The \( N/D \) study on the singularity structure of \( \pi N \)
scattering amplitudes

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The \( N/D \) method is used to study the \( S_{11} \) channel low energy \( \pi N \) scattering amplitude. The input of left cuts are obtained from various phenomenological models. With the aid of the production representation, the total phase shifts can be decomposed into different contributions, and it further reveals that the existence of subthreshold resonance \( N^*(890) \) doesn’t depend on the details of the dynamical input. Additionally, it is found that there exist virtual states in partial waves, which are induced by the \( u \) channel nucleon exchanges. These virtual states accumulate at the end point of the \( u \) channel segment cut. The end point is hence the essential singularity of the full amplitude on the second sheet of complex \( s \) plane.
1. Introduction

The \( \pi N \) scatterings, as one of the most fundamental processes in hadron physics, has been researched systemically for decades. In a series of recent publications [1–3], however, it is suggested that there exists a sub-threshold \( 1/2^- \) nucleon resonance hidden in \( S_{11} \) channel of \( \pi N \) scatterings, with a pole mass \( \sqrt{s} = (0.895 \pm 0.081) - (0.164 \pm 0.023)i \) GeV. The result is obtained by using the production representation (PKU representation) for partial wave amplitudes [4, 5, 8–10]. The \( N^*(890) \) pole may also be generated from a K-matrix fit, though the method suffers from the existence of spurious poles [11]. Properties of \( N^*(890) \) are also investigated, such as its coupling to \( N \gamma \) and \( N \pi \) [12, 13]. It is found that its coupling to \( N \pi \) is considerably larger than that of the \( N^*(1535) \), while its coupling to \( N \gamma \) is comparable to that of the \( N^*(1535) \). These results on couplings look reasonable and are within expectations, hence providing further evidence on the existence of \( N^*(890) \).

This talk reviews how to directly find the pole in the \( S \) matrix element calculated from the N/D method, which comprises the Sec.II-IV. The Sec.V focuses on the essential singularity of full \( \pi N \) amplitude on the second sheet of complex \( s \) plane.

2. A brief introduction to N/D method

The N/D method is derived by Ref. [14] to restore the unitarity of the \( \pi \pi \) partial wave amplitudes. For a single channel scattering, the partial wave amplitudes can be written as:

\[
T(s) = N(s)/D(s) ,
\]

where \( D \) contains only the \( s \)-channel unitarity cut or the right hand cut \( R \), whereas \( N \) only contains the left hand cut (l.h.c.) or \( L \). In single channel approximation, one has:

\[
\text{Im}_R T(s) = \rho(s)|T(s)|^2 ,
\]

where:

\[
\rho(s) = \frac{\sqrt{(s-s_L)(s-s_R)}}{s} ,
\]

with \( s_L = (m_\pi - m_N)^2 \), \( s_R = (m_\pi + m_N)^2 \). This leads to following relations:

\[
\text{Im}_R D(s) = -\rho(s)N(s) ,
\]

\[
\text{Im}_L N(s) = \text{Im}_L[T(s)]D(s) .
\]

Then one can write the dispersion relations:

\[
D(s) = 1 - \frac{(s-s_0)}{\pi} \int_R \frac{N(s')}{s'(s'-s)} ds' ,
\]

\[
N(s) = N(s_0) + \frac{(s-s_0)}{\pi} \int_L \frac{\text{Im}_L[T(s')]D(s')}{s'(s'-s)} ds' .
\]

Noticing that when there appears circular cut in \( T \) on \( s \) plane, We just need to make the following substitution: \( \text{Im}_L \rightarrow \frac{\text{disc}_L}{2i} \) in Eq. (4) and Eq. (5).
To solve the Eq. (5) one may substitute D function into N function and get a integral equation about N function:

\[ N(s) = N(s_0) + \frac{\rho(s)}{\pi} \int_R \frac{\tilde{B}(s', s') \rho(s')N(s')}{(s' - s_0)(s' - s)} ds', \]

(6)

with:

\[ \tilde{B}(s', s) = \frac{s' - s}{2i\pi} \int_L \text{disc}_{\hat{s}}[T(\hat{s})] \frac{d\hat{s}}{(\hat{s} - s)(\hat{s} - s')} \]

(7)

and use the inverse matrix method to obtain a numerical solution.

