On Structural Parameterizations for the 2-Club Problem

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Abstract

The NP-hard 2-Club problem is, given an undirected graph \( G = (V, E) \) and \( \ell \in \mathbb{N} \), to decide whether there is a vertex set \( S \subseteq V \) of size at least \( \ell \) such that the induced subgraph \( G[S] \) has diameter at most two. We make progress towards a systematic classification of the complexity of 2-Club with respect to a hierarchy of prominent structural graph parameters. First, we present the following tight NP-hardness results: 2-Club is NP-hard on graphs that become bipartite by deleting one vertex, on graphs that can be covered by three cliques, and on graphs with domination number two and diameter three. Then, we consider the parameter \( h \)-index of the input graph. This parameter is motivated by real-world instances and the fact that 2-Club is fixed-parameter tractable with respect to the larger parameter maximum degree. We present an algorithm that solves 2-Club in \( |V|^f(k) \) time with \( k \) being the \( h \)-index. By showing \( W[1] \)-hardness for this parameter, we provide evidence that the above algorithm cannot be improved to a fixed-parameter algorithm. Furthermore, the reduction used for this hardness result can be modified to show that 2-Club is NP-hard if the input graph has constant degeneracy. Finally, we show that 2-Club is fixed-parameter tractable with respect to distance to cographs.

1 Introduction

The identification of cohesive subnetworks is an important task in the analysis of social and biological networks, since these subnetworks are likely to represent...
communities or functional subnetworks within the large network. The natural cohesiveness requirement is to demand that the subnetwork is a complete graph, a clique. However, this requirement is often too restrictive and thus relaxed definitions of cohesive graphs such as $s$-cliques [1], $s$-plexes [33], and $s$-clubs [27] have been proposed. In this work, we study the problem of finding large $s$-clubs within the input network. An $s$-club is a vertex set that induces a subgraph of diameter at most $s$. Thus, $s$-clubs are distance-based relaxations of cliques, which are exactly the graphs of diameter one. For constant $s \geq 1$, the problem of finding $s$-clubs is defined as follows.

$s$-Club

**Input:** An undirected graph $G = (V, E)$ and $\ell \in \mathbb{N}$.

**Question:** Is there a vertex set $S \subseteq V$ of size at least $\ell$ such that $G[S]$ has diameter at most $s$?

In this work, we study the computational complexity of 2-Club, that is, the special case of $s = 2$. This is motivated by the following two considerations. First, 2-Club is an important special case concerning the applications: For biological networks, 2-clubs and 3-clubs have been identified as the most reasonable diameter-based relaxations of cliques [30]. Further, Balasundaram et al. [4] also proposed to compute 2-clubs and 3-clubs for analyzing protein interaction networks. 2-Club also has applications in the analysis of social networks [26]. Consequently, the extensive experimental study concentrate on finding 2- and 3-clubs [2, 4, 9, 11, 12, 20, 25]. Second, 2-Club is the most basic variant of $s$-Club that is different from the Clique problem which is equivalent to 1-Club. For example, being a clique is a hereditary graph property, that is, it is closed under vertex deletion. In contrast, being a 2-club is not hereditary, since deleting vertices can increase the diameter of a graph. Hence, it is interesting to spot differences in the computational complexity of the two problems.

In the spirit of multivariate algorithmics [16, 29], we aim to describe how structural properties of the input graph determine the computational complexity of 2-Club. We want to determine sharp boundaries between tractable and intractable special cases of 2-Club, and whether some graph properties, especially those motivated by the structure of social and biological networks, can be exploited algorithmically. By arranging the parameters in a hierarchy (ranging from large to small parameters) we draw a border line between tractability and intractability to obtain a systematic view on “stronger parameterizations” (refer to [24] for further discussion of the parameter hierarchy and its application). A similar approach was followed for other hard graph problems such as Odd Cycle Transversal [22] and for the computation of the pathwidth of a graph [7].

The structural properties that we consider, called structural graph parameters, are usually described by integers; well-known examples of such parameters are the maximum degree or the treewidth of a graph. Our results use the classical framework of NP-hardness as well as the framework of parameterized complexity to show (parameterized) tractability and intractability of 2-Club with respect to the structural graph parameters under consideration. That is,
for some graph parameters we show that 2-Club becomes NP-hard in case of constant parameter values, whereas for other graph parameters we show fixed-parameter (in)tractability.

1.1 Related Work

For all \( s \geq 1 \), \( s \)-Club is NP-complete on graphs of diameter \( s + 1 \) [4]; 2-Club is NP-complete even on split graphs and, thus, also on chordal graphs [4].\(^1\) In contrast, 2-Club is solvable in polynomial time on bipartite graphs, on trees, and on interval graphs [32]. Golovach et al. [19] consider the complexity of \( s \)-Club in special graph classes. For instance they prove polynomial-time solvability of 2-Club on choral bipartite, strongly chordal and distance hereditary graphs. Additionally, it is proven that on a superclass of these graph classes, called weakly chordal graphs, it is polynomial time solvable for odd \( s \) and NP-hard for even \( s \).

\( s \)-Club is well-understood from the viewpoint of approximation algorithms [3]: It is NP-hard to approximate \( s \)-Club within a factor of \( n^{\frac{s}{2} - \epsilon} \) for any \( \epsilon > 0 \). On the positive side, it has been shown that a largest set consisting of a vertex together with all vertices within distance \( \lfloor \frac{s}{2} \rfloor \) is a factor \( n^\frac{s}{4} \) approximation for even \( s \geq 2 \) and a factor \( n^\frac{s}{2} \) approximation for odd \( s \geq 3 \). Several heuristics [8, 11, 12, 12], integer linear programming formulations [2, 4, 9], fixed-parameter algorithms [20, 31], and branch-and-bound algorithms [9] have been proposed and experimentally evaluated [20, 25].

From the viewpoint of parameterized algorithmics, 1-Club is equivalent to Clique and thus \( \text{W}[1] \)-hard with respect to \( \ell \) [14]. In contrast, for all \( s \geq 2 \), \( s \)-Club is fixed-parameter tractable with respect to \( \ell \) [12, 31] and also with respect to the parameter treewidth of \( G \) [32]. Additionally, a search tree-based algorithm that branches into the two possibilities to delete one of two vertices with distance more than \( s \) achieves a running time of \( O(2^n - \ell \cdot nm) \) for the dual parameter \( n - \ell \) which measures the distance to a \( s \)-club [31].\(^2\) This algorithm cannot be improved to \( O((2 - \epsilon)^n - \ell \cdot n^{O(1)}) \) for any \( \epsilon > 0 \) unless the strong exponential time hypothesis fails [20]. Interestingly, Chang et al. [12] proved that with respect to the number of vertices \( n \) the same search tree algorithm runs in \( O(1.62^n) \) time.

The main observation behind the fixed-parameter algorithm for \( \ell \) is that any closed neighborhood \( N[v] \) of a vertex \( v \) is an \( s \)-club for \( s \geq 2 \). Hence, the maximum degree \( \Delta \) in non-trivial instances is less than \( \ell - 1 \). It also holds, however, that \( \ell \leq \Delta + 1 \) in yes-instances. Thus, for constant \( s \), fixed-parameter tractability with respect to \( \ell \) also implies fixed-parameter tractability with respect to the maximum degree of \( G \). Moreover, \( s \)-Club does not admit a polynomial kernel with respect to \( \ell \) (unless \( \text{NP} \subseteq \text{coNP/poly} \)) [31]. Interestingly, taking for

\(^1\)An NP-hardness reduction given by Balasundaram et al. [4, Theorem 1] can be easily modified such that the 2-Club instance is a split graph (make the vertex set \( E \) a clique).

\(^2\)Schäfer et al. [31] considered finding an \( s \)-club of size exactly \( \ell \). The claimed fixed-parameter tractability with respect to \( n - \ell \) however only holds for the problem of finding an \( s \)-club of size at least \( \ell \). The other fixed-parameter tractability results hold for both variants.
each vertex the vertex itself together with all other vertices that are in distance at most $s$ forms a so-called Turing-kernel with at most $k^2$-vertices for even $s$ and at most $k^3$-vertices for odd $s$ [31]. In companion work [20], we considered different structural parameters: We presented a fixed-parameter algorithm for the parameter treewidth and polynomial kernels for the parameters (size of a) feedback edge set and the cluster editing number. Additionally, we showed the non-existence of a polynomial kernel and that the simple search tree-based algorithm for the dual parameter $n - t$ is asymptotically optimal. Somewhat in contrast to this negative result, we showed that an implementation of the branching algorithm for the dual parameter combined with the Turing-kernelization is among the best-performing algorithms on real-world and on synthetic instances.

