No-Arbitrage Pricing, Dynamics and Forward Prices of Collateralized Derivatives

Alessio Calvelli

Financial Engineering
Banca Akros (Banco BPM group)
Viale Eginardo, 29 - 20149 Milan (Italy)
alessio.calvelli@bancaakros.it

Abstract. This paper analyzes the pricing of collateralized derivatives, i.e. contracts where counterparties are not only subject to financial derivatives cash flows but also to collateral cash flows arising from a collateral agreement. We do this along the lines of the brilliant approach of the first part of Moreni and Pallavicini [MP17], in particular we extend their framework where underlyings are continuous processes driven by a Brownian vector, to a more general setup where underlyings are semimartingales (and hence jump processes). First of all, we briefly derive from scratch the theoretical foundations of the main subsequent achievements i.e. the extension of the classical No-Arbitrage theory to dividend paying semimartingale assets, where by dividend we mean any cash flow earned/paid from holding the asset. In this part we merge, in the same treatment and under the same notation, the principal known results with some original ones. Then we extend the approach of [MP17] in different directions and we derive not only the pricing formulae but also the dynamics and forward prices of collateralized derivatives (extending the achievements of the first part of Gabrielli et al. [GPS19]). Finally, we study some important applications (Repurchase Agreements, Securities Lending and Futures contracts) of previously established theoretical frameworks, obtaining some results that are commonly used in practitioners literature, but often not well understood.

Keywords: Arbitrage-Free Pricing; Collateral; Credit Support Annex; ISDA; Collateral Modeling; Initial Margin; Variation Margin; Re-hypothecation; Margin Valuation Adjustment; Repurchase Agreement; Securities Lending; Futures.

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*The opinions here expressed are solely those of the authors and do not represent in any way those of their employers.
1. Introduction

1.1. Motivation

It is well known that, when entering into any financial contract establishing some (conditions for optional) future financial transactions, a risk for a party signing this contract is the risk that her counterparty defaults and fails to pay some due future financial flows specified in the contract. In order to tackle this issue, very often the counterparties agree to sign a collateral agreement, i.e. an annex of the financial contract where they engage themselves to a process referred to as collateralization: the counterparty (or a third independent party acting as custodian) running the credit risk for the financial instrument receives the collateral, i.e. cash or liquid securities, to cover some or all of this risk: the rationale being that in case of default she can sell loaned securities or seize the loaned cash to offset her uncovered positions.

The main objective of this paper is to analyze the pricing of collateralized (financial) derivatives, i.e. contracts where counterparties are not only subject to financial derivatives cash flows but also to collateral cash flows arising from the collateral agreement. We do this along the lines of the brilliant approach of the first part of [MP17], a paper that adapted the results of [PPB12] to multiple currencies in case of perfect collateralization. The findings of [PPB12] were also subsequently obtained in [BFP19] in terms of Backward Stochastic Differential Equations (BSDEs). Finally, we mention the recent publication of [BBFPR22] that compares this approach, settled in practitioners literature, with the more elegant “replication portfolio approach” generally used in academics literature.

First of all, we derive from scratch the theoretical foundations of the main subsequent achievements i.e. the extension of the classical No-Arbitrage theory to dividend paying semimartingale assets, where by dividend we mean any cash flow earned/paid from holding the asset. In this part we merge, in the same treatment and under the same notation, the principal known results with some original ones. Then we extend the framework of [MP17] where underlyings are continuous processes driven by a Brownian vector, to a more general setup where underlyings are semimartingales. Therefore, all processes can jump: this is coherent not only with the fact that the financial derivative price is inherently a jump process with jumps coinciding with intermediate cash flows, but also with the fact that the collateral value process is intrinsically a purely...
discontinuous process (see Remark 3.1). We allow (not only jumps but also all) interest rates be stochastic and we derive both the pricing formulae and also the dynamics and forward prices of collateralized derivatives (extending the achievements of the first part of [GPS19]) and finally we study some important applications (Repurchase Agreements, Securities Lending and Futures contracts) of previously established theoretical frameworks.

As in [MP17] (but this choice is common also to other papers, see e.g. [Pi10]) we do not take into account the residual possibility of a loss on a collateralized contract due to the default of the counterparty: this is a simplifying hypothesis to better understand the mechanics of collateralization. In fact, since the collateralization strongly reduces the bilateral counterparty risk, we take a step forward and assume that it eliminates it: this assumption becomes more realistic the more the collateralization process is performed continuously (we will refer to this case as continuous margin calls). In cases where the underlying assets are continuous and driven by a Brownian vector as in [PPB12], the continuous collateralization implies perfect collateralization, meaning that the collateral perfectly covers the close-out amount (i.e. the residual value of the financial derivative at default time) and hence that the counterparty risk is literally eliminated. As in this paper, when introducing jumps in the underlyings – even in cases of continuous collateralization – since the collateral is by definition a predictable process, it could differ from the close-out amount which is by nature optional (i.e. only adapted but not predictable). In fact, the close-out amount is in someway dependent from the financial derivative price which is an optional process (since the latter is in someway dependent from the underlyings’ quotes). However, the possible difference between the collateral value and the close-out amount could be covered by the fact that, generally, the financial derivatives are over-collateralized (i.e. the collateral value is set to a quantity strictly greater than the financial derivative price: see Section 3). So, even in this paper framework, we could still achieve perfect collateralization if, during the continuous collateralization, the collateral value is set in such a way that the over-collateral covers all the jumps (and particularly the unpredictable ones) of the derivative price. If this is not the case, the residual counterparty risk (when it is non-negligible) should be taken into account and the results of the present paper should be considered as an approximation.

The pros of the aforementioned simplification are that we obtain a clearer formulae and that we rigorously explain some results that are commonly used in practitioners literature, but often not well understood – see Section 4. On the other hand, the introduction of the counterparty risk would be quite straightforward since this subject is broadly explored and the literature dealing with it is well settled. In general, the approach of [MP17] (and hence our approach) already contains the essential elements of the approach of [PPB12] which tackles the calculation of all valuation adjustments – not only the collateral ones.

For obtaining all the achievements described above, the common thread of the paper will be to identify, thanks to No-Arbitrage conditions, Risk-Neutral martingales in progressively more challenging contexts where our intuition could be increasingly lost: as a first example we will see the martingale corresponding to a non-dividend paying asset, then we will strive to recognize the martingale corresponding to a dividend-paying asset, as a third step we will detect a martingale linked to a collateralized derivative, finally we will find martingales in more specific contracts. All these efforts in searching for martingales are motivated from the fact that martingales have some nice properties and, primarily, since they have well defined dynamical features and specific connections with expected values: among all expected values we are particularly interested in pricing ones.

This paper is organized as follows: the next subsection consists in a brief technical setup, Section 2 develops the general theory to be used in the remainder of the paper, Section 3 is dedicated to the presentation and analysis of collateralized derivatives, in Section 4 we describe some specific applications of previous section, the last section outlines some concluding remarks.
1.2. Technical Setup

All processes of the present paper are semimartingales: we refer to Appendix A for semimartingales notation, a list of relevant results and some references. As in [Pr01] we will assume that we are given a filtered complete probability space \((\Omega, \mathcal{F}; (\mathcal{F}_t)_{t \geq 0}; \mathbb{P})\) where \(\mathcal{F}_t \subseteq \mathcal{F}\) for any \(t \geq 0\) and \(\mathbb{P}\) is the so called “real world probability” or “physical measure”. We further assume that \(\mathcal{F}_s \subseteq \mathcal{F}_t\) if \(s < t\); \(\mathcal{F}_0\) contains all the \(\mathbb{P}\)-null sets of \(\mathcal{F}\); and also that \(\bigcap_{t > 0} \mathcal{F}_t \equiv \mathcal{F}_{t+} = \mathcal{F}_t\) by hypothesis. This last property is called the right continuity of the filtration. These hypotheses, taken together, are known as the usual hypotheses (when the usual hypotheses hold, one knows that every martingale has a version which is RCLL, one of the most important consequences of these hypotheses).

Remark 1.1. The reader who is unfamiliar with jump processes and semimartingale theory could read only a subset of the following results interpreting all processes as continuous processes: under this simplifying hypothesis, for any processes \(X, Y\) and time \(u\), one has \(X_{u-} = X_u\) and \(\Delta X_u = 0\) and Quadratic Variation/Covariation equal to their predictable versions: \([X, X]_u = \langle X, X \rangle_u\) and \([X, Y]_u = \langle X, Y \rangle_u\).

Moreover, all vectors of the paper are considered as column vectors and we denote with \(\bar{0}\) the vector with all components equal to zero (its dimension will be clear from the context). We also use the notation \(a \wedge b := \min\{a, b\}\). We define with \(\Theta_T(u) := 1_{u \geq T}\) the Heaviside step function centered at \(T\) and with \(\delta(u - T)\) the Dirac mass centered at \(T\), where \(u, T \in \mathbb{R}\) and we have \(\partial_\Theta(u, T) = \delta(u - T)\) (distributional derivative). Note also that we use the convention \(\int_T^T := \int_{(u, T)}\).

Unless stated otherwise, any interest rate process is stochastic, predictable and bounded: for an interest rate process \(x := (x_t)_{t \geq 0}\) we denote the corresponding bank account with \(B_T^x := \exp\int_0^T x_s \, ds\) (then also any bank account is bounded). Clearly \(B_T^r \equiv B\) where \(r\) is the domestic spot risk-free interest rate process. For \(t \leq T\) we define the \(T\)-zero coupon bond price process associated with interest rate \(x\) as

\[ P^x_t(T) := \mathbb{E}_t\left[ \frac{B_T^x}{B_T^r} \right] \]

where \(\mathbb{E}_t[\cdot]\) stands for the expectation under measure \(\mathbb{Q}\) conditioned to \(\mathcal{F}_t\), and we denote with \(\mathbb{Q}\) the domestic Risk Neutral Measure (the measure with numéraire \(B\)). Of course \(P(T) = P^0(T)\). Finally \(\mathbb{Q}^T\) is the domestic \(T\)-forward measure (the measure with numéraire \(P(T)\)) and \(\mathbb{E}_T^T[\cdot]\) stands for the expectation under measure \(\mathbb{Q}^T\) conditioned to \(\mathcal{F}_T\).

2. General Theory

2.1. Fundamental Theorems of Asset Pricing for Non-Dividend Paying Assets

We start with two cornerstones of Asset Pricing: see [Bj09, Pr01] or Theorem 2.1.4. of [JYC09] and references therein for proofs and an explanation of the condition of No Free Lunch with Local Vanishing Risk (NFLVR): the less interested reader can understand this condition as “no-arbitrage”.

Theorem 2.1 (First Fundamental Theorem of Asset Pricing (First FTAP)).

Consider the market model of non-dividend paying underlying processes \(S^0, S^1, \ldots, S^n\) under the filtered probability space \((\Omega, \mathcal{F}; (\mathcal{F}_t)_{t \geq 0}; \mathbb{P})\) satisfying the usual hypotheses. Assume that \(S^0_0 > 0\) \(\mathbb{P}\)-a.s. for all \(t \geq 0\) and that \(S^0, S^1, \ldots, S^n\) are locally bounded semimartingales. Then the following conditions are equivalent:

i) The model satisfies the NFLVR;
ii) There exists a measure $Q^0 \sim P$ such that the deflated processes
\[
\left\{ \frac{S_u^0}{S_0^0} \right\}_{t \geq 0}, \left\{ \frac{S_u^1}{S_0^1} \right\}_{t \geq 0}, \ldots, \left\{ \frac{S_u^n}{S_0^n} \right\}_{t \geq 0}
\]
are local martingales under $Q^0$, which is called Equivalent Martingale Measure (EMM). Then we call the process $S_0$ as the numéraire of the measure $Q^0$.

The following result establishes that the dynamics of asset prices have to be semimartingales:

**Theorem 2.2 (From [JYC09])**. Let $S$ be an adapted RCLL process. If $S$ is locally bounded and satisfies the NFLVR property for simple integrands, then $S$ is a semimartingale.

We will see (in a more general setting) that not only the discounted prices of securities are local martingales, but also any self-financing strategy and then price, and in particular prices of financial derivatives. In the special case where $S^0 \equiv B$, the bank account process, we call $Q \equiv Q^0$ as the Risk Neutral measure.

**Theorem 2.3 (Second FTAP)**. Assume that the model satisfies the NFLVR condition and consider a fixed numéraire $S^0$. Then the market is complete iff the EMM $Q^0$, corresponding to numéraire $S^0$, is unique.

In case $Q^0$ is unique, then any price process is unique.

### 2.2. Extension to Dividend-Paying Assets

#### 2.2.1. Main Results

We consider a market with underlying assets $S = (S^0, S^1, \ldots, S^n)$ where each $S^i$ is a locally bounded semimartingale. Define with $D := (D^0, D^1, \ldots, D^n)$ the cumulative dividend vector process, where each $D^i$ is a locally bounded semimartingale representing the undiscounted cumulative (net, after taxes) paid by asset $S^i$ from inception. In particular,

\[
D^i_t = D^i_0 + \int_0^t dD^i_u
\]

and $dD^i_u$ are the net dividends paid by $S^i$ in the interval $du$. One can imagine $D^i$ as an account (at zero interest rates) that grows with dividend payments $dD^i_u$. We fix $S^0 \equiv B$ so that clearly $dD^0_t = \Delta D^0_t = 0$ for any $t$ and $Q^0 \equiv Q$ (the Risk Neutral measure with numéraire $B$).

As we read in [DH88], the convention we choose is for the dividend or stock price change at $t$ to be included in the cumulative dividend process or price at $t$. In technical terms, this says that for each asset $i$, both $D^i$ and $S^i$ are assumed to be right-continuous (recall that semimartingales have RCLL paths). The lump net dividend paid out at time $t$ by security $S^i$ is thus $\Delta D^i_t := D^i_t - D^i_{t-}$, and $\Delta S^i_t := S^i_t - S^i_{t-}$ is the jump of the asset: in case $S^i$ jumps only due to the lump dividends we have the following change in price as the stock goes ex-dividend: $\Delta S^i_t = -\Delta D^i_t$ (more on this at Remark 2.17).

If $D^i$ is absolutely continuous, the dividend rate $\partial_t D^i_t$ exists for almost all $t$ and $D^i_t$ is its integral. As another special case, if dividends occur only in lumps, then $D^i$ is a random step function. Because it is of no real use to us to have a dividend payment at $t = 0$, we assume, without loss of generality, that $D^i_0 = D^i_{0-} = 0$ and $S^i_{0-} = S^i_0$ so that $\Delta D^i_0 = \Delta S^i_0 = 0$.

**Remark 2.4.** As we will see later on, we will extend the concept of dividends to any intermediate (after taxes) coupon of the asset $S$, in this sense the coupon is a “dividend” of a financial derivative. Hence, we do not require the positivity of the dividend process as it should be for dividends in a strict sense.
Define also the gain process as
\[ G_t := (G_0^t, \ldots, G_n^t) \]
that is so called since \( dG_i^t = dS_i^t + dD_i^t \) is the gain at time \( t \) one experiences holding the asset \( S_t^i \): it is the sum between the capital gain and the dividend gain (either can be negative).

We build a portfolio of semimartingale vector quantities \( \varphi = (\varphi_0^t, \ldots, \varphi_n^t) \) with value
\[ \Psi_t(\varphi) := \varphi_t \cdot S_t := \sum_{i=0}^{n} \varphi_i^t S_i^t \]
which, with no-dividend jumps (more on this later), is self-financing if we set
\[ d\Psi_t = \varphi_t \cdot dG_t \]
where \( G \) is defined in (1) and \( dG_t := (dG_0^t, \ldots, dG_n^t) \). The self-financing portfolio constraint means that the only change in portfolio value comes from capital gains and dividend gains, whatever the trading strategy. The trading strategy can move value between the stock and cash accounts but not create or destroy value. In fact, heuristically, integrating in the infinitesimally small interval \( [t, t + \delta t] \), we have
\[ \Psi_{t+\delta t} := \varphi_{t+\delta t} \cdot S_{t+\delta t} = \Psi_t + \int_t^{t+\delta t} d\Psi_u \]
self-fin. \[ \approx \Psi_t + \varphi_t \cdot \left[ G_{t+\delta t} - G_t \right] \]
\[ := \varphi_t \cdot \left[ S_{t+\delta t} + (D_{t+\delta t} - D_t) \right] \]
where the first term of the last equation represents the allocation at time \( t \) that naturally evolves due to the underlyings move in \( t + \delta t \), and the second term represents the dividends that drops in the interval \( (t, t + \delta t] \). The wealth that is produced in the last equation due to the allocation at time \( t \) and the market move must be totally reallocated with the new quantities \( \varphi_{t+\delta t} \): see the first equation.

The previous heuristical reasoning has some issues in a semimartingale framework:

i) The integrand of the self-financing condition must be a locally bounded predictable process:

- This is needed from a technical point of view in order to have well posed semimartingale integrals;
- From a Mathematical Finance point of view, this fact has some no-arbitrage implications:
  see Examples 14.1/14.5 in [Pa11] or Example 8.1 in [TC05].

ii) The previous heuristical reasoning does not consider the presence of jumps.

In order to tackle the first issue we modify the above condition with the left limit modification of the trading strategy (recall from Proposition A.6 that the process \( \varphi_- \) is predictable):
\[ d\Psi_t = \varphi_{t-} \cdot dG_t \]
and the left limit is also coherent with the fact that the stock holder earns the lump dividend \( (\Delta D_t := (\Delta D_0^t, \ldots, \Delta D_n^t)) \) and is subject to the asset jump \( (\Delta S_t := (\Delta S_0^t, \ldots, \Delta S_n^t)) \) on her position \( \varphi \) written one instant before these jumps materialize (hence at time \( t^- \)). In order to tackle the second issue we have the following proposition.

**Proposition 2.5.** The system (2)-(3) implies the jump-self-financing condition, i.e. for any \( t \),
\[ \Delta \varphi_t \cdot S_t = \varphi_{t-} \cdot \Delta D_t \]
which says that the (possibly unpredictable) jump dividend gain (rhs of the above equation) must be absorbed by a (jump) increase in asset quantities (left-hand side of the above equation where the asset value has gone ex-dividend).

**Proof.** By (A.1)-(A.3), the self-financing condition (3) implies

\[ \Delta \Psi_t = \varphi_{t-} \cdot \Delta G_t \]  

(5)

where one observes that the trader earns the jump gain \( \Delta G_t = \Delta S_t + \Delta D_t \) under position \( \varphi_{t-} \) (the position on her book one instant before the jumps realization). Therefore

\[ \Psi_t = \Psi_{t-} + \Delta \Psi_t = \varphi_{t-} \cdot S_{t-} + \varphi_{t-} \cdot \Delta G_t = \varphi_{t-} \cdot (S_t + \Delta D_t), \]

on the other hand \( \Psi_t := \varphi_t \cdot S_t \) by (2), then through equating these two equivalent expressions one obtains (4). We explore also an alternative derivation of (4): using (A.1)-(A.2), from (2):

\[ \Delta \Psi_t := \Delta (\varphi_t \cdot S_t) = \varphi_{t-} \cdot \Delta S_t + S_t \cdot \Delta \varphi_t = \varphi_{t-} \cdot \Delta S_t + S_t \cdot \Delta \varphi_t \]

which can be compared to (5) to obtain again condition (4). \( \square \)

**Corollary 2.6.** In case the dividend vector process \( D \) is null or continuous, we have \( \Delta \varphi_t = 0 \) for all \( t \), or \( \varphi_t = \varphi_{t-} \). In this case our first guess of the self-financing condition is correct: we have \( d\Psi_t = \varphi_t \cdot dG_t \), which is the standard self-financing condition in the literature: see e.g. [Pr01] or [JYC09].

**Remark 2.7.** Under condition (4), even if the integrals in (3) are well posed, the trading strategy vector process \( \varphi \) generally looses the predictability feature in cases where either \( S \) or \( D \) are optional (the process \( \varphi_{t-} \) is predictable but \( \varphi \) is optional). This is problematic, since it seems reasonable that the trading strategy be predictable: the trading strategy represents the trader’s holdings at time \( t \), and this should be based on information obtained at times strictly before \( t \), but not \( t \) itself. In other words, the trader cannot be aware of all jumps, even the unpredictable ones.

In order to tackle this issue we could redefine the trading strategy with a LCRL\(^1\) vector process \( \theta \) so that process \( \theta \) is predictable. Then, as in [Du01, DH88], we have this new system for all \( t \geq 0 \):

\[
\begin{align*}
\theta_t &= \theta_{t-} \\
\Psi_t(\theta) := \theta_t \cdot S_t + \theta_t \cdot \Delta D_t \\
d\Psi_t(\theta) &= \theta_t \cdot dG_t
\end{align*}
\]

(6)

In practice one could set \( \theta_t \) in the following way:

\[
\Psi_{t-}(\theta) := \theta_{t-} \cdot (S_{t-} + \Delta D_{t-}) = \theta_t \cdot S_{t-} \stackrel{self-fin.}{=} \Psi_0(\theta) + \int_0^t \theta_u \cdot dG_u
\]

where the second equality is by Remark A.3 and left continuity of the trading strategy: therefore one should set freely the current value of vector \( \theta_t \) redistributing the whole portfolio value at time \( t- \) at the last equality (coherent with the self-financing condition and all past strategies) using the underlying market values \( S_{t-} \) at time \( t- \) (see third equality). Once the trading strategy \( \theta_t \) is set, jumps of the asset and/or of the dividend process may arrive as surprises and perturb the

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\(^1\)Note that a left-continuous trading strategy \( \theta \) could have (predictable) discontinuities of type \( \theta_{t-} - \theta_t \) and this is also coherent with discrete left-continuous rebalancing of the portfolio: e.g. at rebalancing times \( 0 = t_0 < t_1 \lt \ldots \) we could set \( \theta_t = \theta_1 \cdot 1_{t=0} + \sum_{i=1}^{\infty} \theta_i \cdot 1_{t_i-1 < t \leq t_i} \) for some predictable vector random variables \( \theta_i \).
portfolio value from $\Psi_{t-}$ to $\Psi_t$: see the second equation of (6).
Moreover,

$$
\Psi_{t-}(\theta) + \Delta \Psi_t(\theta) = (\theta_{t-} \cdot S_{t-} + \theta_t \cdot \Delta D_{t-}) + (\theta_t \cdot \Delta G_t)
= \theta_{t-} \cdot (S_{t-} + \Delta S_t + \Delta D_t)
= \theta_1 \cdot (S_1 + \Delta D_t) =: \Psi_t(\theta)
$$

where the first equality is by self-financing condition and (A.3), and the second equality is by Remark A.3 and left continuity of the trading strategy: the wealth is conserved even in cases of dividend jumps. In framework (6), one can think to process $\varphi$ as a computation tool, where we set for any $t \geq 0$

$$
\begin{cases}
\varphi_{t-} = \theta_t \\
\Delta \varphi_t \cdot S_t = \theta_t \cdot \Delta D_t
\end{cases}
$$

with initial condition $\varphi_0 = \theta_0$: one can easily check that this system corresponds to (2)-(3)-(4). In the remainder of the section we will have this last framework in mind ($\varphi$ is a computation tool and $\theta$ is the trading strategy) but, with a slight abuse of notation, we will often refer to process $\varphi$ as the trading strategy.

As we read in [Pr01], the we must avoid problems that arise from the classical doubling strategy. Here, a player bets $1$ at a fair bet. If he wins, he stops. If he loses, he next bets $2$. Whenever he wins, he stops, and his profit is $1$. If he continues to lose, he continues to play, each time doubling his bet. This strategy leads to a certain gain of $1$ without risk. However, the player needs to be able to tolerate arbitrarily large losses before he might gain his certain profit. Of course no one has such infinite resources to play such a game. Mathematically one can eliminate this type of problem by requiring trading strategies to give martingales that are bounded below by a constant. Thus the player’s resources, while they can be huge, are nevertheless finite and bounded by a non-random constant. This leads to the next definition.

**Definition 2.8.** A predictable self-financing strategy $\theta$ is said to be admissible on the time interval $[0,T]$ for $T > 0$ if $\theta_0 = 0$ and there exists a constant $a$ such that $\Psi_t(\theta) \geq -a$, a.s. for every $0 \leq t \leq T$.

**Definition 2.9.** An arbitrage opportunity on the time interval $[0,T]$ is an admissible self-financing strategy $\theta$ such that $\Psi_0(\theta) = 0$, $\Psi_T(\theta) \geq 0$ a.s. and $P(\Psi_T(\theta) > 0) > 0$.

Note that, since $P \sim Q$, the last condition is equivalent to $Q(\Psi_T(\theta) > 0) > 0$. A model is arbitrage free if there does not exist arbitrage opportunities in it. As anticipated in the previous section, the NFLVR condition is a slight/technical relaxation of the arbitrage free condition. We now want to re-state the portfolio and the self-financing conditions in a more convenient way.

