Reduction of divisors for classical superintegrable $GL(3)$ magnetic chain

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Abstract

Variables of separation for classical $GL(3)$ magnetic chain obtained by Sklyanin form a typical positive divisor $D$ of degree $n$ on a genus $g$ nonhyperelliptic algebraic curve. Because $n > g$ this divisor $D$ has unique representative $\rho_n(D)$ in the Jacobian which can be constructed using a family of canonical injections. We study properties of the corresponding chain of divisors and prove that the classical $GL(3)$ magnetic chain is superintegrable system with $\text{dim}|D| = 2$ superintegrable Hamiltonians.

1 Introduction

The Riemann-Roch theorem for divisor $D$ on an irreducible genus $g$ algebraic curve $C$ is

$$\text{dim}|D| = \text{deg}(D) - g + i(D). \tag{1.1}$$

In projective construction of the Jacobian proposed by Chow [6] this theorem garanties exactness of the canonical map

$$\rho_n : C(n) \to \text{Jac}(C), \quad n > g,$$

which is compatible with canonical injections

$$j_{n,m} : C(n) \to C(m), \quad n > m.$$

Here $C(n)$ is a $n$-fold symmetric product of $C$ and $\text{Jac}(C)$ is a Jacobi variety. The points of $C(n)$ represent positive divisors of degree $n$ on $C$.

If $\text{deg}(D) = n > g$ and number of independent differentials of the first kind on $C$ having zeros at the points of $D$ is equal to zero $i(D) = 0$, the complete linear system $|D|$ has dimension $\text{dim}|D| = n - g$. In this case there is a unique reduced divisor $\rho_n(D)$

$$\tilde{D} = \rho_n(D), \quad \text{deg}(\tilde{D}) = g, \quad \text{dim}|\tilde{D}| = 0,$$

and a chain of divisors

$$D \to j_{n,n-1}(D) \to j_{n,n-2}(D) \to \cdots \to j_{n,n+1}(D) \to \tilde{D} = \rho_n(D)$$

associated with canonical injections $j_{nm}$.

In classical mechanics positive divisors without matching points

$$D = P_1 + \cdots + P_n, \quad \text{deg}(D) = n \tag{1.2}$$

appear in a study of dynamical systems integrable by Abel’s quadratures

$$\sum_{i=1}^{n} \int P_i \omega_j = t_j, \quad j = 1, \ldots, n. \tag{1.3}$$
Here $\omega_j$ are independent differentials on genus $g$ algebraic curve $C$ defined by a separation relation
\[
C : \quad f(x, y) = 0,
\]
where $f(x, y)$ is an irreducible polynomial of degree $d$.

It is natural to ask a question about meaning of the terms in the Riemann-Roch theorem (1.1) in classical mechanics. According to (1.2) and (1.3)

- $deg(D)$ is a number of variables of separation, i.e., number of degrees of freedom;
- $g$ is a topological genus of algebraic curve $C$ defined by separation relation.

The meaning of terms $dim[D]$ and $i(D)$ depends on a concrete form of Abel’s equations (1.3).

For instance, let us consider Abel’s quadratures for the inhomogeneous classical $GL(N)$ magnetic chain $[8, 16, 17, 18]$. The phase space is a direct product of $N$ classical magnetic chains $[8, 16]$. Our calculations are independent differentials on genus $g$ non-regular differentials associated with Hamiltonians $H_{k,j}$, which commute to each other with respect to the Lie-Poisson bracket which central functions are contained in polynomial $d(x)$.

Thus, we have an integrable system with $n$ degrees of freedom
\[
n = \frac{MN(N-1)}{2},
\]
which is integrable by Abel’s quadratures (1.3) on the nonhyperelliptic curve $C$ (1.4) of genus
\[
g = \frac{(N-1)(MN-2)}{2}.
\]

Variables of separation obtained in $[8, 16, 17, 18]$ form a typical positive divisor $D$ with $deg(D) = n > g$ and $i(D) = 0$. According to the Riemann-Roch theorem dimension of the corresponding linear system is equal to
\[
dim[D] = n - g = N - 1.
\]

