On the joint asymptotic distribution of the restricted estimators in multivariate regression model

Séverien Nkurunziza* and Youzhi Yu†

Abstract

The main Theorem of Jain et al. [Jain, K., Singh, S., and Sharma, S. (2011), Restricted estimation in multivariate measurement error regression model; JMVA, 102, 2, 264–280] is established in its full generality. Namely, we derive the joint asymptotic normality of the unrestricted estimator (UE) and the restricted estimators of the matrix of the regression coefficients. The derived result holds under the hypothesized restriction as well as under the sequence of alternative restrictions. In addition, we establish Asymptotic Distributional Risk for the estimators and compare their relative performance. It is established that near the restriction, the restricted estimators (REs) perform better than the UE. But the REs perform worse than the unrestricted estimator when one moves far away from the restriction.

Keywords: ADR; Asymptotic normality; Measurement error; Multivariate regression model; Restricted estimator; Unrestricted estimator.

1 Introduction

In this paper, we are interested in an estimation problem in multivariate ultrastructural measurement error model with more than one response variable. In particular, as in Jain et al. (2011), we consider the case where the regression coefficients may satisfy some linear restriction. It is practical to use such models in the real world if there is at least two

*University of Windsor, department of Mathematics and Statistics, 401 Sunset Avenue, Windsor, Ontario, N9B 3P4. Email: severien@uwindsor.ca
†University of Windsor, 401 Sunset Avenue, Windsor, Ontario, N9B 3P4. Email: yu13e@uwindsor.ca
correlated response variables. For example, in the field of medical sciences (see Dolby, 1976), more than one body index is often recorded and the interest is to relate these measurements to the amount of different nutrients in the daily diet. Similarly, as described in Bertsch et al. (1974), in the air pollution studies, the observed chemical elements contained in the polluted air are lead, thorium and Uranium etc. It is highly likely that the variables involved in the study may possess some measurement errors. Following Mardia (1980), multivariate regression is applicable in a wide range of situations, such as Economics (see Meeusen, 1997) and Biology (see Mcardle, 1988). We also refer to Stevens (2012) for a discussion about the importance of regression models in education and social-sciences.

In this paper, we derive the asymptotic properties of the unrestricted and the restricted estimators of the regression coefficients in the multivariate regression models with measurement errors, when the coefficients satisfy some restrictions. To give a close reference, we quote Jain et al. (2011) who derived the unrestricted and three restricted estimators for the regression coefficients, and derived a theorem (see Jain et al., Theorem 4.1) which gives the marginal asymptotic distributions of the estimators under the restriction.

To summarize the contribution of this paper, we generalize Theorem 4.1 of Jain et al. (2011) in three ways. First, we derive the joint asymptotic distribution of the unrestricted estimator and any member of the class of the restricted estimators under the restriction. Second, we derive the joint asymptotic distribution of the unrestricted estimator and any member of the class of the restricted estimators under the sequence of local alternative restrictions. Third, we derive the joint asymptotic distribution between the UE and all three restricted estimators given in Jain et al. (2011), under the restriction and under the sequence of local alternative restrictions. In addition, we establish the Asymptotic Distributional Risk (ADR) for the UE and the ADR of any member of the class of restricted estimators. We also compare the relative performance of the proposed estimators. In particular, we prove that in the neighborhood of the restriction, the restricted estimators dominate the unrestricted estimator. We also prove that as one moves far away from the restriction, the unrestricted estimator dominates the restricted estimators. Finally, we generalize Proposition A.10 and Corollary A.2 in Chen and Nkurunziza (2016).

The rest of this paper is organized as follows. Section 2 outlines some preliminary results given in Jain et al. (2011). In Section 3 we present the main results of this paper.
More specifically, in Subsection 3.1, we establish the joint asymptotic distribution between the unrestricted estimator (UE) and any member of the restricted estimators under the restriction. In Subsection 3.2, we derive the joint asymptotic distributions between all estimators under the sequence of the local alternative restrictions. In Subsection 3.3, we derive ADR for the UE and restricted estimators and in Subsection 3.4, we analyse the relative performance of the UE and the restricted estimators. Finally, Section 4 gives some the concluding remark of this paper, and for the convenience of the reader, some technical results are given in the appendix.

2 Model Specifications and preliminary results

In this section, we describe the multivariate regression model with measurement error as well as the assumptions used in order to establish the results of this paper. Following Jain et al. (2011), we consider the multivariate regression model given by

\[ Z = DB + E, \]

where \( Z \) is a \( n \times q \) matrix, \( D \) is a \( n \times p \) matrix, \( B \) is \( p \times q \) matrix of the regression coefficients and \( E \) is a \( n \times q \) matrix of error terms. We assume that \( Z \) is observable but \( D \) is not observable and can be observed only through \( X \) with additional measurement error \( \Delta \) as

\[ X = D + \Delta, \]

where \( X \) and \( \Delta \) are \( n \times p \)-random matrices. Further, we suppose that

\[ D = M + \Psi, \]

where \( M \) is a \( n \times p \)-matrix of fixed components and \( \Psi \) is a \( n \times p \)-matrix of random components. We also suppose that some prior information about the regression coefficient \( B \) is available. In particular, for known matrices \( R_1, R_2 \) and \( \theta \), we suppose that

\[ R_1 B R_2 = \theta, \tag{2.1} \]

where \( R_1 \) is \( r_1 \times p \) matrix, \( R_2 \) is a \( q \times r_2 \) matrix and \( \theta \) is \( r_1 \times r_2 \) matrix. For the interpretation of the restriction in (2.1), \( R_1 \) imposes a linear restriction on the parameters of
individual equations while $R_2$ imposes a linear restriction across equations. For more details about the interpretation of this restriction, we refer for example to Izenman (2008), Jain et al. (2011) and the references therein. To introduce some notations, let $Z(i) = [z_{1i}, z_{2i}, ..., z_{ni}]'$, let $E(i) = [\epsilon_{i1}, \epsilon_{i2}, ..., \epsilon_{iq}]'$, let $Z = [Z(1), Z(2), ..., Z(q)]$, $E = [E(1), E(2), ..., E(n)]'$. Further, let $\Delta = [\Delta(1), \Delta(2), ..., \Delta(n)]'$ and let $\Delta(j) = [\delta_{j1}, \delta_{j2}, ..., \delta_{jp}]'$ for $j = 1, 2, ..., n$, let $M = [M(1), M(2), ..., M(n)]'$ with $M(j) = [M_{j1}, M_{j2}, ..., M_{jp}]'$, and let $\Psi(j) = [\psi_{j1}, \psi_{j2}, ..., \psi_{jp}]'$, $j = 1, 2, ..., n$. We also let $I_p$ to stand for the $p$-dimensional identity matrix. The following assumptions are made in order to derive the proposed estimators and their asymptotic properties. Note that these conditions are similar to that in Jain et al. (2011).

