On mean curvature flow with forcing

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ABSTRACT
This paper investigates geometric properties and well-posedness of a mean curvature flow with volume-dependent forcing. With the class of forcing which bounds the volume of the evolving set away from zero and infinity, we show that a strong version of star-shapedness is preserved over time. More precisely, it is shown that the flow preserves the \( q \)-reflection property, which corresponds to a quantitative Lipschitz property of the set with respect to the nearest ball. Based on this property we show that the problem is well-posed and its solutions starting with \( q \)-reflection property become instantly smooth. Lastly, for a model problem, we will discuss the flow's exponential convergence to the unique equilibrium in Hausdorff topology. For the analysis, we adopt the approach developed by Feldman-Kim to combine viscosity solutions approach and variational method. The main challenge lies in the lack of comparison principle, which accompanies forcing terms that penalize small volume.

1. Introduction

In this paper, we consider the sets \( (\Omega_t)_{t>0} \) in \( \mathbb{R}^n \) moving by the motion law

\[
V = -H + \lambda |[\Omega_t]| \quad \text{on} \quad \partial \Omega_t.
\]

(1.1)

Here \( V = V(x, t) \) and \( H = H(x, t) \) respectively denote the outward normal velocity and the mean curvature of \( \partial \Omega_t \) at \( x \in \partial \Omega_t \), where \( H \) is set to be positive if \( \Omega_t \) is convex at the point. The volume-dependent forcing \( \lambda : \mathbb{R}^+ \to \mathbb{R} \) will be assumed to be locally Lipschitz with growth conditions (see Assumption A below).

We are interested in the global-time description of the flow, including its well-posedness. In general, due to the low-dimensional nature of the interface, finite-time topological singularities are expected even for interfaces starting out with smooth shapes. On the other hand (1.1) is a parabolic flow, and thus parabolic regularity theory applies once we know that the evolving boundary \( \partial \Omega_t \) is locally a graph. Thus our first goal is to establish an \textit{a priori} graph property of \( \partial \Omega_t \) by studying the geometry of the evolution.

The forcing \( \lambda \) we consider in this article keeps the volume of \( \Omega_t \) bounded away from zero and infinity. With such choices of forcing we will show that a strong version of star-shapedness property holds for \( \Omega_t \) at all \( t > 0 \) if initially true, assuming the existence

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of the flow. Let us remark that this geometric result does not extend to the classical mean curvature flow where $\lambda = 0$. With zero forcing and with star-shaped initial set, solutions of (1.1) have been shown to hold certain semi-convexity estimates by Smoczyk [1] and Lin [2]. While these estimates allow classification of possible singularities for the flow in terms of blow-up limits, it remains open whether the initially star-shaped flow stays star-shaped beyond the initial time even with zero forcing.

While the zero forcing case appears to be out of reach at the moment, in the subsequent work [3] we show that our analysis applies to the volume-preserving mean curvature flow, using approximate forcing $\lambda_\delta = \frac{1}{\delta}(|\Omega_0| - |\Omega|)$ with small $\delta > 0$ which satisfies our assumptions. One of the main focus in Ref. [3] will be on obtaining a sufficiently strong convergence of this approximation to preserve the flow’s geometric properties.

With the a priori geometric property of the flow, we next discuss existence and uniqueness of the flow (1.1) based on its variational structure. A formal calculation yields the energy inequality

$$\frac{d}{dt} J(t) = -\int_{\partial \Omega_t} V^2 dS,$$

(1.2)

where $J(t) = \text{Per}(\Omega_t) - \Lambda(|\Omega_t|)$ with $\Lambda$ the anti-derivative of $\lambda$ and $V$ as given in (1.1). From (1.2) one expects $\Omega_t$ to flow toward a stationary point of the energy as time grows. We will make this observation rigorous by generating a discrete-time approximation (or “minimizing movement”) that satisfies the energy dissipation. The aforementioned a priori geometric property enables the uniform convergence of its discrete time approximations, to guarantee that in the continuum limit we recover a smooth solution. For long-time behavior, to illustrate our ideas, we show an exponential convergence of the sets for the following specific flow

$$V = -H + |\Omega_t|^\gamma,$$

(1.3)

which has a simple energy structure with the unique stable equilibrium.

While the variational approach yields the minimizing movements approximation as well as the asymptotic analysis of the flow, viscosity solutions are more suited for geometric arguments. To take advantage of both approaches we will show that the variational flow is, in a sense, a viscosity solution of (1.1). This idea of combining the two approaches were previously used for the mean curvature flow in Ref. [4], but in our problem the standard maximum principle does not apply for (1.1), and thus the notion of viscosity solutions needs to be modified from the standard one. Indeed our main novelty in the analysis is to combine these two approaches to address geometric motions which do not satisfy a comparison principle but still is of parabolic nature. For free boundary problems this combination has been introduced in Ref. [5], where the presence of bulk pressure plays a crucial role in the analysis.

To state the main results, let us begin with discussing the assumptions on the forcing.

**Assumption A.** $\lambda : \mathbb{R}^+ \to \mathbb{R}$ is locally Lipschitz continuous and satisfies $\limsup_{R \to \infty} \frac{\lambda(|B_R|)}{R} < \infty$. In addition, there exists $\rho > 0$ such that $\lambda(|\Omega|) > \frac{n-1}{\rho}$ for all $\Omega \subset B_{5\rho}$.

The first part of the assumption is necessary to show that the evolution is unique and the set does not spread to $\mathbb{R}^n$ in finite time. The second part puts a sufficient penalty
on shrinkage of the evolution, and is used in showing that the evolution always contains a small ball $B_{\rho}(0)$ if initially so (Lemma 3.9). Both $\lambda$ given in (1.3) and $\lambda$ given by a large multiple of $C - |\Omega|$ satisfy Assumption A.

With the parameter $\rho$ given from above assumption, we also assume the following on the initial data.

**Assumption B.** $\Omega_0$ has $\rho$-reflection (see Definition 3.3).

The $\rho$-reflection property should be interpreted as a quantitative smallness requirement on the Lipschitz norm distance between $\Omega_0$ and the nearest ball (see Lemma 3.4).

We are now ready to summarize our results. We adopt Definition 2.6 as the notion of solutions for (1.1). Our first result states the preservation of the $\rho$-reflection property. The proof is based on the reflection maximum principle as well as various barrier arguments based on Assumption A.

**Theorem 1.** [A priori geometric properties, Theorem 3.8] Suppose that $\Omega_0$ has $\rho$-reflection and there is a solution $(\Omega_t)_{t>0}$ of (1.1). Then, $\Omega_t$ has $\rho$-reflection at all times $t > 0$. In particular there exists $r_1 = r_1(\rho) > 0$ such that $\Omega_t$ is star-shaped with respect to a ball $B_{r_1}(0)$ for all $t > 0$.

From above result and the volume bound it follows that $\Omega_t$ has locally Lipschitz boundary which is uniform in time. This fact endows sufficient compactness for the evolution that makes it possible for the discrete-time variational scheme to approximate the flow, in particular establishing the global existence results.

**Theorem 2.** [Well-posedness, Theorem 6.8] Suppose that $\Omega_0$ has $\rho$-reflection. Then, there exists a unique solution $(\Omega_t)_{t>0}$ of (1.1) that is bounded and has smooth boundary for every $t > 0$. $(\Omega_t)_{t>0}$ can be approximated locally uniformly by minimizing movements with constraints.

Lastly we discuss asymptotic convergence of the model flow (1.3) with exponential convergence rate.

**Theorem 3.** [Long-time behavior, Theorem 7.6] Suppose that $\Omega_0$ has $\rho$-reflection. For (1.3), there exists $T > 0$ such that $\partial \Omega_t$ is uniformly $C^{1,1}$ for $t \geq T$. Moreover $\Omega_t$ converges to a ball of radius $r_{\infty}$ exponentially fast as $t \to \infty$, in Hausdorff distance.

We expect that the above results can be extended to flows with more general forcing, such as the volume-preserving mean curvature flow. The main challenge there is in the lack of a priori regularity of $\lambda$. For further discussion we refer to Ref. [3], where Theorems 1 and 2 are shown for the volume preserving mean curvature flow.

1.1. Literature

The viscosity solutions approach for (1.1) with fixed $\lambda$ was introduced by Evans and Spruck ([6–9]), and by Chen, Giga and Goto [10] in more general context. The minimizing movements were first introduced for
Local regularity results are available for (1.4) with the aforementioned unit density hypothesis, using the notion of varifold solutions from geometric measure theory (see Brakke [14]). Escher and Simonett [15] show that if the initial surface is sufficiently close to a sphere in $C^{1,\alpha}$ sense, then the volume preserving mean curvature flow converges to the sphere exponentially fast.

Next we recall results addressing geometric properties of the flow. In Ref. [16], Huisken showed that initially strictly convex surfaces evolving by (1.4) stay convex and shrink to a point in finite time. Parallel results are shown in Ref. [17] for the volume-preserving mean curvature flow, where it is shown that convex surfaces converge into spheres. These convergence results are extended respectively to anisotropic mean curvature flow by Andrews [18], and to its volume-preserving version by Bellettini et al. in Ref. [19].

For non-convex sets, most available results investigates singularities of the flow (1.4) for spatial dimensions larger than two ([1, 2, 20–22]). For (1.4) in space dimension two, Angenent proves in Refs. [23, 24] that the number of intersections of a pair of curves does not increase over the evolution, and in particular simply connected domains stay so until they shrink to a point. Note that, for flows with forcing, topology of an evolving set may not be preserved even in two dimensions (see Figure 1A and B for possible scenarios).

1.2. Outline of the paper

In Section 2, we give a definition on the notion of “viscosity solutions” for (1.1) in terms of its level-set formulation. To do so we first discuss the mean curvature flows with a priori fixed forcing,

$$V = -H + \eta(t).$$

Our solution $\Omega_t$ of (1.1) is then defined as the viscosity solution of (1.5) where $\eta(t)$ coincides with $J[|\Omega_t|].$

In Section 3, we show that (1.1) preserves the $\rho$-reflection property. As in Ref. [5] our arguments are based on reflection comparisons. More precisely, for given $\nu, x_0 \in \mathbb{R}^n$
define $\Pi^+_{\nu, x_0} := \{x + x_0 : x \cdot \nu \geq 0\}$ and $\Pi_\nu := \partial \Pi^+_{\nu, x_0}$. Since the normal velocity law (1.1) is preserved with respect to spatial reflections, comparison principle applies in the region $\Pi^+_{\nu} \times [0, \infty)$ to $\Omega_t$ and $\Omega^\nu_{t, x_0}$, the reflected version of $\Omega_t$ with respect to $\Pi_\nu$. It follows that if

$$\Omega^\nu_{t, x_0} \subset \Omega_0 \text{ in } \Pi^+_{\nu},$$

then such property is preserved for later times. We will show that this property and Assumption B imply that $\partial \Omega_t$ is locally Lipschitz, as long as $\Omega_t$ contains a small neighborhood of the origin. Recall that (1.1) does not satisfy classical comparison principle. This is why we resort exclusively to this particular type of comparison arguments.

Section 4 yields uniqueness of solutions for (1.1). The proof is based on small-time uniqueness for star-shaped solutions of (1.5), and the Lipschitz continuity of $\lambda$ given by Assumption A.

In Section 5, based on the discrete-time minimizing movement, we generate a flat flow of (1.1) characterized as the continuum gradient flow of the energy functional $J(E)$ given in (1.2). Let us mention that, due to the lack of comparison principle, we need strong convergence of the discrete flow to characterize the continuum limit. To this end we impose geometric constraints to the minimizing movement to generate sufficient compactness on the discrete flow: see Definition 5.1 and (5.2).

Section 6 discusses coincidence of the two notions of solutions. Based on Proposition 5.5, we show in Theorem 6.8 that the flat flow is the unique viscosity solution of (1.5) with $\eta(t) = \lambda(\lvert \Omega_t \rvert)$.

Finally in Section 7, we address the large-time behavior for (1.3). The gradient flow structure of (1.1) yields that $\Omega_t$ converges to a ball in Hausdorff distance (Theorem 5.4). Furthermore, we show that the convergence is in almost-$C^{1,1}$ sense (Lemma 7.5). Such regularity result invokes the center manifold approach taken in Ref. [15] to yield the exponential convergence of $\Omega_t$ to a unique ball of radius $r_\infty$.

2. Viscosity solutions

Equation (1.1) can be formulated in terms of level sets, which allows us to introduce the notion of viscosity solutions for the flow. More precisely, for $Q := \mathbb{R}^n \times (0, \infty)$ and $u : Q \to \mathbb{R}$, let us define $\Omega_t = \Omega_t(u) := \{x \in \mathbb{R}^n | u(x, t) > 0\}$ for $t \geq 0$, and consider the following corresponding PDE of mean curvature flows with forcing:

$$\frac{u_t}{|Du|}(x, t) = \nabla \cdot \left( \frac{Du}{|Du|} \right)(x, t) + \lambda(\lvert \Omega_t(u) \rvert) \text{ for } (x, t) \in Q.$$

(MF)

In this section, we introduce a weak notion of solutions for (MF). To this end we first introduce $\eta(\cdot) : [0, + \infty) \to \mathbb{R}$ as an a priori known continuous function of time $t$, and consider

$$\frac{u_t}{|Du|}(x, t) = \nabla \cdot \left( \frac{Du}{|Du|} \right)(x, t) + \eta(t),$$

(2.1)

with initial data
\[ u(x, 0) = u_0(x) := \chi_{\Omega_0} - \chi_{\Omega_0^c} \text{ for } x \in \mathbb{R}^n. \]  

We begin by a list of definitions.

- For a open set \( U \subset \mathbb{R}^n \), we define the parabolic cylinder \( U_T := U \times (0, T] \) and the parabolic boundary of \( U_T, \partial_p U_T := \bar{U}_T - U_T \). We define \( \Omega_T := \mathbb{R}^n \times (0, T] \).

- We denote \( S^{n \times n} \) as the space of \( n \times n \) real symmetric matrices.

- For a function \( h : Q \to \mathbb{R} \) we denote its positive set by \( \Omega_t(h) := \{ x \in \mathbb{R}^n : h(x, t) > 0 \} \) for \( t \geq 0 \).

- For \( f : U \subset \mathbb{R}^d \to \mathbb{R} \) and \( d \in \mathbb{N} \), we denote the lower and upper semi-continuous envelope of \( f \), \( f^- \) and \( f^+ : \overline{U} \to \mathbb{R} \) by

\[
f_-(x) := \liminf_{\varepsilon \to 0} f(y) \quad \text{and} \quad f^+(x) := \limsup_{\varepsilon \to 0} f(y),
\]

where \( \overline{U} \) denotes the closure of \( U \) in \( \mathbb{R}^d \).

- For two evolving sets \( \Omega_t \) and \( \Omega_t \), we say that its boundary \( \partial \Omega_t \) touches \( \partial \Omega_t \) from inside (or outside, respectively) at \( (x_0, t_0) \in Q \) if there exists a neighborhood \( N \) of \( (x_0, t_0) \) such that \( \Omega_t \cap N \subset \Omega_t \cap N \) (or \( \Omega_t \cap N \subset \Omega_t \cap N \), respectively) in \( [0, t_0], (\partial \Omega_t \cap \partial \Omega_t) \cap N = \emptyset \) in \( [0, t_0] \) and \( x_0 \in \partial \Omega_t \cap \partial \Omega_t \).

- For a set \( U \) in \( \mathbb{R}^d \) and \( d \in \mathbb{N} \), we denote the signed distance function by

\[
sd(x, U) := d(x, U) - d(x, U^c). \tag{2.4}
\]

We use the convention that \( \text{sd}(x, U) := \infty \) if \( U \) is empty and \( \text{sd}(x, U) := -\infty \) if \( U^c \) is empty.

For later purposes we also consider the restricted flow

\[
\frac{u_t}{|Du|}(x, t) = \max \left\{ \nabla \cdot \left( \frac{Du}{|Du|} \right)(x, t) + \eta(t), -M \right\} \text{ for } (x, t) \in Q. \tag{2.5}
\]

Here, \( M \) is a given positive constant, which will be fixed large enough later on.

Now we recall the definition viscosity solutions for Eqs. (2.1) and (5.5). Let us denote \( A := (\mathbb{R}^n \setminus \{0\}) \times S^{n \times n} \times [0, \infty) \) and define \( F : A \to \mathbb{R} \) by

\[
F(p, X, t) := \text{trace} \left( \left( I - \frac{p}{|p|} \otimes \frac{p}{|p|} \right) X \right) + \eta(t)|p|.
\]

Then, the Eq. (2.1) can be rewritten in the form of

\[
u_t = F(Du, D^2 u, t).
\]

Since the set \( A \) is dense in \( \mathbb{R}^n \times S^{n \times n} \times [0, \infty) \), the envelopes \( F^- \) and \( F^+ \) are well-defined in \( \mathbb{R}^n \times S^{n \times n} \times [0, \infty) \) with value in \( \mathbb{R} \cup \{\pm \infty\} \).

Recall a test function from Ref. [25, Definition 3.2]. We say that a function \( \phi : Q \to \mathbb{R} \) is a test function on \( Q \) if \( \phi \) is \( C^2 \) with respect to \( x \) and \( C^1 \) with respect to \( t \).

**Definition 2.1.** [10, Definition 2.1], [26, Definition 6.1]

(a) A function \( u : \bar{Q} \to \mathbb{R} \) is a viscosity subsolution of (2.1) if \( u^* < +\infty \) and for any test function \( \phi \) on \( Q \) that touches \( u^* \) from above at \((x_0, t_0)\) we have
\[ \phi_t(x_0, t_0) \leq F^*(D\phi(x_0, t_0), D^2\phi(x_0, t_0), t_0). \]

(b) A function \( u : Q \rightarrow \mathbb{R} \) is a viscosity supersolution of (2.1) if \( u_* > -\infty \) and for any test function \( \phi \) on \( Q \) that touches \( u_* \) from below at \((x_0, t_0)\) we have
\[ \phi_t(x_0, t_0) \geq F_*(D\phi(x_0, t_0), D^2\phi(x_0, t_0), t_0). \]

(c) A function \( u : Q \rightarrow \mathbb{R} \) is a viscosity solution of (2.1) with initial data \( u_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) if \( u^* \) is a viscosity subsolution and \( u_* \) is a viscosity supersolution, and if \( u^* = (u_0)^* \) and \( u_* = (u_0)_* \) at \( t = 0 \).

Parallel definitions can be made for viscosity sub- and supersolutions of (2.5).

**Theorem 2.2.**

(1) [27, Theorem 2.1] Let \( T > 0 \) and \( U \) be a domain in \( \mathbb{R}^n \), not necessarily bounded. Let \( u \) and \( v \) be a bounded subsolution and supersolution, respectively, of (2.1) (or (2.5)). If \( u_* \leq v_* \) on \( \partial_p U_T \), then we have \( u_* \leq v_* \) on \( U_T \).

(2) [28, Theorem 1.1] For a given bounded domain \( \Omega_0 \subset \mathbb{R}^n \) and uniformly continuous initial data \( u_0 : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( \Omega_0 = \{ x \in \mathbb{R}^n : u_0(x) = 0 \} \), there exists a unique viscosity solution \( u \) of (2.1) (or (2.5)), which is uniformly continuous in \( Q \).

