Effect of correlated disorder on the temperature of unconventional Cooper pairing ($^3\text{He}$ in aerogel).

I. A. Fomin

P. L. Kapitza Institute for Physical Problems,
Kosygina 2, 119334 Moscow, Russia

Abstract

Using liquid $^3\text{He}$ in aerogel as an example, it is shown that correlations in positions of impurities affect the temperature $T_c$ of transition of Fermi liquid in an unconventional superfluid or superconductive state. The effect is significant if the correlation length for impurities is greater than the coherence length in the superfluid or superconductive state $\xi_0$. For $^3\text{He}$ in aerogel the suppression of $T_c$ is expressed in terms of the structure factor of aerogel. With the account of the fractal structure of aerogel a simple expression is obtained for the decrease of $T_c$ from its clean value. This expression is in a satisfactory agreement with the experimental data.

1. For unconventional Cooper pairing the temperature of transition into the superfluid (or superconductive) state is suppressed both by magnetic and by non-magnetic impurities [1]. The amount of suppression in most cases is well described by the theory of superconductive alloys of Abrikosov and Gor'kov (AG in what follows) [2]. Superfluid $^3\text{He}$ is the best understood example of unconventional Cooper pairing. The Cooper pairs here have orbital momentum $l = 1$, and spin $s = 1$. Floating impurities can not be introduced in the liquid $^3\text{He}$ – they stick to the walls. To get around the difficulty a high-porosity silica aerogel has been used as an impurity [3, 4]. The aerogel is a self-supporting network of silica strands with a typical thickness 3-4 nm. Rigidity of the network assumes existence of correlations in positions of its elements. The homogeneous scattering model (HSM) [5], which is a generalization of the AG theory for the $p$-wave pairing, does not take into account these correlations. This drawback is essential, since HSM does not provide a quantitative description of properties of superfluid phases.
of $^3\text{He}$ in a presence of aerogel. In particular, HSM does not describe correctly the dependence on pressure of the magnitude of suppression of superfluid transition temperature of $^3\text{He}$ in aerogel $T_a$ with respect to its bulk transition temperature $T_b$.

In the AG-theory, as well as in the HSM, the relative value of the suppression $\tau_{ba} = \frac{T_b - T_a}{T_b}$ is determined by the only parameter $x = \frac{\xi_0}{l_{tr}}$, where $\xi_0 = \hbar v_F / (2\pi T_a)$ is the superfluid coherence length and $l_{tr}$ is a transport mean free path. For $\tau_{ba} \ll 1$ the theory predicts $\tau_{ba} = \frac{\pi^2}{4} \frac{\xi_0}{l_{tr}}$. Both $\xi_0$ and $l_{tr}$ can be found from independent experiments [6]. The measured transition temperature $T_a$ is greater than the value, calculated within the HSM with the use of the known values of $\xi_0$ and $l_{tr}$. The other suggested models [5, 7] invoke an effect of restricted geometry as a mechanism of suppression of $T_a$.

One of the models considers $^3\text{He}$ in a gap between two diffusely scattering planes, and the other – $^3\text{He}$ in a spherical void. Although a better agreement with experiment can be achieved in this way, relation of the models to the real aerogel remains unclear.

In a present paper effect of correlation in position of elements, forming aerogel, on the superfluid transition temperature of $^3\text{He}$ is considered directly. The effect is the stronger the greater is the ratio of correlation length in aerogel $R$ to $\xi_0$. When $R^2 \sim l_{tr} \xi_0$ the change of $\tau_{ba}$ stemming from the correlations can be of the order of the original $\tau_{ba}$. This region is of particular interest for the aerogels, used in experiments. To simplify the argument we consider here only a region $\tau_{ba} \ll 1$.

2. At the $p$-wave Cooper pairing the order parameter is $3 \times 3$ complex matrix $A_{\mu j}$. The first index assumes three values enumerating three projections of spin of the Cooper pair. The second index enumerates three projections of its orbital momentum. To find $T_a$ assuming $\tau_{ba} \ll 1$ one can use Ginzburg and Landau equation:

$$-\tau A_{\mu j} + A_{\mu l} \eta_{lj}(r) - \frac{3}{5} \xi_s^2 \left( \frac{\partial^2 A_{\mu j}}{\partial x_i^2} + 2 \frac{\partial^2 A_{\mu l}}{\partial x_i \partial x_j} \right) = 0. \quad (1)$$

