On the Representation of Intermediate States in the Velocity Basis

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Unstable state furnishes a semigroup irreducible representation of the Poincaré group. The state vector is represented by a superposition of energy eigenkets. As a consequence of this superposition, the state vector can be transformed into the rest frame through a Lorentz transformation only when the eigenkets are labeled by velocity variable, but not momentum variable. We also clarify the meaning of the velocity variable in the state vector with respect to the velocity derived from kinematical consideration of the scattering process.

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\section{I. INTRODUCTION}

Stable particles furnish the unitary irreducible representation of the Poincaré group \cite{1}. The unitary irreducible representation is characterized by two Casimir invariants \(m^2, j\), i.e. the invariant mass square \(m^2\) and spin \(j\) of the particles. Extension of Wigner’s idea to encompass unstable particles has been the subject of many investigations \cite{2,3,4,5,6,7,8}. Since unstable particles decay, probability is not conserved and this results in a non-unitary irreducible representation for unstable particles.

Of the many works carried out in this direction, they differ crucially in the basis employed for the representation space of unstable particles. Refs. \cite{2,3,7} use velocity basis, whereas Refs. \cite{4,5,6} resort to momentum basis for representing the state vector. More specifically, Refs. \cite{2,7} define the 4-velocity \(\hat{p}\) which is related to the 4-momentum \(p\) as \cite{16}

\begin{equation}
 p = \sqrt{s} \hat{p},
\end{equation}

where \(s\) is the invariant energy square (Mandelstam variable) of the scattering process. The invariant energy square may acquire complex value, whereas the 4-velocity is required to remain real. Ref. \cite{2} introduces the additional requirement that the 4-velocity be independent of \(s\) \cite{8}. We shall follow Ref. \cite{8} and refer to the velocity representable in the form \(\hat{p} = p/\sqrt{s}\) as the minimally complex representation. The minimally complex representation has led to a few interesting consequences. It is our objective to clarify the implications of these consequences.

Intuitively, since velocity are kinematically equivalent to momentum, using velocity as variable in the representation space should not lead to profound change in the formulation. However, though the representation is non-unitary, the minimally complex representation are able to keep the velocity \(\hat{p}\) and spin \(j\) real when the variable \(s\) is analytically continued into the complex plane \(\mathbb{C}\). This provides a clean analytic continuation of the scattering amplitude into the complex \(s\)-plane when we study resonance phenomena. On the other hand, complex momentum is inevitable when \(s\) becomes complex, since momentum in principle cannot be rendered independent of \(s\). This leads to many complex representations in the momentum basis \cite{3,4,5,6} which has no obvious physical interpretation. This is an important reason for employing velocity basis in the studies of unstable states.

The requirement that unstable states be transformable by a real Lorentz transformation to a rest system with 4-momentum of the form,

\begin{equation}
 p = \sqrt{s_r}(1,0),
\end{equation}

where \(s_r\) is complex, has motivated Ref. \cite{2} to introduce real velocity to the representation space of unstable states. We shall show that unstable states in the momentum basis in general cannot be transformed into a rest frame by a Lorentz transformation, even though a rest frame exists in principle. This shows that Lorentz covariance is maintained for unstable states in the velocity basis and velocity basis is favorable over momentum basis in the description of unstable states.

However, we have to pay a price for this simplicity of description for unstable states in the velocity basis. By setting the velocity to be independent of the invariant mass square of the scattering process, the unstable states are now endowed with a statistical meaning and devoid of the details of the process (kinematics) from which they are formed. This amounts to information loss. This fact will be explained later in the main text. We will also put the meaning of the velocity variable of the unstable states in proper perspective.

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II. SEMIGROUP REPRESENTATION OF UNSTABLE STATES

Unstable particles have long been known to be associated with the resonance poles of the S-matrix [10]. These poles are complex and their positions in the complex plane can be parameterized in terms of the invariant mass square as 

\[ s_r = (M - i\Gamma/2)^2, \]

where \( M \) is the position of the peak of the resonance with decay width \( \Gamma \).

With S-matrix as the starting point, analytically continuing the scattering amplitude into the complex plane allows the derivation of the Gamow vector, which is the unstable state associated with the complex pole of the S-matrix [7]. Gamow vector furnishes a semigroup irreducible representation of the Poincaré group and is characterized by 2 numbers, \([s_r, j]\), the complex invariant mass square \( s_r \) and the spin of the unstable particle or the partial wave in which the unstable particle is formed, \( j \). In the frame work of Rigged Hilbert Space [11], the Gamow vector is represented by the following vector [13, 7]

\[ |\hat{p}_{j3}[s_rj]^-\rangle = \frac{i}{2\pi} \int_{-\infty}^{\infty} ds \frac{d\hat{p}_{j3}[s_j^-]}{s-s_r}, \tag{3} \]

where \( \hat{p} \) is the space component of the 4-velocity and \( j_3 \) is the third component of the spin. The minus superscript indicates that the Gamow vector is analytically continued into the lower half plane and its time evolution is restricted to the forward light cone. More properties of the Gamow vector can be found in Refs. [7].

