A Modified Mohr Coulomb Criterion for Rocks with Smooth Tension Cut-off

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Abstract. In this study, in order to solve the limitation of the modified Mohr-Coulomb criterion in the numerical implementation induced by the corners or apexes, a modified Mohr-Coulomb criterion with smooth tension cut-off has been proposed. Both the nonlinear meridian shape function and the deviatoric shape function have been studied, and the failure surface of the proposed criterion is smooth and derivable at any stress states. The criterion has been verified through numerical simulations and comparison with the experimental results. Furthermore, it will be easier to adopt this smooth criterion in the numerical algorithm without losing good qualities of modified Mohr-Coulomb criterion.

1. Introduction
Due to its mathematical simplicity, clear physical meaning of the material parameters, and general level of acceptance, the modified Mohr-Coulomb criterion[1] has been widely applied in materials such as rocks, soils and concretes. However, limitations still exist in the numerical implementation of this criterion with apexes or corners comparing to these smooth criterion[2]. In order to overcome this disadvantage, lots of previous studies had been conducted. The smooth Drucker-Prager inner or outer circle had been used to replace the Mohr-Coulomb irregular hexagon[3]. “Round-off” method had been used in the vicinity of the vertices and it resulted in a failure surface which is continuous and differentiable for all stress values[4]. Without removing these singularities, a multi-surface method had been proposed in the numerical implementation[5]. It should be noted that some of these methods may lose the property of the modified Mohr-Coulomb criterion, and the others may be too complicated to apply in the numerical algorithm. Therefore, it is necessary to establish a simple, efficient and smooth modified Mohr-Coulomb criterion. Based on this purpose, a modified Mohr-Coulomb criterion with smooth tension cut-off has been proposed.

2. Background
Coulomb[2] had proposed a simple and effective criterion for the strength of the retaining wall:

\[ \tau = \sigma_N \cdot \tan \phi + c \]  

(1)

where \( \tau \) is the shear strength and \( \sigma_N \) is the normal stress on the failure plane. In this paper compression is taken as positive. In addition, \( c \) is known as the cohesion stress and \( \phi \) is the internal friction. The criterion can be expressed in the Mohr diagram as a straight line inclined to the \( \sigma_N \) axis.
with the angle $\varphi$. Through the trigonometric relation from the Mohr diagram, the alternative form[2] of the criterion by terms of the principal stress can be obtained:

$$f_{MC} = k\sigma_3 - \sigma_1 - \sigma_2 = 0, \quad \text{with} \quad k = (1 + \sin\varphi) / (1 - \sin\varphi), \quad \text{and} \quad \sigma_c = 2c\sqrt{k}$$

(2)

where $\sigma_1$ and $\sigma_3$ are the maximum and minimum principal stress, respectively. This simplest material strength relation is the Mohr-Coulomb criterion, which had been widely applied for the geo-material such as soils, sands, concretes or rocks. The shape of the Mohr-Coulomb criterion is a hexagonal pyramid in the principal space, and it is an irregular hexagon[2] on the $\pi$ plane perpendicular to the hydrostatic axis. It should be noted that the Mohr-Coulomb apex can be found on the $\pi$ plane.

For negative (tensile) values of the minor principal stress, experimental studies revealed that the tensile strength is smaller than the predicted value from the Mohr-Coulomb criterion. In order to solve this discrepancy, the Rankine or “tension cut-off” criterion had been introduced to the negative part:

$$f_r = \sigma_1 - \sigma_t$$

(3)

where $\sigma_t$ is the cut-off value, which can be treated as the tensile strength. Combining the Mohr-Coulomb and Rankine criterion, Equation (2) and (3) is usually referred to as the modified Mohr-Coulomb criterion. In principal stress space, the modified MC criterion with tension cut-off involves the MC pyramid intercepted by a second pyramid with three planes perpendicular to the principal stress axes. For the section through the $\sigma_1 - \sigma_2$ plane, the Mohr-Coulomb apex can also be found.

Due to the existence of the Mohr-Coulomb apex, the modified Mohr-Coulomb criterion is not smooth in the principal space, $\pi$ plane and $\sigma_1 - \sigma_2$ plane. It is difficult to apply this simple criterion in the numerical software as the stress update algorithm for the apex point is usually complicated and inaccurate[4-5]. If the smooth approximation is accepted, for instance, the Drucker-Prager out circle, several important characteristics like Lode effect will be lost.

