ASYMPTOTICALLY HYPERBOLIC MANIFOLD WITH A
HOROSPHERICAL BOUNDARY

XIAOXIANG CHAI

Abstract. We discuss asymptotically hyperbolic manifold with a noncompact boundary which is close to a horosphere in a certain sense. The model case is a horoball or the complement of a horoball in standard hyperbolic space. We show some geometric formulas.

1. Introduction

In the upper half-space model
\[ b = \frac{1}{(2\pi)^n} ((dx^1)^2 + \cdots + (dx^n)^2), \quad x^n > 0 \]
of hyperbolic \( n \)-space \( \mathbb{H}^n \), we fix the horosphere \( \mathcal{H} = \{ x \in \mathbb{H}^n : x^n = 1 \} \).

Definition 1. We say that a manifold \((M^n, g)\) with a noncompact boundary is asymptotically hyperbolic with a horospherical boundary if outside a compact set \( K \subset M \), \( M \) is diffeomorphic to \( \{ x^n \leq 1 \} \) minus a compact set and the metric admits the decay rate
\[ |e|_b + |\bar{\nabla} e|_b + |\bar{\nabla} \bar{\nabla} e|_b = O(e^{-\tau r}) \]
where \( e = g - b \), \( \tau > \frac{n}{2} \) and \( r \) is the \( b \)-geodesic distance to a fixed point \( o \) (in \( \mathbb{H}^n \)).

We pick \( o \) to be \((0, \ldots, 1)\) without loss of generality, the \( b \)-distance to \( o \) is then
\[ 2 \cosh r = \frac{1}{(2\pi)^n} (|\hat{x}|^2 + (x^n)^2 + 1), \]
where \( \hat{x} = (x^1, \ldots, x^{n-1}) \) and \(|\hat{x}|\) is the Euclidean distance of \( \hat{x} \) to the origin \( \hat{o} \). See for example [BP92, Chapter A] for the distance formula. Replacing \( e^r \) with \( \cosh r \) in \( \mathbb{H} \) is somehow more convenient. We are concerned with behaviors near infinity, we assume in this article that the manifold \( M \) is diffeomorphic to \( \{ x^n \leq 1 \} \) with a smooth metric \( g \). The other case \( \{ x^n \geq 1 \} \) is handled in the same way, for clarity of presentation, we omit this case and leave the details to the reader.

For \( \varepsilon \in (0, 1) \), we define
\[ C_\varepsilon = \{ \varepsilon \leq x^n \leq 1 \} \cap \{ |\hat{x}| \leq \rho(\varepsilon) \}. \]
We also write \( C(\varepsilon) \) sometimes for clarity and also for others defined below. We denote \( A_{\varepsilon_1, \varepsilon_2} = C_{\varepsilon_2} \setminus C_{\varepsilon_1} \) with \( \varepsilon_2 < \varepsilon_1 \). Inner boundary \( \partial' C_\varepsilon = \partial C_\varepsilon \setminus \mathcal{H} \) of \( C_\varepsilon \) is the union of
\[ F_\varepsilon = \{ |\hat{x}| \leq \rho(\varepsilon), x^n = \varepsilon \} \]
and
\[ S_\varepsilon = \{ \varepsilon \leq x^n \leq 1, |\hat{x}| = \rho(\varepsilon) \}. \]
We denote by \( \nu \) the \( g \)-normal to \( \partial' C_\varepsilon \).
Along the horosphere $\mathcal{H}$, 

$$c_{\varepsilon} = \{x^n = 1, |\hat{x}| \leq \rho(\varepsilon)\}$$

and $a_{\varepsilon, x_1} = c_{\varepsilon} \setminus c_{x_1}$, and $s_{\varepsilon} = \{x^n = 1, |\hat{x}| = \rho(\varepsilon)\}$. We denote by $\mu$ the normal to $s_{\varepsilon}$ in the horosphere.

Here $\rho(\varepsilon)$ is a smooth decreasing function on $(0, 1)$ satisfying $\rho(\varepsilon) \to \infty$ as $\varepsilon \to 0$. Note that $C_{\varepsilon}$ is a parabolic cylinder in the region $\{x^n \leq 1\}$. Analogously, we can define

$$C_{\varepsilon} = \{1 \leq x^n \leq \varepsilon^{-1}\} \cap \{|\hat{x}| \leq \rho(\varepsilon)\}.$$ 

We also write $C(\varepsilon)$ sometimes for clarity. We denote $A_{\varepsilon, x_2} = C_{\varepsilon} \setminus C_{x_2}$ with $\varepsilon_2 < \varepsilon_1$. Inner boundary $\partial C_{\varepsilon} = \partial C_{x_2} \setminus \mathcal{H}$ of $C_{\varepsilon}$ is the union of

$$\mathcal{F}_{\varepsilon} = \{|\hat{x}| \leq \rho(\varepsilon), x^n = \varepsilon^{-1}\}$$

and

$$S_{\varepsilon} = \{1 \leq x^n \leq \varepsilon^{-1}, |\hat{x}| = \rho(\varepsilon)\}.$$ 

Let $V_0 = \frac{1}{\varepsilon}$ and $V_k = \frac{1}{\varepsilon^k}$ with $k$ ranges from 1 to $n - 1$. We will show later that $V_0$ and $V_k$ is a function satisfying the following condition. Let $\tilde{\eta} = \partial_{\alpha}$, we see that

$$\nabla^2 V = Vb, \partial_{\alpha} V = -V$$

along $\mathcal{H}$.

The functions satisfying the condition $\nabla^2 V = Vb$ is a static potential which is closely related to the static spacetime. The scalar curvature $R_g$ admits the decay rate

$$V(R_g + n(n - 1)) = \tilde{\nabla}_i \tilde{U}^{i} + O(e^{-2\tau + \tau})$$

where

$$U = V \text{div} e - Vd(\text{tr}_b e) + \text{tr}_b edV - e(\tilde{\nabla} V, \cdot)$$

is the mass integrands. See [CH03].