Because the corresponding Abel’s quadratures (1.3) involve $g$ independent regular differentials
\[
w_{k,j} = \frac{\partial f}{\partial h_{k,j}} \frac{dx}{\partial f/\partial y}, \quad \text{where} \quad deg \left( \frac{\partial f}{\partial h_{k,j}} \right) \leq d - 3, \quad \text{at} \quad j < kM - 1,
\]
and $dim[D]$ non-regular differentials associated with Hamiltonians
\[
H_{g+k} = h_{k,kM-1}, \quad k = 1, \ldots, N - 1
\]
we suppose that:

- dimension of linear system $dim[D] = N - 1 > 0$ coincides with the dimension of a space of superintegrable Hamiltonians $H_{g+1}, \ldots, H_n$ commuting simultaneously with $n$ integrals of motion $h_{k,j}$
\[
\{H_i, h_{k,j}\} = 0, \quad i = g + 1, \ldots, n, \quad k = 1, \ldots, N - 1, \quad j = 0, \ldots, kM,
\]

and with $g$ coordinates of the reduced divisor $\rho_n(D)$,
\[
\{H_i, \tilde{x}_j\} = \{H_i, \tilde{y}_j\} = 0;
\]

Here $\rho_n(D) = \tilde{P}_1 + \cdots \tilde{P}_g$, and $\tilde{P}_j = (\tilde{x}_j, \tilde{y}_j)$ is a point on $C$.

- divisors $j_{n,m}(D)$, $m = n - 1, n - 2, \ldots, g$ determine different systems of separation variables for the inhomogeneous classical $GL(N)$ magnetic chain.

In the present note we prove these statements for the case $N = 3$, when $dim[D] = 2$. The $GL(3)$ classical magnetic chain is chosen as a sample toy-model following Sklyanin [10]. Our calculations are quite elementary and do not involve sophisticated algebro-geometric technique of Chow [6] or Marsden-Weinstein technique from [2]. In fact we only repeat original Abel’s calculations from [1].
1.1 Integrals of Abel’s differential equations

Study of reduction of divisors $D \to \tilde{D}$ is closely related to the search of integrals of Abel’s differential equations

$$\text{Tr} \omega(P_1 + \cdots + P_d) = \omega(P_1) + \cdots + \omega(P_d) = 0,$$

was begun in the works of Euler [7], Abel [1], Jacobi [10] and Weierstrass [29]. Here $\omega$ is a differential of the first kind, and points $P_i$ form an intersection divisor of algebraic curve $C$ with a family of auxiliary curves, see discussion and historical remarks in [4, 9].

The Euler main idea in [7] is that existence of algebraic trajectories is intimately connected with the existence of a sufficient number of algebraic integrals of motion. According to [7, 10] algebraic integrals of Abel’s differential equations are suitable for finding algebraic restrictions for the form of trajectories. Indeed, if $g$ differentials in (1.3) form a basis in a space of differential forms of the first kind

$$\omega_j = p_j(x, y) \frac{dx}{f_g(x, y)}, \quad \text{deg}(p_j) \leq d - 3,$$

then

$$\sum_{i=1}^n \int_{P_i} \omega_j + \sum_{i=1}^g \int_{\tilde{P}_i} \omega_j = \text{const}, \quad j = 1, \ldots, g.$$

according to Abel’s theorem [1]. Here points $\tilde{P}_i$ determine a support of the reduced divisor

$$\tilde{D} = \rho_n(D) = \tilde{P}_1 + \cdots + \tilde{P}_g, \quad \text{deg}(\tilde{D}) = g, \quad \text{dim}|\tilde{D}| = 0.$$

Coordinates of divisor $\tilde{D}$ are the so-called Abel’s integrals of equations (1.3) with respect to evolution associated with $\text{dim}|D|$ nonregular differentials, see details in [4]. These coordinates are algebraic functions on coordinates of divisor $D$, i.e. on variables of separation. The corresponding mechanical systems integrable by Abel’s quadratures (1.3) are the so-called superintegrable systems [19, 26, 28].

In [10] Jacobi proposed to use polynomials $U(x)$ and $V(x)$

$$U(x) = \prod (x - \tilde{x}_i) = x^g + \tilde{u}_1 x^{g-1} + \cdots + \tilde{u}_g, \quad V(x) = \tilde{y}_i,$$

as generating functions of Abel’s integrals. In modern terms $U(x)$ and $y - V(x)$ are the so-called Mumford’s coordinates of divisor $\tilde{D}$ [15]. Other generating functions, i.e. other coordinates of divisors, were proposed by Weierstrass in [29]. Now we have a lot of other coordinates of divisors on algebraic curves, which are widely used in modern cryptography [11].

