Relativistic corrections to the Zeeman splitting of hyperfine structure levels in two-fermion bound-state systems

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Abstract

A relativistic theory of the Zeeman splitting of hyperfine levels in two-fermion systems is presented. The approach is based on the variational equation for bound states derived from quantum electrodynamics [1]. Relativistic corrections to the $g$-factor are obtained up to $O((\alpha)^2)$. Calculations are provided for all quantum states and for arbitrary fermionic mass ratio. In the one-body limit our calculations reproduce the formula for the $g$-factor (to $O((Z\alpha)^2)$) obtained from the Dirac equation. The results will be useful for comparison with high-precision measurements.

1. Introduction

In a recent paper [2] we have presented a self-consistent variational method for calculating the non-relativistic Landé $g$ factor of the two-fermion bound-state system. In the lowest-order approximation the linearly dependent part of the energy splitting for a two-fermion system placed in a weak static magnetic field $B$ can be written as [2-6]

$$\Delta E^\text{ext}_{J,m_J,S,\ell,s_1,s_2} = (\mu_{B1}g_1 + \mu_{B2}g_2) B m_J,$$

(1)

where $J$, $m_J$, $S$, $\ell$, $s_1$, $s_2$ are quantum numbers, which characterize the system: $s_1$ and $s_2$ are the spins of the first and second particle respectively, $S = s_1 + s_2$, $s_1 + s_2 - 1$, ..., $|s_1 - s_2|$ is the total spin of the particles, $\ell$ and $J$ represent the orbital and total angular momentum quantum numbers, where $J = \ell + S$, $\ell + S - 1$, ..., $|\ell - S|$. The projection of the total angular momentum on the $B$ field direction is labeled by $m_J = -J$, $-J+1$, ..., $J-1$, $J$. The “Bohr magnetons” for the two particles are defined as $\mu_{B1} = Q_1\hbar/2m_1c$, and $\mu_{B2} = -Q_2\hbar/2m_2c$, where $Q_1$ and $Q_2$ represent the magnitude of the charges. In our notation, $m_1$ and $m_2$ correspond to the masses of the light and heavy particle respectively. The description of the interacting system by the set of quantum numbers $J$, $m_J$, $S$, $\ell$, $s_1$, $s_2$ corresponds to the
LS coupling representation. This representation is used in contrast to the customary $j_1$-$j_2$ coupling scheme (for the case $m_2 >> m_1$), where the states are taken to be the eigenstates of the operators $\hat{J}_1^2 = (\hat{L} + \hat{s}_1)^2$, $\hat{J}_2^2 = \hat{s}_2^2$. As discussed previously [7], for the general case of arbitrary mass ratio, the $j_1$ value is not a good quantum number. Even in the LS representation, for spin-mixed states the orbital angular momentum $L$, and total spin $S$ of the system are not conserved. In this case we designate the states by an additional quantum number $\tilde{s}$, which takes on the values of 0 or 1 for quasisinglet ($sg_q$) and quasitriplet ($tr_q$) states respectively.

In our calculations we assume that the energy-level splitting (1) is smaller than the hyperfine structure (HFS) splitting, $\Delta E^{ext} << \Delta E^{HFS}$, i.e., we treat the interaction with an external magnetic field as a perturbation.

The non-relativistic Landé factors $g_1$ and $g_2$ obtained in [2] can be summarized as follows:

for $\ell = J - 1$:

$$g_i = 1 - \mu_i \frac{J - 1}{J} + \left( \frac{\tilde{g}_{s_i}}{2} - 1 \right) \frac{1}{J}, \quad (2)$$

for $\ell = J + 1$:

$$g_i = 1 - \mu_i \frac{J + 2}{J + 1} - \left( \frac{\tilde{g}_{s_i}}{2} - 1 \right) \frac{1}{J + 1}, \quad (3)$$

for spin-mixed states $\ell = J \neq 0$

$$g_i = \left( 1 - \frac{1 \mp \xi}{2J(J+1)} \right) \mu_j + \frac{\tilde{g}_{s_i}}{2} \left( \frac{1 \mp \xi}{2J(J+1)} \mp (-1)^j 2 |\mu_i - \mu_j| \xi \right), \quad (4)$$

where $i = 1, 2$ is the index designating the particle. The index $j$ is defined as $j = 1$ when $i = 2$, and $j = 2$ when $i = 1$. The quantities $\mu_i$ represent the mass factors:

$$\mu_i = \frac{m_i}{m_1 + m_2}, \quad (5)$$

and $\xi$ is given as

$$\xi = \left( 4 (\mu_1 - \mu_2)^2 J (J + J) + 1 \right)^{-1/2}. \quad (6)$$

The upper and the lower signs in (4) correspond to the quasisinglet $sg_q$ and quasitriplet $tr_q$ states respectively. Note that $\tilde{g}_{s_i}$ are the intrinsic spin magnetic moments of the constituent particles. According to the Dirac theory a free particle at rest has $\tilde{g}_{s_i} = 2$. In QED $\tilde{g}_{s_i}$ is modified by the “anomaly”, which to lowest order is given by the Schwinger correction $\tilde{g}_{s_i} = 2 + \alpha/\pi$ [7], where $\alpha$ is the fine-structure constant.

