The leftmost column of ordered Chinese Restaurant Process up-down chains: intertwining and convergence

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Abstract

Recently there has been significant interest in constructing ordered analogues of Petrov’s two-parameter extension of Ethier and Kurtz’s infinitely-many-neutral-alleles diffusion model. One method for constructing these processes goes through taking an appropriate diffusive limit of Markov chains on integer compositions called ordered Chinese Restaurant Process up-down chains. The resulting processes are diffusions whose state space is the set of open subsets of the open unit interval. In this paper we begin to study nontrivial aspects of the order structure of these diffusions. In particular, for a certain choice of parameters, we take the diffusive limit of the size of the first component of ordered Chinese Restaurant Process up-down chains and describe the generator of the limiting process. We then relate this to the size of the leftmost maximal open subset of the open-set valued diffusions. This is challenging because the function taking an open set to the size of its leftmost maximal open subset is discontinuous. Our methods are based on establishing intertwining relations between the processes we study.

1 Introduction

The construction and analysis of ordered analogues of Petrov’s two-parameter extension of Ethier and Kurtz’s infinitely-many-neutral-alleles diffusion model has recently attracted significant interest in the literature \cite{7, 8, 20, 21, 23}. Recall that the EXP($\alpha, \theta)$ diffusions constructed in \cite{15} are a family of Feller diffusions on the closure of the Kingman simplex

$$\nabla_\infty = \left\{ x = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0, \sum_{i \geq 1} x_i \leq 1 \right\}$$

whose generator acts on the unital algebra generated by $\phi_m(x) = \sum_{i \geq 1} x_i^m$, $m \geq 2$ by

$$G = \frac{1}{2} \left( \sum_{i=1}^\infty x_i \frac{\partial^2}{\partial x_i^2} - \sum_{i,j=1}^\infty x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{i=1}^\infty (\theta x_i + \alpha) \frac{\partial}{\partial x_i} \right).$$

In \cite{20}, for each $\theta \geq 0$, $0 \leq \alpha < 1$, and $\alpha + \theta > 0$, we constructed a Feller diffusion $X^{(\alpha, \theta)}$ whose state space $U$ is the set of open subsets of $(0, 1)$ such that the ranked sequence of lengths of maximal open intervals in $X^{(\alpha, \theta)}$ is an EXP($\alpha, \theta$) diffusion. This was done by considering the scaling limit of up-down chains associated to the ordered Chinese Restaurant Process.

While many interesting properties of $X^{(\alpha, \theta)}$ can be obtained from the corresponding properties for EXP($\alpha, \theta$) diffusions, properties that depend on the order structure cannot be.

In this paper we begin to study nontrivial aspects of the order structure of these diffusions.

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Motivated by [3, Theorem 2 and Theorem 19] and [5, Theorem 5], which consider similar properties in closely related tree-valued processes, we consider the evolution of the left-most maximal open interval of $X^{(\alpha,0)}$ in running in its $(\alpha,0)$-Poisson-Dirichlet interval partition stationarity distribution. Recall that the $(\alpha,0)$-Poisson-Dirichlet interval partition is the distribution of $\{\ell \in (0,1) : V_{\ell-} > 0\}$ where $V_{\ell}$ is a $(2-2\alpha)$-dimensional Bessel process started from $0$. We prove the following result.

**Theorem 1.1.** Define $\xi : \mathcal{U} \to [0,1]$ by $\xi(u) = \inf\{s > 0 : s \in [0,1]\}$. If $X^{(\alpha,0)}$ is running in its $(\alpha,0)$-Poisson-Dirichlet interval partition stationarity distribution, then $\xi(X^{(\alpha,0)})$ is a Feller process. Moreover, the generator of its semigroup $\mathcal{L} : \mathcal{D} \subseteq C[0,1] \to C[0,1]$ is given by

$$\mathcal{L}f(x) = x(1-x)f''(x) - \alpha f'(x)$$

for $x \in (0,1)$, where the domain $\mathcal{D}$ of $\mathcal{L}$ consists of functions $f$ satisfying

1. $f \in C^2(0,1)$ and $\zeta(x) = x(1-x)f''(x) - \alpha f'(x)$ extends continuously to $[0,1]$,
2. $\int_0^1 (f(x) - f(0))x^{-\alpha-1}(1-x)^{n-1} \, dx = 0$, and
3. $f'(x)(1-x)^\alpha \to 0$ as $x \to 1$.

We consider only the $(\alpha,0)$ case because the known stationary distribution of $X^{(\alpha,0)}$ is an $(\alpha,\theta)$-Poisson-Dirichlet interval partition and, except in the $(\alpha,0)$ case, with probability 1 interval partitions with these distributions do not have left-most maximal open intervals. We remark that our theorem statement could be slightly simpler if we knew that $X^{(\alpha,\theta)}$ had a unique stationary distribution, but this is currently an open problem.

Our proof is based on taking the scaling limit of the left-most coordinate in an up-down chain on compositions based on the ordered Chinese Restaurant Process, which are the same chains that were used in [20] to construct $X^{(\alpha,0)}$.

**Definition 1.1.** For $n \geq 1$, a composition of $n$ is a tuple $\sigma = (\sigma_1,\ldots,\sigma_k)$ of positive integers that sum to $n$. The composition of $n = 0$ is the empty tuple, which we denote by $\emptyset$. We write $|\sigma| = n$ and $\ell(\sigma) = k$ when $\sigma$ is a composition of $n$ with $k$ components. We denote the set of all compositions of $n$ by $\mathcal{C}_n$ and their union by $\mathcal{C} = \cup_{n \geq 0} \mathcal{C}_n$.

An up-down chain on $\mathcal{C}_n$ is a Markov chain whose steps can be factored into two parts: 1) an up-step from $\mathcal{C}_n$ to $\mathcal{C}_{n+1}$ according to a kernel $p^\uparrow$ followed by 2) a down-step from $\mathcal{C}_{n+1}$ to $\mathcal{C}_n$ according to a kernel $p^\downarrow$. The probability $T_n(\sigma,\sigma')$ of transitioning from $\sigma$ to $\sigma'$ can then be written as

$$T_n(\sigma,\sigma') = \sum_{\tau \in \mathcal{C}_{n+1}} p^\uparrow(\sigma,\tau)p^\downarrow(\tau,\sigma').$$

(1)

Up-down chains on compositions, and more generally, on graded sets, have been studied in a variety of contexts [2, 6, 9, 10, 11, 15, 16], often in connection with their nice algebraic and combinatorial properties.

In the up-down chains we considered, the up-step kernel $p^\uparrow(\alpha,\theta)$ is given by an $(\alpha,\theta)$-ordered Chinese Restaurant Process growth step [15]. In the Chinese Restaurant Process analogy, we view $\tau = (\tau_1,\ldots,\tau_k) \in \mathcal{C}_n$ as an ordered list of the number of customers at $k$ occupied tables in a restaurant, so that $\tau_i$ is the number of customers at the $i^{th}$ table on the list. During an up-step, a new customer enters the restaurant and chooses a table to sit at according to the following rules:

- The new customer joins table $i$ with probability $(\tau_i - \alpha)/(n + \theta)$, resulting in a step from $\tau$ to $(\tau_1,\ldots,\tau_{i-1},\tau_i + 1,\tau_{i+1},\ldots,\tau_k)$.
- The new customer starts a new table directly after table $i$ with probability $\alpha/(n + \theta)$, resulting in a step from $\tau$ to $(\tau_1,\ldots,\tau_{i-1},\tau_i,1,\tau_{i+1},\ldots,\tau_k)$.
- The new customer starts a new table at the start of the list with probability $\theta/(n + \theta)$, resulting in a step from $\tau$ to $(1,\tau_1,\tau_2,\ldots,\tau_k)$.

We note that, for consistency with [7, 8], this up-step is the left-to-right reversal of the growth step in [13].

The down-step kernel $p^\downarrow$ we consider can also be thought of in terms of the restaurant analogy. During a down-step, a seated customer gets up and exits the restaurant according to the following rule:
The seated customer is chosen uniformly at random, resulting in a step from \(\tau\) to 
\((\tau_1, \ldots, \tau_{i-1}, \tau_i-1, \tau_{i+1}, \ldots, \tau_k)\) with probability \(\tau_i/n\) (the \(i^{th}\) coordinate is to be contracted away if \(\tau_i = 1\), or if the \(i^{th}\) table is no longer occupied).

Note that, in contrast to the up-step, the down-step does not depend on \((\alpha, \theta)\).

