Infinite-dimensional meta-conformal Lie algebras in one and two spatial dimensions

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Abstract. Meta-conformal transformations are constructed as sets of time-space transformations which are not angle-preserving but contain time- and space translations, time-space dilatations with dynamical exponent \( z = 1 \) and whose Lie algebras contain conformal Lie algebras as sub-algebras. They act as dynamical symmetries of the linear transport equation in \( d \) spatial dimensions. For \( d = 1 \) spatial dimensions, meta-conformal transformations constitute new representations of the conformal Lie algebras, while for \( d \neq 1 \) their algebraic structure is different. Infinite-dimensional Lie algebras of meta-conformal transformations are explicitly constructed for \( d = 1 \) and \( d = 2 \) and they are shown to be isomorphic to the direct sum of either two or three centre-less Virasoro algebras, respectively. The form of co-variant two-point correlators is derived. An application to the directed Glauber–Ising chain with spatially long-ranged initial conditions is described.

Keywords: conformal field theory, correlation functions, Boltzmann equation, dynamical processes

* In memoriam Vladimir Rittenberg.
1. Introduction

Conformal invariance has found many brilliant applications, for example to string theory and high-energy physics [78], or to two-dimensional phase transitions [11, 37, 54, 80] the quantum Hall effect [20, 50], or certain stochastic processes [2–4, 27, 71, 81]. These applications are based on a geometric definition of conformal transformations, considered as local coordinate transformations $r \mapsto r' = f(r)$, of spatial coordinates $r \in \mathbb{R}^2$ such that angles are kept unchanged$^4$. The Lie algebra of these transformations is naturally called the ‘conformal Lie algebra’.

1. In order to establish our notation, we briefly recall some basic facts, concentrating on $d = 2$. Use complex light-cone coordinates $z = t + i\mu r$ and $\bar{z} = t - i\mu r$, where the ‘time’ $t$ and the ‘space’ $r$ label the two directions, and $\mu$ is an universal constant with the units of an inverse velocity. The Lie algebra generators read, for $n \in \mathbb{Z}$

$^4$ See [80] and therein for the considerable recent interest into the case $r \in \mathbb{R}^d$ with $d > 2$. 

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\[ \ell_n = -z^{n+1} \partial_z - (n+1)\Delta z^n, \quad \bar{\ell}_n = -\bar{z}^{n+1} \partial_{\bar{z}} - (n+1)\bar{\Delta}\bar{z}^n \quad (1.1) \]

where \( \Delta = \frac{1}{2}(\delta - i\gamma/\mu), \quad \bar{\Delta} = \frac{1}{2}(\delta + i\gamma/\mu) \) are the conformal weights of the scaling operators on which these generators act. The generators (1.1) obey the commutation relations of \( \text{vect}(S^1) \oplus \text{vect}(S^1) \)

\[ [\ell_n, \ell_m] = (n-m)\ell_{n+m}, \quad [\ell_n, \bar{\ell}_m] = 0, \quad [\bar{\ell}_n, \bar{\ell}_m] = (n-m)\bar{\ell}_{n+m}. \quad (1.2) \]

The maximal finite-dimensional Lie sub-algebra is \( \text{conf}(2) := \langle \ell_{\pm 1,0}, \bar{\ell}_{\pm 1,0} \rangle \cong \text{sl}(2, \mathbb{R}) \oplus \text{sl}(2, \mathbb{R}) \). For what follows, we rather consider the generators \( X_n := \ell_n + \bar{\ell}_n \) and \( Y_n := i\mu(\ell_n - \bar{\ell}_n) \) which in ‘time’ and ‘space’ coordinates read

\[ X_n = -\frac{1}{2} \left[(t+i\mu r)^{n+1} + (t-i\mu r)^{n+1}\right] \partial_t + \frac{i}{2\mu} \left[(t+i\mu r)^{n+1} - (t-i\mu r)^{n+1}\right] \partial_r - \frac{n+1}{2} \delta \left[(t+i\mu r)^n + (t-i\mu r)^n\right] - \frac{n+1}{2} \gamma \left[(t+i\mu r)^n - (t-i\mu r)^n\right], \]

\[ Y_n = -\frac{i\mu}{2} \left[(t+i\mu r)^{n+1} - (t-i\mu r)^{n+1}\right] \partial_t - \frac{1}{2} \left[(t+i\mu r)^{n+1} + (t-i\mu r)^{n+1}\right] \partial_r - \frac{n+1}{2} i\mu \delta \left[(t+i\mu r)^n - (t-i\mu r)^n\right] - \frac{n+1}{2} \gamma \left[(t+i\mu r)^n + (t-i\mu r)^n\right]. \quad (1.3) \]

Herein, \( \delta \) and \( \gamma \) denote the scaling dimension and the (rescaled) spin of the respective scaling operator. The commutators (1.2) are recast into

\[ [X_n, X_m] = (n-m)X_{n+m}, \quad [X_n, Y_m] = (n-m)Y_{n+m}, \quad [Y_n, Y_m] = -\mu (n-m)X_{n+m}. \quad (1.4) \]

These conformal transformations also act as dynamical symmetries of differential equations. ‘Dynamical symmetry’ means throughout [76] that the space of solutions of the equation \( \hat{\mathcal{J}} \phi = 0 \) is invariant under the conformal transformations (1.1). The most simple example is the Laplace equation \( \hat{\mathcal{J}} \phi = 0 \), where \( \phi = \phi(z, \bar{z}) = \varphi(t, r) \) and

\[ \hat{\mathcal{J}} = 4\mu^2 \partial_t \partial_r = \mu^2 \partial_t^2 + \partial_r^2. \quad (1.5) \]

The dynamical conformal symmetry of (1.5) follows from the commutators

\[ [\hat{\mathcal{J}}, \ell_n] = -(n+1)z^n \hat{\mathcal{J}} - 4n(n+1)\mu^2 \Delta z^{n-1} \partial_z \]

\[ [\hat{\mathcal{J}}, \bar{\ell}_n] = -(n+1)\bar{z}^n \hat{\mathcal{J}} - 4n(n+1)\mu^2 \bar{\Delta}\bar{z}^{n-1} \partial_{\bar{z}} \quad (1.6) \]

and provided that \( \Delta = \bar{\Delta} = 0 \). Of course, the physical interest in conformal invariance comes from the multitude of systems, beyond the Laplace equation and which are conformally invariant, as mentioned above. Finally, the requirement of co-variance under conformal transformations is sufficient to fix certain \( n \)-point
functions of the scaling operators \( \phi_i(z_i, \bar{z}_i) \). For example, the two-point function \( C(z, \bar{z}) = C(t, r) \) reads, up to normalisation

\[
C(z, \bar{z}) = \langle \phi_1(z, \bar{z})\phi_2(0, 0) \rangle = \delta_{\Delta_1, \Delta_2} \delta_{\Xi_1, \Xi_2} \bar{z}^{-2\Delta_1} z^{-2\Xi_1}
\]

\[
= \langle \varphi_1(t, r)\varphi_2(0, 0) \rangle = \delta_{\delta_1, \delta_2} \delta_{\gamma_1, \gamma_2} (t^2 + \mu^2 r^2)^{-\delta_1} \exp \left( -\frac{2\gamma_1}{\mu} \arctan \left( \frac{\mu r}{t} \right) \right). \tag{1.7}
\]

2. Are there other groups of time-space transformations which can act as dynamical symmetries in certain physical situations? In table 1, several examples of infinite-dimensional Lie groups of time-space transformations are listed. The Schrödinger–Virasoro group \([52, 56, 86]\) is distinct from the conformal group in that the dilatations are of the form \( t \mapsto b^2 t \) and \( r \mapsto br \), with the dynamical exponent \( z = 2 \) for the Schrödinger group, in contrast to \( z = 1 \) for the conformal group. Its maximal finite-dimensional subgroup is the Schrödinger group \([15, 48, 68, 69, 73, 76]\), which acts as dynamical symmetry on the free diffusion/Schrödinger equation. Schrödinger-covariance predicts the form of response functions, as they arise for example in phase-ordering kinetics, notably in non-equilibrium 2D and 3D Ising and Potts models, whose dynamics are not described by free-field theories. See [58] for a review and [65] for a tutorial introduction.

If we concentrate on systems with a dynamical exponent \( z = 1 \), can one find infinite-dimensional groups of time-space transformations distinct from the conformal transformations reviewed above? For the sake of a clear conceptual distinction, those standard conformal transformations, generated from (1.1) or (1.3), will from now on be called ‘ortho-conformal’. It will turn out that alternative sets of time-space transformations exist. In contrast to ortho-conformal transformations, these new transformations are not angle-preserving, neither in a space made from time-space points \( (t, r) \in \mathbb{R}^{1+d} \), nor in space with points \( (r) \in \mathbb{R}^d \). On the other hand, their Lie algebras still contain ortho-conformal Lie algebras as sub-algebras. We shall therefore call them ‘meta-conformal transformations’ \([63, 64]\).

3. In a two-dimensional time-space with points \( (t, r) \in \mathbb{R}^2 \), meta-conformal transformations have the infinitesimal generators \([55]\)

\[
X_n = -t^{n+1} \partial_t - \mu^{-1}[((t + \mu r)^{n+1} - r^{n+1}) \partial_t - (n + 1) \frac{\gamma}{\mu} [(t + \mu r)^n - t^n] - (n + 1) \delta t^n,
\]

\[
Y_n = -(t + \mu r)^{n+1} \partial_r - (n + 1) \frac{\gamma}{\mu} (t + \mu r)^n
\]

(1.8)

where \( \delta, \gamma \) are constants and \( \mu^{-1} \) is a constant universal velocity (‘speed of sound’ or ‘speed of light’). The generators \( X_{-1} = -\partial_t \) and \( Y_{-1} = -\partial_r \) of time- and space-translations, as well as the generator \( X_0 = -t \partial_t - r \partial_r - \delta \) of dilatations are the same as for ortho-conformal transformations (1.3). The other generators are different and the generators (1.8) are in general not angle-preserving. Their Lie algebra \( \langle X_n, Y_n \rangle_{n \in \mathbb{Z}} \) obeys
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Table 1. Examples of infinite-dimensional groups of time-space transformations, with the defining coordinate changes. Herein, $f, f, b, a$ are arbitrary differentiable (vector) functions of their argument and $R(t) \in SO(d)$ is a time-dependent rotation matrix. Physical interpretations of the coordinates $(u, \bar{u})$ and $(\tau, w, \bar{w})$ of the 1D and 2D meta-conformal transformations are listed in tables 2 and 3. The physical interpretation of the co-variant $n$-point functions as either correlators or responses is based on the extension of the Cartan sub-algebra [59, 60, 62].

| Group                  | coordinate changes       | Co-variance |
|------------------------|--------------------------|-------------|
| Ortho-conformal (1 + 1)$D$ | $z' = f(z)$, $\bar{z}' = \bar{f}(\bar{z})$ | Correlator  |
| Schrödinger–Virasoro   | $t' = b(t)$, $r' = (db(t)/dt)^{1/2}r$ | Response    |
| Conformal galilean     | $t' = b(t)$, $r' = (db(t)/dt) r$ | Correlator  |
| Meta-conformal 1D      | $u = f(u)$, $\bar{u}' = \bar{f}(\bar{u})$ | Correlator  |
| Meta-conformal 2D      | $\tau' = \tau$, $w' = f(w)$, $\bar{w}' = \bar{f}(\bar{w})$ | Correlator  |

$[X_n, X_m] = (n - m)X_{n+m}$, $[X_n, Y_m] = (n - m)Y_{n+m}$, $[Y_n, Y_m] = \mu(n - m)Y_{n+m}$. (1.9)

The maximal finite-dimensional Lie sub-algebra is denoted $\text{meta}(1,1) := (X_{\pm 1,0}, Y_{\pm 1,0})$. Indeed, if $\mu \neq 0$, (1.9) is isomorphic to the Lie algebra (1.4). To see this, let $X_n = \ell_n + \bar{\ell}_n$ and $Y_n = \mu \ell_n$. This gives

$$
\ell_n = -\tau^{n+1} \left( \partial_\tau \frac{1}{\mu} \partial_\mu \right) - (n + 1) \left( \delta - \frac{\gamma}{\mu} \right) t^n \\
\bar{\ell}_n = -\frac{1}{\mu} (t + \mu \tau)^{n+1} \partial_\tau - (n + 1) \frac{\gamma}{\mu} (t + \mu \tau)^n
$$

(1.10)

which again satisfy the commutators (1.2). The reduction of (1.10) to the standard form (1.1) in ‘complex’ light-cone coordinates $z, \bar{z}$ is achieved by setting $z = t$ and $\bar{z} = t + \mu r$, and identifying the conformal weights $\Delta = \delta - \gamma/\mu$ and $\bar{\Delta} = \gamma/\mu$. In 1 + 1 time-space dimensions, the meta-conformal transformations (1.8) and the ortho-conformal transformations (1.4) are two representations of the same conformal Lie algebra, see also table 1.

