Inverses of Motzkin and Schröder Paths

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Abstract

We suggest three applications for the inverses: For the inverse Motzkin matrix we look at Hankel determinants, and counting the paths inside a horizontal band, and for the inverse Schrörder matrix we look at the paths inside the same band, but ending on the top side of the band.

1 Introduction

We adopt the convention that lattice paths without restrictions are called “Grand”; the Grand Catalan numbers (step set \{↑, ↓\}) are the number of paths from the origin, taking only \(↑\) and \(↓\) steps, and ending on the x-axis at \((2n, 0)\). The Grand Catalan numbers are the Central Binomial coefficients, \(\binom{2n}{n}\), with generating function \(1/\sqrt{1-4t} = \sum_{n \geq 0} \binom{2n}{n} t^{2n}\). The weighted Grand Motzkin numbers \(G_n\) take steps from \{↑, ↓, →\}, and end on the x-axis in \((n, 0)\). The horizontal steps get the weight \(\omega\). Their generating function is

\[ g(t) := \sum_{n \geq 0} G_n t^n = 1/\sqrt{(1-\omega t)^2 - 4t^2}, \] (1)

and it is seen immediately that for \(\omega = 0\) the Grand Catalan numbers are recovered. If \(\omega = 2\), the \(1/\sqrt{(1-2t)^2 - 4t^2} = 1/\sqrt{1-4t}\) is again a generating function for the Grand Catalan numbers, but we get \(\sum_{n \geq 0} \binom{2n}{n} t^n\). The general Grand Motzkin numbers \(G(n, j)\) enumerate all paths to \((n, j)\), and the first few are given in the following table.
The general Grand Motzkin numbers \( G_n \) is given in row 0.

The lower half of the table is the mirror image of the top half; if we write the table in matrix form, \( G(n, j) \) stands in row \( n \) and column \( j \), and we obtain a Riordan matrix \( G \), because \( G(n+1, j+1) = G(n, j) + \omega G(n, j+1) + G(n, j+2) \) (see Rogers [9], and [6]). It follows that

\[
\sum_{n \geq j} G(n, j) t^n = \frac{1}{\sqrt{1 - \omega t}^2 - 4t^2} \left( \frac{1}{2t} \left( 1 - t\omega - \sqrt{(\omega t - 1)^2 - 4t^2} \right) \right)^j
\]

\[
= g(t) \left( \frac{1}{2t} (1 - \omega t - 1/g(t)) \right)^j
\]

If we restrict the \( \{\nearrow, \searrow, \omega \} \)-paths to the first quadrant, they become Motzkin paths \( M(n, j) \). We will look at the inverse \( (m_{i,j}) \) of the matrix \( M \),
and find it useful in some applications (see also A. Ralston and P. Rabinowitz, 1978 [8, p. 256]). Especially, the bounded Motzkin numbers \( M_{n;w}^{(k)} \), the number of Motzkin paths staying strictly below the parallel to the \( x \)-axis at height \( k \), have a generating function expressed by the inverse \( \binom{m}{i,j} \), through the inverse Motzkin polynomial \( m_k(t) = \sum_{i=0}^{k} m_{k,i} t^{k-i} \),

\[
\sum_{n \geq 0} M_{n;w}^{(k)} t^n = \frac{m_{k-1}(t)}{m_k(t)}
\]

(see [8]). That makes us wonder if paths with different lengths of the horizontal steps \((w,0)\) have similar properties. In the case of \( w = 2 \) (Schröder paths) and \( \omega = 1 \) we have a result,

\[
S_k^{(k)}(t) := \sum_{n \geq 0} S_{n}^{(k)} t^n = \left(1 - t\right) \frac{\sum_{i=0}^{(k-2)/2} (-1)^i s_{k-2-2i}(t) + (k \mod 2) \left( -1 \right)^{(k-1)/2} t^{k-1}}{\left(1 - t\right) \sum_{i=0}^{(k-1)/2} \left( -1 \right)^i s_{k-1-2i}(t) + ((k - 1) \mod 2) \left( -1 \right)^{k/2} t^k}
\]

where the Motzkin terms \((M \text{ and } m)\) are replaced by the corresponding Schröder terms \((S \text{ and } s)\), and \( s_i(t) \) is the inverse Schröder polynomial. Perhaps more interesting is the generating function identity described in Theorem [8]

\[
t^{-k} S_{k-1}^{(k)}(t) = t^{-k} S^{(k)}(t, k - 1)
\]

(as power series) where \( S^{(k)}(t, k - 1) \) is the generating function of the bounded Schröder number ending on \( y = k - 1 \), just below the upper boundary.

2 Motzkin Numbers

Leaving the Grand Motzkin numbers behind, we introduce the restriction of counting only paths that do not go below the \( x \)-axis. A general weighted Motzkin path is counted by the recursion

\[
M(n, m; \omega) = M(n - 1, m + 1; \omega) + \omega M(n - 1, m; \omega) + M(n - 1, m - 1; \omega)
\]

for \( m \geq 0 \), and \( M(n, m; \omega) = 0 \) if \( m < 0 \). The numbers \( M(n, m; \omega) \) are weighted counts of all such path from \((0,0)\) to \((n, m)\), and we give the special name \( M_{n;w} \) to the Motzkin numbers \( M(n, 0; \omega) \). These numbers (with weight \( \omega = 1 \)) have been studied by Th. Motzkin in 1946 [7].
The above table shows that for $\omega = 1$ the original Motzkin numbers are $1, 1, 2, 4, 9, 21, 51, 127, \ldots$ (sequence A001006 in the On-Line Encyclopedia of Integer Sequences (OEIS)).

