Bound State Calculations of the Three-Dimensional Yakubovsky Equations with the inclusion of Three-Body Forces

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Abstract

The four-body Yakubovsky equations in a Three-Dimensional approach with the inclusion of the three-body forces is proposed. The four-body bound state with two- and three-body interactions is formulated in Three-Dimensional approach for identical particles as function of vector Jacobi momenta, specifically the magnitudes of the momenta and the angles between them. The modified three dimensional Yakubovsky integral equations is successfully solved with the scalar two-meson exchange three-body force where the Malfliet-Tjon-type two-body force is implemented. The three-body force effects on the energy eigenvalue and the four-body wave function, as well as accuracy of our numerical calculations are presented.

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I. INTRODUCTION

The topic of three-body forces (3BFs) is as old as nuclear physics [1] and, based on meson exchanges, various processes have been proposed in the past (for a review see [2]). Among them, the Fujita-Miyazawa force [3] with an intermediate ∆ generated by the exchange of two pions is most obvious and is implemented in all modern 3BF models. However the nature of these 3BFs is still not completely understood. In recent years there has been new progress in understanding the form of nuclear forces, because of the application of chiral perturbation theory (χPT) [4]-[8]. From this developments one can expect a more systematic understanding of the form of two-body (2B) and 3B forces. However, χPT implies a priori unknown constants, the low-energy constants, which have to be determined from experimental data. The bound states of few-nucleons seem to be an ideal laboratory to determine 3BF parameters, as the binding energies are sensitive to the 3B interaction and they are expected to be governed by the low-energy regime of nuclear physics [9, 10]. Therefore, the understanding of nuclear few-body bound states is an important contribution to the understanding of the 3BF. To this aim and for their numerical investigations one requires reliable methods leading to the solutions of the non-relativistic Schrödinger equation.

In the past several solution methods have been developed and applied to the four-body bound state problem by using realistic nuclear potentials, the CRCGV [11], the SV [12], the HH [13], the GFMC [14], the NCSM [15], EIHH [16] and the Faddeev-Yakubovsky (FY) [17]-[27]. These calculational schemes are mostly based on a partial wave (PW) decomposition. Stochastic and Monte Carlo methods, however, are performed directly using position vectors in configuration space. One of the most viable approaches appears to be the FY method. The calculations based on FY are performed after a PW expansion with phenomenological potentials in configuration space [17, 18], and in momentum space [19]-[24] and recently with chiral potentials in momentum space [25]-[27].

The FY scheme based on a PW decomposition, which includes spin and isospin degrees of freedom, after truncation leads to two coupled sets of a finite number of coupled equations in three variables for the amplitudes. In PW decomposition the number of channels that must be included grows very rapidly in this case, and a further complication is arisen where there are now six spatial dimensions rather than the three required for three-body calculations. So in a PW decomposition one needs a tremendous number of partial waves to find converged
results. In view of this very large number of interfering terms it appears natural to give up such an expansion and work directly with vector variables.

On this basis recently we have extended the Three-Dimensional (3D) approach, which greatly simplifies the two- and three-body scattering and bound state calculations without using PW decomposition [28]-[38], to the four-body bound state [39, 40]. We have formulated the Yakubovsky equations with only 2BFs as function of vector Jacobi momenta, specifically the magnitudes of the momenta and the angles between them. We have obtained two coupled three-dimensional integral equations in six variables for the amplitudes which greatly simplifies the calculations without using PW decomposition. The obtained three-dimensional integral equations have been solved successfully for simple NN force models. In this paper we follow the same approach and consider the 3BFs in four-body bound state problem. As a simplification we neglect spin and isospin degrees of freedom and study the four-boson bound state problem.

So the purpose of this work is to demonstrate that one can solve the Yakubovsky equations for four-body bound state without using PW decomposition and in the first attempt we have done it by using very simple 2B and 3B model interactions.

In our formulation we work directly with vector variables in the Yakubovsky scheme in momentum space. Here the dependence on momentum vectors shows that our 3D representation in comparison to traditional PW representation avoids the very involved angular momentum algebra occurring for the permutations and especially for the 3BFs and the full solution can be reached exactly and simply whereas the PW representation of the amplitudes leads to rather complicated expressions [22].

We believe that this work is another step forward in the development of 3D approach for studying the few-body systems and it is the first attempt towards the solution of the 4N bound state problem with the inclusion of 3NFs without performing the PW decomposition.

This paper is organized as follows. In section II we briefly represent the coupled Yakubovsky equations for four-body bound state with two- and three-body interactions. In section III we evaluate the matrix elements of 3BFs. In section IV we discuss our choice for independent variables for the unknown amplitudes in the equations and in their kernels. Section V describes details of our algorithm for solving coupled Yakubovsky three-dimensional integral equations. In section VI we present our results for three- and four-body binding energies with and without model 3BFs and we provide the test of our calculation. Finally
we summarize in section VII and provide an outlook.

II. MOMENTUM SPACE REPRESENTATION OF YAKUBOVSKY EQUATIONS WITH 3BFS

The bound state of the four-body (4B) system, in the presence of 3BFs, is described by two coupled Yakubovsky equations:

\[
\begin{align*}
|\psi_1\rangle &= G_0 t P [(1 + P_{34})|\psi_1\rangle + |\psi_2\rangle] + (1 + G_0 t) G_0 W_{123}^{(3)} |\Psi\rangle \\
|\psi_2\rangle &= G_0 t \tilde{P} [(1 + P_{34})|\psi_1\rangle + |\psi_2\rangle]
\end{align*}
\] (1)

where the Yakubovsky components \(|\psi_1\rangle\) and \(|\psi_2\rangle\) belong to ”3 + 1”(123, 4; 12) and ”2 + 2”(12, 34; 12) partitions of the four particles respectively. Here the free four-body propagator is given by \(G_0 = (E - H_0)^{-1}\), and \(H_0\) stands for the free Hamiltonian. The operator \(t\) is the two-body transition matrix determined by a two-body Lippman-Schwinger equation. \(P\), \(\tilde{P}\) and \(P_{34}\) are permutation operators. \(P = P_{12}P_{23} + P_{13}P_{23}\) permutes the particles in three-body subsystem (123) and \(\tilde{P} = P_{13}P_{24}\) interchanges the two two-body subclusters (12) and (34). The quantity \(W_{123}^{(3)}\), as shown in Fig. 1, defines a part of the 3BF in the cluster (123), which is symmetric under the exchange of particles 1 and 2 and which can be related by an interchange of the three particles to two other parts \(W_{123}^{(1)}\) and \(W_{123}^{(2)}\) that sum up to the total 3BF of particles 1, 2 and 3: \(W_{123} = W_{123}^{(1)} + W_{123}^{(2)} + W_{123}^{(3)}\). The total 4B wave function \(|\Psi\rangle\) is given as:

\[
|\Psi\rangle = (1 + P + P_{34} P + \tilde{P}) [(1 + P_{34})|\psi_1\rangle + |\psi_2\rangle]
\] (2)

The symmetry property of \(|\psi_1\rangle\) under exchange of particles 1 and 2, and \(|\psi_2\rangle\) under separate exchanges of particles 1, 2 and 3, 4 guarantee that \(|\Psi\rangle\) is totally symmetric. It can easily be verified that the inclusion of the 3BF component \(W_{123}^{(3)}\) into the definition of the first Yakubovsky component \(|\psi_1\rangle\) does not change its symmetry property.

