ROST’S DEGREE FORMULA IN CHARACTERISTIC TWO

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Abstract. We prove the degree formula expressing the functoriality of the Segre number of a complete variety over a field of characteristic two. This formula has been proved in characteristic not two by Merkurjev and/or Rost. We proceed by studying involutions of varieties in characteristic two.

Introduction

The degree formula provides a connection between arithmetic and geometric properties of algebraic varieties, by relating the existence of closed points of odd degree to the values of characteristic numbers. This formula has applications to questions of algebraic nature, in particular concerning of quadratic forms.

When $X$ is a smooth complete connected variety over a field, its Segre number $s_X$ is the degree of the highest Chern class of the opposite of its tangent bundle:

$$s_X = \deg c_{\dim X}(-T_X) \in \mathbb{Z}.$$  

The index $n_X$ of $X$ is the g.c.d. of the degrees of the closed points of $X$; it is a divisor of $s_X$. The degree formula is the relation in $\mathbb{Z}/2\mathbb{Z}$

$$\frac{s_Y}{n_X} = \deg f \cdot \frac{s_X}{n_X} \mod 2,$$

whenever $f: Y \to X$ is a rational map of connected (smooth complete) varieties of the same dimension. Here $\deg f$ is defined as zero when $f$ is not dominant, and as the degree of the function fields extension otherwise. Thus the degree formula provides a relation between $s_X$ (a geometric, or topological, invariant) and $n_X$ (an arithmetic invariant). The number $n_X$ is known to be a birational invariant of the smooth complete variety $X$; a consequence of the degree formula is that the class $s_X/n_X \mod 2 \in \mathbb{Z}/2\mathbb{Z}$ is a birational invariant of $X$ as well.

A typical application of the degree formula is the following: if the integer $s_X/n_X$ is odd, then the variety $X$ is incompressible, which means that any rational map $X \to X$ is dominant. For example, this is so when $X$ is an anisotropic quadric of dimension $2^n - 1$; this explains why anisotropic non-degenerate quadratic forms of dimension $2^n + 1$ always have first Witt index one.

Variants of the degree formula were initially considered by Voevodsky [Voe96] as a step in the proof of the Milnor conjecture (see [Ros02]). One can prove the

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degree formula using cohomology operations [Mer03], algebraic cobordism [LM07, §4.4], or by more elementary means [Mer01]. But no proof of the formula is valid over a field of characteristic two. In this paper we provide a proof working in characteristic two, in the spirit of the elementary approach of Rost and Merkurjev. It actually requires to work over a field of characteristic two; this contrasts with the characteristic-free approach of [Hau13c, Hau12b, Hau12a, Hau13a, Hau13b] for questions of similar nature.

We prove that the total Segre class \( \text{Sq}(X) \in \text{CH}(X)/2 \text{CH}(X) \) of (the tangent cone of) a variety \( X \) is functorial with respect to morphisms having a degree. The degree formula amounts to the functoriality of the degree of the component of degree zero of \( \text{Sq}(X) \) (when \( X \) is complete). In general the cycle class \( \text{Sq}(X) \) carries much more information than its degree: we expect \( \text{Sq}(X) \) to be the total homological Steenrod square of the fundamental class of \( X \) (Steenrod squares have not been constructed in characteristic two).

One may hope to use similar techniques to prove other degree formulas: those involving other primes than two, or the so-called higher degree formulas. In any case, there is a serious obstruction to generalising our approach to characteristic \( p \) by considering the \( p^m \)-th symmetric product of a variety: the \( n \)-th punctual Hilbert scheme has a more explicit description (in terms of blow-up of the diagonal) when \( n = 2 \) than in general. Another incarnation of this obstruction prevents (until now) a geometric construction of Vishik’s symmetric operations [Vis07] for primes other than two.

The paper is organised as follows. We begin in §1 by providing the notations and conventions used in the paper. Then in §2 we give the main theorem stating the functoriality of the Segre class of the tangent cone, postponing its proof to the last section, and we deduce the degree formula. We give in §3 some classical applications to incompressibility of varieties. In §4, we study \( G \)-actions on varieties over a field of characteristic two, where \( G \) is the group with two elements. We introduce the notions of relative quotient, saturated subschemes and pure actions; we also describe actions on blow-ups. We prove in §5 a key result concerning the \( G \)-action on the principal transform, a scheme sitting between the proper transform and the total transform. Finally, §6 is devoted to the proof of the main theorem, putting together the pieces from the previous sections.

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1. Notations

Throughout the paper, we work over a field of characteristic two.

1.1. Varieties. The letter $k$ will denote a field of characteristic two, and also its spectrum. A variety is a quasi-projective scheme over $k$, and a morphism of varieties is a morphism of $k$-schemes between varieties. We will always implicitly work in the category of varieties. A variety is complete if it is proper over $k$.

When $X$ and $Y$ are varieties, a rational map $Y \dashrightarrow X$ is a morphism of varieties $U \to X$, for some open dense subscheme $U$ of $Y$. The function field of an integral variety $X$ will be denoted $k(X)$.

We say that a morphism $f : Y \to X$ is an isomorphism off a closed subscheme $Z$ of $X$, if the morphism $Y \setminus f^{-1}Z \to X \setminus Z$ obtained by base change is an isomorphism.

A closed subscheme $Z$ of a variety $Y$ is locally principal, resp. an effective Cartier divisor, if its sheaf of ideals is locally generated by a single element, resp. nonzerodivisor.

1.2. Group of cycles. For a variety $X$, we let $Z(X)$ be the free abelian group generated by the elements $[V]$, where $V$ runs over the integral closed subschemes of $X$. When $f : Y \to X$ is a morphism of varieties, we define a morphism $f_* : Z(Y) \to Z(X)$ as follows. Let $W$ be an arbitrary integral closed subscheme of $Y$, and $V$ be the closure of $f(W)$ in $X$. Then we define

$$f_*[W] = \begin{cases} [k(W) : k(V)] \cdot [V] & \text{if } \dim W = \dim V, \\ 0 & \text{otherwise}. \end{cases}$$

One checks that this defines a functor from the category of varieties to the category of abelian groups.

When $T$ is a closed subscheme of $X$, with irreducible components $T_\alpha$ and multiplicities $m_\alpha$, we define the class $[T] = \sum_\alpha m_\alpha[T_\alpha] \in Z(X)$.

1.3. Index of a variety. When $X$ is a variety, we denote by $n_X$ the positive generator of the image of the morphism $p_{X*} : Z(X) \to Z(k) = \mathbb{Z}$ induced by the structural morphism $p_X : X \to k$. The integer $n_X$ is the g.c.d. of the degrees of closed points on $X$.

1.4. Degree of a morphism. Let $f : Y \to X$ a morphism of varieties, and $n$ be an integer. We say that $f$ has degree $n$ and write $n = \deg f$, if we have $f_*[Y] = n \cdot [X]$ in $Z(X)$. More generally, when $f : Y \dashrightarrow X$ is rational map, we say that $f$ has degree $n$ if it induces a morphism of degree $n$ on an open dense subscheme of $Y$.

Some examples of a rational map $f : Y \dashrightarrow X$ with a degree are:

(i) Assume that $f$ is birational, e.g. an open dense embedding. Then $f$ has degree one.
(ii) Let $X$ be a variety with irreducible components $X_\alpha$ and multiplicities $m_\alpha$. Let $Y = \amalg \alpha X'_\alpha$, where $X'_\alpha$ is the disjoint union of $m_\alpha$ copies of $X_\alpha$, and $f: Y \to X$ the natural morphism. Then $f$ has degree one.

(iii) Assume that $Y$ is integral, and that $\dim f(Y) < \dim Y$. Then $f$ has degree zero.

(iv) Assume that $X$ and $Y$ are integral of the same dimension, and that $f$ is dominant. Then $f$ has degree $[k(Y) : k(X)]$.

(v) Assume that $f = g \circ h$, and that $g$ and $h$ both have a degree. Then $f$ has degree $(\deg g)(\deg h)$.

(vi) Assume that $f$ is a flat morphism of rank $n$ (see §1.6 below). Then $f$ has degree $n$ (this follows from §1.6(iii)).

1.5. **Flat pull-back.** When $f: Y \to X$ is flat morphism of constant relative dimension, there is a morphism $f^*: Z(X) \to Z(Y)$ such that $f^*[Z] = [f^{-1}Z]$ for any closed subscheme $Z$ of $X$ (see [Ful98, Lemma 1.7.1]). For a cartesian square

$$
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow{y} & & \downarrow{x} \\
Y' & \xrightarrow{f'} & X' 
\end{array}
$$

with $x$ and $y$ flat of the same constant relative dimension, we have $x^* \circ f'_* = f_* \circ y^*$, as morphisms $Z(Y') \to Z(X)$ (see [EKM08, Proposition 49.20 (1)]). In particular, if $f'$ has a degree, then $f$ has degree equal to $\deg f'$.

1.6. **Flat morphisms of finite rank.** A morphism $f: Y \to X$ is flat of rank $n$ if it is finite, and $f_*\mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-module of rank $n$. Then:

(i) Any base change of $f$ is flat of rank $n$.

(ii) The morphism $f$ is faithfully flat.

(iii) The endomorphism $f_* \circ f^*$ of the group $Z(X)$ is multiplication by $n$ (see [Ful98, Example 1.7.4, Lemma 1.7.1]).