The meta-conformal generators (1.8) are dynamical symmetries of the equation of motion

$$
\mathcal{J} \phi(t, r) = (-\mu \partial_t + \partial_\tau) \phi(t, r) = 0.
$$

(1.11)
Indeed, since (with \( n \in \mathbb{Z} \))

\[
[\mathcal{J}, X_n] = -(n + 1)t^n\mathcal{J} + n(n + 1)\mu \left( \delta - \frac{\gamma}{\mu} \right) t^{n-1}, \quad [\mathcal{J}, Y_n] = 0
\]  

(1.12)
a solution \( \varphi \) of (1.11) with scaling dimension \( \delta_\varphi = \delta = \gamma/\mu \) is mapped onto another solution of (1.11). Hence the space of solutions of the equation (1.11) is meta-conformally invariant. This is the analogue of the ortho-conformal invariance of the 2D Laplace equation. This kind of equation of motion (1.11), with a directional bias, motivates to look for physical applications in the kinetics of spin systems with directed dynamics, as we shall do in section 5.

Meta-conformally co-variant two-point functions have the form [58], up to normalisation

\[
C(t, r) = \langle \varphi_1(t, r)\varphi_2(0, 0) \rangle = \delta_{\delta_1, \delta_2}\delta_{\gamma_1, \gamma_2}^{-2\gamma_1/\mu} \left( 1 + \frac{\gamma_1 r}{\mu t} \right)^{-2\gamma_1/\mu}.
\]  

(1.13)

4. In the limit \( \mu \to 0 \), for both ortho-conformal as well as for meta-conformal transformations, one can make a Lie algebra contraction of (1.4) or (1.9). The result is called ‘conformal galilean algebra’ CGA(1) [51] or ‘BMS-algebra’ BMS3 [14]. Table 1 gives the time-space transformations which follow from CGA(d) for \( d \geq 1 \) (rotations by arbitrary time-dependent angles appear for \( d \geq 2 \)). The generators of CGA(1) can be read off by taking the limit \( \mu \to 0 \) in either (1.3) or else in (1.8)\(^5\). Taking the limit \( \mu \to 0 \) in either (1.7) or else (1.13) gives the CGA(1)-covariant two-point function

\[
C(t, r) = \langle \varphi_1(t, r)\varphi_2(0, 0) \rangle = \delta_{\delta_1, \delta_2}\delta_{\gamma_1, \gamma_2}^{-2\gamma_1/\mu} \exp \left( -2\gamma_1 \frac{r}{t} \right).
\]  

(1.14)

The Lie algebra CGA(d) is not isomorphic to the Schrödinger Lie algebra \( \mathfrak{sch}(d) \) in \( d \) dimensions [32, 56]. An infinite-dimensional extension exists for all dimensions \( d \geq 1 \), see table 1, and is distinct from the Schrödinger–Virasoro group. Applications arise in hydrodynamics [24, 90] or in gravity, e.g. [1, 5–7, 9, 10, 67, 74], and the bootstrap approach has been tried [8, 74].

Two-point functions such as (1.13) and (1.14) display a singularity if \( r/t \) becomes negative enough. This can be avoided by (i) constructing an extension of the Cartan sub-algebra of meta-conformal transformations and (ii) applying the covariance conditions in an extended ‘dual’ space, with respect to the ‘rapidities’ \( \gamma_i \) to considered as additional variables. In 1D, this gives the two-point function \( C(t, r) = C_{12}(t, r) \) as [62]

\(^5\) The conformal galilean generator \( Y_0 = -t\partial_t - \gamma \in \text{CGA}(1) \) is distinct from the ordinary Galilei generator \( Y_{1/2} = -t\partial_t - \mathcal{M}r \in \mathfrak{sch}(1) \) of the Schrödinger algebra, as these imply distinct transformations of the scaling operators.

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Table 2. Possible choices for the ‘complex’ light-cone coordinates $u, \bar{u}$ of the
conformal generators $\ell_u = -u^{n+1} \partial_u = (n + 1) \Delta u^n$ and $\bar{\ell}_u = \bar{u}^{n+1} \partial_{\bar{u}} = (n + 1) \Delta \bar{u}^n$. The meta-conformal representations are equations (1.10) and (2.13) for $\alpha = 0$ and
$\alpha \neq 0$, respectively. The resulting conformal weights $\Delta, \bar{\Delta}$ are also indicated. In
case ii, $\mu = -\beta/3$ and the scaling $\alpha = -\frac{2}{3} \beta^2 = -2\mu^2$ was used.

| Transformation   | $u$  | $\bar{u}$ | $\Delta$ | $\bar{\Delta}$ |
|------------------|------|-----------|----------|----------------|
| Ortho-conformal (1 + 1)$D$ | $z = t + ir$ | $\bar{z} = t - ir$ | $\frac{1}{2} (\delta - \frac{i \gamma}{\mu})$ | $\frac{1}{2} (\delta + \frac{i \gamma}{\mu})$ |
| Meta-conformal 1D i | $\alpha = 0$ | $t$ | $\rho = t + \mu r$ | $\delta - \frac{\gamma}{\mu}$ | $\frac{2}{\mu}$ |
| Meta-conformal 1D ii | $\alpha \neq 0$ | $t + \frac{2\beta}{3} r$ | $t + \frac{\beta}{3} r$ | $\frac{3\gamma}{\mu} - \delta$ | $2\delta - \frac{3\gamma}{\mu}$ |

\[ C(t, r) = \delta_{\delta_1, \delta_2} \gamma_{\gamma_1, \gamma_2} |t|^{-2\delta_1} \left(1 + \frac{\mu}{\gamma_1} \frac{\gamma_1 r}{t} \right)^{-2\gamma_1/\mu} \mu^{-\delta_0} \delta_{\delta_1, \delta_2} \gamma_{\gamma_1, \gamma_2} |t|^{-2\delta_1} \exp \left( -\frac{2\gamma_1 r}{t} \right). \]

(1.15)

One has the symmetry $C_{12}(t, r) = C_{21}(-t, -r)$ under permutation of the scaling
operators $\phi_1$ and $\phi_2$, expected for a correlator. This is analogous to ortho-conformal invariance.

We mention further examples of physical systems with dynamical exponent $z = 1$. First, the dynamical symmetries of the Jeans–Vlassov equation [18, 19, 35, 66, 70, 75, 79, 87, 88] in one space dimension are given by a representation of (1.9), distinct from (1.8) [83]. Second, the non-equilibrium dynamics of open quantum systems after a quantum quench generically has $z = 1$, related to ballistic spreading of signals, see [16, 17, 31] and this apparently holds both for quenches in the vicinity of the quantum critical point [28] as well as for deep quenches into the two-phase coexistence region [89, 91]. Third, effective equations of motion of the form (1.11) arise in recent studies of the generalised hydrodynamics required for the description of strongly interacting non-equilibrium quantum systems [12, 22, 23, 29, 30, 77]. Forth, we shall consider in section 5 the non-equilibrium relaxational dynamics in directed spin systems, such as the directed Glauber–Ising model [42, 45, 46].

5. Can one find meta-conformal transformations in $d \geq 2$ spatial dimensions? We shall require that time- and space-translations, as well as dilatations with $z = 1$, are kept in their form known from ortho-conformal invariance. Table 1 shows several examples of infinite-dimensional time-space transformations groups and how meta-conformal transformations in $d = 1$ or $d = 2$ constructed in this work compare with other known examples. Tables 2 and 3 give the physical interpretations of the formal abstract coordinates $(u, \bar{u})$ or $(\tau, w, \bar{w})$ used in table 1.

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6 For the Schrödinger group, an analogous construction shows that the two-point functions are response functions [56, 60–62]. The scaling form (1.15) of the meta-conformal correlator is the same as the special case $z = 1$ for the conformally co-variant two-time response function $G(t, r)$ [21, equation (3.10)].

7 The 2D meta-conformal case also arises from a systematic extension of Lévy-Leblond’s Carroll group in $(1 + 1)D$, where it is called the ‘conformal $k = \infty$ Carroll Lie group’ [33, 34].
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In this way, the analogies and differences between these distinct groups become apparent, notably concerning the transformation of the spatial coordinates. Only the ortho-conformal transformation include rotations between the ‘time’ and ‘space’ coordinates.

This work is organised as follows. In section 2, a generalisation of the representation (1.8) of 1D meta-conformal transformations will be presented. We shall give a geometrical interpretation of several types of meta-conformal transformations. This allows to formulate an ansatz for the d-dimensional construction which is used in section 3 to find the generic form of the generators of the Lie algebra of meta-conformal transformations, to be denoted by meta(1, d). Particular attention will be devoted to construct the terms which will describe how primary scaling operators will transform under meta-conformal transformations. In section 4 we shall concentrate on the special case of d = 2 dimensions, where stronger results are found. First, we identify two distinct meta-conformal representations which are distinguished by different sets of physical coordinates, as listed in table 3. Second, while for d > 2 this only gives a finite-dimensional Lie algebra, we shall see for d = 2 an infinite-dimensional extension exists which is isomorphic to the direct sum of three Virasoro algebras (without central charge). The corresponding finite (group) transformations are indicated in table 1. The time-dependent transformations might be used to generate the temporal evolution of the physical system. Indeed, the co-variant two-point function is explicitly seen to describe the relaxation towards an ortho-conformally two-point function, which reflects the meta-conformal aspects in this Lie group. Section 5 describes the application to the non-equilibrium relaxation behaviour of the directed Glauber–Ising chain, in the case of spatially longed-ranged initial conditions. We conclude in section 6.

2. Meta-conformal algebras: general remarks

2.1. A generalisation of the one-dimensional case

We begin by reconsidering the dynamical symmetries of equation (1.11), re-written in the form

\[ \hat{H} \phi(t, r) = (\partial_t + c \partial_r) \phi(t, r) = 0 \]  

(2.1)

Table 3. Possible choices for the ‘time’ and ‘complex’ light-cone coordinates τ, w, ¯w of the 2D meta-conformal generators (4.3) in d = 2 spatial dimensions, in terms of the time-space coordinates t, x, y.

| transformation           | τ     | w                         | ¯w         |
|--------------------------|-------|---------------------------|------------|
| meta-conformal 2D i      | α = 0 | t                         | t + β(x + iy)t + β(x − iy) |
| meta-conformal 2D ii     | α ≠ 0 | t + 2β3 x                 | t + β3(x − iy)t + β3(x + iy) |

8 They differ also from Cardy’s proposal \((t, r) \rightarrow (t', r') = (b(r)t, b(r)r)\) [21].
9 The same algebra of dynamical symmetries also arises for diffusion-limited erosion in 1D [63, 64].

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where \( c \) is a constant. For what follows, we need a generalisation of the meta-conformal representation (1.8). By assumption, both time- and space-translations \( X_{-1}, Y_{-1} \), as well as the dilatations \( X_0 \), retain their form given in (1.8). However, the explicit generators \( X_1, Y_{0,1} \) of the finite-dimensional sub-algebra meta\((1,1) := \langle X_n, Y_n \rangle_{n \in \{\pm 1, 0\}} \) admit in general the following form, with the constants \( \alpha, \beta \) [83]:

\[
X_1 = -(t^2 + \alpha r^2) \partial_t - (2\beta \gamma r) \partial_r - 2\delta t - 2\gamma r,
Y_0 = -\alpha r \partial_t - (t + \beta r) \partial_r - \gamma
Y_1 = -\alpha (2t + \beta r^2) \partial_t - (t^2 + 2\beta tr + (\alpha + \beta^2)r^2) \partial_r - 2\gamma t - 2(\alpha \delta + \beta \gamma) r.
\]

For \( n, m, \in \{0, \pm 1\} \) they satisfy the following commutation relations

\[
[X_n, X_m] = (n - m)X_{n+m}, \quad [X_n, Y_m] = (n - m)Y_{n+m}
Y_n, Y_m] = (n - m) (\alpha X_{n+m} + \beta Y_{n+m}).
\]

With respect to the meta-conformal generators (1.8), the new feature is the constant \( \alpha \neq 0 \).

