Mathematical foundations of quantum information: Measurement and foundations

MASANAO OZAWA
Graduate School of Information Science
Nagoya University, Chikusa-ku, Nagoya, 464-8601, Japan

Abstract
The purpose of this paper is to survey mathematical foundations of quantum information. We present an axiomatic construction of quantum measurement theory based on completely positive map valued measures, a universally valid new formulation, the Ozawa formulation, of the uncertainty principle for error and disturbance in measurements, the Wigner-Araki-Yanase limit of measurements and the accuracy limit of quantum computing based on conservation laws, and quantum interpretation based on quantum set theory.

2000 Mathematics Subject Classification: Primary 81-02; Secondary 81P15, 81P68, 81P10

1 Introduction
Quantum mechanics was discovered in the beginning of the 20th century and has revealed that nature is ruled by an uncertain existence called the quantum state. Various paradoxes including Schrödinger’s cat and the Einstein-Podolsky-Rosen (EPR) paradox were derived and overturned our common sense. Nevertheless, quantum mechanics has been crowned with marvelous success in description and prediction for phenomena originated from the microscopic world, and produced an enormous technology of electronics in the latter half of the 20th century.

It is the discovery of laser in 1960 that opened a way of controlling the quantum state that had been a mere hypothesis on the microscopic world to explain our experience such as the stability of atoms. Quantum mechanics started to play a new role of describing the limitation of our ability of controlling the external world. Moreover, by precisely describing the limitation of eavesdropper’s ability unconditionally secure quantum cryptography has recently been developed. This new aspect of quantum mechanics emerges a new research field called quantum information, which has a close connection with information science. Since Shor [46] discovered an algorithm efficiently solving prime factorization by quantum computers, the research on quantum information has made a great progress and produced various proposals on applications to quantum information technology including quantum computing and quantum cryptography.

The purpose of this paper is to survey mathematical foundations of quantum information. In particular, we discuss the most foundational aspect of quantum information centered at quantum measurement theory. It should be emphasized that the new framework of quantum
information has solved not only technological problems relative to computing and communication, but also several problems on foundations of quantum mechanics, having been left unsolved since the emergence of quantum mechanics in the 1920s, and we focus more on the latter aspect of quantum information research.

In Section 2, we discuss quantum measurement, one of the most fundamental notions in quantum information. Von Neumann’s axiomatization [50] of quantum mechanics has answered what are quantum states and what are quantum observables, but left unanswered what are quantum measurements. In the 1970s, a new mathematical theory was emerged on such notions as probability operator-valued measures (POVMs), operations, and instruments for describing various aspects of quantum measurements, and the problem of mathematical characterization of the notion of quantum measurement was completely solved based on those notions [18, 19]. This theory is now an indispensable part of quantum information theory. In Section 3, we discuss the uncertainty principle. In 1927, Heisenberg introduced the uncertainty principle that describes the inevitable amount of disturbance caused by the back action of a measurement and sets a limitation for simultaneous measurements of non-commuting observables. However, his quantitative relation has been revealed not to be universally valid [37], through the debate on the standard quantum limit for gravitational wave detection induced by measurement back action [23, 25]. We discuss the above-mentioned debate and a new universally valid formulation of the uncertainty principle obtained recently [38, 39, 40, 41]. In Section 4, we give an outline of the recent study of the accuracy limits for quantum computation. This result was obtained by quantitatively generalizing the Wigner-Araki-Yanase theorem on the limitation of measurement under conservation laws by using the above new universally valid uncertainty principle [35, 36, 39]. In Section 5, we outline the recent investigation on interpretation of quantum mechanics. We shall discuss simultaneous measurability of non-commuting observables based on the new uncertainty principle [42, 43] and a new interpretation of quantum mechanics based on quantum set theory [44].

2 Quantum measurement theory

2.1 Axioms for quantum mechanics

Axioms for quantum mechanics due to von Neumann [50] are formulated as follows.

Axiom 1 (Axiom for states and observables). To every quantum system S uniquely associated is a Hilbert space $\mathcal{H}$ called the state space of $S$. States of $S$ are represented by density operators, positive operators with unit trace, on $\mathcal{H}$ and observables of $S$ are represented by self-adjoint operators on $\mathcal{H}$.

We follow the convention that the inner product on a Hilbert space is linear in the first variable and conjugate linear in the second. A state of the form $\rho = |\psi\rangle\langle\psi|$ is called a pure state with a state vector $\psi$, where the operator $|\xi\rangle\langle\eta|$ with $\xi, \eta \in \mathcal{H}$ is defined by $|\xi\rangle\langle\eta|\psi = \langle\eta, \psi|\xi$. We denote by $\mathcal{S}(\mathcal{H})$ the space of density operators on $\mathcal{H}$. In this paper, we further assume that every self-adjoint operator on $\mathcal{H}$ has a corresponding observable of $S$; the resulting theory is often called a non-relativistic quantum mechanics without superselection rule.
In what follows, we denote by $\mathcal{B}(\mathbb{R})$ the set of Borel subsets of $\mathbb{R}$ and by $E^A$ the spectral measure of a self-adjoint operator $A$.

**Axiom 2** (Born statistical formula). If an observable $A$ is measured in a state $\rho$, the probability distribution of the outcome $x$ is given by

$$
\Pr\{x \in \Delta \| \rho\} = \text{Tr}[E^A(\Delta) \rho],
$$

(2.1)

where $\Delta \in \mathcal{B}(\mathbb{R})$.

From the above axiom, if $A\rho$ is a trace-class operator, the expectation value is given by $E_x[A \| \rho] = \text{Tr}[A \rho]$, and if $A^{1/2}\rho$ is a Hilbert-Schmidt-class operator, the standard deviation is given by $\sigma(A \| \rho)^2 = \text{Tr}[(A^{1/2}\rho)^2] - \text{Tr}[A\rho]^2$. Henceforth, $\tau_c(\mathcal{H})$ denotes the space of trace-class operators on $\mathcal{H}$ and $\sigma_c(\mathcal{H})$ denotes the space of Hilbert-Schmidt-class operators.

In what follows, $\hbar$ denotes the value of the Planck constant in the unit system under consideration divided by $2\pi$.

**Axiom 3** (Axiom of time evolution). Suppose that a system $S$ is an isolated system with the Hamiltonian $H$ from time $t$ to $t + \tau$. If $S$ is in a state $\rho(t)$ at time $t$, then $S$ is in the state $\rho(t + \tau)$ at time $t + \tau$ such that

$$
\rho(t + \tau) = e^{-iH\tau/\hbar} \rho(t) e^{iH\tau/\hbar}.
$$

(2.2)

**Axiom 4** (Axiom of composition). The state space of the composite system $S = S_1 + S_2$ consisting of the system $S_1$ with the state space $\mathcal{H}_1$ and of the system $S_2$ with the state space $\mathcal{H}_2$ is the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$, every observable $A$ of $S_1$ is identified with the observable $A \otimes 1$ of $S$ and every observable $B$ of $S_2$ is identified with the observable $1 \otimes B$ of $S$.

### 2.2 Von Neumann-Lüders projection postulate

Under the above axioms, we can make a probabilistic prediction on the result of a future measurement from the knowledge of the past state. However, such a prediction applies only to one measurement in the future. If we make many measurements successively, we need another axiom to determine the state after the measurement. In the conventional approach, the following hypothesis has been accepted [50, 14].

**Postulate 1** (Von Neumann-Lüders projection postulate). Under the condition that a measurement of an observable $A$ in a state $\rho$ leads to the outcome $x = x$, the state $\rho_{(x=x)}$ just after the measurement is given by

$$
\rho_{(x=x)} = \frac{E^A(\{x\}) \rho E^A(\{x\})}{\text{Tr}[E^A(\{x\}) \rho]}.
$$

(2.3)

In order to find the state change caused by a measurement, von Neumann discussed a feature of the Compton-Simons experiment [51] pages 212–214 and generalized it to pose the *repeatability hypothesis* [51] page 335.

**Postulate 2** (Repeatability hypothesis). If the physical quantity is measured twice in succession in a system, then we get the same value each time.
Then, from the repeatability hypothesis von Neumann showed that the state change caused by a measurement of an observable with non-degenerate discrete spectrum satisfies the von Neumann-Lüders projection postulate (projection postulate, for short). While von Neumann showed that if the spectrum is degenerate, the repeatability hypothesis is not sufficient to determine the state change uniquely, Lüders [14] later introduced the projection postulate as the state change caused by a sort of canonical measurement.

It is well-known that there are many ways to measure the same observable that do not satisfy the projection postulate. Thus, this hypothesis is not taken to be a universal axiom but a defining condition for a class of measurement. We say that a measurement is projective if it satisfies the projection postulate.

For any sequence of projective measurements, we can determine the joint probability distribution of the outcomes of measurements [53].

**Theorem 1** (Wigner’s formula). Let $A_1, \ldots, A_n$ be observables with discrete spectrum of a system $S$ in a state $\rho$ at time 0. If one carries out projective measurements of observables $A_1, \ldots, A_n$ at times $(0 < t_1 < \cdots < t_n)$ and otherwise leaves the system $S$ isolated with the Hamiltonian $H$, then the joint probability distribution of the outcomes $x_1, \ldots, x_n$ of those measurements is given by

$$
\Pr\{x_1 = x_1, \ldots, x_n = x_n \| \rho\} = \mathsf{Tr}[E_{A_n}(\{x_n\}) \cdots U(t_2 - t_1) E_{A_1}(\{x_1\}) U(t_1) \rho \times U(t_1) \dagger E_{A_1}(\{x_1\}) U(t_2 - t_1) \dagger \cdots E_{A_n}(\{x_n\})],
$$

where $U(t) = e^{-iHt/\hbar}$.

The projection postulate can be applied to quite a restricted class of measurements, and has the following problems if we take it to be a basis of quantum mechanics.

(i) The projection postulate cannot be applied to observables with continuous spectrum [19, 21].

(ii) There exist commonly used measurements of discrete observables that do not satisfy the projection postulate such as photon counting [10].

(iii) There is a useful class of measurements that cannot be represented by observables but by the so-called POVMs (probability operator valued measures) [9].

2.3 Davies-Lewis instruments

State changes induced by measurements have been called quantum state reductions and considered as one of the most difficult notions in quantum mechanics. In order to apply quantum mechanics to the system to be measured sequentially, we need to mathematically characterize all the possible state changes induced by measurements.

