Instanton and Spectral Flow 

in 

Topological Conformal Field Theories

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Abstract

A class of two-dimensional topological conformal field theories (TCFTs) is studied within the framework of gauged WZW models in order to obtain some insights on the global geometrical nature of TCFTs. The BRST quantizations of topological $G/H$ gauged WZW models (the twisted versions of supersymmetric gauged WZW models) are given under fixed back-ground gauge fields. The BRST-cohomology of the system is investigated, and then the correlation functions among these physical observables are considered under the instanton back-grounds. As a consequence, two-dimensional $BF$ gauge theoretical aspects of TCFTs are revealed. Especially, it is shown that two correlation functions under the different instanton back-grounds can change to each other. This process of transmutation is described by the spectral flow. The flow is formulated as a "singular" gauge transformation which creates an appropriate back-ground charge on the physical vacuum of the system. The field identification problem of the system is also discussed from the above point of view.
1 Introduction

Two-dimensional topological conformal field theories (TCFT) play an important role in our recent understanding of string theory and two-dimensional gravity \[1, 2\]. An algebraic method to construct a model of TCFTs was given in \[3\]; the "twist" of a model of \(N = 2\) superconformal field theories (SCFTs). It is known that a large class of \(N = 2\) SCFTs is obtainable by the constructions of Kazama and Suzuki \[4\], that is, the supercoset constructions associated with compact Kähler homogeneous spaces \(G/H\). The corresponding TCFTs were studied in \[5\] from the algebraic point of view.

It is also known that supercoset CFTs can be realized by "supersymmetric gauged WZW models" \[1, 6, 7\]. Much have been learned about coset models by realizing them as gauged WZW models. Especially the global geometry of coset models, which is hidden in the free field realizations \[8\], appears \[9\]. Thus the twisted versions of the supersymmetric gauged WZW models, that is, the topological gauged WZW models give us a chance to study the TCFTs from the geometrical point of view. In spite of this utility the topological gauged WZW models have been little examined. The purpose of this paper is to study these topological models from the global geometrical point.

We begin in section 2 by formulating the topological gauged WZW model for a general compact Kähler homogeneous space \(G/H\). The path-integral quantization of the model is given under a fixed back-ground gauge field, by utilizing the techniques developed in \[11, 12, 13\]. Subsequently, based on the gauge-fixed form, the local operator formulation is studied and the algebraic structure of the model is shown. In section 3 we investigate the spectrum of the physical observables, that is, the BRST-cohomology of the gauge-fixed system. The semi-classical analysis of the system provides a starting point. Then, for example, it arises the question whether this semi-classical approximation is valid in our model. Is the correlation function among these semi-classical observables meaningful under any back-ground gauge field? This question is clarified in the next section. For this purpose we turn our eyes to the BRST-cohomology on the full Hilbert space of the system. We introduce spectral flows \(\mathcal{U}\) as the symmetry transformations of our model.

In section 4 we study the correlation functions among the physical observables \(\{O_a\}_{a \in I}\) under the instanton back-grounds. It is shown that two correlation functions under the different instanton back-grounds change to each other (4.17):

\[
\langle \mathcal{U}_{\gamma_1}(O_{a_1}) \cdots \mathcal{U}_{\gamma_N}(O_{a_N}) \rangle_{c_1} = \langle O_{a_1} \cdots O_{a_N} \rangle_{c_1 + \sum_{i=1}^{N} \gamma_i}.
\]

The vectors \(\gamma_i\) label the spectral flows and the \(c_1\) is the set of the Chern numbers of the back-ground gauge field. This relation tells us that the instantons transmute the physical observables and vice versa. It is described
by the flows $U$. The field identification in the algebraic CFT approach is also discussed from this rather topological view point. Finally section 5 is devoted to our conclusions and speculations. In appendix A the quantizations of the supersymmetric gauged WZW models are summarized. The correspondence between the "twist" of the gauged WZW models and that of $N = 2$ SCFTs are explained. The former is performed by replacing the physical Weyl spinors of the models with unphysical ghost fields, while the latter is done by twisting $N = 2$ SCAs by their $U(1)$-currents, not the fermion number currents. In appendix B the basic properties of spectral flows are described. They are arranged in order to fit our study of the topological gauged WZW models.

2 Topological Gauged WZW Models Associated with General Compact Kähler Homogeneous Spaces

We shall begin with preparing some notations of Lie algebras needed for our discussion.

Let $G$ be a compact simple Lie group, $H$ be a closed subgroup of $G$, and $g, h$ be the corresponding Lie algebras. Assume that the homogeneous space $G/H$ is a compact Kähler space, which implies this space is a so-called "flag manifold". Especially $H$ includes the maximal torus $T$ of $G$. The topological conformal models we investigate in this paper will be defined associated with the coset space of this type, or the pair $(G, H)$. So it may be useful to give here a short digression to survey the structure of $g^C$ and $h^C$.

First $h^C$ can be decomposed as follows;

$$h^C = Z(h^C) \oplus h_0^C = Z(h^C) \oplus h_0^{(1)C} \oplus \cdots \oplus h_0^{(r)C}$$ (2.1)

where $Z(h^C)$ is the center of $h^C$ and $h_0^C$ is a complex semi-simple Lie algebra. We denote its simple factors by $h_0^{(1)C}, \ldots, h_0^{(r)C}$. The corresponding decomposition of $H$ is

$$H = H_0 \times \underbrace{U(1) \times \cdots \times U(1)}_{l \text{ times}}$$

$$= H_0^{(1)} \times \cdots \times H_0^{(r)} \times U(1) \times \cdots \times U(1).$$ (2.2)

$H_0$ is the compact semi-simple Lie group corresponding to $h_0^C$, and $l = \text{rank} g - \text{rank} h \equiv \dim Z(h^C)$. We denote the root system of $g^C$, $h_0^C$ by
\[ \Delta \equiv \Delta^+ \sqcup \Delta^-, \; \Delta_h \equiv \Delta^+_h \sqcup \Delta^-_h, \] respectively. We also set \( \Delta^\pm_h = \Delta^\pm \setminus \Delta^\pm_h \).

\( g^C \) can be decomposed into the following pieces:

\[
g^C = h^C \oplus m_+ \oplus m_-
= Z(h^C) \oplus h_0^C \oplus m_+ \oplus m_-	ag{2.3}
\]

where \( m_\pm = \sum_{\alpha \in \Delta^\pm_m} g_\alpha \). \( g_\alpha \) is the root space for \( \alpha \in \Delta \). This is an example of parabolic decompositions of \( g^C \) (generalizations of the Cartan decomposition);

\[
\left\{ \begin{array}{c}
[h^C, m_\pm] \subset m_\pm \\
[m_\pm, m_\pm] \subset m_\pm
\end{array} \right. \tag{2.4}
\]

If the homogeneous space \( G/H \) is hermitian symmetric, we further obtain the additional commutation relations;

\[
[m_\pm, m_\mp] \subset h^C,
\tag{2.5}
\]

which implies that the subalgebras \( m_+, m_- \) necessarily become abelian. This case is rather easy to handle in many respects.

In the following part of this paper we also assume that the group \( G \) is simply-laced.

Now we can present the model we shall work on, the “topological gauged WZW model for the homogeneous space \( G/H \)”. We may call it the “topological G/H model”. It is no other than the twisted version of the \( N = 2 \) supersymmetric gauged WZW model [11], [13] corresponding to the Kazama-Suzuki supercoset model [4]. Let \( \Sigma \) a Riemann surface. The partition function \( Z \) of the model is given by

\[
Z = \int Dg DAD\bar{D}(\chi, \bar{\chi}, \psi, \bar{\psi}) \exp \left[ -kS_G(g : A) - \frac{1}{2\pi i} \int_\Sigma \left\{ (\bar{\partial}_A \psi, \chi) - (\bar{\chi}, \partial_A \bar{\psi}) \right\} \right].
\tag{2.6}
\]

In this expression the chiral field \( g \) is \( G \)-valued, the gauge field \( A \) is \( h \)-valued, and the ghost system is defined as follows;

ghosts \( \psi : m_+ \)-valued \((0,0)\)-form, \( \bar{\psi} \equiv \psi^\dagger \): \( m_- \)-valued \((0,0)\)-form,

anti-ghosts \( \chi : m_- \)-valued \((1,0)\)-form, \( \bar{\chi} \equiv \chi^\dagger \): \( m_+ \)-valued \((0,1)\)-form.

\( D_A = \partial_A + \bar{\partial}_A \) is the canonical splitting of the covariant exterior derivative \( D_A \) defined by the complex structure “compatible” with the background metric \( g \) on \( \Sigma \) (i.e. such that \( g \) becomes Kähler). The inner product “\( (, ) \)” is the Cartan Killing form normalized by \( (\theta, \theta) = 2 \) (\( \theta \) is the highest root of \( g^C \)). “\( \dagger \)”
is the “hermitian conjugation” so that \( g = \{ u \in g^C \mid u^\dagger = -u \} \). \( S_G(g : A) \) stands for the action of the \( G \)-WZW model gauged by the subgroup \( H \);

\[
S_G(g : A) = \frac{i}{4\pi} \int_{\Sigma} (g^{-1} \partial g, g^{-1} \partial g) - \frac{i}{24\pi} \int_{\Sigma} (\tilde{g}^{-1} d\tilde{g}, [\tilde{g}^{-1} d\tilde{g}, \tilde{g}^{-1} d\tilde{g}])
\]

\[
+ \frac{i}{2\pi} \int_{\Sigma} \{ -(g^{-1} \tilde{\partial} g, A^{10}) + (A^{01}, \partial g g^{-1} - 1) - (A^{01}, A d(g) A^{10}) + (A^{01}, A^{10}) \}.
\]

(2.7)

Besides the corresponding gauge symmetry the model (2.6) satisfies the following BRST symmetry, which is originated in the SUSY of the untwisted model;

\[
\delta_{G/H} \chi = k \Pi_{m_-} (\partial_A g g^{-1} + [\chi, \psi]), \quad \tilde{\delta}_{G/H} \tilde{\chi} = -k \Pi_{m_+} (g^{-1} \partial_A g - [\bar{\chi}, \bar{\psi}]),
\]

\[
\delta_{G/H} g = \psi g, \quad \tilde{\delta}_{G/H} \tilde{g} = -g \tilde{\psi},
\]

\[
\delta_{G/H} \psi = \frac{1}{2} [\psi, \psi], \quad \tilde{\delta}_{G/H} \tilde{\psi} = \frac{1}{2} [\bar{\psi}, \bar{\psi}],
\]

other combinations are defined to vanish,

(2.8)

where \( \Pi_{m_+}, \Pi_{m_-} \) mean the orthogonal projections onto the spaces \( m_+, m_- \) with respect to the inner product \( (\ , \ ) \). \( \delta_{G/H}, \tilde{\delta}_{G/H} \) satisfy the following nilpotencies;

\[
\delta_{G/H}^2 = \tilde{\delta}_{G/H}^2 = 0, \quad \{ \delta_{G/H}, \tilde{\delta}_{G/H} \} = 0.
\]

(2.9)

Because the ghost system \( \psi, \bar{\psi}, \chi, \bar{\chi} \) corresponds to that of the BRST symmetry (2.8), the physical Hilbert space of the model should be restricted by taking the BRST cohomology for this symmetry.

### 2.1 Path Integration

Let us perform the path-integration in (2.6) along the line in (13). For that purpose we should extract the gauge volume of the underlying \( H \)-gauge symmetry from (2.6). We shall parametrize the h-gauge field \( A \) as

\[
A = a^{-1} h \quad (h^{-1} a^{01} = h^{-1} a^{01} h + h^{-1} \partial h, \ h^{-1} a^{10} = -(h^{-1} a^{01})^\dagger),
\]

(2.10)

where \( a \) is a background h-gauge field and \( a^{10}, a^{01} \) are the holomorphic and anti-holomorphic components of \( a \). \( h \) is a \( H^C \)-chiral gauge transformation (11). Then we insert the following identity into (2.6);

\[
1 = \int \mathcal{D}a \mathcal{D}h \mathcal{D}h \delta(A^{-1} a) \Delta_{FP}(A^{-1} a),
\]

(2.11)

where “\( \Delta_{FP}(A^{-1} a) \)” denotes the Fadeev-Popov (FP) determinant obtained by the change of integration variables. Taking account of the decomposition
(2.11) or (2.12), let us express $a$, $h$ as follows:

$$a = (i\omega, a_0), \quad h = (e^{i\frac{1}{2}(X+iY)}, h_0(\equiv \rho^{1/2}U)). \quad (2.12)$$

$i\omega$, $e^{i\frac{1}{2}(X+iY)}$ are the components corresponding to $Z(h^C)$ in (2.11). $\omega$ is a $\mathbb{R}^l$-valued 1-form. $X$, $Y$ are $\mathbb{R}^l$-valued scalar fields, which correspond to the axial and vectorial components of the $Z(H)C$-chiral gauge transformation. $a_0$, $h_0$ are the components corresponding to $h_0^C$ in (2.11). $h_0 = \rho^{1/2}U$ ($\rho \equiv h_0 h_0^\dagger \in H_0^C/H_0, U \in H_0$) is the "polar decomposition" of $H_0^C$.

Under the parametrization $A = h^{-1}a$ the model will suffer the chiral anomaly. We must estimate it for the dynamical variable $g$ and the ghost system $\chi, \bar{\chi}, \psi, \bar{\psi}$ independently.

For $g$, by means of the Polyakov-Wiegmann identity (see [10, 11, 12, 13]) we can get

$$S_G(g; h^{-1}a) = S_G(hG : a) - S_G(hh^\dagger : a)$$

$$= S_G(hG : a) - S_{H_0}(\rho : a_0) - i\frac{1}{4\pi} \int_{\Sigma} (\bar{\partial}X, \partial X) + \frac{1}{2\pi} \int_{\Sigma} (X, F(\omega)), \quad (2.13)$$

where $hG = hh^\dagger$. $F(\omega) \equiv d\omega$ is the $Z(h)$-component of curvature of the background gauge field $a = (i\omega, a_0) \equiv (i\omega, a_0^{(1)}, \ldots, a_0^{(r)})$. $S_{H_0}(\rho, a_0) = \sum_{i=1}^{r} S_{H_0}^{(i)}(\rho(i), a_0^{(i)})$ with $\rho \equiv (\rho^{(1)}, \ldots, \rho^{(r)}) \in \prod_{i=1}^{r} (H_0^{(i)}C/H_0^{(i)})$.

For the ghost system, the same discussion as in [13] gives;

$$Z_{\chi\psi} = \int \mathcal{D}(\chi, \bar{\chi}, \psi, \bar{\psi}) \exp \left[ -\frac{1}{2\pi i} \int_{\Sigma} \left\{ (\bar{\partial}_a \chi, \chi) - (\bar{\chi}, \partial_a \chi) \right\} \right]$$

$$= \int \mathcal{D}(\chi, \bar{\chi}, \psi, \bar{\psi}) \exp \left[ -\frac{1}{2\pi i} \int_{\Sigma} \left\{ (\bar{\partial}_a \psi, \chi) - (\bar{\chi}, \partial_a \psi) \right\} \right]$$

$$\times \prod_{i=1}^{r} \exp \left\{ \left(g^\vee - h_0^\vee\right) S_{H_0^{(i)}}(\rho(i) : a_0^{(i)}) \right\}$$

$$\times \exp \left[ \frac{ig^\vee}{4\pi} \int_{\Sigma} \left\{ (\bar{\partial}X, \partial X) - 2i(X, F(\omega)) + \frac{2}{ig^\vee} \rho_g/h(X)R(g) \right\} \right], \quad (2.14)$$

where $g^\vee(h_0^\vee)$ are the dual Coxeter numbers of $g^C(h_0^{(i)C})$, and we introduce the notation $\rho_{g/h}$ by

$$\rho_{g/h} = \rho_g - \sum_{i=1}^{r} \rho_{h_0^{(i)}}. \quad (2.15)$$

$\rho_g(\rho_{h_0^{(i)}})$ are the Weyl vectors of $g^C(h_0^{(i)C})$. $R(g)$ is the Riemannian curvature tensor, $\chi(\Sigma) = \frac{1}{2\pi} \int R(g)$. The contribution of $\rho$ in (2.14) will be obtained
by calculating the Schwinger term of the underlying $h_0$-current algebra. The contribution of $X$ in (2.14) seems to be a “background charge”. It will be estimated by some direct anomaly calculation or applying the index theorem to the corresponding Dolbeault complex.

We must further estimate the anomaly of the FP determinant $\Delta_{FP}(h^{-1}a)$. It can be rewritten in the local functional form by introducing the additional FP ghosts; $\xi$ (a $h^C$-valued (0,0)-form), $\bar{\xi}$ (a $h^C$-valued (0,0)-form), $\zeta$ (a $h^C$-valued (1,0)-form), $\bar{\zeta}$ (a $h^C$-valued (0,1)-form);

$$\Delta_{FP}(h^{-1}a) = \int \mathcal{D}(\zeta, \bar{\zeta}, \xi, \bar{\xi}) \exp \left[ -\frac{1}{2\pi i} \int_{\Sigma} \left\{ (\bar{\partial}_{h^{-1}a}\xi, \zeta) - (\zeta, \partial_{h^{-1}a}\bar{\xi}) \right\} \right].$$  \tag{2.16}

The chiral anomaly of this ghost system can be computed in the same way as that of the $\chi\psi$-ghosts;

$$\Delta_{FP}(h^{-1}a) = \int \mathcal{D}(\zeta, \bar{\zeta}, \xi, \bar{\xi}) \exp \left[ -\frac{1}{2\pi i} \int_{\Sigma} \left\{ (\bar{\partial}_{a0}\xi, \zeta) - (\zeta, \partial_{a0}\bar{\xi}) \right\} \right] \times \prod_{i=1}^{r} \exp \left\{ 2h_i \gamma S_{H_0^{(i)}}(\rho^{(i)} : a_0^{(i)}) \right\}. \tag{2.17}

Notice that this determinant does not depend on $\omega$ since $Z(h^C) \equiv Z(h^C) \oplus h_0^C$.

By summing up the above estimations of the chiral anomaly and then dropping the gauge volumes $\int \mathcal{D}U, \int \mathcal{D}Y$ off we can obtain the following gauge fixed form of the model (2.6);

$$Z_{g.f.} = \int \mathcal{D}a Z_{g.f.}[a] \equiv \int \mathcal{D}(a_0, \omega) Z_{g.f.}[a_0, \omega],$$

$$Z_{g.f.}[a_0, \omega] = \int \mathcal{D}(g, \rho, X, \chi, \bar{\chi}, \psi, \bar{\psi}, \zeta, \bar{\zeta}, \xi, \bar{\xi}) \times \exp \left\{ -kS_G(g : a) - S_{\chi\psi}(\chi, \bar{\chi}, \psi, \bar{\psi} : a) \right\} \times \exp \left\{ \sum_{i=1}^{r} (k+g^\gamma + h_i^\gamma) S_{H_0^{(i)}}(\rho^{(i)} : a_0^{(i)}) \right\} - S_X(X : \omega) - S_{\zeta\xi}(\zeta, \bar{\zeta}, \xi, \bar{\xi} : a_0). \tag{2.18}

In (2.18) we introduce the following notations;

$$S_{\chi\psi}(\chi, \bar{\chi}, \psi, \bar{\psi} : a) = \frac{1}{2\pi i} \int_{\Sigma} \left\{ (\bar{\partial}_a\psi, \chi) - (\chi, \partial_a\bar{\psi}) \right\} \tag{2.19}$$

$$S_{\zeta\xi}(\zeta, \bar{\zeta}, \xi, \bar{\xi} : a_0) = \frac{1}{2\pi i} \int_{\Sigma} \left\{ (\bar{\partial}_{a_0}\xi, \zeta) - (\zeta, \partial_{a_0}\bar{\xi}) \right\} \tag{2.20}$$

$$S_X(X : \omega) = \frac{1}{4\pi i} \int_{\Sigma} \left\{ (\bar{\partial}X, \partial X) + 2i\alpha_+(X, F(\omega)) + 2i\alpha_+\rho g/h(X)R(g) \right\},$$
\[ \alpha_+ = \sqrt{k + g^\vee}, \quad \alpha_- = -\frac{1}{\sqrt{k + g^\vee}}, \] (2.21)

where we have rescaled the scalar field \( X \) as \( \alpha_+ X \to X \) in (2.21).

