A parabolic model for dimple potentials

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Abstract

We study the truncated parabolic function and demonstrate that it is a representation of the Dirac δ function. We also show that the truncated parabolic function, used as a potential in the Schrödinger equation, has the same bound state spectrum, tunneling and reflection amplitudes as the Dirac δ potential, as the width of the parabola approximates to zero. Dirac δ potential is used to model dimple potentials which are utilized to increase the phase-space density of a Bose–Einstein condensate (BEC) when the initial sample is just above the critical temperature. So, the dimple potentials provide a means of inducing condensation without cooling [1]. Moreover, the improvements in the trapping techniques of cold atoms made it possible to realize BECs in low dimensions [2–6]. Thus, theoretical models of the dimple potentials in one dimension have been extensively studied in the literature, i.e. [7–11]. While some of these models utilize an inverse Gaussian function others adopt Dirac δ functions to describe dimple potentials. We will use a truncated parabolic potential $V(x) = -U_0(1 - x^2/a^2)$ for $x \leq |a|$ to model dimple potentials first proposed in [3] for three dimensions. However, in this study Ma et al used a numeric simulation to explain both the loading process and the subsequent evaporative cooling from a dimple potential.

The advantage of the parabolic model compared to the Gaussian model is that the Schrödinger equation for the truncated parabolic potential is analytically solvable, so one obtains the thermodynamic quantities of a weakly interacting Bose gas at least as a first approximation. Although the Schrödinger equation for the Dirac δ potentials are also analytically solvable, one cannot incorporate both the dimple depth and width together into these models. Only the multiplication of these quantities are used as the coupling coefficient of the Dirac δ potential [12]. In addition to this, Dirac δ models are only valid for tough and narrow dimples. However, experiments with relatively wide dimples are also performed [4]. Therefore, the parabolic model which accommodates both dimple depth and width separately is more useful in describing the dimple potentials. Hence, we will use the truncated parabolic function to investigate BEC in the dimple traps in further studies.

As a special case of point interactions, the Dirac δ potential has been gaining interest for years. Atkinson and Crater [16] investigated the effect of a Dirac δ to the bound states of various potentials in one dimension. Several authors made further studies on this subject [17–19]. Bound states of arbitrary but finite number of Dirac δ shells in higher dimensions were also studied [20]. The resonances of potentials with several Dirac δ were examined by Gadella and co-workers [21, 22]. In physical applications, the Dirac δ potential is mainly used to describe short range effects. For example, Erkol and Demiralp [23, 24] used a δ potential for modeling short range interactions in atomic and nuclear physics. Moreover, it is shown that δ potentials can also be used to find approximate solutions for the tunneling and reflection amplitudes of different potentials [25].

In this study, we show that the truncated parabolic function $f_δ(x) = 3/(4a) (1 - x^2/a^2)$, for $|x| \leq a$ is a representation of the Dirac δ function. Moreover, since the Schrödinger equation of the truncated parabolic potential is analytically solvable, we are able to show that the bound state spectrum, tunneling and reflection coefficients of the truncated parabolic potential go to those of the Dirac δ
potential as the width goes to zero. We also demonstrate that the spectrum of the harmonic trap with a truncated parabolic potential reduces to the spectrum of the harmonic potential with a Dirac \( \delta \) function in the same limit. So, the truncated parabolic potential model of the dimple trap includes all the results obtained for the non-interacting Bose gases by the Dirac \( \delta \) models for the dimple potential as a special case.

The paper is organized as follows. In section 2, we present the solution of the Schrödinger equation for a harmonic trap with a truncated parabolic potential in one dimension and obtain eigenvalue equations and eigenfunctions, for the sake of completeness. In order to find the eigenvalues, the eigenvalue equations are solved numerically. In section 3, we apply the Jeffreys–Wentzel–Kramers–Brillouin (JWKB) approximation to check the validity of numerically found eigenvalues. In section 4, we use the sudden perturbation theory to find the transition amplitudes from the harmonic trap to the harmonic trap with a dimple described by a truncated parabolic potential. In section 5, we show that the truncated parabolic potential function provides a representation of the Dirac \( \delta \) function. Moreover, we also demonstrate that the spectrum, the reflection and transmission coefficients of this potential reduces to those of the Dirac \( \delta \) potential.

2. Harmonic trap with a truncated parabolic potential

In this section, we obtain the eigenvalues and the eigenfunctions of the Hamiltonian

\[
\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})
\]  

\[\text{(1)}\]

in one dimension where the potential function is given as

\[
V(x) = \begin{cases} 
\frac{1}{2}m\omega^2x^2 & \text{for } |x| > a, \\
\frac{1}{2}m\omega^2x^2 - U_0 \left(1 - \frac{x^2}{a^2}\right) & \text{for } |x| \leq a.
\end{cases}
\]  

\[\text{(2)}\]

Here \( U_0 > 0 \) and \( a > 0 \) represent the depth and the width of the truncated parabolic potential, respectively.

The time-independent Schrödinger equation for this potential is

\[
-\frac{\hbar^2}{2m} \frac{d^2\Psi(x)}{dx^2} + V(x)\Psi(x) = E\Psi(x),
\]  

\[\text{(3)}\]

where \( V(x) \) is given in equation (2). For \( U_0 = 0 \), the Hamiltonian reduces to the harmonic oscillator Hamiltonian whose eigenvalues are \( E_n = (n + 1/2)\hbar\omega, \ n = 0, 1, 2, \ldots \) and whose eigenfunctions are given in terms of the Hermite polynomials \( H_n(\sqrt{m\omega/\hbar}x) \). However if \( U_0 \neq 0 \), the eigenvalues are no more \( (n + 1/2)\hbar\omega \) and the eigenfunctions cannot be written in terms of the Hermite polynomials \( H_n(\sqrt{m\omega/\hbar}x), \ n = 0, 1, 2, \ldots \) whose degrees are finite.

