Systematic 1/N corrections for bosonic and fermionic vector models without auxiliary fields.

Robert de Mello Koch and João P. Rodrigues,
Department of Physics and Centre for Non Linear Studies,
University of the Witwatersrand, Wits 2050, South Africa

In this paper, colorless bilocal fields are employed to study the large $N$ limit of both fermionic and bosonic vector models. The Jacobian associated with the change of variables from the original fields to the bilocals is computed exactly, thereby providing an exact effective action. This effective action is shown to reproduce the familiar perturbative expansion for the two and four point functions. In particular, in the case of fermionic vector models, the effective action correctly accounts for the Fermi statistics. The theory is also studied non-perturbatively. The stationary points of the effective action are shown to provide the usual large $N$ gap equations. The homogeneous equation associated with the quadratic (in the bilocals) action is simply the two particle Bethe Salpeter equation. Finally, the leading correction in $1/N$ is shown to be in agreement with the exact $S$ matrix of the model.

PACS numbers:11.10.Lm, 11.10.St, 11.15.Pg, 03.70.+k.

I. INTRODUCTION

One of the most important problems facing high energy physics today is the construction of suitable approximation techniques that will allow an analytic understanding of the long distance properties of non Abelian gauge theories. The large $N$ expansion remains one of the most promising techniques proposed to date. The leading order of this approximation, the master field, is given by the sum over planar diagrams in matrix models [1]. It has up to now been impossible to compute this sum in closed form, except for simple systems [2]. For this reason, not even the leading term in the expansion, in four dimensions, has been computed. In principle, if the leading term could be obtained, one would set up a perturbation theory around asymptotic states consisting of "mesons" and "hadrons" - colorless bound states of the quanta of the matter fields appearing in the Lagrangian. In other words, fluctuations about this master field yield the mass spectrum of the theory.

Vector models by contrast, have soluble large $N$ limits. Indeed, the large $N$ limit of a number of vector models has been studied using an auxiliary field [3], [4], which is not (classically) dynamical. Formally eliminating the auxiliary field, the original model under consideration is regained. The advantage of the auxiliary field is that it is chosen so that it contains no implicit $N$ dependence. Thus, integration over the original field variables yields an effective theory in which all $N$ dependence is fully explicit. There are however, at least two serious objections to this approach. Firstly, it is only viable for Lagrangians consisting of quadratic plus quartic terms. Secondly, the local auxiliary field is a poor substitute for the composite (bilocal) meson field.

Jevicki and Levine [6] have constructed the master fields for a large number of bosonic vector models, working directly with equal time bilocal meson fields, thereby overcoming the difficulties mentioned above. They rewrite the Hamiltonian in terms of these bilocal fields, obtaining an effective theory of mesons in which all $N$ dependence is fully explicit. This change of variables induces a non trivial Jacobian [7], which is obtained by imposing hermiticity on a suitably scaled Hamiltonian. This approach is not easily implemented for fermionic vector models: it is a well known fact that the Dirac hamiltonian describes constrained dynamics. These constraints obscure the Hamiltonian approach. For this reason, the large $N$ limit of fermionic vector models was originally studied using a pseudospin formalism [8]. Although this equal time approach does provide a possible starting point for a systematic expansion in $1/N$, the presence of the pseudospin constraints obscures the relationship between the pseudospin variables and quantities of interest. The possibility of reexpressing a fermionic vector theory in terms of unequal time bilocals has been recently considered by Cavichi et al. [9] who obtained the leading large $N$ form of the effective action directly in terms of these bilocals, and were therefore able to reproduce t’Hooft’s $QCD_2$ equation [10] for the meson spectrum.

In matrix models, higher order corrections in $1/N$ have been systematically computed in the case of the c=1 matrix model string theory [11] directly in terms of invariant variables. These corrections have been shown to be in complete agreement with the exact solution of the model. In this case the effective propagator is simply that of a massless scalar and it also turns out that the effective Hamiltonian is cubic. The invariant variables of the matrix model correspond
to the bilocals in vector models, so the question that we address in this article is whether systematic higher order corrections in $\frac{1}{N}$ can be computed directly in terms of the bilocal variables for vector models.

Higher order calculations have been performed for a Gross-Neveu model with scalar-scalar interactions, using the auxiliary field approach [2], or directly in terms of the original fermionic fields [3]. As a check of the $\frac{1}{N}$ expansion,
given that the exact $S$ matrix of the model is known [4].

In this paper, we develop the necessary formalism needed to systematically compute higher order corrections in $\frac{1}{N}$, by obtaining an exact effective action in terms of unequal time bilocal fields. These bilocal fields are time ordered and we use them to study both bosonic and fermionic vector models.

We obtain the exact effective action by explicitly calculating the full Jacobian associated with the change of variables from the original fields to the time ordered bilocals. This is achieved by requiring that the Schwinger Dyson equations derived directly in terms of the bilocal variables agree with the equations derived in terms of the original fields [5]. This provides a unified approach for both bosonic and fermionic models. The bosonic Jacobian agrees with the result obtained from collective field theory [6]. The result for the fermionic Jacobian is new. It is remarkable that within the context of a path integral quantization such a Jacobian exists and is exactly computable.

This effective action is nonlocal in time but in the context of the functional integral quantization it generates all the "colorless" correlators of the quantum theory. For fermionic theories, since the effective action is exact and also valid for $N = 1$, it provides an exact rewriting of the original theory (although for $N = 1$ of course there is no small expansion parameter). We therefore expect this approach to have interesting applications in condensed matter type fermionic systems. The formalism is also likely to be particularly relevant for problems in which the spectrum is expected to consist only of singlets under the global invariance of the theory, since it then directly describes the dynamics of the physical degrees of freedom. Finally we wish to emphasize that this formalism provides a (nonlocal) bosonization valid for arbitrary dimensions. Although many of these aspects are now under study the main purpose of this article is to obtain the effective action and confirm its validity perturbatively.

So, perturbatively in both $\frac{1}{N}$ and the coupling constant, we consider the linear sigma model and the Gross Neveu model (with scalar-scalar interactions) and verify that typical correlators are in agreement with their Feynman expansions, and in particular verify that the statistics is properly taken into account. This provides a strict test of the validity of our effective action. We go on to describe how to develop a systematic expansion of the effective action and explicitly establish $\frac{1}{N}$ as the perturbative parameter of the expansion.

The paper is organized as follows: in section II the effective action, connection to the collective field theory and check of the perturbative expansion in both $\frac{1}{N}$ and the coupling constant is carried out. In section III we do similarly for fermionic vector models. In section IV we make contact with the (nonperturbative) analysis of the spectrum for Gross-Neveu and Nambu-Jona-Lasinio type model by obtaining the propagator of our effective field theory. We show further that the homogeneous equation for the quadratic fluctuations of our effective field theory is a Bethe-Salpeter equation for these models. In section V we obtain all diagrams contributing to the $\frac{1}{N}$ correction to two particle scattering from the effective field theory and show that they are in precise agreement with the diagrams considered in previous comparisons of the $\frac{1}{N}$ expansion with the exact $S$ matrix [3].

II. BOSONIC VECTOR MODELS AND THE RELATIONSHIP TO THE COLLECTIVE FIELD THEORY.

A. Schwinger Dyson equations and Jacobian.

In this section we will consider $U(N)$ invariant bosonic vector models. By this we mean any theory of $N$ complex scalar fields $\phi^a(x), \ a = 1, 2, ..., N$ ($x$ is a $d$ dimensional parameter) with an action $S$ that is invariant under the global symmetry $\phi^a \rightarrow s^a \phi^a = U_{ab} \phi^b, \ s^a \rightarrow s^a = U_{ab} s^b$ with $U_{ab}$ an arbitrary $U(N)$ element. Furthermore we assume that all coupling constants of the theory have been rescaled appropriately so as to yield a systematic $1/N$ expansion. This is equivalent to the statement that a rescaling of the fields exists under which $S \rightarrow N \tilde{S}$. A specific example that will be examined in detail later in this section is given by the action

$$S = \int d^4 x (\partial_\mu \phi^{*a} \partial^\mu \phi^a - m^2 \phi^{*a} \phi^a - \frac{g^2}{8N} (\phi^{*a} \phi^a)^2),$$  \hspace{1cm} (1)$$

although our discussion applies to arbitrary invariant actions $S$ as described above. These include (non local) effective actions resulting from the explicit integration of intermediate degrees of freedom such as $QCD_2$. This will be the object of further study in another communication [7].
We will assume further that one is only interested in time ordered product expectation values of invariant operators (generically referred to as invariant correlators)

\[ < F[\sigma] > = \int D\phi^* D\phi e^{iS} \prod_{i=1}^{m} \sigma(x_i, y_i) \]

with

\[ \sigma(x, y) = \phi^a(x) \phi^a(y) \]  

It is straightforward to obtain a set of Schwinger-Dyson equations for the invariant correlators. These follow from the identity

\[ 0 = \int D\phi^* D\phi \frac{\delta}{\delta \phi^a(x)} \left[ \phi^a(y) F[\sigma] e^{iS} \right] \]

yielding

\[ < N \delta^d(x - y) F[\sigma] > + < \phi^a(y) \frac{\delta F}{\delta \phi^a(x)} > + i < \phi^a(y) \frac{\delta S}{\delta \phi^a(x)} F[\sigma] > = 0 \]  

The above set of equations involve only invariant correlators as it will be shown in the following.

The philosophy that we wish to adopt here is that there is a Jacobian \( J \) associated with the change of variables from the original variables \( \phi^a \) to the invariant variables (3) inside the functional integral that correctly yields all time ordered product expectation values of invariant operators (7). In other words,

\[ \int D\phi^* D\phi F[\sigma] e^{iS} = \int D\sigma J F[\sigma] e^{iS} \]  

We now follow the procedure described in reference [15] to obtain a differential equation for the Jacobian from the loop equations (5). This results from the identity:

\[ 0 = \int D\sigma \int d^d z \frac{\delta}{\delta \sigma(z, x)} (\sigma(z, y) J F[\sigma] e^{iS}) \]

which implies (5)

\[ < N \delta^d(x - y) L^d F[\sigma] > + \int d^d z \sigma(z, y) \frac{\delta \ln J}{\delta \sigma(z, x)} F[\sigma] + \int d^d z \sigma(z, y) \frac{\delta F[\sigma]}{\delta \sigma(z, x)} + i \int d^d z \sigma(z, y) \frac{\delta S}{\delta \sigma(z, x)} F[\sigma] > = 0. \]  

We can now use the chain rule

\[ \frac{\delta}{\delta \phi^a(x)} = \int d^d z \int d^d y \frac{\delta \sigma(z, y)}{\delta \phi^a(x)} \frac{\delta}{\delta \sigma(z, y)} = \int d^d z \phi^a(z) \frac{\delta}{\delta \sigma(z, x)} \]

in equation (5) which is then equivalently written as

\[ < N \delta^d(x - y) F[\sigma] > + \int d^d z \sigma(z, y) \frac{\delta F}{\delta \sigma(z, x)} > + i \int d^d z \sigma(z, y) \frac{\delta S}{\delta \sigma(z, x)} F[\sigma] > = 0. \]  

Requiring agreement of this last equation with (5) for arbitrary \( F[\sigma] \) we obtain

\[ \int d^d z \sigma(z, y) \frac{\delta \ln J}{\delta \sigma(z, x)} = (N - L^d \delta^d(0)) \delta^d(x - y). \]  

\(^1\)We use \( L^d \) to denote the volume of the system we are studying, i.e. \( L^d = \int d^d x. \)
J is independent of the action, as it should be. The solution to this equation is:

$$
\ln J = (N - L^d \delta^d(0)) Tr \ln \sigma
$$

where the trace is in functional space. For models with "flavor" degrees of freedom $\phi_{a, \alpha}$, $\alpha = 1, 2, ..., m$ the invariants are

$$
\sigma_{\alpha \beta}(x, y) = \phi^a_{\alpha}(x) \phi^a_{\beta}(y)
$$

It is straightforward to generalize the above analysis and to show that the Jacobian is given by

$$
\ln J = (N - mL^d \delta^d(0)) tr Tr \ln \sigma
$$

where the trace is now taken in both functional and flavor space. This result generalizes for an arbitrary number of dimensions the result obtained in [17] for $d = 0$ bosonic vector models.

