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Non-Markovian processes in quantum theory

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Summary. The study of quantum dynamics featuring memory effects has always been a topic of interest within the theory of open quantum system, which is concerned about providing useful conceptual and theoretical tools for the description of the reduced dynamics of a system interacting with an external environment. Definitions of non-Markovian processes have been introduced trying to capture the notion of memory effect by studying features of the quantum dynamical map providing the evolution of the system states, or changes in the distinguishability of the system states themselves. We introduce basic notions in the framework of open quantum systems, stressing in particular analogies and differences with models used for introducing modifications of quantum mechanics which should help in dealing with the measurement problem. We further discuss recent developments in the treatment of non-Markovian processes and their role in considering more general modifications of quantum mechanics.

1.1 Introduction

Quantum theory was born as a new mechanics, capable of providing the correct quantitative assessment of phenomena which could not find their explanation within the usual framework of classical mechanics. About a century after its introduction, many different facets and complementary presentations of the theory have been worked out, putting into evidence in particular that quantum theory indeed provides a new probabilistic framework for the prediction of outcomes of statistical experiments. It is therefore not only a “quantum” version of classical mechanics, it is indeed a “quantum” version of classical probability theory, containing into itself an often non trivial classical limit [1, 2, 3]. One of the most intriguing and delicate aspects of quantum theory is its irreducibly probabilistic structure, conflicting with the deterministic description we are accustomed to, as well as our everyday experience of the realization of definite events. From a classical viewpoint a probabilistic analysis is only necessary if not all degrees of freedom are under control or can be
taken into account in detail. Not so for quantum theory. This state of affairs has led among others to the so-called “measurement problem”, referring to the difficulty in reconciling the classical description for macroscopic objects and the laws of quantum theory, predicting a statistical distribution rather than definite events [4]. On turn, this problem has led to consider alternatives to quantum theory, complying with its successes but leading to a different behavior for the prediction of events, effectively suppressing superposition of macroscopic objects. Among these theories one of the most renowned classes is given by collapse model, also known as dynamical reduction models [5, 6], arisen from the seminal paper [7]. Their distinctive trait is a stochastic non-linear modification of the Schrödinger equation, which on top of the standard evolution allows for the introduction of a collapse or localization mechanism. This mechanism, once accepted, avoids the measurement problem. Importantly, this mechanism has to be implemented at the level of the wavefunction, so as to allow for the suppression of superpositions. Nevertheless, at the level of experimental observations, it usually cannot be distinguished from other effects leading to a vanishing contribution of coherences.

The theory of open quantum system is focused on the description of the reduced dynamics of a system interacting with other degrees of freedom, typically called environment, which are not described in detail [9, 10]. The environment therefore brings in an additional level of randomicity in the dynamics, on top of the unavoidable statistical aspect brought in by quantum theory. In this framework the suppression of superposition states in a given basis is indeed predicted for a class of models known as decoherence models [8]. It thus appears that such models, bringing in another element of probabilistic description, typically provide the same average effect as dynamical reduction models, aimed at overcoming the inherent statistical structure of any quantum dynamics. In this respect, the two fields of dynamical reduction model and open quantum system share some underlying mathematical structure, and we will briefly address recent advancements in open quantum system having this perspective in mind. An important caveat to be mentioned is the fact that decoherence models do not provide a solution of the measurement problem in the sense addressed by collapse models, since the suppression of macroscopic superpositions only takes place in the average and a whole statistical distribution of outcomes is predicted [11].

The contribution is organized as follows. In Sect. 2 we briefly outline the open quantum system viewpoint and address the term quantum process as used in the physical literature. The description of decoherence effects and their relationship to specific collapse models is worked out in Sect. 3. Finally Sect. 4 is devoted to introduce the notion of non-Markovian dynamics for an open system, and its influence on the elaboration of dynamical reduction models.
1.2 Open systems and quantum processes

For the case in which a quantum system is not isolated from other quantum systems, the latter should be taken into account in the description of its dynamics. If the system and the other degrees of freedom, collectively named environment, do not share correlations at the initial time, one can describe the evolution of the system alone by introducing a collection of completely positive trace preserving maps \( \{ \Phi(t) \}_{t \in \mathbb{R}^+} \), which determine the statistics of any local observation once the initial state of the system \( \rho_S(0) \) has been specified according to the formula

\[
\langle A_S \rangle_t = \text{Tr} \{ A_S \Phi(t) | \rho_S(0) \},
\]