Furthermore, if we make the choice [83]

\[
\alpha = \frac{1 + \beta c}{c^2}, \quad \delta = -\gamma c
\]

then the generators (2.2) are indeed dynamical meta-conformal symmetries of the 1D transport equation (2.1). This follows from the commutators

\[
[\hat{\mathcal{J}}, X_1] = -2 \left(t + \frac{1 + \beta c}{c^2} cr\right) \hat{\mathcal{J}}
[\hat{\mathcal{J}}, Y_0] = -\frac{1 + \beta c}{c^2} c \hat{\mathcal{J}}
[\hat{\mathcal{J}}, Y_1] = -2 \left(1 + \frac{\beta c}{c^2} \right) (ct + (1 + \beta c)r) \hat{\mathcal{J}}.
\]

Hence the solution space of \( \hat{\mathcal{J}} \varphi = 0 \) is invariant under the representation (2.2).

While commutators (2.3) and (1.9) look different, the Lie algebra \( \langle X_n, Y_n \rangle_{n \in \{0, \pm 1\}} \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \) is isomorphic to the ortho-conformal algebra conf\((2) [83, proposition 1] and hence also to (1.9). We want to find this isomorphism explicitly.

First, the case \( \alpha = 0 \) reduces the algebra to the usual form (1.9). Then the choices \( c = -1/\beta \) and \( \beta = \mu \) make equations (1.11) and (2.1) coincide.

Second, for \( \alpha \neq 0 \) there is no obvious relation between \( c \) and \( \beta \). Define new generators

\[
Y_n := aX_n + Y_n; \quad \text{for } n \in \{0, \pm 1\}.
\]

Using (2.3), it is easily seen that

\[
[Y_n, Y_m] = (2a + \beta)(n - m)Y_{n+m}
\]

where \( a \) must satisfy the quadratic equation \( a^2 + \beta a - \alpha = 0 \). The solutions are \( a_\pm = (-\beta \pm \sqrt{\beta^2 + 4\alpha})/2 \). Rescaling the generators \( Y_n \), one may effectively rescale one
of the constants $\alpha, \beta$ as desired; we shall take $\alpha = -\frac{2}{9} \beta^2$ in what follows. Then we have the two cases $a_- = -\frac{2}{3} \beta$ and $a_+ = -\frac{1}{3} \beta$.

**Case A:** $a = a_- = -\frac{2}{3} \beta$. Combining equation (2.4) with $\mu = -1/c$ fixes $\mu = -\beta/3$.

We have

$$\mathcal{Y}_n^{(A)} = -\frac{2}{3} \beta X_n + Y_n,$$

(2.7)

and recover the algebra (1.9) in the finite-dimensional case

$$[X_n, \mathcal{Y}_m^{(A)}] = (n - m) \mathcal{Y}_{n+m}^{(A)}, \quad [\mathcal{Y}_n^{(A)}, \mathcal{Y}_m^{(A)}] = -\beta \frac{2}{3} (n - m) \mathcal{Y}_{n+m}^{(A)}.$$  

(2.8)

In addition, from this representation, see (2.2), of the algebra $\langle X_{0,\pm 1}, \mathcal{Y}_{0,\pm 1}^{(A)} \rangle$ with commutators (2.3) and (2.8), an infinite-dimensional extension can be found. To do so, we first define

$$A_n^{(A)} = X_n + \frac{3}{\beta} \mathcal{Y}_n^{(A)} = -X_n + \frac{3}{\beta} Y_n$$

(2.9)

with the following simplified commutators

$$[A_n^{(A)}, A_m^{(A)}] = (n - m) A_{n+m}^{(A)}, \quad [\mathcal{Y}_n^{(A)}, \mathcal{Y}_m^{(A)}] = -\beta \frac{2}{3} (n - m) \mathcal{Y}_{n+m}^{(A)}, \quad [A_n^{(A)}, \mathcal{Y}_m^{(A)}] = 0.$$  

(2.10)

The explicit representation for all $n \in \mathbb{Z}$ will be given below.

**Case B:** $a = a_+ = -\frac{\beta}{3}$. We now have

$$\mathcal{Y}_n^{(B)} = -\frac{\beta}{3} X_n + Y_n, \quad A_n^{(B)} = X_n - \frac{3}{\beta} \mathcal{Y}_n^{(B)}.$$  

(2.11)

However, it is unnecessary to reproduce the commutators, since the cases A and B are not independent. Rather, we have (for $\beta \neq 0, \infty$ and $\alpha \neq 0$)

$$A_n^{(B)} = 2X_n - \frac{3}{\beta} Y_n = -\frac{3}{\beta} \mathcal{Y}_n^{(A)}, \quad \mathcal{Y}_n^{(B)} = -\frac{\beta}{3} X_n + Y_n = \frac{\beta}{3} A_n^{(A)}.$$  

(2.12)

Concentrating on case A, and letting $A_n = A_n^{(A)}$ and $\mathcal{Y}_n = -\mathcal{Y}_n^{(A)}$, we have the following infinite-dimensional representation of meta-conformal transformations, for the chosen rescaling $\alpha = -\frac{2}{9} \beta^2 \neq 0$.

---

10 This choice is motivated from the $d$-dimensional case with $d \geq 2$, see sections 3 and 4.
\[ A_n = -\left( t + \frac{2}{3} \beta r \right)^{n+1} \left( \frac{3}{\beta} \partial_r - \partial_t \right) - (n+1) \left( \frac{3}{\beta} \gamma - \delta \right) \left( t + \frac{2}{3} \beta r \right)^n \]
\[ Y_n = -\left( t + \frac{\beta}{3} r \right)^{n+1} \left( \frac{2}{3} \beta \partial_t - \partial_r \right) - (n+1) \left( \frac{2}{3} \beta \delta - \gamma \right) \left( t + \frac{\beta}{3} r \right)^n \] (2.13)

with the commutation relations, for \( n, m \in \mathbb{Z} \)
\[ [A_n, A_m] = (n - m) A_{n+m}, \quad [Y_n, Y_m] = \frac{\beta}{3} (n - m) Y_{n+m}, \quad [A_n, Y_m] = 0. \] (2.14)

In particular, for \( n = \pm 1, 0 \) the generators (2.2) are reproduced. The generators \( A_n, Y_n \) are the analogues of the generators \( \ell_n, \bar{\ell}_n \) from the representation (1.10) of 1D meta-conformal invariance, see table 2. Indeed, with the light-cone coordinates
\[ u = t + \frac{2\beta}{3} r, \quad \bar{u} = t + \frac{\beta}{3} r \] (2.15)
the generators (2.13) reduce to the usual ortho-conformal form [11]
\[ \ell_n \leftrightarrow A_n = -u^{n+1} \partial_u - (n+1) \Delta u^n, \quad \bar{\ell}_n \leftrightarrow \frac{3}{\beta} Y_n = -\bar{u}^{n+1} \partial_{\bar{u}} - (n+1) \bar{\Delta} \bar{u}^n \]
with the conformal weights \( \Delta = \frac{3\gamma}{\beta} - \delta \) and \( \bar{\Delta} = 2\delta - \frac{3\gamma}{\beta} \).

Summarising, we have have found the distinct kinds of time-space transformations which arise from the conformal algebra in \((1+1)D\), under the assumption stated above.

**Proposition 1.** For \( 1+1 \) time-space dimensions, the distinct representations as time-space transformations of the conformal algebra (1.2) are listed in table 2. The choice of orthogonal coordinates \( u, \bar{u} \) corresponds to ortho-conformal transformations, while meta-conformal transformations are found if non-orthogonal coordinates \( u, \bar{u} \) are used.

The different physical interpretations of the light-cone coordinates \( u, \bar{u} \) are illustrated in figure 1. Clearly, the ‘natural’ coordinates of 1D meta-conformal transformations do not correspond to orthogonal coordinates, while ortho-conformal transformations are obtained for orthogonal coordinates.

Figure 1. Comparison of the light-cone coordinates \( u, \bar{u} \) for (a) 2D ortho-conformal transformations, (b) 1D meta-conformal transformations with \( \alpha = 0 \) and (c) 1D meta-conformal transformations with \( \alpha \neq 0 \).
2.2. Finite 1D meta-conformal transformations

A more clear geometric picture of the meta-conformal transformations of table 2 can be found by constructing the Lie series $F_Y(\varepsilon, t, r) = e^{Y_m} F(0, t, r)$ and $F_X(\varepsilon, t, r) = e^{X_n} F(0, t, r)$ of the corresponding finite transformations. If we use the representation (1.8), they are given as the solutions of the two initial-value problems (herein, $\alpha = 0$)

\[ (\partial_\varepsilon + (t + \mu r)^{m+1} \partial_r + (m + 1) \gamma (t + \mu r)^m) F_Y(\varepsilon, t, r) = 0 \]  
\[ (\partial_\varepsilon + t^{n+1} \partial_r + \mu^{-1} [(t + \mu r)^{n+1} - t^{n+1}] \partial_r + (n + 1) \left( \delta t^n + \frac{\gamma}{\mu} [(t + \mu r)^n - t^n] \right) ) F_X(\varepsilon, t, r) = 0 \]

(2.16a)

subject to the initial conditions $F_X(0, t, r) = F_Y(0, t, r) = \varphi(t, r)$. If we work with the variable $\rho = t + \mu r$ instead of $r$, we write the initial condition $F_X(0, t, \rho) = F_Y(0, t, \rho) = \phi(t, \rho) := \varphi(t, r)$ and find, with the conformal weights $\Delta, \overline{\Delta}$ taken from table 2

\[ Y_m : \quad \phi'(t, \rho) = \left( \frac{d\rho'}{d\rho} \right)^\Delta \phi(t', \rho') \quad t' = t, \quad \rho' = a(\rho) \]  
\[ X_n : \quad \phi'(t, \rho) = \left( \frac{dt'}{dt} \right)^\Delta \left( \frac{d\rho'}{d\rho} \right)^\overline{\Delta} \phi(t', \rho') \quad t' = b(t), \quad \rho' = b(\rho) \]

(2.17b)

where $a = a(\rho)$ and $b = b(t)$ are arbitrary differentiable functions. The transformation of $r$ as generated by $X_n$ reads

\[ r' = \frac{1}{\mu}[b(t + \mu r) - b(t)]. \]  

(2.18)

Re-expanding $b(t) = t - \varepsilon t^{n+1}$ and $a(\rho) = \rho - \varepsilon \rho^{m+1}$ reproduces the differential equations (2.16) for the Lie series. Alternatively, if $\alpha \neq 0$, we can use the representation (2.13), and work with the variables $u$ and $\bar{u}$, as well as with the conformal weights $\Delta$ and $\overline{\Delta}$, as given in table 2 and find

\[ Y_m : \quad \phi'(u, \bar{u}) = \left( \frac{d\bar{u}'}{d\bar{u}} \right)^\Delta \phi(u', \bar{u}) \quad u' = u, \quad \bar{u}' = \bar{f}(\bar{u}) \]  
\[ A_n : \quad \phi'(u, \bar{u}) = \left( \frac{du'}{du} \right)^\Delta \phi(u', \bar{u}) \quad u' = f(u), \quad \bar{u}' = \bar{u} \]

(2.19b)

and where $f = f(u)$ and $\bar{f} = \bar{f}(\bar{u})$ are arbitrary differentiable functions. This is the statement already contained in table 1, which remains valid for $\alpha = 0$ and $\alpha \neq 0$.

Equations (2.17) and (2.19) give the global form of the 1D meta-conformal transformations and show how a primary meta-conformal scaling operator should be defined, generalising the concept from the case of orthogonal coordinates studied in [11] to non-orthogonal coordinates. In this work, we shall concentrate on finding new
meta-conformal transformations and we shall leave the construction of the full conformal field-theory based on (2.17) and (2.19) to future work.

