TRANSITIVE CYLINDER FLOWS
WHOSE SET OF DISCRETE POINTS
IS OF FULL HAUSDORFF DIMENSION

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Abstract. For each irrational \( \alpha \in [0, 1) \) we construct a continuous function \( f: [0, 1) \to \mathbb{R} \) such that the corresponding cylindrical transformation \( [0, 1) \times \mathbb{R} \ni (x, t) \mapsto (x + \alpha, t + f(x)) \in [0, 1) \times \mathbb{R} \) is transitive and the Hausdorff dimension of the set of points whose orbits are discrete is 2. Such cylindrical transformations are shown to display a certain chaotic behaviour of Devaney-like type.

Introduction

Chaotic behaviour in dynamical systems has been of particular interest in topological dynamics since about the second half of the 20th century. Few examples had been studied earlier, thus it must have been surprising to Abram Besicovitch to discover a homeomorphism of the cylinder \( T \times \mathbb{R} \) with both dense and (closed) discrete orbits ([Be2], see also [Be1]). It is an example of a class called now cylindrical transformations or, more generally, skew products. Cylindrical transformation or cylinder is a mapping of the form

\[ T_f: X \times \mathbb{R} \ni (x, t) \mapsto (Tx, t + f(x)) \in X \times \mathbb{R}, \]

where \( T: X \to X \) is, in the most general setting, a homeomorphism of a topological space, and \( f: X \to \mathbb{R} \) is a continuous function. They arise naturally in ergodic theory, as their iterates are the products of respective iterates of \( T \) and the ergodic sums of \( f \) over \( T \). Formally, they were introduced (and even earned their own chapter) in a textbook on topological dynamics [GoHe, Chapter 14]. However, such transformations were already considered before: Besicovitch viewed his cylinder on \([0, 1) \times \mathbb{R}\) as a homeomorphism of the punctured plane \( \mathbb{R}^2 \setminus \{0\} \) and expanded it to a homeomorphism of the plane with dense and discrete orbits. Also, cylinders are sections of the flows derived from some differential equations (studied in [Po], Chapitre XIX, pp. 202ff.; see also [FrLe, Section 8]).

The result of Besicovitch concerned only some particular \( Tx = x + \alpha \) on \( T \) and \( f: T \to \mathbb{R} \). Therefore, a few natural questions arise: which cylinders have both dense and discrete orbits? (Such cylinders are hereafter called Besicovitch cylinders.) For which rotations do such cylinders exist and how common are they? How many discrete orbits do they have? How about

1Let us remind that one of the most popular definitions of chaos, the Devaney chaos, comprises dense orbits (transitivity), dense set of periodic orbits and sensitivity to initial conditions (the last condition usually follows from the first two ones).

2Also called cylinder flow or \( \mathbb{R} \)-extension.
other homeomorphisms \((X,T)\)? These problems were studied, among others, by Frączek and Lemańczyk in \([FrLe]\) and by Kwiatkowski and Siemaszko in \([KwSi]\). In particular, in \([FrLe]\), Besicovitch cylinders over every minimal rotation of tori \(T^d\) were constructed. As for the amount of discrete orbits, it is known that the set of nonrecurrent points in these cases is small in both topological and measure-theoretical sense: it is of first category (albeit dense) and of measure zero. Thus, the authors of \([FrLe]\) used some finer means to analyse the set of points with discrete orbits. Firstly, for every minimal rotation of a torus \(T^d\) they found a Besicovitch cylinder with uncountably many discrete orbits. Secondly, for almost every minimal rotation there is a Besicovitch cylinder for which the points with discrete orbits have altogether the Hausdorff dimension at least \(d + 1/2\) (that is, of codimension at most \(1/2\)). Also, the authors discovered some classes of regular examples (in terms of Hölder continuity, Fourier coefficients or degree of smoothness). They left as an open problem whether higher Hausdorff dimensions can be achieved. The (positive) solution this problem is the main topic of the present paper: by enhancing the techniques from \([FrLe]\) we have constructed Besicovitch cylinders with full Hausdorff dimension of discrete orbits for every minimal rotation of \(T^d\).

The present paper consists of four sections. Section 1 contains some preliminary facts on cylindrical transformations that are relevant to our quest for Besicovitch cylinders; in particular, we show that they form a first category set within some relevant function space. In Sections 2 and 3, we present our construction, define some subsets of \(T\) and prove that their elements have discrete orbits (although there may also exist other discrete orbits). The Hausdorff dimension of these sets is calculated in Section 4. The last section introduces a definition of chaos that some Besicovitch cylinders satisfy, which is also a possible generalization of the Devaney chaos to noncompact dynamical systems.

1. **Cylindrical transformations**

The cylindrical transformations are a special case of the concept of skew product (see \([CoFS]\), Subsection 10.1.3) in ergodic theory, transferred in a natural way to the topological setting. In general, they can be defined for a minimal homeomorphism \(T\) of a compact metric space \(X\) (the base) with a \(T\)-invariant measure \(\mu\) defined on the Borel \(\sigma\)-algebra, and a real continuous function \(f: X \to \mathbb{R}\) (which we will customarily call a cocycle). In the next sections, we will confine ourselves to minimal rotations on tori with Lebesgue measure.\(^3\) Now, \(T\) and \(f\) generate a cylindrical transformation (or a cylinder):

\[
T_f: X \times \mathbb{R} \to X \times \mathbb{R}
T_f(x,t) := (Tx, t + f(x))
\]

