Abstract. Quantization dimension has been computed for many invariant measures of dynamically defined fractals having well separated cylinders, that is, in the cases when the so-called Open Set Condition (OSC) holds. To attack the same problem in case of heavy overlaps between the cylinders, we consider a family of self-similar measures, for which the underlying Iterated Function System satisfies the so-called Weak Separation Property (WSP) but does not satisfy the OSC since complete overlaps occur in between the cylinders. The work in this paper also shows that the quantization dimension determined for the set of overlap self-similar construction satisfying the WSP has a relationship with the temperature function of the thermodynamic formalism.

1. Introduction

The basic goal of quantization for probability distribution is to reduce the number of values, which is typically uncountable, describing a probability distribution to some finite set and thus approximation of a continuous probability distribution by a discrete distribution. It has broad applications in signal processing, telecommunications, data compression, image processing and cluster analysis. Over the years, many useful theorems have been proved and numerous other results and algorithms have been obtained in quantization. For a detailed survey on the subject and comprehensive lists of references to the literature one is referred to [B, BW, G, GG, GKL, GL1, GL2, GL4, GN, Z1, Z2]. Rigorous mathematical treatment of the fundamentals of the quantization theory is provided in Graf-Luschgy’s book (see [GL1]). In general, these theorems almost exclusively involve absolutely continuous probability measures on $\mathbb{R}^d$. Two main goals of the theory are: (1) finding the exact configuration of a so-called ‘$n$-optimal set’ which corresponds to the support of the quantized version of the distribution, and (2) estimating

2000 Mathematics Subject Classification. Primary 28A80; Secondary 37A50, 94A15, 60D05.

Key words and phrases. Quantization dimension, self-similar measure, Weak Separation Property.
the rate at which some specified measure of the error goes to zero as \( n \) goes to infinity. This paper deals with the second kind of problem.

Given a Borel probability measure \( \mu \) on \( \mathbb{R}^d \), a number \( r \in (0, +\infty) \) and a natural number \( n \in \mathbb{N} \), the \( n \)th quantization error of order \( r \) for \( \mu \), is defined by

\[
V_{n,r}(\mu) := V_{n,r}(\mu) = \inf \left\{ \int d(x, \alpha)^r d\mu(x) : \alpha \subset \mathbb{R}^d, 1 \leq \text{card}(\alpha) \leq n \right\},
\]

where \( d(x, \alpha) \) denotes the distance from the point \( x \) to the set \( \alpha \) with respect to a given norm \( \| \cdot \| \) on \( \mathbb{R}^d \). If \( \alpha \) is a finite set, the error \( \int d(x, \alpha)^r d\mu(x) \) is often referred to as the cost, or distortion error of order \( r \) for \( \mu \) and \( \alpha \).

It is known that for a Borel probability measure \( \mu \), if its support contains infinitely many elements and \( \int \|x\|^r d\mu(x) \) is finite, then an optimal set of \( n \)-means always has exactly \( n \)-elements (see [AW, GKL, GL, GL1]). This set \( \alpha \) can then be used to give a best approximation of \( \mu \) by a discrete probability supported on a set with no more than \( n \) points. Such a set \( \alpha \) for which the infimum occurs and contains no more than \( n \) points is called an optimal set of \( n \)-means, or optimal set of \( n \)-quantizers (of order \( r \)). Under suitable conditions this can be done by giving each point \( a \in \alpha \) a mass corresponding to \( \mu(A_a) \), where \( A_a \) is the set of points \( x \in \mathbb{R}^d \) such that \( d(x, \alpha) = d(x, a) \).

So, \( \{A_a : a \in \alpha\} \) is the ‘Voronoi’ partition of \( \mathbb{R}^d \) induced by \( \alpha \). Of course, the idea of ‘best approximation’ is, in general, dependent on the choice of \( r \). For some recent work in the direction of optimal sets of \( n \)-means, one can see [CR, DR1, DR2, GL5, R, R1, R2, R3, R4, R5, R6, RR1]. The set of all optimal sets of \( n \)-means for a Borel probability measure \( \mu \) is denoted by \( C_{n,r}(\mu) \), i.e.,

\[
(1.1) \quad C_{n,r}(\mu) := \{ \alpha \subset \mathbb{R}^d : 1 \leq \text{card}(\alpha) \leq n \text{ and } V_{n,r}(\mu) = \int d(x, \alpha)^r d\mu(x) \}.
\]

Write \( e_{n,r}(\mu) := V_{n,r}^{1/r}(\mu) \). The numbers

\[
(1.2) \quad D_r(\mu) := \liminf_{n \to \infty} \frac{\log n}{-\log e_{n,r}(\mu)}, \quad \text{and} \quad \overline{D}_r(\mu) := \limsup_{n \to \infty} \frac{\log n}{-\log e_{n,r}(\mu)},
\]

are called the lower and the upper quantization dimensions of \( \mu \) of order \( r \), respectively. If \( D_r(\mu) = \overline{D}_r(\mu) \), the common value is called the quantization dimension of \( \mu \) of order \( r \) and is denoted by \( D_r(\mu) \). Quantization dimension measures the speed at which the specified measure of the error goes to zero as \( n \) tends to infinity. For any \( \kappa > 0 \), the numbers \( \liminf_n n^{\frac{\kappa}{r}} V_{n,r}(\mu) \) and \( \limsup_n n^{\frac{\kappa}{r}} V_{n,r}(\mu) \) are called the \( \kappa \)-dimensional lower and upper quantization coefficients for \( \mu \), respectively. The quantization coefficients provide us with
more accurate information about the asymptotics of the quantization error than the quantization dimension. Compared to the calculation of quantization dimension, it is usually much more difficult to determine whether the lower and the upper quantization coefficients are finite and positive. It follows from [GL1, Proposition 11.3] that if

$$0 \leq t < D_r < s,$$

then

$$\lim_{n \to \infty} ne_{n,r}^t = +\infty$$

and

$$\lim_{n \to \infty} ne_{n,r}^s = 0,$$

and if

$$0 \leq t < D_r < s,$$

then

$$\limsup_{n \to \infty} ne_{n,r}^t = +\infty$$

and

$$\lim_{n \to \infty} ne_{n,r}^s = 0.$$

For probabilities with non-vanishing absolutely continuous parts the numbers $D_r$ are all equal to the dimension $d$ of the underlying space, but for singular probabilities the family $\{D_r\}_{r>0}$ gives an interesting description of their geometric (multifractal) structures. A detailed account of this theory can be found in [GL1]. There the quantization dimension is introduced as a new type of fractal dimension spectrum and a formula for its determination is derived in the case of self-similar probabilities with the strong separation property. In [GL2], Graf and Luschgy extended the above result and determined the quantization dimension $D_r$ of self-similar probabilities with the weaker open set condition, but there it remained open whether the $D_r$-dimensional lower quantization coefficient is positive. Later they answered it in [GL3]. Under the open set condition, Lindsay and Mauldin (see [LM]) determined the quantization dimension function $D_r$, where $r \in (0, +\infty)$, of an $F$-conformal measure $m$ associated with a conformal iterated function system determined by finitely many conformal mappings. Subsequently, quantization dimension has been computed for many invariant measures of dynamically defined fractals having well separated cylinders (that is the cases when the so-called Open Set Condition holds), for example, one can see [MR, R7, R8, R9, R10, R11, R12, R13, R14, R15]. From all the known results it can be seen that if the quantization dimension function $D_r(\mu)$ for $r > 0$ of a fractal probability measure $\mu$ exists, it has a relationship with the temperature function of the thermodynamic formalism that arises in multifractal analysis of $\mu$ (see Figure 1).

In a very recent paper, among many other interesting applications, S. Zhu [Zhu] has solved the problem of the computation of quantization dimension for the complete overlapping case in the following very special situation: Let $S = \{S_k\}_{k \leq m}$ be a self-similar IFS with the following properties: (1) there are distinct $i$ and $j$ with $S_i = S_j$. (2) The self-similar IFS $S = \{S_k\}_{k \leq m, k \neq j}$ satisfies the so-called strong separation property. That is, for any two distinct
To determine $D_r$ first find the point of intersection of $y = \beta(q)$ and the line $y = rq$. Then, $D_r$ is the $y$-intercept of the line through this point and the point $(1,0)$.

In this paper, we make a step towards our goal to determine the quantization dimension for self-similar measures on the line in the case when the underlying self-similar system satisfies the WSP (for the definition and basic properties see [Zer]). Namely, we solve this problem for a special family which has the above mentioned properties. Our work also shows that the quantization dimension determined for a set of overlap self-similar construction satisfying the WSP has a relationship with the temperature function of the thermodynamic formalism.

Recently, Kesseböhmer et al. [KNZ, Corollary 1.12] proved that the quantization dimension $D_r$ exists for every self-conformal measure, and it is determined by the intersection point of the $L^q$-spectrum $\beta(q)$ of the measure and the line through the origin with slope $r$ as indicated by Figure 1. In our paper, using a completely different technique for an IFS we also calculated the quantization dimension. Since the $L^q$-spectrum for the self-similar measure studied in our paper has not been described explicitly before, our result is different and has its importance because of the different techniques of the work. Moreover, the combination of [KNZ, Corollary 1.2] and our main result Theorem 3.2 yields an explicit formula (see Corollary 3.3) for the $L^q$-spectrum $\beta(q)$, $q \in (0,1)$ for the self-similar measure with overlaps studied in this paper.
2. AN OVERLAPPING SELF-SIMILAR IFS ON THE LINE

We consider the following self-similar IFS on \( \mathbb{R} \)

\[
S = \left\{ S_i(x) = \frac{1}{3} x + i \right\}_{i \in \{0,1,3\}}.
\]

Naturally the alphabet corresponds to this IFS is \( \mathcal{A} := \{0, 1, 3\} \). We write \( \Sigma (\Sigma^*) \) for the set of infinite (finite) words, respectively, over the alphabet \( \mathcal{A} \). As usual we write \( \sigma \) for the left shift on \( \Sigma \cup \Sigma^* \). We write

\[
i^{-} := (i_1, \ldots, i_{n-1}) \quad \text{for an} \quad i = (i_1, \ldots, i_n) \in \mathcal{A}^n.
\]

We say that \( \Gamma \subset \Sigma^* \) is a maximal finite antichain if for every \( i \in \Sigma \) there exists a unique \( n \) such that \( i|_n \in \Gamma \).

Let \( \Lambda \) be the attractor of \( S \). That is, \( \Lambda \) is the unique non-empty compact set satisfying \( \Lambda = \bigcup_{i \in \mathcal{A}} S_i(\Lambda) \). The smallest interval that contains \( \Lambda \) is \( I = [0, \frac{9}{2}] \).

Put \( I_{i_1 \ldots i_n} := S_{i_1 \ldots i_n}(I) \), where we use the shorthand notation \( S_{i_1 \ldots i_n} = S_{i_1} \circ \cdots \circ S_{i_n} \).

The natural projection \( \Pi : \Sigma \cup \Sigma^* \to \Lambda \) is defined by

\[
\Pi(i) := \sum_{k=1}^{|i|} i_k 3^{-(k-1)},
\]

where \( |i| = n \) if \( i \in \mathcal{A}^n \) and \( |i| = \infty \) if \( i \in \Sigma \). For a finite word \( i \in \Sigma^* \) the projection \( \Pi(i) \) is the left end point of the interval \( I_i \). That is,

\[
\Pi(i) = \Pi(j) \iff I_i = I_j.
\]

We are given a probability vector \( \mathbf{p} := (p_0, p_1, p_3) \). That is, \( p_i > 0 \) and \( \sum_{i \in \mathcal{A}} p_i = 1 \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{\( I_{i_1 \ldots i_n} := S_{i_1 \ldots i_n}(I) \), \((i_1, \ldots, i_n) \in \mathcal{A}^n\), and \( I_{10} = I_{03} \).}
\end{figure}
We form the corresponding product measure \( \mu := \mathbf{p}^N \) and define its push forward measure \( \nu := \Pi_* \mu \). Then, \( \nu \) is the **self-similar measure** corresponding to the probability vector \( \mathbf{p} \). That is, for every maximal finite antichain \( \Gamma \subset \Sigma^* \) we have

\[
(2.5) \quad \nu = \sum_{i \in \Gamma} p_i \cdot \nu \circ S_i^{-1}.
\]

The peculiarity of this IFS \( \mathcal{S} \) is that we have complete overlap: \( I_{0,3} = I_{1,0} \). Moreover, an easy case analysis yields:

**Fact 2.1.** Assume that \( i, j \in \mathcal{A}^n \), with \( i_1 < j_1 \) such that

\[
\Pi(i) = \Pi(j) \quad \text{but} \quad \Pi(i|_k) \neq \Pi(j|_k), \quad \forall k < n.
\]

Then, \( i = (1, \ldots, 1, 0) \) and \( j = (0, 3, \ldots, 3) \).