2.3. Ansatz for the $d$-dimensional case

Meta-conformal transformations are analogues of the conformal algebra $\text{conf}(d)$ and are sought as dynamical symmetries of a ballistic transport equation, with a constant $c \in \mathbb{R}^d$,

$$\mathcal{J}\phi(t, r) = (\partial_t + c \cdot \partial_r)\phi(t, r) = 0. \tag{2.20}$$

This naturally generalises equation (1.11). Our construction starts from two axioms:

(i) The generators of translations and time-space dilatations read in $d$ dimensions

$$X_{-1} = -\partial_t \tag{2.21a}$$

$$Y_{-1}^j = -\partial_{r_j}, \quad j \in \{1, \ldots, d\} \tag{2.21b}$$

$$X_0 = -t\partial_t - r \cdot \partial_r - \delta \tag{2.21c}$$

where $\delta$ stands for a scaling dimension. If $d \leq 3$, we shall also write $j \in \{x, y, z\}$.

(ii) Specifying $X_1$ fixes all further generators. We make the ansatz

$$X_1 := -(t^2 + \alpha r^2)\partial_t - 2t r \cdot \partial_r - p(r \cdot r)\beta \cdot \partial_r - (1 - p)(\beta \cdot r) \cdot \partial_r - 2\delta t - B(r, \beta, \gamma) \cdot \partial_\gamma - k \gamma \cdot r \tag{2.22}$$

where $\alpha$, $p$, and $k$ are scalars, $\beta$, $\gamma$ are vectors and the vector $B$ depends on its arguments. All these must be found self-consistently from the algebra we are going to construct.

By construction, $[X_n, X_m] = (n - m)X_{n+m}$ is obeyed for $n, m \in \{\pm 1, 0\}$. All further generators of the Lie algebra will be obtained from repeated commutators of $X_1$ with $X_{-1}$ and $Y_{-1}^j$, using $[X_1, Y_{m}^j] = (1 - m)Y_{m+1}^j$. The form (2.22) of the generator $X_1$ is motivated as follows.

- For $d = 1$ dimension, one should reproduce $X_1$ in equation (2.2). The 1D generator contains a term $-\beta r^2 \partial_r$, which for $d > 1$ leads to two distinct contributions, as specified in (2.22).
- $X_1$ should be rotation-invariant, that is it should commute with the generators $R_{ij}$ of spatial rotations. However, for the ‘natural’ choice $R_{ij} = r_i \partial_{r_j} - r_j \partial_{r_i}$, the invariance condition $[X_1, R_{ij}] = 0$ does not hold, not even in the special case $B = \gamma = 0$. Therefore, spatial rotations should also include rotations of the vectors $\beta$ and $\gamma$. The rotation generator becomes

$$R_{ij} = (r_i \partial_{r_j} - r_j \partial_{r_i}) + \varepsilon_\gamma (\gamma_i \partial_{r_j} - \gamma_j \partial_{r_i}) + \varepsilon_\beta (\beta_i \partial_{\beta_j} - \beta_j \partial_{\beta_i}) \tag{2.23}$$
where the signatures $\varepsilon_\gamma, \varepsilon_\beta = \pm 1$ allow for a different sense of rotation of $\beta$ or $\gamma$ than of the spatial coordinates $r$. Furthermore, we should allow for the possibility $B \neq 0$. In addition, from the commutation relation of the one-dimensional case (1.9), especially $[[X_1, Y^j_1], Y^j_1] \sim Y^j_1$, and (2.21b), it follows that $B$ can be at most linear in $r$.

Additional restrictions on the form of $X_1$ come from the requirement that it should act as a dynamical symmetry of equation (2.20). By ‘dynamical symmetry’ we mean the following required commutator [76]

$$[\hat{\mathcal{S}}, X_1] \phi = \lambda(t, r) \hat{\mathcal{S}} \phi \quad (2.24)$$

which implies that the space of solutions of $\hat{\mathcal{S}} \phi = 0$ is invariant under the action of $X_1$ (eventually after fixing one or several scaling dimensions of $\phi$ to certain values). As we shall see, this requirement leads to new relations between $\alpha, p$ and $\beta$.

**Example.** The two vectors $\beta$ and $c$ span a two-dimensional space. By rotation-invariance, it is therefore enough to consider the case $d = 2$, since any higher-dimensional situation can be reduced to the present case. Let $B = \gamma = 0$. From (2.22) and (2.24) it follows that $\delta = 0$ and

$$1 + \beta_x c_x + \frac{1-p}{2} \beta_y c_y = \alpha c_x^2 \quad (2.25a)$$

$$p \beta_x c_y + \frac{1-p}{2} \beta_y c_x = \alpha c_x c_y \quad (2.25b)$$

$$1 + \frac{1-p}{2} \beta_x c_x + \beta_y c_y = \alpha c_y^2 \quad (2.25c)$$

$$p \beta_y c_x + \frac{1-p}{2} \beta_x c_y = \alpha c_x c_y \quad (2.25d)$$

We look for a solution of the above system for $\beta \neq 0$. Straightforward calculations show:

1. The case $p = 1$ leads to contradictions between some of the equations in the system (2.25). Then the generator $X_1$ cannot be a symmetry.

2. For $p \neq 1$, we have the following solution of the system (2.25)

$$c_j = \frac{2 \beta_j}{p-1 \beta^2}, \quad j = x, y \quad (2.26a)$$

$$\alpha = \frac{1}{4} (p+1)(p-1) \beta^2 \quad (2.26b)$$
Hence, the condition (2.24) is satisfied, with \( \lambda(t, r) = -2t - (p + 1)(\beta \cdot r) \). In contrast with the 1D case, \( \alpha \) is fixed by (2.26b) in terms of \( \beta \). In particular, \( \alpha = 0 \) is only possible for \( p = -1 \). The solution (2.26) holds true for all dimensions \( d > 1 \).

Equation (2.26a) shows that \( c \) and \( \beta \) are collinear. Calculations are simplified by choosing the orientation of the coordinate axes such that only \( \beta_1 = \beta_x \neq 0 \) and \( \beta_j = 0 \) for all \( j \geq 2 \).

3. For \( d = 1 \), only equation (2.25a) remains, which is equivalent to (2.4). Hence the structure of the 1D meta-conformal algebra is distinct from the one in any other dimension \( d > 1 \).

In general, \( B \neq 0 \) depends linearly on \( r \). Then the sought symmetries generated by \( X_1 \) can become conditional symmetries, that is some auxiliary conditions on the field \( \varphi = \varphi(t, r, \beta, \gamma) \) must be imposed, see [13, 25, 38, 39] and references therein. We shall come back to this at the end of section 3.

3. Meta-conformal algebra in \( d > 1 \) spatial dimensions

We now find the Lie algebra \( \text{meta}(1, d) \) in more than one spatial dimension \( d \). We begin with generic conditions which will hold for any dimension \( d > 1 \). Specific results apply for \( d = 2 \) and will be presented in section 4.

We start from the ansatz (2.22), with \( p \neq 1 \). Throughout, we shall assume \( B = 0 \), unless explicitly stated otherwise. From the defining commutator relation, we have the generator

\[
Y_j^0 := \frac{1}{2} [X_1, Y_j^0] = -\alpha r_j \partial_t - \left( t + \frac{1}{2}(1-p)(\beta \cdot r) \right) \partial r_j - pr_j \beta \cdot \partial r - \frac{1}{2}(1-p)\beta_j r \cdot \partial r - (k/2)\gamma_j .
\]

(3.1)

To be specific, let \( d = 3 \), but the conclusions will apply to any \( d \geq 2 \). Take from (2.26b) the value \( \alpha = \frac{1}{4}(p-1)(p+1)\beta^2 \) and work out \([Y_i^0, Y_j^0]\) for \( i \neq j \). For example

\[
[Y_0^x, Y_0^y] = \frac{(3p-1)(p+1)(p-1)}{8} (\beta_y x - \beta_x y) \partial_t
+ \frac{(3p-1)(p+1)}{4} (\beta_x \beta_y x - \beta_x^2 y) + \frac{(1-p)^2}{4} (\beta_y^2 y - \beta_y \beta_z z) \partial_x
+ \frac{(3p-1)(p+1)}{4} (\beta_x^2 x - \beta_x \beta_y y) - \frac{(1-p)^2}{4} (\beta_z^2 z - \beta_z \beta_x x) \partial_y
+ p^2 (\beta_y \beta_z x - \beta_x \beta_z y) \partial_z
\]

(3.2)
which must be expressed in terms of the generators of the Lie algebra under construction, including the rotation generators \( R_{ij} = r_i \partial_{r_j} - r_j \partial_{r_i} \). From (3.2) we see that the commutator \([Y_0^x, Y_0^y]\) only becomes a linear combination of known generators if the parameter \( p \) obeys

\[
p^2 - \frac{1}{4} (1 - p)^2 = \frac{1}{4} (p + 1)(3p - 1) = 0, \tag{3.3}\]

**Proposition 2.** Consistent representations of meta\((1, d)\) with \( d \geq 1 \) are only possible in the cases (i) \( p_1 = -1 \) and (ii) \( p_2 = \frac{1}{3} \).

In either of these cases\(^{12}\), the commutator (3.2) simplifies to

\[
[Y_0^x, Y_0^y] = -p^2 \left( \beta^2 R_{xy} + \beta_x \beta_z R_{yz} + \beta_y \beta_z R_{zx} \right). \tag{3.4a}
\]

Similarly, for the same values of \( p = -1, \frac{1}{3} \), we find (still for \( d = 3 \))

\[
[Y_0^y, Y_0^z] = -p^2 \left( \beta_x \beta_z R_{xy} + \beta_z^2 R_{yz} + \beta_x \beta_y R_{zx} \right), \tag{3.4b}
\]

\[
[Y_0^z, Y_0^x] = -p^2 \left( \beta_y \beta_z R_{xy} + \beta_x \beta_y R_{yz} + \beta_x^2 R_{zx} \right). \tag{3.4c}
\]

Therefore, a discussion of rotation-invariance is necessary.

1. One might choose to keep full spatial rotation-invariance, with all three generators \( R_{xy}, R_{yz}, R_{zx} \). Since the invariant equation (2.20) contains a vector proportional to \( \beta \), one must include into the rotation generators, viz. \( R_{ij} \mapsto \tilde{R}_{ij} \), terms which describe the simultaneous rotations of the position \( r \) and of \( \beta \). However, changing \( \beta \) then implies changing the invariant equation. The transformations found will map one equation of the type (2.20) to another equation of the same type.

2. Here, we shall use rotation-invariance to orient the coordinate axes such that \( \beta \) is along the \( x \)-axis. In other words, we shall fix, from now on, \( \beta_x = \beta \neq 0 \) and \( \beta_y = \beta_z = \ldots = 0 \). Explicit rotation-invariance will only apply to rotations which leave the \( x \)-axis invariant. These do not exist for \( d = 2 \), but for \( d = 3 \) we have the rotation \( R = R_{yz} \).

### 3.1. Meta-conformal algebra in \( d = 3 \) dimensions with \( \gamma = 0 \)

The case of \( d = 3 \) spatial dimensions gives the generic structure meta-conformal transformations. Throughout, we shall fix \( \beta = (\beta, 0, 0) \). First, we restrict to the more simple case \( \gamma = 0 \). The rotation generator is \( R_{ij} = r_i \partial_{r_j} - r_j \partial_{r_i} \). With (2.26b), we have

\[
X_1 = - \left( t^2 + \alpha (x^2 + y^2 + z^2) \right) \partial_t - \left( 2tx + \beta x^2 + \beta p(y^2 + z^2) \right) \partial_x
- \left( 2t + \frac{1}{2} \beta x \right) y \partial_y - \left( 2t + \frac{1}{2} \beta x \right) z \partial_z - 2\delta t. \tag{3.5}\]

\(^{11}\) The level \( \chi \) of a generator \( \mathcal{X} \) is defined by \([X_0, \mathcal{X}] = \chi \mathcal{X}\). Hence the commutator of the level-zero generators \( Y_0^i \) must itself be of level zero, hence be a linear combination of \( X_0, Y_0^i \) or \( R_{ij} \).

