BANDLIMITED WAVELETS ON THE HEISENBERG GROUP

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Abstract. Let \( H \) be the three-dimensional Heisenberg group. We introduce a structure on the Heisenberg group which consists of the biregular representation of \( \mathbb{H} \times \mathbb{H} \) restricted to some discrete subset of \( \mathbb{H} \times \mathbb{H} \) and a free group of automorphisms \( H \) singly generated and acting semi-simply on \( \mathbb{H} \). Using well-known theorems borrowed from Gabor theory, we are able to construct simple and computable bandlimited discrete wavelets on the Heisenberg group. Moreover, we provide some necessary and sufficient conditions for the existence of these wavelets.

1. Introduction

A wavelet frame is a system generated by the action of translation and dilation of a single function. More precisely, if \( \psi \in L^2(\mathbb{R}) \) and \( a, b \) are some fixed positive numbers and if

\[
W(\psi, a, b) = \{ a^{n/2} \psi(a^n x - bk) : k, n \in \mathbb{Z} \}
\]

is a frame (orthonormal basis) for \( L^2(\mathbb{R}) \) then we call \( W(\psi, a, b) \) a wavelet frame (orthonormal basis). This system provides expansions for functions in \( L^2(\mathbb{R}) \). For example, in the case where \( W(\psi, a, b) \) is a Parseval frame or an orthonormal basis, then it is known that for any function \( \phi \in L^2(\mathbb{R}) \),

\[
\phi = \sum_{k, n \in \mathbb{Z}} \langle \phi, a^{n/2} \psi(a^n \cdot -bk) \rangle a^{n/2} \psi(a^n \cdot -bk).
\]

Wavelet theory does extend to commutative groups of higher dimensions and even to some non-commutative locally compact groups. In fact, the existence of wavelets has been proved on the Heisenberg group and on other type of stratified nilpotent Lie groups (see [13] [15], [12], [11] and [18]). In [11], the authors developed a theory of multiresolution analysis, by applying a concept of acceptable dilations on the Heisenberg group which was used in the commutative case by Gröchenig and Madych in [7]. As a result, they were able to provide a description of Haar wavelets on the Heisenberg group. In [13], Mayeli used multiresolution-like analysis and some sampling theorems for the Heisenberg group obtained by Führ in [6] to prove the existence of Shannon-type wavelets generated by discrete translations and dilations on the Hilbert space of all square-integrable functions on the Heisenberg group. However, there are several complications which make this approach of construction of wavelets

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on the Heisenberg group difficult. Let us be more precise. Since the Fourier and
Plancherel transforms are the central tools employed in the construction of wavelets,
the non-commutative nature of the Heisenberg group represents a major obstruction.
In fact, the left regular representation of the Heisenberg group is decomposed via
the Plancherel transform into a direct integral of Schrödinger representations, each
occurring with infinite multiplicities. It turns out that the structure which consists
of the left regular representation restricted to some discrete subgroup together with
the usual expansive dilation of the Heisenberg group is not a natural structure for
the construction of Shannon-type wavelets on the Heisenberg group. Therefore, in
order to construct simple and computable wavelets on the Heisenberg group, there is
a need to seek a different approach.

In the present work, we introduce a new construction of discrete bandlimited
wavelets on the Heisenberg group generated by a single function. The techniques
developed in this work are quite different from the ones known for commutative
groups and the ones employed by Mayeli in [13]. The structure considered in the
present work consists of a pair $(\tau, D)$ where $D$ is a unitary representation induced
by an automorphism of the Heisenberg group, and $\tau$ is the bi-regular representation
of the Heisenberg group. The advantage of this new approach is that, using avail-
able facts borrowed from Gabor theory, we are able to obtain simple and computable
discrete bandlimited wavelets (Shannon-like) on the Heisenberg group.

Let us summarize our main result. Let $H_0$ be the 3-dimensional Heisenberg group
with Lie algebra $\mathfrak{h}$ spanned by $Z, Y, X$ such that the only non-trivial Lie brackets
are $[X, Y] = Z$. We may think of the Heisenberg group as being isomorphic to the
non-commutative group $(\mathbb{R}^3, \ast)$ with group law defined by

$$(x, y, z) \ast (w, v, u) = (w + x, v + y, u + z + vx).$$

We introduce a convenient faithful finite-dimensional representation of the Heisenberg
group. Let us define an injective homomorphism $\rho : H_0 \to GL(4, \mathbb{R})$ such that

$$\rho (\exp (zZ) \exp (yY) \exp (xX)) = \begin{bmatrix}
1 & 0 & 0 & z \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & 0 & -y & 0 \\
0 & 1 & 0 & y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
1 & x & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
1 & x & -y & z \\
0 & 1 & 0 & y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.$$
\[ Z(\mathbb{H}) = \left\{ \begin{bmatrix} 1 & 0 & 0 & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : z \in \mathbb{R} \right\}. \]

Finally, we define a discrete subgroup \( \Gamma \) of the Heisenberg group as follows:

\[ \Gamma = \left\{ \begin{bmatrix} 1 & k_3 & -k_2 & k_1 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : k_1, k_2, k_3 \in \mathbb{Z} \right\}. \]

Put

\[ \Lambda = \left\{ \begin{bmatrix} 1 & k_3 & -k_2 & k_1 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & m_3 & -m_2 & 0 \\ 0 & 1 & 0 & m_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ m_2 \\ m_3 \end{bmatrix} \in \mathbb{Z}^5 \right\} \subset \Gamma \times \Gamma \]

and let

\[ A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in GL(4, \mathbb{R}) \]

such that \( ab = 2 \). Then the map

\[ M \mapsto AMA^{-1} \]

defines an outer automorphism on the Heisenberg group. Now, let \( \tau : \mathbb{H} \times \mathbb{H} \to U(L^2(\mathbb{H})) \) such that \( \tau(u, v)f = L(u)R(v)f \) where

\[ L(u)f(x) = f(u^{-1}x) \quad \text{and} \quad R(v)f(x) = f(xv). \]

Clearly \( \tau \) is the biregular representation of the Heisenberg group. Next, define a representation \( D \) of the group generated by \( A \) such that \( D :\langle A \rangle \to U(L^2(\mathbb{H})) \) and

\[ D_A^m f(n) = |\delta(A)|^{-m/2} f(A^{-m}nA^m) \]

where

\[ d(A^m n A^{-m}) = |\delta(A)|^m dn \]

and \( dn \) is the canonical Haar measure on the Heisenberg group.