After getting a numerical solution, the next step is to search for the pole on the second sheet, hence analytical continuation to the complex plane is necessary:

\[ D^{II}(s) = D^I(s) + 2i\rho(s)N(s), \quad N^{II}(s) = N^I(s). \]

(8)

3. The PKU representation

Elastic partial wave S matrix elements satisfy a production representation like follows:

\[ S = \prod_i S_i \times e^{2i\rho(s)f(s)}, \]

(9)

Detailed discussions on how to obtain Eq. (9) can be found in Refs. [4, 5], which is fully consistent with unitarity, analyticity and crossing symmetry [6, 7]. For ππ system or πK system, the "spectral" function f(s) reads:

\[ f(s) = \ln\left(\frac{S}{\prod_i S_i}\right)/(2i\rho(s)). \]

(10)

In πN system, however, the situation is different. There exists a point \( s_c = m_N^2 - m_n^4/(2m_N^2) \) in the segment cut \((c_L, c_R)\). The real part of the argument of logarithmic function is negative near the point, but the imaginary part will change sign when s crosses \( s_c \), which leads to that \( f(s) \) has to develop a discontinuity and hence a branch cut emerges crossing \( s_c \). So we redefine \( f(s) \) as the following:

\[ f(s) = \ln\left(\frac{S}{\prod_i S_i}\right)/(2i\rho(s)) - \frac{\pi}{2\tilde{\rho}(s)}. \]

(11)

where the function \( \tilde{\rho}(s) \) is the ‘deformed’ \( \rho(s) \) with its cut \( \rho(\mathbb{c}) \in [s_L, s_R] \), while the cut of the latter is defined on \((-\infty, s_L) \cup (s_R, \infty)\). Notice that \( \tilde{\rho}(s) \) and \( \rho(s) \) are identical in the physical region. In fact, for the same process, Eq. (10) is equal to Eq. (11) in the physical region, but their left cuts are different. The definition of Eq. (11) simplifies the \( f(s) \) cut structure, so using Eq. (11) can simplify PKU decomposition. At last, we emphasize that the S matrix only possesses physical cut and primordial left cuts.

4. Numerical calculations

In Ref. [15], The authors do several calculations with various \( \text{Im}_L T(s) \) as the input of \( N/D \) method. Different input lead the unitary amplitudes \( T(s) \) possessing different \( l.h.c. \), but in every calculation, one can find a pole on the second sheet. The details can be seen in the following.

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1 In fact, we need a cutoff \( \Lambda_R \) and we will set the \( \Lambda_R^2 = 1.48\text{GeV}^2 \) as well as \( s_0 = 1\text{GeV}^2 \) at the following if not specified. Introducing \( \Lambda_R \) means the \( R \) in Eq. (6) presents the interval \((s_R, \Lambda_R^2)\).
4.1 Toy model calculations

Firstly, let’s see a solvable example. Assuming $\text{Im} L T(s)$ is simulated by a set of Dirac $\delta$ functions, or equivalently:

$$ N(s) = \sum_i \frac{\gamma_i}{s - s_i}, \quad (12) $$

which is to be used in the first equation of Eq. (5). We (arbitrarily) choose Case I: one pole at $s_1 = 0$, and Case II: one pole at $s_1 = -m_N^2$, and fit to the “data” obtained from the solutions of Roy Steiner equations[16] by tuning the parameter $\gamma_1$, and search for poles on the $s$-plane. Both cases give a good fit to the data, and a sub-threshold pole emerges in each case with a location listed in Table 1.

| Case I | Case II |
|--------|---------|
| $s_1$ | 0       | $-m_N^2$      |
| $\gamma_1$ (GeV$^2$) | 0.79 | 1.34 |
| $\sqrt{s_{\text{pole}}}$ (GeV) | 0.810 – 0.125i | 0.788 – 0.185i |

Table 1: Subthreshold pole locations using input Eq. (12).

The phase shift and its PKU decomposition are plotted in Ref. [15], and it tells us the background cut ($l.h.c.$) contribution to the phase shift is concave and negative while the subthreshold resonance pole provides a positive and convex phase shift above threshold to counterbalance the former contribution, and the sum of the two reproduces the steadily rising phase shift data.