B. Connection with the collective field theory

The point of view that the large $N$ limit can be understood in terms of a change of variables to invariant variables or subspaces was successfully exploited by Jevicki and Sakita [7] for a large class of models. In the first of [7] these authors are able to obtain the form of the effective Hamiltonian acting on the reduced invariant subspace in terms of the Jacobian associated with the new inner product measure. Remarkably [7] the equation satisfied by the Jacobian can be obtained by the simple requirement that the effective Hamiltonian must be explicitly Hermitian and is given by

$$
\sum_{C, C'} \Omega(C, C') \frac{\delta \ln J}{\delta \phi(C')} = \omega(C) - \sum_{C, C'} \frac{\delta \Omega(C, C')}{\delta \phi(C')}.
$$

In this equation $C$ and $C'$ index the invariant variables. In our case

$$
\Omega(x, y; x', y') = \int d^{d-1}z \frac{\delta}{\delta \phi^a(z)} \sigma^*(x, y) \frac{\delta}{\delta \phi^a(z)} \sigma(x', y') = \delta^{d-1}(y - y')\sigma(x', x)
$$

and

$$
\omega(x, y) = \int d^{d-1}z \frac{\delta^2}{\delta \phi^a(z) \delta \phi^b(z)} \sigma(x, y) = N \delta^{d-1}(x - y).
$$

The above equations only involve equal time correlators as it is appropriate for a Hamiltonian approach. Jevicki and Sakita, in the second of [7] have shown that exactly the same equation must be satisfied in a functional integral description provided the invariants are appropriately labeled i.e. in our case if $x$ and $y$ are $d$ dimensional spacetime points. With this proviso, it is straightforward to show that once equations (16) and (17) are used in equation (15) we reproduce the differential equation (11) for the Jacobian derived in the previous subsection.

C. Perturbative Check

The full Jacobian (12) is not new; it has certainly been written down in [8]. However in all applications that we are aware of only the leading large $N$ term in the Jacobian has been used to obtain the leading contribution to the free energy, leading time ordered correlators and spectrum [3]. It is actually not clear in what sense the second term in the Jacobian is "subleading" due to infinities appearing in it and it is not at all obvious how that it can help generate systematic $1/N$ corrections. It is the purpose of this subsection to show that this is indeed the case, perturbatively in the coupling constant.

---

2With respect to the trivial measure.
1. Exact Effective Action

Any invariant correlator can be calculated from

\[
<F[\sigma]> = \frac{\int D\sigma J e^{iS[\sigma]} F[\sigma]}{\int D\sigma J e^{iS}}
\]

where

\[
S_{\text{eff}} = -i \ln J + S = -i N \text{Tr} \ln \sigma + S + i L^d \delta^d(0) \text{Tr} \ln \sigma
\]

In order to exhibit explicitly the \(N\) dependence we rescale \(\sigma \rightarrow N \sigma\) under which, as explained in subsection A, \(S \rightarrow NS\). Throughout the rest of this section this rescaling is always implicit. Therefore we have:

\[
S_{\text{eff}} = -i N \text{Tr} \ln \sigma + NS + i L^d \delta^d(0) \text{Tr} \ln \sigma = NS_0 + S_1.
\]

We see that as \(N \rightarrow \infty\) the leading ("classical") configuration is determined by

\[
\frac{\delta S_0}{\delta \sigma} |_{\sigma^0} = 0
\]

We can now perturb about \(\sigma^0\) by letting

\[
\sigma(x, y) = \sigma^0(x, y) + \frac{1}{\sqrt{N}} \eta(x, y)
\]

and expanding \(S_{\text{eff}}\) as

\[
S_{\text{eff}} = NS_0(\sigma^0) + S_1(\sigma^0) + \frac{i}{2} B_2 + \frac{1}{2} A_2 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{N}^n} \left[ -i \frac{(-1)^{n+1}}{(n+2)} B_{n+2} + i L^d \delta^d(0) \frac{(-1)^{n+1}}{n} B_n + \frac{1}{(n+2)!} A_{n+2} \right]
\]

where

\[
A_n = \int dx_1 \cdots \int dx_n \int dy_1 \cdots \int dy_n \frac{\delta^n S}{\delta \sigma(x_1, y_1) \cdots \delta \sigma(x_n, y_n)} |_{\sigma^0} \eta(x_1, y_1) \cdots \eta(x_n, y_n)
\]

and

\[
B_n = \text{Tr}(\sigma^{0-1} \eta)^n.
\]

We obtained an effective action with an infinite number of vertices as it is to be expected from any loop expansion. It should however be remembered that in order to calculate any diagram to a given order of \(\frac{1}{\sqrt{N}}\) only a finite number of vertices need to be included. Notice that the subleading term of the Jacobian \(12\) induces tadpole type interactions. In the case of \(c = 1\) strings where the effective Hamiltonian consists of a cubic and a tadpole interaction this tadpole interaction has been shown to be essential for an agreement with exact results \(11\). For \(d = 0\) vector models similar arguments have been presented in reference \(17\). We will see in the following that the tadpole interaction will be essential to obtain agreement with a perturbative Feynman analysis of the \((\phi^{*a} \phi^a)^2\) theory.

\[\text{\textsuperscript{3}}\text{The expansion of Tr} \ln \sigma \text{ in this fashion is justified by the fact that translational invariance requires } \sigma^0 \text{ to be diagonal in momentum space.}\]
2. Perturbative Results for the \((\phi^a \phi^a)^2\) Theory.

We will now be considering in detail the theory defined by the action (1).

\[
S = \int d^d x \left( \partial_{\mu} \phi^a \partial^{\mu} \phi^a - m^2 \phi^a \phi^a - \frac{g^2}{8N} (\phi^a \phi^a)^2 \right)
\]

We will need the following perturbative result for the two-point function:

\[
< \phi^a(x) \phi^a(y) > = \int \frac{d^d p}{(2\pi)^d} G_2(p)
\]

where

\[
G_2(p) = N(-2ig^2\frac{\epsilon^2}{8} + 4(-ig^2)\frac{\epsilon^2}{8} + 4(-ig^2)^2 + O(g^6))
\]

\[
+ (-2ig^2\frac{\epsilon^2}{8} + 8(-ig^2)^2 + 8(-ig^2)^2 + 4(-ig^2)^2 + O(g^6)) + O(\frac{1}{N})
\]

For the connected piece of the four-point function

\[
< \phi^a(x_1) \phi^a(y_1) \phi^b(x_2) \phi^b(y_2) > = \int \frac{dp_1}{(2\pi)^d} \int \frac{dp_2}{(2\pi)^d} \int \frac{dp_3}{(2\pi)^d} \int \frac{dp_4}{(2\pi)^d} \delta(p_1 + p_3 - p_2 - p_4) \times e^{i(p_1 y_1 - p_2 y_2 + p_3 x_2 + p_4 x_1)} G_4^{\text{conn}}(p_1, p_2, p_3, p_4)
\]

where

\[
G_4^{\text{conn}}(p_1, p_2, p_3, p_4) = N(-\frac{ig^2}{4} - \frac{\epsilon^2}{16} - \frac{\epsilon^2}{16} + O(g^6)) + O(1).
\]

In the above expressions, the diagrams have the standard Feynman interpretation (without symmetry factors and coupling constants). For instance

\[
\begin{array}{c}
\QEDbox \\
\frac{i}{p^2 - m^2} \int \frac{d^d k}{(2\pi)^d} \frac{i}{k^2 - m^2} \frac{i}{p^2 - m^2}
\end{array}
\]

We assume that all diagrams have been suitably regularized. Our aim in this section is not to discuss issues related to renormalization, but to confirm that our effective theory systematically reproduces the \(1/N\) expansion, perturbatively in \(g^2\).

3. The Leading Order.

With a translationally invariant ansatz

\[
\sigma(x, y) = \int \frac{dp}{(2\pi)^d} e^{ip(x-y)} \sigma(p)
\]

It is straightforward to verify that the solution to equation (2) \(\sigma^0(p)\) satisfies

\[
\sigma^0(p) = \frac{i}{p^2 - m^2 - \frac{q^2}{4} \int \frac{d^d k}{(2\pi)^d} \sigma^0(k)}
\]
This is the familiar gap equation. Iterating this equation to \(O(g^6)\) one obtains
\[
\sigma^0(p) = \sqrt{-2 \frac{i g^2}{8} \psi_0 + 4 \left(\frac{-i g^2}{8}\right)^2 \sigma_0 + 4 \left(\frac{-i g^2}{8}\right)^2 \sigma_0} + O(g^6)
\] (34)
in agreement with the \(O(N)\) term in equation (28). (We recall that in our effective theory the fields are rescaled by factors of \(N\).)

