MULTIPLE COVERS AND THE INTEGRALITY CONJECTURE FOR RATIONAL CURVES IN CALABI-YAU THREEFOLDS

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Abstract. We study the contribution of multiple covers of an irreducible rational curve $C$ in a Calabi-Yau threefold $Y$ to the genus 0 Gromov-Witten invariants in the following cases.

1. If the curve $C$ has one node and satisfies a certain genericity condition, we prove that the contribution of multiple covers of degree $d$ is given by

$$\sum_{n|d} \frac{1}{n^3}$$

2. For a smoothly embedded contractible curve $C \subset Y$ we define schemes $C_i$ for $1 \leq i \leq l$ where $C_i$ is supported on $C$ and has multiplicity $i$, the number $l \in \{1, \ldots, 6\}$ being Kollár’s invariant “length”. We prove that the contribution of multiple covers of $C$ of degree $d$ is given by

$$\sum_{n|d} \frac{k_{d/n}}{n^3}$$

where $k_i$ is the multiplicity of $C_i$ in its Hilbert scheme (and $k_i = 0$ if $i > l$).

In the latter case we also get a formula for arbitrary genus (Theorem 1.5).

These results show that the curve $C$ contributes an integer amount to the so-called instanton numbers that are defined recursively in terms of the Gromov-Witten invariants and are conjectured to be integers.

1. Motivation and Results

From string theory and M-theory, physicists insist on the existence of instanton numbers. Let $Y$ be a Calabi-Yau threefold, and $\beta \in H_2(Y, \mathbb{Z})$. Then there is supposed to be an integer invariant $n_\beta$, an instanton number, so that the genus 0 Gromov-Witten invariant is given by

$$N_\beta = \sum_{n|\beta} \frac{n_{\beta/n}}{n^3}. \quad (1)$$

A slightly more precise version of this conjecture is stated as Conjecture 7.4.5 in [10]. Since Equation (1) determines the $n_\beta$’s recursively in terms of the $N_\beta$’s, one can regard the equation as the definition of $n_\beta$. The integrality of $n_\beta$ has been verified empirically for $Y$ where the Gromov-Witten invariants are known. However, there is no $Y$ for which the integrality of $n_\beta$ has been proven to hold for all $\beta$. M-theory

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suggests an approach to an intrinsic definition of the $n_\beta$ in terms of sheaves [13], but this has not yet been made precise mathematically.

For smoothly embedded $\mathbb{P}^1$ with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, it has been proven by Aspinwall-Morrison, Voisin, Kontsevich, Manin, Pandharipande, and Lian-Liu-Yau (see [10] [1] [20] [17] [19] [26]) that the contribution to the genus 0 Gromov-Witten invariant of degree $d$ multiple covers is $d^{-3}$. Thus if we imagine that all the rational curves in $Y$ are smoothly embedded with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, then $n_\beta$ is precisely the number of rational curves in the class $\beta$. However this assumption is too optimistic. For example, even if we assume Clemens’ conjecture, a generic quintic 3-fold always has nodal rational curves in degree 5 by Vainsencher [25]. Therefore, to understand the physicists’ instanton numbers and/or to extract more precise enumerative information from the Gromov-Witten invariants, we need to understand how non-generic curves in $Y$ contribute to the invariants.

This problem is open in almost any reasonable situation other than $(−1, −1)$ curves. For example: isolated smooth rational curves with a non-generic normal bundles, and generic nodal curves. In this paper we treat the case of any contractable smooth rational curve (which can have normal bundle $(−1, −1)$, $(0, −2)$, or $(1, −3)$) and the case of a generic irreducible rational curve with one node.

To state our results we make the following definitions.

A $(−1, −1)$ curve has no infinitesimal deformations so the curve is rigid; in fact, no multiple of the curve has infinitesimal deformations so we say the curve is super-rigid. For nodal curves, this is the notion of rigidity that we need. It is an adaptation of a concept due to Pandharipande:

**Definition 1.1** (c.f. Pandharipande [21]). A curve $C$ in a projective variety $Y$ is called $g$-super-rigid if for every map $f : C' \to Y$ with $C'$ a projective curve of arithmetic genus $g$ and $f$ a local immersion, $f(C') = C$ has no infinitesimal deformations as a stable map. A curve that is $g$-super-rigid for all $g$ is simply called super-rigid.

Super-rigidity is a genericity condition for nodal curves in the following sense. One can weaken the super-rigidity condition by only requiring that the defining property holds for maps $f$ with degree at most $N$ onto $C$. Note that for each fixed $N$, this is an open condition in any family of pairs $(Y, C)$ and so super-rigidity is an intersection of a countable number of open conditions. In practice, we will only need that the condition holds for fixed $N$. Our result for nodal curves is the following.

**Theorem 1.2.** Let $C \subset Y$ be an irreducible 0-super-rigid rational curve with exactly one node in a 3-fold $Y$ with $K_Y$ trivial in a neighborhood of $C$. Then the

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1 Aspinwall and Morrison’s calculation predates the definition of Gromov-Witten invariants and so their argument must be regarded as incomplete in the present context. It would be interesting to find a direct comparison between their calculation and the calculations in Gromov-Witten theory. We should also warn the reader that the theorem in Voisin [26] computes a contribution of $d^{-3}$ for multiple covers of immersed $\mathbb{P}^1$’s. As our results show, the contribution of the image curve to the Gromov-Witten invariant need not be $d^{-3}$ if the $\mathbb{P}^1$ is not embedded. Voisin has only considered the contribution of multiple covers which factor through the normalization, so her computation does not compute the full contribution to the Gromov-Witten invariant, as asserted in her abstract.
contribution of degree $d$ multiple covers of $C$ to the genus 0 Gromov-Witten invariant is

$$\sum_{n|d} \frac{1}{n^3}.$$

Unlike the case of a smooth curve, the moduli space of stable maps that multiply-cover a nodal curve has a number of different path components arising from the possible “jumping” behavior of the map at the node. The basic phenomenon is illustrated below for degree two maps. The moduli space has different components depending on whether the map factors through the normalization or not. In the figure below, points labeled by $A$ and $B$ are mapped to corresponding points labeled $A$ and $B$ and the normalization map identifies $A$ and $B$ to the nodal point. Notice that the bottom map cannot be factored through the normalization (the dashed map doesn’t exist) even though the map to the nodal curve is well defined.

Thus we see that the moduli space of genus 0 stable maps that double cover $C$ consists of two connected components (of different dimensions) depending on whether the map factors through the normalization or not. See Proposition 3.2 for the case of arbitrary degree.

An isolated smoothly embedded rational curve $C$ in a Calabi-Yau 3-fold $Y$ can have a normal bundle other than $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. In [23], Reid proves that if the normal bundle is $\mathcal{O} \oplus \mathcal{O}(-2)$, then the curve is isolated if and only if the curve contracts; that is, there exists a birational morphism $f : Y \to X$ with $f(C) = p \in X$ that is an isomorphism of $Y \setminus C$ with $X \setminus p$. If a rational curve contracts, it must be isolated, the normal bundle must be $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $\mathcal{O} \oplus \mathcal{O}(-2)$, or $\mathcal{O}(1) \oplus \mathcal{O}(-3)$.
and $p$ is a compound DuVal (cDV) singularity\textsuperscript{4}. The singularity types of the
generic hyperplane section through $p$ were classified by Katz-Morrison \[10\] who
found that they are determined by Kollár’s invariant “length”.

