An Entanglement-Based Protocol For Strong Coin Tossing With Bias 1/4

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In the literature, strong coin tossing protocols based on bit commitment have been proposed. Here we examine a protocol that instead tries to achieve the task by sharing entanglement securely. The protocol uses only qubits, and has bias $\frac{1}{4}$. This is equal to the best known bias for bit commitment based schemes.

I. INTRODUCTION

Alice and Bob, having recently divorced, want to decide who keeps the car [1]. Both now live separate lives on opposite sides of the country, and to meet in person would be inconvenient and traumatic. A coin tossing protocol seeks to provide a sequence of information exchanges that allow the decision to be fairly made. Whether or not this is possible depends on the physical properties of the systems used for information exchange. Protocols which cannot satisfy the full requirements demanded of coin tossing are given a figure of merit depending on the maximum cheating probability they allow.

Coin tossing comes in two flavours, weak and strong. A weak coin tossing protocol suffices if the parties know which outcome the other prefers. This is the case in the divorcees example above, where both Alice and Bob would like to keep the car. Suppose outcome 0 means Alice keeps the car, and 1 means Bob does. A protocol need not protect against Alice biasing towards 1, nor Bob towards 0, and hence a weak coin toss protocol can be used. In contrast, a strong coin tossing protocol is needed when it is not known which outcome the other party prefers.

A. Previous Results

Coin tossing was introduced by Blum [1] in 1981. There, a variant of our task was discussed in a classical setting using computational assumptions to give security. However, in a classical setting where unconditional security is sought, no protocol can offer any protection against a cheat [2]. That quantum coin tossing protocols offer some advantage over classical ones was realized by Aharonov et al. [3], who introduced a protocol achieving a bias of $\frac{1}{\sqrt{2}}$ [3, 4]. For strong coin tossing, it has been shown by Kitaev that in any protocol, at least one party can achieve a bias greater than $\frac{1}{\sqrt{2}} - \frac{1}{2}$ [5]. It is not known whether this figure represents an achievable bias. The best known bias to date (which is realized via a qutrit-based bit commitment scheme) is $\frac{1}{4}$ [6] and this is optimal for a large set of protocols [7]. For weak coin tossing, Kitaev’s bound is known not to apply and lower biases than $\frac{1}{\sqrt{2}} - \frac{1}{2}$ have been achieved (see for example [8] for the best bias to date). Moreover, Ambainis has shown that a protocol with bias $\epsilon > 0$ must have a number of rounds that grows as $\Omega(\log \log \frac{1}{\epsilon})$ [6].

In this paper, we consider only non-relativistic protocols (in which communications between parties can be effectively taken to be instantaneous). It is known that using two separated sites and exploiting the impossibility of superluminal signalling, ideal coin tossing can be implemented with perfect security [9].

B. Definitions

In a coin tossing protocol, two separated and mistrustful parties, Alice and Bob, wish to generate a shared random bit. We consider a model in which they do not initially share any resources, but have access to trusted laboratories containing trusted error-free apparatus for creating and manipulating quantum states. In general, a protocol for this task may be defined to include one or more security parameters, which we denote $N_1, \ldots, N_r$. 

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2. i.e., security based only in a belief in the laws of physics
If both parties are honest, a coin tossing protocol guarantees that they are returned the same outcome, \( b \in \{0,1\} \) where outcome \( b \) occurs with probability \( \frac{1}{2} + \epsilon_b(N_1, \ldots, N_r) \), or “abort” which occurs with probability \( \zeta_b(N_1, \ldots, N_r) \), where for each \( j \in \{0,1,2\} \), \( \zeta_j(N_1, \ldots, N_r) \to 0 \) as the \( N_i \to \infty \). The bias of the protocol towards party \( P \in \{A,B\} \) is denoted \( \epsilon_P = \max(\epsilon^0_P, \epsilon^1_P) \), where \( P \) can deviate from the protocol in such a way as to convince the other (honest) party that the outcome is \( b \) with probability at most \( \frac{1}{2} + \epsilon^b_P(N_1, \ldots, N_r) \), where the \( \delta^b_P(N_1, \ldots, N_r) \to 0 \) as the \( N_i \to \infty \). We make no requirements of the protocol in the case where both parties cheat.

