TRANSLATING GRAPHS BY MEAN CURVATURE FLOW IN
\( \mathcal{M}^n \times \mathbb{R} \)

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ABSTRACT. In this work, we study graphs in \( \mathcal{M}^n \times \mathbb{R} \) that are evolving by the mean curvature flow over a bounded domain on \( \mathcal{M}^n \), with prescribed contact angle in the boundary. We prove that solutions converge to translating surfaces in \( \mathcal{M}^n \times \mathbb{R} \). Also, for a Riemannian manifold \( \mathcal{M}^2 \) with negative Gaussian curvature at each point, we show non-existence of complete vertically translating graphs in \( \mathcal{M}^2 \times \mathbb{R} \).

1. Introduction

Let \((\mathcal{M}^n, \sigma)\) be a Riemannian manifold with Riemannian metric \(\sigma\), and let \(g = ds^2 + \sigma\) be the product metric on \(\mathcal{M}^n \times \mathbb{R}\). Assume \(\Omega \subset \mathcal{M}^n\) is a compact domain with diameter smaller than the injectivity radius of \(\mathcal{M}^n\), and with smooth boundary \(\partial \Omega\). We denote by \(\gamma\) the inward unit normal vector to \(\partial \Omega\), and by \(\nu\) the upward unit normal to the graph \(u: \Omega \to \mathbb{R}\).

The non-parametric mean curvature flow with specified contact angle in the boundary is a well-known equation, that for \(a^{ij} = g^{ij} - \nabla_i \nabla_j u\) and \(w = \sqrt{1 + |\nabla u|^2}\) is given by

\[
\begin{align*}
u_t &= w \operatorname{div} \left( \frac{\nabla u}{w} \right) = a^{ij} \nabla_i \nabla_j u \quad \text{in} \quad Q_T, \\
u \cdot \gamma &= \Phi(x, u) \quad \text{on} \quad \Gamma_T, \\
u(\cdot, 0) &= u_0(\cdot) \quad \text{on} \quad \Omega_0,
\end{align*}
\]

where \(u_0 \in C^\infty(\overline{\Omega}), Q_T = \Omega \times [0, T], \Gamma_T = \partial \Omega \times [0, T]\) and \(\Omega_T = \Omega \times \{t\}\).

Equation (1.1) has been studied, among others, by Huisken [6] for \(\Phi = 0\), and by Altschuler and Wu [1] for the case \(\mathcal{M}^n = \mathbb{R}^2\) and \(\Omega\) strictly convex. Altschuler and Wu proved that if \(\Omega\) is strictly convex and \(|D\Phi| < \min k(\partial \Omega)\), where \(k\) indicates curvature, the solution \(u(x, t)\) of (1.1) is in \(C^\infty(\Omega \times [0, \infty))\), and converges as \(t \to \infty\) to the graph of a function \(u_\infty \in C^\infty(\Omega)\) (unique up to translation) which moves at a constant speed \(C\) (uniquely determined by the boundary data). Moreover, if \(\int_{\partial \Omega} \Phi = 0\) then \(C = 0\), then \(u_\infty\) is a minimal surface.

In this work, we show that the results of Altschuler and Wu are true in \(\mathcal{M}^n \times \mathbb{R}\) without having the condition over the gradient bound of \(\Phi\). For proving this, we follow Altshuler and Wu’s argument, but use different techniques to prove the main theorems: for the gradient bound of the solution to (1.1), we follow a technique used

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by Guan in [4], where he found a gradient bound of the solution of mean curvature flow when $\Omega \subset \mathbb{R}^n$. Then we prove the convergence of solutions to a translating graph using similar arguments as in [2]. In the last section, we show non-existence of complete translating graphs with constant speed in $\mathcal{M}^2 \times \mathbb{R}$ when the Gaussian curvature at each point of $\mathcal{M}^2$ is less than zero.

