DEFORMATIONS OF PAIRS OF RATIONAL CURVES AND
QUINTIC THREEFOLDS

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Abstract. This is one of a series of papers ([2], [3], [4]) that study the pairs of curves
and hypersurfaces under the same motif: a fundamental property on the first order
deformations of the pairs (Lemma 1.4, (1) below) has an effect on the normal bundle of
the rational curves on the hypersurfaces. Each paper solves an independent problem
with the same condition (Lemma 1.4), which addresses only one aspect of the normal
bundles. In this paper, we let $X_0$ be a generic quintic threefolds in projective
space $\mathbb{P}^4$ over complex numbers, and $C_0$ an immersed rational curve on $X_0$.
We prove

$$H^1(N_{c_0/X_0}) = 0,$$

where $c_0 : \mathbb{P}^1 \to X_0$ is the immersion of $\mathbb{P}^1$.

1. Introduction to the deformation problem.

Let $\mathbb{P}^4$ be the projective space of dimension 4 over complex numbers, and $\mathcal{L} =
\mathcal{O}_{\mathbb{P}^4}(1)$ the very ample line bundle on $\mathbb{P}^4$. Let $f_0 \in H^0(\mathcal{L}^5)$ be a
generic section, and $X_0 = \text{div}(f_0)$.

Let

$$c_0 : \mathbb{P}^1 \to C_0 \subset X_0$$

be an immersion map from $\mathbb{P}^1$ onto an irreducible rational curve $C_0$ of degree
$$d = \text{deg}(c_0^*(\mathcal{L}))$$
on $X_0$. Let $N_{c_0/X_0}$ be the normal bundle of the immersion
$$c_0 : \mathbb{P}^1 \to X_0.$$

Theorem 1.1. Assume the same notations as above. Then

$$(1.1) \quad H^1(N_{c_0/X_0}) = 0.$$

1.1. A general approach-introduction to “blow-up” polynomials. Let
$N_{c_0/\mathbb{P}^4}$ be the normal bundle of the immersion
$$c_0 : \mathbb{P}^1 \to \mathbb{P}^4.$$

We begin the proof by considering an exact sequence.

$$H^0(N_{c_0/\mathbb{P}^4}) \overset{\nu^*}{\longrightarrow} H^0(c_0^*(N_{X_0}/\mathbb{P}^4))$$

$$\downarrow \quad 0 \quad \rightarrow \quad N_{c_0/X_0} \quad \rightarrow \quad N_{c_0/\mathbb{P}^4} \quad \overset{\nu}{\rightarrow} \quad c_0^*(N_{X_0}/\mathbb{P}^4) \quad \rightarrow \quad 0,$$

which is induced from the exact sequence of bundles over $\mathbb{P}^1$, 

$$(1.2) \quad 0 \rightarrow N_{c_0/X_0} \rightarrow N_{c_0/\mathbb{P}^4} \overset{\nu}{\rightarrow} c_0^*(N_{X_0}/\mathbb{P}^4) \rightarrow 0,$$
where $i$ is an isomorphism. The isomorphism $i$ is canonically induced from the vector bundle isomorphism:

$$
c_{0}^{s}(\bar{\alpha})|_{t} \rightarrow c_{0}^{s}((\partial f_{0}/\partial \bar{\alpha})|_{t})
$$

for $\bar{\alpha} \in H^{0}(N_{X_{0}/\mathbb{P}^{4}})$, and it is canonically determined by $c_{0}, f_{0}$.

Because $H^{1}(N_{c_{0}/\mathbb{P}^{4}}) = H^{1}(N_{c_{0}/X_{0}}) = 0$ is equivalent to the surjectivity of $\nu^{s}$. Next our method is to find a special type of an element in the image of $\nu^{s}$ at a general $X_{0}$, which we call the “blow-up polynomial”. In the end in order to verify a “blow-up polynomial”, we reduce $X_{0}$ to a Fermat hypersurface. The following are the definition and its connection to the surjectivity of $\nu^{s}$.

**Definition 1.2.**

(1) We define $B_{(c_{0}, f_{0})} = \text{Image}(\nu^{s})$, where $i$ is the canonical isomorphism above.

(2) $\sigma \in B_{(c_{0}, f_{0})}$ is called a blow-up polynomial in the order $m$ at the set of zeros $t_{1}, \cdots, t_{m}$ of $\sigma$ if for any $s \in H^{0}(\mathcal{O}_{\mathbb{P}^{1}(m)})$,

$$sv \in B_{(c_{0}, f_{0})}$$

where $v$ is in $H^{0}(\mathcal{O}_{\mathbb{P}^{1}(5d - m)})$ such that $\text{div}(v) = \text{div}(\sigma) - \sum_{i=1}^{m} t_{i}$.

**Lemma 1.3.** If $\sigma \in B_{(c_{0}, f_{0})}$ has distinct zeros and it is a blow-up polynomial in the order 1 at each zero of it, then $\sigma$ is a blow-up polynomial in the order 5$d$ at the set of all zeros of $\sigma$.

