A Dynamic Programming Formulation for the Nonlinear Filter
Jin Won Kim and Prashant G. Mehta

Abstract—This paper builds on our recent work where we presented a dual stochastic optimal control formulation of the nonlinear filtering problem [1]. The constraint for the dual problem is a backward stochastic differential equations (BSDE). The solution is obtained via an application of the maximum principle (MP). In the present paper, a dynamic programming (DP) principle is presented for a special class of BSDE-constrained stochastic optimal control problems. The principle is applied to derive the solution of the nonlinear filtering problem.

I. INTRODUCTION

A backward stochastic differential equation (BSDE) is an Itô stochastic differential equation over a finite time-horizon \([0, T]\) where the terminal condition at the terminal time \(T\) is specified. BSDE was first introduced by Bismut as an adjoint equation for the linear-quadratic stochastic optimal control problem [2]. Later, Pardoux and Peng [3] introduced nonlinear BSDEs and proved the existence and uniqueness results for the general Lipschitz cases. BSDEs have several applications in stochastic optimal control [4], mathematical finance [5], and analysis of certain types of partial differential equations [6]. Since BSDE is a dynamical system, it is natural to investigate control problems for BSDEs. After Peng [7] introduced the optimal control problem for forward-backward SDEs, the theoretical framework and applications for the stochastic maximum principle (MP) is widely studied [8], [9], [10], [11], [12].

In a recent paper [1], we introduced a dual model to transform the classical nonlinear filtering problem into a BSDE-constrained stochastic optimal control problem. The model has since been used for the purposes of defining observability of the nonlinear filtering problem [13], and for analysis of the filter stability in the ergodic [14] and non-ergodic [15] settings of the problem. In the present paper [1], the solution of the optimal control problem is obtained by using the MP.

The object of the present paper is to introduce a dynamic programming (DP) approach to solve a class of BSDE-constrained stochastic optimal control problems. After the DP equation is presented in Section III, it is applied to the dual optimal control problem of nonlinear filtering.

The outline of the remainder of this paper is as follows: The nonlinear filtering problem and its dual – the BSDE-constrained optimal control problem – appear in Sec. III. The DP equation and its application to the nonlinear filtering problem are presented in Sec. III. The Appendix contains the proofs.

II. DUALITY FOR NONLINEAR FILTERING

Notation: The state-space \(\mathbb{S} := \{1, 2, \ldots, d\}\) is finite. The set of probability vectors on \(\mathbb{S}\) is denoted by \(\mathcal{P}(\mathbb{S})\); \(\mu \in \mathcal{P}(\mathbb{S})\) if \(\mu(x) \geq 0\) and \(\sum_{x \in \mathbb{S}} \mu(x) = 1\). The space of deterministic functions on \(\mathbb{S}\) is identified with \(\mathbb{R}^d\); Any function \(f : \mathbb{S} \rightarrow \mathbb{R}\) is determined by its value \(f(x)\) at \(x \in \mathbb{S}\). For a measure \(\mu \in \mathcal{P}(\mathbb{S})\) and a function \(f \in \mathbb{R}^d\), \(\mu(f) := \sum_x \mu(x) f(x)\). For two vectors \(f, h \in \mathbb{R}^d\), \(fh\) denotes the element-wise (Hadamard) product: \((fh)(x) := f(x)h(x)\) and similarly \(f^2 = ff\).

A. Filtering problem

Consider a pair of continuous-time stochastic processes \((X, Z)\) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\):

1. The state \(X = \{X_t \in \mathbb{S} : t \geq 0\}\) is a Markov process with initial condition \(X_0 \sim \mu \in \mathcal{P}(\mathbb{S})\) (prior) and the rate matrix \(A \in \mathbb{R}^{d \times d}\). For a function \(f \in \mathbb{R}^d\), the carré du champ operator \(\Gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d\) is defined according to

\[
\Gamma(f)(x) := \sum_{j \in \mathbb{S}} A(x, j)(f(x) - f(j))^2 \quad \text{for } x \in \mathbb{S}
\]

2. The observation process \(Z = \{Z_t \in \mathbb{R} : t \geq 0\}\) is defined according to the following model:

\[
Z_t = \int_0^t h(X_\tau) d\tau + W_t
\]

where \(h : \mathbb{S} \rightarrow \mathbb{R}\) is the observation function and \(W = \{W_t \in \mathbb{R} : t \geq 0\}\) is a Wiener process (w.p.) that is assumed to be independent of \(X\). The scalar-valued observation model is considered for notational ease. The filtration generated by \((X, W)\) is denoted by \(\mathcal{F} := \{\mathcal{F}_t : t \geq 0\}\), and the filtration generated by \(Z\) is denoted as \(\mathcal{Z} := \{\mathcal{Z}_t : t \geq 0\}\) where \(\mathcal{Z}_t = \sigma(Z_s : s \leq t)\).

