Abstract
We extend the validity of Brill’s axisymmetric positive energy theorem to all asymptotically flat initial data sets with positive scalar curvature on simply connected manifolds.

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1 Introduction

In [3] Brill proved a positive energy theorem for a certain class of maximal, axi-symmetric initial data sets on $\mathbb{R}^3$. Brill’s analysis has been extended independently by Moncrief (unpublished), Dain (unpublished), and Gibbons and Holzegel [9] to the following class of metrics:

$$g = e^{-2U+2\alpha} (d\rho^2 +dz^2) + \rho^2 e^{-2U} (d\varphi + \rho B_\rho d\rho + A_z dz)^2 .$$

(1.1)

All the functions are assumed to be $\varphi$–independent.

The above form of the metric, together with Brill’s formula for the mass, are the starting points of the recent work of Dain [7], who proves an upper bound for angular-momentum in terms of the mass for a class of maximal, vacuum, axi-symmetric initial data sets with a metric of the form above.

The aim of this series of papers is to extend the validity of Brill’s positivity theorem, as well as that of Dain’s inequality, to all maximal, asymptotically flat, vacuum, axi-symmetric initial data sets with a metric of the form above.

More precisely, in this paper we prove that any sufficiently differentiable, asymptotically flat, axially symmetric metric on $\mathbb{R}^3$ can be written in the form (1.1). In general the functions appearing in (1.1) will not satisfy the fall-off conditions imposed in [7, 9], but we verify that the proof extends to the more general situation. The result is further extended to include metrics with several asymptotically flat ends provided the manifold is simply connected. In the second paper of this series [6] the constructions of the current paper will be used to extend the validity of Dain’s angular momentum inequality to the class of metrics considered here. We will further allow those non-vacuum models which admit a twist potential $\omega$, see [6] for details.

It is conceivable that, regardless of simple-connectedness and isotropy conditions, axi-symmetric metrics on manifolds obtained by blowing-up a finite number of points in a compact manifold can be represented as in (1.1), with the coordinates $(\rho, z)$ ranging over a subset $\Omega$ of $\mathbb{R}^2$, and with identifications on $\partial \Omega$, but this remains to be seen; in any case it is not clear how to adapt the arguments leading to the mass and angular momentum inequalities to such situations.
2 Axi-symmetric metrics on simply connected asymptotically flat three dimensional manifolds

Let us start with a general discussion of Riemannian manifolds \((M, g)\) with a Killing vector \(\eta\) with periodic orbits; without loss of generality we can assume that the period of principal orbits is \(2\pi\).

Let \(M/U(1)\) denote the collection of the orbits of the group of isometries generated by \(\eta\), and let \(\pi : M \rightarrow M/U(1)\) be the canonical projection. An orbit \(p \in M/U(1)\) will be called non-degenerate if it is not a point in \(M\). Recall that near any \(p \in M/U(1)\) which lifts to an orbit of principal type there exists a canonical metric \(q\) defined as follows: let \(X, Y \in T_p(M/U(1))\), \(\hat{p} \in M\) be any point such that \(\pi(\hat{p}) = p\), and let \(\hat{X}, \hat{Y} \in T_{\hat{p}}M\) be the unique vectors orthogonal to \(\eta\) such that \(\pi_*\hat{X} = X\) and \(\pi_*\hat{Y} = Y\). Then

\[
q(X, Y) := g(\hat{X}, \hat{Y}).
\]  

(2.1)

(The reader will easily check that the right-hand-side of (2.1) is independent of the choice of \(\hat{p} \in \pi^{-1}\{p\}\).)

There exists an open dense set of the quotient manifold \(M/U(1)\) which can, at least locally, be conveniently modeled on smooth submanifolds (perhaps with boundary), say \(N\), of \(M\), which meet orbits of \(\eta\) precisely once; these are called cross-sections of the group action. (For metrics of the form (1.1) there actually exists a global cross-section \(N\), meeting all orbits precisely once.) The manifold structure of \(M/U(1)\) near \(p\) is then, by definition, the one arising from \(N\). For \(p \in \tilde{N} := N \setminus \{\eta = 0\}\) and for \(X, Y \in T_p\tilde{N}\) set

\[
q(X, Y) = g(X, Y) - \frac{g(\eta, X)g(\eta, Y)}{g(\eta, \eta)}. 
\]  

(2.2)

One easily checks that this coincides with our previous definition of \(q\).

The advantage of (2.2) is that it allows us to read-off properties of \(q\) directly from those of \(g\) near \(N\). On the other hand, the abstract definition (2.1) makes clear the Riemannian character of \(q\), and does not require any specific transverse submanifold. This allows to use different \(N\)’s, adapted to different problems at hand, to draw conclusions about \(M/U(1)\); this freedom will be made use of in what follows.
Clearly all the information about \( g \) is contained in \( q \) and in the one-form field
\[
\eta^b := g(\eta, \cdot) ,
\]
since we can invert (2.2) using the formula, valid for any \( X, Y \in TM \),
\[
g(X,Y) = q(P_\eta X, P_\eta Y) + \frac{g(\eta,X)g(\eta,Y)}{g(\eta,\eta)} ,
\]
where \( P_\eta : TM \to T\tilde{N} \) is the projection from \( TM \) to \( T\tilde{N} \) along \( \eta \). (Recall that \( P_\eta \) is defined as follows: since \( \eta \) is transverse to \( T\tilde{N} \), every vector \( X \in TM \) can be uniquely written as \( X = \alpha \eta + Y \), where \( Y \in T\tilde{N} \), then one sets \( P_\eta X := Y \).) In order to establish (2.3) note, first, that this is only a rewriting of (2.2) when both \( X \) and \( Y \) are tangent to \( \tilde{N} \). Next, (2.3) is an identity if either \( X \) or \( Y \) is proportional to \( \eta \), and the result easily follows.

Let \( x^A, A = 1, 2 \) be any local coordinates on \( \tilde{N} \), propagate them off \( \tilde{N} \) by requiring that \( \mathcal{L}_\eta x^A = 0 \), and let \( \varphi \) be a coordinate that vanishes on \( \tilde{N} \) and satisfies \( \mathcal{L}_\eta \varphi = 1 \). Then \( \eta = \partial_\varphi \), and \( P_\eta(X^A \partial_A + X^\varphi \partial_\varphi) = X^A \partial_A \), so that (2.3) can be rewritten as
\[
g = \frac{q_{AB} dx^A dx^B + g(\eta,\eta)(d\varphi + \tilde{\theta}_A dx^A)^2}{q} ,
\]
with
\[
\partial_\varphi g_{AB} = \partial_\varphi \tilde{\theta}_A = \partial_\varphi(g(\eta,\eta)) = 0 .
\]

### 2.1 Global considerations

So far our considerations were completely general, but local. Suppose, however, that \( M \) is simply connected, with or without boundary, and satisfies the usual condition that it is the union of a compact set and of a finite number of asymptotically flat ends. Then every asymptotic end can be compactified by adding a point, with the action of \( U(1) \) extending to the compactified manifold in the obvious way. Similarly every boundary component has to be a sphere [10, Lemma 4.9], which can be filled in by a ball, with the action of \( U(1) \) extending in the obvious way, reducing the analysis of the group action to the boundaryless case. Existence of asymptotically flat regions implies (see, e.g., [2]) that the set of fixed points of the action is non-empty. It is then shown in [13] that, after the addition of a ball to every boundary component if necessary, \( M \) is homeomorphic to \( \mathbb{R}^3 \), with the action of \( U(1) \) conjugate, by a homeomorphism, to the usual rotations of \( \mathbb{R}^3 \). On the other hand, it is shown in [12] that the actions are classified, up to smooth
conjugation, by topological invariants. It follows that the action is in fact smoothly conjugate to the usual rotations of $\mathbb{R}^3$. In particular there exists a global cross-section $\tilde{N}$ for the action of $U(1)$ away from the set of fixed points $\mathcal{A}$, with $\tilde{N}$ diffeomorphic to an open half-plane, with all isotropy groups trivial or equal to $U(1)$, and with $\mathcal{A}$ diffeomorphic to $\mathbb{R}$.\footnote{I am grateful to Joao Costa and Allen Hatcher for discussions or comments on the classification of $U(1)$ actions.}

Somewhat more generally, the above analysis applies whenever $M$ can be compactified by adding a finite number of points or balls. A nontrivial example is provided by manifolds with a finite number of asymptotically flat and asymptotically cylindrical ends, as is the case for the Cauchy surfaces for the domain of outer communication of the extreme Kerr solution.

2.2 Regularity at the axis

In the coordinates of (1.1) the rotation axis

$$\mathcal{A} := \{ g(\eta, \eta) = 0 \}$$

corresponds to the set $\rho = 0$, which for asymptotically flat metrics is never empty, see, e.g., the proof of Proposition 2.4 in [2].

In order to study the properties of $q$ near $\mathcal{A}/U(1) \approx \mathcal{A}$, recall that $\mathcal{A}$ is a geodesic in $M$. It is convenient to introduce normal coordinates $(\hat{x}, \hat{y}, \hat{z}) : \mathcal{U} \to \mathbb{R}^3$ defined on an open neighborhood $\mathcal{U}$ of $\mathcal{A}$, where $\hat{z}$ is a unit-normalized affine parameter on $\mathcal{A}$, and $(\hat{x}, \hat{y})$ are geodesic coordinates on $\exp((T\mathcal{A})^\perp)$. Without loss of generality we can assume that $\mathcal{U}$ is invariant under the flow of $\eta$.

As is well known, we have (recalling that orbits of principal type form an open and dense set of $M$, as well as our normalization of $2\pi$–periodicity of the principal orbits)

$$\eta = \hat{x}\partial_{\hat{y}} - \hat{y}\partial_{\hat{x}}.$$ 

If we denote by $\phi_\eta$ the flow of $\eta$, on $\mathcal{U}$ the map $\phi_\eta$ is therefore the symmetry across the axis $\mathcal{A}$:

$$\phi_\eta(\hat{x}, \hat{y}, \hat{z}) = (-\hat{x}, -\hat{y}, \hat{z}).$$

This formula has several useful consequences. First, it follows that the manifold with boundary

$$N := \{ \hat{x} \geq 0, \hat{y} = 0 \} \subset \mathcal{U}$$

is a cross-section for the action of $U(1)$ on $\mathcal{U}$. This shows that near zeros of $\eta$ the quotient space $M/U(1)$ can be equipped with the structure of a
smooth manifold with boundary. The analysis of the behavior of $q$ near $\partial N \approx \mathcal{A}$ requires some work because of the factor $1/g(\eta, \eta)$ appearing in (2.2).