(3) [28, Theorems 1.1] Let \( u \) and \( v \) be a uniformly continuous subsolution and supersolution, respectively, of (2.1) (or (2.5)) in \( Q \). If \( u(\cdot, 0) \leq v(\cdot, 0) \) in \( \mathbb{R}^n \), then we have \( u \leq v \) in \( Q \).

The following lemma is a consequence of the stability properties of viscosity solutions: see for instance Lemma 6.1 in Ref. [29].

**Lemma 2.3.**

(a) For \( n \in \mathbb{N}, \) let \( u_n := \chi_{\Omega_n^0} - \chi_{(\Omega_n^0)^c} \) be a viscosity solution of (2.1) (or (2.5)) in \( Q \). If \( \partial\Omega_n^0 \) converges to \( \partial\Omega \) as \( n \to \infty \) in Hausdorff distance, uniformly for all \( t > 0 \), then \( u := \chi_{\Omega} - \chi_{\Omega^c} \) is a viscosity solution of (2.1) (or (2.5)).

(b) For \( n \in \mathbb{N}, \) let \( u_n := \chi_{\Omega_n^0} - \chi_{(\Omega_n^0)^c} \) be a viscosity solution of (2.5) in \( Q \) with \( M = M_n \) where \( M_n \to \infty \) as \( n \to \infty \) and with initial data \( u_0 \). If \( \partial\Omega_n^0 \) uniformly converges to \( \partial\Omega_t \) in Hausdorff distance, uniformly for all \( t > 0 \), then \( u \) is a viscosity solution of (2.1).

Note that (2.5) as well as (2.1) are geometric, that is \( F \) satisfies the scaling invariance
\[ F(ap, aX + bp \otimes p, t) = aF(p, X, t), \quad (2.6) \]
for \( a > 0, b \in \mathbb{R}, p \in \mathbb{R}^n, X \in S^{n \times n} \) and \( t > 0 \). So, (2.1) and (2.5) have the following invariance of geometric equations.

**Theorem 2.4.** [30, Theorem 4.2.1] Let \( u \) and \( v \) be a subsolution and supersolution, respectively, of (2.1) (or (2.5)). If \( \phi : \mathbb{R} \to \mathbb{R} \) is upper semicontinuous and nondecreasing, then the composite function \( \phi \circ u \) is also a subsolution. Similarly, if \( \phi : \mathbb{R} \to \mathbb{R} \) is lower semicontinuous and nondecreasing, then \( \phi \circ v \) is also a supersolution.

Let \( v \) be a continuous viscosity solution of (2.1) with uniformly continuous initial data \( u_0 : \mathbb{R}^n \to \mathbb{R} \) such that \( \Omega_0 = \{ x \in \mathbb{R}^n : u_0(x) = 0 \} \). Based on the invariance in
Theorem 2.4 and the stability of viscosity solutions in Ref. [29, Lemma 6.1], we obtain a discontinuous viscosity solution $u$ of (2.1) and (2.2) given by

$$u(x,t) = \chi_{\Omega_t(u)} - \chi_{(\Omega_t(u))^c} \quad \text{and} \quad \Omega_t(u) = \Omega_t(v) \quad \text{for all} \ t \geq 0$$  \hspace{1cm} (2.7)

(See Ref. [28, Theorem 2.1]). Note that $\Omega_t(u)$ satisfies (1.5) if $\partial \Omega_t(u)$ is $C^2$. We will thus consider the set $\Omega_t$ obtained from the above viscosity solutions formulation as a weak notion of sets evolving by (1.5).

**Remark 2.5.** Note that in Theorem 2.2(1), we need $u^* \leq v_*$ at the initial time, so this theorem does not yield the uniqueness for discontinuous solutions. Indeed solutions of the form (2.7) may be non-unique due to the “fattening” of the zero level set, see the discussion in Refs. [28–32] for the flow (1.4). We will show in Section 4 that our solutions are unique under the geometric constraint on the initial data.

**Definition 2.6.** A function $u : \bar{Q} \rightarrow \mathbb{R}$ is a viscosity subsolution (supersolution) of (MF) and (2.2) if $u$ is a viscosity subsolution (supersolution) of (2.1) and (2.2) with continuous and bounded $\eta(t) = \lambda \|[\Omega_t(u)]\|$. A function $u$ is a viscosity solution of (MF) and (2.2) if $u$ is a viscosity solution of (2.1) and (2.2).

**Remark 2.7.** For (MF) and (2.2), the comparison principle fails, and thus viscosity solutions theory cannot be directly applied. Indeed the well-posedness of (MF) and (2.2) will be established later in Section 5.

Next, we introduce a regularization that is often used in free boundary problems (see e.g. [33] and Lemma 3.1 in Ref. [34]). This is useful in our geometric analysis in Sections 4 and 6.

**Lemma 2.8.** Consider a continuous function $l : [0, \infty) \rightarrow \mathbb{R}$ with $L(t) := \int_0^t l(s)ds \leq A$ in $[0,T]$. Let $u$ be a viscosity supersolution of (2.1). Then, the function

$$\bar{u}(x,t) := \inf_{y \in B_{A-l(t)}(x)} u(y,t),$$

is a viscosity supersolution of

$$\bar{u}_t = F(D\bar{u}, D^2\bar{u}, t) + l(t)|D\bar{u}| \quad \text{in} \ Q_T.$$  \hspace{1cm} (2.8)

Similarly, let $u$ be a viscosity subsolution of (2.1). Then, the function

$$\hat{u}(x,t) := \sup_{y \in B_{A-l(t)}(x)} u(y,t)$$

is a viscosity subsolution of

$$\hat{u}_t = F(D\hat{u}, D^2\hat{u}, t) - l(t)|D\hat{u}| \quad \text{in} \ Q_T.$$  \hspace{1cm} (2.9)

**Proof.** Let us show that the function $\tilde{u}$ is a viscosity supersolution of (2.8), the subsolution part can be proved with parallel arguments. For simplicity we will only present the proof for the case $l(t) = c > 0$, in which case $T = A/c$. 

Suppose a test function $\phi$ touches $\tilde{u}_*$ from below at $(x_0, t_0) \in Q_T$. It holds that
\[
\tilde{u}_*(x_0, t_0) - \phi(x_0, t_0) = 0 \quad \text{and} \quad \tilde{u}_*(x, t) - \phi(x, t) \geq 0 \quad \text{in} \quad \mathcal{N}_\delta(x_0, t_0)
\]
\[
B_\delta(x_0) \times (t_0 - \delta, t_0],
\]
for some $\delta > 0$. From the construction of $\tilde{u}_*$, there exists $x_1 \in \mathbb{R}^n$ such that
\[
|x_1 - x_0| \leq A - ct_0 \quad \text{and} \quad \tilde{u}_*(x_0, t_0) = u_*(x_1, t_0).
\]
(2.10)

If $D\phi(x_0, t_0) = 0$, then it suffices to show that
\[
\phi_t(x_0, t_0) \geq F_*(D\phi(x_0, t_0), D^2\phi(x_0, t_0), t_0).
\]
(2.12)

We choose the shifted test function $\psi(x, t) := \phi(x - x_1 + x_0, t)$ and claim that $\psi$ touches $u_*$ from below at $(x_1, t_0)$. As $c > 0$, (2.11) yields that
\[
|x_1 - x_0| \leq A - ct \quad \text{for all} \quad t \in (t_0 - \delta, t_0].
\]
(2.13)

From (2.10), we have $u_*(x_1, t_0) - \psi(x_1, t_0) = 0$. From we have $u_*(x_1, t_0) - \psi(x_1, t_0) = 0$ again and (2.13), it holds that
\[
u_*(x, t) - \psi(x, t) \geq \tilde{u}_*(x - x_1 + x_0, t) - \phi(x - x_1 + x_0, t) \quad \text{for any} \quad (x, t) \in \mathcal{N}_\delta(x_1, t_0),
\]
(2.14)

which yields the claim. Since $u_*$ is a viscosity supersolution of (2.1) we have the corresponding PDE inequality for $\psi$ at $(x_1, t_0)$, which translates to (2.12).

Next, we suppose that $D\phi(x_0, t_0) \neq 0$. If $|x_1 - x_0| < A - ct_0$, then $u(\cdot, t_0)$ is constant in a small neighborhood of $x_0$ in $\mathbb{R}^n$ and it holds that $D\phi(x_0, t_0) = 0$. Thus, we have $|x_1 - x_0| = A - ct_0$. We claim that the shifted test function $\psi(x, t) := \phi(x - (A - ct)\bar{n}, t)$ touches $u_*$ from below at $(x_1, t_0)$ where
\[
\bar{n} := \frac{x_1 - x_0}{|x_1 - x_0|}.
\]

First, note that $x_1 - (A - ct_0)\bar{n} = x_0$ and thus $u_*(x_1, t_0) - \psi(x_1, t_0) = 0$. Furthermore, if we choose $\varepsilon = \frac{1}{2} \min\{\delta, t_0\}$, then
\[
(x - (A - ct)\bar{n}, t) = (x - x_1 + x_0 + c(t - t_0)\bar{n}, t) \in \mathcal{N}_\delta(x_0, t_0) \quad \text{for all} \quad (x, t) \in \mathcal{N}_\varepsilon(x_1, t_0).
\]
(2.15)

(2.10) and (2.15) imply that
\[
u_*(x, t) - \psi(x, t) \geq \tilde{u}_*(x - (A - ct)\bar{n}, t) - \phi(x - (A - ct)\bar{n}, t) \geq 0 \quad \text{for all} \quad (x, t) \in \mathcal{N}_\varepsilon(x_1, t_0),
\]
which yields the claim.

As described in the first case, since $u_*$ is a viscosity supersolution of (2.1) we have the corresponding PDE inequality for $\psi$ at $(x_1, t_0)$, which translates to
\[
\phi_t(x_0, t_0) + cD\phi(x_0, t_0) \cdot \bar{n} \geq F_*(D\phi(x_0, t_0), D^2\phi(x_0, t_0), t_0).
\]
(2.16)

Since $D\phi(x_0, t_0) \neq 0$ and the level set $\{x \in \mathbb{R}^n : \phi(x, t_0) = \phi(x_0, t_0)\}$ touches $\partial\Omega_{t_0}(\bar{u})$ from inside at $x_0$, $-D\phi(x_0, t_0)$ is parallel to the outward normal $\bar{n}$ of $\partial\Omega_{t_0}(\bar{u})$ at $x_0$. Therefore, (2.16) yields
\[ \phi_t(x_0, t_0) \geq F_\ast(D\phi(x_0, t_0), D^2\phi(x_0, t_0), t_0) + c[D\phi(x_0, t_0)]. \]

Now we can conclude that the function \( \tilde{u} \) is viscosity supersolution of (2.8).

In general, we choose the shifted test function \( \psi(x, t) := \phi(x - x_1 + x_0, t) \) or \( \phi(x - L(t)\bar{n}, t) \) and apply the parallel arguments to conclude. \( \square \)

The following lemma will be used in Section 3 to ensure uniform continuity of \( \Omega_t(u) \) over time in Hausdorff distance.

**Lemma 2.9.** Let \( u \) be a bounded viscosity solution of (2.1) or (2.5) given by the form (2.7). Then the following holds for \( 0 < \delta < \frac{1}{\|\eta\|_\infty} \): If \( B_{2\delta}(x_0) \subset (\Omega_{t_0}(u))^C \) (or \( \Omega_{t_0}(u) \)), then \( B_\delta(x_0) \subset (\Omega_t(u))^C \) (or \( \Omega_t(u) \)) for \( t_0 \leq t \leq t_0 + \frac{\delta^2}{n} \).

**Proof.** We will verify the case when \( B_{2\delta}(x_0) \) lies outside of \( \Omega_{t_0}(u) \), since the rest follows from a parallel barrier argument. Let us compare \( u \) with a radial barrier \( \phi \) defined by

\[ \phi := -\chi_{B_{\delta/\delta}(x_0)} + \chi_{B_{\delta}(x_0)} \epsilon, \]

where \( r : [t_0, t_0 + \frac{\delta^2}{n}] \to \mathbb{R} \) solves \( r(t_0) := 2\delta, r'(t) := -\frac{n-1}{\delta} - \|\eta\|_\infty \). By assumption \( u^t(x, t_0) \leq \phi_+(x, t_0) \).

Let us show that \( \phi \) is a viscosity supersolution for \( t_0 \leq t \leq t_0 + \frac{\delta^2}{n} \). Since \( \phi \) is a radial function, the normal velocity on \( \partial\Omega_t(\phi) \) is equal to \( -r'(t) \), and the mean curvature on \( \partial\Omega_t(\phi) \) is \( -\frac{n-1}{r'(t)} \). Moreover, we have

\[ r'(t) = -\frac{n-1}{\delta} - \|\eta\|_\infty \geq -\frac{n}{\delta}. \]

Since \( r(t_0) = 2\delta \), it follows that \( r(t) \geq \delta \) if \( t_0 \leq t \leq t_0 + \frac{\delta^2}{n} \). Therefore, it holds that for \( t_0 \leq t \leq t_0 + \frac{\delta^2}{n} \)

\[ -r'(t) = \frac{n-1}{\delta} + \|\eta\|_\infty \geq \frac{n-1}{r(t)} + \eta(t) \]

and we conclude. Now by Theorem 2.2(1), \( u^t \leq \phi_+ \) for \( t_0 \leq t \leq t_0 + \frac{\delta^2}{n} \) and thus \( B_\delta(x_0 + \delta \nu) \) lies outside of \( \Omega_t(u) \) for \( t_0 \leq t \leq t_0 + \frac{\delta^2}{n} \). \( \square \)

3. Geometry of the flow

In this section, we study geometric properties of evolution of (2.1), following a strong notion of star-shapedness, \( \rho \)-reflection. This property, introduced in Ref. [5], is useful for problems which satisfy the reflection comparison principle (See Theorem 3.5 below).

3.1. Geometric properties

**Definition 3.1.** A bounded set \( \Omega \) in \( \mathbb{R}^n \) is **star-shaped with respect to a ball** \( B_r(0) \) if for any point \( y \in B_r(0) \), \( \Omega \) is star-shaped with respect to \( y \). Let
Sr := \{ \Omega : \text{star-shaped with respect to } B_r(0) \} \text{ and } S_{r,R} := S_r \cap \{ \Omega : \Omega \subset B_R(0) \}.

The following lemma is immediate from the interior and exterior cone properties of sets in Sr.

**Lemma 3.2.** For a continuously differentiable and bounded function \( \phi : \mathbb{R}^n \to \mathbb{R} \), let us denote the positive set of \( \phi \) by \( \Omega(\phi) \). Let us assume that \( \Omega(\phi) \) contains \( B_r(0) \) and \( D\phi \neq 0 \) on \( \partial \Omega(\phi) \). Then the set \( \Omega(\phi) \) is in \( S_r \) if and only if

\[
x \cdot \vec{n}_x = x \cdot \left( -\frac{D\phi}{|D\phi|}(x) \right) \geq r \text{ for all } x \in \partial \Omega(\phi),
\]

where \( \vec{n}_x \) denotes the outward normal of \( \partial \Omega(\phi) \) at \( x \).

Next we proceed to define the reflection property. For a hyperplane \( \Pi = \Pi_\nu(s) := \{ x : x \cdot \nu = s \} \), let \( \Psi_\Pi \) denote the corresponding reflection, i.e.,

\[
\Psi_{\Pi(s)}(x) := x - 2(x - s
u, \nu)\nu. \tag{3.1}
\]

For \( \Pi \) that doesn’t contain the origin, we denote the half-spaces divided by \( \Pi \) by \( \Pi^+ \) and \( \Pi^- \), where \( \Pi^- \) contains the origin.

**Definition 3.3.** [5, Definition 10] bounded, open set \( \Omega \) has \( \rho \)-reflection if

1. \( \Omega \) contains \( \overline{B_\rho(0)} \) and
2. \( \Omega \) satisfies, for all \( \nu \in S^{n-1} \) and all \( s > \rho \),

\[
\Psi_{\Pi_s(\nu)}(\Omega \cap \Pi_\nu^+(s)) \subset \Omega \cap \Pi_\nu^-(s).
\]

The \( \rho \)-reflection property can be viewed as a smallness condition on the Lipschitz norm distance between \( \partial \Omega \) and the nearest ball (see the Appendix in Ref. [5]). The following lemma states several properties and the relationship between the two concepts introduced above, \( \rho \)-reflection and \( S_r \) (See Figures 2 and 3 and [5, Figure 2]).

**Lemma 3.4.** [5, Lemma 3, 9, 10, 24]

1. For a bounded domain \( \Omega \) containing \( B_r(0) \), the followings are equivalent:

   (i) \( \Omega \in S_r \).
There exists \( \varepsilon_0 = \varepsilon_0(r) > 0 \) such that
\[
\Omega \subset \bigcap_{|z| \leq \delta \varepsilon} [(1 + \varepsilon)\Omega + z] \text{ for all } 0 < \varepsilon < \varepsilon_0 \text{ and } 0 < \delta < r, \tag{3.2}
\]

(iii) For all \( x \in \Omega \), there is an interior cone to \( \Omega \):
\[
IC(x, r) := \left( (x + C(-x, \theta_{x,r})) \cap C\left(x, \frac{\pi}{2} - \theta_{x,r}\right) \right) \cup B_r(0) \subset \Omega \text{ for } |x| \geq r, \tag{3.3}
\]
where \( C(x, \theta) \) is a cone in the direction \( x \) with opening angle \( \theta \) for \( x \in \mathbb{R}^n \) and \( \theta \in [0, \pi] \),
\[
C(x, \theta) := \{ y \mid (x, y) > \cos \theta |x||y| \} \text{ and } \theta_{x,r} := \arcsin \frac{r}{|x|} \in \left[0, \frac{\pi}{2}\right]. \tag{3.4}
\]

(iv) For all \( x \in \Omega^C \), there is an exterior cone to \( \Omega \):
\[
EC(x, r) := x + C(x, \theta_{x,r}) \subset \Omega^C \text{ where } \theta_{x,r} = \arcsin \frac{r}{|x|} \in \left[0, \frac{\pi}{2}\right]. \tag{3.5}
\]

2. Suppose that \( \Omega \) has \( \rho \)-reflection. Then \( \Omega \in S_r \) with
\[
r = \left( \inf_{x \in \partial \Omega} |x|^2 - \rho^2 \right)^{1/2}. \tag{3.6}
\]
Moreover
\[
\sup_{x \in \partial \Omega} |x| - \inf_{x \in \partial \Omega} |x| \leq 4 \rho. \tag{3.7}
\]

3. Suppose that \( \Omega \) is in \( S_{r,R} \). If there exists \( \rho > 0 \) such that \( \overline{B_\rho(0)} \subset \Omega \) and \( \rho^2 \geq 5(R^2 - r^2) \), then \( \Omega \) has \( \rho \)-reflection.

