ON ISOMORPHISMS OF BANACH SPACES
OF CONTINUOUS FUNCTIONS

GRZEGORZ PLEBANEK

Abstract. We prove that if $K$ and $L$ are compact spaces and $C(K)$ and $C(L)$ are
isomorphic as Banach spaces then $K$ has a $\pi$-base consisting of open sets $U$ such that
$\overline{U}$ is a continuous image of some compact subspace of $L$. This sheds a new light on
isomorphic classes of the spaces of the form $C([0,1]^\kappa)$ and spaces $C(K)$ where $K$ is
Corson compact.

1. Introduction

Let $K$ and $L$ be compact spaces such that $C(K)$ and $C(L)$ are isomorphic Banach
spaces ($C(K) \sim C(L)$). When the spaces are isometric the classical Banach-Stone theorem
says that $K$ and $L$ are necessarily homeomorphic, see e.g. Semadeni [17], 7.8.4.
Amir [2] and Cambern [6] proved that this also holds when the isomorphism constant is
smaller then 2. In this paper we study what can be said on the relation between $K$ and
$L$ when the isomorphism constant is arbitrary. Some results on the relations between
$K$ and $L$ were also proved by Benyamini [4] and Jarosz [8] when $C(K)$ is only assumed
to be isomorphic — with small constant — to a subspace of $C(L)$. We also study such
relations for arbitrary embeddings.

The isomorphic types of $C(K)$ for metrizable compacta $K$ are known, see Bessaga
and Pełczyński [5] for countable $K$ and Miljutin [12] in the uncountable case (see also
Pełczyński [15], Albiac and Kalton [1] or Semadeni [17]). Outside the class of metric
spaces the isomorphic classification of $C(K)$ spaces exists only in some special cases, see
Galego [7].

The following open question has been around for several years, see e.g. 3.9 in [3], 6.45
in Negrepontis [13] or Question 1 in Koszmider [10].

Problem 1.1. Assume $C(K) \sim C(L)$ and that $L$ is a Corson compact space. Is $K$
necessarily Corson compact?

By the classical Amir-Lindenstrauss theory, the analogous question has a positive
answer for the class of Eberlein compacta (weakly compact subsets of Banach spaces),
since $K$ is Eberlein compact if and only if $C(K)$ is weakly compactly generated (see e.g.

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Negrepontis [13]). Recall also that the answer to [1.1] is positive assuming Martin’s axiom and the negation of continuum hypothesis (MA+¬CH), see Argyros et al. [3].

In [16] we studied similar problems for positive isomorphisms or embeddings. This paper continues this study without the positivity assumption. We now describe the organization of the paper. After preliminaries in section 2 we study embeddings in section 3. The main result is Theorem 3.3 which says that if $T$ is an embedding of $C(K)$ into $C(L)$ then for every $x \in K$ there is $y \in L$ such that the measure $T^*\delta_y$ has a large atom at $x$. As a corollary we obtain that under CH the space $C(2^{\omega_1})$ is not isomorphic to a subspace of $L$ when $L$ is Corson compact. This has been already known under MA+¬CH, see [3]; see also [11].

Given a surjective isomorphism $T : C(K) \to C(L)$, we study in section 4 the function $L \ni y \to ||T^*\delta_y||$ — the main result is stated as Theorem 4.3. Then in section 5 we analyse the set-valued functions that assign to every $y \in L$ the set of large atoms of $T^*\delta_y$. This leads to the main result of the paper, Theorem 6.1, which says that for every nonempty open set $U \subseteq K$ there are an nonempty open set $V$ with $V \subseteq U$ and a compact subset $L_1$ of $L$ such that $V$ is a continuous image of $L_1$. As an application we obtain a partial positive solution to Problem [1.1] namely, we prove that if $C(K) \sim C(L)$, where $L$ is Corson compact and $K$ is homogeneous, then $K$ is also Corson compact.

2. Preliminaries

Let $K$ be a compact space. The dual space $C(K)^*$ of the Banach space $C(K)$ is identified with $M(K)$ — the space of all signed Radon measures of finite variation; we use the symbol $M_1(K)$ to denote the unit ball of $M(K)$. Every $\mu \in M(K)$ can be written as $\mu = \mu^+ - \mu^-$ where $\mu^+$ and $\mu^-$ are mutually singular nonnegative finite Radon measure. Recall that the variation $|\mu|$ of $\mu$ is defined as $|\mu| = \mu^+ + \mu^-$ and the natural norm in $M(K)$ is given by the formula $||\mu|| = |\mu|(K)$.