\(^3\)Minimal rotations on compact groups do not always exist -- groups possessing them are called monoathetic. All tori \(\mathbb{T}^n\) are monoathetic, and a rotation on \(\mathbb{T}^n\) is minimal precisely when its coordinates are irrational and \(\mathbb{Q}\)-linearly independent; moreover, these rotations are uniquely ergodic with respect to Lebesgue measure.
The iterations of $T_f$ are of the form $T_f^n(x, t) = (T^n x, t + f^{(n)}(x))$, where $f^{(n)}$ is given by the formula:

$$f^{(n)}(x) := \begin{cases} 
  f(x) + f(Tx) + \cdots + f(T^{n-1}x), & \text{for } n > 0, \\
  0, & \text{for } n = 0, \\
  -f(T^{-1}x) - f(T^{-2}x) - \cdots - f(T^n x), & \text{for } n < 0.
\end{cases}$$

Observe that the dynamics of a point $(x, t)$ does not depend on $t$, because the mappings

$$\tau_{t_0} : X \times \mathbb{R} \ni (x, t) \mapsto (x, t + t_0) \in X \times \mathbb{R}$$

(for arbitrary $t_0 \in \mathbb{R}$) are in the topological centralizer of $T_f$:

$$T_f(\tau_{t_0}(x, t)) = T_f(x, t + t_0) = (Tx, t + t_0 + f(x)) = \tau_{t_0}(Tx, t + f(x)) = \tau_{t_0}(T_f(x, t)).$$

Unlike the compact case, a homeomorphism of a locally compact space (as here $X \times \mathbb{R}$) need not have a minimal subset. The cylinder on a compact space is never minimal (as proved in [Be2]), thus it is meaningful to study minimal subsets.

From now on, we will usually assume that the base is a torus $X = \mathbb{T}^d$ or even the circle (with a minimal rotation). Then, it is well-known that there are two cases in which the minimal subsets can be easily described:

1. (T1) when $\int_{\mathbb{T}^d} f \, d\mu \neq 0$, all points have closed discrete orbits, or, equivalently: for all $x \in X$: $|f^{(n)}(x)| \xrightarrow{n \to \pm \infty} \infty$.
2. (T2) when the cocycle is a coboundary, i.e. of the form $f = g - g \circ T$ for a (continuous) transfer function $g : \mathbb{T}^d \to \mathbb{R}$, the minimal sets are vertically translated copies of the graph of $g$ in $\mathbb{T}^d \times \mathbb{R}$. Conversely, if some orbit under $T_f$ is bounded, then so are all of them, and the cocycle $f$ is a coboundary (Gottschalk-Hedlund Theorem, [GoHe, Theorem 14.11]).

Notice that if $f$ is a coboundary and $T$ is measure-preserving, then $\int_{\mathbb{T}^d} f \, d\mu = 0$. Also, in both cases [T1] and [T2] the phase space decomposes into minimal sets. In what follows we will call cocycles that fulfil [T1] or [T2] trivial.

**Theorem 1.1** (Lemańczyk, Mentzen). If $X = \mathbb{T}^d$, $T$ is a minimal rotation and a cocycle $f$ is not trivial, then the cylinder $T_f$ is transitive (see [LeMe], Lemmas 5.2, 5.3). Therefore, if $f$ is of average zero, but $T_f$ has a closed discrete orbit, then it is automatically transitive (since by [T2] coboundaries have only bounded orbits).

As in [FlLi3], we consider cylindrical transformations that display both transitive and discrete behaviour, called Besicovitch transformations or Besicovitch cylinders, because their first example was given in [Be2]. Given a homeomorphism of the base, we will also call a cocycle which generates a Besicovitch cylinder a Besicovitch cocycle. For brevity, we will also write ‘discrete’ instead of ‘closed discrete’. By virtue of the condition [T1] and Theorem 1.1, a cocycle is Besicovitch if and only if it has average zero and the resulting cylinder has a discrete orbit. This characterisation will be used in our paper.
Unfortunately, Besicovitch cylinders are not easy to find. When $T$ is an irrational rotation of the circle, too regular cocycles yield no minimal sets at all, as has been proved by Matsumoto and Shishikuro, and, independently, by Mentzen and Siemaszko:

**Theorem 1.2** ([MaSh, Theorem 1], [MeSi, Theorem 2.4]). *If the cocycle on $\mathbb{T}$ is nontrivial and of bounded variation, then the cylinders which it generates have no minimal sets. In particular, they have no discrete orbits.*

Moreover, the set of Besicovitch cocycles for any minimal compact base is first category in the set of all cocycles with zero average – for a proof, see Subsection 1.1.

Given a cylinder $T_f$, we will denote

$$D := \{ x \in X : \text{the } T_f\text{-orbit of } (x, t) \text{ is discrete for every } t \in \mathbb{R} \} = \{ x \in X : \text{the } T_f\text{-orbit of } (x, 0) \text{ is discrete} \}.$$ 

Then the set of points in $X \times \mathbb{R}$ with discrete orbits equals $D \times \mathbb{R}$. Clearly, if $T$ is minimal and $D \neq \emptyset$, then both $D$ and $D \times \mathbb{R}$ are dense in ambient spaces, as $D$ is $T$-invariant.

In [FrLe], the authors construct a Besicovitch cocycle for any minimal rotation of a torus. They also find ones with some special properties, in particular with relatively large $D$.

