Topology of time-reversal invariant energy bands with adiabatic structure

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Abstract
We classify the topology of bands defined by the energy states of quantum systems with scale separation between slow and fast degrees of freedom, invariant under fermionic time reversal. Classical phase space transforms differently from momentum space under time reversal, and as a consequence the topology of adiabatic bands is different from that of Bloch bands. We show that bands defined over a two-dimensional phase space are classified by the Chern number, whose parity must be equal to the parity of the band rank. Even-rank bands are equivalently classified by the Kane–Mele index, an integer equal to one half the Chern number.

Keywords: adiabatic approximation, space adiabatic, topological invariants, time reversal invariance, semiclassical approximation

(Some figures may appear in colour only in the online journal)

1. Introduction
Spectral bands arise in stationary quantum systems in two principal ways. When a system has a non-compact symmetry group, states are labelled by continuous representation indices. A prominent example is a particle in a periodic potential, whose states are labelled by the lattice momentum. The other possibility is that there is a separation of time scales, where the slow degrees of freedom control adiabatically the dynamics of the fast degrees of freedom. The dynamics of molecules in which the nuclei are several of orders of magnitude heavier than the electrons is an example of this type. The limit where the scale separation tends to infinity can be obtained formally by letting the effective Planck constant tend to zero for the slow subsystem, and energy bands (or energy surfaces in the Born–Oppenheimer terminology) are parametrised by the slow positions and momenta, belonging to a classical phase space.

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In both cases the spectral problem reduces to the diagonalisation of a band Hamiltonian $H$ depending parametrically on either the lattice momenta or the slow phase space coordinates, so that an $H$-invariant subspace spanned by a subset of eigenvectors with energies separated from the rest of the spectrum by a parameter-independent gap defines a vector bundle over the space of parameters. Physical observables depending only on the topological properties of these band vector bundles are robust, in the sense that they do not change under continuous variations of the Hamiltonian as long as the gaps do not close.

The Chern number, which classifies vector bundles over two-dimensional manifolds [1], has the physical significance of quantum Hall conductance in systems with a periodic potential [2] and anomalous band multiplicity in systems with slow degrees of freedom [3–5]. In systems with a periodic potential, the Chern number is mapped to its negative under time reversal (TR), and therefore Bloch bands in time-reversal invariant (TRI) systems have zero Chern number. Nevertheless, Kane and Mele have shown [6] that when TR is fermionic, so that the TR operator squares to $-1$, there are TRI bands that can be trivialised only by relaxing TR. Such bands are characterised by a nonzero value of the $\mathbb{Z}_2$-valued Kane–Mele (KM) index, observable experimentally through the presence of conducting edge states. This idea has been generalised to systems with more than two dimensions, other discrete symmetries, and systems with disorder and interactions, giving rise to the field of topological insulators [7, 8].

In contrast, an analogous theory has not been developed for adiabatic systems. Iwai and Zhilinskii [9–11] studied Chern numbers and multiplicity of bands in systems with discrete unitary symmetries, but the role of TR in systems with slow-fast structure is still not well understood. In this paper we address the following fundamental questions regarding fermionic TRI in adiabatic systems:

1. What are the topological classes of TRI vector bundles defined over a classical phase space?
2. Is there a KM index associated with these bundles?

We address these questions for the specific case of adiabatic energy bands over compact two-dimensional phase spaces. For these we show that:

1. TRI bundles are classified by the Chern number, which has to be even for bundles of even rank, and odd for bundles of odd rank. It follows that TRI bundles which are equivalent in the generic sense are also equivalent in the TRI sense. The bundle rank must be even if there is a point in phase space that is fixed by TR.
2. It is possible to define an analog of the KM index for even-rank bundles, but here it is an integer index, equal to half the Chern number.

These results are demonstrated for bundles of arbitrary rank over two phase spaces: a 2-sphere, on which TR is the antipodal map (this is the phase space of a rigid rotor [12]); and a 2-torus, on which TR acts by mirror reflection (the phase space of a particle in a periodic one-dimensional lattice [13]). We conjecture that they are valid for fermionic TRI adiabatic bands over any compact two-dimensional phase space on which TR acts as an orientation-reversing involution.

In the context of Bloch bands, the questions addressed here are answered in part by the ‘periodic table’ of [14] and [15], where it is shown that TRI systems may be assigned an integer invariant, a $\mathbb{Z}_2$ invariant or no invariant, in a manner depending periodically on their symmetry class and dimension. However, as shown by [16], this result relies on the Bloch band coordinates being odd under TR. References [17, 18] generalise the periodic table to systems with defects, whose spatial coordinates are even under TR, showing that dimension
in this case should be replaced by the difference between the number of momentum-like and position-like coordinates. Phase spaces have an equal number of position and momentum coordinates, so that adiabatic bands correspond to zero-dimensional Bloch bands in the periodic table, which carry a \( \mathbb{Z} \) invariant; this is consistent with our results.

For the two-dimensional phase spaces considered here, our results extend those of the periodic table scheme. In particular, we show that the Chern number is a classifying invariant for TRI bundles; that is, equal Chern numbers implies TRI equivalence. We note that the invariants of the periodic table for Bloch bands are not classifying, in general. For example, three-dimensional bundles; that is, equal Chern numbers implies TRI equivalence. We note that the invariants of the periodic table scheme. In particular, we show that the Chern number is a classifying invariant for TRI bundles over spheres and tori of dimension three or less for a family of involutions of variable signature.

The remainder of the paper is organised as follows. The notion of adiabatic energy bands and TRI therein is presented with physical examples in section 2. The main results are derived in section 3 for the 2-sphere phase space and in section 4 for the 2-torus. The summary and conclusions are presented in section 5.

## 2. Time-reversal invariant adiabatic energy bands

### 2.1. Adiabatic energy bands

A composite system has adiabatic structure if some of its degrees of freedom are slow, evolving on a time scale much longer than the fast degrees of freedom. In the adiabatic approximation the fast degrees of freedom evolve under an effective Hamiltonian \( H(p) \) which depends parametrically on the state \( p \) of the slow degrees of freedom. \( H(p) \) is obtained from the full system Hamiltonian \( \mathcal{H} \) as a semiclassical symbol [22], and \( p \) is a point in the classical phase space \( M \) corresponding to the slow degrees of freedom (for example, \( M \) may be taken to be the coset space of the isotropy subgroup of a coherent state in the dynamical symmetry group [23]). The Hilbert space of the full system is then replaced by a trivial bundle \( M \times \mathbb{C}^N \) over \( M \), where \( H(p) \) acts on the \( N_q \)-dimensional fibres that are the Hilbert space of the fast degrees of freedom.