In the sequel, the space $M(K)$ is always equipped with the weak* topology inherited from $C(K)^*$, i.e. the topology making all the functionals $\mu \to \int_K g \, d\mu$ continuous, where $g \in C(K)$. Note that we usually write $\mu(g)$ for $\int_K g \, d\mu$. For any $x \in K$ we denote by $\delta_x \in M(K)$ the corresponding Dirac measure.

The mapping $M(K) \ni \mu \to |\mu| \in M(K)$ is not weak* continuous; nonetheless it has the following semicontinuity properties. Recall that a real-valued function $\varphi$ (defined on some topological space $X$) is lower semicontinuous if the set $\{x \in X : \varphi(x) > r\}$ is open for every $r \in \mathbb{R}$.

**Lemma 2.1.** For every compact space $K$

(i) the mapping $M(K) \ni \mu \to |\mu|(g)$ is lower semicontinuous for every nonnegative function $g \in C(K)$;

(ii) the mapping $M(K) \ni \mu \to |\mu|(U)$ is lower semicontinuous for every open set $U \subseteq K$. 

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For the rest of the paper we fix two compacta $K$ and $L$ such that $C(K)$ is embed-
dable into $C(L)$ and, unless stated explicitly otherwise, we constantly use the following
notation: We fix a linear operator $T : C(K) \to C(L)$ such that
\[ m \cdot ||g|| \leq ||Tg|| \leq ||g||, \]
for all $g \in C(K)$, where $m > 0$. For every $y \in L$ we write $\nu_y = T^*\delta_y$, i.e. $\nu_y \in M(K)$ is
defined by the formula $\nu_y(g) = Tg(y)$ for $g \in C(K)$. Moreover, we write $\theta(y) = ||\nu_y||$;
thus $\theta$ is a real-valued (lower semicontinuous) function on $L$.

Put $E = T[C(K)]$. For every $x \in K$ we let $\mu_x$ be any Hahn-Banach extension of
$(T^{-1})^*\delta_x$, i.e. $\mu_x$ is defined on $E$ by $\mu_x(Tg) = g(x)$ and then extended to a functional on
$C(L)$ with the same norm.

In section 5 we consider set-functions from $L$ into $[K]^{<\omega}$, the family of all finite subsets
of $K$. Recall that a function $\varphi : L \to [K]^{<\omega}$ is said to be upper semicontinuous if the set
$\{y \in L : \varphi(y) \subseteq U\}$ is open for every open $U \subseteq K$. For any set $Y \subseteq L$ we write
$\varphi[Y] = \bigcup_{y \in Y} \varphi(y)$;
we say that $\varphi$ is surjective if $\varphi[L] = K$.

A compact space $K$ is Corson compact if, for some cardinal number $\kappa$, which can be
taken to be equal to the topological weight of $K$, $K$ is homeomorphic to a subset of the
$\Sigma$–product of real lines
\[ \Sigma(\mathbb{R}^\kappa) = \{x \in \mathbb{R}^\kappa : |\{\alpha : x_\alpha \neq 0\}| \leq \omega\}. \]
The class of Corson compacta has been intensively studied for its interesting topological
properties and various connections to functional analysis; we refer the reader to a basic paper [3] by Argyros, Mercourakis and Negrepontis, and to the extensive surveys by Negrepontis [13] and Kalenda [9]. Recall that the class of Corson compacta is stable
under taking closed subspaces and continuous images.

If $h$ is a real-valued function on some topological space $X$ and $A \subseteq X$ then for $x \in A$
we denote by $\text{osc}_x(h, A)$ the oscillation of $h$ at $x$ on the set $A$. i.e.
\[ \text{osc}_x(h, A) = \inf_U \sup \{ |h(x') - h(x'')| : x', x'' \in U \cap A \}, \]
where the infimum is taken over all open neighbourhoods $U$ of $x$.

3. ISOMORPHIC EMBEDDINGS

Lemma 3.1 was noted by Jarosz [8].