The filtering problem is to compute the conditional distribution (posterior), denoted by \(\pi_t \in \mathcal{P}(\mathbb{S})\), of the state \(X_t\) given \(Z_t\). For \(f \in \mathbb{R}^d\), \(\pi_t(f) := \mathbb{E}(f(X_t) | Z_t)\) is the object of interest.

A standard approach [16, Ch. 5] on optimal filtering is to consider a new probability measure \(\tilde{\mathbb{P}}\) on \(\Omega\) such that the Radon-Nikodym derivative with respect to \(\mathbb{P}\) is given by

\[
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}_{\mathcal{F}_t} = \exp\left( -\int_0^t h(X_\tau) dW_\tau - \frac{1}{2} \int_0^t |h(X_\tau)|^2 d\tau \right) =: D_t^{-1}
\]

Then \(Z\) is a \(\tilde{\mathbb{P}}\)-Brownian motion independent of \(X\). Under this new measure, the unnormalized filter is defined by \(\sigma_t(f) :=\)
where $\mathbb{E}(\mathcal{D}_t f(X_t)|Z_t)$ where $\mathbb{E}$ denotes the expectation with respect to $\tilde{P}$.

**Function spaces:** Under $\tilde{P}$, $Z$ is a Brownian motion, and therefore the following Hilbert spaces are defined: The space $L^2_Z([0,T];S)$ is a $Z$-adapted random processes taking values in $S$. Similarly, $L^2_Y([0,T];S)$ is a $S$-valued random element which is $\mathcal{F}_T$-measurable.

In both cases, the $L^2$ norm is unique with respect to $\tilde{P}$.

**B. Dual optimal control problem**

In our recent work [1], a dual optimal control problem is introduced. It is based upon the following linear BSDE:

$$
-dY_t(x) = ((AY_t(x) + h(x)(U_t + V_t(x))) \, dt - V_t(x) \, dZ_t,
Y_T(x) = F(x) \quad \forall x \in \mathcal{S}
$$

The boundary condition prescribed at the terminal time $T$ is allowed to be random, with $F \in L^2_Z(\mathbb{R}^d)$. The control $U = \{U_t : 0 \leq t \leq T\}$ is chosen in the set of admissible control $L^2_Z([0,T];\mathbb{R})$. The solution pair $(Y,V) = \{(Y_t,V_t) : 0 \leq t \leq T\}$ of the BSDE is adapted to the filtration $Z$, and it is uniquely determined in $L^2_Z([0,T];\mathbb{R}^d \times L^2_Z([0,T];\mathbb{R}^d))$ [3].

Define the cost functional

$$
J(U) := \mathbb{E}(\frac{1}{2}|Y_0(X_0) - \mu(X_0)|^2 + \int_0^T l(Y_t,V_t,U_t,t) \, dt)
$$

where the Lagrangian $l$ is given by

$$
l(y,v,u,t,\omega) := \frac{1}{2}\sigma(y)(\Gamma(y)(\omega)) + \frac{1}{2}\sigma_v(|u + v(\omega)|^2)
$$

The dual optimal control problem is to choose a control $U \in L^2_Z([0,T];\mathbb{R})$ such that $J(U)$ is minimized subject to the BSDE constraint (1).

The following proposition is a version of the main result in [1]. The justification appears in Appendix A.

**Proposition 1:** Consider the dual optimal control problem. Define

$$
S_t = \mu(Y_0) - \int_0^t U_t \, dZ_t, \quad t \in [0,T]
$$

Then for all $t \in [0,T]$,

$$
J(U) = \mathbb{E}(\frac{1}{2}D_t|Y_t(X_t) - S_t|^2 + \int_t^T l(Y_t,V_t,U_t,\tau) \, d\tau)
$$

In particular,

$$
J(U) = \frac{1}{2}\mathbb{E}(D_T|Y_T(X_T) - S_T|^2) = \frac{1}{2}\mathbb{E}(|F(X_T) - S_T|^2)
$$

The equation (3) is called duality principle, and it connects the minimum variance estimate problem and the BSDE-constrained stochastic optimal control problem.