Theorem 3.5. (Reflection Comparison) Suppose that \( \Omega_0 \) has \( \rho \)-reflection. Let \( u \) be a bounded viscosity solution of (2.1) given by the form (2.7). Let \( \Pi \) be a hyperplane in \( \mathbb{R}^n \) such that \( \Pi \cap B_\rho(0) = \emptyset \). Then the reflected function \( u(\Psi_\Pi(x), t) \) is also a bounded
viscosity solution in $\Pi^{-} \times (0, \infty)$. Moreover
\[
\Psi_{\Pi}(\Omega_{t} \cap \Pi^{+}) \subset \Omega_{t} \cap \Pi^{-} \text{ for all } t > 0 \text{ if true at } t = 0.
\] (3.8)

Proof. It is easy to see that $u(\Psi_{\Pi}(x), t)$ is also a viscosity solution of (2.1) since $F$ is independent of $x$.

To show (3.8), we will use the comparison principle in $\Pi^{-} \times [0, \infty)$. To do so it is easier for us to consider a continuous version of $u$, i.e. let $\tilde{u}$ be the unique viscosity solution of (2.1) with uniformly continuous initial data $\tilde{u}(x, 0) := -\min\{sd(x, \Omega_{0}), 2R\}$, where $R$ is chosen large enough that $\Omega_{0} \subset \subset B_{R}$. As $u$ is given by the form (2.7), Theorem 2.4 combined with the uniqueness implies that $\Omega_{t}(\tilde{u})$ is equal to $\Omega_{t}(u)$ for all $t > 0$.

Note that Theorem 2.2(2) implies that $\tilde{u}$ is uniformly continuous. As $\tilde{u}(\cdot, 0)$ is bounded in $\mathbb{R}^{n}$, we apply Theorem 2.2(3) to conclude that $\tilde{u}$ is bounded in $\bar{Q}$. Since $\tilde{u}(\Psi_{\Pi}(x), 0) \leq \tilde{u}(x, 0)$ in $\Pi^{-}$ and $\tilde{u}(\Psi_{\Pi}(x), 0) = \tilde{u}(x, 0)$ on $\Pi$, Theorem 2.2(1) applies to $\tilde{u}(x, t)$ and $\tilde{u}(\Psi_{\Pi}(x), t)$ to yield
\[
\tilde{u}(\Psi_{\Pi}(x), t) \leq \tilde{u}(x, t),
\]
for all $x \in \Pi^{-}$ and $t > 0$. Therefore (3.8) follows. \qed

Theorem 3.6. Suppose that $\Omega_{0}$ has $\rho$-reflection. Let $u$ be a bounded viscosity solution of (2.1) given by the form (2.7). Let $I = [0, T)$ be the maximal interval satisfying $\bar{B}_{\rho} \subset \Omega_{t}(u)$. Then, $\Omega_{t}(u)$ has $\rho$-reflection for $t \in I$.

Proof. From the definition of $\rho$-reflection, it is enough show that, for any unit vector $v$ in $\mathbb{R}^{n}$,
\[
\Psi_{\Pi_{v}(\rho)}(\Omega_{t}(u) \cap \Pi^{+}_{v}(\rho)) \subset \Omega_{t}(u) \cap \Pi^{+}_{v}(\rho) \text{ for } t \in I.
\] (3.9)
Since $\Omega_{0}(u)$ has $\rho$-reflection, (3.9) holds at $t = 0$, and we can conclude by Theorem 3.5. \qed

In the next section, we will show that $\Omega_{t}(u) \in S_{r, R}$ in $[0, T]$ if it starts with some geometric restriction for the initial data. This leads to the following regularity of $\Omega_{t}(u)$ over time.

Corollary 3.7. Let $u$ be a bounded viscosity solution of (2.1) given by the form (2.7). If $\Omega_{t}(u) \in S_{r, R}$ and $|\eta(t)| \leq K$ in $[0, T]$, then there exists $C = C(r, R, K, T)$ such that we have
\[
d_{H}(\partial \Omega_{t}(u), \partial \Omega_{s}(u)) \leq C(s - t)^{\frac{n}{2}} \text{ for all } 0 \leq t \leq s \leq T.
\] (3.10)

Proof. Choose $\delta \in \left(0, \min\left\{\frac{1}{r}, \frac{1}{2}\right\}\right)$ and $t \in [0, T]$. We claim that
\[
\sup_{x \in \partial \Omega_{t}(u)} d(x, \partial \Omega_{s}(u)) \leq \frac{2R \delta}{r} \text{ for all } s \in I := \left[t, \min\left\{t + \frac{\delta^{2}}{n}, T\right\}\right].
\] (3.11)
As $\Omega_t(u) \in S_{r,R}$, there exists $x_1 = x_1(s) \in \partial \Omega_t(u)$ such that
\[
\sup_{x \in \partial \Omega_t(u)} d(x, \partial \Omega_t(u)) = d(x_1, \partial \Omega_t(u)) \quad \text{for } s \in I.
\] (3.12)

Let $y = \left(1 - \frac{2\delta}{r}\right)x$ and $z = \left(1 + \frac{2\delta}{r}\right)x$. From the interior and exterior cone properties in Lemma 3.4, it holds that
\[
B_{2\delta}(y) \subset \Omega_t(u) \quad \text{and} \quad B_{2\delta}(z) \subset \Omega_t(u)^C.
\]
As the assumption in Lemma 2.9 is satisfied, we conclude that $y \in \Omega_s(u)$ and $z \in \Omega_s(u)^C$ for all $s \in I$. As $\Omega_s(u) \in S_{r,R}$ in $I$, there exists $x_2 \in \partial \Omega_s(u)$ such that
\[
|x_1 - x_2| \leq \max\{|x_1 - y|, |x_1 - z|\} \leq \frac{2R\delta}{r}.
\] (3.13)

(3.12) and (3.13) imply (3.11). As $\Omega_s(u), \Omega_t(u) \in S_{r,R}$, we apply Lemma C.6 to conclude (3.10).

### 3.2. Preservation of the $\rho$-reflection property

In this subsection, we suppose that there exists a viscosity solution $u$ of our original equation (MF) in the sense of Definition 2.6, and show the preservation of the $\rho$-reflection property. As a consequence, star-shapedness of $\Omega_t(u)$ is preserved for all time. Existence of this solution will be given later in Sections 5 and 6.

**Theorem 3.8.** Suppose that $\Omega_0$ has $\rho$-reflection. Assume that there exists a bounded viscosity solution $u$ given by the form (2.7) of (MF) and (2.2). Then $\Omega_t(u)$ has $\rho$-reflection for all $t > 0$. In particular there exists $r_1 > 0$ such that $\Omega_t$ is star-shaped with respect to a ball $B_{r_1}(0)$ for all $t > 0$.

The proof of above theorem consists of Theorem 3.6 and Lemma 3.9. In Lemma 3.9, we show that the maximal interval $I$ in Theorem 3.6 is $[0, \infty)$.

**Lemma 3.9.** Let $u$ and $\Omega_0$ be as given in above theorem. Then, there exists $a > 0$ depending on $\Omega_0$ such that $B_{(1+a)\rho} \subset \Omega_t(u)$ for all $t > 0$.

**Proof.** Since $\Omega_0$ has $\rho$-reflection, $B_{(1+a)\rho} \subset \Omega_0$ for some $a > 0$. Due to Assumption A and the continuity of $\lambda$, one can choose a small $a > 0$ such that
\[
\lambda(|\Omega|) > \frac{n-1}{\rho} \quad \text{for sets contained in } B_{(5+a)\rho}.
\] (3.14)

Suppose that $B_{(1+a)\rho}$ is not contained in $\Omega_{t_0}(u)$ at some $t_0 > 0$. Then, there exists $t_0 \in (0, t_0)$ such that $\partial \Omega_t(u)$ touches from outside $\partial B_{(1+a)\rho}$ at $(x_0, t_0)$ for the first time. Then, by (3.7) in Lemma 3.4, we have
\[
\sup_{x \in \partial \Omega_{t_0}(u)} |x| \leq 4\rho + \inf_{x \in \partial \Omega_{t_0}(u)} |x| = (5 + a)\rho,
\]
and thus $\Omega_{t_0}(u)$ is contained in $B_{(5+a)\rho}$. Hence it follows from (3.14) that

$$\lambda\left[|\Omega_{t_0}(u)|\right] > \frac{n-1}{\rho} > H[B_{(1+a)\rho}],$$  

(3.15)

where $H[B_{(1+a)\rho}]$ is the mean curvature of $\partial B_{(1+a)\rho}$.

Consider $\phi(x) := -\left(\frac{|x|}{(1+a)\rho}\right)^2$. Note that (3.15) and $|x_0| = (1+a)\rho$ yield

$$\nabla \cdot \left( \frac{D\phi}{|D\phi|} \right)(x_0) + \lambda\left[|\Omega_{t_0}(u)|\right] = -H[B_{(1+a)\rho}] + \lambda\left[|\Omega_{t_0}(u)|\right] > 0. \quad (3.16)$$

Hence $\psi(x,t) := \phi(x)$ is a strict subsolution of (2.1) with $\eta(t) = \lambda[|\Omega_t(u)|]$ in a small neighborhood of $(x_0, t_0)$.

On the other hand, we have $\psi \leq 0$ in $Q$ and $\psi \leq -1$ outside of $B_{(1+a)\rho}$. Recall that $u$ is given by the form (2.7). As $\Omega_{t_0}(u)$ touches $B_{(1+a)\rho}$ at $(x_0, t_0)$ for the first time, $\psi$ touches $u^-$ from below at $(x_0, t_0)$ and we have

$$\nabla \cdot \left( \frac{D\psi}{|D\psi|} \right)(x_0, t_0) + \lambda\left[|\Omega_{t_0}(u)|\right] \leq \psi_t(x_0, t_0) = 0 \quad (3.17)$$

and this contradicts to (3.16). \hfill \Box

**Proof of Theorem 3.8.** First note that $\Omega_t(u)$ has $\rho$-reflection thanks to Lemma 3.9 and Theorem 3.6 applied to $u(x, t)$ and $\eta(t) = \lambda[|\Omega_t(u)|]$. Moreover from (3.6) in Lemma 3.4, $\Omega_t(u) \in S_r$ for

$$r = \left( \inf_{x \in \partial\Omega} |x|^2 - \rho^2 \right)^{1/2} \geq r_1 := \rho(a^2 + 2a)^{1/2}. \quad (3.18)$$

Hence $\Omega_t(u)$ is star-shaped with respect to a ball $B_{r_1}$ for all $t > 0$. \hfill \Box

A particular consequence of Theorem 3.8 is that $\partial \Omega_t(u)$ is a locally Lipschitz graph. This, in combination with Lemma B.1, yields that the evolution is indeed $C^{1,1}$:

**Corollary 3.10.** Let $u$ and $\Omega_0$ be as in Theorem 3.8. Then $\Omega_t(u)$ has $C^{1,1}$-boundary for all $t > 0$. In particular its principal curvatures are bounded by $O(1 + 1/\sqrt{t})$.

Next we note that, with the sublinear growth condition imposed on $\lambda$, $\Omega_t(u)$ is uniformly bounded in finite time.

**Lemma 3.11.** Let $u$ and $\Omega_0$ be as given in Theorem 3.8. Then, there exists $R_1 = R_1(T) > 0$ such that $\Omega_t(u) \subset B_{R_1}$ in $[0, T]$.

**Proof.** By Assumption A, there exists a constant $C_1 > 0$ such that $\lambda[|B_R|] \leq C_1 R$ for all $R \geq \rho$. Since $\Omega_0$ is bounded, there exists $\tilde{R} > \rho$ such that $\Omega_0 \subset B_{\tilde{R}}$. Let us compare $u$ with a radial barrier $\phi : \bar{Q} \to \mathbb{R}$ defined by

$$\phi(x,t) := \chi_{B_{\tilde{R}}}(x) - \chi_{B_{\tilde{R}}^c}(x) \quad \text{for} \quad (x,t) \in \bar{Q},$$
where \( r : [0, T] \rightarrow \mathbb{R} \) is defined by \( r(t) := \hat{R} e^{(C_1 + 1)r(t)} \). Note that \( \Omega_0(u) \subset \Omega_0(\phi) \), and \( r'(t) = (C_1 + 1)r(t) \).

Choose \( \varepsilon \in (0, \hat{R}C_1^{-1}) \) and let us show that \( \Omega_r(u) \subset B_{r(t)+\varepsilon} \) for all time. Suppose it is false, then we have

\[
t_0 := \sup \{ t : \Omega_r(u) \subset B_{r(s)+\varepsilon} \text{ for } 0 \leq s \leq t \} < +\infty. \tag{3.19}
\]

By Corollary 3.7, \( \partial \Omega_r(u) \) evolves continuously in time and thus

\[
\partial \Omega_r(u) \cap \partial B_{r(s)+\varepsilon} \neq \emptyset. \tag{3.20}
\]

Combining (3.19) with Lemma 3.11, we have \( |B_p| \leq |\Omega_r(u)| \leq |B_{r(t)+\varepsilon}| \) in \([0, t_0]\).

Furthermore, as \( r(t) \geq \hat{R} > \rho \), it holds that

\[
\lambda ||\Omega_r(u)|| \leq C_1(r(t) + \varepsilon) \leq r'(t) + \frac{n-1}{r(t)}. \tag{3.21}
\]

Therefore, \( \phi \) is a viscosity supersolution of (2.1) with \( \eta(t) = \lambda ||\Omega_r(u)|| \) in \([0, t_0]\). Note that \( u^* \leq \phi_* \) at \( t = 0 \). From Theorem 2.2(1) we have \( u^* \leq \phi_* \) in \([0, t_0]\) and thus

\[
\Omega_r(u) \subset B_{r(t)} \text{ in } [0, t_0]. \tag{3.22}
\]

By Corollary 3.7 again, \( \partial \Omega_r(u) \) evolves continuously in time and thus we have \( \Omega_r(u) \subset B_{r(t_0)} \) in \([0, t_0]\), which contradicts (3.20).

As a consequence, we conclude that

\[
\Omega_r(u) \subset B_{R_1} \text{ where } R_1 := \hat{R} e^{(C_1+1)T} + \varepsilon \tag{3.23}
\]

in \([0, T]\).

We finish this section with some properties of our solutions that will be used later. The following corollary holds due to the fact that \( \Omega_0 \) has \( \rho \)-reflection and therefore for small \( \varepsilon > 0 \) the sets \( \Omega_0^{\varepsilon+} := (1+\varepsilon)\Omega_0 \) and \( \Omega_0^{\varepsilon-} := (1+\varepsilon)^{-1}\Omega_0 \) satisfy \( \rho(1+O(\varepsilon)) \)-reflection.

**Corollary 3.12.** Let \( u, \Omega_0 \) and \( r_1 \) be as given in Theorem 3.8 and \( R_1 \) as given in Lemma 3.11. Then for sufficiently small \( \varepsilon > 0 \) viscosity solutions \( u^\pm \) of (MF) starting from \( \Omega_0^{\pm} \) have their positive sets \( \Omega_t(u^\pm) \) in \( S_{r_1-O(\varepsilon), r_1+O(\varepsilon)} \) in \([0, T]\).

**Lemma 3.13.** Let \( u, \Omega_0 \) and \( r_1 \) be as given in Theorem 3.8 and \( R_1 \) as given in Lemma 3.11. Then, there exists positive constants \( \tilde{K}_\infty = \tilde{K}_\infty(r_1, R_1, T) \) and \( \tilde{K}_{1/2} = \tilde{K}_{1/2}(r_1, R_1, T) \) such that the following holds for all \( t, s \) in \([0, T]\):

\[
|\lambda ||\Omega_t(u)|| - \lambda ||\Omega_s(u)||| \leq \tilde{K}_{1/2}|t-s|^\frac{1}{2} \tag{3.24}
\]

and

\[
|\lambda ||\Omega_t(u)||| \leq \tilde{K}_\infty.
\]

**Proof.** From Lemma 3.9 and 3.11, \( ||\Omega_t|| \) is bounded away from zero and infinity, and thus \( \lambda \) is bounded. Next, by the Lipschitz continuity of \( \lambda \) and the last inequality of (C.1) in Lemma C.1, there exists \( C_1(r_1, R_1, T) \) such that
From the above inequality and Hölder continuity in Corollary 3.7, we conclude (3.24).

Finally, let us show Lipschitz continuity of $|\Omega_t|$ in time for the later purpose in Lemma 4.12.

**Lemma 3.14.** Let $u$, $\Omega_0$ and $r_i$ be as given in Theorem 3.8, $R_1$ as given in Lemma 3.11, and $\bar{K}_\infty$ as given in Lemma 3.13. Then there exists $C = C(r_1, R_1, \bar{K}_\infty)$ such that we have

$$
|\Omega_t(u)| - |\Omega_s(u)| \leq C \left( 1 + \frac{1}{\sqrt{t}} \right) |s - t| \text{ for } 0 \leq t \leq s \leq T.
$$

(3.25)

**Proof.** First, by Corollary 3.10, all principal curvatures are bounded by $M(t) := C_1 \left( 1 + 1/\sqrt{t} \right)$ for some constant $C_1 = C_1(r_1, R_1, \bar{K}_\infty)$. Thus, there exist interior and exterior balls of radius $M(t)^{-1}$ on each point of $\partial \Omega_t(u)$ for all $t > 0$. As described in Corollary 3.7, we apply Lemma 2.9 in these balls to conclude that

$$
d_H(\partial \Omega_t(u), \partial \Omega_s(u)) \leq C_2 \left( 1 + \frac{1}{\sqrt{t}} \right) |s - t| \text{ for } 0 \leq t \leq s \leq T.
$$

(3.25)

for some $C_2 = C_2(r_1, R_1, \bar{K}_\infty)$. Recall from the first and last inequalities of (C.1) in Lemma C.1 that the volume difference is bounded by the Hausdorff distance. Thus, we conclude that there exists $C = C(r_1, R_1, \bar{K}_\infty)$ satisfying (3.25).

## 4. Uniqueness of the flow

In this section, we show the uniqueness for solutions of (MF) and (2.1) with given initial data (2.2). As pointed out in Remark 2.5, the comparison principle (Theorem 2.2) does not deliver the uniqueness for a discontinuous viscosity solution, due to the possible fattening phenomena of level sets. We show that our flow (1.1) can be uniquely determined when the initial data has $\rho$-reflection. We follow the argument of Ref. [19], where the uniqueness result is shown for convex evolution of volume-preserving flow.

In Section 4.1, we show the short-time uniqueness result for (2.1) in Theorem 4.3 for a star-shaped initial data $\Omega_0$. We define appropriate convolutions to perturb solutions (see Definition 4.4) and show that our perturbation preserves sub- and supersolution properties for (2.1). These perturbations are more delicate than those used in Ref. [30] due to the presence of the time-dependent forcing $\eta$. We use these perturbations to obtain the uniqueness results. At this point, it is crucial to find a uniform interval $[0, t_1]$ where these convolutions are well defined in this interval (see Lemma 4.6). It remains open whether the flow (2.1) stays unique beyond the interval.