**Definition 1.3** (Kollár [9] page 95). Let $C \subset Y$ be a smooth rational curve in
a 3-fold $Y$ with $K_Y$ trivial in a neighborhood of $C$. Suppose that there exists a
contraction $\pi : Y \to X$ with $\pi(C) = p \in X$ and $\pi$ is an isomorphism of $Y\setminus C$
ono onto $X\setminus p$. Define the length of $p$ to be the length (at the generic point of $C$) of
the scheme supported on $C$ with structure sheaf $\mathcal{O}_Y/\pi^{-1}(\mathfrak{m}_{X,p})$ where $\mathfrak{m}_{X,p}$ is the
maximal ideal of the point $p$.

The length of $p$ is a number $l \in \{1, \ldots, 6\}$ and if $l = 1$ then $C$ has normal
bundle $N_{C/Y}$ isomorphic to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ or $\mathcal{O} \oplus \mathcal{O}(-2)$ and if $l > 1$ then
$N_{C/Y} \cong \mathcal{O}(1) \oplus \mathcal{O}(-3)$. When $l > 1$ we can define a sequence of higher order
neighborhoods of $C$:

**Definition 1.4.** Let $C \subset Y$ be as in Definition \[1.3\] and let $l$ be the length of $p$.
Let $X_0 \subset X$ be a generic hyperplane section through $p$ and let $Y_0$ be the proper
transform of $X_0$. For $i = 1, \ldots, l$ define $C_i$ to be the subscheme of $Y_0$ defined by
the symbolic power $\mathcal{I}^{(i)}$ of the ideal sheaf $\mathcal{I}$ defining $C$ in $Y_0$.

Note that $C_1 = C$ and that $C_l$ is the scheme used in the definition of length.

The crucial property of $C_i$ is that under a generic deformation of $Y$, $C_i$ deforms to
$k_i$ smoothly embedded $(-1, -1)$ curves in the homology class $i[C]$ where $k_i$ is the
multiplicity of $C_i$ in its Hilbert scheme (see Proposition \[2.3\] and Lemma \[2.8\]).

Our main result for embedded smooth curves is the following.

**Theorem 1.5.** Let $C \subset Y$ be a smoothly embedded contractable rational curve in
a 3-fold $Y$ with $K_Y$ trivial in a neighborhood of $C$. Then the contribution of degree $d$
multiple covers of $C$ to the genus $g$ Gromov-Witten invariants is

$$\sum_{n|d} \frac{|B_{2g}| n^{2g-3}}{2g \cdot (2g-2)!}$$

where $B_{2g}$ is the $2g$-th Bernoulli number (c.f. \[11\] Theorem 3) and $k_i$ is the multi-
licity of $C_i$ in its Hilbert scheme if $i \leq l$ and $k_i = 0$ otherwise. In particular, the

genus 0 formula is

$$\sum_{n|d} \frac{k_{d/n}}{n^3}$$

For example, if $l = 1$ then $C$ is a $(-1, -1)$ or $(0, -2)$ curve and $k_1$ coincides with
Reid’s invariant “width” (Definition 5.3 of \[22\]).

Our technique for proving Theorem \[1.2\] is to relate the invariants of a super-
rigid 1-nodal curve to the invariants of a contractable chain of rational curves via
a method similar to that used by Bryan-Leung to study contributions of multi-
ple covers of nodal curves in $K3$ surfaces (Section 5 of \[1\]). The invariants of a
contractable chain of curves (as well as the contractable curve in Theorem \[1.5\])
are then computed by deforming a neighborhood of the curve. The deformation
is constructed and analyzed by employing the beautiful subject of deformations

\textsuperscript{4}One might have hoped that all isolated curves contract, but this is false: Clemens has an
example of an isolated $(2, -4)$ curve where no multiple of the curve deforms \[6\] and Jiménez has
an example of a non-contractable, isolated $(1, -3)$-curve \[4\].
DuVal singularities and their simultaneous (partial) resolutions. The subject goes back to Brieskorn \cite{4,5,6} and has been used to study 3-fold singularities with small resolutions by Pinkham \cite{22} and others (c.f. \cite{12,16,23}).

It is crucial that the deformations we construct in this way can be blown down to an open subset (in the analytic topology) of an affine variety. This enables us to conclude that all the complete curves in the deformed 3-fold lie in the exceptional set of the blowdown.

Unfortunately, we do not know how to generalize these techniques to curves with more than one node; in fact, we do not even have a reasonable conjectural formula (see Remark 3.4). There are also problems generalizing the proof of Theorem 1.2 to higher genus (see Remark 3.3). However, considerations of M-theory \cite{13} lead us to the following conjecture:

**Conjecture 1.6.** Let $C \subset Y$ be an irreducible super-rigid rational curve with exactly one node in a 3-fold $Y$ with $K_Y$ trivial in a neighborhood of $C$. Then the contribution of degree $d$ multiple covers of $C$ to the genus $g$ Gromov-Witten invariants is

$$\sum_{n|d} \left( \frac{|B_{2g}|n^{2g-3}}{2g(2g-2)!} + \frac{\delta_{1,g}}{n} \right)$$

where $\delta_{1,g} = 1$ if $g = 1$ and $\delta_{1,g} = 0$ if $g \neq 1$.

Via M-theory, genus $g$ invariants $n^g_\beta$ can be assigned for all $g \geq 0$ and all homology classes $\beta$, from which the 0-point Gromov-Witten invariants can be conjecturally computed by an explicit formula generalizing equation (1), with $n^g_\beta = n_\beta$ (see \cite{13}). Let us consider these invariants (if they exist) to be defined by this formula (c.f. \cite{21}). We can then discuss the contributions of a curve $C$ to $n^g_\beta[\cdot]$. Theorem 1.2 proves that the contribution of $C$ to $n_d[\cdot]$ is 1 for all $d \geq 1$, and Conjecture 1.6 asserts that the contribution to $n^1_d[\cdot]$ is also 1 for all $d \geq 1$ while the contribution to $n^g_d[\cdot]$ is 0 for $g \geq 2$.

In this language, \cite{13} also gives a conjectural formula for these invariants in terms of moduli spaces of $U(d)$ bundles on $C$. These moduli spaces have been computed in \cite{24}, and we have checked that this computation leads to $C$ contributing 1 to $n_d[\cdot]$ for all $d$, as predicted.

Our paper is organized as follows. In Section 2 we study the neighborhood of a contractable curve in a Calabi-Yau 3-fold. We explicitly construct deformations of the neighborhood in order to compute its contribution to the Gromov-Witten invariants. In Section 3, we relate the Gromov-Witten contribution of a 0-super-rigid nodal curve to the contribution of contractable chains of curves. As an example of Theorem 1.3, we work out the length 2 case in detail in Section 4, computing $k_1$ and $k_2$ directly from explicit equations for the 3-fold.

2. **The Neighborhood of a Contractable Curve and its Deformations**

In this section we use the deformation invariance property of Gromov-Witten invariants to compute them for a neighborhood of a contractable curve in a local Calabi-Yau 3-fold. This is achieved by deforming the neighborhood so that all the curves in the neighborhood are smooth ($-1, -1$) curves (c.f. Friedman \cite{12} pg. 678–679). We focus on the two cases of interest: a generic linear chain of curves that contracts to a $cA_n$ singularity and a single contractable smooth curve which
necessarily must contract to a $cA_1$, $cD_4$, $cE_6$, $cE_7$, or $cE_8$ singularity. We will relate the former case to the invariants of a nodal curve in Section 3.