Let \((X^{(\alpha, \theta)}_n(k))_{k \geq 0}\) be a Markov chain on \(\mathcal{C}_n\) with transition kernel \(T^{(\alpha, \theta)}_n\) defined as in Equation (1) using the \(p^\uparrow_{(\alpha, \theta)}\) and \(p^\downarrow_{(\alpha, \theta)}\) just described. A Poissonized version of this chain was considered in [21][23]. It can be shown that \(X^{(\alpha, \theta)}_n\) is an aperiodic, irreducible chain. We denote its unique stationary distribution by \(M^{(\alpha, \theta)}_n\) and note that this is the left-to-right reversal of the \((\alpha, \theta)\)-regenerative composition structures introduced in [12].

The projection \(\phi(\sigma) = \sigma_1\) for \(\sigma \neq \varnothing\) gives rise to the leftmost column processes, defined by \(Y^{(\alpha, \theta)}_n = \phi(X^{(\alpha, \theta)}_n)\). Let \(\nu^{(\alpha, \theta)}_n = M^{(\alpha, \theta)}_n \circ \phi^{-1}\), the distribution of the leftmost column when the up-down chain is in stationarity. The following result, interesting in its own right, is a key step in our proof of Theorem 1.1.

**Theorem 1.2.** For \(n \geq 1\), let \(\mu_n\) be a distribution on \(\{1, \ldots, n\}\). Then, for all \(n\), the up-down chain \(X^{(\alpha, \theta)}_n\) can be initialized so that \(Y^{(\alpha, \theta)}_n\) is a Markov chain with initial distribution \(\mu_n\). Moreover, for any such sequence of initial conditions for \(X^{(\alpha, \theta)}_n\), if the sequence \((n^{-1}Y^{(\alpha, \theta)}_n(0))_{n \geq 1}\) has a limiting distribution \(\mu\), then we have the convergence

\[
(n^{-1}Y^{(\alpha, \theta)}_n(\lfloor n^2t \rfloor))_{t \geq 0} \Rightarrow (F(t))_{t \geq 0}
\]

in the Skorokhod space \(D([0, \infty), [0, 1])\), where \(F\) is a Feller process with generator \(L\) (as in Theorem 1.1) and initial distribution \(\mu\).

While there are many ways to prove a result like Theorem 1.2, we take an approach based on the algebraic properties of the ordered Chinese Restaurant Process up-down chains. In particular, our proof is based on the following surprising intertwining result. For a positive integer \(i\) and composition \(\sigma\), we use the notation \((i, \sigma)\) as a shorthand for the composition \((i, \sigma_1, \sigma_2, \ldots, \sigma_{\ell(\sigma)})\).

**Theorem 1.3.** For \(n \geq 1\), let \(\Lambda_n\) be the transition kernel from \(\{1, \ldots, n\}\) to \(\mathcal{C}_n\) given by

\[
\Lambda_n(i, (i, \sigma)) = M^{(\alpha, \alpha)}_{n-i}(\sigma),
\]

and let \(K_n\) be the transition kernel from \([0, 1]\) to \(\{1, \ldots, n\}\) given by

\[
K_n(x, i) = \binom{n}{i} x^i (1 - x)^{n-i} + \nu^{(\alpha, \alpha)}_n(i)(1 - x)^n.
\]

If the initial distribution of \(X^{(\alpha, \theta)}_n\) is of the form \(\mu \Lambda_n\) for some distribution \(\mu\) on \(\{1, \ldots, n\}\), then the process \(Y^{(\alpha, \theta)}_n\) is Markovian. In this case, the following intertwining relations hold:

(i) \(\Lambda_n T^{(\alpha, \theta)}_n = Q^{(\alpha, \alpha)}_n \Lambda_n\), where \(Q^{(\alpha, \alpha)}_n\) is the transition kernel of \(Y^{(\alpha, \theta)}_n\), and

(ii) \(K_n e^{tn(n+1)}(Q^{(\alpha, \alpha)}_n - 1) = U_t K_n\) for \(t \geq 0\), where \(U_t\) is the semigroup generated by the operator \(L\) defined in Theorem 1.1 and 1 denotes the identity operator.

This paper is organized as follows. In Section 2 we show that the \((\alpha, 0)\) leftmost column process is intertwined with its corresponding up-down chain and describe its transition kernel explicitly. This establishes part of Theorem 1.2. In Section 3 we state a condition under which the convergence of Markov processes can be obtained from some commutation relations involving generators. In Section 4 we analyze the generator of the limiting process. In Section 5 we show that our generators satisfy the commutation relations appearing in the result of Section 3. In Section 6 we verify the convergence condition appearing in the result in Section 3. In Section 7 we provide general conditions under which commutation relations involving generators lead to the corresponding relations for their semigroups. Finally in Section 8 we prove Theorems 1.1, 1.2 and 1.3.

The following will be used throughout this paper. For a compact topological space \(X\), we denote by \(C(X)\) the space of continuous functions from \(X\) to \(\mathbb{R}\) equipped with the supremum norm. Finite topological spaces will always be equipped with the discrete topology. Any
sum or product over an empty index set will be regarded as a zero or one, respectively. The set of positive integers \{1, \ldots, k\} will be denoted by \([k]\). The falling factorial will be denoted using \emph{factorial exponents} — that is, \(x^b = x(x-1) \cdots (x-b+1)\) for a real number \(x\) and nonnegative integer \(b\), and \(0^b = 1\) by convention. The rising factorial will be denoted by \((x)_b = x(x+1) \cdots (x+b-1)\). We denote the gamma function by \(\Gamma(x)\). Multinomial coefficients will be denoted using the shorthand

\[
\binom{\sigma}{\sigma} = \begin{cases} 
|\sigma| & \sigma \neq \emptyset, \\
1 & \sigma = \emptyset.
\end{cases}
\]

2 The Leftmost Column Process

Our study of the leftmost column process will be mainly focused on the \(\theta = 0\) case. However, it will be useful to study the distribution of the \((\alpha, \alpha)\) leftmost column process when the up-down chain is in stationarity. As we will see, this distribution has a role in the evolution of the \((\alpha, 0)\) process.

**Proposition 2.1.** The stationary distribution of \(X_n^{(\alpha, \alpha)}\) is given by

\[
M_n^{(\alpha, \alpha)}(\sigma) = \binom{n}{\sigma} \frac{1}{(\alpha)_n} \prod_{j=1}^{\ell(\sigma)} \alpha \cdot (1-\alpha)_{\sigma_j-1}, \quad \sigma \in \mathcal{C}_n, \; n \geq 0.
\]

Moreover, the following consistency conditions hold:

\[
M_n^{(\alpha, \alpha)} = M_{n-1}^{(\alpha, \alpha)} p_{(\alpha, \alpha)} = M_{n+1}^{(\alpha, \alpha)} p_{(\alpha, \alpha)}^{-1}, \quad n \geq 1.
\]

**Proof.** The stationary distribution of \(X_n^{(\alpha, \theta)}\) is identified in [20, Theorem 1.1] and the formula in the special case \(\alpha = \theta\) follows from [12, Formula 48]. The consistency conditions follows from [18, Proposition 6]. \(\square\)

**Proposition 2.2.** If \(X_n^{(\alpha, \alpha)}\) has distribution \(M_n^{(\alpha, \alpha)}\), then \(Y_n^{(\alpha, \alpha)}\) has distribution

\[
\nu_n^{(\alpha, \alpha)}(i) = \binom{n}{i} \frac{\alpha (1-\alpha)^{i-1}}{(n-i+\alpha)_i} I(1 \leq i \leq n), \quad i \geq 0, \; n \geq 1.
\]

**Proof.** Let \(1 \leq i \leq n\) and \(\sigma \in \mathcal{C}_{n-i}\). It can be verified that

\[
M_n^{(\alpha, \alpha)}(i, \sigma) = \nu_n^{(\alpha, \alpha)}(i) M_{n-i}^{(\alpha, \alpha)}(\sigma).
\]

Summing over \(\sigma\) concludes the proof. \(\square\)

Let \(n \geq i \geq 1\) and \(\sigma \in \mathcal{C}_{n-i}\). Consider taking an \((\alpha, 0)\) up-step from \((i, \sigma)\) followed by a down-step. Let \(U\) be the event in which this up-step stacks a box on the first column of \((i, \sigma)\), and let \(D\) be the event in which the down-step removes a box from the first column of a composition. Then, \(r_{i,i+1} = \mathbb{P}(U \cap D^c)\), \(r_{i,i-1} = \mathbb{P}(U^c \cap D)\), \(p_{(1)}^{(i)} = \mathbb{P}(U^c \cap D^c)\), \(p_{(2)}^{(i)} = \mathbb{P}(U \cap D)\), and \(r_{i,j} = p_{(1)}^{(i)} + r_{i,i}^{(2)}\) do not depend on \(\sigma\). Indeed, we have the formulas

\[
\begin{align*}
   r_{i,i+1} &= \frac{(n-i+\alpha)}{n(n+1)}, & p_{(1)}^{(i)} &= \frac{(n-i+1)(n-i+\alpha)}{n(n+1)}; \\
   r_{i,i-1} &= \frac{(\alpha)(\alpha-i-1)}{n(n+1)}, & r_{i,i}^{(2)} &= \frac{(\alpha)(\alpha+i)}{n(n+1)}.
\end{align*}
\]

We use these formulas to define \(r_{0,-1}, r_{0,1}, r_{0,0}^{(1)}, r_{0,0}^{(2)}\), and \(r_{0,0} = r_{0,0}^{(1)} + r_{0,0}^{(2)}\). Moreover, we extend \(r_{i,j}\) to be zero for all other integer arguments \(i\) and \(j\).