**Definition 2.2.** Let

\[
(2.6) \quad A := \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},
\]

where we index the rows and columns with \( 0, 1, 3 \) in increasing order. We introduce the subshifts of finite types

\[
\Sigma_A := \{ i \in \Sigma : (i_k, i_{k+1}) \neq (0, 3), \forall k \}, \quad \Sigma_B := \{ i \in \Sigma : (i_k, i_{k+1}) \neq (1, 0), \forall k \}.
\]

Analogously, we define

\[
\mathcal{T}_n := \{ i \in \mathcal{A}^n : (i_k, i_{k+1}) \neq (0, 3), \forall k < n \},
\]

and \( \mathcal{U}_n := \{ i \in \mathcal{A}^n : (i_k, i_{k+1}) \neq (1, 0), \forall k < n \} \). Set

\[
\Sigma_A^* := \bigcup_{n=1}^{\infty} \mathcal{T}_n \cup \mathcal{b} \quad \text{and} \quad \Sigma_B^* := \bigcup_{n=1}^{\infty} \mathcal{U}_n \cup \mathcal{b},
\]

where \( \mathcal{b} \) is the empty word.

An obvious case analysis shows (see [BSS, Fact 4.2.18]) that the following fact holds:

**Fact 2.3.** (a) Assume that for the distinct \( i, j \in \mathcal{T}_n \) we have \( I_i \cap I_j \neq \emptyset \). Then, \( |i \wedge j| = n - 1 \) and \( \{i_n, j_n\} = \{0, 1\} \), where \( i \wedge j \) is the common prefix of the words \( i \) and \( j \).

(b) Assume that for the distinct \( i, j \in \mathcal{U}_n \) we have \( I_i \cap I_j \neq \emptyset \). Then, there exists a \( k \leq n - 2 \) and an \( \omega \in \mathcal{T}_k \) such that \( \omega_k \neq 1 \), \( i = \omega \overline{0} \overline{3}^{n-k-1} \) and \( j = \omega \overline{1} \overline{3}^{n-k} \).
For an $i \in T_n$ there can be exponentially many $j \in A^n$ with $I_i \cap I_j \neq \emptyset$. However, the previous Fact implies the following corollary.

**Corollary 2.4.** If $i \in T_n$, then there is at most one $j \in T_n \setminus \{i\}$ such that $I_i \cap I_j \neq \emptyset$. The same remains valid if we replace $T_n$ with $U_n$.

If $p_1 \geq p_3$, then we should work with $\Sigma_A$ and $T_n$, $n \in \mathbb{N}$. On the other hand, if $p_3 > p_1$, then we should work with $\Sigma_B$, and $U_n$, $n \in \mathbb{N}$.

**Principal Assumption 2.5.** We always assume in this note that

$$p_3 \leq p_1. \quad (2.7)$$

For the symmetry pointed out in Corollary 2.4, we may assume without any loss of generality for the rest of the paper that (2.7) holds.

It is immediate from part (a) of Fact 2.3 that

$$i, j \in T_n, \quad i = j \iff S_i = S_j \iff \Pi(i) = \Pi(j). \quad (2.8)$$

### 3. The main result

**Definition 3.1** (Pressure of a potential). We call a continuous function $f : \Sigma_A^* \to [0, \infty)$ a potential. The pressure of the potential $f$ is defined by

$$P(f) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{i \in T_n} f(i), \quad (3.1)$$

if the limit exists, otherwise we replace the limit with $\limsup$.

#### 3.1. The main result. As we mentioned above we assume that $p_1 \geq p_3$. If $p_1 \leq p_3$, then all $T_n$ below should be replaced by $U_n$ and all the results remain unchanged. We define

$$I_i := \{\eta \in A^n : S_\eta = S_i\} \quad \text{and} \quad \psi(i) := \sum_{\eta \in I_i} p_\eta, \quad \text{for every} \ i \in \Sigma_A^*, \quad (3.2)$$

and we define

$$\psi(\emptyset) := 1, \quad \text{where} \ \emptyset \ \text{is the empty word.} \quad (3.3)$$

We will point out in (7.8) that

$$\# I_n \leq n. \quad (3.4)$$

We will prove in Section 4.1 that the limit in the following definition exists:

$$p(t) := P(\psi^t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{i \in T_n} \psi^t(i), \quad t \geq 0. \quad (3.5)$$
It follows from (3.4) that for every \( t \geq 0 \) and \( \varepsilon > 0 \)

\[
\frac{p(t + \varepsilon) - p(t)}{\varepsilon} \in (\log p_{\min}, \log p_{\max}).
\]

Using this and the definition of \( \psi(i) \) we get that the function \( p(t) \) has the following properties:

(i) \( p(t) \) is a continuous and strictly decreasing function on \([0, \infty)\),
(ii) \( p(1) = 0 \) and so \( p(t) > 0 \) for \( t \in [0, 1) \).

This implies that there exists a unique \( t_r \in (0, 1) \) such that

\[
p(t_r) := rt_r \log 3, \quad \text{for every } r \in (0, \infty).
\]

Observe that by the properties (i) and (ii) of \( p(t) \) we obtain that

\[
t_r > 0, \quad \text{for all } r > 0.
\]

We define \( \chi_r \) such that

\[
t_r = \frac{\chi_r}{r + \chi_r}, \quad \text{that is, } \chi_r = \frac{t_r r}{1 - t_r}.
\]

**Theorem 3.2.** The quantization dimension of the measure \( \nu \) is \( D_r(\nu) = \chi_r \).

3.2. **An explicit formula for the \( L^q \)-spectrum of \( \nu \).** As we mentioned, the combination of [KNZ, Corollary 1.12] and Theorem 3.2 yields an explicit formula for the \( L^q \)-spectrum of \( \nu \) for \( q \in (0, 1) \). Namely, the \( L^q \)-spectrum of the measure \( \nu \) is defined for \( q > 0 \) by

\[
\beta(q) := \limsup_{n \to \infty} \frac{\log \sum_{C \in D_n} \nu(C)^q}{\log 2^n},
\]

where \( D_n := \{(k/2^n, (k + 1)/2^n) : k \in \mathbb{Z}\} \) is the partition of \( \mathbb{R} \) into dyadic intervals. In [KNZ] the authors introduced

\[
q_r := \inf \{q > 0 : \beta(q) < qr\}.
\]

In our special case, [KNZ, Corollary 1.12] yields

\[
D_r(\nu) = \frac{rq_r}{1 - q_r}, \quad \text{for every } r > 0.
\]

Putting together (3.9), the assertion of Theorem 3.2 and (3.12) we get

\[
\frac{rt_r}{1 - t_r} = D_r(\nu) = \frac{rq_r}{1 - q_r}, \quad \text{for every } r > 0.
\]

The combination of this and (3.8) yields

\[
q_r = t_r > 0.
\]
We know that the function $\beta(q)$ convex and in this way continuous on $(0, \infty)$. Hence, by the definition of $q_r$ we get

$$
(3.15) \quad \beta(q_r) = rq_r.
$$

**Corollary 3.3.** For every $q \in (0, 1)$ we have

$$
(3.16) \quad \beta(q) = \frac{p(q)}{\log 3}.
$$

**Proof of Corollary 3.3 assuming Theorem 3.2.** Putting together (3.7), (3.14) and (3.15) we get

$$
(3.17) \quad \frac{p(t_r)}{r \log 3} = t_r = q_r = \frac{\beta(q_r)}{r}, \quad \text{for all } r > 0.
$$

Hence,

$$
(3.18) \quad p(q_r) = \beta(q_r) \cdot \log 3, \quad \text{for every } r > 0.
$$

Choose an arbitrary $q' \in (0, 1)$. Let $r := p(q')/q'$. Observe that $r > 0$ by property (ii) of the function $p(t)$. Then, $q_r = q'$. Hence, by (3.17) we get that $\beta(q') = \frac{p(q')}{\log 3}$. \[\square\]

3.3. **The main Proposition.** To prove our main result we need our Main Proposition (Proposition 7.19) below. To state it we need some further notation.

3.3.1. A projection $\Phi : \Sigma \cup \Sigma^* \to \Sigma_A \cup \Sigma_A^*$. Recall that by definition $\Sigma \cup \Sigma^*$ is the collection of finite or infinite words over the alphabet $A$. Similarly, $\Sigma_A \cup \Sigma_A^* \subset \Sigma \cup \Sigma^*$ is the collection of those elements of $\Sigma \cup \Sigma^*$, which do not contain the sequence $(0,3)$. First we define such a mapping $\Phi : \Sigma \cup \Sigma^* \to \Sigma_A \cup \Sigma_A^*$ which has the following properties: For every $\eta \in \Sigma \cup \Sigma^*$

(a) $\Phi$ preserves the length of every word: $|\eta| = |\Phi(\eta)|$,
(b) $\Pi(\eta) = \Pi(\Phi(\eta))$.

In the rest of the paper we frequently use the following notation: For a digit $a$ and $n \in \mathbb{N} \cup \{\infty\}$,

$$
\overline{a^n} := a, a, \ldots, a.
$$

**Definition 3.4.** We define $\Phi : \Sigma \cup \Sigma^* \to \Sigma_A \cup \Sigma_A^*$ as follows: Let $\eta \in \Sigma$. We obtain $i := (i_1, i_2, \ldots) := \Phi(\eta) \in \Sigma_A$ from $\eta \in \Sigma$ by successive substitutions as follows

(a) For every $1 \leq k < \ell < \infty$ such that $(\eta_k, \ldots, \eta_\ell) = (0, 3^{\ell-k})$ and $\eta_{\ell+1} \neq 3$, we define $(i_k, \ldots, i_\ell) := (1^{\ell-k}, 0)$.
(b) If there exists a $1 \leq k < \infty$ such that $(\eta_k, \eta_{k+1}, \eta_{k+1}, \ldots) = (0, 3^{\infty})$, then we define $(i_k, i_{k+1}, \ldots) := (1^\infty)$. 
Then, we get rid of all the $(\eta_k, \eta_{k+1}) = (0, 3)$ in $\eta$, so $\Phi(\eta) \in \Sigma_A \cup \Sigma_A^*$ and by Fact 2.1:

$$\Pi(\Phi(\eta)) = \Pi(\eta), \quad \forall \eta \in \Sigma \cup \Sigma^*.$$  

However, considerable technical difficulties are caused by the fact that

$$\Phi(0, 3) = \Phi(3) = \Phi(\sigma 03).$$

Moreover,

$$\Phi((0, 0, 3)|_2) = 0 = 3 = \Phi(0, 0, 3)|_2.$$  

D.J. Feng [F2] introduced a very important family of potentials. This family was termed quasi-multiplicative potentials by A. Käenmäki and H.W. Reeve [KR]).  

3.3.2. Weak quasi-multiplicative potentials.  

**Definition 3.5.** We say that a function $\phi : \Sigma_A^* \to [0, \infty)$ is a weak quasi-multiplicative potential on $\Sigma_A^*$ if the following two conditions hold:  

(a) There is an $l \in \Sigma_A^*$ which is not the empty word such that $\phi(l) > 0$.  

Moreover, there exist $C_1, C_2 > 0$ such that

$$\phi(ij) \leq C_1 \phi(i) \phi(j), \quad ij \in \Sigma_A^*.$$  

(b) There exists a $z \in \mathbb{N}$ such that

$$\forall i, j \in \Sigma_A^*, \exists k \in \bigcup_{\ell=1}^z T_{\ell} \cup b \text{ such that } ikj \in \Sigma_A^* \text{ and } \phi(i) \phi(j) \leq C_2 \phi(ikj).$$

First we introduce a potential $\hat{\psi} : \Sigma_A^* \to (0, 1]$ as follows:

$$\hat{\psi}(i) := \begin{cases} 
\max \{\psi(i), \psi(i^{-0})\}, & \text{if } i_{|i|} = 1; \\
\psi(i), & \text{if } i_{|i|} \neq 1, \end{cases} \quad \text{for } i \in \Sigma_A^*.$$  

3.3.3. The statement of the Main Proposition.  

**Proposition 3.6.** The following properties hold:  

Property-1 $\hat{\psi}$ is a weak quasi-multiplicative potential on $\Sigma_A^*$.  

Property-2 There exists a $C_3 > 0$ such that for every $n \geq 1$

$$1 \leq \frac{\hat{\psi}(i)}{\psi(i)} \leq C_3 \cdot n, \quad \forall i \in T_n.$$
Property-3 For every \( n \geq 1 \)
\[
\sum_{i \in \mathcal{T}_n} 1_{I_i}(x) \leq 2.
\]

Property-4 For every \( n \in \mathbb{N} \cup \{\infty\} \), \( \eta \in \mathcal{A}^n \), \( 1 \leq z < n \), \( i := \Phi(\eta)|_z \)
\[
(3.26)
\]
\[
\eta|_n \subset \mathcal{I}_1 \cup \mathcal{I}_{1-0}.
\]

Property-2 is an immediate consequence of Fact 7.14,
Property-3 is an immediate consequence of Part (a) of Fact 2.3,
Property-4 is proved in Part (c) of Fact 7.5.

The organization of the rest of the paper
(a) In Section 4 we introduce further pressure functions and finite maximal antichains. Moreover, we prove some of their properties, assuming Proposition 7.19.
(b) In Sections 5 and 6 we prove Theorem 3.2 using only those properties of \( \hat{\psi}(i) \) which are listed in Proposition 7.19.
(c) In Section 7 we prove that \( \hat{\psi}(i) \) satisfies the Properties-1,2 listed in Proposition 7.19.

4. Pressure functions and finite maximal antichains

We will use the following theorem due to D.J. Feng [F2, Theorem 5.5].