We denote by $dv, dw$ and $d\lambda$ respectively the $n$, $n - 1$ and $n - 2$ dimensional volume element. We have the following definition of mass like quantities $M(V)$

$$M(V) = \lim_{\varepsilon \to 0} \left[ \int_{\partial \mathcal{F}_\varepsilon} U^i \tilde{\eta}_i dv - \int_{\partial \mathcal{S}_\varepsilon} V e_{\alpha n} \mu^\alpha d\lambda \right]$$

whenever exists. This is motivated by [Wan01, CH03, AdL20] and [Cha18b]. We will show the existence of $M(V)$ in Theorem 3.

Although not explicitly, [ACG08] show that

**Theorem 1.** On $T^{n-1} \times [0, 1]$, there does not exist a metric $g$ with $R_g \geq -n(n-1)$, $H_g \geq -(n-1)$ on the top face $T^{n-1} \times \{1\}$ and $H_g \geq n-1$ on the bottom face.

However, it seems that a spinorial proof was not available explicitly in the literature. One could use the boundary conditions [CH03 (4.25)] to write down a proof. We see that the quantities (4) we defined is a natural candidate to show a similar positive mass theorem as in [Wan01, CH03, AdL20] thus giving a noncompact version of Theorem 1. Although we have not shown the geometric invariance of $M$, that is the independence of $M$ on the coordinate chart at infinity. The natural conditions should be $R_g \geq -n(n-1)$ and $H_g + n-1 \geq 0$ along the horospherical boundary. Similar conjecture can be stated for the horoball. It is also possible to define an asymptotically hyperbolic manifold with a noncompact boundary which is an equidistant hypersurface and related quantities. Another interesting direction
is to explore the spacetime version of $\mathbf{M}$ in [4]. We will address these questions in a later work.

We show that the $\mathbf{M}(V)$ can be evaluated via the Ricci tensor and the second fundamental form the horosphere $\mathcal{H}$ similar to those in [Cha18b].

**Theorem 2.** We assume that $\rho(\varepsilon) = \varepsilon^{-\alpha}$ where $\alpha > \frac{4}{3}$. We have that

$$
\mathbf{M}(V) = \frac{2}{\pi^2} \left[ \int_{C_\varepsilon} G(X, \nu) d\nu + \int_{\partial C_\varepsilon} W(X, \mu) d\lambda \right] + o(1)
$$

where $G = \text{Ric} - \frac{1}{2} R g - (n - 1)(n - 2) g$ is the modified Einstein tensor and $W = A - H g - (n - 2) g$.

Here the pair $V$ and $X$ is given in (6). The important property of $X$ is that it is a conformal Killing vector and tangent to the horosphere $\mathcal{H}$.

The article is organized as follows:

In Section 2, we show some asymptotics which motivates the Definition 4. We collect basics of conformal Killing vectors in standard hyperbolic space and prove some length estimates under the metric $g$. In Section 3, we give a proof of Theorem 2.

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2. **Background**

The Christoffel symbols of the standard hyperbolic metric $b$ is given by

$$
\bar{\Gamma}_i^j = -(x^n)^{-1} \delta_i^j, \bar{\Gamma}^n_{\alpha\beta} = 0, \bar{\Gamma}^n_{\alpha\beta} = (x^n)^{-1} \delta_{\alpha\beta}
$$

for $i, j$ possibly being $n$ ranging from 1 to $n$ and $\alpha, \beta, \gamma$ ranges from 2 to $n$. We use this convention later as well.

Consider again the half space model and the slice $\{x_n = 1\}$, then (with all quantities evaluated on the slice $\{x_n = 1\}$)

$$
\Gamma^n_{\alpha\beta} = \delta_{\alpha\beta} + \frac{1}{2}(\nabla_{\alpha} e_{\beta n} + \nabla_{\beta} e_{\alpha n} - \nabla_n e_{\alpha\beta}).
$$

The outward normal is $\eta^i = (g^{nn})^{-\frac{1}{2}} g^{ni}$ and the second fundamental form is

$$
A_{\alpha\beta} = -(g^{nn})^{-\frac{1}{2}} \Gamma^n_{\alpha\beta}.
$$

Hence the mean curvature is

$$
H = -h_{\alpha\beta} (g^{nn})^{-\frac{1}{2}} \Gamma^n_{\alpha\beta}.
$$

Note that $h_{\alpha\beta} = \delta_{\alpha\beta} - e_{\alpha\beta} + O(e^{-2\tau r})$ and $(g^{nn})^{-\frac{1}{2}} \Gamma^n_{\alpha\beta} = 1 + \frac{1}{2} e_{nn} + O(e^{-2\tau r})$, we have

$$
2(H + n - 1) = -2h_{\alpha\beta} (g^{nn})^{-\frac{1}{2}} \Gamma^n_{\alpha\beta}
$$

$$
= - \sum_{\alpha} (2\nabla_{\alpha} e_{\alpha n} - \nabla_n e_{\alpha\alpha}) - (n - 1) e_{nn} + 2e_{\alpha\alpha} + O(e^{-2\tau r}).
$$
Write $U$ in coordinates, we have with $E = \text{tr}_b e$
\[ U^i = Vg^{ik}\tilde{\nabla}^j e_{jk} - V\tilde{\nabla}^i E + E\tilde{\nabla}^i V - g^{ik}e_{jk}\tilde{\nabla}^k V. \]

Along $\mathcal{H}$,
\[ U^i\tilde{\eta}_i = \]
\[ = V\tilde{\eta}^i\tilde{\nabla}^j e_{ji} - V\tilde{\eta}^i\tilde{\nabla}_i e_{jj} + E\tilde{\eta}^i\tilde{\nabla}_i V - \tilde{\eta}^i e_{ji}\tilde{\nabla}^j V \]
\[ = V\tilde{\nabla}_i e_{an} - e_{an}\tilde{\nabla}_a V - V\tilde{\nabla}_n e_{an} + e_{aa}\tilde{\nabla}_n V + O(e^{-2\tau r + r}). \]

