NORMAL, COHYPERNORMAL AND NORMALOID
WEIGHTED COMPOSITION OPERATORS ON THE HARDY
AND WEIGHTED BERGMAN SPACES

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Abstract. If $\psi$ is analytic on the open unit disk $\mathbb{D}$ and $\varphi$ is an analytic self-map of $\mathbb{D}$, the weighted composition operator $C_{\psi, \varphi}$ is defined by $C_{\psi, \varphi}f(z) = \psi(z)f(\varphi(z))$, when $f$ is analytic on $\mathbb{D}$. In this paper, we study normal, cohyponormal, hyponormal and normaloid weighted composition operators on the Hardy and weighted Bergman spaces. First, for some weighted Hardy spaces $H^2(\beta)$, we prove that if $C_{\psi, \varphi}$ is cohyponormal on $H^2(\beta)$, then $\psi$ never vanishes on $\mathbb{D}$ and $\varphi$ is univalent, when $\psi \not\equiv 0$ and $\varphi$ is not a constant function. Moreover, for $\psi = K_a$, where $|a| < 1$, we investigate normal, cohyponormal and hyponormal weighted composition operators $C_{\psi, \varphi}$. After that, for $\varphi$ which is a hyperbolic or parabolic automorphism, we characterize all normal weighted composition operators $C_{\psi, \varphi}$, when $\psi \not\equiv 0$ and $\psi$ is analytic on $\mathbb{D}$. Finally, we find all normal weighted composition operators which are bounded below.

1. Introduction

Let $H(\mathbb{D})$ denote the collection of all holomorphic functions on the open unit disk $\mathbb{D}$. A function $f$ is called analytic on a closed set $F$ if there exists an open set $U$ such that $f$ is analytic on $U$ and $F \subseteq U$. The algebra $A(\mathbb{D})$ consists of all continuous functions on the closure of $\mathbb{D}$ that are analytic on $\mathbb{D}$.

For $f$ which is analytic on $\mathbb{D}$, we denote by $\hat{f}(n)$ the $n$-th coefficient of $f$ in its Maclaurin series. The Hardy space $H^2$ is the collection of all such functions $f$ for which
\[
\|f\|^2_2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty.
\]
The space $H^\infty(\mathbb{D})$, simply $H^\infty$, consists of all functions that are analytic and bounded on $\mathbb{D}$. Recall that for $\alpha > -1$, the weighted Bergman space $A^2_\alpha(\mathbb{D}) =$
A^2_\alpha$, is the set of functions $f$ analytic on the unit disk, satisfying the norm condition
\[ \|f\|_2^2 = \int_D |f(z)|^2 w_\alpha(z) dA(z) < \infty, \]
where $w_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha$ and $dA$ is the normalized area measure. When $\alpha = 0$, this gives the Bergman space $A^2(D) = A^2$.

Let $e_w$ be the linear functional for evaluation at $w$, that is, $e_w(f) = f(w)$. Then for functional Hilbert spaces $H$, we let $K_w$ denote the unique function in $H$ which satisfies $(f, K_w) = f(w)$ for every $f \in H$. In this case, the functional Hilbert space $H$ is called a reproducing kernel Hilbert space. The weighted Bergman spaces $A^2_\alpha$ and the Hardy space $H^2$ are all reproducing kernel Hilbert spaces. Let $\gamma = 1$ for $H^2$ and $\gamma = \alpha + 2$ for $A^2_\alpha$. In $H^2$ and $A^2_\alpha$, we have reproducing kernels $K_w(z) = (1 - wz)^{-\gamma}$ with norm $(1 - |w|^2)^{-\gamma/2}$. Moreover, let $k_w$ denote the normalized reproducing kernel.

Let $\varphi$ be an analytic map of the open unit disk $D$ into itself. We define the composition operator $C_\varphi$ by $C_\varphi(f) = f \circ \varphi$, where $f$ is analytic on $D$. If $\psi$ is in $H(D)$ and $\varphi$ is an analytic map of the unit disk into itself, the weighted composition operator with symbols $\psi$ and $\varphi$ is the operator $C_{\psi,\varphi}$ which is defined by $C_{\psi,\varphi}(f) = \psi(f \circ \varphi)$, where $f$ is analytic on $D$. If $\psi$ is a bounded analytic function on $D$, then the weighted composition operator $C_{\psi,\varphi}$ is bounded on $H^2$ and $A^2_\alpha$.

A linear-fractional self-map of $D$ is a map of the form
\[ \varphi(z) = \frac{az + b}{cz + d} \]
for some $a, b, c, d \in \mathbb{D}$ such that $ad - bc \neq 0$, with the property that $\varphi(D) \subseteq D$. We denote the set of those maps by $\text{LFT}(D)$. It is well-known that the automorphisms of the unit disk, that is, the one-to-one analytic maps of the unit disk onto itself, are just the functions $\varphi(z) = \lambda(a - z)/(1 - \bar{a}z)$, where $|\lambda| = 1$ and $|a| < 1$ (see, e.g., [5]). We denote the class of automorphisms of $D$ by $\text{Aut}(D)$.

Let $d\theta$ be the arc-length measure on $\partial D$. The space $L^2(\partial D)$ denotes the Lebesgue space of $\partial D$ induced by $d\theta/(2\pi)$. Also $L^\infty(\partial D)$ is the space of all essentially bounded measurable functions on $\partial D$. Suppose that $dA(z)$ is the area measure on $D$ normalized so that the area of $D$ is 1. For any $\alpha > -1$, let $dA_\alpha$ be the measure on $\mathbb{D}$ defined by
\[ dA_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA(z). \]
The Banach space $L^2(D, dA_\alpha)$ denotes the space of Lebesgue measurable functions $f$ on $\mathbb{D}$ with norm $\left( \int_\mathbb{D} |f(z)|^2 dA_\alpha(z) \right)^{1/2}$. For each $b \in L^\infty(\partial D)$, we define the Toeplitz operator $T_b$ on $H^2$ by $T_b(f) = P(bf)$, where $P$ denotes the orthogonal projection of $L^2(\partial D)$ onto $H^2$. For each $\psi \in L^\infty(D)$, we define the Toeplitz operator $T_\psi$ on $A^2_\alpha$ by $T_\psi(f) = P_\alpha(\psi f)$, where $P_\alpha$ denotes the orthogonal projection of $L^2(D, dA_\alpha)$ onto $A^2_\alpha$. Since an orthogonal projection has
norm 1, clearly $T_\varphi$ is bounded. If $\psi$ is a bounded analytic function on $\mathbb{D}$, then the weighted composition operator can be rewritten as $C_{\psi,\varphi} = T_\psi C_\varphi$.

