MOONSHINE AND DONALDSON INVARIANTS OF CP²

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Abstract. Eguchi, Ooguri, and Tachikawa recently conjectured [9] a new moonshine phenomenon. They conjecture that the coefficients of a certain mock modular form $H(\tau)$, which arises from the $K3$ surface elliptic genus, are sums of dimensions of irreducible representations of the Mathieu group $M_{24}$. We prove that $H(\tau)$ surprisingly also plays a significant role in the theory of Donaldson invariants. We prove that the Moore-Witten [14] $u$-plane integrals for $H(\tau)$ are the SO(3)-Donaldson invariants of $\mathbb{C}P²$. This result then implies a moonshine phenomenon where these invariants conjecturally are expressions in the dimensions of the irreducible representations of $M_{24}$. Indeed, we obtain an explicit expression for the Donaldson invariant generating function $Z(p, S)$ in terms of the derivatives of $H(\tau)$.

1. Introduction and statement of results

This paper concerns the deep properties of the modular forms and mock modular forms which arise from a study of the $K3$ surface elliptic genus. To define these objects, we require Dedekind’s eta-function $\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^\infty (1 - q^n)$ (note. $\tau \in \mathbb{H}$ throughout and $q := e^{2\pi i \tau}$), and the classical Jacobi theta function $\vartheta_{ab}(v|\tau) := \sum_{n \in \mathbb{Z}} q^{\frac{(2n+a)^2}{8}} e^{\pi i (2n+a)(v+b^2)}$, where $a, b \in \{0, 1\}$ and $v \in \mathbb{C}$. We recall some standard identities.

\[
\begin{align*}
\vartheta_1(v|\tau) &= \vartheta_{11}(v|\tau) & \vartheta_1(0|\tau) &= 0 & \vartheta_1'(0|\tau) &= -2\pi \eta^3(\tau) \\
\vartheta_2(v|\tau) &= \vartheta_{10}(v|\tau) & \vartheta_2(0|\tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{(2n+1)^2}{8}} & \vartheta_2'(0|\tau) &= 0 \\
\vartheta_3(v|\tau) &= \vartheta_{00}(v|\tau) & \vartheta_3(0|\tau) &= \sum_{n \in \mathbb{Z}} q^{\frac{n^2}{2}} & \vartheta_3'(0|\tau) &= 0 \\
\vartheta_4(v|\tau) &= \vartheta_{01}(v|\tau) & \vartheta_4(0|\tau) &= \sum_{n \in \mathbb{Z}} (-1)^n q^{\frac{n^2}{2}} & \vartheta_4'(0|\tau) &= 0
\end{align*}
\]

Moreover, for convenience we let $\vartheta_j(\tau) := \vartheta_j(0|\tau)$ for $j = 2, 3, 4$.

The $K3$ surface elliptic genus [7] is given by

\[
Z(z|\tau) = 8 \left[ \left( \frac{\vartheta_2(z|\tau)}{\vartheta_2(\tau)} \right)^2 + \left( \frac{\vartheta_3(z|\tau)}{\vartheta_3(\tau)} \right)^2 + \left( \frac{\vartheta_4(z|\tau)}{\vartheta_4(\tau)} \right)^2 \right].
\]

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This expression is obtained by an orbifold calculation on $T^4/\mathbb{Z}_2$ in [6]. Its specializations at $z = 0$, $z = 1/2$ and $z = (\tau + 1)/2$ gives the classical topological invariants $\chi = 24, \sigma = 16$ and $\hat{A} = -2$ respectively. Here we consider the following alternate representation obtained by Eguchi and Hikami [8] motivated by superconformal field theory:

$$Z(z|\tau) = \frac{\vartheta_1(z|\tau)^2}{\eta(\tau)^3} \left( 24 \mu(z;\tau) + H(\tau) \right).$$

Here $H(\tau)$ is defined by

$$(1.1) \quad H(\tau) := -8 \sum_{w \in \{\frac{1}{2}, \frac{1+i\tau}{2}, \tau^2\}} \mu(w;\tau) = 2q^{-\frac{1}{8}} \left( -1 + \sum_{n=1}^{\infty} A_n q^n \right),$$

where $\mu(z;\tau)$ is the famous function

$$\mu(z;\tau) = \frac{i e^{\pi i z}}{\vartheta_1(z|\tau)} \sum_{n \in \mathbb{Z}} (-1)^n \frac{q^{\frac{1}{16}(n+1)^2}}{1 - q^n} e^{2\pi i n z}$$

defined by Zwegers [17] in his thesis on Ramanujan’s mock theta functions.

