Entanglement monotones and maximally entangled states in multipartite qubit systems

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We present a method to construct entanglement measures for pure states of multipartite qubit systems. The key element of our approach is an antilinear operator that we call comb in reference to the hairy-ball theorem. For qubits (or spin 1/2) the combs are automatically invariant under SL(2,C). This implies that the filters obtained from the combs are entanglement monotones by construction. We give alternative formulae for the concurrence and the 3-tangle as expectation values of certain antilinear operators. As an application we discuss inequivalent types of genuine four-, five- and six-qubit entanglement.

I. INTRODUCTION

Entanglement is one of the most striking features of quantum mechanics, but it is also one of its most counterintuitive consequences of which we still have rather incomplete knowledge. Although the concentrated effort during the past decade has produced an impressive progress, there is no general qualitative and quantitative theory of entanglement.

A pure quantum-mechanical state of distinguishable particles is called disentangled with respect to a given partition \( \mathcal{P} \) of the system iff it can be written as a tensor product of the parts of this partition. In the opposite case, the state must contain some finite amount of entanglement. The question then is to characterize and quantify this entanglement.

As to measuring the amount of entanglement in a given pure multipartite state, the first major step was made by Bennett et al. who discovered that the partial entropy of a party in a bipartite quantum state is a measure of entanglement. It coincides (asymptotically) with the entanglement of formation. Subsequently, the entanglement of formation of a two-qubit state was related to the concurrence. Interestingly, by exploiting the knowledge of the mixed-state concurrence, the so-called 3-tangle \( \tau_3 \) which is a measure for three-party pure states could be derived.

This was a remarkable step since, loosely speaking, it opened the path to studying multipartite entanglement on solid grounds. Further, it was noticed by Uhlmann that antilinearity is an important property of operators that measure entanglement. A particularly interesting consequence of the 3-tangle formula was presented by Dur et al. who found that there are two inequivalent classes of sharing entanglement among three parties.

Another important aspect of the research on entanglement measures was the question regarding the requirements for a function that represents an entanglement monotone. It turned out that the essential property to be satisfied is non-increasing behavior on average under stochastic local operations and classical communication (SLOCC). Later, Verstraete et al. have demonstrated that all homogeneous positive functions of pure-state density matrices that remain invariant under determinant-one SLOCC operations are entanglement monotones.

Despite the enormous effort, the only truly operational entanglement measure for arbitrary mixed states at hand, up to now, is the concurrence. For pure states we have a slightly farther view up to systems of two qutrits and, for three qubits, the 3-tangle. Various multipartite entanglement measures for pure-states have been proposed; but most of these do not yield zero for all possible product states (e.g. Refs. 13, 14, 15, 16). This motivated the quest for an operational entanglement measure emerging from one requirement only: that it be zero for product states (not only for completely separable pure states). In particular, the goal has been to explore the idea that entanglement monotones are related to antilinear operators as pointed out for the concurrence by Uhlmann.

Here we show that it is possible to construct a filter, i.e., an operator that has zero expectation value for all product states. It will turn out that these filters are entanglement monotones by construction. Interestingly, the two-qubit concurrence and the 3-tangle have various equivalent filter representations (see below). In order to illustrate the application of the method to a nontrivial example, we will present filters for up to six-qubit states that are able to distinguish inequivalent types of genuine multipartite entanglement.

Before finding a measure for genuine multipartite entanglement, one first has to agree about a definition of maximal multipartite entanglement:

Definition I.1 A pure \( q \)-qubit state \( |\psi_q\rangle \) has maximal genuine multipartite entanglement, i.e. \( q \)-tangle, if and only if

(i) All reduced density matrices of \( |\psi_q\rangle \) with rank \( \leq 2 \) (this includes all \((q-1)\)-site and single-site ones) are maximally mixed.
(ii) all $p$-site reduced density matrices of $|\psi_q\rangle$, have zero $p$-tangle; $1 < p < q$.

(iii) there is a canonical form of any maximally $q$-tangled state, for which properties (i) and (ii) are unaffected by phase factors, i.e. they are phase invariant.

A stronger form of condition (i) appeared in Ref. [17], where it is demanded that all reduced density matrices be maximally mixed.

Notice that the first condition induces that all reduced density matrices of the state have rank larger than 1. This excludes product states of whatsoever kind. We emphasize that we use the term genuine $q$-qubit entanglement in a more restricted sense than, e.g., in Ref. [7]; in particular, the only class of three-qubit states with genuine three-partite entanglement is represented by the GHZ state.