The following is a useful identity relating the transition kernels of the \((\alpha, 0)\) and \((\alpha, \alpha)\) chains.
Proposition 2.3. For $n \geq 1$ and $(i, \sigma), (j, \sigma') \in \mathcal{C}_n$, we have the identity

$$T_n^{(\alpha,0)}((i, \sigma), (j, \sigma')) = r_{i,j}p_{(\alpha,\alpha)}(\sigma, \sigma')\mathbb{I}(j = i - 1) + r_{i,j}p(\sigma, \sigma')\mathbb{I}(j = i + 1)$$

$$+ (r_{i,j}p_{(\alpha,\alpha)}(\sigma, \sigma') + r_{i,j}(\sigma = \sigma'))\mathbb{I}(j = i)$$

$$+ r_{1,0}p_{(\alpha,\alpha)}(\sigma, (j, \sigma'))\mathbb{I}(i = 1).$$

**Proof.** Fix $(i, \sigma)$ and $(j, \sigma') \in \mathcal{C}_n$. Let $C^\dagger$ be the composition obtained by performing an $(\alpha, 0)$ up-step from $(i, \sigma)$ and $C^\ddagger$ be the composition obtained by performing a down-step from $C^\dagger$. As before, let $U$ be the event in which the up-step adds to the first column of a composition and $D$ be the event in which the down-step removes from the first column of a composition. Then, we have that

$$U = \{C^\dagger = (i + 1, \sigma)\}, \quad U^c = \{\text{other configurations}\}, \quad D^c \subset \{C^\ddagger = C^\dagger\},$$

and

$$D \subset \{C^\dagger > 1, C^\dagger = (C^\dagger - 1, (C^\dagger)_2^{(\ell(C^\dagger))})\} \cup \{(\text{other configurations})\}.$$

To obtain the identity, we note that

$$T_n^{(\alpha,0)}((i, \sigma), (j, \sigma')) = \mathbb{P}\{C^\dagger = (j, \sigma')\},$$

and rewrite this probability by conditioning on the above sets. Of particular importance will be the following observations: the conditional distribution of $(C^\dagger)_2^{(\ell(C^\dagger))}$ given $(C^\dagger)_2^{(\ell(C^\dagger))}$ and $D^c$ is $p^+(((C^\dagger)_2^{(\ell(C^\dagger))}), \cdot)$, and conditionally given $U^c$, $(C^\dagger)_2^{(\ell(C^\dagger))}$ is independent of $D$ and has distribution $p_{(\alpha,\alpha)}(\sigma, \cdot)$. We also make use of the fact that the events $\{C^\dagger = (n + 1 - |\rho|, \rho)\}$ and $\{(C^\dagger)_2^{(\ell(C^\dagger))} = \rho\}$ are identical, since the size of $C^\dagger$ is known to be $n + 1$.

Our first conditional probability is given by

$$\mathbb{P}(C^\dagger = (j, \sigma')|U, D) = \mathbb{P}(C^\dagger = (j, \sigma')|C^\dagger = (i + 1, \sigma), D)$$

$$= \mathbb{P}(i, \sigma) = (j, \sigma')|C^\dagger = (i + 1, \sigma), D)$$

$$= \mathbb{I}(j = i + 1).$$

Next, we will condition on $U \cap D^c$. Notice that this is a null set when $i = n$. When $i < n$, we have

$$\mathbb{P}(C^\dagger = (j, \sigma')|U \cap D^c) = \mathbb{P}(C^\dagger = j, (C^\dagger)_2^{(\ell(C^\dagger))} = \sigma'|C^\dagger = (i + 1, \sigma), D^c)$$

$$= \mathbb{I}(j = i + 1)\mathbb{P}((C^\dagger)_2^{(\ell(C^\dagger))} = \sigma'|U \cap D^c)$$

$$= \mathbb{I}(j = i + 1)\mathbb{P}(\sigma, \sigma').$$

Conditioning on $U^c \cap D$ will require two cases. For $i > 1$, we have

$$\mathbb{P}(C^\dagger = (j, \sigma')|U^c, D) = \mathbb{P}(C^\dagger = (j, \sigma')|C^\dagger = i, D)$$

$$= \mathbb{P}(i, \sigma) = (j, \sigma')|C^\dagger = i, D)$$

$$= \mathbb{I}(j = i - 1)\mathbb{P}((C^\dagger)_2^{(\ell(C^\dagger))} = \sigma'|U^c, D)$$

$$= \mathbb{I}(j = i - 1)\mathbb{P}((C^\dagger)_2^{(\ell(C^\dagger))} = \sigma'|U^c)$$

$$= \mathbb{I}(j = i - 1)\mathbb{P}((C^\dagger)_2^{(\ell(C^\dagger))} = \sigma').$$

And for $i = 1$, we have

$$\mathbb{P}(C^\dagger = (j, \sigma')|U^c, D) = \mathbb{P}(C^\dagger = (j, \sigma')|C^\dagger = 1, D)$$

$$= \mathbb{P}(1, \sigma) = (j, \sigma')|U^c, D)$$

$$= \mathbb{P}((C^\dagger)_2^{(\ell(C^\dagger))} = (j, \sigma')|U^c)$$

$$= \mathbb{P}(1, \sigma) = (j, \sigma')|U^c).$$
Finally, we condition on $U^c \cap D^c$. We have that
\[
P(C^b = (j, \sigma') | U^c, D^c)
= \mathbb{P}(C^b_1 = j, (C^b_2)_i = (\sigma') | C^b_i = i, D^c)
= \mathbb{I}(j = i) \mathbb{P}(C^b_2) = (\sigma') | U^c, D^c)
= \mathbb{I}(j = i) \sum_{\tau \in C_{n+1}} \mathbb{P}(C^b_2) = \tau | U^c, D^c) \mathbb{P}(C^b_2) = \sigma' | (i, \tau), D^c)
= \mathbb{I}(j = i) \sum_{\tau \in C_{n+1}} p_{i,\alpha,\alpha}^\tau(\sigma, \tau)p_j^\tau(\tau, \sigma')
= \mathbb{I}(j = i) T_{n-1}^\tau(\alpha, \sigma').
\]
Collecting the terms above with the appropriate terms in \([1]\) establishes the result. \(\Box\)

Let $n \geq 1$. We define a transition kernel $A_n$ from $[n]$ to $C_n$ by
\[
A_n(i, (i, \sigma)) = M_{n-i}^{(i, \sigma)}(\sigma),
\]
and a transition kernel $\Phi_n$ from $C_n$ to $[n]$ by
\[
\Phi(\sigma, i) = \mathbb{I}(\sigma_1 = i).
\]