In Section 4.2, we show the global-time uniqueness for (MF) when its initial data has $\rho$-reflection (see Theorem 4.9). Here we know that any evolution, if exists, preserves the $\rho$-reflection property, which we use to iterate the short-time uniqueness result from the previous subsection. The key step is to estimate the difference between $\hat{\lambda}[|\Omega_t(u)|]$ and $\hat{\lambda}[|\Omega_t(v)|]$ for two possible solutions (see Lemma 4.12).
4.1. Short-time uniqueness of (2.1)

Definition 4.1. [28, Definition 2.1] For a function \( u : Q \to \mathbb{R} \) and \( t \geq 0 \), we say that \( \Omega_t(u) = \{ u(\cdot, t) > 0 \} \) is regular if the closure of \( \Omega_t(u) \) is \( \{ x \in \mathbb{R}^n : u(x, t) \geq 0 \} \), and the interior of \( \{ x \in \mathbb{R}^n : u(x, t) > 0 \} \) is \( \Omega_t(u) \).

Note that for \( t \geq 0 \), if \( \Omega_t(u) \) is regular, then the interface \( \{ x \in \mathbb{R}^n : u(x, t) = 0 \} \) has an empty interior.

Lemma 4.2. [28, Theorem 2.1] Let \( u : Q \to \mathbb{R} \) be a viscosity solution of (2.1) and (2.2). Then, \( X_t(u) \) is regular for all \( t \geq 0 \) if and only if there exists a unique solution in \( \mathcal{A} \) with initial data \( u(x, 0) = u_0(x) := \chi_{\Omega_0} - \chi_{\Omega_0^c} \).

Recall from Section 2 that
\[
K_\infty := \|\eta\|_{L^\infty([0, \infty))}.
\] (4.1)

We define \( t_1 = t_1(r, K_\infty) \) by
\[
t_1 := \frac{r}{10K_\infty}
\] (4.2)
and we will show the following theorem in this section.

Theorem 4.3. Suppose that the initial set \( \Omega_0 \) is in \( S_r \). Then, there is exactly one bounded viscosity solution \( u \) of (2.1) and (2.2) in \( [0, t_1] \) where \( t_1 \) is given in (4.2). Moreover, \( \Omega_t(u) \) is regular in \( [0, t_1] \).

We begin the proof with some definitions.

Definition 4.4. For \( \varepsilon, r > 0 \) and \( L : [0, +\infty) \to \mathbb{R} \), let us define a maximal time \( T_1 = T_1(\varepsilon, r, L) \) by
\[
T_1 := \sup\{ s > 0 : L(t) < re/2 \text{ for all } t \in [0, s] \};
\] (4.3)
and
\[
u(x, t; \varepsilon, r, L) := \inf\left\{ u\left(\frac{y}{1+\varepsilon}, \frac{t}{(1+\varepsilon)^2}\right) \left| y \in \overline{B}_{re/2-L(t)}(x) \right.\right\};
\]
and
\[
\bar{u}(x, t; \varepsilon, r, L) := \sup\left\{ u\left(\frac{y}{1-\varepsilon}, \frac{t}{(1-\varepsilon)^2}\right) \left| y \in \overline{B}_{re/2-L(t)}(x) \right.\right\}
\]

Lemma 4.5. Let \( u \) be a bounded viscosity solution of (2.1) and (2.2) with forcing \( \eta \) and \( \Omega_0 \in S_r \), and let \( \eta_\varepsilon(t) := (1+\varepsilon)^{-1}\eta(t/(1+\varepsilon)^2) \). Let \( u \) and \( \bar{u} \) be as given above with \( L \in C^1([0, \infty)) \). Then the following holds in \( (0, T_1) \) in the sense of viscosity solutions:
\[
\frac{ut}{|Du|}(x, t) \geq \nabla \cdot \left( \frac{Du}{|Du|} \right)(x, t) + \eta_\varepsilon(t) + L'(t)
\] (4.4)
and
\[
\frac{\ddot{u}_t}{|Du|} (x,t) \leq \nabla \cdot \left( \frac{D\ddot{u}}{|Du|} \right) (x,t) + \eta_\varepsilon(t) - L'(t). \tag{4.5}
\]

Moreover, if \( \varepsilon \leq \varepsilon_0(r) \) for \( \varepsilon_0(r) \) given in (3.2), we have
\[
\Omega_0(\tilde{u}) \subset \subset \Omega_0(u) \subset \subset \Omega_0(u). \tag{4.6}
\]

**Proof.** First, let us denote \( v(x,t) := u \left( \frac{x}{1+t}, \frac{t}{(1+\varepsilon)^2} \right) \). Then, \( v \) is a viscosity solution of
\[
\frac{v_t}{|Dv|} (x,t) = \nabla \cdot \left( \frac{Dv}{|Dv|} \right) (x,t) + \eta_\varepsilon(t).
\]

and thus Lemma 2.8 implies (4.4). Parallel arguments holds for \( \tilde{u} \).

On the other hand, if \( \Omega_0(u) \) is in \( S_r \), then Lemma 3.4 yields, for all \( \varepsilon \leq \varepsilon_0(r) \),
\[
\Omega_0(u) \subset \subset \bigcap_{|z| \leq r/2} [(1 + \varepsilon)\Omega_0(u) + z] = \Omega_0(u), \tag{4.7}
\]

and
\[
\Omega_0(\tilde{u}) = \bigcup_{|z| \leq r/2} [(1 - \varepsilon)\Omega_0(u) + z] \subset \subset \Omega_0(u). \tag{4.8}
\]

\[ \square \]

**Lemma 4.6.** Let \( \eta \) and \( \eta_\varepsilon \) be as given in Lemma 4.5, and let \( t_1 = r/10K_\infty \) be as given in (4.2). Then for the choice of \( L(t) = \int_0^t \eta_\varepsilon(s) + \eta(s)ds \) or \( L(t) = \int_0^t \eta_\varepsilon(s) - \eta(s)ds \) and for \( 0 < \varepsilon \leq 1/4 \), we have
\[
T_1 = T_1(\varepsilon, r, L) \geq t_1 \text{ for } 0 < \varepsilon < 1/4.
\]

**Proof.**

1. First, let us choose \( L(t) = \int_0^t \eta_\varepsilon(s) + \eta(s)ds \) and estimate the function \( L \) by the change of variables.

\[
L(t) = \int_0^t \eta(s) - \frac{1}{1+\varepsilon} \eta \left( \frac{s}{(1+\varepsilon)^2} \right) ds,
\]

\[
= \int_0^t \eta(s) ds - (1+\varepsilon) \int_0^{(1+\varepsilon)^2} \eta(s) ds,
\]

\[
= \int_0^t \eta(s) ds - \varepsilon \int_0^{(1+\varepsilon)^2} \eta(s) ds.
\]

Therefore, we conclude that for \( \varepsilon \in (0,1/4) \)
\[
|L(t)| \leq K_\infty t \left( \frac{\varepsilon^2 + 2\varepsilon}{(1+\varepsilon)^2} \right) + K_\infty \varepsilon t < 5K_\infty \varepsilon t. \tag{4.9}
\]

2. Similarly, let us choose \( L(t) = \int_0^t \eta_\varepsilon(s) - \eta(s)ds \), then for \( \varepsilon \in (0,1/4) \)
\[
|L(t)| = \left| \int_0^{(1+\varepsilon)^2} \eta(s) ds - \varepsilon \int_0^{(1+\varepsilon)^2} \eta(s) ds \right|,
\]

\[
\leq K_\infty t \left( \frac{2\varepsilon - \varepsilon^2}{(1-\varepsilon)^2} \right) + K_\infty \varepsilon t \frac{1}{(1-\varepsilon)^2} < 5K_\infty \varepsilon t.
\]
3. By definition of $T_1$ we have $L(T_1) = r\varepsilon/2$. Thus $5K_\infty\varepsilon t_1 = r\varepsilon/2 = L(T_1) < 5K_\infty\varepsilon t_1$. □

Lemma 4.5 and Lemma 4.6 imply the following.

Lemma 4.7. Let $u$ and $\Omega_0$ be as given in Lemma 4.5 and let $0 < \varepsilon \leq \varepsilon_0(r)$. For $t_1$ given in (4.2), $u$ with the choice of $L(t) = \int_0^t -\eta_+ + \eta$ is a viscosity supersolution of (2.1) in $(0, t_1]$. Similarly, $\bar{u}$ with $L(t) = \int_0^t \eta_- - \eta$ is a subsolution of (2.1) in $(0, t_1]$. Moreover it holds that $\bar{u} \leq u \leq \bar{u}$ in $[0, t_1]$.

Proof. By Lemma 4.6, $\bar{u}$ and $\bar{u}$ are well-defined in $[0, t_1]$. So, we could apply Lemma 4.5 and comparison principle in Theorem 2.2(1) for (2.1) in $[0, t_1]$ to conclude. □

Proof of Theorem 4.3. Suppose that $u$ and $\nu$ are two bounded solutions of (2.1) and $u(\cdot, 0) = \nu(\cdot, 0)$ in $\mathbb{R}^n$. Let us construct $\bar{u}$ and $\bar{\nu}$ as in Lemma 4.7. As $\Omega_0(\bar{u}) \subset \Omega_0(\nu) = \Omega_0(\bar{u}) \subset \Omega_0(u)$ from (4.6), we have $\bar{u}^+(\cdot, 0) \leq \nu^+ (\cdot, 0)$ and $\nu^+ (\cdot, 0) \leq \bar{u}_- (\cdot, 0)$ in $\mathbb{R}^n$. By Lemma 4.7 and the comparison principle in Theorem 2.2(1), it holds that $\bar{u} \leq \nu \leq \bar{u}$ in $[0, t_1]$. Sending $\varepsilon$ to zero, we conclude that $u = \nu$ in $[0, t_1]$. □

Lastly, for the next subsection let us state the following lemma.

Lemma 4.8. Let $u$ and $\Omega_0$ be as given in Lemma 4.5. Then for $0 < \varepsilon \leq \varepsilon_0(r)$ and $0 \leq t \leq t_1$ we have

$$(1 - \varepsilon)\Omega_{t/(1-\varepsilon)^2}(u) \subset \Omega_t(u) \subset (1 + \varepsilon)\Omega_{t/(1+\varepsilon)^2}(u).$$

where $t_1$ is given (4.2).

Proof. Lemma 4.7 implies that $\Omega_t(\bar{u}) \subset \Omega_t(u) \subset \Omega_t(\nu)$ in $[0, t_1]$. Moreover we have, by definition,

$$(1 - \varepsilon)\Omega_{t/(1-\varepsilon)^2}(u) \subset \Omega_t(\bar{u}), \text{ and } \Omega_t(u) \subset (1 + \varepsilon)\Omega_{t/(1+\varepsilon)^2}(u).$$

□

4.2. Uniqueness of mean curvature flows of forcing

In this subsection, we show the uniqueness of our original equation (MF). Here is the main theorem of this subsection.

Theorem 4.9. Suppose that $\Omega_0$ has $\rho$-reflection. Then, there exists at most one bounded viscosity solution of (MF) and (2.2).

Let $u$ and $v$ be two bounded viscosity solutions of (MF) and (2.2), and let $\eta(t; u) := \bar{\lambda}([\Omega_t(u)])$ and $\eta(t; v) := \bar{\lambda}([\Omega_t(v)])$. Fix $T > 0$. Recall from Theorem 3.8 and Lemma 3.11 that both $\Omega_t(u)$ and $\Omega_t(v)$ are in $S_{t_1, R_1}$ in $[0, T]$ where $t_1$ and $R_1$ are given in (3.18) and (3.23), respectively. From Lemma 3.13 that there exists a uniform bound of $\eta(t; u)$ and $\eta(t; v)$ in $[0, T]$,
\[
\tilde{K}_\infty := |||\eta(t; u)| + |\eta(t; v)|||_{L^\infty([0, T])} < \infty. \tag{4.10}
\]

Recall \(\tilde{\eta}(t) := (1 + \varepsilon)^{-1} \eta(t/(1 + \varepsilon)^2)\) and define
\[
L_1(t) := \int_0^t -\eta_\varepsilon(s; u) + \eta(s; v)ds \quad \text{and} \quad L_2(t) := \int_0^t \eta_\varepsilon(s; u) - \eta(s; v)ds \tag{4.11}
\]

**Definition 4.10.** For \(\varepsilon \in (0, \frac{1}{4})\), let us define
\[
\tilde{T}_1 = \tilde{T}_1(\varepsilon, r_1, L_1, L_2) := \sup \Big\{ s \in (0, T): L_1(t), L_2(t) < \frac{r_1 \varepsilon}{2} \quad \text{for all} \ t \in [0, s] \Big\},
\]
where \(r_1\) is given in (3.18). Remind that \(r_1\) is chosen so that \(\Omega_t(u)\) and \(\Omega_t(v)\) are in \(S_{r_1, r_1}\) for all \(t \in [0, T]\).

Let \(u = \mathcal{U}(.; \varepsilon, r_1, L_1)\) and \(\tilde{u} = \tilde{\mathcal{U}}(.; \varepsilon, r_1, L_2)\) be as given in **Definition 4.4**. The construction of \(L_1\) and \(L_2\) and **Lemma 4.5** readily yields the following lemma.

**Lemma 4.11.** \(u\) and \(\tilde{u}\) are a viscosity supersolution, and subsolution, respectively, of (2.1) with \(\eta = \eta(.; v)\) in \((0, \tilde{T}_1)\). Moreover, it holds that \(\tilde{u} \leq v \leq u\) in \([0, \tilde{T}_1]\). Here, \(\tilde{T}_1\) is given in (4.12).

**Lemma 4.12.** There exists \(t_2 > 0\) such that for any \(\varepsilon \in \left(0, \frac{1}{4}\right)\),
\[
\tilde{T}_1 = \tilde{T}_1(\varepsilon, r_1, L_1, L_2) > t_2,
\]
where \(\tilde{T}_1\) is given in (4.12).

**Proof.** Let \(t_1(r_1, \tilde{K}_\infty) = \frac{r_1}{5K_\infty}\) be as given in (4.2). If \(\tilde{T}_1 > t_1\) for all \(\varepsilon \in (0, \frac{1}{4})\), we take \(t_2 = t_1\). If \(\tilde{T}_1 < t_1\) for some \(\varepsilon \in (0, \frac{1}{4})\), **Lemma 4.2** implies that in \([0, \tilde{T}_1]\)
\[
(1 - \varepsilon)\Omega_t/(1 - \varepsilon)^2(u) \subset \Omega_t(u) \subset (1 + \varepsilon)\Omega_t/(1 + \varepsilon)^2(u). \tag{4.14}
\]

**Lemma 4.11** implies that \(\Omega_t(\tilde{u}) \subset \Omega_t(v) \subset \Omega_t(u)\) in \([0, \tilde{T}_1]\). Thus as shown in **Lemma 4.2**, the following holds for \(0 \leq t < \tilde{T}_1:\)
\[
(1 - \varepsilon)\Omega_t/(1 - \varepsilon)^2(u) \subset \Omega_t(\tilde{u}) \subset \Omega_t(v) \subset \Omega_t(u) \subset (1 + \varepsilon)\Omega_t/(1 + \varepsilon)^2(u). \tag{4.15}
\]

By subtracting \(\eta(s; u)\) and adding the same term,
\[
L_1(t) = \int_0^t \eta(s; v) - \eta_\varepsilon(s; u)ds = \int_0^t \eta(s; v) - \eta(s; u)ds + \int_0^t \eta(s; u) - \eta_\varepsilon(s; u)ds. \tag{4.16}
\]

As **Lemma 4.6**, the second term is bounded by \(5\tilde{K}_\infty \varepsilon t\). As for the first term, from Lipschitz continuity of \(\lambda\) for some \(C_1 > 0\),
\[
\mathcal{I}_1 := \left| \int_0^t \eta(s; v) - \eta(s; u)ds \right| \leq \int_0^t |\lambda||\Omega_t(v)|| - \lambda||\Omega_t(u)|||ds \leq C_1 \int_0^t ||\Omega_t(v)|| - ||\Omega_t(u)|||ds
\]

By (4.14)-(4.15) and **Lemma 3.11**,
\[
\mathcal{I}_1 \leq C_1 \int_0^t ||(1 - \varepsilon)\Omega_t/(1 - \varepsilon)^2(u)|| - ||(1 + \varepsilon)\Omega_t/(1 + \varepsilon)^2(u)|||ds
\]
\[
\leq C_1 \int_0^t ||\Omega_t/(1 - \varepsilon)^2(u)|| - ||\Omega_t/(1 + \varepsilon)^2(u)|||ds + C_2 \varepsilon t
\]
for some constant $C_2 = C_2(R_1, T)$. By Lemma 3.14, we conclude that $I_1$ is bounded by $C_3 \varepsilon t$ for some constant $C_3 = C_3(r_1, R_1, T, K_\infty)$. Therefore, we have $L_1(t) < (C_3 + 5K_\infty)\varepsilon t$ in $[0, \tilde{T}_1]$. By similar arguments, the bound holds for $L_2$ as well in $[0, \tilde{T}_1]$.

Finally, by continuity of $L_1$ and $L_2$, we have $L_1(\tilde{T}_1) = r_1\varepsilon/2$ or $L_2(\tilde{T}_1) = r_1\varepsilon/2$. In both cases, it holds that

$$r_1\varepsilon/2 = L_1(\tilde{T}_1) \text{ (or } L_2(\tilde{T}_1)) < (C_3 + 5K_\infty)\tilde{T}_1\varepsilon,$$

so we conclude with

$$\tilde{T}_1 \geq t_2 = t_2(r_1, R_1, T, K_\infty) := \frac{r_1}{2C_3 + 10K_\infty}.$$  

(4.17)

**Proof of Theorem 4.9.** The first part is parallel to the proof of Theorem 4.3. Let $u$ and $v$ be two viscosity solutions of (MF) and (2.2). By Lemma 4.11 and Lemma 4.12, it holds that $\bar{u} \leq v \leq \bar{u}$ in $[0, t_2]$ where $t_2$ is given in (4.13). We can now send $\varepsilon$ to zero to conclude that $u = v$ in $[0, t_2]$.

Next let us consider the corresponding convolutions of $u$ and $\bar{u}$ in the time interval $t_0 + [0, t_2] \subset [0, T]$ for $t_0 > 0$ and $t_2$ given in (4.13). Note that $t_2$ given in (4.13) does not depend on $t_0$ because both $\Omega_t(u)$ and $\Omega_t(v)$ are in $S_{r_1, R_1}$ for all $t \in [0, T]$. Thus, we can iterate the step 1 for $t_0 = kt_2$ on $kt_2 + [0, t_2], k \in \mathbb{N}$ and, conclude that $u = v$ in $[0, T]$.

\[
\square
\]

5. Construction of flat flows

In this section, we construct a flat flow for (MF), which coincides with our notion of viscosity solutions. Our approach is based on minimizing movements first introduced by Almgren-Taylor-Wang [11] (see also Refs. [4, 12, 19]).

As in Ref. [5], we introduce a gradient flow with geometric constraint, corresponding to the preservation of star-shapedness obtained in Theorem 3.8. Our constraint is crucial to ensure the strong (in Hausdorff distance) convergence of the minimizing movements, which enables geometric analysis of the limiting flow. On the other hand, the constraint also poses technical challenges when we show the coincidence of flat flows with viscosity solutions (See Proposition 5.5 and Corollary 5.6). This is why we first approximate our original problem with a “restricted” version (2.5).