The main objective of the present paper is to prove the existence, and to find characteristics of functions \( f \) in \( L^2(\mathbb{H}) \) such that

\[ \{ D_A^m \tau(\gamma, \eta)f : m \in \mathbb{Z}, (\gamma, \eta) \in \Lambda \} \]
is a Parseval frame in $L^2(\mathbb{H})$. That is, given $h \in L^2(\mathbb{H})$,

$$\sum_{m \in \mathbb{Z}} \sum_{(\gamma, \eta) \in \Lambda} |\langle h, D_{A^m} \tau (\gamma, \eta) f \rangle|^2 = \|h\|^2_{L^2(\mathbb{H})}.$$ 

Furthermore

$$h = \sum_{m \in \mathbb{Z}} \sum_{(\gamma, \eta) \in \Lambda} \langle h, D_{A^m} \tau (\gamma, \eta) f \rangle D_{A^m} \tau (\gamma, \eta) f$$

with convergence in the $L^2$-norm.

We recall that the Plancherel transform (see the section titled Preliminaries)

$$\mathbf{P} : L^2(\mathbb{H}) \to \int_{\mathbb{R}^*} L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) |\lambda| d\lambda$$

is a unitary operator obtained by extending the group Fourier transform from $L^2(\mathbb{H}) \cap L^1(\mathbb{H})$ to $L^2(\mathbb{H})$. Let

$$H_S = \mathbf{P}^{-1} \left( \int_S L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) |\lambda| d\lambda \right)$$

such that $S$ a subset of $\mathbb{R}^*$. Then $H_S$ is a $\tau$-invariant Hilbert subspace of $L^2(\mathbb{H})$. Here are the main theorems of this paper which are proved in the third section of this paper.

**Theorem 1.** If $S$ is translation congruent to $(0, 1]$ and $S \subseteq [-1, 1]$ then there is a function $f \in H_S$ such that $\tau(\Lambda) f$ is a Parseval frame in $H_S$ and

$$\|f\|^2_{L^2(\mathbb{H})} \leq \frac{2}{3}.$$ 

Put

$$\Lambda_1 = \left\{ \begin{pmatrix} 1 & k_3 & -k_2 & 0 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & m_3 & -m_2 & 0 \\ 0 & 1 & 0 & m_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : k_i, m_j \in \mathbb{Z} \right\}.$$ 

For a given representation $\pi$, let $\overline{\pi}$ be the corresponding contragredient representation. Let

$$\{\pi_\lambda : \lambda \in \mathbb{R}, \lambda \neq 0\}$$

be a parametrizing set for the unitary dual of the Heisenberg group and let $d\lambda$ be the Lebesgue measure on $\mathbb{R}$.

**Theorem 2.** Assume that $S$ is translation congruent to $(0, 1]$. Let $f \in H_S$. If $\tau(\Lambda) f$ is a Parseval frame in $H_S$ then

$$\left\{ [\pi_\lambda (\kappa) \otimes \overline{\pi}_\lambda (\eta)] (\mathbf{P} f) (\lambda) |\lambda|^{1/2} : (\kappa, \eta) \in \Lambda_1 \right\}$$

is a Parseval frame in $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ for $d\lambda$-almost every $\lambda \in S$. 
Let $a, b$ be non-zero real numbers. Let $f \in L^2(\mathbb{R})$. The family of vectors
\[ \mathcal{G}(f, \mathcal{L}) = \{ e^{2\pi i (k,x)} f(x - n) : k \in b\mathbb{Z}, n \in a\mathbb{Z} \} \]
is called a Gabor system in $L^2(\mathbb{R})$. For any function $f \in L^2(\mathbb{R})$, let $\overline{f}$ be the complex conjugate of $f$.

**Theorem 3.** Assume that $\mathcal{S}$ is dilation congruent to $[-1, -1/2) \cup (1/2, 1]$, translation congruent to $(0, 1]$ and that $\mathcal{S} \subseteq [-1, 1]$. Let $f \in H_\mathcal{S}$ be defined as follows: $P f(\lambda) = u_\lambda \otimes v_\lambda$ such that $G\left(|\lambda|^{1/4} u_\lambda, \mathbb{Z} \times \lambda \mathbb{Z}\right)$ is a Parseval Gabor frame for $d\lambda$-almost every $\lambda \in \mathcal{S}$, and $G\left(|\lambda|^{1/4} v_\lambda, \mathbb{Z} \times \lambda \mathbb{Z}\right)$ is a Parseval Gabor frame for $d\lambda$-almost every $\lambda \in \mathcal{S}$. Then, the system
\[ \{ D_{A^m} \tau(\gamma, \eta) f : m \in \mathbb{Z}, (\gamma, \eta) \in \Lambda \} \]
is a Parseval frame in $L^2(H)$.

**Example 4.** Let us suppose that $a = b = \sqrt{2}$ so that
\[
A = \begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & \sqrt{2} & 0 & 0 \\
0 & 0 & \sqrt{2} & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
Put
\[
\mathcal{S} = \left[-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right].
\]
Then $\mathcal{S}$ is up to a null set translation congruent to $(0, 1]$. Define $f$ such that
\[
P f(\lambda) = |\lambda|^{1/4} \chi_{[0,1)} \otimes |\lambda|^{1/4} \chi_{[0,1)}
\]
where $\chi_{[0,1)}$ is the characteristic function of the set $[0,1)$. According to Proposition 3.1, [16] it is not hard to check that
\[
G\left(|\lambda|^{1/2} \chi_{[0,1)}, \mathbb{Z} \times \lambda \mathbb{Z}\right)
\]
is Parseval frame in $L^2(\mathbb{R})$ for every $\lambda \in \mathcal{S}$. Therefore, appealing to Theorem 1 and Theorem 3, $\tau(\Lambda) f$ is a Parseval frame in $H_\mathcal{S}$ and
\[ \{ D_{A^m} \tau(\gamma, \eta) f : m \in \mathbb{Z}, (\gamma, \eta) \in \Lambda \} \]
is a Parseval frame in $L^2(H)$. Next, put
\[
M(x,y,z) = \begin{bmatrix}
1 & x & -y & z \\
0 & 1 & 0 & y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]
Using the group Fourier inverse transform provided in Theorem 4.15 [6], we obtain

\[ f(\mathbf{M}(x, y, z)) = \int_S \langle \chi_{[0,1)}, \pi_{\lambda}(x) \chi_{[0,1]} \rangle |\lambda|^{3/2} d\lambda. \]

Therefore,

\[
f(\mathbf{M}(x, y, z)) = \begin{cases} 
\int_S \frac{(\exp(2\pi i\lambda(y-z)) - \exp(2\pi i\lambda(yx-z)))|\lambda|^{3/2}}{(8-\sqrt{2})(1-x)} d\lambda & \text{if } x \in [0, 1) \text{ and } y \neq 0 \\
\int_S \frac{(\exp(2\pi i\lambda(y+1-z)) - \exp(-2\pi i\lambda z))|\lambda|^{3/2}}{(8-\sqrt{2})(1+x)} d\lambda & \text{if } x \in (-1, 0] \text{ and } y \neq 0 \\
0 & \text{if } x \in (-1, 0] \text{ and } y = 0 \\
\end{cases}.
\]