**Assumption 1.** $(A_1)$ Elements of vector $E(i) = [\epsilon_{i1}, \epsilon_{i2}, ..., \epsilon_{ni}]$ are independent with mean $0$, variance $\sigma^2\epsilon$, third moment $\gamma_1\sigma_3\epsilon$ and fourth moment $(\gamma_2\epsilon + 3)\sigma^4\epsilon$;

$(A_2)$ $\delta_{ij}$ are independent and identically distributed random variables with mean $0$, variance $\sigma^2\delta$, third moment $\gamma_1\sigma_3\delta$ and fourth moment $(\gamma_2\delta + 3)\sigma^4\delta$;

$(A_3)$ $\Psi_{ij}$ are independent and identically distributed random variables with mean $0$, variance $\sigma^2\phi$, third moment $\gamma_1\sigma_3\phi$ and fourth moment $(\gamma_2\phi + 3)\sigma^4\phi$;

$(A_4)$ $\Delta, \Psi,$ and $E$ are mutually independent;

$(A_5)$ $M(n) \rightarrow \sigma_M$ as $n \rightarrow \infty$ and $\sigma_M$ is finite;

$(A_6)$ Rank($X$)=$p$, Rank($R_1$)=$r_1$ and Rank($R_2$)=$r_2$.

### 2.1 Estimation methods

In this subsection, we outline some results given in Jain et al. (2011) which are used to derive the main results of this paper. Namely, we present the unrestricted estimator (UE) and three restricted estimators (REs) of the regression coefficients. By using the class of objective functions given in Jain et al. (2011), we also present a class of the restricted estimators which includes the three REs. For more details about the content of this subsection, we refer to Jain et al. (2011).
2.1.1 The unrestricted estimator

As in Jain et al. (2011), one considers first the following objective function
\[ G_1 = \text{tr}((Z - XB)'(Z - XB)), \]
which leads to the least squares estimators (LSE)
\[ \hat{B} = (X'X)^{-1}X'Z. \] (2.2)

Under parts \( (A_1) - (A_5) \) of Assumption 1, one can verify that \( \hat{B} \) converges in probability to \( KB \neq B \), where \( K = \Sigma^{-1} (\Sigma - \sigma^2 I_p) \) with \( \Sigma = \sigma_M \sigma_M' + \sigma^2 I_p + \sigma^2 I_p \), and thus, \( \hat{B} \) is not a consistent estimator. Because of that, as in Jain et al. (2011), one replaces \( \hat{B} \) by
\[ \hat{B}_1 = K^{-1}_X \hat{B}, \] (2.3)
where \( K_X = \Sigma_X^{-1} \Sigma_D \) with \( \Sigma_X = n^{-1} M'M + \sigma^2 I_p + \sigma^2 I_p, \Sigma_D = n^{-1} M'M + \sigma^2 I_p \).

Further, as in Jain et al. (2011), one can verify that
\[ \Sigma_X \overset{p}{\to} \Sigma, \quad \text{and} \quad \Sigma_D \overset{p}{\to} \Sigma - \sigma^2 I_p \quad \text{where} \quad \Sigma = [\sigma_M \sigma_M' + \sigma^2 I_p + \sigma^2 I_p]. \]
As given in Jain et al. (2011), note that the estimator \( \hat{B}_1 \) can be obtained directly by minimizing the objective function
\[ \hat{G}_2 = G_1 - \text{tr}[B'(n\Sigma_X)(I_p - K_X)B]. \] (2.4)

For more details, we refer to Jain et al. (2011). In the quoted paper, the authors prove that \( \hat{B}_1 \) is a consistent estimator for \( B \). They also derive the following theorem which gives the asymptotic distribution of \( \sqrt{n}(\hat{B}_1 - B) \). To introduce some notations, let
\[ \bar{K}_X = (\Sigma_X - \Sigma_D)\Sigma_X^{-1}, \quad H = n^{-\frac{1}{2}}X'X - n^{-\frac{1}{2}}\Sigma_X, \quad h = n^{-\frac{1}{2}}(X'[E - \Delta B]) + n^{-\frac{1}{2}}\sigma^2 B, \]
\[ \Lambda = \lim_{n \to \infty} E\{[\text{vec}(h')] + \text{vec}(B'\bar{K}_X H)] [\text{vec}(h') + \text{vec}(B'\bar{K}_X H)'] \} \text{ and let } 0 \text{ be a zero-matrix.} \]

The existence of this matrix is established in Jain et al. (2011).

**Theorem 2.1.** Suppose that Assumptions \((A_1)-(A_6)\) hold hold, we have
\[ n^{\frac{1}{2}}(\hat{B}_1 - B) \overset{d}{\to} n_{p \times q}(0, A_1 \Delta A_1'), \text{ where } A_1 = (\Sigma K)^{-1} \otimes I_q. \]

The proof is similar to that given in Jain et al. (2011, see the proof of Theorem 4.1).

2.1.2 A class of restricted estimators

In this subsection, we present a class of estimators of \( B \) which are consistent and satisfy the restriction in (2.1). As commonly the case in constrained estimation, this is obtained by
minimizing a certain objective function subject to the constraint. In particular, since the objective function \( \hat{g}_2 \) given in (2.3) leads to a consistent estimator, the RE can be obtained by minimizing \( \hat{g}_2 \) subject to the constraint \( R_1 B R_2 = \theta \). The following proposition shows that the above objective function can be seen as a member of a certain class of objective functions. For more details, we refer to Jain et al. (2011).