**Theorem 1.3** ([FrLe]). *For every irrational rotation of $\mathbb{T}$ there exist Besicovitch cocycles ([FrLe, Section 2]). The cocycles can be chosen in such a way that $D$ is uncountable ([FrLe, Proposition 6]). Moreover, for almost every irrational rotation one can find Besicovitch cocycles such that the Hausdorff dimension of $D$ is at least $1/2$ ([FrLe, Theorem 9]).*

It was left as an open problem whether the coefficient $1/2$ could be improved or not. We answer it by developing the techniques from [FrLe]: for every irrational rotation of $\mathbb{T}$ we have obtained Besicovitch cylinders with $D$ of full Hausdorff dimension (Conclusion 4.1). This construction is presented in Section 2.

### 1.1. Nonrecurrent cylinders are first category.

We aim to show that, given a uniquely ergodic homeomorphism of a compact metric space as the base, all cocycles admitting nonrecurrent orbits form a first category set in the space of zero-averaged cocycles (with the uniform topology). In particular, we will prove that Besicovitch cocycles are of first category. Note that a minimal rotation of a compact metric group is uniquely ergodic for the Haar measure.

**Proof.** Denote by $(X, \mu)$ the space, by $T$ a uniquely ergodic homeomorphism thereof, and by $f : X \to \mathbb{R}$ a cocycle. Also, $|p - q|$ will denote the distance between $p, q \in X \times \mathbb{R}$ in the taxicab metric.

Recall that $p \in X \times \mathbb{R}$ is nonrecurrent for $T_f$ if it is not recurrent, i.e. if its positive semi-orbit lies outside some neighbourhood of $p$: there exists $\varepsilon > 0$ such that $|p - T_f^k(p)| \geq \varepsilon$ for every $k > 0$. Thus, all the functions

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4Recall also that the set of discrete orbits is of first category and of measure zero for minimal rotations of tori.
Remark. The proof remains valid for each Banach subspace \( \| \cdot \| \) the supremum distance

\[ N_\varepsilon := \{ f : X \to \mathbb{R} : f \text{ is continuous, } \int_X f \, d\mu = 0, \quad |p - T_k^j(p)| \geq \varepsilon \text{ for some } p \in X \times \mathbb{R} \text{ and all } k > 0 \} \]

To finish the proof, we will show that every \( N_\varepsilon \) is closed and has empty interior – hence their union, by definition, is of first category. From now on, an \( \varepsilon > 0 \) will be fixed.

1. The set \( N_\varepsilon \) has empty interior because the set of coboundaries is dense (by the ergodic theorem for uniquely ergodic homeomorphisms), and the cylinders generated by coboundaries have only recurrent points, so all coboundaries lie outside \( N_\varepsilon \).

2. To prove that \( N_\varepsilon \) is closed, consider a uniformly convergent sequence \( (f_j)_{j=1}^\infty \subset N_\varepsilon \), \( f_j \Rightarrow f \). Let \( p_j \in X \times \mathbb{R} \) be chosen for \( f_j \) as in the definition of \( N_\varepsilon \). We may assume that all \( p_j \) lie in \( X \times \{0\} \), because the dynamic behaviour of a point \( f \) that \( T_f \) does not depend on its second coordinate. Since \( X \) is compact, \( (p_j) \) has an accumulation point, say, \( p_{j_n} \to p \). We will show that this point satisfies the condition from the definition of \( N_\varepsilon \) for \( f \).

By the choice of \( p_{j_n} \), the following holds for all \( k > 0 \) and \( n > 0 \):

\[ \varepsilon \leq |p_{j_n} - T_{f_{j_n}}^k(p_{j_n})| \leq |p_{j_n} - p| + |p - T_j^k(p)| + |T_j^k(p) - T_{f_{j_n}}^k(p_{j_n})| + |T_{f_{j_n}}^k(p_{j_n}) - T_{f_{j_n}}^k(p_{j_n})|. \]

After passing to the limit as \( n \to \infty \) all but the second of the summands vanish. Indeed, this is obvious for the first and the third one. As for the last summand, it follows form the convergence \( f_j \Rightarrow f \): one can easily check that the supremum distance between arbitrary \( T^k_g \) and \( T^k_g' \) equals the supremum distance \( \| g^{(k)} - g'^{(k)} \|_{\text{sup}} \), which is at most \( k\| g - g' \|_{\text{sup}} \), so \( |T_j^k(p_{j_n}) - T_{f_{j_n}}^k(p_{j_n})| \leq k\| f - f_{j_n} \|_{\text{sup}} \to 0 \). This finally proves that \( |p - T_j^k(p)| \geq \varepsilon \) for all \( k > 0 \), and therefore \( f \in N_\varepsilon \). \( \square \)

Remark. The proof remains valid for each Banach subspace \( \mathcal{F} \subset C(X) \), satisfying the ergodic theorem, whose norm is stronger than \( \| \cdot \|_{\text{sup}} \) and on which \( T \) acts as an isometry (in particular, for the space of Hölder continuous functions and for \( \mathcal{C}^k(\mathbb{T}^d) \subset C(\mathbb{T}^d) \)).