**Theorem 4.1** (Feng). Let \( \phi \) be a weak quasi-multiplicative potential on \( \Sigma_A^* \). Then, there exists a unique invariant ergodic measure \( \mathbf{m} \) on \( \Sigma_A \) with the following property
\[
(4.1) \quad \mathbf{m}(i) \approx \frac{\phi(i)}{\sum_{j \in \mathcal{T}_n} \phi(j)} \approx \phi(i) \exp\left(-nP(\phi)\right),
\]
where \( a(i) \approx b(i) \) if there exists a \( c > 0 \) such that \( \frac{1}{c}b(i) \leq a(i) \leq cb(i) \) for all \( i \in \Sigma_A^* \) and
\[
(4.2) \quad P(\phi) = \lim_{n \to \infty} \log \sum_{i \in \mathcal{T}_n} \phi(i).
\]

4.1. Various pressure functions. By Theorem 4.1, we obtain that
\[
(4.3) \quad \text{the limit } \lim_{n \to \infty} \frac{1}{n} \log \sum_{i \in \mathcal{T}_n} \left(\hat{\psi}(i)\right)^t \quad \text{exists.}
\]

Recall that the pressure function \( p(t) = P(\psi^t) \) was defined in (3.5) with the comment that the existence of the limit in (3.5) would be proved later. Using
Property-2 and (4.3) we get that the second equation below holds:

\[(4.4) \quad p(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{i \in \mathcal{T}_n} (\psi(i))^t = \lim_{n \to \infty} \frac{1}{n} \log \sum_{i \in \mathcal{T}_n} \left(\hat{\psi}(i)\right)^t.\]

In particular, the first limit (which was defined as \(p(t)\) in (3.5)) exists.

For every \(t > 0\) we define the potentials

\[(4.5) \quad \phi_t(i) := (\psi(i) \cdot 3^{-|i|r})^t \quad \text{and} \quad \hat{\phi}_t(i) := \left(\hat{\psi}(i) \cdot 3^{-|i|r}\right)^t.\]

We obtain from Property-1 that

**Corollary 4.2.** For every \(t > 0\) the potential \(i \mapsto \hat{\phi}_t(i)\) is also quasi-multiplicative.

We cannot say the same about the potential \(\phi_t\) but by (4.4) the pressure functions of \(\phi_t\) and \(\hat{\phi}_t\) are the same:

\[(4.6) \quad P(t) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{j \in \mathcal{T}_n} \phi_t(j) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{j \in \mathcal{T}_n} \hat{\phi}_t(j) = p(t) - rt \log 3.\]

Using the properties of the function \(p(t)\) stated on page 8 we obtain that \(t \mapsto P(t)\) is also strictly decreasing, \(P(0) = p(0) > 0\) and \(P(1) = -r \log 3\). So, we obtain that

**Fact 4.3.** There is a unique \(t_r \in (0, 1)\) such that

\[(4.7) \quad p(t_r) = rt_0 \log 3, \quad P(t_r) = 0, \quad P(t) > 0 \text{ if } t \in [0, t_r) \text{ and } P(t) < 0 \text{ if } t > t_r.\]

Finally, we introduce the potential \(\hat{\phi} : \Sigma_A^* \to [0, \infty),\)

\[(4.8) \quad \hat{\phi}(i) := \hat{\phi}_{t_r}(i) = \left(\hat{\psi}(i) \cdot 3^{-|i|r}\right)^{t_r}.\]

Then, by definition, the pressure of \(\hat{\phi}\) is equal to 0:

\[P(\hat{\phi}) = \lim_{n \to \infty} \log \sum_{i \in \mathcal{T}_n} \hat{\phi}(i) = 0.\]

Moreover, as a corollary of Feng Theorem (Theorem 4.1) and Corollary 4.2 we obtain:

**Proposition 4.4.** There is a \(C_4 > 1\) and a unique invariant ergodic measure \(m\) on \(\Sigma_A\) such that

\[(4.9) \quad C_4^{-1} \frac{m([i])}{\phi(i)} < C_4, \quad \text{for all } i \in \Sigma_A^*.\]
4.2. Finite maximal antichains of $\Sigma_A$ and $\Sigma$. Let $X$ be either $\Sigma_A$ or $\Sigma$. A finite collection $\Gamma$ of finite words $i$ of $X$ is a finite maximal antichain of $X$ if for every $\omega \in X$ we can find a unique $i \in \Gamma$ such that $\omega \in [i]$.

**Definition 4.5.** Let $\tilde{q}_i := (p, 3^{-r})^t$ and $\varepsilon_0 := \min_{i \in A} \tilde{q}_i$. For an $0 < \varepsilon < \varepsilon_0$ we define

\[
\hat{\Gamma}(\varepsilon) := \left\{ i \in \Sigma_A^* : \hat{\phi}(i) < \varepsilon, \forall p < |i|, \hat{\phi}(i_p) \geq \varepsilon \right\}
\]

and $\hat{\Gamma}^- (\varepsilon) := \{ i^- : i \in \hat{\Gamma}(\varepsilon) \}$.

It is clear that both $\hat{\Gamma}(\varepsilon)$ and $\hat{\Gamma}^- (\varepsilon)$ are maximal antichains of $\Sigma_A$. From this and from (4.9), we get that

\[
\sum_{i \in \hat{\Gamma}(\varepsilon)} \hat{\phi}(i) \leq C_4.
\]

**Lemma 4.6.** There exists a $\gamma' > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, we have

\[
\frac{\sum_{i \in \hat{\Gamma}(\varepsilon)} \hat{\phi}(i)}{\sum_{i \in \hat{\Gamma}(\varepsilon)} \hat{\phi}(i^-)} > \gamma'.
\]

**Proof.** Observe that

\[
\sum_{i \in \hat{\Gamma}(\varepsilon)} \hat{\phi}(i^-) \leq 3 \sum_{j \in \hat{\Gamma}^- (\varepsilon)} \hat{\phi}(j) \leq 3C_4 \sum_{j \in \hat{\Gamma}^- (\varepsilon)} m([j]) \leq 3C_4.
\]

Namely, the first inequality follows from the fact that for every $j \in \hat{\Gamma}^- (\varepsilon)$ there are at most three $i \in \hat{\Gamma}(\varepsilon)$ such that $j = i^-$. The second inequality is immediate from (4.9). The third inequality is a consequence of the fact that $\hat{\Gamma}^- (\varepsilon)$ is a maximal antichain of $\Sigma_A$. On the other hand,

\[
\sum_{i \in \hat{\Gamma}(\varepsilon)} \hat{\phi}(i) \geq C_4^{-1} \sum_{i \in \hat{\Gamma}(\varepsilon)} m(i) = C_4^{-1}.
\]

Putting together (4.13) and (4.14) we obtain that (4.12) holds with the choice of $\gamma' := \frac{1}{3C_4^2}$.

**Corollary 4.7.** Let $0 < \varepsilon < \varepsilon_0$. Then, we have

\[
\#\hat{\Gamma}(\varepsilon) \leq \frac{3C_4^3}{\varepsilon}.
\]

**Proof.** It follows from (4.9) that $C_4 > \sum_{i \in \hat{\Gamma}(\varepsilon)} \hat{\phi}(i)$. Putting together this, (4.12), and the fact that $\hat{\phi}(i^-) \geq \varepsilon$, we obtain

\[
C_4 > \gamma' \sum_{i \in \hat{\Gamma}(\varepsilon)} \hat{\phi}(i^-) > \gamma' \varepsilon \#\hat{\Gamma}(\varepsilon).
\]
This implies that (4.15) holds since γ′ = \frac{1}{3C_4^2}.

Although \(\hat{\Gamma}(\varepsilon)\) is a finite maximal antichain the same is not true (in general) for \(\bigcup_{i \in \hat{\Gamma}(\varepsilon)} \mathcal{I}_i\), where we defined \(\mathcal{I}_i\) in (3.2). So, we need to introduce one more step:

**Definition 4.8.**

\[
\Theta(\varepsilon) := \left\{ i^0 : i \in \hat{\Gamma}(\varepsilon) \text{ and } |i| = 1 \right\} \subset \Sigma_A^*.
\]

Moreover, let

\[
\hat{\Gamma}_E(\varepsilon) := \hat{\Gamma}(\varepsilon) \cup \Theta(\varepsilon) \quad \text{and} \quad \hat{\Gamma}_\Sigma(\varepsilon) := \bigcup_{j \in \hat{\Gamma}_E(\varepsilon)} \mathcal{I}_j.
\]

The following claim states a simple but important property.

**Claim 4.9.** For all \(i \in \hat{\Gamma}_E(\varepsilon)\) we have \(\hat{\phi}(i) < \varepsilon\).

**Proof.** We need to check this only for \(i \in \Theta(\varepsilon)\) since by definition \(\hat{\phi}(i) < \varepsilon\) holds for all \(i \in \hat{\Gamma}(\varepsilon)\). If \(i \in \Theta(\varepsilon)\), then \(i = i^0 - 0\) and \(i^{-1} \in \hat{\Gamma}(\varepsilon)\). By (3.24),

\[
\hat{\psi}(i) = \psi(i) = \psi(i^0) \leq \max\{\psi(i^0), \psi(i^{-1})\} = \psi(i^{-1}).
\]

So, we get that

\[
\forall i \in \Theta(\varepsilon), \quad \hat{\phi}(i) = \left(\hat{\psi}(i) \cdot 3^{-|i|}r\right)^{tr} \leq \left(\hat{\psi}(i^0) \cdot 3^{-|i|}r\right)^{tr} = \hat{\phi}(i^{-1}) < \varepsilon,
\]

where in the last step we used that \(i^{-1} \in \hat{\Gamma}(\varepsilon)\). \(\Box\)

We will use the following immediate consequence of Claim 4.9:

\[
(\psi(i)3^{-|i|}r)^{tr} \leq \left(\hat{\psi}(i)3^{-|i|}r\right)^{tr} = \hat{\phi}(i) < \varepsilon \quad \text{holds for all } i \in \hat{\Gamma}_E(\varepsilon).
\]

**Lemma 4.10.** For every \(0 < \varepsilon < \varepsilon_0\), we have

\[
\Sigma = \bigcup_{\tau \in \hat{\Gamma}_\Sigma(\varepsilon)} [\tau].
\]

**Proof.** Let \(\eta \in \Sigma\). Then, there is a unique \(i \in \hat{\Gamma}(\varepsilon)\) such that \(\Phi(\eta) \in [i]\). Let \(n := |i|\). Then, \(\eta = \Phi(\eta)|_n\). It follows from Property-4 that \(\eta|_n \in \mathcal{I}_i \cup \mathcal{I}_{i-0}\). \(\Box\)

For \(i, j \in \Sigma_A^*\) we say that \(i\) is a proper prefix of \(j\) if \(|i| < |j|\) and \(j|_{|i|} = i\). In this case we write \(i \not\preceq j\). Let

\[
\tilde{\Gamma}_\Sigma(\varepsilon) := \left\{ j \in \hat{\Gamma}_\Sigma(\varepsilon) : \nexists i \in \hat{\Gamma}_\Sigma(\varepsilon) \text{ such that } i \not\preceq j \right\}.
\]

The elements of \(\tilde{\Gamma}_\Sigma(\varepsilon)\) are incomparable and by (4.20) we have \(\Sigma = \bigcup_{\tau \in \tilde{\Gamma}_\Sigma(\varepsilon)} [\tau]\). In this way we have proved that
**Fact 4.11.** The collection of cylinders $\tilde{\Gamma}_\Sigma(\varepsilon)$ is a finite maximal antichain for $\Sigma$. That is, for every $\eta \in \Sigma$ there is a unique $i \in \tilde{\Gamma}_\Sigma(\varepsilon)$ such that $\eta \in [i]$.

**Lemma 4.12.** For an $0 < \varepsilon < \varepsilon_0$, we have

\[(4.22) \quad \sum_{i \in \hat{\Gamma}_E(\varepsilon)} \hat{\phi}(i) \leq 2C_4.\]

**Proof.** Observe that

\[(4.23) \quad \sum_{i \in \hat{\Gamma}_E(\varepsilon)} \hat{\phi}(i) \leq \sum_{i \in \hat{\Gamma}(\varepsilon)} \hat{\phi}(i) + \sum_{i \in \hat{\Gamma}(\varepsilon): |i| = 1} \hat{\phi}(i-0).\]

Putting together formulas (4.5) and (3.24) we get that $S_2 \leq S_1$. This and (4.11) together imply that (4.22) holds.

**Definition 4.13.** Using the notation introduced in Definition 4.5 we choose $m, n \in \mathbb{N}$ satisfying

\[(4.24) \quad \frac{m}{n} < \varepsilon_0^2 \quad \text{and} \quad \varepsilon := \frac{m}{n} C_4^3.\]

Fix such an $m, n$ and $\varepsilon$ for the rest of this section.

Then, we have from Lemma 4.12 and Corollary 4.7 that

\[(4.25) \quad \#\hat{\Gamma}_E(\varepsilon) \leq \frac{n}{m}.\]

Namely, Corollary 4.7 we have $\#\hat{\Gamma}_E(\varepsilon) \leq 2\#\hat{\Gamma}(\varepsilon) \leq \frac{6C_4^3}{\varepsilon} = \frac{n}{m}$ by the choice of $\varepsilon$.

