM | B)$$

Hence for $W_0 \in S(v_B B)$, we have $T_{W_0}S(vB) = \{w \in T_B M \mid g(w, W_0) = 0\}$.

In order to make the computations more transparent it is useful to separate two classes of vector fields $W$ along $\gamma$.

(a) the vertical ones, i.e., those for which $\gamma(s) \equiv b \in B$ is constant and therefore $J(0) = 0$ and $J'(0) = W'(0) \in T_{W_0}S(v_B B)$ is a vertical vector in $T_{W_0}SvB$.

(b) the horizontal ones, i.e., those for which $\gamma'(0) \neq 0$ and $\nabla^v W = (\gamma^*\nabla^v)W \equiv 0$; these are obtained by parallel transporting the initial vector $W_0$ along $\gamma$ in $vB$; notice that the condition $\nabla^v W = 0$ implies that $W'(0) = (\gamma'(0), 0)$ is a horizontal vector in $T_{W(0)}SvB$ such that $d\pi(W'(0)) = \gamma'(0)$; one has $J(0) = \gamma'(0)$ and
\[ J'(0) := \nabla_{\partial_t} J|_{r=0} = \nabla_{\gamma'(0)} W := (\gamma^* \nabla)_{\partial_t} W|_{r=0} \] where \( \nabla \) is the Levi-Civita connection on \( M \). Since the \( 0 = \nabla^\nu_{\gamma'(0)} W = P^B \nabla_{\gamma'(0)} W \) it follows that \( J'(0) \) is a horizontal vector.

By what was just said one has:

(a) when \( W'_1(0), W'_2(0) \) are both horizontal:

\[
\begin{align*}
  f(0) &= g(J_1(0), J_2(0)) = g(\gamma'_1(0), \gamma'_2(0)) = g(W'_1(0), W'_2(0)), \\
  f'(0) &= \partial_{\gamma'_1} g(J_1(r), J_2(r))|_{r=0} \\
  &= g(\nabla_{\gamma'_1(0)} W_1, \gamma'_2(0)) + g(\gamma'_1(0), \nabla_{\gamma'_2(0)} W_2) = -2\Pi W_0(\gamma'_1(0), \gamma'_2(0)), \\
  f''(0) &= \partial_{\gamma'_1}^2 g(J_1(r), J_2(r))|_{r=0} \\
  &= \left[ g(J''_1(r), J_2(r)) + 2g(J'_1(r), J'_2(r)) + g(J_1(r), J''_2(r)) \right]|_{r=0} \\
  &= \left[ g(R^8(\partial_r, J_1(r))\partial_r, J_2(r)) + 2g(J'_1(r), J'_2(r)) + g(J_1(r), R^8(\partial_r, J_2(r))\partial_r) \right]|_{r=0} \tag{7.15}
\end{align*}
\]

where we used that \( J_1 \) and \( J_2 \) are Jacobi.

(b) when \( W'_1(0) \) is horizontal and \( W'_2(0) \) vertical:

\[
\begin{align*}
  f(0) &= g(J_1(0), J_2(0)) = 0, \\
  f'(0) &= g(J'_1(0), J_2(0)) = g(\gamma'_1(0), J'_2(0)) = 0, \\
  f''(0) &= 2g(J'_1(0), J'_2(0)) = 0.
\end{align*}
\]

In the last equality we used that in (7.15), \( J_2(0) = 0 \) and that \( J'(0) \) is horizontal while \( J''_2(0) \) is vertical.

(c) when \( W'_1(0) \) and \( W'_2(0) \) are both vertical:

\[
\begin{align*}
  f(0) &= 0 = f'(0), \\
  f''(0) &= 2g(J'_1(0), J'_2(0)) = 2g(W'_1(0), W'_2(0)), \\
  f'''(0) &= \partial_{\gamma_1'}^2 g(R^8(\partial_r, J_1(r))\partial_r, J_2(r)) + 2g(J'_1(r), J'_2(r)) + g(J_1(r), R^8(\partial_r, J_2(r))\partial_r)|_{r=0} = 0
\end{align*}
\]

because \( J_1(0) = J_2(0) = 0 \). Summarizing:

- for \( W_1, W_2 \in T_{W_0} S(vB) \) both horizontal,

\[
g_1(r)(W_1, W_2) = g(W_1, W_2) - 2r\Pi W_0(W_1, W_2) + O(r^2);
\]

- for \( W_1 \) horizontal and \( W_2 \) vertical, \( g_1(r)(W_1, W_2) = O(r^3) \);

- for \( W_1 \) and \( W_2 \) both vertical, \( g_1(r)(W_1, W_2) = r^2 g(W_1, W_2) + O(r^4) \).

Recall now that \( g_1^i(r)(W_1, W_2) = g_1(r)(P^H(W_1) + r^{-1} P^V(W_1), P^H(W_2) + r^{-1} P^V(W_2)) \). We get that

\[
g_1^i(r)(W_1, W_2) = g(W_1, W_2) - 2r\Pi W_0(P^H(W_1), P^H(W_2)) + O(r^2) = \hat{h}_1(r)^i(r)(W_1, W_2) + O(r^2)
\]
and this corresponds to a second-order perturbation of \( \hat{h}_1 \) according to Definition 7.6.

\[ \square \]

7.2 Gauss–Bonnet for Perturbations of Model Metrics

We will look at perturbation of second order (Definition 7.6) of canonical model edge degenerate metrics. We assume again that \( M \) is an edge manifold.

A canonical model edge degenerate metric \( h \) is uniquely determined by the following data

(a) a collar neighborhood \( U \supset \partial M \) with a diffeomorphism \( R : U \to (-\epsilon, 0] \times N \) that makes the obvious diagram commutative;

(b) an Ehresmann connection on \( \pi : \partial M = N \to B \);

(c) a metric \( g^V \) on \( \text{Ker} \, d\pi \);

(d) a metric \( g^B \) on \( B \).

