Action-Angle Variables for Complex
Projective Space and Semiclassical Exactness

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Abstract

We construct the action-angle variables of a classical integrable model defined on complex projective phase space and calculate the quantum mechanical propagator in the coherent state path integral representation using the stationary phase approximation. We show that the resulting expression for the propagator coincides with the exact propagator which was obtained by solving the time-dependent Schrödinger equation.

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There exist several examples of physical systems in which the semiclassical approximation to a path integral gives an exact quantum mechanical results \[1\]. Recently, this semiclassical exactness gained new interests \[2–6\] in relation to the Duistermaat-Heckman (DH) integration formula \[7–9\] which states that the integral of the exponential of the Hamiltonian $H$ with Liouville measure localizes to the critical points of $H$ if $H$ which is defined on compact phase space $\mathcal{M}$ generates torus action, and subsequently the stationary phase approximation gives the exact results.

The existence of torus action on the phase space $\mathcal{M}$ by symplectic diffeomorphism is one of the essential ingredients of the DH formula. In this respect, it could find some application to completely integrable system \[10\]. Let us consider a completely integrable system on phase space $\mathcal{M}$ of dimension $2N$ in which $N$ conserved quantities $J_m \ (m = 1, \cdots, N)$ with Hamiltonian $H = H(J_m)$ are in involution,

\[
\{H, J_m\} = \{J_m, J_n\} = 0, \quad (1)
\]

where the Poisson bracket is defined by the symplectic structure $\Omega = \frac{1}{2} \Omega_{AB} d\xi^A \wedge d\xi^B$ of $\mathcal{M}$

\[
\{f, g\} = \Omega^{AB} \partial_A f \partial_B g, \quad (2)
\]

with $f, g \in C^\infty(\mathcal{M})$.

The Liouville’s theorem states that if the manifold $\mathcal{M}_J$ defined by the level set of the functions $J_m$

\[
\mathcal{M}_J = \{\xi : J_m(\xi) = j_m\} \quad (3)
\]

with a constant $j_m$ is compact, then it is a smooth $N$-dimensional manifold diffeomorphic to the torus

\[
T^N = \{(\phi_1, \cdots, \phi_N) \mod \ 2\pi\} \quad (4)
\]

and invariant with respect to the symplectic diffeomorphism generated by the Hamiltonian. In this case, we can find the action variables $\vec{I} = (I_1, \cdots, I_N)$ conjugate to the angle $\vec{\phi} = (\phi_1, \cdots, \phi_N)$ so that the original symplectic structure is expressed by the Darboux two form
\[ \Omega = \sum_{m=1}^{N} dI_m \wedge d\phi_m. \quad (5) \]

The original Hamiltonian \( H(J_m) \) is a function of such \( I_m' \)'s, \( H = H(\vec{I}) \) and the system is solved completely:

\[ \vec{I}(t) = \vec{I}(0), \quad \vec{\phi}(t) = \vec{\phi}(0) + \vec{\omega}(\vec{I})t \quad (6) \]

where \( \vec{\omega}(\vec{I}) = \frac{\partial H(\vec{I})}{\partial \vec{I}} \).

The Hamiltonian vector field \( X_m \) associated with the action variable \( I_m \) generates the \( m \)-th circular action:

\[ X_m = \frac{\partial}{\partial \phi_m} \quad (m = 1, \cdots, N) \quad (7) \]

Therefore, if the Hamiltonian is a linear combination of the action variables

\[ H = \vec{\omega} \cdot \vec{I} + H_0 \quad (8) \]

with some constant \( \omega_m \) and \( H_0 \) which are independent of \( \vec{I} \), then \( H \) generates the torus action \( T^N \) and DH formula can be applied to the integral of the exponential of \( H \). In such a case, the stationary phase approximation should yield the exact quantum mechanical results. However, if the Hamiltonian is quadratic or higher order functions of the action variables, the semiclassical exactness does not hold. For example, if \( H = I_1^2 \), \( X_H \) generates \( \phi^1 \) circle action, but in general this circle action does not have a constant angular velocity on the phase space and DH formula does not hold \[7\].

