IN INVARIANTS OF 3-MANIFOLDS AND PROJECTIVE REPRESENTATIONS OF MAPPING CLASS GROUPS VIA QUANTUM GROUPS AT ROOTS OF UNITY

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Abstract. An example of a finite dimensional factorizable ribbon Hopf C-algebra is given by a quotient \( H = U_q(g) \) of the quantized universal enveloping algebra \( U_q(g) \) at a root of unity \( q \) of odd degree. The mapping class group \( M_{g,1} \) of a surface of genus \( g \) with one hole projectively acts by automorphisms in the \( H \)-module \( H^* \otimes g \), if \( H^* \) is endowed with the coadjoint \( H \)-module structure. There exists a projective representation of the mapping class group \( M_{g,n} \) of a surface of genus \( g \) with \( n \) holes labelled by finite dimensional \( H \)-modules \( X_1, \ldots, X_n \) in the vector space \( \text{Hom}_H(X_1 \otimes \cdots \otimes X_n, H^* \otimes g) \). An invariant of closed oriented 3-manifolds is constructed. Modifications of these constructions for a class of ribbon Hopf algebras satisfying weaker conditions than factorizability (including most of \( U_q(g) \) at roots of unity \( q \) of even degree) are described.

After works of Moore and Seiberg [44], Witten [62], Reshetikhin and Turaev [51], Walker [61], Kohno [22, 23] and Turaev [59] it became clear that any semisimple abelian ribbon category with finite number of simple objects satisfying some non-degeneracy condition gives rise to projective representations of mapping class groups of surfaces as well as to invariants of closed 3-manifolds. It was proposed in [38] to get rid of semisimplicity and so extend the class of categories which serves as the set of labels for a modular functor.

In this article we describe (eventually non-semisimple) ribbon Hopf algebras \( H \), whose modules form a category with the required properties, thereby giving representations of mapping class groups. These algebras are called 2-modular. All finite dimensional factorizable ribbon Hopf algebras have those properties.

As a byproduct we obtain a projective representation of the mapping class group \( M_{g,1} \) of a surface of genus \( g \) with one hole in the vector space \( H^* \otimes g \). If \( H^* \) is endowed with the coadjoint \( H \)-module structure, \( M_{g,1} \) acts by automorphisms of the \( H \)-module. For genus 1 and factorizable Hopf algebras this representation was obtained by Majid and the author [40]. In the case of Drinfeld’s doubles another proof

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of modular relations for genus 1 was given by Kerler [16]. The projective representations of $M_{1,1}$ thus obtained for finite dimensional $H = u_q(\mathfrak{s}(2))$ are very close to those of Crivelli, Felder and Wieczerkowski [3], which come from conformal field theories on the torus based on $SU(2)$.

Finite dimensional quotients $u_q(g)$ of quantized universal enveloping algebras $U_q(g)$ at roots $q$ of unity are studied as an example. We show that if the degree $l$ of the root $q$ of unity is odd, the algebra is factorizable, if it is even, then $u_q(g)$ will be 2-modular or not, depending on arithmetical properties of $q$.

We describe also an intermediate class of categories and Hopf algebras between factorizable and 2-modular ones. These categories and Hopf algebras are called 3-modular and they give rise to invariants of closed oriented 3-manifolds. In the case of semisimple factorizable categories this is the Reshetikhin–Turaev invariant [51]. In the case of Hopf algebras it turns out to be the Hennings’ invariant [10] (in the form of Kauffman and Radford [14]).

The results of the paper apply to both main known classes of ribbon categories: semisimple ones and the categories of all modules over a ribbon Hopf algebra. For the former we obtain already known results, the latter gives new representations. Although the language of abelian tensor categories is the most suitable for our purposes, the reader is advised to restrict the consideration to the categories of modules.

Conformal field theories is a powerful source of ribbon categories. Kazhdan and Lusztig constructed non-trivial braided tensor subcategories in the category of modules over an affine Lie algebra [15], motivated by CFT. These categories can be semisimple as well as not. Furthermore, Gaberdiel [9] associates with each CFT a braided tensor category, namely, the category of modules over the chiral symmetry algebra with a non-standard tensor product. By the very nature of CFT one expects [44] appearance of representations of mapping class groups (this is obvious for TQFT). It turns out [38] that such representations can be constructed from ribbon categories even if they are not describing the fusion in some CFT.

Turaev proved that in semisimple case the modular functor extends to a TQFT [59]. Moreover, he constructs the modular functor (that is, representations of mapping class groups) as a part of a bigger functor (TQFT), assigning linear maps to 3-cobordisms. In non-semisimple case the word-by-word repetition of his approach is impossible, which forces one to seek for a direct construction of the modular functor as was done in [38]. Besides, some remnants of TQFT-structure survive in non-semisimple case; this is under consideration now.

We recall basic facts about ribbon abelian categories in Section 1. The quantum Fourier transform is discussed in Section 2. Ordinary ribbon Hopf algebras produce braided Hopf algebras in Section 3. We construct finite dimensional ribbon Hopf algebras $u_q(g)$ in Section 4 and single out factorizable and 3-modular ones in Section 5. Representations of mapping class groups are described in Section 6. New invariants of closed 3-manifolds are proposed in Section 7.
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1. Introduction

1.1. Notations and conventions. \( k \) denotes a field. In this paper a Hopf algebra means a \( k \)-bialgebra with an invertible antipode. Associative comultiplication is denoted \( \Delta x = x(1) \otimes x(2) \), counity is denoted by \( \varepsilon \), antipode in Hopf algebras is denoted \( \gamma \). If \( H \) is a Hopf algebra, \( H^{\text{op}} \) denotes the same coalgebra \( H \) with opposite multiplication, \( H_{\text{op}} \) denotes the same algebra \( H \) with the opposite comultiplication. The category of \( H \)-modules is denoted \( H\text{-Mod} \), and its subcategory of finite dimensional \( H \)-modules is denoted \( H\text{-mod} \). In particular case \( H = k \) we use \( k\text{-Vect} \) and \( k\text{-vect} \) respectively. The category of \( H \)-comodules is denoted \( H\text{-Comod} \), and its subcategory of finite dimensional \( H \)-comodules is denoted \( H\text{-comod} \). The left adjoint action of \( x \in H \) in a Hopf algebra \( H \) means \( \text{ad} \ x.y = x(1)y\gamma(x(2)), \) where \( y \in H \).

Let \( X \) be an \( H \)-module, denote \( X^* \) the space of linear functionals on \( X \). Denote by \( X^\vee \) and \( X^\check{\vee} \) the two different structures of \( H \)-module on \( X^* \), the former being \((h.\xi)(x) = \xi(\gamma^{-1}(h).x)\), the latter being \((h.\xi)(x) = \xi(\gamma(h).x)\) for \( h \in H \), \( \xi \in X^* \), \( x \in X \). Iterating this definition we get \( X^{\vee \vee}, X^{\check{\vee} \check{\vee}} \). Notice that \( (X^\vee)^\vee \) and \( (X^\check{\vee})^{\check{\vee}} \) are naturally identified with \( X \), so we can use the general notation \( X^{(m\vee)} \), \( m \in \mathbb{Z} \), such that

\[ \ldots, X^{(-2\vee)} = X^{\check{\vee}}, X^{(-1\vee)} = X^\vee, X^{(0\vee)} = X, X^{(1\vee)} = X^\vee, X^{(2\vee)} = X^{\check{\vee}}, \ldots \]

\( \mathfrak{g} \) will denote a complex semi-simple Lie algebra of rank \( n \) with Borel and Cartan subalgebras \( \mathfrak{b}_+, \mathfrak{b}_-, \mathfrak{h} \). The root lattice, generated by the simple roots \( \alpha_1, \ldots, \alpha_n \), will be denoted \( Q \). The weight lattice, generated by the fundamental weights \( \omega_1, \ldots, \omega_n \) is denoted \( P \). We write the group \( Q \) also in multiplicative notations \( K_\alpha = \alpha \in Q \), using \( K_i = \alpha_i \) as generators. There is a perfect bilinear pairing

\[ \langle , \rangle : Q \times P \to \mathbb{Z}, \quad \langle \alpha_i, \omega_j \rangle = \delta_{ij}. \]

The Cartan matrix \( a_{ij} \) determines an inclusion

\[ Q \hookrightarrow P, \quad \alpha_j = \sum_{i=1}^{r} a_{ij} \omega_i, \]

and the inner product

\[ (|) : Q \times Q \to \mathbb{Z}, \quad (\alpha_i|\alpha_j) = d_i a_{ij}, \]
where \( d_i = 1, 2, 3 \).

\( q \) will denote an indeterminate or a primitive \( l \)th root of unity \( q = \varepsilon \in \mathbb{C} \). This root of unity is assumed to satisfy \( \varepsilon^{2m} \neq 1 \) for all \( 1 \leq m \leq \max_i d_i \). Let \( l_i \) be the smallest positive integers such that \( \varepsilon^{2d_i l_i} = 1 \). We use the following notations for \( q \)-numbers:

\[
\begin{align*}
(n)_q &= \frac{q^n - 1}{q - 1}, & [n]_q &= \frac{q^n - q^{-n}}{q - q^{-1}} , \\
(n)_q! &= \prod_{m=1}^n (m)_q, & [n]_q! &= \prod_{m=1}^n [m]_q.
\end{align*}
\]

\( U_h(\mathfrak{g}) \) (resp. \( U_q(\mathfrak{g}) \)) is a topological \( \mathbb{C}[[h]] \)-algebra (resp. \( \mathbb{Q}(q) \)-algebra), generated by \( H_i \) (resp. \( K_i^{\pm 1} \)), \( E_i, F_i \), the quantum group of Drinfeld [5] and Jimbo [11]. Equipped with the comultiplication

\[
\begin{align*}
\Delta H_i &= H_i \otimes 1 + 1 \otimes H_i , \\
\Delta E_i &= E_i \otimes 1 + K_i^{-1} \otimes E_i , \\
\Delta F_i &= F_i \otimes K_i + 1 \otimes F_i
\end{align*}
\]

(in \( U_h(\mathfrak{g}) \) \( K_i \) denotes \( e^{h_i H_i} \)) these algebras become Hopf algebras. The Lusztig’s divided powers algebra \( \Gamma(\mathfrak{g}) \) [33] is a \( \mathbb{Z}[q, q^{-1}] \)-subalgebra of \( U_q(\mathfrak{g}) \) generated by \( E_i^{(m)} = E_i^m / [m]_q!, F_i^{(m)} = F_i^m / [m]_q! \) and some Laurent polynomials of \( K_i \), where \( q_i = q^{d_i} \).

Choosing a reduced expression \( s_{i_1} s_{i_2} \ldots s_{i_N} \) of the longest element \( w_0 \) of the Weyl group of \( \mathfrak{g} \), we get a total ordering of the positive part \( \Delta^+ \) of the root system \( \Delta \):

\[
\beta_1 = \alpha_{i_1}, \beta_2 = s_{i_1} \alpha_2, \ldots, \beta_N = s_{i_1} \ldots s_{i_{N-1}} \alpha_{i_N}.
\]

Following [20, 25, 33] introduce the corresponding root vectors in \( \Gamma(\mathfrak{g}) \)

\[
E_{\beta_k} = T_{i_1} \ldots T_{i_{k-1}} E_{i_k}, \quad F_{\beta_k} = T_{i_1} \ldots T_{i_{k-1}} F_{i_k},
\]

where \( T_i : \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g}) \) are Lusztig’s automorphisms [33, 34]. In the products like \( \Pi_{\alpha} E_{\alpha}^m \) we always assume that \( \alpha \) runs over \( \Delta^+ \) according to the above total order. We use also \( q_{\beta_k} = q_{i_k}, l_{\beta_k} = l_{i_k} \) and \( E_{\beta}^{(m)} = E_{\beta}^m / [m]_q!, F_{\beta}^{(m)} = F_{\beta}^m / [m]_q! \in \Gamma(\mathfrak{g}) \).

An \( R \)-matrix will be often denoted \( R = \sum_i R_i \otimes R_i' \).

**1.2. Ribbon abelian categories.** Ribbon (also tortile [56]) category is the following thing: a braided monoidal category \( \mathcal{C} \) [12] with the tensor product \( \otimes \), the associativity \( a : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z \), the braiding (commutativity) \( c : X \otimes Y \to Y \otimes X \) and a unity object \( I \), such that \( \mathcal{C} \) is rigid (for any object \( X \in \mathcal{C} \) there are dual objects \( \vee X \) and \( X^\vee \) with evaluations \( \text{ev} : \vee X \otimes X \to I \), \( \text{ev} : X \otimes X^\vee \to I \) and coevaluations \( \text{coev} : I \to X \otimes \vee X, \text{coev} : I \to X \otimes X \) and possess a ribbon twist \( \nu \). A ribbon twist [12, 50, 56] \( \nu = \nu_X : X \to X \) is a self-adjoint (\( \nu_{X^\vee} = \nu_X^\vee \)) functorial automorphism such that \( \nu^2 = \nu_X^\vee \otimes \nu_Y^\vee \circ \nu_{X \otimes Y} \).
Morphisms constructed from braidings and (co)evaluations are often described by tangles. In conventions of [36] we denote

\[
\text{a morphism } f : X \to Y \text{ by } \begin{array}{c}
\text{X} \\
\text{Y}
\end{array},
\]

the braiding \( c : X \otimes Y \to Y \otimes X \) by \[
\begin{array}{c}
\text{X} \\
\text{Y}
\end{array},
\]

the inverse braiding \( c^{-1} : X \otimes Y \to Y \otimes X \) by \[
\begin{array}{c}
\text{X} \\
\text{Y}
\end{array},
\]

the evaluation \( \text{ev} : X \otimes X^\vee \to k \) by \[
\begin{array}{c}
\text{X} \\
\text{X^\vee}
\end{array},
\]

the coevaluation \( \text{coev} : k \to X^\vee \otimes X \) by \[
\begin{array}{c}
\text{X^\vee} \\
\text{X}
\end{array}.
\]

Consistency of these notations is due to the functor \( \Phi \) from the category of \( \mathcal{C} \)-colored tangles to the category \( \mathcal{C} \) itself [8].

In a ribbon category there are functorial isomorphisms [36]

\[
u_1^2 = \begin{array}{c}
\text{X} \\
\text{X^\vee}
\end{array}, \quad u_{-1}^{-2} = \begin{array}{c}
\text{X^\vee} \\
\text{X}
\end{array},
\]

Changing the category \( \mathcal{C} \) by an equivalent one, we can (and we will) assume that \( \vee X = X^\vee, \ X^\vee X = \vee X = X \) and \( u_0^2 = u_{-1}^{-2} = \text{id}_X \) (see [36]).

**Warning.** In the category \( \mathcal{C} = H\text{-mod} \), where \( H \) is a ribbon Hopf algebra, the equation \( X^\vee = \vee X \) is not satisfied, nevertheless \( X^\vee \) is canonically isomorphic to \( \vee X \). We identify these modules via \( u_0^2 : \vee X \to X^\vee \) (see Section 3.2).
If in addition $C$ is additive, it is $k$-linear with $k = \text{End} I$. We assume in the following that $k$ is a field, in which each element has a square root. (In fact we need a square root only for one element of $k$ which depends on $C$.) In this paper $C$ will be a noetherian abelian category with finite dimensional $k$-vector spaces $\text{Hom}_C(A, B)$. (One more technical condition: isomorphism classes in $C$ form a set.) In such case there exists a coend $F = \int X \otimes X^\vee$ as an object of a cocompletion $\hat{C}$ of $C$. Recall that this coend can be defined via an exact sequence

$$\bigoplus_{f: A \to B \in C} A \otimes B^\vee \xrightarrow{f \otimes B^\vee - A \otimes f^t} \bigoplus_{L \in \hat{C}} L \otimes L^\vee \xrightarrow{\oplus i_L} F \to 0,$$

(1.2.1)

where $f^t : B^\vee \to A^\vee$ is the transposed to a morphism $f : A \to B$. For a general definition of a coend see [41].

$F$ is a Hopf algebra in the category $\hat{C}$ (see [37, 40, 42]). The multiplication $m_F : F \otimes F \to F$ is described in [37] by any of the following $C$-$F$-tangles

The antipode $\gamma_F : F \to F$ is given by

There is a Hopf pairing $\omega : F \otimes F \to I$ [37],

$$\omega = \quad \text{(1.2.5)}$$
such that
\[ \text{Ann } \omega = \text{Ann}^{\text{left }} \omega = \text{Ann}^{\text{right }} \omega \in \hat{C}. \]
The quotient \( f = F/ \text{Ann } \omega \in \hat{C} \) is also a Hopf algebra.

The morphisms called monodromies \( \Omega_l = \Omega_{X,F}^l : X \otimes F \to X \otimes F, \Omega_r = \Omega_{F,X}^r : F \otimes X \to F \otimes F, \Omega = \Omega_{F,F} : F \otimes F \to F \otimes F \) are defined via tangles
\[
\begin{align*}
\Omega_l &= \begin{bmatrix} X & F \\ F & \end{bmatrix}, & \Omega_r &= \begin{bmatrix} F & X \\ F & \end{bmatrix}, & \Omega &= \begin{bmatrix} F & F \\ F & \end{bmatrix}.
\end{align*}
\]

They project to \( f \) as morphisms
\[
\begin{align*}
\Omega_l &= \Omega_{X,f}^l : X \otimes f \to X \otimes f, & \Omega_r &= \Omega_{f,X}^r : f \otimes X \to f \otimes X, & \Omega &= \Omega_{f,f} : f \otimes f \to f \otimes f
\end{align*}
\]
also called monodromies.

1.3. 2-modular categories. The first modular axiom is [37]
(M1) \( f \) is an object of \( C \) (and not only of a cocompletion \( \hat{C} \))
(more scrupulously, it means that there exists an exact sequence \( 0 \to \text{Ann } \omega \to F \to f \to 0 \) in \( \hat{C} \), where \( f \) is an object from \( \mathcal{C} \subset \hat{C} \)).

Being the coend \( \int X \otimes X' \), the object \( F \in \hat{C} \) has an automorphism \( \nu \otimes 1 \overset{\text{def}}{=} f \nu \otimes 1 \) (notations are from [37], see also Section 3.8.1). The second modular axiom says [37]
(M2) \( \nu \otimes 1(\text{Ann } \omega) \subset \text{Ann } \omega \)
(more scrupulously, there exist morphisms \( T' : \text{Ann } \omega \to \text{Ann } \omega \in \hat{C}, T : f \to f \in \mathcal{C} \) such that the diagram
\[
\begin{array}{ccc}
0 & \rightarrow & \text{Ann } \omega \\
&T'\downarrow & \text{\nu} \otimes 1 \downarrow & T \downarrow \\
0 & \rightarrow & \text{Ann } \omega \\
\end{array}
\]
commutes).

An equivalent form of (M2) is [37]
(M2') There exists a morphism \( \theta : I \to f \) such that for any \( X \in \mathcal{C} \) the ribbon twist \( \nu : X \to X \) coincides with the composition
\[
X \simeq I \otimes X \xrightarrow{\theta \otimes X} f \otimes X \xrightarrow{\Omega_r} f \otimes X \xrightarrow{\varepsilon \otimes X} I \otimes X \simeq X.
\]

Definition 1.3.1. A noetherian abelian ribbon category \( \mathcal{C} \) with finite dimensional \( k \)-vector spaces of morphisms \( \text{Hom}_C(A,B) \) is called 2-modular, if axioms (M1), (M2) are satisfied.
Here 2- refers to the dimension of a surfaces which will be the main application.

It was shown in [37] that in the case of a modular category there exists a morphism $\mu : I \rightarrow f$, which is the integral of a dual Hopf algebra $\check{f} \simeq f$, and

$$
\nu^{-1} X \xrightarrow{\nu} = \lambda^{-1} X, \quad \nu X \xrightarrow{\nu} = \lambda X \xrightarrow{\nu^{-1}}
$$

for some invertible constant $\lambda \in k^\times$. The pair $(\mu, \lambda)$ is unique up to a sign. Morphisms $S, S^{-1} : f \rightarrow f$

are inverse to each other. Morphisms $S$ and $T$ (defined via (M2)) yield a projective representation of a mapping class group of a torus with one hole:

$$(ST)^3 = \lambda S^2, \quad S^2 = \gamma_f^{-1},$$

$$T \gamma_f = \gamma_f T, \quad \gamma_f^2 = \nu.$$

Here $\gamma_f : f \rightarrow f$ is the antipode of the Hopf algebra $f$, given by the same tangle as (1.2.4).

2. Modular transformations in $F$

Here we reproduce results of [37] in special assumptions, which permit to prove more. Let $\mathcal{C}$ be a 2-modular category. Fix a morphism $\alpha : I \rightarrow F$ such that

$$\gamma_F \alpha = \alpha : I \rightarrow F \quad \text{and} \quad \mu = (I \xrightarrow{\alpha} F \xrightarrow{\pi} f)$$

(if there is one). In this section we adopt the convention $AB = A \circ B$ for composition.
2.1. The quantum Fourier transform.

Theorem 2.1.1 (cf Theorem 6.3 [37]). For any \( X \in C \) we have

\[
\nu_X \xrightarrow{\alpha} \lambda_{\nu_X^{-1}} \quad (2.1.1)
\]

\[
\nu_X^{-1} \xrightarrow{\alpha} \lambda_{\nu_X} \quad (2.1.2)
\]

Notation. Let \( \beta : I \to F \) be an arbitrary morphism. Introduce morphisms \( F \to F \)

\[
S_{\pm}(\beta) = \left( F \simeq F \otimes I \xrightarrow{F \otimes \beta} F \otimes F \xrightarrow{\Omega^{\pm 1}} F \otimes F \xrightarrow{\varepsilon \otimes F} I \otimes F \simeq F \right),
\]

graphically depicted as

\[
S_{-}(\beta) = \text{Diagram 1}, \quad S_{+}(\beta) = \text{Diagram 2}
\]

We shall use the shorthand \( S_* = S_{\pm}(\beta) : F \to F \) (a sign is chosen arbitrarily) and

\[
S_+ = \lambda S_+(\alpha), \quad S_- = \lambda^{-1} S_-(\alpha).
\]

Proposition 2.1.2 (cf Proposition 6.5 [37]). We have

\[
S_-(\alpha) \gamma_F = S_+(\alpha) = \gamma_F S_-(\alpha).
\]

In particular, \( S_{\pm} \) commute with \( \gamma_F \).