1.7. **External product.** Letting $[W] \times_k [V] = [W \times_k V] \in Z(X \times_k Y)$ when $W$, resp. $V$, is an integral closed subscheme of $Y$, resp. $X$, defines an external product

$$
Z(Y) \otimes_Z Z(X) \to Z(Y \times_k X), \quad y \otimes x \mapsto y \times_k x.
$$

When $Y$ and $X$ are equidimensional varieties, we have $[Y \times_k X] = [Y] \times_k [X]$ (see [EKM08, Proposition 57.12]). If $f: Y \to Y'$, $g: X \to X'$ are two morphisms of varieties, then $(f \times_k g)_*(y \times_k x) = f_*(y) \times_k g_*(x)$ (we may assume that $x$ and $y$ are classes of integral closed subschemes, and use [EKM08, Proposition 50.4]).

1.8. **Chow groups.** The Chow group of a variety $X$ will be denoted by CH$(X)$. We will use its functorialities described in [Ful98] (where CH is written $A_*$).
1.9. **Graded rings.** Let $S$ be a commutative, graded ring. When $s \in S$ is a homogeneous element of degree one, we denote by $S(s)$ the subring of the localisation $S[s^{-1}]$ consisting of the elements $u/s^n$, with $u \in S$ homogeneous of degree $n$. When $I$ is an homogeneous ideal of $S$, we denote by $I(s)$ the ideal of $S(s)$ consisting of elements $i/s^n$ with $i \in I$ homogeneous of degree $n$. Then the rings $(S/I)(s)$ and $S(s)/I(s)$ are naturally isomorphic.

1.10. **Sheaves of ideals.** When $Z$ is the closed subscheme of $X$, we denote by $\mathcal{I}_Z$ the corresponding sheaf of ideals on $X$.

If $f: Y \to X$ is a morphism, and $\mathcal{I}$ a coherent sheaf of ideals on $X$, we denote by $\mathcal{I}_Y$ the image of the morphism $f^*\mathcal{I} \to \mathcal{O}_Y$. Thus when $Z$ is a closed subscheme of $X$, we have $\mathcal{I}_Z\mathcal{O}_Y = \mathcal{I}_{f^{-1}Z}$. When $\mathcal{I}, \mathcal{J}$ are two sheaves of ideals on $X$, we have $(\mathcal{I}\mathcal{O}_Y) \cdot (\mathcal{J}\mathcal{O}_Y) = (\mathcal{I} \cdot \mathcal{J})\mathcal{O}_Y$.

2. **The degree formula**

Let $X$ be a variety. We consider the blow-up $B_X$ of the diagonal $X$ in $X \times_k X$, and its exceptional divisor $i_X: P_X \to B_X$. We let $\pi_X: P_X \to X$ be the natural morphism. We define an operator

$$\xi_X = i_X^* \circ i_{X*}: \text{CH}(P_X) \to \text{CH}(P_X),$$

and consider the cycle class (the reason for the signs is Lemma 2.2),

$$\widetilde{\text{Sq}}(X) = \sum_{n \geq 0} (-1)^n \cdot \pi_{X*} \circ (\xi_X)^n[P_X] \in \text{CH}(X).$$

We let $m_X: X \to X \times_k \mathbb{A}^1$ be the closed embedding given by a rational point of $\mathbb{A}^1$, and define

$$\text{Sq}(X) = m_X^*\widetilde{\text{Sq}}(X \times_k \mathbb{A}^1) \in \text{CH}(X).$$

**Remark 2.1.** The morphism $m_X^*$ is independent of the choice of the rational point. When $X$ is equidimensional of dimension $> 0$, then one may prove that $\text{Sq}(X) = \widetilde{\text{Sq}}(X)$. On the other hand $\widetilde{\text{Sq}}(k) = 0$ while $\text{Sq}(k) = [k] \neq 0$ in $\text{CH}(k)$.

**Lemma 2.2.** Assume that $X$ is smooth over $k$, and let $T_X$ be its tangent bundle. Then we have in $\text{CH}(X)$ the relation

$$\text{Sq}(X) = c(-T_X)$$

where $c$ denotes the total Chern class.

**Proof.** Since $m_X^*c(-T_X \times_k \mathbb{A}^1) = c(-(T_X \oplus 1)) = c(-T_X)$, replacing $X$ by $X \times_k \mathbb{A}^1$, it will suffice to prove that $\widetilde{\text{Sq}}(X) = c(-T_X)$ under the additional assumption that $X$ has no connected component of dimension zero.

The morphism $\pi_X: P_X \to X$ can be identified with the projective bundle $\mathbb{P}(T_X) \to X$. Since the normal bundle to $i_X$ is the tautological bundle $\mathcal{O}(-1)$, we have $\xi_X = c_1(\mathcal{O}(-1)) = -c_1(\mathcal{O}(1))$ [Ful98, Proposition 2.6 (c)]. Since $X$ has no connected component of dimension zero, the cycle is the Segre class of the cone
Let $f: Y \to X$ be a proper morphism of equidimensional varieties. Assume that $f$ has a degree (see §1.4). Then,
\[
f_\ast \operatorname{Sq}(Y) - \deg f \cdot \operatorname{Sq}(X) \in 2 \operatorname{CH}(X).
\]

**Proof.** The main argument will be given in §6; here we merely explain how the theorem follows from Theorem 6.6.

Let $X' = X \times_k \mathbb{A}^1$, $Y' = Y \times_k \mathbb{A}^1$, and consider the morphism $f' = f \times_k \mathbb{A}^1: Y' \to X'$. The closed subscheme $Y' \times_X Y' = Y \times_X Y \times_k \mathbb{A}^1$ of $Y' \times_k Y'$ is contained in the effective Cartier divisor $Y \times_k Y \times \mathbb{A}^1$, and in particular is nowhere dense. Thus we may apply Theorem 6.6 to the morphism $f'$. Taking the alternate sum over all $n \geq 0$, we obtain
\[
f_\ast' \widetilde{\operatorname{Sq}}(Y') - \deg f' \cdot \widetilde{\operatorname{Sq}}(X') \in 2 \operatorname{CH}(X').
\]

We have $\deg f' = \deg f$ by §1.5, and, in $\operatorname{CH}(X)$,
\[
m_X' (f_\ast' \widetilde{\operatorname{Sq}}(Y') - \deg f' \cdot \widetilde{\operatorname{Sq}}(X')) = f_\ast \circ m_X \widetilde{\operatorname{Sq}}(Y') - \deg f' \cdot m_X \widetilde{\operatorname{Sq}}(X')
\]
\[
= f_\ast \operatorname{Sq}(Y) - \deg f \cdot \operatorname{Sq}(X).
\]

**Corollary 2.4.** Let $X$ be an equidimensional variety, with irreducible components $i_\alpha: X_\alpha \to X$ and multiplicities $m_\alpha$. Then
\[
\operatorname{Sq}(X) - \sum_\alpha m_\alpha \cdot i_\ast_\alpha \operatorname{Sq}(X_\alpha) \in 2 \cdot \operatorname{CH}(X).
\]

**Proof.** We apply Theorem 2.3 to the morphism of §1.4 (ii). \qed

**Corollary 2.5.** The association $[V] \mapsto i_\ast \operatorname{Sq}(V)$, where $i: V \to X$ is the closed embedding of an integral subscheme, induces a morphism $\mathcal{Z} \to \operatorname{CH}/2 \operatorname{CH}$ of functors from the category of varieties and proper morphisms to abelian groups.

**Remark 2.6.** To prove that this passes to rational equivalence, i.e. that it descends to a functor $\operatorname{CH} \to \operatorname{CH}/2 \operatorname{CH}$, would give a construction of Steenrod squares.

When $X$ is a complete variety, we have the degree morphism $\deg: \operatorname{CH}(X) \to \mathbb{Z}$. We consider the integer
\[
s_X = \deg \operatorname{Sq}(X) \in \mathbb{Z}.
\]

The next corollary was proved, under the additional assumption that $X$ is smooth, in [Mer02, Ros08, Hau12b].

**Corollary 2.7.** Let $X$ be a complete equidimensional variety, with $\dim X > 0$. Then the integer $s_X$ is even.

**Proof.** We apply Theorem 2.3 to the morphism $X \to k$, which has degree zero. \qed
Taking the degree in Theorem 2.3, we obtain:

**Corollary 2.8** (Degree formula). Let \( f : Y \to X \) be a proper morphism of equidimensional varieties. Assume that \( f \) has a degree. Then we have in \( \mathbb{Z}/2\mathbb{Z} \)

\[
\frac{s_Y}{n_X} = \deg f \cdot \frac{s_X}{n_X} \mod 2.
\]

When \( X \) is a smooth connected complete variety of dimension \( d \), we have by Lemma 2.2 (\( c_d \) denotes the \( d \)-th Chern class, with values in \( \text{CH}_0(X) \))

\[ s_X = \deg c_d(-T_X). \]

**Corollary 2.9.** Let \( f : Y \to X \) be a rational map between smooth, connected, complete varieties of dimension \( d \). Then we have in \( \mathbb{Z}/2\mathbb{Z} \)

\[
\frac{\deg c_d(-T_Y)}{n_X} = \deg f \cdot \frac{\deg c_d(-T_X)}{n_X} \mod 2.
\]

**Proof.** Note that \( f \) automatically has a degree, since we are in one of the situations (iii) or (iv) of §1.4. Let \( Z \) be the closure of the graph of \( f \) in \( Y \times_k X \). The statement follows by applying Corollary 2.8 to the proper morphisms \( Z \to Y \) (of degree one) and \( Z \to X \) (of degree \( \deg f \)). \hfill \Box

**Corollary 2.10.** The residue class \( (\deg c_d(-T_X))/n_X \mod 2 \in \mathbb{Z}/2\mathbb{Z} \) is a birational invariant of a smooth, connected, complete variety \( X \) of dimension \( d \).

