Numeric Spectrum of Relic Gravitational Waves in
Accelerating Universe

Yang Zhang, Wen Zhao, Yefei Yuan, Tianyang Xia
Astrophysics Center
University of Science and Technology of China
Hefei, Anhui, China

Abstract

The accelerating expansion of the Universe at the present stage is a process that will
change the spectrum of relic gravitational waves. Here we present a numerical calculation
for the power spectrum of relic gravitational waves in the accelerating Universe. The
results show that although the overall features of the power spectrum is similar to that
in the non-accelerating models, the amplitude is smaller by an order of $10^{-1}$. We also
find that the spectrum is very sensitive to the index $\beta$ of the inflationary expansion with
the scale factor $a(\tau) \propto |\tau|^{1+\beta}$. With increase of $\beta$, the resulting spectrum is tilted to be
flatter with more power on high frequencies, and the sensitivity of the second science run
of the LIGO detectors puts a restriction on the parameter $\beta < -1.8$. The influence of
reheating following the inflation has been examined.

PACS numbers: 98.80.-k, 98.80.Es, 04.30.-w, 04.62.+v,
Key words: gravitational waves, accelerating universe, dark energy
e-mail: yzh@ustc.edu.cn
In the past the stochastic background of relic gravitational waves has been extensively studied [1] [2] [3]. The details of the spectrum of relic gravitational waves depends not only on the early stage of inflationary expansion, but also on the expansion behavior of the subsequent stages, including the current accelerating expansion. The calculations of spectrum so far [4] [5] [6] [7] [8] [9] have been done for the transitions from the inflationary era to the radiation, or matter dominated state, both being a decelerating expansion. However, the recent observations [10] [11] [12] indicate that the Universe is currently under accelerating expansion, which may be driven by the cosmic dark energy ($\Omega_\Lambda \sim 0.7$) plus the dark matter ($\Omega_m \sim 0.3$) with $\Omega = 1$ [13]. The current accelerating expansion of Universe will have an impact on the relic gravitational waves. We have studied this effect and obtained an approximate spectrum of the relic gravitational waves using an analytic method [14]. In this paper we present a numerical calculation of the spectrum. The resulting spectrum from the numerical calculation agrees with that from the analytic one. Throughout the paper we use units with $c = \hbar = 1$ and adopt notations similar to that of Grishchuk [6] for convenience for comparison.

The history of overall expansion of the Universe can be modelled as the following sequence of successive epochs of power-law expansion.

The initial stage (inflationary)

$$a(\tau) = l_0 \left| \frac{\tau}{\tau_1} \right|^{1 + \beta}, \quad -\infty < \tau \leq \tau_1,$$

(1)

where $\tau$ is the conformal time, $1 + \beta < 0$, and $\tau_1 < 0$. The special case of $\beta = -2$ is the de Sitter expansion.

The z-stage

$$a(\tau) = a_z(\tau - \tau_p)^{1 + \beta_z}, \quad \tau_1 \leq \tau \leq \tau_s.$$

(2)

The reheating process towards the end of inflation can be quite complicated and is model-dependent. This z-stage for various $\beta_z$s can represent a general reheating epoch.

The radiation-dominated stage

$$a(\tau) = a_e(\tau - \tau_e), \quad \tau_s \leq \tau \leq \tau_2.$$

(3)

The matter-dominated stage

$$a(\tau) = a_m(\tau - \tau_m)^2, \quad \tau_2 \leq \tau \leq \tau_E,$$

(4)

where $\tau_E$ is the time when the dark energy density $\rho_\Lambda$ is equal to the matter energy density $\rho_m$. Before the discovery of the accelerating expansion of the Universe, the current expansion was usually taken to be in this matter-dominated stage, which is a decelerating one. The value of the redshift $z_E$ at the time $\tau_E$ is given by $(1 + z_E) = a(\tau_H)/a(\tau_E)$, where $\tau_H$ is the present time. Since $\rho_\Lambda$ is constant and $\rho_m(\tau) \propto a^{-3}(\tau)$, one has $1 = \frac{\rho_\Lambda}{\rho_m(\tau_E)} = \frac{\rho_\Lambda}{\rho_m(\tau_H)(1 + z_E)}$. If the current values $\Omega_\Lambda \sim 0.7$ and $\Omega_m \sim 0.3$ are taken, then it follows that $1 + z_E = (\frac{\Omega_\Lambda}{\Omega_m})^{1/3} \sim 1.33$. 