where \( A_S \) denotes a system observable. The collection of maps \( \{ \Phi(t) \}_{t \in \mathbb{R}^+} \) describes what is usually called a quantum process. The term process is here used in a loose sense, in analogy with the classical situation, hinting at the presence of an irreducible randomicity, here corresponding to the environmental degrees of freedom not accessible or described in detail, but affecting the system dynamics due to a unitary coupling with the environment \( U_{SE}(t) \) as drawn in Fig. 1.1. If system and environment interaction can be neglected, and only in this case, \( \Phi(t) \) is a unitary transformation, implying in particular a group composition law. In all other cases irreversibility is lost, and the general mathematical structure of this collection of maps is not known. Some partial results are however available. A most famous and relevant class of reduced dynamics is obtained if we ask \( \Phi(t) \) to obey a semigroup composition law forward in time. For this case we have \( \Phi(t) = \exp(i t \mathcal{L}) \), with \( \mathcal{L} \) in Lindblad form [9], that is

\[
\mathcal{L}[\rho_S(t)] = -\frac{i}{\hbar} [H, \rho_S(t)] + \sum_k \lambda_k \left[ A_k \rho_S(t) A_k^\dagger - \frac{1}{2} \{ A_k^\dagger A_k, \rho_S(t) \} \right],
\]

where \( \{ A_k \} \) and \( H \) denote system operators, with \( H \) an effective self-adjoint Hamiltonian. A dynamics of this kind has always been called Markovian, since

Fig. 1.1. Illustration of an open system interacting with an environment via a unitary coupling \( U_{SE}(t) \).
it arose as quantum counterpart of classical Markovian semigroups. The implicit idea is that the stochasticity in the dynamics arising due to interaction with the environment does not lead to effects that can be termed memory, making reference to previous history or states of the system. This feature is immediately lost even only considering dynamics which can be obtained as random mixture of unitary evolutions, so-called random unitary dynamics \[12, 13, 14, 15\], which might arise also as a consequence of classical environment noise and can be experimentally engineered \[16, 17\]. The operators \(\{A_k\}\) describe microscopic interaction events, e.g. random localization or momentum transfer events for the case of decoherence as discussed in Sect. 3.

### 1.3 Events and decoherence

Dynamical reduction models and open quantum system theory share a common root in the treatment of measurement in quantum mechanics, to be seen as dealing with a description of the outcomes of statistical experiments in which the interaction with the measurement apparatus is taken into account. Indeed, the first seminal contributions to open quantum systems were intimately connected with the description of measurement processes, and its relevance for the foundations of quantum mechanics \[18, 19, 20\], putting in particular into evidence the relevance of the mathematical notion of complete positivity. Not by chance the original GRW paper, which introduced the first collapse model, was built upon work aimed at the quantum description of continuous measurement in time \[21, 22\], and started the treatment from a master equation describing decoherence in position \[23\].

To better work out this connection, let us consider in more detail how a collapse model can describe in the average a decoherence effect and how a microscopic description of decoherence can be related to a notion of event. In this spirit we briefly recall the formulation of the GRW model in the formulation via stochastic differential equations \[5, 24\].

\[
\frac{\text{d}|\psi(t)\rangle}{\text{d}t} = -\frac{i}{\hbar}\hat{H}_0|\psi(t)\rangle\text{d}t + \int_\mathbb{R} \text{d}y \left( \frac{L(y,\hat{x})}{\|L(y,\hat{x})|\psi(t)\|} - 1 \right) |\psi(t)\rangle\text{d}N(y,t), \tag{1.1}
\]

where \(\psi(t)\) is the system’s wavefunction, \(\hat{H}_0\) denotes the Hamiltonian appearing in the standard Schrödinger equation and the stochastic modification is determined by the collection of operators \(\{L(y,\hat{x})\}_{y\in\mathbb{R}}\), with \(\hat{x}\) the standard position operator, and the family of classical stochastic processes \(\{N(y,t)\}_{y\in\mathbb{R}}\). Note in particular that this modification is non-linear. In order to obtain suppression of spatial superposition of states, the \(L\) operators have to act as localization operators and to recover the original GRW model must be of the form

\[
L(y,\hat{x}) = \frac{1}{\sqrt{\pi r_c}} e^{-\frac{(y-x)^2}{2r_c^2}}. \tag{1.2}
\]
The stochastic modification depends on the field of independent processes \( \{N(y, t)\}_{y \in \mathbb{R}} \) such that \( N(y, t)dy \) is the counting process giving the number of jumps taking place at time \( t \) in the space interval from \( y \) to \( y + dy \). The collection of counting processes satisfies \( dN(x, t)dN(y, t) = \delta(x - y)dN(y, t) \), with rates given by

\[
\mathbb{E}[dN(y, t)] = \lambda \|L(y, \hat{x})|\psi(t)\|_2^2 dt.
\]