The time-independent Schrödinger equation of the Hamiltonian in equation (1) whose potential term is given in equation (2) is different for \( |x| > a \) and for \( |x| \leq a \). For \( |x| > a \) the Schrödinger equation reduces to

\[
-\frac{\hbar^2}{2m} \frac{d^2\Psi(x)}{dx^2} + \frac{1}{2}m\omega^2x^2\Psi(x) = E\Psi(x).
\]  

\[\text{(4)}\]

Although the differential equation (4) is the same as the time-independent Schrödinger equation for the harmonic oscillator potential, the solutions for \( |x| > a \) differ from the harmonic oscillator solution because of the continuity conditions on the wavefunctions and their derivatives at the points \( |x| = a \).

We continue by rewriting equation (4) in terms of the dimensionless variable

\[
z = \left(\frac{m\omega}{\hbar}\right)^{1/2} x
\]  

\[\text{(5)}\]

as

\[
\frac{d^2\Psi(z)}{dz^2} + \left(2\varepsilon - z^2\right)\Psi(z) = 0,
\]  

\[\text{(6)}\]

where \( \varepsilon = E/(\hbar\omega) \). The general solutions of equation (6) can be written in terms of parabolic cylinder functions [26]:

\[
D_\lambda(z) = 2^\lambda e^{-z^2/2} \left\{ \Gamma\left(\frac{1}{2}\right) \Phi(-\lambda/2, 1/2; z^2) \right\}
\]  

\[+\Gamma\left(\frac{1}{2}\right) z\Phi((1-\lambda)/2, 3/2; z^2) \right\},
\]  

\[\text{(7)}\]

\[
D_\lambda(-z) = 2^\lambda e^{-z^2/2} \left\{ \Gamma\left(\frac{1}{2}\right) \Phi(-\lambda/2, 1/2; z^2) \right\}
\]  

\[-\Gamma\left(\frac{1}{2}\right) z\Phi((1-\lambda)/2, 3/2; z^2) \right\},
\]  

\[\text{(8)}\]

where \( \lambda = \varepsilon - 1/2, \Gamma(x) \) is the well known gamma function and \( \Phi(\alpha, \gamma; y) \) is the confluent hypergeometric function [26]. The asymptotic behavior of \( D_\lambda(z) \) and \( D_\lambda(-z) \) are as follows:

\[
\lim_{z \to \infty} D_\lambda(z) = 0, \quad \lim_{z \to \infty} D_\lambda(-z) = \infty,
\]  

\[\lim_{z \to \infty} D_\lambda(-z) = 0, \quad \lim_{z \to \infty} D_\lambda(z) = \infty.
\]  

\[\text{(9)}\]

Because \( D_\lambda(\mp z) \) diverge at \( \pm \infty \), the wavefunctions for \( x < -a \ (z < -\left(\frac{m\omega}{\hbar}\right)^{1/2}/a) \) and \( x > a \ (z > \left(\frac{m\omega}{\hbar}\right)^{1/2}/a) \) are

\[
\psi_1(z) = c_1 D_\lambda(-z),
\]  

\[\text{(10)}\]

\[
\psi_3(z) = c_4 D_\lambda(z),
\]  

\[\text{(11)}\]

where \( c_1 \) and \( c_4 \) are constants that can be determined by the normalization and continuity conditions.

For \( |x| \leq a \), equation (3) takes the form

\[
-\frac{\hbar^2}{2m} \frac{d^2\Psi(x)}{dx^2} + \left\{ \frac{1}{2}m\omega^2x^2 - U_0 \left(1 - \frac{x^2}{a^2}\right) \right\} \Psi(x) = E\Psi(x).
\]  

\[\text{(12)}\]

Defining

\[
\omega_d = \sqrt{\omega^2 + \frac{2U_0}{ma^2}}, \quad \frac{E + U_0}{\hbar\omega_d} = \varepsilon_d, \quad \lambda_d = \varepsilon_d - 1/2
\]  

\[\text{(13)}\]
Hence, the wavefunctions for $\psi_0(x)$, and $\psi_1(x)$ are given by equation (12) takes a similar form with the equation (6)

$$
\frac{d^2\psi(z_d)}{dz_d^2} + \left(2\epsilon_d - z_d^2\right)\psi(z_d) = 0.
$$

Therefore the solutions of this equation are the same as equation (8) with $z$ replaced by $z_d$ and $\lambda$ replaced by $\lambda_d$. Hence, the wavefunctions for $|x| \leq a$ are

$$
\psi_2(z) = c_2 D_{\lambda_d}(z_d) + c_1 D_{\lambda_d}(-z_d),
$$

where $c_2$ and $c_3$ are constants which can be found by normalization and continuity conditions. Since the potential is an even function of $x$, the eigenfunctions are either even or odd (hence $c_4 = \pm c_1$ in equation (11) and $c_3 = \pm c_2$ in equation (15)):

$$
\Psi_\lambda(z) = \begin{cases} 
  c_1 D_{\lambda}(z) & \text{for } z < -\left(\frac{an}{\hbar}\right)^{1/2} a, \\
  c_2 \left[D_{\lambda_d}(z) \pm D_{\lambda_d}(-z_d)\right] & \text{for } |z| < \left(\frac{an}{\hbar}\right)^{1/2} a, \\
  \pm c_1 D_{\lambda}(z) & \text{for } z > \left(\frac{an}{\hbar}\right)^{1/2} a,
\end{cases}
$$

where $+$ and $-$ signs stand for the even and odd eigenfunctions in the second and third regions of the wavefunctions in equation (16). Applying the continuity conditions on the wavefunctions and their derivatives, at $|x| = a$, one can express $c_2$ in terms of $c_1$ and get equations which determine the eigenvalues of the Hamiltonian for the even and odd eigenfunctions, respectively:

$$
\sqrt{\frac{\alpha}{\omega_d}} \left[ D_{\lambda_d}(B) + D_{\lambda_d}(-B) \right] G_\lambda(A) = \left[ G_{\lambda_d}(B) - G_{\lambda_d}(-B) \right] \times D_{\lambda}(A),
$$

$$
\sqrt{\frac{\alpha}{\omega_d}} \left[ D_{\lambda_d}(B) - D_{\lambda_d}(-B) \right] G_\lambda(A) = \left[ G_{\lambda_d}(B) + G_{\lambda_d}(-B) \right] \times D_{\lambda}(A),
$$

where $D_{\lambda}(A) \neq 0$ and $G_\lambda(A) \neq 0$.