1.2 Structural Parameters

We next define the structural parameters under consideration (see Figure 1 for an illustration of their relations). For a set of graphs $\Pi$ (for instance, the set of bipartite graphs) the parameter distance to $\Pi$ measures the number of vertices that have to be deleted in the input graph in order to obtain a graph that is isomorphic to one in $\Pi$. Denoting by $P_t$ an induced path on $t$ vertices, the set of $P_t$-free graphs consists of all graphs not containing any $P_t$. The $P_4$-free graphs are called cographs, $P_3$-free graphs are so-called cluster graphs, and connected $P_3$-free graphs are cliques. A graph where each connected component is an $s$-club is called $s$-club cluster graph. Observe that this is equivalent to requiring that all shortest paths do not contain any $P_{s+2}$. Additionally, deleting all the vertices on a $P_t$ is a factor-$t$ approximation for the parameter distance to $P_t$-free graphs (or even restricted to $P_t$’s on shortest paths) and, thus, we may assume that such a vertex deletion set is provided as an additional input for the corresponding algorithms.

A graph is a co-cluster graph if its complement graph is a cluster graph. The minimum clique cover is the minimum number of cliques in a graph that are needed to cover all vertices, that is, each vertex is contained in at least one of these cliques. The domination number of a graph is the minimum size of a dominating set, this is, a set such that each vertex is contained in it or has at least one neighbor in it. A vertex cover of $G$ is a vertex set whose deletion transforms $G$ in a graph without any edges. An independent set is the complement of a vertex cover. A set of edge insertions and deletions is a cluster editing set if it transforms $G$ into a cluster graph. A set of edges is a feedback edge set if its deletion results in a graph without any cycle. A graph has $h$-index $k$, if $k$ is the largest number such that the graph has at least $k$ vertices of degree at least $k$. The degeneracy of a graph is the smallest number $d$ such that each subgraph has at least one vertex of degree at most $d$. The bandwidth of a graph $G = (V, E)$ is the minimum $k \in \mathbb{N}$ such that there is a function $f : V \rightarrow \mathbb{N}$ with $|f(v) - f(u)| \leq k$ for all edges $\{u, v\} \in E$. 


Figure 1: Overview of the relation between structural graph parameters (see Section 1.3) and of our results\textsuperscript{★} for 2-Club. An edge from a parameter $\alpha$ to a parameter $\beta$ below of $\alpha$ means that $\beta$ can be upper-bounded in a polynomial (usually linear) function in $\alpha$. The box containing the parameter (size of a) vertex cover on top, consists of all parameters for which 2-Club becomes fixed-parameter tractable but does not admit a polynomial kernel [20]. Therein, for all parameters the best performing algorithms run in $2^{O(2^k)} \cdot n^{O(1)}$ time with the only exception distance to 2-club admitting a $2^k \cdot n^{O(1)}$-time algorithm [31]. The box consisting of cluster editing, max leaf $\#$, and feedback edge set contains all parameters admitting a single-exponential time algorithm and a polynomial kernel [20]. The box at the bottom contains those parameters where 2-Club remains NP-hard even for constant values. It is open whether 2-Club is fixed-parameter tractable when it is parameterized by distance to interval or distance to 2-club cluster and whether it admits a polynomial kernel when parameterized by distance to clique.

1.3 Our Contribution

We make progress towards a systematic classification of the complexity of 2-Club with respect to structural graph parameters. Figure 1 gives an overview of our results and their implications. In Section 2, we consider the graph parameters minimum clique cover number, domination number, and some related graph parameters. We show that 2-Club is NP-hard even if the minimum clique cover number of $G$ is three. In contrast, we show that if the minimum clique cover number is two, then 2-Club is polynomial-time solvable. Then, we show that 2-Club is NP-hard even if $G$ has a dominating set of size two. This result is tight in the sense that 2-Club is trivially solvable in case $G$ has a dominating set of size one. In Section 3, we consider the parameter distance to bipartite graphs. We show that 2-Club is NP-hard even if the input graph can be transformed into a bipartite graph by deleting only one vertex. This is somewhat surprising since 2-Club is polynomial-time solvable on bipartite graphs [32]. Then, in Section 4, we consider the graph parameter $h$-index. The study of this parame-
ter is motivated by the fact that the $h$-index is usually small in social networks (see Section 4 for a more detailed discussion). On the positive side, we show that 2-CLUB is polynomial-time solvable for constant $k$. On the negative side, we show that 2-CLUB parameterized by the $h$-index $k$ of the input graph is W[1]-hard. Hence, a running time of $f(k) \cdot n^{O(1)}$ is probably not achievable. Even worse, we prove that 2-CLUB becomes NP-hard even for constant degeneracy. Note that degeneracy is provably at most as large as the $h$-index of a graph.

Finally, in Section 5 we describe a fixed-parameter algorithm for the parameter distance to cographs and show that it can be slightly improved for the weaker parameter distance to cluster graphs. Interestingly, these are rare examples for structural graph parameters, that are unrelated to treewidth and still admit a fixed-parameter algorithm (see Figure 1). Notably, the fixed-parameter algorithm for treewidth and those for distance to cograph both have the same running time characteristic, that is, $O(2^{O(k)} \cdot n^{O(1)})$ and this is, so far, also the best for the much “weaker” parameter vertex cover.

For the sake of completeness, we would like to mention that for the parameters bandwidth and maximum degree, taking the disjoint union of the input graphs is a composition algorithm that proves the non-existence of polynomial kernels [6], under the standard assumption that NP \subseteq coNP/poly does not hold.

### 1.4 Preliminaries

We only consider undirected and simple graphs $G = (V, E)$ where $n := |V|$ and $m := |E|$. For a vertex set $S \subseteq V$, let $G[S]$ denote the subgraph induced by $S$ and $G - S := G[V \setminus S]$. We use $\text{dist}_G(u, v)$ to denote the distance between $u$ and $v$ in $G$, that is, the length of a shortest path between $u$ and $v$. For a vertex $v \in V$ and an integer $t \geq 1$, denote by $N^G_t(v) := \{u \in V \setminus \{v\} \mid \text{dist}_G(u, v) \leq t\}$ the set of vertices within distance at most $t$ to $v$. Moreover, we set $N^G_0[v] := N^G_0(v) \cup \{v\}$, $N^G_t[v] := N^G_t(v)$ and $N^G(v) := N_1(v)$. If the graph is clear from the context, we omit the superscript $G$. Two vertices $v$ and $w$ are twins if $N(v) \setminus \{w\} = N(w) \setminus \{v\}$ and they are twins with respect to a vertex set $X$ if $N(v) \cap X = N(w) \cap X$. The twin relation is an equivalence relation; the corresponding equivalence classes are called twin classes. The following observation is easy to see and it shows that either none or all vertices of a twin class are contained in a maximum-size $s$-club.

**Observation 1.** Let $S$ be an $s$-club in a graph $G = (V, E)$ and let $u, v \in V$ be twins. If $u \in S$ and $|S| > 1$, then $S \cup \{v\}$ is also an $s$-club in $G$.

We briefly recall the relevant notions from parameterized complexity (see [14, 18, 28]). A problem is fixed-parameter tractable (FPT) with respect to a parameter $k$ if there is a computable function $f$ such that any instance $(I, k)$ can be solved in $f(k) \cdot |I|^{O(1)}$ time. A problem is contained in XP if it can be solved in $O(|I|^{f(k)})$ time for some computable function $f$. A kernelization algorithm reduces any instance $(I, k)$ in polynomial time to an equivalent instance $(I', k')$ with $|I'|, k' \leq g(k)$ for some computable $g$. The instance $(I', k')$ is called kernel of size $g$ and in the special case of $g$ being a polynomial it is a polynomial kernel.
A problem that is shown to be $W[1]$-hard by means of a parameterized reduction from a $W[1]$-hard problem is not fixed-parameter tractable, unless FPT $= W[1]$. A parameterized reduction maps an instance $(I, k)$ in $f(k) \cdot |I|^{O(1)}$ time for some function $f$ to an equivalent instance $(I', k')$ with $k' \leq g(k)$ for some functions $f$ and $g$.

## 2 Clique Cover Number and Domination Number

In this section, we prove that on graphs of diameter at most three, 2-CLUB is NP-hard even if either the minimum clique cover number is three or the domination number is two. We first show that these bounds are tight. The size of a maximum independent set is at most two. We describe a reduction from

\begin{proof}

We describe a reduction from CLIQUE. Let $(G = (V, E), k)$ be a CLIQUE instance. We construct a graph $G' = (V', E')$ consisting of three disjoint vertex sets, that is, $V' = V_1 \cup V_2 \cup V_E$. Further, for $i \in \{1, 2\}$, let $V_i = V_i^V \cup V_i^{\text{big}}$, where $V_i^V$ is a copy of $V$ and $V_i^{\text{big}}$ is a set of $n^5$ vertices. Let $u, v \in V$ be
two adjacent vertices in $G$ and let $u_1, v_1 \in V_1$, $u_2, v_2 \in V_2$ be the copies of $u$ and $v$ in $G'$. Then add the vertices $e_{uv}$ and $e_{vu}$ to $G'$ and add the edges $\{v_1, e_{uv}\}, \{e_{vu}, u_2\}, \{u_1, e_{vu}\}, \{e_{uv}, v_2\}$ to $G'$. Furthermore, add for each vertex $v \in V$ the vertex set $V_v^2 = \{e_{v1}^2, e_{v3}^2, \ldots, e_{v3}^n\}$ to $V_E$ and make $v_1$ and $v_2$ adjacent to all these new vertices. Finally, make the following vertex sets to cliques: $V_1$, $V_2$, $E_v^2$, and $V_1^{big} \cup V_2^{big}$. Observe that $G'$ has diameter three and that it has a clique cover number of three.