If the spectrum of \( H(p) \) is discrete with eigenvalues \( \varepsilon^{(1)}(p) \leq \varepsilon^{(2)}(p) \leq \cdots \), the collection of \( j \)-th energy eigen values \( \varepsilon^{(j)}(p), \ p \in M \) defines an energy band when \( \varepsilon^{(j-1)}(p) < \varepsilon^{(j)}(p) < \varepsilon^{(j+1)}(p) \) for all \( p \). If \( \varepsilon^{(j)}(p) \) is \( k \)-fold degenerate, the associated collection of \( k \)-dimensional eigenspaces \( E^{(j)}(p) \) of \( H(p) \) with energy \( \varepsilon^{(j)}(p) \) defines a rank-\( k \) vector bundle over \( M \), which is a subbundle of \( M \times \mathbb{C}^N \). More generally, an energy band consists of a set of consecutive energies \( \varepsilon^{(j)}(p), \varepsilon^{(j+1)}(p), \ldots, \varepsilon^{(j+k-1)}(p) \), \( p \in M \) separated by gaps from the rest of the spectrum.

An example is the Born–Oppenheimer approximation in the quantum mechanics of molecules. The full Hamiltonian is

\[
\mathcal{H} = \sum_n \frac{\mathbf{p}_n^2}{2M_n} + \sum_i \frac{\mathbf{p}_i^2}{2m_i} + \sum_{nm} \frac{e_n e_m}{| \mathbf{Q}_n - \mathbf{Q}_m |} - \sum_n \frac{ee_n}{| \mathbf{Q}_n - \mathbf{q}_n |} + \sum_y \frac{e^2}{| \mathbf{q}_i - \mathbf{q}_j |},
\]

where \( \mathbf{Q}_n, \mathbf{P}_n, M_n, e_n \) are the position, momentum, mass, and charge of the \( n \)-th nucleus, and \( \mathbf{q}_i, \mathbf{p}_i, m, e \) are the position, momentum, mass and charge of the \( i \)-th electron. The nuclear
degrees of freedom are slow with typical time scale $\sqrt{M/m} \sim 10^2$ longer than that of the electronic degrees of freedom, and the corresponding slow phase space is $\mathbb{R}^{2N}$ ($N$ being the number of nuclei) with canonical coordinates $Q_n, P_n$. The fibre Hamiltonian is simply
\[
H(Q_n, P_n) = \sum_n \frac{P_n^2}{2M_n} + \sum_i \frac{\hat{p}_i^2}{2m_i} + \sum_{nm} \epsilon_n \epsilon_m |Q_n - Q_m| - \sum_n \epsilon_n |Q_n - q_n| + \sum_q e^2 |q_n - q_m|,
\]
which operates on the electronic Hilbert space. Since $H(Q_n, P_n)$ depends trivially on the momenta, the energy bands (usually called energy surfaces in this context) are parametrized by the nuclear positions.

2.2. Topological equivalence and time reversal invariance

If the full Hamiltonian $\mathcal{H}$, and therefore the fibre Hamiltonians $H(p)$, are changed continuously without closing the gap between an energy band and the rest of the spectrum, the associated bundle $E(p)$ also changes continuously. Two bundles $E_0(p)$ and $E_1(p)$ that can be connected in this manner are topologically equivalent, which means that there exists a family of subbundles $E(p, s) \subset M \times \mathbb{C}^{2N}$ depending continuously on $s$ such that $E(p, 0) = E_0(p)$ and $E(p, 1) = E_1(p)$.

In this paper we focus on adiabatic Hamiltonians that are invariant under a fermionic time reversal (TR) operation, $T$. That is, $THT^{-1} = \mathcal{H}$, where $T$ is an antiunitary operator with $T^2 = -1$. In the adiabatic approximation, time reversal sends a point $p$ in $M$ to its time-reverse, denoted $p^T$, and a vector $|\psi\rangle$ in the fibre over $p$ to the vector $|Tv\rangle_{p^T}$ in the fibre over $p^T$, where $T: E(p) \rightarrow E(p^T)$ is antiunitary with $T^2 = -1$. Time-reversal invariance (TRI) implies that the fibre Hamiltonians are related by $TH(p)T^{-1} = H(p^T)$.

TRI bundles are classified according to a stronger equivalence concept, which here will be termed TRI equivalence. TRI vector bundles $E_0(p)$ and $E_1(p)$ that are topologically equivalent in the sense defined above are also TRI equivalent if the interpolating bundles $E(p, s)$ can themselves be chosen to be TRI for each $s$. If an adiabatic TRI Hamiltonian is varied continuously while maintaining TRI and without closing spectral gaps, the bands of the new Hamiltonian will be TRI equivalent to those of the original one.

Topological equivalence classes are characterised by invariants. When the base manifold is two-dimensional the only generic invariant is the (first) Chern number, which can be any integer [1, section 11]. The Chern number of TRI Bloch bands is necessarily zero, so that such bands are always equivalent to the trivial bundle. But there are actually two classes of (fermionic) TRI Bloch bands under TRI equivalence [6], distinguished by the value of the KM index. The situation is quite different for adiabatic bands: Since positions and momenta transform differently under TR, the Berry curvature is TR-even rather than TR-odd, and the Chern number can be nonzero.

In the adiabatic setting, energy bands with nonzero Chern numbers have several physically observable consequences for the dynamics of the composite system. If the slow degrees of freedom are treated classically, the band curvature manifests as a gauge force, called geometric magnetism, in the slow dynamics [24, 25]. The corresponding Chern numbers play a role in the quantization of the slow dynamics [4, 5, 22, 26].

2.3. Physical examples

In sections 3 and 4, we prove our principal results for adiabatic TRI bands over the two-dimensional compact phase spaces $S^2$ (genus zero) and $T^2$ (genus 1), namely that (i) the Chern
number classifies the bundles, (ii) the parity of the Chern number is equal to the rank of the bundle, and (iii) for even-rank bundles, the Chern number is twice the Kane–Mele index. Here we present examples of physical systems where these results are applicable.