**5. THE UPPER ESTIMATE FOR THE QUANTIZATION DIMENSION OF THE MEASURE $\nu$**

From now on we follow [GL1] and [R3]. Recall that $\nu = \Pi_\ast \mu$, where $\mu = \mathbf{p}^\mathbb{N}$ is the infinite product measure on $\mathcal{A}^\mathbb{N}$. Also recall that

\[(5.1) \quad V_{n,r}(\nu) := \inf \left\{ \int d(x, \alpha)^r d\nu(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\},\]

where $d(x, \alpha)$ denotes the distance from the point $x$ to the set $\alpha$. Recall that we introduced $C_{n,r}(\mu)$ in (1.1). It was proved in [GL1] that $C_{n,r}(\mu) \neq \emptyset$. In the argument below we will use frequently the potential $\hat{\phi}(i)$ which was defined in (4.8), the set $\mathcal{I}_1$ which was defined in (3.2), the constant $C_4$ defined in (4.9).
5.1. **An upper estimate for** $V_{n,r}(\nu)$. First we need to prove a Fact which is a slight modification of [GL1, Lemma 4.14].

**Fact 5.1.** Let $\mu, \mu_i$ for $i = 1, \ldots, L$ be Borel probability measures on $\mathbb{R}$ such that $\int \|x\|^r d\mu_i(x) < \infty$ for all $i = 1, \ldots, L$. We are given $q_1, \ldots, q_L$ positive numbers such that

$$\mu \leq \sum_{m=1}^{L} q_m \mu_m. \tag{5.2}$$

Moreover, we are given natural numbers $\{n_i\}_{i=1}^{L}$ such that $n_i \geq 1$ and $\sum_{i=1}^{L} n_i \leq n$. Then,

$$V_{n,r}(\mu) \leq \sum_{i=1}^{L} q_i V_{n_i,r}(\mu_i). \tag{5.3}$$

**Proof.** Let $\alpha_i \in C_{n_i,r}(\mu_i)$ for $i = 1, \ldots, L$ and $\alpha := \bigcup_{i=1}^{L} \alpha_i$.

$$V_{n,r}(\mu) \leq \int \min_{a \in \alpha} \|x - a\|^r d\mu(x) \leq \sum_{i=1}^{L} q_i \int \min_{a \in \alpha_i} \|x - a\|^r d\mu_i(x) \tag{5.4}$$

$$\leq \sum_{i=1}^{L} q_i \int \min_{a \in \alpha_i} \|x - a\|^r d\mu_i(x) = \sum_{i=1}^{L} q_i V_{n_i,r}(\mu_i). \tag{5.5}$$

□

To apply this Fact we prove that the condition (5.2) holds in our case:

**Fact 5.2.**

$$\nu \leq \sum_{i \in \hat{\Gamma}_E(\varepsilon)} \psi(i) \nu_i, \tag{5.6}$$

where we write

$$\nu_\eta := (S_\eta)_* \nu, \text{ for an } \eta \in \mathcal{A}^*. \tag{5.7}$$

Note that if $\eta \in \mathcal{I}_i$, then $S_\eta = S_i$. Consequently,

$$\nu_\eta = \nu_i, \text{ for all } \eta \in \mathcal{I}_i. \tag{5.8}$$

**Proof.** Using that $\tilde{\Gamma}_\Sigma(\varepsilon) \subset \hat{\Gamma}_E(\varepsilon)$ is a finite maximal antichain

$$\nu = \sum_{\eta \in \tilde{\Gamma}_\Sigma(\varepsilon)} p_\eta \nu_\eta \leq \sum_{\eta \in \tilde{\Gamma}_\Sigma(\varepsilon)} p_\eta \nu_\eta = \sum_{j \in \hat{\Gamma}_E(\varepsilon)} \sum_{\eta \in \mathcal{I}_j} p_\eta \nu_\eta$$

$$= \sum_{i \in \hat{\Gamma}_E(\varepsilon)} \left( \sum_{\eta \in \mathcal{I}_i} p_\eta \right) \nu_i = \sum_{i \in \hat{\Gamma}_E(\varepsilon)} \sum_{\eta \in \mathcal{I}_i} \nu_i = \sum_{i \in \hat{\Gamma}_E(\varepsilon)} \psi(i) \nu_i. \tag{5.9}$$

□
Fact 5.3. Let us fix an \( n_i \in \mathbb{N} \) for every \( i \in \Gamma_E(\varepsilon) \) such that \( n_i \geq 1 \) and \( \sum_{i \in \Gamma_E(\varepsilon)} n_i \leq n \). Then,

\[
V_{n,r}(\nu) \leq \sum_{i \in \Gamma_E(\varepsilon)} \psi(i) V_{n_i,r}(\nu_i).
\]

**Proof.** This follows from the combination of Fact 5.1 and Fact 5.2. □

Lemma 5.4.

\[
V_{n,r}(\nu) \leq \inf \left\{ \sum_{i \in \Gamma_E(\varepsilon)} 3^{-|i|r} \psi(i) \cdot V_{n_i,r}(\nu_i) : 1 \leq n_i, \sum_{i \in \Gamma_E(\varepsilon)} n_i \leq n \right\}.
\]

**Proof.** Let \( i \in \Gamma_E(\varepsilon), n_i \geq 1 \), and \( \alpha_i \in C_{n_i,r}(\nu) \). Below we prove that

\[
V_{n_i,r}(\nu_i) \leq 3^{-|i|r} V_{n_i,r}(\nu).
\]

We obtain the assertion of the lemma from the combination of Fact 5.3 and (5.11). Now we prove (5.11).

\[
V_{n_i,r}(\nu_i) \leq \int d\left(x, S_i(\alpha_i)^r\nu_i(x) = \int d(x, S_i(\alpha_i)^r d(\nu \circ S_i^{-1})(x)
\]

\[
= \int d(S_i(x), S_i(\alpha_i)^r d\nu(x) = 3^{-|i|r} \int d(x, \alpha_i)^r d\nu(x)
\]

\[
= 3^{-|i|r} V_{n_i,r}(\nu).
\]

Proposition 5.5. There exists a constant \( C_9 > 0 \) such that

\[
\limsup_{n \to \infty} n \cdot e_{n,r}^{\chi}(\nu) \leq C_9 m \cdot e_{m,r}^{\chi}(\nu) < \infty.
\]

**Proof.** Recall the definition of \( n, m, \varepsilon \) from Definition 4.13. For every \( i \in \Gamma_E(\varepsilon) \) we define \( n_i := m \). Then, by (4.25) we have \( \sum_{i \in \Gamma_E(\varepsilon)} n_i = \# \Gamma_E(\varepsilon) m \leq n \). We apply Lemma 5.4 in the first step below:

\[
V_{n,r}(\nu) \leq \sum_{i \in \Gamma_E(\varepsilon)} 3^{-|i|r} \psi(i) V_{m,r}(\nu)
\]

\[
= \sum_{i \in \Gamma_E(\varepsilon)} \left( 3^{-|i|r} \psi(i) \right)^{t_r} \left( 3^{-|i|r} \psi(i) \right)^{1-t_r} V_{m,r}(\nu)
\]

\[
\leq \varepsilon^{1-t_r} \frac{1-t_r}{\varepsilon^{1-t_r}} V_{m,r}(\nu) \sum_{i \in \Gamma_E(\varepsilon)} \phi(i)
\]

The reasoning for the four underbraces above are as follows:
This is immediate from the definition $\hat{\phi}$ and from the fact that $\hat{\psi}(i) \leq \hat{\psi}(i)$ for all $i$.

If $i \in \hat{\Gamma}_E(\varepsilon)$, then $(3^{-|i|}\psi(i))^{tr} < \varepsilon$ (see (4.19)).

This follows from definition: $t_r = \frac{x_r}{r + x_r} \implies \frac{1 - t_r}{t_r} = \frac{r}{x_r}$.

This follows from Lemma 4.12.

So, we have proved that

$$V_{n,r}(\nu) \leq \varepsilon \frac{r}{n} V_{m,r}(\nu) = \left(\frac{m}{n}\right)^{\frac{r}{x_r}} C_9^{\frac{r}{x_r}} V_{m,r}(\nu),$$

where $C_9 := 6C_4^3$. Hence,

$$nV_{n,r}(\nu) \leq C_9 m V_{m,r}(\nu).$$

Letting $n$ approaching to infinity we obtain that (5.13) holds.

Putting together Proposition (5.5) and (1.4) we obtain that

$$D_r(\nu) \leq \chi_r.$$

6. The lower estimate

First we prove a Fact similar to Fact 5.2.

**Fact 6.1.** For every $n$ we have

$$\nu = \sum_{i \in T_n} \psi(i) \nu_1.$$

**Proof.** Fix an $n$. Then, $A^n$ is a finite maximal antichain for $\Sigma$. Hence,

$$\nu = \sum_{\eta \in A^n} p_\eta \cdot \nu_\eta,$$

where $\nu_\eta := \nu \circ S^{-1}_\eta$. Moreover, let $f : \Sigma \to \mathbb{R}$ be a continuous function. Using that $S_\eta = S_i$ if $\eta \in I_i$ we get

$$\int f(x) d\nu(x) = \sum_{\eta \in A^n} p_\eta \int f(x) d\nu_\eta(x) = \sum_{i \in T_n} \sum_{\eta \in I_i} p_\eta \int f(x) d(\nu \circ S^{-1}_\eta)(x)$$

$$= \sum_{i \in T_n} \sum_{\eta \in I_i} p_\eta \int f(x) d(\nu \circ S^{-1}_i)(x) = \sum_{i \in T_n} \psi(i) \int f(x) d\nu_1(x).$$

Let $U := \text{int}(I) = (0, \frac{9}{2})$. Recall that $I = \overline{U} \supset \Lambda$. For an $\eta \in \Sigma^*$ we write $U_\eta := S_\eta(U) \subset U$. Following Graf and Luschgy [GL3] we introduce

$$u_{n,r}(\nu) := \inf \left\{ \int d(x, \alpha \cup U^c)^r d\nu(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\}.$$
As an analogue of [GL3, Lemma 4.4] we need the following assertion:

**Lemma 6.2.** There exists a set $\alpha_n \subset \mathbb{R}^n$ such that $\# \alpha_n \leq n$ and

$$u_{n,r}(\nu) = \int d(x, \alpha_n \cup U^c)^r d\nu(x).$$

**Proof.** Define $f : I^n \to \mathbb{R}$ by $f(x_1, \ldots, x_n) := \int d(x, x_1, \ldots, x_n, 0, 9/2)^r d\nu(x)$. It is easy to see that this function is continuous. Hence, it attains its infimum on $I^n$. Any place where the infimum is attained can be chosen as $\alpha_n$. □

**Lemma 6.3.** Fix an $m$. Then, there exists an $n_0$ such that for all $n \geq n_0$ there exists a sequence $\{n_i\}_{i \in \mathcal{T}_m}$ such that

$$1 \leq n_i < n, \quad \sum_{i \in \mathcal{T}_m} n_i \leq 2n, \quad u_{n,r}(\nu) \geq \sum_{i \in \mathcal{T}_m} \psi(i) 3^{-|i|r} u_{n_i,r}(\nu).$$

**Proof.** Fix an $m$ and to shorten the notation we write $\Gamma := \mathcal{T}_m$. We consider any $\tau \in \Sigma^*$ such that $S_\tau(I) \subset U$. For example $\tau = (1)$ will do since $I_1 = S_1(I) = [1, \frac{5}{2}] \subset (0, \frac{9}{2})$. Let $\varepsilon := \text{dist}(S_\tau(I), U^c)$. If we choose $\tau = 1$, then $\varepsilon = 1$. Let $\delta := 3^{-m}$. Then,

$$d(x, U^c) \geq d(x, S_1(U^c)) \geq \delta \varepsilon, \quad \text{if } x \in I_{1\tau} \text{ and } i \in \mathcal{T}_m.$$ 

For each $n$ let $\alpha_n$ be the optimal set for $u_{n,r}(\nu)$. This exists according to Lemma 6.2. We define

$$\delta_n := \max \{d(x, \alpha_n \cup U^c) : x \in \Lambda\}.$$ 

Since $\delta_n \to 0$ we can choose an $n_0$ such that $\delta_n < \delta \varepsilon$ if $n \geq n_0$. Let $x \in \Lambda_{1\tau}$. Then, $x \in S_1(U) \subset U$, for all $i \in \Gamma$. By compactness, there exists an $a \in \alpha_n \cup U^c$ such that

$$d(x, \alpha_n \cup U^c) = d(x, a) \leq \delta_n < \delta \varepsilon,$$

where the one but last inequality holds since $x \in \Lambda$. So, by (6.7) we have $a \not\in S_i(U^c)$. That is, $a \in S_i(U) \subset U$ but $a \in \alpha_n \cup U^c$, hence $a \in \alpha_n$. Let

$$n_i := \# \alpha_n \cap S_i(U), \quad \alpha_{n_i} := \alpha_n \cap S_i(U).$$

Then, we have just proved that $a \in \alpha_{n_i}$. That is,

$$n_i \geq 1, \quad \forall i \in \Gamma.$$

It follows from the optimal property of $\alpha_n$ that $\alpha_n \not\subset \alpha_{n_i}$. Hence, $n_i < n$ for all $i \in \Gamma$.