In this paper we follow the notation introduced in Ref. [40] and work in a 3D momentum space basis. According to the two types of chains (123, 4; 12) and (12, 34; 12) there are two type of basis states, Fig. 2 which are suitable to represent the two Yakubovsky components \(|\psi_1\rangle\) and \(|\psi_2\rangle\) in the coupled equations (1). The representation of coupled equations (1) in
FIG. 1: Diagrammatic representation of the part $W_{123}^{(3)}$ of a two-meson exchange 3BF. Here particle 3 is single out by the meson-nucleon amplitude described by the blob.

\[ 3+1: (123,4;12) \quad \text{and} \quad 2+2: (12,34;12) \]

FIG. 2: Definition of the $3+1$ and $2+2$ type of Jacobi coordinates.

These basis states will be exactly the same as displayed in Ref. [40] except that an extra term with $W_{123}^{(3)}$ occurs in the first component. This is

\[
\langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | (1 + G_0 t_{12}) G_0 W_{123}^{(3)} | \Psi \rangle = \int D^3 \vec{u} \langle \vec{u}_1 \vec{u}_2 \vec{v}_3 | (1 + G_0 t_{12}) G_0 | \vec{u}_1' \vec{u}_2' \vec{v}_3' \rangle \times \langle \vec{u}_1' \vec{u}_2' \vec{v}_3' | W_{123}^{(3)} | \Psi \rangle
\]

where $D^3 u \equiv d^3 u_1 d^3 u_2 d^3 u_3$. The first matrix element can be handled as described in Ref. [40]. The second matrix element involves the 3BF, which has been worked out in Ref. [33] in a 3D momentum space basis for three-body system. After evaluating the first matrix element in Eq. (3), the coupled three dimensional Yakubovsky integral equations can be rewrite explicitly as:
\begin{align*}
\langle \bar{u}_1 \bar{u}_2 \bar{u}_3 | \psi_1 \rangle &= \frac{1}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}} \\
&\times \left[ \int d^3 u_2' \langle \bar{u}_1 | t_s(\epsilon) | \frac{1}{2} \bar{u}_2 + \bar{u}_2' \rangle \right. \\
&\quad \times \left\{ \langle \bar{u}_2 + \frac{1}{2} \bar{u}_2' \bar{u}_2' \bar{u}_3 | \psi_1 \rangle \\
&\quad + \langle \bar{u}_2 + \frac{1}{2} \bar{u}_2' \frac{1}{3} \bar{u}_2' + \frac{8}{9} \bar{u}_3 \bar{u}_2' - \frac{1}{3} \bar{u}_3 | \psi_1 \rangle \\
&\quad + \langle \bar{u}_2 + \frac{1}{2} \bar{u}_2' - \bar{u}_2' - \frac{2}{3} \bar{u}_3 \frac{1}{2} \bar{u}_2' - \frac{2}{3} \bar{u}_3 | \psi_2 \rangle \right\} \\
&\quad + \left\{ \langle \bar{u}_1 \bar{u}_2 \bar{u}_3 | W_{123}^{(3)} | \Psi \rangle \\
&\quad + \frac{1}{2} \int d^3 \bar{u}_1' \langle \bar{u}_1 | t_s(\epsilon^*) | \bar{u}_1' \rangle \\
&\quad \times \left\{ 2 \langle \bar{v}_3 \frac{2}{3} \bar{v}_2 + \frac{2}{3} \bar{v}_3' \frac{1}{2} \bar{v}_2 - \bar{v}_3' | \psi_1 \rangle + \langle \bar{v}_3 - \bar{v}_2 \bar{v}_3' | \psi_2 \rangle \right\} \right] \\
\langle \bar{v}_1 \bar{v}_2 \bar{v}_3 | \psi_2 \rangle &= \frac{1}{E - \frac{v_1^2}{m} - \frac{v_2^2}{2m} - \frac{v_3^2}{m}} \\
&\times \left\{ 2 \langle \bar{v}_3 \frac{2}{3} \bar{v}_2 + \frac{2}{3} \bar{v}_3' \frac{1}{2} \bar{v}_2 - \bar{v}_3' | \psi_1 \rangle + \langle \bar{v}_3 - \bar{v}_2 \bar{v}_3' | \psi_2 \rangle \right\}
\end{align*}

where the $t_s(\epsilon)$ and $t_s(\epsilon^*)$ are symmetrized two-body $t$-matrices with the two-body subsystem energies $\epsilon = E - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}$ and $\epsilon^* = E - \frac{v_2^2}{2m} - \frac{v_3^2}{m}$. The matrix elements of the 3BF term, $\langle \bar{u}_1 \bar{u}_2 \bar{u}_3 | W_{123}^{(3)} | \Psi \rangle$, are evaluated in the next section.

### III. The Evaluation of 3BF Matrix Elements in a 3D Approach

Each part of a 3BF with two scalar meson exchanges and a constant meson-nucleon amplitude, which is shown in Fig. 1, can be written in the following form

\begin{equation}
W_{123}^{(3)} \propto \frac{F(Q^2)}{Q^2 + m_s^2} \frac{F(Q'^2)}{Q'^2 + m_s^2}
\end{equation}

with a cutoff function

\begin{equation}
F(Q^2) = \left( \frac{\Lambda^2 - m_s^2}{\Lambda^2 + Q^2} \right)^2
\end{equation}
and momentum transfers $\vec{Q}$ and $\vec{Q}'$

$$\vec{Q} = \vec{k}_1 - \vec{k}'_1$$
$$\equiv \left\{ (+\vec{u}_1 - \frac{1}{2}\vec{u}_2) - (+\vec{u}'_1 - \frac{1}{2}\vec{u}'_2) \right\}_{(123,4;12)}$$
$$\equiv \{ \vec{u}_2 - \vec{u}'_2 \}_{(231,4;23)}$$

$$\vec{Q}' = \vec{k}'_2 - \vec{k}_2$$
$$\equiv \left\{ (-\vec{u}'_1 - \frac{1}{2}\vec{u}_2) - (-\vec{u}_1 - \frac{1}{2}\vec{u}_2) \right\}_{(123,4;12)}$$
$$\equiv \{ \vec{u}'_2 - \vec{u}_2 \}_{(312,4;31)}$$

(7)

where the multiple indices for each curly bracket denote the two-body followed by the 3 + 1 fragmentation.