It is well-known that the general $\omega$-weighted Motzkin numbers have the generating function

$$\mu(t; j, \omega) := \sum_{n \geq 0} M(n + j, j; \omega) t^n = \left( \frac{1 - \omega t - \sqrt{(1 - \omega t)^2 - 4t^2}}{2t^2} \right)^{j+1}$$

thus

$$\mu(t) := \sum_{n \geq 0} M_n t^n = \sum_{n \geq 0} M(n, 0; \omega) t^n = \frac{1 - \omega t - \sqrt{(1 - \omega t)^2 - 4t^2}}{2t^2}$$

is the generating function of the Motzkin numbers, satisfying the quadratic equation

$$\mu(t) = 1 + \omega t \mu(t) + t^2 \mu(t)^2$$

Hence

$$M_{n+2; \omega} - \omega M_{n+1; \omega} = \sum_{i=0}^{n} M_{i; \omega} M_{n-i; \omega}$$

a well-known identity, combinatorially shown by using the "First Return Decomposition". The generating function (in $t^2$) of the Catalan numbers $C_n$ is easily obtained by setting $\omega = 0$ in (2), but it also follows from $\omega = 2$

$$\frac{1 - 2t - \sqrt{(1 - 2t)^2 - 4t^2}}{2t^2} = \frac{1 - 2t - \sqrt{1 - 4t}}{2t^2} = \sum_{n \geq 1} C_n t^{n-1}$$
(in \( t \)). Or we can choose \( \omega = 1 \) and get

\[
(1 + t) \sum_{n \geq 1} C_n \left( \frac{t}{1 + t} \right)^{n-1} = \sum_{n \geq 0} M_{n;1} t^n
\]

\[
M_{n;1} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} C_{k+1}
\]

For general \( \omega \) follows from (2) the explicit expression

\[
M_{n;\omega} = \sum_{k=0}^{n/2} \binom{n}{2k} \frac{\omega^{n-2k}}{2k+1} \binom{2k+1}{k}.
\]

### 3 The Inverse

Define \( \phi (t) \) such that \( t/\phi (t) \) is the compositional inverse of \( t \mu (t) \) thus

\[
\phi (t \mu (t)) = \mu (t) = 1 + \omega t \mu (t) + t^2 \mu (t)^2
\]

by (3), and therefore

\[
\phi (t) = 1 + \omega t + t^2
\]

This simple form of the inverse is the reason for many special results for Motzkin numbers. Note that

\[
1/\phi (t) = (1 + \omega t + t^2)^{-1} = \sum_{n \geq 0} U_n (-\omega/2) t^n
\]

the generating function of the \textit{Chebychef polynomials} of the second kind.

Because of the inverse relationship between \( t \mu (t) \) and \( t/\phi (t) \) we have that the matrix inverse of \( (M (i, j; \omega))_{n \times n} \) equals \( (m_{i,j})_{n \times n} \),

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
2 & 2 & 1 & 0 & 0 \\
4 & 5 & 3 & 1 & 0 \\
9 & 12 & 9 & 4 & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 \\
0 & -2 & 1 & 0 & 0 \\
1 & 1 & -3 & 1 & 0 \\
-1 & 2 & 3 & -4 & 1
\end{pmatrix} = (m_{i,j})_{4 \times 4}
\]

Inverse Motzkin matrix when \( \omega = 1 \)

where

\[
\sum_{i \geq 0} m_{i,j} t^i = t^j \phi (t)^{-j-1}
\]

Note that \((m_{i,j})\) is also a \textit{Riordan matrix}. The above generating function for \( m_{i,j} \) implies that

\[
m_{i,j} = [t^j] \frac{1}{1 + \omega t + t^2} \left( \frac{t}{1 + \omega t + t^2} \right)^j = [t^{j-1}] (1 + \omega t + t^2)^{-j-1} = C_{i-j}^{j+1} (-\omega/2).
\]
The polynomials \( C_n^\lambda(x) = \sum_{k=0}^{n/2} \frac{(n-k+\lambda-1)}{n-k} \frac{(n-k)}{n-2k} (-1)^k (2x)^{n-2k} \) are the Gegenbauer polynomials, and therefore

\[
m_{i,j} = \sum_{l=0}^{(i-j)/2} \binom{i-l}{i-j-l} \binom{i-j-l}{l} (-1)^l (-\omega)^{i-j-2l}
\]  

(4)

The recurrence relation for the (orthogonal) Gegenbauer polynomials

\[2x(n+\lambda)C_n^\lambda(x) = (n+2\lambda-1)C_{n-1}^\lambda + (n+1)C_{n+1}^\lambda(x)\]

gives us immediately a recurrence for the inverse numbers \( m_{i,j}, 0 \leq j \leq i-1, \)

\[(i-j) m_{i,j} = -\omega m_{i-1,j} - (i+j) m_{i-2,j}\]

with initial values \( m_{i,j} = \delta_{i,j} \) for \( j \geq i \).

We need later in the paper the following Motzkin polynomial

\[
\sum_{j=0}^{k} m_{k,j} t^{k-j} = \sum_{j=0}^{k} C_j^{k-j+1} (-\omega/2)^j t^{j} \\
= \sum_{l=0}^{k/2} \sum_{j=0}^{k-2l} \binom{k-l}{k-j-l} \binom{k-j-l}{k-j-2l} (-1)^l (-\omega)^{k-j-2l} t^{k-j} \\
= \sum_{l=0}^{k/2} \binom{k-l}{l} (-1)^l 2^l (1-\omega) t^{k-2l}
\]  

(5)

From

\[
(M(i,j))_{0 \leq i,j \leq n}^{-1} = (m_{i,j})_{0 \leq i,j \leq n}
\]

follows

\[
\sum_{k=0}^{n} M(k,i;\omega) m_{k,j} = \delta_{i,j}.
\]

However, in the case of Motzkin matrices more than this simple linear algebra result holds.