In the case when $m_2 >> m_1$ our general result reduces to the previously known result [3-6], in which the orbital motion of the heavy particle is ignored, namely

$$g_1 = g_{j_1} \frac{F(F+1) + j_1(j_1+1) - I(I+1)}{2F(F+1)}, \quad (7)$$
where
\[ g_{j_1} = 1 + (\bar{g}_{s_1} - 1) \frac{j_1(j_1 + 1) + s_1(s_1 + 1) - \ell(\ell + 1)}{2j_1(j_1 + 1)}, \] (8)
and
\[ g_2 = \bar{g}_{s_2} \frac{F(F+1) - j_1(j_1 + 1) + I(I + 1)}{2F(F+1)}. \] (9)

Here \( I \) is the spin of the second particle, and \( F \) is the total angular momentum of the system.

To facilitate the comparison of (4) with (7)-(9) we need to make the following replacement of quantum numbers: \( F \to J, \ J \to j_1, \ \ell_1 = \ell, \ S \to s_1, \ I \to s_2. \)

In this paper we present results for the relativistic case beyond the formulae (2)-(4). Thus, the \( g \)-factors in (1) are written more generally as
\[ g_i = g_{iNR} + \Delta g_{iREL}, \] (10)
where \( g_{iNR} \) are the nonrelativistic Landé factors defined by (2)-(4), and \( \Delta g_{iREL} \) is the relativistic correction. In the next section we calculate \( \Delta g_{iREL} \) to order \( O\left((\alpha')^2\right) \) for all quantum states and arbitrary masses of particles. In most expressions we use natural units, i.e., \( \hbar = c = 1, \ \alpha = e^2/4\pi. \) The coupling constant is defined as \( \alpha' = Q_1Q_2/4\pi. \)

2. Variational wave equation and relativistic corrections to \( g \)-factors of two-fermion systems

The relativistic wave equations for two-fermion systems in the absence of external fields were derived in [1] and [8] on the basis of a modified QED Lagrangian [9]-[10]. In this approach a simple Fock-space trial state of the form
\[ |\psi_{trial}\rangle = \sum_{s_1s_2} \int d^3p_1d^3p_2F_{s_1s_2}(p_1,p_2)b_{p_1s_1}^\dagger D_{p_2s_2}^\dagger |0\rangle, \] (11)
is sufficient to obtain the HFS levels correct to fourth order in the coupling constant \( \alpha' \). Here \( b_{p_1s_1}^\dagger \) and \( D_{p_2s_2}^\dagger \) are creation operators for a free fermion of mass \( m_1 \) and an (anti) fermion of mass \( m_2 \) respectively, and \( |0\rangle \) is the trial vacuum state such that \( b_{q_1s_1}|0\rangle = D_{q_2s_2}|0\rangle = 0. \) The \( F_{s_1s_2} \) are four-component adjustable functions.

The variational principle \( \delta \langle \psi_{trial}|\hat{H} - E|\psi_{trial}\rangle = 0, \) where \( \hat{H} \) is the QED Hamiltonian is invoked to obtain a momentum-space wave equation for the amplitudes \( F_{s_1s_2} \) [1]:
\[ 0 = \sum_{s_1s_2} \int d^3p_1d^3p_2 (\omega_{p_1} + \Omega_{p_2} - E) F_{s_1s_2}(p_1,p_2)\delta F_{s_1s_2}^*(p_1,p_2) \] (12)
\[ - \frac{m_1m_2}{(2\pi)^3} \sum_{\sigma_1\sigma_2s_1s_2} \int \frac{d^3p_1d^3p_2d^3q_1d^3q_2}{\sqrt{\omega_{p_1}\omega_{q_1}\Omega_{p_2}\Omega_{q_2}}} \]
\[ \times F_{\sigma_1\sigma_2}(q_1,q_2) (-i) \tilde{M}_{s_1s_2\sigma_1\sigma_2}(p_1,p_2;q_1,q_2) \delta F_{s_1s_2}^*(p_1,p_2), \]
where \( \omega_{p_1}^2 = p_1^2 + m_1^2 \) and \( \Omega_{p_1}^2 = p_1^2 + m_2^2 \). The inter-particle interaction is represented by the generalized invariant \( M \)-matrix,

