Montonen and Olive’s conjecture in 1977 that some theories possess a duality symmetry that interchanges the electrically charged particle with that of negatively charged t’ Hooft-Polyakov monopoles which relates strong coupling to weak coupling. This will be nearly the same as Shapere et al., who showed that the equation of motion of the coupled Einstein-Maxwell-axion dilaton system, occurs in low energy in String theory, is invariant under electromagnetic duality transformation; that also interchanges the strong and weak coupling limit of the theory. These ideas have been pursued in the $SO(6) \otimes SO(5)$ gauge groups as well as in the supersymmetric group of a four dimensional String. We analyse and find that they give clearer evidence of string theory in two and four dimensional supergravity theory. However, the application to supergravity is beset with the normal difficulties and suggestions have been made how to settle them.

PACS numbers:
Keywords: Superstring, Supergravity, Duality, Supermultiplet, Superspace

I. INTRODUCTION

The classical relativistic string theory, first proposed by Nambu and Goto, turned out to be valid only in 26 dimensions and was raised to the quantum level by Goddard, Goldstone, Rebbi, Thorn and Mandelstam. However, Scherk and Schwarz pointed out that the string theory, in 26 dimensions, can explain the physical phenomena including gravity, but this did not make much progress. It was shown by Deo and collaborators that a four dimensional superstring can easily be constructed and is anomaly free, modular invariant, free from ghosts and unitary. The results are those obtained with the group $SO(3,1)$ and $SO(6) \otimes SO(5)$. These have been enumerated in detail in the papers. However, we shall give some of the features in the next Section.

Like all string theories, the theory had the defect of on being unable to explain the other important features like gauging, duality and supersymmetry, relevant to supergravity theory. Our main purpose is to consider these problems and try to find the solutions. In the Section, we shall discuss the main features of duality as it enters in the domain of superstring theories. Here we use the element of supergravity theory and apply it to the superstring theory. Montonen in 1977 has conjectured that the spontaneously broken gauge symmetry possesses a duality symmetry that interchanges electrically charged particles with negative charged t’ Hooft-Polyakov monopoles. Such symmetry relates strong to weak coupling, because we set the coupling constant to its inverse. The best instance is found in relating the Montonen-Olive duality to globally supersymmetric Yang-Mills-Higgs system. More recently, Shapere et al. observed and showed that the equation of motion of coupled Einstein-Maxwell’s action, which is the effective action in string theory in low energy, is invariant under electromagnetic duality transformation and also interchanges the strong and weak coupling limit of the string theory.

It is very important and useful to construct the dual description of theories, containing antisymmetric tensor fields by opening up a new potential, whose curl gives the dual field strength, and thereby interchanging field equations with Bianchi identities. In this context, we shall discuss and generalise our results in the relativistic region and give our analysis in four dimensions in the subsequent sections.

In Section, we give the results of gauge fields and in Section, we discuss the elementary fermions in detail. In Section, we discuss the action integral for supergravity, following Mishra and Nilles. In this, we find that in the four dimension (not in ten dimension), one can most possibly write down a renormalisable Lagrangian. Our ideas will be somewhat different than those of Mishra and Nilles. Because, they will reflect our motivation of including all aspects of the theory with our proposal for the four dimensional superstring.

We also discuss the renormalisability of the supergravity Lagrangian in Section. This appears renormalisable. In last Section, we give the reasons for having three generation in the Standard Model. It is important therefore that there should be a renormalisable theory of strong, electromagnetic, weak and gravitational interactions at the same energy level.

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II. SUPERSTRING IN FOUR DIMENSIONS

In bosonic string theory, one starts with the Nambu-Goto string \( S_B = -\frac{1}{2\pi} \int d^2 \sigma (\partial_{\alpha} X^\mu \partial^\alpha X_\mu) \), \( \mu = 0, 1, 2, ..., 25 \), \( (2.1) \)

where \( \partial_{\alpha} = (\partial_\sigma, \partial_\tau) \). The general expression for the energy momentum tensors at two world sheet points \( z \) and \( \omega \)

\[
2 < T(z)T(\omega) > = \frac{C}{(z-\omega)^4} + \ldots
\]

\( C \), the coefficient of the most divergent term in the above equation \( (2.2) \), is the central charge. Methods and principles of calculation of \( C \) for a variety of strings, has been given in reference \([9]\). For free bosons, the central charge \( C_B = \delta_\mu^\mu \), with \( \mu = 0, 1, \ldots, 25 \); thereby the string action makes sense only in 26 dimensions. Due to this feature, one has to discard the theory and has to go to 10 dimensional superstring.

However, the 26-dimensional string theory can be made to work as a superstring having four bosonic and forty four fermionic degrees of freedom with SO(44) symmetry. One uses the Mandelstam’s proof of equivalence between one boson and two fermions in 1+1 field theory. This is true in a finite intervals or circles.

The 44 fermions can form 11 Lorentz vectors. The String action \( S_{FB} \), including these fermions, can be written as

\[
S_{FB} = -\frac{1}{2\pi} \int d^2 \sigma \left[ \partial^\alpha X^\mu (\sigma, \tau) \partial_\alpha X_\mu (\sigma, \tau) - i \sum_{j=1}^{11} \bar{\psi}^{\mu,j} \rho^\alpha \partial_\alpha \psi_{\mu,j} \right].
\]

(2.3)

We have

\[
\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \bar{\psi} = \psi^\dagger \rho^0.
\]

(2.4)

Here \( \rho^\alpha \)'s are imaginary, so the Dirac operators \( \rho^\alpha \partial_\alpha \) are real. In this representation of the Dirac algebra, the components of the world sheet spinor \( \psi^{\mu,j} \) are real and they are Majorana spinors. But the action \( (2.3) \) is not supersymmetric. The eleven \( \psi^{\mu,j} \) have to be further divided into two species; \( \psi^{\mu,j}, \ j = 1, 2, ..., 6 \) and \( \phi^{\mu,k}, \ k = 7, 8, ..., 11 \). For the group of six, the positive and negative parts of \( \psi^{\mu,j} = \psi^{(+)} \mu,j + \psi^{(-)} \mu,j \), whereas for the group of five, allowed the freedom of phase of creation operators for Majorana fermions in \( \phi^{\mu,k} = \phi^{(+)} \mu,k - \phi^{(-)} \mu,k \). This is due to the fact that we have Majorana like neutrinos rather than Dirac like ones. In fact ‘neutrinos’ are 6/4 = 24 in number and can be taken as right handed. The left handed ones can be 5 × 4 = 20 in number \([14]\). The action is now

\[
S = -\frac{1}{2\pi} \int d^2 \sigma \left[ \partial_{\alpha} X^\mu \partial^\alpha X_\mu - i \bar{\psi}^{\mu,j} \rho^\alpha \partial_\alpha \psi_{\mu,j} + i \bar{\phi}^{\mu,k} \rho^\alpha \partial_\alpha \phi_{\mu,k} \right],
\]

(2.5)

where \( j=1,2,\ldots,6 \) and \( k=7,8,\ldots,11 \).

Besides the \( SO(3,1) \), the action \( (2.5) \) is invariant under \( SO(6) \otimes SO(5) \). It is also invariant under the transformations

\[
\delta X^\mu = \tilde{\epsilon} (e^j \psi_j^\mu - e^k \phi_k^\mu),
\]

(2.6)

\[
\delta \psi^{\mu,j} = -ie^j \rho^\alpha \partial_\alpha X^\mu \epsilon,
\]

(2.7)

\[
\delta \phi^{\mu,k} = ie^k \rho^\alpha \partial_\alpha X^\mu \epsilon.
\]

(2.8)

Here \( \epsilon \) is a constant anticommuting spinor. \( e^j \) and \( e^k \) are eleven numbers of a row with i.e. \( e^5 = (00001000000) \) and \( e^{10} = (00000000010) \) and \( \sum_{j=1}^{6} e^j e_j = 6 \) and \( \sum_{k=7}^{11} e^k e_k = 5 \). In the formulation of the theory, one must have the proof that the commutator of two supersymmetric transformations gives a world sheet translation. With \( \Psi^\mu = e^j \psi_j^\mu - e^k \phi_k^\mu \), which is the superpartner of \( X^\mu \), we show that

\[
(\delta_1, \delta_2) X^\mu = a^\alpha \partial_\alpha X^\mu,
\]

(2.9)

and

\[
(\delta_1, \delta_2) \Psi^\mu = a^\alpha \partial_\alpha \Psi^\mu,
\]

(2.10)
where $a^\mu = 2i \bar{\epsilon} \rho^a \epsilon_2$ as expected in supersymmetric theories.