2. Construction of a Besicovitch cylinder

Let \( \alpha \) be an irrational number in \([0,1)\) and \((p_n/q_n)_{n \geq 0}\) its sequence of convergents. Recall that then

\[ \frac{1}{2q_nq_{n+1}} < (-1)^n \left( \alpha - \frac{p_n}{q_n} \right) = \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_nq_{n+1}}; \quad (1) \]
For example, we can set $k$. We may also assume that $q$, hence also $f$, superexponentially. Put additionally $n$, $A$, $L$, $q$, $Fib$, $k$, $q$, $n$, and on the other hand $f$, $n$, $A$, $L$, $q$, $Fib$, $k$, $q$, $n$, and the sequences $q_{4n^2+1}$, $Fib_{4n^2+1}$ grow superexponentially. Put additionally

$$A_n := [(3/4)^n q_{1+k_n}] > (3/4)^n q_{1+k_n} - 1 \quad \text{for } n \geq 1. \quad (6)$$

It follows that for $n \geq 2$ on the one hand

$$\frac{A_n q_{1+k_{n-1}}}{A_{n-1} q_{1+k_n}} < \frac{(3/4)^n q_{1+k_{n-1}} - 1}{(3/4)^n q_{1+k_n}} = \frac{3}{4} - \frac{(4/3)^{n-1}}{q_{1+k_n}}$$

and on the other hand

$$\frac{A_{n-1} q_{1+k_n}}{A_n q_{1+k_{n-1}}} > \frac{(3/4)^{n-1} q_{1+k_{n-1}} - 1}{(3/4)^n q_{1+k_{n}} q_{1+k_{n-1}}} = \frac{4}{3} - \frac{(4/3)^{n}}{q_{1+k_{n-1}}}$$

hence altogether

$$1.1 < \frac{q_{1+k_n}}{A_n} : \frac{q_{1+k_{n-1}}}{A_{n-1}} < 25/18. \quad (9)$$

This also proves that the sequence $q_{1+k_n}/A_n$ rises exponentially. \(10\)

For the sake of brevity, we will also denote $L_n := q_{k_n} q_{1+k_n}/n^2$, for $n \geq 1$.

We consider a modification of the example from [FrLe, Section 2]: we define $f_n$ to be $L_n$-Lipschitz, $1/(A_n q_{k_n})$-periodic and even continuous function (hence also $f_n(1/2A_n q_{k_n}) = f_n(x)$) by the formulas:

$$f_n(x) := \begin{cases} 
0, & \text{for } 0 \leq x \leq \frac{1}{12A_n q_{k_n}}, \\
L_n \left(x - \frac{1}{12A_n q_{k_n}}\right), & \text{for } \frac{1}{12A_n q_{k_n}} \leq x \leq \frac{5}{12A_n q_{k_n}}, \\
\frac{q_{1+k_n}}{3A_n n^2}, & \text{for } \frac{5}{12A_n q_{k_n}} \leq x \leq \frac{1}{2A_n q_{k_n}}.
\end{cases}$$
By periodicity:

\[ |f_n(x + \alpha) - f_n(x)| = \left| f_n\left(x + \alpha - \frac{A_n p_{k_n}}{A_n q_{k_n}}\right) - f_n(x)\right| \]

\[ f_n \text{ is } L_n \text{-Lipsch.} \]

\[ \leq L_n \left| \alpha - \frac{p_{k_n}}{q_{k_n}} \right| < \frac{q_{k_n} q_{1+k_n}}{n^2} \left( \frac{1}{q_{k_n} q_{1+k_n}} \right) = \frac{1}{n^2}, \]

so the series

\[ \varphi(x) := \sum_{l=1}^{\infty} (f_l(x + \alpha) - f_l(x)) \]

converges uniformly and yields a continuous cocycle of average zero. Moreover, it is easy to verify that for every \( m \in \mathbb{Z} \):

\[ \varphi^{(m)}(x) = \sum_{l=1}^{\infty} (f_l(x + m\alpha) - f_l(x)), \]

where \( \varphi^{(m)}(x) \) is the second coordinate of \( T_{\varphi}^m(x, 0) \) (we recall that \( T_{\varphi}^m(x, t) = (T^m(x), t + \varphi^{(m)}(x)) \) for every \( x \) and \( t \)).

3. Discrete orbits

Consider, for \( n \geq 1 \) and \( j = 0, \ldots, A_n q_{k_n} - 1 \):

\[ F^{++}_{n,j} := \left[ -\frac{1}{12A_n q_{k_n}}, \frac{1}{12A_n q_{k_n}} \right] + \frac{j}{A_n q_{k_n}}, \]

\[ F^{-+}_{n,j} := \left[ \frac{1}{6A_n q_{k_n}}, \frac{1}{3A_n q_{k_n}} \right] + \frac{j}{A_n q_{k_n}}, \]

\[ F^{--}_{n,j} := \left[ \frac{5}{12A_n q_{k_n}}, \frac{7}{12A_n q_{k_n}} \right] + \frac{j}{A_n q_{k_n}} = F^{++}_{n,j} + \frac{1}{2A_n q_{k_n}}, \]

\[ F^{+-}_{n,j} := \left[ 2 \frac{5}{3A_n q_{k_n}}, \frac{6}{6A_n q_{k_n}} \right] + \frac{j}{A_n q_{k_n}} = F^{-+}_{n,j} + \frac{1}{2A_n q_{k_n}} \]

and for arbitrary \( s_-, s_+ \in \{+,-\} \)

\[ F^{s_--s_+} := \bigcap_{n=1}^{\infty} \bigcup_{j=0}^{A_n q_{k_n} - 1} F^{s_-, s_+}_{n,j}. \]