**Lemma 1** For all nonnegative integers \( i \) and \( k \) holds

\[
M(i,j;\omega) = \sum_{k=0}^{j} m_{j,k} M_{i+k;\omega}
\]

and

\[
m_{i,j} = \sum_{k=0}^{i-j} m_{i+1,j+1+k} M_{k;\omega}
\]
The proof can be done via generating functions. Note that
\[
\sum_{n \geq 0} \sum_{j \geq 0} x^j t^n M(n, j; \omega) = \frac{\mu(t)}{1 - xt\mu(t)} = \frac{1}{1 + \omega x + x^2 - x/t} \left( \mu(t) - \frac{x}{t} \right)
\]
and
\[
\sum_{j \geq 0} x^j \sum_{i \geq j} m_{i,j} t^i = \sum_{j \geq 0} x^j t^j \phi(t)^{j+1} = \frac{\phi(t)}{1 - xt\phi(t)} = \frac{1}{1/\phi(t) - xt} = \frac{1}{1 + \omega t + t^2 - xt}.
\]
Replace \( t \) by \( x \) and \( x \) by \( 1/t \) in the above generating function for the inverse \( m_{i,j} \) to get the Laurent series
\[
\sum_{j \geq 0} t^{-j} \sum_{i \geq j} m_{i,j} x^i = \frac{1}{1 + \omega x + x^2 - x/t}
\]
hence
\[
\sum_{n \geq 0} \sum_{j \geq 0} x^j t^n M(n, j; \omega) = \left( \mu(t) - \frac{x}{t} \right) \sum_{j \geq 0} t^{-j} \sum_{i \geq j} m_{i,j} x^i
\]
Now both sides must be power series in \( x \) and \( t \). This condition gives the Lemma. The Lemma also has the
\[
\sum_{k=0}^{j} m_{j,k} M_{i+k,w} = \delta_{i,j} \text{ for } 0 \leq i \leq j \tag{6}
\]
because \( M(i, j; \omega) = \delta_{i,j} \) for all \( 0 \leq i \leq j \).

4 Two applications of the inverse Motzkin matrix

The Lemma says that
\[
(m_{i,j})_{0 \leq i,j \leq n} = (M_{i+j;\omega})_{0 \leq i,j \leq n} = (M(i, j; \omega))_{0 \leq i,j \leq n}
\]
which gives a direct way of calculating the first Hankel determinant
\[
\det (M_{i+j;\omega})_{0 \leq i,j \leq n} = \frac{1}{\det (m_{i,j})} \det (M(i, j; \omega)) = 1 \tag{7}
\]
However, subsequent Hankel determinants are more complicated; we want to show a way how to calculate a determinant proposed by Cameron and Yip [2]. For a broader theory of Hankel determinants in lattice path enumeration see [3].
4.1 The Hankel determinant \(|\alpha M_{i+j,\omega} + \beta M_{i+j+1,\omega}|_{0 \leq i,j \leq n-1}\)

The Hankel determinant of \((\alpha M_{i+j,\omega} + \beta M_{i+j+1,\omega})_{0 \leq i,j \leq n-1}\) equals for \(\omega = 1\)

\[
\begin{vmatrix}
\alpha + 2\beta & 2\alpha + 4\beta & 4\alpha + 7\beta & \ldots & \alpha M_{n-1;1} + \beta M_{n;1} \\
2\alpha + 4\beta & 4\alpha + 7\beta & 7\alpha + 9\beta & \ldots & M_{n;1} \\
4\alpha + 7\beta & 7\alpha + 9\beta & 4\alpha + 7\beta & \ldots & \vdots & \vdots \\
7\alpha + 9\beta & 9\alpha + 21\beta & 9\alpha + 21\beta & \ldots & \ldots & \ldots \\
\alpha M_{n-1;1} + \beta M_{n;1} & \alpha M_{n;1} + \beta M_{n+1;1} & \alpha M_{n+1;1} + \beta M_{n+2;1} & \alpha M_{2n-2;1} + \beta M_{2n;1} \\
\end{vmatrix}
\]

\[
= (M_{i+j;1})_{0 \leq i,j \leq n-1}
\]

because the last column in the matrix on the right when multiplied with the \(i\)-th row of the matrix on the left gives \(\alpha M_{i+n-1;\omega} - \beta \sum_{k=0}^{n-1} m_{n,k} M_{i+k;\omega} = \alpha M_{i+n-1;\omega} + \beta M_{i+n;\omega} - \beta \delta_{i,n}\) by Corollary 2. Now

\[
= \alpha^{-n} \left( \begin{array}{cccc}
\alpha & 0 & 0 & \ldots & -\beta m_{n,0} \\
\beta & \alpha & 0 & \ldots & -\beta m_{n,1} \\
0 & \beta & \alpha & \ldots & -\beta m_{n,2} \\
\vdots & & & & \\
0 & 0 & \ldots & \alpha & -\beta m_{n,n-2} \\
0 & 0 & \ldots & \beta & \alpha - \beta m_{n,n-1} \\
\end{array} \right)
\]

\[
= \alpha^{-n} \left( \begin{array}{cccc}
\alpha & 0 & 0 & \ldots & -\beta m_{n,0} \\
\alpha \beta & \alpha^2 & 0 & \ldots & -\alpha^2 \beta m_{n,1} \\
0 & \alpha^2 \beta & \alpha^3 & \ldots & -\alpha^3 \beta m_{n,2} \\
\vdots & & & & \\
0 & 0 & \ldots & \alpha^{n-1} \beta & -\alpha^{n-1} \beta m_{n,n-2} \\
0 & 0 & \ldots & \alpha^{n-1} \beta & \alpha^{n-1} - \alpha^{n-1} \beta m_{n,n-1} \\
\end{array} \right)
\]

\[
= \alpha^{-n} \left( \begin{array}{cccc}
\alpha & 0 & 0 & \ldots & -\beta m_{n,0} \\
0 & \alpha^2 & 0 & \ldots & \beta^2 m_{n,0} - \alpha \beta m_{n,1} \\
0 & 0 & \alpha^3 & \ldots & -\beta^3 m_{n,0} + \alpha \beta^2 m_{n,1} - \alpha^2 \beta m_{n,2} \\
\vdots & & & & \\
0 & 0 & \ldots & \alpha^{n-1} & -\sum_{i=0}^{n-2} (-1)^{n-2-i} \beta^{n-1-i} \alpha^i m_{n,i} \\
0 & 0 & \ldots & 0 & \alpha^n - \sum_{i=0}^{n-1} (-1)^{n-1-i} \beta^{n-i} \alpha^i m_{n,i} \\
\end{array} \right)
\]