Here $\tau = (T_b - T)/T_b \ll 1$. Effect of impurities (aerogel) is introduced via a real symmetric tensor $\eta_{lj}(r)$, it describes a local depression of the transition temperature and its possible splitting for different orbital components of $A_{\mu j}$. Interaction with the impurities can be written in Eq.(1) in a local form because tensor $\eta_{lj}(r)$ varies on a distance $\sim \xi_0$, while $A_{\mu j}$ in a vicinity of $T_b$ varies on a distance $\sim \xi(T) = \xi_0 / \sqrt{T} \gg \xi_0$. A coefficient in front of the derivatives is written as in the Ref. [8], i.e. $\xi_s^2 = \frac{7\xi_0}{12} \xi_s^2 \approx 0, 7\xi_0^2$. At $\tau > 0$ Eq.(1) can have physically meaningful solutions. These solutions are generally speaking nonuniform and can be localized. By the definition $T_a$ is a
temperature of the onset of the long-range order in liquid $^3$He. In the present formulation it corresponds to the minimum $\tau = \tau_a$, at which Eq.(1) can have delocalized solution. The equations (1) for different spin components $\mu$ are not coupled and the spin index is not relevant. A problem of finding $\tau_a$ turns out to be analogous to a problem of finding mobility edge of a spin-1 particle in a random potential $\eta_{lj}(\mathbf{r})$, and $\tau$ is analog of energy. To avoid cumbersome calculations let us introduce one more simplification. Instead of the sum of two gradient terms in Eq.(1) we use a model isotropic expression with only one term and an “average” coefficient $\xi^2$:

$$
-\tau A_{\mu j} + A_{\mu l}\eta_{lj}(\mathbf{r}) - \xi^2_s \left( \frac{\partial^2 A_{\mu j}}{\partial x_i^2} \right) = 0.
$$

The average $\xi^2_s$ is defined in the following way. For the longitudinal component of $A_{\mu l}$, which meets the condition $\frac{\partial A_{\mu l}}{\partial x_i} = 0$, the coefficient in front of the derivative in Eq.(1) is $(3/5)\xi^2_s$. For two transverse components, defined as $A_{\mu l} = \frac{\partial \psi_{\mu}}{\partial x_i}$, where $\psi_{\mu}$ are scalars, a proper coefficient is $(9/5)\xi^2_s$. The averaging over three possibilities renders $\xi^2_s = (7/5)\xi^2_0$. Since $\xi^2_s \approx 0, 7\xi^2_0$ with a reasonable accuracy $\xi^2_s \approx \xi^2_0$. In what follows $\xi^2_0$ will be used as a coefficient in front of the gradient term in Eq.(2).

Explicit form of tensor $\eta_{lj}(\mathbf{r})$ depends on the particular structure of aerogel. We assume here, that aerogel consists of spheres of uniform radii $\rho$, distributed with an average density $n$. The values of $\rho$ and $n$ are chosen to provide the required values of porosity and of the mean free path for single-particle excitations. It will became clear later, that a concrete form of the elements is not essential, but for explicit calculations spherical form is preferable. For the porosity 98.3% and $l_{tr} \approx 130$ nm $\rho \approx 2$ nm, i.e. $\rho \ll \xi_0$. For such model

$$
\eta_{lj}(\mathbf{r}) = \sum_s \eta^{(1)}_{lj}(\mathbf{r} - \mathbf{r}_s),
$$

where $\mathbf{r}_s$ is a coordinate of the sphere number $s$, and $\eta^{(1)}_{lj}(\mathbf{r})$ a potential induced by a single sphere, it depends on the cross-section of scattering of the single-particle excitations on the sphere [9]. Assuming for the sake of definiteness diffuse scattering, we arrive at $r \gg \rho$ at the expression [10]:

$$
\eta^{(1)}_{lj}(\mathbf{r}) = -\frac{\rho^2}{r^2} \hat{\nu}_j \hat{\nu}_l \ln \left[ \tanh \left( \frac{r}{2\xi_0} \right) \right].
$$

According to Eq.(4) $\eta^{(1)}_{lj}(\mathbf{r})$ decays on a distance $\sim \xi_0$. At $r \leq \xi_0 \eta^{(1)}_{lj}(\mathbf{r}) \sim (\rho/\xi_0)^2$. Let us treat $\eta_{lj}(\mathbf{r})$ as a perturbation. Green function for Eq.(2) after
averaging over realizations of $\eta_{jl}(r)$ has a form: $\langle G_{mn}(\tau; k, k') \rangle = (2\pi)^3 \delta(k - k') \delta_{mn} G(\tau; k)$, where

$$G(\tau; k) = \frac{1}{\tau - \xi_0^2 k^2 - \Sigma(\tau, k)}.$$  \hspace{1cm} (5)