2.3.1 \textit{S}^2 example: fine structure of Rydberg atoms. Rydberg states of alkali atoms, where an electron is excited to a high principal quantum number, can be described by an effective single-electron Hamiltonian \cite{27}

\[
\mathcal{H}_R = \frac{\mathbf{p}^2}{2m} + V(r) + \frac{\mathbf{L}^2}{2mr^2} + W(r)\mathbf{L} \cdot \mathbf{S},
\]

where \(r, p, L, \) and \(\mathbf{S}\) are the distance from the centre of the atom, radial momentum, angular momentum, and spin of the valence electron, respectively. \(V(r)\) is the effective central potential of the ion core, and \(W(r)\) describes the spin–orbit coupling (the explicit expression is omitted). \(\mathcal{H}_R\) is invariant under TR which sends \(r \rightarrow r, p \rightarrow -p, L \rightarrow -L, \) and \(\mathbf{S} \rightarrow -\mathbf{S},\) and TR is fermionic because the electron has spin \(\frac{1}{2}\).

\(L^2\) and \(\mathbf{L} \cdot \mathbf{S}\) commute with \(\mathcal{H}_R\). We now assume that the angular momentum quantum number \(l\) is large, and then \(\mathcal{H} \equiv \mathbf{L} \cdot \mathbf{S}\) is adiabatic with an \(S^2\) phase space whose points \(\mathbf{n}_L\) signify the direction of a classical angular momentum vector with fixed length \(|\mathbf{L}|\) \cite{4, 12, 23}. The fibre is \(\mathbb{C}^2\), and the fibre Hamiltonian is \(\mathcal{H}(\mathbf{n}_L) = |L|\mathbf{n}_L \cdot \mathbf{S}\). \(\mathcal{H}\) is invariant under fermionic TR that sends \(\mathbf{n}_L\) to \(-\mathbf{n}_L\) and \(\mathbf{S}\) to \(-\mathbf{S}\). The fibre Hamiltonian is diagonalised by

\[
\mathcal{H}(\mathbf{n}_L)|\mathbf{n}_L\rangle_\pm = E_\pm |\mathbf{n}_L\rangle_\pm, \quad E_\pm = \pm \frac{|L|}{2},
\]

The family of + and − eigenvectors form the TRI rank-one bands over \(S^2\). Our results (specifically, (i) above) imply that the Chern numbers of these bands are odd. In this case, it is easy to verify this result directly, since the eigenvectors are those of a spin \(\frac{1}{2}\) in an external magnetic field (whose role is played by \(\mathbf{L}\)), so that the Chern numbers of the + and − bands are \(c_+ = +1\) and \(c_- = -1\), respectively \cite{28}.

2.3.2 \textit{T}^2 Example: cold Fermi atoms in a lattice-ring trap with induced spin–orbit coupling. Our second example draws on the active field of quantum simulation, where a specified Hamiltonian may be realised approximately using light-matter interactions on trapped atoms or ions. In particular, it was shown that effective spin–orbit coupling can be induced from adiabatic non-Abelian gauge fields generated by off-resonant Raman coupling of degenerate energy levels to an additional level \cite{29}. This idea was used in experiments \cite{30} to realize the effective Hamiltonian

\[
\mathcal{H} = \frac{\mathbf{p}^2}{2m} + \frac{\Omega(r)}{2} \sigma_3 + \frac{\delta}{2} \sigma_2 + V_{\text{trap}}(r),
\]

where \(\sigma_{2,3}\) are Pauli matrices acting on a two-dimensional subspace of the atomic hyperfine multiplet, and \(\mathbf{p}_R, \Omega\) and \(\delta\) are the recoil momentum, Rabi frequency, and Raman detuning, respectively. The Rabi frequency is space-dependent because it is proportional to the Raman beam intensity.

We now assume that the trap potential confines the atoms to a ring on which the potential is periodic in the angular coordinate with period \(\frac{2\pi}{N}\), \(\mathcal{N}\) a large integer. The production of a lattice-ring trap potential of this type has been demonstrated experimentally in \cite{31}. Assuming that the atoms are in the transverse ground state, we get an effective angular Hamiltonian
The Hamiltonian is invariant under standard time-reversal on two-component wavefunctions, provided that the coupled-levels subspace is itself time-reversal invariant. Assuming that the atoms are fermions, time reversal squares to $-1$.

The adiabatic structure of this system arises as follows [32]: In an angular interval small enough for $\theta$ to be approximated as constant and big enough for $N\theta$ to make many oscillations, $\mathcal{H}$ can be block-diagonalised in a Bloch basis of exponentials $e^{i k N \theta}$, where $k$ is an integer times $2\pi/N$. The fibre Hamiltonians are diagonal blocks $H(\theta, k)$ that inherit $\theta$ dependence from $\mathcal{H}$, and are $2\pi$-periodic in both $\theta$ and $k$; $(\theta, k)$ can therefore be viewed as points in a two-torus phase space, sent by TR to $(\theta, -k)$.

$H(\theta, k)$ acts on a Hilbert space that is the tensor product of the space of functions defined on the interval $0 \leq \theta \leq 2\pi/N$ with quasi-periodic boundary conditions determined by $k$, with the $\mathbb{C}^2$ hyperfine subspace. The space can be made finite-dimensional by cutting off states with high energies. Since phase space points with $k = 0$ are fixed by TR, bands are even-dimensional (Kramers degeneracy).

Our theory is applicable to the energy bands of $H(\theta, k)$, showing that they have even Chern numbers, equal to twice the KM index. Bands are TRI equivalent if and only if their Chern numbers are equal.

3. Bands on the 2-sphere phase space

3.1. Chern number and rank

We regard $S^2$ as embedded in $\mathbb{R}^3$ in the usual way. As above, let $H(p)$ denote the finite-dimensional $N_A \times N_A$ fast Hamiltonian parameterised by $p \in S^2$, and let $E(p)$ denote an $N_B$-dimensional invariant subspace of $H(p)$ with nonvanishing spectral gap. Let $\mathcal{N}$ denote the closed northern hemisphere. Since $\mathcal{N}$ is contractible, we can choose a continuous orthonormal basis for $E(n)$, $u_j(n) \in \mathbb{C}^{N_A}$, $1 \leq j \leq N_B$, where $n \in \mathcal{N}$. Let $u(n)$ denote the $N_A \times N_B$ matrix whose columns are the $u_j(n)$’s. Similarly, we choose a matrix $\tilde{u}(s)$ whose columns form a continuous orthonormal basis for $E(s)$ when $s$ is in the southern hemisphere. Let $\tilde{u}_E$ and $\tilde{u}_F$ denote the restrictions of $\tilde{u}$ and $\tilde{u}$ to the equator $E = \mathcal{N} \cap S^2$, whose points we parametrize by $0 \leq \phi \leq 2\pi$. The frames $\tilde{u}_E$ and $\tilde{u}_F$ are unitarily related, so that

$$\tilde{u}_E(\phi) = u_F(\phi) U^T(\phi),$$

where $U(\phi) \in U(N_B)$ is an $N_B \times N_B$ unitary transition matrix, and $U(N_B)$ denotes the group of $N_B \times N_B$ unitary matrices (it turns out to be convenient to write (7) in terms of the transpose $U^T$).