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2. Existence of solution

Let $\mathcal{M}^n \subset \mathbb{R}^{n+1}$ be a Riemannian Manifold. We consider the (signed) distance function

$$d(x) = \begin{cases} \min_{y \in \partial \Omega} |x - y|_\mathcal{M} & \text{if } x \in \Omega; \\ -\min_{y \in \partial \Omega} |x - y|_\mathcal{M} & \text{if } x \in \mathcal{M} \setminus \Omega. \end{cases}$$

near $\partial \Omega$, and the inner normal $\gamma = \nabla d$. There exists a neighborhood of radius $\mu > 0$ of points within (unsigned) distance $\mu$ of $\partial \Omega$, on which $d$ is $C^3$ and $\gamma$ is $C^2$.

We will also assume $\Phi$ is extended to a smooth function (still denote by $\Phi$) on $\overline{\Omega}$ with $|\Phi| \leq \Phi_0 < 1$.

Given a solution $u \in C^\infty(\Omega \times [0, T))$ of (1.1), where $T \leq \infty$, we denote by $L$ to the operator

$$Lv = a^{ij} \nabla_i \nabla_j v - v_t,$$

where $v \in C^\infty(\Omega \times [0, T))$, for $a^{ij} = a^{ij}(u)$ defined as above.

**Lemma 2.1.** For a solution $u \in C^\infty(\Omega \times [0, \infty))$ of (1.1),

$$\max_{\Pi \times [0, \infty)} |u_t| = \max_{\Pi} |u_t(\cdot, 0)|$$

where $\Pi = \overline{\Omega}$, and $\Pi = \mathcal{M} \iff x \in \mathcal{M}$.

**Proof.** It suffices to prove that for any fixed $T > 0$, if $u_t(x_0, t_0) = \max_{\Pi \times [0, T]} u_t \geq 0$ for some $(x_0, t_0) \in \Pi \times [0, T]$, then $t_0 = 0$.

Assume $t_0 > 0$. Since

$$Lu_t = \frac{2}{w} a^{ij} \nabla_i w \nabla_j u_t$$

by maximum principle we have $x_0 \in \partial \Omega$.

Let $f_i$ be a coordinate frame so that $f_n = \gamma$ at $x_0$. Then at the point $(x_0, t_0)$ we have:

$$\nabla_{f_i} u_t = (\nabla_{f_i} u)_t = 0 \text{ for } 1 \leq i < n$$

$$w_t = \frac{\nabla_{f_n} u \nabla_{f_n} u_t}{w} = \Phi(x_0) \nabla_{f_n} u_t$$

On the other hand using the contact angle boundary condition we have:

$$\nabla_{f_n} u_t = \Phi(x_0) w_t$$
Since $|\Phi| < 1$ then $\nabla f_n u_t = 0$. This contradicts the Hopf Lemma.

Now we will prove a gradient bound for the solution to the equation (1.1) by applying Guan’s method in \[4\], which is based on Korevaar’s idea in \[7\].

We set
\[
(2.3) \quad \eta = e^{Ku} (Nd + 1 - \Phi v \cdot \nabla d)
\]
where K and N are positive numbers to be determined.

**Lemma 2.2.** For $N > 0$ sufficiently large independent of $K$ and $t$, if for some $t \geq 0$ fixed, $w\eta(., t)$ attains a local maximum value at a point $x_0 \in \partial \Omega$, then $w(x_0, t) \leq K$.