**Proof.** If $\sigma$ is a blow-up polynomial at a zero $t_{1}$ and at a zero $t_{2}$ ($t_{1} \neq t_{2}$), by the linearity of $B_{(c_{0}, f_{0})}$, it must be a blow-up polynomial in the order 2 at the set $\{t_{1}, t_{2}\}$. Then inductively, $\sigma$ is a blow-up polynomial in the order 5$d$ at the set of all zeros of $\sigma$. $\square$

The surjectivity of $\nu^{s}$ is equivalent to the existence of a “blow-up” polynomial in the order 5$d$. By this lemma it is now equivalent to the existence of a “blow-up” polynomial in the order 1 at each zero of $\sigma$. However there are examples of $C_{0}, X_{0}$ where there are no “blow-up” polynomials in order 5$d$. This is where the genericity of $X_{0}$ comes in to provide new types of polynomials in $B_{(c_{0}, f_{0})}$. To see that, we consider another map $\phi^{s}$:

$$
T_{f_{0}}\mathbb{P}^{N} \xrightarrow{\phi^{s}} H^{0}(\mathcal{O}_{\mathbb{P}^{4}(5d)}) \xrightarrow{c_{0}^{s}(\partial F/\partial \alpha)}
$$

where $\mathbb{P}^{N}$ is the projectivization of $H^{0}(\mathcal{O}_{\mathbb{P}^{4}(5)})$ and $F$ is a fixed universal quintic polynomial defining the universal quintic threefold,

$$X = \{(x, f) \in \mathbb{P}^{4} \times \mathbb{P}^{N} : f(x) = 0\}.$$

Next we use the key ingredient of the proof: genericity of $X_{0}$. 


**Lemma 1.4.**

(1) If $X_0$ is generic and $C_0$ is an immersed rational curve, then for any $\alpha \in T_{f_0}\mathbb{P}^N$, there is $\alpha < \alpha > \in H^0(c_0^*(T_{p^*}))$ such that $(\alpha, < \alpha >)$ is tangent to the universal quintic 3-fold $X$ in

\[ \mathbb{P}^4 \times \mathbb{P}^N. \]

$< \alpha >$ is not unique, but we will fix a morphism $\psi$:

$\alpha \mapsto < \alpha >$

at one point $(c_0, f_0) \in M \times \mathbb{P}^N$.

(2)

\[ \text{image}(\phi^s) \subset \mathcal{i} (\text{image}(\nu^s)). \]

**Proof.** (1) This is the proposition 2.1 below.

(2) This is because

\[ c_0^*(\frac{\partial F}{\partial \alpha} + \frac{\partial f_0}{\partial < \alpha >}) = 0. \]

There is another straightforward way to see this. We identify

\[ T_{f_0} U_a = U_a, \]

where $U_a$ is an affine open set of $\mathbb{P}^N$ that contains $[f_0]$ as the origin. For any $f \in U_a$, $\overrightarrow{f}$ denotes the vector corresponding to $f$ in the isomorphism (1.5). Then

\[ c_0^*(f) \]

lies in $B_{(c_0, f_0)}$ because $c_0^*(f) = \phi^s(\overrightarrow{f})$.

Applying the concept of “blow-up polynomial”, we prove that

**Theorem 1.5.** Consider the quintic $x_0x_1x_2x_3x_4$ where

\[ x_i \in H^0(\mathcal{O}_{\mathbb{P}^4}(1)), i = 0, \ldots, 4, \]

are generic. Then $\phi^s(x_0x_1x_2x_3x_4)$ is a “blow-up” polynomial in the order $5d$ in $B_{(c_0, f_0)}$.

We found a specific “blow-up” polynomial in the order $5d$ in $B_{(c_0, f_0)}$. This completes the proof of theorem 1.1. Another expression for $\phi^s(x_0x_1x_2x_3x_4)$ is

\[ \phi^s(x_0x_1x_2x_3x_4) = c_0^*(x_0x_1x_2x_3x_4). \]
1.2. The proof of theorem 1.7–search for a “blow-up” polynomial in the order $5d$. The proof uses two invariants: one is an linear map from a linear space of dimension 4 to $\mathbb{C}$, the other is a subspace of $H^0(\mathcal{L}^4)$. For both invariants we assume $X_0$ is generic, thus the map $\psi$ above, i.e., the map

$$\alpha \rightarrow \langle \alpha \rangle$$

exists.