As explained in [8], $H(\tau)$ is the holomorphic part of a weight 1/2 harmonic Maass form, a so-called mock modular form. Its first few coefficients $A_n$ are:

| $n$ | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   | \cdots |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|---------|
| $A_n$ | 45  | 231 | 770 | 2277| 5796| 13915| 30843| 65550| \cdots  |

Amazingly, Eguchi, Ooguri, and Tachikawa [9] recognized these numbers as sums of dimensions of the irreducible representations of the Mathieu group $M_{24}$. Indeed, the dimensions of the irreducible representations are (in increasing order):

$$1, 23, 45, 45, 231, 231, 252, 253, 483, 770, 770, 990, 990, 1035, 1035, 1035, 1265, 1771, 2024, 2277, 3312, 3520, 5313, 5544, 5796, 10395.$$

One sees that $A_1, A_2, A_3, A_4$ and $A_5$ are dimensions, while

$$A_6 = 3520 + 10395 \quad \text{and} \quad A_7 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771.$$

We have their “moonshine” conjecture — also referred to as “umbral moonshine” [4]:

**Conjecture (Moonshine).** The Fourier coefficients $A_n$ of $H(\tau)$ are given as special sums of dimensions of irreducible representations of the simple sporadic group $M_{24}$.

Here we prove that the coefficients of $H(\tau)$ encode further deep information. We compute the numbers $D_{m,2n}[H(\tau)]$, the Moore-Witten [14] $u$-plane integrals for $H(\tau)$, and we prove that they are, up to a multiplicative factor of 12, the $SO(3)$-Donaldson invariants for $\mathbb{C}P^2$.

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1. This is analogous to the “Monstrous Moonshine” conjecture by Conway and Norton which related the coefficients of Klein’s $j$-function to the representations of the Monster [1]. By work of Borcherds (and others) [2], Monstrous Moonshine is now understood.

2. As in the case of the Monstrous Moonshine Conjecture, there are many representations of the generic $A_n$, and so the proper formulation of this conjecture requires a precise description of these sums [10].
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These invariants are a sequence of rational numbers which together form a diffeomorphism class invariant for \( \mathbb{C}P^2 \) (for background see \([5, 11, 12, 13]\)).

**Theorem 1.1.** For all \( m, n \in \mathbb{N}_0 \), the \( \text{SO}(3) \)-Donaldson invariants \( \Phi_{m,2n} \) for \( \mathbb{C}P^2 \) satisfy

\[
12 \Phi_{m,2n} = D_{m,2n}[H(\tau)] .
\]

**Remark.** The \( u \)-plane integrals \( D_{m,2n}[H(\tau)] \) are given explicitly in terms of the coefficients of \( H(\tau) \) (see \([3.1]\)). Therefore, the Eguchi-Ooguri-Tachikawa Moonshine Conjecture implies that these Donaldson invariants are given explicitly in terms of the dimensions of the irreducible representations of \( M_{24} \). We will discuss the numerical identities implied by Theorem 1.1 in Section 3.2. We also describe the Donaldson invariant generating function in terms of derivatives of \( H(\tau) \).

This paper builds upon earlier work by the authors \([13]\) on the Moore-Witten Conjecture for \( \mathbb{C}P^2 \). We shall make substantial use of the results in that paper, and we will recall the main facts that we need to prove Theorem 1.1.