We now prove that $G$ has a clique of size $k$ $\Leftrightarrow$ $G'$ has a 2-club of size $k' = 2n^5 + kn^3 + 2k + 2(k)^2$.

"$\Rightarrow$": Let $S$ be a clique of size $k$ in $G$. Let $S_c$ contain all the copies of the vertices of $S$. Furthermore, let $S_E := \{e_{uv} \mid u_1 \in S_c \wedge v_2 \in S_c\}$ and $S_b := \{e_{v1}^i \mid v \in S \wedge 1 \leq i \leq n^3\}$. We now show that $S' := S_c \cup S_E \cup S_b \cup V_1^{big} \cup V_2^{big}$ is a 2-club of size $k'$. First, observe that $|V_1^{big} \cup V_2^{big}| = 2n^5$ and $|S_c| = 2k$. Hence, $|S_b| = kn^3$ and $|S_E| = 2(k)^2$. Thus, $S'$ has the desired size. With a straightforward case distinction one can check that $S'$ is indeed a 2-club.

"$\Leftarrow$": Let $S'$ be a 2-club of size $k'$. Observe that $G'$ consists of $|V'| = 2n^5 + 2n + 2\alpha + n^4$ vertices. Since $k' > 2n^5$ at least one vertex of $V_1^{big}$ and of $V_2^{big}$ is in $S$. Since all vertices in $V_1^{big}$ and in $V_2^{big}$ are twins, we can assume by Observation 1 that all vertices of $V_1^{big} \cup V_2^{big}$ are contained in $S'$. Analogously, it follows that at least $k$ sets $V_1^{v1}$, $V_1^{v3}$, $V_1^{v3}$, ..., $V_1^{v3}$ are completely contained in $S'$. Since $S'$ is a 2-club, the distance from vertices in $V_1^{v1}$ to vertices in $V_1^{v3}$ is at most two. Hence, for each set $V_1^{v1}$ in $S'$ the two neighbors $v_1^j$ and $v_2^j$ of vertices in $V_1^{v1}$ are also contained in $S'$. Since the distance of $v_1^j$ and $v_2^j$ for $v_1, v_2 \in S'$ is also at most two, the vertices $e_{v1}^j, e_{v3}^j, e_{v3}^j$ are part of $S'$ as well. Consequently, $v^j$ and $v^j$ are adjacent in $G$. Therefore, the vertices $v^1, \ldots, v^k$ form a size-$k$ clique in $G$.

Since a maximum independent set is also a dominating set, Theorem 1 implies that 2-CLUB is NP-hard on graphs with domination number three and diameter three. In contrast, for domination number one 2-CLUB is trivial. The following theorem shows that this cannot be extended.

**Theorem 2.** 2-CLUB is NP-hard even on graphs with domination number two and diameter three.

**Proof.** We present a reduction from CLIQUE. Let $(G = (V, E), k)$ be a CLIQUE instance and assume that $G$ does not contain isolated vertices. We construct the graph $G'$ as follows. First copy all vertices of $V$ into $G'$. In $G'$ the vertex set $V$ will form an independent set. Now, for each edge $\{u, v\} \in E$ add an edge-vertex $e_{\{u, v\}}$ to $G'$ and make $e_{\{u, v\}}$ adjacent to $u$ and $v$. Let $E_v$ denote the subset of edge-vertices. Next, add a vertex set $C$ of size $n + 2$ to $G'$ and make $C \cup V_{E'}$ a clique. Finally, add a new vertex $v^*$ to $G'$ and make $v^*$ adjacent to all vertices in $V$. Observe that $v^*$ plus an arbitrary vertex from $V_{E'} \cup C$ are a dominating set of $G'$ and that $G'$ has diameter three. We complete the proof by showing that $G$ has a clique of size $k$ $\Leftrightarrow$ $G'$ has a 2-club of size at least $|C| + |V_{E'}| + k$. 

\[ \]
“⇒” Let $K$ be a size-$k$ clique in $G$. Then, $S := K \cup C \cup V_E$ is a size-$|C| + |V_E| + k$ 2-club in $G$: First, each vertex in $C \cup V_E$ has distance two to all other vertices $S$. Second, each pair of vertices $u, v \in K$ is adjacent in $G$ and thus they have the common neighbor $e_{\{u,v\}}$ in $V_E$.

“⇐” Let $S$ be a 2-club of size $|C| + |V_E| + k$ in $G'$. Since $|C| > |V \cup \{v^*\}|$, it follows that there is at least one vertex $c \in S \cap C$. Since $c$ and $v^*$ have distance three, it follows that $v^* \not\in S$. Now since $S$ is a 2-club, each pair of vertices $u, v \in S \cap V$ has at least one common neighbor in $S$. Hence, $V_E$ contains the edge-vertex $e_{\{u,v\}}$. Consequently, $S \cap V$ is a size-$k$ clique in $G$. \hfill \Box

3 Distance to Bipartite Graphs

A 2-club in a bipartite graph is a biclique and, thus, 2-CLUB is polynomial-time solvable on bipartite graphs [32]. However, 2-CLUB is already NP-hard on graphs that become bipartite by deleting only one vertex.

**Theorem 3.** 2-CLUB is NP-hard even on graphs with distance one to bipartite graphs.

**Proof.** We reduce from the NP-hard MAXIMUM 2-SAT problem: Given a positive integer $k$ and a set $C := \{C_1, \ldots, C_m\}$ of clauses over a variable set $X = \{x_1, \ldots, x_n\}$ where each clause $C_i$ contains two literals, the question is whether there is an assignment $\beta$ that satisfies at least $k$ clauses.

Given an instance of MAXIMUM 2-SAT where we assume that each clause occurs only once, we construct an undirected graph $G = (V, E)$. The vertex set $V$ consists of the four disjoint vertex sets $V_C, V_F, V_X, V_\bar{X}$, and one additional vertex $v^*$. The construction of the four subsets of $V$ is as follows.

The vertex set $V_C$ contains one vertex $c_i$ for each clause $C_i \in \mathcal{C}$. The vertex set $V_F$ contains for each variable $x \in X$ exactly $n^5$ vertices $x^1 \ldots x^{n^5}$. The vertex set $V_X^1$ contains for each variable $x \in X$ two vertices: $x_t$ which corresponds to assigning true to $x$ and $x_f$ which corresponds to assigning false to $x$. The vertex set $V_\bar{X}^2$ is constructed similarly, but for every variable $x \in X$ it contains $2 \cdot n^3$ vertices: the vertices $x_1^1 \ldots x_{n^3}^1$ which correspond to assigning true to $x$, and the vertices $x_1^2 \ldots x_{n^3}^2$ which correspond to assigning false to $x$.

Next, we describe the construction of the edge set $E$. The vertex $v^*$ is made adjacent to all vertices in $V_C \cup V_F \cup V_X^1$. Each vertex $c_i \in V_C$ is made adjacent to the two vertices in $V_X^1$ that correspond to the two literals in $C_i$. Each vertex $x^1 \in V_F$ is made adjacent to $x_t$ and $x_f$, that is, the two vertices of $V_X^1$ that correspond to the two truth assignments for the variable $x$. Finally, each vertex $x_t^1 \in V_X^2$ is made adjacent to all vertices of $V_X^1$ except to the vertex $x_f$. Similarly, each $x_f^1 \in V_\bar{X}^2$ is made adjacent to all vertices of $V_\bar{X}^1$ except to $x_t$. This completes the construction of $G$ which can clearly be performed in polynomial time. Observe that the removal of $v^*$ makes $G$ bipartite: each of the four vertex sets is an independent set and the vertices of $V_C$, $V_F$, and $V_\bar{X}$ are only adjacent to vertices of $V_X^1$.\hfill 9
The main idea behind the construction is as follows. The size of the 2-club forces the solution to contain the majority of the vertices in $V_F$ and $V^2_F$. As a consequence, for each $x \in X$ exactly one of $x_i$ or $x_f$ is in the 2-club. Hence, the vertices from $V^2_F$ in the 2-club represent a truth assignment. In order to fulfill the bound on the 2-club size, at least $k$ vertices from $V_C$ are in the 2-club; these vertices can only be added if the corresponding clauses are satisfied by the represented truth assignment. It remains to prove the following claim:

Claim. $(C, k)$ is a yes-instance of MAXIMUM 2-SAT $\Leftrightarrow G$ has a 2-club of size $n^6 + n^4 + n + k + 1$.