We proceed to quantise the theory. Let $b$ and $b'$ be the quanta of $\psi$ and $\phi$ fields in Neveu-Schwarz(NS) formulation and $d$ and $d'$ are the quanta of the above fields in Ramond(R) formulation. Specifically, for the $X$'s, we have,

$$X^\mu(\sigma, \tau) = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-i n \tau} \cos(n \sigma).$$  \hspace{1cm} (2.11)

In terms of complex coordinates $z = \sigma + i \tau$ and $\bar{z} = \sigma - i \tau$, we have,

$$X^\mu(z, \bar{z}) = x^\mu - i \alpha'_n \ln |z| + i \sum_{m \neq 0} \frac{1}{m} \alpha_m^\mu \bar{z}^{-m}.$$  \hspace{1cm} (2.12)

Further, \[
\psi_{\pm}^{\mu, j}(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \sqrt{2} \alpha_r^j e^{-i r (\sigma \pm \tau)}, \quad \text{and} \quad \phi_{\pm}^{\mu, k}(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{r \in \mathbb{Z} + \frac{1}{2}} \sqrt{2} \alpha_r^k e^{-i r (\sigma \pm \tau)} \quad \text{for NS sector},
\] \hspace{1cm} (2.13)

and,

$$\psi_{\pm}^{\mu, j}(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{m = -\infty}^{\infty} d_m^{\mu, j} e^{-im (\sigma \pm \tau)}, \quad \text{and} \quad \phi_{\pm}^{\mu, k}(\sigma, \tau) = \frac{1}{\sqrt{2}} \sum_{m = -\infty}^{\infty} d_m^{\mu, k} e^{-im (\sigma \pm \tau)} \quad \text{for R sector}. \hspace{1cm} (2.14)$$

In the formulation of superstring theory in the above section, a principal role has been played by the proof that the commutator of two supersymmetric transformations gives a world sheet translation. Therefore, it is necessary to have an exact framework in which the super Virasoro conditions can emerge as gauge conditions. For this, the action given in (2.10) should incorporate the superconformal invariance of a full superstring theory. These have been dealt with by one of us in Ref.[9]

Varying the field and Zweibein, the Noether current $J^a$ and the energy momentum tensor $T_{a\beta}$ vanishes,

$$J_a = \frac{\pi}{2e} \delta S = \rho^\beta \rho_\alpha \Psi^\mu \partial_\beta X_\mu = 0,$$  \hspace{1cm} (2.15)

and

$$T_{a\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{i}{2} \bar{\Psi} \rho_\alpha \partial_\beta \Psi = 0.$$  \hspace{1cm} (2.16)

These are the super Virasoro constrain equations. In a light cone basis, the vanishing of the lightcone components are obtained from the variation of the action given in equations (2.15) and (2.16),

$$J_{\pm} = \partial_\pm X_\mu \Psi^\mu_{\pm} = 0,$$  \hspace{1cm} (2.17)

and

$$T_{\pm} = \partial_\pm X^\mu \partial_\pm X_\mu + \frac{i}{2} \bar{\psi}_{\pm}^{\mu, j} \partial_\pm \psi_{\pm \mu, j} - \frac{i}{2} \bar{\phi}_{\pm}^{\mu, k} \partial_\pm \phi_{\pm \mu, k},$$  \hspace{1cm} (2.18)

where $\partial_{\pm} = \frac{1}{2} (\partial_\tau \pm \partial_\sigma)$.

For a brief outline, let $L_m$, $G_r$ and $F_m$ be the Super Virasoro generators of energy, momenta and currents. Then,

$$L_m = \frac{1}{\pi} \int_{-\pi}^{\pi} d\sigma e^{im\sigma} T_{++}$$

$$= \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} : \alpha_{-n} \cdot \alpha_{m+n} : + \frac{1}{2} \sum_{r \in \mathbb{Z} + \frac{1}{2}} (r + \frac{1}{2} m) : (b_{-r} \cdot b_{m+r} - b'_{-r} \cdot b'_{m+r}) : \quad \text{for NS}$$

$$= \frac{1}{2} \sum_{n = -\infty}^{\infty} : \alpha_{-n} \cdot \alpha_{m+n} : + \frac{1}{2} \sum_{n = -\infty}^{\infty} (n + \frac{1}{2} m) : (d_{-n} \cdot d_{m+n} - d'_{-n} \cdot d'_{m+n}) : \quad \text{for R} \hspace{1cm} (2.19)$$

$$G_r = \frac{\sqrt{2}}{\pi} \int_{-\pi}^{\pi} d\sigma e^{ir\sigma} J_+ = \sum_{n = -\infty}^{\infty} \alpha_{-n} \cdot (e^i b_{n+r,j} - e^k b'_{n+r,k}),$$  \hspace{1cm} (2.20)

and

$$F_m = \sum_{n = -\infty}^{\infty} \alpha_{-n} \cdot (e^i d_{n+m,j} - e^k d'_{n+m,k}) = \sum_{n = -\infty}^{\infty} \alpha_{-n} \cdot D_{n+m}.$$  \hspace{1cm} (2.21)
and satisfy the super Virasoro algebra, with central charge $C = 26$ for the action of equation (2.5),

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m,-n},$$  \hspace{1cm} (2.22)

$$[L_m, G_r] = \frac{1}{2}(m - r)G_{m+r}, \hspace{1cm} \text{for NS} \hspace{1cm} (2.23)$$

$$\{G_r, G_s\} = 2L_{s+r} + \frac{C}{3}(r^2 - \frac{1}{4})\delta_{r,-s}, \hspace{1cm} (2.24)$$

$$[L_m, F_n] = \frac{1}{2}(m - n)F_{m+n}, \hspace{1cm} \text{for R} \hspace{1cm} (2.25)$$

$$\{F_m, F_n\} = 2L_{m+n} + \frac{C}{3}(m^2 - 1)\delta_{m,-n}, \hspace{1cm} m \neq 0. \hspace{1cm} (2.26)$$

Equations (2.24) and (2.26) can be obtained using Jacobi identity.

This is also known that the normal ordering constant of $L_0$ is equal to one and we define the physical states, satisfying

$$(L_0 - 1)|\phi > = 0, \hspace{0.5cm} L_m|\phi > = 0, \hspace{0.5cm} G_r|\phi > = 0 \hspace{1cm} \text{for} \hspace{0.5cm} r, m > 0, \hspace{0.5cm} \text{NS} \hspace{0.5cm} \text{Bosonic \hspace{0.5cm} (2.27)}$$

$$L_m|\psi > = F_m|\psi > = 0, \hspace{1cm} \text{for} \hspace{0.5cm} m > 0, \hspace{0.5cm} \text{R \hspace{0.5cm} Fermionic \hspace{0.5cm} (2.28)}$$

and

$$(L_0 - 1)|\psi >_{\alpha} = (F_{\alpha}^2 - 1)|\psi >_{\alpha} = 0. \hspace{1cm} (2.29)$$

So we have,

$$(F_{\alpha} + 1)|\psi >_{\alpha} = 0 \hspace{0.5cm} \text{and} \hspace{0.5cm} (F_{\alpha} - 1)|\psi >_{\alpha} = 0. \hspace{1cm} : \hspace{0.5cm} \text{R. \hspace{0.5cm} (2.30)}$$

These conditions shall make the string model ghost free. It can be seen in a very simple way. Applying $L_0$ condition, the state $\alpha^0_{-1}|0, k >$ is massless. The $L_1$ constraint gives the Lorentz condition $k^\mu|0, k > = 0$, implying a transverse photon and with $\alpha^0_{-1}|\phi > = 0$ as Gupta-Bleuler condition. Applying $L_2, L_3, \ldots, \text{constraints$, one obtains} \alpha^0_m|\phi > = 0.$

Further, since $[\alpha^0_{-1}, G_{r+1}]|\phi > = 0,$ we have $b^0_{r,j}|\phi > = 0$ and $b^0_{r,k}|\phi > = 0.$ All the time components are eliminated from Fock space.