**Proposition 2.10.** Let $\Psi_t(\varphi)$ be defined in (2). Defining also $\tilde{\Psi}_t := \Psi_t/B_t$, then the self-financing conditions (3)-(4) are equivalent to

$$
d\tilde{\Psi}_t = \varphi_{t-} \cdot d\tilde{G}_t
$$

where for any $i$, we define the deflated gain process

$$
\tilde{G}_t^i := \frac{S_t^i}{B_t} + \int_0^t \frac{AD_t^i}{B_u}. 
$$

We also recall that $S^0 \equiv B$ and therefore $\tilde{G}_t^0 = 1$ for all $t$. 

Proof. This a particular case of the more general Lemma 2.20: so we refer to the proof of this Lemma for the case $b = 0$ and then $\beta \equiv B$. Note that in this case some of the calculations are simplified from the fact that $B$ is continuous with finite variation. 

We start by using the bank account as numéraire. We concentrate on asset $S^i$ with $i \in \{1, \ldots, n\}$. Extending the analysis of [Bj09], our program is now as follows:

- Consider the buy-and-hold self-financing portfolio where we hold exactly one unit of the asset $S^i$, and invest all net dividends $D^i$ in the bank account. Denote the value process of this portfolio by $Y^i_t := \Psi_t(\varphi^{bh}(i)) = \varphi^0_t B_t + S^i_t$ with $\varphi^{bh}(i) := (\varphi^0_t, 0, 0, 1, 0, \ldots)$ where the unitary long position corresponds to the $i$-th asset recalling that $i > 0$. Hence, all the dividends dropped by the single asset are continuously rebalanced in the bank account position: at any $t$ in fact $Y^i_t = \varphi^0_t B_t + S^i_t$.
- The point is now that the portfolio $Y^i$ can be viewed as a non-dividend paying asset.
- Thus, the process $Y^i/B$ should be a local martingale under the Risk Neutral measure.

Also, we make a standing assumption that the random variable $X = \int_0^T B_u^{-1} dD^i_u$ is $Q$-integrable.

Lemma 2.11. Recalling the buy-and-hold portfolio definition $Y^i_t := \varphi^0_t B_t + S^i_t$, define also its deflated version $\tilde{Y}^i_t := \frac{Y^i_t}{\varphi_0^i}$. Then the portfolio $Y^i$ respects the self-financing conditions (3)-(4) if

$$\tilde{Y}^i_t = \varphi_0^i + \tilde{G}^i_t$$

(10)

where $\varphi_0^i$ is the initial cash endowment and

$$\varphi_0^i = \varphi_0^i + \int_0^t \frac{dD^i_u}{B_u}$$

(11)

Moreover, $Y^i_t = \theta^0_t B_t + S^i_t + \Delta D^i_t$ where

$$\theta_0^i = \theta_0^i + \int_0^t \frac{dD^i_u}{B_u}$$

Proof. This is a particular case of the more general Lemma 2.21: so we refer to the proof of this Lemma for the case $b = 0$ and then $\beta \equiv B$. 

Remark 2.12. Lemma 2.11 (and its generalization of Lemma 2.20) are measure independent and only due to the construction of a portfolio under the self-financing condition. The measure specification will be crucial instead for Theorem 2.13 (and its generalization in Theorem 2.23).

Theorem 2.13 (First FTAP with Dividends under the RN Measure). For any $i$, we have that $\tilde{G}^i$ is $Q$-local martingale.

Proof. $\tilde{Y}^i$ is a non-dividend paying asset deflated by the $Q$-measure numéraire. Hence, extending the market to this portfolio, due to Theorem 2.1, it must be a $Q$-local martingale. $\tilde{G}^i$ is then equivalent to $\tilde{Y}^i$ for $\varphi_0^i = 0$.

Remark 2.14. We adopted a notation that is quite standardized but could be misleading: for a dividend paying asset $S^i$ we underline that $\tilde{G}^i \neq \frac{G^i}{\varphi_0^i}$ and that neither $\frac{G^i}{\varphi_0^i}$ nor $\frac{G^i}{\varphi_0^i}$ are Risk-Neutral local martingales as they would in cases where the asset had a null dividend process.

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One could naively think that $G^i := S^i + D^i$ represented the position of being long on the asset and reinvesting its dividends: this is not correct since the right way to implement this strategy is described in Lemma 2.11. Moreover, as we will see in Proposition 2.16, under the Risk Neutral measure:

$$dG^i_t := dS^i_t + dD^i_t = (r_t S^i_{-}^t dt + dM^i_t - dD^i_t) + dD^i_t = r_t S^i_{-}^t dt + dM^i_t$$

$\varphi_0^i \neq r_t G^i_{-}^t dt + dM^i_t$
Corollary 2.15. \( \tilde{\Psi}_t \) is a local martingale under \( \mathcal{Q} \).

Proof. Recalling (8), then \( \tilde{\Psi}_t \) is a \( \mathcal{Q} \)-local martingale being a sum of \( \mathcal{Q} \)-local martingales \( \tilde{G}_i^t \) (by the previous theorem).

Proposition 2.16. Define \( M \) as \( \mathcal{Q} \)-local martingale vector process, and \( A \) as a finite variation vector process with \( A_0 = M_0 = \mathbf{0} \). Let \( D \) be a locally bounded semimartingale dividend vector process with \( D_0 = D_0 = \mathbf{0} \). Assume the following general dynamics under \( \mathcal{Q} \):

\[
d S_i^t = dA_i^t + dM_i^t - dD_i^t \quad (12)
\]

Observe that with this dynamics \( S_i^t \) is indeed a semimartingale (being the sum of semimartingales) with dividend dropping \( dD_i^t \) explained at Remark 2.17. Then this dynamics is arbitrage free (more precisely, it respects the NFLVR condition) if we set, for any \( i \),

\[
d A_i^t = r_t S_i^t \, dt \quad (13)
\]

so that \( \tilde{G}_i^t = \frac{dM_i^t}{B_t} \).

Proof. For any scalar semimartingale \( X \) we have from (A.6) and the fact that \( B \) is continuous with finite variation (recalling (A.9)):

\[
d(XB^{-1})_t = B_{t-}^{-1} dX_t + X_t^{-} d(B^{-1})_t + d[X, B^{-1}]_t
\]

\[
= B_{t-}^{-1} dX_t - X_t^{-} \frac{1}{B_t} dB_t
\]

\[
= \frac{dX_t}{B_t} - r_t \frac{X_t^{-}}{B_t} \, dt
\]

that can be used to obtain

\[
d \tilde{G}_i^t := d \left( \frac{S_i^t}{B_t} \right)_t + \frac{dD_i^t}{B_t} = \frac{dA_i^t + dM_i^t - dD_i^t - r_t S_i^t \, dt + dD_i^t}{B_t} = \frac{dA_i^t + dM_i^t - r_t S_i^t \, dt}{B_t}
\]

Now, \( \tilde{G}_i^t \) must be a \( \mathcal{Q} \)-local martingale by Theorem 2.13, hence, with \( M^i \) already being a local martingale, we have the thesis.

Remark 2.17. The dynamics (12) is coherent with the concept that, whenever the dividend is paid to the asset holder, the asset value experiences the opposite absolute shock: precisely

\[
d S_i^t = (\ldots) - dD_i^t
\]

and in particular, in case (\ldots) is continuous, we have \( \Delta S_i^t = -\Delta D_i^t \). However, one would expect that the drop is represented by the gross dividend and not by the net dividend: in fact one could argue that the company pays a part of the dividend to the shareholders and a part to the Government via taxes; the share value should drop by the total (i.e. gross) dividend amount. First of all, we should check if the above result is influenced by guessing (12): let us define the gross dividend process with \( \tilde{D}_i \) and assume

\[
d S_i^t = d\tilde{A}_i^t + dM_i^t - d\tilde{D}_u^t
\]

then we obtain, following the same passages of the proof here above,

\[
d \tilde{A}_i^t = r_t S_i^t \, dt + (d\tilde{D}_u^t - dD_i^t)
\]

and the last dynamics is the one that would guarantee that \( \frac{\tilde{G}_i^t}{t} \) were a \( \mathcal{Q} \)-local martingale.
which is the same result of previous proposition. In fact, the drop with the gross dividend is allowed in the physical measure \( \mathbb{P} \) but not in the Risk Neutral measure \( \mathbb{Q} \): the dividend taxes are (a negative) part of the physical drift that goes away in the passage to the Risk Neutral measure exactly as the expected return of the asset in the physical measure becomes the risk-free rate.

**Remark 2.18.** In [FPP19], the authors prove that there are no sure profits via flash strategies if and only if the deflated gain process components \( \tilde{G}^i \) do not have predictable (resp. fully predictable) jumps, i.e. if there does not exist any predictable time at which the direction (and even the size, in the case of fully predictable jumps) of the jump is known just before the occurrence of the jump. Recalling the last result of Proposition 2.16, this is equivalent to assuming the absence of predictable jumps in all process \( M \) components.

The above results can be extended to any numéraire. From [Pa11] (where the interested reader can refer for a presentation of the numéraire topic) we know that in order to be a numéraire a process should fit the following characteristics.

**Definition 2.19 (From [Pa11]).** The stochastic process \( X \) is a “\( \mathbb{Q} \)-price process”, if

1. \( X_t > 0 \) for all \( t \);
2. \( X_t/B_t \) is a \( \mathbb{Q} \)-strict martingale.

Let \( \beta \equiv S^b \) for \( b \in \{0, 1, \ldots, n\} \) be a non-dividend paying (so \( D^b = 0 \)) semimartingale scalar process with \( \beta_t > 0 \) a.s. for all \( t \). Then \( \beta \) is a \( \mathbb{Q} \)-local martingale due to Theorem 2.13: we assume that \( \beta \) is not only a local martingale but a martingale, so that \( \beta \) is a “\( \mathbb{Q} \)-price process”. Then \( \beta \) is eligible to be a numéraire, i.e. we can define a Radon-Nikodym derivative

\[
L_t := E_t \left[ \frac{d\mathbb{Q}^\beta}{d\mathbb{Q}_t} \right] = \frac{\beta_t B_0}{B_t \beta_0}
\]

where \( L \) is a \( \mathbb{Q} \)-martingale with \( E[L_t] = L_0 = 1 \) for any \( t \geq 0 \). Clearly under the degenerate case in which \( b = 0 \) this is not a change of measure.

**Proposition 2.20.** Define \( \tilde{\Psi}^\beta_t := \Psi_t/\beta_t \) where \( \Psi_t \) is defined in (2). The self-financing conditions (3)-(4) can also be written as:

\[
d\tilde{\Psi}^\beta_t = \varphi_t \cdot d\tilde{G}^\beta_t
\]

where \( \tilde{G}^\beta_t := (\tilde{G}^\beta,0_t, \tilde{G}^\beta,1_t, \ldots, \tilde{G}^\beta,n_t) \) and we define

\[
\tilde{G}^\beta,i_t := \frac{S_t^i}{\beta_t} + \int_0^t \left\{ \frac{dD^i_u}{\beta_u} + d \left[ D^i, \frac{1}{\beta_u} \right]_u \right\}
\]

Clearly \( \tilde{G}^\beta,b_t = 1 \) for all \( t \).

**Proof.** Note that \( 1/\beta \) is strictly positive since \( \beta \) is strictly positive by hypothesis. Define \( \tilde{S}^i_t := S^i/\beta \) for any \( i \), then

\[
d\tilde{S}^i_t = S_t^i - d(\beta^{-1})_t + \beta_t^{-1} dS^i_t + d[S^i, \beta^{-1}]_t
\]
and so, using self-financing condition (3),
\[
\begin{align*}
\Delta \overline{\Psi}_t^\beta &= \Psi_t \cdot d(\beta^{-1})_t + \beta_t^{-1} \cdot d\Psi_t + [d\Psi_t, d(\beta^{-1})_t] \\
&= (\varphi_t \cdot S_t) \cdot d(\beta^{-1})_t + \beta_t^{-1} \cdot \varphi_t \cdot dG_t + \sum_{i=0}^n \varphi_i^t \cdot [dG_i^t, d(\beta^{-1})_t] \\
&= \varphi_t^b \left\{ \beta_t \cdot d(\beta^{-1})_t + \beta_t^{-1} \cdot d\beta_t + [d\beta_t, d(\beta^{-1})_t] \right\} \\
&\quad + \sum_{i=0, i \neq b}^n \varphi_i^t \left\{ S_i^t \cdot d(\beta^{-1})_t + \beta_t^{-1} \cdot d(S^i + D^i)_t + [d(S^i + D^i)_t, d(\beta^{-1})_t] \right\} \\
&= \sum_{i=0, i \neq b}^n \varphi_i^t \left\{ d\tilde{S}_i^t + \beta_t^{-1} \cdot dD_i^t + [dD_i^t, \beta_t^{-1}]_t \right\} \\
&= \sum_{i=0, i \neq b}^n \varphi_i^t \cdot d\tilde{G}_t^{\beta,i}
\end{align*}
\]
where at the forth equality we exploited the fact that in the first curly bracket we have 
\(d(\beta^{-1})_t = 0\) and therefore we obtain (15) – recalling that \(d\tilde{G}_t^{\beta,b} = 0\) for all \(t\). Now we check ex-post that condition (4) holds; from the self-financing condition (15), using (A.1)-(A.3)-(A.8),
\[
\begin{align*}
\Delta \overline{\Psi}_t^\beta &= \varphi_t \cdot \Delta \tilde{G}_t^{\beta,i} := \sum_{i} \varphi_i^t \cdot \Delta \left( S_i^t \cdot \beta_t^{-1} + \int_0^t \left\{ \beta_u^{-1} \cdot dD_u^i + [dD_u^i, \beta_u^{-1}]_u \right\} \right) \\
&= \sum_{i} \varphi_i^t \left( S_i^t \cdot \Delta(\beta_t^{-1})_t + \beta_t^{-1} \cdot \Delta S_i^t + \Delta S_i^t \cdot \Delta(\beta_t^{-1})_t + \beta_t^{-1} \cdot \Delta D_i^t + \Delta D_i^t \cdot \Delta(\beta_t^{-1})_t \right) \\
&= (\varphi_t \cdot S_t) \cdot \Delta(\beta_t^{-1})_t + \beta_t^{-1} \cdot \varphi_t \cdot (\Delta S_t + \Delta D_t)
\end{align*}
\]
on the other hand, using (A.1)-(A.2),
\[
\begin{align*}
\Delta \overline{\Psi}_t^\beta := \Delta \left( \frac{\Psi_t}{\beta_t} \right) := \Delta (\beta_t^{-1} \cdot (\varphi_t \cdot S_t)) &= \beta_t^{-1} \cdot \Delta(\varphi_t \cdot S_t) + (\varphi_t \cdot S_t) \cdot \Delta(\beta_t^{-1}) + \Delta(\varphi_t \cdot S_t) \cdot \Delta(\beta_t^{-1}) \\
&= \beta_t^{-1} \cdot (\varphi_t \cdot \Delta S_t + S_t \cdot \Delta \varphi_t + \Delta S_t \cdot \Delta \varphi_t) + (\varphi_t \cdot S_t) \cdot \Delta(\beta_t^{-1}) \\
&= \beta_t^{-1} \cdot (\varphi_t \cdot \Delta S_t + S_t \cdot \Delta \varphi_t) + (\varphi_t \cdot S_t) \cdot \Delta(\beta_t^{-1})
\end{align*}
\]
equating the last two equations we can observe that condition (4) is respected. \(\square\)

**Lemma 2.21.** Define the buy-and-hold trading strategy \(\mathcal{V}_t^i := \Psi_t(\varphi^{bi}(i)) = \varphi_t^b \beta_t^i + S_t^i\) with \(\varphi^{bi}(i) := (0, 0, 1, \ldots, \varphi_t^i, 0, \ldots)\) where the unitary long position is for the \(i\)-th asset and \(i \neq b\). The discounted wealth \(\tilde{\mathcal{V}}_t^i := \frac{\mathcal{V}_t^i}{\beta_t^i}\) is self-financing if it satisfies
\[
\tilde{\mathcal{V}}_t^i = \varphi_t^b + \tilde{G}_t^{\beta,i}
\]
where
\[
\varphi_t^b = \varphi_0^b + \int_0^t \left\{ \frac{dD_u^i}{\beta_u^{-1}} + [D_u^i, \beta_u^{-1}]_u \right\}.
\]
Moreover, \(\mathcal{V}_t^i = \theta_t^b \beta_t^i + S_t^i + \Delta D_t^i\) where
\[
\theta_t^b = \theta_0^b + \int_0^t \left\{ \frac{dD_u^i}{\beta_u^{-1}} + [D_u^i, \beta_u^{-1}]_u \right\}.
\]
Proof. We proved in the previous proposition the self-financing condition (15) for any self-financing trading strategy \( \varphi \). In particular,

\[
dY_t^i = d\tilde{\varphi}_u^\beta (\varphi^{\beta h(i)}) = d\tilde{G}_t^\beta,i = \rho \left( \frac{S_t^i}{\beta} \right)_u + \frac{dD_t^i}{\beta u} + d \left[ D_t^i, \frac{1}{\beta} \right]_u
\]

recalling that since \( \tilde{G}_t^\beta,b = 1 \), then \( d\tilde{G}_t^\beta,b = 0 \). Integrating in \((0,t]\) on both sides we obtain

\[
\tilde{Y}_t^i = \tilde{Y}_0^i + \tilde{G}_t^\beta,i - \tilde{G}_0^\beta,i = \tilde{G}_t^\beta,i + \tilde{\varphi}_0^b + \frac{S_t^i}{\beta} = \tilde{G}_t^\beta,i + \varphi_0^b
\]

which is the thesis. On the other hand,

\[
d\tilde{Y}_t^i := d \left( \frac{\varphi^b + S_t^i}{\beta} \right)_t = d\varphi_t^b + d \left( \frac{S_t^i}{\beta} \right)_t
\]

Comparing this with the self-financing condition (18) gives

\[
d\varphi_t^b = \frac{dD_t^i}{\beta u} + d \left[ D_t^i, \frac{1}{\beta} \right]_t
\]

so by integrating \( \varphi_t^b = \varphi_0^b + \int_0^t d\varphi_u^b \) we obtain (17). The last statement is, recalling (7), since \( \theta_t^{\beta h(i)} = \varphi_t^{\beta h(i)} \). Also note that, using (A.3)-(A.8),

\[
\Delta \varphi_t^{\beta h(i)} \cdot S_t = \Delta \varphi_t^b \beta_t = \left\{ \frac{\Delta D_t^i}{\beta_t} + \Delta D_t^i \left( \frac{1}{\beta_t} - \frac{1}{\beta_{t-}} \right) \right\} \beta_t
\]

\[
= \beta_t \Delta D_t^i + \Delta D_t^i \left( \frac{\beta_t - \beta_{t-}}{\beta_t \beta_{t-}} \right) = \Delta D_t^i
\]

According to (7), the above quantity must be equal to \( \theta_t^{\beta h(i)} \cdot \Delta D_t = \Delta D_t^i \) which is confirmed (recalling that \( \beta \) does not pay dividends by construction).

\[\square\]

**Corollary 2.22.** In case \( D_t \) has finite variation, we have

\[
\tilde{G}_t^\beta,i = \frac{S_t^i}{\beta_t} + \int_0^t \frac{dD_u^i}{\beta_u} + \sum_{0 < u < t} \Delta D_u^i \Delta (\beta^{-1})_u.
\]

If, in addition, either \( D \) or \( \beta \) are continuous

\[
\tilde{G}_t^\beta,i = \frac{S_t^i}{\beta_t} + \int_0^t \frac{dD_u^i}{\beta_u}
\]

which is exactly the same formulation as that used for the Risk Neutral measure with a different deflator \( \beta \).

**Proof.** Straightforward application of (A.9).

\[\square\]

We can now extend the result of [Bj09] pp. 244-245 – which only tackles the continuous processes case – to the general semimartingale case.

**Theorem 2.23 (First FTAP with Dividends under a General EMM).** Denote by \( Q^\beta \) the measure with numéraire \( \beta \). Then, the deflated gain process \( \tilde{G}_t^\beta,i \), defined in (16), is a \( Q^\beta \) local martingale.

**Proof.** The portfolio \( \tilde{Y}_t^\beta \) of the non-dividend-paying-self-financing trading strategy \( \varphi^{\beta h(i)} \) must be a \( Q^\beta \)-local martingale due to the previous considerations. By setting \( \varphi_0^b = 0 \), one obtains the thesis. 

\[\square\]

**Corollary 2.24.** \( \tilde{Y}_t^\beta \) is a local martingale under \( Q^\beta \).
Proof. We proved (15). Therefore, $\tilde{\Psi}^i_t$ is a $\mathbb{Q}^i$-local martingale being a sum of $\mathbb{Q}^i$-local martingales $\tilde{G}^{\beta,i}_t$ (by the previous theorem).

Corollary 2.25. Under the Risk Neutral measure $\mathbb{Q}$, under the hypothesis that $\tilde{G}^i_t$ is a strict martingale (not only local), one obtains:

$$ S^i_t = B^i_t \mathbb{E}^i \left[ \frac{S^i_T}{B^i_T} + \int_t^T \frac{dD^i_u}{B^i_u} \right] $$  \hspace{1cm} (19) $$

which says that today’s stock price is equal to the expected value of all future discounted earnings arising from holding the stock: the sum of the discounted final value of the asset and of the discounted future flow of dividends. More generally,

$$ S^i_t = \beta_t \mathbb{E}^i \left[ \frac{S^i_T}{B^i_T} + \int_t^T \left\{ \frac{dD^i_u}{\beta_u} + d \left[ D^i_1 \frac{1}{\beta} \right]_u \right\} \right] $$  \hspace{1cm} (20) $$

Proof. All results are straightforwardly derived from the definitions of $\tilde{G}^i$, $\tilde{G}^{\beta,i}$ and from the fact that $\tilde{G}^i = \mathbb{E}^i[\tilde{G}^i_T]$ and $\tilde{G}^{\beta,i} = \mathbb{E}^i[\tilde{G}^{\beta,i}_T]$.

Proposition 2.26. Let us assume (12)-(13) with

$$ dD^i_u = q^i S^i_u dt + d\Phi^i_u $$

where $q^i$ is the stochastic proportional (to the asset value) dividend rate. Moreover, $\Phi^i$ is any residual (with respect to the total dividend $D$) locally bounded semimartingale (but very often a pure jump process representing the lump dividends) with $\Phi_0 = \Phi_{-} = 0$. Defining $\mu := r - q$, we have that (not only $\tilde{G}^i$ but also)

$$ \tilde{X}^i_t := S^i_t + \int_0^t \frac{d\Phi^i_u}{B^i_u} $$

is a $\mathbb{Q}$-local martingale and, in case it is a strict martingale, for any $T > t$

$$ S^i_t = B^i_t \mathbb{E}^i \left[ \frac{S^i_T}{B^i_T} + \int_t^T \frac{d\Phi^i_u}{B^i_u} \right] $$  \hspace{1cm} (21) $$

Proof. Defining $\tilde{S}^i := S^i(B^i)^{-1}$ and using (14)-(12)-(13), we have

$$ d\tilde{S}^i_u = \frac{dS^i_u}{B^i_u} - \mu_u \frac{S^i_u}{B^i_u} du = \frac{1}{B^i_u} \left( (r_u - q^i_u)S^i_{-} dt + dM^i_u - d\Phi^i_u - \mu_u S^i_u du \right) $$

Integrating in $(0,t)$ on both sides

$$ \tilde{X}^i_t := \tilde{S}^i_0 + \int_0^t \frac{1}{B^i_u} d\Phi^i_u = \tilde{S}^i_0 + \int_0^t \frac{1}{B^i_u} dM^i_u = \tilde{X}^i_0 + \int_0^t \frac{1}{B^i_u} dM^i_u $$

where the last equality is since $\Phi^i_0 = \Phi^i_{-} = 0$ by hypothesis. Moreover, taking the conditional expectation we obtain

$$ \mathbb{E}^i[\tilde{X}^i_T] := \tilde{X}^i_0 + \mathbb{E}^i \left[ \int_0^T \frac{1}{B^i_u} dM^i_u \right] = \tilde{X}^i_0 + \int_0^T \frac{1}{B^i_u} dM^i_u = \tilde{X}^i_T $$
where we exploited Property 10 of Proposition A.11. Hence, $\tilde{X}^i$ is a $Q$-local martingale. Substituting on both sides the definition of $\tilde{X}^i$ we obtain the second result. Alternatively, in case the model is Markovian and all processes are continuous (so that all partial derivatives are well defined) and driven by a Brownian vector, this result can be derived applying the Feynman-Kac theorem to expected value (19) to obtain the corresponding PDE. In this PDE perform a change of discounting rate by a change of the continuous payoff and then apply again the Feynman-Kac theorem to this modified PDE to obtain the expected value (21).

Remark 2.27. By (19) and the above proposition

$$S^i_t = B_t E_t \left[ \frac{S^i_T}{B_T} + \int_t^T \frac{d\Phi^i_u}{B_u} + \int_t^T q^i_u S^i_u \frac{d}{B_u} \right] = B_t^\mu E_t \left[ \frac{S^i_T}{B^\mu_T} + \int_t^T \frac{d\Phi^i_u}{B^\mu_u} \right]$$

which is a useful trick for asset pricing (especially when interpreting $S$ as a financial derivative, see Section 2.3): it transforms a continuous dividend cash flow into a change of discounting rate.