**Proof.** The integer \( n_X \) is known to be a birational invariant of such a variety \( X \), and the statement follows from Corollary 2.9. \hfill \Box

**Remark 2.11.** Indeed the group \( \text{CH}_0(X) \) is known to be a birational invariant of such a variety \( X \), and we expect the zero-cycle class \( c_d(-T_X) \in \text{CH}_0(X)/2\text{CH}_0(X) \) to be a birational invariant as well.

### 3. Strong 2-Incompressibility

**Definition 3.1** ([Kar10, p.150]). A complete integral variety \( X \) is called **strongly 2-incompressible** if for every complete integral variety \( Y \) with \( v_2(n_Y) \geq v_2(n_X) \) (\( v_2 \) is the dyadic valuation) and \( \dim Y \leq \dim X \), and such that the index \( n_{Y_k(X)} \) of the \( k(X) \)-variety \( Y_{k(X)} \) is odd, we have \( \dim Y = \dim X \) and the index \( n_{X_k(Y)} \) of the \( k(Y) \)-variety \( X_{k(Y)} \) is odd.

**Proposition 3.2.** Let \( X \) be a complete integral variety such that the integer \( s_X/n_X \) is odd. Then \( X \) is strongly 2-incompressible.

**Proof.** See the proof of [Mer03, Theorem 7.2]. \hfill \Box

According to [Kar10, Example 2.3], the Severi-Brauer variety of a \( p \)-primary central division algebra over a field of characteristic not \( p \) is strongly \( p \)-incompressible, while it is not known whether the corresponding statement holds in characteristic \( p \). We can give a positive answer for \( p = 2 \):
Corollary 3.3. The Severi-Brauer variety of a central division $k$-algebra, whose degree is a power of two, is strongly $2$-incompressible.

Proof. Let $2^n$ be the degree of the division algebra ($n > 0$), and $X$ its Severi-Brauer variety. Since $X$ becomes isomorphic to the projective space $\mathbb{P}^{2^n-1}$ after some extension of the base field, we have

$$s_X = \deg c(-T_X) = \deg c_{2^n-1}(-T_{\mathbb{P}^{2^n-1}}).$$

The dyadic valuation of this integer is $n$ (see [Mer01, Proposition 7.1] which is valid in any characteristic), and we have $n_X = 2^n$ [Mer03, Example 6.1]. It follows that $s_X/n_X$ is odd. $\square$

Corollary 3.4. Any anisotropic smooth complete quadric over $k$ whose dimension is $2^n - 1$ is strongly $2$-incompressible.

Proof. Let $X$ be the quadric. By [EKM08, Lemma 78.1], which is valid in any characteristic, we obtain

$$\text{Sq}(X) = c(-T_X) = (1 + h)^{-2^n-1} \mod 2 \text{CH}(X),$$

where $h \in \text{CH}(X)$ is the hyperplane class, a nilpotent element of the ring $\text{CH}(X)$. Using the relation $\deg(h^{2^n-1}) = 2$, we can compute that $s_X = 2 \mod 2n_X$. Since $n_X = 2$, it follows that $s_X/n_X$ is odd. $\square$

As in [Mer01, §5] (see also [Mer03, §7.5] where a different degree formula is used), we obtain the following two statements, originally due (in characteristic not two) respectively to Hoffmann [Hof95] and Izhboldin [Izh00]. In characteristic two, they have been proved by a different method in [HL06].

Proposition 3.5. Let $\varphi, \psi$ be two anisotropic non-degenerate quadratic forms over $k$. If $\dim \psi < 2^n + 1 \leq \dim \varphi$ for some $n$, then $\psi$ remains anisotropic over the function field of $\varphi$.

Proposition 3.6. Let $\varphi, \psi$ be two anisotropic non-degenerate quadratic forms over $k$. If $\dim \psi = 2^n + 1 \leq \dim \varphi$ for some $n > 0$, and $\psi$ is isotropic over the function field of $\varphi$, then $\varphi$ is isotropic over the function field of $\psi$.

Consequences of the degree formula can also be expressed using the first Witt index $i_1(\varphi)$ (see [EKM08, p.104]) of a non-degenerate quadratic form $\varphi$ such that $\dim \varphi > 1$. In characteristic not two, the statement below can be found in [EKM08, Example 79.7].

Proposition 3.7. Let $\varphi$ be an anisotropic non-degenerate quadratic form over $k$. If $2^n + 1 \leq \dim \varphi$, then $i_1(\varphi) \leq \dim \varphi - 2^n$. In particular if $\dim \varphi = 2^n + 1$, then $i_1(\varphi) = 1$.

Proof. Let $\psi$ be a non-degenerate subform of $\varphi$ of dimension $\dim \varphi - i_1(\varphi) + 1$ (such a subform exists by [EKM08, Corollary 7.32]). By [EKM08, Lemma 74.1 (2)] the form $\psi$ is isotropic over the function field of $\varphi$. If $i_1(\varphi) > \dim \varphi - 2^n$, then $\dim \psi < 2^n + 1$, and we may apply Proposition 3.5 to obtain a contradiction. $\square$
4. Scheme involutions (in characteristic two)

Let $G$ be the group with two elements. A $k$-involution of a variety is called a $G$-action. A $G$-variety is a variety with a $G$-action. A $k$-morphism between $G$-varieties is $G$-equivariant when it is compatible with the involutions. A subscheme of a $G$-variety is $G$-invariant if it is stable under the involution.

4.1. $G$-quotients.

Since a variety is quasi-projective over $k$, and $G$ is finite, any $G$-variety has a cover by affine $G$-invariant open subschemes. Thus we have:

**Proposition 4.1.1.** [SGA I, V, Proposition 1.8, Corollaire 1.5] Let $Y$ be a $G$-variety. There is a finite, surjective morphism of varieties $Y \to Y/G$, which is a categorical $G$-quotient.

**Notation 4.1.2.** When $Y = \text{Spec } A$ is an affine $G$-variety, we will denote by $\sigma_A$ the induced involution of the $k$-algebra $A$, and write $\partial_A = \text{id}_A + \sigma_A$. Then $\ker \partial_A$ is a $k$-algebra, that we denote by $A^G$. We have $Y/G = \text{Spec}(A^G)$ [SGA I, V, Proposition 1.1].

**Definition 4.1.3.** When $Y$ is a $G$-variety, we say that a morphism of varieties $f: Y \to X$ is a $G$-quotient if there is an isomorphism of varieties $X \to Y/G$ making $f$ a morphism over $Y/G$.

**Lemma 4.1.4.** [SGA I, V, Corollaire 1.4] The base change of a $G$-quotient along an open immersion is a $G$-quotient.

4.2. Fixed locus.

**Definition 4.2.1.** Let $Y$ be a $G$-variety. We define the fixed locus $Y^G$, via the cartesian square, where $\delta$ is the diagonal of $Y$ and $\gamma$ the graph of the involution,

$$
\begin{array}{ccc}
Y^G & \longrightarrow & Y \\
\downarrow & & \downarrow \delta \\
Y & \longrightarrow & Y \times_k Y \\
\gamma \\
\end{array}
$$

The two morphisms $Y^G \to Y$ coincide, and allow us to view $Y^G$ as a closed subscheme of $Y$.

**Example 4.2.2.** Let $X \to S$ be a morphism of varieties. The variety $Y = X\times_S X$ has a natural $G$-action, given by the exchange of factors. We claim that the fixed locus $Y^G$ in $Y$ is the diagonal $X$ in $X \times_S X$. Indeed a morphism $h: T \to Y^G$ is the data of $k$-morphisms $g_i: T \to X$ for $i \in \{1, 2, 3, 4\}$, subject to the condition $(g_1, g_2, g_2, g_1) = (g_3, g_4, g_3, g_4)$, or equivalently $g_1 = g_2 = g_3 = g_4$. Thus $h$ corresponds to a single morphism $g: T \to X$, and the composite $T \to Y^G \to Y$ corresponds to $(g, g): T \to X \times_S X$. This proves the claim.
Definition 4.2.3. Let $Y$ be a $G$-variety. We construct a closed subscheme $Y_G$ of $Y/G$ as follows. Let $\varphi: Y \to Y/G$ be the quotient map, and $\sigma$ the involution of the sheaf of $k$-algebras $\varphi_*\mathcal{O}_Y$. We consider the morphism $\text{id} + \sigma: \varphi_*\mathcal{O}_Y \to \varphi_*\mathcal{O}_Y$ of sheaves of $k$-vector spaces on $Y/G$. The subsheaf $\mathcal{O}_{Y/G}$ of $\varphi_*\mathcal{O}_Y$ coincides with $\ker(\text{id} + \sigma)$ by [SGA I, V, Corollaire 1.2], and $(\text{id} + \sigma) \circ (\text{id} + \sigma) = 0$. Thus $\text{id} + \sigma$ induces a morphism $\partial: \varphi_*\mathcal{O}_Y \to \mathcal{O}_{Y/G}$. If $U$ is an open subscheme of $Y/G$, and $a, b \in (\varphi_*\mathcal{O}_Y)(U)$, then we have in $\mathcal{O}_{Y/G}(U)$

$$\partial(ab) = \partial(a)\partial(b) + a\partial(b) + \partial(a)b.$$ 

When $a \in \mathcal{O}_{Y/G}(U) \subset (\varphi_*\mathcal{O}_Y)(U)$, then $\partial(a) = 0$, so that $\partial(ab) = a\partial(b)$. This proves that $\partial: \varphi_*\mathcal{O}_Y \to \mathcal{O}_{Y/G}$ is a morphism of $\mathcal{O}_{Y/G}$-modules. Its image is a quasi-coherent sheaf of ideals of $\mathcal{O}_{Y/G}$, and we define $Y_G$ as the corresponding closed subscheme of $Y/G$.