In Section 2 we recall basic facts about those weight 1/2 harmonic Maass forms whose shadow is the cube of Dedekind’s eta-function. In Section 3 we recall and apply the main results from \([13]\). In particular, we recall the relationship between the \( u \)-plane integrals for such forms and the \( \text{SO}(3) \)-Donaldson invariants for \( \mathbb{C}P^2 \). We then conclude with the proof of Theorem 1.1.

### 2. Certain harmonic Maass forms

We let \( M(\tau) \) be a weight 1/2 harmonic Maass form \(^3\) (for definitions see \([3, 15, 16]\)) for \( \Gamma(2) \cap \Gamma_0(4) \) whose shadow \(^4\) is \( \eta(\tau)^3 \). Namely, we have that

\[
\sqrt{2i} \frac{d}{d\tau} M(\tau) = \frac{1}{\sqrt{\text{Im}\tau}} \eta^3(\tau) .
\]

For such \( M(\tau) \), we write \( M(\tau) = M^+(\tau) + M^-(\tau) \), where the holomorphic part, a mock modular form, is \( M^+(\tau) = q^{-1/8} \sum_{n \geq 0} H_n q^{n/2} \). The non-holomorphic part \( M^-(\tau) \) is

\[
M^-(\tau) = -\frac{2i}{\sqrt{\pi}} \sum_{l \geq 0} (-1)^l \Gamma\left(\frac{1}{2}, \frac{2l + 1}{2} \text{Im}\tau\right) q^{-\frac{(2l+1)^2}{8}} ,
\]

where \( \Gamma(1/2,t) \) is the incomplete Gamma function. This follows from Jacobi’s identity

\[
\eta(\tau)^3 = q^{\frac{1}{8}} \sum_{n=0}^{\infty} (-1)^n (2n + 1) q^{\frac{n^2 + n}{2}} .
\]

**Remark.** Note that the non-holomorphic part \( M^-(\tau) \) is the same for every weight 1/2 harmonic Maass form with shadow \( \eta^3(\tau) \) since this part is obtained as the “Eichler-Zagier” integral of the shadow. However, the holomorphic part is not uniquely determined. It is unique up to the addition of a weakly holomorphic modular form, a form whose poles (if any) are supported at cusps.

\(^3\)These forms were first defined by Bruinier and Funke \([3]\) in their work on geometric theta lifts.

\(^4\)The term shadow was coined by Zagier in \([16]\).
The next result gives families of modular forms from such an $M(\tau)$ using Cohen brackets. To make this precise, we recall the two Eisenstein series

$$E_2(\tau) := 1 - 24 \sum_{n=1}^{\infty} \sum_{d|n} dq^n \quad \text{and} \quad \tilde{E}_2(\tau) := E_2(\tau) - \frac{3}{\pi \Im \tau}.$$ 

The authors proved the following lemma in [13].

**Lemma 2.1.** [Lemma 4.10 of [13]] Assuming the hypotheses above, we have that

$$\mathcal{E}^k_M[\tau] := \sum_{j=0}^{k} (-1)^j \left( \begin{array}{c} k \\ j \end{array} \right) \frac{\Gamma \left( \frac{j}{2} \right)}{\Gamma \left( \frac{1}{2} + j \right)} 2^{2j} 3^j E_2^{k-j}(\tau) \left( q \frac{d}{dq} \right)^j M(\tau)$$

is modular of weight $2k + 1/2$ for $\Gamma(2) \cap \Gamma_0(4)$, and it satisfies

$$\sqrt{2i} \frac{d}{d\tau} \mathcal{E}^k_M[\tau] = \frac{1}{\sqrt{\Im \tau}} \tilde{E}_2^k(\tau) \eta^3(\tau).$$

This lemma implies the following corollary:

**Corollary 2.2.** If $M(\tau)$ and $\tilde{M}(\tau)$ are weight $1/2$ harmonic Maass forms on $\Gamma(2) \cap \Gamma_0(4)$ whose shadow is $\eta(\tau)^3$, then

$$\mathcal{E}^k_M[\tau] - \mathcal{E}^k_M[\tilde{M}(\tau)] = \mathcal{E}^k_M[M(\tau) - \tilde{M}(\tau)] = \mathcal{E}^k_M[M^+(\tau) - \tilde{M}^+(\tau)]$$

is a weakly holomorphic modular form of weight $2k + 1/2$.