For further use we note that the manifold
\[ \tilde{N} := \{ \hat{y} = 0 \} \subset \mathcal{U} \] provides, near $\mathcal{A}$, a natural doubling of $N$ across its boundary $\mathcal{A}$.

In order to understand the smoothness of $q$ on $N$ and $\tilde{N}$, we start by considering the function
\[ f(\hat{x}, \hat{z}) := g(\eta, \eta)(\hat{x}, 0, \hat{z}) \]
Then $f(-\hat{x}, \hat{z}) = f(\hat{x}, \hat{z})$ because $g(\eta, \eta) \circ \phi_t = g(\eta, \eta)$. It follows that all odd $x$–derivatives of $f$ vanish at $\hat{x} = 0$. It is then standard to show, using Borel's summation lemma (cf., e.g., [5, Proposition C1, Appendix C]), that there exists a smooth function $h(s, \hat{z})$ such that
\[ f(\hat{x}, \hat{z}) = \hat{x}^2 h(\hat{x}^2, \hat{z}) \]
Letting $\hat{\rho} = \sqrt{\hat{x}^2 + \hat{y}^2}$, invariance of $g$ under $\phi_t$ allows us to conclude that
\[ g(\eta, \eta)(\hat{x}, \hat{y}, \hat{z}) = g(\eta, \eta)(\hat{\rho}, 0, \hat{z}) = \hat{\rho}^2 h(\hat{\rho}^2, \hat{z}) \] (2.6)
Define $\hat{\varphi}$ via the equations
\[ \hat{x} = \hat{\rho} \cos \hat{\varphi}, \quad \hat{y} = \hat{\rho} \sin \hat{\varphi}, \]
so that
\[ \eta = \partial_{\hat{\varphi}}. \]
Considerations similar to those leading to (2.6) (see Lemma 5.1 of [5]) show that there exist functions $\alpha, \beta, \gamma, \delta, \mu$ and $g_{\hat{z} \hat{z}}$, which are smooth with respect to the arguments $\hat{\rho}^2$ and $\hat{z}^2$, with
\[ \mu(0, \hat{z}) = 1, \quad g_{\hat{z} \hat{z}}(0, \hat{z}) = 1, \]
such that
\[ g = g_{\hat{z} \hat{z}} \hat{z}^2 + 2\alpha \hat{\rho} d\hat{z} d\hat{\rho} + 2\beta \hat{\rho}^2 d\hat{z} d\hat{\varphi} + \gamma \hat{\rho}^2 d\hat{\rho}^2 + 2\delta \hat{\rho}^3 d\hat{\rho} d\hat{\varphi} + \mu(d\hat{\rho}^2 + \hat{\rho}^2 d\hat{\varphi}^2) \]
\[ = \left( g_{\hat{z} \hat{z}} - \frac{\beta^2 \hat{\rho}^2}{\mu} \right) d\hat{z}^2 + 2 \left( \alpha - \frac{\delta \beta \hat{\rho}^2}{\mu} \right) \hat{\rho} d\hat{z} d\hat{\varphi} + \left( \mu + \gamma \hat{\rho}^2 - \frac{\delta^2 \hat{\rho}^2}{\mu} \right) d\hat{\rho}^2 \]
\[ + \mu \hat{\rho}^2 \left( d\hat{\varphi} + \frac{\delta}{\mu} \hat{\rho} d\hat{\varphi} + \frac{\beta}{\mu} \hat{d}\hat{z} \right)^2. \]
(2.7)

\[ ^{2} \text{By this we mean that } \alpha(s, \hat{z}) \text{ is a smooth function of its arguments, and enters (2.7) in the form } \alpha(\hat{\rho}^2, \hat{z}), \text{ etc.} \]
We say that $\tilde{N}$ is a doubling of a manifold $N$ across a boundary $\partial \tilde{N}$ if $\tilde{N}$ consists of two copies of $N$ with points on $\partial \tilde{N}$ identified in the obvious way. From what has been said, by inspection of (2.7) it follows that:

**Proposition 2.1** The quotient space $M/U(1)$ has a natural structure of manifold with boundary near $\mathcal{A}$. The metric $g$ and the one-form $\tilde{\theta}$ are smooth up-to-boundary, and extend smoothly across $\mathcal{A}$ by continuity to themselves when $M/U(1)$ is doubled at $\mathcal{A}$.

For further use we note the formula

\[ g(\eta, \eta) = \hat{\rho}^2 + O(\hat{\rho}^4), \quad (2.8) \]

for small $\hat{\rho}$, which follows from (2.7), where $\hat{\rho}$ is either the geodesic distance from $\mathcal{A}$, or the geodesic distance from $\mathcal{A}$ on $\exp((T\mathcal{A})^\perp)$ (the latter being, for small $\hat{\rho}$, the restriction to $\exp((T\mathcal{A})^\perp)$ of the former).

### 2.3 Asymptotic flatness

We will consider Riemannian manifolds $(M, g)$ that are asymptotically flat, in the usual sense that there exists a region $M_{ext} \subset M$ diffeomorphic to $\mathbb{R}^3 \setminus B(R)$, where $B(R)$ is a coordinate ball of radius $R$, such that in local coordinates on $M_{ext}$ obtained from $\mathbb{R}^3 \setminus B(R)$ the metric satisfies the fall-off conditions, for some $k \geq 1$,

\[ g_{ij} - \delta_{ij} = o_k(r^{-1/2}), \quad (2.9) \]
\[ \partial_k g_{ij} \in L^2(M_{ext}), \quad (2.10) \]
\[ R^i_{jkl} = o(r^{-5/2}), \quad (2.11) \]

where we write $f = o_k(r^\alpha)$ if $f$ satisfies

\[ \partial_{k_1} \ldots \partial_{k_l} f = o(r^{\alpha-\ell}), \quad 0 \leq \ell \leq k. \]

It is well known that (2.9)-(2.10) together with $R(g) \geq 0$ or $R(g) \in L^1$, where $R(g)$ is the Ricci scalar of $g$, guarantees a well-defined ADM mass (perhaps infinite). On the other hand, the condition (2.11) (which follows in any case from (2.9) for $k \geq 2$) is useful when analyzing the asymptotic behavior of Killing vector fields.

We will use (2.9)-(2.11) to construct the coordinate system of (2.3), and also to derive the asymptotic behavior of the fields appearing in (2.3). We start by noting that the arguments of [1, Appendix C] with $N \equiv 0$ there...
show that there exists a rotation matrix $\omega$ such that in local coordinates on $M_{\text{ext}}$ we have

$$\eta^i = \omega^i_j x^j + o_k(r^{1/2}),$$

(2.12)

where $\omega^i_j$ is anti-symmetric. It will be clear from the proof below (see (2.23)) that this equation provides the information needed in the region

$$x^2 + y^2 \geq z^2, \quad x^2 + y^2 + z^2 \geq R^2.$$

(2.13)

However, near the axis a more precise result is required, and we continue by constructing new asymptotically flat coordinates which are better adapted to the problem at hand. The difficulties arise from the need to obtain decay estimates on $q - \delta$, where $\delta$ is the Euclidean metric on $\mathbb{R}^2$, and on $\tilde{\theta}$, which are uniform in $r$ up to the axis $A$.

Let $(\hat{x}^i) \equiv (\hat{x}, \hat{y}, \hat{z})$ be coordinates on $\mathbb{R}^3 \setminus B(R)$, obtained by a rigid rotation of $x^i$, such that $\omega^i_j \hat{x}^j = \hat{y} \partial_{\hat{x}} - \hat{x} \partial_{\hat{y}}$. Set

$$x := \frac{\hat{x} - \hat{x} \circ \phi_s}{2}, \quad y := \frac{\hat{y} - \hat{y} \circ \phi_s}{2}, \quad z := \frac{1}{2\pi} \int_0^{2\pi} \hat{z} \circ \phi_s \, ds. \quad (2.14)$$

Using the techniques in [1, 2] one finds

$$\phi_s(\hat{x}^i) = (\cos(s)\hat{x} - \sin(s)\hat{y} + z\hat{z}(s, \hat{x}^i), \sin(s)\hat{x} + \cos(s)\hat{y} + z\hat{z}(s, \hat{x}^i), \hat{z} + z\hat{z}(s, \hat{x}^i)),

$$

with $z^i$ satisfying

$$z^i = o_{k+1}(r^{1/2}).$$

We then have

$$\frac{\partial z}{\partial \hat{z}} = 1 + \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial z(\phi_s(\hat{x}^i))}{\partial \hat{z}} \, ds = 1 + o_k(r^{-1/2}),$$

Further,

$$\frac{\partial z}{\partial \hat{x}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial z(\phi_s(\hat{x}^i))}{\partial \hat{x}} \, ds = o_k(r^{-1/2}),$$

similarly

$$\frac{\partial z}{\partial \hat{y}} = o_k(r^{-1/2}).$$

The estimates for the derivatives of $x$ and $y$ are straightforward, and we conclude that

$$\frac{\partial x^i}{\partial \hat{x}^i} = \delta^i_j + o_k(r^{-1/2}),$$

8
where, by an abuse of notation, we write again $x^i$ for the functions $(x, y, z)$. Standard considerations based on the implicit function theorem show that, increasing $R$ if necessary, the $x^i$’s form a coordinate system on $\mathbb{R}^3 \setminus B(R)$ in which (2.9)-(2.11) hold. Subsequently, (2.12) holds again.

