CONVERGENCE OF DENSITIES OF SPATIAL AVERAGES OF
STOCHASTIC HEAT EQUATION

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Abstract. In this paper, we consider the one-dimensional stochastic heat equation driven
by a space time white noise. In two different scenarios: (i) initial condition $u_0 = 1$ and
general nonlinear coefficient $\sigma$ and (ii): initial condition $u_0 = \delta_0$ and $\sigma(x) = x$ (Parabolic
Anderson Model), we establish rates of convergence for the uniform distance between the
density of (renormalized) spatial averages and the standard normal density. These results are
based on the combination of Stein method for normal approximations and Malliavin calculus
techniques. A key ingredient in Case (i) is a new estimate on the $L^p$-norm of the second
Malliavin derivative.

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1. Introduction

Consider the one-dimensional stochastic heat equation
\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \sigma(u) W, \quad x \in \mathbb{R}, \ t > 0, \]  \tag{1.1}

with initial condition $u(0, x) = u_0(x)$, where $W$ is a space-time white noise. This is to say,
informally, that $W = \{W(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$ is a centered Gaussian random field with
covariance
\[ \mathbb{E}(W(t_1, x_1)W(t_2, x_2)) = \delta_0(t_1 - t_2)\delta_0(x_1 - x_2), \]
where $\delta_0$ is the Dirac delta measure at zero.

The existence and uniqueness of a mild solution $u(t, x)$ to equation (1.1) has been proved
by Chen and Dalang in [3] (see also the lecture notes by Walsh [15] in the case where $u_0$ is a
bounded function), assuming that $\sigma$ is Lipschitz and $u_0$ is a signed measure that satisfies the
following integrability condition for any $t > 0$
\[ \int_{\mathbb{R}} |u_0|(dx)p_t(x) < \infty. \]  \tag{1.2}

Here and along the paper we will make use of the notation $p_t(x) := \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ for $t > 0$ and
$x \in \mathbb{R}$.

We are interested in the asymptotic behavior of the spatial averages of the solution to
equation (1.1) in the following two particular cases:

Case 1: $u_0 \equiv 1$ and $\sigma : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function such that $\sigma(1) \neq 0$.

Case 2: $u_0 = \delta_0$ and $\sigma(x) = x$.

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We observe that, for any $t > 0$, in Case 1, the process $\{u(t, x) : x \in \mathbb{R}\}$ is stationary and in Case 2, the process $\{U(t, x) : x \in \mathbb{R}\}$, where

$$U(t, x) := \frac{u(t, x)}{p_t(x)},$$

is also stationary (see Amir, Corwin and Quastel [1]).

Fix $R > 0$ and consider the corresponding centered and normalized spatial averages defined by

$$F_{R,t} := \frac{1}{\sigma_{R,t}} \left( \int_{-R}^{R} u(t, x) dx - 2R \right),$$

where $\sigma_{R,t}^2 := \text{Var} \left( \int_{-R}^{R} u(t, x) dx \right)$ (1.3) in Case 1, and

$$G_{R,t} := \frac{1}{\Sigma_{R,t}} \left( \int_{-R}^{R} U(t, x) dx - 2R \right),$$

where $\Sigma_{R,t}^2 := \text{Var} \left( \int_{-R}^{R} U(t, x) dx \right)$ (1.4) in Case 2.

Huang, Nualart and Viitasaari [10] and Chen, Khoshnevisan, Nualart and Pu [7], studied the limiting behavior of $F_{R,t}$ and $G_{R,t}$, respectively, as $R$ tends to infinity. In these papers, functional central limit theorems have been established. Moreover, by the Malliavin-Stein approach, introduced by Nourdin and Peccati (see [12]), upper bounds for the total variation distance have been obtained. More precisely, it has been proven that, for any fixed $t > 0$ and for all $R \geq 1$, 

$$d_{TV}(F_{R,t}, N(0, 1)) \leq \frac{C_t}{\sqrt{R}},$$

and

$$d_{TV}(G_{R,t}, N(0, 1)) \leq \frac{C_t \sqrt{\log R}}{\sqrt{R}},$$

where $d_{TV}$ denotes the total variation distance and $C_t$ is a constant depending on $t$.

The purpose of this paper is to derive upper bounds for the rate of convergence of the uniform distance of densities in the two cases above mentioned. Upper bounds for the uniform distance of densities using techniques of Malliavin calculus were first derived by Hu, Lu and Nualart in [9]. Inspired by the methodology introduced in this reference, we have been able to obtain the following two main results. In Case 1, we will make use of the following hypothesis on $\sigma$:

(H1): $\sigma : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable function with $\sigma'$ bounded and $|\sigma''(x)| \leq C(1 + |x|^m)$, for some $m > 0$.

**Theorem 1.1.** In Case 1, let $u = \{u(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\}$ be the mild solution to the stochastic heat equation (1.1). Assume that $\sigma$ satisfies hypothesis (H1). Suppose also that for some $q > 10$, $E[|\sigma(u(t, 0))|^{-q}] < \infty$. Fix $t > 0$ and let $F_{R,t}$ be defined as in (1.3). Then, for all $R > 0$,

$$\sup_{x \in \mathbb{R}} |f_{F_{R,t}}(x) - \phi(x)| \leq \frac{C_t}{\sqrt{R}},$$

where $f_{F_{R,t}}$ and $\phi$ are the densities of $F_{R,t}$ and $N(0, 1)$, respectively.

We remark that condition $E[|\sigma(u(t, 0))|^{-q}] < \infty$ holds if $\sigma$ is bounded away from zero or if $|\sigma(x)| \leq \Lambda |x|$ for all $x \in \mathbb{R}$ and for some constant $\Lambda > 0$ (see, for instance, [4, Theorem 1.5]).
Theorem 1.2. In Case 2, assume that the random field \( u = \{u(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\} \) solves the stochastic heat equation

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + u \dot{W}, \quad x \in \mathbb{R}, \ t > 0,
\]

with the initial condition \( u_0(x) = \delta_0 \). Let \( G_{R, t} \) be defined as in (1.4). Fix \( \gamma > \frac{19}{2} \). Then, there exists an \( R_0 \geq 1 \) such that for all \( R \geq R_0 \)

\[
\sup_{x \in \mathbb{R}} |f_{G_{R, t}}(x) - \phi(x)| \leq \frac{C_t (\log R)^\gamma}{\sqrt{R}},
\]

where \( f_{G_{R, t}} \) and \( \phi \) are the densities of \( G_{R, t} \) and \( N(0, 1) \), respectively.

The organization of the paper is as follows. In Section 2 we introduce some preliminaries on the definition of mild solution and we recall some basic elements of Malliavin calculus. In Section 3 we establish the basic upper bound for the uniform distance between the density of a given random variable \( F \), which is a functional of an underlying isonormal Gaussian process, and the standard Gaussian density (see Theorem 3.2). In this estimate we assume that the random variable \( F \) has a representation as a divergence, that is, \( F = \delta(v) \). A basic assumption is the existence of negative moments of the projection \( \langle DF, v \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R})} \). Theorem 6.2 in [9] can be considered as a particular case of Theorem 3.2 when \( v = -DL^{-1}F \), where \( L \) is the generator of the Ornstein-Uhlenbeck semigroup. Section 4 is devoted to the proof of Theorem 1.1 and Section 5 provides the proof of Theorem 1.2. Finally, the Appendix contains some technical lemmas used along the paper.

We remark that a fundamental ingredient in the proof of the upper bounds (1.5) and (1.6) is the estimate of the \( p \)-norm of the Malliavin derivative of the solution. More precisely, the upper bounds

\[
\|D_{s, y}u(t, x)\|_p \leq C_{T, p} t^{-s} (x - y)
\]

and

\[
\|D_{s, y}u(t, x)\|_p \leq C_{T, p} t^{-s} (x - y)p_s(y)
\]

for any \( p \geq 2 \) and \( 0 < s < t < T \), \( x, y \in \mathbb{R} \) in Case 1 and Case 2, respectively, play a basic role in proving (1.5) and (1.6). However, when dealing with estimates for the uniform distance of the densities, in view of inequality (3.1) in Theorem 3.2, we need similar estimates but for the \( p \)-norm of the second Malliavin derivative. If \( \sigma(x) = x \) an upper bound for \( \|D_{r, z}D_{s, y}u(t, x)\|_p \) has been obtained by Chen, Khoshnevisan, Nualart and Pu [6, Proposition 3.4] when \( u_0 = 1 \) and by Kuzgun and Nualart in [11] for a general initial condition satisfying (1.2). In Case 1, for a general function \( \sigma \) satisfying Hypothesis (H1), this estimate is established in Proposition 4.1 (see inequality (4.4)). This inequality solves a standing open problem in this topic and it has its own interest.

Along the paper \( C_t \) and \( C_{t, p} \) will denote generic nonnegative and finite constants that depend on \( t \) and \( (t, p) \), respectively and might depend also on \( \sigma \).

2. Preliminaries

We first introduce the white noise on \( \mathbb{R}_+ \times \mathbb{R} \). Let \( \mathcal{H} = L^2(\mathbb{R}_+ \times \mathbb{R}) \) and denote by \( \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}) \) the collection of Borel sets \( A \in \mathbb{R}_+ \times \mathbb{R} \) with finite Lebesgue measure, denoted by \( m(A) \). Consider a family of centered Gaussian random variables \( \{W(A) : A \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R})\} \), defined on a complete probability space \( (\Omega, \mathcal{F}, P) \), with covariance

\[
E[W(A)W(B)] = m(A \cap B) \text{ for all } A, B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}).
\]
The Wiener integrals
\[ W(h) := \int_{\mathbb{R}^+ \times \mathbb{R}} h(s, y)W(ds, dy), \quad h \in \mathcal{H}, \]
define an isornormal Gaussian process on the Hilbert space \( \mathcal{H} \).

For \( t > 0 \), let \( \mathcal{F}_t \) denote the \( \sigma \)-field generated by \( \{ W([0, s] \times A) : 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R}) \} \) and the \( P \)-null sets. A random field \( X = \{ X(s, y) : (s, y) \in \mathbb{R}^+ \times \mathbb{R} \} \) is said to be adapted if \( X(s, y) \) is \( \mathcal{F}_s \)-measurable for each \( (s, y) \in \mathbb{R}^+ \times \mathbb{R} \). For any adapted random field \( X \), that is also jointly measurable and square integrable, i.e.
\[ \int_{\mathbb{R}^+} \int_{\mathbb{R}} E \left[ X^2(s, y) \right] dy ds < \infty, \]
the stochastic integral
\[ \int_{\mathbb{R}^+} \int_{\mathbb{R}} X(s, y)W(ds, dy) \]
is well defined (see [15]).

The following proposition (see [3, 15]) ensures the existence and uniqueness of a mild solution to equation (1.1) in Case 1 and Case 2.

**Proposition 2.1.** In Case 1, there exists a unique measurable and adapted random field \( u = \{ u(t, x) : (t, x) \in \mathbb{R}^+ \times \mathbb{R} \} \) such that for all \( T > 0 \) and \( p \geq 2 \)
\[ \sup_{(t, x) \in [0, T] \times \mathbb{R}} E \left[ |u(t, x)|^p \right] = C_{T, p}, \tag{2.1} \]
and for all \( t \geq 0 \) and \( x \in \mathbb{R} \)
\[ u(t, x) = 1 + \int_{[0, t] \times \mathbb{R}} p_{t-s}(x - y)\sigma(u(s, y))W(ds, dy). \tag{2.2} \]
In Case 2, there exists a unique measurable and adapted random field \( u = \{ u(t, x) : (t, x) \in (0, \infty) \times \mathbb{R} \} \) such that for all \( T > 0 \), \( t \in (0, T) \), \( x \in \mathbb{R} \) and \( p \geq 2 \)
\[ E \left[ |u(t, x)|^p \right] \leq C_{T, p} p_t(x), \tag{2.3} \]
and for all \( t > 0 \) and \( x \in \mathbb{R} \)
\[ u(t, x) = p_t(x) + \int_{[0, t] \times \mathbb{R}} p_{t-s}(x - y)u(s, y)W(ds, dy). \tag{2.4} \]

2.1. **Malliavin Calculus.** We recall some basic facts from Malliavin calculus. For a broader exposition, we refer to [13]. We recall that \( \mathcal{H} = L^2(\mathbb{R}^+ \times \mathbb{R}) \). Let \( \mathcal{S} \) denote the collection of smooth and cylindrical random variables of the form
\[ F = f \left( W(h_1), \ldots, W(h_n) \right), \tag{2.5} \]
where \( f \in C^\infty(\mathbb{R}^n) \), that is, \( f \) is infinitely differentiable and all its partial derivatives are bounded, and \( h_1, \ldots, h_n \in \mathcal{H} \). For a random variable \( F \in \mathcal{S} \) of the form (2.5), we define its Malliavin derivative by putting
\[ D_{t,x}F = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (W(h_1), \ldots, W(h_n)) h_i(t, x) \]
for $t \geq 0$ and $x \in \mathbb{R}$. For any real number $p \geq 1$ and any integer $k \geq 1$, let $\mathbb{D}^{k,p}$ be the closure of $S$ with respect to the norm

$$
\|F\|_{k,p} = \left( \mathbb{E} \left[ \|F\|^p \right] + \sum_{j=1}^k \mathbb{E} \left[ \|D^j F\|^p_{\mathcal{H}^j} \right] \right)^{1/p},
$$

where $D^j$ denotes the $j$th iterated derivative. If $V$ is a real separable Hilbert space, similarly we can introduce the corresponding Sobolev spaces $\mathbb{D}^{k,p}(V)$ of $V$-valued random variables.

The divergence operator $\delta$ is defined as the adjoint of the derivative operator $D$ as an unbounded operator from $L^1(\Omega)$ to $L^1(\Omega; \mathcal{H})$. Namely, an element $v \in L^1(\Omega; \mathcal{H})$ belongs to the domain of $\delta$, denoted by $\text{Dom} \, \delta$, if there exists an integrable random variable $\delta(v)$ verifying

$$
\mathbb{E}[\delta(v)] = \mathbb{E}[[DF, v]_{\mathcal{H}}],
$$

for any $F \in S$. A crucial property of divergence is the fact that any adapted and square integrable process $v$ belongs to $\text{Dom} \, \delta$ and $\delta(v)$ coincides with the stochastic integral:

$$
\delta(v) = \int_{\mathbb{R}_+ \times \mathbb{R}} v(s, y) W(ds, dy).
$$

The following lemma is a factorization property of the divergence operator, obtained in this generality in [2, Lemma 1].