4. Effective Field Theory Spectrum

The leading order \(\sigma^0\) can be used, at the level of quadratic fluctuations, to obtain the mass spectrum of the theory [3]. Perturbing about the leading order, as has been described, we obtain the (leading) quadratic action
\[
S_2 = \frac{i}{2} B_2 + \frac{1}{2} A_2
\]
\[
= \frac{1}{2} \int d^d p_1 d^d p_2 d^d p_3 d^d p_4 \left[-\frac{g^2}{4} \delta(p_1 - p_2 + p_3 - p_4) + i \delta(p_1 - p_2) \delta(p_3 - p_4) \sigma_0^{-1}(p_1) \sigma_0^{-1}(p_3) \right] \eta(p_1, p_2) \eta(p_3, p_4)
\] (35)

After introducing a symmetric Fourier transform
\[
\eta(x_1, x_2) = \int \frac{d^d p_1}{(2\pi)^d/2} \int \frac{d^d p_2}{(2\pi)^d/2} e^{i(p_1 x_1 - p_2 x_2)} \eta(p_1, p_2)
\]
we can write the quadratic action in momentum space
\[
S_2 = \frac{1}{2} \int d^d p_1 d^d p_2 d^d p_3 d^d p_4 \left[-\frac{g^2}{4} \delta(k_1 + p_1 - k_2 - p_2) + i \delta(k_1 - k_2) \sigma_0^{-1}(p_1) \sigma_0^{-1}(p_2) \right] \times
\]
\[
A(p_1, p_2, p_3, p_4) = i \delta(k_2 - p_4) \delta(k_1 - p_3)
\] (36)

This action determines the propagator \(A(p_1, p_2, p_3, p_4)\) of the \(\sigma\) field. Conventional arguments show that \(A(p_1, p_2, p_3, p_4)\) satisfies the equation:
\[
\int d^d p_1 d^d p_2 \left[-\frac{g^2}{4} \delta(k_1 + p_1 - k_2 - p_2) + i \delta(k_1 - k_2) \sigma_0^{-1}(p_1) \sigma_0^{-1}(p_2) \right] \times
\]
\[
A(p_1, p_2, p_3, p_4) = i \delta(k_2 - p_4) \delta(k_1 - p_3)
\] (37)

Now, the physical interpretation of \(A(p_1, p_2, p_3, p_4)\) is as follows: this propagator will (at most) propagate a single two particle bound state and a composite two particle state. The most general ansatz with momentum conservation consistent with this physical picture is given by:
\[
A(p_1, p_2, p_3, p_4) = \delta(p_1 - p_4) \delta(p_2 - p_3) F(p_1, p_2) + \delta(p_1 + p_3 - p_2 - p_4) G(p_1, p_2, p_3, p_4)
\]
\[
= \begin{cases} \text{two particle bound state} \\ \text{composite two particle state} \end{cases}
\] (38)

Inserting this into (38), we immediatly find:
\[
F(p_1, p_2) = \sigma^0(p_1) \sigma^0(p_2)
\]
\[
G(p_1, p_2, p_3, p_4) = \left[-\frac{g^2}{4} \frac{1}{(2\pi)^d} \sigma^0(p_1) \sigma^0(p_2) \sigma^0(p_3) \sigma^0(p_4) - \frac{g^2}{4} \frac{1}{(2\pi)^d} \sigma^0(p_1) \sigma^0(p_2) \int \frac{d^d k}{(2\pi)^d} G(k, p_2 - p_1 + k, p_3, p_4) \right]
\] (39)

Iterating this second equation for \(G(p_1, p_2, p_3, p_4)\), it is not hard to see that we are reproducing the series expansion for:
\[
G(p_1, p_2, p_3, p_4) = \left[-\frac{g^2}{4} \sigma^0(p_1) \sigma^0(p_2) \sigma^0(p_3) \sigma^0(p_4) \right] \left[\frac{1}{1 + i \frac{g^2}{4} \int \frac{d^d k}{(2\pi)^d} \sigma^0(k) \sigma^0(k + p_2 - p_1)}\right]
\] (40)
It is easy to understand this last equation in terms of more familiar approaches to the large $N$ limit: The object $\int \sigma \sigma$ is a bosonic bubble. Thus $G(p_1, p_2, p_3, p_4)$ is simply a sum over chains of bubble diagrams. The factor in square braces is in fact the propagator of the auxiliary field usually introduced to study this model. Notice however, that our $\sigma$ propagator $A(p_1, p_2, p_3, p_4)$ consists of two terms. The first term, which has no analog in the auxiliary field approach, is crucial to obtain a systematic expansion.

The mass spectrum of the theory is determined by searching for the poles of the propagator $A(p_1, p_2, p_3, p_4)$. Rather than performing a full non-perturbative analysis of the spectrum, we content ourselves with constructing an approach, is crucial to obtain a systematic expansion.

In the above, we have employed an obvious matrix notation. Recall that $\sigma^0(k, p)$ is diagonal in momentum space, so that

$$[\sigma^0]_{kp} = \frac{\delta(k - p)}{\sigma^0(p)}$$

5. Corrections to the one and two point functions of the effective theory.

In order to compute the $O(\frac{1}{\sqrt{N}})$ correction to the effective theory one point function, we need to compute the tadpole, and cubic interaction vertices of the effective field theory. These interactions arise from the actions $S_t = iL^d \delta^{(d)}(0) \frac{1}{\sqrt{N}} B_1$ and $S_c = -i \frac{1}{3\sqrt{N}} B_3$

respectively. For completeness, we also present the $\frac{1}{N}$ quartic and subleading quadratic vertices $S_{sq} = -iL^d \delta^{(d)}(0) \frac{1}{2N} B_2$ and $S_q = i \frac{1}{4N} B_4$.

Standard techniques yield the following Feynman rules:

$$-\frac{L^d \delta^{(d)}(0)}{\sqrt{N}} [\sigma^0]_{p_2 p_1}$$

$$\frac{L^d \delta^{(d)}(0)}{2\sqrt{N}} [\sigma^0]_{k_2 p_1} [\sigma^0]_{k_2 - p_1}$$

$$\frac{1}{3\sqrt{N}} ([\sigma^0]_{k_2 p_1} [\sigma^0]_{k_2 p_1} [\sigma^0]_{k_2 p_1} [\sigma^0]_{k_2 p_1} [\sigma^0]_{k_2 p_1} [\sigma^0]_{k_2 p_1} [\sigma^0]_{k_2 p_1} [\sigma^0]_{k_2 p_1})$$

In the above, we have employed an obvious matrix notation. Recall that $\sigma^0(k, p)$ is diagonal in momentum space, so that

$$[\sigma^0]_{kp} = \frac{\delta(k - p)}{\sigma^0(p)}$$
Now, we turn to the computation of the $O(g^4)$ correction to the one point function $<\sigma>$. Two processes contribute: the cubic tadpole and the linear tadpole. Using the expression (42) for the propagator, and the above rules for the vertices, we obtain the following expressions for the cubic tadpole

\[
\mathcal{W}\left(\sigma_{l}^{(0)}\right) = \frac{L^4\delta^4(0)}{\sqrt{N}} \int dp\sigma^0^{-1}(p)A(p,p,p_1,p_2) + \delta(p_1 - p_2)\frac{-ig^2}{4\sqrt{N}}\sigma^0(p_1)\sigma^0(p_2) \left[ \int \frac{d^d k}{(2\pi)^d} \sigma^0(k) - \right. \\
\left. - \frac{ig^2}{4} \int \frac{d^d k}{(2\pi)^d} \sigma^0(k) \int \frac{d^d l}{(2\pi)^d} \sigma^0(l) \right] + \delta(p_1 - p_2)\frac{-ig^2}{4\sqrt{N}}\sigma^0(p_1)\sigma^0(p_2) \left[ \int \frac{d^d l}{(2\pi)^d} \sigma^0(k) \int \frac{d^d l}{(2\pi)^d} \sigma^0(l) \right]
\]

up to $O(g^4)$ and for the linear tadpole

\[
\mathcal{W}(\sigma) = -\frac{L^4\delta^4(0)}{\sqrt{N}} \int bp\sigma^0^{-1}(p)A(p,p,p_1,p_2)
\]

Notice that separately both the linear tadpole and cubic tadpole contain momentum dependent infinities. The sum however is a well defined quantity, with all divergences proportional to $L^d\delta^d(0)$ cancelling and

\[
<\eta> = \delta(p_1 - p_2)\frac{-ig^2}{4\sqrt{N}}\sigma^0(p_1)\sigma^0(p_2) \left[ \int \frac{d^d k}{(2\pi)^d} \sigma^0(k) - \frac{ig^2}{4} \int \frac{d^d l}{(2\pi)^d} \right] \left[ \int \frac{d^d l}{(2\pi)^d} \sigma^0(k) + \sigma^0(k)\sigma^0(l)\sigma^0(k + l - p_1) \right]
\]

Inserting the expression (54) for $\sigma^0$, and recalling that our fields are rescaled by a factor of $N$, one obtains complete agreement with the perturbative result (28). This analysis shows that the linear tadpole is essential to obtain agreement with the Feynman perturbative analysis and to remove momentum dependent infinities.

### III. FERMIONIC VECTOR MODELS.

In this section we show that the approach followed in section 2 for bosonic vector models also applies to fermionic vector models. This is an important observation and this is not only because fermionic systems have more important physical applications as we now explain.

By and large there have been two approaches to the large $N$ limit of vector models: in the first one, auxiliary fields are used. We have already mentioned in the introduction some of the shortcomings of this approach. The other approach is based on the collective field theory idea of changing variables to invariant quantities. However as it has been discussed in section 2b the equation satisfied by the appropriate Jacobian ultimately stems from a Hermiticity requirement of the Hamiltonian. It is not obvious how to generalize this Hamiltonian approach to fermionic systems, although some partial success has been achieved in terms of pseudospin variables [9]. We will not pursue this method in this article but will show that, provided that one is prepared to consider time ordered product expectation values, the fact that the Schwinger-Dyson (loop) equations of the theory imply a differential equation for the Jacobian, as it was demonstrated in section 2a, is straightforwardly generalizable to fermionic systems. Moreover our approach will be fully covariant.

#### A. Schwinger Dyson equations and Jacobian

We assume again that we are dealing with $U(N)$ invariant actions i.e. actions invariant under $\psi^a_\alpha \rightarrow \psi'^a_\alpha = U^{ab}\psi^b_\alpha$, $\bar{\psi}^{*a}_\alpha \rightarrow \bar{\psi}'^{*a}_\alpha = U^{ab}\bar{\psi}^{*b}_\alpha$ with $U^{ab}$ an arbitrary $U(N)$ element. $\alpha$ is a Dirac index. The invariants are now

\[
\sigma_{\alpha\beta}(x,y) = \bar{\psi}^{a}_\alpha(x)\psi^{b}_\beta(y)
\]

and as before we are interested in time ordered product expectation values of invariant operators

\[
<F[\sigma] >= \prod_{i=1}^{m} \sigma_{\alpha_i\beta_i}(x_i,y_i) = \frac{\int D\psi^* D\psi e^{iS} \prod_{i=1}^{m} \sigma_{\alpha_i\beta_i}(x_i,y_i)}{\int D\psi^* D\psi e^{iS}}
\]
A set of Schwinger-Dyson equations for the invariants follow from the identity

\[ 0 = \int D\psi^* D\psi \frac{\delta}{\delta \psi^*_\alpha(x)} \left[ \psi^a_\alpha(y) F[\sigma] e^{iS} \right] \]  

(52)

It is important to remember that the fields \( \psi \) have to be treated as Grassman variables in all manipulations to follow. We obtain

\[ < N \delta_{\beta,\rho} \delta^d(x-y) F[\sigma] > - < \psi^a_\rho(y) \frac{\delta F}{\delta \psi^a_\beta(x)} > - i < \psi^a_\rho(y) \frac{\delta S}{\delta \psi^a_\beta(x)} F[\sigma] > = 0 \]  

(53)

Postulating the existence of a Jacobian \( J \) defined by

\[ \int D\psi^* D\psi F[\sigma] e^{iS} = \int D\sigma J F[\sigma] e^{iS} \]  

(54)

we consider the identity

\[ 0 = \int D\sigma \int d^dz \frac{\partial}{\partial \sigma_{\alpha\beta}(z,x)} (\sigma_{\alpha\rho}(z,y) J F[\sigma] e^{iS}) \]  