Except for the “Chern number dependent term” \( \sim \int_S (X, F(\omega)) \) in (2.18), the similar expression for the gauge fixed form (2.18) is given in [19]. But this “Chern number dependent term” possesses a topological information of the system, and will play an important role in this paper. We will argue on this point in section 4.

2.2 Local Operator Formulation

Nextly we will give some operator formulation of the gauge fixed system (2.18). We shall work on some fixed local holomorphic coordinate patch \((U, z) \subset \Sigma\), where we set the background fields \( g, a_0, \omega \) trivial. The quantization of the system is straightforward, since everything is expressed either in free fields, or in terms of ungauged WZW models on this patch.

First of all, from (2.18), the total energy-momentum (EM) tensor \( T_{\text{tot}} \) of the gauge fixed system is given by:

\[ T_{\text{tot}} = T_g + T_\rho + T_X + T_\chi\psi + T_\zeta\xi, \] (2.22)

where \( T_g, T_\rho \) are the Sugawara EM tensors of the \( G_k, \prod_{i=1}^r \left( H_0^C / H_0 \right)^{-(k+g^\vee+h^\vee_i)} \) WZW models, and \( T_X, T_\chi\psi, T_\zeta\xi \) are those obtained from the actions (2.19)-(2.21). Their explicit forms are given as follows:

\[ T_g = \frac{1}{2(k + g^\vee)} \delta (J_g, J_g), \] (2.23)

\[ T_\rho = \sum_{i=1}^r T_\rho^{(i)}, \] (2.24)

\[ T_\rho^{(i)} = \frac{1}{2\left\{-(k+g^\vee+h^\vee_i) + h^\vee_i \right\}} \delta (J_\rho^{(i)}, J_\rho^{(i)}), \] (2.25)

\[ T_X = -\frac{1}{2} : (\partial_z X, \partial_z X) : + \alpha_- g / h (\partial^2_z X) \] (2.26)

\[ T_\chi\psi = - : (\chi_z, \partial_z \psi) : \] (2.27)

\[ T_\zeta\xi = - : (\zeta_z, \partial_z \xi) : \] (2.28)

where \( : \) denotes the standard normal ordering prescription defined by mode expansions, and \( \delta A(w) B(w) \equiv \frac{1}{2\pi i} \int w \frac{A(z) B(w)}{z-w} \). We denote
the $G_k$, $H_0^{(i)}_{\Delta_k}$-currents of the corresponding WZW models by $J_g = -k \partial g^{-1}$, $J^{(i)}_\rho = (k + g^\vee + h_i^\vee)\partial \rho^{(i)} \rho^{(i)-1}$. One can easily check that $T_{\text{tot}}$ indeed has vanishing central charge:

$$c_{\text{tot}} = c_g + c_\rho + c_X + c_{\chi \psi} + c_{\zeta \xi}$$

$$= \frac{k \text{dimg}}{k + g^\vee} + \sum_{i=1}^r \frac{-(k + g^\vee + h_i^\vee) \text{dim} h_0^{(i)}}{-(k + g^\vee + h_i^\vee) + h_i^\vee} + l + 12\alpha_2 \rho_2^g \rho^{-1}$$

$$+ (-2) \times \frac{1}{2} (\text{dimg} - \text{dim} h_0 - l) + (-2) \times (\text{dim} h_0 + l) = 0. \quad (2.29)$$

This estimation suggests the total system is topologically invariant.

Let us introduce the BRST-charges which will characterize the physical Hilbert space of the gauge fixed system. The BRST symmetry (supersymmetry) (2.8) should correspond to the following BRST charge;

$$Q_{G/H} = \frac{1}{2\pi i} \oint dz G^+_{G/H}, \quad (2.30)$$

where the BRST current $G^+_{G/H}$ is defined as

$$G^+_{G/H} = -\alpha_- : (\psi, J_g + \frac{1}{2} \xi \chi \psi) :. \quad (2.31)$$

We set $J_{\chi \psi} = -[\chi_z, \psi]$. Meanwhile, the BRST-charges for the $H_0^C$ and $Z(H^C)$-chiral gauge transformations are given by

$$Q_{H_0^C} = \frac{1}{2\pi i} \oint dz G^+_{H_0^C}, \quad Q_{Z(H^C)} = \frac{1}{2\pi i} \oint dz G^+_{Z(H^C)}, \quad (2.32)$$

where the BRST currents $G^+_{H_0^C}, G^+_{Z(H^C)}$ are defined by

$$G^+_{H_0^C} = \frac{-\alpha_-}{\sqrt{2}} : (\xi, \hat{J}_h^0 + J_\rho + \frac{1}{2} J_{\zeta \xi}) :, \quad (2.33)$$

$$G^+_{Z(H^C)} = \frac{-\alpha_-}{\sqrt{2}} \left\{ (\xi, \hat{J}_{\chi h}^0 + J_X) - 2 \rho_2 g \rho^{-1} \partial_z \xi \right\}. \quad (2.34)$$

In the above expressions we introduce the current $\hat{J}$ as

$$\hat{J} = J_g + J_{\chi \psi} \quad (2.35)$$

and the notations $\hat{J}_h^0(= \hat{J}_h^0 + \hat{J}_{\chi h}^0)$, $\hat{J}_h^0$, $\hat{J}_{\chi h}^0$ denote the projections of $\hat{J}$ onto the corresponding spaces. This combined current $\hat{J}$ is $Q_{G/H}$-invariant. The current $J_X = \alpha_- \partial_z X$ is obtained from $S_X (2.21)$ by taking the variation of the $Z(h)$-back-ground gauge field $i\omega$. Because the currents $\hat{J}_h^0(= \hat{J}_h^0 + \hat{J}_{\chi h}^0)$, $J_\rho$, $J_{\zeta \xi}, \ldots$
\( j^Z(h) \) and \( J_X \) have levels \( k + g^\gamma - h^\gamma, -(k + g^\gamma + h^\gamma), 2h^\gamma, k + g^\gamma \) and \(-(k + g^\gamma)\) respectively, the "total \( h_0^C, Z(h^C)\)-currents" of the gauge fixed system,

\[
J_{tot}^{h_0} \equiv j^{h_0} + J_\rho + j_{\xi}^{h_0}, \quad J_{tot}^{Z(h)} \equiv j^{Z(h)} + J_X
\] (2. 36)
generate \( h_0^C, Z(h^C)\)-current algebras with vanishing levels. This implies the nilpotency of the BRST-charges \( Q_{h_0^C}, Q_{Z(h^C)} \) (c.f. [12]). These three BRST-charges \( Q_{G/H}, Q_{H_0^C}, Q_{Z(H^C)} \) anti-commute with one another. This fact is indeed natural, since they correspond to independent gauge degrees of freedom.

Set the total BRST charge of the gauge fixed system as

\[
Q_{tot} = Q_{G/H} + Q_{H_0^C} + Q_{Z(H^C)}.
\] (2. 37)

Then we can show that the total EM tensor (2. 22) itself is BRST-exact,

\[
T_{tot} = \{ Q_{tot}, G^\sim_\rho \},
\] (2. 38)

where \( G_{tot} \) has the following factorized form:

\[
G_{tot}^\sim = G_{G/H}^\sim + G_{H_0^C}^\sim + G_{Z(H^C)}^\sim.
\] (2. 43)

The total \( h_0^C, Z(h^C)\)-currents (2. 36) are also BRST-exact;

\[
J_{tot}^{h_0} = \{ Q_{tot}, \sqrt{2}(\alpha_\zeta)^{h_0} \}, \quad J_{tot}^{Z(h)} = \{ Q_{tot}, \sqrt{2}(\alpha_\zeta)^{Z(h)} \},
\] (2. 42)

where \( \zeta_{h_0}, \zeta_{Z(h)} \) are the \( h_0^C, Z(h^C)\)-components of the antighost field \( \zeta \). The BRST-exactness of \( T_{tot}, j^{h_0, Z(h)}_{tot} \) will assure the topological invariance of the system.

To end this section, it is convenient to give several remarks.

Firstly we notice that the total EM tensor (2. 22) can be decomposed into the following three commuting pieces (c.f (2. 38)-(2. 41));

\[
T_{tot} = T_{G/H} + T_{H_0^C} + T_{Z(H^C)}.
\] (2. 43)

Each element in the RHS of (2. 43) is given by;

\[
T_{G/H} \equiv \frac{1}{2(k + g^\gamma)} \left\{ \delta (J_g, J_g)^\circ - \delta (j^{\xi}, j^{\xi})^\circ \right\} + \frac{1}{k + g^\gamma} \rho_{g^\gamma/h}(\partial_{\xi} z^{Z(h)}) (\chi_{\xi, \partial_{\xi} \psi});
\]

\[
T_{H_0^C} \equiv \frac{1}{2(k + g^\gamma)} \left\{ \delta (J^g, J^g)^\circ - \delta (j^{g^\gamma}, j^{g^\gamma})^\circ \right\} + \frac{1}{k + g^\gamma} \rho_{g^\gamma/h}(\partial_{\xi} z^{Z(h)}) (\chi_{\xi, \partial_{\xi} \psi});
\]

\[
T_{Z(H^C)} \equiv \frac{1}{2(k + g^\gamma)} \left\{ \delta (J^g, J^g)^\circ - \delta (j^{g^\gamma}, j^{g^\gamma})^\circ \right\} + \frac{1}{k + g^\gamma} \rho_{g^\gamma/h}(\partial_{\xi} z^{Z(h)}) (\chi_{\xi, \partial_{\xi} \psi});
\]

\[
T_{Z(H^C)} \equiv \frac{1}{2(k + g^\gamma)} \left\{ \delta (J^g, J^g)^\circ - \delta (j^{g^\gamma}, j^{g^\gamma})^\circ \right\} + \frac{1}{k + g^\gamma} \rho_{g^\gamma/h}(\partial_{\xi} z^{Z(h)}) (\chi_{\xi, \partial_{\xi} \psi});
\]
\[ \{Q_{G/H}, G_{G/H}\}, \quad T_{H_0^G} = \frac{1}{2(k + g^\nu)} \left\{ \circ(j^{h_0}, j^{h_0})^* - \circ(J^\rho, J^\rho)^* \right\} - : (\zeta^{h_0}, \partial_z \xi^{h_0}) : \]  
\[ = \{Q_{H_0^G}, G_{H_0^G}\}, \quad (2.44) \]

\[ T_{Z(H^C)} = \frac{1}{2(k + g^\nu)} \circ(j^{Z(h)}, j^{Z(h)})^* - \frac{1}{k + g^\nu} \rho_{g/h} (\partial_z j^{Z(h)}) \]
\[ - \frac{1}{2} : (\partial_j X, \partial_z X) + \alpha_{g/h} (\partial^2 z X) - : (\zeta^{Z(h)}, \partial_z \xi^{Z(h)}) : \]
\[ = \{Q_{Z(H^C)}, G_{Z(H^C)}\}, \quad (2.45) \]

where \( \xi^{h_0} \) and \( \xi^{Z(h)} \) denote the \( h_0 \), \( Z(h) \)-components of the ghost field \( \xi \). By introducing the \( U(1) \)-currents;

\[ J_{G/H} = : (\psi, \chi_z) : + \frac{2}{k + g^\nu} \rho_{g/h} (j^{Z(h)}), \quad (2.47) \]

\[ J_{H_0^G} = : (\zeta^{h_0}, \zeta_z^{h_0}) :, \quad (2.48) \]

\[ J_{Z(H^C)} = : (\zeta^{Z(h)}, \zeta_z^{Z(h)}) : - \frac{2}{k + g^\nu} \rho_{g/h} (J_X), \quad (2.49) \]

one can find out that the pairs \( \{G_{G/H}, T_{G/H}, J_{G/H}\}, \{G_{Z(H^C)}^\pm, T_{Z(H^C)}^\pm, J_{H_0^G}\} \) generate the topological conformal algebras (TCAs) with background charges \( Q_{KS} = \text{dim}^+_n - \frac{4}{k + g^\nu} \rho_{g/h}^2 \) and \( Q_{CG} = l - \frac{4}{k + g^\nu} \rho_{g/h}^2 \) respectively \([1]\).

The OPEs of the “residual sector” \( \{G_{H_0^G}^\pm, T_{H_0^G}^\pm, J_{H_0^G}\} \) also have an almost similar form to the TCA with background charge \( Q = \text{dim}^+_n \). But it is not the completely same, since the nilpotency of the \( G^- \)-operator is broken; \( G_{H_0^G}^\pm (z) G_{H_0^G}^\pm (w) \not\sim 0 \). It may reflect the fact that the manifold \( H_0^G \) is not necessarily Kähler, while \( G/H, Z(H^C) = \mathbb{C}^* \times \cdots \times \mathbb{C}^* \) have natural Kähler structures.

These three algebras commute (anti-commute for fermionic currents) with one another. They correspond to independent degrees of freedom related to the different gauge symmetries. We may call the TCA \( \{G_{G/H}^\pm, T_{G/H}, J_{G/H}\} \) as that of “Kazama-Suzuki sector”, because the field realizations \([2, 31], [2, 44], [2, 47]\) are same as the twisted version of the \( N = 2 \) SCA of the Kazama-Suzuki model \([1]\) for \( G/H \). The TCA \( \{G_{Z(H^C)}^\pm, T_{Z(H^C)}^\pm, J_{Z(H^C)}^\pm\} \) will be called that of Coulomb-gas(CG) sector \([13]\). This is because one can write it in the following convenient form. Introduce a \( iZ(h) \) \( \cong \mathbb{R}^l \)-valued real compact boson \( \varphi \) with the radius \( \alpha_+ \) (normalized by \( (u, \partial_z \varphi(z))(v, \partial_w \varphi(w)) \sim \]

\[ T(z) J(w) \sim - \frac{Q}{(z-w)^3} + \frac{1}{(z-w)^2} J(w) + \frac{1}{z-w} \partial_w J(w). \]

\[ 4\text{A TCA } \{G^\pm, T, J\} \text{ will be called a “TCA with background charge } Q \text{” if they satisfy} \]

\[ T(z) J(w) \sim - \frac{Q}{(z-w)^3} + \frac{1}{(z-w)^2} J(w) + \frac{1}{z-w} \partial_w J(w). \]
Combining the compact and non-compact bosons $\varphi$, $X$ to a $Z(h^C)$-valued complex boson $\hat{\varphi} = \varphi - iX$, the TCA of the CG sector $\{G_{Z(H^C)}^\pm, T_{Z(H^C)}, J_{Z(H^C)}\}$ can be expressed in terms of $\hat{\varphi}$, $\zeta$, $\xi$:

\begin{align*}
T_{Z(H^C)} &= -\frac{1}{2} : (\partial_z \varphi^+ \partial_z \hat{\varphi}) : + i\alpha - \rho \frac{g}{h} (\partial_z \hat{\varphi}) - : (\zeta z^{|h}, \partial_z \zeta z^{|h}) :, \\
G_{Z(H^C)}^+ &= \frac{i}{\sqrt{2}} (\zeta z^{|h}, \partial_z \hat{\varphi}) + \sqrt{2} \alpha - \rho \frac{g}{h} (\partial_z \xi), \\
G_{Z(H^C)}^- &= \frac{i}{\sqrt{2}} (\zeta z^{|h}, \partial_z \hat{\varphi}^+) + \sqrt{2} \alpha - \rho \frac{g}{h} (\partial_z \zeta z^{|h}), \\
J_{Z(H^C)} &= : (\zeta z^{|h}, \zeta z^{|h}) : + i\alpha - \rho \frac{g}{h} (\partial_z \hat{\varphi}) - i\alpha - \rho \frac{g}{h} (\partial_z \hat{\varphi}^+). \quad (2. 51)
\end{align*}

These precisely coincide with those obtained by twisting the $N = 2$ Coulomb gas model [18].

\section{BRST Analysis and the Chiral Primary Ring}

\subsection{The BRST Cohomology on the Semi-Classical Hilbert Space}

Here let us consider the physical states of the gauge fixed system. They are characterized by the total BRST-charge $Q_{\text{tot}} = Q_{G/H} + Q_{H^C_0} + Q_{Z(h^C)}$ [2]. The total Hilbert space $\mathcal{H}$ is spanned by the state vectors having the form $|\text{WZW} (g) \rangle \otimes |\text{WZW} (\rho) \rangle \otimes |X \rangle \otimes |\chi \psi \rangle \otimes |\zeta \xi \rangle$, and the physical Hilbert space $\mathcal{H}_{\text{phys}}$ is defined as the $Q_{\text{tot}}$-BRST cohomology group in the standard manner;

\[ \mathcal{H}_{\text{phys}} = H^*_{Q_{\text{tot}}} (\mathcal{H}). \quad (3. 1) \]

Instead of considering this total cohomology directly, we shall only estimate it “step by step”. Namely we consider $H^0_{Q_{Z(H^C)}^+} \circ H^0_{Q_{H^C_0}} \circ H^0_{Q_{G/H}} (\mathcal{H})$, in order to complete the definition of $\varphi$ we should further define the zero-mode $\varphi_0$ appropriately. We take the following convention; $[a_0, \varphi_0] = -\frac{i}{2}$ with $a_0 = \frac{1}{2\pi i} \oint i\partial_z \varphi dz$, and $[N_{\chi \psi}, \varphi_0] = 0$ with $N_{\chi \psi} = \frac{1}{2\pi i} \oint : (\psi(z), \chi(z)) : dz$. 

\[ \text{To complete the definition of } \varphi \text{ we should further define the zero-mode } \varphi_0 \text{ appropriately.} \]
to make the problem simple. In general this may give only a subspace of
the precise physical Hilbert space \( H^{\ast}_{\text{tot}}(\mathcal{H}) \), but, if we can expect the corresponding spectral sequence degenerates at the 2nd order, it coincides with the physical Hilbert space itself.

First we shall restrict our attention to the states realized by the direct products of the primary states of all the dynamical variables. In other words we replace the total Hilbert space \( \mathcal{H} \) by its “semi-classical subspace”:

\[
\mathcal{H}^{\text{s.c}} \equiv \{ |A\rangle \in \mathcal{H} \mid |A\rangle \text{ is primary} \}
\] (3. 2)

in (3.1).

The most non-trivial part of the cohomology calculation is the estimation of \( H^{r}_{Q_{G/H}}(\mathcal{H}^{\text{s.c}}) \). It is clearly the same algebra as the chiral primary ring in the \( G/H \)-Kazama-Suzuki model, which was fully investigated in the papers [14, 15, 16]. In order to present these results it is convenient to introduce some notations of Lie algebra. Let \( W \) be the Weyl group of \( g^{C} \). For any \( w \in W \) we set \( \Phi_{w} \equiv w(\Delta) \cap \Delta^{\ast} \) and define \( l(w) \equiv \#\Phi_{w} \) (the “minimal length” of \( w \)). We also set \( W^{C}(g/h) \equiv \{ w \in W \mid \Phi_{w} \subset \Delta^{\ast} \} \). With these notations \( H^{r}_{Q_{G/H}}(\mathcal{H}^{\text{s.c}}) \) is spanned by the following elements:

\[
\hat{J}_{a_{1}}^{\ast} \cdots \hat{J}_{a_{n}}^{\ast} |\Lambda, w\rangle_{G/H} \otimes \text{any state vector of } \rho, X, \zeta, \xi,
\]

\[
|\Lambda, w\rangle_{G/H} \equiv |\Lambda, w(\Lambda)\rangle_{g} \otimes \prod_{\alpha \in \Phi_{w}} \psi_{0}^{\alpha} \langle 0\vert_{X_{\psi}},
\] (3. 3)

where \( w \) is any element of \( W(g/h) \) such that \( l(w) = r \) [14, 15, 16]. Notice that \( [Q_{G/H}, \hat{J}] = 0 \) holds. Of course, precisely speaking, (3. 3) only expresses one representative of the corresponding cohomology class. One can always add to it any \( Q_{G/H} \)-exact term. (3. 3) satisfies the condition \( G^{-}_{G/H,0}|\Psi\rangle = 0 \), besides the BRST invariance \( Q_{G/H}|\Psi\rangle = 0 \). These states are called as “chiral primary states” in SCFT [14, 15, 16], and correspond to “harmonic cocycles” in mathematical terminology.

Nextly we should take further the cohomologies with respect to \( Q^{C}_{H^{0}_{C}} \) and \( Q_{Z(\mathcal{H}^{C})} \).