Equations (17) and (18), corresponding to the even and odd eigenfunctions, respectively, have an infinite number of roots and can be solved numerically to determine $\lambda$ and therefore the eigenvalues $E_\lambda = (\lambda + 1/2)\hbar \omega$. When $\lambda$ for an eigenfunction is determined, inserting its value to equation (16) one can obtain the eigenfunction corresponding to this $\lambda$ value. We plot the ground state eigenfunctions for different potential depth values $U_0$ in figure 1 for the dimple sizes $a = 11 \mu m$ (narrow dimple) and $a = 32 \mu m$ (wide dimple) [4]. We notice that the effect of the dimple potential is to condense the ground state wavefunction to the center. Comparing the left and right of figure 1, one can observe that this effect is stronger for a narrow dimple. Finally, we present the first excited state eigenfunctions for different $U_0$ values in figure 2.

3. JWKB approximation

In the previous section we analytically obtained the eigenvalue equations (17) and (18) and the eigenfunctions of the Hamiltonian in equation (1) whose potential term is given in equation (2). However, to determine the eigenvalues one needs to solve equations (17) and (18) numerically. As $U_0$ i.e. the depth of the potential increases, the eigenvalues of the low lying states decrease. As the eigenvalues decrease one needs to check the accuracy of the values obtained by numerical methods. For this aim we apply the JWKB\(^3\) approximation because it gives the exact eigenvalues [28–31] for the harmonic oscillator Hamiltonian whose potential term is one of the shape invariant potentials [29].

The JWKB approximation method gives rise to the well known JWKB quantization formula to find the eigenenergies (e.g. see [31]),

$$
\int_{x_1}^{x_2} [2m (E_n - V(x))]^{1/2} dx = \left(n + \frac{1}{2}\right) \hbar.
$$

where $x_1$ and $x_2$ are classical turning points found by the equality $E_n = V(x_1, 2).$ Since the potential function is different for $|x| \leq a$ and $|x| > a$ the quantization formula differs for $|x| > a$.

\(^3\) We use Mathematica for the numerical solutions of equation (17) and (18).

\(^4\) The JWKB approximation is called as WKB approximation in most physics books. However the mathematical formalism is introduced by Jeffreys [27] so we call it the JWKB approximation.
$E_\alpha < (1/2)m\omega^2a^2$ and for $E_{n} > (1/2)m\omega^2a^2$. We will denote the eigenenergies whose values are less than $V(\alpha) = (1/2)m\omega^2a^2$ by $E_{n}^{(1)}$. For these eigenenergies, equation (20) becomes

$$\int_{x_1}^{x_2} \left[ 2m \left( E_{n}^{(1)} - V(x) \right) \right]^{1/2} dx = \int_{x_1}^{x_2} 2m \left[ E_{n}^{(1)} - \left( \frac{1}{2}m\omega^2x^2 - U_0 \left( 1 - \frac{x^2}{a^2} \right) \right) \right] dx = \int_{x_1}^{x_2} 2m \left[ E_{n}^{(1)} + U_0 - \left( \frac{1}{2}m\omega^2x^2 \right) \right] dx = \left( n + \frac{1}{2} \right) \hbar \pi,$$

(21)

where $\omega_d$ is defined in equation (13) and $x_{1,2} = \pm \sqrt{\left( 2E_{n}^{(1)} + U_0 \right) / (m\omega^2)}$. It is easy to solve equation (21) for $E_{n}^{(1)}$:

$$E_{n}^{(1)} = -U_0 + \left( n + \frac{1}{2} \right) \hbar \omega_d \quad \text{for} \quad n = 0, 1, \ldots, n' - 1.$$

(22)

Here, $(n')$ is the number of the eigenstates whose eigenvalues are less than $V(\alpha)$. It is necessary to determine the lowest level index for the eigenstates in the upperlying region whose eigenvalues are larger than $V(\alpha)$ (see equation (24)). $n'$ is determined by the formula

$$n' = \left[ \frac{1/2m\omega^2a^2 - E_{0}^{(1)}}{\hbar\omega_d} \right] + 1,$$

(23)

where the brackets denote the floor function. The JWKB quantization equation (20) for the energy eigenvalues of the eigenstates in the upperlying region becomes

$$2 \left\{ \int_{x_1}^{x_2} 2m \left[ E_{n}^{(2)} + U_0 - \left( \frac{1}{2}m\omega^2x^2 \right) \right] dx \right\} + m\hbar \omega_d \sqrt{\left( 2E_{n}^{(2)} + U_0 \right) / m\omega^2} = \left( n + \frac{1}{2} \right) \hbar \pi \quad \text{for} \quad n = n', n' + 1, \ldots,$$

(24)

where $x_3 = \sqrt{\left( 2E \right) / (m\omega^2)}$ and $n'$ is given by equation (23). We take the integrals in equation (24) only for positive values of $x$ utilizing the fact that the potential function is an even function. After taking the integrals the JWKB quantization equation for $E_{n}^{(2)} > V(\alpha)$ gives

$$\frac{2(E_{n}^{(2)} + U_0)}{\omega_d} \arcsin \left[ \frac{a}{\sqrt{\frac{2(E_{n}^{(2)} + U_0)}{m\omega^2} - a^2}} \right] + m\hbar \omega_d \sqrt{\left( 2E_{n}^{(2)} + U_0 \right) / m\omega^2} = \frac{E_{n}^{(2)}}{\omega} \left\{ \pi - 2 \arcsin \left[ \frac{a}{\sqrt{\frac{2E_{n}^{(2)}}{m\omega^2} - a^2}} \right] \right\} - m\hbar \omega_d \sqrt{\frac{2E_{n}^{(2)}}{m\omega^2} - a^2} = \left( n + \frac{1}{2} \right) \hbar \pi.$$

(25)

This equation can also be only solved numerically, like equations (17) and (18). However, its solution is much easier than equations (17) and (18) in which one has to find the roots of infinite series.