\[
\widetilde{M}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2) = M^{(1)}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2) + M^{(2)}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2) + \ldots \tag{13}
\]

obtained as part of the derivation. It includes reducible and irreducible effects in all orders of the coupling constant \( \alpha' \), and the sum contains all relevant Feynman diagrams. A discussion of the derivation and structure of the \( M \)-matrix to one-loop level is provided in [1]. This equation allows one to obtain, in principle, all relativistic and QED corrections to the \( g \)-factor.

The lowest-order QED corrections appear within the term \( M^{(2)} \) and can be formally included in the intrinsic factor \( \varphi_{s_1} \), however we shall not do so in this work. In this paper we restrict our consideration to the first term \( M^{(1)} \) of the expansion (13), i.e., only tree level diagrams are included. The term \( M^{(1)} \) contains only relativistic corrections and it can be broken into two parts, namely

\[
M^{(1)}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2) = M^{\text{ope}}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2) + M^{\text{ext}}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2), \tag{14}
\]

where \( M^{\text{ope}}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2) \) is the usual invariant matrix element, corresponding to the one-photon exchange Feynman diagram [1]. The element \( M^{\text{ext}}_{s_1 s_2 \sigma_1 \sigma_2} \) represents the interaction with a given external classical field \( A^{\text{ext}} \),

\[
M^{\text{ext}}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2) = i \left( 2\pi \right)^{3/2} \left( \frac{\sqrt{\Omega_{p_2} \Omega_{q_2}}}{m_2} A^{\text{ext}}_{\mu}(p_1 - q_1) \gamma^{\mu} (p_1, s_1) (-i \Omega_1) \gamma^{\mu} u(q_1, \sigma_1) \delta_{s_2 \sigma_2} \right),
\]

Using the semi-relativistic expansion of the expression \( \gamma^{\mu} u(q_1, \sigma_1) \) up to order \( 1/c^3 \) we obtain for the \( M \)-matrix

\[
M^{\text{ext}}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2) = M^{(1)\text{ext}}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2) + M^{(2)\text{ext}}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2), \tag{16}
\]

where

\[
M^{(1)\text{ext}}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2) = \left( 2\pi \right)^{3/2} \frac{Q_1}{m_1} A_j^{\text{ext}} (p_1 - q_1) \varphi_{s_1}^\dagger \left( i (\sigma_1 \times (p_1 - q_1)) + q_1 + p_1 \right) \varphi_{s_1} \delta_{s_2 \sigma_2} \delta_3 (p_2 - q_2) \tag{17}
\]

is the non-relativistic contribution, and

\[
M^{(2)\text{ext}}_{s_1 s_2 \sigma_1 \sigma_2} (p_1, p_2, q_1, q_2) = \left( 2\pi \right)^{3/2} \frac{Q_2}{m_2} A_j^{\text{ext}} (q_2 - p_2) \lambda_{s_2} \left( i (\sigma_2 \times (p_2 - q_2)) + q_2 + p_2 \right) \delta_{s_1 \sigma_1} \delta_3 (p_1 - q_1) \tag{18}
\]
two categories of relations among the adjustable functions

For a stationary uniform magnetic field \( B = B \hat{z} \) the non-zero Fourier components of the vector potential are

\[
A_1^{\text{ext}}(k) = \frac{(2\pi)^{3/2}}{2} \frac{iB}{\Omega_p} \left( \delta(k_x) \frac{d\delta(k_y)}{dk_y} \delta(k_z) \right), \quad A_2^{\text{ext}}(k) = -\frac{(2\pi)^{3/2}}{2} \frac{iB}{\Omega_p} \left( \delta(k_x) \frac{d\delta(k_y)}{dk_y} \delta(k_z) \right). \quad (19)
\]

The trial state (11) is taken to be an eigenstate of total linear momentum \( \hat{P} \), total angular momentum squared \( \hat{\mathbf{J}}^2 \), its projection \( \hat{J}_3 \), parity \( \hat{P} \), and the Hamiltonian \( \hat{H} \), which corresponds to the hyperfine interaction [8]. In the rest frame, where the total linear momentum vanishes, the adjustable functions \( F_{s_1s_2}(\mathbf{p}_1, \mathbf{p}_2) = F_{s_1s_2}(\mathbf{p}_1) \delta(\mathbf{p}_1 + \mathbf{p}_2) \) can be specified for two categories of relations among the adjustable functions \( F_{s_1s_2}(\mathbf{p}) \):