The Chern number $c$ of the bundle $E$ is given by the winding number of $\det U(\phi)$, i.e.

$$c(E) = \text{winding number of } \det U(\phi) \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{d}{d\phi} \arg \det U(\phi) \, d\phi.$$  

The Chern number is a topological invariant [1, section 11] and is therefore independent of the choice of frame.

Now we suppose that the band $E(p)$ is TRI (with fermionic time reversal). Time reversal on $\mathbb{C}^{N_A}$ is given by $T_V = K \pi$, where $\pi$ denotes the complex conjugate of $\nu$ and $K$ is unitary with $K \bar{K} = -1$. It will be convenient to assume (without loss of generality) that $\det K = 1$. For $M$ a (possibly rectangular) matrix with columns in $\mathbb{C}^{N_A}$, we similarly define $TM$ to be $K \bar{M}$. We note...
that $T(MN) = (TM)\overline{N}$ for $N$ a matrix of appropriate dimension. Time reversal on $S^2$ is the antipodal map, denoted $p^T = -p$ for $p \in S^2$.

Given a frame $\mathbf{u}$ on the northern hemisphere $\mathcal{N}$ as above, we can use time reversal to construct a frame $\hat{\mathbf{u}}$ on the southern hemisphere, as follows:

$$\hat{\mathbf{u}}(n^T) := T\mathbf{u}(n), \quad n \in \mathcal{N}. \tag{9}$$

In this case, the transition matrix $U(\phi)$ in (7) is defined by

$$T\mathbf{u}_s(\phi + \pi) = \mathbf{u}_s(\phi)U(\phi)^\dagger. \tag{10}$$

Applying time reversal to both sides of (10) and replacing $\phi$ by $\phi + \pi$ gives

$$-\mathbf{u}_s(\phi) = T\mathbf{u}_s(\phi + \pi)U(\phi + \pi)^\dagger = \mathbf{u}_s(\phi)U(\phi)^\dagger U(\phi + \pi)^\dagger, \tag{11}$$

which implies that

$$U(\phi + \pi)^\dagger = -U(\phi). \tag{12}$$

Equation (12) implies that $\det U(\phi + \pi) = (-1)^{N_s} \det U(\phi)$, so that the parity of $c$ is equal to that of $N_0$.

### 3.2. Chern number and TRI equivalence

Next, we establish that two TRI bundles on the 2-sphere are TRI-equivalent if and only if they have the same Chern number. The fact that two TRI-equivalent bundles $E_0$ and $E_1$ have the same Chern number is easy to see: Let $E_s, 0 \leq s \leq 1$, be an interpolating bundle, and let $\mathbf{u}_s(n)$ be a unitary frame for $E_s$ on $\mathcal{N}$ which is continuous in $n$ and $s$. For fixed $s$, the Chern number $c(E_s)$ is given by $\mathfrak{w} \det U_s(\phi)$, where $U_s(\phi)$ is defined by the $s$-dependent generalisation of (10). The winding number of $\det U_s(\phi)$ is necessarily continuous in $s$ and therefore constant, so that the Chern numbers of $E_0$ and $E_1$ are the same.

We proceed to establish the converse. Let $E_0$ and $E_1$ be TRI invariant bundles over $S^2$ with orthonormal frames $\mathbf{u}_0(n)$ and $\mathbf{u}_1(n)$ on $\mathcal{N}$ and unitary transition matrices $U_0(\phi)$ and $U_1(\phi)$ respectively. We suppose that they have the same Chern number, so that their transition matrices have the same winding number. Then there exists a unitary $U_s(\phi)$ which is $2\pi$-periodic in $\phi$, obeys (12) for each $s$, and which continuously interpolates between $U_0(\phi)$ and $U_1(\phi)$; the argument is illustrated in figure 1. It is essentially the fact that given a continuous map $\psi : \partial R \to \mathcal{C}$ from the boundary $\partial R$ of a simply connected planar domain $R$ into a connected space $\mathcal{C}$, $\psi$ can be continuously extended into the interior of $R$ if and only if the image of the boundary, $\psi(\partial R)$, is contractible in $\mathcal{C}$. In the present case, we take $R = \{0 \leq s \leq 1, 0 \leq \phi \leq \pi\}$, $\psi = U_s(\phi)$, and $\mathcal{C} = U(N_0)$, the space of unitary matrices of size $N_0$, with fundamental group $\pi_1(U(N_0)) = \mathbb{Z}$ characterised by the winding number of the determinant. On the edges $s = 0$ and $s = 1$ of $\partial R$, $\psi$ is given by $U_0(\phi)$ and $U_1(\phi)$ respectively, whereas on the edge $\phi = 0$, $\psi$ can be taken to be any continuous path $A_s$ in $U(N_0)$ connecting $U_0(0)$ and $U_1(0)$. On the edge $\phi = \pi$, $\psi$ is taken to be $-A_s^t$, in keeping with (12). The definition of $U_s(\phi)$ may then extended to $\pi < \phi < 2\pi$ using (12), so that (12) is everywhere satisfied. We shall make use of the freedom in choosing $A_s$ below.