If $\varphi$ is as in Equation (1), then the adjoint of any linear-fractional composition operator $C_{\varphi}$, acting on $H^2$ and $A_\alpha^2$, is given by $C_{\varphi}^* = T_\gamma C_{\sigma} T_h^*$ (we call it Cowen's adjoint formula), where $\sigma(z) = (\bar{\sigma} z - \tau)/(\bar{\bar{\sigma}} z + \overline{\overline{\tau}})$ is a self-map of $\mathbb{D}$, $\gamma(z) = (-\overline{\bar{\sigma}} z + \overline{\overline{\tau}})^{-\gamma}$ and $h(z) = (cz + d)^\gamma$, with $\gamma = 1$ for $H^2$ and $\gamma = \alpha + 2$ for $A_\alpha^2$ (see [8] and [17]). From now on, unless otherwise stated, we assume that $\sigma$, $h$ and $g$ are given as above.

A point $\zeta$ of $\mathbb{D}$ is called a fixed point of a self-map $\varphi$ of $\mathbb{D}$ if $\lim_{r \to 1} \varphi(r \zeta) = \zeta$. We will write $\varphi'(\zeta)$ for $\lim_{r \to 1} \varphi'(r \zeta)$. Each analytic self-map $\varphi$ of $\mathbb{D}$ that is neither the identity nor an elliptic automorphism of $\mathbb{D}$ has a unique point $w$ in $\mathbb{D}$ that acts like an attractive fixed point in that $\varphi_n(z) \to w$ uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$, where $\varphi_n$ denotes $\varphi$ composed with itself $n$ times ($\varphi_0$ being the identity function). The point $w$, called the Denjoy-Wolff point of $\varphi$, is also characterized as follows:

- if $|w| < 1$, then $\varphi(w) = w$ and $|\varphi'(w)| < 1$;
- if $|w| = 1$, then $\varphi(w) = w$ and $0 < |\varphi'(w)| \leq 1$.

More information about Denjoy-Wolff points can be found in [10, Chapter 2] or [23, Chapters 4 and 5].

A map $\varphi \in \text{LFT}(\mathbb{D})$ is called parabolic if it has a single fixed point $\zeta$ in the Riemann sphere $\hat{\mathbb{C}}$ such that $\zeta \in \partial \mathbb{D}$. Let $\tau(z) = (1 + \zeta z)/(1 - \zeta z)$. The map $\tau$ takes the unit disk onto the right half-plane $\Pi$ and takes $\zeta$ to $\infty$. The function $\phi = \tau \circ \varphi \circ \tau^{-1}$ is a linear-fractional self-map of $\Pi$ that fixes only the point $\infty$, so it must have the form $\phi(z) = z + t$ for some complex number $t$, where $\text{Re}(t) \geq 0$. Let us call $t$ the translation number of either $\varphi$ or $\phi$. Note that if $\text{Re}(t) = 0$, then $\varphi \in \text{Aut}(\mathbb{D})$. Also if $\text{Re}(t) > 0$, then $\varphi \not\in \text{Aut}(\mathbb{D})$. In [23, p. 3], J. H. Shapiro showed that among the linear-fractional transformations fixing $\zeta \in \partial \mathbb{D}$, the parabolic ones are characterized by $\varphi'(\zeta) = 1$. Let $\varphi \in \text{LFT}(\mathbb{D})$ be parabolic with fixed point $\zeta$ and translation number $t$. Therefore,

$$\varphi(z) = \frac{(2 - t)z + t\zeta}{2 + t - t\zeta} \quad (2)$$

Recall that an operator $T$ on a Hilbert space $H$ is said to be normal if $TT^* = T^*T$ and essentially normal if $TT^* - T^*T$ is compact on $H$. Also $T$ is unitary if $TT^* = T^*T = I$. The normal composition operators on $A_\alpha^2$ and $H^2$ have symbol $\varphi(z) = az$, where $|a| \leq 1$ (see [10, Theorem 8.2]). An operator $T$ on a Hilbert space $H$ is said to be binormal if $(T^*T)(TT^*) = (TT^*)(T^*T)$. It is obvious that every normal operator is binormal. In [18], binormal composition operator $C_\varphi$ was characterized when $\varphi$ is a linear-fractional self-map of $\mathbb{D}$. If $A^*A \geq AA^*$ or, equivalently, $\|Ah\| \geq \|A^*h\|$ for all vectors $h$, then $A$ is said to be a hyponormal operator. An operator $A$ is said to be cohyponormal if $A^*$ is hyponormal. An operator $A$ on a Hilbert space $H$ is normal if and only if for all vectors $h \in H$, $\|Ah\| = \|A^*h\|$, so it is not hard to see that an operator $A$ is normal if and only if $A$ and $A^*$ are hyponormal. Recall that an operator
$T$ is said to be normaloid if $\|T\| = r(T)$, where $r(T)$ is the spectral radius of $T$. Then we can see that all normal, hyponormal and cohyponormal operators are normaloid. The normal and unitary weighted composition operators on $H^2$ were investigated in [3] by Bourdon et al. After that, in [21], these results were extended to the bigger spaces containing the Hardy and weighted Bergman spaces. Recently, hyponormal and cohyponormal weighted composition operators have been investigated in [9] and [14]. In this paper, we work on normal, normaloid, cohyponormal and hyponormal weighted composition operators. In the second section, we extend [9, Theorem 3.2] to some weighted Hardy spaces. In the third section, for $\psi = K_a$, where $a \in \mathbb{D}$, we show that if $C_{\psi, \phi}$ is normal, then $|\phi(0)| = |a|$. In the fourth section, we state that for $\phi \in \text{Aut}(\mathbb{D})$ and $\psi \neq 0$ which is analytic on $\overline{\mathbb{D}}$, if $C_{\psi, \phi}$ is normal on $H^2$ or $A^2_{\alpha}$ and $\psi(\zeta) = 0$, then $\zeta \in \partial \mathbb{D}$, $\phi(\zeta) = \zeta$ and $\zeta$ is not the Denjoy-Wolff point of $\varphi$. Also we prove that if $C_{\psi, \phi}$ is normal on $H^2$ or $A^2_{\alpha}$, then 1 is an eigenvalue of $C_{\psi, \phi}$, when $\varphi$ is a parabolic or hyperbolic automorphism, we give a necessary and sufficient condition for $C_{\psi, \phi}$ to be normal on $H^2$ and $A^2_{\alpha}$, when $\psi$ is analytic on $\overline{\mathbb{D}}$. Finally, we show that for a normal weighted composition operator $C_{\psi, \phi}$ on a Hilbert space $H$ which contains all the polynomials, $C_{\psi, \phi}$ is Fredholm if and only if $C_{\psi, \phi}$ has closed range.