From (2.14) one clearly has

$$\forall s \in \mathbb{R} \quad z \circ \phi_s = z,$$

which shows that the planes

$$\mathcal{P}_\tau := \{z = \tau\}, \quad \tau \in \mathbb{R}, \quad |\tau| \geq R,$$

are invariant under the flow of $\eta$; equivalently,

$$\eta^z = 0.$$  

Moreover,

$$x \circ \phi_\pi = -x, \quad y \circ \phi_\pi = -y,$$  

(2.15)

so that all points with coordinates $x = y = 0$ are fixed points of $\phi_\pi$, and that these are the only such points in $M_{\text{ext}}$. Equation (2.15) further implies that $\phi_\pi$ maps the surfaces $\{x = 0\}$ and $\{y = 0\}$ into themselves. Since $\phi_\pi$ is an isometry, we obtain

$$g_{ab}(0, y, z) = g_{ab}(0, -y, z), \quad g_{zz}(0, y, z) = g_{zz}(0, -y, z), \quad g_{za}(0, y, z) = -g_{za}(0, -y, z);$$  

(2.16)

similarly

$$g_{ab}(x, 0, z) = g_{ab}(-x, 0, z), \quad g_{zz}(x, 0, z) = g_{zz}(-x, 0, z), \quad g_{za}(x, 0, z) = -g_{za}(-x, 0, z).$$  

(2.17)

Equation (2.16) leads to

$$\frac{\partial^{2\ell+1} g_{ab}}{\partial y^{2\ell+1}}(0, 0, z) = 0, \quad \frac{\partial^{2\ell+1} g_{zz}}{\partial y^{2\ell+1}}(0, 0, z) = 0, \quad \frac{\partial^{2\ell} g_{za}}{\partial y^{2\ell}}(0, 0, z) = 0$$  

(2.18)

for $\ell \in \mathbb{N}$ (or at least as far as the differentiability of the metric allows). The analogous implication of (2.17) allows us to conclude that

$$\frac{\partial g_{ab}}{\partial x^c}(0, 0, z) = 0, \quad \frac{\partial g_{zz}}{\partial x^a}(0, 0, z) = 0, \quad g_{az}(0, 0, z) = 0.$$  

(2.19)
Incidentally, the last two equations in (2.19) show that \( \{x = y = 0\} \) is a geodesic; this follows in any case from the well-known fact that the set of fixed points of an isometry is totally geodesic.

Consider a point \( p \) lying on the axis of rotation \( \mathcal{A} \), then \( \phi_t(p) = p \) for all \( t \), in particular \( \phi_\pi(p) = p \). From what has been said we obtain that
\[
\mathcal{A} \cap M_{\text{ext}} \subset \{x = y = 0\}.
\] (2.20)

Recall, again, that every connected component of the axis of rotation \( \mathcal{A} \) is an inextendible geodesic in \( (M, g) \). Since the set at the right-hand-side of (2.20) is a geodesic segment, we conclude that equality holds in (2.20). Hence
\[
\eta^i(0, 0, z) = 0 \quad (2.21)
\]
and, for \( |z| \geq R \), the origin is the only point within the plane \( \mathcal{P}_z \) at which \( \eta \) vanishes.

We are ready now to pass to the problem at hand, namely an asymptotic analysis of the fields \( g(\eta, \eta), q \) and \( \tilde{\theta} \) as in (2.4); we start with \( q \). For \( \rho \) sufficiently large the hypersurface \( \{y = 0\} \) is transverse to \( \eta \) (for small \( \rho \) we will return to this issue shortly) and therefore the coordinates
\[
(x^A) := (x, z)
\]
on this hypersurface, with \( x \geq 0 \), can be used as local coordinates on \( M/\text{U}(1) \). The contribution of \( g_{AB} \) to \( q_{AB} \) is of the form \( g_{AB} = \delta_{AB} + o_k(r^{-1/2}) \), which is manifestly asymptotically flat in the usual sense. Next, from (2.9) and (2.12) we obtain
\[
g(\eta, \eta) = \rho^2 + o_k(r^{3/2}) \quad (2.22)
\]
here, as elsewhere, \( \rho^2 = x^2 + y^2 \). Further
\[
\frac{g_{AB} \eta^A \eta^B}{g(\eta, \eta)} \ dx^A \ dx^B = \left( \delta_{Ai} + o_k(r^{-1/2}) \right) \left( \omega^i_a x^a + o_k(r^{1/2}) \right) \times \left( \delta_{Bj} + o_k(r^{-1/2}) \right) \left( \omega^j_b x^b + o_k(r^{1/2}) \right) \rho^2 + o_k(r^{3/2}) \ dx^A \ dx^B
\]
\[
= \frac{o_k(r^{1/2}) dx^A dx^B}{\rho^2 + o_k(r^{3/2})}, \quad (2.23)
\]
because \( \omega^i_a x^a \omega^j_b x^b dx^i dx^j = (x dy - y dx)^2 \), which vanishes when pulled-back to \( \{y = 0\} \). In the region (2.13) we thus obtain
\[
q_{AB} = \delta_{AB} + o_k(r^{-1/2}) \quad (2.24)
\]
which is the desired estimate. However, near the zeros of \( \eta \) this calculation is not precise enough to obtain uniform estimates on \( q \) and its derivatives.

In fact, it will be seen in the remainder of the proof that we need uniform estimates for derivatives up to second order. Since \( g(\eta, \eta) \) vanishes quadratically at the origin we need uniform control of the numerator of (2.23) up to terms \( O(\rho^4) \), in a form which allows the division to be performed without losing uniformity.

So in the region \( \{ \rho \leq |z| \} \cap M_{\text{ext}} \), in which \( |z| \) is comparable with \( r \), we proceed as follows: Let

\[
\lambda_{ab} \equiv \lambda_{ab}(z) := \frac{\partial \eta^a}{\partial x^b}(0, 0, z), \quad \lambda_{ab} := g_{ac}(0, 0, z) \lambda^c_b;
\]

note that \( \lambda_{ab} = \omega_{ab} + o_{k-1}(|z|^{-1/2}) = \omega_{ab} + o_{k-1}(r^{-1/2}) \), similarly for \( \lambda_{bc} \). The Killing equations imply that \( \lambda_{ab} \) is anti-symmetric, hence

\[
\lambda_{xx} = \lambda_{yy} = 0, \quad \lambda_{xy} = -\lambda_{yx} = 1 + o_{k-1}(|z|^{-1/2}) = 1 + o_{k-1}(r^{-1/2}).
\]

From (2.21) we further obtain

\[
\partial_i \eta^i = 0 \implies \nabla_i \eta^i|_{\mathscr{A}} = 0 \implies \nabla_i \eta_i|_{\mathscr{A}} = \nabla_z \eta|_{\mathscr{A}} = \nabla_{z \eta}|_{\mathscr{A}} = 0.
\]

Recall the well known consequence of the Killing equations,

\[
\nabla_i \nabla_j \eta_k = R^\ell_{\ ijk}\eta_\ell,
\]

which implies, at \( \mathscr{A} \),

\[
0 = \nabla_a \nabla_b \eta_c = \partial_a \partial_b \eta_c, \quad (2.25)
0 = \nabla_a \nabla_b \eta_z = \partial_a \nabla_b \eta_z - \Gamma_{ac}^e \lambda_{bc} = \partial_a \partial_b \eta_z - 2\Gamma_{ac}^e \lambda_{bc}. \quad (2.26)
\]

From (2.12) we obtain, by integration of third derivatives of \( \eta_a \) along rays from the origin \( x = y = 0 \) within the planes \( z = \text{const} \),

\[
\frac{\partial^2 \eta_a}{\partial x^b \partial x^c} = o_{k-3}(|z|^{-5/2}) x^c = o_{k-3}(r^{-5/2}) x^c,
\]

and then successive such integrations give

\[
\frac{\partial \eta_a}{\partial x^b} = \lambda_{ab} + o_{k-3}(|z|^{-5/2}) x^c x^d = \lambda_{ab} + o_{k-3}(r^{-5/2}) x^c x^d, \quad \eta_a = \lambda_{ab} x^b + o_{k-3}(r^{-5/2}) x^c x^d x^e. \quad (2.27)
\]
At $y = 0$ we conclude that
\[ \eta_x = o_{k-3}(r^{-5/2})x^c x^d x^e. \]
Similarly we have $\nabla_i \nabla_j \eta^k = R^k_{ij} \eta^l$, hence $\nabla_a \nabla_b \eta^c = \partial_a \partial_b \eta^c = 0$ at $\mathcal{A}$, and we conclude that
\[ \eta^a = \lambda^a b x^b + o_{k-3}(r^{-5/2})x^c x^d x^e. \] (2.28)
This allows us to prove transversality of $\eta$ to the plane \{ $y = 0$ \}. Indeed, from (2.28) at $y = 0$ we have
\[ \eta^y = (1 + o(r^{-1/2}))x + o(r^{-5/2})x^3 = (1 + o(r^{-1/2}))x \]
which has no zeros for $x \neq 0$ and $r \geq R$ if $R$ is large enough. Recall that we have been assuming that $|x| \leq |z|$ in the current calculation; however, we already know that $\eta$ is transverse for $|z| \geq |x|$, and transversality follows.

Increasing the value of the radius $R$ defining $M_{\text{ext}}$ if necessary, we conclude that $\{ y = 0, x \geq 0 \} \cap M_{\text{ext}}$ provides a global cross-section for the action of $U(1)$ in $M_{\text{ext}}$.

Using (2.26), a similar analysis of $\eta_z$ gives
\[ \eta_z = - \left[ \frac{\Gamma_{abc}}{o_{k-1}(r^{-3/2})} \right] \lambda^c b x^a x^b + o_{k-3}(r^{-5/2})x^c x^d x^e. \]

We are now ready to return to (2.22),
\[ g(\eta, \eta) = \eta^i \eta^j = \eta^a \eta^a = \hat{\rho}^2 + o_{k-3}(r^{-5/2})x^a x^b x^c x^d, \] (2.29)
where, at $y = 0$,
\[ \hat{\rho}^2 := \hat{g}_{ab} \lambda^a c x^c \lambda^b d x^d = (1 + o_{k-1}(r^{-1/2}))x^2; \]
it follows that the last equality also holds for $g(\eta, \eta)$ with $k - 1$ replaced by $k - 3$. Instead of (2.23) we write
\[ \frac{g_{A i} \eta^i g_{B j} \eta^j}{g(\eta, \eta)} d x^A d x^B = \frac{\eta_A \eta_B d x^A d x^B}{(1 + o_{k-3}(r^{-1/2}))x^2} \]
\[ = \frac{\eta^2_x d x^2 + 2 \eta_x \eta_z d x d z + \eta^2_z d z^2}{(1 + o_{k-3}(r^{-1/2}))x^2} \]
\[ = \frac{o_{k-3}(r^{-3})x^2 d x^A d x^B}{(1 + o_{k-3}(r^{-1/2}))} \]
\[ = o_{k-3}(r^{-1})x^2 d x^A d x^B. \] (2.30)
We conclude that (2.24) holds throughout \( \{ y = 0 \} \cap M_{\text{ext}} \) with \( k \) replaced by \( k - 3 \).