**III. MAIN RESULTS**

**A. Optimal control problem on BSDE**

Consider the BSDE-constrained stochastic optimal control problem in more general form.

Minimize $J_U(U) := \mathbb{E}(h(Y_0) + \int_0^T l(Y_t,V_t,U_t,t) \, dt)$

Subject to $dY_t = f(Y_t,V_t,U_t,t) \, dt + V_t \, dZ_t, \quad Y_T = \xi$

where $\xi \in L^2_{Z\ast}(\mathbb{R}^d)$. The set of admissible control is $U = L^2_Z([0,T];\mathbb{R})$. The drift term $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times [0,T] \times \Omega \to \mathbb{R}^d$ is assumed to be uniformly Lipschitz with each argument almost surely, for almost every $t$. For every value of $y,v$ and $u$, $f(y,v,u,\cdot)$ is $\mathcal{F}_t$-adapted. Under this condition, [5] admits the unique solution pair $(Y,V) \in L^2_Z([0,T];\mathbb{R}^d \times L^2_Z([0,T];\mathbb{R}^d))$ [3]. For the cost functional, we assume $h : \mathbb{R}^d \to \mathbb{R}$ and $l : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R} \times [0,T] \times \Omega \to \mathbb{R}$ are bounded almost surely. Similar to $f$, $l(y,v,u,\cdot)$ is $\mathcal{F}_t$-adapted.

**B. Value function of stochastic optimal control problems**

To formulate the dynamic programming principle, consider a partial problem up to time $t \leq T$ from $\zeta \in L^2_Z(\mathbb{R}^d)$ defined by:

$$
J_t(U;\zeta) = \mathbb{E}(h(Y_0) + \int_0^t l(Y^\tau_s, V^\tau_s, U, \tau) \, d\tau)
$$

where $\{(Y^\tau_s, V^\tau_s) : \tau \in [0,t]\}$ is the solution to the BSDE:

$$
dY^\tau_s = f(Y^\tau_s, V^\tau_s, U, \tau) \, dt + V^\tau_s \, dW_s, \quad Y^\tau_0 = \zeta
$$

**Definition 1:** Consider the optimal control problem [4], [5].

The value function is a sequence of functions $V := \{V_t : 0 \leq t \leq T\}$ where $V_t : L^2_Z \to \mathbb{R}$ is defined by:

$$
V_t(\zeta) = \inf_{U \in \mathcal{U}} J_t(U;\zeta)
$$

Analogously with the forward-in-time stochastic DP principle, the following theorem whose proof appears in Appendix [3] is proposed:

**Theorem 1:** Let $V$ be the value function of the optimal control problem [4], [5]. Then it satisfies the following:

(i) $V_0(\cdot) = h$

(ii) For any $0 \leq s < t \leq T$ and any $\zeta \in L^2_Z(\mathbb{R}^d)$,

$$
V_t(\zeta) = \inf_{U \in \mathcal{U}} \mathbb{E}(V_s(Y^\tau_s) + \int_s^t l(Y^\tau_s, V^\tau_s, U, \tau) \, d\tau)
$$

For stochastic optimal control problems, an appealing formulation for the value function is to construct a martingale associated with it. The martingale version of DP principle is as follows. The proof appears in Appendix [3].

**Proposition 2:** Let $V$ be the value function of the optimal control problem [4], [5]. Then

(i) $V_0(\cdot) = h$

(ii) Define $M^U_t = \{M^U_t : 0 \leq t \leq T\}$ for any admissible control $U \in \mathcal{U}$ by:

$$
M^U_t = V_t(Y_t) - \int_0^t l(Y_t, V_t, U_t, \tau) \, d\tau
$$

where $(Y,V)$ is the solution to [3]. $M^U$ is a super-martingale for any admissible control $U$, and it is a martingale if and only if $U$ is the optimal solution.
C. Optimal control obtained via the martingale DP principle

The following theorem whose proof appears in Appendix [D] characterizes the optimal control using dynamic programming.

**Theorem 2:** Suppose there exists \( V \) and \( U^* \in \mathcal{U} \) such that:

1. \( V_0(\cdot) = h(\cdot) \).
2. The process \( M^U \) defined by (8) is a super-martingale for each admissible control \( U \in \mathcal{U} \), and a martingale for \( U = U^* \).