### 2.1. The $\mathcal{Q}(q)$ series

Here we recall one explicit example of a harmonic Maass form which plays the role of $M(\tau)$ in the previous subsection. To this end, we define modular forms $A(\tau)$ and $B(\tau)$ by

$$A(\tau) := A(8\tau) = \sum_{n=-1}^{\infty} a(n)q^n := \frac{\eta(4\tau)^8}{\eta(8\tau)^2} = q^{-1} - 8q^3 + 27q^7 - \cdots,$$

$$B(\tau) := B(8\tau) = \sum_{n=-1}^{\infty} b(n)q^n := \frac{\eta(8\tau)^5}{\eta(16\tau)^4} = q^{-1} - 5q^7 + 9q^{15} - \cdots.$$

We sieve on the Fourier expansion of $A(\tau)$ to define the modular forms

$$A_{3,8}(\tau) := A_{3,8}(8\tau) = \sum_{n \equiv 3 \pmod{8}} a(n)q^n = -8q^3 - 56q^{11} + \cdots,$$

$$A_{7,8}(\tau) := A_{7,8}(8\tau) = \sum_{n \equiv 7 \pmod{8}} a(n)q^n = q^{-1} + 27q^7 + 105q^{15} + \cdots.$$

We also recall the definition of the following mock theta function

$$\mathcal{M}(q) := q^{-1} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} q^{4(n+1)^2} \prod_{k=1}^{n} (1 - q^{16k-8})}{\prod_{k=1}^{n+1} (1 + q^{16k-8})} = -q^7 + 2q^{15} - 3q^{23} + \cdots,$$
We define
\[ Q^+(q) = Q^+(\tau) := -\frac{7}{2}A_{3,8}(\tau) + \frac{3}{2}A_{7,8}(\tau) - \frac{1}{2}B(\tau) + 4M(q) , \]
and so we have that
\[ (2.2) \quad Q^+(\tau/8) = \frac{1}{q^{\frac{1}{8}}} \left( 1 + 28q + 39q + 196q^2 + 161q^3 + \ldots \right) . \]

In terms of this \( q \)-series, the authors proved the following theorem in \cite{13}.

**Theorem 2.3.** [Theorem 7.2 of \cite{13}] The function \( Q^+(\tau/8) \) is the holomorphic part of a weight 1/2 harmonic Maass form on \( \Gamma(2) \cap \Gamma_0(4) \) whose shadow is \( \eta(\tau)^3 \).

### 3. \( u \)-plane integrals, Donaldson invariants and the proof of Theorem 1.1

Suppose again that \( M(\tau) \) is a weight 1/2 harmonic Maass form on \( \Gamma(2) \cap \Gamma_0(4) \) whose shadow is \( \eta(\tau)^3 \). For \( m, n \in \mathbb{N}_0 \), the authors proved that the quantities
\[ (3.1) \quad D_{m,2n}[M^+(\tau)] := \sum_{k=0}^{n} \frac{(-1)^{k+1}}{2^{n-k} 3^k} \frac{(2n)!}{(n-k)! k!} \left[ \frac{\partial_0^2(\tau) \left[ \partial_2^4(\tau) + \partial_3^4(\tau) \right]^{m+n-k}}{[\partial_2(\tau) \partial_3(\tau)]^{2m+2n+3}} E_k^{\pm} \right] q^k , \]
where \([.]_{q^k}\) denotes the constant coefficient term, are the Moore-Witten \( u \)-plane integrals for \( M(\tau) \) (cf. \cite{13}). Notice that if \( \tilde{M}(\tau) \) is another such form, then
\[ (3.2) \quad D_{m,2n}[M^+(\tau)] - D_{m,2n}[\tilde{M}^+(\tau)] = D_{m,2n}[M^+(\tau) - \tilde{M}^+(\tau)] . \]

In their seminal paper \cite{14}, Moore and Witten essentially conjectured that the \( u \)-plane integrals in \((3.1)\) for a suitable \( M^+(\tau) \) should give the SO(3)-Donaldson invariants of \( \mathbb{C}P^2 \).

