Uncertainty Principles for Signal Concentrations

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Abstract—Uncertainty Principles for concentration of signals into truncated subspaces are considered. The “classic” Uncertainty Principle is explored as a special case of a more general operator framework. The time-bandwidth concentration problem is shown as a similar special case. A spatial concentration of radio signals example is provided, and it is shown that an Uncertainty Principle exists for concentration of single-frequency signals for regions in space. We show that the uncertainty is related to the volumes of the spatial regions.

I. INTRODUCTION

Signal processing, at its most fundamental level, concerns the extraction of information (signal) from a system under various constraints. Typically, communication engineers consider the signal as being embedded in noise, and although the signal may have spatial, temporal or bandwidth constraints, they are usually concerned with noise as the limiting factor. In the following work, we will consider a fundamental limit to signal extraction without impediments due to noise. We shall pose the following question:

Given a signal (wavefield) which has energy in one volume $V_A$, how well can we constraint that signal to have energy in another volume $V_B$?

This question motivates examination of Uncertainty Principles (UP’s). We may regard uncertainty as a basic limit on signal information extraction, without requiring a model for noise – since any noise can only further hinder our efforts. Uncertainty Principles have gained great popularity since Heisenberg. The famous example is Heisenberg’s Uncertainty Principle: it is impossible to exactly measure the location and momentum of a particle simultaneously. This is a special case of a more general framework, which may be applied (via various Hilbert space techniques) to a range of scenarios.

In a communication theory setting, a similar UP has been well known: that a signal cannot be arbitrarily confined in both time and frequency. The reader is directed to [1] for a discussion. The work of [2]–[4] has formalised this result, although without explicit reference to the operator theoretic nature of the problem. The time-bandwidth concept of a temporal signal provides us with a great deal of intuition and we shall draw heavily on this classic works.

There is a significant distinction between essential dimensionality results such as [1], [6] and uncertainty results [7], [8]. Dimensionality results (eg. $2WT$) may be seen as a counting of the number of degrees of freedom a signal (or function) has within a particular set of constraints – much like the rank of a matrix. Uncertainty tells us to what extent a signal will achieve all constraints, and how uniquely that solution is specified by the constraints.

The remainder of this paper is arranged as follows: In Section II we collate classic results in uncertainty and formulate UP’s in terms of Operator Theoretic forms. Section III develops an Uncertainty Principle for communication between volumes with a particular form of operator channel. We provide a free-space example. Conclusions are drawn, and proofs are consigned to the appendix.

II. UNCERTAINTY PRINCIPLES: A REVIEW

Before explaining the Uncertainty Principle, we develop some relevant notation.

A. Notation

Following Selig [7], let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \| \equiv \langle \cdot, \cdot \rangle^{1/2}$. Further, let $A$ be a linear operator with domain $D(A) \subseteq \mathcal{H}$ and range in $\mathcal{H}$. We then define the normalised expectation value, $\tau_A(f)$ and standard deviation (uncertainty), $\sigma_A(f)$ of the operator $A$ with respect to $f \in D(A)$ to be [7]

$$\tau_A(f) \equiv \frac{\langle Af, f \rangle}{\langle f, f \rangle} \quad (1)$$
$$\sigma_A(f) \equiv \| A - \tau_A(f) f \|. \quad (2)$$

The adjoint of operator $A$, $A^\dagger$ is defined using the following equation [9]

$$\langle Ax, y \rangle = \langle x, A^\dagger y \rangle \quad \forall x \in D(A), y \in D(A^\dagger). \quad (3)$$

Furthermore, $A$ is said to be Hermitian or self-adjoint if $A = A^\dagger$. For a self-adjoint operator

$$\langle Ax, y \rangle = \langle x, Ay \rangle \quad \forall x, y \in D(A). \quad (4)$$

Any operator that obeys equation 4 is said to be symmetric.