Proof. “$\Rightarrow$”: Let $\beta$ be an assignment for $X$ that satisfies $k$ clauses $C_1, \ldots, C_k$ of $C$. Consider the vertex set $S$ that consists of $V_F, v^*$, the vertex set $\{c_1, \ldots, c_k\} \subseteq V_C$ that corresponds to the $k$ satisfied clauses, and for each $x \in X$ of the vertex set $\{x_1, x^*_1, \ldots, x^n\} \subseteq V^2_F \cup V^2_X$ if $\beta(x) = true$ and the vertex set $\{x_f, x^*_f, \ldots, x^n\} \in V^{2}_F \cup V^{2}_X$ if $\beta(x) = false$. Clearly, $|S| = n^6 + n^4 + n + k + 1$. In the following, we show that $S$ is a 2-club. Herein, let $S^1_X := V^1_X \cap S$, $S^2_X := V^2_X \cap S$, and $S_C := V_C \cap S$.

First, $v^*$ is adjacent to all vertices in $S_C \cup V_F \cup S^1_X$. Hence, all vertices of $S \setminus S^2_X$ are within distance two in $G[S]$. By construction, the vertex sets $S^1_X$ and $S^2_X$ form a complete bipartite graph in $G$: A vertex $x^i \in S^2_X$ is adjacent to all vertices in $V^1_X$ except $x_f$ which is not contained in $S^1_X$. The same argument applies to some $x^i_f \in S^1_X$. Hence, the vertices of $S^2_X$ are neighbors of all vertices in $S^1_X$. This also implies that the vertices of $S^2_X$ are in $G[S]$ within distance two from $v^*$ and from every vertex in $V_F$ since each vertex of $V_F \cup \{v^*\}$ has at least one neighbor in $S^1_X$. Finally, since the $k$ vertices in $S_C$ correspond to clauses that are satisfied by the truth assignment $\beta$, each of these vertices has at least one neighbor in $S^1_X$. Hence, every vertex in $S^2_X$ has in $G[S]$ distance at most two to every vertex in $S_C$.

“$\Leftarrow$”: Let $S$ be a 2-club of size $n^6 + n^4 + n + k + 1$, and let $S^1_X := V^1_X \cap S$, $S^2_X := V^2_X \cap S$, $S_F := V_F \cap S$ and $S_C := V_C \cap S$. Clearly, neither $S^2_X = \emptyset$ nor $S_F = \emptyset$.

Since $|V_C| + |V^1_X| + |V^2_X| + 1 \leq n^2 + 2n + 2n^4 + 1 < n^5$ for sufficiently large $n$, $S$ contains more than $n^6 - n^5$ vertices from $V_F$. Consequently, for each $x \in X$ there is an index $1 \leq i \leq n^5$ such that $x^i \in S_F$.

We next show that for each $x \in X$ it holds that either $x_i$ or $x_f$ is contained in $S^1_X$. Towards this, since $S$ is a 2-club, every vertex pair $x^i \in S_F$ and $u \in S^2_X$ has at least one common neighbor in $S$. By construction, this common neighbor is a vertex of $S^1_X$ and thus either $x_i$ or $x_f$. Moreover, by the observation above for each $x \in X$ at least one $x^i$ is contained in $S_F$. Thus, for each $x \in X$ at least one of $x_i$ and $x_f$ is contained in $S^1_X$.

Now observe that, $G[S^1_X \cup S^2_X]$ is a complete bipartite graph, since $S^1_X$ and $S^2_X$ are independent sets and $S^2_X$ has only neighbors in $S^1_X$. This implies that if for some $x \in X$ there exists indices $1 \leq i, j \leq n^3$ with $x^i$ and $x_j^f$ are in $S^2_X$, then $x_i$ and $x_j$ are not in $S^1_X$. This contradicts the above observation that at least one of $x_i$ and $x_j$ is in $S^1_X$. Moreover, since $|V_C| + |V^1_X| + 1 \leq n^2 + 2n + 1 < n^3$ and $|S \setminus V_F| > n^4$, we have $|S^2_X| > n^4 - n^3$. It follows that for each $x \in X$ there is
an index $1 \leq i \leq n^3$ such that either $x^i_t \in S^3_X$ or $x^i_f \in S^3_X$. Finally, this implies that either $x_t$ or $x_f$ is not contained in $S^1_X$.

Summarizing, $S$ has at most $n^6$ vertices from $V_F$, at most $n^4$ vertices belonging to $S^2_X$, exactly $n$ vertices belonging to $S^1_X$, and thus there are $k + 1$ vertices in $S \cup \{v^*\}$. Since $S$ is a 2-club that has nonempty $S^2_X$, every one of the at least $k$ vertices from $S^1_X$ has at least one neighbor in $S^1_X$. Because for each $x \in X$ either $x_f$ or $x_t$ is in $S^1_X$, the $n$ vertices from $S^1_X$ correspond to an assignment $\beta$ of $X$. By the above observation, this assignment satisfies at least $k$ clauses of $C$.

4 Average Degree and $h$-Index

2-Club is fixed-parameter tractable for the parameter maximum degree (the algorithm of Schäfer et al. [31] can be analyzed in that way without any changes). It has been observed that in large-scale biological [23] and social networks [5] the degree distribution often follows a power law, implying that there are some high-degree vertices while most vertices have low degree. This suggests considering stronger, that is, provably smaller, parameters such as $h$-index, degeneracy, and average degree. For any graph it holds that avg. degree $\leq 2 \cdot$ degeneracy $\leq 2 \cdot h$-index, see also Figure 1 for other relationships. Furthermore, analyzing the coauthor network derived from the DBLP dataset$^3$ with more than 715,000 vertices, maximum degree 804, $h$-index 208, degeneracy 113, and average degree 7 shows that also in real-world social networks these parameters are considerably smaller than the maximum degree (see [20] for an analysis of these parameters on a broader dataset).

Unsurprisingly, 2-Club is NP-hard even with constant average degree.

**Proposition 1.** For any constant $\alpha > 2$, 2-Club is NP-hard on connected graphs with average degree at most $\alpha$.

**Proof.** Let $(G, \ell)$ be an instance of 2-Club where $\Delta$ is the maximum degree of $G$. We can assume that $\ell > \Delta + 2$ since, as shown for instance in the proof of Theorem 1, 2-Club remains NP-hard in this case. We add a path $P$ to $G$ and an edge from an endpoint $p$ of $P$ to an arbitrary vertex $v \in V$. Since $\ell > \Delta + 2$, any 2-club of size at least $\ell$ contains at least one vertex that is not in $P$. Furthermore, it cannot contain $p$ and $v$ since in this case it is a subset of either $N[v]$ or $N[p]$ which both have size at most $\Delta + 2$ (v has degree at most $\Delta$ in $G$). Hence, the instances are equivalent. Putting at least $\lceil \frac{2m}{n-2} - n \rceil$ vertices in $P$ ensures that the resulting graph has average degree at most $\alpha$. ⊓⊔

We remark that the bound provided in Proposition 1 is tight: Consider a connected graph $G$ with average degree at most two, that is, $\frac{1}{n} \sum_{v \in V} \deg(v) \leq 2$. Since $\sum_{v \in V} \deg(v) = 2m$, it follows that $n \geq m$ and, thus, the feedback edge set of $G$ contains at most one edge. As 2-Club is fixed-parameter tractable

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$^3$The dataset and a corresponding documentation are available online (http://dblp.uni-trier.de/xml/). Accessed Feb. 2012
with respect to the (size of a) feedback edge set [20], it follows that 2-CLUB can be solved in polynomial time on connected graphs with average degree at most two.

Proposition 1 suggests considering “weaker” parameters such as degeneracy or h-index [15] of G (see Figure 1). Recall that having h-index k means that there are at most k vertices with degree greater than k. Since social networks have small h-index [20], fixed-parameter tractability with respect to the h-index would be desirable. Unfortunately, we show that 2-CLUB is W[1]-hard when parameterized by the h-index and NP-hard with constant degeneracy. Following this result, we show that there is “at least” an XP-algorithm implying that 2-CLUB is polynomial-time solvable for constant h-index.

We reduce from the W[1]-hard MULTICOLORED CLIQUE problem [17].

**Multicolored Clique**

**Input:** An undirected graph \( G = (V,E), k \in \mathbb{N}, \) and a (vertex) coloring \( c : V \rightarrow \{1,\ldots,k\} \).

**Question:** Is there a multicolored clique of size \( k \) in \( G \), that is, a clique \( C \subseteq V \) such that \( c(v) \neq c(v') \) for all \( \{v,v'\} \subseteq V \) with \( v \neq v' \)?