So we have that
\[ 2V(H + n - 1) + U^i\tilde{\eta}_i = \]
\[ = -V\tilde{\nabla}_i e_{an} - e_{an}\tilde{\nabla}_a V - (n - 1)Ve_{nn} + (2V + \tilde{\nabla}_n V)e_{aa} + O(e^{-2\tau r + r}) \]
\[ = -\tilde{\nabla}_a (Ve_{an}) - (n - 1)Ve_{nn} + (2V + \tilde{\nabla}_n V)e_{aa} + O(e^{-2\tau r + r}) \]
\[ = -\partial_a (Ve_{an}) + V\tilde{\Gamma}^i_{an}e_{ai} + V\tilde{\Gamma}^i_{an}a_{in} - V(n - 1)e_{nn} \]
\[ + (2V + \tilde{\nabla}_n V)e_{aa} + O(e^{-2\tau r + r}) \]
\[ (5) = -\partial_a (Ve_{an}) + O(e^{-2\tau r + r}). \]

2.1. Conformal Killing vectors. Since the metric of the hyperbolic space using the upper half space model is conformal to standard metric of the Euclidean metric. We investigate $\tilde{\nabla}_i X^j$ for such $X$ which is a conformal Killing vector with respect to the Euclidean metric. We use repeatedly that the fact
\[ \tilde{\nabla}_i X^j = \tilde{\nabla}^i X_j \]
which follows easily from conformality to $\delta$.

The $X = x^i\partial_i X = x^i\partial_i$, then
\[ \tilde{\nabla}_i X^j = \]
\[ = \partial_i x^j + x^k\tilde{\Gamma}^j_{ik} \]
\[ = \delta^j_i + x^k \frac{1}{2}(x^n)^2 (g_{ij,k} + g_{kj,i} - g_{ik,j}) \]
\[ = \delta^j_i + \frac{1}{2}(x^n)^2 (\frac{\partial}{\partial x^n}((x^n)^{-2})\delta_{ij} + x^j g_{jj,i} - x^i g_{ii,j}) \]
\[ = \frac{1}{2}(x^n)^2 (x^j g_{jj,i} - x^i g_{ii,j}). \]

So $\tilde{\nabla}_i X^j + \tilde{\nabla}^j X_i = 0$. We see that vector $x^i\partial_i$ is actually a Killing vector.

Now we consider the translation vectors $\partial_i$. First, obviously $\partial_i$ for $i \neq n$ is obviously a Killing vector field. For $X = \partial_n$,
\[ \tilde{\nabla}_i X^j = X^k\tilde{\Gamma}^j_{ik} = \Gamma^j_{in} = -(x^n)^{-1}\delta^j_i. \]

So
\[ \tilde{\nabla}_i X^j + \tilde{\nabla}^j X_i = -2(x^n)^{-1}\delta^j_i \]
and $\partial_n$ is a conformal Killing vector. We remark here the recent article of Jang and Miao [JM21] uses $\frac{1}{e^n}$ to express the usual mass along horospheres converging to infinity by taking $\rho(\xi)$ to grow fast. It might be possible to obtain a formula similar to Theorem [2] evaluating the their mass expression only along horospheres by exploiting the special properties of the vector $-\partial_n$.

Now we consider the family $X = X^j\partial_j$ with $X^j$ being
\[ \langle x, a \rangle d^i x^j - \frac{1}{2} \langle x, x \rangle d^i a^j \]
where \(a\) is nonzero constant vector in Euclidean space. We see first that
\[
\partial_i X^j = a_i x^j + \langle x, a \rangle \delta_i^j - x_i a^j.
\]

We consider separately two cases of values of \(i, j\). For \(i \neq n\), we have that
\[
\nabla_i X^j = \partial_i X^j + X^k \Gamma^j_{ik} = \partial_i X^j + X^n \Gamma^j_{in} = a_i x^j + \langle x, a \rangle \delta_i^j - x_i a^j - \frac{1}{2} \langle x, a \rangle \delta_i x_n - \frac{1}{2} \langle x, x \rangle \delta_i \delta_n - \frac{1}{2} \langle x, a \rangle a^n (x^n)^{-1} \delta_i^j = a_i x^j - x_i a^j + \frac{1}{2} \langle x, x \rangle \delta_i \delta_n - \frac{1}{2} \langle x, a \rangle a^n (x^n)^{-1} \delta_i^j.
\]

For \(i = n\), we have that
\[
\nabla_n X^j = \partial_n X^j + X^k \Gamma^j_{nk} = \partial_n X^j - X^k \delta_n^j = \partial_n X^j - X^j (x^n)^{-1} = a_n x^j + \langle x, a \rangle \delta_n^j - x_n a^j - \frac{1}{2} \langle x, a \rangle a^j (x^n)^{-1} = a_n x^j - x_n a^j - \frac{1}{2} \langle x, a \rangle a^j + \frac{1}{2} \langle x, x \rangle \delta_n - \frac{1}{2} \langle x, a \rangle a^n (x^n)^{-1} a^j.
\]

And for \(j \neq n\),
\[
\nabla^j X_n = \nabla^j x^n = \partial^j x^n + X^k \Gamma^j_{nk} = \partial^j x^n - X^n (x^n)^{-1} \delta^j_n = a^j x^n - x_j a^n + \langle x, a \rangle \delta^j a^j - \frac{1}{2} \langle x, a \rangle a^j (x^n)^{-1}.
\]

For \(j = n\), we have that
\[
\nabla_n X^n = \partial_n x^n + X^k \Gamma^n_{nk} = \langle x, a \rangle - X^n (x^n)^{-1} = \frac{1}{2} - \frac{1}{2} |x^2 a^n (x^n)^{-1} = \frac{1}{2} |x^2 a^n (x^n)^{-1} a^n.
\]

To summarize, we have that
\[
\nabla_i X^j + \nabla^j X_i = \frac{1}{2} \langle x, a \rangle \delta^j_i.
\]

We have that \(\langle x, \partial_n \rangle \delta x - \frac{1}{2} \langle x, x \rangle \delta \partial_n \) is a conformal Killing vector and \(\langle x, \partial_i \rangle \delta x - \frac{1}{2} \langle x, x \rangle \delta \partial_i \) is proper Killing vector for all \(i \neq n\).