3. Establishment of a modified Mohr Coulomb criterion with smooth tension cut-off

3.1. General framework

The modified Mohr-Coulomb criterion can be expressed in the invariant space:

$$F_{MC} = \frac{\sqrt{J_2}}{g_{MC}(\theta)} \left( \cos(\pi/6) - (1/\sqrt{3})\sin(\pi/6)\sin\varphi \right) - \frac{I_1}{3} \sin\varphi - c \cdot \cos\varphi = 0$$

(3.1)

$$g_{MC} = \frac{\cos(\pi/6) - (1/\sqrt{3})\sin(\pi/6)\sin\varphi}{\cos\theta + (1/\sqrt{3})\sin\theta\sin\varphi}$$

(3.2)

$$F_r = \frac{1}{\sqrt[3]{J_2}} \frac{\sqrt{J_2}}{g_{MC}(\theta)} - \frac{I_1}{3} - \sigma_t = 0$$

(3.3)

$$g_x = \frac{1}{2\sin(2\pi/3 - \theta)}$$

(3.4)

where $I_1$ is the first stress invariant value, $J_2$ is the second invariant value of the deviatoric stress and $\theta$ is the Lode angle. Equation (3) is the Zienkiewicz-Pande[6] form of the modified Mohr-Coulomb criterion, and $g_x$ and $g_{MC}$ are the shape function for the Rankine and Mohr-Coulomb criterion, respectively.

In this paper, combining the Rankine and Mohr-Coulomb criterion, the general framework for the modified Mohr Coulomb criterion with smooth tension cut-off is proposed as below:

$$F = f(\sqrt{J_2} / g(\theta), I_1)$$

(4)

The establishment for the meridian shape function $f(\sqrt{J_2} / g(\theta), I_1)$ and deviatoric shape function $g(\theta)$ will be discussed in the following two sections.
3.2. Nonlinear meridian shape function

The failure envelope is nonlinear and it can be described as hyperbolic in the meridian plane. To reflect the tensile failure strength a hyperbolic function of the failure criterion is presented, taking Equation (3.1) as the asymptote of the failure, as shown in Figure 1a. The nonlinear relation can be expressed as below:

\[
\begin{align*}
    f\left(\frac{J_2}{g(\theta)}, I_1\right) &= \sqrt{\frac{J_2}{g(\theta)}} - \sqrt{\frac{b^2 (I_1 + d)^2}{a^2} - b^2} \\
    d &= 3c \cdot \cot \phi \\
    b &= \sqrt{\frac{\sin^2 \phi (d - s)^2}{9A}} \\
    a &= 3bA \\
    A &= \cos(\pi/6) - \left(1/\sqrt{3}\right) \sin(\pi/6) \sin \phi
\end{align*}
\]  

(5.1) (5.2) (5.3) (5.4) (5.5)

where \(s\) is the material constant which can be determined by the uniaxial compression strength \(\sigma_t\). For the failure plane, the value of \(f\left(\frac{J_2}{g(\theta)}, I_1\right)\) is larger than zero. However, for Equation (5.1), as the hyperbolic has two branches, the value will be smaller than zero in the red shaded area of Figure 1b. In order to solve this problem, the nonlinear meridian shape function can be expressed in another form of the hyperbolic:

\[
\begin{align*}
    f\left(\frac{J_2}{g(\theta)}, I_1\right) &= |MP_1| - |MP_2| - 2D \\
    |MP_1| &= \sqrt{\left(\frac{J_2}{g(\theta)}\right)^2 + \left(I_1 + d + \sqrt{a^2 + b^2}\right)^2} \\
    |MP_2| &= \sqrt{\left(\frac{J_2}{g(\theta)}\right)^2 + \left(I_1 + d - \sqrt{a^2 + b^2}\right)^2}
\end{align*}
\]

(6.1) (6.2) (6.3)

\[D = a\]  

(6.4)

With the relation in Equation (6.1), only one branch (the red curve in Figure 1b) is considered, and the problem will be solved.

Figure 1. (a) Nonlinear meridian shape function and (b) another form of the hyperbolic function.