The phenomenological parameters \( \lambda \) and \( r_c \) determine intensity and localization strength of the random jumps inducing a dynamical localization in position of the system. Averaging over the realization of the processes one obtains the state determining the statistics of observation on the system, namely

\[
\rho(t) = \mathbb{E}[|\psi(t)\rangle\langle\psi(t)|],
\]

which obeys the master equation

\[
\frac{d}{dt} \rho(t) = -\lambda \left[ \rho(t) - \int dy L(y, \hat{x})\rho(t)L(y, \hat{x}) \right] \quad (1.3)
\]

predicting a reduction of the off-diagonal matrix elements in the position representation according to

\[
\langle x|\rho(t)|y \rangle = \exp \left( -\lambda t \left[ 1 - \int dz L(z, x)L(z, y) \right] \right) \langle x|\rho(0)|y \rangle. \quad (1.4)
\]

The obtained master Eq. (1.3) is in standard Lindblad form [9], describes decoherence in position according to Eq. (1.4), and in particular is characterized by translational invariance. Building on this aspect one realizes that it can be written in an explicit translationally covariant form [25, 26, 27] as follows

\[
\frac{d}{dt} \rho(t) = -\lambda \left[ \rho(t) - \int dq \tilde{L}(q)e^{i\hat{p}q\hat{x}}\rho(t)e^{-i\hat{p}q\hat{x}} \right] \quad (1.5)
\]

with \( \tilde{L}(q) \) Fourier transform of the function \( L^2(y, 0) \), that is again a Gaussian weight. It thus appears that the dynamics that can be observed as a consequence of the localization mechanism, described at the level of trajectories of the wavefunction in Hilbert space by the stochastic differential equation Eq. (1.1), is the same that would arise as a consequence of interaction of the system with an external environment whose effect can be described in terms of localization events as in Eq. (1.3) or in terms of momentum transfers described by the collection of unitaries \( \{e^{i\hat{p}q\hat{x}}\}_{q \in \mathbb{R}} \) as in Eq. (1.5). This viewpoint, connecting the open system based description of decoherence and the measurement based viewpoint of collapse models, implies in particular that the natural benchmark in the assessment of possible modifications of the quantum mechanical predictions due to a collapse mechanism is the estimate
of possible decoherence effects affecting the considered dynamics. Indeed, this is one of the main difficulties in looking for experimental signatures of collapse mechanisms [6]. On the other hand awareness of this relationship has opened the way to consider variants of dynamical reduction models. In particular, it has led to overcome an important intrinsic limitation of models such as Eq. (1.1), which predict an infinite growth of the system energy [28, 24]. A further natural extension of dynamical reduction model arising from analogy and differences shared with open quantum system models is the inclusion of memory effects [29, 30, 31, 32, 33], in view of a definition of non-Markovian dynamics as discussed in Sect. 4.

1.4 Non-Markovian processes

In mentioning some of the basic tenets and results of the theory of open quantum systems, we have put into evidence the notion of quantum process as used and understood in the physical literature. In particular, the time evolutions arising as solutions of master equations in Lindblad form are typically termed quantum Markovian processes, since they provide the natural quantum counterpart of classical semigroup evolutions, arising in connection with homogeneous in time Markovian processes. A next natural step in this respect is considering time evolutions which can provide a quantum realization of a non-Markovian process. Given the looser definition of process considered in the quantum framework, as a collection of time dependent completely positive trace preserving maps describing a continuous quantum dynamics, one might consider a suitable definition of non-Markovian quantum process within this very same framework of dynamical maps. Indeed, providing a notion of non-Markovian quantum process in the same spirit as in the classical case, which gives an exact definition of Markovian process in terms of conditions on the infinite hierarchy of conditional probability densities for the process, appears to be a very difficult task. Already from a conceptual point of view the situation does not appear to be neatly defined, since speaking about values of an observable at a given time calls for a measurement procedure which affects the subsequent values to be assumed by the quantity [34]. On the contrary, focusing on the collection of completely positive trace preserving maps giving the reduced dynamics has allowed to introduce clearcut definitions of Markovian, and in a complementary way non-Markovian, quantum process. Actually, there have been various proposals in this direction. We will here only focus on one of them, based on the behavior of the distinguishability of states in time, which is in direct relationship with a notion of divisibility of the time evolution maps. For more details and a complete treatment we refer the reader to recent reviews [35, 36, 37, 38].