(i) The spin-mixed (quasi-singlet and quasi-triplet) states

In this case we have \( \ell = J \), and the general solution under the condition of well-defined eigenvalues of \( \mathbf{P} \), \( \mathbf{J}^2 \), \( \hat{J}_3 \), and \( \hat{P} \) can be expressed as [1], [8]

\[
F_{s_1s_2}(\mathbf{p}) = C_{Jm_1Jm_2}^{(S_1)m_{s_1s_2}} \hat{f}_J^{(S_1)}(p) Y_J^{m_{s_1s_2}}(\hat{\mathbf{p}}) + C_{Jm_1Jm_2}^{(S_2)m_{s_1s_2}} \hat{f}_J^{(S_2)}(p) Y_J^{m_{s_1s_2}}(\hat{\mathbf{p}}), \quad (20)
\]

where \( m_{11} = 1 \), \( m_{12} = m_{21} = 0 \), \( m_{22} = -1 \). The \( C_{Jm_1Jm_2}^{(S_1)m_{s_1s_2}} = \langle \ell m_1Sm_2 | J m_1Jm_2 \rangle \) are the Clebsch-Gordan (CG) coefficients with total spin \( S \), where \( S = 0 \) (with index \( S_1 \)) for the singlet states and \( S = 1 \) (with index \( S_2 \)) for the triplet states respectively. Here \( \hat{f}_J^{(S_1)}(p) \) and \( \hat{f}_J^{(S_2)}(p) \) are radial functions to be determined. They represent the contributions of spin-singlet and spin-triplet states (the total spin \( S = 0, 1 \) is not conserved in general).

(ii) The \( \ell \)-mixed triplet states

These states occur for \( \ell_{1,2} = J \mp 1 \). Their radial decomposition can be written as

\[
F_{s_1s_2}(\mathbf{p}) = C_{Jm_1Jm_2}^{(\ell_1)m_{s_1s_2}} \hat{f}_{\ell_1}(p) Y_J^{m_{s_1s_2}}(\hat{\mathbf{p}}) + C_{Jm_1Jm_2}^{(\ell_2)m_{s_1s_2}} \hat{f}_{\ell_2}(p) Y_J^{m_{s_1s_2}}(\hat{\mathbf{p}}). \quad (21)
\]

Again, the \( C_{Jm_1Jm_2}^{(\ell_1)m_{s_1s_2}} = \langle \ell_{1,2}m_{\ell_{1,2}}Sm_2 | J m_1Jm_2 \rangle \) are CG coefficients. For these states the system is characterized by \( J \), \( m_1 \), \( m_2 \), and \( P = (-1)^J \). The orbital angular momentum \( \ell = \ell_{1,2} \) is not a good quantum number. Mixing of this type occurs only for states with principal quantum number \( n \geq 3 \).

From the variational principle we obtain a system of coupled radial equations expressed in matrix form as

\[
(\omega_p + \Omega_p - E) \mathbb{F} (p) = \frac{m_1m_2}{(2\pi)^3} \int \frac{q^2 dq}{\sqrt{\omega_p \omega_q \Omega_p \Omega_q}} \mathbb{K} (p, q) \mathbb{F} (q), \quad (22)
\]

where \( \omega_p^2 = p^2 + m_1^2 \) and \( \Omega_p^2 = p^2 + m_2^2 \), and \( q = |q| \). Here \( \mathbb{F} (p) \) is the two-component matrix of radial functions

\[
\mathbb{F} (p) = \begin{bmatrix} f_J^{(S_1)}(p) \\ f_J^{(S_2)}(p) \end{bmatrix}, \quad \begin{bmatrix} f_{\ell_1}(p) \\ f_{\ell_2}(p) \end{bmatrix} \quad (23)
\]
for spin-mixed and $\ell$-mixed states respectively. The kernel of this equation is the $2 \times 2$ matrix $[\mathbb{K}]_{ij} = \mathcal{K}_{ij}$, which has the following form

$$
\mathcal{K}_{ij} = -i \sum_{s_1 s_2 \sigma_1 \sigma_2} C_{J_m j_i}^{s_1 s_2 \sigma_1 \sigma_2} \int d\mathbf{q} d\mathbf{p} \ Y_{\ell_i}^{m_{s_1 s_2}}(\hat{p}) Y_{\ell_j}^{m_{s_2}}(\hat{q}) \times \left( M_{s_1 s_2 \sigma_1 \sigma_2}^{\text{ope}} \left( \mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2 \right) + M_{s_1 s_2 \sigma_1 \sigma_2}^{(1)\text{ext}} \left( \mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2 \right) + M_{s_1 s_2 \sigma_1 \sigma_2}^{(2)\text{ext}} \left( \mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_1, \mathbf{q}_2 \right) \right).
$$