We begin by outlining the theory of deformations of DuVal singularities and their simultaneous resolutions. This is due to Brieskorn [4][5][6] and, in a refined form that we need, Katz-Morrison (see Section 3, especially Theorem 1 of [16]). The application of these ideas to Gorenstein 3-fold singularities with small resolutions was pioneered by Pinkham [22] and Reid [23].

Let $C \subset Y$ be a rational curve (not necessarily irreducible) in a 3-fold $Y$ with $K_Y$ trivial in a neighborhood of $C$ and $\pi: Y \to X$ is a birational morphism such that $\pi(C) = p \in X$ and $\pi|_{Y \setminus C}$ is an isomorphism onto $X \setminus p$. We consider an analytic neighborhood of $p$ (still denoted $X$) and its inverse image under $\pi$ (still denoted $Y$). By a lemma of Reid [23] (1.1,1.14), the generic hyperplane section through $p$ is a surface $X_0$ with an isolated rational double point, and the proper transform of $X_0$ is a partial resolution $Y_0 \to X_0$ (i.e. the minimal resolution $Z_0 \to X_0$ factors through $Y_0 \to X_0$).

The partial resolution $Y_0 \to X_0$ determines combinatorial data $\Gamma_0 \subset \Gamma$ consisting of an ADE Dynkin diagram $\Gamma$ (the type of the singularity $p$) and a subgraph $\Gamma_0$ (the dual graph of the exceptional set of $Y_0$).

Let $Z \to \operatorname{Def}(Z_0)$, $Y \to \operatorname{Def}(Y_0)$, and $X \to \operatorname{Def}(X_0)$ be semi-universal deformations of $Z_0$, $Y_0$, and $X_0$. Following [16], there are identifications

\[
\begin{align*}
\operatorname{Def}(Z_0) & \cong U =: \text{Res}(\Gamma) \\
\operatorname{Def}(Y_0) & \cong U/W_0 =: \text{PRes}(\Gamma, \Gamma_0) \\
\operatorname{Def}(X_0) & \cong U/W =: \text{Def}(\Gamma)
\end{align*}
\]

where $U$ is the complex root space associated to $\Gamma$ and $W$ is its Weyl group. $W_0 \subset W$ is the subgroup generated by reflections of the simple roots corresponding to $\Gamma - \Gamma_0$. Deformations of $Z_0$ or $Y_0$ can be blown down to give deformations of $X_0$ ([23] Theorem 1.4) and the induced classifying maps are given by the natural maps $U \to U/W$ and $U/W_0 \to U/W$ under the above identifications.

Via the defining equation for the hyperplane section $X_0$, we can view $X$ as the total space of a 1-parameter family $X_t$ defined by a classifying map $g: \Delta \to \text{Def}(\Gamma)$. Similarly, we get a compatible family $Y_t$ given by a map $f: \Delta \to \text{PRes}(\Gamma, \Gamma_0)$.

That is, we get the diagram

\[
\begin{array}{cccc}
Z & \overset{\tilde{\sigma}}{\longrightarrow} & Y & \overset{\tilde{\rho}}{\longrightarrow} & X \\
\downarrow & & \downarrow & & \downarrow \\
\text{Res}(\Gamma) & \overset{\sigma}{\longrightarrow} & \text{PRes}(\Gamma, \Gamma_0) & \overset{\rho}{\longrightarrow} & \text{Def}(\Gamma) \\
\end{array}
\]

\[
f \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
where $Y$ is the pullback of $\mathcal{Y}$ by $f$ and $X$ is the pullback of $\mathcal{X}$ by $g$.

Our aim is to deform $Y$ by deforming the map $f$ and to describe the curves in the resulting space. To do this we need to understand the discriminant loci of $\text{PRes}(\Gamma, \Gamma_0)$ and the curves in the corresponding surfaces. We first remark that if $\tilde{f}: \Delta \to \text{PRes}(\Gamma, \Gamma_0)$ is a generic deformation of $f$ (so in particular $\tilde{f}$ is transverse to the discriminant locus), then $\tilde{Y} := \tilde{f}^{-1}(Y)$ is smooth. This is true because the singularities of $Y$ all lie over a set of at least codimension 2 in $\text{PRes}(\Gamma, \Gamma_0)$.

Ultimately, we show that all the curves in $\tilde{Y}$ are smooth $(-1, -1)$ curves. If we then know the number of such curves and their homology class, then the Gromov-Witten invariants of $\tilde{Y}$ can be easily computed from the following multiple cover formula of Faber and Pandharipande.

**Lemma 2.1** (Faber-Pandharipande [11]). If $C$ is a smoothly embedded $(-1, -1)$ curve in a Calabi-Yau 3-fold, then the contribution of degree $d$ multiple covers of $C$ to the genus $g$ Gromov-Witten invariants is given by

$$\frac{|B_{2g}(d^2-3)}{2g \cdot (2g - 2)!}.$$  

The exceptional curve of the minimal resolution $Z_0 \to X_0$ is a rational curve whose dual graph is $\Gamma$. Thus the components of the exceptional divisor are rational curves $C_{e_i}$ which are in one to one correspondence with the simple roots $e_1, \ldots, e_n$ ($n$ is the rank of the root system). A generic deformation of $Z_0$ has no complete curves, but the curves $C_{e_i}$ and various combinations of $C_{e_i}$ will deform in certain codimension one families of deformations. To make this precise, we define the discriminant locus $\mathcal{D} \subset \text{Res}(\Gamma)$ to be those $t \in \text{Res}(\Gamma)$ such that the corresponding surface $Z_t$ has a complete curve. The following proposition is essentially part 3 of Theorem 1 in [16].

**Proposition 2.2.** The irreducible components of the discriminant divisor $\mathcal{D} \subset \text{Res}(\Gamma)$ are in one to one correspondence with the positive roots of $\Gamma$. Under the identification of $\text{Res}(\Gamma)$ with the complex root space $U$, the component $\mathcal{D}_v$ corresponding to the positive root $v = \sum_{i=1}^n a_i e_i$ is $v^\perp \subset U$, i.e. the hyperplane perpendicular to $v$.

Moreover, $\mathcal{D}_v$ corresponds exactly to those deformations of $Z_0$ in $Z$ to which the curve

$$C_v := \bigcup_{i=1}^n a_i C_{e_i}$$

lifts. For a generic point $t \in \mathcal{D}_v$, the corresponding surface $Z_t$ has a single smooth $-2$ curve in the class $\sum_{i=1}^n a_i[C_{e_i}]$ thus there is a small neighborhood $B$ of $t$ such that the restriction of $Z$ to $B$ is isomorphic to a product of $\mathbb{C}^{n-1}$ with the semi-universal family over $\text{Res}(A_1)$.

Katz and Morrison prove this (among other things) by constructing explicit equations for $Z$. Although we do not need the explicit form of these equations for the proofs of our theorems, they are useful for computing the values of $k_i$ in concrete cases (see Section 3).