Some remarks are in order: whereas the first two requirements are well motivated, since the first means a maximal gain of information when a bit of information is read out of a maximally entangled state, and the second excludes hybrids of many different types of entanglement (somewhat following the idea of entanglement as a resource whose amount can be distributed among possibly different types of entanglement only; see e.g. Ref. [5]), we have no good argument in favor of the third, except that maximally entangled states for two and three qubits have such a canonical form.

II. COMBS AND FILTERS

The basic concept is that of the comb. We define a comb of first order as an antilinear operator $A$ with zero expectation value for all states of a certain Hilbert space $\mathcal{H}$. That is,

$$\langle \psi | A | \psi \rangle = \langle \psi | LC | \psi \rangle = \langle L | \psi^* \rangle \equiv 0$$

for all $|\psi\rangle \in \mathcal{H}$, where $L$ is a linear operator and $C$ is the complex conjugation. Here $A$ necessarily has to be antilinear (a linear operator with this property is zero itself). For simplicity we abbreviate

$$\langle \psi | LC | \psi \rangle = : \langle L \rangle_C .$$

Note that the complex conjugation is included in the definition of the expectation value $\langle \ldots \rangle_C$ in Eq. (2).

We will use the comb operators [22] in order to construct the desired filters which are defined as antilinear operators whose expectation values vanish for all product states. While a comb is a local, i.e., a single-qubit operator, a filter is a non-local operator that acts on the whole multi-qubit state. It is worth mentioning already at this point that such a filter is invariant under $\mathcal{P}$-local unitary transformations if the combs have this property. Even more, it is invariant under the complex extension of the corresponding unitary group which is isomorphic to the special linear group. Since the latter represents the SLOCC operations for qubits [7, 9], the filters will be entanglement monotones by construction.

We focus on multipartite systems of qubits (i.e., spin 1/2). The local Hilbert space is $\mathcal{H}_j = \mathbb{C}^2 = \mathfrak{h}$ for all $j$. We need the Pauli matrices $\sigma_0 := 1$, $\sigma_1 := \sigma_x$, $\sigma_2 := \sigma_y$, and $\sigma_3 := \sigma_z$. It is straight forward to verify that the only single-qubit comb is the operator $\sigma_y$:

$$\langle \psi | \sigma_y C | \psi \rangle = \langle \sigma_y \rangle_C \equiv 0 .$$

Since its expectation value is a bi(anti-)linear expression in the coefficients of the state we denote it a comb of order 1. In general we will call a comb to be of order $n$ if its expectation value is $2n$-linear in the coefficients of the state. There is one independent single-qubit comb which is of 2nd order. One can verify that for an arbitrary single-qubit state

$$0 = \langle \sigma_\mu \rangle_C \langle \sigma_\nu \rangle_C := \sum_{\mu,\nu=0}^3 \langle \sigma_\mu \rangle_C g^{\mu,\nu} \langle \sigma_\nu \rangle_C ,$$

with $g^{\mu,\nu} = \text{diag}\{-1, 1, 0, 1\}$ being very similar to the Minkowski metric. [23] Both combs are $SL(2, \mathbb{C})$ invariant [18].

It will prove useful to introduce the embedding

$$\mathcal{E}_n : \mathcal{H} \rightarrow \mathcal{H}_n = \mathcal{H}^\otimes n$$

$$|\psi\rangle \rightarrow \mathcal{E}_n |\psi\rangle = |\psi\rangle^\otimes n .$$
Further define the product • for operators $O, P: \mathcal{H} \rightarrow \mathcal{H}$ such that
\[
O \bullet P : \quad O \bullet P\mathcal{E}_2(|\psi\rangle) = O|\psi\rangle \otimes P|\psi\rangle .
\] (5)

Then we have the single-site ($\mathcal{H} = C^2$) comb $\sigma_y$ for $S_1 = \mathcal{H}$ and $\sigma_\mu \bullet \sigma^\mu$ for $\mathcal{H}^2$. These two one-site combs are sufficient to construct filters for multipartite qubit systems, which are entanglement monotones by construction. For $n$-qubit filters we will use the symbol $\mathcal{F}^{(n)}$. Filters for two qubits are
\[
\mathcal{F}_1^{(2)} = \sigma_y \otimes \sigma_y
\] (6)
\[
\mathcal{F}_2^{(2)} = \frac{1}{3} (\sigma_\mu \otimes \sigma_\nu) \bullet (\sigma^\mu \otimes \sigma^\nu) .
\] (7)