If we are given the probability distribution $\Pr\{x = x \| \rho\}$ of the outcome and the quantum state reduction $\rho \mapsto \rho_{\{x=x\}}$, the “non-selective” state change caused by this measurement is given by

$$
\rho \mapsto T(\rho) = \sum_{x \in \mathbb{R}} \Pr\{x = x \| \rho\} \rho_{\{x=x\}}.
$$

(2.5)
If the measurement is a projective measurement of a discrete observable \( A \), this amounts to a mapping on the space \( \tau_c(\mathcal{H}) \) of trace-class operators on \( \mathcal{H} \) such that

\[
T(\rho) = \sum_{x \in \mathbb{R}} E^A(\{x\}) \rho E^A(\{x\}).
\]  

(2.6)

Nakamura and Umegaki [16] pointed out the analogy between quantum state reductions and the notion of conditional expectation in probability theory by showing that the dual map \( T^* \) of \( T \) is a normal norm one projection, called a conditional expectation [49, 47], from the algebra \( \mathcal{L}(\mathcal{H}) \) of bounded operators on \( \mathcal{H} \) to the commutant \( \{A\}' \) of \( A \), where \( \{A\}' = \{E^A(\Delta) \mid \Delta \in \mathcal{B}(\mathbb{R})\}' \) if \( A \) is unbounded, and suggested that the state change caused by a measurement can be represented by such a conditional expectation from \( \mathcal{L}(\mathcal{H}) \) to \( \{A\}' \) even if the observable \( A \) has continuous spectrum. However, Arveson [2] showed that such a conditional expectation does not exist if \( A \) has continuous spectrum. Based on the above results, Davies and Lewis [7] proposed a general framework for considering all the physically possible state changes caused by measurements by abandoning the repeatability hypothesis as the primary principle. A Davies-Lewis (DL) instrument for a Hilbert space \( \mathcal{H} \) is a measure \( I \) on the \( \sigma \)-field \( \mathcal{B}(\mathbb{R}) \) of Borel subsets of the real line \( \mathbb{R} \) with values in positive linear maps on the space \( \tau_c(\mathcal{H}) \) of trace-class operators on \( \mathcal{H} \), countably additive in the strong operator topology, i.e., \( I(\bigcup_n \Delta_n) = \sum_n I(\Delta_n) \rho \) for all \( \rho \in \tau_c(\mathcal{H}) \) and disjoint sequence \( \{\Delta_n\} \) in \( \mathcal{B}(\mathbb{R}) \), and normalized so that \( I(\mathbb{R}) \) is trace-preserving, i.e., \( \text{Tr}[I(\mathbb{R})\rho] = \text{Tr}[\rho] \) for all \( \rho \in \tau_c(\mathcal{H}) \). A simple example of a DL instrument is given by state change caused by a projective measurement of a discrete observable \( A \) by

\[
I(\Delta)\rho = \sum_{x \in \Delta} E^A(\{x\}) \rho E^A(\{x\})
\]

(2.7)

for all \( \Delta \in \mathcal{B}(\mathbb{R}), \rho \in \tau_c(\mathcal{H}) \).

Let \( \mathcal{S} \) be a system described by a Hilbert space \( \mathcal{H} \). Consider a physically realizable measuring apparatus and denote it by \( A(\mathbf{x}) \). Here, \( \mathbf{x} \) represent the output variable of this apparatus and we assume it is real valued. In quantum mechanics we cannot predict the value of outcome of each measurement and we can only deal with its statistical properties. The statistical properties of the apparatus \( A(\mathbf{x}) \) are determined by (i) the probability distribution \( \text{Pr}\{\mathbf{x} \in \Delta \mid \rho\} \) of the outcome \( \mathbf{x} \) in an arbitrary state \( \rho \) and (ii) the state \( \rho_{\{\mathbf{x} \in \Delta\}} \) just after the measurement under the condition that the outcome satisfies \( \mathbf{x} \in \Delta \), where \( \rho_{\{\mathbf{x} \in \Delta\}} \) is defined for all \( \Delta \in \mathcal{B}(\mathbb{R}) \) with \( \text{Pr}\{\mathbf{x} \in \Delta \mid \rho\} > 0 \), and represents an indefinite state otherwise. Thus, we assume the following postulate.

**Postulate 3** (Statistical properties of apparatuses). To every apparatus \( A(\mathbf{x}) \) for \( \mathcal{H} \) uniquely associated are a probability measure \( \rho \mapsto \text{Pr}\{\mathbf{x} \in \Delta \mid \rho\} \) for any \( \rho \in \mathcal{S}(\mathcal{H}) \) and a density operator \( \rho_{\{\mathbf{x} \in \Delta\}} \) for any \( \rho \in \mathcal{S}(\mathcal{H}) \) and \( \Delta \in \mathcal{B}(\mathbb{R}) \) with \( \text{Pr}\{\mathbf{x} \in \Delta \mid \rho\} > 0 \).

The proposal of Davies and Lewis can be stated as follows.

**Postulate 4** (Davies-Lewis thesis). For every apparatus \( A(\mathbf{x}) \) with output variable \( \mathbf{x} \) there exist uniquely a DL instrument \( I \) satisfying

\[
\text{Pr}\{\mathbf{x} \in \Delta \mid \rho\} = \frac{\text{Tr}[I(\Delta)\rho]}{\text{Tr}[I(\Delta)\rho]},
\]

(2.8)

\[
\rho_{\{\mathbf{x} \in \Delta\}} = \frac{I(\Delta)\rho}{\text{Tr}[I(\Delta)\rho]}.
\]

(2.9)
In what follows, we shall discuss a justification of the Davies-Lewis thesis following [41]. Let \( A(x) \) and \( A(y) \) be two measuring apparatuses with the output variables \( x \) and \( y \), respectively. Consider the successive measurements by \( A(x) \) and \( A(y) \), carried out first by \( A(x) \) for the system \( S \) in a state \( \rho \) immediately followed by \( A(y) \) for the same system \( S \). Then, the joint probability distribution \( \Pr\{x \in \Delta, y \in \Gamma||\rho\} \) \( x \) and \( y \) is given by

\[
\Pr\{x \in \Delta, y \in \Gamma||\rho\} = \Pr\{y \in \Gamma||\rho_{x\in\Delta}\} \Pr\{x \in \Delta||\rho\}. \tag{2.10}
\]

It is natural to assume that the above joint probability distribution has the following property.

**Postulate 5 (Mixing law for joint output probability).** For any successive measurements carried out by apparatuses \( A(x) \) and \( A(y) \) in this order, the joint probability distribution \( \Pr\{x \in \Delta, y \in \Gamma||\rho\} \) of output variables \( x \) and \( y \) is an affine function in \( \rho \).

This assumption can be justified as follows. Since if the system \( S \) is in a state \( \rho_1 \) with probability \( p \) and in a state \( \rho_2 \) with probability \( 1-p \), then the joint probability distribution is given by

\[
P = p \Pr\{x \in \Delta, y \in \Gamma||\rho_1\} + (1-p) \Pr\{x \in \Delta, y \in \Gamma||\rho_2\}.
\]

On the other hand, in this case the system \( S \) is in the state \( \rho = p\rho_1 + (1-p)\rho_2 \), and hence the same probability is given also by

\[
P = \Pr\{x \in \Delta, y \in \Gamma||\rho\}.
\]

This conclude the Mixing law for joint output probability.

From the postulate for statistical properties of apparatuses, to any \( \Delta \in \mathcal{B}(\mathcal{R}) \) and \( \rho \in \tau_c(\mathcal{H}) \) corresponds uniquely a trace-class operator

\[
\mathcal{I}(\Delta, \rho) = \Pr\{x \in \Delta||\rho\}\rho_{x\in\Delta}. \tag{2.11}
\]

Suppose, in particular, that \( A(y) \) is a measuring apparatus for a measurement of a projection \( E \). Then, by Eq. (2.10) and Eq. (2.11), we have

\[
\Pr\{x \in \Delta, y \in \{1\}||\rho\} = \Pr\{y \in \{1\}||\rho_{x\in\Delta}\} \Pr\{x \in \Delta||\rho\} = \Tr[E\rho_{x\in\Delta}] \Pr\{x \in \Delta||\rho\} = \Tr[E \Pr\{x \in \Delta||\rho\}\rho_{x\in\Delta}].
\]

Thus, we have

\[
\Pr\{x \in \Delta, y \in \{1\}||\rho\} = \Tr[E\mathcal{I}(\Delta, \rho)]. \tag{2.12}
\]

Since \( E \) is arbitrary, the mapping \( \rho \mapsto \mathcal{I}(\Delta, \rho) \) is an affine mapping from \( \tau_c(\mathcal{H}) \) to \( \tau_c(\mathcal{H}) \), so that it uniquely extends to a positive linear map from \( \tau_c(\mathcal{H}) \) to \( \tau_c(\mathcal{H}) \). The finite additivity of the function \( \Delta \mapsto \mathcal{I}(\Delta, \rho) \) follows from the countable additivity of \( \Delta \mapsto \Pr\{x \in \Delta, y \in \{1\}||\rho\} \). Let \( \{\Delta_n\} \) be an increasing sequence in \( \mathcal{B}(\mathcal{R}) \) such that \( \bigcup_n \Delta_n = \Delta \). Then, for any \( \rho \in \mathcal{S}(\mathcal{H}) \) we have

\[
\lim_{n \to \infty} \|\mathcal{I}(\Delta, \rho) - \mathcal{I}(\Delta_n, \rho)\|_{\tau_c} = \Tr[\mathcal{I}(\Delta, \rho)] - \lim_{n \to \infty} \Tr[\mathcal{I}(\Delta_n, \rho)] = \Pr\{x \in \Delta||\rho\} - \lim_{n \to \infty} \Pr\{x \in \Delta_n||\rho\} = 0,
\]
where $\| \cdot \|_{\tau c}$ denotes the trace norm on $\tau c(\mathcal{H})$. Thus, the mapping $\Delta \mapsto \mathcal{I}(\Delta, \rho)$ is countably additive in trace norm for any $\rho \in \mathcal{S}(\mathcal{H})$. Since $\tau c(\mathcal{H})$ is linearly generated by $\mathcal{S}(\mathcal{H})$, this is the case for every $\rho \in \tau c(\mathcal{H})$. Letting $\Delta = \mathbf{R}$ and $E = 1$ in Eq. (2.12), we have $\text{Tr}[\mathcal{I}(\mathbf{R}, \rho)] = 1$ for all $\rho \in \tau c(\mathcal{H})$. It follows that $\rho \mapsto \mathcal{I}(\mathbf{R}, \rho)$ is trace-preserving. Letting $\mathcal{I}(\Delta)\rho = \mathcal{I}(\Delta, \rho)$, we have a DL instrument $\mathcal{I}$ satisfying (2.3) and (2.9) for the apparatus $A(x)$. Therefore, we have shown that the Davies-Lewis thesis is a consequence of the Mixing law for joint output probability.

### 2.4 Individual quantum state reduction

It is natural to assume that the output variable $x$ can be read out with arbitrary precision, so that each instance of measurement has the output value $x = x$. Let $\rho_{\{x=x\}}$ be the state of the system $S$ at the time just after the measurement on input state $\rho$ provided that the measurement yields the output value $x = x$. If $\Pr\{x \in \{x\} | \rho\} > 0$, the state $\rho_{\{x=x\}}$ is determined by the relation

$$
\rho_{\{x=x\}} = \rho_{\{x \in \{x\}\}}. \tag{2.13}
$$

However, the above relation determines no $\rho_{\{x=x\}}$, if the output probability is continuously distributed. In order to determine states $\rho_{\{x=x\}}$, the following mathematical notion was introduced in Ref. [21]. A family $\{\rho_{\{x=x\}} | x \in \mathbf{R}\}$ of states is called a family of posterior states for a DL instrument $\mathcal{I}$ and a prior state $\rho$, if it satisfies the following conditions.

(i) The function $x \mapsto \rho_{\{x=x\}}$ is Borel measurable.