4.2 N/D calculations using pure $\chi$PT input

A very natural idea is that using the $\text{Im} L T$ obtained from $\chi$PT as the input of $N/D$. As mentioned in Ref. [15], however, the $\text{Im} L T$ encounters the problem:

$$ \text{Im} L T[O(p^n)](s \to 0) \sim \text{Con.} \times s^{-n-1/2}, \quad n > 1, \quad (13) $$

where $n$ is the chiral order. It’s in contradiction to Froissart bound, but according to the Regge model, when $s \to 0$, the partial wave amplitude should behave as:

$$ T \sim s^{-\alpha(0)}, \quad (14) $$

with $\alpha(0) \approx 0.19$. Though $\text{Im} L T$ of $\chi$PT encounters such a problem, there still exist two ways to avoid this singularity when use it as the input of $N/D$ method.

One way is like the Ref. [15], and let the auxiliary function $\tilde{B}(s', s)$ in eqs. (6) and (7) formally be written as $\tilde{B}(s', s) = T_L(s') - T_L(s)$, where $T_L$ is taken as the $O(p^1)$ partial wave amplitude plus the $O(p^2)$ part of $T^{\alpha=2}_{\text{PT}}$.

At $O(p^2)$ level there are four low energy constants (LECs) $c_i$ with $i = 1, \ldots, 4$. A good fit is obtained with $c_1 = -0.40, \ c_2 = 3.50, \ c_3 = -3.90, \ c_4 = 2.17, \ N(s_0) = 0.47$. Of cause, these LECs satisfy the positivity constraints[17]. The pole locates at:

$$ \sqrt{s} = (1.01 - 0.19i) \text{GeV}. \quad (15) $$

\[2\] The definition can be found in Ref. [15]
Another ways is to introduce intermediate function $\tilde{T}(s)$ and the partial wava amplitude $T(s) = \tilde{T}(s)/s^2$, where $\tilde{T}(s)$ can be written as:

$$\tilde{T}(s) = \frac{N(s)}{D(s)}, \quad (16)$$

According to unitary, the $N(s)$ and $D(s)$ functions should satisfy at this time:

$$\text{Im}_L[N(s)] = D(s)\text{Im}_L[\tilde{T}(s)] = D(s)s^2\text{Im}_L[T(s)],$$

$$\text{Im}_R[D(s)] = N(s)\text{Im}_R[T^{-1}(s)] = -\rho(s)\frac{N(s)}{s^2}. \quad (17)$$

Taking $N(s)$ into $D(s)$ function after writing dispersion relations for them, one can get an integral function about $D(s)$:

$$D_r(s) = \frac{1}{s - s_0} - a_0F(s, s_0) + \frac{1}{\pi} \int_L F(s, s')s^2\text{Im}_L[T(s')]D_r(s')ds' \quad (18)$$

with:

$$D_r = \frac{D(s)}{s - s_0} F(s, s_0) = \frac{1}{\pi} \int_R \frac{\rho(s')ds'}{s'^2(s' - s)(s' - s_0)}. \quad (19)$$

Now we need a cutoff \textsuperscript{3} in Eq. (19) and an extra cutoff $\Lambda_L(\Lambda_L^2 = 1\text{GeV}^2)$ in Eq. (18), which means the $L$ and $R$ denote the interval $(-\Lambda_L^2, s_L)$ and $(s_R, \Lambda_R^2)$, respectively. Then Eq. (18) is solved numerically on the interval $(-\Lambda_L^2, s_L)$. Again we fit the data using the subtraction constant $a_0$ and the LECs in the $O(p^2)\chi PT$ as fit parameters and get:

$$a_0 = 0.88, c_1 = -0.40, c_2 = 3.10, c_3 = -3.50, c_4 = 4.00. \quad (20)$$

One pole locates at:

$$\sqrt{s} = (0.91 - 0.2i)\text{GeV}, \quad (21)$$

even when we push the cutoff $\Lambda_R^2$ to infinity, $N^*(890)$ still exists:

$$\sqrt{s} = (0.93 - 0.21i)\text{GeV}. \quad (22)$$

4.3 N/D calculation using phenomenological models

The ill singularities at $s = 0$ in partial wave chiral amplitudes, as mentioned above, come at least partly from integrating out heavy degrees of freedom, such as: the $\rho$ exchanges and $1^\pm$ baryon exchange etc. Before integrating out heavy degrees of freedom, all these resonance exchange amplitudes contain singularity of $s^{-1/2}$ type at most when $s = 0$. This reveals the fact that partial wave projections and chiral expansions do not commute.