\(^{12}\) In contrast with 1D meta-conformal transformations (see section 2), they are obtained here without any normalisation condition.
All other generators can be found from (3.5). Starting from \( \mathcal{Y}_0 \), we check that \( \mathcal{Y}_0^2 = [X_1, \mathcal{Y}_0] \), we find
\[
[X_0, \mathcal{Y}_0] = -\mathcal{Y}_0^2 \mathcal{R}_{yz}
\]
does not vanish for \( \mathcal{R}_{yz} > 3 \), see (3.4b). Also, \([X_1, \mathcal{R}_{yz}] = 0\), as expected from rotation-invariance. The next family of generators is obtained from \([X_1, \mathcal{Y}_0^2] = [X_1, \mathcal{Y}_0^2] \). If \( n, m \in \{ -1, 0, 1 \} \) and \( j = x, y, z \), the non-vanishing commutators are compactly written as
\[
[X_n, X_m] = (n - m)X_{n+m}, \quad [X_n, \mathcal{Y}_m] = (n - m)\mathcal{Y}_{n+m},
\]
\[
[Y_n, Y_m] = (n - m)(\alpha X_{n+m} + \beta \mathcal{Y}_{n+m}),
\]
\[
[Y_n, Y_m] = (n - m)(\alpha X_{n+m} + \beta \mathcal{Y}_{n+m}),
\]
\[
[Y_n, Y_m] = (n - m)\frac{1 - p}{2} \mathcal{R}_{n+m}; \quad \text{for} \ w = y, z,
\]
\[
[Y_n, Y_m] = \delta_{n+m,0}(n - m - \delta_{nm})\beta^2 \mathcal{R}_{yz},
\]
\[
[Y_n, Y_m] = \mathcal{Y}_{n+m} \mathcal{R}_{yz}; \quad [Y_m, \mathcal{R}_{yz}] = -\mathcal{Y}_m.
\]

**Proposition 3.** If \( p = -1 \) or \( p = \frac{1}{3} \), and \( \gamma = 0 \), the set meta\((1, 3) := (X_{0, \pm 1, \mathcal{Y}_{0, \pm 1}}, \mathcal{R}_{yz})\) of differential operators, as derived from (3.5), closes into a meta-conformal Lie algebra, whose structure is determined by the commutators (3.7), with two distinct representations.

**Proof.** The first part of the proposition follows from the closure of the commutators (3.7). For the second part, let \( p = \frac{1}{3} \) and consider the commutators (3.7). Re-define the generators \( Y_n \mapsto \mathcal{Y}_n := -\frac{2}{3} \mathcal{R}_n + \mathcal{Y}_n \). Again, the commutators (3.7) will hold, where \( Y_n \) is substituted by \( \mathcal{Y}_n \) and if one replaces therein \( \beta \mapsto -(\beta/3) \). □

For clarity, we shall mainly concentrate on the most simple representation, with \( p = -1 \). For easy reference, we repeat below the commutators of the \( p = -1 \) representation of meta\((1, 3) \) (here \( n, m \in \{ -1, 0, 1 \} \) and \( j = x, y, z \))
\[
[X_n, X_m] = (n - m)X_{n+m}, \quad [X_n, \mathcal{Y}_m] = (n - m)\mathcal{Y}_{n+m},
\]
\[
[Y_n, Y_m] = (n - m)\beta \mathcal{Y}_{n+m},
\]
\[
[Y_n, Y_m] = (n - m)\beta \mathcal{Y}_{n+m},
\]
\[
[Y_n, Y_m] = -\frac{1 - p}{2} \mathcal{R}_{n+m}; \quad \text{for} \ w = y, z,
\]
\[
[Y_n, Y_m] = \delta_{n+m,0}(n - m - \delta_{nm})\beta^2 \mathcal{R}_{yz},
\]
\[
[Y_n, \mathcal{R}_{yz}] = \mathcal{Y}_{n+m} \mathcal{R}_{yz}; \quad [\mathcal{Y}_m, \mathcal{R}_{yz}] = -\mathcal{Y}_m.
\]

The structure of meta-conformal algebras can be further simplified. This is in contrast with the semi-direct sums which hold true e.g. for the Schrödinger algebra.

**Proposition 4.** The Lie algebra meta\((1, 3) \) decomposes as a direct sum
\[
\text{meta}(1, 3) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{conf}(3) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus B_2.
\]
**Proof.** The relations (3.8) imply that \( \langle X_{0, \pm 1} \rangle \cong \mathfrak{sl}(2, \mathbb{R}) \). In addition, the action of the \( \mathfrak{sl}(2, \mathbb{R}) \)-subalgebra on the generators of the subalgebra \( \mathfrak{g} := \langle Y_{m}^{x,y,z}, R \rangle_{m \in \{ \pm 1, 0 \}} \), with \( R = R_{y} \), shows a semi-direct structure. Changing the base of the Lie algebra according to \( X_{n} \mapsto A_{n} := X_{n} - Y_{x}^{n} / \beta \), one has \( \langle A_{0, \pm 1} \rangle \cong \mathfrak{sl}(2, \mathbb{R}) \) and the direct sum \( \text{meta}(1, 3) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{g} \), as can be checked through the commutators

\[
[A_{n}, Y_{m}^{x}] = [A_{n}, Y_{m}^{y}] = [A_{n}, Y_{m}^{z}] = [A_{n}, R] = 0.
\]

The structure of the Lie sub-algebra \( \mathfrak{g} \) is made clear by defining the generators

\[
Y_{n}^{+} = Y_{n}^{y} + iY_{n}^{z}, \quad Y_{n}^{-} = Y_{n}^{y} - iY_{n}^{z}.
\]

Then the non-vanishing commutators of the Lie algebra \( \mathfrak{g} \) become

\[
[Y_{n}^{x}, Y_{m}^{x}] = (n - m) \beta Y_{n+m}^{x}, \quad [Y_{n}^{x}, Y_{m}^{y}] = [Y_{n}^{x}, Y_{m}^{z}] = (n - m) \beta Y_{n+m}^{y} \quad [R, Y_{m}^{y}] = \pm iY_{m}^{y} \\quad [R, Y_{m}^{x}] = \pm iY_{m}^{x} \quad [R, Y_{m}^{z}] = \pm iY_{m}^{z}
\]

\[
[Y_{n}^{+}, Y_{m}^{-}] = \begin{cases} 2(n - m) \beta Y_{n+m}^{x} & \text{if } n + m \neq 0 \\ 2i\beta^{2} R & \text{if } n = m = 0 \\ 2(n - m) \beta Y_{n+m}^{x} - 4i\beta^{2} R & \text{if } n \neq m \end{cases}
\]

The correspondence with the roots of the complex Lie algebra \( B_{2} \) is shown in figure 2. □

Although we did not carry out the explicit construction of the generators for \( d > 3 \) dimensions, counting their number allows to formulate the following

**Conjecture.** In 1 \( \leq d \leq 3 \) spatial dimensions, one has for \( \text{meta}(1, d) \) the Lie algebra isomorphisms

\[
\text{meta}(1, 1) \cong A_{1} \oplus A_{1}, \quad \text{meta}(1, d) \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{conf}(d) \cong \begin{cases} A_{1} \oplus D_{n+1} & \text{if } d = 2n \\ A_{1} \oplus B_{n+1} & \text{if } d = 2n + 1 \\
\end{cases}
\]

(3.10)
with \( n = 1, 2, \ldots \), and where \( \text{conf}(d) \) is the ortho-conformal Lie algebra in \( d \) dimensions and \( A_1, B_n, D_n \) are simple complex Lie algebras from Cartan’s classification (and \( D_2 \cong A_1 \oplus A_1 \)).

In stating this, we anticipate results on \( \text{meta}(1, 2) \) from section 4. A further important difference between \( \text{conf}(d) \) when \( d \geq 3 \) dimensions arises from the non-vanishing of commutators such as (3.6) when \( d \geq 3 \). The vanishing of these commutators for \( d = 2 \) is the reason why an infinite-dimensional extension of the 2D metaconformal Lie algebra can be constructed, as we shall show in section 4.

### 3.2. Meta-conformal algebra in \( d = 3 \) dimensions with \( \gamma \neq 0 \)

We first redefine the generator of rotations, which now should include rotations of \( \gamma \)

\[
R_{yz} \mapsto \tilde{R}_{yz} = y \partial_z - z \partial_y + \gamma_y \partial_{\gamma_x} - \gamma_x \partial_{\gamma_y}.
\]

Here, we must also include the term \( B \neq 0 \). To do so, we modify \( X_1 \) of equation (2.5) by the following ansatz, according to (2.22)

\[
X_1 \mapsto X_1 + \tilde{X}_1
\]

\[
\tilde{X}_1 = - (a(b \cdot r) \gamma + b(\gamma \cdot r) \beta + c(\beta \cdot \gamma) r) \cdot \partial_\gamma - k(\gamma \cdot r)
\]

\[
= - \beta \((a + b + c)x \gamma_x + a(y \gamma_y + z \gamma_z)\) \partial_\gamma - k(x \gamma_x + y \gamma_y + z \gamma_z)
\]

\[
- \beta( bx \gamma_y + cy \gamma_z) \partial_{\gamma_y} - \beta(bx \gamma_z + cy \gamma_x) \partial_{\gamma_z} - k(2/\gamma_x)
\]

where \( a, b, c \) and \( k \) are constants to be determined. Next, we construct \( Y_{0}^{x,y,z} \) and \( Y_{1}^{x,y,z} \), as usual\(^{14} \). We find the explicit extra terms beyond (3.1)

\[
Y_0^x \mapsto \tilde{Y}_0^x \quad Y_0^y \mapsto \tilde{Y}_0^y \quad Y_0^z \mapsto \tilde{Y}_0^z
\]

\[
\tilde{Y}_0^x = - (\beta/2) (a(b + c) \gamma_x \partial_\gamma_y + b(\gamma_y \partial_\gamma_x + \gamma_z \partial_\gamma_y) - (k/2) \gamma_x
\]

\[
\tilde{Y}_0^y = - (\beta/2)(a \gamma_y \partial_\gamma_x + c \gamma_z \partial_\gamma_y) - (k/2) \gamma_y
\]

\[
\tilde{Y}_0^z = - (\beta/2)(a \gamma_x \partial_\gamma_y + c \gamma_x \partial_\gamma_y) - (k/2) \gamma_z
\]

The values of the constants \( a, b, c \) are fixed from the requirement that the Lie algebra commutators of \( \text{meta}(1, 3) \) are those of the case \( \gamma = 0 \) treated above\(^{15} \). Equation (3.13) imply

1. the conditions \([Y_0^x, Y_0^y] = [\tilde{Y}_0^x, \tilde{Y}_0^y] = 0\) yield \( a = b = -c \).
2. the condition \([Y_0^y, Y_0^z] = -p^2 \beta^2 R_{yz}\) yields \( a = \pm 2p \).

**Proposition 5.** The time-space 3D meta-conformal transformations with \( \gamma \neq 0 \) can be labelled by the pair \((p, a)\). There are four possibilities: \((p, a) = (-1, -2), (-1, 2), (1, 2), (1, -2)\). The value of the constant \( k \) is not fixed by the commutators.

The commutator \( Y_1^j = [X_1, Y_0^j] \) gives the corresponding extensions of the generators \( Y_1^j \).

---

\(^{14}\) See footnote 13.

\(^{15}\) Clearly, modifying \( X_1 \) and correspondingly \( Y_0^{x,y,z} \) and \( Y_1^{x,y,z} \) by additive terms does not change the commutation relations.
3.3. Symmetries of the ballistic transport equation

Returning to the linear ballistic transport equation, in the form (2.20) and (2.26a)

\[ \hat{J}\varphi_\gamma(t, r) = \left( \partial_t + \frac{2}{\beta(p-1)} \partial_x \right) \varphi_\gamma(t, r) = 0 \]  

we can state the conditions for it having a meta-conformal dynamical symmetry, for \( \gamma \neq 0 \) and \( B \neq 0 \). Herein, the solution \( \varphi = \varphi_\gamma(t, r) \) may also depend on the vector \( \gamma \) of ‘rapidities’.