In framework of the new method of integration, which was proposed by Jacobi in [10] as a generalisation of Euler’s chasing of the algebraic trajectories [7], Abel’s equations (1.3) determine evolution of the integrable system with commuting Hamiltonians

$$H_1, \ldots, H_g; \quad H_{g+1}, \ldots, H_n$$

associated with $g$ regular and $\text{dim}|D| = n - g$ nonregular differentials on $C$. Simultaneously equations (1.3) determine evolution of the second integrable system with commuting Hamiltonians

$$\tilde{u}_1, \ldots, \tilde{u}_g; \quad H_{g+1}, \ldots, H_n.$$

Solving $n + g$ algebraic equations

$$H_i = \alpha_i, \quad \tilde{u}_j = \tilde{\alpha}_j$$

and $\text{dim}|D| = n - g$ differential equations with respect to $2n$ variables we find trajectories and their different parameterizations by time for both of these systems.

In [2] authors propose to consider Hamiltonians

$$\tilde{u}_1, \ldots, \tilde{u}_g; \quad H_{g+1}, \ldots, H_n.$$
as variables of separation which allows us separate solution of the Jacobi’s inversion problem for \( g \) equations

\[
\sum_{i=1}^{g} \int \tilde{F}_j \omega_j = \text{const}, \quad j = 1, \ldots, g
\]

on a Jacby variety \( \text{Jac}(C) \) from solution of the \( \dim|D| = n - g \) equations associated with nonregular differentials, see discussion in textbook [3]. Of course, it is different from the classical Euler-Jacobi construction.

2 \space \text{Abel’s reduction of divisors for } GL(N) \space \text{magnetic chain}

The phase space of this model is a direct product of \( M \) orbits of coadjoint action of the Lie group \( GL(N) \) on \( gl^*(N) \). Variables on this phase space \( S_{\alpha\beta}^{(m)} \) satisfy to the standard linear Lie-Poisson brackets

\[
\{ S_{\alpha\beta}^{(m)}, S_{\gamma\delta}^{(k)} \} = \left( S_{\alpha\beta}^{(m)} \delta_{\alpha\gamma} \delta_{\beta\delta} - S_{\alpha\beta}^{(k)} \delta_{\alpha\gamma} \delta_{\beta\delta} \right) \delta_{mk}.
\] (2.1)

Here \( m, k = 1, \ldots, M, \alpha, \beta = 1, \ldots, N \) and \( \sum_{\alpha=1}^{N} S_{\alpha\alpha}^{(m)} = 0 \). Following Sklyanin [16,17] we study only the generic orbits in fundamental vector representation without introducing any specific representations of algebra \( gl^*(N) \).

Let \( Z \) be an invertible \( N \times N \) number matrix having \( N \) distinct eigenvalues, \( \delta_m, m = 1, \ldots, M \) be some fixed numbers, and \( x \) be a complex parameter (spectral parameter). In order to get Abel’s equations [1,3] we start with the Lax matrix

\[
L(x) = Z \left( x - \delta_M + S^{(M)} \right) \cdots \left( x - \delta_2 + S^{(2)} \right) \left( x - \delta_1 + S^{(1)} \right),
\] (2.2)

and its spectral curve \( C \) on the \( (x, y) \) projective plane

\[
C : \quad \det(L(x) - y) = 0,
\]

which has the form [1,4]

\[
C : \quad f(x, y) = y^N + t_1(x)y^{N-1} + t_2(x)y^{N-2} + \cdots + t_{N-1}(x)y + d(x) = 0.
\]

Using this Lax matrix we can encode linear Lie-Poisson brackets (2.1) into standard \( r \)-matrix form

\[
\{ L(u), L(v) \} = \left[ r_{12}(u - v), \frac{1}{2} \right] L(u) + \frac{1}{2} L(v)], \quad r_{12}(u) = \frac{P}{u} \mathcal{P},
\]

where \( \mathcal{P} \) is the permutation operator in \( \mathbb{C}^N \otimes \mathbb{C}^N \), see [16,17] for details.