On the other hand, when we make an asymptotic expansion of heavy degrees of freedom exchange contributions to $T(S_{11})$ in the vicinity of $s = 0$, we find that the first two most singular terms are of type:

$$\frac{a + bs}{\sqrt{s}}, \quad (23)$$

\textsuperscript{3}For example, Eq. 20 is obtained by fixing $\Lambda_R^2 = 5\text{GeV}^2$. 

which pushes us to use $O(p^1)$ ChPT results plus the $\rho$ meson exchange term and a polynomial as the input of $N/D$ method:

$$\text{disc } T(s) = \text{disc } T^{(1)} + \text{disc } T^{(\rho)} + \text{disc } \left[ \frac{a + bs}{\sqrt s} \right].$$ (24)

where the $\rho$ meson exchange term can be in charge of part the contributions of circular cut. The fit gives $N(s_0) = 0.61$, $a = -7.88 \text{GeV}$, $b = -8.00 \text{GeV}^{-1}$. Finally, one second sheet pole is found located at:

$$\sqrt{s} = (0.90 - 0.20i) \text{GeV}.$$ (25)

At the end of this section, it should be emphasized that the PKU decomposition is done for every calculation and the results are similar: the background cut contribution to the phase shift is negative while the pole gives a positive and phase shift, and the sum of the two reproduces the phase shift data. These calculations also tell us the existence of $N^*(890)$ dose not depend on the model details.

5. Essential Singularities of $\pi N$ Scattering amplitudes

It is pointed out that in the $L_{2I,2J} = S_{11}$ channel partial wave $\pi N$ scattering amplitude, there exist two virtual poles, according to the Ref. [15]. In fact, the existence of virtual poles are quite universal, but different channels behaves rather differently. For simplicity, definitions of notation are ignored there, and one can find them in Ref. [18].

The relations between partial wave helicity amplitudes and invariant amplitudes function $A^I$ as well as $B^I$ (isospin) are expressed as following:

$$T_{++}^I(s) = \frac{1}{64\pi} [2m_N A^I_C(s) + (s - m_N^2 - m_N^2) B^I_C(s)]$$

$$T_{+-}^I(s) = -\frac{1}{64\pi \sqrt{s}} [(s - m_N^2 + m_N^2) A^I_S(s) + m_N (s + m_N^2 - m_N^2) B^I_S(s)]$$

with

$$F^I_C(s) = \int_{-1}^{1} dz_s F^I (s, t) [P_{J+1/2}(z_s) + P_{J-1/2}(z_s)] ;$$

$$F^I_S(s) = \int_{-1}^{1} dz_s F^I (s, t) [P_{J+1/2}(z_s) - P_{J-1/2}(z_s)],$$

where $F$ stands for $A$ or $B$. Meanwhile, the scalar function $A^I$ and $B^I$ can be calculated by ChPT. Here, we presented the results of $O(p^1)$

$$A^{1/2}(s, u) = A^{3/2}(s, u) = \frac{m_N g^2}{F^2},$$

$$B^{1/2}(s, u) = \frac{1 - g^2}{F^2} - \frac{3m_N^2 g^2}{F^2 (s - m_N^2)} - \frac{m_N g^2}{u - m_N^2}$$

$$B^{3/2}(s, u) = \frac{g^2 - 1}{F^2} + \frac{2m_N^2 g^2}{F^2} \frac{1}{u - m_N^2}.$$ (27)
where $F$ and $g$ denote the pion decay constant and the axial coupling constant, respectively. Here we however only need to concern the $1/(u - m_N^2)$ term, which coefficient is immune of any chiral corrections, and its sign determines the existence of the zeros (poles) of the $S$-matrix. Taking Eq. 27 into Eq. 26, and using the Neumann equation:

$$Q_I(x) = \frac{1}{2} \int_{-1}^{1} \frac{P_I(y)}{x - y} dy , \quad (28)$$

where $Q_I$ is called the second Legendre function, one can get

$$B_{C,S}^{1/2,J} = -\frac{m_N^2 g^2}{16 \pi F^2 s \rho(s)^2} \left[ Q_{J+1/2}(y(s)) \pm Q_{J-1/2}(y(s)) \right] + \ldots , \quad (29)$$

