We study the cosmological meaning of duality symmetry by considering a two dimensional model of string cosmology. We find that as seen by an internal observer in this universe, the scale factor rebounds at the self-dual length. This rebound is a consequence of the adiabatic expansion. Furthermore, in this situation there are four mathematically different scenarios which describe physically equivalent universes which are in fact undistinguishable. We also stress that $R$-duality suffices to prove that all the possible evolutions present a maximum temperature.
Duality symmetry is the most important stringy symmetry from many points of view (for a review see [1]). Let us suppose we have a string propagating in a target space $R^{d-1} \times S^1$ where we have set the radius of the compactified dimension equal to $R$. It is a well known fact that every correlation function $A(1,\ldots,N)$ can be written as a topological expansion in the string coupling constant

$$A(1,\ldots,N) = \sum_{g=0}^{\infty} g_{st}^{2(g-1)} A_g(1,\ldots,N),$$

where $A_g$ is the correlator at fixed genus. Duality symmetry means that $A(1,\ldots,N)$ as a function of $R$ and $g_{st}$ is invariant under the replacement

$$R \rightarrow \frac{\alpha'}{R}, \quad g_{st} \rightarrow \frac{\sqrt{\alpha'}}{R} g_{st},$$

(2)

together with an interchange between the momentum and the winding modes of the external states. In other words, we are unable to distinguish between small and large $R$ provided we change the string coupling constant properly. Since this symmetry is preserved by the whole topological expansion, we have that if broken it cannot be within the realm of string perturbation theory.

Since any string scattering experiment is unable to tell us whether we are living in a universe with size $R$ and string coupling constant $g_{st}$ or in a universe with the dual values, it has been argued by some authors [3] that this defines in fact a minimum measurable length at the self-dual distance $\sqrt{\alpha'}$. This has led to a modification of the Heisenberg uncertainty principle in order to include this new feature implied by String Theory

$$\Delta x \sim \frac{\hbar}{\Delta E} + \alpha' \Delta E.$$  

(3)

Thus, independently of the value of $\Delta E$, the uncertainty in the position $\Delta x$ is above some minimum value of the order of $\sqrt{\alpha'}$.

The implications of duality symmetry are very important from a phenomenological point of view (see for example [4]). However, in this work we are only concerned with the meaning of duality symmetry in the cosmological context. This subject has been already investigated in a number of works [5, 6, 7, 8, 9, 10]. In ref. [5] it was argued that duality symmetry together with the existence of a Hagedorn temperature for the string gas filling the universe would imply that the size of the universe as a function of the cosmic time had to rebound at the self-dual size. In this picture one starts with a universe with all spatial dimensions compactified and of the order of the Planck length and ends up with a universe in which only three of the spatial dimensions have grown above the self-dual size while the others remain at the Planck length scale. In ref. [10] the numerical solutions for a two-dimensional space-time filled with a gas
of two-dimensional critical strings was studied and it was found that there exists a class of
solutions for which the scale factor decreases from infinity to zero value (see Fig. 1). The
existence of this type of solutions seems to be contradictory with the scenario depicted by
Brandenberger and Vafa in [5], since no dynamical rebound of the scale factor is seen.

In this brief letter we give an interpretation for the results found in [10] that are compatible
with the image of the scale factor rebounding at the self-dual length. Although our discussion
will be concentrated on a two-dimensional model, the conclusions extracted can be extrapolated
to other scenarios with or without Hagedorn temperature.

Let us suppose we have a string moving in a target space $\mathbb{R}^{d-1} \times S^1$ with metric
\[ ds^2 = ds_{d-1}^2 + R^2 d\theta^2 , \quad (4) \]
where $ds_{d-1}^2$ is the line element in $\mathbb{R}^{d-1}$ and $\theta \in [0, 2\pi)$. Let also the parameter $R$
become dynamical, i.e., a function $R(t)$ of the cosmic time. It is easy to see, and certainly surprising,
that the Brans-Dicke action
\[ S = \int d^d x \sqrt{-g} \left[ \Phi (R - 2\Lambda) - \frac{\omega}{\Phi} \nabla_\mu \Phi \nabla^\mu \Phi \right] , \quad (5) \]
endowed with the ansatz (4) and a space-independent Brans-Dicke field $\Phi(t)$ is invariant under
the following substitution [6, 11]
\[ R(t) \rightarrow \alpha' R(t) , \quad \Phi(t) \rightarrow \frac{R(t)^2}{\alpha'} \Phi(t) , \quad (6) \]
However it is worth stressing that this symmetry, although closely resembles duality in string
theory upon the identification of the square root of the string coupling constant with the inverse
of the dilaton field, it is not the same thing than (2). The reason is that duality is a symmetry
between two backgrounds. In the case discussed before these vacua are two copies of $\mathbb{R}^{d-1} \times S^1$
with dual values for the radius of the compactified dimension. In our case $R(t)$ is no longer a
constant, and the background space is the product of $\mathbb{R}^{d-2}$ by a kind of trumpet with topology
$\mathbb{R} \times S^1$. Duality symmetry will relate this background manifold with other background manifold
which, in general, will not be that obtained by substituting $R(t) \rightarrow \alpha'/R(t)$. There is however
a way in which (2) and (6) can be related. If the variation of $R(t)$ is not very wild, physically,
the string at any instant of time would think that the background space in which it is moving
has a constant radius for the compatified dimension. This implies that, at any instant of time $t$, all physical observables will enjoy duality symmetry (2) with $R = R(t)$ which is exactly the
same symmetry that the Brans-Dicke action has. This means, in particular, that the matter
action to be coupled to (2) will also be invariant under (6) and then all physical quantities
derived from this action functional will present the same symmetry. It is worth remarking
that in this sense duality symmetry has been present in Physics since the sixties waiting for a theory (String Theory) that could provide a matter action with the same symmetry. This matter action in the case at hand can be constructed from the one-loop Helmholtz free energy in the way described in [10] (cf. also [7]).