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Therefore \( \det \left( \alpha M_{i+j;\omega} + \beta M_{i+j+1;\omega} \right)_{0 \leq i,j \leq n-1} = \alpha^n - \sum_{i=0}^{n-1} (-1)^{n-1-i} \beta^{n-i} \alpha^i m_{n,i} = \sum_{i=0}^{n} (-\beta)^{n-i} \alpha^i m_{n,i} = \sum_{i=0}^{n} (-\beta)^{n-i} \alpha^i P^*_{n-i} \left( \frac{-\omega}{2} \right) \). This can be written explicitly as \( \det \left( \alpha M_{i+j;\omega} + \beta M_{i+j+1;\omega} \right)_{0 \leq i,j \leq n-1} = \)

\[
(-\beta)^n \sum_{k=0}^{n/2} (-\alpha/\beta)^k m_{n,k}
= (-\beta)^n U_n \left( \frac{-\alpha/\beta - \omega}{2} \right) = (-\beta)^n \sum_{k=0}^{n/2} \binom{n-k}{k} (-1)^k (-\alpha/\beta - \omega)^{n-2k}
= \sum_{k=0}^{n/2} \binom{n-k}{k} (-1)^k \beta^{2k} (\alpha + \beta \omega)^{n-2k}
= \frac{2^{-n-1}}{\sqrt{(\alpha + \omega \beta)^2 - 4\beta^2}} \times \left( \left( \sqrt{(\alpha + \omega \beta)^2 - 4\beta^2 + \alpha + \omega \beta} \right)^{n+1} + \left( \sqrt{(\alpha + \omega \beta)^2 - 4\beta^2 - \alpha - \beta \omega} \right)^{n+1} \right)
\]

If \( \alpha = \beta = 1 \), then \( \det \left( (M_{i+j;\omega} + M_{i+j+1;\omega})_{0 \leq i,j \leq n-1} \right) = \)

\[
\frac{1}{2^{n+1}(\omega + 1)^2 - 4} \left( \left( 1 + \omega + \sqrt{(\omega + 1)^2 - 4} \right)^{n+1} - \left( 1 + \omega - \sqrt{(\omega + 1)^2 - 4} \right)^{n+1} \right)
= \sum_{k=0}^{n} (-1)^{n-k} \binom{k}{n-k} (\omega + 1)^{2k-n}
\]

which approaches \( n + 1 \) if \( \omega \to 1 \). In the case of Dyck path, we obtain \( \delta_{0,n} \) for this determinant of the sum of matrices. If \( \beta = 1 \) and \( \alpha = 0 \), then the determinant is the second Hankel determinant of the Motzkin numbers,

\[
det \left( (M_{i+j+1;\omega})_{0 \leq i,j \leq n-1} \right) = \sum_{k=0}^{n/2} \binom{n-k}{k} (-1)^k \omega^{n-2k}
\]

If \( \alpha = 1 \) and \( \beta = 0 \) then \( \det (M_{i+j;\omega})_{0 \leq i,j \leq n-1} = 1 \), independent of \( \omega \) (see (7). The same approach also shows the recursion

\[
|M_{i+j+2;\omega}|_{0 \leq i,j \leq n-1} = |M_{i+j+2;\omega}|_{0 \leq i,j \leq n-2} + |M_{i+j+1;\omega}|_{0 \leq i,j \leq n-1}
\]
4.2 Motzkin in a band

The number of Motzkin paths staying strictly below the line \( y = k \) for \( k > 0 \) is known to have the generating function [4, Proposition 12]

\[
\sum_{n \geq 0} M_n^{(k)} t^n = \mu(t) \frac{1 - (t\mu(t))^{2k}}{1 - (t\mu(t))^{2(k+1)}} = \frac{1}{t} \left( \frac{1}{t\mu(t)} \right)^{k+1} - \frac{(t\mu)^{k+1}}{t}
\]

| \( k \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | \( n \) |
|-----|---|---|---|---|---|---|---|---|---|-----|
| 0   | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0   |
| 1   | 0 | 1 | 2 | 4 | 9 | 21 | 51 | 127 | 323 | 835 |
| 2   | 0 | 1 | 3 | 9 | 25 | 69 | 189 | 518 | 1422 |
| 3   | 0 | 1 | 4 | 14 | 44 | 133 | 392 | 1140 |
| 4   | 0 | 1 | 5 | 20 | 69 | 217 | 657 | 2002 |
| 5   | 0 | 1 | 6 | 28 | 102 | 369 | 1257 | 4017 |
| 6   | 0 | 1 | 7 | 38 | 140 | 546 | 1968 | 6725 |

Thus \( M_n^{(k)} \) is given in row 0.

From \( \mu(t) (1 - \omega t) - 1 = t^2 \mu(t)^2 \) (see [4]) follows

\[
\begin{align*}
\mu_{1,2}(t) &= \left( 1 - \omega t \pm \sqrt{(1 - \omega t)^2 - 4t^2} \right) / (2t^2) \\
t\mu_{1,2}(t) &= \left( 1 - \omega t \pm \sqrt{(1 - \omega t)^2 - 4t^2} \right) / (2t)
\end{align*}
\]

thus

\[
\mu_1 + \mu_2 = (1 - \omega t) / t^2 \text{ and } \mu_1 \mu_2 = 1 / t^2
\]

Hence

\[
\sum_{n \geq 0} M_n^{(k)} t^n = \frac{1}{t} \frac{(t\mu_1)^{k} - (t\mu_2)^{k}}{(t\mu_1)^{k+1} - (t\mu_2)^{k+1}} = \frac{1}{t} \frac{(t\mu_2)^{k} - (t\mu_1)^{k}}{(t\mu_2)^{k+1} - (t\mu_1)^{k+1}}
\]

\[
\begin{align*}
&= \sum_{j=0}^{(k-1)/2} \frac{(1)^j (k-1-2j) (1-\omega t)^{k-1-2j}}{1} \\
&= \sum_{j=0}^{k/2} (1)^j (k-j) (1-\omega t)^{k-2j} \\
&= \frac{\sum_{i=0}^{k} m_{k-i} t^{k-i}}{\sum_{i=0}^{k} m_{k-i} t^{k-i}}
\end{align*}
\]

(see [5]). The OEIS lists many special cases for \( k \); here are a few, with \( \omega = 1 \).