2. Preliminaries

2.1. Notations and definitions. The punctured line which is the set of all non-zero real numbers is denoted \( \mathbb{R}^* \). The general linear group which consists of invertible real matrices of order \( d \) is denoted \( GL(d, \mathbb{R}) \). Let \( G \) and \( H \) be two groups. If \( G \) and \( H \) are isomorphic, we write \( G \cong H \). Let \( T \) be a linear operator defined on some Hilbert space. The adjoint of \( T \) is denoted \( T^* \). All sets of interest in this paper should be assumed to be measurable, and we shall identify subsets whose symmetric difference has Lebesgue measure zero. For example, we make no distinction between \([0, 1)\) and \((0, 1]\). Also, all functions mentioned in this paper should be assumed to be measurable functions.

For subsets \( I \) and \( J \) of \( \mathbb{R} \), we say that \( I \) and \( J \) are translation congruent if there is a bijection \( \rho : I \to J \) and an integer-valued function \( k \) on \( I \) such that \( \rho(\lambda) = \lambda + k(\lambda) \). For example, if \( I \) is translation congruent to \([0, 1)\) then \( I \) tiles the real line by \( \mathbb{Z} \). Next, we say that \( I \) and \( J \) are dilation congruent if there exists a bijection \( \delta : I \to J \) and an integer-valued function \( j \) on \( I \) such that \( \delta(\lambda) = 2^j(\lambda)\lambda \).

2.2. Plancherel theory. The facts presented in this subsection are pretty standard. We refer the interested reader to Chapter 7, [5], Chapter 2 [17] and Chapter 4, [3]. Let

\[
\mathbb{P} = \left\{ \begin{bmatrix} 1 & 0 & -y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : z, y \in \mathbb{R} \right\}
\]

be a maximal abelian subgroup of the Heisenberg group. Then the Heisenberg group is isomorphic to a semi-direct product of the type

\[
\mathbb{P} \rtimes \left\{ \begin{bmatrix} 1 & x & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : x \in \mathbb{R} \right\}
\]
which is also isomorphic to \( \mathbb{P} \times \mathbb{R} \). For each \( \lambda \in \mathbb{R} \), we define a corresponding character \( \chi_\lambda \) on \( \mathbb{P} \) by

\[
\chi_\lambda \left( \begin{bmatrix} 1 & 0 & -y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) = e^{-2\pi i \lambda (z)}.
\]

According to the theory of Mackey, or the orbit method (see [3]) it is not too hard to show that the unitary dual of \( \mathbb{H} \) which we denote by \( \hat{\mathbb{H}} \) is up to a null set equal to

\[
\{ \pi_\lambda = \text{Ind}_\mathbb{P}^\mathbb{H} (\chi_\lambda) : \lambda \in \mathbb{R}^* \}.
\]

In fact, there are two families of unitary irreducible representations of the Heisenberg group. The first family of irreducible representations only contains characters and forms a set of Plancherel measure zero, and is therefore negligible. The second family of irreducible representations are infinite-dimensional representations which are parametrized by the punctured line as follows: \( \lambda \mapsto \pi_\lambda = \text{Ind}_\mathbb{P}^\mathbb{H} (\chi_\lambda) \). The reader who is not familiar with the theory of induced representation is invited to refer to the book of Folland [5]. Based on properties of induced representations, each unitary representation \( \pi_\lambda \) is realized as acting in the Hilbert space of square integrable functions defined over \( \mathbb{H}/\mathbb{P} \). More precisely, \( \pi_\lambda \) acts in \( L^2 (\mathbb{R}) \) such that for \( f \in L^2 (\mathbb{R}) \),

\[
\pi_\lambda \left( \begin{bmatrix} 1 & 0 & x & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) f (t) = f (t - x).
\]

Let \( \mathcal{F} \) be the operator-valued Fourier transform defined on \( L^2 (\mathbb{H}) \cap L^1 (\mathbb{H}) \) by

\[
\mathcal{F} (f) (\lambda) = \int_{\mathbb{H}} \pi_\lambda (n) f (n) \, dn.
\]

The Fourier transform can also be defined as follows: for any vectors \( u, v \in L^2 (\mathbb{R}) \),

\[
\langle \mathcal{F} (f) (\lambda) u, v \rangle = \int_{\mathbb{H}} f (n) \langle \pi_\lambda (n) u, v \rangle \, dn.
\]
Thus, for \( f \in L^2(\mathbb{H}) \cap L^1(\mathbb{H}) \)

\[
|\langle \pi_\lambda (n) u, v \rangle| \leq \|u\|_{L^2(\mathbb{R})} \|v\|_{L^2(\mathbb{R})}.
\]

Therefore, \( F \) is a bounded operator on \( L^2(\mathbb{R}) \). For \( f \in L^2(\mathbb{H}) \cap L^1(\mathbb{H}) \), it can also be shown (see Chapter 2, [17]) that \( F(f) (\lambda) \) is actually a Hilbert-Schmidt operator and that the Fourier transform \( F \) extends to the Plancherel transform:

\[
P : L^2(\mathbb{H}) \to \int_{\mathbb{R}^*} L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \ |\lambda| \, d\lambda
\]

such that for any \( f \in L^2(\mathbb{H}) \),

\[
(2.1) \quad \|f\|_{L^2(\mathbb{H})}^2 = \int_{\mathbb{R}^*} \|P f (\lambda)\|_{\mathcal{H}S}^2 \ |\lambda| \, d\lambda.
\]

Let \( d\lambda \) be the Lebesgue measure on \( \mathbb{R}^* \). The Plancherel measure for this group is supported on the punctured line \( \mathbb{R}^* \) and is the weighted Lebesgue measure \(|\lambda| \, d\lambda\). \( ||\cdot||_{\mathcal{H}S} \) denotes the Hilbert-Schmidt norm on \( L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \) and clearly (2.1) implies that \( P \) is a unitary transform.