The self-energy $\Sigma(\tau, k)$ up to the factor $(2\pi)^3 \delta(k - k') \delta_{mn}$ is the averaged sum of the series shown diagrammatically in Fig.1. The arrows in the figure correspond to the unperturbed Green functions

$$G^{(0)}_{mn}(\tau; k) = \frac{\delta_{mn}}{\tau - \xi_0^2 k^2}.$$  \hspace{1cm} (6)

As usual, the integration over momenta of internal lines is assumed. Wavy lines correspond to the Fourier transform of the potential $\eta_{jl}(r)$:

$$\eta_{jl}(k - k') = \eta^{(1)}_{jl}(k - k') \sum_s e^{i(k' - k) r_s},$$  \hspace{1cm} (7)

where $k$ and $k'$ - momenta, corresponding to the arrows coming in and out of the vertex. The averaging is performed over coordinates $r_s$ of particles forming aerogel. The “mobility edge” is found as a pole of the Green function Eq.(5) at $k = 0$:

$$\tau = \Sigma(\tau, 0).$$  \hspace{1cm} (8)

Fourier transform of $\eta^{(1)}_{jl}(k - k')$ can be found directly from Eq.(4). Here we need only $\eta^{(1)}_{jl}(k \to 0) = \delta_{jl} \eta^{(1)}(0)$, where:

$$\eta^{(1)}(0) = \frac{\pi^3}{3} \rho^2 \xi_0.$$  \hspace{1cm} (9)

Consider consecutive terms of the series Fig.1. The first order term is:

$$\langle \eta_{jl}(k - k') \rangle = \eta^{(1)}_{jl}(k - k') \langle \sum_s e^{i(k' - k) r_s} \rangle.$$  \hspace{1cm} (10)
Assuming that aerogel is on the average uniform we have:

$$\langle \sum se_i (k' - k) r_s \rangle = (2\pi)^3 \delta(k - k') n.$$  \hspace{1cm} (11)

As a result, in the first order on the perturbation

$$\tau_{ba}^{(1)} = n\eta^{(1)}(0) = \frac{\pi^2 \xi_0}{4 l_{tr}}.$$  \hspace{1cm} (12)

For the comparison with HSM the answer is expressed in terms of the transport mean free path of the excitations $l_{tr}$, which is defined as:

$$\frac{1}{l_{tr}} = \frac{4}{3} \pi \rho^2 n.$$  \hspace{1cm} (13)

This definition corresponds to a diffuse scattering of quasi-particles by the randomly distributed uniform spheres with radii $\rho$. The first order correction (12) coincides with the result of HSM for small $\tau_{ba}$. The second-order correction is:

$$\Sigma^{(2)}(\tau, k) \delta_{jl} = \int \eta_{jm}^{(1)}(k - k_1) \eta_{ml}^{(1)}(k_1 - k) n \langle \sum t e^{i(k_1 - k)r_{st}} \rangle G_{mn}(\tau, k_1) \frac{d^3k_1}{(2\pi)^3}.$$  \hspace{1cm} (14)

Instead of the unperturbed Green function Eq.(6) the average Green function Eq.(5) is substituted in the r.h.s. of Eq.(14). This substitution is known as the self-consistent Born approximation \[11\]. The summation in Eq. (14) is over the relative coordinate $r_{st} = r_s - r_t$. The averaged sum is the structure factor:

$$\langle \sum t e^{i(k_1 - k)r_{st}} \rangle \equiv S(k_1 - k),$$  \hspace{1cm} (15)

it characterizes correlations in positions of the particles, forming aerogel. A structure factor is directly measured by the small-angle x-ray scattering. For the isotropic aerogel $S(k)$ does not depend on a direction of $k$. Substituting $k = 0$ in Eq. (14) we arrive at:

$$\tau_{ba}^{(2)} \delta_{jl} = n \int \eta_{jm}^{(1)}(-k_1) \eta_{ml}^{(1)}(k_1) S(k_1) G(\tau_{ba}, k_1) \frac{d^3k_1}{(2\pi)^3}.$$  \hspace{1cm} (16)