(24)

Here the $C_{J_m j_i}^{s_1 s_2 \sigma_1 \sigma_2}$ are related to the CG coefficients by: $C_{J_m j_i}^{s_1 s_2 \sigma_1 \sigma_2} = C_{J_m j_i}^{(S_1) s_1 s_2} C_{J_m j_i}^{(S_1) m_{s_1 s_2}}$ and $C_{J_m j_i}^{s_1 s_2 \sigma_1 \sigma_2} = C_{J_m j_i}^{(\ell)_{s_1 s_2} \sigma_1 \sigma_2}$ for the spin- and $\ell$-mixed states respectively. For the spin-mixed states we should take $\ell_i \equiv \ell_j \equiv \ell$.

The solution of equation (22) with kernel (24) including only the first two terms with the $\mathcal{M}^{\text{ope}}$ and $\mathcal{M}^{(1)\text{ext}}$ matrices was discussed in [2]. This solution describes the Zeeman splitting of the HFS energy levels in the non-relativistic limit. These energy levels are given by formulae (1)-(4).

In order to obtain the Landé factors to order $O(\alpha'^2)$ we solve the radial equation (22) with the additional term $\mathcal{M}^{(2)\text{ext}}$ in the kernel (24), which is evaluated perturbatively. The energy eigenvalues can be calculated from the matrix equation (22) as follows:

$$
E \int p^2 dp \mathbb{F}^{\dagger} (p) \mathbb{F} (p) = \int p^2 dp \left( \omega_p + \Omega_p \right) \mathbb{F}^{\dagger} (p) \mathbb{F} (p) - \frac{m_1 m_2}{\pi} \int_0^\infty \frac{p^2 dp}{\sqrt{\omega_p \Omega_p}} \int_0^\infty \frac{q^2 dq}{\sqrt{\omega_q \Omega_q}} \mathbb{F}^{\dagger} (p) \mathbb{K} (p, q) \mathbb{F} (q) .
$$

(25)

In [2] we show that this system decouples for the spin-mixed states if the radial functions $f_j^{(S_1)}$ and $f_j^{(S_1)}$ are taken as $f_j^{(S_1)} = \sqrt{(1 \pm \xi)/2} f_j$ and $f_j^{(S_1)} = \mp \sqrt{(1 \pm \xi)/2} f_j$. Here $f_j \equiv f_{\ell}$ is a common radial function, the upper and lower signs correspond to $s_q$ and $t_r_j$ states respectively. The energy corrections for $\ell$-mixed states can also be calculated independently for $\ell = J - 1$ and $\ell = J + 1$ states with corresponding radial functions $f_{\ell = J \pm 1}$ (see [11]). We evaluate (25) perturbatively using hydrogen-like radial functions (non-relativistic Schrödinger form $f_{\ell} = f_{\ell = J \pm 1}^{\text{Sch}} (p)$) in momentum space [4]).

The calculations are straightforward, and yield the relativistic corrections to the $g$-factor for both particles of the system (the mass factors $\mu_i$ are defined by (5), indexes $i$ and $j$ are defined as in section 1).

For the triplet states $\ell = J - 1$,

$$
\Delta g_i^{\text{REL}} = -\frac{\mu_j^2}{2} \left( \mu_j + \frac{\mu_i}{J} - \frac{1}{2J + 1} \right) \left( \frac{\alpha'}{n} \right)^2 .
$$

(26)

For the triplet states $\ell = J + 1$,

$$
\Delta g_i^{\text{REL}} = -\frac{\mu_j^2}{2} \left( \mu_j - \frac{\mu_i}{J + 1} + \frac{1}{2J + 1} \right) \left( \frac{\alpha'}{n} \right)^2 .
$$

(27)
For the spin-mixed states \( l = J \),

\[
\triangle g_i^{REL} = -\mu_j^2 \left( \mu_j + \mu_i \frac{1 \mp \xi}{4J(J+1)} \right) \left( \frac{\alpha'}{n} \right)^2,
\]

(28)

where the upper and lower sign in (28) corresponds to \( sg_q \) and \( tr_q \) states respectively. For the spin-mixed states the parameter \( \xi \) is given in Eq. (6).