Thus when $Y = \text{Spec } A$, the closed subscheme $Y_G$ is defined by the ideal $\text{im} \partial_A$ of $A^G$ (Notation 4.1.2).

Lemma 4.2.4. Let $Y$ be $G$-variety. We have a cartesian square

$$\begin{array}{ccc}
Y_G & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Y & \longrightarrow & Y/G
\end{array}$$

Proof. To prove that the closed subschemes $Y^G$ and $Y_G \times_{Y/G} Y$ of $Y$ coincide, we may argue locally on $Y/G$, and assume that $Y = \text{Spec } A$.

We let $I$ be the ideal in $A \otimes_k A$ of the diagonal $Y \to Y \times_k Y$. Then the closed subscheme $Y^G$ of $Y$ is defined by the ideal $J$ of $A$ generated by the image of $I$ under the morphism $A \otimes_k A \to A$ mapping $a \otimes_k b$ to $a\sigma_A(b)$. If $a_i, b_i \in A$ are such that $\sum_i a_i b_i = 0$, then

$$\sum_i a_i \otimes_k b_i = \sum_i (a_i \otimes_k 1 + 1 \otimes_k a_i)(1 \otimes_k b_i)$$

proving that the ideal $I$ of $A \otimes_k A$ is generated by the set $\{a \otimes_k 1 + 1 \otimes_k a, a \in A\}$. It follows that the ideal $J$ of $A$ is generated by the set $\{a + \sigma_A(a), a \in A\}$. But this set is the ideal $\text{im} \partial_A$ of $A^G$, which defines $Y_G$ in $Y/G$. This proves the lemma. \qed

Lemma 4.2.5. Let $Y$ be a $G$-variety, $Z$ a $G$-invariant closed subscheme. Then $Z$ inherits a $G$-action, and we have two cartesian squares

$$\begin{array}{ccc}
Z_G & \longrightarrow & Z \\
\downarrow & & \downarrow \\
Y_G & \longrightarrow & Y
\end{array}$$

$$\begin{array}{ccc}
Z^G & \longrightarrow & Z \\
\downarrow & & \downarrow \\
Y^G & \longrightarrow & Y
\end{array}$$
Proof. We obtain the second square using the first and Lemma 4.2.4 (this also follows directly from the definition). To prove that the closed subschemes $Z_G$ and $Y_G \times_{Y/G} (Z/G)$ of $Z/G$ coincide, we may argue locally on $Y/G$, and assume that $Z \to Y$ is given by a surjective $G$-equivariant ring morphism $h: A \to B$. Then the ideal $\text{im} \partial_B = \text{im}(\partial_B \circ h) = \text{im}(h \circ \partial_A)$ of $B^G$ is generated by the image under $h$ of the ideal $\text{im} \partial_A$ of $A^G$. This gives the required cartesian square. \hfill $\square$

4.3. $G$-saturation.

Definition 4.3.1. Let $Y$ be a $G$-variety. A closed subscheme $Z$ of $Y$ is called $G$-saturated if it is the base change of some closed subscheme $W$ of $Y/G$, i.e. if $Z = W \times_{Y/G} Y$. When $Y = \text{Spec} A$, an ideal $I$ of $A$ defines a $G$-saturated closed subscheme if and only if it is generated by $I \cap A^G$.

Remark 4.3.2. A $G$-saturated closed subscheme is $G$-invariant.

Lemma 4.3.3. Let $Y$ be a $G$-variety. The closed subscheme $Y^G$ of $Y$ is $G$-saturated.

Proof. This follows from Lemma 4.2.4. \hfill $\square$

Lemma 4.3.4. The base change of a $G$-saturated closed subscheme along a $G$-equivariant morphism is a $G$-saturated closed subscheme.

Proof. Let $Y \to X$ be a $G$-equivariant morphism and $g: Y/G \to X/G$ be the induced morphism. If $Z$ is a $G$-saturated closed subscheme of $X$, there is a closed subscheme $W$ of $X$ such that $Z = W \times_{X/G} X$. Then $(g^{-1}W) \times_{Y/G} Y = f^{-1}Z$, proving that $f^{-1}Z$ is a $G$-saturated closed subscheme of $Y$. \hfill $\square$

4.4. Relative quotients.

Definition 4.4.1. Let $Y$ be a $G$-variety, and $Z$ a closed subscheme of $Y$. We denote by $\overline{Z}$ the closed subscheme of $Y/G$ defined as the scheme theoretic image of the composite $Z \to Y \to Y/G$. When confusion may arise, we will say that $\overline{Z}$ is the relative quotient of $Z$ with respect to the closed embedding $Z \to Y$.

Proposition 4.4.2. Let $Y \to X$ be a $G$-equivariant morphism. Assume that we have a commutative square, with horizontal arrows closed embeddings,

\[
\begin{array}{ccc}
W & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Z & \longrightarrow & X
\end{array}
\]

Then we have a commutative diagram

\[
\begin{array}{ccc}
W & \longrightarrow & Y/G \\
\downarrow & & \downarrow \\
\overline{W} & \longrightarrow & \overline{Y}/G \\
\downarrow & & \downarrow \\
Z & \longrightarrow & \overline{Z} & \longrightarrow & X/G
\end{array}
\]
Proof. Since $\overline{W}$ is the scheme theoretic image of the morphism $W \to \overline{W}$, it follows from the transitivity of scheme theoretic images [EGA I, I, (9.5.5)] that the morphisms $W \to X/G$ and $\overline{W} \to X/G$ have the same scheme theoretic image. Since the morphism $W \to X/G$ factors through $Z \to X/G$, this image is a closed subscheme of $\overline{Z}$. This proves the proposition. □

Lemma 4.4.3. Let $Y$ be a $G$-variety, and $Z$ a $G$-saturated closed subscheme of $Y$. Then we have a cartesian square

$$
\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\overline{Z} & \longrightarrow & Y/G
\end{array}
$$

Proof. Let $W$ be a closed subscheme of $Y/G$ such that $Z = W \times_{Y/G} Y$, and let $Z' = \overline{Z} \times_{Y/G} Y$. Since $Z \to Y/G$ factors through $W$, we have a closed embedding $\overline{Z} \to W$, hence by base change a closed embedding $Z' \to Z$. But the commutative square of the statement gives rise to a closed embedding $Z \to Z'$.

Lemma 4.4.4. Let $Y$ be a $G$-variety. Consider a cartesian square

$$
\begin{array}{ccc}
Z & \overset{i}{\longrightarrow} & Y \\
\downarrow & & \downarrow \\
W & \overset{j}{\longrightarrow} & Y/G
\end{array}
$$

where $j$ is a locally principal closed embedding, and $i$ is an effective Cartier divisor. Then $W = \overline{Z}$ as closed subschemes of $Y/G$, and $j$ is an effective Cartier divisor.

Proof. We may assume that $Y = \text{Spec} A$ and that the closed subscheme $W$ of $Y/G$ is defined by the principal ideal of $A^G$ generated by an element $t$. By the assumption on $i$, the $A$-module $tA$ is locally free, and in particular torsion-free. Thus $t$ is a nonzerodivisor in $A$, hence in $A^G$. This proves that $j$ is an effective Cartier divisor.

Let $a \in A$ be such that $ta \in A^G$. Then $0 = \partial_A(ta) = t\partial_A(a)$. Since $t$ is a nonzerodivisor in $A^G$, we get $\partial_A(a) = 0$, i.e. $a \in A^G$. Thus we proved that $(tA) \cap A^G = t(A^G)$, which gives the equality $W = \overline{Z}$. □

4.5. Pure actions.

Definition 4.5.1. Let $Y$ be a $G$-variety. We say that the $G$-action on $Y$ is pure if the morphism $Y_G \to Y/G$ (Definition 4.2.3) is an effective Cartier divisor.

Proposition 4.5.2. Let $Y$ be a variety with pure $G$-action. Then the morphism $\varphi : Y \to Y/G$ is flat of rank two (see §1.6).
Proof. Let \( \mathcal{I} \) be the sheaf of ideals defining \( Y_G \) in \( Y/G \). We have an exact sequence of \( \mathcal{O}_{Y/G} \)-modules (see Definition 4.2.3)

\[
0 \to \mathcal{O}_{Y/G} \to \varphi_* \mathcal{O}_Y \xrightarrow{\partial} \mathcal{I} \to 0.
\]

Since by assumption \( \mathcal{I} \) is a locally free \( \mathcal{O}_{Y/G} \)-module of rank one, the sequence splits, and \( \varphi_* \mathcal{O}_Y \) is a locally free \( \mathcal{O}_{Y/G} \)-module of rank two. The morphism \( \varphi \) is finite by Proposition 4.1.1.

Corollary 4.5.3. The \( G \)-action on a variety \( Y \) is pure if and only if:

— the closed embedding \( Y_G \to Y/G \) is locally principal, and

— the closed embedding \( Y^G \to Y \) is an effective Cartier divisor.

Proof. The two conditions are sufficient by Lemma 4.2.4 and Lemma 4.4.4, and necessary by Proposition 4.5.2 (an effective Cartier divisor remains so under flat base change).

Lemma 4.5.4. Let \( Y \) be a variety with pure \( G \)-action, and \( Z \) a closed subscheme of \( Y \). If \( W \) is a closed subscheme of \( Y/G \) whose inverse image in \( Y \) is \( Z \), then \( W = \overline{Z} \).