The orthogonal complement of $E_0$, given by

$$E_0^\perp \subset S^2 \times \mathbb{C}^{N_0} = \{(p, y) \mid y \in E_0(p)^\perp\}, \tag{13}$$

is a TRI bundle in its own right (since $T$ is antiunitary). Let $\mathbf{u}_0^\perp(n)$ denote a unitary frame on $\mathcal{N}$, and $U_0^\perp(\phi)$ the associated $(N_0 - N_0)$-dimensional unitary transition matrix obeying (12).
is a TRI bundle, and we let

\[-(s_1. s_2). \]

We then take

\[\frac{U_1(\phi + \pi)}{\phi} \rightarrow U_1(\phi)\]

This will imply that the first

\[n\]

likewise,

\[\text{by construction, } U_1(\phi + \pi) = -U_1(\phi)\]

It remains to construct \(v_s(n)\). As a preliminary step, we introduce a continuous interpolation \(v_s(n)\) between \(v_0(n)\) and \(v_0(n)\) which, however, fails to satisfy (14). Letting \((\theta, \phi)\) denote polar coordinates on \(N\) with \(0 \leq \theta \leq \pi/2\), we take

\[v_s(n) = \hat{v}_s(n)W_s(n),\]
where $W_t(n)$ is a continuous $N_s$-dimensional unitary matrix that describes the requisite change of frame (it turns out to be convenient to write (17) in terms of $W$). In order that $v_\gamma(n)$ interpolates between $v_0(n)$ and $v_1(n)$, we require that

$$W_0(n) = W_1(n) = 1.$$  

(18)

The requirement (14) that $v_\gamma(n)$ frames a decomposition $E_t(n) \oplus E_t^\perp(n)$ leads to the following condition on $W$ on the equator:

$$V_\gamma(\phi) = W_{sE}(\phi + \pi) \tilde{V}_\gamma(\phi) W_{sE}(\phi),$$  

(19)

where

$$\tilde{V}_\gamma(\phi) := \tilde{V}_{sE}(\phi) T \tilde{V}_{sE}(\phi + \pi).$$  

(20)

The last step is to establish the existence of $W_t(n) \in U(N_s)$ compatible with the conditions (18) and (19). We begin by defining $W$ on the equator. At $\phi = 0$, we take $W_{sE}(0) = 1$, which is obviously compatible with (18). Equation (19) with $\phi = 0$ then implies that

$$W_{sE}(\pi) = V_t(0) \tilde{V}_\gamma(0).$$  

(21)

We note that (21) is consistent with (18), since $V_t$ and $\tilde{V}_\gamma$ coincide at $s = 0$ and $s = 1$. Consistency with (18) further requires that we take $W_{sE}(\phi) = W_{sE}(\phi) = 1$ for all $\phi$. In this way, we have defined $W_{sE}$ on the boundary of the rectangle $R = [0, \pi] \times [0, 1]$ in the $(\phi, s)$-plane. On three edges of the boundary, $W_{sE}$ is the identity. On the fourth edge, namely $\phi = \pi$, we have from (21) that $W_{sE}(\pi) = V_t(0) \tilde{V}_\gamma(0)$, which describes a closed curve in $U(N_s)$. We recall that $V_t(0)$ is block diagonal, and that one of the blocks is $A_s$, which may be taken to be any continuous unitary path from $U_0(0)$ and $U_1(0)$. By choosing its phase appropriately, we can ensure that $\text{wdet}(V_t(0) \tilde{V}_\gamma(0)) = 0$. By an argument like the one in figure 1, it follows that $W_{sE}(\phi)$ can be continuously extended to the interior of $R$. We then use (19) to extend $W_{sE}(\phi)$ to $\pi < \phi \leq 2\pi$ as follows:

$$W_{sE}(\phi + \pi) = V_t(\phi) W_{sE}(\phi) \tilde{V}_\gamma(\phi).$$  

(22)

It remains to check that $W_{sE}(2\pi) = 1$. From (21) and (22),

$$W_{sE}(2\pi) = V_t(\pi) W_{sE}(\pi) \tilde{V}_\gamma(\pi) = V_t(\pi) \tilde{V}_\gamma(0) \left( V_t(\pi) \tilde{V}_\gamma(0) \right)^\dagger. \ \tag{23}$$

From (15), it follows that $V_t(\pi) \tilde{V}_\gamma(0) = -1$. Similarly, we have that $\tilde{V}_\gamma(\pi) V_t(0) = -1$, since $\tilde{V}_\gamma(\phi + \pi) = -V_t(\phi)$ in analogy with (12), as $V_t$ constitutes the transition matrices for the trivial TRI bundle $S^2 \times C^{N_s}$. It follows that $W_{sE}(2\pi) = 1$.

As required by (18), we take $W_0(n) = W_1(n) = 1$. In this way, we have defined $W$ as a continuous map into $U(N_s)$ from the boundary of the three-dimensional domain $\{n \in N\} \times \{0 \leq s \leq 1\}$. This domain is topologically a three-ball, and its boundary is topologically a 2-sphere. Since the second homotopy group of $U(N_s)$, $\pi_2(U(N_s))$, is trivial, it follows that $W_t(n)$ can be continuously extended to the interior of the domain, completing the argument.

3.3. The Kane–Mele index

In the previous section we showed that TRI vector bundles over an $S^2$ phase space are characterised by the Chern number, which can be nonzero, unlike momentum-space TRI where $c = 0$, and bundles are classified by the Kane–Mele index $k$. Still, one may ask whether $k$ can
be defined for phase-space bundles, and what topological information it conveys. We next show that \( k \) makes sense as an integer index on bundles of even rank on the \( S^2 \) phase space, and that \( k = e/2 \).

There are several equivalent definitions of \( k \) for bundles on momentum space. Of these, the one that is applicable to bundles on the \( S^2 \) phase space is the original one given in [6], which does not rely on the existence of TRI points in the base space. It is based instead on the \( N_B \times N_B \) matrix \( M(n) \) given by

\[
M(n) = u^\dagger(n)Tu(n),
\]

where \( u(n) \) is a unitary frame for the bundle in the northern hemisphere.

The fact that \( T^2 = -1 \) implies that \( M \) is skew-symmetric. Therefore, its pfaffian, \( pf M \), is well defined. We will only consider explicitly the generic case where the zeros of \( pf M \) are isolated. In this case, it is always possible to choose an equator \( E \) on \( S^2 \) on which \( pf M_E \) is non-vanishing. Then the Kane–Mele integer is given by

\[
k = wn pf M_E(\phi),
\]

and is equal to the sum of the indices of the zeros of \( pf M \) in \( N \). The positions and indices of these zeros do not change under a gauge transformation, so that \( k \) is gauge invariant. This also follows from the demonstration that \( k = e/2 \), which we present next (compare with appendix A of [33]).