To analyse the fall-off of \( B_\rho \) and \( A_z \), note first that the discussion in the paragraph before (2.4) shows that it suffices to do this at one single surface transverse to the flow of the Killing vector field \( \eta \); unsurprisingly, we choose

\[
N := \{ y = 0, \, x > 0, \, x^2 + z^2 \geq R^2 \},
\]

with \( R \) sufficiently large to guarantee transversality. Next, from (1.1) we find

\[
\eta_i dx^i = g(\eta, \cdot) = g(\partial_\varphi, \cdot) = g(\eta, \eta)(d\varphi + \rho B_\rho d\rho + A_z dz),
\]

which will allow us to relate \( B_\rho \) and \( A_z \) to \( \eta_i \) if we determine, say \( \partial_i \varphi \) and \( \partial_i \rho \) on \( N \). For the sake of clarity of intermediate calculations it is convenient to denote by \( \bar{z} \) the coordinate \( z \) appearing in (1.1), we thus seek a coordinate transformation

\[
(x,y,z) \to (\rho, \varphi, \bar{z}), \quad \text{with} \quad \bar{z} = z \text{ everywhere and } \rho = x \text{ on } N,
\]

which brings the metric to the form (1.1), with \( z \) there replaced by \( \bar{z} \). We wish to show that, on \( N \),

\[
J := \begin{pmatrix}
\partial_\varphi & \partial_z & \partial_\rho \\
\partial_\varphi & \partial_y & \partial_\rho \\
\partial_\varphi & \partial_\rho & \partial_\rho 
\end{pmatrix} = \begin{pmatrix}
1 & \eta^x & 0 \\
0 & \eta^y & 0 \\
0 & 0 & 1
\end{pmatrix}. \tag{2.31}
\]

The second column is immediate from

\[
\eta^x \partial_x + \eta^y \partial_y + \eta^z \partial_z = \eta = \partial_\varphi = \frac{\partial x}{\partial \varphi} \partial_x + \frac{\partial y}{\partial \varphi} \partial_y + \frac{\partial z}{\partial \varphi} \partial_z.
\]

Similarly the third row follows immediately from \( dz = d\bar{z} \). It seems that the remaining entries require considering \( J^{-1} \). Now, \( \varphi \) is a coordinate that vanishes on \( N \), so that \( \partial_x \varphi = \partial_\varphi \rho = 0 \) there. From \( \eta^i \partial_i \varphi = 1 \) we thus obtain \( \partial_y \varphi = 1/\eta^y \). Next, \( \rho = x \) on \( N \), giving \( \partial_x \rho = 1 \) and \( \partial_z \rho = 0 \) there. The equation \( \eta^i \partial_i \rho = 0 \) gives then \( \eta^x + \eta^y \partial_y \rho = 0 \), so that \( \partial_y \rho = -\eta^x/\eta^y \). The derivatives of \( \bar{z} \) are straightforward, leading to

\[
J^{-1} = \begin{pmatrix}
\frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial \rho}{\partial z} \\
\frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial \rho}{\partial z} \\
\frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial \rho}{\partial z}
\end{pmatrix} = \begin{pmatrix}
1 & -\eta^x/\eta^y & 0 \\
0 & 1/\eta^y & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Inverting \( J^{-1} \) leads to (2.31).
From now on we drop the bar on $\bar{z}$. From (2.31) one immediately has

$$A_z = \frac{\eta_z}{g(\eta, \eta)} = \begin{cases} 
  o_{k-1}(r^{-3/2}) + o_{k-3}(r^{-5/2})x, & |x| \leq |z|, \\
  o_k(r^{-3/2}), & \text{otherwise},
\end{cases}$$  \hspace{1cm} (2.32)

Similarly, again on $N$,

$$B_\rho = \frac{\eta_\rho}{\rho g(\eta, \eta)} \frac{\partial x^i}{\partial \rho} = \frac{\eta_x}{x g(\eta, \eta)} = \begin{cases} 
  o_{k-3}(r^{-5/2}), & |x| \leq |z|, \\
  o_k(r^{-5/2}), & \text{otherwise},
\end{cases}$$  \hspace{1cm} (2.33)

Finally, we note that

$$e^{-2U} := \frac{g(\eta, \eta)}{\rho^2} = \begin{cases} 
  1 + o_{k-1}(r^{-1/2}) + o_{k-3}(r^{-5/2})x^2, & |x| \leq |z|, \\
  1 + o_k(r^{-1/2}), & \text{otherwise},
\end{cases}$$  \hspace{1cm} (2.34)

In summary:

**Proposition 2.2** Under (2.9) with $k \geq 3$ the metric $q$ is asymptotically flat. In fact, there exist coordinates $(x, y, z)$ satisfying (2.9) and a constant $R \geq 0$ such that the plane $\{y = 0\} \cap \{r \geq R\}$ is transverse to $\eta$ except at $x = z = 0$ where $\eta$ vanishes and, setting $x^A = (x, z)$ we have

$$q_{AB} - \delta_{AB} = o_{k-3}(r^{-1/2}).$$  \hspace{1cm} (2.35)

Furthermore (2.32)-(2.34) hold.

**2.4 Isothermal coordinates**

We will use the same symbol $q$ for metric on the manifold obtained by doubling $M/\text{U}(1)$ across the axis.

We start by noting the following:

**Proposition 2.3** Let $q$ be an asymptotically flat metric on $\mathbb{R}^2$ in the sense of (2.35) with $k \geq 5$. Then $q$ has a global representation

$$q = e^{2u}(dv^2 + dw^2), \quad \text{with } u \to \sqrt{v^2 + w^2} \to 0.$$

In fact, $u = o_{k-4}(r^{-1/2})$. 

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Remark 2.4 The classical justification of the existence of global isothermal coordinates proceeds by constructing the coordinate $v$ of (2.36) as a solution of the equation $\Delta_q v = 0$. A more careful version of the arguments in the spirit of [16, Lemma 2.3] shows that $v$ has no critical points. However, the approach here appears to be simpler.

Proof: Let $\tilde{q}_{AB} = e^{-2u}q_{AB}$, then $\tilde{q}$ is flat if and only if $u$ satisfies the equation

$$\Delta_q u = -\frac{R(q)}{2}, \tag{2.37}$$

where $R(q)$ is the scalar curvature of $q$. For asymptotically flat metrics $q$, with asymptotically Euclidean coordinates $(x, z)$, this equation always has a solution such that

$$u + \mu \ln(\sqrt{x^2 + z^2}) \to 0,$$

where $\mu = \frac{1}{4\pi} \int_{\mathbb{R}^2} R(q) d\mu_q$, (2.38)

where $d\mu_q$ is the volume form of $q$. More precisely, we have the following:

Lemma 2.5 Consider a metric $q$ on $\mathbb{R}^2$ satisfying

$$q_{AB} - \delta_{AB} = o(\ell^{-1/2})$$

for some $\ell \geq 2$, with $(x^A) = (x, z)$. For any continuous function $R = o_{\ell-2}(r^{-5/2})$ there exists $\hat{u} = o_{\ell-1}(r^{-1/2})$ and a solution of (2.37) such that

$$u = \hat{u} - \mu \ln(\sqrt{x^2 + z^2}),$$

with $\mu$ as in (2.38).

Proof: We start by showing that (2.37) can be solved for $|x|$ large. Indeed, consider the sequence $v_i$ of solutions of (2.37) on the annulus

$$\Gamma(\rho, \rho + i) := D(0, \rho + i) \setminus D(0, \rho),$$

with zero boundary values. Here $\rho$ is a constant chosen large enough so that the functions $\pm C|x|^{-1/2}$, with $C = 8\|R|x|^{5/2}\|_{L^\infty}$, are sub- and supersolutions of (2.37). The usual elliptic estimates show that a subsequence can be chosen which converges, uniformly on compact sets, to a solution $v = O_{\ell^{-1}}(r^{-1/2})$ of (2.37) on $\mathbb{R}^2 \setminus D(0, \rho)$. In the notation of [4] we have in fact $v \in C^{\ell-1,\lambda}_{1/2,0}$ for any $\lambda \in (0, 1)$. Furthermore, using the techniques in [4] one checks that $v = o_{\ell^{-1}}(r^{-1/2})$. 

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We extend $v$ in any way to a $C^{\ell-1,\lambda}$ function on $\mathbb{R}^2$, still denoted by $v$. Let $\hat{q} := e^{-2v}q$, then $\hat{q}$ is flat for $|x| \geq \rho$. Let $\hat{e}^A$ be any $\hat{q}$-parallel orthonormal co-frame on $\mathbb{R}^2 \setminus D(0,\rho)$, performing a rigid rotation of the coordinates if necessary we will have $\hat{e}^A = dx^A + \sum_B \ell_{-1}(|x|^{-1/2}) dx^B$ for $|x|$ large. Let $\hat{x}^A$ be any solutions of the set of equations $d\hat{x}^A = \hat{e}^A$. By the implicit function theorem the functions $\hat{x}^A$ cover $\mathbb{R}^2 \setminus D(0,\hat{\rho})$, for some $\hat{\rho}$, and form a coordinate system there, in which $\hat{q}_{AB} = \delta_{AB}$.

Since (2.37) is conformally covariant, we have reduced the problem to one where $R$ has compact support, and $q$ is a $C^{\ell-1,\lambda}$ metric which is flat outside of a compact set. This will be assumed in what follows.