Given two linear operators $A$ and $B$ with domains $D(A)$ and $D(B)$ respectively, the commutator is defined as

$$[A, B] \equiv AB - BA \quad (5)$$

and the anti-commutator is defined as

$$[A, B]_+ \equiv AB + BA \quad (6)$$

with domains $D(AB) \cap D(BA)$ for either one. Operators $A$ and $B$ are said to commute with each other if $[A, B] = 0$. Otherwise they are called non-commutative operators.
Also, let $L^2[\mathbb{R}]$ be the set of all square integrable functions defined on the real line with norm $\|f\|_2 = (\langle f, f \rangle)^{1/2}$. Here the inner product $\langle \cdot, \cdot \rangle$ is defined as
\[
\langle f, g \rangle = \int_{-\infty}^{\infty} f(t)\overline{g(t)}dt.
\]
(7)
Consider $\|f\|_2^2$ as the energy of the function $f \in L^2[\mathbb{R}]$. The angle between two non-zero functions $f$ and $g$ is defined as
\[
\theta(f, g) = \cos^{-1}\frac{\Re \{\langle f, g \rangle\}}{\|f\|_2 \|g\|_2}.
\]
(8)
We also define
\[
\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx
\]
(9)
to be the Fourier transform of $f(x)$ whenever this integral exists.

B. Classic Uncertainty Principle

The classical uncertainty principle states that the values of two non-commuting observers such as position and momentum cannot be precisely determined in any quantum state. That is, the standard deviation of two non-commuting operators cannot be made arbitrarily small simultaneously. The following theorem is a general statement of this phenomenon.

**Theorem 1 (theorem 3.4, [7]):** If $A$ and $B$ are two symmetric operators on a Hilbert space $\mathcal{H}$, then
\[
\|A - \sigma_A(f)\| \times \|B - \sigma_B(f)\| \geq \frac{1}{2}\left\{\|\langle A, B \rangle f, f \rangle\|^2 + \|\langle A - \sigma_A(f), B - \sigma_B(f) \rangle f, f \rangle\|^2\right\}^{1/2}
\]
for all $f \in D(AB) \cap D(BA)$ and all $a, b \in \mathbb{R}$. Equality holds precisely when $(A - \sigma_A(f))f$ and $(B - \sigma_B(f))f$ are scalar multiples of one another.

A special case of this theorem bounds the standard deviation of two non-commuting operators as explained in the following corollary which we get by substituting $a = \sigma_A(f)$ and $b = \sigma_B(f)$ in theorem 1.

**Corollary 1:** If $A$ and $B$ are two symmetric operators on a Hilbert space $\mathcal{H}$, then
\[
\sigma_A(f)\sigma_B(f) \geq \frac{1}{2} \sqrt{\|\langle A, B \rangle f, f \rangle^2}
\]
(10)
for all $f \in D(AB) \cap D(BA)$ and all $a, b \in \mathbb{R}$. Equality holds precisely when $(A - \sigma_A(f))f$ and $(B - \sigma_B(f))f$ are scalar multiples of one another.

A special case of this corollary is the following Heisenberg Uncertainty Principle which states that a function and its Fourier transform cannot be arbitrarily localised.

**Theorem 2 (theorem 6.1, [7]):** Let $f \in L^2[\mathbb{R}]$, $\|f\|_2 = 1$ and set
\[
x_o = \int x|f(x)|^2dx \quad \omega_o = \int \omega|\hat{f}(\omega)|^2d\omega
\]
\[
\Delta x = \int (x - x_o)^2|f(x)|^2dx \quad \Delta \omega = \int (\omega - \omega_o)^2|\hat{f}(\omega)|^2d\omega
\]
where equality is attained iff
\[
f(x) = (r/\pi)^{1/4}e^{i\omega_o x}e^{-r(x - x_o)^2/2}
\]
for any $r > 0$.

Theorem 2 gives valuable insight into how localized a function can be in both time and frequency. If one defines $\Delta x$ and $\Delta \omega$ to be the measure of approximate time duration and bandwidth of the signal respectively, then Theorem 2 says that the product of time duration and bandwidth of a function is bounded from below by $\pi/2$; if the time-spread gets very small, the frequency-spread must be large and vice-versa.