We can construct a conformal Killing vector
\[
\langle x, \partial_n \rangle \delta x - \frac{1}{2} \langle x, x \rangle \delta \partial_n - x
\]
which has no tangential component to \(H\).

Now we consider the the rotation vectors \(x^i \partial_j - x^j \partial_i\). If both \(i\) and \(j\) is not \(n\), we see easily that \(x^i \partial_j - x^j \partial_i\) is a Killing vector.
We consider now the vector \( X = x^i \partial_k - x^k \partial_i \) with \( k \neq n \), the components \( X^j = x^n \delta^j_k - x^k \delta^j_n \). We compute \( \bar{\nabla}_i X^j \). For \( i \neq n, j \neq n \),
\[
\bar{\nabla}_i X^j = \partial_i X^j + X^k \Gamma^j_{ik} = \delta^j_i \delta^k_n \delta^i_n - \delta^j_i \delta^k_n + x^k (x^n)^{-1} \delta^j_i = x^k (x^n)^{-1} \delta^j_i.
\]
So
\[
\bar{\nabla}_i X^j + \bar{\nabla}_j X^i = X^k \Gamma^j_{ik} + X^k \Gamma^i_{jk} = 2 x^k \delta^j_i.
\]
We have for \( j \neq n \),
\[
\bar{\nabla}_n X^j = \bar{\nabla}_n X^j + X^i \Gamma^j_{nl} = \delta^j_i \delta^k_n \delta^i_n - \delta^j_i \delta^k_n + x^k (x^n)^{-1} \delta^j_i = x^k (x^n)^{-1} \delta^j_i.
\]
Similarly, \( \bar{\nabla}_j X^i = 0 \). And
\[
\bar{\nabla}_n X^n = X^n \Gamma^i_{jn} = X^n \Gamma^n_{nn} = \frac{k}{2n}.
\]
So we see that
\[
\bar{\nabla}_i X^j + \bar{\nabla}_j X^i = \frac{2 x^k}{x^n} \delta^j_i
\]
for all \( i \) and \( j \).

We consider the vectors \( Y = x - \partial_n \) and
\[
Y^{(k)} = (x^n \partial_k - x^k \partial_n) + (\langle x, \partial_k \rangle x - \frac{1}{3} \langle x, x \rangle \delta \partial_k).
\]
Both \( Y \) and \( Y^{(k)} \) are tangent to \( \mathcal{H} \) since \( x^n = 1 \).

The construction is by shifting a conformal Killing vector by a Killing vector. This is motivated by a recent work of the author [Cha21]. We have
\[
(6) \quad \text{div} \ Y = \frac{2k}{x^n}, \quad \text{div} \ Y^{(k)} = \frac{2k}{x^n}.
\]
by previous calculations. Along \( \mathcal{H} \), \( Y = \hat{x} \) and
\[
Y^{(k)} = \frac{1}{2} \partial_k + \langle \hat{x}, \partial_k \rangle \delta - \frac{1}{2} \langle \hat{x}, \hat{x} \rangle \delta \partial_k.
\]
They are conformal Killing vectors along \( \mathcal{H} \), specifically
\[
\partial_{\alpha} Y^\beta + \partial^\beta Y_\alpha = 2 \delta^\beta_\alpha, \quad \text{div}_{\mathcal{H}} Y = n - 1
\]
and
\[
\partial_{\alpha} (Y^{(k)})^\beta + \partial^\beta (Y^{(k)})_\alpha = 2 \langle \hat{x}, \partial_k \rangle \delta \delta^\beta_\alpha, \quad \text{div}_{\mathcal{H}} Y^{(k)} = (n - 1) \langle \hat{x}, \partial_k \rangle \delta.
\]

**Remark 1.** The vectors \( Y \) and \( Y^{(k)} \) constructed here can be used to prove similar results as in [WX19] by considering free boundary hypersurfaces supported on the horosphere.
Lemma 1. These vectors admit the growth rate
\[ |Y|_b + |\nabla Y|_b + |Y^{(k)}|_b + |\nabla Y^{(k)}|_b = O(\cosh r) \]
as \( r \to \infty \).