% DOI: 10.1007/s10955-020-02561-3
The accelerating stage (up to the present)

\[ a(\tau) = l_H(\tau_a - \tau)^{-1}, \quad \tau_E \leq \tau \leq \tau_H. \tag{5} \]

This stage describes the accelerating expansion of the Universe, which is the new feature in our model and will induce some modifications to the spectrum of the relic gravitational waves. It should be mentioned that Eq. (5) is an approximation to the current expansion behavior since the matter component also exists in the current Universe.

Given \( a(\tau) \) for the various epochs, the derivative \( a' = da/d\tau \) and the ratio \( a'/a \) follow immediately. There are eleven constants in the above expressions of \( a(\tau) \), among which the \( \beta \) is imposed upon as the inflationary model parameter. By the continuity conditions of \( a(\tau) \) and \( a(\tau)' \) at the four given joining points \( \tau_1, \tau_s, \tau_2, \) and \( \tau_E \), one can fix only eight constants. Besides, choosing the overall normalization of \( a \) at the present time \( \tau_H \) and taking the expansion rate as the observed Hubble constant will fix two other constants. Specifically, we put \( (\tau_a - \tau_H) = 1 \) as the normalization of \( a(\tau) \) at the present time \( \tau_H \), which fixes the constant \( \tau_a \), and the constant \( l_H \) is fixed by the following calculation

\[ \frac{1}{H} \equiv \left( \frac{a^2}{a'} \right)_{\tau_H} = l_H, \tag{6} \]

so \( l_H \) turns out to be just the Hubble radius at present. Thus after a value of the model parameter \( \beta \) is picked up, all the eleven constants in \( a(\tau) \) are fixed up. In the expanding Robertson-Walker spacetime the physical wave length is related to the comoving wave number by

\[ \lambda \equiv \frac{2\pi a(\tau)}{k}, \tag{7} \]

and the wave number \( k_H \) corresponding to the present Hubble radius is

\[ k_H = \frac{2\pi a(\tau_H)}{l_H} = 2\pi. \tag{8} \]

Incorporating the perturbations to the spatially flat Robertson-Walker space-time, the metric is

\[ ds^2 = a^2(\tau)[d\tau^2 - (\delta_{ij} + h_{ij})dx^idx^j], \tag{9} \]

where the perturbations of spacetime \( h_{ij} \) is a \( 3 \times 3 \) symmetric matrix. The gravitational wave field is the tensorial portion of \( h_{ij} \), which is transverse-traceless \( \partial_i h^{ij} = 0, \delta^{ij} h_{ij} = 0 \). Since the relic gravitational waves are very weak, \( |h_{ij}| \ll 1 \), so one need just study the linearized field equation:

\[ \partial_\mu(\sqrt{-g} \partial^\mu h_{ij}(x, \tau)) = 0. \tag{10} \]

For a fixed wave number \( k \) and a fixed polarization state \( \lambda \), the wave equation reduces to the second-order ordinary differential equation \[14\] \[15\]

\[ h_k^{(\lambda)''}(\tau) + 2\frac{a'}{a} h_k^{(\lambda)'}(\tau) + k^2 h_k^{(\lambda)} = 0, \tag{11} \]
where the prime denotes $d/d\tau$. Since the equation of $h_k^{(\lambda)}(\tau)$ for each polarization is the same, we denote $h_k^{(\lambda)}(\tau)$ by $h_k(\tau)$ in below.

The power spectrum $h(k, \tau)$ of relic gravitational waves is defined by the following equation
\[
\int_0^\infty h^2(k, \tau) \frac{dk}{k} \equiv \langle 0| h^{ij}(x, \tau) h_{ij}(x, \tau) |0 \rangle, \quad (12)
\]
where the right-hand-side is the vacuum expectation value of the operator $h^{ij} h_{ij}$. Taking the contribution from each polarization to be the same, one reads the power spectrum
\[
h(k, \tau) = \frac{4}{\sqrt{\pi}} l_P k |h_k(\tau)|. \quad (13)
\]

Once the mode function $h_k(\tau)$ is given, the spectrum $h(k, \tau)$ follows.