Part (a) of the next result justifies (1.5).

**Theorem 7.10** (a) Let \( g \) be a first-order perturbation of a canonical model edge metric \( h \). Then \( \partial_r \in \Gamma(TM|_{\partial M}) \) is the exterior normal unit of \( \partial M \) with respect to \( g^\varphi \), the Pfaffian \( Pf^g \) is a smooth form on \( M \) and

\[ \lim_{r \to 0} \int_{[r] \times N} Pf^g = Pf(g^\varphi \varphi^{-1}|_{r=0}, \partial_r). \]  

Hence the following holds:

\[ (2\pi)^k \chi(M) = \int_M Pf^g - \int_B \left( \int_{\partial M/B} Pf(g^\varphi \varphi^{-1}|_{r=0}, \partial_r) \right). \]  

(b) Suppose that \( g \) is a second-order perturbation of a canonical model edge degenerate metric \( h \). Then

\[ \lim_{r \to 0} \int_{[r] \times N} Pf^g = \lim_{r \to 0} \int_{[r] \times N} Pf^h. \]  

Consequently, the Gauss–Bonnet formula of Theorem 1.2 holds verbatim where the odd Pfaffian form is associated to the degenerate metric \( h \).

**Proof** We use the notations of Sect. 7. One consequence of the definition of perturbation is that the bilinear form

\[ g^\varphi(\cdot, \cdot) = g(\varphi^{-1}(\cdot), \varphi^{-1}(\cdot)) \]

is a well-defined smooth metric on \( TM \). Due to the fact that \( g^\varphi|_{TM|_{\partial M}} = h^\varphi|_{TM|_{\partial M}} \), the vector \( \partial_r \) has norm 1 also for \( g^\varphi \) at \( r = 0 \).
Moreover if $\nabla^g$ is the Levi-Civita connection of $g$ away from $r = 0$, then $\varphi\nabla^g\varphi^{-1}$ is a $g^\varphi$-metric connection. As proved in Theorem 7.8 this connection is defined everywhere and therefore $\text{Pf}^g$ is smooth on $M$.

It is easy to check that if $\nabla^1$ and $\nabla^2$ are two $g$-metric compatible connections and $\varphi : E \to E$ is a bundle isometry where on the right one uses $g^\varphi$ then

$$\text{TPf}(\nabla^1, \nabla^2) = \text{TPf}(\varphi\nabla^1\varphi^{-1}, \varphi\nabla^2\varphi^{-1}).$$

This is the case for $E = TM|_{\{r\} \times N}$ with $r \not= 0$ and $\nabla^1 = d \oplus P\nabla^g P$ and $\nabla^2 = \nabla^g$ constructed as in Example 2.1, where $P$ is the $g$-orthogonal projection onto $T(\{r\} \times N) \subset E$. The fact that $\varphi\nabla^g\varphi^{-1}$ exists for all values of $r$ implies immediately that the limit in (7.16) exists.

Moreover the limit is entirely determined by $\varphi\nabla^g\varphi^{-1}|_{TM|_{\{r\} \times N}}$ and the orthogonal decomposition

$$TM|_{\partial M} = \mathbb{R}\partial r \oplus TN.$$ 

In fact, if $s_g(r)$ is the $g$-exterior unit normal to $\{r\} \times N$ then $\varphi(s_g(r))$ is the $g^\varphi$ unit normal and it is easy to check that $\varphi(s_g(r))$ is a parallel section with respect to $\varphi\nabla^1\varphi^{-1}$ and therefore $\varphi\nabla^1\varphi^{-1}$, being $g^\varphi$-compatible has a diagonal block-decomposition with respect to $TM|_{\{r\} \times N} \simeq \mathbb{R}\varphi(s_g(r)) \oplus T(\{r\} \times N)$ of type:

$$\varphi\nabla^1\varphi^{-1} = d \oplus P^\varphi \varphi\nabla^g\varphi^{-1} P^\varphi$$

where $P^\varphi$ is the $g^\varphi$-orthogonal projection onto $T(\{r\} \times N)$ and $\varphi$ takes the $g$ unit normal to $\{r\} \times N$ to the $g^\varphi$ unit normal to $\{r\} \times N$.

Therefore

$$\text{TPf}^g|_{\{r\} \times N} = \text{TPf}\left(\varphi\nabla^g\varphi^{-1}|_{\{r\} \times N}, \varphi(s_g(r))\right)$$

and one gets (7.16) when $r \to 0$.

For (b) recall that for second-order perturbations $\varphi\nabla^g\varphi^{-1}|_{r=0} = \varphi\nabla^h\varphi^{-1}|_{r=0}$. □

### 7.3 First Order Perturbations

One can obtain the following alternative form of Theorem 7.10 part (a), stated as Theorem 1.5 in the Introduction:

**Theorem 7.11** Let $g$ be a first-order perturbation of a model edge metric $h = dr^2 \oplus r^2 h^V \oplus \pi^* h^B$. Then

$$(2\pi)^k \chi(M) = \int_M \text{Pf}^g - \sum_{j=0}^{(f-1)/2} \bar{c}\left(\frac{f-1}{2} - j\right) \int_B \left(\text{Pf}(h^B) \int_{N/B} P_{j,f}(h^V)\right)$$

\[- \int_{\partial M} \text{TPf}(\nabla^h_1, \nabla^h_1)\]
where $\nabla^h_1 = \varphi \nabla^h \varphi^{-1}|_{r=0}$ is described in (7.10) and $\nabla^g_1 = \varphi \nabla^g \varphi^{-1}|_{r=0}$. The form $\text{Pf}(h^B)$ is zero, by definition, when $\dim B$ is odd.