Separation of variables in the classical \( GL(N) \) magnetic chain and the corresponding unreduced divisor \( D \) of degree \( \text{deg}(D) = n > g \) with \( \dim|D| = N - 1 \) are discussed in [3,8,16,17,18]. Variables of separation \( (x_i, y_i) = P_i \) form a typical positive divisor without matching points

\[
D = P_1 + \cdots + P_n
\]

having the following Mumford’s coordinates

\[
D = \left( U(x), y - V(x) \right),
\]

where

\[
U(x) = \prod_{i=1}^{n} (x - x_i) = x^n + u_1 x^{n-1} + \cdots + u_n, \quad x_i \not= \pm x_j,
\]

and

\[
V(x) = \frac{P(x)}{Q(x)} + S(x)U(x), \quad V(x_i) = y_i.
\]
Here \( P(x) \) and \( Q(x) \) are polynomials in \( x \) which guarantee that \( V(x_i) = y_i \), and \( S(x) \) is a rational function describing behavior of \( V(x) \) at infinity.

According to Abel [1], we have to consider intersection of spectral curve \( C \) with a family of curves \( y = V(x) \) and identify unreduced divisor \( D \) with some part of the intersection divisor

\[
D + D' + D_\infty = 0, \tag{2.3}
\]

Here \( D_\infty \) is a suitable linear combination of points at infinity, and \( V(x) \) is an interpolation function through points of both divisors \( D \) and \( D_\infty \).

Abscissas of points of the intersection divisor \( D \) are zeroes of Abel’s polynomial \( \psi x \), which is the numerator of function

\[
f(x, V(x)) = V(x)^N + h_1(x)V(x)^{N-1} + h_2(x)V(x)^{N-2} + \cdots + h_N(x) = 0,
\]

so that

\[
\psi x = \theta \prod_{i=1}^{m}(x - x_i) \prod_{j=1}^{m}(x - x'_j), \tag{2.4}
\]

where we have to pick a function \( S(x) \) in such a way that \( n > m \) [1]. As a result we obtain divisor

\[
D' = j_{nm}(D) = P'_1 + \cdots + P'_m, \quad n > m
\]

with coordinates \( (U'(x), y - V'(x)) \), where

\[
U'(x) = \prod_{j=1}^{m}(x - x'_j) = \text{MakeMonic} \left( \frac{\psi x}{U(x)} \right), \quad V'(x) = -V(x) \mod U'(x).
\]

Here the MakeMonic means that we divide the polynomial by its leading coefficient and \( - \) is an inversion on \( C \).

If \( m > g \), we have to repeat Abel’s calculations in order to get divisor \( j_{n,g}(D) \) of degree \( g \) which can be identified with the representative \( \rho_n(D) \) of a given unreduced divisor \( D \) in Jacobian \( \text{Jac}(C) \) [6].

### 2.1 Reduction of divisors for \( GL(3) \) magnetic chain

According to Sklyanin [10,17] separation coordinates \( x_j \) are defined as poles of the vector Baker-Akhiezer function \( \Psi(x) \), which is defined as an eigenvector of the \( N \times N \) Lax matrix \( L(x) \) (2.2)

\[
L(x)\Psi(x) = y(x)\Psi(x). \tag{2.5}
\]

with a suitable normalization \( (\alpha, \Psi(x)) = 1 \).

At \( N = 3 \) the simplest choice is \( \alpha = (0, 0, 1) \), and separation coordinates \( x_j \) are zeroes of polynomial

\[
B(x) = L_{31} \left| \begin{array}{ccc}
L_{11} & L_{12} & L_{13} \\
L_{31} & L_{32} & L_{33}
\end{array} \right| + L_{21} \left| \begin{array}{ccc}
L_{21} & L_{22} & L_{23} \\
L_{31} & L_{32} & L_{33}
\end{array} \right|,
\]

whereas canonically conjugated momenta \( p_j = \ln y_j \) are defined by

\[
y_j = A(x_j), \quad A = \left| \begin{array}{ccc}
L_{11} & L_{12} & L_{13} \\
L_{31} & L_{32} & L_{33}
\end{array} \right| \mod B(x).
\]