Remark 2.28. It is important to underline here that $B^\mu$ (as any other domestic bank account different from $B$) must be only a mathematical tool and not a proper tradable asset, since its dynamics is written in any measure as $dB^\mu_t = \mu_t B^\mu_t dt$ and therefore, by absence of arbitrage, if it were a tradable asset it would drift with the risk-free interest rate $r$.

In particular, there is no way to change the measure to a measure, say $Q^\mu$, with numéraire $B^\mu$ since

$$\frac{dQ^\mu}{dQ} \bigg|_{F_t} = \frac{B^\mu_t B_0}{B_t B_0}$$

has no martingale part, therefore it could be a $Q$-martingale (and hence a “$Q$-price process”) only if it were a constant, i.e. in case $r \equiv \mu$ so that $Q \equiv Q^\mu$.

2.2.2. Forward Price

In this section, in order to simplify the notation, we perform a small abuse of notation and we no longer interpret $S$ as the vector process of all underlyings but as a generic component $S^i$ of this vector. We specify that the interest rates in this section are considered as stochastic.

Definition 2.29. The Forward price $F_t(T)$ is the par rate of a forward contract, i.e. for $t \leq T$ the strike $K$ such that

$$0 = E_t \left[ \frac{B_t}{B_T} (S_T - K) \right]$$

Proposition 2.30. The Forward price can also be written as $F_t(T) = E_t^T [S_T]$.

Proof. A straightforward change of measure,

$$E_t \left[ \frac{B_t}{B_T} (S_T - K) \right] = E_t^T \left[ \frac{P_t(T)}{P_T(T)} (S_T - K) \right] = P_t(T) \left\{ E_t^T [S_T] - K \right\}$$

from which we have the thesis.

Proposition 2.31. The forward price with stochastic dividends and interest rates writes

$$F_t(T) = \frac{1}{P_t(T)} \left\{ S_t - B_t E_t \left[ \int_t^T \frac{dD^i_u}{B_u} \right] \right\}$$

where one should note that inside the curly brackets we have only $F_t$-measurable quantities (the asset value minus the expected value of the discounted future dividends, not earned by the long
forward contract holder) that are capitalized (through the $T$-zero coupon bond) to the date $T$ where the forward contract transaction takes place.

**Proof.** Under the usual change of numéraire techniques, defining

$$L_t = \left. \frac{dQ^T}{dQ} \right|_{F_t} = \frac{P_t(T)}{B_t} \frac{B_0}{P_0(T)}$$

we have

$$F_t(T) = \mathbb{E}_t^T[S_T] = \mathbb{E}_t^T \left[ \frac{L_T S_T}{L_t} \right] = \frac{B_t}{P_t(T)} \mathbb{E}_t \left[ \frac{S_T}{B_T} \right]$$

$$= \frac{B_t}{P_t(T)} \mathbb{E}_t \left[ \tilde{G}_T - \int_0^T \frac{dD_u}{B_u} \right]$$

$$= \frac{B_t}{P_t(T)} \left\{ \tilde{G}_t - \mathbb{E}_t \left[ \int_0^T \frac{dD_u}{B_u} \right] \right\}$$

$$= \frac{B_t}{P_t(T)} \left\{ \frac{S_t}{B_t} - \mathbb{E}_t \left[ \int_t^T \frac{dD_u}{B_u} \right] \right\}$$

which is the thesis. 

The above formula can also be justified by replication arguments. We recall the par long forward contract transactions:

- **At $t$:**
  - Both parties agree to enter into the contract and set the par strike $K$, no cash transaction takes place (since $K$ is the par strike);
- **In the interval $(t, T)$:**
  - No transaction;
- **At $T$:**
  - The long forward contract holder obtains the asset (of value $S_T$) and pays $K$.

A static replication strategy can be built as follows:

- **At $t$:**
  - One buys at $t$ the asset for a value of $S_t$ in order to deliver it at maturity $T$: hence one holds the asset but has an outflow of cash of $S_t$;
  - One sells in the market to a third counterparty (say $MD$) the future dividends of the asset in the interval $(t, T]$ via a dividend swap\(^3\), obtaining as cash the expected value of discounted net dividends, i.e.

  $$V_t^D := B_t \mathbb{E}_t \left[ \int_t^T \frac{dD_u}{B_u} \right]$$

  Note that selling the dividend swap is necessary since, in general, future dividends are stochastic. In case the dividends were deterministic one can simplify the replication strategy,

  $$V_t^D = \int_t^T P(t, u) dD_u$$

\(^3\)In general, in dividend swaps, a leg of future dividends of an asset is exchanged with a fixed rate or floating rate leg. In any case one can immediately sell this last leg and obtain $V^D_t$ of cash.
and hence one can sell at inception, instead of a dividend swap, a strip of \( u \)-Zero Coupon bonds, with \( u \in (t, T] \), and notional \( dD_u \) obtaining at inception an amount \( V_t^D \) of cash (at any expiry \( u \) the money obtained by the dividend \( dD_u \) is given as notional redemption to the \( u \)-Zero Coupon bond counterparty): see in particular formula (B.1) for the deterministic lump dividend case.

- Since the forward contract has a zero cash transaction at \( t \), one has to finance the above cash debt, this is implemented with selling a notional \( N_t \) of a \( T \)-zero-coupon bond for which

\[
S_t - V_t^D = N_t \times P_t(T)
\]

where on the left-hand side we have the cash amount needed and on the right-hand side the asset value of the zero coupon bond (hence the cash obtained at \( t \) from the short position on the bond).

- In the interval \((t, T]\):
  - For any \( u \) in the interval, one receives the (net stochastic) dividend \( dD_u \) from the asset and gives it to the counterparty \( M^D \) (hence all cash transactions offset);
  - At \( T \):
    - One still holds the asset (of value \( S_T \));
    - One has to deliver the notional \( N_t \) to the \( T \)-Zero Coupon buyer.

Since we have built a replication strategy, by arbitrage \( K = N_t \), hence the forward par \( F_t(T) := K = N_t \) and the zero coupon notional coincides with the right-hand side of formula (22).

**Proposition 2.32.** Fix the same setting/notation as in Proposition 2.26 and add the hypothesis that the dividend rate \( q \) is deterministic. We have

\[
F_t(T) = \frac{S_t - \int_t^T P_t^T(u) E^T_t[d\Phi_u]}{P_t^T(T)}
\]

where for any \( u \geq t \) we define \( P_t^u(u) := E_t[B_t^u/B^u_T] = P_t(u) e^{\int_t^u q \, ds} \).

**Proof.** From (21),

\[
(S B^q)_t = B_t \mathbb{E}_t \left[ \frac{(S B^q)_T}{B_T^T} + \int_t^T B^q_u d\Phi_u \frac{B_T}{B^u_u} \right] = P_t(T) B_T^q F_t(T) + \int_t^T B^q_u P_t(u) \mathbb{E}_t^T[d\Phi_u]
\]

and the result is obtained with some algebra. \( \square \)

To be more concrete, the interested reader can refer to Appendix B to explore two basic deterministic dividend models: the continuous proportional dividend and the lump dividend case.

### 2.3. Asset Pricing

Define \( V \) as the price process of a financial derivative with expiry \( T \), written on underlyings \( S^1, \ldots, S^n \) where we denote with \( \Pi \) the cumulative “dividend” (read intermediate cash flows) process of the financial derivative and with \( \phi_T = \phi(S^1_T, \ldots, S^n_T) \) the final payoff at expiry \( T \), where \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \). The financial derivative, being itself a tradable asset, should be priced in a
way for which the extended market $B, S^1, \ldots, S^n, V$ is consistent with the First FTAP, i.e. such that

$$\tilde{G}^V_t := \frac{V_t}{B_t} + \int_0^t \frac{d\Pi_u}{B_u}$$

(24)
is a $Q$-local martingale, where $\tilde{G}^V$ represents the deflated gain process of the financial derivative.

In case $\tilde{G}^V$ is a strict martingale, one can proceed as in (19) to obtain

$$V_t = B_t \mathbb{E}_t \left[ \frac{V_T}{B_T} + \int_t^T \frac{d\Pi_u}{B_u} \right]$$

(25)

This is the main result one can use to price a financial derivative. Without loss of generality, one could set $\phi_T = 0$ and inflate the dividend process $\Pi$ with the expiration cash flow, obtaining a simpler formula

$$V_t = B_t \mathbb{E}_t \left[ \int_t^T \frac{d\Pi_u}{B_u} \right]$$

Specifically, as an example, one could write the dividend process of a strip of vanilla Call options at deterministic fixing times $0 < T_1 < T_2 < \ldots$ and payment time lag $\delta > 0$:

$$d\Pi_u = \sum_j (S_{T_j} - K_j)^+ \ d\Theta_{T_j+\delta}(u)$$

and observe that this process has finite variation (it depends on time only through the Heaviside functions). In particular, we have from Stieltjes integral properties

$$\Pi_t = \int_0^t d\Pi_u = \sum_j (S_{T_j} - K_j)^+ \int_0^t d\Theta_{T_j+\delta}(u) = \sum_j (S_{T_j} - K_j)^+ \sum_{0 < u \leq t} \Delta \Theta_{T_j+\delta}(u)$$

from which it is clear that even if the process $S$ is optional, $\Pi$ is predictable for $\delta > 0$. Moreover,

$$B_t \int_t^T \frac{d\Pi_u}{B_u} = B_t \sum_j (S_{T_j} - K_j)^+ \int_t^T \frac{d\Theta_{T_j+\delta}(u)}{B_u} = B_t \sum_j (S_{T_j} - K_j)^+ \sum_{t < u \leq T} \frac{\Delta \Theta_{T_j+\delta}(u)}{B_u}$$

$$= B_t \sum_{t < T_j < t + \delta} \frac{(S_{T_j} - K_j)^+}{B_{T_j+\delta}}$$

More generally, under measure $Q^\beta$ with numéraire $\beta$, using (20),

$$V_t = \beta_t \mathbb{E}_t^\beta \left[ \frac{\phi_T}{\beta_T} + \int_t^T \left\{ \frac{d\Pi_u}{\beta_u} + d \left[ \Pi, \frac{1}{\beta} \right] \right\} \right]$$

(26)

3. Collateralized Derivatives

3.1. Introduction to Collateralization Modeling

Here we detail the approach of [MP17] for collateralization modeling in the easiest case where all cash flows are in domestic currency. We will remove this hypothesis and be more technical in the next section. We deal with a financial derivative with price $V_u$ at time $u$. 
In order to mitigate the counterparty risk, from contract inception to the time in which the contract is closed, the deal is collateralized (recall Section 1.1 for a brief non-technical presentation of this topic). For this purpose, the most common agreement (we will see other examples in the next section) is a bilateral agreement documented by the International Swaps and Derivatives Association (ISDA), known as Credit Support Annex (CSA). In particular, the agreement also regulates the possibility of re-hypothecation of the collateral assets, namely to use them for funding purposes, as opposed to segregation.

We define with $C_u$ the collateral value at time $u$, with the convention (from the investor point of view) that $C_u > 0$ means that the collateral is held by the investor, otherwise the collateral is held by the counterparty. $C_u > 0$ means that the financial derivative has a positive value for the investor and for this reason the investor holds (with or without re-hypothecation) some cash (or liquid securities) as a loan from the counterparty: in case of default of the counterparty this loan will mitigate the possible losses of the investor on the financial derivative position. In the opposite case, $C_u < 0$ means that the financial derivative has a negative value for the investor, and for this reason the counterparty borrows some collateral from the investor. In the same spirit, $dC_u > 0$ means that the counterparty posts collateral at time $u$, while $dC_u < 0$ means that the investor posts collateral.

We define also the target collateral value with:

$$C_u^* := (1 + \alpha_u)V_u^- \quad (27)$$

where $\alpha_u \geq 0$ is the (predictable) haircut or proportional margin process: to mitigate the risk of loss, borrowers are required to post up collateral in excess of the market value of the primary transaction $V$. The left limit in this formula is to ensure the predictability of the collateral process (the collateral posting party cannot be aware of unpredictable jumps of the financial derivative).

The over-collateralization (i.e. the fact that the collateral value is greater than the financial derivative price) is meant to avoid losses in case of default of one counterparty in the period between two collateral posting times, losses due to changes in market value of $V$ itself and/or of any collateral security different than domestic cash. In particular, a defaulted counterparty stops posting collateral: we denote with $\tau$ the first-to-default time among all counterparties and with $\tau + \delta$ the time of the bankruptcy procedure closeout cash flow payments (generally $\delta$ is about two weeks) and with $t_N$ the last margin call date (see below for details), where $t_N \leq \tau \leq \tau + \delta$. Following [CBB20], one calls the margin period of risk the time lag between $t_N$ and $\tau + \delta$: the cure period constitutes the second part of the margin period of risk (i.e. the time between $\tau$ and $\tau + \delta$), the first part being the time lag between $t_N$ and $\tau$. Moreover, as anticipated above, when securities are used as collateral their market value can fall in the margin period of risk and therefore they are generally devalued to have a cushion for this phenomenon: in this section we will deal only with cash collateralization, see Section 4.1 for a concrete example of stock collateralization.

We have the following specifications:

- **Margin Calls**: at any time $u$ the collateral value $C_u$ can move away from its target value $C_u^*$ because of financial derivative and/or collateral market moves: then with a contractual frequency (generally every business day of contract life) the counterparties check that the absolute value of the difference between the collateral value and its target value is inside of an agreed threshold or percentage: this is in order to reduce the administrative burden. When this limit is broken, the counterparty who is exposed to this breach performs a margin call, meaning that $u = t_i$ for some $i \in \{1, 2, \ldots, N\}$ where we define with $t_0 < t_1 < \ldots < t_N$ the margin (call) times. At $t_i$ the collateral value has a movement $dC_t$ that makes $C_{t_i} = C_{t_i}^*$. By construction, the contract inception date coincides with the first margin date $t_0$. We refer to Section 3.3 of [CBB20] for more precise details on collateralization schemes.
• **Variation Margin (20)**: it is represented by the first addend of (27) meaning that the target value of this margin account is such that \( \mathcal{M}^{\ast} = V_{u-} \) which is the “fair” collateral value (since the collateral is covering the financial derivative counterparty risk). Very often the counterparties agree to use the Variation Margin subject to re-hypothecation.

• **Initial Margin (3) versus Haircut**: it is the second addend of (27) representing a buffer of over-collateralization for all the above mentioned reasons. The over-collateralization target value can be rough and represented by a second addend \( \alpha_{t_0} V_{u-} \) where \( \alpha_{t_0} \) is a fixed percentage decided by the counterparties at contract inception: this is a haircut in a strict sense (see e.g. the Repurchase Agreement contract case at Section 4.1). In other cases, the second addend of (27) represents another (very often segregated, i.e. not subject to re-hypothecation) account called Initial Margin (also known as the Independent Amount) and \( \mathcal{I}^{\ast}_u = \alpha_u V_{u-} \) represents a quite complicated calculation, e.g. in case of ISDA SIMM methodology\(^4\). We do not enter in such technical details and we will consider both the Variation Margin and the Initial Margin (and this is a major simplification) as subject to re-hypothecation (see in particular Remark 3.3).

Let us see a simple example to understand the collateral mechanics: imagine the investor buys a Call option (of price process \( V \)) at \( t_0 \) from the counterparty. The investor pays the counterparty \( V_{t_0} \) of cash and writes in her book the long position on the Call for the same value. At the same time, the counterparty lends an amount \( C_{t_0} \) to the investor to mitigate his counterparty risk. Thus, at \( t_0 \), the investor books the Call option, receives \( C_{t_0} - V_{t_0} \) of cash from the counterparty and writes a debt of \( C_{t_0} \) for the same reason. At \( t_1 \) the Call value is \( V_{t_1} \) hence the investor/counterparty posts the required collateral in order to set \( C_{t_1} = C^*_{t_1} \), moreover, the investor must pay some interests on the debt, linked to the interest rate \( c \). Consequently, the cash to post at \( t_1 \) (if positive by the counterparty, if negative by the investor) is

\[
C_{t_1} - C_{t_0}(1 + c_{t_0}(t_1 - t_0))
\]

At \( t_2 \) the Call value is \( V_{t_2} \) and the amount to post to have \( C_{t_2} = C^*_{t_2} \) is

\[
C_{t_2} - C_{t_1}(1 + c_{t_1}(t_2 - t_1))
\]

Imagine at \( t_2 \) the investor closes the Call position selling back the Call option to the counterparty (in case \( t_2 \) is the expiry of the Call option the option value is \( V_{t_2} = (S_{t_2} - K)^+ \) meaning that the investor obtains the payoff of the Call): the investor erases the Call from her portfolio, she should receive \( V_{t_2} \) of cash from the counterparty but, at the same time, she should reimburse the debt with the counterparty of \( C_{t_2} \). The net cash flow to the investor when closing the financial derivative and the collateral position is thus \( V_{t_2} - C_{t_2} \) of cash (equal to zero in case \( \alpha = 0 \)). For \( i > 0 \),

\[
C_{t_i} - C_{t_{i-1}}(1 + c_{t_{i-1}}(t_i - t_{i-1})) = C_{t_i} - C_{t_{i-1}} - c_{t_{i-1}} C_{t_{i-1}}(t_i - t_{i-1}) \quad \rightarrow \quad dc_u - c_u C_{u-} du.
\]

The implications of the above arrow are discussed in the next remark.

**Remark 3.1.** More precisely, as we will see in the next section, in this paper there is no “continuous time approximation” (i.e. a limit for \( \delta \to 0 \)) on collateral posting: this is an advantage

\(^4\)As we read in Risk.net, the Standard Initial Margin Model (SIMM) is a common methodology to help market participants calculate initial margin on non-cleared derivatives under the framework developed by the Basel Committee on Banking Supervision and the International Organization of Securities Commissions.

The SIMM methodology was developed by ISDA, and is intended to reduce the potential for disputes and create efficiency through netting of exposures. The model applies a sensitivity-based calculation across four product groups: interest rates and foreign exchange, credit, equity and commodities.

The SIMM was officially launched in September 2016 and an updated version, ISDA SIMM 2.0, became effective in December 2017 to include a range of clarifications, enhancements and additional risk factors. We refer to the ISDA website and official documents therein for further details.
with respect to the setting of [MP17]. In fact, in following sections C will be any predictable semimartingale: this is coherent with a discontinuous LCRL process or also with a pure jump RCLL process $C_t = \sum_{0 < u \leq t} \Delta C_u$ with predictable jumps. In the latter case, $dC_u = \Delta C_u$ and we also set $\Delta C_u = 0$ except at collateral posting times $\Delta C_{t_i} = C_{t_i} - C_{t_i-}$ for $i > 0$ where the $t_1, t_2, \ldots$ are the (stochastic) jump times of process $C$. In this case, setting also the collateral interest rate process $c$ as a (stochastic) step process with steps at times $t_i$, we obtain

$$\int_0^t \left\{ dC_u - c u^- d\Pi_u \right\} = \sum_i \left\{ \int_{(t_{i-1}, t_i]} dC_u - c u^- d\Pi_u \right\}$$

$$= \sum_i \left\{ \sum_{t_{i-1} < u \leq t_i} \Delta C_u - c_{t_{i-1}} C_{t_{i-1}} \int_{t_{i-1}}^{t_i} d\Pi_u \right\}$$

$$= \sum_i C_{t_i} - C_{t_{i-1}} \left( 1 + c_{t_{i-1}} (t_i - t_{i-1}) \right)$$

so we have perfect discrete collateral posting. With the same setting, the discounted version

$$\int_0^t \frac{dC_u - c u^- d\Pi_u}{B_u} = \sum_i \left\{ \sum_{t_{i-1} < u \leq t_i} \frac{\Delta C_u}{B_u} - c_{t_{i-1}} C_{t_{i-1}} \int_{t_{i-1}}^{t_i} \frac{1}{B_u} d\Pi_u \right\}$$

$$= \sum_i \left\{ \frac{C_{t_i} - C_{t_{i-1}}}{B_{t_i}} - c_{t_{i-1}} C_{t_{i-1}} \int_{t_{i-1}}^{t_i} \frac{1}{B_u} d\Pi_u \right\}$$

which is different from the desired result of

$$\sum_i \frac{C_{t_i} - C_{t_{i-1}} - c_{t_{i-1}} C_{t_{i-1}} (t_i - t_{i-1})}{B_{t_i}}$$

and this is the only approximation applied in case of discrete collateralization.

We define

$$dD_u^{V-C} := d\Pi_u + dC_u - c u^- d\Pi_u$$

where $d\Pi_u$ represents the financial derivative dividend/cash flow at time $u$, one should note that:

1. The above formulation of dividend process implies that the collateral is subject to re-hypothecation: see in particular Remark 3.3.
2. The first addend has generally finite variation (see next section) but the second addend could have non-finite variation (e.g. in case $C_u = C_u^*$ for any $u$);
3. As outlined in the previous remark, the collateral process is intrinsically discontinuous since the equality $C_u = C_u^*$ is implemented only on margin call times $t_i$’s.

We now write the Profit and Loss ($P&L$) discounted at inception date $t$ of all cash transactions consisting in buying the collateralized derivative at time $t$ and holding it until time $T$ with $T > t$:

$$P&L_t = -(V_t - C_t) + B_t \int_t^T \frac{dD_u^{V-C}}{B_u} + \frac{B_t}{B_T} (V_T - C_T)$$

where the first added is the inception transaction (pay the financial derivative price and obtain the collateral value), the second added represents the discounted intermediate cash flows of the position and the third represents the closing of all positions (sell the financial derivative obtaining its price and give back the collateral). In order to avoid arbitrage, we must have that $\mathbb{E}_t[P&L_t] =$
0 meaning that the inception transaction is the equilibrium transaction corresponding to the $t$-
Risk Neutral discounted expectation of all other future transactions. In particular, $E_t[P & L_t] = 0$
if and only if

$$(V - C)_t = B_t \mathbb{E}_t \left[ \frac{(V - C)_T}{B_T} + \int_t^T \frac{dD_u^Y}{B_u} \right]$$

which in turn corresponds to require that the deflated gain process defined as follows

$$\tilde{G}^V_{\cdot} := \frac{(V - C)_u}{B_u} + \int_0^u \frac{dD_u^Y}{B_s}$$

must be a Risk Neutral martingale. This is the usual condition of Theorem 2.13 for asset $X := (V - C)$ with dividend process $D^X$ (and this explains the weird notation of the dividend process).
The gain process of the collateralized derivative is hence $G^X := X + D^X$.

Now define the buy-and-hold portfolio $\Psi_u := X_u + \varphi^0_u B_u$ in the interval $u \in [t, T]$. As usual, all
the dividends of the position are reinvested in the bank account: if $d\varphi^0_u dX_u / B_u > 0$ of bank account (paying $B_u \varphi^0_u$ of cash). If $d\varphi^0_u dX_u / B_u < 0$ she funds this transaction with selling a
quantity $d\varphi^0_u dX_u / B_u < 0$ of bank account (receiving $B_u \varphi^0_u$ of cash). In this light, we thus
obtain the self-financing condition of the buy-and-hold strategy on the collateralized derivative $X$:

$$d\Psi_u \| \tilde{\nu} = dX_u + \varphi^0_u dB_u + B_u d\varphi^0_u \text{ self-fin. } dX_u + \varphi^0_u dB_u + dD_u^X := dG^X_u + \varphi^0_u dB_u$$

where the first equality is by the Itô Formula and the fact that $B$ is continuous with finite
variation, the second is by substitution of the aforementioned dividend reinvestment strategy,
the third equality is by definition of the deflated gain process here above. Moreover, for $u \geq t$
the bank account quantity writes

$$\varphi^0_u = \varphi^0_t + \int_t^u d\varphi^0_s = \varphi^0_t + \int_t^u \frac{dD_u^X}{B_s}$$

and the bank account position is of value $\varphi^0_u B_u$: be aware of the analogies with Lemma 2.11.

As anticipated in Section 1.1, the common thread of the paper is to identify, thanks to no-
arbitrage conditions, Risk-Neutral martingales in progressively more challenging contexts where
our intuition could be increasingly lost: as a first example we saw the martingale corresponding to
a non-dividend paying asset, then we have attempted to recognize the martingale corresponding
to a dividend-paying asset and here we move a step forward for detecting a martingale linked to
a collateralized derivative. In the next section at (36), we will prove (in a more general multicurrency
environment) that, not only $\tilde{G}^V_{\cdot}$ but also

$$\tilde{G}^\gamma_{\cdot} := \frac{V_u}{B_u} + \int_0^u \frac{dD_u}{B_s}, \quad dD_u := d\Pi_u - (c_u - r_u) \, du$$

is a Risk-Neutral Martingale. We will continue using this demonstration strategy and recognize
new martingales in Section 4.