These invariants are an infinite sequence of rational numbers \( \Phi_{m,2n} \) labeled by integers \( m, n \in \mathbb{N} \) that can be assembled in a generating function in the two formal variables \( p, S \):
\[ Z(p, S) = \sum_{m,n \geq 0} \Phi_{m,2n} \frac{p^m S^{2n}}{m! (2n)!} . \]

This power series is a diffeomorphism invariant for \( \mathbb{C}P^2 \). The main theorem in \cite{13} proved this conjecture for \( Q^+(\tau/8) \).

**Theorem 3.1.** [Theorem 1.1 of \cite{13}] For \( m, n \in \mathbb{N}_0 \) we have that
\[ \Phi_{m,2n} = D_{m,2n}[Q^+(\tau/8)] . \]

Using the work in \cite{13}, we prove the following important theorem.

**Theorem 3.2.** Let \( M(\tau) \) be as above, then for all \( m, n \in \mathbb{N}_0 \) we have:
\[ D_{m,2n}[Q^+(\tau/8)] - D_{m,2n}[M^+(\tau)] = D_{m,2n}[Q^+(\tau/8) - M^+(\tau)] = 0 . \]
Proof. We prove that the constant terms vanish in expressions of the form
\[
\sum_{k=0}^{n} \frac{(-1)^{k+1}}{2^{n-1} 3^n} \frac{(2n)!}{(n-k)! k!} \frac{\vartheta_2^2(\tau) [\vartheta_2^2(\tau) + \vartheta_3^4(\tau)]^{m+n-k}}{[\vartheta_2(\tau) \vartheta_3(\tau)]^{2m+2n+3}} \mathcal{E}_{\frac{k}{2}}^k \left[ Q^+(\tau/8) - M(\tau) \right].
\]
It is sufficient to show that this is the case for each summand. Therefore, after rescaling \( \tau \to 8\tau \) and \( q \to q^8 \) it is enough to show that the constant vanishes in
\[
(3.3) \quad \frac{\Theta_3^4(\tau) [16 \Theta_2^4(\tau) + \Theta_4^4(\tau)]^{m+n-k}}{[\Theta_2(\tau) \Theta_3(\tau)]^{2m+2n+3}} \mathcal{E}_{\frac{k}{2}}^k \left[ Q^+(\tau) - M(8\tau) \right]
\]
Here the classical theta functions are defined by
\[
\Theta_2(\tau) := \frac{\eta(16\tau)^2}{\eta(8\tau)} = \sum_{n=0}^{\infty} q^{(2n+1)^2} = q + q^9 + q^{25} + \cdots,
\]
\[
\Theta_3(\tau) := \frac{\eta(8\tau)^5}{\eta(4\tau)^2 \eta(16\tau)^2} = 1 + 2 \sum_{n=1}^{\infty} q^{4n^2} = 1 + 2q^4 + 2q^4 + 2q^{36} + \cdots,
\]
\[
\Theta_4(\tau) := \frac{\eta(4\tau)^2}{\eta(8\tau)} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{4n^2} = 1 - 2q^4 + 2q^4 - 2q^{36} + \cdots.
\]
These are related to the theta functions \( \vartheta_2(\tau), \vartheta_3(\tau) \) and \( \vartheta_4(\tau) \) by
\[
\vartheta_2(\tau) = 2 \Theta_2 \left( \frac{\tau}{8} \right), \quad \vartheta_3(\tau) = \Theta_3 \left( \frac{\tau}{8} \right), \quad \vartheta_4(\tau) = \Theta_4 \left( \frac{\tau}{8} \right).
\]
We define a weakly holomorphic modular function by
\[
(3.4) \quad \widehat{Z}_0(q) = \widehat{Z}_0(\tau) := \frac{\Theta^*(4\tau)}{\Theta_2(\tau)^2 \Theta_3(\tau)^2}
\]
where \( \Theta^*(4\tau) \) is the weight 2 Eisenstein series with
\[
\Theta^*(4\tau) = 16\Theta_2(\tau)^4 + \Theta_4(\tau)^4 = 1 + 24q^4 + 24q^2 + \cdots.
\]
A calculation shows that
\[
q \frac{d}{dq} \widehat{Z}_0(q) = \frac{\Theta_4(\tau)^9}{\Theta_2(\tau) \Theta_3(\tau) \eta(8\tau)^3}.
\]
Equation (3.3) becomes
\[
(3.5) \quad q \frac{d}{dq} \widehat{Z}_0(q) \cdot \widehat{Z}_0(q)^{m+n-k} \cdot \mathcal{H}_k(q)
\]
where
\[
(3.6) \quad \mathcal{H}_k(q) := \frac{\eta(8\tau)^3}{(\Theta_2(\tau) \Theta_3(\tau))^{2k+2}} \mathcal{E}_{\frac{k}{2}}^k \left[ Q^+(\tau) - M(8\tau) \right].
\]
To prove the theorem, it suffices to show that the constant term in \( \Phi \) vanishes. Hence, it is enough to show that \( \mathcal{H}_k(q) \) is a polynomial in \( \hat{Z}_0(q) \). To this end, we define \( \mathcal{M}_0(\Gamma_0(8)) \) to be the space of modular function on \( \Gamma_0(8) \) which are holomorphic away from infinity, and is a subspace of \( \mathbb{C}((q^2)) \). One can easily verify that \( \mathcal{M}_0(\Gamma_0(8)) \) is precisely the set of polynomials in \( \hat{Z}_0(q) \). From Corollary \ref{cor:modular} we can observe that \( \mathcal{H}_k(q) \) is modular with weight 0. A calculation shows that \( \Theta_2(\tau)\Theta_3(\tau)^{-2} = q^{-2} f(q^4) \) is holomorphic away from infinity, and \( f(q) \in \mathbb{Z}[[q]] \). We also have \( \eta(8\tau)^3 = q g(q^8) \) and \( \mathcal{E}_2^k(\mathcal{Q}^+(\tau) - M(8\tau)) = q^{-1} h(q^4) \), where \( g(q), h(q) \in \mathbb{Z}[[q]] \). Hence, \( \mathcal{H}_k(q) \in \mathbb{C}((q^2)) \) is modular of weight 0 on \( \mathcal{M}_0(\Gamma_0(8)) \), and so is a polynomial in \( \hat{Z}_0(\tau) \).