Theorem 2.1.3 (cf Theorem 6.7 [37]). We have

\[
S_{\pm} T^{-1} S_{\pm} = T S_{\pm} T, \quad S_{\pm} T S_{\pm} = T^{-1} S_{\pm} T^{-1}.
\]

Corollary 2.1.4. \( S_- T S_- = T^{-1} S_- T^{-1} \).
Lemma 2.1.5 (cf Lemma 6.9 [37]). There are identities
\[ S_+ T S_- S_+ = S_+ T, \quad S_+ T S_- = T S_+, \]
\[ S_+ T^{-1} S_- S_+ = S_+ T^{-1}, \quad S_+ T^{-1} S_- = T^{-1} S_. \]

Lemma 2.1.6 (cf Lemma 6.10 [37]). The morphism \( T \) commutes with \( S_+ S_- \) and \( S_+ S_+ \). We have
\[ S_+ S_+ = S_+ S_- = S_+ S_- = S_+ S_- . \]

Corollary 2.1.7. The morphisms \( P_1 = S_- S_+ \) and \( P_2 = S_+ S_- \) are projections such that \( P_1 P_2 = P_1 \) and \( P_2 P_1 = P_2 \).

Proposition 2.1.8. The following kernels and images coincide
\[ \mathrm{Ker} \ S_+ = \mathrm{Ker} \ S_- = \mathrm{Ker} \ P_1 = \mathrm{Ker} \ P_2 = \mathrm{Ker} \pi = \mathrm{Ann} \omega, \quad (2.1.3) \]
\[ \mathrm{Im} \ S_+ = \mathrm{Im} \ S_- = \mathrm{Im} \ P_1 = \mathrm{Im} \ P_2. \quad (2.1.4) \]

In particular, \( P_1 = P_2 \).

Proof. We get by Corollary 2.1.7 \( \mathrm{Ker} \ S_+ \subset \mathrm{Ker} \ P_1, \ \mathrm{Ker} \ S_- \subset \mathrm{Ker} \ P_2 \), \( \mathrm{Ker} \ P_1 = \mathrm{Ker} \ P_2 \). Lemma 2.1.6 gives \( \mathrm{Ker} \ P_1 \subset \mathrm{Ker} \ S_- \), \( \mathrm{Ker} \ P_2 \subset \mathrm{Ker} \ S_+ \). Therefore, \( \mathrm{Ker} \ S_+ = \mathrm{Ker} \ S_- = \mathrm{Ker} \ P_1 = \mathrm{Ker} \ P_2 \).

Since
\[ \begin{array}{ccc}
\mathrm{Ann} \omega & \Xrightarrow{X} & \mathrm{Ann} \omega \\
\_ & \_ & \_ \\
\end{array} = \begin{array}{ccc}
\mathrm{Ann} \omega & \Xrightarrow{X} & \mathrm{Ann} \omega \\
\_ & \_ & \_ \\
\end{array} = 0 \]
we have \( \mathrm{Ann} \omega \subset \mathrm{Ker} \ S_- \). This identity also implies
\[ \left( F \simeq F \otimes I \xrightarrow{F \otimes \alpha} F \otimes F \xrightarrow{\Omega} F \otimes F \xrightarrow{\epsilon \otimes F} I \otimes F \simeq F \xrightarrow{\pi} f \right) = \]
\[ = \left( F \xrightarrow{\pi} f \simeq f \otimes I \xrightarrow{f \otimes \alpha} f \otimes F \xrightarrow{\Omega} f \otimes F \xrightarrow{\epsilon \otimes f} I \otimes f \simeq f \right) = \]
\[ = \left( F \xrightarrow{\pi} f \simeq f \otimes I \xrightarrow{f \otimes \mu} f \otimes f \xrightarrow{\Omega} f \otimes f \xrightarrow{\epsilon \otimes f} I \otimes f \simeq f \right) = \]
\[ = \left( F \xrightarrow{\pi} f \xrightarrow{Sf} f \right), \]
hence \( \pi S_- = S \pi \). Thus, \( \mathrm{Ker} (\pi S_-) = \mathrm{Ker} (S \pi) = \mathrm{Ker} \pi \) and \( \mathrm{Ker} S_- \subset \mathrm{Ker} \pi = \mathrm{Ann} \omega \), whence (2.1.3) follows.

Proposition 2.1.2 implies \( \mathrm{Im} S_- (\alpha) = \mathrm{Im} S_+ (\alpha) \), so \( \mathrm{Im} S_- = \mathrm{Im} S_+ \). Corollary 2.1.7 gives that \( \mathrm{Im} S_- \supset \mathrm{Im} P_1, \ \mathrm{Im} S_+ \supset \mathrm{Im} P_2 \). Since \( S_- S_+ S_- = S_- \) and \( S_+ S_- S_+ = S_+ \),
by Lemma 2.1.6, we get \( \mathrm{Im} S_- \subset \mathrm{Im} P_1, \ \mathrm{Im} S_+ \subset \mathrm{Im} P_2 \). Therefore, (2.1.4) holds. □
Notation. Denote \( P = P_1 = P_2 = S_+S_- = S_-S_+ \).

**Theorem 2.1.9.** The morphisms \( S_+(\alpha), S_-(-\alpha), T, \gamma_F : F \to F \) commute with the projection \( P : F \to F \). The restrictions of \( S_\pm(\alpha) \) to \( \ker P = \text{Ann} \omega \) vanish. The restrictions to \( \text{Im} P \) (which depends on \( \alpha \))

\[
S = S_-(-\alpha) \bigg|_{\text{Im} P}, \quad S^{-1} = S_+(\alpha) \bigg|_{\text{Im} P},
\]

inverse to each other, are identified with \( S^\pm_1 : f \to f \) by an isomorphism \( \text{Im} P \overset{\pi}{\to} f \). Therefore,

\[
(ST)^3 = \lambda S^2, \quad S^2 = \gamma_F^{-1}
\]

on \( \text{Im} P \).

**Proof.** Lemma 2.1.6 implies that \( T \) commutes with \( P \) and \( PS_- = S_-S_- = S_+ = S_+P \). Proposition 2.1.2 implies \( P\gamma_F = \gamma_F P \). Other statements follow by Proposition 2.1.8, Corollary 2.1.4 and Proposition 2.1.2.

A certain similarity of the properties of \( S \) with the properties of the ordinary Fourier transform [39] suggests the name of quantum Fourier transform for this morphism.

### 2.2. 3-modular categories.

Here 3- refers to applications to 3-manifolds. Recall that \( \mu : I \to f \) is a two-sided integral and \( \gamma_F \mu = \mu \). If \( F \in C \) (analogue of being finite dimensional), then it has an invertible object of left integrals \( \text{Int}_l \) [37]. The canonical projection \( \pi : F \to f \) sends \( \text{Int}_l \) to \( \text{Im} \mu \).

**Definition 2.2.1.** A 3-modular category is a 2-modular category \( C \) with an additional property

\[(M3) \ F \in C \quad \text{and it has a two-sided integral} \quad \sigma : I \to F \quad \text{such that} \quad \pi \sigma = \mu. \]

The assumption \( F \in C \) is not needed in this paper and can be consistently omitted. However, it holds in all known examples.

**Proposition 2.2.1.** Suppose \((M3)\) holds. Then \( \gamma_F \sigma = \sigma \).

**Proof.** Clearly, \( \gamma_F \sigma \) is another two-sided integral in \( F \). Therefore, it must be proportional to \( \sigma \) [37]. Projecting the equation \( \gamma_F \sigma = C \sigma, \ C \in k^\times \), to \( f \) we get \( \mu = \gamma_F \mu = C \mu \), hence, \( C = 1 \). \( \square \)

**Definition 2.2.2.** A perfect modular category is a 2-modular category with a condition

\[(PM) \text{ Ann} \omega = 0 \quad \text{(equivalently,} \quad \pi : F \to f \quad \text{is an isomorphism)}. \]

A perfect modular category is a particular case of a 3-modular category (and the easiest to deal with). The reader is advised to assume \( C \) perfect for the first reading whenever appropriate.
3. RIBBON AND BRAIDED HOPF ALGEBRAS

In this section we reformulate the results obtained in the abstract setting of ribbon abelian categories in the case of category of finite dimensional modules $\mathcal{C} = H\text{-mod}$ over a Hopf algebra $H$. This job was already partially done by Majid and the author [40], so we shall omit most of the proofs.

3.1. Quasitriangular Hopf algebras. Let $H$ be a Hopf $k$-algebra with an invertible antipode. There are several (more or less equivalent) ways to make $\mathcal{C} = H\text{-mod}$ into a braided category, if it permits this structure. We choose the most direct one. Assume that $H$ has an $R$-matrix, which is an element $R \in H \otimes H$ (algebraic tensor product!) satisfying the relations of Drinfeld [5]

\[
(\Delta \otimes 1)R = R^{13}R^{23}
\]
\[
(1 \otimes \Delta)R = R^{13}R^{12}
\]
\[
\Delta^{op}a = R\Delta(a)R^{-1}
\]

for any $a \in H$, so $(H, R)$ is quasitriangular.

For a finite dimensional $H$-module $V$ as for any vector space there is a canonical linear map $v_0^2 : V \to V \vee \vee$ such that $\langle v, y \rangle = \langle y, v_0^2(v) \rangle$ for $v \in V$, $y \in V^\vee$. Its square gives

\[
v_0^4 = (V \xrightarrow{v_0^2} V \vee \vee \xrightarrow{v_0^2} V^{(4\vee)}).
\]

On the other hand, in $\mathcal{C}$ as in any rigid braided category there are morphisms $u_0^{-4} = (V \xrightarrow{u_0^{-1}} V \vee V \xrightarrow{u_0^{-1}-1} V^{(-4\vee)})$. Composing them we get linear bijections

\[
g : V \xrightarrow{u_0^{-4}} V^{(-4\vee)} \xrightarrow{v_0^4} V.
\]

They are decomposable into a product of the two bijections

\[
u_1 : V \xrightarrow{v_0^2} V \vee \vee \xrightarrow{u_0^{-2}} V,
\]
\[
u_4 : V \xrightarrow{v_0^2} V \vee \vee \xrightarrow{u_0^{-2}} V.
\]

Theorem 3.1.1 (Drinfeld [6]). The maps $u_1, u_4, g$ are given by the action of the following elements

\[
u_1 = \gamma(R''R'), \quad u_4 = \gamma^2(R')R'' = \gamma(u_1)^{-1}, \quad g = u_1u_4.
\]

The element $g$ is grouplike ($\varepsilon(g) = 1$, $\Delta g = g \otimes g$) and for any $a \in H$ we have $gag^{-1} = \gamma^4(a)$.

To find $g$ we can use the following. Let $h_+$ (resp. $h_-$) be the minimal subspace of $H$ such that $R \in h_+ \otimes H$ (resp. $R \in H \otimes h_-$). Then $h_+, h_-$ are Hopf subalgebras. The finite dimensional subspace spanned by their products $h = h_+h_-$ coincides with $h_-h_+$, therefore, $h$ is also a Hopf subalgebra. Moreover, it is a minimal quasitriangular Hopf subalgebra of $(H, R)$ [45]. All elements $u_1, u_4, g$ belong to $h$. 
Pick a basis \((a_i) \subset h_+\) and a basis \((b_i) \subset h_-\) for which \(R = \sum_i a_i \otimes b_i\). Introduce a non-degenerate pairing \(\pi : h_- \otimes h_+ \rightarrow h\). Clearly, \(g : h \rightarrow h\) has the property \(g(b_i) = \delta_j^i\). Axioms of \(R\)-matrix imply that \(\pi : h_{op} \otimes h_+ \rightarrow k\) is a Hopf pairing, i.e.

\[
\begin{align*}
\pi(ab, c) &= \pi(a, c(1))\pi(b, c(2)), \\
\pi(a, cd) &= \pi(a(1), d)\pi(a(2), c).
\end{align*}
\]

Now construct the double \(D(h_+)\) generated by its Hopf subalgebras \(h_+, h_-.\) The natural projection \(j : D(h_+) \rightarrow h\) is a homomorphism of quasitriangular Hopf algebras. Clearly, \(g_H = g_h = j(g_{D(h_+)}\) is the relationship between the elements \(g\) for the three algebras. To find \(g_{D(h_+)}\) use the following.

**Theorem 3.1.2** (Drinfeld [6], Kauffman and Radford [13]). Let \(\delta_+ \in h_+, \delta_- \in h_-\) be left integrals, that is, \(x\delta_\pm = \varepsilon(x)\delta_\pm\) for any \(x \in h_\pm\). Then \(g_{D(h_+)} = a_+^{-1}a_-\) for grouplike elements (called moduli) \(a_+ \in h_+, a_- \in h_-\) such that

\[
\begin{align*}
delta_+y &= \pi(a_-, y)\delta_+ \quad \text{for any } y \in h_+, \\
delta_-y &= \pi(y, a_+)\delta_- \quad \text{for any } y \in h_-.
\end{align*}
\]

### 3.2. Ribbon Hopf algebras.

Assume now that \(C = H\text{-mod}\) has a ribbon structure. Then there is a morphism \(u_0^{-2} = u_-^2\nu = u_1^{-2}\nu^{-1} : V \rightarrow \otimes V\) for any finite dimensional \(H\)-module \(V\). One can prove that the map

\[
\kappa_V : V \xrightarrow{u_0^{-2}} \otimes V \xrightarrow{u_0^2} V
\]

commutes with all morphisms and satisfies \(\kappa_{X \otimes Y} = \kappa_X \otimes \kappa_Y\) and \(\kappa_X^2 = g_X\). If \(H\) is finite dimensional, we deduce that \(\kappa_V\) is the action of a grouplike element \(\kappa\) of \(H\).

**Definition 3.2.1** (comp. [50]). A *ribbon Hopf algebra* \((H, R, \kappa)\) is a quasitriangular Hopf algebra \((H, R)\) and a grouplike element \(\kappa \in H\) such that

\[
\begin{align*}
\kappa^2 &= g, \\
\kappa\kappa^{-1} &= \gamma^2(a)
\end{align*}
\]

for any \(a \in H\).

In the category of finite dimensional modules over a ribbon Hopf algebra we have canonical isomorphisms \(u_0^2 : V \rightarrow V^\otimes\), \(u_0^2(v) = u_0^2(\kappa v) = \kappa v_0^2(v)\), which we use to identify these modules.

The following is essentially proved by Kauffman and Radford.

**Theorem 3.2.1** (cf [13]). If \((H, R, \kappa)\) is a ribbon Hopf algebra, then the category \(H\text{-mod}\) is a ribbon braided category with the ribbon twist given by multiplication by the central element

\[
\nu = \gamma^2(R')R''\kappa^{-1} = R''\gamma^2(R')\kappa = R''\kappa R' = R'\kappa^{-1}R''.
\]
The following holds
\[ \nu^{-1} = R' \gamma(R'') \kappa = \gamma(R'') R' \kappa^{-1}, \]
\[ \varepsilon(\nu) = 1, \]
\[ \gamma(\nu) = \nu, \]
\[ \Delta \nu = (R^{21} R^{12}) \cdot \nu \otimes \nu. \]  

Remark 3.2.1. Definition 3.2.1 is equivalent to the definition of a ribbon Hopf algebra of Reshetikhin and Turaev [50]. The element \( \nu^{-1} \) was denoted \( v \) in [50].

### 3.3. A braided Hopf algebra.

Here we describe explicitly the braided Hopf algebra \( F \) and its dual algebra \( U \) for the case of \( \mathcal{C} = H\text{-mod} \). Let \( H \) be a ribbon Hopf algebra and let \( H^\circ \) be its dual [57]. Assume that \( H \) has enough finite dimensional modules, so that the pairing \( H \otimes H^\circ \to k \) is non-degenerate. Define the Hopf algebra \( \text{Fun} = (H^\circ)_{op} \) as \( H^\circ \) with the opposite coproduct (note that usually the algebra of functions is a subalgebra of \( H^\circ \), not of \( (H^\circ)_{op} \)). We have an equivalence of categories \( \mathcal{C} = H\text{-mod} \simeq \text{Fun}\text{-comod} \). The pairing \( \langle \cdot, \cdot \rangle : H \otimes \text{Fun} \to k \) satisfies
\[ \langle x, fg \rangle = \langle x(1), f \rangle \langle x(2), g \rangle \]
\[ \langle xy, f \rangle = \langle x, f(2) \rangle \langle y, f(1) \rangle \]
for \( x, y \in H, f, g \in \text{Fun} \), where \( \Delta f = f(1) \otimes f(2) \) is the coproduct in \( \text{Fun} \).

Consider linear maps \( i_L : L \otimes L^\vee \to \text{Fun}, l_a \otimes l_b \mapsto t_{La}^b \), where \( t_{La}^b \) is the matrix element of the \( H \)-module \( L \) with a basis \( \{ l_a \} \), that is, \( t_{La}^b \) is a linear function on \( H \) given by \( \langle u, t_{La}^b \rangle = \langle u.l_a, l^b \rangle \) for \( u \in H \). The maps \( i_L \) become homomorphisms of \( H \)-modules if \( \text{Fun} \) is given the coadjoint \( H \)-module structure
\[ u \triangleright f = \langle u, f(1) \gamma(f(3)) \rangle f(2) \]
for \( u \in H, f \in \text{Fun} \). The vector space \( \text{Fun} \) with this \( H \)-module structure will be denoted \( F \).

Theorem 3.3.1 ([4, 54, 63]). The family \( (i_L : L \otimes L^\vee \to F)_{L \in \mathcal{C}} \) is a coend of the bifunctor \( \mathcal{C} \times \mathcal{C}^{op} \to \mathcal{C}, (A, B) \mapsto A \otimes B^\vee \), so we can write \( F = \int^L L \otimes L^\vee \). In other words, the sequence (1.2.1) is exact.

Being the coend, \( F \) is a Hopf algebra in the category \( \tilde{\mathcal{C}} = \text{Fun}\text{-Comod} \subset H\text{-Mod} \) (braided Hopf algebra), as we have seen in Section 1.2. The Hopf structure of \( F \) described by tangles in [37] converts to the following. As the coalgebra \( F \) coincides with \( \text{Fun} \). The multiplication in \( F \) is expressed via the multiplication in \( \text{Fun} \) as
\[ m_F(f \otimes g) = \rho(\gamma(f(2)) \otimes g(1) \gamma(g(3)))f(1)g(2) \]
\[ = \rho(f(2) \otimes g(3) \gamma^{-1}(g(1)))f(1)g(2) \]
\[ = \rho(f(1) \gamma(f(3)) \otimes g(1))g(2)f(2) \]
by (1.2.2) and (1.2.3), where
\[ \rho(a \otimes b) = \langle R, a \otimes b \rangle = \langle R', a \rangle \langle R'', b \rangle. \]

The unity of \( F \) is the same as the unity of \( \text{Fun} \). The antipode \( \gamma_F \) of \( F \) is expressed via the antipode \( \gamma \) of \( \text{Fun} \): 
\[ \gamma_F(f) = \rho(f(1) \otimes \gamma(f(4))) \langle \nu \kappa^{-1}, f(2) \rangle \gamma(f(3)) \]
for \( f \in F \). The inverse antipode is 
\[ \gamma_F^{-1}(f) = \rho(f(4), f(1)) \langle \kappa^{-1} \nu^{-1}, f(3) \rangle \gamma^{-1}(f(2)). \]

All the structure maps \( \Delta, \varepsilon, m_F, \gamma_F \) are homomorphisms of \( H \)-modules.

3.4. The dual braided Hopf algebra. In order to define a Hopf algebra dual to \( F \) we shall not consider the rational part \( \circ F \) of the \( H \)-module \( \ast F \) of all linear functionals on \( F \). Instead we denote by \( U \) the \( H \)-submodule \( H \subset \ast F \). This amounts to consider the adjoint action
\[ \text{ad} \ a. x = a(1) x \gamma(a(2)) \]
for \( a \in H, x \in U \). Since \( \ast F \) is an algebra in the category \( H \text{-Mod} \) (being dual to the coalgebra \( F \in H \text{-Mod} \), so is \( U \) with the usual multiplication \( m \) of \( H \). We want to introduce a comultiplication \( \nabla : U \rightarrow U \otimes U \) which would be dual to \( m_F \) in the proper sense:
\[ \langle \nabla u, f \otimes g \rangle \equiv \langle u^{(1)} \otimes u^{(2)}, f \otimes g \rangle \equiv \langle u^{(2)}, f \rangle \langle u^{(1)}, g \rangle = \langle u, m_F(f \otimes g) \rangle \]
for \( u \in U, f, g \in F \). By dualising the formulae for \( m_F \) one arrives to the following
\[ \nabla u = \text{ad} R'' u^{(2)} \otimes R' u^{(1)} \]  
(3.4.1)
\[ = u^{(1)} \gamma(R'') \otimes \text{ad} R'. u^{(2)}. \]  
(3.4.2)

The counity of \( U \) coincides with the counity of \( H \). The operations \( m, \nabla, \varepsilon \) make \( U \) into a braided Hopf algebra in \( H \text{-Mod} \), the antipode \( \gamma_U : U \rightarrow U \) being 
\[ \gamma_U(u) = \nu \kappa^{-1} \gamma(R'') \gamma(u) R' \]
and the inverse antipode being 
\[ \gamma_U^{-1}(u) = R' \kappa^{-1} \nu^{-1} \gamma(u) R'' \]  
\[ = \gamma^2(R') \gamma^{-1}(u) R'' \kappa^{-1} \nu^{-1}. \]

This is the unique Hopf algebra structure dual to \( F \) on the \( H \)-module \( H \).
3.5. The algebra $\mathbf{f}$. The algebra $\text{Fun}^*$ acts in every finite dimensional $H$-module $X$. Consider special elements $l^−_V(v \otimes w) \in \text{Fun}^*$ determined for any $w \in W^\vee$, $v \in V \in H\text{-mod}$ as the operators in $X$:

$$l^−_V(v \otimes w)(x) = (ev \otimes 1)(1 \otimes c^2)(v \otimes w \otimes x) = \sum_{i,j} \langle v, R''_j R'_i w \rangle R'_j R''_i x.$$  \hspace{1cm} (3.5.1)

The subspace $u$ spanned by $l^−_V(v \otimes w)$ is contained in $H$ and even in $h$, so it is finite dimensional. As shown in [37] $u$ is a braided Hopf subalgebra of $\mathcal{O}F$, therefore, it is closed under the operations of $U$ and constitutes a finite dimensional braided Hopf subalgebra of $U$.