2. Cohyponormal weighted composition operators

In this section, we provide the generalized result of [9, Theorem 3.2] on some weighted Hardy spaces. We first state the following well-known lemma which was proved in [13, p. 1211] and [19, p. 1524].

**Lemma 2.1.** Let $C_{\psi, \phi}$ be a bounded operator on $H^2$ and $A^2_{\alpha}$. For each $w \in \mathbb{D}$, $C_{\psi, \phi}^* K_w = \overline{\psi(w)} K_{\overline{\phi(w)}}$.

Let $H$ be a Hilbert space. The set of all bounded operators from $H$ into itself is denoted by $B(H)$. Now assume that $H$ is a Hilbert space of analytic functions on $\mathbb{D}$. For $f \in H$, let $[f]$ denote the smallest closed subspace of $H$ which contains $\{z^n f\}_{n=0}^{\infty}$. If $S \in B(H)$ is the unilateral shift $Sf = zf$, then $[f]$ is the smallest closed subspace of $H$ containing $f$ which is invariant under $S$; moreover, if $[f] = H$, then the function $f$ is called cyclic. Also for $\psi \in H$, we define a multiplication operator $M_{\psi} : H \to H$ that for each $f \in H$, $M_{\psi}(f) = \psi f$. In this section, we assume that $M_{\psi}$ and $S$ are bounded operators, but in general every multiplication operator is not bounded.

**Theorem 2.2.** Assume that $H$ is a Hilbert space of analytic functions on $\mathbb{D}$ and the polynomials are dense in $H$. Assume that $\psi \neq 0$ and $\varphi$ is not a constant function. If $C_{\psi, \phi}$ is cohyponormal on $H$, then $\psi$ is cyclic in $H$.

**Proof.** Suppose that $C_{\psi, \phi}$ is cohyponormal. By the Open Mapping Theorem, $\ker C_{\psi, \phi} = \{0\}$. Then $C_{\psi, \phi}^* = \{0\}$. Since $\ker M_{\psi} \subseteq \ker C_{\psi, \phi}^*$, we have
ker \( M^*_ψ = \{0\} \). [6, Theorem 2.19, p. 35] and [6, Corollary 2.10, p. 10] imply that \( \text{ran} \, M^*_ψ = H \). Then \( ψH \) is dense in \( H \). Because the polynomials are dense in \( H \), it is easily seen that this is equivalent to saying that polynomial multiples of \( ψ \) are dense in \( H \), that is, to \( ψ \) being a cyclic vector. □

Note that by [16, Corollary 1.5, p. 15], if \( f \in H^2 \) is cyclic, then it is an outer function. Then under the conditions of Theorem 2.2, if \( C_{ψ, ϕ} \) is cohyponormal on \( H^2 \), then \( ψ \) is an outer function (see [9, Theorem 3.2]).

**Lemma 2.3.** Let \( H \) be a reproducing kernel Hilbert space of analytic functions on \( \mathbb{D} \). Assume that for each \( w \in \mathbb{D} \), there is \( g \in H \) such that \( g(w) \neq 0 \). Let \( ψ \) be cyclic in \( H \). Then \( ψ \) never vanishes on \( \mathbb{D} \).

**Proof.** Since \( ψ \) is cyclic in \( H \), \( \{ψ : p \text{ is a polynomial}\} \) is dense in \( H \). Let \( f \in H \). Then there is a sequence \( \{p_n\} \) of polynomials such that \( p_n \psi \to f \) as \( n \to \infty \). Suppose that \( ψ(w) = 0 \) for some \( w \in \mathbb{D} \). We can see that \( f(w) = \langle f, K_w \rangle = \lim_{n \to \infty} \langle p_n, K_w \rangle = 0 \). Then for each \( f \in H \), \( f(w) = 0 \) and it is a contradiction. □

Let \( H \) be a Hilbert space of analytic functions on the unit disk. If the monomials \( 1, z, z^2, \ldots \) are an orthogonal set of non-zero vectors with dense span in \( H \), then \( H \) is called a weighted Hardy space. We will assume that the norm satisfies the normalization \( ||1|| = 1 \). The weight sequence for a weighted Hardy space \( H \) is defined to be \( β(n) = ||z^n|| \). The weighted Hardy space with weight sequence \( β(n) \) will be denoted by \( H^2(β) \). The inner product on \( H^2(β) \) is given by

\[
\langle \sum_{j=0}^{∞} a_j z^j, \sum_{j=0}^{∞} c_j z^j \rangle = \sum_{j=0}^{∞} a_j \overline{c_j} β(j)^2.
\]

We require the following corollary, which is a generalization of [9, Theorem 3.2]. The proof which shows that \( ϕ \) is univalent of the following corollary relies on some ideas from [9, Theorem 3.2].

**Corollary 2.4.** Let \( H^2(β) \) be a weighted Hardy space. Suppose that \( \sup β(j+1)/β(j) \) is finite. Assume that \( ψ \neq 0 \) and \( ϕ \) is not a constant function. If \( C_{ψ, ϕ} \) is cohyponormal on \( H^2(β) \), then \( ψ \) never vanishes on \( \mathbb{D} \) and \( ϕ \) is univalent.

**Proof.** By Theorem 2.2, Lemma 2.3, [10, Proposition 2.7] and [10, Theorem 2.10], \( ψ \) never vanishes on \( \mathbb{D} \). Assume that there are points \( w_1 \) and \( w_2 \) in \( \mathbb{D} \) such that \( ϕ(w_1) = ϕ(w_2) \) and \( w_1 \neq w_2 \). Hence

\[
C_{ϕ, ψ} (\overline{ϕ(w_2)K_{w_1}} - ϕ(w_1)K_{w_2}) = \overline{ϕ(w_2)ϕ(w_1)}K_{ϕ(w_1)} - ϕ(w_1)ϕ(w_2)K_{ϕ(w_2)} = 0.
\]

We conclude that \( 0 \in σ_p(C_{ϕ, ψ}^*) \). Therefore, by [7, Proposition 4.4, p. 47], \( 0 \in σ_p(C_{ϕ, ψ}) \) and \( C_{ϕ, ψ}(ψ(w_2)K_{w_1} - ϕ(w_1)K_{w_2}) = 0 \). Since \( ψ \) never vanishes on \( \mathbb{D} \), \( C_{ϕ, ψ}(ψ(w_2)K_{w_1} - ϕ(w_1)K_{w_2}) = 0 \). Setting \( h = K_{w_2}/K_{w_1} \), we find

\[
h ◦ ϕ \equiv \frac{ψ(w_2)}{ψ(w_1)}.
\]
Since $\varphi$ is not a constant function, $\varphi(\mathbb{D})$ is an open set by the Open Mapping Theorem. It follows that $K_{w_2}/K_{w_1}$ is a constant function and it is a contradiction. □

Suppose that $T$ belongs to $B(H^2)$ or $B(A^2_{\alpha})$. Through this paper, the spectrum of $T$, the essential spectrum of $T$ and the point spectrum of $T$ are denoted by $\sigma(T)$, $\sigma_e(T)$ and $\sigma_p(T)$, respectively.