(55)

which implies

\[ < m \delta_{\beta,\rho} \delta^d(x-y) \delta^d(0) L^d F[\sigma] > + \int d^dz \sigma_{\alpha\rho}(z,y) \frac{\partial \ln J}{\partial \sigma_{\alpha\beta}(z,x)} F[\sigma] + \int d^dz \sigma_{\alpha\rho}(z,y) \frac{\partial F[\sigma]}{\partial \sigma_{\alpha\beta}(z,x)} + i \int d^dz \sigma_{\alpha\rho}(z,y) \frac{\partial S}{\partial \sigma_{\alpha\beta}(z,x)} F[\sigma] > = 0. \]  

(56)

In the above equation \( m \) is the dimension of the representation of the Clifford algebra. Using the chain rule

\[ \frac{\delta}{\delta \psi^*_\alpha(x)} = \int dz \int dy \frac{\delta \beta}{\delta \psi^*_\alpha(y)} \frac{\delta}{\delta \sigma_{\alpha\beta}(y,z)} = - \int dy \bar{\psi}^a_\beta(y) \frac{\delta}{\delta \psi^a_{\beta}(y,z)} \]  

(57)

in equation (53) it becomes

\[ < N \delta_{\beta,\rho} \delta^d(x-y) F[\sigma] > - < \int d\sigma \sigma_{\alpha\rho}(z,y) \frac{\delta F}{\delta \sigma_{\alpha\beta}(z,x)} > - i < \int d\sigma \sigma_{\alpha\rho}(z,y) \frac{\delta S}{\delta \sigma_{\alpha\beta}(z,x)} F[\sigma] > = 0. \]  

(58)

Comparing this last equation with equation (56) for arbitrary \( F[\sigma] \) we obtain

\[ \int d^dz \sigma_{\alpha\rho}(z,y) \frac{\delta \ln J}{\delta \sigma_{\alpha\beta}(z,x)} = -(N + mL^d \delta^d(0)) \delta_{\beta,\rho} \delta^d(x-y), \]  

(59)

The solution to this equation is

\[ \ln J = -(N + mL^d \delta^d(0)) \text{tr Tr} \ln \sigma \]  

(60)

The trace in the above equation runs over both Dirac and functional spaces. This is the main result of this article. As mentioned earlier the leading term of this Jacobian has been obtained in [9]. This Jacobian should be compared to the bosonic one [14].

It is known that in models such as the Gross Neveu model an enlarged \( O(2N) \) symmetry is present that is better exhibited in terms of Majorana components [18]. One could have considered a set of invariants

\[ \sigma'_{\alpha\beta}(x,y) = \psi^a_\alpha(x) \psi^a_\beta(y) \]  

(61)

for which the Jacobian is easily seen to be

\[ \ln J = - \frac{1}{2} (N + mL^d \delta^d(0)) \text{tr Tr} \ln \sigma \]  

(62)

Finally if flavor degrees of freedom are added as in Nambu Jona-Lasinio type models then one needs only include a further trace in flavor space in the Jacobian [61].
B. Effective action and large $N$ configuration.

Our effective action is given by

$$S_{\text{eff}} = -i\ln J + NS = iN\text{Tr}ln\sigma + NS + iM\delta^d(0)\text{Tr}ln\sigma = NS_0 + S_1.$$  \hfill (63)

We have assumed that we have again rescaled the fields so that an overall factor of $N$ multiplies the action. The leading $N \to \infty$ configuration is determined by

$$\frac{\delta S_0}{\delta \sigma} |_{\sigma^0} = 0$$  \hfill (64)

Shifting about $\sigma^0$ as:

$$\sigma_{\alpha\beta}(x,y) = \sigma^0_{\alpha\beta}(x,y) + \frac{1}{\sqrt{N}}\eta_{\alpha\beta}(x,y)$$  \hfill (65)

we expand

$$S_{\text{eff}} = NS_0(\sigma^0) + S_1(\sigma^0) - \frac{i}{2}C_2 + \frac{1}{2}D_2 + \sum_{n=1}^{\infty} \frac{1}{\sqrt{N}}[\text{(63)}] + \frac{1}{(n+2)!}D_{n+2}$$  \hfill (66)

where

$$D_n = \int d^d x_1 \ldots d^d x_n \int d^d y_1 \ldots d^d y_n \delta^n S_{\sigma_0\beta_1 (x_1, y_1) \ldots \sigma_\alpha \beta_n (x_n, y_n)} \eta_{\sigma_0 \beta_1 ... \eta_{\sigma_n \beta_n}}$$  \hfill (67)

and

$$C_n = \text{trTr}((\sigma^0)^{-1})^n$$  \hfill (68)

At this point, a few comments are in order. The original fields $\psi$ and $\bar{\psi}$ are fermionic fields, and consequently they become Grassman variables under path integral quantization. Using the Grassman nature of the original variables, it follows that $\gamma_{\sigma\alpha,\sigma\beta}$ is an antihermmitian bosonic bilocal; all knowledge of the original fermionic statistics is now coded in this property and the specific form of the interactions in \[63\]. This provides a non trivial check of the effective field theory that can be carried out with perturbation theory.

1. Perturbative Results

We specialize now to the following Lagrangian density (written before rescaling):

$$\mathcal{L} = \bar{\psi}(i\not\partial - m)\psi + \frac{\lambda^2}{2N}(\bar{\psi}\psi)^2 + \frac{\lambda_5^2}{2N}(\bar{\psi}\gamma_5\psi)^2$$  \hfill (69)

For now $\lambda$ and $\lambda_5$, assumed to be of the same order, are left arbitrary, although later the cases $\lambda_5 = 0$ (the Gross Neveu model) and the case $\lambda^2 = -\lambda_5^2$ (a Nambu-Jona-Lasinio type model) will be considered in detail.

By standard diagrammatic techniques, one can obtain for the two point function

$$<\psi_\alpha(x)\bar{\psi}_\beta(y)> = \int \frac{d^d k}{(2\pi)^d} C^{(2)}_{\alpha\beta}(k)e^{-ik(x-y)}$$  \hfill (70)

with

$$G^{(2)}_{\alpha\beta}(p) = N(\text{leading terms}) + O(\lambda^6)$$

\hspace{1cm} + (\text{diagrams as shown}) + O(\lambda^6) + O(\lambda^8)$$  \hfill (71)
We have introduced a dot into our notation, so as to indicate the contraction of Dirac and color indices. Since the summation over color indices is taken into account in the overall factor of $N$, each dot should be thought of as a sum over the Dirac matrices $\mathbf{1}$ and $\gamma^5$, multiplied by the interaction strengths $\lambda$ and $\lambda_5$ respectively. For instance

$$\bigotimes = -i \sum_{M=\lambda_1,\lambda_5} \sum_{\rho} \frac{i}{p-m} \int \frac{d^d k}{(2\pi)^d} \text{Tr}[M \gamma^\rho \gamma^5 \gamma^\rho M \gamma^5] \frac{i}{p-m}$$

(72)

For completeness, every single diagram appearing in (71) is explicitly written down in appendix A.

The connected piece of the four point function is given by:

$$<\bar{\psi}_{\alpha'}^a(x_1)\psi_{\rho'}^a(y_1)\bar{\psi}_{\alpha}^b(x_2)\psi_{\rho}^b(y_2)> = \int \frac{dp_1}{(2\pi)^d} \int \frac{dp_2}{(2\pi)^d} \int \frac{dp_3}{(2\pi)^d} \int \frac{dp_4}{(2\pi)^d} (2\pi)^d \delta(p_1 + p_3 - p_2 - p_4) \times e^{i(p_3 x_1 - p_4 y_1 + p_1 x_2 - p_2 y_2)} G^{(4)_{\text{conn}}}_{\alpha';\rho'}(p_3, p_4, p_1, p_2)$$

(73)

where

$$G^{(4)_{\text{conn}}}_{\alpha';\rho'}(p_1, p_2, p_3, p_4) = N(\bigotimes + \bigotimes + \bigotimes + \text{permutations} + O(\lambda^0)) + O(1).$$

(74)

Explicitly:

$$\bigotimes = -i \sum_{M=\lambda_1,\lambda_5} \sum_{\rho} \frac{i}{p_4 - m} \int \frac{d^d k}{(2\pi)^d} \text{Tr}[M \gamma^\rho \gamma^5 \gamma^\rho M \gamma^5] \frac{i}{p_4 - m}$$

(75)

$$\bigotimes = -(i)^2 \sum_{M,N=\lambda_1,\lambda_5} \sum_{\rho} \frac{i}{p_4 - m} \int \frac{d^d k}{(2\pi)^d} \text{Tr}[M \gamma^\rho \gamma^5 \gamma^\rho M \gamma^5] \frac{i}{p_4 - m} \frac{i}{p_3 - m} \frac{i}{p_2 - m} N \frac{i}{p_2 - m}$$

(76)

$$\bigotimes = -(i)^2 \sum_{M,N=\lambda_1,\lambda_5} \sum_{\rho} \frac{i}{p_4 - m} \int \frac{d^d k}{(2\pi)^d} \text{Tr}[M \gamma^\rho \gamma^5 \gamma^\rho M \gamma^5] \frac{i}{p_4 - m} \frac{i}{p_3 - m} \frac{i}{p_2 - m} N \frac{i}{p_2 - m}$$

(77)

2. The Leading Configuration

With a translationally invariant ansatz:

$$\sigma_{\alpha\beta}(x, y) = \int \frac{d^d k}{(2\pi)^d} e^{ik(x-y)} \sigma_{\alpha\beta}(k)$$

(78)

we obtain from (74)

$$-i \sigma^{-1}_{\rho\alpha}(k) = (\not{k} - m + \lambda^2 \tilde{\sigma} + \lambda_5^2 \gamma_5 \tilde{\sigma}_5)_{\alpha\rho}$$

(79)

where

$$\tilde{\sigma} = \int \frac{d^d k}{(2\pi)^d} \tilde{\sigma}_{\alpha\rho}(k) = -\tilde{G} = -\int \frac{d^d k}{(2\pi)^d} \text{Tr}[G^{(2)}(k)]$$

(80)

and
\[ \sigma_5 = \int \frac{d^4k}{(2\pi)^d} \gamma^5 \sigma_{\alpha\beta}(k) = -\tilde{G}_5 = - \int \frac{d^4k}{(2\pi)^d} Tr[\gamma^5 G^{(2)}(k)] \] (81)

This is the standard gap equation. We again assume that all integrals are regularized. Using (79), we can write, to \(O(\lambda^4)\)

\[ G_{\alpha\beta}(p) = -\sigma^0_{\beta\alpha}(p) = i(p - m - \lambda^2 \tilde{G} - \lambda^2 \gamma^5 \tilde{G})^{-1} \]

\[ = \frac{i}{p - m} - \frac{i}{p - m} (\lambda^2 \tilde{G} + \lambda^2 \gamma^5 \tilde{G}) + (i)^2 \frac{i}{p - m} (\lambda^2 \tilde{G} + \lambda^2 \gamma^5 \tilde{G}) \]

\[ + (i)^2 \frac{i}{p - m} \left( \lambda^2 \tilde{G} + \lambda^2 \gamma^5 \tilde{G} \right) \frac{i}{p - m} \] (82)

Since

\[ \lambda^2 \tilde{G} + \lambda^2 \gamma^5 \tilde{G} = \sum_{M,N = \lambda_1, \ldots, \lambda_5} \int \frac{d^4k}{(2\pi)^d} Tr[M - i \frac{k}{k - m}] M - i \sum_{M,N = \lambda_1, \ldots, \lambda_5} \int \frac{d^4k}{(2\pi)^d} \int \frac{d^4k'}{(2\pi)^d} \times \]

\[ \times Tr[M \frac{i}{k - m}] N \frac{i}{k' - m}] Tr[N \frac{i}{k' - m}] M + O(\lambda^6), \] (83)

it is straightforward to substitute this expression into (82) to obtain

\[ G_{\alpha\beta}^{(2)}(p) = \boxed{\text{diagram}} + \boxed{\text{diagram}} + \boxed{\text{diagram}} + \boxed{\text{diagram}} + O(\lambda^6) \] (84)

in agreement with the leading order term in the diagrammatic expansion for the two point function. Notice that in this case, not all diagrams contribute with the same sign, due to the appearance of closed fermion loops in the leading order. It is remarkable that the original fermion statistics is completely accounted for by the change in sign of the leading order term in the Jacobian by comparison to the bosonic case. Recall that \(\gamma_{\alpha\beta}^0\sigma_{\alpha\beta}\) is an antihermitian bilocal bosonic field. We will see that this change of sign correctly reproduces the Fermi statistics to all orders of the \(1/N\) expansion.