Since the Weyl group acts transitively on the positive roots, we see that the discriminant locus in $\text{Def}(\Gamma)$, which is the image of $\mathcal{D}$, is irreducible. It corresponds to those deformations of $X_0$ which are singular.
The two cases relevant to this paper are when $\Gamma = \Gamma_0 = A_n$ (so $\text{PRes}(\Gamma, \Gamma_0) = \text{Res}(\Gamma$)) and when $\Gamma_0$ is a single vertex. In this latter case, $\Gamma$ must be one of $A_1$, $D_4$, $E_6$, $E_7$, or $E_8$ and $\Gamma_0$ must be the central vertex or, in the case of $E_8$, it can also be the vertex one away from the center on the long branch. These six possibilities correspond to the six possible values of the length (this is the main theorem of [16]).

2.1. The case of a contractable smooth $\mathbf{P}^1$. Assume then that $\Gamma_0$ is a single vertex so that $(\Gamma, \Gamma_0)$ is one of the above six possibilities. Let $D \subset \text{PRes}(\Gamma, \Gamma_0)$ denote the discriminant locus, i.e. the image of $D$ under the quotient of $W_0$. The corresponding surfaces are those which are singular or contain complete curves (or both). We write $D = D^{\text{sing}} \cup D^{\text{curv}}$ where the surfaces corresponding to the points in $D^{\text{curv}}$ contain a complete curve and $D^{\text{sing}}$ is the union of the remaining components. A generic point of $D^{\text{curv}}$ corresponds to a smooth surface with a smooth (-1,-1) curve and a generic point of $D^{\text{sing}}$ corresponds to a surface with no curves and a single $A_1$ singularity.

For the purposes of computing the Gromov-Witten invariants of $Y$ we need to understand the components of $D^{\text{curv}}$. Since the family $\mathcal{Y}$ is obtained from $Z$ by simultaneously blowing down the curves in $Z_t$ corresponding to $\Gamma - \Gamma_0$ followed by taking the quotient by $W_0$, we can study $D^{\text{curv}}$ by studying the orbit structure of $W_0$ acting on $D = \cup \mathcal{D}_v$.

**Proposition 2.3.** The divisor $D^{\text{curv}} \subset \text{PRes}(\Gamma, \Gamma_0)$ consists of $l$ irreducible components $D_1, \ldots, D_l$. Moreover, if $t$ is a generic point in $D_i$ then the surface $\mathcal{Y}_t$ has a smooth $-2$ curve in the class $i[C]$. These curves form the fibers of a divisor $\mathcal{C}_t$ inside the restriction of $\mathcal{Y}$ over $D_i$. The scheme-theoretic fiber of $\mathcal{C}_t$ over the origin is precisely the subscheme $C_t \subset Y_0$.

**Proof:** Consider the action of $W_0$ on the roots $\{\nu_i\}$ and the induced action on the hyperplanes $\{\mathcal{D}_v\}$. Let $e_1$ denote the simple root corresponding to $\Gamma_0$. The coefficient $\alpha_1$ of any root $\nu = \sum_{i=1}^n \alpha_i e_i$ is preserved by the generators of $W_0$ and so it is an invariant of the orbits of $W_0$ acting on $\{\mathcal{D}_v\}$. For orbits with $\alpha_1 \neq 0$ it turns out that $\alpha_1$ is a complete invariant:

**Lemma 2.4.** If $v = \sum_{i=1}^n \alpha_i e_i$ is a root with $\alpha_1 \neq 0$, then $v' = \sum_{i=1}^n \alpha'_i e_i$ is in the $W_0$-orbit of $v$ if and only if $\alpha'_1 = \alpha_1$.

Since the groups and root systems are finite and there are only a finite number of cases, this can easily be checked by hand. It is a fun exercise (really!) which we encourage the reader to carry out.

The $W_0$ orbits of $\{\mathcal{D}_v\}$ with $\alpha_1 \neq 0$ are exactly the components of $D^{\text{curv}}$ and the $W_0$ orbits of $\{\mathcal{D}_v\}$ with $\alpha_1 = 0$ are the components of $D^{\text{sing}}$. By inspection, the possible non-zero values of $\alpha_1$ are $1, \ldots, l$, and so define $D_k$ to be the image of $\{\mathcal{D}_v : v = \sum \alpha_i e_i \text{ has } \alpha_1 = k\}$ in $\text{PRes}(\Gamma, \Gamma_1)$. Since the surface $\mathcal{Z}_t$ for $t \in \mathcal{D}_v$ has a curve in the class $\sum_{i=1}^n \alpha_i [C_{e_i}]$, the surface $\mathcal{Y}_t$ (where $t \mapsto t'$) has a curve in the class $\alpha_1 [C]$. For generic $t \in \mathcal{D}_v$ this curve is a single smooth $(-1,-1)$ curve and $\mathcal{Z}_t \cong \mathcal{Y}_{t'}$.

For the claim about $C_t$, note that after pulling back to $\text{Res}(\Gamma)$, we can pick a discriminant component $\mathcal{D}_v$ mapping to $D_i$, and a corresponding (possibly reducible) divisor $\mathcal{C}_v$ in the restriction of $\mathcal{Z}$ to $\mathcal{D}_v$ whose fibers are the (possibly reducible) curves corresponding to the discriminant component $\mathcal{D}_v$. The divisor $\mathcal{C}_v$ is the scheme-theoretic image of $\mathcal{C}_v$ under the restriction of $\tilde{\sigma}$ over the relevant discriminant components. The fiber $C_v$ of $\mathcal{C}_v$ over the origin is itself an exceptional ADE
configuration, and its scheme structure is defined by the pullback of the maximal ideal of the singularity being resolved. Writing its cycle class as \( C_v = \sum_{j \geq 1} \alpha_j C_{e_j} \), we have that \( \alpha_1 = i \). We have that \( C_v \) is scheme-theoretically defined by the ideal of functions vanishing to order \( \alpha_j \) at each \( C_{e_j} \). We now look at the scheme-theoretic image of \( C_v \) after contracting the \( C_{e_j} \) for \( j > 1 \) via \( \tilde{\sigma} \). Its ideal consists of those functions \( f \) such that \( \text{ord}_{C_{e_j}} \tilde{\sigma}^*(f) \geq \alpha_j \), where \( \text{ord}_{C_{e_j}} \) denotes the order of vanishing along \( C_{e_j} \). It remains to show that if \( f \in \mathcal{I}^{(i)} \), then \( \text{ord}_{C_{e_j}} \tilde{\sigma}^*(f) \geq \alpha_j \) for all \( j \). To expedite this task we use the following lemma.

**Lemma 2.5.** Let \( f \) be a function on a neighborhood of \( C \) in \( Y_0 \). Put \( m_k = \text{ord}_{C_{e_j}} \tilde{\sigma}^*(f) \). Then for each \( \tilde{\sigma} \)-exceptional curve \( C_{e_k} \) we have \( 2m_k \geq \sum_j m_j \).

The lemma follows immediately from the obvious assertion: for all \( \tilde{\sigma} \)-exceptional curves \( D \), we have \( \tilde{\sigma}^* (f) \cdot D \leq 0 \).

It is now a simple matter to check in each case that the inequalities of Lemma 2.5 together with \( m_1 \geq \alpha_1 = i \) imply that \( m_k \geq \alpha_k \) for all \( k \).

**Remark 2.6.** The proof of Proposition 2.3 shows that the scheme theoretic exceptional set of \( Y_0 \to X_0 \) is defined by the ideal \( \mathcal{I}(l) \).