Though a good qualitative tool, this is inadequate for the purposes of signal processing. Theorem 2 does not answer the question: given a bandlimited function (i.e. $\hat{f}(\omega) = 0$ for $\omega \notin [-\Omega, \Omega]$) how much of the energy of $f$ is ‘concentrated’ in finite duration of time. This would be useful in answering the question: “given a bandlimited channel, how much of the transmitted signal can a receiver measure over a finite period of time?”

C. Uncertainty principle for energy concentrations

Landau and Pollak [3] proposed that for the purposes of signal processing, a more relevant uncertainty principle should use sharper measures of concentrations in time and frequency than the ones used in Heisenberg’s principle [3]. To help derive their uncertainty principle [3] define $D = \{f : f \in L^2[\mathbb{R}], f(t) = 0 \forall |t| > T/2\}$ to be the class of all time-limited functions and $B = \{f : f \in L^2[\mathbb{R}], f(\omega) = 0 \forall |\omega| > \Omega\}$ to be the class of all band limited functions. Here, $\hat{f}(\omega)$ is the Fourier transform of $f(t)$ as defined in equation 9. Also, $T$, the time duration and $W = \Omega/2\pi$, the bandwidth are fixed for the remainder of this paper. It is easy to prove [2] that $D$ and $B$ are complete subspaces of $L^2[\mathbb{R}]$.

Define the projection operators $B : L^2[\mathbb{R}] \rightarrow B$ and $D : L^2[\mathbb{R}] \rightarrow D$ as follows
\[
Df(t) = \begin{cases} f(t), & |t| \leq T/2 \\ 0, & |t| > T/2 \end{cases}
\]
(12)
\[
Bf(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \hat{f}(\omega)e^{i\omega t}d\omega
\]
(13)
Using these operators we can calculate the fraction of energy, $\alpha^2$ of any function $f \in L^2[\mathbb{R}]$ in the finite duration of time $[-T/2, T/2]$,
\[
\alpha^2 = \frac{\|Df\|_2^2}{\|f\|_2^2}
\]
(14)
Similarly, we can calculate, $\beta^2$, the fraction of energy of a function in a finite bandwidth $[-\Omega, \Omega]$,
\[
\beta^2 = \frac{\|Bf\|_2^2}{\|f\|_2^2}
\]
(15)
[3] show that $\alpha$ and $\beta$ cannot be arbitrarily large. Specifically, they prove

**Theorem 3 (theorem 2, [3]):** Let $0 \leq \alpha, \beta \leq 1$. Then there exists a function $f \in L^2[\mathbb{R}]$, $\|f\|_2 = 1$ with $\|Df\|_2 = \alpha$ and $\|Bf\|_2 = \beta$ if and only if $(\alpha, \beta) \neq (0, 1)$ or $(1, 0)$ and
\[
\cos^{-1}\alpha + \cos^{-1}\beta \leq \cos^{-1}\sqrt{\lambda_0}
\]
where, $\lambda_0$ is the largest eigenvalue of the equation

$$\lambda \psi = B D \psi$$

This theorem constrains the possible values of $\alpha$ and $\beta$ because $\lambda_0 < 1$ [2]. So any function cannot have arbitrarily large fractions of energy in both a finite-time duration and a finite-frequency bandwidth. We will show in the next section that this theorem is a special case of a more general theorem just like the Heisenberg Principle is a special case of the classical uncertainty principle as alluded to by [3]. We show that this more general theory can be used to understand communication through arbitrary channels.

III. COMMUNICATION BETWEEN FINITE VOLUMES AND THE UNCERTAINTY PRINCIPLE

The derivations above are for a function defined on the real line and its Fourier transform, however the principle can be extended to arbitrary transforms defined on $\mathbb{R}^n$. The key property is that the subspaces $B$ and $D$ form nonzero minimum angles. Before deriving these results, we explain the physical model.