Then \( U = U^* \) is an optimal control with cost \( \mathbb{E}(V_T(\xi)) \).

**Remark 1:** From the definition of super-martingale, the second condition is equivalent to write for any \( 0 \leq s \leq t \leq T \),

\[
V_s(Y_s) \geq \mathbb{E}(V_t(Y_t) - \int_s^t l(Y_r, V_r, U_r, r) \, dr \mid Z_r)
\]  
(9)

For forward-in-time Markovian stochastic control problems, the counterpart of (9) is exactly the dynamic programming principle and \( V \) is the value function (cf. [17, Remark 6.1.5]). However for BSDE problems, conditioning on \( Z_r \) is not the same as fixing on \( Y_r \). Therefore, the conditions in Theorem 2 do not yield an interpretation of \( V_t \) as the value function at time \( t \) in this case but only concludes that \( U^* \) is optimal.

D. Application to nonlinear filtering

The Proposition 1 suggests that the optimal solution yields \( \delta_t = \pi_t(Y_t) \). Hence, consider \( V_t(\xi) \) to be

\[
V_t(\xi) = \frac{1}{2} \mathbb{E}(D_t|\xi(X_t) - \pi_t(\xi)|^2)
\]  
(10)

The following proposition whose proof appears in Appendix [E] allows to obtain the optimal solution in a DP approach.

**Proposition 3:** Consider the dual optimal control problem (1), (2). Let \( M^U \) be defined by (5) where \( V_t \) is defined as (10). Then \( M^U \) is a \( \mathbb{P} \)-super-martingale, and \( M^U \) is a \( \mathbb{P} \)-martingale if and only if for all \( t \),

\[
U_t = -\left( \pi_t \left( h(Y_t) \right) - \pi_t \left( h(X_t) \right) \right) - \pi_t(V_t)
\]  
(11)

**Remark 2:** By the Theorem 2 we conclude (11) is the optimal solution to the dual optimal control problem. In [1], stochastic maximum principle is used to derive the optimal control. A similar super-martingale is also considered using the innovation process. The change of measure is introduced to make \( Z \) be a filtration generated by a Brownian motion \( Z \) in this paper.

IV. Conclusion

The DP approach provides a sufficient condition to obtain the optimal control directly, while MP provides necessary condition. However, a verification theorem to obtain the value function for control problems on BSDEs is still an open question. Although the function \( V \) plays a role like the value function in its forward-in-time counterpart, it is challenging to prove that \( V \) is in fact the value function due to the information structure. This is subject to future research.

REFERENCES

[1] J. W. Kim, P. G. Mehta, and S. P. Meyn, “What is the Lagrangian for nonlinear filtering?” in 2019 IEEE 58th Conference on Decision and Control (CDC). Nice, France: IEEE, Dec 2019, pp. 1607–1614.
[2] J.-M. Bismut, “An introductory approach to duality in optimal stochastic control,” SIAM Review, vol. 20, no. 1, pp. 62–78, 1978.
[3] E. Pardoux and S. Peng, “Adapted solution of a backward stochastic differential equation,” Systems & Control Letters, vol. 14, no. 1, pp. 55–61, 1990.
[4] M. Kohlmann and X. Y. Zhou, “Relationship between backward stochastic differential equations and stochastic controls: a linear-quadratic approach,” SIAM Journal on Control and Optimization, vol. 38, no. 5, pp. 1392–1407, 2000.
[5] N. El Karoui, S. Peng, and M. C. Quenez, “Backward stochastic differential equations in finance,” Mathematical finance, vol. 7, no. 1, pp. 1–71, 1997.
[6] É. Pardoux, “Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic pdes of second order,” in Stochastic Analysis and Related Topics VI. Springer, 1998, pp. 79–127.
[7] S. Peng, “Backward stochastic differential equations and applications to optimal control,” Applied Mathematics and Optimization, vol. 27, no. 2, pp. 125–144, 1993.
[8] J. J. Huang, G. Wang, and J. Xiong, “A maximum principle for partial information backward stochastic control problems with applications,” SIAM journal on Control and Optimization, vol. 48, no. 4, pp. 2106–2117, 2009.
[9] G. Wang, Z. Wu, and J. Xiong, An introduction to optimal control of FBsDE with incomplete information. Springer, 2018.
[10] J. W. Kim and P. G. Mehta, “A dual characterization of observability for stochastic systems,” in 24th International Symposium on Mathematical Theory of Networks and Systems (MTNS), vol. 54, no. 9, Cambridge, UK, 2021, pp. 659–664.
[11] J. W. Kim, P. G. Mehta, and S. P. Meyn, “The conditional Poincaré inequality for filter stability,” in 2021 IEEE 60th Conference on Decision and Control (CDC), Austin, TX, Dec 2021.
[12] J. W. Kim and P. G. Mehta, “A dual characterization of the stability of the Wonham filter,” in 2021 IEEE 60th Conference on Decision and Control (CDC), Austin, TX, Dec 2021.
[13] J. Xiong, An Introduction to Stochastic Filtering Theory. Oxford University Press on Demand, 2008, vol. 18.
[14] R. Van Handel, “Stochastic calculus, filtering, and stochastic control.” Course notes, 2007. [Online]. Available: http://www.princeton.edu/~rvan/acm217/ACM217.pdf