5.1. Constrained minimizing movements

Recall the following energy functional associated with (MF),

$$J(E) = \text{Per}(E) - \Lambda(|E|).$$

(5.1)

where the function $\Lambda(s)$ is an anti-derivative of $\lambda(s)$, and $\text{Per}(E)$ denotes the perimeter of $E$. For the sets $E$ and $F$ in $\mathbb{R}^n$, we use the pseudo-distance defined by

$$\tilde{d}(F, E) := \left( \int_{E \Delta F} d(x, \partial E)dx \right)^{\frac{1}{2}}, \quad E \Delta F := (E \setminus F) \cup (F \setminus E).$$
We consider minimizing movements for the restricted Eq. (2.5) in a finite time interval \([0, T]\) with initial data (2.2) with the admissible sets

\[ A_M(E) := \{ F \in S_{r_0, R_0} | d_H(\partial(F \cap E), \partial E) \leq Mh \} \text{ for } E \subset \mathbb{R}^n, \]

with

\[ r_0 < r_1 = r_1(\rho, a) = \rho(a^2 + 2a)^{1/2} \text{ and } R_0 > R_1, \]

where \( r_1 \) is given in (3.18) and \( R_1 = R_1(T) \) in (3.23). Recall that \( \rho \) is given in Definition 3.3 and \( a \) is given in Lemma 3.9. The dependence of \( R_1 \) on \( T \) is the reason why we restrict the discussion in this and next section to the finite time interval. For simplicity we will omit the time dependence in \( R_1 \) and thus in \( R_0 \).

**Definition 5.1.** For \( h > 0, T_{h,M} \) is defined by

\[ T_{h,M}(E) = \arg \min_{F \in A_M(E)} I_h(F; E), \quad I_h(F; E) := J(F) + \frac{1}{h} d^2(F, E), \]

The existence of a minimizer, \( T_{h,M}(E) \) follows from Lemma C.1, C.2 and C.3.

The constrained minimizing movement \( E_{t,M}^h \) of \( J \) for \( t \in [0, T] \) with initial set \( E_0 \) can be defined by

\[ E_{t,M}^h := T_{h,M}^{[t/h]}(E_0). \]

Here, \( T^m \) for \( m \in \mathbb{N} \) is the \( m \)-th functional power.

**Definition 5.2.** A function \( \omega_M := \chi_{E_0} - \chi_{E_1}^c \) is a restricted flat flow of (2.5) and (2.2) if \( E_0 = \Omega_0 \) and there exists a sequence \( h_k \to 0 \) such that

\[ d_H(E_t, E_{t,M}^h) \to 0 \]

locally uniformly in time as \( k \) goes to infinity.

To show the existence of a restricted flat flow, let us show compactness property of the constrained minimizing movements.

**Lemma 5.3.** The constrained minimizing movement \( E_{t,M}^h \) in Definition 5.1 satisfies the following inequality for \( 0 < t < s \leq T \):

\[ \tilde{d}^2(E_t^h, E_s^h) \leq d_H(E_t, E_s) (J(E_s^h) - J(E_t^h)) \]

and, as a consequence,

\[ d_H(E_t^h, E_s^h) \leq d_H(E_t, E_s) \]

**Proof.** We will use the triangle-like inequality (see e.g. Lemma 17, [5]):

\[ \frac{\tilde{d}^2(F_{k+1}, F_k)}{k} \leq \frac{\sum_{j=1}^{k} \tilde{d}^2(F_{j+1}, F_j)}{k} \text{ for } F_1, ..., F_{k+1} \in S_{r, R}. \]

Suppose that \( t \in [Kh, (K+1)h) \) and \( s \in [(K+L)h, (K+L+1)h) \) for some \( K \) and \( L > 0 \). By the construction of \( E_{t,M}^h \) in Definition 5.1 for \( k \in \mathbb{N} \),
\[
J(E_{(k-1)h}^{h,M}) - J(E_{kh}^{h,M}) \geq \frac{1}{h} d^2(E_{kh}^{h,M}, E_{(k-1)h}^{h,M}).
\]

By summing both sides from \( k = K + 1 \) to \( k = K + L \),
\[
J(E_{kh}^{h,M}) - J(E_{(K+L)h}^{h,M}) \geq \sum_{k=K+1}^{K+L} \frac{1}{h} d^2(E_{kh}^{h,M}, E_{(k-1)h}^{h,M}),
\]
\[
\geq r,R \frac{1}{Lh} d^2(E_{(K+L)h}^{h,M}, E_{kh}^{h,M}),
\]
where the last inequality follows from (5.6). (5.5) follows from Lemma C.2. \( \Box \)

One can apply Lemma 5.3 and compactness of star-shaped sets (Lemma C.1, C.2 and C.3) to obtain the following:

**Theorem 5.4.** There exists at least one restricted flat flow \( w_M \) of (2.5) and (2.2) in the sense of Definition 5.2.

### 5.2. Barrier property under star-shapedness

Next we establish a “restricted barrier property” for a restricted flat flow with respect to a classical subsolution and supersolution of (2.5) with \( \eta(t) = \lambda[|\Omega_t(w_M)|] \). The proof of this proposition is rather technical and follows that of Ref. [35]: see Appendix A. In a different setting, similar results are shown in Refs. [13, 36].

**Proposition 5.5.** (Restricted barrier property) Let \( w_M \) be a restricted flat flow of (2.5) with the admissible set constraint parameters \( r_0 \) and \( R_0 \) satisfying (5.3). For any \( r > r_0 \) and \( R < R_0 \), suppose that there exists a test function \( \phi \) on \( Q_T \) such that \( \phi \) is a classical subsolution in \( Q_T \) of (2.5) with \( \eta(t) = \lambda[|\Omega_t(w_M)|] \), \( |D\phi| \neq 0 \) on \( \partial \Omega_t(\phi) \) and \( \Omega_t(\phi) \in S_{r,R} \) in \([0,T] \). If \( \Omega_0(\phi) \subset \subset \Omega_0(w_M) \), then
\[
\Omega_t(\phi) \subset \subset \Omega_t(w_M) \quad \text{for all } t \in [0,T].
\]

Similarly, suppose that there exists a test function \( \psi \) on \( Q_T \) such that \( \psi \) is a classical supersolution in \( Q_T \) of (2.5) with \( \eta(t) = \lambda[|\Omega_t(w_M)|] \), \( |D\psi| \neq 0 \) on \( \partial \Omega_t(\psi) \) and \( \Omega_t(\psi) \in S_{r,R} \) in \([0,T] \). If \( \Omega_0(w_M) \subset \subset \Omega_0(\psi) \), then
\[
\Omega_t(w_M) \subset \subset \Omega_t(\psi) \quad \text{for all } t \in [0,T].
\]

In the proof of Proposition 5.5, we only use the properties of the classical solution \( \phi \) in small neighborhood of \( (x_0, t_0) \), thus we can deduce the following localized barrier property of the flat flow.

**Corollary 5.6.** Let \( w_M \) be a restricted flat flow of (2.5) with the admissible set constraint parameter \( r_0 \) and \( R_0 \) satisfying (5.3). If there exists a test function \( \phi \) on \( Q_T \) such that \( \phi \) touches \( w \) from below at \( (x_0, t_0) \), \( |x_0| < R_0 \), \( |D\phi|(x_0, t_0) \neq 0 \) and \( -x_0 \cdot \frac{D\phi}{|D\phi|}(x_0, t_0) > r_0 \). then
\[
\frac{\phi_t}{|D\phi|}(x_0, t_0) \geq \max \left\{ \nabla \cdot \left( \frac{D\phi}{|D\phi|} \right)(x_0, t_0) + \eta(t_0), -M \right\}.
\]
Similarly, if there exists a test function \( \psi \) on \( QT \) such that \( \psi \) touches \( w \) from above at 
\((x_0, t_0), |x_0| < R_0, |D\psi|(x_0, t_0) \neq 0 \) and 
\(-x_0 \cdot \frac{D\psi}{|D\psi|}(x_0, t_0) > r_0 \) then
\[
\frac{\psi_t}{|D\psi|}(x_0, t_0) \leq \max\left\{ \nabla \cdot \left( \frac{D\psi}{|D\psi|} \right)(x_0, t_0) + \eta(t_0), -M \right\}.
\]

6. Existence of the flow: Coincidence between flat flows and viscosity solutions

Our goal in this section is to show the existence of a viscosity solution for (MF).

Let us give a brief summary of this section. We will show that a restricted flat flow coincides with the corresponding viscosity solution as long as the viscosity solution is star-shaped (Proposition 6.1). Ensuring this star-shaped property for the viscosity solution is the last step leading to the coincidence result (Proposition 6.4): this is where we need the lower bound \( M \) on the velocity of the flow imposed by (2.5). After we show the coincidence between a restricted flat flow and the corresponding viscosity solution, we address removing the bound \( M \) to obtain our desired result (Theorem 6.8).

6.1. Coincidence for the restricted problem

In this section, our goal is to show coincidence between flow flows and viscosity solutions for the restricted problem (2.5). To this end we first show a comparison result between a flat flow and the corresponding viscosity solution of (2.5). We use the doubling argument in Refs. [29, 37] which preserves the star-shaped geometry of the level sets of the solutions.

**Proposition 6.1.** Let \( w_M \) be a restricted flat flow of (2.5) with the admissible set constraint parameter \( r_0 \) and \( R_0 \) satisfying (5.3). Suppose that there exists a viscosity subsolution \( u : QT \to \mathbb{R} \) of (2.5) with \( \eta(t) = \lambda \|[\Omega_t(w_M)]\| \) such that \( \Omega_t(u) \) is in \( S_{r,R} \) for all \( t \in [0, T] \) for some \( r > r_0 \) and \( R < R_0 \). If \( \Omega_0(u) \subset \subset \Omega_0(w_M) \), then
\[
\Omega_t(u) \subset \subset \Omega_t(w_M) \quad \text{for all } t \in [0, T].
\]

Similarly, suppose that there exists a viscosity supersolution \( u : QT \to \mathbb{R} \) of (2.5) with \( \eta(t) = \lambda \|[\Omega_t(w_M)]\| \) such that \( \Omega_t(u) \) is in \( S_{r,R} \) for all \( t \in [0, T] \) for some \( r > r_0 \) and \( R < R_0 \). If \( \Omega_0(w_M) \subset \subset \Omega_0(u) \), then
\[
\Omega_t(w_M) \subset \subset \Omega_t(u) \quad \text{for all } t \in [0, T].
\]

**Proof.** The proof follows the outline of Ref. [37], where the comparison principle is shown for a nonlocal mean-curvature flow.

For \( c, \delta > 0 \), let us consider
\[
Z(x, t) := \sup_{|z| \leq c^{-\delta t}} u(x + z, t) \text{ and } 0 \leq t \leq \frac{c}{\delta},
\]
where \( c \) is chosen sufficiently small so that \( \Omega_0(Z) \subset \subset \Omega_0(w_M) \). Due to Lemma 2.8, the function \( Z \) is a viscosity subsolution of
\[ u_t = F(Du, D^2u, t) - \delta |Du|. \]

We will show Proposition 6.2 by showing that for any \( \delta > 0 \) and \( 0 \leq t \leq c/\delta \) we have

\[ \Omega_t(Z) \subset \subset \Omega_t(w_M). \quad (6.1) \]

Note that for any \( z \in \mathbb{R}^n \) such that \( |z| \leq c \), the interior cone \( IC(x, r) \) given in (3.3) satisfies \( IC(x + z, r - c) \subset IC(x, r) + z \) (See Lemma C.5). Thus, by the equivalence relation in Lemma 3.4, \( \Omega_t(u) \in S_{r, R} \) implies that \( \Omega_t(u) + z \in S_{r_c, R+c} \) for all \( |z| \leq c \) and thus

\[ \Omega_t(Z) = \bigcup_{|z| \leq \epsilon} [\Omega_t(u) + z] \in S_{r_c, R+c}. \]

Thus, \( \Omega_t(Z) \in S_{r_0+c, R_0-c} \) for \( 0 < \epsilon \leq \min\{t, \frac{R_0-R}{2}\} \).

Suppose (6.1) is false, then we have

\[ t_0 := \sup\{t : \Omega_t(Z) \subset \subset \Omega_t(u) \text{ for } 0 \leq s \leq t \} \in (0, c/\delta). \]

Due to Lemma 3.13 and Lemma 5.3, both sets \( \partial \Omega_t(Z) \) and \( \partial \Omega_t(w_M) \) evolve continuously in time. Hence, \( \partial \Omega_t(Z) \) touches \( \partial \Omega_t(w_M) \) from inside for the first time at \( t = t_0 \in (0, \frac{c}{\delta}) \).

For \( \epsilon \in \left(0, \frac{\delta}{2n}\right) \), let us define \( \tilde{Z} := \chi_{\Omega_t(Z)} \) and \( \tilde{W} := \chi_{\Omega_t(w_M)} \) and

\[ \Phi_\epsilon(x, y, t) := \tilde{Z}(x, t) - \tilde{W}(y, t) - \frac{|x - y|^4}{4\epsilon} - \frac{\epsilon}{2(t_0 - t)}. \]

Let \( d_0 \) be distance between \( \partial \Omega_0(Z) \) and \( \partial \Omega_0(w_M) \). Since \( \tilde{Z} - \tilde{W} \) is bounded, we can choose a sufficiently small \( \epsilon \ll d_0 \) such that \( \Phi(x, y, 0) < 0 \) for all \( x \) and \( y \).

Since the function \( \tilde{Z} - \tilde{W} \) is upper semicontinuous and bounded above by zero for all \( t < t_0 \), the function \( \Phi_\epsilon(x, y, t) \) has a local maximum at \( (x_\epsilon, y_\epsilon, t_\epsilon) \) in \( \mathbb{R}_n \times [0, t_0) \) for any \( \epsilon \). By Hölder continuity of \( \partial \Omega_t(Z) \) and \( \partial \Omega_t(w_M) \) from Lemma 3.13 and Lemma 5.3, there exists \( x_1 \in \partial \Omega_{t_0-\epsilon}(\tilde{Z}) \) and \( y_1 \in \partial \Omega_{t_0-\epsilon}(\tilde{W}) \) such that \( |x_1 - y_1| \leq K\epsilon^2 \) where \( K \) depends on Hölder constants of \( \partial \Omega_t(Z) \) and \( \partial \Omega_t(w_M) \). For \( \epsilon \ll K^{-4} \), it holds that \( \Phi(x_\epsilon, y_\epsilon, t_\epsilon) > \Phi(x_1, y_1, t_0 - \epsilon) > \frac{1}{2}, \) and thus \( t_\epsilon \in (0, t_0) \). Also, \( \Phi(x_\epsilon, y_\epsilon, t_\epsilon) \) is uniformly bounded from below in \( \epsilon \), and thus it holds that \( |x_\epsilon - y_\epsilon| = O(\epsilon^2) \).

Moreover, since \( \tilde{Z} - \tilde{W} > \Phi > \frac{1}{3} \) at \( (x_\epsilon, y_\epsilon, t_\epsilon) \), we conclude that \( x_\epsilon \in \Omega_{t_\epsilon}(\tilde{Z}), y_\epsilon \in \Omega_{t_\epsilon}(\tilde{W})^C \). As \( t_0 \) is the first touching point and \( t_\epsilon < t_0 \), it holds that \( |x_\epsilon - y_\epsilon| > 0 \). On the other hand, \( \tilde{Z}(x, t_\epsilon) - \tilde{W}(y, t_\epsilon) = 1 \) for all \( (x, y) \in \Omega_{t_\epsilon}(\tilde{Z}) \times \Omega_{t_\epsilon}(\tilde{W})^C \), and thus \( (x_\epsilon, y_\epsilon) \) is a maximizer of the third term \( -\frac{|x_\epsilon - y_\epsilon|^4}{4\epsilon} \) in \( \Omega_{t_\epsilon}(\tilde{Z}) \times \Omega_{t_\epsilon}(\tilde{W})^C \). We conclude that \( x_\epsilon \) and \( y_\epsilon \) are on \( \partial \Omega_{t_\epsilon}(\tilde{Z}) \) and \( \partial \Omega_{t_\epsilon}(\tilde{W}) \), respectively.

Then, as Eq. (2.9) in Ref. [37], there exist quadratic test functions \( \phi^\epsilon(x, t) \) and \( \psi^\epsilon(x, t) \) such that
\[
\begin{align*}
\phi^\varepsilon(x, t) &:= \left[ a_\varepsilon(t - t_\varepsilon) + p_\varepsilon \cdot (x - x_\varepsilon) + \frac{1}{2} (x - x_\varepsilon)^T X_\varepsilon (x - x_\varepsilon) \right]_+ \geq \tilde{Z}(x, t) \quad \text{in } N_1, \\
\psi^\varepsilon(y, t) &:= \left[ b_\varepsilon(t - t_\varepsilon) + q_\varepsilon \cdot (y - y_\varepsilon) + \frac{1}{2} (y - y_\varepsilon)^T Y_\varepsilon (y - y_\varepsilon) \right]_+ \leq \tilde{W}(y, t) \quad \text{in } N_2,
\end{align*}
\]

where constants \( a_\varepsilon, b_\varepsilon \in \mathbb{R}, p_\varepsilon, q_\varepsilon = \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + O(\varepsilon^2) \in \mathbb{R}^n \setminus \{0\}, X_\varepsilon, Y_\varepsilon \in S^{n \times n}, \) neighborhoods \( N_1 \) of \((x_\varepsilon, t_\varepsilon)\) and \( N_2 \) of \((y_\varepsilon, t_\varepsilon)\) satisfying the inequalities:

\[
\begin{align*}
& a_\varepsilon - b_\varepsilon \geq 0, \\
& |p_\varepsilon - q_\varepsilon|^2 \leq \varepsilon^2 \min\{1, |p_\varepsilon|^2\}, \\
& |p_\varepsilon - q_\varepsilon| \leq \varepsilon^2 \min\{1, |p_\varepsilon|^2\}.
\end{align*}
\]

(6.2)

For simplicity we carry out the computations in steps 4–5. for the case \( M = \infty. \) Since \( \tilde{Z} \) is a viscosity solution and \( \phi^\varepsilon \) touches \( \tilde{Z} \) from above at \((x_\varepsilon, t_\varepsilon)\), it holds that

\[
\frac{a_\varepsilon}{|p_\varepsilon|} = \frac{\phi^\varepsilon_t}{|D\phi^\varepsilon|} (x_\varepsilon, t_\varepsilon) \leq \nabla \cdot \left( \frac{D\phi^\varepsilon_t}{|D\phi^\varepsilon|} \right) (x_\varepsilon, t_\varepsilon) + \eta(t_\varepsilon) - \delta = \frac{1}{|p_\varepsilon|} \left( \text{trace}(X_\varepsilon) - \frac{p_\varepsilon^T X_\varepsilon p_\varepsilon}{|p_\varepsilon|^2} \right) + \eta(t_\varepsilon) - \delta.
\]

By inequalities (6.3) and the ellipticity of the operator, \( \text{trace}(X) - \frac{p^T X p}{|p|^2} \), it can be seen that

\[
\frac{b_\varepsilon}{|q_\varepsilon|} \leq \frac{a_\varepsilon}{|p_\varepsilon|} \leq \frac{1}{|p_\varepsilon|} \left( \text{trace}(X_\varepsilon) - \frac{p_\varepsilon^T X_\varepsilon p_\varepsilon}{|p_\varepsilon|^2} \right) + \eta(t_\varepsilon) - \delta, \\
\leq \frac{1}{|p_\varepsilon|} \left( \text{trace}(Y_\varepsilon) - \frac{p_\varepsilon^T Y_\varepsilon p_\varepsilon}{|p_\varepsilon|^2} \right) + \eta(t_\varepsilon) - \delta.
\]

Thus, by (6.3), for sufficiently small \( \varepsilon > 0, \) it holds that

\[
\frac{b_\varepsilon}{|q_\varepsilon|} \leq \frac{1}{|q_\varepsilon|} \left( \text{trace}(Y_\varepsilon) - \frac{q_\varepsilon^T Y_\varepsilon q_\varepsilon}{|q_\varepsilon|^2} \right) + \eta(t_\varepsilon) - \frac{\delta}{4}.
\]

(6.4)

Moreover, as \( \Omega_\varepsilon(\tilde{Z}) \in S_{r_0 + \varepsilon, r_0 - \varepsilon}, |x_\varepsilon| < R_0 - c \) and Lemma 3.2 implies that

\[
x_\varepsilon \cdot \left( - \frac{p_\varepsilon}{|p_\varepsilon|} \right) \geq r_0 + c.
\]

There exists sufficiently small \( \varepsilon_0 \) such that for any \( \varepsilon \in (0, \varepsilon_0), \)

\[
|y_\varepsilon| < R_0, \quad \text{and } y_\varepsilon \cdot \left( - \frac{q_\varepsilon}{|q_\varepsilon|} \right) > r_0.
\]

(6.5)

This contradicts Corollary 5.6. since \( \psi^\varepsilon \) touches \( \tilde{W} \) from below at \((y_\varepsilon, t_\varepsilon)\), but satisfies (6.4) and (6.5). \qed
Next, we will show that viscosity solutions \( u \) of (2.5) has a short time star-shapedness property.