A. Physical Problem and Notation

Consider communication using scalar waves between two volumes $V_T$, $V_R \subset \mathbb{R}^3$, where $\mathbb{R}^3$ is the standard three dimensional Euclidian space. Let $L^2(V)$ be the space of all square integrable functions defined on volume $V \subset \mathbb{R}^3$ with the standard inner product. Following [10], assume there are sources $f \in L^2(\mathbb{R}^3)$ that generate waves $\tilde{f} \in L^2(\mathbb{R}^3)$ governed by some transformation $\Gamma: L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$:

$$\tilde{f} = \Gamma f.$$  

Here, $L^2(\mathbb{R}^3)$ is the set of all square integrable functions defined on $\mathbb{R}^3$ with the standard inner product. If we only have sources in the transmitting volume and can only measure waves within the receiving volume we will need to consider restrictions of $f$ and $\tilde{f}$ to $V_T$ and $V_R$ respectively. Let $\chi_V$ be the characteristic (truncation) operator:

$$\chi_V f(r) = \begin{cases} f(r) & r \in V, \\ 0 & \text{otherwise.} \end{cases}$$

We say that a function $f$ is $\epsilon$-concentrated in volume $V$ in some norm $\| \cdot \|$ if

$$\frac{\| \chi_V f \|^2}{\| f \|^2} \geq 1 - \epsilon^2.$$  

With this notation in place we can now look at the set of functions $D$ that a receiver can measure and the set of functions $B$ a transmitter can generate:

$$D = \{ f : f = \chi_{V_R} g, g \in L^2(\mathbb{R}^3) \}$$

$$B = \{ f : f = \Gamma \chi_{V_T} g, g \in L^2(\mathbb{R}^3) \}.$$  

If the spaces $D$ and $B$ are complete, we can define projection operators $D: L^2(\mathbb{R}^3) \rightarrow D = \chi_{V_R}$ and $B: L^2(\mathbb{R}^3) \rightarrow B$. Note that $B \neq \Gamma \chi_{V_T}$. We can also define an angle between these two subspaces as follows:

$$\theta(B, D) = \inf_{f \in B, g \in D} \theta(f, g).$$

B. Uncertainty Principle for arbitrary subspaces

We can again think of $\| f \|_2^2$ as the energy of a function. Then $\alpha^2 = \| Df \|_2^2 / \| f \|_2^2$ is the fraction of energy of $f$ in the space of receiver functions and $\beta^2 = \| Bf \|_2^2 / \| f \|_2^2$ is the fraction of the energy of $f$ in the space of transmitter functions. We can now prove the uncertainty principle that constrains the possible range of values that $\alpha$ and $\beta$ can take provided the subspaces $B$ and $D$ form a non-zero minimum angle.

Theorem 4: Let subspaces $B$ and $D$ form a non-zero minimum angle $\theta_0$ then

$$\cos^{-1} \alpha + \cos^{-1} \beta \geq \theta_0.$$  

This theorem has a very simple physical interpretation. If the space of all the functions that a transmitter can generate and the space of all the functions a receiver can receive form a non-zero minimum angle then there exist no functions that can have arbitrarily large fractions of energy in these two spaces of functions. We can calculate this minimum angle by calculating the norm of the operator $\| BD \|_2$ and this is the subject of our next theorem.

Theorem 5: The angle between two complete subspaces $B$ and $D$ with projection operators $B$ and $D$ is

$$\theta(B, D) = \cos^{-1} \| BD \|_2.$$  

C. General Uncertainty principle

We prove a slightly modified version of the general uncertainty theorem proved in [11]. In the following $L^1(V)$, $L^2(V)$ and $L^\infty(V)$ are spaces of functions defined on $V \subset \mathbb{R}^3$ with finite $L_1$ ($\| \cdot \|_1 = \int_V \cdot \, dV$), $L_2$ ($\| \cdot \|_2 = \| \cdot \|_2$) and $L_\infty$ ($\| \cdot \|_\infty = \sup_{V} \| \cdot \|$) norms.