**Proposition 2.1.** We have \( \hat{g}_2 = \text{tr}(Z'Z) + \text{tr}[(\hat{B}_1 - B)'(X'X)K_X(\hat{B}_1 - B)] \).

The proof follows directly from algebraic computations. From Proposition 2.1, as in Jain et al. (2011) one considers below a more general class of objective functions. To this end, let \( P_{p \times p} \) denote the set of all observable \( p \times p \)-symmetric and positive definite matrices and let

\[
\left\{ \hat{g}_3 \left( \hat{\Sigma} \right) = \text{tr}((\hat{B}_1 - B)'\hat{\Sigma}(\hat{B}_1 - B)) : \hat{\Sigma} \in P_{p \times p} \right\}. \tag{2.5}
\]

Thus, \( \hat{g}_2 \) is a member of this class with \( \hat{\Sigma} = (X'X)K_X \). Other members of objective functions correspond to the cases where \( \hat{\Sigma} = S = X'X \) and \( \hat{\Sigma} = n I_p \). For further details about the objective function in (2.5), we refer to Jain et al. (2011). From the above class of objective function, one obtains a class of restricted estimators \{\( \hat{B}(\hat{\Sigma}) : \hat{\Sigma} \in P_{p \times p} \)\} which satisfies the constraint \( R_1 B R_2 = \theta \). Namely, by using the Lagrangian method, we get

\[
\hat{B}(\hat{\Sigma}) = \hat{B}_1 - (\hat{\Sigma})^{-1}R_1'\left[R_1(\hat{\Sigma})^{-1}R_1\right]^{-1}\left(R_1\hat{B}_1 R_2 - \theta\right)(R_2'R_2)^{-1}R_2', \tag{2.6}
\]

where \( \hat{\Sigma} \) is a known symmetric and positive definite matrix. In particular, from (2.6), by replacing \( \hat{\Sigma} \) by \((X'X)K_X, X'X \) and \( n I_p \), respectively, one gets

\[
\hat{B}_2 = \hat{B}_1 - (X'XK_X)^{-1}R_1'[R_1(X'XK_X)^{-1}R_1]^{-1}(R_1\hat{B}_1 R_2 - \theta)(R_2'R_2)^{-1}R_2', \tag{2.7}
\]

\[
\hat{B}_3 = \hat{B}_1 - (X'X)^{-1}R_1'[R_1(X'X)^{-1}R_1]^{-1}\left(R_1\hat{B}_1 R_2 - \theta\right)(R_2'R_2)^{-1}R_2', \tag{2.8}
\]

\[
\hat{B}_4 = \hat{B}_1 - R_1'[R_1R_1']^{-1}\left(R_1\hat{B}_1 R_2 - \theta\right)(R_2'R_2)^{-1}R_2'. \tag{2.9}
\]

Note that the estimators \( \hat{B}_2, \hat{B}_3 \) and \( \hat{B}_4 \) are derived in Jain et al. (2011). Here, their derivation is given for the paper to be self-contained.
3 Main results

In this section, we derive the joint asymptotic distribution of all estimators, under the restriction as well as under the sequence of local alternative restrictions. In particular, we generalize Theorem 4.1 in Jain et al. (2011) which gives the marginal asymptotic distributions under the restriction.

3.1 Asymptotic properties under the restriction

In this subsection, we derive the joint asymptotic normality of the UE and any member of the restricted estimators, under the restriction. We suppose that the weighting matrix \( \hat{\Sigma} \) satisfies the following assumption.

**Assumption 2.** \( \hat{\Sigma} \) is such that \( \frac{1}{n} \hat{\Sigma} \xrightarrow{P} Q_0 \) where \( Q_0 \) is nonrandom and positive definite matrix.

Note that the matrices \( X'XK \), \( X'X \) and \( nI_p \) satisfy Assumption 2 with the matrix \( Q_0 \) equals to \( \Sigma K \), \( \Sigma \) and \( I_p \) respectively. To set up some notations, let

\[
A_1 = (\Sigma K)^{-1} \otimes I_q,
\]

\[
A_2 (Q_0) = A_1 - (Q_0)^{-1} R_1 (R_1 (Q_0)^{-1} R_1')^{-1} R_1 (\Sigma K)^{-1} \otimes R_2 (R_2 R_2)^{-1} R_2',
\]

\[
\Sigma_{11} = A_1 \Lambda A_1', \quad \Sigma_{22} (Q_0) = A_2 (Q_0) \Lambda A_2' (Q_0), \quad \Sigma_{21} (Q_0) = \Sigma_{12} (Q_0), \quad (3.1)
\]

\[
\Sigma_{12} (Q_0) = A_1 \Lambda A_2' (Q_0).
\]

**Theorem 3.1.** If Assumptions 1-2 hold and \( R_1 BR_2 = \theta \), we have

\[
\left( n^{\frac{1}{2}} \left( \hat{B} - B \right), n^{\frac{1}{2}} \left( \hat{B} (\Sigma) - B \right) \right)' \xrightarrow{d} \left( \eta', \zeta^* \right)' \quad \text{where}
\]

\[
\begin{pmatrix}
\eta \\
\zeta
\end{pmatrix} \sim N_{2p \times q} \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} (Q_0) \\ \Sigma_{21} (Q_0) & \Sigma_{22} (Q_0) \end{pmatrix} \right), \quad (3.2)
\]

where \( \Sigma_{11}, \Sigma_{12} (Q_0), \Sigma_{21} (Q_0) \) and \( \Sigma_{22} (Q_0) \) are defined in (3.1).

The proof of this theorem is given in the Appendix. The above theorem generalizes Theorem 4.1 in Jain et al. (2011) in two ways. First, the estimator \( \hat{B} (\Sigma) \) encloses as special cases the restricted estimators \( \hat{B}_2, \hat{B}_3 \) and \( \hat{B}_4 \). Second, the above result gives the joint asymptotic distribution between the UE and any member of the class of restricted
estimators; from which the marginal asymptotic distribution follows directly. Indeed, if \( Q_0 \) is taken as \( K\Sigma, \Sigma \) and \( I_p \), respectively, the above result gives the asymptotic distribution of \( n^{1/2} \left( \hat{B}_2 - B \right), n^{1/2} \left( \hat{B}_3 - B \right), \) and \( n^{1/2} \left( \hat{B}_4 - B \right) \) given in Jain et al. (2011). Below, we give another generalization of the limiting distributions given in Jain et al. (2011). In particular, we establish the joint asymptotic normality between the estimators \( \hat{B}_1, \hat{B}_2, \hat{B}_3 \) and \( \hat{B}_4 \), under the sequence of local alternative restrictions. On the top of this result, as intermediate step, we also generalize Proposition A.10 and Corollary A.2 in Chen and Nkurunziza (2016).

3.2 Asymptotic results under local alternative

In this subsection, we present the asymptotic properties of the UE and the restricted estimators under the following sequence of local alternative restrictions

\[
R_1B R_2 = \theta + \frac{\theta_0}{\sqrt{n}}, \quad n = 1, 2, \ldots
\]

where \( \theta_0 \) is fixed with \( ||\theta_0|| < \infty \). Note that if \( \theta_0 = 0 \) in (3.3), then (3.3) becomes (2.1). Thus, the results established under (3.3) generalize the results given in Jain et al. (2011), which are established under (2.1).