(ii) For any Borel set $\Delta$, we have

$$
\mathcal{I}(\Delta)\rho = \int_\Delta \rho_{\{x=x\}} \text{Tr}[d\mathcal{I}(x)\rho]. \tag{2.14}
$$

The following theorem ensures the existence of a family of posterior states [21].

**Theorem 2** (Existence of posterior states). For any DL instrument $\mathcal{I}$ and prior state $\rho$, there exists a family of posterior states essentially uniquely with respect to the probability measure $\text{Tr}[\mathcal{I}(\cdot)\rho] = 0$.

We define the *individual quantum state reduction* to be the correspondence from the input state $\rho$ to the family $\{\rho_{\{x=x\}} | x \in \mathbf{R}\}$ of posterior states for DL-instrument $\mathcal{I}$ of $A(x)$ and prior state $\rho$. For distinction, we shall call the previously defined quantum state reduction $\rho \mapsto \rho_{\{x \in \Delta\}}$ as the *collective quantum state reduction*.

The operational meaning of the individual quantum state reduction is given as follows. Suppose that a measurement using the apparatus $A(x)$ on input state $\rho$ is immediately followed by a measurement using another apparatus $A(y)$. Then, the joint probability distribution $\Pr\{x \in \Delta, y \in \Delta' | \rho\}$ of the output variables $x$ and $y$ is given by

$$
\Pr\{x \in \Delta, y \in \Gamma | \rho\} = \int_\Delta \Pr\{y \in \Gamma | \rho_{\{x=x\}}\} \Pr\{x \in dx | \rho\}. \tag{2.15}
$$

Thus, $\Pr\{y \in \Gamma | \rho_{\{x=x\}}\}$ is the conditional probability distribution of the output variable $y$ of the $A(y)$ measurement immediately following the $A(x)$ measurement carried out on the input state $\rho$ given that the $A(x)$ measurement leads to the outcome $x = x$. 

7
2.5 Complete positivity

Since the postulate for statistical properties of apparatuses (Postulate 4) and the mixing law for joint output probability (Postulate 5) are considered to be universally valid, we can conclude that every physically realizable apparatus has a DL instrument representing its statistical properties (Postulate 4). Thus, the problem of mathematically characterizing all the physically possible quantum measurements is reduced to the problem as to what class of DL instruments really can be considered to arise, in principle, from a physically realizable process.

A linear map \( T \) from a *-algebra \( \mathcal{A} \) to a *-algebra \( \mathcal{B} \) is called completely positive if \( T \otimes \text{id}_n : \mathcal{A} \otimes M_n \to \mathcal{B} \otimes M_n \) is a positive map for every finite number \( n \), where \( M_n \) is the matrix algebra of order \( n \) and \( \text{id}_n \) is the identity map on \( M_n \). The above condition is equivalent to requiring the relation

\[
\sum_{i,j=1}^n B_i T(A_i A_j^\dagger) B_j^\dagger \geq 0 \tag{2.16}
\]

for any finite sequences \( A_1, \ldots, A_n \in \mathcal{A} \) and \( B_1, \ldots, B_n \in \mathcal{B} \).

Let \( \mathcal{H} \) be a Hilbert space. A contractive completely positive map on the space \( \tau_c(\mathcal{H}) \) of trace-class operators is called an operation for \( \mathcal{H} \). The dual map \( T^* : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \) of a completely positive map \( T : \tau_c(\mathcal{H}) \to \tau_c(\mathcal{H}) \) is defined by \( \text{Tr}[T^*(A)\rho] = \text{Tr}[AT(\rho)] \) for any \( A \in \mathcal{L}(\mathcal{H}) \) and \( \rho \in \tau_c(\mathcal{H}) \). This is a normal completely positive map on \( \mathcal{L}(\mathcal{H}) \). A DL instrument for \( \mathcal{H} \) is called a completely positive (CP) instrument, or simply an instrument, if \( I(\Delta) \) is completely positive for every \( \Delta \in \mathcal{B}(\mathbb{R}) \).

Just like different reference frames may describe the same physical process, different mathematical models may describe the same measuring process. For instance, an apparatus measuring an observable \( A \) of the system described by the Hilbert space \( \mathcal{H} \) is also considered as an apparatus for measuring the observable \( A \otimes 1 \) of the system described by the Hilbert space \( \mathcal{H} \otimes \mathcal{H}' \) with Hilbert space \( \mathcal{H}' \) describing another system. The above consideration naturally leads to the following postulate.

**Postulate 6** (Trivial extendability principle). For any apparatus \( A(x) \) measuring a system \( S \) and any quantum system \( S' \) not interacting with \( A(x) \) nor \( S \), there exists an apparatus \( A(x') \) measuring system \( S + S' \) with the following statistical properties:

\[
\Pr\{x' \in \Delta\| \rho \otimes \rho'\} = \Pr\{x \in \Delta\| \rho\}, \tag{2.17}
\]

\[
(\rho \otimes \rho')_{\{x' \in \Delta\}} = \rho_{\{x \in \Delta\}} \otimes \rho', \tag{2.18}
\]

for any Borel set \( \Delta \), any state \( \rho \) of \( S \), and any state \( \rho' \) of \( S' \).

Now, suppose that \( A(x) \) is an apparatus measuring a system \( S \) described by Hilbert space \( \mathcal{H} \), and let \( I \) be the DL instrument corresponding to \( A(x) \). Then, according to the above postulate the physically identical measuring process can be described mathematically by another apparatus \( A(x') \) measuring the system \( S + S' \) with another system \( S' \) but without interacting with \( S' \). Let \( I' \) be the DL instrument corresponding to \( A(x') \). Then, we have

\[
I'(\Delta) = I(\Delta) \otimes \text{id}, \tag{2.19}
\]
for all $\Delta \in \mathcal{B}(\mathcal{H})$, where $\text{id}$ is the identity map on $\tau c(\mathcal{H}')$. We say that an DL instrument $\mathcal{I}$ has the trivial extendability if $\mathcal{I}(\Delta) \otimes \text{id}$ defines another instrument for an arbitrary Hilbert space $\mathcal{H}'$. Thus, according to the trivial extendability postulate, $\mathcal{I}(\Delta) \otimes \text{id}$ is required to be a positive map. This means that the DL instrument $\mathcal{I}$ should be a CP instrument. Thus, the trivial extendability postulate leads to the following postulate \[41\].

**Postulate 7.** For any apparatus $A(x)$ the corresponding instrument $\mathcal{I}$ is completely positive.

Now, we have shown that the set of posulates \{Postulate 3, Postulate 5, Postulate 6\} is equivalent to the set \{Postulate 3, Postulate 4, Postulate 7\}.

The transpose map on the matrix algebra is a typical example of positive maps that are not completely positive. The transpose map on $\tau c(\mathcal{H})$ relative to an orthonormal basis $\{\phi_n\}$ of $\mathcal{H}$ is a bounded linear map on $\tau c(\mathcal{H})$ defined by

$$T(|\phi_n\rangle\langle\phi_m|) = |\phi_m\rangle\langle\phi_n| \quad (2.20)$$

for all $n, m$. This is a trace-preserving positive map on $\tau c(\mathcal{H})$, but not completely positive. For any observable $A = \sum_n n|\phi_n\rangle\langle\phi_n|$, we have a DL instrument $\mathcal{I}$ defined by

$$\mathcal{I}(\Delta)\rho = \sum_{n \in \Delta} T(E^A(\{n\})\rho E^A(\{n\})).$$

According to the Davies-Lewis thesis this DL instrument should correspond to the following measurement statistics:

$$\Pr\{x = n||\rho\} = \text{Tr}[E^A(\{n\})\rho],$$

$$\rho_{\{x=n\}} = \frac{T(E^A(\{n\})\rho E^A(\{n\}))}{\text{Tr}[E^A(\{n\})\rho]}.$$

However, according to the trivial extendability postulate we can conclude that we have no measuring apparatus that physically realizes the above measurement statistics.

From the above, we conclude that physically realizable measurement statistics is necessarily described by a CP-instrument. We say that two measuring apparatuses are statistically equivalent if they have the same statistical properties. Our main objective is to determine the set of statistical equivalence classes of physically realizable measurements. Since every statistical equivalence class of physically realizable measurements uniquely corresponds to a CP-instrument, the problem is reduced to the problem as to which CP-instrument is physically realizable. The purpose of the following argument is to show that every CP-instrument can be considered, in principle, to be physically realizable.

### 2.6 Measuring processes

Von Neumann \[50\] showed that the projection postulate is consistent with axioms of quantum mechanics. Though von Neumann actually discussed the repeatability hypothesis for discrete observables with non-degenerate spectrum, his argument can be easily generalized to the projection postulate for discrete observables not necessarily with non-degenerate spectrum. The process of a measurement always includes the interaction between the object and
the apparatus, and after the interaction the outcome of the measurement is obtained by measuring the meter in the apparatus. Since the latter process can be done without the interaction between the object and the apparatus, the process of the measurement can be divided, at least, into the above two stages. Von Neumann showed that the statistical properties of the projective measurement can be obtained by such a description of the measuring process with an appropriate choice of the interaction, and the consistency of the projection postulate with quantum mechanics follows.

By generalizing von Neumann’s argument the standard models of measuring processes was introduced in [19]. According to that formulation, a measuring process for (the system described by) a Hilbert space $\mathcal{H}$ is defined as a quadruple $(\mathcal{K}, \rho_0, U, M)$ consisting of a Hilbert space $\mathcal{K}$, a density operator $\rho_0$, a unitary operator $U$ on the tensor product Hilbert space $\mathcal{H} \otimes \mathcal{K}$, and a self-adjoint operator $M$ on $\mathcal{K}$. A measuring process $(\mathcal{K}, \rho_0, U, M)$ is said to be pure if $\rho_0$ is a pure state, and it is said to be separable if $\mathcal{K}$ is separable.

The measuring process $(\mathcal{K}, \rho_0, U, M)$ mathematically models the following description of a measurement. The measurement is carried out by the interaction, referred to as the measuring interaction, between the object system $S$ and the probe system $P$, a part of the measuring apparatus $A(x)$ that actually takes part in the interaction with the object $S$. The probe system $P$ is described by the Hilbert space $\mathcal{K}$ and prepared in the state $\rho_0$ just before the measurement. The time evolution of the composite system $P + S$ during the measuring interaction is represented by the unitary operator $U$. The outcome of the measurement is obtained by measuring the observable $M$ called the meter observable in the probe $P$.