**Proposition 2.4.** For $n \geq 1$, the transition kernel $Q_n(\alpha, 0) = A_n T_n(\alpha, 0) \Phi_n$ satisfies
\[
A_n T_n(\alpha, 0) = Q_n(\alpha, 0) A_n.
\] (5)
Consequently, if the initial distribution of $X_n^{(\alpha, 0)}$ is of the form $\mu A_n$, then $Y_n^{(\alpha, 0)}$ is a time-homogeneous Markov chain with transition kernel $Q_n^{(\alpha, 0)}$. Moreover, the transition kernel $Q_n^{(\alpha, 0)}$ is given explicitly by
\[
Q_n^{(\alpha, 0)}(i, j) = r_{i,j} + r_{1,0} \nu_{\alpha,0}^{(\alpha, 0)}(j) \mathbb{I}(i = 1).
\]
**Proof.** Let $C_n$ be the kernel on $[n]$ defined by the right side of the above equation. Fix $i, j \in [n]$ and $\alpha' \in C_{n-j}$. Using Proposition 2.3 and the identities (2) and (3), we compute
\[
(A_n T_n^{(\alpha, 0)})(i, (j, \sigma')) = \sum_{\sigma \in C_{n-i}} A_n(i, (i, \sigma)) T_n^{(\alpha, 0)}((i, \sigma), (j, \sigma'))
= r_{i,j} \sum_{\sigma \in C_{n-i}} M^{(\alpha, \alpha)}(\sigma) p_{i,\alpha,\alpha}(\sigma, (\alpha') \mathbb{I}(j = i - 1) + p^\tau(\alpha, \alpha') \mathbb{I}(j = i + 1))
+ \mathbb{I}(j = i) \sum_{\sigma \in C_{n-i}} M^{(\alpha, 0)}(\sigma) T_n^{(\alpha, 0)}(\sigma, (\alpha') r^1_{i,\alpha}) + \mathbb{I}(\sigma = \alpha') r^2_{i,\alpha}
+ r_{1,0} \mathbb{I}(i = 1) \sum_{\sigma \in C_{n-i}} M^{(\alpha, 0)}(\sigma) p_{i,\alpha,\alpha}(\sigma, (j, \sigma'))
= r_{i,j} \left(M^{(\alpha, 0)}(\alpha') \mathbb{I}(j = i - 1) + M^{(\alpha, 0)}(\alpha') \mathbb{I}(j = i + 1)\right)
+ \mathbb{I}(j = i) \left(M^{(\alpha, 0)}(\alpha') r^1_{i,\alpha} + M^{(\alpha, 0)}(\alpha') r^2_{i,\alpha}\right) + r_{1,0} \mathbb{I}(i = 1) M^{(\alpha, 0)}(j, \sigma')
= r_{i,j} M^{(\alpha, 0)}(\alpha') + r_{1,0} \mathbb{I}(i = 1) \nu_{\alpha,0}^{(\alpha, 0)}(j) M^{(\alpha, 0)}(\sigma')
= C_n(i, j) A_n(j, (j, \sigma'))
= (C_n A_n)(i, (j, \sigma')).
\]
The final equality follows from the fact that \( \Lambda_n(j, \cdot) \) is supported on \( \{ \sigma \in C_n : \sigma_1 = j \} \). This establishes the identity \( \Lambda_n \Phi_n = C_n \Lambda_n \). Observing that \( \Lambda_n \Phi_n \) is the identity kernel on \([n]\), we find that

\[
Q_n^{(\alpha, 0)} = \Lambda_n T_n^{(\alpha, 0)} \Phi_n = C_n \Lambda_n \Phi_n = C_n,
\]

from which we obtain \([5]\) and the explicit description of \( Q_n^{(\alpha, 0)} \). The final claim follows from applying Theorem 2 in \([22]\).

3 Convergence from Commutation Relations

In this section, we provide a condition under which commutation relations between operators implies the convergence of those operators in an appropriate sense. In the interest of generality, we first state this condition in the setting of Banach spaces, but we then reformulate it in the context of Markov processes to suit our purposes. The general setting is as follows.

Let \( V, V_1, V_2, \ldots \) be Banach spaces and \( \pi_1, \pi_2, \ldots \) be uniformly bounded linear operators with \( \pi_n : V \to V_n \). These spaces will be equipped with the following mode of convergence.

**Definition 3.1.** A sequence \( \{f_n\}_{n \geq 1} \) with \( f_n \in V_n \) converges to an element \( f \in V \) (and we write \( f_n \to f \)) if

\[
\|f_n - \pi_n f\|_{n \to \infty} \to 0,
\]

where for convenience, we denote every norm by the same symbol \( \| \cdot \| \).

**Proposition 3.1.** For \( n \geq 1 \), let \( L_n : D_n \subset V \to V_n \) and \( A_n : V_n \to V_n \) be linear operators in addition to \( A : D \subset V \to D \). Suppose that for every \( f \in D \),

1. \( A_n L_n f = L_n A f \) for large \( n \), and
2. \( (L_n - \pi_n)f \to 0 \) as \( n \to \infty \) (the sequence need only be defined for large \( n \)).

Then for \( f \in D \), the sequence \( f_n = L_n f \) (defined for large \( n \)) satisfies

\[
f_n \to f \quad \text{and} \quad A_n f_n \to A f.
\]

**Proof.** Let \( f \in D \) and \( n \) be large enough so that \( (i) \) holds. In particular, we can define \( f_n = L_n f \). Writing

\[
\|f_n - \pi_n f\| = \|L_n f - \pi_n f\|
\]

it is clear that \( f_n \to f \). Writing

\[
\|A_n f_n - \pi_n A f\| = \|A_n L_n f - \pi_n A f\|
\]

\[
= \|L_n A f - \pi_n A f\|
\]

\[
= \|(L_n - \pi_n) A f\|
\]

and noting that \( A f \in D \), we obtain the other convergence.

In the probabilistic context, the above result has some additional consequences.

**Theorem 3.1.** Let \( E \) be a compact, separable metric space, \( A \) be the generator of the Feller semigroup \( S(t) \) on \( C(E) \), and \( D \) be a core for \( A \) that is invariant under \( A \). For each \( n \geq 1 \), let \( E_n \) be a finite set endowed with the discrete topology, \( Z_n \) be a Markov chain on \( E_n \), \( \gamma_n : E_n \to E \) be any function, and \( L_n : D_n \subset C(E) \to C(E_n) \) be a linear operator. Denote the transition operator of \( Z_n \) by \( S_n \) and the projection \( f \mapsto f \circ \gamma_n \) by \( \pi_n : C(E) \to C(E_n) \).

Let \( \{ \delta_n \}_{n \geq 1} \) and \( \{ \varepsilon_n \}_{n \geq 1} \) be positive sequences converging to zero such that \( \varepsilon_n^{-1} \delta_n \to 1 \). Suppose that for \( f \in D \), the following statements hold:

1. \( \delta_n^{-1}(S_n - 1)L_n f = L_n A f \) for large \( n \), and
2. \( (L_n - \pi_n)f \to 0 \) as \( n \to \infty \) (the sequence need only be defined for large \( n \)).

Then,
(i) the discrete semigroups \( \{1, S_n, S_n^2, \ldots \}_{n \geq 1} \) converge to \( \{S(t)\}_{t \geq 0} \) in the following sense: for all \( f \in C(E) \) and \( t \geq 0 \),
\[
S_n^{(t/\varepsilon_n)} \pi_n f \xrightarrow{n \to \infty} S(t)f
\]

(ii) the above convergence is uniform in \( t \) on bounded intervals, and

(iii) if \( A \) is conservative and the distributions of \( \gamma_n(Z_n(0)) \) converge, say to \( \mu \), then we have the convergence of paths
\[
\gamma_n(Z_n(t/\varepsilon_n)) \xrightarrow{\varepsilon_n \to 0} F(t)
\]
in the Skorokhod space \( D([0, \infty), E) \), where \( F(t) \) is a Feller process with initial distribution \( \mu \) and generator \( A \).

Proof. This is a combination of Proposition 3.1 and standard convergence results. In particular, for \( f \in D \), we can define the sequence \( f_n = L_n f \) for large \( n \) and obtain the convergence
\[
f_n \to f \quad \text{and} \quad \delta_n^{-1}(S_n - 1)f_n \to Af.
\]
Recalling that \( \varepsilon_n^{-1}\delta_n \to 1 \), we then obtain the convergence \( \varepsilon_n^{-1}(S_n - 1)f_n \to Af \). Applying Chapter 1 Theorem 6.5 in [4] then yields the convergence of semigroups in \([i]\) and \([iii]\).

Applying Chapter 4 Theorem 2.12 in [4] yields the path convergence in \([iii]\).

4 The Limiting Generator

In this section, we introduce the generator of a Feller process on \([0, 1]\) that will be identified as the limiting process. We describe this generator both on a core of polynomials and on its full domain. However, the core description is sufficient for the analysis that will follow.