The map

$$l^− : \bigoplus_{V \in \mathcal{C}} V \otimes V^\vee \xrightarrow{\otimes l^−_V} U$$

factors through the coend (1.2.1), therefore, determines a map $l^− : \mathcal{O}F \rightarrow U$, which is a homomorphism of Hopf algebras in $H\text{-Mod}$ [37]. The image of $l^−$ is $u$. The form $\omega : \mathcal{O}F \otimes \mathcal{O}F \rightarrow k$ (1.2.5) can be presented as

$$\omega(f \otimes t_{La} b) = \langle l^−(f), l^a b \rangle = \langle l^−(f), t_{La} b \rangle,$$

hence, $\text{Ann}^\text{left} \omega = \text{Ker} l^−$. Therefore, the braided Hopf algebras $\mathbf{f} = \mathcal{O}F/\text{Ann} \omega$ and $u$ are isomorphic [37, Corollary 3.10]. Thus the first modular axiom (M1) is always satisfied for the considered algebras $H$.

By definition the subspace $u \subset \mathbf{u}$ is the minimal subspace such that $R_{12} R_{21}^\tau \in u \otimes \mathbf{u}$. Since $(\gamma \otimes \gamma)(R_{12} R_{21}^\tau) = R_{21} R_{12}$, the minimal subspace $B \subset \mathbf{h}$ such that $R_{12} R_{21}^\tau \in \mathbf{h} \otimes B$ is $\gamma(u)$. It does not coincide necessarily with $u$, for $u$ is not an ordinary Hopf subalgebra. Repeating the reasoning we get $\gamma^2(u) = u$ and conclude that $R_{12} R_{21}^\tau \in u \otimes \gamma(u)$. Similarly $(R_{12} R_{21}^\tau)^{-1} \in u \otimes \gamma(u)$.

3.6. 2-modular Hopf algebras. We already know by (3.5.1) the image

$$l^−(f) = \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), f \rangle R'_j R''_i$$  \hspace{1cm} (3.6.1)

for any $f \in \mathcal{O}F$. Notice that $\gamma^{-1}(R''_j R'_i) \otimes R'_j R''_i \in u \otimes u$. The pairing $u \otimes \mathbf{f} \hookrightarrow U \otimes \mathcal{O}F \rightarrow k$ factorizes through a perfect pairing $u \otimes \mathbf{f} \rightarrow k$ due to $\text{Ann}^\text{left} \omega = \text{Ann}^\text{right} \omega$. Therefore, an element $x \in H$ is representable in the form $l^−(f)$ for some $f \in \mathbf{f}$ iff $x \in u$.

**Theorem 3.6.1.** A ribbon Hopf algebra $(H, R, \kappa)$ is 2-modular (that is, $H\text{-mod}$ is 2-modular) if and only if $\nu \in u$, or equivalently, $\nu^{-1} \in u$. 
Proof. If the axiom (M2') holds, then $\nu = l^-(\theta(1))$ for some $\theta : k \to f$. Hence, $\nu \in u$ by the above discussion.

If $\nu \in u$, then for some $f \in f$ we have $\nu = l^-(f)$. Since $\nu$ is central, the subspace $k\nu \subset u$ is a trivial $H$-submodule. Hence, its preimage $kf \subset f$ by an isomorphism $l^- : f \to u$ is also a trivial $H$-submodule. Now $\theta : k \to f, 1 \mapsto f$ is the homomorphism required in the axiom (M2').

The condition $\nu \otimes 1(\text{Ann}\omega) \subset \text{Ann}\omega$ is equivalent to $\nu^{-1} \otimes 1(\text{Ann}\omega) \subset \text{Ann}\omega$ \cite[Corollary 5.12]{37}. Therefore, $\nu \in u$ if and only if $\nu^{-1} \in u$. \(\square\)

**Corollary 3.6.2.** If a ribbon Hopf algebra $(H, R, \kappa)$ is 2-modular, then $\kappa \in h$. Therefore, its minimal quasitriangular subalgebra $(h, R)$ equiped with $\kappa$ will be also a 2-modular ribbon Hopf algebra.

**Remark 3.6.1.** Let $H$ be a ribbon Hopf algebra. The category $H$-mod is perfect modular if and only if $F = f$, or equivalently, $H = u$, so $H$ is called factorizable \cite{49}.

### 3.7. 3-modular Hopf algebras.

**Proposition 3.7.1.** Let $\alpha \in \text{Fun}$ be an element. Denote the corresponding linear functional $H \to k$ and the linear map $k \to \text{Fun}, 1 \mapsto \alpha$, also by $\alpha$. The following conditions are equivalent:

(i) $\alpha : k \to F$ is a morphism of $H$-modules;

(ii) $\alpha_{(1)}\gamma(\alpha_{(3)}) \otimes \alpha_{(2)} = 1 \otimes \alpha$;

(iii) $\alpha(x_{(1)}u\gamma(x_{(2)})) = \varepsilon(x)\alpha(u)$ for any $x, u \in H$;

(iv) $\alpha : U \to k$ is a morphism of $H$-modules;

(v) $\alpha(xy) = \alpha(y\gamma^2(x))$ for any $x, y \in H$.

Proof is a straightforward check and it is left to the reader.

**Theorem 3.7.2.** An element $\sigma \in F$ is a two-sided integral on the algebra $U$ if and only if the equivalent conditions (i)–(v) of Proposition 3.7.1 are satisfied and $\sigma$ is a left integral on the algebra $H$, that is, $(1 \otimes \sigma)\Delta u = \sigma(u)$ for any $u \in H$.

Proof. Conditions (i) and (iv) are clearly necessary. Assume now that (iii) holds. $\sigma \in F$ is a two-sided integral if and only if

$$m_F(f \otimes \sigma) = \varepsilon(f)\sigma = m_F(\sigma \otimes f)$$

$$\iff \langle u, m_F(f \otimes \sigma) \rangle = \varepsilon(f)\langle u, \sigma \rangle = \langle u, m_F(\sigma \otimes f) \rangle$$

$$\iff \langle \nabla u, f \otimes \sigma \rangle = \varepsilon(f)\sigma(u) = \langle \nabla u, \sigma \otimes f \rangle$$

$$\iff \langle (\sigma \otimes 1)\nabla u, f \rangle = \varepsilon(f)\sigma(u) = ((1 \otimes \sigma)\nabla u, f)$$

$$\iff (\sigma \otimes 1)\nabla u = \sigma(u) = (1 \otimes \sigma)\nabla u \quad (3.7.1)$$

for any $f \in F, u \in U$. 
Now we calculate
\[(\sigma \otimes 1)\nabla u = \sigma(\text{ad } R'' u(2)) R' u(1)\]
\[= \sigma(u(2)) \varepsilon(\text{ad } R') R' u(1)\]
\[= u(1) \sigma(u(2)),\]
\[(1 \otimes \sigma)\nabla u = u(1) \gamma(R'') \sigma(\text{ad } R' u(2))\]
\[= u(1) \gamma(R'') \varepsilon(R') \sigma(u(2))\]
\[= u(1) \sigma(u(2))\]
by (3.4.1) and (3.4.2). Thus, all equations (3.7.1) hold if one equation
\[(1 \otimes \sigma)\Delta u = \sigma(u)\]
holds.  

Let the algebra \(F\) have a two-sided integral \(\sigma : I \to F\). Then \(\pi \circ \sigma : k \to f\) is a two-sided integral, therefore it is proportional to \(\mu\). If the proportionality constant does not vanish, then \(\sigma\) can be rescaled to satisfy \(\pi \circ \sigma = \mu\). Non-vanishing of \(\pi \circ \sigma\) is equivalent to non-vanishing of \(l^-(\sigma) \in u\) or of \(\gamma^{-1}(l^-(\sigma)) \in u\). Formula (3.6.1) gives
\[\gamma^{-1}(l^-(\sigma)) = \sigma(R'_i R''_j R' R'_j)\]

So we get

**Theorem 3.7.3.** A finite dimensional ribbon Hopf algebra \((H, R, \kappa)\) is 3-modular (that is, \(H\)-mod is 3-modular) if and only if the following conditions hold
(M2) \(\nu \in u\)
(M3) \(\int(xy) = \int(y \gamma^2(x))\), \(\int(R'_i R''_j R'_i R'_j) \neq 0\)
where \(\int : H \to k\) is a left integral on the algebra \(H\).

The property \(\int(xy) = \int(y \gamma^2(x))\) above is equivalent to unimodularity\(^1\) of \(H\), which means that each left integral \(\Lambda \in H\) is a right integral as well. This follows from the Radford’s formula \(\int(y \gamma^2(x)) = \int(x(1) \alpha(x(2)) y)\) [46], where \(\alpha \in G(H^*)\) is the modulus relating left and right integrals for \(H\).

**Proposition 3.7.4.** A factorizable ribbon Hopf algebra is unimodular.

**Proof.** This a corollary of the above theorem. Or, notice simply that in factorizable case the map \(l^- : F \to U = H\) is an isomorphism of algebras, preserving the counity, and \(F\) is unimodular.

For Drinfeld doubles unimodularity was proven by Hennings [10] and Radford [45].

### 3.8. Some operators in \(F\) and \(U\).

\(^1\)I am grateful to the referee for this remark.
3.8.1. 3-modular case. Assume that \((H, R, \kappa)\) is 3-modular. We find explicitly the linear maps \(S, T : F \to F\) and their transposed maps \(S^\dagger, T^\dagger : U \to U\).

Let \(y \in H\). There are maps \(F \to F\)

\[
y \otimes 1 = \begin{bmatrix} y \\ F \end{bmatrix}, \quad 1 \otimes y = \begin{bmatrix} F \\ y \end{bmatrix}.
\]

The first is obtained as the projection of \((y \otimes 1)(l_a \otimes l^b) = \langle y, t_{La} \rangle l_c \otimes l^b\) in the form \(y \otimes 1(t_{La}^b) = \langle y, t_{La} \rangle t_{Lc}^b\). Therefore,

\[
y \otimes 1(f) = \langle y, f(1) \rangle f(2)
\]

and similarly

\[
1 \otimes y(f) = \langle \gamma^{-1}(y), f(2) \rangle f(1).
\]

The transposed operators in \(U\) are

\[
u^\dagger(y \otimes 1) = uy, \quad u^\dagger(1 \otimes y) = \gamma^{-1}(y)u
\]

for any \(u \in U\). In particular, we have \(T = \nu \otimes 1 = 1 \otimes \nu\) and its transpose \(T^\dagger\)

\[
T(f) = \langle \nu, f(1) \rangle f(2) = f(1) \langle \nu, f(2) \rangle \quad \text{for } f \in F \quad (3.8.1)
\]

\[
T^\dagger(u) = u\nu = \nu u \quad \text{for } u \in U \quad (3.8.2)
\]

Similarly other maps are constructed by projection:

\[
\Omega_l : X \otimes F \to X \otimes F
\]

\[
\Omega_l(x \otimes f) = \sum_{i,j} R''_j R'_i x \otimes \langle R''_j R'_i, f(1) \rangle f(2)
\]

\[
\Omega_r : F \otimes X \to F \otimes X
\]

\[
\Omega_r(f \otimes x) = \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), f(2) \rangle f(1) \otimes R''_j R'_i x
\]

\[
\Omega : F \otimes F \to F \otimes F
\]

\[
\Omega(h \otimes f) = \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), h(2) \rangle h(1) \otimes \langle R''_j R'_i, f(1) \rangle f(2)
\]

\[
\Omega^{-1} : F \otimes F \to F \otimes F
\]

\[
\Omega^{-1}(h \otimes f) = \sum_{i,j} \langle \gamma^{-2}(R''_j R'_i), h(2) \rangle h(1) \otimes \langle R''_j R'_i, f(1) \rangle f(2)
\]
Assume according to (M3) that $\sigma \in F$ is a two-sided integral with $\pi \circ \sigma = \mu$. Then

$$S_-(\sigma) : F \rightarrow F$$

$$S_-(\sigma)(f) = \rho(\sigma(1) \otimes \gamma(f(1)))\rho(f(2) \otimes \gamma(\sigma(3)))\sigma(2)$$

$$= \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), f \rangle \langle R'_j R''_i, \sigma(1) \rangle \sigma(2)$$

$$= \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), \sigma(1) \rangle \langle R'_j R''_i, f \rangle \sigma(2)$$

(3.8.7)

$$S_+(\sigma) : F \rightarrow F$$

$$S_+(\sigma)(f) = \rho(f(1) \otimes \sigma(1))\rho(\sigma(3) \otimes f(2))\sigma(2)$$

$$= \sum_{i,j} \langle \gamma^{-2}(R''_j R'_i), f \rangle \langle R'_j R''_i, \sigma(1) \rangle \sigma(2)$$

$$= \sum_{i,j} \langle \gamma^{-2}(R''_j R'_i), \sigma(1) \rangle \langle R'_j R''_i, f \rangle \sigma(2)$$

(3.8.11)

$$^tS_-(\sigma) : U \rightarrow U$$

$$^tS_-(\sigma)(u) = \sum_{i,j} \sigma(\gamma^{-1}(R''_i u R'_j))R'_i R''_j$$

$$= \sum_{i,j} \sigma(u \gamma(R''_i R'_j))R'_i R''_j$$

$$= \sum_{i,j} \sigma(\gamma^{-1}(R''_i R'_j) u) R'_i R''_j$$

(3.8.9)

$$^tS_+(\sigma) : U \rightarrow U$$

$$^tS_+(\sigma)(u) = \sum_{i,j} \sigma(R'_j u R''_i)R'_i R''_j$$

$$= \sum_{i,j} \sigma(u R''_i R'_j) \gamma^{-2}(R''_j R'_i)$$

$$= \sum_{i,j} \sigma(\gamma^{-2}(R''_i R'_j) u) R''_i R'_j$$

(3.8.13)

We know by Theorem 2.1.9 that $P = S_+(\sigma)S_-(\sigma)$ is a projection in $F$ with $\text{Ker} \ P = \text{Ann} \ \omega$. Therefore,

$$^tP = ^tS_-(\sigma)^tS_+(\sigma) = ^tS_+(\sigma)^tS_-(\sigma) = ^tS_+(\sigma)^2 \gamma^{-1}$$

is a projection in $U$ with $\text{Im} \ ^tP = u$. 

Let us determine now the normalization for the integral $\sigma$. Using the pairing $\langle \cdot, \cdot \rangle : U \otimes F \to k$ one compute

$$\langle 1, \gamma_F^{-1} S_+(\sigma)^2(1) \rangle = \varepsilon(P(1)) = \varepsilon(1) = 1$$

since $1 - P(1) \in \text{Ann } \omega \subset \text{Ker } \varepsilon$. On the other hand

$$\langle 1, \gamma_F^{-1} S_+(\sigma)^2(1) \rangle = \langle \iota S_+(\sigma)^2 \gamma_U^{-1}(1), 1 \rangle = \langle \iota S_+(\sigma)^2(1), 1 \rangle$$

so we have to satisfy $\varepsilon(\iota S_+(\sigma)^2(1)) = 1$. Using (3.8.13) we get

$$\iota S_+(\sigma)(1) = \sum_{i,j} \sigma(R'_j R''_i) R''_j R'_i$$

$$\iota S_+(\sigma)^2(1) = \sum_{i,j,k,l} \sigma(R'_j R''_i) \sigma(R''_k (R''_j R'_i) R''_l R'_k) R''_j R'_i$$

$$\varepsilon(\iota S_+(\sigma)(1)) = \sum_{i,j} \sigma(R'_j R''_i) \sigma(R''_i R'_j)$$

Therefore, $\sigma$ is normalized so that

$$(\sigma \otimes \sigma)(R^{12} R^{21}) = 1. \tag{3.8.16}$$

3.8.2. 2-modular case. Let us assume less now, namely, that $(H, R, \kappa)$ is 2-modular. Then the formulae (3.8.3)–(3.8.12), (3.8.14), (3.8.15) with $F$ replaced by $f$, $U$ replaced by $u$ and $\sigma$ replaced by $\mu \in f$ or $\mu : u \to k$ give the correct operators $\Omega^x_f$, $\Omega^x_r$, $(\Omega f, t) \pm 1$, $S = S_-(\mu) : f \to f$, $S^{-1} = S_+(\mu) : f \to f$, $t S : u \to u$ and $t S^{-1} : u \to u$.

Notice that

$$\sum_{i,j} \gamma^{-2}(R''_j) R'_i \otimes R''_j R'_i = (\gamma^{-1} \otimes 1)(R^{21} R^{12})^{-1} \in u \otimes u.$$

To determine the normalisation constant for $\mu$ we use (3.8.9). As above, using the perfect pairing $\langle \cdot, \cdot \rangle : u \otimes f \to k$ we get

$$1 = \langle 1, \gamma_f S^2(1) \rangle = \varepsilon(\iota S^2(1)) = \sum_{i,j} \mu(R''_j R'_i) \mu(\gamma(R''_j R'_i)),$$

hence,

$$(\mu \otimes \mu \gamma)(R^{12} R^{21}) = 1.$$

In 3-modular case we can use either this formula or (3.8.16) in the form

$$\sum_{i,j} \mu(\gamma^{-2}(R''_j) R'_i) \mu(R''_j R'_i) = 1.$$

**Proposition 3.8.1.** Let $(H, R, \kappa)$ be a 2-modular Hopf algebra. Then

$$\iota S(\nu \pm 1) = \lambda^{\pm 1} \nu^{\pm 1}, \quad \iota S^{-1}(\nu \pm 1) = \lambda^{\pm 1} \nu^{\pm 1},$$

$$\lambda = \mu(\nu), \quad \lambda^{-1} = \mu(\nu^{-1}).$$
Moreover, if \((H, R, \kappa)\) is 3-modular, then
\[
^tS_+(\sigma)(\nu^\pm) = \lambda^\pm \nu^\mp, \quad ^tS_-(\sigma)(\nu^\pm) = \lambda^\pm \nu^\mp, \quad (3.8.17)
\]
\[
\lambda = \sigma(\nu), \quad \lambda^{-1} = \sigma(\nu^{-1}).
\]

The perfect modular case was considered in [40].

Proof. Both cases being similar, we prove the second statement. Equations (2.1.1), (2.1.2) mean that for any \(H\)-module \(X\) and any vector \(x \in X\)
\[
(\varepsilon \otimes 1)\Omega_r(T^\pm(\sigma) \otimes x) = \lambda^\pm \nu^\mp x.
\]
Using (3.8.1) and (3.8.4)
\[
(\varepsilon \otimes 1)\Omega_r(T^\pm(\sigma) \otimes x) = (\varepsilon \otimes 1)\Omega_r((\nu^\pm, \sigma(1)) \otimes x)
\]
\[
= (\nu^\pm, \sigma(1)) \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i), \sigma(2) \rangle R'_j R''_i x
\]
\[
= \sum_{i,j} \langle \nu^\pm, \sigma(1) \rangle \langle \gamma^{-1}(R''_j R'_i), \sigma(2) \rangle R'_j R''_i x
\]
\[
= \sum_{i,j} \langle \gamma^{-1}(R''_j R'_i) \nu^\pm, \sigma \rangle R'_j R''_i x
\]
\[
= ^tS_-(\sigma)(\nu^\pm) x.
\]
Therefore, \( ^tS_-(\sigma)(\nu^\pm) = \lambda^\pm \nu^\mp \). Applying \( \gamma \nu \) we get the same result for \( ^tS_+(\sigma) \).
Applying \( \varepsilon \) to (3.8.17) we get \( \sigma(\nu^\pm) = \lambda^\pm \) by (3.2.1). \( \square \)

4. Quantum groups

4.1. R-matrices for quantum groups.

Theorem 4.1.1 ([20, 21, 26]). The expression \( R = \hat{R} \hat{R} \in U_h(\mathfrak{g}) \hat{\otimes} U_h(\mathfrak{g}) \) is an R-matrix in the topological Hopf algebra \( U_h(\mathfrak{g}) \), where
\[
\hat{R} = \prod_{\beta \in (\beta_1, \ldots, \beta_N)} \exp_{q_\beta}( (q_\beta - q_\beta^{-1}) E_\beta \otimes F_\beta ), \quad (4.1.1)
\]
\[
\hat{R} = \exp\left( \frac{\hbar}{2} \sum c_{ij} H_i \otimes H_j \right),
\]
where \( q_\beta = e^{h(\beta|\beta)/2} \) and \( (c_{ij}) \) is the inverse matrix to \( (d_{aij}) \).