All these efforts in searching for martingales are motivated from the fact that martingales have
some nice properties and, primarily, since they have well defined dynamical features and specific
connections with expected values: among all expected values we are particularly interested in
pricing ones. For example in fact, from $G_1 = \mathbb{E}_t[G_T]$, we have

$$V_t = B_t \mathbb{E}_t \left[ \frac{V_T}{B_T} + \int_t^T \frac{dD_u}{B_u} \right]$$

which corresponds to (34) in the present single currency environment.
3.2. Multi-Currency Collateralized Derivatives

We can generalize the previous section results to a multi-currency setup: denote with $t \geq 0$ today’s time and with $V^f_t$ today’s value of the financial derivative with start date $0^f$ and expiry $T$ (for $t \leq T$) denominated in the $f$-currency as the cumulative dividend account

$$\Pi^f_t := \int_0^t \psi^f_u \, du + \Phi^f_t$$

with continuous dividend rate process $\psi^f$ and (RCLL/finite variation process) lump dividends/cash-flow account

$$\Phi^f_t := \sum_i \phi^f_{T_i} \mathbf{1}_{t \geq T_i}$$

both denominated in the $f$-currency. Note that, thanks to the arguments of Section 2.3, even if we describe $V^f$ as the price of a financial derivative, all the results of this section will be valid even if $V^f$ were the value of a collateralized underlying of the model where $T$ is not the expiry of the underlying but a generic future time. This is why we keep final condition $V^f_T = \phi^f_T$ as a placeholder of the value of the underlying at a future time, even if we could have set $\phi^f_T = 0$ and charge the last cash-flow on the lump dividend flow. In particular, the dividend paid at time $u$ writes

$$d\Pi^f_u = \psi^f_u \, du + d\Phi^f_u = \psi^f_u \, du + \sum_i \phi^f_{T_i} \, d\Theta_{T_i}(u)$$

The collateral account $C^g$ is instead in the $g$-currency. Denote with $d$ the domestic currency and, for currencies $x, y \in \{d, f, g, \ldots\}$ denote with $X^{xy}$ the FX rate to convert an amount denoted in currency $x$ into currency $y$, and with $X^x := X^{xd}$ the FX rate to convert an amount denoted in foreign currency $x$ into the domestic currency $d$: we adopt the convention of [MP17] for which the currency $d$ is “silent”, hence $B \equiv B^d$ is the domestic bank account and $Q \equiv Q^d$ the domestic measure. Clearly $X^{xx} \equiv 1$ and $X^{xy} \equiv 1/X^{yx}$. Moreover, for any foreign currency $x$, $X^x$ is a semimartingale with dynamics under $Q$

$$dX^x_t = \mu^x_t X^x_t \, dt + dM^x_t$$

where for all $x$ we have that $M^x$ is a RCLL square-integrable $Q$-martingale with $M^x_0 = 0$. Clearly $X^d_t = 1$ for all $t$, so $\mu^d_t = M^d_t = 0$ for all $t$.

Again from [MP17], for currency $x$, recall the definition of the $x$-currency basis measure $Q^{xb}$ as the measure with the numéraire corresponding to the $x$-currency basis bank account $B^{xb} := B^{xb}$ where we define the $x$-basis spot risk-free interest rate

$$r^{xb} := r - \mu^x$$

and by direct substitution $B := B^r = B^{rb}$ so that clearly $Q \equiv Q^d \equiv Q^{db}$. Define also for any $t \leq U$ the $x$-currency basis zero coupon bond

$$P^{xb}_t(U) := \mathbb{E}^{xb}_t \left[ \frac{B^{xb}_U}{B^{xb}_t} \right]$$

where we denote with $\mathbb{E}^{xb}_t[\cdot]$ the expectation under measure $Q^{xb}$. In case $x \neq d$ this the curve $U \mapsto P^{xb}_t(U)$ is often called “Forex (Discount) Curve”: see in particular (41). Observe that without FX market dislocations (see the second part of [MP17] for an explanation of these

\[^5\]In order to avoid burdening the notation, we analyze the case of a spot/past start deal: the forward start case would bring only slight modifications of the following results. In fact, if one is willing to tackle all cases, then one should let the start date be $t_0 < T$ and change all integration domains from $(t, T]$ to $(\max(t, t_0), T]$.  

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frictions on the FX market), we obtain the classical FX drift rate \( \mu^x = r - r^x \) (where the right-hand side corresponds to the difference of spot risk-free interest rates of corresponding currencies) and hence \( r^{xb} = r^x \), then also \( Q^{xb} = Q^x \) (the Risk Neutral measure of currency \( x \)): in practice all \( b \)-superscripts can be canceled out.

For any currency couple \( x, y \) define with
\[
\beta^{yx} := B^{yb} X^{yx} = \frac{B^{yb}}{X^{xy}} \tag{30}
\]
the \( y \)-basis bank account converted in currency \( x \), and the Radon–Nikodym derivative
\[
\E^x \left[ \frac{dQ^{yb}}{dQ^{xy}} \right] = \frac{\beta^{yx}}{y^{\frac{\partial}{\partial x}} \beta^{xy}} \tag{31}
\]
i.e. \( \beta^{yx} \) is the deflator to “export” an \( x \)-denoted security in the \( Q^{yb} \)-economy: the passage \( Q^{xb} \mapsto Q^{yb} \) is equivalent to the deflator passage \( B^{xb} \mapsto \beta^{yx} \). In particular, recalling Definition 2.19, the fact that \( \beta^{ad} = B^{yb} X^y \) is a “\( Q \)-price process” (hence eligible to be a numéraire) can be directly proved by (28) and straightforward applications of (A.6): we will confirm the more general result that \( \beta^{yx} \) is a “\( Q^{xb} \)-price process” (and hence that the above change of numéraire is valid) ex-post at Corollary 3.9.

We are now ready to extend the heuristical reasoning of the previous section to the current setup with a more precise methodology: the following proposition corresponds to Proposition 2.1 of [MP17] in a more general semimartingale setting.

**Proposition 3.2.** With the above notation, defining also
\[
V_t := X_t^f V_{t}^f, \quad \tilde{V}_t := V_t \frac{B_t}{g_t}, \quad C_t := X_t^g C_t^g, \quad \tilde{C}_t := C_t \frac{B_t}{B_t}
\]
and
\[
dD^{V-C}_u := X_u^f d\Pi_u^f + X_u^g dC_u^g - C_u^g \left( \frac{dC_u^g}{X_u^g} \right) du \tag{32}
\]
with \( D^{V-C}_0 = D^{V-C}_T = 0 \), then the gain process of the collateralized financial derivative \((V - C)\) under collateral re-hypothecation is defined as \( G^{V-C}_t := (V - C)_t + D^{V-C}_t \) and its deflated version
\[
\tilde{G}_t^{V-C} := \tilde{V}_t - \tilde{C}_t + \int_0^t \frac{dD^{V-C}_u}{B_u} \tag{33}
\]
is a martingale under the domestic Risk Neutral measure \( Q \). We obtain the following pricing formula:
\[
V_t^f = \frac{B_t}{X_t^f} \E_t \left[ \frac{\phi_T^f X_T^f}{B_T} + \int_t^T \frac{dD_u}{B_u} \right] \tag{34}
\]
where
\[
dD_u = X_u^f dD_u^f, \quad dD_u^f := d\Pi_u^f - X_u^g C_u^g \left( \frac{dC_u^g}{X_u^g} - r_{yb}^u \right) du \tag{35}
\]
with \( D^f_0 = D^f_T = 0 \). In particular, the gain process of the financial derivative is \( G_t := V_t + D_t \) and its deflated version
\[
\tilde{G}_t := \tilde{V}_t + \int_0^t \frac{dD_u}{B_u} \tag{36}
\]
is a \( Q \)-martingale.

**Proof.** See Appendix C.1. \hfill \Box
Remark 3.3 (Re-hypothecation). In dividends formulation (32) we have an implicit hypothesis of collateral re-hypothecation: in fact all positive and negative collateral cash flows enter in the collateralized financial derivative gain process. Instead, in case of no-hypothecation, meaning that the collateral is segregated, the negative collateral cash flows should always be there but the positive cash flows should not: these last cash flows should only let the collateral credit grow, but this benefit will be on the disposal of the investor only at collateral closing date. See also Section 4.3 for an example of a segregated collateral account.

Remark 3.4 (Non-linearity). In pricing formula (34) the price of the financial derivative depends (via the dividends (35)) on future realizations of the collateral value process $C^g$ which in turn will depend (in a potentially very complicated way) on future realizations of the price of the financial derivative itself: the pricing formula is in some sense recursive. This phenomenon is known in the literature as non-linearity of the pricing, where this terminology has a precise meaning arising from the theory of BSDEs (i.e. the non-linearity of the generator of the BSDE describing the price process of the financial derivative): see [BCR18] and references therein (where the authors achieve broader and more ambitious results). In some cases, see Propositions 3.13-3.16 and corresponding corollaries, we will add some simplifying hypotheses that will let us tackling the non-linearity issue.

Remark 3.5. As anticipated before, without FX market dislocations one has the usual drift of the FX rate $X^g$, i.e. $\mu^g_u = r_u - r^g_u$ where $r^g_u$ is the spot risk-free interest rate of currency $g$. Therefore, $\mu^g_u := r_u - \mu^u_u = r^g_u$. In this case, the contribution of $r$ is canceled out in the last addend of formula (35). Moreover, if we remunerate the collateral at risk-free rate $c^g = r^g$ the last addend of the above formula disappears ending up with the usual pricing formula.

We now derive the dynamics of the principal objects of our interest.

Proposition 3.6. The no-arbitrage dynamics of $(V - C)$ under $Q$ is

$$d(V - C)_t = r_t(V - C)_{t-} \, dt + d\mathcal{M}^{V-C}_{t} - dD^{V-C}_{t}$$

where $\mathcal{M}^{V-C}$ is a RCLL $Q$-martingale with $\mathcal{M}^{V-C}_0 = 0$. Then, under $Q$

$$dV_t = r_t V_{t-} \, dt + d\mathcal{M}_t - dD_t$$

$$d\mathcal{M}_t := d\mathcal{M}^{V-C}_t + C^g_{t-} \, dM^g_t$$

(37)

where $\mathcal{M}$ is a RCLL $Q$-martingale with $\mathcal{M}_0 = 0$.

Proof. The dynamics of $(V - C)$ is a direct consequence of the fact that $\tilde{G}^{V-C}_t$ is a $Q$-martingale, see Proposition 2.16 from which one also obtains that $d\tilde{G}^{V-C}_t = B_t^{-1} \, d\mathcal{M}^{V-C}_t$. Moreover, by (33),

$$d\tilde{G}^{V-C}_t = \tilde{V}_t - \tilde{C}_t + \frac{dD^{V-C}_u}{B_u}$$

so, recalling (35),

$$d\tilde{V}_t = d\tilde{G}^{V-C}_t - \left( \frac{dD^{V-C}_u}{B_u} - d\tilde{C}_t \right)$$

$$= \frac{1}{B_t} \left\{ d\mathcal{M}^{V-C}_t - X^f_t \, d\Pi^f_t - X^g_u C^g_{u-} \left\{ (r_u - c^g_u) \, du - \frac{dX^g_u}{X^g_{u-}} \right\} \right\}$$

$$= \frac{1}{B_t} \left\{ d\mathcal{M}^{V-C}_t - X^f_u \, d\Pi^f_t + X^g_u C^g_{u-} \left\{ (c^g_u - r^g_u) \, du + \frac{dM^g_u}{X^g_{u-}} \right\} \right\}$$

$$= \frac{1}{B_t} \left\{ d\mathcal{M}^{V-C}_t + C^g_{u-} \, dM^g_u - dD_u \right\}$$
where we used (C.1) at the second line and Proposition A.4 in the last equation. Now,
\[ dV_t = dB_t V_t = dB_t + b_t V_t dt = dM_{V-C}^t + C_u^g dM_u^g - dD_u + r_t V_t dt \]
which is the dynamics of \( V \).

**Remark 3.7.** One could deduce the dynamics of \( C \) from \( dV_u - d(V - C)_u \):
\[ dC_u = r_u C_{u-} du + (dM_u - dM_{u-C}^u) - (dD_u - dD_{u-C}^u) \]
however, this is not an explicit dynamics since in the term \( dD_{u-C}^u \) there is indeed a \( dC_u \) term: in fact, substituting
\[ dC_u = r_u C_{u-} du + C_u^g dM_u^g + X_u^g dC_u^g + d(C^g X_u^g)_u - (r_u^b X_u^g C_u^g)_u dD_u - dX_u^g \]
so, canceling \( dC_u \) on both sides,
\[ 0 = \mu_u^g C_u^g dM_u^g + C_u^g dD_u + (dX_u^g - \mu_u^b X_u^g dD_u) - C_u^g dX_u^g \]
and also the right-hand side is equal to zero.

**Proposition 3.8.** Recalling (28), under \( Q^{fb} \), we have
\[ dX^x_t = (r_t - r_t^{fb}) X_{x-}^t dt + \frac{d[M^x, M^{fb}]_t}{X_t^x} + dM^{x, fb}_t \]  
where \( M^{x, fb} \) is \( Q^{fb} \)-martingale corresponding to \( M^x \) after measure change to \( Q^{fb} \). Moreover, under \( Q^{fb} \),
\[ dX^{xf}_t = X^{xf}_{x-} \left\{ (r_t^{fb} - r_t^{fb}) dt - X^{xf}_{x-} dM^{fb}_t \right\} \]
\[ x = d \]
\[ dX^{xf}_t = X^{xf}_{x-} \left\{ (r_t^{fb} - r_t^{fb}) dt - \frac{dM^{fb}_t}{X_{x-}^t} + dM^{x, fb}_t \right\} \]
\[ x \neq d \]
where \( M^{fb} \) is a \( Q^{fb} \)-martingale corresponding to \( M^f \) after measure change to \( Q^{fb} \).

**Proof.** See Appendix C.2.

**Corollary 3.9.** Recalling definition (30), we confirm that \( \beta^{x,f} := B^{fb} X^{x,f} \) is a "\( Q^{fb} \)-price process".

**Proof.** We have to prove that \( \beta^{x,f}/B^{fb} \) is a \( Q^{fb} \)-martingale. We have, using (A.6) and that \( B^{fb}, B^{zfb} \) are continuous with finite variation:
\[ d(\beta^{x,f} B^{-1}) = (B^{fb})^{-1} d\beta^{x,f} + (B^{fb})^{-1} d((B^z)^{-1}) \]
\[ = (B^{fb})^{-1} \left\{ B^{zfb} dX^{x,f} + X^{x,f} dB^{zfb}_t - r_t^{fb} \beta^{x,f}_t dt \right\} \]
\[ = B^{zfb} \left\{ dX^{x,f} + r_t^{zfb} X^{x,f} dt - r_t^{fb} X^{x,f}_t dt \right\} \]
\[ = B^{zfb} \left\{ dX^{x,f} - (\mu_t^z - \mu_t^b) X^{x,f}_t dt \right\} \]
and the result is obtained substituting (39) or (40) and recalling that \( \mu^d = 0 \).
Proposition 3.10. The FX forward (i.e. the par rate of an FX forward contract) is written as
\[ X_{t}^{f}(U) := \mathbb{E}_{t}^{U,f_{b}}[X_{U}^{f}] = X_{t}^{f} \frac{P_{t}^{f_{b}}(U)}{P_{t}^{f}(U)} \]  

**Proof.** We have, recalling definition (30) and that the passage \( Q_{f_{b}} \mapsto Q_{x_{b}} \) is equivalent to the deflator passage \( B_{f_{b}} \mapsto B_{x_{f}} :\)
\[ X_{t}^{f} \frac{P_{t}^{f_{b}}(U)}{P_{t}^{f}(U)} X_{U}^{f} = X_{t}^{f} \frac{B_{t}^{x_{f}}}{B_{t}^{f_{b}}} \]  
from which we have the thesis. \( \square \)

Proposition 3.11. Defining \( \tilde{V}_{t}^{f} := \frac{V_{t}^{f}}{B_{t}^{f_{b}}} \), \( C_{t}^{f} := C_{t}^{x_{f}} X_{t}^{x_{f}} \), \( \tilde{C}_{t}^{f} := \frac{C_{t}^{f}}{B_{t}^{f_{b}}} \), and
\[ dD_{u}^{V,C} := d\Pi_{u}^{f} + X_{u}^{g_{f}} \left\{ dC_{u}^{a} - e_{u}^{a} C_{u}^{a} + \left[ dC_{u}^{a} \frac{dX_{u}^{g_{f}}}{X_{u}^{g_{f}}} \right] \right\} , \]
with \( D_{0}^{V,C} = D_{0}^{V,C} = 0 \), then the gain process of the collateralized financial derivative \( (V^{f} - C_{t}^{f}) \) in currency \( f \) is \( G_{t}^{f,V,C} := (V^{f} - C_{t}^{f}) + D_{t}^{V,C} \) and the deflated gain process \( \tilde{G}_{t}^{f,V,C} \) is a martingale under \( Q_{f_{b}} \). We also have the pricing formula:
\[ V_{t}^{f} = B_{t}^{f_{b}} \mathbb{E}_{t}^{f_{b}} \left[ \frac{\phi_{T}^{f}}{B_{T}^{f_{b}}} + \int_{t}^{T} \frac{dD_{u}^{f}}{B_{u}^{f_{b}}} \right] \]  
recalling definition (35). In particular, the gain process of the financial derivative is \( G_{t}^{f} := V_{t}^{f} + D_{t}^{f} \) and its deflated version
\[ \tilde{G}_{t}^{f} := \tilde{V}_{t}^{f} + \int_{0}^{t} \frac{dD_{u}^{f}}{B_{u}^{f}} \]
is a \( Q_{f_{b}} \)-martingale. \( \square \)

**Proof.** See Appendix C.3.

We now derive the Risk-Neutral drift of \( V^{f} \), expanding the results of Section 2 in \cite{GPS19}.

Proposition 3.12. The no-arbitrage dynamics of \( (V^{f} - C_{t}^{f}) \) under \( Q^{f_{b}} \) is
\[ d(V^{f} - C_{t}^{f}) = r_{t}^{f_{b}}(V^{f} - C_{t}^{f}) dt + dM_{t}^{f,V,C} - dD_{t}^{f,V,C} \]
where \( M^{f,V,C} \) is a RCLL \( \mathbb{Q}^{fb} \)-martingale with \( M^{f,V,C}_0 = 0 \). Then, under \( \mathbb{Q}^{fb} \), recalling definition (35),

\[
dV_t = r_t^{fb} V_t^f \, dt + dM_t^f - dD_t^f
\]

(46)

where \( M^f \) is a RCLL \( \mathbb{Q}^{fb} \)-martingale. Moreover, under \( \mathbb{Q} \), the quanto-d dynamics of \( V^f \) writes:

\[
dV_t = V_t^f \left\{ r_t^{fb} \, dt - \frac{d[M^f,X]^f_t}{X_t^{fb}} \right\} + dM_t^{fd} - dD_t^f
\]

(47)

where \( M^{fd} \) is a \( \mathbb{Q} \)-martingale corresponding to \( M^f \) after a change of measure to \( \mathbb{Q} \).

**Proof.** See Appendix C.4. \( \square \)

We extend here the results of Corollary 2.1 of [MP17], in different directions: firstly, in this reference the authors do not consider over-collateralization, secondly they work in a setting where all market risks are described by a vector of continuous Itô processes driven by a Brownian vector. Coherently with (27), define the target value of collateral account as:

\[
C_n^\alpha := (1 + \alpha_n) V_u^{f} X_u^{fb}
\]

where the haircut \( \alpha_n \geq 0 \) is a predictable process. As promised in Remark 3.4, the following Proposition tackles the issue of a recursive pricing formula.

**Proposition 3.13.** We add the hypotheses that the continuous dividend is proportional, i.e. \( \psi^f \equiv \ell^f \, V_{u-}^f \) for some predictable process \( \ell^f \) and also that we have continuous margin calls i.e. \( C_u^\alpha = C_u^{\alpha*} \) for any \( u \in [0,T] \). Then

\[
V_t^f = B_t^{fb} \mathbb{E}^{fb} \left[ \frac{\phi^f_T}{B_T^f} + \int_t^T \frac{d\Phi^f_u}{B_u^{2f}} \right]
\]

(48)

where, the blended discounting interest rate

\[
z_u^f := (1 + \alpha_u)(c_u^g - r_u^{gb} + r_u^{fb}) - \left\{ \alpha_u r_u^{fb} + \ell_u^f \right\}
\]

(49)

**Proof.** Substituting \( \psi^f = \ell^f \, V_{u-}^f \) and \( C_u^\alpha = C_u^{\alpha*} \), and defining

\[
q_u := \ell_u^f - (1 + \alpha_u)(c_u^g - r_u^{gb}),
\]

(50)

the financial derivative dividend process (35) becomes

\[
dD_t^f = d\Phi^f_u + q_u V_u^f \, du
\]

and using the result of Proposition 2.26, we obtain the thesis since

\[
z_u^f := r_u^{fb} - q_u
\]

\[
= (1 + \alpha_u - \alpha_u) r_u^{fb} + (1 + \alpha_u)(c_u^g - r_u^{gb}) - \ell_u^f
\]

\[
= -\alpha_u r_u^{fb} + (1 + \alpha_u)(c_u^g - r_u^{gb} + r_u^{fb}) - \ell_u^f
\]

Alternatively, as in Corollary 2.1 of [MP17], we can use the Feynman-Kac Theorem in case all processes are continuous (so that all PDEs are well defined) and driven by a Brownian vector. \( \square \)

**Remark 3.14.** Formula (48) is sometimes referred among practitioners to as “CSA discounting”. This formula is in the most general form: all rates are stochastic and cash flows and collateral are under different foreign currencies. One could simplify the first hypothesis: see Corollary
3.15 also in order to better understand the formula. Moreover, since \( d, f \) and \( g \) are only placeholders for currencies, one could also simplify the multi-currency framework: e.g. \( g = f \) (cash flows and collateral both in currency \( f \)) or \( f = d \) (cash flows in domestic currency, collateral in currency \( g \)), or the other way round \( g = d \) (cash flows in currency \( f \) and collateral in domestic currency), etc.

We try to gain intuition on the crucial pricing formula (48) and in particular on the first addend of \( z^f \), the second being a payoff increase (in case both \( r^{fb}, \ell^f \geq 0 \), remark the minus sign before the curly brackets) for over-collateralization and proportional dividends.

**Corollary 3.15.** In case \( \alpha = \ell^f = 0 \) and all rates are deterministic, (48) becomes

\[
V_t^f = P_t^f(T) E_t^f [\phi_T^f] + \int_t^T P_t^f(u) E_t^f [d\Phi_u^f]
\]

where, for \( U \geq t \)

\[
P_t^f(U) = \frac{P_t^g(U) X_t^{fg}(U)}{X_t^{gf}}
\]

which has a clear financial meaning: the forward FX converts the time-\( U \) cash flow from currency \( f \) to currency \( g \), then the cash flow is discounted to \( t \) with rate \( c^g \) and finally converted back in currency \( f \) with \( 1/X_t^{fg} = X_t^{gf} \).

**Proof.** Straightforward from hypotheses, (41)-(48).

**Proposition 3.16.** In the same setting as in Proposition 3.13 we have

\[
V_t^f = \frac{B_t^f}{X_t^f} \mathbb{E}_t^f \left[ \phi_T^f X_T^f \frac{B_T^f}{B_T^{fb}} + \int_t^T X_u^f \frac{d\Phi_u^f}{B_u^f} \right]
\]

where

\[
z_u := (1 + \alpha_u)(c_u^g - r_u^{gb} + r_u) - \{\alpha_u r_u + \ell_u^f\}
\]

**Proof.** Recalling (50) and that \( z^f := r^{fb} - q \), we have

\[
B_t^z := e^{\int_t^u (r^{fb} - q) du} = \frac{B_t^f}{B_t^f}
\]

So using (20), and defining \( z := r - q \), the (48) can be rewritten as

\[
V_t^f = \frac{B_t^f}{B_t^f} \mathbb{E}_t^f \left[ \phi_T^f B_T^g \frac{B_T^f}{B_T^{fb}} + \int_t^T B_u^g \frac{d\Phi_u^f}{B_u^{fb}} \right] = \frac{\phi_T^f}{B_t^f} \mathbb{E}_t^f \left[ \phi_T^f B_T^g \frac{B_T^f}{B_T^{fb}} + \int_t^T B_u^g \frac{d\Phi_u^f}{B_u^{fb}} + B_u^g d \left[ \Phi_u^f \frac{1}{\beta_u^{df}} \right] \right] = \frac{B_t^f}{X_t^f} \mathbb{E}_t^f \left[ \phi_T^f X_T^f \frac{B_T^f}{B_T^{gb}} + \int_t^T X_u^f \frac{d\Phi_u^f}{B_u^f} + \sum_{t < u \leq T} \frac{\Delta \Phi_u^f \Delta X_u^f}{B_u^f} \right] = \frac{B_t^f}{X_t^f} \mathbb{E}_t^f \left[ \phi_T^f X_T^f \frac{B_T^f}{B_T^{gb}} + \sum_{t < u \leq T} \frac{\Delta \Phi_u^f \Delta X_u^f}{B_u^f} \right]
\]

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recalling that $\Phi^f$ is a pure jump process. Moreover,
\begin{align*}
z_u := r_u - q_u \\
= (1 + \alpha_u - \alpha_u) r_u + (1 + \alpha_u)(e_u^g - r_u^g) - \ell^f_u \\
= -\alpha_u r_u + (1 + \alpha_u)(e_u^g - r_u^g + r_u) - \ell^f_u
\end{align*}
which gives the thesis.