**Proof** We use the notation of the proof of Theorem 7.10. As in Sect. 2.2, we consider the metric $ds^2 + (1 - s)h^\varphi + sg^\varphi$ on $[0, 1] \times U$, where $U$ is the collar. Parallel transport induces a bundle isometry

$$\tau^{-1}_1 : (TM|_U, g^\varphi) \rightarrow (TM|_U, h^\varphi).$$

While parallel transport $\tau^{-1}_1$ need not take $\varphi(s_g(r))$ to $\varphi(s_h(r))$ since at $r = 0$ $(\varphi(s_g(r))|_{r=0} = (\varphi s_h(r))|_{r=0} = \partial_r$ and $\tau_1|_{r=0} = \text{id}$ it is clear that for $r$ small one can find a smooth homotopy between $\tau^{-1}_1 \circ \varphi(s_g(r))$ and $\varphi(s_h(r))$ within $(S(TM), h^\varphi)$.

Then we can apply Proposition 2.7 to conclude that

$$\text{TPf}(\varphi \nabla^g \varphi^{-1}, \varphi(s_g(r))) - \text{TPf}(\varphi \nabla^h \varphi^{-1}, \varphi(s_h(r)))$$

$$= \text{TPf}(\tau^{-1}_1 \varphi \nabla^g \varphi^{-1} \tau_1, \tau^{-1}_1 (\varphi(s_g(r)))) - \text{TPf}(\varphi \nabla^h \varphi^{-1}, \varphi(s_h(r)))$$

$$= -\text{TPf}(\tau^{-1}_1 \varphi \nabla^g \varphi^{-1} \tau_1, \varphi \nabla^h \varphi^{-1})|_{[r] \times N} + d\gamma.$$

From here we deduce immediately that at $r = 0$

$$\int_{\partial M} \text{TPf}(\varphi \nabla^g \varphi^{-1}|_{r=0}, \partial_r) = \int_{\partial M} \text{TPf}(\varphi \nabla^h \varphi^{-1}|_{r=0}, \partial_r) + \int_{\partial M} \text{TPf}(\nabla^h_1, \nabla^g_1).$$

**Remark 7.12** For horizontal variations of the metric that are constant along the fiber, more precise computations are given in Theorem 5.9.

The sum in the previous Theorem also has an alternative characterization.

**Proposition 7.13** If $\nabla'$ is the connection from (7.2), then $\varphi \nabla' \varphi^{-1} = d \oplus \pi_2^* \nabla^V \oplus \pi_2^* \pi^* \nabla^B$ and

$$\sum_{j=0}^{(f-1)/2} \tilde{c} \left( \frac{f-1}{2} - j \right) \int_B \left( \text{Pf}(h^B) \int_{N/B} P_{j, f} \left( h^V \right) \right) = \int_{[0] \times N} \text{TPf}(\varphi \nabla' \varphi^{-1}, \varphi \nabla^h \varphi)(7.18)$$

**Proof** The first statement is a simple computation. The left hand side of (7.18) is equal, by the proof of Theorem 1.2 to

$$\lim_{r \to 0} \int_{[r] \times N} \text{TPf}(\nabla^h, s_h(r)) = \int_{\partial M} \text{TPf} \left( \varphi \nabla^h \varphi^{-1}|_{r=0}, \partial_r \right).$$

From (7.10) we see that at $r = 0$ the projection of $\varphi \nabla^h \varphi^{-1}$ on $TN$ gives the connection $\pi_2^* \nabla^V \oplus \pi_2^* \pi^* \nabla^B$, hence the induced block-diagonal connection from $\varphi \nabla^h \varphi^{-1}$ with $\varphi \nabla^g \varphi^{-1}$.
respect to the decomposition \( TM = \mathbb{R} \partial_r \oplus TN \) (as in Example 2.1) is exactly \( \varphi \nabla' \varphi^{-1} \). Hence, by definition:

\[
\operatorname{TPf} \left( \varphi \nabla' \varphi^{-1} \big|_{r=0}, \varphi \nabla^h \varphi^{-1} \big|_{r=0} \right) = \operatorname{TPf} \left( \varphi \nabla^h \varphi^{-1} \big|_{r=0}, \partial_r \right)
\]

\( \square \)

We use this result in order to give a more geometric expression to the boundary contribution for first-order perturbations of conical model metrics.

**Definition 7.14** Let \( g \) be a first-order perturbation of the conical metric

\[
h = dr^2 \oplus r^2 g^N
\]
on \((-\epsilon, 0] \times N\). Define the (asymptotic) second fundamental form \( \Pi^g \) of \( \partial M := \{0\} \times N \) as follows:

\[
\Pi^g(X, Y) := h^\phi \left( (\nabla^g_1)^X Y, \partial_r \right) = g^\phi \left( (\nabla^g_1)^X Y, \partial_r \right)
\]

where \( \nabla^g_1 = \varphi \nabla^g \varphi^{-1} \big|_{TM} \big|_{\partial M} \) is the connection resulting from Theorem 7.8.