Notation: Let $\frac{\partial}{\partial y}$ be the map (partial derivative map):

$$\frac{\partial}{\partial y} : (H^0(\mathcal{L}^r))^* \times H^0(\mathcal{L}^r) \rightarrow H^0(\mathcal{L}^{r-1})$$

where $H^0(\mathcal{L}^r) = \otimes_r H^0(\mathcal{L})$. We denote the image of $(y, g)$ by $\frac{\partial g}{\partial y}$. For $(y_1, \cdots, y_k) \in \prod_k (H^0(\mathcal{L}))^*$, $k \leq r$, the following composition map

$$H^0(\mathcal{L}^r) \rightarrow H^0(\mathcal{L}^{r-1}) \rightarrow \cdots \rightarrow H^0(\mathcal{L}^{r-k})$$

is denoted by $\frac{\partial^k}{\partial y_1 \cdots \partial y_k}$, and $\frac{\partial^k}{\partial y_i}$ is the short for $\frac{\partial}{\partial y_i} \cdots \frac{\partial}{\partial y_i}$. Let

$$Q = \frac{\partial}{\partial ((H^0(\mathcal{L}))^* \times \{ f_0 \})}.$$ 

Notice $Q$ is the linear space spanned by the generators of the Jacobian ideals of $f_0$ in $\mathbb{P}^n$.

(1) The construction of a subspace in $H^0(\mathcal{L}^4)$.

For each fixed $L, q$ with $L \in H^0(\mathcal{L})$, $L(c_0(q)) = 0$, let

$$V_{L,q} = \{ Q \in H^0(\mathcal{L}^4) : \phi^*(\overline{Q}) \text{ is a blow-up polynomial at } q \}.$$ 

Actually $Q \subset V_{L,q}$ directly follows from the definitions. Also note $V_{L,q}$ depends on the pair $c_0, f_0$.

Let $L \in H^0(\mathcal{L})$ and $t \in \text{div}(c_0^*(L))$, then the sections of divisor

$$\text{div}(c_0^*(L)) - \{ t \}$$

are denoted by $L_t$. So $L_t \in H^0(\mathcal{O}_P(1))$.

The immediate lemma is

**Lemma 1.6.** Let $Q \in H^0(\mathcal{L}^4)$. Also let $L \in H^0(\mathcal{L})$, $t \in \text{div}(c_0^*(L))$ and $w \in H^0(\mathcal{O}_P(1))$ such that $\text{div}(w) \neq t$. Then $Q \in V_{L,q}$ if and only if

$$wL_t c_0^*(Q) \in B(c_0, f_0).$$

**Proof.** Because $B(c_0, f_0)$ is linear, then the line through $wL_t c_0^*(Q)$ and $c_0^*(LQ)$ in $H^0(\mathcal{O}_P(5d))$
lies in $B(c_0, f_0)$ if and only if $wLc_0^5(Q)$ is also in $B(c_0, f_0)$. □

(2) The construction of the linear map.  
Let $x = (x_0, \cdots, x_4)$ be a coordinate system for $(H^0(L))^*$. For any $Q \in H^0(L^4)$, the Taylor expansion

\[ Q = Q_0(\bar{x}) + Q_1(\bar{x}) \cdots + Q_4(x_0)^4, \]

where $\bar{x} = (x_1, \cdots, x_4)$ and

\[
\begin{align*}
Q_i &= \frac{\partial^i Q(\bar{x})}{i! \partial x^i} x_0^i, & \text{for } i \neq 4 \\
Q_4 &= \frac{\partial^4 Q}{4! \partial x_0^4},
\end{align*}
\]

gives a linear map $\xi$:

\[
\begin{array}{c}
H^0(L^4) \\
\uparrow \\
Q \\
\downarrow \\
Q_4.
\end{array}
\]

This is well-defined because $Q_4$ is a constant independent of $\bar{x}$. Next we restrict $\xi$ to a hyperplane $H(x_0, x, L, q, f_0)$ of $Q$ that consists of all quartics $Q \in Q$ satisfying

\[ Q_0(\bar{x}) + Q_1(\bar{x}) + \cdots + Q_3(\bar{x}) \in V_{L,q}. \]

We denote

\[ \xi|_{H(x_0, x, L, q, f_0)} \]

by

\[ z(x_0, x, L, q, f_0) \]

which also depends on $c_0$.

The purpose of these two invariants is simple. Notice in the definition we have

\[ z(x_0, x, L, q, f_0)(Q) \cdot x_0^4 \in V_{L,q}. \]

for $Q \in H(x_0, x, L, q, f_0)$. The map $z(x_0, x, q, L, f_0)$ being non-zero will imply

\[ x_0^4 \in V_{L,q}. \]

Then theorem 1.4 directly follows from it (see section 4). So the key question in this paper is: when is $z(x_0, x, q, L, f_0)$ a non-zero map?