We now wish to compute the Gromov-Witten invariants of \( Y \) in the class \( d[C] \). Let \( \tilde{f} : \Delta \to \text{PRes}(\Gamma, \Gamma_0) \) be a small generic deformation of \( f \) and define \( \tilde{Y} \) to be the total space of the induced one parameter family of surfaces, \( \tilde{f}^{-1}(Y) \).

It makes sense to speak of the Gromov-Witten invariants of a local manifold like \( Y \) since for any closed 3-fold \( \tilde{Y} \) containing \( Y \), the moduli space of (non-constant) stable maps has a connected component consisting of maps whose image lies entirely in \( Y \). This is because a curve that lies in \( Y \) must be in the exceptional set of \( Y \to X \) since \( X \) is an open set in an affine variety. We understand the “Gromov-Witten invariants of \( Y \)” to mean the virtual fundamental cycle restricted to that component. Alternatively, one can apply the analytic definition of the Gromov-Witten invariants directly to \( Y \) since the above argument shows that the moduli space of stable maps is compact, the (analytic) virtual class constructions of Li and Tian apply.

**Lemma 2.7.** The Gromov-Witten invariants of \( \tilde{Y} \) are well defined and equal to the Gromov-Witten invariants of \( Y \).

By the above argument, all of the complete curves of our deformations of \( Y \) lie in the exceptional set of \( Y \to X \) which in turn lies over \( D^{\text{curv}} \subset \text{PRes}(\Gamma, \Gamma_0) \). Let \( F : \Delta \times [0, \epsilon] \to \text{PRes}(\Gamma, \Gamma_0) \) be a smooth map so that \( f_s := F|_{\Delta \times \{s\}} \) is analytic, \( f_0 = f \), and \( F \) is transverse to \( D^{\text{curv}} \). Furthermore, we may assume, by making \( \epsilon \) smaller if necessary, that \( F^{-1}(D^{\text{curv}}) \) is bounded away from \( \partial \Delta \). Let \( \tilde{f} = f_\epsilon \). The parameterized moduli spaces are then compact and so the deformation invariance arguments in [18] apply.

Since \( \tilde{f} \) is transverse to \( D^{\text{curv}} \), by Proposition 2.3, we see that the complete curves of \( \tilde{Y} \) are all smooth \((-1, -1)\) curves in the classes \( [C], 2[C], \ldots, l[C] \). The number of irreducible curves in the class \( i[C] \) is the intersection number of \( \Delta \) with \( D_i \), i.e. the cardinality of \( \tilde{f}^{-1}(D_i) \). Thus Theorem 1.5 follows easily from Lemma 2.4 once we show that the intersection numbers of \( \Delta \) with \( D_i \) are equal to the Hilbert scheme multiplicities.
Lemma 2.8. The intersection number $\#\{f^{-1}(D_i)\}$ is equal to $k_i$, the multiplicity of $C_i$ (Definition 1.4) in its Hilbert scheme.

Proof: For each point in $D_i$, the corresponding surface has a unique irreducible complete curve. Let $C_i \rightarrow D_i$ be the corresponding family of curves. We claim this family is a flat family of rational curves $C_i \subset Y$.

From the explicit construction in [16], we see that $C_i \rightarrow D_i$ is projective so for flatness, it suffices to check that the genus and degree of the fibers are constant where any convenient polarization can be chosen for the purpose of defining the degree. The fibers are all rational, since they are the exceptional set of the map of a surface to a surface with a rational singularity. To see that the degree is constant, consider the pullback of $C_i \rightarrow D_i$ to $\text{Res}(\Gamma)$. We get a family $\sigma^{-1}(C_i) \rightarrow D_i$ where $D_i = \sigma^{-1}(D_i)$ is the union of the components $D_v$ mapping to $D_i$. It is enough to show that $\sigma^{-1}(C_i) \rightarrow D_i$ has constant fiberwise degree. By restricting $(\rho \circ \sigma)^{-1}(\mathcal{X})$, $\hat{\sigma}^{-1}(\mathcal{Y})$, and $\mathcal{Z}$ to $D_i$, we get the families of singular, partially resolved, and minimally resolved surfaces over $D_i$ which we denote by $\mathcal{X}'$, $\mathcal{Y}'$, and $\mathcal{Z}'$ respectively. From [16], these surfaces all have projective compactifications and hence polarizations $L_{\mathcal{X}}'$, $L_{\mathcal{Y}}'$, and $L_{\mathcal{Z}}'$. We need to show that $\sigma^{-1}(C_i) \subset \mathcal{Y}'$ has constant degree, i.e. the fibers of $\sigma^{-1}(C_i)$ have constant intersection with $L_{\mathcal{Y}}'$, which we will refer to as $\sigma^{-1}(C_i) \cdot L_{\mathcal{Y}}'$ is constant on the fibers". Let $C \subset \mathcal{Z}'$ be the exceptional set for the blowdown $\mathcal{Z}' \rightarrow \mathcal{X}'$ and let $\mathcal{C} = \mathcal{C}' + \mathcal{C}''$ where $\mathcal{C}''$ is the exceptional set for $b : \mathcal{Z}' \rightarrow \mathcal{Y}'$. Then the exceptional set $C \rightarrow D_i$ is flat. This was written explicitly for all $v$ in [16] in the cases of $A_n$ (page 469) and $D_n$ (page 473). This also holds true for $E_n$ as can be checked directly case by case. It follows that $C \cdot L_{\mathcal{Z}}'$ is constant on fibers. Furthermore, $L_{\mathcal{Z}}'$ can be chosen to be the pullback of $L_{\mathcal{Y}}'$ plus an appropriate multiple $m$ of $\mathcal{C}''$. Since $b(C'')$ is of lower dimension, $b_s(C'') = 0$, and we have an identity on the fibers

$$\sigma^{-1}(C_i) \cdot L_{\mathcal{Y}}' = b_s(C') \cdot L_{\mathcal{Y}}' = b_s(C) \cdot L_{\mathcal{Y}}' = C \cdot (L_{\mathcal{Z}}' - mC''),$$

which is constant since the intersection number of the fibers of $C$ and $\mathcal{C}''$ is constant.

Thus $C_i \rightarrow D_i$ is flat.

Now consider an analytic family $f_\epsilon$ of deformations of $f$ parameterized by a disk $B$, i.e. an analytic map

$$F : \Delta \times B \rightarrow \text{PRes}(\Gamma, \Gamma_0)$$

where $F|_{\Delta \times \epsilon}$ is denoted $f_\epsilon$. Assume that $\tilde{f}$ is given by $f_1$. We get a finite map $\tilde{B} \rightarrow B$ whose degree is the intersection number $\#\{f^{-1}(D_i)\}$ by projecting $F^{-1}(D_i)$ to $B$. By the flatness of $C_i \rightarrow D_i$, we get a flat family of curves $F^{-1}(C_i) \rightarrow B$. This family sit inside the family of 3-folds $F^{-1}(\mathcal{Y}) \rightarrow B$ (whose fiber over $\epsilon$ is the 3-fold $f_\epsilon^{-1}(\mathcal{Y})$) and so determines a subscheme of the relative Hilbert scheme (or Douady space) of $F^{-1}(\mathcal{Y})/B$ that is finite over $B$. This family is isomorphic to $\tilde{B} \rightarrow B$ and since its degree is exactly the multiplicity of $C_i$ in $Y = f^{-1}(\mathcal{Y})$, the lemma is proved.