Theorem 3.2. Suppose that Assumptions 1 and 2 hold along with the sequence of local alternative in (3.3), then

\[
n^{1/2} \begin{pmatrix}
(\hat{B}_1 - B)' \\
(\hat{B}(\hat{\Sigma}) - B)'
\end{pmatrix} \xrightarrow{d} \left( \eta_1', \eta^* \right)'
\]

where

\[
\eta_1 = N_{2p \times q} \left( \begin{pmatrix} 0 \\
\mu(Q_0) \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12}(Q_0) \\
\Sigma_{21}(Q_0) & \Sigma_{22}(Q_0) \end{pmatrix} \right)
\]

and

\[
\eta^* = N_{2q \times q} \left( \begin{pmatrix} \Sigma_{12}(Q_0) \\
\Sigma_{22}(Q_0) \end{pmatrix} \right)
\]

are defined as in Theorem 3.1.

The proof of this theorem is given in the Appendix. By using the similar techniques, we establish the joint distribution of the UE and the restricted estimators given in (2.7), (2.8)
If Assumption 1 holds along with \( (\Sigma K)^{-1} R_1 (\Sigma K)^{-1} R_1' - 1 R_1 (\Sigma K)^{-1} \otimes R_2 (R_2' R_2)^{-1} R_2' \),
\[ A_2 = A_1 - (\Sigma K)^{-1} R_1 (\Sigma K)^{-1} R_1' - 1 R_1 (\Sigma K)^{-1} \otimes R_2 (R_2' R_2)^{-1} R_2', \]
\[ A_3 = A_1 - (\Sigma K)^{-1} R_1 (\Sigma K)^{-1} R_1' - 1 R_1 (\Sigma K)^{-1} \otimes R_2 (R_2' R_2)^{-1} R_2', \]
\[ A_4 = A_1 - R_1 (R_1 R_1')^{-1} R_1 (\Sigma K)^{-1} \otimes R_2 (R_2' R_2)^{-1} R_2', \]
\[ \Sigma_{ij} = A_i A_j', i = 1, 2, 3, 4, j = 1, 2, 3, 4, \]
\[ \mu_2 = - (\Sigma K)^{-1} R_1 (\Sigma K)^{-1} R_1' - 1 \theta_0 (R_2' R_2)^{-1} R_2', \]
\[ \mu_3 = - \Sigma^{-1} R_1 (\Sigma K)^{-1} \theta_0 (R_2' R_2)^{-1} R_2', \mu_4 = - R_1 (R_1 R_1')^{-1} \theta_0 (R_2' R_2)^{-1} R_2'. \]

**Theorem 3.3.** If Assumption 1 holds along with \((3.3)\), we have
\[ \left( n^{\frac{1}{2}} \left( \bar{B}_1 - B \right)' , n^{\frac{1}{2}} \left( \bar{B}_2 - B \right)' , n^{\frac{1}{2}} \left( \bar{B}_3 - B \right)' , n^{\frac{1}{2}} \left( \bar{B}_4 - B \right)' \right) \xrightarrow{d_{n \to \infty}} \eta \]
where
\[ \eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} \sim N_{4p \times q} \begin{pmatrix} 0 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \Sigma_{34} \\ \Sigma_{41} & \Sigma_{42} & \Sigma_{43} & \Sigma_{44} \end{pmatrix}. \tag{3.5} \]

The proof of this theorem is given in the Appendix. Since the sequence of local alternative includes as a special case the restriction, one deduces the following corollary.

**Corollary 3.1.** If Assumption 1 holds and \( R_1 B R_2 = \theta \), we have
\[ n^{\frac{1}{2}} \left( \left( \bar{B}_1 - B \right)' , \left( \bar{B}_2 - B \right)' , \left( \bar{B}_3 - B \right)' , \left( \bar{B}_4 - B \right)' \right) \xrightarrow{d_{n \to \infty}} \left( \eta_1', \zeta_2', \zeta_3', \zeta_4' \right)' \]
where
\[ \begin{pmatrix} \eta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{pmatrix} \sim N_{4p \times q} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} & \Sigma_{14} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} & \Sigma_{24} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} & \Sigma_{34} \\ \Sigma_{41} & \Sigma_{42} & \Sigma_{43} & \Sigma_{44} \end{pmatrix}. \tag{3.6} \]

The proof follows directly from Theorem \((3.3)\) by taking \( \theta_0 = 0 \).

### 3.3 Asymptotic Distributional Risk

Asymptotic Distributional Risk (ADR) is one of the important statistical tools to compare different estimators. In this subsection, we derive ADR of the UE and that of any member of the proposed class of the restricted estimators, i.e. ADR of \( \bar{B}_1 \) and \( \bar{B}(\Sigma) \). Recall that,
if $\sqrt{n}(\hat{\theta} - \theta) \overset{d}{\to} U$, where $\hat{\theta}$, $\theta$ and $U$ are matrices. The ADR is defined as

$$\text{ADR}(\hat{\theta}, \theta; W) = \text{E}[\text{tr}(U' W U)]$$

where $W$ is a weighting matrix. For more details about the ADR, we refer for example to Saleh (2006), Chen and Nkurunziza (2015, 2016) and references therein. To introduce some notations, let $C_3(Q_0) = Q_0^{-1} R_1^{-1} (R_4 Q_0^{-1} R_1^{-1})^{-1}$, $C_4 = (R_2 R_2)^{-1} R_2'$, $J_1(Q_0) = C_3(Q_0) R_1 Q_0^{-1}$ and $J = R_2 C_4$.

**Theorem 3.4.** Suppose that the conditions of Theorem 3.2 hold, then

$$\text{ADR}(\hat{\dot{B}}, B; W) = \text{tr}((W \otimes I_q)(A_1 \Lambda A_1')),$$

$$\text{ADR}(\hat{\ddot{B}}, B, W) = \text{ADR}(\hat{\dot{B}}, B, W) - f_1(Q_0) + (\text{vec}(\theta_0))' F_1(Q_0) \text{vec}(\theta_0),$$

with

$$f_1(Q_0) = \text{tr}(((J_1'(Q_0) W \otimes J) \otimes I_{pq}) \text{vec}((\Sigma K)^{-1} \otimes I_q))(\text{vec}(\Lambda))'$$

and

$$F_1(Q_0) = C_3(Q_0) W C_3(Q_0) \otimes (R_2 R_2)^{-1}. $$

The proof of this theorem follows from Theorem 3.1. For the convenience of the reader, it is also outlined in the Appendix.