Denote by \( R^N \) the curvature form of the metric \( g^N \) and set

\[
\mathcal{G}^\partial M_{j, 2k-1} := \frac{1}{j!(2k - 1 - 2j)!} \mathcal{B}_{h^\phi} \left( (R^N)^j \wedge (\Pi^g)^{2k-1-2j} \right).
\]

**Theorem 7.15** For first-order perturbations \( g \) of conical metrics \( dr^2 \oplus r^2 g^N \) the following holds

\[
(2\pi)^k \chi(M) = \int_M \operatorname{Pf}^g - \sum_{j=0}^{k-1} (-1)^j (2j - 1)!! \int_{\partial M} \mathcal{G}^\partial M_{k-1-j, 2k-1}.
\]

**Proof** Proposition 7.13 and Theorem 7.11 together say that the contribution of the boundary is

\[
\int_{\partial M} \operatorname{TPf}(\varphi \nabla' \varphi^{-1}, \varphi \nabla^h \varphi^{-1}) + \int_{\partial M} \operatorname{TPf}(\varphi \nabla^h \varphi^{-1}, \varphi \nabla^g \varphi^{-1}) = \int_{\partial M} \operatorname{TPf}(\varphi \nabla' \varphi^{-1}, \varphi \nabla^g \varphi^{-1}).
\]

In the conical case

\[
\varphi \nabla' \varphi^{-1} = d \oplus \pi_2^* \nabla^N.
\]

and these are also the block-diagonal components of \( \varphi \nabla^g \varphi^{-1} \) at \( r = 0 \). In order to justify this let us take another look at (7.13). When \( X \) is tangent to \( \partial M \) then since \( C^\varphi \) is the identity on \( \partial M \) we get that at \( r = 0 \) one has

\[
h^\phi \left( (\nabla^g_X C^\varphi)(Y'), Z' \right) = 0.
\]
On the other hand, \( \varphi^{-1}(Y') = r^{-1}Y' \) and \( \varphi(X) = rX \) for \( X, Y' \in \Gamma(TN) \). Then the factors \( r^{-1} \) and \( r \) cancel each other out and one has:

\[
h^\varphi \left( \left( \nabla^\varphi_{\varphi^{-1}(Y')} C^\varphi \right) (\varphi(X)), Z' \right) = 0.
\]

The last term of (7.13) is similar and therefore also vanishes. We conclude that for \( X \in T\partial M, Y', Z' \in T\partial M \)

\[
h^\varphi(\omega^\varphi(\varphi(X)) \varphi^{-1}(Y'), X') = 0.
\]

One sees easily that the same holds for \( Y', Z' = \partial_r \). This justifies the claim that the block diagonal components of \( \varphi \nabla^g \varphi^{-1} \) and \( \varphi \nabla^h \varphi^{-1} \) when restricted to \( r = 0 \) are the same. But the block diagonal components of \( \varphi \nabla^h \varphi^{-1} \) are the same as those of \( \varphi \nabla' \varphi^{-1} \). Finally, the off-diagonal components of \( \varphi \nabla^g \varphi^{-1} \) are precisely the components of \( \Pi^g \).

The situation is similar now to the proof of the Gauss–Bonnet formula 1.1, and the transgression \( TPf(\varphi \nabla' \varphi^{-1}, \varphi \nabla^g \varphi^{-1}) \) can be computed accordingly. \qed

8 Perturbations of Manifolds with Fibered Boundary

Recall that an end of a manifold with fibered boundary is modeled on \((1, \infty) \times N \) with the metric

\[
dr^2 \oplus g^V \oplus r^2\pi^*g^B
\]

It is convenient to set \( u = r^{-1} \). In the new coordinate, the metric on \( U^c = (0, 1) \times N \) is of type:

\[
h = (d(u^{-1}))^2 \oplus g^V \oplus u^{-2}\pi^*g^B.
\]

This leads us to consider, in the spirit of the previous section, the following endomorphism \( \varphi : \mathbb{R} \oplus TN|_U : \)

\[
\varphi(s, v, w) = (u^{-2}s, v, u^{-1}w), \quad s \in \mathbb{R}, \ v \in \pi^*_2VN, \ w \in \pi^*_2\pi^*TB
\]

where we use \( \partial_u \) as the coordinate vector on \( \mathbb{R} \). Then clearly

\[
h^\varphi(X', Y') := h(\varphi^{-1}(X'), \varphi^{-1}(Y'))
\]

extends to a smooth metric on \((-1, 0] \times N =: U \).

**Theorem 8.1** Let \( \nabla^h \) be the Levi-Civita connection of \( h \) on \( U^c \). Then \( \varphi \nabla^h \varphi^{-1} \) extends to a smooth connection on \( U \) compatible with \( h^\varphi \).
Proof The proof follows closely that of Theorem 7.1. The auxiliary connection $\nabla'$ is

$$\nabla' = \left[ d - \frac{2}{u} du \right] \oplus \pi_2^* \nabla^{VN} \oplus \left[ \left( \frac{\partial}{\partial u} - \frac{1}{u} \right) du + \pi_2^* \pi_2^* \nabla^B \right]$$

where $d$ is the trivial connection on $\mathbb{R}$ and $\nabla^{VN}$ is the projection of the Levi-Civita connection of a slice $u = \text{constant}$ to $VN$. We notice that

(a) $\varphi \nabla' \varphi^{-1}$ extends smoothly;
(b) $d - \frac{2}{u} du$ and $\pi_2^* \nabla^{VN}$ are the projections of the Levi-Civita connection $\nabla^h$ to $\mathbb{R}$ and to $\pi_2^* VN$ respectively.
(c) $\partial_u$ is orthogonal to the slices and the unit normal vector is $u^2 \partial_u$; the vector field $X = u^2 \partial_u$ satisfies the conditions of Lemma 3.1 and this allows the computation of the second fundamental form of the slices in the same vein we did before.

One then carefully analyzes the blocks of the 1-form $\varphi (\nabla^h - \nabla') \varphi^{-1}$ and sees that they extend as well.

Remark 8.2 One might prefer to work directly with the $r$ coordinate. In that case one first needs to turn $(1, \infty)$ into a manifold and this can be done via the unique chart $(-1, 0] \rightarrow (1, \infty)$ where $u \mapsto -1/u$ for $u \neq 0$ and $0 \rightarrow \infty$. Then the vector field that trivializes the tangent bundle of $(1, \infty)$ (using the standard coordinate of $(1, \infty)$ is $\tilde{\partial}_r := r^2 \partial_r$ which makes sense also at $\infty$ and corresponds to $\partial_u$. Consequently the metric on $(1, \infty) \times N$ in these coordinates can be written as $r^4 \tilde{\partial}_r^2 \oplus g^V \oplus r^2 g^B$ and $\varphi(s, v, w) = (r^2 s, v, rw)$, etc.