**Proposition 6.** For generic dimension \( d > 2 \), and \( p = \frac{1}{3} \) or \( p = -1 \), we have

(i) For \( \gamma = 0 \), the meta-conformal representations constructed above leave invariant the solution space of the equation (3.14), under the condition \( \delta = 0 \).

(ii) For \( \gamma \neq 0 \) and if \( \varphi_\gamma(t, r) = \varphi(t, r) \) does not explicitly depend on \( \gamma \), the corresponding meta-conformal representation leaves the solution space of (3.14) invariant, if \( k = 1 \) and \( \gamma_x = (1 - p)\beta\delta \).

(iii) If the solution \( \varphi_\gamma(t, r) \) does also depend on \( \gamma \), invariance of the solution space of (3.14) is only obtained under the conditions \( k = 1 \) and

\[ \left( \delta + \frac{1}{\beta(p-1)} \gamma_x + \frac{a}{p-1} \gamma \cdot \partial_\gamma \right) \varphi_\gamma(t, r) = 0. \]  

In case (iii) we have an on-shell or a of equation (3.14), see e conditional symmetry.g. [13, 25, 39].

**Proof.** This follows from the non-vanishing commutators

\[ \left[ \hat{J}, X_0 \right] = -\hat{J}, \quad \left[ \hat{J}, Y_0^x \right] = -\frac{p+1}{2} \beta \hat{J} \]

\[ \left[ \hat{J}, X_1 \right] = -(2t + (p+1)\beta x) \hat{J} - 2 \left( \delta + \frac{k}{\beta(p-1)} \gamma_x + \frac{a}{p-1} \gamma \cdot \partial_\gamma \right) \]

\[ \left[ \hat{J}, Y_1^x \right] = -\frac{\beta(p+1)}{2} (2t + (p+1)\beta x) \hat{J} \]

\[ - (p+1) \beta \left( \delta + \frac{2 + (p-1)k}{(p-1)(p+1)\beta} \gamma_x + \frac{a}{p-1} \gamma \cdot \partial_\gamma \right) \]

and recalling that \( a = \pm 2p \). \( \square \)

4. Meta-conformal algebra in \( d = 2 \) spatial dimensions

Finding meta-conformal transformations in \( d = 2 \) space dimensions (with points \( (t, x, y) \in \mathbb{R}^3 \)) proceeds as follows. As in sections 2 and 3, the generators of translations and dilatations read
The relationship with the usual time-space coordinates in one and two spatial dimensions

\[X_{-1} = -\partial_t\]

\[Y_{x-1} = -\partial_x, \quad Y_{y-1} = -\partial_y\]

\[X_0 = -t\partial_t - x\partial_x - y\partial_y - \delta.\]

The form of \(X_1\) is given by (2.22) where for simplicity we set \(B = 0\) and fixed \(k = 2\). Explicitly

\[X_1 = -(t^2 + \alpha(x^2 + y^2))\partial_t - (2tx + \beta_x x^2 + (1 - p)\beta_y xy + p\beta_x y^2)\partial_x - (2ty + p\beta_y y^2 + (1 - p)\beta_x xy + \beta_y y^2)\partial_y - 2\Delta t - 2\gamma_x x - 2\gamma_y y\]  

and we kept the vector \(\beta = (\beta_x, \beta_y)\) arbitrary. In complete analogy with the case of \(d \geq 3\) dimensions, the generators of the nine-dimensional Lie algebra \(\text{meta}(1,2) = \langle X_n, Y_n \rangle_{n \in \{\pm 1, 0\}, j \in \{x,y\}}\) are found\(^{16}\), for the two admissible cases (i) \(p = -1\) and (ii) \(p = \frac{1}{3}\). However, the Lie algebra of 2D meta-conformal transformations is infinite-dimensional, as we shall now show.

### 4.1. Infinite-dimensional extension

In what follows, we shall choose coordinate axes such that \(\beta = (\beta, 0)\).

**Proposition 7.** Consider the set of generators, with \(n \in \mathbb{Z}\)

\[A_n = -\tau^{n+1}\partial_\tau - (n + 1)\partial \tau^n\]

\[B_n^+ = -w^{n+1}\partial_w - (n + 1)\Delta w^n\]

\[B_n^- = -\bar{w}^{n+1}\partial_{\bar{w}} - (n + 1)\bar{\Delta} \bar{w}^n\]

which act on the time-space coordinates \((\tau, w, \bar{w})\). Their non-vanishing commutators are

\[[A_n, A_m] = (n - m)A_{n+m}, \quad [B_n^+, B_m^\pm] = (n - m)B_{n+m}^\pm\]

The two possible cases of 2D meta-conformal transformations correspond to (i) \(p = -1\) with \(\alpha = 0\) and (ii) \(p = \frac{1}{3}\) with \(\alpha = -\frac{2}{3}\beta^2 \neq 0\). The relationship with the usual time-space coordinates \((t, x, y)\) is given, for both cases, in Table 3 and the meta-conformal weights are in Table 4.

\(^{16}\) See footnote 13.
The Lie algebra (4.4) is isomorphic to the direct sum $\text{vect}(S^1) \oplus \text{vect}(S^1) \oplus \text{vect}(S^1)$, the direct sum of three centre-less Virasoro algebras. The corresponding meta-conformal weights $\vartheta$, $\Delta$, $\bar{\Delta}$ are expressed in table 4 in terms of the scaling dimension $\delta$ and the two 'rapidities' $\gamma_x, \gamma_y$.

The finite transformations associated with the generators $A_n, B_n^+, B_n^-$ with $n \in \mathbb{Z}$ are given by the corresponding Lie series. With the definition $\varphi(\tau, w, \bar{w}) = \phi(t, z, \bar{z})$, the final result is

\[
\begin{align*}
B_n^+ : & \quad \varphi'(\tau, w, \bar{w}) = \left( \frac{dw'}{dw} \right)^\Delta \varphi(\tau', w', \bar{w}') \quad ; \quad \tau' = \tau, \quad w' = f(w), \quad \bar{w}' = \bar{w} \\
B_n^- : & \quad \varphi'(\tau, w, \bar{w}) = \left( \frac{dw'}{dw} \right)^\Delta \varphi(\tau, w', \bar{w}') \quad ; \quad \tau' = \tau, \quad w' = w, \quad \bar{w}' = \bar{f}(\bar{w}) \\
A_n : & \quad \varphi'(\tau, w, \bar{w}) = \left( \frac{d\tau'}{d\tau} \right)^\vartheta \varphi(\tau, w', \bar{w}') \quad ; \quad \tau' = b(\tau), \quad w' = w, \quad \bar{w}' = \bar{w}
\end{align*}
\]

where $f = f(w)$, $\bar{f} = \bar{f}(\bar{w})$ and $b = b(\tau)$ are arbitrary differentiable functions. The differential equations for the Lie series are recovered by expanding $b(t) = t - \varepsilon t^{n+1}$, and analogously for $f(z)$ and $\bar{f}(\bar{z})$.

Equation (4.5) show that the relaxational behaviour described by the 2D meta-conformal symmetry is governed by three independent conformal transformations, rather than two as it is the case for 2D ortho-conformal invariance at equilibrium.

**Proposition 8.** For a vanishing meta-conformal weight $\vartheta = \vartheta_0 = 0$, the Lie algebra acts as a dynamical symmetry on the linear ballistic transport equation $\hat{\mathcal{J}} \phi(t, z, \bar{z}) = 0$ with

\[
\hat{\mathcal{J}} = \begin{cases} 
-\partial_t + \frac{4}{3} (\partial_z + \partial_{\bar{z}}) & ; \text{ case I with } p = -1 \\
\partial_t - \frac{2}{3} (\partial_z + \partial_{\bar{z}}) & ; \text{ case II with } p = \frac{1}{3} 
\end{cases}
\]

**Proof.** This follows from $[A_n, \hat{\mathcal{J}}] = ((n + 1)\tau^n \hat{\mathcal{J}} - (n + 1)n\vartheta \tau^n$ and $[B_n^\pm, \hat{\mathcal{J}}] = 0$. □

### 4.2. The case $p = -1$

*4.2.1. Lie algebra generators.* The tables 3 and 4 give the relationship between the coordinates $(\tau, w, \bar{w})$ and the usual time-space coordinates $(t, x, y)$ and also the explicit meta-conformal weights. In this sub-section, we consider the case $\alpha = 0$. The correspondence between the generators is as follows

\[
A_n = X_n - \frac{1}{3} Y_n^x, \quad B_n^\pm = \frac{1}{2\beta} (Y_n^x \pm iY_n^y).
\]
Infinite-dimensional meta-conformal lie algebras in one and two spatial dimensions

For illustration, we write down explicitly the generators of $\text{meta}(1,2)$ in the original, physically motivated, basis. In addition, although the discussion above was done for the choice $\beta = (\beta_x, \beta_y)$, we give here the generators for the generic situation where $\beta = (\beta, \beta)$. They read\(^{17}\)

\begin{align}
X_1 &= -t^2 \partial_t - (2tx + \beta_x x^2 + 2\beta_x xy - \beta_y y^2) \partial_x \\
&\quad - (2ty - \beta_y y^2 + 2\beta_x xy + \beta_y y^2) \partial_y - 2\delta t - 2\gamma_x x - 2\gamma_y y \\
Y^0_0 &= -(t + \beta_x x + \beta_y y) \partial_x - (\beta_x y - \beta_y x) \partial_y - \gamma_x \\
Y^y_0 &= -(\beta_y x - \beta_x y) \partial_x - (t + \beta_x y + \beta_x x) \partial_y - \gamma_y \\
Y^t_1 &= -(t^2 + 2t\beta_x x + 2t\beta_y y + (\beta_x^2 - \beta_y^2) x^2 + 4\beta_x \beta_y xy - (\beta_x^2 - \beta_y^2) y^2) \partial_x \\
&\quad - (2\beta_x y - 2\beta_y x - 2\beta_x \beta_y x^2 + 2(\beta_x^2 - \beta_y^2) xy + 2\beta_x \beta_y y^2) \partial_y - 2\gamma_x (t + \beta_x x + \beta_y y) - 2\gamma_y (\beta_x y - \beta_y x) \\
Y^x_1 &= -(2t\beta_y y - 2t\beta_x x + 2\beta_x \beta_y x^2 - 2(\beta_x^2 - \beta_y^2) y^2 - 2\beta_x \beta_y y^2) \partial_x \\
&\quad - (t^2 + 2t\beta_x x + 2t\beta_y y + (\beta_x^2 - \beta_y^2) x^2 + 4\beta_x \beta_y xy - (\beta_x^2 - \beta_y^2) y^2) \partial_y - 2\gamma_y (t + \beta_x x + \beta_y y) - 2\gamma_x (\beta_x y - \beta_y x).
\end{align}

They satisfy the following non-vanishing commutation relations, with $n, m \in \{0, \pm 1\}$

\begin{align}
[X_n, X_m] &= (n-m)X_{n+m}, \\
[X_n, Y^x_{m}] &= (n-m)Y^x_{n+m}, \quad [X_n, Y^y_{m}] = (n-m)Y^y_{n+m}, \\
[Y^x_{n}, Y^y_{m}] &= (n-m)(\beta_y Y^x_{n+m} + \beta_x Y^y_{n+m}), \\
[Y^x_{n}, Y^x_{m}] &= -[Y^y_{n}, Y^y_{m}] = (n-m)(\beta_x Y^x_{n+m} - \beta_y Y^y_{n+m}).
\end{align}

**Proposition 9.** The set of generators $\{X_{0, \pm 1}, Y^x_{0, \pm 1}, Y^y_{0, \pm 1}\}$ defined in (4.1) and (4.8) closes into the Lie algebra $\text{meta}(1,2)$ if $\beta_x, \beta_y$, are fixed constants.

At first sight, this looks as if one could extend this further by including spatial rotations, with a generator

\begin{equation}
R_{xy} := x \partial_y - y \partial_x + \gamma_x \partial_{\gamma_y} - \gamma_y \partial_{\gamma_x} + \beta_x \partial_{\beta_y} - \beta_y \partial_{\beta_x}
\end{equation}

which would add the non-vanishing commutators $[Y^x_n, R_{xy}] = Y^y_n$ and $[Y^y_n, R_{xy}] = -Y^x_n$ to the Lie algebra (4.9). However, doing so the the components of $\beta$ would have been considered as *variables*. Then objects such as $\beta_x Y^x_n$ on the right-hand-side in (4.9) can no longer be considered as Lie algebra generators. Hence, it is necessary to give up spatial rotation-invariance and to fix the values of the components of $\beta$ (as we already did above). From a physical point of view, the absence of rotation-invariance is natural, since the dynamical equation has a preferred direction (in this work chosen along the $x$-axis).