The initial condition is taken to be during the inflationary stage. For a given wave number $k$, its wave crossed over the horizon at a time $\tau_i$, i.e. when the wave length $\lambda_i = 2\pi a(\tau_i)/k$ is equal to $1/H(\tau_i)$, the Hubble radius at time $\tau_i$. Eq.(1) yields $1/H(\tau_i) = l_0 |\tau_i|^2 + |1 + \beta|$, and for the exact de Sitter expansion with $\beta = -2$ one has $H(\tau_i) = l_0$. Note that a different $k$ corresponds to a different time $\tau_i = 2\pi |1 + \beta|/k$. Now choose the initial amplitude of mode function $h_k(\tau)$ as
\[
|h_k(\tau_i)| = \frac{1}{a(\tau_i)}. \quad (14)
\]
Then the initial amplitude of the power spectrum is
\[
h(k, \tau_i) = A(\frac{k}{k_H})^{2 + \beta}, \quad (15)
\]
where the constant
\[
A = \frac{8\sqrt{\pi}}{|1 + \beta|^{1 + \beta} l_0}. \quad (16)
\]
The power spectrum for the primordial perturbations of energy density is $P(k) \propto |h(k, \tau_H)|^2$, and its spectral index $n$ is defined as $P(k) \propto k^{n-1}$. Thus one reads off the relation $n = 2\beta + 5$. The exact de Sitter expansion with $\beta = -2$ will yield the so-called scale-invariant spectral index $n = 1$.

The spectral energy density parameter $\Omega_g(\nu)$ of the gravitational waves is defined through the relation $\rho_g/\rho_c = \int \Omega_g(\nu) \frac{d\nu}{\nu}$, where $\rho_g = \frac{1}{32\pi G} h_{ij,0} h^{ij,0}$ is the energy density of the gravitational waves, and $\rho_c$ is the critical energy density. One reads
\[
\Omega_g(\nu) = \frac{\pi^2}{3} h^2(k, \tau_H) \left(\frac{\nu}{\nu_H}\right)^2. \quad (18)
\]

The spectrum $h(k, \tau_H)$ has an overall factor $A$, which is fixed as follows. If the CMB anisotropies at low multipoles are induced by the gravitational waves, or, if the contributions from the gravitational waves and from the density perturbations are of the same order of magnitude, we may assume $\Delta T/T \simeq h(k, \tau_H)$. The observed CMB anisotropies [12] at lower multipoles is $\Delta T/T \simeq 0.37 \times 10^{-5}$ at $l \sim 2$. Taking this to be the perturbations at the Hubble radius $1/H$ yields
\[
h(k_H, \tau_H) = A \frac{1}{(1 + z_E)^3} = 0.37 \times 10^{-5}. \quad (17)
\]
To facilitate our numerical computation, we introduce a time variable $t$ by

$$e^t = |\tau - \tau_0|,$$

(18)

then the wave equation of $k$-mode is written as

$$\ddot{h}_k + \left(\frac{2\dot{a}}{a} - 1\right)\dot{h}_k + k^2 e^{2t}h_k = 0,$$

(19)

where the dot denotes $d/dt$. This is a second order differential equation with the function $\dot{a}/a$ occurring in the coefficient of $\dot{h}$ being given piecewise for each expansion stage. For each given $k$ the equation can be solved by using the Runge-Kutta method, yielding the $k$-mode wave function $h_k(t)$. We have done all this for various values of $\log_{10} k$ in the range $(-25, 20)$, in each time we take an increase $\Delta \log k = 0.1$. Switching back to the conformal time $\tau$ as given in Eq.(19) we arrive at $h_k(\tau)$ as well as $h(k, \tau)$ through Eq.(13). Collecting all the values of $h(k, \tau)$ at the fixed time $\tau = \tau_H$ yields the spectrum function $h(k, \tau_H)$ with $k$ being its variable now. In the numeric calculation the following specifications are made.