Both forms are explicitly permutation invariant, and they are filters since, if the state were a product, the combs would annihilate its expectation value. From the filters we obtain the pure-state concurrence in two different equivalent forms:
\[
C = \left| \left\langle \mathcal{F}_1^{(2)} \right\rangle_C \right|
\] (8)
\[
C^2 = \left| \left\langle \mathcal{F}_2^{(2)} \right\rangle_C \right| = \frac{1}{3} \left| \left\langle \sigma_\mu \otimes \sigma_\nu \right\rangle_C \left\langle \sigma^\mu \otimes \sigma^\nu \right\rangle_C \right| .
\] (9)

While the first form in Eq. (8) has the well-know convex-roof extension of the pure-state concurrence via the matrix
\[
R = \sqrt{\rho} \sigma_y \otimes \sigma_y \rho^* \sigma_y \otimes \sigma_y \sqrt{\rho}
\] (10)

it can be shown that the convex roof extension of the second form in Eq. (9) is related to
\[
Q = \sqrt{\rho} \sigma_\mu \otimes \sigma_\nu \rho^* \sigma_\kappa \otimes \sigma_\lambda \rho \sigma^\mu \otimes \sigma^\nu \rho^* \sigma^\kappa \otimes \sigma^\lambda \sqrt{\rho} .
\] (11)

and we find that $Q \equiv R^2$.

Now let us consider the 3-tangle. For states of three qubits we find, e.g.,
\[
\mathcal{F}_1^{(3)} = (\sigma_\mu \otimes \sigma_y \otimes \sigma_y) \bullet (\sigma^\mu \otimes \sigma_y \otimes \sigma_y)
\] (12)
\[
\mathcal{F}_2^{(3)} = \frac{1}{3} (\sigma_\mu \otimes \sigma_\nu \otimes \sigma_\lambda) \bullet (\sigma^\mu \otimes \sigma^\nu \otimes \sigma^\lambda) .
\] (13)

Both $\mathcal{F}_1^{(3)}$ and $\mathcal{F}_2^{(3)}$ are filters and the latter is explicitly permutation invariant. From these operators the pure-state 3-tangle is obtained in the following way:
\[
\tau_3 = \left| \left\langle \mathcal{F}_1^{(3)} \right\rangle_C \right| = \left| \left\langle \mathcal{F}_2^{(3)} \right\rangle_C \right| .
\] (14)

Interestingly, all three-qubit filters are powers of the 3-tangle as entanglement measure. We mention, however, that there is no immediate extension to mixed states as in the case of the 'alternative' two-qubit concurrence, Eq. (10).

### III. FILTERS FOR FOUR-QUBIT STATES

Classifications of four-qubit states with respect to their entanglement properties have been studied, e.g., in Refs. [19, 20, 21]. Here we introduce three four-qubit filter operators and study the three classes of entangled states they are measuring.

A four-qubit filter has the property that its expectation value for a given state is zero if the state is separable, i.e., if there is a one-qubit or a two-qubit part which can be factored out (note that for a three-qubit filter it is enough to extract one-qubit parts only). An expression that obeys this requirement for any single qubit and any combination of qubit pairs is given by
\[
\mathcal{F}_1^{(4)} = (\sigma_\mu \sigma_\nu \sigma_y \sigma_y) \bullet (\sigma^\mu \sigma_y \sigma_\lambda \sigma_y) \bullet (\sigma_y \sigma^\nu \sigma^\lambda \sigma_y) .
\] (15)
Recall that any combination of the type $\sigma_{\mu}\sigma_{y}$ ($\mu \neq 2$) represents a two-qubit comb. Note that the expectation value of an $n$th-order four-qubit filter has to be taken with respect to the corresponding $\mathcal{H}_n$, see Ref. \[18\]. It is straightforward to check that for a four-qubit GHZ state

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$$

we have $\langle \Phi_1 | F_1^{(4)} | \Phi_1^* \rangle = 1$. However, there is another state for which $\langle F_1^{(4)} \rangle_C$ does not vanish. For

$$|\Phi_5\rangle = \frac{1}{\sqrt{6}}(\sqrt{2} |1111\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle)$$

we find $\langle \Phi_5 | F_1^{(4)} | \Phi_5^* \rangle = 8/9$. Interestingly, \[10\] is the only maximally entangled state measured by the four qubit hyperdeterminant \[21\], which is a homogeneous function of degree 24. Its value for this state is $(\frac{8}{9})^4$, i.e. exactly the same as of the 24th order homogeneous invariant $F_1^{(4)}$, which however also measures the GHZ state. The hyperdeterminant of the four qubit GHZ state is zero.