Using a similarity transformation \( WL(x)W^{-1} \) we can transform number matrix \( Z \) to low-diagonal form

\[
Z = \left( \begin{array}{ccc}
z_{11} & 0 & 0 \\
z_{21} & z_{22} & 0 \\
z_{31} & z_{32} & z_{33}
\end{array} \right),
\]
which guarantees that polynomial
\[ U(x) = \text{MakeMonic} \, B(x) = x^n + u_1 x^{n-1} + \cdots + u_g \]
and rational approximating function
\[ V(x) = A(x) \]
can be considered as Mumford’s coordinates of the typical positive unreduced divisor
\[ D = \left( U(x), y - V(x) \right), \quad \deg(D) = n, \quad \dim|D| = 2. \] (2.6)

This divisor is an input of the Abel’s reduction algorithm.

**First round of the reduction algorithm**
Substituting \( y = V(x) \) into equation \( f(x, y) = 0 \) we obtain Abel’s polynomial \( \psi_x \) (2.4) and Mumford’s coordinates of divisor
\[ D' = j_{n,n-1}(D) = \left( U'(x), y - V'(x) \right), \quad \deg(D') = n - 1, \quad \dim|D'| = 1, \]
where
\[ U'(x) = \text{MakeMonic} \left( \begin{array}{ccc} L_{32} & L_{12} & L_{13} \\ L_{32} & L_{33} & -L_{12} \\ -L_{12} & L_{31} & L_{12} \end{array} \right) = x^{n-1} + u'_1 x^{n-2} + \cdots + u'_{n-1} \]
and
\[ V'(x) = - \frac{L_{12} \, L_{32} \, L_{13} \, L_{33}}{L_{12}} \mod U'(x). \]

Now we have to repeat Abel’s calculations starting with the Mumford’s coordinates of divisor \( D' = j_{n,n-1}(D) \).

**Second round of the reduction algorithm**
Substituting \( y = V'(x) \) into equation \( f(x, y) = 0 \) we obtain Abel’s polynomial \( \psi'_x \) (2.3) and Mumford’s coordinates of divisor
\[ D'' = j_{n,n-2}(D) = \left( U''(x), y - V''(x) \right), \quad \deg(D'') = g, \quad \dim|D''| = 0, \]
where
\[ U''(x) = \text{MakeMonic} \left( \begin{array}{ccc} -L_{12} & L_{12} & L_{13} \\ L_{22} & L_{23} & -L_{13} \\ -L_{13} & L_{32} & L_{13} \end{array} \right) = x^g + u''_1 x^{g-1} + \cdots + u''_g \]
and
\[ V''(x) = - \frac{L_{12} \, L_{22} \, L_{23} \, L_{13}}{L_{13}}. \]

It is the output of Abel’s reduction algorithm for the \( GL(3) \) magnetic chain.

**End of the reduction algorithm.**
Following the Chow projective construction of the Jacobian now we have to identify divisor \( D'' = j_{n,n-2} \) with representative \( \rho_n(D) \) of unreduced \( D \) in \( \text{Jac}(C) \) [6].

For the Gaudin magnets Abel’s reduction of divisor is equivalent to reduction of generic orbits of the corresponding loop algebra [2]. In a quantum case a counterpart of this reduction of divisors is discussed in [13, 14].
2.2 Properties of divisors for $GL(3)$ magnetic chain

Using Abel’s reduction we obtained a chain of divisors

$$D \rightarrow D' = j_{n,n-1}(D) \rightarrow D'' = j_{n,n-2}(D) \sim \rho_n(D) \in Jac(C).$$

Now we want to study properties of these divisors. It is easy to calculate Poisson brackets between Mumford’s coordinates of divisors

$$\{U(x), U(\tilde{x})\} = \{U'(x), U'(\tilde{x})\} = \{U''(x), U''(\tilde{x})\} = 0,$$

$$\{V(x), V(\tilde{x})\} = \{V'(x), V'(\tilde{x})\} = \{V''(x), V''(\tilde{x})\} = 0,$$

and

$$\{V(x), U(\tilde{x})\} = \frac{1}{x - \tilde{x}} \left( V(x)U(\tilde{x}) - U(x)V(\tilde{x}) \right) \frac{L_{32}(\tilde{x})^2}{L_{32}(x)^2},$$

$$\{V'(x), U'(\tilde{x})\} = -\frac{1}{x - \tilde{x}} \left( V'(x)U'(\tilde{x}) - U'(x)V'(\tilde{x}) \right),$$

$$\{V''(x), U''(\tilde{x})\} = \frac{1}{x - \tilde{x}} \left( V''(x)U''(\tilde{x}) - U''(x)V''(\tilde{x}) \right),$$

from which the wanted Poisson brackets for variables of separation are derived immediately using Sklyanin’s method [16, 17]. Here $x$ and $\tilde{x}$ are spectral parameters.