It follows from Property-3 that

$$\# \{i \in \Gamma : x \in S_{n_i}(I)\} \leq 2, \quad \forall x \in I.$$
Thus,
\[
(6.11) \quad \sum_{i \in \Gamma} n_i \leq 2n.
\]
Finally we prove that
\[
(6.12) \quad u_{n,r}(\nu) \geq \sum_{i \in \Gamma} \psi(i) 3^{-mr} u_{n_i,r}(\nu).
\]
To verify this we will use the following trivial observation
\[
(6.13) \quad \alpha_n \cup S_i(U^c) = \alpha_n \cup (S_i(U))^c = (\alpha_n \cap S_i(U)) \cup (S_i(U))^c.
\]
Using this we get
\[
\int d(S_i(x), \alpha_n \cup S_i(U^c))^r \, d\nu(x) = \int d(S_i(x), (\alpha_n \cap S_i(U)) \cup S_i(U)^c)^r \, d\nu(x)
\]
\[
= 3^{-mr} \int d \left( x, S_i^{-1} \left( \frac{1}{\alpha_n} \bigg( \alpha_n \cap S_i(U) \bigg) \cup S_i(U)^c \right) \right)^r \, d\nu(x)
\]
\[
= 3^{-mr} \int d \left( x, S_i^{-1}(\alpha_n) \cup U^c \right)^r \, d\nu(x) \geq 3^{-mr} u_{n_i,r}(\nu).
\]
Now we put all of these together:
\[
u_{n,r}(\nu) = \int d(x, \alpha_n \cup U^c)^r \, d\nu(x) = \sum_{i \in \Gamma} \psi(i) \int d(S_i(x), \alpha_n \cup U^c)^r \, d\nu(x)
\]
\[
\geq \sum_{i \in \Gamma} \psi(i) \int d(S_i(x), \alpha_n \cup S_i(U^c))^r \, d\nu(x) \geq \sum_{i \in \Gamma} \psi(i) 3^{-mr} u_{n_i,r}(\nu),
\]
where at the second step we used (6.3), at the third step we used that $S_i(U^c) \supset U^c$. At the fourth step we used the previous inequality. 

**Proposition 6.4.** Let $0 < \ell < \chi_r$. Then,
\[
(6.15) \quad \liminf_{n \to \infty} n \cdot (\nu_{n,r}(\nu))^\ell > 0.
\]
Our proof follows the line [GL3, Lemma 4.4] and [R3, Proposition 3.12].

**Proof.** Fix an $\ell \in (0, \chi_r)$. Then, $\frac{\ell}{r+\ell} < t_r$. Recall the definition of the pressure function $P(t)$ from (4.6). It follows from (4.7) that
\[
(6.16) \quad P \left( \frac{\ell}{r+\ell} \right) = \lim_{m \to \infty} \frac{1}{m} \log \sum_{\mathcal{T}_m} (\psi(i) 3^{-mr})^{\frac{\ell}{r+\ell}} > 0.
\]
Fix an $m$ such that
\[
(6.17) \quad \sum_{i \in \mathcal{T}_m} (\psi(i) 3^{-mr})^{\frac{\ell}{r+\ell}} > 2.
\]
For this $m$ we choose $n_0$ as in Lemma 6.3. Let
\begin{equation}
C := \min \{ q^\frac{r}{\tau} u_{q,r}(\nu) : q \leq n_0 \}.
\end{equation}
Clearly, $u_{n,r}(\nu) > 0$. Hence, $C > 0$. Choose an $n$ such that
\begin{equation}
n \geq n_0 \quad \& \quad k < n \implies k^{\frac{r}{\tau}} u_{k,r}(\nu) \geq C.
\end{equation}
Below we prove that
\begin{equation}
n^{\frac{r}{\tau}} u_{n,r}(\nu) \geq C.
\end{equation}
For this $m$ and $n$ we choose $n_i$ for every $i \in \mathcal{T}_m$, as in Lemma 6.3, such that the inequalities of (6.6) hold.
\begin{equation}
n^{\frac{r}{\tau}} u_{n,r}(\nu) \geq n^{\frac{r}{\tau}} \sum_{i \in \Gamma} \psi(i) 3^{-mr} u_{n_i,r}(\nu)
\end{equation}
\begin{equation}
= n^{\frac{r}{\tau}} \sum_{i \in \Gamma} \psi(i) 3^{-mr} u_{n_i,r}(\nu)(n_i)^{-\frac{r}{\tau}} \left( n_i^{\frac{r}{\tau}} u_{n_i,r}(\nu) \right)_{\geq C}
\end{equation}
\begin{equation}
\geq C \sum_{i \in \Gamma} \psi(i) 3^{-mr} \left( \frac{n_i}{n} \right)^{-\frac{r}{\tau}} a_n,
\end{equation}
where at first step we used (6.14), and at the last step we used that $n_i < n$ so by (6.19) we have $u_{n_i,r}(\nu) \geq C$. Set $a_n := \sum_{i \in \Gamma} \psi(i) 3^{-mr} \left( \frac{n_i}{n} \right)^{-\frac{r}{\tau}}$. To verify (6.20), it is enough to prove that
\begin{equation}
a_n \geq 1.
\end{equation}
To see this we use the so-called reversed Hölder inequality: Let $\{x_k\}_{k=1}^M$ and $\{y_k\}_{k=1}^M$ be finite sequences of positive numbers and let $p \in (1, \infty)$. Then,
\begin{equation}
\sum_{k=1}^M x_k y_k \geq \left( \sum_{k=1}^M x_k^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^M y_k^{p-1} \right)^{-\frac{1}{p-1}}.
\end{equation}
In our case the summation is taken for $i \in \Gamma$, $x_i = \psi(i) 3^{-mr}$ and $y_i = (\frac{n_i}{n})^{-\frac{r}{\tau}}$. Finally, $p := 1 + \frac{r}{\tau}$. That is, $\frac{1}{p} = \frac{\ell}{r+\ell}$, $\frac{1}{p-1} = \frac{\ell}{r}$ and $-(p-1) = -\frac{r}{\tau}$.

Hence, from the reversed Hölder inequality we get
\begin{equation}
a_n \geq \left( \sum_{i \in \Gamma} \psi(i) 3^{-mr} \right)^{\frac{r}{r+\ell}} \left( \sum_{i \in \Gamma} \left( \frac{n_i}{n} \right)^{-(\frac{r}{\tau})} \right)^{-\frac{r}{\tau}}.
\end{equation}
Putting together this, (6.11) and (6.17) we get that
\begin{equation}
a_n \geq 2^{\frac{1}{1+\frac{r}{\tau}}} \cdot 2^{-\frac{r}{\tau}} = 2.
\end{equation}
In this way we have proved that (6.23) holds which implies that $n^{\frac{r}{\tau}} u_{n,r}(\nu) > C$ for all $n$. Thus, (6.15) holds.
Clearly, \( u_{n,r}(\nu) \leq V_{n,r}(\nu) \). Hence, we get from Proposition 6.4 that
\[
\liminf_{n \to \infty} n \cdot e^\ell_{n,r} > 0.
\]
Combining this with (1.3) we obtain that
\[
\chi_r \leq D_r(\nu).
\]

Proof of Theorem 3.2. Putting together (5.15) and (6.27) we obtain that
\[
D_r(\nu) = \chi_r.
\]

\[\square\]

7. Checking Properties 1-4

7.1. Good blocks and bad blocks. Let \( n \in \mathbb{N} \cup \{\infty\} \). For arbitrary \( 1 \leq k < \ell \leq n \) and we use the shorthand notation
\[
[n] := \{1, \ldots, n\}, \quad [k, \ell] := \{k, \ldots, \ell\}.
\]

Definition 7.1. Bad blocks, Good blocks

Let \( n \in \mathbb{N} \cup \{\infty\} \) and \( \eta = (\eta_1, \ldots, \eta_n) \). For a \( k < \ell \leq n \) we say that \( [k, \ell] \) is a bad block of \( [n] \) with respect to an \( \eta = (\eta_1, \ldots, \eta_n) \in \mathcal{A}^n \) if
\( (\eta_k, \ldots, \eta_{\ell}) = (0, 3, 3, \ldots, 3) \). We say that \( [k, \ell] \) is a maximal bad block if either \( \ell = \infty \) or \( \eta_{\ell+1} \neq 3 \). Similarly, \( [k, \ell] \) is a good block of \( [n] \) with respect to \( \eta \) if \( (\eta_k, \ldots, \eta_{\ell}) = (1, 1, 1, \ldots, 1, 0) \). We say that \( [k, \ell] \) is a maximal good block if either \( k = 1 \) or \( \eta_{k-1} \neq 1 \). We write \( B(\eta), (G(\eta)) \) for the collection of maximal bad (good) blocks with respect to \( \eta \), respectively. That is,
\[
B(\eta) := \bigcup \{[k, \ell] : 1 \leq k < \ell < \infty, \, \eta_k = 0, \, \eta_{k+1} = \cdots = \eta_{\ell} = 3, \, \eta_{\ell+1} \neq 3\}
\]
\[
G(\eta) := \bigcup \{[k, \ell] : 1 \leq k < \ell < \infty, \, \eta_k = \cdots = \eta_{\ell-1} = 1, \, \eta_{\ell} = 0, \, \eta_{k-1} \neq 1\}
\]

Similarly,
\[
B(\eta) := \bigcup \{[k, \infty] : 1 \leq k < \infty, \, \eta_k = 0, \, \eta_{k+1} = \eta_{k+2} = \eta_{k+3} = \cdots = 3\}
\]

Definition 7.2. A "bad" and a "good" partition of \([n]\)

Given an \( n \in \mathbb{N} \cup \{\infty\} \) and a \( \eta \in \mathcal{A}^n \), the following definitions are meant to be with respect to \( \eta \).

1. \( A_{\text{Good}}(\eta) := [n] \setminus \bigcup_{[k, \ell] \in G(\eta)} [k, \ell] \), and \( A_{\text{Bad}}(\eta) := [n] \setminus \bigcup_{[k, \ell] \in B(\eta)} [k, \ell] \),
2. \( B_{\text{Good}}(\eta) := \bigcup_{[k, \ell] \in G(\eta)} \{\ell\} \), and \( B_{\text{Bad}}(\eta) := \bigcup_{[k, \ell] \in B(\eta)} \{\ell\} \).
(3) $C_{\text{Good}}(\eta) := \bigcup_{[k, \ell - 1]} [k, \ell - 1]$, and $C_{\text{Bad}}(\eta) := \bigcup_{[k, \ell] \in B(\eta)} [k, \ell - 1]$

(4) $D_{\text{Good}}(\eta) := A_{\text{Good}}(\eta) \cup B_{\text{Good}}(\eta)$, and $D_{\text{Bad}}(\eta) := A_{\text{Bad}}(\eta) \cup B_{\text{Bad}}(\eta)$

We use most frequently $C_{\text{Bad}}(\eta)$ and $D_{\text{Bad}}(\eta)$, so we also explain their meaning in words:

$C_{\text{Good}}(\eta)$ is the collection of indices which are in a good box of $[n]$ with respect to $\eta$ but not as a right endpoint of a maximal good box (with respect to $\eta$).

$D_{\text{Good}}(\eta)$ is the collection of indices which are either the right endpoint of a maximal good box of $[n]$ or not contained in any good boxes of $[n]$ (with respect to $\eta$).

In this way we partition $[n]$ into $C_{\text{Good}}(\eta) \cup D_{\text{Good}}(\eta)$. The indices in $D_{\text{Good}}(\eta)$ cause less problem than the ones in $C_{\text{Good}}(\eta)$. This is indicated by the following fact. Before stating it recall that $I_i$ was defined in (3.2). Using this definition we get that for an $i \in \Sigma^*_{A}$

$$I_i = \{ \eta \in A^{[i]} : \Pi(\eta) = \Pi(i) \} .$$

**Fact 7.3.** Let $i \in \mathcal{T}_n$ and let $z \in D_{\text{Good}}(i)$. Then, for every $\eta \in I_i$ we have

$$\eta|_z \in I_{i|z}, \quad \sigma^{z} \eta \in I_{\sigma^{z} i}$$

where $\tau|_z := (\tau_1, \ldots, \tau_z)$ if $\tau = (\tau_1, \ldots, \tau_n) \in A_n$ and $z \leq n$.

Note that for all $a, b, c, d \in \Sigma^*_{A}$

$$\Pi(a) = \Pi(b) \& \Pi(c) = \Pi(d) \implies \Pi(a, 0, 3, \ldots, 3, c) = \Pi(b, 1, \ldots, 1, 0, d).$$

This is immediate from the definition (2.3) of the natural projection $\Pi$ since

$$\Pi(0, 3, \ldots, 3) = 0 \cdot 3^{-1} + \sum_{k=2}^{\ell-1} 3 \cdot 3^{-(k-1)} = \sum_{k=1}^{\ell-1} 1 \cdot 3^{-(k-1)} + 0 \cdot 3^{-(\ell-1)} = \Pi(1, \ldots, 1, 0).$$

Now we summarize some important properties of the mapping $\Phi$ introduced in Definition 3.4:

**Fact 7.4.** (a) For every $\eta \in A^n$ we have $\Phi(\eta) \in \mathcal{T}_n$,

(b) $\Pi(\eta) = \Pi(\Phi(\eta))$ if $\eta \in A^n$,

(c) $I_i = \{ \eta \in A^n : \Phi(\eta) = i \}$, for any $i \in \mathcal{T}_n$.