Suppose that the measurement carried out by an apparatus $A(x)$ is described by a measuring process $(\mathcal{K}, \rho_0, U, M)$. Then, it follows from axioms 1 to 4 that the statistical properties of the apparatus $A(x)$ is given by

$$
\Pr\{x \in \Delta | \rho\} = \frac{\text{Tr} \left[(1 \otimes E^M(\Delta)) U (\rho \otimes \rho_0) U^\dagger\right]}{\text{Tr} \left[(1 \otimes E^M(\Delta)) U (\rho \otimes \rho_0) U^\dagger\right]},
$$

and

$$
\rho\{x \in \Delta\} = \frac{\text{Tr}_{\mathcal{K}} \left[(1 \otimes E^M(\Delta)) U (\rho \otimes \rho_0) U^\dagger\right]}{\text{Tr} \left[(1 \otimes E^M(\Delta)) U (\rho \otimes \rho_0) U^\dagger\right]},
$$

where $\text{Tr}_{\mathcal{K}}$ stands for the partial trace on the Hilbert space $\mathcal{K}$; see [19] for the detailed justification. Thus, if the measurement by the apparatus $A(x)$ is described by the measuring process $(\mathcal{K}, \rho_0, U, M)$, the apparatus $A(x)$ indeed has the instrument $I$ determined by

$$
I(\Delta)\rho = \text{Tr}_{\mathcal{K}} \left[(1 \otimes E^M(\Delta)) U (\rho \otimes \rho_0) U^\dagger\right].
$$

In this case, we call $I$ the instrument of the measuring process $(\mathcal{K}, \rho_0, U, M)$. Here, it is important to note that we never appeal to the projection postulate in order to derive the above relation [19]. In fact, Eq. (2.21) holds even in the case where the measurement of the meter observable $M$ is not a projective measurement; for detailed discussion on this point see [19, 25, 30, 29, 31, 32, 34].

Now, we have shown that if the apparatus $A(x)$ is described by the measuring process $(\mathcal{K}, \rho_0, U, M)$, the statistical properties of $A(x)$ is determined by the instrument $I$ specified by Eq. (2.21). Then, the problem is whether the converse is true. The following theorem solves this problem [18, 19].

**Theorem 3** (Realization theorem for instruments). For any instrument $I$ for a Hilbert space $\mathcal{H}$, there exists a pure measuring process $(\mathcal{K}, \rho_0, U, M)$ for $\mathcal{H}$ such that $I$ is the instrument for $(\mathcal{K}, \rho_0, U, M)$. If $\mathcal{H}$ is separable, $\mathcal{K}$ can be made separable.
From the above theorem, we conclude the following. If we are given a physical measuring apparatus, that apparatus is considered to have its own statistical properties, which is mathematically described by a DL-instrument from the mixing law of the joint output probability. On the other hand, mathematical description of a physical measuring apparatus should satisfy the trivial extendability, so that the DL-instrument must be a CP-instrument. It is a difficult problem to generally consider all the physically realizable measuring processes, but for our purpose it suffices to consider a special class of measuring processes, which we consider as physically realizable and call “measuring processes” with a rigorous mathematical definition. What is concluded by the realization theorem of instruments is that for every physically realizable measuring apparatus $A(x)$, there exists at least one measuring apparatus $A(x')$ in the above class that is statistically equivalent to $A(x)$. Therefore, it is concluded that a universal or an existential statement on all the physically realizable measurements is justified if it is valid over the measurements carried out by measuring apparatus in that class as long as the statement concerns only statistical properties of measurements. This gives an important approach to establishing the universally valid uncertainty principle.

Now, we have justified the general measurement axiom formulated as follows.

**Axiom 5** (General measurement axiom). To every apparatus $A(x)$ for the system $S$ with the state space $H$, there corresponds an instrument $I$ such that the probability of the outcome $x \in \Delta$, where $\Delta \in \mathcal{B}(\mathbb{R})$, of the measurement in a state $\rho \in \mathcal{S}(H)$ is given by

$$
\Pr\{x \in \Delta \parallel \rho\} = \text{Tr}[I(\Delta)\rho],
$$

(2.22)

and if $\Pr\{x \in \Delta \parallel \rho\} > 0$ the state $\rho_{\{x \in \Delta\}}$ just after the measurement under the condition that the measurement leads to the outcome $x \in \Delta$ is given by

$$
\rho_{\{x \in \Delta\}} = \frac{I(\Delta)\rho}{\text{Tr}[I(\Delta)\rho]}.
$$

(2.23)

Conversely, to every instrument $I$ there exists at least one apparatus $A(x)$ with the above probability of the outcome.

A probability operator-valued measure (POVM) for a Hilbert space $H$ is a measure $\Pi$ on $\mathcal{B}(\mathbb{R})$ with values in positive operators on $H$, countably additive in strong operator topology, i.e., $\Pi(\bigcup_n \Delta_n \psi) = \sum_n \Pi(\Delta_n)\psi$ for all $\psi \in H$ and disjoint sequence $\{\Delta_n\}$ in $\mathcal{B}(\mathbb{R})$, and normalized so that $\Pi(\mathbb{R}) = 1$. Let $I$ be an instrument for $H$. The dual map $I(\Delta)^*$ of $I(\Delta)$ is a normal completely positive map on the space $\mathcal{L}(H)$ of bonded operators on $H$. The relation

$$
\Pi(\Delta) = I(\Delta)^* 1,
$$

(2.24)

where $\Delta \in \mathcal{B}(\mathbb{R})$, defines a POVM $\Pi$, called the POVM of $I$. Conversely, every POVM arises in this way. From Axiom 5, Axiom 2 can be generalized as follows.

**Axiom 6** (Generalized statistical formula). To every apparatus $A(x)$ for the system $S$ with the state space $H$, there corresponds a POVM $I$ such that the probability of the outcome $x \in \Delta$, where $\Delta \in \mathcal{B}(\mathbb{R})$, of the measurement in a state $\rho \in \mathcal{S}(H)$ is given by

$$
\Pr\{x \in \Delta \parallel \rho\} = \text{Tr}[\Pi(\Delta)\rho].
$$

(2.25)

Conversely, to every POVM $\Pi$ there exists at least one apparatus $A(x)$ with the above probability of the outcome.
An apparatus \( A(x) \) is said to measure an observable \( A \) if its POVM is the spectral measure of \( A \). Axiom \( 2 \) is derived from Axiom \( 6 \) under the additional condition \( \Pi = E^A \). Let \( A \) be a discrete observable. The relation 

\[
\mathcal{I}^A(\Delta) \rho = \sum_{x \in \Delta} E^A(\{x\}) \rho E^A(\{x\}),
\]

where \( \rho \in \tau_c(H) \), defines an instrument \( \mathcal{I}^A \), called the instrument of the projective measurement of \( A \). In this case, the POVM of \( \mathcal{I}^A \) is \( E^A \), and the projection postulate is derived from Axiom \( 5 \) under the additional condition \( \mathcal{I} = \mathcal{I}^A \).

The Wigner formula is generalized to the following.

**Theorem 4 (Generalized Wigner’s formula).** Let \( \mathcal{I}_1, \ldots, \mathcal{I}_n \) be instruments for system with the state space \( \mathcal{H} \) in a state \( \rho \) at time \( 0 \). If one carries out measurements described by \( \mathcal{I}_1, \ldots, \mathcal{I}_n \) at times \( 0 < t_1 < \cdots < t_n \) and otherwise leaves the system \( S \) isolated, then the joint probability distribution of the outcomes \( x_1, \ldots, x_n \) of those measurements is given by

\[
\Pr\{x_1 \in \Delta_1, x_2 \in \Delta_2, \ldots, x_n \in \Delta_n \| \rho\} = \text{Tr}\left[ \mathcal{I}_n(\Delta_n) \alpha(t_n - t_{n-1}) \cdots \mathcal{I}_2(\Delta_2) \alpha(t_2 - t_1) \mathcal{I}_1(\Delta) \alpha(t_1) \rho \right],
\]

where \( \alpha \) is defined by \( \alpha(t) \rho = e^{-iHt/\hbar} \rho e^{iHt/\hbar} \) for all \( t \in \mathbb{R} \) and \( \rho \in (\mathcal{H}) \).

Foundations of quantum measurement theory based on the notion of instruments have been developed in \cite{18, 19, 21, 20, 22, 24, 27, 28}.

### 3 Uncertainty principle

#### 3.1 Heisenberg’s proof

In 1927, by considering the famous thought experiment of the \( \gamma \) ray microscope, Heisenberg \cite{8} showed the relation

\[
\epsilon(Q) \eta(P) \sim \hbar
\]

for the measurement error \( \epsilon(Q) \) of a position measurement and the disturbance \( \eta(P) \) of the momentum caused by that measurement. He further stated that this is a straightforward mathematical consequence of the canonical commutation relation \( [Q, P] = i\hbar \), and attempted to give a formal proof based on the Dirac-Jordan theory. In that proof he used the fact that the product of the spread of the position and the spread of the momentum in a Gaussian wave function amounts to the Planck constant. Immediately afterward, Kennard \cite{13} reformulated this relation in terms of the standard deviations \( \sigma(Q) \) and \( \sigma(P) \) of the position and the momentum, respectively, as

\[
\sigma(Q) \sigma(P) \geq \frac{\hbar}{2},
\]

which he proved in any state \( \psi \). In 1929 Robertson \cite{45} further generalized and proved this relation to arbitrary pairs of observables \( A \) and \( B \) as

\[
\sigma(A) \sigma(B) \geq \frac{1}{2} |\langle \psi, [A, B] \psi \rangle|.
\]
Since then, most of text books have shown the derivation of Robertson’s relation Eq. (3.3) in terms of the Schwarz inequality and then explained its physical meaning to be the quantitative relation such that if one measures the position more precisely then the momentum is more disturbed as in the γ ray thought experiment.

However, it is obvious that neither the Kennard inequality nor the Roberson inequality expresses the relation between the measurement error and the disturbance, since the notion of standard deviations is nothing to do with the properties of measuring apparatuses but determined solely by the state of the measured object. In fact, Heisenberg’s proof of Eq. (3.1) runs as follows. Heisenberg assumes that the measurement of the position with the error $\epsilon(Q)$ leaves the object in a state $\psi$ with the standard deviation $\sigma(Q)$ satisfying $\sigma(Q) = \epsilon(Q)$. Then, he uses the relation Eq. (3.2) to obtain $\epsilon(Q)\sigma(P) \sim \hbar$, and concludes that it is because the disturbance $\eta P$ satisfies (3.1) that a measurement with small $\epsilon(Q)$ always increases the standard deviation $\sigma(P)$ of the momentum.

The assumption used here is not correct that the measurement of the position with the error $\epsilon(Q)$ leaves the object in a state $\psi$ with the standard deviation $\sigma(Q)$ satisfying $\sigma(Q) = \epsilon(Q)$. This was revealed in the 1980s through the debate over the problem as to whether there exists a detection limit derived from the uncertainty principle. In the rest of this section, we discuss this debate and the correct formulation of the uncertainty principle.

### 3.2 Gravitational wave detection and the uncertainty principle

In the 1970s, from a simple quantum mechanical analysis on the performance of gravitational wave detectors it was generally accepted that a theoretical limit, called the standard quantum limit (SQL), of the sensitivity of gravitational wave detectors is derived from the uncertainty principle, and in particular that the SQL can be escapable by resonator type detectors but not escapable by non-resonator type detectors [4, 6]. However, in the 1980s a dispute arose among theorists on the validity of the SQL [55, 5, 23, 25].

A typical non-resonator type detector is an apparatus that estimates the existence or the strength of gravitational waves by detecting the change in the difference of the lengths of two orthogonal optical paths caused by the tidal force carried by the gravitational waves. The measurement of the small change of the position of the mirror is assumed to obey quantum mechanics. Thus, the problem is how accurately one can predict the position of the mirror as a free mass in the absence of gravitational waves. If there is an inevitable error, the detectable force should give the displacement greater than the error, and the gravitational waves cannot be detected if they are weaker than those which give such a minimum displacement.