In [10] we studied numerically the cosmological solutions for a two-dimensional universe $S^1 \times \mathbb{R}$ filled with string matter. As we said, there are solutions that describe a universe contracting from infinite size down to $R = 0$. From a mathematical point of view we find that duality symmetry does not imply the existence of a dynamical rebound at the self-dual radius. The only meaning of duality from a mathematical point of view is that if we find a solution described by $R(t)$ and $\Phi(t)$ the dualized functions are also solutions of the system of differential equations. However these solutions present a strange feature, at least strange from a quantum field theoretical point of view: although the contraction is adiabatic, as the size of the universe monotonously goes to zero the temperature reaches a maximum and then decreases and is actually zero for zero size. Comparing $T(t)$ with $R(t)$ one finds that the maximum is gotten when the size is the self-dual point of the duality transformation. What we really then have is that the number of excited degrees of freedom below the self-dual point does not increase; on the contrary it decreases and we have the same number at $R = 0$ than at $R = \infty$. These are the workings of the equation of state which differ from the field theoretical case $\rho = p$. The entropy of a system measures, roughly speaking, the number of degrees of freedom that can be excited at a given temperature. For an entropy function which enjoys duality as a function of the volume, it implies the absence of excitable degrees of freedom at small radius. So duality supports the suspicion that some authors have had about the scarce number of degrees of freedom that can be excited below the Planck length (see for example [12]).

Let us elaborate further the analysis of duality symmetry from a purely physical point of view. We have to consider an observer inside our two-dimensional universe performing a set of experiments in order to measure the size $R$, the temperature $\beta^{-1}$ and the strength of the coupling $g_{st}$ (or equivalently $\Phi$). Of course we have to assume that the time required by these experiments is much less than the characteristic time of expansion, in order to regard the universe as locally static. In that case, what duality symmetry would imply is that our one-dimensional experimentalist is unable to decide whether he/she is living in a universe with radius $R$ and a gravitational constant given by $g_{st}$ or in the dual universe (let us remind that the inverse temperature $\beta$ is not affected by duality symmetry). Let us assume that he/she chooses one of the two possible branches arbitrarily; say branch 1 in Fig. 1 defined by $t \leq t_{sd}$, $R(t) \geq \sqrt{\alpha'}$. Since the universe that we are considering (branches 1 and 4 of Fig. 1)
has a monotonously decreasing scale factor, for every time $t_0$ there is a time $t_0^*$ such that

$$R(t_0^*) = \frac{\alpha'}{R(t_0)}.$$  \hfill (7)

The question now is whether the internal observer is able to decide if the state of the universe at $t_0$ is different from that in $t_0^*$. The only way in which this can be accomplished is if both the field $\Phi$ and the inverse temperature $\beta$ at time $t_0^*$ are different from the dual values of $\Phi(t_0)$ and $\beta(t_0)$. Let us consider the case of the inverse temperature $\beta$. To show that $\beta(t_0) = \beta(t_0^*)$ it is enough the fact that the entropy $S(\beta, R)$ is a dual function, i.e.,

$$S(\beta, R) = S\left(\beta, \frac{\alpha'}{R}\right),$$  \hfill (8)

and that the scale factor $R(t)$ is a monotonous function of time. In fact, since $R(t)$ is a monotonous function, we can parametrize the evolution of our universe by $R$ instead of $t$. This means that the entropy is a function of $R$, but since the expansion is adiabatic (i.e., $S(R)$ is constant), we have

$$S(R) = S[\beta(R), R] = S\left[\beta\left(\frac{\alpha'}{R}\right), \frac{\alpha'}{R}\right] = S\left(\frac{\alpha'}{R}\right).$$  \hfill (9)

But, because of $R$-duality we have also

$$S\left(\frac{\alpha'}{R}\right) = S\left[\beta\left(\frac{\alpha'}{R}\right), R\right].$$  \hfill (10)