1. \( \sum_{n \geq 0} M_n^{(1)} t^n = \frac{1}{1-t} \leftrightarrow 1, 1, 1, 1, \ldots \)

2. \( \sum_{n \geq 0} M_n^{(2)} t^n = \frac{1-t}{(1-t)^2} = 1 + t + 2t^2 + 4t^3 + 8t^4 + 16t^5 \ldots \) thus \( 1, 1, 2, 4, 8, 16, 32, 64, \ldots \) the powers of 2.

3. \( \sum_{n \geq 0} M_n^{(3)} t^n = \frac{2t-1}{(1-t)(2t+2t^2-t^3)} \) thus \( 1, 1, 2, 4, 9, 21, 50, 120, \ldots \) (A171842)
4. \( \sum_{n \geq 0} M^{(4)}_{n,1} t^n = \frac{(1 - 3t + t^2 + t^3)}{(1 - 4t + 3t^2 + 2t^3 - t^4)} \), thus 1, 1, 2, 4, 9, 21, 51, 127, 322, 826, \ldots: (A005207), generating function by Alois P. Heinz.

The special form of the generating function
\[
\sum_{n \geq 0} M^{(k)}_{n,\omega} t^n = \sum_{i=0}^{k} \frac{m_{k-1,i} t^{k-i}}{m_{k,i} t^{k-i}}
\] (8)
works with weight \( \omega \), for all \( k = 1, 2, \ldots \). It is equivalent to the recursion
\[
\sum_{j=0}^{n} M^{(k)}_{n-j} m_{k,k-j} = 0 \quad \text{for all } n \geq k, \text{ with initial values } \sum_{j=0}^{n} M^{(k)}_{n-j} m_{k,k-j} = m_{k-1,k-1-n} \text{ for all } n = 0, \ldots, k - 1.
\]

5 **Horizontal steps of length** \( w \)

A “natural” generalization of Motzkin paths is a lattice path \( W \) that takes horizontal steps of some positive length \( w \), weighted by \( \omega \). We would like to see similar results as (8) in such cases. However, we have a result only for the case \( w = 2 \), the Schröder paths.

| \( n \) | \( \omega \) | \( \mu_w (t; \omega) \) |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 6 |
| 2 | 10 | 26 |
| 3 | 14 | 52 |
| 4 | 20 | 108 |
| 5 | 28 | 192 |
| 6 | 35 | 288 |
| 7 | 48 | 384 |
| 8 | 63 | 512 |

\( \mu_w (t; \omega) \) is well known,
\[
\sum_{n \geq 0} W_{n,\omega} t^n = \frac{1 - \omega t^w - \sqrt{(1 - \omega t^w)^2 - 4t^2}}{2t^2} := \mu_w (t; \omega)
\] (9)
The recursion can be reformulated as
\[ W(n, j; \omega) = W(n + 1, j - 1; \omega) - W(n, j - 2; \omega) - \omega W(n + 1 - w, j - 1; \omega) \quad \text{for} \ m \geq n \]

We find the generating function identity
\[ \sum_{i \geq 0} W(i, j; \omega) t^i = \]
\[-\omega \sum_{i \geq w-1} W(i + 1 - w, j - 1; \omega) t^i \]
\[= \sum_{i \geq 0} W(i + 1, j - 1; \omega) t^i - \omega \left( \sum_{i \geq 1} W(i + 1, j - 1; \omega) t^{i+1+w-1} \right) \]
\[-\sum_{i \geq 0} W(i, j - 2; \omega) t^i \]
\[= t^{-1} \sum_{i \geq 0} W(i + 1, j - 1; \omega) t^{i+1} - \omega t^{w-1} \left( \sum_{i \geq 1} W(i + 1, j - 1; \omega) t^{i+1} \right) \]
\[-\sum_{i \geq 0} W(i, j - 2; \omega) t^i \]
\[= (t^{-1} - \omega t^{w-1}) \left( \sum_{i \geq 0} W(i, j - 1; \omega) t^i - \delta_{j,1} \right) - \sum_{i \geq 0} W(i, j - 2; \omega) t^i \]

Let \( \mathcal{W}(t, j; \omega) = \sum_{i \geq 0} W(i, j; \omega) t^i \). In this notation,
\[ \mathcal{W}(t, j; \omega) = \frac{1 - \omega t^w}{t} \mathcal{W}(t, j - 1; \omega) - \mathcal{W}(t, j - 2; \omega) \quad \text{for} \ j > 1 \quad (10) \]
\[ \mathcal{W}(t, 1; \omega) = \frac{1}{t} \left( (1 - \omega t^w) \mathcal{W}(t, 0; \omega) - 1 \right) \]

For example,
\[ \mathcal{W}(t, 2; \omega) = \frac{(1 - \omega t^w)}{t} \mathcal{W}(t, 1; \omega) - \mathcal{W}(t, 0; \omega) = \frac{(1 - \omega t^w)}{t} \left( (1 - \omega t^w) \mathcal{W}(t, 0; \omega) - 1 \right) - \mathcal{W}(t, 0; \omega) \]
\[= \left( \frac{(1 - \omega t^w)^2}{t^2} - 1 \right) \mathcal{W}(t, 0; \omega) - \frac{(1 - \omega t^w)}{t^2}, \quad \text{and} \quad \mathcal{W}(t, 0; \omega) = \mu_w(t; \omega) \text{ is given in} \] (10).

\[ \mathcal{W}(t, 3; \omega) = \frac{(1 - \omega t^w)}{t} \mathcal{W}(t, 2; \omega) - \mathcal{W}(t, 1; \omega) \]
\[= \frac{(1 - \omega t^w)}{t^2} \left( \left( \frac{(1 - \omega t^w)^2}{t^2} - 1 \right) \mathcal{W}(t, 0; \omega) - \frac{(1 - \omega t^w)}{t^2} \right) - \frac{t}{t^2} \left( (1 - \omega t^w) \mathcal{W}(t, 0; \omega) - 1 \right) \]
\[= \left( \frac{(1 - \omega t^w)^2}{t^2} - 2 \right) \frac{(1 - \omega t^w)}{t^2} \mu_w(t; \omega) + \frac{1}{t^2} - \frac{(1 - \omega t^w)^2}{t^2} \]

We find an explicit expression for \( \mathcal{W}(t, j; \omega) \) in the next section.