The bias of the protocol is defined to be \( \max(\epsilon_A, \epsilon_B) \). A protocol is said to be balanced if \( \epsilon^b_A = \epsilon^b_B \), for \( b = 0 \) and \( b = 1 \).

We define the following types of coin tossing:

**Ideal Coin Tossing:** A coin tossing protocol is ideal if it has \( \epsilon_A = \epsilon_B = 0 \), that is, no matter what one party does to try to bias the outcome, their probability of successfully doing so is strictly zero. It is then said to be perfectly secure if for some finite values of \( N_1, \ldots, N_r \), the quantities \( \zeta^b_N(N_1, \ldots, N_r) \) and \( \delta^b_P(N_1, \ldots, N_r) \) are strictly zero, and otherwise is said to be secure.

**Strong Coin Tossing:** A strong coin tossing protocol is parameterized by a bias, \( \gamma \). The protocol has the property that \( \epsilon^b_P \leq \gamma \) for all \( P \in \{A,B\} \) and \( b \in \{0,1\} \), with equality for some \( P, b \).

**Weak Coin Tossing:** A weak coin tossing protocol is also parameterized by a bias, \( \gamma \). It has the property that \( \epsilon^0_A \leq \gamma \) and \( \epsilon^1_B \leq \gamma \), with equality in one of the two inequalities.

In the next section we give a new protocol for strong coin tossing and show that it is balanced and has bias \( \frac{1}{4} \).

## II. THE PROTOCOL

1. Alice creates 2 copies of the state \( |\psi\rangle = N^{-\frac{1}{2}}(|00\rangle + |11\rangle) \) and sends the second qubit of each to Bob.

2. Bob randomly selects one of the states to be used for the coin toss. He informs Alice of his choice.

3. Alice and Bob measure their halves of the chosen state in the \( \{0,1\} \) basis to generate the result of the coin toss.

4. Alice sends her half of the other state to Bob who tests whether it is the state it should be by measuring the projection onto \( |\psi\rangle \). If his test fails, Bob aborts.

### A. Alice’s Bias

Assume Bob is honest. We will determine the maximum probability that Alice can achieve outcome 0, \( p_A \) (an analogous result follows by symmetry for the case that Alice wants to bias towards 1). Alice’s most general strategy is as follows. She can create a state in an arbitrarily large Hilbert space, \( |\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{A_1} \otimes \mathcal{H}_{B_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_{B_2} \), where \( \mathcal{H}_A \) represents the space of an ancillary system Alice keeps, \( \mathcal{H}_{B_1} \) and \( \mathcal{H}_{B_2} \) are qubit spaces sent to Bob in the first step of the protocol, and \( \mathcal{H}_{A_1} \) and \( \mathcal{H}_{A_2} \) are qubit spaces, one of which will be sent to Bob for verification. On receiving Bob’s choice of state in step 2 Alice can do one of two local operations on the states in her possession, before sending Bob the relevant qubit for verification. Alice should choose her state and local operations so as to maximize the probability that Bob obtains outcome 0 and does not detect her cheating.

Let us denote the state of the entire system by

\[
|\Psi\rangle = a_{00} |\phi_{00}\rangle_{AA_1A_2} |00\rangle_{B_1B_2} + a_{01} |\phi_{01}\rangle_{AA_1A_2} |01\rangle_{B_1B_2} + a_{10} |\phi_{10}\rangle_{AA_1A_2} |10\rangle_{B_1B_2} + a_{11} |\phi_{11}\rangle_{AA_1A_2} |11\rangle_{B_1B_2},
\]

where \( \{ |\phi_{ij}\rangle\}_{i,j} \) are normalized states in Alice’s possession, and \( \{a_{ij}\}_{i,j} \) are coefficients. Suppose Bob announces that he will use the first state for the coin toss. There is nothing Alice can subsequently do to affect the probability of Bob measuring 0 on the qubit in \( \mathcal{H}_{B_1} \). We can assume that Bob makes the measurement on this qubit immediately on making his choice. Let us also assume that Alice discovers the outcome of this measurement so that she knows the pure state of the entire system (we could add a step in the protocol where Bob tells her, for example\(^2\)). If Bob