3. Effective Field Theory Spectrum.

In this section, we use the leading order \(\sigma^0_{\alpha\beta}\), at the level of quadratic fluctuations, to obtain the mass spectrum of the effective field theory [3]. From (60), the leading quadratic action is

\[ S_2 = -\frac{i}{2} C_2 + \frac{1}{2} D_2 \]

\[ = \frac{1}{2} \int dx_1 dx_2 dx_3 dx_4 \left[ \lambda^2 \delta_{\alpha\beta} \delta_{\rho\sigma} \delta(x_1 - x_2) \delta(x_1 - x_3) \delta(x_1 - x_4) \right. \]

\[ - i \sigma^0_{\alpha\beta}(x_1, x_2) \sigma^0_{\alpha\beta}(x_3, x_4) \eta_{\alpha\beta}(x_1, x_2) \eta_{\rho\sigma}(x_3, x_4) \] (85)

Now, using a symmetric Fourier transform

\[ \eta_{\alpha\beta}(x_1, x_2) = \int \frac{d^4p_1}{(2\pi)^d/2} \frac{d^4p_2}{(2\pi)^d/2} e^{i(p_1 x_1 - p_2 x_2)} \eta_{\alpha\beta}(p_1, p_2) \] (86)

the quadratic action is

\[ S_2 = \int dp_1 dp_2 dp_3 dp_4 \left[ \frac{\lambda^2}{(2\pi)^d} \delta_{\alpha\beta} \delta_{\rho\sigma} \delta(p_1 + p_3 - p_2 - p_4) + \frac{\lambda^2}{(2\pi)^d} \delta_{\alpha\beta} \delta_{\rho\sigma} \delta(p_1 + p_3 - p_2 - p_4) \right. \]

\[ - i \delta(p_1 - p_2) |\sigma^0_{\alpha\beta}(p_1)| \delta(p_3 - p_4) |\sigma^0_{\alpha\beta}(p_3)| \eta_{\alpha\beta}(p_1, p_2) \eta_{\rho\sigma}(p_3, p_4) \] (87)

in momentum space. Using this action we find that the propagator \(A_{\alpha\beta\rho\sigma}\) of the field \(\sigma_{\alpha\beta}\) satisfies:
The diagonal elements of $\Gamma$ \cite{MN} are reflected in the above propagator: The first term corresponds to an exchange of the two ingoing field theories. We use these vertices in this section to obtain agreement with the perturbative results quoted in a later section.

Inserting this ansatz into (88), we find the solution

$$A_{\mu
u\rho\sigma}(p_1, p_2, p_3, p_4) = \delta(p_1 - p_4)\delta(p_2 - p_3)F_{\mu\nu\rho\sigma}(p_1, p_2) + \delta(p_1 + p_3 - p_2 - p_4)G_{\mu\nu\rho\sigma}(p_1, p_2, p_3, p_4)$$

(89)

This last term corresponds to summing up the bubble diagrams, which we know dominate the large $N$ limit. The diagonal elements of $\Gamma_{MN}$ are identical to the propagators of the auxiliary fields which are usually employed to study the Fermi statistics is reflected in the above propagator: The first term corresponds to an exchange of the two ingoing fermions, with respect to the leading configuration. It thus appears with the opposite sign to the corresponding term in the bosonic propagator \cite{A}. Also, the fermionic bubble in the denominator of the second term appears with a minus sign, reflecting the fact that closed fermion loops come with a factor of $-1$.

A full nonperturbative treatment of the spectrum will be discussed in a later section. At this point we simply construct the connected piece of $A_{\alpha'\beta'\alpha\rho}$ to $O(\lambda^4)$. First, expand the second (connected) term of (90) as

$$A_{\alpha'\beta'\alpha\rho}(p_3, p_4, p_1, p_2) = -\delta(p_3 - p_2)\delta(p_1 - p_4)\sigma^0_{\alpha'\rho}(p_2)\sigma^0_{\alpha\rho}(p_1) + \frac{i}{(2\pi)^d}\delta(p_1 - p_2 + p_3 - p_4) \times \sum_{M, N = \lambda_1, \lambda_5 \gamma_5} [\sigma^0(M\sigma^0(p_3))_{\rho'\alpha}\Gamma_{MN}(p_2 - p_1)[\sigma^0(p_2)N\sigma^0_T(p_1)]_{\rho\alpha}]$$

(90)

where

$$\Gamma_{MN}(p_2 - p_1) = [1 + iG(p_2 - p_1)]^{-1}$$

(91)

and

$$G_{MN}(p_2 - p_1) = \int \frac{d^d k}{(2\pi)^d} Tr[M\sigma^0_T(k + p_2 - p_1)N\sigma^0_T(k)] = G_{MN}(p_2 - p_1)$$

(92)

This last term corresponds to summing up the bubble diagrams, which we know dominate the large $N$ limit. The diagonal elements of $\Gamma_{MN}$ are identical to the propagators of the auxiliary fields which are usually employed to study the Fermi statistics is reflected in the above propagator: The first term corresponds to an exchange of the two ingoing fermions, with respect to the leading configuration. It thus appears with the opposite sign to the corresponding term in the bosonic propagator \cite{A}. Also, the fermionic bubble in the denominator of the second term appears with a minus sign, reflecting the fact that closed fermion loops come with a factor of $-1$.

A full nonperturbative treatment of the spectrum will be discussed in a later section. At this point we simply construct the connected piece of $A_{\alpha'\beta'\alpha\rho}$ to $O(\lambda^4)$. First, expand the second (connected) term of (90) as

$$A_{\alpha'\beta'\alpha\rho}^{\text{conn}}(p_3, p_4, p_1, p_2) = \frac{i}{(2\pi)^d}\delta(p_1 - p_2 + p_3 - p_4) \sum_{M = \lambda_1, \lambda_5 \gamma_5} [\sigma^0(p_4)M\sigma^0_T(p_3)]_{\rho'\alpha}[\sigma^0(p_2)M\sigma^0_T(p_1)]_{\rho\alpha}$$

(93)

Using the expansion for $(\sigma^0)_{\beta_0} = \sigma^0_{\alpha'\beta} = -G_{\alpha'\beta}$ to $O(\lambda^2)$ obtained in a previous section, one readily verifies that the above expression is in exact agreement with the expansion (74-77).

4. Corrections to the one and two point functions.

In this section we compute the tadpole, (subleading) quadratic, cubic and quartic interaction vertices of the effective field theories. We use these vertices in this section to obtain agreement with the perturbative results quoted in subsection 2 for the (fermion) two point function. These vertices will be used in a calculation in a later section.

From the following terms in the action

$$\int dp_1 dp_2 \left[ (\lambda^2 \delta_{\mu\nu} \delta_{\rho\sigma} + \lambda^2 \gamma_{\mu\nu} \gamma_{\rho\sigma}) \delta(k_1 + p_1 - k_2 - p_2) \right] \frac{-i\delta(k_1 - p_2)\sigma^0_{\alpha\rho} - 1(p_1)\delta(p_1 - k_2)\sigma^0_{\rho\mu} - 1(p_2)}{(2\pi)^d} \times A_{\alpha\rho\sigma\mu}(p_1, p_2, p_3, p_4) = i\delta(k_1 - p_4)\delta(k_1 - p_3)\delta_{\alpha\mu}\delta_{\rho\nu}$$

(88)
we easily derive the rules

\[
\begin{align*}
\sigma^{\lambda} & = \frac{1}{3} \sqrt{N} \left( \sigma^{0^{-1}} \right)_{p_2 \beta p_1 \alpha} \frac{m \Lambda^4 \sigma^0(0)}{\sqrt{N}} \\
\sigma^{\lambda} & = \frac{1}{3} \sqrt{N} \left( \sigma^{0^{-1}} \right)_{p_2 \beta p_1 \alpha} \frac{m \Lambda^4 \sigma^0(0)}{\sqrt{N}}
\end{align*}
\]

We have again gone over to an obvious matrix notation. Since \( \sigma^0 \) is diagonal in momentum space, we have

\[
\left[ \sigma^{0^{-1}} \right]_{p a k b} = \delta(p - k) \left[ \sigma^{0^{-1}}(p) \right]_{\alpha \beta}
\]

As in the bosonic case, two processes contribute to the \( O(\frac{1}{\sqrt{N}}) \) correction to the one point function: the cubic tadpole and the linear tadpole. Using the expression (B6) for the propagator, and using the expansion (93) (for the second connected term of the propagator), together with the above rules, we write down expressions for the cubic tadpole

\[
\begin{align*}
\sigma^{\lambda} & = \frac{1}{3} \sqrt{N} \left( \sigma^{0^{-1}} \right)_{p_2 \beta p_1 \alpha} \frac{m \Lambda^4 \sigma^0(0)}{\sqrt{N}} \\
\sigma^{\lambda} & = \frac{1}{3} \sqrt{N} \left( \sigma^{0^{-1}} \right)_{p_2 \beta p_1 \alpha} \frac{m \Lambda^4 \sigma^0(0)}{\sqrt{N}}
\end{align*}
\]

(94)

(95)

(96)

(97)

(98)
< \eta_{\alpha \beta}(k_1, k_2) > = \frac{1}{\sqrt{N}} \delta(k_1 - k_2) \left[ i \int \frac{dp}{(2\pi)^d} \sum_{M=\lambda_1, \lambda_5} [\sigma^0T(k_2)M\sigma^0T(p)M\sigma^0T(k_1)]_{\beta\alpha} + \sum_{M,N=\lambda_1, \lambda_5} \int \frac{dp}{(2\pi)^d} \times \right.
\left. \times [\sigma^0T(k_2)N\sigma^0T(p)M\sigma^0T(k_1)]_{\beta\alpha} G_{MN}(k_2 - p) + \int \frac{dp_1}{(2\pi)^d} \int \frac{dp_2}{(2\pi)^d} \text{Tr} [\sigma^0T(p_1)M\sigma^0T(p_1)N\sigma^0T(p_2)] \times \right.
\left. \times [\sigma^0T(k_2)M\sigma^0T(k_1)]_{\beta\alpha} + O(\lambda^6) \right]

Substituting the expressions for \sigma^0 and G_{MN} given in (44) and (42), we find complete agreement with the O(1) term of (73). Notice that once again all diagrams enter with the correct signs - thus corrections computed using our Jacobian tool have Fermi statistics correctly accounted for! (It should be recalled that < \sigma_{\alpha\beta}(k_1, k_2) > = -G_{\beta\alpha}^{(2)}(k_1, k_2)).