The sets \( F^{s_--s_+} \) are nonempty and uncountable; indeed, every interval \( F^{s_--s_+}_{n-1,j} \) contains at least

\[ \left| \frac{F^{s_--s_+}_{n-1,j}}{A_n q_{k_n}} \right| - 1 = \left| \frac{A_n q_{k_n}}{6A_n q_{k_n}} \right| - 1 \]

\[ = \left| \frac{1}{6} \frac{A_n q_{1+k_n}}{A_n q_{1+k_n}} \frac{q_{1+k_n}}{q_{1+k_n}} \frac{q_{k_n}}{q_{k_n}} \right| - 1 \geq \left| \frac{1}{6} \frac{18}{25} \frac{25}{25} - 1 \right| = 2 \]

of the intervals \( F^{s_--s_+}_{n,j} \), since the intervals from the \( n \)-th union are uniformly distributed with period \( 1/(A_n q_{k_n}) \); therefore, the intersections \( F^{s_--s_+} \) are topological Cantor sets.
We will now show that the products $F^{s-s_+} \times \mathbb{R}$ consist of discrete points, i.e. points with discrete orbits, which proves that $T_\varphi$ is a Besicovitch cylinder. More precisely, we will show that for every $x \in F^{s-s_+}$:

- if $s_+ = s_-$, then
  $$\varphi^{(m)}(x) \xrightarrow{m \to \pm \infty} s_+ \infty,$$

- if $s_+ \neq s_-$, then
  $$\varphi^{(m)}(x) \xrightarrow{m \to \pm \infty} (-1)^{s_n} s_\pm \infty,$$

where the coefficient $(-1)^{s_n}$ is constant (cf. (9)).

Later, in the next section, we will verify that these sets are of full Hausdorff dimension.

3.1. The case of $F^{++}$. Fix an element $x \in F^{++}$ and an integer $|m| > q_{1+k_1}/(3A_1)$. We wish to bound the summands $f_l(x + m\alpha) - f_l(x)$ from below. To this end, recall that $x$ determines a sequence $(j_l)_{l=1}^\infty$ such that $x \in F^{++}_l$ for every $l \in \mathbb{N}$ and let $x_l$ be given by $x_l := x - j_l/(A_lq_l)$; then $|x_l| \leq 1/(12A_lq_l)$. Now, by the properties of $f_l$

$$f_l(x + m\alpha) - f_l(x) = f_l(x_l + m\alpha) - f_l(x_l) = f_l(x_l + m\alpha)$$

$$= f_l \left( x_l + m\alpha - \frac{mA_l}{A_lq_l} \right) = f_l \left( x_l + m \left( \alpha - \frac{p_{k_l}}{q_{k_l}} \right) \right), \quad (13)$$

which implies that

$$f_l(x + m\alpha) - f_l(x) \geq 0. \quad (14)$$

Because of (13), there exists a unique $n = n(m)$ which satisfies

$$q_{1+k_{n-1}}/(2A_{n-1}) \leq |m| < q_{1+k_n}/(2A_n),$$

and when $|m|$ tends to infinity, so does $n(m)$. Such assumption enables us to estimate the $n$-th summand of $\varphi^{(m)}$:

- if $m \left( \alpha - \frac{p_{k_n}}{q_{k_n}} \right) < q_{1+k_n}/2A_n$, then
  $$\left| f_n(x_n + m\alpha - \frac{mp_{k_n}}{q_{k_n}})A_nq_k \right| < \frac{1}{12} \frac{q_{1+k_n}}{2A_nq_k} < \frac{1}{2} \frac{q_{1+k_n}}{2A_nq_k};$$

- if $m \left( \alpha - \frac{p_{k_n}}{q_{k_n}} \right) > q_{1+k_{n-1}}/2A_{n-1}$, then
  $$\left| f_n(x_n + m\alpha - \frac{mp_{k_n}}{q_{k_n}})A_nq_k \right| > \frac{1}{4A_nq_k} \frac{q_{1+k_{n-1}}}{A_{n-1}q_k} = \frac{9}{50}A_nq_k.$$

Therefore, owing to the bound for $x_n$,

$$\frac{1}{12} + \frac{1}{75} = \frac{9}{50} - \frac{1}{12} < \left| x_n + m\alpha - \frac{mp_{k_n}}{q_{k_n}} \right| A_nq_k \left| < \frac{1}{12} + \frac{1}{2} < 1 - \frac{1}{12} \frac{1}{75} \right.$$

This leads to the bound we seek, since $f_n$ is even, symmetrical and unimodal on $[0, 1/(A_nq_k)]$:

$$f_n \left( x_n + m \left( \alpha - \frac{p_{k_l}}{q_{k_l}} \right) \right) = f_n \left( x_n + m \left( \alpha - \frac{p_{k_l}}{q_{k_l}} \right) \right)$$

$$> f_n \left( \frac{1}{12A_nq_k} + \frac{1}{75A_nq_k} \right) = L_n \cdot \frac{1}{75A_nq_k} = \frac{q_{1+k_n}}{75A_nq_k} \to \infty,$$
3.2. The case of $F^{-\infty}$. The behaviour of functions $f_n$ on the set $F_{n,j}^{-\infty}$ is symmetrical to the situation on $F_{n,j}^{++}$, and the calculations are analogous.