Let \( U(\phi) \) be the unitary transition matrix for the frame \( u(n) \). Then using (10), we have that

\[
M_E(\phi + \pi) = u^\dagger_E(\phi + \pi)Tu_E(\phi + \pi) = (-T^2u_E(\phi + \pi))\dagger Tu_E(\phi + \pi) \\
= -(Tu_E(\phi)U'(\phi)))\dagger u_E(\phi)U'(\phi) = -U(\phi)M_E^\dagger(\phi)U'(\phi) = U(\phi)M_E(\phi)U'(\phi).
\]

Therefore,

\[
\begin{align*}
&pf M_E(\phi + \pi) = det U(\phi)\overline{pf M_E(\phi)}. \\
&\text{It follows from (12) that } det U(\phi + \pi) = det U(\phi), \text{ so that}
\end{align*}
\]

\[
k = \int_0^{2\pi} \frac{d\phi}{2\pi} \partial_{\phi}(\pi \arg det M_E(\phi) + \arg pf M_E(\phi + \pi)) = \int_0^{2\pi} \frac{d\phi}{2\pi} \partial_{\phi} \arg det U(\phi) = \frac{e}{2}.
\]

4. Bands on the 2-torus phase space

In the following treatment of the 2-torus phase space, we emphasise those aspects which differ from the case of the 2-sphere; arguments that carry over are summarised briefly.

4.1. Chern number and rank

We parametrize \( T^2 \) by \( p = (\psi, \phi) \), where \( \psi \) and \( \phi \) are taken between 0 and \( 2\pi \). As above, let \( H(p) \) denotes the \( N_A \)-dimensional fast Hamiltonian parameterised by \( p \in T^2 \), and let \( E(p) \) denote an \( N_B \)-dimensional invariant subspace of \( H(p) \) with nonvanishing spectral gap. Let \( N \) denote the set for which \( \pi < \psi < 2\pi \), and \( S \) the set for which \( 0 < \psi < \pi \). Since \( N \) has trivial second homology group, we can construct a continuous \( N_B \times N_A \) matrix \( u(n) \) for \( n \in N \) orthonormal basis for \( E(n) \); similarly let \( \hat{u}(s) \) denote an \( N_B \times N_A \) matrix defined continuously on \( s \in S \) whose columns form an orthonormal basis for \( \hat{E}(s) \). The sets \( N \) and \( S \) intersect in
two disjoint circles, namely $\psi = \pi$ and $\psi = 2\pi$ (this is in contrast to the case of $S^2$, where $N$ and $S$, the northern and southern hemispheres, intersect in a single circle, the equator). We will use an index $j = 1, 2$ to label the circles $C_j = \{ \psi = j\pi \}$. Let $u_j(\phi)$ and $\hat{u}_j(\phi)$ denote the restrictions of $u$ and $\hat{u}$ to $C_j$. The frames $u_j$ and $\hat{u}_j$ are unitarily related, so that
\begin{equation}
\hat{u}_j(\phi) = u_j(\phi)U_j(\phi),
\end{equation}
where $U_j(\phi)$ is an $N_B \times N_B$ unitary transition matrix. The Chern number $c$ of the bundle $E$ is given by the difference of the winding numbers of $U_1(\phi)$ and $U_2(\phi)$, ie
\begin{equation}
c(E) = \text{wn } U_1(\phi) - \text{wn } U_2(\phi).
\end{equation}

Regarding $T^2$ as a phase space with $\phi$ as coordinate and $\psi$ as quasi-momentum, time reversal is taken to be $p^T = (2\pi - \psi, \phi)$. Like time reversal on $S^2$, the map is orientation-reversing. Unlike time reversal on $S^2$, the map has fixed points; indeed, the circles $C_1$ and $C_2$ are point-wise-invariant under $p \rightarrow p^T$, and $N$ is mapped onto $S$ (and vice versa). With time reversal on $\mathbb{C}^{N_B}$ defined as above by $v \rightarrow K v$, we assume that the band $E(p)$ is TRI (with fermionic time reversal).

Given a frame $u$ on $N$ as above, we can use time reversal to construct a frame $\hat{u}$ on $S$ via
\begin{equation}
\hat{u}(n^T) = T u(n), \quad n \in N.
\end{equation}
In this case, the transition matrix $U_j(\phi)$ in (29) is defined by
\begin{equation}
T u_j(\phi) = u_j(\phi)U_j(\phi)^\dagger.
\end{equation}
Applying time reversal to both sides of (32) gives
\begin{equation}
- u_j(\phi) = T u_j(\phi)U(\phi)^\dagger = u_j(\phi)U_j(\phi)^\dagger U_j(\phi)^\dagger,
\end{equation}
which implies that
\begin{equation}
U_j(\phi)^\dagger = -U_j(\phi),
\end{equation}
ie $U_j(\phi)$ is unitary antisymmetric. This implies in turn that $N_B$ is even, and that $\text{wn } \det U_j(\phi)$ is even (the winding number of $\det U_j$ is the sum of the winding numbers of its eigenvalues, which necessarily come in signed pairs on the unit circle), so that the Chern number is even. As for TRI bundles on $S^2$, the rank and the Chern number have the same parity. But for $T^2$, only even Chern numbers are possible.

4.2. Chern number and TRI equivalence

As in the case of $S^2$, it is easy to see that TRI-equivalent bundles have the same Chern numbers; the argument is similar and therefore omitted. We proceed to establish the converse. Let $E_0$ and $E_1$ be TRI bundles over $T^2$ with unitary frames $u_0(n)$ and $u_1(n)$ on $N$ and antisymmetric unitary transition matrices $U_{j0}(\phi)$ and $U_{j1}(\phi)$ respectively. As above, $j = 1, 2$ labels the circles $C_j$ bounding $N$. Without loss of generality (for example, by changing the phases of the vectors in the frame $u_0(n)$ appropriately), we can assume that
\begin{equation}
\text{wn } \det U_{j0}(\phi) = \text{wn } \det U_{j1}(\phi) \text{ for } j = 1.
\end{equation}
Let us assume that $E_0$ and $E_1$ have the same Chern number. It then follows from this assumption and from (30) that
\begin{equation}
\text{wn } \det U_{j0}(\phi) = \text{wn } \det U_{j1}(\phi) \text{ for } j = 2.
\end{equation}
We will require some facts about antisymmetric unitary matrices. Let $AU(2N)$ denote the space of antisymmetric unitary matrices of dimension $2N$. If $U \in AU(2N)$, then $U$ is unitarily congruent to the $2N$-dimensional Poisson matrix,

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

(37)

where $I$ denotes the $N$-dimensional identity matrix. That is, given $U \in AU(2N)$, there exists $X \in U(2N)$ such that $X^*UX = J$ [34, p 270]. Thus, as $U(2N)$ is connected, so is $AU(2N)$. It follows that if $U(\phi) \in AU(2N)$ is continuous in $\phi$, then we can find $X(\phi) \in U(2N)$ continuous in $\phi$ such that

$$U(\phi) = X(\phi)JX(\phi).$$

(38)