Proof. The base change of the morphism \( \overline{Z} \to W \) along \( Y \to Y/G \) is the identity of \( Z \) by Lemma 4.4.3. The statement follows, since the morphism \( Y \to Y/G \) is faithfully flat by Proposition 4.5.2 and §1.6 (ii).

Proposition 4.5.5. Let \( Y \to X \) be a \( G \)-equivariant morphism. Assume that we have a cartesian square

\[
\begin{array}{ccc}
W & \to & Z \\
\downarrow & & \downarrow \\
Y & \to & X
\end{array}
\]

with vertical arrows closed embeddings. Assume that \( Z \) is \( G \)-saturated in \( X \), and that the \( G \)-action on \( Y \) is pure. Then the square

\[
\begin{array}{ccc}
\widetilde{W} & \to & \overline{Z} \\
\downarrow & & \downarrow \\
Y/G & \to & X/G
\end{array}
\]

given by Proposition 4.4.2, is cartesian.

Proof. Let \( \widetilde{W} = \overline{Z} \times_{X/G} (Y/G) \), so that we have by Proposition 4.4.2 a commutative diagram

\[
\begin{array}{ccc}
W & \to & \widetilde{W} & \to & \overline{Z} \\
\downarrow & & \downarrow & & \downarrow \\
Y & \to & Y/G & \to & X/G
\end{array}
\]

(1)
where the rightmost square is cartesian. The commutative squares

\[
\begin{array}{ccc}
W & \longrightarrow & Z \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
& & X/G
\end{array}
\]  

are cartesian by Lemma 4.4.3, hence so is the exterior square of (2). It coincides with the exterior square of (1), and therefore the leftmost square of (1) is cartesian. Since \( \overline{W} = \overline{W} \) as closed subschemes of \( Y/G \), proving the proposition. □

**Proposition 4.5.6.** Let \( Y \) be a variety with pure \( G \)-action, and \( Z \) a \( G \)-invariant closed subscheme of \( Y \) such that \( Y^G \cap Z \to Z \) is an effective Cartier divisor. Then the morphism \( Z \to \overline{Z} \) is a \( G \)-quotient.

**Proof.** Using Lemma 4.2.5, we see that the closed subscheme \( Z_G \) of \( Z/G \) is locally principal, being a base change of the locally principal closed subscheme \( Y_G \) of \( Y \), and that the morphism \( Z^G \to Z \) is an effective Cartier divisor, being equal to the effective Cartier divisor \( Y^G \cap Z \to Z \). Thus by Corollary 4.5.3, the \( G \)-action on \( Z \) is pure. By Proposition 4.5.2, it follows that the morphisms \( Z \to Z/G \) and \( Y \to Y/G \) are flat of rank two. So is the morphism \( Z \to \overline{Z} \), being a base change of the morphism \( Y \to Y/G \). Applying Lemma 4.5.7 below to the morphisms \( Z \to \overline{Z} \), we conclude that \( Z/G \to \overline{Z} \) is an isomorphism. □

**Lemma 4.5.7.** Let \( Z \xrightarrow{g} Y \xrightarrow{f} X \) be two morphisms of varieties. If each of the morphisms \( f \circ g \) and \( g \) is flat of rank \( n \), then \( f \) is an isomorphism.

**Proof.** The morphism \( f \) is faithfully flat and finite. We check locally that \( f \) is an isomorphism, and assume that \( f_* O_Y \) is a free \( O_X \)-module of rank \( r \), and that \( g_* O_Z \) is a free \( O_Y \)-module of rank \( n \). Then \( f_* \circ g_* O_Z \) is a free \( O_X \)-module of rank \( nr \). By assumption, it is also locally free of rank \( n \), so that \( r = 1 \), and \( f \) is an isomorphism. □

4.6. **Actions on blow-ups.**

Let \( Y \) be a \( G \)-variety, and \( B \) the blow-up of a \( G \)-invariant closed subscheme in \( Y \). Then the universal property of the blow-up provides a \( G \)-action on \( B \) such that the morphism \( B \to Y \) is \( G \)-equivariant. In the sequel, \( B \) will be implicitly endowed with this \( G \)-action.

**Lemma 4.6.1.** Let \( Y' \to Y \) be a \( G \)-equivariant morphism, and \( Z \) a \( G \)-invariant closed subscheme of \( Y \), with inverse image \( Z' \) in \( Y' \). Let \( B \), resp. \( B' \), be the blow-up of \( Z \) in \( Y \), resp. \( Z' \) in \( Y' \). Then the natural morphism \( B' \to B \) is \( G \)-equivariant.
Lemma 4.6.2. Let $B$ be the blow-up of $Z$ in $Y$. Then we have two cartesian squares

$$
\begin{array}{ccc}
B_G & \longrightarrow & B/G \\
\downarrow & & \downarrow \\
Y_G & \longrightarrow & Y/G
\end{array} \quad \quad \quad
\begin{array}{ccc}
B^G & \longrightarrow & B \\
\downarrow & & \downarrow \\
Y^G & \longrightarrow & Y
\end{array}
$$

Proof. We obtain the second square using the first and Lemma 4.2.4. To prove that the closed subschemes $B_G$ and $Y_G \times_{Y/G} (B/G)$ of $B/G$ coincide, we may argue locally on $Y/G$, and assume that $Y = \text{Spec } A$. Let $J$ be the ideal of $A$ defining the closed subscheme $Z$ in $Y$. We consider the Rees algebra $S = \bigoplus_{n \geq 0} J^n$. Using the notations of §1.9, the variety $B = \text{Proj } S$ is covered by the open subschemes $\text{Spec } S(s)$, for $s$ belonging to a set of generators of the ideal $J$ of $A$. Since $Z$ is $G$-saturated in $Y$, we may choose $s \in J \cap A^G$ (we use Notation 4.1.2). Then $\text{Spec } S(s)$ is a $G$-invariant subscheme of $B$, and we have $\partial_{S(s)}(y/s^n) = \partial_A(y)/s^n$. We claim that the ideal $\text{im } \partial_{S(s)}(S(s))^G$ is generated by the elements $\partial_A(a)/s^0$ for $a \in A$; this will conclude the proof.

Since the ideal $J$ of $A$ is generated by $J \cap A^G$, for $n \in \mathbb{N}$ the ideal $J^n$ of $A$ is generated by $J^n \cap A^G$. Thus the group $S(s)$ is generated by the elements $(ja)/s^n$, with $n \in \mathbb{N}$ and $j \in J^n \cap A^G$ and $a \in A$. The claim follows from the computation

$$
\partial_{S(s)}((ja)/s^n) = \partial_A(ja)/s^n = (j\partial_A(a))/s^n = (\partial_A(a)/s^0) \cdot (j/s^n).
$$

Lemma 4.6.3. Let $Y$ be a $G$-variety, and $Z$ a $G$-saturated closed subscheme of $Y$. Let $B$ the blow-up of $Z$ in $Y$, and $P$ its exceptional divisor. Then $\overline{P} \to B/G$ is an effective Cartier divisor.

If $W$ is a closed subscheme of $Y/G$ whose inverse image in $Y$ is $Z$, then the inverse image of $W$ in $B/G$ is $\overline{P}$.

Proof. Since $Z$ is $G$-saturated, we can always find $W$ as in the statement. Let $B'$ be the blow-up of $W$ in $Y/G$, and $P'$ its exceptional divisor. By the universal
property of the blow-up, there is a morphism $b: B \to B'$ such that $b^{-1}P' = P$. If we endow $B'$ with the trivial $G$-action, then $b$ is $G$-equivariant, hence factors as $B \to B/G \to B'$. The base change $\tilde{P} \to B/G$ of $P' \to B'$ remains a locally principal closed embedding. The base change of $\tilde{P} \to B/G$ along $B \to B/G$ is the effective Cartier divisor $P \to B$. By Lemma 4.4.4, it follows that $\tilde{P} = \tilde{P} \to B/G$ is an effective Cartier divisor.

**Proposition 4.6.4.** Let $Y$ be a $G$-variety, and $B$ the blow-up of $Y^G$ in $Y$. Then the $G$-action on $B$ is pure, and $B^G$ is the exceptional divisor in $B$.

**Proof.** By Lemma 4.6.2, the exceptional divisor $P$ in $B$ coincides with $B^G$. Applying Lemma 4.6.3 with $W = Y_G$ (and thus $Z = Y^G$ by Lemma 4.2.4), and using Lemma 4.6.2, we obtain that $B_G = \tilde{P} \to B/G$ is an effective Cartier divisor. □

**Proposition 4.6.5.** Let $Y$ be a variety with pure $G$-action, and $Z$ a $G$-saturated closed subscheme of $Y$. Then the $G$-action on the blow-up $B$ of $Z$ in $Y$ is pure.

**Proof.** We use Lemma 4.6.2. The closed embedding $B_G \to B/G$ is locally principal, being a base change of $Y_G \to Y/G$. Since $Y^G \to Y$ is an effective Cartier divisor, so is its base change $B^G \to B$, by Lemma 4.6.6 below. □

**Lemma 4.6.6.** The inverse image of an effective Cartier divisor (resp. a nowhere dense closed subscheme) under a blow-up morphism is an effective Cartier divisor (resp. a nowhere dense closed subscheme).