In this paper, we present another example of physical system in which the semiclassical approximation gives the exact quantum mechanical results using the above observation. It is a classical integrable model of \( SU(N+1) \) isospin defined on complex projective phase space in the external magnetic field where the action-angle variables are explicitly constructed \[11\]. We calculate the quantum mechanical propagator in the coherent state path integral method using the semiclassical approximation and show that the result agree with the exact propagator which was obtained by solving the time-dependent Schrödinger equation \[11\].
We note that our result is the $SU(N+1)$ generalization of the semiclassical exactness of $SU(2)$ spin models which already exist in the literature [2,12–14].

The explicit construction of the action-angle variables for $CP(N)$ manifold can be made traceable by performing the symplectic reduction of $CP(N)$ from $S^{2N+1}$, $CP(N) \simeq S^{2N+1}/U(1)$ [15]. In terms of a complex column vector $z = (z_0, z_1, \cdots, z_N)^T \in \mathbb{C}^{N+1}$ and its complex conjugate $\bar{z} = (z_0^*, z_1^*, \cdots, z_N^*)$, $S^{2N+1}$ is defined by the constraint

$$\phi = \bar{z}z - 1 = 0. \quad (9)$$

Let us define the symplectic structure on $\mathbb{C}^{N+1}$ by

$$\Omega_{\mathbb{C}} = 2iJd\bar{z} \wedge dz \quad (10)$$

with a constant $J$. Then for $f, g \in C^\infty(\mathbb{C}^{N+1})$, we have

$$\{f, g\} = -\frac{i}{2J} \sum_{l=0}^{N} \left( \frac{\partial f}{\partial \bar{z}_l} \frac{\partial g}{\partial z_l} - \frac{\partial f}{\partial z_l} \frac{\partial g}{\partial \bar{z}_l} \right) \quad (11)$$

The next step is to introduce standard coordinates of $CP(N)$ by $\xi_m = z_m/z_0 (z_0 \neq 0, m = 1, 2, \cdots, N)$ and to make coordinate transformation from $z_I$ to $(z_0, \xi_m)$. To make reduction to the $CP(N)$ manifold, we choose a gauge condition such that $\bar{z}_0 = z_0$. We note that similar gauge condition was chosen in reduction to the maximal orbits of $SU(N+1)$ group [16]. Then, the solution to the constraint Eq.(3) is given by

$$\bar{z}_0 = z_0 = \frac{1}{\sqrt{1 + |\xi|^2}} \quad (12)$$

where $|\xi|^2 = \sum_{m=1}^{N} \xi_m^* \xi_m = \bar{\xi} \xi$. By substituting $z_0 = \frac{1}{\sqrt{1 + |\xi|^2}}$ and $z_m = \frac{\xi_m}{\sqrt{1 + |\xi|^2}}$ into Eq.(11), we find that the symplectic structure Eq.(10) descends to the one which is given by the Fubini-Study metric on $CP(N)$

$$\Omega = 2iJ \left[ \frac{d\bar{\xi} \wedge d\xi}{1 + |\xi|^2} - \frac{(\xi d\bar{\xi}) \wedge (\bar{\xi} d\xi)}{(1 + |\xi|^2)^2} \right]. \quad (13)$$

Also, the Poisson bracket Eq.(11) descends to $CP(N)$ by

$$\{F, H\} = -i \sum_{m,n} g^{mn} \left( \frac{\partial F}{\partial \bar{\xi}_m^*} \frac{\partial H}{\partial \xi_n} - \frac{\partial F}{\partial \xi_m} \frac{\partial H}{\partial \bar{\xi}_n^*} \right), \quad (14)$$
where $g^{mn}$ is the inverse of the Fubini-Study metric given by
\[
g^{mn} = \frac{1}{2J} (1 + |\xi|^2)(\delta_{mn} + \xi^*_m\xi_n). \tag{15}\]