**Proof.** Part (a) is obvious from Definition 3.4 since we kill all bad blocks of $\eta$.

To prove part (b) we apply (7.5) with $a = c$ and $b = d$ in every step of the construction of $\Phi(\eta)$ in Definition 3.4.
To prove part (c), observe that the inclusion "⊃" follows from part (b) and (7.3). In order to verify the inclusion "⊂" in (c), we assume the opposite to get a contradiction. That is, we assume that there exists an \( \eta \in A^{|i|} \) such that \( \eta \in I_i \) but \( \Phi(\eta) \neq i \). But as we saw in part (b), \( \Pi(\eta) = \Pi(\Phi(\eta)) \). In this way the distinct \( i, \Phi(\eta) \in T_n \) satisfy \( \Pi(\Phi(\eta)) = \Pi(i) \). This is impossible by part (a) of Fact 2.3.

Part (c) of Fact 7.6 and the definition of \( \Phi \) (Definition 3.4) imply that

\[
\text{(7.7)} \quad i \in T_n \quad \& \quad G(i) = \emptyset \quad \implies \quad I_i = \{i\}.
\]

Similarly, the following inequality is an immediate consequence of the definition of \( \Phi \) and part (c) of Fact 7.4

\[
\text{(7.8)} \quad \#I_n \leq n, \quad \text{for every } n.
\]

We partition \( A^n \) as follows

\[
\text{(7.9)} \quad A^n = \bigcup_{i \in T_n} I_i, \quad \text{where } I_i := \{\eta \in A^n : \Pi(\eta) = \Pi(i)\} = \{\eta \in A^n : \Phi(\eta) = i\}.
\]

Using Fact 7.4 and (2.8) this is a partition indeed.

**Proof of Fact 7.3.** Let \( \eta \in A^n \) for an \( n \geq 2 \). By part (c) of Fact 7.4 we have \( \Phi(\eta) = i \). Then, by the definition of the mapping \( \Phi \) we get \( i \) from \( \eta \) by replacing all maximal bad blocks of \( \eta \) by the corresponding good blocks. That is, if \( [k, \ell] \in B(\eta) \), then we define \( i_k = \cdots = i_{\ell - 1} = 1 \) and \( i_\ell = 0 \). Every bad block in \( \eta \) is a good block in \( i \). If we stop at an index \( z \in D_{\text{Good}}(i) \), then the collection of maximal good blocks of \( i|_z \) are the same as the collection of those maximal good blocks of \( i \) which intersect \( [z] = \{1, \ldots, z\} \). So, if we apply the definition of \( \Phi \) to \( \eta|_z \) we get that \( \Phi(\eta|_z) = i|_z \). The second part follows from the first part and the definition of \( \Pi \).

Using a little modification of this argument we can also prove that

**Fact 7.5.** Let \( \eta \in A^n \), where \( n \in \mathbb{N} \cup \{\infty\} \). Let \( z < n \) and \( i := \Phi(\eta)|_z \).

\( \text{(a)} \) If \( z \in D_{\text{Bad}}(\eta) \), then

\[
\Phi(\eta|_z) = \Phi(\eta)|_z \quad \text{that is } \quad \eta|_z \in I_i.
\]

\( \text{(b)} \) If \( z \in C_{\text{Bad}}(\eta) \), then the \( z \)-th coordinate of \( \Phi(\eta) \) is equal to 1 and

\[
\Phi(\eta|_z) = \Phi(\eta)|_{z-1} 0 \neq \Phi(\eta)|_z \quad \text{that is } \quad \eta|_z \in I_{i-0}.
\]

\( \text{(c)} \) Consequently, we get that for all \( z < n \) we have

\[
\eta|_z \in I_i \cup I_{i-0}.
\]
Example 7.6. Let $\eta = (0, 3, \ldots, 3)_{n-1}, i = (1, \ldots, 1, 0)$ and let $k = n - 1$. Moreover, let $\tau^\ell := (1, \ldots, 1, 0, 3, \ldots, 3)_{n-1-\ell}$, where $0 \leq \ell \leq n - 1$. Then,

$$I_i = \{\tau^0, \ldots, \tau^{n-1}\} \quad \text{and} \quad I_{i^-} = \{(1, \ldots, 1)\},$$

where $i^-$ was defined in (2.2).

That is, there exists $n - 1$ elements $\eta \in I_i$ such that $\Phi(\eta|_{n-1}) \not\in I_{i|_{n-1}}$.  

Fact 7.7. Let $i \in T_n, \eta \in I_i$. Let $[k, \ell] \in G(i)$ and $[k', \ell'] \in B(\eta)$ such that $[k, \ell] \cap [k', \ell'] \neq \emptyset$. Then, $\ell = \ell'$.

Namely, $\ell' > \ell$ is not possible since this would follow that $i_\ell = 0, i_{\ell+1} = 3$. $\ell' < \ell$ is not possible since this would follow $k \leq \ell' < \ell$ and $i_{\ell'} = 0$ which is not possible since by definition $i_{\ell'} = 1$.

7.2. The cases when $\psi(i)$ is multiplicative. For the rest of the paper we fix some notation. Let

$$\chi := \chi(i) := \begin{cases} \max \{\ell : [k, \ell] \in G(i)\}, & \text{if } G(i) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, let

$$\xi := \xi(j) := \min \{k : \forall \ell > k, j_\ell = 1\}.$$

Clearly,

$$G(i) \neq \emptyset \implies (i_{\chi-1}, i_\chi) = (1, 0).$$

We also get by definition that

$$\chi(\ell) \leq \xi(\ell), \quad \forall \ell \in T_p, \quad \forall p.$$

Fact 7.8. For $\ell \geq \chi$

$$\psi(i) = \psi(i|_\ell) \cdot p_{i_{\ell+1} \ldots i_n}.$$  

Proof. If $G(i) \neq \emptyset$, then the proof is immediate from Definition 3.4. In the case when $G(i) = \emptyset$, then $\chi = 0, \psi(i|_0) = \psi(\emptyset) = 1$. Then, the fact follows from (7.7). \qed

Lemma 7.9. Let $n \geq 2, i \in T_n$ and $z \in D_{\text{Good}}(i)$. Then,

$$\psi(i) = \psi(i|_z) \cdot \psi(\sigma^z i).$$
Proof.
\[ \psi(i) = \sum_{\eta \in I_i} p_\eta = \sum_{\eta \in I_i} p_{\eta|z} \cdot p_{\sigma^z \eta} \leq \sum_{\omega \in I_{\eta|z}} p_\omega \cdot \sum_{\tau \in I_{\sigma^z \eta}} p_\tau = \psi(i|z) \cdot \psi(\sigma^z i), \]
where in the third step we used that by Fact 7.3 for an \( \eta \in I_i \) we have \( \eta|z \in I_{\eta|z} \) and \( \sigma^z \eta \in I_{\sigma^z \eta} \) since \( z \in D_{\text{Good}}(i) \). On the other hand,
\[ \psi(i|z) \cdot \psi(\sigma^z i) = \sum_{\omega \in I_{i|z}} p_\omega \cdot \sum_{\tau \in I_{\sigma^z \eta}} p_\tau = \sum_{\omega \in I_{i|z}} p_{\omega|z} \cdot \sum_{\tau \in I_{\sigma^z \eta}} p_{\tau} \leq \sum_{\eta \in I_i} p_\eta = \psi(i), \]
where in the one but last step we used that \( (i, i_{z+1}) \neq (0, 3) \) since \( i \in T_n \). Moreover, we also used that by (2.3),
\[ (7.19) \quad \Pi(\eta) = \Pi(\eta|z) + 3^{-z} \Pi(\sigma^z \eta), \quad \forall \eta \in A. \]
Hence, \( \omega \in I_{i|z} \) and \( \tau \in I_{\sigma^z i} \) implies that \( \eta = \omega \tau \in I_i \). \( \square \)

7.3. The properties of the sequence \( \{a_w\}_w \). Observe that for a \( w \geq 1 \)
\[ (7.20) \quad I_1^w = \{1^w\} \quad \text{and} \quad I_{1^w-1}^0 = \left\{1^{w-1}0^3\right\}_{\ell=0}^{w-1}. \]
Hence,
\[ (7.21) \quad \psi(1^w) = p_1^w \quad \text{and} \quad \psi(1^{w-1}0) = a_w := \frac{p_0}{p_1} p_1^w \sum_{\ell=0}^{w-1} \left(\frac{p_3}{p_1}\right)^\ell. \]
So,
\[ (7.22) \quad \psi(1^{w-1}0) > \psi(1^w) \iff \sum_{\ell=0}^{w-1} \left(\frac{p_3}{p_1}\right)^\ell > \frac{p_1}{p_0}. \]
This motivates the following definition. For an \( n \geq 1 \) and \( i \in T_n \) we define
\[ (7.23) \quad i^* := \begin{cases} i^{-1}, & \text{if } n - \xi(i) \geq 1 \text{ and } \sum_{\ell=0}^{n-\xi(i)-1} \left(\frac{p_3}{p_1}\right)^\ell > \frac{p_1}{p_0}; \\ i, & \text{otherwise}. \end{cases} \]
We introduce
\[ (7.24) \quad A := \left\{ q \in \mathbb{N} : q \geq 1 \& \sum_{\ell=0}^{q-1} \left(\frac{p_3}{p_1}\right)^\ell > \frac{p_1}{p_0} \right\}. \]
Observe that
\[ (7.25) \quad i^* \neq i \iff |i| - \xi(i) \in A. \]
Recall that in (2.7) we assumed that \( p_3 \leq p_1 \). A simple calculation shows that
\[ (7.26) \quad A \neq \emptyset \iff p_1 - p_3 < p_0. \]
Putting together (7.22) and (7.24) we get that
\begin{equation}
q \in \mathcal{A} \iff \hat{\psi}(\mathcal{A}) > \psi(\mathcal{A}).
\end{equation}

We define
\begin{equation}
q_0 := \begin{cases} 
\min \mathcal{A}, & \text{if } p_1 - p_3 < p_0; \\
\infty, & \text{otherwise}.
\end{cases}
\end{equation}

Now we prove the sub-multiplicative property of the sequence \(\{a_w\}_{w=1}^\infty\).

**Fact 7.10.** Let \(u, v \geq 1\). Then, there are constants \(C_6, C_7 > 0\) such that
\begin{equation}
a_{u+v} < C_6 a_u a_v,
\end{equation}
and
\begin{equation}
\sum_{\ell=0}^{u+v} \left( \frac{p_3}{p_1} \right)^\ell < C_7 \cdot \sum_{\ell=0}^{u-1} \left( \frac{p_3}{p_1} \right)^\ell \cdot \sum_{\ell=0}^{v-1} \left( \frac{p_3}{p_1} \right)^\ell.
\end{equation}

**Proof.** First we prove (7.30). Recall that we assumed that \(0 < p_3 \leq p_1\). If \(p_1 = p_3\), then we can clearly choose a constant \(C_7 > 0\) such that \(u + v + 1 < C_7 \cdot u \cdot v\). If \(p_3 < p_1\), then (7.30) holds with the choice of \(C_7 = \frac{1}{1 - \frac{p_3}{p_1}}\). It is immediate that (7.29) follows from (7.30).

The following fact is an important but trivial consequence of the definitions.

**Fact 7.11.** Given a \(k \in \mathcal{T}_n\) with \(k_n \neq 1\) and a natural number \(w \geq 1\), we have
\begin{equation}
\psi((k\mathcal{T}^w)^*) = \psi(k)\psi((\mathcal{T}^w)^*).
\end{equation}

**Proof.** By definition, \(\xi(k\mathcal{T}^w) = n\). Therefore,
\[(k\mathcal{T}^w)^* = k\mathcal{T}^{w-1}0 \iff w \geq q_0 \iff (\mathcal{T}^w)^* = \mathcal{T}^{w-1}0.
\]
Hence, \((k\mathcal{T}^w)^* = k(\mathcal{T}^w)^*\). Observe that by \(k_n \neq 1\) we have \(n \in D_{\text{Good}}(k(\mathcal{T}^w)^*)\). Using this and Fact 7.9 we get
\[\psi((k\mathcal{T}^w)^*) = \psi(k(\mathcal{T}^w)^*) = \psi(k)\psi((\mathcal{T}^w)^*).\]

The second condition in first line of (7.23) is to guarantee that \(\hat{\psi}(i) > \psi(i)\).

We know by (3.24) that \(\hat{\psi}(i) = \max\{\psi(i), \psi(i-0)\}\).