Let us use the stereographic projection, say $\psi$, to map $\mathbb{R}^2$ to a sphere, then (2.37) becomes an equation for $\hat{u} := (u - v) \circ \psi^{-1}$ on $S^2 \setminus \{i^0\}$, where $i^0$ is the north pole of $S^2$, of the form

$$\Delta_h \hat{u} = |x|^4 f,$$

where $h_{AB} := |x|^{-4} q_{AB}$ is a $C^{\ell-1,\lambda}$ metric on $S^2$, similarly $f$ is a $C^{\ell-2}$ function on $S^2$ supported away from the north pole. In fact, in a coordinate system

$$y^A = x^A/|x|^2$$

near $i^0 = \{y^A = 0\}$, where the $x^A$’s are the explicitly flat coordinates on $\mathbb{R}^2 \setminus D(0,R)$ for the metric $q$, we have

$$h_{AB} = \delta_{AB}.$$

Let $H_k(S^2)$ be the usual $L^2$-type Sobolev space of functions on $S^2$ and set

$$H_k = \left\{ \chi \in H_k(S^2) \mid \int_{S^2} \chi \, d\mu_h = 0 \right\},$$

where $d\mu_h$ is the measure associated with the metric $h$. We have

**Proposition 2.6** Let $h$ be a twice-differentiable metric on $S^2$, then $\Delta_h : H_2 \rightarrow H_0$ is an isomorphism.

**Proof:** Injectivity is straightforward. To show surjectivity, let $X \subset L^2$ be the image of $H_2$ by $\Delta_h$, by elliptic estimates $X$ is a closed subspace of $L^2(S^2)$. Let $\varphi \in L^2$ be orthogonal to $X$, then

$$\forall \chi \in H_2 \int \varphi \Delta_h \chi \, d\mu_h = 0.$$
Thus \( \varphi \) is a weak solution of \( \Delta_h \varphi = 0 \), by elliptic estimates \( \varphi \in H^2 \). But setting \( \chi = \varphi \) and integrating by parts one obtains \( d \varphi = 0 \), hence \( \varphi \) is constant, which shows that \( X = \mathbb{H}_0 \). \( \square \)

Returning to the proof of Lemma 2.5, we have seen that (2.37) can be reduced to solving the problem

\[
\Delta_{\bar{h}} \bar{u} = \bar{f} ,
\]

where \( \bar{h} \) is flat outside of a compact set. Let

\[
\mu := -\frac{1}{2\pi} \int_{\mathbb{R}^2} \bar{f} \, d\mu_{\bar{h}} ,
\]

then

\[
\int_{\mathbb{R}^2} \Delta_{\bar{h}} \left( \mu \ln \sqrt{1 + x^2 + z^2} \right) \, d\mu_{\bar{h}} = \lim_{\rho \to \infty} \mu \oint_{C(0,\rho)} D \left( \ln \sqrt{1 + x^2 + z^2} \right) \cdot n
\]

\[
= 2\pi \mu = -\int_{\mathbb{R}^2} \bar{f} \, d\mu_{\bar{h}}
\]

Thus (2.42) is equivalent to the following equation for the function \( \tilde{u} := \bar{u} + \mu \ln \sqrt{1 + x^2 + z^2} \):

\[
\Delta_{\bar{h}} \tilde{u} = \bar{f} + \Delta_{\bar{h}} \left( \mu \ln \sqrt{1 + x^2 + z^2} \right) ,
\]

and the right-hand-side has vanishing average. Transforming to a problem on \( S^2 \) as in (2.39), we can solve the resulting equation by Proposition 2.6. Transforming back to \( \mathbb{R}^2 \), and shifting \( u \) by a constant if necessary, the result follows. \( \square \)

Returning to the proof of Proposition 2.3, we claim that \( \mu = 0 \); that is,

\[
\int_{\mathbb{R}^2} R(q) \, d\mu_q = 0 . \tag{2.43}
\]

This is the simplest version of the Gauss-Bonnet theorem, we give the proof for completeness: consider any metric on \( \mathbb{R}^2 \) satisfying

\[
\begin{align*}
q_{AB} - \delta_{AB} &= o_1(1) , & R(q) &\in L^1 .
\end{align*}
\]

Let \( \tilde{\theta}^a, a = 1, 2, \) be an orthonormal co-frame for \( q \) obtained by a Gram-Schmidt procedure starting from \((dx^1, dx^2)\), with connection coefficients \( \omega^a_b \). Then \( \omega^a_b = o(r^{-1}) \). It is well known that, in dimension two,

\[
R(q) \, d\mu_q = 2d\omega^1 \, 2 . \tag{2.44}
\]

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Equation (2.43) immediately follows by integration on $B(R)$, using Stokes’ theorem, and passing to the limit $R \rightarrow \infty$.

To finish the proof, note that the metric $\tilde{q}$ is a complete flat metric on $\mathbb{R}^2$, and the Hadamard-Cartan theorem shows the existence of global manifestly flat coordinates, say $(v, w)$ so that $q$ can be written as in (2.36).

Returning to the problem at hand, recall that the metric $q$ on $\mathbb{R}^2$ has been obtained by doubling $M/U(1)$ across $\mathscr{A}$. Let us denote by $\phi$ the corresponding isometry; note that in $M_{\text{ext}}/U(1)$, in the coordinates $(x, z)$ constructed in Section 2.3, the isometry $\phi$ is the symmetry around the $z$-axis: $\phi(x, z) = (-x, z)$. Similarly, in geodesic coordinates centred on $\mathscr{A}$, $\phi(x, z) = (-x, z)$.

As $\phi$ is an isometry of $q$, preserving the boundary conditions satisfied by $u$, uniqueness of solutions of (2.37) implies that $u \circ \phi = u$. Smoothness on the doubled manifold shows that on $\mathscr{A}$ the gradient $\nabla u$ has only components tangential to $\mathscr{A}$. This implies that $\mathscr{A}$ is totally geodesic both for $q$ and $\tilde{q}$.

Choose any point $p$ on $\mathscr{A}$. By a shift of $(v, w)$ we can arrange to have $(v(p), w(p)) = (0, 0)$. Let $(\rho, z)$ be coordinates obtained by a rigid rotation of $(v, w)$ around the origin so that the vector tangent to $\mathscr{A}$ at $p$ coincides with $\partial_z$. Then the axis $\{(0, z)\}_{z \in \mathbb{R}}$ is a geodesic of $\tilde{q}$, sharing a common direction at $p$ with $\mathscr{A}$, hence

$$\mathscr{A} \equiv \{(0, z)\}_{z \in \mathbb{R}}.$$ 

Since $\phi$ is an isometry of $\tilde{q}$ which is the identity on $\mathscr{A}$, it easily follows that

$$\phi(\rho, z) = (-\rho, z),$$

so that $M/U(1) = \{\rho \geq 0\}$. We have thus obtained the representation (1.1) of $g$.

The reader might have noticed that the function $u$ constructed in this section is a solution of a Neumann problem with vanishing data on the axis.

For further use, we note that from (1.1), on $\exp((T \mathscr{A})^\perp)$ the geodesic distance $\hat{\rho}$ from the origin equals

$$\hat{\rho} = e^{-(U-\alpha)(0, z)} \rho + O(\rho^3),$$

and comparing with (2.8) we obtain

$$\alpha(0, z) = 0.$$ 

(2.45)

Now, the function $u = o_{k-4}(r^{-1/2})$ of Proposition 2.3 equals $u = 2(\alpha - U)$ (compare (1.1)). By (2.45) and an analysis of Taylor expansions as in
Section 2.3 we infer that, at \( y = 0 \),
\[
\alpha = o_{k-5}(r^{-3/2})x .
\] (2.46)

From Proposition 2.2 we conclude:

**Theorem 2.7** Let \( k \geq 5 \). Any Riemannian metric on \( \mathbb{R}^3 \) invariant under rotations around a coordinate axis and satisfying
\[
g_{ij} - \delta_{ij} = o_k(r^{-1/2})
\] (2.47)

admits a global representation of the form (1.1), with the functions \( U, \alpha, B_\rho \) and \( A_z \) satisfying
\[
A_z = o_{k-3}(r^{-3/2}) ; \quad B_\rho = o_{k-3}(r^{-5/2}) ; \quad U = o_{k-3}(r^{-1/2}) ; \quad \alpha = o_{k-4}(r^{-1/2}) .
\] (2.48)

Furthermore (2.46) holds.

**Remark 2.8** The decay rate \( o(r^{-1/2}) \) in (2.47) has been tailored to the requirement of a well-defined ADM mass; the result remains true with decay rates \( o(r^{-\alpha}) \) or \( O(r^{-\alpha}) \) for any \( \alpha \in (0, 1) \), with the decay rate carrying over to the functions appearing in (1.1) in the obvious way, as in (2.48).

### 2.4.1 Several asymptotically flat ends

The above considerations generalize to several asymptotically flat ends:

**Theorem 2.9** Let \( k \geq 5 \), and consider a simply connected three-dimensional Riemannian manifold \( (M, g) \) which is the union of a compact set and of \( N \) asymptotically flat ends, and let \( M_{\text{ext}} \) denote the first such end. If \( g \) is invariant under an action of \( U(1) \), then \( g \) admits a global representation of the form (1.1), where the coordinates \( (z, \rho) \) cover \( (\mathbb{R} \times \mathbb{R}^+) \setminus \{ \bar{a}_i \}_{i=2}^N \), with the punctures \( \bar{a}_i = (0, a_i) \) lying on the \( z \)-axis, each \( \bar{a}_i \) representing “a point at infinity” of the remaining asymptotically flat regions. The functions \( U, \alpha, B_\rho \) and \( A_z \) satisfy (2.48) in \( M_{\text{ext}} \).