Since the following discussion on the Lorentz transformation property of the unstable states is general and need not be restricted to the Gamow vectors, we shall consider the phenomenological generalization of Eq. (3) to intermediate states in a scattering process, cf. Eq. (6) and (7) below. Intermediate states are transient states in scattering processes, which may or may not correspond to resonant states [20]. For instance, a state \( C \) may form temporarily when particles \( a \) and \( b \) scatter into particles \( d \) and \( e \), i.e.

\[ a + b \rightarrow C \rightarrow d + e. \tag{4} \]

In a decay chain with 3-body final states, an intermediate state \( D \) may be formed as follows,

\[ a + b \rightarrow c + D \rightarrow c + e + f. \tag{5} \]

Let us consider the superposition of energy eigenkets weighted by a function \( w(s) \) in the velocity basis,

\[ \int ds \, w(s) \, |\hat{p}_{j3}[s_j^-]\rangle. \tag{6} \]

The function \( w(s) \), for example, may be proportional to the scattering amplitude. For the Gamow vector, \( w(s) \sim 1/(s - s_r) \), which is proportional to the relativistic Breit-Wigner amplitude. Eq. (6) integrates over all appropriate values of invariant energy square \( s \) according to \( w(s) \). The corresponding superposition of energy eigenkets in the momentum basis is

\[ \int ds \, w'(s) \, |p_{j3}[s_j^-]\rangle, \tag{7} \]

which is weighted by the function \( w'(s) \).

III. LORENTZ COVARIANCE OF UNSTABLE STATES IN THE VELOCITY BASIS

As pointed out in Ref. [2], a desirable property of the vectors representing unstable states is their ability to transform to the rest system under a Lorentz transformation. The vector in the velocity basis Eq. (3) satisfies this requirement. To see this, we parameterize the standard boost as [12]

\[ L_{\mu}^\nu(\hat{p}) = \left( \begin{array}{cc} \hat{p}_0/\sqrt{\hat{p}_0^2 + \hat{p}_r^2} & -\hat{p}_j/\sqrt{\hat{p}_0^2 + \hat{p}_r^2} \\ \hat{p}_r/\sqrt{\hat{p}_0^2 + \hat{p}_r^2} & \hat{p}_0/\sqrt{\hat{p}_0^2 + \hat{p}_r^2} \end{array} \right), \quad i, j = 1, 2, 3. \tag{8} \]

Under the Lorentz transformation \( L^{-1}(\hat{p}) \) (a rotation free boost), the velocity vector Eq. (6) is transformed into its rest frame [7, 8],

\[ U[L^{-1}(\hat{p})] \int ds \, w(s) \, |\hat{p}_{j3}[s_j^-]\rangle = \int ds \, w'(s) \, |0_{j3}[s_j^-]\rangle. \tag{9} \]

where \( U \) is the unitary representation of the Lorentz transformation. The 4-momentum of the transformed state is given by Eq. (2). Note that the form of Eq. (6) is retained under the transformation. Hence we say that the velocity vector is Lorentz covariant.

Complication arises when we consider the Lorentz transformation of the momentum vector in Eq. (7). Let us apply the Lorentz transformation \( L^{-1}(\hat{p}) \) to the state vector. Even though all the constituent momentum kets under the \( s \) integration in Eq. (7) have identical momenta \( \hat{p} \) (the space components of the 4-momentum vector \( p \)), they have different invariant energy square when \( s \) changes. By definition \( \hat{p} = \sqrt{s}, \) hence for each different \( s_1 \neq s_2 \neq... \), the corresponding velocities are different too, i.e. \( \hat{p}_1 \neq \hat{p}_2 \neq... \). Therefore, transforming a vector with momentum \( p \) to its rest frame

\[ \int ds \, w'(s) \, |p_{j3}[s_j^-]\rangle \rightarrow \int ds \, w'(s) \, |0_{j3}[s_j^-]\rangle, \tag{10} \]

can only be achieved with a series of different Lorentz transformations, i.e. \( L^{-1}(\hat{p}_i) \), where \( i = 1, 2, 3, ... \), each acting separately on \( |p_{j3}[s_j]\rangle \), where \( p = \sqrt{s} \hat{p}_i \). This violates the requirement of having only a single Lorentz transformation to bring the momentum vector into its rest frame. Therefore, Eq. (7) is not a desirable representation vector for intermediate states.
IV. CONSISTENCY OF THE MINIMALLY COMPLEX REPRESENTATION WITH THE KINEMATICS

When we define the velocity variable through the minimally complex representation in Eq. (11), we have implicitly assumed that the velocity is independent of the invariant energy square s, i.e., velocity is not a function of s. This choice is mathematically legitimate since out of the set of 5 variables \{p_0, p, s\} and with the off mass-shell constraints for unstable states \(p_0^2 - p^2 = s\), we choose the set of 4 variables \{\hat{p}, s\} to be independent. The time component of the velocity is determined through the constraint \(p_0^2 - \hat{p}^2 = 1\).