There are several works based on the Breit equation (e.g. [12]-[15]), where the relativistic corrections to the \( g \)-factor of two-fermion bound states were considered. Some of the calculations were obtained only for small mass ratio [12]; in others the orbital motion of the heavy particle was ignored [13]. These calculations did not take into account the mixed nature of the quantum states. We emphasize, that only mixed states diagonalize the Hamiltonian of HFS [2], [8]. Our result (26)-(28) is new and overcomes the above-mentioned shortcomings.

For the relatively simple case of the ground state \( 1S_{1/2} \) \((J = 1, \ P = -1)\) our results agree with the previous result of Hegstrom [14] and Grotch [15], namely \( \triangle g_i^{REL} = -\mu_j^2 \alpha'^2 / 3 \). Note that their definition of \( g_e \) based upon \( \Delta E = \mu B_1 g_e B m_s \), Eq. (12) of [14], differs by a factor of 2 from our definition (1).

We provide our result (26), (28) in numerical form for the lighter particle in atomic hydrogen (Table 1), muonium (Table 2), and muonic hydrogen (Table 3), for which \( \alpha' = \alpha \). We consider only states with principal quantum number \( n = 1, 2 \). For comparison, the nonrelativistic Landé factors \( g_1^{NR} \) ((2), (4)) with \( g_{s1} = 2 \) are also included in the tables. We used the following values for the mass ratios: \( m_e/m_p = 5.4461702 \times 10^{-4} \) and \( m_p/m_\mu = 8.8802433 \). The fine-structure constant is taken as \( \alpha = 7.2973523 \times 10^{-3} \).

Table 1. \( g \) factor of the electron \( e^- \) in atomic hydrogen for \( n = 1, 2 \) states.

| \( e^- \) | \( 1S_{1/2}(J=1) \) | \( 2S_{1/2}(J=1) \) | \( 2P_{1/2}(J=1) \) | \( 2P_{3/2}(J=1) \) | \( 2P_{3/2}(J=2) \) |
|----------|----------------|----------------|----------------|----------------|----------------|
| \( g_1^{NR} \) | 1.00000000 | 1.00000000 | 0.33305135 | 1.66613230 | 0.99972780 |
| \( \triangle g_i^{REL} \) | -0.0000177 | -0.0000044 | -0.0000133 | -0.0000133 | -0.0000053 |

Table 2. \( g \) factor of the electron \( e^- \) in muonium for \( n = 1, 2 \) states.

| \( \mu^+e^- \) | \( 1S_{1/2}(J=1) \) | \( 2S_{1/2}(J=1) \) | \( 2P_{1/2}(J=1) \) | \( 2P_{3/2}(J=1) \) | \( 2P_{3/2}(J=2) \) |
|----------|----------------|----------------|----------------|----------------|----------------|
| \( g_1^{NR} \) | 1.00000000 | 1.00000000 | 0.33085048 | 1.66193000 | 0.99759350 |
| \( \triangle g_i^{REL} \) | -0.0000176 | -0.0000044 | -0.0000131 | -0.0000131 | -0.0000053 |

Table 3. \( g \) factor of the antimuon \( \mu^- \) in muonic hydrogen for \( n = 1, 2 \) states.

| \( p^+\mu^- \) | \( 1S_{1/2}(J=1) \) | \( 2S_{1/2}(J=1) \) | \( 2P_{1/2}(J=1) \) | \( 2P_{3/2}(J=1) \) | \( 2P_{3/2}(J=2) \) |
|----------|----------------|----------------|----------------|----------------|----------------|
| \( g_1^{NR} \) | 1.00000000 | 1.00000000 | 0.28805620 | 1.56093737 | 0.94960333 |
| \( \triangle g_i^{REL} \) | -0.0000143 | -0.0000036 | -0.0000099 | -0.0000098 | -0.0000040 |
For the systems considered in Tables 1-3 the relativistic corrections for the heavier particle \( \Delta g^2_{REL} \) are negligible in comparison with \( \Delta g^1_{REL} \) due to the small mass factor \( \mu_1 \). The relativistic corrections shown in the tables are small. They would be higher for the corresponding states in high-\( Z \) ions. For a realistic comparison of the \( g \)-factor with experiment one would also need to calculate the QED corrections up to second order in \( \alpha \), that is the higher order matrix element \( \mathcal{M}^{(2)} \) of Eq. (14) would have to be included [3].