The Poisson brackets between superintegrable Hamiltonians [1,3]

$$H_{g+1} = h_{1,M-1} \quad \text{and} \quad H_{g+2} = h_{2,2M-1}$$

and Mumford’s coordinates of divisors $D' = j_{n,n-1}(D)$ and $D'' = j_{n,n-2}(D)$ are equal to zero according to Abel’s theorem. Using Poisson brackets (2.1) it is easy to directly verify this fact

$$\{H', U'(x)\} = 0, \quad \{H'', V''(x)\} = 0, \quad \text{where} \quad H' = z_{11}H_{g+1} + H_{g+2},$$

and

$$\{H_{g+1}, U''(x)\} = \{H_{g+2}, U''(x)\} = 0, \quad \{H_{g+1}, V''(x)\} = \{H_{g+2}, V''(x)\} = 0.$$  

Thus, the $GL(3)$ magnetic chain is a superintegrable system with $\dim(D) = 2$ superintegrable Hamiltonians.

**Proposition 1** Spectral curve $C$ [1,4] of the Lax matrix $L$ (2.4) contributes $n$ functionally independent Hamiltonians $h_{k,j}$, which commute to each other with respect to the Lie-Poisson bracket (2.7)

$$h_{1,1}, \ldots, H_{g+1} = h_{1,M-1}, h_{2,1}, \ldots, H_{g+2} = h_{2,2M-1}.$$  

Divisors of poles of its Baker-Akhiezer function $\Psi(x)$ (2.5)

$$D \rightarrow D' \rightarrow D'' \sim \rho_n(D)$$

give rise to $\dim|D| = 2$ families of $n$ functionally independent Hamiltonians which also commute to each other

$$u_1', \ldots, u_1'', H', \quad \text{and} \quad u_g', \ldots, u_g'', H_{g+1}, H_{g+2}.$$  

According to Euler and Jacobi we can use coordinates of divisors $D'$ and $D''$ for algebraic definition of forms of trajectories. Following [2] we can also use coordinates of reduced divisor $D''$ for reduction of $n = g + 2$ Abel’s equations (1.3) to Jacobi’s canonical inversion problem on $Jac(C)$ of genus $g$ curve $C$.  

7
3 Conclusion

In [24, 25, 26, 27, 28] we considered superintegrable systems associated with elliptic and hyperelliptic curves for which reduction of divisors is a well-studied standard part of modern cryptography on algebraic curves [11]. In this note we use Lax matrices and classical r-matrix theory in order to perform reduction of divisors and to study properties of the obtained divisors on nonhyperelliptic curves.

After quite trivial calculations involving a Lax matrix, we obtain chain of divisors $D \rightarrow D' \rightarrow D''$ of degree $n$, $n - 1$ and $n - 2 = g$ on the nonhyperelliptic spectral curve and the corresponding superintegrable Hamiltonians $H', H_{q+1}, H_{g+2}$. It allows us to apply the classical Euler-Jacobi method to solve Abel’s equations. Moreover, these divisors and superintegrable Hamiltonians determine three different systems of canonical coordinates, which are variables of separation for the classical $GL(3)$ magnetic chain. Substituting these different canonical variables into new separation relations we can get various integrable systems having integrals of motion of third, fourth and sixth order in original variables, see examples in [20, 21, 22, 23].

In our opinion, the main result of our mathematical experiment is that reduction of divisors produces variables of separation. It allows us to suppose that the magic Sklyanin’s recipe "Take the poles of the properly normalized Baker-Akhiezer function and the corresponding eigenvalues of the Lax operator and you obtain variables of separation” can be changed. Because any unreduced divisor has a unique representative according to the Riemann-Roch theorem we want to propose "take the reduced divisor of poles of the arbitrary normalized Baker-Akhiezer function” instead of the key words ”the properly normalized”. We plan to discuss this conjecture in further publications.

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