Consider the \( H^{0}_{C} \)-part. Under the action of \( \hat{J}_{0}^{\ast} \) (2. 33) we can extract irreducible \( h^{0}_{C} \)-modules (with respect to \( \hat{J}_{0} \)) from \( H^{r}_{Q_{G/H}}(\mathcal{H}^{\text{s.c}}) \):

\[
H^{r}_{Q_{G/H}}(\mathcal{H}^{\text{s.c}}) = \sum_{\Lambda, w} H^{G/H}_{\Lambda, w} \otimes \mathcal{H}_{\rho, X, \zeta, \xi},
\] (3. 4)

where \( \mathcal{H}_{\rho, X, \zeta, \xi} \) denotes the space of the primary states of \( \rho, X, \zeta \) and \( H^{G/H}_{\Lambda, w} \equiv \sum_{\{a_{i}\}} C\hat{J}_{a_{1}}^{\ast} \cdots \hat{J}_{a_{n}}^{\ast} |\Lambda, w\rangle_{G/H} \) is the irreducible \( h^{0}_{C} \)-module with the highest weight vector \( |\Lambda, w\rangle_{G/H} \) having the highest weight \( w \ast \Lambda \mid_{h^{0}_{C}} \) [14, 15, 16]. Here we introduce the notation:

\[
w \ast \Lambda = w(\Lambda + \rho_{g}) - \rho_{g},\]

(3. 5)
and $|h_0^C\rangle$ denotes the projection to $h_0^C \cap t^C$, i.e. the Cartan subalgebra (CSA) of $h_0^C$. $t^C$ is the CSA of $g^C$. $w * \Lambda|_{h_0^C}$ is dominant integral with respect to $h_0^C$ as is shown from the definition of $W(g/h)$. Fix one of the pair $(\Lambda, w)$ and consider $Q_{H_0^C}$-cohomology on the corresponding space $\mathcal{H}_{\Lambda, w}^{G/H} \otimes \mathcal{H}_{p, X, \xi \zeta}^{s.c}$, which is also a $h_0^C$-module with respect to the total $h_0$-current $J_{\Lambda, w}^h = J_{h_0}^h + J_{\rho} + J_{\xi \zeta}^h$ (2.37), and then the desired $Q_{H_0^C}$-cohomology is nothing but the Lie algebra cohomology of $h_0^C$. In order to proceed further it is necessary to fix an appropriate $h_0^C$-module as the (semi-classical) Hilbert space of $\rho$. Here we shall take a $h_0^C$-module with the highest weight:\[
abla (\Lambda, w) \overset{\text{def}}{=} \overline{w * \Lambda|_{h_0^C}} \ (\equiv \text{the conjugate to } w * \Lambda|_{h_0^C}), \quad (3.6)\]
and denote it by $\mathcal{H}_{\rho}(\nabla (\Lambda, w))$. In this choice the $Q_{H_0^C}$-cohomology is easily solved:\[
abla^q (\mathcal{H}_{\Lambda, w}^{G/H} \otimes \mathcal{H}_{p, X, \xi \zeta}^{s.c}) \equiv H^q(h_0^C; C) \otimes \text{Inv}_{h_0^C} [\mathcal{H}_{\Lambda, w}^{G/H} \otimes \mathcal{H}_{\rho}(\nabla (\Lambda, w))] \otimes \mathcal{H}_{X, \xi \zeta}^{s.c}(h_0^C) \otimes \mathcal{H}_{X, \xi \zeta}^{s.c}(h_0^C) \quad (3.7)\]
This is because $h_0^C$ is semi-simple and $\mathcal{H}_{\Lambda, w}^{G/H} \otimes \mathcal{H}_{p, X, \xi \zeta}^{s.c}$ is finite dimensional. In the R.H.S of (3.7) the first factor $H^q(h_0^C; C)$ corresponds to the contribution of the $h_0$-component of $\xi \zeta$-ghost system, and the second piece is the singlet tensors for the global $H_0$-rotations. $\mathcal{H}_{X, \xi \zeta}^{s.c}(h_0^C)$ is the semi-classical Hilbert space of $X$ and the $Z(h_0^C)$-component of $\xi \zeta$-ghost system.

Finally we should take the cohomology for the BRST-charge $Q_{Z(h_0^C)}$. First let us consider the sector with no $\xi \zeta$-ghost. Obviously all we have to do is to construct the singlet states for global $Z(h_0^C)$-rotation. We find that any element of the $Q_{G/H}$ and $Q_{H_0^C}$-cohomology space above constructed has a definite $Z(h_0^C)$-charge; $w * \Lambda|_{Z(h_0^C)} + \text{charge of } X$. Hence the desired result is simple; if and only if the $Z(h_0^C)$-charge of $X$ is equal to the value $-w * \Lambda|_{Z(h_0^C)}$, we get the non-trivial BRST-cohomology. The sector with the $\xi \zeta$-ghosts is also simple. We point out the following fact: Let $\xi^1, \ldots , \xi^l$ be the $Z(h_0^C)$-components of the ghost field $\xi$. Assume $|I|$ satisfies $J_{\text{tot}, 0}^Z|I| = I^i|I|$, $I^i \neq 0$ for $\forall i \in S \subset \{1, \ldots , l\}$, then, for $\forall S' \supset S$, $\prod_{i \in S'} \xi^i |I|$ is $Q_{Z(h_0^C)}$-invariant.\footnote{In the ref. [19] a different choice is taken for this representation of $h_0^C$. Because the total BRST cohomology depends on this choice the results given in [19] do not fully coincide with these we present in this paper.}
But it is BRST-trivial except only the case $S = \emptyset$. In fact we find that

\[ \prod_{i \in S'} \xi^i_0|I\rangle = Q_{Z(H^C)} \frac{1}{I_{i_0}} \prod_{i \in S\setminus \{i_0\}} \xi^i_0|I\rangle \tag{3.8} \]

for any $i_0 \in S$. This observation leads to the fact that the $\zeta|\xi$-ghost sector is completely factorized like as the $H^0_{0C}$-part, namely, the physical state with the $Z(H^C)$-ghost number $p$ can be explicitly written as\footnote{One might think that, because $\xi^i_0 = \{Q_{Z(H^C)}, \alpha, X^i_0\}$ holds, any state including the zero-modes of $\xi$ becomes BRST-trivial, even if it has the form of (3.3). But, actually it is not the case, since the operator $X^i_0$ cannot act on the Hilbert space of the states possessing the definite $Z(H^C)$-charge.}

an element of $H^0_{Q_{Z(H^C)}} \circ H^q_{Q_{HT}} \circ H^r_{Q_{G/H}} (\mathcal{H}^{e,c}) \otimes \prod_{i \in S} \xi^i_0|0\rangle \zeta \xi$,

\[ S \subset \{1, \ldots, l\}, \quad \forall S = p. \tag{3.9} \]

To sum up, the desired physical states can be written as follows: Let us denote the cohomology state corresponding to $\Lambda$, $w$ and having no $\zeta|\xi$-ghosts (of both the $H^0_{0C}$ and the $Z(H^C)$-part) by $|\Lambda, w\rangle$, which has $l(w)$ as the $\chi|\psi$-ghost number and invariant under the chiral $H^C \equiv H^0_{0C} \times Z(H^C)$-gauge transformations. The semi-classical physical Hilbert space can be expressed as

\[ \mathcal{H}_{\text{phys}} = \sum_{\Lambda, w} \mathcal{C}|\Lambda, w\rangle \otimes \sum_S \mathcal{C} \prod_{i \in S} \xi^i_0|0\rangle \zeta \xi \otimes H^r(h^C_C; C). \tag{3.10} \]

Now let us turn our interests to the physical observables. We shall write the “chiral primary operator” corresponding to the physical state $|\Lambda, w\rangle$ as $O_{\Lambda, w}(x)$;

\[ O_{\Lambda, w}(0)|0\rangle = |\Lambda, w\rangle. \tag{3.11} \]

What ring structure do these operators have? Because it reflects only the local structure of the model, we may describe it by using some technique of CFT. Defining its structure constant by

\[ O_{\Lambda_1, w_1} O_{\Lambda_2, w_2} = \sum_{\Lambda_3, w_3} C_{(\Lambda_1, w_1), (\Lambda_2, w_2)}^{(\Lambda_3, w_3)} O_{\Lambda_3, w_3} \quad \text{(modulo BRST-exact terms)}, \tag{3.12} \]

we will get the following result;

\[ C_{(\Lambda_1, w_1), (\Lambda_2, w_2)}^{(\Lambda_3, w_3)} \propto F(G_k)^{\Lambda_3}_{\Lambda_1, \Lambda_2} \prod_{i=1}^{r} F(H^{(i)}_{0, k + g^\gamma - h^\gamma})^{w_3 \ast \Lambda_3}_{w_1 \ast \Lambda_1, w_2 \ast \Lambda_2} \times \delta(w_1|Z + w_2 \Lambda_2|Z - w_3 \Lambda_3|Z) \delta(w_1 \ast 0|Z + w_2 \ast 0|Z - w_3 \ast 0|Z). \tag{3.13} \]
where $F(G_k)$, $F(H^{(i)}_{0,k+g'-h_i'})$ mean the fusion coefficients of the corresponding current algebras, and “delta function” is defined by

$$
\delta(x) = \begin{cases} 
1 & x = 0 \\
0 & x \neq 0.
\end{cases}
$$

(3.14)

The notations $\Lambda_{i}$, $\Lambda_{z}$ mean the orthogonal projections to $h_{0}^{(i)C}$, $Z(h_{0}^{C})$ respectively. The appearance of $F(G_{k})_{\Lambda_{1}\Lambda_{2}}$ in (3.13) is due to the current algebra $J_{g}$, that is, $G_{k}$-WZW model. The current algebras $j_{h_{0}^{(i)}}^i, \ldots, j_{h_{0}^{(r)}}^i$ will give the factor $\prod_{i=1}^{r} F(H^{(i)}_{0,k+g'-h_i'}) w_{1}^{r\times \Lambda_{1}} |_{h_{1}}$, $w_{2}^{r\times \Lambda_{2}} |_{h_{2}}$, $w_{3}^{r\times \Lambda_{3}} |_{h_{3}}$ in (3.13), which will include the contribution of the $\rho$-sector ($\prod_{i} H^{(i)}_{-(k+g'+h_i')}^r$ WZW model). $\prod_{i} \delta(w_{1}(\Lambda_{1})|_{z}+w_{2}(\Lambda_{2})|_{z}-w_{3}(\Lambda_{3})|_{z})$ is due to the conservation of the $Z(h_{0}^{C})$-charge of $g$-sector. $\delta(w_{1} * 0 |_{z}+w_{2} * 0 |_{z}-w_{3} * 0 |_{z})$ reflects the conservation of the $Z(h_{0}^{C})$-charge of the $\chi_{\psi}$-sector, which we can derive by using the identity $: -w * 0 |_{z} = \{\rho g - w(\rho g)\} |_{z} = \sum_{\alpha \in \Phi_{w}} \alpha |_{z}$. The $Z(h_{0}^{C})$-charge conservation of the $X$-sector is automatically ensured by those of $g$ and $\chi_{\psi}$-sector because of the BRST-invariance. We also note that the conservation of the $N=2$ $U(1)$-charge (the eigenvalue of $J_{G/H,0}$) is included in the above $Z(h_{0})$-charge conservations.

The structure constant (3.13) has a complicated form, but if we only consider some suitable subring, we can get more simple results. For example, let us consider the no-ghost sector, i.e. the physical operators of the form $O_{\Lambda_{1}}$, (“1” means the identity in the Weyl group, which trivially belongs to $W(g/h)$);

$$
C^{(\Lambda_{3},1)}_{(\Lambda_{1},1), (\Lambda_{2},1)} \propto F(G_{k})_{\Lambda_{1}^{3}, \Lambda_{2}} \delta(\Lambda_{1}|_{z}+\Lambda_{2}|_{z}-\Lambda_{3}|_{z}),
$$

(3.15)

which is the structure constant introduced by Gepner in the case of $G/H = CP^{N}$. In particular, in the case of $G/H = G/T$ ($T$ is the maximal torus of $G$), we get

$$
C^{(\Lambda_{3},1)}_{(\Lambda_{1},1), (\Lambda_{2},1)} \propto \delta(\Lambda_{1}+\Lambda_{2}-\Lambda_{3}),
$$

(3.16)

since in this case $Z(H^{C}) = T^{C}$ holds and $H_{0}^{C}$ is absent.

In this subsection we only considered the semi-classical physical observables which will correspond to the solutions of the equations of motion ; $\bar{\partial}(\partial g g^{-1}) = 0$, $\bar{\partial} \psi = 0$ etc. But, under some non-trivial background, i.e. $c_{1} \equiv \frac{i}{2\pi} \int F(a) \neq 0$, new observables other than the above semi-classical ones will appear. They may be interpreted as “instanton-sectors” which will

\footnote{Because the fusion rule of a current algebra is deeply connected with the structure of null vectors the negative level current algebra $H^{(i)}_{0,k+g'-h_i'}$ does not give stronger condition than its positive level counter part $H^{(i)}_{0,k+g'-h_i'}$.}
correspond to the solutions of equations of motion; \( \partial_a(\partial_a g g^{-1}) = 0, \partial_a \psi = 0 \)
etc. with \( c_1 \neq 0 \). To study these instanton contributions we should take the full Hilbert space into account.

### 3.2 The BRST-Cohomology on the Total Hilbert Space and Spectral Flow

Let us return to the problem of solving the BRST-cohomology on the total Hilbert space. We will adopt the same strategy as for the semi-classical case. Namely we will consider \( H^0_{Q_{(H \subset C)}_{G/H}} \circ H^N_{Q_{H_0 \subset C}} \circ H^r_{Q_{G/H}}(H) \) instead of studying the cohomology \( H^{p+q+r}_{Q_{tot}}(H) \) directly.

The results in this section will be described by using the terminology of affine Lie algebra. Besides the notations for affine Lie algebra some mathematical formulae which we need in this section are summarized in appendix B. For example we will denote the sets of positive (negative) roots of the \( g^C, h_0^C \)-current algebras by \( \Delta^+(\Delta^-) \), \( \hat{\Delta}^+_{h_0} \) \( \hat{\Delta}^-_{h_0} \) respectively. \( \hat{\Delta}^+_{(h_0)} \) consists of elements, \( \alpha (\alpha \in \Delta^+_{(h_0)} \) and \( \alpha + n\delta (\alpha \in \Delta^-_{(h_0)}, n \in \mathbb{Z}_{>0} \). \( \delta \) is the generator of imaginary roots. The modes of the coset, \( H^C_0 \)-ghost fields will be labelled by the elements of \( \hat{\Delta}^+_{h_0} \setminus \hat{\Delta}^+_{h_0} \). \( \hat{\Delta}^+_{h_0} = \hat{\Delta}^+_{h_0} \parallel \hat{\Delta}^-_{h_0} \). The (affine) Weyl groups of \( g^C, h_0^C \)-current algebras will be denoted by \( \hat{W}, \hat{W}(h_0) \). Any element \( \hat{w} \in \hat{W} \) can be uniquely expressed as \( t_{\alpha}w (w \in W, \alpha \in Q : \) the root lattice of \( g^C \). \( t_{\alpha}(\alpha \in Q) \) is the “translation” by \( \alpha \).

Firstly we pay attention to the estimation of \( H^r_{Q_{G/H}} \). This cohomology problem was fully studied in the papers [14, 15]. It was shown that the cohomology elements are labelled by

\[
(\hat{\Lambda}, \hat{w}) \in \hat{P}^k_+ \times \hat{W}(g/h), \quad (3.17)
\]

where \( \hat{P}^k_+ \) is the set of dominant integral weights of \( g^C \)-current algebra with level \( k \). \( \hat{W}(g/h) \) is the subset of \( \hat{W} \) which elements satisfy the condition; \( \Phi_{\hat{w}} \subset \hat{\Delta}^+_m \), where we set \( \Phi_{\hat{w}} = \hat{w}(\Delta^-) \cap \hat{\Delta}^+ \). \( H^r_{Q_{G/H}}(H) \) can be described as follows;

\[
H^r_{Q_{G/H}}(H) = \sum_{\hat{\Lambda}, \hat{w}} \mathcal{H}^{G/H}(\hat{\Lambda}, \hat{w}) \otimes \mathcal{H}_{|X, \xi, \zeta}. \quad (3.18)
\]

\( \hat{w} = t_{\alpha}w \in \hat{W}(g/h) \) in the R.H.S of (3.18) are those elements which satisfy \( r = l(w) - 2(\rho, \alpha) \). \( \mathcal{H}^{G/H}(\hat{\Lambda}, \hat{w}) \) is spanned by the following vectors \(^9\)

\[
\hat{\mathcal{J}}^m_{\alpha, m_1} \cdots \hat{\mathcal{J}}^m_{\alpha, m_n} |\hat{\Lambda}, \hat{w}|_{G/H} \quad (m_1, \cdots, m_n \in \mathbb{Z}_{\geq 0}), \quad (3.19)
\]

\(^9\) We label the modes of \( \chi_\alpha(z) = \sum_n \chi_{\alpha, n} z^{-n-1}, \psi^\alpha(z) = \sum_n \psi_{\alpha, n} z^{-n} (\alpha \in \Delta^+_m) \) by \( \psi_\alpha \)
\[ |\hat{\Lambda}, \hat{\varpi} \rangle_{G/H} = |\hat{\Lambda}, \hat{\varpi}(\Lambda) \rangle_g \otimes \prod_{\hat{\alpha} \in \Phi_w} \psi_{-\hat{\alpha}} |0\rangle_{\chi_{\psi}}. \] (3.20)

Under the action of the \( h_0^C \)-current algebra \( \mathcal{J}^{h_0^C} \) (2.35), \( \mathcal{H}^{G/H}(\hat{\Lambda}, \hat{\varpi}) \) is the irreducible \( h_0^C \)-module with the highest weight vector \( |\hat{\Lambda}, \hat{\varpi} \rangle_{G/H} \) which has the weight for the \( h_i^C \)-direction;

\[ \hat{\varpi} \ast \hat{\Lambda}_{|_i} + (k + g^\vee - h_i^\vee)\Lambda_0 \quad (\hat{\varpi} \ast \hat{\Lambda} \equiv \hat{\varpi}(\hat{\Lambda} + \hat{\rho}_g) - \hat{\rho}_g), \] (3.21)

where \( \hat{\rho}_g = \rho_g + g^\vee \Lambda_0 \). "\( |_i \)" means taking the classical part of \( \hat{\varpi} \ast \hat{\Lambda} \) and then projecting it to the \( h_i^C \)-component. \( \mathcal{H}_{\rho,X,\zeta,\xi} \) in the R.H.S of (3.18) is the Hilbert space of \( \rho, X, (\zeta, \xi) \) fields.