**Remark 2.5.** Suppose that $C_{\psi,\varphi}$ is cohypnormal on $H^2$ or $A^2_{\alpha}$ and $\psi \not= 0$. First, assume that $\varphi$ is not a constant function. Since $H^2$ and $A^2_{\alpha}$ are weighted Hardy spaces, Theorem 2.2 and Corollary 2.4 imply that $\psi$ is cyclic and $\psi$ never vanishes on $\mathbb{D}$. Now suppose that $\varphi \equiv c$, where $c$ is a constant number and $|c| < 1$. Assume that there are points $w_1$ and $w_2$ in $\mathbb{D}$ such that $\psi(w_1) = 0$ and $\psi(w_2) \neq 0$. From Lemma 2.1, we observe that $C_{\psi,\varphi}^*(K_{w_1}) = \overline{\psi(w_1)K_c} \equiv 0$. Since $C_{\psi,\varphi}$ is cohypnormal, we have $C_{\psi,\varphi}(K_{w_1}) = 0$. Since $\psi \cdot K_{w_1} \varphi \equiv 0$, we obtain that $\psi(w_2)K_{w_1}(c) = 0$ and so $\psi(w_2) = 0$. It is a contradiction. Hence we conclude that $\psi$ never vanishes on $\mathbb{D}$.

3. Normaloid weighted composition operators

Let $\alpha$ be a complex number of modulus 1 and $\varphi$ be an analytic self-map of $\mathbb{D}$. Since $\text{Re} \left( \frac{\varphi(z)}{\alpha - \varphi(z)} \right)$ is a positive harmonic function on $\mathbb{D}$, this function is the poisson integral of a finite positive Borel measure $\mu_\alpha$ on $\partial \mathbb{D}$. Let us write $E(\varphi)$ for the closure in $\partial \mathbb{D}$ of the union of the closed supports of the singular parts $\mu^s_\alpha$ of the measures $\mu_\alpha$ as $|\alpha| = 1$. In the next lemma and proposition, the set of points which $\varphi$ makes contact with $\partial \mathbb{D}$ is $\{ \zeta \in \partial \mathbb{D} : \varphi(\zeta) \in \partial \mathbb{D} \}$.

**Lemma 3.1** ([14, Lemma 3.2]). Let $\varphi$ be an analytic self-map of $\mathbb{D}$. Suppose that $\varphi \in A(\mathbb{D})$ and the set of points which $\varphi$ makes contact with $\partial \mathbb{D}$ is finite. Assume that there are a positive integer $n$ and $\zeta \in \partial \mathbb{D}$ such that $E(\varphi^n) = \{ \zeta \}$, where $\zeta$ is the Denjoy-Wolff point of $\varphi$. Let $\psi \in H^\infty$ be continuous at $\zeta$. Then

$$r_\gamma(C_{\psi,\varphi}) = |\psi(\zeta)| |\varphi'(\zeta)|^{-\gamma/2}.$$  

**Proposition 3.2.** Let $\varphi$ be an analytic self-map of $\mathbb{D}$. Suppose that $\varphi \in A(\mathbb{D})$ and the set of points which $\varphi$ makes contact with $\partial \mathbb{D}$ is finite. Assume that there are a positive integer $n$ and $\zeta \in \partial \mathbb{D}$ such that $E(\varphi^n) = \{ \zeta \}$, where $\zeta$ is the Denjoy-Wolff point of $\varphi$. Suppose that $\psi = K_\alpha$ for some $\alpha \in \mathbb{D}$. Let $C_{\psi,\varphi}$ be normaloid on $H^2$ or $A^2_{\alpha}$. Then

$$\frac{(1 - |\varphi(\alpha)|^2)(1 + |\alpha|)}{1 - |\alpha|} \geq \varphi'(\zeta).$$

In particular, if $\varphi'(\zeta) = 1$, then $2|\alpha| \geq |\varphi(\alpha)|^2(1 + |\alpha|)$. 


Proof. Assume that $C_{\psi,\varphi}$ is normaloid. Let $\gamma = 1$ for $H^2$ and $\gamma = \alpha + 2$ for $A^2_{\alpha}$. By Lemmas 2.1 and 3.1, we can see that

\[
\left| \frac{1}{1 - \overline{a}\zeta} \right|^{2\gamma} (\varphi'(\zeta))^{-\gamma} = \|C_{\psi,\varphi}\|^2 
\geq \langle C^*_{\psi,\varphi}k_a, C^*_{\psi,\varphi}k_a \rangle 
= (1 - |a|^2)^\gamma \langle C^*_{\psi,\varphi}K_a, C^*_{\psi,\varphi}K_a \rangle 
= \frac{1}{(1 - |a|^2)^{2\gamma}} \left( \frac{1}{1 - |\varphi(a)|^2} \right)^\gamma.
\]

Then

\[
\frac{1}{(1 - |a|^2)^{2\gamma}} \varphi'(\zeta)^{-\gamma} \geq \frac{1}{(1 - |a|^2)^{2\gamma}} \left( \frac{1}{1 - |\varphi(a)|^2} \right)^\gamma,
\]

so the result follows. Now suppose that $\varphi'(\zeta) = 1$. In this case, after some computation, we can see that $2|a| \geq |\varphi(a)|^2 + |a||\varphi(a)|^2$. □

**Corollary 3.3.** Let $\varphi$ satisfy the hypotheses of Proposition 3.2. If $C_{\varphi}$ is normaloid on $H^2$ or $A^2_{\alpha}$, then $1 - |\varphi(0)|^2 \geq \varphi'(\zeta)$. Moreover, if $\varphi'(\zeta) = 1$, then $C_{\varphi}$ is not normaloid.

**Proof.** Let $\psi \equiv K_0$. By Proposition 3.2, we can see that if $C_{\varphi}$ is normaloid, then $1 - |\varphi(0)|^2 \geq \varphi'(\zeta)$. Now assume that $\varphi'(\zeta) = 1$. Suppose that $C_{\varphi}$ is normal. Then $1 - |\varphi(0)|^2 \geq 1$. Hence $\varphi(0) = 0$ and it is a contradiction. □

In Proposition 3.4, we only prove the third part. Proofs of the other parts is similar to part (c) and follows from the definitions of hyponormal and cohyponormal operators.

**Proposition 3.4.** Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $\psi = K_a$ for some $a \in \mathbb{D}$. The following statements hold on $H^2$ or $A^2_{\alpha}$.

(a) If $C_{\psi,\varphi}$ is cohyponormal, then $|\varphi(0)| \geq |a|$.