The basic insight can be summarized as follows. By interacting with the environmental degrees of freedom the system gets correlated with the environment and possibly leads to a change in time of the reduced state of the
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evironment itself. As a consequence of the dynamics therefore, the capability of distinguishing two different initial system states, by performing measurements on the system degrees of freedom only, changes in time. Indeed, taking the partial trace necessary to define the reduced system state, which is all that is necessary in order to provide the statistics of measurements on the system, the whole information about correlations is no more available. To exploit this fact one can introduce a suitable quantifier of the distinguishability between states, such as the trace distance, given by the trace norm of the difference of the states

$$D(\rho_1^S(t), \rho_2^S(t)) = \frac{1}{2} \| \rho_1^S(t) - \rho_2^S(t) \|_1$$  \hspace{1cm} (1.6)

and consider its behavior in time. Being a contraction under the action of completely positive trace preserving transformations, the trace distance always diminishes with respect to its initial value, that is

$$D(\rho_1^S(t), \rho_2^S(t)) \leq D(\rho_1^S(0), \rho_2^S(0)).$$

In particular for the semigroup case, considered in Sect. 2 for the case of a quantum Markovian process, due to the composition law one has a monotonous reduction of the distance among states with time. In such a situation the distance between states, and therefore their distinguishability [39], gets smaller and smaller with elapsing time. The failure of this monotic decreasing behavior for at least a couple of possible initial states has been taken as indication of non-Markovian dynamics in the seminal paper [40], since it amounts to a revival in the distinguishability between the states that can only arise as a consequence of previously established correlations with the environment or changes in the environmental state that affect the subsequent reduced system dynamics. This fact is schematically drawn in Fig. 1.2.

![Fig. 1.2. Open system interacting with an environment via a unitary coupling $U_{SE}(t)$. Markovian effects (M) are depicted as an information flow from system to environment, while an information flow from environment to system (NM) is identified with memory effects.](image)

The validity of this interpretation is substantiated by the inequality [41, 42, 43]
\[ D(\rho_1^S(t), \rho_2^S(t)) \leq D(\rho_1^{SE}(s), \rho_2^{SE}(s)) \leq D(\rho_1^S(s) \otimes \rho_1^E(s)) + D(\rho_2^{SE}(s), \rho_2^S(s) \otimes \rho_2^E(s)) \]

where it is assumed that \( t \geq s \). The term at the lhs when positive provides a signature of non-Markovianity, so that the positivity of the rhs is a precondition for non-Markovianity, to be traced back to the effects mentioned above: correlations and influence of the system on the environment. While the notions of distinguishability, contractivity of the used distinguishability quantifier upon the action of a quantum transformation, and connection of the distinguishability revivals to the imprint of the system dynamics left in correlations or environment, provide the basic traits of this approach to the description of memory effects in quantum mechanics, many more subtle issues are involved in the definition of this framework. Importantly, there is a stringent mathematical connection between this viewpoint and divisibility properties of the time evolution, corresponding to the fact that the evolution over a finite time can always be split into evolutions over shorter times, each described by a proper quantum transformation \[44, 45, 46\].

Dynamics allowing for non-Markovian effects have also been considered in the above-mentioned framework of a decoherence dynamics driven by random events \[47, 48\], as well as in the introduction of more general dynamical reduction models \[49, 31\]. While in the context of decoherence allowing for non-Markovian dynamics is a way to consider more general and accurate description of the reduced dynamics, within the framework of dynamical reduction models non-Markovian models lead to possibly more stringent exclusion regions of the parameter values which characterise the model.

### 1.5 Conclusions and Outlook

In recent times a lot of work in the field of open quantum system has been devoted to characterization and study of non-Markovian dynamics. This research has involved both the very definition and clarification of what can be meant as quantum dynamics featuring memory effects, as well as the possible relevance of non-Markovian dynamics in the description of the reduced dynamics of non isolated quantum systems as well as related fields. In this contribution we have recalled in particular the relationship between the description of decoherence in open quantum system and modifications of quantum mechanics such as dynamical reduction models introduced for the sake of better grasping the so-called quantum measurement problem. We have briefly discussed a natural physical interpretation of non-Markovian dynamics as related to information exchange between system and environment, and pointed to the use of the formalism of non-Markovian dynamics to consider more general collapse model which might help in improving the known bounds on the parameters characterizing the possible deviations from standard quantum mechanics. The
relevance of the classification of non-Markovian dynamics itself as well as the role of memory effects in collapse mechanisms remain two open questions that will surely involve future research.

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