Under the same hypotheses of Corollary 3.15, the payoff multiplier becomes
\begin{align*}
P_t^z(U) X^f_U = \frac{P_t^x(U) X^g(U)}{X^d_y(U) X^f_U}
\end{align*}
i.e., counterclockwise from the right: convert the payoff from currency $f$ to the domestic currency with the corresponding stochastic future FX rate, then convert it to currency $g$ with the corresponding forward FX rate, discount to reference time $t$ with rate $c^g$ and then convert again the flow in the domestic currency with the corresponding FX spot.

**Corollary 3.17 (Domestic Collateral and Cash Flows).** In case of continuous collateralization and when both collateral and cash-flows are denoted in domestic currency (i.e. $g = f = d$), we have
\begin{align*}
V_t = B_t^z \mathbb{E}_t \left[ \frac{\phi_T}{B_T^z} + \int_t^T \frac{d\Phi_u}{B_u^z} \right]
\end{align*}
where in this case $z = (1 + \alpha)c - \{\alpha r + \ell \}$ and $c$ is the domestic collateral rate. In case $\ell = \alpha = 0$ we have obtained the usual pricing formula with a new discounting rate passing from $r$ to the collateral discounting rate $c$: the price of the financial derivative no longer depends on the domestic spot risk-free rate.

**Proof.** In this case the FX rate is equal to one for all $t$ and its drift is equal to zero for all $t$. \qed

The following proposition extends the results of [GPS19] to the case of stochastic rates. Let $Q_U^{f:b}$ correspond to the $U$-forward basis measure of currency $f$ for domestic collateral rate $c = r$:
\begin{align*}
\mathbb{E}_t^{Q_U^{f:b}} \left[ dQ_U^{f:b} \right] = \frac{P_t^{f:b}(U) B_t^{f:b}}{P_U^{f:b}(U) B_U^{f:b}}
\end{align*}
i.e. $Z_t^f(T; r)$ in the notation of [MP17] (see their equation (2.23)).

**Proposition 3.18.** For $t \leq U \leq T$, the forward of $V^f$ writes
\begin{align}
F_t^f(U) := \mathbb{E}_t^{U^{f:b}}[V_U^f] = \frac{1}{P_t^{f:b}(U)} \left\{ V_t^f - B_t^{f:b} \mathbb{E}_t^{f:b} \left[ \int_t^U \frac{dD_u^f}{B_u^{f:b}} \right] \right\} \quad (52)
\end{align}
where the dividends $D^f$ are defined in (35). The $d$-forward quanto is instead written:
\begin{align}
F_t^{f:qd}(U) := \mathbb{E}_t^{U^{f:b}}[V_U^{f:qd}] = \frac{1}{P_t(U)} \left\{ V_t^{f:qd} - B_t \mathbb{E}_t \left[ \int_t^U \frac{dD_u^{f:qd}}{B_u} \right] \right\} \quad (53)
\end{align}
where
\begin{align*}
dD_u^{f:qd} := dD_u^f + V_u^{f:qd} \left\{ \mu_u^f du + \frac{d[M^f, X^f]_u}{X_u^{f:qd} V_u^{f:qd}} \right\}
\end{align*}
Proof. Recalling definition (44), we have

\[
F^f_t(U) := \mathbb{E}^{U,f_b}[V^f_U]
\]

\[
= \frac{1}{P^f_{t}(U)} \mathbb{E}^{U,f_b} \left[ \frac{P^f_{t}(U)}{P^f_{f_b}(U)} V^f_U \right]
\]

\[
= \frac{B^f_{t}}{P^f_{t}(U)} \mathbb{E}^{f_b} \left[ \frac{V^f_U}{B^f_{t}} \right]
\]

\[
= \frac{B^f_{t}}{P^f_{t}(U)} \mathbb{E}^{f_b} \left[ \tilde{G}^f_t - \int_0^t d\tilde{D}^f_u \right]
\]

\[
= \frac{B^f_{t}}{P^f_{t}(U)} \left\{ V^f_t - B^f_{t} \mathbb{E}^{f_b} \left[ \int_0^t d\tilde{D}^f_u \right] \right\}
\]

where we exploited the martingale property of \( \tilde{G}^f_t \). For the forward quanto: rewrite (47) with

\[
dV^f_t = r_t V^f_t \, dt + d\mathcal{M}^q_t - d\tilde{D}^q_t
\]

so that

\[
\tilde{G}^{q}_t := V^f_t / B^f_t + \int_0^t d\tilde{D}^q_u / B^q_u
\]

is a \( \mathbb{Q} \)-martingale. Then one can proceed analogously as the proof of \( F^f_t(U) \)-formula in the domestic currency case.

Remark 3.19. Formula (52) has the usual appearance of the capitalization from \( t \) to date \( T \) thanks to \( P^f_{t}(U) \) of the asset value \( V^f_t \) plus the expected discounted future dividend flow: it extends the concept of forward price to the new context with multi-currency collateralization and FX market dislocations.

Corollary 3.20. In case

\[
\frac{d[\mathcal{M}^f_t,X^f_u]}{X^f_u-V^f_u} = \rho_u \, du
\]

and all interest rates are deterministic, the forward quanto writes

\[
F^q_t(U) = \frac{1}{P^q_t(U)} \left\{ V^f_t - B^q_t \mathbb{E}_t \left[ \int_t^T d\tilde{D}^q_u / B^q_u \right] \right\}
\]

where \( \eta := r^{q} - \rho \) and \( P^q_t(U) := \mathbb{E}_t[B^q_t / B^q_U] = B^q_t / B^q_U \).

Proof. From dynamics (54) and Remark 2.27, defining \( q := \mu^f + \rho \),

\[
V^f_t = B^f_t \mathbb{E}_t \left[ \frac{\phi^f_t}{B^f_T} + \int_t^T d\tilde{D}^f_u / B^f_u + \int_t^T \rho_u V^f_u \, du \right] = B^{\eta-q}_t \mathbb{E}_t \left[ \frac{\phi^f_t}{B^{\eta-q}_T} + \int_t^T d\tilde{D}^f_u / B^{\eta-q}_u \right]
\]
and \( r - q := r - (\mu^f + \rho) = r^{fb} - \rho =: \eta \). Using this result, the tower property of conditional expectation and the fact that we have deterministic rates:

\[
F_t^f(U) := E_t^U[V_t^f] = E_t[Q][E_t^U[V_t^f]] = E_t \left[ B_t^q \frac{\phi^f_{T^f}}{B_T^q} + B_t^q \int_t^T \frac{dD_u^f}{B_u^q} \right] \\
= \frac{B_t^q}{B_T^q} \left[ B_t^q \frac{\phi^f_{T^f}}{B_T^q} + B_t^q \int_t^T \frac{dD_u^f}{B_u^q} \right] \\
= \frac{B_t^q}{B_T^q} \left\{ V_t^f - B_t^q \mathbb{E}_t \left[ \int_t^T \frac{dD_u^f}{B_u^q} \right] \right\}
\]

which gives the thesis. \( \square \)

In the following proposition we add some simplifying hypotheses in order to have a sharper formula for the forward price. The result is the same as in [GPS19] in the new, more general, semimartingale framework.

**Proposition 3.21.** With the same hypotheses as in Proposition 3.13, in case also all interest rates are deterministic, we have

\[
F_t^f(U) = \frac{1}{P_t^{fb}(U)} \left[ V_t^f - \sum_i P_t^{z^f}(T_i) E_t^{fb}[\phi^f_{T_i}] \mathbb{1}_{t \leq T_i \leq U} \right]
\]

where we recall definition of rate \( z_U^f \) at (49) and we defined \( P_t^{z^f}(u) = \mathbb{E}_t^{fb}[B_t^{z^f}/B_u^{z^f}] = B_t^{z^f}/B_u^{z^f} \).

**Proof.** From (52), recalling the definition of rate \( q \) at (50),

\[
F_t^f(U) P_t^{fb}(U) = V_t^f - \int_t^U E_t^{fb} \left[ \frac{d\Phi^f_q + q_u V_t^f}{B_u^{fb}} \right] du \\
= V_t^f - \int_t^U P_t^{fb}(u) \mathbb{E}_t^{u:fb}[\Phi^f_q + q_u V_t^f] du \\
= V_t^f - \int_t^U P_t^{fb}(u) \sum_i \phi^f_{T_i} \delta(u - T_i) + q_u V_t^f \right] du \\
= V_t^f - \int_t^U P_t^{fb}(u) \sum_i \mathbb{E}_t^{u:fb} \left[ \phi^f_{T_i} \right] \delta(u - T_i) + \mathbb{E}_t^{u:fb} [q_u V_t^f] du
\]

where we recall that \( \delta(u - x_0) \) is the Dirac mass centered at \( x_0 \). With deterministic rates \( Q^{fb} = Q^{u:fb} \) for any \( u \), we have

\[
\partial_U \left( F_t^f(U) P_t^{fb}(U) \right) = -P_t^{fb}(U) \left( \sum_i \mathbb{E}_t^{fb} \left[ \phi^f_{T_i} \right] \delta(U - T_i) + q_u F_t^f(U) \right)
\]

while, on the other hand,

\[
\partial_U \left( F_t^f(U) P_t^{fb}(U) \right) = P_t^{fb}(U) \partial_U F_t^f(U) + F_t^f(U) P_t^{fb}(U) \partial_U P_t^{fb}(U) = P_t^{fb}(U) \left( \partial_U F_t^f(U) - r_t^{fb} \right)
\]

so substituting we end up with the following ODE, for \( U \in [t, T] \)

\[
\partial_U F_t^f(U) = z_t^f F_t^f(U) + b(U)
\]
where

\[ b(U) := -\sum_i \mathbb{E}_i^{fb} \left[ \phi_{T_i}^f \right] \delta(U - T_i) \]

with initial condition \( F_t^f(t) = V_t^f \). The linear ODE can be solved with:

\[
F_t^f(U) = \frac{1}{B^f_{U}} \left[ F_t^f(t) + \int_t^U b(u) \frac{B^f_{t}}{B^f_{u}} \, du \right]
\]

\[
= \left( \frac{B^f_{t}}{B^f_{U}} \right)^{-1} \left[ V_t^f + \int_t^U b(u) \left( \frac{B^f_{t}}{B^f_{u}} \right) \, du \right]
\]

which can be rearranged to obtain the thesis.

4. Applications

4.1. Repurchase Agreement

We follow [Ch10] for deal description. The so-called classic Repurchase Agreement (classic repo for short) is a contract between two parties, let us say \( R^S \) (the “repo seller”: the seller of securities \( S \)) and \( R^B \) (the “repo buyer”: the buyer of securities \( S \)). In essence, a repo agreement is a secured loan (or collateralized loan): on the trade date the two counterparties sign an agreement whereby on a future date (the value or settlement date) \( R^S \) will sell to \( R^B \) a nominal amount of securities in exchange for cash. The price received for the securities is the market price of the stocks on the value date. The agreement also demands that on the termination date \( R^B \) will sell identical stock back to \( R^S \) at a previously agreed price: the cash exchanged at value date plus interests calculated at an agreed repo rate. Note that although legal title to the collateral passes to the repo buyer, economic costs and benefits of the collateral (e.g. dividends, coupons, capital gains, etc.) remain with the repo seller.

The transaction described above is a specific classic repo, that is, one in which the collateral supplied is specified as a particular stock, as opposed to a general collateral (GC) trade in which a basket of collateral can be supplied, of any particular issue, as long as it is of the required type and credit quality. If the collateral asset is particularly on demand – that is, it is special – the repo rate may be significantly below the GC rate. Special status will push the repo rate downwards: zero rates and even negative rates are possible when dealing in specials. In fact, we distinguish in the repo market between the cash-driven players (players that demand/invest cash) and security-driven players (players that are interested in securities loans e.g. for selling them).

To counter all risks of repo transactions (credit risk, market risk of collateral, issuer risk of collateral, etc.), the repo loan is not only secured but also subject to a margination process. In particular, we denote with \( t_i \) all margin call dates, and in particular \( t_0 \equiv t \) where \( t \) is the repo value date. Moreover, we denote with \( T \) the repo termination date and with \( \kappa_T \) the repo rate of the transaction, that fixes at time \( t \) and is valid for the repo contract of interval \([t, T]\). Moreover, we denote with \( C_u \) for any \( u \in [t, T] \) the value of the collateral (stock and cash) held by the repo buyer. We define also the collateral target value

\[
C^*_u := (1 + \alpha_t) \left[ H_t (1 + \kappa_T \times yf(t, u)) \right]
\]

where \( yf \) is the year fraction according to the termsheet accrual rule, \( \alpha_t \geq 0 \) is the haircut and \( H_t := NS_t \) represents the cash loaned by the seller for a reference number of securities \( N \) and a repo collateral stock denoted by \( S \). Hence the collateral target value not only encompasses the cash loan but also the repo interests already due at date \( u \). We have the following items:
\* $u = t$: the start cash proceeds of a repo can be less than the market value of the collateral by an agreed amount or percentage known as the haircut so that at value date $t$ we have the following equalities:

$$Q_t S_t = C_t \overset{\text{def}}{=} (1 + \alpha_t)H_t =: C_t^*$$

meaning that the seller receives $H_t$ of cash and at the same time sells to the buyer an effective number $Q_t$ of securities $S$ with $Q_t = N(1 + \alpha_t)$. In practice, the collateral value is a multiple (greater than one) of the cash: the deal is over-collateralized. We precise that the repo interests are calculated with respect to the cash loan, so the total repo interests writes $H_t \times \kappa_t^* \times y[t, T]$.

\* $u \in (t, T)$: The market value of the collateral is maintained through the life of repo contract.

So if the market value of the collateral falls, the buyer calls for extra cash or collateral. If the market value of the collateral rises, the seller calls for extra cash or for obtaining back part of the collateral stocks. In order to reduce the administrative burden, margin calls can be limited to changes in the market value of the collateral in excess of an agreed amount, or percentage, which is called a margin maintenance limit. Imagine that at date $u$ the collateral value $C_u$ falls below the original cash loan $H_t$, then the buyer, under a margin arrangement, can call margin from the counterparty in the form of securities or cash setting $u = t_i$ for some $i > 0$. We have two cases:

i) The original repo contract is maintained. The margin adjustment at time $t_i$ is such that the collateral value reaches its target value, i.e. $C_{t_i} = C_{t_i}^*$. In a cash-driven trade, additional shares are delivered to restore the haircut level. In this case, the terms of the original trade remain unchanged, but the transfer of shares becomes a separate repo of stock against zero cash, and is unwound at the maturity date of the original trade. Where a repo is stock-driven, cash may be supplied as collateral. The market convention is for this cash to earn interest for the collateral supplier at the trade repo rate, or at a rate agreed between the two parties (that we will denote $c$). The interest accrues at ACT/365 or on a basis agreed between parties.

ii) The original repo is closed-out, and then re-opened under new terms. The changed terms of the trade reflect the movement in collateral required to restore margin. See [Ch10] p. 382 for further details on this case.

We now formalize case i) above. For any margin call date $t_i$, we define with $F_t$, the residual quantity (with respects to collateral securities) of cash collateral (equal to zero at inception date $F_t = 0$): $F_u$ has values in $\mathbb{R}$ and represents the collateral cash held by the repo buyer at date $u$, (the sign convention is due to the buyer being the stock collateral taker), $F_u > 0$ means that the repo buyer holds cash collateral, while $F_u < 0$ means that the repo seller holds cash collateral. For any $i \geq 0$

$$C_{t_i} = Q_{t_i} S_{t_i} + F_{t_i} = C_{t_i}^*$$

where either $Q$ or $F$ are set such that the last equality is respected. For any $i > 0$ then the margin call received by the repo buyer is therefore

$$C_{t_i} - \{Q_{t_{i-1}} S_{t_i} + F_{t_{i-1}}\} - c_{t_{i-1}} F_{t_{i-1}} (t_i - t_{i-1})$$

$$= (Q_{t_i} - Q_{t_{i-1}}) S_{t_i} + F_{t_i} - F_{t_{i-1}} \left(1 + c_{t_{i-1}} (t_i - t_{i-1})\right)$$

(56)

where the addend in curly bracket is the value of collateral pre-rebalancing (the natural evolution of $C_{t_{i-1}}$ in the interval $(t_{i-1}, t_i]$: note that the cash remains flat since there is no other cash inflow/outflow but the value of the stock evolves with its market movement), the last addend of the first equation ensures that the collateral cash be remunerated via interest rate $c$.

We now summarize all transactions of the repo contract.
• At time $t$:
  - $R^S$ obtains $H_t = NS_t$ of cash from $R^B$.
  - $R^S$ repo sells to $R^B$ a number $Q_t = N(1+\alpha_t)$ of assets $S$ (of value $C_t = Q_tS_t = (1+\alpha_t)H_t$).
  - $R^S$ and $R^B$ agree on the repurchase price of
    $$H_t\left(1 + \kappa_t^T \times y(t,T)\right)$$
    which is fixed at $t$ but will be paid at $T$.
• In the interval $(t,T)$:
  - All the dividends/coupons paid by the collateral assets $S$ are collected from $R^B$ and transferred to $R^S$.
  - In case of margin maintenance breach, the value of the collateral account is restored as explained previously.
• At time $T$:
  - $R^S$ pays to $R^B$ the repurchase price agreed at $t$.
  - $R^B$ closes the collateral account – giving back to $R^S$ all the assets $S$ and the cash (if any) she detains as collateral (in case $R^B$ detains negative cash collateral, she receives it back from the seller). In case of no margin calls, $R^B$ gives back to $R^S$ the quantity $Q_t$ of asset $S$ (of value $N(1 + \alpha_t) \times S_T$).

Now we focus on a stock-driven repo for a fixed number of stocks $Q_t = N(1 + \alpha_t) = 1$, i.e.

$$N = \frac{1}{1 + \alpha_t}$$

and assume that all margin calls are performed via (domestic currency) cash: hence the non-cash part of the collateral is represented by a single stock $S$ for the entire contract period, i.e. $Q_u = 1$ for all $u \in [t,T]$. We have that at inception, the collateral is of value

$$C_t = Q_tS_t = S_t$$

which corresponds to the single stock $S$ repo-sold by $R^S$. For $u \in [t,T]$ we define the cash loan value process:

$$X_u := S_t \left\{ \frac{1}{1 + \alpha_t} \right\} \left(1 + \kappa_t^T(u-t)\right) = S_t \left\{ \frac{1}{1 + \alpha_t} \right\} \frac{B_u}{B_t^\kappa}$$

where the last equality is by a bootstrap of a deterministic term structure of repo rate $\kappa_u$: observe that since at the first order $1 + \kappa_T^T(u-t) \approx e^{\kappa_T^T(u-t)}$ then $\kappa_u \approx \kappa_T^T$ for any $u \in [t,T]$. Observe that $X_t = H_t$ represents the cash loan received by the repo seller at value date, and $X_T$ is the cash that the seller should pay at maturity to close the loan position. By Itô formula,

$$dX_u = \frac{S_t}{B_t^\kappa} \left\{ \frac{1}{1 + \alpha_t} \right\} \kappa_u B_u \, du = \kappa_u X_u \, du \quad (57)$$

Consistent with this paper’s simplifying hypothesis that the collateralization eliminates the bilateral counterparty risk, we write the P&L discounted at date $t$ of the repo seller:

$$P\&L_t = S_t \left\{ \frac{1}{1 + \alpha_t} \right\} - C_t + B_t \left\{ \int_t^T \frac{\Phi_u - (dF_u - c_uF_u\, du)}{B_u} + \frac{-S_t}{1 + \alpha_t} \left(1 + \kappa_T^T(T-t)\right) + C_T}{B_T} \right\}$$

$$= - (S_t + F_t - X_t) + B_t \left\{ \int_t^T \frac{\Phi_u - (dF_u - c_uF_u\, du)}{B_u} + \frac{S_T + F_T - X_T}{B_T} \right\}$$

35
where \( d\Phi_u \) is the dividend (e.g. in case of \( S \) being an equity stock) or coupon (e.g. bond) of the asset \( S \) paid at time \( u \) (recall that the repo seller receives the dividend from the repo buyer); the margin calls are by (56), recalling that this equation is written from the repo buyer point of view, hence the negative sign. Recall that all margin calls are by domestic cash and that \( F_t = 0 \) by construction.

In order to avoid arbitrage, we must have \( \mathbb{E}_t[P_t L_t] = 0 \) meaning that the inception transaction is the equilibrium transaction coinciding with the \( t \)-expected value under the Risk Neutral Measure of all discounted future transactions: defining \( S := S - X \) and \( C := -F \) this condition is equivalent to the fact that

\[
\tilde{S}_u \cdot C := \frac{S_u - C_u}{B_u} + \int_0^u \frac{dD_u^S \cdot C}{B_u}
\]

is a \( \mathbb{Q} \)-martingale where

\[
dD_u^S \cdot C := d\Phi_u + dC_u - c_u C_u \cdot du.
\]

Therefore, we can exploit the results of Proposition 3.2 for \( f = g = d \), and in particular (34), to obtain that

\[
S_t = B_t \mathbb{E}_t \left[ \frac{S_T}{B_T} + \int_t^T \frac{dD_u^S \cdot C}{B_u} \right]
\]

where

\[
dD_u := d\Phi_u - C_u \cdot (c_u - r_u) \cdot du = d\Phi_u + F_u - (c_u - r_u) \cdot du
\]

with \( D_0 = D_{0-} = 0 \). In case of continuous margin calls for any \( u \in [t, T] \) we have \( C_u = S_u + F_u = C_u^* \), where

\[
C_u^* = S_t (1 + \kappa_t (u - t)) = (1 + \alpha_t) X_u,
\]

so

\[
F_u = C_u^* - S_u = (1 + \alpha_t) X_u - S_u = -S_u + \alpha_t X_u
\]

and

\[
dD_u = d\Phi_u + (-S_u - + \alpha_t X_u - (c_u - r_u)) \cdot du = d\Phi_u + \alpha_t X_u - (c_u - r_u) \cdot du + S_u - (r_u - c_u) \cdot du
\]

so

\[
F_u = C_u^* - S_u = (1 + \alpha_t) X_u - S_u = -S_u + \alpha_t X_u
\]

and

\[
dD_u = d\Phi_u + (-S_u - + \alpha_t X_u - (c_u - r_u)) \cdot du = d\Phi_u + \alpha_t X_u - (c_u - r_u) \cdot du + S_u - (r_u - c_u) \cdot du
\]

where \( q_u := r_u - c_u \) and \( d\Pi_u := d\Phi_u + \alpha_t X_u - (c_u - r_u) \cdot du \), and we can also apply Proposition 2.26 with obvious substitutions \((S', q', \Phi') \Rightarrow (S, q, \Pi)\) and, since \( \mu := r - q = c \), writing also

\[
S_t = X_t + B_t^c \mathbb{E}_t \left[ \frac{S_T - X_T}{B_T} + \int_t^T \frac{d\Pi_u}{B_u} \right]
\]

\[
= (X_t - P^c_f(T) X_T) + B_t^c \mathbb{E}_t \left[ \frac{S_T}{B_T} + \int_t^T \frac{d\Pi_u}{B_u} \right]
\]

\[
= S_t \left\{ \frac{1}{1 + \alpha_t} \right\} \left[ 1 - \frac{P^c_f(T)}{P^c_f(T)} \right] + B_t^c \mathbb{E}_t \left[ \frac{S_T}{B_T} + \int_t^T \frac{d\Pi_u}{B_u} \right]
\]

since \( X_T \) is \( \mathcal{F}_T \)-measurable. With some algebra we also obtain

\[
S_t = \frac{1 + \alpha_t}{P^c_f(T)} B_t^c \mathbb{E}_t \left[ \frac{S_T}{B_T} + \int_t^T \frac{d\Pi_u}{B_u} \right] (59)
\]
If we also add the hypothesis of deterministic rates, we obtain a simplified forward price formula of asset $S$, for any $U \in (t,T)$:

$$F_t(U) := \mathbb{E}^U_t[S_U] = \mathbb{E}_t[S_U] = X_U + \mathbb{E}_t[S_U]$$

$$= X_U + \frac{1}{P^c_t(U)} \left\{ S_t - \int_t^U P^c_t(u) \mathbb{E}_t [d\Pi_u] \right\}$$

$$= \frac{S_t}{1 + \alpha_t} \left[ \frac{1}{P^c_t(U)} + \frac{\alpha_t}{P^c_t(U)} \right] - \frac{1}{P^c_t(U)} \left\{ \int_t^U P^c_t(u) \left\{ \mathbb{E}_t [d\Phi_u] + \alpha_t X_u - (\kappa_u - r_u) du \right\} \right\}$$

where the second equality is since interest rates are deterministic, the third since $X_U$ is $\mathcal{F}_t$-measurable, the second line is by (23) with the same mapping $(S^i, q^i, \Phi^i) \mapsto (\mathcal{S}, q, \Pi)$. Setting $c_u = \kappa_u$ for any $u \in [t,T],

$$F_t(U) = \frac{1}{P^c_t(U)} \left\{ S_t - \int_t^U P^c_t(u) \left\{ \mathbb{E}_t [d\Phi_u] + \alpha_t X_u - (\kappa_u - r_u) du \right\} \right\}$$

and if $\alpha_t \equiv 0$ we finally obtain:

$$F_t(U) = \frac{1}{P^c_t(U)} \left\{ S_t - \int_t^U P^c_t(u) \mathbb{E}_t [d\Phi_u] \right\}$$

which is the formula of the dirty forward price of a bond (see e.g. [TS12, ch. 13]) under repo rate continuous compounding, interpreting $S_t$ as the dirty price of the bond at time $t$ and $d\Phi_u$ as the (stochastic) bond coupon at time $u$. Under the same assumptions, using (37)-(58)-(57) and the continuity of process $X$, we have that for any $u \in [t,T]$:

$$dS_u = d\mathbb{S}_u + dX_u = r_u \mathbb{S}_u - du + dM_u - dD_u + dX_u$$

$$= r_u \mathbb{S}_u - du + dM_u - (d\Phi_u + \mathbb{S}_u - (r_u - \kappa_u) du) + dX_u$$

$$= \kappa_u \mathbb{S}_u - du + dM_u - d\Phi_u + (dX_u - \kappa_u X_u du)$$

$$= \kappa_u \mathbb{S}_u - du + dM_u - d\Phi_u$$

for some RCLL $\mathbb{Q}$-martingale driver $M$ with $M_0 = 0$. This dynamics is also coherent with the last forward price formula and with (59) in case $c = \kappa$: as a result of the repo transaction, in all these cases the repo rate substitutes the spot risk-free rate (a similar result is obtained also in [Pi10] with different arguments). Remark however that this dynamics is the result of all the simplifications above.