**Proof.** Let $D \to Y$ be the effective Cartier divisor (resp. nowhere dense closed subscheme), and $b: B \to Y$ the blow-up of closed subscheme $Z$ in $Y$. Any associated (resp. generic) point $x$ of $B$ is contained in the open complement $U$ of the exceptional divisor $P$, since $P \to B$ is an effective Cartier divisor. Since $b$ maps $U$ isomorphically to an open subscheme of $Y$, it follows that $b(x)$ is an associated (resp. generic) point of $Y$, and therefore cannot be inside the effective Cartier divisor (resp. nowhere dense closed subscheme) $D$. Thus no associated (resp. generic) point of $B$ is mapped to $D$, and the statement follows. □

5. **Principal transform and proper transform**

Let $X$ be a variety, and $D \to Z \to X$ be closed embeddings. Assume that $D \to X$ is an effective Cartier divisor. The residual scheme to $D$ in $Z$ on $X$ is the unique closed subscheme $R$ of $X$ such that $\mathcal{I}_R \cdot \mathcal{I}_D = \mathcal{I}_Z$, where we use the notations of §1.10 (see [Ful98, Definition 9.2.1]).

**Remark 5.1.** The following facts follow at once from the definition.

(i) $R = X$ if and only if $Z = X$.

(ii) $R \to X$ is an effective Cartier divisor if and only if $Z \to X$ is so, in which case $Z = D + R$ as Cartier divisors on $X$. 
Lemma 5.2. Consider a commutative diagram with cartesian squares

\[
\begin{array}{ccc}
D' & \longrightarrow & Z' \\
\downarrow & & \downarrow \\
D & \longrightarrow & Z \\
\end{array}
\]

Assume that the horizontal arrows are closed embeddings, and that the horizontal composites are effective Cartier divisors. Let \( R, \) resp. \( R' \), be the residual scheme to \( D \) in \( Z \) on \( X \), resp. to \( D' \) in \( Z' \) on \( X' \). Then there is a unique morphism \( R' \to R \) fitting into the commutative diagram with cartesian squares

\[
\begin{array}{ccc}
R' & \longrightarrow & Z' \\
\downarrow & & \downarrow \\
R & \longrightarrow & Z \\
\end{array}
\]

Proof. Unicity follows from the fact that \( R \to X \) is a monomorphism. In view of §1.10, we have

\[
I_{Z'} = I_Z O_{X'} = (I_D \cdot I_R) O_{X'} = (I_D O_{X'}) \cdot (I_R O_{X'}) = I_{D'} \cdot (I_R O_{X'}).
\]

But \( I_{R'} \) is the unique sheaf of ideals of \( O_{X'} \), such that \( I_{Z'} = I_{D'} \cdot I_{R'} \), hence

\[
I_{R'} = I_R O_{X'},
\]

proving the statement. \( \square \)

Lemma 5.3. Let \( D \to Z \to X \) be closed embeddings, with \( D \to X \) an effective Cartier divisor. Let \( R \) be the residual scheme to \( D \) in \( Z \) on \( X \).

(i) Let \( T \) be a closed subscheme of \( Z \), and assume that \( D \cap T \to T \) is an effective Cartier divisor. Then \( T \) is a closed subscheme of \( R \).

(ii) Assume that \( X \) has a \( G \)-action, and that \( \overline{D} \to X/G \) is an effective Cartier divisor. If both \( D \) and \( Z \) are \( G \)-saturated in \( X \), then so is \( R \), and moreover \( \overline{R} \) is the residual scheme to \( \overline{D} \) in \( \overline{Z} \) on \( X/G \).

Proof. (i) : We apply Lemma 5.2 with \( D' = D \cap T \) and \( Z' = X' = T \), and note that \( R' = T \) by Remark 5.1 (i).

(ii) : We apply Lemma 5.2 to the diagram with cartesian squares

\[
\begin{array}{ccc}
D & \longrightarrow & Z \\
\downarrow & & \downarrow \\
\overline{D} & \longrightarrow & \overline{Z} \\
\end{array}
\]

and obtain that \( R = h^{-1} \tilde{R} \), where \( \tilde{R} \) is the residual scheme to \( \overline{D} \) in \( \overline{Z} \) on \( X/G \). Thus \( R \) is a \( G \)-saturated closed subscheme of \( X \). Since \( R \to X/G \) factors through \( \tilde{R} \), it follows that \( I_{\tilde{R}} \subset I_{\overline{R}} \) (we use the notations of §1.10). Let \( T \) be the closed subscheme of \( X/G \) defined by the sheaf of ideals \( I_{\overline{R}} \cdot I_{\overline{R}} \). Then \( T \) is a closed
subscheme of $\mathcal{Z}$ (the latter is defined by the sheaf of ideals $I_{\mathcal{R}} \cdot I_{\mathcal{D}}$), and $\mathcal{R}$ is the residual scheme to $\mathcal{D}$ in $T$ on $X/G$. We have $h^{-1}T = Z$, since

$$I_{h^{-1}T} = (I_{\mathcal{R}} \cdot I_{\mathcal{D}}) \mathcal{O}_X = (I_{\mathcal{R}} \mathcal{O}_X) \cdot (I_{\mathcal{D}} \mathcal{O}_X) = I_{\mathcal{R}} \cdot I_{\mathcal{D}} = I_Z.$$  

Thus $Z \to X/G$ factors through $T$, hence $\mathcal{Z} \subset T$, so that $T = \mathcal{Z}$, and $\mathcal{R} = \mathcal{R}$. □

**Definition 5.4.** Let $Z \to Y \to X$ be closed embeddings. Let $b: B_{Z \to X} \to X$ be the blow-up of $Z$ in $X$, and $P_{Z \to X}$ its exceptional divisor. The *principal transform* of $Y$, denoted by $R_{Z \to Y \to X}$, is the residual scheme to $P_{Z \to X}$ in $b^{-1}Y$ on $B_{Z \to X}$. Thus $R_{Z \to Y \to X}$ is defined in $B_{Z \to X}$ by the sheaf of ideals $(I_Y \mathcal{O}_{B_{Z \to X}})(-1)$.

There are closed embeddings $B_{Z \to Y} \to b^{-1}Y \to B_{Z \to X}$ with the property that $B_{Z \to Y} \cap P_{Z \to X} = P_{Z \to Y}$. Therefore by Lemma 5.3 (i), there is a closed embedding $B_{Z \to Y} \hookrightarrow R_{Z \to Y \to X}$.

**Lemma 5.5.** Let $Y \to X$ be a closed embedding and $Z \to Y$ a regular closed embedding. Then the closed embedding $B_{Z \to Y} \to R_{Z \to Y \to X}$ is an isomorphism.

*Proof.* This can extracted from [Alu04]; let us nonetheless give a proof. We may assume that $X = \text{Spec} \ A$, and that $Y$, resp. $Z$, is defined by the ideal $I$, resp. $J$, of $A$. The closed subscheme $b^{-1}Y$ of $B_{Z \to X}$ is given by the homogeneous ideal $L = \bigoplus_{n \geq 0} I \cdot J^n$ of the Rees algebra $\bigoplus_{n \geq 0} J^n$. The closed subscheme $R_{Z \to Y \to X}$ of $B_{Z \to X}$ is given by the homogeneous ideal $\bigoplus_{n \geq 0} I \cdot J^{n-1}$ (we write $J^1 = A$), since its degree $n$ component agrees with that of $L(-1) = \bigoplus_{n \geq 1} I \cdot J^{n-1}$ for $n \geq 1$. We have surjections of graded $A/I$-algebras (Sym denotes the symmetric algebra)

$$\text{Sym}_{A/I}(J/I) \to \bigoplus_{n \geq 0} J^n/(I \cdot J^{n-1}) \to \bigoplus_{n \geq 0} (J/I)^n.$$  

The map on the right induces the closed embedding $B_{Z \to Y} \to R_{Z \to Y \to X}$. If $Z \to Y$ is a regular closed embedding, then the composite above is an isomorphism [Bou07, §5, N°2, Théorème 1, (i) ⇒ (iii)], whence the statement. □

Now we assume given a morphism of varieties $Y \to X$. We write

$$B_Y = B_{Y \to Y \times_k Y}, \quad B_{Y/X} = B_{Y \to Y \times_X Y}, \quad R_{Y/X} = R_{Y \to Y \times_X Y \to Y \times_k Y}.$$  

The natural $G$-actions on $Y \times_k Y$ and $Y \times_X Y$ have fixed locus the diagonal $Y$ (Example 4.2.2), and $B_Y$ and $B_{Y/X}$ inherit a $G$-action by §4.6. There is a natural closed embedding $B_{Y/X} \to B_Y$, which is $G$-equivariant by Lemma 4.6.1. In particular $B_{Y/X}$ is a $G$-invariant closed subscheme of $B_Y$.

**Lemma 5.6.** The morphism $B_{Y/X} \to X$ factors through the relative quotient map $B_{Y/X} \to \overline{B_{Y/X}}$ with respect to the embedding into $B_Y$.

*Proof.* The $G$-action on $B_Y$ is pure by Proposition 4.6.4. The closed embedding of the fixed locus $(B_Y)^G = P_Y$ into $B_Y$ restricts to an effective Cartier divisor on $B_{Y/X}$, viz. the exceptional divisor for blow-up of $Y$ in $Y \times_X Y$. Thus by
Proposition 4.5.6, the morphism \( B_{Y/X} \to \overline{B_{Y/X}} \) is a \( G \)-quotient. Since \( B_{Y/X} \to X \) is \( G \)-equivariant and \( X = (X \times_k X)^G \) has trivial \( G \)-action, the lemma follows. □

**Proposition 5.7.** The morphism \( R_{Y/X} \to X \) factors through the relative quotient map \( R_{Y/X} \to \overline{R_{Y/X}} \) with respect to the embedding into \( B_Y \).