Nextly we should take the cohomology with respect to \( Q_{H_0^C} \) (2.32). For this purpose we should take an appropriate representation for \( \rho \) field. As is the semi-classical case we may choose, as the Hilbert space of \( \rho \), the \( \hat{h}_0^C \)-module which highest weight is given by

\[ \hat{\lambda}^{(i)}(\hat{\Lambda}, \hat{\varpi}) \overset{\text{def}}{=} \hat{\varpi} \ast \hat{\Lambda}_{|_i} + (-k - g^\vee - h_i^\vee)\Lambda_0 \] (3.22)

for the \( h_i^C \)-component. We write it as

\[ \mathcal{H}_{\rho}(\hat{\lambda}(\hat{\Lambda}, \hat{\varpi})) = \sum \mathbf{C} J_{\rho,-m_1}^{a_1} \cdots J_{\rho,-m_n}^{a_n} |\hat{\lambda}(\hat{\Lambda}, \hat{\varpi}), \hat{\lambda}(\hat{\Lambda}, \hat{\varpi}) \rangle_{\rho}. \] (3.23)

With this choice there exist the following elements of \( H^0_{Q_{H_0^C}} \circ H^0_{Q_{G/H}}(\mathcal{H}) \); \( \text{Inv}_{h_0^C} [\mathcal{H}^{G/H}^{(s,c)}(\hat{\Lambda}, \hat{\varpi}) \otimes \mathcal{H}^{(s,c)}_{\rho}(\hat{\lambda}(\hat{\Lambda}, \hat{\varpi}))] \otimes |0\rangle_{\zeta_{\xi}}, \) (3.24)

where

\[ \mathcal{H}^{G/H}^{(s,c)}(\hat{\Lambda}, \hat{\varpi}) = \sum \mathbf{C} \hat{J}_{0,-m_1}^{a_1} \cdots \hat{J}_{0,-m_n}^{a_n} |\hat{\Lambda}, \hat{\varpi} \rangle_{G/H}, \]
\[ \mathcal{H}^{(s,c)}_{\rho}(\hat{\lambda}(\hat{\Lambda}, \hat{\varpi})) = \sum \mathbf{C} J_{\rho,0}^{a_1} \cdots J_{\rho,0}^{a_n} |\hat{\lambda}(\hat{\Lambda}, \hat{\varpi}), \hat{\lambda}(\hat{\Lambda}, \hat{\varpi}) \rangle_{\rho} \] (3.25)

are the semi-classical Hilbert space of \( \mathcal{H}^{G/H}(\hat{\Lambda}, \hat{\varpi}), \mathcal{H}^{(s,c)}_{\rho}(\hat{\lambda}(\hat{\Lambda}, \hat{\varpi})) \) respectively. The global \( h_0^C \)-invariance in (3.24) ensures the \( Q_{H_0^C} \)-invariance of \( (\hat{\alpha} = \alpha + n\delta \in \Delta_m) \),

\[ \psi_{\hat{\alpha}=\alpha+n\delta} = \begin{cases} \psi^{-\alpha}_n & \text{for } \alpha \in \Delta_m^- \\ \chi_{\alpha,n} & \text{for } \alpha \in \Delta_m^+ \end{cases} \]
the state since the \((\zeta, \xi)\)-vacuum state \(|0\rangle_{\zeta \xi}\) is characterized by the conditions\[10\]:

\[\zeta_{\alpha + n\delta}|0\rangle_{\zeta \xi} = (h, \zeta^t_n)|0\rangle_{\zeta \xi} = 0 \quad (\text{for } n \geq 0, \alpha \in \Delta_{h_0}, h \in t),\]

\[\xi_{\alpha + n\delta}|0\rangle_{\zeta \xi} = (h, \xi^t_n)|0\rangle_{\zeta \xi} = 0 \quad (\text{for } n > 0, \alpha \in \Delta_{h_0}, h \in t).\]

It is the CSA of \(g\). Lastly we should take the cohomology with respect to \(Q_{Z(H^C)} \circ H^0_{\mathcal{Q}_{Z(H^C)}} \circ H^0_{\mathcal{Q}_{\mathcal{G}/H}} (\mathcal{H})\):

\[|\hat{\Lambda}, \hat{w}\rangle \overset{\text{def}}{=} \text{Inv}_{h_0} \mathcal{H}_G^{C, s.c} (\hat{\Lambda}, \hat{w}) \otimes \mathcal{H}_F^{s.c} (\hat{\lambda}(\hat{\Lambda}, \hat{w}))) \otimes | - \hat{w} * \hat{\Lambda}|_X \otimes |0\rangle_{\zeta \xi}.\]

(3. 26)

Here “\(|Z\rangle\)” stands for the similar meaning as \(|i\rangle\), i.e. the \(\hat{w} * \hat{\Lambda}|_Z\) is the \(Z(h^C)\)-projection of the classical part of \(\hat{w} * \hat{\Lambda}\).

To proceed further let us introduce a powerful tool - the “spectral flow” \(U \overset{[14, 15]}{=} \cdots\), which is a family of infinite symmetry transformations of our topological model in the sense that they make the BRST charge \(Q_{tot} \overset{(2. 37)}{=} \cdots\) invariant:

\[U Q_{tot} U^{-1} = Q_{tot},\]

(3. 27)

and that they change the background gauge fields of the gauge fixed model appropriately. The second point will be discussed in the next section. These transformations are essentially induced from the “translations” in the (affine) Weyl group of \(g^C\)-current algebra.

Let us describe them explicitly. For this purpose we introduce the following subset of the weight lattice \(P\) of \(g^C\) which labels the spectral flow \(U\):

\[P(g/h) = \{ \gamma \in P ; \exists \sigma \in W(h_0) \text{ s.t } \sigma(C^\text{aff}_{0,h_0} + \gamma) = C^\text{aff}_{0,h_0} \},\]

(3. 28)

where \(W(h_0)\) is the Weyl group of \(h_0\) and \(C^\text{aff}_{0,h_0}\) is the subdomain of \(t^*\) which contains the Weyl alcove of \(h_0\):

\[C^\text{aff}_{0,h_0} = \{ u \in t^* ; (u, \forall \alpha^{(i)}_{h} ) \geq 0, (u, \forall \theta^{(i)}) \leq 1 \}.\]

(3. 29)

\(^{10}\)We label the modes of \(\zeta(z) = \sum_n \zeta_n z^{-n-1}, \xi(z) = \sum_n \xi_n z^{-n}\) by \(\zeta_{\alpha + \delta}, \xi_{\alpha + \delta} (\alpha \in \Delta_{h_0}), \zeta^t_n\) and \(\xi^t_n\):

\[
\zeta_{\alpha + \delta} = \begin{cases}
\zeta_{\alpha, n} & \text{for } \alpha \in \Delta^+_{h_0} \\
\zeta^{-\alpha}_n & \text{for } \alpha \in \Delta^-_{h_0}
\end{cases}, \quad \xi_{\alpha + \delta} = \begin{cases}
\xi_{\alpha, n} & \text{for } \alpha \in \Delta^+_{h_0} \\
\xi^{-\alpha}_n & \text{for } \alpha \in \Delta^-_{h_0}
\end{cases}
\]
Moreover, by expressing $\hat{w}$ the equivalent class of $P$ should be identified [14, 15]. Because there exists the isomorphism between states $B.7$ in appendix B.) In the standpoint of the coset CFT it is claimed that any $(\hat{w})$ should be transformed into $U_{\gamma} |\hat{w}\rangle = |\hat{w}'\rangle G/H$ if it exists, and we shall denote it by $\hat{w}_\gamma$. We further introduce the notation;

$$\hat{w}(\gamma) = \sigma_\gamma t_\gamma \quad (3.30)$$

for any element of $\gamma \in \mathcal{P}(g/h)$. The spectral flow $U_{\gamma}$ will be defined as the action of $\hat{w}(\gamma)$.

We are now in a position to write the definition of spectral flow. Under the action of $U_{\gamma}$ ($\gamma \in \mathcal{P}(g/h)$) the $g^C$-current algebra $J_g$ and the ghost fields $(\chi, \psi)$ should be transformed into $U_{\gamma} J_g U_{\gamma}^{-1}, U_{\gamma} \chi U_{\gamma}^{-1}, U_{\gamma} \psi U_{\gamma}^{-1};$

$$U_{\gamma} J_{g,\alpha}(z) U_{\gamma}^{-1} = J_{g,\sigma_\gamma(\alpha)}(z) z^{-(\alpha,\gamma)}$$
$$U_{\gamma} J'_{g}(z) U_{\gamma}^{-1} = J_{g}(z) z^{(\alpha,\gamma)} \quad (3.31)$$
$$U_{\gamma} (h, J_g)(z) U_{\gamma}^{-1} = (\sigma_\gamma(h), J_g)(z) - h(\gamma, h) \delta_{n,0}$$
$$U_{\gamma} \chi_\alpha(z) U_{\gamma}^{-1} = \chi_{\sigma_\gamma(\alpha)}(z) z^{-(\alpha,\gamma)}$$
$$U_{\gamma} \psi^\alpha(z) U_{\gamma}^{-1} = \psi^{\sigma_\gamma(\alpha)}(z) z^{(\alpha,\gamma)} \quad (3.32)$$

Moreover, by expressing $\hat{w}(\gamma)$ as $\hat{w}(\gamma) = \hat{\tau}_\gamma \hat{\omega}_\gamma \in D \times \hat{W}$ ($\hat{\omega}_\gamma \in \hat{W}$, $\hat{\tau}_\gamma \in D$ (= the group of extended Dynkin diagram automorphisms of $g^C$)), the transformations of the primary states are given by

$$U_{\gamma} |\hat{A}, \hat{A}\rangle_g = |\hat{\tau}_\gamma(\hat{A}), \hat{w}(\gamma)(\hat{A})\rangle_g$$
$$U_{\gamma} |0\rangle_{\chi\psi} = \prod_{\hat{\delta} \in \Phi_{\hat{w}(\gamma)}} |\hat{\psi}_{-\hat{\delta}}|_\chi\psi. \quad (3.33)$$

From the transformation rule (3.34) the state $|\hat{A}, \hat{w}\rangle G/H$ (3.20) will be transformed into another physical state $|\hat{A}', \hat{w}'\rangle G/H$ by $U_{\gamma};$

$$U_{\gamma} |\hat{A}, \hat{w}\rangle G/H = |\hat{A}', \hat{w}'\rangle G/H \quad (3.34)$$

Remark that this $\hat{w}'$ is indeed an element of $\hat{W}(g/h)$. (See the proposition B.7 in appendix B.) In the standpoint of the coset CFT it is claimed that any states $|\hat{A}, \hat{w}\rangle G/H$ which are transformed into each other by some spectral flow should be identified [14, 15]. Because there exists the isomorphism between $\mathcal{P}(g/h)$ and $P/Q(h_0);$
$W(g/h) \times Q/Q(h_0)$ (see the proposition B.3 and the proposition B.4 in appendix B) and $D \cong P/Q$.

Nextly we will describe the action of $\mathcal{U}_\gamma$ ($\gamma \in \mathcal{P}(g/h)$) on the $H^C/H$-WZW sector (the $\rho$-sector) and the $H^C$-ghost sector. The $h^C_0, Z(h^C)$-current algebras $J_\rho, J_X$ and the ghost fields $(\zeta, \xi)$ are transformed into $\mathcal{U}_\gamma J_\rho \mathcal{U}_\gamma^{-1}, \mathcal{U}_\gamma J_X \mathcal{U}_\gamma^{-1}$ and $(\mathcal{U}_\gamma \zeta \mathcal{U}_\gamma^{-1}, \mathcal{U}_\gamma \xi \mathcal{U}_\gamma^{-1})$. Their explicit forms are given in (B. 26), (B. 27), (B. 29), (B. 30) in appendix B. From the definition (3. 30) $\hat{w}(\gamma)$ acts on $h^C_0$ as an extended Dynkin diagram automorphism of $h^C_0$. We denote its action as $\hat{\tau}_{h^C_0} \gamma \in D(h^C_0)$ (the group of the extended Dynkin diagram automorphisms of $h^C_0$). Then $\mathcal{U}_\gamma$ transforms the primary states into

$$\mathcal{U}_\gamma |\hat{\tau}_{h^C_0} \gamma \rangle = |\hat{\tau}_{h^C_0} \gamma \rangle$$ (3. 36)

$$\mathcal{U}_\gamma |\beta \rangle_X = |\beta - (k + g^\vee) \gamma \rangle_X$$ (3. 37)

$$\mathcal{U}_\gamma |0 \rangle_{\zeta \xi} = |\hat{\tau}_{h^C_0} \gamma \rangle_{\zeta \xi}$$ (3. 38)

where the state $|\hat{\tau}_{h^C_0} \gamma \rangle_{\zeta \xi}$ is defined by $\prod_{\alpha \in \Delta^+_{h^C_0}} \zeta_{-\hat{\tau}_{h^C_0} \gamma(\alpha)} \xi_\alpha |0 \rangle_{\zeta \xi}$. Especially the $(\zeta, \xi)$ Fock vacuum can be written as $|\hat{\tau}_{h^C_0} \gamma = id \rangle_{\zeta \xi}$.

Since the BRST cohomology state (3. 20) is composed of the primary states of all the sectors or their descendents, its transformation rule under the actions of spectral flows can be derived from the above formulae. Moreover, because of the property (3. 23), this transformed state is also BRST-invariant (and not BRST-trivial). If taking care of the transformation rules of the $\rho$-sector (3. 37) and the $H^C$-ghost sector (3. 38), we find that the transformed state does not necessarily have the form of (3. 20). Namely we will obtain new cohomology classes by making the spectral flows act on (3. 20). We rewrite the BRST-cohomology state (3. 20) as $|\hat{\Lambda}, \hat{w}, \hat{\tau}_{h^C_0} \gamma = id \rangle$. Then $\mathcal{U}_\gamma$ transforms this states into $\mathcal{U}_\gamma |\hat{\Lambda}, \hat{w}, id \rangle$, which we denote by $|\hat{\Lambda}', \hat{w}', \hat{\tau}_{h^C_0} \gamma \rangle$. $\hat{\Lambda}', \hat{w}'$ are those given in (3. 34). In this way we have obtained a family of the physical states labelled by

$$|\hat{\Lambda}', \hat{w}', \hat{\tau}_{h^C_0} \gamma \rangle \in \hat{P}_+^k \times \hat{W}(g/h) \times D(h_0).$$ (3. 39)

The actions of the spectral flows are closed among them. This is because the flows have the following property: For $\gamma_1, \gamma_2 \in \mathcal{P}(g/h)$,

$$\mathcal{U}_{\gamma_1} \mathcal{U}_{\gamma_2} = \mathcal{U}_{\gamma_2} \mathcal{U}_{\gamma_1} = \mathcal{U}_{\gamma_1 \circ \gamma_2},$$ (3. 40)

where $\gamma_1 \circ \gamma_2 \in \mathcal{P}(g/h)$ is defined via the isomorphism (3. 35);

$$P/Q(h_0) \ni [\gamma_1 + \gamma_2] \longmapsto \gamma_1 \circ \gamma_2 \in \mathcal{P}(g/h).$$ (3. 41)
For \( \gamma \in \mathcal{P}(g/h) \), we also define \( \simeq \gamma \in \mathcal{P}(g/h) \) by the image of \([\gamma]\).

We denote the physical observable corresponding to the physical state \(|\hat{\Lambda}, \hat{w}, \hat{\tau}h_0\rangle\) by \( O_{\hat{\Lambda}, \hat{w}, \hat{\tau}h_0} \) as in the semi-classical case (3.11). The ring structure of these operators will be given by the generalization of that of the semi-classical operators (3.13);

\[
C^{(\hat{\Lambda}_3, \hat{w}_3, \hat{\tau}h_0)}_{(\hat{\Lambda}_1, \hat{w}_1, \hat{\tau}_1h_0), (\hat{\Lambda}_2, \hat{w}_2, \hat{\tau}_2h_0)} \propto \prod_{i=1}^{r} F(G_k)^{\hat{\Lambda}_3}_{\hat{\Lambda}_1, \hat{\Lambda}_2} F(H^{(i)}_{0,k+g^r-h^r_0})^{\hat{\lambda}^{(i)}_3}_{\hat{\lambda}^{(i)}_1, \hat{\lambda}^{(i)}_2} \times \delta(\hat{\tau}_1 h_0 \hat{\tau}_2 h_0 \hat{\tau}_3 h_0^{-1}),
\]

(3.42)

where \( \hat{\lambda}^{(i)}_j \equiv \hat{\lambda}^{(i)}(\hat{\Lambda}_j, \hat{w}_j) \) and \( \delta(\hat{\tau}_1 h_0 \hat{\tau}_2 h_0 \hat{\tau}_3 h_0^{-1}) \) stands for the ”delta function on \( D(h_0)\)” defined by

\[
\delta(\hat{\tau}h_0) = \begin{cases} 1 & \hat{\tau}h_0 = \text{id} \\ 0 & \hat{\tau}h_0 \neq \text{id}. \end{cases}
\]

(3.43)

It is an important question whether the field identification by the spectral flow is compatible with this ring structure (3.42). We rewrite the physical operators as \( O_a \) \( (a \in I) \). \( I \) is the index set (3.39). Define the action of spectral flow on \( \{O_a\}_{a \in I} \) by the standard field-state correspondence, that is, the operator \( \mathcal{U}_\gamma(O_a) \equiv O_{\gamma a} \) is given by

\[
\mathcal{U}_\gamma(O_a)|0\rangle = \mathcal{U}_\gamma|a\rangle,
\]

(3.44)

where \(|a\rangle\) is the physical state corresponding to \( O_a \). Then we can show the following identity with respect to the structure constant \( C^{c}_{ab} \):

\[
C^{(\gamma_1 + \gamma_2 - c)}_{\gamma_1 - a, \gamma_2 - b} = C^{c}_{ab},
\]

(3.45)

or equivalently,

\[
\mathcal{U}_{\gamma_1}(O_a)\mathcal{U}_{\gamma_2}(O_b) = \sum_{c \in I} C^{c}_{ab} \mathcal{U}_{\gamma_1 + \gamma_2}(O_c).
\]

(3.46)

This can be proved by observing the explicit form of \( C^{c}_{ab} \) (3.42). The most non-trivial part of the proof is the following relation for the G-WZW sector;

\[
F(G_k)^{\hat{\tau} \hat{\gamma} + \hat{\gamma}'}_{\gamma, (\hat{\Lambda}_1), \gamma, (\hat{\Lambda}_2)} = F(G_k)^{\hat{\Lambda}_1}_{\hat{\Lambda}_1, \hat{\Lambda}_2},
\]

(3.47)

and the similar relations of the \( H^{(i)}_0 \)-parts. This relation (3.47) is shown by making use of the Verlinde’s formula on the fusion coefficients [20] and
the modular transformation properties of the affine characters. (Refer for example \[15, 21, 26\].) In the next section we will derive this formula (3.10) from a more physical viewpoint, that is, as a direct consequence of the topological invariance of the system.

The identity (3.45) implies that the field identifications are consistent with the ring structure of the BRST-cohomology. Especially if we set

\[ [O_a] = \{ U_\gamma(O_a) ; \forall \gamma \in \mathcal{P}(g/h) \}, \quad (3.48) \]

we can consistently introduce a product on them by

\[ [O_a][O_b] \overset{\text{def}}{=} [O_aO_b] \quad (3.49) \]

without depending on the choice of representatives. Hence, on the level of local properties of our model, these field identifications completely work and one may always extract at most finite physical degrees of freedom.

However, once we turn our attention to the global structure of the system, we will face with some subtle problem. Especially, if we calculate the correlation functions with some fixed back-ground topology (Euler number and Chern classes in our case), we will find that

\[ \langle U_\gamma(O_a)O_b \ldots O_c \rangle_{g,c_1} \neq \langle O_aO_b \ldots O_c \rangle_{g,c_1}, \quad (3.50) \]

because \( O_a \) and \( U_\gamma(O_a) \) have different ghost numbers although they have the same \( N = 2 \ U(1) \)-charges. In the next section we will discuss this problem in detail.

### 4 Correlation Functions of Physical Observables and a Geometrical Interpretation of Spectral Flow

In this section we will study the correlators of the physical observables constructed in section 3, (the correlators among the 0-form components). We shall perform this estimation with the topology fixed, that is, the genus \( g \) of \( \Sigma \) and the “vector of Chern numbers” \( c_1 \equiv \frac{i}{2\pi} \int F(a) \) fixed to be some definite values. For the genus \( g \), the higher genus correlation functions \( (g \geq 2) \) always become zero. This is a simple result obtained from counting of the anomaly for the \( N = 2 \ U(1) \)-charge (2.47) (of the Kazama-Suzuki sector). Because all the physical observables obtained above have non-negative \( N = 2 \)
$U(1)$-charges, while its back-ground charge is equal to $Q_{KS}(1 - g)$, which is negative when $g \geq 2$. ($Q_{KS} = \dim m_+ - \frac{4}{k + g} \rho_0^2 / h > 0$.) We shall consider the genus 0 case. We write the correlator on sphere as $\langle \cdots \rangle_{c_1}$. We should manifestate the domain in which $c_1$ takes its value. Notice that $c_1$ must belong to the weight lattice $P$.\footnote{Precisely speaking, $c_1 \in \nu^{-1}(P)$, where $\nu : t^C \rightarrow t^{C*}$ is the isomorphism defined by the inner product. (See appendix B.) But we shall here omit it to avoid the complexity of notations.}

The back-ground charges in (4.2) can be determined from the estimations of the ghost number or the chiral anomalies. Let $N^{(\alpha)}_{\chi \psi} \equiv \frac{1}{2\pi i} \oint dz : \psi^\alpha \chi_\alpha : (\alpha \in \Delta^+_0)$, $N^{(\alpha)}_{\zeta \xi} (\alpha \in \Delta^+_0)$, and $N^{(j)}_{\zeta \xi} (j = 1, \ldots, \text{rank} g)$ be the ghost number operators for $(\chi, \psi), (\zeta, \xi)$-ghost sectors. The out-vacuum $\langle 0, c_1 \rangle$ should be characterized by the conditions;

$$\langle 0, c_1 | N^{(\alpha)}_{\chi \psi} = \langle 0, c_1 | \{1 + (\alpha, c_1)\}$$

Let us study the correlation function in the operator formalism. In this approach the correlation function may be regarded as the matrix element under the following configuration of the back-ground gauge field $a = (a_0, i\omega)$ (2.12):

$$\mathcal{F}(a) = -\pi c_1 \delta^{(2)}(z - \infty) d\bar{z} \wedge dz. \quad (4.1)$$

We normalize the delta function as $\int \delta^{(2)}(z) d\bar{z} \wedge dz = 2i$. So we can define the correlator by

$$\langle O_{a_1}(x_1) O_{a_2}(x_2) \ldots O_{a_3}(x_3) \rangle_{c_1} = \langle 0, c_1 | O_\zeta(x_0) O_{a_1}(x_1) O_{a_2}(x_2) \ldots O_{a_n}(x_n) | 0 \rangle. \quad (4.2)$$

The in-vacuum $|0\rangle$ has no back-ground charge. The out-vacuum $\langle 0, c_1 \rangle$ is the BRST-invariant state having the suitable back-ground charges corresponding to $c_1$. This is an analogous situation as that in the Coulomb gas realization of CFT [8]. $O_\zeta(x)$ denotes the BRST-invariant operator which state is $\prod \xi_\alpha |0\rangle_{\zeta \xi} \otimes |0\rangle_{\text{other}}$. This operator should be inserted in order to cancel the $\zeta \xi$-ghost number anomalies. In the following we may omit to write it explicitly. We also notice that, since our EM tensor is BRST-exact (2.38), the correlation function (4.2) does not depend on the operator insertion points. So we may also omit to write these insertion points.