\(^2\) Such a step can only make it easier for Alice to cheat, so security under this weakened protocol implies security under the original one.
probability of winning via measures 0 on her part of the state he announces is maximized. And Alice can win if she can pass Bob’s test in the final step of the protocol. Since entanglement cannot be increased by local operations, the system Alice sends to Bob in this case can be no more entangled than this state. Alice therefore cannot fool Bob into thinking she was honest with probability greater than \(\frac{(a_{00} + a_{01})^2}{2(a_{00} + a_{01})^2} = \frac{1}{2}\). Using a similar argument for the case that Bob chooses the second state for the coin toss shows that Alice’s overall success probability is at most \(\frac{1}{2}(2a_{00}^2 + 2a_{00}a_{01} + 2a_{00}a_{10} + a_{01}^2 + a_{10}^2)\). Maximizing this subject to the normalization condition gives a maximum of \(\sqrt{\frac{2}{3}}\), hence we have the bound \(p_A \leq \frac{\sqrt{2}}{3}\). Equality is achievable within the original protocol (i.e., without the additional step we introduced) by having Alice use the state

\[
\sqrt{\frac{2}{3}}|0000\rangle_{A_1A_2B_2} + \frac{1}{\sqrt{6}}(|0011\rangle + |1100\rangle),
\]

and simply sending \(H_{A_1}\) or \(H_{A_2}\) to Bob in the final step, depending on Bob’s choice.

The protocol is cheat-sensitive towards Alice—any strategy which increases her probability of obtaining one outcome gives her a non-zero probability of being detected.

### B. Bob’s Bias

Assume Alice is honest. We will determine the maximum probability that Bob can achieve the outcome 0, \(p_B\). The maximum probability for outcome 1 follows by symmetry. Bob seeks to take the qubits he receives, perform some local operation on them, and then announce one of them to be the coin-toss state such that the probability that Alice measures 0 on her part of the state he announces is maximized.

Suppose that we have found the local operation maximizing Bob’s probability of convincing Alice that the outcome is 0. Having performed this operation and sent the announcement to Alice, the outcome probabilities for Alice’s subsequent measurement on the state selected by Bob in the \(|0\rangle, |1\rangle\) basis are fixed. Bob’s probability of winning depends only on this. It is therefore unaffected by anything Alice does to the other qubit, and, in particular, is unaffected if Alice measures both of her qubits in the \(|0\rangle, |1\rangle\) basis before looking at Bob’s choice. Such a measurement commutes with Bob’s local operation, so could be done by Alice prior to Bob’s operation without changing any outcome probabilities. If Alice does this measurement she gets outcome 1 on both qubits with probability \(\frac{1}{4}\). In such a case, Bob cannot convince Alice that the outcome is 0. Therefore, we have bounded Bob’s maximum probability of winning via \(p_B \leq \frac{1}{4}\).

To achieve equality, Bob can measure each qubit he receives in the \(|0\rangle, |1\rangle\) basis, and if he gets one with outcome 0, choose this state as the one to use for the coin toss. There is no cheat sensitivity towards Bob; he can use this strategy without fear of being caught.

### III. DISCUSSION

We have introduced a new protocol for strong coin tossing which achieves a bias of \(\frac{1}{4}\). Whilst this bias is not an improvement over existing protocols, our protocol uses a conceptually different approach to previous ones. Rather than being built on bit-commitment, the protocol works by attempting to share entanglement between two parties, and then exploiting the resulting quantum correlations to implement a coin toss. This provides a further illustration of the power of entanglement as a resource. Furthermore, our protocol requires only qubits for its implementation, whereas bit-commitment based protocols cannot achieve such a bias without using higher dimensional systems [7].

**Additional Notes:** The protocol we present is a special case of a protocol for random bit string generation found in [10], where only one bit is sought\(^3\). However, the security analysis in [10] does not extend to the single bit case. We have also learned that this protocol was independently discovered by Louis Salvail [11] whose work on this was not published.

\(^3\) We thank the anonymous referee for pointing this out.
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