More specifically, theorem 1.5 follows from the thread of the following propositions:

**Proposition 1.7.** Let $L \in H^0(L)$, $q \in \text{div}(c_0^5(L))$. Then there is a coordinate system $x$ such that

\[ z(x_0, x, L, q, f_0) \neq 0. \]
Proposition 1.8. Let $L \in H^0(\mathcal{L})$, $q \in \text{div}(c_0^*(L))$. If there is a coordinate system $x$ such that

$$z(x_0, x, L, q, f_0) \neq 0$$

then

$$V_{L, q} = H^0(\mathcal{L}^4).$$

Proposition 1.9. If $V_{L, q} = H^0(\mathcal{L}^4)$ for any $q \in \mathbb{P}^1$ with $L(c_0(q)) = 0$, then

$$\phi^*(\bar{x}_0 \cdots \bar{x}_4) = c_0^*(x_0 \cdots x_4)$$

in theorem 1.5 is a blow-up polynomial in the order $5d$ at the set of all zeros of it.

All these propositions are under the assumption that $X_0$ is a generic quintic. The last two propositions, propositions 1.8, 1.9 do not require deformations of $X_0$. They easily follow from the definitions. Proposition 1.7 is the only place where deformations of $X_0$ are needed.

In the rest of the paper, we are going to concentrate on proving theorem 1.1. In section 2, we give a description of first order deformation of a pair of a quintic and an immersed rational curve. In section 3, we give a description of the deformation of the space $B_{(c_0, f_0)}$. The main purpose of this section is to prepare the set-up for the deformation of $f_0$. In section 4, we provide very easy proofs of propositions 1.8, 1.9. The importance of this section lies in the idea. In section 5, we use a deformation of $f_0$ to prove proposition 1.7, thus theorem 1.1.

2. First order deformations of a pair.

In this section, we give a description of the first order deformations of a pair which will be used throughout. Let $c_0$ be as above and

$$\bar{c}_0 : \mathbb{P}^1 \to X_0 \times \{f_0\} \subset \mathcal{X}$$

be the morphism that lifts the image $C_0$ to $\mathcal{X}$. The projection

$$P : \mathcal{X} \to \mathbb{P}^N$$

induces a map on the sections of bundles over $\mathbb{P}^1$,

$$P^* : H^0(\bar{c}_0^*(T\mathcal{X})) \to T_{f_0} \mathbb{P}^N,$$

where $T_{f_0} \mathbb{P}^N \simeq H^0(T_{f_0} \mathbb{P}^N \otimes \mathcal{O}_{\mathbb{P}^1})$ is the space of global sections of the trivial bundle whose each fibre is $T_{f_0} \mathbb{P}^N$.

Proposition 2.1. If $X_0$ is generic, $P^*$ is surjective.

Remark $P^*$ being surjective is an condition on the first order deformation of the pair $c_0, f_0$. 

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Proof. Let $M_d$ be the parameter space of immersions $\mathbb{P}^1 \to \mathbb{P}^4$, whose image has degree $d$. So $M_d$ is an open set of

$$\mathbb{P}(\oplus_5 \mathcal{O}_{\mathbb{P}^1}(d)).$$

The map $c_0$ represents a point in $M_d$ which is still denoted by $c_0$. Let $\mathcal{X}$ be the universal hypersurface. Let

$$\Gamma \subset M_d \times \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(5)))$$

be the incidence scheme containing the point $(c_0, f_0)$. Let $T_{(c_0, f_0)} \Gamma$ be the Zariski tangent space of $\Gamma$. Since the $c_0$ is an immersion, the pullback of the normal sheaf $N_{C_0/\mathbb{P}^4}$ to $\mathbb{P}^1$ is the same as the normal bundle $N_{C_0/\mathbb{P}^4}$. Then the sections of the sheaf $N_{C_0/\mathbb{P}^4}$ can be pulled-back to a section of the normal bundle $N_{c_0/\mathbb{P}^4}$ over $\mathbb{P}^1$. Thus $(e_{\Gamma})^*_*$, the differential of the evaluation $e_{\Gamma}$:

$$(f, c, t) \to c(t) \times \{f\}$$

induces a linear map, still denoted by $(e_{\Gamma})^*_*$:

$$T_{(f_0, c_0)} \Gamma \to H^0(\mathcal{L}_0(T_X)).$$

$(e_{\Gamma})^*_*$ does not exist if $c_0$ is a non-immersion map. This gives us a commutative diagram

$$\begin{array}{c}
T_{(f_0, c_0)} \Gamma \xrightarrow{(e_{\Gamma})^*_*} H^0(\mathcal{L}_0(T_X)) \\
\downarrow (\pi_{\Gamma})^*_* \quad \downarrow \mathcal{P}^* \\
T_{f_0} \mathbb{P}^N = \mathbb{T}_{f_0} \mathbb{P}^N,
\end{array}$$

where $\pi_{\Gamma}$ is the projection. Since $X_0$ (i.e. $f_0$) is generic, so $\pi_{\Gamma}$ is dominant, then $(f_0, c_0) \in \Gamma$ is a generic point in $\Gamma$. Then the dominance of $\pi_{\Gamma}$ implies the surjectivity of $(\pi_{\Gamma})^*_*$. Thus $\mathcal{P}^*$ is surjective. \]  

3. The deformation of $B_{(c_0, f_0)}$. We need to deform $X_0$ to prove proposition 1.7. The structure to control during the deformation is the subspace $B_{(c_0, f_0)}$ in the fixed vector space $H^0(\mathcal{O}_{\mathbb{P}^1}(5d))$.