In the standard argument [4, 6], the time $t = 0$ is set as the instant just after the first measurement and let the time $t = \tau$ be the instant of the time just before the second measurement. Then, it is claimed that according to Kennard’s inequality (3.2) applied to the standard deviations $\sigma(\hat{x}(0))$ and $\sigma(\hat{p}(0))$ of the position and the momentum just after the first measurement, the variance of the position $\hat{x}$ increases until the time $\tau$ of the second measurement as

$$\sigma(\hat{x}(\tau))^2 \geq \sigma(\hat{x}(0))^2 + \sigma(\hat{p}(0))^2 \tau^2 / m^2 \geq 2 \sigma(\hat{x}(0))\sigma(\hat{p}(0))\tau / m \geq \frac{\hbar \tau}{m}.$$  (3.4)
From the above, we obtained the SQL
\[ \sigma(\hat{x}(\tau)) \geq \sqrt{\frac{\hbar \tau}{m}}. \] (3.5)

In this way, the SQL has been explained as a straightforward consequence of Kennard’s inequality [3.2].

Now, we suppose that a constant classical force \( f \) acts on a mass \( m \) from time \( t = 0 \) to \( t = \tau \). If \( \Delta f \) is the minimum detectable force, then we have \( \Delta f \tau^2 / 2m \geq \Delta \hat{x}_{\text{SQL}} \), since the displacement at the time \( \tau \) caused by this force should be more than \( \Delta \hat{x}_{\text{SQL}} \). Thus, the standard quantum limit (SQL) for the detection of weak classical force is obtained as
\[ \Delta f_{\text{SQL}} = \sqrt{\frac{4\hbar m}{\tau^3}}. \] (3.6)

In 1983, Yuen [55] pointed out a serious flaw in this standard argument. Since the evolution of a free mass is given by
\[ \hat{x}(t) = \hat{x}(0) + \hat{p}(0)t/m \] (3.7)
the variance of \( \hat{x} \) at time \( \tau \) is given by
\[ \sigma(\hat{x}(\tau))^2 = \sigma(\hat{x}(0))^2 + \sigma(\hat{p}(0))^2\tau^2/m^2 \]
\[ + \langle \delta \hat{x}(0)\delta \hat{p}(0) + \delta \hat{p}(0)\delta \hat{x}(0) \rangle \tau/m, \] (3.8)
where \( \Delta \hat{x} = \hat{x} - \langle \hat{x} \rangle \) and \( \Delta x^2 = \langle \Delta \hat{x}^2 \rangle \), etc. Thus the standard argument implicitly assumes that the last term — we shall call it the correlation term — in Eq. (3.9) is non-negative. Yuen’s assertion [55] is that some measurements of \( \hat{x} \) leave the free mass in a state with the negative correlation term.

In other words, the measurement of the position of a free-mass at \( t = \tau \) has no uncertainty, if the state at \( t = 0 \) is an eigenstate of \( \hat{x}(\tau) \). Any eigenstate of \( \hat{x}(\tau) \) is not normalizable but there are (normalized) wave functions arbitrarily near to it, and the contractive states are among them.

However, if the measurement is only approximately accurate, namely, the measurement outcome at time \( \tau \) includes the additional error to the actual position \( \hat{x}(\tau) \), then the expected uncertainty of the measurement outcome is considered to include the measurement error in addition to the quantum mechanical uncertainty. Thus, the problem is reduced to the problem as to whether it is possible to realize, in principle, the measurement such that its measurement error for the position \( \hat{x}(0) \) is negligibly small but the mass is left in a state arbitrarily near to an eigenstate of the observable \( \hat{x}(\tau) \).

The existence of such a measurement contradicts the Heisenberg type inequality (3.1). In fact, by Eq. (3.1) we have
\[ \epsilon[\hat{x}(0)]\eta[\hat{x}(\tau)] \geq \frac{\tau \hbar}{2m} \] (3.9)
and hence if the measurement error is \( \epsilon[\hat{x}(0)] \approx 0 \), the disturbance of \( \hat{x}(\tau) \) satisfies \( \eta[\hat{x}(\tau)] \sim \infty \), so that it is impossible to have the relation \( \Delta \hat{x}(\tau) \approx 0 \) in the state after the measurement.
A dispute arose as to whether such a measurement is possible or not, and the theoretical aspect of the dispute was settled by the result [23] showing such a measurement can be carried out by a model that obtained by a straightforward modification of the von Neumann’s model [50] of position measurement [15].

3.3 Noise and disturbance in quantum measurement

Let \((K, \rho_0, U, M)\) be a measuring process for a system \(S\) described by a Hilbert space \(H\). For this measuring process and an observable \(A\) of \(S\), we define the noise operator \(N(A)\), and disturbance operator \(D(A)\) by

\[
N(A) = U^\dagger (1 \otimes M)U - A \otimes 1, \quad (3.10)
\]

\[
D(A) = U^\dagger (A \otimes 1)U - A \otimes 1. \quad (3.11)
\]

Their means, \(\langle N(A) \rangle\) and \(\langle D(A) \rangle\), in the state \(\rho \otimes \rho_0\) are called the mean noise and mean disturbance, respectively, for observable \(A\) in a state \(\rho\). Their root-mean-squares (rms’s), \(\langle N(A)^2 \rangle^{1/2}\) and \(\langle D(A)^2 \rangle^{1/2}\), in the state \(\rho \otimes \rho_0\) are called the (rms) noise and (rms) disturbance, respectively, for observable \(A\) in a state \(\rho\), and denoted by \(\epsilon(A)\) and \(\eta(A)\).

We also define mean noise operator \(n(A)\) and mean disturbance operator \(d(A)\) by

\[
n(A) = \text{Tr}_K[N(A)(1 \otimes \rho_0)], \quad (3.12)
\]

\[
d(A) = \text{Tr}_K[D(A)(1 \otimes \rho_0)]. \quad (3.13)
\]

The \(n\)th moment operator \(\Pi^{(n)}\) of a POVM \(\Pi\) is defined by

\[
\langle \eta, \Pi^{(n)} \xi \rangle = \int_R x^n \langle \eta, \Pi(dx)\xi \rangle, \quad (\xi \in \text{dom}(\Pi^{(n)}), \eta \in H)
\]

\[
\text{dom}(\Pi^{(n)}) = \{\xi \in H \mid \int_R x^{2n} \langle \xi, \Pi(dx)\xi \rangle < \infty\}.
\]

Let \(T\) be an operation for \(H\). For any observable \(A\), denote by \(T^*E^A\) the POVM defined by \((T^*E^A)(\Delta) = T^*(E^A(\Delta))\). If \(A\) is bounded, it is easy to see that \(T^*(A^n)\) is the \(n\)th moment operator of \(T^*E^A\). If \(A\) is unbounded, we define \(T^*(A^n)\) as the \(n\)th moment operator of \(T^*E^A\), i.e., \(T^*(A^n) = (T^*E^A)^{(n)}\).

The following theorem shows that the mean noise, the rms noise, and the mean noise operator are determined by the POVM of the measuring process and the mean disturbance, the rms disturbance, and the mean disturbance operator are determined by the operation of the measuring process.

**Theorem 5.** Let \((K, \rho_0, U, M)\) be a measuring process for a Hilbert space \(H\), and let \(T\) and \(\Pi\) be the corresponding POVM and operation. Then, we have

\[
n(A) = \Pi^{(1)} - A, \quad (3.14)
\]

\[
d(A) = T^*(A) - A, \quad (3.15)
\]

\[
\langle N(A) \rangle = \text{Tr}[\Pi^{(1)} \rho] - \text{Tr}[A \rho], \quad (3.16)
\]

\[
\langle D(A) \rangle = \text{Tr}[AT(\rho)] - \text{Tr}[A \rho], \quad (3.17)
\]

\[
\epsilon(A)^2 = \text{Tr}[\Pi^{(2)} \rho] - \text{Tr}[\Pi^{(1)} \rho A] - \text{Tr}[\Pi^{(1)} A \rho] + \text{Tr}[A^2 \rho], \quad (3.18)
\]

\[
\eta(A)^2 = \text{Tr}[A^2 T(\rho)] - \text{Tr}[AT(\rho A)] - \text{Tr}[AT(A \rho)] + \text{Tr}[A^2 \rho]. \quad (3.19)
\]
Here, we assume $\rho$ to satisfy $A\sqrt{\rho} \in \sigma_c(\mathcal{H})$ and that the all relevant trace is convergent.

Following the proposal introduced in by Heisenberg [8], we call the relation

$$\epsilon(A)\eta(B) \geq \frac{1}{2}|\text{Tr}([A, B]\rho)|$$

(3.20)

the Heisenberg type inequality.

### 3.4 Von Neumann’s measurement

Von Neumann [50] introduce the following measuring process of a position measurement. Caves [5] showed that this measurement satisfies the SQL. Here, we shall show that this measurement satisfies the Heisenberg inequality.

The measured object $S$ is a one-dimensional quantum system with position $\hat{x}$, momentum $\hat{p}_x$, satisfying $[x, p_x] = i\hbar$, and Hamiltonian $H_S$. Suppose that the object $S$ interacts with the probe $P$ in the apparatus $A(x)$ from time $t$ to $t + \Delta t$ and becomes free from time $t + \Delta t$. In von Neumann’s measuring process, the probe $P$ is a one-dimensional quantum system with position $\hat{y}$, momentum $\hat{p}_y$, satisfying $[y, p_y] = i\hbar$, and Hamiltonian $H_P$. The meter observable in the probe $P$ is the position $\hat{y}$ of $P$. The interaction between the object $S$ and the probe $P$ is given by

$$H_{SP} = \hat{x}\hat{p}_y,$$

(3.21)

so that the total Hamiltonian of the composite system $S + P$ is given by

$$H_{S+P} = H_S \otimes 1 + 1 \otimes H_P + KH_{SP},$$

(3.22)

where the coupling constant $K$ is so large that free Hamiltonians can be neglected. The time duration $\Delta t$ is assumed to satisfy $K\Delta t = 1$. Thus, the time evolution of the composite system $S + P$ in the time duration $(t, t + \Delta t)$ is given by

$$U = e^{-i\hat{x}\hat{p}_y/\hbar}.$$ 

(3.23)

Let $\xi$ be the initial state of the probe. Then, the von Neumann’s model corresponds to the measuring process $(L^2(\mathbb{R}), \xi, e^{-i\hat{x}\hat{p}_y/\hbar}, \hat{y})$, and its instrument is given by

$$I(\Delta)\rho = \int_{\Delta} \xi(\hat{x} - x)\rho\xi(\hat{x} - x)^\dagger dx.$$ 

(3.24)

Solving Heisenberg’s equation of motion, we have

$$\dot{x}(t + \Delta t) = \dot{x}(t),$$

(3.25)

$$\dot{y}(t + \Delta t) = \dot{x}(t) + \dot{y}(t),$$

(3.26)

$$\dot{p}_x(t + \Delta t) = \dot{p}_x(t) - \dot{p}_y(t),$$

(3.27)

$$\dot{p}_y(t + \Delta t) = \dot{p}_y(t).$$

(3.28)
Thus, the noise operator and the disturbance operator are given by
\begin{align*}
N(\hat{x}) &= \hat{y}(t + \Delta t) - \hat{x}(t) = \hat{y}(t), \\
D(\hat{p}_x) &= \hat{p}_x(t + \Delta t) - \hat{p}_x(t) = -\hat{p}_y(t).
\end{align*}
(3.29) (3.30)

Let \(\sigma(\hat{y})\) and \(\sigma(\hat{p}_y)\) be the standard deviations of the position and the momentum of the probe, respectively, at the time \(t\) of the measurement. Then, by the Kennard inequality, (3.2), we have
\[\epsilon(\hat{x})\eta(\hat{p}_x) \geq \sigma(\hat{y}) \sigma(\hat{p}_y) \geq \frac{\hbar}{2}.\]
(3.31)

Thus, the Heisenberg type inequality (3.20) holds for von Neumann’s measuring process [37].