Let \( \mathcal{P} \) denote the space of polynomials on \([0, 1]\) equipped with the supremum norm. We will study the operator \( \mathcal{B}: \mathcal{P} \to \mathcal{P} \) and the functional \( \eta: \mathcal{P} \to \mathbb{R} \) given by
\[
(\mathcal{B}f)(x) = x(1 - x)f''(x) - \alpha f'(x), \quad x \in [0, 1],
\]
and
\[
\eta(f) := \int_0^1 (f(x) - f(0))x^{-\alpha - 1}(1 - x)^{\alpha - 1} \, dx
\]
\[
= \int_0^1 f'(x)x^{-\alpha - 1}(1 - x)^{\alpha - 1} \, dx.
\]

Letting \( \mathbb{N} = \{0, 1, 2, \ldots\} \), we define a family of polynomials \( \{h_n\}_{n \in \mathbb{N} \setminus \{1\}} \) by
\[
h_n(x) = \sum_{s=0}^{n} x^s (-1)^{n-s} \binom{n-1}{s} \binom{s - \alpha}{n-s} x^{n-s}, \quad x \in [0, 1].
\]
Note that \( h_0 \equiv 1 \) and \( h_n \) has degree \( n \). Moreover, these polynomials are related to the Jacobi polynomials \( P_n^{(a, b)} \) and the shifted Jacobi polynomials \( J_n^{(a, b)} \) by the identity
\[
h_n(x) = J_n^{(\alpha - 1, -\alpha - 1)}(x) = P_n^{(\alpha - 1, -\alpha - 1)}(2x - 1), \quad x \in [0, 1].
\]

Proposition 4.1. Let \( \mathcal{H} = \ker \eta \) and \( \omega_n = -n(n-1) \) for \( n \in \mathbb{N} \setminus \{1\} \). The following statements hold:

(i) \( \mathcal{B}h_n = \omega_n h_n \) for all \( n \in \mathbb{N} \setminus \{1\} \),

(ii) the family \( \{h_n\}_{n \in \mathbb{N} \setminus \{1\}} \) is a Hamel basis for \( \mathcal{H} \), and

(iii) \( \mathcal{H} \) is a dense subspace of \( C[0, 1] \).

Proof. The claim in \([i]\) can be obtained from the classical theory of Jacobi polynomials (e.g. (4.1.3), (4.21.2), and (4.21.4) in [24]).

Noting that \( h_n \) has degree \( n \) shows that the family \( \{h_n\}_{n \in \mathbb{N} \setminus \{1\}} \) is independent. Since \( h_0 \equiv 1 \), it clearly lies in \( \mathcal{H} \). To see that the other \( h_n \) also lie in \( \mathcal{H} \), we use \([i]\) to identify
them as elements in the range of $B$ and observe that this range lies in $H$. Indeed, this can be verified using (6): for $f \in P$, we have that

$$\eta(Bf) = \int_0^1 (x(1-x)f''(x) - \alpha f'(x) + \alpha f'(0))x^{-\alpha-1}(1-x)^{\alpha-1} \, dx$$

$$= \int_0^1 f''(x)x^{-\alpha}(1-x)^{\alpha} \, dx - \alpha \int_0^1 (f'(x) - f'(0))x^{-\alpha-1}(1-x)^{\alpha-1} \, dx$$

$$= \alpha \eta(f') - \alpha \eta(f')$$

$$= 0.$$ 

To obtain equality from the containment $\text{span}\{h_n\}_{n \in \mathbb{N}\setminus\{1\}} \subset H$, we observe that the former space is a maximal subspace of $P$ (it has codimension one) while the latter is a proper subspace of $P$.

The claim in (iii) will follow from showing that $\eta$ is not continuous (see Chapter 3 Theorem 2 in [11]). To see that this holds, notice that the functions $f_j(x) = (1-x)^j$, $j \geq 1$, have norm 1 but their images under $\eta$ are unbounded:

$$\eta(f_j) = -\int_0^1 jx^{-\alpha}(1-x)^{j-1+\alpha} \, dx$$

$$= -\frac{j(1-\alpha)\Gamma(j+\alpha)}{\alpha \Gamma(j)}.$$ 

**Proposition 4.2.** The operator $B|_H$ is closable and its closure, $\overline{B|_H}$, is the generator of a Feller semigroup on $C[0,1]$.

**Proof.** We show that $B|_H$ satisfies the conditions of the Hille-Yosida Theorem. For $\lambda > 0$, Proposition 4.1(iii) show that the range of $\lambda - B|_H$ is exactly $H$. Proposition 4.1(ii) then tells us that this range, as well as the domain of $B|_H$, is dense in $C[0,1]$.

To establish the positive-maximum principle, suppose that $f \in H$ has a nonnegative maximum at $y \in [0,1]$. If $y \neq 0$, the tools of differential calculus show that $(B|_H)f(y) \leq 0$, as desired. When $y = 0$, consider the element $F \in L^1[0,1]$ given by

$$F(x) = (f(x) - f(0))x^{-\alpha-1}(1-x)^{\alpha-1}$$

almost everywhere. Since $f(x) \leq f(0)$ on $[0,1]$, the norm of $F$ is given by

$$\|F\|_1 = \int_0^1 |f(x) - f(0)|x^{-\alpha-1}(1-x)^{\alpha-1} \, dx$$

$$= -\int_0^1 (f(x) - f(0))x^{-\alpha-1}(1-x)^{\alpha-1} \, dx$$

$$= -\eta(f).$$

Recalling that $f \in H = \ker \eta$, it follows that $F = 0$ almost everywhere. Together with the continuity of $f$, this implies that $f \equiv f(0)$, and consequently, $(B|_H)f(y) \leq 0$. 

The final result in this section is the explicit description of the generator $\overline{B|_H}$ and its domain $\text{Dom}(\overline{B|_H})$.

To begin, we define an operator $\hat{L} : C[0,1] \cap C^2(0,1) \to C(0,1)$ by

$$\hat{L}f(x) = x(1-x)f''(x) - \alpha f'(x).$$

We will write $\hat{L}f \in C[0,1]$ whenever $\hat{L}f$ can be continuously extended to $[0,1]$. Recalling the definition of $L$ and $D$ from Theorem 1.4, we see that $L$ is the restriction of $\hat{L}$ to $D$. We also define functions $m : (0,1] \to \mathbb{R}$ and $s : (0,1] \to \mathbb{R}$ by

$$m(x) = \int_1^x t^{-1-\alpha}(1-t)^{\alpha-1} \, dt = -\alpha^{-1}x^{-\alpha}(1-x)^{\alpha-1}$$

9
and 
\[ s(x) = \int_1^x t^n(1-t)^{-\alpha} dt. \]

Note that \( \hat{L} \) admits the factorization
\[ \hat{L}f = \frac{1}{m'} \left( \frac{f'}{s'} \right)', \]
from which we obtain the formula
\[ f(x) - f(c) = \frac{f'(c)}{s'(c)}(s(x) - s(c)) + \int_c^x \int_c^y \hat{L}f(z)m'(z)dz\, s'(y)dy, \quad x, c \in (0, 1). \]  
(7)

Another identity that will be useful is
\[ \int_1^y m'(z)dz\, s'(y) = m(y) s'(y) = -\alpha^{-1}, \quad y \in (0, 1). \]  
(8)

Proposition 4.3. The identity \( B|_H = L \) holds, where \( L \) is as defined in Theorem 1.1

Proof. We begin by showing that the following holds:
\[ f(x) - f(1) = \int_1^x \int_1^y Lf(z)m'(z)dz\, s'(y)dy, \quad f \in D, \, x \in [0, 1]. \]  
(9)

To do this, we will take limits in (7). First we take the limit \( c \to 1 \). The term \( \frac{f'(c)}{s'(c)} \) converges to zero due to (D3) (see Theorem 1.1). The limit of the integral is handled by the dominated convergence theorem. A preliminary bound can be obtained from (8) and (11):
\[ \lim_{n \to \infty} \int_1^x \int_1^y \hat{L}f(z)m'(z)dz\, s'(y)dy = 0 \]
(10)

Now we show that \( \text{Dom}(B|_H) \subseteq D \). Fixing \( f \in \text{Dom}(B|_H) \), there exists a sequence \( \{f_n\}_{n \geq 1} \) of functions in \( H \) such that
\[ f_n \to f \quad \text{and} \quad Bf_n \to B|_H f. \]  
(10)

Noting that \( f_n \in D \) for all \( n \), we can apply (9). In this case, the identity \( Bf_n = Lf_n \) yields
\[ f_n(x) - f_n(1) = \int_1^x \int_1^y Bf_n(z)m'(z)dz\, s'(y)dy, \quad x \in [0, 1]. \]  
(11)

Using (10) and the dominated convergence theorem, we can take the limit \( n \to \infty \) above. A suitable bound follows from the boundedness of the sequence \( \{Bf_n\} \) and (8). We obtain
\[ f(x) - f(1) = \int_1^x \int_1^y B|_H f(z)m'(z)dz\, s'(y)dy, \quad x \in [0, 1]. \]  
(12)

Together with the fact that \( B|_H f \in C(0,1), m \in C^1(0,1) \) and \( s \in C^2(0,1) \), this expression implies that \( f \in C^2(0,1) \). Differentiating the expression yields the identity
\[ B|_H f = \frac{1}{m'} \left( \frac{f'}{s'} \right)' = \hat{L}f \quad \text{on} \,(0,1). \]  
(13)

This shows that \( f \) satisfies (D1). To obtain (D2), we recall that
\[ \int_0^1 (f_n(x) - f_n(0))x^{-\alpha^{-1}}(1-x)^{-\alpha^{-1}} dx = 0 \]
for all \( n \) and extend this to \( f \) by taking the limit \( n \to \infty \). Once again, we apply the dominated convergence theorem. A preliminary bound can be obtained from (8) and (11):
\[ |x^{-1}(f_n(x) - f_n(0))| = x^{-1} \left| \int_0^x \int_1^y Bf_n(z)m'(z)dz\, s'(y)dy \right| \leq x^{-1} \|Bf_n\| \int_0^x \int_1^y m'(z)dz\, s'(y)dy = \|Bf_n\|s^{-1}. \]  
(13)
The boundedness of the sequence \( \{Bf_n\} \) then provides a suitable bound.