For the evaluation of Eq. (4) matrix elements of the form $\langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | W_{123}^{(3)} | \Psi \rangle$ need to be calculated. From Fig. 1 we see that $W_{123}^{(3)}$ can be considered as a sequence of meson exchanges in the subsystem (23), where it is called for convenience subsystem 1, and subsystem (31), is called 2. Since the structure of the 3BF we consider is specified by two momentum transfers of consecutive meson exchanges, it is convenient to insert a complete set of states of the type 2 between $W_{123}^{(3)}$ and $|\Psi\rangle$ and another complete set of states of type 1 between the two meson exchanges. Then the matrix element of $W_{123}^{(3)}$ is rewritten as

$$3\langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | W_{123}^{(3)} | \Psi \rangle$$
$$= \int_1 D^3u' \langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | \vec{u}'_1 \vec{u}'_2 \vec{u}'_3 \rangle_1$$
$$\times \int_1 D^3u'' \langle \vec{u}'_1 \vec{u}'_2 \vec{u}'_3 | F(Q^2) \frac{1}{Q^2 + m_s^2} \vec{u}''_1 \vec{u}''_2 \vec{u}''_3 \rangle_1$$
$$\times \int_2 D^3u''' \langle \vec{u}''_1 \vec{u}''_2 \vec{u}''_3 | \vec{u}'''_1 \vec{u}'''_2 \vec{u}'''_3 \rangle_2$$
$$\times \int_2 D^3u'''' \langle \vec{u}'''_1 \vec{u}'''_2 \vec{u}'''_3 | F(Q'^2) \frac{1}{Q'^2 + m_s^2} | \vec{u}''''_1 \vec{u}''''_2 \vec{u}''''_3 \rangle_2$$
$$\times \langle \vec{u}''''_1 \vec{u}''''_2 \vec{u}''''_3 | \Psi \rangle$$

(8)

Here the subscripts 1, 2, 3 of the bra and ket vectors and in integrals stand for the different types of three-body coordinate systems of (3 + 1)-type fragmentation $(ijk,4;ij)$. Both meson-exchange propagators in the 3BF term only depend on the momentum transfer in a
two-body subsystem, as indicated in Eq. (7), i.e.

\[ 1 \langle \vec{u}'_1 \vec{u}'_2 \vec{u}'_3 | \frac{F(Q^2)}{Q^2 + m_s^2} | \vec{u}''_1 \vec{u}''_2 \vec{u}''_3 \rangle_1 \]

\[ = \left\{ \delta^3(\vec{u}'_1 - \vec{u}''_1)\delta^3(\vec{u}'_3 - \vec{u}''_3) \frac{F((\vec{u}''_2 - \vec{u}''_2)^2)}{(\vec{u}'_2 - \vec{u}'_2)^2 + m_s^2} \right\}_1 \]

\[ 2 \langle \vec{u}'''_1 \vec{u}'''_2 \vec{u}'''_3 | \frac{F(Q'^2)}{Q'^2 + m_s^2} | \vec{u}''''_1 \vec{u}''''_2 \vec{u}''''_3 \rangle_2 \]

\[ = \left\{ \delta^3(\vec{u}'''_1 - \vec{u}'''_1)\delta^3(\vec{u}'''_3 - \vec{u}'''_3) \frac{F((\vec{u}''''_2 - \vec{u}''''_2)^2)}{(\vec{u}'_2 - \vec{u}'_2)^2 + m_s^2} \right\}_2 \]

(9)

Using Eq. (9), one can rewrite Eq. (8) as:

\[ 3 \langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | W^{(3)}_{123} | \Psi \rangle \]

\[ = \int_1 D^3 u_3 \langle \vec{u}'_1 \vec{u}'_2 \vec{u}'_3 | \vec{u}''_1 \vec{u}''_2 \vec{u}''_3 \rangle_1 \]

\[ \times \int_1 d^3 u'' \left[ \frac{F((\vec{u}''_2 - \vec{u}''_2)^2)}{(\vec{u}'_2 - \vec{u}'_2)^2 + m_s^2} \right]_1 \]

\[ \times \int_2 D^3 u''' \langle \vec{u}'''_1 \vec{u}'''_2 \vec{u}'''_3 | \vec{u}'''_1 \vec{u}'''_2 \vec{u}'''_3 \rangle_2 \]

\[ \times \int_2 d^3 u'''' \left[ \frac{F((\vec{u}''''_2 - \vec{u}''''_2)^2)}{(\vec{u}''''_2 - \vec{u}''''_2)^2 + m_s^2} \right]_2 \]

\[ \times 2 \langle \vec{u}''''_1 \vec{u}''''_2 \vec{u}''''_3 | \Psi \rangle \]

(10)

We would like to point out that in our vector based method the calculation of the transformations from one three-body subsystem to another, i.e. \( 3 \langle \| \rangle_1 \) and \( 1 \langle \| \rangle_2 \), are efficiently five-dimensional interpolations, whereas in calculation of the coordinate transformations via a PW decomposition, there is a complicated angular momentum recoupling algebra involved.

Also we would like to mention that we do not follow the explicit evaluation of the coordinate transformations in Eq. (10) leading to expressions with meson propagators which contain linear combinations of three or four momentum vectors. Thus direct integrations for evaluating the matrix element of the 3BF would involve magnitudes of momentum vectors and angles between all of them, which can be very complicated and involved. We therefore follow the method proposed in Ref. [33] and do not carry out the coordinate transformation.
analytically, we evaluate the integration of Eq. (10) in separate steps where in each step we only integrate over one vector variable at a time. Thus we define an auxiliary function

\[ F_2(\vec{u}_1', \vec{u}_2', \vec{u}_3') = \int_2 d^3u_2' \left[ \frac{F((\vec{u}_2' - \vec{u}_2'')^2)}{(u_2''^2 - u_2'^2)^2 + m^2_s} \right]_2 \]

the integration of the meson exchange between particles 2 and 3 in Eq. (11) is carried out completely in the coordinate system of type 2. Once \( F_2(\vec{u}_1', \vec{u}_2', \vec{u}_3') \) is obtained, it needs to be expressed in terms of momenta in a coordinate system of type 1 in order to carry out the integration over the remaining meson exchange. This transformation, labeled \( F_{21}(\vec{u}_1', \vec{u}_2', \vec{u}_3') \) is explicitly given as

\[ F_{21}(\vec{u}_1', \vec{u}_2', \vec{u}_3') = \int_1 D^3u_3'' \langle \vec{u}_1' \vec{u}_2' \vec{u}_3' | \vec{u}_1'' \vec{u}_2'' \vec{u}_3'' \rangle_1 F_2(\vec{u}_1'', \vec{u}_2'', \vec{u}_3'') \]