We compare the values of the eigenenergies obtained by the JWKB approximation and those found by the numerical solutions of equations (17) and (18) for different $U_0$ and $a$ values in tables 1 and 2. In table 1, we take $\hbar = 1, 2m = 1$. In table 2, we take the true value for $\hbar$ and more realistic values for the other parameters [2]. In these tables the first $n'$ eigenenergies, denoted by $E_{n}^{(1)}$, are less than $V(\alpha)$, and their JWKB values are calculated by equation (22) and $E_{n}^{(2)}$'s are larger than $V(\alpha)$ and their JWKB values are calculated by equation (25).

The potential energy function given in equation (2) changes its form at $|x| = a$ where its value is $V(\alpha)$. Unless the eigenenergies are close to this value, the JWKB approximation gives accurate results. However, as the values of the eigenenergies get close to the $V(\alpha)$, the accuracy of the JWKB approximation decreases. It is still useful to apply the JWKB approximation in this interval because in numerical calculations one may skip some of the eigenstates since the difference between the successive eigenvalues are not uniform in this interval. The reason that the JWKB approximation gives accurate results for the ground and first few excited eigenstates, but not for the eigenstates whose eigenenergy values are around $V(\alpha)$, is the following: the eigenfunctions of the ground and first few excited states go...
All the energies are in units of $h\omega$. Therefore, they are not much affected by the dimple potential. For the excited states whose eigenvalues are much larger than $\omega$, so, these eigenfunctions behave like they are the eigenfunctions of the shifted harmonic oscillator potential $V(x) = -U_0 + (1/2) m \omega^2 x^2$ for which the JWKB approximation gives exact results. In other words, they do not ‘feel’ the change of form of the potential. For the excited states whose eigenvalues are much larger than $\omega$, the JWKB approximation again becomes accurate. We can also explain this using the eigenfunctions of these states. These eigenfunctions are nonzero up to very large values compared to $|x| = a$. So, the region where the dimple potential exists $|x| \leq a$ is very narrow compared to the width of these eigenfunctions. Therefore, they are not much affected by the dimple potential.

| The eigenenergies | The eigenstate no. | Analytic | JWKB | Difference (Analytic-JWKB) |
|-------------------|-------------------|---------|------|---------------------------|
| $E_n^{(1)}$       | 0                 | -8.8333 | -8.8333 | 0.0000                   |
|                   | 1                 | -6.5001 | -6.5000 | 0.0001                   |
|                   | 2                 | -4.1681 | -4.1667 | 0.0014                   |
|                   | 3                 | -1.8447 | -1.8333 | 0.114                    |
|                   | 4                 | 0.4355  | 0.5000  | 0.0645                   |
|                   | 5                 | 2.5432  | 2.6852  | 0.1420                   |
|                   | 6                 | 4.1711  | 4.0504  | 0.1207                   |
|                   | 7                 | 5.2756  | 5.2638  | 0.0118                   |
|                   | 8                 | 6.3845  | 6.4181  | 0.0336                   |
|                   | 9                 | 7.5823  | 7.5390  | 0.0433                   |
|                   | 10                | 8.6277  | 8.6381  | 0.1040                   |
|                   | 11                | 9.7171  | 9.7217  | 0.0046                   |

| Table 1. The eigenenergies of the first few eigenstates for $\hbar = 1$, $m = 1/2$, $\omega = 1$, $a = 3$, $U_0 = 10$. For these values $n' = 5$ and $V(\alpha) = 2.25$. All the energies are in units of $h\omega$. |

| The eigenenergies | The eigenstate no. | Analytic | JWKB | Difference (Analytic-JWKB) |
|-------------------|-------------------|---------|------|---------------------------|
| $E_n^{(2)}$       | 0                 | -72.7948 | -72.7948 | 0.0000                   |
|                   | 1                 | -67.4650 | -67.4650 | 0.0000                   |
|                   | 2                 | -62.1353 | -62.1353 | 0.0000                   |
|                   | 3                 | -56.8055 | -56.8055 | 0.0000                   |
|                   | 4                 | -51.4758 | -51.4758 | 0.0000                   |
|                   | 5                 | -46.1461 | -46.1461 | 0.0000                   |
|                   | 6                 | -40.8163 | -40.8163 | 0.0000                   |
|                   | 7                 | -35.4866 | -35.4866 | 0.0000                   |
|                   | 8                 | -30.1570 | -30.1569 | 0.0001                   |
|                   | 9                 | -24.8278 | -24.8271 | 0.0007                   |
|                   | 10                | -19.5005 | -19.4974 | 0.0031                   |
|                   | 11                | -14.1807 | -14.1676 | 0.0131                   |
|                   | 12                | -8.8874  | -8.8379  | 0.0495                   |
|                   | 13                | -3.6816  | -3.5082  | 0.1734                   |
|                   | 14                | 1.2167   | 1.8216   | 0.6049                   |
|                   | 15                | 4.8125   | 4.4790   | 0.3335                   |
|                   | 16                | 6.0940   | 6.0210   | 0.0730                   |
|                   | 17                | 7.3188   | 7.4137   | 0.0949                   |
|                   | 18                | 8.8970   | 8.7300   | 0.1670                   |
|                   | 19                | 9.8991   | 9.9980   | 0.0989                   |
|                   | 20                | 11.3352  | 11.2316  | 0.1036                   |
|                   | 21                | 12.4303  | 12.4396  | 0.0093                   |
|                   | 22                | 13.6232  | 13.6273  | 0.0041                   |
|                   | ...               | ...      | ...      | ...                      |
|                   | 499               | 497.1836 | 497.1835 | 0.0001                   |
|                   | 500               | 498.1856 | 498.1857 | 0.0001                   |

| Table 2. The eigenenergies of low lying eigenstates for $\omega = 2\pi 20 \text{Hz}$, $m = 23 \text{amu}$, $a = 11 \mu \text{m}$, $U_0 = 1.0 \times 10^{-30} \text{J}$ and $\hbar$ is in SI units. For these values $n' = 15$ and $V(\alpha) = 2.75\hbar\omega$. All the energies are in units of $h\omega$. |