This, together with the single-valuedness and the monotonous character of the entropy with respect to $\beta$ (the second is the result of the fact that the specific heat at constant volume is positive) implies that

$$\beta(R) = \beta\left(\frac{\alpha'}{R}\right),$$  \hfill (11)

so the temperature at $t_0$ and $t_0^*$ are exactly the same. Duality also holds for the internal energy and then the whole thermodynamics for the universe of size $R$ is indistinguishable from that at $\alpha'/R$. By the way, it is easy to see that this relation implies that the temperature has a extreme at $R = \sqrt{\alpha'}$ (indeed a maximum). Then the only way in which the internal observer could distinguish the universe in $t_0$ from the universe in $t_0^*$ is if

$$\Phi(t_0^*) \neq \frac{R^2(t_0)}{\alpha'}\Phi(t_0).$$  \hfill (12)

Then we have to address the problem of measuring $\Phi$ (or equivalently $g_{st}$). Since we are working at one (thermal) loop level, we are neglecting any interaction between strings. This means that
in this approximation the strings are free and then there is no hope of measuring the string coupling constant, so $g_{st}$ is not an observable for our internal observer. The question about whether (12) holds seems to be irrelevant at one loop. It would be necessary to make the computation to two loops in order $g_{st}$ to be a measurable parameter. Nevertheless it is well known that static duality including the transformation of the dilaton field arises in the sigma model even computing at world-sheet tree level [13]. To better understand the situation, let us summarize the way in which the observer would describe the evolution of his/her universe. We have assumed that in the initial measurement at $t_0$ the observer has chosen one of the two possible branches (say, $R \geq \sqrt{\alpha'}$). Then, at successive instants $t_1, \ldots, t_{n-1}$ he/she performs different measurements of $R$ and plot them. When the scale factor as measured from outside decreases below the self-dual distance (i.e., when $t > t_{sd}$) the result of the measurements will be exactly the same as those for a given $t' < t_{sd}$ and this is interpreted by our experimentalist as the indication that the scale factor has suffered a rebound after reaching the self-dual size. The observer, travelling forward in time, has actually jumped from the solution coming from infinity to the dual one, i.e. $\alpha'/R(t)$, heading back to infinity (he has jumped from branch 1 to branch 2 in Fig 1). If, after jumping, he/she tries to put to test Einstein-Brans-Dicke equations the result will be that they hold, provided that the dilaton is changed from $\Phi$ to $R^2/\alpha' \Phi$. Indeed these equations are the conditions up to the lowest order in $\alpha'$ for the sigma model to define a conformal theory. Both branches (1 and 2 in Fig. 1) differ because of the initial conditions but these conditions are dual between them. The branches are glued together at the self-dual point in a non smoothly way; the first derivative of the scale factor at $t_{sd}$ jumps by a finite quantity.

We have seen that the observer travelling forward in time has the possibility of interpreting his/her Universe either as coming from infinity and going to a cold end without size or coming from infinity and bouncing back to expand at the Planck length. Once he/she is in a universe which expands it is possible by inverting time to go back to the universe of self-dual size (of course theoretically). After moving a lapse back in time from $t = t_{sd}$ the observer also have two different branches to choose. He/she can either go back to infinite size (branch 1 in Fig.1) or take another path smoothly going to zero size (branch 3 in Fig.1). It is clear that a solution $R(t)$ and its dual $\alpha'/R(t)$ define four physically equivalent scenarios given by taking together branch 1 and 4 or 2 and 3 or 1 and 2 or 3 and 4 (see Fig. 1). This is precisely the number of different ways we can represent the solitonic contribution to the partition function computed in a static background [14]: as a sum over a pair of windings, as a sum over windings and momenta (we have two possibilities of this kind), and as a sum over a pair of momenta, all of them are related by applying Poisson summation formula to the solitonic sum in the partition function. Each time our observer jumps at the self-dual distance, he/she performs a Poisson
summation to redefine the eigenstates of the position operator.

An important consequence of the combined effect of duality and the adiabatic expansion is the fact that for every solution of those described in [10] we find a maximum value of the temperature in the universe. In particular, for those solutions which go through the self-dual size this maximum can be easily seen to be located at $t_{sd}$ [10]. The absence of a Hagedorn temperature in this case makes the analysis more transparent than for critical strings. However we see the similitude with the case discussed in [3]. There, since all the spatial dimensions were compactified, the Hagedorn temperature was the maximum temperature of the universe and was reached at the self-dual size. The rebound of the scale factor was there, as here, a consequence of the change between momentum and winding modes to describe localized states. In our scenario there is a maximum temperature although we do not have a Hagedorn one. This indicates that the scenario depicted in [3] is independent of the existence of a Hagedorn temperature, and can be extended to other situations as that studied in [10].

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Figure 1: Scale factor $R(t)$ versus time showing the four possible branches.