Here we used that \( F_2(\vec{u}_1', \vec{u}_2', \vec{u}_3') \) is a scalar function due to the total wave function \( \Psi(\vec{u}_1 \vec{u}_2 \vec{u}_3) \) being a scalar in the ground state. In our vector based method, this transformation is effectively a five dimensional interpolation on \( F_2 \) in Eq. (11), which can be handled by the cubic Hermitian splines of Ref. [41]. The integration over the second meson exchange between particle 3 and 1 in the coordinate system of type 1 is now given by

\[ F_1(\vec{u}_1', \vec{u}_2', \vec{u}_3') = \int_1 d^3u_3'' \left\{ \frac{F((\vec{u}_2' - \vec{u}_2'')^2)}{(u_2''^2 - u_2'^2)^2 + m^2_s} \right\}_1 F_{21}(\vec{u}_1', \vec{u}_2', \vec{u}_3') \]

The matrix element \( 3\langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | W^{(3)}_{123} | \Psi \rangle \) is finally obtained by integrating \( F_1(\vec{u}_1', \vec{u}_2', \vec{u}_3') \) over \( \vec{u}_1', \vec{u}_2' \) and \( \vec{u}_3' \), i.e. carrying out the final coordinate transformation from the system of type 1 back to the one of type 3,

\[ 3\langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | W^{(3)}_{123} | \Psi \rangle = \int_1 D^3u' \langle \vec{u}_1' \vec{u}_2' \vec{u}_3' | \vec{u}_1' \vec{u}_2' \vec{u}_3' \rangle_1 F_1(\vec{u}_1', \vec{u}_2', \vec{u}_3') \]

\[ = F_1(-\frac{1}{2} \vec{u}_1 - \frac{3}{4} \vec{u}_2, \vec{u}_1 - \frac{1}{2} \vec{u}_2, \vec{u}_3) \]
IV. CHOOSING THE COORDINATE SYSTEMS

In order to solve the coupled three dimensional Yakubovsky integral equations, Eq. (4), directly without employing PW projection, we have to define suitable coordinate systems. The Yakubovsky components are given as a function of Jacobi momenta vectors and as a solution of integral equations. Since we ignore spin and isospin dependencies, the both Yakubovsky components are scalars and thus only depend on the magnitudes of Jacobi momenta and the angles between them. The first important step for an explicit calculation is the selection of independent variables. As indicated in Ref. [33] one needs six variables to uniquely specify the geometry of the three vectors. The coupled three dimensional Yakubovsky integral equations, Eq. (4), with only 2BFs was solved successfully in Ref. [40]. For the evaluation of the 3BF term in the first Yakubovsky component in Eq. (4), \( 3 \langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | W_{123}^{(3)} | \Psi \rangle \), we start with calculating first \( F_2(\vec{u}_1', \vec{u}_2', \vec{u}_3') \), Eq. (11), and realize that for this integration we can choose \( \vec{u}_3' \) parallel to the \( z \)-axis and \( \vec{u}_2' \) in the \( x-z \) plane. This leads to the simplification of the azimuthal angles. The explicit expression is

\[
F_2(u_1', u_2', u_3', x_1', x_2', x_3', x_1''u_1'', u_2'', u_3'') = \int_0^\infty du_2'' u_2''^2 \int_{-1}^{+1} dx_2'' \int_0^{2\pi} d\phi_2'' \Gamma(u_2'', u_2', y_2''x_2'') \times \Psi(u_1', u_2', u_3', x_1', x_2', x_3', x_1''u_1'', u_2'', u_3'')
\]

with

\[
\begin{align*}
    u_1' &= |\vec{u}_1'| \\
    u_2' &= |\vec{u}_2'| \\
    u_3' &= |\vec{u}_3'| \\
    u_2'' &= |\vec{u}_2''| \\
    x_1'' &= \hat{u}_3' \cdot \hat{u}_1'' \equiv \cos(\hat{\theta}_1'') \\
    x_2'' &= \hat{u}_3' \cdot \hat{u}_2'' \equiv \cos(\hat{\theta}_2'') \\
    x_3'' &= \hat{u}_3' \cdot \hat{u}_3'' \equiv \cos(\hat{\theta}_3'')
\end{align*}
\]
This leads to the explicit expression which is functionally the same as Eq. (15):

\[ y_{1m}^{2m} = u_1^{m} \cdot u_2^{m} \]
\[ \equiv x_1^{m} x_2^{m} + \sqrt{1 - x_1^{m} r} \sqrt{1 - x_2^{m} r} \cos(\varphi_1^{m}) \]
\[ y_{1m}^{2m} = u_1^{m} \cdot u_2^{m} \]
\[ \equiv x_1^{m} x_2^{m} + \sqrt{1 - x_1^{m} r} \sqrt{1 - x_2^{m} r} \cos(\varphi_1^{m} - \varphi_2^{m}) \]
\[ y_{2m}^{2m} = u_1^{m} \cdot u_2^{m} \]
\[ \equiv x_2^{m} x_2^{m} + \sqrt{1 - x_2^{m} r} \sqrt{1 - x_2^{m} r} \cos(\varphi_2^{m}) \]
\[ x_{u_1^{m} u_2^{m}} = \frac{y_{1m}^{2m} - x_1^{m} x_2^{m}}{\sqrt{1 - x_1^{m} r} \sqrt{1 - x_2^{m} r}} \]
\[ x_{u_3^{m} u_2^{m}} = \frac{y_{1m}^{2m} - x_1^{m} x_2^{m}}{\sqrt{1 - x_1^{m} r} \sqrt{1 - x_2^{m} r}} \]
\[ \Pi^2 = u_2^{m} r^2 + u_2^{m} r^2 + 2 u_2^{m} u_2^{m} y_{2m}^{2m} \]
\[ \Gamma(u_2^{m}, u_2^{m}, y_{2m}^{2m}) = \frac{F(u_2^{m}, u_2^{m}, y_{2m}^{2m})}{\Pi^2 + m_s^2} \]
(16)