IV. A NON PERTURBATIVE ANALYSIS OF THE EFFECTIVE FIELD THEORY SPECTRUM

In this section, we obtain the non-perturbative effective field theory spectrum and propagator for the Gross-Neveu model and a Nambu-Jona-Lasinio type model (a model with a continuous chiral symmetry) and relate them to previously obtained results.

A. The Gross-Neveu Model

For the Gross-Neveu model, we set \lambda_5 = m = 0 and work in d = 1 + 1 dimensions

\[ \mathcal{L} = \bar{\psi}^\alpha \gamma_i \partial_i \psi^\alpha + \frac{\lambda^2}{2N} (\bar{\psi}^\alpha \psi^\alpha)^2 \] (100)

Notice that this model has a discrete chiral symmetry

\[ \psi^\alpha \rightarrow \gamma_5^\delta \psi^\alpha, \quad \bar{\psi}^\alpha \rightarrow -\bar{\psi}^\gamma \gamma_5^\alpha \] (101)

Using the formalism developed in section III, we find that the leading configuration \sigma^0 satisfies

\[ -i[S^{-1}]_{\rho\alpha}(k) = (k + \lambda^2 \sigma)_{\alpha\rho} \] (102)

where

\[ \sigma = \int \frac{d^2 k}{(2\pi)^2} \text{Tr}[\sigma(k)] \] (103)

Inserting (102) into (103) we find the standard result

\[ \sigma = \int \frac{d^2 k}{(2\pi)^2} \text{Tr}[(k + \lambda^2 \sigma)^{-1}] = \frac{\lambda^2 \sigma}{2\pi} \log(\lambda^2 \sigma)^2 \] (104)

where \Lambda is an ultraviolet cut off. Therefore

\[ \lambda^2 \sigma = \Lambda e^{-\frac{\sigma}{\lambda^2}} \] (105)

Thus the fermions have acquired a dynamically generated mass \lambda^2 \sigma and the discrete chiral symmetry (101) has been spontaneously broken, as is well known [1]. This mass can be made finite as \Lambda \rightarrow \infty by adding the counter term

\[ S_{cf} = -\frac{c}{2} \int \frac{d^2 p_1}{2\pi} \frac{d^2 p_2}{2\pi} d^2 p_3 d^2 p_4 \delta(p_1 + p_3 - p_2 - p_4) \eta_{\alpha\rho}(p_1, p_2) \eta_{\rho\nu}(p_3, p_4) \] (106)

to our effective action, corresponding to a coupling constant renormalization of the original theory. Fixing the renormalized mass to some value \( m_r \)

\[ m_r = \Lambda e^{-\frac{\sigma}{\lambda^2}} = \Lambda e^{-\frac{\sigma}{\lambda^2}} \] (107)

we can fix \( c \)
\[ c = \lambda^2 + \left[ \frac{1}{2\pi} \log \frac{m_0^2}{\Lambda^2} \right]^{-1} \] (108)

Using the arguments constructed in section B3 we know that the effective field theory propagator
\[ A_{\mu\nu\mu'\nu'}(p_1, p_2, p_3, p_4) = \varrho_{\mu\nu}(p_1, p_2)\varrho_{\mu'\nu'}(p_3, p_4) > \] (109)
can be expressed in terms of the leading configuration \( \sigma^0 \) as
\[ A_{\mu\nu\mu'\nu'}(p_1, p_2, p_3, p_4) = -\delta(p_3 - p_2)\delta(p_1 - p_4)\sigma^0_{\mu\nu}(p_1)\sigma^0_{\mu'\nu'}(p_3) \]
\[ + \frac{i\lambda_0^2}{(2\pi)^2} \delta(p_1 + p_3 - p_2 - p_4) \frac{\sigma^0_{\mu\nu}(p_1)\sigma^0_{\mu'\nu'}(p_3)\sigma^0_{\tau'\nu}(p_4)}{1 + i\lambda_0^2 \int \frac{d^4k}{(2\pi)^4} \sigma^0_{\tau\gamma}(k)\sigma^0_{\gamma\sigma}(k - p_1 + p_2)} \] (110)

To construct the propagator explicitly, we perform the integral [112]:
\[ I = i\lambda_0^2 \int \frac{d^2l}{(2\pi)^2} \sigma^0_\gamma(l)\sigma^0_\sigma(l - p_1 + p_2) \]
\[ = -\frac{\lambda_0^2}{2\pi} \frac{4m_0^2 - k^2}{-k^2} \cos \left[ \frac{4m_0^2 - k^2}{2} \right] \right] \right] - \frac{\lambda_0^2}{\pi} \log \frac{\Lambda}{m_\tau}. \] (111)
where \( k = p_1 - p_2 \). Now, from (107) we see that
\[ \frac{\lambda_0^2}{\pi} \log \frac{\Lambda}{m_\tau} = 1 \] (112)
so that
\[ I = -\frac{\lambda_0^2}{2\pi} \sqrt{\frac{4m_0^2 - k^2}{-k^2}} \cos \left[ \frac{4m_0^2 - k^2}{2} \right] \right] \right] - 1. \] (113)
Inserting this into the propagator (110), we obtain
\[ A_{\mu\nu\mu'\nu'}(p_1, p_2, p_3, p_4) = -\delta(p_2 - p_3)\delta(p_4 - p_1) \left[ \frac{i}{p_1 - m_\tau} \right]_{\mu'\nu} \left[ \frac{i}{p_3 - m_\tau} \right]_{\nu'\mu} \]
\[ - \frac{i}{2\pi} \left[ \frac{i}{p_2 - m_\tau} \frac{i}{p_1 - m_\tau} \right]_{\nu'\mu} \left[ \frac{i}{p_4 - m_\tau} \frac{i}{p_3 - m_\tau} \right]_{\nu'\mu'} \delta(p_1 + p_3 - p_2 - p_4) \]
\[ \frac{\delta(p_1 + p_3 - p_2 - p_4)}{\sqrt{\frac{4m_0^2 - k^2}{-k^2}}} \cos \left[ \frac{4m_0^2 - k^2}{2} \right] \right] \right] \right] - 1. \] (114)
Notice that all dependence on \( \Lambda \) has disappeared as it must.

The interpretation of the above propagator is clearer in terms of rapidity variables \( p_0^0 = mcosh(\theta), p_1^1 = msinh(\theta) \). If \( \phi = \theta_1 - \theta_2 \) the connected piece of the propagator (after the removal of the external legs) is given by
\[ D(\phi) = 2\pi i \frac{\tanh(\frac{\phi}{2})}{\phi}. \] (115)
For on-shell particle-antiparticle scattering (corresponding to time flowing from right to left in (89)) we let \( \phi \to \phi - i\pi \) \((p_1 \to p_1, p_2 \to -p_2)\) and obtain the amplitude
\[ D(\phi) = 2\pi i \frac{\coth(\frac{\phi}{2})}{\phi - i\pi}. \] (116)
Since \( s = (p_1 + p_2)^2 = 4m^2cosh^2(\frac{\phi}{2}) \) we see that as \( \phi \to 0 \) the above amplitude displays a cut at \( s = 4m^2. \)


B. A Nambu-Jona-Lasinio type model

The model we consider in this section has \( \lambda_5^2 = -\lambda^2 \) and \( m = 0 \)

\[
\mathcal{L} = \bar{\psi}^a \gamma^\mu D_\mu \psi^a + \frac{\lambda^2}{2N} (\bar{\psi}^a \psi^a)^2 - \frac{\lambda_5^2}{2N} (\bar{\psi}^a \gamma^5 \psi^a)^2
\] (117)

We again work in \( d = 1 + 1 \) dimensions. This model is invariant under the global continuous chiral transformation

\[
\psi_\alpha \rightarrow e^{i\gamma_5} \gamma_\alpha \psi_\alpha \quad \bar{\psi}_\alpha \rightarrow -\bar{\psi}_\alpha e^{i\gamma_5} \gamma_\alpha
\] (118)

The leading configuration \( \sigma^0 \) satisfies

\[
- i [\sigma^{0-1}]_{\rho\alpha}(k) = (k + \lambda^2 \sigma + \lambda_5^2 \gamma^5 \bar{\sigma})_{\alpha\rho}
\] (119)

where

\[
\sigma = \int \frac{d^2 k}{(2\pi)^2} Tr[\sigma(k)]
\] (120)

and

\[
\bar{\sigma} = i \int \frac{d^2 k}{(2\pi)^2} Tr[\gamma^5 \bar{\sigma}(k)]
\] (121)

Inserting (119) into (120) yields the standard gap equation (13)

\[
\sigma = \int \frac{d^2 k}{(2\pi)^2} \left[ k + \lambda^2 \gamma^5 \bar{\sigma} + \lambda^2 \bar{\sigma} \right]^{-1} = \int \frac{d^2 k}{(2\pi)^2} \frac{4i \lambda^2 \sigma}{k^2 - 4\lambda^4 (\sigma^2 + \bar{\sigma}^2)}
\] (122)

Similarly, the gap equation for \( \bar{\sigma} \) (13) is obtained by inserting (119) into (121)

\[
\bar{\sigma} = \int \frac{d^2 k}{(2\pi)^2} \left[ k + \lambda^2 \gamma^5 \bar{\sigma} + \lambda^2 \bar{\sigma} \right]_{\nu\rho} = \int \frac{d^2 k}{(2\pi)^2} \frac{4i \lambda^2 \bar{\sigma}}{k^2 - 4\lambda^4 (\sigma^2 + \bar{\sigma}^2)}
\] (123)

Fixing the mass to some value \( m_r \), and parametrizing \( \bar{\sigma} \)

\[
2\lambda^2 \sigma = m_r \cos \theta \quad 2\lambda^2 \bar{\sigma} = m_r \sin \theta,
\] (124)

we find

\[
\frac{1}{2\lambda^2} = \frac{1}{2\pi} \log \frac{\Lambda^2}{m_r^2}
\] (125)