4. Transition amplitudes

In a recent paper, Garrett et al. [4] report the formation of a BEC without cooling in an anisotropic harmonic trap using dimple potentials. In this study, they investigated the behavior of a Bose gas in a harmonic trap with ‘narrow’ and ‘wide’ dimples both for adiabatic and sudden processes. For the sudden turn of the dimple, sudden perturbation theory can be used to calculate the condensate fraction. Describing the harmonic trap with a dimple potential by equation (2), it is possible to apply sudden perturbation theory since one can calculate the eigenfunctions for both cases (only a harmonic trap $V(x) = (1/2) m \omega^2 x^2$ for $-\infty < x < \infty$ and a harmonic trap with a dimple potential).
In this section, we will present the sudden perturbation calculations for the one dimensional case. Before the turn on of the dimple, the potential is only the harmonic potential and the eigenfunctions for this case are \[30, 31\]

\[ \psi_n(z) = \left( \frac{(m \omega)^{1/2}}{(\pi \hbar)^{1/2} 2^n n!} \right)^{1/2} \exp\left(-\frac{z^2}{2}\right) H_n(z), \quad (26) \]

where \( z \) is given by equation (5). The eigenfunctions of the potential given by equation (2) are found in section 2 and given in equation (16). Using the sudden perturbation theory \[30\], we calculate the transition amplitudes numerically as

\[ t_{0\lambda}(z) = \sqrt{\frac{\hbar}{m \omega}} \int_{-\infty}^{\infty} \psi_n(z) \psi_\lambda(z) \, dz. \quad (27) \]

We determine the ground state eigenfunction in equation (27) for a given \( U_0 \) first by finding the \( \lambda \) value corresponding to the ground state eigenenergy using equation (17). Then we insert this \( \lambda \) value into equation (16) and calculate the normalization constant \( c_1 \) and the integral in equation (27), numerically. Finally, we find the transition probabilities:

\[ P_{n\lambda} = |t_{n\lambda}|^2. \quad (28) \]

We present the change of the transition probabilities with respect to \( U_0 \) from the ground state and from the second excited state of the harmonic oscillator to the ground state of the harmonic oscillator with a dimple for the values \( \hbar = 1 \), \( m = 1/2, \omega = 1, a = 3 \) in figures 3 and 4, respectively. The transitions from the even eigenstates to odd eigenstates are zero since the wavefunctions of these states are even and odd functions.

In following studies we aim to use the truncated parabolic potential to model dimple potentials used in Bose–Einstein condensation experiments. Similar transition probability calculations from higher levels to the new ground state can be used for the calculation of the condensate fraction of a BEC in a harmonic trap with a dimple when the dimple is turned on suddenly \[4\].

5 The fact that \( H_1(x), H_2(x) \) and \( H_3(x) \) are not unique does not affect the result.

5. Dirac \( \delta \) potential as a limiting case of the parabolic potential

Recent papers show that modeling dimple potentials by Dirac \( \delta \) functions reveal basic properties of a BEC in a harmonic trap with a dimple \[12, 13\]. In this section we will present that the \( \delta \) potential is included in the truncated parabolic potential as a special case.

We will first show that the function

\[ f_\delta(x) = \begin{cases} \frac{\lambda}{2\pi} (1 - \frac{x^2}{\sigma^2}) & \text{for } |x| \leq a, \\ 0 & \text{for } |x| > a \end{cases} \quad (29) \]

represents the Dirac \( \delta \) function. There is an advantage of this representation compared to other representations in the literature (see e.g. \[32, \text{appendix II}\]). When the function \( f_\delta(x) \) in equation (29) is used as a potential for a Hamiltonian like in equation (2), the resulting time-independent Schrödinger equation is analytically solvable.

It is clear that the representation given in equation (29) satisfies the condition \( \int_{-\infty}^{\infty} \delta(x) \, dx = 1 \) used in defining the Dirac \( \delta \) function. We will show that this definition satisfies also the sampling property of the \( \delta \) function \[33\]: \( \int_{-\infty}^{\infty} \delta(x) h(x) \, dx = h(0) \) for \( a \to 0 \) where \( h(x) \) denotes an analytical function. That is

\[ \lim_{a \to 0} \int_{-\infty}^{\infty} f_\delta(x) h(x) = \lim_{a \to 0} \int_{-a}^{a} f_\delta(x) h(x) = h(0). \quad (30) \]

For this aim we define the functions \( H_1(x), H_2(x), H_3(x) \) as

\[ \frac{dH_1(x)}{dx} = h(x), \quad \frac{d^2H_2(x)}{dx^2} = h(x), \quad \frac{d^3H_3(x)}{dx^3} = h(x). \quad (31) \]

We apply three times integration by parts to the expression \( \lim_{a \to 0} \int_{-a}^{a} f_\delta(x) h(x) \) and get

\[ \lim_{a \to 0} \int_{-a}^{a} f_\delta(x) h(x) = \lim_{a \to 0} \left\{ \frac{3}{2a^2} [H_2(a) + H_2(-a)] \right\}
\]

\[ - [H_3(a) - H_3(-a)] \right\}. \quad (32) \]
Using the Maclaurin series expansion of $H_2(a)$, $H_2(-a)$ and $H_1(a)$, $H_1(-a)$ up to $a^3$ one can see that the right hand side of the equation (32) is equal to $h(0)$ in the limit $a \to 0$.