### 3.4 Risk Analysis

In this section, we compare $\text{ADR}(\hat{\dot{B}}, \Sigma, B, W)$ and $\text{ADR}(\hat{\dot{B}}, B; W)$ in order to evaluate the relative performance of $\hat{\dot{B}}, \Sigma$ and $\hat{\dot{B}}, B$. To simply some notations, for a given symmetric matrix $A$, let $\text{ch}_{\min}(A)$ and $\text{ch}_{\max}(A)$ be, respectively, the smallest and largest eigenvalues of $A$.

**Theorem 3.5.** Suppose that the conditions of Theorem 3.4 hold. If $||\theta_0||^2 < \frac{f_1(Q_0)}{\text{ch}_{\max}(F_1(Q_0))}$, then $\text{ADR}(\hat{\ddot{B}}, B, W) \leq \text{ADR}(\hat{\dot{B}}, B; W)$. If $||\theta_0||^2 > \frac{f_1(Q_0)}{\text{ch}_{\min}(F_1(Q_0))}$, then $\text{ADR}(\hat{\ddot{B}}, B, W) > \text{ADR}(\hat{\dot{B}}, B; W)$.

The proof of this theorem is given in the Appendix.

**Remark 3.1.** Since $\text{ADR}(\hat{\dot{B}}, \Sigma, B, W)$ and $\text{ADR}(\hat{\dot{B}}, B; W)$ are positive real numbers, $\text{ADR}(\hat{\ddot{B}}, B, W) \leq \text{ADR}(\hat{\dot{B}}, B; W)$ iff $\text{ADR}(\hat{\dot{B}}, B; W)/\text{ADR}(\hat{\ddot{B}}, \Sigma, B, W) \geq 1$. This ratio is known as the mean squares relative efficiency (RE). In presenting the simulation results, we compare the estimators by using the RE.
4 Concluding Remarks

In this paper, we study the asymptotic properties of the UE and the restricted estimators of the regression coefficients of multivariate regression model with measurement errors, when the coefficients may satisfy some restrictions. In comparison with the findings in literature, we generalize Proposition A.10 and Corollary A.2 in Chen and Nkurunziza (2016). Further, we generalize Theorem 4.1 of Jain et al. (2011) in three ways. First, we derive the joint asymptotic distribution between the UE and any member of the class of the restricted estimators under the restriction. Recall that, in the quoted paper, only the marginal asymptotic normality is derived under the restriction. Second, we derive the joint asymptotic normality between the UE and any member of the class of the restricted estimators under the sequence of local alternative restrictions. Third, we establish the joint asymptotic distribution between the UE and the three restricted estimators, given in Jain et al. (2011), under the restriction and under the sequence of local alternative restrictions. Further, we establish the ADR of the UE and the ADR of any member of the class of restricted estimators under the sequence of local alternative restrictions. We also study the risk analysis and establish that the restricted estimators perform better than the unrestricted estimator in the neighborhood of the restriction.

Acknowledgement

The authors would like to acknowledge the financial support received from the Natural Sciences and Engineering Research Council of Canada (NSERC).

A Some technical results

In this appendix, we give technical results and proofs which are underlying the established results. The following lemma is useful in establishing the asymptotic distributions.

Lemma A.1. Let $Y$ be a $p \times q$ random matrix and $Y \sim N_{p \times q}(O, \Lambda)$, with $\Lambda$ a $pq \times pq$ matrix. For $j = 1, 2, \ldots, m$, let $\kappa_j$ and $\alpha_j$ be $p \times p$-nonrandom matrices, let $\iota_j$ and $\beta_j$ be $q \times q$-nonrandom matrices, and let $\varphi_j$ be $p \times q$-nonrandom matrices. Then
for some algebraic computations, this completes the proof.

Be sequences of random matrices such that

$$\begin{pmatrix}
\kappa_1 Y_{11} + \alpha_1 Y_{11} + \varrho_1 \\
\kappa_2 Y_{12} + \alpha_2 Y_{12} + \varrho_2 \\
\vdots \\
\kappa_m Y_{1m} + \alpha_m Y_{1m} + \varrho_m
\end{pmatrix} \sim \mathcal{N}_{mq \times p}
\begin{pmatrix}
\varrho_1 \\
\varrho_2 \\
\vdots \\
\varrho_m
\end{pmatrix}
\begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1m} \\
A_{21} & A_{22} & \cdots & A_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
A_{m1} & A_{m2} & \cdots & A_{mm}
\end{pmatrix},
$$

where $A_{ij} = (A_{ij})', i, j = 1, 2, \ldots, m$, and

$$A_{ij} = (\kappa_i \otimes \iota_j')\Lambda(\kappa_j' \otimes \iota_j) + (\kappa_i \otimes \iota_j')\Lambda(\alpha_j' \otimes \beta_j) + (\alpha_i \otimes \beta_j')\Lambda(\alpha_j' \otimes \beta_j).$$

Proof. We have

$$\text{vec}
\begin{pmatrix}
\kappa_1 Y_{11} + \alpha_1 Y_{11} + \varrho_1 \\
\kappa_2 Y_{12} + \alpha_2 Y_{12} + \varrho_2 \\
\vdots \\
\kappa_m Y_{1m} + \alpha_m Y_{1m} + \varrho_m
\end{pmatrix} =
\begin{pmatrix}
\kappa_1 \otimes \iota_1' + \alpha_1 \otimes \beta_1' \\
\kappa_2 \otimes \iota_2' + \alpha_2 \otimes \beta_2' \\
\vdots \\
\kappa_m \otimes \iota_m' + \alpha_m \otimes \beta_m'
\end{pmatrix}
\text{vec}(Y) +
\begin{pmatrix}
\text{vec}(\varrho_1) \\
\text{vec}(\varrho_2) \\
\vdots \\
\text{vec}(\varrho_m)
\end{pmatrix},$$

then the rest of the proof follows from the properties of normal random vectors along with some algebraic computations, this completes the proof. \hfill \square

Note that this result is more general than Corollary A.2 in Chen and Nkurunziza (2016). By using this lemma, we establish the following lemma, which is more general than Proposition A.10 and Corollary A.2 in Chen and Nkurunziza (2016). The established lemma is particularly useful in deriving the joint asymptotic normality between $\hat{B}_1$, $\hat{B}_2$, $\hat{B}_3$ and $\hat{B}_4$.