Proposition 4.1.2 (cf [58]). The Cartan part of the R-matrix, \( \hat{R} \in U_h(\mathfrak{h}) \hat{\otimes} U_h(\mathfrak{h}) \) satisfies
\[
\hat{R} \cdot E_i \otimes 1 \cdot \hat{R}^{-1} = E_i \otimes K_i, \quad \hat{R} \cdot F_i \otimes 1 \cdot \hat{R}^{-1} = F_i \otimes K_i^{-1},
\]
\[
\hat{R} \cdot 1 \otimes E_i \cdot \hat{R}^{-1} = K_i \otimes E_i, \quad \hat{R} \cdot 1 \otimes F_i \cdot \hat{R}^{-1} = K_i^{-1} \otimes F_i.
\]
Consider an embedding of Hopf algebras $U_q(\mathfrak{g}) \hookrightarrow U_h(\mathfrak{g}) \otimes_{\mathbb{C}[h]} \mathbb{C}[h^{-1}, h]$, $K_i \mapsto e^{h_i H_i}$, with respect to a field extension $\mathbb{Q}(q) \hookrightarrow \mathbb{C}[h^{-1}, h]$, $q \mapsto e^h$. The image of Lusztig’s divided power algebra $\Gamma(\mathfrak{g})$ is contained in $U_h(\mathfrak{g})$. Let

$$U_h(\mathfrak{g})^\alpha = \{ x \in U_h(\mathfrak{g}) \mid [H_i, x] = \alpha(H_i)x \}$$

$$\Gamma(\mathfrak{g})^\alpha = \{ x \in \Gamma(\mathfrak{g}) \mid K_i x K_i^{-1} = q^{\alpha(\mathfrak{g})}x \}$$

be natural gradings, $\alpha \in \mathbb{Q}$. Note that $\Gamma(\mathfrak{g})^\alpha \subset U_h(\mathfrak{g})^\alpha$. We have $R, \tilde{R} \in \prod_{\alpha \in Q_+} (U_h(\mathfrak{b}_+)^\alpha \otimes U_h(\mathfrak{b}_-)^{-\alpha})$. Denote $\tilde{R} = \sum_{\alpha \in Q_+} \tilde{R}_\alpha$, then $\tilde{R}_\alpha \subset U_h(\mathfrak{b}_+)^\alpha \otimes U_h(\mathfrak{b}_-)^{-\alpha}$. Combining Theorem 4.1.1 and Proposition 4.1.2 we get

**Proposition 4.1.3 (cf [58]).** In $U_h(\mathfrak{g})^{\otimes 3}$ (or $U_h(\mathfrak{g})^{\otimes 2}$) the following equations are satisfied

$$(\Delta \otimes 1) \tilde{R} = \tilde{R}^{13} \cdot \sum_{\alpha \in Q_+} K_{-\alpha} \otimes \tilde{R}_\alpha$$  \hspace{1cm} (4.1.2)

$$(1 \otimes \Delta) \tilde{R} = \tilde{R}^{13} \cdot \sum_{\alpha \in Q_+} \tilde{R}_\alpha \otimes K_{\alpha}$$  \hspace{1cm} (4.1.3)

$$\Delta^{op} \cdot \tilde{R} \tilde{R} \tilde{R} = \tilde{R} \cdot \tilde{\Delta} x$$  \hspace{1cm} (4.1.4)

where $x \in U_h(\mathfrak{g})$ and the new coproduct $\tilde{\Delta} x = \tilde{R} \Delta x \tilde{R}^{-1}$ satisfies

$$\tilde{\Delta} H_i = H_i \otimes 1 + 1 \otimes H_i$$

$$\tilde{\Delta} E_i = E_i \otimes K_i + 1 \otimes E_i$$

$$\tilde{\Delta} F_i = F_i \otimes 1 + K_i^{-1} \otimes F_i$$

**Proof.** Since $\tilde{R} \tilde{R}$ is the $R$-matrix we have

$$(\Delta \otimes 1)(\tilde{R} \tilde{R}) = \tilde{R}^{13} \tilde{R}^{13} \tilde{R}^{23} \tilde{R}^{23}$$

$$= \tilde{R}^{13} \cdot (\sum_{\alpha \in Q_+} K_{-\alpha} \otimes \tilde{R}_\alpha) \cdot \tilde{R}^{13} \tilde{R}^{23}$$

$$= \tilde{R}^{13} \cdot (\sum_{\alpha \in Q_+} K_{-\alpha} \otimes \tilde{R}_\alpha) \cdot (\Delta \otimes 1)(\tilde{R})$$

whence (4.1.2). Other properties are similar. \( \square \)

Moreover, $\tilde{R}_\alpha \in \Gamma(\mathfrak{b}_+)^\alpha \otimes_{\mathbb{Z}[q, q^{-1}]} \Gamma(\mathfrak{b}_-)^{-\alpha} \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{C}[h]$, due to the formula

$$\exp_{q^{-2}}((q_{\beta} - q_{\beta}^{-1}) E_{\beta} \otimes F_{\beta}) = \sum_{m \geq 0} q_{\beta}^{m(m-1)}(q_{\beta} - q_{\beta}^{-1})^m (m)_{q_{\beta}^{-2}}! E_{\beta}^m \otimes F_{\beta}^m.$$  \hspace{1cm} (4.1.5)

Since $\Gamma(\mathfrak{g})$ is a free $\mathbb{Z}[q, q^{-1}]$-module [33], the equations (4.1.2)–(4.1.4) with $x \in \Gamma(\mathfrak{g})$ can be interpreted as follows. All terms are elements of the $\mathbb{Z}[q, q^{-1}]$-module
\( \pi_{\alpha,\beta,\gamma} \in Q \Gamma(g)^\alpha \otimes \Gamma(g)^\beta \otimes \Gamma(g)^\gamma \) (or of \( \prod_{\alpha,\beta} Q \Gamma(g)^\alpha \otimes \Gamma(g)^\beta \supseteq \hat{R} \)) and the equations state in particular that these elements can be multiplied.

Let \( \varepsilon \in \mathbb{C} \) be a root of unity. Change the base by the homomorphism \( \mathbb{Z}[q, q^{-1}] \to \mathbb{C}, q \mapsto \varepsilon \) and denote

\[
\Gamma_{\varepsilon}(g) = \Gamma(g) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{C}.
\]

The equations (4.1.2)–(4.1.4) still hold for \( \Gamma_{\varepsilon}(g) \) in place of \( \Gamma(g) \). But the sum (4.1.5) is finite, so \( \hat{R}_\alpha \) vanish for all \( \alpha \) except finite number and \( \hat{R} \in \Gamma_{\varepsilon}(b_+ \otimes \Gamma_{\varepsilon}(b_-)) \). Therefore, (4.1.2)–(4.1.4) hold in \( \Gamma_{\varepsilon}(g)^{\otimes 3} \) or \( \Gamma_{\varepsilon}(g)^{\otimes 2} \) with \( x \in \Gamma_{\varepsilon}(g) \).

Reversing the proof of Proposition 4.1.3 we see that if we find a symmetric tensor \( \hat{R} \in \Gamma_{\varepsilon}(h)^{\otimes 2} \) satisfying \( (\Delta \otimes 1)\hat{R} = \hat{R}^{13}\hat{R}^{23} \) and

\[
\begin{align*}
\hat{R} \cdot 1 \otimes E_i^{(p)} \cdot \hat{R}^{-1} &= K_i^p \otimes E_i^{(p)} \quad (4.1.6) \\
\hat{R} \cdot 1 \otimes F_i^{(p)} \cdot \hat{R}^{-1} &= K_i^{-p} \otimes F_i^{(p)} \quad (4.1.7)
\end{align*}
\]

for all \( p \in \mathbb{Z}_{>0} \), we could construct an \( \hat{R} \)-matrix \( R = \hat{R}\hat{R} \) for \( \Gamma_{\varepsilon}(g) \). It turns out that such element is easy to find not in \( \Gamma_{\varepsilon}(g) \) but in its quotient (which sometimes coincide with \( \Gamma_{\varepsilon}(g) \)). Consider a symmetric bilinear (bimultiplicative) form

\[
\pi : Q \times Q \to \mathbb{C}^\times, \quad \pi(K_i, K_j) = q_i^{a_{ij}}, \quad \pi(K_\alpha, K_\beta) = q^{(\alpha|\beta)}.
\]

Let \( \text{Ann } \pi = \{g \in Q \mid \forall h \quad \pi(g, h) = 1\} \) be its annihilator. Since for \( h \in Q \)

\[
\begin{align*}
hE_i^{(p)} &= \pi(K_i, h)^p E_i^{(p)} h \\
hF_i^{(p)} &= \pi(K_i, h)^{-p} F_i^{(p)} h
\end{align*}
\]

the elements of \( \text{Ann } \pi \) lie in the centre of \( \Gamma_{\varepsilon}(g) \). Introduce the Hopf \( \mathbb{C} \)-algebra

\[
\Gamma'_{\varepsilon}(g) = \Gamma_{\varepsilon}(g)/(g-1)_{g \in \text{Ann } \pi}.
\]

The subgroup generated by \( K_i \) in \( \Gamma'_{\varepsilon}(g) \) is denoted \( \mathbb{G} = Q/\text{Ann } \pi \). The form \( \pi \) factorizes through a non-degenerate form on \( \mathbb{G} \) denoted also \( \pi \) by abuse of notations.

Clearly, \( K_i^{2l_i} = 1 \) in \( \mathbb{G} \), where \( l_i \) are minimal positive integers such that \( q_i^{2l_i} = 1 \). More elements from \( \text{Ann } \pi \) can be found via the following

**Lemma 4.1.4.** For any \( d = 1, 2, 3 \) let \( l(d) \) be the minimal positive integer \( l \) such that \( q_i^{2dl_i} = 1 \) and let \( \rho(d) = \sum_{|\alpha|^2 = 2dl_i} \alpha \). Then for any \( i \)

\[
q_i^{l(d)(\alpha_i, \rho(d))} = 1.
\]

**Proof.** The equation to prove is

\[
q^{l(d)(\alpha_i, \rho(d))} = 1.
\]
Assume first that $||\alpha||^2 \neq 2d$. Let $s_i$ be the simple reflection corresponding to $\alpha_i$. The well known equation $s_i(\Delta^+ - \alpha_i) = \Delta^+ - \alpha_i$ implies $s_i(\Delta^+_d) = \Delta^+_d$, where $\Delta^+_d = \{ \alpha \in \Delta^+ \mid ||\alpha||^2 = 2d \}$. Hence, $s_i(\rho(d)) = \rho(d)$ and
\[
(\alpha_i|\rho(d)) = (s_i(\alpha_i)|s_i(\rho(d))) = -(\alpha_i|\rho(d))
\]
vanishes.

Assume now that $||\alpha||^2 = 2d$. Then $s_i(\Delta^+_d - \alpha_i) = \Delta^+_d - \alpha_i$ and $s_i(\rho(d) - \alpha_i) = \rho(d) - \alpha_i$. Hence,
\[
(\alpha_i|\rho(d)) = (s_i(\alpha_i)|s_i(\rho(d))) = -(\alpha_i|\rho(d) - 2\alpha_i)
\]
implies $(\alpha_i|\rho(d)) = (\alpha_i|\alpha_i) = 2d$ and the lemma follows. \qed

**Corollary 4.1.5.** For any $i$
\[
\frac{\langle \alpha_i, \sum_{\alpha \in \Delta^+} t_\alpha \alpha \rangle}{q_i} = 1.
\]

**Corollary 4.1.6.** For any $d = 1, 2, 3$ we have $K^{l(d)}_{\rho(d)} = 1$ in $G$. In particular,
\[
\prod_{\alpha \in \Delta^+} K^{l_\alpha}_{\alpha} = 1.
\]

The $R$-matrix from the following theorem was already obtained by Rosso [53] in the case of $l$ relatively prime to $\det(d_{ij}a_{ij})$ using the Drinfeld’s double.

**Theorem 4.1.7.** The Hopf algebra $H = \Gamma'_{\varepsilon}(g)$ is quasitriangular with the $R$-matrix $R = \tilde{R}R$ where $\tilde{R}$ is given by (4.1.1) and
\[
\tilde{R} = \frac{1}{|G|} \sum_{g,h \in G} \pi(g,h)^{-1} g \otimes h.
\]

**Proof.** We check the property (4.1.6) of $\tilde{R} \in \Gamma'_{\varepsilon}(g)^{\otimes 2}$
\[
\tilde{R} \cdot 1 \otimes E^{(p)}_i = \frac{1}{|G|} \sum_{g,h \in G} \pi(g,h)^{-1} g \otimes h E^{(p)}_i
\]
\[
= \frac{1}{|G|} \sum_{g,h \in G} \pi(g,h)^{-1} \pi(K_i, h) g \otimes E^{(p)}_i h
\]
\[
= \frac{1}{|G|} \sum_{g,h \in G} \pi(gK_i^{-p}, h)^{-1} g \otimes E^{(p)}_i h
\]
\[
= \frac{1}{|G|} \sum_{g,h \in G} \pi(f, h)^{-1} K_i^p f \otimes E^{(p)}_i h
\]
\[
= K_i^p \otimes E^{(p)}_i \cdot \tilde{R}
\]
Similarly for (4.1.7).
\( \hat{R} \) is an \( R \)-matrix for \( \Gamma'_\varepsilon(\mathfrak{h}) \). Indeed,

\[
\hat{R}^{13} \hat{R}^{23} = \frac{1}{|G|^2} \sum_{g,h,k,l \in G} \pi(g,h)^{-1} \pi(k,l)^{-1} g \otimes k \otimes h l
\]

\[
= \frac{1}{|G|^2} \sum_{g,h,k,l \in G} g \otimes k \otimes f \sum_{h \in G} \pi(g,h)^{-1} \pi(k,fh^{-1})^{-1}
\]

\[
= \frac{1}{|G|^2} \sum_{g,h,k,l \in G} \pi(k,f)^{-1} g \otimes k \otimes f \sum_{h \in G} \pi(kg^{-1},h)
\]

\[
= \frac{1}{|G|^2} \sum_{g,h,k,l \in G} \pi(k,f)^{-1} g \otimes k \otimes f |G| \delta_{k,g}
\]

\[
= \frac{1}{|G|} \sum_{g,h,k \in G} \pi(g,f)^{-1} g \otimes g \otimes f
\]

\[
= (\Delta \otimes 1) \hat{R}
\]

and by symmetry \((1 \otimes \Delta) \hat{R} = \hat{R}^{13} \hat{R}^{12}\). It follows from the above discussion that \( \hat{R} \hat{R} \)

is an \( R \)-matrix for \( \Gamma'_\varepsilon(\mathfrak{g}) \). \( \square \)

### 4.2. The algebra \( u_q(\mathfrak{g}) \)

Assume that for \( q = \varepsilon \) we have \( \varepsilon^{2m} \neq 1 \) for all \( 0 < m \leq d \).

Rewrite Theorem 4.1.7 as

\[
R = \sum_{0 \leq m_0 \leq \ell_0} \prod_{\alpha} \left( \frac{q_\alpha - q_\alpha^{-1}}{m_\alpha} \right)^{m_\alpha} \prod_{\alpha} F_{\alpha}^{m_\alpha} \otimes \prod_{\alpha} F_{\alpha}^{m_\alpha} \cdot \hat{R}
\]  \hspace{1cm} (4.2.1)

(\( \alpha \) runs over \( \beta_1, \ldots, \beta_N \)). Non-degeneracy of \( \pi : G \times G \to \mathbb{C}^\times \) implies that the minimal subspaces \( A, B \subset \mathbb{C}[G] \) such that \( \hat{R} \in A \otimes B \) are \( A = B = \mathbb{C}[G] \). Thus, \( (4.2.1) \) implies that the smallest subspaces \( h_+, h_- \subset \Gamma'_\varepsilon(\mathfrak{g}) \) such that \( \hat{R} \in h_+ \otimes h_- \) have bases

\[
h_1 \prod_{\beta \in \{\beta_1, \ldots, \beta_N\}} \mathbb{E}_{\beta}^{k_\beta}, \quad r_1 \prod_{\beta \in \{\beta_1, \ldots, \beta_N\}} \mathbb{F}_{\beta}^{m_\beta}
\]

where \( h \in \mathbb{G} \), \( 0 \leq k_\beta < l_\beta \). As discussed in Section 3.1 (cf. [45]) \( h_+ \) and \( h_- \) are Hopf subalgebras. They coincide with the subalgebras \( u_q(b_+) = \mathbb{C}(K_i^{1},E_i)_{1 \leq i \leq n} \) and \( u_q(b_-) = \mathbb{C}(K_i^{-1},F_i)_{1 \leq i \leq n} \) since \( E_i \in h_+, \quad F_i \in u_q(b_-) \) by e.g. [34, Proposition 40.1.3]. Therefore, by general theory \( u_q(\mathfrak{g}) = \mathbb{C}(K_i^{1},E_i,F_i)_{1 \leq i \leq n} = h_- h_+ = h_- h_+ \) is a quasitriangular Hopf subalgebra of \( \Gamma'_\varepsilon(\mathfrak{g}) \). It has the basis \( \prod_{\alpha \in \{\beta_1, \ldots, \beta_N\}} \mathbb{E}_{\alpha}^{k_\alpha} \cdot h \cdot \prod_{\beta \in \{\beta_1, \ldots, \beta_N\}} \mathbb{F}_{\beta}^{m_\beta} \), where \( h \in \mathbb{G} \), \( 0 \leq k_\beta, m_\beta < l_\beta \). The \( R \)-matrix \( \hat{R} \in u_q(b_+) \otimes u_q(b_-) \) is the dual tensor to the non-degenerate Hopf pairing

\[
\pi : h_- \times h_+^\text{op} \to \mathbb{C}
\]

\[
\pi(h, E_i) = 0 \quad \text{for} \quad h \in \mathbb{G},
\]

\[
\pi(F_i, h) = 0 \quad \text{for} \quad h \in \mathbb{G},
\]

\[
\pi(F_i, E_j) = \delta_{ij}(q_i - q_i^{-1})^{-1} \quad \text{for} \quad 1 \leq i, j \leq n.
\]
Table 1. An estimate for the constants $N(\mathfrak{g})$

| $\mathfrak{g}$ | $A_n$ | $B_n$ | $C_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|---------------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| $N(\mathfrak{g}) \leq$ | 1 | 4 | 4 | 2 | 3 | 4 | 6 | 8 | 9 |

4.2.1. A presentation of $u_q(\mathfrak{g})$. A lemma of Levendorskii and Soibelman [27] states: for any positive roots $\alpha < \beta$ there are unique constants $c_{n_1, \ldots, n_j} \in \mathbb{C}(q)$, such that in $U_q(\mathfrak{g}) \otimes_{\mathbb{Q}(q)} \mathbb{C}(q)$

$$E_\alpha E_\beta = q^{-(\alpha|\beta)} E_\beta E_\alpha = \sum_n c_{n_1, \ldots, n_j} E^n_{\gamma_1} \cdots E^n_{\gamma_j}, \quad (4.2.2)$$

where $j$ is the number of all positive roots lying between $\alpha$ and $\beta$, which are denoted $\alpha < \gamma_1 < \cdots < \gamma_j < \beta$. When this lemma is combined with Lusztig’s basis theorem for $\Gamma(\mathfrak{g})$ [34, Propositions 41.1.4, 41.1.7] we see that in fact

$$c_{n_1, \ldots, n_j} \in \mathbb{K} = \mathbb{Z}[q, q^{-1}, (q - q^{-1})^{-1}, [N(\mathfrak{g})]_1],$$

where the constant $N(\mathfrak{g}) \geq \max_i d_i$ is minimal possible. An upper bound for it is given in Table 1. It is obtained as the product of $\max_i d_i$ with the maximal coefficient $c_i$ in decomposition $\alpha = \sum c_i \alpha_i$ of the highest root $\alpha \in \Delta$. (Apply $T_{i_k}^{-1} T_{i_{k-1}}^{-1} \cdots T_{i_1}^{-1}$ to (4.2.2), if $E_{\alpha} = T_{i_1} \cdots T_{i_{k-1}} E_{i_k}$.) This estimate is rather rough, and can be essentially improved. The conjectured value of $N(\mathfrak{g})$ is $\max_i d_i$.

Introduce a $\mathbb{K}$-subalgebra $U_q(\mathfrak{g}) \subset U_q(\mathfrak{g})$ generated by $K_i, E_i, F_i$. It is closed under the automorphisms $T_i$ and all relations (4.2.2) make sense in $U_q(\mathfrak{g})$, thus $U_q(\mathfrak{g})$ has a Poincaré–Birkhoff–Witt basis $\prod_i E^{n_i}_\alpha \cdot K^\cdot \prod_j F^{n_j}_\beta$ (compare [25, 34]). Clearly, $U_q(\mathfrak{g})$ is a subalgebra of $\Gamma(\mathfrak{g}) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{K}$.

Assume in this subsection that $\varepsilon^{2m} \neq 1$ for all $1 \leq m \leq N(\mathfrak{g})$. Then we have a homomorphism of Hopf algebras

$$\phi : U_q(\mathfrak{g}) \hookrightarrow U_q(\mathfrak{g}) \otimes_{\mathbb{Z}[q, q^{-1}]} \mathbb{K} \to \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g})$$

via the homomorphism $\mathbb{K} \to \mathbb{C}$, $q \mapsto \varepsilon$. Since $\mathbb{C}$ is the image of $\phi = u_q(\mathfrak{g})$ the relations (4.2.2) are valid also in $u_q(\mathfrak{g})$.

The algebra $U_{\varepsilon}(\mathfrak{g}) = U_q(\mathfrak{g}) \otimes_{\mathbb{K}} \mathbb{C}$ with $q$ set to $\varepsilon$. The obvious homomorphism $\tilde{\phi} : U_{\varepsilon}(\mathfrak{g}) \to \Gamma(\mathfrak{g})$ induced by (4.2.3) has $u_q(\mathfrak{g})$ as its image. Its kernel contains $E^{l_\alpha}_\alpha, F^{l_\alpha}_\alpha$ and $h - 1$ for $h \in \Ann(\pi : Q \times Q \to \mathbb{C}^\times)$. Indeed, $E^{l_\alpha}_\alpha = [l_\alpha] q! T_{i_1} \cdots T_{i_{k-1}} (E^{(l_\alpha)})$ in $U_q(\mathfrak{g})$ and $\Gamma(\mathfrak{g})$ for some sequence $(i_1, \ldots, i_{k-1}, i)$, and the factorial vanishes in $\Gamma(\mathfrak{g})$. Let $I \subset U_{\varepsilon}(\mathfrak{g})$ be a two-sided ideal generated by these elements. By the explicit form of bases we get $\dim U_{\varepsilon}(\mathfrak{g})/I \leq \dim u_q(\mathfrak{g})$. Since $\tilde{\phi}$ induces an epimorphism $U_{\varepsilon}(\mathfrak{g})/I \to u_q(\mathfrak{g})$, this is in fact an isomorphism, and a presentation of $u_q(\mathfrak{g})$ by generators and relations follows. Namely, the new relations

$$E^{l_\alpha}_\alpha = 0, \quad F^{l_\alpha}_\alpha = 0, \quad h = 1 \text{ for } \alpha \in \Delta^+, \quad h \in \Ann(\pi : Q \times Q \to \mathbb{C}^\times)$$
are added to the standard presentation of $U_q(\mathfrak{g})$.

4.2.2. The special grouplike element for $u_q(\mathfrak{g})$. Let us find the grouplike element $g = u \gamma(u)^{-1}$ for the algebras $H$ and $h$, where $u = \sum_n \gamma(b^n) a_n$, $R = \sum_n a_n \otimes b^n$. Clearly,

$$\delta_+ = \sum_{h \in G} h \cdot \prod_{\beta \in (\beta_1, \ldots, \beta_N)} E_{\beta}^{l_{\beta} - 1} \in h_+,$$

$$\delta_- = \sum_{h \in G} h \cdot \prod_{\beta \in (\beta_1, \ldots, \beta_N)} F_{\beta}^{l_{\beta} - 1} \in h_-$$

are non-zero left integrals in the algebras $h_+$ and $h_-$. Indeed, it suffices to check this for $h \in G$

$$\delta_+ y = \sum_{h \in G} h \cdot \prod_\beta E_{\beta}^{l_{\beta} - 1} y = \sum_{h \in G} h y \cdot \prod_\beta \pi(K_\beta, y)^{-l_{\beta} + 1} E_{\beta}^{l_{\beta} - 1} = \pi(\prod_\beta K_\beta^{-l_{\beta} + 1}, y) \delta_+.$$

Similarly, for any $y \in h_-$ and $a_- = \prod_\beta K_\beta^{-l_{\beta} + 1} = K_{-2\rho} \in h_-$ (cf Corollary 4.1.6) we have

$$\delta_- y = \pi(y, a_+) \delta_-.$$

By Drinfeld’s theorem [6] (see Section 3.1) we find

$$g = a_- a_+^{-1} = K_{1_{\mathfrak{g}}} \in h.$$

4.3. Ribbon structure of $\Gamma'_\varepsilon(\mathfrak{g})$ and $u_q(\mathfrak{g})$. By definition a ribbon structure is a choice of a group-like element $\kappa$ such that $\kappa^2 = K_{1_{\mathfrak{g}}}$ and $\kappa a = \gamma^2(a)\kappa$ for all $a \in \Gamma'_\varepsilon(\mathfrak{g})$. The only group-like elements of $\Gamma'_\varepsilon(\mathfrak{g})$ are $K_a \in \mathfrak{g}$. Commuting $\kappa$ with $E_i$ we get $\pi(\kappa, K_i^l) = q_i^2$, which holds for the only element $\kappa = K_{2\rho}$. Therefore, the quasitriangular Hopf algebra $\Gamma'_\varepsilon(\mathfrak{g})$ (or $u_q(\mathfrak{g})$) admits the unique ribbon structure $\kappa = K_{2\rho}$.