Similarly for the integration over the second meson exchange, i.e., the evaluation of \( F_1(\bar{u}_1^{m}, \bar{u}_2^{m}, \bar{u}_3^{m}) \) of Eq. (13), we can choose \( \bar{u}_3^{m} \) parallel to the \( z \)-axis and \( \bar{u}_2^{m} \) in the \( x-z \) plane. This leads to the explicit expression which is functionally the same as Eq. (15):

\[ F_1(u_1^{m}, u_2^{m}, u_3^{m}, x_1^{m}, x_2^{m}, u_3^{m} u_1^{m} u_2^{m}) \]
\[ = \int_{0}^{\infty} du_1^{m} u_2^{m} \int_{-1}^{1} dx_2^{m} \int_{0}^{2\pi} d\phi_2^{m} \Gamma(u_2^{m}, u_2^{m}, y_{2m}^{2m}) \]
\[ \times F_{21}(u_1^{m}, u_2^{m}, u_3^{m}, x_1^{m}, x_2^{m}, x_3^{m} u_1^{m} u_2^{m}) \]
(17)

with the same variables as Eq. (16) with \( u_1^{m}, u_2^{m}, u_3^{m}, x_1^{m}, x_2^{m}, x_3^{m}, \varphi_1^{m}, \varphi_2^{m} \) instead of \( u_1^{m}, u_2^{m}, u_3^{m}, x_1^{m}, x_2^{m}, x_3^{m}, \varphi_1^{m}, \varphi_2^{m} \). The evaluation of \( F_{21}(\bar{u}_1^{m}, \bar{u}_2^{m}, \bar{u}_3^{m}) \), Eq. (12), is not an integration but rather a five dimensional interpolation and explicitly is given by

\[ F_{21}(u_1^{m}, u_2^{m}, u_3^{m}, x_1^{m}, x_2^{m}, x_3^{m} u_1^{m} u_2^{m}) \]
\[ = F_2(\Pi_1, \Pi_2, u_3^{m}, x_1^{m} u_3^{m}, x_2^{m} u_3^{m}) \]
\[ = F_2(\Pi_1, \Pi_2, u_3^{m}, x_1^{m} u_3^{m}, x_2^{m} u_3^{m}) \]
(18)
with

\[
\Pi_1 = \left| -\frac{1}{2} \vec{u}_1' - \frac{3}{4} \vec{u}_2'' \right| = \frac{1}{2} \sqrt{u_1'^2 + \frac{9}{4} u_2''^2 + 3 u_1' u_2' y_1' y_2''} \\
\Pi_2 = \left| \vec{u}_1' - \frac{3}{4} \vec{u}_2'' \right| = \sqrt{u_1'^2 + \frac{1}{4} u_2''^2 - u_1' u_2' y_1' y_2''} \\
x_{\Pi_1 u_3'} = (-\frac{1}{2} \vec{u}_1' - \frac{3}{4} \vec{u}_2''). \hat{u}_3' = \frac{1}{\Pi_1} (-\frac{1}{2} u_1' x_1' - \frac{3}{4} u_2' x_2') \\
x_{\Pi_2 u_3'} = (\vec{u}_1' - \frac{3}{4} \vec{u}_2''). \hat{u}_3' = \frac{1}{\Pi_2} (u_1' x_1' - \frac{1}{2} u_2'' x_2') \\
x_{\Pi_1 \Pi_2} = \frac{1}{\Pi_1 \Pi_2} \left( -\frac{1}{2} u_1'^2 + \frac{3}{8} u_2''^2 - \frac{1}{2} u_1' u_2' y_1' y_2'' \right) \\
x_{\Pi_1 \Pi_2}^{u_3'} = \frac{x_{\Pi_1 \Pi_2} - x_{\Pi_1 u_3'} x_{\Pi_2 u_3'}}{\sqrt{1 - x_{\Pi_1 u_3'}^2} \sqrt{1 - x_{\Pi_2 u_3'}^2}}
\]

(19)

Finally, the matrix element \(3 \langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | W_{123}^{(3)} | \Psi \rangle\) is explicitly obtained by a five dimensional interpolation as

\[
3 \langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | W_{123}^{(3)} | \Psi \rangle = F_1(\pi_6, \pi_7, u_3, x_{\pi_6 u_3}, x_{\pi_7 u_3}, x_{\pi_6 \pi_7})
\]

(20)

with

\[
\pi_6 = \left| -\frac{1}{2} \vec{u}_1 - \frac{3}{4} \vec{u}_2 \right| = \frac{1}{2} \sqrt{u_1^2 + \frac{9}{4} u_2^2 + 3 u_1 u_2 y_1 12} \\
\pi_7 = \left| \vec{u}_1 - \frac{1}{2} \vec{u}_2 \right| = \sqrt{u_1^2 + \frac{1}{4} u_2^2 - u_1 u_2 y_1 12} \\
x_{\pi_6 u_3} = (-\frac{1}{2} \vec{u}_1 - \frac{3}{4} \vec{u}_2). \hat{u}_3 = \frac{1}{\pi_6} (-\frac{1}{2} u_1 x_1 - \frac{3}{4} u_2 x_2) \\
x_{\pi_7 u_3} = (\vec{u}_1 - \frac{1}{2} \vec{u}_2). \hat{u}_3 = \frac{1}{\pi_7} (u_1 x_1 - \frac{1}{2} u_2 x_2) \\
x_{\pi_6 \pi_7} = (-\frac{1}{2} \vec{u}_1 - \frac{3}{4} \vec{u}_2). (\vec{u}_1 - \frac{1}{2} \vec{u}_2) \\
= \frac{1}{\pi_6 \pi_7} (-\frac{1}{2} u_1^2 + \frac{3}{8} u_2^2 - \frac{1}{2} u_1 u_2 y_1 12) \\
x_{\pi_6 \pi_7}^{x_3} = \frac{x_{\pi_6 \pi_7} - x_{\pi_6 u_3} x_{\pi_7 u_3}}{\sqrt{1 - x_{\pi_6 u_3}^2} \sqrt{1 - x_{\pi_7 u_3}^2}}
\]

(21)

The last term of first Yakubovsky component in Eq. (11) requires an additional integration of the matrix element \(\langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | W_{123}^{(3)} | \Psi \rangle\) and the fully off-shell two-body \(t\)-matrix. Again, with
choosing \( \vec{u}_3 \) parallel to the \( z \)-axis we only have four vectors to consider, \( \vec{u}_1, \vec{u}_2, \vec{u}_3 \) and \( \vec{u}_1' \), thus the integration is of a similar type as the one of the first three terms of first Yakubovsky component in Eq. (4),

\[
\frac{1}{2} \int d^3 \tilde{u}' \frac{\langle \vec{u}_1 | t_s(\epsilon) | \vec{u}' \rangle}{E - \tilde{u}'^2 m - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}} \langle \vec{u}_1' \vec{u}_2 \vec{u}_3 | W_{123}^{(3)} | \Psi \rangle
\]

\[
= \frac{1}{2} \int_0^{\infty} d\tilde{x}_1' d\tilde{\varphi}_1' \int_0^{2\pi} d\tilde{\varphi}_1 \int_0^{1} d\tilde{x}_1 \int_0^{1} d\tilde{x}_2 \frac{t_s(u_1, \tilde{u}'_1, y_{i1}; \epsilon)}{E - \tilde{u}_2^2 m - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}}
\]

\[
\times \langle \vec{u}_1' \vec{u}_2 \vec{u}_3 | W_{123}^{(3)} | \Psi \rangle
\]  