The search for explicit expression for the action-angle variables on $CP(N)$ is facilitated by the observation that the Hamiltonian function which generates the circle action of $z_m \in \mathbb{C}^{N+1}$ coordinate is given by the quadratic function $2J z_m^* z_m$ (no sum on $m$) which can be easily checked

\[
\{z_n, \epsilon 2J z_m^* z_m\} = i\epsilon z_m \delta_{mn} \tag{16}\]

using the Eq.(11). Hence on $CP(N)$, the Hamiltonian function $I_m$ which generates the circle action on $\xi_m$ plane can be expressed as

\[
I_m = \frac{2J \xi_m^* \xi_m}{1 + |\xi|^2}. \tag{17}\]

The explicit form of the action-angle variables can be made more palpable by the use of stereographical projection. Let us introduce the polar angles $(\theta_1, \cdots, \theta_N)(0 \leq \theta \leq \pi)$ and $(\phi_1, \cdots, \phi_N)$ via

\[
\begin{align*}
\xi_1 &= \tan(\theta_1/2) \cos(\theta_2/2)e^{-i\phi_1} \\
\xi_2 &= \tan(\theta_1/2) \sin(\theta_2/2) \cos(\theta_3/2)e^{-i\phi_2} \\
&\vdots \\
\xi_{N-1} &= \tan(\theta_1/2) \sin(\theta_2/2) \cdots \sin(\theta_{N-1}/2) \cos(\theta_N/2)e^{-i\phi_{N-1}} \\
\xi_N &= \tan(\theta_1/2) \sin(\theta_2/2) \cdots \sin(\theta_{N-1}/2) \sin(\theta_N/2)e^{-i\phi_N} \tag{18}\end{align*}
\]

Using Eq.(17), we find the following expression for the action variables $I_m$

\[
\begin{align*}
I_m &= 2J \sin^2(\theta_1/2) \cdots \sin^2(\theta_m/2) \cos^2(\theta_{m+1}/2) \quad (m < N) \\
I_N &= 2J \sin^2(\theta_1/2) \cdots \sin^2(\theta_{N-1}/2) \sin^2(\theta_N/2) \tag{19}\end{align*}
\]

One can check that substitution of Eq.(18) into Eq.(13) produces Eq.(5) with $I_m$ given by the above formula. So our model Hamiltonian is given by a linear combination of action
variables given by Eq.(8) and (17). The Hamiltonian vector field associated to it generates $T^N$ torus action given by

$$X_H = i \sum_{m=1}^{N} \omega_m (\xi_m \frac{\partial}{\partial \xi_m} - \xi^*_m \frac{\partial}{\partial \xi^*_m}) = \sum_{m=1}^{N} \omega_m \frac{\partial}{\partial \phi_m}. \quad (20)$$

The classical solution describes a *conditionally-periodic motion* [10]:

$$\vec{\theta}(t) = \vec{\theta}(0), \quad \vec{\phi}(t) = \vec{\phi}(0) + \tilde{\omega} t. \quad (21)$$

We perform the path integral of our model using the coherent state method [17–19]. Let us consider $|0\rangle$, the highest weight state annihilated by all positive roots of $SU(N+1)$ algebra in Cartan basis. Then for $CP(N)$ with given $P \equiv 2J$ ($P \in \mathbb{Z}^+$) we have an irreducible representation $(P,0,\cdots,0)$ of $SU(N+1)$ group [20] and there are precisely $N$ negative roots $E_{\alpha}, \alpha = 1, 2, \cdots, N$ such that $E_{\alpha}|0\rangle \neq |0\rangle$. Let us label $\{E_{\alpha}\} = \{E_m\}$. We define a coherent state on $CP(N)$ corresponding to the point $\xi = (\xi_1, \cdots, \xi_N)$ by [18,19,21]