Following Eq. (8) and neglecting a possible small change of the spectrum we arrive at $G(\tau_{ba}, k_1) = -1/(\xi_0 k_1)^2$. At evaluation of the integral in Eq.(16) an account must be taken of the fact that in an interval of scales $\rho < r < R$, where $R$ is an upper limit of the interval, aerogel has fractal structure. It means, that in the corresponding interval of wave vectors $k_{min} < k < k_{max}$ its
structure factor $S$ has a power dependence on $k$: $S \sim k^{-D}$. The exponent $D$ is referred as a fractal dimension. In particular for the samples of 98% aerogel used in Ref.[12] $k_{\text{min}} \simeq 5 \div 7 \cdot 10^{-3} \text{Å}^{-1}$, $k_{\text{max}} \simeq 1 \div 2 \cdot 10^{-1} \text{Å}^{-1}$, $D \approx 1,8$. This rate of decay of $S(k)$ secures convergence of the integral in the infinity. A principal contribution to the integral is provided by the region of small $k$. That allows to substitute in the integrand $\eta_{jm}^{(1)}(k_1) \simeq \eta_{jm}^{(1)}(0) = \delta_{j} \eta_{1}^{(1)}(0)$, then

\[ Q = n \left[ \delta_{j} \right] \int S(k_1)G(0,k_1) \frac{d^3k_1}{(2\pi)^3}. \quad (17) \]

Therefore a particular shape of the elements, forming aerogel is not essential. Within the fractal interval the integral in Eq.(17) is $\sim \int dk/k^D$ and at $k_{\text{min}} \to 0$ it diverges at small $k$. Actually the integral converges since at $k < k_{\text{min}}$ dependence $S(k)$ saturates. That reflects an absence of correlations on a distances exceeding $R \sim 1/k_{\text{min}}$. A smooth cut-off can be introduced by assuming a simple model law of decay of correlations, depending on $R$. Then, by definition, $R$ is a correlation radius. The structure factor $S(k)$ can be expressed in terms of the pair correlation function $C(r)$:

\[ S(k) = \langle \sum_{l} e^{i kr_{st}} \rangle = n \int C(r)e^{i kr}d^3r. \quad (18) \]

At $r \ll R$ for a fractal with the dimension $D$ correlation function behaves as: $C(r) \approx A(R/r)^{3-D}$, where $A$ - is a coefficient. At $r \gg R$ correlations vanish and $C(r) \to 1$, i.e. the integral in Eq.(18) diverges. It converges if $C(r)$ is substituted by $v(r) = C(r) - 1$. This substitution corresponds to separation of the effect of correlations from that of uncorrelated distribution of elements. The unity gives a contribution to $S(k)$, which is proportional to $\delta(k)$ and does not affect the following calculations. The integral in Eq. (17), denoted as $Q$ in what follows, with the aid of straightforward transformations can be expressed in terms of $v(r)$:

\[ Q = \int S(k_1)G(k_1) \frac{d^3k_1}{(2\pi)^3} = -\frac{n}{\xi_0^2} \int v(r)rdr. \quad (19) \]

At $r \ll R$ function $v(r) \sim A(R/r)^{3-D}$. For $D > 1$ the integral in Eq.(19) converges at small $r$, therefore the fractal asymptotic can be used up to $r = 0$. At $r \geq R$ vanishing of correlations has to be taken into account. Following Ref. [13] we assume here the exponential decay of correlations, i.e. substitute for $v(r)$:

\[ v(r) = [A \left( \frac{R}{r} \right)^{3-D} - 1]e^{r/R}. \quad (20) \]
Coefficient $A$ is found from the normalization condition: $4\pi n \int v(r)r^2dr = -1$. For aerogels in question $nR^3 \gg 1$. Then $A = 2/\Gamma(D)$, where $\Gamma(D)$ is Euler Gamma-function. With this $A$ we arrive at: $\int v(r)rdr = R^2 (3-D)/(D-1)$. As a result

$$\tau_{ba}^{(2)} = -\left( n\eta^{(1)}(0) \right)^2 \frac{R^2}{\xi_0^2} \frac{3-D}{D-1} = -\frac{\pi^2}{4} \frac{\xi_0}{l_{tr}} \left( \frac{\pi^2}{4} \frac{R^2}{\xi_0 l_{tr}} \frac{3-D}{D-1} \right).$$

(21)

Suppression of the transition temperature with the account of the second order term

$$\tau_{ba} = \frac{\pi^2}{4} \frac{\xi_0}{l_{tr}} - \frac{\pi^2}{4} \frac{\xi_0}{l_{tr}} \left( \frac{\pi^2}{4} \frac{R^2}{\xi_0 l_{tr}} \frac{3-D}{D-1} \right)$$