### 3.5 Contractive state measurement

The notion of contractive state measurements proposed by Yuen [55] has been shown to be realized by the following measuring process [23, 25, 26, 33].

The measured object \(S\), the probe \(P\), and the time of interaction are described in the same way as von Neumann’s model. The interaction \(H_{SP}\) is given by
\[H_{SP} = \frac{K\pi}{3\sqrt{3}} \{2(\hat{x}\hat{p}_y - \hat{p}_x\hat{y}) + (\hat{x}\hat{p}_x - \hat{y}\hat{p}_y)\}.\]
(3.32)

Thus, this model of measurement corresponds to the measuring process
\[
(L^2(R), \xi, \exp[-i\pi/3\sqrt{3}\hat{p}_x]\{2(\hat{x}\hat{p}_y - \hat{p}_x\hat{y}) + (\hat{x}\hat{p}_x - \hat{y}\hat{p}_y)\}], \hat{y}),
\]
and its instrument is given by
\[I(\Delta)\rho = \int_{\Delta} e^{-ix\hat{p}_x/\hbar}\xi(\xi)e^{ix\hat{p}_x/\hbar}\text{Tr}[E^x(dx)\rho].\]
(3.33)

Solving Heisenberg’s equation of motion, we have
\begin{align*}
\hat{x}(t + \Delta t) &= \hat{x}(t) - \hat{y}(t), \\
\hat{y}(t + \Delta t) &= \hat{y}(t), \\
\hat{p}_x(t + \Delta t) &= -\hat{p}_y(t), \\
\hat{p}_y(t + \Delta t) &= \hat{p}_x(t) + \hat{p}_y(t).
\end{align*}
(3.34) (3.35) (3.36) (3.37)

Thus, the noise operator and the disturbance operator are given by
\begin{align*}
N(\hat{x}) &= \hat{y}(t + \Delta t) - \hat{x}(t) = 0, \\
D(\hat{p}_x) &= \hat{p}_x(t + \Delta t) - \hat{p}_x(t) = -\hat{p}_y(t) - \hat{p}_x(t),
\end{align*}
(3.38) (3.39)

and hence
\[\epsilon(\hat{x})\eta(\hat{p}_x) = 0.\]
(3.40)

Thus, this model does not satisfy the Heisenberg type inequality, (3.20) [37].
3.6 Universal uncertainty principle

What relation between the error and the disturbance holds for arbitrary measurements? The following theorem generally holds [40, 38, 41].

**Theorem 6** (Universal uncertainty principle). The rms error \( \epsilon(A) \), the rms disturbance \( \eta(B) \), and the standard deviations \( \sigma(A), \sigma(B) \) satisfy the relation

\[
\epsilon(A) \eta(B) + \epsilon(A) \sigma(B) + \sigma(A) \eta(B) \geq \frac{1}{2} |\text{Tr}([A, B]\rho)|
\]

(3.41)

for any observables \( A, B \), state \( \rho \), and instrument \( I \).

**Theorem 7** (Condition for the Heisenberg type inequality). The rms error \( \epsilon(A) \) and the rms disturbance \( \eta(B) \) satisfy the relation

\[
\epsilon(A) \eta(B) + \frac{1}{2} |\text{Tr}([n(A), B]) + \text{Tr}([A, d(B)])| \geq \frac{1}{2} |\text{Tr}([A, B]\rho)|
\]

(3.42)

for any observables \( A, B \), state \( \rho \), and instrument \( I \). Moreover, if the mean error \( \langle N(A) \rangle \) of \( A \) and the mean disturbance \( \langle D(B) \rangle \) of \( B \) are independent of the object state, then the Heisenberg type inequality (3.20) holds.

In fact, if \( \langle N(A) \rangle \) and \( \langle D(B) \rangle \) are independent of the object state, then \( n(A) \) and \( d(B) \) scalar operators, so that we have \([n(A), B] = [A, d(B)] = 0\).

From the universal uncertainty principle, there are two typical cases where the Heisenberg type inequality fails and we have a new trade-off relation in each case.

(i) Constraint for error-free measurements: In the case where \( \eta(B) = 0 \), the relation

\[
\epsilon(A) \sigma(B) \geq \frac{1}{2} |\text{Tr}([A, B])|
\]

(3.43)

holds for the error of \( A \) and the standard deviation of \( B \).

(ii) Constraint for non-disturbing measurements: In the case where \( \eta(B) = 0 \), the relation

\[
\sigma(A) \eta(B) \geq \frac{1}{2} |\text{Tr}([A, B])|
\]

(3.44)

holds for the disturbance of \( B \) and the standard deviation of \( A \).

The model of the contractive state measurement (3.32) is an instance of error-free measurements and reveals the possibility of a measurement breaking the standard quantum limit for gravitational wave detection. In the next section, we shall show that the new constraint for non-disturbing measurements leads to a quantitative generalization of the Wigner-Araki-Yanase theorem and an accuracy constraint for quantum computing.

4 Accuracy limits of quantum computing

4.1 Decoherence and conservation laws in quantum computing

The prime factorization problem has been used for public key cryptography such as the RSA protocol, since no efficient algorithm has been found for this problem. However, Shor [40]...
found an efficient algorithm for quantum computers solving prime factoring in 1994. Since then, active researches have been developed as to the realizability of quantum computers.

A major part of the problem of realizability of a quantum computer is the problem of decoherence. In general, decoherence in quantum computer components can be classified into two classes: (i) the environment induced decoherence, arising from the interaction between the computer memory and the environment, and (ii) the controller induced decoherence, arising from the interaction between the computer register and the control system of the quantum logic gate operation. According to the theory of fault-tolerant quantum computing, provided the noise in individual quantum gates is below a certain threshold it is possible to efficiently perform arbitrarily large quantum computing \[17\]. The environment induced decoherence may be overcome by using materials with long decoherence time. On the other hand, the controller induced decoherence poses a dilemma between controllability and decoherence; the control needs coupling, whereas the coupling causes decoherence. Thus, the problem is reduced to the problem as to whether the controller induced decoherence is derived to be inevitable from fundamental physical laws and the problem of its quantitative evaluation.

One of the reasons why the controller induced decoherence is considered to be inevitable in quantum state control is the existence of conservation laws in nature. The Wigner-Araki-Yanase (WAY) theorem \[52, 1\] is a starting point of the research as to how conservation laws impede quantum state control. The WAY theorem states that no measuring interaction realizes a measurement with absolute precision for an observables not commuting with additive conserved quantity.

### 4.2 Quantitative generalization of the Wigner-Araki-Yanase theorem

We show that the above new constraint on the accuracy of non-disturbing measurements \[3.43\] can be used to derive the quantitative expression of the WAY theorem as follows \[35, 39\].

**Theorem 8** (Quantitative generalization of the WAY theorem). For any measuring process \((\mathcal{K}, \xi, U, M)\), if observables \(L_1\) and \(L_2\) on Hilbert spaces \(\mathcal{H}\) and \(\mathcal{K}\), respectively, satisfy \([U, L_1 \otimes 1 + 1 \otimes L_2] = 0\) and \([M, L_2] = 0\), then for any observable \(A\) on \(\mathcal{H}\) we have

\[\epsilon(A)^2 \geq \frac{|\langle[A, L_1] \rangle|^2}{4\sigma(L_1)^2 + 4\sigma(L_2)^2}, \tag{4.1}\]

where the mean and standard deviations are taken for the initial states of the system and the apparatus.

The proof runs as follows. By the relation \([U, L_1 \otimes 1 + 1 \otimes L_2] = 0\), the interaction between the system \(S\) and the probe \(P\) does not disturb \(L_1 \otimes 1 + 1 \otimes L_2\). Moreover, by the relation \([M, L_2] = 0\), the subsequent measurement of the probe observable \(M\) can be done without disturbing \(L_1 \otimes 1 + 1 \otimes L_2\). Thus, a measuring process describing the same measurement, in which we regard \(S + P\) as the measured object, satisfies \(\eta(L_1 \otimes 1 + 1 \otimes L_2) = 0\). By substituting \(B = L_1 \otimes 1 + 1 \otimes L_2\) in inequality \[3.43\], we have

\[\epsilon(A)^2 \geq \frac{|\langle[A \otimes 1, L_1 \otimes 1 + 1 \otimes L_2] \rangle|}{4\sigma(L_1 \otimes 1 + 1 \otimes L_2)^2}, \tag{4.2}\]
and hence we have Eq. (4.1) from the relations $\langle [A \otimes 1, L_1 \otimes 1 + 1 \otimes L_2] \rangle = [A, L_1]$ and $\sigma(L_1 \otimes 1 + 1 \otimes L_2)^2 = \sigma(L_1)^2 + \sigma(L_2)^2$.

Yanase [54] derived the accuracy limit for measurements of a spin component under the angular momentum conservation law. Let $A$ be the $z$-component $S_z \otimes 1$ of a spin $1/2$ particle $S$, let $L_1$ be the $x$-component $S_x \otimes 1$ of $S$, and let $L_2$ be the $x$-component $1 \otimes S_x$ of the probe $P$. Yanase showed that the error probability $P_e$ satisfies $P_e \sim \hbar^2/16\langle L_2^2 \rangle$. In this case, we have $|\langle [A, L_1] \rangle| = |\langle [S_z, S_x] \rangle| = \hbar|\langle S_y \rangle| \leq \hbar^2/2$, and hence by Theorem 8 we have

$$\max_{\psi} \epsilon(A)^2 \geq \frac{\hbar^4}{4\hbar^2 + 16(\Delta L_2)^2}, \quad (4.3)$$

where max is taken over all the possible state $\psi$ of the object $S$. From the relation $P_e = \epsilon(S_z)^2/\hbar^2$, we have

$$\max_{\psi} P_e \geq \frac{1}{4 + 16(\Delta L_2/\hbar)^2}. \quad (4.4)$$

Therefore, inequality (4.1) for that case improves Yanase’s result.

From Eq. (4.4), it is concluded that the angular momentum conservation law prevents the interaction for a precise spin measurement. However, this result does not imply the unmeasurability of spin. It is clear from Eq. (4.1) that the inevitable error is inversely proportional to the variance of the conserved quantity included in the apparatus. An apparatus for a high precision measurements is usually of macroscopic size and have a large amount of the conserved quantity, and hence the practical apparatus circumvents the present limitation. On the other hand, as discussed in the next section, it is an interesting problem how an elementary quantum logic gate in a small integrated circuit can operate with very high precision demanded for fault-tolerant computing.