**Proof.** Assume $x_0 \in \partial \Omega$ and $t \geq 0$ fixed. In the following all the derivatives are evaluated at $(x_0, t)$. We choose a coordinate basis $f_i, 1 \leq i \leq n$ in $\mathcal{M}$ so that $f_n$ is the interior normal direction to $\partial \Omega$ at $x_0$, and such that
\[
(2.4) \quad \nabla f_i u \geq 0, \quad \nabla f_n u = 0 \quad \text{for } 2 \leq \alpha \leq n - 1
\]
We have $\nabla f_i w\eta = 0$ at $x_0$ on $\partial \Omega$, therefore:
\[
(2.5) \quad 0 = (1 - \Phi^2) \nabla f_i w - 2w\Phi \nabla f_i \Phi + Kw (1 - \Phi^2) \nabla f_i u
\]
Thus:
\[
(2.6) \quad \nabla f_i w = - Kw \nabla f_i u + \frac{2 \Phi w \nabla f_i \Phi}{1 - \Phi^2}
\]
Also at $x_0$ on $\partial \Omega$ we have $\nabla f_n w\eta \leq 0$:
\[
(2.7) \quad Nw + \nabla f_n w - w\Phi \nabla f_n \Phi - \Phi \nabla f_n \nabla f_n u + Kw (1 - \Phi^2) \nabla f_n u \leq 0
\]
Using the definition of $w$ and the coordinate basis at the point $x_0$ we have:
\[
(2.8) \quad \nabla f_n w = \frac{1}{2} \frac{\nabla f_n (\nabla f_i u f_1 + \nabla f_n u f_1 + \nabla f_i u f_n)}{w} = \frac{\langle \nabla f_i, \nabla f_i u, \nabla f_i u \rangle}{w}
\]
Using the boundary contact angle condition we have $\nabla f_n u = \Phi w$, thus
\[
(2.9) \quad \nabla f_n \nabla f_n u = w \nabla f_i \Phi + \Phi \nabla f_i w
\]
and
\[
(2.10) \quad w^2 (1 - \Phi^2) = w^2 - |\nabla f_n u|^2 = 1 + |\nabla f_i u|^2
\]
From (2.8), (2.9) and (2.10) we have:
\[
(2.11) \quad \nabla f_n w = \langle \nabla f_i, \nabla f_i u \rangle - K\Phi |\nabla f_n u|^2 + \frac{\Phi^2}{1 - \Phi^2} \langle \nabla f_i, \nabla f_i u \rangle + \frac{1}{w} \langle \nabla f_n, \nabla f_n u, \nabla f_n u \rangle
\]
By substituting (2.11) and (2.10) in (2.7) we have:
\[
0 \geq N + \frac{1 + \Phi^2}{w (1 - \Phi^2)} \langle \nabla f_i, \nabla f_i u \rangle + Kw\Phi (1 - \Phi^2) - \frac{K\Phi}{w} - \Phi \nabla f_n \Phi + Kw\Phi (1 - \Phi^2)
\]
\[
\geq N - C - \frac{K\Phi}{w} + wK\Phi (1 - \Phi^2)
\]
Taking $N$ large enough we have:

\begin{align}
\frac{K\Phi}{w} & \geq wK\Phi \left(1 - \Phi^2\right) \\
\frac{w^2}{1 - \Phi^2} & \leq 0
\end{align}

Let $N$ fixed so that the lemma 2.2 holds.

**Lemma 2.3.** There exists $K > 0$ sufficiently large so that if $w\eta(x_0, t_0) = \max_{\Omega \times [0, T]} w\eta$ for some $(x_0, t_0) \in \Omega \times [0, T]$, then $w(x_0, t_0) \leq C$, for some constant $C$.

**Proof.** By lemma 2.2 we can assume $x_0 \in \Omega$ and $t_0 > 0$. Let $f_i$, $1 \leq i \leq n$ be a coordinate basis for $M$ orthonormal at $x_0$.

At $(x_0, t_0)$ we have:

\begin{align}
0 = \nabla f_i w\eta &= \eta \nabla f_i w + w\nabla f_i \eta & \text{for } 1 \leq i \leq n \\
0 \leq (w\eta)_t &= w_t \eta + \eta w
\end{align}

and

\begin{align}
0 \geq L(w\eta) &= wL\eta + \eta \left(Lw - \frac{2}{w} a^{ij} \nabla f_i w \nabla f_j w\right)
\end{align}

Let $\eta = e^{Ku} h$ and $h = Nd + 1 - \Phi \nabla d \cdot v$. We have

\begin{align}
\frac{1}{\eta} L\eta &= K^2 a^{ij} \nabla f_i u \nabla f_j u + KLu + \frac{1}{h} Lh + \frac{2K}{h} a^{ij} \nabla f_i u \nabla f_j h
\end{align}