Ghosts do not couple to physical states, therefore these states must be of the form (up to null states)$^{12}$

$$|\{m\}p\rangle_M \otimes c_1|0\rangle_G, \hspace{1cm} (2.31)$$

where $|\{m\}p\rangle_M$ denotes the occupation numbers and momentum of physical matter states. The operator $c_1$ lowers the energy of the states by one unit and is necessary for BSRT invariance. The ghost is responsible for lowering the ground state energy producing shiftable tachyon($F_2$ picture)

$$(L_0^M - 1)|\psi >_{\text{phys}} = 0. \hspace{1cm} (2.32)$$

Therefore, the mass shell condition is

$$\alpha^0 M^2 = N^B + N^F_{NS} - 1 : \hspace{0.5cm} \text{NS,} \hspace{1cm} (2.33)$$

$$\alpha^0 M^2 = N^B + N^F_R - 1 : \hspace{0.5cm} \text{R, \hspace{0.5cm} (2.34)}$$

where

$$N^B = \sum_{m=1}^{\infty} \alpha_m \alpha_{-m} \hspace{1cm} (2.35)$$

$$N^F_{NS} = \sum_{r=1/2}^{\infty} r(b_r \cdot b_r + b'^r \cdot b'^r) : \hspace{0.5cm} \text{NS,} \hspace{1cm} (2.36)$$

and

$$N^F_R = \sum_{m=1}^{\infty} m(d_m \cdot d_m + d'_m \cdot d'_m) : \hspace{0.5cm} \text{R.} \hspace{1cm} (2.37)$$

In general,

$$\alpha^0 M^2 = -1, -1/2, 0, 1/2, 1, 3/2, \ldots, \hspace{1cm} (2.38)$$
in the NS sector. This, in the shifted Hilbert state, is
\[ \alpha' M^2 = -1/2, 0, 1/2, 1, 3/2, .... \] (2.39)

Due to presence of Ramond and Neveu-Schwartz sectors, with periodic and anti-periodic boundary conditions, we can effect a GSO projection on the mass spectrum on NSR model. The projection, as desired, is
\[ G = \frac{1}{2} \left( 1 + (-1)^{F+F'} \right), \]
where \( F = \sum b_r \cdot b_r \) and \( F' = b'_r \cdot b'_r \). This will eliminate the half integral value from the mass spectrum by choosing
\[ G = 1, \]
We have, therefore
\[ \alpha' M^2 = -1, 0, 1, 2, 3, .... \] (2.41)

The G.S.O. projection eliminates the half integral values. The tachyonic self energy of bosonic sector
\[ <0 | (L_0 - 1)^{-1} | 0 > \]
is cancelled by \( - <0 | (F_0 + 1)^{-1} (F_0 - 1)^{-1} | 0 > \), the negative sign being due to the fermionic loop. Such tadpole cancellations have been noted also by Chattaraputi et al in reference [18]. One can proceed a step further and write down the world sheet charge. To be more sure, consider the supersymmetric charge
\[ Q = \frac{1}{\pi} \int_0^{\pi} \rho^0 \rho^1 \partial \alpha X^\mu \Psi_\mu d\sigma. \]
(2.42)
The supersymmetric result is
\[ \sum \{ Q^\dagger, Q \} = 2H \quad \text{and} \quad \sum |Q_\alpha | \phi_\alpha > |^2 = 2 < \phi_\alpha | H | \phi_\alpha >. \] (2.43)

The ground state is of zero energy. There is no tachyon. The physical mass spectrum in both sectors are integral numbers of Regge trajectory, \( \alpha M^2 = 0, 1, 2, .... \)

III. POLCHINSKI SUPERSTRING

The last section summarises what have been done earlier. Now we propose to construct a theory in four dimension within the following observation of Polchinski, with a link between superstring states and generator in Conformal Field Theory(CFT). For example, for any conserved charge \( Q \), the operator equation \( QA \) is \( Q(\psi_A) \). If \( A \) is a unit operator \( \hat{I} \) and
\[ Q = \alpha_m = (2\pi)^{-1} \int dt \ z^m \ \partial X \quad \text{for} \ m > 0, \]
(3.1)
so that \( \partial X \) is analytic and the integral vanishes for \( m \geq 0 \). We get \( \alpha_m |\psi_I \rangle = 0 \), for \( m \geq 0 \). The exact correspondance of the unit operator and string operators \( |0, 0 \rangle \) is thus
\[ \hat{I} \leftrightarrow |0, 0 \rangle. \] (3.2)
Similarly,
\[ : e^{k \cdot X(z)} : \leftrightarrow |0, k \rangle. \] (3.3)

Unitarity gives the normalisation
\[ <0, k' |0, k > = 2\pi \delta(k - k'). \] (3.4)
This can be generalised to momentum states, with \( k_0 = \bar{k} \),
\[ <0, k' |0, k > = (2\pi)^3 (2k_0) \delta^3(k - k'). \] (3.5)

With these remarks, we shall follow the work of Li, where \( 1/2 n(n - 1) \) generators of \( O(n) \) are represented by
\[ L_{ij} = X_i \frac{\partial}{\partial X_j} - X_j \frac{\partial}{\partial X_i}, \quad i,j = 1, \ldots, n. \] (3.6)

The commutation among the generators, called Lie algebra can be worked out. The rule is to write
\[
\left[ \frac{\partial}{\partial X_i}, X_j \right] = \delta_{ij},
\] (3.7)
so that we have
\[
[L_{ij}, L_{kl}] = \delta_{jk} L_{il} + \delta_{il} L_{jk} - \delta_{ik} L_{jl} - \delta_{jl} L_{ik}.
\] (3.8)

Hence one must have \( \frac{1}{2} n(n - 1) \) vector gauge bosons \( W_{ij}^\mu \) with the transformation law
\[
W_{ij}^\mu \rightarrow W_{ij}^\mu + \epsilon_{ik} W_{kj}^\mu + \epsilon_{jl} W_{li}^\mu, \quad W_{ij}^\mu = -W_{ji}^\mu,
\] (3.9)
where \( \epsilon_{ij} = -\epsilon_{ji} \) is the infinitesimal parameters which characterise such rotation in \( O(n) \). Under gauge transformation of the second kind, we have
\[
W_{ij}^\mu \rightarrow W_{ij}^\mu + \epsilon_{ik} W_{kj}^\mu + \epsilon_{jl} W_{li}^\mu + \frac{1}{g} \partial^\mu \epsilon_{ij}.
\] (3.10)

The Yang-Mills Lagrangian is then written as
\[
L = -\frac{1}{4} |F_{ij}^{\mu\nu}|^2,
\] (3.11)
with
\[
F_{ij}^{\mu\nu} = \partial^\mu W_{ij}^{\nu} - \partial^\nu W_{ij}^{\mu} + g \left( W_{ik}^{\mu} W_{kj}^{\nu} - W_{ik}^{\nu} W_{kj}^{\mu} \right),
\] (3.12)
\( F_{ij}^{\mu\nu} \) has the obvious properties, namely,
\[
\Box F_{ij}^{\mu\nu} = 0, \quad \partial_\mu F_{ij}^{\mu\nu} = \partial_\nu F_{ij}^{\mu\nu} = 0, \quad \text{and} \quad F_{ij}^{\mu\mu} = 0.
\] (3.13)

There are two sets of field strength tensors which are found in the model, one for \( SO(6) \) and the other for \( SO(5) \). Since \( \Box F_{ij}^{\mu\nu} = 0 \), one can take plane wave solution and write
\[
F_{ij}^{\mu\nu}(x) = F_{ij}^{\mu\nu}(p) \ e^{ipx},
\] (3.14)
so that,
\[
p^2 F_{ij}^{\mu\nu}(p) = p_\mu F_{ij}^{\mu\nu}(p) = p_\nu F_{ij}^{\mu\nu}(p) = 0, \quad F_{ij}^{\mu\nu}(p) = 0 \quad \text{and} \quad F_{ij}^{\mu\nu}(p) = -F_{ji}^{\mu\nu}(p).
\] (3.15)

These are physical state conditions \( (3.27)-(3.28) \) as well and
\[
L_0 F_{ij}^{\mu\nu}(p) = 0, \quad G_2 F_{ij}^{\mu\nu}(p) = 0, \quad \text{and} \quad L_1 F_{ij}^{\mu\nu}(p) = 0.
\] (3.16)