If $U(\phi)$ in (38) is $2\pi$-periodic, it is not necessarily the case that $X(\phi)$ is $2\pi$-periodic. Indeed, (38) does not determine $X(\phi)$ uniquely, but rather up to left multiplication by an element of the unitary (or compact) symplectic group $USp(2N)$. We recall that $USp(2N)$ consists of the $2N$-dimensional unitary matrices $S$ for which $S'S = J$. Since $USp(2N)$ is connected [35, p 384], we may choose $S(\phi)$ continuous in $\phi$ such that $S(0) = I$ and $S(2\pi) = X(0)X^{-1}(2\pi)$. After replacing $X(\phi)$ by $S(\phi)X(\phi)$, we may therefore assume without loss of generality that if $U(\phi)$ in (38) is $2\pi$-periodic, then so is $X(\phi)$. It follows that the fundamental group of $AU(2N)$ is $2\pi$, with the homotopy class of a closed curve $U(\phi)$ characterised by $\text{wv} \det U(\phi) = 2 \text{wv} \det X(\phi)$.

With these facts in hand, the previous argument from the case of $N_A$ proceeds as in the $S^2$-case; we find an $NA \times NA$ unitary matrix $v_j(n)$ which continuously interpolates between $U_{\mu}(\phi)$ and $U_{\lambda}(\phi)$. Moreover, $A_{\mu} := U_{\mu}(0)$, $0 \leq \mu \leq 1$, may be chosen to be any continuous path in $AU(N_A)$ between $U_{\rho}(0)$ and $U_{\lambda}(0)$. We shall make use of this freedom below.

Noting that $E_1^j$ is a TRI bundle, let $u_1^j(n)$ denote a unitary frame on $N$ with $(N_A - N_B)$-dimensional antisymmetric unitary transition matrices $U_{1j}^j(\phi)$. Similarly, let $u_1^j(n)$ denote a unitary frame for $E_1^j$ on $N$ with transition matrices $U_{1j}^j(\phi)$. Since $E_0 \oplus E_0^\perp$ is the trivial bundle, it follows that $c(E_0^\perp) = -c(E_0)$, and similarly that $c(E_1^\perp) = -c(E_1)$, and therefore that $c(E_1^\perp) = c(E_1^\perp)$. Therefore, arguing as above, we can construct $U_{1j}^j(\phi) \in AU(N_B - N_A)$ which is $2\pi$-periodic in $\phi$ and which interpolates between $U_{1j}^j(\phi)$ and $U_{1j}^j(\phi)$, with $A_{1j} := U_{1j}^j(0)$, $0 \leq \mu \leq 1$, chosen to be any continuous path in $AU(N_A - N_B)$ between $U_{1j}^j(0)$ and $U_{1j}^j(0)$. The demonstration that $E_0$ and $E_1$ are TRI-equivalent proceeds as in the $S^2$-case; we find an $NA \times NA$ unitary matrix $v_j(n)$ which continuously interpolates between unitary frames $v_0(n)$ for $E_0(n) \oplus E_0^\perp(n)$ and $v_1(n)$ for $E_1(n) \oplus E_1^\perp(n)$ such that

$$v_j^j(\phi) \sigma_j(\phi) v_j^j(\phi) = v_j^j(\phi), \quad j = 1, 2,$$

(39)

where $v_j^j(\phi)$ is antisymmetric block-diagonal unitary with blocks of dimension $N_B$ and $(N_A - N_B)$ given by $U_{\mu}(\phi)$ and $U_{\lambda}(\phi)$. We first introduce a continuous interpolation $v_\psi(n)$ between $v_0(n)$ and $v_1(n)$, given by

$$v_\psi(n) = \begin{cases} v_0((1 - 2s)\psi + 4\pi s, \phi) v_0(2\pi, \phi)^{-1}, & 0 \leq s \leq \frac{1}{2} \\ v_1((2s - 1)\psi + 4\pi(1 - s), \phi) v_1(2\pi, \phi)^{-1}, & \frac{1}{2} \leq s \leq 1 \end{cases}$$

(40)

which, however, fails to satisfy (39). We then take $v_\psi(n)$ to be of the form

$$v_\psi(n) = v_\psi(n)W_\psi(n).$$

(41)
where $W_j(n)$ is a continuous $N_A$-dimensional unitary matrix. In order that $\psi_j(n)$ interpolates between $\psi_0(n)$ and $\psi_1(n)$, we require that

$$W_0(n) = W_1(n) = 1. \quad (42)$$

In addition, the requirement (39) leads to the following condition on $W$ restricted to the $\mathcal{C}_j$'s:

$$V_{js}(\phi) = W^\prime_j(\phi) \tilde{\mathcal{V}}_{js}(\phi) W_j(\phi), \quad (43)$$

where

$$\tilde{\mathcal{V}}'_j(\phi) := \tilde{\mathcal{V}}_j^1(\phi) T \mathcal{V}_j(\phi). \quad (44)$$

That is, $W_j(\phi)$ should provide a congruence between $\tilde{\mathcal{V}}_j(\phi)$ and $V_j(\phi)$.

As in the case of $S^1$, we aim to construct $W_j(n)$ continuous on the boundary $\partial \mathcal{D}$ of a three-dimensional domain $\mathcal{D}$ topologically equivalent to the three-ball. In the present case of $T^3$, $\mathcal{D}$ is taken to be

$$\mathcal{D} := N \times \{0 < s < 1\} = \{\pi < \psi < 2\pi\} \times \{0 < \phi < 2\pi\} \times \{0 < s < 1\}. \quad (45)$$

In addition to the conditions (42) and (43), we require periodicity in $\phi$, i.e., $W_j(\psi, 0) = W_j(\psi, 2\pi)$. The existence of a continuous extension of $W$ to the interior of $\mathcal{D}$ then follows from the fact that $\pi_2(U(N_A))$ is trivial.