Besides the 3rd-order filter $F_1^{(3)}$ there exist also filters of 4th order and of 6th order. Examples are

$$F_2^{(4)} = (\sigma_{\mu}\sigma_{\nu}\sigma_{y}\sigma_{y}) \cdot (\sigma_{\mu}\sigma_{y}\sigma_{\nu}\sigma_{\lambda}) \cdot (\sigma_{y}\sigma_{\nu}\sigma_{\mu}\sigma_{\lambda}) \cdot (\sigma_{y}\sigma_{\nu}\sigma_{\lambda}\sigma_{\tau})$$

$$F_3^{(4)} = \frac{1}{2}(\sigma_{\mu}\sigma_{\nu}\sigma_{y}\sigma_{y}) \cdot (\sigma_{\mu}\sigma_{y}\sigma_{\nu}\sigma_{y}) \cdot (\sigma_{\mu}\sigma_{y}\sigma_{\nu}\sigma_{\nu}) \cdot (\sigma_{y}\sigma_{y}\sigma_{\nu}\sigma_{\lambda}) \cdot (\sigma_{y}\sigma_{y}\sigma_{\lambda}\sigma_{\tau}) \cdot (\sigma_{y}\sigma_{y}\sigma_{\lambda}\sigma_{\tau}) .$$

While $F_2^{(4)}$ measures only GHZ-type entanglement ($\langle \Phi_2 | F_2^{(4)} | \Phi_2^* \rangle = 1$) the 6th-order filter $F_3^{(4)}$ has the non-zero expectation values $1/2$ for the GHZ state and $1$ for yet another state,

$$|\Phi_4\rangle = \frac{1}{2}(|1111\rangle + |1000\rangle + |0010\rangle + |0001\rangle) .$$

$F_1^{(4)}$ and $F_2^{(4)}$ have zero expectation value for this state (as well as the hyperdeterminant). Finally, all four-qubit filters $F_j^{(4)}$ ($j=1,2,3$) have zero expectation value for the four-qubit $W$ state $1/2(0111) + |1011\rangle + |1101\rangle + |1110\rangle$.

The states $|\Phi_j\rangle$ are the maximally entangled states for four qubits; they satisfy all three requirements in Def. \[14\] including the stronger condition (i) from Ref. \[17\]. Note that they cannot be transformed into one another by SLOCC operations: A state with a finite expectation value for one filter cannot be transformed by means of SLOCC operations into a state with zero expectation value for the same filter. For example, $F_2^{(4)}$ detects the GHZ state $|\Phi_1\rangle$ but gives zero for the other two states. Therefore, the four-qubit entanglement in those states must be different from that of the GHZ state.

Hence, there are at least three inequivalent types of genuine entanglement for four qubits \[24\]. We mention that the three maximally entangled states $|\Phi_j\rangle$ are not distinguished by the classification for pure four-qubit states of Ref. \[19\]. This can be seen by computing the expectation values of the four-qubit filters and the reduced one-qubit density matrices for each of the nine class representatives of Ref. \[19\]. Only the classes 1–4 and 6 have non-vanishing “$4$-tangle”. The corresponding local density matrices can be completely mixed only for class 1. Therefore, all three states $|\Phi_j\rangle$ must belong to that class.