If we set
\[
r_i = \sqrt{\rho^2 + (z - a_i)^2} ,
\]
then we have the following asymptotic behavior near each of the punctures
\[
U = 2 \ln r_i + o_{k-4}(r_i^{1/2}) , \quad \alpha = o_{k-4}(r_i^{1/2}) ,
\] (2.49)
where \( f = o_\ell(r_i^{1/2}) \) means that \( \partial_{A_1} \ldots \partial_{A_j} f = o_\ell(r_i^{1/2-j}) \) for \( 0 \leq j \leq \ell \). Finally, (2.45) holds.
Proof: As discussed in Section 2.1, $M$ is diffeomorphic to $\mathbb{R}^3$ minus a finite set of points and, after performing a diffeomorphism if necessary, the action of the group is that by rotations around a coordinate axis of $\mathbb{R}^3$. As in the proof of Theorem 2.7 there exists a function $v = o_k - 4(r^{-1/2})$ so that the metric $e^{-2v}q$ is flat for $|x|$ large enough in each of the asymptotic regions. Equation (2.37) is then equivalent to the following equation for $u - v$,

$$
\Delta_{e^{-2v}q}(u - v) = -e^{2v}\left(\frac{R(q)}{2} + \Delta_q v\right), \quad (2.50)
$$

where the right-hand-side is compactly supported on $M/U(1)$. Let $M_{\text{ext}}/U(1)$ be the orbit space associated to the first asymptotically flat region and let $\psi$ be any smooth strictly positive function on $M/U(1)$ which coincides with $|\vec{y}|^{-4}$ in each of the remaining asymptotically flat regions of $M/U(1)$, where the $y^A$s are the manifestly flat coordinates there, with $\psi$ equal to one in $M_{\text{ext}}/U(1)$. Then (2.50) is equivalent to

$$
\Delta_{\psi e^{-2v}q}(u - v) = -\psi^{-1}e^{2v}\left(\frac{R(q)}{2} + \Delta_q v\right). \quad (2.51)
$$

Both the metric $\psi e^{-2v}q$ and the source term extend smoothly through the origins, say $i_0^j$, $j = 1, \ldots, N$, of each of the local coordinate systems $x^A := y^A/|\vec{y}|^2$. Simple connectedness of the two-dimensional manifold

$$
\mathcal{N} := M/U(1) \cup \{i_0^j\}_{j=2}^N
$$

implies that $\mathcal{N} \approx \mathbb{R}^2$, so that (2.51) is an equation to which Lemma 2.5 applies. We thus obtain a solution, say $w$, of (2.51), and subsequently a solution $v + w$ of (2.37) which tends to a constant in each of the asymptotically flat regions (possibly different constants in different ends), except (as will be seen shortly) in $M_{\text{ext}}$ where it diverges logarithmically. Note that at large distances in each of the asymptotically flat regions the function $w$ is harmonic with respect to the Euclidean metric, hence approaches its asymptotic value as $|y|^{-1}$, with gradient falling-off one order faster. Similarly $v$ has controlled asymptotics there, as in the proof of Lemma 2.5. Integrating (2.37) over $M/U(1)$ one finds that the coefficient of the logarithmic term is again as in (2.38).

In order to determine that coefficient, we note that since $\mathcal{N} \approx \mathbb{R}^2$ there exists a global orthonormal coframe for $g$, e.g. obtained by a Gram-Schmidt procedure from a global trivialization of $T^*\mathbb{R}^2$. As a starting point for this procedure one can, and we will do so, use a holonomic basis $dx^A$ with the coordinate functions $x^A$ equal to the manifestly flat coordinates in $M_{\text{ext}}/U(1)$.
Furthermore, after a rigid rotation of the $y^A$’s if necessary, where the $y^A$’s are the manifestly flat coordinates for the metric $e^{-2(w+v)q}$ in the asymptotically flat regions other than $M_{\text{ext}}/U(1)$, we can also assume that the $dx^A$’s coincide with $d(y^A/|y|^2)$ near each $i_0^j$. By (2.44) and by what is said in the paragraph following that equation we have

$$
\mu = \frac{1}{4\pi} \int_{M/U(1)} R(q) d\mu_q = \sum_{j=2}^{N} \lim_{\epsilon \to 0} \frac{1}{2\pi} \oint_{C(i_j^0, \epsilon)} \omega^1_2 .
$$

where the $C(i_j^0, \epsilon)$’s are circles of radius $\epsilon$ centred at the $i_j^0$’s. Near each $i_j^0$ the metric $q$ takes the form $e^{2(v+w)}\delta_{AB}dy^Ady^B = e^{2(v+w)}|\vec{x}|^{-4}\delta_{AB}dx^Adx^B$. The co-frame $\hat{\theta}^A$ is given by $\hat{\theta}^A = e^{(v+w)}|\vec{x}|^{-2}dx^A$, leading to

$$
\omega^1_2 = \frac{2}{|\vec{x}|^2} (x^1dx^2 - x^2dx^1) + o(|\vec{x}|^{-1/2})dx^A,
$$

so that

$$
\lim_{\epsilon \to 0} \oint_{C(i_j^0, \epsilon)} \omega^1_2 = 4\pi .
$$

We note that we have proved:

**Proposition 2.10** Let $q$ be a Riemannian metric on a simply connected two-dimensional manifold which is the union of a compact set and $N$ ends which are asymptotically flat in the sense of (2.35), then

$$
\mu := \frac{1}{4\pi} \int R(q) d\mu_q = 2(N - 1) .
$$

□

Since $\mu \neq 0$, the function $v + w$ obtained so far needs to be modified to get rid of the logarithmic divergence. In order to do that for $j = 2, \ldots, N$ we construct functions $u_j$, $q$-harmonic on $M/U(1)$, such that, in coordinates $x^A$ which are manifestly conformally flat in each of the asymptotic regions,

$$
u_j = \begin{cases} 
\ln |\vec{x}| + o(1), & \text{in } M_{\text{ext}}/U(1); \\
-\ln |\vec{x}| + O(1), & \text{in the } j\text{’th asymptotic region}; \\
O(1), & \text{in the remaining asymptotic regions.}
\end{cases} \quad (2.52)
$$

This can be done as follows: let $\hat{u}_j$ be any smooth function which in local manifestly conformally flat coordinates both near $i_j^0$ and on $M_{\text{ext}}/U(1)$
equals \( \ln |\vec{x}| \), and which equals one at large distances in the remaining asymptotically flat regions. Let \( \psi \) be as in (2.51), then \( \Delta_{\psi e^{-2(v+w)}} \hat{u}_j \) is compactly supported in \( M/\mathrm{U}(1) \). Further

\[
\int_{M/\mathrm{U}(1)} \Delta_{\psi e^{-2(v+w)}} \hat{u}_j \, d\mu_{\psi e^{-2(v+w)}} = \int_{M/\mathrm{U}(1)} \Delta_{\psi e^{-2v}} \hat{u}_j \, d\mu_{\psi e^{-2v}} = \lim_{R \to \infty} \int_{C(0,\rho)} D \ln |\vec{x}| \cdot n - \lim_{\epsilon \to 0} \int_{C(0,\epsilon)} D \ln |\vec{x}| \cdot n = 0.
\]

We can therefore invoke Lemma 2.5 to conclude that there exists a uniformly bounded function \( \hat{v} \), approaching zero as one recedes to infinity in \( M_{\text{ext}}/\mathrm{U}(1) \), such that

\[
\Delta_{\psi e^{-2(v+w)}} \hat{v} = -\Delta_{\psi e^{-2(v+w)}} \hat{u}_j.
\]

Subsequently the function \( u_j := \hat{u}_j + \hat{v} \) is \( q \)-harmonic and satisfies (2.52).

The function

\[
u := v + w + 2 \sum_{j=2}^N \alpha
\]

where \( \alpha \) is an appropriately chosen constant (compare [4]), defines the desired conformal factor approaching one as one tends to infinity in \( M_{\text{ext}}/\mathrm{U}(1) \) so that \( e^{-2u} \) is flat. This conformal factor further compactifies each of the asymptotic infinities except the first one to a point, so that \( e^{-2u} \) extends by continuity to a flat complete metric on the simply connected manifold \( \overrightarrow{\mathcal{N}} \). By the Hadamard-Cartan theorem there exists on \( \overrightarrow{\mathcal{N}} \) a global manifestly flat coordinate system for \( e^{-2u} \). The axis of rotation can be made to coincide with a coordinate axis as in the proof of Theorem 2.7. It should be clear that the points at infinity \( i_0^j \) lie on that axis.

In order to prove (2.49), note that the construction above gives directly.

\[
U - \alpha = u = C_i + 2 \ln r_i + o_{k-4}(r_i^{1/2})
\]

Next, \( U \) can be determined by applying an inversion

\[
y^A \mapsto (\rho, z - a_i) = (x^A) = (y^A/|\vec{y}|^2)
\]

to (2.34),

\[
\rho^2 e^{-U} = g(\eta, \eta) = \frac{\rho^2}{(\rho^2 + (z - a_i)^2)^2} \left( 1 + o_{k-3}((\rho^2 + (z - a_i)^2)^{1/4}) \right).
\]

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Since $\alpha$ vanishes on the axis $(y^1)^2 + (y^2)^2 = 0$ in each of the asymptotic regions, we conclude that $C_i = 0$, and (2.49) follows. \hfill \Box

3 ADM mass

Let $m$ be the ADM mass of $g$,

$$m := \lim_{R \to \infty} \frac{1}{16\pi} \int_{S_R} (g_{ij,j} - g_{jj,i}) dS_i ,$$

where $dS_i = \partial_i (dx \wedge dy \wedge dz)$. This has to be calculated in a coordinate system satisfying (2.9). Typically one takes $S_R$ to be a coordinate sphere $S(R)$ of radius $R$; however, as is well-known, under (2.9) $S_R$ can be taken to be any piecewise differentiable surface homologous to $S(R)$ such that

$$\inf \{ r(p)| p \in S_R \} \to R \to \infty \infty . \quad (3.1)$$

We will exploit this freedom in our calculation to follow.

