On a Combinatorial Property of Families of Sequences Converging to $\infty$

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Abstract. We consider families $\Phi$ of sequences converging to $\infty$, with the property that for every open set $U \subseteq \mathbb{R}$ that is unbounded above there exists a sequence belonging to $\Phi$, which has an infinite number of terms belonging to $U$.

In this paper we consider families $\Phi$ of sequences converging to $\infty$ that satisfy the following condition (C):

(C): for every open set $U \subseteq \mathbb{R}$ that is unbounded above there exists a sequence belonging to $\Phi$, which has an infinite number of terms belonging to $U$.

The origin of this consideration is the observation that if a family $\Phi$ of sequences converging to $\infty$ satisfies the condition (C) and $f : \mathbb{R} \to \mathbb{R}$ is continuous, then convergence of $\{f(x_n)\}_{n \in \omega}$ to $d \in \mathbb{R} \cup \{-\infty, \infty\}$ for all sequences $\{x_n\}_{n \in \omega}$ belonging to $\Phi$ implies that $\lim_{x \to \infty} f(x) = d$.

We present below necessary definitions from [1] (see also [2], [3]). For the functions $f, g : \omega \to \omega$ we define:

\[ f \leq_* g \iff \{i : f(i) > g(i)\} \text{ is finite (g dominates f)}. \]

We say that $F \subseteq \omega^\omega$ is bounded if $\exists g : \omega \to \omega \ \forall f \in F \ f \leq_* g$ (briefly: $F \leq_* g$, $g$ dominates $F$). Every countable set $F = \{f_j : j \in \omega\}$ is bounded, because the function $g(n) = \max \{f_j(n) : j \leq n\} + 1$ dominates $F$ (i.e. $F \leq_* g$). Let $b = \min \{\text{card } F : F \text{ is unbounded}\}$.
Theorem 1. If $\Phi$ is a family of sequences converging to $\infty$ and $\text{card } \Phi < b$, then $\Phi$ does not satisfy condition (C).

Proof. Let $\text{card } \Phi = \kappa$, $\Phi = \{\phi_\alpha : \alpha < \kappa\}$. There exists such a sequence $\{a_n\}_{n \in \omega}$ strictly increasing and converging to $\infty$, that for every $\alpha < \kappa$ the set $\{a_n : n \in \omega\}$ is disjoint from the set of terms of the sequence $\phi_\alpha$. Let $h_\alpha(n)$ be the least positive integer for which $(a_n - 1/h_\alpha(n), a_n + 1/ h_\alpha(n))$ is disjoint from the set of terms of the sequence $\phi_\alpha$. Since the family $\{h_\alpha : \alpha < \kappa\}$ is bounded, there exists such a function $h : \omega \to \omega$ that for every $\alpha < \kappa h_\alpha \leq h$. In the open set $\bigcup_{n \in \omega} (a_n - 1/h(n), a_n + 1/ h(n))$ every sequence $\phi_\alpha$ has only a finite number of terms.

Corollary. Every family of sequences converging to $\infty$ which satisfies the condition (C) is uncountable, Martin’s axiom implies that each such family has cardinality $c$ (because Martin’s axiom implies that $b = c$, see [1], [3]).

For a sequence $\{a_n\}_{n \in \omega}$ converging to $\infty$ we define the non-decreasing function $f_{\{a_n\}} : \omega \to \omega$: if $\bigcup_{n \in \omega} (a_n - 1, a_n + 1) \supseteq (i, \infty)$ then

$$f_{\{a_n\}}(i) = \max \left\{ j \in \omega \setminus \{0\} : \bigcup_{n \in \omega} \left( a_n - \frac{1}{j}, a_n + \frac{1}{j} \right) \supseteq (i, \infty) \right\}$$

else $f_{\{a_n\}}(i) = 0$. The following Lemma is obvious.

Lemma. If for a family $\Phi$ of sequences converging to $\infty$ the family of functions $\{f_{\{a_n\}} : \{a_n\} \in \Phi\}$ is unbounded, then $\Phi$ satisfies condition (C).