(22)

turns out to be smaller then that, predicted by HSM. The factor $\frac{\pi^2}{4} \frac{R^2}{\xi_0 l_{tr}} \frac{3-D}{D-1}$ grows when $\xi_0$ decreases. Compensation of the small parameter $\xi_0/l_{tr}$ by a big ratio $(R/\xi_0)^2$ can result in a product of the order of unity. In this case higher order terms in the expansion of $\tau_{ba}$ over $\eta_{jl}$ have to be taken into account as well. The third order term is:

$$\tau_{ba}^{(3)} = n(\eta^{(1)}(0))^3 \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} G(k_1)G(k_2) \langle \sum_{t,u} e^{i k_1 r_{st}} e^{i k_2 r_{us}} \rangle.$$

(23)

The averaged sum here depends on three-particle correlations. The higher order terms respectively depend on the higher order correlation functions. The sum of the series can be found if the assumption is made, that the higher order correlation function can be decoupled in products of the two-particle correlation functions. In particular:

$$\langle \sum_{t,u} e^{i k_1 r_{st}} e^{i k_2 r_{us}} \rangle = S(k_1)S(k_2).$$

(24)

In this case $\tau_{ba}^{(3)}$ is a product of $\tau_{ba}^{(2)}$ by $\eta^{(1)}(0)Q$ etc.. The consecutive terms form geometric series with the sum:

$$\tau_{ba} = \frac{\tau^{(1)}}{1 - \tau^{(1)}Q}.$$  

(25)

With the given above expressions for $\tau^{(1)}$ and $Q$ we arrive at

$$\tau_{ba} = \frac{\pi^2 \xi_0}{4 l_{tr}} \frac{\pi^2}{4} \frac{R^2}{\xi_0 l_{tr}} \frac{3-D}{D-1}.$$  

(26)

The obtained expression for $\tau_{ba}$ in both limits of large and small $R$ gives physically natural results. At $R \ll \xi_0$ the result of HSM is reproduced. If,
on the other hand, $R$ is so large that $\frac{\xi_0}{l_{tr}} \left( \frac{R}{\xi_0} \right)^2 \gg 1$, then the mean free path $l_{tr}$ drops out of the expression for the transition temperature: $\tau_{ba} \sim \left( \frac{\xi_0}{R} \right)^2$. Such behavior is in line with the qualitative argument, based on the presence in aerogel of the low-density regions, or ‘voids’, with a characteristic size $\xi_a$. According to the argument for $\xi_0 \geq \xi_a$ superfluidity sets on starting from interior of the ‘voids’, then $\tau_{ba} \sim (\xi_0/\xi_a)^2$, while at $\xi_0 \gg \xi_a$ homogeneous limit is recovered $\tau_{ba} \sim \xi_0/l_{tr}$. For interpolation between the two limits Sauls and Sharma suggested to substitute in the formula of HSM for $T_a$ instead of the pairbreaking parameter $x = \xi_0/2l_{tr}$ an effective pairbreaking parameter $\tilde{x} = x/(1 + \xi_a^2/x)$ with $\xi_a = \xi_a/l_{tr}$. The obtained expression turns out to be in a good agreement with the data for 98% aerogel. Eq.(26) can also be rewritten as the HSM formula (12) with a substitution of the parameter $\tilde{x}$ instead of $x$. To do this one has to set $\xi_a = R\pi \sqrt{(3 - D)/(8(D - 1))}$ in the definition of $\tilde{x}$. Therefore Eq.(26) with a proper choice of $R$ has also agree with the data for $\tau_{ba}$, when $\tau_{ba}$ is small. The constraint is not crucial.

The method of correlation function makes possible generalization of the obtained result for finite $\tau_{ba}$. Good agreement with the data for $T_a$ in a 98% aerogel is achieved at $\xi_a \approx 500 \text{Å}$ and $l_{tr} \approx 1400 \text{Å}$. With these values of parameters at pressure above $\approx 20 \text{ bar}$ $\xi_a^2/x \approx 1.6$, i.e. effect of correlation is essential.

In conclusion, one can see that the account of the correlations in positions of the elements forming aerogel explains suppression of the temperature of transition of $^3\text{He}$ in the superfluid state by aerogel. The observed difference in properties of superfluid $^3\text{He}$ in aerogel from the predictions, based on the HSM, indicates that correlations are essential for a proper interpretation of these properties as well.

The above argument can be applied to unconventional superconductors as well. In particular, for some of high-$T_c$ superconductors the AG-theory does not properly describe the suppression of $T_c$ by impurities. The numerical analysis of the Ref. demonstrates that the discrepancy originates from the fact that the correlation length $\xi_0$ in these materials is significantly smaller than in conventional superconductors and can be comparable with the average distance between the impurities.

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