**Lemma 2.2.** Fix $p, p' > 1$ with $1/p + 1/p' = 1$. Let $F \in \mathbb{D}^{1,p'}, v \in \text{Dom} \, \delta$, be such that $v \in L^p(\Omega; \mathcal{H})$ and $\delta(v) \in L^p(\Omega)$. Then $Fv \in \text{Dom} \, \delta$, and

$$
\delta(Fv) = F\delta(v) - \langle DF, v \rangle_{\mathcal{H}}.
$$

Because $\delta$ is a continuous linear operator from $D^{1,p}(\mathcal{H})$ to $L^p(\Omega)$, Lemma (2.2) holds true provided $F \in \mathbb{D}^{1,p'}$ and $v \in \mathbb{D}^{1,p}(\mathcal{H})$.

The operators $D$ and $\delta$ satisfy the following commutation relation

$$
D_{s,y}(\delta(v)) = v(s, y) + \delta(D_{s,y}v),
$$

for all almost all $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$, provided $v \in \mathbb{D}^{1,2}(\Omega; \mathcal{H})$ such that for almost all $(s, y) \in \mathbb{R}_+ \times \mathbb{R}$, $D_{s,y}v$ belongs to the domain of the divergence in $L^2$ and $\mathbb{E} \left[ \int_{\mathbb{R}_+ \times \mathbb{R}} |\delta(D_{s,y}v)|^2 dsdy \right] < \infty$ (see [13, Proposition 1.3.2]).

2.2. **Notation.** Recall that $p(t) := \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ for $t > 0$ and $x \in \mathbb{R}$. We will make use of the following identity. For $0 < s < t$, $a, b \in \mathbb{R}$,

$$
p_{t-s}(a)p_s(b) = p_t(a + b)p_{t-s}(b - \frac{s}{t}(a + b)).
$$

(2.7)

For a Lipschitz function $f$, let $\text{Lip}_f := \sup_{x \neq y} |f(x) - f(y)| < \infty$. We set

$$
\Sigma_{t,x} := \sigma(u(t,x)), \quad \Sigma_{t,x}^{(1)} := \sigma'(u(t,x)), \quad \Sigma_{t,x}^{(2)} := \sigma''(u(t,x)).
$$

(2.8)

Set $Q_R := [-R, R]$, and define for $0 < s < t$ and $y \in \mathbb{R}$

$$
\phi_{R,t}(s, y) := \frac{1}{\sigma_{R,t}} \int_{Q_R} p_{t-s}(x - y) dx,
$$

(2.9)
and
\[
\varphi_{R,t}(s,y) := \frac{1}{\Sigma_{R,t}} \int_{Q_R} p_{s(t-s)/t}(y - \frac{s}{t}x)dx.
\]
(2.10)

For an \( \mathcal{H} \)-valued random variable \( v : \Omega \to \mathcal{H} \) and \( F \in \mathbb{D}^{1,1} \), we put
\[
D_vF := \langle DF, v \rangle_{\mathcal{H}}.
\]

3. Existence and estimates of densities

In this section, we recall the basic density formula for one-dimensional random variables, and state and prove an estimate on the uniform distance between the density of a random variable and the standard normal density, see Theorem 3.2. The proof of this theorem follows the lines of the proof Theorem 6.2 of [9]. For the sake of completeness, we include here some details of the proof. These results will be later applied in the context of spatial averages of the stochastic heat equation.

The results of this section are valid in the framework of a general isonormal Gaussian process \( W = \{ W(h) : h \in \mathcal{H} \} \) on a Hilbert space \( \mathcal{H} \). The following density formula under general assumptions on the random variable has been proved in [2, Proposition 1].

**Proposition 3.1.** Let \( F \in \mathbb{D}^{1,1} \) and \( v \in L^1(\Omega; \mathcal{H}) \) be such that \( D_vF \neq 0 \) a.s. Assume that \( v/D_vF \in \text{Dom } \delta \). Then the law of \( F \) has a continuous and bounded density given by
\[
f_F(x) = \mathsf{E} \left[ 1_{\{ F > x \}} \delta \left( \frac{v}{D_vF} \right) \right].
\]

Using Lemma 2.2, in the context of Proposition 3.1, the following constitute sufficient conditions for \( v/D_vF \in \text{Dom } \delta \), for some \( p, p' \) with \( 1/p + 1/p' = 1 \) (see [2, Lemma 3]):

(i) \( (D_vF)^{-1} \in L^{p'}(\Omega) \).

(ii) \( v \in \mathbb{D}^{1,p}(\mathcal{H}) \).

In view of [2, Lemma 4], a sufficient condition for (i) is \( (D_vF)^{-1} \in L^{p'}(\Omega) \) and
\[
(D_vF)^{-2} \left[ \| D^2F \|_{\mathcal{H} \otimes \mathcal{H}} \| v \|_\mathcal{H} + \| Dv \|_{\mathcal{H} \otimes \mathcal{H}} \| DF \|_\mathcal{H} \right] \in L^{p'}(\Omega).
\]

Therefore, assuming that \( F \in \mathbb{D}^{2,p} \) and \( (D_vF)^{-1} \in L^q(\Omega) \), then condition (i) holds if \( 2/q + 3/p = 1 \) for some \( p > 3 \) and \( q > 2 \). In particular, we can take \( q = 4 \) and \( p = 6 \).

**Theorem 3.2.** Assume that \( v \in \mathbb{D}^{1,6}(\Omega; \mathcal{H}) \) and \( F = \delta(v) \in \mathbb{D}^{2,6} \) with \( \mathsf{E}[F] = 0 \), \( \mathsf{E}[F^2] = 1 \) and \( (D_vF)^{-1} \in L^4(\Omega) \). Then, \( v/D_vF \in \text{Dom } \delta \), \( F \) admits a density \( f_F(x) \) and the following inequality holds true
\[
\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq \left( \| F \|_4 \| (D_vF)^{-1} \|_4 + 2 \right) \| 1 - D_vF \|_2
\]
\[
+ \| (D_vF)^{-1} \|_4^2 \| D_v (D_vF) \|_2,
\]
(3.1)

where \( \phi(x) \) is the density of the law \( \mathcal{N}(0,1) \).

**Proof.** First, note that, by Proposition 3.1, \( F \) admits a density \( f_F(x) = \mathsf{E} \left[ 1_{\{ F > x \}} \delta(\bar{v}) \right] \), where \( \bar{v} = v/D_vF \). As a consequence, we can write
\[
\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| = \sup_{x \in \mathbb{R}} \left| \mathsf{E} \left[ 1_{\{ F > x \}} \delta(\bar{v}) \right] - \mathsf{E} \left[ 1_{\{ \mathcal{N} > x \}} \mathcal{N} \right] \right|,
\]
(3.2)
where \( N \) denotes a \( N(0,1) \) random variable. We have
\[
\delta(\bar{v}) = \delta \left( \frac{v}{D_v F} \right) = \frac{F}{D_v F} - D_v \left( \frac{1}{D_v F} \right) = \frac{F}{D_v F} + \frac{D_v (D_v F)}{(D_v F)^2}. \tag{3.3}
\]
Indeed, the second equality follows from Lemma 2.2 together with \( F = \delta(v) \), and the third one follows from the chain rule. Then, substituting (3.3) into (3.2), yields
\[
\Phi_x := \left| E \left[ 1_{\{F>x\}} \delta(\bar{v}) \right] - E \left[ 1_{\{N>x\}} N \right] \right|
= \left| E \left[ 1_{\{F>x\}} \frac{F}{D_v F} \right] - E \left[ 1_{\{F>x\}} \frac{D_v F}{(D_v F)^2} \right] \right| - E \left[ 1_{\{N>x\}} N \right]. \tag{3.4}
\]
Adding and subtracting \( E \left[ 1_{\{F>x\}} F \right] \) in (3.4), we get
\[
\Phi_x \leq E \left[ \left| \frac{(1 - D_v F)}{D_v F} \right| \right] + E \left[ \frac{D_v (D_v F)}{(D_v F)^2} \right] + \left| E \left[ F 1_{\{F>x\}} - N 1_{\{N>x\}} \right] \right|. \tag{3.5}
\]
Applying Hölder’s inequality to the first term, we obtain
\[
E \left[ \left| \frac{(1 - D_v F)}{D_v F} \right| \right] \leq \| F \|_4 \left\| (D_v F)^{-1} \right\|_4 \| 1 - D_v F \|_2. \tag{3.6}
\]
Meanwhile, applying Hölder’s inequality to the second term, we get
\[
E \left[ \frac{D_v (D_v F)}{(D_v F)^2} \right] \leq \left\| (D_v F)^{-1} \right\|_4^2 \left( D_v (D_v F) \right). \tag{3.7}
\]
Finally, applying Stein’s method with \( h(y) = y 1_{\{y>x\}} \) (see [12, Chapter 3]), and using the integration by parts formula \( E[F f(F)] = E[f'(F)(DF,v)_H] \), we have
\[
\left| E \left[ F 1_{\{F>x\}} - N 1_{\{N>x\}} \right] \right| = E \left[ f'_h(F) - F f_h(F) \right]
= E \left[ f'_h(F) (1 - D_v F) \right] \leq \| f'_h(F) \|_2 \| 1 - D_v F \|_2, \tag{3.8}
\]
where \( f_h \) is the solution to the Stein equation associated to the function \( h \). The estimate in [9, Lemma 2.1] states that \( \| f'_h(y) \| \leq \| y \| + 1 \). Therefore,
\[
\| f'_h(F) \|_2 \leq 2.
\]
Then, substituting (3.6), (3.7) and (3.8) into (3.5) yields the desired estimate.

\[
\square
\]

4. Proof of Theorem 1.1

Before proving Theorem 1.1, we will show two fundamental technical ingredients: Moment estimates for the second Malliavin derivative of the solution and negative moments of the projection of \( DF_{R,t} \) on \( u_{R,t} \), where \( F_{R,t} = \delta(v_{R,t}) \).

4.1. Moment estimates of the second derivative of \( u \). Let \( u \) be the solution to the stochastic heat equation (1.1) with initial condition \( u_0 = 1 \). We know (see [14, Proposition 5.1] or [13, Proposition 2.4.4]) that for all \( (t, x) \in \mathbb{R}_+ \times \mathbb{R} \), \( u(t, x) \in \cap_{p \geq 2} \mathbb{D}^{1,p} \) and the Malliavin derivative \( D_{s,y} u(t, x) \) solves the following linear stochastic integral equation:
\[
D_{s,y} u(t, x) = p_{t-s}(x-y) \sigma(u(s, y)) + \int_{[s,t] \times \mathbb{R}} p_{t-r}(x-\xi) \sigma'(u(\tau, \xi)) D_{s,y} u(\tau, \xi) W(d\tau, d\xi) \tag{4.1}
\]
for almost all \( s \in [0, t]\) and \( y \in \mathbb{R} \). Moreover, the following estimate holds true:

\[
\|D_{s,y} u(t,x)\|_p \leq C_{T,p} p_{t-s}(x-y)
\]  

(4.2)

for all \( 0 \leq s < t \leq T \) and \( x, y \in \mathbb{R} \).

The following proposition provides the corresponding bound for the moments of the second derivative.

**Proposition 4.1.** Let \( u \) be the solution to the stochastic heat equation (1.1) with initial condition \( u_0 = 1 \), and assume hypothesis (H1). Fix \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\). Then \( u(t, x) \in \cap_{p \geq 2} D_{p}^{2} \) and for almost all \( 0 < r < s < t, y, z \in \mathbb{R} \), the second derivative \( D_{r,z} D_{s,y} u(t,x) \) satisfies the following linear stochastic differential equation:

\[
D_{r,z} D_{s,y} u(t,x) = p_{t-s}(x-y)\sigma'(u(s,y)) D_{r,z} u(s,y)
\]

\[
+ \int_{[s,t] \times \mathbb{R}} p_{t-\tau}(x-\xi) \sigma''(u(\tau,\xi)) D_{r,z} u(\tau,\xi) D_{s,y} u(\tau,\xi) W(\tau, d\xi)
\]

\[
+ \int_{[s,t] \times \mathbb{R}} p_{t-\tau}(x-\xi) \sigma'(u(\tau,\xi)) D_{r,z} D_{s,y} u(\tau,\xi) W(\tau, d\xi).
\]  

(4.3)

Moreover, for all \( 0 \leq r < s < t \leq T \) and \( x, y, z \in \mathbb{R} \), we have

\[
\|D_{r,z} D_{s,y} u(t,x)\|_p \leq C_{T,p} \Phi_{r,z,s,y}(t,x),
\]  

(4.4)

where \( C_{T,p} \) is a constant that depends on \( T, p \) and \( \sigma \) and

\[
\Phi_{r,z,s,y}(t,x) := p_{t-s}(x-y)
\]

\[
\times \left( p_{s-r}(y-z) + \frac{p_{t-r}(z-y) + p_{t-r}(z-x) + 1_{\{|y-x|>|z-y|\}}}{(s-r)^{1/4}} \right).
\]  

(4.5)

**Proof.** We will make use of the Picard iteration scheme. For any \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\) we put \( u_0(t, x) = 1 \), and for \( n \in \mathbb{N} \) we inductively define

\[
u_{n+1}(t,x) = 1 + \int_{[0,t] \times \mathbb{R}} p_{t-\tau}(x-\xi) \sigma''(u(\tau,\xi)) W(\tau, d\xi).
\]

Then, for any \( p \geq 2 \), there exists a constant \( c_{T,p} \) such that for all \((t, x) \in [0, T] \times \mathbb{R}\)

\[
\sup_{n \in \mathbb{N}} \|u_n(t,x)\|_p \leq c_{T,p}.
\]  

(4.6)

This result is proved in [13, Theorem 2.4.3] for the case of the stochastic heat equation on \([0, 1]\) with Dirichlet boundary conditions and the proof works similarly for the equation on \(\mathbb{R}\).