**Lemma 2.** There are two polynomial-time computable reductions that compute for any instance \( (G,c,k) \) of MULTICOLORED CLIQUE an equivalent 2-CLUB-instance \( (G',\ell) \) such that \( G' \) has diameter three and, additionally, in reduction i) \( G' \) has h-index at most \( k + 7 \) and in reduction ii) \( G' \) has degeneracy five.

**Proof.** The only difference between both reductions is the construction of a so-called coloring gadget. We first describe the common part. Let \( (G,c,k) \) with \( G = (V,E) \) and \( c : V \rightarrow \{1,\ldots,k\} \) be an instance of MULTICOLORED CLIQUE. We construct a graph \( G' \) and choose \( \ell \in \mathbb{N} \) such that \( (G',\ell) \) is a yes-instance for 2-CLUB if and only if \( (G,c,k) \) is a yes-instance for MULTICOLORED CLIQUE. We will first construct some structures in \( G' \) which allow to describe the basic ideas: For each vertex \( v \in V \) create a vertex gadget by adding the \( \alpha \)-vertices \( \{\alpha_1^v,\ldots,\alpha_n^v\} \), the \( \beta \)-vertices \( \{\beta_1^v,\ldots,\beta_{n+1}^v\} \), and the \( \gamma \)-vertices \( \{\gamma_1^v,\ldots,\gamma_n^v\} \), and \( \{\omega_1^v,\omega_2^v\} \). Add edges such that \( (\alpha_1^v,\beta_1^v,\gamma_1^v,\alpha_2^v,\beta_2^v,\gamma_2^v,\ldots,\alpha_n^v,\beta_n^v,\gamma_n^v,\omega_1^v,\omega_2^v,\alpha_1^v) \) induces a cycle. Add the three vertices

\[
U = \{u_\alpha,u_\beta,u_\gamma\}
\]

and add edges from all \( \alpha \)- (\( \beta \), \( \gamma \))-vertices to \( u_\alpha \) (\( u_\beta \), \( u_\gamma \)), respectively. Add the edges \( \{\omega_1^u,u_\alpha\} \) and \( \{\omega_2^u,u_\gamma\} \). Furthermore, for a fixed ordering \( V = \{v_1,\ldots,v_n\} \) add for each edge \( \{v_i,v_j\} \in E \) an edge-vertex \( e_{v_i,v_j} \) that is adjacent to each of \( \{\alpha_j^v,\beta_j^v,\gamma_j^v\} \). (Observe that the \( \alpha \)- and \( \gamma \)-vertex neighbor are in different vertex gadgets.) The following property is fulfilled:

1. For each vertex \( v \) in a vertex gadget it holds that \( |N(v) \cap U| = 1 \) and \( |N(v) \cap N(u)| = 1 \) for each \( u \in U \setminus N(u) \).
The idea of the construction is that \( U \) will be forced to be contained in any 2-club \( S \) of size at least \( \ell \). Hence by Property 1 it follows that if an \( \alpha \)-vertex in a vertex gadget is contained in \( S \), then the unique \( \beta \)- and \( \gamma \)-vertex in its neighborhood has to be contained in \( S \) as well. Since this argument symmetrically holds for \( \beta \)- and \( \gamma \)-vertices, it follows that either all or none of the vertices from a vertex gadget are contained in \( S \). Observe that, in this context, \( \omega^\alpha_v (\omega^\gamma_v) \) behaves like a “normal” \( \alpha \)- (\( \gamma \)-) vertex. Analogously, each edge-vertex \( e_{v_i,v_j} \in S \) needs to have a common neighbor with each vertex of \( U \). Thus, \( e_{v_i,v_j} \in S \) implies that all vertices in the two vertex gadgets that correspond to \( v_i \) and \( v_j \) are contained in \( S \). By connecting the vertices \( \{\omega^\alpha_v, \omega^\gamma_v\} \) appropriately we will ensure that for each color \( c \) at most one vertex gadget whose vertex in \( G \) is colored with \( c \) can have a non-empty intersection with \( S \). (The construction of the corresponding coloring gadget is the only part where the two reductions differ.) Furthermore, we choose the value of \( \ell \) such that \( S \) contains vertices from at least \( k \) vertex gadgets and at least \( \binom{k}{2} \) edge-vertices. Hence, there are exactly \( k \) vertex gadgets together with \( \binom{k}{2} \) edge-vertices that contribute to \( S \). Since the vertices corresponding to the vertex gadgets have different colors and since the endpoints of the edges corresponding to the \( \binom{k}{2} \) edge-vertices are all within this set of \( k \) vertices, the set \( S \) corresponds to a multicolored clique in \( G \).

To complete the construction and to ensure the properties discussed above, we next add the anchor gadget and the coloring gadget. To argue about their correctness we claim that, eventually,

\[
|V'| = \frac{n(3n+3)}{n \text{ vertex gadgets}} + \frac{4n^3 + 7}{\text{anchor gadget}} + \frac{m}{m \text{ edge-vertices}} + \frac{|V_C|}{\text{coloring gadget}} \quad (1)
\]

and we set

\[
\ell := \frac{k(3n+3)}{k \text{ vertex gadgets}} + \frac{4n^3 + 7}{\text{anchor gadget}} + \frac{\binom{k}{2}}{\binom{k}{2} \text{ edge-vertices}} + \frac{|V_C|}{\text{coloring gadget}} \quad . \quad (2)
\]

**Anchor Gadget:** We denote by \( V_A \) the set of all vertices in the anchor gadget including \( U \) and it will have size \( 4n^3 + 7 \). Besides \( U \) the anchor gadget will contain only four other vertices, namely \( \{l_U, l, r_1, r_2\} \), that have neighbors outside the gadget. Before describing the construction we will list some properties of it that will be used in the argumentation later on.

2. The set \( U \) is contained in any 2-club in \( G' \) of size at least \( \ell \).

3. A 2-club of size at least \( \ell \) contains either all or none of the vertices of a vertex gadget.

4. For any two vertices \( u \in U \) and \( v \in \{r_1, r_2\} \) it holds that \( (N(u) \cup N(v)) \cap V_A = V_A \setminus (U \cup \{v\}) \), that \( (N(l) \cup N(v)) \cap V_A = V_A \setminus U \), and \( (N(l_U) \cup N(v)) \cap V_A = V_A \).
Informally, Property 4 ensures that if a vertex is adjacent to one of \( \{l_U, l\} \cup U \) and to one of \( \{r_1, r_2\} \), then it has distance at most two to all vertices in \( V_\alpha \setminus U \) and also distance at most two to all of \( U \) if it is adjacent to \( l_U \).

The anchor gadget is constructed as follows (see Figure 2): Add four sets \( V_\alpha, V_\beta, V_\gamma, V_{\alpha,\beta,\gamma} \) each of size \( n^3 \) and add edges from each vertex in \( V_{\alpha,\beta,\gamma} \) to each in \( U \cup \{l, l_U\} \). Additionally, add edges from each vertex in \( V_\alpha \) to each of \( \{u_\alpha, r_1, r_2\} \), from each vertex in \( V_\beta \) to each of \( \{u_\beta, r_1, r_2\} \), and from each vertex in \( V_\gamma \) to each of \( \{u_\gamma, r_1, r_2\} \). Finally, add edges such that \( \{l_U, l, r_1, r_2\} \) is a clique and an edge from \( l_U \) to each vertex in \( U \).

By the construction above, Property 4 is fulfilled and the anchor gadget is a 2-club. Observe that \( u_\alpha \) (\( u_\beta, u_\gamma \)) is the only common neighbor of any vertex in \( V_{\alpha,\beta,\gamma} \) and any vertex in \( V_\alpha \) (\( V_\beta, V_\gamma \), resp.) and hence if at least one vertex from each set \( V_\alpha, V_\beta, V_\gamma, V_{\alpha,\beta,\gamma} \) is contained in a 2-club, then also \( U \) is contained.

To prove Property 2, let \( S \subseteq V' \) be a 2-club of size \( \ell \) that is disjoint to at least one of \( \{V_\alpha, V_\beta, V_\gamma, V_{\alpha,\beta,\gamma}\} \). The number of vertices that are not in \( S \) is at most \( |V'| - \ell \), which is (see Equations (1) and (2)):

\[
|V'|- \ell = (n-k)(3n+3) + m - \binom{k}{2} < n^3.
\]

This implies a contradiction and proves Property 2. As argued above Properties 1 and 2 imply the correctness of Property 3.

Recall that so far only \( U \) has neighbors outside the anchor gadget, namely all \( \alpha- (\beta-, \gamma-) \)-vertices are adjacent to \( u_\alpha \) (\( u_\beta, u_\gamma \), resp.) and \( \omega_\alpha^n \) (\( \omega_\beta^n \) resp.) is adjacent to \( u_\alpha \) (\( u_\gamma \), resp.). We describe via properties how to connect the anchor gadget to the vertex gadgets.
5. \( \omega_a^v \) is adjacent to \( r_1 \) and \( \omega_a^v \) is adjacent to \( r_2 \) for all \( v \in V \).