It is well known that there is only one bound state of the Dirac delta potential (see e.g. [20])

$$V_\delta = -\frac{\hbar^2}{2m} \sigma \delta(x). \quad (33)$$

The solution of the eigenvalue equation for this potential is $\kappa = \sigma/2$ where $\kappa = \sqrt{-2mE}/\hbar$. Therefore one finds [20]

$$E = -\frac{\hbar^2 \sigma^2}{8m}. \quad (34)$$

Here the coupling coefficient of the $\delta$ function is taken as $-\hbar^2 \sigma/(2m)$ for calculational convenience. We will now calculate the bound state eigenvalues for the potential

$$V(x) = \begin{cases} -U_0 \left(1 - \frac{x^2}{\nu^2}\right) & \text{for } |x| \leq a, \\ 0 & \text{for } |x| > a \end{cases} \quad (35)$$

and present that for small $a$ the eigenvalue of the ground state energy of this potential approximates to the ground state energy of the Dirac $\delta$ potential given in equation (33). The even and odd eigenfunctions of the bound states for the potential given in equation (35) are:

$$\psi_e(x) = \begin{cases} A_e e^{i \gamma x} & \text{for } x < -a, \\ B_e \left[ D_{\nu} \left(\frac{\nu x}{H}\right) + D_{\nu} \left(-\frac{\nu x}{H}\right)\right] & \text{for } |x| \leq a, \\ A_e e^{-i \gamma x} & \text{for } x > a \end{cases}$$

and

$$\psi_o(x) = \begin{cases} A_o e^{i \gamma x} & \text{for } x < -a, \\ B_o \left[ D_{\nu} \left(\frac{\nu x}{H}\right) - D_{\nu} \left(-\frac{\nu x}{H}\right)\right] & \text{for } |x| \leq a, \\ -A_o e^{-i \gamma x} & \text{for } x > a \end{cases} \quad (36)$$

where $\nu = \sqrt{2U_0/(ma^3)}$, $\gamma = (E + U_0)/(\hbar \nu) - 1/2$ and the coefficients $A_e$, $A_o$, $B_e$ and $B_o$ are to be determined from normalization and continuity conditions of the wavefunction and its derivative. Applying these continuity conditions one gets the eigenvalue equation for the potential given in equation (35) of the even and odd eigenfunctions, respectively as

$$-\kappa \sqrt{\frac{\hbar}{mv}} = -\sqrt{(2\gamma + 1)} - G_{\nu} \left(\frac{\nu x}{H}\right) - G_{\nu} \left(-\frac{\nu x}{H}\right),$$

$$-\kappa \sqrt{\frac{\hbar}{mv}} = -\sqrt{(2\gamma + 1)} + G_{\nu} \left(\frac{\nu x}{H}\right) + G_{\nu} \left(-\frac{\nu x}{H}\right), \quad (38)$$

where $\gamma = E/(\hbar \nu) - 1/2$; $D_{\nu}(z)$, $D_{\nu}(-z)$ and $G_{\nu}(z)$ are defined in equations (7), (8) and (19), respectively. By taking into account the definition of the representation for $\delta$ in equation (29), the comparison of equations (33) and (35) reveals that $\sigma = 8maU_0/(3h^2)$. As $a \to 0$, the eigenvalue equation (38) reduces to equation (34), which gives the bound state eigenvalue of the $\delta$ potential. One can show this by expanding the right hand side of equation (38) into a Taylor series. For constant $U_0a$, the left hand side of the equality in equation (38) is of the order of $a^{3/4}$. Therefore, we expand the right hand side in terms of $\sqrt{mv}/\hbar$ so that the expansion terms in the right hand side include terms up to $a^{3/4}$. Doing this, we get the equality

$$\sqrt{-(2\gamma + 1)} = -(2\gamma + 1) \frac{\sqrt{mv}}{\hbar} a + \frac{7}{6} \left(\frac{\sqrt{mv}}{\hbar} a\right)^3. \quad (40)$$

Substituting $\nu = \sqrt{2U_0/(ma^3)}$, $\gamma = (E + U_0)/(\hbar \nu) - 1/2$ and $\gamma = E/(\hbar \nu) - 1/2$ into this equation we find

$$E = -\frac{8mU_0^2a^2}{9 \hbar^2}. \quad (41)$$

Since $\sigma = 8maU_0/(3h^2)$, equation (41) is identical to equation (34), which reveals that as $a \to 0$ the representation given in equation (29) gives the same ground state eigenvalue with $\delta$ potential for the solution of the Schrödinger equation. Doing a similar expansion to equation (39), we see that this equation does not have a solution in the limit $a \to 0$. That means there is no odd bound state which proves there is only one bound state for the potential given in equation (35) in the limit $a \to 0$.

We also calculate the reflection and tunneling probabilities for the truncated parabolic potential given in equation (35) for the scattering states. Then, we show that as $a \to 0$, the tunneling and reflection amplitudes for this potential reduce to the transition and reflection amplitudes for the Dirac $\delta$ potential given in equation (33). The wavefunction for a scattering state of a $\delta$ potential with an energy $E > 0$ can be written as

$$\psi_s(x) = \begin{cases} c_1 D_{\nu} \left(\frac{\nu x}{H}\right) + c_2 D_{\nu} \left(-\frac{\nu x}{H}\right) & \text{for } |x| \leq a, \\ T e^{i \nu x} & \text{for } x > a \end{cases} \quad (45)$$

where $k = \sqrt{2mE}/\hbar$ and we take the coefficient of the wave incoming from $-\infty$ equal to unity for calculational convenience. The transition and reflection amplitudes for this potential are easily found to be

$$T_\delta = \frac{2ik}{2ik + \sigma}, \quad (43)$$

$$R_\delta = -\frac{\sigma}{2ik + \sigma}. \quad (44)$$

The wavefunction for a scattering state incoming from $-\infty$ with an energy $E > 0$ for the potential in equation (35) can be written as

$$\psi_s(x) = \begin{cases} e^{i \nu x} + R e^{-i \nu x} & \text{for } x < -a, \\ T e^{i \nu x} & \text{for } x > a \end{cases} \quad (45)$$