5.2 Solution to Recursion for \( \mathcal{W} \) and \( \mathcal{W}^{(k)} \)

The linear recursion (10) is called Fibonacci-like. It is of the form
\[ \sigma_n = u \sigma_{n-1} + v \sigma_{n-2} \]
with \( u = \frac{1 - \omega t^w}{t^w} \) and \( v = -1 \), for \( n > 1 \). We know the initial values \( \sigma_0 \) and \( \sigma_1 = u \sigma_0 - 1 / t \).

Hence \( \sigma_n = [t^n] \frac{\sigma_0 + (\sigma_1 - u \sigma_0)t}{1 - u - v t} = [t^n] \frac{\sigma_0 - \sigma_1}{1 - u - v t} \) in this case, or \( \sigma_n = [t^n] \left( \frac{\sigma_0 - \sigma_1}{t} \right) \sum_{i=0}^{\infty} \binom{i}{j} (-1)^j \omega^{i-j} t^{j+2j} \)

\[
= \sigma_0 \sum_{j=0}^{n} \binom{n-j}{j} (-1)^j \left( \frac{1 - \omega t^w}{t} \right)^{n-2j} - \frac{1}{t} \sum_{j=0}^{n-1} \binom{n-1-j}{j} (-1)^j \left( \frac{1 - \omega t^w}{t} \right)^{n-1-2j}
\]

Let us define

\[
p_n(t) := \sum_{j=0}^{n} \binom{n-j}{j} \left( \frac{1 - \omega t^w}{t} \right)^{n-2j} (-1)^j
\]

Hence

\[
\mathcal{W}(t, j; \omega) = \left( \frac{1 - \omega t^w - \sqrt{(1 - \omega t^w)^2 - 4t^2}}{2t^2} \right) p_j(t) - p_{j-1}(t) / t
\]

where \( p_j = 0 \) for all \( j < 0 \).

The generating function \( \mathcal{W}^{(k)}(t, j; \omega) = \sum_{n \geq 0}^{(k)} W^{(k)}(n, j; \omega) t^n \) is generating the case where the lattice paths stay strictly below \( y = k \); the numbers \( W^{(k)}(n, j; \omega) \) are the number of paths with \( \omega \)-weighted horizontal steps of length \( w \), and diagonal up and down steps, that do not reach the line \( y = k \), and stay above the \( x \)-axis. That means, \( 0 \leq j < k \). We also know \( \mathcal{W}^{(k)}(t, 0; \omega) \)

\[
= \sum_{n \geq 0} W^{(k)}_n t^n = \mu_w(t; \omega) \frac{1 - (t\mu_w(t; \omega))^{2k}}{1 - (t\mu_w(t; \omega))^{2(k+1)}}
\]

\[
= \frac{1 - \omega t^w - \sqrt{(1 - \omega t^w)^2 - 4t^2}}{2t^2} \frac{1 - \left( \frac{1 - \omega t^w - \sqrt{(1 - \omega t^w)^2 - 4t^2}}{2t} \right)^{2(k+1)}}{1 - \left( \frac{1 - \omega t^w - \sqrt{(1 - \omega t^w)^2 - 4t^2}}{2t} \right)^{2(k+1)}}
\]

The recursion is the same as for \( \mathcal{W}(t, j; \omega) \). Only the initial values have changed (see \( \mathcal{W}^{(k)}(t, 0; \omega) \) above).

We get

\[
\mathcal{W}^{(k)}(t, j; \omega) = \left( \mu_w(t; \omega) \frac{1 - (t\mu_w(t; \omega))^{2k}}{1 - (t\mu_w(t; \omega))^{2(k+1)}} \right) p_j(t) - p_{j-1}(t)
\]

and \( \sum_{n \geq 0} W^{(k)}_n t^n \)
The power series \( \sum_{n=0}^{\infty} S(n,j;\omega) t^j \) is given in (12). For the compressed Schröder numbers this equation says

\[
S(t,j;\omega) = \sum_{n=0}^{\infty} S(n,j;\omega) t^j = \left( \frac{1 - \omega t - \sqrt{(1-\omega t)^2 - 4t}}{2t} \right) p_j(t) - p_{j-1}(t)/t
\]
where

\[ p_n(t) = t^{-n} \sum_{j=0}^{n} \binom{n-j}{j} t^j (1 - \omega t)^{n-2j} (-1)^j \]  \hspace{1cm} (15)

All references to Schröder numbers will from now on mean the compressed Schröder numbers. Note that

\[ S^{(k)}(t; \omega) = \sum_{n \geq 0} S_n^{(k)} t^n = \frac{p_{k-1}(t)}{t p_k(t)} \]  \hspace{1cm} (16)

by (14).

### 6.1 Inverse Schröder numbers

From (14) we see that

\[ \mu_s(t) = 1 + \omega t^2 \mu_s(t) + t^2 \mu_s(t)^2. \]

Hence

\[ \phi(t \mu_s(t)) = \mu_s(t) = 1 + \omega t^2 \mu_s(t) + t^2 \mu_s(t)^2 \]

\[ \phi(t) = 1 + \frac{\omega t^2}{\phi(t)} + t^2 \]

thus \( \phi(t) = \frac{1}{2} + \frac{1}{2} t^2 + \frac{1}{2} \sqrt{(1 + t^2)^2 + 4t^2 \omega}, \) a power series in \( t^2. \) We let \( \xi = t^2 \)

and get

\[ \phi(\xi) = \frac{1}{2} + \frac{1}{2} \xi + \frac{1}{2} \sqrt{(1 + \xi)^2 + 4\xi \omega} \]

\[ \mu_s(\xi) = 1 + \omega \xi \mu_s(t) + \xi \mu_s(\xi)^2 \]

\[ = \frac{1 - \omega \xi - \sqrt{(1 - \omega \xi)^2 - 4\xi}}{2 \xi} \]