(22)

with

\[
\tilde{u}'_1 = |\vec{u}'_1| \\
\tilde{x}'_1 = \hat{u}_3. \hat{u}_1 \equiv \cos(\hat{u}_1') \\
y_{i1} = \hat{u}_1'. \hat{u}_1 \equiv \tilde{x}'_1 x_1 + \sqrt{1 - \tilde{x}'_1^2} \sqrt{1 - x_1^2} \cos(\varphi') \\
y_{i2} = \hat{u}_1'. \hat{u}_2 \equiv \tilde{x}'_2 x_2 + \sqrt{1 - \tilde{x}'_2^2} \sqrt{1 - x_2^2} \cos(\varphi')
\]  

(23)

These considerations lead to the explicit representation for the Yakubovsky components \( |\psi_1\rangle \) and \( |\psi_2\rangle \):
\[
\psi_1(u_1 u_2 u_3 x_1 x_2 x_3) = \frac{1}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}}
\times \left[ \int_0^\infty du_2' u_2'^2 \int_{-1}^{+1} dx_2' \int_0^{2\pi} d\varphi_2' t_s(u_1, \hat{\pi}, \hat{x}; \epsilon)
\times \left\{ \psi_1(\pi_1 u_2' u_3 x_{12} x_{13} x_{\pi_1 u_2'})
+ \psi_1(\pi_1 \pi_2 \pi_3 x_{22} x_{23} x_{\pi_1 \pi_2})
+ \psi_2(\pi_1 \pi_4 \pi_5 x_{32} x_{33} x_{\pi_1 \pi_4}) \right\}
+F_1(\pi_6, \pi_7, u_3, x_{\pi_6 u_3}, x_{\pi_7 u_3}, x_{u_3}^{\pi_6 \pi_7})
+ \frac{1}{2} \int_0^\infty dv_1' u_1'^2 \int_{-1}^{+1} dX_1' \int_0^{2\pi} d\varphi_1' t_s(v_1, \hat{u}_1', \hat{y}_{13}; \epsilon)
\times F_1(\pi_6', \pi_7', u_3, x_{\pi_6' u_3}, x_{\pi_7' u_3}, x_{u_3}^{\pi_6' \pi_7}) \right] \right.
\]

\[
\psi_2(v_1 v_2 v_3 X_1 X_2 X_3) = \frac{1}{2} \left[ \int_0^\infty dv_3' v_3'^2 \int_{-1}^{+1} dX_3' \int_0^{2\pi} d\varphi_3' t_s(v_1, v_3', Y_{13'}; \epsilon^*)
\times \left\{ 2 \psi_1(v_3 \Sigma_1 \Sigma_2 X_{12} X_{13} X_{\Sigma_2 \Sigma_1})
+ \psi_2(v_3 v_2 v_3' X_{22} X_{23} X_{v_3' v_2}) \right\} \right] \right. \tag{24}
\]

The coupled equations, Eq. (24), is the starting point for numerical calculations and the details will be described in the next section. The 3D representation of total wave function \(|\Psi\rangle\) which directly appears in Eqs. (15) and (24) is represented in Ref. [40], where we have presented it as function of vector Jacobi momenta.

In a standard PW representation Eq. (4) is replaced by two coupled sets of a finite number of coupled integral equations [20], where the evaluation of two-body \(t\)-matrices and permutation operators \(P, \tilde{P}\) and \(P_{34}\) as well as coordinate transformations due to considering angular momentum quantum numbers instead of angle variables leads to more complicated expressions in comparison to our 3D representation.
V. NUMERICAL TECHNIQUES

In this section we describe some details of the numerical algorithm for solving the coupled Yakubovsky three dimensional integral equations, and more details can be found in Ref. [40]. The Yakubovsky components are given as a function of Jacobi momenta vectors and as a solution of coupled three dimensional integral equations, Eq. (24). The both Yakubovsky components are scalars and thus only depend on the magnitudes of Jacobi momenta and the angles between them. The dependence on the continuous momentum and angle variables should be replaced in the numerical treatment by a dependence on certain discrete values. For this purpose we use the Gaussian quadrature grid points. The coupled Yakubovsky equations represent a set of three dimensional homogenous integral equations, which after discretization turns into a huge matrix eigenvalue equation. The huge matrix eigenvalue equation requires an iterative solution method. We use a Lanczos-like scheme that is proved to be very efficient for nuclear few-body problems [42]. The momentum variables have to cover the interval $[0, \infty]$. In practice we limit the intervals to suitable cut-offs and their values are chosen large enough to achieve cut-off independence. The functional behavior of the kernel of eigenvalue equation is determined by the two-body $t-$matrices. We also solve the Lippman-Schwinger equation for the fully-off-shell two-body $t-$matrices directly as function of the Jacobi vector variables [28]. Since the coupled integral equations require a very large number of interpolations, we use the cubic Hermitian splines of Ref. [41] for its accuracy and high computational speed. It should be mentioned that by adding the additional grid points, 0 to all momentum and $\pm 1$ to all angle grid points, we avoid the extrapolation outside the Gaussian grids.