We apply the properties of the divergence operator, namely using (2.6), to get that for almost all \((s, y) \in [0, t] \times \mathbb{R}\) and \(x \in \mathbb{R}\),

\[
D_{s,y} u_{n+1}(t, x) = p_{t-s}(x-y)\sigma'(u_n(s,y)) + \int_{[s,t] \times \mathbb{R}} p_{t-\tau}(x-\xi) \Sigma_n^{(1)} u_n(\tau,\xi) W(\tau, d\xi),
\]  

(4.7)

and for almost all \( s > t \), \( D_{s,y} u_{n+1}(t, x) = 0 \), where we made use of the notation (2.8) with

\[
\Sigma_n(s,y) := \sigma(u_n(s,y)) \quad \text{and} \quad \Sigma_n^{(1)}(s,\tau,\xi) := \sigma'(u_n(\tau,\xi)).
\]

It has also been proven in [10, Lemma A.1] that there is a constant \( c_{T,p} \), depending on \( T \) and \( p \), such that for almost all \((s, y) \in [0, t] \times \mathbb{R}\) and for all \((t, x) \in [0, T] \times \mathbb{R}\),

\[
\sup_{n \in \mathbb{N}} \|D_{s,y} u_n(t,x)\|_p \leq c_{T,p} p_{t-s}(x-y).
\]  

(4.8)
Once again using (2.6) and (4.7) together with the Leibniz rule for derivatives, we have, for almost every \( r, z \) such that \( 0 < r < s < t \) and \( z \in \mathbb{R} \),

\[
D_{r, z} D_{s, y} u_{n+1}(t, x) = p_{t-s}(x - y) \Sigma^{(1)}_{t, x} D_{r, z} u_n(s, y) + \int_{[s, t] \times \mathbb{R}} p_{t-r}(x - \xi) \Sigma^{(2)}_{t, x \xi} D_{r, z} D_{s, y} u_n(t, \xi) W(d\tau, d\xi)
\]

\[
+ \int_{[s, t] \times \mathbb{R}} p_{t-r}(x - \xi) \Sigma^{(1)}_{t, x \xi} D_{r, z} D_{s, y} u_n(t, \xi) W(d\tau, d\xi), \tag{4.9}
\]

where \( \Sigma^{(2)}_{t, x \xi} := \sigma''(u_n(t, \xi)) \). Applying Burkholder-Davis-Gundy inequality in (4.9), the estimate (4.8), Hypothesis \((\text{H}1)\) and the moment estimates (4.6), for any \( p \geq 2 \) we have for all \((t, x) \in [0, T] \times \mathbb{R} \),

\[
\|D_{r, z} D_{s, y} u_{n+1}(t, x)\|^2_p \leq C_{T, p} p^2_{t-s}(x - y) p^2_{r-t}(y - z)
+ C_{T, p} \int_{[s, t] \times \mathbb{R}} p^2_{t-r}(x - \xi) p^2_{r-t}(\xi - z) p^2_{t-s}(\xi - y) d\xi d\tau
+ C_{T, p} \int_{[s, t] \times \mathbb{R}} p^2_{t-r}(x - \xi) \|D_{r, z} D_{s, y} u_n(t, \xi)\|_p^2 d\tau d\xi, \tag{4.10}
\]

for some constant \( C_{T, p} > 0 \) which depends on \( T, p \) and \( \sigma \). Let \( J \) be the measure on \([s, t] \times \mathbb{R}\) defined by

\[
J(d\tau, d\xi) := p^2_{r-t}(\xi - z) \delta_{s, y}(d\tau, d\xi) + p^2_{r-t}(\xi - z) p^2_{t-s}(\xi - y) d\tau d\xi.
\]

Then, we can put the first two summands in (4.10) together and rewrite this inequality as follows:

\[
\|D_{r, z} D_{s, y} u_{n+1}(t, x)\|^2_p \leq C_{T, p} \int_{[s, t] \times \mathbb{R}} p^2_{t-s}(x - y) J(d\tau, d\xi)
+ C_{T, p} \int_{[s, t] \times \mathbb{R}} p^2_{t-r}(x - \xi) \|D_{r, z} D_{s, y} u_n(t, \xi)\|_p^2 d\tau d\xi.
\]

After one iteration, this leads to

\[
\|D_{r, z} D_{s, y} u_{n+1}(t, x)\|^2_p \leq C_{T, p} \int_{s}^{t} \int_{\mathbb{R}} p^2_{t-s}(x - y_1) J(ds_1, dy_1)
+ C_{T, p} \int_{s}^{t} \int_{\mathbb{R}} \int_{s_1}^{s} \int_{\mathbb{R}} p^2_{t-s_1}(x - y_1) p^2_{s_1-s_2}(y_1 - y_2) J(ds_2, dy_2) dy_1 ds_1
+ C_{T, p} \int_{s}^{t} \int_{\mathbb{R}} \int_{s_1}^{s} \int_{\mathbb{R}} p^2_{t-s_1}(x - y_1) p^2_{s_1-s_2}(y_1 - y_2) \|D_{r, z} D_{s, y} u_{n-1}(s_2, y_2)\|_p^2 dy_2 dy_1 ds_2 ds_1.
\]

If we perform \( n - 1 \) iterations, taking into account that \( \|D_{r, z} D_{s, y} u_1(s, y)\|_p^2 = 0 \), we obtain

\[
\|D_{r, z} D_{s, y} u_{n+1}(t, x)\|^2_p \leq C_{T, p} \int_{s}^{t} \int_{\mathbb{R}} p^2_{t-s_1}(x - y_1) J(ds_1, dy_1)
+ \sum_{k=1}^{n-1} C_{T, p}^{k+1} \int_{s}^{t} \int_{\mathbb{R}} \int_{s_1}^{s} \int_{\mathbb{R}} \cdots \int_{s_{k-1}}^{s_k} \int_{\mathbb{R}} p^2_{t-s_1}(x - y_1) p^2_{s_1-s_2}(y_1 - y_2) \cdots
\times p^2_{s_{k-1}-s_{k+1}}(y_k - y_{k+1}) J(ds_{k+1}, dy_{k+1}) dy_k ds_k \cdots dy_1 ds_1.
\]
For $0 \leq r < s < t$, $x, y, z \in \mathbb{R}$, set

$$K_{r,z,s,y}^2(t, x) := \int_s^t \int_{\mathbb{R}} p_{t-s}^2(x - y_1)J(ds_1, dy_1). \quad (4.11)$$

For the sake of simplicity, we use $K^2(t, x)$ to denote $K_{r,z,s,y}^2(t, x)$. The identity $p_t(x) = \frac{1}{\sqrt{2\pi t}} p_{t/2}(x)$ now implies

$$\|D_{r,z}D_{s,y}u_{n+1}(t, x)\|_p^2 \leq C_{T,p}K^2(t, x)$$

$$+ \sum_{k=1}^{n-1} \frac{C_{T,p}^{k+1}}{(2\pi)^{k+1/2}} \int_{s<s_{k+1}<\cdots<s_2<s_1<t} ds_k \cdots ds_1 \int_{\mathbb{R}^{k+1}} dy_1 \cdots dy_k$$

$$\times [(t-s_1)(s_1-s_2)\cdots(s_k-s_{k+1})]^{-\frac{1}{2}}$$

$$\times p_{t-s_k}(x-y_1)p_{t-s_{k+1}}(y_1-y_2)\cdots p_{t-s_{k+1}}(y_k-y_{k+1})J(ds_{k+1}, dy_{k+1}).$$

Integrating in the variables $y_1, \ldots, y_k$ and using the semigroup property of the heat kernel yields

$$\|D_{r,z}D_{s,y}u_{n+1}(t, x)\|_p^2 \leq C_{T,p}K^2(t, x) \cdot \left( \prod_{k=1}^{n-1} \frac{C_{T,p}^{k+1}}{(2\pi)^{k+1/2}} \int_{0<r_k<\cdots<r_2<r_1<1} dr_k \cdots dr_1 \right)$$

$$\times [(1-r_1)(r_1-r_2)\cdots r_k]^{-\frac{1}{2}} \int_{\mathbb{R}} \int_s^t (t-\tau)^{-\frac{k+1}{2}} p_{t-\tau}(x-\xi)J(d\tau, d\xi)$$

$$\leq C_{T,p}K^2(t, x) \cdot \left( \prod_{k=1}^{n-1} \frac{\Gamma(1/2)^k C_{T,p}^{k+1}}{(2\pi)^{k+1/2} \Gamma(k/2)} \int_{\mathbb{R}} \int_s^t (t-\tau)^{-\frac{k+1}{2}} p_{t-\tau}(x-\xi)J(d\tau, d\xi) \right)$$

$$\leq C K^2(t, x) \cdot \left( \prod_{k=1}^{n-1} \frac{\Gamma(1/2)^k C_{T,p}^{k+1} T^{k+1}}{(2\pi)^{k+1/2} \Gamma(k/2)} \right) K^2(t, x)$$

$$=: \tilde{C}_{T,p}^2 K^2(t, x).$$

Using Lemma A.1, we arrive at the upper-bound

$$\sup_{n \in \mathbb{N}} \|D_{r,z}D_{s,y}u_n(t, x)\|_p \leq \tilde{C}_{T,p} \Phi_{r,z,s,y}(t, x).$$
As a consequence, applying Minkowski’s inequality we can write
\[
\sup_{n \in \mathbb{N}} E \left[ \|D^2u_n(t, x)\|_{\Omega \otimes \mathcal{H}}^p \right] \leq \sup_{n \in \mathbb{N}} \left( \int_{[0,t]^2} \|D_{r,z}D_{s,y}u_n(t, x)\|_p^2 dy dz dr ds \right)^{\frac{p}{2}} 
\leq \tilde{C}_{T,p}^p \left( 2 \int_0^t \int_0^s \Phi_{r,z,s,y}(t, x) dz dy dr ds \right)^{\frac{p}{2}} < \infty.
\]
Since \( u_n(t, x) \) converges in \( L^p(\Omega) \) to \( u(t, x) \) for all \( p \geq 2 \), we deduce that \( u(t, x) \in \cap_{p \geq 2} D_{2,p}^2 \).

4.2. Negative moments. Recall that \( u \) satisfies equation (2.2). For any fixed \( t > 0 \), the random variable \( F_{R,t} \) defined in (1.3) is given by
\[
F_{R,t} = \frac{1}{\sigma_{R,t}} \left( \int_{Q_R} \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(u(s, y))W(ds, dy) dx \right)
= \int_0^t \int_{\mathbb{R}} \frac{1}{\sigma_{R,t}} \left( \int_{Q_R} p_{t-s}(x-y)\sigma(u(s, y)) dx \right) W(ds, dy),
\]
where we recall that \( Q_R = [-R, R] \). So, taking into account that the Itô-Walsh stochastic integral coincides with the divergence operator, we obtain the representation
\[
F_{R,t} = \delta(v_{R,t}),
\]
where
\[
v_{R,t}(s, y) = 1_{[0,t]}(s) \frac{1}{\sigma_{R,t}} \int_{Q_R} p_{t-s}(x-y)\sigma(u(s, y)) dx.
\] (4.12)

In order to apply Theorem 3.2 we need estimates on the negative moments of \( D_{v_{R,t},F_{R,t}} \). The next proposition provides the desired estimates.

**Proposition 4.2.** Let \( u \) be the solution to the integral equation (2.2) and assume that \( \sigma \) is Lipschitz. Fix \( p \geq 2 \), \( t > 0 \) and assume that there exists \( q > 5p \) such that \( E \left[ |\sigma(u(t, 0))|^{-2q} \right] < \infty \). Then, there exists \( R_0 > 0 \) such that
\[
\sup_{R \geq R_0} E \left[ |D_{v_{R,t},F_{R,t}}|^{-p} \right] < \infty.
\] (4.13)

**Proof.** Consider the Malliavin derivative of \( F_{R,t} \) given by
\[
D_{r,z}F_{R,t} = \frac{1}{\sigma_{R,t}} \int_{Q_R} dx D_{r,z}u(t, x).
\]
From (4.12) and (1.3), we can write
\[
D_{v_{R,t}}F_{R,t} = \int_0^t \int_{\mathbb{R}} v_{R,t}(r,z)D_{r,z}F_{R,t}dzdr
= \frac{1}{\sigma_{R,t}^2} \int_{Q^2_R} \int_0^t \int_{\mathbb{R}} p_{t-r}(x_1 - z)\sigma(u(r,z))D_{r,z}u(t,x_2)dzdrdx_1dx_2
= \frac{1}{\sigma_{R,t}^2} \int_{Q^2_R} \int_0^t \int_{\mathbb{R}} p_{t-r}(x_1 - z)\sigma^2(u(r,z))\Psi^{r,z}(t,x_2)dzdrdx_1dx_2, \quad (4.14)
\]
with the notation
\[
\Psi^{r,z}(t,x) = \frac{D_{r,z}u(t,x)}{\sigma(u(r,z))},
\]
for any \( r < t \). Notice that \( \sigma(u(r,z)) \neq 0 \) almost surely because \( \mathbb{E} \left[ |\sigma(u(r,z))|^{-2q} \right] < \infty \) due to our hypothesis and the stationarity of the process \( \{u(r,z) : z \in \mathbb{R}\} \).

We claim that
\[
\Psi^{r,z}(t,x) \geq 0. \quad (4.15)
\]
Indeed, from equation (4.1), it follows that \( \{\Psi^{r,z}(t,x) : (t,x) \in [r,\infty) \times \mathbb{R}\} \) satisfies:
\[
\Psi^{r,z}(t,x) = p_{t-r}(x-z) + \int_{[r,t] \times \mathbb{R}} p_{t-s}(x-y)\sigma'(u(s,y))\Psi^{r,z}(s,y)W(ds,dy).
\]
That means, \( \Psi^{r,z}(t,x) \) solves the heat equation
\[
\frac{\partial \Psi^{r,z}}{\partial t} = \frac{1}{2} \frac{\partial^2 \Psi^{r,z}}{\partial x^2} + \sigma'(u)\Psi^{r,z} \dot{W}, \quad x \in \mathbb{R}, \ t \in [r,\infty),
\]
with initial condition \( \Psi^{r,z}(t,x) \big|_{t=r} = \delta_x(x) \) and, in particular, \( \Psi^{r,z}(t,x) \) is nonnegative.