### 4.2. Securities Lending

We follow [Ch10] for the transaction description. The securities lending contracts are generally set under the master agreement of the International Securities Lending Association (ISLA); in particular we also refer to [ISLA10] for further contract details.

Securities lending or stock lending is defined as a temporary transfer of securities in exchange for collateral. It is not a repo in the normal sense; there is no sale or repurchase of the securities. The temporary use of the desired asset (the stock that is being borrowed) is reflected in a fixed fee payable by the party temporarily taking the desired asset, usually accruing daily as a basis point charge on the market value of the stock being lent, and generally payable in arrears on a monthly basis. The most common type of collateral is cash; however, it can happen that the collateral is represented by other securities to be given as collateral. In case of cash collateralization, the stock lender must pay the interest rates for the cash loan: these interest rates accrue and are paid with the same conventions as the lending fee, so that these payments can offset.
Most stock loans are on an “open” basis, meaning that they are confirmed (or terminated) each morning, although term loans also occur. As in a classic repo transaction, coupon or dividend payments that become payable on a security or bond during the term of the loan will be transferred from the stock borrower to the stock lender. Analogously, any coupon or dividend payments that become payable on the collateral (in case of collateral assets) during the term of the loan will be transferred from the stock lender to the stock borrower.

At the maturity of the deal (or in case of default by one of the counterparties), the stock lender receives back the stock and returns back the collateral to the stock borrower. The counterparty risk is reduced not only by the presence of the collateral, but also due to the margination process.

We write below the P&L discounted at date \( t \) of the stock lender in cases of cash collateralization for a number of stocks \( N = 1 \), in cases where both the asset and the collateral are quoted in domestic currency and that the collateralization process eliminates the counterparty risk:

\[
P\&L_t = -(S_t - C_t) + B_t \int_t^T \frac{d\Phi_u + \ell_u S_u - du + dC_u - cu C_{u-} du}{B_u} + B_t (S_T - C_T)
\]

where \( \ell \) is the stock loan fee, \( c \) is the interest rate of the cash collateralization, \( d\Phi_u \) is the dividend/coupon of the asset \( S \) paid at time \( u \) (recall that the stock lender receives the dividend from the stock borrower), \( C \) is the cash collateral value process (specifically, \( C_u \) is the cash loan of the stock seller at time \( u \)) and \( dC_u \) (with values in \( \mathbb{R} \)) represents the possible margin call of the collateral at time \( u \). We have three terms in the above formula: the first represents all transactions at inception date \( t \), the second represents all intermediate flows (discounted at \( t \)) in the interval \( [t, T] \) and the third is the closing of all cash/borrowing positions at maturity \( T \) (discounted at \( t \)). See also Remark 3.1 for encompassing the case of discrete collateralization: we do not take into account the fact that collateral interests and lending fees accumulates in a period (e.g. one month) and then are paid at the end of this period.

In order to avoid arbitrage, we must have \( \mathbb{E}_t[P\&L_t] = 0 \) meaning that the inception transaction is the equilibrium transaction coinciding with the \( t \)-expected value under the Risk Neutral Measure of all future transactions. It is easy to see that condition \( \mathbb{E}_t[P\&L_t] = 0 \) is verified if and only if

\[
\tilde{G}_t := \frac{(S_t - C_t)}{B_t} + \int_0^t \frac{dD^S_{u-}}{B_u}
\]

is a Risk-Neutral martingale, where we defined

\[
dD^S_{u-} := d\Phi_u + \ell_u S_u - du + dC_u - cu C_{u-} du
\]

and \( D^S_{0-} = 0 \). We can directly apply the results of Proposition 3.2 obtaining

\[
S_t = B_t \mathbb{E}_t \left[ \frac{S_T}{B_T} + \int_t^T \frac{dD_u}{B_u} \right]
\]

where, recalling (35),

\[
dD_u := d\Phi_u + \ell_u S_u - du - C_u - cu - (r_u - r_u) du
\]

with \( D_0 = D_{0-} = 0 \) and also

\[
\tilde{G}_t := \frac{S_t}{B_t} + \int_0^t \frac{dD_u}{B_u}
\]

is a Risk-Neutral martingale. Moreover, due to Proposition 3.6, the dynamics of \( S \) under the Risk Neutral Measure must be

\[
dS_u = r_u S_u - du + dM_u - dD_u
\]

where \( M \) is a \( \mathbb{Q} \)-RCLL martingale with \( M_0 = 0 \).
Moreover, due to (52), for $t \leq U \leq T$, the forward price of $S$ is:

$$F_t(U) := \frac{1}{P_t(U)} \left\{ S_t - B_t \mathbb{E}_t \left[ \int_t^U \frac{dD_u}{B_u} \right] \right\}.$$

In case of continuous margin calls $C_u = C_u^\ast := (1 + \alpha_u)S_u$ for any $u$, then we can also apply Proposition 3.16 obtaining

$$S_t = B_t^\ast \mathbb{E}_t \left[ \frac{S_T}{B_T^\ast} + \int_t^T \frac{d\Phi_u}{B_u^\ast} \right],$$

where

$$z_u = -\alpha_u r_u + (1 + \alpha_u) c_u - \ell_u$$

With deterministic rates, we obtain from (23) a simplified forward price formula

$$F_t(U) = \frac{1}{P_t(U)} \left\{ S_t - \int_t^U P_t^\ast(u) \mathbb{E}_t[|\Phi_u|] \right\}.$$

**Proposition 4.1.** In case of continuous proportional dividends $d\Phi_u = q_u S_u$ and in case $r = c$ (or with no collateralization) one obtains:

$$dS_u = (r_u - q_u - \ell_u) S_u - du + dM_u$$

Assuming also that $q, \ell$ are deterministic (while $r$ remains stochastic), we have

$$F_t(U) = \frac{S_t e^{-\int_t^U (q_u + \ell_u) du}}{P_t(U)}.$$

**Proof.** The SDE is by direct substitution. The forward price formula is derived as (23) with dividend rate $\tilde{q} := q + \ell$ and $\Phi = 0$. \qed

**Remark 4.2.** As a result, we obtained the well known dynamics of the asset with proportional dividends and lending fee. In market jargon, the rate $\ell$ is generally called the “repo rate”, even if this terminology is quite misleading.

### 4.3. Futures

We follow [HK04] for the deal description. We work on a Futures contract on underlying $S$. We define once for all a set of times $0 \leq t_0 \leq t \leq T$, where

- $0$ is the inception date of the Futures contract;
- $t_0$ is the date at which the investor enters into the Futures contract;
- $t$ is today, a generic time between $t_0$ and $T$;
- $T$ is the Futures contract expiry.

The Futures contract has the following properties:

- At every point in time $t \in [0, T]$, a Futures price process $f_t^T$ is quoted on the market;
- At reset date $f_T^T = S_T = F_T(T)$;
- As in the case of forward contracts, since the Futures contract holder does not own the underlying $S$, she does not receive any dividend eventually paid by the underlying.
- The deal is centralized: i.e. the counterparty of any Futures contract is the Clearing House (or central counterparty) and the deal is subject to margination in order to mitigate the investor’s default risk for the House. In fact, the investor opens at inception $t_0$ an account at the Clearing House paying to her some cash $C_{t_0}$ and opening a credit $C$ position with her of the same amount. We distinguish between:
− **Initial Margin account** (3): the movements of the investor to the credit \( C \), in particular \( C_{u} = 0 \) since at \( u = u_{o} \) the investor pays \( C_{u} \) and opens the account writing the credit \( C_{u} \) in her book. Whenever the balance drops below the *maintenance margin level*, the investor must bring the balance back up to the initial margin level again: hence at time \( u \) she enhances her credit of \( d \mathcal{I}_{u} > 0 \) receiving \( -d \mathcal{I}_{u} < 0 \) (so paying) the same amount of cash. On the other hand, whenever the balance exceeds the initial margin level, the investor has the right to (and therefore should) withdraw any excess credit amount above the initial margin: hence at time \( u \) the investor credit decreases of \( d \mathcal{I}_{u} < 0 \) and the investor receives an amount of cash of \( -d \mathcal{I}_{u} > 0 \).

− A **Variation Margin account** (2\( \mathfrak{M} \)): the movements of the Clearing House to \( C \), in particular \( 2 \mathfrak{M}_{t_{o}} = 0 \). Over the time the Futures price will change from the initial value \( f_{t_{0}}^{T} \), sometimes going up \( f_{t_{0}+\delta_{u}}^{T} - f_{t_{0}}^{T} > 0 \), sometimes going down \( f_{t_{0}+\delta_{u}}^{T} - f_{t_{0}}^{T} < 0 \): at the end of every day the Clearing checks the closing Futures price. If the price has gone up, the exchange credits to the investor this increase; if it has gone down it debits this decrease.

− Interest rate accrual: the Clearing must remunerate its debt at a rate of \( -c \) (possibly zero).

− Position closing: when the investor closes the contract at \( u \) she obtains back all remaining funds \( C_{u} \) of her credit with respect to the Clearing. Futures contacts are generally cash settled, but in the non-interest rates case they could be deliver underlying: in this case the investor should hold the contract until expiry \( u = T \), receiving back as usual \( C_{T} \), but also paying \( f_{T}^{T} \) of cash in order to obtain the underlying asset, whose value is \( f_{T}^{T} = F_{T}(T) = S_{T} \).

We will see in a moment that \( C_{T} \) can be calculated from (63), but forgetting here the initial margin account and the interest rate accrual, so setting \( C_{T} = 2 \mathfrak{M}_{T} \), the investor’s net cash position is \( C_{T} - f_{T}^{T} = 2 \mathfrak{M}_{T} - f_{T}^{T} = f_{T}^{T} - f_{t_{0}}^{T} - f_{T}^{T} = -f_{t_{0}}^{T} \) versus the stock obtained (similar to the long forward contract case where the holder pays \( F_{T}^{T} \) in order to obtain the stock).

To our knowledge, the best references that deal with Futures pricing are [Bj09]-[Du01] who do not take into account the initial margin and represents the margin account flows as a continuous cash flows (dividend) of the financial derivative: instead, we adapt the framework of [MP17] to this contract case obtaining different results (in particular the price of the Futures contract being different from zero).

Denoting with \( t_{i} \) the margin call times, the collateral account value can be written as:

\[
\begin{cases}
C_{t_{i}} = C_{t_{i-1}}(1 + c_{t_{i-1}}(t_{i} - t_{i-1})) + (3_{t_{i}} - 3_{t_{i-1}}) + (f_{t_{i}}^{T} - f_{t_{i-1}}^{T}) & i > 0 \\
C_{t_{0}} = 3_{t_{0}}
\end{cases}
\]  

(61)

The first line describes the dynamics of the collateral account: it accrues with interest rate \( c \) and grows with cash inflows/outflows of the investor (\( d \mathcal{I}_{t} \)) and of the Clearing (\( d 2 \mathfrak{M}_{t} \)). Note that, setting \( c = 0 \), we can prove recursively that we obtain

\[
\begin{cases}
C_{t_{i}} = 3_{t_{i}} + (f_{t_{i}}^{T} - f_{t_{i}}^{T}) & i > 0 \\
C_{t_{0}} = 3_{t_{0}}
\end{cases}
\]

which says that the collateral account value at margin date is exactly equal to the initial margin value plus the difference between the current and the inception quoted Futures price.

First, one can describe the dynamics of \( C \) thinking that all processes \( 3, c, C, 2 \mathfrak{M} \) are RCLL pure jump predictable processes with predictable jumps at margin call times \( t_{i} \); in particular

\[
2 \mathfrak{M}_{t} = \sum_{t_{o} < u \leq t} \Delta 2 \mathfrak{M}_{u} = f_{T}^{T}(t) - f_{t_{0}}^{T}
\]  

(62)

where \( \Delta 2 \mathfrak{M}_{u} = 2 \mathfrak{M}_{u} = 0 \) except on jump times \( t_{i} \), where \( \Delta 2 \mathfrak{M}_{t_{i}} = f_{t_{i}}^{T} - f_{t_{i-1}}^{T} \) and we denoted with \( \gamma(u) \) the greater margin call date \( t_{i} \) such that \( t_{i} \leq u \). The left limit is in order to guarantee
that \( \mathfrak{M} \) is predictable. Therefore, \( \text{d}\mathfrak{M}_u \) is different from \( \text{d}f_u^T \) which is a market quantity quoted in continuous time and not a simple pure jump process. In particular, we have from (61):

\[
C_t = C_{t_0}, \quad \mathfrak{M}_{t_0} = 0
\]

\[
dC_u = c_u C_u - \text{d}u + \text{d}M_u + \text{d}\mathfrak{M}_u
\]

Integrating in the interval \([t_0, t]\),

\[
C_t = C_{t_0} + \int_{t_0}^t c_u C_u - \text{d}u + \text{d}M_u - \mathfrak{M}_u
\]

\[
= \int_{t_0}^t c_u C_u - \text{d}u + \mathfrak{M}_u - f_{\gamma(t)} - f_{t_0}^T
\]

(63)

Now, defining \( \tilde{C}_t := C_t/B_t \), using (14) we have

\[
d\tilde{C}_u = \frac{\text{d}M_u + \text{d}\mathfrak{M}_u}{B_u} + \frac{C_u - (c_u - r_u) \text{d}u}{B_u}
\]

(64)

and integrating in \([t_0, \tau]\)

\[
\tilde{C}_\tau = \tilde{C}_{t_0} + \int_{t_0}^\tau \frac{\text{d}M_u + \text{d}\mathfrak{M}_u}{B_u} + \frac{C_u - (c_u - r_u) \text{d}u}{B_u}
\]

The P&L discounted at inception time \( t_0 \) of all transactions of the investor until target expiry \( \tau \) is:

\[
P\&L_{t_0} = -C_{t_0} + B_{t_0} \mathbb{E}_t \left[ \frac{\tilde{C}_\tau}{B_\tau} + \int_{t_0}^\tau \frac{-\text{d}M_u}{B_u} \right]
\]

\[
= B_{t_0} \mathbb{E}_t \left[ \int_{t_0}^\tau \frac{\text{d}M_u + C_u - (c_u - r_u) \text{d}u}{B_u} \right]
\]

where the first addend at the first line represents the opening of the credit at \( t_0 \), the second addend is the closing of the credit (the investor withdraws the cash in the account) and the third represents the initial margin account (possible) calls: remember that \( \text{d}\mathfrak{M}_u \) is not a cash flow for the investor (it is operated by the Clearing House on a segregated account). In the second line we substituted the expression of \( \tilde{C}_\tau \).

For a generic time \( t \) with \( t_0 \leq t \leq \tau \), the value of the contract is represented by the expected value of all discounted future transactions:

\[
V_t = B_t \mathbb{E}_t \left[ \frac{\tilde{C}_\tau}{B_\tau} + \int_{t}^{\tau} \frac{-\text{d}M_u}{B_u} \right]
\]

\[
= \tilde{C}_t + B_t \mathbb{E}_t \left[ \int_{t}^{\tau} \frac{\text{d}M_u + C_u - (c_u - r_u) \text{d}u}{B_u} \right]
\]

(65)

where the reference to \( t_0 \) is hidden both in the value of \( \tilde{C}_t \) (see (63)) and in the value of \( \mathfrak{M}_u \), recalling (62).

As in the previous section, we work under the hypothesis that the margination eliminates the counterparty risk for the Clearing House and that the House is default free. Since we assume continuous margin call, we assume also that \( f^T \) is continuous so that \( \mathfrak{M} \) stays predictable.

**Proposition 4.3.** In the above setting, we add the hypothesis of continuous margin calls so that \( \text{d}\mathfrak{M}_u = \text{d}f_u^T \) and \( \gamma(u) = u \). We assume that \( (f_u^T)_{u \in [0,T]} \) is a continuous semimartingale with decomposition \( f_u^T = f_0^T + A_u + M_u \) where \( A \) is a continuous process with finite variation and \( M \) is a continuous strict \( Q \)-martingale with \( A_0 = M_0 = 0 \). Then absence of arbitrage implies:

i) Perfect collateralization, i.e. \( V_u = C_u \) for all \( u \in [t_0, T] \);

ii) Collateral rate equal to the risk-free interest rate, i.e. \( c \equiv \gamma \);
iii) The Futures price \( f_u^T \) must be a \( \mathbb{Q} \)-martingale (i.e. \( A_u = 0 \) for all \( u \in [0, T] \)).

**Proof.** At inception, by construction \( V_{t_0} = C_{t_0} = \mathcal{J}_{t_0} \), which is the cash to enter in the Futures contract. At any time \( \tau \in [t_0, T] \) the investor can exit the contract obtaining \( C_\tau \), so \( V_\tau = C_\tau \). Since \( \tau \) is an arbitrary time by absence of arbitrage we have \( V_t = C_t \) for any \( t \). Hence, from this and from (65) we must have that \( \mathbb{E}_t[\tilde{G}_\tau] = \tilde{G}_t \), i.e. that \( \tilde{G} \) is a \( \mathbb{Q} \)-martingale, where we define

\[
\tilde{G}_t := \int_0^t \frac{df_u^T + C_u - (c_u - r_u)}{B_u} \, du
\]

In case \( \tilde{G} \) is a \( \mathbb{Q} \)-martingale, note also that \( \mathbb{E}_u[P\&L_{t_0}] = 0 \). In order for \( \tilde{G} \) to be a \( \mathbb{Q} \)-martingale we must have \( dA_u = -C_u - (c_u - r_u) \, du \). In this case

\[
A_t = \int_0^t dA_u = - \int_0^t C_u - (c_u - r_u) \, du.
\]

Now, in the market we have many Futures contracts on the same underlying and expiry but with different inception dates \( t_0^1, t_0^2, \ldots \) since all market participants can enter in the Futures contract at any time before the expiry. For all of them, the above condition should be valid. However recalling (63), for the same time \( u \) the value of \( C_u \) with inception time \( t_0^i \) is different from the value of \( C_u \) with inception time \( t_0^j \) for \( i \neq j \), while \( f_u^T \) is the unique market quote among all inception dates. Hence we should set \( c_u = r_u \) in order to make the second addend of \( \tilde{G} \) disappear. This implies that \( A_t = 0 \) for all \( t \), and so \( f_u^T = f_0^T + M_t \) is a \( \mathbb{Q} \)-martingale.

**Remark 4.4.** We have an example of perfect collateralization: if a counterparty wants to enter exactly in the same position as the investor, she must pay \( C_t \). In particular, under all hypotheses/constraints of the above proposition we have

\[
V_t = C_t + B_t \mathbb{E}_t \left[ \int_0^T \frac{df_u^T}{B_u} \right] = C_t = \int_0^t r_u C_u \, du + \mathcal{J}_t + (f_u^T - f_0^T)
\]

where the last equality is by (63), in particular: the first addend is the interest carry accrued from collateral, the second addend is the current level of the initial margin account and the third addend is the level of the margin account due to market moves. This result can also be compared with the price of an uncollateralized forward contract (with no counterparty risk) struck at par at issue date \( t_0 \) that has value at time \( t > t_0 \):

\[
V_t = P_t(T) [F_t(T) - F_{t_0}(T)]
\]

that should be compared with the long uncollateralized Futures contract value which is \( V_t = f_u^T - f_0^T \).

Finally, the result does not change if \( \tau \) becomes a (stochastic) stopping time of an American option:

\[
V_t = \sup_{\tau \in \mathcal{T}} B_t \mathbb{E}_t \left[ \frac{C_\tau}{B_\tau} + \int_\tau^T \frac{-d\mathcal{J}_u}{B_u} \right] = C_t + \sup_{\tau \in \mathcal{T}} B_t \mathbb{E}_t \left[ \int_\tau^T \frac{df_u^T}{B_u} \right] = C_t
\]

where \( \mathcal{T} \) is the set of all exercise strategies with values in \([t_0, T]\) and we used (65) for \( r = c \): since \( f_u^T \) is a \( \mathbb{Q} \)-martingale, all exercise strategies must give the same price \( V_t = C_t \).

Moreover, since \( f_u^T \) is a \( \mathbb{Q} \)-martingale,

\[
f_t^T = \mathbb{E}_t [f_T^T] = \mathbb{E}_t [S_T] = \mathbb{E}_t [F_T(T)]
\]
and therefore the Futures price can be seen as a convexity adjusted Forward price: in fact the last expected value is taken under the “unnatural” measure $\mathcal{Q}$ instead of the natural $\mathcal{Q}^{T}$ one (the measure under which $F$ is a martingale).

5. Conclusion

We refer to Section 1.1 for a list of the achievements of this paper which is mainly targeted for practitioners in the financial industry: we tried to make it the most linear, rigorous, self-contained and (hopefully) didactic as possible. Moreover, we tried to never take anything for granted and to make a (perhaps small) step further in understanding the topics of the paper. We hope that the efforts of this approach were visible for the reader especially for Section 2 (a brief but operationally comprehensive tour of the No-Arbitrage theory for dividend paying assets) and for Section 4 (where we worked on applications studying from scratch the technicalities of the termsheets). A further development would be to extend the same approach to all other valuation adjustments.

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Appendix

A. Semi-martingales

We briefly enumerate some results of semimartingales theory in order to let the paper be almost self-contained: only few propositions with a detailed proof and no indication of a source are, to the best of our knowledge, original.

A function $f$ is RCLL ("right continuous with left limits") or càdlàg (from French acronym “continue à droite, limitée à gauche”) in a domain $D$ if it is right continuous and has finite left limits in $D$. A function $f$ is LCRL ("left continuous with right limits") or càglàd (“continue à gauche, limitée à droite”) in a domain $D$ if it is left continuous and has finite right limits in $D$.

For a RCLL function, we define also, for $a,b \in \mathbb{R}$ and RCLL functions $f,g$:

$$\Delta f(t) := f(t) - f(t-)$$

where

$$f(t-) := \lim_{h \to 0} f(t-h),$$

Remark also that the jump operator is linear (straightforward by definition): for $a,b \in \mathbb{R}$ and RCLL functions $f,g$:

$$\Delta \left( af(t) + bg(t) \right) = a \Delta f(t) + b \Delta g(t)$$  \hspace{1cm} (A.1)

and (again straightforward by definition)

$$\Delta \left( f(t)g(t) \right) = f(t-) \Delta g(t) + g(t-) \Delta f(t) + \Delta f(t) \Delta g(t)$$  \hspace{1cm} (A.2)

A jump processes is a process with RCLL paths. We then enounce the following theorem, which is a cornerstone of jump process theory, stating that the number of big jumps of RCLL functions is finite and that their number of jumps is finite or countable.

**Theorem A.1 (From [Ap09]).** If $f : D \mapsto \mathbb{R}$ with $D \subseteq \mathbb{R}$ is a RCLL function then

i) For each $k > 0$, the set $S_k = \{ t \in D, \Delta f(t) > k \}$ is finite.

ii) The set $S = \{ t \in D, \Delta f(t) \neq 0 \}$ is at most countable.

**Proposition A.2 (From math.stackexchange.com).** Let $f : [a, b] \mapsto \mathbb{R}$. Define $g : (a, b] \mapsto \mathbb{R}$ with $g(t) := f(t-)$, then $g$ is left continuous for any $t \in (a, b]$.

**Proof.** $g(t) = g(t-)$ can be proved using the sequential definition of of left continuity, i.e. we should prove that (from [Ap09]) for all sequences $(s_n, n \in \mathbb{N})$ in $(a, b)$ with each $s_n < t$ and $\lim_{n \to \infty} s_n = t$ we have that $\lim_{n \to \infty} g(s_n) = g(t)$.