Now we need to compute $Lw$.

\begin{align}
\nabla f_i w &= \frac{\langle \nabla \nabla f_i u, \nabla u \rangle}{\omega} \\
\nabla f_i \nabla f_j w &= \frac{\langle \nabla \nabla f_i \nabla f_j u, \nabla u \rangle}{\omega} + \frac{\langle \nabla \nabla f_i u, \nabla \nabla f_j u \rangle}{\omega} - \frac{\nabla f_j w \nabla f_i w}{\omega} \\
w_t &= a^{ij} \frac{\langle \nabla \nabla f_i \nabla f_j u, \nabla u \rangle}{\omega} - 2 a^{ij} \frac{\nabla f_j u \nabla f_i \nabla f_j u}{\omega^3} \langle \nabla f_i u, \nabla u \rangle \\
&+ 2 a^{ij} \frac{\nabla f_i u \nabla f_j u \nabla f_j u}{\omega^4} \langle \nabla w, \nabla u \rangle
\end{align}

\begin{align}
Lw &= a^{ij} \frac{\langle \nabla \nabla f_i u, \nabla \nabla f_j u \rangle}{\omega} - a^{ij} \frac{\nabla f_i \nabla f_j w}{\omega} - 2 a^{ij} \frac{\nabla f_j u \nabla f_i \nabla f_j u}{\omega^3} \langle \nabla w, \nabla u \rangle \\
&- 2 a^{ij} \frac{\nabla f_i \nabla f_j u \nabla f_j u \nabla f_j u}{\omega^4} \langle \nabla w, \nabla u \rangle
\end{align}

Since at the point $x_0$ the basis $f_i$ is orthonormal we have at the point $x_0$: 

\[ Lw = \frac{a^{ij} \left< \nabla \nabla f_i u, \nabla \nabla f_j u \right>}{w} - a^{ij} \nabla_{f_i} w \nabla_{f_j} w + 2 \left| \nabla w \right|^2 - \frac{2}{w^3} \nabla_{f_i} u \nabla_{f_j} u \nabla_{f_i} w \nabla_{f_j} w \]

\[ = \frac{a^{ij} \left< \nabla \nabla f_i u, \nabla \nabla f_j u \right>}{w} + a^{ij} \nabla_{f_i} w \nabla_{f_j} w \]

\[ = \frac{|A|^2}{w} - \frac{\nabla_{f_i} u \nabla_{f_j} u \left( \nabla \nabla f_i u, \nabla \nabla f_j u \right)}{w^3} + a^{ij} \nabla_{f_i} w \nabla_{f_j} w \]

\[ = \frac{|A|^2}{w} - \left| \nabla w \right|^2 + a^{ij} \frac{\nabla_{f_i} w \nabla_{f_j} w}{w} \]

\[ = w |A|^2 + \frac{2}{w} a^{ij} \nabla_{f_i} w \nabla_{f_j} w \]

Using Cauchy-Schwarz and the absorbing inequality we have:

\[ a^{ij} \nabla_{f_i} \left( \Phi \nabla f_k d \right) \nabla_{f_j} v^k = \frac{1}{w} a^{ij} a^{kl} \nabla_{f_i} \left( \Phi \nabla f_k d \right) \nabla_{f_j} \nabla_{f_i} u \]

\[ \leq \frac{1}{w} \sqrt{a^{ij} a^{kl} \nabla_{f_i} \nabla_{f_k} u \nabla_{f_j} u} \sqrt{a^{ij} a^{kl} \nabla_{f_i} \left( \Phi \nabla f_k d \right) \nabla_{f_j} \left( \Phi \nabla f_k d \right)} \]

\[ \leq \frac{1}{4w^2} a^{ij} a^{kl} \nabla_{f_i} \nabla_{f_k} u \nabla_{f_j} \nabla_{f_i} u + a^{ij} a^{kl} \nabla_{f_i} \left( \Phi \nabla f_k d \right) \nabla_{f_j} \left( \Phi \nabla f_k d \right) \]

\[ \leq \frac{1}{4} |A|^2 + C \]