As a consequence, from (4.14) and (4.15) it follows that \( D_{v_{R,t}}F_{R,t} \geq 0 \) and we can write
\[
D_{v_{R,t}}F_{R,t} \geq \frac{1}{\sigma_{R,t}^2} \int_{Q^2_R} \int_{t-\varepsilon}^t \int_{\mathbb{R}} p_{t-r}(x_1 - z)\sigma(u(r,z))D_{r,z}u(t,x_2)dzdrdx_1dx_2,
\]
for any \( \varepsilon < t \) and \( \alpha < 1 \). Set \( t_{\alpha} := t - \varepsilon^\alpha \). Using this estimate, we get
\[
P \left( D_{v_{R,t}}F_{R,t} < \varepsilon \right) \leq P \left( \frac{1}{\sigma_{R,t}^2} \int_{Q^2_R} \int_{t_{\alpha}}^t \int_{\mathbb{R}} p_{t-r}(x_1 - z)\sigma(u(r,z))D_{r,z}u(t,x_2)dzdrdx_1dx_2 < \varepsilon \right).
\]
With the notation (2.9), using (4.1) we obtain
\[
= \int_{t_{\alpha}}^t \int_{\mathbb{R}} \phi_{R,t}(r,z)\sigma^2(u(r,z))dzdr
+ \int_{t_{\alpha}}^t \int_{[r,t] \times \mathbb{R}} \phi_{R,t}(s,y)\sigma'(u(s,y))D_{r,z}u(s,y)W(ds,dy) \sigma(u(r,z))dzdr,
\]
\[=: I_1 + I_2.\]
From
\[ P (I_1 + I_2 < \varepsilon) \leq P (I_1 < 2\varepsilon) + P (I_1 + I_2 < \varepsilon, I_1 \geq 2\varepsilon) \] (4.16)
we have
\[
P \left( \frac{1}{\sigma_R^2} \int_t^{t_a} \int_R p_{t-r}(x_1 - z)\sigma(u(r, z))D_{r,z}u(t, x_2)dzdx_1dx_2 < \varepsilon \right)
\leq P (I_1 < 2\varepsilon) + P (|I_2| > \varepsilon).
\]

We shall next estimate these probabilities, starting with the first one:
\[
P (I_1 < 2\varepsilon) = P \left( \int_t^{t_a} \int_R \phi_{R,t}^2(r, z)\sigma^2(u(r, z))dzdr < \varepsilon \right)
\leq P \left( \int_t^{t_a} \int_R \phi_{R,t}^2(r, z)\left( \sigma(u(r, z)) - \sigma(u(t, z)) + \sigma(u(t, z)) \right)^2 dzdr < 2\varepsilon \right).
\]

Using the inequality \((a + b)^2 \geq a^2/2 - b^2\) for \(a, b \in \mathbb{R}\), and an estimate similar to (4.16), we get
\[
P \left( \int_t^{t_a} \int_R \phi_{R,t}^2(r, z)\left( \sigma(u(r, z)) - \sigma(u(t, z)) + \sigma(u(t, z)) \right)^2 dzdr < 2\varepsilon \right)
\leq P \left( \int_t^{t_a} \int_R \phi_{R,t}^2(r, z)\sigma^2(u(t, z))dzdr + 6\varepsilon \right)
+ P \left( \int_t^{t_a} \int_R \phi_{R,t}^2(r, z)\left( \sigma(u(r, z)) - \sigma(u(t, z)) \right)^2 dzdr > \varepsilon \right)
=: K_1 + K_2. \tag{4.17}
\]

For the term \(K_1\) in (4.17), by Chebyshev’s inequality, for \(q > 5p\) we obtain
\[
K_1 = P \left( \left[ \int_t^{t_a} \int_R \phi_{R,t}^2(r, z)\sigma^2(u(t, z))dzdr \right]^{-1} > \frac{1}{6\varepsilon} \right)
\leq (6\varepsilon)^q E \left[ \left( \int_t^{t_a} \int_R \phi_{R,t}^2(r, z)\sigma^2(u(t, z))dzdr \right)^{-q} \right]. \tag{4.18}
\]

Set
\[
m(\varepsilon, R) := \int_t^{t_a} \int_R \phi_{R,t}^2(r, z)dzdr.
\]
Then, taking into account that the function \( x \rightarrow x^{-q} \) is convex and applying Jensen’s inequality, we can write
\[
E \left[ \left( \int_{t_0}^t \int_R \phi_{R,t}(r,z) \sigma^2(u(t,z))dzdr \right)^{-q} \right] = m(\varepsilon,R)^{-q}E \left[ \left( \frac{1}{m(\varepsilon,R)} \int_{t_0}^t \int_R \phi_{R,t}(r,z) \sigma^2(u(t,z))dzdr \right)^{-q} \right] \leq m(\varepsilon,R)^{-q-1} \int_{t_0}^t \int_R \phi_{R,t}(r,z)E \left[ |\sigma(u(t,z))|^{-2q} \right] drdz. \tag{4.19}
\]

Since the solution is stationary in space, the factor \( C_t := E \left[ |\sigma(u(t,z))|^{-2q} \right] \) does not depend on \( z \) and we assume it is finite. Therefore, from (4.18) and (4.19), we get
\[
K_1 \leq C_t(6\varepsilon)^q m(\varepsilon,R)^{-q} \tag{4.20}
\]
for some constant \( C_t > 0 \). Moreover,
\[
m(\varepsilon,R) = \frac{1}{\sigma_{R,t}} \int_0^{\varepsilon} \int_{-R}^R \int_{-R}^R p_{2s}(x_1 - x_2)dx_1dx_2ds \geq \frac{\sqrt{2}R}{\sigma_{R,t}} \int_0^{\varepsilon} \int_{-R/\sqrt{2}}^{R/\sqrt{2}} p_{2s}(y)dyds. \tag{4.21}
\]

Then, assuming \( \varepsilon \leq 1 \) and \( R \geq R_0 \), we obtain
\[
m(\varepsilon,R) \geq \frac{\sqrt{2}R}{\sigma_{R,t}} \int_0^{\varepsilon} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} p_{2s}(y)dyds \geq C_t \varepsilon^\alpha, \tag{4.21}
\]
where in the last inequality we have used Lemma A.3 part (a). Hence, from (4.20) and (4.21), we have
\[
K_1 \leq C_t \varepsilon^{(1-\alpha)}. \tag{4.22}
\]

In order to estimate the term \( K_2 \) in (4.17), we use Chebyshev’s inequality followed by Minkowski’s inequality, as follows:
\[
K_2 \leq \varepsilon^{-q}E \left[ \left( \int_{t_0}^t \int_R \phi_{R,t}(r,z) (|\sigma(u(r,z)) - \sigma(u(t,z)))^2dzdr \right)^q \right] \leq \varepsilon^{-q} \left( \int_{t_0}^t \int_R \phi_{R,t}(r,z) \left( E \left[ |\sigma(u(r,z)) - \sigma(u(t,z))|^{2q} \right] \right)^{1/2}dzdr \right)^{q}. \tag{4.23}
\]

The Lipschitz continuity of \( \sigma \) and the 1/4-Hölder continuity of the solution \( u(t,x) \) in \( L^{2q}(\Omega) \) allow us to write for any \( r \in [t_0,t] \)
\[
\|\sigma(u(r,z)) - \sigma(u(t,z))\|_{2q} \leq \text{Lip}_\sigma \|u(r,z) - u(t,z)\|_{2q} \leq C_t \text{Lip}_\sigma |t - r|^{1/4} \leq C_t \text{Lip}_\sigma \varepsilon^{\alpha/4}. \tag{4.24}
\]
Using Fubini’s theorem and Chebyshev’s inequality, we have
\[
\int_{t_n}^t \int_{\mathbb{R}} \phi_{R,t}(r,z) dz dr \leq \frac{1}{\sigma_{R,t}^2} \int_0^\infty \int_{Q_R^2} p_r(x_1 - z)p_r(z - x_2)dz dx_1 dx_2 dr
\]
\[
\leq \frac{1}{\sigma_{R,t}^2} \int_0^\infty \int_{Q_R^2} p_{2r}(x_1 - x_2)dx_1 dx_2 dr \leq \frac{2R}{\sigma_{R,t}^2} \varepsilon^\alpha \leq C_t \varepsilon^\alpha.
\]
(4.25)

Substituting (4.24) and (4.25) into (4.23), yields
\[
K_2 \leq C_t \varepsilon^{\frac{\sigma_{R,t}^2 - 1}{2}}.
\]
(4.26)

We are left to estimate the following probability:
\[
K_3 := \mathbb{P}(\varepsilon | I_2 | > \varepsilon).
\]

Using Fubini’s theorem and Chebyshev’s inequality, we have
\[
K_3 \leq \frac{1}{\varepsilon^q} \mathbb{E} \left[ \left( \int_{[t_n,t]} \int_{\mathbb{R}} \phi_{R,t}(r,z) \phi_{R,t}(s,y) \Sigma_{1,y}^{(1)} D_{1,y} u(s,y) \Sigma_{1,z} dr dz W(ds,dy) \right)^q \right].
\]
Then, applying Burkholder-Davis-Gundy inequality, followed by Minkowski’s inequality, we get
\[
K_3 \leq \frac{C_q}{\varepsilon^q} \mathbb{E} \left[ \left( \int_{[t_n,t]} \int_{\mathbb{R}} \phi_{R,t}(r,z) \phi_{R,t}(s,y) \Sigma_{1,y}^{(1)} D_{1,y} u(s,y) \Sigma_{1,z} dr dz \right)^{\frac{q}{2}} ds dy \right]^q
\]
\[
= \frac{C_q}{\varepsilon^q} \mathbb{E} \left[ \left( \int_{[t_n,t]} \int_{\mathbb{R}} \phi_{R,t}(r_1,z_1) \phi_{R,t}(r_2,z_2) \phi_{R,t}^2(s,y) \right) ds dy \right]^q
\]
\[
\times \left( \int_{[t_n,t]} \int_{\mathbb{R}} \int_{[t_n,t]} \phi_{R,t}(r_1,z_1) \phi_{R,t}(r_2,z_2) \phi_{R,t}^2(s,y) \right)^{\frac{q}{2}} ds dy \right]^q
\]
\[
\leq \frac{C_q}{\varepsilon^q} \left( \int_{[t_n,t]} \int_{\mathbb{R}} \int_{[t_n,t]} \phi_{R,t}(r_1,z_1) \phi_{R,t}(r_2,z_2) \phi_{R,t}^2(s,y) \right)^{\frac{q}{2}} ds dy \right]^q \quad \text{(4.27)}
\]
where
\[
X_{r_1,r_2,z_1,z_2}(s,y) := \left( \Sigma_{1,y}^{(1)} \right)^2 D_{r_1,z_1} u(s,y) D_{r_2,z_2} u(s,y) \Sigma_{r_1,z_1} \Sigma_{r_2,z_2}.
\]

Using Hölder’s inequality, the Lipschitz property of \(\sigma\), the estimate (4.2) and the fact that \(\sup_{(r,z) \in [t_n,t] \times \mathbb{R}} \|\sigma(u(r,z))\|_p < \infty\) for all \(p \geq 2\), we have
\[
\|X_{r_1,r_2,z_1,z_2}(s,y)\|_{q/2} \leq C_t p_{s-r_1}(y-z_1) p_{s-r_2}(y-z_2).
\]

Plugging this bound in the estimate (4.27), we see that
\[
K_3 \leq \frac{C_t}{\varepsilon^q} \left( \int_{[t_n,t]} \int_{\mathbb{R}} \int_{[t_n,t]} \phi_{R,t}(r_1,z_1) \phi_{R,t}(r_2,z_2) \phi_{R,t}^2(s,y) \right)^{\frac{q}{2}} ds dy \right]^q \quad \text{(4.28)}
\]
\[
\times p_{s-r_1}(y-z_1) p_{s-r_2}(y-z_2) dz_1 dz_2 dr_1 dr_2 ds^2.
\]
Integrating in $z_1$ and $z_2$, and using the semigroup property, we have for $t_\alpha < s < t$ and for $R \geq R_0$,  
\[ 
\int_{\mathbb{R}^3} \left( \prod_{i=1,2} \phi_{R,t}(r_i, z_i) \phi_{R,t}(s, y)p_{s-r_i}(y-z_i) \right) \, dz_1 dz_2 dy
\]
\[ = \frac{1}{\sigma_{R,t}} \int_{\mathbb{R}} \int_{\mathbb{R}} q_{R,t}^2 \prod_{i=1,2} \phi_{R,t}(s, y)p_{t+s-2r_i}(y-x_i) \, dx_1 dx_2 dy
\]
\[ \leq \frac{1}{\sigma_{R,t}^2} \int_{\mathbb{R}} \phi_{R,t}^2(s, y) \left( \prod_{i=1,2} \int_{\mathbb{R}} p_{t+s-2r_i}(y-x) \, dx \right) \, dy
\]
\[ = \frac{1}{\sigma_{R,t}^2} \int_{\mathbb{R}} \phi_{R,t}^2(s, y) dy \leq \frac{C_t}{\sigma_{R,t}^2} \leq C_t,
\]
where we use Lemma A.4 part (a) of the Appendix. Now, plugging this estimate in (4.28), we get
\[ K_3 \leq C_T \varepsilon q/5.
\]
(4.29)

Now, choosing $\alpha = 4/5$, we get from (4.22), (4.26) and (4.29),
\[ \sup_{R \geq R_0} \mathbb{P} \left( D_{v_{R,t}} F_{R,t} < \varepsilon \right) \leq C_T \varepsilon^{q/5}.
\]
Finally, using this estimate we get
\[ \sup_{R \geq R_0} \mathbb{E} \left[ (D_{v_{R,t}} F_{R,t})^{-p} \right] = \sup_{R \geq R_0} p \int_0^\infty \varepsilon^{-p-1} \mathbb{P} \left( D_{v_{R,t}} F_{R,t} < \varepsilon \right) \, d\varepsilon
\]
\[ \leq 1 + \sup_{R \geq R_0} p \int_0^1 \varepsilon^{-p-1} \mathbb{P} \left( D_{v_{R,t}} F_{R,t} < \varepsilon \right) \, d\varepsilon
\]
\[ \leq 1 + C_T p \int_0^1 \varepsilon^{-p-1+q/5} \, d\varepsilon < \infty
\]
for $q > 5p$, which completes our proof.