To obtain (D3), we differentiate (12) and compute

\[
\left| \frac{f'(x)}{s'(x)} \right| = \left| \int_1^x \overline{B|H}f(z)m'(z)dz \right|
\]

\[
\leq \|\overline{B|H}f\| \int_x^1 m'(z)dz
\]

\[
= \|\overline{B|H}f\|((-m(x)) \to 0).
\]

We have shown that \( \text{Dom}(\overline{B|H}) \subset D \) and \( \overline{B|H} = L \) on \( \text{Dom}(\overline{B|H}) \) (see (13)). Therefore, it only remains to show that \( \text{Dom}(\overline{B|H}) = D \).

Recall the Bernstein polynomials

\[
b_{i,k}(x) = \binom{k}{i} x^i (1-x)^{k-i}, \quad i \in \mathbb{Z}, \ k \geq 0.
\]

Note that \( b_{i,k} \equiv 0 \) whenever \( i < 0 \) or \( i > k \). For each \( k \geq 0 \), the collection \( \{b_{i,k}\}_{i=0}^k \) forms a basis of \( \mathcal{P}_k \) and a partition of unity – that is, \( \sum_{i=0}^k b_{i,k} \equiv 1 \).

Similarly, define

\[
\mathcal{H}_k = \mathcal{H} \cap \mathcal{P}_k, \quad k \geq 0.
\]

Recall the Bernstein polynomials

\[
b_{i,k}(x) = \binom{k}{i} x^i (1-x)^{k-i}, \quad i \in \mathbb{Z}, \ k \geq 0.
\]

Note that \( b_{i,k} \equiv 0 \) whenever \( i < 0 \) or \( i > k \). For each \( k \geq 0 \), the collection \( \{b_{i,k}\}_{i=0}^k \) forms a basis of \( \mathcal{P}_k \) and a partition of unity – that is, \( \sum_{i=0}^k b_{i,k} \equiv 1 \). We also have the relations

\[
b'_{i,k} = k(b_{i-1,k-1} - b_{i,k-1}),
\]

\[
b_{i,k} = \frac{k+1-i}{k+1} b_{i,k+1} + \frac{i+1}{k+1} b_{i+1,k+1},
\]

and

\[
x(1-x) b_{i,k} = \frac{(i+1)(k-i)}{(k+1)(k+2)} b_{i+1,k+2},
\]

which hold whenever the relevant quantities are defined.

For \( n \geq 1 \), we define a transition kernel from \([0,1]\) to \([n]\) by

\[
K_n(x,i) = b_{i,n}(x) + \varphi_n^{(a,a)}(i) b_{0,n}(x).
\]
Proposition 5.1. Let \( n \geq 1 \). As an operator from \( C([n]) \) to \( C[0,1] \), \( K_n \) is injective and
\[
\mathcal{H}_n = \left\{ \sum_{j=0}^{n} c_j b_{j,n} : c_0, \ldots, c_n \in \mathbb{R}, \quad c_0 = \sum_{j=1}^{n} \nu_{\alpha,\alpha}^{(1)}(j)c_j \right\}
\]
(17)
\[
= \text{range } K_n.
\]
(18)

Proof. Let \( n \geq 1 \). From the independence of the Bernstein polynomials and the identity
\[
\text{range } K_n = \text{span}\{b_{i,n}(x) + \nu_{\alpha,\alpha}^{(1)}(i)b_{0,n}(x)\}_{i=1}^{n},
\]
it follows that the range of \( K_n \) is an \( n \)-dimensional space. As a result, \( K_n \) is injective. Observing that the right hand side of (17) has dimension at most \( n \) and contains the range of \( K_n \), it follows that these two spaces are equal. Since \( \mathcal{H}_n \) also has dimension \( n \) (see Proposition 4.15), it only remains to show that the range of \( K_n \) is contained in \( \mathcal{H}_n \). The containment in \( \mathcal{H}_n \) is clear. For the containment in \( \mathcal{H} \), we simply compute, for \( i \in [n] \),
\[
\eta(b_{i,n}(x) + \nu_{\alpha,\alpha}^{(1)}(i)b_{0,n}(x))
\]
\[
= \binom{n}{i} \int_{0}^{1} x^{i-\alpha}(1-x)^{n-i-1} dx - na^{-1}\nu_{\alpha,\alpha}^{(1)}(i) \int_{0}^{1} x^{-\alpha}(1-x)^{n-i-1} dx
\]
\[
= \binom{n}{i} \frac{\Gamma(i-\alpha)\Gamma(n-i+\alpha)}{\Gamma(n)} - na^{-1}\nu_{\alpha,\alpha}^{(1)}(i) \frac{\Gamma(1-\alpha)\Gamma(n+\alpha)}{\Gamma(n+1)}
\]
\[
= 0.
\]
\]

Proposition 5.2. The action of \( B \) on the Bernstein polynomials is given by
\[
Bb_{i,n} = n(n+1) \sum_{k=0}^{n} (r_{k,i} - 1(k = i)) b_{k,n}, \quad 0 \leq i \leq n.
\]

Proof. Let \( n \geq 2 \) and \( 0 \leq i \leq n \). Applying (14) twice, we see that
\[
b''_{i,n} = n(b'_{i-1,n-1} - b'_{i,n-1})
\]
\[
= n(n-1) (b_{i-2,n-2} - 2b_{i-1,n-2} + b_{i,n-2}).
\]

Applying now (16), we have that
\[
x(1-x)b''_{i,n}(x)
\]
\[
= n(n-1) \left( \frac{(i-1)(n+1-i)}{(n-1)n} b_{i-1,n}(x) - \frac{n(n-2)}{(n-1)n} b_{i,n}(x) + \frac{i(n+1-i)}{(n-1)n} b_{i+1,n}(x) \right)
\]
(19)
\[
= (i-1)(n+1-i) b_{i-1,n}(x) - 2i(n-i) b_{i,n}(x) + (i+1)(n-1-i) b_{i+1,n}(x)
\]

Using (14) and (15), we find that
\[
b'_{i,n} = n(b_{i-1,n-1} - b_{i,n-1})
\]
\[
= n \left( \frac{n+1-i}{n} b_{i-1,n} + \frac{i}{n} b_{i,n} - \frac{n-1}{n} b_{i,n} - \frac{i+1}{n} b_{i+1,n} \right)
\]
(20)
\[
= (n+1 - i) b_{i-1,n} + (2i - n) b_{i,n} - (i + 1) b_{i+1,n}.
\]

As a result,
\[
Bb_{i,n} = (i-1 - \alpha)(n+1-i) b_{i-1,n} - (\alpha(2i - n) + 2i(n-i)) b_{i,n}
\]
\[
+ (i+1)(n-1-i+\alpha) b_{i+1,n}
\]
\[
= n(n+1) \left( r_{i-1,n} b_{i-1,n} + (r_{i,i} - 1) b_{i,n} + r_{i+1,i} b_{i+1,n} \right)
\]
\[
= n(n+1) \sum_{k=i-1}^{i+1} (r_{k,i} - 1(k = i)) b_{k,n}.
\]