The field strength tensor, satisfying \( (3.16) \), is found to be
\[
F_{ij}^{\mu\nu}(p) = b_i^\alpha b_j^\beta b_\alpha^\mu b_\beta^\nu |0,p> + \epsilon_{ij}(p^\mu \alpha_{\nu-1} - p^\nu \alpha_{\mu-1})|0,p>.
\] (3.17)

with \( \epsilon_{ij} = \epsilon_{i\alpha}^\alpha \epsilon_{j\beta}^\beta \epsilon_{\alpha\beta} \), \((i,j) = 1,\ldots,6\) for \( O(6) \) and \( 1,\ldots,5 \) for \( O(5) \) with \( b' \) replaced by \( b \). For simplicity, we drop the \( ^\dagger \)'s. In terms of the excitation quanta of the string, the vector generators are
\[
W_{ij}^\mu = \frac{1}{\sqrt{2ng}} n_\kappa \epsilon^{\kappa\mu\nu\sigma} b_{\nu,i} b_{\sigma,j} + \epsilon_{ij} \alpha_{\mu}^\nu, \quad \text{with} \quad \epsilon_{ij} = \epsilon_i^\alpha \epsilon_j^\beta \epsilon_{\alpha\beta},
\] (3.18)
where \( n_\kappa \) is the time-like four vector and can be taken as \((1,0,0,0)\). One finds that
\[
\partial^\mu W_{ij}^\mu - \partial^\nu W_{ij}^\nu = p^\mu W_{ij}^\nu - p^\nu W_{ij}^\mu
\]
\[
= \frac{1}{\sqrt{2ng}} \left( n_\kappa \epsilon^\nu\mu\alpha_{\nu} - n_\kappa \epsilon^\kappa\mu\nu \alpha_{\kappa}^\nu \right) b_{\lambda,i} b_{\sigma,j} + \epsilon_{ij}(p^\mu \alpha_{\nu-1} - p^\nu \alpha_{\mu-1}).
\] (3.19)
As $\mu$ must be equal to $\nu$, if $\kappa, \lambda$ and $\sigma$ are the same, then the first term vanishes and we have

$$g \left( W_{ik}^\mu W_{kj}^\nu - W_{ik}^\nu W_{kj}^\mu \right) = \frac{1}{2n} \left( n_\kappa \epsilon^{\kappa\lambda\mu\nu} b_{\lambda,i} b_{\sigma,k} n_\kappa' \epsilon^{\kappa'\nu\lambda'\sigma'} b_{\lambda',k} b_{\sigma,j} - \mu \leftrightarrow \nu \right) + \epsilon_{ik} \epsilon_{kj} \left[ \alpha_{-1}^\mu, \alpha_{-1}^\nu \right]$$

(3.20)

In the above, we have used

$$\{b_{\lambda,k}, b_{\sigma,k}\} = \eta_{\lambda\sigma} \delta_{kk} = n \eta_{\lambda\sigma},$$

(3.21)

and the creation operators $\alpha^\mu$ and $\alpha^\nu$ commute. Equation (3.17) is referred as the field strength. Since the product of pairs of $b$ and $b'$ commute, the gauge group of the action (2.5) is the product group $SO(6) \otimes SO(5)$. This is same as the symmetry group of the action (2.23). For O(6), we have $i, j = 1, \ldots, 6$ and for O(5), we have to replace $b$ by $b'$ with $i, j = 1, \ldots, 5$. Now, we denote

$$F^{(\alpha)}_{\mu\nu} = \partial_\mu A^{(\alpha)}_\nu - \partial_\nu A^{(\alpha)}_\mu,$$

(3.22)

as a 12-dimensional vector which can be found out from (3.17) and that

$$F^{\mu\nu(\alpha)}(x) = \int d^3x \frac{1}{\sqrt{2p^0}} \epsilon^{\alpha i j} (b^{(\mu)}_i b^{(\nu)}_j)|0, p > e^{ipx}, \alpha = 2, \ldots, 7$$

(3.23)

$$= \int d^3x \frac{1}{\sqrt{2p^0}} \epsilon^{\alpha i j} (b^{(\mu)}_i b^{(\nu)}_j)|0, p > e^{ipx}, \alpha = 8, \ldots, 12,$$

(3.24)

and

$$F^{\mu\nu(1)}(x) = \int d^3x \frac{1}{\sqrt{2p^0}} (p^{(\mu)} \alpha^\nu_{-1} - p^{(\mu)} \alpha^{\nu}_1)|0, p > e^{ipx} \alpha = 1.$$  

(3.25)

It should be noted that we have not taken account of the degrees of freedom arising out of SO(3,1) vectors. We shall do what follows, by means of a complex operator field $\lambda = \lambda_1 \pm i \lambda_2$.

### IV. INVARIANCE AND DUALITY

The basic idea is to prove that the fields we have observed are dual [21, 22, 23]. With the pair of field variables $E^{(\beta)i}$ and $B^{(\alpha)i}$, the action $S$, in flat spacetime,

$$S = \frac{1}{2} \int d^4x \left( B^{(\alpha)i} \mathcal{L}_{\alpha\beta} E^{(\beta)i} + B^{(\alpha)i} B^{(\alpha)i} \right),$$

(4.1)

where

$$E^\alpha_i = \partial_0 A^{(\alpha)}_i - \partial_i A^{(\alpha)}_0, \quad B^{(\alpha)i} = e^{ijk} \partial_j A^{(\alpha)}_k, \quad 1 \leq i, j, k \leq 3,$$

(4.2)

with

$$\mathcal{L} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

(4.3)

In defining (4.2), we have used (3.22). The action $S$ in (4.1) is invariant under the following transformations

$$\delta A^{(\alpha)}_0 = \Psi^{(\alpha)}, \quad \delta A^{(\alpha)}_i = \partial_i A^{(\alpha)}.$$  

(4.4)

For the $\Psi^{(\alpha)}$, we can set $A^{(\alpha)}_0 = 0$. The equation of motion of the field $A^{(2)}_i$ is

$$\epsilon^{ijk} \partial_k (B^{(j)k} - E^{(j)k}) = 0.$$  

(4.5)

Since no time derivative is involved, $A^{(2)}_i$ can be treated as an auxiliary field. We can eliminate this so that we are left with

$$B^{(2)i} = E^{(1)i} + \partial_i \phi,$$  

(4.6)
for some $\phi$ which we shall take to be zero so that we have

$$B^{(2)k} = E^{(1)}_{k},$$

and on substitution, the action becomes

$$S = -\frac{1}{2} \int d^4x \left( B^{(1)i} B^{(1)i} - E^{(1)}_{i} E^{(1)}_{i} \right),$$

in the gauge $A^{(1)}_{0} = 0$. The action in equation (4.11) is manifestly invariant under duality symmetry

$$A^{(a)}_{\mu} \to L_{\alpha\beta} A^{(b)}_{\mu},$$

which implies the transformation

$$\left( \begin{array}{c} B^{(1)i} \\ E^{(1)}_{i} \end{array} \right) = L \left( \begin{array}{c} B^{(1)i} \\ E^{(1)}_{i} \end{array} \right),$$

when we use the equation of motion for (4.7) of $A^{(2)}_{0}$. The action (4.1) becomes much simpler in curved space time for the field $A^{(1)}_{\mu}$,

$$S = -\frac{1}{4} \int d^4x \sqrt{-g} g^{\mu\rho} g^{\nu\sigma} F^{(1)}_{\mu\nu} F^{(1)}_{\rho\sigma}.$$  

This can be quantised in the form

$$S = -\frac{1}{8} \int d^4x \left( B^{(a)i} \mathcal{L}_{\alpha\beta} F^{(b)i} - \frac{g_{ij}}{\sqrt{-g}} B^{(a)ij} B^{(b)ij} + \epsilon_{ijk} g^{00} B^{(a)ij} L_{\alpha\beta} B^{(b)jk} \right).$$

We now generalise the above (4.12) to our case and get

$$S_{A} = \int d^4x \sqrt{-g} \left( R - \frac{1}{2\lambda_{2}} \partial_{\mu} \lambda \partial^{\mu} \lambda - \frac{1}{4} \lambda_{2} F^{(a)}_{\mu\nu} (LML)_{ab} F^{(b)\mu\nu} + \frac{1}{4} \lambda_{2} F^{(a)}_{\mu\nu} L_{ab} F^{(b)\mu\nu} + \frac{1}{8} g^{\mu\rho\nu\sigma} Tr(\partial_{\mu} M L \partial_{\nu} M L) \right).$$

Here $\lambda = \lambda_{1} \pm i\lambda_{2}$ and $A^{(a)}_{\mu}$, $3 \leq a \leq 12$, are 10 abelian gauge fields i.e.

$$F^{(a)}_{\mu\nu} = \partial_{\mu} A^{(a)}_{\nu} - \partial_{\nu} A^{(a)}_{\mu}, \quad \tilde{F}^{(a)}_{\mu\nu} = \frac{1}{2} (\sqrt{-g})^{-1} \epsilon^{\mu\nu\rho\sigma} F^{(a)}_{\rho\sigma},$$

and

$$L = \begin{pmatrix} 0 & I_{6} \\ I_{6} & 0 \end{pmatrix}.$$  

$I_{n}$ is a $n \times n$ unitary matrix, $M$ is the $12 \times 12$ matrix valued scalar fields satisfying the constraint,

$$M^{T} = M, \quad \text{and} \quad M^{T} LM = L,$$

and is given by

$$M = \begin{pmatrix} G^{-1} & -B G^{-1} \\ -B G^{-1} & G - B G^{-1} \end{pmatrix}.$$  