$$B_{C,S}^{3/2,J} = -\frac{m_N^2 g^2}{8 \pi F^2 s \rho(s)^2} \left[ Q_{J+1/2}(y(s)) \pm Q_{J-1/2}(y(s)) \right] + \ldots , \quad (29)$$

in which the neglected terms . . . denote the parts regular at $c_L = (m_N^2 - m_\pi^2)/m_N^2$ and $c_R = m_N^2 + 2m_\pi^2$ (see discussions below), which receive chiral corrections. The definition of $y(s)$ is

$$y(s) = \frac{2m_\pi^2 s - s^2 + s_L s_R}{(s - s_L)(s - s_R)} . \quad (30)$$

Using the explicit expressions of $Q_I(s)$

$$Q_0(x) = \frac{1}{2} \ln \frac{x + 1}{x - 1} , \quad (31)$$

$$Q_I(x) = \frac{1}{2} P_I(x) \ln \frac{x + 1}{x - 1} - O_{I-1}(x) , \quad (31)$$

with

$$O_{I-1}(x) = \sum_{r=0}^{[\frac{l-1}{2}]} \frac{(2l - 4r - 1)}{(2r + 1)(l - r)} P_{l-2r-1}(x) . \quad (32)$$

One can find that the function $B_{C,S}^{1/2,J}(s)$ has the term

$$\ln \frac{y(s) + 1}{y(s) - 1} = \ln \frac{m_N^2}{s} + \ln \frac{s - c_L}{s - c_R} . \quad (33)$$

This term leads to that there exist three branch points at $s = 0, c_L, c_R$ in function $B_{C,S}^{1/2,J}$, and then in parity eigenstates amplitudes $T_{\pm}^{1/2,J}$. When $s$ tends to $c_R$, it's easy to find that

$$s \to c_R, \quad T_{\pm}^{1/2,J} \to \frac{g^2 m_N^2 (m_N^2 + 2m_\pi^2)}{16 \pi F^2 (4m_N^2 - m_\pi^2)} \ln \frac{s - c_L}{s - c_R} \to \infty . \quad (34)$$

$$s \to c_R, \quad T_{\pm}^{3/2,J} \to \frac{g^2 m_N^2 (m_N^2 + 2m_\pi^2)}{8 \pi F^2 (4m_N^2 - m_\pi^2)} \ln \frac{s - c_L}{s - c_R} \to -\infty . \quad (34)$$

The sign is independent of the angular momentum $J$. Things will be different, however, when $s$ tends to $c_L$. It turns out that $T_{\pm}^{1/2,J}$ tends to $\mp(1)^{J+1/2} \infty$, and $T_{\pm}^{3/2,J}$ tends to $\pm(1)^{J+1/2} \infty$. Taken the definition of $S$-matrix and the fact that $2i\bar{\rho}(s)$ is negative at $s = c_R, c_L$ into consideration, conclusions can be draw that:
1) In the region \( s \in (c_R, s_R) \), the S-matrix \( S_{\pm}^{1/2, J}(s) \) must exist a zero point, which corresponds to a virtual state.

2) In the region \( s \in (s_L, c_L) \), the S-matrix \( S_{\pm}^{1/2, J}(s) \) and \( S_{\pm}^{3/2, J}(s) \) exist zero points for \( J = 1/2, 5/2, 9/2, \ldots \); the S-matrix \( S_{\pm}^{1/2, J}(s) \) and \( S_{\pm}^{3/2, J}(s) \) exist zero points for \( J = 3/2, 7/2, 11/2, \ldots \).

The following discussions dedicate to determining the location of those zeros. Let us focus on \( I = 1/2 \) for the moment. The explicit expressions of S-matrix reads

\[
S_{\pm}^{1/2, J}(s) = A_{\pm}^{1/2, J}(s) + B_{\pm}^{1/2, J}(s) \ln \frac{s - c_L}{s - c_R}, \quad J \geq \frac{3}{2},
\]

with

\[
A_{\pm}^{1/2, J}(s) = 1 - \frac{im_\pi^2 g^2}{4\pi F^2 s \rho(s)} \left\{ (W \pm m_N)(E_N \pm m_N) \left[ \frac{P_{J+1/2}(y)}{2} \ln \frac{m_N^2}{s} - O_{J-1/2}(y) \right] + (W \mp m_N)(E_N \pm m_N) \left[ \frac{P_{J-1/2}(y)}{2} \ln \frac{m_N^2}{s} - O_{J-3/2}(y) \right] \right\}, \quad (36)
\]

where \( W \equiv \sqrt{s} \) denotes the center-of-mass frame energy and \( E_N = \frac{s + m_N^2 - m_N^2}{2\sqrt{s}} \) is the nucleon energy. It’s worth stressing that the function \( B_{\pm}^{1/2, J}(s) \) is immune of chiral perturbation corrections.