**Lemma 3.1.** If $\mu = \mu_x$ for some fixed $x \in K$ then $||\nu_y|| \geq m$ for $\mu$-almost all $y \in L$.

**Proof.** Let $N = \{y \in L : ||\delta_y|| < 1\}$; then $\mu(N) = 0$.

Indeed, we have $N = \bigcup_{r < 1} N_r$, where the sets
$N_r = \{y \in L : ||\delta_y|| \leq r\}$,
are closed; it is therefore sufficient to check that \(|\mu|(N_r) = 0\) for every \(r < 1\).

Take any \(\varepsilon > 0\) and \(g \in C(K)\) such that \(|Tg| \leq 1\) and \(\mu(Tg) > ||\mu|| - \varepsilon\). Then

\[
||\mu|| - \varepsilon = ||\mu|| + ||\mu|| - |F| - \varepsilon < \mu(Tg) =
\]

\[
= \int_{N_r} Tg \, d\mu + \int_{L \setminus N_r} Tg \, d\mu \leq r||\mu|| + ||\mu|| - |F| - \varepsilon < ||\mu|| - \varepsilon,
\]

which gives \(|\mu|(N_r) \leq \varepsilon/(1 - r)\) and, consequently, \(|\mu|(N_r) = 0\), as \(\varepsilon > 0\) is arbitrary.

Now for any \(y \in L \setminus N\) and any \(\varepsilon > 0\) there is \(g \in C(K)\), \(|Tg| \leq 1\) such that \(|Tg(y)| > 1 - \varepsilon\). Then \(|g| \leq 1/m\) and \(|\nu_y(g)| = |Tg(y)| > 1 - \varepsilon\). This implies \(||\nu_y|| \geq m\), and we are done.

\textbf{Lemma 3.2.} Consider a fixed \(x \in K\) and the measure \(\mu = \mu_x\). Assume that \(\varepsilon > 0\) and that there is a compact subset \(F \subseteq L\) such that

(i) \(|\nu_y| \geq m\) for every \(y \in F\);
(ii) \(\text{osc}_y(\theta, F) \leq \varepsilon\) for every \(y \in F\);
(iii) \(|\mu|(L \setminus F) < \varepsilon\).

Then there is \(y \in F\) such that \(|\nu_y\{x\}| \geq m - 2\varepsilon\).

Proof. Let \(H\) be any neighbourhood of \(x\) and \(f_H : K \to [0, 1]\) be a continuous function such that \(f_H(x) = 1\) and \(f_H = 0\) outside \(H\). We shall check that there is \(y_H \in F\) such that \(Tf_H(y_H) \geq m - \varepsilon\).

Indeed, otherwise \(|\nu_y(f_H)| < m - \varepsilon\) for \(y \in F\) which, together with \(|\mu|(F) \leq ||\mu|| \leq 1/m\) and \(||Tf_H|| = 1\), would give

\[
1 = f_H(x) = \mu(Tf_H) = \int_F Tf_H \, d\mu + \int_{L \setminus F} Tf_H \, d\mu <
\]

\[
< (m - \varepsilon)||\mu|(F) + \varepsilon \leq \frac{m - \varepsilon}{m} + \varepsilon \leq 1,
\]
a contradiction. In particular, it follows that \(|\nu_y|\{x\} \geq m - \varepsilon\).

The net \(y_H\) ordered by the reverse inclusion of the \(H\)’s has a converging subnet \((y_H)_{H \in \mathcal{H}}\); denote its limit by \(y\). By (ii) we may assume that \(||\nu_y|| \leq ||\nu_y|| + \varepsilon\) for every \(H \in \mathcal{H}\).

We shall prove that \(|\nu_y|\{x\} \geq m - 2\varepsilon\); by regularity it suffices to check that \(|\nu_y|(U) \geq m - 2\varepsilon\) for every open set \(U \ni x\).