3.3. The case of $F^{++}$ and $F^{-\infty}$. Choose an $x \in F^{++} \cup F^{-\infty}$. Again, there is $(j_l)_{l=1}^\infty$ such that $x \in F_{i,l}^{++} \cup F_{i,l}^{-\infty}$ for every $l \in \mathbb{N}$, and we denote by $x_l$ the respective “reductions” $x - j_l/(A_l q_k_l)$; then

$$x_l \in \left[\frac{1}{6A_l q_k_l}, \frac{1}{3A_l q_k_l}\right] \quad (s_+ = +) \quad \text{or} \quad x_l \in \left[\frac{2}{3A_l q_k_l}, \frac{5}{6A_l q_k_l}\right] \quad (s_+ = -).$$

Additionally, fix an integer $|m| > q_{1+k_l}/(12A_1)$. It follows from the periodicity of $f_l$

$$f_l(x + m \alpha) - f_l(x) = f_l(x_l + m \alpha) - f_l(x_l) = f_l(x_l + m(\alpha - p_{k_l}/q_{k_l})) - f_l(x_l). \quad (15)$$

We remind that $\text{sign}(\alpha - p_{k_l}/q_{k_l}) = (-1)^{k_l}$ and $(-1)^{k_l}$. Take now $n = n(m) > 1$ for which

$$q_{1+k_n}/(12A_n) \leq |m| < q_{1+k_{n+1}}/12A_{n+1}. \quad (16)$$

These constraints along with the inequalities (11) imply that for $l > n$

$$\left| m \left( \alpha - \frac{p_{k_l}}{q_{k_l}} \right) \right| < \frac{q_{1+k_{n+1}}}{12A_{n+1}} \cdot \frac{1}{q_{k_l} q_{1+k_l}} \leq \frac{q_{1+k_l}}{12A_l} \cdot \frac{1}{q_{k_l} q_{1+k_l}} = \frac{1}{12A_l q_{k_l}}. \quad (17)$$

Therefore, both arguments $x_l + m(\alpha - p_{k_l}/q_{k_l})$ and $x_l$ lie in the same interval of linearity (and monotonicity) of $f_l$, so the sign of the difference (15) equals $(-1)^{k_l} s_+ \text{sign} m$ (it does not depend on $l$) and the expression (15) can be estimated:

$$\left| f_l \left( x_l + m \left( \alpha - \frac{p_{k_l}}{q_{k_l}} \right) \right) - f_l(x_l) \right| = L_l \left| m \left( \alpha - \frac{p_{k_l}}{q_{k_l}} \right) \right| \quad \geq \frac{q_{k_l} q_{1+k_l}}{l^2} \cdot \frac{q_{1+k_n}}{12A_n} \cdot \frac{1}{2q_{k_l} q_{1+k_l}} = \frac{q_{1+k_n}}{24A_n l^2}. \quad \text{(18)}$$

Since all these differences are of the same sign, this yields an estimate for the part of the sum (11) with $l > n$:

$$\left| \sum_{l>n} \left( f_l \left( x_l + m \left( \alpha - \frac{p_{k_l}}{q_{k_l}} \right) \right) - f_l(x_l) \right) \right| > \frac{q_{1+k_n}}{24A_n} \sum_{l>n} \frac{1}{l^2} \geq \frac{q_{1+k_n}}{25A_n n^2}. \quad (19)$$
where the inequality (⋆) holds for \( n \) large enough, which results form the 
fact that the remainder \( \sum_{l>n} 1/l^2 \) is asymptotically equivalent to \( 1/n \) (thus 
greater than \( 24/(25n) \) for large \( n \))\(^{5}\).

As it occurs, we do not have to work hard to take the remaining summand into account – it suffices to subtract the upper bounds of the functions \( f_l \):

\[
\left| \sum_{l \leq n} (f_l(x_l + m\alpha) - f_l(x_l)) \right| \leq \sum_{l \leq n} 2 \max_{x \in T} f_l = \frac{2}{3} \sum_{l \leq n} \frac{q_{l+k_l}}{A_l l^2} \quad (18)
\]

Note that this sum behaves roughly like the sum of a finite geometric series:

since \( q_{1+k_l}/A_l \) grows exponentially and, asymptotically, \( l^2 \) grows slower, the 
quotient for large \( l \) also grows exponentially, say:

\[
\frac{q_{1+k_l}}{A_l l^2} \geq C \frac{q_{1+k_{l-1}}}{A_{l-1} (l-1)^2} \quad \text{for some } C > 1 \text{ and } l \text{ large enough}
\]

(e.g. when \( l^2/(l-1)^2 < 1.1/C \)). Then, indeed, the sum \((18)\) is of order of its 
largest term, and therefore we arrive at a satisfactory bound:

\[
\frac{2}{3} \sum_{l \leq n} \frac{q_{1+k_l}}{A_l l^2} \leq \frac{2}{3} \sum_{l \leq n} \frac{1}{C^{n-1}} \cdot \frac{q_{1+k_{n}}}{A_n n^2} \leq \frac{2}{3} \cdot \frac{C}{C - 1} \cdot \frac{q_{1+k_{n}}^{(⋆⋆)}}{50 A_n n} \quad (19)
\]

where the inequality (⋆⋆) also holds for large \( n \). Combining the estimations \((17)\), \((18)\) and \((19)\), we eventually obtain the required divergence:

\[
|\varphi^{(m)}(x)| \quad (11,15) \sum_{l \geq 1} \left| (f_l (x_l + m (\alpha - p_{k_l}/q_{k_l})) - f_l(x_l)) \right| \\
\geq \sum_{l>n(m)} \cdots \left| 25 A_n n \right| / (25 A_n n) - q_{1+k_{n}} / 50 A_n n = \frac{q_{1+k_{n}}}{50 A_n n} \quad (19) \\
\rightarrow m \rightarrow \infty.
\]

Also, the sign of \( \varphi^{(m)} \) is correct, because the prevailing part has correct sign.