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. We will apply Theorem 3.2 to the random variable $F_{R,t} = \delta(v_{R,t})$. Fix $t > 0$. From the proof of Theorem 1.1 in [10], we have
\[ \left\| 1 - D_{v_{R,t}} F_{R,t} \right\|_2^2 \leq \frac{C_t}{\sqrt{R}}.
\]
(4.30)

We are only left to estimate the term $\left\| D_{v_{R,t}} (D_{v_{R,t}} F_{R,t}) \right\|_2^2$. Recall that
\[ D_{v_{R,t}} F_{R,t} = \frac{1}{\sigma_{R,t}} \int_0^t \int_{\mathbb{R}} \phi_{R,t}(s, y) \sigma(u(s, y)) D_{s,y} u(t, x) \, dxdyds.
\]
Taking the Malliavin derivative, we get
\[
D_{r,z}(D_{v_{R,t}}F_{R,t}) = \frac{1}{\sigma_{R,t}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{Q_R} \phi_{R,t}(r,z)\phi_{R,t}(s,y)(u(s,y))D_{r,z}u(t,x)dx dy dz ds dr + \frac{1}{\sigma_{R,t}} \int_{0}^{t} \int_{\mathbb{R}} \int_{Q_R} \phi_{R,t}(s,y)\sigma(u(s,y))D_{r,z}D_{s,y}u(t,x)dx dy dz ds dr,
\]
and, using the notation (2.8), we get
\[
D_{v_{R,t}}(D_{v_{R,t}}F_{R,t}) = \frac{1}{\sigma_{R,t}} \int_{0}^{t} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{Q_R} \phi_{R,t}(r,z)\phi_{R,t}(s,y)\Sigma_{r,z}\Sigma_{s,y}D_{r,z}u(t,x)dx dy dz ds dr + \frac{2}{\sigma_{R,t}} \int_{0}^{t} \int_{\mathbb{R}} \int_{\mathbb{R}^2} \int_{Q_R} \phi_{R,t}(r,z)\phi_{R,t}(s,y)\Sigma_{r,z}\Sigma_{s,y}D_{r,z}D_{s,y}u(t,x)dx dy dz ds dr.
\]
Now using (4.1) and (4.3) for \(D_{s,y}u(t,x)\) and \(D_{r,z}D_{s,y}u(t,x)\), respectively, we have
\[
D_{v_{R,t}}(D_{v_{R,t}}F_{R,t}) = 2Y_{R,t}^1 + Y_{R,t}^2 + 2Y_{R,t}^3 + 2Y_{R,t}^4,
\]
where:
\[
Y_{R,t}^1 = \int_{0}^{t} \int_{\mathbb{R}} \int_{\mathbb{R}^2} dy dz dr dr dz dr d \xi d \eta d \zeta,
\]
\[
Y_{R,t}^2 = \int_{0}^{t} \int_{\mathbb{R}} \int_{\mathbb{R}^2} dy dz dr dr dz dr d \xi d \eta d \zeta \times \int_{[s,t] \times \mathbb{R}} \phi_{R,t}(r,z)\Sigma_{r,z}\Sigma_{s,y}D_{r,z}u(t,x)W(d \tau, d \xi),
\]
\[
Y_{R,t}^3 = \int_{0}^{t} \int_{\mathbb{R}} \int_{\mathbb{R}^2} dy dz dr dr dz dr d \xi d \eta d \zeta \times \int_{[s,t] \times \mathbb{R}} \phi_{R,t}(r,z)\Sigma_{r,z}\Sigma_{s,y}D_{r,z}D_{s,y}u(t,x)W(d \tau, d \xi),
\]
\[
Y_{R,t}^4 = \int_{0}^{t} \int_{\mathbb{R}} \int_{\mathbb{R}^2} dy dz dr dr dz dr d \xi d \eta d \zeta \times \int_{[s,t] \times \mathbb{R}} \phi_{R,t}(r,z)\Sigma_{r,z}\Sigma_{s,y}D_{r,z}D_{s,y}u(t,x)W(d \tau, d \xi).
\]
Putting together the terms \(Y_{R,t}^i\) for \(i = 2, 3, 4\), we can write
\[
D_{v_{R,t}}(D_{v_{R,t}}F_{R,t}) = 2Y_{R,t}^1 + Y_{R,t}^5,
\]
where
\[
Y_{R,t}^5 = \int_{0}^{t} \int_{\mathbb{R}} \left( \int_{0}^{t} \int_{\mathbb{R}^2} \phi_{R,t}(s,y)\phi_{R,t}(r,z)Z_{r,z,s,y}(r,\xi)dz dr dr dz \right) \phi_{R,t}(r,\xi)W(d \tau, d \xi),
\]
and we are using the notation
\[
Z_{r,z,s,y}(\tau, \xi) =: \sum_{r,z} \sum_{s,y} \sum_{\tau, \xi} D_{r,z} u(s, y) D_{s,y} u(\tau, \xi) \\
+ 2 \sum_{r,z} \sum_{s,y} \sum_{\tau, \xi} D_{r,z} u(\tau, \xi) D_{s,y} u(\tau, \xi) \\
+ 2 \sum_{r,z} \sum_{s,y} \sum_{\tau, \xi} D_{r,z} D_{s,y} u(\tau, \xi).
\] (4.31)

Therefore,
\[
\|D_{v,R,t} \left( D_{v,R,t} F_{R,t} \right) \|_2 \leq 2 \|Y_{R,t}^1\|_2 + \|Y_{R,t}^5\|_2.
\]

**Estimation of** \(\|Y_{R,t}^1\|_2\): Note that using the estimates (2.1) and (4.2) and Hölder’s inequality we have, for \(r < s\),
\[
\left\| \sum_{r,z,s,y} \sum_{\tau, \xi} D_{r,z} u(s, y) \right\|_2 \leq C_1 p_{s-r}(z - y).
\]

As a consequence,
\[
\|Y_{R,t}^1\|_2 \leq C_1 \int_0^t \int_r^t \int_R^2 \phi_{R,t}^2(s, y) \phi_{R,t}(r, z) p_{s-r}(z - y) dy dz ds dr.
\]

Integrating in \(z\) and using the semigroup property, we have
\[
\int_R^2 \phi_{R,t}(r, z) p_{s-r}(z - y) dz = \frac{1}{\sigma_{R,t}} \int_{Q_R} \int_R p_{r-t}(x - z) p_{s-r}(z - y) dz dx \leq \frac{1}{\sigma_{R,t}}.
\]

Using the above estimate, and Lemma A.4 part (a), Lemma A.3 part (a) and we get, for \(R \geq R_0\),
\[
\|Y_{R,t}^1\|_2 \leq \frac{C_1}{\sigma_{R,t}} \int_0^t \int_r^t \int_R^2 \phi_{R,t}^2(s, y) dy dz ds dr \leq \frac{C_1}{\sqrt{R}}.
\]

**Estimation of** \(\|Y_{R,t}^5\|_2\): Using the Itô-Walsh isometry of the stochastic integral and Cauchy-Schwarz inequality, we obtain
\[
\|Y_{R,t}^5\|_2^2 = \int_0^t \int_R E \left[ \left( \int_0^\tau \int_{Q_R} \phi_{R,t}(s, y) \phi_{R,t}(r, z) Z_{r,z,s,y}(\tau, \xi) d\sigma + d\tau \right) dy dz \right]^2 d\tau d\xi
\]
\[
= \int_0^t \int_R \int_{0 \leq r_1 \leq s_1 \leq \tau} \int_{0 \leq r_2 \leq s_2 \leq \tau} \prod_{i=1}^4 \int_{0 \leq \gamma_1 \leq \omega_1 \leq \tau} \int_{0 \leq \gamma_2 \leq \omega_2 \leq \tau} d\gamma_1 d\omega_1 d\gamma_2 d\omega_2 \phi_{R,t}^2(\tau, \xi) d\tau d\xi
\]
\[
\times \|Z_{r_1, z_1, s_1, y_1}(\tau, \xi)\|_2 \phi_{R,t}^2(\tau, \xi) d\tau d\xi.
\]

From the decomposition (4.31), using Hölder’s inequality and the estimates (2.1), (4.2) and (4.4), we can write
\[
\|Y_{R,t}^5\|_2^2 \leq C_1 \int_0^t \int_R d\tau \phi_{R,t}^2(\tau, \xi) \int_{0 \leq r_1 \leq s_1 \leq \tau} \int_{0 \leq r_2 \leq s_2 \leq \tau} \prod_{i=1}^4 \int_{0 \leq \gamma_1 \leq \omega_1 \leq \tau} \int_{0 \leq \gamma_2 \leq \omega_2 \leq \tau} d\gamma_1 d\omega_1 d\gamma_2 d\omega_2 \times [p_{s_1-r_1}(y_1 - y_1) + p_{s_2-r_1}(\xi - y_1) p_{s_1-r}(\xi - y_1) + \Phi_{r_1, z_1, s_1, y_1}(\tau, \xi)].
\]
The estimates $\phi_{R,t}(r_i, z_i), \phi_{R,t}(r_i, z_i) \leq \frac{1}{\sigma_{R,t}}$ imply
\[
\|Y_{R,t}^5\|_2^2 \leq \frac{C_l}{\sigma_{R,t}} \int_0^t \int_\mathbb{R} d\xi d\tau \phi_{R,t}^2(\tau, \xi) \int_{0 \leq r_1 \leq s_1 \leq \tau} \int_{\mathbb{R}^n} dy_z dz_i dr_i ds_i \times [p_{s_i-r_i}(y_i - z_i)p_{r-r_i}(\xi - y_i) + \Phi_{r_i,s_i}(\xi - y_i)].
\]
Integrating the variables $z_i$ and $y_i$ for $i = 1, 2$ and using Lemma A.2, we have
\[
\|Y_{R,t}^5\|_2^2 \leq \frac{C_l}{\sigma_{R,t}} \int_0^t \int_\mathbb{R} \phi_{R,t}^2(\tau, \xi) \left( \int_{0 < r < s < \tau} \left( 1 + (s - r)^{-1/4} \right) dr ds \right)^2 d\xi d\tau.
\]
Using the above estimate, Lemma A.4 part (a), and Lemma A.3 part (a) we finally have for $R \geq R_0$
\[
\|Y_{R,t}^5\|_2^2 \leq \frac{C_l}{\sqrt{R}}. \tag{4.32}
\]
Finally, plugging the estimates (4.13), (4.30) and (4.32) into (3.1) we complete the proof. □

5. PROOF OF THEOREM 1.2

Let $u = \{u(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R} \}$ be the solution to equation (2.4). The process $u$ is no longer stationary in the space variable, but if we define $U$ as
\[
U(t, x) := \frac{u(t, x)}{p(t, x)}
\]
for $(t, x) \in (0, \infty) \times \mathbb{R}$, then for any $t > 0$, the process $\{U(t, x) : x \in \mathbb{R}\}$ is stationary, see [1].

It has been proven in [7] that $\lim_{t \to 0} U(t, x) = 1$ in $L^p(\Omega)$ for all $x \in \mathbb{R}$ and $p \geq 2$. Moreover, equation (2.4) can be reformulated in terms of $U$ as follows
\[
U(t, x) = 1 + \int_{[0, t] \times \mathbb{R}} p_{s, x}(\xi - \frac{\tau}{t})U(s, y)W(d\tau, d\xi), \tag{5.1}
\]
where we used the identity (2.7).

According to Chen, Hu and Nualart [4, Proposition 5.1], for any $t > 0$ and any $x \in \mathbb{R}$, the random variable $u(t, x)$ belongs to the Sobolev space $\mathbb{D}^{k,p}$ for any $k \geq 1$ and $p \geq 2$. As a consequence, for all $t > 0$ and $x \in \mathbb{R}$, $U(t, x) \in \bigcap_{k \geq 1} \bigcap_{p \geq 2} \mathbb{D}^{k,p}$. Furthermore, for almost all $(s, y) \in (0, t) \times \mathbb{R}$, using (2.6) and (5.1), we have,
\[
D_{s,y}U(t, x) = p_{s, x}(\xi - \frac{\tau}{t})U(s, y) + \int_{[s, t] \times \mathbb{R}} p_{s, x}(\xi - \frac{\tau}{t})D_{s,y}U(\tau, \xi)W(d\tau, d\xi), \tag{5.2}
\]
and for almost all $r \leq s \leq t$ and $y, z \in \mathbb{R}$,
\[
D_{r,z}D_{s,y}U(t, x) = p_{s, x}(\xi - \frac{\tau}{t})D_{r,z}U(s, y) + \int_{[s, t] \times \mathbb{R}} p_{s, x}(\xi - \frac{\tau}{t})D_{r,z}D_{s,y}U(\tau, \xi)W(d\tau, d\xi). \tag{5.3}
\]
Let $G_{R,t}$ and $\Sigma_{R,t}$ be as defined in (1.4). Then, for any fixed $t > 0$, $G_{R,t} = \delta(w_{R,t})$, where
\[
w_{R,t}(s, y) = 1_{[0, t]}(s) \frac{1}{\Sigma_{R,t}} \int_{Q_R} p_{s, x}(\xi - \frac{\tau}{t})U(s, y)dx = 1_{[0, t]}(s)\varphi_{R,t}(s, y)U(s, y), \tag{5.4}
\]
and \( \varphi_{R,t}(s, y) \) has been defined in (2.10). Finally, we also note that

\[
D_{s,y}G_{R,t} = \frac{1}{\Sigma_{R,t}} \int_{Q_R} D_{s,y}U(t, x),
\]

and using (5.4)

\[
D_{w_R,t}G_{R,t} = \frac{1}{\Sigma_{R,t}} \int_0^t \int_{Q_R} \varphi_{R,t}(s, y)U(s, y)D_{s,y}U(t, x)dxdyds.
\] (5.5)

The following moment estimates hold for all \( p \geq 2 \) and for all \((t, x) \in (0, T] \times \mathbb{R}\) and almost all \(0 < r < s < t\) and \(y, z \in \mathbb{R}\):

\[
\|u(t, x)\|_p \leq c_{T,p}p_t(x), \tag{5.6}
\]

\[
\|D_{s,y}u(t, x)\|_p \leq c_{T,p}p_{t-s}(x-y)p_s(y), \tag{5.7}
\]

and

\[
\|D_{r,z}D_{s,y}u(t, x)\|_p \leq c_{T,p}p_{t-s}(x-y)p_{t-r}(y-z)p_r(z), \tag{5.8}
\]

where \( c_{T,p} \) is a constant depending only on \( T \) and \( p \). We refer to Chen and Dalang [3, Theorem 2.4] for the proof of (5.6), to Chen, Khoshnevisan, Nualart and Pu [7, Lemma 2.1] for the proof of (5.7) and Kuzgun and Nualart [11, Corollary 1.2] for the proof of (5.8). Dividing by the factor \( p_t(x) \) and using the identity (2.7) we derive the corresponding estimates for the process \( U(t, x) \):

\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}} \|U(t, x)\|_p \leq c_{T,p}, \tag{5.9}
\]

\[
\|D_{s,y}U(t, x)\|_p \leq c_{T,p}p_{s(t-s)}(y - \frac{s}{t}x). \tag{5.10}
\]

and

\[
\|D_{r,z}D_{s,y}u(t, x)\|_p \leq c_{T,p}p_{s(t-s)}(y - \frac{s}{t}x)p_{r(t-r)}(z - \frac{r}{s}y). \tag{5.11}
\]

The next proposition ensures the existence of negative moments required in the application of Theorem 3.2.