(b) If $C_{\psi,\varphi}$ is hyponormal, then $|\varphi(0)| \leq |a|$.

(c) If $C_{\psi,\varphi}$ is normal, then $|\varphi(0)| = |a|$.

**Proof.** (c) Let $\gamma = 1$ for $H^2$ and $\gamma = \alpha + 2$ for $A^2_{\alpha}$. Since $K_0 \equiv 1$, it follows from Lemma 2.1 that

\[
\frac{1}{(1 - |\varphi(0)|^2)^\gamma} = \langle \overline{\varphi(0)}K_{\varphi(0)}, \overline{\varphi(0)}K_{\varphi(0)} \rangle 
= \langle C_{\psi,\varphi}C^*_{\psi,\varphi}K_0, K_0 \rangle 
= \langle C^*_{\psi,\varphi}C_{\psi,\varphi}K_0, K_0 \rangle 
= \langle \psi, \psi \rangle 
= \frac{1}{(1 - |a|^2)^\gamma},
\]

which implies that $|\varphi(0)| = |a|$. □
Let $\varphi \in \text{LFT}(\mathbb{D})$. It is easy to see that $\varphi$ must belong to one of the following three disjoint classes:

- Automorphism of $\mathbb{D}$.
- Non-automorphism of $\mathbb{D}$ with $\varphi(\mathbb{D}) \subseteq \mathbb{D}$.
- Non-automorphism of $\mathbb{D}$ with $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial \mathbb{D}$.

Let $\varphi \in \text{LFT}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$. Then by [10, Theorem 2.48], $\varphi$ has a fixed point $p \in \mathbb{D}$. Suppose that $\varphi \in \text{LFT}(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subseteq \mathbb{D}$ or $\varphi$ is the identity or an automorphism of $\mathbb{D}$ with a fixed point in $\mathbb{D}$. All normal weighted composition operators $C_{\psi,\varphi}$ of these types were found (see [3, Theorem 10], [9, Theorem 3.7] and [21, Theorem 4.3]). Also suppose $\varphi \in \text{Aut}(\mathbb{D})$ which has no fixed point in $\mathbb{D}$ and $\psi = K_a$ for some $a \in \mathbb{D}$; all normal weighted composition operators $C_{\psi,\varphi}$ on $H^2$ and $A^2_\alpha$ are characterized in Theorem 4.5. Bourdon et al. in [3, Proposition 12] obtained a condition that reveals what is required for normality of a weighted composition operator $C_{\psi,\varphi}$ on $H^2$, where $\varphi$ is a linear-fractional and $\psi = K_{\sigma(0)}$ (also by the similar proof, an analogue result holds on $A^2_\alpha$). In the following corollary, for $\psi = K_a$ and $\varphi \in \text{LFT}(\mathbb{D})$, where $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial \mathbb{D}$ and $a \in \mathbb{D}$, we investigate normal weighted composition operators $C_{\psi,\varphi}$ on $H^2$ and $A^2_\alpha$.

**Corollary 3.5.** Suppose that $\varphi \in \text{LFT}(\mathbb{D})$ is not an automorphism and $\varphi(\zeta) = \eta$ for some $\zeta, \eta \in \partial \mathbb{D}$. Assume that $\psi = K_a$ for some $a \in \mathbb{D}$. If $C_{\psi,\varphi}$ is normal on $H^2$ or $A^2_\alpha$, then $\varphi$ is a parabolic non-automorphism and $|a| = |t/(2 + t)| = |\sigma(0)|$, where $t$ is the translation number of $\varphi$.

**Proof.** Let $C_{\psi,\varphi}$ be normal on $H^2$ or $A^2_\alpha$. Then $C_{\psi,\varphi}$ is essentially normal. Since $\psi$ never vanishes on $\partial \mathbb{D}$, we conclude from [13, Theorem 2.6] and [13, Remark 2.7] that $\varphi$ is a parabolic non-automorphism and the result follows from Proposition 3.4 and Equation (2). □

4. Normal weighted composition operators

Each disk automorphism $\varphi$ has two fixed points on the sphere, counting multiplicity. The automorphisms are classified according to the location of their fixed points: elliptic if one fixed point is in $\mathbb{D}$ and a second fixed point is in the complement of the closed disk, hyperbolic if both fixed points are in $\partial \mathbb{D}$, and parabolic if there is one fixed point in $\partial \mathbb{D}$ of multiplicity two (see [10] and [23]). Let $\varphi$ be an automorphism of $\mathbb{D}$. In [13] and [15], the present authors investigated essentially normal weighted composition operator $C_{\psi,\varphi}$, when $\psi \in A(\mathbb{D})$ and $\psi(z) \neq 0$ for each $z \in \overline{\mathbb{D}}$. In this section, we just assume that $\psi$ is analytic on $\overline{\mathbb{D}}$ and we attempt to find all normal weighted composition operators $C_{\psi,\varphi}$. Also we will show that $\psi$ never vanishes on $\overline{\mathbb{D}}$.

**Lemma 4.1.** Let $\varphi \in \text{Aut}(\mathbb{D})$ and $f \in A(\mathbb{D})$. Then $T_f^* C_{\varphi} - C_{\varphi} T_{f \circ \varphi^{-1}}^*$ is compact on $H^2$ and $A^2_\alpha$. 


Proof. We know that \( \sigma = \varphi^{-1} \). It is not hard to see that
\[
C_{\varphi}^* T_f = T_g C_{\sigma} T_h^* T_f = C_{\sigma} T_g \varphi T_h^* T_f
\]
and
\[
T_f \varphi^{-1} C_{\varphi}^* = T_f \varphi^{-1} C_{\sigma} T_g \varphi T_h^* = C_{\sigma} T_f \varphi T_h^*.
\]
Since \( C_{\varphi} C_{\sigma} = I \), by [11, Proposition 7.22] and [22, Corollary 1(c)], \( C_{\varphi}^* T_f - T_f \varphi^{-1} C_{\varphi}^* \) is compact and the result follows.

In this section, we assume that \( \varphi(z) = \lambda(a - z)/(1 - \bar{a}z) \) and \( w(z) = (1 - \bar{a}z)^{\gamma} \psi(z) \), where \( a \in \mathbb{D}, |\lambda| = 1, \psi \in A(\mathbb{D}) \) and \( \gamma = 1 \) for \( H^2 \) and \( \gamma = \alpha + 2 \) for \( A_\alpha^2 \). We will use the notation \( A \equiv B \) to indicate that the difference of the two bounded operators \( A \) and \( B \) is compact.