First we extend the pull-back map $e_{\mathfrak{m}}^*$. Consider map $\rho$:

$$(3.1) \quad \mathbb{P}(\oplus_5 H^0(\mathcal{O}_{\mathbb{P}^1}(d))) \times H^0(\mathcal{O}_{\mathbb{P}^1}(m)) \xrightarrow{\rho} H^0(\mathcal{O}_{\mathbb{P}^1}(md))$$

where

$$s(x_0, \ldots, x_4) \in H^0(\mathcal{O}_{\mathbb{P}^1}(m)) = Sym^m(H^0(\mathcal{O}_{\mathbb{P}^1}(1))).$$

If $[l_0, \ldots, l_4] = c$, we denote $\rho(c, f)$ by

$$c^*(f),$$
which is the original $c_0^*(f)$ if $c_0$ is a regular map.

Let $\bar{\Gamma}$ be the closure of $\Gamma$ in

$$P(\oplus_5 H^0(\mathcal{O}_{P^1}(d))) \times P^N.$$ 

Let $S \subset \bar{\Gamma}$ be a curve such that the general points of it are the general points of $\Gamma$.

**Lemma 3.1.** In definition 1.2 we defined

$$B(c,f) \subset H^0(\mathcal{O}_{P^1}(5d)),$$

for generic $(c,f) \in S$. Let $I$ be the closure of the collection

$$\{(c,f,B(c,f))\}_{\text{generic}} \subset S_U \times H^0(\mathcal{O}_{P^1}(5d))$$

where $S_U$ is an open set of $S$. Then $I$ is an algebraic curve.

**Proof.** Let $S \in \bar{\Gamma}$ be a curve such that the generic points of $S$ are generic points of $\Gamma$. Let $S_U$ be an open set of $S$ such that at any

$$(c,f) \in S_U$$

both $S_U$ and $\text{div}(f)$ are smooth.

Let $C_U$ be the universal curve

$$C_U = \{(y, (c,f)) : y \in c(P^1)\} \subset P^n \times S_U,$$

$X_U$ be the universal hypersurface

$$X_U = \{(x, (c,f)) : x \in \text{div}(f)\} \subset P^n \times S_U.$$

Consider the projection maps

$$C_U \rightarrow X_U \downarrow \pi_1 \downarrow \pi_2 \leftarrow S_U \leftarrow S_U.$$ 

From the quotient maps of sheaves,

$$N_{C_U/(P^4 \times U)} \rightarrow N_{X_U/(P^4 \times U)}|_{C_U},$$

we obtain a morphism

$$(\pi_1)_*(N_{C_U/(P^4 \times U)}) \rightarrow (\pi_2)_*(N_{X_U/(P^4 \times U)}|_{C_U})$$

of direct image sheaves. Because $\pi_i$ are flat and both normal bundles are coherent sheaves, by the semicontinuity after shrink $S_U$, we may assume the dimensions

$$\text{dim}(H^0(N_{c/P^4})), \text{dim}(H^0(c^*(N_{\text{div}(f_0)/P^4})))$$

\footnote{This compatification of $M_d$ is different from Kontsevich's compatification of stable maps. So we are not going to use Kontsevich's moduli space of stable maps.}
are constants with respect to the variation of \((c, f) \in \mathcal{S}_U\). Thus \((\pi_1)_* (N_{C_U / (\mathbb{P}^4 \times U)})\) and \((\pi_2)_* (N_{X_U / (\mathbb{P}^4 \times U)}|_{C_U})\) both are sheaves of sections of trivial vector bundles (after shrink \(S_U\)),

\[
\mathcal{S}_U \times H^0(N_{c_0/P^4}), \mathcal{S}_U \times H^0(\mathcal{O}_{P^1}(5d)),
\]

where the pair \((c_0, f_0) \in S_U\) is fixed. We may assume that the trivialization of \((\pi_2)_* (N_{X_U / (\mathbb{P}^4 \times S_U)}|_{C_U})\) is induced from \(i\) in the formula (1.3).

Thus \(\mu\) corresponds to a vector bundle morphism, which is still denoted by \(\mu\):

\[
U \times H^0(N_{c_0/P^4}) \overset{\mu}{\rightarrow} U \times H^0(\mathcal{O}_{P^1}(5d)).
\]

Then let \(\mathcal{I}\) be the closure of the image

\[
\mu(S_U \times H^0(N_{c_0/P^4})) = \{(c, f, B_{(c, f)})\} \in S_U
\]
in

\[
S \times H^0(\mathcal{O}_{P^1}(5d)).
\]

Thus \(\mathcal{I}\) is algebraic. We complete the proof.