The back-ground charges in (4.2) can be determined from the estimations of the ghost number or the chiral anomalies. Let $N^{(\alpha)}_{\chi \psi} \equiv \frac{1}{2\pi i} \oint dz : \psi^\alpha \chi_\alpha : (\alpha \in \Delta^+_0)$, $N^{(\alpha)}_{\zeta \xi} (\alpha \in \Delta^+_0)$, and $N^{(j)}_{\zeta \xi} (j = 1, \ldots, \text{rank} g)$ be the ghost number operators for $(\chi, \psi), (\zeta, \xi)$-ghost sectors. The out-vacuum $\langle 0, c_1 \rangle$ should be characterized by the conditions;

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$$\langle 0, c_1 | N^{(\alpha)}_{\chi \psi} = \langle 0, c_1 | \{1 + (\alpha, c_1)\}$$

We normalize the delta function as $\int \delta^{(2)}(z) d\bar{z} \wedge dz = 2i$. So we can define the correlator by

$$\langle O_{a_1}(x_1) O_{a_2}(x_2) \ldots O_{a_3}(x_3) \rangle_{c_1} = \langle 0, c_1 | O_\zeta(x_0) O_{a_1}(x_1) O_{a_2}(x_2) \ldots O_{a_n}(x_n) | 0 \rangle. \quad (4.2)$$

The in-vacuum $|0\rangle$ has no back-ground charge. The out-vacuum $\langle 0, c_1 \rangle$ is the BRST-invariant state having the suitable back-ground charges corresponding to $c_1$. This is an analogous situation as that in the Coulomb gas realization of CFT [8]. $O_\zeta(x)$ denotes the BRST-invariant operator which state is $\prod \xi_\alpha |0\rangle_{\zeta \xi} \otimes |0\rangle_{\text{other}}$. This operator should be inserted in order to cancel the $\zeta \xi$-ghost number anomalies. In the following we may omit to write it explicitly. We also notice that, since our EM tensor is BRST-exact (2.38), the correlation function (4.2) does not depend on the operator insertion points. So we may also omit to write these insertion points.

The back-ground charges in (4.2) can be determined from the estimations of the ghost number or the chiral anomalies. Let $N^{(\alpha)}_{\chi \psi} \equiv \frac{1}{2\pi i} \oint dz : \psi^\alpha \chi_\alpha : (\alpha \in \Delta^+_0)$, $N^{(\alpha)}_{\zeta \xi} (\alpha \in \Delta^+_0)$, and $N^{(j)}_{\zeta \xi} (j = 1, \ldots, \text{rank} g)$ be the ghost number operators for $(\chi, \psi), (\zeta, \xi)$-ghost sectors. The out-vacuum $\langle 0, c_1 \rangle$ should be characterized by the conditions;

$$\langle 0, c_1 | N^{(\alpha)}_{\chi \psi} = \langle 0, c_1 | \{1 + (\alpha, c_1)\}$$

Precisely speaking, $c_1 \in \nu^{-1}(P)$, where $\nu : t^C \rightarrow t^{C*}$ is the isomorphism defined by the inner product. (See appendix B.) But we shall here omit it to avoid the complexity of notations.
\[ \langle 0, c_1 | N^{(i)}_{\xi\xi} \rangle = \langle 0, c_1 | \{1 + (\alpha, c_1)\}, \quad (4.4) \]
\[ \langle 0, c_1 | \xi \rangle = \langle 0, c_1 \rangle, \quad (4.5) \]
\[ \langle 0, c_1 | (J_{g, 0}, h) \rangle = \langle 0, c_1 | \{-k \langle c_1, h \rangle\}, \quad (\forall h \in t^C) \quad (4.6) \]
\[ \langle 0, c_1 | (J^{(i)}_{\rho, 0}, h) \rangle = \langle 0, c_1 | \{(k + g^\vee + h_i^\vee)\langle c_1, h \rangle\}, \quad (\forall h \in t(h^{(i)}_0)^C) \quad (4.7) \]
\[ \langle 0, c_1 | (J_{X, 0}, h) \rangle = \langle 0, c_1 | \{2\rho_{g/h}, h\} + (k + g^\vee)\langle c_1, h \rangle\}, \quad (\forall h \in Z(h^C)). \quad (4.8) \]

It is worthwhile to notice that the relation,
\[ \langle 0, c_1 | J G/H, 0 = \langle 0, c_1 | Q_{KS}, \quad (4.9) \]
holds without depending on \( c_1 \), which suggests some geometrical meaning of the Kazama-Suzuki \( U(1) \)-current \( J_G/H \) \((2.47)\). The charge of it generates a global \( U(1) \)-rotation on both the bosonic part \( g \) (and necessarily \( X \)) and the fermionic part \( \chi \psi \) along the direction which does not depend on \( c_1 \).

In order to give the explicit form of the out-vacuum state we first notice that the state \( \langle 0, c_1 = 0 \rangle \) still has the non-zero back-ground charges which are due to the Fegin-Fucks term \( \sim \int \rho_{g/h}(X)R(g) \) in \( S_X \) \((2.21)\). This out-vacuum \( \langle 0, c_1 = 0 \rangle \) is realized by
\[
\langle 0, c_1 = 0 | = \langle k\Lambda_0, k\Lambda_0|_g \otimes \bigotimes_{i=1}^{r} (-(k + g^\vee + h_i^\vee)\Lambda_0, -\langle k + g^\vee + h_i^\vee\rangle\Lambda_0|_{\rho^{(i)}} \otimes \langle 2\rho_{g/h}|_x \otimes \langle 0|_{\chi\psi} \prod_{\alpha \in \Delta^a_{0}} \chi_{\alpha, 0} \otimes \langle 0|_{\xi\xi} \prod_{a \in h} \zeta_{a}^a \quad (4.10) \]
\[
\equiv \langle k\Lambda_0, w_0, 0 | ,
\]
in the notation of the previous section: \( (k\Lambda_0, w_0, id) \in \hat{P}^k_{+} \times \hat{W}(g/h) \times D(h_0) \).

\( w_0 \) is the element having the maximal length in \( W(g/h) \). For the description of \( \langle 0, c_1 \rangle \) with general values of \( c_1 \) we utilize the relation;
\[ \langle 0, c_1 | U_\gamma = \langle 0, c_1 \hat{\gamma} \rangle, \quad (4.11) \]
which follows from the facts that the spectral flow \( U_\gamma \) preserves the BRST-invariance and that the state \( \langle 0, c_1 | U_\gamma \) satisfies the same conditions \((4.3)-(4.11)\) as those of \( \langle 0, c_1 \hat{\gamma} \rangle \). This identity \((4.11)\) is significant from the view of topological theory because it enables us to interpret the spectral flows as the transformations which connect the sectors with different Chern numbers, that is, the different instanton sectors. By applying the equality \((4.11)\) to the state \( \langle 0, c_1 = 0 | \) \((4.10)\) we can obtain the following general expression of \( \langle 0, c_1 \rangle : \)
\[ \langle 0, c_1 = \gamma | = \langle 0, c_1 = 0 | U_\gamma \]
\[ = \langle \hat{\gamma}^{-1}k\Lambda_0 \rangle, \hat{w}^{-1}\hat{w}_0\hat{w}_0^{-1}, \hat{\gamma}^{-1}h_0^{-1}, \hat{\gamma}^{-1}, \hat{\gamma}^{-1}h_0^{-1} \rangle, \quad (4.12) \]
where \( \tilde{\tau}_\gamma, \tilde{w}(\gamma) \) and \( \tilde{\tau}_\gamma^{\text{H}_0} \) are, as in the previous section, the elements uniquely defined by \( \gamma \).

To begin with, we shall consider the one point function. There exists a unique physical operator \( O_{\text{max}}^{c_1}(x) \) defined by the condition

\[
\langle O_{\text{max}}^{c_1} \rangle_{c_1} = 1. \tag{4. 13}
\]

This is the operator which state is dual to the out-vacuum \( \langle 0, c_1 = \gamma | \) [12], that is, the state \( | \tilde{\tau}_\gamma^{-1}(k\Lambda_0), \tilde{w}(\gamma)^{-1}w_0\tilde{\tau}_\gamma, \tilde{\tau}_\gamma^{\text{H}_0}^{-1} \rangle \). It has the maximum \( U(1) \)-charge of our model, which is equal to \( Q_{\text{KS}} \). For example \( O_{\text{max}}^{c_1=0} \) is the operator corresponding to \( | k\Lambda_0, w_0, \text{id} \rangle \). It includes the term \( \prod_{\alpha \in \Delta^+_m} \psi_\alpha^0 | 0 \rangle \chi_{\psi} \) for the \( \chi_{\psi} \)-sector. This makes us interpret \( O_{\text{max}}^{c_1=0} \) as the top cohomology class of \( H^*_{\text{DR}}(G/H) \) defined by the volume element of \( G/H \). How about a case of \( c_1 \neq 0 \)? The physical state corresponding to \( O_{\text{max}}^{c_1} \) includes the elements,

- \( \psi_0^\alpha, \ldots, \psi_{(\alpha,c_1)}^\alpha \) if \( \alpha \in \Delta^+_m \) satisfies \( (\alpha, c_1) \geq 0 \),
- \( \chi_{\alpha,-1}, \ldots, \chi_{\alpha,(c_1)+1} \) if \( \alpha \in \Delta^+_m \) satisfies \( (\alpha, c_1) \leq -2 \),
- no modes of \( \chi_{\alpha}, \psi^\alpha \) if \( \alpha \in \Delta^+_m \) satisfies \( (\alpha, c_1) = -1 \).

These modes have the following geometrical meaning. Let \( \mathcal{L}_\alpha \) be the line bundle in which \( \psi^\alpha \) lives. Then \( \chi_{\alpha} \) is a section of \( K \otimes \mathcal{L}_\alpha^{-1} \). \( K \equiv T^{1,0} \Sigma \) is the canonical bundle of \( \Sigma \). The space of the solutions of the equation of motion for \( \psi^\alpha \) is \( H^0(\Sigma, \mathcal{L}_\alpha) \), and that for \( \chi_{\alpha} \) is \( H^0(\Sigma, K \otimes \mathcal{L}_\alpha^{-1}) \). These spaces can be realized by

\[
H^0(\mathbb{C}P^1, \mathcal{L}) \cong \begin{cases} 0, & \deg \mathcal{L} < 0, \\ P_n(X_0, X_1), & \deg \mathcal{L} = n \geq 0, \end{cases} \tag{4. 14}
\]

where \( X_0, X_1 \) denote the homogeneous coordinates of \( \mathbb{C}P^1 \), and \( P_n(X_0, X_1) \) is the set of homogeneous polynomials of order \( n \). Since we are now considering the situation of \( F(a) \sim c_1 d(2)/(z - \infty) \), we may regard as \( \mathcal{L} = \mathcal{O}(\infty)^n \), so the \( n + 1 \) independent elements of \( H^0(\mathbb{C}P^1, \mathcal{L}) \) behave as \( 1, z, \ldots, z^n \) around \( z = 0 \). Moreover, by recalling \( \deg \mathcal{L}_\alpha = (\alpha, c_1) \), \( \deg K \otimes \mathcal{L}_\alpha^{-1} = -2 - (\alpha, c_1) \) and the mode expansions

\[
\psi^\alpha(z) = \sum_{n \in \mathbb{Z}} \frac{\psi_n^\alpha}{z^n}, \quad \chi_{\alpha}(z) = \sum_{n \in \mathbb{Z}} \frac{\chi_{\alpha,n}}{z^{n+1}},
\]

we can find that \( O_{\text{max}}^{c_1} \) just includes all the solutions of the equations of motion for \( \psi^\alpha \), \( \chi_{\alpha} \). Therefore we can say \( O_{\text{max}}^{c_1} \) corresponds to the “top cohomology class” on the instanton moduli space. Because of the relation

\[
O_{\text{max}}^{c_1=\gamma} = U_{\gamma}(O_{\text{max}}^{c_1=0}), \tag{4. 15}
\]

which is easily shown from [4, 12] \( U_{\gamma} \) maps the top cohomology class on the \( c_1 = 0 \) instanton moduli space to that on the \( c_1 = \gamma \) moduli space.
Nextly we shall study the correlator of the form \( \langle \prod_{i=1}^{N} O_{a_i} \rangle_{c_1} \). We first notice that this correlator is non-zero if and only if the following relation holds;

\[
\prod_{i=1}^{N} O_{a_i} \sim \text{const} O_{\text{max}}^{c_1} + \sum_b O_b \pmod{\text{BRST}}. \tag{4.16}
\]

Making use of this observation and combining the relations (4.15), (3.46) successively, we can show the formula;

\[
\langle U_{\gamma_1}(O_{a_1}) U_{\gamma_2}(O_{a_2}) \ldots U_{\gamma_n}(O_{a_n}) \rangle_{c_1} = \langle O_{a_1} O_{a_2} \ldots O_{a_n} \rangle_{c_1} + \sum_{i=1}^{n} \gamma_i. \tag{4.17}
\]

This means that the field identification rule (3.48) by the spectral flow is still consistent at the level of correlators if summing them up with respect to the Chern numbers \( c_1 \). Namely the correlation function among the identified observables should be defined by;

\[
\langle [O_{a_1}] [O_{a_2}] \ldots [O_{a_n}] \rangle \overset{\text{def}}{=} \sum_{c_1 \in \mathcal{P}(g/h)} \langle O_{a_1} O_{a_2} \ldots O_{a_n} \rangle_{c_1}, \tag{4.18}
\]

or equivalently, by fixing \( c_1 \) to be some definite value, one may take the next;

\[
\langle [O_{a_1}] [O_{a_2}] \ldots [O_{a_n}] \rangle \overset{\text{def}}{=} \sum_{\gamma \in \mathcal{P}(g/h)} \langle U_{\gamma}(O_{a_1}) O_{a_2} \ldots O_{a_n} \rangle_{c_1}. \tag{4.19}
\]

It is interesting to derive the identity (4.17) from somewhat different viewpoints. We restrict ourselves to a simple case \( G/H = G/T \). We start with the definition of correlator by path-integration;

\[
\left\langle \prod_{i=1}^{N} O_{a_i}(z_i) \right\rangle_{c_1} = \int \mathcal{D}(g, X, \chi, \psi, \zeta) \prod_{j=1}^{N} O_{a_j}(z_j) e^{-kS_G(g, \omega) - S_X(X, \omega) - S_{\chi, \psi}(\chi, \psi, \omega) - S_{\zeta, \xi}(\zeta, \xi, \omega)}. \tag{4.20}
\]

\( \omega \) in (4.20) is the real valued back-ground gauge field; \( a = i\omega \) and \( c_1 = -\frac{1}{2\pi} \int_{\Sigma} F(\omega) \). Since we study the holomorphic part only, we may set \( \omega^{10} = 0 \) by treating \( \omega^{10} \) and \( \omega^{01} \) as independent variables.

Let us consider the case of \( \left\langle \prod_{j=1}^{N} O_{a_j}(z_j) \right\rangle_{c_1} \) with \( c_1 = \sum_{j=1}^{N} \gamma_j, \gamma_j \in \mathcal{P} \) and \( a_j \equiv (\hat{A}_j, w_j) \in \hat{P}_k \times W(g/h) \) for \( \forall j \). Namely, we set all the inserted

\[\text{12}^\text{Strictly speaking, we should treat the holomorphic and the anti-holomorphic parts simultaneously by setting } \omega^{10} = \omega^{01} \].
observables semi-classical. First we point out that the topological invariance
of our theory implies that the correlation function (4.20) does not depend
on any configuration of the back-ground gauge field \( \omega \) as far as the Chern
numbers \( c_1 \) are fixed. It is assured by the BRST-exactness (2.42) of the
total current \( J_{\text{tot}}^t = J_g^t + J_X^t + J_{\chi\psi}^t \). Hence one may choose the following
configuration;

\[
\omega^{01}|_{U_0} = -i \sum_{j=1}^{N} \gamma_j \bar{\partial} \log(z - z_j), \quad \omega^{01}|_{U_{\infty}} = 0, \tag{4.21}
\]

where \( U_0, U_{\infty} \) are the coordinate patches around \( 0, \infty \) such that \( \Sigma = U_0 \cup U_{\infty}, \{z_j\} \subset U_0, \{z_j\} \notin U_{\infty} \). This configuration gives

\[
F(\omega) = i\pi \sum_{j=1}^{N} \gamma_j \delta^{(2)}(z - z_j) d\bar{z} \wedge dz, \tag{4.22}
\]

and then \( c_1 = \sum_{j=1}^{N} \gamma_j \). We are now considering the situation such that the
curvature of the back-ground gauge field has some delta function singularities
at the points \( z_j \) where the observables \( O_{a_i} \) have been inserted, while in the
situation considered before we took \( F(\omega) \sim \sum_{j} \gamma_j \delta^{(2)}(z - \infty) \). These curva-
ture singularities will give some non-trivial modifications to the observables
inserted at the points of singularities.

To recognize the above mentioned effect directly it may be helpful to
consider first the \( X \)-sector. Recall that the action of \( X \)-sector (2.21) has
the term \( \sim \int (X, F(\omega)) \). Then the substitution of (4.22) into the action (2.21) is equivalent to the insertion of some vertex operators at the points \( z_j \).
This effect will add the value \( -(k + g^{\nu}) \gamma_j \) on to the \( X \)-momentum of \( O_{a_j}(z_j) \).
It signals the possibility that the instanton contribution modifies the physical
observables. This story is, however, too naive to justify completely. Because
the vertex operator insertions from the curvature singularities are at the
same points as that \( O_{a_j} \) are inserted, we should treat carefully the OPE
singularities among them. We will later discuss these OPE singularities.