The five time instants $\tau_1, \tau_s, \tau_2, \tau_E$, and $\tau_H$ determine the overall expansion of the Universe in various stages, and they must be given as the parameters of our cosmological model. In this paper we fix them by the following equations:

$$a(\tau_H)/a(\tau_E) = 1.33,$$

(20)

$$a(\tau_E)/a(\tau_2) = 3454,$$

(21)

$$a(\tau_2)/a(\tau_s) = 10^{24},$$

(22)

$$a(\tau_s)/a(\tau_1) = 300.$$

(23)

The explanation to these relations are given as follows. Here we assume that $\Omega_\Lambda = 0.7$ and $\Omega_m = 0.3$, and that $a(\tau_H)/a(\tau_E) = (1 + z_E) = (\Omega_\Lambda/\Omega_m)^{1/3} \simeq 1.33$. The time instant $\tau_2$ is the moment when the matter domination started off, and the WMAP observation has indicated that the corresponding redshift is $(1+z_2) \simeq 3454$, so the second relation above follows. For the radiation dominated stage from $\tau_s$ to $\tau_2$, we assume that the starting temperature is $T_s \simeq 10^{15}$ Gev, a typical energy scale of grand unified theories, and that the ending temperature is $T_2 \simeq 1$ ev. Then from $a(\tau_2)/a(\tau_s) = T_s/T_2$ the third relation follows. The $z-$ stage from $\tau_1$ to $\tau_s$ represents the reheating of the early Universe, which has not been properly understood so far by cosmologists. For definiteness in numerical computation, we have chosen the relation $a(\tau_s)/a(\tau_1) = 300$, whose implications are not in conflict with the observations on relic gravitational waves. The other nine constants are $l_0$, $a_z$, $a_e$, $a_m$, $\tau_p$, $\tau_e$, $\tau_m$, $\beta_s$, and $\beta$, which are fixed by smoothly joining $a(\tau)$ and $a'/a(\tau)$ at the four jointing points, giving a relation between $\beta_s$ and $\beta$. For the primordial perturbation spectrum to be close to the scale-invariant one, we take
\[ \beta = -1.8, \ -1.9, \ -2.0, \] and obtain \( \beta_s = 0.598, \ -0.552, \ -1.689 \), respectively. The initial condition is taken at the horizon-crossing during the i-stage of inflation, and it consists of the initial values of \( h_k(\tau_i) \) and \( h'_k(\tau_i) \) as well. We have already specified \( h_k(\tau_i) \) as is implied in Eq.(15), and the other is taken to be \( h'_k(\tau_i) = 0 \), which reflects the fact that \( h_k(\tau) \) is a constant outside the horizon 1/\( H \).

The resulting spectrum of relic gravitational waves is presented in Fig.1 for the accelerating Universe, and in Fig.2 for the matter-dominated non-accelerating Universe. These spectra have an oscillating behavior caused by the Bessel functions contained in \( h_k(\tau) \) as has been expected. We also plotted in Fig.3 their fitting curves, which are smoother and reflect the corresponding spectral amplitudes. As the figures show, for a given \( \beta \), the amplitude is smaller by a factor \( \sim 10^{-1} \) in the accelerating Universe than in the non-accelerating Universe. The spectrum depends on \( \beta \), and a small \( \beta \) gives a small spectrum amplitude of gravitational waves. The recent second science run of the LIGO interferometric detectors [16] gives a best sensitivity \( 3 \times 10^{-24} \) near a frequency \( \sim 300\text{Hz} \). Our calculation for the \( \beta = -1.9 \) case yields an amplitude \( h \simeq 10^{-26} \), much smaller than the sensitivity. However, in the model of \( \beta = -1.8 \), the amplitude of the gravitational waves is just to fall into the sensitivity. Since the second run of LIGO has not yet observed any signal of stochastic gravitational waves in this frequency range, we arrive at a constraint \( \beta < -1.8 \) on the inflationary model. Although relic gravitational waves are still difficult to detect directly at present, future observations on CMB polarizations may reveal some information on it [17]. Fig. 4 shows the effects of the existence of reheating z-stage during very early universe, which tends to increase the amplitude of relics gravitational waves in the high frequencies \( \nu > 10^8 \text{Hz} \). In the range \( \nu < 10^8 \text{Hz} \), both spectra are almost the same.