**Fact 7.12.** For every \(n \geq 1\) and \(i \in \mathcal{T}_n\)
\begin{equation}
i^* = \begin{cases} 
i-0, & \text{if } \hat{\psi}(i) > \psi(i); \\
i, & \text{if } \hat{\psi}(i) = \psi(i).
\end{cases}
\end{equation}
If $i_n \neq 1$, then by (3.24) we have $\hat{\psi}(i) = \psi(i)$. In this case $n - \xi(i) = 0$. Hence, it follows from (7.23) that $i^* = i$. So, we may assume that $i_n = 1$. That is, $n - \xi(i) \geq 1$. Hence, $i = i|_{\xi(i)}\overline{1}^{\overline{n} - \xi(i)}$. Using Fact 7.11, we get

$$
\psi(i^*) = \psi(i|_{\xi(i)}) \cdot \psi\left(\overline{1}^{\overline{n} - \xi(i)}\right).
$$

Using that $\xi(\ell) \in D_{\text{Good}}(i)$ we obtain from Lemma 7.9 that

$$
\psi(i) = \psi(i|_{\xi(i)}) \cdot \psi\left(\overline{1}^{\overline{n} - \xi(i)}\right).
$$

Putting together the last two displayed formulas with (7.25) and (7.27) we get the assertion of Fact. □

Observe that in virtue of Fact 7.12 we have

$$
\hat{\psi}(i) = \psi(i^*).
$$

### 7.4. The Proof of Property-2

**Lemma 7.13.** There exists a constant $C_6 > 1$ such that the following holds. Let $q \geq 1$ and let $j \in T_q$ be arbitrary. Let $u, v \in A$ such that $ju, jv \in T_{q+1}$. Then,

$$
C_6^{-1}(|j| + 1)^{-1} \times \frac{\psi(jv)}{\psi(ju)} < C_6(|j| + 1).
$$

**Proof.** We may assume that $u \neq v$ and

$$
\text{if } 0 \in \{u, v\}, \text{ then } v = 0.
$$

Using that any good block ends with a 0, this implies that $\chi(jv) \geq \chi(ju)$. More precisely,

$$
(G(jv) \neq \emptyset \& \chi(ju) \neq \chi(jv)) \implies \chi(ju) < \chi(jv) = q + 1.
$$

Recall from (7.14) that

$$
\xi := \xi(j) = \min \{k \in [0, q] : \forall \ell > k, \ j_\ell = 1\},
$$

where $\xi = q$ means that $j$ does not end with a block of 1s. If $\xi < q$, then we can write

$$
j = j|_{\xi}\overline{1}^{q - \xi}.
$$

By definition, we get that

$$
\xi \in \{0\} \cup D_{\text{Good}}(j).
$$

By Fact 7.3 we get that

$$
\omega \in I_{ju}, \ \tau \in I_{jv} \implies \Pi(\omega|_{\xi}) = \Pi(\tau|_{\xi}) = \Pi(j|_{\xi}) = \Pi(ju|_{\xi}) = \Pi(jv|_{\xi}).
$$
Moreover,
(7.42)\[
\mathcal{I}_{j\mathbf{u}} = \left\{ \tau \mathbf{T}^{q-\xi} \mathbf{u} : \tau \in \mathcal{I}_{j|\xi} \right\}, \quad \text{if } u \in \{1, 3\}, \quad \mathcal{I}_{j0} = \bigcup_{\ell=0}^{q-\xi} \left\{ \tau \mathbf{T}^{q-\xi-\ell} \mathbf{0}^3 : \tau \in \mathcal{I}_{j|\xi} \right\}.
\]
Using that for \( u \in \{1, 3\}, \chi(ju) \leq \xi \) it follows from Fact 7.8 that
(7.43)\[
\psi(ju) = \psi(j|\xi)p_1^{q-\xi}p_u.
\]
Using the second part of (7.42) we get
(7.44)\[
\psi(j0) = \psi(j|\xi) \cdot \sum_{\ell=0}^{q-\xi} p_1^{q-\xi-\ell} p_3^\ell p_0.
\]
Hence,
(7.45)\[
\frac{\psi(jv)}{\psi(ju)} = \begin{cases} \sum_{\ell=0}^{q-\xi} p_1^{q-\xi-\ell} p_3^\ell p_0, & \text{if } v = 0; \\ \frac{p_0 p_1}{p_3 (p_1 - p_3)}, & \text{if } v \neq 0. \end{cases}
\]
That is, by (2.7) we obtain
(7.46)\[
\frac{p_{\min}}{p_{\max}} \leq \frac{\psi(jv)}{\psi(ju)} \leq C_{14}(|j|+1-\xi), \quad u \in \{1, 3\} \text{ and } v \in \{0, 1, 3\} \setminus \{u\},
\]
where \( C_{14} := \begin{cases} \frac{p_0 p_1}{p_3 (p_1 - p_3)}, & \text{if } p_1 > p_3; \\ \frac{p_0}{p_3}, & \text{if } p_0 = p_3. \end{cases} \]

We obtain from (7.44) that
(7.47)\[
\xi(i) \leq |i|-1 \implies \psi(i^0) = \psi(i|\xi(i)) \cdot \frac{p_0}{p_1} p_1^{\xi(i)-|i|-1} \sum_{\ell=0}^{|i|-\xi(i)-1} \left( \frac{p_3}{p_1} \right)^\ell = \psi(i|\xi(i)) \cdot a_{|i|-\xi(i)},
\]
where the sequence \( \{a_q\}_{q=1}^\infty \) was defined in (7.21). Hence, by Fact 7.12
(7.48)\[
i^* \neq i \implies \widehat{\psi}(i) = \psi(i^0) = \psi(i|\xi(i)) \cdot a_{|i|-\xi(i)}.
\]

**Fact 7.14.** Let \( i \in \mathcal{T}_n \). If \( \xi(i) = n \), then \( i_n \neq 1 \). So, by definition \( \psi(i) = \widehat{\psi}(i) \). There exists a \( C_3 > 0 \) such that whenever \( \xi(i) \leq n-1 \), then we have
(7.49)\[
\psi(i) \leq \psi(i) \leq C_3 \cdot (n - \xi(i)) \cdot \psi(i).
\]
Moreover,
(7.50)\[
i \neq i^* \iff \psi(i) < \widehat{\psi}(i) \iff \max \{1, q_0\} \leq n - \xi(i).
\]
Observe that Fact 7.14 implies that Property-2 holds.
Proof. Let \( i \in \mathcal{T}_n \) and \( j := i^- \). It is assumed that \( \xi(i) \leq n - 1 \). This implies that \( i = j1 \). If \( \xi(i) = n - 1 \), then \( \hat{\psi}(i) = \psi(i) \) by definition. We may assume that \( \xi(i) \leq n - 2 \). Then, either \( \hat{\psi}(i) = \psi(i) \) or if not then by Fact 7.12 \( i^* = j0 \) and by (7.23) and (7.45)

\[
1 \leq \frac{\hat{\psi}(i)}{\psi(i)} \leq \frac{p_0}{p_1}(n - \xi).
\]

This completes the proof of the first part with the choice of \( C_3 := \frac{p_0}{p_1} \).

To verify the second part we first observe that if \( \xi(i) \geq n - 1 \), then \( i = i^* \) so \( \hat{\psi}(i) = \psi(i) \). So, we may assume that \( 2 \leq n - \xi(i) \). In this case \( \hat{\psi}(i) = \psi(j0) \) and \( \psi(i) = \psi(j1) \). We write \( \xi := \xi(j) \). Then, by (7.43) and (7.44) we get that

\[
\psi(i) < \hat{\psi}(i) \iff p_1^{n-\xi} < \sum_{\ell=0}^{n-1-\xi} p_1^{n-\xi-\ell} p_3^0 p_0 \iff n - \xi \geq q_0.
\]

\[\Box\]

7.5. The Proof of Property-1.

Lemma 7.15. There exists a \( C_{10} > 1 \) such that for any \( n, m \geq 1, k \in \mathcal{T}_{n+m} \)

\[
\hat{\psi}(k) \leq C_{10} \hat{\psi}(k|n) \hat{\psi}(\sigma^n k).
\]

Proof. Fix a \( k \in \mathcal{T}_{n+m} \). Let

\[
i = (i_1, \ldots, i_n) := k|n \in \mathcal{T}_n \quad \text{and} \quad j = (j_1, \ldots, j_m) := \sigma^n k \in \mathcal{T}_m.
\]

We distinguish two cases:

Case 1 \( \xi(k) \leq n - 1 \). Then, \( \xi(k) = \xi(i) \) and \( 1 = i_n = j_1 = \cdots = j_m \). That is, \( j = \Gamma^m \). We distinguish three cases again:

(a) \( i^* \neq i \) (this implies that \( k^* \neq k \)). Then, \( \xi(i) = \xi(k) \). Hence,

\[
\frac{\psi(k^*)}{\psi(i^*)} = \frac{\psi[i|\xi(i)] \cdot a_{n+m-\xi(i)}}{\psi[i|\xi(i)] \cdot a_{n-\xi(i)}} = p_1^m \cdot \sum_{\ell=0}^{n-\xi(i)-1} \frac{p_3^\ell}{p_1} \cdot \sum_{\ell=0}^{n-\xi(i)-1} \frac{p_3^\ell}{p_1} \cdot r(i,j).
\]

(I) If \( m \leq q_0 \), then by definition \( j^* = j \) and then, by Fact 7.12, \( \hat{\psi}(j) = \psi(j) = p_1^m \) and \( r(i,j) \leq C_5 \cdot q_0 \) for a \( C_5 > 0 \). So, in this case (7.52) holds since in this situation, \( \hat{\psi}(i) = \psi(i^-0) \) and \( \hat{\psi}(k) = \psi(k^-0) \).
(II) Assume that \( m > q_0 \). Then, as we have said, \( j = \mathbb{T}^m \) and

\[
\hat{\psi}(j) = \psi(j^0) = a_m = \frac{p_0}{p_1} p_1^m \sum_{\ell=0}^{m-1} \left( \frac{p_3}{p_1} \right)^\ell.
\]

So, in order to verify (7.52) for this case, we have to show that

\[
(7.55) 
\quad r(i,j) \leq \text{Const} \cdot \sum_{\ell=0}^{m-1} \left( \frac{p_3}{p_1} \right)^\ell.
\]

This follows from (7.30).

(b) \( i^* = i \) but \( k^* \neq k \). Then,

\[
(7.56) 
\quad n - \xi(i) < q_0 \leq n + m - \xi(i).
\]

We write \( \xi := \xi(i) \). Then, \( \hat{\psi}(k) = \psi(k^0) \) and \( \hat{\psi}(i) = \psi(i) \),

\[
(7.57) 
\quad \frac{\psi(k^*)}{\psi(i^*)} = \frac{\psi(i|\xi)p_0}{\psi(i|\xi)p_1} \frac{1}{\frac{p_0}{p_1} p_1^n} \frac{1}{\frac{p_1}{p_3}} \sum_{\ell=0}^{n-\xi+m-1} \left( \frac{p_1}{p_3} \right)^\ell = \frac{p_0}{p_1} p_1^n \sum_{\ell=0}^{n-\xi+m-1} \left( \frac{p_1}{p_3} \right)^\ell.
\]

(II) If \( m > q_0 \), then \( \hat{\psi}(j) = p_1^n \). Then, \( n - \xi + m < 2q_0 \). Hence, (7.52) holds in this case.

(c) \( i^* = i \) and \( k^* = k \) (this implies that \( j^* = j \) since \( \xi(k) \leq n - 1 \)). Then, \( \hat{\psi}(i) = \psi(i) \), \( \hat{\psi}(j) = \psi(j) \) and \( \hat{\psi}(k) = \psi(k) \). Hence,

\[
\frac{\hat{\psi}(k)}{\hat{\psi}(i)} = \frac{\psi(i^\ell)}{\psi(i^\ell)} \frac{p_1^{n+m-\xi}}{p_1^{n-\xi}} = p_1^n = \hat{\psi}(j).
\]

This verifies (7.52) for this case.

Case 2 \( \xi(k) \geq n \). In this case \( m - \xi(j) = n + m - \xi(k) \). This means that \( k^* \neq k \) if and only if \( j^* \neq j \). In particular, \( \xi(k) \geq n \) implies that

\[
(7.58) 
\quad k^* = ij^*.
\]

Now we distinguish two cases:

(a) \( n \in D_{\text{Good}}(k^*) \). Then, it follows from Lemma 7.9 that \( \psi(k^*) = \psi(i)\psi(j^*) \).

Here we used that by (7.58), we have \( \sigma^n(k^*) = (\sigma^n k)^* \). Using that \( \psi(i) \leq \hat{\psi}(i) \) we obtain that (7.52) holds in this case.

(b) \( n \in C_{\text{Good}}(k^*) \). This means that there is a good block \([u,v+1] \in \mathcal{G}(k^*) \) such that

\[
(7.59) 
\quad 1 \leq u \leq n \leq v \leq n + m - 1, \quad \text{and} \quad k_{v+1} = 0.
\]
Using that either \( \xi(k^*) = n + m \) or \( k_{\xi(k^*)} \neq 1 \), we get that

\[
\xi(k^*) \in D_{\text{Good}}(k^*).
\]

Moreover,

\[
u - 1 < v + 1 \leq \xi(k^*), \quad \text{and} \quad u - 1, v + 1 \in D_{\text{Good}}(k^*).
\]

Hence, by Lemma 7.9 we have

\[
\psi(k^*) = \psi(i_{|u-1}) \psi(T^{v-u+1}0) \psi(\sigma^{v+1}k|_{\xi(k)}) \psi\left(\left(T^{n+m-\xi(k)}\right)^*\right),
\]

where we remark that the first and last words in (7.62), \( i_{|u-1} \) and \( (T^{n+m-\xi(k)})^* \), respectively, can be the empty words \( b \). In this case we recall that \( \psi(b) = 1 \). Using Fact 7.10 in the third step we get

\[
\psi(T^{v-u+1}0) = \psi(T^{n-u+1}1v^n) = a_{(n-u+1)+(v-n+1)} \leq C_6 a_{n-u+1}a_{v-n+1}
\]

\[
C_6 \psi(T^{n-u}0) \psi(T^{v-n}0) \leq C_6 \psi\left(\left(T^{n+u+1}\right)^*\right) \psi(T^{v-n}0).
\]

Now we substitute this into (7.62) and get

\[
\psi(k^*) \leq C_6 \psi(i_{|u-1}) \psi\left(\left(T^{n-u+1}\right)^*\right) \psi(T^{v-n}0) \psi(\sigma^{v+1}k|_{\xi(k)}) \psi\left(\left(T^{n+m-\xi(k)}\right)^*\right),
\]

where we used Fact 7.11 and in the last step we used that \( n + m - \xi(k) = m - \xi(j) \). That is, (7.52) holds also in this last possible case.

\[\square\]

We will need the following Claim:

**Claim 7.16.** For an arbitrary \( i \in T_n \) we have

\[
\psi(i0) \geq p_0 \cdot \hat{\psi}(i).
\]

**Proof.** Fix an \( i \in T_n \).

(a) Assume that \( i_n \neq 1 \). Then, \( i^* = i \) and \( n \in D_{\text{Good}}(i0) \). Then, by Lemma 7.9, we have \( \psi(i0) = \psi(i)\psi(0) = p_0 \hat{\psi}(i) \). So, (7.65) holds in this case.