However, when we consider the kinematics of a scattering event, there seems to be an apparent contradiction with the assumption of minimally complex representation \[13\]. Consider the kinematics of the resonance formation process \[21\] in Eq. (11). The rest mass of the stable particles \(a, b, d, e\) are labeled by \(m_i\), where \(i = a, b, d, e\) respectively. Suppose in the rest frame of particle \(a\), it is bombarded by particle \(b\) which has velocity \(\hat{p}_b = \gamma v = (\gamma v, 0, 0)\), where \(v = |v|\) and \(\gamma\) is the usual dilation factor \(\gamma = 1/\sqrt{1 - v^2}\). The 4-momentum of the unstable state \(C\), which equals the total 4-momentum of the system, is given by \(p_C = p_a + p_b = (m_b \gamma + m_a, m_b \gamma v, 0, 0)\). The invariant energy square of the system is \(s = p_C^2 = m_a^2 + m_b^2 + 2m_b m_a \gamma\). The velocity of the intermediate state can then be brought into the following form,

\[
|\hat{p}_C| = \frac{|p_C|}{\sqrt{s}} = \frac{\lambda^2(s, m_a^2, m_b^2)}{2m_a \sqrt{s}},
\]

where the coefficient

\[
\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ca)
\]

frequently occurs in kinematical expression \[8, 13\]. Since we obtain a \(s\)-dependent velocity \(\hat{p}_C\), one may question the validity of introducing a \(s\)-independent velocity \(\hat{p}\) that is consistent with the kinematics.

This controversy arises in the carelessness of identifying the velocity vector Eq. (9) with the unstable state occurred in a specific scattering event. As is clear from the Gamow vector, its construction requires information on the scattering amplitude encoded in \(w(s)\), which is derived from a series of different scattering events that probes through various values of \(s\) for the response of the scattering amplitude. Thus, the Gamow vector is a statistical representation (average sum weighted by \(w(s)\)) for the whole collection of scattering events, not just a single one.

A statistical meaning is ingrained implicitly in the construction of the unstable states. Once the unstable states are formed, information on the details of the kinematics of each event prior to the formation of the unstable states are lost. However, the overall conservation of total momentum, total angular momentum and other conserved quantum numbers of the scattering process are still inherited by the unstable states \[8\]. It is in this sense that the Gamow vector is a universal unstable state. Cares must be taken in interpreting the velocity variable to avoid apparent contradiction with the kinematics of a scattering event.

V. CONCLUSION

We show that the semigroup irreducible representation of the Poincaré group for unstable states in the velocity basis can be transformed into its rest system by a real Lorentz transformation, while this is not true for the irreducible representation in the momentum basis. This is the consequence of 2 facts: (1) the Lorentz transformation is parameterized by velocity, and (2) to properly describe unstable states, a superposition of energy eigenkets over a range of invariant energy square is necessary. As a result, unstable state transforms covariantly under Lorentz transformation only in the velocity basis. This distinction between velocity and momentum basis does not exist when asymptotic states are considered, since the invariant energy square \(s = m^2\) of stable particles are constant. This finding advocates using velocity basis in the studies of unstable particles, or intermediate states in general.

In scattering experiments, the existence of unstable states can be reviewed only if the experiments probe across an appropriate range of invariant energy square. Disconnected isolated events cannot review the underlying unstable state. Likewise, a mathematical description of unstable states has to involve the superposition of eigenkets across a range of energy, weighted by a function proportional to the scattering amplitude. This construct endows the Gamow vector with a statistical meaning, and its velocity variable cannot be identified to the velocity
obtained from the kinematical consideration of a single scattering event.

The minimally complex representation has avoided complex momentum and complex spin representation of unstable particles. This simplicity in the description of unstable state has caused the loss of some information. Except the overall conservation of the total 4-momentum, total angular momentum and conserved quantum numbers, the details of the process from which the unstable states are formed are lost. This conforms to the notion of unstable states as indistinguishable particles that forget its history upon formation, and the formulation is able to put it on equal footing with stable particles.

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[16] We do not consider massless particles.
[17] There has been controversy on the parameterizations of the decay width and invariant energy square, we refer the interested reader to Ref. [9].
[18] The integration range includes the unphysical negative values of s in the lower half plane of the second sheet. This range is important to ensure an exact exponential decay of the Gamow vector.
[19] We do not consider bound states.
[20] For example, the energy may be well below the threshold to form the unstable particle.
[21] We consider resonance formation process though in the original version of the apparent contradiction, resonance production process Eq. (5) is considered [13]. The apparent contradiction in resonance production process is considered from a different point of view in [14].