Let us find the ribbon twist element using the formula $\nu = \sum_n a_n K_{2\rho}^{-1} b^n$, $R = \sum_n a_n \otimes b^n$. Equation (4.2.1) can be rewritten as

$$R = \sum_{0 \leq m_{\alpha} < l_{\alpha}} \left( \prod_\alpha \left( \frac{q_{\alpha} - q_{\alpha}^{-1} m_{\alpha}}{(m_{\alpha})_{q_{\alpha} - 2}!} E_{\alpha}^{m_{\alpha} a_{\alpha}} \otimes 1 \right) \cdot \check{R}_{\alpha} \cdot \left( \prod_\alpha K_{\alpha}^{m_{\alpha}} \otimes F_{\alpha}^{m_{\alpha}} \right) \right)$$

$$= \sum_{0 \leq m_{\alpha} < l_{\alpha}} \left( \prod_\alpha \left( \frac{q_{\alpha} - q_{\alpha}^{-1} m_{\alpha}}{(m_{\alpha})_{q_{\alpha} - 2}!} E_{\alpha}^{m_{\alpha} a_{\alpha}} \right) \check{R}_{\alpha} \left( \prod_\alpha K_{\alpha}^{m_{\alpha}} \right) \otimes \check{R}_{\alpha} \left( \prod_\alpha F_{\alpha}^{m_{\alpha}} \right) \right).$$
where $\hat{R} = \sum_a \hat{R}_a \otimes \hat{R}_a'$. Therefore,

$$\nu = \sum_{0 \leq m_\alpha < l_\alpha} \left( \prod_\alpha \left( \frac{q_\alpha - q_\alpha^{-1}}{(m_\alpha)_{q_\alpha^{-2}}} \right)^{m_\alpha} \right) \cdot \hat{R}_a \hat{R}_a' K_{-2\rho - 1} \left( \prod_\alpha F^{m_\alpha} \right) \cdot \left( \prod_\alpha F^{m_\alpha} \right). \quad (4.3.1)$$

### 4.3.1. The subalgebra $u$

Recall that the subalgebra $u \subset h$ is defined as the smallest subspace such that $R_{12} R_{21} \in u \otimes h$. Let us find it.

Theorem 4.1.7 gives also an alternative expression for

$$R = \hat{R} \cdot \prod_{\beta \in (\beta_1, \ldots, \beta_N)} \exp_{q_\beta^{-2}} ((q_\beta - q_\beta^{-1}) E_{\beta} K_{\beta} \otimes K_{-1 \beta}^{-1} F_{\beta})$$

which yields

$$R_{12} R_{21} = \sum_{n,m} \left( \prod_\alpha \left( \frac{q_\alpha - q_\alpha^{-1}}{(m_\alpha)_{q_\alpha^{-2}}} \right)^{m_\alpha} \right) \hat{R}_a \hat{R}_b' \left( \prod_{\beta} (K_{-1 \beta}^{-1} F_{\beta})^{n_\beta} \right) \otimes \left( \prod_\alpha F^{m_\alpha} \right) \hat{R}_a' \hat{R}_b' \left( \prod_{\beta} \left( \frac{q_\beta - q_\beta^{-1}}{(n_\beta)_{q_\beta^{-2}}} \right)^{n_\beta} (E_{\beta} K_{\beta})^{n_\beta} \right). \quad (4.3.2)$$

### Proposition 4.3.1

The algebra $u$ has the basis

$$\left( \prod_\alpha E^{m_\alpha} \right) K_\lambda \left( \prod_{\beta} F^{n_\beta} \right),$$

where $K_\lambda = K_{2 \mu} \cdot \prod_{\beta} K_{\beta}^{n_\beta}$ in $G$ for some $\mu \in Q$.

This follows from the formula (4.3.2) above and

### Lemma 4.3.2

The smallest subspace $A \subset C[G]$ such that $\hat{R}^2 \in A \otimes C[G]$ is $A = C[2G]$, where $2G = \{ x^2 \mid x \in G \} \subset G$ is the subgroup of squares.

**Proof.** Note that $\hat{R}$ is a symmetric tensor and

$$\hat{R}^2 = \frac{1}{|G|^2} \sum_{g,h \in G} \pi(g, h)^{-1} g \otimes h \cdot \frac{1}{|G|} \sum_{a,b \in G} \pi(a, b)^{-1} a \otimes b$$

$$= \frac{1}{|G|^2} \sum_{c,d \in G} c \otimes d \sum_{g,b \in G} \pi(g, bd^{-1}) \pi(gc^{-1}, b)$$

$$= \frac{1}{|G|^2} \sum_{c,d \in G} c \otimes d \sum_{g,b \in G} \pi(g, b^2 d^{-1}) \pi(c^{-1}, b)$$

$$= \frac{1}{|G|} \sum_{c,d \in G} c \otimes d \sum_{b \in G} \delta_{d,b} \pi(c, b)^{-1}.$$
This implies that $\hat{R}^2 \in \mathbb{C}[2G] \otimes \mathbb{C}[2G]$. Introduce a symmetric bilinear (bimultiplicative) pairing

$$\pi_2 : 2G \times 2G \to \mathbb{C}^\times, \quad \pi_2(c, d) = \pi(c, b), \text{ where } b^2 = d.$$ 

Clearly, $\pi(c, b)$ has the same value for all $b \in G$ such that $b^2 = d$. All such $b$ differ by an element of $X = \{b \in G \mid b^2 = 1\}$. Therefore,

$$\hat{R}^2 = \frac{1}{|G|} \sum_{c,d \in 2G} \pi_2(c, d)^{-1} c \otimes d \sum_{b \in G} \delta_{d,b^2} = \frac{1}{|X|} \sum_{c,d \in 2G} \pi_2(c, d)^{-1} c \otimes d$$

and the statement follows. □

**Corollary 4.3.3.** $u_q(g)$ is factorizable if and only if $2G = G$. In particular, it is factorizable if the degree $l$ of the root $\varepsilon$ is odd.

### 4.4. 2-modular structure of $u_q(g)$ and $\Gamma'_\varepsilon(g)$

Now we can answer the question when $u_q(g)$ is 2-modular.

**Theorem 4.4.1.** The ribbon Hopf algebras $u_q(g)$ and $\Gamma'_\varepsilon(g)$ are 2-modular, that is $\nu \in u$, if and only if for any $x \in G$ such that $x^2 = 1$ we have

$$\pi(x, x) = \pi(x, K_{2\rho}). \quad (4.4.1)$$

**Remark 4.4.1.** Both sides of (4.4.1) are characters $X \to \{1, -1\}$, where $X$ is the subgroup $\{x \in G \mid x^2 = 1\}$.

**Proof.** Equation (4.3.1) together with Proposition 4.3.1 imply that $\nu \in u$ if and only if $\hat{R}'_a \hat{R}''_a K_{-2\rho} \in \mathbb{C}[2G]$. We have

$$\hat{R}'_a \hat{R}''_a K_{-2\rho} = \frac{1}{|G|} \sum_{g,h \in G} \pi(g, h)^{-1} ghK_{-2\rho} = \frac{1}{|G|} \sum_{b \in G} \phi(b) b, \quad (4.4.2)$$

where the coefficient $\phi(b)$ is the Gaussian sum

$$\phi(b) = \sum_{g \in G} \pi(g, g) \pi(g, b^{-1} K_{-2\rho}). \quad (4.4.3)$$
Its absolute value is found by the standard procedure
\[ |\phi(b)|^2 = \sum_{h,x \in G} \pi(hx, hx) \pi(hx, b^{-1}K_{-2p}) \pi(h, h)^{-1} \pi(h, b^{-1}K_{-2p})^{-1} \]
\[ = \sum_{x \in G} \pi(x, x) \pi(x, b^{-1}K_{-2p}) \sum_{h \in G} \pi(h, x^2) \]
\[ = |G| \sum_{x \in X} \pi(x, x) \pi(x, b^{-1}K_{-2p}). \quad (4.4.4) \]

This formula suggests to consider a function on \( Y = \mathbb{G}/2\mathbb{G} \)
\[ \psi(y) = \sum_{x \in X} \pi(x, x) \pi(x, y). \]

Here the pairing \( \pi \) restricts to a non-degenerate pairing \( \pi: X \times Y \to \{1, -1\} \). Thus the algebra \( u_q(\mathfrak{g}) \) is 2-modular iff \( \psi(y) = \text{const} \delta_{y[\mathbb{G}/2\mathbb{G}]} \). This condition means that the Fourier coefficients \( \pi(x, x) \) of the function \( \psi \) are proportional to the Fourier coefficients \( \pi(x, K_{2p}) \) of the delta function \( \delta_{y[\mathbb{G}/2\mathbb{G}]} \). Equivalently, \( \pi(x, x) = \pi(x, K_{2p}) \) for all \( x \in X \) since the proportionality constant is 1. \( \square \)

4.5. The integral on \( u_q(\mathfrak{g}) \).

**Proposition 4.5.1.** The functional \( \int: u_q(\mathfrak{g}) \to \mathbb{C} \)
\[ \int \left( \prod_{\alpha} E_{\alpha}^{m_{\alpha}} \cdot K_\lambda \cdot \prod_{\beta} F_{\beta}^{n_{\beta}} \right) = \prod_{\alpha} \delta_{m_{\alpha}, l_{\alpha} - 1} \cdot \delta_{K_\lambda, K_{-2p}} \cdot \prod_{\beta} \delta_{n_{\beta}, l_{\beta} - 1} \]
is a left integral on the Hopf algebra \( u_q(\mathfrak{g}) \). It is independent of the choice of the reduced expression for \( w_0 \).

**Proof.** Notice that for \( \tau = \sum_{\alpha \in \Delta^+} (l_{\alpha} - 1) \alpha \)
\[ (1 \otimes \int) \Delta \left( \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1} \cdot K_\lambda \cdot \prod_{\beta} F_{\beta}^{l_{\beta} - 1} \right) = K_{-\tau} K_\lambda \int \left( \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1} \cdot K_\lambda \cdot \prod_{\beta} F_{\beta}^{l_{\beta} - 1} \right) \]
\[ = K_{2p} K_{-2p} \int \left( \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1} \cdot K_\lambda \cdot \prod_{\beta} F_{\beta}^{l_{\beta} - 1} \right) \]
since \( K_{-\tau} = K_{2p} \) by Corollary 4.1.6. This implies \( (1 \otimes f) \Delta x = f x \).

All left integrals are proportional to each other, whence all functionals \( f \) defined for different reduced expressions are proportional to each other. The elements of the highest grade \( \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1} \otimes \prod_{\beta} F_{\beta}^{l_{\beta} - 1} \) are also proportional to each other. Since
\[ \pi \left( \prod_{\beta} F_{\beta}^{l_{\beta} - 1} \otimes \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1} \right) = \prod_{\alpha} \left( (l_{\alpha} - 1)_{q_{\alpha} - 1} (q_{\alpha} - q_{\alpha}^{-1})^{1 - l_{\alpha}} \right) \]
(see [20, 21, 26], compare with (4.2.1)) these elements are all equal. Therefore all integrals are equal. \( \square \)
4.5.1. *Invariance of the integral.* When \( u_q(g) \) is factorizable, it is unimodular by Proposition 3.7.4. We will prove it also in non-factorizable case.

**Proposition 4.5.2.** The algebra \( u_q(g) \) is unimodular.

*Proof.* A left integral \( \delta_+ \in u_q(b_+) \) and a right integral \( \omega_- \in u_q(b_-) \) are given by

\[
\delta_+ = \sum_{h \in \mathbb{G}} h \cdot \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1}, \quad \omega_- = \prod_{\beta} F_{\beta}^{l_{\beta} - 1} \cdot \sum_{h \in \mathbb{G}} h.
\]

by a result of Hennings [10] and Radford [45] the double \( D(u_q(b_+)) \) is unimodular with the two-sided integral \( \delta_+ \omega_- \). Its projection to \( u_q(g) \) via the epimorphism \( j : D(u_q(b_+)) \to u_q(g) \) is

\[
\delta = j(\delta_+ \omega_-) = |\mathbb{G}| \sum_{h \in \mathbb{G}} h \cdot \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1} \cdot \prod_{\beta} F_{\beta}^{l_{\beta} - 1}.
\]

This is a two-sided non-zero integral in \( u_q(g) \). \( \square \)

4.6. *3-modular structure of \( u_q(g) \).*

**Proposition 4.6.1.** If \( u_q(g) \) is a 2-modular Hopf algebra then it is 3-modular as well.

*Proof.* We have to check the remaining condition (M3) from Theorem 3.7.3, namely

\[
(\int \otimes 1)(R^{12} R^{21}) \neq 0.
\]

Only maximal powers will contribute to this expression (see (4.3.2))

\[
(\int \otimes 1)(R^{12} R^{21}) =
\]

\[
= \text{const} \int \{ \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1} \hat{R}_i^{\alpha} \hat{R}_j^{\alpha} \} \{ \prod_{\beta} F_{\beta}^{l_{\beta} - 1} \hat{R}_i^{\beta} \hat{R}_j^{\beta} \} \prod (K_{E_{\beta}}^{l_{\beta} - 1} - 1)
\]

\[
= \text{const} \int \{ \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1} \hat{R}_i^{\alpha} \hat{R}_j^{\alpha} K_{2\rho} \} \prod_{\beta} F_{\beta}^{l_{\beta} - 1} \hat{R}_i^{\beta} \hat{R}_j^{\beta} \prod (E_{\beta}^{l_{\beta} - 1} - 1) K_{-2\rho}
\]

\[
= \text{const} \sum_{c,d \in 2\mathbb{G}} \pi_2(c,d)^{-1} \int \{ \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1} c K_{2\rho} \} \prod_{\beta} F_{\beta}^{l_{\beta} - 1} \} \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1} d \prod_{\beta} E_{\beta}^{l_{\beta} - 1} K_{-2\rho}
\]

\[
= \text{const} \sum_{c,d \in 2\mathbb{G}} \prod_{\alpha} E_{\alpha}^{l_{\alpha} - 1} \prod_{\beta} E_{\beta}^{l_{\beta} - 1} K_{-2\rho}
\]

where we used (4.3.3). Since

\[
\text{const} = \left( \prod_{\alpha} \frac{(q_{\alpha} - q_{\alpha}^{-1})^{l_{\alpha} - 1}}{(l_{\alpha} - 1) q_{\alpha}^{-2!}} \right)^2
\]

this obviously does not vanish. \( \square \)
Now we find the unique (up to a sign) normalization of the integral which will be used for constructing switching operators.

**Proposition 4.6.2.** Let \( \int'_u \colon u_q(g) \to \mathbb{C} \) be the renormalized left integral

\[
\int' = \sqrt{|2G|} q^{-2(\rho|\rho)} \prod_{\alpha} \frac{(l_\alpha - 1)_{q_\alpha - 2}!}{(q_\alpha - q_\alpha^{-1})_{l_\alpha - 1}}
\]

considered also as an element \( \mu \in \text{fun}_q(G) \). If \( u_q(g) \) is 3-modular, then \( P = S_2^\gamma F : F \to F \) is a projection for \( S = S_-(\mu) \).

**Proof.** The claim is equivalent to equation (3.8.16)

\[
(\int' \otimes \int') (R^{12} R^{21}) = 1.
\]

Substituting the expression for \( (\int \otimes \int) (R^{12} R^{21}) \) from Proposition 4.6.1 we get

\[
(\int \otimes \int) (R^{12} R^{21}) = \left( \prod_{\alpha} \frac{(q_\alpha - q_\alpha^{-1})_{l_\alpha - 1}}{(l_\alpha - 1)_{q_\alpha - 2}!} \right)^2 \frac{1}{|2G|} \int (\prod_{\alpha} F_{\alpha}^{l_{\alpha} - 1}) (\prod_{\beta} E_{\beta}^{l_{\beta} - 1}) K_{-2\rho}
\]

\[
= \left( \prod_{\alpha} \frac{(q_\alpha - q_\alpha^{-1})_{l_\alpha - 1}}{(l_\alpha - 1)_{q_\alpha - 2}!} \right)^2 \frac{1}{|2G|} q^{(2\rho|2\rho)} K_{-2\rho} \prod_{\alpha} F_{\alpha}^{l_{\alpha} - 1}.
\]

due to \( (\prod_{\alpha} F_{\alpha}^{l_{\alpha} - 1}) \cdot K_{-2\rho} = q^{(2\rho|2\rho)} K_{-2\rho} \prod_{\alpha} F_{\alpha}^{l_{\alpha} - 1} \). This fixes the normalization. \( \square \)

**Proposition 4.6.3.** The exponential central charge \( \lambda = \int' \nu \) for 2-modular \( u_q(g) \) is a root of unity

\[
\lambda = q^{-2(\rho|\rho)} \sqrt{|2G|} \sum_{g \in G} \pi(g, g) \pi(g, K_{-2\rho})
\]

\[
= q^{-2(\rho|\rho)} \sqrt{|2G|} \frac{1}{n} \sum_{\alpha \in Q / lQ} q^{(\alpha|\alpha) - (\alpha|2\rho)},
\]

where \( n \) is the rank of \( g \) and \( l \) is the degree of the root of unity \( q = \varepsilon \).

**Proof.** From (4.3.1) we find

\[
\nu = \prod_{\alpha} \frac{(q_\alpha - q_\alpha^{-1})_{l_\alpha - 1}}{(l_\alpha - 1)_{q_\alpha - 2}!} \int (\prod_{\alpha} F_{\alpha}^{l_{\alpha} - 1}) \cdot \tilde{R}_a^\rho \tilde{R}_a^\rho K_{-4\rho} \cdot \prod_{\alpha} F_{\alpha}^{l_{\alpha} - 1}.
\]

Substituting \( \tilde{R}_a^\rho \tilde{R}_a^\rho K_{-2\rho} = \frac{1}{|G|} \sum_{b \in G} \phi(b) b \) by (4.4.2) we find

\[
\lambda = \int' \nu = \sqrt{|2G|} q^{-2(\rho|\rho)} \frac{1}{|G|} \phi(1),
\]
where by (4.4.3)
\[ \phi(1) = \sum_{g \in G} \pi(g, g) \pi(g, K_{-2\rho}). \] (4.6.3)

Its absolute value is determined by (4.4.4)
\[ |\phi(1)|^2 = |G| \sum_{x \in X} \pi(x, x) \pi(x, K_{-2\rho}). \] (4.6.4)
\[ = |G| \cdot |X| \]
since all summands equal 1 by Theorem 4.4.1. Therefore,
\[ |\lambda|^2 = \frac{2|G|}{|G|} |X| = 1 \]
due to an exact sequence \(0 \to X \to G \to 2G \to 0\).

By (4.6.1) \(\lambda\) satisfies an algebraic equation of degree \(2l\). Since \(\lambda\) is an algebraic number of absolute value 1, it is a root of unity.

If we sum up in (4.6.3) not over \(G\), but over its covering group \(Q/\ell Q\), the sum will multiply by \(\ell^r/|G|\). This proves (4.6.2).

**Remark 4.6.1.** The condition (4.4.1) implies that
\[ \pi(gx, gx) \pi(gx, K_{-2\rho}) = \pi(g, g) \pi(g, K_{-2\rho}) \]
for all \(g \in G, x \in X\). Therefore, we can introduce a function on \(2G = G/x\),
\[ s : 2G \to \mathbb{C}^\times, \quad s(h) = \pi(g, g) \pi(g, K_{-2\rho}) \text{ if } g^2 = h \]
and write
\[ \lambda = q^{-2(\rho, \rho)} \frac{1}{\sqrt{|2G|}} \sum_{h \in 2G} s(h). \] (4.6.5)

One easily recognizes in (4.6.1), (4.6.2) and (4.6.5) generalized quadratic Gauss sums.

Notice also that if \(u_q(\mathfrak{g})\) is not 2-modular, then \(\int \nu = 0\) by (4.6.4).

### 5. Examples of 3-modular algebras

We assume that \((a_{ij})\) is the Cartan matrix of a simple Lie algebra \(\mathfrak{g}\) and \(q = \varepsilon\) is a primitive root of unity of degree \(l\) such that \(\varepsilon^{2p} \neq 1\) for \(1 \leq p \leq N(\mathfrak{g})\). We explore case by case when \(u_q(\mathfrak{g})\) is 3-modular or equivalently 2-modular.

In general, \(G = Q/\text{Ann } \pi\), \(Q\) being the root lattice and \(\text{Ann } \pi = Q \cap l\text{ Cow}\), where \(\text{Cow} = \mathbb{Z}\{\frac{1}{d_i} \omega_i\}\) is the coweight lattice and \(\omega_i\) is the basis of the weight lattice \(P\).
This implies \( \text{Ann } \pi = Q \cap \mathbb{Z}\{l'_i \omega_i\} \), where \( l'_i = l \) if \( d_i \mid l \) and \( l'_i = l/d_i \) if \( d_i \nmid l \). The inclusion \( Q \hookrightarrow P, \alpha_j = \sum_{i=1}^n a_{ij} \omega_i \), determines the annihilator:

\[
\text{Ann } \pi = \{ u = \sum_j u_j \alpha_j \mid \forall i \sum_j a_{ij} u_j \equiv 0 \pmod{l'_i} \}
\]

Similarly for \( X = \ker \{ 2 : G \to G \} \)

\[
X = \{ u = \sum_j u_j \alpha_j \mid \forall i \sum_j a_{ij} u_j \equiv 0 \pmod{l'_i} \}/\text{Ann } \pi.
\]

2- or 3-modularity of \( u_q(g) \) is equivalent to the property

\[
(u|u) - (2|u) \equiv 0 \pmod{l}
\]

for all \( u \in X \).