### 4.3 Quantum limits for the realization of quantum computing

In the current paradigm the strategies for the realization of quantum computing can be summarized as follows [17].

1. To physically represent computational qubits by spin components of spin $1/2$ systems, for the feasibility of initialization and read-out.

2. To physically realize elementary logic gates by $1$ qubit rotation operation and controlled not (CNOT) operation between $2$ qubits, and any quantum circuit can be built up from those two sorts of unitary operations.

3. To clear the accuracy threshold, every operation should be implemented with the error probability below $10^{-5} - 10^{-6}$.

From the above it can be concluded that since rotations of the spin and the CNOT do not conserve the spin, it has been shown from the above strategies that if those gates are implemented by physical interactions obeying the angular momentum conservation law then the unavoidable noise similar to the WAY theorem arises [36]. However, not every quantum gates will play the same role as the measuring apparatus, and in fact there are quantum gates that obey the angular momentum conservation law like the SWAP gate. Thus, it is not always possible to estimate the error probability from inequality (4.4) quantifying the WAY theorem.
but some useful arguments have been known for estimating the error probability for several gates [36, 39, 12].

Along with this line, we have now established a method for estimating the error probability for arbitrary unitary gates. Let $S$ be a spin 1/2 system described by a Hilbert space $\mathcal{H}_S$, and $\{\ket{0}, \ket{1}\}$ the eigenbasis of the z-component of the spin. An arbitrary unitary gate $U_S$ on $\mathcal{H}_S$ can be represented by

$$U_S = e^{i\phi} \left( \cos \frac{\theta}{2} 1 + i \sin \frac{\theta}{2} \vec{n} \cdot \vec{\sigma} \right)$$

with uniquely determined angles $\phi, \theta$ with $0 \leq \phi < 2\pi$, $0 \leq \theta \leq \pi$ and a unit vector $\vec{n} = (l_x, l_y, l_z)$, where $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector consisting of the Pauli operators. An implementation of $U_S$ is a pair $\alpha = (U, \ket{\xi})$ consisting of a unitary operator $U$ of the composite system $S + A$ with a system $A$, called the ancilla, described by a Hilbert space $\mathcal{H}_A$. An implementation $\alpha = (U, \ket{\xi})$ defines a trace-preserving quantum operation $\mathcal{E}_\alpha$ by

$$\mathcal{E}_\alpha(\rho) = \text{Tr}_A[U(\rho \otimes |\xi\rangle\langle\xi|)U^\dagger].$$

for any density operator $\rho$ of the system $S$, where $\text{Tr}_A$ stands for the partial trace over $\mathcal{H}_A$. On the other hand, the gate $U_S$ defines a unitary operation $\text{ad}H$ by $\text{ad}U_S(\rho) = U_S \rho U_S^\dagger$. The gate error probability $P_e$ of the implementation $\alpha = (U, \ket{\xi})$ is defined as the completely bounded distance between $\mathcal{E}_\alpha$ and $\text{ad}U_S$, i.e.,

$$D_{CB}(\mathcal{E}_\alpha, U_S) = \sup_{n,\rho} D(\mathcal{E}_\alpha \otimes \text{id}_n(\rho), \text{ad}H \otimes \text{id}_n(\rho)), \quad (4.6)$$

where $n$ runs over positive integers, $\text{id}_n$ is the identity operation on the matrix algebra $M_n$, $\rho$ is a density operator on $\mathcal{H} \otimes \mathbb{C}^n$, and $D$ stands for the trace distance $D(\rho_1, \rho_2) = \frac{1}{2}\text{Tr}||\rho_1 - \rho_2||$. On the other hand the gate fidelity of the implementation $\alpha = (U, \ket{\xi})$ is defined by

$$F(\mathcal{E}_\alpha, U_S) = \inf_{|\psi\rangle} F(\psi),$$

where $|\psi\rangle$ varies over the state vectors of $S$ and $F(\psi)$ is the fidelity between the two states $\mathcal{E}_\alpha(|\psi\rangle\langle\psi|)$ and $\text{ad}U_S(|\psi\rangle\langle\psi|)$ given by $F(\psi) = \langle \psi | U_S^\dagger \mathcal{E}_\alpha(|\psi\rangle\langle\psi|) U_S | \psi \rangle^{1/2}$. The above measures of imperfection of the implementation $\alpha = (U, \ket{\xi})$ satisfy the relation [17]

$$1 - F(\mathcal{E}_\alpha, U_S)^2 \leq D_{CB}(\mathcal{E}_\alpha, U_S).$$

(4.8)

The left-hand-side is called the gate infidelity of the implementation $\alpha = (U, \ket{\xi})$. Now, we assume that the implementation $(U, \ket{\xi})$ is rotationally invariant; namely, it satisfies the spin conservation law for $j = x, y, z$ components, $[U, S_j \otimes 1 + 1 \otimes L_j] = 0$, and that the spin quantum number of the ancilla is $N/2$. Then, a lower bound of the gate infidelity is given as follows [11]. If $0 \leq \theta \leq \pi/2$, we have

$$\frac{\sin^2 \theta}{4 + 4N^2} \leq 1 - F(\mathcal{E}_\alpha, U_S)^2,$$

(4.9)

and if $\pi/2 \leq \theta \leq \pi$, we have

$$\frac{1}{4 + 4N^2} \leq 1 - F(\mathcal{E}_\alpha, U_S)^2.$$  

(4.10)
5 Interpretation of quantum theory

5.1 Simultaneous measurements of noncommuting observables

It has long been accepted that two observables are simultaneously measurable if and only if their corresponding operators commute. However, this is true only when we take it as the statement that two observables are simultaneously measurable in any state if and only if their corresponding operators commute. In fact, in the singlet state of a system consisting of two spin-1/2 particles any two components of the spin of the first particle is simultaneously measurable. In order to do so, we have only to measure one component indirectly through the measurement of the same component of the second particle, which is strictly anti-correlated with the same component of the first particle, and to measure the other component directly in the same time.

In what follows we present a mathematical theory of simultaneous measurability, and give a theoretical basis for simultaneous measurability of noncommuting observables.

We say that two observables \( A, B \) are commuting in a state \( \rho \) if for every Borel sets \( \Delta, \Gamma \) we have

\[
\text{Tr}[E^A(\Delta)E^B(\Gamma)\rho] = 0.
\]

In this case, the joint probability distribution \( \mu \) of observables \( A, B \) in the state \( \rho \) is defined by

\[
\mu^{A,B}_\rho(\Delta \times \Gamma) = \text{Tr}[E^A(\Delta)E^B(\Gamma)\rho].
\]  (5.1)

We say that two observables \( A, B \) have a quantum identical correlation in a state \( \rho \), and write \( A \equiv \rho B \), if they are commuting in \( \rho \) and the joint probability distribution satisfies

\[
\mu^{A,B}_\rho(\{(x, y) \in \mathbb{R}^2 \mid x = y\}) = 1.
\]  (5.2)

In this case, two observables \( A, B \) are considered to be simultaneously measurable in \( \rho \) and their measurement outcomes are always identical.

Let \( f, g \) be Borel functions. A measuring process \((\mathcal{K}, \xi, U, M)\) for a Hilbert space \( \mathcal{H} \) is said to simultaneously measure observables \( A, B \) with \( f, g \) in a state \( \rho \) if we have

\[
U^\dagger(1 \otimes f(M))U \equiv_{\rho \otimes |\xi\rangle\langle\xi|} A \otimes 1,
\]  (5.3)

\[
U^\dagger(1 \otimes g(M))U \equiv_{\rho \otimes |\xi\rangle\langle\xi|} B \otimes 1.
\]  (5.4)

Two observables \( A, B \) are said to be simultaneously measurable in a state \( \rho \) if there is a measuring process \((\mathcal{K}, \xi, U, M)\) together with Borel functions \( f, g \) such that \((\mathcal{K}, \xi, U, M)\) simultaneously measures observables \( A, B \) with \( f, g \) in a state \( \rho \). From the following theorem, the notion of simultaneous measurement is determined by the POVM of a measuring process [42].

**Theorem 9.** A measuring process \((\mathcal{K}, \xi, U, M)\) for a Hilbert space \( \mathcal{H} \) simultaneously measures observables \( A, B \) with Borel functions \( f, g \) in a state \( \rho \) if and only if the POVM \( \Pi \) of the measuring process \((\mathcal{K}, \xi, U, M)\) satisfies

\[
\text{Tr}[\Pi(f^{-1}(\Delta))E^A(\Gamma)\rho] = \text{Tr}[\Pi(g^{-1}(\Delta))E^A(\Gamma)\rho] = 0
\]  (5.5)

for every disjoint Borel subsets \( \Delta, \Gamma \).
Let $C(A_1, A_2, \rho)$ be the projection onto the minimum invariant subspace of $\mathcal{H}$ of $A_1$ and $A_2$ including the range of $\rho$. Let $C(A_1, \rho) = C(A_1, I, \rho)$. The conceptual difference between the commutativity and simultaneous measurability is given by the following theorems [43]; see also M. Ozawa, *Quantum reality and measurement: A quantum logical approach*, Found. Phys. 41 (2011), 592–607.

**Theorem 10.** Two observables $A, B$ are commuting in a state $\rho$ if and only if there exists a POVM $\Pi$ on $\mathbb{R}^2$ such that for every Borel subset $\Delta$ we have

$$
\Pi(\Delta \times \mathbb{R})C(A, B, \rho) = E^A(\Delta)C(A, B, \rho),
$$

$$
\Pi(\mathbb{R} \times \Delta)C(A, B, \rho) = E^B(\Delta)C(A, B, \rho).
$$

**Theorem 11.** Two observables $A, B$ are simultaneously measurable in a state $\rho$ if and only if there exists a POVM $\Pi$ on $\mathbb{R}^2$ such that for every Borel subset $\Delta$ we have

$$
\Pi(\Delta \times \mathbb{R})C(A, \rho) = E^A(\Delta)C(A, \rho),
$$

$$
\Pi(\mathbb{R} \times \Delta)C(B, \rho) = E^B(\Delta)C(B, \rho).
$$

### 5.2 Quantum reality and quantum set theory

Let $S$ be a quantum system described by a Hilbert space $\mathcal{H}$. For any observable $A$ of $S$ and an interval $\Delta$, we denote by $A \in \Delta$ the proposition that the value of the observable $A$ is in the interval $\Delta$, and call it an atomic observational proposition. Observational propositions are those constructed from atomic observational propositions using logical symbols of negation $\neg$, conjunction $\land$, disjunction $\lor$, and implication $\rightarrow$. The lattice $\mathcal{Q}$ of projections on the Hilbert space $\mathcal{H}$ is called the quantum logic of the system $S$; symbols $\land$, $\lor$, and $\perp$ denote meet, join, and orthogonal complement, respectively. We define the $\mathcal{Q}$-valued truth value $\llbracket \phi \rrbracket$ of an observational proposition $\phi$ by the following rules.