The internal metric $G_{\alpha\beta} = E_{\alpha}^{(a)} \delta_{ab} E_{\beta}^{(b)}$ and $(G_{\alpha\mu} B_{\mu\alpha})$ have limit $0 \leq (\mu, \nu) \leq 3$ and $1 \leq a \leq 6$ and the transformation can be realised as transformation of $G$ and $B$. $B_{\mu\nu}$ is determined from the equation

$$H_{\mu\nu\rho} = \partial_{\mu} B_{\nu\rho} + A^{(a)}_{\mu} \mathcal{L}_{ab} F^{(b)}_{\nu\rho} = -\left( \sqrt{-g} \right)^{-1} \epsilon_{\mu\nu\rho\sigma} \partial^{\sigma} \lambda_{1}.$$  

$H_{\mu\nu\rho}$ is the tensor which gives rise to pseudotensor $B_{\mu\nu}$. The equation of motion have the symmetry

$$\lambda - \frac{a\lambda + b}{c\lambda + d} : \quad F^{(a)}_{\mu\nu} \to c\lambda_{2} (ML)_{ab} \tilde{F}^{(b)}_{\mu\nu} + (c\lambda_{1} + d) F^{(a)}_{\mu\nu},$$  

where $c\lambda_{2}$.
with

\[ ad - bc = 1. \quad (4.20) \]

Particularly, if \( a=0, b=1, c=1 \) and \( d=0 \), then the transformations are

\[ \lambda \rightarrow \frac{1}{\lambda}, \quad F^{(a)}_{\mu\nu} \leftrightarrow -\lambda_1 F^{(a)}_{\mu\nu} - \lambda_2 (ML)_{ab} F^{(b)}_{\mu\nu}. \quad (4.21) \]

If \( \lambda_1 = 0 \), then the transformation takes electric field to magnetic field and vice versa. It also takes \( \lambda_2 \) to \( \frac{1}{\lambda_2} \) i.e. this duality transformation takes a strong coupling to a weak coupling theory and vice versa. The transformation is referred to as the strong-weak coupling or electric-magnetic duality transformation.

We further note that the Ricci tensor is

\[ R_{\mu\nu} = \frac{\partial \lambda \partial \nu \lambda + \partial \mu \partial \lambda \lambda}{4(\lambda_2)^2} + 2\lambda_2 F^{(a)}_{\rho\mu}(ML)_{ab} F^{(b)}_{\rho\nu} - \frac{1}{2} \lambda_2 (ML)_{ab} F^{(b)}_{\rho\nu}, \quad (4.22) \]

\[ D_{\mu}\lambda - (ML)_{ab} F^{(b)\mu\nu} + \lambda_1 \overline{F}^{(a)\mu\nu} = 0, \quad (4.23) \]

\[ \frac{D^\mu D^\nu \lambda}{(\lambda_2)^2} + i \frac{D^\mu \lambda D^\nu \lambda}{(\lambda_2)^2} - iF^{(a)}_{\mu\nu}(ML)_{ab} F^{(b)\mu\nu} + \overline{F}^{(a)}_{\mu\nu} (L)_{ab} F^{(b)\mu\nu} = 0, \quad (4.24) \]

where \( D_{\mu} \) is the standard covariant derivative with metric \( g^{\mu\nu} \). The Bianchi identity of the field strength tensor \( F^{(a)}_{\mu\nu} \) are given as

\[ D_{\mu} \overline{F}^{(a)\mu\nu} = 0. \quad (4.25) \]

This complete our discussion on this aspect of the theory. The last equation \( (4.25) \) is due to duality.

**V. GAUGE FIELDS**

Whether in supersymmetry or supergravity, there will be need for gauge fields. The total number of these depend on the particular case under consideration. In our case, the hypercharge vector \( B_\mu \) is given by the state

\[ |\phi(p)\rangle = \alpha_{-1}^\mu |0, p > B_\mu. \quad (5.1) \]

\( L_0 \) condition fixes the mass of \( B_\mu \) as zero and \( L_1 \) satisfy the Lorentz condition \( p^\mu B_\mu(x) = 0 \). We have also tensors \( b^\mu_+, b^\mu_-, b^\mu_+ \), which are massless and they have vector fields \( A^\mu_i \) in terms of which

\[ G^\mu_{ij} = \int \frac{d^3p}{\sqrt{2p_0}} e^{ipx} \left( b^\mu_i b^\nu_j - b^\nu_i b^\mu_j - \frac{2}{3} \delta_{ij} b^\mu_j b^\nu_i \right) |0, p > \quad (5.2) \]

\[ = \partial^\mu A^\nu_{ij} - \partial^\nu A^\mu_{ij} + \left( A^\mu_i A^\nu_{ji} - A^\nu_i A^\mu_{ji} \right). \quad (5.3) \]

Here \( i, j = 1, 2, ..., 6 \) and we have omitted \(-\frac{1}{2} \). The eight gluons are obtained by using the eight Gell-Mann \( \lambda_i \) matrices.

\[ V^\mu_i = (\lambda_i)_{ij} A^\mu_j. \quad (5.4) \]

Similarly for \( b^\nu_+ \), we get the \( W^\mu_i \)-mesons

\[ W^\mu_{ij} = \int \frac{d^3p}{\sqrt{2p_0}} e^{ipx} \left( b^\nu_i b^\mu_j - b^\nu_j b^\mu_i - \delta_{ij} b^\nu_j b^\nu_i \right) |0, p >, \quad (5.5) \]

\[ = \partial^\nu W^\mu_{ij} - \partial^\nu W^\mu_{ij} + \left( W^\mu_i W^\nu_j - W^\mu_j W^\nu_i \right). \quad (5.6) \]

with

\[ W^\mu_i = (\tau_i)_{ij} W^\mu_j. \quad (5.7) \]
We have got \( i, j = 1, 2, \ldots, 5 \). Here \( \tau_l, l = 1, 2, 3 \) are the \( 2 \times 2 \) isospin matrices. Thus there are 8 gluons and 3 W-bosons and one hypercharge vector \( B_{\mu} \). All together, we have 12 vector mesons. We call all of them as the components of \( W_\mu \). However, these vector mesons should be the part of the standard model group \( SU_C(3) \otimes SU_L(2) \otimes U_Y(1) \). Without breaking the symmetry, one can take the other method where we take the Wilson loop

\[
U_\gamma = P \exp \left( \oint_\gamma A_\mu \, dx^\mu \right).
\] (5.8)

\( P \) represents the ordering of each term with respect to the closed path \( \gamma \). This breaking can be accomplished by one element \( U_0 \) of SU(4), such that

\[
U_0^2 = 1,
\] (5.9)

with SO(6) = SU(4), so that the descendant groups are

\[
SU(6) \otimes SU(5) \otimes Z_2 \rightarrow SU_C(3) \otimes B_{-L}(1) \otimes U_Y(1),
\] (5.10)

making an identification with the usual low energy phenomenology. But this is not the standard model. We have an additional U(1). However, there is an instance in \( E_6 \), where there is a reduction of rank by one and several U(1)'s. Following the same idea, we may take

\[
U_\gamma = (\alpha_\gamma) \otimes \left( \begin{array}{cc} \beta_\gamma & 0 \\ 0 & \beta_\gamma^{-1} \end{array} \right) \otimes \left( \begin{array}{c} \delta_\gamma \\ \delta_\gamma^{-1} \end{array} \right).
\] (5.12)

\( \alpha_\gamma^3 = 1 \) such that \( \alpha_\gamma \) is the cube root of unity. The structure of \( U \) inevitably lowers the rank by one and we get,

\[
SU(6) \otimes SU(5) \otimes Z_3 \rightarrow SU_C(3) \otimes SU_L(2) \otimes U_Y(1),
\] (5.13)

which is our supersymmetric standard model.

To use this idea, let the color vector fields denoted by \( V_\mu^l, l=1, \ldots, 8 \), are gluon fields. \( l=9 \) is the \( U_Y(1) \) field and \( l=10, 11, 12 \) stand for W-mesons fields. We shall use the temporal gauge \( [24] \), where \( V_0^l = 0 \). For each \( V_\mu^l \), there are electric field strength \( E_\mu^l \) and magnetic field strength \( B_\mu^l \). Following Nambu [25], the combination

\[
\mathcal{F}_\mu^l = \frac{1}{\sqrt{2}} \left[ E_\mu^l + B_\mu^l \right],
\] (5.14)

satisfies the only nonvanishing equal time commutation relation

\[
\left[ \mathcal{F}_\mu^l(x), \mathcal{F}_\nu^m(y) \right] = i\delta^{lm} \epsilon_{ijk} \partial^k (x-y).
\] (5.15)

We then construct Wilson’s loop line integrals to convert the ordinary derivatives acting on fermion fields to respective gauge covariant derivatives. In the above, we note that the equations (5.14) and (5.15) are obtained if \( E_\mu^l \rightarrow -B_\mu^l \) and \( B_\mu^l \rightarrow E_\mu^l \).