We recall that given \( T, P \in L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \),

\[
\langle T, P \rangle_{\mathcal{H}S} = \sum_k \langle Te_k, Pe_k \rangle
\]

where \( \{e_k\}_k \) is an orthonormal basis for \( L^2(\mathbb{R}) \). Also, given a Hilbert-Schmidt operator \( T : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \), we say that \( T \) is a finite-rank operator if the range of \( T \) is a finite dimensional subspace of \( L^2(\mathbb{R}) \). It is well-known that finite-rank operators form a dense subspace of \( L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \). Moreover, if \( T \) is a rank-one operator, then \( T = u \otimes v \) for some \( u, v \in L^2(\mathbb{R}) \), and the inner product of arbitrary rank-one operators in \( L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \) is given by

\[
\langle u \otimes v, w \otimes y \rangle_{\mathcal{H}S} = \langle u, w \rangle_{L^2(\mathbb{R})} \langle v, y \rangle_{L^2(\mathbb{R})}.
\]

Let \( \tau \) be the biregular representation \( \tau : \mathbb{H} \times \mathbb{H} \to \mathcal{U}(L^2(\mathbb{H})) \) (See Page 233, [5]) defined by

\[
\tau (u, v) f = L (u) R (v) f
\]

where \( L (u) f (x) = f (u^{-1}x) \) and \( R (v) f (x) = f (xv) \). The Plancherel transform intertwines the biregular representation with a direct integral of tensor representations as follows:

\[
P \circ \tau (x, y) \circ P^{-1} = \int_{\mathbb{R}^*} \pi_\lambda (x) \otimes \pi_\lambda (y) \ |\lambda| \, d\lambda.
\]
Furthermore, for $\lambda \in \mathbb{R}^*$

$$
P(\tau (x, y) \phi)(\lambda) = \pi_\lambda(x) (P \phi)(\lambda) (\pi_\lambda(y))^* = [\pi_\lambda(x) \otimes \pi_\lambda(y)] (P \phi)(\lambda)
$$

with $\pi_\lambda$ being the contragredient of the representation $\pi_\lambda$ acting in the dual of $L^2(\mathbb{R})$.

We remind the reader that $\pi_\lambda(x) = \pi_\lambda(x^{-1})^t$, where $t$ denotes the transpose of an operator.

2.3. A class of dilations. Now, we will present a class of dilations which will be important in the construction of wavelets. Also, in the proof of Lemma 5, we will clarify why this particular realization of the Heisenberg group as a subgroup of $GL(4, \mathbb{R})$ is convenient. In fact, this finite-dimensional representation of the Heisenberg group allows us to define a natural automorphism of the Heisenberg group (obtained by conjugations) which generates a class of dilations. Put

$$
A = \begin{bmatrix}
\frac{b}{d} & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & c & 0 \\
0 & 0 & 0 & d \\
\end{bmatrix} \in GL(4, \mathbb{R})
$$

where $b, c, d \in \mathbb{R}^*$. Let $\varphi_A : \mathbb{H} \to \mathbb{H}$ such that

$$
\varphi_A(M) = AMA^{-1}.
$$

**Lemma 5.** $\langle \varphi_A \rangle$ is a subgroup of $\text{Aut}(\mathbb{H})$.

**Proof.** Let

$$
X = \begin{bmatrix}
a & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & c & 0 \\
0 & 0 & 0 & d \\
\end{bmatrix}.
$$

With some simple calculations, it is easy to see that

$$
(2.2) \quad X \begin{bmatrix}
1 & x & -y & z \\
0 & 1 & 0 & y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} X^{-1} = \begin{bmatrix}
1 & \frac{a}{d}x & -\frac{a}{d}y & \frac{a}{d}z \\
0 & 1 & 0 & \frac{a}{d}y \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
$$

Therefore, (2.2) induces an action of the group generated by $X$ on $\mathbb{H}$ if and only if $\frac{b}{d} = \frac{a}{c}$. Next, solving the above equation for $a$, we obtain

$$
X = A = \begin{bmatrix}
\frac{bc}{d} & 0 & 0 & 0 \\
0 & b & 0 & 0 \\
0 & 0 & c & 0 \\
0 & 0 & 0 & d \\
\end{bmatrix}.
$$
Now, to show that $A$ defines an automorphism, we observe that

$$A \begin{bmatrix} 1 & x_1 & -y_1 & z_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x_2 & -y_2 & z_2 \\ 0 & 1 & 0 & y_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A^{-1}$$

is equal to

$$\begin{bmatrix} 1 & \frac{x}{d} (x_1 + x_2) & -\frac{b}{d} (y_1 + y_2) & b \frac{c}{d} (z_1 + z_2 + x_1 y_2) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Finally,

$$\begin{bmatrix} 1 & x_1 & -y_1 & z_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A^{-1} \begin{bmatrix} 1 & x_2 & -y_2 & z_2 \\ 0 & 1 & 0 & y_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A^{-1}$$

is equal to (2.3) as well. Thus conjugation by $A$ defines a bijective homomorphism of the Heisenberg group.

Now, we will define the dilation action which will play an essential role in the construction of discrete wavelets in $L^2(\mathbb{H})$. Let $(a, b) \in \mathbb{R}^* \times \mathbb{R}^*$ and define

$$A_{(a,b)} = A = \begin{bmatrix} ab & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in GL(4, \mathbb{R})$$

so that

$$A \begin{bmatrix} 1 & x & -y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & ax & -by & abz \\ 0 & 1 & 0 & by \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Moreover, we assume that $(a, b) \in \mathbb{R}^* \times \mathbb{R}^*$ are chosen so that

$$ab = 2. \quad (2.4)$$

We acknowledge that (2.4) seems rather peculiar at first. However, the need to impose this condition will be clarified in the proof of Theorem 1 and Theorem 3. Moreover, we also observe that in general the group $A^{-1} \Gamma A$ is not a subgroup of $\Gamma$. In fact, $A^{-1} \Gamma A$ is not generally even a discrete subgroup of the Heisenberg group. Therefore, we remark that the structure that we are interested in this paper does not fit the definition of affine structure given in [2].
Put
\[ H = \langle A \rangle \cong \mathbb{Z} \]
and define \( D : H \to \mathcal{U}(L^2(\mathbb{H})) \) such that
\[ D_A f(n) = |\text{det } A|^{-1/2} f(A^{-1}nA) = \frac{1}{2} f(A^{-1}nA). \]

Then \( D \) is a dilation representation which will play an important role in the construction of bandlimited wavelets on the Heisenberg group.