Since the derivative of the metric is zero, using (2.16) we will have

\[ Lh = a^{ij} \left( N \nabla f_i \nabla f_j d - v^k \nabla_{f_j} \nabla_{f_i} \Phi \right) - 2 a^{ij} \nabla f_i \left( \Phi \nabla f_k d \right) \nabla f_j v^k - \Phi \left( \nabla f_k d \right) L v^k \]

\[ \geq \Phi \left( \nabla f_k d \right) v^k |A|^2 - \frac{1}{2} |A|^2 - C \]

(2.17)

Using (2.13), Cauchy-Schwarz and absorbing inequalities at the point \((x_0, t_0)\) we will get:

\[ a^{ij} \nabla_{f_i} u \nabla_{f_j} h = \frac{1}{w^2} \nabla_{f_i} u \nabla_{f_j} h \]

\[ \geq -C - \frac{\Phi}{w^2} \nabla f_i d \nabla f_j u \nabla f_j v^k = -C - \frac{\Phi}{w^2} \nabla f_i d a^{kl} \nabla_{f_i} w \]

\[ = -C + \frac{\Phi}{w} \nabla f_i d a^{kl} \left( K \nabla_{f_i} u + \frac{\nabla_{f_i} h}{h} \right) \]

\[ \geq -C - \frac{K}{w^2} - \frac{\Phi^2}{h w} a^{kl} \nabla f_i d \nabla f_j d \nabla f_j \nabla_{f_i} u \]

\[ = -C - \frac{K}{w^2} - \frac{\Phi^2}{h w^2} a^{kl} a^{ij} \nabla f_i d \nabla f_j d \nabla f_j \nabla_{f_i} u \]

(2.18)

\[ \geq -C - \frac{C K}{w^2} - \frac{1}{4K} |A|^2 \]

Plugging the (2.18) and (2.17) inequalities in (2.15) we get:
\[ (2.19) \quad \frac{1}{\eta} L \eta \geq K^2 \left( 2 - \frac{1}{w^2} \right) + \frac{\Phi \nabla f_k \cdot \nu_k - 1}{h} |A|^2 - \frac{C(K+1)}{h} \leq \frac{C K^2}{h w^2} \]

Inequality (2.19) and (2.14) implies:

\[ 1 - \Phi_0 - \frac{C}{w^2} \leq \frac{C}{K^2}(K+1) \]

(2.20)

\[ w^2 \leq \frac{C}{1 - \Phi_0 - \frac{C(K+1)}{K^2}} \]

So by choosing \( K \) large enough, there is a constant \( C \) such that \( w(x_0, t_0) \leq C \).  \( \square \)

Thus the gradient bound follows from Lemma (2.3), since

\[ (2.21) \quad w(x, t) \leq \frac{w(x_0, t_0) \eta(x_0, t_0)}{\eta(x, t)} \leq C_1 e^{C_2(w(x_0, t_0) - w(x, t))} \leq C_1 e^{C_3 M_T}, \]

where \( M_T = \max_{\Omega \times [0, T]} |u - u_0| \).

**Theorem 2.4.** A unique, smooth solution to the equation (1.1) exists.

**Proof.** The gradient bound (2.21) implies long-time existence for the solutions of the equation (1.1) using standard theory. \( \square \)

### 3. Extension of Translating Graphs

The elliptic version of the mean curvature equation (1.1) is

\[ (3.1) \quad \text{div} \left( \frac{\nabla u}{w} \right) = \frac{C}{w} \quad \text{in} \quad \Omega \]

\[ v \cdot \gamma = \Phi(x, u) \quad \text{on} \quad \partial \Omega \]

where using divergence theorem we can compute

\[ (3.2) \quad C = \int_{\partial \Omega} \frac{\Phi ds}{\int_{\Omega} w dx} \]

Notice that if \( u(x) \) is a solution of (3.1) then \( \tilde{u} = u - Ct \) is the solution of the mean curvature equation (1.1) which is translating upward with speed \( C \).