**Lemma A.2.** For $j = 1, 2, \ldots, m$, let \{\kappa_{jn}\}_{n=1}^{\infty}, \{\iota_{jn}\}_{n=1}^{\infty}, \{\alpha_{jn}\}_{n=1}^{\infty}, \{\beta_{jn}\}_{n=1}^{\infty}, \{\varrho_{jn}\}_{n=1}^{\infty}$ be sequences of random matrices such that $\kappa_{jn} \xrightarrow{P} \kappa_j, \iota_{jn} \xrightarrow{P} \iota_j, \alpha_{jn} \xrightarrow{P} \alpha_j, \beta_{jn} \xrightarrow{P} \beta_j, \varrho_{jn} \xrightarrow{P} \varrho_j$, where, for $j = 1, 2, \ldots, m$, $\kappa_j, \iota_j, \alpha_j$ and $\beta_j, \varrho_j$ are non-random matrices as defined in Lemma A.1. If a sequence of $p \times q$ random matrices \{\textbf{Y}_n\}_{n=1}^{\infty} is such that $\textbf{Y}_n \xrightarrow{d} \textbf{Y} \sim \mathcal{N}_{pq \times q}(0, \Lambda)$, where $\Lambda$ is a $pq \times pq$ matrix. We have

$$\begin{pmatrix}
\kappa_{1n} Y_{n11} + \alpha_{1n} Y_{n11} + \varrho_{1n} \\
\kappa_{2n} Y_{n22} + \alpha_{2n} Y_{n22} + \varrho_{2n} \\
\vdots \\
\kappa_{mn} Y_{nmn} + \alpha_{mn} Y_{nmn} + \varrho_{mn}
\end{pmatrix} \xrightarrow{d} U \sim \mathcal{N}_{mq \times p}(\varrho, A)$$

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with \( \varrho = \begin{pmatrix} \varrho_1 \\ \varrho_2 \\ \vdots \\ \varrho_m \end{pmatrix}, \quad A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}, \)

where \( A_{ij}, i = 1, 2, \ldots, m; j = 1, 2, \ldots, m \) are as defined in Lemma \( \text{A.1} \).

Proof. We have

\[
\begin{pmatrix}
\vec{\kappa}_1 Y_{n\ell_1 n} + \alpha_1 Y_{n\beta_1 n} + \varrho_{1n} \\
\vec{\kappa}_2 Y_{n\ell_2 n} + \alpha_2 Y_{n\beta_2 n} + \varrho_{2n} \\
\vdots \\
\vec{\kappa}_m Y_{n\ell_m n} + \alpha_m Y_{n\beta_m n} + \varrho_{mn}
\end{pmatrix}
= \begin{pmatrix}
\kappa_1 \otimes \ell_1' \ + \ \alpha_1 \otimes \beta_1' \\
\kappa_2 \otimes \ell_2' + \alpha_2 \otimes \beta_2' \\
\vdots \\
\kappa_m \otimes \ell_m' + \alpha_m \otimes \beta_m'
\end{pmatrix} \vec{Y}_n
+ \begin{pmatrix}
\vec{\varrho}_{1n} \\
\vec{\varrho}_{2n} \\
\vdots \\
\vec{\varrho}_{mn}
\end{pmatrix},
\]

where \( \vec{Y}_n \xrightarrow{d, n \to \infty} \vec{Y} \sim \mathcal{N}_{pq}(0, \Lambda), \)

and

\[
\begin{pmatrix}
\kappa_1 \otimes \ell_1' + \alpha_1 \otimes \beta_1' \\
\kappa_2 \otimes \ell_2' + \alpha_2 \otimes \beta_2' \\
\vdots \\
\kappa_m \otimes \ell_m' + \alpha_m \otimes \beta_m'
\end{pmatrix}
\xrightarrow{p, n \to \infty}
\begin{pmatrix}
\kappa_1 \otimes \ell_1' + \alpha_1 \otimes \beta_1' \\
\kappa_2 \otimes \ell_2' + \alpha_2 \otimes \beta_2' \\
\vdots \\
\kappa_m \otimes \ell_m' + \alpha_m \otimes \beta_m'
\end{pmatrix}.
\]

Then, by using Slutsky’s theorem, we have

\[
\begin{pmatrix}
\vec{\kappa}_1 Y_{n\ell_1 n} + \alpha_1 Y_{n\beta_1 n} + \varrho_{1n} \\
\vec{\kappa}_2 Y_{n\ell_2 n} + \alpha_2 Y_{n\beta_2 n} + \varrho_{2n} \\
\vdots \\
\vec{\kappa}_m Y_{n\ell_m n} + \alpha_m Y_{n\beta_m n} + \varrho_{mn}
\end{pmatrix}
\xrightarrow{d, n \to \infty}
\begin{pmatrix}
\kappa_1 Y_{\ell_1} + \alpha_1 Y_{\beta_1} + \varrho_{1} \\
\kappa_2 Y_{\ell_2} + \alpha_2 Y_{\beta_2} + \varrho_{2} \\
\vdots \\
\kappa_m Y_{\ell_m} + \alpha_m Y_{\beta_m} + \varrho_{m}
\end{pmatrix}
\]

and then

\[
\begin{pmatrix}
\kappa_1 Y_{\ell_1} + \alpha_1 Y_{\beta_1} + \varrho_{1} \\
\kappa_2 Y_{\ell_2} + \alpha_2 Y_{\beta_2} + \varrho_{2} \\
\vdots \\
\kappa_m Y_{\ell_m} + \alpha_m Y_{\beta_m} + \varrho_{m}
\end{pmatrix}.
Then, the proof follows directly from Lemma A.1.

Proof of Theorem 3.2. We have

\[
\begin{pmatrix}
\kappa_1 \nu n^1 n + \alpha_1 \nu n^2 n + \varrho_1 n \\
\kappa_2 \nu n^2 n + \alpha_2 \nu n^2 n + \varrho_2 n \\
\vdots \\
\kappa_m \nu n^m n + \alpha_m \nu n^m n + \varrho_m n
\end{pmatrix}
\xrightarrow{\frac{d}{n \to \infty}}
\begin{pmatrix}
\kappa_1 \nu \nu_1 + \alpha_1 \nu \beta_1 + \varrho_1 \\
\kappa_2 \nu \beta_1 + \alpha_2 \nu \beta_2 + \varrho_2 \\
\vdots \\
\kappa_m \nu \beta_m + \alpha_m \nu \beta_m + \varrho_m
\end{pmatrix}
\equiv U.
\]

Hence, the proof follows directly from Lemma A.1.