6. All \( \alpha-, \beta-, \) and \( \gamma- \) vertices and all edge-vertices are adjacent to each of \( \{r_1, r_2\} \). Additionally, each edge-vertex is adjacent to \( l \).

Observe that Property 6 does not violate the correctness of Property 3 since the vertices \( r_1, r_2 \) are not neighbors of any vertex in \( U \) (see Property 4). Properties 4 to 6 together imply that all vertex pairs in \( G' \) except \( \{\omega_a^v, \omega_a^{v'}\} \) with \( v \neq v' \) have distance at most two. We next construct the so-called coloring gadget that guarantees that only those vertex pairs \( \{\omega_a^v, \omega_a^{v'}\} \) have a common neighbor (and thus can be contained in any 2-club) for which \( c(v) \neq c(v') \). We will give two different constructions of the coloring gadget where the first guarantees an \( h \)-index of at most \( k + 7 \) and the second guarantees degeneracy five. Denoting the set of vertices in the coloring gadget by \( V_C \) both constructions fulfill the following properties:

7. Each vertex in \( V_C \) is adjacent to each of \( \{l_U, r_1, r_2\} \).

8. Any pair \( \{\omega_a^v, \omega_a^{v'}\}, v \neq v' \), has a common neighbor in \( V_C \) if and only if \( c(v) \neq c(v') \).

The two properties above are sufficient to prove the correctness of both reductions.

**Coloring gadget i):** For each color \( i \in \{1, \ldots, k\} \) add a vertex \( c_i \) and let \( V_C = \{c_1, \ldots, c_k\} \) be the vertex set containing these vertices. Add an edge between a vertex \( \omega_a^v \) and \( c_i \) if \( c(v) = i \) and an edge from \( \omega_a^v \) to \( c_i \) if \( c(v) \neq i \) (Property 8). Finally, add edges such that each vertex in \( V_C \) is adjacent to each vertex in \( \{l_U, r_1, r_2\} \) (Property 7).

Note that the \( h \)-index of \( G' \) is at most \( |V_C| + |U| + |\{l_U, l, r_1, r_2\}| = k + 7 \), as the vertices in \( V_C \cup U \cup \{l_U, l, r_1, r_2\} \) are the only ones that might have degree at least \( k + 7 \).

**Coloring gadget ii):** For each pair \( \{\omega_a^v, \omega_a^{v'}\} \) with \( c(v) \neq c(v') \) add a vertex \( c_{v, v'} \) that is adjacent to each of \( \{\omega_a^v, \omega_a^{v'}\} \) (Property 8). Finally, denoting all these new vertices by \( V_C \) we add an edge from each vertex in \( V_C \) to each vertex in \( \{l_U, r_1, r_2\} \) (Property 7).

We next prove that \( G' \) has degeneracy five by giving an elimination ordering, that is, an order of how to delete vertices of degree at most five that results in an empty graph: In the anchor gadget each of the vertices in \( V_\alpha, V_\beta, V_\gamma, V_{\alpha, \beta, \gamma} \) has maximum degree five and hence they can be deleted. Then, delete all vertices in \( V_C \), as each of them also has degree five. Delete all edge-vertices (they also have degree five). In the remaining graph each vertex in a vertex gadget (see Property 6) is adjacent to its two neighbors in its vertex gadget, adjacent to one of \( U \), and one or two neighbors in \( \{r_1, r_2\} \). Hence, all vertices in vertex gadgets can be removed as they have degree at most five. The remaining vertices are \( U \cup \{l_U, l, r_1, r_2\} \) and all vertices in \( U \cup \{l, r_1, r_2\} \) have maximum degree four.

It remains to prove the correctness of the two reductions:
Claim. \((G, c, k)\) is a yes-instance of MULTICOLORED CLIQUE \iff \((G', \ell)\) is a yes-instance of 2-CLUB.

\("\Rightarrow\" Let \(C\) be a multicolored clique in \(G\) of size \(k\). We construct a set \(S \subseteq V'\) of size \(\ell\) and prove that it is a 2-club in \(G'\). The set \(S\) contains each vertex gadget that corresponds to some vertex in \(C\), the coloring gadget, the anchor gadget, and any edge vertex \(e_{v_i, v_j}\) with \(v_i, v_j \in C\). See Equation (2) to verify that \(|S| = \ell\). To verify that \(S\) is a 2-club, note that for each vertex \(v\) in a vertex gadget it holds that its unique neighbor with any vertex in \(U \setminus N(v)\) is contained in \(S\) and thus from Properties 1 and 4 to 6 it follows that in \(G'[S]\) the vertex \(v\) has distance at most two to any anchor gadget vertex. Additionally, Properties 5 to 8 imply that \(v\) has distance at most two to all other vertex gadget vertices in \(S\), all coloring gadget vertices, and all edge vertices in \(S\). Properties 4, 6 and 7 imply that any coloring gadget vertex has distance at most two to all anchor vertices, coloring gadget vertices, and edge vertices. Finally, Properties 4 and 6 show that each edge vertex has distance two to all anchor vertices.

\("\Leftarrow\" Let \(S\) be a 2-club of size at least \(\ell\). By Property 2 it follows that \(U \subseteq S\) and by Property 3 it follows that each vertex gadget is either fully contained in \(S\) or is disjoint to \(S\). Denote by \(C\) the vertices in \(G\) that correspond to the vertex gadgets that are fully contained in \(S\). First, since two vertices \(\omega_v^\alpha\) and \(\omega_{v'}^\gamma\), \(v \neq v'\), do not have a common neighbor if \(c(v) = c(v')\) (Property 8) and there are only \(k\) colors, it follows that \(|C| \leq k\). Hence by Equations (1) and (2) it follows that \(S\) contains at least \(\binom{k}{2}\) edge vertices. Since each edge vertex \(e_{v_i, v_j}\) needs to have a common neighbor with each vertex in \(U\) and the \(\alpha\)- and the \(\gamma\)- vertex neighbors of \(e_{v_i, v_j}\) are in different vertex gadgets, it follows that \(\{v_i, v_j\} \subseteq C\). From this, since \(|C| \leq k\) it follows that \(|C| = k\) and that \(S\) contains exactly \(\binom{k}{2}\) edge vertices, implying that \(|C|\) induces a clique in \(G\). Finally, note that this clique is multicolored because of Property 8.

Lemma 2 imply several consequences.

Corollary 1. 2-Club is NP-hard on graphs with degeneracy five.

Corollary 2. 2-Club parameterized by h-index is W[1]-hard.

Since the reduction in Lemma 2 is from Multicolored Clique and in the reduction the new parameter is linearly bounded in the old one, the results of Chen et al. [13] imply the following.

Corollary 3. 2-Club cannot be solved in \(n^{o(k)}\)-time on graphs with h-index \(k\) unless the exponential time hypothesis fails.

We next prove that there is an XP-algorithm for the parameter h-index.

Theorem 4. 2-Club can be solved in \(O(2^{k^4} \cdot n^{2k} \cdot n^2m)\) time where \(k\) is the h-index of the input graph.

Proof. We give an algorithm that finds a maximum 2-club in a graph \(G' = (V', E')\) in \(O(2^{k^4} \cdot n^{2k} \cdot n^2m)\) time where \(k\) denotes the h-index of \(G'\). Let
Let \( X' \subseteq V' \) be the set of all vertices in \( G' \) with degree greater than \( k \). By definition of the \( h \)-index, \( |X'| \leq k \). For the proof of correctness fix any maximum 2-club \( S \) in \( G' \). Throughout the algorithm via branching we will guess some vertices contained in \( S \) and we will collect them in the set \( P \). Then, cleaning the graph means to exhaustively remove all vertices that do not have distance at most two to all vertices in \( P \). These vertices cannot be contained in \( S \) and, clearly, if this requires to delete some vertex in \( P \) we will abort this branch.  

First, branch into the at most \( 2^k \) cases to guess the set \( X = X' \cap S \) (potentially \( X = \emptyset \)). Delete all vertices from \( X' \setminus X \), initialize \( P \) with \( X \), and clean the graph. Denoting the resulting graph by \( G = (V,E) \), we next describe how to find a maximum 2-club in \( G \) that contains \( X \). Towards this, consider the at most \( 2^k \) twin classes of the vertices in \( V \setminus X \) with respect to \( X \). Branch into the \( O(n^{2k}) \) cases to guess for each twin class \( T \) any vertex from \( T \cap S \), called the center of \( T \). Clearly, if \( T \cap S = \emptyset \), then there is no center and we delete all vertices in \( T \). Add all the centers to \( P \) and clean the graph.