Given such an open set \(U \ni x\), choose a continuous function \(g : K \to [0, 1]\) such that \(g = 0\) outside \(U\) and \(g = 1\) on an open set \(V\) containing \(x\). For any \(H \in \mathcal{H}\) with \(H \subseteq V\) we have \(|\nu_y|(g) \geq |\nu_{y_H}|(H) \geq m - \varepsilon\). Hence

\[
|\nu_{y_H}|(1 - g) \leq |\nu_{y_H}|(K) - (m - \varepsilon) \leq |\nu_y|(K) + \varepsilon - (m - \varepsilon) = |\nu_y|(K) - m + 2\varepsilon.
\]

Since \(\nu_{y_H} \to \nu_y\), it follows from Lemma 2.7.1(i) and the above inequality that

\[
|\nu_y|(1 - g) \leq |\nu_y|(K) - m + 2\varepsilon.
\]

We conclude that \(|\nu_y|(U) \geq |\nu_y|(g) \geq m - 2\varepsilon\) and the proof is complete.
We are ready for the main result of this section.

**Theorem 3.3.** If $T : C(K) \rightarrow C(L)$ is an isomorphic embedding then, writing $\nu_y = T^* \delta_y$ for $y \in L$, for every $x \in K$ we have

$$\sup \{|\nu_y(\{x\})| : y \in L\} \geq \frac{1}{||T|| ||T^{-1}||}.$$  

**Proof.** Clearly we can assume that $||T|| = 1$ and denote $m = 1/||T^{-1}|| > 0$. The function $\theta : L \ni y \rightarrow ||\nu_y||$ is lower semicontinuous hence Borel. Given $x \in K$, by Lemma 3.1 $||\nu_y|| \geq m \mu_x$-almost everywhere. By the Lusin theorem we can therefore find for any $\varepsilon > 0$ a compact set $F \subseteq L$ with $|\mu|(L \setminus F) < \varepsilon$ and such that $\theta$ is continuous on $F$ and $\theta(y) \geq m$ for $y \in F$. Applying Lemma 3.2 we finish the proof. \qed

We conclude this section by showing the following result on Corson compacta.

**Corollary 3.4.** Let $K$ be such a compact space that $\text{card} K > \mathfrak{c} = \text{card}[C(K)]$ and that $L$ is Corson compact. Then $C(K)$ cannot be embedded into $C(L)$.

**Proof.** Suppose otherwise and let $T : C(K) \rightarrow C(L)$ be an embedding, where $L$ is Corson compact. Since the class of Corson compacta is closed under taking continuous images, by passing to a quotient of $L$ we can additionally assume that the functions from $E = T[C(K)]$ distinguish points of $L$. As the space $C(K)$ has cardinality $\mathfrak{c}$, this implies that the topological weight of $L$ is at most $\mathfrak{c}$. Thus $L$ is homeomorphic to a subspace of $\Sigma(\mathbb{R}^n)$ so in particular $\text{card} L \leq \mathfrak{c}$.

On the other hand, the cardinality of the sets $\{x \in K : |\nu_y(\{x\})| \geq m/2\}$ is at most $2/m$ and they cover all of $K$ by Theorem 3.3. It follows that $\text{card} K \leq \mathfrak{c}$, contrary to our assumption. \qed

**Corollary 3.5.** Assuming CH, $C(2^{\omega_1})$ cannot be embedded into $C(L)$ with $L$ being Corson compact.

**Proof.** Under CH the Cantor cube $K = 2^{\omega_1}$ has cardinality $2^\mathfrak{c} > \mathfrak{c}$. Moreover, there are only $\mathfrak{c}$ many continuous functions on $K$ because every such a function is determined by countably many coordinates. Hence we can apply Corollary 3.4. \qed

Note that in Corollary 3.3 CH can be relaxed to $2^{\omega_1} > \mathfrak{c}$. At this point it is worth recalling that under MA$+\neg$CH, if $L$ is Corson compact then $M_1(L)$ is also Corson compact in its weak* topology, Argyros et al. \cite{3}. Consequently, if $T : C(K) \rightarrow C(L)$ is an embedding then $T^*[M_1(L)]$ is Corson compact and so is $K$ ($T^*[M_1(L)]$ contains a ball in $M(K)$ because $T^*$ is onto and $K$ can be embedded into the space of measures via the mapping $K \ni x \rightarrow \delta_x$). We do not know if Corollary 3.3 can be proved without any extra set-theoretic axioms, i.e. we do not know what happens when $2^{\omega_1} = \mathfrak{c}$ but MA does not hold. It will become clear in section 6 that spaces such as $C(2^{\omega_1})$ cannot be isomorphic to a space $C(L)$ whenever $L$ is Corson compact.
4. Isomorphisms

Keeping the notation from section 2, we shall now consider the case when $T : C(K) \to C(L)$ is an isomorphism. Note that in this case the measure $\mu_x \in M(L)$ is uniquely determined by the condition $\mu_x(Tg) = g(x)$, $g \in C(K)$.