Proof. The proof is by direct calculation for each term. The length of \( Y = x - \partial_n \) is
\[
|Y|_b = \frac{1}{x^n} \sqrt{\hat{x}^2 + (x^n - 1)^2} \\
\leq \frac{\sqrt{2}}{x^n} (|\hat{x}| + |x^n - 1|) \\
\leq \frac{2}{x^n} (|\hat{x}|^2 + x^n + 1)
\]
= \( O(\cosh r) \),
according to (2). For \( i \neq n, j \neq n \), \( \nabla_i Y_j = \frac{x_i}{x^n} \), \( \nabla_i Y^n = -\nabla_n Y^i = -\frac{x_i}{x^n} \); \( \nabla_n Y^n = \frac{1}{x^n} \). So the length of \( \nabla Y \) is
\[
|\nabla Y|_b = \frac{1}{x^n} (n + 2|\hat{x}|^2)^{\frac{1}{2}} = O\left(\frac{\sqrt{n}}{x^n}\right) = O(\cosh r).
\]
The length of \( Y^{(k)} \) is
\[
|Y|_b^2 = (x^n)^{-2}[(x^n)^2 + (x^k)^2 + \frac{1}{2} \langle x, x \rangle \delta x^n] \\
= \frac{1}{x^n} [(x^k)^2 + (x^n - \frac{1}{2} \langle x, x \rangle \delta)^2] \\
\leq \frac{1}{x^n} [|\hat{x}|^2 + 2(x^n)^2 + \frac{1}{2} \langle x, x \rangle] \\
\leq \frac{1}{x^n} [|\hat{x}|^2 + 2(x^n)^2 + |\hat{x}|^4 + (x^n)^4].
\]
We see then that \( |Y|_b = O(\cosh r) \). We write \( U = Y^{(k)} \), we have that for \( i \neq n, j \neq n \),
\[
\nabla_i U^j = a_i x^j - x_i a^j + \frac{x^k}{x^n} \delta_i^j.
\]
For \( j \neq n \),
\[
\nabla_j U^n = -\nabla_n U^j = a_j x^n + \frac{x^j x^k}{x^n} - \frac{1}{2x^n} \langle x, x \rangle \delta a^j
\]
and \( \nabla_n U^n = \frac{x^k}{x^n} \). Here \( a \) is the vector \( \partial_k \). Summing up these entries of \( \nabla U \), we have that
\[
|\nabla U|_b^2 = \frac{(x^n)^2}{(x^n)^2} + \sum_{i \neq n, j \neq n} (a_i x^j - x_i a^j + \frac{x^k}{x^n} \delta_i^j)^2 \\
+ 2 \sum_{j \neq n} (a_j x^n + \frac{x^j x^k}{x^n} - \frac{1}{2x^n} \langle x, x \rangle \delta a^j)^2
\]
\[
= (x^n)^2 + [|\hat{x}|^2 + (x^n)^2 + (n - 1) \frac{(x^k)^2}{(x^n)^2} - 2(x^k)^2] \\
+ 2[(x^n)^2 + \frac{|\hat{x}|^2 (x^k)^2}{x^n} + \frac{(x^n)^2}{4(x^n)^2} + 2(x^k)^2 - \langle x, x \rangle \delta - \frac{1}{x^n} (x^k)^2 \langle x, x \rangle \delta].
\]
Using the relation that \( \langle x, x \rangle \delta = (x^n)^2 + |\hat{x}|^2 \), the last line reduces to
\[
|\nabla U|_b^2 = (x^n)^{-2} \left( \frac{1}{2} \langle x, x \rangle_\delta + (x^k)^2 \right).
\]
We see then \( |\nabla U|_b = O(\cosh r) \). \( \square \)
3. Proof

3.1. Finiteness of $M(V)$. First, we show that the quantity $M(V)$ is well defined under natural conditions.

Theorem 3. If $(M, g)$ is an asymptotically hyperbolic manifold with a horospherical boundary and if $e^{\varepsilon}(R_{g} + n(n - 1)) \in L^{1}(M)$ and $e^{\varepsilon}(H + n - 1) \in L^{1}(M)$, then the quantity $M(V)$ defined in (3) exists and is finite.

We are concerned only with behavior near infinity, so we can assume that $M$ is diffeomorphic to $\mathbb{H}^{n}$ minus a horoball. We then set up notations.

Before going to the proof of Theorem 3, we have the following elementary lemma concerning the integral of $\cosh^{-2\tau+1} r$ on the regions defined as

$I_1 = \{\varepsilon \leq x \leq 1\} \cap \{|\hat{x}| \leq \rho(\varepsilon)\}$
$I_2 = \{\varepsilon \leq x \leq \varepsilon\} \cap \{|\hat{x}| \leq \rho(\varepsilon)\}$
$I_3 = \{1 \leq x \leq \varepsilon^{-1}\} \cap \{|\hat{x}| \leq \rho(\varepsilon)\}$
$I_4 = \{\varepsilon^{-1} \leq x \leq \varepsilon^{-1}\} \cap \{|\hat{x}| \leq \rho(\varepsilon)\}$

Lemma 2. Assume that $\rho(\varepsilon) \to \infty$ as $\varepsilon \to 0$, we have that for $k \in \{1, 2, 3, 4\}$,

$$\int_{I_k} \cosh^{-2\tau+1} r d\tilde{v} \to 0$$

if $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

Proof. We deal with $I_1$ first. From (2),

$$\int_{I_1} \cosh^{-2\tau+1} r d\tilde{v}$$

$$= \int_{\varepsilon_1}^{1} dx^n \int_{\rho(\varepsilon_1) \leq |\hat{x}| \leq \rho(\varepsilon_2)} (x^n)^{-n+1} \cosh^{-2\tau+1} r d\hat{x}$$

$$= \int_{\varepsilon_1}^{1} (x^n)^{-n+1} dx^n \int_{\rho(\varepsilon_1) \leq |\hat{x}| \leq \rho(\varepsilon_2)} \frac{|\hat{x}|^2 + (x^n)^2 + 1}{2x^n}^{-2\tau+1} d\hat{x}$$

$$\leq 2^{1-2\tau} \int_{\varepsilon_1}^{1} (x^n)^{-n+2\tau} dx^n \int_{\rho(\varepsilon_1) \leq |\hat{x}| \leq \rho(\varepsilon_2)} |\hat{x}|^2 d\hat{x}$$

$$\leq C \int_{\rho(\varepsilon_1)}^{\rho(\varepsilon_2)} s^{2-4\tau+n-2} ds$$

Note that $-n + 2\tau > 0$ and $2 - 4\tau + n - 2 < -n < -1$, this is $o(1)$ obviously as long as $\varepsilon \to 0$, $\rho(\varepsilon) \to \infty$. Similarly, on the region $I_2$,

$$\int_{I_2} \cosh^{-2\tau+1} r d\tilde{v}$$

$$\leq 2^{1-2\tau} \int_{\varepsilon_2}^{\varepsilon_1} (x^n)^{-n+2\tau} dx^n \int_{0}^{\rho(\varepsilon_2)} (\rho^2 + 1)^{-2\tau+1} \rho^{-n-2} d\rho$$

which is also $o(1)$ as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$.