Lagrange inversion tells us that for all \( 0 \leq i \leq k \) holds

\[ (i + 1) \left[ \mu_s^{-k-1} \right]_{k-i} = (k + 1) \left[ \phi^{-i-1} \right]_{k-i} = (k + 1) s_{k,i} \]
and therefore

\[ s_{k,j} = [\mu_{s}^{-k-1}]_{k-j} = \frac{j+1}{k+1} \sum_{m=0}^{k-j} \left( \frac{1 - \omega t - \sqrt{(1 - \omega t)^2 - 4t}}{2t} \right)^{k-j} \]

\[ = \frac{j+1}{k+1} \sum_{m=0}^{k-j} \left( \frac{1 - \omega t + \sqrt{(1 - \omega t)^2 - 4t}}{4t} \right)^{k-j} \]

\[ = \frac{j+1}{k+1} \left( -1 \right)^{k-j} \sum_{m=0}^{k-j} \frac{(k+1-2m)}{(k-j-m)} \frac{j+1}{k-m+1} \left( \frac{1 - \omega t}{n} \right)^{k-j-m}, \]

the compressed weighted inverse Schröder numbers. We need the following polynomials: \( s_{n,k} t^{n-k} \)

\[ = \sum_{n=0}^{\infty} \frac{1}{n-m+1} \left( \frac{n-m+1}{n-k} \right) \left( \frac{n+1-2m}{n-k-m} \right) t^{n-k} (-1)^{n-k} \frac{1}{n-m+1} \left( \frac{n-m+1}{n-k-m} \right) \left( \frac{n+1-2m}{n-k-m} \right) t^{n-k} \]

\[ = t^{n} \sum_{m=0}^{n} \frac{1}{n-m+1} \left( \frac{n-m+1}{n-k} \right) \left( \frac{n+1-2m}{n-k-m} \right) (-1)^{n+1} \left( 1 - \omega t \right)^{n-2m} t^{m} \]

Hence

\[ s_{n}(t) = \sum_{k \geq 0} s_{n,k} t^{n-k} \]

\[ = \sum_{m=0}^{n} \left( \frac{t \omega m}{n-m+1} - 1 \right) \left( \frac{n-m+1}{m} \right) (-1)^{m+1} \left( 1 - \omega t \right)^{n-2m} t^{m} \]

\[
\begin{array}{cccccc}
1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 \\
2 & -4 & 1 & 0 & 0 & 0 \\
-2 & 8 & -6 & 1 & 0 & 0 \\
2 & -12 & 18 & -8 & 1
\end{array}
\]

The compressed inverse \((s_{n,k})\) for \(\omega = 1\)
This matrix is A080246 in the OEIS. At the same reference we find the generating function of the $k$-th column,

$$\sum_{n \geq k} s_{n,k} t^n = \left(\frac{1 - t}{1 + t}\right)^k.$$  

Also,

$$\sum_{n \geq 0} \sum_{k=0}^n s_{n,k} t^{n-k} = \sum_{k=0}^\infty t^{-k} \left(\frac{1 - t}{1 + t}\right)^k = \frac{t(1 + t)}{2t + t^2 - 1}.$$  

Example: $\sum_{n \geq 0} s_{4,k} t^{4-k} = s_4(t) = 1 - 8t + 18t^2 - 12t^3 + 2t^4$.

6.2 Delannoy numbers

The numbers $D(n,k) = \sum_{l=0}^n \binom{k}{l} \binom{n+k-l}{k} \omega^l$ are the Delannoy numbers; the numbers $D(n,n+j)$ are counting all weighted Grand Schröder paths to $(2n+j,j)$. Hence they satisfy the recursion

$$D(n+1,n+j) = \omega D(n-1,n-1+j) + D(n,n+j-1) + D(n-1,n+j)$$  \hspace{1cm} (18)

| $j$ | 1 | 0 | 7 + 6$\omega$ | 6 + 5$\omega$ | 21 + 30$\omega + 10\omega^2$ | 129 |
|-----|---|---|---------------|-------------|------------------|-----|
| 5   | 1 | 0 | 4 + 3$\omega$  | 0           | 15 + 20$\omega + 6\omega^2$ | 0   |
| 4   | 1 | 0 | 3 + 2$\omega$  | 0           | 10 + 12$\omega + 3\omega^2$ | 0   |
| 3   | 1 | 0 | 5 + 4$\omega$  | 0           | 6 + 6$\omega + \omega^2$ | 63  |
| 2   | 1 | 0 | 0             | 0           | 0                | 7   |
| 1   | 1 | 0 | 0             | 0           | 0                | 0   |
| 0   | 1 | 2 | 2 + $\omega$  | 0           | 6 + $\omega + \omega^2$ | 0   |
| $n \to$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |

Uncompressed Grand Schröder numbers ($\omega = 1$)

The generating function

$$\sum_{n=0}^\infty \sum_{l=0}^n \binom{k}{l} \binom{n+k-l}{n-l} \omega^l t^n = \frac{1}{1 - t} \left(\frac{1 + t\omega}{1 - t}\right)^k$$

shows that $D(n,k)$ is a Sheffer polynomial of degree $n$ in $k$. The Delannoy polynomial is of the form

$$d_k(t) = \sum_{j=0}^{k} D(k-j,j) t^j = \sum_{j=0}^{k} \sum_{l=0}^{k-j} \binom{j}{l} \binom{k-l}{j} \omega^l t^j = \sum_{l=0}^{k} \binom{k-l}{l} \omega^l t^l (1 + t)^{k-2l}$$  \hspace{1cm} (17)
and has generating function
\[
\sum_{k=0}^{\infty} d_k(t)x^k = \frac{1}{1 - x - t(x + \omega x^2)}.
\]

From (18) follows for \(\omega = 1\) that
\[
d_{k-1}(t) = td_{k-1}(t) + td_{k-2}(t) + d_k(t)
\]
Also for \(\omega = 1\) holds
\[
p_k(t) = t^{k-1} \sum_{l=0}^{k} \binom{k-l}{l} t^l (1 - t)^{k-2l} \omega = t^{k-1} d_k(-t)
\]
(see (15)). Hence
\[
S^{(k)}(t; 1) = \sum_{n \geq 0} S_n^{(k)} t^n = \frac{d_{k-1}(-t)}{tp_k(t)} = \frac{d_{k-1}(-t)}{d_k(-t)}
\]
(19)
by (16). This shows an intimate connection between the generating function of the Schröder numbers in a band and the Delannoy polynomials, when \(\omega = 1\).