We start by the following general observation on lower semicontinuous (lsc) functions.

Lemma 4.1. Let $f$ be a bounded lsc function on $K$, let $U$ be a nonempty open subset of $K$ and fix $\eta > 0$. Then there is a nonempty open set $V \subseteq U$ such that the oscillation of $f$ on $V$ is $\leq \eta$. The same is true for a finite collection of bounded lsc functions or for differences of such functions.

Proof. By our assumption on $f$, the set $C_i = \{x \in K : f(x) \leq i\eta\}$ is closed for every integer $i$. As $f$ is bounded there is minimal $i$ such that $U \cap \text{int}C_i \neq \emptyset$. By minimality, $U \cap \text{int}C_i$ is not contained in $C_{i-1}$ so $V = (U \cap \text{int}C_i) \setminus C_{i-1} \neq \emptyset$. The oscillation of $f$ is smaller than $\eta$ on $C_1 \setminus C_{i-1}$ so certainly on $V$.

Given bounded lsc $f_1, \ldots, f_k$ and an open set $U = V_0$ we just iterate the step above to find open sets $V_i \subseteq V_{i-1}$ such that the oscillation of $f_i$ is smaller than $\eta$.

If $f_j = g_j - g_j'$ where $g_j, g_j'$ are bounded lsc functions we can apply the above argument to $g_j$'s with $\eta/2$.

Lemma 4.2. Let $Y_1 \subseteq Y_2 \subseteq Y_3 = L$ be closed subsets of $L$ and let $\eta > 0$. Suppose that $U \subseteq K$ is an open neighbourhood of $x_0 \in K$ such that for $j = 1, 2, 3$ and every $x \in U$

\begin{equation}
\tag{\ast} \left| \mu_x(Y_j) - \mu_{x_0}(Y_j) \right| < \eta.
\end{equation}

Then there is an open set $V$ with $x_0 \in V \subseteq U$ and a compact set $L_1 \subseteq Y_2 \setminus Y_1$ such that for every $x \in V$ we have

$$|\mu_x(L_1)| > |\mu_x(Y_2 \setminus Y_1)| - 4\eta.$$

More generally, if $Z_1 \subseteq Z_2 \subseteq \ldots \subseteq Z_N = L$ are closed sets, $\eta > 0$ and (\ast) holds for every $1 \leq j \leq N$ then for any open set $U$ with $x_0 \in U \subseteq K$ there are an open set $x_0 \in V \subseteq U$ and compact sets $L_j \subseteq Z_{j+1} \setminus Z_j$, $j \leq N - 1$ such that

$$|\mu_x(L_j)| > |\mu_x(Z_{j+1} \setminus Z_j)| - 4\eta,$$

for every $x \in V$.

Proof. Note that there is a continuous function $h : L \to [0, 1]$ such that its support $S = \{y \in L : h(y) > 0\}$ is disjoint from $Y_1$ and

$$\mu_{x_0}(h) > |\mu_{x_0}(L \setminus Y_1)| - \eta.$$

We put $L_1 = S \cap Y_2$ and define $V$ as

$$V = \{x \in U : |\mu_x(h) - \mu_{x_0}(h)| < \eta\};$$

then $V$ is open since the mapping $x \to \mu_x(h)$ is continuous.
Now for any \( x \in V \)
\[
|\mu_x|(S) \geq \mu_x(h) > \mu_{x_0}(h) - \eta > |\mu_{x_0}|(L \setminus Y_1) - 2\eta > |\mu_x|(L \setminus Y_1) - 4\eta,
\]
where the last inequality follows from (*) applied to \( l = 1 \) and \( l = 3 \). It follows that
\[
|\mu_x|(L_1) = |\mu_x|(S \cap Y_2) \geq |\mu_x|(Y_2 \setminus Y_1) - 4\eta,
\]
for \( x \in V \), and this shows the first assertion.