3.3. Deviatoric shape function

As shown in Figure 2a, the deviatoric shape of the Mohr-Coulomb criterion is the irregular hexagon, and the Rankine criterion is the regular triangle. One of the famous smooth deviatoric shape functions is the Willam-Warnke criterion[7] as shown in Figure 2a, which can be expressed as below:
The slope is a criterion coefficient, which is usually larger than 0.5 and smaller than 1.0. Furthermore, K is the ratio of between generalized tensile strength \( \sigma_1 = \sigma_2 \) to compressive strength \( \sigma_3 = \sigma_3 \). For the Willam-Warnke criterion, increasing \( K \) from 0.5 to 1.0, the deviatoric shape turns from the regular triangle to the circle in Figure 2b. If an appropriate \( K \) value is taken, the shape will be close to the Mohr-Coulomb criterion. This value can be calculated from Mohr-Coulomb criterion as below:

\[
K_{\text{mc}} = \frac{3 - \sin \varphi}{3 + \sin \varphi}
\]  

\[(8)\]

Figure 2. (a) The deviatoric shape of Mohr-Coulomb, Rankine and Willam-Warnke criterion and (b) The shape at different values of \( K \).

Similarly, the value for the Rankine criterion can be calculated as below:

\[
K_r = 0.5
\]  

\[(9)\]

As materials such as rock and concrete show pressure dependent strength properties, then the \( K \) value will depend on the value of \( I_1 \):

\[
K = R \frac{K_{\text{mc}} - K_r}{s} (I_1 + s) + K_r
\]  

\[(10)\]

where \( R \) is the material constant. Actually, Equation (10) is a transitional relations, and it means the deviatoric shape turns from the Rankine regular triangle to the Mohr-Coulomb irregular hexagon. When \( K \) reaches \( K_{\text{mc}} \), it keeps constant as \( K_{\text{mc}} \). \( R \) is the changing ratio, and the larger it is, the transform will be more violent as shown in Figure 3.

Figure 3. The deviatoric shape turns from the Rankine regular triangle to the Mohr-Coulomb irregular hexagon (with different \( R \) values).
3.4. Determination of the Parameter in the Criterion

Combining Equation (4), (6), (7) and (10), a modified Mohr Coulomb criterion with smooth tension cut-off can be established. The failure surface of the new criterion is shown in Figure 4, and it is smooth and derivable at any points. Except several traditional parameters such as the cohesion, friction angle and tensile strength, only one parameter $R$ need to be determined in the new criterion. After acquiring the traditional parameters, one can determine the parameter $R$ with several conventional triaxial compression strengths after inputting them into the criterion.

$$R = 0.5$$

$$R = 2$$

Figure 4. The failure surface of the proposed criterion with different $R$ values.

4. Validation of the Proposed Criterion

The failure criterion had been implemented in a finite software as a yielding criterion for an associate model. The geometric information and loading condition is shown in Figure 5. Two cases, including uniaxial compression (A of Figure 5) and uniaxial tension (B of Figure 5) are illustrated. The elastic modulus, Poisson’s ratio, cohesion, friction angle and tensile strength are 4.78GPa, 0.25, 15MPa, 30° and 5MPa. The result of plastic strain contour are also shown in Figure 5.

Figure 5. The geometric information and loading condition of plan A and B. Meanwhile, the results of A and B are also provided. Besides, $\varepsilon^P$ is the plastic strain.

The failure angle of Result A is about 60°, which equals with $45° + \phi / 2$, indicating the Mohr-Coulomb criterion is effective. Moreover, the crack is vertical to the tensile direction, indicating the Rankine criterion is also effective. Results in Figure 5 indicate that the validation of the proposed criterion.
Comparison between the test data and the theoretical result of the proposed criterion of Yunnan sandstone[8] has been conducted in Figure 6. The fitting coefficient is 99.65%, which indicates that the modified Mohr-Coulomb criterion with smooth tension cut-off matches well with the experimental data.

5. Conclusions
In this paper, a modified Mohr-Coulomb criterion with smooth tension cut-off has been proposed. Both the nonlinear meridian shape function and the deviatoric shape function have been studied, and the failure surface of the proposed criterion is smooth and derivable at any stress states. The proposed criterion has been verified through numerical simulating and comparison with the laboratory data, and good results have been acquired. Furthermore, it will be easier to adopt this smooth criterion in the numerical algorithm without losing good qualities of modified Mohr-Coulomb criterion.

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