Corollary A.1. Suppose that the conditions Lemma A.2 hold. We have

\[
(Y', (Y + \alpha_2 Y \beta_2 + \varrho_2)^') \xrightarrow{\frac{d}{n \to \infty}} (Y', (Y + \alpha_2 Y \beta_2 + \varrho_2)^'),
\]

with

\[
\begin{pmatrix}
Y \\
Y + \alpha_2 Y \beta_2 + \varrho_2
\end{pmatrix}
\sim N_{2q \times p}
\begin{pmatrix}
0 \\
\varrho_2
\end{pmatrix},
\begin{pmatrix}
V_{11} & V_{12} \\
V_{21} & V_{22}
\end{pmatrix}
\]

where \(V_{11} = \Lambda; \quad V_{12} = \Lambda + \Lambda(\alpha_2 \otimes \beta_2); \quad V_{21} = (V_{12})'; \quad V_{22} = (I_{pq} + \alpha_2 \otimes \beta_2) \Lambda(I_{pq} + \alpha_2 \otimes \beta_2).\)

The proof follows directly from Lemma A.2 by taking \(m = 2, \kappa_{jn} = I_p, \nu_{jn} = I_q, \alpha_{1n} = 0, \beta_{1n} = 0\) and \(\varrho_{1n} = 0.\)

Proof of Theorem 3.2. We have

\[
(\hat{B} - B) = (\hat{B}_1 - B) + (\hat{\Sigma} - R)R_1[R_1(\hat{\Sigma} - R)]^{-1}(\theta - R_1 \hat{B}_1 R_2)(R_2 R_2)^{-1}R_2
\]

\[
= (\hat{B}_1 - B) - G_{2n}(W_0)R_1(\hat{B}_1 - B)R_2 P_n + G_{2n}(W_0)(\theta - R_1 \hat{B}_1 R_2)P_n.
\]

with \(G_{2n}(W_0) = (\hat{\Sigma} - R)(R_1(\hat{\Sigma} - R)]^{-1}(\theta - R_1 \hat{B}_1 R_2)\) and \(P_n = (R_2^{-1} R_2)^{-1}R_2.\) Then, since

\[
R_1 B R_2 = \theta + \theta_0 / \sqrt{n},
\]

this last relation gives

\[
n \frac{1}{2} (\hat{B} - B) = n \frac{1}{2} (\hat{B}_1 - B) - G_{2n}(W_0)R_1(n \frac{1}{2} (\hat{B}_1 - B) - G_{2n}(W_0)\theta_0 P_n).
\]

Hence,

\[
\begin{pmatrix}
\sqrt{n}(\hat{B}_1 - B) \\
\sqrt{n}(\hat{B} - B)
\end{pmatrix}
\Delta
\begin{pmatrix}
n \frac{1}{2} (\hat{B}_1 - B) \\
n \frac{1}{2} (\hat{B}_1 - B) - G_{2n}(W_0)R_1(n \frac{1}{2} (\hat{B}_1 - B) - G_{2n}(W_0)\theta_0 P_n)
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
0 \\
-G_{2n}(W_0)\theta_0 P_n
\end{pmatrix}.
\]
Note that \( G_{2n} \xrightarrow{P} G_2(Q_0) \) and \( P_n \xrightarrow{P} P \), with \( G_2(Q_0) = (Q_0)^{-1}R'_1(R_1(Q_0)^{-1}R'_1)^{-1} \) and \( P = (R_2R_2)^{-1}R'_2 \). Further, let \( \alpha = G_2(Q_0)R_1 \), let \( \beta = R_2P \) and let \( \varrho = G_2(Q_0)\theta_0P \). By using Corollary \([4.1]\) we have

\[
\left( \begin{array}{c}
\sqrt{n}(\hat{B}_1 - B) \\
\sqrt{n}(B(\Sigma) - B)
\end{array} \right) \xrightarrow{d} \left( \begin{array}{c}
Y \\
Y + \alpha Y \beta + \varrho
\end{array} \right)
\]

\[
\sim N_{2q \times P} \left( \begin{array}{c}
0 \\
\mu(Q_0)
\end{array} \right), \left( \begin{array}{c}
\Sigma_{11} \\
\Sigma_{12}(Q_0)
\end{array} \right)
\]

this completes the proof. \( \square \)

**Proof of Theorem 2.4** The proof follows from Theorem 3.2 by taking \( \mu(Q_0) = 0 \). \( \square \)

**Proof of Theorem 3.3** From \([2.7], [2.8] \) and \([2.9] \), we have

\[
n^{\frac{1}{2}}(\hat{B}_2 - B) = n^{\frac{1}{2}}(\hat{B}_1 - B) - n^{\frac{1}{2}}G_{2n}R_1(\hat{B}_1 - B)R_2P_n + n^{\frac{1}{2}}G_{2n}(\theta - R_1BR_2)P_n
\]

\[
n^{\frac{1}{2}}(\hat{B}_3 - B) = n^{\frac{1}{2}}(\hat{B}_1 - B) + n^{\frac{1}{2}}G_{3n}(\theta - R_1B_1R_2)P_n
\]

\[
n^{\frac{1}{2}}(\hat{B}_4 - B) = n^{\frac{1}{2}}(\hat{B}_1 - B) - n^{\frac{1}{2}}G_{4n}R_1(\hat{B}_1 - B)R_2P_n + n^{\frac{1}{2}}G_{4n}(\theta - R_1BR_2)P_n.
\]

with \( G_{2n} = (SK_X)^{-1}R'_1[R_1(SK_X)^{-1}R'_1]^{-1} \), \( S = X'X \), \( G_{3n} = S^{-1}R'_1[R_1S^{-1}R'_1]^{-1} \), \( G_{4n} = R'_1[R_1R'_1]^{-1} \), \( P_n = (R_2R_2)^{-1}R'_2 \). Then, since \( R_1BR_2 = \theta + \theta_0/\sqrt{n} \), we have

\[
n^{\frac{1}{2}}(\hat{B}_2 - B) = n^{\frac{1}{2}}(\hat{B}_1 - B) - n^{\frac{1}{2}}G_{2n}R_1(\hat{B}_1 - B)R_2P_n + G_{2n}\theta_0P_n
\]

\[
n^{\frac{1}{2}}(\hat{B}_3 - B) = n^{\frac{1}{2}}(\hat{B}_1 - B) - n^{\frac{1}{2}}G_{3n}R_1(\hat{B}_1 - B)R_2P_n + G_{3n}\theta_0P_n
\]

\[
n^{\frac{1}{2}}(\hat{B}_4 - B) = n^{\frac{1}{2}}(\hat{B}_1 - B) - n^{\frac{1}{2}}G_{4n}R_1(\hat{B}_1 - B)R_2P_n + G_{4n}\theta_0P_n.
\]