Recalling that \( r_{k,i} - 1(k = i) \) is zero unless \( i-1 \leq k \leq i+1 \) and \( b_{k,n} \equiv 0 \) unless \( 0 \leq k \leq n \), we can change the lower and upper limits of the sum to 0 and \( n \), respectively. This establishes the \( n \geq 2 \) case. When \( n = 1 \), we observe that (20) still holds and the first and last quantities of (19) are still equal. When \( n = 0 \), the claim is trivial. \( \square \)
Proposition 5.3. For \( n \geq 1 \), the following relation holds on \( C([n]) \):

\[
\mathcal{B}K_n = K_n n(n+1)(Q_n^{(\alpha,0)} - 1).
\]

Proof. Let \( n \geq 1 \) and \( i \in [n] \). Define \( e_i : [n] \to \mathbb{R} \) by \( e_i = \mathbb{1}(i = \cdot) \). From Proposition 5.2, we have that

\[
n^{-1}(n+1)^{-1}\mathcal{B}K_n e_i = n^{-1}(n+1)^{-1}\mathcal{B}(b_{k,n} + b_{0,n} \nu_n^{(\alpha,\alpha)}(i))
\]

\[
= \sum_{k=0}^{n}(r_{k,i} - \mathbb{1}(k = i) + \nu_n^{(\alpha,\alpha)}(i)(r_{k,0} - \mathbb{1}(k = 0))) b_{k,n}
\]

\[
= (r_{0,i} + \nu_n^{(\alpha,\alpha)}(i)(r_{0,0} - 1)) b_{0,n} + \sum_{k=1}^{n}(r_{k,i} - \mathbb{1}(k = i) + \nu_n^{(\alpha,\alpha)}(i)r_{1,0} \mathbb{1}(k = 1)) b_{k,n}.
\]

On the other hand, Proposition 2.4 gives us that

\[
K_n(Q_n^{(\alpha,0)} - 1)e_i = \sum_{k=1}^{n}(b_{k,n} + b_{0,n} \nu_n^{(\alpha,\alpha)}(k))(Q_n^{(\alpha,0)} - 1)e_i(k)
\]

\[
= \sum_{k=1}^{n}(b_{k,n} + b_{0,n} \nu_n^{(\alpha,\alpha)}(k))(Q_n^{(\alpha,0)} - 1)(k,i)
\]

\[
= \sum_{k=1}^{n}(b_{k,n} + b_{0,n} \nu_n^{(\alpha,\alpha)}(k))(r_{k,i} - \mathbb{1}(i = k) + \nu_n^{(\alpha,\alpha)}(i)r_{1,0} \mathbb{1}(k = 1)).
\]

To show that the two expressions are equal, it will suffice to show that the coefficients of \( b_{k,n} \) are the same in each. For \( k \geq 1 \), this is immediate. For \( k = 0 \), we observe that each of the above functions lies in \( H_n \) (see Proposition 4.1 and (18)) and apply (17). \( \square \)

6 The Convergence Argument

In this section, we verify the convergence condition appearing in Theorem 5.1. We rely on a description of the inverse of the transition operator \( K_n \) in terms of a variant of the Bernstein polynomials.

These variants fall into the class of degenerate Bernstein polynomials [14] and are given by

\[
b_{i,k,n}(x) = {k \choose i} \frac{(nx)^i(n-x)^{i(k-i)}}{n^k}, \quad 0 \leq i \leq k \leq n.
\]

Proposition 6.1. For \( k \geq i \geq 0 \), we have the expansions

\[
b_{i,k} = \sum_{j=0}^{n} b^*_{i,k,n}(\frac{j}{n})b_{j,n}, \quad n \geq k.
\]

Proof. The expansions of a Bernstein polynomial in the Bernstein bases are given in Equation (2) in [19]. Let us verify that the coefficients in those expansions match the coefficients in the above expansions. Fix \( n \geq k \geq i \geq 0 \). The coefficient of \( b_{j,n} \) in the above expansion is given by

\[
b^*_{i,k,n}(\frac{j}{n}) = {k \choose i} \frac{j^i(n-j)^{i(k-i)}}{n^k}.
\]

When \( j < i \) or \( j > n-k+i \), it is clear that this coefficient is zero. If instead \( i \leq j \leq n-k+i \),
this coefficient is reduces to
\[
\binom{k}{i} j^{i} (n - j)^{i (k - i)} \frac{n!}{j^{i} (n-j)^{i (k-j)}} = \binom{k}{i} \prod_{j=1}^{\min(n,k)} \left( \frac{n-j}{j} \right) ^{i}
\]

In either case, this coefficient agrees with the coefficient in \([19]\). \(\square\)

Let \(\iota_n: [n] \to [0,1]\) be defined by \(j \mapsto \frac{j}{n}\) and \(\rho_n: C[0,1] \to C[n]\) be the associated projection, \(f \mapsto f \circ \iota_n\).

**Proposition 6.2.** For \(n \geq k \geq i \geq 1\), we have the identity
\[
K_n \rho_n (b^*_{i,k,n} + \nu^{(\alpha,\alpha)}_k(i) b^*_{0,k,n}) = b_{i,k} + \nu^{(\alpha,\alpha)}_k(i) b_{0,k}.
\]

**Proof.** It follows from definition that
\[
K_n \rho_n (b^*_{i,k,n} + \nu^{(\alpha,\alpha)}_k(i) b^*_{0,k,n}) = \sum_{j=1}^{n} (b_{j,n} + \nu^{(\alpha,\alpha)}_n(j) b_{0,n}) (b^*_{i,k,n}(\frac{j}{n}) + \nu^{(\alpha,\alpha)}_k(i) b^*_{0,k,n}(\frac{j}{n})).
\]

Meanwhile, Proposition 6.1 gives us the expansion
\[
b_{i,k} + \nu^{(\alpha,\alpha)}_k(i) b_{0,k} = \sum_{j=0}^{n} (b^*_{i,k,n}(\frac{j}{n}) + \nu^{(\alpha,\alpha)}_k(i) b^*_{0,k,n}(\frac{j}{n})) b_{j,n}.
\]

Upon comparison, we find that the coefficient of \(b_{j,n}\) is the same in both expressions whenever \(j \geq 1\). Since both functions lie in \(H_n\), the coefficients of \(b_{0,n}\) must agree as well (see \([17]\)). As a result, the two functions are equal. \(\square\)

**Proposition 6.3.** For \(k \geq i \geq 0\), we have the convergence
\[
b^*_{i,k,n} \xrightarrow{n \to \infty} b_{i,k}.
\]

**Proof.** We write
\[
b^*_{i,k,n}(x) = \binom{k}{i} \frac{1}{n^k} \prod_{r=0}^{i-1} (nx - r) \prod_{s=0}^{k-i-1} (n - nx - s) \]
\[
= \binom{k}{i} \frac{n^k}{n^{k-i}} \prod_{r=0}^{i-1} \frac{x - r}{n} \prod_{s=0}^{k-i-1} \frac{1 - x - s}{n},
\]

and handle each factor separately. The constants \(\frac{n^k}{n^{k-i}}\) converge to 1 and each factor in a product converges to either \(u(x) = x\) or \(v(x) = 1 - x\). \(\square\)

**Proposition 6.4.** Let \(f \in \mathcal{H}\) and fix \(m \geq 1\) such that \(f \in \mathcal{H}_m\). Then we have the convergence
\[
(K^{-1}_n - \rho_n) f \xrightarrow{n \geq m \to \infty} 0
\]
in the sense of Definition 3.1.

**Proof.** It suffices to consider the case when \(f = b_{i,k} + \nu^{(\alpha,\alpha)}_k(i) b_{0,k}\) for some \(i\) and \(k\) satisfying \(1 \leq i \leq k\). Defining \(f_n = b^*_{i,k,n} + \nu^{(\alpha,\alpha)}_k(i) b^*_{0,k,n}\) for \(n \geq 1\), it follows from Proposition 6.2 that
\[
(K^{-1}_n - \rho_n) f = \rho_n (f_n - f).
\]

Since the \(\rho_n\) are uniformly bounded, the result follows from Proposition 6.3. \(\square\)
7 Semigroup Relations from Generator Relations

In this section, we provide general conditions under which commutation relations involving generators lead to the corresponding relations for their semigroups.

**Theorem 7.1.** Let \( A \) and \( B \) be the generators of the Feller semigroups \( V_t \) and \( W_t \), respectively, and let \( \mathcal{E} \) and \( \mathcal{F} \) denote their respective domains. Suppose that there is a subspace \( E \subset \mathcal{E} \), a linear operator \( L : E \to \mathcal{F} \), and a set \( I \subset (0, \infty) \) such that

1. \( L \) is bounded,
2. \( I \) is unbounded,
3. \( E \subset (\lambda - A)E \) for \( \lambda \in I \), and
4. \( LA = BL \) on \( E \).