In order to proceed further we shall consider the following chiral gauge
transformation\(^{13}\):

\[
\Omega(t)(z) = e^{tu(z)}, \quad u(z) = \frac{h}{z - w}, \quad (\forall h \in t), \tag{4.23}
\]

and we replace \( -kS_g(g : \omega) \) by \( -kS_g(g : \Omega(t)^{-1}\omega) \) in (4.20). Then, by

\(^{13}\)We set \( \Omega^t = 1 \) formally, since we are here dealing with the holomorphic sector only.
changing the integration variable \( g \) to \( \Omega(t)^{-1}g \), we can get the identity;

\[
\int \mathcal{D}(g, X, \chi, \psi, \zeta, \xi) \prod_{j=1}^{N} O_{a_j}(z_j) \exp\{-kS_{G}(g : \Omega(t)^{-1}\omega) - S_{X}(X : \omega) - S_{X\psi}(\chi, \psi : \omega) - S_{\zeta\xi}(\zeta, \xi : \omega)\} = \int \mathcal{D}(g, X, \chi, \psi, \zeta, \xi) \prod_{j=1}^{N} e^{t(w_j(\Lambda_j), u(z_j))} O_{a_j}(z_j) \exp\{-kS_{G}(g : \omega) + kS_{G}(\Omega(t) : \omega)\} \tag{4. 24}
\]

Here we have used the fact; \( \mathcal{D}(\Omega(t)^{-1}g) = \mathcal{D}g, \quad O_{a_j}(z_j)[\Omega(t)^{-1}g] = e^{t(w_j(\Lambda_j), u(z_j))} O_{a_j}(z_j)[g] \) and the Polyakov-Wiegmann identity; \( S_{G}(\Omega(t)^{-1}g : \Omega(t)^{-1}\omega) = S_{G}(g : \omega) - S_{G}(\Omega(t) : \omega) \). Differentiating the both hand sides of (4. 24) with respect to \( t \) and then setting \( t = 0 \), we obtain the Ward identity;

\[
\int \mathcal{D}(g, X, \ldots)(h, J_{g}^{t}(w)) \prod_{j=1}^{N} O_{a_j}(z_j) e^{-S_{\text{tot}}(g, X, \ldots; \omega)} = \sum_{j=1}^{N} \frac{\langle w_{j}(\Lambda_{j}) + k\gamma_{j}, h \rangle}{w - z_{j}} \int \mathcal{D}(g, X, \ldots) \prod_{j=1}^{N} O_{a_j}(z_j) e^{-S_{\text{tot}}(g, X, \ldots; \omega)}. \tag{4. 25}
\]

Notice that the net effect of the curvature singularity \( \sim \gamma_{j} \delta^{(2)}(z - z_{j}) \) is the shift of the weight of \( g \)-sector; \( w_{j}(\Lambda_{j}) \rightarrow w_{j}(\Lambda_{j}) + k\gamma_{j} \). The similar arguments also work for the \( X \)-sector and the \( \chi \psi \)-sector. By replacing \( S_{X}(X : \omega) \) (or \( S_{X}\psi(\chi, \psi : \omega) \)) by \( S_{X}(X : \Omega(t)^{-1}\omega) \) (or \( S_{X}\psi(\chi, \psi : \Omega(t)^{-1}\omega) \)), and making use of the next formulas

\[
\begin{align*}
\mathcal{D}(\Omega(t)^{-1}X) &= \mathcal{D}(X), \\
S_{X}(\Omega(t)^{-1}X : \Omega(t)^{-1}\omega) &= S_{X}(X : \omega) - (k - g^{\gamma})S_{G}(\Omega(t) : \omega), \\
\mathcal{D}(\Omega(t)^{-1}X, \Omega(t)^{-1}\psi) &= \mathcal{D}(X, \psi) e^{g^{\gamma}S_{G}(\Omega(t) : \omega)}, \\
S_{X\psi}(\Omega(t)^{-1}X, \Omega(t)^{-1}\psi : \Omega(t)^{-1}\omega) &= S_{X\psi}(X, \psi : \Omega(t)^{-1}\omega),
\end{align*}
\tag{4. 26}
\]

one can show the Ward identities;

\[
\int \mathcal{D}(g, X, \ldots)(h, J_{X}^{t}(w)) \prod_{j=1}^{N} O_{a_j}(z_j) e^{-S_{\text{tot}}(g, X, \ldots; \omega)} = \sum_{j=1}^{N} \frac{\langle w_{j}(\rho_{g}G - G\rho_{g}) + g^{\gamma}\gamma_{j}, h \rangle}{w - z_{j}} \int \mathcal{D}(g, X, \ldots) \prod_{j=1}^{N} O_{a_j}(z_j) e^{-S_{\text{tot}}(g, X, \ldots; \omega)}, \tag{4. 27}
\]

\[
\int \mathcal{D}(g, X, \ldots)(h, J_{X\psi}^{t}(w)) \prod_{j=1}^{N} O_{a_j}(z_j) e^{-S_{\text{tot}}(g, X, \ldots; \omega)} = \sum_{j=1}^{N} \frac{\langle w_{j}(\rho_{g} - \rho_{g}G - g^{\gamma}\gamma_{j}, h \rangle}{w - z_{j}} \int \mathcal{D}(g, X, \ldots) \prod_{j=1}^{N} O_{a_j}(z_j) e^{-S_{\text{tot}}(g, X, \ldots; \omega)}. \tag{4. 28}
\]
What do the identities (4.25), (4.27), (4.28) mean? These identities suggest that, under the singular background $F(\omega) \sim \gamma_0 \delta(z - z_j)$, the chiral primary field $O_{a_j}(z_j)$ behaves as $\mathcal{U}_j(O_{a_j})(z_j)$ in the flat background. Because the BRST-invariance is not broken even in the non-flat background, the BRST-invariant operator $O_{a_j}(z_j)$ should be changed to another BRST-invariant operator by the effect of curvature singularity. Comparing the spectrum of physical observables studied in the previous section with the data of the Ward identities, we can conclude that $O_{a_j}(z_j)$ should be changed to the operator $\mathcal{U}_j(O_{a_j})(z_j)$. So, we have again obtained the relation (4.17). Notice that this derivation of (4.17) does not need the result (3.46) which played an important role to obtain the relation (4.17) in the previous section. We can rather show this formula (3.46) very easily by means of the above result. In fact, if noting that

$$\langle \mathcal{U}_{\gamma_1}(O_{a_{1}})\mathcal{U}_{\gamma_2}(O_{a_{2}})\prod_{\text{others}} O_{b} \rangle_{c_{1}} = \langle O_{a_{1}} O_{a_{2}} O_{a_{3}} \prod_{\text{others}} O_{b} \rangle_{c_{1}},$$

(4.29)

which is a trivial identity derived from (4.17), the desired result (3.46) follows. Thus the relation (3.46) is a natural consequence of the topological invariance of our model.

Related with the above discussion based on the path integration it may be helpful to study the local operator formulation around the point where the background gauge field is singular. Consider again (4.21) as the configuration of the background gauge field and pay attention to a particular point, say, $z_1$. We take a holomorphic coordinate $\{z, U\}$ around $z_1$ such that $z_1 = 0$, $z_j \not\in U$ $(j \neq 1)$ and simply write as $\gamma_1 = \gamma$, namely,

$$\omega^{01}|_U = -i \gamma \bar{\partial} \log z.$$

(4.30)

Let us consider the operator formalism on $U$. Notice that, by setting $\Omega(\gamma) = z^{-\nu-1(\gamma)}$, $\omega^{01}$ can be trivialized on $U$;

$$i\omega^{01}|_U = -\bar{\partial} \Omega(\gamma) \Omega^{-1} \Omega(\gamma)|_U.$$

(4.31)

Thanks to this relation we can consistently give the operators appropriate to the back-ground (4.30). Let $\mathcal{O} = \mathcal{O}[g, X, \chi, \psi]$ be a general operator under the flat back-ground, then the corresponding operator $\mathcal{O}^{(\gamma)}$ which is appropriate to the singular back-ground (4.30) (or (4.31)) should be defined by the gauge transform $\mathcal{O}^{(\gamma)} = \Omega(\gamma)\mathcal{O}$, that is,

$$\mathcal{O}^{(\gamma)}[g, X, \chi, \psi] \overset{\text{def}}{=} \mathcal{O}[\Omega^{-1}(\gamma)g, \Omega^{-1}(\gamma)X, \Omega^{-1}(\gamma)\chi, \Omega^{-1}(\gamma)\psi]$$

$$\equiv \mathcal{O}[\Omega^{-1}(\gamma)g, X + \alpha_+ \log \Omega^{-1}(\gamma)\chi, \Omega^{-1}(\gamma)\psi].$$

(4.32)
In particular the current $J_g$ is transformed to $J_g^{(\gamma)}$;

$$
J_g^{(\gamma)}(z) = J_g(z)[\Omega_{(\gamma)}^{-1} g] = \Omega_{(\gamma)}^{-1} J_g(z) \Omega_{(\gamma)} + k \Omega_{(\gamma)}^{-1} \partial \Omega_{(\gamma)}(z). \tag{4.33}
$$

If we substitute $\Omega_{(\gamma)}(z) = z^{-\nu_{-1}(\gamma)}$ into (4.33) and compare it with the definition of spectral flows (3.31), we can find the following identity:

$$
J_g^{(\gamma)}(z) = U_{\gamma} J_g(z) U_{\gamma}^{-1}. \tag{4.34}
$$

It is easy to observe that the same relations also hold for other fields;

$$
X^{(\gamma)}(z) = U_{\gamma} X(z) U_{\gamma}^{-1}, \quad \chi^{(\gamma)}(z) = U_{\gamma} \chi(z) U_{\gamma}^{-1}, \quad \psi^{(\gamma)}(z) = U_{\gamma} \psi(z) U_{\gamma}^{-1}. \tag{4.35}
$$

These mean that the singular gauge transformation $\Omega_{(\gamma)}$ which connects the flat background (on $U$) to the singular background (4.30) is realized by the spectral flow $U_{\gamma}$. These transformed operators will characterize the state vector which is inserted at the origin of the coordinate patch $U$. Especially the vacuum vector under the singular background (4.30) will be given by

$$
|0\rangle_{(\gamma)} = U_{\gamma}|0\rangle, \tag{4.36}
$$

where $|0\rangle$ is the usual vacuum with the locally flat background on $U$, that is, $\tilde{\omega}$ such that $\tilde{\omega} = \Omega_{(\gamma)}^{-1} \omega$. The gauge transformed physical observables can be written as

$$
O_a^{(\gamma)}(z) = U_{\gamma} O_a(z) U_{\gamma}^{-1}. \tag{4.37}
$$

Hence the next readily follows;

$$
O_a^{(\gamma)}(0)|0\rangle_{(\gamma)} = U_{\gamma}(O_a)(0)|0\rangle. \tag{4.38}
$$

This relation can be read as follows: In the L.H.S of (4.38) the “topological charge” $\gamma$ is attached to the vacuum vector $|0\rangle_{(\gamma)}$, which correspond to the existence of curvature singularity. While, in the RHS of (4.38) the vacuum vector has no charge and the charge $\gamma$ is absorbed into the operator $O_a$, which means $O_a$ is changed to $U_{\gamma}(O_a)$.

For a general operator $O(z)$ its gauge transformed partner $O^{(\gamma)}(z)$ will have a singularity at $z = 0$. This is because, in general, $O^{(\gamma)}$ has the same singularity as that of $\Omega_{(\gamma)}$. Can we define the operator $O^{(\gamma)}(0)$ in the L.H.S of (4.38)? This issue has its origin in the singular configuration of the background gauge field (4.30) and the same difficulty has appeared in the earlier discussion of the X-sector, where it is necessary to insert some vertex operators at the points other operators have been inserted. We cannot avoid this difficulty if we study each sector independently. However, fortunately we can find, from the definitions of spectral flow (3.31), ..., that the singularity
of each sector precisely cancels in all. This is not surprising, since the BRST-invariance (in particular, the $Q_T$-invariance) of the operator $O_a$ leads to the chiral gauge invariance. Actually $O_a^{\gamma}(z)$ should define the same cohomology class as that of $O_a(z)$, namely,

$$O_a^{\gamma}(z) = O_a(z) + \{Q_{\text{tot}}, *\}$$  \hspace{1cm} (4. 39)

should hold. So we may rewrite (4. 38) as

$$O_a(0)|0\rangle_{(\gamma)} = \mathcal{U}_t(O_a)(0)|0\rangle + Q_{\text{tot}}|\ast\rangle.$$  \hspace{1cm} (4. 40)

To sum up, one may say: When (and only when) treating the $g, X, \chi\psi$-sectors as a combined system and considering only the BRST-invariant operators, one can construct a consistent operator formalism even under the singular background (4. 30). This reflects the fact that the total system is topologically invariant but each sector is not.

It may be also interesting to consider the following configuration of the back-ground gauge field instead of (4. 30);

$$\omega|_U = -\gamma \omega(g)|_U,$$  \hspace{1cm} (4. 41)

where $\omega(g)$ is the Levi-Civita connection. Since the Euler number of $U \simeq \text{semisphere}$, is equal to 1, $-\frac{1}{2\pi} \int_U F(\omega) = \gamma$ holds. Under this configuration of $\omega$ (4. 41) the conformal fields in our model will gain some extra spins. For example, the ghost $\psi^\alpha(z)$ will be transformed to

$$\psi^{(\gamma)\alpha}(z)(\equiv \mathcal{U}_t, \psi^\alpha(z)\mathcal{U}^{-1}_t) = \sum_{n \in \mathbb{Z}} \frac{\psi_n^{\alpha}}{z^{n-(\alpha,\gamma)}},$$

$$= \sum_{n \in \mathbb{Z}} \psi_n^{(\gamma)\alpha} z^n,$$  \hspace{1cm} (4. 42)

where

$$\psi_n^{(\gamma)\alpha} (\equiv \mathcal{U}_t, \psi_n^\alpha \mathcal{U}_t^{-1}) = \psi_n^{\alpha}_{n+(\alpha,\gamma)}.$$  \hspace{1cm} (4. 43)

The first line of (4. 42) and the vacuum condition; $\psi_n^{\alpha}|0\rangle_{(\gamma)} = 0$ iff $n > (\alpha, \gamma)$, mean that the field $\psi^{(\gamma)\alpha}(z)$ behaves as a spin $-(\alpha, \gamma)$ primary field on the vacuum $|0\rangle_{(\gamma)}$ with respect to the modes $\psi_n^\alpha$. This aspect corresponds to the configuration (4. 41). One may say that the spectral flow is the transformation changing the spins of the elementary fields as is expected from the back-ground (4. 41). On the other hand, the 2nd line of (4. 42) and the vacuum condition; $\psi_n^{(\gamma)\alpha}|0\rangle_{(\gamma)} = 0$ iff $n > 0$, say that it is a spin 0 primary field with respect to the gauge transformed modes $\psi_n^{(\gamma)\alpha}$, which corresponds to the configuration (4. 30) considered before. In this way, we obtain the two physical interpretations of the spectral flow corresponding to the choice of the back-ground (4. 30) or (4. 41); one of them is a singular gauge transformation and the other is a spin changing transformation. They are of course consistent with each other.
5 Conclusions

We have investigated the topological gauged WZW models for the cases of general Kähler homogeneous spaces. The gauge fixing and the operator formalism based on it were presented in such a way that they are natural extensions of those in [13]. We investigated the BRST-cohomology of our system, defined by the total BRST-charge

\[ Q_{\text{tot}} = Q_{G/H} + Q_{Z(H)} + Q_{H^C}. \]

The cornerstone of this study is the concept “spectral flows” [14, 15], which are symmetry transformations preserving the BRST-charge \( Q_{\text{tot}} \). Because of the relation (3.46): \( U_1(\phi_{a, b}) U_2(\phi_{c, d}) = \sum_{c \in I} e^{i R_{\phi_{c, d}}(\phi_j)} U_1(\phi_{b, c}), \) the identifications of the physical observables by the spectral flows are compatible with the chiral ring structure. We further argued the problem of these field identifications from the global geometrical standpoint. The correlation function among the identified observables, which is valid under any back-ground gauge field, is described in (4.18) (or (4.19)).

Under these studies the geometrical and physical pictures of the spectral flows appeared. They give the following two insights on the roles of the gauge field.

Firstly, the spectral flow is capable to connect two arbitrary instanton sectors in the system which are labelled by different Chern numbers. This is recognized as a phenomenon that the physical observables are transmuted by the effect of the back-ground curvature singularity, and it gives the relation (4.17). Moreover, the consistency condition (3.45) of the ring structure under the field identifications (3.48) was shown to be a natural consequence of the relation (4.17). In this respect it may be important to remark the analogy with the two-dimensional BF gauge theory [22]:

\[ S_{\text{BF}} = \frac{i}{2\pi} \int (\phi, F(\omega)). \]

In this BF theory the correlator among the vertex operators \( e^{i(\beta, \phi)} \) satisfies the same relation as in (4.17). The above BF theoretical aspect of topological SU(2)/U(1) gauged WZW model played an important part in the study of two-dimensional topological gravity [4].

Secondly, related with the path-integral approach, it was shown that two physical interpretations were possible for the spectral flow. It can be interpreted as a transformation changing the spins of the elementary fields in the system, while it can also work as a singular gauge transformation which creates an appropriate back-ground charge on the physical vacuum. In this respect it is useful to note the analogy with the following Coulomb gas system:

\[ S_{\text{CG}} = \frac{1}{4\pi} \int \{ (\partial \phi, \partial \phi) + 2\alpha_+ (\phi, F(\omega)) + 2\alpha_\phi (\phi) R(\phi) \}. \]

If we set \( \omega = -\omega(g) g \), the EM tensor of the system \( T_{\text{CG}} \) will suffer the twist by the \( U(1) \)-current \( J_{\text{CG}} = 2i\alpha_+ \partial \phi ; T_{\text{CG}} \to \tilde{T}_{\text{CG}} = T_{\text{CG}} + \frac{i}{2} \partial J_{\text{CG}}. \) This causes the changes of the spins of the vertex operators \( e^{i(\beta, \phi)} \). The important difference between this Coulomb gas system and our topological system is that,
in the topological model, the procedure of changing the spins is performed BRST-invariantly, because both the EM tensor and the $U(1)$-current are BRST-exact. The above "twist" does not change the physical contents of the topological system, while it does in the Coulomb gas system.

In order that spectral flows work completely in a given model, the underlying current algebras must be integrable. (See appendix B.) Therefore, if they are non-integrable, the symmetry of spectral flow may be lost. In these cases the state identification (in the sense of this paper) will no longer work, and so the infinite BRST states will be left for us. It seems interesting to understand these infinite physical states in a unifying standpoint, for example, by some appropriate generalization of spectral flow. In \cite{23} we plan to study the case of $SU(2)/U(1)$ with fractional levels. Especially the infinite physical states in the system will be studied from the above point of view. The correspondence between this infinite dimensional cohomology and the Lian-Zuckerman cohomology \cite{24} of Liouville theory in two-dimensional gravity will be discussed.

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Appendix

A  Quantization of the $N = 2$ SUSY Gauged WZW Model

In this appendix we present the quantizations of $N = 2$ supersymmetric (SUSY) gauged WZW models \[1, 6\] in order to comment the relation between the ”twist” of the SUSY gauged WZW models and that of $N = 2$ SCFTs \[3\]. We note that these quantizations are essentially a special case of the work in \[7\], in which general (not necessarily of $N = 2$) SUSY gauged WZW models were considered and it was shown that they describe $N = 1$ supercoset CFTs.

$N = 2$ SUSY gauged WZW model associated with $G/H$, where $G/H$ is assumed to be Kähler, is given by

$$Z = \int DgDAD\psi D\bar{\psi} \exp \left[ -kS_G(g, A) - \frac{1}{\pi} \int_\Sigma dv(g) \{ (\psi, \bar{\partial}_{A^g} \psi) + (\bar{\psi}, \partial_{A^\psi} \bar{\psi}) \} \right].$$

(A. 1)

In this expression the Weyl fermions $\psi, \bar{\psi}$ are $m_+ \oplus m_- \text{-valued}$, where we set $m_\pm$ be those introduced in section 2 in the text. The gauge field $A$ is h-valued. $dv(g)$ denotes the canonical volume element defined by a fixed Kähler metric $g$.