Two twin classes \( T \) and \( T' \) are in conflict if \( N^G(T) \cap N^G(T') \cap X = \emptyset \). Now, the crucial observation is that, if \( T \) and \( T' \) are in conflict, then all vertices in \( (T \cup T') \cap S \) are contained in the same connected component of \( G[S \setminus X] \), since otherwise they would not have pairwise distance at most two. However, this implies that all vertices in \( T \cap S \) have pairwise distance at most four in \( G[S \setminus X] \). Hence, for each twin class \( T \) with center \( c \) that is in conflict to any other twin class it holds that \( T \cap S \subseteq N^G_{4-X}[c] \) and since \( G-X \) has maximum degree at most \( k \), one can guess \( N^S_{4-X}[c] := N^G_{4-X}[c] \cap S \) by branching into at most \( 2^{4k} \) cases. Delete all vertices in \( T \) guessed to be not contained in \( N^S_{4}[c] \), add \( N^S_{4}[c] \) to \( P \), and clean the graph. Note that the remaining graph is a 2-club, since \( P \) contains \( X \) and the intersection of \( S \) with each twin class that is in conflict to any other twin class. By definition of twin classes that are in conflict, it holds that all other twin classes share a common neighbor in \( X \).

5 Distance to (Co-)Cluster Graphs and Cographs

In this section we present fixed-parameter algorithms for 2-CLUB parameterized by distance to co-cluster graphs, by distance to cluster graphs, and by distance to cographs. All these algorithms have running time \( 2^{O(2^k)} \cdot n^{O(1)} \) which is roughly similar to the one obtained for treewidth [20]. For the weaker parameters the constants in the exponential part of the running time are smaller. Hence, none of the algorithms “dominates” one of the other algorithms even with distance to cographs being a provably smaller parameter than distance to cluster graphs or distance to co-cluster graphs (see Figure 1). As already mentioned, even for the considerably weaker parameter vertex cover the best known algorithm has running time \( 2^{O(2^k)} \cdot n^{O(1)} \). In contrast, the parameter distance to clique which is unrelated to vertex cover admits a trivial \( O(2^k \cdot nm) \)-time algorithm, even in case of the general s-CLUB. This is implied by the \( O(2^k \cdot nm) \)-time algorithm for the dual parameter \( n - \ell \) [31] which can be interpreted as distance to 2-clubs and the fact that each clique is a 2-club.
5.1 Distance to Co-Cluster Graphs and Distance to Cluster Graphs

To complete the picture drawn in Figure 1, we also give short description of an algorithm for 2-CLUB parameterized by the distance to co-cluster graphs. A graph is a co-cluster graph if its complement graphs is cluster graph.

**Theorem 5.** 2-CLUB is solvable in \(O(2^k \cdot 2^{2^k \cdot nm})\) time where \(k\) denotes the distance to co-cluster graphs.

**Proof.** Let \((G, X, \ell)\) be an 2-CLUB instance where \(X\) has \(|X| = k\) and \(G - X\) is a co-cluster graph. Note that the co-cluster graph \(G - X\) is either a connected graph or an independent set. In the case that \(G - X\) is an independent set, the set \(X\) is a vertex cover and we thus apply the algorithm we gave in companion work [20] to solve the instance in \(O(2^k \cdot 2^{2^k \cdot nm})\) time.

Hence, assume that \(G - X\) is connected. Since \(G - X\) is the complement of a cluster graph, this implies that \(G - X\) is a 2-club. Thus, if \(\ell \leq n - k\), then we can trivially answer yes. Hence, assume that \(\ell > n - k\) or, equivalently, \(k > n - \ell\). Schäfer et al. [31] showed that 2-club can be solved in \(O(2^{n-\ell \cdot nm})\) (simply choose a vertex pair having distance at least three and branch into the two cases of deleting one of them). Since \(k > n - \ell\) it follows that 2-club can be solved in \(O(2^k \cdot 2^{2^k \cdot nm})\) time in this case.

Next, we present a fixed-parameter algorithm for the parameter distance to cluster graphs.

**Theorem 6.** 2-CLUB is solvable in \(O(2^k \cdot 3^{2^k \cdot nm})\) time where \(k\) denotes distance to cluster graphs.

**Proof.** Let \((G, X, \ell)\) be a 2-CLUB instance where \(G - X\) is a cluster graph and \(|X| = k\). First, branch into all possibilities to choose the subset \(X' \subseteq X\) that is contained in the desired 2-club \(S\). Then, remove \(X \setminus X'\) and all vertices that are not within distance two to all vertices in \(X'\), and let \(G' = (V', E')\) denote the resulting graph.

Let \(T = T_1, \ldots, T_p\) be the set of twin classes of \(V' \setminus X'\) with respect to \(X'\) and let \(C_1, \ldots, C_q\) denote the clusters of \(G' - X'\). Two twin classes \(T_i\) and \(T_j\) are in conflict if \(N(T) \cap N(T') \cap X' = \emptyset\). The three main observations exploited in the algorithm are the following: First, if two twin classes \(T_i\) and \(T_j\) are in conflict, then all vertices of \(T_i\) that are in a 2-club and all vertices from \(T_j\) that are in a 2-club must be in the same cluster of \(G' - X'\). Second, every vertex from \(G' - X'\) can reach all vertices in \(X'\) only via vertices of \(X'\) or via vertices in its own cluster. Third, if one 2-club-vertex \(v \in S\) is in a twin class \(v \in T_i\) and in a cluster \(v \in C_j\), then all vertices that are in \(T_i\) and in \(C_j\) can be added to \(S\) without violating the 2-club property.

We exploit these observations in a dynamic programming algorithm. In this algorithm, we create a two-dimensional table \(A\) where an entry \(A[i, T']\) stores the maximum size of a set \(Y \subseteq \bigcup_{1 \leq j \leq i} C_j\) such that the twin classes of \(Y\) are
exactly $T' \subseteq T$ and all vertices in $Y$ have in $G[Y \cup X']$ distance at most two to each vertex from $Y \cup X'$.

Before filling the table $A$, we calculate a value $s(i, T')$ that stores the maximum number of vertices we can add from $C_i$ that are from the twin classes in $T'$ and fulfill the requirements in the previous paragraph. This value is defined as follows. Let $C_i^{T'}$ denote the maximal subset of vertices from $C_i$ whose twin classes are exactly $T'$. Then, $s(i, T') = |C_i^{T'}|$ if $C_i^{T'}$ exists and every pair of non-adjacent vertices from $C_i^{T'}$ and from $X'$ have a common neighbor. Otherwise, set $s(i, T') = -\infty$. Note that as a special case we set $s(i, \emptyset) = 0$.

Furthermore, for two subsets $T''$ and $\tilde{T}$ define the predicate $\text{conf}(T'', \tilde{T})$ as true if there is a pair of twin classes $T_i \in T''$ and $T_j \in \tilde{T}$ such that $T_i$ and $T_j$ are in conflict, and as false, otherwise.

Using these values, we now fill $A$ with the following recurrence:

$$A[i, T'] = \max_{T'' \subseteq T', \tilde{T} \subseteq T'} \left\{ \begin{array}{ll} A[i - 1, \tilde{T}] + s(i, T'') & \text{if } \tilde{T} \cup T'' = T' \land \neg \text{conf}(\tilde{T}, T''), \\ -\infty & \text{otherwise.} \end{array} \right.$$  

This recurrence considers all cases of combining a set $Y$ for the clusters $C_1$ to $C_{i-1}$ with a solution $Y'$ for the cluster $C_i$. Herein, a positive table entry is only obtained when the twin classes of $Y \cup Y'$ is exactly $T'$ and the pairwise distances between $Y \cup Y'$ and $Y \cup Y' \cup X'$ in $G[Y \cup Y' \cup X']$ are at most two. The latter property is ensured by the definition of the $s()$ values and by the fact that we consider only combinations that do not put conflicting twin classes in different clusters.

Now, the table entry $A[q, T']$ contains the size of a maximum vertex set $Y$ such that in $G'[Y \cup X']$ every vertex from $Y'$ has distance two to all other vertices. It remains to ensure that the vertices from $X'$ are within distance two from each other. This can be done by only considering a table entry $A[q, T']$ if each non-adjacent vertex pair $x, x' \in X'$ has either a common neighbor in $X'$ or in one twin class contained in $T'$. The maximum size of a 2-club in $G'$ is then the maximum value of all table entries that fulfill this condition.

The running time can be bounded roughly as $O(2^k \cdot 3^2 \cdot n m)$: We try all $2^k$ partitions of $X$ and for each of these partitions, we fill a dynamic programming table with $2^k \cdot n$ entries. The number of overall table lookups and updates is $O(3^2 \cdot n)$ since there are $3^2$ possibilities to partition $T$ into the three sets $T''$, $\tilde{T}$, and $T \setminus T'$. Since each $C_i$ is a clique, the entry $s(i, T')$ is computable in $O(n m)$ time and the overall running time follows.

5.2 Distance to Cographs

We proceed by describing the fixed-parameter algorithm with respect to the parameter distance to cographs. Recall that since cographs are exactly the $P_4$-free graphs, any connected component of a cograph is a 2-club.
Theorem 7. 2-Club is solvable in $O(2^k \cdot 8^k \cdot n^4)$ time where $k$ denotes the distance to cographs.