\(^{17}\) See footnote 13.

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4.2.2. Two-point function. A simple application of dynamical symmetries is the computation of covariantly transforming two-point functions. Non-trivial results can be obtained from so-called \textit{\textquoteleft quasi-primary\textquoteright} scaling operators \( \phi(t, z, \bar{z}) \), which transform co-variantly under the finite-dimensional sub-algebra \( \langle A_{\pm 1, 0}, Y_{\pm 1, 0} \rangle \). Because of temporal and spatial translation-invariance, we can directly write

\[
F(t, z, \bar{z}) = \langle \phi_1(t, z, \bar{z}) \phi_2(0, 0, 0) \rangle \tag{4.11}
\]

where the brackets indicate a thermodynamic average which will have to be carried out when such two-point functions are to computed in the context of a specific statistical mechanics model. Extending the generators of \textbf{meta}(1, \( d \)) constructed above to two-body operators, the covariance is then expressed through the Ward identities

\[
X_0^{[2]} F = X_1^{[2]} F = Y_0^{[\pm 2]} F = Y_1^{[\pm 2]} F = 0.
\]

Each scaling operator is characterised by three constants \((\delta, \gamma, \bar{\gamma})\). Standard calculations (along the well-known lines of ortho- or meta-conformal invariance) then lead to

\[
F(t, z, \bar{z}) = F_0 \delta_{\delta_1, \delta_2} \delta_{\gamma_1, \gamma_2} \delta_{\bar{\gamma}_1, \bar{\gamma}_2} t^{-2\delta_1} \left( 1 + \beta \frac{z}{t} \right)^{-2\gamma_1/\beta} \left( 1 + \beta \frac{\bar{z}}{t} \right)^{-2\bar{\gamma}_1/\beta} \tag{4.12}
\]

where \( F_0 \) is a normalisation constant. This shows a cross-over between an ortho-conformal two-point function when \( t \ll z, \bar{z} \) and a novel scaling form in the opposite case \( t \gg z, \bar{z} \). We illustrate this for scalar quasi-primary scaling operators, where \( \gamma_1 = \bar{\gamma}_1 \)

\[
F(t, z, \bar{z}) \sim \begin{cases} 
  t^{-2\delta_1} \left( \frac{z}{t} \right)^{-2\gamma_1/\beta} & \text{if } t \ll z, \bar{z} \\
  t^{-2\delta_1} \exp \left[ -2\gamma_1 \frac{z + \bar{z}}{t} \right] & \text{if } t \gg z, \bar{z}.
\end{cases}
\]

If the time-difference is small compared to the spatial distance, the form of the correlator reduces to the one of standard, ortho-conformal invariance. For increasing time-differences \( t \), the behaviour becomes increasingly close to the known one of effectively 1D meta-conformal invariance.\(^{19}\)

The two-point function (4.12) can be written in the scaling form

\[
F(t, z, \bar{z}) = t^{-2\delta_1} f(z/t, \bar{z}/t).
\]

Using the algebraic construction described in \([60–62]\), and restricting to the \textquoteleft scalar\textquoteright case \( \gamma_1 = \bar{\gamma}_1 \) for notational simplicity, the scaling function \( f(u, v) \) can be extended from the sector \( u \geq 0, v \geq 0 \) to the full plane \((u, v) \in \mathbb{R}\) in the following form (setting \( \beta = 1 \))

\[
f(u, v) = \left( (1 + |u|)^2 + v^2 \right)^{-2\gamma_1} \tag{4.14}
\]

Figure 3 displays \( f(u, v) \). The change from the cusp, characteristic for 1D meta-conformal symmetry, along the \( v = 0 \) axis to the rounded form of 1D ortho-conformal symmetry, along the \( u = 0 \) axis, is clearly seen.

In figure 4, the variation of the scaling function (4.14) is shown in polar coordinates, viz. \( f = f(r \cos \psi, r \sin \psi) \), over against the length amplitude \( r \), for fixed values of the angle \( \psi \). The value \( \psi = 0 \) corresponds to the 1D meta-conformal case with its characteristic cusp at \( r = 0 \). The value \( \psi = \pi/2 \) corresponds to the 1D ortho-conformal case.

\(^{18}\) See footnote 13.

\(^{19}\) We did not yet carry out explicitly the full algebraic procedure which should in the \( t \gg z, \bar{z} \) limit produce the non-diverging behaviour \( F \sim t^{-2\delta_1} \exp \left[ -2\gamma_1 \frac{|z + \bar{z}|}{t} \right] \), see [62].
with is rounded profile near to \( r = 0 \). For larger values of \( r \), the decay of the scaling function becomes independent of \( \psi \). The other three quadrants look analogously.

4.3. The case \( p = 1/3 \)

Again, we refer to tables 3 and 4 for the rendering of the canonical coordinates \((\tau, w, \bar{w})\) in terms of the usual time-space coordinates \((t, x, y)\) and the meta-conformal weights. The correspondence between the generators is now as follows.
\[ A_n = \frac{1}{3} X_n + \frac{3}{\beta} Y^x_n, \quad B_n^\pm = X_n - \frac{3}{2\beta} Y^x_n \pm i\frac{3}{2\beta} Y^y_n. \] (4.15)

Once more, the co-variant two-point function is build from the quasi-primary scaling operators \( \phi(t, x, y, \gamma, \bar{\gamma}) \), where we recall \( \gamma = \frac{1}{2}(\gamma_x + i\gamma_y) \) and \( \bar{\gamma} = \frac{1}{2}(\gamma_x - i\gamma_y) \). Taking into the account the covariance of time- and space-translations, we write
\[
F = F(t, x, y, \gamma_1, \gamma_2, \bar{\gamma}_1, \bar{\gamma}_2) = \langle \phi(t_1, x_1, y_1, \gamma_1, \bar{\gamma}_1) \phi(t_2, x_2, y_2, \gamma_2, \bar{\gamma}_2) \rangle, \tag{4.16}
\]
where \( t = t_1 - t_2, x = x_1 - x_2, y = y_1 - y_2 \). In the canonical coordinates, and writing \( \tau = \tau_1 - \tau_2, w = w_1 - w_2 \) and \( \bar{w} = \bar{w}_1 - \bar{w}_2 \), the co-variant two-point function reads, up to normalisation
\[
F = \tau^{-2\vartheta_1} w^{-2\Delta_1} \bar{w}^{-2\bar{\Delta}_1}, \tag{4.17}
\]
and with the constraints \( \vartheta_1 = \vartheta_2, \Delta_1 = \Delta_2 \) and \( \bar{\Delta}_1 = \bar{\Delta}_2 \). One may re-express this in the original variables, see table 3, with a qualitative behaviour quite similar to the case \( p = -1 \) treated above.

5. Application: the directed Glauber–Ising chain

We now discuss how a meta-conformal dynamical symmetry is realised in the relaxational dynamics of the directed Glauber–Ising chain. On an infinitely long chain, Ising spins \( \sigma_n = \pm 1 \) are attached to each site \( n \), such that to each configuration \( \{\sigma\} \) of spins the energy \( \mathcal{H}[\sigma] = -\sum_n \sigma_n \sigma_{n+1} \) is associated. The dynamics proceed through flips of...
individual spins and is described by a markovian master equation [85]. The rates for a flip of the spin $\sigma_n$ is given by [42, 45]

$$w_n(\sigma_n) = \frac{1}{2} \left[ 1 - \frac{\gamma}{2} (1-v) \sigma_{n-1} \sigma_n - \frac{\gamma}{2} (1+v) \sigma_n \sigma_{n+1} \right]$$

(5.1)

where $\gamma = \tanh(2/T)$ parametrises the temperature and the left-right bias of the dynamics is described by the parameter $v$20. The influence of the parameter $v$ on the transition rates is illustrated in figure 5. Such a directed dynamics does no longer obey the condition of detailed balance, although global balance still holds. Therefore, with the rates (5.1), the equilibrium Gibbs–Boltzmann state is still a stationary state of the dynamics [42]. For either a fully disordered or else a thermalised initial state, the consequences of a non-vanishing bias $v \neq 0$ on the long-time relaxational properties, especially on the precise way how the equilibrium fluctuation-dissipation theorem is broken, have been studied in great detail [42, 45]. Analogous studies have also been carried out in a 2D directed kinetic Ising model [44, 46, 47] and the directed $d$-dimensional spherical model [43]. In particular, a 2D directed kinetic Ising model quenched to $T = 0$ from a fully disordered initial state shows strong evidence for a relaxational behaviour with a dynamical exponent $z = 1$ [46]. Important observables of interest are the two-time and single-time spin-spin correlators

$$C_n(t, s) := \langle \sigma_n(t) \sigma_0(s) \rangle, \quad C_n(t) := C_n(t, t) = \langle \sigma_n(t) \sigma_0(t) \rangle$$

(5.2)

where spatial translation-invariance will be admitted throughout. At present, we shall merely focus on how a meta-conformal dynamical symmetry is realised in this model. As we shall see, it will be essential to consider initial states with spatially long-ranged correlations, viz. $C_n(0) \sim |n|^{-N}$ for $|n| \gg 1$21.

From the rates (5.1), the equations of motion of the correlators are readily found [42]

$$\partial_t C_n(t) = -2(1 - \gamma)C_n(t) + \gamma \left( C_{n-1}(t) + C_{n+1}(t) - 2C_n(t) \right) + \delta_{n,0} Z(t) \quad (5.3a)$$

$$\partial_s C_n(\tau + s) = -(1 - \gamma)C_n(\tau + s) + \frac{\gamma}{2} \left( C_{n-1}(\tau + s, s) + C_{n+1}(\tau + s, s) - 2C_n(\tau + s, s) \right) + \frac{\gamma \nu}{2} \left( C_{n+1}(\tau + s, s) - C_{n-1}(\tau + s, s) \right) \quad (5.3b)$$

20 A bias might arise from the effect of an external electric field acting on charged particles or else particles moving on an inclined lattice in a gravitational field.

21 For unbiased dynamics with $v = 0$, it is known that long-ranged initial conditions with $N > 0$ do not modify the leading long-time relaxation behaviour of the Glauber–Ising chain [57].
where \( \tau = t - s \), the Lagrange multiplier \( Z(t) \) is fixed by the condition \( C_0(t) = 1 \), one has the compatibility condition \( C_n(t, t) = C_n(t) \) and the initial correlator \( C_n(0) \) must yet be specified.

It is known that the requirement of meta-conformal co-variance determines the scaling form of correlators \([62]\), rather than response functions as it is the case, e.g. for Schrödinger-invariance. Concentrating on the correlators \((5.2)\), from \((5.3a)\) it follows that the single-time correlator \( C_n(t) \) is independent of the bias \( v \) and that one should study the two-time correlators \( C_n(t, s) \). For illustration, consider first the infinite-temperature limit \( \gamma \rightarrow 0 \) but such that \( \gamma v \rightarrow \nu \) remains finite \([45]\). Take the continuum limit of \((5.3b)\) and let \( C(\tau + s, s; r) = e^{-\tau} \tilde{C}(\tau + s, s; r) \). This gives the equation \( (\partial_t - \nu \partial_r) \mathcal{C}(\tau + s, s; r) = 0 \), analogous to \((1.11)\), and with the solution \( \mathcal{C}(\tau + s, s; r) = \mathcal{C}(s; r + \nu \tau) \). In the special case \( s = 0 \) of a vanishing waiting time, one has \( C(0, 0; r) = \tilde{C}(0; r) \), \( C(0, r) \) is indeed the form predicted by meta-conformal invariance, up to an exponential prefactor.