Theorem 2. There exists a family of sequences converging to $\infty$ which satisfies condition (C) and has cardinality $b$.

Proof. Let $F \subseteq \omega^\omega$ be an unbounded family and $\text{card } F = b$. Replacing every function $f \in F$ by the function $\bar{f}(n) = \max\{f(0), f(1), \ldots, f(n)\}$ we obtain an unbounded family $\bar{F}$ of non-decreasing functions. Hence, $\text{card } \bar{F} = b$. To every function $g \in \bar{F}$ we assign a sequence converging to $\infty$:

$$0, \frac{1}{g(0)+1}, \frac{2}{g(0)+1}, \ldots, \frac{g(0)}{g(0)+1}, 1, \frac{1}{g(1)+1}, \frac{2}{g(1)+1}, \ldots, \frac{g(1)}{g(1)+1}, 2, \frac{2}{g(2)+1}, \ldots, \frac{2}{g(2)+1}, \ldots$$
We obtain the family of sequences which has cardinality b, according to the Lemma this family satisfies the condition (C).

**Theorem 3.** If \( X \subseteq \mathbb{R} \) is a set of second category, then the family of sequences \( \{x + \log(n + 1) : x \in X\} \) satisfies condition (C).

**Proof.** Transformation \( x \rightarrow \exp(-x) \) allows us to formulate this theorem in an equivalent form:

If \( X \subseteq (0, \infty) \) is a set of second category, then for every open set \( U \subseteq (0, \infty) \) with \( 0 \in \overline{U} \) there exists an \( x \in X \) such that an infinite number of terms of the sequence \( \{\frac{x}{n}\}_{n \in \omega \setminus \{0\}} \) belong to \( U \).

Using operations on sets we can formulate this condition as follows:

\[
\left[ \bigcap_{n \in \omega \setminus \{0\}} \bigcup_{k \geq n} kU \right] \cap X \neq \emptyset.
\]

Every set \( \bigcup_{k \geq n} kU \) is an open and dense subset of \([0, \infty)\), so we establish the assertion using the Baire category theorem.

The origin of Theorem 3 is the problem number 107 from the problem book [5]:

Assume that \( f : (0, \infty) \rightarrow \mathbb{R} \) is continuous and for every \( x > 0 \) the limit \( \lim_{n \to \infty} f \left( \frac{x}{n} \right) \) exists and equals 0. Is it possible to conclude that \( \lim_{n \to 0^+} f(x) = 0 \)?

From Theorem 3 follows a positive solution of this problem under the weaker assumption that the set \( \{x \in (0, \infty) : \lim_{n \to \infty} f \left( \frac{x}{n} \right) = 0\} \) is a set of second category.

**Remark** [4]. If a sequence \( \{a_n\}_{n \in \omega} \) converging to \( \infty \) satisfies

\[
\lim_{n \to \infty} (a_{n+1} - a_n) = 0
\]

then

for every open set \( U \subseteq \mathbb{R} \) that is unbounded above the set

\[
\{r \in \mathbb{R} : \text{card} \{r + a_n : n \in \omega\} \cap U \geq \omega\}
\]

is a dense subset of \( \mathbb{R} \);
if we additionally assume that a sequence \( \{a_n\}_{n \in \omega} \) converging to \( \infty \) is non-decreasing then (2) implies (1).

The author suggests that if a sequence \( \{a_n\}_{n \in \omega} \) converging to \( \infty \) satisfies (1) then a dense subset considering in (2) is a countable intersection of open dense sets, so if \( X \subseteq \mathbb{R} \) is a set of second category, then the family of sequences \( \{\{x + a_n\}_{n \in \omega} : x \in X\} \) satisfies the condition (C).

Using the Baire category theorem we can prove that if \( X \subseteq \mathbb{R} \) is a set of the second category and if a continuous function \( h : \mathbb{R} \to [0, 1] \) is periodic and surjective, then the family of sequences \( \{\{n + h(nx)\}_{n \in \omega} : x \in X\} \) satisfies condition (C). Using the Baire category theorem for the space of irrationals we can replace \( h(nx) \) by the fractional part of \( nx \) in the statement above.

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