**Definition 6.2.** (Inf-Sup Convolutions with Space Scaling) For \( \delta, \varepsilon > 0 \) and \( M \) as in (2.5), define \( \tilde{u}^S, \tilde{u}^S : \mathbb{R}^n \times [0, \frac{\delta}{5M}] \rightarrow \mathbb{R} \) as

\[
\tilde{u}^S(x, t; \delta, \varepsilon) := \inf \left\{ u \left( \frac{y}{1 + \varepsilon}, t \right) \mid y \in B_{\delta \varepsilon - 5M \varepsilon}(x) \right\}, \tag{6.6}
\]

and

\[
\tilde{u}^S(x, t; \delta, \varepsilon) := \sup \left\{ u \left( \frac{y}{1 - \varepsilon}, t \right) \mid y \in B_{\delta \varepsilon - 5M \varepsilon}(x) \right\}, \tag{6.7}
\]

Note that \( \Omega_0(\tilde{u}^S) \subset \Omega_0(u) \subset \Omega_0(\tilde{u}^S) \) due to (4.7) and (4.8).

**Lemma 6.3.** Let \( u \) be a viscosity solution of (2.5). Suppose that \( \Omega_0(u) \in S_r \) and \( |\eta(t)| < M \) in \([0, T]\). Then, for any fixed \( 0 < \delta < r \), there exists \( \varepsilon_0 \in (0, 1) \) such that \( \tilde{u}^S(\tilde{u}^S) \) is a viscosity supersolution (subsolution) of (2.5) for all \( 0 < \varepsilon < \varepsilon_0 \).

**Proof.** We only prove for \( \tilde{u}^S \), the proof for \( \tilde{u}^S \) is parallel. Let us define \( v(x, t) := u(\frac{x}{1 + \varepsilon}, t) \) in \( \tilde{Q} \). Since \( u \) is a viscosity solution of (2.5), \( v \) solves

\[
\frac{v_t}{|Dv|} = (1 + \varepsilon)\max \left[ (1 + \varepsilon)\nabla \cdot \left( \frac{Dv}{|Dv|} \right) + \eta(t), -M \right]
\]

in the viscosity sense. Proceeding as in the proof of Lemma 2.8 one can verify that \( \tilde{u}^S \) satisfies

\[
\frac{\tilde{u}^S_t}{|D\tilde{u}^S|} \geq (1 + \varepsilon)\max \left[ (1 + \varepsilon)\nabla \cdot \left( \frac{D\tilde{u}^S}{|D\tilde{u}^S|} \right) + \eta(t), -M \right] + 5M\varepsilon.
\]

For simplicity, let us denote \( H := -\nabla \cdot \left( \frac{D\tilde{u}^S}{|D\tilde{u}^S|} \right) \). Then, the right hand side is

\[
(1 + \varepsilon)\max[-(1 + \varepsilon)H + \eta(t), -M] + 5M\varepsilon = \max[-(1 + \varepsilon)^2H + (1 + \varepsilon)^2\eta(t) + 5M\varepsilon, -H + 4M\varepsilon].
\]

First suppose that \( -H + \eta \geq -M \). Since \( |\eta(t)| < M \) and \( \varepsilon < 1 \), it holds that

\[
-(1 + \varepsilon)^2H + (1 + \varepsilon)\eta + 5M\varepsilon \geq -H + (\varepsilon^2 + 2\varepsilon)(-\eta - M) + (1 + \varepsilon)\eta + 5M\varepsilon,
\]

\[
\geq -H + \eta - (\varepsilon^2 + \varepsilon)\eta + (3\varepsilon - \varepsilon^2)M \geq -H + \eta.
\]

If \( -H + \eta < -M \), then \(-M + 4M\varepsilon \geq -M = \max\{ -H + \eta, -M \} \), so we conclude that

\[
\frac{\tilde{u}^S_t}{|D\tilde{u}^S|} \geq \max \left[ \nabla \cdot \left( \frac{D\tilde{u}^S}{|D\tilde{u}^S|} \right) + \eta(t), -M \right].
\]

\( \square \)
**Proposition 6.4.** (Short-time star-shapedness) Let \( u \) be a viscosity solution of (2.5) and (2.2). Suppose that \( \Omega_0(u) \in S_{r,R} \) and \( |\eta(t)| < M \) for \( 0 \leq t \leq \frac{r}{5M} \). Then, \( \Omega_t(u) \in S_{r-5Mt,R+Mt} \) for \( t \in \left[ 0, \frac{r}{5M} \right) \).

**Proof.** Applying Theorem 2.2 for (2.5) to \( u \) and \( \tilde{u}^S \), we have
\[
\Omega_t(u) \subset \Omega_t(\tilde{u}^S)
\]
By Definition 6.2 of \( \tilde{u}^S \), it holds that
\[
\Omega_t(u) \subset \bigcap_{|z| \leq (\eta-5Mt)\varepsilon} [(1+\varepsilon)\Omega_t(u) + z],
\]
for all \( 0 < \varepsilon < \varepsilon_0, 0 < \delta < \frac{\delta}{5M} \) where \( \varepsilon_0 \) is given in Lemma 6.3. Then, by (3.2) in Lemma 3.4, we conclude that \( \Omega_t(u) \in S_{r-5Mt} \).

On the other hand, let us compare \( u \) with a radial barrier \( \phi \) defined by
\[
\phi := \chi_{B_{\varepsilon(t)}} - \chi_{B_{\varepsilon(t)}^c},
\]
where \( r : [0, T] \to \mathbb{R} \) is defined by \( r(t) := R + Mt. \) Since \( \phi \) is a viscosity supersolution, comparison principle implies \( \Omega_t(u) \subset B_{R+Mt} \). Thus, \( \Omega_t(u) \in S_{r-5Mt,R+Mt} \) for \( t \in \left[ 0, \frac{r}{5M} \right) \). \( \square \)

Now we are ready to prove our main theorem in this subsection.

**Theorem 6.5.** Suppose that \( \Omega_0 \) has \( \rho \)-reflection. Let \( r_0 \) and \( R_0 \) satisfy (5.3), and \( \tilde{K}_\infty = \tilde{K}_\infty(r_0,R_0) \) be as in Lemma 3.13. For \( M > \tilde{K}_\infty \), consider a restricted flat flow \( w_M \) of (2.5) and (2.2) in the sense of Definition 5.2. If \( u \) is a unique viscosity solution of (2.5) and (2.2) with \( \eta(t) = \lambda[\Omega_t(w_M)] \), then \( w_M = u \) in \( \bar{Q} \).

**Proof.** The existence and short time uniqueness of \( u \) for the above choice of \( \eta(t) \) follows by Theorem 2.2 and Theorem 4.3.

Recall that \( \Omega_0 \in S_{r_1,R_1} \) where \( r_1 \) and \( R_1 \) are given in (3.18) and (3.23). Let us first show that \( u = w_M \) in the small time interval \( I = [0, t_0] \), where \( t_0 := \min\{r_1-r_0 \frac{R_1-R_0}{10M}, \frac{R_0-R_1}{2M} \} \).

As Corollary 3.12, we can make \( \Omega_0 \) strictly smaller \( \Omega_0^- \) or bigger \( \Omega_0^+ \) by dilation and can still make it stay in \( S_{r_0,R_0} \) with \( r_\varepsilon = r_1 - O(\varepsilon) > r_0 \) and \( R_\varepsilon = R_1 + O(\varepsilon) > R_0 \), where \( \varepsilon \) can be chosen arbitrarily small such that \( r_\varepsilon - r_0 > \frac{2}{5} \frac{r_0}{R_0} \) and \( R_\varepsilon - R_0 > \frac{2}{5} \frac{R_0}{R_1} \). Let us choose to make the domain strictly bigger, \( \Omega_0^+ \), we can apply Proposition 6.4 to ensure that the corresponding viscosity solution \( u^\varepsilon \) of (2.5) satisfies, for some \( r > r_0 \) and \( R < R_0 \),
\[
\Omega_t(u^\varepsilon) \in S_{r,R} \text{ for } t \in I.
\]
We can then apply Proposition 6.1 to \( u^\varepsilon \) and \( w_M \) to yield that
\[
\Omega_t(w_M) \subset \Omega_t(u^\varepsilon) \text{ for } t \in I.
\]
Now to send \( \varepsilon \to 0 \), note that \( \Omega_t(u^\varepsilon) \) satisfies Hölder continuity, Corollary 3.7. Thus along a sequence \( \varepsilon = \varepsilon_n \to 0, \Omega_t(u^\varepsilon) \) converges to a domain \( \Omega_t \in S_{r,R} \) uniformly with respect to \( d_H \) in the time interval \( I \). Lemma 2.3 then yields that the corresponding level set function \( u \) for \( \Omega_t \) is the unique viscosity solution of (2.5) with the initial data \( u_0 \). From (6.9) we have
Similarly, using $\Omega_0^{-}$ instead of $\Omega_0^{+}$ we can conclude that $\Omega_t(u) \subset \Omega_t(w_M)$ and thus it follows that they are equal sets for the time interval $I$.

3. Once we know that $u = w_M$ in $I$, we know that $\eta(t)$ equals $\lambda[||\Omega_t(u)||]$ in $I$, and thus Theorem 3.8 and Lemma 3.11 applies and now we know that $\Omega_t(u) \in S_{R_1, R_1}$ for $t \in I$. Now we can repeat the argument at $t = t_0$ over the time interval $t_0 + I$, using the fact that $\Omega_{t_0}(u) \in S_{R_1, R_1}$. Now we can repeat above arguments to obtain that $w_M = u$ for all times.

6.2. Existence

Let us define the notion of flat flows for our original problem.

**Definition 6.6.** A function $w := \chi_{E_t} - \chi_{E^c_t}$ for $E_t \subset \mathbb{R}^n, t > 0$ is a flat flow of (MF) and (2.2) if $E_0 = \Omega_0$ and there exists a sequence $M_k \to \infty$ such that

$$d_H(E_t, \Omega_t(w_{M_k})) \to 0$$

locally uniformly in time as $k$ goes to infinity. Here, $w_{M_k}$ is a restricted flat flow with $M = M_k$ in the sense of Definition 5.2.

Let us first show existence of the flat flow.

**Lemma 6.7.** Suppose that $\Omega_0$ has $\rho$-reflection. There exists at least one flat flow $w$ of (MF) and (2.2).

**Proof.** Due to Corollary 3.7 and Theorem 6.5, we have

$$d_H(\Omega_t(w_M), \Omega_t(w_{M_k})) \leq C|s - t|^{1/2},$$

where $C$ does not depend on $M$. Thus along a sequence $M_k \to \infty$ such that $\Omega_t(w_{M_k})$ converges with respect to $d_H$ to $\Omega_{t_0}$ locally uniformly in time. We conclude that $w := \chi_{\Omega_t} - \chi_{\Omega^c_t}$ of (MF) is a flat flow in the sense of Definition 6.6.

Now, let us show existence and uniqueness of viscosity solution of our original equation (MF).

**Theorem 6.8.** Suppose that $\Omega_0$ has $\rho$-reflection. Let $w$ be a flat flow of (MF) and (2.2) and let $u$ be the unique viscosity solution of (2.1) and (2.2) with $\eta(t) = \lambda[||\Omega_t(w)||]$. Then $w = u$ in $\bar{Q}$. In other words, $w$ is the unique viscosity solution of (MF) and (2.2).

**Proof.** By the construction of a flat flow, there exists a sequence $M_k \to \infty$ such that

$$d_H(\Omega_t(w), \Omega_t(w_{M_k})) \to 0$$

uniformly in time as $k$ goes to infinity. Note that by Theorem 6.5, any restricted flat flow $w_{M_k}$ is the unique viscosity solution of (2.5) with $\eta(t) = \lambda[||\Omega_t(w_{M_k})||]$. Then, Lemma 2.3 implies that $w$ is a viscosity solution of (2.1) with $\eta(t) = \lambda[||\Omega_t(w)||]$. By the uniqueness of the viscosity solution of (2.1) (Theorem 2.2), we conclude that $w = u$ in $\bar{Q}$.
Thus, \( w \) is a viscosity solution of \((MF)\) with \( \eta(t) = \lambda[|\Omega_r(w)|] = \lambda[|\Omega_r(u)|] \). From Theorem 4.9, it is unique. \( \square \)

We expect that the unconstrained minimizing movement scheme gives the parallel results in Theorem 6.8. In the subsequent work [3, Proposition 3.4], we show that a viscosity solution of \((MF)\) can be approximated by a minimizing movement scheme with an admissible set \( S_{r_0, R_0} \) instead of \( A_M \) in (5.2).

7. Regularity and convergence

In this final section, we discuss the large time behavior and exponential convergence to equilibrium for solutions of (1.3), for which the corresponding energy is for \( \gamma < -\frac{1}{n} \)

\[
J(E) := \begin{cases} 
\text{Per}(E) - \frac{|E|^\gamma+1}{\gamma + 1} & \text{if } \gamma \neq -1 \\
\text{Per}(E) - \ln |E| & \text{if } \gamma = -1
\end{cases}
\]

(7.1)

Let us point out that here the dissipation of energy is crucial to guarantee that any subsequential limit of the evolving set over time variable must correspond to the unique equilibrium solution.

Let \( u \) be the unique viscosity solution of \((MF)\) and (2.2) obtained from Theorem 6.8. First, we show uniform boundedness for sets \( X_t(u) \) for all times.

**Lemma 7.1.** There exists \( R > 0 \) such that \( X_t(u) \subset B_R \) for all \( t > 0 \).

**Proof.** For simplicity we denote \( X_t(u) \) by \( \Omega_t \). Recall that the energy \( J \) decreases through the flow,

\[
J(\Omega_0) \geq J(\Omega_t) = \text{Per}(\Omega_t) - \Lambda|[\Omega_t]|
\]

By the isoperimetric inequality, it holds that

\[
\text{Per}(\Omega_t) - \Lambda|[\Omega_t]| \geq nw_n \frac{|\Omega_t|^{\frac{1}{n}}}{|\Omega_t|^{\frac{n}{n}} - \Lambda|[\Omega_t]|} = |\Omega_t|^{\frac{1}{n}} \left( nw_n \frac{\Lambda|[\Omega_t]|}{|\Omega_t|^{\frac{n}{n}}} \right),
\]

where \( w_n \) is the volume of \( B_1 \). Since \( \lambda(s) = s^\gamma \) with \( \gamma < -1/n \), we have

\[
\lim_{s \to -\infty} \frac{\Lambda[s]}{s^{\frac{1}{n}}} = \frac{n}{n - 1} \lim_{s \to -\infty} s^{\frac{1}{n}} \lambda[s] = 0
\]

which yields that \( |\Omega_t| \) is uniformly bounded. As \( \Omega_t \) has \( \rho \)-reflection, we conclude. \( \square \)

Next, we improve the regularity of \( \Omega_t(u) \). Due to the fact that the support of \( u \) is uniformly bounded (See Lemma 7.1) and is in \( S \), for all times, it follows that there exists \( s_0, L_0 > 0 \) such that for any point \( x \in \Omega_t(u) \) and \( (t - s_0^2)_+ \leq s \leq t, \partial \Omega_s(u) \) can be represented as a Lipschitz graph in \( B_{s_0}(x) \) with its Lipschitz constant less than \( L_0 \). This fact, the \( a \) priori estimates obtained in Lemma B.1 in the Appendix, and the regularization procedure given in Theorem 3.4 of Ref. [38] yields the following short-time result.

**Lemma 7.2.** Suppose that \( \Omega_0 \) has \( \rho \)-reflection. For \( t \geq 1 \), \( \partial \Omega_t(u) \) is uniformly \( C^{1,1} \). More precisely there exists \( s_0, L_0 > 0 \) such that for any point \( x \in \Omega_t(u) \) and for \( (t - s_0^2)_+ \leq s \leq t, \partial \Omega_s(u) \) can be represented as a \( C^{1,1} \) graph in \( B_{s_0}(x) \) with its \( C^{1,1} \) norm less than \( L_0 \).
Next we show the convergence of $\Omega_t(u)$ in terms of the Hausdorff distance. This is due to the fact that the ball is the unique critical point of the perimeter energy among the class of $C^{1,1}$, star-shaped sets with given volume.

**Lemma 7.3.** $J(E)$ given in (7.1) has a unique minimizer $B_{r_\infty}$ among sets in $S_{\rho}$, up to translation.

**Proof.** By the usual re-arrangement argument one can show that the minimizer is a ball (See for instance [39, Theorem 14.1]). By differentiating the energy $J(B_r)$ with respect to radius $r$ and recalling (1.3), we have

$$
\frac{dJ(B_r)}{dr} = n(n-1)w_n r^{n-2} - nw_n r^{n-1} \lambda ||B_r|| = w_n r^{n-2}(n - r\lambda ||B_r||),
$$

which is zero if and only if $r\lambda ||B_r|| = n - 1$. Note that such $r$ is unique since $r\lambda ||B_r|| = w_n r^{n+1}$ is a decreasing function on $r$. Let’s denote this by

$$
r_\infty := \left(\frac{n - 1}{w_n}\right)^{n+1}. (7.2)
$$

As we have $\lambda ||B_\rho|| > \frac{n-1}{\rho}$ due to Assumption A, it follows that $r_\infty > \rho$.