$^6$ Since we take $U_{0\sigma}$ as equal to a constant $c$, $\nu = \sqrt{2U_0/(ma^3)}$ becomes $\nu = \sqrt{2c/(ma^3)}$. Hence $\nu$ is of order $a^{-3/2}$.
The norm squares of $R$ and $T$ given in equation (45) give the probability of reflection and tunneling, respectively. These coefficients are calculated using the continuity of the wavefunction and its derivative at points $x = |a|$. In order to present the result for $R$, we define

$$F_1(\lambda, a) = -\Phi \left( -\frac{\lambda}{2}, 1, a^2 s^2 \right) + 2(1+\lambda)\Phi \left( -\frac{\lambda}{2}, 3, a^2 s^2 \right),$$

(46)

$$F_2(\lambda, a) = (k^2 + s^2 + a^2 s^4) \Phi \left( -\frac{\lambda}{2}, 1, a^2 s^2 \right) - 2s^2(1 + a^2 s^2)(1 + \lambda) \Phi \left( -\frac{\lambda}{2}, 3, a^2 s^2 \right),$$

(47)

$$F_3(\lambda, a) = 3(1 + iak + a^2 s^2) \Phi \left( 1 - \frac{\lambda}{2}, 3, 2a^2 s^2 \right) + 2a^2 s^2(1 + \lambda) \Phi \left( 3 - \frac{\lambda}{2}, 5, 2a^2 s^2 \right),$$

(48)

$$F_4(\lambda, a) = 2as^2 \lambda \Phi \left( 1 - \frac{\lambda}{2}, 3, 2a^2 s^2 \right) + (ik + as)^2 \times \Phi \left( -\frac{\lambda}{2}, 1, a^2 s^2 \right),$$

(49)

where $\Phi(\alpha, \gamma; \eta)$ is the confluent hypergeometric function [26]. In terms of the functions $F_1(\lambda, a)$ to $F_4(\lambda, a)$ the reflection amplitude is

$$R = -\left[ a e^{-2iak} \frac{2a^2 s^4(2 + \lambda) \Phi \left( 1 - \frac{\lambda}{2}, 3, 2a^2 s^2 \right) F_1(\lambda, a) + 3\Phi \left( 1 - \frac{\lambda}{2}, \frac{3}{2}, a^2 s^2 \right) F_2(\lambda, a)}{F_3(\lambda, a) F_4(\lambda, a)} \right].$$

(50)

In order to demonstrate $T$, we additionally define

$$F_3(\lambda, a) = -2a^2 s^2(2 + \lambda) \Phi \left( 1 - \frac{\lambda}{2}, 3, 2a^2 s^2 \right) \times \Phi \left( -\frac{\lambda}{2}, 1, a^2 s^2 \right),$$

(51)

$$F_6(\lambda, a) = \Phi \left( -\frac{\lambda}{2}, 1, a^2 s^2 \right) + 2a^2 s^2(1 + \lambda) \times \Phi \left( -\frac{\lambda}{2}, 3, 2a^2 s^2 \right).$$

(52)

In terms of the functions $F_3(\lambda, a)$ to $F_6(\lambda, a)$

$$T = -ie^{-2iak} k \frac{F_3(\lambda, a) + 3\Phi \left( 1 - \frac{\lambda}{2}, \frac{3}{2}, a^2 s^2 \right) F_6(\lambda, a)}{F_3(\lambda, a) F_4(\lambda, a)}.$$  

(53)

In the appendix, we show the change of the tunneling and reflection probabilities with respect to energy $E$, width $a$ and depth $U_0$ of the truncated parabolic potential. Now we will show that the tunneling amplitude $T$ for the potential given in equation (35) goes to $T_4$ in equation (43). For this reason, assuming $U_0a$ is constant, we expand the numerator and the denominator of the right hand side of equation (53) with respect to $a$ and since we are interested in the limit $a \to 0$, we keep only the constant terms which are independent of $a$. Doing this, we find

$$\lim_{a \to 0} T = \frac{3i k}{3ik + 4mU_0a/b^2}.$$  

(54)

Since $\sigma = 8maU_0/(3h^2)$, we see that $T$ reduces to $T_4$ in equation (43) for constant $U_0a$.

Finally, we show that the eigenvalues of the Hamiltonian with the potential given in equation (2) approximates to the eigenvalues of the Hamiltonian with the potential

$$V(x) = \frac{1}{2}ma^2 x^2 - \hbar^2 \frac{\sigma \delta (x)}{2m}$$

(55)

for $U_0 = 3h^2/2m$ as $a \to 0$. The eigenvalues of even states of the Hamiltonian with the potential of equation (55) is equal to $E_n = (\lambda + 1/2)\hbar\omega$ where $\lambda$ values are the roots of the equation [12, 17]

$$\frac{\Gamma(x - \frac{1}{2})}{\Gamma(-\frac{1}{2})} = \frac{\Lambda}{4}.$$  

(56)

Here $\Lambda = \sigma/\sqrt{m}\omega(\gamma/m\omega)$. The odd eigenvalues are equal to the odd eigenvalues of the harmonic oscillator potential i.e. they are equal to $E_n = (n + 1/2)\hbar\omega$ for $n = 1, 3, 5, \ldots$ [12]. As $a \to 0$, for constant $U_0a = c$, equation (17) reduces to equation (56) and the roots $\lambda$ of equation (18) go to $\lambda = 2n + 1$ where $n = 0, 1, 2, 3, \ldots$. In order to show this, we first expand $\sqrt{\omega/\omega_d}$ with respect to $a$ and find

$$\frac{\omega}{\omega_d} = 1 + ma^2 \frac{\sigma^2}{2c},$$

(57)

for $U_0a = c$. Then we expand $D_{as}(B), D_{as}(-B), G_{as}(B), G_{as}(-B)$ in terms of $B = \sqrt{m}\omega/\hbar a; G_{as}(A)$ and $D_{as}(A)$ in terms of $A = \sqrt{m}\omega/\hbar a$ in equation (17) and obtain for the eigenvalue equation of the even eigenfunctions

$$-\frac{8}{3} \frac{(\Gamma(1/2))^2}{\Gamma(1/2)^2} \frac{(mc/2)^{3/4}}{2^{1/4} \hbar^{3/2} a^{3/4}}$$

$$= 2 \left( \frac{m}{2c} \right)^{1/4} \omega^{1/2} \Gamma(1/2) \Gamma(-1/2) \left( \frac{1/2}{\sqrt{2}} \right) a^{3/4}.$$  