Let \( v_{R, \pm}^J \in (c_R, s_R) \) is the zero point of \( S_{\pm}^{1/2, J}(s) \), which gives

\[
S_{\pm}^{1/2, J}(v_{R, \pm}^J) = A_{\pm}^{1/2, J}(v_{R, \pm}^J) + B_{\pm}^{1/2, J}(v_{R, \pm}^J) \ln \frac{v_{R, \pm}^J - c_L}{v_{R, \pm}^J - c_R} = 0. \quad (37)
\]

Supposing \( v_{R, \pm}^J \approx c_R \) when \( J \) greater than some \( J_N \), the solution of Eq. (37) reads

\[
v_{R, \pm}^J = c_R + (c_R - c_L)e^{A_{\pm}^{1/2, J}(c_R)/B_{\pm}^{1/2, J}(c_R)} . \quad (38)
\]

Using \( P_{1}(y(c_R)) = P_{1}(1) = 1 \), one can find that \( B_{\pm}^{1/2, J}(c_R) \) is negative and independent of \( J \). The function \( A_{\pm}^{1/2, J}(c_R) \) increases as \( J \) increases, since \( O_{J+1/2}(1) - O_{J-1/2}(1) \) is greater than zero according to Eq. (32). Further, when \( J \) tends \( \infty \), the \( A_{\pm}^{1/2, J}(c_R) \) also goes to the infinity, which can be seen through

\[
\lim_{n \to \infty} O_n(1) = \lim_{k \to \infty} O_{2k}(1) = \lim_{k \to \infty} \sum_{r=0}^{k} \frac{(4k - 4r + 1)}{(2r + 1)(2k + 1 - r)} = \lim_{k \to \infty} \sum_{r=0}^{k} \frac{2}{2r + 1} - \frac{1}{2k + 1 - r} \frac{1}{2k + 1} = \lim_{k \to \infty} \sum_{r=0}^{k} \frac{2}{2r + 1} - \frac{1}{k + 1} \to \infty . \quad (39)
\]

It means the zero point, \( v_{R, \pm} \), tends \( c_R \) with the \( J \) increasing so that \( v_{R, \pm} = c_R \) when \( J \to \infty \) according to Eq. (38). The zero point of partial wave S-matrix on the first sheet will become the pole
of partial wave amplitude on the second sheet according to the analytic continuation:

\[ T_{II}^R(s) = \frac{T(s)}{S(s)}. \]  

(40)

Moreover, full amplitudes on the second sheet can be written as:

\[ T_{II}^R(s, t) = 8\pi \sum_{J=1/2} (2J + 1) \left[ \frac{T_+^J(s)}{S_+^J(s)} \pm \frac{T_-^J(s)}{S_-^J(s)} \right] d_J^{1/2, \pm 1/2}(\cos \theta). \]  

(41)

Now one can find that the full amplitude \( T_{II}^R \) possesses infinite poles on a finite interval, so these poles will accumulate and \( c_R \) is the accumulation point on the second sheet. Similarly analyses can be made in the situation in the line segment \( (s_L, c_L) \). Further, one finds that both \( v_{L+}^{1/2, J} \) and \( v_{L-}^{1/2, J} \) approaches \( c_L \) when \( J \to \infty \). Therefore, following the steps of Ref. [19], we prove that \( s = c_L, c_R \) are two essential singularities of \( T^{1/2}(s, t) \), on the second sheet of complex \( s \)-plane. Unlike Ref. [19], where the proof is fully non-perturbative, our proof given here is valid to all orders of perturbation chiral expansions. As for \( I = 3/2 \), one can prove that only \( c_L \) is an essential singularity of \( T^{3/2}(s, t) \) on the second sheet of \( s \)-plane.

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