**Proposition 4.2.** Let \( \psi \in A(\mathbb{D}) \) and \( \varphi \in \text{Aut}(\mathbb{D}) \). If \( C_{\varphi, \psi} \) is hyponormal on \( H^2 \) or \( A_\alpha^2 \), then for each \( \zeta \in \partial \mathbb{D}, |w(\zeta)| - |w(\varphi(\zeta))| \geq 0 \). Moreover, if \( C_{\varphi, \psi} \) is ccohyponormal on \( H^2 \) or \( A_\alpha^2 \), then for each \( \zeta \in \partial \mathbb{D}, |w(\varphi(\zeta))| - |w(\zeta)| \geq 0 \).

Proof. Suppose that \( C_{\varphi, \psi} \) is hyponormal. By [11, Proposition 7.22], [22, Corollary 1(c)] and the preceding lemma, we can see that
\[
C_{\psi, \varphi} C_{\psi, \varphi}^* = T_\psi C_{\psi, \varphi} C_\sigma T_{\psi}^* T_\psi^*
= T_\psi C_\sigma T_{\psi} \varphi T_{\psi}^* T_\psi^*
= T_{|\psi|^2 T_{\varphi} \sigma}
\]
and
\[
C_{\psi, \varphi}^* C_{\psi, \varphi} = T_g C_{\psi, \varphi}^* T_{\psi}^* T_\psi C_{\psi, \varphi}
= T_g C_{\sigma} T_{\psi}^* T_\psi C_{\sigma} T_{\varphi} \sigma^{-1}
= T_g C_{\sigma} T_{\psi}^* T_\psi C_{\sigma} T_{\varphi} \sigma^{-1}
= T_{|\psi|^2 T_{\varphi} \sigma^{-1}}.
\]
Let \( \varphi(a) = 0 \) for \( a \in \mathbb{D} \). After some computation, we see that \( \overline{h(z)} g(\varphi(z)) = |1 - \overline{a}z|^{2\gamma} / (1 - |\alpha|^2)^\gamma \), where \( \gamma = 1 \) for \( H^2 \) and \( \gamma = \alpha + 2 \) for \( A^2_\alpha \). Hence by [12, Corollary 2.6] and [4, Corollary 1.3], we have
\[
\sigma(e_C_{\psi, \varphi} C_{\psi, \varphi} - C_{\psi, \varphi} C_{\psi, \varphi}^*) = \frac{1}{(1 - |\alpha|^2)^\gamma} \sigma(e_{T_{|\psi|^2 T_{\varphi} \sigma^{-1}}} |2 - |w|^2| z))
\]
\[
= \left\{ \frac{|w(\varphi^{-1}(\zeta))|^2 - |w(\zeta)|^2}{(1 - |\alpha|^2)^\gamma} : \zeta \in \partial \mathbb{D} \right\}.
\]
Since \( C_{\psi, \varphi} C_{\psi, \varphi} \geq C_{\psi, \varphi} C_{\psi, \varphi}^* , |w(\varphi^{-1}(\zeta))|^2 - |w(\zeta)|^2 \geq 0 \) for each \( \zeta \in \partial \mathbb{D} \), and so \( |w(\zeta)|^2 - |w(\varphi(\zeta))|^2 \geq 0 \) for any \( \zeta \in \partial \mathbb{D} \). Therefore, the conclusion follows.

The idea of the proof of the result for ccohyponormal operator \( C_{\psi, \varphi} \) is similar to hyponormal operator, so it is left for the reader. \( \square \)
Suppose $\psi$ is a non-constant analytic function on $\overline{\mathbb{D}}$ and $\psi$ never vanishes on $\mathbb{D}$. By [5, Exercise 1, p. 129], we see that $|\psi|$ assumes its minimum value on $\partial\mathbb{D}$. Assume that $C_{\psi,\varphi}$ is essentially normal on $H^2$ or $A_2$. Now let $\psi(z) = 0$ for some $z \in \partial\mathbb{D}$. In the following proposition, we show that $\varphi(z) = \zeta$ and $\zeta$ is not the Denjoy-Wolff point of $\varphi$.

**Proposition 4.3.** Let $\varphi \in \text{Aut}(\mathbb{D})$. Suppose that $\psi$ is analytic on $\overline{\mathbb{D}}$ and $\psi$ never vanishes on $\mathbb{D}$. Let $C_{\psi,\varphi}$ be essentially normal on $H^2$ or $A_2$. If $\psi(z) = 0$ for some $z \in \partial\mathbb{D}$, then $\varphi(z) = \zeta$ and $\zeta$ is not the Denjoy-Wolff point of $\varphi$.

**Proof.** Assume that there is $\zeta \in \partial\mathbb{D}$ such that $\psi(\zeta) = 0$. According to [13, Theorem 3.2] and [15, Theorem 3.3], the map $\psi$ is zero on $B = \{\zeta, \varphi(\zeta), \varphi_2(\zeta), \ldots\}$. It is trivial that $B \subset \partial\mathbb{D}$. Since $\psi$ is an analytic function on $\overline{\mathbb{D}}$ never vanishing on $\mathbb{D}$, we get that $B$ is a finite set. Suppose that $B$ contains $N$ elements. If $\varphi(z) \neq \zeta$, then one can write $B = \{\zeta, \varphi(\zeta), \varphi_2(\zeta), \ldots, \varphi_{N-1}(\zeta)\}$. Observe that $\varphi_N(b) = b$ for each $b \in B$, which ensures that $N \leq 2$. If $N = 1$, then it is straightforward to see that $\varphi(z) = \zeta$ but this is a contradiction. If $N = 2$, i.e., $B = \{\zeta, \varphi(z)\}$, then $B$ is precisely the set of all fixed points of $\varphi_2$. Since $\varphi_2(\zeta) \neq \varphi(z)$, the point $\varphi(z)$ cannot be fixed by $\varphi$. Since all fixed points of $\varphi$ belong to $B$, we obtain that $\zeta$ must be a fixed point of $\varphi$, which is a contradiction. Thus $\varphi(z) = \zeta$.

It remains to show that $\zeta$ is not the Denjoy-Wolff point of $\varphi$. Assume that $\zeta$ is the Denjoy-Wolff point of $\varphi$. Since $|w| = |w \circ \varphi|$ on $\partial\mathbb{D}$ from [13, Theorem 3.2] and [15, Theorem 3.3], choose a constant $\lambda \in \partial\mathbb{D}$ such that $w \circ \varphi = \lambda w$. Denjoy-Wolff Theorem implies that

$$0 = |w(\zeta)| = \lim_{n \to \infty} |w(\varphi_n(z))| = \lim_{n \to \infty} |\lambda^n w(z)| = |w(z)|$$

for each $z \in \mathbb{D}$. Then $\psi \equiv 0$ on $\mathbb{D}$, which is a contradiction. Hence, $\zeta$ is not the Denjoy-Wolff point of $\varphi$. \qed

If $C_{\psi,\varphi}$ is normal on $H^2$ or $A_2$, then by Remark 2.5, $\psi$ never vanishes on $\mathbb{D}$. In Proposition 4.3, we saw that for $\varphi \in \text{Aut}(\mathbb{D})$ which is not a hyperbolic automorphism and $\psi$ that is analytic on $\overline{\mathbb{D}}$, if $C_{\psi,\varphi}$ is normal on $H^2$ or $A_2$, then $\psi$ never vanishes on $\mathbb{D}$.