(58)

After rearranging the terms we get

$$\frac{2}{3} \frac{U_0a m^{1/2}}{\hbar^{1/2} a^{1/2}} = \frac{\Gamma(1/2)}{\Gamma(-1/2)}.$$  

(59)

When we insert $\sigma = 8maU_0/(3h^2)$ and $\Lambda = \sigma/\sqrt{m}\omega(\gamma/m\omega)$ into equation (59), we see that this equation reduces to equation (56). Applying a similar procedure to the eigenvalue equation (18) of the odd eigenfunctions, we see that it reduces to

$$\frac{\Gamma(1/2)}{\Gamma(-1/2)} = 0.$$  

(60)
as $a \to 0$ for constant $U_0 a$. In this limit $\lambda_d = -1/2$, therefore the right hand side of equation (60) can be equal to zero if and only if $\Gamma\left(\frac{1-a}{2}\right) = \infty$. Since $\Gamma(-n) = \infty$ when $n = 0, 1, 2, \ldots$, we get for $\lambda$

$$\frac{1-\lambda}{2} = -n \implies \lambda = 2n + 1 \text{ for } n = 0, 1, 2, \ldots \quad (61)$$

Therefore, similar to the Dirac $\delta$ case, in the limit $a \to 0$ the odd eigenvalues of the potential given in equation (2) reduces to the odd eigenvalues of the harmonic oscillator potential with a Dirac $\delta$ potential at the origin.

The fact that the eigenvalues of the harmonic potential with a symmetric truncated parabolic potential around the origin reduce to the eigenvalues of the harmonic potential with a Dirac $\delta$ at the origin, in the limit $a \to 0$ for fixed $U_0 a$, shows that the results obtained modeling the dimple potential with a Dirac $\delta$ for a non-interacting BEC in a harmonic trap with a dimple [12, 13] are included in the parabolic model of the dimple as a special case.

6. Conclusion

We propose, in this study, that the parabolic potential defined in equation (2) is more appropriate for modeling the dimple potentials than the Dirac $\delta$ potential used in the literature. To this aim, we showed that the parabolic potential includes the Dirac $\delta$ potential as a special case and the wavefunctions, eigenvalues, tunneling and reflection coefficients of the Dirac $\delta$ potential can be obtained from those of the parabolic potential as a limiting case.

In section 2, we have summarized the solution of the Schrödinger equation for a harmonic trap with a truncated parabolic potential in one dimension and obtained eigenvalue equations and eigenfunctions. Then, we presented the numerical solutions of the eigenvalue equations. When the depth of the parabolic potential increases the eigenenergies of the ground and first few excited states decrease to more negative values, as shown in table 2. As the eigenenergy values decrease the numerical solutions become instable. Therefore, in section 3, we applied the JWKB approximation to check the validity of the numerically found eigenvalues. The results in tables 1 and 2 show that the JWKB approximation and the numerical solution for the eigenenergies agree very well except in the transition region of the potential given in equation (2). So, we conclude that one can use the JWKB approximation to find the eigenvalues for this potential when the numerical solutions fail.

In section 4, we obtained the formula for the transition amplitudes and found the transition probabilities from the ground and second excited eigenstates of the harmonic trap to the ground state of the harmonic trap with a dimple described by the truncated parabolic potential for different dimple depths, using the sudden perturbation theory. In the following studies we aim to use these and similar results in three dimensions for modeling the sudden turn on of the dimple potential experiments [4].

In section 5, we have first shown that the truncated parabolic function provides a representation of the Dirac $\delta$ function. Moreover, we have also demonstrated that the bound state spectrums of the potentials given in equations (35) and (2) reduce to the spectrum of the Dirac $\delta$ potential and harmonic potential with a Dirac $\delta$ as $a \to 0$, respectively, for fixed $U_0 a$. As one can see from the Bose–Einstein distribution $\langle n_1 \rangle = 1/(e^{\beta(E_1-\mu)} - 1)$ for a non-interacting Bose gas the effect of the trapping potential to the thermodynamic properties of the gas comes only through the eigenvalues of the potential. So, we come to the following conclusion: the fact that the eigenvalues of the truncated parabolic potential reduce to the eigenvalues of the Dirac $\delta$ potential in the limit $a \to 0$, for fixed $U_0 a$, shows that the results obtained modeling the dimple potential with a Dirac $\delta$ for a non-interacting BEC in a harmonic trap with a dimple [12, 13] are included in the parabolic model of the dimple as a special case.

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Appendix

We present in this appendix, the change of the tunneling and reflection probabilities with respect to energy $E$ of the incoming wave, width $a$ and depth $U_0$ of the potential given in equation (35). The scattering wavefunction, reflection ($R$) and tunneling ($T$) amplitudes for this potential are given in equations (45), (50) and (53), respectively. We first present the change of the tunneling and reflection probabilities with increasing energy $E$ of the incoming wave in figure A.1, for fixed potential width $a = 3$ and depth $U_0 = 10$. Then, in figure A.2, we show the variation of the tunneling and reflection amplitudes for increasing $a$ and fixed $U_0 = 10$, $E = 1$. Finally, we demonstrate the change of the $|T|^2$ and $|R|^2$ for increasing $U_0$ but for fixed $E = 1$ and $a = 3$ in figure A.3. In all figures, we use natural units $\hbar = 1$, $m = 1/2$. One can recognize resonances with varying widths and amplitudes for all these cases.

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