3. Relativistic corrections to the \( g \)-factor in the one-body (Dirac) limit

In this section we show the validity of our results for the \( g \) factor in the one-body limit. Note that the applicability of the formulae (2)-(4), (26)-(28) is restricted by the condition \( \Delta E^{ext} \ll \Delta E^{HFS} \) (i.e., a weak magnetic field \( B \)). To lowest order in \( \alpha \) the HFS energy splitting for all states [8] (for \( m_2 \gg m_1 \)) is given as \( \Delta E^{HFS} \approx \alpha^4 m_1 m_2 \). In the limit \( m_2 \to \infty \) the HFS disappears, and in this case, the condition \( \Delta E^{ext} \ll \Delta E^{HFS} \) can not be satisfied for a nonzero magnetic field.

In this case we need to go back to the original variational equation (12) and rewrite it in a form acceptable for the one-body limit. It is not difficult to show that with the trial state

\[
|\psi_{trial}\rangle = \sum_s \int d^3p F_s(p) b_s^\dagger |0\rangle ,
\]
equation (12) reduces to the integral equation

\[
0 = \sum_s \int d^3p \left( \omega_p - E \right) F_s(p) F_s^\dagger(p) - \frac{m}{(2\pi)^3} \sum_s \int \frac{d^3p d^3q}{\sqrt{\omega_p \omega_q}} F_s(q) \left(-i\right) \tilde{\mathcal{M}}_{ss'}(p, q) \delta F_{s'}^\dagger(p),
\]

where \( E \) is the total one-body energy. In analogy to the two-body case, the matrix \( \tilde{\mathcal{M}}_{ss'}(p, q) \) is made up of two parts (up to \( O(\alpha^4) \)). The first part corresponds to the one-photon exchange term \( \mathcal{M}^{\text{ope}}_{s_1s_2s_1s_2}(p_1, q_1, q_2) \) taken in the limit \( m_2 \to \infty \). This part describes the interaction of the particle with a static Coulomb potential \( \mathcal{M}^{\text{Coulomb}}_{s_1s_2s_1s_2}(p_1, q_2, q_1, q_2) \to \mathcal{M}^{\text{Coulomb}}_{ss'}[8] \). The second part represents the interaction with an external magnetic field, namely \( \mathcal{M}^{\text{ext}}_{ss'}(p, q) = \mathcal{M}^{(1)\text{ext}}_{ss'}(p, q) + \mathcal{M}^{(2)\text{ext}}_{ss'}(p, q) \), which can be obtained from (17)-(18)

\[
\mathcal{M}^{(1)\text{ext}}_{ss'}(p, q) = \frac{(2\pi)^{3/2}}{2mc} A_{j}(p - q) \varphi_s^\dagger(i(\sigma \times (p - q)) + q + p)_{j} \varphi_{s'}
\]

\[
\mathcal{M}^{(2)\text{ext}}_{ss'}(p, q) = \frac{(2\pi)^{3/2}}{16m^3c^3} A_{j}(p - q) \varphi_s^\dagger \left((p^2 - q^2)(q - i(\sigma \times q)) - (p^2 - q^2)(p + i(\sigma \times p))\right)_{j} \varphi_{s'}
\]

Further calculations require a classification of the states. The trial state (29) is taken to be an eigenstate of total angular momentum squared \( \hat{\mathbf{j}}^2 \) and its projection \( \hat{\mathbf{j}}_3 \). These conditions can be satisfied if the adjustable two-component functions \( F_s(p) \) are taken in the following form:
For states $\ell = j - 1/2$

$$F_1 (p) = f_{j-\frac{1}{2}} (p) C_{jm_j}^{\ell m_j} \frac{1}{\sqrt{2}} Y_{j-1/2} (\hat{p}), \quad F_2 (p) = f_{j-\frac{1}{2}} (p) C_{jm_j}^{\ell m_j} \frac{1}{\sqrt{2}} Y_{j+1/2} (\hat{p})$$

For states $\ell = j + 1/2$

$$F_1 (p) = f_{j+\frac{1}{2}} (p) C_{jm_j}^{\ell m_j} \frac{1}{\sqrt{2}} Y_{j-1/2} (\hat{p}), \quad F_2 (p) = f_{j+\frac{1}{2}} (p) C_{jm_j}^{\ell m_j} \frac{1}{\sqrt{2}} Y_{j+1/2} (\hat{p}),$$

where $C_{jm_j}^{\ell m_j}$ are the CG coefficients for $s = 1/2$, $m_s = \pm 1/2$.