**Remark 3.1.** Since \( w \leq C_1 e^{C_2 M_T} \) and \( |\Phi| < 1 \), we have

\[ C = \int_{\partial \Omega} \frac{\Phi ds}{\int_{\Omega} w dx} \leq C_1 e^{C_2 M_T} \frac{|\partial \Omega|}{|\Omega|} \]

**Theorem 3.2.** Equation (3.1) has a unique solution, for a unique value of \( C \) given by (3.2).
Proof. Let $\epsilon > 0$. Now we look at the equation

\begin{equation}
\text{div} \left( \frac{\nabla u}{w} \right) = \frac{\epsilon u}{w} \quad \text{in} \quad \Omega
\end{equation}

\begin{equation}
v \cdot \gamma = \Phi(x, u) \quad \text{on} \quad \partial \Omega
\end{equation}

We can prove existence of a solution $u_\epsilon$ to the equation (3.3) by replacing $\Psi$ by $\frac{\epsilon u}{w}$ in the proof of existence of solutions to the prescribed mean curvature equation in [2]. Moreover, from that argument we also get a gradient bound $|\nabla u_\epsilon| < M$ for some constant $M$ independent of $\epsilon$. This implies $|\nabla (\epsilon u_\epsilon)| \to 0$ when $\epsilon \to 0$. So $\epsilon u_\epsilon \to C$ when $\epsilon \to 0$.

Now assume there exist two solutions $u_1$ and $u_2$ solving (3.3) with $C_1$ and $C_2$. Suppose $C_1 < C_2$ and $u_1 \geq u_2$. Then $u = u_1 - u_2$ is a solution of a linear elliptic differential inequality $L(u) < 0$. By the maximum principle, the minimum of $u$ must occur at the point $b \in \partial \Omega$. Then $|\nabla u_1(x)| = |\nabla u_2(x)|$.

However strict monotonicity in $q$ of the function $\sqrt{1 + |\nabla u_1|^2 + |\nabla u_2|^2}$ implies that $\nabla u_1(b) = \nabla u_2(b)$. Thus $\nabla u(b) = 0$ which yields contradiction to the Hopf boundary point Lemma. So $C_1 \geq C_2$. By reversing the roles of $u_1$ and $u_2$ we will get the opposite inequality. Thus $C_1 = C_2$. The proof of $u_1 = u_2$ is similar. \hfill □

Corollary 3.3. For a solution $u(x, t)$ of (1.1) there exists a constant $M$ such that

\begin{equation}
|u(x, t) - Ct| \leq M
\end{equation}

Proof. It will follow by sandwiching the solution of the mean curvature equation 1.1 between two translating elliptic solutions and applying a maximum principle. \hfill □

Now we show that the limit of solutions of the mean curvature equation 1.1 is a translating graph if $C \neq 0$ and is a minimal surface if $C = 0$.

Theorem 3.4. Let $u_1$ and $u_2$ be any two solutions of (1.1) and let $u = u_1 - u_2$. Then $u$ becomes a constant function at $t \to \infty$. In particular since $\tilde{u} = u - Ct$ solves (1.1) when $u$ solves (1.1), all limit solutions of (1.1) are $\tilde{u}$ up to translation.

Proof. Same as the proof of Theorem 3.1 in [1]. \hfill □

Theorem 3.5. If $C = 0$ for the elliptic barrier then $\lim_{t \to 0} u_t = 0$. That is the solutions converge to the corresponding minimal surfaces.