2.4. **Short survey on frame theory.** Given a countable sequence \( \{f_i\}_{i \in I} \) of vectors in a Hilbert space \( \mathbf{H} \), we say \( \{f_i\}_{i \in I} \) forms a frame if and only if there exist strictly positive real numbers \( A, B \) such that for any vector \( f \in \mathbf{H} \)
\[ A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2. \]

In the case where \( A = B \), the sequence of vectors \( \{f_i\}_{i \in I} \) forms what we call a tight frame, and if \( A = B = 1 \), \( \{f_i\}_{i \in I} \) is called a Parseval frame or a normalized tight frame. Let us suppose that \( \{f_i\}_{i \in I} \) is a Parseval frame in \( \mathbf{H} \). Then for any vector \( h \in \mathbf{H} \), we have the following remarkable expansion formula:
\[ h = \sum_{i \in I} \langle h, f_i \rangle f_i. \]

A lattice \( \mathcal{L} \) in \( \mathbb{R}^{2d} \) is a discrete additive subgroup of \( \mathbb{R}^{2d} \). A lattice \( \mathcal{L} \) is called a full-rank lattice if \( \mathcal{L} = M\mathbb{Z}^{2d} \) for some invertible matrix \( M \) of order \( 2d \). We say \( \mathcal{L} \) is separable if \( \mathcal{L} = A\mathbb{Z}^d \times B\mathbb{Z}^d \) and \( A, B \) are invertible matrices of order \( d \). Let \( f \in L^2(\mathbb{R}^d) \). The family of functions
\[ \mathcal{G}(f, \mathcal{L}) = \{ e^{2\pi i \langle k, x \rangle} f(x - n) : k \in B\mathbb{Z}^d, n \in A\mathbb{Z}^d \} \]
is called a Gabor system. The volume of \( \mathcal{L} = M\mathbb{Z}^{2d} \) is defined as \( \text{vol}(\mathcal{L}) = |\text{det } M| \) and the density of \( \mathcal{L} \) is defined as \( d(\mathcal{L}) = |\text{det } M|^{-1} \).

**Lemma 6.** Given a separable full-rank lattice \( \mathcal{L} = A\mathbb{Z}^d \times B\mathbb{Z}^d \) in \( \mathbb{R}^{2d} \), the following statements are equivalent.

1. There exists \( f \in L^2(\mathbb{R}^d) \) such that \( \mathcal{G}(f, \mathcal{L}) \) is a Parseval frame in \( L^2(\mathbb{R}^d) \).
2. \( \text{vol}(\mathcal{L}) = |\text{det } A \text{ det } B| \leq 1. \)
3. There exists \( f \in L^2(\mathbb{R}^d) \) such that \( \mathcal{G}(f, \mathcal{L}) \) is complete in \( L^2(\mathbb{R}^d) \).

See theorem 3.3 in [8].

**Lemma 7.** Let \( \mathcal{L} \) be a full rank lattice in \( \mathbb{R}^{2d} \). If \( \mathcal{G}(f, \mathcal{L}) \) is a Parseval frame for \( L^2(\mathbb{R}^d) \), then \( \|f\|^2 = \text{vol}(\mathcal{L}) \).

For a complete proof of the Lemma above, we refer the reader to [8]. The following lemma is due to Khosravi and Asgari (see Theorem 2.3 [1]).
Lemma 8. Let \( \{x_n\}_{n \in I} \) and \( \{y_m\}_{m \in J} \) be two Parseval frames for Hilbert spaces \( H \) and \( K \), respectively. Then \( \{x_n \otimes y_m\}_{(n,m) \in I \times J} \) is a Parseval frame for the Hilbert space \( H \otimes K \).

3. Proofs of the main results

Lemma 9. Let \( v \in L^2(\mathbb{R}) \), and \( \overline{v} \) its conjugate. If \( G(\pi, \mathbb{Z} \times \lambda \mathbb{Z}) \) is a Parseval Gabor frame then

\[
\sum_{\eta \in \mathbb{I}} \left| \langle \pi_\lambda(\eta) v, u \rangle_{L^2(\mathbb{R})} \right|^2 = \|u\|_{L^2(\mathbb{R})}^2
\]

for all \( u \in L^2(\mathbb{R}) \) and

\[
\mathbb{I} = \left\{ \begin{bmatrix} 1 & m_3 & -m_2 & 0 \\ 0 & 1 & 0 & m_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : (m_2, m_3) \in \mathbb{Z}^2 \right\}.
\]

Proof. Let us suppose that \( G(\pi, \mathbb{Z} \times \lambda \mathbb{Z}) \) is a Parseval Gabor frame. Now, let

\[
v = \sum_{k \in \mathbb{J}} \alpha_k e_k \quad \text{and} \quad u = \sum_{j \in \mathbb{J}} \beta_j e_j, \quad \alpha_k, \beta_j \in \mathbb{C}
\]

where \( \{e_k : k \in \mathbb{J}\} \) is an orthonormal basis for \( L^2(\mathbb{R}) \) such that \( \overline{e_k} = e_k \) for all \( k \in \mathbb{J} \). Then

\[
\sum_{\eta \in \mathbb{I}} \left| \langle \pi_\lambda(\eta) v, u \rangle_{L^2(\mathbb{R})} \right|^2 = \sum_{\eta \in \mathbb{I}} \left| \langle \pi_\lambda(\eta) \left( \sum_{k \in \mathbb{J}} \alpha_k e_k \right), \sum_{j \in \mathbb{J}} \beta_j e_j \rangle \right|^2
\]

\[
= \sum_{\eta \in \mathbb{I}} \sum_{k \in \mathbb{J}} \sum_{j \in \mathbb{J}} \overline{\beta_j} \alpha_k \left| \langle \pi_\lambda(\eta) e_k, e_j \rangle \right|^2.
\]

Next, since

\[
\pi_\lambda(\eta) e_k = \pi_\lambda(\eta^{-1})^t e_k,
\]

then

\[
\sum_{\eta \in \mathbb{I}} \left| \langle \pi_\lambda(\eta) v, u \rangle \right|^2 = \sum_{\eta \in \mathbb{I}} \sum_{k \in \mathbb{J}} \sum_{j \in \mathbb{J}} \overline{\beta_j} \alpha_k \left| \langle \pi_\lambda(\eta^{-1})^t e_k, e_j \rangle \right|_{L^2(\mathbb{R})}^2
\]

\[
= \sum_{\eta \in \mathbb{I}} \sum_{k \in \mathbb{J}} \sum_{j \in \mathbb{J}} \overline{\beta_j} \alpha_k \left| \langle \pi_\lambda(\eta^{-1}) e_j, e_k \rangle \right|_{L^2(\mathbb{R})}^2.
\]