After substitution of these formulae into (30) and completion of the variational procedure we obtain the radial equation

$$(\omega_p - E) f_\ell (p) = \frac{m}{(2\pi)^3} \int \frac{q^2 dq}{\sqrt{\omega_p \omega_q}} K_\ell (p, q) f_\ell (p),$$

where the kernel $K_\ell (p, q)$ is expressed through the matrix $\mathcal{M}$ and coefficients $C_{jm_j}^{\ell m_j} = C_{jm_j}^{\ell m_j} C_{jm_j}^{\ell m_j}$, namely

$$K_\ell (p, q) = -i \sum_{\sigma' \sigma} C_{jm_j}^{\ell m_j} \int dq d\hat{p} \left( \mathcal{M}_{\sigma' \sigma}^{\text{Coulomb}} + \mathcal{M}_{\sigma' \sigma}^{(1) \text{ext}} (p, q) + \mathcal{M}_{\sigma' \sigma}^{(2) \text{ext}} (p, q) \right) Y_{\ell m_s}^{m_s} (\hat{p}) Y_{\ell m_s}^{m_s} (\hat{q}),$$

with $\sigma' = 1, 2$, $\sigma = 1, 2$, and $m_{1,2} = m_j \mp 1/2$.

In the absence of an external magnetic field, equation (35) represents the relativistic radial equation in momentum space for a bound one-body system. As was shown in [8], the solution of the two-body equation (22) reduces to the solution of the one-body equation (35).

We now evaluate the contribution of the next two terms $\mathcal{M}_{\sigma' \sigma}^{(1) \text{ext}} (p, q)$ and $\mathcal{M}_{\sigma' \sigma}^{(2) \text{ext}} (p, q)$ of equations (31) and (32). The relevant energy $\Delta E_{\ell, m_j, \ell}^{\text{ext}}$ can be calculated perturbatively like in the two-body case (cf. Eq. (25)). We obtain

$$\Delta E_{\ell, m_j, \ell}^{\text{ext}} = \mu_B g_B B m_j,$$

where

$$g_D = g_L (\ell, j) + \Delta g_D^{\text{REL}} (n, j).$$

Here $g_L (\ell)$ is the usual result for the anomalous Zeeman effect, being the Landé $g$ factor, whose value is ([4]):

$$g_L (\ell, j) = \frac{2j + 1}{2\ell + 1}.$$  

The second term is the relativistic correction to the $g_D$ factor

$$\Delta g_D^{\text{REL}} (n, j) = -\frac{(2j + 1)^2}{8j (j + 1)} \left( \frac{\alpha'}{n} \right)^2.$$  

Formulae (39) and (40) coincide with the result of the expansion (up to $O (\alpha'^2)$) of the general formula for the $g$-factor obtained by Margenau [16] on the basis of the Dirac equation with $\alpha' = Z \alpha$. 

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4. Concluding remarks

We considered the relativistic theory of the Zeeman splitting and the $g$-factor of the hyperfine structure in the two-fermion system on the basis of variational relativistic equations derived from quantum electrodynamics [1]. Relativistic corrections to the $g$-factor beyond the non-relativistic formulae (2)-(4) (from Ref. [2]) were calculated up to order $O(\alpha'^2)$ for all quantum states, and are given in Eqs. (26)-(28). The $g$-factor corrections take into account the mixed nature of the states (spin-mixing and $\ell$-mixing), and the orbital motion of the heavy particle. They are obtained for arbitrary mass ratio and are symmetrical with respect to the masses of the constituent particles. For the ground state our result reduces to the well-known formula obtained some time ago by Hegstrom [14] and Grotch [15]. We show that in the one-body case the solution of the variational relativistic equation reproduces the result for the $g$-factor obtained from the Dirac equation.

In our calculations we assumed that the trial state (11) is an eigenstate of the total linear momentum operator $\hat{P}$. However this is only an approximation, because one can show that the operator $\hat{P}$ does not commute with the HFS Hamiltonian. To fix this problem we need to modify the trial state, or use an appropriate unitary transformation for the HFS Hamiltonian. The latter approach was discussed in the literature [12], [13], [17]. An analysis shows, that in our case the unitary transformation leads to the appearance of additional terms in the invariant $M$-matrix. This is a technically difficult problem which we postpone for the future.

We note that, in contrast to the Breit approach, the anomalous magnetic moment is not introduced from another calculation. As discussed in section 2 (below equation (13)) all QED effects are contained in the $M$-matrix. The anomalous magnetic moment will appear naturally in our calculations if we include the next term $M^{(2)}$ of the expansion of $M$-matrix in (13).

Concerning comparison with experiment we note that measurements so far appear to be restricted to states where the spin structure of the heavier particle can be ignored. This includes data for the $1S_{1/2}$ state in ions [18], [19]. The present work will be of practical importance when these measurements will be extended to all $n = 2$ (or higher-$n$) levels.

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