If \( l \) is odd, \( X = 0 \) and \( u_q(g) \) is perfect modular (we shall see that this is not the only case).

5.1. The algebra \( u_q(\mathfrak{sl}(2)) \). This example was already considered in [40] for odd \( l \). Set \( l_1 = l \) for \( l \) odd and \( l_1 = l/2 \) for \( l \) even, then it is the degree of \( \varepsilon^2 \). Since \( \pi(K, K) = \varepsilon^2 \) for \( K = K_1 \), the group \( \mathbb{G} \) is isomorphic to \( \mathbb{Z}/l_1\mathbb{Z} \). Several cases emerge.

(1) \( l_1 = 2m + 1 \) is odd. Then \( u_q(\mathfrak{sl}(2)) \) is perfect and

\[
\lambda = \varepsilon^{-1} \frac{1}{\sqrt{l_1}} \sum_{k=1}^{l_1} \varepsilon^{2k^2-2k} = \left( \frac{2a}{l_1} \right) e^{-\frac{\pi a}{2} \varepsilon^{-1} - 2m^2}
\]

where \( a \) is determined by \( \varepsilon^2 = e^{2\pi i a/l_1} \). Here for any relatively prime integers \( c \) and odd positive \( b \left( \frac{c}{b} \right) = \Pi_j \left( \frac{a}{p_j} \right) \) denotes the Jacobi quadratic symbol, where \( b = p_1p_2 \ldots p_r \) with prime \( p_j \) and \( \left( \frac{a}{p_j} \right) = \pm 1 \) is the Legendre symbol (see e.g. [24]).

(2) \( l_1 = 2(2m + 1) \). Then \( u_q(\mathfrak{sl}(2)) \) is 3-modular since \( X = \{ 1, K^{2m+1} \} \) and

\[
\pi(K^{2m+1}, K^{2m+1})\pi(K^{2m+1}, K^{-1}) = \varepsilon^{2(2m+1)2m} = 1.
\]

\( 2\mathbb{G} \) is isomorphic to \( \mathbb{Z}/(2m + 1)\mathbb{Z} \) and according to (4.6.5)

\[
\lambda = \varepsilon^{-1} \frac{1}{\sqrt{2m + 1}} \sum_{k=0}^{2m} \varepsilon^{2k^2-2k} = \left( \frac{a}{2m + 1} \right) e^{\frac{\pi a}{2} \varepsilon^{2m^2+2m-1}}
\]
where \( a \) is determined by \( \varepsilon^2 = e^{2\pi ia/l_1} \). The value of the generalized Gauss quadratic sum was found here by a method of Lang [24] combined with results of Chandrasekharan [2].

(3) If \( l_1 = 4m \). Then \( u_\varepsilon(\mathfrak{sl}(2)) \) is not 2-modular since \( X = \{ 1, K^{2m} \} \) and 
\[
\pi(K^{2m}, K^{2m-1}) = \varepsilon^{4m(2m-1)} = \varepsilon^{\frac{4}{2}(2m-1)} = -1.
\]

5.2. The Cartan matrix \( A_n \). The determinant of the Cartan matrix is \( n + 1 \). Let \( p = (l, n + 1) \) be the greatest common divisor. Then

\[
\text{Ann } \pi = \left\{ \frac{L}{p} u \mid \forall i \sum_j a_{ij} v_j \equiv 0 \pmod{p} \right\}.
\]

Various possibilities appear here as we already have seen for \( n = 1 \).

5.3. The Cartan matrix \( B_n \). Here \( g = \mathfrak{o}(2n + 1) \), \( d_i = 2 \) for \( 1 \leq i < n \) and \( d_n = 1 \). Index of connection \( \det A = 2 \). Let \( l' = l \) for \( l \) odd and \( l' = l/2 \) for \( l \) even. Then

\[
\text{Ann } \pi = \left\{ \sum_j u_j \alpha_j \mid \forall i < n \sum_j a_{ij} u_j \equiv 0 \pmod{l'}, \sum_j a_{nj} u_j \equiv 0 \pmod{l} \right\}.
\]

In particular, \( 2u = (\det A)u \equiv 0 \pmod{l'} \) if \( u \in \text{Ann } \pi \). Solving the equations we find that \( \text{Ann } \pi = l'Q \) and \( G = Q/l'Q \).

(1) \( l' \) is odd. Then \( u_q(g) \) is perfect modular. \( l \) might be even in this case.

(2) \( l' = 2m \) is even. Then \( l = 4m, X = \mathbb{F}_2\{m\alpha_i\} \) and

\[
||m\alpha_i||^2 - (2\rho|m\alpha_i|) = 4m(m - 1) \equiv 0 \pmod{l} \quad \text{for } i < n
\]

\[
||m\alpha_n||^2 - (2\rho|m\alpha_n|) = 2m(m - 1).
\]

The algebra \( u_\varepsilon(g) \) is 2-modular iff \( m \) is odd.

5.4. The Cartan matrix \( C_n \). Here \( g = \mathfrak{sp}(2n) \), \( d_i = 1 \) if \( 1 \leq i < n \) and \( d_n = 2 \). The index of connection \( \det A = 2 \). Let \( l' = l \) for \( l \) odd and \( l' = l/2 \) for \( l \) even. Then

\[
\text{Ann } \pi = \left\{ \sum_j u_j \alpha_j \mid \forall i < n \sum_j a_{ij} u_j \equiv 0 \pmod{l}, \sum_j a_{nj} u_j \equiv 0 \pmod{l'} \right\}.
\]

(1) \( l \) is odd. Then \( u_q(g) \) is perfect modular with \( G = Q/lQ \).

(2) \( l = 2(2m + 1) \). Then \( u_q(g) \) is 3-modular and

\[
\text{Ann } \pi = \begin{cases} lQ + \mathbb{Z}(2m + 1)\alpha_n & \text{for } n \text{ odd} \\ lQ + \mathbb{Z}(2m + 1)\alpha_n + \mathbb{Z}(2m + 1)(\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1}) & \text{for } n \text{ even} \end{cases}
\]

In both cases \( X = \mathbb{F}_2\{(2m + 1)\alpha_i\} \).
(3) \( l = 4m \). Then
\[
\text{Ann } \pi = lQ + \mathbb{Z}2m\alpha_n + \mathbb{Z}m(2\alpha_1 + 2\alpha_3 + \cdots + 2\alpha_{2\frac{m}{l}} + n\alpha_n).
\]
A) \( n \) is odd. Then \( u_q(\mathfrak{g}) \) is 3-modular and
\[
X = F_2\{2m\alpha_i, m\alpha_n\}.
\]
B) \( n = 2n' \) is even. Then
\[
X = F_2\{2m\alpha_i, m\alpha_n, m(\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} + n'\alpha_n)\}
\]
and
\[
||m(\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} + n'\alpha_n)||^2 - (2\rho|m(\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1} + n'\alpha_n)) \equiv 2mn'(m - 3) \pmod{l}.
\]
Therefore, if \( n' \) even or \( m \) odd, then \( u_q(\mathfrak{g}) \) is 3-modular. If \( n' \) odd and \( m \) even, it is not.

5.5. The Cartan matrix \( D_n \). Here \( \mathfrak{g} = \mathfrak{o}(2n) \) and \( d_i = 1 \). The index of connection \( \det A = 4 \). Since \( Au \equiv 0 \pmod{l} \) implies \( 4u \equiv 0 \pmod{l} \), we have for \( p = (4,l) \), \( u = \frac{l}{4}b, b = (b_1, \ldots, b_n), Ab \equiv 0 \pmod{l} \). This enables one to find all \( u \in \text{Ann } \pi = Q \cap lP \).

(1) \( l \) is odd. \( u_q(\mathfrak{g}) \) is perfect modular with \( \mathbb{G} = Q/lQ \).
(2) \( l = 2(2m + 1) \). Then \( u_q(\mathfrak{g}) \) is 3-modular with
\[
\begin{align*}
\mathbb{G} &= Q/(lQ + \mathbb{Z}(2m + 1)(\alpha_{n-1} + \alpha_n)) \quad \text{for } n \text{ odd or} \\
\mathbb{G} &= Q/(lQ + \mathbb{Z}(2m + 1)(\alpha_{n-1} + \alpha_n) + \mathbb{Z}(2m + 1)(\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1})) \\
\end{align*}
\]
for \( n \) even. In both cases \( X = F_2\{(2m + 1)\alpha_i\} \).
(3) \( l = 4m \) and \( n \) is odd. \( u_q(\mathfrak{g}) \) is 3-modular with
\[
\text{Ann } \pi = lQ + \mathbb{Z}2m(\alpha_{n-1} + \alpha_n) + \mathbb{Z}m(2\alpha_1 + 2\alpha_3 + \cdots + 2\alpha_{n-2} - \alpha_{n-1} + \alpha_n),
\]
\[
X = F_2\{2m\alpha_i, m(\alpha_{n-1} + \alpha_n)\}
\]
since
\[
||m(\alpha_{n-1} + \alpha_n)||^2 - (2\rho|m(\alpha_{n-1} + \alpha_n)) = 4m^2 - 4m \equiv 0 \pmod{l}.
\]
(4) \( l = 4m \) and \( n = 2n' \). Here
\[
\text{Ann } \pi = lQ + \mathbb{Z}2m(\alpha_{n-1} + \alpha_n) + \mathbb{Z}2m(\alpha_1 + \alpha_3 + \cdots + \alpha_{n-3} + \alpha_{n-1}),
\]
\[
X = F_2\{2m\alpha_i, m(\alpha_{n-1} + \alpha_n), m(\alpha_1 + \alpha_3 + \cdots + \alpha_{n-3} + \alpha_{n-1})\}. 
\]
Since
\[
||m(\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1})||^2 - (2\rho|m(\alpha_1 + \alpha_3 + \cdots + \alpha_{n-1})) = 2mn'(m - 1)
\]
the algebra $u_q(\mathfrak{g})$ is 3-modular for $n'$ even or $m$ odd, and it is not if $n'$ is odd and $m$ is even.

5.6. The Cartan matrix $E_6$. Here $d_i = 1$ and $\det A = 3$. The equation $Au \equiv 0 \pmod{l}$ implies $3u \equiv 0 \pmod{l}$.

(1) $l$ is odd. Then $u_q(\mathfrak{g})$ is perfect modular with

$$G = \begin{cases} \mathbb{Q}/l\mathbb{Q}, & \text{if } 3 \nmid l \\ \mathbb{Q}/(l\mathbb{Q} + \mathbb{Z}p(\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6)), & \text{if } l = 3p \end{cases}$$

(2) $l = 2m$ and $3 \nmid m$. Then $u_q(\mathfrak{g})$ is 3-modular with $G = \mathbb{Q}/l\mathbb{Q}$ and $X = \mathbb{F}_2\{m\alpha_i\}$.

(3) $l = 6m$. Again $u_q(\mathfrak{g})$ is 3-modular with

$$G = \mathbb{Q}/(l\mathbb{Q} + \mathbb{Z}2m(\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6)),$$

$$X = \mathbb{F}_2\{3m\alpha_i, m(\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6)\},$$

since

$$||m(\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6)||^2 - (2p|m(\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6)) = 12m^2 \equiv 0 \pmod{l}.$$

5.7. The Cartan matrix $E_7$. Here $d_i = 1$ and $\det A = 2$. The equation $Au \equiv 0 \pmod{l}$ implies $2u \equiv 0 \pmod{l}$.

(1) $l$ is odd. Then $u_q(\mathfrak{g})$ is perfect modular with $G = \mathbb{Q}/l\mathbb{Q}$.

(2) $l = 2(2m + 1)$. Then $u_q(\mathfrak{g})$ is 3-modular with

$$G = \mathbb{Q}/(l\mathbb{Q} + \mathbb{Z}(2m + 1)(\alpha_2 + \alpha_5 + \alpha_7)),$$

$$X = \mathbb{F}_2\{(2m + 1)\alpha_i\}.$$

(3) $l = 4m$. Then

$$G = \mathbb{Q}/(l\mathbb{Q} + \mathbb{Z}2m(\alpha_2 + \alpha_5 + \alpha_7)),$$

$$X = \mathbb{F}_2\{2m\alpha_i, m(\alpha_2 + \alpha_5 + \alpha_7)\}.$$

Since

$$||m(\alpha_2 + \alpha_5 + \alpha_7)||^2 - (2p|m(\alpha_2 + \alpha_5 + \alpha_7)) = 6m(m - 1)$$

the algebra $u_q(\mathfrak{g})$ is 3-modular iff $m$ is odd.

5.8. The Cartan matrix $E_8$. Here $d_i = 1$ and $\det A = 1$, therefore, $G = \mathbb{Q}/l\mathbb{Q}$.

(1) $l$ is odd. Then $u_q(\mathfrak{g})$ is perfect modular.

(2) $l = 2m$. Then $u_q(\mathfrak{g})$ is 3-modular with $X = \mathbb{F}_2\{m\alpha_i\}$. 
5.9. The Cartan matrix $F_4$. Here $d_1 = d_2 = 2$, $d_3 = d_4 = 1$ and $\det A = 1$, which implies $Q = P$. Let $l' = l$ for $l$ odd and $l' = l/2$ for $l$ even. Then

$$\text{Ann } \pi = \mathbb{Z}\{l'\omega_1, l'\omega_2, l\omega_3, l\omega_4\},$$

where

$$\omega_1 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$$
$$\omega_2 = 3\alpha_1 + 6\alpha_2 + 8\alpha_3 + 4\alpha_4$$
$$\omega_3 = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 3\alpha_4$$
$$\omega_4 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$$

are the fundamental weights. Therefore,

1. $l$ is odd. Then $u_q(g)$ is perfect modular with $G = Q/lQ$.
2. $l = 2l'$. Then $u_q(g)$ is 3-modular with $G \simeq (\mathbb{Z}/l\mathbb{Z})^2 \times (\mathbb{Z}/l\mathbb{Z})^2$

$$G = Q/(l'\alpha_1, l'(3\alpha_2 + 4\alpha_3 + 2\alpha_4), l(\alpha_1 + 2\alpha_2 + 3\alpha_3), l\alpha_4),$$

$$X = \begin{cases} 
\mathbb{F}_2\{l'(\alpha_1 + 2\alpha_2 + 3\alpha_3), l'\alpha_4\} & \text{for } l' \text{ odd} \\
\mathbb{F}_2\{m\alpha_1, m(3\alpha_2 + 4\alpha_3 + 2\alpha_4), l'(\alpha_1 + 2\alpha_2 + 3\alpha_3), l'\alpha_4\} & \text{for } l' = 2m
\end{cases}$$

5.10. The Cartan matrix $G_2$. Here $d_1 = 1$, $d_2 = 3$ and $\det A = 1$, which implies $Q = P$. Let $l' = l$ if $3 \nmid l$ and $l' = l/3$ if $3 | l$. Then $\text{Ann } \pi = \mathbb{Z}\{l\omega_1, l'\omega_2\}$, where

$$\omega_1 = 2\alpha_1 + \alpha_2$$
$$\omega_2 = 3\alpha_1 + 2\alpha_2$$

are the fundamental weights. Therefore,

$$G = Q/\mathbb{Z}\{l\omega_1, l'\omega_2\} \simeq \mathbb{Z}/l\mathbb{Z} \times \mathbb{Z}/l'\mathbb{Z}.$$

1. $l$ is odd. Then $u_q(g)$ is perfect modular.
2. $l = 2m$. Then $u_q(g)$ is 3-modular with

$$X = \mathbb{F}_2\{m\omega_1, \frac{l'}{2}\omega_2\}$$

since

$$||\frac{l'}{2}\omega_2||^2 - (2\rho)|\frac{l'}{2}\omega_2| = 3l'(\frac{l'}{2} - 3) \equiv 0 \pmod{l}.$$
6. Representations of mapping class groups

According to [38] for any 2-modular category \( \mathcal{C} \) there are projective representations of mapping class groups of \( \mathcal{C} \)-labedled surfaces in vector spaces \( \text{Hom}_\mathcal{C}(\cdot, \cdot) \). Here we pay attention mainly to the categories \( \mathcal{C} = \mathcal{H} \text{-mod} \), where \( \mathcal{H} \) is a 2-modular Hopf algebra, and we describe the representations explicitly. In the particular case of closed surfaces or surfaces with one hole these representations are closely related to the representations of mapping class groups in a category of tangles obtained by Matveev and Polyak [43].

By a labeled surface we shall understand the following: a compact oriented surface with a labeling of boundary circles \( L : \pi_0(\partial \Sigma) \to \text{Ob} \mathcal{C} \), \( i \mapsto X_i \) and with a chosen point \( x_i \) on \( i \)th boundary circle, i.e. a section \( x : \pi_0(\partial \Sigma) \to \partial \Sigma \) of the projection \( \partial \Sigma \to \pi_0(\partial \Sigma) \) is fixed. By a homeomorphism of labeled surfaces we mean an orientation and labeling preserving homeomorphism, which sends the set of chosen points to itself. A mapping class group \( \text{MCG}(\Sigma) \) is defined as the group of homeomorphisms \( \Sigma \to \Sigma \) modulo isotopy equivalence relation. An isotopy is supposed not to move the chosen points.

We use the convention \( AB = A \cdot B = B \circ A \) in this section.

6.1. A central extension of the category of surfaces. Projective representations of the mapping class groups are constructed in [38] as follows. The category \( \text{Surf} \) of labeled surfaces and their homeomorphisms is embedded (non canonically) into a category \( \text{ON} \) of oriented nets, which are a sort of labeled oriented graphs with 1-, 2-, or 3-valent vertices. Morphisms of \( \text{ON} \) are generated by natural maps of graphs and some extra generators called fusing, braiding, twists and switches. In particular, the homeomorphisms \( S, T : \mathbb{T}^2 \to \mathbb{T}^2 \) of the torus go to generators \( S, T \) in \( \text{ON} \). The generators of \( \text{ON} \) are subject to several relations, including

\[
(ST)^3 = S^2. \tag{6.1.1}
\]

A central extension \( \text{EN} \) of the category \( \text{ON} \) is defined [38] by adding one more generator \( C \) commuting with other generators and deforming the relation (6.1.1) to

\[
(ST)^3 = CS^2.
\]

Lift the central extension to \( \text{Surf} \) as in the diagram

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & E\text{Surf} & \longrightarrow & \text{Surf} & \longrightarrow & 1 \\
\| & & & & \downarrow \phi & & & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \text{EN} & \xrightarrow{j} & \text{ON} & \longrightarrow & 1
\end{array}
\]

The meaning of this diagram is the following. Objects of \( E\text{Surf} \) are objects of \( \text{Surf} \). For any two homeomorphic surfaces \( \Sigma_1, \Sigma_2 \in \text{Ob} \text{Surf} \) let \( E\text{Surf}(\Sigma_1, \Sigma_2) \) be
determined as a central extension of $\text{Surf}(\Sigma_1, \Sigma_2)$ as in pull-back

\[
\begin{array}{cccccc}
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & E\text{Surf}(\Sigma_1, \Sigma_2) & \longrightarrow & \text{Surf}(\Sigma_1, \Sigma_2) & \longrightarrow & 1 \\
\| & & \downarrow & & \downarrow & & \phi & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & E\text{N}(\phi(\Sigma_1), \phi(\Sigma_2)) & \longrightarrow & \text{ON}(\phi(\Sigma_1), \phi(\Sigma_2)) & \longrightarrow & 1
\end{array}
\]

Simply set $E\text{Surf}(\Sigma_1, \Sigma_2) = \frac{1}{j}(\phi(\text{Surf}(\Sigma_1, \Sigma_2)))$ and set $E\text{Surf}(\Sigma_1, \Sigma_2)$ empty if $\Sigma_1$ and $\Sigma_2$ are not homeomorphic in $\text{Surf}$.

The central extension $E\text{Surf}$ of the groupoid $\text{Surf}$ (as any other central extension by $\mathbb{Z}$) can be described by a cohomology class in $H^2(\text{Surf}, \mathbb{Z})$ uniquely up to equivalence of categories. Namely, choose arbitrarily a lifting $\tilde{f} \in E\text{Surf}$ for any morphism $f \in \text{Surf}$ and for any composable $f, g \in \text{Surf}$ set $\theta(f, g) = m$ if $\tilde{f}^{-1}(fg)\tilde{g}^{-1} = C^m$. For any $f, g, h \in \text{Surf}$ such that $fg$ and $gh$ exist we have

\[
\theta(f, g) + \theta(f, gh) = \theta(f, g) + \theta(g, h).
\]

Restricting to $f, g \in MCG(\Sigma)$ we get a 2-cocycle $\theta \in Z^2(MCG(\Sigma), \mathbb{Z})$, whose cohomology class $[\theta] \in H^2(MCG(\Sigma), \mathbb{Z})$ does not depend on the lifting.

A functor $E\text{N} \rightarrow k\text{-vect}$ sending $C$ to $\lambda \in k^\times$ was constructed in [38]. So a summary of this work can be given by the following commutative diagram of functors

\[
\begin{array}{ccc}
Z' : E\text{Surf} & \longrightarrow & E\text{N} \\
\downarrow p & & \downarrow j \\
\text{Surf} & \longrightarrow & \text{ON} \phi
\end{array}
\]

Any section of the projection $p$ would give a projective representation $Z : \text{Surf} \rightarrow k\text{-vect}$ satisfying $Z(fg) = \theta(f, g)Z(f)Z(g)$ for any composable $f, g \in \text{Surf}$ with a 2-cocycle $\theta$.

Here we shall describe the projective representation of $M'_{g,n} = MCG(\Sigma_{g,n})$ in $Z(\Sigma_{g,n}) = \text{Hom}(X_1 \otimes \ldots \otimes X_n, f^{\otimes q})$ for various genera $q$ and number of holes $n$. The group $M'_{g,n}$ depends on coincidence of labels $X_1, \ldots, X_n$. To describe the representations uniformly we allow for homeomorphisms which do not preserve the labels, obtaining a larger group $M_{g,n} \supset M'_{g,n}$. It has a projection $p : M_{g,n} \rightarrow \mathfrak{S}_n$ to the symmetric group. We represent elements $h \in M_{g,n}$ by operators

\[
Z(h) : \text{Hom}(X_1 \otimes \ldots \otimes X_n, f^{\otimes q}) \rightarrow \text{Hom}(X_{p(h)^{-1}(1)} \otimes \ldots \otimes X_{p(h)^{-1}(n)}, f^{\otimes q})
\]

satisfying $Z(hf) = \lambda^k Z(h)Z(f)$ for some $k \in \mathbb{Z}$.