2.2. The case of a generic contractable $A_n$ curve. We now assume that $Y$ is a neighborhood of a generic, contractable chain of $n$ rational curves $C = C_{e_1} \cup \cdots \cup C_{e_n}$. Thus we are in the case where $\Gamma_0 = \Gamma = A_n$, and $Y = f^{-1}(\mathcal{Y})$ where $f : \Delta \rightarrow \text{Res}(A_n)$. Our genericity assumption is that $f$ is transverse to the components $D_{ij} \subset \text{Res}(A_n)$ of the discriminant locus. That is, although $f$ intersects the discriminant locus only at the origin, it meets each hyperplane $D_{ij}$
transversely. Here the hyperplanes \( \{ \mathcal{D}_{ij} : 1 \leq i \leq j \leq n \} \) are perpendicular to the roots \( v_{ij} = \sum_{k=i}^{j} e_k \). This condition is related to the non-existence of infinitesimal deformations:

**Lemma 2.9.** Let \( f : \Delta \to \text{Res}(A_n) \) be as above so that \( f \) intersects each hyperplane \( \mathcal{D}_{ij} \) transversely exactly once at \( 0 \in \text{Res}(A_n) \). Let \( Y = f^{-1}(\mathcal{Y}) \) and let \( C_i \cup \cdots \cup C_n = C \subset Y \) be the corresponding contractable curve. For any \( 1 \leq i \leq j \leq n \), let \( C_{ij} = C_{e_i} \cup \cdots \cup C_{e_j} \) and let \( N_{ij} \) be the normal bundle of \( C_{ij} \) in \( Y \). Then \( H^0(C_{ij}, N_{ij}) = 0 \) for all \( i \) and \( j \); i.e. \( C_{ij} \) has no infinitesimal deformations. Conversely, if \( H^0(C_{ij}, N_{ij}) \neq 0 \), then \( f \) meets \( \mathcal{D}_{ij} \) non-transversely.

**Proof:** This follows directly from the description of deformations of the curves \( C_{ij} \) in Chapter 6 of [28], or from the characterization of \( \mathcal{D}_n \) in terms of deformations of \( C_n \) that was given in Proposition 2.2. Suppose that for some \( i \) and \( j \), \( C_{ij} \) has a non-trivial infinitesimal deformation. It defines a morphism \( h : C \to Y \) where \( \pi : C \to I = \text{Spec}(C[e]/e^2) \) and \( h_{|\pi^{-1}(0)} : C_{ij} \to Y \). Since \( \mathcal{D}_{ij} \cap \text{Res}(A_n) \) classifies those deformations to which \( C_{ij} \) lifts, the map \( C \to Y \) is induced by a classifying map \( I \to \mathcal{D}_{ij} \) so we get a commutative diagram

\[
\begin{array}{ccccccc}
\mathcal{C} & \longrightarrow & Y & \longrightarrow & \mathcal{Y}_{|\mathcal{D}_{ij}} & \longrightarrow & \mathcal{C} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
I & \longrightarrow & \Delta & \longrightarrow & \text{Res}(A_n) & \longrightarrow & \mathcal{D}_{ij} & \longrightarrow & I.
\end{array}
\]

and so \( \Delta \) lies in \( \mathcal{D}_{ij} \) to first order and we conclude that \( f \) is not transverse to \( \mathcal{D}_{ij} \). Conversely, if \( f \) is not transverse to \( \mathcal{D}_{ij} \), then the reverse of the above argument gives an infinitesimal deformation of \( C_{ij} \).

We can now compute the Gromov-Witten invariants of \( Y \) as in the previous case by deforming \( Y = f^{-1}(\mathcal{Y}) \) to \( \tilde{Y} = \tilde{f}^{-1}(\mathcal{Y}) \). The Gromov-Witten invariants of \( Y \) and \( \tilde{Y} \) are well defined and equal by the same argument as Lemma 2.7 and all the curves of \( \tilde{Y} \) are smooth \((-1, -1)\) curves arising from the transverse intersection of \( \tilde{f} \) with \( \mathcal{D}_{ij} \) at generic points. Thus we see that there is exactly one \((-1, -1)\) curve in the class \( \sum_{k=1}^{n} [C_{e_k}] \) for each \( 1 \leq i \leq j \leq n \) and these are all of the complete curves in \( \tilde{Y} \). Combining this observation with Lemma 2.1 we get the following proposition:

**Proposition 2.10.** Assume that \( d_i > 0 \) for \( i = 1, \ldots, n \). Let \( N_g(d_1, \ldots, d_n) \) denote the contribution of the genus \( g \) Gromov-Witten invariant of \( Y \) in the class \( \sum_{i=1}^{n} d_i [C_{e_i}] \). Then

\[
N_g(d_1, \ldots, d_n) = \begin{cases} 
\frac{B_{2g}}{2g(2g-2)!} d^{2g-3} & \text{if } d_k = d \text{ for all } k \\
0 & \text{otherwise.}
\end{cases}
\]

It is straightforward to formulate and prove a generalization of Proposition 2.10 to the non-generic case, where each \( C_{ij} \) is assumed isolated but not necessarily infinitesimally isolated. The multiplicity of each \( C_{ij} \) in its Hilbert scheme appears in the formula.
3. Relating nodal curves to chains.

In this section we relate the Gromov-Witten invariants of a super-rigid nodal rational curve in a Calabi-Yau 3-fold to the Gromov-Witten invariants of a neighborhood of a contractable chain of curves in a Calabi-Yau 3-fold.

Let $C_0 \subset Y$ be a super-rigid, irreducible, rational curve $C_0$ with exactly one node in a projective 3-fold $Y$. We consider a neighborhood (in the analytic topology) of $C_0$ which we denote by $Y$ that is sufficiently small enough so that all non-constant stable maps to $Y$ whose image is contained in $Y$ have image $C_0$. Then the moduli space $\overline{M}_0(Y, d[C_0])$ of genus 0 stable maps of degree $d[C_0]$ to $Y$ contains the moduli space $\overline{M}_0(Y, d[C_0])$ as a connected component. The restriction of the virtual class $[\overline{M}_0(Y, d[C_0])]^{vir}$ to $\overline{M}_0(Y, d[C_0])$ defines a rational number $N_0$ which is the contribution of $C_0$ to the degree $d[C_0]$ genus 0 Gromov-Witten invariant of $Y$.

We construct a local Calabi-Yau 3-fold $Y_n$ with a contractable chain of curves $C_n = C_{e_1} \cup \cdots \cup C_{e_n} \subset Y_n$ and a local isomorphism

$$g_n : Y_n \rightarrow Y$$

restricting to the local immersion

$$g_n : C_n \rightarrow C_0.$$ 

To construct this, let $g_\infty : Y_\infty \rightarrow Y$ be the universal cover of $Y$. We may assume that $Y$ is a sufficiently small enough neighborhood of $C_0$ so that $\pi_1(Y) \cong \pi_1(C_0) \cong \mathbb{Z}$ and so $Y_\infty$ is an open neighborhood of $C_\infty$, a linear chain of a countable number of rational curves. Fix a subchain $C_n$ of length $n$ and let $Y_n \subset Y_\infty$ be an open neighborhood of $C_n$ and let $g_n$ be the restriction of $g_\infty$ to $Y_n$. Since $K_{Y_\infty} = g_\infty^* K_Y$, we see that $K_{Y_n}$ is trivial.