$$|P, \xi\rangle = \exp(\sum_{m} \xi_m E_m)|0\rangle \quad (22)$$

Notice that this definition differs from the usual one by the normalization factor. We have chosen this definition here because in the subsequent analysis, $\bar{\xi}$ and $\xi$ can be treated independently and the overspecification problem can be side-stepped [12,22,23]. We denote $|P, \xi\rangle = |\xi\rangle$ from now on. The coherent states which we have defined on $CP(N)$ have the following two properties which are essential in the path integral formulation. One is the resolution of unity,

$$\int D\mu(\bar{\xi}, \xi) \frac{|\xi\rangle \langle \xi|}{(1 + |\xi|^2)^{2J}} = I, \quad (23)$$

where $D\mu(\bar{\xi}, \xi) = c d\bar{\xi} d\xi / (1 + |\xi|^2)^{N+1}$ with a constant $c$ is the Liouville measure [19]. The other is reproducing kernel,

$$\langle \xi^" | \xi' \rangle = (1 + \bar{\xi}^" \xi')^{2J}. \quad (24)$$

We are interested in evaluating the propagator
\[ G(\tilde{\xi}', \xi'; T) = \langle \xi'' | e^{-i\hat{H}T} | \xi' \rangle. \] (25)

Let us divide the time interval \( T = t'' - t' \) by \( P + 1 \) steps with \( \bar{\xi}(P + 1) = \tilde{\xi}'' \) and \( \xi(0) = \xi' \), and let \( \epsilon = T/(P + 1) \). Inserting Eq. (23) repeatedly

\[
G(\tilde{\xi}'', \xi'; T) = \lim_{\epsilon \to 0} \int \cdots \int \prod_{p=1}^{P+1} \frac{D\mu(p)}{(1 + |\xi(p)|^2)^{2J}} \prod_{p=1}^{P+1} \langle \xi(p) | \xi(p-1) \rangle \left( 1 - i\epsilon \frac{\langle \xi(p) | \hat{H} | \xi(p-1) \rangle}{\langle \xi(p) | \xi(p-1) \rangle} \right)
\]

and using Eq. (24), we have

\[
\prod_{p=1}^{P+1} \langle \xi(p) | \xi(p-1) \rangle = \prod_{p=1}^{P+1} (1 + |\xi(p)|^2)^{2J} \exp \left\{ \epsilon \left( \frac{-2J \bar{\xi}(p)d\xi(p)}{1 + |\xi(p)|^2} \right) \right\}
\]

Hence in the limit \( \epsilon \to 0 \) we obtain the following expression

\[
G(\tilde{\xi}'', \xi'; T) = \int D\mu \exp \left\{ 2J \log(1 + \tilde{\xi}''(t'')) + i \int_{t'}^{t''} dt \left[ \frac{2J \bar{\xi}\dot{\xi}}{1 + |\xi|^2} - H(\tilde{\xi}, \xi) \right] \right\}
\]

where \( H(\tilde{\xi}, \xi) = \langle \xi | \hat{H}(\tilde{\xi}, \xi) | \xi \rangle / \langle \xi | \xi \rangle \) is the classical Hamiltonian given by the Eq.(8) and (17). The boundary conditions in the path integral is given by \( \xi(t') = \xi' \) and \( \tilde{\xi}(t'') = \tilde{\xi}'' \). Also \( G(\tilde{\xi}'', \xi'; T)|_{T \to 0} = (1 + \tilde{\xi}'' \xi')^{2J} \). We introduced \( \xi(t'') \) which is only a superfluous variable because the result of path integral Eq.(28) does not depend on this variable. It depends only on \( \tilde{\xi}'' \) and \( \xi' \). The equations of motion are

\[
i\dot{\xi} = g^{*mn} \frac{\partial H(\tilde{\xi}, \xi)}{\partial \xi^*_n}, \quad i\dot{\xi}^* = -g^{*mn} \frac{\partial H(\tilde{\xi}, \xi)}{\partial \xi_n}.
\]