Using the definition of $g(t) := f(t-)$, given $\epsilon > 0$ there is $\delta > 0$ such that

$$|f(s) - g(t)| \leq \epsilon$$

for all $t - \delta \leq s < t$. Since $s_n \to t-$, there is $n^*$ such that $t - \delta \leq s_n < t$ for all $n \geq n^*$. Fix $n \geq n^*$. Then for $t - \delta < s_n$ we have

$$|f(s) - g(t)| \leq \epsilon$$

or equivalently,

$$g(t) - \epsilon \leq f(s) \leq g(t) + \epsilon$$

Letting $s \to s_n-$ in the previous inequality, we get

$$g(t) - \epsilon \leq g(s_n) \leq g(t) + \epsilon$$

so we have $|g(s_n) - g(t)| \leq \epsilon$ for all $n \geq n^*$.
Remark A.3. From the previous proposition, we have, for $h > 0$:
\[
f(t -) := \lim_{h \to 0} g(t - h) := g(t) = f(t-)
\]
where used the left continuity of $g$. Then also:
\[
\Delta f(t-) := f(t-)-f(t-\cdot) = f(t-)-f(t-) = 0.
\]

Proposition A.4. Let $f$ be a RCLL function such that $f : D \mapsto \mathbb{R}$ with $D \subseteq \mathbb{R}$. Then, for $t, T \in D$
\[
\int_t^T f(u-) \, du = \int_t^T f(u) \, du
\]

Proof. See Remark 11.25 of [Ru76]; this is a more general result saying that the Lebesgue integral does not “see” discontinuities of null Lebesgue-measure. □

Definition A.5 (From [JYC09]). Let $\mathcal{F}$ be a given filtration.

i) The optional $\sigma$-algebra $\mathcal{O}$ is the smallest $\sigma$-algebra on $\mathbb{R}^+ \times \Omega$ generated by RCLL $\mathcal{F}$-adapted processes (considered as mappings on $\mathbb{R}^+ \times \Omega$).

ii) The predictable $\sigma$-algebra $\mathcal{P}$ is the smallest $\sigma$-algebra generated by the $\mathcal{F}$-adapted LCRL processes. The inclusion $\mathcal{P} \subset \mathcal{O}$ holds.

A process is said to be predictable (resp. optional) if it is measurable with respect to the predictable (resp. optional) $\sigma$-field.

If a process is left continuous (specifically LCRL) then it is predictable but the opposite is not always true: as a very simple example of a predictable right continuous (specifically RCLL) process, think to process $X := (X_t)_{t \geq 0}$ with $X_t = Z 1_{t \geq \tau}$ where $\tau \in [0, +\infty)$ and $Z$ is a random variable $\mathcal{F}_{\tau-}$-measurable (the example is valid even in cases where $\tau$ is a predictable stopping time).

Proposition A.6 (From [JS03]). If $X$ is a RCLL adapted process, then $X_- := (X_{u-})_{u \geq 0}$ is a predictable process; moreover, if $X$ is predictable, then $\Delta X_u$ is predictable for any $u \geq 0$.

Definition A.7 (From [JYC09]). An $\mathcal{F}$-semimartingale is a RCLL process $X$ which can be written as $X_t = X_0 + M_t + A_t$ where $M$ is an $\mathcal{F}$-local martingale and where $A$ is an $\mathcal{F}$-adapted RCLL process with finite variation, and $M_0 = A_0 = 0$.

In general, the decomposition of a semimartingale is not unique; we shall speak about decompositions of semimartingales. It is necessary to add some conditions on the finite variation process to get the uniqueness.

Definition A.8 (From [JYC09]). A special semimartingale is a semimartingale where $A$ (the finite variation part) is predictable. Such a decomposition $X = M + A$ with $A$ predictable, is unique. We call it the canonical decomposition of $X$, if it exists.

Definition A.9 (From [JYC09]). Let $X$ be a semimartingale such that $\forall t \geq 0$, $\sum_{0 < s \leq t} |\Delta X_s| < \infty$. Then process $\widetilde{X}_t := X_t - \sum_{0 < s \leq t} \Delta X_s$ is a continuous semimartingale with unique decomposition $\tilde{X} = M + A$ where $M$ is a continuous local martingale and $A$ is a continuous process with bounded variation. The continuous martingale $M$ is called the continuous local martingale part of $X$ and it is denoted by $X^c$.

Definition A.10 (From [JYC09]). A process $H$ is locally bounded if there exists a sequence of stopping times $(\tau_n)_{n \geq 1}$ with $\tau_n > 0$ increasing to $\infty$ a.s. such that for each $n \geq 1$, $(H_{t \wedge \tau_n})_{t \geq 0}$ is bounded.
As we read in [JYC09], if $X$ is a semimartingale with decomposition $M + A$, then for any predictable locally bounded process $H$, we can define the process
\[
(H \ast X)_t := \int_{[0,t]} H_u \, dX_u = \int_0^t H_u \, dM_u + \int_0^t H_u \, dA_u
\]
where the second addend is a Stieltjes integral and the integral is not dependent on the decomposition of $X$ (see [Pr04] for a more technical and detailed construction of this integral).

**Proposition A.11 (From [Pr04, JYC09]).** Let $H, K$ be locally bounded predictable processes and $X, Y$ semimartingales. Note that:

1. The process $(H \ast X)_t$ is a semimartingale, in particular it is a RCLL adapted process.
2. The process $(H \ast X)_t$ does not depend on the decomposition of the semimartingale $X$.
3. Bilinearity:
   \[
   (H + K) \ast X = H \ast X + K \ast X
   \]
   and
   \[
   H \ast (X + Y) = H \ast X + H \ast Y
   \]
4. Associativity:
   \[
   H \ast (K \ast X) = (HK) \ast X
   \]
   which is, defining $Z_t := \int_0^t K_u \, dX_u$, then
   \[
   \int_0^t H_u \, dZ_u = \int_0^t H_uK_u \, dX_u
   \]
5. The jump process of the stochastic integral of $H$ with respect to $X$ is equal to $H$ times the jump process of $X$:
   \[
   \Delta(H \ast X)_t = H_t \Delta X_t. \tag{A.3}
   \]
6. We have $H \ast X^c = (H \ast X)^c$.
7. We also have that
   \[
   \int_{[0,t]} H_u \, dX_u = H_0 \Delta X_0 + \int_{(0,t]} H_u \, dX_u = H_0 X_0 + \int_{(0,t]} H_u \, dX_u
   \]
   since $X_{0-} = 0$ by convention.
8. Let $\tau$ be a stopping time
   \[
   (H \ast X)_{t \land \tau} = (H \mathbb{1}_{[0,\tau]}) \ast X = H \ast (X^\tau)
   \]
   where we denoted the stopped process $X_t^\tau := X_{t \land \tau}$.
9. Let $M$ be a local martingale, then $H \ast M$ is a local martingale (Theorem 29 of [Pr04]).
10. Let $M$ be a square integrable martingale, and $Z$ be a bounded predictable process. Then $Z \ast M$ is a square integrable martingale (Theorem 11 of [Pr04]).
11. Let $M$ be a square integrable martingale, and $Z$ be predictable such that $\int_0^t Z_u^2 \, d[M,M]_u < \infty$ a.s. for each $t$. Then $Z \ast M$ is a square integrable martingale (Lemma p. 171 of [Pr04]).

**Definition A.12 (Quadratic Covariation, from [Pr04]).** Given two semimartingales $X, Y$, the Quadratic Covariation process $[X,Y]$ is the semimartingale defined by
\[
[X,Y]_t := X_t Y_t - X_0 Y_0 - \int_0^t Y_u^+ \, dX_u - \int_0^t X_u^- \, dY_u. \tag{A.4}
\]
Since by definition \([X,Y]_0 = 0\), integrating both \([X,Y]_t\) and \((XY)_t\) allows one to have the differential form

\[
d[X,Y]_t = (XY)_t - Y_{t-} dX_u - X_{t-} dY_t
\]  

(A.5)

We will also denote the Quadratic Variation \([X,X] \equiv [X]\).

**Corollary A.13 (Product differentiation rule).** If \(X, Y\) are semimartingales then

\[
d(\text{XY})_t = Y_{t-} dX_t + X_{t-} dY_t + d[X,Y]_t
\]  

(A.6)

**Proof.** Restatement of (A.5), it could also be proved from (A.14). \(\square\)

**Proposition A.14 (from \([\text{Pr04, JYC09}]\)).** The Quadratic Covariation has the following properties:

1. The map \((x,y) \mapsto f(x,y)\) with \(f(x,y) \equiv [x,y]\) is symmetric (direct application of the definition).
2. The map \((x,y) \mapsto f(x,y)\) with \(f(x,y) \equiv [x,y]\) is bilinear, i.e. for semimartingales \(X^i, Y^j\) and \(a_i, b_j \in \mathbb{R}\)

\[
\left[ \sum_{i} a_i X^i, \sum_{j} b_j Y^j \right]_t = \sum_{i} \sum_{j} a_i b_j [X^i, Y^j]_t
\]

(by direct application of (A.7)).

3. From the properties above we have the two polarization identities:

\[
[X,Y] = \frac{1}{4} \left( [X+Y,Y+Y] - [X-Y,X-Y] \right) = \frac{1}{2} \left( [X+Y,Y+Y] - [X,Y] - [Y,Y] \right)
\]

4. \([X,Y]\) is a nonanticipating RCLL process with paths of finite variation (this follows from the polarization identity, as \([X,Y]\) is the difference of two increasing functions).

5. Take a time grid \(\pi^k = \{0 = t^k_0 < t^k_1 < \cdots < t^k_{n+1} = T\}\), the discrete approximation below converges in probability to \([X,Y]\) uniformly on \([0,T]\):

\[
\sum_{t^k_i \in \pi^k \cap \left] t^k_{i-1}, t^k_i \right]} (X_{t^k_{i+1}} - X_{t^k_i})(Y_{t^k_{i+1}} - Y_{t^k_i}) \xrightarrow{i\to \infty} \frac{1}{2} \sum_{i=0}^{n} \Delta X_i \Delta Y_i
\]  

(A.7)

over all partitions \(\pi^k\). Some reference presents this limit as the definition of Quadratic Covariation.

6. The jumps of the Quadratic Covariation process occur only at points where both processes have jumps,

\[
\Delta [X,Y]_t = \Delta X_t \Delta Y_t
\]  

(A.8)

7. If one of the processes \(X\) or \(Y\) is of (locally) finite variation, then the sum

\[
\sum_{0 < u \leq t} |\Delta X_u| |\Delta Y_u|
\]

is almost surely finite and

\[
[X,Y]_t = \sum_{0 < u \leq t} \Delta X_u \Delta Y_u.
\]  

(A.9)

8. Defining \(\tilde{X}_t := \int_0^t H_u \, dX_u\) and \(\tilde{Y}_t := \int_0^t K_u \, dY_u\) then

\[
[\tilde{X}, \tilde{Y}]_t = \left[ \int_0^t H_u \, dX_u, \int_0^t K_u \, dY_u \right]_t = \int_0^t H_u K_u \, d[X,Y]_u.
\]

For this reason, the following formal calculation rules can be applied:

\[
d[\tilde{X}, \tilde{Y}]_t = [d\tilde{X}_t, d\tilde{Y}_t] = [H_t \, dX_t, K_t \, dY_t] = H_t K_t \, d[X,Y]_t
\]  

(A.10)
Proposition A.15 (From [JYC09]). Let $A$ be a finite variation process and $X$ a semimartingale:

$$\int_0^t \Delta X_u \, dA_u = \sum_{0<u\leq t} \Delta X_u \Delta A_u = [X,A]_t$$

Then

$$d(A_t X_t) = A_t \, dX_t + X_t \, dA_t$$

Proposition A.16. Define with $\tilde{X}_t := \sum_i \int_0^t H^i_u \, dX^i_u$ and $\tilde{Y}_t := \sum_j \int_0^t K^j_u \, dY^j_u$ where $X^i, Y^j$ are semimartingales and $H^i, K^j$ are locally bounded predictable processes. Then

$$[\tilde{X}, \tilde{Y}]_t = \sum_{i,j} \sum_{i<j} \int_0^t H^i_u K^j_u \, d[X^i,Y^j]_u$$

or

$$d[\tilde{X}, \tilde{Y}]_u = \left[ \sum_i H^i_u \, dX^i_u, \sum_j K^j_u \, dY^j_u \right] = \sum_{i,j} H^i_u K^j_u \, d[X^i,Y^j]_u$$

Proof. We have:

$$[\tilde{X}, \tilde{Y}]_t := \left[ \sum_i \int_0^t H^i_u \, dX^i_u, \sum_j \int_0^t K^j_u \, dY^j_u \right]_t$$

$$= \sum_{i,j} \left[ \int_0^t H^i_u \, dX^i_u, \int_0^t K^j_u \, dY^j_u \right]_t$$

$$= \sum_{i,j} \int_0^t H^i_u K^j_u \, d[X^i,Y^j]_u$$

where in the second line we used the bilinearity of Quadratic Covariation A.14 and in the third line (A.10).

Proposition A.17 (Covariation of local martingales, from [Pr04]). If $X$ and $Y$ are two locally square integrable local martingales. Then $[X,Y]$ is the unique adapted RCLL process $A$ with paths of finite variation on compacts satisfying:

i) $XY - A$ is a local martingale;

ii) $\Delta A = \Delta X \Delta Y$

iii) $A_0 = X_0 Y_0$.

Definition A.18 (Compensator, from [JYC09]). An adapted increasing process $A$ is said to be a compensator for the semimartingale $X$ if $X - A$ is a local martingale.

For example if $X$ is a local martingale, the process $[X,X]$ is a compensator for $X^2$. In general, a semimartingale admits many compensators. If there exists a predictable compensator, then it is unique (among predictable compensators).

We now introduce the Predictable Quadratic Covariation, in some references also called Sharp Brackets/Angle Brackets/Conditional Quadratic Covariation.

Definition A.19 (Predictable Quadratic Covariation, from [Pr04]). Let $X$ be a semimartingale such that its Quadratic Variation process $[X,X]$ is of locally integrable variation. Then the Predictable Quadratic Variation of $X$, denoted $\langle X,X \rangle$ exists and it is defined to be the compensator of $[X,X]$. The Predictable Quadratic Covariation $\langle X,Y \rangle$ can be defined as the compensator of $[X,Y]$ provided of course that $[X,Y]$ is of locally integrable variation.
Proposition A.20 (From [Kl05]). If \( X \) is a continuous semimartingale with integrable Quadratic Variation, then \( \langle X, X \rangle = [X, X] \), and there is no difference between the sharp and the square bracket processes.

The Predictable Variation is inconvenient since, unlike the Quadratic Variation, it doesn’t always exist. Moreover, while \([X, X], [X, Y], \) and \([Y, Y]\) all remain invariant with a change to an equivalent probability measure, the sharp brackets, in general, change with a change to an equivalent probability measure and may even no longer exist.

**Proposition A.21** (Covariation process decomposition, from [Pr04]). If \( X \) and \( Y \) are semimartingales, their Quadratic Covariation is

\[
[X, Y]_t = [X, Y]_t^c + \sum_{0 < u \leq t} (\Delta X_u)(\Delta Y_u)
\]

or

\[
d[X, Y]_t = d[X, Y]_t^c + \Delta X_u \Delta Y_u.
\]

(A.11)

where \([X, Y]^c\) denotes the continuous local martingale part of \([X, Y]\): see Definition A.9.

**Proposition A.22** (From [Pr04]). For semimartingales \( X, Y \), we have the following equalities:

\[
[X, Y]^c = [X^c, Y^c] = \langle X^c, Y^c \rangle
\]

(A.12)

If both \( X \) and \( Y \) are continuous

\[
[X, Y]^c = [X, Y] = \langle X, Y \rangle
\]

Lemma A.23 (Itô Formula, from [JYC09]). Let \( X = (X_1, \ldots, X_n) \) be a semimartingale vector process and \( f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R} \) with \( f \in C^{1,2} \). Then,

\[
f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial}{\partial x} f(u, X_u) \, du + \int_0^t \sum_{i=1}^n \partial_i f(u, X_u) \, dX_i^u
\]

(A.13)

\[
+ \frac{1}{2} \int_0^t \sum_{i,j=1}^n \partial_{i,j} f(u, X_u) \, d[X_i, X_j]_u + \sum_{0 < u \leq t} \left\{ f(u, X_u) - f(u, X_u^-) - \sum_{i=1}^n \partial_i f(u, X_u) \Delta X_i^u \right\}
\]

or in differential form

\[
df(u, X_u) = \partial_x f(u, X_u^-) \, du + \sum_{i=1}^n \partial_i f(u, X_u) \, dX_i^u
\]

(A.14)

\[
+ \left\{ f(u, X_u) - f(u, X_u^-) - \sum_{i=1}^n \partial_i f(u, X_u) \Delta X_i^u \right\}
\]

\[
+ \left\{ f(u, X_u) - f(u, X_u^-) - \sum_{i=1}^n \partial_i f(u, X_u) \Delta X_i^u \right\}
\]

Proposition A.24. Define \( Y_t := f(t, X_t) \) where \( f : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R} \) with \( f \in C^{1,2} \) and let \( X = (X_1, \ldots, X_n) \) be a semimartingale vector process. Then, for another scalar semimartingale process \( Z \)

\[
d[Y, Z]_t = \sum_{i=1}^n \partial_i f(t, X_t) \, d[X^i, Z]_t^c + \left\{ f(u, X_u) - f(u, X_u^-) \right\} \Delta Z_u
\]

(A.15)

which is the continuous Quadratic Variation result plus the co-jump term.
Proof. Denoting with \( f^- := f(u, X_{u-}) \) and using the Itô formula

\[
[Y, Z]_t = \left[ Z, \int_0^t \sum_{i=1}^n \partial_{x_i} f^- \, dX^i_u + \frac{1}{2} \int_0^t \sum_{i,j=1}^n \partial^2_{x_i x_j} f^- \, d[X^i, X^j]_u + \sum_{0 < u \leq t} \left\{ f^- - \sum_{i=1}^n \partial_{x_i} f^- \Delta X^i_u \right\} \right]_t
\]

\[
= \int_0^t \sum_{i=1}^n \partial_{x_i} f^- \, d[X^i, Z]_u + \frac{1}{2} \int_0^t \sum_{i,j=1}^n \partial^2_{x_i x_j} f^- \, d([X^i, X^j]_u, Z]_u
\]

\[
+ \left[ \sum_{0 < u \leq t} \left\{ f^- - \sum_{i=1}^n \partial_{x_i} f^- \Delta X^i_u \right\}, Z \right]_t
\]

where in the first equality we used that the time integral is continuous with finite variation, in the second equality we used the Proposition A.16, while in the third equality \([X^i, X^j]_u\) is continuous with finite variation. Define with

\[
\Delta A_u := f(u, X_u) - f(u, X_{u-}) - \sum_{i=1}^n \partial_{x_i} f(u, X_{u-}) \Delta X^i_u
\]

and, under Stieltjes integration

\[
A_t := \sum_{0 < u \leq t} \Delta A_u = \int_0^t dA_u
\]

is a (finite variation) pure jump RCLL process with jumps \( \Delta A_u \) equal to zero except on \( X \) jump times (recall that \( f \in C^{1,2} \)). Therefore, the last addend of the above derivation is

\[
[A, Z]_t = \left[ \int_0^t dA_u, \int_0^t dZ_u \right]_t
\]

\[
= \int_0^t d[A, Z]_u
\]

\[
= \sum_{0 < u \leq t} \Delta A_u \Delta Z_u
\]

where the last line is since \( A \) is of finite variation. The result follows from (A.11). \( \Box \)

We then enounce the Girsanov Theorem generalized to semimartingales, following [Pr04] p. 131 (see also [JYC09] p. 534). Let \( X \) be a semimartingale on a space \((\Omega, \mathcal{F}, \mathbb{P})\) satisfying the usual hypotheses. Let \( Q \sim \mathbb{P} \), then there exist a \( \mathbb{P} \)-integrable random variable \( \frac{dQ}{d\mathbb{P}} \) such that \( \mathbb{E}_\mathbb{P}^\mathbb{P}[\frac{dQ}{d\mathbb{P}}] = 1 \). We let

\[
L_t := \mathbb{E}_\mathbb{P}^\mathbb{P} \left[ \frac{dQ}{d\mathbb{P}} \right]
\]

be the right continuous version. Then \( (L_t)_{t \geq 0} \) is a uniformly integrable martingale, hence a semimartingale. Note that since \( Q \) is equivalent to \( \mathbb{P} \) then \( \frac{d\mathbb{P}}{dQ} \) is \( Q \)-integrable and \( \frac{d\mathbb{P}}{dQ} = (\frac{dQ}{d\mathbb{P}})^{-1} \).

Lemma A.25. Let \( Q \sim \mathbb{P} \), and \( L \) defined as above. An adapted RCLL process \( M \) is a \( Q \)-local martingale if and only if \( ML \) is a \( \mathbb{P} \)-local martingale.

Theorem A.26 (Generalized Girsanov Theorem). Let \( Q \sim \mathbb{P} \). Let \( X \) be a semimartingale under \( \mathbb{P} \) with decomposition \( X = M^\mathbb{P} + A^\mathbb{P} \). Then \( X \) is also a semimartingale under \( Q \) and has a decomposition \( X = M^Q + A^Q \) with

\[
M^\mathbb{P}_0 = A^\mathbb{P}_0 = M^Q_0 = A^Q_0 = 0
\]

and where \( M^Q \) is a \( Q \)-local martingale and \( A^Q \) is a \( Q \)-finite variation process.
i) General result, optional version:

\[ dM_t^Q = dM_t^P - \frac{d[M^P, L_t]}{L_t} \]

and

\[ dA_t^Q = dA_t^P + \frac{d[M^P, L_t]}{L_t} \]

ii) Predictable version: If \([X, L]\) is \(\mathbb{P}\)-locally integrable (which implies that \(\langle X, L \rangle\) exists) then

\[ dM_t^Q = dM_t^P - \frac{d\langle M^P, L_t \rangle}{L_t} \]

and

\[ dA_t^Q = dA_t^P + \frac{d\langle M^P, L_t \rangle}{L_t} \]

**Corollary A.27.** A semimartingale \(X\) of finite variation with null local martingale part, i.e. a semimartingale with decomposition \(X = 0 + A\), has the same dynamics under \(\mathbb{P}\) and under \(\mathbb{Q}\).

### B. Basic Dividend Models

As a follow up of Section 2.2.2 we analyze two basic deterministic dividend models: the continuous proportional case and the lump dividend case. We calculate the forward price and check that the deflated gain process is effectively a local martingale in both cases.

#### B.1. Deterministic Continuous Proportional Dividends

We set

\[ dD_t = q_t S_t \, dt \]

with \(q\) deterministic, this means that the dividend in the \(dt\) interval is proportional to the asset value at time \(t\) and to the length of the interval itself. This is the easy case: the Risk Neutral dynamics, recalling (12)-(13) becomes

\[ dS_t = (r_t - q_t) S_t \, dt + dM_u \]

Moreover, using (23), the forward price becomes

\[ F_t(T) = \frac{S_t e^{-\int_0^T q_u \, du}}{F_t(T)} \]

As an exercise, we can check that the deflated gain process (9) is indeed a martingale with the chosen dividend model. In this setting, the process \(Z_t := e^{\int_0^t q_u \, du} S_t\) has dynamics

\[ dZ_t = r_t Z_{t-} \, dt + dM_u \]

and therefore \(\frac{Z_t}{T}\) is a \(\mathbb{Q}\)-martingale (being driftless and by item 10 of Proposition A.11). We can use this property, and the fact that \(q\) is deterministic, to prove that \(\tilde{G}_t = \mathbb{E}_t[\tilde{G}_T]\) for this specific
case. We have

\[
E_t \left[ \int_t^T \frac{dD_u}{B_u} \right] = E_t \left[ \int_t^T \frac{q_u S_{u-} \, du}{B_u} \right] \\
= \int_t^T q_u e^{-\int_u^T q_v \, dv} E_t \left[ \frac{Z_{u-}}{B_u} \right] \, du \\
= \frac{Z_t}{B_t} \int_t^T q_u e^{-\int_u^T q_v \, dv} \, du \\
= \frac{Z_t}{B_t} \left[ e^{-\int_u^T q_v \, dv} \right]_{u=t}^{u=T} \\
= \frac{S_t}{B_t} \left( 1 - e^{-\int_t^T q_v \, dv} \right)
\]

and since

\[
E_t \left[ \frac{S_T}{B_T} \right] = E_t \left[ \frac{e^{-\int_0^T q_u \, du} Z_T}{B_T} \right] \\
= e^{-\int_0^T q_u \, du} E_t \left[ \frac{Z_T}{B_T} \right] \\
= e^{-\int_0^T q_u \, du} \frac{Z_t}{B_t} \\
= e^{-\int_0^T q_u \, du} \frac{S_t}{B_t}
\]

we have that

\[
E_t \left[ \tilde{G}_T \right] = E_t \left[ \frac{S_T}{B_T} + \int_t^T \frac{dD_u}{B_u} \right] + \int_0^t \frac{dD_u}{B_u} \\
= e^{-\int_0^T q_u \, du} \frac{S_t}{B_t} + \int_0^t \frac{dD_u}{B_u} + e^{-\int_0^T q_u \, du} \frac{S_t}{B_t} \\
= \frac{S_t}{B_t} + \int_0^t \frac{dD_u}{B_u} =: \tilde{G}_t
\]

B.2. Deterministic Discrete Lump Dividends

We set

\[
D_t = \sum_{i=1}^{+\infty} \phi_{\tau_i} \mathbb{1}_{0 < \tau_i \leq t}
\]

where \(\phi_{\tau_i}\) is the dividend paid at time \(\tau_i\) for an increasing sequence of stopping times \(\tau_1 < \tau_2 < \ldots\). We have

\[
dD_u = \sum_{i=1}^{+\infty} \phi_u \, d\Theta_{\tau_i}(u)
\]

where we recall that \(\Theta_T(u)\) is the Heaviside function centered at \(T\): this is a Stieltjes integral (see e.g. [Pa11]). In order to compact the notation we define the counting process \(N_t := \sum_{i=1}^{+\infty} \mathbb{1}_{\tau_i \leq t}\) and rewrite

\[
D_t = \sum_{i=1}^{N_t} \phi_{\tau_i} \quad dD_t = \phi_u \, dN_u
\]
Notice that \( D_t = \sum_{0 < u \leq t} \Delta D_u \) is a RCLL pure jump process with jumps

\[
\Delta D_u = \sum_{i=1}^{+\infty} \phi_u \Delta \Theta_{\tau_i}(u) = \phi_u \Delta N_u
\]

In this case \( \Delta S_t = (\ldots) - \Delta D_t \) from which it is clear that \( S_t \) drops with dividends \( \phi_t \) at times \( t = \tau_i \) for some \( i \).