**Proposition 4.4.** Let $\varphi \in \text{Aut}(\mathbb{D})$ and $\varphi(a) = 0$ for some $a \in \mathbb{D}$. Assume that $\psi$ is analytic on $\overline{\mathbb{D}}$ and $\psi \neq 0$. If $C_{\psi,\varphi}$ is normal on $H^2$ or $A_2$, then $w$ is an $\alpha$-eigenvector for the operator $C_{\varphi}$ and the corresponding $C_{\varphi}$-eigenvalue for $w$ is 1.

**Proof.** Suppose that $C_{\psi,\varphi}$ is normal. Proposition 4.2 implies that $|w| = |w \circ \varphi|$ on $\partial\mathbb{D}$. Since $\psi$ never vanishes on $\mathbb{D}$, by [5, Exercise 6, p. 130], we conclude that $C_{\varphi}(w) = \lambda w$, where $|\lambda| = 1$. If $\varphi$ is an elliptic automorphism with a fixed point $t \in \mathbb{D}$, then $w(t) = w(\varphi(t)) = \lambda w(t)$. Since $\psi$ never vanishes on $\mathbb{D}$, $\lambda = 1$. Now suppose that $\varphi$ is a parabolic or hyperbolic automorphism with Denjoy-Wolff...
point $\zeta$. Then
\[
w(\zeta) = \lim_{r \to 1} w(\varphi(r\zeta)) = \lambda \lim_{r \to 1} w(r\zeta) = \lambda w(\zeta).
\]
By Proposition 4.3, $\psi(\zeta) \neq 0$ and so $\lambda = 1$. Therefore, we conclude that $C_\varphi(w) = w$. \qed

Let $\varphi$ be an elliptic automorphism or the identity. As we stated before
Corollary 3.5, all normal weighted composition operators $C_{\psi, \varphi}$ of these types
were found. Also we must say that in this case $\psi$ never vanishes on $\overline{\mathbb{D}}$.
In the next theorem, for $\varphi$, not the identity and not an elliptic automorphism of
$\mathbb{D}$, which is in $\text{Aut}(\mathbb{D})$, we show that constant multiples of $K_{\sigma(0)}$ are the only
examples for $\psi$ that again $\psi$ never vanishes on $\overline{\mathbb{D}}$. It is interesting
that again $\psi$ never vanishes on $\overline{\mathbb{D}}$ and these weighted composition operators are
actually a constant multiples of unitary weighted composition operators (see
[3, Theorem 6] and [21, Corollary 3.6]).

**Theorem 4.5.** Assume that $\varphi$, not the identity and not an elliptic automor-
phism of $\mathbb{D}$, is in $\text{Aut}(\mathbb{D})$. Suppose that $\psi$ is analytic on $\overline{\mathbb{D}}$ and $\psi \neq 0$. Then $C_{\psi, \varphi}$ is normal on $H^2$ or $A^2_k$ if and only if $\psi = \psi(0)K_{\sigma(0)}$; hence $\psi$ never
vanishes on $\overline{\mathbb{D}}$.

**Proof.** Suppose that $C_{\psi, \varphi}$ is normal and $\varphi(a) = 0$. By Proposition 4.4, $w \circ \varphi = w$ on $\mathbb{D}$.
Then for each integer $n$, $\psi(z)(1 - \overline{\psi(z)})^\gamma = \psi(\varphi_n(z))(1 - \overline{\varphi_n(z)})^\gamma$ on
$\mathbb{D}$, where $\gamma = 1$ for $H^2$ and $\gamma = \alpha + 2$ for $A^2_k$. Let $\zeta$ be the Denjoy-Wolff point
of $\varphi$. Applying Denjoy-Wolff Theorem, we have
\[
\psi(\zeta)(1 - \overline{\psi(\zeta)})^\gamma = \lim_{n \to \infty} \psi(\varphi_n(z))(1 - \overline{\varphi_n(z)})^\gamma = \psi(z)(1 - \overline{\psi(z)})^\gamma
\]
for each $z \in \mathbb{D}$. Hence $\psi(z) = \frac{\psi(z)(1 - \overline{\psi(z)})^\gamma}{(1 - \overline{\psi(z)})^\gamma}$ and so $\psi = \psi(0)K_{\sigma(0)}$.

Conversely, it is not hard to see that the fact that for a constant number $c$,
$C_{\psi, \varphi}$ is normal implies that $C_{\psi, \varphi}$ is also. Then without loss of generality, we
assume that $\psi = K_{\sigma(0)}$. Observe that $g \circ \varphi = \frac{1}{(1 - |\varphi|^2)^{1/2}}$ and $h \psi \equiv 1$. Since
$\sigma = \varphi^{-1}$, we obtain from Cowen’s adjoint formula that
\[
C_{\psi, \varphi}^* C_{\psi, \varphi} = T_g C_\sigma T_{h \psi} C_\varphi T_g C_\sigma = T_g C_\sigma C_\varphi T_{\psi \sigma} \equiv T_g \psi \varphi^{-1} - 1
\]
and
\[
C_{\psi, \varphi}^* C_{\psi, \varphi} = T_g C_\sigma C_\varphi T_{h \psi} C_\sigma = T_g C_\sigma C_\varphi T_{\psi \sigma} \equiv T_g \psi \varphi.
\]
After some computation, one can see that $\psi \cdot g \circ \varphi = g \cdot \psi \circ \varphi^{-1} = \frac{1}{(1 - |\varphi|^2)^{1/2}}$. Hence $C_{\psi, \varphi}$ is normal. \qed

**Lemma 4.6.** Assume that $\varphi \in \text{Aut}(\mathbb{D})$. Suppose that for some $\zeta \in \partial\mathbb{D}$,
$\{\varphi_n(\zeta) : n$ is a positive integer} is a finite set. If $\varphi$ is parabolic or hyperbolic,
then $\zeta$ is a fixed point of $\varphi$. 