Definition 8.3 A perturbation of the model fibered boundary metric $h$ is a metric $g$ such that $g^\varphi$ extends smoothly to a metric on $TM|_U$ and

$$g^\varphi = h^\varphi + f(u) \alpha$$

for some smooth function $f$ on $(-1, 0]$ that vanishes at 0, and some smooth bilinear symmetric form $\alpha$ on $TM|_U$. It is called of first, respectively second order if $f(u) = O(u)$, respectively $f(u) = O(u^2)$.

Lemma 8.4 A perturbation of first, resp. second order for the metric $h = dr^2 \oplus r^2 g^N$ on $(1, \infty) \times N$ is a metric $g$ such that

$$g = h + O(r^{-1}) \cdot \gamma_N(r) dr^2 + O(1) \cdot (d r \otimes \beta_N(r) + \beta_N(r) \otimes dr) + O(r) \cdot \alpha_N(r)$$

respectively

$$g = h + O(r^{-2}) \cdot \gamma_N(r) dr^2 + O(r^{-1}) \cdot (d r \otimes \beta_N(r) + \beta_N(r) \otimes dr) + O(1) \cdot \alpha_N(r)$$

where $\gamma_N(r)$, $\beta_N(r) \in \Omega^1(N)$ and $\alpha_N(r) \in \Gamma^+(T^*N \otimes T^*N)$ are smooth families of 0 and 1-forms, resp. symmetric $(1, 1)$ double forms on $N$ which extend smoothly at $\infty$, i.e., when composed with $-1/u$ they extend smoothly to $u = 0$. 
Theorem 8.5 For a first-order perturbation g of h, the connection $\varphi \nabla^g \varphi^{-1}$ extends to a smooth connection, while for a second order perturbation the restriction of $\varphi \nabla^g \varphi^{-1}$ to $u = 0$ (or $r = \infty$) coincides with the restriction of $\varphi \nabla^h \varphi^{-1}$ to $u = 0$.

Proof Almost identical to Theorem 7.8. Notice that in formula (7.13), $\nabla^\varphi \varphi^{-1} (Y') C^\varphi$ makes sense at $u = 0$ as $\varphi^{-1} (s, v, w) = (u^2 s, v, u w)$ while $\nabla^\varphi$ and $C^\varphi$ extend by Theorem 8.1 and Definition 8.3 respectively.

Corollary 8.6 The Gauss–Bonnet formula of Theorem 1.3 holds for second-order perturbations of a metric with fibered boundary.

Example 8.7 A catenoid in $\mathbb{R}^3$ has the following parametrization

$$C = \{(\cosh (v) \theta, v) \in \mathbb{R}^3 \mid \theta \in S^1, \ v \in \mathbb{R}\}.$$ 

Use the change of coordinates $v = \arcsinh(r)$ in order to write the metric as

$$dr^2 + (1 + r^2)d\theta^2$$

where $\partial_\theta$ is the unit tangent vector on $S^1$ with the round metric. This is a second-order perturbation of the flat metric $dr^2 + r^2d\theta^2$. The catenoid is a minimal surface with two ends, its total Gaussian curvature is $-4\pi$, Euler characteristic 0, and each end contributes to the Gauss–Bonnet formula by 1, which is the integral of the odd Pfaffian $(2\pi)^{-1}\text{TPf}(S^1, g_{\text{round}}, 1)$.

The rest of the sections are concerned with applications.

9 Exact b-metrics

In Melrose et al. (1993), as a by-product of his b-Index Theorem, Melrose shows that for exact b-metrics (see Definition 2.8) on a manifold with boundary, Gauss–Bonnet Theorem holds with no correction term on the boundary (Lemma 9.2). We explain how this result fits in with the ideas of this paper.

Let $M$ be a manifold with boundary of dimension $2k$ with a fixed collar neighborhood $U \simeq (-\epsilon, 0] \times \partial M$. The model degenerate metrics we consider here are the exact b-metrics of type:

$$g := \frac{dr^2}{r^2} + \pi_2^* g_{\partial M}(r).$$

where $g_{\partial M} : (-\epsilon, 0] \to \partial M$ is a smooth family of metrics on $\partial M$ and $\pi_2 : (-\epsilon, 0] \times \partial M \to \partial M$. 

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Proposition 9.1  Gauss–Bonnet for metrics \( g \) of type \((9.1)\) holds in the following form:

\[
(2\pi)^k \chi(M) = \int_M \text{Pf}^g
\]

Proof  We let \( u := \ln(-r) \) and with this coordinate \( U \setminus \partial M \simeq (-\infty, \ln(\epsilon)) \times \partial M \).

The metric then on this infinite cylinder takes the form:

\[
du^2 + g_{\partial M}(-e^u)
\]

Using the fact that \( X := \partial_u \) is a gradient vector field we get by Lemma 3.1 that the \( II_u \), the second fundamental form of the slice \( u = \text{const} \) converges to 0 when \( u \to -\infty \) since \( L_X(g_{\partial M}(-e^u)) = -e^uh(u) \) with \( h \) bounded when \( u \to \infty \). Since the Riemannian curvature double form of the slice \( u = \text{const} \) converges to the corresponding double form of \( g_{\partial M}(0) \) we get that \( \int_{[u] \times \partial M} \text{Pf}^u \) converges to 0 where \( \text{Pf}^u \) is the transgression of the slice.

As far as the convergence of the Pfaffian is concerned, recall that the Levi-Civita connection of any exact \( b \)-metric extends to a connection on the \( b \)-tangent bundle (see Proposition 2.37 in Melrose et al. 1993) defined everywhere and then we are in the same situation as in the conical/edge case treated before.