For the integral over $I_3$,

$$\int_{I_3} \cosh^{-2\tau+1} r d\tilde{v}$$

$$\leq 2^{1-2\tau} \int_{\varepsilon_1}^{\varepsilon_1^{-1}} dx^n \int_{\rho_2 \leq |\hat{x}| \leq \rho_2} (x^n)^{-n+1} \cosh^{-2\tau+1} r d\hat{x}$$

$$\leq C \int_{\rho(\varepsilon_1)}^{\rho(\varepsilon_2)} s^{2-4\tau+n-2} ds$$

Similarly, on the region $I_4$,

$$\int_{I_4} \cosh^{-2\tau+1} r d\tilde{v}$$

$$\leq 2^{1-2\tau} \int_{\varepsilon_2}^{\varepsilon_2^{-1}} dx^n \int_{\rho_1 \leq |\hat{x}| \leq \rho_2} (x^n)^{-n+1} \cosh^{-2\tau+1} r d\hat{x}$$

$$\leq C \int_{\rho(\varepsilon_1)}^{\rho(\varepsilon_2)} s^{2-4\tau+n-2} ds$$

Note that $-n + 2\tau > 0$ and $2 - 4\tau + n - 2 < -n < -1$, this is $o(1)$ obviously as long as $\varepsilon \to 0$, $\rho(\varepsilon) \to \infty$. Similarly, on the region $I_2$,
We have that
\[ \int_1^{\varepsilon_1} \int_{p_1 \leq |\hat{x}| \leq p_2} \left( \frac{|\hat{x}|^2 + (x^n)^2 + 1}{\hat{x}^n} \right)^{-2\tau + 1} (x^n)^{n+1} d\hat{x} dx^n \]
\[ = 2^{1-2\tau} \int_1^{\varepsilon_1} \int_{\rho(\varepsilon_1)} (x^n)^{-n+2\tau} (s^2 + (x^n)^2 + 1)^{-2\tau + 1} ds dx^n \]
\[ \leq C \int_1^{\varepsilon_1} \int_{\rho(\varepsilon_1)} (x^n)^{-n+2\tau} (s^2 + (x^n)^2 + 1)^{n-2\tau - p} - \frac{e^{-\tau + 1 - n\tau - p}}{2} s^{n-2} ds dx^n \]
\[ \leq C \int_1^{\varepsilon_1} t^{-p} dt \int_{\rho(\varepsilon_1)} (s^2 + 1)^{-2\tau + 1 - n\tau - p} s^{n-2} ds. \]

We fix some \( p \) with \( 1 < p < 2\tau - 1 \). Then power
\[ n - 2 + 2(-2\tau + 1 - \frac{n\tau - p}{2}) \]
is less than \( -1 \). We see the integral is \( o(1) \) as \( \varepsilon_1 \to 0 \). With a similar argument, on the region \( I_4 \),
\[ \int_1^{\varepsilon_2} \int_{|\hat{x}| \leq p_2} (x^n)^{-n+1} \cosh^{-2\tau + 1} r d\hat{x} dx^n \]
\[ \leq C \int_1^{\varepsilon_2} t^{-p} dt \int_0^{\rho(\varepsilon_2)} (s^2 + 1)^{-2\tau + 1 - n\tau - p} s^{n-2} ds \]
\[ \leq C \int_1^{\varepsilon_2} t^{-p} dt \int_0^{\infty} (s^2 + 1)^{-2\tau + 1 - n\tau - p} s^{n-2} ds. \]

we fix the same \( p \) as before, this integral is also \( o(1) \).

\[ \square \]

**Proof of Theorem 3.** The proof is basically a restatement of the expansion we derived earlier. See [CH03]. We have the expansion of the scalar curvature \( R_g \) near \( b \) that
\[ R_g = -n(n-1) + DR(e) + O(e^{-2\tau r}). \]
The specific form of \( O(e^{-2\tau r}) \) is \( O(|e|^2 + |\nabla e|^2 + |e||\nabla^2 e|) \) (See for example [Cha18a]). Here,
\[ DR(e) = \text{div}(\text{div} e - dE) + (n-1)E \]
is the linearization operator of the scalar curvature. We have
\[ VDR(e) = (D^* R(V), e) + \nabla_i U^i \]
where \( D^* R = \nabla^2 V - Vb \) is the formal \( L^2 \) adjoint of \( DR \). Since \( V \) is the static potential \( \hat{b} \), we have
\[ V(R_g + n(n-1)) = \nabla_i U^i + O(e^{-2\tau r + r}). \]
Now we integrate (8) over the region $A = A_{\varepsilon_2, \varepsilon_1}$, we see that

$$\int_A V(R_g + n(n - 1)) = \int_{\partial C(\varepsilon_2)} \psi^j \tilde{\eta}_i - \int_{\partial C(\varepsilon_1)} \psi^j \tilde{\eta}_i + \int_A O(e^{-2\tau r + r}) + \int_{a_{\varepsilon_2, \varepsilon_1}} \psi^j \tilde{\eta}_i.$$

Using (9), so

$$\int_A V(R_g + n(n - 1)) = \int_{\partial C(\varepsilon_2)} \psi^j \tilde{\eta}_i - \int_{\partial C(\varepsilon_1)} \psi^j \tilde{\eta}_i + \int_A O(e^{-2\tau r + r}) + \int_{a_{\varepsilon_2, \varepsilon_1}} [-\partial_\alpha (V\alpha) - 2V(H_g + n - 1)].$$

Using divergence theorem on $a_{\varepsilon_2, \varepsilon_1}$, we have that

$$\left( \int_{\partial C(\varepsilon_2)} \psi^j \tilde{\eta}_i - \int_{\partial C(\varepsilon_1)} \psi^j \tilde{\eta}_i \right) - \left( \int_{\partial C(\varepsilon_2)} \psi^j \tilde{\eta}_i - \int_{\partial C(\varepsilon_2)} V\alpha \theta^\alpha \right)$$

$$= \int_A V(R_g + n(n - 1)) + 2 \int_{a_{\varepsilon_2, \varepsilon_1}} V(H_g + n - 1) + \int_A O(e^{-2\tau r + r})$$

$$= \int_A O(e^{-2\tau r + r}) + \int_{a_{\varepsilon_2, \varepsilon_1}} O(e^{-2\tau r + r}) = o(1)$$

by Lemma 2, and the integrability of $V(R_g + n(n - 1))$ and $V(H_g + n - 1)$. Therefore, we have shown that $M(V)$ exists and is finite.

Now we turn to the proof of Theorem 2.