The second equality above is due to the fact that

\[
\langle \pi_\lambda(\eta)^t e_j, e_k \rangle_{L^2(\mathbb{R})} = \langle \pi_\lambda(\eta) e_k, e_j \rangle_{L^2(\mathbb{R})}.
\]
Next,\[
\sum_{\eta \in I} \left| \langle \pi_{\lambda}(\eta) v, u \rangle_{L^2(\mathbb{R})} \right|^2 = \sum_{\eta \in I} \left| \left\langle \sum_{j \in J} \beta_j e_j, \sum_{k \in J} \alpha_k e_k \right\rangle_{L^2(\mathbb{R})} \right|^2
\]
\[
= \sum_{\eta \in I} \left| \left\langle \sum_{j \in J} \beta_j e_j, \sum_{k \in J} \frac{\pi}{\lambda(\eta)} e_k \right\rangle_{L^2(\mathbb{R})} \right|^2
\]
\[
= \sum_{\eta \in I} \left| \left\langle \sum_{j \in J} \beta_j e_j, \pi_{\lambda}(\eta) e_\tau \right\rangle_{L^2(\mathbb{R})} \right|^2
\]
\[
= \left\| \sum_{j \in J} \beta_j e_j \right\|_{L^2(\mathbb{R})}^2
\]
\[
= \|u\|_{L^2(\mathbb{R})}^2.
\]

3.1. Proof of Theorem 1. Let
\[
(\gamma, \eta) = \begin{pmatrix}
1 & k_3 & -k_2 & k_1 \\
0 & 1 & 0 & k_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
1 & m_3 & -m_2 & 0 \\
0 & 1 & 0 & m_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
where \(k_i\) and \(m_j\) are integers. Also, we define
\[
\Lambda_1 = \left\{ \begin{pmatrix}
1 & k_3 & -k_2 & 0 \\
0 & 1 & 0 & k_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & m_3 & -m_2 & 0 \\
0 & 1 & 0 & m_2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} : k_i, m_j \in \mathbb{Z} \right\}.
\]
We will show that there is a function \(f\) such that for any vector \(h \in H_S\),
\[
\sum_{(\gamma, \eta) \in \Lambda} \left| \langle h, \tau(\gamma, \eta) f \rangle_{L^2(H)} \right|^2 = \|h\|_{L^2(H)}^2.
\]
Using the fact that \(P\) is a unitary map, and that
\[
P(\tau(x, y) \phi)(\lambda) = \pi_{\lambda}(x)(P \phi)(\lambda)(\pi_{\lambda}(y))^* = [\pi_{\lambda}(x) \otimes \pi_{\lambda}(y)](P \phi)(\lambda),
\]
we have

\[
\sum_{(\gamma, \eta) \in \Lambda} \left| \langle h, \tau (\gamma, \eta) f \rangle_{L^2(\mathbb{H})} \right|^2 = \sum_{(\gamma, \eta) \in \Lambda} \left| \int_{\mathcal{S}} \langle \mathbf{P} h(\lambda), \left[ \pi_\lambda (\gamma) \otimes \pi_\lambda (\eta) \right] \mathbf{P} f(\lambda) | \lambda \rangle |_{\mathcal{HS}} d\lambda \right|^2.
\]

Next, we define

\[
F_{\kappa, \eta} (\lambda) = \langle \mathbf{P} h(\lambda), \left[ \pi_\lambda (\kappa) \otimes \pi_\lambda (\eta) \right] \mathbf{P} f(\lambda) | \lambda \rangle |_{\mathcal{HS}}.
\]

Using (3.1), we obtain:

\[
\sum_{(\gamma, \eta) \in \Lambda} \left| \langle h, \tau (\gamma, \eta) f \rangle_{L^2(\mathbb{H})} \right|^2 = \sum_{(\kappa, \eta) \in \Lambda_1} \sum_{k_1 \in \mathbb{Z}} \left| \int_{\mathcal{S}} e^{2\pi i \langle \lambda, k_1 \rangle} F_{\kappa, \eta} (\lambda) d\lambda \right|^2.
\]

Since \( \mathcal{S} \) is translation congruent to \((0, 1] \) then \( \{ e^{2\pi i \langle \lambda, k \rangle} \chi_{\mathcal{S}} (\lambda) : k \in \mathbb{Z} \} \) is an orthonormal basis for \( L^2 (\mathcal{S}) \). Since \( F_{\kappa, \eta} \) belongs to \( L^2 (\mathcal{S}) \) then the function \( k \mapsto \int_{\mathcal{S}} e^{2\pi i \langle \lambda, k \rangle} F_{\kappa, \eta} (\lambda) d\lambda \) is the Fourier transform of \( F_{\kappa, \eta} \). Thus

\[
\sum_{(\gamma, \eta) \in \Lambda} \left| \langle h, \tau (\gamma, \eta) f \rangle_{L^2(\mathbb{H})} \right|^2 = \sum_{(\kappa, \eta) \in \Lambda_1} \sum_{k_1 \in \mathbb{Z}} \left| \widehat{F_{\kappa, \eta}} (k_1) \right|^2.
\]

Now, using Plancherel theorem on \( L^2 (\mathcal{S}) \), we obtain

\[
\sum_{(\gamma, \eta) \in \Lambda} \left| \langle h, \tau (\gamma, \eta) f \rangle_{L^2(\mathbb{H})} \right|^2 = \sum_{(\kappa, \eta) \in \Lambda_1} \left\| \widehat{F_{\kappa, \eta}} \right\|_{L^2(\mathcal{S})}^2
\]

\[
= \sum_{(\kappa, \eta) \in \Lambda_1} \int_{\mathcal{S}} \left| \langle \mathbf{P} h(\lambda), \left[ \pi_\lambda (\kappa) \otimes \pi_\lambda (\eta) \right] \mathbf{P} f(\lambda) | \lambda \rangle |_{\mathcal{HS}} \right|^2 d\lambda.
\]
Letting $P f(\lambda) = u_\lambda \otimes v_\lambda$, so that $(P f(\lambda))_{\lambda \in \mathcal{S}}$ is a measurable field of rank-one operators

$$
\sum_{(\gamma, \eta) \in \Lambda} \left| \langle h, \tau (\gamma, \eta) f \rangle_{L^2(\mathbb{H})} \right|^2
$$

$$
= \sum_{(\kappa, \eta) \in \Lambda_1} \int_{\mathcal{S}} \left| \langle P h(\lambda), [\pi_\lambda (\kappa) \otimes \overline{\pi}_\lambda (\eta)] u_\lambda \otimes v_\lambda |\lambda| \rangle_{HS} \right|^2 d\lambda
$$

$$
= \int_{\mathcal{S}} \sum_{(\kappa, \eta) \in \Lambda_1} \left| \langle P h(\lambda), [\pi_\lambda (\kappa) \otimes \overline{\pi}_\lambda (\eta)] (u_\lambda \otimes v_\lambda) |\lambda| \rangle_{HS} \right|^2 d\lambda
$$

$$
= \int_{\mathcal{S}} \sum_{(\kappa, \eta) \in \Lambda_1} \left| \left\langle P h(\lambda), \pi_\lambda (\kappa) \left( |\lambda|^{1/4} u_\lambda \right) \otimes \overline{\pi}_\lambda (\eta) \left( |\lambda|^{1/4} v_\lambda \right) \right\rangle_{HS} \right|^2 |\lambda| d\lambda.
$$