VI. THE FERMIONS

Now we consider the fermionic sector. Let \( \Gamma_\mu \) represent the Dirac matrices and let \( u(p) \) is the fermionic wave function with \( \Gamma_\mu d_{-1,j,\mu} |0, p > u(p) \) and \( \Gamma_\mu d_{-1,k,\mu} |0, p > u(p) \) are all massless states. Broadly \( j=1, \ldots, 6 \) are the
colour sector and k=7,...,11 are the electroweak sector. The current generator condition gives \( \Gamma^\mu p_\mu(p) = 0 \), the Dirac equation for each of them. We have just that \( SO(6) \otimes SO(5) \rightarrow \mathbb{Z}_2 \otimes SU_C(3) \otimes SU_L(2) \otimes U_Y(1) \) and we have eleven fermions of zero mass. We must find a method of putting all the fermions in the electroweak and color groups in one generation. Choosing any four d’s, we can write

\[
\{d_i, d_j\} = 2\delta_{ij},
\]

(6.1)

omitting the suffix -1, and prefix primes and supply a factor 2

Since \( SO(4) \) can come from either \( SO(6) \) or \( SO(5) \). We can choose any four of them and define new operators b’s as follows

\[
b_1 = \frac{1}{2}(d_1 + id_2), \quad b_1^* = \frac{1}{2}(d_1 - id_2), \quad b_2 = \frac{1}{2}(d_3 + id_4), \quad \text{and} \quad b_2^* = \frac{1}{2}(d_3 - id_4).
\]

(6.2)

They satisfy the algebra \((i, j=1,2)\),

\[
\{b_i, b_j\} = 0, \quad \{b_i^*, b_j^*\} = 0 \quad \text{and} \quad \{b_i, b_j^*\} = 2\delta_{ij}.
\]

(6.3)

This U(2) group is identified here as the isospin group of the strong color sector. We can further define

\[
b_3 = \frac{1}{2}(d_5 + id_6), \quad \text{and} \quad b_3^* = \frac{1}{2}(d_5 - id_6).
\]

(6.4)

Then we give the necessary assignments of all observed fermions in Table I.

---

**Table I:**

| Particles | States | \( I_3 \) | \( Y \) | \( Q \) | States Assigned |
|-----------|--------|----------|-------|-------|-----------------|
| \( \nu_L \) | \( \frac{1}{2} \) | 0 | -1 | -1 | \( \psi_0 \) |
| \( \nu_R \) | \( \frac{1}{2} \) | 1/2 | 1 | 0 | \( \psi_1, \psi_2 \) |
| \( \nu_L \) | \( \frac{1}{2} \) | 0 | 3/2 | 2/3 | \( \psi_3, \psi_4, \psi_5 \) |
| \( \nu_R \) | \( \frac{1}{2} \) | 0 | -2/3 | -1/3 | \( \psi_6, \psi_7, \psi_8 \) |

The next problem is to supersymmetrize the duality invariance Maxwell action, so that the manifest duality invariance is preserved. A Majorana field shall be represented by a pair of complex two component spinors \( \psi^{(\alpha)}(1 \leq \alpha \leq 2) \) satisfying the condition

\[
\psi^{(\alpha)*} = \sigma_2 \mathcal{L}_{\alpha\beta} \psi^{(\alpha)},
\]

(6.5)

with \( \sigma_i \) is the Pauli matrices. Along with the action for the fermions,

\[
S_F = \int d^4x \left[ i\psi^{(\alpha)*} \partial_\alpha \psi^{(\alpha)} - \psi^{(\alpha)*} \mathcal{L}_{\alpha\beta} \sigma_k \partial_k \psi^{(\beta)} \right],
\]

(6.6)

the full action is

\[
S_C = \int d^4x \left[ i\psi^{(\alpha)*} \partial_\alpha \psi^{(\alpha)} - \psi^{(\alpha)*} \mathcal{L}_{\alpha\beta} \sigma_k \partial_k \psi^{(\beta)} - \frac{1}{2} B^{(\alpha)i} \mathcal{L}_{\alpha\beta} E^{(\beta)i} + B^{(\alpha)i} B^{(\alpha)i} \right].
\]

(6.7)

The above action is invariant under the following supersymmetric transformation,

\[
\delta \psi^{(\alpha)} = \frac{1}{2} (\mathcal{L}_{\alpha\beta} \sigma_k B^{(\beta)k} \epsilon - \sigma_k B^{(\alpha)k} \sigma_2 \epsilon^*),
\]

(6.8)
\[ \delta A_i^{(\alpha)} = i\psi^{(\alpha)\dagger} \sigma^i \epsilon - i\psi^{(\beta)\dagger} \mathcal{L}_{\alpha\beta} \sigma_i \sigma^2 \epsilon, \tag{6.9} \]

where \( \epsilon \) is the arbitrary two component complex spinor. One can write the action \( S \), where fermions can couple to gravity thereby achieving general coordinate invariance and Lorentz invariance,

\[ S = \int d^4 x \sqrt{-g} \left( i e^{\mu}_{\alpha} \psi^{(\alpha)\dagger} D_{\mu} \psi^{(\alpha)} - e^{\mu}_{k} \psi^{(\alpha)\dagger} \mathcal{L}_{\alpha\beta} \sigma_k D_{\mu} \psi^{(\beta)} \right), \tag{6.10} \]

where \( D_{\mu} \) is the covariant derivative involving the spin connection. We can rewrite the total fermionic part of the action as

\[ S_F = \int d^4 x \sqrt{-g} \left( i e^{\mu}_{\alpha} \psi^{(\alpha)\dagger} D_{\mu} \psi^{(\alpha)} - e^{\mu}_{k} \psi^{(\alpha)\dagger} \mathcal{L}_{\alpha\beta} \sigma_k D_{\mu} \psi^{(\beta)} \right), \tag{6.11} \]

where 'a' run from 1 to 12 as shown in Table IV.

Let \( U's \) be the Wilson loop integrals to construct the ordinary derivative from covariant derivative. For color, this is

\[ U_C(x) = \exp \left( ig \int_0^x \sum_l \lambda_l V_i^l dy_i \right). \tag{6.12} \]

The isospin phase functions can be given, if we define

\[ Y(x) = g' \int_0^x B_i dy_i, \tag{6.13} \]

by the following that

\[ U_Q(x) = \exp \left( \frac{ig}{2} \int_0^x \tau \cdot W_i dy_i - \frac{i}{6} Y(x) \right), \tag{6.14} \]

\[ U(x) = \exp \left( \frac{ig}{2} \int_0^x \tau \cdot W_i dy_i - \frac{i}{2} Y(x) \right), \tag{6.15} \]

\[ U_1(x) = \exp \left( \frac{2i}{3} Y(x) \right), \tag{6.16} \]

\[ U_2(x) = \exp \left( \frac{i}{3} Y(x) \right), \tag{6.17} \]

and

\[ U_R(x) = \exp (iY(x)). \tag{6.18} \]

The fermionic and vector fields can then be put in a simpler form. If, let

\[ \psi^a = \begin{cases} U_Q U_C \left( \begin{array}{c} u_q^a \\ a_L + 3 \end{array} \right), & a = 1, 2, 3; \\ U_1 U_C u^a_L, & a = 4, 5, 6; \\ U_2 U_C u^a_R, & a = 7, 8, 9; \\ U_R u^a_R, & a = 10; \\ U \left( \begin{array}{c} u_c^a \\ e^- \end{array} \right), & a = 11, 12 \end{cases} \]

one can write, with the above \( \psi^a \), the action \( S_{CF} \) for supergravity and Yang Mills fields as

\[ S_{CF} \sim \int d^4 x \sqrt{-g} \left( i e^{\mu}_{\alpha} \psi^{(\alpha)\dagger} (F_1 \cdot F_i)^a D_{\mu} \psi^{(\alpha)} - e^{\mu}_{k} \psi^{(\alpha)\dagger} (F_1 \cdot F_i)^a \mathcal{L}_{\alpha\beta} \sigma_k D_{\mu} \psi^{(\beta)} \right). \tag{6.19} \]
VII. ACTION INTEGRAL FROM SUPERGRAVITY

Supergravity is a generalisation of supersymmetry. We shall use the formalism of Cremmer et al [26]. The action is local and we get [12].

\[ A = \int d^4x \ d^4\theta \ e \phi(\tilde{S}e^{2\tilde{g}V}S) + \int d^4x \ d^4\theta \left[ \Re(1/R)g(S) + \Re(1/R)f_{ab}(S)W^{a\alpha}W_{a\alpha} \right]. \]  

(7.1)

\( \theta \) is Grassmann variable, \( e \) is the superspace vielbein determinant needed for getting the invariant volume in superspace. \( R \) is the chiral scalar curvature superfield derived from the curvature 2-form in superspace. \( f_{ab}(S) \) is the function of chiral multiplets so that \( f_{ab}(S)W^{a\alpha}W_{a\alpha} \) remains gauge invariant. \( a \) and \( b \) are the gauge indices for the adjoint representation. We have here three independent functions \( \phi, f_{ab} \) and \( g(S) \). The Lagrangian so constructed is invariant under supergravity transformation. But the usual Lagrangian for supergravity is not equation (7.1). Subsequently we will see that the lagrangian of theory of relativity will emerge from it automatically because of local supersymmetry.