**Proof of Theorem 2.** We use the method of [Her16]. We define a cutoff function $\chi$ which vanish inside $C(\varepsilon^{\frac{3}{4}})$, equals 1 outside $C(\varepsilon^{\frac{3}{4}})$ and

$$\hat{g} = \chi g + (1 - \chi)b$$

The cutoff function is a product of two cutoff functions $\chi = \chi_1(x^n)\chi_2(|\tilde{x}|)$. The function $\chi_1(x^n) = f(-\log x^n)$ where $f(t)$ is the cutoff vanish inside $[0, -\frac{3}{4}\log \varepsilon]$ and equal to 1 in $[-\frac{3}{4}\log \varepsilon, \infty)$ with the estimate

$$f + (\log \varepsilon)|f'| + (\log \varepsilon)^2|f''| \leq C.$$  

We find then

$$|\nabla \chi_1|_b = x^n|\frac{\partial \chi_1}{\partial x^n}| = x^n \cdot \frac{1}{x^n} f'(-\log x^n) \leq \frac{C}{\log \varepsilon} \leq C$$

and similarly $|\nabla^2 \chi_1|_b \leq \frac{C}{(\log \varepsilon)^2} \leq C$. We define $\chi_2(t)$ to be the standard cutoff which vanishes inside $[0, \varepsilon^{-\frac{3}{4}}]$ and is equal to 1 in $[\varepsilon^{-\frac{3}{4}}, \infty)$ with the estimate

$$0 \leq \chi_2 \leq 1, |\chi_2| \leq C(\varepsilon^{-\frac{3}{4}} - \varepsilon^{-\frac{1}{2}}), |\chi_2'| \leq C(\varepsilon^{-\frac{3}{4}} - \varepsilon^{-\frac{1}{2}})^{-1}, |\chi_2''| \leq C(\varepsilon^{-\frac{3}{4}} - \varepsilon^{-\frac{1}{2}})^{-2}.$$

It is easy to check that

$$|\chi_2| + |\nabla \chi_2| + |\nabla^2 \chi_2| \leq C$$
when \( x^n \in (0, \varepsilon^{-1}] \) due to the condition \( \alpha > \frac{4}{3} \). From the construction of the cutoff function, \( \hat{g} \) is an asymptotically hyperbolic metric, that is the difference \( \hat{\epsilon} = \hat{g} - b \) satisfies also the decay rate (1).

The reader might want to visit the the model \( e^{2t}|d\hat{x}|^2 + dt^2 \) where \( t = -\log x^n \) for a better understanding of the construction of the cutoff function. We use mostly the upper half space model because of the conformality used in calculation of conformal Killing vectors.

We shall also denote by \( \hat{g} \) the complete metric obtained by gluing the hyperbolic metric inside \( \mathbb{C}(\varepsilon^{-\frac{1}{2}}) \) and the metric \( g \) outside \( \mathbb{C}_\varepsilon \).

By divergence theorem, we have on \( A = \mathbb{C}_\varepsilon \setminus \mathbb{C}(\varepsilon^{-\frac{1}{2}}) \) by the divergence theorem and the second Bianchi identity that

\[
\int_{\partial A} \hat{G}(X, \hat{\nu}) d\sigma = \int_A \nabla^i (\hat{G}_{ij} X^j) d\hat{v} = \int_A \hat{G}_{ij} \nabla^i X^j d\hat{v}.
\]

We first analyze the term \( \int_A \hat{G}_{ij} \nabla^i X^j d\hat{v} \). We use

\[
\nabla^i X^j = h^{il} (\nabla_l X^j + X^k (\hat{\Gamma}^l_{ik} - \hat{\Gamma}^l_{ik})) = \nabla^i X^j + (\hat{g}^{il} - b^{il}) \nabla_l X^j + h^{il} X^k (\hat{\Gamma}^l_{ik} - \hat{\Gamma}^l_{ik}).
\]

In fact \( \int_A \hat{G}_{ij} (\hat{g}^{il} - b^{il}) \nabla_l X^j d\hat{v} \) is \( o(1) \) by noting that the decay of \( \hat{G}^{ij}_l (\hat{g}^{il} - b^{il}) \nabla_l X^j = O(\cosh^{-2\alpha} r |\nabla X|) \) and the growth rate (1). Similarly \( \int_A \hat{G}_{ij} g^{il} (\hat{\Gamma}^l_{ik} - \hat{\Gamma}^l_{ik}) d\hat{v} \) is \( o(1) \).

\[
\int_A \hat{G}_{ij} \nabla^i X^j d\hat{v} = \frac{1}{2} \int_A \hat{G}_{ij} (\nabla^i X^j + \nabla^j X^i) d\hat{v} = \frac{1}{n} \int_A \text{div}^b X \hat{G}_{ij} b^{ij} d\hat{v} = \frac{1}{n} \int_A \text{div}^b X \hat{G}_{ij} \hat{g}^{ij} d\hat{v} - \frac{1}{n} \int_A \text{div}^b X \hat{G}_{ij} (\hat{g}^{ij} - b^{ij}) d\hat{v} = \frac{2-n}{2n} \int_A \text{div}^b X (\hat{\mathcal{R}} + n(n-1)) d\hat{v} - \frac{1}{n} \int_A \text{div}^b X \hat{G}_{ij} (h^{ij} - b^{ij}) d\hat{v}.
\]

We have that similarly the two terms \( \int_A \text{div}^b X \hat{G}_{ij} (h^{ij} - b^{ij}) d\hat{v} \) and \( \int_A \text{div}^b X (\hat{\mathcal{R}} + n(n-1)) (d\hat{v} - d\hat{\nu}) \) are \( o(1) \). Therefore,

\[
\int_A \hat{G}_{ij} \nabla^i X^j d\hat{v} = \frac{2-n}{2n} \int_A \text{div}^b X (\hat{\mathcal{R}} + n(n-1)) d\hat{v} + o(1).
\]