As in the topological case we must perform the gauge fixing for the $H^C$-chiral gauge transformations, and for this aim we need to estimate the chiral anomalies of the ingredients. The solution of this problem is almost the same as the topological case in the text. The only exception is the abelian part (i.e. $Z(H^C)$-part) of the anomaly of the coset fermions $\psi, \bar{\psi}$. It is due to the fact that $\psi, \bar{\psi}$ are spinor fields and that we have to use the index theorem for the spin complex rather than the Dolbeault complex. Therefore we must replace the estimation of the anomaly of $\chi_{\psi}$-system (2. 14) by the following one;

$$Z_\psi = \int D(\psi, \bar{\psi}) \exp \left[ -\frac{1}{\pi} \int_\Sigma dv(g) \{ (\psi, \bar{\partial}_{a^g} \psi) + (\bar{\psi}, \partial_{a^\psi} \bar{\psi}) \} \right]$$

$$= \int D(\psi, \bar{\psi}) \exp \left[ -\frac{1}{\pi} \int_\Sigma dv(g) \{ (\psi, \bar{\partial}_{a^g} \psi) + (\bar{\psi}, \partial_{a^\psi} \bar{\psi}) \} \right]$$

$$\times \prod_{i=1}^r \exp \left\{ (g^\psi - h^\psi_i)S_{H_0}^{(i)}(\rho^{(i)}_0, a^{(i)}_0) \right\}$$

$$\times \exp \left[ \frac{i g^\psi}{4\pi} \int_\Sigma \{(\bar{\partial}X, \partial X) + 2i(X, F(\omega))\} \right].$$

(A. 2)

Notice that a distinction from (2. 14) is the absence of the “back-ground
charge term” \( \sim \int \rho g/h (X) R \). In this way we can finally arrive at the following expression of the gauge-fixed system:

\[
Z_{g.f.}[g] = \int \mathcal{D} a Z_{g.f.}[a, g] \equiv \int \mathcal{D}(a_0, \omega) Z_{g.f.}[a_0, \omega, g],
\]

\[
Z_{g.f.}[a_0, \omega, g] = \int \mathcal{D}(g, \rho, X, \psi, \bar{\psi}, \zeta, \bar{\zeta}, \xi, \bar{\xi})
\times \exp \left\{ -kS_G(g, a) - S_\psi(\psi, \bar{\psi}, a) \right\}
\times \exp \left\{ \sum_{i=1}^{r} (k + g^\vee + h_i^\vee) S_{H_0^{(i)}}(\rho^{(i)}, a_0^{(i)}) \right\}
- S'_X(X, \omega, \omega(g)) - S_{\zeta\xi}(\zeta, \bar{\zeta}, \xi, \bar{\xi}, a_0) \right\} (A. 3)
\]

where we set

\[
S_\psi(\psi, \bar{\psi}, a) = \frac{1}{\pi} \int_{\Sigma} dv(g) \{(\psi, \bar{\partial}_a z \psi) + (\bar{\psi}, \partial_a z \psi)\} \tag{A. 4}
\]

\[
S_{\zeta\xi}(\zeta, \bar{\zeta}, \xi, \bar{\xi}, a_0) = \frac{1}{2\pi i} \int \{(\bar{\partial}_a \xi, \zeta) - (\bar{\zeta}, \partial_a \xi)\} \tag{A. 5}
\]

\[
S'_X(X, \omega, \omega(g)) = \frac{1}{4\pi i} \int \{(\bar{\partial} X, \partial X) + 2i\alpha_+(X, F(\omega))\} \tag{A. 6}
\]

\[
(\alpha_+ = \sqrt{k + g^\vee}, \quad \alpha_- = -\frac{1}{\sqrt{k + g^\vee}}).
\]

The fields \( X, \rho, \xi, \bar{\xi}, \zeta, \bar{\zeta} \) are the counterparts of those in (2.18).

We can read the total EM tensor of the gauge fixed system from (A.3);

\[
T_{tot}' = T_g + T_\rho + T'_X + T_\psi + T_{\zeta\xi}, \tag{A. 7}
\]

where \( T_g, T_\rho, T_{\zeta\xi} \) are those given in (2.23), (2.25), (2.28) and

\[
T'_X = -\frac{1}{2} : (\partial_z X, \partial_z X) :, \tag{A. 8}
\]

\[
T_\psi = -\frac{1}{2} : (\psi, \partial_z \psi) :, \tag{A. 9}
\]

The total central charge is easily calculated;

\[
c_{tot}' = c_g + c_\rho + c'_X + c_\psi + c_{\zeta\xi}
= \frac{k \dim g}{k + g^\vee} + \sum_{i=1}^{r} \frac{-(k + g^\vee + h_i^\vee) \dim h_0^{(i)}}{-(k + g^\vee + h_i^\vee) + h_i^\vee} + l
+ \frac{1}{2} \dim (m_+ \oplus m_-) + (-2) \times (\dim h_0 + l)
= 3 \dim m_+ - \frac{12}{k + g^\vee} \rho g^2 / h. \tag{A. 10}
\]
It is equal to the central charge of the Kazama-Suzuki model for \( G/H \). In fact the total system is equivalent to the Kazama-Suzuki model modulo a BRST exact term. \( T'_\text{tot} \) can be factorized to

\[
T'_\text{tot} = T_{\text{KS}} + T'_{Z(HC)} + T_{H^0C},
\]

(A. 11)

where \( T_{\text{KS}} \) is the EM tensor of the Kazama-Suzuki model [4]

\[
T_{\text{KS}} = \frac{1}{2(k + g^\vee)} \left\{ \circ (J_g, J_g) - \circ (J_h, J_h) \right\} - \frac{1}{2} : (\psi, \partial_z \psi) :,
\]

(A. 12)

and \( T'_{Z(HC)} \) is defined by

\[
T'_{Z(HC)} = \frac{1}{2(k + g^\vee)} \circ (\hat{J}_Z(h), \hat{J}_Z(h)) - \frac{1}{2} : (\partial_z X, \partial_z X) :.
\]

(A. 13)

\( T_{H^0C} \) is that given in (2. 43). The physical degrees of freedom in this SUSY model are characterized by the BRST-charge \( Q'_\text{tot} = Q_{Z(HC)} + Q_{H^0C} \) (2. 32). It is easy to see

\[
T'_\text{tot} = T_{\text{KS}} + \{ Q'_\text{tot}, G^+_{Z(HC)} \} + G^-_{H^0C} \}
\]

(A. 14)

Here \( G^+_{H^0C} \) is that given in (2. 44), and \( G^-_{Z(HC)} \) is

\[
G^-_{Z(HC)} = \frac{\alpha}{\sqrt{2}} (\zeta_z, \hat{J}_Z(h) - J_X).
\]

(A. 15)

In this way we can see that \( \{ G^\pm_{G/H}, T_{\text{KS}}, J_{G/H} \} \) are non-trivial physical observables (i.e. \( Q'_\text{tot} \)-invariant and not exact). They generates a \( N = 2 \) SCA. This means that the model is equivalent to the Kazama-Suzuki model.

By comparing \( T'_\text{tot} \) with \( T_{\text{tot}} \) (2. 43) we can find

\[
T_{\text{tot}} - T'_\text{tot} = T_{G/H} + T_{Z(HC)} - T_{\text{KS}} - T'_{Z(HC)} = \frac{1}{2} \partial_z J_{G/H} + \{ Q_{Z(HC)}, * \}.
\]

(A. 16)

Thus the "twist" of the EM tensor of this SUSY gauged WZW model is given by adding the term,

\[
\frac{1}{2} \partial_z J_{G/H} \equiv \frac{1}{4} \partial_z : (\psi, \psi) : + \frac{1}{k + g^\vee} \partial_z \rho_{G/H}(\hat{J}_Z(h)).
\]

The first component reflects the difference of spin between \( \psi \) and \( \chi \psi \) and the second one is due to the distinction of the abelian anomaly commented above.
B Notes on Algebra Automorphisms of Affine Lie Algebras and the Spectral Flow

The purpose of this appendix is to give the precise definitions of the spectral flow \[14, 15\] which is introduced in section 4. We will present several results needed for our main subjects without proof. Refer the ref. \[15, 25, 26\] for the proofs and more complete discussions.

Let us start with preparing some notations and conventions of affine Lie algebras.

B.1 Notations for Affine Lie Algebras

Let \( g \) be a (complex) simple Lie algebra (rank \( l \)), and \( \hat{g} \equiv Lg \oplus CK \oplus Cd \) be the corresponding affine Lie algebra. \( Lg \equiv g \otimes \mathbb{C} [z, z^{-1}] \) is the loop algebra of \( g \). \( K \) is the canonical central element and \( d(\equiv \frac{d}{dz}) \) is the scaling element.

We assume that \( g \) is simply-laced for simplicity. Let \( t, \hat{t} \equiv t \oplus CK \oplus Cd \) be the Cartan subalgebras of \( g, \hat{g} \) respectively, \( \Delta = \Delta^+ \uplus \Delta^- \), \( \hat{\Delta} = \hat{\Delta}^+ \uplus \hat{\Delta}^- \) be the root systems of \( g, \hat{g} \) and \( \Pi = \{ \alpha_1, \ldots, \alpha_l \} \), \( \hat{\Pi} = \{ \alpha_0 \equiv \delta - \theta, \alpha_1, \ldots, \alpha_l \} \) (\( \theta \) is the highest root of \( g \) and \( \delta \) is defined by \( \delta(d) = 1, \delta(t) = \delta(K) = 0 \) ) be the simple roots of \( g, \hat{g} \).

We denote by \((\ ,\ ,)\) the Cartan-Killing metric normalized so that the square length of each root is 2. It is well-known that this metric is naturally extended to an invariant metric on \( \hat{g} \) \[25\], which we shall denote by \((\ |\ )\); \( (u(z)|v(z)) = \frac{1}{2\pi i} \oint \frac{1}{z}(u(z), v(z)) dz \ (\forall u(z), v(z) \in Lg), \)
\( (K|d) = 1, \)
\( (K|u(z)) = (d|u(z)) = (K|K) = (d|d) = 0 \ (\forall u(z) \in Lg). \)

This metric induces the dual metric on \( \hat{g}^* \), especially on \( \hat{t}^* \equiv t^* \oplus CA_0 \oplus C\delta \), which we express by the same notation \((\ |\ )\);
\( (\alpha|\beta) = (\alpha, \beta) \ (\forall \alpha, \beta \in t^*), \)
\( (\Lambda_0|\delta) = 1, \)
\( (\alpha|\Lambda_0) = (\alpha|\delta) = (\Lambda_0|\Lambda_0) = (\delta|\delta) = 0, \ (\forall \alpha \in t^*). \)

\( \hat{\Delta}_{\text{im}} = \hat{\Delta}^+_{\text{im}} \uplus \hat{\Delta}^-_{\text{im}} \) respectively;
\begin{align*}
\hat{\Delta}^+_{\text{real}} &= \hat{\Delta}^+_{\text{real}} \uplus \{ \alpha + n\delta ; \alpha \in \Delta, n \in \mathbb{Z}_{>0} \}, \\
\hat{\Delta}^-_{\text{real}} &= -\hat{\Delta}^+_{\text{real}}, \\
\hat{\Delta}^+_{\text{im}} &= \{ n\delta ; n \in \mathbb{Z}_{>0} \}, \quad \hat{\Delta}^-_{\text{im}} = -\hat{\Delta}^+_{\text{im}}; \quad (B. 3)
\end{align*}
where $\Delta(\Delta^+)$ are the sets of (positive) roots of $g$.

We introduce the root lattice $Q$ and the weight lattice $P$ of $g$;

$$Q = \sum_{i=1}^{l} \mathbb{Z}\alpha_i, \quad P = \sum_{i=1}^{l} \mathbb{Z}\Lambda_i,$$

(B. 4)

where $\Lambda_1, \ldots, \Lambda_l$ are the fundamental weights of $g$ which satisfy,

$$(\Lambda_i|\alpha_j) = \delta_{ij}, \quad (\Lambda_i|\Lambda_0) = (\Lambda_i|\delta) = 0,$$

for $1 \leq i, j \leq l$. The set of dominant integral weights with level $k$ of $\hat{g}$ is given by

$$\hat{P}_+^k = \{ \hat{\Lambda} = \Lambda + k\Lambda_0 : \Lambda \in P_+, (\Lambda|\theta) \leq k \},$$

(B. 5)

where $P_+$ is the set of dominant integral weights of $g$.

Let $W, \hat{W}$ be the Weyl groups of $g, \hat{g}$. $\hat{W}$ has the structure $\hat{W} = W \times Q$, where the root lattice $Q$ acts on $\hat{t}^*$ by “translations”;

$$t_\gamma(\hat{\mu}) = \hat{\mu} + (\hat{\mu}|\gamma)\gamma - \left(\hat{\mu}|\gamma\right)\frac{1}{2}|\gamma|^2(\hat{\mu}|\delta)\delta,$$

(B. 6)

$$\forall \gamma \in Q, \forall \hat{\mu} \in \hat{t}^*$$

Under the linear isomorphism $\nu : \hat{t} \xrightarrow{\cong} \hat{t}^*$ defined by $\langle \nu(h), h' \rangle = (h|h')$, the root lattice $Q$ also acts on $\hat{t}$ by

$$t_\gamma(\hat{h}) = \hat{h} + (\hat{h}|K)\nu^{-1}(\gamma) - \left((\hat{h}|\nu^{-1}(\gamma)) + \frac{1}{2}|\gamma|^2(\hat{h}|K)\right)K,$$

(B. 7)

$$\forall \gamma \in Q, \forall \hat{h} \in \hat{t}.$$

The Weyl group $W$ of $g$ is generated by the “reflections in simple roots” $r_{\alpha_i}(1 \leq i \leq l)$ which act on $\hat{\lambda} \in \hat{t}^*$ by

$$r_{\alpha_i}(\hat{\lambda}) = \hat{\lambda} - (\hat{\lambda}|\alpha_i)\alpha_i \quad (1 \leq i \leq l).$$

The Weyl group $W$ may be realized as $N(T)/T$, where $T$ is the Cartan torus of $G$ and $N(T)$ is the normalizer of $T$. $r_{\alpha_i} \in W(1 \leq i \leq l)$ will correspond to $e^{\frac{\pi i}{2}(e_{\alpha_i} + e^{\alpha_i})} \in N(T)$, where $e_{\alpha} \in g_{\alpha}$ and $e^{\alpha} \in g_{-\alpha}(\alpha \in \Delta^+)$ are the Cartan-Weyl base of $g$. Hence $r_{\alpha_i} \in W(1 \leq i \leq l)$ will act on $\hat{x} \in \hat{g}$ by

$$r_{\alpha_i}(\hat{x}) = e^{\frac{\pi i}{2}(e_{\alpha_i} + e^{\alpha_i})}\hat{x}e^{-\frac{\pi i}{2}(e_{\alpha_i} + e^{\alpha_i})}.$$

$\langle \cdot, \cdot \rangle$ is the dual pairing, i.e $\langle \nu(h), h' \rangle = \nu(h)(h')$. Especially $\nu(K) = \delta, \nu(d) = \Lambda_0$. 
B.2 Some Automorphisms of Affine Lie Algebras

According to \[15, 26\], we shall extend the affine Weyl group \( \hat{W} \) to the following group:

\[
\tilde{W} = W \ltimes P \supset \hat{W}, \tag{B. 8}
\]

where the classical weight lattice \( P \) acts on \( \hat{t}^\ast (\hat{t}) \) in the completely same way as the root lattice \( Q \) (see (B. 6), (B. 7).) Moreover we introduce the following subgroup of \( \tilde{W} \):

\[
D = \{ \hat{w} \in \tilde{W} \mid \hat{w}(\hat{\Delta}^+) = \hat{\Delta}^+ \}. \tag{B. 9}
\]

With this preparation the next lemma holds;

**Lemma B.1**

1. \( \tilde{W} = D \ltimes \hat{W} \), namely, any element \( \hat{w} \) of \( \tilde{W} \) is uniquely expresseible as \( \hat{w} = \hat{\tau}\hat{w}_0, \hat{\tau} \in D, \hat{w}_0 \in \hat{W} \), and \( \hat{W} \) is a normal subgroup of \( \tilde{W} \).

2. For \( \forall \gamma \in P \), define the element \( \hat{\tau}_\gamma \) of \( D \) by the above unique decompo-

sition \( t_\gamma = \hat{\tau}_\gamma \hat{w}_\gamma, \hat{\tau}_\gamma \in D, \hat{w}_\gamma \in \hat{W} \), then it holds that

\[
\begin{align*}
\hat{\tau}_{\gamma+\alpha} &= \hat{\tau}_\gamma \quad \text{for } \forall \alpha \in Q, \\
\hat{\tau}_{\gamma_1+\gamma_2} &= \hat{\tau}_{\gamma_1}\hat{\tau}_{\gamma_2}, \\
\text{the map } \gamma \in P &\mapsto \hat{\tau}_\gamma \in D \text{ is onto.}
\end{align*}
\]

This lemma implies that the group \( D \) is isomorphic to \( P/Q(\cong Z(G)) \) as abelian group. It is further known \[26\] that \( \text{Aut}(\Pi) \cong \text{Aut}(\Pi) \ltimes D \) holds, so one may call \( D \) as the "group of proper extended Dynkin dyagram automorphisms".

One can think \( \tilde{W} \) as a subgroup of \( \text{Aut}(\hat{g}) \). In fact, the action of \( \tilde{W} \) on the CSA \( \hat{t} \) (and its dual \( \hat{t}^\ast \)) is already defined. Introduce a Cartan-Weyl base of \( \hat{g} \): \( e_{\alpha+n\delta} = e_{\alpha}z^n \in \hat{g}_{\alpha+n\delta}, e^{\alpha+n\delta} = e^\alpha z^{-n} \in \hat{g}_{-\alpha-n\delta} \) for \( \alpha + n\delta \in \hat{\Delta}_\text{real}^+ \) and \( h z^n \in g_{n\delta} \) for \( n \in \mathbb{Z} \setminus \{0\} \). Here \( e_{\alpha} \in g_{\alpha}, e^\alpha \in g_{-\alpha}(\alpha \in \Delta^+) \) are the Cartan-Weyl base of \( g \) and \( h \in t \). Then \( \hat{w} = wt_\gamma \in \tilde{W} \) \( (w \in W, t_\gamma \in P) \) will act on them as

\[
\begin{align*}
\hat{w}(e_{\alpha}z^n) &= e_{w(\alpha)}z^{-(\alpha,\gamma)} \\
\hat{w}(e^\alpha z^{-n}) &= e^{w(\alpha)}z^{-(\alpha,\gamma)} \\
\hat{w}(hz^n) &= w(h)z^n. \tag{B. 10}
\end{align*}
\]

It is easy to check that these definitions besides (B. 7), (B. 8) give an automorphism of \( \hat{g} \).
Let us turn our interests to the g-current algebra with level $k \in \mathbb{Z}_{>0}$ (the integrable representations of $\hat{\mathfrak{g}}$); The current $J(z) = J^k(z) + \sum_{\alpha \in \Delta^+} \{J^{\alpha}(z) e_{\alpha} + J_{\alpha}(z) e_{-\alpha}\}$ is acting on the Hilbert space $\mathcal{H}_k = \sum_{\hat{\Lambda} \in \hat{P}_k^+} L(\hat{\Lambda})$, where $L(\hat{\Lambda})$ is the integrable $\hat{\mathfrak{g}}$-module with the highest weight $\hat{\Lambda}$. We prepare the next notation;

for $\forall \mathbf{x} = u(z) + aK$, $(u(z) \in Lg)$ $J[\mathbf{x}] \overset{\text{def}}{=} \frac{1}{2\pi i} \oint (J(z), u(z)) \, dz + ak$. (B. 11)

The following theorem is important for our purpose;

**Theorem B.2** For an arbitrary element $\hat{w} = \text{wt}_{\gamma}$ of $\hat{\tilde{W}}$ ($w \in W$, $\gamma \in P$), there exists a unitary transformation $U_{\hat{w}}$ on $\mathcal{H}_k$ such that

$$U_{\hat{w}} \, J[\mathbf{x}] \, U_{\hat{w}}^{-1} = J[\hat{w}(\mathbf{x})],$$

namely,

$$U_{\hat{w}} \, J_{\alpha}(z) \, U_{\hat{w}}^{-1} = z^{-\langle \alpha, \gamma \rangle} J_{w(\alpha)}(z), \quad (\forall \alpha \in \Delta^+)$$

$$U_{\hat{w}} \, J^{\alpha}(z) \, U_{\hat{w}}^{-1} = z^{\langle \alpha, \gamma \rangle} J^{w(\alpha)}(z), \quad (\forall \alpha \in \Delta^+)$$

$$U_{\hat{w}} (h, J^l(z)) \, U_{\hat{w}}^{-1} = (\hat{w}(h), J^l(z)) - k\langle \gamma, h \rangle \frac{1}{z}, \quad (\forall h \in t)$$

$$U_{\hat{w}} |\hat{\Lambda}, \hat{\Lambda}\rangle = |\hat{\tau}_{\gamma}(\hat{\Lambda}), \hat{w}(\hat{\Lambda})\rangle$$

(B. 13)

where $\hat{\tau}_{\gamma} \in D$ is determined by the unique decomposition $t_{\gamma} = \hat{\tau}_{\gamma} \hat{\tilde{w}}_{\gamma} \in D \times \hat{\tilde{W}}$ in lemma [B.1].

For the Sugawara EM tensor $T(z) \equiv \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}} \overset{\text{def}}{=} \frac{1}{2(k + g)} \circ (J(z), J(z)) \circ$, one can obtain the following transformation formula by direct computations.