4. Application of blow-up polynomials.

Proof. of proposition 1.9. Applying the assumption

\[V_{L,q} = H^0(\mathcal{O}_{P^4}(4)),\]

for

\[L = c_0(x_k), \quad \text{for each } k = 0, \ldots, 4\]

and a zero \(q\) of \(c_0(x_k)\), we obtain that \(\prod c_0(x_i)\) must be a blow-up polynomial in order \(5d\). We complete the proof.

Proposition 4.1.

\(V_{L,q}\) is a subspace in the linear space \(H^0(\mathcal{O}_{P^4}(4))\).

Proof. This directly follows from the definition of \(V_{L,q}\).

Proof. of proposition 1.8. If \(z(x, L, q, f_0)\) is non-zero, this must be true for a generic coordinate’s system \(x\). Since \(z(x, L, q, f_0)(Q)x_0^4 \in V_{L,q}\),

\[x_0^4 \in V_{L,q}.\]

Now consider the Veronese map \(v_4\)

\[(\mathbb{P}^4)^* \overset{v_4}{\rightarrow} \mathbb{P}(H^0(\mathcal{O}_{P^4}(4)))\]

whose image is the Veronese variety ([1], pg 25). Since Veronese variety is non-degenerated, \(v_4((\mathbb{P}^4)^*)\) does not lie in any hyperplane. Thus

\[
\text{span}\{x_0^4\} = H^0(\mathcal{O}_{P^4}(4)).
\]

By proposition 3.1 (linearity of \(V_{L,q}\)),

\[
\text{span}\{x_0^4\} \subset V_{L,q}.
\]

Therefore

\[V_{L,q} = H^0(\mathcal{O}_{P^4}(4)).\]

We complete the proof.
5. Non-vanishing of \( z(x_0, x, L, q, f_0) \). In this section, we use the following special notations:

1. Let \( x_0, \cdots, x_4 \) be the homogeneous coordinates for \( \mathbb{P}^4 \), and \( Q \in H^0(\mathcal{O}_{\mathbb{P}^4}(4)) \), we denote the truncated Taylor expansion as in formula (1.8),

\[
Q_0(\bar{x}) + Q_1(\bar{x})x_k + \cdots + Q_r(\bar{x})(x_k)^r
\]

by

\[
(Q)_{(r,x_k)}
\]

for the coordinates \( x_0, \cdots, x_4 \) and the choice of \( x_k \). Notice this notation depends on the choice of \( x_k \).

2. Let \( L \in H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \) and \( t \in \text{div}(c^*(L)) \), then the sections of divisor

\[
\text{div}(c^*(L)) - \{t\}
\]

are denoted by \( L_t \). So \( L_t \in H^0(\mathcal{O}_{\mathbb{P}^1}(d-1)) \).

**Proof.** of proposition 1.7. The proof uses a deformation of \( f_0 \) to the Fermat hypersurface. Specifically, we construct a family of a set of linear maps \( \xi|_{\mathcal{O}} \) (see the formula (1.9)) over a smooth curve. Let’s divide the proof into 4 steps:

1. Define a base variety \( \tilde{C} \)—a smooth curve that represents the parameter space of the deformation.

2. Define a variety \( \tilde{I}_k \) and a proper, flat morphism \( \tilde{I}_k \to \tilde{C} \)—the fibres of it represent the deformations of the set of linear maps \( \xi|_{\mathcal{O}} \).

3. Using the first method to find the fibres of \( \tilde{I}_k \)—calculation via

\[
z(x_k, x, L, q, f_0) = 0.
\]

4. Using the second method to find the fibres of \( \tilde{I}_k \)—calculation via

\[
Q_0(\bar{x}) + Q_1(\bar{x}) + \cdots + Q_3(\bar{x}) \in V_{L,q}.
\]

5.1. Definition of the base variety \( \tilde{C} \). Fix a coordinate system \( [x_0, \cdots, x_4] \) for \( \mathbb{P}^4 \). Let \( f \in H^0(\mathcal{O}_{\mathbb{P}^4}(5)) \) be a generic quintic. Consider the one parameter family of quintics \( X_\theta \) over the plane

\[
\theta \in \mathbb{C}
\]

where \( X_\theta \) is defined by a family of quintics in the form:

\[
f_\theta = \sum_{i=0}^{4} (x_i)^5 + (\theta + 1)f,
\]

for the generic section \( f \) in \( H^0(\mathcal{L}^5) \).