This is a newer feature of supergravity. The meaning of chiral multiplet 12 in number are denoted by \( (A_\mu^{(a)}), (\chi_\mu^{(a)}) \), the gauge fermions will be called \( \psi^a \) and the field strength by \( W^{\mu
u}_{(a)} \).

The functions \( \phi(\tilde{S}, S) \) and \( g(S) \) can be combine to a single function \( \phi' \),

\[ \phi'(\tilde{S}, S) = \frac{1}{3} \frac{\phi(\tilde{S}, S)}{\left(\phi(\tilde{S}, S)^2\right)^{1/3}}, \]  

(7.2)

and we shall consider a single function \( G(A_\mu^{(a)}) \) given by

\[ \phi'(A_\mu^{(a)}) = e^{4G(A_\mu^{(a)})}, \]  

(7.3)

then the Lagrangian will be given by the two functions \( G(A_\mu^{(a)}) \) and \( f_{ab} \). When coupled to supergravity, the kinetic function and the superpotential loose their independent meaning. They enter only through the function \( G \) and emphasizing the fact that the scalar field space in supergravity is very much like Kahler Manifold, \( G \) denotes the Kahler like potential and Kahler like metric is \( G_{\mu\nu}^{(a)} \). We thus have now the action depending on two functions \( G \) and \( f_{ab} \), whereas the globally supersymmetric action has three functions \( f, \phi \) and \( g \).

With \( e \) as the determinant of the vielbein, the bosonic part of the Lagrangian is given as [12]

\[ e^{-1}L_B = \exp(-G) \left( \left(3 + G_{a}^{\mu}(G_{(a)}^\mu)^{-1}\nu G^{\nu\mu}\right) \right) - \frac{1}{2}g^2 \Re(f^{-1})_{ab}(G^{\mu\nu}T_{\alpha}^{(a)}A^{(a)}_{\nu})(G^{\lambda\rho}T_{\lambda}^{(a)}A^{(d)}_{\rho}) - \frac{1}{2}R + \frac{1}{2}G^{ab\mu}D_\lambda A_\mu^{(a)}D^\lambda A_\nu^{(b)} + \frac{1}{4}f_{ab}W_{\mu\nu}^{(a)}W^{(b)\mu\nu} + \frac{i}{4}3f_{ab}W_{\mu\nu}^{(a)}\bar{W}^{(b)\mu\nu}, \]  

(7.4)

with

\[ G_{a\mu} = \frac{\partial G}{\partial A_\mu^{(a)}}, \quad G_{\mu}^{a} = \frac{\partial G}{\partial A_{(a)}^{(a)}} \quad \text{and} \quad G^{ab\mu\nu} = \frac{\partial^2 G}{\partial A_{\mu}^{(a)} \partial A_{\nu}^{(b)}}. \]  

(7.5)

\( G^{-1} \) is the inverse of the matrix obtained from the second order differentiation with respect to fields and their complex conjugates. The first term in equation (7.4) contains the potential due to matter sector and the second term due to gauge sector. We shall introduce the complex scalar field \( \lambda \) which will form a part of \( G \) with \( \lambda = \lambda_1 + i\lambda_2 \) as before. The \( F' \)s have also the same properties as before so that

\[ \tilde{F}'_{(a)\mu\nu} = \frac{1}{2}(\sqrt{-g})^{-1}e^{\mu\rho\sigma}F_{\rho\sigma}^{(a)}, \]  

(7.6)

and

\[ \exp(-G)(3 + G_{a}^{\mu}(G_{(a)}^\mu)^{-1}\nu (G^{(b)})^\nu \]  

(7.7)

becomes proportional to

\[ g^{\mu\nu} \left[ \frac{1}{2\lambda_2^2} \partial_\mu \lambda_2 \lambda_2 + \frac{i}{8} Tr(\partial_\mu ML \partial_\nu ML) \right]. \]
The term
\[ G^{ab\nu} D_{\lambda} A_{\mu}^{(a)} D^{\lambda} A_{\nu}^{(b)} = - \frac{1}{4} \Re f_{ab} F_{\mu
u}^{(a)} F_{\lambda\mu\nu}^{(b)} - i \frac{1}{4} \Im f_{ab} F_{\mu
u}^{(a)} \tilde{F}_{\lambda\mu\nu}^{(b)}, \] (7.8)
becomes
\[ \rightarrow - \frac{1}{2} \lambda_2 F_{\mu\nu}^{(a)} (LML)_{ab} F_{(b)\mu\nu} + \frac{1}{2} \lambda_2 F_{\mu\nu}^{(a)} L_{ab} F_{\lambda\mu\nu}. \] (7.9)

So, we have, on the whole, the same expression as equation (4.24). Only the last two terms left to be considered for bosons.

Classical gravity term will be \( \frac{1}{2} R \) and the gauge field Lagrangian will be \(- \frac{1}{4} W_{\mu\nu} \cdot W^{\mu\nu}\).

Now we write the \( \mathcal{L}_{F,kin} \) as
\[ \mathcal{L}_{F,kin} = \frac{1}{2} \Re f_{ab} \left( - \frac{1}{2} \psi^a \gamma^\mu D_{\mu} \psi^b + \frac{1}{2} \bar{\psi}^a \gamma^\mu \sigma^a \eta \bar{\psi}^b W_{\alpha\eta}^{(b)} - \frac{1}{2} \bar{\psi}^a \gamma^\mu \psi^b G^{\alpha\beta} D_{\mu} A_{\alpha}^{(c)} \right) \]
\[ - i \frac{1}{4} \Im f_{ab} e^{-1} D_{\mu} \left( e \bar{\psi}^a \gamma^\mu \gamma^\nu \psi^b \right) - \frac{1}{2} f_{ab} \left( \bar{\psi}^a \gamma^\mu \sigma^a \mu W_{\nu}^{(b)} \psi^b \right) - \frac{1}{4} \epsilon^\mu \nu \lambda \sigma^\mu \lambda D_{\sigma} \bar{\psi} \psi \]
\[ + \frac{1}{8} \epsilon^\mu \nu \lambda \sigma^\mu \lambda G^{\alpha\sigma} D_{\lambda} A_{\alpha}^{(a)} - G^{ab\nu} \bar{\psi} \lambda \sigma^\mu \lambda D_{\alpha} A_{\mu} \gamma^\beta \chi \]
\[ - \frac{1}{2} \epsilon^\alpha \mu \gamma^\beta \chi D_{\alpha} A_{\mu}^{(b)} \chi_{R} \left( G_{\alpha\beta\mu} + \frac{1}{2} G_{abc} \psi_{\alpha} \psi_{c} \right) + G_{\mu \nu} \bar{\psi} \psi \gamma^\alpha \lambda D_{\psi} \psi \chi_{R} + h.c. \] (7.10)

where
\[ f_{a\beta} = \frac{\partial f_{ab}}{\partial A_{b}^{(a)}} \] (7.14)

Here \( \varphi \) denotes the gravitino and \( \chi \) is the gaugino. We proceed to simplify the above equation (7.13). Taking the gravitational counterpart as zero, we get the value
\[ - \frac{1}{4} \Re \bar{\varphi} \chi \gamma^\mu D_{\mu} \varphi, \] (7.15)

plus the term which is not a physical contribution. Next, the two terms of the gravitino kinetic energy are
\[ - \frac{1}{4} \epsilon^\mu \nu \lambda \sigma^\mu \lambda \psi_{\alpha} \psi_{\beta}, + \frac{1}{8} \epsilon^\mu \nu \psi \psi \psi \psi \] (7.16).

There still remains another term, which is the first term and is needed for the supergravity action to be invariant under duality transformation. This goes like
\[ - \frac{1}{4} \Re \bar{\varphi} \chi \gamma^\mu D_{\mu} \psi. \] (7.17)

Essentially, the above are the most important terms and we can proceed to write the part of the fermion action \( e^{-1} \mathcal{L}_{F} \) that does not contain covariant derivative
\[ e^{-1} \mathcal{L}_{F} = e^{-G/2} \bar{\varphi} \bar{\varphi} R \sigma^{\mu \nu} \varphi_{\mu} \varphi_{\nu} + \frac{1}{4} e^{-G/2} G^{\mu \nu} \left( G^{(1)} \right)_{\mu \nu} f_{aba} \bar{\psi} \psi + e^{-G/2} \left( G_{\mu \nu} \bar{G}_{\mu \nu} - G_{\mu} \bar{G}_{\mu} \right) \chi \chi \] (7.18)
This is the most general form of the Supergravity Lagrangian. We have taken the help of Refs.-[12] and [13] who have written this Lagrangian for a simpler processes. At a first sight, this Lagrangian appears nonrenormalisable. However, we shall show that there exists a possibility that it will lead to normalisable theory.