Prefer to stay simple, we decide that jump times \( \tau_i \)'s and jump size \( \phi \) are deterministic, hence \( N \) is a deterministic process: the generalization to a Compound Poisson Process would be straightforward. We have

\[
V^D_t := B_t E_t \left[ \int_t^T \frac{dD_u}{B_u} \right] = B_t E_t \left[ \sum_{t < u \leq T} \frac{\Delta D_u}{B_u} \right]
\]

\[
= B_t E_t \left[ \sum_{t < u \leq T} \frac{\phi_u \Delta N_u}{B_u} \right]
\]

\[
= B_t E_t \left[ \sum_{j=N_t+1}^{N_T} \frac{\phi_{\tau_j}}{B_{\tau_j}} \right]
\]

\[
= \sum_{i=N_t+1}^{N_T} \phi_{\tau_i} P_i(\tau_i)
\]

hence the forward price writes

\[
F_t(T) = \frac{S_t - \sum_{i=N_t+1}^{N_T} \phi_{\tau_i} P_i(\tau_i)}{P_t(T)}
\]

which is the correct forward with stochastic interest rates: here the replication strategy of the general case can be simplified. Instead of directly selling the future dividend flow, the trader can sell at time \( t \) a number \( N_T - N_t \) of zero coupon bonds with the same expiries as the dividends \( (\tau_i) \) and with each notional equal to the future (known) lump dividend \( (\phi_{\tau_i}) \): when the dividend is paid at \( \tau_i \) it is directly given to the \( \tau_i \)-Zero coupon holder to close the contract at its maturity \( \tau_i \).

C. Technical Proofs

C.1. Proof of Proposition 3.2

From the previous section and (16):

\[
\tilde{G}^{V,C}_t := \frac{V^f_t}{\beta^{diff}_t} - \frac{C^o_t}{\beta^{diff}_t} + \int_0^t \frac{d\Pi^f_u}{\beta^{diff}_u} + d \left[ \Pi^f_t, \frac{1}{\beta^{diff}} \right] - \int_0^t \frac{dC^g_u - c_u^g C^g_u}{\beta^{diff}_u} - d \left[ C^g_t, \frac{1}{\beta^{diff}} \right] - \frac{X^f_u}{B_t} \left( \frac{X^g_u}{B_t} \right) + \int_0^t \frac{\Delta X^f_u \Delta \Pi^f_u}{B_u} + \sum_{0 < u \leq t} \Delta X^g_u \Delta C^g_u - d \left( \frac{X^g_u}{B_u} \right) - \frac{C^g_u}{B_u} \left( \frac{X^g_u}{B_u} \right)
\]

\[
+ d \left[ C^g_t, \frac{1}{\beta^{diff}} \right]_u
\]
where we used (A.9). Now,

\[ Y := \int_0^t \frac{X_u^{f}}{B_u} \, d\Pi_u^f + \sum_{0 < u \leq t} \frac{\Delta X_u^{f} \Delta \Phi_u^f}{B_u} \]

\[ = \int_0^t \frac{X_u^{f}}{B_u} (d\Phi_u^f + \psi_u^f \, du) + \sum_{0 < u \leq t} \frac{\Delta X_u^{f} \Delta \Phi_u^f}{B_u} \]

\[ = \int_0^t \frac{X_u^{f}}{B_u} \psi_u^f \, du + \sum_{0 < u \leq t} \frac{X_u^{f} \Delta \Phi_u^f}{B_u} \]

\[ = \int_0^t \frac{X_u^{f}}{B_u} \, d\Pi_u^f \]

where the third equality by Proposition A.4. Moreover, using Proposition A.16, and that \( B \) is continuous with finite variation (see (A.9)).

\[ \Delta \Phi_u^f = \Delta \Phi_u^f (X_u^{f} + \Delta X_u^{f}) \]

\[ \text{hence, from this result and from (32)} \]

\[ \frac{dD_{u}^{V-C}}{B_u} = \frac{X_u^{f}}{B_u} \, d\Pi_u^f + \frac{X_u^{g}}{B_u} \left\{ (r_u - \gamma_u^g) \, du + d\mu_u^g \right\} \]

(3.1)

which can be substituted in (33):

\[ \tilde{G}_{t}^{V-C} = \tilde{V}_t - \tilde{C}_0 + \int_0^t \frac{X_u^{f}}{B_u} \, d\Pi_u^f + \int_0^t \frac{X_u^{g}}{B_u} \left\{ (r_u - \gamma_u^g) \, du + d\mu_u^g \right\} \]

and using the above expression twice in the equality \( \mathbb{E}_t [\tilde{G}_{T}^{V-C}] = \tilde{G}_{t}^{V-C} \), one obtains

\[ \tilde{V}_t - \tilde{C}_0 = \mathbb{E}_t \left[ \tilde{V}_t - \tilde{C}_0 + \int_0^T \frac{X_u^{f}}{B_u} \, d\Pi_u^f + \int_0^T \frac{X_u^{g}}{B_u} \left\{ (r_u - \gamma_u^g) \, du + d\mu_u^g \right\} \right] \]

from which we have the thesis (using Proposition A.4), since from (28),

\[ \mathbb{E}_t \left[ \int_0^T \frac{C_u^{g}}{B_u} \, dX_u^g \right] = \mathbb{E}_t \left[ \int_0^T \frac{C_u^{g}}{B_u} \mu_u^g X_u^g \, du \right] + \mathbb{E}_t \left[ \int_0^T \frac{C_u^{g}}{B_u} \, dM_u^g \right] \]

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and second addend is equal to zero by Property 10 or 11 of Proposition A.11 in case the integrand is well behaved: the proof of the pricing formula (34) is concluded thanks to Proposition A.4. The last statement is a direct application of the definition of $\tilde{G}$ to the pricing formula.

C.2. Proof of Proposition 3.8

We change the measure from $Q^f$ to $Q$, recalling (30):

$$L_t := E^Q_t \left[ dQ^{f,b} \right] = \frac{\beta_t^{fd}}{B_t} B_0 = \frac{B_t^{f,b} X^f_t (B^{-1})_t}{X^f_0}$$

We have under $Q$, since all bank accounts are continuous with finite variation,

$$dL_t = \frac{1}{X^f_0} \left\{ \beta_t^{f,b} d((B^{-1})_t) + (B^{-1})_t \cdot d\beta_t^{f,b} \right\}$$

$$= \frac{1}{X^f_0} B_t \left\{ -r_t \beta_t^{f,b} dt + B_t^{f,b} dX^f_t + X^f_t dB_t^{f,b} \right\}$$

$$= L_{t-} \left\{ (r^f_t - r_t) dt + \frac{dX^f_t}{X^f_t} \right\}$$

$$= L_{t-} \left\{ \frac{dM^f_t}{X^f_t} \right\} = L_t \left\{ \frac{dM^f_t}{X^f_t} \right\}$$

where the last equality is since the FX rate is the only jump component of $L$: we obtained that $L$ is a $Q$-martingale as expected. We now apply the Girsanov Theorem A.26 (and use Proposition A.16):

$$dM^f_t = dM^f_t - \frac{d[M^f, L]_t}{L_t}$$

$$= dM^f_t - \frac{d[M^f, M^f]_t}{X^f_t}$$

is a local martingale under $Q^{f,b}$. In particular, from (A.3) and (A.8),

$$\Delta M^x_t = \Delta M^f_t = \Delta M^x_t \frac{\Delta M^f_t}{X^f_t}$$

(C.2)

Therefore, recalling (28),

$$dX^x_t \overset{Q}{=} \mu^x_t X^x_{t-} dt + dM^x_t$$

$$\overset{Q^{f,b}}{=} \mu^x_t X^x_{t-} dt + \left( dM^f_t + \frac{d[M^f, L]_t}{L_t} \right)$$

where we used Proposition A.16: the first result is proved. Observing (38), under $Q^{f,b}$,

$$\Delta X^x_t = \frac{\Delta [M^x, M^f]_t}{X^f_t} + \Delta M^x_t$$

so the FX jumps do not change measure.

For the second result, given (38) for $x = f$, we apply the Itô formula (A.14) to $X^{df} = (X^f)^{-1}$:

$$dX^{df} = -(X^f)^{-2} dX^f + (X^f)^{-3} d[X^f]^c + (X^f)^{-1} - (X^f)^{-1} + (X^f)^{-2} \Delta X^f$$

$$= -(X^{df})^{-2} dX^f + (X^{df})^{-3} d[X^f]^c + (X^f)^{-1} - (X^f)^{-1} + (X^{df})^{-2} \Delta X^f$$

$$= (X^{df})^{-2} \left\{ -\mu^x_t X^f_{t-} dt - \frac{d[M^f]}{X^f_t} - dM^{f,b} + X^{df} d[M^f]^c + \Delta M^f \right\} - \frac{\Delta X^f}{X^f_t X^{df}_t}$$

$$= (X^{df})^{-2} \left\{ -\mu^x_t X^f_{t-} dt - X^{df} d[M^f] - dM^{f,b} + X^{df}_t d[M^f]^c + \Delta M^f - \frac{X^{df}}{X^{df}_t} \Delta M^f \right\}$$

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Now, using (A.11), Proposition A.15 and that the Quadratic Covariation has finite variation:

\[-X^{df} d[M_f] + X^{df} d[M_f]^c + \Delta M_f - \frac{X^{df}}{X_f} \Delta M_f\]

\[= -(X^{df} + \Delta X^{df}) \left\{ d[M_f]^c + (\Delta M_f)^2 \right\} + X^{df} d[M_f]^c + \Delta M_f \left( 1 - \frac{X_f}{X_f} \right)\]

\[= -X^{df} (\Delta M_f)^2 - \Delta X^{df} (\Delta M_f)^2 + \Delta M_f \frac{\Delta X_f}{X_f}\]

\[= (\Delta M_f)^2 \left\{ -X^{df} - \Delta X^{df} + X^{df} \right\} = 0\]

that completes the second result. For the last result, under \(Q^{fb}\),

\[dX^{xf} := d(X^x X^{df}) = X^x dX^{df} + X^{df} dX^x + d[X^x, X^{df}]\]

\[= X^x \left\{ -\mu X^{df} dt - (X^{df})^2 dM_f^{fb} \right\} + X^{df} \left\{ \mu X^x dt + X^{df} d[M^x, M_f] + dM^{xz, fb} \right\} + d[X^x, X^{df}]\]

\[= X^x X^{df} \left\{ (\mu - \mu^f) dt - X^{df} dM_f^{fb} + \frac{dM^{xz, fb}}{X^x} \right\} + X^{df} X^{df} d[M^x, M_f] + d[X^x, X^{df}] \quad (C.3)\]

Now, from Proposition A.16 and (C.2),

\[d[X^x, X^{df}] = [X^{df} d[M^x, M_f] + dM^{xz, fb} - (X^{df})^2 dM_f^{fb}]\]

\[= -(X^{df})^2 \left\{ X^{df} \Delta M^x \Delta M_f + X^{df} d[M^x, M_f] + d[M^{xz, fb}, M_f^{fb}] \right\} \]

\[= -(X^{df})^2 \left\{ X^{df} \Delta M^x \Delta M_f \left( \Delta M_f - \frac{\Delta M^f \Delta M_f}{X_f} \right) + d[M^{xz, fb}, M_f^{fb}] \right\} \]

\[= -(X^{df})^2 \left\{ X^{df} \Delta M^x (\Delta M_f)^2 (1 - X^{df} \Delta M_f) + d[M^{xz, fb}, M_f^{fb}] \right\}.\]

Now, using again Proposition A.16:

\[d[M^{xz, fb}, M_f^{fb}] = \left[ d[M^x - \frac{d[M^x, M_f]}{X_f}, dM_f - \frac{d[M_f, M_f]}{X_f} \right] \]

\[= d[M^x, M_f] - X^{df} [dM^x, d[M_f, M_f]] - X^{df} [d[M^x, M_f], dM_f] \]

\[+ (X^{df})^2 [d[M^x, M_f], d[M_f, M_f]] \]

\[= d[M^x, M_f] - 2 X^{df} \Delta M^x (\Delta M_f)^2 + (X^{df})^2 \Delta M^x (\Delta M_f)^3\]

then

\[d[X^x, X^{df}] = -(X^{df})^2 \left\{ -X^{df} \Delta M^x (\Delta M_f)^2 + d[M^x, M_f] \right\}\]

and, recalling (C.3) and using (A.11) and Proposition A.15,

\[X^{df} X^{df} d[M^x, M_f] + d[X^x, X^{df}]\]

\[= X^{df} \left( X^{df} + \Delta X^{df} \right) d[M^x, M_f] - (X^{df})^2 \left\{ -X^{df} \Delta M^x (\Delta M_f)^2 + d[M^x, M_f] \right\} \]

\[= X^{df} \left\{ \Delta X^{df} (d[M^x, M_f]^c + \Delta M^x \Delta M_f) + X^{df} \Delta M^x (\Delta M_f)^2 \right\} \]

\[= X^{df} \left\{ \Delta X^{df} \Delta M^x \Delta M_f + X^{df} \Delta M^x (\Delta M_f)^2 \right\} \]

\[= X^{df} \left\{ -\frac{\Delta M_f}{X_f} \Delta M^x \Delta M_f + X^{df} \Delta M^x (\Delta M_f)^2 \right\} = 0\]

so we have the thesis.
C.3. Proof of Proposition 3.11

Recalling (30), we have:

\[ \tilde{G}_t^{f,V,C} := \frac{V_t^f}{B_t^{f_b}} - \frac{C^g_t}{\beta_t^{f_g}} + \int_0^t \frac{d\Pi_u^f}{B_u^{f_b}} + \int_0^t \frac{dC^g_u - c^g_u C^g_{u-}}{\beta_t^{f_g}} + d \left[ C^g, \frac{1}{\beta_t^{f_g}} \right]_u \]

\[ = \tilde{V}_t^f - \tilde{C}_t^f + \int_0^t \frac{d\Pi_u^f}{B_u^{f_b}} + \int_0^t \frac{X^g_f}{B_u^{f_b}} \left( dC^g_u - c^g_u C^g_{u-} \right) du + d \left[ C^g, \frac{1}{\beta_t^{f_g}} \right]_u \]

using that \( B^{f_b} \) is continuous with finite variation

\[ d \left[ C^g, \frac{1}{\beta_t^{f_g}} \right]_u = d \left[ C^g, X^g f (B^{f_b})^{-1} \right]_u \]

\[ = (B^{f_b})^{-1} \left[ dC^g_u, dX^g_f \right] \]

\[ = \frac{X^g_f}{B_u^{f_b}} \left[ dC^g_u, \frac{dX^g_f}{X^g_u} \right] \]

which, together with the previous equation, recalling (44), gives the definition of \( D^{f,V,C} \). The easiest way for getting the pricing equation is to directly change measure from (34) to measure \( Q^{f_b} \), i.e. from numéraire \( B \mapsto \beta^{f_d} := B^{f_b} X^f \) using (20):

\[ V_t^f = \beta^{f_d}_t X_t^f \left[ \phi^f_t X_T \right] + \int_t^T \frac{X^f_u d\Pi_u^f}{\beta_t^{f_d}} + X^f_0 \left[ \Pi_t^f, \frac{1}{\beta_t^{f_d}} \right] - \int_t^T \frac{X^g_u C^g_{u-}}{\beta_t^{f_d}} (c^g_u - r^{g_b}_u) du \]

and, using the fact that \( \Pi_t^f \) has finite variation and (A.9), recalling also Proposition A.4,

\[ A := \int_t^T \frac{X^f_u d\Pi_u^f}{\beta_u^{f_d}} + X^f_0 \left[ \Pi_t^f, \frac{1}{\beta_t^{f_d}} \right] \]

\[ = \int_t^T \frac{X^f_u (\psi^f_u du + d\Phi_u^f)}{B_u^{f_b} X^g_u} + \sum_{t<u \leq T} X^f_u B_u^{f_b} \frac{\Delta \Pi_u^f}{B_u^{f_b}} \left( \frac{1}{X^g_u} - \frac{1}{X^g_{u-}} \right) \]

\[ = \int_t^T \frac{d\Pi_u^f}{B_u^{f_b}} + \sum_{t<u \leq T} X^f_u \frac{\Delta \Pi_u^f}{B_u^{f_b}} \left( \frac{1}{X^g_u} + \frac{1}{X^g_{u-}} \right) \]

\[ = \int_t^T \frac{d\Pi_u^f}{B_u^{f_b}} \]

so that the pricing equation is confirmed. The last statement is a direct application of the \( \tilde{G} \) definition to the pricing formula and the proof is concluded. One could also obtain the analogous of (C.1), to be used in the next proposition: we have,

\[ d\tilde{C}_t^f := d \left( C^g X^g f (B^{f_b})^{-1} \right)_u = (C^g X^g f)_u - d( (B^{f_b})^{-1} )_u d(C^g X^g f)_u \]

\[ = (B^{f_b})^{-1} \left\{ -r^{f_b}_t X^g_{u-} C^g_{u-} du + X^g_{u-} dC^g + C^g_{u-} dX^g_f + d[X^g f, C^g]_u \right\} \]

\[ = \frac{X^g_{u-}}{B_u^{f_b}} \left\{ dC^g - r^{f_b}_{u-} C^g_{u-} du + C^g_{u-} \frac{dX^g_f}{X^g_{u-}} + \left[ dC^g, \frac{dX^g_f}{X^g_{u-}} \right] \right\} \]

so

\[ dD_t^{f,V,C} = \frac{d\Pi_t^f}{B_u^{f_b}} + \frac{X^g_{u-} C^g_{u-}}{B_u^{f_b}} \left( r^{f_b}_{u-} - c^g_{u-} \right) du - \frac{dX^g_f}{X^g_{u-}} \]

which will be useful afterwards.

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C.4. Proof of Proposition 3.12

The dynamics of \((V_f - C_f)\) is a direct consequence of the fact that \(\tilde{G}_{t}^{f.V.C}\) is a \(Q^{fb}\)-martingale, see Proposition 2.16 from which one also obtains that \(d\tilde{G}_{t}^{f.V.C} = (B_{t}^{fb})^{-1} d\mathcal{M}_{t}^{f.V.C}\). Moreover, by (44),

\[
d\tilde{G}_{t}^{f.V.C} = d\tilde{V}_{t}^{f} - d\tilde{C}_{t}^{f} + \frac{d\mathcal{M}_{t}^{f.V.C}}{B_{t}^{fb}}
\]

so

\[
d\tilde{V}_{t}^{f} = d\tilde{G}_{t}^{f.V.C} - \left(\frac{d\mathcal{M}_{t}^{f.V.C}}{B_{t}^{fb}} - d\tilde{C}_{t}^{f}\right)
\]

\[
= \frac{1}{B_{t}^{fb}} \left\{ d\mathcal{M}_{t}^{f.V.C} - d\Pi_{t}^{f} - X_{t^{-}}^{g_f} C_{t^{-}}^{g_f} \left\{ (r_{t}^{fb} - c_{t}^{gb}) dt - \frac{dX_{t}^{g_f}}{X_{t}^{g_f}} \right\} \right\}
\]

where we used (C.4) at the second line. Now, using (40),

\[
dV_{t}^{f} = d(B_{t}^{fb}\tilde{V}_{t}^{f}) = B_{t}^{fb} d\tilde{V}_{t}^{f} + \tilde{V}_{t}^{f} dB_{t}^{fb}
\]

\[
= d\mathcal{M}_{t}^{f.V.C} - d\Pi_{t}^{f} - X_{t^{-}}^{g_f} C_{t^{-}}^{g_f} \left\{ (r_{t}^{fb} - c_{t}^{gb}) dt - \frac{dM_{t}^{g.f.b}}{X_{t}^{g_f}} \right\}
\]

that can be rearranged to obtain the dynamics of \(V_f\).

We then change the measure to this from \(Q^{fb}\) to \(Q\), where

\[
L_{t} := \mathbb{E}_{t}^{fb} \left[ \frac{dQ^{fb}}{dQ} \right] = \frac{\beta_{t}^{df} B_{t}^{fb}}{\beta_{0}^{df} B_{0}^{fb}} = \frac{\beta_{t}^{df}}{\beta_{0}^{df}}
\]

We have, using Proposition A.4,

\[
dL_{t} = \frac{1}{\beta_{0}^{df}} \left\{ \beta_{t}^{df} \left( d((B^{fb})^{-1})_{t} + ((B^{fb})^{-1})_{t} d\beta_{t}^{df} \right) \right\}
\]

\[
= L_{t} \left\{ -r_{t}^{fb} dt + \frac{d\beta_{t}^{df}}{\beta_{t}^{df}} \right\}
\]

We now apply the Girsanov Theorem A.26: first we observe that \(D_f\) is a semimartingale with zero local martingale part, so its dynamics does not change with the change of measure, see Corollary A.27. Moreover,

\[
d\mathcal{M}_{t}^{pdf} = d\mathcal{M}_{t}^{f} - \frac{dM_{t}^{f,L}}{L_{t}}
\]

is a local martingale under \(Q\). Therefore

\[
dV_{t}^{f} \overset{Q^{fb}}{=} r_{t}^{fb} V_{t}^{f} dt + d\mathcal{M}_{t}^{f} - dD_{t}^{f}
\]

\[
\overset{Q}{=} r_{t}^{fb} V_{t}^{f} dt + \left\{ d\mathcal{M}_{t}^{pdf} + \frac{dM_{t}^{f,L}}{L_{t}} \right\} - dD_{t}^{f}
\]
so we have to calculate the Covariation term: using Proposition A.16 and that $B$ is continuous with finite variation

\[
\frac{d[M^f, L]}{L_t} = \left[ \frac{dM^f_t, -r^f_t \, dt + \frac{d\beta^f_t}{\beta^f_t}}{\beta^f_t} \right] = \left[ \frac{dM^f_t, d\beta^f_t}{\beta^f_t} \right] = \frac{d \left[ M^f_t, X^g_f \right]}{X^g_f}
\]

Moreover, defining $g(x) := x^{-1}$ and using (A.15)

\[
d \left[ M^f_t, g(X^f) \right] = g'(X^f) \cdot d[M^f, X^f] + \left\{ g(X^f) - g(X^f) \right\} \Delta M^f
\]

\[
= - \frac{1}{(X^f)^2} \cdot d[M^f, X^f] + \left\{ \frac{1}{X^f} - \frac{1}{X^f} \right\} \Delta M^f
\]

\[
= - \frac{1}{(X^f)^2} \cdot d[M^f, X^f] - \frac{\Delta X^f \Delta M^f}{X^f X^f}
\]

\[
= - \frac{1}{X^f} \left\{ \frac{1}{X^f} \cdot d[M^f, X^f] + \frac{\Delta X^f \Delta M^f}{X^f} \right\}
\]

Therefore, using Proposition A.15 and the fact that the Quadratic Covariation has finite variation,

\[
\frac{d[M^f, L]}{L} = \frac{X^f \cdot d \left[ M^f_t, g(X^f) \right]}{X^f}
\]

\[
= - \frac{1}{X^f} \left\{ \frac{X^f + \Delta X^f}{X^f} \cdot d[M^f, X^f] + \Delta X^f \Delta M^f \right\}
\]

\[
= - \frac{1}{X^f} \left\{ d[M^f, X^f] + \Delta X^f \Delta M^f \right\}
\]

\[
= - \frac{d[M^f, X^f]}{X^f}
\]

which concludes the proof.