Hence by the classical Gauss–Bonnet we get the claim when \( u \to \infty \).

Definition 9.2  A second order perturbation of a metric of type \((9.1)\) is a metric of type:

\[
\frac{dr^2}{r^2} + O(1)dr^2 + O(r)(dr \otimes \pi^*_2 \beta_{\partial M}(r) + \pi^*_2 \beta_{\partial M}(r) \otimes dr) + \pi^*_2 g_{\partial M}(r) + O(r^2)\pi^*_2 \alpha_{\partial M}(r)
\]

where \( \beta_{\partial M} \) and \( \alpha_{\partial M} \) are smooth families of 1, respectively (1, 1)-forms on \( \partial M \) up to \( r = 0 \).

Notice that the fixed collar neighborhood which is part of our initial data induces a splitting \( T(M|_{(-\epsilon,0) \times \partial M}) \simeq \mathbb{R} \oplus \pi^*_2 T\partial M \). We introduce the morphism \( \varphi \in \text{End}(T(M|_{(-\epsilon,0) \times \partial M})) \) which has the block matrix decomposition relative to the splitting

\[
\varphi := \begin{pmatrix} r^{-1} & 0 \\ 0 & \text{id}_{T\partial M} \end{pmatrix}, \quad \varphi^{-1} = \begin{pmatrix} r & 0 \\ 0 & \text{id}_{T\partial M} \end{pmatrix}
\]

Remark 9.3  The morphism \( \varphi^{-1} \) which is well-defined up to \( r = 0 \) is in a sense a concrete representation of the canonical morphism \( bTM \to TM \) whose image at \( r = 0 \) is \( T\partial M \). The bilinear form \( g^\varphi(\cdot, \cdot) := g(\varphi^{-1}(\cdot), \varphi^{-1}(\cdot)) \) is a non-degenerate metric on \( (-\epsilon, 0) \times \partial M \). In other words, \( g^\varphi \) is the genuine metric on \( bTM \) via the identification of \( bTM \) with \( TM \) induced by the collar diffeomorphism.

One can consider perturbations of \( g \) in the sense of Sect. 7. One notices immediately the following
Lemma 9.4  The notions of perturbations of g of second order in the sense of Definitions 9.2 and 7.6 coincide.

The fact that \( \phi \nabla g \phi^{-1} \) extends on \( \{0\} \times \partial M \), where \( \nabla g \) is the Levi-Civita connection induced by \( g \) on \((-\epsilon, 0) \times \partial M \) is the content of Proposition 2.37 in Melrose et al. (1993) and will not be repeated. We also notice the following

Proposition 9.5  The extensions of \( \phi \nabla g \phi^{-1} \) and \( \phi \nabla h \phi^{-1} \) where \( h \) is a second order perturbation of \( g \) coincide when restricted to \( T M|_{\{0\} \times \partial M} \).

Proof  The main idea of the proof of Theorem 7.8, namely relation (7.13) holds in this context as well. Therefore, using the same notation \( \nabla \phi := \phi \nabla g \phi^{-1} \), \( C^\phi \) uniquely identified by \( g^\phi(C^\phi \cdot, \cdot) = h^\phi \) we need only be concerned that the next limits are 0 for all \( X \) and \( Y' \):

\[
\lim_{r \to 0} \nabla_{X}^{\phi} C^{\phi} \quad \lim_{r \to 0} \left( \nabla_{\phi^{-1}(Y')} C^{\phi} \right) (\phi(X))
\]

where \( C^{\phi} = I + r^2 D \) for some smooth \( D \) all the way to \( r = 0 \). This can be checked immediately. \( \square \)

The definition of a general exact \( b \)-metric differs from a second order perturbation of a model metric in the sense that it does not impose the vanishing of the mixed terms when \( r = 0 \). However we still have the following.

Proposition 9.6  Every exact \( b \)-metric on \( M \) is a second order perturbation of a model metric of type (9.1).

Proof  Immediate corollary to the model form for exact \( b \)-metrics proved by Joshi (2000). \( \square \)

Now the proof of relation (7.17) from Theorem 7.10 goes through unaltered and we get:

Theorem 9.7  (Melrose) For an exact \( b \)-metric, Gauss–Bonnet holds in the form

\[
(2\pi)^k \chi(M) = \int_M \text{Pf}^g.
\]

10 Riemannian Orbifolds with Simple Singularities

Let \( M \) be a Riemannian manifold and suppose \( G \) is a finite group that acts by isometries on \( M \) such that the following properties are satisfied:

(i) \( \text{Fix}_G(M) \) is a (necessarily closed) submanifold of \( M \);
(ii) \( G \) acts freely on \( M \setminus \text{Fix}_G(M) \).

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The quotient $\hat{\mathcal{M}} := M/G$ is an example of a Riemannian orbifold. We use the following definition (see Borzelino 1992):

**Definition 10.1** A Riemannian orbifold $\hat{\mathcal{M}}$ is a Hausdorff topological space endowed with a countable basis of open charts $U_i$, closed under finite intersection such that each chart $U_i$ is homeomorphic with the quotient of an open set $\tilde{U}_i \subset \mathbb{R}^n$ endowed with a Riemannian metric $g_i$ (that turns $\tilde{U}_i$ into a geodesically convex set) modulo the action of a finite group $G_i$ that acts effectively by isometries on $\tilde{U}_i$. Moreover, the following data is part of the structure:

For each inclusion $\iota : U_i \subset U_j$ there exist

(i) an injective group morphism $\phi_{ij} : G_i \to G_j$;
(ii) an isometric embedding $\tilde{f}_{ij} : \tilde{U}_i \to \tilde{U}_j$, equivariant with respect to $\phi_{ij}$

fitting a commutative diagram

\[
\begin{array}{ccc}
\tilde{U}_i & \xrightarrow{\tilde{f}_{ij}} & \tilde{U}_j \\
\downarrow & & \downarrow \\
\tilde{U}_i/G_i & \xrightarrow{\phi_{ij}} & \tilde{U}_j/G_j \\
\end{array}
\]

Clearly, every open subset of an orbifold is an orbifold.