The expression for the effective field theory propagator in terms of the leading configuration is

\[
A_{\mu\nu'}(p_1, p_2, p_3 p_4) = -\delta(p_3 - p_2)\delta(p_1 - p_4)\sigma^{0\mu}(p_1)\sigma^{0\nu'}(p_3)
\]

\[
+ i \lambda_0^2 \delta(p_1 + p_3 - p_2 - p_4) \sum_{M, N=1, \gamma}
\]

\[
\left[ \sigma^{0T}(p_1) M^{0T}(p_2) \right]_{\mu\nu} \left[ \sigma^{0T}(p_3) N^{0T}(p_4) \right]_{\nu'\mu'}
\]

\[
+ i \lambda_0^2 \int \frac{d^2 k}{(2\pi)^2} Tr[\sigma^{0T}(k)M^{0T}(k-p_1+p_2)N]
\] (126)

To make the following arguments simple and transparent, we pick \( \theta = 0 \) in \( \sigma^{0}_{\alpha\beta} \). To obtain an explicit expression for the propagator, we need to compute three different integrals. The first integral was considered in the previous section, so we simply quote the result (3)

\[
I_1 = i \lambda_0^2 \int \frac{d^2 k}{(2\pi)^2} \sigma^{0}_{\gamma\alpha}(k)\sigma^{0}_{\gamma\alpha}(k-p_1+p_2)
\]

\[
= -\frac{\lambda_0^2}{2\pi} \log \left[ \frac{4m_{r}^2 - k^2}{-k^2} \right] - 1
\] (127)
The second integral

$$I_2 = \int \frac{d^2 k}{(2\pi)^2} Tr[\sigma^{0T}(k - p_1 + p_2)i\sigma^{0T}(k)\gamma^5]$$  \hspace{1cm} (128)$$
vanishes, due to the trace over Dirac indices. The third integral is easily computed

$$I_3 = i\lambda_0^2 \int \frac{d^2 k}{(2\pi)^2} Tr[\sigma^{0T}(k - p_1 + p_2)i\gamma^5\sigma^{0T}(k)i\gamma^5]$$
$$= -\frac{\lambda_0^2}{2\pi} \sqrt{\frac{4m_r^2 - k^2 - \sqrt{-k^2}}{-k^2}} \log \left[ \frac{\sqrt{4m_r^2 - k^2 - \sqrt{-k^2}}}{\sqrt{4m_r^2 - k^2 + \sqrt{-k^2}}} \right] \frac{\lambda_0^2}{2\pi} \log \frac{\Lambda^2}{m_r^2} + \frac{\lambda_0^2}{\pi}$$  \hspace{1cm} (129)$$

where \(\Lambda\) is an ultraviolet cut off. Using the renormalization condition, in the form  \[[125]\], we may rewrite this as

$$I_3 = -\frac{\lambda_0^2}{2\pi} \sqrt{\frac{4m_r^2 - k^2 - \sqrt{-k^2}}{-k^2}} \log \left[ \frac{\sqrt{4m_r^2 - k^2 - \sqrt{-k^2}}}{\sqrt{4m_r^2 - k^2 + \sqrt{-k^2}}} \right] - 1 + \frac{\lambda_0^2}{\pi}$$  \hspace{1cm} (130)$$

Thus, the effective field theory propagator reads explicitly

$$A_{\mu\nu;\rho\sigma}(p_1, p_2, p_3, p_4) = -\delta(p_2 - p_3)\delta(p_4 - p_1) \left[ \frac{i}{p_4 - m_r} \right]_{\nu;\beta} \left[ \frac{i}{p_3 - m_r} \right]_{\rho;\beta'}$$
$$- \frac{i}{2\pi} \left[ \frac{i}{p_2 - m_r} \right]_{\nu;\beta} \left[ \frac{i}{p_1 - m_r} \right]_{\rho;\beta'} \frac{\delta(p_1 + p_3 - p_2 - p_4)}{\sqrt{4m_r^2 - k^2 - \sqrt{-k^2}}} \log \left[ \frac{\sqrt{4m_r^2 - k^2 - \sqrt{-k^2}}}{\sqrt{4m_r^2 - k^2 + \sqrt{-k^2}}} \right]$$
$$- \frac{i}{2\pi} \left[ \frac{i}{p_2 - m_r} \right]_{\nu;\beta} \left[ \frac{i}{p_1 - m_r} \right]_{\rho;\beta'} \frac{\delta(p_1 + p_3 - p_2 - p_4)}{\sqrt{4m_r^2 - k^2 - \sqrt{-k^2}}} \log \left[ \frac{\sqrt{4m_r^2 - k^2 - \sqrt{-k^2}}}{\sqrt{4m_r^2 - k^2 + \sqrt{-k^2}}} \right] - 2$$  \hspace{1cm} (131)$$

The first term again represents the crossed propagation of two free fermions. The second term has been discussed earlier. The third term above, is however new. It has the property that

$$\sqrt{\frac{4m_r^2 - k^2 - \sqrt{-k^2}}{-k^2}} \log \left[ \frac{\sqrt{4m_r^2 - k^2 - \sqrt{-k^2}}}{\sqrt{4m_r^2 - k^2 + \sqrt{-k^2}}} \right] |_{k^2 = 0} = 2.$$  \hspace{1cm} (132)$$

This has been interpreted as the signature of a massless scalar in the spectrum of the model  \[[20]\]. This massless particle is associated with the fact that the continuous chiral symmetry of the model, is "nearly broken"  \[[21]\].

Finally, we remark that the Lagrangian  \[[17]\] with added flavor degrees of freedom

$$\mathcal{L} = i\bar{\psi} \gamma^5 \psi + \frac{1}{2} \lambda_0^2 (\bar{\psi} \gamma^5 \psi)^2 - (\bar{\psi} \gamma^5 \tau_i \psi)(\bar{\psi} \gamma^5 \tau_i \psi)$$  \hspace{1cm} (133)$$

is a popular candidate to study the low energy phenomenology of the light mesons. In the above, the \(\tau_i\) are taken as the generators of SU(2) for the two flavor model, and as the generators of SU(3) for the three flavor model. The invariant correlators now carry four labels

$$\sigma^{ij}_{\alpha\beta}(x, y) = \bar{\psi}^{ij}_\alpha(x) \psi^{ij}_\beta(y)$$  \hspace{1cm} (134)$$

where \(i, j\) are flavor indices. It is not hard to construct the leading configuration

$$\sigma^{ij}_{\alpha\beta}(x, y) = \delta^{ij} \int \frac{d^2 k}{(2\pi)^2} e^{-ik(x-y)} \left[ \frac{-i}{k - m_r} \right]_{\alpha\beta}$$  \hspace{1cm} (135)$$

where the mass \(m_r^2 = \lambda^4(\sigma^2 + \bar{\sigma}^2)\) satisfies the gap equation

$$\frac{1}{4i\lambda_0^2} = \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 - 2\lambda_0^2 m_r^2}$$  \hspace{1cm} (136)$$

The effective field theory propagator is easily written in terms of this leading configuration.
we find that \( \bar{\eta} \) channel, corresponds to the leading order mass shell condition for the fermion-antifermion bound state.

This condition requires \( k^2 = 4m^2 \) which, since \( k \) is the physical momentum transfer for the particle antiparticle channel, corresponds to the leading order mass shell condition for the fermion-antifermion bound state.

\[ A^{ij,j'}_{\mu
u}(p_1, p_2, p_3 p_4) = -\delta(p_3 - p_2)\delta(p_1 - p_4)\sigma^{0ij}_{\mu
u}(p_1)\sigma^{0j'i}_{\mu'\nu'}(p_3) + \frac{i\lambda^2}{(2\pi)^2} \delta(p_1 + p_3 - p_2 - p_4) \sum_{M,N=1,1, i, \gamma, \tau} \frac{[\sigma^T(p_1) M \sigma^T(p_2)] \pi_{ij}^{\mu} [\sigma_0^T(p_3) N \sigma_0^T(p_4)] \pi_{j'i'}^{\mu'} }{1 + i\lambda^2 \int \frac{d^4k}{(2\pi)^4} \text{Tr} \text{Tr} [\sigma_0^T M \sigma_0^T(k - p_1 + p_2) N]} \]  

(137)

where \( \text{Tr} \) denotes a trace in Dirac space, \( tr \) denotes a trace in flavor space and \( M, N \) are now direct products of matrices belonging to Dirac space with matrices belonging to flavor space. Notice that in this case a triplet of massless pseudoscalar bosons appears in the spectrum. These are usually interpreted as the \( \pi^0 \), the \( \pi^- \) and the \( \pi^0 \) particles.

In the context of this phenomenological model, our effective field theory is nothing but the theory of the mesons built from quark anti-quark pairs. The couplings of the various meson-meson interactions are proportional to \( N \) (the number of quarks) raised to some negative (integer or half integer) power.

### C. Homogeneous Bethe Salpeter Equation.

We have shown how to compute the Feynman rules and propagator for our effective field theory and have verified their correctness perturbatively, and non perturbativeley to first nontrivial order in \( \frac{1}{N} \). However before any scattering amplitudes can be computed, we have to supply a set of asymptotic states. \( S \) matrix elements are then taken using these asymptotic states, as usual.

In any field theory, only the quadratic term in the action provides a harmonic oscillator and thus a spectrum consistent with a set of free particles. Thus, it is the quadratic term that codes the asymptotic states of the theory, as solutions of the corresponding homogeneous equation. This should always correspond to the large \( N \) approximation to the homogenous Bethe Salpeter equation.

For example, the effective field theory wave functions for the Gross Neveu model satisfy

\[ i(\not{p} - m)_{\rho'\alpha} \bar{\eta}_{\rho\alpha}(p_3, p_2)(\not{p} - m)_{\rho\alpha} + \frac{\lambda^2}{4\pi^2} \int dp_1 dp_4 \delta(p_1 + p_3 - p_2 - p_4) \eta_{\rho\alpha}(p_4, p_1) \delta_{\rho'\alpha} = 0. \]  

(138)

It is possible to obtain a particular solution to this equation. Making the ansatz

\[ \eta_{\rho\alpha}(p_3, p_2) = \bar{\eta}_{\rho\alpha}(p_1, p_3 - p_2), \]  

(139)

we find that \( \bar{\eta} \) satisfies the equation

\[ (\not{p} - m)_{\rho'\alpha} \bar{\eta}_{\rho\alpha}(p_2, p_3 - p_2)(\not{p} - m)_{\rho\alpha} = i \frac{\lambda^2}{4\pi^2} \int dp_1 \bar{\eta}_{\rho\alpha}(p_1, p_3 - p_2) \delta_{\rho'\alpha}. \]  

(140)

Rewriting this last equation as

\[ \bar{\eta}_{\mu\nu}(p_2, p_3 - p_2) = \frac{i}{\not{p} - m} \frac{i}{\not{p} - m} \mu\nu i \frac{\lambda^2}{4\pi^2} \int dp_1 \bar{\eta}_{\rho\alpha}(p_1, p_3 - p_2) \]  

(141)

taking the trace of both sides and integrating over \( p_2 \) keeping \( p_3 - p_2 = k \) fixed, we are lead to the consistency condition

\[ 1 = i \frac{\lambda^2}{4\pi^2} \int dp_2 \left( \frac{i}{\not{p} - m} \frac{i}{\not{p} + k - m} \right) \mu\nu \]  

(142)

This condition requires \( k^2 = 4m^2 \) which, since \( k \) is the physical momentum transfer for the particle antiparticle channel, corresponds to the leading order mass shell condition for the fermion-antifermion bound state.

\[ \text{There is a slight abuse of notation in (137), please refer to (90-92) for clarification.} \]
V. SUBLEADING $1/N$ CORRECTIONS.