Theorem 6 (Generalized Uncertainty): Suppose $\Gamma(f) = \tilde{f}$ and $f \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ and $\tilde{f} \in L^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ and satisfies

1) $\| f \|_2 = \alpha \| \tilde{f} \|_2$

2) $\| \tilde{f} \|_\infty \leq \beta \| f \|_1$.  

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Also, suppose $f$ is $\epsilon_f$-concentrated to $V_T$ in the $L_1$ norm and $f$ is $\epsilon_R$-concentrated to $V_R$ in the $L_2$ norm. Then,

$$|V_T||V_R|\alpha^2\beta^2 \geq (1 - \epsilon_f)^2(1 - \epsilon_R)^2$$

The theorem has a very simple physical interpretation for communication between finite volumes. Firstly, by requiring $\|f\|_2 = \alpha\|f\|_2$, we ensure that the energy of the received signal is proportional to that of the transmitting signal. So $\alpha$ determines the attenuation in the signal and we expect it to be greater than 1. Secondly, $\|f\|_\infty \leq \beta\|f\|_1$ can be thought of as a stability condition (i.e. bounded input gives bounded output). As a direct corollary to Theorem 6 we obtain an uncertainty principle for communicating volumes

**Corollary 2 (Volumetric Communication Uncertainty):** Let the transmitting volume $V_T$ be finite, and the conditions for Theorem 6 hold. Then

$$|V_T||V_R|\alpha^2\beta^2 \geq (1 - \epsilon_R)^2$$

Since $f$ must be perfectly concentrated in $V_T$ and so $\epsilon_f = 0$, i.e. the maximum fraction of energy that can be inside the receiving volume is bounded from above.

**IV. CONCLUSIONS**

We investigated Uncertainty Principles for the transmission of waves between volumes, using a generalised version of the classic Heisenberg UP. We have provided a framework for future UP developments where volumetric constraints on signals induce a limit to signal resolution. Application of operator-theoretic tools has provided a means to develop concet

**APPENDIX**

**Proof:** [outline theorem 1] Let $A, B, a, b$ and $f$ be as stated in the theorem. From the Cauchy-Schwarz inequality we have

$$2\|\langle B - b \rangle f\|\|\langle A - a \rangle f\| = 2|\Re\{\langle B - b \rangle f, \langle A - a \rangle f\}|^{1/2} + \Re\{\langle B - b \rangle f, \langle A - a \rangle f\}^2$$

$$|2\Re\{\langle B - b \rangle f, \langle A - a \rangle f\}^2|^{1/2}$$

Using symmetry of the operators and the fact that scalar multiplication commutes with all linear operators. Similarly

$$2\Re\{\langle B - b \rangle f, \langle A - a \rangle f\} = |\langle A - a I, B - b I \rangle f, f\|$$

Re-arranging (24), (25), (23) completes the proof.

**Proof:** [outline Theorem 2] The theorem is proved trivially by noting that $\sigma_A(f) = \Delta x$ and $\sigma_B(f) = \Delta \omega/2\pi$ [7]; where, $A f = -f$ and $B f = -if$ [7].

**Lemma 1:** Let $f, g, h \in L^2[\mathbb{R}^3]$. Then,

$$\theta(f,g) \leq \theta(f,h) + \theta(g,h).$$

**Proof:** [Proof of Lemma 1] Let $\hat{f} = f/\|f\|_2$. Then

$$\theta(f,g) = \cos^{-1}\left(\frac{\Re\{\langle \hat{f}, \hat{g} \rangle\}}{\|\hat{f}\|_2\|\hat{g}\|_2}\right) = \cos^{-1}\Re\{\langle \hat{f}, \hat{g} \rangle\}$$