**Definition 10.2** Let $M$ and $N$ be two Riemannian orbifolds. Then a homeomorphism $f : M \to N$ is an isometry if it is a local isometry, i.e., if for every pair $m \in M$, $n = f(m) \in N$ there exist

(a) charts $m \in U \subset M$, $n \in D \subset N$ with corresponding open sets $\tilde{U} \subset \mathbb{R}^n$ and $\tilde{D} \subset \mathbb{R}^n$ and groups $G_U$ and $G_D$
(b) a group isomorphism $\phi : G_U \to G_D$, and
(c) an isometry $\tilde{f} : \tilde{U} \to \tilde{D}$ which is equivariant with respect to $\phi$

such that the next diagram commutes

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\tilde{f}/G_U} & U \\
\downarrow & & \downarrow \\
\tilde{D} & \xrightarrow{\tilde{f}/G_D} & D \\
\end{array}
\]

For every point $p \in M$, the isomorphism class of the isotropy group $G_p$ is unambiguously defined. In a chart $U_i \ni p$ the group $G_p$ is represented by the conjugacy class of the isotropy group of a lift $\tilde{p} \in \tilde{U}_i$.

**Definition 10.3** The singular locus $Z$ of an orbifold is:

\[ Z := \{ p \in M \mid G_p \neq \{e\} \}. \]
From the above definitions it is clear that \( \hat{M} \setminus Z \) inherits a Riemannian manifold structure and we denote the metric by \( g \). We will consider Riemannian orbifolds \( \hat{M} \) for which the singular strata have a “nice” structure.

**Definition 10.4** A Riemannian orbifold \( \hat{M} \) is called with simple singularities if each connected component \( Z_i \) of \( Z \) has the property that there exists

- an open neighborhood \( D_i \) of \( Z_i \),
- a finite group \( \Gamma_i \), and
- a Riemannian manifold \( M_i \)

such that

(i) \( \Gamma_i \) acts by isometries on \( M_i \), \( \text{Fix}_{\Gamma_i}(M_i) \) is a compact submanifold in \( M_i \) and \( \Gamma_i \) acts freely on \( M_i \setminus \text{Fix}_{\Gamma_i}(M_i) \);
(ii) There exists an isometry of Riemannian orbifolds \( h_i : D_i \to M_i / \Gamma_i \) such that

\[
    h_i(Z_i) = \text{Fix}_{G_i}(M_i).
\]

Any Riemannian orbifold with isolated singularities satisfies the previous definition. Denote by \( \text{Fix}(\hat{M}) \) the set of connected components of the singular locus \( Z \). The following is a particular case of Satake’s Gauss–Bonnet from Satake (1957).

**Theorem 10.5** Let \( \hat{M} \) be a compact Riemannian orbifold with simple singularities of dimension \( 2k \) and let \( g \) be the Riemannian metric on \( \hat{M} \setminus Z \). Then

\[
    \chi(\hat{M}) = \frac{1}{(2\pi)^k} \int_{\text{Int} \hat{M}} \text{Pf}^S + \sum_{Z_i \in \text{Fix}(\hat{M})} \chi(Z_i) \left| G_i \right|^{-1} - 1.
\]

**Proof** Fix a connected component \( Z \in \text{Fix}(\hat{M}) \) and let \( D \) be the neighborhood of \( Z \) from Definition 10.4 such that \( D \simeq M / \Gamma \). Let \( B := \text{Fix}_\Gamma(M) \). Since the action of \( \Gamma \) on \( M \) is via isometries in the induced action \( \Gamma \times TM \to TM \) via differentials, the subset \( S(vB) \) is invariant. Moreover, the action is free and linear in every fiber \( S(vB) \).

Now let \( \Gamma \) act trivially on \( (-\epsilon, 0] \). Then it is straightforward to see that

\[
    \exp : (-\epsilon, 0] \times S(vB) \to M, \quad (r, p, v) \mapsto \exp_p(rv)
\]

is a \( \Gamma \)-equivariant map since every isometry \( g \in \Gamma \) will take a geodesic with initial conditions \( (p, v) \) to a geodesic with initial conditions \( (gp, d_p g(v)) \).

It follows that we can find an (equivariant) tubular neighborhood for every \( Z_i \in \text{Fix}(\hat{M}) \) whose boundary is diffeomorphic to a quotient \( N = S(vB_i) / \Gamma_i \). By additivity of \( \chi \) (or Mayer–Vietoris),

\[
    \chi(\hat{M}) = \chi(\hat{M} \setminus Z) + \sum_{Z_i \in \text{Fix}(\hat{M})} \chi(Z_i) \quad (10.2)
\]

One applies Gauss–Bonnet for manifolds with boundary in the complement of these tubular neighborhoods in \( \hat{M} \) and then passes to limit \( r \to 0 \) in order to obtain a
formula for $\chi(\hat{M}\setminus Z)$. We can therefore restrict our attention to what happens in the neighborhood $D$ with the limits of the integrals of the transgression forms.

Recall now that the manifold $M \coloneqq (-\epsilon, 0] \times S(vB)$ has a model degenerate metric, left invariant by $\Gamma$ (it is obvious that $\Gamma$ leaves invariant the splitting $TS(vB) = VS(vB) \oplus HS(vB)$).