We introduce new coordinates $x$ and $y$ so that $\rho$ and $\varphi$ in (1.1) become the usual polar coordinates on $\mathbb{R}^2$:

$$x = \rho \cos \varphi , \quad y = \rho \sin \varphi .$$

This implies

$$\rho d\rho = \frac{1}{2} d(\rho^2) = xdx + ydy ,$$

$$\rho^2 d\varphi = xdy - ydx ,$$

$$\rho^2 d\varphi^2 = dx^2 + dy^2 - d\rho^2 .$$

Inserting the above in (1.1) one obtains

$$g = e^{-2U} \underbrace{(dx^2 + dy^2)}_{d\rho^2 + \rho^2 d\varphi^2} + \frac{e^{-2U} (e^{2\alpha} - 1)}{\rho^2} (xdx + ydy)^2 + e^{-2U} 2\alpha dz^2$$

$$+ 2e^{-2U} (xdy - ydx) \left( B_\rho (xdx + ydy) + A_z dz \right) + \text{terms quadratic in } (B_\rho, A_z) . \quad (3.2)$$

This will satisfy (2.9) if we assume that

$$U , \frac{(e^{2\alpha} - 1)x^2}{\rho^2} , \frac{(e^{2\alpha} - 1)xy}{\rho^2} , \frac{(e^{2\alpha} - 1)y^2}{\rho^2} = o_1(r^{-1/2}) , \quad (3.3)$$

$$B_\rho x^2 , B_\rho xy , B_\rho y^2 , A_z x , A_z y = o_1(r^{-1/2}) , \quad (3.4)$$
consistently with Theorem 2.7. Then the term occurring in the last line of (3.2) will not give any contribution to the mass integral:

\[
g = e^{-2U}(dx^2 + dy^2) + \frac{e^{2\alpha} - 1}{\rho^2}(xdx + ydy)^2 + e^{-2U + 2\alpha}dz^2
\]

\[
+ 2(xy - ydx)(B_{\rho}(xdx + ydy) + A_zdz)
\]

\[
+ o_1(r^{-1})dx^i dx^j.
\]  

(3.5)

Let us denote by \(x^n\) the variables \(x, y\). As an example, consider the contribution of (c) to the mass integrand:

\[
(c) \rightarrow g_{zz,z}dS_z - g_{zz,i}dS_i = -g_{zz,a}dS_a = \left(2(U - \alpha),a + o(r^{-2})\right) dS_a.
\]

A similar calculation of (a) easily leads to

\[
(a) + (c) \rightarrow (4U, i + o(r^{-2})) dS_i - 2\alpha_a dS_a.
\]

The contribution of (b) to the mass integrand looks rather uninviting at first sight:

\[
(b) \rightarrow \left[\left(\frac{e^{2\alpha} - 1}{\rho^2}\right)_{,y} xy - \left(\frac{e^{2\alpha} - 1}{\rho^2}\right)_{,x} y^2 + \frac{e^{2\alpha} - 1}{\rho^2} x\right] dS_x
\]

\[
+ \left[\left(\frac{e^{2\alpha} - 1}{\rho^2}\right)_{,x} xy - \left(\frac{e^{2\alpha} - 1}{\rho^2}\right)_{,y} x^2 + \frac{e^{2\alpha} - 1}{\rho^2} y\right] dS_y
\]

\[
- \left(\frac{e^{2\alpha} - 1}{\rho^2}\right)_{,z} (x^2 + y^2) dS_z.
\]

Fortunately, things simplify nicely if \(S_R\) is chosen to be the boundary of the solid cylinder

\[
C_R := \{-R \leq z \leq R, \ 0 \leq \rho \leq R\}.
\]

(3.6)

Then \(S_R\) is the union of the bottom \(B_R = \{z = -R, \ 0 \leq \rho \leq R\}\), the lid \(L_R = \{z = R, \ 0 \leq \rho \leq R\}\), and the wall \(W_R = \{-R \leq z \leq R, \ \rho = R\}\). On the bottom and on the lid we only have a contribution from \(dS_z\), which equals

\[-\left(2\alpha_z + o(r^{-2})\right) dx \wedge dy\]
on the lid, and minus this expression on the bottom. On the wall $dS_z$ gives no contribution, while

$$dS_x|_{W_R} = (dy \wedge dz)|_{W_R} = x|W_R d\varphi \wedge dz, \quad dS_y|_{W_R} = -(dx \wedge dz)|_{W_R} = y|W_R d\varphi \wedge dz.$$  

Surprisingly, the terms in $(b)|_{W_R}$ containing derivatives of $\alpha$ drop out, leading to

$$(b)|_{W_R} \longrightarrow (2\alpha + o(r^{-2})) d\varphi \wedge dz$$

We continue with the contribution of $B_\rho$ to $(d)$:

$$\left[ \left( (x^2 - y^2)B_\rho \right)_y - (2xyB_\rho)_x \right] dS_x + \left[ \left( (x^2 - y^2)B_\rho \right)_x + (2xyB_\rho)_y \right] dS_y.$$  

It only contributes on the wall $W_R$, giving however a zero contribution there:

$$\left[ \left( (x^2 - y^2)(x \partial_y + y \partial_x) + 2xy(y \partial_y - x \partial_x) \right) B_\rho \right] d\varphi \wedge dz = 0.$$  

Finally, $A_z$ produces the following boundary integrand:

$$-y \partial_z A_z dS_x + x \partial_z A_z dS_y + \left[ (x \partial_y - y \partial_x) A_z \right] dS_z,$$

and one easily checks that the $dS_x$ and $dS_y$ terms cancel out when integrated upon $W_R$, while giving no contribution on the bottom and the lid.

Collecting all this we obtain

$$m = \lim_{R \to \infty} \frac{1}{16\pi} \left[ 4 \int_{S_R} \partial_i U dS_i + 2 \int_{W_R} (\alpha - \frac{x^a}{\rho} \partial_a \alpha) d\varphi dz + 2 \int_{L_R} \partial_z \alpha dx dy + 2 \int_{B_R} \partial_z \alpha dx dy \right]$$

$$= \lim_{R \to \infty} \frac{1}{4\pi} \left[ \int_{S_R} \partial_i (U - \frac{1}{2} \alpha) dS_i + \frac{1}{2} \int_{W_R} \alpha d\varphi dz \right].$$

We have the following formula for the Ricci scalar $(3) R$ of the metric (1.1)
(the details of the calculation can be found in [9]).

\[
- \frac{e^{-2U+2\alpha}}{4} (3)^R = -\Delta_\delta (U - \frac{1}{2} \alpha) + \frac{1}{2} (DU)^2 - \frac{1}{2\rho} \partial_\alpha + \frac{\rho^2 e^{-2\alpha}}{8} (\rho B_{\rho,z} - A_{z,\rho})^2 .
\]

(3.7)

The Laplacian \( \Delta_\delta \) and the gradient \( D \) are taken with respect to the flat metric \( \delta \) on \( \mathbb{R}^3 \).

Now,

\[
\lim_{R \to \infty} \frac{1}{4\pi} \left[ \int_{S_R} \partial_i (U - \frac{1}{2} \alpha) dS_i + \frac{1}{2} \int_{W_R} \alpha d\phi dz \right] = \lim_{R \to \infty} \left[ \frac{1}{4\pi} \int_{C_R} \left[ \Delta_\delta (U - \frac{\alpha}{2}) + \frac{1}{2\rho} \frac{\partial \alpha}{\partial \rho} \right] d^3x + \frac{1}{4} \int_{-R}^R \alpha (\rho = 0, z) dz \right].
\]

(3.8)

The last integral vanishes by (2.45). Equations (3.7)-(3.8) and the dominated convergence theorem yield now

\[
m = \frac{1}{16\pi} \int \left[ (3)^R + \frac{1}{2} \rho^2 e^{-4\alpha+2U} (\rho B_{\rho,z} - A_{z,\rho})^2 \right] e^{2(\alpha-U)} d^3x + \frac{1}{8\pi} \int (DU)^2 d^3x .
\]

(3.9)

Since \( (3)^R = 16\pi \mu + K_{ab} K^{ab} \geq 0 \) for initial data sets satisfying \( \text{tr}_g K = 0 \), this proves positivity of mass for initial data sets as considered above.

Suppose that \( m = 0 \) with \( (3)^R \geq 0 \), then (3.9) gives

\[
(3)^R = \rho B_{\rho,z} - A_{z,\rho} = DU = 0 .
\]

(3.10)

The last equality implies \( U \equiv 0 \), and from (3.7) we conclude that

\[
\Delta_\delta \alpha - \frac{1}{2\rho} \frac{\partial \alpha}{\partial \rho} = 0 .
\]

The maximum principle applied on the set

\[
B(R) \setminus \{ \rho \leq 1/R \}
\]

\[^3\text{In the time-symmetric case (3.7) can be viewed as a PDE for } U \text{ given the remaining functions and the matter density. Assuming that this equation can indeed be solved, this allows us to prescribe freely the functions } \alpha, B_\rho \text{ and } A_z \text{. In such a rough analysis there does not seem to be any constraints on } \alpha, B_\rho \text{ and } A_z \text{ (in particular they can be chosen to satisfy (3.3)-(3.4)), while } U, \text{ and hence its asymptotic behavior, is determined by (3.7).}\]
gives $\alpha \equiv 0$ after passing to the limit $R \to \infty$. The before-last equality in (3.10) shows that the form $\rho B_\rho d\rho + A_z dz$ is closed, and simple-connectedness implies that there exists a function $\lambda$ such that $\rho B_\rho d\rho + A_z dz = d\lambda$, bringing the metric (1.1) to the form

$$d\rho^2 + dz^2 + \rho^2 (d(\varphi + \lambda))^2.$$  \hspace{1cm} (3.11)

Hence $g$ is flat. One could now attempt to analyse $\varphi + \lambda$ near the axis of rotation to conclude that $(\rho, \varphi + \lambda, z)$ provide a new global polar coordinate system, and deduce that $g$ is the Euclidean metric. However, it is simpler to invoke the Hadamard-Cartan theorem to achieve that conclusion.

Summarizing, we have proved:

**Theorem 3.1** Consider a metric of the form (1.1) on $M = \mathbb{R}^3$, where $(\rho, \varphi, z)$ are polar coordinates, with Killing vector $\partial_\varphi$, and suppose that the decay conditions (3.3)-(3.4) hold. If

$$^3 R \geq 0$$

then $0 \leq m \leq \infty$. Furthermore, we have $m < \infty$ if and only if

$$^3 R \in L^1(\mathbb{R}^3), \quad DU, \rho B_{\rho,\rho} - A_{z,\rho} \in L^2(\mathbb{R}^3).$$

Finally, $m = 0$ if and only if $g$ is the Euclidean metric. \hfill $\Box$

**Remark 3.2** Theorem 2.7 shows that the coordinates required above will exist for a general asymptotically flat axisymmetric metric on $\mathbb{R}^3$ if (2.9) holds with $k = 6$.