Let $\theta(f,g) \neq 0$. Otherwise, there is nothing to prove. Also, let $S = \Span\{f, g\}$ be the space of all functions spanned by $f$ and $g$. Then this space is complete and we can write $h$ as [9]

$$h = h^\perp + h_\parallel,$$

where, $h_\parallel \in S$ and $h^\perp$ is orthogonal to both $f$ and $g$. Because $\|h_\parallel\|_2 \leq \|\hat{h}\|_2$ we have

$$\theta(f,h) = \theta(\hat{f}, \hat{h}) = \cos^{-1}\Re\{\langle \hat{f}, \hat{h} \rangle\} = \cos^{-1}\Re\{\langle \hat{f}, \hat{h} \rangle\}$$

$$= \theta(\hat{f}, \hat{h})$$

Similarly,

$$\theta(g,h) = \theta(\hat{g}, h_\parallel)$$

Now, if $\theta(\hat{g}, h_\parallel) = 0$ the proof is trivial as

$$\theta(f,g) = \theta(\hat{f}, \hat{g}) = \theta(\hat{f}, \hat{h}) \leq \theta(f,h) + \theta(g,h).$$

If $\theta(\hat{g}, h_\parallel) \neq 0$, let

$$\hat{h}_1 = h_\parallel/\|h_\parallel\|_2,$$

$$\hat{h}_2 = \frac{g - \hat{h}_1 \langle g, \hat{h}_1 \rangle}{\|g - \hat{h}_1 \langle g, \hat{h}_1 \rangle\|_2}$$

be two unit vectors that are orthogonal to each other and whose span is $S$. We can therefore write,

$$\hat{f} = a_1 \hat{h}_1 + a_2 \hat{h}_2$$

$$\hat{g} = b_1 \hat{h}_1 + b_2 \hat{h}_2$$

Where, $a_1 = a_1' + ia_1''$, $a_2 = a_2' + ia_2''$, $b_1 = b_1' + ib_1''$ and $b_2 = b_2' + ib_2''$ are complex numbers. From the orthogonality of $\hat{h}_1$ and $\hat{h}_2$ we have

$$\cos\theta(\hat{f}, \hat{h}_\parallel) = a_1'$$

$$\cos\theta(\hat{g}, h_\parallel) = b_1'$$

$$\cos\theta(\hat{f}, \hat{g}) = \Re\{a_1' b_1 + a_2' b_2\}$$

$$= a_1' b_1' + a_2' b_2' + a_1'' b_1'' + a_2'' b_2''$$

$a_1'$ is the complex conjugate of $a_1$. From orthogonality of $\hat{h}_1$ and $\hat{h}_2$ and $\hat{f}$ and $\hat{g}$ have unit norm

$$a_1'^2 + a_1''^2 + a_2'^2 + a_2''^2 = 1$$

$$b_1'^2 + b_1''^2 + b_2'^2 + b_2''^2 = 1$$

Consider $(a_1', a_2', b_1')$ and $(b_1', b_2', b_2')$ as two three dimensional vectors, the Cauchy-Schwarz inequality gives

$$\sqrt{a_1'^2 + a_2'^2 + b_1'^2} \geq |a_1' b_1' + a_2' b_2'|$$

Here, $|\cdot|$ denotes the absolute value of a real number.

$$a_1' b_1' + a_2' b_2' \geq -\sqrt{a_1'^2 + a_2'^2 + b_1'^2} \sqrt{a_1'^2 + a_2'^2 + b_1'^2} \sqrt{b_1'^2 + b_2'^2 + b_2'^2}$$

$$\geq \sqrt{a_1'^2 + a_2'^2 + b_1'^2} \sqrt{a_1'^2 + a_2'^2 + b_1'^2} \sqrt{b_1'^2 + b_2'^2 + b_2'^2}$$