We will now assume that $f$ is defined such that for each $\lambda \in \mathcal{S}$, the set of systems

$$
(3.2) \quad \left\{ |\lambda|^{1/4} \pi_\lambda (I_1) u_\lambda, |\lambda|^{1/4} \pi_\lambda (I_2) \overline{v}_\lambda \right\}
$$

is a set of Parseval frames where $\Lambda_1 = I_1 \times I_2$ and

$$
I_1 = \left\{ \begin{bmatrix} 1 & k_3 & -k_2 & 0 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : (k_2, k_3) \in \mathbb{Z}^2 \right\},
$$

$$
I_2 = \left\{ \begin{bmatrix} 1 & m_3 & -m_2 & 0 \\ 0 & 1 & 0 & m_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} : (m_2, m_3) \in \mathbb{Z}^2 \right\}.
$$

We recall that by the density condition (see Lemma 6), and Lemma 9 the above fact is possible since

$$
|\lambda|^{1/4} \pi_\lambda (I_1) u_\lambda = \mathcal{G} \left( |\lambda|^{1/4} u_\lambda, \mathbb{Z} \times \lambda \mathbb{Z} \right),
$$

$$
|\lambda|^{1/4} \pi_\lambda (I_2) \overline{v}_\lambda = \mathcal{G} \left( |\lambda|^{1/4} \overline{v}_\lambda, \mathbb{Z} \times \lambda \mathbb{Z} \right),
$$

and $\text{vol}(\mathbb{Z} \times \lambda \mathbb{Z}) = |\lambda| \leq 1$. Next, we would like to prove that $f$ has finite norm. Applying the Plancherel theorem, we obtain

$$
\|f\|^2_{L^2(\mathbb{R})} = \int_{\mathcal{S}} \|P f(\lambda)\|^2_{HS} |\lambda| d\lambda = \int_{\mathcal{S}} \|u_\lambda\|^2_{L^2(\mathbb{R})} \|v_\lambda\|^2_{L^2(\mathbb{R})} |\lambda| d\lambda.
$$

Since we assume that

$$
\mathcal{G} \left( |\lambda|^{1/4} u_\lambda, \mathbb{Z} \times \lambda \mathbb{Z} \right), \text{ and } \mathcal{G} \left( |\lambda|^{1/4} \overline{v}_\lambda, \mathbb{Z} \times \lambda \mathbb{Z} \right)
$$
are Parseval frames in $L^2(\mathbb{R})$, by Lemma 7

$$|\lambda|^{1/2} \|u_\lambda\|_{L^2(\mathbb{R})}^2 = |\lambda|^{1/2} \|v_\lambda\|_{L^2(\mathbb{R})}^2 = \text{vol}(\mathbb{Z} \times \lambda \mathbb{Z}) = |\lambda|.$$ 

Thus,

$$\|u_\lambda\|_{L^2(\mathbb{R})}^2 = |\lambda|^{1/2} \|v_\lambda\|_{L^2(\mathbb{R})}^2 = |\lambda|^{1/2}$$

and

$$\|f\|_{L^2(\mathbb{H})}^2 \leq \int_{-1}^{1} \lambda^2 d\lambda = \frac{2}{3}.$$ 

Finally, applying Lemma 8,

$$\sum_{(\gamma, \eta) \in \Lambda} |\langle h, \tau(\gamma, \eta) f \rangle|_2^2$$

$$= \int_{S} \|P_h(\lambda)\|_{H_S}^2 \langle h, \pi(\lambda) \left( |\lambda|^{1/4} u_\lambda \right) \otimes \pi(\eta) \left( |\lambda|^{1/4} v_\lambda \right) \rangle_{H_S}^2 |\lambda| d\lambda$$

$$= \int_{S} \|P_h(\lambda)\|_{H_S}^2 |\lambda| d\lambda = \|h\|_{L^2(\mathbb{H})}^2.$$ 

### 3.2. Proof of Theorem 2

We recall that

$$\Lambda_1 = \left\{ \begin{pmatrix} 1 & k_3 & -k_2 & 0 \\ 0 & 1 & 0 & k_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \begin{pmatrix} 1 & m_3 & -m_2 & 0 \\ 0 & 1 & 0 & m_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} : k_i, m_j \in \mathbb{Z} \right\}.$$ 

Let $h \in H_S$. Let us suppose that $\tau(\Lambda) f$ is a Parseval frame in $H_S$. Then

$$\sum_{(\gamma, \eta) \in \Lambda} \left| \langle h, \tau(\gamma, \eta) f \rangle_{L^2(\mathbb{H})} \right|^2 = \|h\|_{L^2(\mathbb{H})}^2.$$
Now, using the fact that $S$ is translation congruent to $(0, 1]$, it is not too hard to check that

$$\sum_{(\gamma, \eta) \in \Lambda} \left| \langle h, \tau (\gamma, \eta) f \rangle_{L^2(\mathbb{H})} \right|^2 - \|h\|_{L^2(\mathbb{H})}^2$$

$$= \sum_{(\kappa, \eta) \in \Lambda_1} \int_S |\langle Ph(\lambda), [\pi_\lambda (\kappa) \otimes \bar{\pi}_\lambda (\eta)] (P f) (\lambda) |\lambda\rangle_{\mathcal{H}_S}|^2 d\lambda - \|h\|_{L^2(\mathbb{H})}^2$$

$$= \int_S \sum_{(\kappa, \eta) \in \Lambda_1} |\langle Ph(\lambda), [\pi_\lambda (\kappa) \otimes \bar{\pi}_\lambda (\eta)] (P f) (\lambda) |\lambda\rangle_{\mathcal{H}_S}|^2 d\lambda - \int_S \|Ph(\lambda)\|_{\mathcal{H}_S}^2 |\lambda| d\lambda$$

$$= \int_S \left( \sum_{(\kappa, \eta) \in \Lambda_1} |\langle Ph(\lambda), [\pi_\lambda (\kappa) \otimes \bar{\pi}_\lambda (\eta)] (P f) (\lambda) |\lambda\rangle_{\mathcal{H}_S}|^2 - \|Ph(\lambda)\|_{\mathcal{H}_S}^2 |\lambda| \right) d\lambda$$

$$= 0.$$

Next, replacing $h$ with $g$ such that $Pg(\lambda) = \chi_B (\lambda) Ph(\lambda)$ where $B$ is any Borel subset, then