### IV. FILTERS FOR MORE QUBITS

In this section we will continue the discussion from the previous section and demonstrate how general multipartite filters are constructed. It is not the scope of this work to discuss independence and completeness of a given set of filters, nor to “taylor” a filter for a given single class of entanglement. We only emphasize that every filter is an invariant and that linear homogeneous combinations and in fact any homogeneous function of them is an invariant, as well. Thus, when a sufficient set of independent filters is known together with their weights for the corresponding entanglement classes, such a taylored invariant can be constructed. This invariant, though, is not expected to be simply the modulus square of some filter.
For five qubits we find four independent filters, their independence becoming clear from their values on a set of maximally entangled states. In order to compactify the formulas, the tensor product symbol \( \otimes \) will be omitted.

\[
\mathcal{F}_1^{(5)} = (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_y \sigma_y) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y) \right) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y) \right)
\]

\[
\mathcal{F}_2^{(5)} = (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_y \sigma_y) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y) \right) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y) \right)
\]

\[
\mathcal{F}_3^{(5)} = (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_y \sigma_y) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y) \right) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y) \right)
\]

\[
\mathcal{F}_4^{(5)} = \frac{1}{8} \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y) \right) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y) \right)
\]

The set of maximally entangled states distinguished by these filters is

\[
|\Psi_2\rangle = \frac{1}{\sqrt{2}} (|11111\rangle + |00000\rangle)
\]

\[
|\Psi_4\rangle = \frac{1}{2} (|11111\rangle + |11100\rangle + |00010\rangle + |00001\rangle)
\]

\[
|\Psi_5\rangle = \frac{1}{\sqrt{6}} (\sqrt{2} |11111\rangle + |11000\rangle + |01001\rangle + |00001\rangle)
\]

\[
|\Psi_6\rangle = \frac{1}{2\sqrt{2}} (\sqrt{3} |11111\rangle + |10000\rangle + |01000\rangle + |00100\rangle + |00001\rangle)
\]

The index of the state indicates the number of Fock-states in the normal form of the state and will be termed its length; a deeper discussion of the maximally entangled states and their connection to the filters that measure them is beyond the scope of this article and will be reported elsewhere.

The states \(|\Psi_2\rangle - |\Psi_5\rangle\) satisfy all three requirements of definition I.1 for being a maximally entangled states. It is interesting that \(|\Psi_6\rangle\) instead satisfies only the 1st and the 3rd requirement but contains four-tangle as measured by the filter \(\mathcal{F}_3^{(4)}\).

For six qubits we only exemplarily write two independent filters but indicate how to construct filters for a general number of qubits.

\[
\mathcal{F}_1^{(6)} = (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_y \sigma_y \sigma_y) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y \sigma_y) \right) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y \sigma_y) \right)
\]

\[
\mathcal{F}_2^{(6)} = (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_y \sigma_y \sigma_y \sigma_y) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y \sigma_y) \right) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y \sigma_y) \right)
\]

\[
\vdots
\]

\[
\mathcal{F}_i^{(6)} = (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_y \sigma_y \sigma_y \sigma_y) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y \sigma_y) \right) \cdot \left( (\sigma_{\mu_1}^{\mu_2} \sigma_y \sigma_{\mu_3} \sigma_y \sigma_y) \right)
\]

where in the latter formula all the \(\mu_i\) are to be contracted properly; in the \(\sigma_i\) the “•” either have to be substituted by indices which then have to be contracted properly or by \(\sigma_y\). This also indicates how higher filters can be constructed and suggests that for a filter of an \(n\)-qubit system, at least \(\mathcal{F}_{n-1}\) be needed. It is worthwhile to mention that the above list is not meant to be exhaustive, nor did we explicitly check for permutation invariance, which eventually
could help crystallizing the “proper” filters. The set of maximally entangled states to be distinguished by the six qubit filters is

\[
|\Xi_2\rangle = \frac{1}{\sqrt{2}}(|111111\rangle + |000000\rangle)
\]

(30)

\[
|\Xi_4\rangle = \frac{1}{2}(|111111\rangle + |111100\rangle + |000010\rangle + |000001\rangle)
\]

(31)

\[
|\Xi_5\rangle = \frac{1}{\sqrt{6}}(\sqrt{2}|111111\rangle + |111000\rangle + |000100\rangle + |000010\rangle + |000001\rangle)
\]

(32)

\[
|\Xi_6\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|1\ldots1\rangle + |110000\rangle + |00\rangle \otimes |W_4\rangle)
\]

(33)

\[
|\Xi_7\rangle = \frac{1}{2\sqrt{2}}(\sqrt{3}|111111\rangle + |W_6\rangle)
\]

(34)

where \(|W_4\rangle := |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle\) and \(|W_6\rangle\) analogously are the W state for four and six qubits. We want to mention that only the states up to length 5 are free of any subtangle and that only for states up to length 4 all reduced density matrices are maximally mixed \([17]\).