Then \( LV_t = W_t L \) on \( E \) for each \( t \geq 0 \).

**Proof.** Fix \( \lambda \in I \) and let \( R_A^\lambda \) and \( R_B^\lambda \) be the resolvent operators corresponding to \( A \) and \( B \) respectively. It follows from (iii) that \( E \) is invariant under \( R_A^\lambda \). Combining this with (iv), we obtain the following relation on \( E \):

\[
R_B^\lambda L = R_B^\lambda (\lambda - B)L R_A^\lambda = LR_A^\lambda.
\]

It then follows easily that

\[
L\lambda(\lambda R_A^\lambda - I) = \lambda(\lambda R_B^\lambda - I)L \quad \text{on } E,
\]

or equivalently, \( LA_\lambda = B_\lambda L \) on \( E \), where \( A_\lambda \) and \( B_\lambda \) are the Yosida approximations of \( A \) and \( B \) respectively. Noting that \( E \) is invariant under \( A_\lambda \), this extends to nonnegative integers \( k \):

\[
LA_\lambda^k = B_\lambda^k L \quad \text{on } E.
\]

Applying now (i), we have for \( f \in E \) and \( t \geq 0 \) the identity

\[
Le^{tA_\lambda} f = \sum_{k=0}^{\infty} \frac{t^k}{k!} (A_\lambda^k f)
= \sum_{k=0}^{\infty} \frac{t^k}{k!} (LA_\lambda^k f)
= \sum_{k=0}^{\infty} \frac{t^k}{k!} (B_\lambda^k L f)
= e^{tB_\lambda} L f.
\]

Letting \( \lambda \) become arbitrarily large (see (ii)) yields \( LV_t f = W_t L f \). This establishes the result on \( E \). The extension to \( \overline{E} \) follows from the boundedness of \( L \).

**Corollary 7.1.** Let \( A \) and \( B \) be the generators of the Feller semigroups \( V_t \) and \( W_t \), respectively, and let \( \mathcal{E} \) and \( \mathcal{F} \) denote their respective domains. Suppose that there is a subspace \( E \subset \mathcal{E} \), a linear operator \( L : E \to \mathcal{F} \), and a filtration of \( E \) by finite dimensional spaces \( \{ E_k \}_{k \geq 1} \) such that

1. \( AE_k \subset E_k \) for all \( k \), and
2. \( LA = BL \) on \( E \).

Then \( LV_t = W_t L \) on \( E \) for each \( t \geq 0 \).

**Proof.** Let \( k \geq 1 \). It follows from (i) that \( E_k \) is invariant under the injective operators \( \{ \lambda - A \}_{\lambda > 0} \). Together with the fact that \( E_k \) is finite-dimensional, this implies that

\[
(\lambda - A)E_k = E_k, \quad \lambda > 0.
\]
Letting $L_k: E_k \rightarrow \mathcal{F}$ denote the restriction of $L$ to $E_k$, it follows from (i) and (ii) that

$$L_k A = B L_k \quad \text{on } E_k.$$ 

Since $E_k$ is finite-dimensional, $L_k$ is bounded and $E_k = E_k$. Applying Theorem 7.1 we find that $LV_t = W_t L$ on $E_k$ for each $t \geq 0$. Taking a union over $k$ extends the identity to $E$. 

8 Proofs of Main Results

Proof of Theorem 1.3 The first claim was proved in Proposition 2.4. For the second claim, we appeal to Corollary 4.1. We take $A = n(n+1)(Q_n^{(0,0)} - 1)$, $B = \mathcal{L}$, $L = K_n$, and $E = C([0]) = E_k$ for all $k$. The containment $AE_k \subset E_k$ holds trivially and the identity $LA = BL$ was established in Proposition 5.3. Applying Corollary 7.1 we obtain the desired identity in terms of transition operators, which implies the same relation in terms of transition kernels.

Proof of Theorem 1.2 The claim about the existence of initial distributions for $X_n^{(a,0)}$ follows from Theorem 1.3. The second claim follows from applying Theorem 3.1 with $E = [0,1]$, $A = \mathcal{L}$, $D = H_n$, $E_n = [1]$, $Z_n = Y_n^{(a,0)}$, $\gamma_n(j) = \frac{1}{n}$, $D_n = H_n$, $L_n = K_n$, $\delta_n^{-1} = n(n+1)$, and $\epsilon_n^{-1} = n^2$. To verify that $A$ is the generator of a conservative Feller semigroup on $C[0,1]$, $D$ is a core for $A$, and $D$ is invariant under $A$, we appeal to Propositions 3.1, 4.2 and 4.1. Condition (a) can be obtained from the identity in Proposition 5.3 by recalling that $K_n$ is injective (see Proposition 5.1) and that each $f$ in $D = H$ lies in $D_n = H_n$ for large $n$. Condition (b) is exactly the result of Proposition 6.4.

Proof of Theorem 1.4 Define $\iota : C \rightarrow \mathcal{U}$ by

$$\iota(\sigma) = \left(0, \frac{\sigma_1}{|\sigma|}, \ldots, \frac{\sigma_n}{|\sigma|}, 1\right).$$

From 20 Theorem 1.3, we have that if

$$\iota(X_n^{(a,\theta)}(0)) \Rightarrow X^{(a,\theta)}(0),$$

then

$$\left(\iota(X_n^{(a,\theta)}([n^2 t]))\right)_{t \geq 0} \Rightarrow \left(X^{(a,\theta)}(t)\right)_{t \geq 0},$$

where $|a|$ is the integer part of $a$ and the convergence is in distribution on the Skorokhod space $D([0,\infty), \mathcal{U})$, where the metric on $\mathcal{U}$ is given by the Hausdorff distance between the complements (complements being taken in $[0,1]$). If $\xi$ were continuous, the result would follow immediately, but $\xi$ is discontinuous. However, it is straightforward to show that if $u_n \rightarrow u$ in $\mathcal{U}$ and $\xi(u_n) \rightarrow c > 0$, then $\xi(u) = c$.

Assuming now that $X_n^{(a,0)}$ is running in stationarity, the fact that $\iota(X_n^{(a,0)}(0))$ converges in distribution to an $(a,0)$ Poisson-Dirichlet interval partition distribution follows from [18] and the fact that $\phi(X_n^{(a,0)})$ is a Markov chain follows from Theorem 1.3. Observe that $(p_n^{(a,0)})^{n-1}((1), \cdot)$ is the stationary distribution of $X_n^{(a,0)}$ and, in the $(a,0)$ ordered Chinese Restaurant Process growth step, no new table is ever created at the start of the list. Thus, for every $k$, $\phi(X_n^{(a,0)}(k))$ is distributed like the size of the table containing 1 in the usual $(a,0)$ Chinese Restaurant Process after $n$ customers are seated, see [17]. Consequently, since our chain is stationary, for each $t$,

$$\frac{1}{n} \phi(X_n^{(a,0)}([n^2 t])) = \xi(\iota(X_n^{(a,0)}([n^2 t]))) = d \xi(\iota(X_n^{(a,0)}(0))) \Rightarrow W,$$

where $W$ has a Beta$(1 - a, a)$ distribution, see [17].

Therefore, from Theorem 1.2 with $F$ as defined there and $F(0) = d W$, passing to a subsequence if necessary, and using the Skorokhod representation theorem, we may assume that

$$\left(\left(\iota(X_n^{(a,0)}([n^2 s])), \xi(\iota(X_n^{(a,0)}([n^2 s])))\right)\right)_{t,s \geq 0} \Rightarrow \left((X^{(a,0)}(t), F(s))\right)_{t,s \geq 0}.$$
in $D([0, \infty), \mathcal{U}) \times D([0, \infty), [0, 1])$. Fix $t \geq 0$. Since Feller processes have no fixed discontinuities, $F$ is almost surely continuous at $t$ and, therefore, since convergence in $D([0, \infty), \mathcal{U})$ implies convergence at continuity points,

$$
\xi(t(X^{(a,0)}_n([n^2t]))) \xrightarrow{a.s.} F(t).
$$

Since $F(t) = dW$, $P(F(t) > 0) = 1$ and, since

$$
\xi(t(X^{(a,0)}_n([n^2t]))) \xrightarrow{a.s.} X^{(a,0)}(t),
$$

it follows that $F(t) = a.s. \xi(X^{(a,0)}(t))$. Consequently, $F(t)$ is a modification of $\xi(X^{(a,0)}(t))$ and since $F$ has a Feller semigroup, so does $\xi(X^{(a,0)}).$ \hfill $\square$

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