The second part follows by iteration of the first part to \( Y_1 = Z_j, \ Y_2 = Z_{j+1} \) and with the resulting open sets \( U = V_0 \supseteq V_1 \supseteq \ldots \supseteq V_{N-1} = V \).

**Theorem 4.3.** Fix \( \varepsilon > 0 \). Then for every nonempty open set \( U \subseteq K \) there is a nonempty open set \( V \subseteq U \) and a compact set \( F \subseteq L \) such that

(a) osc\(_y\)(\( \theta, F \)) \( \leq \varepsilon \) for every \( y \in F \);

(b) for every \( x \in V \) there is \( y \in F \) such that \( |\nu_y(\{x\})| \geq m - 2\varepsilon \).

**Proof.** We use Lemma 3.2. Condition (i) of the lemma holds trivially for isomorphisms and we construct \( F \) satisfying 3.2(ii)-(iii).

Let \( Z_i = \{ y \in L : ||\nu_y|| \leq m + \varepsilon i \} \). Then \( Z_1 \subseteq Z_2 \subseteq \ldots \subseteq L \) are closed and \( Z_N = L \) for some \( N \leq 1/\varepsilon \). Take \( \eta > 0 \) such that \( \eta = \varepsilon/(4N) \).

The functions \( x \rightarrow |\mu_x|(Z_i) = ||\mu_x|| - |\mu_x|(L \setminus Z_i) \) are differences of lsc functions so by Lemma 4.1 there is a nonempty open set \( W \subseteq U \) such that their oscillations on \( W \) are smaller than \( \eta \).

By Lemma 4.2 there are disjoint compact sets \( L_i \) and a nonempty open set \( V \subseteq W \) such that
\[
|\mu_x|(L_i) > |\mu_x|(Z_{i+1} - Z_i) - 4\eta,
\]
for every \( i \) and \( x \in V \). It follows that, writing \( L_0 = Z_1 \), the compact set \( F = \bigcup_{0 \leq i < N} L_i \) satisfies \( |\mu_x|(F) > |\mu_x|(L) - \varepsilon \). As the oscillation of \( y \rightarrow ||\nu_y|| \) is smaller than \( \varepsilon \) on each of the closed disjoint sets \( L_0, \ldots, L_{N-1} \), (a) holds and now Lemma 3.3 gives (b). \( \square \)

5. **Finite valued maps**

We shall consider now set-valued mappings related to the isomorphism \( T : C(K) \rightarrow C(L) \). For any \( r > 0 \) and \( y \in L \) we define
\[
\varphi_r(y) = \{ x \in K : |\nu_y(\{x\})| \geq r \}.
\]
Note that \( \varphi_r(y) \) has at most \( 1/r \) elements since \( ||\nu_y|| \leq 1 \) for every \( y \in L \).

**Lemma 5.1.** Let \( \varepsilon > 0 \) and let \( F \subseteq L \) be a closed set such that osc\(_y\)(\( \theta, F \)) \( < \varepsilon \) for \( y \in F \).

(i) If \( U \subseteq K \) is open and \( \varphi_{r-\varepsilon}(y) \subseteq U \) for some \( y \in F \) then there is a neighbourhood \( W \) of \( y \) in \( F \) such that \( \varphi_r(z) \subseteq U \) for every \( z \in W \).

(ii) \( \varphi_{r}(F) \subseteq \varphi_{r-\varepsilon}[F] \).
Proof. As $\varphi_{r-\varepsilon}(y) \subseteq U$, for any $x \in K \setminus U$ we have $|\nu_y(\{x\})| < r - \varepsilon$ and therefore there is an open set $U_x \ni x$ such that $|\nu_y(U_x)| < r - \varepsilon$. There are $x_i, i \leq i_0$, such that the sets $U_i = U_{x_i}$ form a finite cover of $K \setminus U$. Let

$$\eta = \min\{r - \varepsilon - |\nu_y(U_i) : i \leq i_0\}.$$ 

Using Lemma 2.1(ii) we can find a set $W \ni y$ open in $F$ and such that if $z \in W$ then

$$|\nu_z|(K \setminus U_i) > |\nu_y|K \setminus U_i) - \eta,$$

for every $i \leq i_0$; by our assumption on $F$ we can also demand that $|\nu_z| < |\nu_y| + \varepsilon$ for $z \in W$.