Proof. Let $G'$ be the input graph of a 2-Club instance. Moreover, let $X'$ be a vertex subset of $G'$ whose deletion results in a cograph. We next describe a fixed-parameter algorithm with respect to $k = |X'|$ that finds a maximum-size 2-club in $G$. For our correctness proof we fix any maximum 2-club $S$ in $G'$.

First branch into the at most $2^k$ cases to guess $X = X' \cap S$. Delete all vertices in $X' \setminus X$. Denoting by $G = (V, E)$ the remaining graph, observe that $G - X$ is a cograph. The remaining task is to find a maximum 2-club in $G$ that contains $X$.

Before proceeding to describe the algorithm we introduce the following characterization of cographs [10]: A graph is a cograph if it can be constructed from single vertex graphs by a sequence of parallel and series compositions. Given $t$ vertex-disjoint graphs $G_i = (V_i, E_i)$, the series composition is the graph $(\bigcup_{i=1}^t V_i, \bigcup_{i=1}^t E_i \cup \{v, u \mid v \in G_i \land u \in G_j \land 1 \leq i < j \leq t\})$ and the parallel composition is $(\bigcup_{i=1}^t V_i, \bigcup_{i=1}^t E_i)$. The corresponding cotree of a cograph $G$ is the tree whose leaves correspond to the vertices in $G$ and each inner node represents a series or parallel composition of its children up to a root which represents $G$.

We next describe a dynamic programming algorithm that proceeds in a bottom-up manner on the cotree of $G - X$ and finds a maximum 2-club in $G$ that contains $X$. We may assume that $t = 2$ for all series and parallel compositions, as otherwise we can simply split up the corresponding nodes in the cotree. For each node $P$ in the cotree let $V(P) \subseteq V \setminus X$ be the vertices corresponding to the leaves of the subtree rooted in $P$. Furthermore, consider the (at most $2^k$ many) twin classes of $V \setminus X$ with respect to $X$ and for a subset of twin classes $T$ let $V(T) = \bigcup_{T \in T} T$ denote the union of all vertices in the twin classes of $T$. We compute a table $\Gamma$ where for any subset of twin classes $T$ and any node $P$ of the cotree the entry $\Gamma(P, T)$ is the size of a largest set $L \subseteq V(P) \cap V(T)$ that fulfills the following properties:

1. for all $T \in T : T \cap L \neq \emptyset$ and
2. for all $v \in L \cup X$ and $u \in L$: dist$_{G[L \cup X]}(u, v) \leq 2$

The intention of the definition above is that the graph $G[L \cup X]$ is a “2-club-like” structure that contains a vertex from each twin class in $T$ (Item 1) and for any pair of vertices, except those where both vertices are from $X$, have distance at most two (Item 2). Denoting the root of the cotree by $r$ and by $T_r$ the set of all twin classes that have a non-empty intersection with $S$, $\Gamma(r, T_r) \geq |S \setminus X|$ as $S \setminus X$ trivially fulfills all properties. Reversely, for any subset of twin class $T$ that contains for each pair of vertices $\{u, v\} \in X$ with dist$_{G[X]}(u, v) > 2$ a twin class $T \in T$ with $\{u, v\} \subseteq N(T)$, any set corresponding to $\Gamma(r, T)$ forms together with $X$ a 2-club.

We now describe the dynamic programming algorithm. Let $P$ be a leave node of the cotree with $V(P) = \{x\}$ and let $T$ be any subset of twin classes. The two sets $\{x\}$ and $\emptyset$ are the only candidates for $L$. Hence we set $\Gamma(P, T) := 1$ if $x$
fulfills both properties, \( \Gamma(P, \emptyset) := 0 \) (\( \emptyset \) fulfills both properties), and \( \Gamma(P, T) = -\infty \) otherwise.

Next we describe the dynamic programming algorithm for inner nodes of the cotree. Let \( P \) be any node of the cotree with children \( P_1, P_2 \) and let \( T \) be any subset of twin classes. We construct a graph \( G^P \) by exhaustively deleting in \( G[(V(P) \cap V(T)) \cup X] \) all vertices from \( V(P) \cap V(T) \) that have distance more than two to any vertex in \( X \). (Clearly, such a vertex has to be deleted because of Item 2.) If the resulting graph \( G^P \) violates Item 1, then there is no set corresponding to \( \Gamma(P, T) \) and thus we set the entry to be \( -\infty \). Additionally, if \( G^P \) fulfills all properties, then set \( \Gamma(P, T) = |V(G^P)| - |X| \). To handle the remaining case where \( G^P \) violates only Item 2 we make a case distinction on the node type of \( P \).

Case 1: \( P \) is a series node.

Let \( \{u, v\} \subseteq V(G^P) \setminus X \) be a vertex pair with \( \text{dist}_{G^P}(u, v) > 2 \). Since a series composition introduces an edge between each vertex in \( V(P_1) \) and each vertex in \( V(P_2) \) and \( V(G^P) \subseteq V(P) = V(P_1) \cup V(P_2) \), it follows that either \( V(G^P) \cap V(P_1) = \emptyset \) or \( V(G^P) \cap V(P_2) = \emptyset \). This implies that \( \Gamma(P, T) = \max(\Gamma(P_1, T), \Gamma(P_2, T)) \).

Case 2: \( P \) is a parallel node.

Consider any set \( L \) that corresponds to \( \Gamma(P, T) \). By the definition of a parallel node there is no edge between a vertex from \( V(P_1) \) to a vertex in \( V(P_2) \). Consequently, any pair of vertices in \( L \) with one vertex in \( V(P_1) \) and the other in \( V(P_2) \) have a common neighbor in \( X \). Correspondingly, we say that two twin classes are consistent if they have at least one common neighbor in \( X \) and two sets of twin classes are consistent if any twin class of the first set is consistent with any twin class of the second set. Denoting by \( T_1^S \) (\( T_2^S \)) the set of twin classes with a non-empty intersection with \( L \cap V(P_1) \) (\( L \cap V(P_2) \)), by the argumentation above it follows that \( T_1^S \) is consistent with \( T_2^S \). Additionally, it is straightforward to verify that \( L \cap V(P_1) \) (\( L \cap V(P_2) \)) fulfills all properties (except being a largest set) for the entry \( \Gamma(T_1^S, P_1) \) (\( \Gamma(P_2, T_2^S) \)).

Reversely, for any two consistent sets of twin classes \( T_1, T_2 \) let \( L_1 \) (\( L_2 \)) be any vertex set that corresponds to \( \Gamma(P_1, T_1) \) (\( \Gamma(P_2, T_2) \)). It holds that \( L_1 \cup L_2 \) fulfills all properties for \( \Gamma(P, T_1 \cup T_2) \) and hence \( \Gamma(T_1 \cup T_2) \geq |L_1 \cup L_2| \). Hence it is correct to set \( \Gamma(T, P) \) to be the largest value of \( \Gamma(P_1, T_1) + \Gamma(P_2, T_2) \) where \( T_1, T_2 \) are consistent and \( T_1 \cup T_2 = T \). This completes the description of the algorithm.

The table \( \Gamma \) contains \( O(n \cdot 2^k) \) entries as there are at most \( 2^k \) twin classes. Each entry can be computed in \( O(n^3 + 4^k) \) time. In total, together with the factor of \( 2^k \) needed guess \( X \), the running time of the above algorithm is \( O(2^k \cdot 8^k \cdot n^k) \).

\[ \square \]

### 6 Conclusion

We have resolved the complexity status of 2-CLUB for most of the parameters in the complexity landscape shown in Figure 1. Still, several open questions
remain. First, there are obviously parameters for which the parameterized complexity is still open. For example, is 2-CLUB parameterized by distance to interval graphs or by distance to 2-club cluster graphs in XP or even fixed-parameter tractable? In this context, also parameter combinations could be of interest. Clearly a complete investigation of the parameter space is infeasible. Hence, one should focus only on practically relevant parameter combinations. One example could be the following question that is left open by the hardness results for h-index and degeneracy. Is 2-CLUB also W[1]-hard with respect to the parameter h-index if the input graph has constant degeneracy? Second, it remains open whether there is a polynomial kernel for the parameter distance to clique or to identify further non-trivial structural parameters for which polynomial kernels exist. Third, for many of the presented fixed-parameter tractability results it would be interesting to either improve the running times or to obtain tight lower bounds. For example, is it possible to solve 2-CLUB parameterized by distance to clique in \(\delta^k \cdot n^{O(1)}\) time for some \(\delta < 2\)? Similarly, is it possible to solve 2-CLUB parameterized by vertex cover in \(2^{O(2^k)} \cdot n^{O(1)}\) time? An answer to the latter question could be a first step towards improving the (also doubly exponential) running time of the algorithms for the parameters treewidth or distance to cographs. Finally, it would be interesting to see which results carry over to 3-CLUB [25, 30] or to the related 2-CLIQUE problem [4].

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