Remark 3.1. Observe that the calculations for \( F^{+-} \) and \( F^{-+} \) (in this and 
the previous section) do not require all the assumptions on \( k_n \) and \( A_n \) that 
we have made initially. Actually, we only need that \( k_n \) are all of the same 
parity, \( q_{1+k_{n}}/A_n \) grows at least geometrically, and \( A_n q_{k_n} \geq 18 A_n - 1 \). \( q_{k_n} \) \(- \) for example, we may put \( A_n := 1 \) for every \( n \) (then we have to ensure the 
equality \( q_{k_n} \geq 18 q_{k_{n-1}} \)). Moreover, we do not use the pieces of constant 
value of the functions \( f_n \). Summarizing, the sets \( F^{+-} \) and \( F^{-+} \) also consist of 
discrete points in the following example from [FrLe] Section 2]:

\[
\varphi(x) := \sum_{n=1}^{\infty} (g_n(x + \alpha) - g_n(x))
\]

\(^{5}\)This follows from the termwise equivalence to a telescoping series of \( 1/n \):

\[
\sum_{l \geq n+1} \frac{1}{(l+1)^2} - \sum_{l \geq n+2} \frac{1}{(l+2)^2} = 1/(n+1)^2 \approx (1/n) - 1/(n+1),
\]

and from an analogue of the Stolz-Cesàro Theorem.
where \( g_n \) are \( L_n \)-Lipschitz, \( 1/q_k \)-periodic continuous functions:

\[
g_n(x) := \begin{cases} 
L_n x, & \text{for } 0 \leq x \leq \frac{1}{2q_k}, \\
L_n \left( \frac{1}{q_k} - x \right), & \text{for } \frac{1}{2q_k} \leq x \leq \frac{1}{q_k}, 
\end{cases}
\]  

(20)

and \( q_k \geq 18q_{k-1} \) (this coefficient can be decreased by widening \( F_{n,j}^{s-s+} \) appropriately).

4. Hausdorff dimension of \( F_{n,j}^{s-s+} \)

To compute the Hausdorff dimension of \( F_{n,j}^{s-s+} \), we will use methods from [Fa] (Example 4.6 and Proposition 4.1):

Consider a sequence of unions of a finite number of disjoint closed intervals in \([0,1]\) (here: the sequence \((\bigcup_{j=0}^{n-1} F_{n,j}^{s-s+})_{n \geq 1}\)). Suppose that the intervals of the \( n \)-th union \((n \geq 1)\)

- are of length at most \( \delta_n \) and \( \delta_n \to 0 \),
- are separated by gaps of length at least \( \varepsilon_n \) (with \( \varepsilon_n > \varepsilon_{n+1} > 0 \)),
- contain at least \( m_{n+1} \geq 2 \) and at most \( m_{n+1} \) intervals of the \((n+1)\)-st union.

Then the Hausdorff dimension of the intersection of this sequence lies between the following two numbers:

\[
\liminf_{n \to \infty} \frac{\log(2m_2 \cdots m_{n+1})}{\log(m_1 \cdots m_{n+1})} \leq \liminf_{n \to \infty} \frac{\log(m_1 \cdots m_{n+1})}{\log(m_2 \cdots m_{n+1})}.
\]

First, note that \( \delta_n = |F_{n,j}^{s-s+}| = 1/(6A_n q_k) \to 0 \). Next, observe that

\[
\varepsilon_n = \frac{1}{A_n q_k} - |F_{n,j}^{s-s+}| > \frac{1}{A_n q_k} - \frac{1}{6A_n q_k} > \frac{1}{2A_n q_k}.
\]

As for \( m_n \) and \( m_{n+1} \), we have already checked that \( m_n \geq 2 \) (see (12)), but we need a more precise estimate. Using the inequality \( |t| - 1 > t/2 \) for \( t \geq 3 \), we conclude that:

\[
m_n \geq \frac{|F_{n-1,j}^{s-s+}|}{1/A_n q_k} - 1 = \frac{A_n q_k}{6A_{n-1} q_{k_n+1}} - 1 \geq \frac{A_n q_k}{12A_{n-1} q_{k_n+1}}.
\]

On the other hand, only one more interval can fit into:

\[
m_n \leq \frac{|F_{n-1,j}^{s-s+}|}{1/A_n q_k} \leq \frac{A_n q_k}{6A_{n-1} q_{k_n+1}}.
\]

Consequently:

\[
m_2 \cdots m_n \geq \frac{A_2 q_{k_2}}{12A_1 q_{k_1}} \cdots \frac{A_n q_{k_n}}{12A_{n-1} q_{k_{n-1}}} = \frac{A_n q_{k_n}}{12^{n-1} A_1 q_{k_1}},
\]

\[
m_{n+1} \varepsilon_{n+1} \geq \frac{1}{12} \frac{A_{n+1} q_{k_{n+1}}}{A_n q_{k_n}} \cdot \frac{1}{2A_{n+1} q_{k_{n+1}}} = \frac{1}{24A_n q_{k_n}},
\]

\[
\overline{m_2} \cdots \overline{m_n} \leq \frac{A_n q_{k_n}}{6^{n-1} A_1 q_{k_1}}.
\]
hence eventually
\[
\dim_H F^{s-s+} \geq \liminf_{n \to \infty} \frac{\log A_n q_{k_n} - (n - 1) \log 12 - \log A_1 q_1}{\log A_n q_{k_n} + \log 24} = 1 - \limsup_{n \to \infty} \frac{n}{n \log A_n q_{k_n}} \cdot \log 12,
\]
\[
\dim_H F^{s-s+} \leq \liminf_{n \to \infty} \frac{\log A_n q_{k_n} - (n - 1) \log 6 - \log A_1 q_1}{\log A_n q_{k_n} + \log 6} = 1 - \limsup_{n \to \infty} \frac{n}{n \log A_n q_{k_n}} \cdot \log 6.
\]

Let us remark that the coefficient 12 can be lowered nearly to 6, if \(m_n\) are larger. Nevertheless, under the assumption (4) the dimension equals 1.