Proof. Let \( \{ \zeta, \varphi(\zeta), \varphi_2(\zeta), \ldots \} \) be a finite set. Then there is an integer \( N \) such that \( \varphi_N(\zeta) = \zeta \), so \( \zeta \) is a fixed point of \( \varphi_N \). It is not hard to see that \( \varphi \) is parabolic or hyperbolic if and only if \( \varphi_N \) is parabolic or hyperbolic, respectively and fixed points of \( \varphi \) and \( \varphi_N \) are the same. Hence \( \zeta \) is the fixed point of \( \varphi \). \( \square \)

Let \( \psi \in A(D) \). If \( C_{\psi, \varphi} \) is cohyponormal on \( H^2 \) or \( A^2_\alpha \), then \( \psi \) never vanishes on \( D \) or \( \psi \equiv 0 \) (see Remark 2.5). Therefore, by Maximum Modulus Theorem and [5, Exercise 1, p. 129], there are \( \zeta_1, \zeta_2 \in \partial D \), with \( |w(\zeta_1)| \leq |w(\zeta)| \) and \( |w(\zeta)| \leq |w(\zeta_2)| \) for all \( \zeta \in D \). In the following theorem, we assume that \( \zeta_1 \) and \( \zeta_2 \) are given as above.

**Theorem 4.7.** Suppose that \( \psi \in A(D) \). Let \( \{ \eta \in \partial D : |w(\eta)| = |w(\zeta_1)| \} \) and \( \{ \eta \in \partial D : |w(\eta)| = |w(\zeta_2)| \} \) be finite sets. Suppose that \( C_{\psi, \varphi} \) is cohyponormal on \( H^2 \) or \( A^2_\alpha \). The following statements hold.

(a) If \( \psi \) is a parabolic automorphism, then \( |w| \) is a constant function on \( \partial D \). Moreover, if \( \psi \not\equiv 0 \), then \( C_{\psi, \varphi} \) is normal and \( \psi = \psi(0)K_{\sigma(0)} \).

(b) If \( \varphi \) is a hyperbolic automorphism, then \( \zeta_1 \) and \( \zeta_2 \) are the fixed points of \( \varphi \).

Proof. (a) Suppose that \( \varphi \) is a parabolic automorphism. Since \( C_{\psi, \varphi} \) is cohyponormal, by Proposition 4.2, for each positive integer \( n \), we have

\[
|w(\varphi_n(\zeta_2))| \geq |w(\varphi_{n-1}(\zeta_2))| \geq |w(\varphi_{n-2}(\zeta_2))| \geq \cdots \geq |w(\zeta_2)|
\]

and

\[
|w(\zeta_1)| \geq |w(\varphi^{-1}(\zeta_1))| \geq |w(\varphi_2^{-1}(\zeta_1))| \geq \cdots \geq |w(\varphi_n^{-1}(\zeta_1))|.
\]

It is not hard to see that \( \varphi^{-1} \) is parabolic and \( \varphi \) and \( \varphi^{-1} \) have the same fixed point. Then by Lemma 4.6 and the statement which was stated before Theorem 4.7, we can see that \( \zeta_1 = \zeta_2 \). Then \( |w| \) is constant on \( \partial D \). Now assume that \( \psi \not\equiv 0 \). We know that \( \psi \) never vanishes on \( D \) (see Corollary 2.4), so by [5, Exercise 2, p. 129], \( w \) is a constant function. Then \( \psi = \psi(0)K_{\alpha} \) and by Theorem 4.5, \( C_{\psi, \varphi} \) is normal.

(b) Suppose that \( \varphi \) is a hyperbolic automorphism. By the proof of part (a) and Lemma 4.6, we can see that \( \zeta_1 \) and \( \zeta_2 \) are the fixed points of \( \varphi \). \( \square \)

For \( \psi \in H^\infty \) and \( \varphi \) which is an elliptic automorphism of \( D \), cohyponormality and normality of a weighted composition operator \( C_{\psi, \varphi} \) on \( H^2 \) are equivalent (see [9, Proposition 3.17]). In the previous theorem, we showed that if \( \varphi \) is a parabolic automorphism and \( \psi \) and \( w \) satisfy the hypotheses of this theorem, then \( C_{\psi, \varphi} \) is cohyponormal on \( H^2 \) or \( A^2_\alpha \) if and only if \( C_{\psi, \varphi} \) is normal and we saw that \( \psi = \psi(0)K_{\sigma(0)} \).

Recall that a bounded linear operator \( T \) between two Banach spaces is Fredholm if it is invertible modulo compact operators. We say that an operator \( A \in B(H) \) is bounded below if there is a constant \( c > 0 \) such that \( c\|h\| \leq \|A(h)\| \) for all \( h \in H \). Moreover, we know that a normal operator \( N \) on a Hilbert space \( H \) is bounded below if and only if \( N \) is invertible (see [6,
Exercise 15, p. 36). By this fact, the statements (a) and (c) in Theorem 4.8 are equivalent.

**Theorem 4.8.** Suppose that $C_{\psi, \varphi}$ is a normal operator on a Hilbert space $H$ of analytic functions on $D$. Assume that all the polynomials belong to $H$. The following statements are equivalent.

(a) The operator $C_{\psi, \varphi}$ is bounded below.

(b) The operator $C_{\psi, \varphi}$ is Fredholm.

(c) The operator $C_{\psi, \varphi}$ is invertible.

**Proof.** (b) implies (c). Suppose that $C_{\psi, \varphi}$ is Fredholm. Since by [6, Corollary 2.4, p. 352], $\dim \ker C_{\psi, \varphi} < \infty$, it is not hard to see that $\psi \not\equiv 0$. We claim that $\varphi$ is not a constant function. Assume that $\varphi \equiv c$, where $|c| < 1$. It is not hard to see that for each $n$, $z^n(z - c) \in \ker C_{\psi, \varphi}$, so $\dim \ker C_{\psi, \varphi} = \infty$ and it is a contradiction. By the Open Mapping Theorem, we can see that $0 \notin \sigma_p(C_{\psi, \varphi})$. Assume that $C_{\psi, \varphi}$ is not invertible. Then by [6, Proposition 4.6, p. 359], $0 \in \sigma_p(C_{\psi, \varphi})$ and it is a contradiction.

(c) implies (b). This is clear. □

Assume $\psi \not\equiv 0$ and $\varphi$ is not a constant function. By the Open Mapping Theorem, it is clear that $\ker C_{\psi, \varphi} = \{0\}$. Then $C_{\psi, \varphi}$ has closed range if and only if $C_{\psi, \varphi}$ is bounded below. We know that $C_{\psi, \varphi}$ on $H^2$ or $A^2_\alpha$ is invertible if and only if $\varphi \in \text{Aut}(D)$ and $\psi \in H^\infty$ is bounded away from zero on $D$ (see [2, Theorem 3.4]). As we stated before if $\varphi$ is an elliptic automorphism or the identity, all normal weighted composition operators were found; moreover others were characterized in Theorem 4.8. Then closed range weighted composition operators on $H^2$ and $A^2_\alpha$ which are normal were found.

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