(b) Assume that \( i_n = 1 \). Then, \( \xi := \xi(i) \leq n - 1 \) and \( \xi \in D_{\text{Good}}(i0) \). Then, by Lemma 7.9 we have

\[
\psi(i0) = \psi(i|_{\xi}) \psi(\sigma^{\xi}(i0)) = \psi(i|_{\xi}) \psi\left(T^{n-\xi}0\right) = \psi(i|_{\xi}) \cdot a_{n-\xi+1}.
\]

On the other hand, by Fact 7.11

\[
\hat{\psi}(i) = \begin{cases} 
\psi(i|_{\xi}) p_i^{n-\xi}, & \text{if } n - \xi < q_0; \\
\psi(i|_{\xi}) a_{n-\xi}, & \text{if } n - \xi \geq q_0.
\end{cases}
\]

Putting together (7.66) and (7.67) we get that (7.65) holds.
Claim 7.17. Let \( j \in T_m \) for an \( m \geq 1 \). Then, we have
\[
\hat{\psi}(1j) \geq p_1 \hat{\psi}(j).
\]

Proof. First we consider the case when \( j = \Gamma^m \). This case can be subdivided into the three cases when \( m < q_0 - 1, m = q_0 - 1 \) and \( m \geq q_0 \). Using formula (7.21), one can easily point out in each of these three cases that (7.68) holds. So, from now on we may assume that there exists a \( p \in \{1, \ldots, m\} \) such that \( j_p \neq 1 \). Using this, one can easily see in the same way as in Case 2 of the proof of Lemma 7.15 that
\[
(7.69) \quad (1j)^* = 1j^*.
\]
Observe that
\[
(7.70) \quad \eta \in I_{j^*} \implies 1\eta \in I_{1j^*}.
\]
Namely,
\[
(7.71) \quad \eta \in I_{j^*} \iff \Pi(\eta) = \Pi(j^*) \implies S_1(\Pi(\eta)) = S_1(\Pi(j^*)) \implies \Pi(1\eta) = \Pi(1j^*) \implies 1\eta \in I_{1j^*}.
\]
By definition
\[
(7.72) \quad \psi(1j^*) = \sum_{\omega \in I_{1j^*}} p_\omega \geq \sum_{\eta \in I_{j^*}} p_{1\eta} = p_1 \sum_{\eta \in I_{j^*}} p_{1\eta} = p_1 \hat{\psi}(j^*).
\]

Lemma 7.18. There exist a \( C_{11} > 0 \) such that
\[
(7.73) \quad \hat{\psi}(i01j) \geq C_{11} \hat{\psi}(i) \hat{\psi}(j), \quad \forall i, j \in \Sigma_A^*.
\]

Proof. Given \( i \in T_n \) and \( j \in T_m \). Clearly, \( \tilde{k} := i01j \in T_{n+m+2} \). Using that \( \xi(\tilde{k}) \leq n + 1 \). Hence, by the same argument that we used in Case 2, part of the proof of Lemma 7.15, we obtain that
\[
(7.74) \quad \tilde{k}^* = i0(1j)^*.
\]
Observe that \( n + 1 \in D_{\text{Good}}(\tilde{k}^*) \). Hence, first using Lemma 7.9 and then using Claims (7.16) and (7.17) we get that
\[
(7.75) \quad \hat{\psi}(\tilde{k}) = \psi(\tilde{k}^*) = \psi(i0) \psi((1j)^*) \geq p_0 \hat{\psi}(i)p_1 \hat{\psi}(j).
\]

Putting together Lemmas 7.15 and 7.18 we obtain that

Corollary 7.19. For every \( t > 0 \), \( \hat{\psi}^t \) is a quasi-multiplicative potential in the sense of Definition 3.5.
This proves that Property-1 holds.

**Acknowledgements** We would like to thank Aljoscha Niemann for his numerous valuable comments, and for calling our attention to the fact that putting together [KNZ, Corollary 1.12] and our Theorem 3.2 yields an expression for the $L^q$-spectrum over $(0,1)$ for the measures studied in this paper.

**References**

[AW] E.F. Abaya and G.L. Wise, *Some remarks on the existence of optimal quantizers*, Statistics & Probability Letters, Volume 2, Issue 6, December 1984, pp. 349-351.

[B] J.A. Bucklew, *Two results on the asymptotic performance of quantizers*, IEEE Trans. Infor. Theory, Vol. IT-30, No. 2 (1984), 341-348.

[BW] J.A. Bucklew and G.L. Wise, *Multidimensional asymptotic quantization theory with $r$th power distortion measures*, IEEE Transactions on Information Theory, 1982, Vol. 28 Issue 2, 239-247.

[BSS] B. Barany and K. Simon and B. Solomyak, *Self-similar and self-affine sets and measures* Book in preparation, 2022.

[CR] D. Comez and M.K. Roychowdhury, *Quantization for uniform distributions on stretched Sierpinski triangles*, Monatshefte für Mathematik, Volume 190, Issue 1, 79-100 (2019).

[DR1] C.P. Dettmann and M.K. Roychowdhury, *Quantization for uniform distributions on equilateral triangles*, Real Analysis Exchange, Vol. 42(1), 2017, pp. 149-166.

[DR2] C.P. Dettmann and M.K. Roychowdhury, *An algorithm to compute CVTs for finitely generated Cantor distributions*, Southeast Asian Bulletin of Mathematics (2021) 45: 173-188.

[F1] D.J. Feng, *Smoothness of the $L^q$-spectrum of self-similar measures with overlaps*, Journal of London Mathematical Society (2) 68 (2003), 102-118.

[F2] D.J. Feng, *Equilibrium states for factor maps between subshifts*, Advances in Mathematics 226 (2011) 2470-2502.

[G] P.M. Gruber, *Optimum quantization and its applications*, Advances in Mathematics 186 (2004) 456-497.

[GG] A. Gersho and R.M. Gray, *Vector quantization and signal compression*, Kluwer Academy publishers: Boston, 1992.

[GKL] R.M. Gray, J.C. Kieffer and Y. Linde, *Locally optimal block quantizer design*, Information and Control, 45 (1980), pp. 178-198.

[GL] A. György and T. Linder, *On the structure of optimal entropy-constrained scalar quantizers*, IEEE transactions on information theory, 48, 416-427 (2002).

[GL1] S. Graf and H. Luschgy, *Foundations of quantization for probability distributions*, Lecture Notes in Mathematics 1730, Springer, Berlin, 2000.

[GL2] S. Graf and H. Luschgy, *The quantization dimension of self-similar probabilities*, Math. Nachrichten 241, 103-109 (2002).

[GL3] S. Graf and H. Luschgy, *Asymptotics of the Quantization Errors for Self-Similar Probabilities*, Real Anal. Exchange, Volume 26, Number 2 (2000), 795-810.

[GL4] S. Graf and H. Luschgy, *Quantization for probability distribution with respect to the geometric mean error*, Math. Proc. Camb. Phil. Soc., vol. 136 (2004), pp. 687-717.
[GL5] S. Graf and H. Luschgy, *The Quantization of the Cantor Distribution*, Math. Nachr., 183, 113-133 (1997).

[GN] R. Gray and D. Neuhoff, *Quantization*, IEEE. Trans. Inform Theory, 44 (1998), 2325-2383.

[KNZ] M. Kesseböhmer, A. Niemann, S. Zhu *Quantization dimensions of compactly supported probability measures via Rényi dimensions*, preprint arXiv:2205.15776.

[KR] A. Kaenmaki and H.W. Reeve, *Multifractal analysis of birkhoff averages for typical infinitely generated self-affine sets*, Journal of Fractal Geometry 1 83-152 (2014).

[LN] K. S. Lau, Sz.M. Ngai, *Multifractal measures and weak separation condition*, Advances in Mathematics, 141 (1999) 45-96.

[LM] L.J. Lindsay and R.D. Mauldin, *Quantization dimension for conformal iterated function systems*, Institute of Physics Publishing, Nonlinearity 15 (2002), 189-199.

[MR] E. Mihailescu and M.K. Roychowdhury, *Quantization coefficients in infinite systems*, Kyoto Journal of Mathematics, Vol. 55, No. 4 (2015), 857-873.

[R] L. Roychowdhury, *Optimal quantization for nonuniform Cantor distributions*, Journal of Interdisciplinary Mathematics, Vol 22 (2019), pp. 1325-1348.

[R1] M.K. Roychowdhury, *Quantization and centroidal Voronoi tessellations for probability measures on dyadic Cantor sets*, Journal of Fractal Geometry, 4 (2017), 127-146.

[R2] M.K. Roychowdhury, *Optimal quantizers for some absolutely continuous probability measures*, Real Analysis Exchange, Vol. 43(1), 2017, pp. 105-136.

[R3] M.K. Roychowdhury, *Optimal quantization for the Cantor distribution generated by infinite similitudes*, Israel Journal of Mathematics 231 (2019), 437-466.

[R4] M.K. Roychowdhury, *Least upper bound of the exact formula for optimal quantization of some uniform Cantor distributions*, Discrete and Continuous Dynamical Systems- Series A, Volume 38, Number 9, September 2018, pp. 4555-4570.

[R5] M.K. Roychowdhury, *Center of mass and the optimal quantizers for some continuous and discrete uniform distributions*, Journal of Interdisciplinary Mathematics, Vol. 22 (2019), No. 4, pp. 451-471.

[R6] M.K. Roychowdhury, *Optimal quantization for mixed distributions*, Real Analysis Exchange, Vol. 46(2), 2021, pp. 451-484.

[R7] M.K. Roychowdhury, *Quantization dimension function and ergodic measure with bounded distortion*, Bulletin of the Polish Academy of Sciences Mathematics, 57 (2009), 251-262.

[R8] M.K. Roychowdhury, *Quantization dimension for some Moran measures*, Proc. Amer. Math. Soc., 138 (2010), 4045-4057.

[R9] M.K. Roychowdhury, *Quantization dimension function and Gibbs measure associated with Moran set*, J. Math. Anal. Appl., 373 (2011), 73-82.

[R10] M.K. Roychowdhury, *Quantization dimension and temperature function for recurrent self-similar measures*, Chaos, Solitons & Fractals, 44 (2011), 947-953.

[R11] M.K. Roychowdhury, *Quantization dimension and temperature function for bi-Lipschitz mappings*, Israel Journal of Mathematics, 192 (2012), 473-488.

[R12] M.K. Roychowdhury, *Quantization dimension estimate of inhomogeneous self-similar measures*, Bulletin of the Polish Academy of Sciences Mathematics, Vol. 61, No. 1 (2013), 35-45.
[R13] M.K. Roychowdhury, *Quantization dimension estimate for condensation systems of conformal mappings*, Real Analysis Exchange, 2013, Vol. 38 Issue 2, 317-335.

[R14] M.K. Roychowdhury, *Quantization dimension for Gibbs-like measures on cookie-cutter sets*, Kyoto Journal of Mathematics, Volume 54, Number 2 (2014), 239-257.

[R15] M.K. Roychowdhury, *Quantization dimension estimate of probability measures on hyperbolic recurrent sets*, Dynamical Systems, Vol. 29, No. 2, 225-238, 2014.

[RR1] J. Rosenblatt and M.K. Roychowdhury, *Optimal quantization for piecewise uniform distributions*, Uniform Distribution Theory 13 (2018), no. 2, 23-55.

[Z1] P.L. Zador, *Asymptotic quantization error of continuous signals and the quantization dimension*, IEEE Trans. Inform. Theory, Vol. 28, No.2 (1982), 139-149.

[Z2] P.L. Zador, *Development and evaluation of procedures for quantizing multivariate distributions*, Ph.D. thesis, Stanford University, 1964.

[Zer] M.P.W. Zerner, *Weak separation properties for self-similar sets*, Proc. Amer. math. Soc, 124, vol. 124, (1996), 3529-3539.

[Zhu] S. Zhu, *Asymptotics of the quantization errors for Markov-type measures with complete overlaps*, preprint, arXiv: 2202.07109, 2022.

Mrinal Kanti Roychowdhury, University of Texas Rio Grande Valley, 1201 West University Drive, Edinburg, TX 78539-2999, USA

Email address: mrinal.roychowdhury@utrgv.edu

Károly Simon, Budapest University of Technology and Economics, MTA-BME Stochastics Research Group, P.O. Box 91, 1521 Budapest, Hungary

Email address: simonk@math.bme.hu