The reader might want to consider $C = H\text{-mod}$ as the category of labels (set $I = k$ in this case), although the results are valid for an arbitrary 2-modular category $C$. 

6.2. Sphere. Consider a sphere $\Sigma_{0,n}$ with $n$ disks removed, boundary circles are labeled by $X_1, X_2, \ldots, X_n$. The mapping class group $M_{0,n}$ is generated by the braiding homeomorphisms $\omega_i$ interchanging the $i$th and $i+1$st holes and the inverse Dehn twists $R_i$ performed in a collar neighbourhood of $i$th boundary circle. The relations

$$R_i R_j = R_j R_i,$$

$$\omega_i R_j = R_j \omega_i \text{ if } j \neq i, i+1,$$

$$\omega_i R_i = R_{i+1} \omega_i, \quad \omega_i R_{i+1} = R_i \omega_i,$$

$$\omega_i \omega_j = \omega_j \omega_i \text{ if } |i - j| > 1$$

$$\omega_i \omega_i+1 \omega_i = \omega_i+1 \omega_i \omega_i+1$$

$$\omega_1 \omega_2 \ldots \omega_{n-1} \omega_2 \omega_1 = R_1^{-2}$$

$$(\omega_1 \ldots \omega_{n-1})^n = R_1^{-1} \ldots R_n^{-1}.$$ are defining relations of $M_{0,n}$.

The group $M_{0,n}$ is represented in

$$Z(\Sigma_{0,n}) = \text{Hom}(X_1 \otimes X_2 \otimes \ldots \otimes X_n, I)$$

by the operators

$$Z(\omega_i) = \text{Hom}(1 \otimes c_{X_{i+1},X_i} \otimes 1, I),$$

$$Z(R_i) = \text{Hom}(1 \otimes \nu_{X_i} \otimes 1, I).$$

This is a usual representation not only projective.

6.3. Torus. Consider a torus $\Sigma_{1,n}$ with $n$ disks removed, boundary circles are labeled by $X_1, X_2, \ldots, X_n$. The mapping class group $M_{1,n}$ is generated by the homeomorphisms $S, T = Tw_k, R_i, a_j = Tw_{\gamma_j} Tw_k$, where cycles $\gamma_j$ separate $j - 1$st and $j$th holes as shown at Figure 1, and the braiding homeomorphisms $\omega_i$ interchanging the $i$th and $i+1$st holes. Denote $b_j = S^{-1} a_j S$ and for $j < k$ denote

$$B_{jk}^{-1} = (\omega_{k-1} \omega_{k-2} \ldots \omega_j)(\omega_k \omega_{k-1} \ldots \omega_{j+1}) \ldots (\omega_{n-1} \omega_{n-2} \ldots \omega_{n+j-k})$$

$$(\omega_{j-k+n} \omega_{j-k+n-1} \ldots \omega_j)(\omega_{j-k+n+1} \omega_{j-k+n} \ldots \omega_{j+1}) \ldots (\omega_{n-1} \omega_{n-2} \ldots \omega_{k-1}).$$

Figure 1.
The relations

\[(ST)^3 = S^2,\]
\[S^4 = B_{12}B_{23}B_{34} \ldots B_{n-1,n}R_1^{-1} \ldots R_n^{-1}\]
\[S^{-1}b_jS = b_ja_j^{-1}b_j^{-1}\]
\[T^{-1}a_jT = a_j, \quad T^{-1}b_jT = b_ja_j\]
\[\omega_iS = S\omega_i, \quad \omega_iT = T\omega_i\]
\[\omega_iR_i = R_{i+1}\omega_i, \quad \omega_iR_{i+1} = R_i\omega_i\]
\[\omega_iR_j = R_j\omega_i \text{ for } j \neq i, i + 1\]
\[\omega_i\omega_j = \omega_j\omega_i \text{ for } |i - j| > 2\]
\[a_1 = b_1 = 1\]
\[a_ja_k = a_ka_j, \quad b_jb_k = b_kb_j\]
\[a_ja_k^{-1}b_ja_j^{-1}a_kb_k^{-1} = B_{jk} \text{ for } j < k\]
\[a_k^{-1}b_j^{-1}b_ka_kb_k^{-1} = B_{jk} \text{ for } j < k\]
\[a_iB_{jk} = B_{jk}a_i, \quad b_iB_{jk} = B_{jk}b_i \text{ for } i \leq j < k\]

are defining relations of \(M_{1,n}\) ([44], extending [1]).

Set

\[Z(\Sigma_{1,n}) = \text{Hom}(X_1 \otimes \ldots \otimes X_n, f),\]
\[Z(S) = \text{Hom}(X_1 \otimes \ldots \otimes X_n, S),\]
\[Z(T) = \text{Hom}(X_1 \otimes \ldots \otimes X_n, T),\]
\[Z(R_i) = \text{Hom}(X_1 \otimes \ldots \otimes \nu_{X_i} \otimes \ldots \otimes X_n, f),\]
\[Z(\omega_i) = \text{Hom}(X_1 \otimes \ldots \otimes c_{X_{i+1}, X_i} \otimes \ldots \otimes X_n, f),\]
\[Z(a_j) = \left(\text{Hom}(X_1 \otimes \ldots \otimes X_{j-1} \otimes X_j \otimes \ldots \otimes X_n, f)\right)\]
\[\xrightarrow{\sim} \text{Hom}(X_1 \otimes \ldots \otimes X_{j-1}, f \otimes (\nu X_n \otimes \ldots \otimes \nu X_j)) \xrightarrow{\text{Hom}(X_1 \otimes \ldots \otimes X_{j-1}, \Omega_{r^{-1}})} \text{Hom}(X_1 \otimes \ldots \otimes X_{j-1} \otimes (X_j \otimes \ldots \otimes X_n), f)\]
\[\xrightarrow{\text{Hom}(X_1 \otimes \ldots \otimes X_{j-1} \otimes \nu X_j \otimes \ldots \otimes X_n, f)} \text{Hom}(X_1 \otimes \ldots \otimes X_{j-1} \otimes X_j \otimes \ldots \otimes X_n, f)\).

Here \(S\) and \(T\) in the right hand side mean \(S : f \to f, T : f \to f\) described in Section 2.1. Equation (2.1.5) implies that

\[(Z(S)Z(T))^3 = \lambda Z(S)^2.

Moreover, the above is a projective representation of \(M_{1,n}\) [38].
Notice that in particular case $n = 0$ we have

$$Z(S)^4 = 1 : \text{Hom}(I, f) \rightarrow \text{Hom}(I, f)$$

since $S^4 = \nu^{-1} : f \rightarrow f$.

**Remark 6.3.1.** The map $Z(b_j) \overset{\text{def}}{=} Z(S)^{-1}Z(a_j)Z(S)$ has another presentation, which is an important part of the proof [38]. Consider for simplicity the case $n = 2$. Introduce a map $B_2$ as the composition

$$\text{Hom}(X_1 \otimes X_2, F) \xrightarrow{\text{Hom}(c_{X_2 \otimes X_1}, F)} \text{Hom}(X_2 \otimes X_1, F) \xrightarrow{\sim} \text{Hom}(X_1, X_2 \otimes \int^N N \otimes N^\vee)$$

$$\xrightarrow{- \circ \text{coev}_{X_1}} \text{Hom}(X_1, \int^N X_2 \otimes N \otimes N^\vee \otimes X_2^\vee \otimes X_2^\vee)$$

$$\sim \text{Hom}(X_1, \int^N (X_2 \otimes N) \otimes (X_2 \otimes N)^\vee \otimes X_2^\vee) \rightarrow \text{Hom}(X_1, \int^P P \otimes P^\vee \otimes X_2^\vee)$$

$$\sim \text{Hom}(X_1 \otimes X_2^\vee, F) \xrightarrow{\text{Hom}(X_1 \otimes u_2^2, F)} \text{Hom}(X_1 \otimes X_2, F).$$

It covers $Z(b_2)$ in the sense that the diagram

$$\text{Hom}(X_1 \otimes X_2, F) \xrightarrow{B_2} \text{Hom}(X_1 \otimes X_2, F)$$

$$\downarrow \quad \downarrow$$

$$\text{Hom}(X_1 \otimes X_2, f) \xrightarrow{Z(b_2)} \text{Hom}(X_1 \otimes X_2, f)$$

(6.3.1)

is commutative. Notice that in perfect modular case $F = f$ and $Z(b_2)$ simply equals $B_2$.

If $\mathcal{C}$ is semisimple, the vertical arrows in (6.3.1) are surjective. The same holds if $\mathcal{C} = H\text{-mod}$ for finite dimensional $H$ and one of the $H$-modules $X_1$ or $X_2$ is projective (then $X_1 \otimes X_2$ is also projective). In these assumptions we can represent $B_2$ in a different way as a composition of bijections

$$\text{Hom}(X_1 \otimes X_2, F) \xrightarrow{\text{Hom}(c_{X_2 \otimes X_1}, F)} \text{Hom}(X_2 \otimes X_1, \int^N N \otimes N^\vee)$$

$$\simeq \text{Hom}(X_2 \otimes X_1, \int^N N \otimes N^\vee) \simeq \int^N \text{Hom}(X_2 \otimes (X_1 \otimes N), N)$$

$$\simeq \int^{N, P} \text{Hom}(X_2 \otimes P, N) \otimes \text{Hom}(X_1 \otimes N, P) \simeq \int^P \text{Hom}(X_1 \otimes (X_2 \otimes P), P)$$

$$\simeq \int^P \text{Hom}(X_1 \otimes X_2, P \otimes P^\vee) \simeq \text{Hom}(X_1 \otimes X_2, \int^P P \otimes P^\vee) = \text{Hom}(X_1 \otimes X_2, F).$$

Knowing $Z(b_2)$ by (6.3.1) for projective $X_1$ or $X_2$, one recovers this map for arbitrary $X_1, X_2$ using short projective resolutions. In the calculations above we used
Lemma 6.3.1 ([38]). Let $F : \mathcal{C} \to k\text{-Vect}$, $G : \mathcal{C}^{\text{op}} \to k\text{-Vect}$ be functors. Then
\[
\int^X F(X) \otimes \text{Hom}(X, B) \to F(B), \quad v \otimes f \mapsto F(f).v
\]
\[
\int^X \text{Hom}(B, X) \otimes G(X) \to G(B), \quad f \otimes v \mapsto G(f).v
\]
are isomorphisms of vector spaces.

6.4. Closed surfaces and surfaces with one hole.

6.4.1. Surfaces with one hole. Let $\Sigma = \Sigma_{g,1}$ be a surface of genus $g$ with one disk removed and the boundary labeled by an $H$-module $X$. Lickorish [29] proved that its mapping class group $M_{g,1}$ is generated by inverse Dehn twists in a neighbourhood of the following cycles: $a_k, b_k, d_k, e_k$ $(a_1 = d_1 = e_1)$ (see Figure 2). These Dehn twists are denoted by the same letters as cycles. Wajnryb [60] found a system of defining relations for $M_{g,1}$. An equivalent system of relations is the following:

(A) $a_1b_1a_i = b_1a_1b_i$, $a_{i+1}b_ia_{i+1} = b_ia_{i+1}b_i$, $d_id_i = b_id_ib_i$, $e_ie_i = b_ie_ib_i$ and all other pairs of generators commute.

(B) $(a_1b_1a_2)^4 = d_2e_2$

(C) $e_kG_k = G_kd_k$, where $G_k = b_ka_k \ldots b_ia_1a_1b_1 \ldots a_kb_k$.

(D) $J_kd_{k+1} = d_kJ_k$, where
\[
J_k = b_ka_ka_{k+1}b_kb_{k+1}a_{k+1}^{-1}b_k^{-1}d_kb_ka_kb_{k-1}d_{k-1}(a_k^{-1}b_k^{-1}a_k^{-1}b_k^{-1})d_k^{-1}b_k^{-1}a_k^{-1}b_k^{-1}
\]

(E) $hb_1^{-1}a_2^{-1}b_1^{-1}hb_1a_2a_1b_1d_2 = d_3a_3a_2a_1$, where
\[
h = b_2^{-1}a_2^{-1}a_3^{-1}b_2^{-1}d_2b_1a_2a_3b_2.
\]

To construct $Z(\Sigma)$ we follow the recipee of [38]. First of all we choose an oriented net (roughly this is an oriented graph, for precise definition see [38]) which encodes the structure of the surface. Several graphs can be chosen, for instance, Figures 3–5. They are all isomorphic in the category of oriented nets [38]. Next step is to compute the functor $\mathcal{C}^{\text{op}} \to k\text{-vect}$ corresponding to these graphs, obtained by taking coends.
Figure 3.

Figure 4.

Figure 5.
over internal edges. The functor is obtained in different forms, which are isomorphic due to associativity isomorphisms in $C$. By Lemma 6.3.1 we find that to Figure 3 corresponds

$$\int^{C_i, D_i, L_i} \text{Hom}(X, L_g \otimes C_g) \otimes \text{Hom}(C_g, L_{g-1} \otimes C_{g-1}) \otimes \ldots \otimes \text{Hom}(C_3, L_2 \otimes C_2) \otimes \text{Hom}(L_g \otimes D_g, D_g) \otimes \ldots \otimes \text{Hom}(L_2 \otimes D_2, D_2) \otimes \text{Hom}(C_2 \otimes D_1, D_1) \simeq$$

$$\simeq \int^{D_i, L_i} \text{Hom}(X, L_g \otimes L_{g-1} \otimes \ldots \otimes L_2 \otimes C_2) \otimes \text{Hom}(L_g, D_g \otimes \vee D_1) \otimes \ldots \otimes \text{Hom}(L_2, D_2 \otimes \vee D_2) \otimes \text{Hom}(C_2, D_1 \otimes \vee D_1) \simeq$$

$$\simeq \int^{D_i} \text{Hom}(X, (D_g \otimes D_g \vee) \otimes \ldots \otimes (D_2 \otimes D_2 \vee) \otimes (D_1 \otimes D_1 \vee)). \quad (6.4.1)$$

To Figure 4 corresponds

$$\int^{C_i, D_i, E_i} \text{Hom}(C_2 \otimes D_1, D_1) \otimes \text{Hom}(D_2 \otimes E_2, C_2) \otimes \text{Hom}(C_3, D_2 \otimes E_2) \otimes \ldots \otimes \text{Hom}(C_g, D_{g-1} \otimes E_{g-1}) \otimes \text{Hom}(D_g \otimes E_g, C_g) \otimes \text{Hom}(X, D_g \otimes E_g) \simeq$$

$$\simeq \int^{C_i, D_i} \text{Hom}(C_2, D_1 \otimes D_1 \vee) \otimes \text{Hom}(C_3, D_2 \otimes D_2 \vee \otimes C_2) \otimes \ldots \otimes \text{Hom}(C_g, D_{g-1} \otimes D_{g-1} \vee \otimes C_{g-1}) \otimes \text{Hom}(X, D_g \otimes D_g \vee \otimes C_g) \simeq$$

$$\simeq \int^{D_i} \text{Hom}(X, (D_g \otimes D_g \vee) \otimes \ldots \otimes (D_2 \otimes D_2 \vee) \otimes (D_1 \otimes D_1 \vee)). \quad (6.4.2)$$

To Figure 5 corresponds

$$\int^{A_i, D_i, E_i} \text{Hom}(E_2 \otimes D_1, A_2) \otimes \text{Hom}(D_2 \otimes A_2, D_1) \otimes \text{Hom}(E_3, A_3 \otimes E_2) \otimes \ldots \otimes \text{Hom}(E_g, A_g \otimes E_{g-1}) \otimes \text{Hom}(D_g \otimes A_g, D_{g-1}) \otimes \text{Hom}(X, D_g \otimes E_g) \simeq$$

$$\simeq \int^{D_i, E_i} \text{Hom}(E_2 \otimes D_1, D_2 \vee \otimes D_1) \otimes \text{Hom}(E_3, D_3 \vee \otimes D_2 \otimes E_2) \otimes \ldots \otimes \text{Hom}(E_g, D_g \vee \otimes D_{g-1} \otimes E_{g-1}) \otimes \text{Hom}(X, D_g \otimes E_g) \simeq$$

$$\simeq \int^{D_i} \text{Hom}(X, (D_g \otimes D_g \vee) \otimes \ldots \otimes (D_2 \otimes D_2 \vee) \otimes (D_1 \otimes D_1 \vee)). \quad (6.4.3)$$

If $X$ is projective this space is

$$\text{Hom}(X, F \otimes \ldots \otimes F \otimes F).$$
The same answer will be obtained if we calculate the coend in the category of left exact functors $C^\text{op} \to k\text{-vect}$. The final step is taking a quotient and setting

$$Z(\Sigma_{g,1}) = \text{Hom}(X, f^\otimes g).$$

The generators $a_k, d_k, e_k$ of $M_{g,1}$ are represented by applying ribbon twist to internal variables $A_k, D_k, E_k$. Thus from any presentation (6.4.1)–(6.4.3) we get

$$Z(d_k) = \text{Hom}(X, f^\otimes g \otimes T \otimes f^\otimes k^{-1}).$$

From the third presentation (6.4.3) we get that $a_k$ acts by applying ribbon twist to $D_k^\vee \otimes D_{k-1}$, which induces $T \otimes T \cdot \Omega : F \otimes F \to F \otimes F$. Hence,

$$Z(a_k) = \text{Hom}(X, f^\otimes g^{-k} \otimes T \otimes T \otimes f^\otimes k^{-2}) \cdot \text{Hom}(X, f^\otimes g^{-k} \otimes \Omega \otimes f^\otimes k^{-2}).$$

From the second presentation (6.4.2) we get that $e_k$ acts by applying ribbon twist to $D_k^\vee \otimes C_k$ or, equivalently, to $D_k^\vee \otimes D_{k-1} \otimes D_{k-1}^\vee \otimes \ldots \otimes D_1 \otimes D_1^\vee$. This induces $\Omega^r \cdot T \otimes \nu : F \otimes F^\otimes k^{-1} \to F \otimes F^\otimes k^{-1}$, hence,

$$Z(e_k) = \text{Hom}(X, f^\otimes g^{-k} \otimes \Omega f^\otimes k{-1}) \cdot \text{Hom}(X, f^\otimes g^{-k} \otimes T \otimes \nu f^\otimes k{-1}).$$

Finally, $b_k$ is conjugate to $d_k$ by a homeomorphism of the type $S$, and we set

$$Z(b_k) = \text{Hom}(X, f^\otimes g^{-k} \otimes STS^{-1} \otimes f^\otimes k{-1})$$

$$= \text{Hom}(X, f^\otimes g^{-k} \otimes S^{-1}TS \otimes f^\otimes k{-1}).$$

As shown in [38] these operators define a projective representation $M_{g,1} \to PGL(Z(\Sigma_{g,1}))$ with the 2-cocycle whose values are powers of $\lambda$. Therefore, this representation is induced by a projective representation $z : M_{g,1} \to \text{Aut}_C(f^\otimes g)/\{\lambda^n\}_{n \in \mathbb{Z}}$

$$a_k \mapsto f^\otimes g^{-k} \otimes (T \otimes T \cdot \Omega) \otimes f^\otimes k{-2},$$

$$b_k \mapsto f^\otimes g^{-k} \otimes STS^{-1} \otimes f^\otimes k{-1},$$

$$d_k \mapsto f^\otimes g^{-k} \otimes T \otimes f^\otimes k{-1},$$

$$e_k \mapsto f^\otimes g^{-k} \otimes (\Omega f^\otimes k{-1} \cdot T \otimes \nu f^\otimes k{-1}).$$

Indeed, set $X = f^\otimes g$ and apply both parts of relations (A)-(E) to the vector $\text{id}_X \in Z(\Sigma_{g,1})$.

6.4.2. Closed surfaces. Now let $\Sigma_{g,0}$ be a closed surface of genus $g$. The mapping class group $M_{g,0}$ is the quotient of $M_{g,1}$ by two extra relations [60]:

(F) $H_g^2 = 1$, where

$$H_g = b_g a_g \ldots b_2 a_2 b_1 a_1 b_1 a_2 b_2 \ldots a_g b_g d_g e_g;$$

(G) $d_g = e_g$. 

Setting

\[ Z(\Sigma_{g,0}) = \text{Hom}(C, f^\otimes g) \]
\[ Z(a_k) = \text{Hom}(C, f^\otimes g_k \otimes (T \otimes T \cdot \Omega) \otimes f^\otimes k^{\otimes 2}) \]
\[ Z(b_k) = \text{Hom}(C, f^\otimes g_k \otimes STS^{-1} \otimes f^\otimes k^{\otimes 1}) \]
\[ Z(d_k) = \text{Hom}(C, f^\otimes g_k \otimes T \otimes f^\otimes k^{\otimes 1}) \]
\[ Z(e_k) = \text{Hom}(C, f^\otimes g_k \otimes (\Omega_{f, f^{\otimes k-1}} \cdot T \otimes \nu_{f^{\otimes k-1}})) \]

we get a projective representation \( M_{g,0} \rightarrow PGL(Z(\Sigma_{g,0})) \) [38].

6.4.3. \textbf{Exercises.} The reader might want to check some of the above results straightforwardly. Several exercises will help in doing this. They present minor generalizations of some of the quoted results and specify the power of \( \lambda \) involved in relations.

\textbf{Easy exercise 1.} All maps \( z(a_i), z(d_j), z(e_k) : f^\otimes g \rightarrow f^\otimes g \) commute. \( z(b_i) \) commutes with all maps except \( z(a_i), z(a_{i+1}), z(d_i), z(e_i) \). Also \( z(d_i)z(b_i)z(d_i) = z(b_i)z(d_i)z(b_i) \) (see relation (A)).

\textbf{Exercise 2.} For any \( Y \in C \) we have

\[ \Omega_{f, Y} \cdot S^{-1} \otimes \nu_Y \cdot \Omega_{f, Y} \]
\[ = \lambda(T^{-1}S^{-1}T^{-1}) \otimes Y \cdot \Omega_{f, Y} \cdot (S^{-1}T^{-1}) \otimes Y : f \otimes Y \rightarrow f \otimes Y, \]

\[ \Omega_{f, Y} \cdot \nu_Y \otimes S^{-1} \cdot \Omega_{Y, f} \]
\[ = \lambda Y \otimes (T^{-1}S^{-1}T^{-1}) \cdot \Omega_{Y, f} \cdot Y \otimes (S^{-1}T^{-1}) : Y \otimes f \rightarrow Y \otimes f. \]

\textbf{Easy exercise 3.} Deduce from the previous exercise that \( z(a_i)z(b_i)z(a_i) = z(b_i)z(a_i) \) \( z(b_i), z(a_{i+1})z(b_i)z(a_{i+1}) = z(b_i)z(a_{i+1})z(b_i), z(e_i)z(b_i)z(e_i) = z(b_i)z(e_i)z(b_i) \) (see relation (A)).