Take any $z \in W$ and $x \in K \setminus U$. Then $x \in U_i$ for some $i \leq i_0$ and hence

$$|\nu_z|\{x\}) \leq |\nu_z|(U_i) = |\nu_z|(K) - |\nu_z|(K \setminus U_i) <$$

$$< |\nu_y|(K) + \varepsilon - |\nu_y|(K \setminus U_i) + \eta = |\nu_y|(U_i) + \eta \leq r.$$

Hence $\varphi_r(z) \subseteq U$ for $z \in W$, and this shows (i).

To check (ii) suppose that $x \notin \varphi_{r-\varepsilon}(y)$ for any $y \in F$. Then for every $y \in F$ there is $U_y \ni x$ such that $\varphi_{r-\varepsilon}(y) \subseteq K \setminus U_y$. By (i) $\varphi_r(z) \subseteq K \setminus U_y$ for $z$ from some set $V_y \ni y$ open in $F$. Take a finite cover $V_y$, $y \leq i_0$ of $F$ and let $U = \bigcap_{y \leq j_0} U_y$. Then $U$ is disjoint from $\varphi_r[F]$, so $x \notin \varphi_r[F]$, as required. \hfill $\Box$

**Lemma 5.2.** If $U \subseteq K$ is a nonempty open set then there are $\varepsilon > 0$, a compact set $K_1 \subseteq U$ having nonempty interior, a closed set $F \subseteq L$, and $s > 0$ such that

1. $K_1 \subseteq \varphi_s[F]$ and $K_1 \cap \varphi_{3s/2}[F] = \emptyset$;
2. every $x \in \text{int}K_1$ has a neighbourhood $H$ such that $\text{card}(\varphi_{s-2\varepsilon}(y) \cap H) \leq 1$ for every $y \in F$.

**Proof.** Let $\varepsilon = m/20$ and let $F$ be as in Theorem 4.3. Then

$$r_0 = \sup\{r > 0 : U \cap \overline{\varphi_r[F]} \neq \emptyset\},$$

satisfies $r_0 \geq m - 2\varepsilon$ by 4.3.

If we now take $s$ such that $s + \varepsilon < r_0 < 3s/2$ then

$$U \cap \left(\varphi_{s+\varepsilon}[F] \setminus \varphi_{3s/2}[F]\right),$$

contains a compact set $K_1$ with nonempty interior. Thus $K_1 \cap \varphi_{3s/2}[F] = \emptyset$ and $\varphi_s[F] \supseteq \varphi_{s+\varepsilon}[F] \supseteq K_1$ by Lemma 5.1(ii).

To verify the second part, fix $x \in L_1$, take any $y \in F$ and choose $U_y \ni x$ such that $|\nu_y|(U_y) < (3/2)s$. There is open $V_y \ni y$ such that for every $z \in V$

$$|\nu_y|(K) > |\nu_z|(K) - \varepsilon;$$

$$|\nu_z|(K \setminus U_y) > |\nu_y|(K \setminus U_y) - \varepsilon.$$
Indeed, the first requirement can be fulfilled by the property of $F$ while the second by Lemma 5.2(ii). It follows that for any $z \in V_y$ we have
\[
|\nu_2|(U_y) = |\nu_2|(K) - |\nu_2|(K \setminus U_y) \leq \\
\leq |\nu_2|(K) - |\nu_2|(K \setminus U_y) + 2\varepsilon = |\nu_2|(U_y) + 2\varepsilon < (3/2)s + 2\varepsilon.
\]
Note that $(3/2)s + 2\varepsilon < 2(s - 2\varepsilon)$, which is equivalent to $12\varepsilon < s$: indeed, $3/2s > m - 2\varepsilon$ so $s > 2/3m - 4/3\varepsilon = 2/3 \cdot 20\varepsilon - 4/3\varepsilon = 12\varepsilon$. Therefore $\varphi_{s-\varepsilon}(y)$ cannot intersect $U_y$ at two points.