Using the Hamiltonian given in Eq.(8) and (17), we get the following

\[
i\dot{\xi}^* - i\omega_m \xi^* = 0, \quad \dot{\xi} + i\omega_m \xi = 0,
\]

where no summation on the index \( m \) is assumed. We note that our model is equivalent to a collection of \( N \) harmonic oscillator in coherent state representation [22,23] although the Hamiltonian appears to be highly nonlinear. The solutions are given by

\[
\xi^*_m(t) = \xi^*_m e^{i\omega_m (t-t'')} \quad \xi_m(t) = \xi'_m e^{-i\omega_m (t-t')} \]

(31)
Now, we evaluate the propagator Eq. (28) around the classical solution using stationary phase method [24]. Denoting the above classical solution by \( \bar{\xi}_{\text{cl}} \) and \( \xi_{\text{cl}} \) and expanding around the classical solutions

\[
\bar{\xi} = \xi_{\text{cl}} + \delta \bar{\xi}, \quad \xi = \xi_{\text{cl}} + \delta \xi
\]  

with boundary conditions \( \delta \bar{\xi}(t'') = \delta \xi(t') = 0 \), we find the following expression for the propagator:

\[
G(\bar{\xi}'', \xi'; T) = K(T)(1 + \bar{\xi}''_{\text{cl}} \xi_{\text{cl}}(t''))^{2J} \exp(iS(\bar{\xi}_{\text{cl}}, \xi_{\text{cl}}, T))
\]  

Here \( K(T) \) is the Van Vleck-Pauli-Morette determinant [24] coming from the Gaussian integration of the fluctuations \( \delta \bar{\xi} \) and \( \delta \xi \). Its explicit form is given by \( K = C_0/\sqrt{\det M} \) (with \( C_0 \) a constant) where

\[
M = \begin{pmatrix}
  -\delta^{mn} \frac{d}{dt} - i\partial^m g^{*nl} \partial^l H & -i\partial^m g^{*nl} \partial^l H \\
  -i\partial^m g^{nl} \partial^l H & \delta^{mn} \frac{d}{dt} - i\partial^m g^{nl} \partial^l H
\end{pmatrix}.
\]  

Using the classical equations of motion Eq. (30), we have

\[
M = \begin{pmatrix}
  -\delta^{mn} \left( \frac{d}{dt} + i\omega_m \right) & 0 \\
  0 & \delta^{mn} \left( \frac{d}{dt} - i\omega_m \right)
\end{pmatrix}.
\]  

From the above equation, we see that \( \det M \) is equal to \( 2N \) product of harmonic oscillator factor (\( N \) harmonic oscillator with frequency \( \omega_m \) and the other \( N \) with \( -\omega_m \)) and a simple lattice calculation shows that each harmonic oscillator factor is equal to one. So we have \( K(T) = C_0 \). Substituting the classical solution Eq. (31) and \( K(T) = C_0 \) into Eq. (33), we obtain

\[
G(\bar{\xi}'', \xi'; T) = C_0 \left( 1 + \sum_{m=1}^{N} \xi_m^{''} \xi'_m \exp(-i\omega_m T) \right)^{2J} \exp(iH_0 T).
\]  

The above expression is equal to the exact propagator obtained by solving the time-dependent schrödinger equation [11] (with \( C_0 = 1 \) which is fixed by the boundary condition \( G(\bar{\xi}'', \xi'; T) \bigg|_{T \to 0} = (1 + \bar{\xi}'' \xi')^{2J} \)).
In summary, we constructed the action-angle variables of the completely integrable model defined on $CP(N)$ manifold by using the symplectic reduction of $CP(N)$ from $S^{2N+1}$. Then, adopting the coherent state path integral method, we showed that the stationary phase approximation of the integrable model yields the exact quantum mechanical propagator, thus providing the $SU(N+1)$ generalization of the semiclassical exactness of $SU(2)$ spin models [2,12–14]. Extension to other coadjoint orbits of Lie group is also conceivable and will be reported elsewhere.

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