The Delannoy polynomials at negative argument, \(d_k(-t)\), satisfy for \(\omega = 1\) the same recursion as \(d_k(t)\),
\[
d_{k-1}(-t) = td_{k-1}(-t) + td_{k-2}(-t) + d_k(-t).
\]
This follows again from (18).

Another connection exists with the inverse polynomial \(s_n(t)\); from (17)
\[
s_n(t) = \sum_{k \geq 0} s_{n,k} t^{n-k} =
\]
t\(\omega \sum_{m=1}^{n} \binom{n-m}{m-1} (-1)^{m+1} (1 - \omega t)^{n-2m} t^m - \sum_{m=0}^{n} \binom{n-m+1}{m} (-1)^{m+1} (1 - \omega t)^{n-2m} t^m
\]
follows for \(\omega = 1\)
\[
s_n(t) = \frac{t^2}{1-t} d_{n-1}(-t) + d_{n+1}(-t) / (1-t).
\]
(20)
and vice-versa,
\[
d_{n+1}(-t) = (1-t) s_n(t) - t^2 d_{n-1}(-t)
\]
\[
= (1-t) \sum_{i=0}^{n/2} t^{2i} (-1)^{i} s_{n-2i}(t) + (n \mod 2) (-1)^{(n+1)/2} t^{n+1}
\]
Hence the generating function of the bounded Schröder numbers can for \(\omega = 1\) be written as
\[
S^{(k)}(t; 1) = \sum_{n \geq 0} S_n^{(k)} t^n = \frac{(1-t) \sum_{i=0}^{(k-2)/2} t^{2i} (-1)^{i} s_{k-2i-2}(t) + (k \mod 2) (-1)^{(k-1)/2} t^{k-1}}{(1-t) \sum_{i=0}^{(k-1)/2} t^{2i} (-1)^{i} s_{k-1-2i}(t) + ((k-1) \mod 2) (-1)^{k/2} t^{k}}
\]
7 Schröder in a Band

From (19) follows for \( \omega = 1 \) the generating function of the (compressed) bounded (by \( k \)) Schröder numbers,

\[
S^{(k)}(t; 1) = \frac{d_{k-1}(-t)}{d_k(t)}
\]

(21)

Example: \( S^{(4)}(t; 1) = \frac{d_3(-t)}{d_4(t)} = \frac{1-5t+5t^2-t^3}{1-7t+13t^2-7t^3+t^4} \)

\[
= 1 + 2t + 6t^2 + 22t^3 + 89t^4 + 377t^5 + 1630t^6 + 7110t^7 + 31130t^8 + 136513t^9 + 599041t^{10} + 2629418t^{11} + 11542854t^{12} + 50674318t^{13} + 222470009t^{14} + 976694489t^{15} + 4287928678t^{16} + O(t^{17})
\]

The compressed bounded (\( k = 4 \)) Schröder numbers (\( \omega = 1 \))

\[
\begin{array}{cccccccccc}
\uparrow m \\
k=4 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
3 & 1 & 7 & 36 & 168 & 756 & 3353 \\
2 & 1 & 6 & 29 & 132 & 588 & 2597 & 11430 \\
1 & 1 & 4 & 16 & 67 & 288 & 1253 & 5480 & 24020 \\
0 & 1 & 2 & 6 & 22 & 89 & 377 & 1630 & 7110 & 31130 \\
\end{array}
\]

From the recursion (13) follows that

\[
d_{k-1}(-t) = td_{k-1}(-t) + td_{k-2}(-t) + d_k(-t)\]

and therefore

\[
d_{k-1}(-t) = \frac{t}{1-t}d_{k-2}(-t) + \frac{1}{1-t}d_k(-t)
\]

Hence

\[
d_{k-1}(-t) - td_{k-2}(-t) = \frac{t^2}{1-t}d_{k-2}(-t) + \frac{1}{1-t}d_k(-t) = s_{k-1}(t)
\]

(see (20)) and

\[
\frac{d_{k-1}(-t)}{d_k(-t)} \left( d_{k-1}(-t) - td_{k-2}(-t) \right) - \frac{d_{k-1}(-t)}{d_k(-t)} s_{k-1}(t) = 0
\]

Therefore

\[
\frac{d_{k-1}(-t)}{d_k(-t)} s_{k-1}(t) = S^{(k)}(t, k-1; 1) - (d_{k-2}(-t) - td_{k-3}(-t))
\]

(see (21)).

**Theorem 3** The power series part of \( t^{-k}S^{(k)}(t; 1) s_{k-1}(t) \) equals \( t^{-k}S^{(k)}(t, k-1; 1) \).

Example: (a) \( t^{-4}S^{(4)}(t; 1) s_3(t) = \frac{(t-1)(2t^2-8t^2+6t-1)(1-4t^2)}{(1-4t^2+7t^3-7t^4+7t^5)t^4} \)

\[
= (t^{-4} - 4t^{-3} + 2t^{-2}) + 1 + 7t + 36t^2 + 168t^3 + 756t^4 + 3353t^5 + 14783t^6 +
\]
\[ 65016t^7 + 285648t^8 + 1254456t^9 + 5508097t^{10} + 24183271t^{11} + 106173180t^{12} + O(t^{13}) \]
\[ (b) \ t^{-4} S^{(4)}(t, 4 - 1; 1) = t^{-4} \frac{1-5t+5t^2-t^3}{1-7t+13t^2-7t^3+t^4} \left( 1 - 6t + 8t^2 - 2t^3 \right) - \frac{2t^2+1-4t}{t^4} = \]
\[ 1 + 7t + 36t^2 + 168t^3 + 756t^4 + 3353t^5 + 14783t^6 + 65016t^7 + 285648t^8 + 1254456t^9 + 5508097t^{10} + 24183271t^{11} + 106173180t^{12} + O(t^{13}) \]

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