\textbf{Exercise 4.} For any \( Y \in C \) we have

\[ (\nu_Y \otimes S^{-1} \cdot \Omega_{Y, f}^4) = \nu_Y \otimes f \cdot \nu_Y \otimes f : Y \otimes f \rightarrow Y \otimes f, \]

What does it mean for \( Y = I \)?

\textbf{Easy exercise 5.} Deduce from the previous exercise that

\[ (z(d_1)z(b_1)z(a_2))^4 = \lambda^4 z(d_2)z(e_2) \]

(relation (B)).

\textbf{Exercise 6.} For any \( Y \in C \) we have

\[ (T^{-1}S^{-1}) \otimes Y \cdot \Omega_{f, Y} \cdot (S^{-1}TS) \otimes Y \cdot (\Omega_{f, Y}^*)^{-1} \cdot (ST) \otimes Y = \Omega_{f, Y} \cdot T \otimes \nu_Y : f \otimes Y \rightarrow f \otimes Y. \]
**Exercise 7.** Deduce from the previous exercise that \( z(e_2) = z(G_2)z(d_2)z(G_2)^{-1} \), where \( z(G_2) \overset{\text{def}}{=} z(b_2)z(a_2)z(d_1)^2z(a_2)z(b_2) \).

**Easy exercise 8.** Show that

\[
Z(d_g) = Z(e_g): \text{Hom}(\mathbb{C}, f^{\otimes g}) \to \text{Hom}(\mathbb{C}, f^{\otimes g}).
\]

**6.5. General case.** Let \( \Sigma_{g,n} \) be a surface of genus \( g \) with \( n \) disks removed, boundary circles are labelled by \( X_1, \ldots, X_n \). Figure 6 suggests an embedding \( \Sigma_{g,1} \hookrightarrow \Sigma_{g,n} \), which induces a homomorphism \( M_{g,1} \to M_{g,n} \). The mapping class group \( M_{g,n} \) is generated by images \( a_k, b_k, d_k, e_k \) of generators of \( M_{g,1} \), the braidings \( \omega_i \), the twists \( R_i \), new generators \( S_k \), which are homeomorphisms of the type \( S \) inside the \( T^2 - D^2 \) region \( F_l \) and identity outside, and inverse Dehn twists \( t_{j,k} \) in tubular neighbourhood of the cycles \( t_{j,k} \). This is not a minimal system of generators, since \( S_k \) can be expressed through \( b_k \) and \( d_k \). All of these generators are easily representable.

Set

\[
\begin{align*}
Z(\Sigma_{g,n}) & = \text{Hom}(X_1 \otimes X_2 \otimes \ldots \otimes X_n, f^{\otimes g}) \\
Z(a_k) & = \text{Hom}(X_1 \otimes \ldots \otimes X_n, f^{\otimes g-k} \otimes (T \otimes T \cdot \Omega) \otimes f^{\otimes k-2}), \\
Z(b_k) & = \text{Hom}(X_1 \otimes \ldots \otimes X_n, f^{\otimes g-k} \otimes STS^{-1} \otimes f^{\otimes k-1}), \\
Z(d_k) & = \text{Hom}(X_1 \otimes \ldots \otimes X_n, f^{\otimes g-k} \otimes T \otimes f^{\otimes k-1}), \\
Z(e_k) & = \text{Hom}(X_1 \otimes \ldots \otimes X_n, f^{\otimes g-k} \otimes (\Omega_{f^{\otimes k-1}} \cdot T \otimes \nu_{f^{\otimes k-1}})), \\
Z(\omega_i) & = \text{Hom}(X_1 \otimes \ldots \otimes c_{X_{i+1},X_i} \otimes \ldots \otimes X_n, f^{\otimes g}), \\
Z(R_i) & = \text{Hom}(X_1 \otimes \ldots \otimes \nu_{X_i} \otimes \ldots \otimes X_n, f^{\otimes g}), \\
Z(S_k) & = \text{Hom}(X_1 \otimes \ldots \otimes X_n, f^{\otimes g-k} \otimes S \otimes f^{\otimes k-1}).
\end{align*}
\]
Set $Z(t_{j,k})$ to be the composition

$$\text{Hom}(X_1 \otimes \ldots \otimes X_j \otimes X_{j+1} \otimes \ldots \otimes X_n, f^g)$$

Then the above is a projective representation of $M_{g,n}$ [38].

I don’t know any defining system of relations of $M_{g,n}$. Perhaps, it was never written explicitly. Proof of the result [38] used the exact sequence [1, 55]

$$1 \rightarrow \bar{B}_{g,n} \rightarrow M_{g,n} \rightarrow M_{g,0} \rightarrow 1,$$

$$1 \rightarrow \mathbb{Z}^n \rightarrow \bar{B}_{g,n} \rightarrow B_{g,n} \rightarrow 1,$$

where $B_{g,n}$ is the braid group of a surface of genus $g$, whose presentation was given by Scott [55].

7. INVARIANTS OF CLOSED 3-MANIFOLDS

I have nothing to add to the method of obtaining invariants of closed 3-manifolds via Kirby calculus invented by Reshetikhin and Turaev [51]. It was used afterwards by many authors including Lickorish [30, 31], Kirby and Melvin [19]. Turaev [59] have shown that a semisimple modular category serves well to define a 3-manifold invariant. In this paper I propose to use (eventually non-semisimple) 3-modular categories $\mathcal{C}$ as starting data for the method. When $\mathcal{C} = H$-mod this invariant coincides with the one defined by Hennings [10] translated to unoriented setting by Kauffman and Radford [14].

7.1. 3-MANIFOLDS FROM LINKS IN $S^3$. It is well known that any closed connected oriented 3-manifold can be obtained via surgery along a framed tame link in $S^3$ [28]. Namely, given a framed tame link $L = L_1 \sqcup L_2 \sqcup \cdots \sqcup L_m \subset S^3 = \partial B^4$ we construct a compact oriented 4-manifold $W_L$ glueing $m$ 2-handles $D^2 \times D^2$ ($D^2$ is a disk) to the ball $B^4$ along closed tubular neighbourhoods $U_i \subset S^3$ of $L_i$. The part of the
boundary \( S^1 \times D^2 = \partial D^2 \times D^2 \) of \( i \)-th 2-handle is identified with \( U_i \simeq L_i \times D^2 \) so that the linking number of \( S^1 \times 1 \subset \partial D^2 \times \partial D^2 \) with \( L_i \) coincides with the self-linking number of \( L_i \). The boundary \( M_L = \partial W_L \) is an oriented compact closed connected 3-manifold.

Kirby proved [17] that two manifolds \( M_L \) and \( M_{L'} \) obtained by surgery from two framed links \( L, L' \subset S^3 \) are homeomorphic iff the link \( L' \) can be obtained from \( L \) by a finite sequence of transformations, which we call Kirby moves:

1. The elimination or insertion of an unknotted component labelled \( \pm 1 \), unlinked with other components.
2. The band (or handle slide) move—making a connected sum of one of the components \( L_i \) with a parallel copy \( \tilde{L}_j \) of another component \( L_j \).

Fenn and Rourke [7] proved a similar result with a subset of transformations introduced by Kirby, which we call Kirby–Fenn–Rourke moves (see Figures 7–8).

\[ \begin{array}{c}
\text{Figure 7.} \\
\end{array} \]

\[ \begin{array}{c}
\text{Figure 8.} \\
\end{array} \]

Therefore, to give an invariant \( \tau(M) \in k \) for 3-manifolds \( M = M_L \) amounts to give an invariant \( \tau(L) \) for framed links \( L \) in \( S^3 \), which would not change under Kirby or Kirby–Fenn–Rourke moves. We may and we shall consider framed links in \( \mathbb{R}^3 \) instead of \( S^3 \).
7.2. An invariant determined by a 3-modular category. Let $C$ be a 3-modular category. The reader might want to assume that $C = H\text{-mod}$ for simplicity. First of all we construct an invariant of framed links in $\mathbb{R}^3$.

Let $L$ be a framed tame link in $\mathbb{R}^3$. It can be deformed to a smooth link so that its projection $p(L)$ to the standard plane $\mathbb{R}^2$ were a smooth generic link diagram $D_L$ and the normal vector field on $L$ determining the framing were parallel to $\mathbb{R}^2$. Choose a straight line $\mathbb{R}^1$ in $\mathbb{R}^2$ so that $D_L$ lies in one half-plane with respect to this line (we shall draw it as a lower half-plane). Choose a point $e_i \in L_i$ for each $i$ and connect it in $\mathbb{R}^3$ with a point $x_i$ of $\mathbb{R}^1$ by a curve $\gamma_i$, which projects to the lower half-plane. We assume $p(\gamma_i)$ smooth and transversal to $p(L_i)$ in the point $p(e_i)$. Make it generic, so that $x_i \neq x_j$ and the curves $\gamma_j$ don’t intersect each other and $L$ except at the ends. Draw the corresponding plane diagram $\bar{D}_L$ assigning necessary signs of overcrossing or undercrossing to each double point of the projection to $\mathbb{R}^2$. Duplicate the projections of the curves $\gamma_i$, replacing them by two parallel curves $\gamma_i^-$ and $\gamma_i^+$ (e.g. parts of the boundary of a small neighbourhood $V_i$ of $p(\gamma_i)$). Remove the connected component of $L_i \cup V_i$ containing $e_i$. The result will be a diagram of a tangle looking like Figure 9. Make this diagram into a $C$-tangle $T_L$ [37] assigning a color $X_i \in \text{Ob}\, C$ to the point $x_i$ and $X_i^\vee$ to the point $x_i'$ and inserting to $\gamma_i^-$ such a morphism $u_0^{2a} : X_i \to X_i^{(2a\vee)}$, $a \in \mathbb{Z}$, as consistency requires. The tangle $T_L$ represents a morphism in $C$

$$\Phi(T_L; X_1, \ldots, X_m) : X_1 \otimes X_1^\vee \otimes X_2 \otimes X_2^\vee \otimes \ldots \otimes X_m \otimes X_m^\vee \to I.$$ 

Fix all objects $X_i, X_i^\vee$ except for $i = j$ and vary $X_j, X_j^\vee$. We see that the following
diagram commute for any morphism \( f : Y_j \to Z_j \in \mathcal{C} \)

\[
\begin{array}{cccc}
X_1 \otimes X_1 \otimes \cdots \otimes f \otimes Y_j \otimes Z_j \otimes \cdots \otimes X_m \otimes X_m^p \\
\downarrow \otimes f \otimes Z_j^p \otimes \cdots \\
X_1 \otimes X_1 \otimes \cdots \otimes Z_j \otimes Z_j^p \otimes \cdots \otimes X_m \otimes X_m^p \\
\Phi(T_L; X_1, \ldots, Y_j, \ldots, X_m) \\
\end{array}
\xrightarrow{\Phi(T_L; X_1, \ldots, Z_j, \ldots, X_m)} I
\]

Indeed, insert a morphism \( f \) to a point of \( \gamma_j^- \) in the Figure 9 and push it along the curve through all crossings. It will appear on \( \gamma_j^+ \) as \( f^t \).

Commutativity of the above diagram means that \( \Phi(T_L; \ldots) \) factorizes through the canonical mappings \( i_{x_j} : X_j \otimes X_j^p \to F \) on \( j \)th place (compare with (1.2.1)). Therefore, it defines a morphism

\[ \Phi(T_L) : F^\otimes m \to I. \]

Take an invariant element of \( F \), that is, a morphism \( \alpha : I \to F \in \mathcal{C} \). Define a number in \( k \)

\[ \phi(T_L, \alpha) : I = I^\otimes m \xrightarrow{\alpha^\otimes m} F^\otimes m \xrightarrow{\Phi(T_L)} I. \]

**Proposition 7.2.1.** Assume that \( \gamma_F(\alpha) = \alpha \). Then the number \( \phi(T_L, \alpha) \in k \) depends only on \( L \) and \( \alpha \) and not on choices made in constructing \( T_L \).

**Notation.** \( \tau(L, \alpha) = \phi(T_L, \alpha) \).

**Proof.** Since the source of \( \alpha \) is a unity object, we can change all overcrossings with \( \gamma_j \) to undercrossings and vice versa without changing the value of \( \phi(T_L, \alpha) \). Since all tensor factors in \( \alpha^\otimes m \) coincide, the result does not depend on the order of \( x_j = \gamma_j(1) \) on the line. Indeed, an interchange of the two neighbour ends \( x_i \) and \( x_j \) is equivalent to adding of a crossing which is interpreted as \( c : I \otimes I \to I \otimes I \), and this is an identity morphism.

The tangent vector \( \hat{p}(\gamma_i)'(0) \) can be chosen normal to \( p(L_i) \) at the point \( p(e_i) \). The following diagram explains that the change from one normal vector to another is equivalent to composing \( \alpha \) with \( \gamma_F \) or \( \gamma_F^{-1} \) on \( j \)th place (Figure 10). Since \( \gamma_F \alpha = \alpha = \gamma_F^{-1} \alpha \) the resulting \( \phi(T_L, \alpha) \) will not change.
Also the value of $\phi(T_L, \alpha)$ does not depend on the choice of $e_i$. Indeed, make $e_i$ slide along $L_i$ and deform $\gamma_i$ simultaneously. This will not change $\Phi(T_L; X_1, \ldots, X_m)$ and a fortiori $\phi(T_L, \alpha)$.

Therefore, $\phi(T_L, \alpha)$ depends only on the plane diagram $D_L$ and $\alpha$. It is invariant under the three Reidemeister moves performed on $D_L$, namely, $\Omega_2$, $\Omega_3$ [47] and $\Omega_{1F}$

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{reidemeister_moves.png}}
\end{array}
\]

Indeed, these moves are local, and we can always choose the points $e_i \in L_i$ outside the changed pieces, hence, the corresponding moves for $C$-tangles apply without changing the morphism $\Phi(T_L; X_1, \ldots, X_m)$. For $\Omega_{1F}$ move notice that $u_1^2 : X \to X^{\vee\vee}$ (resp. $u_1^{-2} : X \to \vee\vee X$) in the left (resp. right) hand side will be accompanied by a power of $u_0^2$. Thus, invariance under $\Omega_{1F}$ follows from the equation

\[
u u_0^{-2} u_1^2 = u_0^2 \nu u_1^{-2} : X \to X.
\]

Since the set of equivalence classes of plane diagrams under $\Omega_{1F}$, $\Omega_2$, $\Omega_3$ is the same as the set of equivalence classes of framed links in $\mathbb{R}^3$ ([36] after [47]), we deduce that $\phi(T_L, \alpha)$ depends only on $L$ and $\alpha$. □

7.2.1. A 3-manifold invariant. By the very definition of a 3-modular category $C$ the Hopf algebra $F \in C$ has a two-sided integral $\sigma : I \to F$. In the following we shall consider the link invariant $\tau(L, \sigma) = \phi(T_L, \sigma)$.

Denote by $s(L)$ the signature of the intersection form in $H^2(W_L; \mathbb{R})$, where $W_L$ is the 4-manifold obtained by surgery along $L$ and $M_L = \partial W_L$. It is the same as the signature of the linking matrix of $L$ [18].

**Theorem 7.2.2.** The number

\[
\tau(M_L) = \lambda^{-s(L)} \tau(L, \sigma)
\]

is an invariant of 3-manifolds, that is, it depends only on the homeomorphism class of $M_L$.

Proof. The equations (2.1.1), (2.1.2) show that application of one of the Kirby–Fenn–Rourke moves multiplies $\tau(L, \sigma)$ by $\lambda^{\pm 1}$. Simultaneously $s(L)$ increases by $\pm 1$. □

It is worth understanding why $\tau(M_L)$ is invariant under Kirby moves. The answer is because $\sigma$ is an integral in $F$. Graphically this property is expressed by the following
Here the multiplication morphism $\xi$ is determined from the commutative diagram

$$
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
X \otimes Y \otimes Y^\vee \otimes X^\vee \\
\downarrow i
\end{array}
\end{array}
\xrightarrow{X \otimes Y \otimes Y^\vee \otimes X^\vee}
\begin{array}{c}
X \otimes F \otimes X^\vee
\downarrow \xi
\end{array}
\end{array}

= \\
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
X \otimes X^\vee \otimes I
\end{array}
\end{array}
\end{array}
\xrightarrow{: X \otimes X^\vee \otimes I \to F.}
\end{array}
\end{array}
\end{array}
$$

Let us give another proof of Theorem 7.2.2, using only the Kirby moves. For particular categories related with $SL(2)$ this was done by Lickorish [32] through a different approach. Choose a pair of components $L_i, L_j$ in $L$ and perform a Kirby band move as shown at Figure 11. The points $a \in L_i$ and $b \in L_j$ become “connected” by a double line $\beta$. Take the point $b$ as $e_j$ and choose $\gamma_j$ so that $\gamma_j(0)$ pointed out in the direction of $\beta$. Duplicate all $\gamma_k$ as required by the recipe. Since the curves $L_j$ and a new-made part of $L'_i$ go parallely, we can insert the morphism $\xi$ to this $C$-$F$-tangle without changing the morphism $\ldots \otimes X \otimes X^\vee \otimes \ldots \otimes F \otimes \ldots \to k$ it represents (see Figure 12). In the vicinity of $e_j$ the picture looks like the left hand side of (7.2.1). Therefore we can change it to the right hand side without changing $\tau(L', \sigma)$. The resulting tangle at Figure 13 is isotopic to the initial one, so its value $\tau(L', \sigma)$ equals to $\tau(L, \sigma)$. Finally, the signature $s(L)$ will not change under Kirby band move [18], whence Theorem 7.2.2 follows.
Figure 12.

Figure 13.
7.3. Lens spaces. We calculate as an example the invariant $\tau$ for lens spaces $L(p, q)$. Consider the $n$-component chain link $L \subset S^3$ and the value of its invariant

$$
\begin{pmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_{n-1} \\
\alpha_n
\end{pmatrix}
$$

The surgery at $L$ gives the lens space $L(p, q)$, where prime to each other $p, q \in \mathbb{Z}$ satisfy

$$
pq = a_n - 1
$$

(see Rolfsen [52]). The right hand side of (7.3.1) can be presented through the maps $S, T : F \rightarrow F$ as

$$
\tau(L, \sigma) = \varepsilon(T^{a_n}ST^{a_{n-1}}S \ldots T^{a_2}ST^{a_1}S(1)) = \langle 1, T^{a_n}ST^{a_{n-1}}S \ldots T^{a_2}ST^{a_1}S(1) \rangle
$$

The last formula uses only the maps $^{t}T, ^{t}S = ^{t}S_{-}(\sigma)$ in $U$, determined by (3.8.2) and (3.8.9). We can also use $^{t}S_{+}(\sigma)$ (see (3.8.13)) instead of $^{t}S$.

In particular case $n = 1, a_1 = 0$ we get $\tau(L, \sigma) = \varepsilon(\mu) = f'1$, which vanishes for all $u_{q}(\mathfrak{g})$, hence, $\tau(S^2 \times S^1) = 0$. This contrasts sharply with the case of a perfect modular semisimple $C$, where $\varepsilon(\mu) = \sqrt{\sum_i (\dim_C X_i)^2}$ is invertible (see [35, 59]). Here $X_i$ runs over the set of isomorphism classes of simple objects of $C$. Their categorical dimensions

$$
\dim_C X_i = \left( I \xrightarrow{\text{coev}} X_i \otimes X_i \xrightarrow{1 \otimes u^0_i} X_i \otimes X_i \xrightarrow{\vee} I \right)
$$

are real numbers if the ground field is $\text{End} I = \mathbb{C}$ [35], so $\varepsilon(\mu)$ is positive.
Finally, $s(L)$ is the signature of the linking matrix
\[
\begin{pmatrix}
a_1 & 1 & & & \\
1 & a_2 & 1 & & \\
& 1 & \ddots & \ddots & \\
& & \ddots & 1 & \\
0 & 1 & \cdots & 1 & a_n-1 \\
& & & 1 & a_n
\end{pmatrix}
\]
which gives $\tau(L(p,q))$.

### 7.4. The Hennings invariant

Let $H$ be a finite dimensional 3-modular Hopf algebra (see Theorem 3.7.3). Construct a 3-manifold invariant $\tau(M)$ taking the 3-modular category $\mathcal{C} = H\text{-mod}$ as data. Calculating $\tau(L,\sigma)$ for some link $L$ we get an expression involving $R$-matrices (as many as there are crossings in $\bar{D}_L$), elements $\sigma$ and the powers of $\kappa$ (as many as there are components in $L$) under the sign of counity. It turns out that the result of calculation coincides with the Hennings invariant [10] defined in unoriented setting by Kauffman and Radford [14] up to change of conventions. Indeed, the calculation of $\tau(L,\sigma)$ can be performed using the graphical rules of Hennings–Kauffman–Radford as follows:

1. change all crossings in $\bar{D}_L$ to the composite of $R$-matrix and the permutation

2. add to each component such power of the morphism $u_0^2 = v_0^2 \cdot \kappa = \kappa \cdot v_0^2$ or its inverse $u_0^{-2} = v_0^{-2} \cdot \kappa^{-1}$

3. that after cancellations of $v_0^2$’s with $v_0^{-2}$’s the diagram becomes a disjoint union of unknots;

4. slide all elements put on the diagram to the left of the chosen points $e_i$ via the rule

Now each component looks alike
with $x_1, \ldots, x_n \in H$ attached to it, and its contribution is
\[
\varepsilon(x_n \ldots x_1 \otimes 1(\sigma)) = \varepsilon(\langle x_n \ldots x_1, \sigma(1) \rangle \sigma(2)) = \langle x_n \ldots x_1, \sigma \rangle
\]
(see Section 3.8.1). We can view $\sigma$ as the left integral $\sigma : H \to k$ on the algebra $H$. So
\[
\tau(L, \sigma) = \sum \sigma(x_n \ldots x_1) \ldots \sigma(z_m \ldots z_1)
\]
where the arguments are either powers of $\kappa$ or come from $R$-matrices acted upon by some powers of the antipode. This is the knot invariant used in the definition of Hennings invariant [10] up to change of the sign of the braiding.

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