**Conclusion 4.1.** For every irrational rotation of \(\mathbb{T}\) there exists a Besicovitch cocycle such that the set \(D \times \mathbb{R} (\supset F^{s-s+} \times \mathbb{R})\) of discrete points of the respective cylinder has Hausdorff dimension two.

## 5. Discrete Devaney Chaos

The sole property of transitivity is enough for some dynamicists to call a dynamical system chaotic. However, over the years multiple definitions for chaos have been proposed. Let us recall the notion of the Devaney chaos, one of the most popular ones: a dynamical system \((X, T)\) on a metric space \((X, d)\) is chaotic in the sense of Devaney if:

1. it is transitive,
2. the set of periodic points is dense,
3. the system is sensitive, i.e. there are points around every point \(x \in X\) (arbitrarily close) whose orbits at least once diverge far enough from the orbit of \(x\): there is \(\varepsilon > 0\) such that for every \(x \in X\) and \(\delta > 0\) there are \(n > 0\) and \(y\) with \(d(x, y) < \delta\) and \(d(T^n(x), T^n(y)) > \varepsilon\).

We remind that the the last condition follows from the remaining ones, if \(X\) is infinite (\cite{BaBC}, main theorem, or \cite{GlWe}, Corollary 1.4).

It occurs that the dynamical systems we consider in this article satisfy a bit more general condition, namely, with “periodic orbits” replaced by “discrete orbits” (note that both notions are equivalent in compact spaces). We will call this property discrete Devaney chaos and check this fact in a moment. A similar generalization was proposed in \cite{GlWe}, with “almost periodic” (that is, contained in a minimal set) instead of “periodic” and it was shown that this, combined with transitivity, implies sensitivity, if \(X\) is compact. Note also, that there are no periodic points in cylinders over minimal rotations, so they cannot be Devaney chaotic.

Recall first that a space or a set is **boundedly compact** if bounded closed subsets are always compact.\(^6\) In particular, closed subsets of Euclidean

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\(^6\)Such spaces are also given other names in the literature: they are called proper, finitely compact, totally complete, Heine-Borel or having the Heine-Borel property (not to be confused with the Heine-Borel [covering] property, or precompactness, that is, “every open cover has a finite subcover”).
spaces are boundedly compact. All such spaces are complete and separable. Also, a system is called maximally sensitive, if it sensitive with every $\varepsilon < \frac{\text{diam}(X)}{2}$, and maximally chaotic, if it is Devaney chaotic and maximally sensitive (definitions introduced in [AlPr]).

**Theorem 5.1.** Let $X$ be an infinite, boundedly compact space without isolated points, and let $T$ be transitive with dense set of discrete points. Then the system is sensitive. If, moreover, the set of discrete nonperiodic points is dense, then the system is maximally sensitive.

**Proof.** Since $X$ is complete, separable and without isolated points, the system is even positively transitive (it has a dense semi-orbit – see [Ox], p. 70; the proof was recalled in [Dy], Proposition 2.1). The set of discrete points consists of periodic points and nonperiodic discrete points, both of which are invariant. Thus, one of these sets contains a positively transitive point in its closure, and so it is dense. If periodic points are dense, then, by [BaBC] or [GIMW], the system is sensitive. The new result is when the second set is dense, what we assume henceforth.

Any infinite (= nonperiodic) discrete orbit, by bounded compactness, has no bounded subsequence, so $\text{diam}(X) = \infty$. Fix then any $x \in X$, $\varepsilon > 0$ and $\delta > 0$. In the $\delta$-neighbourhood of $x$ there is a point $y_1$ with dense semi-orbit and a discrete nonperiodic point $y_2$. Then, for infinitely many $n > 0$ the orbit of $y_1$ returns to $x$: $d(T^n(y_1), x) < \varepsilon$, and on the other hand, for $n$ large enough the orbit of $y_2$ stays far away from $x$: $d(T^n(y_2), x) > 3\varepsilon$ (by bounded compactness again). Consequently, for some $n > 0$: $d(T^n(y_1), T^n(y_2)) > 2\varepsilon$, and hence $d(T^n(x), T^n(y_1)) > \varepsilon$ or $d(T^n(x), T^n(y_2)) > \varepsilon$. □

**Remark 5.2.** The Besicovitch cylinders that we consider are of course transitive and have a dense set of discrete points (we have found discrete points in $F^{s-8} \times \mathbb{R}$, but their orbits are dense). Therefore, there are examples of maximally discretely chaotic systems with full-dimensional set of relatively “regular” (almost periodic, discrete) points. This feature seems not to be studied so far. However, there are results about full Hausdorff dimension of the set of points with nondense orbits, although they are rather concerned with bounded orbits – see e.g. [KL, Ur].

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