2National ICT Australia is funded through the Australian Government’s Backing Australia’s Ability initiative, in part through the Australian Research Council.
Lemma 1. Therefore

$$\cos(\theta(\hat{f}, h)) + \cos(\theta(\hat{g}, h))$$

$$= \cos(\theta(\hat{f}, h)) \cos(\theta(\hat{g}, h)) - \sin(\theta(\hat{f}, h)) \sin(\theta(\hat{g}, h))$$

$$= a_1 b_1 - \sqrt{1 - a_1^2} \sqrt{1 - b_1^2}
\quad (41)$$

$$= a_1 b_1 - \sqrt{a_1^2 + a_2^2 + b_1^2 \sqrt{b_1^2 + b_2^2}}
\quad (42)$$

$$\leq a_1 b_1 + a_2 b_1' + a_2 b_2' + a_1 b_2'
\quad (43)$$

$$= \cos(\theta(\hat{f}, \hat{g}))$$

We get equation (41) from equations (34) and (35). Equations (38) and (39) are used to get (42) and finally we use inequality (40) to get (43). Now, from the monotonicity of cos, we have

$$\theta(\hat{f}, \hat{g}) \leq \theta(\hat{f}, h) + \theta(\hat{g}, h). \quad (44)$$

Substituting inequalities (28) and (29) into the above and using equation (27) proves the lemma.

Proof: [Proof outline Theorem 4] From the definition of $\theta(f, g)$

$$\cos(\theta(f, Df)) = \frac{\mathcal{R}\{\langle f, Df \rangle \}}{\|Df\|_2 \|f\|_2} = \frac{\mathcal{R}\{\langle Df, Df \rangle \}}{\|Df\|_2 \|f\|_2} = \|Df\|_2 \|f\|_2 = \alpha.$$  

Noting $f = Df + f - Df$ and $\langle f - Df, Df \rangle = 0$. Similarly $\beta = \cos(\theta(f, Bf))$. In order to complete our proof we use Lemma 1. Therefore

$$\cos^{-1} \alpha + \cos^{-1} \beta = \theta(f, Df) + \theta(f, Bf)$$

$$\geq \theta(Df, Bf)$$

$$\geq \theta_0 \quad (47)$$

where (47) is from $Df \in \mathcal{D}$ and $Bf \in \mathcal{B}$ and these two subspaces have the minimum angle $\theta_0$.

Proof: [Proof of Theorem 5] The angle between two subspaces is

$$\theta(\mathcal{B}, \mathcal{D}) = \inf_{f \in \mathcal{B}, g \in \mathcal{D}} \theta(f, g)
\quad (48)$$

Using $\cos \theta(f, g)$ from the proof of the last theorem, we can write

$$\cos \theta(\mathcal{B}, \mathcal{D}) = \sup_{f \in \mathcal{B}, g \in \mathcal{D}} \mathcal{R}\{\langle f, g \rangle \}
\quad (49)$$

$$\quad \|f\|_2 = 1, \|g\|_2 = 1$$

$$\quad = \sup_{f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R})}
\quad \|f\|_2 = 1, \|g\|_2 = 1
\quad \mathcal{R}\{\langle f, Dg \rangle \}
\quad (50)$$

$$\quad = \sup_{f \in L^2(\mathbb{R}), g \in L^2(\mathbb{R})}
\quad \|f\|_2 = 1, \|g\|_2 = 1
\quad \mathcal{R}\{\langle f, Bg \rangle \}
\quad (51)$$

$$\quad = \|BD\|_2 \quad (52)$$

where (50) is from self-adjointness of $B$ and (51) from the definition of the operator norm.

Proof: [Proof of Theorem 6]

$$\|f\|_2^2 \leq \alpha^2 \|f\|_2^2$$

$$\leq \alpha^2 \left(1 - \frac{e_i}{2}\right)^{-1} |V_R| \|\hat{f}\|_2^2$$

$$\leq \alpha^2 \left(1 - \frac{e_i}{2}\right)^{-1} |V_R| \|\hat{f}\|_2^2$$

$$\leq \alpha^2 \left(1 - \frac{e_i}{2}\right)^{-1} |V_R| \|\hat{f}\|_2^2$$

We get the last step using the Cauchy-Schwarz inequality. Rearranging the above inequality gives the required result.

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