### 3.1. Proof of Theorem \ref{thm:main}.

Since \( H(\tau) \) is the mock modular part of a weight 1/2 harmonic Maass form on \( \Gamma(2) \cap \Gamma_0(4) \) whose shadow is the \( 8 \cdot 3 \cdot \eta(\tau)^3/2 = 12 \eta(\tau)^3 \), it follows from Theorem \ref{thm:main} that for \( m, n \in \mathbb{N}_0 \) we have:

\[
\Phi_{m,2n} = D_{m,2n}[Q^+(\tau/8)] = D_{m,2n}[H(\tau)/12] + D_{m,2n}\bigg[ Q^+(\tau/8) - H(\tau)/12 \bigg] = D_{m,2n}[H(\tau)/12] = \frac{1}{12} D_{m,2n}[H(\tau)].
\]

### 3.2. Discussion of the identities implied by Theorem \ref{thm:main}.

In the table below we list the first non-vanishing \( SO(3) \)-Donaldson invariants \( \Phi_{m,2n} \) on \( \mathbb{C}P^2 \) as well as the coefficients \( D_{m,2n}[M^+(\tau)] \) when the mock modular form is given as \( M^+(\tau) = q^{-1/8} \sum_{k \geq 0} H_k q^{k/2} \). In general, \( D_{m,2n}[M^+(\tau)] \) is nonvanishing for \( m + n \equiv 0 \pmod{2} \) and a rational linear combination of the first \((m+n)/2+1\) coefficients of \( M^+(\tau) \).