Assume first that the exponential map $\exp : D_\epsilon(vB) \to M$ is an isometry onto its image. Then the induced map:

$$\exp / \Gamma : (D_\epsilon(vB) / \Gamma) \setminus \{0\} \to (M / \Gamma) \setminus B$$

is an isometry onto its image where $\{0\}$ is the zero section of the disk bundle $D(vB)$.

Use Theorem 1.2, Examples 5.2 together with (10.2) in order to conclude that formula (10.1) holds in this case since the fiber-integral equals the integral over the Riemannian manifold $S(vB) / \Gamma$ of the integrand that appears in (5.3). That integrand is invariant under the action of rotations and therefore descends to $S(vB) / \Gamma$. The result of fiber integration is therefore $\frac{1}{|\Gamma|}$.

In the general case (without any assumption about the exponential map), by Theorem 7.9, the degenerate metric $g$ on $(-\epsilon, 0] \times S(vB)$ is a first-order perturbation of the degenerate model metric. It is easy to see that the transgression form on the slice $\{r\} \times S(vB) / \Gamma$, $r \neq 0$ pulls-back to the transgression form induced by $g$ on $\{r\} \times S(vB)$. Since $\Gamma$ acts freely, the map $\{r\} \times S(vB) \to \{r\} \times S(vB) / \Gamma$ is a covering with $|\Gamma|$ sheets. Therefore in the limit $r \to 0$, the integral that interests us amounts to $\frac{1}{|\Gamma|}$ of the corresponding integral over $S(vB)$. But the latter equals $\chi(B)$, by the concluding remarks of Example 5.2.

Remark 10.6 This formula fits with Gauss–Bonnet for $V$-manifolds as follows. Satake (1957) proved that

$$\int_{\operatorname{Int} \hat{M}} \operatorname{Pf}^g = \chi_V(\hat{M})$$

where $\chi_V(\hat{M})$ is the Euler–Satake characteristic of the orbifold\(^3\). Following Satake (1957), Gusein-Zade, Luengo and Mello-Fernandez define $\chi_V(\hat{M})$ in Gusein-Zade et al. (2018) as:

$$\chi_V(\hat{M}) = \sum_{G \in \mathcal{G}} \frac{1}{|G|} \chi(\hat{M}^G)$$

(10.3)

where $\mathcal{G}$ is the set of isomorphism classes of finite groups. In our case:

$$\chi_V(\hat{M}) = \chi(\operatorname{Int} \hat{M}) - \sum_{Z_i \in \operatorname{Fix}(\hat{M})} \frac{1}{|G_i|} \chi(Z_i)$$

and Theorem 10.5 follows by the additivity of $\chi$. Note that Satake’s Gauss–Bonnet does not assume the orbifolds to be simple. On the other hand, proving that (10.3)\(^3\) Not to be confused with the “physicists” orbifold Euler characteristic.
equals the $V$-total index of a $V$-vector field with isolated singularities needs a triangulability theorem (see for example Moerdijk and Pronk 1999, Proposition 1.2.1 and the references therein).

11 Other Applications

The Gauss–Bonnet formulæ proved here imply some global obstructions for the existence of flat cobordisms with prescribed ends of fibered boundary- or incomplete edge type. The simplest instance of such an obstruction arises for even-dimensional cones modeled by quotients of the round sphere, for instance lens spaces.

**Corollary 11.1** There do not exist flat metrics on a compact manifold with one cone singularity modeled on $\Gamma \setminus S^{2k-1}$ for a nontrivial group of isometries $\Gamma$ acting freely on the round sphere.

**Proof** When we remove a point from a smooth manifold $M$, the Euler characteristic decreases by 1, and this is reflected in the transgression form of Theorem 4.6 on the odd round sphere: the integral of this local transgression form must equal $(2\pi)^k$ (Remark 4.8). We deduce that on the quotient of $S^{2k-1}$ by a finite group of isometries $\Gamma$ acting freely, this transgression form integrates to $(2\pi)^k/|\Gamma|$. The Pfaffian form of a flat metric vanishes, hence $1/|\Gamma| = \chi(M) \in \mathbb{Z}$, thus $\Gamma$ must be trivial. $\square$

More generally, for edge metrics Theorems 1.2 and 7.10 imply some restrictions for the existence of a flat manifold bounding an edge singularity modeled on spherical fibrations:

**Corollary 11.2** Let $N \to B$ be a locally trivial fibration of closed manifolds with fiber type $F$. If there exists a compact flat Riemannian manifold $(M, g)$ bounding $N$ endowed with a second-order perturbation of a model edge metric (1.2) where all the fibers have constant sectional curvature 1, then the order of $\pi_1(F)$ must divide $\chi(B)$.

**Proof** Each fiber is isometric to the quotient of the round sphere by the free action of a finite group $\Gamma = \pi_1(F)$ of isometries of $S^{2k-1}$, hence by Remark 4.8 the integral of the transgression form along each fiber is constant equal to $(2\pi)^k/|\Gamma|$. The conclusion follows from this remark and from the Gauss–Bonnet formula of Theorem 1.2, which by Theorem 7.10 remains valid also for second-order perturbations of model edge singularities. Of course, the Pfaffian term vanishes by the flatness assumption on $g$. $\square$

Finally, exactly the same argument using Theorem 1.3 instead of Theorem 1.2 implies an obstruction for the existence of flat manifolds with fibered boundary ends:

**Corollary 11.3** Assume that $(M, g)$ is a flat manifold which near the boundary $N$ is a second-order perturbation of a fibered boundary metric modeled by a fibration $N \to B$, where $B$ is the quotient of the round sphere $S^{2h-1}$ by the free action of a finite group $\Gamma$ of isometries. Then the order of $\Gamma$ must divide $\chi(F)$.

The proof is identical to that of Corollary 11.2, applying Theorem 1.3 instead of Theorem 1.2.
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