**Proof.** Since we are working with quasi-projective varieties, we may factor \( f \) as \( Y \to W \to X \) with \( W \) smooth over \( X \), and \( Y \) a closed subscheme of \( W \). We let \( Y_X \), resp. \( W_X \), be the inverse image of \( Y \times_X Y \) in \( B_Y \), resp. \( W \times_X W \) in \( B_W \). There is a natural morphism \( B_Y \to B_W \), which is \( G \)-equivariant by Lemma 4.6.1.

Applying Lemma 5.2 to the commutative diagram with cartesian squares

\[
\begin{array}{ccc}
P_Y & \longrightarrow & Y_X \longrightarrow B_Y \\
\downarrow & & \downarrow & & \downarrow \\
P_W & \longrightarrow & W_X \longrightarrow B_W
\end{array}
\]

we obtain a commutative diagram

\[
\begin{array}{ccc}
R_{Y/X} & \longrightarrow & Y_X \longrightarrow B_Y \\
\downarrow & & \downarrow & & \downarrow \\
R_{W/X} & \longrightarrow & W_X \longrightarrow B_W
\end{array}
\]

We let \( \overline{R_{Y/X}} \), resp. \( \overline{R_{W/X}} \), be the relative quotient of \( R_{Y/X} \), resp. \( R_{W/X} \), with respect to the embedding into \( B_Y \), resp. \( B_W \). Now since \( W \) is smooth over \( X \), the closed embedding \( W \to W \times_X W \) is regular, and therefore by Lemma 5.5 we have \( R_{W/X} = B_{W/X} \) as closed subschemes of \( W_X \). Applying Lemma 5.6 with \( Y = W \), we see that \( R_{W/X} \to X \) factors through \( R_{W/X} \to \overline{R_{W/X}} \). The morphism \( R_{Y/X} \to X \) factors through \( R_{Y/X} \to R_{W/X} \), hence through \( R_{Y/X} \to \overline{R_{Y/X}} \). Finally, it factors through \( R_{Y/X} \to \overline{R_{Y/X}} \) by Proposition 4.4.2. □

6. **Proof of the main theorem**

Let \( f: Y \to X \) be a morphism of equidimensional varieties. We construct a commutative diagram with cartesian squares,

\[
\begin{array}{ccc}
Y & \longrightarrow & P_Y \longrightarrow S_f \\
\downarrow & & \downarrow \beta & & \downarrow \\
Y \times_X Y & \longrightarrow & Y_X \longrightarrow P_X \longrightarrow \overline{P_X} \longrightarrow \overline{X} \\
\downarrow & & \downarrow \delta & & \downarrow i_f & & \downarrow i_X \\
Y \times_k Y & \longrightarrow & B_Y \longrightarrow B_X \longrightarrow X \times_k X
\end{array}
\]
The variety $B_Y$, resp. $B_X$, is the blow-up of the diagonal $Y$ in $Y \times_k Y$, resp. $X$ in $X \times_k X$. We denote by $P_Y$, resp. $P_X$, the exceptional divisor. We let $Y_X \rightarrow B_Y$ be the base change of $Y \times_k Y \rightarrow Y \times_k Y$, and $B_f$ be the blow-up of $Y_X$ in $B_Y$, with exceptional divisor $P_f$. Since $Y_X$ is the inverse image of the diagonal $X$ under the morphism $B_Y \rightarrow X \times_k X$, we obtain, by the universal property of the blow-up, a morphism $γ: B_f \rightarrow B_X$ such that $γ^{-1}P_X = P_f$.

We endow $Y \times_k Y$ and $X \times_k X$ with the $G$-actions induced by the exchange of factors (Example 4.2.2), so that the morphism $f \times f$ is $G$-equivariant. There are $G$-actions on $B_Y$, $B_X$, $B_f$ such that the morphisms $B_Y \rightarrow Y \times_k Y$, $B_X \rightarrow X \times_k X$ and $α: B_f \rightarrow B_Y$ are $G$-equivariant (see §4.6). So is the morphism $γ: B_f \rightarrow B_X$ by Lemma 4.6.1. Each variety in the diagram (3) is a $G$-saturated closed subscheme of the variety of the bottom row lying in the same column, by Example 4.2.2, Lemma 4.3.3 and Lemma 4.3.4. The $G$-actions on $B_Y$ and $B_X$ are pure by Proposition 4.6.4, and so is the $G$-action on $B_f$ by Proposition 4.6.5.

For $† \in \{X, Y, f\}$, we use the effective Cartier divisor $i_†: P_† \rightarrow B_†$ to define an operator

$$ξ_† = i_†^* \circ i_†*: \text{CH}(P_†) \rightarrow \text{CH}(P_†).$$

The morphism $j_† = i_† \circ s: S_† \rightarrow B_f$ is an effective Cartier divisor by Lemma 4.6.6. We define the operator

$$σ_† = j_†^* \circ j_†*: \text{CH}(S_†) \rightarrow \text{CH}(S_†).$$

Since $Y$, resp. $X$, is equidimensional, so is $Y \times_k Y$, resp. $X \times_k X$ [EGA IV$_2$, (4.2.6) and (4.2.4, (iii))]. Since $B_f$, resp. $B_X$, contains as a dense subscheme $B_f − P_f$, resp. $B_X − P_X$, which is an open subscheme of $Y \times_k Y$, resp. $X \times_k X$, it follows that $B_f$, resp. $B_X$, is equidimensional. Thus by [Ful98, Proposition 2.6 (d)], we have in $\text{CH}(P_X)$ and $\text{CH}(P_f)$ and $\text{CH}(S_†)$

$$i_†^*[B_X] = [P_X] \text{ and } i_†^*[B_f] = [P_f] \text{ and } j_†^*[B_f] = [S_†].$$

We consider the principal transform $R_{Y/X} = R_Y \rightarrow Y \times_k Y \rightarrow Y \times_k Y$ (Definition 5.4) and form the cartesian square

$$\begin{array}{ccc}
R_f & \rightarrow & B_f \\
\downarrow & & \downarrow \\
R_{Y/X} & \rightarrow & B_Y
\end{array}$$

The variety $R_{Y/X}$ is a $G$-saturated closed subscheme of $B_Y$ by Lemma 5.3 (ii). It follows from Lemma 4.3.4 that $R_f$ is a $G$-saturated closed subscheme of $B_f$. Thus the morphism $φ: R_f \rightarrow \overline{R_f}$ is a base change of the pure $G$-quotient $B_f \rightarrow B_f/G$, and in particular is flat of rank two (§1.6) by Proposition 4.5.2.
Lemma 6.1. We have in $\text{CH}(P_f)$, for any $n$,

$$s_*(\sigma_f)^n[S_f] = (\xi_f)^n[P_f] \mod \text{im}(r_* \circ \varphi^*),$$

where $r: R_f \to P_f$ is the natural closed embedding.

Proof. It follows from Lemma 5.2 that $R_f$ is the residual scheme to $S_f$ in $P_f$ on $B_f$. By Remark 5.1 (ii), we have $P_f = R_f + S_f$ as Cartier divisors on $B_f$. By Proposition 4.5.5, the commutative diagram

$$\begin{array}{ccc}
S_f & \rightarrow & P_f \\
\downarrow & & \downarrow \\
\overline{R_f} & \rightarrow & B_f/G
\end{array}$$

has cartesian squares. The morphisms $\overline{Y} \rightarrow B_f/G$ and $\overline{P_f} \rightarrow B_f/G$ are effective Cartier divisors by Lemma 4.6.3. The closed embedding $\overline{S_f} \rightarrow \overline{P_f}$ is locally principal (being a base change of $\overline{Y} \rightarrow B_f/G$). Since $S_f \rightarrow B_f$ is an effective Cartier divisor, so is $\overline{S_f} \rightarrow \overline{B_f/G}$ by Lemma 4.4.4. By Lemma 5.3 (ii), the residual scheme to $\overline{S_f}$ in $\overline{P_f}$ on $B_f/G$ is $\overline{R_f}$. It follows from Remark 5.1 (ii) that $\overline{R_f} \rightarrow B_f/G$ is an effective Cartier divisor.

Thus, for $A \in \{R, S, P\}$, the closed embeddings $A_f \rightarrow B_f$ and $\overline{A_f} \rightarrow B_f/G$ are effective Cartier divisors. We consider the restrictions of the associated line bundles $L_A = \mathcal{O}_{B_f}(A_f)|_{P_f}$ and $\overline{L_A} = \mathcal{O}_{B_f/G}(A_f)|_{\overline{P_f}}$. We have $\psi^*(\overline{L_A}) = L_A$, where $\psi$ is the morphism fitting in the commutative diagram with cartesian squares

$$\begin{array}{ccc}
R_f & \rightarrow & P_f \\
\downarrow & \varphi & \downarrow \\
\overline{R_f} & \rightarrow & \overline{P_f} \rightarrow B_f/G
\end{array}$$

We have $\xi_f = c_1(L_P) = c_1(L_R) + c_1(L_S)$. Since the line bundle $\mathcal{O}_{B_f/G}(\overline{R_f})$ is trivial off $\overline{R_f}$, it follows that $\text{im} c_1(\overline{L_R}) \subset \text{im} \tau_*$ (here we use that $c_1(L) = 0$ when $L$ is trivial, compatibility of $c_1$ with restriction to open subschemes, and the localisation sequence). Therefore, as subgroups of $\text{CH}(P_f)$,

$$\text{im}(c_1(L_R) \circ \varphi^*) = \text{im}(\psi^* \circ c_1(\overline{L_R})) \subset \text{im}(\psi^* \circ \tau_* \circ \varphi^*) = \text{im}(r_* \circ \varphi^*).$$