As $\frac{dJ(B_r)}{dr}$ is positive for $r > r_\infty$ and negative for $r < r_\infty$, we conclude that $r_\infty$ is a minimizer of $J(B_r)$.

We show the uniform convergence of $\Omega_t(u)$ when $\Omega_0$ has $\rho$-reflection. Note that without the assumption on the initial set, the topology of $\Omega_t(u)$ may not be preserved as described in Figure 1. Thus it is unclear whether the flow converges to a ball.

**Theorem 7.4.** Suppose that $\Omega_0$ has $\rho$-reflection. Then, the set $\Omega_t(u)$ uniformly converges to a ball as $t \to \infty$, modulo translation. More precisely

$$
\inf \{d_H(\Omega_t(u), B_{r_\infty}(x)) : x \in B_{\rho}(0)\} \to 0 \text{ as } t \to \infty,
$$

where $r_\infty$ is given in (7.2).

**Proof.** For simplicity we denote $\Omega_t(u)$ by $\Omega_t$. Recall $R > 0$ from Lemma 7.1. Define $U_n : [0, \infty) \to S_{r, R}$ by $U_n(t) := \Omega_{t + n}$. Due to Corollary 3.7 the maps $U_n$ are a sequence of equicontinuous maps into $S_{r, R}$ in Hausdorff topology, and thus along a subsequence $U_n$ locally uniformly converges to $U_\infty : [0, \infty) \to S_{r, R}$. From the standard stability theory of the viscosity solutions, it follows that $\lambda u_{\infty} - \lambda u_{\infty}^C$ is a viscosity solution of (MF). Let us recall that the energy $J(\Omega_t)$ is monotone decreasing in time and $J : S_{r, R} \to \mathbb{R}$ is bounded from below. Thus, we have

$$
\lim_{n \to \infty} J(\Omega_n) = \lim_{n \to \infty} J(U_n(t)) = \lim_{n \to \infty} J(U_n(s)) \text{ for any } t, s > 0. (7.3)
$$

Also recall from Lemma 5.3 that we have

$$
\tilde{d}^2(\Omega_t, \Omega_s) \leq_{r, R}|t - s|(|J(\Omega_t) - J(\Omega_s)|) \text{ for any } s > t > 0. (7.4)
$$
Combining (7.3) and (7.4) with the uniform convergence of $U_n$ in Hausdorff distance yields that for all $t, s > 0$
\[
\tilde{d}^2(U_\infty(t), U_\infty(s)) \leq r, r|t - s| \left( \lim_{n \to \infty} J(U_n(t)) - \lim_{n \to \infty} J(U_n(s)) \right) = 0,
\]
and thus $U_\infty$ is independent of time, and $\chi_{U_\infty}$ solves the prescribed mean curvature problem
\[
H = \lambda(|E|).
\]

The only viscosity solution of the above problem, among sets in $S_{r,R}$ with $C^{1,1}$ boundaries, is smooth due to the fact that the corresponding level set PDE in the graph setting is uniformly elliptic with Lipschitz coefficients (See also Lemma B.1). From the Soap Bubble Theorem by Alexandrov in Refs. [40, 41], it follows that the only possible solution is radial, and so we conclude that $U_\infty = B_r$.

To obtain exponential convergence, it is necessary to observe further regularity properties of $\Omega_t(u)$. To this end, note that the following holds as a consequence of Lemma 7.2 and Theorem 5.4:

**Lemma 7.5.** Suppose that $\partial \Omega_t$ has $\rho$-reflection. For any $\epsilon > 0$ and $0 < \alpha < 1$ there exists $T$ and $C > 0$ such that for any $t > T$ we have the following:

(a) There exists $x_t \in B_\rho(0)$ such that for all $x \in \partial \Omega_t(u)$, the outward unit normal $\nu_x$ at $\partial \Omega_t(u)$ satisfies
\[
|\nu_x - \frac{(x - x_t)}{|x - x_t|}| \leq 2C_\epsilon^\alpha \text{ on } x \in \partial \Omega_t(u). \tag{7.5}
\]

(b) For all $x, y \in \partial \Omega_t(u)$,
\[
\left| \left( \nu_x - \frac{(x - x_t)}{|x - x_t|} \right) - \left( \nu_y - \frac{(y - x_t)}{|y - x_t|} \right) \right| \leq C_\epsilon^{1 - \alpha} |x - y|^\alpha. \tag{7.6}
\]

**Proof.** Choose a sufficiently small $\epsilon$ depending on $\rho$. By Theorem 5.4, we can find $T > 1$ such that for any $t > T$ there exists $x_t \in B_\rho(0)$ such that
\[
B_{r_\infty - \epsilon^2}(x_t) \subset \Omega_t \subset B_{r_\infty + \epsilon^2}(x_t). \tag{7.7}
\]

Due to Lemma 7.2, the outward normal vector $\nu_x$ at $x \in \partial \Omega_t$ satisfies
\[
|\nu_x - \nu_y| \leq C_0 |x - y|, \tag{7.8}
\]
where $C_0$ is independent of $t > 1$. This means that if (7.5) fails at $x_0 \in \partial \Omega_t$ with sufficiently large $C$, let’s say $C > C_0 + 2$, then $\nu_x$ stays different from $\frac{(x_0 - x_t)}{|x_0 - x_t|}$ by at least $2\epsilon$ in $\epsilon$-neighborhood of $x_0$. As a result the boundary part $\partial \Omega_t \cap \partial B_\epsilon(x_0)$ intersects the complement of the $2\epsilon^2$-neighborhood of the tangential hyperplane of $B_r(x_t)$ at $x_0$, where $r = |x_0 - x_t|$. This violates (7.7), and thus we conclude that (7.5) holds.
To show (b), recall that due to (7.8) we have
\[
\frac{|\nu_x - \nu_y|}{|x - y|^2} \leq Cd^{1 - \alpha} \quad \text{if} \quad |x - y| \leq d \quad \text{for} \quad 0 < \alpha < 1.
\]
On the other hand the same quantity is bounded by $\frac{2C}{d^2}$ if $|x - y| \geq d$ due to (7.5). Hence choosing $d = \varepsilon$ we arrive at (7.6).

Lemma 7.5 states that after a finite time $\Omega_t$ gets arbitrarily close to a ball in $C^{1,\alpha}$ norm. For the volume preserving mean curvature flow, Ref. [15] proves that when the initial domain is close to a ball in the sense of Lemma 7.5 with sufficiently small $\varepsilon$, it converges to a unique round ball exponentially fast. Their analysis can be also applied to our problem with minor modifications:

**Theorem 7.6.** Suppose that $\Omega_0$ has $\rho$-reflection. The set $\Omega_t(u)$ exponentially converges to a unique ball of radius $r_\infty$ whose center depends only on the initial set $\Omega_0$, as $t \to \infty$.

**Proof.** Parallel (center-manifold analysis) argument as in Ref. [15], posed in the same function space, applies here since the difference between our problem and the volume preserving flow lies in the Lagrange multiplier $\lambda(t)$, which is a lower order term compared to the mean curvature term. \qed

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Proof of Proposition 5.5.

1. We will prove the case \( w_M < \psi \) at \( t = 0 \), parallel proof holds for the other case.

2. First, let us assume that \( \Omega_t(\phi) \) touches \( \Omega_t(w_M) \) from inside for the first time at \( t = t_0 \) at \( x_0 \in \Omega_{t_0}(w_M) \). Our goal is to make a perturbation of \( \Omega_t(w_M) \) using \( \Omega_t(\phi) \), which leads to a contradiction with the gradient flow property of \( w_M \). To this end, let \( \tilde{\phi} \) be a parallel translation of \( \phi \) in the direction of normal vector at \( x_0, \tilde{n}_{x_0} \), so that \( \Omega_{t_0}(\tilde{\phi}) \) has nonempty intersection with the complement of \( \Omega_t(w_M) \):

   \[
   \tilde{\phi}(x,t) := \phi\left(x - \delta\left(e + (t - t_0)\tilde{n}_{x_0}\right),t\right). \tag{A.1}
   \]

   Here, \( \delta > 0 \) will be chosen in next step. Then, \( U_t := \Omega_t(\tilde{\phi}) \setminus \Omega_t(w_M) \) is nonempty at \( t_0 \) and we have

   \[
   \left| \frac{\partial \tilde{\phi}}{\partial t} \right|_{(x_0,t_0)} \leq \max\left\{ \nabla \cdot \left( \frac{D\tilde{\phi}}{|D\tilde{\phi}|} \right)(x_0,x) + \eta(t_0), -M \right\} - \delta. \tag{A.2}
   \]

3. Let us first assume that

   \[
   \left| \frac{\partial \tilde{\phi}}{\partial t} \right|_{(x_0,t_0)} \leq \nabla \cdot \left( \frac{D\tilde{\phi}}{|D\tilde{\phi}|} \right)(x_0,x) + \eta(t_0) - \delta. \tag{A.3}
   \]

   The other case will be discussed in step 4.
For any \( \varepsilon \in \left(0, \frac{\delta}{2\pi + \pi} \right) \) where \( C \) is defined in (A.9), there exists sufficiently small \( \varepsilon \in \left(0, \frac{\delta}{2\pi} \right) \) such that (a) \( \varepsilon \leq d_H(\Omega(\phi), \Omega(\omega_M)) \) in \([t_0 - 4\varepsilon, t_0 - 2\varepsilon] \), (b) \( |U_t| < \varepsilon \) in \([t_0 - 4\varepsilon, t_0] \), and (c)

\[
\left| \frac{\partial \phi_t}{|D\phi|} (x, t) \right| \leq \nabla \cdot \left( \frac{D\phi}{|D\phi|} \right) (x, t) + \eta(t) - \frac{\delta}{4} \text{ and } \left| \frac{\partial \phi_t}{|D\phi|} (x, t) - \frac{\partial \phi_t}{|D\phi|} (x, t_0) \right| < \frac{\varepsilon}{2}
\]  

(A.4)

in \( \mathcal{N}_\varepsilon \times [t_0 - 4\varepsilon, t_0] \) where \( \mathcal{N}_\varepsilon := \{ x : d(x, U_t) < \varepsilon \text{ for all } t_0 - 4\varepsilon \leq t \leq t_0 \} \).

Note that (a) implies \( \Omega(\phi) \subset \Omega(\omega_M) \) in \([t_0 - 4\varepsilon, t_0 - 2\varepsilon] \). By definition of \( \omega_M \) and Lemma C.1, there exists sufficiently small \( h \in (0, \varepsilon) \) such that the constrained minimizing movements \( U^{h, M}_{t_0} \) starting from \( \Omega(\omega_M) \) satisfies the following relations: \( \Omega(\phi) \subset E^{h, M}_{t_0} \) in \([t_0 - 4\varepsilon, t_0 - 2\varepsilon] \) and

\[
|U^{h}_{t_0}| < \varepsilon, \lambda \left[ E^{h, M}_{t_0} \right] - \lambda \left[ \Omega(\omega_M) \right] < \varepsilon, \text{ and } d_H(U^{h}_{t_0}, U) < \varepsilon \text{ in } [t_0 - 4\varepsilon, t_0]
\]  

(A.5)

where \( U^{h}_{t_0} := \Omega(\phi) - E^{h, M}_{t_0} \).

Then, there exists \( k \in \mathbb{N} \) such that \( \Omega_{t_0 - hk} \subset E_{t_0 - hk}^{h, M} \) and \( U^{h}_{t_0 - hk} \) is nonempty. By (A.5), \( U^{h}_{t_0} \subset \mathcal{N} \) and thus (A.4) holds in \( U^{h}_{t_0} \).

4. For simplicity let us denote sets

\[
F_0 := E_{t_0 - \varepsilon}^{h, M}, \quad F_t := E_{t_0 - \varepsilon}^{h, M}, \quad \tilde{U} := U^{h}_{t_0} \text{ and } \tilde{F}_t := E_{t_0 - \varepsilon}^{h, M} \cup \tilde{U}.
\]

(L.6)

Let us show that \( \tilde{F}_t \in A_{\lambda} \). First, as \( \varepsilon \leq \frac{\eta - \eta_{\lambda}}{2} \), \( \Omega^{h}_{t_0} \subset S_0 \). Moreover, \( E_{t_0}^{h} \subset S_0 \), and thus \( \tilde{F}_t \in S_0 \). On the other hand, since \( \tilde{F}_t \subset F_t \),

\[
d_H(\partial(F_t \cap F_0), \partial F_0) \leq d_H(\partial(F_t \cap F_0), \partial F_0) \leq M_h
\]  

(A.7)

Next, let us show that \( I_h(F_t; F_0) > I_h(\tilde{F}_t, F_0) \). Let us write out the difference of the energies:

\[
I_h(F_t; F_0) - I_h(\tilde{F}_t, F_0) = \left( \text{Per}(F_t) - \text{Per}(\tilde{F}_t) \right) + \left( -\lambda |F_t| + \lambda |\tilde{F}_t| \right) + \frac{1}{h} \left( \tilde{d}^2(F_0, F_0) - \tilde{d}^2(\tilde{F}_t, F_0) \right).
\]

Let us estimate the first term

\[
I_1 := \text{Per}(F_t) - \text{Per}(\tilde{F}_t) \geq \int_{\partial F_t \setminus \partial \tilde{F}_t} d\sigma - \int_{\partial \tilde{F}_t \setminus \partial F_t} d\sigma.
\]

Let \( \tilde{n} \) be the outward normal vector at each point of \( \partial F_t / \partial \tilde{F}_t \) and \( \partial \tilde{F}_t / \partial F_t \). Note that, \( -\frac{D\phi}{|D\phi|} \cdot \tilde{n} \leq 1 \) on \( \partial F_t / \partial \tilde{F}_t \) and \( -\frac{D\phi}{|D\phi|} \cdot \tilde{n} = 1 \) on \( \partial \tilde{F}_t / \partial F_t \), and thus

\[
I_1 \geq \int_{\partial F_t \setminus \partial \tilde{F}_t} -\frac{D\phi}{|D\phi|} (x, t_1) \cdot \tilde{n} d\sigma - \int_{\partial \tilde{F}_t \setminus \partial F_t} -\frac{D\phi}{|D\phi|} (x, t_1) \cdot \tilde{n} d\sigma = \int_{\partial \tilde{F}_t \setminus \partial F_t} \frac{D\phi}{|D\phi|} (x, t_1) \cdot \tilde{n} d\sigma.
\]

Note that outward normal of \( \tilde{U} \) is opposite to that of \( \partial F_t / \partial \tilde{F}_t \). Finally, by divergence theorem, we conclude that

\[
I_1 \geq \int_{\tilde{F}_t} \nabla \cdot \frac{D\phi}{|D\phi|} (x, t_1) dx.
\]  

(A.8)

Next, since \( \Lambda(\cdot) \) is \( C^{1,1} \), we have

\[
I_2 := -\Lambda |F_t| + \Lambda |\tilde{F}_t| \geq \lambda |F_t| \tilde{U} - C|\tilde{U}|^2 \quad \text{where } C := \sup_{|B_0| \leq \varepsilon \subset \tilde{B}_h} |\lambda'(z)|.
\]  

(A.9)

Lastly we have

\[
I_3 := \frac{1}{h} \tilde{d}^2(F_0, F_0) - \frac{1}{h} \tilde{d}^2(\tilde{F}_t, F_0) = -\frac{1}{h} \int_{U} \delta(x, F_0) dx,
\]  

(A.10)
where $sd(x, \Omega)$ is the signed distance function given in (2.4). Since $\Omega_{t-h}(\bar{\phi}) \subset F_0$, it holds that $sd(x, F_0) \leq sd(x, \Omega_{t-h}(\bar{\phi}))$ for all $x \in \mathbb{R}^n$. Moreover, since (A.4) holds in $U$, we have

$$I_3 \geq -\frac{1}{h} \int_U sd(x, \Omega_{t-h}(\bar{\phi})) dx \geq -\int_U \left[ \frac{\phi_t}{|D\phi|} (x, t_1) + \epsilon dx, \right]$$  

(A.11)

Putting all terms together, we have

$$I_4 := I_h(F_h; F_0) - I_h(\bar{F}_h; F_0) \geq \int_U \left( \nabla \cdot \frac{D\phi}{|D\phi|} (x, t_1) - \frac{\phi_t}{|D\phi|} (x, t_1) + \lambda ||F_h|| \right) dx - \epsilon |\bar{U}| - C|\bar{U}|^2,$$

Applying (A.4) and (A.5), it holds that

$$I_4 \geq \int_U \left( \frac{\delta}{4} - \lambda [\Omega_t(w_M)] + \lambda ||F_h|| \right) dx - \epsilon |\bar{U}| - C|\bar{U}|^2 \geq |\bar{U}| \left( \frac{\delta}{4} - 2\epsilon - C|\bar{U}| \right) > 0,$$

where the last inequality follows from the fact that $\epsilon < \frac{\delta}{8+4\epsilon}$ and $|\bar{U}| \leq \epsilon$.

5. Lastly consider the case

$$\frac{\phi_t}{|D\phi|} (x_0, t_0) \leq -M - \frac{\delta}{2}$$

in the Eq. (A.2).

By parallel argument in step 2–4, for any $\epsilon > 0$, we can choose $\epsilon$ and $h$ sufficiently small in the definition of $\bar{\phi}$ in (A.1) and the constrained minimizing movement $E^{h,M}_t$ such that for $F_0$ and $F_h$ as defined in step 4, $\Omega_{t-h}(\bar{\phi})$ is contained in $F_0$, $\Omega_t(\phi) \cap F_h$ is nonempty, and $\bar{\phi}$ satisfies

$$\frac{\phi_t}{|D\phi|} (x, t) \leq -M - \frac{\delta}{2}$$

and

$$\left| \frac{\phi_t}{|D\phi|} (x, t) - \frac{\phi_t}{|D\phi|} (x, t_0) \right| < \frac{\epsilon}{2},$$

(A.12)

for $x \in \Omega_t(\bar{\phi}) \setminus F_h$ and $t \in [t_1 - h, t_1]$. Note that (A.7) holds for $F_h := F_h \cup \Omega_t(\phi)$.

Let $x^*$ be a point in $(\partial F_h) \setminus (\partial F_h)$. As $\Omega_{t-h}(\bar{\phi})$ is contained in $F_0$ and $\Omega_t(\phi)$ has a negative normal velocity, the point $x^* \in \Omega_{t-h}(\bar{\phi}) \subset \Omega_{t-h}(\hat{\phi}) \subset F_0$. Thus, $x^*$ is on $\partial F_h \cap F_0$, and

$$d_H(\partial(F_h \cap F_0), \partial F_0) \geq \sup_{x \in \partial(F_h \cap F_0)} d(x, \partial F_0) \geq d(x^*, \partial F_0).$$

Moreover, as $x^* \in \Omega_{t-h}(\bar{\phi}) \subset F_0$, and (A.12), we have

$$d(x^*, \partial F_0) \geq d(x^*, \partial \Omega_{t-h}(\hat{\phi})) > \left( M + \frac{\delta}{2} \right) h - \frac{\epsilon}{2} h > Mh,$$

(A.13)

and this contradicts (A.7).