| \((m, n)\) | \( \Phi_{m,2n} \) | \( D_{m,2n}[M^+(\tau)] \) |
|---|---|---|
| (0, 0) | -1 | \(-\frac{1}{7} H_1 + 6 H_0\) |
| (0, 2) | \(-\frac{3}{16}\) | \(-\frac{49}{64} H_2 + \frac{9}{4} H_1 - \frac{2133}{64} H_0\) |
| (1, 1) | \(-\frac{5}{10}\) | \(-\frac{79}{64} H_2 + \frac{1}{4} H_1 - \frac{105}{64} H_0\) |
| (2, 0) | \(-\frac{19}{16}\) | \(-\frac{19}{64} H_2 - \frac{1}{4} H_1 + \frac{411}{64} H_0\) |
| (0, 4) | \(-\frac{232}{256}\) | \(-\frac{14}{128} H_3 + \frac{2041}{128} H_2 + \frac{44631}{1024} H_1 + \frac{108741}{128} H_0\) |
| (1, 3) | \(-\frac{152}{256}\) | \(-\frac{1331}{128} H_3 - \frac{49}{128} H_2 + \frac{10341}{1024} H_1 - \frac{1749}{128} H_0\) |
| (2, 2) | \(-\frac{136}{256}\) | \(-\frac{121}{128} H_3 - \frac{91}{128} H_2 + \frac{2895}{1024} H_1 - \frac{3687}{128} H_0\) |
| (3, 1) | \(-\frac{184}{256}\) | \(-\frac{11}{128} H_3 - \frac{29}{128} H_2 + \frac{569}{1024} H_1 - \frac{753}{128} H_0\) |
| (4, 0) | \(-\frac{680}{256}\) | \(-\frac{1}{128} H_3 - \frac{7}{128} H_2 + \frac{565}{1024} H_1 + \frac{1725}{128} H_0\) |

Theorem \ref{thm:main} states that choosing \( M^+(\tau) = Q^+(\tau/8) \) from \ref{eq:modular} we find equality of the Donaldson invariants \( \Phi_{m,2n} \) and the \( u \)-plane integral \( D_{m,2n}[M^+(\tau)] \). In fact, setting \( H_0 = 1, H_1 = 28, H_2 = 39, H_3 = 196 \) in the third column of the table above gives the Donaldson invariants of the second column.

On the other hand, the choice \( M^+(\tau) = H(\tau)/12 \) from \ref{eq:main} implies that \( H_0 = -1/6, H_{2k} = A_k/6, H_{2k+1} = 0 \) for \( k \in \mathbb{N} \). Theorem \ref{thm:main} states that choosing \( M^+(\tau) = H(\tau)/12 \) we still find equality of the Donaldson invariants \( \Phi_{m,2n} \) and the \( u \)-plane integral \( D_{m,2n}[M^+(\tau)] \).
In fact, setting \( H_0 = -1/6, H_1 = 0, H_2 = 45/6, H_3 = 0 \) in the third column of the table gives the Donaldson invariants of the second column as well.

The proof of Theorem 1.1 implies the following form for the generating function \( Z(p, S) \) of the SO(3)-Donaldson invariants of \( \mathbb{C}P^2 \) in terms of the mock modular form \( H(\tau) \):

\[
Z(p, s) = - \sum_{m,n \geq 0} \frac{p^m S^{2n}}{2^{2m+3n+4} \cdot 3^{n+1} \cdot m! \cdot n!} \times \left[ q \frac{d}{dq} \tilde{Z}_0(q) \sum_{k=0}^n (-1)^k \binom{n}{k} \tilde{Z}_0(q)^{m+n-k} \tilde{E}^k[H(8\tau)] \right]_{\varphi^0},
\]

where \( \tilde{Z}_0(q) \) was defined in (3.4) and we have set

\[
\tilde{E}^k[H(8\tau)] = \frac{\eta(8\tau)^3}{(\Theta_2(\tau)\Theta_3(\tau))^{2k+2}} \tilde{E}^k[H(8\tau)].
\]

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