VI. NUMERICAL RESULTS

A. Three- and Four-Body Binding Energies

In our calculations for 2BF we employ the spin-averaged Malfliet-Tjon V potential [43]. This force is a superposition of a short-ranged repulsive and long-ranged attractive Yukawa interactions. We use the same parameters as given in Ref. [40] where the nucleon mass is defined by $\sqrt{\hbar^2/m} = 41.470 \text{MeV fm}^2$. With this interaction we solve the Lippman-Schwinger equation for the fully-off-shell two-body $t-$matrices directly as function of the Jacobi vector
TABLE I: Three-body binding energies with and without three-body forces in MeV. The numbers in parenthesis are binding energies calculated in Ref. 33 for three-body bound state with a modified version of Malfliet-Tjon by a cutoff function of dipole type. Also the number in bracket is calculated in FY scheme in PW representation, Ref. [19].

| Potential       | Three-body Binding Energy |
|-----------------|---------------------------|
| MT-V            | -7.74 [-7.73]             |
| MT-V+MT3-I      | -8.92                     |
| MT-V+MT3-II     | -8.70                     |
| MT2-II          | -7.69 (-7.70)             |
| MT2-II+MT3-I    | -8.87 (-8.87)             |
| MT2-II+MT3-II   | -8.64 (-8.65)             |

TABLE II: Four-body binding energies with and without three-body forces in MeV. The number in bracket is binding energy calculated in FY scheme in PW representation, Ref. [19].

| Potential       | Four-body Binding Energy |
|-----------------|--------------------------|
| MT-V            | -31.3 [-31.36]           |
| MT-V+MT3-I      | -38.8                    |
| MT-V+MT3-II     | -37.5                    |

variables as described in Ref. [28]. The so obtained $t-$matrices are then symmetrized to get $t_s(u_1, \tilde{\pi}, \tilde{x}; \epsilon)$ and $t_s(v_1, v'_3, Y_{13}; \epsilon^*)$.

For four-body (three-body) binding energy calculations thirty (forty) grid points for Jacobi momentum variables and twenty (thirty two) grid points for angle variables have been used respectively. As demonstrated in tables I and II, the calculations of the three- and four-body binding energies using only the MT-V 2BF yield the values $E = -7.74$ and $-31.3$ MeV, Ref. [40].

In our calculations for 3BF we use a model of 3BF which is based on multi-meson exchanges. We study two different types of 3BFs, a purely attractive and a superposition of attractive and repulsive, which are named MT3-I and MT3-II respectively Ref. [33]. As shown in Ref. [33] The parameters of these 3BFs are chosen so that the correction due to these 3BFs to the three-body binding energy calculated with the modified Malfliet-Tjon
2BF (MT2-II) is small, and they lead to binding energies near to the triton binding energy.

The three- and four-body binding energies calculated in 3D approach are given in tables I and II. Our results for three-body binding energies with the addition of the MT3-I and MT3-II 3BFs, while MT-V is used as 2BF, are $-8.92$ and $-8.70$ [MeV] and while MT2-II is used as 2BF are $-8.87$ and $-8.64$ respectively. Our results agree with corresponding values presented in Ref. [33] and [29]. Our results for four-body binding energies with the addition of the MT3-I and MT3-II 3BFs, while MT-V is used as 2BF, are $-38.8$ and $-37.5$ [MeV] respectively. Unfortunately we could not compare our results for four-body binding energies with other calculations, since to the best of our knowledge no comparable work with scalar two-meson exchange 3BFs exists. So in order to test the accuracy of our calculations we carried out two numerical tests which are presented in next section.

According to our experience for four-body bound state calculations with 2BF alone [40], we expect that our results with 3BF provide the same accuracy in comparison to other calculations of the four-body binding energy based on PW decomposition, while the numerical procedure are actually easier to implement.

**B. Test of Calculations**

In this section we investigate the numerical stability of our algorithm and our 3D representation of Yakubovsky components. We specially investigate the stability of the eigenvalue of the Yakubovsky kernel with respect to the number of grid points for Jacobi momenta, polar and azimuthal angle variables. We also investigate the quality of our representation of the Yakubovsky components and consequently wave function by calculating the expectation value of the Hamiltonian operator.

In table III we present the obtained eigenvalue results for binding energies given in tables I and II for different grids. We choose the number of grid points for Jacobi momenta as $N_{jac}$, for spherical angles as $N_{sph}$ and for polar angles as $N_{pol}$. As demonstrated in this table, the calculation of the eigenvalues $\lambda$ convergence to the value one for $N_{jac} = 30$ and $N_{sph} = N_{pol} = 20$. It should be clear that the solution of coupled Yakubovsky three-dimensional integral equations, with six independent variables for the amplitudes, is much more time-consuming with respect to the solution of three-dimensional Faddeev integral equation [33], with three variables for the amplitude.
TABLE III: Stability of the eigenvalue $\lambda$ of Yakubovsky kernel with respect to the number of grid points in Jacobi momenta $N_{jac}$, spherical angles $N_{sph}$ and polar angles $N_{pol}$. $E_{MT-V} = -31.3$, $E_{MT-V+MT3-I} = -38.8$, $E_{MT-V+MT3-II} = -37.5$ MeV and $\lambda_1, \lambda_2$ and $\lambda_3$ are corresponding eigenvalues.

| $N_{jac}$ | $N_{sph} = N_{pol}$ | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ |
|-----------|----------------------|-------------|-------------|-------------|
| 20        | 20                   | 0.987       | 0.988       | 1.010       |
| 26        | 20                   | 0.995       | 0.996       | 1.004       |
| 30        | 12                   | 0.997       | 0.997       | 1.003       |
| 30        | 16                   | 0.999       | 0.999       | 1.001       |
| 30        | 20                   | 1.000       | 1.000       | 1.000       |

The solution of coupled Yakubovsky three-dimensional integral equations in momentum space allows to estimate numerical errors reliably. With the binding energy $E$ and the Yakubovsky components $|\psi_1\rangle$ and $|\psi_2\rangle$ available, we are able to calculate the total wave function $|\Psi\rangle$ from Eq. (2) by considering the choice of coordinate systems which are represented in Ref. [40]. So in order to demonstrate the reliability of our calculations we can evaluate the expectation value of the Hamiltonian operator $H$ and compare this value to the previously calculated binding energy of the eigenvalue equation, Eq. (24). Explicitly we evaluate the following expression:

$$
\langle \Psi | H | \Psi \rangle = \langle \Psi | H_0 | \Psi \rangle + \langle \Psi | V | \Psi \rangle + \langle \Psi | W | \Psi \rangle = 12 \langle \psi_1 | H_0 | \Psi \rangle + 6 \langle \psi_2 | H_0 | \Psi \rangle + 6 \langle \Psi | V_{12} | \Psi \rangle + 4 \langle \Psi | W_{123} | \Psi \rangle
$$

(25)

where $V$ represents the 2BFs ($\sum_{i<j} V_{ij}$) and $W$ the 3BFs ($\sum_{i<j<k} W_{ijk}$). The expectation value of the kinetic energy $\langle H_0 \rangle$ and the 2B potential energy $\langle V_{12} \rangle$ have been evaluated in
Ref. [40]. The expectation value of the 3B potential energy, \( \langle W_{123} \rangle \), is given by

\[
\langle \Psi | W_{123} | \Psi \rangle = 3 \langle \Psi | W_{123}^{(3)} | \Psi \rangle \\
= 3 \times 8 \pi^2 \int_0^\infty du_1 u_1^2 \int_{-1}^{+1} dx_1 \int_0^{2\pi} d\varphi_1 \\
\times \int_0^\infty du_2 u_2^2 \int_{-1}^{+1} dx_2 \int_0^\infty du_3 u_3^2 \\
\times \Psi(u_1 u_2 u_3 x_1 x_2 \varphi_1) W_{123}^{(3)} \Psi(u_1 u_2 u_3 x_1 x_2 \varphi_1)
\]