**Lemma 3.1.** Super-rigidity of $C_0 \subset Y$ implies that $C_n \subset Y_n$ is a generic contractible curve in the sense of subsection 2.2.

**Proof:** Let $N_{C_0/Y}$ be the normal bundle of $C_0$ in $Y$ (note that $C_0$ and $C_n$ are all local complete intersections). By super-rigidity,

$$H^0(C_n, g_n^*(N_{C_0/Y})) = 0.$$ 

Since $g_n$ is a local isomorphism, $g_n^*(N_{C_0/Y}) \cong N_{C_n/Y_n}$. For any subchain $C_{ij} \subset C_n$, we have a injective sheaf map

$$0 \rightarrow N_{C_{ij}/Y_n} \rightarrow N_{C_n/Y_n}|_{C_{ij}}$$

and so $H^0(C_{ij}, N_{C_{ij}/Y_n}) = 0$ as well.

In particular, each component of $C_n \subset Y_n$ is a $(-1, -1)$-curve. Thus $C_n$ is a contractible curve and each subchain $C_{ij}$ has no infinitesimal deformations. Thus by Lemma 2.9, we then have that $C_n \subset Y_n$ is generic. \qed

The connected components of $\overline{M}_0(Y, d[C_0])$ are identified by the following proposition:

**Proposition 3.2.** The connected components of $\overline{M}_0(Y, d[C_0])$ are indexed by $n$-tuples $(d_1, \ldots, d_n)$ of positive integers with $\sum d_i = d$ and they are isomorphic to $\overline{M}_0(Y_n, (d_1, \ldots, d_n))$ where $(d_1, \ldots, d_n) \in H_2(Y_n, \mathbb{Z}) \cong H_2(C_n, \mathbb{Z})$ via the natural basis of $H_2(C_n, \mathbb{Z})$ indexed by the components of $C_n$. Furthermore, the virtual class on $\overline{M}_0(Y, d[C_0])$ agrees with the class induced by $Y_n$ on $\overline{M}_0(Y_n, (d_1, \ldots, d_n))$. 

It follows immediately from this proposition and the genus 0 case of Proposition 2.10 that

\[ N_d = \sum_{n|d} \frac{1}{n^3} \]

which proves Theorem 1.2.

**Proof:** Let \( f : \Sigma \to Y \) be a genus 0 stable map. Since \( \pi_1(\Sigma) = 0 \), \( f \) lifts to a map to the universal cover 

\[ \tilde{f} : \Sigma \to Y_\infty. \]

This map is unique up to deck transformations. Since \( \Sigma \) is compact, the image of \( \tilde{f} \) is supported on some finite chain. After a deck transformation, we can assume that the image of \( \tilde{f} \) is exactly \( C_n \) for some \( n \). The same argument works equally well for families of stable maps and so we get an isomorphism of moduli functors

\[ \bigoplus_{(d_1, \ldots, d_n), \sum d_i = d} \overline{M}_0(Y_n, (d_1, \ldots, d_n)) \to \overline{M}_0(Y, d[C_0]) \]

where the isomorphism is given by composition with the appropriate \( g_n \).

The virtual class \([\overline{M}_0(Y_n, (d_1, \ldots, d_n))]_{vir}\) is determined (as in [3]) by the perfect relative obstruction theory

\[ [R^\bullet \pi_* \tilde{f}^* (T_{Y_n})]^\vee \to L^\bullet_{\overline{M}_0(Y_n, (d_1, \ldots, d_n))/\mathcal{M}_0} \]

where \( \pi : C \to \overline{M}_0(Y_n, (d_1, \ldots, d_n)) \) is the universal curve and \( \tilde{f} : C \to Y_n \) is the universal map. Since \( g_n \) is a local isomorphism, \( T_{Y_n} \cong g_n^* T_Y \) and so \( \tilde{f}^* (T_{Y_n}) = \tilde{f}^* g_n^* (T_Y) \). Since \( g_n \circ \tilde{f} \) is the universal map for \( \overline{M}_0(Y, d[C_0]) \) under the above isomorphism, the two obstruction theories are isomorphic and thus the virtual classes agree.

**Remark 3.3.** The difficulty in generalizing this proof to higher genus Gromov-Witten invariants is that when \( \Sigma \) is no longer simply connected, we must consider maps that factor through the finite covers of \( Y \) as well. The two terms in Conjecture 1.6 correspond to the contributions of (1) maps factoring through the universal cover (which we know), and (2) maps that don’t factor where the conjecture predicts the contribution is the same as if \( C_0 \) were a smooth super-rigid elliptic curve. Note that for the genus 0 invariants, we only need to assume 0-super-rigidity whereas to extend the argument to higher genus we would need 1-super-rigidity as well.

**Remark 3.4.** The difficulty in generalizing this proof to a rational curve with more nodes is that the universal cover is a curve whose dual graph is a complicated tree. One thus has to deal with curves not necessarily of ADE type and so our deformation technique does not apply. However, for degree five or less, the maps will factor through a map to the universal cover that either has an ADE image or is degree one on each component. Thus under the appropriate genericity conditions we can derive a formula for multiple covers of \( n \)-nodal irreducible rational curves for multiple covers of degree 5 or less.
4. Explicit computation in the length two \((cD_4)\) case

In this section, we show how to compute the multiplicities \(k_1\) and \(k_2\) (see Theorem \[15\]) directly from explicit equations for the 3-fold in the case of a curve contracting to a \(cD_4\) singularity.

Recall that in the case of contractable a smooth rational curve \(C\) inside \(Y\), the length of the singular point \(p = \pi(C)\) inside \(X\) is at most six. When the length equals one, \(p\) is a \(cA_1\) singularity and the divisor \(D_{\text{curv}}\) inside \(\text{PRes}(\Gamma, \Gamma_0)\) is just the origin of the affine line.

In this section we discuss in detail the length two case which corresponds to \(p\) being a \(cD_4\) singularity. First we want to describe the divisor \(D_{\text{curv}} = D_1 \cup D_2\) in a suitable coordinate system of \(\text{PRes}(\Gamma, \Gamma_0)\). To start, let \(x_1, x_2, x_3\) and \(x_4\) be any coordinate system for \(\text{PRes}(\Gamma, \Gamma_0)\) and the threefold \(Y\) is given by a map \(f\):

\[
\begin{align*}
  f : & \quad \Delta \to \text{PRes}(\Gamma, \Gamma_0) \\
  f(t) & = (x_1(t), x_2(t), x_3(t), x_4(t)).
\end{align*}
\]

Notice that \(x_i(t)\) vanishes at \(t = 0\) for any \(i\). In fact we can choose a good coordinate system for \(\text{PRes}(\Gamma, \Gamma_0)\) in such a way that \(X\) can be described in term of an explicit equation as follows (see \[13\] and \[14\]):

\[
X = \left\{ x^2 + \frac{1}{2} (yz + x_2 x_4) = \frac{1}{2} F(z, t) \right\} \subset \mathbb{C}^3 \times \Delta,
\]

where \(F(z, t) = (z - x_2)^2 + x_1^2 z (z^2 + x_3 z + x_4^2)\) and \((x, y, z, t)\) are coordinates on \(\mathbb{C}^3 \times \Delta\).