Let \( L \in H^0(\mathcal{O}_{\mathbb{P}^4}(1)) \). Choose an algebraic curve

\[
\mathcal{C} = \{(c_\theta, f_\theta, t) \in M_d \times \Delta \times \mathbb{P}^1 : c_\theta(f_\theta) = 0, c_\theta(L)(t) = 0\}
\]
where $\Delta$ is an open set of the parameter space $C$ of the family $f_{\theta}$. Let $\tilde{C}$ be the closure of $C$ in
\[ P(\oplus_5 O_{P^1}(d)) \times C \times P^1. \]
Let
\[ \pi : \tilde{C} \to \tilde{C} \]
be a smooth resolution of $\tilde{C}$.
Assume the above $\tilde{C} \subset \tilde{C}$ is an open set, that is isomorphic to the image $\pi(C)$. To abuse the notations, we'll denote an element in $\tilde{C}$ by the triple
\[ (c_{\theta}, f_{\theta}, t), \]
where $\theta \in C$ (and $c_{\theta}, t$ can not be uniquely determined by $\theta$). We also assume the projection of $\tilde{C}$ to $\tilde{\Gamma}$ is the curve $S$ in section 3. So we may apply the results in section 3.

5.2. Definition of the variety $\tilde{I}_k$ over $\tilde{C}$. We first choose the same coordinates $x_0, \cdots, x_4$ for the definition of $z(x_0, x, L, q, f_{\theta})$. Let $L \in H^0(O_{P^1}(1))$, and $w \in H^0(O_{P^1}(1))$ be fixed. Then we fix an integer $k$ between 0 and 4. Define two subvarieties $J_1, J_2$ of
\[ C \times P(H^0(O_{P^1}(5d))) \times C^5 \times C^5, \]
where the first $C^5$ is $Q$ viewed as the domain of the linear map $\xi$ (see the formula (1.9)), and the second $C^5$ collects the images of 5 linear maps $\xi$.

\[ J_1 = \{(c_{\theta}, f_{\theta}, t, [g], (\epsilon_0, \cdots, \epsilon_4), (z_0, \cdots, z_4)) : [g] = [wL_4 \sum_1^4 \epsilon_0^i \left( \frac{\partial f_{\theta}}{\partial x_i} \right) \partial_{x_i}^4 z_j)] \}
\]
(Notice $J_1$ is a graph of a regular map).

\[ J_2 = I_\Delta \times C^5 \times C^5, \]
where
\[ I_\Delta \subset C \times P(H^0(O_{P^1}(5d))) \]
and its projection to $\tilde{\Gamma} \times P(H^0(O_{P^1}(5d)))$ lies in the projectivized $I$ defined in lemma 3.1.
At last we let
\[ I_k = J_1 \cap J_2 \]
and $\tilde{I}_k$ be its closure in
\[ \tilde{C} \times P(H^0(O_{P^1}(5d))) \times C^5 \times C^5 \supset I_k. \]

Now $\tilde{I}_k$ for each $k = 0, \cdots, 4$ is a well-defined algebraic variety over a smooth curve $\tilde{C}$. $I_k$ is a family of the graphs of 5 linear maps $\xi|_Q$ restricted to hyperplanes. One of those maps is restricted to
\[ z(x_k, x, L, q, f_0). \]
5.3. The fibres of $\bar{I}_k$—first method. The projection $\bar{I}_k \to \bar{C}$ is proper and flat. Let’s see its fibre over a generic point $(f_\theta, c_\theta, t) \in C$. This step is to show the fibre is a non-zero subspace.

The fibre is determined by the equations

$$w L_t c_\theta^* \left( \sum_i \epsilon_i \frac{\partial f_\theta}{\partial x_i} (3, x_k) \right) \in B_{(c_\theta, f_\theta)}.$$  

Let

$$(\mathcal{H}_\theta)^k = \{ \epsilon \in C^5 : w L_t c_\theta^* \left( \sum_i \epsilon_i \frac{\partial f_\theta}{\partial x_i} (3, x_k) \right) \in B_{(c_\theta, f_\theta)} \},$$

$$H_\theta^k = \{ \epsilon : \sum_{i=0}^n \epsilon_i \frac{\partial^5 f_\theta}{\partial x_i \partial x_k^4} = 0 \} \subset C^5.$$  

Then

$$H_\theta^k \subset (\mathcal{H}_\theta)^k \subset H_{(x_k, x, L, t, f_\theta)}.$$  

Since

$$H_\theta^k \subset C^5$$

has codimension 1, we obtain that

(1) either $H_{(x_k, x, L, t, f_\theta)} = Q$,

(2) or

$$(\mathcal{H}_\theta)^k = H_\theta^k = H_{(x_k, x, L, t, f_\theta)}, \quad \text{and}$$

$$z_k = z(x_k, x, L, t, f_\theta)(\sum_i \epsilon_i \frac{\partial f_\theta}{\partial x_i}),$$

for $(\epsilon_0, \ldots, \epsilon_4) \in H_{(x_k, x, L, t, f_\theta)}$. Now if (1) is true, then because $Q_4$ is always non-zero for some $Q \in Q$, the proposition is proved. So we may assume (2), i.e., $H_{(x_k, x, L, t, f_\theta)} \neq Q$. The argument shows two things:

(I) the fibre of $\bar{I}_k$ over a generic point in $C$ is a sublinear space of dimension 4, which is a graph of a regular map from the linear space

$$(\mathcal{H}_\theta)^k = H_\theta^k = H_{(x_k, x, L, t, f_\theta)},$$  

(II) to show the $z(x_k, x, L, q, f_\theta)$ is non-zero for some $k$, it suffices to show $z_k$-coordinate for some points of $I_k$ is non-zero.