(i) $\llbracket A \in \Delta \rrbracket = E^A(\Delta)$.

(ii) $\llbracket \neg \phi \rrbracket = \llbracket \phi \rrbracket^\perp$.

(iii) $\llbracket \phi_1 \land \phi_2 \rrbracket = \llbracket \phi_1 \rrbracket \land \llbracket \phi_2 \rrbracket$.

(iv) $\llbracket \phi_1 \lor \phi_2 \rrbracket = \llbracket \phi_1 \rrbracket \lor \llbracket \phi_2 \rrbracket$.

(v) $\llbracket \phi_1 \rightarrow \phi_2 \rrbracket = \llbracket \phi_1 \rrbracket^\perp \lor (\llbracket \phi_1 \rrbracket \land \llbracket \phi_2 \rrbracket)$.

Then, the Born statistical formula can be extended to the following relation:

$$
\Pr\{A_1 \in \Delta_1, \ldots, A_n \in \Delta_n | \rho\} = \text{Tr}[[A_1 \in \Delta_1 \land \cdots \land A_n \in \Delta_n] \rho].
$$

(5.10)

However, by this method we cannot determine the truth value or the probability of some observational proposition such as $A = B$, meaning that the value of the observable $A$ and the observable $B$ are identical. In the recent investigation [44], it becomes clear that quantum set theory is quite useful for such a problem on extending the probability interpretation of quantum mechanics; in fact, the notion of quantum identical correlations between two observables, which plays an important role in the theory of quantum measurements as mentioned in the preceding subsection, has been shown to be equivalent with the notion of equality between two real numbers in quantum set theory, and hence that notion has acquired a natural and independent motivation.
In 1963 P. J. Cohen proved that the continuum hypothesis is independent from the axioms of ZFC set theory by inventing a new method, called forcing, to construct a new model of ZFC. In 1966 Scott and Solovay reformulated forcing by the method of Boolean-valued models of set theory, which is eventually widely accepted as tractable approach to Cohen’s forcing. In 1981 G. Takeuti introduced quantum set theory by extending the construction of Boolean-valued models from Boolean logic to quantum logic.

In what follows, we survey quantum set theory based on the recent development; see also M. Ozawa, Orthomodular-valued models for quantum set theory, arXiv:0908.0367. Let $Q$ be a complete orthomodular lattice, in which the orthogonal complementation $\perp$ corresponds to negation, the infimum operation $\wedge$ corresponds to disjunction, and the supremum operation $\vee$ corresponds to conjunction. Although the operation $\rightarrow$ corresponding to implication is ambiguous in general, here we define $a \rightarrow b = a^{\perp} \lor (a \land b)$ for all $a, b \in Q$; the operation $\rightarrow$ so defined is often called the Sasaki arrow. The $Q$-valued universe $V^{(Q)}$ of set theory is defined by a transfinite recursion on subclasses $V^{(Q)}_\alpha$ as follows, where $On$ stands for the class of ordinal numbers.

(i) $V^{(Q)}_0 = \emptyset$. (ii) $V^{(Q)}_{\alpha+1} = \{ u \mid u : \text{dom}(u) \rightarrow Q, \ \text{dom}(u) \subseteq V^{(Q)}_\alpha \}$. (iii) For limit ordinal $\alpha$, $V^{(Q)}_\alpha = \bigcup_{\beta < \alpha} V^{(Q)}_\beta$. (iv) $V^{(Q)} = \bigcup_{\alpha \in On} V^{(Q)}_\alpha$.

If $Q$ is a complete Boolean algebra $B$, the model $V^{(Q)}$ coincides with the Scott-Solovay Boolean-valued model $V^{(B)}$. If $Q$ is the projection lattice on a Hilbert space, $V^{(Q)}$ coincides with Takeuti’s model. If $Q = 2 = \{ 0, 1 \}$, this reduces to the usual interpretation of set theory in the two-valued logic.

An element of $V^{(Q)}$ is called a $Q$-valued set. From the above definition, $Q$-valued set $u$ is a function on the set $\text{dom}(u)$, a subset consisting of $Q$-valued sets in some $V^{(Q)}_\alpha$, with values in $Q$, and $u(x)$ essentially represents the truth value in $Q$ of the relation $x \in u$ with an appropriate modification, if necessary. For any $Q$-valued sets $u, v$, the truth values of atomic propositions $u = v$ and $u \in v$ are defined as follows.

(i) $[u = v] = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow [u' \in v]) \land \bigwedge_{v' \in \text{dom}(v)} (v(v') \rightarrow [v' \in u])$.
(ii) $[u \in v] = \bigvee_{u' \in \text{dom}(u)} (v(v') \land [u = v'])$.

Any well-formed formula $\phi$ is constructed from atomic propositions and logical symbols $\neg, \land, \lor, \rightarrow, (\forall x \in y), (\exists x \in y)$, and $(\forall x), (\exists x)$, by well-known composition rules. The quantifiers $(\forall x \in y)$ and $(\exists x \in y)$ are called bounded quantifiers and the quantifiers $(\forall x)$ and $(\exists x)$ are called unbounded quantifiers. Any formula without unbounded quantifiers is called a bounded formula. The truth value of a statement $\phi$ is defined as follows.

(i) $[\neg \phi] = [\phi]^\perp$. (ii) $[\phi_1 \land \phi_2] = [\phi_1] \land [\phi_2]$. (iii) $[\phi_1 \lor \phi_2] = [\phi_1] \lor [\phi_2]$. (iv) $[\phi_1 \rightarrow \phi_2] = [\phi_1] \rightarrow [\phi_2]$.
(v) $[(\forall x \in u) \phi(u')] = \bigwedge_{u' \in \text{dom}(u)} [\phi(u')].$ (vi) $[(\exists u \in u) \phi(u')] = \bigvee_{u' \in \text{dom}(u)} [\phi(u')]$. (vii) $[(\forall x) \phi(x)] = \bigwedge_{u \in V^{(Q)}} [\phi(u)].$
(viii) $[(\exists x) \phi(x)] = \bigvee_{u \in V^{(Q)}} [\phi(u)].$

Let $V$ be the universe of ZFC set theory. For any $a \in V$, the $Q$-valued set $\bar{a}$ is defined by $\text{dom}(\bar{a}) = \{ x \mid x \in a \}$ and $\bar{a}(\bar{x}) = 1$ for all $x \in a$. The correspondence $a \mapsto \bar{a}$ embeds the universe $V$ into the $Q$-valued universe $V^{(Q)}$. Then, the relation between sets $a, b$ is equivalent to the relation between $Q$-valued sets $\bar{a}, \bar{b}$; namely, $a \in b \iff b \in a \iff b = bCa = bCa \neq b$ are equivalent to $[\bar{a} \in \bar{b}] = 1C [\bar{a} \in \bar{b}] = 0C[\bar{a} \in \bar{b}] = 1C [\bar{a} = \bar{b}] = 0$, respectively.

It is an important problem to investigate what statements hold in the $Q$-valued universe. If $Q$ is a complete Boolean algebra $B$, it is well-known that the following transfer principle
holds: for any formula $\phi(x_1, \ldots, x_n)$ provable in ZFC, we have

$$[\phi(u_1, \ldots, u_n)] = 1$$

for every $u_1, \ldots, u_n \in V(B)$. If $Q$ is not distributive, the above transfer principle does not hold in general. For instance, the transitivity of equality nor the substitution law for equality do not hold. However, we can see that the $Q$-valued universe has a rich structure, since it includes many Boolean-valued universes as subuniverses.

A subset $S$ of $Q$ is called a commuting system if every two elements of $S$ commute. For any $Q$-valued sets $u_1, \ldots, u_n$, let $L(u_1, \ldots, u_n)$ be the subset of $Q$ consisting of all elements of $Q$ which are used to construct $u_1, \ldots, u_n$. Let $\bigvee(u_1, \ldots, u_n)$ be the maximum element $p \in Q$ such that $p$ commutes with all elements of $L(u_1, \ldots, u_n)$ and that $p \land L(u_1, \ldots, u_n)$ is a commuting system. Then, for any complete orthomodular lattice $Q$, the following transfer principle holds [44]: for any bounded formula $\phi(x_1, \ldots, x_n)$ provable in ZFC, we have

$$[\phi(u_1, \ldots, u_n)] \geq \bigvee(u_1, \ldots, u_n)$$

for every $u_1, \ldots, u_n \in V(B)$.

It can be seen that the set of natural numbers in $V(Q)$ is $\tilde{\omega}$, and that the set of rational numbers in $V(Q)$ is $\tilde{\mathbb{Q}}$. However, the set of real numbers in $V(Q)$ is not necessarily corresponds to $\mathbb{R}$. Here, the set of real numbers in $V(Q)$ is defined as the $Q$-valued set consisting of the Dedekind cuts of $\tilde{Q}$ in $V(Q)$. Then, we have $[\tilde{\mathbb{R}} \subseteq \mathbb{R}_Q] = 1$.

Suppose that $Q$ is the quantum logic of the system $S$ described by the Hilbert space $\mathcal{H}$, namely, the lattice of projections on $\mathcal{H}$. Then, for every $u$ such that $[u \in \mathbb{R}_Q] = 1$ the projections $E\lambda$ with $\lambda \in \mathbb{R}$ defined by $E\lambda = [u \leq \lambda]$ form as a resolution of the identity, and hence $u$ corresponds to the self-adjoint operator $\hat{u}$ defined by

$$\hat{u} = \int_{\mathbb{R}} \lambda dE\lambda.$$  

The above relation sets up a one-to-one correspondence between the real numbers in $V(Q)$ and the observables of the quantum system $S$. Let $\tilde{A}$ be the real number in $V(Q)$ corresponding to an observable $A$ and let $\tilde{\Delta}$ be the interval in $V(Q)$ corresponding to an interval $\Delta$ in the real line. Then, we have

$$[\tilde{A} \in \tilde{\Delta}] = E^A(\Delta).$$  

Thus, quantum observables are nothing but real numbers in quantum set theory, and observational propositions can be embedded in propositions on the real numbers in quantum set theory without changing their truth values [44].

In quantum set theory the truth value of the equality relation has been defined. Using this, for any observables $A, B$ we can determine the truth value of the observational proposition $A = B$. Namely, we define

$$[A = B] = [\tilde{A} = \tilde{B}].$$  

Then, it can be seen that this equality relation is equivalent to the quantum identical correlation. In fact, we can see that $A \equiv_\rho B$ holds for a state $\rho$ if and only if $\text{Tr}[[\tilde{A} = \tilde{B}] | \rho] = 1$, i.e.,

$$\text{Tr}[[\tilde{A} = \tilde{B}] | \rho] = 1.$$
or equivalently $[\hat{A} = \hat{B}]$ coincides with the projection onto the subspace generated by vector states $\psi$ for which the relation $A \equiv_\psi B$ holds \[4\]. Thus, the notion of the identical correlation between two observables is nothing but the equality relation between two reals in quantum set theory.

As above, quantum set theory is a useful machinery to systematically extend the interpretation of quantum mechanics. We can expect that quantum set theory will play an important role in describing a consistent image of quantum reality, which has been a long standing mystery in modern physics.

Acknowledgements

This work is supported in part by KAKENHI, No.21244007 and No.22654013.

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