**Proposition 5.1.** Fix \( t \in (0, T] \), \( p \geq 2 \) and \( \gamma > 5 \). Then, there exist \( R_0 > 1 \) and a constant \( c_{t,p,\gamma} \), depending on \( t, p \) and \( \gamma \), such that

\[
\left\| \left( D_{w_R,t}G_{R,t} \right)^{-1} \right\|_p \leq c_{t,p,\gamma}(\log R)^\gamma
\]

for all \( R \geq R_0 \).

**Proof.** Using (5.5) and (5.2), we have

\[
D_{w_R,t}G_{R,t} = \frac{1}{\Sigma_{R,t}} \int_0^t \int_{Q_R} \varphi_{R,t}(s, y)U(s, y)D_{s,y}U(t, x)dxdyds.
\]

\[
= \int_0^t \int_{Q_R} \varphi_{R,t}^2(s, y)U^2(s, y)dyds
\]

\[
+ \int_0^t \int_{Q_R} \varphi_{R,t}(s, y)U(s, y) \left( \int_{[s,t] \times \mathbb{R}} \varphi_{R,t}(\tau, \xi)D_{s,y}U(\tau, \xi)W(d\tau, d\xi) \right) dyds.
\]
Since $U$ and $DU$ are non-negative, $D_{w_{R,t}} G_{R,t} \geq 0$ and we have
\[
D_{w_{R,t}} G_{R,t} \geq \int_{t_0}^t \int_R \varphi^2_{R,t}(s,y) U^2(s,y) dy ds
\]
\[
+ \int_{t_0}^t \int_R \varphi_{R,t}(s,y) U(s,y) \left( \int_{[s,t]} \varphi_{R,t}(\tau,\xi) D_{s,y} U(\tau,\xi) W(d\tau,d\xi) \right) dy ds
\]
\[
=: I_1 + I_2,
\]
where $t_\alpha = t - \varepsilon^\alpha$, with $\varepsilon \in (0, \frac{t}{2}]$ and $\alpha \in (0, 1]$. As in the proof of Proposition 4.2, we can write
\[
P \left( D_{w_{R,t}} G_{R,t} < \varepsilon \right) \leq P \left( I_1 < 2\varepsilon \right) + P \left( |I_2| > \varepsilon \right). \tag{5.12}
\]
We now estimate these probabilities in two steps.

**Step 1:** By Chebyshev inequality, for any $q \geq 2$,
\[
P \left( I_1 < 2\varepsilon \right) \leq P \left( I_1^{-1} > \frac{1}{2\varepsilon} \right) \leq (2\varepsilon)^q \mathbb{E} \left[ \left( \int_{t_0}^t \int_R \varphi^2_{t,R}(s,y) U^2(s,y) dy ds \right)^{-q} \right]. \tag{5.13}
\]
Set
\[
m(\varepsilon, R) = \int_{t_0}^t \int_R \varphi^2_{t,R}(s,y) dy ds.
\]
Using Lemma A.4 part (b), taking into account that $s > \frac{2}{\varepsilon}$, for all $R \geq R_0$, we have
\[
m(\varepsilon, R) \geq \frac{c_1 \varepsilon^\alpha}{\log R}. \tag{5.14}
\]
Then, because the function $x \to x^{-q}$ is convex, applying Jensen’s inequality, we can write
\[
\mathbb{E} \left[ \left( \int_{t_0}^t \int_R \varphi^2_{t,R}(s,y) U^2(s,y) dy ds \right)^{-q} \right]
\]
\[
\leq m(\varepsilon, R)^{-q-1} \int_{t_0}^t \int_R \varphi_{R,t}(s,y) \mathbb{E} \left[ U^{-2q}(s,y) \right] dy ds. \tag{5.15}
\]
Since $\{U(s,y) : y \in \mathbb{R}\}$ is stationary, we have for all $s \in [\frac{2}{\varepsilon}, 2]$
\[
\mathbb{E} \left[ (U(s,y))^{-2q} \right] = \mathbb{E} \left[ (U(s,0))^{-2q} \right] = (p_s(0))^{-2q} \mathbb{E} \left[ (u(s,0))^{-2q} \right]
\]
\[
\leq (\pi t)^{-q} \mathbb{E} \left[ \inf_{s \in [\frac{2}{\varepsilon}, 2]} u(s,0) \right]^{-2q} = c_{t,q} < \infty, \tag{5.16}
\]
where $c_{t,q}$ is a constant depending on $q$ and $t$ and the last equality follows from [8, Theorem 1.4]. In what follows, $c_{t,q}$ will denote a generic constant depending on $q$ and $t$. Substituting (5.16) into (5.15) and using Lemma A.4 part (b) and (5.14) yields
\[
\mathbb{E} \left[ \left( \int_{t_0}^t \int_R \varphi^2_{t,R}(s,y) U^2(s,y) dy ds \right)^{-q} \right] \leq c_{t,q} m(\varepsilon, R)^{-q-1} \int_{t_0}^t \int_R \varphi_{R,t}(s,y) dy ds
\]
\[
\leq c_{t,q} m(\varepsilon, R)^{-q-1} \int_{t_0}^t \frac{1}{s \log R} ds
\]
\[
\leq c_{t,q} e^{-\alpha q}(\log R)^q, \tag{5.17}
\]
for $R \geq R_0$. Finally, from (5.13) and (5.17), we get
\[ P (I_1 < 2\varepsilon) \leq c_{t,q} (\log R)^q \varepsilon^{q(1-\alpha)}. \] (5.18)

**Step 2:** Set $\Pi = P (|I_2| > \varepsilon)$. Using Fubini's theorem and Chebyshev's inequality for any $q \geq 2$, we have
\[ \Pi \leq \frac{1}{\varepsilon^q} E \left[ \int_{[t_0,t] \times \mathbb{R}} \int_{R} \varphi_{R,t}(\tau,\xi) \varphi_{R,t}(s,y) U(s,y) D_{s,y} U(\tau,\xi) ds dy \right] \] \[ \leq \frac{c_t}{\varepsilon^q} \left( \int_{t_0}^{t} \int_{R} \varphi_{R,t}^{\frac{q}{2}}(\tau,\xi) \varphi_{R,t}(s_1,y_1) \varphi_{R,t}(s_2,y_2) \right. \] \[ \times Y_{s_1,1,s_2,2}(\tau,\xi) dy_1 dy_2 ds_1 ds_2 d\tau \right) \] \[ \leq \frac{c_t}{\varepsilon^q} \left( \int_{t_0}^{t} \int_{R} \varphi_{R,t}(\tau,\xi) \varphi_{R,t}(s_1,y_1) \varphi_{R,t}(s_2,y_2) \right. \] \[ \times \left. \| Y_{s_1,1,s_2,2}(\tau,\xi) \|_{q/2} dy_1 dy_2 ds_1 ds_2 d\tau \right)^{\frac{q}{2}}, \] (5.19)

where
\[ Y_{s_1,1,s_2,2}(\tau,\xi) := U(s_1,y_1) D_{s_1,y_1} U(\tau,\xi) U(s_2,y_2) D_{s_2,y_2} U(\tau,\xi). \]

Note that using the estimates (5.9) and (5.10) and Hölder's inequality, we can write
\[ \| Y_{s_1,1,s_2,2}(\tau,\xi) \|_{q/2} \leq c_{t,q} p_{\frac{2}{\tau}}(\frac{r-\alpha}{\tau}) (y_1 - \frac{s_1}{\tau}) p_{\frac{\alpha}{2}\frac{r-\alpha}{\tau}}(y_2 - \frac{s_2}{\tau}). \] (5.20)

Substituting the estimate (5.20) into (5.19), we obtain
\[ \Pi \leq \frac{c_{t,q}}{\varepsilon^q} \left( \int_{t_0}^{t} \int_{R} \varphi_{R,t}(\tau,\xi) \left( \int_{t_0}^{t} \int_{R} \varphi_{R,t}(s,y) p_{\frac{2}{\tau}}(\frac{r-\alpha}{\tau}) (y - \frac{s}{\tau}) dy ds \right)^{\frac{q}{2}} d\tau d\xi \right)^{\frac{q}{2}}. \] (5.21)

Using the semigroup property, we have
\[ \int_{R} \varphi_{R,t}(s,y) p_{\frac{2}{\tau}}(\frac{r-\alpha}{\tau}) (y - \frac{s}{\tau}) dy \leq \frac{1}{\Sigma_{R,t}} \int_{R} p_{\frac{2}{\tau}}(\frac{r-\alpha}{\tau}) (y - \frac{t}{\tau}) dy \] \[ = \frac{1}{\Sigma_{R,t}} \int_{R} \frac{p_{\frac{2}{\tau}}(\frac{r-\alpha}{\tau}) (y - \frac{s}{\tau})}{s} dy = \frac{t}{s\Sigma_{R,t}} \int_{R} p_{\frac{2}{\tau}}(\frac{r-\alpha}{\tau}) (x - \frac{t}{\tau}) dx, \]
where we used the identity $p_t(ax) = \frac{1}{a} p_{\frac{t}{a}}(x)$. Hence, taking into account that $t_{\alpha} > \frac{t}{\frac{r}{2}}$, we can write
\[ \int_{t_0}^{t} \int_{R} \varphi_{R,t}(s,y) p_{\frac{2}{\tau}}(\frac{r-\alpha}{\tau}) (y - \frac{s}{\tau}) dy ds \leq \frac{1}{\Sigma_{R,t}} \int_{t_0}^{t} \frac{t}{s} ds \leq \frac{2 e^\alpha}{\Sigma_{R,t}}. \] (5.22)
Finally, plugging the estimate (5.22) into (5.21), and using Lemma A.3 part (b) and Lemma A.4 part (b), we get for $R \geq R_0$,

$$\Pi \leq c_{t,q} R^{-q/2} \varepsilon (\frac{\alpha}{2} - 1)^{q/2} (\int_{t_0}^{T} \int_{R}^{2R} \varphi_{R,t}^2(\tau, \xi) d\xi d\tau)^{q/2}$$

(5.23)

Now, choosing $\alpha = 4/5$, we get, substituting (5.23) and (5.18) into (5.12),

$$P \left( D_{w_{R,t}} G_{R,t} < \varepsilon \right) \leq c_{t,q} (\log R)^{q/5}. $$

Using this estimate, we get

$$E \left[ (D_{w_{R,t}} G_{R,t})^{-p} \right] = p \int_{0}^{\infty} \varepsilon^{-p-1} P \left( D_{w_{R,t}} G_{R,t} < \varepsilon \right) d\varepsilon \leq 1 + p \int_{0}^{1} \varepsilon^{-p-1} P \left( D_{w_{R,t}} G_{R,t} < \varepsilon \right) d\varepsilon \leq 1 + c_{t,q} (\log R)^{q/5} \int_{0}^{1} \varepsilon^{-p-1+q/5} d\varepsilon. $$

Finally, for $q = \gamma p > 5p$, and for $R \geq R_0$, we obtain

$$\left\| (D_{w_{R,t}} G_{R,t})^{-1} \right\|_p \leq c_{t,p,\gamma} (\log R)^{\gamma}, $$

which completes our proof.

Now, we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** We will apply Theorem 3.2 to the random variable $G_{R,t} = \delta(w_{R,t})$. Proposition 5.1 provides the estimate

$$\left\| (D_{w_{R,t}} G_{R,t})^{-1} \right\|_4 \leq c_{t,4,\gamma} (\log R)^{\gamma}, $$

(5.24)

for any $\gamma > 5$, and for $R$ large enough. Moreover, from the proof of Theorem 1.1 in [7], we have

$$\left\| 1 - D_{w_{R,t}} G_{R,t} \right\|_2 \leq \frac{C_1 \sqrt{\log R}}{\sqrt{R}}. $$

(5.25)

We are only left to estimate the term $\left\| D_{w_{R,t}} (D_{w_{R,t}} G_{R,t}) \right\|_2$. Recall that from (5.5) we have

$$D_{w_{R,t}} G_{R,t} = \frac{1}{\Sigma_{R,t}} \int_{0}^{t} \int_{Q_R} \varphi_{R,t}(s,y) U(s,y) D_{s,y} U(t,x) dxdyds. $$

Applying again the derivative operator, we obtain

$$D_{r,z} (D_{w_{R,t}} G_{R,t}) = \frac{1}{\Sigma_{R,t}} \int_{0}^{t} \int_{Q_R} \varphi_{R,t}(s,y) \left( D_{r,z} U(s,y) D_{s,y} U(t,x) + U(s,y) D_{s,y} U(t,x) \right) dxdyds, $$
so that,
\[
D_{w_{R,t}}(D_{w_{R,t}} G_{R,t}) = \frac{1}{\Sigma_{R,t}} \int_{0<r<s<t} \int_{\mathbb{R}^2} dxdydzdsdr \varphi_{R,t}(s,y)\varphi_{R,t}(r,z)U(r,z) \\
\times \left( D_{r,z}U(s,y)D_{s,y}U(t,x) + 2U(s,y)D_{r,z}D_{s,y}U(t,x) \right).
\]

Now using (5.2) and (5.3) for \( D_{s,y}U(t,x) \) and \( D_{r,z}D_{s,y}U(t,x) \), we get
\[
D_{w_{R,t}}(D_{w_{R,t}} G_{R,t}) = 2\mathcal{X}_{R,t}^{1} + \mathcal{X}_{R,t}^{2} + 2\mathcal{X}_{R,t}^{3},
\]
where:
\[
\mathcal{X}_{R,t}^{1} = \int_{0}^{t} \int_{\mathbb{R}} \int_{\mathbb{R}} dxdydzdsdr \varphi_{R,t}^{2}(s,y)\varphi_{R,t}(r,z)U(r,z)U(s,y)D_{r,z}U(s,y),
\]
\[
\mathcal{X}_{R,t}^{2} = \int_{0}^{t} \int_{\mathbb{R}} \int_{\mathbb{R}} dxdydzdsdr \varphi_{R,t}(s,y)\varphi_{R,t}(r,z)U(r,z)D_{r,z}U(s,y) \\
\times \int_{(s,t)\times \mathbb{R}} \varphi_{R,t}(\tau,\xi)D_{s,y}U(\tau,\xi)W(d\tau,d\xi),
\]
\[
\mathcal{X}_{R,t}^{3} = \int_{0}^{t} \int_{\mathbb{R}} \int_{\mathbb{R}} dxdydzdsdr \varphi_{R,t}(s,y)\varphi_{R,t}(r,z)U(r,z)U(s,y) \\
\times \int_{(s,t)\times \mathbb{R}} \varphi_{R,t}(\tau,\xi)D_{r,z}D_{s,y}U(\tau,\xi)W(d\tau,d\xi).
\]

As a consequence, we have
\[
\|D_{w_{R,t}}(D_{w_{R,t}} G_{R,t})\|_{2} \leq 2\|\mathcal{X}_{R,t}^{1}\|_{2} + \|\mathcal{X}_{R,t}^{2} + 2\mathcal{X}_{R,t}^{3}\|_{2}. \tag{5.26}
\]

We will further estimate the two terms in the right-hand side of the previous display.