### 3.1 Several asymptotically flat ends

Theorem 3.1 proves positivity of mass for axi-symmetric metrics on $\mathbb{R}^3$. More generally, one has the following:

**Theorem 3.3** Let $(M, g)$ be a simply connected three dimensional Riemannian manifold which is the union of a compact set and of a finite number of asymptotic regions $M_i, i = 1, \ldots, N$, which are asymptotically flat in the sense of (2.9)-(2.10) with $k \geq 6$. If $g$ is invariant under an action of $U(1)$ and

$$^3 R \geq 0,$$

then the ADM mass $m_i$ of each of the ends $M_i$ satisfies $0 < m_i \leq \infty$, with $m_i < \infty$ if and only if

$$^3 R \in L^1(M_i), \quad DU, \rho B_{\rho,\rho} - A_{z,\rho} \in L^2(M_i).$$

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Proof: The result follows immediately from the calculations in this section together with Theorem 2.9: Indeed, one can integrate (3.7) on a set
\[ \hat{C}_R := C_R \setminus C_{1/R} = \{ -R \leq z \leq R , 1/R \leq \rho \leq R \} , \]
where \( C_R \) is as in (3.6). The asymptotics (2.49) implies that the boundary integrals over the boundary of \( C_{1/R} \) give zero contribution in the limit \( R \to \infty \), so that (3.9) remains valid by the monotone convergence theorem in spite of the (mildly) singular behavior at the punctures \( \vec{a}_i \) of the functions appearing in the metric. \( \square \)

3.2 Nondegenerate instantaneous horizons

In order to motivate the boundary conditions in this section, recall that in Weyl coordinates the Schwarzschild metric takes the form (cf., e.g., [14, Equation (20.12)])
\[ 4g = -e^{2U_{\text{Schw}}} dt^2 + e^{-2U_{\text{Schw}}} \rho^2 d\varphi^2 + e^{2\lambda_{\text{Schw}}} (d\rho^2 + dz^2) , \]
where
\[ U_{\text{Schw}} = \ln \rho - \ln \left( m \sin \tilde{\theta} + \sqrt{\rho^2 + m^2 \sin^2 \tilde{\theta}} \right) , \]
\[ \lambda_{\text{Schw}} = -\frac{1}{2} \ln \left[ \frac{(r_{\text{Schw}} - m)^2 - m^2 \cos^2 \tilde{\theta}}{r_{\text{Schw}}^2} \right] , \]
\[ = -\frac{1}{2} \ln \left[ \frac{4z(z + m)^2 + \rho^2 \sqrt{(z + m)^2 + \rho^2}}{2m \sqrt{(z - m)^2 + \rho^2 + (z + m)^2 + \rho^2}} \right] . \]

In (3.13) the angle \( \tilde{\theta} \) is a Schwarzschild angular variable, with the relations
\[ 2m \cos \tilde{\theta} = \sqrt{(z + m)^2 + \rho^2} - \sqrt{(z - m)^2 + \rho^2} , \]
\[ 2(r_{\text{Schw}} - m) = \sqrt{(z + m)^2 + \rho^2} + \sqrt{(z - m)^2 + \rho^2} , \]
\[ \rho^2 = r_{\text{Schw}} (r_{\text{Schw}} - 2m) \sin^2 \tilde{\theta} , \quad z = (r_{\text{Schw}} - m) \cos \tilde{\theta} , \]

where \( r_{\text{Schw}} \) is the usual Schwarzschild radial variable such that \( e^{2U_{\text{Schw}}} = 1 - 2m/r_{\text{Schw}} \). As is well known, and in any case easily seen, \( U_{\text{Schw}} \) is smooth.
on $\mathbb{R}^3$ except on the set $\{\rho = 0, -m \leq z \leq m\}$. From (3.13) we find, at fixed $z$ in the interval $-m < z < m$ and for small $\rho$,

$$U_{\text{Schw}}(\rho, z) = \ln \rho - \ln(2\sqrt{(m + z)(m - z)}) + O(\rho^2) \quad (3.17)$$

(with the error term not uniform in $z$). This justifies our definition: an interval $[a, b] \subset \mathcal{A}$ will be said to be a nondegenerate instantaneous horizon if for fixed $z \in (a, b)$ and for small $\rho$ we have

$$U(\rho, z) = \ln \rho + \tilde{U}(z) + o(1) \quad (3.18)$$

for a smooth function $\tilde{U}$. As in the Schwarzschild case the function $U - \alpha$ is assumed to be smooth across $I$. Thus, to compensate for the logarithmic singularity of $U$, we further assume, again for fixed $z \in (a, b)$ and for small $\rho$, that there exists a function $\tilde{\lambda}(z)$ such that

$$\alpha(\rho, z) = U(\rho, z) + \tilde{\lambda}(z) + o(1) \quad (3.19)$$

Under those conditions the calculation of the mass formula proceeds as follows. For $k = 1, \ldots, N$ let

$$I_k = [c_k, d_k] \subset \mathcal{A}$$

be pairwise disjoint intervals at which the nondegenerate instantaneous horizon boundary conditions hold. Denote by $\tilde{U}$ the function, harmonic on $\mathbb{R}^3 \setminus \bigcup_k I_k$, which is the sum of Schwarzschild potentials $U_{\text{Schw}}$ as in (3.14), each with mass $(d_k - c_k)/2$ and a logarithmic singularity at $I_k$. As in [9], the term $|DU|^2$ in (3.7) is rewritten as:

$$|DU|^2 = |D(U - \tilde{U})|^2 = |D(U - \tilde{U})|^2 + D_i \left[(2U - \tilde{U})D^i \tilde{U}\right].$$

Denote by $I_{\epsilon}$ the set of points which lie a distance less than or equal to $\epsilon$ from the singular set $\bigcup_k I_k$:

$$I_{\epsilon} = \{p \mid d(p, \cup_k I_k) \leq \epsilon\}.$$ 

By inspection of the calculations so far one finds that (3.9) becomes now

$$m = \frac{1}{16\pi} \int \left[(3)R + \frac{1}{2}\rho^2 e^{-4\alpha + 2U} (\rho B_{\rho, z} - A_{z, \rho})^2 \right] e^{2(\alpha - U)} d^3x + \frac{1}{8\pi} \int \left(D(U - \tilde{U})\right)^2 d^3x$$

$$+ \frac{1}{8\pi} \lim_{\epsilon \to 0} \int_{\partial I_{\epsilon}} \left[D^i(2U - \alpha) - (2U - \tilde{U})D^i \tilde{U} + \alpha \frac{D^i \rho}{\rho}\right] n_i d^2S \quad (3.20)$$

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In the last line of (3.20) the normal $n_i$, taken with respect to the flat metric, has been chosen to point away from $I_k$.

Away from the end points of the intervals $I_k$ the logarithmic terms in $U$, $\tilde{U}$ and $\alpha$ cancel out, leaving a contribution

$$\frac{1}{4} \sum_k \left( |I_k| + \int_{I_k} (\tilde{\lambda} + \tilde{\beta}) dz \right),$$

where $|I_k|$ is the length of $I_k$, and where we have denoted by $\tilde{\beta}$ the limit at $\cup_k I_k$ of $\tilde{U} - U$,

$$\tilde{\beta}(z) := \lim_{\rho \to 0, z \in \cup_k I_k} \left( \tilde{U}(\rho, z) - U(\rho, z) \right).$$

As already pointed out, the error term in (3.17) is not uniform in $z$, and therefore it is not clear whether or not there will be a separate contribution from the end points of $I_k$ to the limit as $\epsilon$ tends to zero of the integral over $\partial I_\epsilon$. Assuming that no such contribution arises\(^4\), we conclude that the following formula for the mass holds:

$$m = \frac{1}{16\pi} \int \left[ (3) R + \frac{1}{2} \rho^2 e^{-4\alpha + 2U} (\rho B_{\rho, z} - A_{z, \rho})^2 \right] e^{2(\alpha - U)} d^3 x$$

$$+ \frac{1}{8\pi} \int (D(U - \tilde{U}))^2 d^3 x$$

$$+ \frac{1}{4} \sum_k \left( |I_k| + \int_{I_k} (\tilde{\lambda} + \tilde{\beta}) dz \right). \quad (3.21)$$

In the Schwarzschild case the volume integrals vanish, $\tilde{\beta} = 0$, for $z \in (-m, m)$ the function $\tilde{\lambda}$ equals

$$\tilde{\lambda}(z) = -\frac{1}{2} \ln \left[ \frac{(m - z)(z + m)}{(2m)^2} \right],$$

and one can check (3.21) by a direct calculation of the integral over $I_1$.

### 3.3 Conical singularities

So far we have assumed that the metric is smooth across the rotation axis $\mathcal{A}$. However, in some situations this might not be the case. One of the simplest

\(^4\)Note that this assumption, asymptotic flatness, finiteness of the volume integral in (3.20), and the boundary condition (3.18) on $U$ essentially enforce the boundary condition (3.19) on $\alpha$.\[\]
examples is the occurrence of conical singularities, when the regularity condition (2.45) fails to hold. It is not clear what happens with the construction of the coordinates (1.1) in such a case, and therefore it appears difficult to make general statements concerning such metrics. Nevertheless, there is at least one instance where conical singularities occur naturally, namely in the usual construction of stationary axisymmetric solutions: here one assumes at the outset that the space-time metric takes a form which reduces to (1.1) after restriction to slices of constant time; and the components of the metric are then obtained by various integrations starting from a solution of a harmonic map equation; cf., e.g., [8, 11, 15].

So consider a metric of the form (1.1) on $\mathbb{R}^3 \setminus \{\vec{a}_i\}$, where each puncture $\vec{a}_i$ corresponds to either an asymptotically flat region or to asymptotically cylindrical regions (which, typically, correspond to degenerate black holes). Assuming that $d\alpha$ is bounded at the axis and does not give any supplementary contribution at the punctures, (3.9) becomes instead

$$m = \frac{1}{16\pi} \int_{\mathbb{R}^3 \setminus \{\vec{a}_i\}} \left[ (3)R + \frac{1}{2} \rho^2 e^{-4\alpha+2U} \left( \rho B_{\rho,z} - A_{z,\rho} \right)^2 \right] e^{2(\alpha-U)} \, d^3x$$

$$+ \frac{1}{8\pi} \int_{\mathbb{R}^3 \setminus \{\vec{a}_i\}} (DU)^2 \, d^3x + \frac{1}{4} \oint_{\partial S \setminus \{\vec{a}_i\}} \dot{\alpha} \, dz, \quad (3.22)$$

where $\dot{\alpha}$ denotes the restriction of $\alpha$ to $\mathcal{A}$.

Using (3.22) and (3.21), the reader will easily work out a mass formula when both conical singularities and nondegenerate instantaneous horizons occur.

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