In summary, we have presented a numerical spectrum of relic gravitational waves in the present accelerating Universe. The result confirms our approximate analytic one. In the frequency range \( \nu > 10^{-20} \text{Hz} \) the spectral amplitude is smaller by \( \sim 10^{-1} \) as compared with the non-accelerating model. A larger value of \( \beta \) yields a flatter spectrum with more power on the higher frequencies. The resulting sensitivity of the LIGO detectors has put a restriction on the model parameter \( \beta < -1.8 \).

ACKNOWLEDGMENT: We like to thank Prof. T.Y. Huang at Nanjing University and Profs. X.M. Zhang and C.G. Huang at Institute of High Energy Physics for stimulating discussions. Y. Zhang’s work has been supported by the Chinese NSF (10173008) and by NKBRSF (G19990754). Y.F. Yuan is supported by the Special Funds for Major State Research Projects. W.Zhao’s work is partially supported by Graduate Student Research Funding from USTC.

References

[1] Starobinsky A A 1979 JEPT Lett. 30 682.

[2] Rubakov V A, Sazhin M V and Veryaskin A V 1982 Phys.Lett.115B, 189.
[3] Abbott L F and Harari D D 1986 Nucl.Phys. B264 487 .
[4] Allen B 1988 Phys.Rev.D 37 2078.
[5] Sahni V 1990 Phys.Rev.D 42 453.
[6] Grishchuk L 1997 Class.Quant.Grav. 14 1445; Grishchuk L 2001 Lecture Notes Physics 562, 164 , in “Gyros, Clocks, Interferometers...: Testing Relativistic Gravity in Space”, Lammerzahl et al (Eds).
[7] Rizuelo A and Uzan J-P 2000 Phys.Rev.D 62 083506.
[8] Tashiro H, Chiba K, and Sasaki M 2004 Class.Quant.Grav. 21 1761.
[9] Henriques A B 2004 Class.Quant.Grav. 21 3057.
[10] Riess A, et. al. 1998 AJ 116 1009.
[11] Perlmutter S, et al. 1999 Astrophys.J. 517 565.
[12] Spergel D N, et al. 2003 ApJ Suppl. 148 175 .
[13] Zhang Y 2002 Gen. Rel. Grav. 34 2155; Zhang Y 2003 Gen. Rel. Grav. 35 689; Zhang Y 2003 Chin.Phys.Lett. 20 1899; Zhang Y 2004 Chin.Phys.Lett. 21 1183 .
[14] Zhang Y et al. 2005 Class.Quant.Grav. 22 1383-1394
[15] Grishchuk L 1974 Sov.Phys. JETP, Vol. 40, No.3, 409.
[16] Abbott R, et al. gr- qc/0410007.
[17] Zhang Y, Hao H, and Zhao W 2005 Acta. Astron. Sinica 46 1.
Figure 1: The numerical spectra $h(k, \tau_H)$ in the accelerating universe. The top (dot), middle (solid), bottom (dash dot dot) lines correspond $\beta = -1.8, -1.9$ and $-2.0$, respectively.

Figure 2: The numerical spectra $h(k, \tau_H)$ in the non-accelerating universe. The top (dot), middle (solid), bottom (dash dot dot) lines correspond $\beta = -1.8, -1.9$ and $-2.0$, respectively.
Figure 3: The fitting curves of the numerical spectra $h(k, \tau_H)$. The two curves on top correspond to $\beta = -1.8$, the two curves in middle are $\beta = -1.9$, and the two curves on bottom are $\beta = -2.0$. For each fixed $\beta$ the dash curve is for the non-accelerating model, and the solid is for the accelerating model.

Figure 4: The influence of reheating $z$-stage on the spectra $h(k, \tau_H)$ in the accelerating universe for the $\beta = -1.9$ case. The two models with or without the $z$-stage give almost the same spectra, except that at very high frequencies $\nu > 10^8$ Hz the existence of a reheating $z$-stage enhances the amplitude.