The filter values for the maximally entangled states are reported in the table. The states are classified by the length of their normal form. An “X” indicates that the corresponding state does not occur. Whereas the tangles for four/five qubits discriminate all three/four maximally entangled states, the two six-tangles we explicitly wrote only attribute to those states with minimal length (the GHZ) and with maximal length. This table shows that the indicated states correspond to different entanglement SLOCC classes. In fact there is a relation between the length of the state and the degree of multilinearity of the filter, which will be reported on in another publication.

| length | \(|\mathcal{F}_4^{(4)}\rangle\) | \(|\mathcal{F}_4^{(6)}\rangle\) | \(\frac{1}{2}\) | \(\frac{1}{3}\) | \(\frac{1}{6}\) | \(\frac{1}{3}\) | \(\frac{1}{2}\) | \(\frac{1}{3}\) | \(\frac{1}{6}\) | \(\frac{1}{3}\) | \(\frac{1}{2}\) |
|--------|-----------------|-----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 2      | 1               | 1               | \(\frac{1}{2}\) | 1     | \(\frac{1}{3}\) | 1     | \(\frac{1}{6}\) | 1     | \(\frac{1}{3}\) |
| 4      | 0               | 0               | 1     | 0     | 0     | 0     | \(\frac{2}{3}\) | 0     | 0     |
| 5      | \(\frac{8}{9}\) | 0               | 0     | 0     | 0     | \(\frac{2}{3}\) | 0     | 0     |
| 6      | X               | X               | X     | \(\frac{3\sqrt{3}}{32}\) | 0     | 0     | 0     | 0     |
| 7      | X               | X               | X     | X     | X     | X     | 0     | \(\frac{2^6}{5^2}\) |

V. CONCLUSIONS

We have presented a new and efficient way of generating entanglement monotones. It is based on operators which we called filters. The expectation values of these operators are zero for all possible product states, not only for the completely factoring case. The building blocks of the filters (denoted combs) guarantee invariance under \(SL(2,\mathbb{C})^\otimes N\) for qubits. As a consequence, all filters are automatically entanglement monotones. They are measures of genuine multipartite entanglement. This circumvents the difficult task to construct entanglement monotones from the essentially known (linear) local unitary invariants.

As an immediate result of our method the concurrence for pure two-qubit states is reproduced. Moreover, we have found an alternative expression for the concurrence with the corresponding convex roof extension based on the corresponding filter operator. The application of the method to pure three-qubit states yields several operator-based expressions for the 3-tangle, including an explicitly permutation-invariant form.

Further advantages of this approach are the feasibility of constructing specific monotones that vanish for certain separable (pure) states and the applicability of this concept to partitions into subsystems other than qubits (i.e. qutrits . . .). The methods permits in a direct manner quantification and classification of multipartite entanglement. We demonstrate this with the explicit expressions for four- up to six-qubit entanglement measures that for the first time detect three different types of genuine four-qubit entanglement and four different types of five-qubit entanglement; the types of genuine four-qubit entanglement are not distinguished by the classification of four-qubit states in Ref. [19].

As to \(N\)-qubit systems, there remain various interesting questions. Clearly, it would be desirable to have a recipe how to build invariant combs for more complicated systems (e.g. higher spin). It would also be interesting to know what characterizes a complete set of filters for any given \(N\). While it is not obvious how the convex roof construction
for two qubits can be generalized, we believe that the operator form of the \( N \)-tangles in terms of filters makes it easier to solve this problem. The question is whether there is a systematic way to obtain a convex-roof construction for a given filter with general multi-linearity.

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22. If the antilinear operator \( A = LC \) is a comb (with the complex conjugation \( C \)), for the sake of brevity we will also call the linear operator \( L \) a comb.
23. Note that the only \( SL(2, \mathbb{C}) \) invariant linear operator of order two is very similar to its antilinear counterpart: \( 0 = \left\langle \sigma_\mu \right| \left. \sigma_\nu \right\rangle := \sum_{\mu, \nu = 0}^{3} \left\langle \sigma_\mu \right| g^{\mu, \nu} \left\langle \sigma_\nu \right| \) with \( g^{\mu, \nu} = \text{diag}\{ -1, 1, 1, 1 \} \).
24. In fact, there are \textit{exactly} three maximally entangled states for four qubits. This will be discussed in a forthcoming publication.