$$\sum_{(\kappa, \eta) \in \Lambda_1} |\langle Ph(\lambda), [\pi_\lambda (\kappa) \otimes \bar{\pi}_\lambda (\eta)] (P f) (\lambda) |\lambda\rangle_{\mathcal{H}_S}|^2 - \|Ph(\lambda)\|_{\mathcal{H}_S}^2 |\lambda| = 0$$

for every $h$ and for $d\lambda$-almost every $\lambda \in S$. For every $h$, there exists a null set $N_h$ such that for every $\lambda \in S - N_h$, (3.3) holds. However, we must show that (3.3) holds for all $h$ and for all $\lambda$ in a conull subset which does not depend on $h$. In order to do so, we pick a countable dense subset $A$ of $L^2 (\mathbb{H})$ such that $\{h (\lambda) : h \in A\}$ is dense in $L^2 (\mathbb{R}) \otimes L^2 (\mathbb{R})$. Then for

$$\lambda \in S - \left( \bigcup_{h \in A} (N_h) \right)$$

and for $m \in A$ we have that

$$\sum_{(\kappa, \eta) \in \Lambda_1} \left| \left\langle Pm(\lambda), [\pi_\lambda (\kappa) \otimes \bar{\pi}_\lambda (\eta)] (P f) (\lambda) |\lambda\rangle_{\mathcal{H}_S}^{1/2} \right\rangle_{\mathcal{H}_S}^2 = \|Pm(\lambda)\|_{\mathcal{H}_S}^2.$$
3.3. Proof of Theorem 3. Let \( f \in H_S \) be defined such that \( P f = (u_\lambda \otimes v_\lambda)_{\lambda \in \mathcal{S}} \) such that \( G \left( |\lambda|^{1/4} u_\lambda, Z \times \lambda \mathbb{Z} \right) \) is a Parseval Gabor frame for \( d\lambda \)-almost every \( \lambda \in \mathcal{S} \), and \( G \left( |\lambda|^{1/4} \overline{\nu}_\lambda, Z \times \lambda \mathbb{Z} \right) \) is a Parseval Gabor frame for \( \lambda \in \mathcal{S} \), so that (due to Theorem II) the system \( \tau (\Lambda) f \) is a Parseval frame in \( H_S \). Let us define a unitary representation of \( H \) by \( C : H \to U(L^2(\mathbb{R})) \) such that for \( \phi \in L^2(\mathbb{R}) \)
\[
C(A)\phi(t) = |a|^{-1/2} \phi \left( a^{-1} t \right).
\]
Notice that \( C(A) \) is just a unitary operator acting on \( L^2(\mathbb{R}) \) by dilation. Let \( h \) be an arbitrary element of the Hilbert space \( H_S \). It is fairly easy to see that
\[
P(D_{A^m} h)(\lambda) = |\det A|^{m/2} C(A^m) \circ Ph(2^m \lambda) \circ C(A^m)^{-1}.
\]
To see that this holds, it suffices to perform the following computations. Given arbitrary \( u, v \in L^2(\mathbb{R}) \) we have
\[
\langle P(D_A h)(\lambda) u, v \rangle = \int \! \! \! \int D_A h(r) \langle \pi_\lambda(r) u, v \rangle dr
\]
\[
= \int \! \! \! \int |\det A|^{1/2} h(r) \langle \pi_\lambda(ArA^{-1}) u, v \rangle dr
\]
\[
= |\det A|^{1/2} \int \! \! \! \int h(r) \langle C(A) \pi_{2\lambda}(r) C(A)^{-1} u, v \rangle dr.
\]
The last equality above is justified because for any \( u \in \mathbb{L}^2(\mathbb{R}) \),
\[
\pi_{2\lambda}(x, y, z) \left[ C(A)^{-1} u \right] (t) = e^{2\pi i (2\lambda) z} e^{-2\pi i 2\lambda y t} \left[ C(A)^{-1} u \right] (t - x) = |a|^{1/2} e^{2\pi i (2\lambda) z} e^{-2\pi i 2\lambda y t} u(a(t - x)) = |a|^{1/2} e^{2\pi i \lambda (2z)} e^{-2\pi i \lambda y t} u(at - ax) = |a|^{1/2} \left[ \pi_\lambda(ArA^{-1}) u \right] (at) = C(A)^{-1} \left[ \pi_\lambda(A(x, y, z)A^{-1}) u \right] (t).
\]
So,
\[
P(D_{A^m} h)(\lambda) = |\det A|^{m/2} C(A^m) Ph(2^m \lambda) C(A^m)^{-1}.
\]
Assuming that \( Ph(\lambda) = u_\lambda \otimes v_\lambda \) is a rank-one operator, we check that
\[
P(D_{A^m} h)(\lambda) w = |\det A|^{m/2} C(A^m) (u_{2m,\lambda} \otimes v_{2m,\lambda}) C(A^m)^{-1} w
\]
\[
= |\det A|^{m/2} \left\langle C(A^m)^{-1} w, v_{2m,\lambda} \right\rangle C(A^m) u_{2m,\lambda}
\]
\[
= |\det A|^{m/2} \left\langle w, C(A^m) v_{2m,\lambda} \right\rangle C(A^m) u_{2m,\lambda}
\]
\[
= |\det A|^{m/2} \left( C(A^m) u_{2m,\lambda} \otimes C(A^m) v_{2m,\lambda} \right) w.
\]
As a result,
\[
P(D_{A^m} h)(\lambda) = |\det A|^{m/2} (C(A) u_{2m,\lambda} \otimes C(A) v_{2m,\lambda}).
\]
Thus, $D_{A^m}(H_S) = H_{2^{-m}(S)}$ and for $m \neq k$, $H_{2^{-m}(S)}$ is orthogonal to $H_{2^{-k}(S)}$. Also, since the system $\tau(\Lambda)f$ is a Parseval frame (see Theorem 1) for $H_S$ and since $D_{A^m}$ is unitary, then

$$D_{A^m}(\tau(\Lambda)f) = \tau(A^m(\Lambda))D_{A^m}f$$

is a Parseval frame in the Hilbert space $H_{2^{-m}(S)}$. Finally, since $S$ is dilation congruent to the set $[-1, -1/2) \cup (1/2, 1]$ then $\bigcup_{m \in \mathbb{Z}} H_{2^{-m}(S)}$ is dense in $L^2(\mathbb{H})$ and

$$\bigcup_{m \in \mathbb{Z}} H_{2^{-m}(S)} = L^2(\mathbb{H}).$$

We conclude that the given system

$$\{D_{A^m}\tau(\gamma, \eta)f : m \in \mathbb{Z}, (\gamma, \eta) \in \Lambda\}$$

is a Parseval frame for $L^2(\mathbb{H})$.

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