Take a finite set $F_0 \subseteq F$ such that the sets $V_y$, for $y \in F_0$ form a cover of $F$. Then $H = \bigcap_{y \in F_0} U_y$ is as required. \hfill \Box

6. Results

Now we are ready to state and prove our main result. Recall that a family of nonempty open subsets $\mathcal{V}$ is a $\pi$-base of $K$ if for every nonempty open $U \subseteq K$ there is $V \in \mathcal{V}$ such that $V \subseteq U$.

**Theorem 6.1.** Let $K$ and $L$ be compact spaces such that $C(K)$ is isomorphic to $C(L)$. Then $K$ has a $\pi$-base $\mathcal{V}$ such that for every $V \in \mathcal{V}$, $\overline{V}$ is a continuous image of some compact subspace of $L$.

**Proof.** Given nonempty open $U \subseteq K$, we take $K_1$, $F$, $s$ and $\varepsilon$ as in Lemma 5.2. Let $K_2 = \overline{H} \subseteq K_1$, where $H \neq \emptyset$ is as in part (2) of Lemma 5.2. Put
\[
Y = \{ y \in F : \varphi_s(y) \cap K_2 \neq \emptyset \}.
\]
Then $\varphi_{s-\varepsilon}(y) \cap K_2 \neq \emptyset$ for every $y \in Y$ by Lemma 5.1(ii).

We define $h : \overline{Y} \to K_2$ so that $h(y)$ is the unique point in $K_2 \cap \varphi_{r-\varepsilon}(y)$. Then $h$ maps $\overline{Y}$ onto $K_2$ so it remains to check that $h$ is continuous.

Let the set $C \subseteq K_2$ be closed. Then $h^{-1}[C] = A$, where
\[
A = \{ y \in \overline{Y} : \varphi_{r-\varepsilon}(y) \cap C \neq \emptyset \},
\]
and $A$ is closed. Indeed, if $y \in \overline{Y} \setminus A$, i.e. $\varphi_{r-\varepsilon}(y) \cap C = \emptyset$ then $\varphi_{r-2\varepsilon}(y) \cap C = \emptyset$ as well by 5.2(2), and it follows from Lemma 5.1 that $\varphi_{\varepsilon}(z) \cap C = \emptyset$ for all $z$ from some neighbourhood of $y$. \hfill \Box

Of course the above theorem gives no information for spaces $K$ having dense sets of isolated points. On the other hand, the result has the following consequences.

**Corollary 6.2.** Given a cardinal number $\kappa$ and a compact space $L$, if $C[0,1]^\kappa$ is isomorphic to $C(L)$ then $L$ maps continuously onto $[0,1]^\kappa$.

**Proof.** Clearly, every nonempty open subset of $[0,1]^\kappa$ contains a subset homeomorphic to the whole space. Hence if $C(K) \sim C(L)$ then Theorem 6.1 implies that there is a compact subspace $L_1 \subseteq L$ and a continuous surjection $h : L_1 \to [0,1]^\kappa$; in turn such $h$ can be extended to a continuous mapping $L \to [0,1]^\kappa$ by the Tietze extension theorem. \hfill \Box

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Recall that a topological space $X$ is said to be homogeneous if for every $x, x' \in X$ there is a homeomorphism $f$ of $X$ onto itself such that $f(x) = x'$.

**Corollary 6.3.** If $L$ is Corson compact and $C(K) \sim C(L)$ for some compact $K$ then $K$ has a $\pi$-base of sets with Corson compact closures. In particular, $K$ is Corson compact itself whenever $K$ is homogeneous.

*Proof.* The first assertion follows from Theorem 6.1 since the class of Corson compacta is closed under taking compact subspaces and continuous images.

If $K$ is homogeneous then it follows that $K$ can be covered by a finite family $\{V_i : i \leq i_0\}$ of open sets where $V_i$ is Corson compact for every $i$. Then a disjoint union $K' = \bigoplus_{i \leq i_0} V_i$ is Corson compact and so is $K$ since $K$ is a continuous image of $K'$. $\square$

Let us note that in Theorem 6.1 cannot be strengthen by replacing a $\pi$-base with a base. This can be seen using a result due to Okunev [14] showing that isomorphisms of $C(K)$ spaces do not preserve the Frechet property; cf. [16], section 5.

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Instytut Matematyczny, Uniwersytet Wrocławski

E-mail address: grzes@math.uni.wroc.pl