By direct computations, \(D_{\text{curv}} = D_1 \cup D_2\) can be described explicitly as follows:

\[
\begin{align*}
D_1 & = \left\{ (x_2^2 + x_2 x_3 + x_4^2)^2 + x_1^2 (x_1^2 x_4^2 - 4 x_2 x_4^2 - x_3 x_4^2 - x_3 x_4^2) = 0 \right\}, \\
D_2 & = \{ x_1 = 0 \}.
\end{align*}
\]

From a local analysis near the singular points of the partial resolution of a \(D_4\) singularity, it is shown in \[13\] that the smoothness of \(Y\) implies that the order of vanishing of \(x_i(t)\) at \(t = 0\) is exactly one for \(i = 2, 3\) or 4. On the other hand, the order of vanishing of \(x_1(t)\) at \(t = 0\) is precisely \(k_2\) via the above description of \(D_2\).

To determine \(k_1\) we must compute the intersection number of \(f(\Delta)\) with the hypersurface \(D_1\) at the origin. In fact \(k_1\) is at least four since \(D_1\) is defined by a polynomial which vanishes at the origin to the fourth order. We expect that \(k_1 = 4\) in generic situation. We can describe this generic condition rather easily in terms of either \(F(z, t)\) or \(f\). If we set \(z = x_2(t)\) then we have

\[
F(x_2(t), t) = x_1^2 x_2 (x_2^2 + x_2 x_3 + x_4^2)
\]

Since \(x_1(t)\) vanishes at \(t = 0\) to order \(k_2\) and the order of vanishing of the other \(x_i(t)\)'s is one, the order of vanishing of \(F(x_2(t), t)\) at \(t = 0\) is at least \(2k_2 + 3\). If it is exactly \(2k_2 + 3\), then \(f(\Delta)\) is transverse to the hypersurface \(x_2^2 + x_2 x_3 + x_4^2 = 0\).

By the description of \(D_1\), this implies that \(k_1 = 4\). An alternative way to see this is as follows. Let us write \(x_i(t) = c_i t + h_i(t) t^2\) for \(i = 2, 3\) or 4 for nonzero \(c_i\)’s. Then \(c = [c_2, c_3, c_4]\) determines a point in \(\mathbb{P}^2\). Then we have \(k_1 = 4\) as long as this point does not lie inside the conic defined by \(x_2^2 + x_2 x_3 + x_4^2 = 0\). Therefore in such generic situation, we have \(k_1 = 4\) and \(k_2\) equals of order of vanishing of \(x_1(t)\)

---

\(^3\) The coordinate \(x_1\) (resp. \(x_2, x_3\) and \(x_4\)) we use here corresponds to \(\sigma_1\) (resp. \(\sigma_2, \sigma_3\) and \(\sigma_4\)) in \([3]\).
Multiple covers and integrality

at \( t = 0 \). When \( c \) does lie inside the above conic, then, in order to determine \( k_1 \), we also need to consider the fifth order terms in the defining equation of \( D_1 \). We would still obtain an explicit description of \( k_1 \) but it will be not as simple as in the generic case.

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**References**

[1] Paul S. Aspinwall and David R. Morrison. Topological field theory and rational curves. *Comm. Math. Phys.*, 151(2):245–262, 1993.

[2] W. Barth, C. Peters, and A. Van de Ven. *Compact Complex Surfaces*, volume 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag, 1984.

[3] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.

[4] E. Brieskorn. Singular elements of semi-simple algebraic groups. pages 279–284, 1971.

[5] Egbert Brieskorn. Über die Auflösung gewisser Singularitäten von holomorphen Abbildungen. *Math. Ann.*, 166:76–102, 1966.

[6] Egbert Brieskorn. Die Auflösung der rationalen Singularitäten holomorpher Abbildungen. *Math. Ann.*, 178:255–270, 1968.

[7] Jim Bryan and Naichung Conan Leung. The enumerative geometry of \( K3 \) surfaces and modular forms. [alg-geom/9711031](http://arxiv.org/abs/alg-geom/9711031), 1997.

[8] Herbert Clemens. The infinitesimal Abel-Jacobi mapping and moving the \( \mathcal{O}(2) \oplus \mathcal{O}(-4) \) curve. *Duke Math. J.*, 59(1):233–240, 1989.

[9] Herbert Clemens, János Kollár, and Shigefumi Mori. Higher-dimensional complex geometry. *Astérisque*, (166):144 pp. (1989), 1988.

[10] David A. Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry*. American Mathematical Society, Providence, RI, 1999.

[11] C. Faber and R. Pandharipande. Hodge integrals and Gromov-Witten theory, 1998. Preprint, [math.AG/981017](http://arxiv.org/abs/math.AG/981017).

[12] Robert Friedman. Simultaneous resolution of threefold double points. *Math. Ann.*, 274(4):671–689, 1986.

[13] Rajesh Gopakumar and Cumrun Vafa. M-theory and topological strings–II, 1998. Preprint, [hep-th/9812](http://arxiv.org/abs/hep-th/9812).

[14] Jesús Jiménez. Contraction of nonsingular curves. *Duke Math. J.*, 65(2):313–332, 1992.

[15] Sheldon Katz. Small resolutions of Gorenstein threefold singularities. In *Algebraic geometry: Sundance 1988*, pages 61–70. Amer. Math. Soc., Providence, RI, 1991.

[16] Sheldon Katz and David R. Morrison. Gorenstein threefold singularities with small resolutions via invariant theory for Weyl groups. *J. Algebraic Geom.*, 1(3):449–530, 1992.

[17] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves* (Texel Island, 1994), volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995.

[18] Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds. In *Topics in symplectic 4-manifolds* (Irvine, CA, 1996), pages 47–83. Internat. Press, Cambridge, MA, 1998.

[19] Hong H. Lian, Kefeng Liu, and Shing-Tung Yau. Mirror principle. I. *Asian J. Math.*, 1(4):729–763, 1997.

[20] Yu. I. Manin. Generating functions in algebraic geometry and sums over trees. In R. Dijkgraaf, C. Faber, and G. van der Geer, editors, *The moduli space of curves*, pages 401–417. Birkhäuser, Boston, 1995.

[21] R. Pandharipande. Hodge integrals and degenerate contributions, 1998. Preprint, [math.AG/9811141](http://arxiv.org/abs/math.AG/9811141).

[22] Henry C. Pinkham. Factorization of birational maps in dimension 3. In *Singularities, Part 2 (Arcata, Calif., 1981)*, pages 343–371. Amer. Math. Soc., Providence, RI, 1983.

[23] Miles Reid. Minimal models of canonical 3-folds. In *Algebraic varieties and analytic varieties (Tokyo, 1981)*, pages 131–180. North-Holland, Amsterdam, 1983.

[24] Titus Teodorescu. *Semistable torsion-free sheaves over curves of arithmetic genus one*. PhD thesis, Columbia University, 1999.
[25] Israel Vainsencher. Enumeration of n-fold tangent hyperplanes to a surface. *J. Algebraic Geom.*, 4(3):503–526, 1995.

[26] Claire Voisin. A mathematical proof of a formula of Aspinwall and Morrison. *Compositio Math.*, 104(2):135–151, 1996.

[27] Jonathan M. Wahl. Equisingular deformations of normal surface singularities. I. *Ann. of Math. (2)*, 104(2):325–356, 1976.

[28] Thomas Zerger. Contracting rational curves on smooth complex threefolds, 1996. Thesis, Oklahoma State University.

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