We now analyse the long-time behaviour in more detail, and for any temperature \( T \gg 0 \). The equation of motion \((5.3b)\) is solved through a Fourier transformation

\[
\tilde{C}(\tau + s, s; k) = \sum_{n \in \mathbb{Z}} C_n(\tau + s, s) e^{-ink}, \quad C_n(\tau + s, s) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk e^{ikn} \tilde{C}(\tau + s, s; k) \tag{5.4}
\]

which in Fourier space leads to

\[
\tilde{C}(\tau + s, s; k) = \tilde{C}(s, k) \exp \left( - \left[ 1 - \gamma \cos k - i \gamma v \sin k \right] \tau \right). \tag{5.5}
\]

Tauberian theorems \([36]\) state that the long-time behaviour follows from the form of \( \tilde{C}(\tau + s, s; k) \) around \( k \approx 0 \). Here, we want to look at a ‘ballistic’ scaling regime where \( k \tau \) is being kept fixed, rather than that regime \( k^2 \tau = \text{cste.} \) typical for diffusive motion. Indeed, for diffusive scaling, the momenta \( k \sim \tau^{-1/2} \gg \tau^{-1} \) are much larger that the ones to be considered here. From now on, we consider a long-ranged initial correlator of the form \( C_n(0) \sim |n|^{-R} \), for \( |n| \rightarrow \infty \) and with \( R > 0 \). A simple explicit form \([53]\), which is symmetric in \( n \), has the required asymptotic behaviour and is normalised to \( C_0(0) = 1 \) reads, along with its Fourier transform \([49]\)

\[
C_n(0) = \frac{\Gamma(|n| + (1 - R)/2)}{\Gamma(|n| + (1 + R)/2)} \frac{\Gamma((1 + R)/2)}{\Gamma((1 - R)/2)}, \quad \tilde{C}_n(0; k) = \frac{\Gamma((1 + R)/2)^2}{\Gamma(R)} \left( 2 \sin \frac{|k|}{2} \right)^{R-1} \tag{5.6}
\]

such that indeed \( \tilde{C}_n(0; k) \approx \tilde{C}_0|k|^{R-1} \), for \( |k| \) sufficiently small.

(a) The most simple case arises when the waiting time \( s = 0 \). We can directly insert the initial correlator \((5.6)\) into \((5.5)\) and read off the two-point correlator in the requested scaling limit, and for the range \( 0 < R < 1 \),

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\[ C_n(\tau, 0) \approx \frac{\tilde{C}_0}{2\pi} \int dk \left| k \right|^{R-1} \left( 1 - \frac{\gamma}{2} k^2 \tau + \ldots \right) e^{ik(n+\gamma v)\tau} e^{-(1-\gamma)\tau} \]

\[ \approx \frac{\tilde{C}_0 \Gamma(\mathcal{N}) \cos(\pi \mathcal{N}/2)}{\pi} \frac{1}{(n + \gamma v \tau)^R} e^{-(1-\gamma)\tau} \]

\[ = \frac{\Gamma\left((1 + \mathcal{N}/2\right)^2 \cos(\pi \mathcal{N}/2)}{\pi} \frac{1}{(n + \gamma v \tau)^R} e^{-(1-\gamma)\tau} \]  

(5.7)

where the integral is taken from [40, equation (2.3.12)], see also [26], and (5.6) was used. The unbiased diffusive terms merely lead to corrections to scaling. Equation (5.7) reproduces indeed the prediction (1.13) of meta-conformal invariance, with \( \delta_1 = \frac{\gamma}{\mu} = \frac{8}{3} \), and up to an exponentially decaying prefactor \( 2^2 \) and a choice of scale of spatial distances. Clearly, both the bias \( v \neq 0 \) as well as long-ranged initial conditions with \( 0 < \mathcal{N} < 1 \) are necessary ingredients for the meta-conformal dynamical symmetry to arise.

(b) For arbitrary waiting times \( s > 0 \), we must now show, under suitable conditions and at least for \( s \) sufficiently large and for \( |k| \) sufficiently small, that \( \tilde{C}(s; k) \approx \tilde{C}(s)|k|^{R-1} \). If that is so, then the two-time correlator \( C_\mathcal{N}(\tau + s, s) \), see equation (5.5), will be of the same form as in (5.7), with a prefactor \( \tilde{C}(s) \) which might still depend on the waiting time \( s \).

The proof of this property requires to solve (5.3a). Define the Laplace transform \( \tilde{C}(p; k) := \int_0^\infty dt \ e^{-ps} \tilde{C}(s; k) \). The solution of (5.3a) reads in Laplace-Fourier space

\[ \tilde{C}(p; k) = \frac{\tilde{Z}(p) + \tilde{C}(0; k)}{p + 2(1 - \gamma \cos k)}. \]  

(5.8)

The Laprange multiplier \( \tilde{Z}(p) \) is found from the condition \( \tilde{C}_0(p) = 1/p \). Explicitly

\[ \frac{1}{p} = \frac{1}{2\pi} \int_{-\pi}^{\pi} dk \left[ \frac{\tilde{Z}(p)}{p + 2(1 - \gamma \cos k)} + \frac{\tilde{C}(0; k)}{p + 2(1 - \gamma \cos k)} \right]. \]  

(5.9)

Herein, the first integral can be taken from [42]. To analyse the second integral, we use again the explicit form (5.6) and consider the leading small-\( p \) behaviour of

\[ J(p; \gamma, \mathcal{N}) := \frac{\Gamma\left((1 + \mathcal{N}/2\right)^2}{\pi \Gamma(\mathcal{N})} \int_0^\pi dk \frac{\left(2 \sin k/2\right)^{\mathcal{N}-1}}{p + 2(1 - \gamma \cos k)} \approx \begin{cases} J(0; \gamma, \mathcal{N}) & \text{if } \gamma < 1 \\ J_\infty p^{\mathcal{N}/2-1} & \text{if } \gamma = 1 \end{cases} \]  

(5.10)

where \( J_\infty = \Gamma\left((1 + 1/2\right) \Gamma(1 - \mathcal{N}/2)/2^{\mathcal{N}} \sqrt{\pi} \). For \( \gamma < 1 \), \( J(0; \gamma, \mathcal{N}) \) is a finite constant. From the constraint (5.9), and since \( \mathcal{N} > 0 \), this implies for the leading small-\( p \) behaviour of the Laprange multiplier

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\(^{22}\) Such non-universal exponential factors also arise in other problems, for example the number \( \mathcal{N}_{\text{av}} \approx e^{N \ln 2} N^{\gamma-1} \) of a self-avoiding random walk (SAW) of \( N \gg 1 \) steps contains a non-universal fugacity \( \varepsilon \) and an universal exponent \( \gamma \) [41].
where the estimates (5.10) for $J(p; \gamma, \mathcal{R})$ were used. We see that the leading behaviour of $Z(p)$ is independent of the initial condition.

Using (5.11), we now examine the correlator (5.8) in the asymptotic double limit $p \to 0$ and $k \to 0$. Because of the dynamical exponent $z = 1$ of meta-conformal invariance, we expect that this limit should be taken such that $p/k$ is being kept fixed. First, for $\gamma < 1$, we find

$$\bar{C}(p; k) \simeq \frac{2\sqrt{1 - \gamma^2} \gamma^{-1} + C_0 |k|^{8-1}}{2(1 - \gamma) + o(p, k^2)} \simeq \sqrt{1 + \frac{\gamma}{1 - \gamma}} p^{-1} (1 + o(1)) \quad \text{if } \gamma < 1$$

(5.12)

because for $\mathcal{R} > 0$, the second term in the numerator is less singular than the first one. Hence, going back to sufficiently long waiting times $s \gg 1$, we obtain $\bar{C}(s; k) \simeq \sqrt{1 + \frac{\gamma}{1 - \gamma}}$ which is constant and independent of the long-range initial conditions. Hence for $\gamma < 1$ there is no meta-conformal invariance of the two-time correlator in the limit of large waiting times. Second, for $\gamma = 1$ we have instead

$$\bar{C}(p; k) \simeq \frac{2p^{-1/2} + C_0 |k|^{8-1}}{p + k^2} \simeq \left\{ \begin{array}{ll}
\Gamma\left(\frac{1+\gamma}{2}\right)^2 \Gamma\left(\frac{8}{2}\right) |k|^{8-1} p \quad & \text{if } \mathcal{R} < \frac{1}{2} \\
2p^{-3/2} & \text{if } \mathcal{R} > \frac{1}{2}
\end{array} \right.$$  

(5.13)

Hence, if $\mathcal{R} < \frac{1}{2}$, we have the leading long-time behaviour $\bar{C}(s; k) \simeq C_0 |k|^{8-1}$, with $C_0$ given in (5.6), for the single-time correlator. We have therefore verified a sufficient condition that the form of the two-time correlator $C_n(\tau + s, s)$ is in agreement with the expected form (1.13) of meta-conformal invariance. On the other hand, if $\mathcal{R} > \frac{1}{2}$, no clear evidence for such an invariance is found. Therefore, for large waiting times $s \to \infty$, meta-conformal invariance of the two-time correlator can only be established under more restrictive conditions than for $s = 0$ (or $s$ finite and sufficiently small).

We summarise the results of this section as follows.

**Proposition 10.** At zero temperature $T = 0$, the two-time spin-spin correlator $C_n(\tau, s)$ in the directed Glauber–Ising chain, with long-ranged initial correlators of the form $C_n(0) \sim |n|^{-\mathcal{R}}$ with $0 < \mathcal{R} < \frac{1}{2}$, takes for large waiting times $s \gg 1$ and large time differences $\tau = t - s \gg 1$ the form (1.13), predicted by meta-conformal invariance.

While we gave here an example of 1D meta-conformal invariance, we point out that the Lie algebra of 2D meta-conformal transformations is isomorphic to the dynamical symmetry [64] of the spatially non-local stochastic process of 1D diffusion-limited erosion [72] or the terrace-step-kink model [71, 82].
6. Conclusions

We have explored the construction of time-space transformations, with a dynamical exponent $z = 1$, which may have physical applications as dynamical symmetries. Ortho-conformal transformations have been the well-known standard example of such transformations, with spectacular applications to conformal field-theory, especially in 2D equilibrium phase transitions. Our main result is stated in table 1: there are infinite-dimensional Lie groups of time-space transformations, both for $d = 1$ and $d = 2$, which contain the same temporal and spatial translations as well as dilatations, as the ortho-conformal group, yet these transformations are in general not angle-preserving and hence cannot be ortho-conformal. The relationship between ortho- and meta-conformal transformations for any $d$ is stated in (3.10). The 1D meta-conformal case illustrates the interest in working with representations of the conformal group which uses non-orthogonal coordinates. For the 2D case, the associated Lie algebra is isomorphic to the direct sum of three Virasoro algebras, rather than two as one is used to from 2D ortho-conformal invariance. Tables 2, 3 and 4 show how the generic generators (4.3) are related to the physically motivated time-space transformations.

The meta-conformal transformations as constructed here are well-known to act as dynamical symmetries of a simple linear equation of ballistic transport. A new class of applications has been described here: the long-time, large-distance relaxation of non-equilibrium spin systems whose dynamics contains a directional bias. If in addition sufficiently long-ranged initial spatial correlations occur, then the dynamical scaling regime with $z = 1$ is described by meta-conformal invariance. We have shown this explicitly for the two-time spin-spin correlator of the directed Glauber–Ising chain, at vanishing temperature and for a decay exponent $0 < \kappa < \kappa_c = \frac{1}{2}$ of the initial spin-spin correlator $C_n(0) \sim |n|^{-\kappa}$.

While this kind of application merely uses the finite-dimensional sub-algebra of meta-conformal invariance, the full theory based on the infinite-dimensional symmetry remains to be constructed. On the other hand, one still must demonstrate that meta-conformal symmetries arise in systems which are not described by linear equations of motion. Previous experience from the phase-ordering kinetics of non-equilibrium spin systems (where \(z = 2\)), provides evidence that dynamical Schrödinger-invariance applies generically \[58\], for example to kinetic Ising and Potts models, although the Schrödinger group was originally constructed as the dynamical symmetry of the free diffusion equation. Therefore, by analogy a naturally-looking path for identifying meta-conformally invariant systems appears to be the study of directed spin systems in $d > 1$ spatial dimensions. Our results on the Glauber–Ising chain suggest that meta-conformal invariance might be found for directed systems quenched to temperatures $T \leq T_c$, that is below or onto the critical temperature $T_c$. The existence of dynamical scaling with $z = 1$ in such higher-dimensional models has already been demonstrated \[46\].

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