**Estimation of \( \|\mathcal{X}_{R,t}^{1}\|_{2} \):** Using the estimates (5.9) and (5.10) and applying Hölder’s inequality, we can write
\[
\|U(s,y)U(r,z)D_{r,z}U(s,y)\|_{2} \leq C_{t}p_{\frac{r}{s},z}(z - \frac{r}{s}y).
\]

Therefore,
\[
\|\mathcal{X}_{R,t}^{1}\|_{2} \leq \int_{0}^{t} \int_{\mathbb{R}} \int_{\mathbb{R}} dxdydzdsdr \varphi_{R,t}^{2}(s,y)\varphi_{R,t}(r,z) \|U(s,y)U(r,z)D_{r,z}U(s,y)\|_{2} \\
\leq C_{t} \int_{0}^{t} \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_{R,t}^{2}(s,y)\varphi_{R,t}(r,z)p_{\frac{r}{s},z}(z - \frac{r}{s}y)dzdydsdr =: I_{1}. \tag{5.27}
\]

To estimate \( I_{1} \), we first integrate in \( z \) and use the semigroup property, to obtain
\[
\int_{\mathbb{R}} \varphi_{R,t}(r,z)p_{\frac{r}{s},z}(z - \frac{r}{s}y)dz = \frac{1}{\Sigma_{R,t}} \int_{Q_{R}} \int_{\mathbb{R}} p_{\frac{r}{s},z}(z - \frac{r}{s}y)p_{\frac{r}{s},z}(z - \frac{r}{s}y)dzdx \\
= \frac{1}{\Sigma_{R,t}} \int_{Q_{R}} p_{\frac{r}{s},z}(z - \frac{r}{s}y)dy - \frac{r}{s}x)dx \\
= \frac{s}{r\Sigma_{R,t}} \int_{Q_{R}} p_{\frac{r}{s},z}(z - \frac{r}{s}y)dy - \frac{r}{s}x)dx. \tag{5.28}
\]
Now using the estimate $\varphi_{R,t}(s, y) \leq \frac{t}{r\Sigma_{R,t}}$ for one of the factors together with (5.28) and then applying the semigroup property in $y$, we get

$$
\int_{\mathbb{R}^2} \varphi_{R,t}^2(s, y)\varphi_{R,t}(r, z)p_{\frac{2(s-r)}{r}}(z - \frac{r}{s}y)dzdy
\leq \frac{t}{r\Sigma_{R,t}} \int_{\mathbb{R}^2} \int_{\mathbb{R}} p\left(s - \frac{s}{r}x\right)p\left(2\left(s - \frac{s}{r}y\right)\right)dydx_1dx_2
= \frac{t}{r\Sigma_{R,t}} \int_{\mathbb{R}^2} p_{\frac{2(s-r)}{s}}(s - \frac{s}{r}x)r^{s-r}(x_1 - x_2)dx_1dx_2
= \frac{t^2}{sr\Sigma_{R,t}} \int_{\mathbb{R}^2} p_{\frac{2(s-r)}{r}}(x_1 - x_2)dx_1dx_2
= \frac{4Rt^2}{\pi sr\Sigma_{R,t}} \int_{\mathbb{R}} \varphi(\xi)e^{-\frac{2(s-r)}{r}\xi^2}d\xi,
$$

(5.29)

where the last equality follows from Lemma A.5. So, substituting (5.29) into (5.27), we get

$$
I_1 \leq C_t \frac{R}{\Sigma_{R,t}} \int_{\mathbb{R}} \varphi(\xi) \left( \int_0^t \int_0^s \frac{1}{s}e^{-\frac{2(s-r)}{r}\xi^2}drdsd\xi \right).
$$

By Lemma A.6, we can write

$$
I_1 \leq C_t \frac{R\log R}{\Sigma_{R,t}} \left( \int_{\mathbb{R}} \varphi(\xi) \log(e + \frac{1}{\sqrt{2}|\xi|})d\xi \right) \left( \int_0^t \log(e + \frac{1}{s})ds \right).
$$

Finally Lemma A.3 part (b) yields

$$
I_1 \leq C_t (R\log R)^{-1/2}.
$$

(5.30)

**Estimation of $\|X_{R,t}^2 + 2X_{R,t}^3\|_2$:** Define

$$
V_{r,z,s,y}(\tau, \xi) = U(r, z)D_{r,z}U(s, y)D_{s,y}u(\tau, \xi) + 2U(r, z)U(r, z)D_{r,z}D_{s,y}U(\tau, \xi).
$$

With this notation in mind, we can write

$$
X_{R,t}^2 + 2X_{R,t}^3 = \int_{[0, t] \times \mathbb{R}} \left( \int_0^t \int_{\mathbb{R}^2} \varphi_{R,t}(s, y)\varphi_{R,t}(r, z)V_{r,z,s,y}(\tau, \xi)dsdrdydz \right) \varphi_{R,t}(\tau, \xi)W(d\tau, d\xi).
$$
Using the Ito-Walsh isometry of the stochastic integral and Cauchy-Schwarz inequality, we obtain

\[
I_2 := \|\mathcal{X}^2_{R,t} \|_2 + 2\|\mathcal{X}^3_{R,t}\|_2 = \int_0^t \int \mathbb{E} \left[ \left( \int_0^t \int \varphi_{R,t}(s, y) \varphi_{R,t}(r, z) V_{r, z, s, y}(\tau, \xi) \text{d}s \text{d}r \text{d}y \text{d}z \right)^2 \right] \times \varphi^2_{R,t}(\tau, \xi) \text{d}\xi \text{d}\tau
\]

\[
= \int_0^t \int \int_{r_1 \leq r_2 \leq \tau} \int \int \text{d}y_i \text{d}z_i \text{d}r_i \text{d}s_i \varphi_{R,t}(s_i, y_i) \varphi_{R,t}(r_i, z_i)
\]

\[
\times \|V_{r_1, z_1, s_i, y_i}(\tau, \xi)\|_2 \|2\varphi^2_{R,t}(\tau, \xi)\|_2 \text{d}\tau.
\]

Using (5.9) (5.10) and (5.11), we see that, for \( i = 1, 2, \)

\[
\|V_{r_1, z_1, s_i, y_i}(\tau, \xi)\|_2 \leq C_i p_{s_i(t-s_i)}(y_i - \frac{s_i}{\tau}) p_{r_i(t-s_i)}(z_i - \frac{r_i}{s_i})
\]

and hence

\[
I_2 \leq C_i \int_0^t \int \int_{s_1 \leq \tau \leq \tau_1} \int \int \text{d}y_i \text{d}z_i \text{d}r_i \text{d}s_i \varphi_{R,t}(s_i, y_i) \varphi_{R,t}(r_i, z_i) p_{s_i(t-s_i)}(y_i - \frac{s_i}{\tau}) p_{r_i(t-s_i)}(z_i - \frac{r_i}{s_i}) \text{d}z_i \text{d}y_i.
\]

Integrating in the variable \( z_i \) and using the semigroup property, we have

\[
\int_R \varphi_{R,t}(r_i, z_i) p_{s_i(t-s_i)}(z_i - \frac{r_i}{s_i}) \text{d}z_i = \frac{1}{\Sigma_{R,t}} \int_{Q_R} p_{s_i(t-s_i)}(z_i - \frac{r_i}{s_i}) \text{d}z_i \text{d}x_i
\]

\[
= \frac{1}{\Sigma_{R,t}} \int_{Q_R} \frac{p_{s_i(t-s_i)} \left( \frac{r_i}{s_i} + x_i \right) \text{d}x_i - \frac{r_i}{s_i} y_i}{s_i} \text{d}x_i
\]

\[
= \frac{1}{\Sigma_{R,t} r_i} \int_{Q_R} p_{\frac{s_i(t-s_i)}{r_i} + x_i \frac{r_i}{s_i}} \left( \frac{s_i}{t} x_i - \frac{s_i}{r_i} y_i \right) \text{d}x_i.
\]

From (5.32), using the estimate \( \varphi_{R,t}(s_i, y_i) \leq \frac{1}{\Sigma_{R,t}}, \) and applying the semigroup property, we see that

\[
\int_{Q_R} \varphi_{R,t}(s_i, y_i) \varphi_{R,t}(r_i, z_i) p_{s_i(t-s_i)} \left( y_i - \frac{s_i}{\tau} \right) p_{r_i(t-s_i)} \left( z_i - \frac{r_i}{s_i} \right) \text{d}y_i \text{d}x_i \]

\[
\leq \frac{t}{\Sigma_{R,t} r_i} \int_{Q_R} \frac{p_{s_i(t-s_i)} \left( \frac{r_i}{s_i} + x_i \frac{r_i}{s_i} \right) \text{d}x_i - \frac{s_i}{t} y_i}{s_i} \text{d}x_i
\]

\[
= \frac{t}{\Sigma_{R,t} r_i} \int_{Q_R} \frac{p_{\frac{s_i(t-s_i)}{r_i} + x_i \frac{r_i}{s_i}} \left( \frac{s_i}{t} x_i - \frac{s_i}{r_i} y_i \right) \text{d}x_i}{s_i}
\]

\[
= \frac{t \tau}{\Sigma_{R,t} r_i s_i} \int_{Q_R} \frac{p_{\frac{s_i(t-s_i)}{r_i} + x_i \frac{r_i}{s_i}} \left( \frac{s_i}{t} x_i - \frac{s_i}{r_i} y_i \right) \text{d}x_i}{s_i}
\]

(5.33)
Substituting the estimate (5.33) into (5.31), together with bound \( \varphi_{R,t}(\tau, \xi) \leq \frac{t}{\tau \Sigma_{R,t}} \), and then integrating in \( \xi \) this time, we get
\[
\int_{\mathbb{R}} \varphi_{R,t}^{2}(\tau, \xi) \prod_{i=1,2} \int_{\mathbb{R}^{2}} \varphi_{R,t}(s_{i}, y_{i}) \varphi_{R,t}(r_{i}, z_{i}) p_{\Sigma_{R,t}}(y_{i} - \frac{s_{i}}{\tau} \xi, r_{i} - \frac{s_{i}}{\tau} \xi) (z_{i} - \frac{r_{i}}{s_{i}} y_{i}) dz_{i}dy_{i} d\xi \\
\leq \frac{t^{4}}{\Sigma_{R,t} \tau^{1} r^{2} s^{1} s^{2}} \int_{\mathbb{R}} \prod_{i=1,2} \int_{\mathbb{R}^{2}} \frac{p_{\Sigma_{R,t}}^{2}(s_{i}, y_{i}) + p_{\Sigma_{R,t}}^{2}(s_{i}, y_{i}) + r_{i}^{2}(s_{i} - r_{i})}{r_{i}^{2} + s_{i}^{2}} \left( \frac{\tau}{t} x_{i} - \xi \right) dx_{i} d\xi \\
= \frac{t^{4}}{\Sigma_{R,t} \tau^{1} r^{2} s^{1} s^{2}} \int_{\mathbb{R}} \prod_{i=1,2} \int_{\mathbb{R}^{2}} \frac{p_{\Sigma_{R,t}}^{2}(s_{i}, y_{i}) + p_{\Sigma_{R,t}}^{2}(s_{i}, y_{i}) + r_{i}^{2}(s_{i} - r_{i})}{r_{i}^{2} + s_{i}^{2}} \left( \frac{\tau}{t} x_{i} - \xi \right) dx_{i} d\xi \\
= \frac{t^{5}}{\Sigma_{R,t} \tau^{1} r^{2} s^{1} s^{2}} \int_{\mathbb{R}} \prod_{i=1,2} \int_{\mathbb{R}^{2}} \frac{p_{\Sigma_{R,t}}^{2}(s_{i}, y_{i}) + p_{\Sigma_{R,t}}^{2}(s_{i}, y_{i}) + r_{i}^{2}(s_{i} - r_{i})}{r_{i}^{2} + s_{i}^{2}} \left( \frac{\tau}{t} x_{i} - \xi \right) dx_{i} d\xi \\
= \frac{4t^{5} R}{\pi \Sigma_{R,t} \tau^{1} r^{2} s^{1} s^{2}} \int_{\mathbb{R}} \varphi(\xi) e^{-2t(\frac{1}{\tau} + \frac{1}{r} + \frac{1}{s} - 1)\frac{\xi^{2}}{\tau^{2}}} d\xi, \tag{5.34}
\]
where in the last inequality we have used Lemma (A.5). Moreover, using the bound
\[
\frac{t}{r_{1}} + \frac{t}{r_{2}} - \frac{t}{\tau} - 1 \geq \frac{t - r_{1}}{2r_{1}} + \frac{t - r_{2}}{2r_{2}},
\]
and substituting (5.34) into (5.31), we obtain
\[
I_{2} \leq \frac{C_{1} t^{5} R}{\Sigma_{R,t} \tau^{1} r^{2} s^{1} s^{2}} \int_{\mathbb{R}} \prod_{i=1,2} \int_{0}^{t} \int_{0}^{\tau} \int_{r_{i} s_{i} \vee s_{2}}^{t} \varphi(\xi) e^{-t(\frac{r_{i} s_{i}}{r_{1}} + \frac{s_{i} s_{2}}{r_{2}})} \frac{\xi^{2}}{R^{2}} d\tau ds_{1} ds_{2} dr_{1} dr_{2} d\xi \\
\leq \frac{C_{1} t^{5} R}{\Sigma_{R,t} \tau^{1} r^{2} s^{1} s^{2}} \int_{\mathbb{R}} \varphi(\xi) d\xi \int_{0}^{\tau} \frac{d\tau}{\tau} \left( \int_{0}^{\tau} \frac{1}{r} e^{-s(s - \frac{r}{\tau})\frac{\xi^{2}}{R^{2}}} dr_{1} \int_{0}^{\tau} \frac{1}{s} e^{-s(s - \frac{r}{\tau})\frac{\xi^{2}}{R^{2}}} dr_{2} ds_{1} ds_{2} dr_{1} dr_{2} d\xi \right)^{2}.
\]
By Lemma A.6, we get
\[
I_{2} \leq \frac{C_{1} t^{5} R(\log R)^{2}}{\Sigma_{R,t} \tau^{1} r^{2} s^{1} s^{2}} \int_{\mathbb{R}} \varphi(\xi) \int_{0}^{t} \frac{d\tau}{\tau} \left( \int_{0}^{\tau} \log(e + \frac{1}{s}) \log(e + \frac{1}{|\xi|}) ds_{1} \right)^{2}.
\]
which implies, in view of Lemma A.3 part (b),
\[
I_{2} \leq C_{1} \frac{1}{R^{2} \log R}, \tag{5.35}
\]
for all \( R \geq R_{0} \). Plugging (5.30) and (5.35) into (5.26), yields, for all \( R \geq R_{0} \),
\[
\| D_{w_{R,t}}(D_{w_{R,t}}G_{R,t}) \|_{2} \leq C_{1}(R \log R)^{-1/2}. \tag{5.36}
\]
Finally, from (5.24), (5.25) and (5.36), applying Theorem 3.2 we get
\[
\sup_{x \in \mathbb{R}} \left| f_{G_{R,t}}(x) - \phi(x) \right| \leq \frac{C_{1} \tau(\log R)^{2(\gamma - \frac{1}{2})}}{\sqrt{R}},
\]
for all \( R \geq R_{0} \), which yields the desired estimate. \( \square \)
**Lemma A.1.** For $0 < r < s < t$ and $y, z, x \in \mathbb{R}$, we have
\[ K_{r, s, y}(t, x) \leq C(t)\Phi_{r, s, y}(t, x), \]
where $\Phi$ and $K$ are defined in (4.5) and (4.11) respectively.