**Corollary B.3** Let $\hat{w}$ be as above,

$$U_{\hat{w}} \, L_n \, U_{\hat{w}}^{-1} = L_n - \langle w(\gamma), J^l_n \rangle + \frac{k}{2} |\gamma|^2 \delta_{n,0}. \quad (B. 14)$$

Generally $U_{\hat{w}}$ maps $L(\hat{\Lambda})$ to another integrable module $L(\hat{\tau}_{\gamma}(\hat{\Lambda}))$, and $U_{\hat{w}}$ makes $L(\hat{\Lambda})$ invariant if $\hat{w} \in \hat{\tilde{W}}$, or equivalently, $\gamma \in Q$. It is worth pointing out that the integrability (or locally nilpotency) of the representation is crucial for the existence of $U_{\hat{w}}$ (see [24]). So, in non-integrable cases we cannot construct the unitary transformations $U_{\hat{w}}$.
B.3 Spectral Flow

Now we would like to introduce the concept of spectral flow which gives rise to automorphisms of TCA being investigated in section 3, 4 and has its origin in the Lie algebra automorphisms given in theorem B.2.

First of all, recall the parabolic decomposition (2.3);

\[ g = h + m_+ + m_- = Z(h) \oplus h_0 + m_+ + m_- = Z(h) \oplus (h_0^{(1)} \oplus \cdots \oplus h_0^{(r)}) \oplus m_+ \oplus m_- . \]  

(B.15)

According to the above decomposition let us introduce the following root lattices;

\[ Q(h_0) = \bigoplus_{i=1}^{r} Q(h_0^{(i)}) , \quad Q(h_0^{(i)}) = \sum_{\alpha \in \Pi_{h_0^{(i)}}} \mathbb{Z} \alpha , \]  

where \( \Pi_{h_0^{(i)}} \subset \Pi \) is the set of simple roots of \( h_0^{(i)} \). The corresponding weight lattices are defined by;

\[ P(h_0) = \bigoplus_{i=1}^{r} P(h_0^{(i)}) , \quad P(h_0^{(i)}) = \bigoplus_{j=1}^{\text{rank} h_0^{(i)}} \mathbb{Z} \Lambda_j^{(i)} , \]  

where \( \{ \Lambda_j^{(i)} \}_{j=1}^{\text{rank} h_0^{(i)}} \) is the set of fundamental weights of \( h_0^{(i)} \).

By using the root and weight lattices \( Q(h_0^{(i)}) \) and \( P(h_0^{(i)}) \) we can also define \( \hat{W}(h_0) \equiv \hat{W}(h_0^{(1)}) \times \cdots \times \hat{W}(h_0^{(r)}) \), \( \hat{W}(h_0) \equiv \hat{W}(h_0^{(1)}) \times \cdots \times \hat{W}(h_0^{(r)}) \) as \( \hat{W}(h_0^{(i)}) = W(h_0^{(i)}) \otimes Q(h_0^{(i)}) \), \( \hat{W}(h_0^{(i)}) = W(h_0^{(i)}) \otimes P(h_0^{(i)}) \) respectively.

Naively we would like to define the spectral flow associated with the automorphism group \( \hat{W} \). However, there is a technical problem; because the symmetry of the \( \rho \)-sector in (2.18) is described by a \( h_0 \)-current algebra with negative level, we cannot construct the automorphism as that given in theorem B.2. In order to avoid this difficulty we must define the transformations of spectral flow as the actions of a suitable subgroup of \( \hat{W} \), which we will denote by \( \hat{D} \), rather than \( \hat{W} \) itself. This subgroup should be defined as;

\[ \hat{D} = \{ \hat{w} = \sigma t_\gamma \in \hat{W} ; \hat{w}(\hat{\Delta}^+_h_0) \subset \hat{\Delta}^+_h_0 , \sigma \in W(h_0) , \gamma \in P \} . \]  

(B.18)

Namely \( \hat{w} \in \hat{D} \) means that \( \hat{w} \) acts along the \( \hat{h}_0 \)-direction as a diagram automorphism, and also notice that \( \hat{w}(\hat{\Delta}_m) \subset \hat{\Delta}_m \) holds. So, by considering the
action of $\tilde{D}$, we will be able to get a well-defined spectral flow without the difficulty of negative level.

For our purpose it is more convenient to consider the following subset $\mathcal{P}(g/h)$ of $P$ instead of $\tilde{D}$ itself;

$$\mathcal{P}(g/h) = \{ \gamma \in P ; \exists \sigma \in W(h_0), \text{s.t } \sigma t_\gamma \in \tilde{D} \}.$$  \hspace{1cm} (B. 19)

It is easy to see that, for $\forall \gamma \in \mathcal{P}(g/h)$, $\sigma$ in the R.H.S of (B. 19) is unique, and we will denote it by $\sigma_\gamma$. In other words, the set $\mathcal{P}(g/h)$ has a one-to-one correspondence with $\tilde{D}$. We write this correspondence by

$$\mathcal{P}(g/h) \ni \gamma \mapsto \hat{w}(\gamma) \overset{\text{def}}{=} \sigma_\gamma t_\gamma \in \tilde{D}.$$  \hspace{1cm} (B. 20)

An equivalent definition of $\mathcal{P}(g/h)$ which is more geometrical is given by

$$\mathcal{P}(g/h) = \{ \gamma \in P ; \exists \sigma \in W(h_0) \text{ s.t } \sigma(C_{\alpha_0}^{\text{aff}} + \gamma) = C_{\alpha_0}^{\text{aff}} \},$$

where $C_{\alpha_0}^{\text{aff}}$ is the subdomain of $t^*$ which contains the Weyl alcove of $h_0$;

$$C_{\alpha_0}^{\text{aff}} = \{ u \in t^* ; (u, \alpha) \geq 0 (\forall \alpha \in \Delta_{h_0}^+), (u, \forall \theta^{(i)}) \leq 1 \}.$$  \hspace{1cm} (B. 21)

Here we denote the maximal root of $h_0^{(i)}C$ by $\theta^{(i)}$ ($1 \leq i \leq r$). Notice that $\sigma \in W(h_0)$ in the R.H.S of (B. 21) is the same element as that of (B. 19), i.e. $\sigma = \sigma_\gamma$. For instance, in the case of $G = SU(N)$ (and $H$ is arbitrary), we can find

$$\mathcal{P}(g/h) = \{ \gamma \in P ; (\gamma, \forall \alpha_{l_i}^{(i)}) = 0, -1, \text{ and } \sum_{l_i=1}^{\text{rank}h_0} (\gamma, \alpha_{l_i}^{(i)}) \geq -1 \text{ for each } h_0^{(i)} \}$$

where $\alpha_{l_i}^{(i)} \in \Pi_{h_0^{(i)}}$ ($1 \leq l_i \leq \text{rank}h_0^{(i)}$).

To proceed further let us consider the quotient weight lattice $P/Q(h_0)$. Take any element $[\gamma] \in P/Q(h_0)$, it is easy to show from (B. 19) (or (B. 21)) that the set $[\gamma] \cap \mathcal{P}(g/h)$ necessarily includes a unique non-zero element. We will denote it by $\gamma_0$. In this way one can obtain a one-to-one correspondence $[\gamma] \longleftrightarrow \gamma_0$ between $P/Q(h_0)$ and $\mathcal{P}(g/h)$. Making use of this correspondence, we will define an addition (we express it by “$\diamond$”) on $\mathcal{P}(g/h)$ induced from that of $P/Q(h_0)$. Notice that, on $P(Z(h)) \cap P(\subset \mathcal{P}(g/h))$, $\diamond$ coincides with the usual addition $+$, but generally does not. Under these preparations we can prove;

$$\hat{w}(\gamma_1 \diamond \gamma_2) = \hat{w}(\gamma_1)\hat{w}(\gamma_2).$$

We summarize the above results as a proposition.
Proposition B.4 The three abelian groups $\tilde{D}$, $\mathcal{P}(g/h)$ and $P/Q(h_0)$ are isomorphic with one another. These isomorphisms are given by

$$\gamma \in \mathcal{P}(g/h) \mapsto \hat{w}(\gamma) \in \tilde{D} \quad (B.22)$$

$$\gamma \in \mathcal{P}(g/h) \mapsto [\gamma] \in P/Q(h_0). \quad (B.23)$$

In addition, the subgroups $\tilde{D} \cap \hat{W}$, $\mathcal{P}(g/h)$ and $P/Q(h_0)$ are also isomorphic.

We have arrived at the stage to present the explicit definition of spectral flow. As was already mentioned, we shall give it associated with each element of $\mathcal{P}(g/h)$ or equivalently $\tilde{D}$. Recall that the ingredients of the model in the text are $g, \rho, X, \chi, \psi, \zeta, \xi$, $g, \rho$ and $X$ are the variables of $G$, $H_0^C/H_0$ and $Z(H)^C/Z(H)$-WZW models. $\{(\chi_{\alpha}, \psi_{\alpha})\}_{\alpha \in \Delta^+_{\text{in}}}$ are $G/H$ ghost systems. $\{(\zeta_{\alpha}, \xi_{\alpha})\}_{\alpha \in \Delta^+_{\text{Bo}}}$ and $\{(\zeta^t, \xi^t)\}$, are the $h^C = h_0^C \oplus Z(h)^C$ ghost systems. Their mode expansions are given by $\chi_{\alpha}(z) = \sum_n \chi_{\alpha,n} z^{-n-1}, \psi_{\alpha}(z) = \sum_n \psi_{\alpha,n} z^{-n}$ etc.

Fix an arbitrary element $\gamma$ of $\mathcal{P}(g/h)$. The spectral flow $\mathcal{U}_\gamma$ should be defined as follows;

$\gamma$-sector;

$$\mathcal{U}_\gamma = U_{\hat{w}(\gamma)} \quad \text{in theorem B.2.} \quad (B.24)$$

$\rho$-sector; Introduce the similar notations $J^{(i)}(\cdot, \cdot)$ as (B.11) for the current algebra $J^{(i)}(\rho)$ (the $H^0_{(i)} - (k + g^0 + h^0)$-current algebra);

$$\text{for } x = u(z) + aK^{(i)} (u(z) \in Lh_0^{(i)}, K^{(i)} \text{ is the central element of } \hat{h}_0^{(i)}),$$

$$J^{(i)}(\rho)[x] = \frac{1}{2\pi i} \oint (J^{(i)}(\rho)(z), u(z)) dz + a(-k - g^0 - h^0).$$

We then define

$$\mathcal{U}_\gamma J^{(i)}(\rho)[x] \mathcal{U}_\gamma^{-1} = J^{(i)}(\rho)[\hat{h}_0^{(i)}(x)], \quad (B.25)$$

where $\hat{h}_0^{(i)} \equiv \hat{h}_0^{(i)} \cdots \hat{h}_0^{(i)}$ is the element of $D(h_0)$ given by

$$\hat{h}_0^{(i)} = \hat{w}(\gamma)|_{h_0^{(i)}},$$

or more explicitly,

$$\mathcal{U}_\gamma J^{(i)}(\rho,\alpha)(z) \mathcal{U}_\gamma^{-1} = z^{-\alpha,\gamma} J^{(i)}(\rho,\sigma,\alpha), \quad (\gamma \alpha \in \Delta^+_{\text{Bo}})$$

$$\mathcal{U}_\gamma J^{(i)}(\rho,\alpha)(z) \mathcal{U}_\gamma^{-1} = z^{\alpha,\gamma} J^{(i)}(\rho,\sigma,\alpha), \quad (\gamma \alpha \in \Delta^+_{\text{Bo}})$$

$$\mathcal{U}_\gamma (h, J^{(i)}(\rho)(z)) \mathcal{U}_\gamma^{-1} = (\sigma, h, J^{(i)}(\rho)(h)) + (k + g^0 + h^0) \frac{1}{z}, \quad (\gamma h \in t(h_0^{(i)})).$$

$$\mathcal{U}_\gamma (\hat{\lambda}, \hat{\lambda}) \mathcal{U}_\gamma^{-1} = (\hat{h}_0^{(i)}(\hat{\lambda}), \hat{h}_0^{(i)}(\hat{\lambda})). \quad (B.26)$$
\(X\)-sector:

\[
\mathcal{U}_\gamma (h, J_X(z)) \mathcal{U}_\gamma^{-1} = (h, J_X(z)) + (k + g^\gamma) \langle \gamma, h \rangle \frac{1}{z}, \quad (\forall h \in Z(h))
\]

\[
\mathcal{U}_\gamma |\beta\rangle_X = |\beta - (k + g^\gamma) \gamma\rangle_X,
\]

where the primary state \(|\beta\rangle_X \ (\beta \in Z(h)^*)\) is defined by

\[
h, J_{X,n})|\beta\rangle_X = \delta_{n,0} \langle \beta, h \rangle |\beta\rangle_X, \quad (n \in \mathbb{Z}_{\geq 0}, \forall h \in Z(h)).
\]

\(\chi\psi\)-sector: Define the following sets of affine roots;

\[
\hat{\Delta} = \{ \alpha + n\delta \ ; \ \alpha \in \Delta, n \in \mathbb{Z} \}, \quad \hat{\Delta} = \hat{\Delta} + \hat{\Delta} -.
\]

and introduce the following notation for the mode expansions of \(\chi\psi\);

\[
\psi = \begin{cases} 
\psi_n^\alpha & \alpha \in \Delta_m^- \\
\chi_{\alpha,n} & \alpha \in \Delta_m^+ \end{cases} \quad (\forall \hat{\alpha} \equiv \alpha + n\delta \in \hat{\Delta})
\]

We then define

\[
\mathcal{U}_\gamma \psi \hat{\alpha} \mathcal{U}_\gamma^{-1} = \psi \hat{\alpha} (\hat{\alpha}), \quad (\forall \hat{\alpha} \in \hat{\Delta})
\]

\[
\mathcal{U}_\gamma |0\rangle_{\chi\psi} = \prod_{\hat{\alpha} \in \Phi_{\psi(\gamma)}} \psi_{-\hat{\alpha}} |0\rangle_{\chi\psi},
\]

where \(\Phi_{\psi(\gamma)}\) is the subset of \(\hat{\Delta}^+\) given by

\[
\Phi_{\psi} = \hat{\psi}(\hat{\Delta}^-) \cap \hat{\Delta}^+ \quad \text{for} \quad \forall \hat{\psi} \in \hat{\mathbb{W}}.
\]

The Fock vacuum \(|0\rangle_{\chi\psi}\) is the state which satisfies

\[
\psi_{\hat{\alpha}} |0\rangle_{\chi\psi} = 0, \quad (\forall \hat{\alpha} \in \hat{\Delta}_m^+).
\]

\(\zeta\xi\)-sector: Introduce the following notations;

\[
\zeta_{\alpha+n\delta} = \begin{cases} 
\zeta_{\alpha,n} & \alpha \in \Delta^+_{h_0} \\
\zeta_{\alpha,-n} & \alpha \in \Delta^-_{h_0} \end{cases}
\]

\[
\xi_{\alpha+n\delta} = \begin{cases} 
\xi_{\alpha,n} & \alpha \in \Delta^+_{h_0} \\
\xi_{\alpha,-n} & \alpha \in \Delta^-_{h_0} \end{cases}
\]

and then we define

\[
\mathcal{U}_\gamma \zeta \hat{\alpha} \mathcal{U}_\gamma^{-1} = \zeta \hat{\alpha} (\hat{\alpha}) , \quad (\forall \hat{\alpha} \in \hat{\Delta}_{h_0})
\]

\[
\mathcal{U}_\gamma \xi \hat{\alpha} \mathcal{U}_\gamma^{-1} = \xi \hat{\alpha} (\hat{\alpha}) , \quad (\forall \hat{\alpha} \in \hat{\Delta}_{h_0}).
\]

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For the Cartan directions we set the actions of $U_\gamma$ as
\[
U_\gamma(h, \zeta)^t U_\gamma^{-1} = (\sigma_\gamma(h), \zeta^t_n), \quad (\forall h \in t).
\] (B. 30)

The Fock vacuum $|0\rangle_{\zeta\xi}$ is the state which satisfies
\[
\zeta^{\alpha+}n\delta |0\rangle_{\zeta\xi} = (h, \zeta)^t_{n} |0\rangle_{\zeta\xi} = 0 \quad (\forall n \geq 0, \forall \alpha \in \Delta_{h_0}, \forall h \in t),
\]
\[
\xi^{\alpha+}n\delta |0\rangle_{\zeta\xi} = (h, \xi)^t_{n} |0\rangle_{\zeta\xi} = 0 \quad (\forall n > 0, \forall \alpha \in \Delta_{h_0}, \forall h \in t),
\]
and its transformation law which is consistent with (B. 29), (B. 30) is as follows;
\[
U_\gamma |0\rangle_{\zeta\xi} = \prod_{\alpha \in \Delta_{h_0}} \zeta^{-\hat{\tau}_h(\alpha)} \xi_\alpha |0\rangle_{\zeta\xi}. \tag{B. 31}
\]

For the coset part $(g, \chi \psi)$ these definitions coincide with those given in [14, 15]. But one must also introduce the concept of spectral flow for the $H^C_0$ and $Z(H^C)$-parts in order to formulate the model as a Lagrangian field theory.

Our main results in this appendix are included in the following theorem;

**Theorem B.5**  
1. All the BRST charges $(Q_{G/H}, Q_{Z(H^C)}, Q_{H^C}, Q_{tot})$ which are defined in section 2 are invariant under the spectral flows, that is, for an arbitrary element $\gamma$ of $\mathcal{P}(g/h)$,
\[
U_\gamma Q_* U_\gamma^{-1} = Q_*, \tag{B. 32}
\]
where “$*$” means “$G/H$” or ... “tot”.

2. The total EM tensor (2. 22) is invariant under the spectral flow modulo BRST-exact terms, explicitly written as;
\[
U_\gamma L_{tot,n} U_\gamma^{-1} = L_{tot,n} - \langle \sigma_\gamma(\gamma), J^{t}_{tot,n} \rangle
\]
\[
\quad = L_{tot,n} - \{ Q_{tot}, \sqrt{2} \alpha^+ \langle \sigma_\gamma(\gamma), \zeta_n \rangle \}. \tag{B. 33}
\]

3. \{${G}_{G/H}^\pm, T_{G/H}, J_{G/H}$\}, the TCA of Kazama-Suzuki sector given in section 2, is “strictly” invariant under the spectral flow;
\[
U_\gamma T_{G/H}(z) U_\gamma^{-1} = T_{G/H}(z), \quad U_\gamma G_{G/H}^\pm(z) U_\gamma^{-1} = G_{G/H}^\pm(z),
\]
\[
U_\gamma J_{G/H}(z) U_\gamma^{-1} = J_{G/H}(z). \tag{B. 34}
\]
This theorem implies that the BRST-cohomology states of our model are invariant under the spectral flow. This fact plays a crucial role in our discussions in section 4. Any physical state $|\Psi\rangle$ and its transformed state $\mathcal{U}_\gamma|\Psi\rangle$ possess the same KS $U(1)$-charges.

We will summarize a few useful statements as propositions.

**Proposition B.6** The map $\gamma \in \mathcal{P}(g/h) \mapsto \mathcal{U}_\gamma$ gives a homomorphism of abelian group. Namely,

$$\mathcal{U}_{\gamma_1 \circlearrowleft_\gamma_2} = \mathcal{U}_{\gamma_1} \mathcal{U}_{\gamma_2}, \quad (\forall \gamma_1, \gamma_2 \in \mathcal{P}(g/h)). \quad (B. 35)$$

Let us introduce the following subset of $\hat{W}$;

$$\hat{W}(g/h) = \{ \hat{w} \in \hat{W} ; \Phi_{\hat{w}} \subset \hat{\Delta}^+_m \}, \quad (B. 36)$$

and then the next propositions hold;

**Proposition B.7** If $\hat{w} \in \hat{W}(g/h)$, $\gamma \in \mathcal{P}(g/h)$, then $\hat{w}(\gamma) \hat{w}^{-1} \in \hat{W}(g/h)$ holds. Especially, for any $\alpha \in \mathcal{P}(g/h) \cap Q$, $\hat{w}(\alpha) \hat{w} \in \hat{W}(g/h)$.

**Proposition B.8**

$$\hat{W}(g/h) = (\hat{D} \cap \hat{W}) W(g/h), \quad (B. 37)$$

where we set

$$W(g/h) = \{ w \in W ; \Phi_w \subset \Delta^+_m \}.$$  

(Refer [15] for the proof) This means that any element $\hat{w}$ of $\hat{W}(g/h)$ can be uniquely decomposed as $\hat{w} = \omega w$, where $\omega \in \hat{D} \cap \hat{W}$ and $w \in W(g/h)$, or equivalently one can uniquely express it as $\hat{w} = \hat{w}(\alpha)w$ by an element $\alpha \in \mathcal{P}(g/h) \cap Q$.  

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