Proof. By Corollary 3.3 and because $C = 0$, we can estimate:

\begin{equation}
\left| \int_{\partial \Omega} u \Phi ds \right| \leq M \int_{\partial \Omega} |\Phi| ds \leq M \text{Length}(\partial \Omega) \quad \forall t \in [0, T].
\end{equation}

Also we have

\begin{equation}
\frac{d}{dt} \int_{\Omega} wdx = \int_{\Omega} \frac{\nabla u_t, \nabla u}{w} dx = -\int_{\Omega} \frac{u^2}{w} dx + \int_{\partial \Omega} u_t \Phi ds
\end{equation}
This implies
\[
\frac{d}{dt} \left( \int_{\Omega}wdx - \int_{\partial \Omega} u \Phi ds \right) = - \int_{\Omega} \frac{u_t^2}{w} dx
\]
Therefore
\[
\int_{0}^{T} \int_{\Omega} \frac{u_t^2}{w} dx dt = - \int_{\partial \Omega} u(x,T) \Phi ds - \int_{\partial \Omega} u(x,0) \Phi ds
\]
\[
- \int_{\Omega} w(x,T) dx + \int_{\Omega} w(x,0) dx \leq C.
\]
Thus \( \lim_{t \to \infty} \int_{\Omega} \frac{u_t^2}{w} dx = 0 \). Since \( w \) is bounded then \( \lim_{t \to \infty} u_t = 0 \).

\[\square\]

4. Complete translating graphs in \( M^2 \times \mathbb{R} \)

In this section we consider the case of a surface \( M^2 \) such that \( c(M) := \inf K < 0 \), where \( K \) is the Gauss curvature of \( M^2 \). Let \( \Sigma \) be a complete continuous translating graph with constant speed \( C \) in \( M^2 \times \mathbb{R} \). Let \( \pi : \Sigma \to M^2 \equiv M^2 \times \{0\} \), be the horizontal projection and define \( c(\Sigma) = \inf \{ K(\pi(p)) : p \in \Sigma \} \).

We will show that if \( c(M^2) < 0 \), then there is no vertically complete translating graph with constant speed by mean curvature flow in \( M^2 \times \mathbb{R} \). This result is inspired and follows the arguments in [3] and [5]. In [5], Hauswirth, Rosenberg and Spruck proved that a complete immersed surface in \( H^2 \times \mathbb{R} \) of constant mean curvature \( \frac{1}{2} \) which is transverse to the vertical killing field must be an entire graph. Also, Espinar and Rosenberg in [3] proved that when \( c(\Sigma) < 0 \) and \( H > \sqrt{-c(\Sigma)/2} \), then \( \Sigma \) is a vertical cylinder over a complete curve of \( M^2 \) of constant geodesic curvature \( 2H \).

**Lemma 4.1.** There is no complete entire vertical graph \( \Sigma \) with \( \inf_{\Sigma} H > \frac{\sqrt{-c(M^2)}}{2} \) in \( M^2 \times \mathbb{R} \) when \( c(M^2) < 0 \).

**Proof.** Suppose that such an entire graph exists. Let \( \Sigma = \text{graph}(u) \) be such a graph, where \( u : M^2 \to \mathbb{R} \). Then we have:

\[
(4.1) \quad \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 2H
\]

Let \( \Omega \subset M^2 \) be an open domain with compact closure and smooth boundary \( \partial \Omega \), let \( \gamma \) denote the outward normal to \( \partial \Omega \). From (4.1) and divergence theorem we have:

\[
2 \inf H \ Vol(\Omega) \leq \int_{\Omega} \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) dV
\]
\[
\leq \int_{\partial \Omega} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}, \gamma \right) dA
\]
\[
\leq \text{Area}(\partial \Omega)
\]
Thus $2 \inf H \leq J(M^2)$, where

$$J(M^2) = \inf \{ \frac{\text{Area}(\partial \Omega)}{\text{Vol}(\Omega)} : \Omega \subset M^2 \text{ open, } \overline{\Omega} \text{ compact, } \partial \Omega \text{ is } C^\infty \}.$$

By the proof of Lemma 4.1 in [3], we have $J(M^2) \leq \sqrt{-c(M^2)}$. Hence

$$\sqrt{-c(M^2)} < 2 \inf H \leq J(M^2),$$

which is a contradiction. \hfill \Box

**Theorem 4.2.** There is no complete immersed translating graph $\Sigma$ in $M^2 \times \mathbb{R}$ so that $c(\Sigma) < 0$ and $\inf \Sigma H > \sqrt{-c(\Sigma)}/2$.