\section*{Appendix B: Regularity}

In this section, we use notation from Refs. [16, 17]. Let $\partial \Omega_0$ be represented locally by some diffeomorphism, $F_0 : U \subset \mathbb{R}^{n-1} \to F_0(U) \subset \partial \Omega_0$. Then, (1.5) can be formulated into

$$\begin{cases} 
\frac{\partial}{\partial t} F(x, t) = (\eta(t) - H(x, t)) \cdot \bar{n}(x, t), & \text{for } x \in U, t \geq 0 \\
F(\cdot, 0) = F_0 
\end{cases}$$

(B.1)

The induced metric, its inverse matrix, and the second fundamental form are denoted by $\{g_{ij}\}, \{g^{ij}\}$ and $A = \{a_{ij}\}$. Note that $g_{ij}$ and $h_{ij}$ can be computed as follows:
\[ g_{ij} = \left( \frac{\partial F}{\partial x_i} \right) \left( \frac{\partial F}{\partial x_j} \right), \quad h_{ij} = - \left( \bar{h}, \frac{\partial^2 F}{\partial x_i \partial x_j} \right), \]  

(B.2)

We use the following notion for the trace of the second fundamental from,

\[ H = g^{ij} h_{ij}, \quad |A|^2 = g^{ij} g^{kl} h_{ik} h_{lj}, \quad \text{and} \quad C = g^{ij} g^{kl} g^{mn} h_{ik} h_{lm} h_{nj}. \]

The following lemma is parallel to Theorem 3.1 in Ref. [38] and Lemma 3.2 in Ref. [42].

**Lemma B.1.** Let \( u(x, t) \) be a solution of

\[ \frac{\partial u}{\partial t} = \sqrt{1 + |Du|^2} \text{ div} \left( \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) + \eta(t) \sqrt{1 + |Du|^2} \right) \]  

in \( Q_R = B_R(0) \times [0, R^2] \). Then for \( 0 < t \leq R^2 \), we have the interior gradient estimate

\[ |D^2 u|^2(0, t) \leq K \left( 1 + \sup_{Q_t} |Du|^6 \right) \left( \frac{1}{R^2} + \frac{1}{t} \right) \]  

where the constant \( K = K(||u||_{L^\infty(Q_0)}, ||\eta||_{L^\infty([0, R^2]))}. \)

**Proof.** First, by Corollary 1.2 in Ref. [17], it holds that \( \left( \frac{\partial}{\partial t} - \Delta \right) |A|^2 = -2|\nabla A|^2 + 2|A|^4 - 2\eta C. \)

Let us denote \( v = \sqrt{1 + |Du|^2} \). As Lemma 1.1 in Ref. [38] and Lemma 3.2 in Ref. [42], the function \( v \) satisfies the equation

\[ v_t = \Delta v - |A|^2 v - \frac{2}{v} \nabla v^2. \]  

(B.5)

Let us define \( \phi(r) := \frac{r}{1 - \delta r} \) and \( g := |A|^2 \phi(v^2) \). Then, by the direct computation motivated from Lemma 3.2 in Ref. [42] and Theorem 3.1 in Ref. [38], we have

\[ \mathcal{I}_1 := \left( \frac{\partial}{\partial t} - \Delta \right) g = (-2|\nabla A|^2 + 2|A|^4 - 2\eta C) \phi(v^2) + \left( -|A|^2 v - \frac{2}{v} |\nabla v|^2 \right) \times \frac{2|A|^2}{(1 - \delta v^2)^2} \]

Note that \( \delta \phi(\phi^2) = \frac{1}{1 - \delta \phi^2} = 1 \), it holds that

\[ \mathcal{I}_1 = -2\delta g^2 - 2|\nabla A|^2 \phi(\phi^2) + \frac{-4|A|^2 |\nabla v|^2}{(1 - \delta v^2)^2} - 2\eta C \phi(\phi^2), \]

\[ = -2\delta g^2 - 2|\nabla A|^2 \phi(\phi^2) + \left( -2\delta |\nabla v|^2 g + \frac{-2|A|^2 |\nabla v|^2}{(1 - \delta v^2)^2} \right) + \frac{-2|A|^2 |\nabla v|^2}{(1 - \delta v^2)^2} - 2\eta C \phi(\phi^2). \]

Now, choose \( \delta := \frac{1}{2} \inf_{Q_t} v^2. \) Applying Young’s inequality and \( \nabla g = 2A \nabla A \phi(\phi^2) + 2v|A|^2 \phi'(\phi^2) \nabla \phi \),

\[ \phi v^{-3} (\nabla g, \nabla \phi) \leq |\nabla A|^2 \phi(\phi^2) + \frac{|A|^2 |\nabla v|^2}{1 - \delta v^2} + \frac{|A|^2 |\nabla v|^2}{(1 - \delta v^2)^2}. \]

Finally, from Young’s inequality and \( \phi(\phi^2) \geq v^2 \), the last term of \( \mathcal{I}_1 \) is bounded by

\[ | - 2\eta C \phi(v^2) | \leq 2K_1 g^3 / 2|v| \leq \delta g^2 + \frac{K_1^2 g^2}{\delta} \]  

(B.6)

for some constant \( K_1 := K_1(||\eta||_{L^\infty([0, R^2]))} > 0. \)
Putting all together, it holds that
\[
\left( \frac{\partial}{\partial t} - \Delta \right) g \leq -2\delta g^2 + \frac{-2\delta|\nabla v|^2 g}{(1 - \delta v^2)} - 2\phi v^{-3} \langle \nabla g, \nabla v \rangle + \delta g^2 + \frac{K_0^2 g v^2}{\delta}.
\]

The rest of the proof is parallel to Theorem 3.1 in Ref. [38] and Lemma 3.2 in Ref. [42]. Taking a cutoff function as in Ref. [38], \( \psi = \psi(r) = (R^2 - r^2) \) where \( r = r(X, t) \) satisfies \( r(X, 0) \leq \frac{R^t}{2}, \)

\[
\left| \left( \frac{\partial}{\partial t} - \Delta \right) r \right| \leq K_2 \quad \text{and} \quad |\nabla r|^2 \leq K_2 r
\]
on \( X = F(x, t) \) for some constant \( K_2 = K_2(||u||_{L^\infty(\Omega_d)}, ||\eta||_{L^\infty(\partial(\Omega_d))}) > 0. \) It holds that
\[
\left( \frac{\partial}{\partial t} - \Delta \right) [tg\psi] \leq -\delta g^2 \psi t - \tilde{b} \cdot \nabla (tg\psi) + c \left( \left( 1 + \frac{1}{\delta v^2} \right) r + R^2 \right) t g + g\psi + \frac{K_0^2 g v^2}{\delta} \psi t
\]
where \( \tilde{b} = \tilde{b}(v, \psi, \phi) \) and \( c = c(K_2) \) is a constant (See Eqs. (21) and (23) in Ref. [38] for details).

Let \( t_0 \) be a maximizer of \( m(T) := \sup_{0 \leq t \leq T} \sup_{r(x, t) \leq R} |tg\psi| \). Then, by parallel computation in Theorem 3.1 in Ref. [38], we conclude that
\[
\delta g^2 \psi t_0 \leq c \left( \left( 1 + \frac{1}{\delta v^2} \right) r + R^2 \right) t_0 g + g\psi + \frac{K_0^2 g v^2}{\delta} \psi t_0.
\]
Note that \( \frac{R^t}{2} \leq \psi \leq R^t \) at \( t = 0, \phi(v^2) \geq v^2 \geq 1, \) and \( v^2 \leq \frac{1}{\delta} \). Thus, it holds that
\[
|A|^2 \leq \frac{2}{\delta R^t} \left( c R^t \left( 2 + \frac{1}{\delta} \right) + \frac{R^t}{T} + \frac{K_0^2 R^t}{\delta^2} \right) \leq K \left( 1 + \frac{1}{\delta^2} \right) \left( \frac{1}{T} + \frac{1}{R^t} \right),
\]
where \( K = K(K_1, c) \), thus we conclude.

\[\square\]

Appendix C: Geometric properties

**Lemma C.1.** [5, Lemma 23 and 24] Let us consider two sets \( \Omega_1, \Omega_2 \in \mathbb{S}_{r, R} \) for \( R > r > 0. \) Then the following holds:
\[
\begin{align*}
&d_H(\Omega_1, \Omega_2) \leq d_H(\partial \Omega_1, \partial \Omega_2), \quad d_H(\partial \Omega_1, \partial \Omega_2) \leq r, \quad d_H(\Omega_1, \Omega_2), \quad |\Omega_1 \Delta \Omega_2| \leq r, \quad d_H(\Omega_1, \Omega_2), \quad \\
&|\hat{d}(\Omega_1, E) - \hat{d}(\Omega_2, E)|, |\hat{d}(E, \Omega_1) - \hat{d}(E, \Omega_2)| \leq r, \quad d_H(\Omega_1, \Omega_2) \text{ for any } E \in \mathbb{S}_{r, R}.
\end{align*}
\]

**Lemma C.2.** Let us consider two sets \( \Omega_1, \Omega_2 \in \mathbb{S}_{r, R} \) for \( R > r > 0. \) Then the following holds:
\[
d_H(\Omega_1, \Omega_2)^{n+1} \leq w_1^{-1} \left( \frac{4R}{r} \right)^{n+1} \hat{d}(\Omega_1, \Omega_2) \text{ and } d_H(\Omega_1, \Omega_2)^{n+1} \leq w_1^{-1} \left( \frac{4R}{r} \right)^{n+1} \hat{d}(\Omega_2, \Omega_1)
\]

**Proof.** Due to the first inequality of (C.1) in Lemma C.1, it is enough to show that
\[
d_H(\partial \Omega_1, \partial \Omega_2)^{n+1} \leq w_1^{-1} \left( \frac{4R}{r} \right)^{n+1} \hat{d}(\Omega_1, \Omega_2) \text{ and } d_H(\partial \Omega_1, \partial \Omega_2)^{n+1} \leq w_1^{-1} \left( \frac{4R}{r} \right)^{n+1} \hat{d}(\Omega_2, \Omega_1).
\]
Without loss of generality, let us assume that \( d_H(\partial \Omega_1, \partial \Omega_2) = \sup_{x \in \partial \Omega_1} d(x, \partial \Omega_2) \). Since \( \partial \Omega_1 \) and \( \partial \Omega_2 \) are compact, there exists \( x_1 \in \partial \Omega_1 \) and \( x_2 \in \partial \Omega_2 \) such that \( \sup_{x \in \partial \Omega_1} d(x, \partial \Omega_2) = d(x_1, \partial \Omega_2) = |x_1 - x_2| \). Since \( \Omega_2 \in \mathbb{S}_r \), there exists \( y \in \partial \Omega_2 \) such that \( x_1 \) and \( y \) are parallel. Note that we have \( d(x_1, \partial \Omega_2) \leq |x_1 - y| \). We argue for the case \( |x_1| < |y| \). Since \( x_1 \in \partial \Omega_1 \) and \( y \in \partial \Omega_2 \), there exists an exterior cone \( EC(x_1, r) \) and an interior cone \( IC(y, r) \) given in (3.3) and (3.5) such that \( EC(x_1, r) \cap IC(y, r) \subset \Omega_2 \setminus \Omega_1 \). Note that, for \( \theta \in (0, \frac{\pi}{2}) \) such that \( \sin(\theta) = \frac{1}{r} \), we have
\((x_1 + C(x_1, \theta)) \cap (y + C(-y, \theta)) \subset EC(x_1, r) \cap IC(y, r)\).

Note also that there is \(\delta = \delta(r, R)\) such that
\[
B_{2\delta(x_1-y)}(x_1 + y)/2 \subset (x_1 + C(x_1, \theta)) \cap (y + C(-y, \theta)).
\]
Specifically, as \(x_1\) and \(y\) are parallel, the above inequality holds for
\[
\delta(r, R) = \frac{\sin(\theta)}{4} = \frac{r}{4R}.
\]
Then, it holds that
\[
\tilde{d}^2(\Omega_1, \Omega_2) \geq \int_{\Omega_1 \Delta \Omega_2} d(x, \partial \Omega_2) dx \geq \int_{B_{\delta(x_1-y)}((x_1 + y)/2)} \delta|x_1 - y| dx = w \delta^{n+1} |x_1 - y|^{n+1}.
\]
The same inequality holds for \(\tilde{d}^2(\Omega_2, \Omega_1)\) and thus we can conclude. Lastly, if \(|x_1| < |y|\), then we can apply the parallel arguments in \((x_1 + C(-x_1, \theta)) \cap (y + C(y, \theta)) \subset \Omega_1 \setminus \Omega_2\).

**Lemma C.3.** [5, Lemma 24] The metric space \((S_r, d_H)\) is compact:

1. Let us consider a sequence of sets \(F_k \subset S_r\) for \(k \in \mathbb{N}\). Then \(\{F_k\}_{k \in \mathbb{N}}\) has a subsequence that converges and any subsequential limit is also in \(S_r\).
2. Let \(I\) be a compact interval in \([-\infty, \infty)\). Let us consider a sequence of evolving sets \(F_k(\cdot) : I \rightarrow S_r\) for \(k \in \mathbb{N}\). Assume that \(\{F_k\}_{k \in \mathbb{N}}\) is equicontinuous in time, that is for all \(\varepsilon > 0\), there exists \(\delta > 0\) such that
   \[
d_H(F_k(t), F_k(s)) \leq \varepsilon,
   \]
   for all \(|t - s| \leq \delta\) and \(k \in \mathbb{N}\). Then \(\{F_k\}_{k \in \mathbb{N}}\) has a subsequence that converges uniformly on \(I\) to an evolving set \(F(\cdot) : I \rightarrow S_r\).

**Lemma C.4.** For \(r > 0\) and \(x \in \mathbb{R}^n\) such that \(|x| \geq r\), it holds that
\[
IC(x, r) = \{zx + (1-z)y : z \in (0,1), y \in B_r(0)\}.
\]
Here, \(IC(\cdot, \cdot)\) is given in (3.3).

**Proof.** The proof is based on the geometry of interior cones describe in Figure 3. Let us show that
\[
\mathcal{N} := \{zx + (1-z)y : z \in (0,1), y \in B_r(0)\} \subset IC(x, r).
\]
For \(z \in \mathcal{N}\), we fix \(z \in (0,1)\) and \(y \in B_r(0)\) satisfying \(z := zx + (1-z)y\). If \(z \in B_r(0)\), then it can be checked that \(z \in IC(x, r)\). Let us assume that \(z \in B_r(0)^C\) and show that
\[
z \in (x + C(-x, \theta_{x,r})) \cap C\left(x, \frac{\pi}{2} - \theta_{x,r}\right).
\]
Note that \(x + C(-x, \theta_{x,r})\) is a convex set and \(y \in B_r(0) \subset x + C(-x, \theta_{x,r})\) (See Figure 3) and thus \(z \in x + C(-x, \theta_{x,r})\). It remains to show that
\[
z \in C\left(x, \frac{\pi}{2} - \theta_{x,r}\right).
\]
As \(y \in B_r(0)\) and \(z \in B_r(0)^C\), there two intersection points \(z_1\) and \(z_2\) between \(\partial B_r(0)\) and the line passing through \(y\) and \(z\) such that
\[
z_i := \alpha_i (x + (1-\alpha_i)y) \in \partial B_r(0)\] for \(i = 1, 2\) and \(|x - z_1| < |x - z_2|\).
for some \( z_1 \in (0, x] \) and \( z_2 < 0 \). As \( z_1 \) and \( z_2 \) are intersection points between a circle and a line, it holds that

\[
|x|^2 - r^2 = |x - z_1||x - z_2| \quad \text{and thus} \quad |x - z_1| < \sqrt{|x|^2 - r^2}.
\]

As \( x \in C(x, \frac{\pi}{2} - \theta_{x,r}) \) and \( d(x, \partial C(x, \frac{\pi}{2} - \theta_{x,r})) = \sqrt{|x|^2 - r^2} \) (See Figure 3), we conclude that \( z_1 \in C(x, \frac{\pi}{2} - \theta_{x,r}) \). As \( C(x, \frac{\pi}{2} - \theta_{x,r}) \) is a convex set, we conclude (C.7) and thus (C.5) holds.

The opposite relation can be shown by similar geometric arguments. As \( B_r(0) \subseteq \mathcal{N} \), it suffices to show that

\[
z \in \mathcal{N} \quad \text{for all} \quad z \in \left\{ (x + C(-x, \theta_{x,r})) \cap C \left( x, \frac{\pi}{2} - \theta_{x,r} \right) \right\} \setminus B_r(0).
\]

Consider a line passing through \( x \) and \( z \), we can find a point \( y \in B_r(0) \) such that \( z = zx + (1 - x)y \) for some \( x \in (0, 1) \). \( \square \)

**Lemma C.5.** For \( x, z \in \mathbb{R}^n \) and \( r > c > 0 \), assume that \( |x| \geq r \) and \( |z| < c \). Then, it holds that

\[
IC(x + z, r - c) \subseteq IC(x, r) + z.
\]

Here, \( IC(\cdot, \cdot) \) is given in (3.3).

**Proof.** We claim that for \( x \in (0, 1) \) and \( y \in B_{r-c}(0) \), it holds that

\[
z(x + z) + (1 - x)y \in IC(x, r) + z.
\]

Note that

\[
z(x + z) + (1 - x)y - z = zx + (1 - x)(y - z).
\]

As \( y \in B_{r-c}(0) \) and \( z \in B_c(0) \), we have \( y - z \in B_r(0) \). From Lemma C.4, we have (C.9). From Lemma C.4 again, we conclude (C.8). \( \square \)

**Lemma C.6.** Let us consider two sets \( \Omega_1, \Omega_2 \subseteq S_{r,R} \) for \( R > r > 0 \). Then the following holds:

\[
\sup_{x \in \partial \Omega_1} d(x, \partial \Omega_1) \leq \frac{R}{r} \sup_{x \in \partial \Omega_2} d(x, \partial \Omega_2). \tag{C.10}
\]

**Proof.** If \( \sup_{x \in \partial \Omega_1} d(x, \partial \Omega_1) = 0 \), then (C.10) holds. We suppose that \( \sup_{x \in \partial \Omega_1} d(x, \partial \Omega_1) > 0 \). As \( \Omega_2 \subseteq S_{r,R} \), there exists \( x_2 \in \partial \Omega_2 \) such that

\[
\sup_{x \in \partial \Omega_2} d(x, \partial \Omega_1) = d(x_2, \partial \Omega_1) =: l > 0.
\]

As a consequence, we have

\[
B_1(x_2) \subseteq \Omega_1^C \quad \text{and} \quad B_l(x_2) \subseteq \Omega_1. \tag{C.12}
\]

Let us assume the former one. As \( \Omega_1 \subseteq S_{r,R} \), there exists \( x_1 \in \partial \Omega_1 \) such that \( x_1 \) is in the line segment between the origin and \( x_2 \). From (C.12), \( |x_1 - x_2| \geq l \). From the interior cone property of \( S_{r,R} \), in Lemma 3.4, it holds that

\[
d(x_1, \partial \Omega_1) \geq d(x_1, \partial IC(x_2, r)) \geq \frac{lr}{R} \tag{C.13}
\]

and we conclude (C.10). The latter case in (C.12) can be shown by the parallel arguments. \( \square \)