VIII. RENORMALISABILITY OF GRAVITON, GRAVITINO AND FUTURE OUTLOOK

We note that the symmetric, traceless tensor, which is non zero, physical and having zero mass, is

\[ h_{\mu\nu}(p) = \sum_{i,j} c_{ij} \left( b_{\mu}^{i\dagger} b_{\mu}^{j\dagger} + b_{\mu}^{j\dagger} b_{\mu}^{i\dagger} - \frac{1}{2} \eta_{\mu\nu} b_{\alpha}^{i\dagger} b_{\alpha}^{j\dagger} \right) |0, p\rangle. \]  

This can be taken as the graviton with the commutator;

\[ [h_{\mu\nu}(p), h_{\lambda\sigma}(p')] = f^{\mu\nu\lambda\sigma} |c|^2 \delta^{(4)}(p - p'), \]  

where

\[ f^{\mu\nu\lambda\sigma} = g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda} - g^{\mu\nu} g^{\lambda\sigma}. \]  

\( c_{ij} \) and \(|c|^2\) include renormalisation factors. This traceless tensor also satisfies

\[ L_0 h_{\mu\nu}(p) = -ph_{\mu\nu} = 0, \]  

and in flat space time, we have \( \Box h_{\mu\nu}(x) = 0 \), so that, we can take plane wave solution i.e. \( h_{\mu\nu}(x) = h_{\mu\nu}(p)e^{ipx} \). Then, we should have \( h_{\mu\nu} h_{\mu\lambda} = \delta^\lambda_\nu \).

The covariant derivative \( D_\mu \), in terms of \( \omega^{ab}_\mu \), is such that

\[ D_\mu e^a_\lambda = 0 = (\partial_\mu + \omega^{ab}_\mu \sigma_{bc}) e^c_\lambda, \]  

where \( \sigma_{ab} \) is usual the antisymmetric product of two \( \gamma \) matrices.

We have \( g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ij} \) and \( e^a_\mu e^{a\mu} = \delta^{ij} \), so that in the tangent space,

\[ h_{\mu\nu} = e^a_\mu e^b_\nu T_{ab}. \]  

After some algebra, we have

\[ [D_\mu, D_\lambda] h^\lambda_\nu = e^a_\mu e^b_\nu R^a_{\mu\lambda} T^{cb}, \]  

so the Riemannian Curvature Tensor is

\[ R^{ab}_{\mu\lambda} = \partial_\mu \omega^{ab}_\lambda + \omega^{ac}_\mu \omega^{cb}_\lambda - (\mu \leftrightarrow \lambda). \]  

Inversion of equation (8.7),

\[ [D_\mu, D_\lambda] h_{\mu}^\lambda = R_{\mu\lambda} h_{\mu}^\lambda \]  

is the parallel transport equation.

In flat space time \( [\partial_\mu, \partial_\nu] = 0 \), this means \( [\partial_\mu, \partial_\nu] h_{\mu\nu} = 0 \). Using the relation \( \Box h_{\mu\nu} = 0 \), one can easily find that

\[ R_{\mu\nu} = h^\lambda_\nu R_{\mu\sigma} h^\sigma_\lambda = h^\lambda_\nu [D_\mu, D_\nu] h^\lambda_\sigma, \]  

so that \( [D_\mu, D_\nu] = 0 \) everywhere in flat or curved space time,

\[ R_{\mu\nu} = 0. \]  

The graviton action, with \( R = g^{\mu\nu} R_{\mu\nu} \), is

\[ S_{\text{graviton}} = -\frac{1}{2\kappa} \int d^4x \ e \ R. \]  

This is given by the Lagrangian

\[ L_G = -\frac{1}{2\kappa} \sqrt{-g} \ R. \]  

Here \( \kappa = \sqrt{8\pi G} \), where \( G \) is the universal gravitational constant.
Similarly, we can take, for the flat space-time dimensions, the Rarita-Schwinger equation for the objects

$$\epsilon^{\mu\nu\lambda\sigma} \psi_{\mu} \gamma^5 \gamma_{\lambda} \partial_{\nu} \psi_{\sigma} = 0,$$

which is equivalent to in the "off shell momentum space" and is given by the equation

$$p^\mu \psi_{\mu} = \gamma^\mu \psi_{\mu} = \gamma \cdot \psi = 0.$$  \hspace{1cm} (8.13)

Let the ground state in Ramond-sector be

$$|\phi_0 >= \alpha_{-1}^\mu |0, p > u_{\mu} \quad \text{or} \quad |\phi_0 >= D_{-1}^\mu |0, p > u_{1\mu};$$ \hspace{1cm} (8.14)

which satisfy

$$\gamma^\mu \psi_{\mu} = p^\mu \psi_{\mu} = 0.$$ \hspace{1cm} (8.15)

Furthermore, since $$\gamma \cdot p \sim F_0$$, we also have

$$F_0 \psi_{\mu} \sim (\gamma \cdot p) \psi_{\mu} = 0.$$ \hspace{1cm} (8.18)

So, $$\psi_{\mu}$$ is the vectorial spinor of the Rarita-Schwinger equation. This equation, or equation (8.16) is in accordance to calculating the renormalisability condition with spin-$3/2$. These equations generalise to

$$D_{\mu} \psi_{\mu} = (\partial_{\mu} + \omega_{\mu}^{ab} \sigma_{ab}) \psi_{\mu}$$ \hspace{1cm} (8.19)

and the Rarita-Schwinger Lagrangian becomes

$$L_{RS} = \frac{1}{2} \psi_{\alpha} \gamma^5 D_{\sigma} \psi_{\mu} \epsilon^{\alpha \sigma \mu}.$$ \hspace{1cm} (8.20)

The sum of the two terms $$L_{RS}$$ and $$L_{G}$$ is invariant under local supersymmetric transformation

$$\delta_{\mu}^{\mu} = \frac{1}{2} \kappa \xi \gamma^m \psi_{\mu},$$

$$\delta \psi_{\mu} = \frac{1}{\kappa} (\partial_{\mu} + \omega_{\mu}^{mn} \sigma_{mn}) \xi = \frac{1}{\kappa} D_{\mu} \xi,$$

$$\delta \omega_{mn}^{\mu} = 0.$$ \hspace{1cm} (8.21)

Thus, the Lagrangian appears to be renormalizable. This will be taken up further in a future work.

**IX. CONCLUSION**

We also note that we have arrived at the group $$Z_3 \otimes SU_c(3) \otimes SU_L(2) \otimes U_Y(1)$$ to explain the experimental result. It is of the interest to note that the number of generations $$n_G$$ will be given by the Euler number $$\chi$$ which is 6 in our consideration of 4-dimensional theory, so that $$n_G = \frac{\chi}{2} = n_+ - n_- = 3$$, where $$n_-$$ and $$n_+$$ are the number of negative and positive chiralities respectively. The zero mass modes of Dirac objects in standard model are grouped into three families(generations). They are

$$\begin{pmatrix} u \\ d \\ e \end{pmatrix}, \begin{pmatrix} c \\ s \\ \nu_e \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} t \\ b \\ \nu \end{pmatrix}.$$  \hspace{1cm} (9.1)
The left handed ones are doublets and right handed ones are singlets. The quarks are colored. There are several fermionic zero modes, 24 are with +ve helicity and 21 with negative helicity. Thus $n_+ - n_- = 3$ as per topological findings. So there have to be only three fermions (neutrinos with unpaired helicity) in the Standard Model.

Thus, we have explored the four dimensional supergravity as far as practicable. We have started and also ended with four dimensions and never taken any more dimensions, specifically, nothing has been done to involve ten dimensions, as four dimensions appear to be good enough for all purposes. We have examined the difficulties of the four dimensional methods. But we find that it is more efficient than ten dimensions or heteretic string theory. Our approach to quantise gravity, using physical states of a superstring, has been quite successful. It has been noted that spin-2 graviton and spin zero dilaton are obtained from the metric field strength $g_{\mu\nu}$, whereas spin zero part can be eliminated in superstring theory. With a four dimensional theory, we have considered the question of renormalisability of gravitational field and spin-$\frac{3}{2}$ fermions afresh and find that it is quite possible for a total 4-d theory which includes all interactions.

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