**Proof.** We know that $\Sigma$ can not be an entire graph by Lemma 4.1. So the proof is completed when we prove that this graph is an entire graph. Suppose $\Sigma$ is a complete immersed translating graph over a domain $\Omega \subset M$.

As $\Sigma$ has bounded geometry, there exists $\delta > 0$ such that for each $p \in \Sigma$, $\Sigma$ is a graph in exponential coordinates over the disk $D_\delta \subset T_p\Sigma$ of radius $\delta$, centered at the origin of $T_p\Sigma$. This graph, denoted by $G(p)$, has bounded geometry. $\delta$ is independent of $p$ and the bound on the geometry of $G(p)$ is uniform as well.

We denote by $F(p)$ the surface $G(p)$ translated to height zero $M^2 = M^2 \times \{0\}$, i.e., let $\phi_p$ be the isometry of $M^2 \times \mathbb{R}$ which takes $p$ to $\pi(p)$, we denote $F(p) = \phi_p(\Sigma)$.

Now, let $p \in \Sigma$ and assume $\Sigma$ in a neighborhood of $p$ is a vertical graph of a function $f$ defined on $B_R$, $B_R$ the open geodesic ball of radius $R$ of $M^2$ centered at $\pi(p) = O \in M^2$. If $\Sigma$ is not an entire graph then we let $R$ be the largest such $R$ so that $f$ exists. Since $\Sigma$ is translating graph by mean curvature flow, $f$ has bounded gradient on relatively compact subsets of $B_R$. Let $q \in \partial B_R$ be such that $f$ does not extend to any neighborhood of $q$ to a translating graph with $\inf H > \sqrt{-c(\Sigma)}/2$.

Let $q_n$ be a sequence in $B_R$ converging to $q$, and let $p_n = (q_n, f(q_n)) \in \Sigma$ be images of $q_n$ in $\Sigma$. Let $F(n)$ denote the image of $G(p_n)$ under the vertical translation taking $p_n$ to $q_n$. Observe that $T_{q_n}(F(n))$ converges to the vertical plane, for any subsequence of the $q_n$. Otherwise the graph of bounded geometry $G(p_n)$, would extend to a vertical graph beyond $q$, for $q_n$ close enough to $q$, hence $f$ would extend; a contradiction.

Thus, for any sequence $q_n \in B_R$ converging to $q$ we have $w(q_n) \to +\infty$. That means $H(q_n) \to 0$. Which is contradiction by $\inf \Sigma H > \sqrt{-c(\Sigma)}/2$.

\hfill \Box

**Theorem 4.3.** If $c(M^2) < 0$, then there is no complete vertically translating graph with non-zero constant speed in $M^2 \times \mathbb{R}$.
Proof. Assume there is a graph $u$ which is vertically translating with constant speed $C \neq 0$. By theorem (4.2), we have:

$$\frac{C}{\sup w} = 2 \inf H \leq \sqrt{-c(M)} \text{ i f } C > 0$$

$$\frac{C}{\inf w} = 2 \inf H \leq \sqrt{-c(M)} \text{ i f } C < 0$$

Let $(x_0, t_0)$ be the same as in Lemma (2.3), from the formula (2.21) we get:

$$C \sqrt{-c(M)} \leq C_1 e^{C_2 (u(x_0, t_0) - u(x, t))}$$

(4.2) $$u(x, t) \leq u(x_0, t_0) - \frac{1}{C_2} \ln \left( \frac{C}{C_1 \sqrt{-c(M)}} \right)$$

Since $C$ and $\sqrt{-c(M^2)}$ are fixed constants, we have $C \neq C_1 \sqrt{-c(M)}$ which is a contradiction by continuity of $u(x, t_0)$. \qed

Corollary 4.4. There is no complete translating graph with non-zero speed in $\mathbb{H}^2 \times \mathbb{R}$.

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