Therefore,

\[
\left( \begin{array}{c}
n^{\frac{1}{2}}(\hat{B}_1 - B) \\
n^{\frac{1}{2}}(\hat{B}_2 - B) \\
n^{\frac{1}{2}}(\hat{B}_3 - B) \\
n^{\frac{1}{2}}(\hat{B}_4 - B)
\end{array} \right) \xrightarrow{d} \eta_1 \sim N_{p \times q}(O, \Sigma_{11}).
\]

\[
G_{2n} \xrightarrow{P} G_2 = (\Sigma K)^{-1}R'_1(R_1(\Sigma K)^{-1}R'_1)^{-1}, \quad G_{3n} \xrightarrow{P} G_3 = \Sigma^{-1}R'_1(R_1\Sigma^{-1}R'_1)^{-1},
\]

\[
G_{4n} \xrightarrow{P} G_4 = R'_1(R_1R'_1)^{-1}, \quad P_n \xrightarrow{P} P = (R_2R_2)^{-1}R_2. \quad \text{Therefore, by using Lemma \([4.2]\) we get the statement of the proposition.} \quad \square
\]
Proof of Theorem 3.4. The first statement follows from Theorem 3.2. Further, we have,

\[
\text{ADR}\left(\hat{B}(\Sigma), B, W\right) = \text{tr}\left[(W \otimes I_q)E\left[\text{vec}(\eta^*)\left(\text{vec}(\eta^*)\right)\right]\right]
\]

\[
= \text{tr}((W \otimes I_q)(\Sigma_{22}(Q_0))) + \text{tr}\left(\mu'(Q_0)W\mu(Q_0)\right).
\]

This gives

\[
\text{ADR}(\hat{B}(\Sigma), B, W) = \text{ADR}(\hat{B}_1, B; W) - \text{tr}((W \otimes I_q)(A_1A(J_1(Q_0) \otimes J))
\]

\[-\text{tr}((W \otimes I_q)((J_1(Q_0) \otimes J)A_1A_1^\prime) + \text{tr}((W \otimes I_q)((J_1(Q_0) \otimes J)A(J_1(Q_0) \otimes J))
\]

\[+ \text{tr}((C_3C_4') \otimes (C_3'Q_0W)C_3(Q_0))\text{vec}(\theta_0)(\text{vec}(\theta_0))').
\]

Further, one can verify that \(\text{vec}(J_1(Q_0) \otimes J) = \text{vec}((\Sigma K)^{-1} \otimes I_q)\). Then, the rest of the proof follows from some algebraic computations. \(\square\)

Proof of Theorem 3.5. From Theorem 3.4 we have

\[
\text{ADR}(\hat{B}(\Sigma), B, W) = \text{ADR}(\hat{B}_1, B, W) - f_1(Q_0) + (\text{vec}(\theta_0))^\prime F_1(Q_0)\text{vec}(\theta_0). \quad (A.1)
\]

Note that \(f_1(Q_0) \geq 0\) and obviously, if \(f_1(Q_0) = 0\), \(\text{ADR}(\hat{B}(\Sigma), B; W) > \text{ADR}(\hat{B}_1, B, W)\) provided that \((\text{vec}(\theta_0))^\prime F_1(Q_0)\text{vec}(\theta_0) > 0\). Thus, we only consider the case where \(f_1(Q_0) > 0\). From (A.1), \(\text{ADR}(\hat{B}(\Sigma), B; W) > \text{ADR}(\hat{B}_1, B, W)\) if and only if

\[
-f_1(Q_0) + (\text{vec}(\theta_0))^\prime F_1(Q_0)\text{vec}(\theta_0) > 0.
\]

If \(f_1(Q_0) < (\text{vec}(\theta_0))^\prime F_1(Q_0)\text{vec}(\theta_0)\), we have

\[
\frac{f_1(Q_0)}{(\text{vec}(\theta_0))^\prime F_1(Q_0)\text{vec}(\theta_0)} < 1, \text{ and } (\text{vec}(\theta_0))^\prime \text{vec}(\theta_0) \frac{f_1(Q_0)}{(\text{vec}(\theta_0))^\prime F_1(Q_0)\text{vec}(\theta_0)} < ||\theta_0||^2.
\]

Further, since \(f_1(Q_0) > 0\), we have

\[
\frac{(\text{vec}(\theta_0))^\prime F_1(Q_0)\text{vec}(\theta_0)}{(\text{vec}(\theta_0))^\prime \text{vec}(\theta_0)f_1} > \frac{1}{||\theta_0||^2}. \quad (A.2)
\]

Further, by using Courant Theorem, we have

\[
\text{ch}_{\text{min}}(F_1(Q_0)) < \frac{(\text{vec}(\theta_0))^\prime F_1(Q_0)\text{vec}(\theta_0)}{(\text{vec}(\theta_0))^\prime \text{vec}(\theta_0)} < \text{ch}_{\text{max}}(F_1(Q_0)). \quad (A.3)
\]

Therefore, for the inequality in (A.2) to hold, it suffices to have

\[
\frac{1}{||\theta_0||^2} < \frac{\text{ch}_{\text{min}}(F_1(Q_0))}{f_1(Q_0)}.
\]

That is if \(||\theta_0||^2 > \frac{f_1(Q_0)}{\text{ch}_{\text{min}}(F_1(Q_0))}\), we have \(\text{ADR}(\hat{B}(\Sigma), B, W) > \text{ADR}(\hat{B}_1, B; W)\). Further, if \(f_1(Q_0) > (\text{vec}(\theta_0))^\prime F_1(Q_0)\text{vec}(\theta_0)\), by using (A.3), we establish the condition that if \(||\theta_0||^2 < \frac{f_1(Q_0)}{\text{ch}_{\text{max}}(F_1(Q_0))}\), then \(\text{ADR}(\hat{B}(\Sigma), B, W) < \text{ADR}(\hat{B}_1, B; W)\), this completes the proof. \(\square\)
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