APPENDIX

A. Proof of the Proposition 1

The claim is essentially [1, Prop. 1 and 2]. Therefore the only justification is that the optimal control formulation is identical. The original formulation in [1] is:

\[
J(U) = \mathbb{E} \left( \frac{1}{2} Y_0(X_0) - \mu(X_0) \right)^2 \\
+ \int_0^T \left( \frac{1}{2} l(Y_r, U_r, V_r) \right)^2 \, dr
\]

Apply the change of measure from \( P \) to \( \mathbb{P} \):

\[
J(U) = \tilde{\mathbb{E}} \left( \frac{1}{2} Y_0(X_0) - \mu(X_0) \right)^2 \\
+ \int_0^T \frac{1}{2} l(Y_r, U_r, V_r) \, dr
\]
By the tower property of conditional expectation,
\[
J(U) = \mathbb{E}\left(\frac{1}{2}|Y_0(X_0) - \mu(Y_0)|^2 \right) + \int_0^T \frac{1}{2}\mathbb{E}(D_t \Gamma(Y_t)(X_t) | Z_t) dt \\
+ \int_0^T \frac{1}{2}\mathbb{E}(D_t U + V_t(X_t))^2 | Z_t) dt \\
= \mathbb{E}\left(\frac{1}{2}|Y_0(X_0) - \mu(Y_0)|^2 + \int_0^T l(Y_t, V_t, U_t, t) dt \right)
\]
Therefore, (2) is identical to the cost functional considered in [1], and the claim follows.

B. Proof of the Theorem
We start from the definition of the value function:
\[
\mathcal{V}_i(\xi) = \inf_{U \in \mathcal{U}_t} \mathbb{E}\left( h(Y^U_t) + \int_0^T l(Y^U_t, V^U_t, U_t, \tau) d\tau \right)
\]
Recall the integral formulae:
\[
Y^U_t = \xi - \int_0^t f(Y_s, V_s, U_s, \tau) d\tau - \int_0^t V_s dZ_t
\]
Note that it depends only on \(U_\tau : \tau \in [s,t]\). Meanwhile,
\[
Y^U_s = Y^U_t - \int_s^t f(Y_s, V_s, U_s, \tau) d\tau - \int_s^t V_s dZ_t, \quad u \leq s
\]
depends only on \(U_\tau : \tau \in [0, s]\) given \(Y^U_t\), and therefore
\[
(Y^U_s, Y^U_s) = (Y^U_s, Y^U_s)
\]

C. Proof of the Proposition
We start from (7):
\[
\mathcal{V}_i(\xi) \leq \mathbb{E}\left( \mathcal{V}_i(Y^U_s) + \int_s^T l(Y^U_t, V^U_t, U_t, \tau) d\tau \right)
\]
Note that both sides are a map a random variable \(\xi\) to a scalar. For \(\xi = Y_t\),
\[
(Y_t, V_t) = (Y^U_t, V^U_t)
\]
and therefore
\[
\mathbb{E}(\mathcal{V}_i(Y_t) | Z_t) \leq \mathbb{E}(\mathcal{V}_i(Y_t) + \int_0^T l(Y_t, V_t, U_t, \tau) d\tau | Z_t)
\]
Upon subtracting \(\mathbb{E}(\int_0^T l(Y_t, V_t, U_t, \tau) d\tau | Z_t)\) on both sides, we have
\[
\mathbb{E}(\mathcal{V}_i(Y_t) - \int_0^T (Y_t, V_t, U_t, \tau) d\tau | Z_t)
\]
\[
\leq \mathbb{E}(\mathcal{V}_i(Y_t) - \int_0^T l(Y_t, V_t, U_t, \tau) d\tau | Z_t)
\]
Since the right-hand side is \(Z_t\)-measurable, we may drop conditional expectation, and hence
\[
\mathbb{E}(M^U_t | Z_t) \leq M^U_t
\]
Therefore, \(M^U_t\) is a super-martingale. The inequality becomes equality upon choosing the optimal control.