Since, as morphisms $\text{CH}(\overline{R_f}) \rightarrow \text{CH}(P_f)$,

$$\xi_f \circ r_* \circ \varphi^* = c_1(L_P) \circ r_* \circ \varphi^* = r_* \circ c_1(r^*L_P) \circ \varphi^* = r_* \circ \varphi^* \circ c_1(\overline{L_P}),$$

it follows that, as subgroups of $\text{CH}(P_f)$,

$$\text{im}(\xi_f \circ r_* \circ \varphi^*) \subset \text{im}(r_* \circ \varphi^*).$$
We now prove the lemma by induction on \( n \). The case \( n = 0 \) is clear, since we have, in \( \text{CH}(P_f) \),
\[
[P_f] = s_*[S_f] + r_*[R_f] = s_*[S_f] + r_* \circ \varphi^*[R_f].
\]
If \( n > 0 \), using induction, we have in \( \text{CH}(P_f) \),
\[
(\xi_f)^n[P_f] = \xi_f \circ (\xi_f)^{n-1}[P_f] = \xi_f \circ s_* \circ (\sigma_f)^{n-1}[S_f] \mod \text{im}(\xi_f \circ r_* \circ \varphi^*).
\]
In view of (6), we have in \( \text{CH}(P_f)/\text{im}(r_* \circ \varphi^*) \)
\[
(\xi_f)^n[P_f] = \xi_f \circ s_* \circ (\sigma_f)^{n-1}[S_f] \mod \text{im}(r_* \circ \varphi^*)
\]
\[
= (c_1(L_S) + c_1(L_R)) \circ s_* \circ (\sigma_f)^{n-1}[S_f] \mod \text{im}(r_* \circ \varphi^*)
\]
\[
= s_* \circ (\sigma_f)^n[S_f] + c_1(L_R) \circ c_1(L_S)^n \circ \psi^*[\xi_{\overline{S_f}}] \mod \text{im}(r_* \circ \varphi^*)
\]
\[
= s_* \circ (\sigma_f)^n[S_f] + c_1(L_R) \circ \psi^* \circ c_1(L_S)^n[S_f] \mod \text{im}(r_* \circ \varphi^*)
\]
\[
= s_* \circ (\sigma_f)^n[S_f] \mod \text{im}(r_* \circ \varphi^*),
\]
where we used (5) for the last equality.

\[\square\]

**Lemma 6.2.** We have \( \text{im}(\pi_X \circ \delta \circ r) \subset 2 \text{CH}(X) \).

**Proof.** The morphism \( \pi_X \circ \delta \circ r : R_f \to X \) factors through \( R_{Y/X} \), and therefore through \( \overline{R_{Y/X}} \) in view of Proposition 5.7. Since \( R_f \to \overline{R_{Y/X}} \) factors through \( \varphi : R_f \to \overline{R_f} \) (Proposition 4.4.2), so does \( \pi_X \circ \delta \circ r \). Thus \( \text{im}(\pi_X \circ \delta \circ r_* \circ \varphi^*) \subset \text{im}(\varphi_* \circ \varphi^*) \). But, since \( \varphi \) is flat of rank two, the endomorphism \( \varphi_* \circ \varphi^* \) of the group \( \text{CH}(\overline{R_f}) \) is multiplication by two (§1.6 (iii)).

\[\square\]

**Lemma 6.3.** Assume that \( Y \times_X Y \) is nowhere dense in \( Y \times_k Y \). We have in \( \text{CH}(P_Y) \), for any \( n \),
\[
\beta_* \circ (\sigma_f)^n[S_f] = (\xi_Y)^n[P_Y].
\]

**Proof.** By Lemma 4.6.6, the subscheme \( Y_X \) is nowhere dense in \( B_Y \). Therefore the blow-up morphism \( \alpha : B_f \to B_Y \) is birational, and in particular has degree one. In view of the cartesian square
\[
\begin{array}{ccc}
S_f & \xrightarrow{j_f} & B_f \\
\downarrow{\beta} & & \downarrow{\alpha} \\
P_Y & \xrightarrow{i_Y} & B_Y
\end{array}
\]
we have, as morphisms \( \text{CH}(B_f) \to \text{CH}(P_Y) \),
\[
(7) \quad \beta_* \circ j_f^* = i_Y^* \circ \alpha_*
\]
and therefore, as morphisms \( \text{CH}(P_f) \to \text{CH}(P_Y) \),
\[
(8) \quad \beta_* \circ \sigma_f = \beta_* \circ j_f^* \circ j_{f*} = i_Y^* \circ \alpha_* \circ j_{f*} = i_Y^* \circ i_{Y*} \circ \beta_* = \xi_Y \circ \beta_*.
\]
Finally, we compute, in $\text{CH}(P_Y)$,
\[
\beta_* \circ (\sigma_f)^n[S_f] = \beta_* \circ (\sigma_f)^n \circ j_f^*[B_f] \quad \text{by (4)}
\]
\[
= (\xi_Y)^n \circ \beta_* \circ j_f^*[B_f] \quad \text{by (8)}
\]
\[
= (\xi_Y)^n \circ i_Y^* \circ \alpha_*[B_f] \quad \text{by (7)}
\]
\[
= (\xi_Y)^n \circ i_Y^*[B_Y] \quad \text{since } \deg \alpha = 1
\]
\[
= (\xi_Y)^n [P_Y] \quad \text{by (4)}. \square
\]

**Lemma 6.4.** Assume that $f$ has a degree. Then the morphism $\gamma: B_f \to B_X$ has degree $(\deg f)^2$.

**Proof.** In view of §1.7, we have in $Z(X \times_k X)$
\[
(f \times_k f)_*[Y \times_k Y] = (f_*[Y]) \times_k (f_*[Y])
\]
\[
= (\deg f \cdot [X]) \times_k (\deg f \cdot [X])
\]
\[
= (\deg f)^2 \cdot [X \times_k X],
\]
and therefore $f \times_k f$ has degree $(\deg f)^2$.

Consider the commutative digram, with horizontal arrows open embeddings,
\[
\begin{array}{c}
Y \times_k Y \leftarrow B_f - P_f \xrightarrow{\nu} B_f \\
\downarrow f \times_k f \quad \quad \quad \downarrow h \\
X \times_k X \leftarrow B_X - P_X \xrightarrow{u} B_X
\end{array}
\]

Since the square on the left is cartesian, the morphism $h$ has degree equal to $(\deg f)^2$ (see §1.5). The morphisms $u$ and $v$ are dense open immersions, being the complements of effective Cartier divisors, hence have degree one by §1.4 (i). Then we have in $Z(B_X)$ the equalities
\[
\gamma_*[B_f] = \gamma_* \circ v_*[B_f - P_f] = u_* \circ h_*[B_f - P_f] = (\deg f)^2 \cdot u_*[B_X - P_X] = (\deg f)^2 \cdot [B_X],
\]
so that the morphism $\gamma$ has degree equal to $(\deg f)^2$. \square

**Lemma 6.5.** Assume that $f$ is proper and has a degree. Then we have in $\text{CH}(P_X)$, for any $n$,
\[
\delta_* \circ (\xi_f)^n[P_f] = (\deg f)^2 \cdot (\xi_X)^n[P_X].
\]

**Proof.** Since $f$ is proper, so are $\gamma$ and $\delta$. Using the relevant cartesian square of (3), we see that, as morphisms $\text{CH}(B_f) \to \text{CH}(P_X)$,
\[
\delta_* \circ i_f^* = i_X^* \circ \gamma_*,
\]
and, as morphisms $\text{CH}(P_f) \to \text{CH}(P_X)$,
\[
\delta_* \circ \xi_f = \delta_* \circ i_f^* \circ i_{f*} = i_X^* \circ \gamma_* \circ i_{f*} = i_X^* \circ i_{X*} \circ \delta_* = \xi_X \circ \delta_*.
\]
Thus we have in $\text{CH}(P_X)$,
\[
\delta_\ast \circ (\xi_f)^n[P_f] = \delta_\ast \circ (\xi_f)^n \circ i_\ast^f[B_f] \\
= (\xi_X)^n \circ \delta_\ast \circ i_\ast^f[B_f] \\
= (\xi_X)^n \circ i_X^\ast \circ \gamma_\ast[B_f] \\
= (\xi_X)^n \circ i_X^\ast((\deg f)^2 \cdot [B_X]) \\
= (\deg f)^2 \cdot (\xi_X)^n[P_X]
\]
by Lemma 6.4
\[
\square
\]

**Theorem 6.6.** Let $f : Y \rightarrow X$ be a proper morphism of equidimensional varieties. Assume that $f$ has a degree (see §1.4), and that $Y \times_X Y$ is nowhere dense in $Y \times_k Y$. Then we have, for any $n$,
\[
f_\ast \circ \pi_{Y,\ast} \circ (\xi_Y)^n[P_Y] - \deg f \cdot \pi_{X,\ast} \circ (\xi_X)^n[P_X] \in 2 \text{CH}(X).
\]

**Proof.** We have in $\text{CH}(X)/2 \text{CH}(X)$,
\[
f_\ast \circ \pi_{Y,\ast} \circ (\xi_Y)^n[P_Y] = f_\ast \circ \pi_{Y,\ast} \circ \beta_\ast \circ (\sigma_f)^n[S_f] \\
= \pi_{X,\ast} \circ \delta_\ast \circ \sigma_\ast \circ (\sigma_f)^n[S_f] \\
= \pi_{X,\ast} \circ \delta_\ast \circ (\xi_f)^n[P_f] \\
= (\deg f)^2 \cdot \pi_{X,\ast} \circ (\xi_X)^n[P_X]
\]
by Lemma 6.1, Lemma 6.2
\[
\square
\]

We conclude using the relation $(\deg f)^2 = \deg f \mod 2.$

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