5.4. Fibres of $\bar{I}_k$—second method, the direct calculation. Now we use the direct calculation to find the fibres of $\bar{I}_k$. The purpose of this calculation is to show the fibres of $\bar{I}_k$ over the Fermat quintic is independent of $k$.

Consider the fibre over a generic point $(c_\theta, f_\theta, t) \in \bar{C}$, where $\theta \neq -1$ but near $-1$. We take the partial derivative of $f_\theta$ to obtain that

$$z_j = h \epsilon_j + (\theta + 1)(\frac{\partial^4}{4! \partial x_j^4} \sum_i \epsilon_i \frac{\partial f_\theta}{\partial x_i}).$$
At last it suffices to determine the domain $H(x_k, x, L, t, f_\theta)$ of $z(x_k, x, L, t, f_\theta)$, that varies with $\theta$. Notice that the key equations in the intersection

$$J_1 \cap J_2$$

are

$$g = wL_t c_\theta^* \left( \left( \sum_i \epsilon_i \frac{\partial f_\theta}{\partial x_i} \right)(3, x_k) \right) \in B_{(c_\theta, f_\theta)}.$$  \hfill (5.5)

Thus first we calculate $g$ directly. Note

$$\left( \sum_i \epsilon_i \frac{\partial f_\theta}{\partial x_i} \right)(3, x_k) = (\theta + 1) \left( \sum_i \epsilon_i \frac{\partial f}{\partial x_i} \right)(3, x_k).$$

Hence the equations (5.5) become

$$wL_t c_\theta^* \left( \left( \sum_i \epsilon_i \frac{\partial f}{\partial x_i} \right)(3, x_k) \right) \in B_{(c_\theta, f_\theta)}.$$  \hfill (5.6)

Let

$$\frac{\partial f}{\partial x_i} = a_i x^4_k + \left( \frac{\partial f}{\partial x_k} \right)(3, x_k)$$

for some complex number $a_i$. Also notice

$$5x^4_k + (\theta + 1) \frac{\partial f}{\partial x_k} \in V_{L,t},$$

which means

$$wL_t c_\theta^* \left( 5x^4_k + (\theta + 1) \frac{\partial f}{\partial x_k} \right) \in B_{(c_\theta, f_\theta)}.$$  \hfill (5.7)

Then we simplify the constraint (5.6) to linear equations for $\epsilon$:

$$wL_t \sum_{i=0}^4 \epsilon_i c_\theta^* \left( \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial x_k} \right) \in B_{(c_\theta, f_\theta)}.$$  \hfill (5.8)

Let $V$ be a generic hyperplane containing $B_{(c_{-1}, f_{-1})}$. Then for all different $k$, the hyperplanes $(H_{-1})^k$ are the same and are equal to $W_{(c_{-1}, f_{-1}, t) \epsilon \in \mathbb{C}}$, where

The most important part of this argument is that the hyperplane (5.8) is independent of $k$.

So the fibres $(\bar{I}_k)_{(c_{-1}, f_{-1}, t)}$ for each $k, k = 0, \ldots, n$ are the same and are equal to the set

$$\left\{ \left( f_{-1}, c_{-1}, t, [wL_t \sum_{i=0}^4 \epsilon_i c_\theta^* \left( \frac{\partial f}{\partial x_i} \right)(3, x_k)], (\epsilon_0, \ldots, \epsilon_4), (h\epsilon_0, \ldots, h\epsilon_4) \right) \right\}.$$
where

\[(5.9) \quad (\epsilon_0, \cdots, \epsilon_4) \in W(c-1,f-1,t).\]

Because \(W(c-1,f-1,t)\) is linear, one of \(\epsilon_i, i = 0, \cdots, 4\) is non-zero. Thus there is \(k\) such that \((\bar{I}_k(c-1,f-1,t))\) contains a point with the non-zero coordinate \(\epsilon_k\). This contradicts the result from section 5.3. Alternatively this shows when \(\theta\) is near \(-1\), there is a \(k\) among \(0, \cdots, 4\), such that the fibre of \(I_k\) over the point \((\theta, c, t)\) must contain the non-zero \(z_k\). Hence \(z(x_k, x, L, t, f_\theta)\) is a non-zero linear map for the special choice of \(x_k\). Thus for a generic coordinates \(y\) of \(P^4\), \(z(y_0, y, L, t, f_\theta)\) is also non-zero. Note \(f_\theta\) is a generic hypersurface. We complete the proof of proposition 1.7.

\[\bx\]

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