The formalism which was developed in sections II and III can be used to obtain subleading corrections to any correlator of interest. For instance, the $1/N$ correction to the propagator results from the following diagrams in the effective field theory

$$<\delta\sigma\delta\sigma> = \times \times + \times \times \times \times + \times \times \times \times + \times \times \times \times$$

(143)

with the Feynman rules given by (95) in terms of the leading configuration $\sigma_0$ and where the propagators in the diagrams refer to (114). It is essential that both the disconnected and connected pieces of this propagator are included. One observes that the contributions from the subleading term of the Jacobian (proportional to $L^d\delta^d(0)$) precisely cancel similar infinite contributions generated in the other diagrams. All the remaining expressions then have an interpretation in terms of original Feynman diagrams. In this precise sense the subleading term in the Jacobian provides a normal ordering. Diagrammatically, we find:

$$\text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} + \text{Diagram 5}$$

(144)

In the above, we have only included basic skeletons; permutations of these basic skeletons need to be included. For example, the first diagram appearing would be repeated with another diagram in which the second fermion is dressed.

It is well known that the Gross Neveu model is exactly integrable with no particle production (although it has a rich spectrum [18]) and the exact $S$ matrices are known [14]. For two particle scattering the $S$ matrix element is given by

$$out < P_b(\tilde{p}_1)P_d(\tilde{p}_2)|P_a(p_1)P_c(p_2)>^{in} =_{ac} S_{bd}(\theta)\delta(\tilde{p}_1^1 - p_1^1)\delta(\tilde{p}_2^1 - p_2^1) -_{ac} S_{db}(\theta)\delta(\tilde{p}_1^2 - p_1^2)\delta(\tilde{p}_2^2 - p_1^2).$$

(145)

where
In the above equations \( \theta = \theta_1 - \theta_2 \) where \( \theta_1 \) and \( \theta_2 \) are the rapidity variables of the incoming particles. The above \( U(N) \) symmetric two particle \( S \) matrix is consistent with the underlying \( O(2N) \) symmetry of the Gross Neveu model [22]. \( \sigma_2 \) and \( \sigma_3 \) are the standard symbols used to describe the exact \( S \) matrix with \( O(N) \) symmetry.

It turns out that the simplest way to verify the \( \frac{1}{N} \) correction to the propagator is to consider the contributions of the diagrams in (144) to forward two particle scattering corresponding to time flowing from top to bottom. The corresponding \( S \) matrix is then

\[
S_{ab}(\text{forward}) = N^2(\sigma_2 + \frac{1}{N}\sigma_3).
\]  

(147)

By looking at the diagrams in (144) we see that the first two diagrams can not contribute to forward scattering. The third diagram is clearly the leading \( (O(\frac{1}{N})) \) contribution to \( \sigma_3 \). The remaining diagrams should sum up to the \( \frac{1}{N} \) contribution to \( \sigma_2 \). They are precisely the diagrams considered by Berg et. al. [13] who have indeed confirmed that this is the case.

**VI. CONCLUSIONS**

In this paper, we have used colorless bilocal fields to study the large \( N \) limit of both fermionic and bosonic vector models. By requiring that the Schwinger Dyson equations in terms of the original variables agree with the equations derived directly in terms of the invariant variables led to a functional differential equation for the Jacobian. The equation was solved exactly, leading to an exact effective action. This effective action was then shown to reproduce the familiar perturbative expansion for the two and four point functions. In particular, in the case of fermionic vector models, the effective action correctly accounts for the Fermi statistics. The theory was then studied non-perturbatively. The stationary points of the effective action provide the usual large \( N \) gap equations. The homogeneous equation associated with the quadratic (in the bilocals) action is simply a two particle Bethe Salpeter equation. Finally, the leading correction in \( \frac{1}{N} \) was shown to be in agreement with the exact \( S \) matrix of the model.

There are a number of interesting questions which can now be asked. Firstly, it is clear that the above invariant variables are classically commuting functions. This change of variables therefore provides a bosonization valid in an arbitrary number of dimensions. It is interesting to ask if a link can be drawn to more conventional bosonization schemes. Also, this bosonization may prove to be a powerful tool for analyzing many body condensed matter systems. Secondly, it is well known that the large \( N \) limit of the Gross-Neveu model posses a number of interesting configurations, corresponding to many particle bound states. The present formalism is particularly well suited to studying two particle bound states. It would thus be interesting to determine the Feynman rules of the effective theory in a two particle bound state background. One would expect that these rules may be simpler than the rules obtained in this paper. Finally, the possibility of obtaining an equal time approach for fermionic vector models remains of great interest.
In this appendix, explicit expressions are supplied for the fermionic diagrams appearing in the text (see equation (71)):

\[
\begin{align*}
\bigcirc_{i} &= \frac{i}{\not{p} - m} \\
\bigcirc_{i} &= -i \sum_{M=1,i\gamma^5} \frac{i}{\not{p} - m} \int \frac{d^dk}{(2\pi)^3} Tr[M \frac{i}{\not{p} - m}] M \frac{i}{\not{p} - m} \\
\bigcirc_{i} &= i^2 \sum_{M,i, N=1,i\gamma^5} \frac{i}{\not{p} - m} \int \frac{d^dk}{(2\pi)^3} Tr[M \frac{i}{\not{p} - m}] M \frac{i}{\not{p} - m} \int \frac{d^dk'}{(2\pi)^3} Tr[N \frac{i}{\not{p} - m}] N \frac{i}{\not{p} - m} \\
\bigcirc_{i} &= i \sum_{M=1,i\gamma^5} \frac{i}{\not{p} - m} \int \frac{d^dk}{(2\pi)^3} M \frac{i}{\not{p} - m} M \frac{i}{\not{p} - m} \\
\bigcirc_{i} &= -i^2 \sum_{M,i, N=1,i\gamma^5} \frac{i}{\not{p} - m} \int \frac{d^dk}{(2\pi)^3} Tr[M \frac{i}{\not{p} - m}] M \frac{i}{\not{p} - m} \int \frac{d^dk'}{(2\pi)^3} N \frac{i}{\not{p} - m} N \frac{i}{\not{p} - m} \\
\bigcirc_{i} &= -i^2 \sum_{M,i, N=1,i\gamma^5} \frac{i}{\not{p} - m} \int \frac{d^dk}{(2\pi)^3} M \frac{i}{\not{p} - m} M \frac{i}{\not{p} - m} \int \frac{d^dk'}{(2\pi)^3} Tr[N \frac{i}{\not{p} - m}] N \frac{i}{\not{p} - m} \\
\bigcirc_{i} &= -i \sum_{M=1,i\gamma^5} \frac{i}{\not{p} - m} \int \frac{d^dk}{(2\pi)^3} M \frac{i}{\not{p} - m} N \frac{i}{\not{p} - m} \int \frac{d^dk'}{(2\pi)^3} Tr[N \frac{i}{\not{p} - m}] M \frac{i}{\not{p} - m} \\
\bigcirc_{i} &= -i^2 \sum_{M,i, N=1,i\gamma^5} \frac{i}{\not{p} - m} \int \frac{d^dk}{(2\pi)^3} \int \frac{d^dk'}{(2\pi)^3} Tr[M \frac{i}{\not{p} - m} N \frac{i}{\not{p} - m} N \frac{i}{\not{p} - m}] M \frac{i}{\not{p} - m} \\
\bigcirc_{i} &= -i^2 \sum_{M,i, N=1,i\gamma^5} \frac{i}{\not{p} - m} M \int \frac{d^dk}{(2\pi)^3} \int \frac{d^dk'}{(2\pi)^3} Tr[N \frac{i}{\not{p} - m}] \frac{i}{\not{p} - m} 
\end{align*}
\]
[1] G. t’Hooft, Nucl. Phys. B72, (1974) 461.
[2] E. Brezin, C. Itzykson, G. Parisi and J.B. Zuber, Commun. Math. Phys. 59 (1978) 35; M.L. Mehta, Commun. Math. Phys. 79 (1981) 327.
[3] A. Jevicki and J.P. Rodrigues, Nucl. Phys. B230 [FS10] (1984) 317.
[4] D. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235;
[5] L. Dolan and R. Jackiw, Phys. Rev. D9 (1974) 3235; H.J. Schnitzer, Phys. Rev. D10 (1974) 1800; R. Jackiw, S. Coleman and H.D. Politzer, Phys. Rev. D10 (1974) 2491; E. Brezin and J. Zinn-Justin, Phys. Rev. B14 (1976) 3110.
[6] A. Jevicki and H. Levine, Ann. Phys. 136 (1981) 113.
[7] A. Jevicki and B. Sakita, Nucl. Phys. B165 (1980) 511; B185 (1981) 89.
[8] A. Jevicki and N. Papanicolaou, Nucl. Phys. B171 (1980) 362.
[9] M. Cavicchi, P. Di Vecchia and I. Pesando, Mod. Phys. Lett. A8 (1993) 2427 9306091.
[10] G. t’Hooft, Nucl. Phys. B75 (1974) 461.
[11] K. Demeterfi, A. Jevicki and J.P. Rodrigues, Nucl. Phys. B362 (1991) 173; B365 (1991) 499; Mod. Phys. Lett. A6 (1991) 3199.
[12] J.F. Schonfeld, Nucl. Phys. B95 (1975) 148. R.G. Root, Phys. Rev. D11 (1975) 831. R.W. Haymaker and F. Cooper, Phys. Rev D19 (1979) 562.
[13] B.Berg, M. Karowski, V. Kurak and P. Weisz, Phys. Lett. 76B (1978) 502.
[14] A.B. Zamolodchikov and AL.B. Zamolodchikov, Phys. Lett. B72 (1978) 481; Ann. Phys 120 (1979) 253; R. Shankar and E. Witten, Nucl. Phys. B141 (1978) 349. M. Karowski and H.J. Thun, Nucl. Phys. B190 [FS3] (1981) 61.
[15] A. Jevicki and J.P. Rodrigues, Nucl. Phys. B421 (1994) 278, 9312118.
[16] R.de Mello Koch and J.P. Rodrigues, work in progress.
[17] J.P. Rodrigues and A. Welte, Mod. Phys. Lett. A8 (1993) 4175.
[18] R. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D12 (1975) 2443.
[19] Y. Nambu and G. Jona-Lasinio, Phys. Rev. 122 (1961) 345; 124 (1961) 246.
[20] N. Andrei and J.H.Lowenstein, Phys. Rev. Lett. 43 (1979) 1698.
[21] R. Köberle, V. Kurak and J.A. Sweica, Phys. Rev. D20 (1979) 897, A.J. da Silva, M. Gomes and R. Köberle, Phys. Rev. D20 (1979) 895.
[22] B.Berg, M.Karowski, P.Weisz and V.Kurak, Nucl. Phys. B134 (1978) 125.