We are concerned about \( \int_A \hat{G}(X, \hat{\nu}) d\sigma \). Denote \( a = c_a \setminus c(a^{-\frac{1}{2}}) \). Since \( X \) is tangent to the \( \mathcal{H} \), we use Gauss-Codazzi equation, we find that

\[
\int_a \hat{G}(X, \hat{\nu}) d\sigma = \int_a X^\beta \hat{D}^\alpha (\hat{A}_{\alpha\beta} - \hat{H} \hat{\rho}_{\alpha\beta}) d\hat{\sigma}.
\]

We add an extra term to the tensor \( \hat{A} - \hat{H} \hat{\rho} \) so that it is trace free if \( g \) is just the hyperbolic metric \( b \). The modification is in spirit similar to the modified Einstein tensor \( G \). So
\[
\int_a \dot{G}(X, \dot{\nu})d\dot{\sigma} = \int_a X^\alpha \dot{D}^\alpha (\dot{A}_{\alpha \beta} - \dot{H} \rho_{\alpha \beta} - (n - 2)h_{\alpha \beta})d\dot{\sigma}.
\]

We apply the divergence theorem, we obtain

(12) \[
\int_a \dot{G}(X, \dot{\nu})d\dot{\sigma} = \int_{\partial A} \dot{W}(X, \dot{\nu})d\lambda - \int_a \dot{W}_{a\beta} \dot{D}^\alpha X^\beta d\dot{\sigma}.
\]

Now we analyze the term \( \int_a \dot{W}_{a\beta} \dot{D}^\alpha X^\beta d\dot{\sigma} \). We use

\[
\dot{D}^\alpha X^\beta = \rho^{\alpha \gamma} (\dot{D}_\gamma X^\beta + X^\xi (\dot{\Lambda}_\xi^\beta - \ddot{\Lambda}_\xi^\beta))
\]

\[
= \dot{D}^\alpha X^\beta + (\ddot{\rho}^{\alpha \gamma} - \ddot{\rho}^{\alpha \gamma}) D_\gamma X^\beta + \dot{\rho}^{\alpha \gamma} X^\xi (\dot{\Lambda}_\xi^\beta - \ddot{\Lambda}_\xi^\beta).
\]

The terms \( \int_a \dot{W}_{a\beta} (\ddot{\rho}^{\alpha \gamma} - \ddot{\rho}^{\alpha \gamma}) D_\gamma X^\beta d\dot{\sigma} \) and \( \int_a \dot{\rho}^{\alpha \gamma} X^\xi (\dot{\Lambda}_\xi^\beta - \ddot{\Lambda}_\xi^\beta)d\dot{\sigma} \) are \( o(1) \).

We know that

\[
\int_a \dot{W}_{a\beta} \dot{D}^\alpha X^\beta d\dot{\sigma}
\]

\[
= \frac{1}{2} \int_a \dot{W}_{a\beta} (\dot{D}^\alpha X^\beta + \ddot{D}^\alpha X^\beta) d\dot{\sigma}
\]

\[
= \frac{1}{n-1} \int_a \text{div}^b X \rho^{\alpha \beta} \dot{W}_{a\beta} d\dot{\sigma}
\]

\[
= \frac{2-n}{n-1} \int_a \text{div}^b X \rho^{\alpha \beta} \dot{W}_{a\beta} d\dot{\sigma} - \frac{1}{n-1} \int_a \text{div}^b X (\dot{\rho}^{\alpha \beta} - \ddot{\rho}^{\alpha \beta}) \dot{W}_{a\beta} d\dot{\sigma}
\]

We have that \( \int_a \text{div}^b X (\dot{\rho}^{\alpha \beta} - \ddot{\rho}^{\alpha \beta}) \dot{W}_{a\beta} d\dot{\sigma} \) and \( \int_a \text{div}^b X \dot{\rho}^{\alpha \beta} \dot{W}_{a\beta} (d\dot{\sigma} - d\dot{\sigma}) \) are \( o(1) \). Therefore,

(13) \[
\int_a \dot{W}_{a\beta} \dot{D}^\alpha X^\beta d\dot{\sigma} = \frac{2-n}{n-1} \int_a \text{div}^b X (\dot{H} + n - 1) d\dot{\sigma} + o(1).
\]

We have from (10), (11), (12) and (13) that

\[
\int_{\partial A} \dot{G}(X, \dot{\nu})d\dot{\sigma}
\]

\[
= \int_{(\partial A)\setminus a} \dot{G}(X, \dot{\nu})d\dot{\sigma} + \int_a \dot{G}(X, \dot{\nu})d\dot{\sigma}
\]

\[
= \int_{(\partial A)\setminus a} \dot{G}(X, \dot{\nu})d\dot{\sigma} + \int_a \dot{W}(X, \dot{\nu})d\lambda - \frac{2-n}{n-1} \int_a \text{div}^b X (\dot{H} + n - 1) d\dot{\sigma}
\]

\[
= \frac{2-n}{2n} \int_A \text{div}^b X (\dot{R} + n(n - 1)) d\nu + o(1).
\]

Since \( \text{div}^b X = nV \) and \( \text{div}^b X = (n - 1)V \) along \( \mathcal{H} \), by the same argument arriving (9), we have

\[
\frac{1}{n}(2-n)M(V) + o(1)
\]

\[
= \frac{2-n}{2n} \int_A \text{div}^b X (\dot{R} + n(n - 1)) d\nu + \frac{2-n}{n-1} \int_a \text{div}^b X (\dot{H} + n - 1) d\dot{\sigma}
\]

\[
= \int_{(\partial A)\setminus a} \dot{G}(X, \dot{\nu})d\dot{\sigma} + \int_{\partial a} \dot{W}(X, \dot{\nu})d\lambda.
\]
Now since the metric $\hat{g}$ is standard inside $C(\varepsilon^{1/2})$ and $\hat{g} = g$ outside $C(\varepsilon^{3/2})$, we have the desired formula.

Remark 2. It might be possible to drop the condition that $\rho(\varepsilon) = \varepsilon^{-\alpha}$ where $\alpha > \frac{4}{3}$ using the expansion and the method developed by the author [Cha18a].

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