D. Proof of the Theorem
By assumption that \(M^U_t\) is a super-martingale,
\[
\mathbb{E}(M^U_{t^*}^2) \leq M^U_{t^*} = \mathbb{V}_0(Y_0) + h(Y_0)
\]
Take expectation on the right-hand side and expand the left-hand side as
\[
\mathbb{E}(\mathcal{V}_T(\xi) - \int_0^T l(Y_t, V_t, U_t, t) dt) \leq \mathbb{E}(h(Y_0))
\]
Therefore we have
\[
\mathbb{E}(\mathcal{V}_T(\xi)) \leq J(U), \quad \forall U
\]
where equality holds for \(U = U^*\).

E. Proof of the Proposition
By the tower property of the conditional expectation,
\[
\mathcal{V}_i(Y_t) = \mathbb{E}\left( D_t | Y_t(X_t) - \pi(Y_t)|^2 | Z_t \right)
\]
The term inside the expectation equals to
\[
\mathbb{E}(D_t | Y_t(X_t) - \pi(Y_t)|^2 | Z_t)
\]
where we used \(\pi(Y_t) = \pi(1) \pi(Y_t)\).
The first term requires \(d(Y^U_t)^2\), which is
\[
dY^U_t = -2Y_t(A_y + h(U_t + V_t)) + V_t^2 dt + 2Y_tV_t dZ_t
\]
Use Zakai equation [16, Theorem 5.5] to take differential form of each term:
\[
d\sigma(Y^U_t)^2 = \sigma(A_y^2) dt + \sigma(h Y_t^2) dZ_t
\]
\[
\quad + \sigma(-2Y_t(A_y + h(U_t + V_t)) + V_t^2) dt
\]
\[
\quad + 2\sigma(Y_t V_t) dZ_t + 2\sigma(h Y_t V_t) dt
\]
\[
\quad = \sigma(Y_t) dY_t dt + \sigma(V_t^2) dt - 2\sigma(h V_t) U_t dt
\]
Again use Zakai equation to compute:
\[
d\sigma(Y_t) = \sigma(A_y) dt + \sigma(h Y_t) dZ_t - \sigma(A_y + h(U_t + V_t)) dt
\]
\[
\quad + \sigma(Y_t) dZ_t + \sigma(h V_t) dt
\]
\[
\quad = -\sigma(h) U_t dt + \sigma(h Y_t + \sigma(V_t)) dZ_t
\]
Upon using the nonlinear filter in a similar way, one can obtain
\[
d\sigma(Y_t) = -\pi(h) (U_t - U^*_t) dt + U^*_t dZ_t
\]
where \(U^*_t = -\pi(h) (U_t - U^*_t) \pi(h) \pi(Y_t) - \pi(Y_t)\) as given in (11). Therefore by Itô product rule,
\[
d\sigma(Y_t) \pi(Y_t) = -\sigma(h) U_t \pi(Y_t) dt - \sigma(Y_t) \pi(h) (U_t - U^*_t) dt
\]
\[
\quad + (\sigma(h Y_t) + \sigma(V_t)) U_t dt + (\cdots) dZ_t
\]
where the martingale term is omitted. \(dM^U_t\) can now be simplified by collecting terms, and
\[
dM^U_t = \frac{1}{2}\sigma(1) (U_t - U^*_t)^2 dt + (\cdots) dZ_t
\]
Since \(-\frac{1}{2}\sigma(1) (U_t - U^*_t)^2 \leq 0\) and \(Z\) is a \(\mathcal{P}^U\)-martingale, \(M^U_t\) is a \(\mathcal{P}\)-super-martingale, and it is a martingale if and only if \(U_t = U^*_t\) for all \(t\).