(26)

Here the integrations need the evaluation of the matrix element \( 3 \langle \vec{u}_1 \vec{u}_2 \vec{u}_3 | W_{123}^{(3)} | \Psi \rangle \) of Eq. (10). The expectation values of the kinetic energy \( \langle H_0 \rangle \), the 2B interaction \( \langle V \rangle \), the 3B interaction \( \langle W \rangle \) and the Hamiltonian operator \( \langle H \rangle \) for three- and four-body bound states are given in tables IV and V respectively. In the same tables the corresponding binding energies calculated in 3D scheme are also shown for comparison to the expectation values of the Hamiltonian operator. One can see that the energy expectation value and eigenvalues \( E \) agree with high accuracy. All these numbers are not meant to provide insight into the physics of three and four interacting nucleons, but serve only as a demonstration that this technique allows a very accurate and easy handling of typical nuclear forces consisting of attractive and repulsive (short range) parts. In addition, they will serve as benchmarks for future studies.

3BF effects have a stronger impact on four-body bound state than for three-body bound state as can be seen for instance by comparing expectation values of the potential energies for the two systems. We find in case of MT-V \( \langle V \rangle = -37.51 \text{ (} -101.0 \text{)} \) MeV for three (four)-body bound state without 3BF and \( \langle V \rangle = -40.63, -40.02 \text{ (} -110.1, -107.6 \text{)} \) MeV with MT3-I and MT3-II 3BFs correspondingly. In the latter case the expectation values for the 3BFs are \( \langle W \rangle = -1.41, -1.07 \text{ (} -7.5, -6.0 \text{)} \) MeV for three (four)-body bound state. Already the trivial fact that there are four triplets in four-body bound state makes it clear that one has to expect 3BF effects to be more pronounced in the four-body bound state than in the three-body bound state.
TABLE IV: Expectation values with respect to the three-body wave functions for various potential combinations. We present the expectation values of the kinetic energy ⟨$H_0$⟩, the 2B interaction ⟨$V$⟩ and the three-body interaction ⟨$W$⟩. Additionally the expectation values of the Hamiltonian operator ⟨$H$⟩ are compared to the binding energy results from the Faddeev equations. All energies are given in MeV.

| Potential        | ⟨$H_0$⟩ | ⟨$V$⟩ | ⟨$W$⟩ | ⟨$H$⟩ | E  |
|------------------|---------|-------|-------|-------|----|
| MT-V             | 29.77   | -37.51| -      | -7.74 | -7.74 |
| MT-V+MT3-I       | 33.13   | -40.63| -1.41 | -8.91 | -8.92 |
| MT-V+MT3-II      | 32.38   | -40.02| -1.07 | -8.71 | -8.70 |
| MT2-II           | 28.64   | -36.33| -      | -7.69 | -7.69 |
| MT2-II+MT3-I     | 31.88   | -39.40| -1.34 | -8.86 | -8.87 |
| MT2-II+MT3-II    | 31.17   | -38.78| -1.04 | -8.65 | -8.64 |

TABLE V: The same as table IV but for four-body case.

| Potential        | ⟨$H_0$⟩ | ⟨$V$⟩ | ⟨$W$⟩ | ⟨$H$⟩ | E  |
|------------------|---------|-------|-------|-------|----|
| MT-V             | 69.7    | -101.0| -      | -31.3 | -31.3 |
| MT-V+MT3-I       | 78.8    | -110.1| -7.5  | -38.8 | -38.8 |
| MT-V+MT3-II      | 76.1    | -107.6| -6.0  | -37.5 | -37.5 |

VII. SUMMARY AND OUTLOOK

Instead of solving the coupled Faddeev-Yakubovsky equations in a partial wave basis, we introduce an alternative approach for four-body bound state calculations which implement directly momentum vector variables. We formulated the coupled Yakubovsky equations for identical spinless particles, interacting by two- and three-body forces, as function of vector Jacobi momenta, specifically the magnitudes of the momenta and the angles between them. We expect that coupled three-dimensional Yakubovsky equations for a bound state can be handled in a straightforward and numerically reliable fashion. In comparison to an angular momentum decomposition which is commonly used [19]-[27], this direct approach has great advantages. In our Three-Dimensional case there is only two coupled three-dimensional integral equations to be solved, whereas in the partial wave case one has two coupled sets of a finite number of coupled equations with kernels containing relatively complicated geometr-
rical expressions. The comparison of 3D and PW formalisms shows that our 3D formalism avoids the very involved angular momentum algebra occurring for the permutations and transformations and it is more efficient especially for the three-body forces.

The three dimensional Yakubovsky integral equations was successfully solved using Malfliet-Tjon type 2BF alone, and its numerical feasibility and accuracy established [40]. Here we present results including the scalar two-meson exchange three-body force and study its effects on the energy eigenvalue and the four-body wave function. The stability of our algorithm and our Three-Dimensional representation of Yakubovsky components have been achieved with the calculation of the eigenvalue of Yakubovsky kernel, where different number of grid points for Jacobi momenta and angle variables have been used. Also we have calculated the expectation value of the Hamiltonian operator. This test of calculation represents good agreement between the obtained eigenvalue energy and expectation value of the Hamiltonian operator.

This is very promising and nourishes our hope that calculations with realistic two and three-nucleon forces, namely considering spin and isospin degrees of freedom, will most likely be more easily implemented than the traditional partial wave based method.

To this aim the first step for realistic calculations of three-nucleon bound state in a realistic Three-Dimensional approach has already been taken by calculation of Triton binding energy with Bonn-B potential [44, 45] and formulation of four-nucleon bound state is currently underway and it will be reported elsewhere [46]. They will be the first steps for realistic calculations of three- and four-nucleon bound states in a Three-Dimensional scheme.

It should be mentioned that the input to such calculations is the NN $t$-matrix which is calculated in an approach based on a helicity representation and depends on the magnitudes of the initial and final momenta and the angle between them [31]. Consequently the calculation of NN $t$-matrix in helicity representation needs the NN potentials in an operator form which can be incorporated in 3D formalism. As indicated in sec. 3.2 of Ref. [47] (or sec. III of Ref. [31]) the general structure of the NN potential operator which fits well to the helicity representation is given, and on this representation both Bonn-B and AV18 NN potentials are given in operator form, see appendixes C and D (or sec. IV of Ref. [31]).
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