**Proof.** Using the identity $p_t^2(a) = \frac{1}{(2\pi t)^{3/2}}p_t(a)$, we see that the first term in $K_{r, s, y}(t, x)$ is bounded by a constant depending on $t$ times the first term in $\Phi_{r, s, y}(t, x)$. So, we estimate the integral term in $K_{r, s, y}(t, x)$ that we denote by $I$. Using the above identity for the square of the Gaussian together with the identity (2.7) we get
\[
I = \int_s^t \int_R p_t^2(x-w)p_{t-s}^2(w-y)p_{t-r}^2(w-z) \, dw \, d\theta
\]
\[
= \int_s^t \int_R \frac{1}{(2\pi)^{3/2}(t-\theta)(\theta-s)(\theta-r)}p_{t-\theta}^2(x-w)p_{s-r}^2(w-y)p_{r-s}^2(w-z) \, dw \, d\theta
\]
\[
= \frac{1}{2\pi}p_{t-r}(x-y) \int_s^t \int_R \frac{1}{(2\pi)^{3/2}(t-\theta)(\theta-s)(\theta-r)}p_{(t-\theta)(\theta-s)}^2(z-y-\frac{\theta-s}{t-s}(x-y)) \, d\theta.
\]
Since for $r < s < \theta < t$
\[
\frac{\theta-r}{2} \leq \frac{(t-\theta)(\theta-s)}{2(t-s)} + \frac{\theta-r}{2} \leq \frac{t-r}{2},
\]
we have
\[
p_{(t-\theta)(\theta-s)}^2(z-y-\frac{\theta-s}{t-s}(x-y)) \leq \frac{\sqrt{t-r}}{\sqrt{\theta-r}}p_{t-r}^2(z-y-\frac{\theta-s}{t-s}(x-y))
\]
and
\[
I \leq \frac{1}{2\pi}p_{t-r}(x-y) \int_s^t \frac{\sqrt{t-r}}{\sqrt{(t-\theta)(\theta-r)^2(\theta-s)}}p_{t-r}^2(z-y-\frac{\theta-s}{t-s}(x-y)) d\theta
\]
\[
\leq \frac{\sqrt{t-r}p_{t-r}(x-y)}{(2\pi)^{3/2}}J,
\]
where
\[
J = \int_s^t \frac{\sqrt{t-r}}{\sqrt{(t-\theta)(\theta-s)}}p_{t-r}^2(z-y-\frac{\theta-s}{t-s}(x-y)) d\theta.
\]
Making the change of variables $\frac{\theta-s}{t-s} = \gamma$ and putting $\beta = \frac{r-s}{t-s} > 0$ yields $\theta-r = \theta-s-r = (t-s)(\gamma + \beta)$ and
\[
J = \frac{1}{t-s} \int_0^1 (1-\gamma)^{-\frac{1}{2}}\gamma^{-\frac{1}{2}}(\gamma + \beta)^{-1}p_{t-r}(z-y+\gamma(y-x)) d\gamma.
\]
We consider two cases:
Case 1: If \( z - y \) and \( z - x \) have same sign, then
\[
p_{t-z}(z - y + \gamma(y - x)) \leq p_{t-z}(z - y) + p_{t-z}(z - x).
\]

Case 2: If \( z - y \) and \( z - x \) have different sign, suppose firstly that \( z - y > 0 \) and \( z - x = z - y + y - x < 0 \). Then, \( 0 < z - y < -(y - x) \); so \( |z - y| < |y - x| \) and
\[
p_{t-z}(z - y + \gamma(y - x)) \leq \frac{1}{\sqrt{\pi(t - r)}} 1_{\{y - x > |z - y|\}}.
\]

Similarly, if \( z - y < 0 \) and \( z - x = z - y + y - x > 0 \), then \( 0 > z - y > -(y - x) \), which implies \( |z - y| < |y - x| \) and we end up with the same inequality.

Finally, noting that for \( \beta = \frac{t - s}{t - r} > 0 \)
\[
\int_0^1 (1 - \gamma)^{-1/2} \gamma^{-1/2} (\gamma + \beta)^{-1} d\gamma = \frac{1}{\sqrt{\beta(\beta + 1)}} = \frac{t - s}{\sqrt{(t - r)(s - r)}},
\]
we get
\[
I \leq \frac{\sqrt{1 - r} p_{t-z}(x - y)}{(2\pi)^{3/2}} J
\]
\[
\leq C_T \frac{p_{t-s}(x - y)}{\sqrt{s - r}} \left( p_{t-z}(z - y) + p_{t-z}(z - x) + 1_{\{|y-x|>|z-y|\}} \right)
\]
\[
\leq C_T' \frac{p_{t-s}^2(x - y)}{\sqrt{s - r}} \left( p_{t-r}^2(z - y) + p_{t-r}^2(z - x) + 1_{\{|y-x|>|z-y|\}} \right),
\]
which then completes our proof by taking the square roots on both sides. \( \square \)

**Lemma A.2.** Let \( \Phi \) be as in (4.5). For fixed \( 0 < r < s < t \) and \( x \in \mathbb{R} \),
\[
\int_{\mathbb{R}^2} \Phi_{r,z,s,y}(t, x) dy dz \leq C_t \left( 1 + \frac{1}{(s - r)^{1/4}} \right).
\]

**Proof.** Fix \( 0 < r < s < t \) and \( x \in \mathbb{R} \), using the semigroup property and Gaussian integrals, we have
\[
\int_{\mathbb{R}^2} p_{t-s}(x - y) \left( p_{s-r}(y - z) + \frac{p_{t-r}(z - y) + p_{t-r}(z - x) + 1_{\{|y-x|>|z-y|\}}}{(s - r)^{1/4}} \right) dy dz
\]
\[
= 1 + \frac{1}{(s - r)^{1/4}} + \int_{\mathbb{R}^2} p_{t-s}(x - y) 1_{\{|y-x|>|z-y|\}} dy dz
\]
\[
\leq C_t \left( 1 + \frac{1}{(s - r)^{1/4}} \right).
\]
\( \square \)

**Lemma A.3.** Let \( \sigma_{R,t}^2 \) and \( \Sigma_{R,t}^2 \) be as defined in (1.3) and (1.4) respectively. Then
(a) \( \lim_{R \to \infty} \frac{\sigma_{R,t}^2}{R} = 2 \int_0^t \xi(s) ds \) where \( \xi(s) = E \left[ (\sigma(u(s,y)))^2 \right] \).
(b) \( \lim_{R \to \infty} \frac{\Sigma_{R,t}^2}{R \log R} = 2t \).

**Proof.** See proposition 3.1 in [10] for part (a) and proposition 4.1 in [10] for part (b). \( \square \)
Lemma A.4. Fix $t > 0$. Let $\phi_{R,t}$ and $\varphi_{R,t}$ be defined as in (2.9) and (2.10). Then, there exists $R_0 \geq 1$, depending on $t$, such that for all $0 < s < t$ and $R \geq R_0$:

(a) $c_t \leq \int_R \phi_{R,t}^2(s,y)dy \leq C_t$, where the lower bound holds for $t/2 < s < t$.

(b) $\frac{c_t}{s \log R} \leq \int_R \varphi_{R,t}^2(s,y)dy \leq \frac{C_t}{s \log R}$, where the lower bound holds for $t/2 < s < t$.

Proof. (a) We start with the upper bound. Using the semigroup property, we see that

$$
\int_R \phi_{R,t}^2(s,y)dy = \frac{1}{\sigma_{R,t}^2} \int_{Q_R^2} p_{t-s}(y-x_1)p_{t-s}(x_2 - y)dxdy
$$

$$
= \frac{1}{\sigma_{R,t}^2} \int_{Q_R^2} p_{2(t-s)}(x_1 - x_2)dxdy
$$

$$
\leq \frac{1}{\sigma_{R,t}^2} \int_{Q_R^2} p_{2(t-s)}(x_1 - x_2)dxdy = \frac{2R}{\sigma_{R,t}^2} \leq C_t,
$$

where the last bound follows from Lemma A.3 part (a). To see the lower bound, let $R \geq 1$, and $t/2 < s < t$. Then,

$$
\int_R \phi_{R,t}^2(s,y)dy = \frac{1}{\sigma_{R,t}^2} \int_{Q_R^2} p_{2(t-s)}(x_1 - x_2)dxdy \geq \frac{1}{2\sigma_{R,t}^2} \int_{Q_R^2/\sqrt{2}} p_{2(t-s)}(y_1)dxdy
$$

$$
\geq \frac{R}{\sqrt{2}\sigma_{R,t}} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} p_{2(t-s)}(y)dy \geq c_t,
$$

where the last bound follows from Lemma A.3 part (a).

(b) Similarly, using the semigroup property, we see that

$$
\int_R \varphi_{R,t}^2(s,y)dy = \frac{1}{\Sigma_{R,t}^2} \int_{Q_R^2} p_{s(t-x)}(y - \frac{s}{t}x_1)p_{s(t-x)}(y - \frac{s}{t}x_2)dxdy
$$

$$
= \frac{1}{\sigma_{R,t}^2} \int_{Q_R^2} p_{2s(t-x)}(y_1 - y_2)dxdy
$$

$$
= \frac{s^2 \Sigma_{R,t}^2}{t^2} \int_{Q_{sR/t}^2} p_{2s(t-x)}(y_1 - y_2)dxdy
$$

$$
\leq \frac{2Rt}{s \Sigma_{R,t}^2} \leq \frac{C_t}{s \log R}.
$$
for all \( R \geq R_0 \), where the last bound follows from Lemma A.3 part (b). To see the lower bound, let \( t/2 < s < t \). Then, assuming \( R \geq 1 \),
\[
\int_R \varphi_{R,t}^2(s,y)dy = \int_R \frac{t^2}{s^2 \Sigma_{R,t}^2} p_{2s(t-s)}(y_1 - y_2)dy_1 dy_2
\]
\[
\geq \frac{\sqrt{2tR}}{s \Sigma_{R,t}} \int_{Q_{R,t}} p_{2s(t-s)}(z)dz
\]
\[
\geq \frac{\sqrt{2tR}}{s \Sigma_{R,t}} P\left( |N| \leq \frac{R}{2} \sqrt{\frac{s}{t(t-s)}} \right)
\]
\[
\geq \frac{\sqrt{2tR}}{s \Sigma_{R,t}} P\left( |N| \leq \frac{1}{2 \sqrt{t}} \right) \geq \frac{c_t}{s \log R},
\]
where the last bound follows from Lemma A.3 part (b) and \( N \) denotes a \( N(0,1) \) random variable. \( \square \)

**Lemma A.5.** For all \( R, t > 0 \),
\[
\int_{Q_{R}} p_t(x_1 - x_2)dx_1 dx_2 = \frac{4R}{\pi} \int_{\mathbb{R}} \varphi(\xi)e^{-\frac{\xi^2}{R^2}}d\xi,
\]
where
\[
\varphi(\xi) = 1 - \frac{\cos \xi}{\xi^2}.
\]

**Proof.** See Appendix in [6]. \( \square \)

**Lemma A.6.** For all \( R \geq e \) and all \( s > 0 \),
\[
\frac{1}{s} \int_0^s \frac{1}{r} e^{-s\left(\frac{r^2}{R^2}\right)} dr \leq 7 \log R \log(e + \frac{1}{s}) \log(e + \frac{1}{|N|}).
\]

**Proof.** See [6, Lemma A.1]. \( \square \)

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