From the previous discussion of (38), we can find $X_j(\phi), X_j(\phi) \in U(N_A)$ continuous in $s$ and $\phi$ and $2\pi$-periodic in $\phi$ such that

$$\tilde{\mathcal{V}}_j(\phi) = X_j^\prime(\phi) J X_j(\phi), \quad V_j(\phi) = X_j^\prime(\phi) J X_j(\phi), \quad (46)$$

Since $V_j(\phi)$ and $V_j(\phi)$ coincide for $s = 0$, it is clear that we may assume that

$$X_j(\phi) = \tilde{X}_j(\phi). \quad (47)$$

In fact, since $\tilde{\mathcal{V}}_j(\phi)$ and $V_j(\phi)$ also coincide for $s = 1$, the following argument shows that we may also assume, without loss of generality, that

$$X_1(\phi) = \tilde{X}_1(\phi). \quad (48)$$

For suppose that (48) is not satisfied. Then $\tilde{X}_1(\phi) X_1^{-1}(\phi)$ describes a continuous closed curve in $USp(N_A)$, the $N_A$-dimensional unitary symplectic group, as $\phi$ varies between 0 and $2\pi$. Since $USp(N_A)$ is simply connected [35, p. 384], we may construct $S_j(\phi) \in USp(N_A)$ continuous in $s$ and $\phi$ and periodic in $\phi$ such that $S_j(\phi) = X_j(\phi) X_1^{-1}(\phi)$ and $S_0(\phi) = 1$ (that is, $S_j(\phi)$ describes a contraction of the closed curve $X_j(\phi) X_1^{-1}(\phi)$ to the identity). Then we may replace $X_j(\phi)$ by $S_j(\phi) X_j(\phi)$, which ensures that (48) is satisfied while preserving (46) as well as periodicity in $\phi$.

We proceed to define $W$ on $\partial \mathcal{D}$. On the faces of $\mathcal{D}$ given by $\psi = j\pi$, we take

$$W_j(j\pi, \phi) := W_j(\phi) = \tilde{X}^{-1}_j(\phi) X_j(\phi). \quad (49)$$

By construction, $W_j(\phi)$ is periodic in $\phi$. Also, (46) implies that (43) is satisfied. Moreover, (47) and (48) imply that $W_0(\phi) = W_1(\phi) = 1$, in keeping with (42).

It remains to define $W$ on the face of $\partial \mathcal{D}$ with $\phi = 0$; by requiring $W$ to be periodic in $\phi$, this determines $W$ on the face with $\phi = 2\pi$. In fact, $W$ has already been defined on the four edges of the $(\phi = 0)$-face. On the two edges with $s = 0$ and $s = 1$, we have that $W = I$. On the edges with $\psi = j\pi$, $j = 1, 2$, we have from (49) that
\[ W_s(j\pi, 0) = \hat{X}_s^{-1}(0)X_s(0). \] (50)

Therefore, the winding number \( m \) of \( \det W_s(n) \) on the (appropriately oriented) boundary of the face \( \mathcal{F} \) of \( \partial D \) given by \( \phi = 0 \) is given by

\[ m = \text{wn} \det \hat{X}_1^{-1}(0)X_1(0) - \text{wn} \det \hat{X}_2^{-1}(0)X_2(0). \] (51)

From (46),

\[ (\det X_s(0))^2 = \det V_s(0) = \det A_s \det A_s^\dagger. \] (52)

It follows that by choosing \( A_s \) and \( A_s^\dagger \) appropriately, we can ensure that

\[ \text{wn} \det \hat{X}_1^{-1}(0)X_1(0) = 0, \] (53)

and therefore that \( m = 0 \). It follows that \( W_s(n) \) can be continuously extended from the boundary of \( \mathcal{F} \) to its interior.

### 4.3. The Kane–Mele index

As in the case of \( S^2 \), we can define an analogue of the Kane–Mele index, \( k \), as the winding number of the pfaffian of \( M_j(\pi) = u_j^T(n)Tu_j(n) \) on the boundary of \( \mathcal{N} \), which turns out to be half the Chern number. The argument is simpler in the case of \( T^2 \). Indeed, from (32) and (34),

\[ M_j(\pi) = -U_j(\phi), \] (54)

where \( M_j(\phi) := M(j\psi, \phi) \). It follows that

\[ c = \text{wn} \det M_1(\phi) - \text{wn} \det M_2(\phi) = \text{wn} (\text{pf} M_1(\phi)^2) - \text{wn} (\text{pf} M_2(\phi)^2) = 2(\text{wn pf} M_1(\phi) - \text{wn pf} M_2(\phi)) = 2k. \] (55)

### 5. Conclusions

The problem of the topological classification of band structure, which has been extensively studied in condensed matter systems with lattice translational symmetry, may also be formulated for the bands that arise in quantum systems with a separation of time scales. The slow degrees of freedom, regarded as classical, constitute the base manifold, and have the structure of a phase space rather than the Brillouin zone of quasi-momenta. The fast degrees of freedom constitute the fibres, and are the analogue of the space-periodic Bloch states. The single-particle symmetry classes—generic, bosonic TRI (time reversal invariant), and fermionic TRI—are well defined in this adiabatic context.

We have studied fermionic TRI adiabatic bands that arise physically in TRI composite systems with a separation between fast and slow degrees freedom, where the fast degrees of freedom include an odd number of fermions. For the cases where the base manifold is an \( S^2 \) phase space on which TR acts by inversion, or a \( T^2 \) phase space on which TR acts by mirror reflection, we have shown that the bands are characterised by two topological invariants familiar from the study of Bloch bands, namely the Chern number and (when the band rank is even) the KM (Kane–Mele) index.

Because TR is orientation-reversing on phase spaces, the Chern number for adiabatic bands need not vanish, in contrast to the case of Bloch bands. Rather, it is constrained to have the same parity as the band rank. The band rank need not be even (as is the case for fermionic TRI
Bloch bands), when there are no TR-invariant points in the slow phase space. Furthermore, the KM index can be defined as an invariant integer (rather than an integer modulo 2), because the orientation-reversing nature of TR excludes the possibility of deformations and gauge transformations that change the KM index by 2. The two invariants are not independent, however; the Chern number is twice the KM index, whenever the latter is defined.

The results are consistent with the prediction from the topological theory of Bloch bands that the fermionic TRI class over spaces with equal number of positions and momenta are characterised by an integer invariant. For the cases we consider, our results demonstrate that this invariant, which is the Chern number, is TRI classifying—that is, adiabatic bands are TRI equivalent if and only if their Chern numbers are the same. Moreover, the parity of the Chern number is determined by the rank of the bundle.

This work demonstrates that concepts and results from the theory of topological insulators can be beneficially applied to adiabatic systems. Further development of these results, including generalisations to slow phase spaces of arbitrary genus and higher dimensions, raises a number of interesting questions, including characterising TR involutions on a general phase space, the analogue of edge states for adiabatic systems, and, if one were to realise the extended symmetry classes of [36] in this setting, the many-body analogue of the adiabatic bands.

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